Abstract

In this work, we initiate the study of fault tolerant Max-Cut, where given an edge-weighted undirected graph $G = (V, E)$, the goal is to find a cut $S \subseteq V$ that maximizes the total weight of edges that cross $S$ even after an adversary removes $k$ vertices from $G$. We consider two types of adversaries: an adaptive adversary that sees the outcome of the random coin tosses used by the algorithm, and an oblivious adversary that does not. For any constant number of failures $k$ we present an approximation of $(0.878 - \epsilon)$ against an adaptive adversary and of $\alpha_{GW} \approx 0.8786$ against an oblivious adversary (here $\alpha_{GW}$ is the approximation achieved by the random hyperplane algorithm of [Goemans-Williamson J. ACM ’95]). Additionally, we present a hardness of approximation of $\alpha_{GW}$ against both types of adversaries, rendering our results (virtually) tight.

The non-linear nature of the fault tolerant objective makes the design and analysis of algorithms harder when compared to the classic Max-Cut. Hence, we employ approaches ranging from multi-objective optimization to LP duality and the ellipsoid algorithm to obtain our results.

1 Introduction

In this work, we initiate the study of fault tolerant Max-Cut. In the classic Max-Cut problem, we are given an undirected graph $G = (V, E)$ equipped with non-negative edge weights $w : E \to \mathbb{R}_+$. The goal is to find a cut $S \subseteq V$ that maximizes the total weight of edges that cross $S$. Max-Cut is one of Karp’s 21 NP-complete problems [37] and has been for close to three decades a case study for the introduction of new approaches both in the theory of algorithms and the complexity theory. Perhaps the two most prominent examples of the above are: (1) the random hyperplane rounding method of Goemans and Williamson for semi-definite programs [29], which yields an approximation of $\alpha_{GW} \approx 0.8786$ for Max-Cut; and (2) the Unique Games Conjecture of Khot [38]. The former has opened an entirely new area in the field of approximation algorithms with applications to a wide range of problems, e.g., Max-DiCut [26,42,44], Max-Bisection [5,53], Max-Agreement [19,56], Max-2SAT [26,42], Max-SAT [17], and Cut Norm [2], to name a few. The latter has been a dominant method for proving hardness of approximation results in the last two decades, e.g., the celebrated tight hardness for Max-Cut [39,45], and Vertex Cover [40].

Motivated by large scale real life systems, fault tolerant algorithms seek to find a solution to a given optimization problem that is resilient to failures of some parts of the input. The above can be intuitively formulated as a two step process: (1) the algorithm finds a solution to the problem at hand; and (2) an adversary removes parts of the input. The goal of the algorithm is that no matter which part of the input the adversary removes, the remaining solution after removal still retains some desired properties despite the
Typically, the focus of fault tolerance has been network design problems, e.g., BFS \cite{33,48,50,52} and spanners \cite{15,17,25,47,55}. Additional related algorithmic problems for which fault tolerant algorithms were studied include, e.g., single source reachability \cite{9,10}, connected dominating set \cite{18,59}, and facility location \cite{23,32,36,57}.

In this work, we initiate the study of fault tolerant Max-Cut, where the adversary can remove vertices from the graph (all edges touching the removed vertices are also deleted). Intuitively, fault tolerant Max-Cut can be seen as a two players game, in which one player (the algorithm) chooses a cut and the other player (the adversary) removes up to a prespecified number $k$ of vertices. The algorithm desires to maximize the total weight of edges crossing the cut, while the adversary aims to minimize the total weight of edges crossing the cut.

We study two types of adversaries. The first is an adaptive adversary that chooses which $k$ vertices to fail after seeing the cut the algorithm produces. Specifically, the adaptive adversary knows the input, how the algorithm operates, and if the algorithm is randomized, the adaptive adversary also knows the outcome of all random coin tosses used by the algorithm. The second type of adversary is an oblivious adversary. Similarly to the adaptive adversary, the oblivious adversary knows the input and how the algorithm operates. However, in contrast to the adaptive adversary, the oblivious adversary does not know the outcome of the random coin tosses used by the algorithm, in case the latter is randomized (equivalently, the oblivious adversary only knows the distribution over cuts the algorithm produces). Thus, the oblivious adversary is required to choose which $k$ vertices to fail without the knowledge of which cut was sampled. To the best of our knowledge only adaptive adversaries were studied in the fault tolerance literature.

**The Challenges.** The fault tolerant Max-Cut problem differs considerably from classic Max-Cut for several reasons. First, the structure of the solutions may be different. Specifically, there are instances for which an optimal solution to fault tolerant Max-Cut is not an optimal solution to classic Max-Cut, and vice versa. Furthermore, it might be the case that the ratio between the values of the optimal solutions is large or even unbounded.

Second, the application of known techniques (which can be successfully applied to Max-Cut) to fault tolerant Max-Cut imposes some obstacles that arise from the non-linear nature of the fault tolerant objective. For example, the random hyperplane rounding method of Goemans and Williamson cannot be analyzed in a straightforward manner as one is required to lower bound the expectation of the minimum value (over all possible actions of the adversary) of the cut the random hyperplane defines, as opposed to just the expected value of the cut the random hyperplane defines. Moreover, even analyzing the simplest known algorithm for Max-Cut, i.e., choosing a uniform random cut, requires great care (refer to Section \ref{sec:related} for further details). Hence, the design and analysis of algorithms for fault tolerant Max-Cut requires some new insights into the problem.

**1.1 Our Contributions**

**Adaptive Adversary.** When focusing on an adaptive adversary, our main result is an (almost) tight approximation of $0.878 - \epsilon$, for any constant number $k$ of failures and unweighted graphs. This is summarized in the following theorem (it is important to note that the constant in the theorem is slightly smaller than the Goemans-Williamson approximation factor $\alpha_{GW}$).

**Theorem 1.1.** For every constant $k > 0$ and $\epsilon > 0$, there is a polynomial time $(0.878 - \epsilon)$-approximation algorithm for fault tolerant Max-Cut on unweighted graphs against an adaptive adversary and $k$ faults.

Our algorithm is based on viewing fault tolerant Max-Cut against an adaptive adversary as a multi-objective optimization problem, where for every possible subset of $k$ vertices the adversary can fail, one can define a different objective. The goal is to maximize the worst, i.e., minimum, objective. This approach does not suffice, since all known results for the multi-objective variant of Max-Cut (formally known as Simultaneous Max-Cut \cite{12,13}) can handle only a constant number of objectives. In our case, even when a single failure is allowed, the number of objectives equals $n$. Hence, to overcome this difficulty, we incorporate local search into the above multi-objective approach to obtain the claimed result in Theorem 1.1.
Oblivious Adversary. When focusing on an oblivious adversary, our main result is a tight approximation of $\alpha_{GW}$ for any constant number $k$ of failures. However, in contrast to the adaptive adversary setting, this result holds for general weighted graphs and achieves the $\alpha_{GW}$-approximation guarantee exactly. This is summarized in the following theorem.

**Theorem 1.2.** For every constant $k > 0$, there is a polynomial time $\alpha_{GW}$-approximation algorithm for fault tolerant Max-Cut on general weighted graphs against an oblivious adversary and $k$ faults.

The approach we adopt for approximating fault tolerant Max-Cut against an oblivious adversary significantly differs from the approach taken against an adaptive adversary. Surprisingly, our algorithm is based on an approximation-preserving reduction from fault tolerant Max-Cut to the classic Max-Cut problem. This reduction uses LP duality alongside the ellipsoid algorithm and is achieved by presenting a suitable approximate dual separation oracle for a configuration LP that encodes the distribution over cuts that the algorithm produces.

**Hardness of Approximation.** We prove that fault tolerant Max-Cut in unweighted graphs, against both adaptive and oblivious adversaries, cannot be approximated better than $\alpha_{GW}$ without breaking well-known hardness assumptions. It is important to note that this settles the approximability of the oblivious adversary setting (see Theorem 1.2 above), and almost settles the approximability of the adaptive adversary setting (see Theorem 1.1 above) as the constant in Theorem 1.1 is slightly smaller than $\alpha_{GW}$.

**Theorem 1.3.** Assuming the Unique Games Conjecture and $NP \nsubseteq BPP$, there is no polynomial time $(\alpha_{GW} + \epsilon)$-approximation algorithm for fault tolerant Max-Cut in unweighted graphs, for any constant $\epsilon > 0$. This holds for both adaptive and oblivious adversaries.

**Simple Purely Combinatorial Algorithms.** While Theorem 1.1 provides an (almost) tight result against an adaptive adversary, and Theorem 1.2 provides a tight result against an oblivious adversary, the techniques we employ yield algorithms which are polynomial but not simple. For example, the work of [12] for approximating Simultaneous Max-Cut, an important ingredient in the design of our algorithm against an adaptive adversary, is based on SDP hierarchies and the running time is exponential in the number of objectives. In contrast, the classic Max-Cut problem admits some very simple and fast heuristics, e.g., choosing a random uniform cut. Thus, we also aim to study simple and purely combinatorial algorithms for fault tolerant Max-Cut.

We prove that fault tolerant Max-Cut does yield a simple purely combinatorial local search algorithm with a provable approximation guarantee against an adaptive adversary. Unfortunately, the classic local search for Max-Cut, that in each step moves a single vertex from one side of the cut to the other side, fails in the fault tolerant setting. Nonetheless, we prove that a local search that allows for a slightly richer family of local improvement steps suffices. This is summarized in the following theorem (refer to Section 3.2 for additional details).

**Theorem 1.4.** There is a purely combinatorial polynomial time $1/2$-approximation algorithm for fault tolerant Max-Cut on unweighted input graphs against an adaptive adversary and a single fault.

We further study how a uniform random cut performs against both types of adversaries, and prove that this performance depends on the type of the adversary. Specifically, for an oblivious adversary an approximation of $1/2$ is achieved, by a uniform random cut. However, this is not the case when considering an adaptive adversary, since we prove that a uniform random cut cannot achieve an approximation better than $1/4$.

**1.2 Related Work**

The weighted version of Max-Cut is one of Karp's NP-complete problems [37], and the unweighted version is also known to be NP-complete [27]. In general graphs, one cannot obtain an approximation factor better
than 16/17 for the undirected version, or better than 12/13 for the directed version, unless $P = NP$. The best known approximation for Max-Cut is the celebrated random hyperplane algorithm of Goemans and Williamson that obtains an approximation factor of roughly 0.8786 by rounding the natural semi-definite programming relaxation \cite{GW95}. This is the best approximation that one can achieve, assuming the Unique Games Conjecture of Khot \cite{K02} and $P \neq NP$.

The problem of fault tolerant Max-Cut against an adaptive adversary that we introduce in this paper can be viewed as a special case of Simultaneous Max-Cut, in which the input is a collection of $\tau$ weighted graphs on the same vertex set and the goal is to partition the vertices into two parts, such that the size of the cut is large in every given graph. In a straightforward manner, our problem would imply $\tau = \binom{n}{2}$, which is unacceptable since the known approximations for Simultaneous Max-Cut are for a constant number of instances only \cite{12,13}. Nonetheless, we do use the algorithm from \cite{12} to obtain an algorithm that achieves an approximation of 0.878 for fault tolerant Max-Cut against an adaptive adversary. The state-of-the-art for Simultaneous Max-Cut is a polynomial 0.878-approximation for any constant number of input graphs \cite{12}, which is nearly optimal since assuming the Unique Games conjecture, Simultaneous Max-Cut cannot be approximated better than $(\alpha_{GW} - \delta)$ (where $\delta \geq 10^{-5}$) \cite{11}.

One more notion of resilience is that of robust submodular maximization, see, e.g., \cite{6,46}. Given a submodular function $f$ and, e.g., a cardinality constraint $k$, a set $A$ is robust against $\tau$ failures if $A = \arg\max_{A \subseteq V, |A| \leq k} \min_{S \subseteq A, |S| \leq \tau} f(A - S)$, i.e., a subset of size at most $k$ that achieves the maximal value after at most $\tau$ elements are removed from the solution. Note that this notion of robustness differs from fault tolerance. The reason is that the failed elements are removed from the solution, as opposed to removed from the instance. Specifically, when considering the cut function of an undirected graph (which is submodular) the removal of a vertex from $S$ (as in robust) differs from removing the same vertex from the graph (as in fault tolerant).

Due to the importance of coping with failures, the fault tolerance of many additional fundamental problems has been extensively studied. Prime examples are replacement paths \cite{1,21,22,30,54}, BFS trees \cite{35,48,50,52}, spanners \cite{15,17,25,41,47,55}, connected dominating sets \cite{18,50}, and more \cite{8,10,11,23,32,36,57}.

Fault tolerance was also studied in the distributed setting, such as for BFS trees \cite{28}, MST \cite{28}, and spanners \cite{25,49}.

Paper Organization. Section \ref{sec:preliminaries} contains all required formal definitions and preliminary lemmas used throughout the paper. Section \ref{sec:adaptive} deals with the adaptive adversary, whereas Section \ref{sec:oblivious} deals with the oblivious adversary. Section \ref{sec:hardness} consists of the hardness results, and Section \ref{sec:uniform} deals with a uniform random cut.

## 2 Preliminaries

**Graph Notations.** We consider only edge-weighted graphs $G = (V, E, w)$ with positive integer weights $w_e$ assigned to the edges $e \in E$. By unweighted graphs we mean graphs with $w_e = 1$, for all $e \in E$. A cut $S$ in a graph $G = (V, E, w)$ is a subset of vertices $S \subseteq V$. We let $\delta(S, G) = \{ e \in E : |e \cap S| = 1 \}$ denote the set of all crossing edges in the graph $G$. The size or weight of a cut $S$, denoted by $C_{S, G}$, is the total weight of the crossing edges: $C_{S, G} = \sum_{e \in \delta(S, G)} w_e$. When $G$ is clear from the context, we use $C_S$ and $\delta(S)$.

For a set $F \subseteq V$ of vertices, the degree $d(F)$ of $F$ is the total weight of edges adjacent to $F$: $d(F) = \sum_{e \in E \cap F \times \bar{F}} w_e$. For a subset $F \subseteq V$ and cut $S \subseteq V$, the crossing degree $d_S(F)$ of $F$ is the total weight of edges adjacent to $F$ that cross $S$: $d_S(F) = \sum_{e \in \delta(S) \cap F \times \bar{F}} w_e$. We use $d(v)$ and $d_S(v)$, if $F = \{ v \}$. We also let $n = |V|$, $m = |E|$, and $\Delta = \max_{v \in V} d(v)$. Finally, we let $2^V$ and $\binom{V}{k}$ denote the collection of all and all size-$k$ subsets of $V$, respectively.

