Translation invariant pure states and its split property

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Abstract

A translation invariant state in quantum spin chain is determined uniquely upto isomorphism by a Markov map on the support projection of an associated Cuntz’s state. We prove that Kolmogorov’s property of the Markov map is a necessary and sufficient condition for such a state to be pure. Kolmogorov’s property naturally give rise to a Mackey’s system of imprimitivity for the group of integers. A duality argument originated from non-commutative probability theory is employed to prove an elegant alternative necessary and sufficient condition for pureness. Main result of this theory made it possible to prove Haag duality property of any translation invariant lattice symmetric pure state. Further such a real state is split if special correlation function decays exponentially. The last statement proves T Matsui’s conjecture on split property for a translation invariant real lattice symmetric pure state.
1 Introduction:

Let \((A_R, \theta_R, \omega_R)\) be a \(C^*\)-dynamical system of endomorphism, where \(A_R\) be a \(C^*\)-algebra, \(\theta\) be an injective unital endomorphism and \(\omega_R\) be a \(\theta_R\)-invariant state. Let \((A, \theta, \omega)\) be the \(C^*\) inductive limit of \(A_R \rightarrow^{\theta_R} A_R\) with inductive limit state as \(\omega\) on \(A\) and \(\theta\) be the associated automorphism. The map \(\omega_R \rightarrow \omega\) is one to one and affine on the convex set of \(\theta_R\)-invariant states on \(A_R\) and thus an \(\theta_R\) invariant factor state goes to an \(\theta\)-invariant factor state. However such a statement for pure states is not true. One general mathematical problem [Pow1,Pow2,Ar2,BJP,BJKW] in non-commutative ergodic theory in \(C^*\)-dynamical systems is addressed on these questions and looked for an useful criteria for the inductive limit state \(\omega\) to be pure. There is a striking similarity between the construction of inductive limit states [Sa] associated with an invaraint state of an injective unital endomorphism on a \(C^*\)-algebra and Kolmogorov’s construction of stationary Markov processes [AcM] associated with an invariant state of a Markov map. Exploring this in [Mo2] we have obtained a sufficient condition in terms of Kolmogorov’s property of an associated Markov semigroup \((M, \tau, \phi_0)\) on the support projections of the state \(\omega_R\) in the GNS space associated with \(\omega_R\). It seems that such a criteria is also new to classical situation where \(A_R\) is a commutative \(C^*\)-algebra. Such a classical situation includes classical interacting particle systems [Li].

In this paper our interest is to deal with the same problem however in a special case where \(A_R = \otimes_{\mathbb{Z}_+} M_d(C)\) is the \(C^*\) -completion of the infinite tensor product of the algebra \(M_d(C)\) of \(d\) by \(d\) complex matrices and \(\theta_R\) be the right shift on \(A_R\). \(A_R\) is a UHF (Uniformly Hyper Finite) \(C^*\) algebra of Glimm type \((d^n : n \geq 1)\). Pure states on a UHF algebra are studied in the general framework of [Pow1],[Pow2] also in [Ar2],[Mo2]. The \(\ast\)-automorphism group
of UHF acts transitively on the set of pure states [Pow1]. Such a situation has been investigated in details at various degrees of generality in [BJP] and [BJKW] with primary motivation to develop a $C^*$ algebraic method to study iterative function systems and its associated wavelet theory. One interesting result in [BJP] says that any translation invariant pure state on $\mathcal{A}_R$ is also a product state and the canonical endomorphism associated with two such states are unitarily equivalent. However such a statement is not true for two translation invariant pure states on $\mathcal{A}$ as their restriction to $\mathcal{A}_R$ need not be isomorphic. Thus the problem, classification of all translation invariant pure state on $\mathcal{A}$ upto unitary isomorphism, is a delicate one. Since a $\theta$ invariant states $\omega$ on $\mathcal{A}$ is completely determined by its restriction $\omega_R$ to $\mathcal{A}_R$, in principle it is possible to describe various property of $\omega$ including pureness by studying certain properties of their restriction $\omega_R$. Since pureness of $\omega_R$ is not necessary for pureness of $\omega$, a valid fundamental question that arise here: What are the parameters that determines both $\omega$ and $\omega_R$ uniquely and how these parameters determines properties of $\omega$ and $\omega_R$?

The papers on translation invariant ‘quantum Markov states’ [Ac] or ‘finitely correlated states’ [FNW1,FNW2] prove existence of a canonical quantum dynamical semigroup on matrix algebra. Exploring Popescu’s work on representation theory of Cuntz algebra [Cu], Bratteli, Jorgensen and Price [BJP] developed a general method valid for any translation invariant state. In a follow up paper Bratteli, Jorgensen, Kishimoto and Werner [BJKW] studied associated Popescu systems in details and a duality argument is used towards the same problem for $\mathcal{A}$. Theorem 7.1 in [BJKW] has aimed towards a sufficient condition on the associated Popescu elements for purity of the translation invariant state. However the proof is faulty as certain argument is not symmetric. Lemma 7.6 in [BJKW] needs additional assumption related to the support pro-
jection of the dual Cuntz's elements. Besides this additional structure proof of Lemma 7.8 in [BJKW] is also not complete unless we find a proof for \( \tilde{\mathcal{M}} = \mathcal{M}' \) (we retained same notations here in the text) for such a factor state \( \omega \). Thus main body of the proof for Theorem 7.1 in [BJKW] is incomplete.

Here we will retain the same framework but deviate from the main argument used in [BJKW] to prove that one of the additional condition namely \( \tilde{\mathcal{M}} = \mathcal{M}' \) has no relation with purity of \( \omega \). Our main mathematical results in Theorem 3.5 include a necessary and sufficient condition on support projections of associated Cuntz's states for purity of \( \omega \). We arrived at this criteria by proving first that Kolmogorov's property of the canonical Markov map is necessary and sufficient for purity of \( \omega \) and thus finds it's natural relation to Mackey's systems of imprimitivity for the group of integers. Using uniqueness part of Mackey's theory we prove necessary part of the refined criteria which has originated from [BJKW]. The argument used in the proof is rather delicate as it involves Hausdorff maximality theorem.

Further we prove that Haag duality property for such a pure state holds if and only if \( \mathcal{M}' = \tilde{\mathcal{M}} \). Such an equality i.e. \( \mathcal{M}' = \tilde{\mathcal{M}} \) can be ensured if \( \omega \) is also lattice symmetric. This additional symmetry helps us to reformulate relations between split property and decaying property of special correlation functions of a real state as a problem in the GNS Hilbert space (see details below).

We briefly set the standard notation and known relations in the following text. The quantum spin chain that we consider here is described by a UHF \( C^* \)-algebra denoted by \( \mathcal{A} = \otimes_{\mathbb{Z}} M_d(C) \). Here \( \mathcal{A} \) is the \( C^* \) -completion of the infinite tensor product of the algebra \( M_d(C) \) of \( d \) by \( d \) complex matrices, each component of the tensor product element is indexed by an integer \( j \). Let \( Q \) be a matrix in \( M_d(C) \). By \( Q^{(j)} \) we denote the element \( \ldots \otimes 1 \otimes 1 \ldots 1 \otimes Q \otimes 1 \otimes \ldots 1 \otimes \ldots , \ldots \).
where $Q$ appears in the $j$-th component. Given a subset $\Lambda$ of $Z$, $\mathcal{A}_\Lambda$ is defined as the $C^*$-subalgebra of $\mathcal{A}$ generated by all $Q^{(j)}$ with $Q \in M_d(C)$, $j \in \Lambda$. We also set

$$\mathcal{A}_{loc} = \bigcup_{\Lambda: |\Lambda| < \infty} \mathcal{A}_\Lambda$$

where $|\Lambda|$ is the cardinality of $\Lambda$. Let $\omega$ be a state on $\mathcal{A}$. The restriction of $\omega$ to $\mathcal{A}_\Lambda$ is denoted by $\omega_\Lambda$. We also set $\omega_R = \omega_{[1,\infty)}$ and $\omega_L = \omega_{(-\infty,0]}$. The translation $\theta_k$ is an automorphism of $\mathcal{A}$ defined by $\theta_k(Q^{(j)}) = Q^{(j+k)}$. Thus $\theta_1, \theta_{-1}$ are unital $*$-endomorphism on $\mathcal{A}_R$ and $\mathcal{A}_L$ respectively. We say $\omega$ is translation invariant if $\omega \circ \theta_k = \omega$ on $\mathcal{A}$ ( $\omega \circ \theta_1 = \omega$ on $\mathcal{A}$ ). In such a case $(\mathcal{A}_R, \theta_1, \omega_R)$ and $(\mathcal{A}_L, \theta_{-1}, \omega_L)$ are two unital $*$-endomorphisms with invariant states. It is well known [Pow1] that translation invariant state $\omega$ is a factor (i.e. the GNS representation is a factor representation ) if and only if

$$\lim_{|k| \to \infty} \omega(Q_1 \theta_k(Q_2)) \to \omega(Q_1)\omega(Q_2)$$

for all $Q_1, Q_2$ in $\mathcal{A}$. Similar statement with appropriate direction of limit is valid for $\omega_L, \omega_R$. Thus for a translation invariant factor state $\omega$ of $\mathcal{A}$, states $\omega_R$ and $\omega_L$ are factors too. Converse is also true.

However for a translation invariant pure state $\omega$ on $\mathcal{A}$, the factor state $(\mathcal{A}_R, \omega_R)$ need not be pure or even type-I in general. The unique ground state $\omega$ for XY model [Ma1] is a pure state on $\mathcal{A}$ for which $(\mathcal{A}_R, \omega_R)$ is type-III$_1$. A general question that is central here when can we guarantee that $\omega_R(\omega_L)$ are type-I factors? To that end we recall [BR,Ma2] a standard definition of a state to be split in the following.

**DEFINITION 1.1:** Let $\omega$ be a translation invariant state on $\mathcal{A}$. We say that $\omega$ is *split* if the following condition is valid: Given any $\epsilon > 0$ there exists a $k \geq 1$ so that
where the above supremum is taken over all local elements $Q \in \mathcal{A}_{(-\infty,-k] \cup [k,\infty)}$ with the norm less than 1.

Here we recall few simple facts from [Pow1,BR,Ma1,Ma2]. The uniform cluster condition is valid if and only if the state $\omega$ is quasi-equivalent to the product state $\psi_L \otimes \psi_R$ of a state $\psi_L$ of $\mathcal{A}_L$ and another state $\psi_R$ of $\mathcal{A}_R$. Thus a Gibbs state of a finite range interaction is split. On the other hand if $\omega$ is a pure translation invariant state, then $\omega$ is a factor state. Furthermore in such a case $\omega_R(\omega_L)$ is type-I if and only if $\omega$ is also a split state. The canonical trace is a non-split pure state and unique ground state of XY model [AMa,Ma2] is a non-split pure state. One central aim is to find a criteria for a pure translation invariant state to be split. To that end we present a precise definition for exponential decay.

**DEFINITION 1.2:** Let $\omega$ be a translation invariant state on one dimensional spin chain $\mathcal{A}$. We say the two point spacial correlation functions for $\omega$ decay exponentially if there exists a $\delta > 0$ so that

$$e^{\delta k}|\omega(Q_1 \theta_k(Q_2)) - \omega(Q_1)\omega(Q_2)| \to 0 \quad (1.2)$$

as $|k| \to \infty$ for any local elements $Q_1, Q_2 \in \mathcal{A}$.

A translation invariant state $\omega$ is said to be in detailed balance if $\omega$ is lattice symmetric and real (for details see section 3). The canonical trace on $\mathcal{A}$ is both real and lattice symmetric. This notion of detailed balance state is introduced as an reminiscence of Onsager’s reciprocal relations explored in recent articles [AM,Mo1,Mo2] on non-commutative probability theory. Here we also recall well known notion [DLS] that a state $\omega$ on $\mathcal{A}$ is called reflection
positive if \( \omega(xJ(x)) \geq 0 \) for all \( x \in A_R \) where \( J \) is the reflection map from \( A_R \) onto \( A_L \). As an application of our main result Theorem 3.5 we have the following theorem.

**THEOREM 1.3:** Let \( \omega \) be a pure translation invariant lattice symmetric state on \( A \). Then the following hold:

(a) \( \omega \) admits Haag duality property i.e. \( \pi_\omega(A_R)' = \pi_\omega(A_L)'' \);

(b) If \( \omega \) is also real then

(i) \( \omega \) is reflection positive;

(ii) If two point spatial correlation function for \( \omega \) decays exponentially then \( \omega \) is a split state i.e. \( \pi_\omega(A_R)'' \) is a type-I factor.

The paper is organized as follows. In section 2 we study Popescu systems associated with a translation invariant state on Cuntz algebra \( O_d \) and review ‘commutant lifting theorem’ investigated in [BJKW]. The proof presented here remove the murky part of the proof of Theorem 5.1 in [BJKW]. In section 3 we explore both the notion of Kolmogorov’s shift and intimate relation with Mackey’s imprimitivity systems to explore a duality argument introduced in [BJKW]. We find a useful necessary and sufficient condition in terms of support projection of Cuntz’s state for a translation invariant factor state \( \omega \) on \( A \) to be pure. The criteria on support projection is crucial to prove our main mathematical result Theorem 3.5 and its application Proposition 3.9. Proof of Theorem 1.3 appeared in Theorem 3.10 and Theorem 4.4. The criteria also find its non-trivial application towards an elegant criteria for phase transition in ground states [Mo3] for some prime Hamiltonian such as half-odd integer XXX anti-ferromagnetic Heisenberg model in one dimensional lattice quantum spin chain. However we defer the application and refer the interested reader to drafts [Mo3] where we deal with a general theory motivated by results [AL],[AKLT]
towards Haldane's conjecture on ground states of anti-ferromagnets. Appendix includes central results used here from non-commutative probability theory.

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2 States on $\mathcal{O}_d$ and the commutant lifting theorem

First we recall that the Cuntz algebra $\mathcal{O}_d (d \in \{2, 3, \ldots, \})$ is the universal $C^*$-algebra generated by the elements $\{s_1, s_2, \ldots, s_d\}$ subject to the relations:

\[ s_i^*s_j = \delta_{ij}1 \]
\[ \sum_{1 \leq i \leq d} s_is_i^* = 1. \]

There is a canonical action of the group $U(d)$ of unitary $d \times d$ matrices on $\mathcal{O}_d$ given by

\[ \beta_g(s_i) = \sum_{1 \leq j \leq d} g_{ij}^*s_j \]

for $g = (g_{ij}^*) \in U(d)$. In particular the gauge action is defined by

\[ \beta_z(s_i) = zs_i, \quad z \in \mathbb{D} = S^1 = \{z \in \mathbb{C} : |z| = 1\}. \]

If UHF$_d$ is the fixed point subalgebra under the gauge action, then UHF$_d$ is
the closure of the linear span of all wick ordered monomials of the form
\[ s_{i_1} \cdots s_{i_k} s_{j_k}^* \cdots s_{j_l}^* \]
which is also isomorphic to the UHF\(_d\) algebra
\[ M_{d^\infty} = \otimes_1^\infty M_d \]
so that the isomorphism carries the wick ordered monomial above into the matrix element
\[ e_{i_1}^1(1) \otimes e_{i_2}^2(2) \otimes \cdots \otimes e_{i_k}^k(k) \otimes 1 \otimes 1 \cdots \]
and the restriction of \( \beta_g \) to UHF\(_d\) is then carried into action
\[ \text{Ad}(g) \otimes \text{Ad}(g) \otimes \text{Ad}(g) \otimes \cdots \]

We also define the canonical endomorphism \( \lambda \) on \( \mathcal{O}_d \) by
\[ \lambda(x) = \sum_{1 \leq i \leq d} s_i x s_i^* \]
and the isomorphism carries \( \lambda \) restricted to UHF\(_d\) into the one-sided shift
\[ y_1 \otimes y_2 \otimes \cdots \to 1 \otimes y_1 \otimes y_2 \cdots \]
on \( \otimes_1^\infty M_d \). Note that \( \lambda \beta_g = \beta_g \lambda \) on UHF\(_d\).

Let \( d \in \{2, 3, \ldots\} \) and \( \mathbb{Z}_d \) be a set of \( d \) elements. \( \mathcal{I} \) be the set of finite sequences \( I = (i_1, i_2, \ldots, i_m) \) where \( i_k \in \mathbb{Z}_d \) and \( m \geq 1 \). We also include empty set \( \emptyset \in \mathcal{I} \) and set \( s_\emptyset = 1 = s_\emptyset^* \), \( s_I = s_{i_1} \cdots s_{i_m} \in \mathcal{O}_d \) and \( s_I^* = s_{i_m}^* \cdots s_{i_1}^* \in \mathcal{O}_d \). In
the following we recall a commutant lifting theorem (Theorem 5.1 in [BJKW]), crucial for our purpose.

**THEOREM 2.1:** Let \( v_1, v_2, \ldots, v_d \) be a family of bounded operators on a Hilbert space \( \mathcal{K} \) so that \( \sum_{1 \leq k \leq d} v_k v_k^* = I \). Then there exists a unique up to
isomorphism Hilbert space $\mathcal{H}$, a projection $P$ on $\mathcal{K}$ and a family of isometries \( \{S_k : 1 \leq k \leq d, P\} \) satisfying Cuntz’s relation so that

$$PS_k^*P = S_k^*P = v_k^*$$

for all $1 \leq k \leq d$ and $\mathcal{K}$ is cyclic for the representation i.e. the vectors \( \{S_I\mathcal{K} : |I| < \infty\} \) are total in $\mathcal{H}$.

Moreover the following hold:

(a) $\Lambda_n(P) \uparrow I$ as $n \uparrow \infty$;

(b) For any $D \in \mathcal{B}_\tau(\mathcal{K})$, $\Lambda_n(D) \to X'$ weakly as $n \to \infty$ for some $X'$ in the commutant \( \{S_k, S_k^* : 1 \leq k \leq d\}' \) so that $PX'P = D$. Moreover the self adjoint elements in the commutant \( \{S_k, S_k^* : 1 \leq k \leq d\}' \) is isometrically order isomorphic with the self adjoint elements in $\mathcal{B}_\tau(\mathcal{K})$ via the surjective map $X' \to PX'P$, where $\mathcal{B}_\tau(\mathcal{K}) = \{x \in \mathcal{B}(\mathcal{K}) : \sum_{1 \leq k \leq d} v_k x v_k^* = x\}$.

(c) \( \{v_k, v_k^*, 1 \leq k \leq d\}' \subseteq \mathcal{B}_\tau(\mathcal{K}) \) and equality hold if and only if $P \in \{S_k, S_k, 1 \leq k \leq d\}''$.

**PROOF:** Following Popescu [Po] we define a completely positive map $R : \mathcal{O}_d \to \mathcal{B}(\mathcal{K})$ by

$$R(s_I s_j^*) = v_I v_j^*$$

for all $|I|, |J| < \infty$. The representation $S_1, \ldots, S_d$ of $\mathcal{O}_d$ on $\mathcal{H}$ thus may be taken to be the Stinespring dilation of $R$ [BR, vol-2] and uniqueness up to unitary equivalence follows from uniqueness of the Stinespring representation. That $\mathcal{K}$ is cyclic for the representation follows from the minimality property of the Stinespring dilation. For (a) let $Q$ be the limiting projection. Then we have $\Lambda(Q) = Q$, hence $Q \in \{S_k, S_k^*\}'$ and $Q \geq P$. In particular $QS_1 f = S_1 f$ for all $f \in \mathcal{K}$ and $|I| < \infty$. Hence $Q = I$ by the cyclicity of $\mathcal{K}$. For (b) essentially we defer from the argument used in Theorem 5.1 in [BJKW]. We fix any $D \in \mathcal{B}_\tau(\mathcal{K})$ and note that $P \Lambda_k(D)P = \tau_k(D) = D$ for any $k \geq 1$. Thus
for any integers \( n > m \) we have

\[
\Lambda_m(P)\Lambda_n(D)\Lambda_m(P) = \Lambda_m(P\Lambda_{n-m}(D)P) = \Lambda_m(D)
\]

Hence for any fix \( m \geq 1 \) limit \( \langle f, \Lambda_n(D)g \rangle \) as \( n \to \infty \) exists for all \( f, g \in \Lambda_m(P) \). Since the family of operators \( \Lambda_n(D) \) is uniformly bounded and \( \Lambda_m(P) \uparrow I \) as \( m \to \infty \), a standard density argument guarantees that the weak operator limit of \( \Lambda_n(D) \) exists as \( n \to \infty \). Let \( X' \) be the limit. So \( \Lambda(X') = X' \), by Cuntz’s relation, \( X' \in \{S_k, S_k^* : 1 \leq k \leq d\}' \). Since \( P\Lambda_n(D)P = D \) for all \( n \geq 1 \), we also conclude that \( PX'P = D \) by taking limit \( n \to \infty \). Conversely it is obvious that \( P\{S_k, S_k^* : k \geq 1\}'P \subseteq B_r(K) \). Hence we can identify \( P\{S_k, S_k^* : k \geq 1\}'P \) with \( B_r(K) \).

Further it is obvious that \( X' \) is self-adjoint if and only if \( D = PX'P \) is self-adjoint. Now fix any self-adjoint element \( D \in B_r(K) \). Since identity operator on \( K \) is an element in \( B_r(K) \) for any \( \alpha \geq 0 \) for which \( -\alpha P \leq D \leq \alpha P \), we have \( \alpha\Lambda_n(P) \leq \Lambda_n(D) \leq \alpha\Lambda_n(P) \) for all \( n \geq 1 \). By taking limit \( n \to \infty \) we conclude that \( -\alpha I \leq X' \leq \alpha I \), where \( PX'P = D \). Since operator norm of a self-adjoint element \( A \) in a Hilbert space is given by

\[
||A|| = \inf_{\alpha \geq 0} \{\alpha : -\alpha I \leq A \leq \alpha I\}
\]

we conclude that \( ||X'|| \leq ||D|| \). That \( ||D|| = ||PX'P|| \leq ||X'|| \) is obvious, \( P \) being a projection. Thus the map is isometrically order isomorphic taking self-adjoint elements of the commutant to self-adjoint elements of \( B_r(K) \).

We are left to prove (c). Inclusion is trivial. For the last part note that for any invariant element \( D \) in \( B(K) \) there exists an element \( X' \) in \( \{S_k, S_k^* , 1 \leq k \leq d\}' \) so that \( PX'P = D \). In such a case we verify that \( Dv_k^* = PX'PS_k^*P = PX'S_k^*P = PS_k^*X'P = PS_k^*PX'P = v_k^*D \). We also have \( D^* \in B_r(K) \) and thus \( D^*v_k^* = v_k^*D^* \). Hence \( D \in \{v_k, v_k^* : 1 \leq k \leq d\}' \). Since \( P\pi_\omega(\mathcal{O}_d)'P = B(K)_r \),
we conclude that $B(K)_r \subseteq \mathcal{M}'$. Thus equality hold whenever $P \in \{S_k, S^*_k, 1 \leq k \leq d\}''$. For converse note that by commutant lifting property self-adjoint elements of the commutant $\{S_k, S^*_k, 1 \leq k \leq d\}'$ is order isometric with the algebra $\mathcal{M}'$ via the map $X' \rightarrow PX'P$. Hence $P \in \{S_k, S^*_k, 1 \leq k \leq d\}''$ by Proposition 4.2 in [BJKW].

A family $(v_k, 1 \leq k \leq d)$ of contractive operators on a Hilbert space $K$ is called Popescu’s elements and dilation $(\mathcal{H}, P, \mathcal{K}, S_k, 1 \leq k \leq d)$ in Theorem 2.1 is called Popescu’s dilation to Cuntz elements. In the following proposition we deal with a family of minimal Popescu elements for a state on $\mathcal{O}_d$.

**THEOREM 2.2:** There exists a canonical one-one correspondence between the following objects:

(a) States $\psi$ on $\mathcal{O}_d$

(b) Function $C : \mathcal{I} \times \mathcal{I} \rightarrow \mathbb{C}$ with the following properties:
   (i) $C(\emptyset, \emptyset) = 1$;
   (ii) for any function $\lambda : \mathcal{I} \rightarrow \mathbb{C}$ with finite support we have
       $$\sum_{I,J \in \mathcal{I}} \lambda(I)C(I,J)\lambda(J) \geq 0$$
   (iii) $\sum_{i \in \mathbb{Z}^d} C(Ii, Ji) = C(I, J)$ for all $I, J \in \mathcal{I}$.

(c) Unitary equivalence class of objects $(\mathcal{K}, \Omega, v_1, \ldots, v_d)$ where
   (i) $\mathcal{K}$ is a Hilbert space and $\Omega$ is an unit vector in $\mathcal{K}$;
   (ii) $v_1, \ldots, v_d \in B(\mathcal{K})$ so that $\sum_{i \in \mathbb{Z}^d} v_i v_i^* = 1$;
   (iii) the linear span of the vectors of the form $v_i^* \Omega$, where $I \in \mathcal{I}$, is dense in $\mathcal{K}$.

Where the correspondence is given by a unique completely positive map $R : \mathcal{O}_d \rightarrow B(\mathcal{K})$ so that
\begin{itemize}
\item[(i)] \( R(s_I s_J^*) = v_I^* v_J^*; \)
\item[(ii)] \( \psi(x) = \langle \Omega, R(x)\Omega \rangle; \)
\item[(iii)] \( \psi(s_I s_J^*) = C(I, J) = \langle v_I^* \Omega, v_J^* \Omega \rangle. \)
\item[(iv)] For any fix \( g \in U_d \) and the completely positive map \( R_g : \mathcal{O}_d \to \mathcal{B}(K) \) defined by \( R_g = R \circ \beta_g \) give rises to a Popescu system given by \( (K, \Omega, \beta_g(v_i), ..., \beta_g(v_d)) \) where \( \beta_g(v_i) = \sum_{1 \leq j \leq d} g_i^j v_j. \)
\end{itemize}

**PROOF:** For a proof we simply refer to Proposition 2.1 in [BJKW].

The following is a simple consequence of Theorem 2.1 valid for a \( \lambda \)-invariant state \( \psi \) on \( \mathcal{O}_d \). This proposition will have very little application in main body of this paper but this gives a clear picture explaining the delicacy of the present problems.

**PROPOSITION 2.3:** Let \( \psi \) be a state on \( \mathcal{O}_d \) and \( (\mathcal{H}, \pi, \Omega) \) be the GNS space associated with \( (\mathcal{O}_d, \psi) \). We set \( S_i = \pi(s_i) \) and normal state \( \psi_{\Omega} \) on \( \pi(\mathcal{O}_d)'' \) defined by
\[
\psi_{\Omega}(X) = \langle \Omega, X\Omega \rangle
\]
Let \( P \) be the projection on the closed subspace \( K \) generated by the vectors \( \{S_i^* \Omega : |I| < \infty\} \) and
\[
v_k = PS_k P \tag{2.3}
\]
for \( 1 \leq k \leq d \). Then following hold:
\begin{itemize}
\item[(a)] \( \{v_I^* \Omega : |I| < \infty\} \) is total in \( K \).
\item[(b)] \( \sum_{1 \leq k \leq d} v_k v_k^* = I; \)
\item[(c)] \( S_k^* P = PS_k^* P \) for all \( 1 \leq k \leq d; \)
\item[(d)] For any \( I = (i_1, i_2, ..., i_k), J = (j_1, j_2, ..., j_l) \) with \( |I|, |J| < \infty \) we have
\[
\psi(s_I s_J^*) = \langle \Omega, v_I^* v_J^* \Omega \rangle \tag{2.4}
\]
\end{itemize}
and the vectors \( \{ S_I f : f \in \mathcal{K}, |I| < \infty \} \) are total in the GNS Hilbert space associated with \((\mathcal{O}_d, \psi)\). Further such a family \((\mathcal{K}, v_k, 1 \leq k \leq d, \omega)\) satisfying (a) to (d) are determined uniquely up to isomorphism.

Conversely given a Popescu system \((\mathcal{K}, v_k, 1 \leq k \leq d, \Omega)\) satisfying (a) and (b) there exists a unique state \(\psi\) on \(\mathcal{O}_d\) so that (c) and (d) are satisfied.

Furthermore the following statements are valid:

(e) If the normal state \(\phi_0(x) = \langle \Omega, x \Omega \rangle\) on the von-Neumann algebra \(\mathcal{M} = \{v_i, v_i^*\}''\) is invariant for the Markov map \(\tau(x) = \sum_{1 \leq k \leq d} v_i x v_i^*, x \in \mathcal{M}\) then \(\psi\) is \(\lambda\) invariant and \(\phi_0\) is faithful on \(\mathcal{M}\).

