Abstract

Chiral symmetry breaking in a purely fermionic theory is investigated by the help of the renormalization group method. The RG equation for the running mass $m_k$ admits a solution with vanishing bare mass and finite physical mass. The running Fermi coupling constant, $G_k$, converges to a finite (renormalized) physical value. It is also shown that the RG equation for $\tilde{G}$, the dimensionless Fermi coupling, has an UV fixed point $\tilde{G}_{UV}$. Contrary to a previous result however, it is proven that the chiral symmetry breaking point $\tilde{G}_c$ does not coincide with $\tilde{G}_{UV}$.

The problem of the dynamical generation of a fermion mass has been studied over the years by several authors[1]. Nambu and Jona-Lasinio (NJL) considered a purely fermionic model in the mean field approximation and found a non trivial solution for the fermion mass, for values of the Fermi constant larger than a critical one [2]. The non perturbative nature of this approximation is obvious as the chiral symmetry of the model forbids the appearance of a mass term at any order of perturbation theory.

Here I study a fermionic model by the help of a different non-perturbative technique, the renormalization group method. It is widely recognized that the approach pioneered by Wilson[3] provides a powerful method to study quantum and statistical field theories. In the Wegner-Houghton[4] realization it is implemented by establishing an exact integro-differential flow equation

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for the Wilsonian effective action $S_k$. If $\Lambda$ is an UV scale, in the $k \to \Lambda$ limit (and eventually $\Lambda \to \infty$) $S_k$ corresponds to the bare UV action, while $S_{k=0}$ is the physical effective action. The Wegner-Houghton equation as it stands is actually an intractable one, a systematic approximation scheme can be built by the help of the derivative expansion. At its lowest order it gives the so-called Local Potential Approximation (LPA).

By considering the LPA for a fermionic model with discrete $\gamma_5$ symmetry of the bare action and potential truncated to the quadrifermionic interaction term, I show (for the first time at the best of my knowledge) that:

- The RG equation for the running mass, $m_k$, admits a solution vanishing in the $k \to \infty$ limit and finite at $k = 0$, i.e. a solution breaking the discrete $\gamma_5$ symmetry of the bare theory;
- The RG equation for the running Fermi coupling constant, $G_k$, admits a solution having the canonical scaling in the UV region, $G_k \sim \frac{1}{k^2}$, and flowing to a finite value in the IR, i.e. the theory can be renormalized.

By freezing the dimensionless Fermi coupling constant to a given fixed value, I also consider the RG equation for the running mass alone. According to the value of the Fermi constant, I find that a non vanishing physical mass, $m_{ph} = m_{k=0} \neq 0$, may or may not be generated from the chirally invariant bare theory. Clearly with this additional approximation we get close to the approach of the original NJL paper. There are however important differences between our approach and that of Ref.[2].

In this latter paper the quantum fluctuations (responsible for the dynamical generation of the fermion mass) are taken into account by the help of a mean field approximation that allows to establish the gap equation for a constant (momentum independent) mass function. As they were dealing with a (perturbatively) non renormalizable theory, the authors of Ref.[2] restricted themselves to consider the cut-off theory. Being the mass function approximated by a constant, the one-loop integral obviously receives a quadratically divergent contribution.

In our approach the quantum fluctuations are taken into account through the resummation operated by the RG equation, the momentum dependence of the mass function being given by the running scale. The solution to our RG equation, $m_k$, vanishes in the UV and the $k \to \infty$ limit does not generate divergent contributions. A quantitative and detailed comparison between our approach and the NJL approximation will be presented in a forthcoming paper[4]. It is sufficient to add here that, within a certain approximation,
our RG equation for the running mass \( m_k \) reproduces the NJL result.

A related aspect of our analysis is that of the existence of a continuum limit.

In massless QED the lowest order approximation to the gap equation, for values of the fine structure constant \( \alpha \) larger than a critical one, \( \alpha_c = \frac{\pi}{3} \), gives for the dynamically generated fermion mass [6] [7] :

\[
m = \Lambda f(\alpha),
\]

(1)

where \( \Lambda \) is the UV cut-off and \( f(\alpha) \) is a known function of \( \alpha \). Miransky suggested [8] that this equation should be regarded as defining the UV flow of the running coupling constant \( \alpha = \alpha(\Lambda) \), the fermion mass \( m \) being fixed. From the specific form of the function \( f(\alpha) \) it is immediately found that \( \alpha(\Lambda) \) flows toward \( \alpha_c \) for \( \Lambda \to \infty \). This interpretation has received lot of attention as it seemed the only possibility to evade the conclusion that the physics of the dynamical generation of a fermion mass actually occurs at the cut-off scale [9]. Miransky also extended it to other gauge theories as well as to the quadrifermionic theory [10]. Following this suggestion several models have been constructed and the phase structure of certain theories investigated.

