Many Thomassen Faces on High-Representativity Embeddings:

Part I

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Abstract

This is the first in a sequence of six papers in which we prove the following: Given a graph $G$ embedded in an orientable surface $\Sigma$ of genus $g(\Sigma)$, $G$ can be $L$-colored, where $L$ is a list-assignment for $G$ in which every vertex has a 5-list except for a collection of arbitrarily large faces which have 3-lists, as long as those faces are at least distance $2^{\Omega(g(\Sigma))}$ apart and the face-width of $G$ is at least $2^{\Omega(g(\Sigma))}$. This is a generalization of the result in Thomassen’s 5-choosability proof in which a) arbitrarily many faces, rather than just one face, are permitted to have 3-lists and b) the embedding $G$ is not planar but rather locally planar in the sense that it has high representativity. Our result has useful applications in proving that drawings on surfaces graph with arbitrarily large pairwise far-apart crossing structures are 5-choosable under certain conditions. In particular, in a follow-up to this sequence of five papers, we prove a generalization of a result of Dvořák, Lidický, and Mohar which states that if a graph drawn in the plane so that all crossings in $G$ are pairwise of distance at least 15 apart, then $G$ is 5-choosable. In our generalization, we prove an analogous result in which planar drawings with pairwise far-apart crossings have been replaced by drawings on arbitrary surfaces with pairwise far-apart matchings with many crossings, where the graph obtained by deleting these matching edges is a high-representativity embedding.

1 Background

Given a graph $G$, a list-assignment for $G$ is a family of sets $\{L(v) : v \in V(G)\}$ indexed by the vertices of $G$, such that $L(v)$ is a finite subset of $\mathbb{N}$ for each $v \in V(G)$. The elements of $L(v)$ are called colors. A function $\phi : V(G) \rightarrow \bigcup_{v \in V(G)} L(v)$ is called an $L$-coloring of $G$ if $\phi(v) \in L(v)$ for each $v \in V(G)$, and, for each pair of vertices $x, y \in V(G)$ such that $xy \in E(G)$, we have $\phi(x) \neq \phi(y)$. Given a set $S \subseteq V(G)$ and a function $\phi : S \rightarrow \bigcup_{v \in S} L(v)$, we call $\phi$ an $L$-coloring of $S$ if $\phi(v) \in L(v)$ for each $v \in S$ and $\phi$ is an $L$-coloring of the
induced graph $G[S]$. Given an integer $k \geq 1$, a graph $G$ is called $k$-choosable if, for every list-assignment $L$ for $G$ such that $|L(v)| \geq k$ for all $v \in V(G)$, $G$ is $L$-colorable.

In 1994, Thomassen demonstrated in [11] that all planar graphs are 5-choosable, settling a problem that had been posed in the 1970’s. Actually, Thomassen proved something stronger.

**Theorem 1.1.** Let $G$ be a planar graph with facial cycle $C$. Let $xy$ be an edge of $G$ with $x, y \in V(C)$. Let $L$ be a list-assignment for $V(G)$ such that each vertex of $V(G \setminus C)$ has a list of size at least five and each vertex of $V(C) \setminus \{x, y\}$ has a list of size at least three, where $xy$ is $L$-colorable. Then $G$ is $L$-colorable.

In 2003, the author proved a “many-faces” form of Thomassen’s result as part of his PhD thesis.

**Theorem 1.3.** There exists a constant $c$ such that the following holds: Let $G$ be a planar graph and let $F_1, \ldots, F_m$ be a collection of facial subgraphs of $G$ such that $d(F_i, F_j) \geq c$ for each $1 \leq i < j \leq m$. Let $x_1y_1, \ldots, x_my_m$ be a collection of edges in $G$, where $x_iy_i \in E(F_i)$ for each $i = 1, \ldots, m$. Let $L$ be a list-assignment for $G$ such that the following hold.

1) For each $v \in V(G) \setminus (\bigcup_{i=1}^m V(C_i))$, $|L(v)| \geq 5$; AND

2) For each $i = 1, \ldots, m$, $x_iy_i$ is $L$-colorable, and, for each $v \in V(F_i) \setminus \{x_i, y_i\}$, $|L(v)| \geq 3$.

Then $G$ is $L$-colorable.

Postle and Thomas also have an independent proof of Theorem 1.3 with a different distance constant, which consists of a sequence of papers, where [8] is the last paper in the sequence.

Theorem 1.3 gives a positive answer to a conjecture posed at the very end of [5] and also gives a positive answer to a list-coloring version of the following conjecture from [10] for ordinary colorings, albeit with a different distance constant.

**Conjecture 1.4.** Let $G$ be a planar graph and $W \subseteq V(G)$ such that $G[W]$ is bipartite and any two components of $G[W]$ have distance at least 100 from each other. Can any coloring of $G[W]$ such that each component is 2-colored be extended to a 5-coloring of $G$?
A positive answer to Conjecture 1.4 in the special case where each component of $W$ is a lone vertex was provided by Albertson in [1]. Note that Conjecture 1.4 generalizes the setting of pairwise far-apart precolored vertices to that of precolored far-apart 2-colored bipartite components. The main result of [3] provides a positive answer to a list-coloring version of Conjecture 1.4 in the case where each component of $W$ is a lone vertex, albeit with a different distance constant.

The purpose of this sequence of papers is to modify the proof of Theorem 1.3 in the author’s doctoral thesis so as to produce a more general, “locally planar” version of Theorem 1.3 for embeddings on arbitrary surface. We begin with the following preliminaries. Given an embedding $G$ on an orientable surface $\Sigma$, the deletion of $G$ partitions $\Sigma$ into a collection of disjoint, open path-connected components called the faces of $G$. Our main objects of study are the subgraphs of $G$ bounding the faces of $G$. Given a subgraph $H$ of $G$, we call $H$ a facial subgraph of $G$ if there exists a path-connected component $U$ of $\Sigma \setminus G$ such that $H = \partial(U)$. We call $H$ is called a cyclic facial subgraph (or, more simply, a facial cycle) if $H$ is both a facial subgraph of $G$ and a cycle.

Given an orientable surface $\Sigma$, an embedding $G$ on $\Sigma$, and a cycle $C \subseteq G$, we say that $C$ is contractible if it can be contracted on $\Sigma$ to a point, otherwise we say it is noncontractible. More generally, given a subgraph $H$ of $G$, we say that $H$ is contractible if it does not contain any noncontractible cycles, and otherwise $H$ is noncontractible. We now introduce two standard parameters that measure the extent to which a graph embedded on a surface deviates from planarity.

**Definition 1.5.** Let $\Sigma$ be an orientable surface, let $G$ be an embedding on $\Sigma$. We associate to $G$ the following two parameters.

1) The *edge-width* of $G$, denoted by $\text{ew}(G)$, is the length of the shortest noncontractible cycle in $G$.

2) The *face-width* of $G$, denoted by $\text{fw}(G)$, is the smallest integer $k$ such that there exists a noncontractible closed curve of $\Sigma$ which intersects with $G$ on $k$ points.

If $G$ has no noncontractible cycles, then we define $\text{ew}(G) = \infty$, and if $g(\Sigma) = 0$, then we define $\text{fw}(G) = \infty$.

We now have the following definition.

**Definition 1.6.** Let $\Sigma$ be an orientable surface, let $G$ be an embedding on $\Sigma$. We say that $G$ is a 2-cell embedding if each component of $\Sigma \setminus G$ is homeomorphic to an open disc.

If $G$ is a 2-cell embedding, then $\text{fw}(G)$ can be alternatively regarded as the smallest integer $k$ such that there exists a collection of $k$ facial subgraphs of $G$ whose union contains a noncontractible cycle of $G$. In practice, we are usually working with a 2-cell embedding, and in that case, we usually use the above definition of $\text{fw}(G)$ rather than that of Definition 1.5 since it is usually easier to work with for our purposes. We have the following simple standard
Observation 1.7. Let $\Sigma$ be an orientable surface and let $G$ be an embedding on $\Sigma$. Then the following hold.

1) $\text{ew}(G) \geq \text{fw}(G)$; \AND

2) For any subgraph $H$ of $G$, we have $\text{ew}(H) \geq \text{ew}(G)$.

Our main result is Theorem 1.8 below, the proof of which makes up this series of papers and generalizes Theorem 1.3 to an embedding $G$ on an orientable surface $\Sigma$ such that representativity of $G$ is at least $2^{\Omega(g(\Sigma))}$ and the special faces of $G$ are of distance at least $2^{\Omega(g(\Sigma))}$ apart. Note that the pairwise-distance lower bound does not depend on the number of special faces or their sizes.

Theorem 1.8. There exist constants $c,c' \geq 0$ such that the following hold. Let $\Sigma$ be an orientable surface of genus $g := g(\Sigma)$, let $G$ be an embedding on $\Sigma$ of face-width at least $c \cdot 3^g$, and let $F_1,\ldots,F_m$ be a collection of facial subgraphs of $G$ such that $d(F_i,F_j) \geq c' \cdot 3^g$ for each $1 \leq i < j \leq m$. Let $x_1y_1,\ldots,x_my_m$ be a collection of edges in $G$, where $x_iy_i \in E(F_i)$ for each $i = 1,\ldots,m$. Let $L$ be a list-assignment for $G$ such that the following hold.

1) For each $v \in V(G) \setminus \left( \bigcup_{i=1}^m V(F_i) \right)$, $|L(v)| \geq 5$; \AND

2) For each $i = 1,\ldots,m$, $x_iy_i$ is $L$-colorable, and, for each $v \in V(F_i) \setminus \{x_i,y_i\}$, $|L(v)| \geq 3$.

Then $G$ is $L$-colorable.

In [2], Dvořák, Lidický, and Mohar proved the following result.

Theorem 1.9. Let $G$ be a graph drawing in the plane such that the crossings of $G$ are pairwise of distance at least 15 apart. Then $G$ is 5-choosable.

In the statement of Theorem 1.9, the distance between two crossings in a drawing $G$ refers to the graph-theoretic distance between the two pairs of edges which are incident to the respective crossings. In a paper which is a follow-up to the sequence of papers making up the proof of Theorem 1.8 we prove the following result about drawings on surfaces, which is an application of Theorem 1.8. This result is a generalization of Theorem 1.9 to the situation where planar drawings with pairwise far-apart crossings have been replaced by drawings on arbitrary surfaces with pairwise far-apart matchings with many crossings, where the graph obtained by deleting these matching edges in a high-representativity embedding.

Theorem 1.10. There exist constants $C,C' \geq 0$ such that the following hold. Let $\Sigma$ be an orientable surface of genus $g := g(\Sigma)$, let $G$ be a drawing on $\Sigma$ and let $\mathcal{F}$ be a collection of contractible cycles in $G$ which are pairwise of distance at least $C' \cdot 3^g$ apart. Suppose that, for each $F \in \mathcal{F}$, there is a connected component $U_F$ of $\Sigma \setminus F$ such that

1) For each crossing point $x$ of $G$, there is an $F \in \mathcal{F}$ such that $x \in U_F$; \AND
2) The embedding obtained from \( G \) by deleting all the edges in \( \bigcup_{F \in \mathcal{F}} U_F \) has face-width at least \( C \cdot 3^g \); AND

3) For each \( F \in \mathcal{F} \) and \( v \in V(F) \), there is at most one chord of \( F \) in \( U_F \) incident to \( v \), and \( U_F \cap V(G) = \emptyset \).

Then \( G \) is \( L \)-colorable.

### 2 Conventions and Notation of This Paper Sequence

Unless otherwise specified, all graphs are regarded as embeddings on a previously specified orientable surface. If we want to talk about a graph \( G \) as an abstract collection of vertices and edges, without reference to sets of points and arcs on a surface then we call \( G \) an abstract graph. All graphs we consider are simple, (that is, free of loops or repeated edges), finite, and undirected.

When we deal with a graph \( G \) drawn or embedded in \( \mathbb{R}^2 \), it is useful that we can make reference to the outer face of \( G \), particularly if we are working with a minimal counterexample argument in which we want the outer face to satisfy some special properties. We want to do something similar when we talk about graphs drawn or embedded on arbitrary orientable surfaces. For the purposes of this sequence of papers, an orientable surface \( \Sigma \) is regarded as a sphere with a specified north pole and a collection of handles, where each handle of \( \Sigma \) is obtained by cutting out of the sphere two open discs whose closures do not contain the north pole of the sphere, and then gluing the handle onto the sphere along the boundaries of these discs. We adopt the convention that, for any embedding \( G \) on \( \Sigma \), none of the vertices is the north pole and none of points of \( \Sigma \) contained in any the edges of \( G \) is the north pole. This is always permissible by some appropriate deformation within a sufficiently small open ball around the north pole. In this way, it is possible to define an analogue of an outer face and the interior and exterior of a cycle in the context of an arbitrary orientable surface.

**Definition 2.1.** Let \( \Sigma \) be an orientable surface, let \( G \) be an embedding on \( \Sigma \), and let \( C \) be a contractible cycle in \( G \). Let \( U_0, U_1 \) be the two open connected components of \( \Sigma \setminus C \), where \( U_0 \) contains the north pole of \( \Sigma \). We let \( \text{Int}_G(C) \) denote the subgraph of \( G \) consisting of all the edges and vertices in \( \text{Cl}(U_1) \), and we let \( \text{Ext}_G(C) \) denote the subgraph \( G \setminus (\text{Int}_G(C) \setminus C) \) of \( G \). Given a pair of subgraphs \( G_0, G_1 \) of \( G \), we refer to \( \{G_0, G_1\} \) as the natural \( C \)-partition of \( G \) if there exists an \( i \in \{0, 1\} \) such that \( G_i = \text{Int}_G(C) \) and \( G_{1-i} = \text{Ext}_G(C) \).

Given the orientation introduced above, it is natural to introduce the following terminology.

**Definition 2.2.** Let \( \Sigma \) be an orientable surface, let \( C \subseteq \Sigma \) be a contractible cycle, and let \( U_0, U_1 \) be the two connected components of \( \Sigma \setminus C \), where \( U_0 \) contains the north pole of \( \Sigma \). We say that \( U_0 \) is internally bounded by \( C \) and \( U_1 \) is externally bounded by \( C \). We adopt the same terminology for the closed regions \( \text{Cl}(U_0) \) and \( \text{Cl}(U_1) \). If \( U_1 \) is homeomorphic to an open disc, then we say that \( D \) is inward contractible. Likewise, if \( U_0 \) is homeomorphic to an
open disc, then we say that $D$ is outward contractible.

We also introduce the following standard notation.

**Definition 2.3.** Given a graph $G$, a subgraph $H$ of $G$, a subgraph $P$ of $G$, and an integer $k \geq 0$, we call $P$ a $k$-chord of $H$ if $|E(P)| = k$ and $P$ is of the following form.

1) $P := v_1 \ldots v_kv_1$ is a cycle with $v_1 \in V(H)$ and $v_2, \ldots, v_k \notin V(H)$; OR
2) $P := v_1 \ldots v_{k+1}$, and $P$ is a path with distinct endpoints, where $v_1, v_{k+1} \in V(H)$ and $v_2, \ldots, v_k \notin V(H)$.

Given a $k \geq 1$ and a $k$-chord $P$ of $H$, $P$ is called a proper $k$-chord of $H$ if $P$ is not a cycle, i.e $P$ intersects $H$ on two distinct vertices. Otherwise it is called a cyclic or an improper $k$-chord. Note that, for any $1 \leq k \leq 2$, any $k$-chord of $H$ is a proper $k$-chord of $H$, since $G$ has no loops or duplicated edges. A 1-chord of $H$ is simply referred to as a chord of $H$. In some cases, we are interested in analyzing $k$-chords of $H$ in $G$ where the precise value of $k$ is not important.

We thus introduce the following definition. We call $P$ a generalized chord of $H$ if there exists an integer $k \geq 1$ such that $P$ is a $k$-chord of $H$. We call $P$ a proper generalized chord of $H$ if there exists an integer $k \geq 1$ such that $P$ is a proper $k$-chord of $H$. We define improper (or cyclic) generalized chords of $H$ analogously.

Given an orientable surface $\Sigma$, an embedding $G$ on $\Sigma$, a cyclic facial subgraph $C$ of $G$, and a proper generalized $Q$ of $C$, there is a natural way to talk about one or the other “side” of $Q$ in $G$. That is, analogous to Definition 2.1 there is a natural topological way to partition $G$ into two sides of $Q$, which is made precise below.

**Definition 2.4.** Let $\Sigma$ be an orientable surface, let $G$ be an embedding on $\Sigma$, let $C$ be a cyclic facial subgraph of $G$ and let $Q$ be a generalized chord of $C$, where each cycle in $C \cup Q$ is contractible. The unique natural $(C, Q)$-partition of $G$ is a pair $\{G_0, G_1\}$ of subgraphs of $G$ such that the following hold.

1) $G = G_0 \cup G_1$ and $G_0 \cap G_1 = Q$; AND
2) For each $i \in \{0, 1\}$, there is a unique open path-connected region $U$ of $\Sigma \setminus (C \cup Q)$ such that $G_i$ consists of all the edges and vertices of $G$ in the closed region $\text{Cl}(U)$.

If the facial cycle $C$ is clear from the context then we usually just refer to $\{G_0, G_1\}$ as the natural $Q$-partition of $G$.