(f) If \(P \in \pi(\mathcal{O})''\) then following are equivalent:

(i) \(\psi\) is an ergodic state for \((\mathcal{O}_d, \lambda)\);

(ii) \((\mathcal{M}, \tau, \phi_0)\) is ergodic.

In such a case \(\mathcal{M}\) is a factor.

**PROOF:** We fix a state \(\psi\) and consider the GNS space \((\mathcal{H}, \pi, \Omega)\) associated with \((\mathcal{O}_d, \psi)\) and set \(S_i = \pi(s_i)\). It is obvious that \(S_k^* P \subseteq P\) for all \(1 \leq k \leq d\), thus \(P\) is the minimal subspace containing \(\Omega\) and invariant by all \(\{S_k^*; 1 \leq k \leq d\}\) i.e.

\[
PS_k^* P = S_k^* P
\]  

(2.5)

Thus \(v_k^* = PS_k^* P = S_k^* P\) and so \(\sum_k v_k v_k^* = \sum_k PS_k S_k^* P = P\) which is identity operator in \(\mathcal{K}\). This completes the proof of (a) (b) and (c).

For (d) we note that

\[\psi(s_i s_j^*) = \langle \Omega, S_i S_j^* \Omega \rangle\]

\[= \langle \Omega, PS_i S_j^* P \Omega \rangle = \langle \Omega, v_i v_j^* \Omega \rangle.\]
Since $\mathcal{H}$ is spanned by the vectors \{\(S_I S_J^* \Omega : |I|, |J| < \infty\)\} and $\mathcal{K}$ is spanned by the vectors \(\{S_I^* \Omega = v_I^* \Omega : |I| < \infty\}\), $\mathcal{K}$ is cyclic for $S_I$ i.e. the vectors \(\{S_I \mathcal{K} : |I| < \infty\}\) spans $\mathcal{H}$. Uniqueness up to isomorphism follows as usual by total property of vectors $v_I^* \Omega$ in $\mathcal{K}$.

Conversely for a Popescu systems ($\mathcal{K}, v_i, \Omega$) satisfying (a) and (b), we consider the family ($\mathcal{H}, S_k$, 1 \(\leq k \leq d, P\) of Cuntz’s elements defined as in Theorem 2.1. We claim that $\Omega$ is a cyclic vector for the representation $\pi(s_i) \rightarrow S_i$. Note that by our construction vectors \(\{S_I f, f \in K : |I| < \infty\}\) are total in $\mathcal{H}$ and $v_I^* \Omega = S_I^* \Omega$ for all $|J| < \infty$. Thus by our hypothesis that vectors \(\{v_I^* \Omega : |I| < \infty\}\) are total in $\mathcal{K}$, we verify that vectors \(\{S_I S_J^* \Omega : |I|, |J| < \infty\}\) are total in $\mathcal{H}$. Hence $\Omega$ is a cyclic for the representation $s_i \rightarrow S_i$ of $\mathcal{O}_d$.

We left to prove (e) and (f). It simple to note by (d) that $\psi \lambda = \psi$ i.e.

$$\sum_i < \Omega, S_i S_I S_J^* S_i^* \Omega > = \sum_i < \Omega, v_i v_I^* v_J^* v_i^* \Omega >$$

$$= < \Omega, v_I^* v_J^* v_i^* \Omega > = < \Omega, S_I S_J^* \Omega >$$

for all $|I|, |J| < \infty$ where in the second equality we have used our hypothesis that the vector state $\phi_0$ on $\mathcal{M}$ is $\tau$-invariant. In such case we aim now to show that $\phi_0$ is faithful on $\mathcal{M}$. To that end let $p'$ be the support projection in $\mathcal{M}$ for $\tau$ invariant state $\phi_0$. Thus $\phi_0(1-p') = 0$ i.e. $p' \Omega = \Omega$ and by invariance we also have $\phi_0(p' \tau (1-p') p') = \phi_0(1-p') = 0$. Since $p' \tau (1-p') p' \geq 0$ and an element in $\mathcal{M}$, by minimality of support projection, we conclude that $p' \tau (1-p') p' = 0$. Hence $p' \Omega = \Omega$ and $p' v_k^* p' = v_k^* p'$ for all $1 \leq k \leq d$. Thus $p' v_I^* \Omega = v_I^* \Omega$ for all $|I| < \infty$. As $\mathcal{K}$ is the closed linear span of the vectors \(\{v_I^* \Omega : |I| < \infty\}\), we conclude that $p' = p$. In other words $\phi_0$ is faithful on $\mathcal{M}$. This completes the proof for (e).

We are left to show (f) where we assume that $P \in \pi(\mathcal{O}_d)''$. $\Omega$ being a cyclic
vector for $\pi(O_d)''$, the weak* limit of the increasing projection $\Lambda^k(P)$ is $I$. Thus by Theorem 3.6 in [Mo1] we have $\pi(O_d)''$, $\Lambda$, $\psi_\Omega$ is ergodic if and only if the reduced dynamics $(M, \tau, \phi_0)$ is ergodic. Last part of the statement is an easy consequence of a Theorem of D. E. Evans [Ev], (also see [Fr], [Mo1], [BJKW]).

Before we move to next result we comment here that in general for a $\lambda$ invariant state on $O_d$ the normal state $\phi_0$ on $M = \{v_k, v_k^* : 1 \leq k \leq d\}''$ need not be invariant for $\tau$. To that end we consider ([BR] vol-II page 110) the unique KMS state $\psi = \psi_\beta$ for the automorphism $\alpha_t(s_i) = e^{it}s_i$ on $O_d$. $\psi$ is $\lambda$ invariant and $\psi|UHF_d$ is the unique faithful trace. $\psi$ being a KMS state for an automorphism, the normal state induced by the cyclic vector on $\pi\psi(O_d)''$ is also separating for $\pi(O_d)''$. As $\psi \beta_z = \psi$ for all $z \in S^1$ we have $\langle \Omega, \pi(s_I)\Omega \rangle = \langle \Omega, \beta_z(s_I)\Omega \rangle = z^{|I|} \langle \Omega, \pi(s_I)\Omega \rangle$ for all $z \in S^1$ and so $\langle \Omega, \pi(s_I)\Omega \rangle = 0$ for all $|I| \geq 1$. In particular $\langle \Omega, v_i^*\Omega \rangle = 0$ where $(v_i)$ are defined as Proposition 2.3 and thus $\langle v_i\Omega, v_i^*\Omega \rangle = \langle \Omega, v_i^*v_I\Omega \rangle = 0$ for all $1 \leq i \leq d$. Hence $v_i\Omega = 0$. By Proposition 2.3 (e), $\Omega$ is separating for $M$ and so we get $v_i = 0$ for all $1 \leq i \leq d$ and this contradicts that $\sum v_i v_i^* = 1$. Thus we conclude by Proposition 2.3 (e) that $\phi_0$ is not $\tau$ invariant on $M$. This example also indicates that the support projection of a $\lambda$ invariant state $\psi$ in $\pi(O_d)''$ need not be equal to the minimal sub-harmonic projection $P$ i.e. the closed span of vectors $\{S_i^*\Omega : |I| < \infty\}$ containing $\Omega$ and $\{v_I v_J^* : |I|, |J| < \infty\}$ need not be even an algebra.

Now we aim to deal with another class of Popescu elements associated with an $\lambda$-invariant state on $O_d$. In fact this class of Popescu elements will play a significant role for the rest of the text and we will repeatedly use this proposition!
PROPOSITION 2.4: Let $\mathcal{H}, \pi, \Omega$ be the GNS representation of a $\lambda$ invariant state $\psi$ on $\mathcal{O}_d$ and $P$ be the support projection of the normal state $\psi\Omega(X) = <\Omega, X\Omega>$ in the von-Neumann algebra $\pi(\mathcal{O}_d)''$. Then the following hold:

(a) $P$ is a sub-harmonic projection for the endomorphism $\Lambda(X) = \sum_k S_k XS_k^*$ on $\pi(\mathcal{O}_d)''$ i.e. $\Lambda(P) \geq P$ satisfying the following:
   (i) $\Lambda_n(P) \uparrow I$ as $n \uparrow \infty$;
   (ii) $PS_k^*P = S_k^*P$, $1 \leq k \leq d$;
   (iii) $\sum_{1 \leq k \leq d} v_kv_k^* = I$
       where $S_k = \pi(s_k)$ and $v_k = PS_kP$ for $1 \leq k \leq d$;

(b) For any $I = (i_1, i_2, ..., i_k), J = (j_1, j_2, ..., j_l)$ with $|I|, |J| < \infty$ we have $\psi(s_is_j^*) = <\Omega, v_Iv_J^*\Omega>$ and the vectors $\{S_if : f \in \mathcal{K}, |I| < \infty\}$ are total in $\mathcal{H}$;

(c) The von-Neumann algebra $\mathcal{M} = P\pi(\mathcal{O}_d)''P$, acting on the Hilbert space $\mathcal{K}$ i.e. range of $P$, is generated by $\{v_k, v_k^* : 1 \leq k \leq d\}''$ and the normal state $\phi_0(x) = <\Omega, x\Omega>$ is faithful on the von-Neumann algebra $\mathcal{M}$.

(d) The self-adjoint part of the commutant of $\pi(\mathcal{O}_d)'$ is norm and order isomorphic to the space of self-adjoint fixed points of the completely positive map $\tau$. The isomorphism takes $X' \in \pi(\mathcal{O}_d)'$ onto $PX'P \in \mathcal{B}_r(\mathcal{K})$, where $\mathcal{B}_r(\mathcal{K}) = \{x \in \mathcal{B}(\mathcal{K}) : \sum_k v_kxv_k^* = x\}$. Furthermore $\mathcal{M}' = \mathcal{B}_r(\mathcal{K})$.

Conversely let $\mathcal{M}$ be a von-Neumann algebra generated by a family $\{v_k : 1 \leq k \leq d\}$ of bounded operators on a Hilbert space $\mathcal{K}$ so that $\sum_k v_kv_k^* = 1$ and the commutant $\mathcal{M}' = \{x \in \mathcal{B}(\mathcal{K}) : \sum_k v_kxv_k^* = x\}$. Then the Popescu dilation $(\mathcal{H}, P, S_k, 1 \leq k \leq d)$ described in Theorem 2.1 satisfies the following:

(i) $P \in \{S_k, S_k^*, 1 \leq k \leq d\}''$;
(ii) For any faithful normal invariant state $\phi_0$ on $\mathcal{M}$ there exists a state $\psi$ on $\mathcal{O}_d$ defined by

$$\psi*(s_I s_J^*) = \phi_0(v_I v_J^*), \quad |I|, |J| < \infty$$

so that the GNS space associated with $(\mathcal{M}, \phi_0)$ is the support projection for $\psi$ in $\pi(\mathcal{O}_d)^\prime\prime$ satisfying (a)-(d).

Further for a given $\lambda$-invariant state $\psi$, the family $(\mathcal{K}, \mathcal{M}, v_k \leq k \leq d, \phi_0)$ satisfying (a)-(d) is determined uniquely up to unitary conjugation.

(e) $\phi_0$ is a faithful normal $\tau$-invariant state on $\mathcal{M}$. Furthermore the following statements are equivalent:

(i) $(\mathcal{O}_d, \lambda, \psi)$ is ergodic;
(ii) $(\mathcal{M}, \tau, \phi_0)$ is ergodic;
(iii) $\mathcal{M}$ is a factor.

**PROOF:** $\Lambda(P)$ is also a projection in $\pi_\psi(\mathcal{O}_d)^\prime\prime$ so that $\psi_\Omega(\Lambda(P)) = 1$ by invariance property. Thus we have $\Lambda(P) \geq P$ i.e. $P\Lambda(I - P)P = 0$. Hence we have

$$PS_k^*P = S_k^*P$$

(2.6)

Moreover by $\lambda$ invariance property we also note that the faithful normal state $\phi_0(x) = < \Omega, x\Omega >$ on the von-Neumann algebra $\mathcal{M} = P\pi_\psi(\mathcal{O}_d)^\prime\prime P$ is invariant for the reduce Markov map [Mo1] on $\mathcal{M}$ given by

$$\tau(x) = P\Lambda(PxP)P$$

(2.7)

We claim that $\lim_{n \to \infty} \Lambda^n(P) = I$. That $\{\Lambda^n(P) : n \geq 1\}$ is a sequence of increasing projections follows from sub-harmonic property of $P$ and endomorphism property of $\Lambda$. Let the limiting projection be $Y$. Then $\Lambda(Y) = Y$ and so $Y \in \{S_k, S_k^*\}'$. Since by our construction GNS Hilbert space $\mathcal{H}_{\pi_\omega}$ is generated
by $S_IS\Omega$, $Y$ is a scaler, being a non-zero projection, it is the identity operator in $\mathcal{H}_{\pi_0}$.

Now it is routine to verify (a) (b) and (c). For the first part of (d) we appeal to Theorem 2.2. For the last part note that for any invariant element $D$ in $\mathcal{B}(\mathcal{K})$ there exists an element $X'$ in $\pi(\mathcal{O}_d)'$ so that $PX'P = D$. Since $P \in \pi(\mathcal{O}_d)^{''}$ we note that $(1 - P)X'P = 0$. Now since $X' \in \{S_k, S^*_k\}'$, we verify that $Dv^*_k = PXPS^*_kP = PXS^*_kXP = P \pi(\hat{\omega})S^*_kX^*P = v^*_kD$. Since $D^* \in \mathcal{B}_r(\mathcal{K})$ we also have $D^*v^*_k = v^*_kD^*$. Thus $D \in \{v_k, v^*_k : 1 \leq k \leq d\}' = \mathcal{M}'$. Since $P\pi(\hat{\omega})P = \mathcal{B}(\mathcal{K})_r$, we conclude that $\mathcal{B}(\mathcal{K})_r \subseteq \mathcal{M}'$. The reverse inclusion is trivial. This completes the proof for (d).

For the converse part of (i), since by our assumption and commutant lifting property self-adjoint elements of the commutant $\{S_k, S^*_k, 1 \leq k \leq d\}'$ is order isometric with the algebra $\mathcal{M}'$ via the map $X' \rightarrow PX'P$, $P \in \{S_k, S^*_k, 1 \leq k \leq d\}''$ by Proposition 4.2 in [BJKW]. For (ii) without loss of generality assume that $\phi_0(x) = \langle \Omega , x\Omega \rangle$ for all $x \in \mathcal{M}$ and $\Omega$ is a cyclic and separating vector for $\mathcal{M}$. ( otherwise we set state $\psi(s_is^*_j) = \phi_0(vivy^*_j)$ and consider it’s GNS representation ) We are left to show that $\Omega$ is a cyclic vector for the representation $\pi(s_i) \rightarrow S_i$. To that end let $Y \in \pi(\mathcal{O}_d)'$ be the projection on the subspace generated by the vectors $\{S_IS^*_j\Omega : |I|, |J| < \infty\}$. Note that $P$ being an element in $\pi(\mathcal{O}_d)^{''}$, $Y$ also commutes with all the element $P\pi(\mathcal{O}_d)^{''}P = P\mathcal{M}P$. Hence $Yx\Omega = x\Omega$ for all $x \in \mathcal{M}$. Thus $Y \geq P$. Since $\Lambda_n(P) \uparrow I$ as $n \uparrow \infty$ by our construction, we conclude that $Y = \Lambda_n(Y) \geq \Lambda_n(P) \uparrow I$ as $n \uparrow \infty$. Hence $Y = I$. In other words $\Omega$ is cyclic for the representation $s_i \rightarrow S_i$. This completes the proof for (ii).

Uniqueness up to unitary isomorphism follows as GNS representation is determined uniquely up to unitary conjugation and so its support projection.
The first part of (e) we note that $PS_I S_J^* P = v_I v_J^*$ for all $|I|, |J| < \infty$ and thus $\mathcal{M} = P \pi(O_d)'' P$ is the von-Neumann algebra generated by $\{v_k, v_k^* : 1 \leq k \leq d\}$ and thus $\tau(x) = P A (P x P) P$ for all $x \in \mathcal{M}$. That $\phi_0$ is $\tau(x) = \sum_k v_k x v_k^*$ invariant follows as $\psi$ is $\lambda$-invariant. We are left to prove equivalence of statements (i)-(iii).

By Theorem 3.6 in [Mo1] Markov semigroup $(\mathcal{M}, \tau, \phi_0)$ is ergodic if and only if $(\pi(O_d)'' ,\Lambda, \psi\Omega)$ is ergodic (here we need to recall by (a) that $\Lambda_n(P) \uparrow I$ as $n \uparrow \infty$). By a standard result [Fr, also BJKW] $(\mathcal{M}, \tau, \phi_0)$ is ergodic if and only if there is no non trivial projection $e$ invariant for $\tau$, i.e. $\mathcal{I} = \{e \in \mathcal{M} : e^* = e, e^2 = e, \tau(e) = e\} = \{0, 1\}$. If $\tau(e) = e$ for some projection $e \in \mathcal{M}$ then $(1-e)\tau(e)(1-e) = 0$ and so $ev_k^*(1-e) = 0$. Same is true if we replace $e$ by $1-e$ as $\tau(1) = 1$ and $\tau(1-e) = 1-\tau(e) = 1-e$ and thus $(1-e)v_k^*(1-e) = 0$. Thus $e$ commutes with $v_k, v_k^*$ for all $1 \leq k \leq d$. Hence $\mathcal{I} \subseteq \mathcal{M} \cap \mathcal{M}'$. Inequality in the reverse direction is trivial and thus $\mathcal{I}$ is trivial if and only if $\mathcal{M}$ is a factor. Thus equivalence of (ii) and (iii) follows by a standard result [Ev,Fr] in non-commutative ergodic theory. This completes the proof. 

The following two propositions are essentially easy adaptations of results appeared in [BJKW, Section 6 and Section 7], crucial in our present framework.

**PROPOSITION 2.5:** Let $\psi$ be a $\lambda$ invariant factor state on $O_d$ and $(\mathcal{H}, \pi, \Omega)$ be its GNS representation. Then the following hold:

(a) The closed subgroup $H = \{z \in S^1 : \psi \beta_z = \psi\}$ is equal to $\{z \in S^1 : \beta_z \text{ extends to an automorphism of } \pi(O_d)''\}$.

(b) Let $O_d^H$ be the fixed point sub-algebra in $O_d$ under the gauge group $\{\beta_z : z \in H\}$. Then $\pi(O_d^H)'' = \pi(UHF_d)''$.

(c) If $H$ is a finite cyclic group of $k$ many elements and $\pi(UHF_d)''$ is a factor, then $\pi(O_d)'' \cap \pi(UHF_d)' \equiv \mathcal{L}^m$ where $1 \leq m \leq k$. 


**PROOF:** It is simple that $H$ is a closed subgroup. For any fix $z \in H$ we define unitary operator $U_z$ extending the map $\pi(x)\Omega \to \pi(\beta_z(x))\Omega$ and check that the map $X \to U_zXU_z^*$ extends $\beta_z$ to an automorphism of $\pi(O_d)''$. For the converse we will use the hypothesis that $\psi$ is a $\lambda$-invariant factor state and $\beta_z^*\lambda = \lambda\beta_z$ to guarantee that $\psi\beta_z(X) = \frac{1}{n} \sum_{1 \leq k \leq n} \psi \lambda^k \beta_z(X) = \frac{1}{n} \sum_{1 \leq k \leq n} \psi \beta_z \lambda^k(X) \to \psi(X)$ as $n \to \infty$ for any $X \in \pi(O_d)''$, where we have used the same symbol $\beta_z$ for the extension. Hence $z \in H$.

For any $z_1, z_2 \in S^1$ we extend both $\psi_{\beta_{z_1}}$ and $\psi_{\beta_{z_2}}$ to its inductive limit state on $O_d^*$ using the canonical endomorphism $O_d \to^\lambda O_d$. Inductive limit state being an affine map, their inductive limit states are also factors. The inductive limit of the canonical endomorphism became an automorphism. $(A, \theta)$ is asymptotically abelian i.e. $||x^{\theta^n}(y) - \theta^n(y)x|| \to 0$ as $n \to \infty$ for all $x, y \in A$ (see also page 240 in [BR, vol2]). Thus in particular $(A, \mathbb{Z}, \omega)$ is $\mathbb{Z}$-central for any translation invariant ergodic state $\omega$ (see page 380 in [BR vol-2]). Thus we may appeal to a general result in $C^*$-non-commutative ergodic theory to conclude that their inductive limit, being translation invariant factor states, are either same or orthogonal (Theorem 4.3.19 in [BR vol2]).

In the following instead of working with $O_d$ we should be working with the inductive limit $C^*$ algebra and their inductive limit states. For simplicity of notation we still use $UHF_d, O_d$ for its inductive limit of $O_d \to^\lambda O_d$ and $UHF_d \to^\lambda UHF_d$ respectively and so for its inductive limit states.

Now we aim to prove (b). $H$ being a closed subgroup of $S^1$, it is either entire $S^1$ or a finite subgroup $\{exp(\frac{2i\pi l}{k})|l = 0, 1, ..., k-1\}$ where the integer $k \geq 1$. If $H = S^1$ we have nothing to prove for (b). When $H$ is a finite closed subgroup, we identify $[0, 1)$ with $S^1$ by the usual map and note that if $\beta_t$ is restricted to $t \in [0, \frac{1}{k})$, then by scaling we check that $\beta_t$ defines a representation of $S^1$ in
automorphisms of $O^H_d$. Now we consider the direct integral representation $\pi'$ defined by

$$\pi' = \int_{[0, 1]}^{\oplus} \, dt \pi|_{O^H_d} \beta_t$$

of $O^H_d$ on $\mathcal{H}_{|O^H_d} \otimes L^2([0, 1/\kappa])$, where $\mathcal{H}_{|O^H_d}$ is the cyclic space of $\pi(O^H_d)$ generated by $\Omega$. That it is indeed direct integral follows as states $\psi_{\beta_{t_1}}$ and $\psi_{\beta_{t_2}}$ are either same or orthogonal for a factor state $\psi$ (see the above paragraph). Interesting point here to note that the new representation $\pi'$ is $(\beta_t)$ covariant i.e. $\pi' \beta_t = \beta_t \pi'$, hence by simplicity of the $C^*$ algebra $O_d$ we conclude that

$$\pi'((UHF_d)'') = \pi'(O^H_d)'\beta_t$$

By exploring the hypothesis that $\psi$ is a factor state, we also have as in Lemma 6.11 in [BJKW] $I \otimes L^\infty([0, 1/\kappa]) \subset \pi'(O^H_d)'\beta_t$. Hence we also have

$$\pi'(O^H_d)'\beta_t = \pi(O^H_d)'\otimes L^\infty([0, 1/\kappa]).$$

Since $\beta_t$ is acting as translation on $I \otimes L^\infty([0, 1/\kappa])$ which being an ergodic action, we have

$$\pi'((UHF_d)'') = \pi(O^H_d)'\otimes 1$$

Since $\pi'((UHF_d)'') = \pi((UHF_d)'\otimes 1$, we conclude that $\pi((UHF_d)'\otimes = \pi(O^H_d)'\otimes 1$.

A proof for the statement (c) follows from Lemma 7.12 in [BJKW]. The original idea of the proof can be traced back to Arveson’s work on spectrum of an automorphism of a commutative compact group [Ar1].

Let $\omega'$ be an $\lambda$-invariant state on the UHF$_d$ sub-algebra of $O_d$. Following [BJKW, section 7], we consider the set

$$K_{\omega'} = \{ \psi : \psi \text{ is a state on } O_d \text{ such that } \psi \lambda = \psi \text{ and } \psi|_{UHF_d} = \omega' \}$$

By taking invariant mean on an extension of $\omega'$ to $O_d$, we verify that $K_{\omega'}$ is non empty and $K_{\omega'}$ is clearly convex and compact in the weak topology. In
case $\omega'$ is an ergodic state ( extremal state ) $K_{\omega'}$ is a face in the $\lambda$ invariant states. Before we proceed to the next section here we recall Lemma 7.4 of [BJKW] in the following proposition.

**Proposition 2.6:** Let $\omega'$ be ergodic. Then $\psi \in K_{\omega'}$ is an extremal point in $K_{\omega'}$ if and only if $\psi$ is a factor state and moreover any other extremal point in $K_{\omega'}$ have the form $\psi\beta_z$ for some $z \in S^1$.

**Proof:** Though Proposition 7.4 in [BJKW] appeared in a different set up, same proof goes through for the present case. We omit the details and refer to the original work for a proof.

3 Dual Popescu system and pure translation invariant states:

In this section we review the amalgamated Hilbert space developed in [BJKW] and prove a powerful criteria for a translation invariant factor state to be pure.

To that end let $\mathcal{M}$ be a von-Neumann algebra acting on a Hilbert space $\mathcal{K}$ and $\{v_k, 1 \leq k \leq d\}$ be a family of bounded operators on $\mathcal{K}$ so that $\mathcal{M} = \{v_k, v_k^*, 1 \leq k \leq d\}''$ and $\sum_k v_k v_k^* = 1$. Furthermore let $\Omega$ be a cyclic and separating vector for $\mathcal{M}$ so that the normal state $\phi_0(x) = \langle \Omega, xv\Omega \rangle$ on $\mathcal{M}$ is invariant for the Markov map $\tau$ on $\mathcal{M}$ defined by $\tau(x) = \sum_k v_k xv_k^*$ for $x \in \mathcal{M}$.

Let $\omega$ be the translation invariant state on $\text{UHF}_d = \otimes_{\mathbb{Z}} M_d$ defined by

$$\omega(e_{j_1}^i(l) \otimes e_{j_2}^i(l+1) \otimes \ldots \otimes e_{j_n}^i(l+n-1)) = \phi_0(v_l v_l^*)$$

where $e_{j}^i(l)$ is the elementary matrix at lattice sight $l \in \mathbb{Z}$.

We set $\bar{v}_k = \mathcal{J}\sigma_\mathcal{J}(v_k^*) \mathcal{J} \in \mathcal{M}'$ ( see [BJKW] for details ) where $\mathcal{J}$ and $\sigma =$
\((\sigma_t, \ t \in IR)\) are Tomita’s conjugation operator and modular automorphisms associated with \(\phi_0\).

By KMS relation [BR vol-1] we verify that
\[
\sum_k \tilde{v}_k \tilde{v}_k^* = 1
\]
and
\[
\phi_0(v_I v_J^*) = \phi_0(\tilde{v}_I \tilde{v}_J^*) \tag{3.1}
\]
where \(\tilde{I} = (i_n, \ldots, i_2, i_1)\) if \(I = (i_1, i_2, \ldots, i_n)\). Moreover \(\tilde{v}_I^* \Omega = J \sigma^*_I (v_I)^* J \Delta I \Omega = v_I^* \Omega\). We also set \(\tilde{\mathcal{M}}\) to be the von-Neumann algebra generated by \(\{\tilde{v}_k : 1 \leq k \leq d\}\). Thus \(\tilde{\mathcal{M}} \subseteq \mathcal{M}'\). A major problem that we will have to address when equality holds.

Let \((\mathcal{H}, P, S_k, 1 \leq k \leq d)\) and \((\tilde{\mathcal{H}}, P, \tilde{S}_k, 1 \leq k \leq d)\) be the Popescu dilation described as in Theorem 2.1 associated with \((\mathcal{K}, v_k, 1 \leq k \leq d)\) and \(\mathcal{K}, \tilde{v}_k, 1 \leq k \leq d)\) respectively. Following [BJKW] we consider the amalgamated tensor product \(\mathcal{H} \otimes_{\mathcal{K}} \tilde{\mathcal{H}}\) of \(\mathcal{H}\) with \(\tilde{\mathcal{H}}\) over the joint subspace \(\mathcal{K}\). It is the completion of the quotient of the set
\[
\mathcal{C} \tilde{I} \otimes \mathcal{C} I \otimes \mathcal{K},
\]
where \(\tilde{I}, I\) both consist of all finite sequences with elements in \(\{1, 2, \ldots, d\}\), by the equivalence relation defined by a semi-inner product defined on the set by requiring
\[
< \tilde{I} \otimes I \otimes f, \tilde{I} \tilde{J} \otimes IJ \otimes g > = < f, \tilde{v}_I v_J g >,
\]
\[
< \tilde{I} \tilde{J} \otimes I \otimes f, \tilde{I} \otimes IJ \otimes g > = < \tilde{v}_I f, v_J g >
\]
and all inner product that are not of these form are zero. We also define two commuting representations \((S_i)\) and \((\tilde{S}_i)\) of \(\mathcal{O}_d\) on \(\mathcal{H} \otimes_{\mathcal{K}} \tilde{\mathcal{H}}\) by the following prescription:
\[
S_I \lambda(\tilde{J} \otimes J \otimes f) = \lambda(\tilde{J} \otimes IJ \otimes f),
\]
\[ \tilde{S}_j \lambda(\bar{J} \otimes J \otimes f) = \lambda(\bar{J} \otimes J \otimes f), \]

where \( \lambda \) is the quotient map from the index set to the Hilbert space. Note that the subspace generated by \( \lambda(\emptyset \otimes I \otimes \mathcal{K}) \) can be identified with \( \mathcal{H} \) and earlier \( S_I \) can be identified with the restriction of \( S_I \) defined here. Same is valid for \( \tilde{S}_I \). The subspace \( \mathcal{K} \) is identified here with \( \lambda(\emptyset \otimes \emptyset \otimes \mathcal{K}) \). Thus \( K \) is a cyclic subspace for the representation

\[ \tilde{s}_j \otimes s_i \rightarrow \tilde{S}_j S_i \]

of \( \tilde{\mathcal{O}}_d \otimes \mathcal{O}_d \) in the amalgamated Hilbert space. Let \( P \) be the projection on \( \mathcal{K} \). Then we have

\[ S_i^* P = P S_i^* = v_i^* \]
\[ \tilde{S}_i^* P = P \tilde{S}_i^* = \tilde{v}_i^* \]

for all \( 1 \leq i \leq d \).