In the quadrifermionic theory the result of the mean field approximation can be written again as \( m = \Lambda f(\tilde{G}) \), where \( \Lambda \) is the UV cut-off, \( \tilde{G} \) is the dimensionless Fermi coupling and \( f(\tilde{G}) \) a known function of \( \tilde{G} \). The UV flow of the Fermi constant \( \tilde{G} = \tilde{G}(\Lambda) \) is defined as before and accordingly it is found that the critical point \( \tilde{G}_c \) coincides with an UV fixed point \( \tilde{G}_{UV} \) [10].

This interpretation however is not the result of a RG analysis. It rather comes from the attempt to define a continuum limit out of the results of the gap equation.

By studying for the first time this problem in a real RG framework, I show that contrary to the Miransky suggestion the two points \( \tilde{G}_c \) and \( \tilde{G}_{UV} \) are well separated, \( \tilde{G}_c \) being smaller than \( \tilde{G}_{UV} \).

The (euclidean) wilsonian action for our model at the scale \( k \) in the LPA has the form :

\[
S_k = \int d^4x \left[ \overline{\psi} \gamma_\mu \partial_\mu \psi + U_k (\overline{\psi} \psi) \right].
\]

(2)

Expanding the potential \( U_k (\overline{\psi} \psi) \) in powers of \( \overline{\psi} \psi \) and retaining terms up to \( (\overline{\psi} \psi)^2 \), i.e. up to the quadrifermionic interaction term,
the RG equation for $S_k$ actually reduces to a system of differential equations for the running mass $m_k$ and the running Fermi coupling constant $G_k$.

The goal is to see whether a solution for $m_k$ exists such that for a vanishing bare mass $m_B$, the UV limit of the running mass, a finite physical mass, $m_{ph} = m_{k=0} = \text{finite}$, is obtained.

It is often convenient to move to dimensionless variables. Introducing the dimensionless scale parameter $t = \ln \mu_k$, where $\mu$ is a given boundary value of $k$, together with the dimensionless running mass $\tilde{m}_t$ and running Fermi coupling constant $\tilde{G}_t$,

$$\tilde{m}_t = \frac{m_k}{k} \quad \text{and} \quad \tilde{G}_t = k^2 G_k,$$  

the flow equations for $\tilde{m}_t$ and $\tilde{G}_t$, obtained after integration of the degrees of freedom in the momentum shell $[t, t + \delta t]$ and taking the $\delta t \to 0$ limit, are:

$$\frac{d\tilde{m}_t}{dt} = \tilde{m}_t \left[ 1 + \frac{3}{8\pi^2} \frac{\tilde{G}_t}{(1 + m_t^2)} \right]$$  \hspace{1cm} (5)

$$\frac{d\tilde{G}_t}{dt} = -2\tilde{G}_t \left[ 1 - \frac{1}{8\pi^2} \frac{\tilde{G}_t}{(1 + m_t^2)^2} \right].$$  \hspace{1cm} (6)

At a first sight it could seem from Eq.(5) that given the boundary $m_{t_0} = 0$ at an UV scale $t_0$, the only possible solution to this equation is $\tilde{m}_t = 0$ for any value of $t$. This in turn would imply that $m_{ph} = 0$, the symmetry is not broken and no mass term is generated.

The complete analysis of the system (5)-(6) can be performed only numerically. However the UV ($t \to -\infty$) and the IR ($t \to +\infty$) asymptotic regions can be studied analytically.

