Neutrino index of refraction in a magnetized two-stream electron background

José F. Nieves*
Laboratory of Theoretical Physics
Department of Physics,
University of Puerto Rico
Río Piedras, Puerto Rico 00936

Yaithd D. Olivas†
Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de Mexico,
Circuito Exterior, C. U., A. Postal 70-543, 04510 Mexico DF, Mexico

Sarira Sahu‡
Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de Mexico
Circuito Exterior, C. U., A. Postal 70-543, 04510 Mexico DF, Mexico and
Astrophysical Big Bang Laboratory, RIKEN,
Hirosawa, Wako, Saitama 351-0198, Japan
(Dated: June 2017)

We study the propagation of a neutrino in a medium that consists of two or more thermal backgrounds of electrons and nucleons moving with some relative velocity, in the presence of a static and homogeneous electromagnetic field. We calculate the neutrino self-energy and dispersion relation using the linear thermal Schwinger propagator, we give the formulas for the dispersion relation and discuss general features of the results obtained, in particular the effects of the stream contributions. We discuss in detail specifically a medium that consists of a normal electron background plus another electron stream background that is moving with a velocity four-vector \( v^\mu \), as a whole, relative to the normal background, in the presence of a static and homogeneous magnetic field \( \vec{B} \) in the reference frame of the normal background. For a neutrino propagating with momentum \( \vec{k} \), in the presence of the stream the neutrino dispersion relation acquires an anisotropic contribution of the form \( \hat{k} \cdot \vec{v} \) in addition to the well known term \( \hat{k} \cdot \vec{B} \), as well as an additional contribution proportional to
$\vec{B} \cdot \vec{v}$. We consider the contribution from a nucleon stream background as an example of other possible stream backgrounds, and comment on possible generalizations to take into account the effects of inhomogeneous fields. We explain why a term of the form $\hat{k} \cdot (\vec{v} \times \vec{B})$ does not appear in the dispersion relation in the constant field case, while a term of similar form can appear in the presence of an inhomogeneous field involving its gradient.
I. INTRODUCTION AND SUMMARY

Since the discovery of the MSW effect[1], for many years a lot of attention has been
given to the calculation of the properties of neutrinos in a matter background under various
conditions. The matter background modifies the neutrino dispersion relations[2], and also
induces electromagnetic couplings that can lead to effects in several astrophysical and/or
cosmological settings[3]. In supernova environments the presence of the neutrino background
leads to neutrino collective oscillations[4] that have been the subject of significant work in
the context of instabilities in supernovas[5].

It is now well known that the presence of a magnetic field produces an angular asymmetry
in the neutrino dispersion relation when it propagates in an otherwise isotropic background
medium[6]. Since many of the physical environments of interest in the contexts mentioned
include the presence of a magnetic field, a significant amount of work has been dedicated
to study the calculation of the neutrino self-energy in the presence of a magnetic field, or
a magnetized background medium[7], and the study of the properties and propagation of
neutrinos in such media[8].

In the previous calculations related to the propagation of neutrinos in a background
medium, the background is taken to be at rest since there is no other reference frame defined
in the problem at hand. In the present work we consider the propagation of a neutrino in
a medium that consists of a normal matter background and a constant magnetic field, as
considered in the previous works already cited, plus a stream background (superimposed on
the normal background) which has a non-zero velocity relative to the normal background.
In the context of plasma physics the propagation of photons in magnetized or unmagnetized
two stream plasma systems is a well studied subject[9]. Here we initiate the analogous study
for the case of neutrinos. Our goal is to determine the neutrino dispersion relation for a
neutrino that propagates in such stream systems, and in particular including the effects of
a magnetic field. Beyond the intrinsic interest, the results are of practical application in
astrophysical contexts in which the asymmetric neutrino propagation is believed to produce
important effects such as the dynamics of pulsars[10, 11] and supernovas[12].

Before embarking on the details, we make these statements more precise. In the common
notation, the velocity four-vector of the background is denoted by $u^\mu$, and its reference frame
is defined by setting
\[ u^\mu = (1, \vec{0}). \quad (1.1) \]

We will refer to it as the *normal* background.

We assume that in the rest frame of the normal background, there is a constant magnetic field \( \vec{B} = B\hat{b} \). In that frame we define
\[ B^\mu = B b^\mu, \quad b^\mu = (0, \hat{b}). \quad (1.2) \]

We can then write the corresponding EM tensor in the form
\[ F_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta} u^\alpha B^\beta, \quad (1.3) \]
and its dual, defined as usual by \( \tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta} \), is given by
\[ \tilde{F}_{\mu\nu} = B_\mu u_\nu - u_\mu B_\nu. \quad (1.4) \]

\( F_{\mu\nu} \) and \( \tilde{F}_{\mu\nu} \) are such that
\[ F_{\mu\nu} u^\nu = 0, \]
\[ \tilde{F}_{\mu\nu} u^\nu = B_\mu. \quad (1.5) \]

In the present work we assume that there are other backgrounds, to which we refer as the *stream* backgrounds, which are superimposed on the normal matter background. A stream background is a thermal background that has a non-zero velocity relative to the normal matter background. For definiteness, we consider only the contributions from the electrons and nucleons \( (N = n, p) \) in the normal and stream backgrounds, and we use the symbols \( s = e, N \) and \( s' = e', N' \) to refer to them, respectively. We also use \( f_e = e, e' \) to refer to the electrons in either background, and similarly for the nucleons \( f_N = N, N' \). The symbol \( f \) stands for any fermion in either background. In particular, \( u_f^\mu \) denotes the velocity four-vector of any of the backgrounds.

