ON STEIN’S CONJECTURE ON THE POLYNOMIAL CARLESON OPERATOR

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ABSTRACT. We prove that the generalized Carleson operator $C_d$ with polynomial phase function is of strong type $(p, r)$, $1 < r < p < \infty$; this yields a positive answer in the $1 < p < 2$ case to a conjecture of Stein which asserts that for $1 < p < \infty$ we have that $C_d$ is of strong type $(p, p)$. A key ingredient in this proof is the further extension of the relational time-frequency perspective (introduced in [5]) to the general polynomial phase.

1. Introduction

We define the (generalized) polynomial Carleson operator as

$$C_d f(x) := \sup_{Q \in Q_d} \left| p.v. \int_{\mathbb{T}} \frac{1}{y} e^{iQ(y)} f(x - y) \, dy \right|,$$

where here $d \in \mathbb{N}$, $Q_d$ is the class of all real polynomials $Q$ with $\deg(Q) \leq d$, and $f \in C^1(\mathbb{T})$ ($\mathbb{T} = [-\frac{1}{2}, \frac{1}{2}]$). Remark that in the case $d = 1$, $C_1$ corresponds to the classical Carleson operator.

The main result of this paper is:

**Theorem.** Let $1 < r < p < \infty$; then

$$\|C_d f\|_{L^r(\mathbb{T})} \lesssim_{p, r, d} \|f\|_{L^p(\mathbb{T})}.$$

Furthermore, combining this result with the methods in [6] and with some general interpolation techniques, we obtain:

**Corollary.** i) If $1 < p \leq 2$ then $C_d$ is of weak type $(p, p)$.

ii) If $1 < p < 2$ then $C_d$ is of strong type $(p, p)$.

As one may observe, i) extends Carleson’s theorem on the pointwise convergence of Fourier series, which asserts that $C_1$ is of weak type $(2, 2)$ ([1], [2], [4]). Also, for $1 < p < 2$, ii) recovers (for $d = 1$) the further extension of...
Hunt ([3]) and in the same range of exponents gives (for general $d$) a positive answer to:

**Conjecture (Stein [7],[8]).** If $1 < p < \infty$ then $C_d$ is of strong type $(p,p)$.

Our results are heavily based on the intuition and methods developed in [5], which further were significantly influenced by the powerful geometric and combinatorial ideas presented in [2].

We recall here that one key geometric ingredient in the proof in [5] was to regard the quadratic symmetry from a *relational* perspective. As the name suggests, this perspective stresses the importance of *interactions* between objects rather than simply treating them independently (for details, see [5], Section 2). This approach had as a consequence the splitting of the operator $C_2$ into “small pieces” with time-frequency portrait (morally) localized near parallelograms (tiles) of area one. In this article, following the above-mentioned perspective, our tiles (that will reflect the time-frequency localization of the “small pieces” of $C_d$) will be some “curved regions” representing neighborhoods\(^1\) of polynomials in the class $Q_{d-1}$.

In Section 2 we present the notations and the general procedure of constructing our tiles, in Section 3 we elaborate on the discretization of our operator $C_d$, while Section 4 is dedicated to the study of the interaction between tiles. The main ingredients that are presented in Section 5 - their proofs will be postponed to Section 7 - will help us to prove our Theorem in Section 6. In the Appendix we include several useful results regarding the distribution and growth of polynomials.

Finally, we mention that this paper is closely connected to [5]. Indeed, its entire conception and realization refer to and require knowledge from the latter. Following this, we will maintain the same structure of the presentation and stress only the sensitive points that differ and/or bring some further significant new insight into the treated topic.

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2. Notations and construction of the tiles

As mentioned in the introduction, we denote by $Q_d$ the class of all real polynomials of degree smaller than or equal to $d$. If not specified, $q$ will always designate an element of $Q_{d-1}$, while $Q$ will refer to an element of $Q_d$. When appearing together in a proof $q$ will designate the derivative of $Q$.

Take now the canonical dyadic grid in $[0, 1] = T^2$ and in $\mathbb{R}$. Throughout the paper the letters $I, J$ will refer to dyadic intervals corresponding to the

\(^1\)For the exact meaning of this description, see Section 2.

\(^2\)The reader should not be confused by the fact that, depending on our convenience, the symbol $T$ may refer to a different unit interval from that mentioned in the statement of our Theorem.
grid in \( \mathbb{T} \) while \( \alpha^1, \ldots, \alpha^d \) will represent dyadic intervals associated with the grid in \( \mathbb{R} \). Now, if \( I \) is any (dyadic) interval we denote by \( c(I) \) the center of \( I \). Let \( I_r \) be the “right brother” of \( I \), with \( c(I_r) = c(I) + |I| \) and \( |I_r| = |I| \); similarly, the “left brother” of \( I \) will be denoted \( I_l \) with \( c(I_l) = c(I) - |I| \) and \( |I_l| = |I| \). If \( a > 0 \) is some real number, by \( aI \) we mean the interval with the same center \( c(I) \) and with length \( |aI| = a|I| \); the same conventions apply to intervals \( \{\alpha^k\}_k \).

A tile \( P \) is a \((d + 1)\)-tuple of dyadic (half open) intervals, i.e. \( P = [\alpha^1, \alpha^2, \ldots, \alpha^d, I] \), such that \( |\alpha^j| = |I|^{-1}, j \in \{1, \ldots, d\} \). The collection of all tiles \( P \) will be denoted by \( \mathbb{P} \).

Now, for each tile \( P = [\alpha^1, \alpha^2, \ldots, \alpha^d, I] \) we will associate a geometric (time-frequency) representation, denoted \( \hat{P} \). The exact procedure is as follows: for \( I \) as before we first set \( x_I = (x^1_I, x^2_I, \ldots, x^d_I) \in \mathbb{T}^d \) to be the \( d \)-tuple defined as follows: \( x^1_I, x^2_I \) are the endpoints of the interval \( I \), \( x_3^I = \frac{x^1_I + x^2_I}{2} \), \( x^4_I = \frac{x^1_I - x^2_I}{2} \) and inductively (in the obvious manner) we continue this procedure until we reach the \( d \)-th coordinate. Then, define

\[
Q_d-1(P) = \{ q \in Q_{d-1} \mid q(x^j_I) \in \alpha^j \ \forall j \in \{1, \ldots, d\} \}.
\]

We will say that \( q \in P \) iff \( q \in Q_{d-1}(P) \).

Finally, we set

\[
\hat{P} = \{ (x, q(x)) \mid x \in I \& q \in P \}.
\]

The collection of all geometric tiles \( \hat{P} \) will be denoted with \( \hat{\mathbb{P}} \).

In the following we will also work with dilates of our tiles: for \( a > 0 \) and \( P = [\alpha^1, \alpha^2, \ldots, \alpha^d, I] \) we have \( aP = [a\alpha^1, a\alpha^2, \ldots, a\alpha^d, I] \). Similarly, we write

\[
a\hat{P} := \hat{aP} = \{ (x, q(x)) \mid x \in I \& q \in Q_{d-1}(aP) \}.
\]

Also, if \( \mathcal{P} \subseteq \mathbb{P} \) then by convention \( a\mathcal{P} := \{ aP \mid P \in \mathcal{P} \} \); similarly, if \( \hat{\mathcal{P}} \subseteq \hat{\mathbb{P}} \) then \( a\hat{\mathcal{P}} := \{ a\hat{P} \mid \hat{P} \in \hat{\mathcal{P}} \} \).

For each tile \( P = [\alpha^1, \alpha^2, \ldots, \alpha^d, I] \in \mathbb{P} \) we associate the “central polynomial” \( q_P \in Q_{d-1} \) given by the La Grange interpolation polynomial:

\[
q_P(y) := \sum_{j=1}^{d} \frac{\prod_{k=1 \atop k \neq j}^{d} (y - x^k_I)}{\prod_{k=1}^{d} (x^k_I - x^j_I)} c(\alpha^j).
\]

For \( P = [\alpha^1, \alpha^2, \ldots, \alpha^d, I] \) we denote the collection of its neighbors by \( N(P) = \{ P' = [\alpha^{1'}, \alpha^{2'}, \ldots, \alpha^{d'}, I] \mid \alpha^{k'} \in \{\alpha^k, \alpha^r_k, \alpha^l_k\} \ \forall k \in \{1, \ldots, d\} \} \).

For any dyadic interval \( I \subseteq [0, 1] \), define the (non-dyadic) intervals

\[
I^*_r = [c(I) + \frac{7}{2}|I|, c(I) + \frac{11}{2}|I|] \ \& \ I^*_l = [c(I) - \frac{11}{2}|I|, c(I) - \frac{7}{2}|I|]
\]

and set \( I^* = I^*_r \cup I^*_l \) and \( \bar{I} = 13I \).
Similarly, for $P = [\alpha^1, \alpha^2, \ldots, \alpha^d, I]$ we define
\[ \hat{P}^* = \{(x, q(x)) \mid x \in I^* & q \in P\} \]
and further repeat the same procedure for $\hat{P}^*, \hat{P}^*$ and $\hat{P}$.

