Dressed-state approach to a simple radiating atom

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We consider a system consisting of an atom in the dipole approximation, coupled to the electromagnetic field. Using recently introduced renormalized coordinates and dressed states, we give a non-perturbative solution to the atom radiation process. From this formalism a non-exponential decay law naturally emerges. However, at least for low frequencies and weak coupling, there is only a very small, practically non-detectable, deviation from a purely exponential decay, in agreement with all previous results in the literature.

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I. INTRODUCTION

Actually, apart from computer calculations, the most currently used method to solve coupled equations describing the physics of interacting systems is perturbation theory. This is a common feature to different branches of physical sciences, such as celestial mechanics, field theory and statistical physics. The method, originally introduced to deal with orbital problems in celestial mechanics, has been extremely successful since then, giving remarkably accurate results in Quantum Electrodynamics and in Weak interactions. In high energy physics, asymptotic freedom allows to apply Quantum Chromodynamics in its perturbative form and very important results have been obtained in this way in the last decades.

However, in spite of its wide applicability, there are situations where the use of perturbation theory is not possible, as in the low energy domain of Quantum Chromodynamics, where confinement of quarks and gluons takes place. In this branch of physics, few analytical approaches in the context of Quantum field theory are available up to the present moment (in this particular subject the interested reader can find a recent account in [1]). There are also other situations, in the domain of Cavity Electrodynamics and Quantum Optics, where perturbation methods are of little usefulness, for instance, the observation of resonant effects associated to the coupling of atoms with strong radio-frequency fields. The theoretical understanding of these effects on perturbative grounds requires the calculation of very high-order terms in perturbative series, which makes the standard Feynman diagrams technique practically unreliable [2]. Methods to non-perturbatively deal with such kind of systems, have been introduced originally in Refs. [3] and [4], and since then they have been used to investigate several situations involving the interaction of atoms and electromagnetic fields [5–7].

In this paper we give a non-perturbative treatment to the system composed of an atom (approximated by an harmonic oscillator) interacting with the electromagnetic field. In the dipole approximation, we reduce the problem to a system of a mechanical oscillator (the bare atom) coupled to the infinite set of the harmonic modes of the (bare) field. The emergence of an exponential decay of an excited state of the oscillator and also renormalization aspects of this problem have been thoroughly discussed in the literature. See for instance Refs. [8–15]. Here we use an alternative approach, based on precisely defined renormalized coordinates introduced and employed recently to describe the coupling of a mechanical oscillator with a scalar field in previous papers [16–18]. In fact, in the quoted papers these coordinates have been denoted as dressed coordinates but, as explained in the paragraphs following Eq. (26), we presently think that the denomination of renormalized coordinates is more appropriate for them. It should be noticed that our definition of dressed states is not the same of the collective dressed states introduced in previous works in the literature [3,4]. In particular our dressed states are not eigenfunctions of the interacting Hamiltonian.

To describe the radiation process, having as initial condition that only the mechanical oscillator (the atom) be excited, we do not proceed as usual, considering the interaction term in the Hamiltonian written in terms of the bare oscillator coordinates and field modes as a perturbation, which induces transitions among the eigenstates of the free Hamiltonian. We introduce renormalized coordinates for, respectively the mechanical oscillator and the

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electromagnetic field modes. In terms of these new coordinates, dressed states can be consistently defined in section III and in terms of these a non-perturbative treatment of the radiation process is possible. Unless explicitly stated all along the paper we use natural units $c = 1, \hbar = 1$.

II. THE COUPLED SYSTEM

In this section we review the basic aspects of the method of reducing the system composed of a harmonic oscillator (the atom) interacting with the electromagnetic field to a system of coupled harmonic oscillators. Readers familiar with this derivation can skip the calculations and go directly to the comments at the end of this section.