The Adaptive Adversary. We define the $k$-FT value of a cut against an adaptive adversary to be the minimal size of the cut, subsequent to a failure of any $k$ vertices. Formally, for a cut $S$ in a graph $G = (V, E, w)$ and a constant $k > 0$, the $k$-FT value of $S$ is defined as $\varphi(S, k, G) = \min_{F \subseteq \binom{V}{k}} C_{S - F, G - F}$. 


Definition 2.1 \((k\text{-AFTcut})\). Given an edge-weighted graph \(G = (V, E, w)\) and a number \(k \in \mathbb{N}\), a cut \(S\) is a \(k\)-adaptive fault tolerant cut, or \(k\text{-AFTcut}\) for short, if \(\varphi(S, k, G) = \max_{S' \subseteq V} \{\varphi(S', k, G)\}\).

We usually omit \(G\) and/or \(k\) from \(\varphi(S, k, G)\) when \(G\) is clear from the context and \(k = 1\). The Max-Cut problem, i.e., that of finding a cut with the largest size, corresponds to the special case \(k = 0\).

The Oblivious Adversary. We represent a randomized algorithm that finds a cut in a graph \(G = (V, E, w)\) by a probability distribution \(D\) over all possible cuts \(2^V\). For a distribution \(D\) over cuts, we define the \(k\)-FT value of \(D\) to be the minimal expected size of the cut, subsequent to the failure of any \(k\) vertices. Formally, for a graph \(G = (V, E, w)\), a distribution \(D\) over cuts and a constant \(k > 0\), we define the \(k\)-FT value of \(D\), denoted by \(\mu(D, k, G)\), as
\[
\mu(D, k, G) = \min_{F \subseteq V : |F| = k} \mathbb{E}_{S \sim D} [C_{S \setminus F, G \setminus F}].
\]

Definition 2.2 \((k\text{-OFTcut})\). Given an edge-weighted graph \(G = (V, E, w)\) and a number \(k \in \mathbb{N}\), a distribution \(D\) over all cuts \(2^V\) is a \(k\)-oblivious fault tolerant cut, or \(k\text{-OFTcut}\) for short, if
\[
\mu(D, k, G) = \max_{D' \subseteq 2^V} \{\mu(D', k, G)\}.
\]

Note that here we assume the adversary chooses the set \(F\) of faults deterministically; it easily follows from the linearity of expectation that the adversary always has a deterministic best choice – a subset that has the largest expected crossing degree.

Dissimilarity of \(AFT\text{cut} and Max-Cut\). In the two following observations we show that the problem of finding an \(AFT\text{cut}\) differs from finding a Max-Cut, that is, there exists a solution for Max-Cut which is not a solution for \(AFT\text{cut}\), and vice versa.

Observation 2.3. There exists a solution for Max-Cut in \(G\), which is not a solution for \(AFT\text{cut}\).

Proof. Consider a graph that consists two triangles that share a vertex, and label the shared vertex by 1 (see Figure 1). It holds that \(\{1\}\) is a Max-Cut. In addition, \(\varphi(\{1\}) = 0\) because when vertex 1 fails there are no crossing edges. There is a better solution for \(AFT\text{cut}\), for example \(\{1, 2, 5\}\). It holds that \(\varphi(\{1, 2, 5\}) = 2\), thus \(\{1\}\) is not an \(AFT\text{cut}\) even tough it is a Max-Cut. Note that we can generalize the example by having \(t\) triangles that share a vertex, labeled by 1. It then holds that \(\{1\}\) is a Max-Cut, while \(\varphi(\{1\}) = 0\). However, a cut that consists of 1 and an single vertex from each triangle, is an \(AFT\text{cut}\) wit FT value \(t\).

![Figure 1: A Max-Cut that is not an AFTcut](image)

Observation 2.4. There exists a solution for \(AFT\text{cut}\), which is not a solution for Max-Cut in \(G\).

![Figure 2: An AFTcut that is not a Max-Cut](image)
Proof. Let $G$ be a 5-path, $1, 2, 3, 4, 5$, with an additional leaf 6 connected to vertex 4 (Figure 2). It holds that $\{1, 3, 4\}$ is an AFTcut. $C_{\{1,3,4\}} = 4$, while there exists larger cuts, for example $C_{\{2,4\}} = 5$. Thus, $\{1, 3, 4\}$ is not a Max-Cut even tough it is an AFTcut.

Greedy moves and stable cuts. We assume here that we are given an unweighted graph $G = (V, E)$. A key observation in our algorithms against an adaptive adversary is that any solution can be transformed into another one where each vertex contributes many of its edges to the cut. If a vertex contributes too little, we can just move it to the opposite side of the cut: while this could increase the crossing degree of some vertices (negative contribution to the FT value), it increases the cut size by more, giving a positive net contribution to the FT value. We prove this formally in Lemma 2.6 after some formal definitions.

For every $v \in V$ and $S \subseteq V$, let $S \oplus v$ denote the cut obtained from $S$ by switching $v$ to its opposite side, that is, $S \oplus v = S - v$, if $v \in S$, and $S \oplus v = S \cup \{v\}$, otherwise. Given a subset $S \subseteq V$, a constant $k \in \mathbb{N}$, and a vertex $v \in V$, we say that replacing $S$ with $S \oplus v$, i.e., moving $v$ to its opposite side w.r.t. $S$, is a k-greedy step if $d_{S}(v) \leq (d(v) - k)/2$. A cut $S$ is k-stable if it has no k-greedy step, that is, for every $v \in V$, it holds that $d_{S}(v) > (d(v) - k)/2$. For $k = 1$, we use stable instead of 1-stable.

Observation 2.5. For every cut $S$ and a vertex $v$, it holds that $C_{S \oplus v} - C_{S} = d_{S}(v) - d_{S}(v)$.

Lemma 2.6. Let $v \in V$ be a vertex, $S \subseteq V$ be a cut, and $k > 0$ be an integer, such that $d_{S}(v) \leq (d(v) - k)/2$; then $C_{S \oplus v} \geq C_{S} + k$, and $\varphi(S \oplus v, k) \geq \varphi(S, k)$.

Proof. Assume, without loss of generality, that $v \in S$ (otherwise, we swap $S$ and $V - S$). Observation 2.5 implies that $C_{S \oplus v} \geq C_{S} + k$, since $d_{S}(v) + k \leq d(v) - d_{S}(v) / d_{S}(v)$.

For the second claim, we show that for every $F \in \binom{V}{k}$, $C_{S \oplus v, F} - C_{S, F} \leq C_{S \oplus v, F, G} - C_{S, F, G}$. Assume that $v \notin F$, as otherwise $S - F = S \oplus v - F$, and the claim holds trivially. Recall that $C_{S \oplus v} \geq C_{S} + k$. In addition, $d_{S}(F) \leq d_{S}(F) + k$, since for every $u \in F$, at most one crossing edge is added to the cut (the edge $\{u, v\}$). Putting those together, we have that: $C_{S \oplus v, F} - C_{S} - d_{S}(F) \leq C_{S \oplus v} - d_{S}(F) = C_{S \oplus v, F, G} - C_{S, F, G}$. Since this holds for every $F$, we have that $\varphi(S \oplus v, k) \geq \varphi(S, k)$.

By repeatedly applying a k-greedy step to a cut, we keep increasing the cut value, while not decreasing the k-FT value; thus, after at most $m$ greedy steps, we have a k-stable cut with a k-FT value at least as good as the original one. We let STABILIZECUT$(G, S, k)$ denote this procedure, which takes as input a graph $G$, a cut $S$ in $G$, and a number $k$, then starting with $S$, repeatedly applies a (arbitrary) k-greedy step, while there is one, and returns the obtained k-stable cut. The following corollary follows from the reasoning above (the second claim follows by applying STABILIZECUT to an optimal k-AFTcut).

Corollary 2.7. Let $S$ be a cut in graph $G = (V, E)$, and $k$ be a positive integer. Let $S' = \text{STABILIZECUT}(G, S, k)$. It holds that $S'$ is k-stable, $C_{S'} \geq C_{S}$ and $\varphi(S', k) \geq \varphi(S, k)$. In particular, every unweighted graph $G = (V, E)$ has a k-stable optimal k-AFTcut.

3 Fault Tolerance Against an Adaptive Adversary

3.1 A 0.878-Approximation for Multiple Faults

In this section, we give a $(0.878 - \epsilon)$-approximation algorithm for k-AFTcut on unweighted graphs, for constants $k, \epsilon > 0$. A core tool that we use in our algorithm is an algorithm for the Simultaneous Max-Cut problem, where given several graphs defined over the same vertex set, the goal is to find a cut that is large for all graphs simultaneously. A 0.878-approximation algorithm for this problem with a constant number of graphs has been given in [12]. The algorithm is based on semidefinite programming techniques.

The main idea behind our algorithm is to separate a constant number of “heavy” (high-degree) vertices for which the following holds; given a cut which is large subsequent to any failure of $k$ heavy vertices, the cut is large even if light (non-heavy) vertices fail as well. For such a heavy set, a good approximation for Simultaneous Max-Cut on the instances obtained by removing each possibility of $k$ heavy vertices from $G$. 


should be a good approximation for \( k\text{-AFTcut} \) on \( G \). We give a greedy algorithm that selects the set of heavy vertices. We then consider two cases. We show that if the heavy vertices do not cover most of the edges in the graph (the “non-shallow” case), then an approximate solution for Simultaneous Max-Cut with respect to the heavy set gives an approximate solution for \( k\text{-AFTcut} \). Otherwise (the “shallow” case), we identify a set of “super-heavy” vertices, which is shown to fail in any near-optimal solution. Therefore, finding a near-optimal solution for the original graph reduces to finding a near-optimal solution on the graph remaining by removing the “super-heavy” vertices. We show that it can be solved via brutforce, or by finding a good solution to Max-Cut (e.g., obtained via \cite{29}). We prove the following theorem.

**Theorem 1.1.** For every constant \( k > 0 \) and \( \epsilon > 0 \), there is a polynomial time \((0.878 - \epsilon)\)-approximation algorithm for fault tolerant Max-Cut on unweighted graphs against an adaptive adversary and \( k \) faults.

Before proceeding to the algorithm, we introduce the Simultaneous Max-Cut framework.

**Definition 3.1** (Simultaneous Max-Cut). Let \( V \) be a vertex set. We are given \( k \) edge-weighted graphs, \( G_i = (V, E_i), i = 1, \ldots, k, \) on the vertex set \( V \), where the weights are normalized, so that \( \sum_{e \in E_i} w_e = 1 \), for each \( i \). In the (Pareto) Simultaneous Max-Cut problem, given the graphs \( G_i \) together with thresholds \( c_i \in [0,1] \), the goal is to find a cut \( S^* \subseteq V \) such that \( C_{S^*,G_i} \geq c_i \), for every \( i \). We say that an algorithm is an \( \alpha \)-approximation algorithm for the problem if for every input \( G_i, c_i, i = 1, \ldots, k \), where there exists a cut \( S^* \) such that \( C_{S^*,G_i} \geq c_i \), for every \( i \), the algorithm returns a cut \( \tilde{S} \) such that \( C_{\tilde{S},G_i} \geq \alpha c_i \), for every \( i \).

**Theorem 3.2.** \cite{12} For every constant \( k \geq 1 \) and parameter \( n \geq 1 \), there is a polynomial-in-\( n \) algorithm that computes an \( \alpha_{SMC} \)-approximate solution to any Simultaneous Max-Cut instance with \( k \) weighted graphs on a vertex set of size \( n \), in which all non-zero edge-weights are lower-bounded by \( \exp(n^{-\epsilon}) \), for constants \( k \) and \( c \), and \( \alpha_{SMC} = 0.878 \).

We apply the Simultaneous Max-Cut framework for unweighted graphs \( G_i \). We let \textsc{SimultaneousMC} denote the algorithm that gets as input a constant number of unweighted graphs \( G_i, i = 1, \ldots, k \), and returns a cut \( \tilde{S} \) with the following property: for every cut \( S^* \) and number \( c \) such that \( C_{S^*,G_i} \geq c \), for all \( i \), it holds that \( C_{\tilde{S},G_i} \geq \alpha_{SMC} \cdot c \), for all \( i \). This can be achieved by combining the algorithm given in Theorem 3.2 (by appropriately scaling the edge-weights and the thresholds) with a binary search on \( c \).

In addition to the Simultaneous Max-Cut algorithm, we use the \( \alpha_{GW} \)-approximation for Max-Cut due to Goemans and Williamson \cite{29}, for \( \alpha_{GW} \approx 0.8786 \). We use Goemans-Williamson (with input \( G \)) to denote this algorithm. Note that the actual value of the approximation factor \( \alpha_{SMC} \) is slightly larger than 0.878 but is less than \( \alpha_{GW} \).

**The Main Algorithm.** The inputs to the algorithm (see the pseudocode in Algorithm 1) are an unweighted graph \( G \), and parameters \( k \) (number of faults) and \( \epsilon \) (precision). First, it computes the set \( H \) of heavy vertices via the subroutine \textsc{HeavyVertices}, then applies \textsc{SimultaneousMC} on a collection \( \{G_{-F} : F \in \binom{H}{k} \} \) of subgraphs containing one subgraph for every failure of \( k \) heavy vertices. The following notation is used: for a subset \( F \subseteq V \) of vertices, we let \( G_{-F} = (V, E_{-F}) \), where \( E_{-F} = \{ e \in E : e \cap F = \emptyset \} \). Note that in \( G_{-F} \), we do not remove the vertices of \( F \) from the graph, as opposed to \( G - F \), but only the edges adjacent to \( F \).