As already stated the normal background can be taken to be at rest, so that for all the species in the normal background we set
\[ u_s^\mu = u^\mu, \quad (1.6) \]
with the components of \( u^\mu \) given in Eq. (1.1). But for the stream backgrounds
\[ u_{s'}^\mu = (u_{s'}^0, \vec{u}_{s'}), \quad (1.7) \]
The main objective of the present work is the calculation of the neutrino dispersion relation, taking into account the simultaneous presence of the stream background and the magnetic field. Our work is based on the calculation of the thermal self-energy diagrams shown in Fig. 1, using the thermal Schwinger propagator, linearized in $B$, including only the electrons in both backgrounds, and to the leading order $O(1/m_W^2)$ terms. The results of the calculation are summarized in Eqs. (4.25)-(4.28) for the self-energy, and in Eqs. (5.5)-(5.9) for the corresponding dispersion relations. The main result is that for a neutrino propagating with momentum $\vec{k}$ in the presence of a stream, the neutrino dispersion relation acquires an anisotropic contribution of the form $\hat{k} \cdot \vec{u}_s'$ in addition to the well known term $\hat{k} \cdot \vec{B}$, and the standard isotropic term receives an additional contribution proportional to $\vec{B} \cdot \vec{u}_s'$. The term involving $\hat{k} \cdot (\vec{u}_s' \times \vec{B})$ does not appear in the dispersion relation, due to time-reversal invariance.

In Section II we summarize the general parametrization of the self-energy, review the relevant formulas for the electron thermal propagator and the main ingredients involved in the calculation. The formulas for the parameter coefficients that appear in the neutrino thermal self-energy are obtained and summarized in Section IV. The calculation of the contribution of a nucleon stream is also given there as an illustration of possible generalizations. In Section V we discuss and summarize the main features of the results obtained for the neutrino dispersion relation, and comment on related work, in particular the calculation in the case of an inhomogeneous external field.

### II. GENERAL CONSIDERATIONS

We denote by $\Sigma_{eff}$ the background-dependent contribution to the neutrino self-energy, determined from the calculation of the diagrams in Fig. 1. Chirality of the neutrino interactions then imply that

$$\Sigma_{eff} = R \Sigma L,$$

and the dispersion relation for a given neutrino flavor $\nu_\ell$ is then obtained by solving the equation

$$(\vec{k} - \Sigma) \psi_L = 0.$$
Fig. 1. The diagrams that contribute to the neutrino self-energy in a background of electrons and nucleons to the lowest order for a given neutrino flavor $\nu_\ell$ ($\ell = e, \mu, \tau$). Diagram (a) contributes only to the $\nu_e$ self-energy, while Diagram (b) contributes for the three neutrino flavors. In our calculation we consider two sets of these two diagrams, one set with the normal background ($s = e, n, p$) and another set with the stream backgrounds ($s' = e', n', p'$).

In the lowest (1-loop) order each background gives a separate contribution $\Sigma_f$ to the total self-energy. In the presence of a constant electromagnetic field each term $\Sigma_f$ is a function of $k^\mu$, $u^\mu_f$ as well as $F^{\mu\nu}$, and its general form is

$$
\Sigma_f = a_f k^\mu + b_f u^\mu_f + c_f \tilde{F}^{\mu\nu} u_f \gamma_\mu + d_f F^{\mu\nu} u_f \gamma_\mu + g_f F^{\mu\nu} k_{\nu} \gamma_\mu + \tilde{g}_f \tilde{F}^{\mu\nu} k_{\nu} \gamma_\mu .
$$

In the present calculation we restrict ourselves to the contact $O(1/m_W^2)$ term of the $W$ propagator, and we do not consider the momentum dependent terms nor its dependence on the magnetic field. To this order in $1/m_W^2$ the $a_f, g_f, \tilde{g}_f$ terms in Eq. (2.3) vanish and we do not consider them any further. Regarding the other terms, for our particular case in which the field is a pure $B$ field in the rest frame of the normal background, Eq. (1.5) implies that
\[ \Sigma_s, \ (s = e, n, p), \]  
is reduced to  
\[ \Sigma_s = b_s \gamma + c_s B, \]  
(2.4)  
which is the form used in Ref. [6]. However, for the stream backgrounds, using Eq. (1.4),  
\[ \Sigma_{s'} = b_{s'} \gamma_{s'} + c_{s'} \left[ (u \cdot u_{s'}) B - (B \cdot u_{s'}) \gamma \right] - d_{s'} B_{s'}, \]  
(2.5)  
where  
\[ E^\mu_{s'} = F^{\mu\nu} u_{s'\nu} = \epsilon^{\mu\alpha\beta} u_{s'\nu} u_{\alpha} B_{\beta}. \]  
(2.6)  
In the rest frame of the normal background \( E^\mu_{s'} \) has components  
\[ E^\mu_{s'} = (0, \bar{u}_{s'} \times \bar{B}), \]  
(2.7)  
which can be interpreted as the electric field that the stream background particles “see”. Thus the \( d_{s'} \) term represents an electric dipole type of coupling of the stream background particles. As we will see, the \( d_{s'} \) is actually not present in our final result for \( \Sigma_{s'} \), which we understand as a consequence of the fact that such couplings require time-reversal violating effects for which there is no source in the context of our calculation. We will discuss this in further detail in Section V.

In summary, the contribution from each background to the self-energy can be parametrized in the form  
\[ \Sigma_f = b_f \gamma_f + c_f \tilde{F}^{\mu\nu} u_{f\nu} \gamma_{f\mu}. \]  
(2.8)  
Therefore for the total self-energy we can write  
\[ \Sigma = V, \]  
(2.9)  
with  
\[ V^\mu = \sum_f \left( b_f u_f^\mu + c_f \tilde{F}^{\mu\nu} u_{f\nu} \right), \]  
(2.10)  
which using Eq. (1.4) can be expressed in the equivalent form  
\[ V^\mu = \sum_f \left\{ b_f u_f^\mu + c_f \left[ (u \cdot u_f) B^\mu - (B \cdot u_f) u^\mu \right] \right\}. \]  
(2.11)
III. THERMAL PROPAGATORS