Throughout the paper $p$ will be the index of the Lebesgue space $L^p$ and, unless otherwise mentioned, will obey $1 < p < \infty$. Also, $p'$ will be its Hölder conjugate (i.e. $\frac{1}{p} + \frac{1}{p'} = 1$), while $p^* \overset{\text{def}}{=} \min(p, p')$.

For $f \in L^p(T)$, we denote by
\[ Mf(x) = \sup_{x \in I} \frac{1}{|I|} \int_I |f| \]
the Hardy-Littlewood maximal function associated to $f$.

If $\{I_j\}$ is a collection of pairwise disjoint intervals in $[0, 1]$ and $\{E_j\}$ a collection of sets such that for a fixed $\delta \in (0, 1)$
\[(4) \quad E_j \subset I_j \quad \& \quad \frac{|E_j|}{|I_j|} \leq \delta \quad \forall \ j \in \mathbb{N}, \]
then we denote
\[(5) \quad M_\delta f(x) := \begin{cases} \sup_{I \supseteq I_j} \frac{1}{|I|} \int_I |f|, & \text{if } x \in E_j \\ 0, & \text{if } x \notin E_j \end{cases} . \]

The symbol $c(d)$ will designate a positive constant depending only on $d$; furthermore, this constant is allowed to change from line to line.

For $A, B > 0$ we say $A \lesssim B$ (resp. $A \gtrsim B$) if there exist an absolute constant $C > 0$ such that $A < CB$ (resp. $A > CB$); if the constant $C$ depends on some quantity $\delta > 0$ then we may write $A \lesssim_\delta B$. If $C^{-1}A < B < CA$ for some (positive) absolute constant $C$ then we write $A \approx B$.

As in [5], for $x \in \mathbb{R}$ we set $[x] := \frac{1}{1+|x|}$.

The exponents $\eta$ and $\epsilon$ may change throughout the paper.

In what follows, for notational simplicity we will refer to the operator $C_d$ as $T$.

### 3. Discretization

We first express $T$ in terms of its elementary building blocks:
\[ Tf(x) = \sup_{a_1, \ldots, a_d \in \mathbb{R}} |M_{1,a_1} \ldots M_{d,a_d} HM^*_1 a_1 \ldots M^*_d a_d f(x)| = \sup_{Q \in Q_d} |T_Q f(x)| , \]
where $\{M_{j,a_j}\}_{j \in \{1, \ldots, d\}}$ is the family of (generalized) modulations given by
\[ M_{j,a_j} f(x) := e^{ia_j x^j} f(x) \quad j \in \{1, \ldots, d\} \]
(here $f \in L^p$, $a_j \in \mathbb{R}$ & $x \in \mathbb{T}$) and
\[ T_Q f(x) = \int_{\mathbb{T}} \frac{1}{y} e^{i(Q(x) - Q(x-y))} f(x - y) \, dy \]
with $Q \in Q_d$ given by $Q(y) = \sum_{j=1}^d a_j y^j$. 

Equivalently
\[ T_Q f(x) = \int_{\mathbb{T}} \frac{1}{x-y} e^{i (J^T_y q)} f(y) \, dy, \]
where here, as mentioned in the previous section, \( q \) stands for the derivative of \( Q \).

Now linearizing \( T \) we write
\[ T f(x) = T_Q f(x) = \int_{\mathbb{T}} \frac{1}{x-y} e^{i (J^T_y q_x)} f(y) \, dy, \]
where \( Q_x(y) := \sum_{j=1}^d a_j(x) y^j \) with \( \{ a_j(\cdot) \}_{j \in \{1, \ldots, d\}} \) measurable functions and \( q_x \) refers to the derivative of \( Q_x \) (i.e., \( q_x(t) = \frac{d}{dt} Q_x(t) \)).

Further, proceeding as in [2] and [5], we define \( \psi \) to be an odd \( C^\infty \) function such that \( \text{supp} \, \psi \subseteq \{ y \in \mathbb{R} : 2 < |y| < 8 \} \) and
\[ \frac{1}{y} = \sum_{k \geq 0} \psi_k(y) \quad \forall \ 0 < |y| < 1, \]
where by definition \( \psi_k(y) := 2^k \psi(2^k y) \) (with \( k \in \mathbb{N} \)). Using this, we deduce that
\[ T f(x) = \sum_{k \geq 0} T_k f(x) := \sum_{k \geq 0} \left( \int_{\mathbb{T}} e^{i (J^T_y q_x)} \psi_k(x-y) f(y) \, dy \right). \]

Now for each \( P = [\alpha^1, \alpha^2, \ldots, \alpha^d, I] \in \mathbb{P} \) let \( E(P) = \{ x \in I \mid q_x \in P \} \). Also, if \( |I| = 2^{-k} \ (k \geq 0) \), we define the operators \( T_P \) on \( L^2(\mathbb{T}) \) by
\[ T_P f(x) = \left\{ \int_{\mathbb{T}} e^{i (J^T_y q_x)} \psi_k(x-y) f(y) \, dy \right\} \chi_{E(P)}(x). \]

As expected, if \( \mathbb{P}_k := \{ P = [\alpha^1, \alpha^2, \ldots, \alpha^d, I] \in \mathbb{P} \mid |I| = 2^{-k} \} \), for fixed \( k \) the \( \{ E(P) \} \) form a partition of \([0,1]\), and so
\[ T_k f(x) = \sum_{P \in \mathbb{P}_k} T_P f(x). \]

Consequently, we have
\[ T f(x) = \sum_{k \geq 0} T_k f(x) = \sum_{P \in \mathbb{P}} T_P f(x). \]

This ends our decomposition.

Finally, note that (as in [5]) we may assume that
\[ \text{supp} \, \psi \subseteq \{ y \in \mathbb{R} : 4 < |y| < 5 \}. \]

Consequently, for a tile \( P = [\alpha^1, \alpha^2, \ldots, \alpha^d, I] \), the associated operator has the properties
\[ \text{supp} \, T_P \subseteq I \quad \& \quad \text{supp} \, T_P^* \subseteq \{ x \mid 3|I| \leq \text{dist}(x, I) \leq 5|I| \} = I^*, \]
where here \( T_P^* \) denotes the adjoint of \( T_P \).

Also, in what follows, (splitting \( \mathbb{P} = \bigcup_{j=0}^{D-1} \bigcup_{k \geq 0} \mathbb{P}_{kD+j} \) where \( D \) is the smallest integer larger than \( 2d \log_2(2d) \)) we can suppose that if
$P_j = [\alpha_j^1, \alpha_j^2, \ldots, \alpha_j^d, I_j] \in \mathbb{P}$ with $j \in \{1, 2\}$ such that $|I_1| \neq |I_2|$, then $|I_1| \leq 2^{-D} |I_2|$ or $|I_2| \leq 2^{-D} |I_1|$.

4. Quantifying the interactions between tiles

In this section we will focus on the behavior of the expression

$$|(T_{P_1} f, T_{P_2} g)|.$$

Before this, we will need to introduce some quantitative concepts that are adapted to the information offered by the localization of \{T_P\}.

4.1. Properties of $T_P$ and $T_P^*$

For $P = [\alpha^1, \alpha^2, \ldots, \alpha^d, I] \in \mathbb{P}$ with $|I| = 2^{-k}$, $k \in \mathbb{N}$, we have

$$T_P f(x) = \left\{ \int_T e^{i \left( f_y q_x \right)} \psi_k(x - y) f(y) dy \right\} \chi_{E(P)}(x),$$

$$T_P^* f(x) = \int_T e^{-i \left( f_y q_x \right)} \psi_k(y - x) \left( \chi_{E(P)} f \right)(y) dy .$$

Based on the relational approach developed in [5] we have:

(8)

- the time-frequency localization of $T_P$ is “morally” given by the tile $\hat{P}$;
- the time-frequency localization of $T_P^*$ is “morally” given by the (bi)tile $\hat{P}^s$.

(Remark that, due to Lemma C (see the Appendix), one may think of $\hat{P}$ as the $|I|^{-1}/2$ neighborhood of the graph of the “central polynomial” $q_P$ restricted to the interval $I$.)

4.2. Factors of a tile

For a tile $P = [\alpha^1, \alpha^2, \ldots, \alpha^d, I]$ we define two quantities:

a) an absolute one (which may be regarded as a self-interaction); we define the density (analytic) factor of $P$ to be the expression

$$A_0(P) := \frac{|E(P)|}{|I|} .$$

Notice that $A_0(P)$ determines the $L^2$ norm of $T_P$.

b) a relative one (interaction of $P$ ($\hat{P}$) with something exterior to it) which is of geometric type.

Suppose first that we are given $q \in \mathcal{Q}_{d-1}$ and $J$ a dyadic interval; we introduce the quantity

$$\Delta_q(J) := \frac{\text{dist}^J(q, 0)}{|J|^{-1}} ,$$
where we used the notations (for $q_1, q_2 \in \mathbb{Q}_{d-1}$):
\[
\text{dist}^A(q_1, q_2) = \sup_{y \in A} \{\text{dist}_y(q_1, q_2)\} \quad \& \quad \text{dist}_y(q_1, q_2) = |q_1(y) - q_2(y)|.
\]

Now we define the geometric factor of $P$ ($\hat{P}$) with respect to $q$ to be the term
\[
[\Delta_q(P)] = \frac{1}{1 + |\Delta_q(P)|},
\]
where
\[
\Delta_q(P) := \inf_{q_1 \in P} \Delta_{q-q_1}(I_P).
\]

**4.3. The resulting estimates**

We conclude this section by observing how the above quantities relate in controlling the interaction in (6).