We start with a simple proposition.

**PROPOSITION 3.1:** The following hold:

(a) For any \( 1 \leq i, j \leq d \) and \( |I|, |J| < \infty \) and \( |\bar{I}|, |\bar{J}| < \infty \)

\[ \langle \Omega, \tilde{S}_j \tilde{S}_j^* S_i S_i^* S_j S_j^* \Omega \rangle = \langle \Omega, \tilde{S}_i \tilde{S}_i^* S_j \tilde{S}_j^* S_i S_i^* S_j S_j^* \Omega \rangle; \]

(b) The vector state \( \psi_\Omega \) on

\[ \mathrm{UHF}_d \otimes \mathrm{UHF}_d \equiv \odot_{-\infty}^0 M_d \otimes_{1}^\infty M_d \equiv \odot_{\mathbb{Z}}^\infty M_d \]

is equal to \( \omega \);

(c) \( \pi(\tilde{\mathcal{O}}_d \otimes \mathcal{O}_d)^{\prime\prime} = B(\tilde{\mathcal{H}} \otimes_{\mathcal{K}} \mathcal{H}) \) if and only if \( \{ x \in \mathcal{B}(\mathcal{K}) : \tau(x) = x, \tilde{\tau}(x) = x \} = \{ zI : z \in \mathcal{L} \} \).

**PROOF:** By our construction \( \tilde{S}_i^* \Omega = \tilde{v}_i^* \Omega = v_i^* \Omega = S_i^* \Omega \). Now (a) and (b) follows by repeated application of \( \tilde{S}_i^* \Omega = S_i^* \Omega \) and commuting property of the
two representation $\pi(O_d \otimes I)$ and $\pi(I \otimes \tilde{O}_d)$. The last statement (c) follows from a more general fact proved below that the commutant of $\pi(O_d \otimes \tilde{O}_d)^{\prime\prime}$ is order isomorphic with the set $\{x \in B(\mathcal{K}) : \tau(x) = x, \tilde{\tau}(x) = x\} = \{zI : z \in \mathcal{C}\}$ via the map $X \to PXP$ where $X$ is the weak$^*$ limit of $\Lambda^m \tilde{\Lambda}^n(x)$ as $(m,n) \to (\infty,\infty)$. For details let $Y$ be the strong limit of increasing sequence of projections $(\Lambda \tilde{\Lambda})^n(P)$ as $n \to \infty$. Then $Y$ commutes with $S_i \tilde{S}_j, S_i^* \tilde{S}_j^*$ for all $1 \leq i, j \leq d$. As $\Lambda(P) \geq P$, we also have $\Lambda(Y) \geq Y$. Hence $(1-Y)S_i^*Y = 0$. As $Y$ commutes with $S_i \tilde{S}_j$ we get $(1-Y)S_i^*S_i \tilde{S}_jY = 0$ i.e. $(1-Y)\tilde{S}_jY = 0$ for all $1 \leq j \leq d$. By symmetry of the argument we also get $(1-Y)S_iY = 0$ for all $1 \leq i \leq d$. Hence $Y$ commutes with $\pi(O_d)^{\prime\prime}$ and by symmetry of the argument $Y$ commutes as well with $\pi(\tilde{O}_d)^{\prime\prime}$. As $Yf = f$ for all $f \in \mathcal{K}$ and $\mathcal{K}$ is cyclic for the representation $\pi(\tilde{O}_d \otimes O_d)$ we conclude that $Y = I$ in $\tilde{H} \otimes \mathcal{K} \mathcal{H}$.

Let $x \in B(\mathcal{K})$ so that $\tau(x) = x$ and $\tilde{\tau}(x) = x$ then as in the proof of Theorem 2.1 we also check that $(\Lambda \tilde{\Lambda})^k(P)\Lambda^m \tilde{\Lambda}^n(x)(\Lambda \tilde{\Lambda})^k(P)$ is independent of $m, n$ as long as $m, n \geq k$. Hence weak$^*$ limit $\Lambda^m \tilde{\Lambda}^n(x) \to X$ exists as $m, n \to \infty$. Furthermore limiting element $X \in \pi(O_d \otimes \tilde{O}_d)'$ and $PXP = x$. That the map $X \to PXP$ is an order-isomorphic on the set of self adjoint elements follows as in Theorem 2.1. This completes the proof.

**REMARK:** Proposition 3.1 in brief says that $(\tilde{H} \otimes_\mathcal{K} \mathcal{H}, S_i \tilde{S}_j, 1 \leq i, j \leq d, P)$ is the noting but the Popescu dilation associated with Popescu elements $(\mathcal{K}, v_i \tilde{v}_j, 1 \leq i, j \leq d)$. In fact one can give an alternative construction for amalgamated representation of $\tilde{O}_d \otimes O_d$.

Now we will be more specific in our starting Popescu systems in order to explore the representation $\pi$ of $\tilde{O}_d \otimes O_d$ in the amalgamated Hilbert space $\tilde{H} \otimes_\mathcal{K} \mathcal{H}$. To that end let $\omega$ be a translation invariant extremal state on $\mathcal{A}$ and we fix any extremal point $\psi \in K_\omega$. In the following we will eventually consider
the Popescu systems \((\mathcal{K}, \mathcal{M}, v_k, 1 \leq k \leq d, \Omega)\) described as in Proposition 2.4 and associated dual Popescu systems \((\mathcal{K}, \tilde{\mathcal{M}}, \tilde{v}_k, 1 \leq k \leq d)\) where \(\tilde{\mathcal{M}}\) is the von-Neumann algebra generated by \(\{\tilde{v}_k : 1 \leq k \leq d\}\). Thus in general \(\tilde{\mathcal{M}} \subseteq \mathcal{M}'\) and an interesting question: when do we have \(\mathcal{M}' = \tilde{\mathcal{M}}\)? Going back to our starting example of unique KMS state for the automorphisms \(\beta_t(s_i) = ts_i, t \in S^1\), we check that \(v_k^* = S_k^*\) and \(\mathcal{J} \tilde{v}_k \mathcal{J} = \frac{1}{d} S_k\) and thus equality hold i.e. \(\hat{\mathcal{M}} = \mathcal{M}'\). But the corner vector space \(\tilde{\mathcal{M}}_c = P \pi(\tilde{\mathcal{O}}_d)^\prime P\) generated by the elements \(\{\tilde{v}_I \tilde{v}_J^* : |I|, |J| < \infty\}\) fails to be an algebra. Thus two questions sounds reasonable here.

(a) Does equality \(\mathcal{M}' = \tilde{\mathcal{M}}\) holds in general for an extremal element \(\psi \in K_{\omega'}\) and a factor state \(\omega\)?

(b) When can we expect \(\tilde{\mathcal{M}}_c\) to be a \(*\)-algebra and so equal to \(\tilde{\mathcal{M}}\)?

The dual condition on support projection and equality \(\hat{\mathcal{M}} = \mathcal{M}'\) are rather deep and will lead us to a far reaching consequence on the state \(\omega\). In the paper [BJKW] these two conditions are implicitly assumed to give a criteria for a translation invariant factor state to be pure. Apart from this refined interest, we will address the converse problem that turns out to be crucial for our main results. In the following we prove a crucial step towards that goal and fixing basic structure which will be repeatedly used in the computation using Cuntz relations.

**PROPOSITION 3.2:** Let \(\omega\) be an extremal translation invariant state on \(\mathcal{A}\) and \(\psi\) be an extremal point in \(K_{\omega'}\). If the amalgamated representation \(\pi\) of \(\tilde{\mathcal{O}}_d \otimes \mathcal{O}_d\) in \(\tilde{\mathcal{H}} \otimes \mathcal{K} \mathcal{H}\) of the Popescu systems \((\mathcal{K}, \mathcal{M}, v_k, 1 \leq k \leq d)\) are taken as in Proposition 2.4 then following are true:

(a) \(\pi(\tilde{\mathcal{O}}_d \otimes \mathcal{O}_d)^\prime = B(\tilde{\mathcal{H}} \otimes \mathcal{K} \mathcal{H})\). Furthermore \(\pi(\mathcal{O}_d)^\prime\) and \(\pi(\tilde{\mathcal{O}}_d)^\prime\) are factors and the following sets are equal:
(i) $H = \{ z \in S^1 : \psi \beta_z = \psi \};$

(ii) $H_\pi = \{ z : \beta_z \text{ extends to an automorphism of } \pi(O_d)'' \};$

(iii) $\tilde{H}_\pi = \{ z : \beta_z \text{ extends to an automorphism of } \pi(\tilde{O}_d)'' \}.$ Moreover $\pi(U\tilde{HF}_d \otimes I)''$ and $\pi(I \otimes UHF_d)''$ are factors.

(b) $z \rightarrow U_z$ is the unitary representation of $H$ in the Hilbert space $\tilde{H} \otimes_K H$ defined by $U_z(\pi(\tilde{s_j} \otimes s_i)\Omega) = \pi(z\tilde{s_j} \otimes z s_i)\Omega$

(c) The commutant of $\pi(UHF_d \otimes UHF_d)''$ is invariant by the canonical endomorphisms $\Lambda(X) = \sum_i S_i X S_i^*$ and $\tilde{\Lambda}(X) = \sum_i \tilde{S}_i X \tilde{S}_i^*.$ Same is true for each $i$ that the surjective map $X \rightarrow \tilde{S}_i^* X S_i$ keeps the commutant of $UHF_d \otimes UHF_d)''$ invariant. Same holds for the map $X \rightarrow \tilde{S}_i^* X \tilde{S}_i.$

(d) The centre of $\pi(UHF_d \otimes UHF_d)''$ is invariant by the canonical endomorphisms $\Lambda(X) = \sum_i S_i X S_i^*$ and $\tilde{\Lambda}(X) = \sum_i \tilde{S}_i X \tilde{S}_i^*.$ Moreover for each $i$ the surjective map $X \rightarrow \tilde{S}_i^* X S_i$ keeps the centre of $\pi(UHF_d \otimes UHF_d)''$ invariant. Same holds for the map $X \rightarrow \tilde{S}_i^* X \tilde{S}_i.$

**PROOF:** $P$ being the support projection by Proposition 2.4 we have $\{ x \in B(K) : \sum_k v_k x v_k^* = x \} = \mathcal{M}'.$ That $(\mathcal{M}', \tilde{\tau}, \phi_0)$ is ergodic follows from a general result [Mo1] (see also [BJKW] for a different proof) as $(\mathcal{M}, \tau, \phi_0)$ is ergodic for a factor state $\psi$ being extremal in $K_{\omega'}$ (Proposition 2.6). Hence $\{ x \in B(K) : \tau(x) = \tilde{\tau}(x) = x \} = \mathcal{L}'$. Hence by Proposition 3.1 we conclude that $\pi(\tilde{O}_d \otimes O_d)'' = B(\tilde{\mathcal{H}} \otimes_K \mathcal{H})$

That both $\pi(O_d)''$ and $\pi(\tilde{O}_d)''$ are factors follows trivially as $\pi(\tilde{O}_d \otimes O_d)'' = B(\tilde{\mathcal{H}} \otimes_K \mathcal{H})$ and $\pi(O_d)'' \subseteq \pi(\tilde{O}_d)'.$

By our discussion above we first recall that $\Omega$ is a cyclic vector for the representation of $\pi(O_d).$ Let $G = \{ z = (z_1, z_2) \in S^1 \times S^1 : \beta_z \text{ extends to an automorphism of } \pi(O_d)'' \}$ be the closed subgroup where

$$\beta_z(\tilde{s_j} \otimes s_i) = z_1 \tilde{s}_j \otimes z_2 s_i.$$
By repeated application of the fact that $\pi(O_d)^{\prime\prime}$ commutes with $\pi(\tilde{O}_d)^{\prime\prime}$ and $S_i^*\Omega = \tilde{S}_i^*\Omega$ as in Proposition 3.1 (a) we verify that $\psi_\beta(z,z) = \psi$ on $O_d \otimes \tilde{O}_d$ if $z \in H$. For $z \in H$ we set unitary operator $U_z\pi(x \otimes y)\Omega = \pi(\beta_z(x) \otimes \beta_z(y))\Omega$ for all $x \in \tilde{O}_d$ and $y \in O_d$. Thus we have $U_z\pi(s_i)U_z^* = z\pi(s_i)$ and also $U_z\pi(\tilde{s}_i)U_z^* = z\tilde{s}_i$. By taking it’s restriction to $\pi(O_d)^{\prime\prime}$ and $\pi(\tilde{O}_d)^{\prime\prime}$ respectively we check that $H \subseteq \tilde{H}_\pi$ and $H \subseteq H_\pi$.

For the converse let $z \in H_\pi$ and we use the same symbol $\beta_z$ for the extension to an automorphism of $\pi(O_d)^{\prime\prime}$. By taking the inverse map we check easily that $\tilde{z} \in H_\pi$ and in fact $H_\pi$ is a subgroup of $S^3$. Since $\lambda$ commutes with $\beta_z$ on $O_d$, the canonical endomorphism $\Lambda$ defined by $\Lambda(X) = \sum_k S_k XS_k^*$ also commutes with extension of $\beta_z$ on $\pi(O_d)^{\prime\prime}$. Note that the map $\pi(x)|_H \to \pi(\beta_z(x))|_H$ for $x \in O_d$ is a well defined linear $*$-homomorphism. Since same is true for $\tilde{z}$ and $\beta_z\beta_{\tilde{z}} = I$, the map is an isomorphism. Hence $\beta_z$ extends uniquely to an automorphism of $\pi(O_d)^{\prime\prime}$ commuting with the restriction of the canonical endomorphism on $\pi(O_d)|_H$. Since $\pi(O_d)|_H$ is a factor, we conclude as in Proposition 2.5 (a) that $z \in H$. Thus $H_\pi \subseteq H$. As $\pi(\tilde{O}_d)^{\prime\prime}$ is also a factor, we also have $\tilde{H}_\pi \subseteq H$. Hence we have $H = H_\pi = \tilde{H}_\pi$ and $\{(z,z) : z \in H\} \subseteq G \subseteq H \times H$.

For the second part of (a) we will adopt the argument used for Proposition 2.5. To that end we first note that $\Omega$ being a cyclic vector for the representation $\tilde{O}_d \otimes O_d$ in the Hilbert space $\tilde{H} \otimes K H$, by Lemma 7.11 in [BJKW] (note that the proof only needs the cyclic property) the representation of $UHF_d$ on $\tilde{H} \otimes K H$ is quasi-equivalent to it’s sub-representation on the cyclic space generated by $\Omega$. On the other hand by our hypothesis that $\omega$ is a factor state, Power’s theorem [Po1] ensures that the state $\omega'$ (i.e. the restriction of $\omega$ to $A_R$ which is identified here with $UHF_d$) is also a factor state on $UHF_d$. Hence quasi-equivalence ensures that $\pi(I \otimes UHF_d)^{\prime\prime}$ is a factor. We also note that the
argument used in Lemma 7.11 in [BJKW] is symmetric i.e. same argument is also valid for $\tilde{UHF}_d$. Thus $\pi(UHF_d \otimes I)$" is also a factor.

As $\Lambda(E)$ commutes with $\pi(\lambda(UHF_d \otimes UHF_d))$" and $\{S_iS_j^* : 1 \leq i, j \leq d\}$ we verify by Cuntz’s relation that $\Lambda(E)$ is also an element in the commutant of $\pi(UHF_d \otimes UHF_d)$" once $E$ is so. It is obvious that $\Lambda(E)$ is an element in $\pi(UHF_d \otimes UHF_d)$" if $E$ is. Thus $\Lambda(E)$ is an element in the commutant/centre of $\pi(UHF_d \otimes UHF_d)$" once $E$ is so.

For the last statement consider the map $X \rightarrow S_i^*XS_i$. Clearly it preserves $\pi(UHF_d \otimes UHF_d)$" and onto. Hence we need to show that $S_i^*ES_i$ is an element in the commutant whenever $E$ is so. To that end note that $S_i^*ES_iS_i^*EXS_i = S_i^*ES_i = S_i^*XES_i = S_i^*XS_iS_i^*ES_i$. Thus onto property of the map ensures that $S_i^*ES_i$ is an element in the commutant of $\pi(UHF_d \otimes UHF_d)$" once $E$ is so. This completes the proof of (c) and (d).

One interesting problem here how to describe the von-Neumann algebra $I$ consists of invariant elements of the gauge action $\{\beta_z : z \in H\}$ in $B(\tilde{H} \otimes K H)$. A general result due to E. Stormer [So] says that the algebra of invariant elements are von-Neumann algebra of type-I with centre completely atomic. Here the situation is much simple because $I = \{U_z : z \in H\}'$ and if we write spectral decomposition as $U_z = \sum_{k \in \hat{H}} z^k F_k$ for $z \in H$, $\hat{H}$ is the dual group of $H$ either $\hat{H} = \{z : z^n = 1\}$ or $\mathbb{Z}$. Thus centre of $I$ is equal to $\{F_k : k \in \hat{H}\}$.

As a first step we describe the center $Z$ of $\pi(UHF_d \otimes UHF_d)$" by exploring Cuntz relation that it is also non-atomic even for a factor state $\omega$. In fact we will show that the centre $Z$ is a sub-algebra of the centre of $I$. In the following proposition we give an explicit description.

**PROPOSITION 3.3 :** Let $\omega, \psi$ be as in Proposition 3.2 with Popescu
system \((\mathcal{K}, \mathcal{M}, v_k, \Omega)\) be taken as in Proposition 2.4 i.e. on support projection. Then the centre of \(\pi(\tilde{\text{UHF}}_d \otimes \text{UHF}_d)''\) is completely atomic and the element \(E_0 = [\pi(\tilde{\text{UHF}}_d \otimes \text{UHF}_d)' \vee \pi(\tilde{\text{UHF}}_d \otimes \text{UHF}_d)'\Omega]\) is a minimal projection in the centre of \(\pi(\tilde{\text{UHF}}_d \otimes \text{UHF}_d)''\) and centre is invariant for both \(\Lambda\) and \(\tilde{\Lambda}\). Furthermore the following hold:

(a) The centre of \(\pi(\tilde{\text{UHF}}_d \otimes \text{UHF}_d)''\) has the following two disjoint possibilities:

(i) There exists a positive integer \(m \geq 1\) such that the centre is generated by the family of minimal orthogonal projections \(\{\Lambda_k(E_0) : 0 \leq k \leq m - 1\}\) where \(m \geq 1\) is the least positive integer so that \(\Lambda^m(E_0) = E_0\). In such a case \(\{z : z^m = 1\} \subseteq H\).

(ii) The family of minimal nonzero orthogonal projections \(\{E_k : k \in \mathbb{Z}\}\) where \(E_k = \Lambda^k(E_0)\) for \(k \geq 0\) and \(E_k = S_I^*E_0S_I\) for \(k < 0\) where \(|I| = -k\) and independent of multi-index \(I\) generates the centre. In such a case \(H = S^1\).

(b) \(\Lambda(E) = \tilde{\Lambda}(E)\) for any \(E\) in the centre of \(\pi(\tilde{\text{UHF}}_d \otimes \text{UHF}_d)''\)

(c) If \(\Lambda(E_0) = E_0\) then \(E_0 = 1\).

**PROOF:** Let \(E' \in \pi(\tilde{\text{UHF}}_d \otimes \text{UHF}_d)'\) be the projection on the subspace generated by the vectors \(\{S_I^*S_{I'}^*\tilde{S}_{I'}S_I^*\Omega, \ |I| = |J|, |I'| = |J'| < \infty\}\) and \(\pi_\Omega\) be the restriction of the representation \(\pi\) of \(\tilde{\text{UHF}}_d \otimes \text{UHF}_d\) to the cyclic subspace \(\mathcal{H}_\Omega\) generated by \(\Omega\). Identifying \(\mathcal{A}\) with \(\tilde{\text{UHF}}_d \otimes \text{UHF}_d\) we check that \(\pi_\omega\) is unitary equivalent with \(\pi_\Omega\). Thus \(\pi_\Omega\) is a factor representation.

For any projection \(E\) in the centre of \(\pi(\tilde{\text{UHF}}_d \otimes \text{UHF}_d)''\), via the unitary equivalence we note that \(EE' = E'EE'\) is an element in the centre of \(\pi_\Omega(\tilde{\text{UHF}}_d \otimes \text{UHF}_d)''\). \(\omega\) being a factor state we conclude that \(EE'\) is a scaler multiple of \(E'\) and so we have

\[EE' = \omega(E)E' \tag{3.2}\]
Thus we also have $EYE' = \omega(E)YE'$ for all $Y \in \pi(UHF_d \otimes UHF_d)'$ and so

$$EE_0 = \omega(E)E_0 \quad (3.3)$$

Since $EE'$ is a projection and $E' \neq 0$, we have $\omega(E) = \omega(E)^2$. Thus $\omega(E) = 1$ or 0. So for such an element $E$ the following is true:

(i) If $E \leq E_0$ then either $E = 0$ or $E = E_0$ i.e. $E_0$ is a minimal projection in the centre of $\pi(UHF_d \otimes UHF_d)''$

(ii) $\omega(E) = 1$ if and only if $E \geq E_0$

(iii) $\omega(E) = 0$ if and only if $EE_0 = 0$.

As $\Lambda(E_0)$ is a projection in the centre of $\pi(UHF_d \otimes UHF_d)''$ by our last proposition i.e. Proposition 3.2 (c), we have either $\omega(\Lambda(E_0)) = 1$ or 0. Since $\Lambda(E_0) \neq 0$ by injective property of the endomorphism, we have either $\Lambda(E_0) \geq E_0$ or $\Lambda(E_0)E_0 = 0$. In case $\Lambda(E_0) \geq E_0$ we have $S_i^*E_0S_i \leq S_i^*\Lambda(E_0)S_i = E_0$ for all $1 \leq i \leq d$. If so $S_i^*E_0S_i$ being a non-zero projection in the centre of $\pi(UHF_d \otimes UHF_d)''$ (Proposition 3.2 (c)), by (i) we have $E_0 = \Lambda(E_0)$. Thus we have either $\Lambda(E_0) = E_0$ or $\Lambda(E_0)E_0 = 0$.

If $\Lambda(E_0)E_0 = 0$, we have $\Lambda(E_0) \leq I - E_0$ and by Cuntz’s relation we check that $E_0 \leq I - S_i^*E_0S_i$ and $S_j^*S_i^*E_0S_iS_j \leq I - S_j^*E_0S_j$ for all $1 \leq i, j \leq d$. So we also have $E_0S_j^*S_i^*E_0S_iS_jE_0 \leq E_0 - E_0S_j^*E_0S_jE_0 = E_0$. Thus we have either $E_0S_j^*S_i^*E_0S_iS_jE_0 = 0$ or $E_0S_j^*S_i^*E_0S_iS_jE_0 = E_0$ as $S_j^*S_i^*E_0S_iS_j$ is an element in the centre by Proposition 3.2 (c). So either we have $\Lambda^2(E_0)E_0 = 0$ or $\Lambda^2(E_0) \leq E_0$. $\Lambda$ being an injective map we either have $\Lambda^2(E_0)E_0 = 0$ or $\Lambda^2(E_0) = E_0$.

More generally we check that if $\Lambda(E_0)E_0 = 0, \Lambda^2(E_0)E_0 = 0, \ldots \Lambda^k(E_0)E_0 = 0$ for some $k \geq 1$ then either $\Lambda^{k+1}(E_0)E_0 = 0$ or $\Lambda^{k+1}(E_0) = E_0$. To verify that first we check that in such a case $E_0 \leq I - S_i^*E_0S_i$ for all $|I| = n$ and then
following the same steps as before to check that \( S_i^* S_i E_0 S_i S_i \leq I - S^* E_0 S_i \) for all \( i \). Thus we have \( E_0 S_i^* S_i E_0 S_i S_i E_0 \leq E_0 \) and arguing as before we complete the proof of the claim that either \( \Lambda^{k+1}(E_0) E_0 = 0 \) or \( \Lambda^{k+1}(E_0) = E_0 \).

We summarize now by saying that \( E_0, \Lambda(E_0), \ldots, \Lambda^{m-1}(E_0) \) are mutually orthogonal projections with \( m \geq 1 \) possibly be infinite if not then \( \Lambda^m(E_0) = E_0 \).

Let \( \pi_k, k \geq 0 \) be the representation \( \pi \) of \( \text{UHF}_d \otimes \text{UHF}_d \) restricted to the subspace \( \Lambda^k(E_0) \). The representation \( \pi_0 \) of \( \text{UHF}_d \otimes \text{UHF}_d \) being isomorphic and so is quasi-equivalent to the representation \( \pi \) of \( \text{UHF}_d \otimes \text{UHF}_d \) restricted to \( E' \). For a general discussion on quasi-equivalence we refer to section 2.4.4 in [BR vol-1]. \( \omega \) being a factor state, \( \pi_0 \) is a factor representation. We claim that each \( \pi_k \) is a factor representation. We fix any \( k \geq 1 \) and let \( X \) be an element in the centre of \( \pi_k(\text{UHF}_d \otimes \text{UHF}_d) \). Then for any \( |I| = k, S_i^* E_k S_I = E_0 \) and so \( S_i^* X S_I \) is an element in the centre of \( \pi_0(\text{UHF}_d \otimes \text{UHF}_d) \) by Proposition 3.2 (d). Further \( S_i^* X S_I = S_i^* X S_J S_J^* S_i \) for all \( |J| = |I| = k \). \( \pi_0 \) being a factor representation, we have \( S_i^* X S_I = c E_0 \) for some scaler \( c \) independent of the multi-index we choose \( |I| = k \). Hence \( c \Lambda_k(E_0) = \sum_{|J|=k} S_J S_i^* X S_J S_J^* = \sum_{|J|=k} S_J S_i^* S_I S_J X = X \) as \( X \) is an element in the centre of \( \pi(\text{UHF}_d \otimes \text{UHF}_d) \).

Thus for each \( k \geq 1 \), \( \pi_k \) is a factor representation as \( \pi_0 \) is.

We also note that \( \Lambda(E_0) \tilde{\Lambda}(E_0) \neq 0 \). Otherwise we have \( < S_i \Omega, \tilde{S}_j \Omega > = 0 \) for all \( i, j \) and so \( < \Omega, \tilde{S}_j S_i^* \Omega > = 0 \) for all \( i, j \) as \( \pi(O_d)'' \) commutes with \( \pi(\tilde{O}_d)'' \). However \( \tilde{S}_i^* \Omega = S_i^* \Omega \) and \( \sum_i \tilde{S}_i \tilde{S}_i^* = 1 \) which leads a contradiction. Hence \( \Lambda(E_0) \tilde{\Lambda}(E_0) \neq 0 \). As \( \pi \) restricted to \( \Lambda(E_0) \) is a factor state and both \( \Lambda(E_0) \) and \( \tilde{\Lambda}(E_0) \) are elements in the centre of \( \pi(\text{UHF}_d \otimes \text{UHF})'' \) by Proposition 3.2 (d), we conclude that \( \Lambda(E_0) = \tilde{\Lambda}(E_0) \). Using commuting property of the endomorphisms \( \Lambda \) and \( \tilde{\Lambda} \), we verify by a simple induction method that \( \Lambda^k(E_0) = \tilde{\Lambda}^k(E_0) \) for all \( k \geq 1 \). Thus the sequence of orthogonal projections \( E_0, \tilde{\Lambda}(E_0), \ldots \)
are also periodic with same period or aperiodic according as the sequence of orthogonal projections $E_0, \Lambda(E_0), \ldots$.