A. Electron propagator

The internal fermion lines in the diagrams in Fig. 1 stand for the thermal fermion propagator in an external electromagnetic field, for which we will adopt the linearized Schwinger propagator used in Ref. [13]. We consider first the propagator for the electron, for either the normal or stream background, and use the notation $f_e = e, e'$ to refer to any of them. Following that reference, we write the Schwinger propagator (in the vacuum) in the form

$$S_F^{(e)} = S_0^{(e)} + S_B^{(e)}, \quad (3.1)$$

where $S_0^{(e)}$ is the free propagator

$$S_0^{(e)} = \frac{\not{p} + m_e}{p^2 - m_e^2 + i\epsilon}, \quad (3.2)$$

and $S_B^{(e)}$ is the linearized $B$-dependent part of the Schwinger propagator for the electron[14],

$$S_B^{(e)} = \frac{eBG_e}{(p^2 - m_e^2 + i\epsilon)^2}, \quad (3.3)$$

with

$$G_e(p) = \gamma_5 \left[ (p \cdot b) \not{u} - (p \cdot u) \not{b} + m_e \not{b} \right]. \quad (3.4)$$

Ordinarily the thermal propagator is then constructed by the rule[15],

$$S_{11}^{(e)} = S_F^{(e)} - \left[ S_F^{(e)} - \bar{S}_F^{(e)} \right] \eta(p \cdot u), \quad (3.5)$$

with $\eta$ defined, as usual (see below), in terms of the distribution function of the background electrons, and

$$\bar{S}_F^{(e)} = \gamma^0 S_F^{(e)\dagger} \gamma^0. \quad (3.6)$$

For the calculation in this work the propagator for each electron background ($f_e = e, e'$) is taken to be similar to Eq. (3.5), but with $\eta(p \cdot u) \rightarrow \eta_{f_e}(p \cdot u_{f_e})$, i.e.,

$$S_{11}^{(f_e)} = S_F^{(e)} - \left[ S_F^{(e)} - \bar{S}_F^{(e)} \right] \eta_{f_e}(p \cdot u_{f_e}), \quad (3.7)$$

with $\bar{S}_F^{(e)}$ as defined in Eq. (3.1). For any background fermion $f$ the function $\eta_f(p \cdot u_f)$ is given by

$$\eta_f(p \cdot u_f) = \theta(p \cdot u_f) f_f(p \cdot u_f) + \theta(-p \cdot u_f) f_f(-p \cdot u_f), \quad (3.8)$$
with

\[
f_f(x) = \frac{1}{e^{\beta_f(x-\mu_f)} + 1},
\]

\[
f_f(x) = \frac{1}{e^{\beta_f(x+\mu_f)} + 1},
\]

(3.9)

\(\beta_f\) and \(\mu_f\) being the inverse temperature and the chemical potential of the background. \(S_{11}^{(f)}\) can be written in the form

\[
S_{11}^{(f)} = S_0^{(e)} + S_B^{(e)} + S_T^{(f)} + S_{TB}^{(f)}.
\]

(3.10)

where \(S_0^{(e)}\) and \(S_B^{(e)}\) are the background-independent terms, given above, while \(S_T^{(f)}\) is the thermal, but \(B\)-independent, part

\[
iS_T^{(f)} = -2\pi\delta(p^2 - m_e^2)\eta_{f_e}(p \cdot u_{f_e})(\not{p} + m_e),
\]

(3.11)

and

\[
iS_{TB}^{(f)} = (eB)2\pi\delta'(p^2 - m_e^2)\eta_{f_e}(p \cdot u_{f_e})G_e(p),
\]

(3.12)

which is the part that is of most interest to us. Notice that the factor \(G_e(p)\) that appears here is the same for both the normal and stream backgrounds, defined in Eq. (3.4), since it refers to the \(B\)-dependent part of the vacuum Schwinger propagator. It is useful to note that

\[
BG_e(p) = \tilde{F}^{\mu\nu}p_\nu\gamma_\mu\gamma_5 + \frac{i}{2}m_e\tilde{F}^{\mu\nu}\sigma_{\mu\nu}\gamma_5,
\]

(3.13)

where \(\tilde{F}_{\mu\nu}\) is given in Eq. (1.4) and \(\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]\).

**B. Nucleon propagator**

A nucleon (\(N = n, p\)) has an anomalous magnetic moment coupling that also contributes to the \(B\)-dependent part of the neutrino thermal self-energy. The formula analogous to Eq. (3.10) for the thermal Schwinger propagator for a nucleon including the anomalous magnetic moment coupling, was obtained in Ref. [13]. Adapting that result to our case, the thermal Schwinger propagator for a nucleon in either the normal or stream background (\(f_N = N, N'\)) is

\[
S_{11}^{(f_N)} = S_0^{(N)} + S_B^{(N)} + S_T^{(f_N)} + S_{TB}^{(f_N)},
\]

(3.14)
where $S_0^{(N)}$ is the free nucleon propagator, and

$$S_B^{(N)} = \frac{e_NBG_N(p) + \kappa_NBH_N(p)}{(p^2 - m_N^2 + i\epsilon)^2};$$

$$iS_T^{(fN)} = -2\pi\delta(p^2 - m_N^2)\eta_{fN}(p \cdot u_{fN})(\not{p} + m_N),$$

$$iS_{TB}^{(fN)} = 2\pi\delta'(p^2 - m_N^2)\eta_{fN}(p \cdot u_{fN})[e_NBG_N(p) + \kappa_NBH_N(p)], \tag{3.15}$$

Here we denote by $u_{fN}^\mu$ the velocity four-vector of the nucleon background, while $e_N$ and $\kappa_N$ stand for the nucleon electric charge and anomalous magnetic moment, respectively. As in the electron case above, our working rule is that Eq. (3.15) holds for either a normal or stream nucleon background, with the corresponding choice of $u_{fN}^\mu$. $G_N(p)$ is the same function given by Eqs. (3.4) and (3.13), with the substitution $m_e \rightarrow m_N$, while

$$H_N(p) = (\not{p} + m_N)\gamma_5\not{p}(\not{p} + m_N). \tag{3.16}$$

In analogy with Eq. (3.13), here we note that $H_N(p)$ can be rewritten in the form

$$BH_N(p) = (\not{p} + m_N)\frac{i}{2}\tilde{F}^{\mu\nu}\sigma_{\mu\nu}\gamma_5(\not{p} + m_N). \tag{3.17}$$