Note that this is consistent with Definition 2.1 in the sense that, if $Q$ is not a proper generalized chord of $C$ (i.e $Q$ is a cycle) then the natural $Q$-partition of $G$ is the same as the natural $(C, Q)$-partition of $G$. If $\Sigma$ is the sphere (or plane) then the natural $(C, Q)$-partition of $G$ is always defined for any $C, Q$.

We adopt some standard notation relating to graphs and list-assignments.

**Definition 2.5.** For any graph $G$, vertex set $X \subseteq V(G)$, integer $j \geq 0$, and real number $r \geq 0$, we have the following.

1) We set $D_j(X, G) := \{v \in V(G) : d(v, X) = j\}$. 6
2) We set \( B_r(X, G) := \{ v \in V(G) : d(v, X) \leq r \} \).

3) For any subgraph \( H \) of \( G \), we usually just write \( D_j(H, G) \) to mean \( D_j(V(H), G) \), and likewise, we usually write \( B_r(H, G) \) to mean \( B_r(V(H), G) \).

If the underlying graph \( G \) is clear from the context, then we drop the second coordinate from the above notation in order to avoid clutter. We now introduce some additional notation related to list-assignments for graphs. We very frequently analyze the situation where we begin with a partial \( L \)-coloring \( \phi \) of a subgraph of a graph \( G \), and then delete some or all of the vertices of \( \text{dom}(\phi) \) and remove the colors of the deleted vertices from the lists of their neighbors in \( G \setminus \text{dom}(\phi) \). We thus make the following definition.

**Definition 2.6.** Let \( G \) be a graph, let \( \phi \) be a partial \( L \)-coloring of \( G \), and let \( S \subseteq V(G) \). We define a list-assignment \( L^S_{\phi} \) for \( G \setminus (\text{dom}(\phi) \setminus S) \) as follows.

\[
L^S_{\phi}(v) :=
\begin{cases}
\{\phi(v)\} & \text{if } v \in \text{dom}(\phi) \cap S \\
L(v) \setminus \{\phi(w) : w \in N(v) \cap (\text{dom}(\phi) \setminus S)\} & \text{if } v \in V(G) \setminus \text{dom}(\phi)
\end{cases}
\]

If \( S = \emptyset \), then \( L^S_{\phi} \) is a list-assignment for \( G \setminus \text{dom}(\phi) \) in which the colors of the vertices in \( \text{dom}(\phi) \) have been deleted from the lists of their neighbors in \( G \setminus \text{dom}(\phi) \). The situation where \( S = \emptyset \) arises so frequently that, in this case, we simply drop the superscript and let \( L_{\phi} \) denote the list-assignment \( L^S_{\phi} \) for \( G \setminus \text{dom}(\phi) \). In some cases, we specify a subgraph \( H \) of \( G \) rather than a vertex-set \( S \). In this case, to avoid clutter, we write \( L^H_{\phi} \) to mean \( L^V(H)_{\phi} \).

### 3 Charts and Tessellations

Over the course of the proof of Theorem [18], we primarily deal with the following structures.

**Definition 3.1.** Let \( k, \alpha \geq 1 \) be integers. A tuple \( T = (\Sigma, G, C, L, F^*_\alpha) \) is called an \( (\alpha, k) \)-chart if \( \Sigma \) is an orientable surface, \( G \) is an embedding on \( \Sigma \) with list-assignment \( L \), and \( C \) is a family of facial subgraphs of \( G \), where \( C^*_\alpha \) is the outer face of \( G \) and the following conditions are satisfied.

1) \( C^*_\alpha \in C \) and, for any distinct \( H_1, H_2 \in C \), we have \( d(H_1, H_2) \geq \alpha \); AND

2) \( |L(v)| \geq 5 \) for all \( v \in V(G) \setminus \left( \bigcup_{H \in C} V(H) \right) \); AND

3) For each \( H \in C \), there exists a connected subgraph \( P_{T, H} \) of \( H \) satisfying the following.
   
   i) \( |E(P_{T, H})| \leq k \) and \( P_{T, H} \) is induced in \( H \); AND
   
   ii) \( |L(v)| \geq 3 \) for all \( v \in V(H) \setminus V(P_{T, H}) \); AND
iii) \( V(P_{T,H}) \) is \( L \)-colorable and \( |L(v)| = 1 \) for each \( v \in V(P_{T,H}) \).

The definition above entails that \( C \neq \emptyset \), since the outer face is an element of \( C \), but, given an \( H \in C \), we possibly have \( P_{T,H} = \emptyset \). We now introduce some more natural terminology.

**Definition 3.2.** Given an orientable surface \( \Sigma \), we have the following.

1) A tuple \( T \) is called a **chart** if there exist integers \( k, \alpha \geq 1 \) such that \( T \) is an \((\alpha, k)\)-chart.

2) A chart \( T = (\Sigma, G, C, L) \) is called **colorable** if \( G \) is \( L \)-colorable. We call \( G \) the **underlying graph** of the chart and we call \( \Sigma \) the **underlying surface** of the chart.

3) Given a chart \( T = (\Sigma, G, C, L) \) is a chart, we have the following notation.

a) For each \( H \in C \), the uniquely specified subgraph \( P_{T,H} \) of \( H \) satisfying 3) of Definition 3.1 is called the **precolored subgraph** of \( H \).

b) The elements of \( C \) are called the **rings** of the chart. In particular, the elements of \( C \setminus \{C_s\} \) are called the **internal** rings of the chart.

Note that, in the setting above, \( P_{T,H} \) is induced in \( H \), but not necessarily induced in \( G \). The terminology in Definition 3.2, together with the notation \( C \), is suggestive of the fact that our primary interest is the case where the rings of the chart are cyclic facial subgraphs of \( G \), but in general, the definition of facial subgraphs does not require the elements of \( C \) to be cyclic facial subgraphs of \( G \), or even connected subgraphs of \( G \). In most applications of the terminology above, we are dealing with the case where \( G \) has high face-width and each \( H \in C \) is a contractible cycle of \( G \). If the underlying chart is clear from the context, we usually drop the \( T \) from the notation \( P_{T,H} \) to avoid clutter, i.e. we just write \( P_H \). The bold-font \( P \) is only ever used to refer to these paths in charts so there is no danger of confusing them with other paths. For our purposes, it is essential to distinguish the special case in which the entirety of an element of \( C \) is precolored, so we introduce the following terminology.

**Definition 3.3.** Let \( T = (\Sigma, G, C, L) \), be a chart and let \( H \in C \). We say that \( H \) is a **closed** \( T \)-ring if \( P_H = H \). Otherwise, we say that \( C \) is an **open** \( T \)-ring. If the chart \( T \) is clear from the context then we just call \( C \) a closed ring or open ring respectively.

We now restate Theorem 3.4 in the language of charts.

**Theorem 3.4.** There exist constants \( c, c' \) such that the following hold. Let \( \Sigma \) be an orientable surface of genus \( g : = g(\Sigma) \), let \( G \) be an embedding on \( \Sigma \) with \( \text{fw}(G) \geq c \cdot 3^g \), and let \( T \) be a \((c' \cdot 3^g, 1)\)-chart with underlying surface \( \Sigma \) and underlying graph \( G \). Then \( T \) is colorable.

Of particular importance to us over the course of the proof of Theorem 3.4 are embeddings which do not have sepa-
rating cycles of length 3 or 4, so we give them a special name.

**Definition 3.5.** Let $\Sigma$ be an orientable surface and let $G$ be an embedding on $\Sigma$.

1) A **separating cycle** in $G$ is a contractible cycle $C$ in $G$ such that each of the two connected components of $\Sigma \setminus C$ has nonempty intersection with $V(G)$.

2) We call $G$ **short-separation-free** if $G$ does not contain any separating cycle of length 3 or 4. Likewise, given a chart $T$, we call $T$ a **short-separation-free chart** if the underlying graph of $T$ is a short-separation-free.

### 4 An Overview of the Proof of Theorem 1.8

We now provide a brief overview of the proof of Theorem 1.8. We first introduce the following terminology. One of the key ingredients in the proof of Theorem 1.8 is the reduction to a particular class of embeddings on an orientable surface which is easier to study.

**Definition 4.1.** Let $T = (\Sigma, G, C, L, C_\ast)$ be a chart.

1) We call $T$ **near-triangulated** if, for every facial subgraph $H$ of $G$, with $H \notin C$, $H$ is a triangle.

2) We call $T$ a **tessellation** if it is near-triangulated and short-separation-free.

3) Given integers $k, \alpha \geq 1$, we call $T$ an $(\alpha, k)$-**tessellation** if it is both a tessellation and an $(\alpha, k)$-chart.

In Papers I-V, we show that Theorem 3.4 holds for tessellations. Actually, we prove something stronger by defining a structure called a **mosaic** and showing that all mosaics are colorable, where a mosaic is a special kind of a tessellation satisfying some additional properties. This result, which is the main step in the proof of Theorem 3.4, is stated in Theorem 6.5 and Papers I-V consists of the proof of Theorem 6.5. The key to the proof of Theorem 6.5 is to choose the right definition of mosaics, i.e. to choose the right induction hypothesis. We show that Theorem 6.5 holds by a minimal counterexample argument. In this paper, we gather some basic structural properties of minimal counterexamples, and we also analyze $k$-chords of the rings of a minimal counterexample for small values of $k$. In particular, we show that, for sufficiently small values of $k$, there is no $k$-chord of $C$ which separates the faces of $C \setminus \{C\}$ (by “sufficiently small” we mean “small relative to the pairwise distance between the special faces of the mosaic with lists of size less than five”). This means that, given a $k$-chord of $C$ in $G$, there is a “small” side of the $k$-chord in $G$ in which all vertices outside of $C$ have $L$-lists of size at least five. This fact is essential to our construction of a smaller counterexample from a minimal counterexample.

Papers II and III build directly on the work of Paper I. In particular, in Paper III, we show that the structure of the graph near each ring is very regular and easy to work with. Paper IV consists of a lone result about planar graphs,
and is outside the context of charts. We use the result of Paper IV as a black box in papers V and VI. In particular, Paper IV does not rely on any of the language of charts (or even embeddings on surfaces other than the plane/sphere) and it can be read independently of the rest of the sequence. In Paper V we use the work of Papers I-IV to produce a smaller counterexample by carefully coloring and deleting a contractible subgraph of a minimal counterexample. More precisely, given a minimal counterexample $(\Sigma, G, C, L, C_*)$, the subgraph of $G$ which we color and delete is “almost” a path between $C_*$ and a $C \in C \setminus \{C_*\}$, in the sense that it differs from such a path only in some regions which are on the small sides of some short generalized chords of either $C_*$ or $C$.

We have to be careful when we perform the deletion described above because the resulting smaller embedding still needs to have high representativity in order to be the underlying graph of a smaller counterexample. While edge-width is monotone with respect to subgraphs in the sense of 2) of Observation 1.7, this is not true of face-width. When we delete a subgraph of an embedding on some orientable surface, the face-width of the resulting smaller embedding can actually go down. By choosing our induction hypothesis correctly, we ensure that the subgraph obtained from a minimal counterexample in the manner described above has high-enough face-width to be a smaller counterexample. Having high edge-width is a weaker condition than having high face-width, and the discussion above suggests that it might be easier to work with edge-width than face-width in a minimal counterexample argument, because of the monotonicity of edge-width, but the conditions on the face-width cannot be dropped. In particular, the counterexample below in Figure 4.1 shows that Theorem 1.8 does not hold if we replace the words “face-width” with “edge-width”.

Figure 4.1: Theorem 1.8 does not hold if the $2^{\Omega(g)}$-face-width condition is replaced by a weaker $2^{\Omega(g)}$-edge-width condition
In Figure 4.1, we have an embedding $G$ on the torus in which the thick edge comes out of the page, and a list-assignment $L$ for $V(G)$ in which $x$ is precolored by $a$ and every vertex of $G - x$ has the list $\{a, b, c\}$. Any contractible cycle in $G$ contains the edge $xy$, and the distance between $x$ and $y$ in $G - xy$ is $k + 1$, so the edge width of $G$ can be made arbitrarily large. It is straightforward to check that, for any $L$-coloring $\phi$ of $G - xy$, and any integer $j \in \{4, \ldots, k\}$ with $j \equiv 1 \mod 3$, we have $\phi(q_{j-1}) = a$ and $\phi$ colors the endpoints of the chord $p_j q_j$ with $b, c$. Thus, if $k \equiv 1 \mod 3$, then $G$ is not $L$-colorable. One of the things we do in Paper II is show that, given a minimal counterexample to our induction hypothesis, the underlying embedding has sufficiently high representativity near the special cases that the embedding obtained by the deletion described above is still the underlying embedding of a mosaic. In Paper VI, we complete the proof of Theorem 3.4 by showing that Theorem 6.5 implies Theorem 3.4, and thus the equivalent Theorem 1.8. Each paper in the sequence depends at most on the results of the previous papers in the sequence as black boxes, and relies on definitions introduced in the previous papers, which are restated in introductory sections of each paper, but otherwise does not depend on the details of the proofs in the previous papers.

5 The Results of Paper I

As indicated above, in Papers I-V, we prove by induction that Theorem 3.4 holds for a special restricted class of charts. In Section 6, we introduce this special class of charts. In particular, in Section 6 we introduce our induction hypothesis and, in the remainder of Paper I, we show that a minimal counterexample with respect to this induction hypothesis satisfies some desirable properties. Indeed, our stronger induction hypothesis has been chosen precisely so that a minimal counterexample satisfies these desirable properties. More precisely, given a minimal counterexample $T := (\Sigma, G, C, L, C_*)$, the high-representativity conditions we impose on $G$ imply that, for each ring $C \in C$ and each generalized chord $P$ of $C$ of sufficiently short length, at least one side of $P$ is contractible. In rough terms, the purpose of the remainder of Paper I is to show that, with $C, P$ as above, the following properties are satisfied.

1) $P$ does not separate any two elements of $C \setminus \{C\}$; AND
2) If one side of $P$ contains at least one (and thus every) element of $C \setminus \{C\}$, then the other side is contractible.

Each section of Paper I after Section 5 consists of either one or two main results, as indicated in Table 5.1.

| Section | Main Result(s)       | Section | Main Result(s)       |
|---------|----------------------|---------|----------------------|
| 6       | Proposition 6.8      | 9       | Theorem 9.1          |
| 7       | Facts 7.3 and 7.4    | 10      | Theorem 10.4         |
| 8       | Theorems 8.12 and 8.16 | 11      | Theorems 11.2 and 11.5 |

Table 5.1: Remaining sections of this paper and their main results
6 Mosaics and Their Properties

In this section, we describe our special class of charts for which we prove Theorem 3.4 before proving the full Theorem 3.4. We begin by fixing constants $N_{mo}, \beta$, where $N_{mo} \geq 100$ and $\beta := 100N_{mo}^2$. It is convenient to also require that $N_{mo}$ is a multiple of 3. The subscript of the notation $N_{mo}$ refers to mosaics, which is the term we use for our special charts. In particular, we define a mosaic to be a tessellation which satisfies some additional properties satisfied below, where a mosaic is allowed to contain some precolored faces (i.e., closed rings) of length at most $N_{mo}$. In Papers I-V, we then show that any tessellation satisfying these properties is colorable by showing that no minimal counterexample to colorability exists, in a sense which is made precise in the section below. In order to state our stronger induction hypothesis, we begin with the following definitions.

**Definition 6.1.** Let $\Sigma$ be an orientable surface, let $G$ be an embedding on $\Sigma$, and let $L$ be a list-assignment for $V(G)$. Given a facial subgraph $H$ of $G$, we say that $H$ is $L$-predictable if, letting $S := \{v \in V(H) : |L(v)| = 1\}$, the following hold.

1) For every subgraph $K \subseteq G[V(S)]$ and every $v \in D_1(K) \setminus V(H)$, where $K$ is induced in $G$ and $K$ is either a path or a cycle, the graph $G[N(v) \cap V(K)]$ is either a proper subpath of $K$ or all of $K$; AND

2) There is a vertex $v \in D_1(S) \setminus V(H)$ such that, for any proper $L$-coloring $\phi$ of $V(S)$, every vertex of $D_1(S) \setminus (V(H) \cup \{v\})$ has an $L_{\phi}$-list of size at least three, and $v$ has an $L_{\phi}$-list of size at least two.

We say that $H$ is highly $L$-predictable if it satisfies 1) above and Condition 2) has been replaced by the following slightly stronger condition.

3) For any proper $L$-coloring $\phi$ of $V(S)$, every vertex of $D_1(S) \setminus V(H)$ has an $L_{\phi}$-list of size at least three.

In order to state the distance conditions we impose on our tessellations, we introduce the following notation.

**Definition 6.2.** Let $T = (G, C, L, C^*)$ be a chart. We define a rank function $Rk(T|\cdot) : C \to \mathbb{R}$, and for each $H \in C$, a subset $w_T(H)$ of $V(C)$ as follows.

$$w_T(H) := \begin{cases} V(H) & \text{if } H \text{ is a closed } T\text{-ring} \\ V(H \setminus P_H) & \text{if } H \text{ is an open } T\text{-ring} \end{cases} \quad \text{Rk}(T|H) := \begin{cases} |V(H)| & \text{if } H \text{ is a closed } T\text{-ring} \\ 2N_{mo} & \text{if } H \text{ is an open } T\text{-ring} \end{cases}$$

If the underlying chart $T$ is clear from the context then we drop the symbol $T$ from the notation above. We also introduce the following notation, which is a generalization of standard notation used to denote the subpath of a path consisting of its internal vertices.