If $\Lambda^m(E_0) = E_0$ for some $m \geq 1$ then we check that $\sum_{0 \leq k \leq m-1} \Lambda^k(E_0)$ is an $\Lambda$ and as well $\tilde{\Lambda}$ invariant projection and thus equal to 1 by cyclic property of $\Omega$ for $\pi(O_d \otimes \tilde{O}_d)^\nu$. In such a case we set $V_z = \sum_{0 \leq k \leq m-1} z^k E_k$ for $z \in S^1$ for which $z^m = 1$ and check that $\Lambda(V_z) = \sum_{0 \leq k \leq m-1} z^k \Lambda(E_k) = \sum_{0 \leq k \leq m-1} z^k E_{k+1} = \bar{z} V_z$ where $E_m = E_0$ and so by Cuntz relation we have $V_z^* S_i V_z = \bar{z} S_i$ for all $1 \leq i \leq d$. Following the same steps we also have $\tilde{\Lambda}(V_z) = \bar{z} V_z$ and so $V_z^* \tilde{S}_i V_z^* = \bar{z} \tilde{S}_i$ for $1 \leq i \leq d$. Thus $V_z = U_z$ for all $z \in H_0 = \{z : z^m = 1\} \subseteq H$.

Now we consider the case where $E_0, \Lambda(E_0), \ldots \Lambda^k(E_0), \ldots$ is a sequence of aperiodic orthogonal projections. We extend family of projections $\{E_k : k \in \mathbb{Z}\}$ to all integers by

$$E_k = \Lambda^k(E_0) \quad \text{for all} \quad k \geq 1$$

and

$$E_k = S_I^* E_0 S_I \quad \text{for all} \quad k \leq 1, \quad \text{where} \quad |I| = -k$$

We claim that the definition of $\{E_k ; k \leq -1\}$ does depends only on length of the multi-index $I$ that we choose. We may choose any other $J$ so that $|J| = |I|$ and check the following identity:

$$S_I^* E_0 S_I = S_J^* E_0 S_I S_J^* S_J = S_I^* S_I S_J^* E_0 S_J = S_J^* E_0 S_J$$

where $E_0$, being an element in the centre of $\pi(UHF_d \otimes UHF_d)^\nu$, commutes with $S_I S_J^*$ as $|I| = |J|$. Further $\Lambda^k(E_0) = \tilde{\Lambda}^k(E_0)$ ensures that $S_I \tilde{S}_J^*$ commutes with $E_0$ for all $|I| = |J| = k$ and $k \geq 1$. Hence we also have

$$E_{-k} = S_I^* E_0 S_I \tilde{S}_J^* \tilde{S}_J = \tilde{S}_J^* E_0 \tilde{S}_J$$

for all $|J| = |I| = k$ and $k \geq 1$. 

Now we claim that
\[ \Lambda(E_k) = \tilde{\Lambda}(E_k) = E_{k+1} \]
for all \( k \in \mathbb{Z} \). For \( k \geq 0 \) we have nothing to prove. For \( k \leq -1 \) we check that the following steps
\[
\Lambda(S^*_i E_0 S_I) = \sum_k S_k S^*_i S^*_r E_0 S_r S_i S^*_k
\]
\[
= \sum_k S^*_r E_0 S_r S_k S^*_i S_i S^*_k = S^*_r E_0 S_r
\]
where we wrote \( I = (I', i) \) and used elements \( S_k S^*_i \) commutes with \( \{E_k : k \in \mathbb{Z}\} \), elements in the centre of \( \pi(UHF_d \otimes UHF_d)'' \). For a proof that \( \tilde{\Lambda}(E_k) = E_{k+1} \) we may follow the same steps as \( E_k = \tilde{S}^*_i X \tilde{S}_I \) where \( |I| = -k \) and \( k \leq -1 \).

We also claim that \( \{E_k : k \in \mathbb{Z}\} \) is an orthogonal family of non-zero projections. To that end we choose any two elements say \( E_k, E_m, k \neq m \) and use endomorphism \( \Lambda^n \) for \( n \) large enough so that both \( n + k \geq 0, n + m \geq 0 \) to conclude that \( \Lambda^n(E_k E_m) = E_{k+n} E_{k+m} = 0 \) as \( k + n \neq k + m \). \( \Lambda \) being an injective map we get the required orthogonal property. Thus \( \sum_{k \in \mathbb{Z}} E_k \) being an invariant projection for both \( \Lambda \) and \( \tilde{\Lambda} \) we get by cyclicity of \( \Omega \) that \( \sum_{k \in \mathbb{Z}} E_k = I \).

Let \( \pi_k, k \leq -1 \) be the representation \( \pi \) of \( UHF_d \otimes UHF_d \) restricted to the subspace \( E_k \). Going along the same line as above, we verify that for each \( k \leq -1 \), \( \pi_k \) is a factor representation of \( UHF_d \otimes UHF_d \). We also set \( V_z = \sum_{-\infty < k < \infty} z^k E_k \) for all \( z \in S^1 \) and check that \( \Lambda(V_z) = \bar{z} V_z \) and also \( \tilde{\Lambda}(V_z) = \bar{z} V_z \). Hence \( S^1 = H \) and as \( H \) is a closed subset of \( S^1 \). This completes the proof of (a).

Proof for (b) and (c) are now simple consequence of the proof of (a). \( \blacksquare \)

It is clear that \( \mathcal{I} \) contains \( \mathcal{I}_0 := \pi(UHF_d \otimes UHF_d)'' \lor \{U_z : z \in H\}'' \). By the last proposition the centre of \( \mathcal{I} \), which is equal to \( \{U_z : z \in H\}'' \),
contains the centre of \( \pi(UHF_d \otimes UHF_d)^\prime\prime \) and thus by taking commutant we also have \( \mathcal{I} \subseteq \pi(UHF_d \otimes UHF_d)^\prime\prime \vee \pi(UHF_d \otimes UHF_d)' \). In the last proposition we have described explicitly the factor decomposition of the representation \( \pi \) of \( \pi(UHF_d \otimes UHF_d)^\prime\prime \). One central issue when such an factor decomposition is also an extremal decomposition. A clear answer at this stage seems to be bit hard. However the following proposition makes an attempt for our purpose. To that end we set few more notations and elementary properties.

For each \( k \in \hat{H} \), let \( \pi'_k \) be the representation \( \pi \) of \( \tilde{UHF}_d \otimes UHF_d \) restricted to \( F_k \). We claim that each \( \pi'_k \) is pure once \( \pi'_0 \) is pure. Fix any \( k \in \hat{H} \) and let \( X \) be an element in the commutant of \( \pi'_k(UHF_d \otimes UHF_d)^\prime\prime \) then \( S_i^* F_k S_I = S_i^* \Lambda^k(F_0) S_I = F_0 \) as \( S_i^* S_I \) commutes with \( F_0 \) for \( |I| = |J| = k \) and further for any \( |I| = |J| \), \( S_i^* S_I S_i^* S_J = S_i^* S_I S_j^* S_J = S_j^* S_J \) as \( X \) commutes with \( S_I S_J^* \) with \( |I| = |J| \). Thus by Proposition 3.2 (c) \( S_i^* X S_J \) is an element in commutant of \( \pi'_0(UHF_d \otimes UHF_d)^\prime\prime \) for any \( |I| = k \) and thus \( S_i^* X S_I = c F_0 \) for some scaler \( c \) independent of \( |I| = k \) as \( \pi'_0 \) is pure. We use commuting property of \( X \) with \( \pi(UHF_d \otimes UHF_d)^\prime\prime \) to conclude that \( X = c \Lambda^k(E_0) \) for some scaler \( c \). If \( k \leq -1 \) we employ the same method but with endomorphism \( \Lambda^{-k} \) so that \( \Lambda^{-k}(X) \) is an element in the commutant of \( \pi'_0(UHF_d \otimes UHF_d)^\prime\prime \). Thus \( \sum_{I:|I|=-k} S_I X S_I^* = c I \) and by injective property of the endomorphism we get \( X \) is a scaler. Thus we conclude that each \( \pi'_k \) is a pure once \( \pi'_0 \) is pure.

Next we claim that for each fix \( k \in \hat{H}_0 \), representation \( \pi_k \) of \( \tilde{UHF}_d \otimes UHF_d \) defined in Proposition 3.3 is quasi-equivalent to representation \( \pi'_k \) ( here we recall \( \hat{H}_0 \subseteq \hat{H} \) as \( H_0 \subseteq H \) ). That \( \pi'_0 \) is quasi-equivalent to \( \pi_0 \) follows as they are isomorphic. More generally for any \( k \in \hat{H} \), we abuse the notation and extend \( \pi_k \) for the restriction of \( \pi \) to the minimal central projections \( E_k \) on the subspace span by \( \{ \pi(UHF_d \otimes UHF_d)' f : \forall \ f \in \mathcal{H} \otimes_K \tilde{H}, \ F_k f = f \} \). Each
$E_k$ is a minimal central element containing $F_k$ and however two such elements i.e. $E_k$ and $E_j$ are either equal or mutually orthogonal. Thus $\{E_k : k \in \hat{H}\} = \{E_k : k \in \hat{H}_0\}$ and quasi-equivalence follows as $\pi_k$ is isomorphic with $\pi_k'$ for all $k \in \hat{H}_0$.

At this stage we also set for the time being

$$F'_0 = [\pi(UHF_d \otimes UHF_d)''\Omega]$$

It is obvious that $F'_0 \leq F_0$. We prove in following text that equality holds if $\omega$ is pure.

First we consider the case when $H = \{z : z^n = 1\}$. Projections $\Lambda(F'_0)$ and $\tilde{\Lambda}(F'_0)$ are elements in $\pi(UHF_d \otimes UHF_d)'$ by Proposition 3.3 (*). The representation $\pi(UHF_d \otimes UHF_d)''$ restricted to both the projections $\Lambda(F'_0)$, $\tilde{\Lambda}(F'_0)$ are as well pure. A pull back by the map $X \rightarrow S_i^*XS_i$ with any $1 \leq i \leq d$ will do the job for the projection $\Lambda(F'_0)$. Thus $\Lambda(F'_0)\tilde{\Lambda}(F'_0)\Lambda(F'_0) = c\Lambda(F'_0)$ for some scaler.

By pulling back with the action $X \rightarrow S_i^*XS_i$ we get

$$c = \langle \Omega, S_i^*\tilde{\Lambda}(F'_0)S_i\Omega \rangle = \sum_k \langle \Omega, S_kS_i^*F'_0S_iS_k^*\Omega \rangle$$

as $S_k^*\Omega = S_k^*\Omega$ and further $F_0$ commutes with $\pi(UHF_d)$ and thus $c = \sum \langle \Omega, S_kS_i^*F'_0S_iS_k^*\Omega \rangle$. This shows that $\tilde{\Lambda}(F'_0) \geq \Lambda(F'_0)$. Interchanging the role of $\Lambda$ and $\tilde{\Lambda}$ we conclude that $\Lambda(F'_0) = \tilde{\Lambda}(F'_0)$. Proof essentially follows along the same line for $\Lambda^k(F'_0) = \tilde{\Lambda}^k(F'_0)$ for all $k \geq 1$. By Proposition 2.5 we also note that $\Lambda^n(F'_0) = F'_0 = \tilde{\Lambda}^n(F'_0)$ as $H = \{z : z^n = 1\}$.

Thus $F' = \sum_{0 \leq k \leq n-1} \Lambda(F'_0)$ is a $\Lambda$ and as well $\tilde{\Lambda}$ invariant projection. Since $F'\Omega = \Omega$ we conclude by the cyclic property of $\Omega$ for $\pi(O_d \otimes \tilde{O}_d)''$ that $F' = 1$. Since $\Lambda^k(F'_0) \leq F_k$ and $\sum_k F_k = 1$ we conclude that $\Lambda^k(F'_0) = F_k$. In such a
case we may check that

$$F_k = [\pi(UHF_d \otimes UHF_d)''S_\Omega^* : |I| = n - k]$$

for $1 \leq k \leq n - 1$.

Similarly in case $H = S^1$ and $\omega$ is pure we also have $F_0 = F'_0$ and for $k \geq 1$

$$F_k = [\pi(UHF_d \otimes UHF_d)''S_\Omega^* : |I| = k]$$

$$F_{-k} = [\pi(UHF_d \otimes UHF_d)''S_\Omega^* : |I| = k]$$

Thus we have got an explicit description of the complete atomic centre of $I$ when $\omega$ is a pure state.

**PROPOSITION 3.4:** Let $\omega, \psi$ and Popescu system $(\mathcal{K}, \mathcal{M}, v_k, \Omega)$ be as in Proposition 3.3. Then

(a) $\{\beta_z : z \in H\}$ invariant elements in $\pi(UHF_d \otimes O_d)''$ (as well as in $\pi(\tilde{O}_d \otimes UHF_d)''$) are equal to $\pi(UHF_d \otimes UHF_d)''$.

(b) $I = I_0$ if and only if $\omega$ is pure.

Further the following statements are equivalent:

(c) $I = \pi(UHF_d \otimes UHF_d)''$;

(d) $\pi(UHF_d \otimes O_d)'' = B(\tilde{H} \otimes_{\mathcal{K}} \mathcal{H})$;

(e) $\pi(\tilde{O}_d \otimes UHF_d)'' = B(\tilde{H} \otimes_{\mathcal{K}} \mathcal{H})$;

In such a case (if any of (c), (d) and (e) is true) the following statements are also true:

(f) $\pi(UHF_d \otimes UHF_d)''$ is a type-I von-Neumann algebra with centre equal to $\{U_z : z \in H\}''$ where $U_z$ is defined in Proposition 3.2.

(g) $\omega$ is a pure state on $\mathcal{A}$.

Conversely if $\omega$ is a pure state then $\pi(UHF_d \otimes UHF_d)''$ is a type-I von-Neumann algebra with centre equal to $\{U_z : z \in H_0\}''$ where $H_0$ is a subgroup
of $H$.

**PROOF:** Along the same line of the proof of Proposition 2.5 (b) we get \( \{ \beta_z : z \in H \} \) invariant elements in \( \pi(O_d \otimes \text{UHF}_d)^\prime \prime \) is \( \pi(\text{UHF}_d \otimes \text{UHF}_d)^\prime \prime \) where factor property of \( \pi(O_d)^\prime \prime \) is crucial as in proof of Proposition 2.5 (b). Same holds for \( \pi(\text{UHF}_d \otimes \tilde{O}_d)^\prime \prime \) as \( \pi(\tilde{O}_d)^\prime \prime \) is a factor. Here we comment that factor property of \( \pi(\tilde{O}_d)^\prime \prime \) can be ensured whenever \( \psi \) is an extremal element in \( K_{\omega} \). (See Proposition 3.2 (a)).

For (b) we will first prove \( I_0 = I \) if \( \omega \) is pure. As by definition \( I_0 \subseteq I \), it is enough if we show \( I_0' \subseteq I' \). Let \( X \in I_0' \) i.e. \( X \) commutes with \( \{ U_z : z \in H \}'' \) and \( \pi(\text{UHF}_d \otimes \text{UHF}_d)^\prime \prime \). For each \( k \in \hat{H} \), \( F_kXF_k \) is an element in the commutant of \( F_k\pi(\text{UHF}_d \otimes \text{UHF}_d)^\prime \prime F_k \). \( \omega \) being pure each representation \( \pi \) restricted to \( F_k \) is irreducible and thus \( F_kXF_k = c_kF_k \) for some scalers \( c_k \). Hence \( X = \sum_k c_kF_k \in I' = \{ U_z : z \in H \}'' \).

For the converse we need to show that the restriction of \( \pi(\text{UHF}_d \otimes \text{UHF}_d)^\prime \prime \) to \( F_0' \) is pure. Let \( X \) be an element on the subspace \( F_0' \) and in the commutant of \( F_0'\pi(\text{UHF}_d \otimes \text{UHF}_d)^\prime \prime F_0' \), ( which in our earlier notation \( E' \) in Proposition 3.3 ). Then \( X \) commutes with each \( F_k \) for \( k \in \hat{H} \) and \( \pi(\text{UHF}_d \otimes \text{UHF}_d)^\prime \prime F_k \) as \( F_0' \leq F_0 \) and \( F_k \) are orthogonal to \( F_0' \) for \( k \neq 0 \). So \( X \) commutes with \( \{ U_z : z \in H \}'' \) and \( \pi(\text{UHF}_d \otimes \text{UHF}_d)^\prime \prime \) i.e. \( X \in I_0' \). By our assumption \( I_0 = I \), we have now \( X \in I' \) which is equal to \( \{ F_k : k \in \hat{H} \}'' \) and so \( X = cF_0 \) for some scaler \( c_0 \). This shows that \( F_0' = F_0 \) and \( \omega \) is pure.

(c) implies (d): \( \{ U_z : z \in H \} \) is a commuting family of unitaries such that \( \beta_z(X) = U_zXU_z^* \) and thus by (c) \( \{ U_z : z \in H \}'' \subseteq \pi(\text{UHF}_d \otimes \text{UHF}_d)^\prime \prime \). Let \( X \) be an element in the commutant of \( \pi(\text{UHF}_d \otimes O_d)^\prime \prime \). Then \( X \) commutes also with \( \{ U_z : z \in H \}'' \) and thus \( X \in \pi(\text{UHF}_d \otimes \text{UHF}_d)^\prime \prime \) by (c). Hence \( X \)
is an element in the centre of $\pi(\tilde{\text{UHF}}_d \otimes \text{UHF}_d)$ and so $X = \sum_k c_k E_k$ where $E_k$ are the minimal projections in the centre of $\pi(\tilde{\text{UHF}}_d \otimes \text{UHF}_d)$ given in Proposition 3.3. However $X$ also commutes with $\pi(O_d)$ by our assumption (c) and $\Lambda(E_k) = E_{k+1}$ for $k \in \hat{H}$. So $c_k = c_{k+1}$ and $X$ is a scaler multiple of unit operator. Hence (d) follows from (c). Along the same line we prove (c) implies (e). For a proof for (d) implies (c) and (e) implies (c), we simply apply (a).

Now we will prove (f) and (g). That $\pi(\tilde{\text{UHF}}_d \otimes \text{UHF}_d)$ is a type-I von-Neumann algebra (with completely atomic centre) follows by a theorem of [So] once we use (c). In the proof of Proposition 3.3 we have proved that the centre of $\pi(\tilde{\text{UHF}}_d \otimes \text{UHF}_d)$ is $\{U_z : z \in H_0\}$ where $H_0 \subseteq H$. For equality in the present situation we simply use (c), as $\beta_w(U_z) = U_z$ for all $w,z \in H$, to conclude that $U_z$ is in the centre of $\pi(\tilde{\text{UHF}}_d \otimes \text{UHF}_d)$.

If (c) hold then $\mathcal{I}_0 = \mathcal{I}$ and thus (g) follows by (b). Here we will give another proof using the same idea to prove (f). Let $X$ be an element in the commutant of $\pi_0(\tilde{\text{UHF}}_d \otimes \text{UHF}_d)$, where $\pi_0$ is the factor representation on the minimal central projection $E_0$ defined in Proposition 3.3. Then $X$ commutes with $\{U_z : z \in H\}$ and so by (c) $X$ is in an element in $\pi(\tilde{\text{UHF}}_d \otimes \text{UHF}_d)$. So $X$ is in the centre of $\pi_0(\tilde{\text{UHF}}_d \otimes \text{UHF}_d)$. $\pi_0$ being a factor representation $X$ is a scaler multiple of $E_0$. Thus $\pi_0$ is an irreducible representation and so $\omega$ is pure.

By Proposition 3.1 we recall that $\pi_0'$ is unitarily equivalent to GNS representation of $(\mathcal{A}, \omega)$. Thus $\pi_0'$ is irreducible if and only if $\omega$ is pure. So for a pure state $\omega$, for each $k \in \hat{H}_0$, $\pi_k$ being quasi-equivalent to $\pi'_k$, $\pi_k$ is a type-I factor representation of $\pi(\text{UHF}_d \otimes \tilde{\text{UHF}}_d)$. This completes the proof.
The following theorem is the central step that will be used repeatedly.

**THEOREM 3.5:** Let $\omega$ be an extremal translation invariant state on $\mathcal{A}$ and $\psi$ be an extremal element $\psi$ in $K_\omega$. We consider the Popescu elements $(K, v_k : 1 \leq k \leq d, \mathcal{M}, \Omega)$ as in Proposition 2.4 for the dual Popescu elements and associated amalgamated representation $\pi$ of $\mathcal{O}_d \otimes \tilde{\mathcal{O}}_d$ as described in Proposition 3.1. Then the following hold:

(a) $\pi(\tilde{\mathcal{O}}_d \otimes \mathcal{O}_d)'' = \mathcal{B}(\mathcal{H} \otimes_K \mathcal{H})$;
(b) $Q = E \tilde{E}$ is the support projection of the state $\psi$ in $\pi(\mathcal{O}_d)'' E$ and also in $\pi(\tilde{\mathcal{O}}_d)'' E$ where $E$ and $\tilde{E}$ are the support projections of the state $\psi$ in $\pi(\mathcal{O}_d)''$ and $\pi(\tilde{\mathcal{O}}_d)''$ respectively;
(c) $\omega$ is pure on $\mathcal{A}$ if and only if $\phi_0(\tau_n(x)\tau_n(y)) \to \phi_0(x)\phi_0(y)$ as $n \to \infty$ for all $x, y \in \mathcal{M}$ where $\mathcal{M} = \{v_i = PS_i P : 1 \leq i \leq d\}''$ and $\tau(x) = \sum_{1 \leq k \leq d} v_k x v_k^*$, $x \in \mathcal{M}$;
(d) $\omega$ is pure on $\mathcal{A}$ if and only if $P$ is also the support projection of $\psi$ in $\pi(\tilde{\mathcal{O}}_d)'' \tilde{\mathcal{H}}$;
(e) $\omega$ is pure on $\mathcal{A}$ if and only if $H$ is a fine subgroup of $S^1$.

PROOF: (a) is a restatement of Proposition 3.2 (a).

By our construction in general $\pi(\tilde{\mathcal{O}}_d)'' \subseteq \pi(\mathcal{O}_d)'$. Before we aim to prove
the reverse inclusion we state few general points now. To that end let \( E \) and \( \tilde{E} \) be the support projections of the state \( \psi \) in \( \pi(\mathcal{O}_d)''' \) and \( \pi(\tilde{\mathcal{O}}_d)''' \) respectively.

We set von-Neumann algebras \( \mathcal{N}_1 = \pi(\mathcal{O}_d)'E \) and \( \mathcal{N}_2 = \pi(\tilde{\mathcal{O}}_d)'E \). Note that it is enough for \( \pi(\mathcal{O}_d)' = \pi(\tilde{\mathcal{O}}_d)''' \) if we prove that \( \mathcal{N}_2 = \mathcal{N}_1 \). That it is enough follows as \( E \) being the support projection of the state on the factor \( \pi(\mathcal{O}_d)''' \), we have \( E \geq [\pi(\tilde{\mathcal{O}}_d)\Omega] \) and hence \( \Lambda^n(E) \uparrow I \) as \( n \to \infty \) because \( \Omega \) is cyclic for \( \pi(\mathcal{O}_d \otimes \tilde{\mathcal{O}}_d) \) in \( \mathcal{H} \otimes \tilde{\mathcal{H}} \). Hence two operators in \( \pi(\mathcal{O}_d)' \) are same if their actions are same on \( E \). Further we note that \( Q = E\tilde{E} \in \mathcal{N}_2 \subseteq \mathcal{N}_1 \) and claim that \( Q \) is the support projection of the state \( \psi \) in \( \mathcal{N}_2 \). To that end let \( xE \geq 0 \) for some \( x \in \pi(\tilde{\mathcal{O}}_d)''' \) so that \( \psi(QxQ) = 0. \) As \( \Lambda^k(xE) \geq 0 \) for all \( k \geq 1 \) and \( \Lambda^k(E) \to I \) we conclude that \( x \geq 0 \). As \( E\Omega = \Omega \) and thus \( \psi(E\tilde{E}E) = \psi(QxQ) = 0 \), we conclude \( \tilde{E}x\tilde{E} = 0 \), \( \tilde{E} \) being the support projection for \( \pi(\tilde{\mathcal{O}}_d)''' \). Hence \( QxQ = 0. \) As \( \psi(Q) = 1 \), we complete the proof of the claim that \( Q \) is the support of \( \psi \) in \( \mathcal{N}_2 \). Similarly \( Q \) is also the support projection of the state \( \psi \) in \( \pi(\mathcal{O}_d)''' \tilde{E} \). This completes the proof of (b).

As \( E \in \pi(\mathcal{O}_d)''' \) and \( \tilde{E} \in \pi(\tilde{\mathcal{O}}_d)''' \) we check that von-Neumann algebras \( \mathcal{M}_1 = Q\pi(\mathcal{O}_d)'''Q \) and \( \mathcal{M}'_1 = Q\pi(\tilde{\mathcal{O}}_d)Q \) acting on \( Q \) satisfies \( \mathcal{M}_1 \subseteq \mathcal{M}'_1 \). Now we check that \( \pi(\tilde{\mathcal{O}}_d \otimes \mathcal{O}_d)''' = \mathcal{B}(\mathcal{H} \otimes_{\mathcal{K}} \tilde{\mathcal{H}}) \) and note that in such a case \( Q\pi(\tilde{\mathcal{O}}_d \otimes \mathcal{O}_d)'''Q \) is the set of all bounded operators on the Hilbert subspace \( Q \).

As \( E \in \pi(\mathcal{O}_d)''' \) and \( \tilde{E} \in \pi(\tilde{\mathcal{O}}_d)''' \) we check that together \( \mathcal{M}_1 = Q\pi(\mathcal{O}_d)'''Q \) and \( \mathcal{M}'_1 = Q\pi(\tilde{\mathcal{O}}_d)Q \) generate all bounded operators on \( Q \). Thus both \( \mathcal{M}_1 \) and \( \mathcal{M}'_1 \) are factors. The canonical states \( \psi \) on \( \mathcal{M}_1 \) and \( \mathcal{M}'_1 \) are faithful and normal.

We set \( l_k = QS_kQ \) and \( \tilde{l}_k = Q\tilde{S}_kQ \), \( 1 \leq k \leq d \) and recall that \( v_k = PS_kP \) and \( \tilde{v}_k = P\tilde{S}_kP \), \( 1 \leq k \leq d \). We note that \( Pl_kP = v_k \) and \( P\tilde{l}_kP = \tilde{v}_k \) where we recall by our construction \( P \) is the support projection of the state \( \psi \) in \( \pi(\mathcal{O}_d)'''[\pi(\mathcal{O}_d)\Omega] \). \( Q \) being the support projection of \( \pi(\mathcal{O}_d)\tilde{E} \), by Theorem 2.4 applied to Cuntz elements \( \{S_i\tilde{E} : 1 \leq i \leq d\} \), \( \pi(\mathcal{O}_d)'\tilde{E} \) is order isomorphic with
\( \mathcal{M}'_1 \) via the map \( X \to QXQ \). As the projection \( F = [\pi(O_d)^n\Omega] \in \pi(O_d)' \), we check that the element \( QF\tilde{E}Q \in \mathcal{M}'_1 \). However \( QF\tilde{E}Q = E\tilde{E}F\tilde{E}E = PQ = P \) and thus \( P \in \mathcal{M}'_1 \). We also check that \( \mathcal{M}_1\Omega = \mathcal{M}_1P\Omega = P\mathcal{M}_1\Omega = \mathcal{M}\Omega \) and thus \( P = [\mathcal{M}_1\Omega] \). We set \( \tilde{\mathcal{M}} \) for the von-Neumann algebra generated by \( \{\tilde{v}_k : 1 \leq k \leq d\} \). Note that \( P\tilde{\mathcal{M}}_1P \subseteq \tilde{\mathcal{M}} \) and unless \( P \) is an element in \( \tilde{\mathcal{M}}_1 \), equality is not guaranteed for a factor state \( \omega \). The major problem is to show that \( P \) is indeed an element in \( \tilde{\mathcal{M}}_1 \) when \( \omega \) is a pure state.

To that end we need to set few additional notations. We set

\[
\mathcal{M}_0 = \{x \in \mathcal{M} : \beta_z(x) = x; z \in H\}
\]

and

\[
(\mathcal{M}')_0 = \{x \in \mathcal{M}' : \beta_z(x) = x, \ z \in H\}.
\]

Similarly we also set \( \tilde{\mathcal{M}}_0 \) and \( (\tilde{\mathcal{M}})'_0 \) as \( \{\beta_z : z \in H\} \) invariant elements of \( \tilde{\mathcal{M}} \) and \( (\tilde{\mathcal{M}})'_0 \). Notice that as a set \( (\tilde{\mathcal{M}})'_0 \) could be different from \( (\tilde{\mathcal{M}})'_0 \).