The pair \( (H, \tilde{S}) \) is shallow if all vertices in \( V - H \) have degree at most \( 3k \), and there are \( k \) vertices in \( H \) whose removal reduces the weight of \( \tilde{S} \) below \( 3k^2/\epsilon \). To state this formally, let us introduce a notation that will be useful later too. For a cut \( S \subseteq V \), we use \( C_{S,kH} \) to denote the smallest size of the cut after the failure of any \( k \) vertices from \( H \), i.e., \( C_{S,kH} = \min \{ C_{S,F,G_{-F}} : F \in \binom{H}{k} \} \). Thus, \((H, \tilde{S})\) is shallow if we have \( \max_{v \in V-H} d(v) \leq 3k \) and \( C_{\tilde{S},kH} < 3k^2/\epsilon \). If \((H, \tilde{S})\) is not shallow, the algorithm simply returns \( \tilde{S} \). Otherwise, we recompute the cut via \textsc{ShallowFtcut}, using alternative methods.

The proof of Theorem 1.1 is split into two parts, addressing shallow and non-shallow cases separately. The running time is dominated by Simultaneous Max-Cut. Before specifying further details, let us mention how the proof follows from the main lemmas addressing those cases.
Algorithm 1: \((\alpha_{SMC} - \epsilon)\)-approximation for \(k\)-AFTcut

1. Input: \(G = (V, E), k, \epsilon\)
2. Output: \((\alpha_{SMC} - \epsilon)\)-approximation for \(k\)-AFTcut
3. \(H \leftarrow \text{HEAVYVERTICES}(G, k, \epsilon)\)
4. \(\tilde{S} \leftarrow \text{SIMULTANEOUSMC}((G_{\neq F} : F \in \binom{H}{k}))\)
5. If \((H, \tilde{S})\) is shallow then
6. Return \(\text{SHALLOWFTCUT}(G, H, \tilde{S}, k, \epsilon)\)
7. Else
8. Return \(\tilde{S}\)

Proof of Theorem 1.1. Let \(G\) be a graph and let \(S^*\) be an optimal \(k\)-AFTcut on \(G\). Let \(\tilde{S}\) be the output of Algorithm 1 on \(G, k, \epsilon\). We show that \(\varphi(\tilde{S}, k) \geq (\alpha_{SMC} - \epsilon) \cdot \varphi(S^*, k)\). Lemma 3.6 provides this for the non-shallow case, while Lemma 3.10 provides it in the shallow case. The algorithm is indeed polynomial, since the sub-routines are such, and the input to \text{SIMULTANEOUSMC} consists of \(\binom{|H|}{k} = O(k/\epsilon)^k = O(1)\) subsets, where \(|H| = O(k^2/\epsilon)\) is proven in Lemma 3.3.

Algorithm 2: \text{HEAVYVERTICES}

1. Input: \(G = (V, E), k, \epsilon\)
2. Output: \(H \subseteq V\), the set of heavy vertices
3. Let \(v_1, \ldots, v_n\) be an ordering of vertices by non-increasing degree
4. \(\sigma \leftarrow 0, i \leftarrow 1, H \leftarrow \{v_1, \ldots, v_k\}\)
5. While \(d(v_{k+i}) > (\epsilon \cdot \alpha_{SMC}/k) \cdot \sigma\) and \(d(v_{k+i}) > 3k\) do
6. \(\sigma \leftarrow \sigma + (d(v_{k+i}) - 3k)/4\)
7. \(H \leftarrow H \cup \{v_{k+i}\}\)
8. \(i \leftarrow i + 1\)
9. Return \(H\)

The selection of heavy vertices (Algorithm 2) is done by a simple greedy procedure, where we sequentially select vertices in the heavy set \(H\) in a non-increasing order by degree. The selection stops either when the remaining vertices \((V - H)\) have a small degree (at most \(3k\)) or when \(H\) has sufficiently many incident edges (used in Lemma 3.6). By Corollary 2.7, any cut can be transformed into one with a similar \(k\)-FT value, where every vertex \(v\) has crossing degree at least \((d(v) - k)/2\), and at least \((d(v) - 3k)/2\), after \(k\) failures. Thus, heavy vertices are guaranteed to contribute \(\sigma\) in the “stable version” of every cut. The degree constraint ensures that we do not select vertices that are unnecessary, according to this logic, which helps us keep the size of \(H\) bounded.

In the analysis below, we often use the notation \(\sigma_i\) to denote the value of \(\sigma\) after the \(i\)-th iteration.

**Lemma 3.3.** Algorithm 2 terminates within \(t = 4(k^2 + k)/(\epsilon \cdot \alpha_{SMC})\) iterations. In particular, \(|H| \leq t + k\).

Proof: If \(d(v_{k+i}) \leq 3k\), then by the condition in Line 5, the algorithm terminates before the \(t\)-th iteration; therefore, assume \(d(v_{k+i}) > 3k\). For every \(i \leq t\), after the \(i\)-th iteration, it holds that \(\sigma_i = \sum_{j=1}^{i} (d(v_{k+j}) - 3k)/4\); thus, after \(t\) iterations,

\[
\sigma_t = \sum_{j=1}^{t} \frac{d(v_{k+j}) - 3k}{4} \geq t \cdot \frac{d(v_{k+i}) - 3k}{4} = \frac{3k^2 + k}{\epsilon \cdot \alpha_{SMC}} \cdot (d(v_{k+i}) - 3k)
\]

\[
= \frac{k}{\epsilon \cdot \alpha_{SMC}} \cdot d(v_{k+i}) + \frac{3k^2}{\epsilon \cdot \alpha_{SMC}} \cdot d(v_{k+i}) - \frac{3k^2(3k + 1)}{\epsilon \cdot \alpha_{SMC}} \geq \frac{k}{\epsilon \cdot \alpha_{SMC}} d(v_{k+i})
\]
where in the first inequality, we use the fact that the vertices are processed in a non-increasing order of degrees, and in the last inequality, we use the assumption that \(d(v_{k+1}) \geq 3k+1\). It follows that \(d(v_{k+1}) \leq (\alpha_{\text{SMC}}/k)\sigma_t\), and using \(d(v_{k+1}) \leq d(v_{k+1})\), we get that the algorithm terminates within the first \(t\) iterations, by the condition in Line 5.

\[\square\]

### 3.1.1 The Non-shallow Case

Recall that in the non-shallow case, the cut \(\tilde{S}\) and the set \(H\) of heavy vertices are such that either \(d_{\text{max}} = \max_{v \in V-H} d(v) > 3k\) or \(C_{S_{\text{smc}}-k \times H} \geq 3k^{2/\epsilon}\) holds. Let \(S_{\text{smc}}^*\) be an optimal solution of Simultaneous Max-Cut for the instances \(\{G_{-F} : F \in \binom{H}{k}\}\). Let \(S_{fi}^*\) be an optimal solution for \(k\)-AFTcut on \(G\).

Let us begin with an observation connecting the three cuts \(\tilde{S}, S_{\text{smc}}^*, \) and \(S_{fi}^*\).

**Observation 3.4.** It holds that \(C_{\tilde{S} \times H} \geq \alpha_{\text{SMC}} \cdot C_{S_{\text{smc}} \times H} \geq \alpha_{\text{SMC}} \cdot \phi(S_{fi}^*, k)\).

**Proof.** The first inequality holds because \(\tilde{S}\) is an \(\alpha_{\text{SMC}}\)-approximation to the Simultaneous Max-Cut problem as described in Theorem 3.2 while the second one holds since by definition, \(S_{\text{smc}}^*\) is the cut \(S\) optimizing \(C_{S \times H}\), and \(S_{fi}^*\) is the one optimizing \(C_{S \times H}\).

We also use the following lower bound on \(C_{S_{\text{smc}} \times H}\) in terms of node degrees, in order to show that the degree of light (non-heavy) vertices is small in comparison with the cut size even after a heavy failure, implying that light vertex faults can be tolerated.

**Lemma 3.5.** \(C_{S_{\text{smc}} \times H} \geq \sum_{i=1}^{n-k} (d(v_{k+1}) - 3k)/4\) where \(v_1, \ldots, v_n\) are sorted by degree, in descending order, and \((x) = \max\{x, 0\}\) for any argument \(x\).

**Proof.** By Corollary 2.7 there is an optimal solution \(S \subseteq V\) for \(k\)-AFTcut that satisfies \(d_S(v) > (d(v) - k))/2\), for every \(v \in V\). Let \(F \in \binom{H}{k}\). Since every vertex \(v \notin F\) has at most \(k\) neighbors in \(F\), we have, in \(G - F\), that \(d_{S-F}(v) > (d(v) - k)/2 - k = (d(v) - 3k)/2\), and also \(d_{S-F}(v) \geq 0\). Hence, we have

\[
C_{S-F,G-F} = \sum_{v \in V-F} d_{S-F,G-F}(v)/2 \geq \frac{1}{2} \sum_{v \in V-F} (d(v) - 3k)/4 \geq \frac{1}{4} \sum_{i=1}^{n-k} (d(v_{k+1}) - 3k)/4 ,
\]

where the last inequality holds by the assumption on the ordering of vertices (and since \(|V - F| = n - k|\)). Since \(F\) is an arbitrary \(k\)-subset of \(H\), we conclude that \(C_{S \times H} \geq \sum_{i=1}^{n-k} (d(v_{k+1}) - 3k)/4\), and the claim now follows from \(C_{S_{\text{smc}} \times H} \geq C_{S \times H}\) (by the definition of \(S_{\text{smc}}^*\)).

We are now ready to prove that in the non-shallow case, \(\tilde{S}\) is a \((\alpha_{\text{SMC}} - \epsilon)\)-approximation for \(k\)-AFTcut.

**Lemma 3.6.** If \((H, \tilde{S})\) is not shallow, then it holds that \(\phi(\tilde{S}, k) \geq (\alpha_{\text{SMC}} - \epsilon) \phi(S_{fi}^*, k)\), for an optimal \(k\)-AFTcut \(S_{fi}^*\).

**Proof.** Note that in this case, we have either \(d_{\text{max}} = \max_{v \in V-H} d(v) > 3k\) or \(C_{\tilde{S} \times H} \geq 3k^{2/\epsilon}\). Consider an arbitrary \(F \in \binom{V}{k}\). It suffices to show that \(C_{\tilde{S} \times F,G-F} \geq (1 - \epsilon) \alpha_{\text{SMC}} \cdot \phi(S_{fi}^*, k)\). Let \(H' = F \cap H\) and \(L' = F \cap (V - H)\) be the heavy and light (non-heavy) vertices in \(F\), respectively. Observe that \(C_{\tilde{S} \times F,G-F}\) is obtained from \(C_{\tilde{S} \times H',G-H'}\) by removing the set \(L'\) of at most \(k\) light vertices. Since each vertex in \(L'\) has degree at most \(d_{\text{max}}\),

\[
C_{\tilde{S} \times F,G-F} \geq C_{\tilde{S} \times H',G-H'} - kd_{\text{max}} .
\]

On the other hand, we have, from Observation 3.4 (and using \(|H'| \leq k|\) that

\[
C_{\tilde{S} \times H',G-H'} \geq C_{\tilde{S} \times k \times H} \geq \alpha_{\text{SMC}} \cdot C_{S_{\text{smc}} \times k \times H} \geq \alpha_{\text{SMC}} \cdot \phi(S_{fi}^*, k) .
\]

In the following, we show that \(d_{\text{max}} \leq (\epsilon/k)C_{\tilde{S} \times H',G-H'}\), which implies the claim by combining 1 and 2.

9
If \( d_{\text{max}} \leq 3k \) and \( C_{S-k^*H} \geq 3k^2/\epsilon \), then the claim holds, since \( C_{S-H',G-H'} \geq C_{S-k^*H} \geq (k/\epsilon)d_{\text{max}} \). Thus, we may henceforth focus on the case \( d_{\text{max}} > 3k \). Using (2) and Lemma 3.7, we have

\[
C_{S-H',G-H'} \geq \alpha_{\text{SMC}} \cdot C_{S_{smc}-k^*H} \geq \alpha_{\text{SMC}} \cdot \sum_{i=1}^{n-k} (d(v_{k+i}) - 3k)/4.
\]

Let \( t \) be the number of iterations after which Algorithm 2 terminates. Since \( d_{\text{max}} > 3k \), it follows from the algorithm description that the vertex \( v_{k+t+1} \) satisfies \( d(v_{k+t+1}) \leq (\epsilon/\alpha_{\text{SMC}}/k)\sigma_t \) and that \( d_{\text{max}} = d(v_{k+t+1}) \). Hence \( d_{\text{max}} \leq (\epsilon/\alpha_{\text{SMC}}/k)\sigma_t \). On the other hand, we have

\[
\sigma_t = \sum_{i=1}^{t} (d(v_{k+i}) - 3k)/4 \leq \sum_{i=1}^{n-k} (d(v_{k+i}) - 3k)/4 \leq (1/\alpha_{\text{SMC}})C_{S-H',G-H'},
\]

where we used (3). This gives us the bound \( d_{\text{max}} \leq (\epsilon/\alpha_{\text{SMC}}/k)\sigma_t \), as claimed, which completes the proof.

3.1.2 The Shallow Case

Recall that \( \tilde{H} = \{v \in V | (d(v) - 3k)/2 > C_{S-k^*H}/\alpha_{\text{SMC}}\} \), \( G_R = G - \tilde{H} \) is the graph obtained by removing the super-heavy vertices. We let \( m_R \) be the number of edges in \( G_R \), \( n_R \) be the number of non-isolated vertices in \( G_R \), and \( \ell = 6k^2/(\alpha_{\text{SMC}}) + 3k \) be a parameter. We show that the algorithm SHALLOWFTCUT, as described in Algorithm 3, returns a cut that in the shallow case is a \( (\alpha_{\text{SMC}} - \epsilon) \)-approximation.

**Algorithm 3: SHALLOWFTCUT**

1. **Input:** \( G = (V, E) \), \( H \), \( \tilde{S} \), \( k \), \( \epsilon \)
2. **Output:** Cut \( \tilde{S} \subseteq V \)
3. if \( m_R < 2\ell/\alpha_{\text{SMC}} \) then
   4. for every \( S' \subseteq V_R \) (\( V_R \) is of constant size) do
      5. Compute \( \varphi(S', k - |\tilde{H}|) \)
      6. \( \tilde{S} \leftarrow \arg \max_{S' \subseteq V_R} \varphi(S', k - |\tilde{H}|) \)
   7. else
      8. \( \tilde{S} \leftarrow \text{GOEMANS-WILLIAMSON}(G_R) \)
9. while \( \exists v \in \tilde{H} \) such that \( d_{\tilde{S}}(v) \leq (d(v) - k)/2 \) do
10. \( \tilde{S} \leftarrow \tilde{S} \oplus v \)
11. return \( \tilde{S} \)

Let us begin with two observations, which show that \( \tilde{H} \) is indeed small and is contained in every worst-case fault set in stable cuts, and that the vertices outside \( \tilde{H} \) have small degree, bounded by \( \ell \).