**Definition 6.3.** Given a graph $H$, we let $\hat{H}$ be the graph obtained from $H$ by deleting the vertices of degree at most one. In particular, if $H$ is a path of length at least two, then $\hat{H}$ is the path obtained by deleting the endpoints of $H$, and
if $H$ is a cycle, then $\hat{H} = H$.

We now state our induction hypothesis.

**Definition 6.4.** A chart $T := (\Sigma, G, C, L, C_*)$ is called a *mosaic* if $T$ is a tessellation such that, letting $g$ be the genus of $\Sigma$, the following additional conditions hold.

M0) For each $H \in C$, if $H$ is open, then $|E(P_H)| \leq \frac{2N_{mo}}{3}$, and if $H$ is closed, then $|E(P_H)| \leq N_{mo}$; AND

M1) For each open ring $H$, $P_H$ is a path and there is no chord of $H$ with an endpoint in $\hat{P}_H$, and furthermore, $H$ is highly $L$-predictable; AND

M2) Each closed ring is $L$-predictable; AND

M3) For each $C \in C \setminus \{C_*\}$, we have $d(w(C_*), w(C)) \geq \beta \cdot 3^g - 1 + Rk(C) + Rk(C_*)$; AND

M4) For any distinct $C_1, C_2 \in C \setminus \{C_*\}$, we have $d(w_T(C_1), w_T(C_2)) \geq \beta \cdot 3^g + Rk(C_1) + Rk(C_2)$; AND

M5) $fw(G) \geq 1.1 \beta \cdot 3^{g-2}$ and $ew(G) \geq 2.4 \beta \cdot 3^{g-2}$.

Note that the outer face is “special” in the sense that the internal rings satisfy a stronger pairwise distance condition between each other. Papers I-V consist of the proof of the following result.

**Theorem 6.5.** All mosaics are colorable.

We begin by introducing the following terminology.

**Definition 6.6.** Let $T = (\Sigma, G, C, L, C_*)$ be a mosaic. We say that $T$ is *critical* if the following hold.

1) $G$ is not $L$-colorable; AND

2) Any mosaic $(\Sigma', G', C', L', D)$ with $|V(G')| < |V(G)|$ is colorable; AND

3) Any mosaic $(\Sigma', G', C', L', D)$ with $|V(G')| = |V(G)|$ and $\sum_{v \in V(G')} |L(v)| < \sum_{v \in V(G)} |L(v)|$ is colorable.

**Definition 6.7.** Given a chart $T = (\Sigma, G, C, L, C_*)$ and a subgraph $H$ of $G$, we let $C \subseteq H$ denote $\{C \in C : C \subseteq H\}$.

The remainder of Section 6 consists of the following simple facts about critical mosaics.

**Proposition 6.8.** Let $T = (\Sigma, G, C, L, C_*)$ be a critical mosaic. Then the following hold:

1) For each $C \in C$ and $v \in V(C \setminus P_C)$, $|L(v)| = 3$; AND

2) For each $v \in V(G) \setminus (\bigcup_{C \in C} V(C))$, $|L(v)| = 5$ and $\deg(v) \geq 5$.

3) $G$ is 2-connected and a 2-cell embedding

4) For each open ring $H \in C$, $|V(H)| \geq 5$ and $|E(P_H)| \geq 1$. 

13
Proof. Both 1) and the first part of 2) both follow directly from the minimality of $\sum_{v \in V(G)} |L(v)|$, or else we can remove colors from the lists of some vertices of $G$. Now suppose toward a contradiction that there is a $v \in V(G) \setminus (\bigcup_{C \in \mathcal{C}} V(C))$ such that $\deg_G(v) \leq 4$. Note that every facial of $G$ containing $v$ is a triangle, so $N(v)$ induces a cycle of length at most four and $G[N(v)] = K_4$. Since $G$ is short-separation-free, we have $G = K_4$, and $G$ is trivially $L$-colorable, contradicting the fact that $T$ is a counterexample. Note that the proof of 2) is somewhat atypical. In a minimal counterexample argument involving list-assignments, we usually deal with a vertex $v$ such that $\deg(v) < |L(v)|$ by deleting $v$ to produce a smaller counterexample, but in the context above, the graph $G - v$ does not satisfy the triangulation conditions of Definition 4.1, i.e it is not the underlying graph of a tessellation. In general, some care must be taken when constructing smaller counterexamples from critical mosaics.

We now prove 3). This is the most involved part of the proof of Proposition 6.8. It suffices to prove that every element of $C$ is a cycle. If that holds, then, since $fw(G) > 0$ and every facial subgraph of $G$ other than those of $C$ is a triangle, it follows that $G$ is 2-connected and a 2-cell embedding. So suppose toward a contradiction that there is an $H \in \mathcal{C}$ which is not a cycle. Since $H$ is a facial subgraph of $G$, it follows that either there is a vertex of $H$ which is a cut-vertex of $G$, or $H$ has nonempty intersection with more than one connected component of $G$. In any case, there is an $S \subseteq V(H)$ with $|S| \leq 1$ such that $G \setminus S$ has more than one connected component.

Claim 6.9. Let $K$ be a connected component of $G \setminus S$ and let $L'$ be a list-assignment for $V(K + S)$ where each $x \in S$ is precolored with a color of $L(x)$, and otherwise $L' = L$. Then the following hold.

i) $K + S$ is $L$-colorable; AND

ii) If $V(K \cap P_H) = \emptyset$ then $K + S$ is $L'$-colorable.

Proof: Let $H^K := H \cap (K + S)$ and $C^K := \{H' \in C \setminus \{H\} : H' \subseteq K + S\}$. Furthermore, let $C^K_*$ be the outer face of $K + S$. We first prove 2). Suppose that $V(K \cap P_H) = \emptyset$ and let $T' := (\Sigma, K + S, C^K \cup \{H^K\}, L', C^K_*)$. We claim that $T'$ is a mosaic. Firstly, since $(K \cap P_H) = \emptyset$, $H$ is an open $T$-ring. Since $H$ is an open $T$-ring and $H^K$ has a precolored path (with respect to $T'$) consisting of one vertex, it follows that $H^K$ is an open $T'$-ring and that $T'$ satisfies all of M0)-M4). Note that $fw(H^K) \geq fw(G)$ and $ew(H^K) \geq ew(G)$, so M5) holds for $T'$ as well. Thus, $T'$ is a mosaic, and since $|V(K + S)| < |V(G)|$, it follows that $K + S$ is $L'$-colorable. This proves ii).

Now we prove i). If $V(K \cap P_H) = \emptyset$ then, by ii), $K + S$ is $L'$-colorable and thus $L$-colorable, so we are done in that case, so now suppose that $V(K \cap P_H) \neq \emptyset$. Let $T^\dagger := (\Sigma, K + S, C^K \cup \{H^K\}, L, C^K_*)$. Recall that, by Definition 3.1, $P_H$ is an induced subgraph of $H$. Since $S \subseteq V(H)$ and $S$ is either empty or consists of a cut-vertex of $G$, it follows that $P_H \cap (K + S)$ is an induced subgraph of $H^K$. Suppose toward a contradiction that $K + S$ is not $L$-colorable. Since $|V(K + S)| < |V(G)|$, it immediately follows from the criticality of $T$ that $T^\dagger$ is not a mosaic. It
is clear that each component of $G \setminus S$ has edge-width at least $\text{ew}(G)$ and, since $|S| \leq 1$, each component of $G \setminus S$ also has face-width at least $\text{fw}(G)$, so $T^\dagger$ satisfies M5). If $H$ is an open $T$-ring, then $|E(P_H \cap H^K)| \leq |E(P_H)| \leq \frac{2N_{mo}}{3}$, and if $H$ is a closed $T$-ring, then $H^K$ is a closed $T^\dagger$-ring as well, and $|E(P_H \cap H^K)| \leq |E(P_H)| \leq N_{mo}$. Thus, $T^\dagger$ satisfies M0) in any case, so it violates one of M1)-M4). Now consider the following cases.

**Case 1:** $V(H^K) \not\subseteq V(P_H)$.

In this case, $H$ is an open $T$-ring and $H^K$ is an open $T^\dagger$-ring. Since $P_H$ is a path, $P_H \cap H^K$ is also a (possibly empty) path, and furthermore, we have $\omega_{T^\dagger}(H^K) = V(H^K) \setminus V(P_H)$. Since $\omega_T(H) = V(H) \setminus V(P_H)$, it follows that $T^\dagger$ satisfies the distance conditions of Definition 6.4. Possibly $H^K$ is still the outer face of $K + S$, but in any case, since the distance conditions between the internal rings of $C$ are stronger than the conditions on the distance from the internal rings to the outer face of $C$, we get that $T^\dagger$ satisfies M3)-M4), and M1)-M2) are immediate, so we contradict the fact that one of M1)-M4) is violated.

**Case 2:** $V(H^K) \subseteq V(P_H)$.

In this case, if $H^K$ is a path, then we have $V(H^K) = V(K + S)$, since $H^K$ is a facial subgraph of $K + S$. If that holds, then $K + S$ is $L$-colorable, since $P_H$ is already precolored, so we contradict our assumption. Thus, $H^K$ is not a path, and since $P_H \cap H^K$ is an induced subgraph of $H^K$, it follows from M1) applied to $T$ that $H$ is not an open $T$-ring. Thus, $H$ is a closed $T$-ring, and $H^K$ is a closed $T^\dagger$-ring. It is again immediate that $T^\dagger$ is a mosaic. In particular, the rank of $H^K$ is less than that of $H$, so $T^\dagger$ satisfies M3) and M4). Possibly $H^K$ is still the outer face of $K + S$, but in any case, since the distance conditions between the internal rings of $C$ are stronger than the conditions on the distance from the internal rings to the outer face of $C$, we have in any case that $T^\dagger$ satisfies M3)-M4), and M1)-M2) are immediate, contradicting the fact that one of M1)-M4) is violated. This completes the proof of Claim 6.9.

Now we return to the proof of 3) of Proposition 6.8. It immediately follows from i) of Claim 6.9 that $S \neq \emptyset$, or else we $L$-color each component of $G \setminus S$, and the union of these $L$-colorings is an $L$-coloring of $G$, contradicting the fact that $T$ is critical. Thus, we have $|S| = 1$, so let $S = \{x\}$ for some $x \in V(H)$. If $x \in V(P_H)$, then $x$ is already precolored by $L$, and in that case, we again $L$-color $K + x$ for each connected component $K$ of $G - x$, and the union of these $L$-colorings is an $L$-coloring of $G$, contradicting the fact that $T$ is critical. Thus, we have $x \notin V(P_H)$. Since $P_H$ is connected, there is a unique connected component $K$ of $G - x$ with $P_H \subseteq K$. By i) of Claim 6.9, $K + x$ admits an $L$-coloring $\phi$. By 2) of Claim 6.9, applied to each of the remaining connected components of $G - x$, $\phi$ extends to an $L$-coloring of $G$, contradicting the fact that $T$ is critical. This completes the proof of 3).

We now prove 4) of Proposition 6.8. Suppose toward a contradiction that $|V(H)| \leq 4$. By 3), $H$ is a cycle. Since $G$ is short-separation-free and $H$ is a cycle of length at most four, $G$ contains no chord of $H$, so $V(H)$ admits an $L$-coloring $\psi$. Let $L'$ be a list-assignment for $V(G)$ where the vertices of $H$ are precolored by $\psi$ and otherwise $L' = L$. 

\[15\]
By 3), $H$ is a cycle, and $T' := (\Sigma, G, C, L', C_*)$ is a tessellation in which $H$ is a closed ring. We claim that $T'$ is a mosaic. M5) is immediate, since the embedding is the same as in $T$. Since $H$ is a chordless cycle of length at most four, it is immediate that $H$ is an $L'$-predictable facial cycle of $G$, so M2) is satisfied. Furthermore, $T'$ still satisfies the distance conditions of Definition 6.4, as the rank of $H$ has only decreased. It is immediate that the other conditions are satisfied. Thus, $T'$ is a mosaic, and, by the minimality of $\sum_{v \in V(G)} |L(v)|$, it follows that $G$ is $L'$-colorable and thus $L$-colorable, contradicting the fact that $T$ is critical. Now suppose toward a contradiction that $|E(P_C)| < 1$. Thus, $P_C$ is either empty or consists of a lone vertex, so let $L'$ be a list-assignment for $V(G)$ obtained from $L$ by properly $L$-precoloring a path $P$ of length one in $H$, where this path includes the lone vertex of $P_C$, if it exists. Since $H$ is a cycle, we have $V(H) \neq V(P)$, and it is immediate that $T' := (\Sigma, G, C, L', C_*)$ is a tessellation where $H$ is an open ring with a precolored path $P$, and since this path has length one, it is immediate that $C$ is a highly $L'$-predictable facial subgraph of $G$, and since $V(\hat{P}) = \emptyset$, it follows that $T'$ satisfies M1) and is thus a mosaic. It follows from the minimality of $\sum_{v \in V(G)} |L(v)|$ that $G$ is $L'$-colorable and thus $L$-colorable, a contradiction.

7 Face-Width Bounds

The usefulness of imposing both face-width conditions and stronger edge-width conditions in M5) of Definition 6.4 lies in the following simple results, which makes precise the intuition that, if an embedding has high edge-width and consists mostly of triangles, then it also has high face-width.

Proposition 7.1. Let $\Sigma$ be an orientable surface and let $G$ be a 2-cell embedding on $\Sigma$. Then the following hold.

1) Let $F \subseteq G$ be a noncontractible cycle and let $\mathcal{F}$ be a family of facial subgraphs of $G$, each of which is a triangle, where $F$ is contained in the union of the elements of $\mathcal{F}$. Then $\text{ew}(G) \leq |\mathcal{F}| + 2$: AND

2) Let $\alpha \geq 2$ be a constant and let $\mathcal{C}$ be a family of facial subgraphs of $G$, where each facial subgraph of $G$, other than those of $\mathcal{C}$, is a triangle. Let $D \in \mathcal{C}$ be of distance at least $\alpha$ from each element of $\mathcal{C} \setminus \{D\}$. Let $F' \subseteq G$ be a cycle with $V(F' \cap D) \neq \emptyset$. Then either $F'$ contains no vertices of any element of $\mathcal{C} \setminus \{D\}$ or $F'$ cannot be contained in the union of fewer than $2(\alpha - 1)$ facial subgraphs of $G$.

Proof. Suppose that 1) does not hold and let $F$ be a vertex-minimal counterexample to 1). Thus, there is a collection $\mathcal{F}$ of facial triangles of $G$ such that $F$ is contained in the union of the elements of $\mathcal{F}$, but $\text{ew}(G) > |\mathcal{F}| + 2$. Since $\mathcal{F}$ is nonempty, we have $\text{ew}(G) > 3$, and thus $F$ is not a triangle. Let $F' := \{ T \in \mathcal{F} : |E(T) \cap E(F)| = 2 \}$. For each $T \in \mathcal{F}'$, $T \cap F$ is a subpath of $F$ of length two. Furthermore, since $F$ is not a triangle, each element of $\mathcal{F} \setminus \mathcal{F}'$ intersects with $F$ on a subpath of $F$ of length at most one. Thus, if $\mathcal{F}' = \emptyset$, then we have $\text{ew}(G) \leq |E(F)| \leq |\mathcal{F}|$, contradicting our assumption.
Since $F' \neq \emptyset$, let $T \in F'$, and let $F^*$ be the cycle obtained from $F$ by replacing the 2-path $T \cap F$ with the lone remaining edge of $T$. Since $\text{ew}(G) > 3$, $T$ is contractible, so $F^*$ is not contractible. Furthermore, $F^*$ is still contained in the union of the elements of $\mathcal{F}$. Since $|V(F^*)| = |V(F)| - 1$, it follows from the minimality of $|V(F)|$ that $\text{ew}(G) \leq |\mathcal{F}| + 2$, a contradiction. This proves 1). To prove 2), we first prove the following intermediate result.

**Claim 7.2.** Let $\{P_1, \ldots, P_m\}$ be collection of pairwise vertex-disjoint paths in $G$, each of which contains no edges of $\bigcup_{H \in C} E(H)$. For each $i = 1, \ldots, m$, let $x_i, y_i$ be the endpoints of $P_i$. Then $P_1 \cup \ldots P_m$ cannot be contained in the union of fewer than $\sum_{i=1}^m d_G(x_i, y_i)$ facial subgraphs of $G$.

**Proof:** Let $\mathcal{F}$ be a minimum-cardinality family of facial subgraphs of $G$ whose union contains $\bigcup_{i=1}^m P_i$. We suppose that each path in $\{P_1, \ldots, P_m\}$ has length at least one, since any path which is just an isolated vertex can be removed without changing the quantity $\sum_{i=1}^m d_G(x_i, y_i)$. In particular, since each edge of $\bigcup_{i=1}^m P_i$ is contained in an element of $\mathcal{F}$, it follows from the minimality of $\mathcal{F}$ that every element of $\mathcal{F}$ contains an edge of $\bigcup_{i=1}^m P_i$ and is thus a triangle.