Existence of a \( \phi_0 \) preserving norm one projection \( \int_{z \in H} \beta_zdz \) ensures that modular operator of \( \phi_0 \) preserves \( \mathcal{M}_0 \) [Ta] and so does on \( (\mathcal{M}')_0 \). However there is no reason to take it granted for \( \tilde{\mathcal{M}}_0 \) to be invariant by the modular group of \( ((\mathcal{M})'_0, \phi_0) \). By Takesaki’s theorem such a property is true if and only if there exists a \( \phi_0 \)-invariant norm one projection from \( (\mathcal{M})'_0 \) onto \( \tilde{\mathcal{M}}_0 \). In the following we avoid this tempted route.

Before we proceed towards that goal, we will first prove that Kolmogorov’s property [Mo2] is also a necessary condition for the state \( \omega \) to be pure.

We claim that

\[
\bigcap_{n \geq 1} \Lambda^n(\pi(UHF_d)') = \pi(UHF_d)' \cap \pi(UHF_d)'.
\]
That $\tilde{\Lambda}^n(\pi(UHF_d)') \subseteq \{\tilde{S}_i \tilde{S}_j^* : |I| = |J| < \infty\}'$ follows by Cuntz relation and thus $\bigcap_{n \geq 1} \tilde{\Lambda}^n(\pi(UHF_d)') \subseteq \pi(UHF_d)' \cap \pi(UHF_d)'$. For the reverse inclusion let $X \in E\pi(UHF_d)' \cap \pi(UHF_d)'_0 E$. For $n \geq 1$, we choose $|I| = n$ and set $Y_n = \tilde{S}_i^* X \tilde{S}_j$. We check that it is independent of the index that we have chosen as $Y_n = \tilde{S}_i^* X \tilde{S}_j \tilde{S}_j^* \tilde{S}_i = \tilde{S}_i^* \tilde{S}_j X \tilde{S}_i \tilde{S}_j$ where in second equality we have used $X \in \pi(UHF_d)'$ and also $\Lambda^n(Y_n) = \sum_{|J| = n} \tilde{S}_j \tilde{S}_i^* X \tilde{S}_i \tilde{S}_j^* = X$. This proves the equality in the claim.

Since $E \in \pi(UHF_d)^n$ and $F_k \in \pi(UHF_d \otimes UHF_d)'$, we also have $F_k E \cap_{n \geq 1} \tilde{\Lambda}^n(\pi(UHF_d)') E F_k = F_k E \pi(UHF_d \otimes UHF_d)' E F_k = \{c_k F_k E : c_k \in \mathcal{L}\}$ by purity of the state $\omega$ (This the first time we have used that $\omega$ is pure ). So $E \cap_{n \geq 1} \tilde{\Lambda}^n(\pi(UHF_d)') E = E\{U_z : z \in H\}^n E$.

In particular we have $F_0 \cap_{n \geq 1} \tilde{\Lambda}^n(\pi(UHF_d)'_0 E) F_0 = \{c_0 F_0 E, c_0 \in \mathcal{L}\}$. Since support projection of the state $\psi$ in $\pi(UHF_d)'_0 E$ is $P$, the support projection in the restriction to $F_0$ is $PF_0$. But $P_0 = PF_0$ and is the cyclic space $[\mathcal{M}_0 \Omega]$. Thus once more by Proposition 1.1 in [Ar2], we get $||\Psi^n - \phi_0|| \to 0$ as $n \to \infty$ for any normal state on $\mathcal{M}_0' = P\pi(UHF_d)' EP$ ( Here we recall by Proposition 2.4 $P\pi(O_d)' P' = \mathcal{M}$ as $P$ is also the support projection in $\pi(O_d)^n F$ ), where commutant is taken in $\mathcal{B}(\mathcal{K}_0)$. By duality [Mo1] we conclude that $\phi_0(\tau_n(x) \tau_n(y)) \to \phi_0(x) \phi_0(y)$ as $n \to \infty$ for all $x, y \in \mathcal{M}_0 \subseteq \mathcal{B}(\mathcal{K}_0)$. This completes the proof that Kolmogorov property is also necessary for the state $\omega$ to be pure [Mo2]. This completes the proof of (c).

First we will prove that $\omega$ is pure if $P$ is also the support projection of the state $\psi$ in $\pi(\tilde{O}_d)^n \tilde{F}$, where $\tilde{F} = [\pi(\tilde{O}_d)^n \Omega]$. The support projection of $\psi$ in $\pi(\tilde{O}_d)^n \tilde{F}$ is $E\tilde{F}$ and thus we also have $P = E\tilde{F}$ by our hypothesis. Since $\Lambda^n(P) = \Lambda^n(E) F \uparrow F$ and now $\Lambda^n(P) = E\Lambda^n(\tilde{F}) \uparrow \tilde{E}$ as $n \uparrow \infty$, we also have $F = \tilde{E}$. Similarly we also have for each $n$, $E\Lambda^n(F) = \tilde{\Lambda}^n(E) \tilde{F}$ and thus taking
limit we also get $E = \tilde{F}$.

So we have $P = EF = E\tilde{E} = Q$. $\tilde{M} = P\pi(\tilde{O}_d)''P$ is cyclic in $\mathcal{K}$ i.e. $[\tilde{M}\Omega] = [P\pi(\tilde{O}_d)''P\Omega] = P\tilde{F} = PE = P$ as $\tilde{F} = E$.

However $\bigcap_{n \to \infty} \tilde{\Lambda}^n(\pi(\text{UHF}_d)) = \pi(\text{UHF}_d)'' \cap \pi(\text{UHF}_d)'$ (for a proof which is simple application of Cuntz relation, we refer to section 5 of [Mo2]) Further $\psi$ being a factor state in $K_{\omega'}$, by Proposition 3.2 $\pi(\text{UHF}_d)''$ is a factor. In particular we have $\bigcap_{n \to \infty} \tilde{\Lambda}^n(\pi(\text{UHF}_d))\tilde{F} = \mathcal{L}\tilde{F}$. Thus by Proposition 1.1 in [Ar2] we conclude that $||\Psi \circ \tilde{T}_n - \phi_0|| \to 0$ as $n \to \infty$ for all normal state $\Psi$ on $\tilde{M}_0$ where $\tilde{M}_0 = P\pi(\text{UHF}_d)''\tilde{F}P$ as $\tilde{F} = E$. Note that $\tilde{M}_0 \subseteq M_0$.

For $x \in \tilde{M}, y \in \tilde{M}'$ we have

$$\phi_0(\tilde{T}(x)y) = \sum_k <\tilde{v}_k\Omega, xv_k^*y\Omega> = \sum_k <\tilde{v}_k\Omega, xyv_k^*\Omega>$$

(as $v_k^*\Omega = \tilde{v}_k^*\Omega$ )

$$= \sum_k <\Omega, xv_kyv_k^*\Omega> = \phi_0(x\tilde{T}(y))$$

The KMS dual of $(\tilde{M}, \tilde{T}, \phi_0)$ is given by $(\tilde{M}', \tau, \phi_0)$ where $\tau(x) = \sum_k v_kxv_k^*$ for $x \in \tilde{M}'$, where commutant is taken in $B(\mathcal{K})$. Now moving to $\{\beta_z : z \in H\}$ invariant elements in the duality relation above, we verify that KMS-adjoint Markov map of $(\tilde{M}_0, \tilde{T}, \phi_0)$ is given by $(\tilde{M}_0', \tau, \phi_0)$ where $\tilde{M}_0'$, the commutant of $\tilde{M}_0$ is taken in $B(\mathcal{K}_0)$ and $\mathcal{K}_0$ is the Hilbert subspace $P_0$.

Thus by Proposition 3.4 in [Mo2] we have $\phi_0(\tau_n(x)\tau_n(y)) \to \phi_0(x)\phi_0(y)$ as $n \to \infty$ for all $x, y \in \tilde{M}_0' \subseteq B(\mathcal{K}_0)$. In particular same holds for $x, y \in M_0 \subseteq \tilde{M}_0' \subseteq B(\mathcal{K}_0)$ and by Theorem 5.5 in [Mo2], we conclude that the inductive
limit state $\omega$ of $(\text{UHF}_d, \lambda, \omega') \rightarrow^\lambda (\text{UHF}_d, \lambda, \omega')$ i.e. $(A, \theta, \omega)$ is pure on $A$. This completes the proof for “if part” of (d).

Now we get back to the main body aiming to prove (e) and converse part of (d). As a first step we claim that $H$ is a finite cyclic group if $\omega$ is pure. Suppose not i.e. $H = S^1$ and $K_\omega$ is a singleton set by Proposition 2.6. Further as a first step we check that auto-morphism

$$\beta_{S^1 \times S^1} : \mathcal{O}_d \otimes \mathcal{O}_d \rightarrow \mathcal{O}_d \otimes \mathcal{O}_d$$

defined by $\beta_{z_1, z_2}(\tilde{s}_i \otimes s_j) = z_1 \tilde{s}_i \otimes z_2 s_j$ is $\psi$-invariant:

$$\psi(\beta_{z_1, z_2}(\tilde{s}_i \tilde{s}_j^* \pi S_{i} S_{j}^*)) = \psi((z_1^{|J'| - |J|}\tilde{s}_2^{|J'| - |J|} \tilde{s}_j S_{i} S_{j}^*)$$

$$= \psi(z_1^{|J'| - |J|}\tilde{s}_2^{|J'| - |J|} \tilde{s}_j^* S_{i} S_{j}^*)$$

$$= \psi(z_1^{|J| + |J'| - |J|}\tilde{s}_2^{|J| + |J'| - |J|} \tilde{s}_j S_{i} S_{j}^*)$$

for some $z \in S^1$ so that $z_1^{|J'| - |J|}\tilde{s}_2^{|J'| - |J|} = z_1^{|J| - |J|}\tilde{s}_2^{|J| - |J'|}$. Thus we verify our claim as $H = S^1$ by our starting assumption. As $\psi$ be $\beta_{S^1 \times S^1}$ invariant, there exist unitary operators $U_{z_1, z_2}$ on $\tilde{\mathcal{H}} \otimes_{\mathcal{K}} \mathcal{H}$ satisfying

$$U_{z_1, z_2} \pi(X) U_{z_1, z_2}^* = \pi(\beta_{z_1, z_2}(X)), \quad U_{z_1, z_2} \Omega = \Omega.$$

By Proposition 2.5 applied to Cunzt algebra $\mathcal{O}_d \otimes \mathcal{O}_d$ with factor state $\psi$, we get $\beta_{S^1 \times S^1}$ invariant elements are $\pi(\text{UHF}_d \otimes \text{UHF}_d)''$ i.e. $\pi(\mathcal{O}_d \otimes \mathcal{O}_d)'' \cap \{U_{z_1, z_2} : z_1, z_2 \in S^1\}' = \pi(\text{UHF}_d \otimes \text{UHF}_d)''$. $\psi$ being a factor state we also have $\pi(\mathcal{O}_d \otimes \mathcal{O}_d)'' = \mathcal{B}(\tilde{\mathcal{H}} \otimes_{\mathcal{K}} \mathcal{H})$ by (a) of this proposition. Thus $\pi(\text{UHF}_d \otimes \text{UHF}_d)'' = \{U_{z_1, z_2} : z_1, z_2 \in S^1\}'$. Note by our definition that $U_z = U_{z, z}$ for all $z \in S^1$ and also $\omega$ being pure by Proposition 3.4 $\mathcal{I} = \mathcal{I}_0$. Since $\{U_z : z \in S^1\}$ commutes with $U_{z_1, z_2} : z_1, z_2 \in S^1$ we get from above that $\{U_z : z \in S^1\} \subseteq \pi(\text{UHF}_d \otimes \text{UHF}_d)''$ and thus $\mathcal{I} = \pi(\text{UHF}_d \otimes \text{UHF}_d)''$ by Proposition 3.4 (b).
Now we conclude by Proposition 3.4 (a) and equivalence statements (c) and (d) of Proposition 3.4 that

\[ \{ U_z : z \in S^1 \}'' = \{ U_{z_1, z_2} : z_1, z_2 \in S^1 \}'' \]

This brings a contradiction as shown below.

We consider the restriction of \( \pi \) to \( \tilde{UHF}_d \otimes UHF_d \) and the Fourier decomposition of \( \mathcal{H} \) with respect to it’s commutant i.e. \( \{ U_z : z = (z_1, z_2) \in S^1 \times S^1 \}'' \) which is a commutative algebra with \( \pi \)-irreducible spectral projections \( E_{m,n} \) on the subspace \( \mathcal{H}_{m,n} \). i.e.

\[ U_{z_1, z_2} = \sum_{m,n \in \mathbb{Z}} z_1^m z_2^n E_{m,n}, \]

so that \( \mathcal{H} = \oplus_{m,n \in \mathbb{Z}} \mathcal{H}_{m,n} \) and \( \pi = \oplus_{m,n \in \mathbb{Z}} \pi_{m,n} \) Fourier co-efficients ensures that \( \pi_{k,l} \) and \( \pi_{m,n} \) are unitary equivalent if and only if \( k = m \) and \( l = n \).

Since \( U_{z_1, z_2} S_i U_{z_1, z_2}^* = z_2 S_i \) and \( U_{z_1, z_2} \tilde{S}_i U_{z_1, z_2}^* = z_1 \tilde{S}_i \), we may write \( \Lambda(U_{z_1, z_2}) = z_2 U_{z_1, z_2} \) and \( \tilde{\Lambda}(U_{z_1, z_2}) = z_1 \tilde{U}_{z_1, z_2} \). Thus we have \( \Lambda(E_{m,n}) = E_{m,n-1} \) and so by Cuntz relation \( E_{m,n} S_i^* = S_i^* E_{m,n-1} \). Similarly we have \( \tilde{\Lambda}(E_{m,n}) = E_{m-1,n} \) and \( E_{m,n} \tilde{S}_i^* = \tilde{S}_i^* E_{m,n-1} \).

Now we recall by Proposition 3.4 that \( F_0 = [\pi(\pi(\tilde{UHF}_d \otimes UHF_d)'' \Omega)] \) as \( \omega \) is a pure state and thus \( F_0 \leq E_{0,0} \). Also recall explicit description \( F_k \) for pure \( \omega \) to conclude \( F_k \leq E_{0,k} \). \( \omega \) being pure by Proposition 3.4 we also have \( \sum_k F_k = I \) and thus we arrive at \( \sum_k E_{0,k} = I \). This brings a contradiction as by our construction \( E_{m,n} \) are all mutually orthogonal non-zero projections. Hence we arrived at a contradiction to our starting assumption that \( H = S^1 \) and \( \omega \) is pure. This completes the proof that \( H \) is a finite subgroup of \( S^1 \) when \( \omega \) is pure.

Conversely if \( H \) is finite say trivial we get by (a) that \( \omega \) is pure. For \( H = \)
\{z : z^n = 1\} finite but not trivial, we consider \( \psi \) restricted to Cuntz algebra \( \mathcal{O}_d^H \equiv \mathcal{O}_d \) and state \( \omega \) to \( C^* \) algebra \( \overline{\text{UHF}}_d^n \otimes \text{UHF}_d^m \) which is isomorphic to \( \mathcal{A} \). \( \omega \) is a factor state on \( \overline{\text{UHF}}_d^H \otimes \text{UHF}_d^H \) by Power’s criteria where we used shift \( \theta^n \). Thus the case is reduced to the situation where gauge group for \( \psi|_{\mathcal{O}_d^H} \) is trivial. We conclude that \( \omega \) is a pure state on \( \overline{\text{UHF}}_d^H \otimes \text{UHF}_d^H \) which is isomorphic to \( \overline{\text{UHF}}_d \otimes \text{UHF}_d \) by Glimm’s invariance. This shows that \( \omega \) is pure on \( \mathcal{A} \) and completes the proof of (e).

Now we aim to prove \( \tilde{F} = E \) if \( \omega \) is pure. \( \omega \) being pure, \( H \) is not equal to \( S^1 \) by the paragraph above. Thus without loss of generality we may assume for the proof of the equality that \( H \) is the trivial subgroup. Otherwise we replace \( \mathcal{O}_d \) by \( \mathcal{O}_d^H \) and \( \tilde{\mathcal{O}}_d \) by \( \tilde{\mathcal{O}}_d^H \). We set unitary operator \( V = \sum_k S_k \tilde{S}_k^* \).

Since \( \beta_z(V) = V \) for all \( z \in H \), \( V \in \pi(\text{tildeUHF}_d \otimes \text{UHF}_d)^n \) by Proposition 3.4 as \( \omega \) is pure. That \( V \) is a unitary operator follows by Cuntz’s relations and commuting property of \((S_i)\) and \((\tilde{S}_i)\). Further a simple computation shows that \( V \pi(x) V^* = \pi(\theta(x)) \) for all \( x \in \mathcal{A} = \mathcal{A}_L \otimes \mathcal{A}_R \), identified with \( \overline{\text{UHF}}_d \otimes \text{UHF}_d \) and \( \theta \) is the right shift.

\[
V E V^* = \sum_{k,k'} S_k \tilde{S}_k^* E S_{k'} \tilde{S}_{k'}^* \\
= \Lambda(E) \geq E
\]

So \( V(I - E)V^* \leq I - E \), i.e. \( (I - E)V^*E = 0 \). Also for any \( X \in \pi(\tilde{\mathcal{O}}_d)^n \) we have \( V^* F X \Omega = \sum_k \tilde{S}_k X \tilde{S}_k^* \Omega \) and thus \( V \tilde{F} V^* \geq \tilde{F} \). We set two family of increasing projections for all natural numbers \( n \in \mathbb{Z} \) as follows

\[
E_n = V^n E (V^n)^*, \quad \tilde{F}_n = V^n \tilde{F} (V^n)^*
\]

We claim that \( (V^n, E_n - |\Omega > < \Omega|, n \in \mathbb{Z}) \) and \( (V^n, \tilde{F}_n - |\Omega > < \Omega|, n \in \mathbb{Z}) \) both admits Mackey’s imprimitivity relation for the group \( \mathbb{Z} \) if \( \omega \) is pure. All
that we need to prove $E_n \to |\Omega><\Omega|$ as $n \to -\infty$ and $\tilde{F}_n \to I$ as $n \to \infty$ (since $\tilde{F}_n \leq E_n$). $E_n$ is the support projection of the state $\omega$ in $\theta^n(\pi(UHF_d)''$) and the main result in [Mo1] shows that $E_n \to |\Omega><\Omega|$ as $n \to -\infty$ if Kolmogorov’s property (i.e. $\phi_0(\tau^n(x)\tau^n(y)) \to \phi_0(x)\phi_0(y)$ for all $x, y \in M_0$ which is now equal to $M$ as we assumed $H$ is trivial). Thus we are left to prove $\tilde{F}_n \to I$ as $n \to \infty$. $\tilde{F}_n$ is equal to $[\theta^n(\pi(\tilde{UHF}_d))\Omega]$ and thus $\tilde{F}_n \to I$ as $n \to \infty$ follows as $[\pi(UHF_d \otimes \tilde{UHF}_d)\Omega] = I$.

For a cardinal number $n \in \aleph_0$, we amplify the representation $\pi$ to $n$ many copies: $n\pi = \oplus_{1 \leq k \leq n} \pi_k$ is acting on $nH = \oplus_{1 \leq k \leq n} H_k$ defined by

$$n\pi(x)(\oplus k \zeta_k) = \oplus (\pi(x)\zeta_k)$$

where $\pi_k = \pi$ is the representation of $A = UHF_d \otimes \tilde{UHF}_d$ on $H_k = H$ where $H = [\pi(UHF_d \otimes UHF_d)\Omega]$. We also extend $\tilde{F} = \oplus \tilde{F}_\alpha$, $\tilde{E} = \oplus E_\alpha$ and $\tilde{V} = \oplus_{1 \leq k \leq n} V_k$ respectively. We also set notation $\Omega_k = \oplus_{1 \leq k \leq n} \delta^k_j \Omega$.

Thus by Mackey’s theorem, there exists a cardinal number $n \in \aleph_0$ and an unitary operator $U : nH \to nH$ so that $\tilde{V} = U\tilde{V}U^*$ and $\tilde{E} = U\tilde{F}U^*$. We set a representation $\pi^n : A \to B(nH)$ by $\pi^n(x) = Un\pi(x)U^*$ and rewrite the above identity as

$$\oplus_{1 \leq k \leq n} [\pi_k(UHF_d)'\Omega_k] = \oplus_{1 \leq k \leq n} [\pi^n_k(UHF_d)''\Omega_k]$$

where $\pi^n_k(x) = U\pi_k(x)U^*$. Note that by our construction we can ensure $U\Omega_k = \Omega_k$ for all $1 \leq k \leq n$ as the operator intertwining between two imprimitivity systems are acting on the orthogonal subspace of the projection generated by vectors $\{\Omega_k : 1 \leq k \leq n\}$.

We claim $E = \tilde{F}$. Suppose not i.e. $\tilde{F} < E$. In such a case we have

$$\oplus_{1 \leq k \leq n} [\pi_k(UHF_d)'\Omega_k] < \oplus_{1 \leq k \leq n} [\pi^n_k(UHF_d)''\Omega_k]$$
Thus in principle we can repeat our construction now with $\pi_u$ and so we get a strict partial ordered set of quasi-equivalent representation of $A$. In the following we now aim to employ formal set theory to bring a contradiction on our starting assumption that $\tilde{F} < E$.

To that end we need to deal with more then one representation of $A$. For the rest of the proof we reset notation $\pi_0$ for $\pi$ used for the pure representation of $A$ in $H_0 = [\pi_0(A)\Omega_0]$ where $\Omega_0$ is the cyclic vector, the reset notation for $\Omega$.

Let $P$ be the collection of representation $(\pi, H_\pi, \Omega)$ quasi-equivalent to $\pi_0 : A \to B(H_0)$ with a shift invariant vector state $\omega(x) = (\Omega, \pi(x)\Omega) >$ i.e. $\omega(\pi(\theta(x))) = \omega(\pi(x))$.

So there exists minimal cardinal numbers $n_\pi, n_0(\pi) \in \aleph_0$ so that $n_\pi H_\pi$ is unitary equivalent to $n_0(\pi)\pi_0$. Thus given an element $(\pi, H_\pi, \Omega)$ we can associate two cardinal numbers $n_\pi$ and $n_0(\pi)$ and without loss of generality we assume that $H_\pi \subseteq n_0(\pi)H_0$ and $n_\pi H_\pi = n_0(\pi)H_0$. $\pi_0$ being a pure representation, any element $\pi \in P$ is a type-I factor representation of $A$. The interesting point here that $\oplus_{\pi \in P} \pi$ is also an element in $P$ with associated cardinal numbers $\sum_\pi n_\pi$ and $\sum_\pi n_0(\pi)$.

We say $(\pi_1, H_1, \Omega_1') << (\pi_2, H_2, \Omega_2')$ if there exists an isometry $U : n_{\pi_1}H_1 \to n_{\pi_2}H_2$ so that

(a) For each $1 \leq \alpha \leq n_{\pi_1}$ we have $U\Omega_1^\alpha = \Omega_2^{\alpha'}$ for some $1 \leq \alpha' \leq n_{\pi_2}$;

(b) $n_{\pi_2}\pi_2(x)E_2' = Un_{\pi_1}\pi_1(x)U^*$ where $E_2' \in n_{\pi_2}\pi_2(A)'$;

(c) $\oplus_{1 \leq \alpha \leq n_{\pi_1}}[\pi_1^\alpha(UHF_d)'\Omega_1^\alpha] < \oplus_{1 \leq \alpha \leq n_{\pi_2}}[\pi_2^\alpha(UHF_d)'\Omega_2^{\alpha'}]E_2'$.

In the inequality we explicitly used that both Hilbert spaces are subspaces of $nH_0$ for some larger cardinal number $n$. That the partial order is non-reflexive follows as $(\pi, H, \Omega) << (\pi, H, \Omega)$ contradicts (c) as $I = E_2'$. 
By our starting assumption that $\tilde{F} \neq E$ we check that $\pi_0 << \pi_u$. Thus going via the isomorphism we also check that for a given element $\pi \in \mathcal{P}$ there exists an element $\pi' \in \mathcal{P}$ so that $\pi << \pi'$. Thus $\mathcal{P}_0$ is a non-empty set and has at least one infinite chain.

Partial order property follows easily. If $\pi_1 << \pi_2$ and $\pi_2 << \pi_3$ then $\pi_1 << \pi_3$. If $U_{12}$ and $U_{23}$ are isometric operators that satisfies (a)-(c) respectively, then $U_{13} = U_{23}U_{12}$ will do the job for $\pi_1$ and $\pi_3$.

However by Hausdorff maximality theorem there exists a non-empty maximal totally ordered subset $\mathcal{P}_0$ of $\mathcal{P}$. We claim that $\pi_{max} = \oplus_{\pi \in \mathcal{P}_0} \pi$ on $\mathcal{H}_{\pi_{max}} = \oplus_{\pi \in \mathcal{P}_0} \mathcal{H}_\pi$ is an upper bound in $\mathcal{P}_0$. That $\pi_{max} \in \mathcal{P}$ is obvious. Further given an element $(H_1, \pi_1, \Omega_1) \in \mathcal{P}_0$ there exists an element $(H_2, \pi_2, \Omega_2) \in \mathcal{P}_0$ so that $\pi_1 << \pi_2$ by our starting remark as $\pi_0 << \pi_u$. By extending isometry $U_{12}$ to an isometry from $H_1 \rightarrow n_{\pi_{max}} \mathcal{H}_{\pi_{max}}$ trivially we get the required isometry that satisfies (a),(b) and (c) where cardinal numbers $n_{\pi_{max}} = \sum_{\pi \in \mathcal{P}_0} n_\pi \in \aleph_0$. Thus by maximal property of $\mathcal{P}_0$ we have $\pi_{max} \in \mathcal{P}_0$. This brings a contradiction as by our construction $(\pi_{max}, \mathcal{H}_{\pi_{max}}, \Omega) << (\pi_{max}, \mathcal{H}_{\pi_{max}}, \Omega)$ as $\pi_{max} \in \mathcal{P}_0$ but partial order is strict. This contradicts our starting hypothesis that $\tilde{F} < E$.

This completes the proof that $\tilde{F} = E$.

By the symmetry of the argument we also conclude that $F = \tilde{E}$ where we need to change the direction of the automorphism $\theta$ to its reverse direction. Hence $P = Q$.

Now we remove that assumption that $H$ is trivial. By our construction $Q_0 = [\pi_0(UHF)'\Omega][\pi_0(UHF_d)'\Omega]$ as $E = [\pi(UHF)'\Omega]$ and $\tilde{E} = [\pi(UHF_d)'\Omega]$ and $Q = E\tilde{E}$. On the other hand $P = EF$ where $F = [\pi(O_d)''\Omega]$ and thus $P_0 = [\pi_0(UHF)'\Omega][\pi_0(UHF_d)''\Omega]$. 

So by the proof above we have \( P_0 = Q_0 \). Now we write the equality \( P_0 = Q_0 \) as \( \text{EFF}_0 = E\tilde{E}\text{FF}_0 \) and apply \( \Lambda \) on both side to conclude that \( \Lambda(E)\text{FF}_1 = \Lambda(E)\tilde{E}\text{FF}_1 \) and multiplying by \( E \) from left we get \( E\text{EFF}_1 = E\tilde{E}\text{FF}_1 \) as \( \Lambda(E)E = E \) and thus we get \( PF_1 = QF_1 \). By repeated application of \( \Lambda \), we get \( PF_m = QF_m \). Thus we get \( P = \sum_k PF_k = \sum_k QF_k = Q \). This completes the proof for \( P = Q \). Similarly \( \tilde{F} = \sum_k \tilde{F}F_k = \sum_k EF_k = E \) and thus we get \( PF_1 = QF_1 \). By repeated application of \( \Lambda \), we get \( PF_m = QF_m \). Thus we get \( P = \sum_k PF_k = \sum_k QF_k = Q \). This completes the proof for \( P = Q \). Similarly \( \tilde{F} = \sum_k \tilde{F}F_k = \sum_k EF_k = E \) and also \( F = \tilde{E} \).

Now we are left to prove the equivalence of those three statements given in (g). We appeal to the commutant lifting Theorem 2.1 for Cuntz elements \((S,E)\) to conclude that the self adjoint elements in \( E\pi(\hat{O}_d)'E \) are order isomorphic with self adjoint elements of \( Q\pi(\hat{O}_d)'Q \) (here we have used that \( Q \) is the support projection of the state \( \psi \) in \( N_1 \)) which is equal to \( M_1' = \tilde{M}' \) as \( P = Q \). However \( QE\pi(\mathcal{O}_d)''EQ = M_1 = M \) and \( E\pi(\mathcal{O}_d)''E \subseteq E\pi(\hat{O}_d)'E \).