In the Coulomb gauge,$$
\phi = 0 \quad \nabla . \vec{A} = 0 ,
$$
the Lagrangean for the oscillator-electromagnetic field system is given by

$$
L = \frac{m}{2} \ddot{\vec{q}}^2 - \frac{m\omega^2}{2} \vec{q}^2 + \frac{1}{2} \int d^3 x \left[ \left( \frac{\partial}{\partial t} \vec{A} \right)^2 - \left( \nabla \times \vec{A} \right)^2 \right] + \int d^3 x J . \vec{A} ,
$$

where the dots mean derivative with respect to time,

$$
J = e \delta(\vec{x} - \vec{q}) \dot{\vec{q}} ,
$$

and $e$ is the electric charge of the oscillator. We consider the particle-field system in a cube of side $a$, that is, $-a/2 < (x, y, z) < a/2$ and impose periodic boundary conditions for the field: $\vec{A}(-a/2, y, z, t) = \vec{A}(a/2, y, z, t)$, $\vec{A}(x, -a/2, z, t) = \vec{A}(x, a/2, z, t)$ and $\vec{A}(x, y, -a/2, t) = \vec{A}(x, y, a/2, t)$. At the end we will take the continuum limit $a \to \infty$.

Expanding the electromagnetic field in real modes we have

$$
\vec{A}(\vec{x}, t) = \sum_k u_k(\vec{x}) \dot{q}_k(t) ,
$$

where the $u_k(\vec{x})$'s form a real orthonormal basis and satisfy

$$
-\nabla^2 u_k(\vec{x}) = \omega_k^2 .
$$

Replacing Eq. (4) in Eq. (2), using the identity $\nabla . (\vec{F} \times \vec{G}) = \vec{G} \times \vec{F} - \vec{F} \times \vec{G}$ and the orthonormality relations of the $u_k(\vec{x})$'s, we find easily,

$$
L = \frac{m}{2} \ddot{\vec{q}}^2 - \frac{m\omega^2}{2} \vec{q}^2 - \frac{1}{2} \sum_k \left( \dot{q}_k^2 - \omega_k^2 \dot{q}_k^2 \right) + e \sum_k u_k(\vec{q}) \dot{q}_k \dot{q}_k .
$$

For all dynamical purposes the above Lagrangian is equivalent to

$$
L = \frac{m}{2} \ddot{\vec{q}}^2 - \frac{m\omega^2}{2} \vec{q}^2 - \frac{1}{2} \sum_k \left( \dot{q}_k^2 - \omega_k^2 \dot{q}_k^2 \right) - e \sum_k u_k(\vec{q}) \dot{q}_k \dot{q}_k ,
$$

since the difference between Eqs. (6) and (7) is a total time derivative.

Solving Eq. (5) with periodic boundary conditions we find

$$
u_k^{(1)}(\vec{x}) = \sqrt{\frac{2}{V}} \sin(\vec{k} . \vec{x}) , \quad 
u_k^{(2)}(\vec{x}) = \sqrt{\frac{2}{V}} \cos(\vec{k} . \vec{x}) ,
$$