For each $T \in \mathcal{F}$, since $T$ is a triangle, $E(T)$ intersects with $\bigcup_{i=1}^m P_i$ on a subpath of $P_1 \ldots P_m$ of length either one or two. Let $\mathcal{F}_i := \{T \in \mathcal{F} : E(T) \cap E(P_i) \neq \emptyset\}$ for each $i = 1, \ldots, m$. Since $P_1, \ldots, P_m$ are pair-wise-disjoint, it follows that $\mathcal{F} = \bigcup_{i=1}^m \mathcal{F}_i$ as a disjoint union. Thus, it suffices to show that $|\mathcal{F}_i| \geq d_G(x_i, y_i)$ for each $i = 1, \ldots, m$.

Let $j \in \{1, \ldots, m\}$ and let $\mathcal{F}_j' := \{T \in \mathcal{F}_j : |E(T) \cap \mathcal{F}_j| = 2\}$. Each element of $\mathcal{F}_j \setminus \mathcal{F}_j'$ intersects with $P_j$ precisely on an edge of $P_j$. For each $T \in \mathcal{F}_j'$, since $T$ is a triangle, it follows that the 2-path $T \cap P_j$ contributes a $+1$ to the deficit $|E(P_j)| - d_G(x_j, y_j)$. That is, we have $|E(P_j)| - d_G(x_j, y_j) \geq |\mathcal{F}_j'|$. Since $|E(P_j)| \leq |\mathcal{F}_j \setminus \mathcal{F}_j'| + 2|\mathcal{F}_j'|$, we have $d_G(x_j, y_j) \leq |\mathcal{F}_j \setminus \mathcal{F}_j'| + |\mathcal{F}_j'|$, and thus $d_G(x_j, y_j) \leq |\mathcal{F}_j|$, as desired.

Now we prove 2). Let $F'$ be as in the statement of 2). Note that $F'$ contains two edge-disjoint paths $P_0, P_1$, each of length at least $\alpha$. By deleting at most a terminal edge from each of $P_0, P_1$ if necessary, we produce a subgraph $K \subseteq P_0 \cup P_1$ such that $K$ consists of two vertex-disjoint paths, where each component of $K$ has endpoints which are of distance at least $\alpha - 1$ apart in $G$. Applying Claim 7.2, we get that $K$ (and thus $F'$) cannot be contained in the union of fewer than $2(\alpha - 1)$ facial subgraphs of $G$, so we are done.

Many of the arguments in the subsequent sections (and papers) involve starting with a critical mosaic $(\Sigma, G, C, L, C_s)$, and then showing some desirable property holds by constructing a smaller mosaic either from one side of a separating cycle in $G$ or one side of a generalized chord of a ring of $C$. When we do this, we need to make sure the resulting smaller embedding still has high enough face-width to satisfy M5). This the purpose of the two facts below, which are consequences of Proposition 7.1. In Fact 7.3, we put a lower bound on the face-width of one side of a separating cycle in a high-face-width embedding. In Fact 7.4 we do the same for one side of a generalized chord of a facial cycle in a high-face-width embedding.
**Fact 7.3.** Let \( \Sigma \) be an orientable surface and let \( G \) be a 2-cell embedding on \( \Sigma \). Let \( D \subseteq G \) be contractible cycle and let \( \alpha \geq 2 \) be a constant. Let \( U, U' \) be the components of \( \Sigma \setminus D \), where \( U \) is an open disc. Suppose further that \( D \) is of distance at least \( \alpha \) from each facial subgraph of \( G \) in \( \text{Cl}(U') \) which is distinct from \( D \) and not a triangle. Then, for any 2-cell embedding \( H \) on \( \Sigma \) obtained from \( G \cap \text{Cl}(U') \) by adding some edges and vertices to \( \text{Cl}(U) \), we have 
\[
\text{fw}(H) \geq \min\{\text{fw}(G), 2(\alpha - 1), \text{ew}(G) - |E(D)| - 1 \}.
\]

**Proof.** Let \( G' := G \cap \text{Cl}(U') \). If \( G' \) is contractible, then \( H \) is also contractible, so Fact 7.3 holds trivially in that case. Now suppose that \( G' \) is not contractible. Since \( D \) is contractible, we have \( G' \neq D \). Let \( F \subseteq H \) be a noncontractible cycle and let \( F \) be a family of facial subgraphs of \( H \) whose union contains \( F \). We claim now that 
\[
|F| \geq \min\{\text{fw}(G), 2(\alpha - 1), \text{ew}(G) - |E(D)| - 1 \}.
\]
Suppose not. Since \( U \) is an open disc and \( D \) is a 2-cell subgraph of \( G' \), it follows that there is a noncontractible cycle \( F^* \subseteq G' \) and a family \( F^* \) of facial subgraphs of \( G' \) whose union contains \( F^* \), where \( |F^*| \leq |F| \) and \( F^* \) is obtained from \( F \) by replacing the set of elements of \( F \) which are not facial subgraphs of \( G' \) with \( \{D\} \) if any such elements exist.

Now, we have \( |F^*| < \min\{\text{fw}(G), 2(\alpha - 1), \text{ew}(G) - |E(D)| - 1 \} \). Since \( |F^*| < \text{fw}(G) \) and every facial subgraph of \( G' \), other than possibly \( D \), is also a facial subgraph of \( G \), we have \( V(F \cap D) \neq \emptyset \), and, in particular, \( D \in F^* \). Since \( |F^*| < 2(\alpha - 1) \), it follows from 2) of Proposition 7.1 that \( F \) shares no vertices with any facial subgraph of \( G \) in \( \text{Cl}(U') \) which is distinct from \( D \) and not a triangle. Thus, each element of \( F^* \setminus \{D\} \) is a triangle, and also a facial subgraph of \( G \). Since \( G' \neq D \) and \( \alpha \geq 2 \), each edge of \( D \) is contained in a facial triangle of \( G \), so \( F \) can be contained in the union of at most \( |F^* \setminus \{D\}| + |E(D)| \) facial triangles of \( G \). Since \( D \in F^* \), it follows from 1) of Proposition 7.1 that \( \text{ew}(G) \leq (|F^*| - 1) + |E(D)| + 2 \), and thus \( |F^*| \geq \text{ew}(G) - |E(D)| - 1 \), contradicting our assumption. 

Our second fact is the following.

**Fact 7.4.** Let \( \Sigma \) be an orientable surface and let \( G \) be a 2-cell embedding on \( \Sigma \). Let \( C \subseteq G \) be cyclic facial subgraph of \( G \) and let \( Q \) be a proper generalized chord of \( C \), where each of the three cycles of \( C \cup Q \) is contractible. Let \( \alpha \geq 2 \) be a constant and let \( G = G_0 \cup G_1 \) be the natural \( Q \)-partition of \( G \), and, for each \( j = 0, 1 \), let \( U_j \) be the unique component of \( \Sigma \setminus (C \cup Q) \) such that \( G_j = G \cap \text{Cl}(U_j) \), and suppose that \( U_1 \) is an open disc. Suppose further that \( C \cup Q \) is of distance at least \( \alpha \) from each facial subgraph of \( G \) in \( \text{Cl}(U_0) \) which is distinct from \( C \) and not a triangle. Let \( H \) be a 2-cell embedding on \( \Sigma \) obtained from \( G_0 \) by adding some edges and vertices to \( \text{Cl}(U_1) \). Then 
\[
\text{fw}(H) \geq \min\{\text{fw}(G), 2(\alpha - 1), \text{ew}(G) - 2|E(Q)| - 2 \}.
\]

**Proof.** Let \( x, y \) be the endpoints of \( Q \) and let \( G' \) be a graph obtained from \( G \) by adding to \( G \) a 2-path \( P \) with endpoints \( x, y \), where this 2-path is otherwise disjoint to \( \text{Cl}(U_0 \cup U_1) \). Likewise, with \( P \) as above, we let \( H' := H + P \). Note that, since \( G, H \) are 2-cell embeddings, each of \( G', H' \) is also a 2-cell embedding. Since \( C \) is a facial subgraph of \( G \).
and $\Sigma \setminus \text{Cl}(U_0 \cup U_1)$ is homeomorphic to a disc, we have $\text{fw}(G') = \text{fw}(G)$ and $\text{fw}(H') = \text{fw}(H)$. Now, the cycle $D := Q + P$ is contained in each of $G', H'$, and, applying Fact 7.3 to $D$, we have $\text{fw}(H') \geq \min\{\text{fw}(G'), 2(\alpha - 1), \text{ew}(G') - |E(D)| - 1\}$. Note that the addition of $P$ to $G$ has decreased the edge-width by at most $|E(Q)| - 1$, i.e. we have $\text{ew}(G') \geq \text{ew}(G) - (|E(Q)| - 1)$ and thus $\text{ew}(G') - |E(D)| - 1 \geq \text{ew}(G) - 2|E(Q)| - 2$. We conclude that $\text{fw}(H) \geq \min\{\text{fw}(G), 2(\alpha - 1), \text{ew}(G) - 2|E(Q)| - 2\}$, as desired. □

8 Obstructing Cycles

The purpose of this section is to show that one side of a separating cycle of length at most $N_{mo}$ is “obstructed” in a sense that one of the rings of the chart has to be close enough to the separating cycle to prevent us from coloring one side and then extending the precoloring by constructing of a smaller mosaic from the other side. This is made precise in the statement of Theorem 8.16. We first show that, under certain conditions, one side of a separating cycle in a critical mosaic is colorable. We first introduce the following notation.

**Definition 8.1.** Let $\Sigma$ be an orientable surface, let $G$ be an embedding on $\Sigma$, and let $C \subseteq G$ be a contractible cycle. We let $\text{Ext}_G^+(C)$ denote the subgraph of $G$ induced by $V(\text{Ext}_G(C))$, i.e. the graph consisting of $\text{Ext}_G(C)$ and all the chords of $C$ in the open region externally bounded by $C$. Likewise, we set $\text{Int}_G^+(C) := G[V(\text{Int}_G(C))]$.

Roughly speaking, we perform the following procedure to show that one side of a separating cycle in a critical mosaic is colorable: Let $T$ be a critical mosaic with underlying graph $G$ and let $D \subseteq G$ be a separating cycle. Let $\{G_0, G_1\}$ be the natural $D$-partition of $G$ and let $i \in \{0, 1\}$. To show that $V(G_i)$ is $L$-colorable, we need to construct a mosaic which has fewer vertices than $G$ and contains $G_i$ as a subgraph. This requires some care because our new graph still needs to be short-separation-free and needs to satisfy the high face-width conditions and triangulation conditions. We thus introduce the following two definitions.

**Definition 8.2.** Let $\Sigma$ be an orientable surface and let $D$ be an embedding of a contractible cycle on $\Sigma$. Let $U_0, U_1$ be the two components of $\Sigma \setminus D$, where $U_0$ contains the north pole of $\Sigma$. We define $\Sigma_D$ to be the surface obtained from $\Sigma$ by replacing $\text{Cl}(U_1)$ with a closed disc externally bounded by $D$. In particular, $\Sigma_D$ has the same north pole as $\Sigma$, and, if $D$ is inward-contractible, then $\Sigma_D = \Sigma$.

**Definition 8.3.** Let $\Sigma$ be an orientable surface and let $D^0$ be an embedding of an inward-contractible cycle on $\Sigma$ with $|E(D^0)| \geq 5$. Let $U$ be the open disc externally bounded by $D$, and let $n := \left\lceil \frac{|E(D^0)|}{4} \right\rceil$. A $D^0$-web is an embedding on $\Sigma$ obtained from $D^0$ by adding to $U$ a sequence of $n$ concentric cycles $D^1, \ldots, D^n$, an additional vertex $x$, where $x$ is adjacent to all the vertices of $D^n$, and, for each $i = 1, \ldots, n$, adding some edges between $D^i$ and $D^{i-1}$, such that the following hold.
1) $D^1$ is a cycle $v_1 \ldots v_{2n}$ of length $2n$, where, for each $v_i$ of odd index, the neighborhood of $v_i$ on $D^0$ is a lone vertex, and, for each $v_i$ of even index, the neighborhood of $v_i$ on $D^0$ is an edge of $D^0$, and furthermore, the $D^0$-neighborhoods of any two consecutive $D^1$-vertices have nonempty intersection; AND

2) $D^2$ is a cycle of length $n$ lying in the open disc externally bounded by $D^1$, where, for each $v \in V(D^2)$, the neighborhood of $v$ on $D^1$ is a path of length two whose midpoint has even index in $v_1 \ldots v_{2n}$. Furthermore, the $D^1$-neighborhoods of consecutive $D^2$-vertices intersect on precisely a common endpoint of the two paths; AND

3) For each $i = 3, \ldots, n$, the following hold.
   a) $D^i$ separates $x$ from $D^{i-1}$; AND
   b) For each $w \in V(D^i)$, the neighborhood of $w$ in $D^{i-1}$ is an edge of $D^{i-1}$; AND
   c) For any distinct $w, w' \in V(D^i)$, the neighborhoods of $w, w'$ in $D^{i-1}$ are distinct edges, and these edges share an endpoint of $D^{i-1}$ if and only if $ww'$ is an edge of $D^i$.

When we construct a smaller mosaic from one side of a separating cycle in the underlying graph of a critical mosaic, we need to check that the resulting graph satisfies the required properties, which is the purpose of Proposition 8.4 below.

**Proposition 8.4.** Let $\Sigma$ be an orientable surface and let $n \geq 3$ be an integer. Let $G$ be an embedding on $\Sigma$ and let $D \subseteq G$ be an inward-contractible cycle of length $n$. Let $\mathcal{D}$ be the set of induced cycles of $G[V(D)] \cap \text{Int}(D)$ and let $G'$ be an embedding obtained from $\text{Ext}^+(D)$, where, for each $D' \in \mathcal{D}$ of length at least five, we add a $D'$-web to the open disc externally bounded by $D'$. Then the following hold.

1. $G'$ is short-separation-free; AND
2. For any $x, y \in V(G)$, we have $d_G(x, y) \leq d_{G'}(x, y)$; AND
3. $|V(\text{Int}^{G'}(D))| \leq 9n^2$.

**Proof.** Let $\mathcal{D}'$ be the set of $D \in \mathcal{D}$ of length at least five. For each $F \in \mathcal{D}'$, since $\left\lceil \frac{|E(F)|}{4} \right\rceil \geq 2$, we are adding at least two cycles to the open disc externally bounded by $F$ before we add a lone neighbor to all the vertices of the innermost cycle, and the innermost cycle has length $|E(F)|$. Possibly $G$ has some chords of $F$ in $\text{Ext}(F)$, or there is a possibly a vertex of $V(\text{Ext}(F)) \setminus V(F)$ with two nonadjacent neighbors in $D$, but in any case, since $G$ is short-separation-free, it is clear that $G'$ is still short-separation-free. Furthermore, for each $F \in \mathcal{D}'$, since we have added $\left\lceil \frac{|E(F)|}{4} \right\rceil$ concentric cycles to the open disc bounded by $F$, where the first one has length $2|E(F)|$ and the others all have length $|E(F)|$, it is clear from our construction that, for any $x, y \in V(F)$, there is no $(x, y)$-path in $G'$ of length strictly less than $d_G(x, y)$, so 2) follows. Now we prove 3). We first note the following.
Claim 8.5. For each $D' \in D$, we have $|V(\text{Int}_{G'}(D'))| \leq |E(D')|^2$.

Proof: This is immediate if $|E(D')| \leq 4$ so let $|E(D')| \geq 5$. By Definition 8.3, we have $|V(\text{Int}_{G'}(D'))| \leq \frac{|E(D')|^2}{4} + 3|E(D')| + 1$, where the $3|E(D')|$-term is due to the contribution from the edges of the outermost cycle in the open disc externally bounded by $D'$ and the edges of $D'$ itself. Since $|E(D')| \geq 5$, we have $3|E(D')| + 1 \leq \frac{3|E(D')|^2}{4}$, so $|V(\text{Int}_{G'}(D'))| \leq |E(D')|^2$. □

Let $m := |D|$. Note that the subgraph of $G'$ consisting of $D$ and all of the chords of $D$ in the open disc externally bounded by $D$ can be regarded as an outerplanar embedding, since $D$ is inward-contractible. In particular, we have $m \leq n - 4$, and furthermore, $\sum_{D' \in D} |E(D')| = n + 2(m - 1) \leq 3n$, since, in the sum on the left, each chord of $D$ in the open disc bounded by $D$ is counted twice and each edge of $D$ is counted once. Applying Claim 8.5 we have the following.

$$|V(\text{Int}_{G'}(D))| \leq \sum_{D' \in D} |V(\text{Int}_{G'}(D'))| \leq \sum_{D' \in D} |E(D')|^2 \leq \left( \sum_{D' \in D} |E(D')| \right)^2$$

Thus, we get $|V(\text{Int}_{G'}(D))| \leq 9n^2$. □

We now introduce the following strengthening of Definition 6.1.