If (i) of (g) is true i.e. \( \tilde{M}' = M \) then by order isomorphic property that self adjoint elements in \( E\pi(\hat{O}_d)'E \) are equal to the self adjoint elements in \( E\pi(\mathcal{O}_d)''E \). Since both are von-Neumann algebras in their own right, we conclude \( N_1 = N_2 \). This completes the proof for (i) implies (ii) in (g) by our starting general observation stated at the beginning of the proof. That statement (ii) implies (i) in (g) is obvious as \( P = Q \in \pi(\mathcal{O}_d)''F \) once we look at the corner von-Neumann algebras.

We are left to prove equivalence with (iii). We recall pure representation \( \pi_0 \) on \( F_0 \). We have \( \pi_0(\text{UHF}_d)'' \subseteq \pi_0(\text{UHF}_d)' \) and the above argument says that self-adjoint elements in \( \pi_0(\text{UHF}_d)'' \) is order isomorphic to self-adjoint elements of \( \tilde{M}_0 \) and self-adjoint elements of \( \pi_0(\text{UHF}_d)' \) is order isomorphic to that of \( M_0' \). That (i) of (g) implies \( \tilde{M}_0 = M_0' \) is obvious. Thus by order isomorphic property (i) implies (iii). Conversely we claim (iii) ensures that \( \pi_k(\text{UHF}_d)' = \pi_k(\text{UHF})' \) where \( \pi_k \) are pure representations of \( \text{UHF}_d \otimes \tilde{\text{UHF}}_d \) defined as in
Proposition 3.4. It is good enough if we show for \( k = 0 \) as endomorphism \( \tilde{\Lambda} \) will push forward the property as in Proposition 3.4. However for \( k = 0 \) it is precisely Haag duality property (iii). Thus we have \( P\pi_k(UHF_d)^\prime P = P\pi_k(UHF)'P \) i.e. \( \{ x \in \mathcal{M} : \beta_z(x) = z^k x, \ z \in H \} = \{ x \in \tilde{\mathcal{M}} : \beta_z(x) = z^k x, \ z \in H \} \). Since \( \mathcal{M}' \) and \( \tilde{\mathcal{M}} \) are the von-Neumann algebra generated by the elements in these respective subsets, we conclude that \( \mathcal{M}' = \tilde{\mathcal{M}} \). This completes the proof.

The merit of the results in Proposition 3.5 is well understood once we note that for a factor state \( \omega \) if \( \mathcal{M}'_1 = \tilde{\mathcal{M}}_1 \), then \( \Omega \) is also cyclic for \( \mathcal{M}_1 \) and as \( P \in \mathcal{M}'_1 \), we have \( P \in \tilde{\mathcal{M}} \). \( Q \) being the support projection of normal state in \( \pi(\tilde{O}_d)E \), we get \( P = Q \). However the canonical trace on \( \mathcal{A} \) indeed gives an example where \( \mathcal{M}'_1 \) is not equal to \( \tilde{\mathcal{M}}_1 \) though \( \mathcal{M}' = \tilde{\mathcal{M}} \) in such a spacial case. Thus pureness of \( \omega \) is crucial in Proposition 3.5.

Since \( \bigcap_{n \geq 1} \Lambda^n(UHF_d)^\prime F = \pi(UHF_d)^\prime \bigcap \pi(UHF_d)'F \) by Cuntz's relation. \( P_0 \) being the support projection of \( \psi \) on \( \pi(UHF_d)^\prime F \), we also have by Proposition 1.1 in [Ar2] that \( ||\Psi \tau^n - \phi_0|| \to 0 \) as \( n \to \infty \) for any normal state \( \Psi \) on \( \mathcal{M}_0 \subseteq \mathcal{B}(\mathcal{K}) \) if and only if \( \omega \) is a factor state. Thus the asymptotic property studied extensively in [Ar2] is a weaker property then Kolmogorov’s property. Further Kolmogorov’s property is a necessary and sufficient condition for the Markov shift to be unitary equivalent to free shift on \( L^2(\mathbb{Z}) \) upto a multiplicity (\( H \) being separable it is countable).

We warn here an attentive reader in general for a factor state \( \omega \), the set \( F\pi(\tilde{O}_d)^\prime F \), which is a subset of \( F\pi(O_d)'F \), need not be an algebra. However by commutant lifting theorem applied to dilation \( v_i \to S_i F, \ \pi(O_d)'F \) is order isomorphic to \( \mathcal{M}' \) as \( P = EF \) is the support projection. Thus the von-Neumann sub-algebra generated by the elements \( F\pi(\tilde{O}_d)^\prime F \) is order iso-
morphic to \( \tilde{\mathcal{M}} \). However \( \tilde{\mathcal{M}}_{00} \) may properly include \( \tilde{\mathcal{M}}_{00} = \{ P\pi(\text{UHF}_d)P \}'' \) (as an example take \( \psi \) to be the unique KMS state on \( \mathcal{O}_d \) and \( \omega \) be the unique trace on \( \mathcal{A} \) for which we get \( \tilde{\mathcal{M}}_{00} = \mathcal{I} \) and \( P\pi(\tilde{\mathcal{O}}_d)''P \) is the linear span of \( \{ \tilde{v}^*_J, I, \tilde{v}_J : |J| < \infty \} \). Thus in such an example we have proper inclusion. In fact our main theorem and order isomorphism property says that the inclusion is proper if and only if \( \omega \) is not pure.

Proposition 3.5 (g) says that Haag duality for \( \omega \) is true if and only \( \mathcal{M}' = \tilde{\mathcal{M}} \). \( \Omega \) is cyclic and separating for both \( \mathcal{M}' \) and \( \tilde{\mathcal{M}} \). Further relative commutant of \( \tilde{\mathcal{M}} \) in \( \mathcal{M}' \) is trivial. If the modular automorphism group \( (\sigma_t) \) of \( \mathcal{M}' \) also preserves \( \tilde{\mathcal{M}} \) then modular operators of \( \tilde{\mathcal{M}} \) is the restriction of the modular operators for \( \mathcal{M}' \). Thus analytic elements \( \sigma_z(x) \in \tilde{\mathcal{M}} \) if \( x \in \tilde{\mathcal{M}} \). Thus we get \( \mathcal{J}v_k\mathcal{J} = \sigma^\frac{1}{2}(\tilde{v}_k) \in \tilde{\mathcal{M}} \). Since \( \mathcal{M} = \{ v_k : 1 \leq k \leq d \}'' \) and \( \mathcal{J}\tilde{\mathcal{M}}\mathcal{J} = \mathcal{M}' \) we arrived at \( \tilde{\mathcal{M}} = \mathcal{M}' \). At this stage it is not clear how we can ensure existence of a norm one projection from \( \mathcal{M}' \) to \( \tilde{\mathcal{M}} \) directly and so the equality \( \mathcal{M}' = \tilde{\mathcal{M}} \) even when \( \omega \) is a pure state. Further interesting point here that the equality \( \tilde{\mathcal{M}} = \mathcal{M}' \) hold when \( \omega \) is the unique trace on \( \mathcal{A} \) as \( v^*_k = S^*_k \) and \( \mathcal{J}\tilde{v}_k^*\mathcal{J} = \frac{1}{d}S_k \) for all \( 1 \leq k \leq d \).

In the following we investigate \( \omega \) with some additional natural symmetry sufficient for equality \( \tilde{\mathcal{M}} = \mathcal{M}' \).

Let \( \psi \) be a \( \lambda \)-invariant state on \( \mathcal{O}_d \) and \( \tilde{\psi} \) be the state on \( \mathcal{O}_d \) defined by

\[
\tilde{\psi}(s_is^*_j) = \psi(s_js^*_i)
\]

for all \( |I|, |J| < \infty \) and \( (\mathcal{H}_{\tilde{\psi}}, \pi_{\tilde{\psi}}, \Omega_{\tilde{\psi}}) \) be the GNS space associated with \( (\mathcal{O}_d, \tilde{\psi}) \). That \( \tilde{\psi} \) is well defined follows once we check by (3.1) that

\[
\psi(s_js^*_i) = \phi_0(v_jv^*_i) = \phi_0(\tilde{v}_j\tilde{v}^*_i)
\]

and appeal to Proposition 2.2 by observing that cyclicity condition i.e. the
closed linear span \( P_0 \) of the set of vectors \( \{ \tilde{v}_i^* \Omega : |I| < \infty \} \) is \( \mathcal{K} \), can be ensured if not true already by taking a new set of Popescu elements \( \{ P_0 \tilde{v}_k P_0 : 1 \leq k \leq d \} \). Otherwise one may also recall that the map \( s_I s_J^* \rightarrow \tilde{\psi}_I \tilde{\psi}_J^* \) being unital and completely positive [Po] (in particular positive), \( \tilde{\psi} \) is a well defined state on \( \mathcal{O}_d \).

Similarly for any translation invariant state \( \omega \) on \( \mathcal{A} \) we set translation invariant state \( \tilde{\omega} \) by reflecting around the point \( \frac{1}{2} \) on \( \mathcal{A} \) by

\[
\tilde{\omega}(Q^{(-l)} \otimes Q^{(-l+1)} \otimes ... \otimes Q_{-1}^{(-1)} \otimes Q_0^{(0)} \otimes Q_1^{(1)} \otimes ... \otimes Q_n^{(n)}) = \omega(Q_n^{(-n+1)} \otimes Q_1^{(0)} \otimes Q_0^{(1)} \otimes Q_{-1}^{(2)} \otimes ... Q_{-l+1}^{(l)} \otimes Q_{-l}^{(l+1)})
\] (3.4)

for all \( n, l \geq 1 \) and \( Q_{-l}, .. Q_{-1}, Q_0, Q_1, .., Q_n \in M_n(\mathcal{L}) \) where \( Q^{(k)} \) is the matrix \( Q \) at lattice point \( k \). We define \( \tilde{\omega} \) on \( \mathcal{A} \) by extending linearly to any \( Q \in \mathcal{A}_{loc} \).

Note first that the map \( \psi \rightarrow \tilde{\psi} \) is a one to one and onto affine map in the convex set of \( \lambda \) invariant state on \( \mathcal{O}_d \). In particular the map \( \psi \rightarrow \tilde{\psi} \) takes an element from \( K_\omega \) to \( K_{\tilde{\omega}} \) and the map is once more one to one and onto. Hence for any extremal point \( \psi \in K_\omega \), \( \tilde{\psi} \) is also an extremal point in \( K_{\tilde{\omega}} \). Using Power’s criteria we also verify here that \( \omega \) is an extremal state if and only if \( \tilde{\omega} \) is an extremal state. However such a conclusion for a pure state \( \omega \) is not so obvious. We have the following useful proposition.

**PROPOSITION 3.6:** Let \( \omega \) be an extremal translation invariant state on \( \mathcal{A} \) and \( \psi \rightarrow \tilde{\psi} \) be the map defined for \( \lambda \) invariant states on \( \mathcal{O}_d \). Then the following hold:

(a) \( \psi \in K_\omega \) is a factor state if and only if \( \tilde{\psi} \in K_{\tilde{\omega}} \) is a factor state.

(b) A Popescu systems \( (\mathcal{K}, \mathcal{M}, v_k, \Omega) \) of \( \psi \) satisfies Proposition 2.4 with \( \pi_\psi(s_k), 1 \leq k \leq d, P, \Omega \) i.e. the projection \( P \) on the subspace \( \mathcal{K} \) is the support projection of the state \( \psi \) in \( \pi(\mathcal{O}_d)^\prime\prime \) and \( v_i = P \pi_\psi(s_i) P \) for all.
1 \leq i \leq d$, then the dual Popescu systems \((\mathcal{K}, \mathcal{M}', \tilde{v}_k, \Omega)\) satisfies Proposition 2.4 with \((\pi_{\tilde{\psi}}(s_k), 1 \leq k \leq d, P, \Omega)\) i.e. the projection \(P\) on the subspace \(\mathcal{K}\) is the support projection of the state \(\tilde{\psi}\) in \(\pi_{\tilde{\psi}}(\mathcal{O}_d)'\) and \(\tilde{v}_i = P\pi_{\tilde{\psi}}(s_i)P\) for all \(1 \leq i \leq d\), if and only if \(\{x \in \mathcal{B}(\mathcal{K}) : \sum_k \tilde{v}_kx\tilde{v}_k^* = x\} = \mathcal{M}\).

(c) \(\omega\) is pure if and only if \(\tilde{\omega}\) is pure.

**PROOF:** Since \(\omega\) is an extremal translation invariant state, by Power’s criteria \(\tilde{\omega}\) is also an extremal state. As an extremal point of \(K_{\omega}\) is map to an extremal point in \(K_{\tilde{\omega}}\) by one to one property of the map \(\psi \rightarrow \tilde{\psi}\), we conclude by Proposition 2.6 that \(\psi\) is a factor state if and only if \(\tilde{\psi}\) is a factor state. (b) follows by the converse part of the Proposition 2.4 applied to the dual Popescu systems \((\mathcal{K}, \mathcal{M}', \tilde{v}_k, \Omega)\). This completes the proof for (b).

For (c) we recall that pureness of \(\omega\) ensures that \(\tilde{E}\tilde{F} = EF = P\) is also the support projection for the state \(\tilde{\psi}\) (in our notation in section 3 we used \(\psi\) instead \(\tilde{\psi}\) in \(\pi(\tilde{\mathcal{O}}_d)'\tilde{F}\). Thus pure property of \(\tilde{\omega}\) will follow once we verify Kolmogorov’s property of \((\tilde{\mathcal{M}}_0, \tilde{\tau}, \phi_0)\) which is equivalent to Arveson’s criteria \(||\Psi_{\tau_n} - \phi_0|| \rightarrow 0\) as \(n \rightarrow \infty\) of the KMS adjoint map [Mo2, see also Theorem 5.3 in Appendix]. Factor property of \(\omega\) says that \(\bigcap_{n \geq 1} \Lambda^n(\pi(\text{UHF}_d)) = \mathcal{L}\) and thus by Proposition 1.1 in [Ar2] we complete the proof. Note in the proof pureness is used only to ensure that \(P = \tilde{E}\tilde{F}\).

Thus the state \(\tilde{\omega}\) is translation invariant, ergodic, factor state, pure if and only if \(\omega\) is translation invariant, ergodic, factor state, pure respectively.

We say \(\omega\) is **lattice symmetric** if \(\tilde{\omega} = \omega\).

For a \(\lambda\) invariant state \(\psi\) on \(\mathcal{O}_d\) we define as before a \(\lambda\) invariant state \(\tilde{\psi}\) by

\[
\tilde{\psi}(s_Is_J^*) = \psi(s_Is_J^*)
\]

for all \(|I|, |J| < \infty\). It is obvious that \(\psi \in K_{\omega}\) if and only if \(\tilde{\psi} \in K_{\tilde{\omega}}\) and the
map $\psi \to \tilde{\psi}$ is an affine map. In particular an extremal point in $K_\omega$ is also mapped to an extremal point of $K_{\tilde{\omega}}$. It is also clear that $\tilde{\psi} \in K_\omega$ if and only if $\omega$ is lattice symmetric. Hence a lattice symmetric state $\omega$ determines an affine map $\psi \to \tilde{\psi}$ on the compact convex set $K_\omega$. Furthermore, if $\omega$ is also extremal on $\mathcal{A}$, then the affine map, being continuous on the set of extremal elements in $K_\omega$, which can be identified with $S^1/H \equiv S^1$ or $\{1\}$ (by Proposition 2.6) $z \to \psi_\beta z$ being fixing an extremal element $\psi \in K_\omega$ for the time being).

There exists $z_0 \in S^1$ so that $\tilde{\psi} = \psi_\beta z_0$ and as $\beta \tilde{\beta}_z = \tilde{\beta}_z$ for all $z \in S^1$, we get the affine map taking $\psi_\beta z \to \psi_\beta z_0 \beta$, and thus determines a continuous one to one and onto map on $S^1/H$ and as $\tilde{\psi} = \psi$ its inverse is itself. Thus either the affine map has a fixed point or $z_0^2 = 1$ i.e. it is a rotation map by an angle $2\pi$ (Here we have identified $S^1/H$ with $S^1$ in case $H \neq S^1$). Thus there exists an extremal element $\psi \in K_{\tilde{\omega}}$ so that either $\tilde{\psi} = \psi_\beta \zeta$ where $\zeta$ is either 1 or $-1$ where we recall that we have identified $S^1/H = S^1$ when $H \neq S^1$. Note that if we wish remove the identification, then for $H = \{z : z^n = 1\}$ for some $n \geq 1$, $\zeta$ is either 1 or $\exp \frac{2\pi i}{n}$. Note that in case $H = S_1$ then $\tilde{\psi} = \psi$ for $\psi \in K_{\tilde{\omega}}$ as $K_\omega$ is a singleton set by Proposition 2.6.

**PROPOSITION 3.7:** Let $\omega$ be a translation invariant lattice symmetric state on $\mathcal{A}$. Then the following hold:

(a) If $\omega$ is also an extremal translation invariant state on $\mathcal{A}$ then $H = \{z \in S^1 : \psi_\beta z = \psi\}$ is independent of $\psi \in K_\omega$.

(b) If $H = \{z : z^n = 1\}$ for some $n \geq 0$ then $\tilde{\psi} = \psi_\beta \zeta$ for all $\psi \in K_{\omega'}$ where $\zeta$ is fixed either 1 or $\exp \frac{2\pi i}{n}$. Let $(\mathcal{H}, S_k, 1 \leq k \leq d, \Omega)$ be the GNS space associated with $(\mathcal{O}_d, \psi)$, $P$ be the support projection of the state $\psi$ in $\pi(\mathcal{O}_d)$ and $\mathcal{K} = PH$ with Popescu systems $(\mathcal{K}, \mathcal{M}, v_k, 1 \leq k \leq d, \Omega)$ as in Proposition 2.4 where $v_k = PS_k P$ and associated normal state $\phi_0$ on $\mathcal{M} = \{v_k, v_k^* : 1 \leq k \leq d\}$.
is invariant for \( \tau(x) = \sum_k v_k x v_k^* \). Let \( (\tilde{\mathcal{H}}, S_k, 1 \leq k \leq d, \Omega) \) be the Popescu minimal dilation in Theorem 2.1 of the dual Popescu systems \( (\mathcal{K}, \tilde{\mathcal{M}}, \tilde{v}_k, 1 \leq k \leq d, \Omega) \) defined in Proposition 3.2. Then there exists a unitary operator \( U_\zeta : \mathcal{H} \otimes_{\mathcal{K}} \tilde{\mathcal{H}} \) so that

\[
U_\zeta^* = U_\zeta, U_\zeta \Omega = \Omega, \quad U_\zeta S_k U_\zeta^* = \beta_\zeta(\tilde{S}_k) \tag{3.6}
\]

for all \( 1 \leq k \leq d \). Furthermore if \( \omega \) is also pure then there exists a unitary operator \( u_\zeta : \mathcal{K} \rightarrow \mathcal{K} \) so that

\[
u_\zeta = \beta_\zeta(\tilde{v}_k) \tag{3.7}
\]

for all \( 1 \leq k \leq d \) and \( u_\zeta J u_\zeta^* = J, \quad u_\zeta \Delta^\frac{1}{2} u_\zeta^* = \Delta^{-\frac{1}{2}}, \quad u_\zeta^* = u_\zeta \) and \( u_\zeta \mathcal{M} u_\zeta^* = \mathcal{M}', \quad u_\zeta^* \mathcal{M} u_\zeta = \tilde{\mathcal{M}} \). Moreover \( \mathcal{M}' = \tilde{\mathcal{M}} \).

Further if \( \zeta = 1 \) then \( u_\zeta \) is self-adjoint and otherwise if \( \zeta \neq 1 \) then \( u_\zeta^{2n} \) is self adjoint.

(c) Further if \( H = S^1 \) then \( K_{\omega'} \) is having only one element \( \psi \), so \( \psi = \tilde{\psi} \) and (3.6) is valid with \( \zeta = 1 \).

**PROOF:** (a) follows by Proposition 2.6. Now we aim to prove (b). For existence of an extremal state \( \psi \in K_{\omega'} \) so that \( \tilde{\psi} = \psi \beta_\zeta \) we refer to the paragraph preceding the statement of this proposition. As \( (\psi \beta_z) = \tilde{\psi} \beta_z \) for all \( z \in S^1 \), a simple application of Proposition 2.6 says that \( \tilde{\psi} = \psi \beta_\zeta \) for all extremal points in \( K_{\omega'} \) if it holds for one extremal element. Hence existence part in (b) is true by Krein-Millmann theorem.

\( \Omega \) is a cyclic vector for \( \pi(\mathcal{O}_d \otimes \tilde{\mathcal{O}}_d) \) and thus we define \( U_\zeta : \mathcal{H} \otimes_{\mathcal{K}} \tilde{\mathcal{H}} \rightarrow \mathcal{H} \otimes_{\mathcal{K}} \tilde{\mathcal{H}} \) by

\[
U_\zeta : S_1 S_j^* \tilde{S}_j S_1^* \Omega \rightarrow \beta_\zeta(S_1 S_j^* \tilde{S}_j S_1^*) \Omega
\]

That \( U_\zeta \) is an unitary operator follows from (3.1) and the dual relation (3.5) along with our condition that \( \tilde{\psi} = \psi \beta_\zeta \). By our construction we also have
\[ U_\zeta S_k = \beta_\zeta(\tilde{S}_k)U_\zeta \] for all \( 1 \leq k \leq d \). In particular \( U_\zeta \pi(O_d)^* U_\zeta^* = \pi(\tilde{O}_d)^* \).

Now for a pure state \( \omega \) by Proposition 3.5 we have \( P = Q \) and hence \( UPU^* = UQU^* = Q = P \) which ensures an unitary operator \( u_\zeta = PU_\zeta P \) on \( K \) and a routine calculation shows that
\[ u_\zeta v_k^* u_\zeta^* = \beta_\zeta(\tilde{v}_k^*) \quad (3.8) \]
for all \( 1 \leq k \leq d \). As \( U_\zeta^* = U_\zeta \) we have \( u_\zeta^* = u_\zeta \). If \( \zeta \neq 1 \), then \( \zeta^{2n} = 1 \) and thus \( U_\zeta^{2n} \) is inverse of its own. Thus \( u_\zeta^{2n} \) is self-adjoint.

In the following we consider the case \( \zeta = 1 \) for simplicity of notation and otherwise for the case \( \zeta \neq 1 \) very little modification is needed in the symbols or simply reset temporary notation \( \tilde{v}_k \) i.e. include the phase factor.

We denote \( u_1 = u \) in the following for simplicity. It is simple to verify now the following steps \( uSv_i v_j^* \Omega = uv_i v_j^* \Omega = \tilde{v}_i \tilde{v}_j^* \Omega = F \tilde{v}_i \tilde{v}_j^* \Omega \) where \( Sx\Omega = x^*\Omega \), \( xM \) and \( Fx'\Omega = x'^*\Omega \), \( x' \in \mathcal{M}' \) are the Tomita’s conjugate operator. Hence \( u\mathcal{J} \Delta^{\frac{1}{2}} = \mathcal{J} \Delta^{-\frac{1}{2}} u \), i.e. \( u\mathcal{J} \Delta^\frac{1}{2} u^* = \mathcal{J} \Delta^{-\frac{1}{2}} \) and by uniqueness of polar decomposition we conclude that \( u\mathcal{J} u^* = \mathcal{J} \) and \( u\Delta^\frac{1}{2} u^* = \Delta^{-\frac{1}{2}} \). That \( u\mathcal{M} u^* = \tilde{\mathcal{M}} \) is obvious. For \( u\mathcal{M} u^* = \tilde{\mathcal{M}} \) we note that by our construction \( U\tilde{S}_k U^* = S_k \) and so \( U\pi(\tilde{O}_d) U^* = \pi(O_d) \) and hence projecting to its support projection we get the required relation.

We are left to prove the crucial relation \( \tilde{\mathcal{M}} = \mathcal{M}' \). It is obvious that \( \tilde{\mathcal{M}} \subseteq \mathcal{M}' \). However \( \mathcal{M}' = \mathcal{J}\mathcal{M}\mathcal{J} = \mathcal{J}u\tilde{\mathcal{M}}u^* \mathcal{J} = u\mathcal{J}\tilde{\mathcal{M}}\mathcal{J} u^* \subseteq u\mathcal{J}\mathcal{M}'\mathcal{J} u^* = u\mathcal{M} u^* = \tilde{\mathcal{M}} \). Note that in the third identity we have used that \( u \) commutes with \( \mathcal{J} \). Hence \( \mathcal{M}' = \tilde{\mathcal{M}} \).

Before we move to the main result of this section we need to introduce another useful concept. If \( Q = Q_0^{(l)} \otimes Q_1^{(l+1)} \otimes \ldots \otimes Q_m^{(l+m)} \) we set \( Q^t = \)
\( Q_0^{(l)} \otimes Q_1^{(l+1)} \otimes \ldots \otimes Q_m^{(l+m)} \) where \( Q_0, Q_1, \ldots, Q_m \) are arbitrary elements in \( M_d \) and \( Q_0^t, Q_1^t, \ldots \) stands for transpose with respect to an orthonormal basis \((e_i)\) for \( \mathcal{U}^d \) (not complex conjugate) of \( Q_0, Q_1, \ldots \) respectively. We define \( Q^t \) by extending linearly for any \( Q \in \mathcal{A}_{loc} \). For a state \( \omega \) on \( \text{UHF}_d C^* \) algebra \( \otimes \mathbb{Z} M_d \) we define a state \( \bar{\omega} \) on \( \otimes \mathbb{Z} M_d \) by the following prescription

\[
\bar{\omega}(Q) = \omega(Q^t)
\]  

(3.9)

Thus the state \( \bar{\omega} \) is translation invariant, ergodic, factor state if and only if \( \omega \) is translation invariant, ergodic, factor state respectively. We say \( \omega \) is real if \( \bar{\omega} = \omega \). In this section we study a translation invariant real state.

For a \( \lambda \) invariant state \( \psi \) on \( \mathcal{O}_d \) we define a \( \lambda \) invariant state \( \bar{\psi} \) by

\[
\bar{\psi}(s_Is_J^*) = \psi(s_Js_I^*)
\]  

(3.10)

for all \(|I|, |J| < \infty \) and extend linearly. That it defines a state follows as for an element \( x = \sum c(I,J)s_Is_J^* \) we have \( \bar{\psi}(x^*x) = \psi(y^*y) \geq 0 \) where \( y = \sum c(I,J)s_Js_I^* \). It is obvious that \( \psi \in K_{\omega'} \) if and only if \( \bar{\psi} \in K_{\bar{\omega}'} \) and the map \( \psi \to \bar{\psi} \) is an affine map. In particular an extremal point in \( K_{\omega'} \) is also mapped to an extremal point in \( K_{\bar{\omega}'} \). It is also clear that \( \bar{\psi} \in K_{\omega'} \) if and only if \( \omega \) is real. Hence a real state \( \omega \) determines an affine map \( \psi \to \bar{\psi} \) on the compact convex set \( K_{\omega'} \). Furthermore, if \( \omega \) is also extremal on \( \mathcal{A} \), then the affine map, being continuous on the set of extremal elements in \( K_{\omega'} \), which can be identified with \( S^1/H \equiv S^1 \) or \( \{1\} \) (by Proposition 2.6) by fixing an extremal element \( \psi_0 \in K_{\omega'} \). In such a case there exists a unique \( z_0 \in S^1 \) so that \( \bar{\psi}_0 = \psi_0\beta_{z_0} \).

Now \( \psi_0\beta_z = \bar{\psi}_0\beta_z \) for all \( z \in S^1 \), the affine map takes \( \psi_0\beta_z \to \psi_0\beta_{z_0z} \). If \( z_0 = 1 \) we get that the map fixes two point namely \( \psi_0 \) and \( \psi_0\beta_{-1} \).

Even otherwise we can choose \( z \in S^1 \) so that \( z^2 = z_0 \) and for such a choice we get an extremal element namely \( \psi_0\beta_z \) gets fixed by the map. What is also
crucial here that we can as well choose \( z \in S^1 \) so that \( z^2 = -z_0 \), if so then \( \psi_0 \beta_z \) gets mapped into \( \psi_0 \beta_{z_0} \beta_z = \psi_0 \beta_{-z} = \psi_0 \beta_z \beta_{-1} \). Thus in any case we also have an extremal element \( \psi \in K_{\omega'} \) so that \( \bar{\psi} = \psi \beta_\zeta \) where \( \zeta \in \{1, -1\} \).