where $V = a^3$, $\omega_k = |\vec{k}|$ and

$$
\vec{k} = \frac{2\pi}{a} (l, m, n) , \quad l, m, n = \pm 1, \pm 2, \pm ...
$$
Notice that there are positive and negative values of \( l, m, n \). However only one half of the total set of possible values of \( k \)'s will be admitted since \( \sin(k \vec{x}) \) and \( \cos(k \vec{x}) \) are not, respectively, linearly independent of \( \sin(-k \vec{x}) \) and \( \cos(-k \vec{x}) \). Replacing Eq. (8) in Eq. (4) and using the transversal condition \( \nabla \cdot \vec{A} = 0 \) we find
\[
\vec{k} \cdot \dot{\vec{q}}^{(\alpha)}(t) = 0 , \quad \alpha = 1, 2 ,
\]
where \( \vec{q}^{(1)}_k(t) \) and \( \vec{q}^{(2)}_k(t) \) are associated, respectively to \( u^{(1)}_k(\vec{x}) \) and \( u^{(2)}_k(\vec{x}) \). From Eq. (10) we can write \( \vec{q}^{(2)}_k(t) = \vec{\epsilon}^{(2)}_{k1} \vec{\epsilon}^{(1)}_{k2} \) and \( \vec{\epsilon}^{(1)}_{k1} \) and \( \vec{\epsilon}^{(2)}_{k2} \) are the unit orthogonal polarization vectors. Next we consider the dipole approximation, that is, we assume that the electromagnetic field does not changes significantly for points around the oscillator equilibrium position \( \vec{q} = 0 \). This allows us to make in the last term of Eq. (7) the substitution \( u_k(\vec{q}) \approx u_k(0) \). From Eq. (8) we find \( u^{(1)}_k(0) = 0 \), from which we conclude that, in the dipole approximation, the modes \( \vec{q}^{(1)}_k(t) \), do not interact with the oscillator. On the other hand \( u^{(2)}_k(0) = \sqrt{2} \). Also, if we choose one of the polarization vectors to lie on the plane defined by \( \vec{k} \) and \( \vec{q} \) we can see that only the mode corresponding to this polarization vector interacts with the oscillator since the other mode will be orthogonal to \( \vec{q} \). If we choose \( \vec{\epsilon}^{(1)}_{k1} \) as orthogonal to \( \vec{q} \), then the terms different from zero in the last sum of Eq. (7) will contain terms proportional to \( \vec{q} \cdot \vec{\epsilon}^{(2)}_{k2} q \) where \( \theta_k \) is the angle between \( \vec{q} \) and \( \vec{k} \). Then Eq. (7) can be cast in the form,
\[
L = \frac{m}{2} \dot{\vec{q}}^2 - \frac{m\omega^2}{2} q^2 - \frac{1}{2} \sum_k (\vec{q}^2_k - \omega_k^2 \dot{q}^2_k) - e \sqrt{\frac{2}{V}} \sum_k \sin \theta_k q_k q ,
\]
where we have written only the field modes that interact with the the oscillator (without labels, since there is just only one) and we have suppressed the vectorial symbol in the oscillator coordinates since now only appears its modulus. From Eq. (11) we obtain the Hamiltonian,
\[
H = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2 + \frac{e}{2} \sum_k \sin \theta_k q_k q + \frac{e^2}{m} \sum_k \sin \theta_k q^2 .
\]
Notice that in the above Hamiltonian the last term proportional to \( q^2 \) is infinity. This infinite term guarantees the positivity of the Hamiltonian. In the case of an oscillator coupled to a massless scalar field, considered in Refs. [16–18], this term is absent at the beginning and is introduced afterwards by renormalizing the oscillator frequency. Indeed, also in the present situation we can define a bare frequency \( \omega_B \), such that,
\[
\omega_B^2 = \omega^2 + \frac{2e^2}{mV} \sum_k \sin \theta_k .
\]
By performing a canonical transformation,
\[
q \to \frac{q_0}{\sqrt{m}}, \quad p \to \sqrt{mp_0} \quad \text{and} \quad q_k \to \frac{p_k}{\omega_k} , \quad p_k \to -\omega_k q_k
\]
in Eq. (12) we get
\[
H = \frac{p_0^2}{2} + \frac{\omega_B^2}{2} q_0^2 + \frac{1}{2} \sum_k \left( \frac{p_k^2}{\omega_k^2} + \omega_k^2 q_k^2 \right) - \sum_k c_k q_k q ,
\]
where
\[
c_k = e \sqrt{\frac{2}{mV}} \sin \theta_k \omega_k .
\]
The Hamiltonian given by Eq. (15) is almost identical to the corresponding Hamiltonian in the scalar field case [16–18], with the differences that the physical oscillator frequency arises naturally (that is, we do not need to renormalize the frequency oscillator), the coupling involves an angle \( \theta_k \) and the sum is in three dimensions rather than in one. We have to mention that a Hamiltonian of the form given in Eq. (15) has been used many times in the literature as a simplified model to the quantum Brownian motion [19,12] and also to study the decoherence problem [10]. However we would like to remark that in all early treatments no use of renormalized coordinates (to be defined below) has been done in the way we employ them. Basically we introduce our renormalized coordinates in order to guarantee the stability of the ground state of the oscillator (the atom) in the absence of field quanta (the vacuum), (see Refs. [16,17] for details). If these coordinates were not introduced, as explained below, the oscillator in its ground state would decay to other states, in contradiction with experimental facts.
III. RENORMALIZED COORDINATES AND DRESSED STATES