**Lemma 3.7.** Let \( S \subseteq V \) be a cut such that \( d_{S}(v) \geq (d(v) - k)/2 \), for every \( v \in \tilde{H} \). If \( F \in \binom{V}{k} \) is such that \( C_{S,F,G-F} = \varphi(S,k) \), then \( \tilde{H} \subseteq F \). In particular, \( |\tilde{H}| \leq k \).

**Proof.** Let \( v \in \tilde{H} \), and assume, towards a contradiction, that \( v \notin F \). We have \( d_{S}(v) \geq (d(v) - k)/2 \), and since \( |F| \leq k \), \( d_{S,F,G-F}(v) \geq (d(v) - k)/2 - k = (d(v) - 3k)/2 \). Since \( v \notin F \), \( C_{S,F,G-F} \geq d_{S,F,G-F}(v) \); hence,

\[
C_{S,F,G-F} \geq (d(v) - 3k)/2 > C_{S-k^*H}/\alpha_{\text{SMC}} \geq C_{S_{smc}-k^*H} \geq \varphi(S_{ft}^*, k) .
\]

where we use \( v \in \tilde{H} \) in the second inequality, and Observation 3.4 in the last two inequalities. This implies that \( \varphi(S,k) = C_{S,F,G-F} > \varphi(S_{ft}^*, k) \), which is a contradiction to the definition of \( S_{ft}^* \).

**Observation 3.8.** \( d(v) \leq \ell \) holds for all \( v \in V - \tilde{H} \).
**Proof.** For every \( v ∈ \tilde{H} \), it holds that \((d(v)−3k)/2 ≤ C_{S−k*H}/α_{SMC}\), which implies that \(d(v) ≤ 2C_{S−k*H}/α_{SMC} + 3k < 6k²/(α_{SMC}ε) + 3k = ℓ\), since, by our assumption, \(C_{S−k*H} < 3k²/ε\).

Let \( \tilde{S} \) be the output of Algorithm 3. First, we show that \( \tilde{S} \) is a good solution in \( G_R \). Then, we prove the main claim of this subsection, that is, that \( \tilde{S} \) is a \((α_{SMC}−ε)\)-approximation for \( k−AFTcut \) in the shallow case.

**Lemma 3.9.** If \( d_{max} ≤ 3k \) and \( C_{S−k*H} < 3k²/ε \), then \( \tilde{S} \) is a \((1−ε)α_{SMC}\)-approximation for \( k−|\tilde{H}|\)-\( AFTcut \) in \( G_R \).

**Proof.** Indeed, in the case \( m_R < 2kℓ/(α_{SMC}ε) \), \( \tilde{S} \) is an optimal solution for \( k−|\tilde{H}|\)-\( AFTcut \) by the description of the algorithm, and the claim follows. If \( m_R ≥ 2kℓ/(α_{SMC}ε) \), note that \( C_{S−k,R,G−R} ≥ α_{GW}(m_R/2) ≥ kℓ/ε \), since \( \tilde{S} \) is an \( α_{GW} \)-approximation for Max-Cut in \( G_R \). By Observation 3.8, \( d(v) ≤ ℓ \) holds for each vertex in \( V − \tilde{H} \). If \( k−|\tilde{H}| \) vertices fail, the cut size is still at least \( C_{S−k,R,G−R} − kℓ ≥ (1−ε)C_{S−k,R,G−R} \). The claim then follows from the fact that the optimal FT value is bounded by the optimal Max-Cut size, \( \tilde{S} \) is an \( α_{GW} \)-approximation for Max-Cut in \( G_R \), and \( α_{GW} ≥ α_{SMC} \).

**Lemma 3.10.** If \( (H,\tilde{S}) \) is shallow, then it holds that \( ϕ(\tilde{S},k) ≥ (α_{SMC}−ε)⋅ϕ(S_{f1},k) \), for an optimal \( k-AFTcut \) \( S_{f1} \).

**Proof.** Note that in this case, \( d_{max} ≤ 3k \), and \( C_{S−k*H} < 3k²/ε \). Let \( S^* \) be an optimal solution for \( k-AFTcut \) on \( G \) such that for every \( v ∈ V \), \( d_S(v) ≥ (d(v) − k)/2 \); such cut exists, by Corollary 2.7. Note that \( \tilde{S} \) satisfies \( d_S(v) ≥ (d(v) − k)/2 \), for every \( v ∈ \tilde{H} \). By Lemma 3.7, it holds that \( \tilde{H} \) belongs to every worst-case failure set for those cuts, that is, if \( F^* ∈ (S^*) \) is such that \( C_{S−F^*,G−F^*} = ϕ(S^*,k) \), then \( \tilde{H} ∈ F^* \), and similarly, if \( F ∈ (S) \) is such that \( C_{S,F,G−F} = ϕ(S,k) \), then \( \tilde{H} ∈ F \). We can conclude that \( ϕ(S^*−\tilde{H},k−|\tilde{H}|,G_R) = ϕ(S^*,k) \), and similarly, \( ϕ(\tilde{S}−\tilde{H},k−|\tilde{H}|,G_R) ≥ (1−ε)α_{SMC}⋅ϕ(S^*,k−|\tilde{H}|,G_R) \). Combining these together, we see that \( ϕ(\tilde{S},k) ≥ (1−ε)α_{SMC}⋅ϕ(S^*,k) ≥ (1−ε)α_{SMC}−ε)⋅ϕ(S_{f1},k) \).

### 3.2 A Combinatorial 1/2-Approximation for a Single Fault

In the case of a single fault, we have the following result, that is, a simple and efficient 1/2-approximation for the case of a single fault. Moreover, we show that an FT value of \((m − Δ)/2\) can be achieved, for \( Δ ≥ 3 \), while \( m − Δ \) is an (easy) upper bound.

**Theorem 1.4.** There is a purely combinatorial polynomial time 1/2-approximation algorithm for fault tolerant Max-Cut on unweighted input graphs against an adaptive adversary and a single fault.

**The Challenge.** In the discussion below, we call a vertex \( v \) critical for a cut \( S \) if \( C_{S−v,G−v} = ϕ(S) \).

It is well-known (and easy to show) that every stable cut is a 1/2-approximate Max-Cut. This even holds for \( AFTcut \), with \( Δ = 2 \) (see Lemma 3.14). However, in general, while we know that greedy steps (moving a vertex \( v \) with \( d(v) < d_S(v)/2 \)) never decrease the FT value (Lemma 2.6), a stable cut can be a poor approximation for \( AFTcut \). Consider, for example, a graph that consists of \( ℓ \) triangles with a single common vertex \( u \). Note that \( d(u) = Δ = 2\ell \), \( d(v) = 2 \), for every \( v ≠ u \), and \( m = 3\ell \). The cut \( S^* = \{u\} \) is a stable cut, with \( ϕ(S^*) = 0 \). In order to transform \( S^* \) into a 1/2-approximation, we have to decrease the crossing degree of the critical vertex \( u \) without decreasing the size of the cut. This can be done by moving a neighbor \( v \) of \( u \) from the opposite side of the cut, since \( d_{S^*}(v) = d(v)/2 \).

In general, moving such vertex \( v \) (which we call a neutral move below) does not change the size of the cut, and decreases the crossing degree of \( u \). Nevertheless, it does not always imply that the FT value increases, as there can be an additional critical vertex \( u' \) in \( S \) that is not affected, or that moving \( v \) creates a new critical vertex \( u'' \) with the same crossing degree as \( u \).
Our algorithm is based on some key structural properties of stable cuts that we prove. Essentially, we show that any given cut \( S \) with FT value less than \((m - \Delta)/2\) either admits a greedy move, or a neutral move followed by a greedy move, or a neutral move that increases the FT value (see Lemma 3.16). Our algorithm is then a repeated application of such steps until the cut has the desired FT value; thus, it can be seen as a local search over two-move combinations, for maximizing the sum of the cut size and FT value.

Our key technical observation is that in a balanced cut \( S \) with an FT value less than \((m - \Delta)/2\), the critical vertex is unique. Moreover, letting \( x_S(v) = d_S(v) - d(v)/2 \) denote the excess contribution of a vertex \( v \) to the cut, it holds for the critical vertex \( u \) that \( x_S(u) = \sum_{v \in u} x_S(v) + \Delta - d(u) \) (see Lemma 3.13). Note that in a stable cut \( S \), \( x_S(v) \) is a non-negative multiple of 1/2, for all \( v \). In most typical cases (e.g., when \( d(u) < \Delta \), or when there are not too few nodes \( v \) with \( x_S(v) > 0 \)), the inequality above quickly gives us the properties we claimed. However, covering all cases turns out to be quite tedious (see Lemma 3.16).

**Outline of the Algorithm.** We give the pseudocode of the algorithm in Algorithm 4. If \( \Delta \leq 2 \) then the algorithm returns an arbitrary stable cut. For \( \Delta > 2 \), the algorithm initializes a solution \( \tilde{S} \) to be the empty set, and then updates it in iterations, until \( \varphi(\tilde{S}) \geq (m - \Delta)/2 \). In each iteration, the algorithm chooses a vertex \( v \) and moves it to the other side of the cut, as follows. First, if there is a vertex \( v \) such that \( \varphi(\tilde{S} \oplus v) \geq (m - \Delta)/2 \) then the algorithm moves \( v \). We call this a type-0 step. Note that after applying a type-0 step, the algorithm terminates. Otherwise, if there is a vertex \( v \) with \( d_S(v) < d(v)/2 \), then the algorithm moves it to the other side of the cut. This step is called a type-1 step. Otherwise, if there is a vertex \( v \) with \( d_S(v) = d(v)/2 \) such that \( \varphi(\tilde{S} \oplus v) \geq \varphi(\tilde{S}) \) and a type-1 step can be applied to \( \tilde{S} \oplus v \), then the algorithm moves \( v \) to the other side of \( \tilde{S} \). This is called a build-up step. Finally, if none of the above conditions hold, the algorithm takes a vertex \( v \) with \( d_S(v) = d(v)/2 \) that satisfies \( \varphi(\tilde{S} \oplus v) > \varphi(\tilde{S}) \), and moves \( v \) to the other side of \( \tilde{S} \). We prove that in this case, such a vertex exists, and hence this covers all possibilities. The latter step is called a type-2 step.

**Algorithm 4:** Combinatorial 1/2-approximation for \( AFTcut 

1. **Input:** \( G = (V, E) \)
2. **if** \( \Delta \leq 2 \) **then** return STABILIZE\(CUT(G, \emptyset, 1) \)
3. \( \tilde{S} \leftarrow \emptyset \)
4. **while** \( \varphi(\tilde{S}) < (m - \Delta)/2 \) **do**
5.   **if** \( \exists v, \varphi(\tilde{S} \oplus v) \geq (m - \Delta)/2 \) **then**
6.     \( \tilde{S} \leftarrow \tilde{S} \oplus v \)  // type-0 step
7. **else if** \( \exists v, d_S(v) < d(v)/2 \) **then**
8.     \( \tilde{S} \leftarrow \tilde{S} \oplus v \)  // type-1 step
9. **else if** \( \exists v, w, (d_S(v) = d(v)/2 \text{ and } \varphi(\tilde{S} \oplus v) \geq \varphi(\tilde{S}) \text{ and } d_{\tilde{S} \oplus v}(w) < d(w)/2) \) **then**
10. \( \tilde{S} \leftarrow \tilde{S} \oplus v \)  // build-up for another type-1 step
11. **else**
12. \( v \leftarrow \text{a vertex such that } d_S(v) = d(v)/2 \text{ and } \varphi(\tilde{S} \oplus v) > \varphi(\tilde{S}) \)
13. \( \tilde{S} \leftarrow \tilde{S} \oplus v \)  // type-2 step
14. return \( \tilde{S} \)

**Outline of the Proof.** The approximation is based on the a simple observation, that the optimal FT value is bounded by \( m - \Delta \), which holds since after failing a degree-\( \Delta \) vertex, only \( m - \Delta \) edges remain in the graph. Thus, in the proof of Theorem 1.4 our aim is to get a cut with FT value \((m - \Delta)/2\). If \( \Delta \leq 2 \), this is not always achievable (consider, e.g., a triangle). We show that nevertheless, any stable cut is a 1/2-approximation (see Lemma 3.14). If \( \Delta > 2 \), we show that in every two consecutive iterations, either the size of the cut increases or the FT value of the cut increases, while both never decrease. Since \( \varphi(\tilde{S}) \) and \( C_{\tilde{S}} \) are bounded, we get that the algorithm terminates. By the pseudocode of the algorithm it follows that the
algorithm terminates only when \( \varphi(\tilde{S}) \geq (m - \Delta)/2 \), i.e., \( \tilde{S} \) is a 1/2-approximation.

The approximation is based on the following simple observation, which holds since after failing a degree-\( \Delta \) vertex, only \( m - \Delta \) edges remain in the graph.

**Observation 3.11.** Let \( S^* \) be an optimal AFT-cut in a graph \( G \). It holds that \( \varphi(S^*) \leq m - \Delta \).

We use the following notation.

**Definition 3.12 (excess).** The excess of a vertex \( v \) in a cut \( S \subseteq V \) is \( x_S(v) = d_S(v) - d(v)/2 \).

Note that for a stable cut \( S \) (see Section 2), \( x_S(v) \geq 0 \) for every \( v \in V \). In addition, if \( d(v) \) is even, then \( x_S(v) \) is an integer. Otherwise, \( x_S(v) = a + 1/2 \) for some integer \( a \). Now, we prove the properties that are required for showing the correctness of our algorithm.

**Lemma 3.13.** Let \( S \) be a stable cut in a graph \( G = (V, E) \) such that \( \varphi(S) < (m - \Delta)/2 \). Then \( S \) has a unique critical vertex \( u \), and \( u \) satisfies

\[
d_S(u) > \sum_{v \in u} x_S(v) + \Delta - d(u)/2.
\]  

Moreover, \( u \) has a neighbor \( w \) in its opposite side of the cut, which satisfies \( x_S(w) = 0 \).