Thus going back to the original set up, we sum up the above by saying that if \( H = \{z : z^n = 1\} \subseteq S^1 \) and \( \zeta \in \{1, \exp \frac{i\pi}{n}\} \) then there exists an extremal element \( \psi \in K_{\omega'} \) so that \( \bar{\psi} = \psi \beta_\zeta \).

**PROPOSITION 3.8:** Let \( \omega \) be a translation invariant real factor state on \( \bigotimes_{\mathbb{Z}} M_d \). Then the following hold:

(a) if \( H = \{z : z^n = 1\} \subseteq S^1 \) and \( \zeta \in \{1, \exp \frac{i\pi}{n}\} \) then there exists an extremal element \( \psi \in K_{\omega'} \) so that \( \bar{\psi} = \psi \beta_\zeta \). Let \( (\mathcal{H}, \pi_\psi(s_k) = S_k, 1 \leq k \leq d, \Omega) \) be the GNS representation of \( (\mathcal{O}_d, \psi) \), \( P \) be the support projection of the state \( \psi \) in \( \pi(\mathcal{O}_d)'' \) and \( (\mathcal{K}, \mathcal{M}, v_k, 1 \leq k \leq d, \Omega) \) be the associated Popescu systems as in Proposition 2.4. Let \( \bar{v}_k = \mathcal{J} v_k \mathcal{J} \) for all \( 1 \leq k \leq d \) and \( (\bar{\mathcal{H}}, \bar{S}_k, P, \Omega) \) be the Popescu minimal dilation as described in Theorem 2.1 associated with the systems \( (\mathcal{K}, \mathcal{M}', \bar{v}_k, 1 \leq k \leq d, \Omega) \). Then there exists a unitary operator \( W_\zeta : \mathcal{H} \to \bar{\mathcal{H}} \) so that

\[
W_\zeta \Omega = \Omega, \quad W_\zeta S_k W_\zeta^* = \beta_\zeta(S_k)
\]

for all \( 1 \leq k \leq d \). Furthermore \( P \) is the support projection of the state \( \bar{\psi} \) in \( \pi(\mathcal{O}_d)'' \) and there exists a unitary operator \( w_\zeta \) on \( \mathcal{K} \) so that

\[
w_\zeta \Omega = \Omega, \quad w_\zeta v_k w_\zeta^* = \beta_\zeta(v_k) = \mathcal{J} \beta_{\zeta'}(v_k) \mathcal{J}
\]

for all \( 1 \leq k \leq d \) and \( w_\zeta \mathcal{J} w_\zeta^* = \mathcal{J} \) and \( w_\zeta \Delta^{\frac{1}{2}} w_\zeta^* = \Delta^{-\frac{1}{2}} \). Moreover if \( \mathcal{M} \) is a factor then \( w_\zeta^* = w_\zeta \).

(b) If \( H = S^1, K_{\omega'} \) is a set with unique element \( \psi \) so that \( \bar{\psi} = \psi \) and relations (3.11) and (3.12) are valid with \( \zeta = 1 \).
PROOF: For existence part in (a) we refer the paragraph above preceded the statement of the proposition. We fix a state \( \psi \in K_{\omega'} \) so that \( \bar{\psi} = \psi \beta_\zeta \) and define \( W : \mathcal{H} \rightarrow \bar{\mathcal{H}} \) by

\[
W_\zeta : S_I S_J^* \Omega \rightarrow \beta_\zeta (\bar{S}_I \bar{S}_J^*) \Omega
\]

That \( W_\zeta \) is a unitary operator follows from (3.10) and thus \( W_\zeta S_k = \beta_\zeta (\bar{S}_k) W_\zeta \) for all \( 1 \leq k \leq d \). For simplicity of notation we take the case \( \zeta = 1 \) as very little modification is needed to include the case when \( \zeta \neq 1 \) or reset Cuntz elements by absorbing the phase factor in the following computation and use notation \( W \) for \( W_\zeta \).

\( P \) being the support projection we have by Proposition 2.4 that \( \mathcal{M}' = \{ x \in \mathcal{B}(\mathcal{H}) : \sum_k v_k x v_k^* = x \} \) and thus \( \mathcal{M} = \{ x \in \mathcal{B}(\mathcal{K}) : \sum_k J v_k J x J v_k^* J = x \} \). Hence by the converse part of Proposition 2.4 we conclude that \( P \) is also the support projection of the state \( \bar{\psi} \) in \( \bar{\pi}(\mathcal{O}_d)'' \). Hence \( W_\zeta PW_\zeta^* = P \). Thus we define an unitary operator \( w_\zeta : \mathcal{K} \rightarrow \mathcal{K} \) by \( w_\zeta = PW_\zeta P \) and verify that

\[
\bar{v}_k^* = P \bar{S}_k^* P
\]

\[
= PW_\zeta \beta_\zeta (S_k^*) W_\zeta^* P = PW_\zeta P \beta_\zeta (S_k^*) PW_\zeta^* P
\]

\[
= PW_\zeta P \beta_\zeta (v_k^*) PW_\zeta^* P = w_\zeta \beta_\zeta (v_k^*) w_\zeta^*.
\]

We recall that Tomita’s conjugate linear operators \( S, F \) [BR] are the closure of the linear operators defined by \( S : x \Omega \rightarrow x^* \Omega \) for \( x \in \mathcal{M} \) and \( F : y \Omega \rightarrow y^* \Omega \) for \( y \in \mathcal{M}' \). We check the following relations for \( \zeta = 1 \) with simplified notation \( w_1 = w \),

\[
w S v_I v_J^* \Omega = w v_I v_J^* \Omega = \bar{v}_I \bar{v}_J^* \Omega
\]

\[
= F \bar{v}_I \bar{v}_J^* \Omega = F w v_I v_J^* \Omega
\]
for $|I|, |J| < \infty$. Since such vectors are total, we have $wS = Fw$ on the domain of $S$. Thus $wSw^* = F$ on the domain of $F$. We write $S = J\Delta^{\frac{1}{2}}$ as the unique polar decomposition. Then $F = S^* = \Delta^{\frac{1}{2}}J = J\Delta^{-\frac{1}{2}}$. Hence $wJw^*w\Delta^{\frac{1}{2}}w^* = J\Delta^{-\frac{1}{2}}$. By the uniqueness of polar decomposition we get $wJw^* = J$ and $w\Delta^{\frac{1}{2}}w^* = \Delta^{-\frac{1}{2}}$. Same algebra is valid in case $\zeta \neq 1$ if we reset the notations $\tilde{v}_k$ on the right hand side absorbing the phase factor. In the following we repeat it for completeness of the proof as the phase factor could be delicate.

$$w_\zeta Sv_Iv_J^*\Omega = w_\zeta v_Iv_J^*\Omega = w_\zeta v_Iv_J^*w_\zeta^*\Omega$$

$$= \zeta^{|I|-|J|}\bar{v}_Iv_J^*\Omega = \zeta^{|I|-|J|}F\bar{v}_Iv_J^*\Omega$$

$$= F\zeta^{-|I|+|J|}\bar{v}_Iv_J^*\Omega \text{ for } |I|, |J| < \infty$$

$$= Fw_\zeta v_Iv_J^*\Omega$$

for all $|I|, |J| < \infty$.

Now we are going to show that $w_\zeta$ is self-adjoint if $\zeta = 1$ and also if $\zeta \neq 1$, $w_\zeta^{2n}$ is self-adjoint. We give the proof in the following for the case $\zeta = 1$ as the proof follows same steps for $\zeta \neq 1$ where we need to replace $w_\zeta^{2n}$ in the place of $w_1$. In the following we take $\zeta = 1$ and use notation $w$ for $w_1$. We claim that

$$w\bar{v}_k^*w^* = v_k^*$$

(3.13)

for all $1 \leq k \leq d$.

Note by (3.12) and Tomita’s theorem that $wMw^* = M'$. However by Tomita’s theorem we also have $JwMw^*J = M$ and as $J$ commutes with $w$, we conclude that $wM'w^* = M$. Further the separating property of the vector $\Omega$ for $M$ ensures that (3.13) hold if we verify the following identities:

$$w\bar{v}_k^*w^*\Omega = w\bar{v}_k^*\Omega$$
\[
= w \mathcal{J} v_k^* \Omega = \mathcal{J} w w_k^* w \Omega = \mathcal{J} \bar{v}_k^* \Omega = \bar{v}_k^* \Omega
\]

Thus \( w \bar{v}_k w^* = v_k^* \). Hence \( w^2 \in \mathcal{M}' \) and as \( w \) commutes with \( \mathcal{J} \), \( w^2 \in \mathcal{M} \). \( \omega \) being an extremal element in \( K_{\omega'} \) we have \( \mathcal{M} \vee \mathcal{M} = \mathcal{B}(\mathcal{K}) \) by Proposition 3.5 and as \( \mathcal{M} \subseteq \mathcal{M}' \), we get that \( \mathcal{M} \) is a factor. Thus for a factor \( \mathcal{M} \), \( w^2 \) is a scaler. Since \( w \Omega = \Omega \) we get \( w^* = w \). This completes the proof. \[ \square \]

A state \( \omega \) on \( \otimes_{\mathbb{Z}} M_d \) is said be in detailed balance if \( \omega \) is both lattice symmetric and real. In the following proposition as before we identified once more \( S^1/H \equiv S^1 \) in case \( H \neq S^1 \) and set \( \zeta \) be the least value in \( S^1 H \equiv S^1 \) so that \( \zeta^2 \in H \) (see Proposition 3.7 for details).

**Proposition 3.9:** Let \( \omega \) be a translation invariant extremal state on the UHF \( d \) algebra \( \otimes_{\mathbb{Z}} M_d \). Then the following are equivalent:

(a) \( \omega \) is real and lattice symmetric;

(b) There exists an extremal element \( \psi \in K_{\omega'} \) so that \( \bar{\psi} = \psi \beta_{\zeta} \) and \( \bar{\psi} = \psi \bar{\beta}_{\zeta} \), where \( \zeta \) is either 1 or \( \exp^{i\pi n} \).

Furthermore if \( \omega \) is a pure state then the following hold:

(c) There exists a Popescu elements \( (\mathcal{K}, v_k, 1 \leq k \leq d, \Omega) \) so that \( \omega = \omega_v \) with \( v_k = \mathcal{J} \bar{v}_k \mathcal{J} \) for all \( 1 \leq k \leq d \).

(d) The map \( \mathcal{J} : \mathcal{H} \otimes_{\mathcal{K}} \bar{\mathcal{H}} \to \mathcal{H} \otimes_{\mathcal{K}} \bar{\mathcal{H}} \) defined by \( \pi(s_1 s_j^* \bar{s}_j \bar{s}_j^*) \Omega \to \pi(s_1 s_j^* \bar{s}_j \bar{s}_j^*) \Omega \), \( |I|, |J|, |I'|, |J'| < \infty \) extends the map \( \mathcal{J} : \mathcal{K} \to \mathcal{K} \) to an anti-unitary map so that \( \mathcal{J} \pi(s_i) \mathcal{J} = \bar{\pi}(\bar{s}_i) \) for all \( 1 \leq i \leq d \) where \( \bar{\pi} \) is the conjugate linear extension of \( \pi \) from the generating set \( (\bar{s}_i) \), i.e. \( \bar{\pi}(\bar{s}_i \bar{s}_j^*) = \pi(\bar{s}_i \bar{s}_j^*) \) for \( |I|, |J| < \infty \) and then extend it anti-linearly for its linear combinations.

**Proof:** Since \( \omega \) is lattice symmetric, by Proposition 3.7 \( \bar{\psi} = \psi \beta_{\zeta} \) for all \( \psi \in K_{\omega'} \) where \( \zeta \) is fixed number either 1 or \( \exp^{i\pi} \) for some \( n \geq 1 \). Now we use real property of \( \omega \) and choose by Proposition 3.8 an extremal element \( \psi \in K_{\omega'} \).
so that $\bar{\psi} = \psi\beta_\zeta$. This proves that (a) implies (b). That (b) implies (a) is obvious.

Now we aim to prove the last statements which is the main point of the proposition. For simplicity of notation we consider the case $\zeta = 1$ and leave it to reader to check that a little modification needed to include the case $\zeta \neq 1$ and all the algebra stays valid if $\tilde{v}_k$ is replaced by $\beta_\zeta(\tilde{v}_k)$. We consider the Popescu system $(\mathcal{K},\mathcal{M},v_k, 1 \leq k \leq d, \Omega)$ as in Proposition 2.4 associated with $\psi$. Thus by Proposition 3.7 and Proposition 3.8 there exists unitary operators $u_\zeta, w_\zeta$ on $\mathcal{K}$ so that

$$u_\zeta v_k u_\zeta^* = \beta_\zeta(\tilde{v}_k)$$
$$w_\zeta v_k w_\zeta^* = \beta_\zeta(v_k) = \mathcal{J}\beta_\zeta(v_k)\mathcal{J}$$

where $u_\zeta^* = u_\zeta$, $u_\zeta\mathcal{J}u_\zeta^* = \mathcal{J}$, $w_\zeta^* = w_\zeta$, $w_\zeta\mathcal{J}w_\zeta^* = \mathcal{J}$ and $u_\zeta\Delta^\frac{1}{2}u_\zeta^* = w_\zeta\Delta^\frac{1}{2}w_\zeta^* = \Delta^{-\frac{1}{2}}$. Thus

$$u_\zeta w_\zeta v_k w_\zeta^* u_\zeta^* = \beta_\zeta(\tilde{v}_k)\mathcal{J}\beta_\zeta(v_k)\mathcal{J} = \mathcal{J}\beta_\zeta(\tilde{v}_k)\mathcal{J} = \mathcal{J}\tilde{v}_k\mathcal{J} = \mathcal{J}$$

(3.14)

We also compute that

$$w_\zeta u_\zeta v_k u_\zeta^* w_\zeta^* = \beta_\zeta(\tilde{v}_k)w_\zeta^* = \mathcal{J}\tilde{v}_k\mathcal{J}$$

(3.15)

and

$$w_\zeta u_\zeta v_k u_\zeta^* w_\zeta^* = \beta_\zeta(\tilde{v}_k)w_\zeta^* = \mathcal{J}\tilde{v}_k\mathcal{J}$$

(3.16)

By Theorem 3.2 for a factor state $\omega$ we also have $\mathcal{M} \lor \tilde{\mathcal{M}} = \mathcal{B}(\mathcal{K})$. As $\tilde{\mathcal{M}} \subseteq \mathcal{M}'$, in particular we note that $\mathcal{M}$ is a factor. So $u_\zeta w_\zeta^* u_\zeta^* w_\zeta \in \mathcal{M}'$ commuting also with $\mathcal{J}$ and thus a scaler as $\mathcal{M}$ is a factor. As $u_\zeta\Omega = w_\zeta\Omega = \Omega$, we conclude that $u_\zeta$ commutes with $w_\zeta$. 
Now we set \( v_\zeta = u_\zeta w_\zeta \) which is an unitary operator commuting with both \( \mathcal{J} \) and \( \Delta^\frac{1}{2} \). That \( v_\zeta \) commuting with \( \Delta^\frac{1}{2} \) follows as \( u_\zeta^* w_\zeta \Delta^\frac{1}{2} = u_\zeta^* \Delta^{-\frac{1}{2}} w_\zeta = \Delta^\frac{1}{2} u_\zeta^* w_\zeta \).

Next claim that we make now that \( v_\zeta \) is a self-adjoint element. To that end note that the relations (3.15) and (3.16) together says that \( v_\zeta^2 \in \mathcal{M}' \) and as \( v_\zeta \) commutes with \( \mathcal{J} \), \( v_\zeta^2 \) is an element in the centre of \( \mathcal{M} \). The centre of \( \mathcal{M} \) being trivial as \( \omega \) is a factor state (here we have more namely pure) and \( v_\zeta \Omega = \Omega \), we conclude that \( v_\zeta^2 \) is the unit operator. Hence \( v_\zeta \) is a self-adjoint element.

Our next claim \( v_\zeta = 1 \). As a first step we aim to show that \( v_\zeta \) is an element in the centre of \( \mathcal{M} \) and for simplicity we use the notation \( v \) for \( v_\zeta \) in the following computation. To that end let \( \theta \) be an unitary element in \( \mathcal{M}' \) and by (3.14) we also have

\[
\theta v \theta^* v_k \theta v \theta^* = \mathcal{J} \tilde{v}_k \mathcal{J} \quad (3.17)
\]

By symmetry we also have

\[
\theta v \theta^* \mathcal{J} \tilde{v}_k \mathcal{J} \theta v \theta^* = v_k \quad (3.18)
\]

The automorphism \( \alpha_\theta : x \rightarrow \theta v \theta^* x \theta v \theta^* \) on \( \mathcal{M} \) is independent of \( \theta \) and equal to \( \alpha_1(x) = vxv \). Since the automorphism \( \alpha_1 \) preserves \( \phi_0 \), it commutes with Tomita’s modular automorphism group and conjugation action. Thus in particular (3.17) can be rewritten as

\[
\theta v \theta^* \tilde{v}_k \theta v \theta^* = \mathcal{J} v_k \mathcal{J} \quad (3.19)
\]

Thus the unitary operator \( v^* \theta v \theta^* \) commutes with both \( \{v_k : 1 \leq k \leq d\} \) and \( \{\tilde{v}_k : 1 \leq k \leq d\} \). \( \omega \) being a factor state by Proposition 3.2 and our starting remark we have \( \mathcal{M} \vee \tilde{\mathcal{M}} = \mathcal{B}(\mathcal{K}) \) and thus \( \theta v \theta^* = \mu v \) where \( \mu \) is a scaler of modulus 1. However \( v^* = v \) and so we get \( \mu = \bar{\mu} = 1 \). \( \theta \) being an arbitrary
unitary element in $M'$, we conclude that $v \in M$. As $v = JVJ \in M'$ and $M$ is a factor, $v$ is a scaler multiple of 1. As $v\Omega = \Omega$, we get $v = 1$. This completes the proof for (c). The last statement (d) follows by a routine calculation as shown below for a special vectors.

\[
<\Omega, \pi(s_I s_{I'}^* s_{J'} \tilde{s}_J \tilde{s}_J^* \Omega) > \\
= <\Omega, v_I v_{I'}^* \tilde{v}_{I'} \tilde{v}_{J'} \Omega > \\
= <\Omega, JVJ v_I v_{I'}^* v_{J'} \Omega > \\
(\text{as } JVJ = \tilde{v}_i) \\
= <\tilde{v}_I \tilde{v}_{I'}^* v_I v_{I'}^* \Omega, \Omega > \\
(\mathcal{J} \text{ being anti-unitary}) \\
= <\pi(s_I s_{I'}^* s_{J'} \tilde{s}_J \tilde{s}_J^*) \Omega, \Omega > 
\]

For anti-unitary relation involving for more general vectors, we use Cuntz relations and the above special cases. The statement is obvious as $\mathcal{J}$ is anti-linear. This completes the proof.

We set an anti-automorphism $\mathcal{J} : \mathcal{O}_d \otimes \tilde{\mathcal{O}}_d \to \mathcal{O}_d \otimes \tilde{\mathcal{O}}_d$ defined by $\mathcal{J}(s_I s_{I'}^* \otimes \tilde{s}_J \tilde{s}_J^*) = s_I s_{I'}^* \otimes \tilde{s}_J \tilde{s}_J^*$ for $|I|, |J|, |I'|, |J'| < \infty$ by extending anti-linearly. We say a state $\psi$ on $\mathcal{O}_d \otimes \tilde{\mathcal{O}}_d$ is reflection positive if $\psi(\mathcal{J}(x)x) \geq 0$ for all $x \in \mathcal{O}_d$. Similarly for a state $\omega$ on $\mathcal{A}$. Note that this notion is an abstract version of the concept “reflection positivity” of a state introduced in [FILS].

**THEOREM 3.10:** Let $\omega$ be a translation invariant factor state on $\mathcal{A}$ and $\psi$ be an extremal point $K_{\omega'}$. We consider the representation $\pi$ of $\mathcal{O}_d \otimes \tilde{\mathcal{O}}_d$ described as in Proposition 3.2. If $\omega$ is lattice symmetric pure state then we have the duality relation

\[
\pi_\omega(\mathcal{A}_R)^{''} = \pi_\omega(\mathcal{A}_L)' 
\]
where \( A_L, A_R \) are \( C^* \)-subalgebras of \( A \) defined as in section 1.

Further if \( \omega \) is also real then

(a) \( H \subseteq \{1, -1\} \) and \( \omega = \omega_v \) where Popescu elements \( (\mathcal{K}, v_k; 1 \leq k \leq d, \Omega) \) satisfies the dual relation

\[
\mathcal{J} v_k \mathcal{J} = \tilde{v}_k
\]
i.e. on the domain of the modular operator

\[
v_k = \Delta^{\frac{1}{2}} v_k^* \Delta^{-\frac{1}{2}}
\]
for all \( 1 \leq k \leq d \).

(b) \( \omega \) is reflection positive.

(c) \( \Delta = I \) if and only if \( v_k = v_k^* \), \( 1 \leq k \leq d \). In such a case \( M \) is finite type-I and spacial correlation functions of \( \omega \) decays exponentially.

**PROOF:** For Haag duality we refer to the proof of Proposition 3.5 (f) where we have shown that duality is equivalent to \( \tilde{M} = M' \). Thus the result follows from Proposition 3.7.

For (a) we recall from Proposition 3.9 that we have an extremal element \( \psi \) so that associated Popescu elements satisfies the following:

\[
v_k = \mathcal{J} \tilde{v}_k \mathcal{J} = \Delta^{\frac{1}{2}} v_k^* \Delta^{-\frac{1}{2}}
\]
for all \( 1 \leq k \leq d \). As \( \beta_z(v_k) = z v_k \) for all \( z \in H \) and \( \beta_z(v_k^*) = \bar{z} v_k^* \) for all \( z \in H \) and \( (\beta_z : z \in H) \) commutes with modular automorphism, we compute the following:

\[
z v_k = \beta_z(v_k) = \Delta^{\frac{1}{2}} \beta_z(v_k^*) \Delta^{-\frac{1}{2}} = \Delta^{\frac{1}{2}} \bar{z} v_k^* \Delta^{-\frac{1}{2}} = \bar{z} v_k
\]
for all \( z \in H \).

Thus \( z = \sum_k z v_k v_k^* = \sum_k \bar{z} v_k v_k^* = \bar{z} \) for all \( z \in H \) i.e. \( H \subseteq \{1, -1\} \).
For (b) we recall from Proposition 3.5 that $P_0 \pi(UHF_d)^{''} P_0 = \mathcal{M}_0$ and $P_0 \pi(\tilde{UHF}_d)^{''} P_0 = \tilde{\mathcal{M}}_0$. Thus for any $x \in \mathcal{A}_R$ we write

$$\omega(x, J(x)) = \langle \Omega, \pi(x) \pi(J(x)) \Omega \rangle$$

$$= \langle \Omega, P \pi(x)PP \pi(J(x))P \Omega \rangle = \langle \Omega, P \pi(x)P \pi(J(x))P \Omega \rangle \geq 0,$$

where we have used equality $\pi(J(x)) = J \pi(x)J$ from Proposition 3.9. In fact this also shows that $\omega$ is positive definite i.e. $\omega(x, J(x)) = 0$ if and only if $x = 0$ (the self-dual Tomita’s positive cone \{ $J a J a \Omega : a \in \mathcal{M}$ \} [BR1] being pointed).

We will show the non-trivial part. Assume $v_k = v_k^*$ for all $1 \leq k \leq d$. So $\Delta$ is affiliated to $\mathcal{M}'$. As $J \Delta J = \Delta^{-1}$, $\Delta$ is also affiliated to $\mathcal{M}$. Hence $\Delta = I$ as $\mathcal{M}$ is a factor.

In such a case $\phi_0$ is a tracial state on $\mathcal{M}$. $\mathcal{M}$ being hyper-finite $\phi_0$ is the unique trace. Thus $\mathcal{M}$ is either a type-I finite factor or a type-II$_1$ factor. Now we will rule out the possibility for $\mathcal{M}$ to be a type-II$_1$ factor. It is enough if we show that $\mathcal{M}_0$ acting on $[\mathcal{M}_0 \Omega]$ is a type-I finite factor. We appeal to Corollary 4.4 in [Mo4] to conclude that $\mathcal{M}_0$ acting on $[\mathcal{M}_0 \Omega]$ can not be a type-II$_1$ factor as $\phi_0(\tau_n(x) \tau_n(x)) \rightarrow \phi_0(x) \phi_0(y)$ for all $x, y \in \mathcal{M}_0$. This completes the proof of the first part of (c).

For the last part of (c) is rather elementary. We note that purity of $\omega$ ensures that the point spectrum of the self-adjoint contractive operator $T$, defined by $Tx \Omega = \tau(x) \Omega$ on the KMS Hilbert space, in the unit circle is trivial i.e. \{ $z \in S^1 : T f = z f$ for some non zero $f \in \mathcal{K}$ \} is the trivial set \{1\} (as a consequence of strong mixing property). Thus $T$ being a contractive matrix on a finite dimensional Hilbert space, the spectral radius of $T - |\Omega \rangle \langle \Omega|$ is $\alpha$ for some $\alpha < 1$. Now we use Proposition 3.1 and Proposition 3.9 for any
$X_l \in A_L$ and $X_r \in A_R$ to verify the following

$$e^{\delta k} |\omega(X_l \theta_k(X_r)) - \omega(X_l)\omega(X_r)|$$

$$= e^{\delta k} |\phi_0(\mathcal{J} x_l \mathcal{J} \tau_k(x_r)) - \phi_0(x_l)\phi_0(x_r)| \to 0$$

as $k \to \infty$ for any $\delta > 0$ so that $e^\delta \alpha < 1$ where $\mathcal{J} x_l \mathcal{J} = PX_l P$ and $x_r = PX_r P$ for some $x_l, x_r \in \mathcal{M}$. As $\alpha < 1$ such a $\delta > 0$ exists. This completes the proof for (c).

\[\blacksquare\]

4 Detailed balance translation invariant pure state and split property:

Let $\omega$ be a translation invariant real lattice symmetric pure state on $A$ as in last section Proposition 3.9. We fix an extremal element $\psi \in K_\omega$ so that $\tilde{\psi} = \psi \beta_\zeta$ and consider the Popescu elements $(\mathcal{K}, \mathcal{M}, v_i, \Omega)$ as in Proposition 3.9. $P$ being the support projection of a factor state $\psi$ we have $\mathcal{M} = P\pi_\Omega(\mathcal{O}_d)'' P = \{v_k, v^*_k : 1 \leq k \leq d\}''$ (Proposition 2.4). So the dual Popescu elements $(\mathcal{K}, \mathcal{M}', \tilde{v}_k, 1 \leq k \leq d, \Omega)$ satisfy the relation $\tilde{v}_k = \mathcal{J} v_k \mathcal{J}$ (recall that the factor $\zeta$ won’t show up as two symmetry will kill each other as given in Proposition 3.9) for all $1 \leq k \leq d$ and so we have

$$\phi_0(\mathcal{J} x \mathcal{J} \tau(y)) = \phi_0(\mathcal{J} \tau(x) \mathcal{J} y) \quad (4.1)$$

for all $x, y \in \mathcal{M}$. A direct proof for (4.1) also follows from Proposition 3.1 (b) as $P = EE\tilde{E}$ and

$$< \Omega, \tilde{v}_l v_j^* v_l v_j^* \Omega > = < \Omega, E\tilde{E}S_l \tilde{S}_l^* E\tilde{E}S_l S_j^* S_l^* \tilde{E}E \Omega >$$

$$= < \Omega, \tilde{S}_l S_j \tilde{S}_l^* S_l S_j^* \Omega >$$
for all $|I|, |J|, |I'|, |J'| < \infty$.

We quickly recall as $\mathcal{M}_0$ is the $\{\beta_z : z \in H\}$ invariant elements of $\mathcal{M} = P\pi(\mathcal{O}_d''P)$, the norm one projection $x \rightarrow \int_{z \in H} \beta_z(x)dz$ from $\mathcal{M}$ onto $\mathcal{M}_0$ preserves the faithful normal state $\phi_0$. So by Takesaki's theorem modular group associated with $\phi_0$ preserves $\mathcal{A}_0$. Further since $\beta_z(\tau(x)) = \tau(\beta_z(x))$ for all $x \in \mathcal{M}$, the restriction of the completely positive map $\tau(x) = \sum_k v_kxv_k^*$ to $\mathcal{M}_0$ is a well defined map on $\mathcal{M}_0$. Hence the completely positive map $\tau(x) = \sum_k v_kxv_k^*$ on $\mathcal{M}_0$ is also KMS symmetric i.e.

$$<< x, \tau(y) >> = << \tau(x), y >>$$

where $x, y \in \mathcal{M}_0$ and $<< x, y >> = \phi_0(x^*\sigma\tau(y))$ and $\sigma_i$ is the modular automorphism group on $\mathcal{M}_0$ associated with $\phi_0$ and $[\mathcal{M}_0\Omega] = P_0$ where $P_0 = PF_0$. However the inclusion $\mathcal{M}_0 \subseteq \mathcal{M}$ need not be an equality in general unless $H$ is trivial. The unique ground state of $XY$ model in absence of magnetic field give rise to a non-split translation invariant real lattice symmetric pure state $\omega$ and further $H = \{1, -1\}$ [Mo3].