Definition 8.6. Let $\Sigma$ be an orientable surface and let $G$ be an embedding on $\Sigma$. Given a cyclic facial subgraph $C \subseteq G$, we say that $C$ is expectable if $C$ is an induced cycle and the following hold.

1) For every $v \in D_1(C)$, the graph $G[N(v) \cap V(C)]$ is a path of length at most two; AND

2) There is at most one $v \in D_1(K) \setminus V(C)$ such that $|N(v) \cap V(K)| = 3$.

The following is immediate.

Observation 8.7. Let $\Sigma$ be an orientable surface, let $G$ be an embedding on $\Sigma$ and let $C \subseteq G$ be a cyclic facial subgraph of $G$. If $C$ is expectable, then, for any list-assignment $L$ of $V(G)$, $C$ is $L$-predictable.

With the definitions and results above in hand, we prove the following.

Proposition 8.8. Let $\mathcal{T} = (\Sigma, G, C, L, C_\ast)$ be a critical mosaic, and let $D \subseteq G$ be a cycle, where $|V(D)| \leq N_{\text{mo}}$ and at least one $C \in C$ lies in $\text{Int}(D)$. Suppose further that each such $C$ is of distance at least $10N_{\text{mo}}^2$ from $D$. Then $V(\text{Ext}(D))$ is $L$-colorable.

Proof. Note that, since $\text{ew}(G) \geq \text{fw}(G) > N_{\text{mo}}$, $D$ is contractible, so the above is well-defined. It suffices to show that $\text{Ext}^+(D)$ is $L$-colorable. Let $D$ be the set of the induced cycles of $G[V(D)] \cap \text{Int}(D)$. Possibly $D = \{D\}$, if $D$
has no chords in Int(D). It follows from M5) applied to T that there is a D† ∈ D such that each cycle of D \ {D†} is inward-contractible (possibly D† is inward-contractible as well). Now, Ext+(D) can be regarded as an embedding on ΣD† in the obvious way, where this embedding has outer face C+. Note that each cycle in D, including D†, is inward-contractible in ΣD†, and D is also inward-contractible on ΣD†. Let G′ be an embedding on ΣD† obtained from Ext+(D), where, for each D′ ∈ D of length at least five, we add an inward-facing D′-web to the open disc externally bounded by D′.

**Claim 8.9.** |V(G′)| < |V(G)|.

**Proof:** Since D is inward-contractible on ΣD†, it follows from 3) of Proposition 8.4 that |V(IntG′(D))| ≤ 9|E(D)|^2. By assumption, there is a C ∈ C with C ⊆ IntG(D), and dl(C, D) ≥ 10N^2 mo, so |V(IntG(D))| ≥ 10N^2 mo. Since D has length at most N mo, we have |V(IntG′(D))| < |V(IntG(D))|, and thus |V(G′)| < |V(G)|. □

Let C′ := {C ∈ C : C ⊆ ExtG(D)}. Let L′ be a list-assignment for V(G′) where each vertex of G′ \ G is given an arbitrary 5-list and otherwise L′ = L. By assumption, each C ∈ C lying in the closed region externally bounded by D is of distance at least 10N^2 mo from D, i.e there is no C ∈ C \ C′ which has nonempty intersection with D, so every vertex of G′ with an L′-list of size less than five lies in ∪C∈C′ V(C).

Let T′ := (ΣD†, G′, C′, L′, C∗). It follows from 1) of Proposition 8.4 that G′ is short-separation free. Since D is a separating cycle and G is short-separation-free, there is no induced cycle of G[V(D)] of length precisely four, i.e each element of D is either a triangle or has length at least five. In particular, it follows from our construction that every facial subgraph of G′, except those of C′, is a triangle, so T′ is a tessellation. As the genus has not increased, it also follows from 2) of Proposition 8.4 that T′ satisfies M3), M4) of Definition 6.4 and M0)-M2) are trivially satisfied. Now we just check M5). Note that g(ΣD†) ∈ {0, g(Σ)}. If g(ΣD†) = 0, then M5) is trivially satisfied, so we are done in that case, so now suppose that g(ΣD†) = g(Σ). It follows from 2) of Proposition 8.4 that ew(G′) ≥ ew(G). Note that, since G is a 2-cell embedding on Σ, G′ is a 2-cell embedding on ΣD†. Since ew(G) ≥ 2.4β · 3^g(Σ)−2 and fw(G) ≥ 1.1β · 3^g(Σ)−2, it follows from our distance conditions, together with Fact 7.3 that fw(G′) ≥ 1.1β · 3^g(Σ)−2. Thus, T′ is a mosaic. Since T is critical, it follows from Claim 8.9 that G′ is L′-colorable, so ExtG′(D) is L-colorable, as desired. □

Note that Proposition 8.8 is slightly stronger than the statement that Int(D) is L-colorable, since it states that the subgraph of G induced by V(Int(D)) is L-colorable. We have an analogous fact for the other side.

**Proposition 8.10.** Let T = (Σ, G, C, L, C∗) be a critical mosaic, and let D ⊆ G, where |V(D)| ≤ N mo and at least one C ∈ C lies in Int(D). Suppose further that each C ∈ C lying in Ext(D) is of distance at least 10N^2 mo from D. Then V(Int(D)) is L-colorable.
Proof. Since \( \text{ew}(G) \geq \text{fw}(G) > N_{\text{mo}} \), \( D \) is contractible, so the above is well-defined. It suffices to prove that \( \text{Int}^+(D) \) is \( L \)-colorable. By assumption, we have \( d(C_s, D) \geq 10N_{\text{mo}}^2 \), and there is a \( C^\dagger \in C \) with \( C^\dagger \subseteq \text{Int}(D) \). Thus, \( D \) is a separating cycle. In particular, \( D \) separates \( C^\dagger \) from \( C_s \). Since \( \text{fw}(G) > 1 \), \( C^\dagger \) is contractible. Let \( U^\dagger \) be the connected component of \( \Sigma \setminus C^\dagger \) externally bounded by \( C^\dagger \). Since \( C^\dagger \neq C_s \) and \( C^\dagger \) is a facial subgraph of \( G \), we have \( U^\dagger \cap V(G) = \emptyset \). We begin by “inverting” the orientation of \( G \) in the following way. Let \( \Sigma^\dagger \) be a surface obtained from \( \Sigma \) by moving the north pole to \( U^\dagger \). Let \( G^\dagger \) be the resulting embedding of \( \text{Int}^+(D) \) on \( \Sigma^\dagger \) in the obvious way, where this embedding has outer face \( C^\dagger \).

Let \( \mathcal{D} \) be the set of the induced cycles of \( G^\dagger \)[\( V(D) \] \( \cap \) \( \text{Int}^+(D) \) (the chords of \( D \) in \( \text{Int}^+(D) \) are precisely the chords of \( D \) in \( \text{Ext}^+(D) \)). Since \( \text{ew}(G^\dagger) = \text{ew}(G) > N_{\text{mo}} \), there is a unique \( F \in \mathcal{D} \) such that every element of \( \mathcal{D} \setminus \{F\} \) is inward-contractible. Regarding \( G^\dagger \) as an embedding on \( \Sigma^\dagger_p \), each element of \( \mathcal{D} \cup \{D\} \) is inward-contractible on \( \Sigma^\dagger_p \).

As in Proposition \( 8.8 \), we augment \( G^\dagger \) to an embedding \( K \) on \( \Sigma^\dagger_p \) in the following way. For each \( D \in \mathcal{D} \) of length at least five, we add a \( D \)-web to the open region of \( \Sigma^\dagger_p \) externally bounded by \( D \).

Claim 8.11. \( |V(K)| < |V(G)| \).

Proof: It just suffices to show that \( |V(\text{Int}_K(D))| < |V(\text{Ext}_G(D))| \). Since \( D \) is inward-contractible on \( \Sigma^\dagger_p \), it follows from 3) of Proposition \( 8.4 \) that \( |V(\text{Int}_K(D))| \leq 9|E(D)|^2 \). By assumption, we have \( d_G(C_s, D) \geq 10N_{\text{mo}} \), so \( |V(\text{Ext}_G(D))| \geq 10N_{\text{mo}}^2 \) and thus \( |V(\text{Int}_K(D))| < |V(\text{Ext}_G(D))| \), as desired. ■

Let \( C' := \{ C \in C : C \subseteq K \} \). Let \( L' \) be a list-assignment for \( V(K) \) where each vertex of \( K \setminus G \) is given an arbitrary 5-list and otherwise \( L' = L \). By assumption, each \( C \in C \) lying in the closed region of \( \Sigma \) internally bounded by \( D \) is of distance at least \( 10N_{\text{mo}}^2 \) from \( D \), i.e there is no such \( C \) which has nonempty intersection with \( D \), so every vertex of \( K \) with an \( L' \)-list of size less than five lies in \( \bigcup_{C \in C'} V(C) \).

Let \( \mathcal{T}' := (\Sigma^\dagger_p, K, C', L', C^\dagger) \). By 1) of Proposition \( 8.4 \), \( K \) is short-separation free. Since \( G \) is short-separation-free and \( D \) is a separating cycle of \( G \), there is no cycle of \( \mathcal{D} \) of length precisely four, i.e each element of \( \mathcal{D} \) is either a triangle or has length at least five. In particular, every facial subgraph of \( K \), except those of \( C' \), is a triangle, so \( \mathcal{T}' \) is a tessellation. Since \( C^\dagger \) is an internal ring of \( C \) but the outer face of \( K \), and the genus has not increased, the distance conditions that \( C^\dagger \) needs to satisfy have only weakened, so it also follows from 2) of Proposition \( 8.4 \) that \( \mathcal{T}' \) satisfies M3), M4) of Definition \( 6.4 \) and M0)-M2) are trivially satisfied, so we just need to check M5). We have \( g(\Sigma^\dagger_p) \in \{0, g(\Sigma)\} \), and if \( g(\Sigma^\dagger_p) = 0 \), then M5) is trivially satisfied, so suppose that \( g(\Sigma^\dagger_p) = g(\Sigma) \). It follows from 2) of Proposition \( 8.4 \) that \( \text{ew}(K) \geq \text{ew}(G) \). Note that, since \( G \) is a 2-cell embedding on \( \Sigma \), it follows from our construction of \( K \) that \( K \) is a 2-cell embedding on \( \Sigma^\dagger_p \). Since \( \text{ew}(G) \geq 2.4\beta \cdot 3^{g(\Sigma) - 2} \) and \( \text{fw}(G) \geq 1.1\beta \cdot 3^{g(\Sigma) - 2} \), it follows from Fact \( 7.3 \) together with our distance conditions, that \( \text{fw}(K) \geq 1.1\beta \cdot 3^{g(\Sigma) - 2} \), so \( \mathcal{T}' \) satisfies M5). Thus,
$T'$ is a mosaic. Since $T$ is critical, it follows from Claim 8.11 that $K$ is $L'$-colorable, so $\text{Int}_C(D)$ is $L$-colorable, as desired. 

Now we prove the two main results of Section 6, the first of which is stated below in Theorem 8.12. Note that, in the statement of Theorem 8.12, $D$ is contractible by our edge-width conditions, so $\text{Int}(D)$ is well-defined.

**Theorem 8.12.** Let $T = (\Sigma, G, C, L, C^*)$ be a critical mosaic and let $D \subseteq G$ be a separating cycle of length at most $N_{\text{mo}}$. Then either $D$ is inward-contractible or $\text{Int}(D)$ contains at least one element of $C$.

**Proof.** Suppose there is a separating cycle $D \subseteq G$ of length at most $N_{\text{mo}}$ for which neither of these hold, and, among all such cycles, we choose $D$ so that $|V(\text{Int}(D))|$ is minimized. Let $g := g(\Sigma)$ and let $U$ be the component of $\Sigma \setminus D$ externally bounded by $D$. By our assumption on $D$, we have $g > 0$ and, since $D$ is contractible, every noncontractible closed curve of $\Sigma$ intersects with $\text{Cl}(U)$.

**Claim 8.13.** Let $D'$ be a cycle of length at most $N_{\text{mo}}$ with $D' \subseteq \text{Int}(D)$ and let $U'$ be the open connected component of $\Sigma \setminus D'$ externally bounded by $D'$. If $U'$ contains a noncontractible closed curve of $\Sigma$, then $D'$ is a separating cycle of $G$.

**Proof:** By our edge-width conditions, $D'$ is contractible so $U'$ is well-defined, and since $G$ is a 2-cell embedding, there is a noncontractible cycle of $G$ lying in $G \cap \text{Cl}(U')$. Since $\text{ew}(G) \geq 2.4\beta \cdot 3^{9/2}$, we have $|V(\text{Int}(D')) \setminus V(D')| > 0$. Since $D$ is a separating cycle of $G$ and $D' \subseteq \text{Int}(D)$, we have $|V(\text{Ext}(D')) \setminus V(D')| > 0$ as well, so $D'$ is a separating cycle of $G$. ■

Since $C \subseteq \text{Int}(D) = \emptyset$ and $D$ has been chosen to minimize $|V(\text{Int}(D))|$, Claim 8.13 immediately implies the following.

**Claim 8.14.** There are no chords of $D$ in $U$ and, for any $v \in U$, either $|N(v) \cap U| \leq 1$ or $N(v) \cap V(D)$ consists of two consecutive vertices of $D$. In particular, $D$ is an expectable facial subgraph of $\text{Int}(D)$.

Now we have the following.

**Claim 8.15.** $V(\text{Ext}(D))$ is $L$-colorable.

**Proof:** Since $D$ is contractible and all of noncontractible closed curves of $\Sigma$ have nonempty intersection with $\text{Cl}(U)$, it follows that $\text{Ext}(D)$ can be regarded as an embedding on $\mathbb{S}^2$ with outer face $C_*$, and there is an embedding $G^\dagger$ on $\mathbb{S}^2$ which is obtained from $\text{Ext}(D)$ by adding a $D$-web to the open disc of $\mathbb{S}^2$ externally bounded by $D$. Let $L^\dagger$ be a list-assignment for $V(G^\dagger)$ where each vertex of $\setminus D$ is assigned an arbitrary 5-list, and otherwise $L^\dagger = L$. Let $\mathcal{T}^\dagger := (\mathbb{S}^2, G^\dagger, C, L^\dagger, C_*)$ and note that $\mathcal{T}^\dagger$ is a tessellation.
We claim now that $\mathcal{T}^\dagger$ is a mosaic. M5) is trivially satisfied, since the underlying surface is $S^2$, and M0)-M2) are immediate as well. Since the genus has only decreased, it follows from 2) of Proposition 8.4 that $\mathcal{T}^\dagger$ also satisfies M3)-M4), so $\mathcal{T}^\dagger$ is indeed a mosaic. By 3) of Proposition 8.4, we have $|V(\text{Int}_G(D))| \leq 9N^2_{\text{mo}}$. Since $G$ is a 2-cell embedding, $\Sigma$ contains a noncontractible cycle of $G$ in $\text{Cl}(U)$, so $|V(\text{Int}_G(D))| \geq \text{ew}(G) \geq 2.4\beta \cdot 3^{g-2}$. Thus, we have $|V(G^\dagger)| < |V(G)|$. Since $\mathcal{T}$ is critical, it follows that $G^\dagger$ is $L^1$-colorable, so $\text{Ext}(D)$ is $L$-colorable. It follows from Claim 8.14 that any $L$-coloring of $\text{Ext}(D)$ is also an $L^1$-coloring of the subgraph of $G$ induced by $V(\text{Ext}(D))$. □

By Claim 8.15 there is an $L$-coloring $\psi$ of $V(\text{Ext}(D))$. Let $\mathcal{T}' := (\Sigma, \text{Int}(D), \{D\}, L^D_\psi, D)$. Every facial subgraph of $\text{Int}(D)$, other than $D$, is a triangle, so $\mathcal{T}'$ is a tessellation, where $D$ is a closed ring. We claim now that $\mathcal{T}'$ is a mosaic. M0) and M1) are trivially satisfied, since $|E(D)| \leq N_{\text{mo}}$, and $D$ is a closed ring. By Claim 8.14, $\mathcal{T}'$ satisfies M2) as well. Since $\mathcal{T}'$ has only a lone ring, the distance conditions M3)-M4) are trivially satisfied, so we just need to check that M5) holds. Since edge-width is monotone, we have $\text{ew}(\text{Int}(D)) \geq 2.4\beta \cdot 3^{g-2}$. Since $\text{fw}(G) \geq 1.1\beta \cdot 3^{g-2}$ and $|E(D)| \leq N_{\text{mo}}$, it follows from Fact 7.3 that $\text{fw}(\text{Int}(D)) \geq 1.1\beta \cdot 3^{g-2}$. We conclude that $\mathcal{T}'$ is indeed a mosaic. Since $D$ is a separating cycle of $G$, we have $|V(\text{Int}(D))| < |V(G)|$. Since $\mathcal{T}$ is critical, it follows that $V(\text{Int}(D))$ is $L^D_\psi$-colorable, so $\psi$ extends to an $L$-coloring of $G$, contradicting the fact that $T$ is not colorable. □

In the second of the two main results of Section 8, which is stated below in Theorem 8.16, we establish some very useful bounds on the distance between separating cycles and rings in a critical mosaic.