**Proof.** First, we show that \( S \) has a unique critical vertex. Since for every vertex \( v \), \( d_S(v) = d(v)/2 + x_S(v) \) and \( \sum_{v \in V} d(v)/2 = m \), we get that

\[
C_S = \frac{1}{2} \sum_{v \in V} d_S(v) = \frac{1}{2} \sum_{v \in V} \left( \frac{d(v)}{2} + x_S(v) \right) = \frac{m}{2} + \sum_{v \in V} x_S(v) / 2.
\]  

Let \( u \) be a critical vertex, and assume, without loss of generality, that \( u \in S \) (otherwise we swap \( S \) and \( V - S \)). On one hand, we have \( \varphi(S) = C_S - d_S(u) = m/2 + \sum_{v \in V} x_S(v)/2 - d_S(u) \), and on the other hand, we have \( \varphi(S) < (m - \Delta)/2 \), which together imply:

\[
m/2 + \sum_{v \in V} x_S(v)/2 - d_S(u) < (m - \Delta)/2.
\]

After a rearrangement, the latter implies (5). Using \( d_S(u) = d(u)/2 + x_S(u) \) in (5) and simplifying, we get

\[
x_S(u) > \sum_{v \in u} x_S(v) + \Delta - d(u)/2 \geq \sum_{v \in u} x_S(v).
\]

Since \( u \) is an arbitrary critical vertex, this implies that \( u \) is the only critical vertex of \( S \).

Next, let us show that there is a neighbor \( w \in V - S \) of \( u \) (recall that \( u \in S \)) with \( x_S(w) = 0 \). Assume to the contrary that for every \( v \notin S \) such that \( \{u, v\} \in E \), it holds that \( x_S(v) \geq 1/2 \) (recall that \( S \) is stable, and hence \( x_S(v) \) is a non-negative integer multiple of 1/2). Using (5), this implies:

\[
C_S = \frac{m}{2} + \sum_{v \in V} x_S(v)/2 \geq \frac{m}{2} + \frac{1}{2} \left( x_S(u) + \sum_{v \in \{u, v\} \in \delta(S)} x_S(v) \right)
\]

\[
geq \frac{m}{2} + \frac{1}{2} \left( x_S(u) + \frac{d_S(u)}{2} \right) \geq \frac{m}{2} + x_S(u),
\]

where we use \( d_S(u) = |\{v : \{u, v\} \in \delta(S)\}| \) in the second inequality, and \( d_S(u) = d(u)/2 + x_S(u) \geq 2x_S(u) \), in the third one. Since \( u \) is the critical vertex of \( S \), this gives that

\[
\varphi(S) = C_S - d_S(u) \geq \frac{m}{2} + x_S(u) - d_S(u) = \frac{m}{2} - \frac{d(u)}{2} \geq (m - \Delta)/2,
\]

in contradiction to \( \varphi(S) < (m - \Delta)/2 \). This completes the proof.  

\[\Box\]
We can now show that in the case of $\Delta \leq 2$, any stable cut is a 1/2-approximation.

**Lemma 3.14.** Every stable cut $S$ in a graph $G = (V, E)$ with $\Delta \leq 2$ is a 1/2-approximation for $\text{AFTcut}$.

**Proof.** We assume w.l.o.g. that $G$ is connected, i.e., it is either a path-graph or a cycle-graph. If $\varphi(S) \geq (m - \Delta)/2$, then Observation 3.11 implies that $S$ is a 1/2-approximation. Otherwise, the conditions of Lemma 3.13 apply, and hence there is a unique critical vertex vertex $u$, which satisfies $x_S(u) > \sum_{v \in u} x_S(v)$ (where we use that Lemma 3.14.

Every stable cut has a unique critical vertex $S_u \bar{w}$, so that there is only one (odd number) vertex with an odd degree. In addition, if there are two vertices $u, v' \in V - u$ such that $x_S(u) = x_S(v') = 1/2$, we get a contradiction to $x_S(u) > \sum_{v \in u} x_S(v)$. Therefore $x_S(u) = 1$, and $x_S(v) = 0$, for every $v \neq u$, which implies that all degrees are even, i.e., $G$ is a cycle graph. The uniqueness of the critical vertex also implies that $d_S(u) = 2$ and $d_S(v) = 1$, for all $v \neq u$; therefore, we have $C_S = (2 + n - 1)/2 = (n + 1)/2$, $\varphi(S) = C_S - d_S(u) = (n - 3)/2$, and $n$ is odd. Note that for every cut $S'$ of an odd cycle, there is at least one edge that does not cross the cut, which means that $\varphi(S') \leq m - \Delta - 1$. Since $\varphi(S) = (n - 3)/2 = (m - \Delta - 1)/2$, we have that $S$ is a 1/2-approximation. \qed

In the following lemma, we show that given a cut $S$ with a unique critical vertex $u$, it holds that $\varphi(S \oplus w) \geq \varphi(S)$ for every vertex $w$ with $x_S(w) = 0$ which is not in the same side of the cut as $w$. Therefore, if the algorithm cannot apply a build-up step, it means that for every $w$ as described above, $S \oplus w$ is a stable cut.

**Lemma 3.15.** Let $S$ be a stable cut in a graph $G = (V, E)$ with a unique critical vertex $u \in S$. Let $w \notin S$ with $x_S(w) = 0$. Then, $\varphi(S \oplus w) \geq \varphi(S)$.

**Proof.** By definition, it holds that

$$\varphi(S \oplus w) = C_{S \oplus w} - \max_{v \in V} d_{S \oplus w}(v) = C_S - \max_{v \in V} d_{S \oplus w}(v),$$

(7)

where we use $C_{S \oplus w} = C_S - d_S(u) + d_{S \oplus w}(w)$ (Observation 2.5), and $d_S(w) = d_{S \oplus w}(w) + 2x_S(w) = d_{S \oplus w}$. We have that $d_{S \oplus w}(u) \leq d_S(u)$, since if $\{u, w\} \in E$ then $d_{S \oplus w}(u) = d_S(u) - 1$, and otherwise $d_{S \oplus w}(u) = d_S(u)$. For every other vertex $v \neq u$, we have that $d_{S \oplus w}(v) \leq d_S(v) + 1 \leq d_S(u)$, where the last inequality holds since $u$ is the unique critical vertex of $S$, i.e., $d_S(v) < d_S(u)$. Altogether, we have that $\max_{v \in V} d_{S \oplus w}(v) \leq d_S(u)$, which, together with (7) implies that $\varphi(S \oplus w) \geq C_S - d_S(u) = \varphi(S)$, as claimed. \qed

Now, we show that if the conditions for the type-0, type-1 and build-up steps do not hold, then Algorithm 4 can apply the type-2 step. It holds that $\varphi(\bar{S}) < (m - \Delta)/2$ because otherwise the algorithm terminates. Since the algorithm does not apply a type-1 step, we get that $\bar{S}$ is a stable cut and Lemma 3.13 implies that $\bar{S}$ has a unique critical vertex $u$. In addition, for every vertex $w$ which is not in the same side of the cut as $u$ with $x_{\bar{S}}(w) = 0$, it holds that $\bar{S} \oplus w$ is a stable cut that satisfies $\varphi(\bar{S} \oplus w) < (m - \Delta)/2$ (otherwise the algorithm can apply a type-0 or a build-up step). We show that there is a neighbor $v$ of $u$ with $x_{\bar{S}}(v) = 0$, such that moving $v$ to the other side increases the FT value of $\bar{S}$, therefore Algorithm 4 can apply a type-2 step by choosing $v$.

**Lemma 3.16.** Let $S$ be a stable cut in a graph $G = (V, E)$ with $\Delta > 2$, such that $\varphi(S) < (m - \Delta)/2$. Denote the unique critical vertex of $S$ by $u$, assume w.l.o.g. that $u \in S$ and assume that for every vertex $w \notin S$ with $x_S(w) = 0$, $S \oplus w$ is a stable cut and satisfies $\varphi(S \oplus w) < (m - \Delta)/2$. Then there is a vertex $v$ with $x_S(v) = 0$ such that $\varphi(S \oplus v) > \varphi(S)$.

**Proof.** Let us choose a vertex $v \in N(u) \setminus S$ with $x_S(v) = 0$ (where $N(u)$ is the neighborhood of $u$), as follows: if there is a vertex $w \in V$ such that $x_S(u), x_S(w) > 0$, while $x_S(w') > 0$, for all $w' \notin \{u, w\}$, and there is a vertex $w' \in (N(u) \setminus N(w)) \setminus S$ ($w'$ is a neighbor of $u$ on its opposite side, and a non-neighbor of $w$), then let $v = w'$ (note that $x_S(v') = 0$). Otherwise, pick an arbitrary $v \in N(u) \setminus S$ with $x_S(v) = 0$ (as provided by Lemma 3.13). Let $u'$ be the unique critical vertex in $S \oplus v$ (by Lemma 3.13) We claim that
$u' = u$. Before proving the claim, let us see how it implies the lemma. By Observation \[25\] $C_S \geq C_S$, and $d_{S \oplus v}(u) = d_S(u) - 1$, thus $u' = u$ implies that $\varphi(S \oplus v) = C_S - d_{S \oplus v}(u) > C_S - d_S(u) = \varphi(S)$, and the lemma follows.

Assume henceforth, towards a contradiction, that $u' \neq u$. By Lemma 3.13 we have

$$d_S(u) > \sum_{w \in u} x_S(w) + \Delta - \frac{d(u)}{2}. \quad (8)$$

Note that since the cut $S$ is stable, $x_S(w) \geq 0$ holds for all vertices, and if $x_S(w) > 0$, then $x_S(w) \geq 1/2$. Let us study the value of $X = \sum_{w \in u} x_S(w)$.

**Case 1.** If $X \geq x_S(u') + 1$, then we have, via (8), that

$$d_S(u) > x_S(u') + 1 + \Delta - d(u)/2 \geq x_S(u') + 1 + d(u')/2 = d_S(u') + 1 \geq d_{S \oplus v}(u'),$$

where we use $\Delta - d(u)/2 \geq \Delta/2 \geq d(u')/2$, and $d_S(u') = x_S(u') + d(u')/2$. This, however, implies that $d_{S \oplus v}(u) = d_S(u) - 1 \geq d_{S \oplus v}(u')$, which contradicts that $u'$ is the unique critical vertex of $S \oplus v$; therefore, $X \leq x_S(u') + 1/2$.

**Case 2.** If $X = x_S(u') + 1/2$, then there is a vertex $u'' \notin \{u, u'\}$ such that $x_S(u'') = 1/2$, and for every $w \notin \{u, u', u''\}$, $x_S(w) = 0$. Using this in (8), we have

$$d_S(u) > x_S(u') + 1/2 + \Delta - d(u)/2.$$

Let us show that $\Delta - d(u)/2 \geq d(u')/2 + 1/2$, which would imply that the right-hand side above is at least $d_S(u') + 1$. Assume, towards a contradiction, that $\Delta < (d(u') + d(u) + 1)/2$, which implies that $d(u) = d(u') = \Delta$. Recall that $x_S(u'') = 1/2$, implying that $d(u'')$ is odd. For every vertex $w \notin \{u, u', u''\}$, we have $x_S(w) = 0$, hence $d(w)$ is even. This implies that we have an odd number of odd-degree vertices (since $d(u) = d(u')$), which is impossible. Thus, we again have $d_S(u) > d_S(u') + 1 \geq d_{S \oplus v}(u')$, which contradicts the fact that $u'$ is the unique critical vertex of $S \oplus v$.

**Case 3.** We conclude that $X = x_S(u')$. It immediately follows that all vertices $w \notin \{u, u'\}$ have $x_S(w) = 0$. We also have the following property.

**Claim 3.17.** Every crossing edge is adjacent to either $u$ or $u'$, the latter only when $x_S(u') > 0$.

**Proof.** Let $w \notin S \cup \{u, u'\}$. We have that $x_S(w) = 0$ and $w \notin S$, so by our assumption, $S \oplus w$ is a stable cut. The latter implies that there cannot be an edge $\{w, u'\} \in E$ for $w \notin S$, $w' \in S$, such that $x_S(u') = 0$, since we would then have $d_{S \oplus v}(w') < d(w')/2$, contradicting the assumption that $S \oplus w$ is stable. However, each vertex $w$ with $x_S(w) = 0$ has exactly half of its neighbors in the opposite side of the cut. This can only happen if $w$ is isolated or its only neighbors in the opposite side are $u$ and, possibly, $u'$. The latter can only happen when $x_S(u') > 0$.

To proceed, we consider two cases for $x_S(u')$.

**Case 3.1.** Consider the case $x_S(u') = 0$; then, the only vertex $w$ with $x_S(w) > 0$ is $u$. The argument above implies that there is no vertex $w$ on the same side of the cut as $u$, since otherwise $w$ would be an isolated vertex. Thus, we have $S = \{u\}$, and each vertex in $V - S$ is adjacent to $u$ and has exactly one neighbor in $V - S$, i.e., for every $w \neq u$, $d(w) = 2$. Since $\Delta > 2$, we have that $d_S(u) = d(u) > 2$, therefore $d_{S \oplus v}(u) \geq 2$ and it follows that $d_{S \oplus v}(u) = \max_{w \in V} d_{S \oplus v}(w)$. This, however, contradicts the assumption that $u'$ is the unique critical vertex in $S \oplus v$, and that $u' \neq u$.

**Case 3.2.** If $x_S(u') > 0$, then there are two possibilities: either $u'$ is on the same side of the cut as $u$ or it is on the other side.

**Case 3.2.1.** If $u, u'$ are on the same side of the cut, then by a similar reasoning as above, $S = \{u, u'\}$. In $S \oplus v$, we have $d_{S \oplus v}(u) = d_S(u) - 1$, and $d_{S \oplus v}(u') \leq d_S(u') \leq d_S(u) - 1$, where the last inequality holds since $u$ is the unique critical vertex of $S$. We conclude that $d_{S \oplus v}(u') \leq d_{S \oplus v}(u)$, which contradicts the assumption that $u'$ is the unique critical vertex of $S \oplus v$, and $u' \neq u$.  