We can diagonalize the Hamiltonian given by Eq. (15) as in Refs. [16–18] (see also appendix of Ref. [20]), defining

\[ q_\mu = \sum_r t^{r\mu}_r Q_r, \quad p_\mu = \sum_r t^{r\mu}_r P_r, \quad \mu = (0,k), \]

where \{t^{r\mu}_r\} is an orthogonal matrix. We obtain,

\[
H = \frac{1}{2} \sum_r (P^2_r + \Omega^2_r Q^2_r). \tag{17}
\]

The matrix elements \( t^{r\mu}_r \) are given by

\[ t^{r\mu}_0 = \left[ 1 + \sum_k \frac{c_k^2}{(\omega_k^2 - \Omega_r^2)} \right]^{-\frac{1}{2}}, \tag{18} \]

\[ t^{r\mu}_k = \frac{c_k}{(\omega_k^2 - \Omega_r^2)} t^{r\mu}_0, \tag{19} \]

and the normal frequencies \( \Omega_r \), are given as solutions of the equation

\[
\omega^2 - \Omega^2_r = \sum_k \frac{c_k^2}{\omega_k^2(\omega_k^2 - \Omega_r^2)}. \tag{20}
\]

Using Eq. (13) and (16) we can write Eq. (20) as

\[
\omega^2 - \Omega^2_r = \Omega^2_r \sum_k \frac{c_k^2}{\omega_k^2(\omega_k^2 - \Omega_r^2)}. \tag{21}
\]

In order to define the renormalized coordinates and dressed states we start from the eigenstates of our system, represented by the normalized eigenfunctions,

\[
\phi_{n_0n_1n_2...}(Q) = \prod_r \left[ \frac{H_{n_r}(\sqrt{\Omega_r}Q_r)}{\sqrt{2^{n_r}n_r!}} \right] \Gamma_0(Q), \tag{22}
\]

where \( \Gamma_0(Q) \) has the same functional form as \( \Gamma_0(q') \), that is,

\[
\Gamma_0(q') \propto e^{-\frac{1}{2} \sum_r \omega_r Q^2_r}. \tag{26}
\]

It is worthwhile to notice that our renormalized coordinates are new objects, different from both the bare coordinates \( q \) and the normal coordinates \( Q \), and have been used for the first time in Refs. [16–18]. In particular, the renormalized coordinates and dressed states although being collective objects, should not be confused with the normal coordinates.
\( Q \), and the eigenstates (22). Differently from the eigenstates (22) which are stable, our dressed states are all unstable, except the ground dressed state obtained by setting \( \{ m_\mu = 0 \} \) in Eq. (25).

The idea, introduced in Ref. [16], (for a recent clear explanation see Ref. [21]) is that the dressed states given by Eq. (25) are supposed to be the physically meaningful states, instead of the ones written in terms of the bare coordinates \( q_\mu \) that appear in the original Lagrangean. This can be seen as an analogous of the wave function renormalization in quantum field theory, which justifies the denomination of renormalized to the new coordinates \( q' \). Thus, the dressed state given by Eq. (25) describes the atom in its \( m_0 \)-th excited level and \( m_k \) field quanta (photons) of frequencies \( \omega_k \). It should be noticed that the introduction of the renormalized coordinates guarantees the stability of the dressed vacuum state, since by definition it is identical to ground state of the system. The fact that the definition given by Eq. (24) assures this requirement, can be easily seen replacing Eq. (24) in Eq. (26). We obtain \( \Gamma_0(q') \propto \Gamma_0(Q) \), which shows that the dressed vacuum state given by Eq. (26) is the same ground state of the interacting Hamiltonian given by Eq. (17).