Let’s start with the UV region and suppose that a solution exists such that for $t \to -\infty$, $m_t^2$ vanishes more rapidly than $\tilde{G}_t$. Under this assumption the system becomes:

$$U_k(\bar{\psi}\psi) = m_k\bar{\psi}\psi - \frac{G_k}{2}(\bar{\psi}\psi)^2,$$  \hspace{1cm} (3)
\[
\frac{d\tilde{m}_t}{dt} = \tilde{m}_t \left[ 1 + \frac{3\tilde{G}_t}{8\pi^2} \right] \tag{7}
\]

\[
\frac{d\tilde{G}_t}{dt} = -2\tilde{G}_t \left[ 1 - \frac{\tilde{G}_t}{8\pi^2} \right] \tag{8}
\]

Eq. (8) has the solution:

\[
\tilde{G}_t = \frac{8\pi^2}{1 + Ce^{2t}},
\tag{9}
\]

where \(C\) is an integration constant. We note here that, in terms of the dimensionful Fermi constant, this UV flow is:

\[
G_k \sim_{k \to \infty} \frac{8\pi^2}{k^2}.
\tag{10}
\]

To study the behavior of Eq. (7) in the \(t \to -\infty\) region, we can now replace in this equation the asymptotic value \(\tilde{G}_t \sim 8\pi^2\). Eq. (7) then becomes:

\[
\frac{d\tilde{m}_t}{dt} - 4\tilde{m}_t = 0
\tag{11}
\]

whose solution is trivially (A is an integration constant):

\[
\tilde{m}_t = Ae^{4t}.
\tag{12}
\]

The assumption that lead us to the approximated system (7)-(8) is consistent with Eqs. (9) and (12), consequently these are asymptotic solutions to the original system (5)-(6). We also see that the system (5)-(6) possesses the UV \((t = -\infty)\) fixed point:

\[
\tilde{m}_{UV} = 0, \quad \tilde{G}_{UV} = 8\pi^2.
\tag{13}
\]

Moreover, as \(k = \mu e^{-t}\), from Eq. (12) we have:

\[
m_k \sim_{k \to \infty} A\frac{\mu^4}{k^3},
\tag{14}
\]

that gives:

\[
m_B = \lim_{k \to \infty} m_k = 0.
\]
Eq. (14) is a potentially interesting result, we have found a running mass $m_k$ vanishing in the $k \to \infty$ limit.

Let’s move now to the IR region. By simple inspection we see that in the asymptotic region $t \to \infty$, a solution to the system (5)-(6) exists such that:

\[
\begin{align*}
\tilde{m}_t &\sim_{t \to \infty} e^t \\
\tilde{G}_t &\sim_{t \to \infty} e^{-2t}.
\end{align*}
\]

(15) (16)

This again is a potentially interesting result. Moving to dimensionful parameters, Eqs. (15) and (16) in fact give:

\[
\begin{align*}
m_{ph} &= \lim_{k \to 0} m_k = \text{finite} \\
G_{ph} &= \lim_{k \to 0} G_k = \text{finite}.
\end{align*}
\]

(17) (18)

We don’t know yet however whether a solution $\{\tilde{m}_t, \tilde{G}_t\}$ to the system (5)-(6) exists possessing both the IR and the UV behavior respectively given by Eqs. (15)-(16) and (9)-(12). That such a solution exists is the main result of this paper and will be proven numerically later.

Before moving to the numerical solution of the system (5)-(6) however, we want to consider an additional approximation under which it is possible to find an analytical solution. We freeze the value of the Fermi constant to a given fixed value and restrict ourselves to consider the evolution of the running mass $\tilde{m}_t$ alone.

This approximation to our RG equations is already worth to study. In the RG framework, the flow equation for the running mass plays the same role as the gap equation for the mass function in the Schwinger-Dyson approach. Moreover, as we have pointed out before, it can be proven [5] that in a certain limit it reproduces the mean field result.

Let’s freeze $\tilde{G}_t$ to its UV fixed point value found above, namely $\tilde{G}_t = \tilde{G}_{UV} = 8\pi^2$. Under this approximation our original system (5)-(6) reduces to the differential equation for $\tilde{m}_t$:

\[
\frac{d\tilde{m}_t}{dt} = \tilde{m}_t [1 + \frac{3}{1 + \tilde{m}_t^2}],
\]

(19)

that we can conveniently rewrite as,
\[
\frac{d\tilde{m}_t^2}{dt} = 2\tilde{m}_t^2\left[1 + \frac{3}{1 + \tilde{m}_t^2}\right]. \tag{20}
\]

We have already seen that a solution to Eq.\,(20) exists such that in the UV limit, i.e. for \( t \to -\infty \),
\[
\tilde{m}_t^2 \sim e^{8t} \tag{21}
\]
(see Eq. \((12)\) above).