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Case 3.2.2. It remains to consider the case when \( x_S(u') > 0 \), and \( u \) and \( u' \) are on opposite sides of the cut \( S \). Recall that in this case, \( u, u' \) are the only vertices with positive excess, hence, by the choice of \( v \), if \( Q = (N(u) \setminus N(u')) \setminus S \neq \emptyset \), then \( v \in Q \). We show that \( Q = \emptyset \). Assume the opposite, i.e., \( Q \neq \emptyset \), and \( v \in Q \). Since \( d(u) \) and \( d(u') \) have the same parity (all other degrees in the graph are even), we have that \( x_S(u) \geq x_S(u') + 1 \). Since \( \{v, u'\} \notin E \) and \( \{v, u'\} \notin E \), we have \( x_{S \oplus v}(u') = x_S(u') \), and \( x_{S \oplus v}(u) = x_S(u) - 1 \), implying that \( x_{S \oplus v}(u) \geq x_{S \oplus v}(u') \), which contradicts to the assumption that \( u \neq u' \) and \( u' \) is the unique critical vertex in \( S \oplus v \). Thus, we have \( Q = \emptyset \): every neighbor of \( u \) in \( V \setminus S \) is also a neighbor of \( u' \). Since for every node \( w \notin \{u, u'\} \), \( x_S(w) = 0 \), it follows from Claim 3.17 that \( w \) has a single crossing edge: \( \{w, u\} \), if \( w \notin S \) or \( \{w, u'\} \), if \( w \in S \). Thus, every vertex in \( S \) is adjacent to \( u' \), and every vertex in \( V \setminus S \) is adjacent to \( u \). Since \( Q = \emptyset \), we see that every vertex in \( V \setminus S \) is also adjacent to \( u' \) (and to no other vertex in \( V \setminus S \)); hence \( \Delta = d(u') \geq d(u) \).

Claim 3.18. \( d(u) = d(u') = \Delta \), and for every \( w \notin \{u, u'\} \), \( N(w) = \{u, u'\} \).

Proof. We only need to show that \( d(u) = \Delta \): then, both \( u \) and \( u' \) are adjacent to all other vertices, and By Claim 3.17 every such vertex has degree 2. Assume, towards a contradiction, that \( d(u) < \Delta \). By 3.14, we have that \( x_S(u) > x_S(u') + \Delta - d(u) \), hence \( x_S(u) > x_S(u') + 1 \). Since \( d(u), d(u') \) have the same parity (all other degrees in the graph are even) \( x_S(u) - x_S(u') \) is an integer, and hence \( x_S(u) - x_S(u') \geq 2 \). It holds that \( x_{S \oplus v}(u) \geq x_S(u) - 1 \), and \( x_{S \oplus v}(u') \leq x_S(u') + 1 \), i.e., \( x_{S \oplus v}(u) \geq x_{S \oplus v}(u') \), which contradicts the assumption that \( u' \) is the unique critical vertex of \( S \oplus v \), and that \( u' \neq u \).

Recall that for every \( w \notin \{u, u'\} \), exactly one of \( \{w, u\} \) and \( \{w, u'\} \) crosses \( S \). Since \( u \) is the single critical vertex in \( S \), we have \( d_S(u) > d_S(u') \). If \( u \) and \( u' \) are not adjacent, then \( d_S(u) + d_S(u') = \Delta \). The latter implies that \( d_S(u') < \Delta/2 = d(u')/2 \), contradicting to the assumption that \( S \) is a stable cut. Thus, \( \{u, u'\} \notin E \), and \( d_S(u) + d_S(u') = \Delta + 1 \), since the edge \( \{u, u'\} \) is counted in both \( d_S(u) \) and \( d_S(u') \). By the reasoning above, \( d_S(u') \geq \Delta/2 \), and hence \( d_S(u) \leq \Delta/2 + 1 \). Hence, \( \Delta \) is even, and \( d_S(u') = \Delta/2 \), implying that \( x_S(u') = 0 \), which is a contradiction.

Since we got a contradiction in all cases, we conclude that \( u = u' \), as claimed at the beginning. This completes the proof.

We are now ready to prove the main statement about our algorithm.

Proof of Theorem 1.4. By Lemma 3.14 StabilizeCut gives a 1/2-approximation when \( \Delta \leq 2 \), so we focus on \( \Delta \geq 3 \). First, let us show correctness of the algorithm, that is, if neither of type-0, type-1 or build-up steps applies, then a type-2 step can be applied. Let \( S_i \) denote the cut in the beginning of iteration \( i \). Let \( i \) be an iteration where none of the first three steps applies. Assume w.l.o.g. that the critical vertex of \( S_i \) is in \( S_i \). Then, \( S_i \) is stable, for every \( w \notin S_i \) with \( x_{S_i}(w) = 0 \), \( \varphi(S_i \oplus w) < (m - \Delta)/2 \), and \( S \oplus w \) is stable (the latter holds by Lemma 3.15, as no build-up step applies). Thus, Lemma 3.16 holds, so there is a type-2 step.

We show that the algorithm terminates within \( 4m + 2 \) iterations, giving a cut \( \bar{S} \) with \( \varphi(\bar{S}) \geq (m - \Delta)/2 \), which by Observation 3.11 is a 1/2-approximation. Assume, towards a contradiction, that the algorithm does not terminate within \( 4m + 2 \) iterations, i.e., no type-0 is applied. Since we always have \( S_{i+1} - S_i \oplus v \) with \( d_{S_i}(v) \leq d(v)/2 \), Observation 2.5 implies that \( C_{S_{i+1}} \leq C_{S_{i+2}} \). We also have \( \varphi(S_{i+1}) \leq \varphi(S_{i+2}) \), which in type-2 and build-up steps holds by definition, and in type-1 steps holds by Lemma 2.6.

Next, we show that in every consecutive pair of iterations, either the cut size or the FT value strictly increases. Formally, for every \( i \), either \( C_S < C_{S_{i+2}} \) or \( \varphi(S_i) < \varphi(S_{i+2}) \) holds. If either one of the two iterations is a type-1 step then \( C_{S_{i+2}} < C_{S_{i+2}} \), since the cut size never decreases, while it increases in a type-1 step, by Lemma 2.7. If neither of the two steps is a type-1 step, then the first one, \( i \), must be a type-2 step: otherwise, it would be a build-up step, which has to be followed by a type-1 step. By the definition of a type-2 step, \( \varphi(S_i) < \varphi(S_{i+2}) \).

It follows that after \( 4m + 2 \) iterations, \( \max\{C_{\bar{S}}, \varphi(\bar{S})\} > m \), which gives a contradiction.

\[\square\]
4 Fault Tolerance Against an Oblivious Adversary

We give an algorithm that approximates the fault tolerant Max-Cut against the oblivious adversary with (constant) \( k \) faults within an \( \alpha_{GW} \)-approximation factor. The main idea is to frame the problem as a linear program (LP) with an exponential number of variables, then reduce the number of variables using a solution of its dual (with an exponential number of constraints but a polynomial number of variables). The dual is approximately solved by the ellipsoid algorithm together with an approximate separation oracle that is given by a Max-Cut algorithm. A similar approach has been used, e.g. in \cite{35}, for an unrelated problem.

Theorem 1.2. For every constant \( k > 0 \), there is a polynomial time \( \alpha_{GW} \)-approximation algorithm for fault tolerant Max-Cut on general weighted graphs against an oblivious adversary and \( k \) faults.

For simplicity, we present the algorithm for a single fault, and then show how to extend it to any constant number \( k \) of faults. The \( OFTcut \) problem can be formulated as the following LP, \( \text{Primal}_1 \), with an exponential number of variables.

\[
\begin{align*}
\text{max} & \quad \sum_{S \subseteq V} P_S \cdot \sum_{e \in \delta(S)} w_e - Z \\
n\text{s.t.} & \quad \sum_{S \subseteq V} P_S \cdot \sum_{v \in \delta(S)} w_{\{u,v\}} \leq Z \quad \forall u \in V \\
 & \quad \sum_{S \subseteq V} P_S \leq 1 \\
 & \quad 0 \leq P_S \quad \forall S \subseteq V
\end{align*}
\]

The variable \( P_S \) represents the probability assigned to the cut \( S \subseteq V \). The variable \( Z \) represents the expected weight that the adversary removes from the graph. Constraints (9-11) make \( P_S \) a probability distribution. In (9), for each vertex \( u \), we bound by \( Z \) the expected weight that is removed from the cut when \( u \) fails. To see that the left hand side is indeed the expected removed weight, note that it equals \( \sum_{S \subseteq V} P_S \cdot d_S(u) \).

Consider the dual problem of the LP above, \( \text{Dual}_1 \):

\[
\begin{align*}
\text{min} & \quad Y \\
n\text{s.t.} & \quad \sum_{\{u,v\} \in \delta(S)} w_{\{u,v\}} - \sum_{v \in \delta(S)} X_u \sum_{\{u,v\} \in \delta(S)} w_{\{u,v\}} \leq Y \quad \forall S \subseteq V \\
 & \quad \sum_{u \in V} X_u \leq 1 \\
 & \quad 0 \leq X_u \quad \forall u \in V
\end{align*}
\]

The dual LP captures the following problem: The adversary picks a distribution over the vertices, and the algorithm picks a cut (depending on the choice of the adversary). The goal of the adversary is to choose its distribution (without knowing the cut choice of the algorithm) so as to minimize the expected cut size after a random failure from its distribution.

The dual LP \( \text{Dual}_1 \) has an exponential number of constraints but only \( |V| + 1 \) variables. Such LPs can be solved efficiently via the ellipsoid method \cite{31}, given an efficient separation oracle. The latter is an algorithm that given an assignment of values to the variables of the LP, reports a violated constraint if the assignment is infeasible, or otherwise reports that it is feasible. For the particular case of \( \text{Dual}_1 \), the ellipsoid algorithm can be viewed as a binary search over the values of \( Y \), such that in each stage (fixed \( Y \)), a black-box procedure does a polynomial number of queries to a given separation oracle, and either reports the first solution \( \{X_u\}_{u \in V} \) it finds such that \( \{X_u\}_{u \in V}, Y \) is feasible according to the oracle, or reports that there is no such solution.
In Algorithm 5, it holds that:

Let us see what a separation oracle looks like in our case. For given values \( \{ X_u \}_{u \in V} , Y \), let \( G' = (V' , E' , w' ) \) be the graph with weights \( w'_{(u,v)} = (1 - X_u - X_v) w_{(u,v)} \). With this notation, constraint (12) becomes \( C_{S,G'} \leq Y \).

In order to see if a given assignment of variables is feasible, it thus suffices to find a maximum weight cut \( G' \) using a derandomized variant of the Goemans-Williamson algorithm \([29, 43]\), which we denote by \( \text{DERANDOMIZED-GOEMANS-WILLIAMSON} \). If the size of the cut is larger than \( Y \), it returns the violated constraint [13] corresponding to \( S_{\text{ALG}} \), otherwise it reports that the solution is feasible. In Algorithm 5 we give the pseudocode of our approximate separation oracle.

**Algorithm 5: Approximate separation oracle**

1. **Input:** \( \{ X_u \}_{u \in V} , Y, G \)
2. if \( \sum_{u \in V} X_u > 1 \) then
   3. return violated constraint \( \sum_{u \in V} X_u \leq 1 \)
4. \( S_{\text{ALG}} \leftarrow \text{DERANDOMIZED-GOEMANS-WILLIAMSON}(G') \)
5. if \( C_{S_{\text{ALG}},G'} > Y \) then
   6. return violated constraint for subset \( S_{\text{ALG}} \)
   7. else
   8. return feasible

The following lemma shows that the oracle always answers correctly if the assignment is feasible, and even if it incorrectly outputs **feasible**, the assignment is nearly feasible.

**Lemma 4.1.** Given an assignment \( \{ X_u \}_{u \in V} , Y \) to the variables in \([\text{Dual}1]\) as input to the separation oracle in Algorithm 5 it holds that:

1. if the assignment is feasible, then the oracle returns **feasible**,
2. if the assignment is infeasible, then either the oracle outputs a violated constraint, or reports **feasible**, in which case \( \{ X_u \}_{u \in V} , Y/\alpha_{\text{GW}} \) is feasible.

**Proof:** Let \( \{ X_u \}_{u \in V} , Y \) be an assignment to the variables of \([\text{Dual}1]\). If it is feasible, then it holds that \( \sum_{u \in V} X_u \leq 1 \), and in addition every \( S \subseteq V \) satisfies \( C_{S,G'} \leq Y \), therefore the oracle returns **feasible**.

If the assignment is infeasible, there are two cases. If \( \sum_{u \in V} X_u > 1 \), the oracle returns this violated constraint. Otherwise, there is a subset \( S' \subseteq V \) such that \( C_{S',G'} > Y \). Let \( S^* \) be an optimal solution for Max-Cut on \( G' \), and note that \( C_{S^*,G'} > Y \). If \( \alpha_{\text{GW}} \cdot C_{S'^*,G'} > Y \), then we also have that \( C_{S_{\text{ALG}},G'} > Y \) (since \( S_{\text{ALG}} \) is an \( \alpha_{\text{GW}} \)-approximate Max-Cut), and the oracle returns the violated constraint for \( S_{\text{ALG}} \).

Otherwise, \( C_{S^*,G'} \leq Y/\alpha_{\text{GW}} \). Since \( S^* \) is an optimal solution for Max-Cut on \( G' \), it follows that for every \( S \subseteq V \), it holds that \( C_{S,G'} \leq Y/\alpha_{\text{GW}} \), i.e., the solution \( \{ X_u \}_{u \in V} , Y/\alpha_{\text{GW}} \) is feasible. \( \square \)

It is not hard to see that the application of the ellipsoid algorithm on \([\text{Dual}1]\) takes a polynomial time (i.e., at most as much time as it would take with an exact separation oracle), since our approximate oracle is (possibly) incorrect only on the last call from the ellipsoid algorithm (for a given \( Y \), when it incorrectly reports a solution as feasible.