As expected, we need to quantify the relative position of $\hat{P}_1^*$ with respect to $\hat{P}_2^*$. (We consider only the nontrivial case $I_{\hat{P}_1} \cap I_{\hat{P}_2}^* \neq \emptyset$; also, throughout this section we suppose that $|I_1| \geq |I_2|$.)

**Definition 1.** Given two tiles $P_1$ and $P_2$, we define the geometric factor of the pair $(P_1, P_2)$ by
\[
[\Delta(P_1, P_2)],
\]
where
\[
\Delta(P_1, P_2) = \Delta_{1,2} := \sup_{y \in I_2} \{\inf_{q_1 \in P_1} \text{dist}_y(q_1, q_2)\}.
\]

With these notations, remark (using the results in the Appendix) that we have
\[
[\Delta_{1,2}] \approx_d \max \left\{ \left[ \Delta_{q_{P_1}}(P_2) \right], \left[ \Delta_{q_{P_2}}(P_1) \right] \right\}.
\]

For $P_1$ and $P_2$ as above, we define the “interaction polynomial”
\[
q_{1,2} := q_{P_1} - q_{P_2}.
\]

Fix now an interval (not necessarily dyadic) $\bar{J} \subseteq \mathbb{T}$, a polynomial $q \in \mathbb{Q}_{d-1}$ and three positive constants $\eta, v, w$. In what follows we will present a general procedure for constructing two types of critical sets associated with $\bar{J}, q, \eta, v$ and $w$, denoted $\mathcal{I}_s(\eta, v, q, \bar{J})$ and respectively $\mathcal{I}_c(\eta, w, q, \bar{J})$.

Suppose for the moment that $q \not\in \mathbb{Q}_0$; let $J$ be the largest\(^3\) dyadic interval contained in $\bar{J}$. We define
\[
\mathcal{M}_q^0(\bar{J}) = \{x \in \bar{J} \mid x \text{ is a local minimum for } |q| \& |q|(x) < \eta\}.
\]

\(^3\)If there are two such intervals just pick either of them.
Now since $q \in \mathbb{Q}_{d-1} \setminus \mathbb{Q}_0$ we have that $\mathcal{M}_d^0(\bar{J})$ is a finite set of the form $\mathcal{M}_d^0(\bar{J}) = \{x_j\}_{j \in \{1, \ldots, r\}}$ with $r \leq 2d$. Without loss of generality we suppose that the $x_j$ are arranged in increasing order.

Further, for each $j \in \{1, \ldots, r\}$ define $\tilde{q}_j(x) := q(x) - q(x_j)$ and construct the dyadic intervals $I^1_j$, $I^2_j$ and $I^3_j$ as follows: $I^1_j$ is the smallest dyadic interval $I$ for which $x_j \in I$ and $\Delta \tilde{q}_j(I) > c(d)v$; $I^2_j$ is the smallest dyadic interval $I$ having the left end point equal with the right end point of $I^1_j$ and for which $\Delta \tilde{q}_j(I) > c(d)v$; similarly, $I^3_j$ is the smallest dyadic interval $I$ having the right end point equal with the left end point of $I^1_j$ and for which $\Delta \tilde{q}_j(I) > c(d)v$.

Now set

$$S_j(\eta, v, q, \bar{J}) = \bigcup_{k=1}^3 I^k_j$$

and define

$$S(\eta, v, q, \bar{J}) = \bigcup_{j=1}^r S_j(\eta, v, q, \bar{J}).$$

Also set

$$C_j(\eta, w, q, \bar{J}) = [x_j - w, x_j + w] \cap \bar{J}$$

and further take

$$C(\eta, w, q, \bar{J}) = \bigcup_{j=1}^r C_j(\eta, w, q, \bar{J}).$$

We now need to do one more step before ending our construction; suppose that $A \subseteq \bar{J}$ is a finite union of (closed) intervals: $A = \bigcup_{j=1}^l A_j$ with $l \in \mathbb{N}$, $A_j = [u_j, v_j]$ (pairwise disjoint) and $\{u_j\}$ monotone increasing. Then, setting $A_0 = A_{l+1} = \emptyset$ we define

$$\mathcal{E}(\bar{J}, A) = \left( \bigcup_{j=1}^l A_j \right) \cup \left( \bigcup_{j \in \{1, \ldots, l+1\}, |C_j| < |A_j|} C_j \right),$$

where here the intervals $C_j$ obey the partition condition

$$\bar{J} = \bigcup_{j=1}^{l+1} (A_j \cup C_j).$$

Finally if $q \not\in \mathbb{Q}_0$ then define

$$\mathcal{I}_s(\eta, v, q, \bar{J}) = \mathcal{E}(\bar{J}, S(\eta, v, q, \bar{J}))$$

and

$$\mathcal{I}_c(\eta, w, q, \bar{J}) = \mathcal{E}(\bar{J}, C(\eta, w, q, \bar{J})).$$

Otherwise, if $q \in \mathbb{Q}_0$, just set $\mathcal{I}_s(\eta, v, q, \bar{J}) = \mathcal{I}_c(\eta, w, q, \bar{J}) = \emptyset$. 

Fix $\epsilon_0 \in (0,1)$. Set $w(J) = c(d) |J| \left[ \Delta_q(J) \right]^{\frac{1}{2}-\epsilon_0}$, $v(J) = c(d) \left[ \Delta_q(J) \right]^{-2\epsilon_0}$ and $\eta(J) = v(J) w(J)^{-1}$.

Then (using the results in the Appendix) we deduce

\begin{equation}
\mathcal{I}_s(\eta(J), v(J), q, J) \subseteq \mathcal{I}_s(\eta(J), w(J), q, J).
\end{equation}

We now define the $(\epsilon_0)$-critical intersection set $I_{1,2}$ of the pair $(P_1, P_2)$ as

\begin{equation}
I_{1,2} := \mathcal{I}_c(\eta_{1,2}, w_{1,2}, q_{1,2}, I_1 \cap I_2),
\end{equation}

where $\eta_{1,2} := \eta(I_1 \cap I_2)$ and $w_{1,2} := w(I_1 \cap I_2)$.

Notice that, based on (11) and Lemma C of the Appendix, we have that

\begin{equation}
\bigcup_{q_j \in P_j \ \text{and} \ \ j \in \{1,2\}} \left\{ y \in \tilde{I}_2 \mid \frac{|q_1(y) - q_2(y)|}{|I_2|^{-1}} \leq [\Delta_{1,2}]^{-\frac{2}{\sqrt{d}}-\epsilon_0} \right\} \subseteq I_{1,2}.
\end{equation}

Now using (13) together with the principle of (non-)stationary phase, one deduces the following:

**Lemma 0.** Let $P_1, P_2 \in \mathbb{P}$; then we have

\begin{equation}
\left| \int \tilde{\chi}_{I_{1,2}} T_{P_1}^* f T_{P_2}^* g \right| \lesssim_{n, d, \epsilon_0} \Delta(P_1, P_2) \frac{\int \mathcal{E}(P_1) |f| \mathcal{E}(P_2) |g|}{\max(|I_1|, |I_2|)} \quad \forall n \in \mathbb{N},
\end{equation}

\begin{equation}
\left| \int_{I_{1,2}} T_{P_1}^* f T_{P_2}^* g \right| \lesssim_d \Delta(P_1, P_2) \frac{\int \mathcal{E}(P_1) |f| \mathcal{E}(P_2) |g|}{\max(|I_1|, |I_2|)},
\end{equation}

where $\tilde{\chi}_{I_{1,2}}$ is a smooth variant of the corresponding cut-off.

Applying the same methods for the limiting case $\epsilon_0 = 0$, we obtain

\begin{equation}
\| T_{P_1} T_{P_2} \|_2 \lesssim_d \min \left\{ \frac{|I_2|}{|I_1|}, \frac{|I_1|}{|I_2|} \right\} \left[ \Delta(P_1, P_2) \right]^\frac{2}{d} A_0(P_1) A_0(P_2).
\end{equation}

**Proof.** We first notice that relation (15) is straightforward; indeed, to see this we just use the relation

\[ |T_{P_j} f| \lesssim \frac{\mathcal{E}(P_j) |f|}{|I_j|} \chi_{I_j} \quad \forall \ j \in \{1,2\} \]

together with the definition of $I_{1,2}$.

We turn now our attention towards (14). First, for notational convenience we set $\varphi = \tilde{\chi}_{I_{1,2}}$; with this, we have:

\[ \int \varphi T_{P_1}^* f T_{P_2}^* g = \int f T_{P_1} (\varphi T_{P_2}^* g) = \int \int (f \chi_{E(P_1)}(x)) (g \chi_{E(P_2)}(s)) \mathcal{K}(x, s) \, dx \, ds, \]
where

\[ K(x, s) = \int e^{i [f_s q_s - f^*_s q_s]} \psi_k_1(x - y) \varphi(y) \psi_k_2(y - s) \, dy. \]

(Here we use the conventions \(|I_1| = 2^{-k_1}, |I_2| = 2^{-k_2}\) with \(k_2 \geq k_1\) positive integers.)