We can also immediately verify that in the IR, i.e. for \( t \to \infty \), Eq.\,(20) has the asymptotic solution (\( B \) is an integration constant)
\[
\tilde{m}_t^2 = -3 + B e^{2t} \sim B e^{2t}. \tag{22}
\]

Moving to the dimensionful running mass we have:
\[
m^2_{ph} = \lim_{k \to 0} m^2_k = B \mu^2 = finite. \tag{23}
\]

Once more, we don’t know yet whether a solution to Eq.\,(20) exists with both the UV and the IR behavior of Eqs.\,(21) and \,(22) respectively. Fortunately the analytical solutions to Eq.\,(20) can be found and one of them is relevant to our problem. It has a quite long expression that we can write as:
\[
\tilde{m}_t^2 = \frac{\sqrt{3}}{6} \left[ b_t^\frac{1}{2} + \frac{24 a_t^\frac{1}{2} b_t^\frac{3}{2} - 3 a_t^\frac{2}{3} b_t^\frac{1}{3} + 4 e^{8(t+\alpha)} b_t^\frac{1}{3} + 32 c_t}{a_t^\frac{1}{3} b_t^\frac{2}{3}} \right]^\frac{1}{2} - 3 \tag{24}
\]

where
\[
a_t = 8 e^{8(t+\alpha)} \left[ \frac{\sqrt{3}}{9} \left( e^{8(t+\alpha)} + 27 \right)^\frac{1}{3} - 1 \right],
b_t = 12 a_t^\frac{1}{3} + 3 a_t^\frac{2}{3} - 4 e^{8(t+\alpha)},
c_t = \sqrt{6} e^{4(t+\alpha)} \left( \sqrt{3} (e^{8(t+\alpha)} + 27)^\frac{1}{4} - 9 \right), \tag{25}
\]

and \( \alpha \) is an integration constant.

A long but straightforward computation shows that the above solution, Eq.\,(24), has the required IR and UV asymptotic behavior.
This is one of our main results. The RG equation for the running mass \( m_k \) admits a solution such that \( \lim_{k \to \infty} m_k = 0 \) and \( \lim_{k \to 0} m_k = \text{finite} \). In other words the RG equation for \( m_k \) generates a physical mass from the massless bare theory, thus breaking dynamically the chiral symmetry. Through a crossover region the UV \( \frac{1}{k^3} \) flow of the mass function is converted into an IR scaling giving rise to a finite value at \( k = 0 \).

Of course the question arises concerning the existence of a critical value for the Fermi coupling constant.

We have just seen that for \( \tilde{G} = 8\pi^2 \) we have a symmetry breaking solution. As compared to the self consistent approach\(^2\) we do not have here an algebraic equation for \( m_{ph} \) but rather a differential equation for \( \tilde{m}_t \) (or, what amounts to the same thing, for \( m_k \)). We expect that there exists a critical value \( \tilde{G}_c \) of the dimensionless Fermi constant \( \tilde{G} \) such that for \( \tilde{G} < \tilde{G}_c \) we have \( m_{ph} = 0 \), while for \( \tilde{G} > \tilde{G}_c \) it is \( m_{ph} \neq 0 \). To find \( \tilde{G}_c \) we should in principle replace \( \tilde{G} = 8\pi^2 \) in Eq.(20) with a generic value of \( \tilde{G} \) and seek for the solution \( \tilde{m}_t \). Unfortunately it is not a trivial task to look for analytical solutions of Eq.(20) for arbitrary values of \( \tilde{G} \). Analytical solutions can only be obtained for certain specific values of \( \tilde{G} \).