The output of the ellipsoid algorithm/binary search is an assignment \( \{ X_u \}_{u \in V} , Y \) to the variables of \([\text{Dual}1]\) such that \( \{ X_u \}_{u \in V} , Y \) is feasible according to the oracle, while \( Y - \epsilon \) is infeasible with every assignment to the \( X \) variables, where \( \epsilon \) is the precision of the binary search. As observed above, we have that \( \{ X_u \}_{u \in V} , Y/\alpha_{\text{GW}} \) is feasible, and it follows that if \( Y^* \) is the optimal value of \([\text{Dual}1]\), then \( Y^* \leq Y/\alpha_{\text{GW}} \). Since the ellipsoid algorithm queries the oracle a polynomial number of times, there
is a set $\mathcal{H} \subseteq 2^V$ of a polynomial number of cuts $S$, for which constraint (12) is queried. Consider a modified variant of (Dual1), called (Dual2), where only constraints of cuts in $\mathcal{H}$ are present:

$$
\min \ Y \\
\text{s.t. } \sum_{\{u,v\} \in \delta(S)} w_{\{u,v\}} - \sum_{u \in V} X_u \sum_{v \in \{u,v\} \in \delta(S)} w_{\{u,v\}} \leq Y \quad \forall S \in \mathcal{H} \\
\sum_{u \in V} X_u \leq 1 \\
0 \leq X_u \quad \forall u \in V
$$

Let $Y_2^*$ be the optimal value of (Dual2). Note that $Y_2^* \leq Y^*$. Note also that the ellipsoid algorithm returns exactly the same solution $\{X_u\}_{u \in V}, Y$, when executed on (Dual1) and (Dual2) (since our algorithm is deterministic, and only constraints in $\mathcal{H}$ are queried); hence, we have $Y - \epsilon \leq Y^*_2$. Finally, let us consider the primal LP corresponding to (Dual2):

$$
\max \ \sum_{S \in \mathcal{H}} P_S \cdot \sum_{e \in \delta(S)} w_e - Z \\
\text{s.t. } \sum_{S \in \mathcal{H}} P_S \cdot \sum_{v \in \{u,v\} \in \delta(S)} w_{\{u,v\}} \leq Z \quad \forall u \in V \\
\sum_{S \in \mathcal{H}} P_S \leq 1 \\
0 \leq P_S \quad \forall S \in \mathcal{H}
$$

Note that (Primal2) is obtained from (Primal1) by removing variables $P_S$ with $S \notin \mathcal{H}$ (i.e., setting $P_S = 0$).

The new primal has polynomially many constraints and variables, so can be solved in polynomial time. From the arguments above, we have that its optimal value $Y_2^*$ satisfies $Y - \epsilon \leq Y_2^* \leq Y^* \leq Y/\alpha_{GW}$. Recalling that $Y^*$ is the optimal value for the original LP, we see that $Y_2^*$ is a $\alpha_{GW}$-approximation (with any polynomial precision $\epsilon$).

**Extending the proof of Theorem 1.2 for $k$ failures** In order to extend the algorithm to $k$ failures, for a constant $k \in \mathbb{N}$, all we need to do is to slightly generalize the primal and dual LPs, while the overall structure stays the same. The extension of (Primal1) to the case of $k$ faults is as follows:

$$
\max \ \sum_{S \subseteq V} P_S \cdot \sum_{e \in \delta(S)} w_e - Z \\
\text{s.t. } \sum_{S \subseteq V} P_S \cdot \sum_{v \in \{u,v\} \in \delta(S)} w_{\{u,v\}} \leq Z \quad \forall u \in V \\
\sum_{S \subseteq V} P_S \leq 1 \\
0 \leq P_S \quad \forall S \subseteq V
$$

The corresponding dual problem is as follows.

$$
\min \ Y \\
\text{s.t. } \sum_{e \in \delta(S)} w_e - \sum_{F \in \binom{V}{k}} X_F \sum_{e \in \delta(S) \cap F \neq \emptyset} w_e \leq Y \quad \forall S \subseteq V \\
\sum_{F \in \binom{V}{k}} X_F \leq 1 \\
0 \leq X_F \quad \forall F \in \binom{V}{k}
$$
The separation oracle is similar to Algorithm 5, but defines the weight function of $G'$ as $w'_{\{u,v\}} = \left(1 - \sum_{\{u,v\} \cap F \neq \emptyset} X_F\right)w_{\{u,v\}}$. The rest of the algorithm is the same, and the proof is similar.

5 Hardness of Approximation

In this section we show that assuming the Unique Games Conjecture, one cannot approximate $AFTcut$ and $OFTcut$ within a factor greater than $\alpha_{GW}$. Formally, we prove the following:

**Theorem 1.3.** Assuming the Unique Games Conjecture and $NP \not\subseteq BPP$, there is no polynomial time $(\alpha_{GW} + \epsilon)$-approximation algorithm for fault tolerant Max-Cut in unweighted graphs, for any constant $\epsilon > 0$. This holds for both adaptive and oblivious adversaries.

In both cases, given an unweighted instance $G$ of Max-Cut, we construct an unweighted graph $G'$, according to Algorithm 6: we take the disjoint union of $G$ with a star with $n = \left\lfloor \frac{\sqrt{V}}{2} \right\rfloor$ leaves and a center $u^*$, and add an edge joining $u^*$ to an arbitrary vertex $v_1 \in V$. This completes the construction of $G'$ (see Figure 3). Clearly, this is a polynomial construction.

**Algorithm 6:** Approximate Max-Cut Using AFTcut

1. **Input:** $G = (V,E)$, $V = \{v_1, \ldots, v_n\}$
2. Let $G' = (V', E')$ for $V' = V \cup \{u^*, u_1, \ldots, u_n\}$, $E' = E \cup \{\{u^*, u_i\} : i \in [n]\} \cup \{\{u^*, v_1\}\}$
3. **return** $G'$

Below, we show for each kind of adversary how to translate a given (approximate) solution to $AFTcut$ or $OFTcut$ in $G'$ into a solution to Max-Cut in $G$, which would imply the corresponding inapproximability results, using the fact that Max-Cut is hard to approximate within a factor better than $\alpha_{GW}$. We use the following simple observation.

**Observation 5.1.** Let $S \subseteq V$ be a cut in $G$, and $S' = S \cup \{u^*\}$. It holds that in $G'$, $u^*$ is a critical vertex of $S'$, i.e., $\varphi(S') = C_{S'-u^*, G'-u^*}$. For every cut $S'' \subseteq V'$, we have $C_{S''-u^*, G''} = C_{S'' \cap V, G}$.

The proof follows from the fact that for every vertex $v \in V'$, $d_{S'}(v) \leq n \leq d_{S''}(u^*)$, and that all edges in $G' - u^*$ belong to $G$.

5.1 Adaptive Adversary

First, let us observe that the optimal values of Max-Cut in $G$ and $AFTcut$ in $G'$ are equal.

![Figure 3: The Construction of $G'$](image)
Lemma 5.2. Let $S_{ft}^*$ be an optimal AFTcut in $G'$ and $S_{mc}^*$ be an optimal Max-Cut in $G$. It holds that $\varphi(S_{ft}^*, G') \leq C_{S_{mc}^*, G}$.

Proof. First, we show that $\varphi(S_{ft}^*, G') \leq C_{S_{mc}^*, G}$. This follows since $C_{S_{ft}^*, G'-u^*} \geq \varphi(S_{ft}^*, G')$, and by Observation 5.1, $C_{S_{ft}^*, G'-u^*} \geq C_{S_{ft}^*, G'} \leq C_{S_{mc}^*, G}$.

Next, let us show that $\varphi(S_{ft}^*, G') \geq C_{S_{mc}^*, G}$. Let $S = S_{mc}^* \cup \{u^*\}$. By Observation 5.1, we have that $\varphi(S, G') = C_{S-G', G-V} = C_{S_{mc}^*, G}$. This completes the proof.

5.2 Oblivious Adversary

Here, we rely on the fact that under the assumption of the Unique Games Conjecture and $NP \neq BPP$, there is no randomized algorithm that outputs a better than $\alpha$-approximation for Max-Cut, for $\alpha > \alpha_{GW}$. Let us construct an $\alpha$-approximation algorithm for Max-Cut. For any input graph $G$, construct the graph $G'$, as per Algorithm 6. As assumed, we can compute an $\alpha$-approximate AFTcut $S_{ft}$ in $G'$. Let $S_{ft}'$ and $S_{mc}'$ be optimal solutions for AFTcut in $G'$ and Max-Cut in $G$ (resp.). By Observation 5.1 and Lemma 5.2, we have $C_{S_{ft}^*, G'} \leq \alpha \cdot \varphi(S_{ft}^*, G') = \alpha \cdot C_{S_{mc}^*, G}$. Thus, we get an $\alpha$-approximation algorithm for Max-Cut, which is impossible under the Unique Games Conjecture and $NP \neq P$. The proof extends to randomized algorithms as well (here we also assume $NP 
eq BPP$, which together with the Unique Games Conjecture excludes better than $\alpha_{GW}$-approximation algorithms, including randomized, for Max-Cut).

5.3 Oblivious Adversary

Here, we rely on the fact that under the assumption of the Unique Games Conjecture and $NP \neq BPP$, there is no randomized algorithm that outputs a better than $\alpha_{GW}$-approximation for Max-Cut with constant probability.

Again, we begin by showing that the optimal values for OFTcut in $G'$ and Max-Cut in $G$ are equal.

Lemma 5.3. Let $D^*$ be the distribution of an optimal OFTcut in $G'$, and $S_{mc}^*$ be an optimal Max-Cut in $G$. It holds that $\mu(D^*, G') = C_{S_{mc}^*, G}$.

Proof. Let $\tilde{S} = S_{mc}^* \cup \{u^*\}$, and let $D$ be the distribution that assigns probability 1 to $\tilde{S}$ and probability 0 to all other cuts. By Observation 5.1, $u^*$ is a critical vertex, hence for every vertex $v \in G'$, we have $\mathbb{E}_{D}[C_{S_{-u^*, G'-u^*}}] = C_{S_{-u^*, G'-u^*}} \leq C_{S_{mc}^*, G} = \mathbb{E}_{\tilde{D}}[C_{S_{mc}^*, G}]$. Using Observation 5.1 again, we have $\mu(D^*, G') = \mathbb{E}_{D}[C_{S_{mc}^*, G}] = C_{S_{mc}^*, G}$.

Thus, by Observation 5.1, it holds that $C_{S_{ft}^*, G'-u^*} \leq C_{S_{mc}^*, G}$, for every cut $S$ from the support of $D^*$, which implies that $\mu(D^*, G') \leq \mathbb{E}_{D}[C_{S_{mc}^*, G}]$. This completes the proof.

Proof of Theorem 5.3 for an Oblivious Adversary. Assume, for a contradiction, that we have an $\alpha$-approximation algorithm for AFTcut, for $\alpha > \alpha_{GW}$. We design a randomized approximation algorithm for Max-Cut. Let $G$ be an input to Max-Cut. Construct the graph $G'$ as per Algorithm 6. Let $D$ be the distribution of an $\alpha$-approximate AFTcut in $G'$. By Lemma 5.3, we have $\mathbb{E}_{D}[C_{S_{mc}^*, G}] = \mu(D, G') \geq \alpha \cdot C_{S_{mc}^*, G}$, where $S_{mc}^*$ is a Max-Cut in $G$. By Observation 5.1, it holds that $C_{S_{ft}^*, G'-u^*} \geq C_{S_{mc}^*, G}$ for every cut $S_{ft}$ in the support of $D$. Setting $p = \mathbb{P}[C_{S_{mc}^*, G} = (1 - \epsilon/2)C_{S_{mc}^*, G}]$, we have

$$\alpha \cdot C_{S_{mc}^*, G} \leq \mathbb{E}_{D}[C_{S_{mc}^*, G}] = \alpha \cdot C_{S_{mc}^*, G} \cdot (1 - p) \cdot (\alpha - \epsilon/2)C_{S_{mc}^*, G},$$

implying that $p \geq \epsilon/2$. Thus, for a random cut $S_{ft}$ sampled from $D$, it holds that $S_{ft} \cap V$ is an $(\alpha - \epsilon/2)$-approximation to Max-Cut, with probability $\epsilon/2$, where $\alpha - \epsilon/2 > \alpha_{GW}$. This contradicts to our assumption about the Unique Games Conjecture and $NP \neq BPP$.

6 The Approximation Factor of a Random Cut

In this section, we study the approximation provided by a random cut for the fault tolerant Max-Cut problem, where a random cut is obtained by including each vertex in the cut independently, with probability $1/2$. 

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In the case of an adaptive adversary, we show that a random cut cannot achieve an $\alpha$-approximation for weighted graphs, with $\alpha > 1/4$. Nonetheless, we show that if the input is a connected unweighted graph with sufficiently many vertices, then a random cut gives a $(1/2 - \epsilon)$-approximation. For an oblivious adversary, we show that the uniform distribution over all cuts, which can be seen as a randomized algorithm that outputs a random cut as described above, gives a $1/2$-approximation for weighted instances and many faults.

### 6.1 A Negative Result for Adaptive Adversary and Weighted Instances

Here we show that a random cut cannot achieve an approximation factor that is greater than $1/4$ for weighted $\text{AFTcut}$.

**Theorem 6.1.** For every $\epsilon > 0$ and $n \geq 3/4\epsilon$, there is an $n$-vertex graph $G$ such that $\mathbb{E}[\varphi(S)] \leq (1/4 + \epsilon)\varphi(S^*)$, where $S$ is a uniformly random cut $S$, and $S^*$ is an optimal $\text{AFTcut}$ in $G$.

**Proof.** Let $n \geq 4$, and consider the weighted graph $G = (V, E, w)$ with $V = \{v_0, v_1, \ldots, v_{n-1}\}$, $E = \{e_i = \{v_i, v_{i+1 \mod n} : i = 0, 1, \ldots, n-1\}$, and the following weight function: $w_{v_0} = w_{v_2} = n(n-3)$ (we call these edges heavy), and $w_{e_i} = 1$, for $i \notin \{0, 2\}$ (we call these edges light). Thus, $G$ is a weighted cycle with edges of weight 1, except for two non-adjacent edges of weight $n(n-3)$ (See Figure 4).