Now making the change of variables \(y = |I_2| u\) and using the way in which we defined \(I_{1,2}\) and \(\varphi\), we deduce that

\[ |K(x, s)| \lesssim \left| |I_1|^{-1} \int_T e^{i\phi(u)} r(u) \, du \right| \]

with \(r \in C_0^\infty(\mathbb{R})\) such that \(|\partial^l r(u)| \lesssim_d (|\Delta(P_1, P_2)|^{\epsilon_0 - \frac{1}{2}})^l (l \in \mathbb{N})\) and \(\|\partial^\phi\|_{L^\infty(supp \, r)} \gtrsim_d [\Delta(P_1, P_2)]^{-\epsilon_0 - \frac{1}{2}}\). Using the non-stationary phase principle we thus obtain (14).

For (16), we set \(\epsilon_0 = 0\) in the previous argument.

\[ \square \]

5. Main ingredients

In this section, we will present the concepts and results that we need for proving our theorem.

**Definition 2.** For \(P = [\alpha^1, \alpha^2, \ldots, \alpha^d, I] \in \mathbb{P}\) we define the **mass** of \(P\) to be

\[ A(P) := \sup_{P' = [\alpha'^1, \alpha'^2, \ldots, \alpha'^d, I'] \in \mathbb{P}} \frac{|E(P')|}{|P'|} \left[ \Delta(2P, 2P') \right]^N, \]

where \(N\) is a fixed large natural number.

Next, we introduce a qualitative concept that characterizes the overlapping relation between tiles.

**Definition 3.** Let \(P_j = [\alpha^1_j, \alpha^2_j, \ldots, \alpha^d_j, I_j] \in \mathbb{P}\) with \(j \in \{1, 2\}\). We say that

- \(P_1 \leq P_2\) iff \(I_1 \subset I_2\) and \(\exists \ q \in P_2\) s.t. \(q \in P_1\),
- \(P_1 \prec P_2\) iff \(I_1 \subset I_2\) and \(\forall \ q \in P_2\) \(\Rightarrow q \in P_1\).

**Definition 4.** We say that a set of tiles \(\mathcal{P} \subset \mathbb{P}\) is a tree (relative to \(\leq\)) with top \(P_0\) if the following conditions are satisfied:

1) \(\forall \ P \in \mathcal{P} \Rightarrow \frac{3}{2}P \leq 10P_0\)
2) if \(P \in \mathcal{P}\) and \(P' \in N(P)\) such that \(\frac{3}{2}P' \leq P_0\) then \(P' \in \mathcal{P}\)
3) if \(P_1, P_2 \in \mathcal{P}\) and \(P_1 \leq P \leq P_2\) then \(P \in \mathcal{P}\).

Now we can state the main results of this section; their proofs will be postponed until Section 7.

**Proposition 1.** There exists \(\eta \in (0, 1/2)\) (depending only on the degree \(d\)) s.t. if \(\mathcal{P}\) is any given family of disjoint tiles (i.e. no two of them can be related through \(\leq\)) with the property that

\[ A(P) \leq \delta \quad \forall \ P \in \mathcal{P} \]
then, for $1 < p < \infty$, we have

$$\|TP\|_p \lesssim_{p,d} \delta^\eta (1 - \frac{1}{p^*}).$$

**Proposition 2.** Let $\{P_j\}_j$ be a family of trees with tops $P_j = [\alpha_j^1, \alpha_j^2, \ldots, \alpha_j^d, I_j]$. Suppose that

1) $A(P) < \delta$ \quad $\forall j, P \in P_j$,
2) $\forall k \neq j \& \forall P \in P_j \quad 2P \not\sim 10P_k$,
3) No point of $[0,1]$ belongs to more than $K\delta^{-(1+\rho)}$ of the $I_j$ (here $\rho$ is a fixed number with $0 < \rho \leq \min\{1, \frac{1}{2}\}$).

Then there exist a constant $\eta \in (0, \frac{1}{2})$ (depending only on $d$) and a set $F \subset T$ with $|F| \lesssim \delta^50K^{-M}$ (here $M \in \mathbb{N}$ is fixed) such that $\forall f \in L^p(T)$ we have

$$\left\| \sum_j T^P_j f \right\|_{L^p(F^c)} \lesssim_{p,d} \left( \delta^{\eta(1 - \frac{1}{p^*})} M \log K + K^{\frac{1}{p^*} - \frac{1}{p}} \delta^{\frac{1}{2} - (1+\rho)(\frac{1}{p^*} - \frac{1}{p})} \right) \|f\|_p.$$

(Remark: Any collection of tiles $\mathcal{P}$ that can be represented as $\cup_j P_j$ with the family $\{P_j\}$ respecting the conditions mentioned above will be called a "forest".)

6. Proof of “pointwise convergence”

We now present the proof of our Theorem.

The main challenge will be to organize the collection $\mathcal{P}$ into a “controlled number” of forests. For this, the first step is to split our family of tiles into a union of subfamilies having uniform density. More exactly, (as in [2], [5]) we decompose

$$\mathcal{P} = \bigcup_{n=0}^{\infty} \mathcal{P}_n \quad \text{with} \quad \mathcal{P}_n = \{ P \in \mathcal{P} \mid 2^{-n-1} < A(P) \leq 2^{-n} \}.$$

In what follows, writing $T = \sum_{n=0}^{\infty} T_{\mathcal{P}_n}$, we will show that

$$\exists \quad \eta = \eta(p, d) > 0 \quad \text{such that} \quad \|T_{\mathcal{P}_n}\|_{L^p(E_n)} \lesssim_{p,d} 2^{-\eta n},$$

where $E_n$ are some exceptional (“small”) sets inside the torus.

Now, for a fixed $n$, we need to search for clustered families of tiles - the natural environments for the future trees - living inside $\mathcal{P}_n$. Consequently, the next step is to identify some preferred tiles that will help determine the tops of these trees. For this, we select the tiles $\{\bar{P}_k\}$ ($\bar{P}_k = [\alpha_k^1, \alpha_k^2, \ldots, \alpha_k^d, \bar{I}_k]$) to be those $(d + 1)$-tuples which are maximal with respect to $\leq$ and obey $|E(P)| \geq 2^{-n-1}$. Proceeding as in [5], (and also maintaining the same notations) we define

$$\mathcal{P}_n^0 := \{ P \in \mathcal{P}_n \mid \exists k \in \mathbb{N} \; s.t. \; 4P \not\sim \bar{P}_k \}$$
and set
\[ C_n := \left\{ P \in \mathcal{P}_n \mid \text{there are no chains } P \leq P_1 \leq \ldots \leq P_n & \{P_j\}_{j=1}^n \subseteq \mathcal{P}_n \right\}. \]

With this done, it is easy to see that
\[ \mathcal{P}_n \setminus C_n \subseteq \mathcal{P}_0^n. \]

Now defining the set \( \mathcal{D}_n \subseteq C_n \) with the property
\[ \mathcal{P}_n \setminus \mathcal{D}_n = \mathcal{P}_0^n, \]
we remark that \( \mathcal{D}_n \) breaks up as a disjoint union of a most \( n \) sets \( \mathcal{D}_{n1} \cup \mathcal{D}_{n2} \cup \ldots \cup \mathcal{D}_{nn} \) with no two tiles in the same \( \mathcal{D}_{nj} \) comparable. Applying Proposition 1, we see that
\[
\left\| T^{\mathcal{D}_n} \right\|_p \leq \sum_{j=1}^{n} \left\| T^{\mathcal{D}_{nj}} \right\|_p \lesssim_{p,d} \sum_{j=1}^{n} 2^{-n\eta/(1 - \frac{1}{p})} \lesssim 2^{-n\eta/(1 - \frac{1}{p})},
\]
and so, in what follows, it will be enough to limit ourselves to the set of tiles \( \mathcal{P}_0^n \).

Once we have established a certain structure of our collection of tiles - all our elements are clustered near some maximal elements \( \{\bar{P}_k\} \) - to move closer towards the concept of “forest” we need to learn how to control the number of these maximal elements (see 3) Proposition 2). For this, we make the following reasoning: Define the counting function
\[ N(x) = \sum_k \chi_{\bar{I}_k}(x) \]
and set
\[ G_n := \left\{ x \in \mathbb{T} \mid x \text{ is contained in more than } 2^{(1+\rho)nK} \text{ of the } |\bar{I}_k| \right\}, \]
where here the parameter \( \rho \) is as defined in the statement of Proposition 2.

In contrast with the \( L^2 \) case, to obtain \( L^p \) bounds for our operator \( T \), we need to be more precise in estimating the measure of \( G_n \), or equivalently to better control the size of the level sets corresponding to \( N \):
\[ G_n = \left\{ x \in \mathbb{T} \mid N(x) \geq 2^{(1+\rho)nK} \right\}. \]

For this task we may proceed in two ways: the first (“modern”) one uses \( BMO \)-estimates and is shorter, more efficient but less descriptive; the second (“classical”) one, may be found in [2] (Section 8) and uses a vector-valued variant of the Hardy-Littlewood maximal theorem.