We now fix a translation invariant real lattice symmetric pure state $\omega$ and explore KMS-symmetric property of $(\mathcal{M}_0, \tau, \phi_0)$ and the extended Tomita’s conjugation operator $J$ on $\mathcal{H} \otimes_K \mathcal{H}$ defined in Proposition 3.9 to study the relation between split property and exponential decaying property of spacial correlation functions of $\omega$.

For any fix $n \geq 1$ let $Q \in \pi(\mathcal{A}_{[-k+1,k]})$. We write

$$Q = \sum_{|I| = |J| = |I'| = |J'| = n} q(I', J'|I, J)\tilde{S}_{I'}\tilde{S}_{J'}S_{I}S_{J}^*$$

and $q$ be the matrix $q = ((q(I', J'|I, J)))$ of order $d^{2n} \times d^{2n}$.

PROPOSITION 4.1: The matrix norm of $q$ is equal to operator norm of $Q$
in $\pi(A_{[-n,1, n]}).$

**PROOF:** $C^*$ completions of $\pi(UHF_d \otimes UHF_d)$ is isomorphic to $A$. Thus we note that the operator norm of $Q$ is equal to the matrix norm of $\hat{q}$ where $\hat{q} = ((\hat{q}(I', I|J', J)))$ is a $d^{2n} \times d^{2n}$ matrix with $\hat{q}(I', I|J', J) = q(I', J'|I, J)$. However the map $L(q) = \hat{q}$ is linear and identity preserving. Moreover $L^2(q) = q$. Thus $||L|| = 1$. Hence $||q|| = ||\hat{q}||$. This completes the proof.

**PROPOSITION 4.2:** Let $\omega$ be a translation invariant real lattice symmetric pure state on $UHF_d \otimes \mathbb{R}M_d$. Then there exists an extremal point $\psi \in K_\omega$ so that $\psi \beta_\zeta = \tilde{\psi} = \bar{\psi}$ where $\zeta \in \{1, \exp \frac{i\pi}{2}\}$ and the associated Popescu systems $(\mathcal{H}, S_k, 1 \leq k \leq d, \Omega)$ and $(\mathcal{H}, \tilde{S}_k, 1 \leq k \leq d, \Omega)$ described in Proposition 3.2 and Proposition 3.9 satisfies the following:

(a) For any $n \geq 1$ and $Q \in \pi(A_{[-n,1, n]})$ we write

$$Q = \sum_{|I'|=|J'|=|I|=|J|=n} q(I', J'|I, J)\tilde{S}_{I'}\tilde{S}_{J'}S_I S_J$$

and set a notation for simplicity as

$$\hat{\theta}_k(Q) = \sum_{|I|=|J|=|I'|=|J'|=n} q(I', J'|I, J)\tilde{S}_{I'}\tilde{S}_{J'} S_I S_J S_{I'} S_{J'}.$$

Then $\hat{\theta}_k(Q) \in A_{(-\infty,-k]\cup[k+1,\infty)}$.

(b) $Q = \mathcal{J}Q\mathcal{J}$ if and only if $q(I', J'|I, J) = q(I, J|I', J')$;

(c) If the matrix $q = ((q(I', J'|I, J)))$ is non-negative then there exists a matrix $b = ((b(I', J'|I, J)))$ so that $q = b^* b$ and then

$$q = PQP = \sum_{|K|=|K'|=n} \mathcal{J}x_{K,K'}\mathcal{J}x_{K,K'},$$

where $x_{K,K'} = \sum_{I,J: |I|=|J|=n} b(K,K'|I,J) v_I v_J^*$ $\in \mathcal{M}_0$.

(d) In such a case i.e. if $Q = \mathcal{J}Q\mathcal{J}$ the following hold:

(i) $\omega(Q) = \sum_{|K|=|K'|=n} \phi_0(\mathcal{J}x_{K,K'}\mathcal{J}x_{K,K'})$
(ii) $\omega(\hat{\theta}_k(Q)) = \sum_{|K|=|K'|=n} \phi_0(Jx_KK'J\tau_{2k}(x_{K,K'})).$

**PROOF:** Since the elements $\tilde{S}_I^*S_{I'}\tilde{S}_J^*S_J : |I| = |J| = |I'| = |J'| = n$ form an an linear independent basis for $\pi(A_{[-n+1,n]})$, (a) follows. (b) is also a simple consequence of linear independence of the basis elements and the relation $\mathcal{J}\tilde{S}_I^*S_{I'}\mathcal{J} = S_{I'}^*S_I\tilde{S}_J^*S_J$ as described in Proposition 3.9.

For (c) we write $Q = \sum_{|K|=|K'|=n} JQ_KK'JQ_{K,K'}$ where $Q_{K,K'} = \sum_{I,J: |I|=|J|=n} b(K,K'|I,J)S_I^*S_J$. $\omega$ being pure we have (Proposition 3.5) $P = EE\bar{E}$ where $E$ and $\bar{E}$ are support projection of $\psi$ in $\pi(\mathcal{O}_d)''$ and $\pi(\tilde{\mathcal{O}}_d)''$ respectively. So for any $X \in \pi(\mathcal{O}_d)''$ and $Y \in \pi(\tilde{\mathcal{O}}_d)''$ we have $PXYP = \bar{E}EX\bar{E} = \bar{E}YE\bar{E}X\bar{E} = PXYP$. Thus (c) follows as $\omega(Q) = \phi_0(q)$ by Proposition 3.1 (b). For (d) we use (a) and (c). This completes the proof.

**PROPOSITION 4.3:** Let $\omega$, a translation invariant pure state on $\mathcal{A}$, be in detailed balance. Then the following are equivalent:

(a) $\omega$ is decaying exponentially.

(b) The spectrum of $T - |\Omega><\Omega|$ is a subset of $[-\alpha,\alpha]$ for some $0 \leq \alpha < 1$ where $T$ is the self-adjoint contractive operator defined by

$$Tx\Omega = \tau(x)\Omega, \ x \in M_0$$

on the KMS-Hilbert space $<< x,y >> = \phi_0(x^*\sigma_2(y))$.

**PROOF:** Since $T_kx\Omega = \tau_k(x)\Omega$ for $x \in M_0$ and for any $L \in A_L$ and $R \in A_R$ we have $\omega(L\theta_k(R)) = \phi_0(JyJ\tau_k(x)) = << y, T_kx >>$ where $x = P\pi(R)P$ and $y = \mathcal{J}P\pi(L)P\mathcal{J}$ are elements in $M_0$. Since $P\pi(A_R)''P = M_0$ and $P\pi(A_L)''P = \tilde{M}_0 = M'_0$ as $\tilde{M} = M'$ by Proposition 3.9, we conclude that (a) hold if and only if $e^{k\delta} |< f, T_kg > - < f, \Omega > < \Omega, g > | \to 0$ as $k \to \infty$ for any vectors $f, g$ in a dense subset $\mathcal{D}$ of the KMS Hilbert space.
That (b) implies (a) is now obvious since $e^{k\delta} \alpha^k = (e^{\delta} \alpha)^k \to 0$ whenever we choose a $\delta > 0$ so that $e^{\delta} \alpha < 1$ where $\alpha < 1$.

For the converse suppose that (a) hold and $T^2 - |\Omega| < \Omega|$ is not bounded away from 1. Since $T^2 - |\Omega| < \Omega|$ is a positive self-adjoint contractive operator, for each $n \geq 1$, we find a unit vector $f_n$ in the Hilbert space so that $E_{[1-1/n,1]} f_n = f_n$ and $f_n \in D$, where $s \to E_{[s,1]}$ is the spectral family of the positive self-adjoint operator $T^2 - |\Omega| < \Omega|$ and in order to ensure $f_n \in D$ we also note that $E_{[s,1]} D = \{E_{s,1} f : f \in D\}$ is dense in $E_{[s,1]}$ for any $0 \leq s \leq 1$.

Thus by exponential decay there exists a $\delta > 0$ so that $e^{2k\delta} (1 - \frac{1}{n})^k \leq e^{2k\delta} \int_{[0,1]} s^k < f_n, dE_s f_n > = e^{2k\delta} < f_n, [T^{2k} - |\Omega| < \Omega]| f_n > \to 0$ as $k \to \infty$ for each $n \geq 1$. Hence $e^{2\delta} (1 - \frac{1}{n}) < 1$. Since $n$ is any integer, we have $e^{2\delta} \leq 1$. This contradicts that $\delta > 0$. This completes the proof.

Now we are set to state our main result in this section. For any $Q \in \pi(A)$ we set $\mathcal{J}(Q) = \mathcal{J}Q\mathcal{J}$. Recall that $\mathcal{J}^2 = I$. Any element $Q = \frac{1}{2}(Q + \mathcal{J}(Q)) + \frac{1}{2}(Q - \mathcal{J}(Q))$ is a sum of an even element in $\{Q : \mathcal{J}(Q) = Q\}$ and an odd element in $\{Q : \mathcal{J}(Q) = -Q\}$. Moreover $iQ$ is an even element if $Q$ is an odd element. Also note that $||Q_{\text{even}}|| \leq ||Q||$ and $||Q_{\text{odd}}|| \leq ||Q||$. Hence it is enough if we verify (1.1) for all even elements for split property. We fix any $n \geq 1$ and an even element $Q \in \mathcal{A}_{[-n+1,n]}$. We write as in Proposition 4.2 $Q = \sum_{|I'| = |J'| = |I| = |J| = n} q(I', J'| I, J) \tilde{S}_{I'} \tilde{S}_{J'} S_I S_J$. The matrix $q = (q(I', J'| I, J)$ is symmetric and thus $q = q_+ - q_-$ where $q_+$ and $q_-$ are the unique non-negative matrix contributing it's positive and negative parts of $q$. Hence $||q_+|| \leq ||q||$ and $||q_-|| \leq ||q||$. We set a notation for simplicity that

$$\hat{\theta}_k(Q) = \sum_{|I'| = |J'| = |I| = |J| = n} q(I', J'| I, J) \tilde{\Lambda}^k(\tilde{S}_{I'} \tilde{S}_{J'}) \Lambda^k(S_I S_J)$$
which is an element in $\mathcal{A}_{(-\infty,-k]} \bigcup (k,\infty)$ and by Proposition 4.2 (d)

$$\omega(\hat{\theta}_k(Q)) = \sum_{|K|=|K'|=n} \phi_0(\mathcal{J}x_{K,K'} \mathcal{J}\tau_2k(x_{K,K'}))$$

provided

$q = (q(I',J'|I,J)$ is positive, where $PQP = \sum_{|K|=|K'|=n} \mathcal{J}x_{K,K'} \mathcal{J}x_{K,K'}$ and $x_{K,K'} = \sum_{I,J} b(K,K'|I,J)v Iv^*$ and $q = b^*b$. Thus in such a case we have by Proposition 4.2 (d) that

$$|\omega(\hat{\theta}_k(Q)) - \omega_L \otimes \omega_R(\hat{\theta}_k(Q))| = \sum_{|K|=|K'|=n} \phi_0(\mathcal{J}x_{K,K'} \mathcal{J}(\tau_2k - \phi_0)(x_{K,K'}))$$

$$= \sum_{|K|=|K'|=n} << x_{K,K'}, (T - |\Omega| < |\Omega|)^{2k} x_{K,K'} >>$$

$$\leq \alpha^{2k} \sum_{|K|=|K'|=n} << x_{K,K'}, x_{K,K'} >>$$

provided $||T - |\Omega| < |\Omega|| \leq \alpha$ and so

$$\leq \alpha^{2k} \omega(Q) \leq \alpha^{2k} ||\hat{q}|| = \alpha^{2k} ||q||$$

In the last identity we have used Proposition 4.1.

Hence for an arbitrary $Q$ for which $\mathcal{J}(Q) = Q$ we have

$$|\omega(\hat{\theta}_k(Q)) - \omega_L \otimes \omega_R(\hat{\theta}_k(Q))| \leq \alpha^{2k}(||q_+|| + ||q_-||) \leq 2\alpha^{2k} ||q|| = 2\alpha^{2k} ||Q||$$

where in the last identity we have used once more Proposition 4.1. Thus we have arrived at our main result.

**THEOREM 4.4:** Let $\omega$ be a translation invariant real lattice symmetric pure state on $\mathcal{A}$. If the special correlation function of $\omega$ decays exponentially then $\omega$ is split.

For a pure translation invariant split state $\omega$, it is well known [Ma2,BJP] that $\mathcal{M} = \mathcal{M}_0$ and $\mathcal{M}$ is a type-I factor. In case $\mathcal{M}$ is a finite type-I factor, it
is evident that the spacial correlation function decays exponentially as the criteria given in Proposition 4.3 can be verified easily as the contractive operator $T$ acting on a finite dimensional Hilbert space $\mathcal{K}$ and $\{Tf = f : f \in \mathcal{K}\} = \mathcal{L}\Omega$. Such a pure state is a quantum Markov state [Ac], which is also known widely in the literature as a valence bound state [AKLT]. We defer illustration of our main result as non-trivial examples within the frame work in statistical mechanics [Ru,BR2] particularly in condensed matter physics [AL,AKLT] deserve a detail analysis leading to some theoretical predictions. For the time being interested reader are referred to preprints [Mo3].

5 Appendix:

Let $\mathbb{I}$ be either $\mathbb{R}$, set of real numbers or $\mathbb{Z}$, set of integers and $\mathbb{I}^+$ is non-negative numbers of $\mathbb{I}$. Let $\tau_t : \mathcal{A}_0 \rightarrow \mathcal{A}_0$, $\mathbb{I}^+$ be a semigroup of completely positive unital maps on a von-Neumann algebra $\mathcal{A}_0 \subseteq \mathcal{B}(\mathcal{H}_0)$ with a faithful normal invariant state $\phi_0$. Here we review the construction of stationary Markov processed given as in [AM] in order to fix the notations and important properties.

We consider the class $\mathcal{M}$ of $\mathcal{A}_0$ valued functions $\underline{x} : \mathbb{I} \rightarrow \mathcal{A}_0$ so that $x_r \neq I$ for finitely many supported points and equip with the point-wise multiplication $(\underline{x\ y})_r = x_{r}y_{r}$. We define the map $L : (\mathcal{M}, \mathcal{M}) \rightarrow \mathcal{C}$ by

$$L(\underline{x\ y}) = \phi_0(x^*_{r_n} \tau_{r_n-1-r_n}(x^*_{r_{n-1}}(......x^*_{r_2} \tau_{r_2-r_2}(x^*_{r_1} y_{r_1})y_{r_2})...y_{r_{n-1}})y_{r_n})$$ (5.1)

where $\underline{r} = (r_1, r_2, ..r_n)$ $r_1 \leq r_2 \leq .. \leq r_n$ is a finite collection of points in $\mathbb{Z}$ containing both the support sets of $\underline{x}$ or $\underline{y}$. That this kernel is well defined follows from our hypothesis that $\tau_t(I) = I$, $t \geq 0$ and the invariance of the state $\phi_0$ for $\tau$. The complete positiveness of $\tau$ implies that the map $L$ is a
non-negative definite form on the set $\mathcal{M}$. Thus there exists a Hilbert space $\mathcal{H}$ and a map $\lambda: \mathcal{M} \rightarrow \mathcal{H}$ such that

$$<\lambda(x), \lambda(y)> = L(x, y).$$

Often we will omit the symbol $\lambda$ to simplify our notations unless more then one such maps are involved.

We use the symbol $\Omega$ for the unique element in $\mathcal{H}$ associated with $x = (x_r = I, r \in \mathbb{N})$ and the associated vector state $\phi$ on $B(\mathcal{H})$ defined by $\phi(X) = <\Omega, X\Omega>.$

For each $t \in \mathbb{N}$ we define shift operator $S_t : \mathcal{H} \rightarrow \mathcal{H}$ by the following prescription:

$$(S_t x)_r = x_{r+t}$$

(5.2)

It is simple to note that $S = ((S_t, t \in \mathbb{N}))$ is a unitary group of operators on $\mathcal{H}$ with $\Omega$ as an invariant element.

For any $t \in \mathbb{N}$ we set

$$\mathcal{M}_t = \{x \in \mathcal{M}, x_r = I \forall r > t\}$$

and $F_t$ for the projection onto $\mathcal{H}_t$, the closed linear span of $\{\lambda(\mathcal{M}_t)\}$. For any $x \in A_0$ and $t \in \mathbb{R}$ we also set elements $i_t(x), \in \mathcal{M}$ defined by

$$i_t(x)_r = \begin{cases} x, & \text{if } r = t \\ I, & \text{otherwise} \end{cases}$$

We also note that $i_t(x) \in \mathcal{M}_t$ and set $*$-homomorphisms $j^0_0: A_0 \rightarrow B(\mathcal{H}_0)$ defined by

$$j^0_0(x)y = i_0(x)y$$
for all $y \in M$. That it is well defined follows from (5.1) once we verify that it preserves the inner product whenever $x$ is an isometry. For any arbitrary element we extend by linearity. Now we define $j_0^f : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ by

$$j_0^f(x) = j_0^0(x)F_0\|.$$  

(5.3)

Thus $j_0^f(x)$ is a realization of $\mathcal{A}_0$ at time $t = 0$ with $j_0^f(I) = F_0$. Now we use the shift $(S_t)$ to obtain the process $j^f = (j_t^f : \mathcal{A}_0 \to \mathcal{B}(\mathcal{H}), t \in \mathbb{Z})$ and forward filtration $F = (F_t, t \in \mathbb{R})$ defined by the following prescription:

$$j_t^f(x) = S_tj_0^f(x)S_t^* \quad F_t = S_tF_0S_t^*, \quad t \in \mathbb{Z}. \quad (5.4)$$

So it follows by our construction that $j_{r_1}^f(y_1)j_{r_2}^f(y_2)\ldots j_{r_n}^f(y_n)\Omega = y$ where $y_r = y_{r_1}$, if $r = r_1$ otherwise $I$, ($r_1 \leq r_2 \leq \ldots \leq r_n$). Thus $\Omega$ is a cyclic vector for the von-Neumann algebra $\mathcal{A}_{[-\infty)}$ generated by $\{j_r^f(x), r \in \mathbb{R}, x \in \mathcal{A}_0\}$. From (5.4) we also conclude that $S_tXS_t^* \in \mathcal{A}_{[-\infty)}$ whenever $X \in \mathcal{A}_{[-\infty)}$ and thus we can set a family of automorphism $(\alpha_t)$ on $\mathcal{A}$ defined by

$$\alpha_t(X) = S_tXS_t^*.$$  

Since $\Omega$ is an invariant element for $(S_t)$, $\phi$ is an invariant state for $(\alpha_t)$.

We also claim that

$$F_sj_t^f(x)F_s = j_s^f(\tau_{t-s}(x)) \quad \forall s \leq t. \quad (5.5)$$

For that purpose we choose any two elements $\underline{y}, \underline{y}' \in \lambda(\mathcal{M}_s)$ and check the following steps with the aid of (5.1) and (5.2):

$$<\underline{y}, F_sj_t^f(x)F_s\underline{y}'> = <\underline{y}, i_t(x)\underline{y}'> = <\underline{y}, i_s(\tau_{t-s}(x))\underline{y}'>.$$  

Since $\lambda(\mathcal{M}_s)$ spans $\mathcal{H}_s$ it complete the proof of our claim.
For any element \( x \in \mathcal{M} \), we verify by the relation \(< y, F_t x > = < y, x >\) for all \( y \in \mathcal{M}_t \) that

\[
(F_t x)_r = \begin{cases} 
  x_r, & \text{if } r < t; \\
  \tau_{t-r}(\ldots \tau_{r_{n-2}}(\tau_{r_{n-1}}(x_{r_{n-1}})x_{r_{n-1}})\ldots x_t), & \text{if } r = t \\
  I, & \text{if } r > t
\end{cases}
\]

where \( r_1 \leq \ldots \leq r_k \leq t \leq \ldots \leq r_n \) is the support of \( x \). The result follows once we note that for any fix \( x, y \in \mathcal{H} \) if \( t \leq r_1, r'_1 \), where \( r_1, r'_1 \) are the lowest support of \( x \) and \( y \) respectively,

\[
< x, F_t y > = < F_t x, F_t y >
\]

\[
= \phi_0((\tau_{r_1-t}(\ldots \tau_{r_{n-2}}(\tau_{r_{n-1}}(x_{r_{n-1}})x_{r_{n-1}})\ldots x_{r_1}))^*)
\]

\[
\tau_{r'_1-t}(\ldots \tau_{r'_m-2}(\tau_{r'_m-r'_m}(y_{r'_m})y_{r'_m})\ldots y_{r'_1})]
\]

Thus \( F_t \to \Omega > < \Omega \) as \( t \to -\infty \) if and only if \( \phi_0(\tau_t(x)\tau_t(y)) \to \phi_0(x)\phi_0(y) \) as \( t \to \infty \) for all \( x, y \in \mathcal{A}_0 \). In such a case we have \( \mathcal{A}_{[\cdot, \infty)} = \mathcal{B}(\mathcal{H}) \). Such a property is called Kolmogorov’s property for \((\tau_t)\) and associated imprimitivity systems are called Kolmogorov shift. For details we refer to [Mo1]. So far we have not used the faithfulness property of \( \phi_0 \). In [Mo2] we have shown that the converse i.e. \( \mathcal{A}_{[\cdot, \infty)} = \mathcal{B}(\mathcal{H}) \) implies \( F_t \to \Omega > < \Omega \) as \( t \to -\infty \) provided \( \phi_0 \) is faithful.

Following [AM] once more, we will consider the time reverse process associated with the KMS-adjoint (Or Petz adjoint) quantum dynamical semigroup \((\mathcal{A}, \bar{\tau}, \phi_0)\). First we recall from [AM] time reverse process associated with the KMS-adjoint (Petz-adjoint) semigroup in the following paragraph.

Let \( \phi_0 \) be a faithful state and without loss of generality let also \((\mathcal{A}_0, \phi_0)\) be in the standard form \((\mathcal{H}_0, \mathcal{A}_0, \mathcal{J}, \Delta, \mathcal{P}, \omega_0)\) [BrR] where \( \omega_0 \in \mathcal{H}_0 \), a cyclic and
separating vector for $A_0$, so that $\phi_0(x) = \langle \omega_0, x\omega_0 \rangle$ and the closer of the closeable operator $S_0 : x\omega_0 \rightarrow x^*\omega_0$, $S$ possesses a polar decomposition $S = J\Delta^1/2$ with the self-dual positive cone $P$ as the closure of $\{JxJx\omega_0 : x \in A_0\}$ in $H_0$. Tomita’s [BR] theorem says that $\Delta^{it}A_0\Delta^{-it} = A_0$, $t \in \mathbb{R}$ and $JA_0J = A_0'$, where $A_0'$ is the commutant of $A_0$. We define the modular automorphism group $\sigma = (\sigma_t, t \in \mathbb{I})$ on $A_0$ by

$$\sigma_t(x) = \Delta^{it}x\Delta^{-it}$$

which satisfies the KMS relation

$$\phi_0(x\sigma_{-\frac{\pi}{2}}(y)) = \phi_0(\sigma_{\frac{\pi}{2}}(y)x)$$

for any two analytic elements $x, y$ for the automorphism. A more useful form for KMS relation here

$$\phi_0(\sigma_{-\frac{\pi}{2}}(x)^*\sigma_{-\frac{\pi}{2}}(y^*)) = \phi_0(y^*x)$$

which shows that $Jx\Omega = \sigma_{-\frac{\pi}{2}}(x^*)\Omega$. Furthermore for any normal state $\psi$ on $A_0$ there exists a unique vector $\zeta \in P$ so that $\psi(x) = \langle \zeta, x\zeta \rangle$.

We consider the unique Markov semigroup $(\tau'_t)$ on the commutant $A'_0$ of $A_0$ so that $\phi(\tau_t(x)y) = \phi(x\tau'_t(y))$ for all $x \in A_0$ and $y \in A'_0$. Proof follows a standard application of Dixmier lemma a variation of Radon-Nikodym theorem. We define weak* continuous Markov semigroup $(\tilde{\tau}_t)$ on $A_0$ by $\tilde{\tau}_t(x) = J\tau'_t(JxJ)J$. Thus we have the following adjoint relation

$$\phi_0(\tau_t(x)\sigma_{-\frac{\pi}{2}}(y)) = \phi_0(\sigma_{\frac{\pi}{2}}(x)\tilde{\tau}_t(y))$$

(5.6)

for all $x, y \in A_0$, analytic elements for $(\sigma_t)$.

We consider the forward weak Markov processes $(\mathcal{H}, S_t, j^t, F_t, F'_t, t \in \mathbb{I}, \Omega)$ associated with $(A_0, \tau_t, t \geq 0, \phi_0)$ and the forward weak Markov processes $(\tilde{\mathcal{H}}, \tilde{S}_t, \tilde{j}^t, \tilde{F}_t, \tilde{F}'_t, t \in \mathbb{I}, \tilde{\Omega})$ associated with $(A_0, \tilde{\tau}_t, t \geq 0, \phi_0)$.
Now following a basic idea of G.H. Hunt, as in [AM], here we consider the transformation \( \mathcal{H} \to \tilde{\mathcal{H}} \) generalizing Tomita’s conjugate operator given by \( \tilde{x}(t) = \sigma_{-\frac{2}{\pi}}(x(-t)^*) \) for \( x \in \mathcal{H} \) supported on analytic elements of the modular automorphism group of \( \phi_0 \). We recall that KMS condition and duality property will ensure that such a transformation is anti-inner product preserving and thus extends to an anti-unitary operator \( U_0 : \mathcal{H} \to \tilde{\mathcal{H}} \) which takes \( x \) to \( \tilde{x} \).

Further there exists a unique backward weak Markov processes \((j^b_t), (\tilde{j}^b_t)\) which generalizes Tomita’s representation so that

\[
F_s[j^b_t(x)] = j^b_s(\tilde{\tau}_{s-t}(x))
\]

for \(-\infty < t \leq s < \infty\).

\[
\tilde{F}_s[\tilde{j}^b_t(x)] = \tilde{F}_s[j^b_{s-t}(x))
\]

for \(-\infty < t \leq s < \infty\). We have the following theorem.

**THEOREM 5.1:** [AM] There exists an unique anti-unitary operator \( U_0 : \mathcal{H} \to \tilde{\mathcal{H}} \) so that

(a) \( U_0\Omega = \tilde{\Omega} \);
(b) \( U_0S_tU_0^* = \tilde{S}_{-t} \) for all \( t \in \mathbb{R} \);
(c) \( U_0j^f_t(x)U_0 = \tilde{j}^b_{-t}(x^*) \), \( U_0j^b_t(x)U_0 = \tilde{j}^f_{-t}(x^*) \) for all \( t \in \mathbb{R} \);
(d) \( U_0F_tU_0^* = \tilde{F}_{-t}, \ U_0F_tU_0^* = \tilde{F}_{-t} \) for all \( t \in \mathbb{R} \);

Exploring Markov property of both forward and backward processes, Tomita’s duality also ensures the following duality theorem.

**THEOREM 5.2:** [Mo2] For each \( t \in \mathbb{Z}, \mathcal{A}'_t = \mathcal{A}_t^b \), where \( \mathcal{A}_t = \{j^f_s(x) : x \in \mathcal{A}_0, s \geq t\}'' \) and \( \mathcal{A}_0^b = \{j^b_s(x) : x \in \mathcal{A}_0, s \leq t\}'' \).

**THEOREM 5.3:** \( \phi_0(\tau_t(x)\tau_t(y)) \to \phi_0(x)\phi_0(y) \) as \( t \to \infty \) for all \( x, y \in \mathcal{A}_0 \) if and only if \( ||\Psi_{\tilde{\tau}_t} - \phi_0|| \to 0 \) as \( t \to \infty \) for any normal state \( \Psi \) on \( \mathcal{A}_0 \).
**PROOF:** Note by Theorem 5.2 we have $\forall A|t| = B(\mathcal{H})$ if and only if $\bigcap A|t| = C$. Thus the result follows by Proposition 1.1 in [Ar2] and the results on Kolmogorov’s property proved above.

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