The necessity of introducing renormalized coordinates can be understood by considering what would happen if we write Eq. (25) in terms of the bare coordinates \( q_\mu \). In the absence of interaction, the bare states are stable since they are eigenfunctions of the Hamiltonian obtained from Eq. (15) by setting \( c_k = 0 \). But when we consider the interacting term they become all unstable. With the excited states there is no problem since we known from experiment that this is indeed the case. On the other hand, we know from experiment that the atom in its ground state is stable, in contradiction with what our simplified model for the system describes in terms of the bare coordinates. Obviously our simplified model is only approximate, the correct theory that describes the system atom-electromagnetic field is Quantum Electrodynamics (QED), but a rigorous, non-perturbative description using QED would be extremely complicated. So, if we wish to have a non-perturbative approach in terms of our simplified model we have to modify something in order to remedy the above mentioned problem. The solution is just the introduction of the renormalized coordinates \( q'_\mu \), as the physically meaningful ones. On the other hand if, in terms of creation and annihilation operators, we use the so called rotating wave approximation [22] for the interacting term in Eq. (15) (the Lee-Friedrichs model), we would remain, neglecting part of the interaction Hamiltonian, only with terms involving \( \hat{a}_0^\dagger \hat{a}_k \) and \( \hat{a}_k^\dagger \hat{a}_0 \) [this is not the case for the whole Hamiltonian (15) which we consider here]. In this situation there is no problem with the stability of the atom ground state described in terms of the bare coordinates \( q_\mu \), since in this case the action of the interacting term on the bare oscillator ground state (obtained from Eq. (25) setting \( q'_\mu = q_\mu \) and \( m_\mu = 0 \)), gives zero. That is, the free oscillator ground state still remain as the ground state of the interacting system. This is the reason why in early treatments (see for example [23] and references therein) of the simplified model presented here, there has been no need to introduce renormalized coordinates.

We can write the Hamiltonian (15) in terms of dressed coordinates. Replacing Eq. (24) in Eq. (15) we get the renormalized Hamiltonian,

\[
H = \frac{1}{2} \left( \alpha_{00}(p'_0)^2 + \beta_{00} \omega_0^2 (q'_0)^2 \right) + \frac{1}{2} \sum_k \left( \alpha_{kk}(p'_k)^2 + \beta_{kk} \omega_k^2 (q'_k)^2 \right) + \sum_{k_1,k_2} \left( \alpha_{k_1,k_2} p'_k p'_k + \beta_{k_1,k_2} \omega_{k_1} \omega_{k_2} q'_k q'_k \right),
\]

where

\[
\alpha_{\mu \nu} = \sqrt{\omega_\mu \omega_\nu} \sum_r \frac{t_r \Gamma_r}{\Omega_r} \tag{28}
\]

and

\[
\beta_{\mu \nu} = \frac{1}{\sqrt{\omega_\mu \omega_\nu}} \sum_r \Omega_r t_r \Gamma_r. \tag{29}
\]

Expression (27) seems complicated and without clear physical significance. In particular, it appears that the dressed atom and the dressed field modes are in the same foot of equality, that is, the dressed atom, described by \( q'_0 \) and \( p'_0 \), appears to play no central role as in the original Hamiltonian. However this is not the case. Since the expressions for \( t_0' \) and \( t_k' \), given by Eqs. (18)-(19), are very different, the renormalized coordinate \( q'_0 \) and momentum \( p'_0 \) still play a central role, because they are multiplied respectively by \( \alpha_{00} \) and \( \beta_{00} \) that contain terms of the type \( (t_0')^2 \), see Eqs. (28)-(29). On the other hand the dressed coordinates \( q'_k \) and momenta \( p'_k \) are multiplied by \( \alpha_{kk} \) and \( \beta_{kk} \) than contain terms of the type \( t_k' t_k' \). The interacting terms in Eq. (27) involve both dressed coordinates and momenta different from the original Hamiltonian, that involve only explicit interaction among the coordinates \( q_0 \) and \( q_k \). The last term
in the first line of Eq. (27) expresses explicit interaction between the dressed atom and the dressed field modes. The second line in Eq. (27) expresses the explicit interaction among the dressed field modes. However the couplings of these two type of interactions are very different, the dressed atom-photon and dressed photon-photon interactions contain, respectively, terms like $t_{q}^{\mu}t_{q}^{\nu}$ and $t_{q}^{\mu}t_{q}^{\nu}$ that are very different.