**Theorem 8.16.** Let $\mathcal{T} = (\Sigma, G, \mathcal{C}, L, C_\ast)$ be a critical mosaic and let $D \subseteq G$ be a cycle of length at most $N_{\text{mo}}$, where $D$ is a separating cycle of $G$ and at least one element of $\mathcal{C}$ lies in $\text{Int}(D)$. Let $g := g(\Sigma)$ and $g' := g(\Sigma_D)$. Then the following hold.

1. There exists a $C \in C_{\subseteq \text{Int}(D)}$ with $\max\{d(v, w(C)) : v \in V(D)\} < \beta \cdot 3^{(g-g')-1} + \frac{3|E(D)|}{2} + \text{Rk}(C)$; AND

2. For each $C \in C_{\subseteq \text{Int}(D)}$, we have $d(D, w(C)) > \beta \cdot (3^{g-2} - 3^{g'-2}) + \text{Rk}(C) - \frac{3|E(D)|}{2}$.

**Proof.** Let $\mathcal{F}$ be the set of separating cycles of $G$ at length at most $N_{\text{mo}}$ such that, for each $D \in \mathcal{F}$, $\text{Int}(D)$ contains an element of $\mathcal{C}$. By M5), each element of $\mathcal{F}$ is contractible. We first prove 1). Suppose that 1) does not hold, and let $D \in \mathcal{F}$ be a counterexample to 1) which minimizes the quantity $|V(\text{Int}(D))|$. We claim 17. Each element of $C_{\subseteq \text{Int}(D)}$ is of distance at least $\beta \cdot 3^{(g-g')-1}$ from $D$, and furthermore, $V(\text{Ext}(D))$ is $L$-colorable.

**Proof:** Since $D \in \mathcal{F}$, we have $C_{\subseteq \text{Int}(D)} \neq \emptyset$. Note that, for each $C \in \mathcal{C}$, each vertex of $C$ is of distance at most $\frac{\text{Rk}(C)}{6}$ from $w(C)$. Since $D$ is a counterexample to 1), it follows that every element of $C_{\subseteq \text{Int}(D)}$ is of distance at
least \( \beta \cdot 3^{q(g-g')} - 1 \) from \( D \). Since \( g - g' \geq 0 \) and \( \beta \geq 100N_{\text{max}}^2 \), it follows from Proposition[8,3] that \( V(\text{Ext}(D)) \) is \( L \)-colorable. 

Let \( U, U^* \) be the connected component of \( \Sigma \setminus D \), where \( U^* \) contains the north pole of \( \Sigma \). We now have the following.

Claim 8.18.

i) \( D \) has no chords in \( U \); AND

ii) For each \( v \in U \), we have \( |N(v) \cap V(D)| \leq 3 \), and the vertices of \( N(v) \cap V(D) \) are consecutive on \( D \); AND

iii) There is at most one \( v \in U \) with more than two neighbors on \( D \)

Proof: Firstly, for any cycle \( D^\dagger \subseteq \text{Int}(D) \) of length at most \( N_{\text{max}} \), \( D^\dagger \) is contractible, and furthermore, we have \( \Sigma_{D^\dagger} \in \{g, 0\} \). In particular, if \( g' = g \), then \( g(\Sigma_{D^\dagger}) = g \) as well. Thus, we have \( g - g' \geq g - g(\Sigma_{D^\dagger}) \). This observation, together with the minimality of \( |V(\text{Int}(D))| \), immediately implies i).

Now suppose that there is a \( v \in U \) such that \( N(v) \cap V(D) \neq \emptyset \) and \( N(v) \cap V(D) \) does not consist of at most three consecutive vertices of \( D \). Since \( G \) is short-separation-free and \( D \) is a separating cycle, we have \( |V(D)| \geq 5 \). Furthermore, since \( D \) has no chords in \( U \), it follows from our triangulation conditions that, if \( v \) has two neighbors of distance precisely two apart on \( D \), then \( v \) is also adjacent to the unique common neighbor of these vertices on \( D \). Since \( N(v) \cap V(D) \) does not consist of at most three consecutive vertices of \( D \), it follows that \( \text{Int}(D) \) contains a 2-chord \( wvw' \) of \( D \) with midpoint \( v \), where \( w, w' \) are of distance at least three apart in \( D \). Thus, \( \text{Int}(D) \) contains two contractible cycles \( D^0, D^1 \) such that \( D^0 \cup D^1 = D + wvw' \). For each \( i = 0, 1 \), we have \( |E(D_i)| < |E(D)| \).

Furthermore, for each \( i = 0, 1 \), we have \( g - g' \geq g - g(\Sigma_{D^i}) \). Since each of \( D_0, D_1 \) lies in the ball of distance one from \( D \), it follows that at least one of \( D_0, D_1 \) is a counterexample to 1) with strictly fewer vertices in its interior than \( D \), contradicting our assumption. This proves ii).

Now we prove iii). Suppose that iii) does not hold and let \( w_0, w_1 \) be two distinct vertices of \( U \) which each ahve more than two neighbors on \( D \). By ii), there exist two subpaths \( P_0, P_1 \) of \( D \), each of length two, such that \( N(w_i) \cap V(D) = V(P_i) \) for each \( i = 0, 1 \). Thus, for each \( i = 0, 1 \), we let \( x_i \) be the midpoint of \( P_i \) and let \( D_i \) be the cycle of length \( |E(D_i)| \) obtained from \( D \) by replacing \( v_i \) with \( w_i \). Let \( m := |E(D)| = |E(D_0)| = |E(D_1)| \). Each of \( D_0, D_1 \) is contractible, and since \( G \) is short-separation-free, we have \( \text{Int}(D_i) = \text{Int}(D) - v_i \) for each \( i = 0, 1 \). In particular, by Claim[8,1] there is a \( C^\dagger \in C \) lying in the common interior of \( D_0, D_1 \). Furthermore, since all triangles of \( G \) are contractible, we have \( g - g(\Sigma_{D_0}) = g - g(\Sigma_{D^1}) = g - g' \).

By the minimality of \( |V(\text{Int}(D))| \), we have \( \max\{d(v, w(C^\dagger)) : v \in V(D_i)\} \geq \beta \cdot 3^{g-g'} + \frac{3\alpha}{\beta} + \text{Rk}(C^\dagger) \) for each \( i = 0, 1 \). But then, since \( D \) is a counterexample to 1), it follows that, for each \( i = 0, 1 \), \( x_i \) is the unique vertex of \( D \)
which maximizes the quantity \(d(v, w(C^1))\) among the vertices of \(D\). Since \(x_0 \neq x_1\), we have a contradiction. ■

By Claim 8.17, \(V(\text{Ext}(D))\) admits an \(L\)-coloring \(\phi\). Let \(\Sigma^*_D\) be the surface obtained from \(\Sigma\) by replacing \(U^*\) with an open disc, where the north pole of \(\Sigma^*_D\) lies in this disc. Now, \(\text{Int}(D)\) can be regarded as an embedding on \(\Sigma^*_D\) in the obvious way, where this embedding has outer face \(D\). Note that \(g(\Sigma^*_D)) = g - g'\). We claim now that \(T^* := (\Sigma^*_D, \text{Int}(D), \mathcal{C}^{|\text{Int}(D)|} \cup \{D\}, L^D_\phi, D)\) is a mosaic, where \(D\) is a closed \(T'^*-\text{ring}\). M0) and M1) are clearly satisfied, and it follows from Claim 8.18 that \(D\) is expectable and thus \(L^D_\phi\)-predictable, so \(T'\) satisfies M2) as well.

Claim 8.19. \(T'\) satisfies M5).

**Proof:** This is trivial if \(\Sigma^*_D = \mathbb{S}^2\) so suppose that \(\Sigma^*_D \neq \mathbb{S}^2\). Since \(D\) is contractible, all the noncontractible closed curves of \(\Sigma\) have nonempty intersection with \(\text{Cl}(U)\), and thus \(\Sigma^*_D = \Sigma\) and \(g' = 0\). Since edge-width is monotone, we have \(\text{ew}(\text{Int}(D)) \geq 2.4\beta \cdot 3g^2 - 2\). Because \(D\) is a counterexample to 1), each element of \(\mathcal{C}^{|\text{Int}(D)|}\) is of distance at least \(\beta \cdot 3g^2 - 1\) from \(D\). Since \(G\) is a 2-cell embedding, \(\text{Int}(D)\) is also a 2-cell embedding. Since \(|E(D)| \leq N_{mo}\) it follows from Fact 7.23 that \(\text{fw}(\text{Int}(D)) \geq 3.1\beta \cdot 3g^2 - 2\). ■

Since \(D\) is a counterexample to 1) and any two vertices of \(D\) are of distance at most \(\frac{\beta}{4}\) apart, it follows that, for each \(C \in \mathcal{C}^{\text{Int}(D)}\), we have \(d(w(C), w(D)) \geq \frac{\beta}{4} + |V(D)| + \text{Rk}(T|C)|\). Since \(g(\Sigma^*_D) \leq g\) and all the rings of \(\mathcal{C}^{\text{Int}(D)}\) have the same rank in \(T\) and \(T'\), it follows that \(T'\) satisfies M3)-M4) as well. Thus, \(T'\) is a mosaic. Since \(D\) is a separating cycle of \(G\), we have \(|V(\text{Int}(D))| < |V(G)|\), and since \(T\) is critical, \(\text{Int}(D)\) admits an \(L^D_\phi\)-coloring, so \(\phi\) extends to an \(L\)-coloring of \(T\), a contradiction. This proves 1).

Now we prove 2). We follow a similar argument to 1). Suppose that 2) does not hold, and let \(D \in \mathcal{F}\) be a counterexample to 2) which minimizes the quantity \(|V(\text{Ext}(D))|\). Since \(D\) is a counterexample to 2), there is a \(C^1 \in \mathcal{C}\) with \(C^1 \subseteq \text{Int}(D)\) and \(d(D, w(C^1)) \leq \beta \cdot (3g^2 - 1 - 3g^2 - 1) + \text{Rk}(C^1) - \frac{3|E(D)|}{2}\).

Claim 8.20.

i) \(d(D, w(C_*)) \geq \beta \cdot 3g^2 - 1 + \text{Rk}(C) + |E(D)|; \text{AND} \)

ii) For each \(C \in \mathcal{C}^{\text{Ext}(D)} \setminus \{C_*\}\), we have \(d(D, w(C)) \geq \beta \cdot 3g^2 + \text{Rk}(C) + |E(D)|\).

**Proof:** Let \(C \in \mathcal{C}\). Since any two vertices of \(D\) are of distance at most \(\frac{|E(D)|}{2}\) apart, we have \(d(D, w(C)) \geq d(w(C), w(C^1)) - d(D, w(C^1)) - \frac{|E(D)|}{2}\). Consider the following cases.

Case 1: \(C = C_*\)

In this case, we have \(d(w(C), w(C^1)) \geq \beta \cdot 3g^2 - 1 + \text{Rk}(C) + \text{Rk}(C^1)\) by the distance conditions of Definition 6.4 and thus \(d(D, w(C)) \geq \beta \cdot 3g^2 - 1 + \text{Rk}(C) + |E(D)|\), as desired.
Claim 8.21.

Let $U \subseteq D$ be a separating cycle of $G$. Since edge-width is monotone, we have $\text{ew}(\text{Ext}(D)) \geq \text{ew}(G) \geq 2.4 \beta \cdot 3^{g-2}$. Since $G$ is a 2-cell embedding $\text{Ext}(D)$ is also a 2-cell embedding, and since $|E(D)| \leq N_{\text{mo}}$, it follows from Fact 7.3 that $\text{fw}(\text{Ext}(D)) \geq 1.1 \beta \cdot 3^{g-2}$.

We conclude that $\mathcal{T}''$ is a mosaic. Since $D$ is a separating cycle of $G$, we have $|V(\text{Ext}(D))| < |V(G)|$. Since $\mathcal{T}$ is critical, $\mathcal{T}''$ is colorable, so $\psi$ extends to an $L$-coloring of $G$, a contradiction. This completes the proof of Theorem 8.16.

To conclude Section 8 we prove the following two useful corollaries to Theorems 8.12 and 8.16.

Corollary 8.23. Let $\mathcal{T} = (\Sigma, G, \mathcal{C}, L, C_\ast)$ be a critical mosaic and let $D \subseteq G$ be a cycle of length at most $N_{\text{mo}}$, where $D$ is a separating cycle of $G$. Let $g' := g(\Sigma_D)$ and suppose further that there is a $C \in \mathcal{C}$ such that one of the following holds.
a) \( C \neq C_\ast \) and 
\[ d(D, w(C)) \leq \beta \cdot (3^{g - 1} - 3^{g' - 1}) + \text{Rk}(C) - \frac{3|E(D)|}{2}, \text{ OR} \]

b) \( C = C_\ast \) and 
\[ d(D, w(C)) \leq \beta \cdot (3^{g - 1} - 3^{3(g' - 1)}) + \text{Rk}(C) - \frac{3|E(D)|}{2} \]

Then every element of \( C \) lies in \( \text{Ext}(D) \), and furthermore, \( g = g' \), i.e \( D \) is inward-contractible.

**Proof.** It suffices to show that every element of \( C \) lies in \( \text{Ext}(D) \). If this holds, then, since \( D \) is a separating cycle of \( G \), it follows from Theorem 8.12 that \( D \) is also inward-contractible. Suppose toward a contradiction that not every element of \( C \) lies in \( \text{Ext}(D) \). Since each element of \( C \) is a facial subgraph of \( G \), it follows that \( C^{\text{Int}(D)} \neq \emptyset \). By assumption, there is a \( C \in \mathcal{C} \) satisfying either a) or b). By 1) of Theorem 8.16 there is a \( C^\dagger \subseteq \text{Int}(D) \) with 
\[ \max \{ d(v, w(C^\dagger)) : v \in V(D) \} < \beta \cdot 3^{3(g' - 1)} + \frac{3|E(D)|}{2} + \text{Rk}(C) \]

Consider the following cases.

**Case 1:** \( C \neq C_\ast \)

In this case, we have 
\[ d(D, w(C)) \leq \beta \cdot (3^{g - 1} - 3^{g' - 1}) + \text{Rk}(C) - \frac{3|E(D)|}{2} \]. By 2) of Theorem 8.16 we have \( C \subseteq \text{Ext}(D) \). Since \( D \) is a separating cycle and \( C, C^\dagger \) lie on opposite sides of \( D \), we have \( C \neq C^\dagger \). Furthermore, we have 
\[ d(w(C), w(C^\dagger)) < \beta \cdot (3^{g - 1} - 3^{g' - 1}) + \beta \cdot 3^{3(g' - 1)} + \text{Rk}(C) + \text{Rk}(C^\dagger) \]. It follows that 
\[ d(w(C), w(C^\dagger)) < \beta \cdot 3^g + \text{Rk}(C) + \text{Rk}(C^\dagger) \], contradicting the distance conditions of Definition 6.4.

**Case 2:** \( C = C_\ast \)

In this case, we have 
\[ d(D, w(C)) \leq \beta \cdot (3^{g - 1} - 3^{3(g' - 1)}) + \text{Rk}(C) - \frac{3|E(D)|}{2} \]. Since \( D \) is a separating cycle and \( C \) is the outer face of \( G \), we have \( C^\dagger \neq C \). Furthermore, we have 
\[ d(w(C), w(C^\dagger)) < \beta \cdot 3^g + \text{Rk}(C) + \text{Rk}(C^\dagger) \], contradicting the distance conditions of Definition 6.4.

Our second corollary to Theorems 8.12 and 8.16 is the following.

**Corollary 8.24.** Let \( D \) be a separating cycle of length at most \( N_{\text{max}} \) and let \( C \in \mathcal{C} \setminus \{C_\ast\} \), where \( V(C \cap D) \neq \emptyset \). Then the following hold.

1) \( D \) is inward-contractible; AND

2) If \( C \) is an open ring, then \( C^{\subseteq \text{Int}(D)} = \emptyset \); AND

3) If \( C \) is a closed ring, then either \( C \subseteq \text{Int}(D) \) or \( C^{\subseteq \text{Int}(D)} = \emptyset \).

**Proof.** Let \( g := g(\Sigma) \) and \( g' := g(\Sigma_D) \). We first show that \( D \) is inward-contractible. Suppose not. Thus, \( g > 0 \), and since \( D \) is contractible, every noncontractible closed curve of \( \Sigma \) intersects with the open region externally bounded by \( D \), i.e \( g' = 0 \) and \( g' < g \). Note that every vertex of \( C \) has distance at most \( \frac{N_{\text{max}}}{3} \) from \( w(C) \), so 
\[ d(D, w(C)) \leq \frac{N_{\text{max}}}{3} \]. Possibly \( C \) is a closed ring with \( \text{Rk}(C) - \frac{3|E(D)|}{2} < 0 \), but, in any case, we have 
\[ d(D, w(C)) \leq \frac{2g}{3} - \frac{3N_{\text{max}}}{2} \], so it follows from a) of Corollary 8.23 that \( D \) is inward-contractible, a contradiction. This proves 1).
Now we prove 2) and 3) together. Suppose that \( C \) violates either 2) or 3). In either case, we have \( C \subseteq \text{Int}(D) \neq \emptyset \). Since \( D \) is inward-contractible, we have \( g = g' \), and, thus, by a) of Corollary 8.23 we have \( \text{Rk}(C) - \frac{3|E(D)|}{2} < \frac{\beta}{3} + \frac{7N_{\text{mo}}}{3} \). If \( C \) is an open ring, then \( \text{Rk}(C) = 2N_{\text{mo}} \geq \frac{4N_{\text{mo}}}{3} \), so \( C \) is an closed ring and 3) is violated.