IV. CALCULATION

A. W-diagram

For a given background $f_e = e, e', \gamma$ the W diagram in Fig. 1 gives a contribution to the neutrino thermal self-energy

$$-i\Sigma_{f_e}^{(W)} = \left(\frac{-ig}{\sqrt{2}}\right)^2 \frac{i}{m_W^2} \int \frac{d^4p}{(2\pi)^4} \gamma^\mu LiS_{11}^{(f_e)}(p)\gamma_\mu. \tag{4.1}$$

Using Eq. (3.7) and retaining only the background-dependent part,

$$\Sigma_{f_e}^{(W)} = \left(\Sigma_{f_e}^{(W)}\right)_T + \left(\Sigma_{f_e}^{(W)}\right)_{TB}, \tag{4.2}$$

where

$$-i\left(\Sigma_{f_e}^{(W)}\right)_T = \left(\frac{-ig}{\sqrt{2}}\right)^2 \frac{i}{m_W^2} \int \frac{d^4p}{(2\pi)^4} \gamma^\mu LiS_{TT}^{(f_e)}(p)\gamma_\mu,$$

$$-i\left(\Sigma_{f_e}^{(W)}\right)_{TB} = \left(\frac{-ig}{\sqrt{2}}\right)^2 \frac{i}{m_W^2} \int \frac{d^4p}{(2\pi)^4} \gamma^\mu LiS_{TB}^{(f_e)}(p)\gamma_\mu. \tag{4.3}$$
which correspond to the $B$-independent and $B$-dependent contribution to the neutrino thermal self-energy, respectively. By simple Dirac algebra they can be expressed in the form

$$
\left(\Sigma^{(W)}_{f_c}\right)_T = \left(\frac{g^2}{m_W^2}\right)J_{f_c},
$$

$$
\left(\Sigma^{(W)}_{f_c}\right)_{TB} = -\left(\frac{eg^2}{m_W^2}\right)\tilde{\Gamma}^{\mu\nu}J_{f_\nu\gamma}\mu, \tag{4.4}
$$

where we have used Eq. (3.13), and

$$
I_{f\mu} = \int \frac{d^4p}{(2\pi)^3}\delta(p^2 - m_f^2)\eta_f(p \cdot u_f)p_\mu, \tag{4.45}
$$

$$
J_{f\mu} = \int \frac{d^4p}{(2\pi)^3}\delta'(p^2 - m_f^2)\eta_f(p \cdot u_f)p_\mu. \tag{4.5}
$$

The integrals $I_{f\mu}, J_{f\mu}$ must be of the form

$$
I_{f\mu} = \tilde{I}_f u_{f\mu}, \tag{4.6}
$$

$$
J_{f\mu} = \tilde{J}_f u_{f\mu}, \tag{4.6}
$$

with the coefficients given by

$$
\tilde{I}_f = \int \frac{d^4p}{(2\pi)^3}\delta(p^2 - m_f^2)\eta_f(p \cdot u_f)p \cdot u_f, \tag{4.7}
$$

$$
\tilde{J}_f = \int \frac{d^4p}{(2\pi)^3}\delta'(p^2 - m_f^2)\eta_f(p \cdot u_f)p \cdot u_f, \tag{4.7}
$$

which can then be evaluated in any reference frame since they are scalar integrals. A convenient one to use is the rest frame of each background $f$. Denoting the energy and momentum of the background particles in that reference frame by $\mathcal{E}_f$ and $\vec{P}$, a straightforward evaluation yields

$$
\tilde{I}_f = \frac{1}{4}(n_f - n_{\bar{f}}), \tag{4.8}
$$

$$
\tilde{J}_f = -\frac{1}{2} \int \frac{d^3P}{(2\pi)^32\mathcal{E}_f} d\mathcal{E}_f \left(f_f(\mathcal{E}_f) - f_f(\mathcal{E}_f)\right), \tag{4.8}
$$

where

$$
n_{f,\bar{f}} = 2 \int \frac{d^3P}{(2\pi)^3} f_{f,\bar{f}}(\mathcal{E}_f), \tag{4.9}
$$

and

$$
\mathcal{E}_f = \sqrt{\vec{P}^2 + m_f^2}. \tag{4.10}
$$
Therefore

\[
\begin{align*}
\left( \Sigma_f^{(W)} \right)_T &= b_f^{(W)} \phi_f, \\
\left( \Sigma_f^{(W)} \right)_{TB} &= c_f^{(W)} F^{\mu\nu} u_{f\nu} \gamma_\mu,
\end{align*}
\]

where

\[
\begin{align*}
b_f^{(W)} &= \frac{g^2}{4m_W^2} (n_{f_e} - n_{\bar{f}_e}), \\
c_f^{(W)} &= \left( \frac{e g^2}{2m_W^2} \right) \left\{ \frac{d^3 P}{(2\pi)^3} \frac{d}{dE_e} \left( f_{f_e}(E_e) - f_{\bar{f}_e}(E_e) \right) \right\}.
\end{align*}
\]

\[
(4.11)
\]

\[
B. \ Z\text{-diagram}
\]

For the \(Z\) diagram we need the following neutral current couplings,

\[
L_Z = -\frac{g}{2 \cos \theta_W} Z^\mu \left[ \sum_{s=e,n,p} \bar{s} \gamma_\mu (X_s + Y_s \gamma_5) s + \sum_\ell \bar{\nu}_\ell \gamma_\mu \nu_\ell \right],
\]

where

\[
\begin{align*}
X_e &= -\frac{1}{2} + 2 \sin^2 \theta_W, \\
Y_e &= \frac{1}{2},
\end{align*}
\]

and (Verify these, and give the reference)

\[
\begin{align*}
X_p &= -X_e, \\
X_n &= -\frac{1}{2}, \\
Y_n &= -Y_p = \frac{1}{2} g_A.
\end{align*}
\]

The parameters \(X_N, Y_N\) are the vector and axial vector form factors of the nucleon neutral-current at zero momentum transfer. In Eq. (4.15), \(g_A\) stands for the normalization constant of the axial charged vector of the nucleon, \(g_A = 1.26\).