We can show that the Hamiltonian written in dressed coordinates, given by Eq. (27), approximately decouples into the dressed atom and the dressed field modes. That is, we show that in Eq. (27), $\alpha_{\mu\mu} \gg \alpha_{\mu\nu}|_{\mu\neq\nu}$ and $\beta_{\mu\mu} \gg \beta_{\mu\nu}|_{\mu\neq\nu}$. For this end note that for $c_k = 0$ (or equivalently for $e = 0$, where $e$ is the electric charge) $q_\mu = q_\mu'$. In this case, $\alpha_{\mu\mu} = \beta_{\mu\mu} = 1$ and $\alpha_{\mu\nu}|_{\mu\neq\nu} = \beta_{\mu\nu}|_{\mu\neq\nu} = 0$. Then, because $e \ll 1$, we must have $\alpha_{\mu\mu} = 1 + \mathcal{O}(e)$, and $\beta_{\mu\mu} = 1 + \mathcal{O}(e)$ whereas $\alpha_{\mu\nu}|_{\mu\neq\nu} = \mathcal{O}(e)$ and $\beta_{\mu\nu}|_{\mu\neq\nu} = \mathcal{O}(e)$, showing that $\alpha_{\mu\mu} \gg \alpha_{\mu\nu}$ and $\beta_{\mu\mu} \gg \beta_{\mu\nu}$. This shows that the introduction of the renormalized coordinates allows, approximately, to divide the system into the dressed atom and the dressed field modes.

The decomposition given in Eq. (27) shows that the renormalized Hamiltonian takes into account explicitly some of the implicit interactions that are preset in the original Hamiltonian. For example, a photon-photon interaction term, of the type given by the second line of Eq. (27) can be obtained from the interaction term $q_\mu q_\nu$, that appears in the original Hamiltonian, at second order in perturbation theory. Also we have to note that in Eq. (27) the ”uncoupled” terms, are not identical to the uncoupled ones in Eq. (15), the differences are just self-interacting terms. From these discussions we can see that our dressed states, defined by Eq. (25), are not free, they are collective states, that take into account the interactions. The dressed atom, for example is the bare atom ”surrounded” by bare (virtual) photons. The same picture applies to the dressed photons. That these claims are correct can be seen naturally, writing any dressed state given by Eq. (25) in the basis of bare states, the bare states being identical to the states given by Eq. (25) with the dressed coordinates $q_\mu'$ replaced by the bare ones $q_\mu$.

With the formalism of dressed coordinates and dressed states we can compute the probability that the atom (understood as the dressed mechanical oscillator $q_0'$) prepared at $t = 0$ in its first excited state, still remain at the time $t$ in the same state, or from it the probability of decay of the atom from its first excited state. For this end we consider the initial dressed state $\psi_{100...}(q')$, in which only the dressed atom is in the first excited level, all the others oscillators being in the ground state. Then it is shown in Ref. [16] the following expression for its time evolution,

$$\psi_{100...}(q',t) = \sum_{\mu} f_{0\mu}(t)\psi_{00...01\mu...}(q')$$

(30)

where

$$f_{0\mu}(t) = \sum_{s} t_{q}^{\mu}s^{\nu}e^{-i\Omega_{s}t}.$$  

(31)

From Eq. (30) we see that the initially excited dressed atom naturally distributes its energy among itself and all other dressed oscillators (the dressed field quanta) as time goes on, with well defined probability amplitudes given by the quantities $f_{0\mu}(t)$ in Eq. (30). From Eq. (30) we obtain for the probability amplitude that the dressed mechanical oscillator (the atom) remain in its first excited state the following expression,

$$f_{00}(t) = \sum_{r} (t_{0}^{r})^{2}e^{-i\Omega_{r}t}.$$  

(32)

As already mentioned above, we do not make explicit use of the concepts of interacting bare oscillator and field, described by the bare coordinates $q_0$ and $\{q_k\}$.