By 1) of Theorem 8.16 there is a \( C' \in C \subseteq \text{Int}(D) \) such that \( \max\{d(v, w(C')) : v \in V(D)\} < \frac{\beta}{3} + \frac{7N_{\text{mo}}}{3} \). Since each of \( C, C' \) is an internal ring, we have \( C = C' \), or else we contradict M4). Thus, \( C \subseteq \text{Int}(D) \), which is false, since 3) is violated.

**9 Generalized Chords of Closed Rings in Critical Mosaics**

Roughly speaking, the purpose of the remaining sections of Paper I is to show that, given a critical mosaic \( T \), a ring \( C \) of \( T \), and a generalized chord \( P \) of \( C \) whose length is small relative to the pairwise-distance bound on the rings of the mosaic, \( P \) does not separate the remaining rings of \( T \), and furthermore, if one side of \( P \) contains a noncontractible closed curve of \( \Sigma \), then that side also contains all the elements of \( C \setminus \{C\} \). We now state and prove the lone theorem which makes up Section 9. Note that, in the statement below, the natural \( Q \)-partition of \( G \) is well-defined by the face-width conditions on mosaics.

**Theorem 9.1.** Let \( T = (\Sigma, G, C, L, C_*) \) be a critical mosaic and let \( C \in C \) be a closed ring. Let \( Q \) be a generalized chord of \( C \) with \( |V(Q) \setminus V(C)| \leq \frac{N_{\text{mo}}}{3} - 2 \) and let \( G = G_0 \cup G_1 \) be the natural \( Q \)-partition of \( G \). Then there exists a \( j \in \{0, 1\} \) such that \( G_{1-j} \) is contractible and each element of \( C \setminus \{C\} \) lies in \( G_j \).

We break the proof of Theorem 9.1 into two lemmas, the first of which is as follows deals with the case of a proper generalized chord. The case of an improper generalized chord requires a slightly different argument.

**Lemma 9.2.** Let \( T = (\Sigma, G, C, L, C_*) \) be a critical mosaic and let \( C \in C \) be a closed ring. Let \( Q \) be a proper generalized chord of \( C \) with \( |E(Q)| \leq \frac{N_{\text{mo}}}{3} \) and let \( G = G_0 \cup G_1 \) be the natural \( Q \)-partition of \( G \). Then there exists a \( j \in \{0, 1\} \) such that \( G_{1-j} \) is contractible and each element of \( C \setminus \{C\} \) lies in \( G_j \).

**Proof.** Given a path \( Q \subseteq G \), we say that \( Q \) is unacceptable if \( Q \) is a proper generalized chord of \( C \) with \( |E(Q)| \leq \frac{N_{\text{mo}}}{3} \), but \( Q \) violates Lemma 9.2. Suppose toward a contradiction that there is an unacceptable path, and let \( g := g(\Sigma) \).

**Claim 9.3.** \( C \neq C_* \).

**Proof:** Suppose toward a contradiction that \( C = C_* \). Let \( G = G_0 \cup G_1 \) be the natural \( Q \)-partition of \( G \). For each \( j = 0, 1 \), let \( U_j \) be the unique component of \( \Sigma \setminus (C_* \cup Q) \) such that \( G \cap \text{Cl}(U_j) = G_j \). By assumption, there is an \( \ell \in \{0, 1\} \) such that. For each \( j = 0, 1 \), let \( C_j := G_j \cap (C_* \cup Q) \). Since \( Q \) is a proper generalized chord of \( C \), each of \( C_0, C_1 \) is a cycle. Since \( C_* \) is the outer face of \( G \), we have \( G_j \subseteq \text{Int}(C_j) \) for each \( j = 0, 1 \).
Subclaim 9.3.1. Each of \( C_0, C_1 \) has length at most \( N_{mo} \)

Proof: Suppose not, and suppose without loss of generality that \( |E(C_0)| > N_{mo} \). Since \( |E(C_0)| + |E(C_1)| = |E(C_*)| + 2|E(Q)| \leq \frac{4N_{mo}}{3} \), we have \( |E(C_1)| < |E(C_*)| - \frac{2N_{mo}}{3} \). We note now that at least one element of \( C \) lies in \( \text{Int}(C_1) \). If \( C_1 \) is not inward-contractible, then this just follows from Theorem 8.12. On the other hand, if \( C_1 \) is not inward-contractible, then, since \( Q \) is unacceptable, at least one element of \( C \) lies in \( \text{Int}(C_1) \), so we have \( C \subseteq \text{Int}(C_1) \neq \emptyset \) in any case. Let \( g' := g(\Sigma C_1) \). It now follows from 1) of Theorem 8.16 that there is \( C' \in C \) with \( C' \subseteq \text{Int}(C_1) \) and \( \max\{d(v, w(C')) : v \in V(C_1)\} < \beta \cdot 3^{(q - q_0) - 1} + \frac{3|E(C_1)|}{2} \cdot \text{Rk}(C') \). By our distance conditions on \( T \), we have \( d(C_*, w(C')) \geq \beta \cdot 3^{q - 1} + |E(C_*)| + \text{Rk}(C') \), so we get \( \frac{3|E(C_1)|}{2} > |E(C_*)| \) and thus \( |E(C_0)| < \frac{|E(C_1)|}{2} + 2|E(Q)| \leq \frac{|E(C_1)|}{2} + \frac{2N_{mo}}{3} \). Since \( |E(C_0)| > N_{mo} \), it follows that \( |E(C_1)| > \frac{2N_{mo}}{3} \). Since \( |E(C_1)| < |E(C_*)| - \frac{N_{mo}}{3} \leq \frac{2N_{mo}}{3} \), we have a contradiction. ■

We now have the following.

Subclaim 9.3.2. \( g = 0 \).

Proof: Suppose that \( g > 0 \). Since each of \( C_0, C_1 \) is contractible, there is an \( \ell \in \{0, 1\} \) such that all the non-contractible closed curves of \( \Sigma \) intersect with \( \text{Cl}(U_i) \), say \( \ell = 0 \) without loss of generality. Thus, \( C_1 \) is inward-contractible. Since \( Q \) is unacceptable, at least one element of \( C \) lies in \( \text{Int}(C_1) \). Let \( g' := g(\Sigma C_1) \). Since \( C_1 \) is inward-contractible, we have \( g = g' \). By Subclaim 9.3.1, we have \( |E(C_1)| \leq N_{mo} \). Since \( g > 0 \), we have \( \beta \cdot (3^{q - 1} - 3^{(q - q_0) - 1}) \geq \frac{2g}{\beta} \). Since \( G_1 = \text{Int}(C_1) \) and \( V(C_1 \cap C_1) \neq \emptyset \), it follows from b) of Corollary 8.23 that every element of \( C \) lies in \( \text{Ext}(C_1) \), a contradiction. ■

Since \( Q \) is unacceptable and \( g = 0 \), it follows that, for each \( j = 0, 1 \), there is a \( C_j \in C \) with \( C_j' \subseteq \text{Int}(C_j) \) such that \( d(C_j, w(C_j')) \leq \frac{g}{\beta} + 4N_{mo} \). Since \( |E(C_*)| \leq N_{mo} \), we have \( d(w(C_*'), w(C_*')) \leq \frac{2g}{\beta} + 5N_{mo} \). Since \( C_*' \neq C_1' \) and each of \( C_*' \) is an internal ring, we contradict M4). This proves Claim 9.3. ■

For any proper generalized chord \( Q \) of \( C \) with \( |E(Q)| \leq \frac{N_{mo}}{3} \), we have \( Q \subseteq \text{Ext}(C) \), since \( C \neq C_* \). Thus, the natural \( Q \)-partition of \( G \) is of the form \( G = G_{in} \cup G_{out} \), where there exist cycles \( C_{in} \) and \( C_{out} \) such that the following hold.

1) \( C_{in} := (C \cup Q) \cap G_{in} \) and \( C_{out} := (C \cup Q) \cap G_{out} \); AND

2) \( G_{in} = \text{Int}(C_{in}) \) and \( G_{out} = \text{Ext}(C_{out}); \) AND

3) \( \text{Int}(C_{out}) = C \cup \text{Int}(C_{in}) \)

Among all unacceptable paths of \( C \), we now choose an unacceptable path \( Q \) so that, in the notation above, \( |V(G_{out})| \) is minimized. We also let \( G_{in}, C_{in}, C_{out} \) be as above with respect to \( Q \).
Claim 9.4. At least one element of \( C \setminus \{C\} \) lies in \( \mathsf{Int}(C) \), and furthermore, \(|E(C_{in})| > N_{mo}\) and \(|E(C_{out})| < |E(C)| - \frac{N_{mo}}{3} \leq \frac{2N_{mo}}{3}\).

**Proof:** Note that \( C \not\subseteq \mathsf{Int}(C_{in}) \). Suppose toward a contradiction that no element of \( C \setminus \{C\} \) lies in \( \mathsf{Int}(C_{in}) \). Since \( Q \) is unacceptable and \( C_{in} \) is noncontractible, it follows that \( g > 0 \) and all the noncontractible closed curve of \( \Sigma \) intersect with the closed region externally bounded by \( C_{in} \), contradicting Theorem 8.12. Now suppose toward a contradiction that \(|E(C_{in})| \leq N_{mo}\). Since \( V(C \cap C_{in}) \neq \emptyset \) and \( C \not\subseteq \mathsf{Int}(C_{in}) \), it follows from 3) of Corollary 8.24 that no element of \( C \) lies in the closed region externally bounded by \( C_{in} \), which is false, as shown above. Since \(|E(C)| \leq N_{mo}\) and \(|E(C_{in})| + |E(C_{out})| = |E(C)| + 2|E(Q)|\), we have \(|E(C_{out})| < |E(C)| - \frac{N_{mo}}{3} \leq \frac{2N_{mo}}{3}\). \( \blacksquare \)

We now have the following.

Claim 9.5. \( \mathsf{Ext}(C_{out}) \) contains no chords of \( C_{out} \) with an endpoint in \( Q \). Furthermore, for any vertex \( v \) of \( \mathsf{Ext}(C_{out}) \setminus C_{out} \), if \( v \) has a neighbor in \( Q \), then \( N(v) \cap V(C_{out}) \) consists of either at most one vertex or two consecutive vertices of \( C_{out} \).

**Proof:** By Claim 9.4, at least one element of \( C \setminus \{C\} \) lies in \( \mathsf{Int}(C) \), so it follows from the minimality of \(|V(\mathsf{Ext}(C_{out}))|\) that \( \mathsf{Ext}(C_{out}) \) contains no chord of \( C_{out} \) with precisely one endpoint in \( Q \) and no 2-chord of \( C_{out} \) which violates Claim 9.5 and has precisely one endpoint of \( Q \). Thus, we just need to deal with chords and 2-chords of \( C_{out} \) which have both endpoints in \( Q \). Suppose toward a contradiction that Claim 9.5 does not hold and let \( P \subseteq \mathsf{Ext}(C_{out}) \) be a path which is a chord or 2-chord of \( C_{out} \), where \( P \) has both endpoints in \( Q \) and \( P \) violates Claim 9.5. Since \( P \) has both endpoints in \( Q \), let \( F_1, F_2 \) be the two cycles of \( C_{out} \cup P \) which contain \( P \), where \( F_1 \cap Q \) has one connected component and \( F_2 \cap Q \) has two connected components.

Let \( Q' \) be the proper generalized chord of \( C_{out} \) obtained from \( Q \) by replacing the path \( Q \cap F_1 \) with \( P \). Since \( P \) violates Claim 9.5, we have \(|E(Q')| \leq |E(Q)|\). By Claim 9.4, \( Q \) separates an element of \( C \setminus \{C\} \) from \( C_s \). If \( C_{in} \subseteq \mathsf{Int}(F_2) \), then \( Q' \) also separates an element of \( C \setminus \{C\} \) from \( C_s \), so \( Q' \) is unacceptable, and since \(|E(Q')| \leq |E(Q)|\), we contradict the minimality of \(|V(\mathsf{Ext}(C_{out}))|\). Thus, we have \( C_{in} \not\subseteq \mathsf{Int}(F_2) \).

Since \( C_{in} \not\subseteq \mathsf{Int}(F_2) \), we have \( C_{in} \subseteq \mathsf{Int}(F_1) \), and \( C \subseteq \mathsf{Int}(F_1) \) as well. Since \( C \) is a closed ring, we have \( w(C) = V(C) \) and \( \text{Rk}(C) = |E(C)| \), so it follows from 2) of Theorem 8.16 that \( d(F_1, C) > |E(C)| - \frac{3|E(F_1)|}{2} \). By Claim 9.4, we have \(|E(C_{in})| > N_{mo}\), and since \(|E(Q)| \leq \frac{N_{mo}}{3}\), we have \(|E(C)| > \frac{2N_{mo}}{3}\). Now, at least endpoint of \( P \) has distance at most \( \frac{|E(Q)|}{2} \) from \( C \), so we get \( \frac{|E(Q)|}{2} > |E(C)| - \frac{3|E(F_1)|}{2} \). Since we have two strict inequalities, we get

\[
3|E(F_1)| + |E(Q)| \geq 2 \left( \frac{2N_{mo}}{3} + 2 \right)
\]
Now, since at least one endpoint of \( P \) is an internal vertex of \( Q \), we have \(|E(F_1)| \leq |E(Q)| + 1\), so we get \( 4|E(Q)| + 3 \geq 2 \left( \frac{2N_{\text{mo}}}{3} + 2 \right) \), contradicting the fact that \(|E(Q)| \leq \frac{N_{\text{mo}}}{3} \). ■

Since every vertex of \( C_{\text{out}} \) has distance at most \( \frac{N_{\text{mo}}}{3} \) from \( V(C) \), it follows from M4) that every element \( C_{\text{out}} \subseteq \text{Ext}(C_{\text{out}}) \) has distance at least \( 10N_{\text{mo}}^2 \) from \( C_{\text{out}} \). Since \( C \subseteq \text{Int}(C_{\text{out}}) \), it follows from Proposition 8.10 that there is an \( L \)-coloring \( \phi \) of \( V(\text{Int}(C_{\text{out}})) \). Let \( C' := C_{\text{out}} \cap \text{Int}(C_{\text{out}}) \) and consider the chart \( T' := (\Sigma, \text{Ext}(C_{\text{out}}), C', P_{\phi}^{\text{int}}(C_{\text{out}})) \). We claim now that \( T' \) is a mosaic. Firstly, \( T' \) is a tessellation in which \( C_{\text{out}} \) is a closed ring. By Claim 9.4 we have \(|E(C_{\text{out}})| \leq N_{\text{mo}} \), so M0) is satisfied, and M1) is immediate. Since \( C \) is an \( L \)-colorable \( T \)-ring, and the vertices of \( C_{\text{out}} \setminus \hat{Q} \) are precolored, it follows from Claim 9.3 that \( C_{\text{out}} \) is an \( L^{C_{\text{out}}}_{\phi} \)-predictable \( T' \)-ring. Thus, \( T' \) also satisfies M2).

Since \(|E(Q)| \leq \frac{N_{\text{mo}}}{3} \), every vertex of \( C_{\text{out}} \) is of distance at most \( \frac{N_{\text{mo}}}{3} \) from \( C \). By Claim 9.4, we have \(|E(Q)| - \frac{N_{\text{mo}}}{3} \leq \text{Rk}(T') - |C_{\text{out}}| - \frac{N_{\text{mo}}}{3} \). Thus, \( T' \) satisfies the distance conditions M3)-M4). By the monotonicity of edge-width, we have \( \text{ew}(\text{Ext}(C_{\text{out}})) \geq 2.4\beta \cdot 3^{p-2} \). As \( G \) is a 2-cell embedding, \( \text{Ext}(C_{\text{out}}) \) is also a 2-cell embedding, and since \( T' \) satisfies the distance conditions M3)-M4) and \(|E(C_{\text{out}})| \leq N_{\text{mo}} \), it follows from Fact 7.3 that \(|\text{fw}(\text{Ext}(C_{\text{out}}))| \geq 1.1\beta \cdot 3^{p-2} \). Thus, \( T' \) also satisfies M5), so \( T' \) is a mosaic. Since \(|V(\text{Ext}(C_{\text{out}}))| < |V(G)| \), it follows from the criticality of \( T \) that \( \text{Ext}(C_{\text{out}}) \) is \( L_{\phi}^{C_{\text{out}}} \)-colorable, so \( \phi \) extends to an \( L \)-coloring of \( G \), a contradiction. This completes the proof of Lemma 9.2. ■

Lemma 9.2 has the following consequence.

**Proposition 9.6.** Let \( T = (\Sigma, G, C, L, C_\ast) \) be a critical mosaic and let \( C \in C \) be a closed ring. Then \( C \) is an induced cycle.

**Proof.** Suppose toward a contradiction that \( C \) is not an induced cycle, and let \( xy \in E(G) \setminus E(C) \) be a chord of \( C \).