The \(Z\) diagram contribution is

\[
-\frac{1}{2} \Sigma_f^{(Z)} = \left( \frac{-ig}{2 \cos \theta_W} \right)^2 \left( \frac{i}{m_Z^2} \right) \left\{ \int \frac{d^4 p}{(2\pi)^4} (-1) \text{Tr} \gamma_\mu (X_f + Y_f \gamma_5) i S_{11}^{(f)}(p) \right\} \gamma^\mu,
\]

and retaining only the background-dependent part,

\[
\Sigma_f^{(Z)} = \left( \Sigma_f^{(Z)} \right)_T + \left( \Sigma_f^{(Z)} \right)_{TB},
\]

\[
(4.17)
\]
where
\[
\left(\Sigma_f^{(Z)}\right)_T = -\left(\frac{g^2}{4m_W^2}\right) \left\{ \frac{1}{(2\pi)^4} \int \frac{d^4p}{(2\pi)^4} \text{Tr} \gamma_\mu (X_f + Y_f) iS_f^{(f)}(p) \right\} \gamma_\mu ,
\]
\[
\left(\Sigma_f^{(Z)}\right)_{TB} = -\left(\frac{g^2}{4m_W^2}\right) \left\{ \frac{1}{(2\pi)^4} \int \frac{d^4p}{(2\pi)^4} \text{Tr} \gamma_\mu (X_f + Y_f) iS_f^{(f)}(p) \right\} \gamma_\mu .
\] (4.18)

We consider the contributions from the electron and the nucleon backgrounds separately.

1. Electron background contribution

In terms of the integrals \(I_{f\mu}, J_{f\mu}\) defined in Eq. (4.5),
\[
\left(\Sigma_f^{(Z)}\right)_T = \left(\frac{g^2 X_e}{m_W^2}\right) I_{f_e},
\]
\[
\left(\Sigma_f^{(Z)}\right)_{TB} = \left(\frac{eg^2 Y_e}{m_W^2}\right) \tilde{F}^{\mu\nu} J_{f_e\nu} \gamma_\mu .
\] (4.19)

Comparing with Eq. (4.4), we then obtain
\[
\left(\Sigma_f^{(Z)}\right)_T = X_e \left(\Sigma_f^{(W)}\right)_T ,
\]
\[
\left(\Sigma_f^{(Z)}\right)_{TB} = -Y_e \left(\Sigma_f^{(W)}\right)_{TB} ,
\] (4.20)
with the final expressions for \(\left(\Sigma_f^{(W)}\right)_T\) and \(\left(\Sigma_f^{(W)}\right)_{TB}\) given in Eq. (4.11).

2. Nucleon background contribution

We consider here the nucleon backgrounds. As with the electron case, what interests us is the contribution to the neutrino self-energy arising from \(S_T^{(fN)}\) and \(S_{TB}^{(fN)}\), corresponding to the \(B\)-independent and \(B\)-dependent contributions of each background \((f_N = N, N')\). Denoting them by \(\left(\Sigma_f^{(Z)}\right)_T\) and \(\left(\Sigma_f^{(Z)}\right)_{TB}\), respectively. From Eq. (4.18),
\[
-i \left(\Sigma_f^{(Z)}\right)_X = \left(\frac{-i g}{2 \cos \theta_W}\right) \left(\frac{i}{m_Z^2}\right) \left\{ \frac{1}{(2\pi)^4} \int \frac{d^4p}{(2\pi)^4} (-1) \text{Tr} \gamma_\mu (X + Y_N) iS_X^{(fN)}(p) \right\} \gamma_\mu ,
\] (4.21)
where \(X\) stands for either subscript, \(T\) or \(TB\). The calculation involving \(S_T^{(fN)}(p)\) and the \(G_N(p)\) term of \(S_{TB}^{(fN)}(p)\) follows the steps that lead to Eq. (4.19). On the other hand, using Eq. (3.17) it follows that
\[
\text{Tr} \gamma_\mu (X + Y_N) BH_N(p) = -8m_N Y_N \tilde{F}^{\mu\nu} p_\nu .
\] (4.22)
Thus we obtain

\[
\begin{align*}
\left( \Sigma^{(Z)}_{f_N} \right)_T &= \left( \frac{g^2 X_N}{m_W^2} \right) I_{f_N}, \\
\left( \Sigma^{(Z)}_{f_N} \right)_{TB} &= \left( \frac{g^2 Y_N}{m_W^2} \right) (e_N + 2m_N \kappa_N) \tilde{F}^{\mu\nu} J_{f_N,\nu\gamma_\mu},
\end{align*}
\]  

where \( I_{f_N} \) and \( J_{f_N} \) are the integrals given by Eq. (4.5), with \( f = f_N \). Thus, the contribution of a nucleon background to the neutrino self-energy is given by

\[
\Sigma^{(Z)}_{f_N} = b_{f_N,\phi} f_N + c_{f_N} \tilde{F}^{\mu\nu} u_{f_N,\nu\gamma_\mu},
\]  

with

\[
\begin{align*}
b_{f_N} &= \frac{g^2 X_N}{4m_W^2} (n_{f_N} - n_{\bar{f}_N}), \\
c_{f_N} &= -\left( \frac{g^2 Y_N}{2m_W^2} \right) (e_N + 2m_N \kappa_N) \left\{ \int \frac{d^3 P}{(2\pi)^3 2E_N} dE_N \left( f_{\Phi_N}(E_N) - f_{\bar{f}_N}(E_N) \right) \right\}.
\end{align*}
\]

C. Summary

Using the results given in Eqs. (4.11), (4.20) and (4.24), the thermal self-energy for each neutrino flavor

\[
\Sigma = \sum_{f=e,e',n,n',p,p'} b_{f_N} \phi_{f_N} + c_{f_N} \tilde{F}^{\mu\nu} u_{f_N,\nu\gamma_\mu},
\]  

where, for \( f_e = e, e' \),

\[
\begin{align*}
b_{f_e} &= \frac{g^2}{4m_W^2} (n_{f_e} - n_{\bar{f}_e}) \chi_{f_e} (\nu_e) \chi_{f_e} (\nu_{\mu,\tau}), \\
c_{f_e} &= \left( \frac{eg^2}{2m_W^2} \right) \left\{ \int \frac{d^3 P}{(2\pi)^3 2E_N} dE_N \left( f_{\Phi_N}(E_N) - f_{\bar{f}_e}(E_N) \right) \right\} \chi_{f_e} (\nu_e) \chi_{f_e} (\nu_{\mu,\tau})
\end{align*}
\]