It is now not too difficult to check that for \( \tilde{G} = 4\pi^2 \) and \( \tilde{G} = \frac{8\pi^2}{3} \) a solution \( \tilde{m}_t \) vanishing in the UV and converging to a finite value in the IR exists, while for the value \( \tilde{G} = 2\pi^2 \) this solution is lost. Moreover, while for \( \tilde{G} = 8\pi^2 \) we have found that the UV behavior of the mass function is \( m_k \sim \frac{1}{k^3} \), for \( \tilde{G} = 4\pi^2 \) it is \( m_k \sim \frac{1}{k^2} \) and for \( \tilde{G} = \frac{8\pi^2}{3} \) it is \( m_k \sim \frac{1}{k} \). We are then lead to the conclusion that \( \tilde{G}_c \) lies in the range \( [2\pi^2, \frac{8\pi^2}{3}] \).\(^3\)

This is another important result. Comparing with our previous result \( \tilde{G}_{UV} = 8\pi^2 \), we conclude that the UV fixed point \( \tilde{G}_{UV} \) does not coincide with the critical point \( \tilde{G}_c \).

Needless to say the existence and the location of an UV fixed point for \( \tilde{G}_t \) can only be established from the RG equation for \( \tilde{G}_t \) itself. Indications coming from other arguments have to be taken with caution. Our RG analysis has shown that the theory possesses an UV fixed point, \( \tilde{G}_{UV} \), that however does not coincide with the Miransky limit \( [10] \). In addition the correct UV scale dependence of \( \tilde{G}_t \), Eq.(19), is different from the one deduced from this result that the UV fixed point, \( \tilde{G}_{UV} \), and the critical point, \( \tilde{G}_c \), do not coincide.

\(^2\)From the specific form of the solution at \( \tilde{G} = \frac{8\pi^2}{3} \), I believe that \( \tilde{G}_c = \frac{8\pi^2}{3} \). In any case the precise location of the critical point is not our main concern here. What is important is the result that the UV fixed point, \( \tilde{G}_{UV} \), and the critical point, \( \tilde{G}_c \), do not coincide.
Figure 1: (a): In this figure we see the running of the dimensionful mass $m_k$, the boundary condition is given in the text. (b): Here we focus on the flow well on the right of the crossover region where we recognize the UV scaling, $m_k \sim \frac{1}{k^3}$.

interpretation of the gap equation. As the above criticism obviously extends to any other theory where this interpretation was applied, we claim that those phase diagrams and models based on it have to be reconsidered.

We conclude now by presenting the numerical solution of the system (5)-(6). As it is preferable to work directly with the dimensionful parameters, in Figs.(1) and (2) we show the running of $m_k$ and $G_k$ versus $k$. As boundary values we have taken $m_k = 10^{-6}$ and $G_k = 7.89510^{-5}$ at $k = 10^3$.

In Fig.(1.a) we see the running of the mass parameter $m_k$ and observe the transition from the UV regime where $m_k \to 0$ to the IR regime where $m_k$ converges to a finite value, the physical mass $m_{ph}$. The UV and IR asymptotic flows both coincide with our previous analytical results, Eqs.(14) and (22) respectively. Fig.(1.b) presents a magnification of the UV region where the $1/k^3$ UV behavior, found analytically in Eq. (14), is easily recognized.

Fig.(2.a) shows the running of $G_k$ versus $k$. In the UV region its flow is nothing but the UV $1/k^2$ canonical scaling, already found in Eq. (10). Through a crossover region this flow is converted to an IR scaling and $G_k$ converges to a finite value at $k = 0$, as seen in Eq. (18). As for $m_k$ we have
Figure 2: (a): In this figure we show the running of the dimensionful Fermi coupling constant $G_k$. (b): We focus here on the flow on the right of the crossover region and observe the UV scaling, $G_k \sim \frac{1}{k^2}$.

magnified the UV region to better show the UV $1/k^2$ scaling.

In summary we have found that the RG equation for the running mass $m_k$ admits a solution that breaks the original discrete $\gamma_5$ chiral symmetry of the bare theory and that the running Fermi coupling constant $G_k$ has the canonical scaling, $G_k \sim \frac{8\pi^2}{k^2}$, in the UV and flows to a renormalized value, $G_{k=0} = \text{finite}$, in the IR. It should not be underestimated here that for a theory in $d = 4$ dimensions we have established non-perturbative renormalization group equations allowing to follow the renormalization flow of the running coupling constants all the way down from the UV to the IR. The theory, at least within the approximation considered in this paper, can be renormalized. There is certainly no need to remind the kind of (UV or IR) pathologies that are typically encountered in RG equations, think for instance of QED, $\lambda\phi^4$ or QCD.

Finally we have also shown that an old result, concerning the coincidence between the chiral symmetry breaking point and the UV fixed point of the theory, actually turns out to be incorrect.

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