First approach:
We first observe that from the definition of \( \{\bar{I}_k\} \) we have that \( N \in BMO(D)^4 \) with \( \|N\|_{BMO(D)} \leq 2^n \). Applying now the John-Nirenberg inequality, we have \( (\gamma > 0) \)
\[
\left\{ x \in \mathbb{T} \mid |N(x) - \int_{\mathbb{T}} N| > \gamma \right\} \lesssim e^{-\frac{\gamma}{\|N\|_{BMO(D)}}},
\]
where \( c > 0 \) is an absolute constant.

Now taking into account the fact that \( \|N\|_1 \lesssim 2^n \) and setting \( \gamma := 2^{(1+\rho)nK} \) we deduce that
\[ |G_n| \lesssim e^{-c2^{\rho n}K}. \]

4Throughout the paper we will denote with \( BMO(D) \) the dyadic \( BMO \) on the torus.
Second approach:
Since by Chebyshev's inequality
\[(20) \quad |\{x \mid N(x) \geq \gamma\}| \leq \gamma^{-r} \int N^r \quad \forall \ r > 0,\]
we are led to the study of the $L^r$ bounds for $N$, where now our interest is for $r$ to be large.

The strategy for this is to relate the behavior of $\{\chi_{\bar{I}_k}\}$ with that of $\{\chi_{E(P_k)}\}$, since the latter has better disjointness properties. Indeed, set $h_k(x) := \chi_{E(P_k)}$ and $h_k^* = M(h_k)$ the Hardy-Littlewood maximal operator applied to $h_k$; then we have
\[(21) \quad N(x) \lesssim \rho' 2^{n(1+\rho')} \sum_k (h_k^*)^{1+\rho'}(x) \quad \forall \ x \in \mathbb{T},\]
where $\rho' < \rho$ is a fixed positive constant.

Now applying the $l^p$ vector-valued variant of the Hardy-Littlewood maximal theorem ($r > 1$), we deduce
\[(22) \quad \int \left[ \sum_k (h_k^*)^{1+\rho'} \right]^r \lesssim r \rho' \int \left[ \sum_k (h_k)^{1+\rho'} \right]^r = \sum_k |E(P_k)| \leq 1,\]
and so, by (20), (21) and (22) we have
\[|\{x \mid N(x) \geq \gamma\}| \leq \gamma^{-r} 2^{nr(1+\rho')}.\]

Now choosing $\gamma = 2^{(1+\rho)n^2}K$ and $r = M$ we conclude that there exists $\eta = \eta(p, d) > 0$ such that
\[|G_n| \lesssim_p \frac{2^{-n\eta}}{K^M}.\]

Now, to control the number of tops that lie one upon the other, we will use the estimates (of the set $G_n$) just obtained to erase a few more tiles from $P^0_n$. More exactly if we set
\[P^G_n = \left\{ P = [\alpha^1, \alpha^2, \ldots, \alpha^d, I] \in P^0_n \mid I \not\subseteq G_n \right\},\]
we have that
\[(23) \quad T^{P^G_n} f(x) = T^{P^0_n} f(x) \quad \forall \ f \in L^p(\mathbb{T}) \ & \ x \in G_n^c.\]

Now, since we have good control on the measure of the set $G_n$, we will erase from $\{\bar{P}_k\}$ all $\bar{P}_k$ with $\bar{I}_k \subseteq G_n$ and focus further on estimating $T^{P^0_n}$ only on $G_n^c$.

The point is that now the set $P^G_n$ has the following properties:
1) $A(P) \leq 2^{-n} \quad \forall \ P \in P^G_n$,
2) $\forall \ P \in P^G_n \Rightarrow \exists \ k \in N \ st \ 4P \subseteq \bar{P}_k$,
3) No $x \in \mathbb{T}$ belongs to more than $2^{(1+\rho)n}K$ of the $\bar{I}_k$'s.
We remark that we are coming closer to the definition of a forest - see Proposition 2. One ingredient that is still missing refers to the separation condition 2). This will force us to rearrange the tiles inside $P_n^G$ into collections that are clustered near some dilated maximal elements. The exact procedure follows the same lines as those described in the section 7.2. of [5]; for convenience, we will sketch the major steps in the next section, leaving the details for the reader.

6.2. Decomposing into forests

As announced, in this section we intend to reorganize the set $P_n^G$; more precisely, we will show that up to a negligible set of tiles

$$P_n^G \cong \bigcup_{j=1}^{M} B_{nj},$$

with each $B_{nj}$ a forest and $M$ some constant less than $2n \log K$. As in [5] we set

$$B(P) := \# \{ j \mid 4P \leq \bar{P}_j \} \quad \forall \, P \in P_n^G,$$

and

$$P_{nj} := \{ P \in P_n^G \mid 2^j \leq B(P) < 2^{j+1} \} \quad \forall \, j \in \{ 0, \ldots, M \}.$$  

Fixing now a family $P_{nj}$ we look for candidates for the tops of the future trees. More exactly, take $\{P_r\}_{r \in \{1, \ldots, s\}} \subseteq P_{nj}$ to be those tiles with the property that $4P_r$ are maximal elements with respect to the relation \( \leq \) inside the set $4P_{nj}$.

Now, in all the reasonings that we will make further, we will use the following four essential properties:

(A) \( 4P^l \leq 4P^k \Rightarrow I_l = I_k \);

(B) \( \forall P \in P_{nj} \exists \ P^l \text{ s.t. } 4P \leq 4P^l \);

(C) If \( P \in P_{nj} \text{ s.t. } \exists k \neq l \) with \( \{ 4P^l \leq 4P^k \} \), then \( \{ 4P^k \leq 4P^l \} \);

(D) If \( P_j = [\alpha_j^1, \alpha_j^2, \ldots, \alpha_j^d, I_j] \in \mathbb{P} \) with \( j \in \{ 1, 2 \} \) s.t. \( |I_1| \neq |I_2| \), then

\[
|I_1| \leq 2^{-D} |I_2| \text{ or } |I_2| \leq 2^{-D} |I_1| .
\]

(While (A), (B) and (C) are derived from the definition of $P_{nj}$ and the way in which we have chosen $\{P_r\}_{r \in \{1, \ldots, s\}}$, property (D) follows from the assumption made at the end of Section 3.)

The next step, is to discard the tiles that are not “close” to our new maximal elements (remember that our final goal is to construct “well separated”

---

5 The set can be written as a union of at most $c(d)$ families of disjoint tiles.

6 Here we use the following convention: let be $\mathcal{D}$ a collection of tiles; $P$ is maximal (relative to $\leq$) in $\mathcal{D}$ iff $\forall \, P' \in \mathcal{D}$ such that $P \leq P'$ we also have $P' \leq P$. 

trees); for technical reasons, we also get rid of the maximal elements together with their neighbors and respectively of the minimal elements; more exactly, we define:

\[ A_{nj} := \left\{ P \in \mathcal{P}_{nj} \mid \forall P^l \Rightarrow \frac{3}{2} P \not\leq P^l \right\} \cup \left\{ P \in \mathcal{P}_{nj} \mid \exists l \ | \ I_P = |I_{P^l}| , \frac{3}{2} P \leq P^l \right\} \cup \left\{ P \in \mathcal{P}_{nj} \mid P \text{ minimal with respect of } " \leq " \right\} . \]

Also, we set

\[ B_{nj} := \mathcal{P}_{nj} \setminus A_{nj} . \]

Now, using the properties (B) and (D), it is easy to see that \( A_{nj} \) forms a negligible set of tiles.

For the remaining set \( B_{nj} \), one should follow the steps bellow (here we use the properties (A)-(D)):

- Set \( S_k = \{ P \in B_{nj} \mid \frac{3}{2} P \leq P^k \} \); without loss of generality we may suppose \( B_{nj} = \bigcup_{k=1}^{s} S_k \);
- Introduce the following relation among the sets \( \{ S_k \}_k \) :

\[ S_k \propto S_l \text{ if } \exists P_1 \in S_k \text{ and } \exists P_2 \in S_l \text{ such that } 2P_1 \leq 10P^l \text{ or } 2P_2 \leq 10P^k ; \]

- Deduce that " \( \propto \) " becomes an order relation and that

\[ S_k \propto S_l \Rightarrow 4P^k \leq 4P^l \Rightarrow I_k = I^l ; \]

- Let \( \hat{k} := \{ m \mid S_m \propto S_k \} \); then the cardinality of \( \hat{k} \) is at most \( c(d) \), and for

\[ \hat{S}_k := \bigcup_{m \in \hat{k}} S_m , \]

one has that \( \hat{S}_k \) is a tree having as a top any \( P^l \) with \( l \in \hat{k} \).
- Conclude that the relation " \( \propto \) ", can be meaningfully extended\(^7\) among the sets \( \{ \hat{S}_k \}_k \) and that

\[ \hat{S}_k \propto \hat{S}_l \Rightarrow k = l . \]

Consequently, from the algorithm just described, we deduce that the set

\[ B_{nj} = \bigcup_k \hat{S}_k \]

is a forest as defined in Proposition 2.