The formulas for \( b_{f_N} \) and \( c_{f_N} \) \((f_N = N, N') \) are given in Eq. (4.25) and they hold for any neutrino flavor. Thus, \( \Sigma \) is of the expected form discussed in Section II [e.g., Eq. (2.8)], in particular with \( d_{\nu} = 0 \) as anticipated there, with the coefficients \( b_f, c_f \) given above in Eqs. (4.28) and (4.25).
V. DISCUSSION AND CONCLUSIONS

A. Dispersion relations

For the purpose of determining the dispersion relations we use the expression for $\Sigma$ in terms of $V^\mu$, Eq. (2.9). The equation for the propagating neutrino modes, Eq. (2.2), then becomes

\begin{equation}
(\vec{k} - \vec{V}) \psi_L = 0, \quad (5.1)
\end{equation}

and the dispersion relations are obtained by solving

\begin{equation}
k^0 - V^0 = \pm |\vec{k} - \vec{V}|. \quad (5.2)
\end{equation}

Remembering that $V^\mu$ does not depend on $k$ (to the order $1/m_W^2$ that we are considering in this work), the solutions are $k^0 = \omega_\pm(\vec{k})$, where

\begin{equation}
\omega_\pm(\vec{k}) = V^0 \pm [|\vec{k}| - \hat{k} \cdot \vec{V}], \quad (5.3)
\end{equation}

with $\hat{k}$ being the unit vector along the direction of propagation. The dispersion relation for the neutrino and the antineutrino are identified as usual,

\begin{align*}
\omega_\nu(\vec{k}) &\equiv \omega_+(\vec{k}), \\
\omega_{\bar{\nu}}(\vec{k}) &\equiv -\omega_-(\vec{k}),
\end{align*}

which to the lowest order yield

\begin{equation}
\omega_{\nu,\bar{\nu}}(\vec{k}) = |\vec{k}| \pm \delta, \quad (5.5)
\end{equation}

where the upper(lower) sign holds for the neutrino(antineutrino) and

\begin{equation}
\delta = V^0 - \hat{k} \cdot \vec{V}. \quad (5.6)
\end{equation}

Explicitly, using Eq. (2.11),

\begin{equation}
\delta = \sum_f \delta_f \quad (5.7)
\end{equation}

with

\begin{equation}
\delta_f = b_f u_f^0 + c_f \vec{B} \cdot \vec{u}_f - b_f \hat{k} \cdot \vec{u}_f - c_f u_f^0 \hat{k} \cdot \vec{B}. \quad (5.8)
\end{equation}

The fact that the dispersion relation in the presence of a magnetic field has an anisotropic term proportional to $\hat{k} \cdot \vec{B}$ is well known. As we have already mentioned many of its possible
effects have been studied and more complete calculations involving higher order contributions have been performed in the references cited. The above results show that in the presence of a stream (with a velocity four-vector $u^\mu_f$ relative to the normal background), the dispersion relation acquires another anisotropic term of the form $\hat{k} \cdot \vec{u}_f$. Furthermore, the standard isotropic (Wolfenstein) term receives an additional contribution proportional to $\vec{B} \cdot \vec{u}_f$ that involves the stream velocity and the magnetic field.

It has been suggested repeatedly in the literature that the anisotropic terms in the neutrino dispersion relations can have effects in several astrophysical environments including pulsars[10] and the dynamics of supernovas[12]. The resonance condition for neutrino oscillations in a magnetic field depends on $\hat{k} \cdot \vec{B}$, and therefore is satisfied at different depths, corresponding to different densities and temperatures. This difference results in an asymmetry in the momentum distributions of the neutrinos. In the presence of a stream background, the neutrino asymmetry will depend on the relative orientation of the three vectors $\vec{k}, \vec{B}, \vec{u}_f$.

As an example, let us consider specifically the two-stream electron background. Denoting the velocity four-vector of the stream by $v^\mu$, then

$$\delta = b_e + c_e' \vec{B} \cdot \vec{v} + b_e v^0 - b_e' \hat{k} \cdot \vec{v} - \left( c_e + c_e' v^0 \right) \hat{k} \cdot \vec{B}. \quad (5.9)$$

We wish to compare the size of the term proportional to $\hat{k} \cdot \vec{v}$ relative to $\hat{k} \cdot \vec{B}$, thus we consider the quantity

$$r = \frac{b_e'}{(c_e + c_e' v^0) B}. \quad (5.10)$$

For simplicity we will take $v^0 \sim 0$, and for definiteness we will assume that the two backgrounds are described by the classical thermal distribution functions. In that case,

$$\frac{df_f}{dE_f} = -\beta_f f_f, \quad (5.11)$$

and similarly for $f_{\bar{f}}$, and therefore,

$$c_{e'} \sim \frac{g^2}{m_W^2} \frac{\Delta N_{f}}{B_e} \begin{cases} \frac{m_e}{T_{f_e}} & (\text{NR limit } T_{f_e} \ll m_e) \\ \left( \frac{m_{\bar{f}}}{T_{f_{\bar{f}}}} \right)^2 & (\text{ER limit } T_{f_{\bar{f}}} \gg m_e) \end{cases} \quad (5.12)$$

where we have defined $\Delta N_f = n_f - n_{\bar{f}}$ and $B_e = m_e^2/e$. On the other hand,

$$b_{e'} \sim \frac{g^2}{m_W^2} \Delta N_{e'}, \quad (5.13)$$
in any case. We can consider two possibilities, according to whether \( c_e \gg c_e' \) or the way around. For definiteness let us consider the case \( c_e \gg c_e' \). This situation can occur, for example, if the temperature of the normal background is greater than the temperature of the stream. In this case

\[
r = \frac{b_N}{Bc_e} \sim \frac{1}{B/B_c} \left( \frac{\Delta N_e}{\Delta N_e'} \right) \begin{cases} \left( \frac{T_e}{m_e} \right) & (T_e \ll m_e) \\ \left( \frac{T_e}{m_e} \right)^2 & (T_e \gg m_e) \end{cases}
\]

(5.14)

The indication is that it is possible that \( r \sim 1 \) for acceptable values of the parameters involved. In other words, it is conceivable that there are environments where the conditions are such that the asymmetries due to the \( \hat{k} \cdot \nu \) and \( \hat{k} \cdot \vec{B} \) terms can be comparable.