We have introduced renormalized coordinates $q_0'$ and $\{q_k'\}$ for, respectively the dressed atom and the dressed field modes. In terms of these new coordinates dressed states can be rigorously defined. A non-perturbative approach to the radiation process is possible in terms of the dressed states for the system contained in a cavity of arbitrary size, including microcavities [17]. Also free space emission starting from periodic boundary conditions, which we examine in this paper can be treated using our dressed states. This unified treatment is not possible in the conventional framework. Our dressed states Eq. (25) are collective but non stable states, linear combinations of the eigenstates (22) defined in terms of the normal modes. The coefficients of these combinations can be calculated and explicit formulas for these coefficients for the interesting states are given in Refs. [16,17]. Moreover, as noticed above, they have the very interesting property, used in Refs. [16–18] for the case of a scalar field, of distributing the energy initially in a particular dressed state, among itself and all other dressed states with precise and calculable time-dependent probability amplitudes. We choose these dressed states as physically meaningful to the study of the radiation process.
IV. THE RADIATION PROCESS

Now, to compute Eq. (32) in the continuum limit $a \to \infty$, let us define the complex valued function,

$$W(z) = z^2 - \omega^2 + z^2 \sum_k \frac{e_k^2}{\omega_k^2(z - z_k^2)}.$$  \hspace{1cm} (33)

Notice, from Eqs. (21) and (33), that the $\Omega_r$’s are given by the real roots of $W(z)$, and that in the continuum limit these real roots will extend over all the positive real axis in the complex $z$-plane. Deriving Eq. (33) with respect to $z$ and comparing with Eq. (18) we find,

$$(t_0^2) = \left. \frac{2z}{W'(z)} \right|_{z = \Omega_r}. \hspace{1cm} (34)$$

Replacing Eq. (34) in Eq. (32) and using Cauchy theorem we get

$$f_{00}(t) = \frac{1}{\pi \ell} \oint_{C} dz \frac{z}{W(z)}, \hspace{1cm} (35)$$

where $C$ is a counterclockwise contour in the $z$-plane that encircles the real positive axis. The integral in Eq. (35) can be computed choosing a contour $C$ that lies just below and above the real positive axis. Below the real positive axis we have $z = \Omega - i\epsilon$ and above $z = \Omega + i\epsilon$. Then we have

$$f_{00}(t) = \frac{1}{i\pi} \int_0^\infty d\Omega \left[ \frac{(\Omega - i\epsilon)e^{-(i\Omega + \epsilon)t}}{W(\Omega - i\epsilon)} - \frac{(\Omega + i\epsilon)e^{-(i\Omega - \epsilon)t}}{W(\Omega + i\epsilon)} \right]. \hspace{1cm} (36)$$

The next step is to compute $W(\Omega + i\epsilon)$ and $W(\Omega - i\epsilon)$. Using Eq. (16) in Eq. (33) we get in the continuum limit

$$W(z) = z^2 - \omega^2 + \frac{e^2}{4m} \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta_k \int_0^\infty dk \frac{k^2}{k^2 - z^2} \sin \frac{\theta_k^3}{2\pi m} \hspace{1cm} (37)$$

where we integrate over the $\theta_k$ angle only up to $\pi/2$ instead to $\pi$, since as we remarked above, only one half of the modes are linearly independent. The integral over $k$ in Eq. (37) is linearly divergent. Then in order to make meaningful the above equation we will need to regularize it. We do this by analytic regularization, that is, we introduce the integral

$$I(\alpha) = \mu^{\alpha-1} \int_0^\infty dk \frac{k^2}{(k^2 - z^2)^\alpha}, \hspace{1cm} (38)$$

which is well defined for $\alpha > 2$; then we can give a meaning to the integral in Eq. (37) by analytically extending the integral given by Eq. (38) to $\alpha = 1$. The parameter $\mu$ in Eq. (38) is introduced in order to maintain the correct dimension of the original integral. Using Cauchy theorem we obtain for $I(\alpha)$,