\(^7\)In this case, the role played by the maximal element \( P^k \) in the initial definition is now taken by the top of the corresponding tree.
6.3. Ending the proof

To complete the proof we proceed as follows:

We first apply Proposition 2 for each family $B_{nj}$ and obtain that

$$
\|T_{b_{nj}} f\|_{L_p(F_n)} \lesssim_{p,d} 2^{-n} \eta(p,d) \left( M \log K + K \frac{1}{p^*} - \frac{1}{p} \right) \|f\|_p ,
$$

where $\eta(p,d)$ is a constant depending only on $p, d$ and $F_{nj}$ is a small set with measure $|F_{nj}| \lesssim \frac{2^n}{K^{n'}}$. As a result, denoting $F_n = \bigcup F_{nj}$, we have that

$$
T_{F_n} f \lesssim_{p,d} 2^{-n} \eta(p,d) \left( M \log K + K \frac{1}{p^*} - \frac{1}{p} \log K \right) \|f\|_p ,
$$

with $|F_n| \lesssim \frac{n \log K}{2^n K^{n'}}$.

Therefore, we deduce

$$
\|T_{F_n} f\|_{L_p(E_n)} \lesssim_{p,d} n 2^{-n} \eta(p,d) \left( M \log K \right)^2 + K \frac{1}{p^*} - \frac{1}{p} \log K \right) \|f\|_p ,
$$

where $E_n = F_n \cup G_n$ still has measure $\lesssim \frac{n \log K}{2^n K^{n'}}$.

Summing now over $n$, we obtain

$$
\|T f\|_{L_p(E)} \lesssim_{p,d} \left( M \log K \right)^2 + K \frac{1}{p^*} - \frac{1}{p} \log K \right) \|f\|_p
$$

with $E = \bigcup_n E_n$ and $|E| \lesssim \frac{\log K}{K^{n'}}$.

In conclusion, given $\gamma > 0$, we have that for all $K, M > 100$

$$
\{|T f(x)| > \gamma\} \leq \frac{\|T f\|_{L_p(E)}^p}{\gamma^p} + |E|

\lesssim_{p,d} \left( M \log K \right)^2 + K \frac{1}{p^*} - \frac{1}{p} \log K \right) \frac{\|f\|_p}{\gamma^n} + \frac{\log K}{K^{n'}}.
$$

Now, if we pick $K$ to minimize the right-hand side and make $M$ large enough, we arrive at the relation

$$
\{|T f(x)| > \gamma\} \lesssim_{p,d,\epsilon} \left( \frac{\|f\|_p}{\gamma^\epsilon} \right)^{p-\epsilon} \quad \forall \ 1 < p < \infty \ & \ \epsilon \in (0,1) ,
$$

which further implies (using interpolation)

$$
\|T f\|_r \lesssim_{r,p,d} \|f\|_p \quad \forall \ 1 < r < p < \infty ,
$$

ending the proof of our theorem.
7. Some technicalities - the proofs of Propositions 1 and 2

Proof of Proposition 1

Following exactly the same steps from the corresponding proof of Proposition 1 in [5], one can show that there exists $\eta$ (depending only on the degree $d$) such that

$$\|TP\|_2 \lesssim d^{\eta}.$$

Now, for the case $2 < p < \infty$, we observe that:

$$\|TP\|_{\infty} \to \infty = \sup_{\|f\|_\infty \leq 1} \|TPf\|_\infty \lesssim \sup_{p \in P} \frac{\int_{I_p^*} |f|}{|I_p|} \lesssim 1,$$

so interpolating between $L^2$ and $L^\infty$ we obtain the desired conclusion.

For the case $1 < p < 2$ we need to focus on the behavior of $TP^*$. Indeed, on the one hand we know that

$$\|TP^*\|_{2 \to 2} = \|TP\|_{2 \to 2} \lesssim d^\frac{\eta}{2}.$$

On the other hand, for $f \in L^\infty$ we have

$$|TP^*f| \leq \sum_{P \in P} |TP^*f| \lesssim \sum_{P \in P} \frac{\int_{E(P)} |f|}{|I_p|} \chi_{I_p^*} \lesssim \|f\|_\infty \sum_{P \in P} \frac{|E(P)|}{|I_p|} \chi_{I_p^*}.$$

If we now add the fact

$$\left\| \sum_{P \in P} \frac{|E(P)|}{|I_p|} \chi_{I_p^*} \right\|_{BMO(D)} \lesssim 1,$$

we conclude that

$$\|TP^*\|_{\infty \to BMO(D)} \lesssim 1.$$

The claim now follows by interpolation.$^9$

The remainder of the section will be dedicated to proving Proposition 2. The strategy here is as follows: for the single tree estimate (Lemma 1) we prove the $L^p$ bounds directly; after that, we will show only $L^2$ statements (Lemma 2, 3 and 4) that will be just enough to prove Proposition 2 in the case $p = 2$. Once we get this, by applying interpolation techniques we will recover the entire range of $p$.

\[8\] The proof is based on two cases: for the diagonal term, since we don’t have oscillatory behavior, one uses maximal methods, while for the off-diagonal term we use the $TT^*$-method together with Lemma 0.

\[9\] We use here the fact that $\|TP^*\|_{p' \to p'} = \|TP\|_{p \to p'}$. 


Lemma 1. Let $\delta > 0$ be fixed and let $\mathcal{P} \subseteq \mathbb{P}$ be a tree such that
\[ A(P) < \delta \quad \forall \ P \in \mathcal{P}. \]

Then
\[ (24) \quad \| T^P \|_p \lesssim_{p,d} \delta^{\frac{1}{p}}. \]

Proof. We start by setting the parameters of our tree; more precisely, we fix the top $P_0 = [\alpha_0^1, \alpha_0^2, \ldots, \alpha_0^d, I_0]$, and frequency polynomial $q_0$. Since we have specified the polynomial $q_0$ we also know the form of $Q_0$; suppose now that
\[ Q_0(y) = \sum_{j=1}^d a_j^0 y^j. \]

Then, denoting with $g(x) = M^*_{1,a_0^1} \cdots M^*_{d,a_0^d} f(x)$ and reasoning\(^\text{10}\) as in the proof of Lemma 1 in [5], one can show that:
\[ | T^P f(x) | \lesssim_d M_d(R * g)(x) + M_\delta g(x), \]
where we set $R(y) = \sum_{k \in \mathbb{N}} \psi_k(y)$ (here without loss of generality we suppose that $\mathcal{P} \subset \bigcup_{k \in \mathbb{N}} \mathbb{P}_{kd}$).

Now, taking into account the fact that $\| M_\delta g \|_p \lesssim_p \delta^{\frac{1}{p}} \| g \|_p$ and $\| R * g \|_p \lesssim_p \| g \|_p$, we conclude that (24) holds. \( \square \)

Definition 5. Fix a number $\delta \in (0,1)$. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two trees with tops $P_1 = [\alpha_1^1, \alpha_1^2, \ldots, \alpha_1^d, I_1]$ and respectively $P_2 = [\alpha_2^1, \alpha_2^2, \ldots, \alpha_2^d, I_2]$: we say that $\mathcal{P}_1$ and $\mathcal{P}_2$ are ($\delta$-)separated if either $I_1 \cap I_2 = \emptyset$ or else
i) $P = [\alpha^1, \alpha^2, \ldots, \alpha^d, I] \in \mathcal{P}_1$ and $I \subseteq I_2 \Rightarrow | \Delta(P, P_2) | < \delta,$
ii) $P = [\alpha^1, \alpha^2, \ldots, \alpha^d, I] \in \mathcal{P}_2$ and $I \subseteq I_1 \Rightarrow | \Delta(P, P_1) | < \delta.$

Notation: Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two trees as in Definition 5. Take $q_j$ to be the central polynomial of $P_j$ ($j \in \{1,2\}$), set $q_{1,2} = q_1 - q_2$, and then define
\begin{itemize}
  \item $I_s$ - the separation set (relative to the intersection) of $\mathcal{P}_1$ and $\mathcal{P}_2$ by
  \[ I_s = I_s \left( \eta_{1,2}, c(d)\delta^{-1}, q_{1,2}, I_1 \cap I_2 \right), \]
  where $\eta_{1,2} = \eta(I_1 \cap I_2)$;
  \item $I_c$ - the ($\epsilon$-)critical intersection set (between $\mathcal{P}_1$ and $\mathcal{P}_2$) by
  \[ I_c = \bigcup_{j=1}^r I_c \left( \eta(I^j_s), w(I^j_s), q_{1,2}, I^j_s \right), \]
  where $\epsilon$ is some small fixed positive real number and $I_s = \bigcup_{j=1}^r I^j_s$ is the decomposition of $I_s$ into maximal disjoint intervals $\{I^j_s\}_{j \in \{1, \ldots, r\}}$ with $r \leq 2d$.\(^\text{10}\)
\end{itemize}

\(^\text{10}\)Here the key element is the following perspective: “A tree behaves like a maximal truncated Hilbert transform”.\]
Observation 5. It is important to notice the following three properties of our above-defined sets; these facilitate the adaptation of the reasonings involved in the proofs of Lemmas 2 and 4 to those of the corresponding lemmas in [5]:

1) for all dyadic \( J \subset \tilde{I}_1 \cap \tilde{I}_2 \) such that \( I_s \cap 5J = \emptyset \) we have \((c(d) \leq d^n)\)
\[
\inf_{x \in J} |q_{1,2}(x)| \leq \sup_{x \in J} |q_{1,2}(x)| \leq c(d) \inf_{x \in J} |q_{1,2}(x)| .
\]

2) \( \forall \ P \in \mathcal{P}_1 \cup \mathcal{P}_2 \) and \( j \in \{1, \ldots r\} \) if \( I_s^j \cap 5I_P \neq \emptyset \) then \( |I_P| > |I_s^j| .\)

3) \( \forall \ P \in \mathcal{P}_1 \cup \mathcal{P}_2 \) we have (for \( \epsilon \) properly chosen) \( |\tilde{I}_P \cap I_c| < \delta \frac{\epsilon^{100}}{n(d,n)}|I_P|.\)

Indeed, these facts are an easy consequence of the results mentioned in the Appendix and the way in which \( I_s \) and \( I_c \) are defined.