Similar results are obtained in other cases as well. To include other backgrounds we just have to add to \( \delta \) the corresponding \( \delta_f \). For example, for a stream nucleon background,

\[
\delta_{N'} = b_{N'} u_{N'}^0 + c_{N'} \vec{B} \cdot \vec{u}_{N'} - b_{N'} \hat{k} \cdot \vec{u}_{N'} - c_{N'} u_{N'}^0 \hat{k} \cdot B.
\]

(5.15)

The quantitative estimates of the effects in realistic situations of the additional asymmetric terms that we have reported above involve stellar astrophysics studies that are beyond the scope of the present work. But as we have suggested they are subjects worth of further study.

B. Comment on the \( F^{\mu\nu} u_{f\nu} \gamma^\mu \) term

The calculations of Section IV confirm explicitly that the \( d_f \) term in the general expression for the thermal self-energy [Eq. (2.3)] is zero, as it was anticipated in Section II. This result can be understood by making reference to previous work[16] where the conditions under which such dipole-type couplings may appear in the neutrino effective Lagrangian were studied. To establish contact with that reference, notice that the terms involving \( c_f, d_f \) in Eq. (2.3) are represented by the operators

\[
O_M' = c_f \tilde{F}^{\mu\nu} u_{f\nu} \bar{\nu}_L \gamma^\mu \nu_L, \quad O_E' = d_f F^{\mu\nu} u_{f\nu} \bar{\nu}_L \gamma^\mu \nu_L,
\]

(5.16)

in the neutrino effective Lagrangian. The coefficients \( c_f, d_f \) here correspond to the coefficients that were denoted by \( d_{M,E}' \) there, respectively (evaluated at \( k = 0 \)). Borrowing the results of that reference [e.g., Eqs. (14) and (16b)] the presence of \( O_E' \) requires time-reversal violation at some level. Since there is no source of \( T \) violation in the context of our calculation, the
$O_E'$ term is not generated. On the other hand $O_M'$ is even under time-reversal but odd under $CP$, and therefore it can be generated if the background is $CP$ asymmetric.

Here we would like to point out the following. In the presence of non-constant fields (non-static and/or non-homogeneous) there can be additional terms involving the derivatives of $F_{\mu\nu}$ and/or $\tilde{F}_{\mu\nu}$. For example, limiting ourselves to terms with first derivatives, consider the following

$$O_E'' = h_f \left( \partial^{\lambda} F_{\lambda\mu} \right) u_f \bar{u}_f \bar{\nu}_L \gamma_{\mu} \nu_L.$$  \hspace{1cm} (5.17)

This term is even under $CP$ and even under time-reversal. Therefore, it can be present in the effective Lagrangian without implying time-reversal violation and even if the background and the interactions are $CP$-symmetric. This contrasts with $O_M'$ which is $CP$-odd and therefore does not exist if the background is $CP$-symmetric (neglecting the $CP$ violating effects of the weak interactions). $O_E''$ can give additional anisotropic contributions to the neutrino dispersion relation [e.g., Eq. (5.9)] that are not present otherwise, with different kinematic properties from the constant field case. For example, in the presence of a static but inhomogeneous field, it gives a term involving the gradient of $\hat{k} \cdot (\vec{v} \times \vec{B})$.

Of course this type of term (with derivatives of the electromagnetic field) will not appear in the approach we are using in the present work based on the electron thermal propagator in a constant $B$ field. Instead we would have to resort to the type of approach employed in Ref. [6], which is based on calculating the electromagnetic vertex first, and then taking the static limit in a suitable way to obtain the self-energy in the (inhomogeneous) external field. This calculation is underway and the results will be presented separately.

\section*{C. Conclusions}

To summarize, in this work we have studied the propagation of a neutrino in a magnetized two stream plasma system. Specifically, we considered a medium that consists of a normal electron background plus another electron stream background that is moving as a whole relative to the normal background. In addition, we assume that in the rest frame of the normal background there is a constant magnetic field.

Using the thermal Schwinger propagator for the electrons in the medium we have calculated the neutrino self-energy in such environment, linearized in $B$ and to the leading order $O(1/m_W^2)$ terms. The results of the calculation are summarized in Eqs. (4.25)-(4.28).
From the self-energy the dispersion relations were obtained in the standard way, and the corresponding formulas are summarized in Eqs. (5.5)-(5.9).

In the presence of the stream (with velocity $\vec{v}$ relative to the normal background), the dispersion relation acquires an anisotropic term of the form $\hat{k} \cdot \vec{v}$ in addition to the well known term of the form $\hat{k} \cdot \vec{B}$, and the standard isotropic term receives an additional contribution proportional to $\vec{B} \cdot \vec{v}$ that involves the stream velocity and the magnetic field. We explained why a term of the form $\hat{k} \cdot (\vec{v} \times \vec{B})$ does not appear in the dispersion relation, due to time-reversal invariance, and why a term of similar kinematic form can appear in the presence of an inhomogeneous magnetic field, involving the derivative of the field. We have given the explicit formulas for the dispersion relations and outlined possible generalizations, for example to include the nucleon contribution or the case of non-homogeneous fields. We have made simple estimates of the magnitude of the asymmetric terms proportional to $\hat{k} \cdot \vec{v}$ and $\hat{k} \cdot \vec{B}$, and found that they can be comparable for acceptable values of the parameters involved.