Let $S^*$ be an optimal $\text{AFTcut}$ in $G$. Note that $\varphi(S^*) \geq n(n-3)$, since, e.g., $\varphi(\{v_1, v_2\}) = n(n-3)$. Let $S$ be a random cut, and let $\mathcal{E}$ be the event that both edges $e_0$ and $e_2$ cross $S$. We have $\mathbb{P}[\mathcal{E}] = 1/4$, since the probability for each edge to cross is 1/2, and $e_0, e_2$ are disjoint, hence independent. Given $\neg\mathcal{E}$, i.e., at least one of $e_0, e_2$, say, $e_0$, does not cross $S$, we have that $\varphi(S) \leq n-3$, since, when we fail an endpoint of $e_2$, say, $v_2$, there are no heavy crossing edges left. Thus, we have

$$\mathbb{E}[\varphi(S)] = \mathbb{E}[\varphi(S) | \mathcal{E}] \cdot \mathbb{P}[\mathcal{E}] + \mathbb{E}[\varphi(S) | \neg \mathcal{E}] \cdot \mathbb{P}[\neg \mathcal{E}]$$

$$\leq (1/4)\varphi(S^*) + (3/4)(n-3)$$

$$= \varphi(S^*) \cdot (1/4 + 3(n-3)/4\varphi(S^*))$$

$$\leq \varphi(S^*) \cdot (1/4 + 3/4n) .$$

We get the desired approximation, since $n \geq 3/4\epsilon$.

### 6.2 Adaptive Adversary and Unweighted Instances

Here we show that for every $0 < \epsilon < 1/16$, a random cut can get a $(1/2 - \epsilon)$-approximation for unweighted $\text{AFTcut}$ for every graph $G$ with large enough $n$. 

---

**Figure 4:** The Construction for Theorem 6.1
Theorem 6.2. Let $0 < \epsilon < 1/16$, and let $S$ be a random cut in an unweighted graph $G = (V, E)$. If $G$ is connected and $n = |V|$ is large enough (w.r.t. $\epsilon$) then $S$ is a $(1/2 - \epsilon)$-approximation for $AFTcut$ with a single fault in $G$.

Proof. Fix a constant $0 < \epsilon < 1/16$. We assume that $n$ is sufficiently large w.r.t. $\epsilon$. Let $c = 48/\epsilon^4$. We also assume that $G$ is connected. In particular, $m \geq n - 1$.

Let $S$ be a random cut obtained by sampling every vertex $v \in V$ independently with probability $1/2$. Every edge $e \in E$ crosses $S$ with probability exactly $1/2$, so $E[S,G] = m/2$.

**Case 1:** $\Delta < c \log n$. In this case we have $\Delta < cm/2$, since $m \geq n - 1 \geq (2c/\epsilon) \log n$, for large enough $n$. Thus, we have $E[\varphi(S)] = E[S,G|\epsilon] E[\varphi(S)|\epsilon] \geq E[S,G] - E[\max_{e \in E} d_S(v)] \geq m/2 - \Delta \geq (1 - \epsilon)m/2$, hence we get a $(1 - \epsilon)/2$-approximation.

**Case 2:** $\Delta \geq c \log n$. Note that for every vertex $v \in V$, $E[d_{S}(v)] = d(v)/2$. Thus, for $v$ with $d(v) > \Delta/2$, we have $E[d_{S}(v)] > \Delta/4 \geq (\epsilon/4) \log n$. By a Chernoff bound (note that $d_{S}(v)$ is a sum of independent Bernoulli random variables, one for each adjacent edge of $v$) and the choice of $c$, we have:

$$
P\left[d_{S}(v) - d(v)/2 \geq \epsilon^2 \cdot (d(v)/2)\right] \leq 2 \cdot \exp(-\epsilon^4/3 E[d_{S}(v)]) \leq 2 \cdot \exp(-\epsilon^4 c/12) \log n \leq 2 \cdot n^{-4}.
$$

Let $E$ be the event that $d_{S}(v) = (1 \pm \epsilon^2/2)d(v)/2$ holds for all $v \in V$ with $d(v) \geq \Delta/2$. By the union bound, we have that $P[E] \geq 1 - n^{-2}$. We consider two cases.

**Case 2.1:** $m > (1 + 2\epsilon)\Delta$. Recall that $E[S,G] = m/2$. Thus,

$$
m/2 = E[S,G] = E[S,G|E] P[E] + E[S,G|\neg E] P[\neg E] \leq E[S,G|E] P[E] + m/n^2,
$$

where the last inequality follows because $P[\neg E] \leq 1/n^2$, and $S,G$ is bounded by $m$. Hence, $E[S,G|E] P[E] \geq m/2 - m/n^2 \geq m/2 - 1$. It then follows that

$$
E[\varphi(S)] \geq E\left[S,G - \max_v d_{S}(v)\right] \\
\geq E\left[S,G|E\right] P[E] \\
= E\left[S,G\right] P[E] - E\left[\max_v d_{S}(v)\right] P[E] \\
\geq m/2 - 1 - (1 + \epsilon^2)\Delta/2 \\
= (m - \Delta)/2 - 1 - \epsilon^2\Delta/2,
$$

where the last inequality follows because given $E$ the maximal degree is $(1 + \epsilon^2)\Delta/2$. Using $2\epsilon\Delta \leq m - \Delta$ and $\Delta \geq c \log n$, we get that $1 \leq \epsilon^2\Delta/2 \leq (\epsilon/2)(m - \Delta)/2$; hence, $1 + \epsilon^2\Delta/2 \leq \epsilon(m - \Delta)/2$, and we get the claimed $(1 - \epsilon)/2$-approximation:

$$
E[\varphi(S)] \geq (m - \Delta)/2 - \epsilon(m - \Delta)/2 \geq (1 - \epsilon)(m - \Delta)/2.
$$

**Case 2.2:** $m \leq (1 + 2\epsilon)\Delta$. It follows that $m < 2\Delta - 1$, therefore there is a unique vertex $v$ with $d(v) = \Delta$. Let $E'$ be the set of edges not adjacent to $v$. By the assumption, $|E'| \leq 2\epsilon\Delta$; hence, the set $V'$ of vertices adjacent to an edge in $E'$ has size $|V'| \leq 2|E'| \leq 4\epsilon\Delta$. Note that for every $w \in V'$, $d(w) = 1$. Let $E'$ be the event that $X = d_{S\setminus V',G\setminus V'}(v) > 2\Delta$ (note that $X$ is a sum of independent Bernoulli random variables, one for each adjacent edge of $v$ in $G - V'$). Since $d_{G\setminus V'}(v) > (1 - 4\epsilon)\Delta$, $E[X] > (1 - 4\epsilon)\Delta/2$. First we show that $P[E''] \geq 1 - n^{-2}$. Using $\epsilon < 1/16$ and a Chernoff bound, we have that

$$
P[X \leq 2\epsilon\Delta] \leq P\left[X \leq \frac{4\epsilon}{1 - 4\epsilon} E[X]\right] \\
\leq P[X \leq (1/3) \cdot E[X]] \\
\leq \exp\left(-\frac{(2/3)^2 E[X]}{2}\right) \\
\leq \exp(-\frac{(1 - 4\epsilon)\Delta/9}) \leq \exp(-\Delta/12).
$$
Using \( \Delta \geq c \log n \) with \( c \geq 24 \), we get that \( \Pr[X \leq 2\epsilon\Delta] \leq n^{-2} \), thus \( \Pr[\mathcal{E}] \geq 1 - n^{-2} \). Note that given \( \mathcal{E}' \), \( d_{S,G}(v) \geq 2\epsilon\Delta \), while \( d_{S,G}(u) \leq 2\epsilon\Delta \), for every \( u \in V-v \); hence \( v \) is a critical vertex for \( S \), i.e., \( \varphi(S) = C_{S-v,G-v} \). Hence, we have:

\[
\mathbb{E}[\varphi(S)] \geq \mathbb{E}[\varphi(S) | \mathcal{E}'] \cdot \Pr[\mathcal{E}'] = \mathbb{E}[C_{S-v,G-v} | \mathcal{E}'] \cdot \Pr[\mathcal{E}']
\]

where to get the second row, we use the fact that \( C_{S-v,G-v} \) is independent of \( \mathcal{E}' \), since \( \mathcal{E}' \) only conditions on vertices that can never contribute to the cut \( S-v \) in \( G-v \), because all of their edges are adjacent to \( v \). This completes the proof.

\[ \square \]

**Remark 6.3.** Note that as opposed to Max-Cut, a random cut gives a \((1/2 - \epsilon)\)-approximation for large \( n \) only. For example, we show that for an unweighted 4-cycle, it holds that \( \mathbb{E}[\varphi(S)] = \frac{1}{4} \cdot \varphi(S^*) \), where \( S^* \) is an optimal \( \text{AFTcut} \).

Let \( v_1, v_2, v_3, v_4 \) be the 4-cycle. Overall, it has 16 cuts, so the probability of each cut being output by the algorithm is \( 1/16 \). For every cut \( S \) such that \( |S| \in \{0,1,3,4\} \) (there are 10 such cuts), it holds that \( \varphi(S) = 0 \), since when \( |S| \in \{0,4\} \) it holds that \( C_S = 0 \), and when \( |S| \in \{1,3\} \), all the crossing edges are adjacent to one vertex, therefore when that vertex fails, no crossing edges remain. For \( |S| = 2 \), we have 6 options. When \( S \) consists of two adjacent vertices (there are 4 options), it holds that \( \varphi(S) = 1 \). When \( S \) is \( \{v_1,v_3\} \) or \( \{v_2,v_4\} \), it holds that \( \varphi(S) = 2 \). Altogether, we get that

\[
\mathbb{E}[\varphi(S)] = 0 \cdot 10/16 + 1 \cdot 4/16 + 2 \cdot 2/16 = 1/2.
\]

However, an optimal \( \text{AFTcut} \) is for example \( S^* = \{v_1,v_3\} \), for which \( \varphi(S^*) = 2 \); hence, \( \mathbb{E}[S] = \frac{1}{4} \cdot \varphi(S^*) \).

### 6.3 Oblivious Adversary, Weighted Instances, and Many Faults

Here we show that the uniform distribution (output every cut with probability \( 1/2^n \)) is a \( 1/2 \)-approximation for \( k-\text{OFTcut} \), for every \( k \) (not necessarily constant).

**Theorem 6.4.** Let \( G = (V,E,w) \) be a weighted graph and \( k \in \mathbb{N} \) be a number. Let \( \mathcal{U} \) be the uniform distribution over all cuts in \( G \). Then, \( \mathcal{U} \) is a \( 1/2 \)-approximation for \( k-\text{AFTcut} \) against an oblivious adversary.

Let \( \Delta_k \) denote the maximum degree (total weight of adjacent edges) of a subset of size \( k \), that is, \( \Delta_k = \max_{F \in \binom{V}{k}} \delta(F) \).

Before proving the theorem, we make two observations: 1. failing a subset \( F \) removes \( \delta(F) \) edges from a cut, and 2. there is a subset \( F \) whose removal always limits the remaining cut size by \( m - \Delta_k \).

**Observation 6.5.** For every cut \( S \subseteq V \) and subset \( F \in \binom{V}{k} \), it holds that \( C_{S,F,G-F} = C_S - \delta(F) \).

**Observation 6.6.** There is a subset \( F \in \binom{V}{k} \) such that for every distribution \( D \), it holds that \( \mathbb{E}_{S \sim D} [C_{S,F,G-F}] \leq m - \Delta_k \).

**Proof.** For every cut \( S \subseteq V \) and subset \( F \in \binom{V}{k} \) with \( d(F) = \Delta_k \), it holds that \( C_{S,F,G-F} \leq m - \Delta_k \), since the right side is the total edge-weight in \( G-F \). Thus, \( \mathbb{E}_{S \sim D} [C_{S,F,G-F}] \leq \max_S C_{S,F,G-F} \leq m - \Delta_k \), as claimed.

**Proof of Theorem 6.4.** Let \( F \in \binom{V}{k} \) be arbitrary. Note that for \( S \sim \mathcal{U} \), every edge \( e \in E \) crosses \( S \) with probability \( 1/2 \); hence, by linearity of expectation, it holds that \( \mathbb{E}_{S \sim \mathcal{U}} [d_S(F)] = d(F)/2 \), and \( \mathbb{E}_{S \sim \mathcal{U}} [C_S] = m/2 \).

By Observation 6.6, we have

\[
\mathbb{E}_{S \sim \mathcal{U}} [C_{S,F,G-F}] = \mathbb{E}_{S \sim \mathcal{U}} [C_S] - \mathbb{E}_{S \sim \mathcal{U}} [d_S(F)] = (m - d(F))/2 \geq (m - \Delta_k)/2,
\]

since, by definition, \( d(F) \leq \Delta_k \). Since this holds for every \( F \in \binom{V}{k} \), Observation 6.6 implies that \( \mathcal{U} \) is a \( 1/2 \)-approximation, as claimed.

\[ \square \]
7 Discussion

Our work leaves several open questions regarding fault tolerant Max-Cut. An immediate question is to bridge the (rather small) gap between our approximation of $(0.8780 - \epsilon)$ and our hardness of $\alpha_{GW}$ for $k$-AFTcut.

The central bottleneck is that Simultaneous Max-Cut, a main ingredient in our algorithm, has hardness of approximation that is slightly below $\alpha_{GW}$ and equals $(\alpha_{GW} - \delta)$ (where $\delta \geq 10^{-5}$) [11]. Thus, either one finds a different algorithm for $k$-AFTcut that does not rely on Simultaneous Max-Cut and achieves an approximation of $\alpha_{GW}$, or one can extend the hardness result of [11] to $k$-AFTcut and thus rule out an approximation of $\alpha_{GW}$ for $k$-AFTcut. Another question is what approximation factors can be obtained for AFTcut on general weighted graphs.

Another interesting question is how to deal with a non-constant number of faults, for both of the adversaries. Since the number of all possible cases of failure is not polynomial, a new approach may be needed. There are techniques that are used to deal with a non-constant number of faults, e.g., failure sampling, that is presented in [24]. It would be interesting to see whether these techniques can be used for fault tolerant Max-Cut as well.

One more important and intriguing open question is what happens in other fault tolerant problems when an oblivious adversary is considered. We are unaware of previous algorithms for an oblivious adversary in the fault-tolerance literature. Since an oblivious adversary is arguably more realistic in its nature, and since it is likely that one can get improved algorithms for this case, pursuing this line of research could be crucial for many additional fundamental problems involving fault tolerance.

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