(Remark that property 1) above implies the following relation:

\[
\forall \ P \in \mathcal{P}_1 \text{ such that } I_s \cap 5\tilde{I}_P = \emptyset \text{ and } I_P \subset I_2 \text{ we have } \text{Graph}(q_2) \cap (c(d)\delta^{-1}) \tilde{P} = \emptyset .
\]

Of course, the same is true for the symmetric relation, i.e. replacing the index 1 with 2 and vice versa.)

Lemma 2. Let \( \{P_j\}_{j \in \{1,2\}} \) be two separated trees with tops \( P_j = [\alpha^1_j, \alpha^2_j, \ldots \alpha^d_j, I_0]. \) Then, for any \( f, g \in L^2(\mathbb{T}) \) and \( n \in \mathbb{N}, \) we have that

\[
\left( \int T^{P^*_1} f, T^{P^*_2} g \right)_n \lesssim \|f \|_{L^2(I_0)} \|g \|_{L^2(I_0)} + \|\chi_{I_0} T^{P^*_1} f \|_2 \|\chi_{I_c} T^{P^*_2} g \|_2 .
\]

Proof. In what follows we intend to adapt the methods described in the proof of Lemma 2 of [5] to our context. For this, we need first to modify the definition of the sets \( \{A_l\}_l; \) more exactly we follow the procedure below:

Let \( I_s = \bigcup_{j=1}^r I_s^j \) be the decomposition of \( I_s \) into maximal disjoint intervals \( (r \leq 2d); \) without loss of generality we may suppose\(^{11}\) that \( \{I_s^j\}_j \) are placed in consecutive order with \( I_s^{j+1} \) located to the right of \( I_s^j \). For a fixed \( j \in \{0, \ldots r\}, \) let \( W_j \) be the standard Whitney decomposition of the set \([0,1] \cap (I_s^j \cap (I_s^{j+1}))\) with respect to the set \( I_s^j \cup I_s^{j+1}; \) we take \( \tilde{W}_j \) to be the “large scale” version of \( W_j, \) which is obtained as follows:

- take the union of all the intervals in \( W_j \) of length strictly smaller than \( c(d)|I_s^j| \) that approach \( I_s^j \) and denote it by \( R_j \) (we can do this in such a way that \( R_j \) can be written as a union of at most two dyadic intervals, each one of length \( c(d)|I_s^j| \));
- apply the same procedure to obtain \( R_{j+1}; \)
- the rest of the intervals belonging to \( W_j \) remain unchanged and are transferred to \( \tilde{W}_j. \)

Define \( \mathcal{W}_j := \tilde{W}_j \cup I_s^j \cup I_s^{j+1} \) and observe that this is a partition of \([0,1].\)

\(^{11}\) We made here the convention \( I_s^0 = I_s^{r+1} = \emptyset. \)
Finally, we take $\mathcal{W}$ to be the common refinement of the partitions $\mathcal{W}_j$, $j \in \{0, \ldots, r\}$. Take now $A_0 = \bigcup_{I \in \mathcal{I}_0, I \neq \emptyset} I$ and set

$$\mathcal{W} = A_0 \cup \bigcup_{l=1}^k A_l.$$ 

Then, for $l \in \{1, \ldots, k\}$ and $m \in \{1, 2\}$ we define the sets

$$S_{m,l} := \left\{ P \in \mathcal{P}_m \mid I_P \subset A_l \text{ and } |I_P| \leq \frac{|A_l|}{20} \right\}.$$ 

Also, we take $S_{m,0} := \mathcal{P}_m \setminus \left( \bigcup_{l=1}^k S_{m,l} \right)$. With this done, for $l \in \{0, \ldots, k\}$, we set

$$T^*_m,l = \sum_{P \in S_{m,l}} T^*_P$$

and deduce that

$$\langle T^{P_1}_*, T^{P_2}_* \rangle = \sum_{n,l=0}^k \langle T_{1,n}^*, T_{2,n}^* \rangle.$$ 

Now, as intended, we may follow the same steps as in [5], Lemma 2 and show that

$$\sum_{l=0}^k \sum_{n=1}^k |\langle T_{1,n}^*, T_{2,n}^* g \rangle| \lesssim_{n,d} \delta^n \|f\|_{L^2(I_0)} \|g\|_{L^2(I_0)};$$

$$|\langle T_{0,n}^* f, T_{0,n}^* g \rangle| \lesssim_{n,d} \delta^n \|f\|_{L^2(I_0)} \|g\|_{L^2(I_0)} + \left\| \chi_{I_0} T^{P_1}_* f \right\|_2 \left\| \chi_{I_0} T^{P_2}_* g \right\|_2.$$ 

finishing our proof. \(\Box\)

**Definition 6.** A tree $\mathcal{P}$ with top $P_0 = [\alpha_0^1, \alpha_0^2, \ldots, \alpha_0^d, I_0]$ is called normal if $P = [\alpha^1, \alpha^2, \ldots, \alpha^d, I] \in \mathcal{P} \Rightarrow |I| \leq \frac{\delta^{100}}{K} |I_0| \& \text{ dist}(I, \partial I_0) > 20 \frac{\delta^{100}}{K^2} |I_0|$. (Here $K, M \in \mathbb{N}$ are some fixed large constants and $\partial I_0$ designates the boundary of $I_0$.)

**Observation 6.** Notice that if $\mathcal{P}$ is a normal tree then

$$\text{supp} \ T^{P_0}_* f \subseteq \left\{ x \in I_0 \mid \text{dist}(x, \partial I_0) > 10 \frac{\delta^{100}}{K^2 M} |I_0| \right\}.$$ 

Also, it is worth mentioning that the constant $M$ introduced above is exactly that appearing in the variational estimates used in Section 6.

**Definition 7.** A row is a collection $\mathcal{P} = \bigcup_{j \in \mathbb{N}} \mathcal{P}_j$ of normal trees $\mathcal{P}_j$ with tops $P_j = [\alpha_j^1, \alpha_j^2, \ldots, \alpha_j^d, I_j]$ such that the $\{I_j\}$ are pairwise disjoint.

The proofs of the next two lemmas require no nontrivial modifications from the corresponding proofs in [5].
Lemma 3. Let \( \mathcal{P} \) be a row as above, let \( \mathcal{P}' \) be a tree with top \( \mathcal{P}' = [\alpha_0', \cdots, \alpha_d', I_0] \) and suppose that \( \forall j \in \mathbb{N}, I_0 \subseteq I_j' \) and \( \mathcal{P}^j, \mathcal{P}' \) are separated trees; denote by \( I_0' \) the critical intersection set between each \( \mathcal{P}^j \) and \( \mathcal{P}' \).

Then for any \( f, g \in L^2(\mathbb{T}) \) and \( n \in \mathbb{N} \) we have that

\[
\left| \left< T^{\mathcal{P}'*} f, T^{\mathcal{P}^*} g \right> \right| \lesssim_{n,d} \delta^n \| f \|_2 \| g \|_2 + \left\| \sum_j \chi_{I_0'} I^{\mathcal{P}^*} f \right\|_2 \left\| \sum_j \chi_{I_0'} I^{\mathcal{P}'*} g \right\|_2.
\]

Lemma 4. Let \( \mathcal{P} \) be a tree with top \( \mathcal{P}_0 = [\alpha_0^1, \alpha_0^2, \cdots, \alpha_0^d, I_0] \); suppose also that we have a set \( \mathcal{A} \subseteq \tilde{I}_0 \) with the property that

\[
\exists \delta \in (0,1) \text{ st } \forall \mathcal{P} = [\alpha^1, \alpha^2, \cdots, \alpha^d, I] \in \mathcal{P} \text{ we have } |I' \cap A| \leq \delta |I|.
\]

Then \( \forall f \in L^2(\mathbb{T}) \) we have

\[
\left\| \chi_{\mathcal{A}} T^{\mathcal{P}^*} f \right\|_2 \lesssim_{d} \delta^n \| f \|_2.
\]

Proof of Proposition 2

As in [5], we can ignore the behavior of our operator on the set \( F = \bigcup_j \left\{ x \in I_j \mid \text{dist}(x, \partial I_j) \leq 100 \frac{\delta^{100}}{K^{2M}} |I_j| \right\} = \bigcup_j F_j \) since

\[
|F| \lesssim \sum_j |F_j| \lesssim \sum_j |I_j| \lesssim \frac{\delta^{100}}{K^{2M}} \lesssim \frac{\delta^{50}}{K^M}.
\]

Now on \( F^c \) we will use the estimates obtained in Lemma 3. But before this, we first need to remove a few tiles\(^{12} \) from each tree \( \mathcal{P}_j \).