In the context of plasma physics the propagation of photons in two stream plasma systems is a well studied subject. Here we have started to carry out an analogous study for the case of neutrinos. The present work is limited in several ways, for example by restricting ourselves to an electron background and stream, the linear approximation in the $B$ field, and the calculation of only the leading $O(1/m^2_{\nu})$ terms. However, the results reveal interesting effects that are potentially important in several physical contexts, such as supernova dynamics and gamma-ray bursts physics where the effects of such systems are a major focus of current research, and in this sense our work motivates and paves the way for further calculations without these simplifications.

**ACKNOWLEDGMENTS**

S.S is thankful to Japan Society for the promotion of science (JSPS) for the invitation fellowship. The work of S.S. is partially supported by DGAPA-UNAM (México) Project
No. IN110815.

[1] L. Wolfenstein, Phys. Rev. D 17, 2369 (1978); P. Langacker, J. P. Leville, and J. Sheiman, ibid. 27, 1228 (1983); S. P. Mikheyev and A. Yu. Smirnov, Sov. J. Nucl. Phys. 42, 913 (1985).

[2] D. Nötzold and G. Raffelt, Nucl. Phys. B, 924 (1988); P. B. Pal and T. N. Pham, Phys. Rev. D 40, 714 (1989); José F. Nieves, “Neutrinos in a medium,” ibid. 40, 866 (1989); J. C. D’Olivo, José F. Nieves, and Manuel Torres, “Finite temperature corrections to the effective potential of neutrinos in a medium,” ibid. 46, 1172 (1992).

[3] See, Georg G. Raffelt, Stars as Laboratories for Fundamental Physics (University of Chicago Press, Chicago, 1996) and references therein.

[4] J. T. Pantaleone, “Neutrino oscillations at high densities,” Phys. Lett. B 287, 128 (1992), Neutrino oscillations at high densities; S. Samuel, “Neutrino oscillations in dense neutrino gases,” Phys. Rev. D 48, 1462 (1993), Neutrino oscillations in dense neutrino gases; V. A. Kostelecky, J. T. Pantaleone, and S. Samuel, “Neutrino oscillation in the early universe,” Phys. Lett. B 315, 46 (1993), Neutrino oscillation in the early universe; H. Duan, G. M. Fuller, and Y. Z. Qian, “Collective neutrino oscillations,” Ann. Rev. Nucl. Part. Sci. 60, 569 (2010), http://arXiv.org/abs/1001.2799.

[5] See for example, Evgeny Akhmedov and Alessandro Mirizzi, “Another look at synchronized neutrino oscillations,” (2016), Another look at synchronized neutrino oscillations, http://arxiv.org/abs/1601.07842, and references therein.

[6] J. C. D’Olivo, José F. Nieves, and P. B. Pal, “Electromagnetic properties of neutrinos in a background of electrons,” Phys. Rev. D 40, 3679 (1989).

[7] A. Erdas, “Neutrino self-energy in an external magnetic field,” Phys. Rev. D 80, 113004 (2009).

[8] N. Kazarian A. Ioannisian, “Transition radiation by neutrinos at an edge of magnetic field,” (2017), Transition Radiation by Neutrinos at an Edge of Magnetic Field, https://arxiv.org/abs/1702.00943.

[9] See for example, R. Shaisultanov, Y. Lyubarsky, and D. Eichler, “Stream instabilities in relativistically hot plasma,” Astrophys. J. 744, 182 (2012), http://arxiv.org/abs/1104.0521; A. Yalinewich and M. Gedalin, “Instabilities of relativistic counter-streaming proton beams
in the presence of a thermal electron background,” Physics of Plasmas 17, 062101 (2010); A. R. Soto-Chavez, S. M. Mahajan, and R. D. Hazeltine, “Two-fluid temperature-dependent relativistic waves in magnetized streaming pair plasmas,” Phys. Rev. E 81, 026403 (2010); H. Che, J. F. Drake, M. Swisdak, and P. H. Yoon, “Nonlinear development of streaming instabilities in strongly magnetized plasmas,” Phys. Rev. Lett. 102, 145004 (2009), http://arxiv.org/abs/0903.1311; V. N. Oraevsky and V. B. Semikoz, “Neutrino-driven streaming instability of spin waves in dense magnetized plasma,” Phys. Atom. Nucl. 66, 466 (2003).

[10] A. Kusenko and G. Segre, “Velocities of pulsars and neutrino oscillations,” Phys. Rev. Lett. 77, 4872 (1996).

[11] T. Maruyama, T. Maruyama, J. Hidaka, T. Kajino, N. Yasutake, T. Kuroda, M.K. Cheoun, C.Y. Ryu, and Grant J. Mathews, “Rapid spin deceleration of magnetized proto-neutron stars via asymmetric neutrino absorption,” Phys. Rev. C 89, 035801 (2014), http://arxiv.org/abs/1301.7495.

[12] See for example, S. Sahu and V. M. Bannur, “Effect of random magnetic field on active sterile neutrino conversion in the supernova core,” Phys. Rev. D 61, 023003 (2000); H. Duan and Y. Z. Qian, “Neutrino processes in strong magnetic fields and implications for supernova dynamics,” ibid. 69, 123004 (2004); A. A. Gvozdev and I. S. Ognev, “Influence of a strong magnetic field on the neutrino heating of a supernova shock,” Astron. Lett. 31, 442 (2005).

[13] José F Nieves, “Nucleon contribution to the induced charge of neutrinos in a matter background and a magnetic field,” Phys. Rev. D 70, 073001 (2004).

[14] We follow the convention that \( e \) stands for the electric charge of the electron (\( e < 0 \)).

[15] See for example, P. Elmfors, D. Grasso, and G. Raffelt, Nucl. Phys. B 479, 3 (1996), and references therein. Since we will restrict ourselves to calculate the real part of the self-energy and dispersion relation, we need only the 11 element of the thermal propagator.

[16] José F. Nieves and P. B. Pal, “Electromagnetic properties of neutrinos in a medium,” Phys. Rev. D 40, 1693 (1989).