$$I(\alpha) = i\pi \mu^{\alpha-1}(\alpha - 1)! \left. \frac{\partial^{\alpha-1}}{\partial w^{\alpha-1}} \frac{w^2}{(w + z)^\alpha} \right|_{w = z}, \hspace{1cm} \text{Im}(z) > 0, \hspace{1cm} (39)$$

$$I(\alpha) = -i\pi \mu^{\alpha-1}(\alpha - 1)! \left. \frac{\partial^{\alpha-1}}{\partial w^{\alpha-1}} \frac{w^2}{(w + z)^\alpha} \right|_{w = z}, \hspace{1cm} \text{Im}(z) < 0. \hspace{1cm} (40)$$

Then by analytic extension we find $I(1) = iz/2$ for $\text{Im}(z) > 0$ and $I(1) = -iz/2$ for $\text{Im}(z) < 0$. Replacing these values in the integral of Eq. (37) we find

$$W(\Omega + i\epsilon) = \Omega^2 - \omega^2 + \frac{e^2\Omega^3 \epsilon}{6\pi m} + O(\epsilon), \hspace{1cm} (41)$$

$$W(\Omega - i\epsilon) = \Omega^2 - \omega^2 - \frac{e^2\Omega^3 \epsilon}{6\pi m} + O(\epsilon). \hspace{1cm} (42)$$
Substituting the above expressions in Eq. (36) we get in the $\epsilon \to 0^+$ limit,

$$f_{00}(t) = \frac{2\tau}{\pi} \int_0^\infty d\Omega \frac{\Omega^4 e^{-i\Omega t}}{((\Omega^2 - \omega^2)^2 + \tau^2 \Omega^6)},$$

(43)

where we have introduced the time dimensional parameter $\tau = e^2/(6\pi m)$. The integral given by Eq. (43) can not be evaluated analytically. We can check the validity of Eq. (43) taking $t = 0$ and noticing that for $\tau \omega \ll 1$ the integrand is sharply peaked around $\Omega = \omega$. Then we obtain from Eq. (43),

$$f_{00}(0) = \frac{\tau \omega^3}{\pi} \int_{-\infty}^{\infty} dx \frac{1}{x^2 + \tau^2 \omega^6} = 1,$$

(44)

as expected from the definition given by Eq. (32). One can check the above result also for any value of $\tau$ using Cauchy theorem for $t = 0$ in Eq. (43). We see from Eq. (43) that the probability $|f_{00}(t)|^2$ will not be a pure exponential. However, at least for low frequencies as we will see below, there will be only a small deviation from an exponential law.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{Plot on the vertical axis, of the probability that the oscillator remain excited at time $t$ in units $\omega t = s$.}
\end{figure}

Considering $t > 0$ we can integrate Eq. (43) numerically and from the result compute the probability $P(t) = |f_{00}(t)|^2$ that the atom still remain excited at time $t$. For instance, taking $\tau \omega = 0.05$ and using for $e$ and $m$ respectively the proton charge and the mass of the hydrogen atom, we get $\omega \approx 5.9 \times 10^{10}/s$. For these values we obtain the plot showed in Fig. 1 where we can see an almost exponential behavior for the decaying process.

V. CONCLUDING REMARKS

In this paper we have analysed a simplified version of an atom-electromagnetic field system and we have tried to give the more exact and rigorous treatment we could to the problem. We have adopted a general physicist’s point of view, in the sense that we have renounced to approach very closely to the real behavior of a complicated non-linear system. As a counterpart, an exact solution has been possible. We have used a formalism (renormalized coordinates and dressed states) that allows a non-perturbative approach to the time evolution of the system. In particular, we have applied this formalism to study non perturbatively the time evolution of the excited atom. In the free space limit, we get the result that the renormalized coordinate approach describing the atom modified by the presence of the field in an indissoluble way, gives an exact result for emission in free space, generalizing the well known exponential decay law.

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