For \( \mathcal{P} = \bigcup_j \mathcal{P}_j \) and \( L = \log (K^{100M} \delta^{-100M}) \) we let

\[
\mathcal{P}^+ := \{ P \in \mathcal{P} \mid \text{there is no chain } P < P_1 < \cdots < P_L \text{ with all } P_j \in \mathcal{P} \}
\]

and

\[
\mathcal{P}^- := \{ P \in \mathcal{P} \mid \text{there is no chain } P_1 < P_2 < \cdots < P_L < P \text{ with all } P_j \in \mathcal{P} \}.
\]

Now, it is easy to see that each such set can be split into at most \( L \) subsets with no two comparable tiles inside the same subset. Consequently, using Proposition 1, we deduce that\(^{13} \)

\[
\left\| T^{\mathcal{P}^+} \right\|_p, \left\| T^{\mathcal{P}^-} \right\|_p \lesssim_{p,d} L \delta^{\gamma(1-\frac{1}{p'})} \lesssim \delta^{\gamma(1-\frac{1}{p'})} \log K.
\]

\(^{12} \)If for a tree \( \mathcal{P}_j \) there are multiple tiles with the same time interval, then we will take their (geometric) union and consider it as a single tile.

\(^{13} \)\( \eta \) may change from line to line.
We remove all the above-mentioned sets from our collection $\mathcal{P}$ and decompose this new set as follows:

$$\mathcal{P} = \bigcup_j \mathcal{P}_j^0$$

where $\mathcal{P}_j^0 = \mathcal{P}_j \cap \mathcal{P}$.

This new collection $\mathcal{P}$ has the following properties:

1) \( \forall P \in \mathcal{P}_j^0, |I_P| \leq \frac{100M}{K} |I_j| \)

2) \( \forall j \neq k \), the trees $\mathcal{P}_j^0$ and $\mathcal{P}_k^0$ are $\delta'$-separated where $\delta' = \frac{100M}{K}$.

Splitting now each $\mathcal{P}_j^0 = \mathcal{P}_j^N \cup \mathcal{P}_j^C$, with

$$\mathcal{P}_j^C = \text{def} \{ P \in \mathcal{P}_j^0 | I_P \subseteq F_j \},$$

we conclude that $\{ \mathcal{P}_j^N \}$ represents a collection of normal, $\delta'$-separated trees, while for the remaining parts of the trees we have the relation

$$\text{supp} \mathcal{T}^C \subset F_j.$$

Consequently, on $F^c$ we have that

$$\mathcal{T}^P f = \sum_j \mathcal{T}^N_j f,$$

and so our conclusion reduces to

$$\left\| \sum_j \mathcal{T}^N_j f \right\|_p \lesssim_{p,d} \left( K \frac{1}{p'} \delta^{\frac{1}{p'}} (1+\rho)(\frac{1}{p'} - \frac{1}{p''}) \right) \|f\|_p.$$

Now, as in [5], we may divide $\bigcup_j \mathcal{P}_j^N$ into a union of at most $S = K \delta^{-1+\rho}$ rows, $\mathcal{R}_1$, $\mathcal{R}_2$, ..., $\mathcal{R}_S$. Using the same techniques as in [2] and further [5], one can show (applying the Cotlar-Stein Lemma together with Lemmas 1 - 4) that

$$\left\| \sum_j \mathcal{T}^N_j f \right\|_2 \leq \left\| \sum_{j=1}^S \mathcal{T}^{R_j}_j f \right\|_2 \lesssim_d \left( \sum_{j=1}^S \left\| \mathcal{T}^{R_j}_j f \right\|_2^2 \right)^{1/2} \leq \left\| \sum_{j=1}^S \mathcal{T}^{R_j}_j f \right\|_2 \lesssim_d \delta^{\frac{1}{2}} \|f\|_2.$$

On the other hand, from the triangle inequality, we trivially have

$$\left\| \sum_{j=1}^S \mathcal{T}^{R_j}_j f \right\|_1 \leq \sum_{j=1}^S \left\| \mathcal{T}^{R_j}_j f \right\|_1 \leq \sum_{j=1}^S \left\| \mathcal{T}^{R_j}_j f \right\|_\infty \leq \left\| \sum_{j=1}^S \mathcal{T}^{R_j}_j f \right\|_\infty.$$

Now, using interpolation between (31) and (32), we obtain

$$\left\| \sum_{j=1}^S \mathcal{T}^{R_j}_j f \right\|_{p'} \lesssim_{p,d} \left( \sum_{j=1}^S \left\| \mathcal{T}^{R_j}_j f \right\|_{p''}^p \right)^{1/p}.$$
If we denote $E_j := \text{supp } T^R_j$ then, by Lemma 1, we deduce

$$\left\| T^R_j f \right\|_{p',d} \lesssim \delta^{1/p'} \left( \int_{E_j} |f|^{p'} \right)^{1/p'}.$$  

Now, taking into account the fact that $\{E_j\}_j$ are disjoint, we conclude (using Hölder)

$$\left\{ \sum_{j=1}^{S} \left( \int_{E_j} |f|^{p'} \right)^{\frac{p}{p'}} \right\}^{\frac{1}{p'}} \lesssim_{p,d} S^{\frac{1}{p'}} \left( \int |f|^{p'} \right)^{1/p'},$$

which ends our proof.

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8. Appendix - Results on the $L^\infty$-distribution of polynomials

**Lemma A.** If $q \in \mathcal{Q}_{d-1}$ and $I, J$ are some (not necessarily dyadic) intervals obeying $I \supseteq J$, then there exists a constant $c(d)$ such that

$$\|q\|_{L^\infty(I)} \leq c(d) \left( \frac{|I|}{|J|} \right)^{d-1} \|q\|_{L^\infty(J)}.$$  

*Proof.* Let $\{x^k_j\}_{k \in \{1, \ldots, d\}}$ be obtained as in the procedure described in Section 2. Then, since $q \in \mathcal{Q}_{d-1}$, for any $x \in I$ we have that

$$q(x) := \sum_{j=1}^{d} \prod_{k \neq j}^d (x - x_j^k) q(x_j^k).$$

As a consequence,

$$\|q\|_{L^\infty(I)} \leq d \|q\|_{L^\infty(J)} \sup_{x \in I} \left| \prod_{k \neq j}^d (x - x_j^k) \left( \prod_{k \neq j}^d (x_j^k - x^k_j) \right)^{-1} \right| \leq d \|q\|_{L^\infty(J)} \frac{|I|^{d-1}}{(|J|/d)^{d-1}}.$$  

**Lemma B.** If $q \in \mathcal{Q}_{d-1}$, $\eta > 0$ and $I \subset \mathbb{T}$ some (dyadic) interval, then

$$|\{y \in I \mid |q(y)| < \eta\}| \leq c(d) \left( \frac{\eta}{\|q\|_{L^\infty(I)}} \right)^{\frac{1}{d-1}} |I|.$$  

*Proof.* The set $A_\eta = \{y \in I \mid |q(y)| < \eta\}$ is the pre-image of $(-\eta, \eta)$ (an open set) under a polynomial of degree $d - 1$, so it can be written as

$$A_\eta = \bigcup_{k=1}^{r} J_k(\eta),$$

where $r \in \mathbb{N}$, $r \leq d - 1$ and $\{J_k(\eta)\}_k$ are open intervals. Now all that remains is to apply the previous lemma with $J = J_k(\eta)$ for each $k$.  

$\square$
Lemma C. If $P = [\alpha^1, \alpha^2, \ldots, \alpha^d, I] \in \mathbb{P}$ and $q \in P$, then
\[ \|q - qp\|_{L^\infty(I)} \leq c(d) |I|^{-1}. \]

Proof. Set $u := q - qp$; then, since both $q, qp \in P$, we deduce (for all $k \in \{1, \ldots, d\}$):
\[ u(x^k_I) \in [-|I|^{-1}, |I|^{-1}]. \]

On the other hand,
\[ u(x) := \sum_{j=1}^{d} \frac{\prod_{k \neq j}^d (x - x^k_I)}{\prod_{k \neq j}^d (x^j_I - x^k_I)} u(x^j_I) \quad \forall \ x \in I. \]

Then, proceeding as in Lemma A, we conclude
\[ \|u\|_{L^\infty(I)} \leq d |I|^{-1} \frac{|I|^{d-1}}{(|I|/d)^{d-1}} \leq d^d |I|^{-1}. \]

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