Let \( G = G_0 \cup G_1 \) be the natural \( xy \)-partition of \( G \). Applying Lemma 9.2 there is a \( j \in \{0, 1\} \), say \( j = 0 \) without loss of generality, such that \( G_1 \) is contractible and \( C' \subseteq G_0 \) for each \( C' \in C \setminus \{C\} \). Let \( C_0 := (C \cap G_0) + xy \), let \( C_{\text{out}} \) be the outer face of \( G_0 \), and let \( T_0 := (\Sigma, G_0, C \setminus \{C\} \cup \{C_0\}, L, C_{\text{out}}) \). Note that, since \(|V(C_0)| < |V(C)| \), and \( C_0 \) is a closed \( T_0 \)-ring, \( T_0 \) satisfies the distance conditions of Definition 6.4 and it is immediate that M0)-M2) are also satisfied. By the monotonicity of edge-width, we have \( \text{ew}(G_0) \geq \text{ew}(G) \geq 2.4\beta \cdot 3^{p-2} \). Since \( C_0 \) has distance at least \( \beta \cdot 3^{p-1} \) from all the rings of \( C \setminus \{C\} \), it follows from Fact 7.3 that \( \text{fw}(G_0) \geq 1.1\beta \cdot 3^{p-2} \). Thus, \( T_0 \) satisfies M5) as well, so \( T_0 \) is a mosaic.

Since \( T \) is critical and \(|V(G_0)| < |V(G)| \), it follows that \( G_0 \) admits an \( L \)-coloring \( \phi \). Let \( C_1 := (G_1 \cap Q) + xy \).

By definition, we have \(|L(p)| = 1 \) for each \( p \in V(C) \), and since \( V(C) \) is \( L \)-colorable, \( \phi \) extends to an \( L \)-coloring \( \phi' \) of \( V(G_0) \cup V(C_1) \). Now, we have \(|L(v)| \geq 5 \) for all \( v \in V(G_1) \setminus V(C_1) \). Since \( C \) is \( L \)-predictable and all the
vertices of \( V(C_1) \) are already \( L \)-precolored, it follows that each connected component \( H \) of \( G_1 \setminus C_1 \) contains a facial subgraph \( F \) which contains every vertex of \( H \) with an \( L_{\phi'} \)-list of size less than five, and furthermore, there is a vertex \( w \in V(F) \) such that \( |L_{\phi'}(v)| \geq 3 \) for all \( v \in V(F - w) \), and \( |L_{\phi}(w)| \geq 2 \). Since \( G_1 \) is contractible and \( G \) is a 2-cell embedding, \( G_1 \) can be regarded as an embedding on a disc. Thus, by Theorem 1.1 each connected component of \( G_1 \setminus C_1 \) is \( L_{\phi'} \)-colorable, so \( \phi \) extends to an \( L \)-coloring of \( G \), contradicting the fact that \( T \) is critical.

We now complete the proof of Theorem 9.1 by proving the following lemma, which deals with improper generalized chords.

**Lemma 9.7.** Let \( T = (\Sigma, G, C, L, C_* ) \) be a critical mosaic and let \( C \subseteq \mathcal{C} \) be a closed ring. Let \( D \) be an improper generalized chord of \( C \) with \( |E(D)| \leq \frac{N_{\Sigma}}{3} - 1 \) and let \( G = G_0 \cup G_1 \) be the natural \( D \)-partition of \( G \). Then there exists a \( j \in \{0, 1\} \) such that \( G_{1-j} \) is contractible and each element of \( \mathcal{C} \setminus \{C\} \) lies in \( G_j \).

**Proof.** Suppose toward a contradiction that Lemma 9.7 does not hold. Given a cycle \( D \), we say that \( D \) is **undesirable** if \( D \) is an improper generalized chord of \( C \) with \( |E(D)| \leq \frac{N_{\Sigma}}{3} - 1 \), but \( D \) violates Lemma 9.7. Since \( C \) is a facial subgraph of \( G \) and any cycle of length at most \( \frac{N_{\Sigma}}{3} - 1 \) is contractible, we choose an undesirable cycle cycle \( F \) which minimizes the size of the side of \( F \) containing \( C \). More precisely, we choose \( F \) so that, letting \( G = G_0 \cup G_1 \) be the natural \( F \)-partition of \( G \), where \( C \subseteq G_0 \), the quantity \( |V(G_0)| \) is minimized among all undesirable cycles. Note that, if \( C = C_* \) then \( G_0 = \text{Ext}(F) \). Let \( g := g(\Sigma) \) and \( g' := g(\Sigma,F) \) and let \( w^* \) be the unique vertex of \( C \cap F \). Let \( U \) be the unique component of \( \Sigma \setminus F \) which does not contain \( C \setminus w^* \), and let \( \Sigma' \) be the surface obtained from \( \Sigma \) by replacing \( U \) with an open disc.

**Claim 9.8.**

1. \( \Sigma' = S^2 \) and \( F \) separates two elements of \( \mathcal{C} \setminus \{C\} \); AND

2. If \( C \neq C_* \), then \( C \subseteq \text{Int}(F) \) and \( F \) is inward-contractible. On the other hand, if \( C = C_* \), then \( C \subseteq \text{Ext}(F) \) and \( F \) is outward-contractible.

**Proof:** We deal with the two cases.

**Case 1:** \( C \neq C_* \)

In this case, since \( C \) is a closed ring, it immediately follows from Corollary 8.24 that \( C \subseteq \text{Int}(F) \) and \( F \) is inward-contractible. In particular, \( G_0 = \text{Int}(F) \) and \( g' = g \). Since \( F \) is undesirable, there is at least one element of \( \mathcal{C} \setminus \{C\} \) in \( \text{Int}(F) \), so \( F \) separates an element of \( \mathcal{C} \setminus \{C\} \) from \( C_* \). Since \( G_0 = \text{Int}(F) \), \( U \) is the component of \( \Sigma \setminus F \) which is internally bounded by \( F \), so \( \Sigma' = S^2 \).
Case 2: $C = C_*$

In this case, we have $G_1 = \text{Int}(F)$ and $G_0 = \text{Ext}(F)$.

Subclaim 9.8.1. $F$ is outward-contractible.

Proof: Suppose not. Since $F$ is contractible, it is inward-contractible. Since $F$ is inward-contractible but also undesirable, we have $C \subseteq \text{Int}(F) \neq \emptyset$. Since $g > 0$ and $g = g'$, we have $\beta \cdot (3^g - 3^{g-g'}) \geq \frac{2g}{3}$. Since $V(F \cap C_*) \neq \emptyset$, it follows from b) of Corollary 8.23 that every element of $C$ lies in $\text{Ext}(F)$, contradicting the fact that $C \subseteq \text{Int}(F) \neq \emptyset$.

Since $G_1 = \text{Int}(F)$, we have $\Sigma^\dagger = \Sigma_F$. Since $F$ is outward-contractible, we have $\Sigma_F = S^2$, so $\Sigma^\dagger = S^2$. Suppose toward a contradiction that $F$ does not separate any two elements of $C \setminus \{C\}$. Since $F$ is undesirable, we have $g > 0$ and $C \setminus \{C\} \neq \emptyset$, and furthermore, every noncontractible closed curve of $\Sigma$ which intersects an element of $C \setminus \{C\}$ also intersects with $F$. Since $g > 0$ and $F$ is outward-contractible, it is not inward-contractible, so we contradict Theorem 8.12.

We now have the following.

Claim 9.9. $F$ has no chords in $G_0$, and furthermore, for any $v \in V(G_0) \setminus V(C \cup F)$, either $N(v) \cap V(F) \leq 1$ or $N(v) \cap V(F)$ consists of two consecutive vertices of $F$.

Proof: Suppose not, and, for some $k = 1, 2$, let $P \subseteq G_0$ be a $k$-chord of $F$ which violates Claim 9.9. Since $P$ violates Claim 9.9, there is a pair of cycles $K, K' \subseteq F \cup P$ such that the following hold.

1) $K$ is an improper generalized chord of $C$ and $V(C \cap K) = \{w_s\}$; AND

2) $K \cap K' = P$ and $C \subseteq \text{Ext}(K')$

By the minimality of $|V(G_0)|$, $K$ is not an undesirable cycle. By 1) of Claim 9.8, $F$ separates two elements of $C \setminus \{C\}$. Since $K$ is not undesirable, it follows that $\text{Int}(K')$ contains an element of $C \setminus \{C\}$. Since $|E(K')| \leq N_{\text{mo}}$, it follows from 1) of Theorem 8.16 that there is a $C_1 \in C$ with $C_1 \subseteq \text{Int}(K')$ with $d(w(C_1), K') < \beta \cdot 3^{g-1} + 4N_{\text{mo}}$. Each vertex of $K'$ has distance at most $\frac{N_{\text{mo}}}{3}$ from $V(C)$, so if $C \neq C_*$, then $C, C_1$ are distinct internal rings and we contradict M4. Thus we have $C = C_*$, and $\text{Int}(K') \setminus V(K')$ is disjoint to $\text{Int}(F)$. Since $F$ separates two elements of $C \setminus \{C\}$, it follows from 1) of Theorem 8.16 that there is a $C_2 \in C$ with $C_2 \subseteq \text{Int}(F)$ and $d(w(C_2), F) < \beta \cdot 3^{g-1} + 4N_{\text{mo}}$. Since $C_1 \neq C_2$ and each of $C_1, C_2$ is an internal ring, we again contradict M4.

We now have the following.
Claim 9.10. For each $k = 1, 2$, there is no $k$-chord of $C \cup F$ with one endpoint in $C - w^*$ and the other endpoint in $F - w^*$.

Proof: Suppose not, and, for some $k = 1, 2$, let $P \subseteq G_0$ be a $k$-chord of $F$ which violates Claim 9.10. Let $x$ be the unique endpoint of $P$ lying in $C - w_*$. Note that $F \cup P$ contains two distinct proper generalized chords $Q_0, Q_1$ of $C$, each of which has endpoints $x, w_*$. Furthermore, since $|E(F)| < \frac{N_{inv}}{3}$, we have $|E(Q_i)| \leq \frac{N_{inv}}{3}$ for each $i = 0, 1$. By 1) of Claim 9.8, $F$ separates two elements of $C \setminus \{C\}$, so at least one of $Q_0, Q_1$ separates two elements of $C \setminus \{C\}$, contradicting Lemma 9.2.

Since $C$ is an $L$-predictable $T$-ring, we immediately have the following consequence of Claims 9.9 and 9.10.

Claim 9.11. For any $L$-coloring $\phi$ of $V(C \cup F)$, there is an $x \in V(G_0) \cap D_1(C \cup F)$ such that $|L_\phi(x)| \geq 2$ and $|L_\phi(y)| \geq 3$ for each $y \in (V(G_0) \cap D_1(C \cup F)) \setminus \{x\}$.

We now have the following.

Claim 9.12. $V(G_1 \cup C)$ is $L$-colorable.

Proof: By Proposition 9.6, $C$ is an induced subgraph of $G$. Combining this with Claims 9.9 and 9.10, we get that $G_1 \cup C$ is an induced subgraph of $G$, so it suffices to show that $G_1 \cup C$ is $L$-colorable. To show this, we construct a mosaic whose underlying graph has fewer vertices than $G$ and contains $G_1 \cup C$ as a subgraph.

Let $U'$ be the unique component of $\Sigma \setminus (C \cup F)$ such that $G_0 = G \cap C(U')$. By 1) of Claim 9.8, we have $\Sigma^4 = S^2$.

Let $G'_1$ be a graph obtained from $G_1 \cup C$ by adding to $U'$ a cycle of length $E(C \cup Q)$, where, for each $x \in V(K)$, the graph $G_1[N(x) \cap V(C \cup Q)]$ is an edge of $C \cup Q$, and, for any two vertices $x, y \in V(K)$, $xy$ is an edge of $K$ if and only if $G_1[N(x) \cap V(C \cup Q) \cap G_1[N(y) \cap V(C \cup Q)]$ are adjacent edges of $C \cup Q$, at most one of which lies in $E(C) \cap \delta(w^*)$. Now, $K$ is contractible by our edge-width conditions, and it follows from Claim 9.8 that, if $C = C_*$ then $K$ is inward-contractible, and otherwise $K$ is outward-contractible.

Let $\Sigma^4_1$ be a surface defined as follows. If $C = C_*$ then $\Sigma^4_1 = \Sigma$, and otherwise $\Sigma^4_1$ is obtained from $\Sigma$ by moving the north pole so that $C$ is the outer face of $G'_1$. Regarding $G'_1$ as an embedding on $\Sigma^4_1$, we define an embedding $G'_1$ on $\Sigma^4_1$ which is obtained from $G'_1$ by adding a $K$-web in the open disc of $\Sigma^4_1$ externally bounded by $K$.

Subclaim 9.12.1. $|V(G'_1)| < |V(G)|$.

Proof: Since $|E(K)| \leq \frac{4N_{inv}}{9}$, we just need to check that $|V(G_0) \setminus V(C \cup Q)| > 8N_{inv}^2$, and then it follows from 3) of Proposition 8.4 that $|V(G'_1)| < |V(G)|$. If $G_0$ contains a noncontractible cycle, then we are done by our edge-width conditions. If $G_0$ does not contain any noncontractible cycle, then, since $F$ is undesirable, $F$
separates an element $C'$ of $C \setminus \{C\}$ from $C$, and since \( \frac{2}{3} = \frac{100N^2}{3} \), it follows from our distance conditions on $T$ that $|V(G_0) \setminus V(C \cup Q)| > 8N^2_{\text{mo}}$, as desired. ■

By definition, $C$ is the outer face of $G_1^\dagger$. Let $L^\dagger$ be a list-assignment for $V(G_1^\dagger)$, where all the vertices of $G_1^\dagger \setminus G$ are given arbitrary 5-lists, and otherwise $L^\dagger = L$. Let $T^\dagger := (\Sigma^\dagger, G_1^\dagger \subseteq G_0, L^\dagger, C)$. It follows from 1) of Proposition 8.4 that $T^\dagger$ is a tessellation, where $C$ is a closed $T^\dagger$-ring, and it follows from our construction of $G_1^\dagger$ that $C$ is an expectable, and thus $L^\dagger$-predictable, facial subgraph of $G_1^\dagger$, so $T^\dagger$ satisfies M(0)-M(2). It also follows from 2) of Proposition 8.3 that $T^\dagger$ still satisfies the distance conditions M(3) and M(4). In particular, if the north pole of $\Sigma$ has moved so that $C$ is the outer face, then, since $C$ is an internal ring of $T$, the distance conditions that $C$ needs to satisfy have only weakened.

Now we just need to check M(5). All the noncontractible cycles of $G_1^\dagger$ lie in Cl($U$), so we get $\text{ew}(G_1^\dagger) \geq \text{ew}(G_1) \geq \text{ew}(G)$. Furthermore, since $G$ is a 2-cell embedding, it follows from our construction of $G_1^\dagger$ that $G_1^\dagger$ is also a 2-cell embedding. Since $\text{ew}(G) \geq 2.4.\beta \cdot 3^{g(S)} - 2$, it follows from Fact 7.3 together with our distance conditions on $T$, that $\text{fw}(G_1^\dagger) \geq \beta \cdot 3^{g-2}$. We conclude that $T^\dagger$ satisfies M(5), so $T^\dagger$ is a mosaic. Since $|V(G_1^\dagger)| < |V(G)|$, it follows that $G_1^\dagger$ is $L^\dagger$-colorable, and thus $G_1 \cup C$ is $L$-colorable, as desired. ■

Applying Claim 9.12 let $\phi$ be an $L$-coloring of $V(G_1 \cup C)$. Applying Claim 9.11 let $x \in V(G_0) \cap D_1(C)$ such that $|L_\phi(x)| \geq 2$ and $|L_\phi(y)| \geq 3$ for each $y \in (V(G_0) \cap D_1(C \cup F)) \setminus \{x\}$. Let $G' := G \setminus (C \cup G_0)$ and let $L'$ be a list-assignment for $V(G')$ where $L'(x)$ consists of a lone color of $L_\phi(x)$ and otherwise $L' = L_\phi$. Let $F'$ be the outer face of $G'$. By Claim 9.8 if $C \neq C_*$, then $G_0 = \text{Int}(C)$. It follows that, regardless of whether $C = C_*$ or not, $F'$ consists of all the vertices of $V(G_0) \cap D_1(F \cup C)$. Let $T' := (\Sigma^\dagger, G' \subseteq G_0 \setminus \{C\}, L', F')$. Now, $T'$ is a tessellation, where the outer face is an open $T'$-ring with precolored path $x$. We claim now that $T'$ is a mosaic. M(2) is immediate, and since the precolored path of $F'$ is a lone vertex, M(0) and M(1) are also immediate. By 1) of Claim 9.8 $\Sigma^\dagger = S^2$, so M(5) is trivially satisfied. Since $F'$ is outer face of $G'$, M(4) is immediate as well so we just need to check that $T'$ satisfies M(3).

**Claim 9.13.** $T'$ satisfies M(3).

**Proof:** Firstly, every vertex of $F'$ has distance at most $\frac{N_{\text{mo}}}{3} + 1$ from $C$. We have $\text{Rk}(T'|F') = 2N_{\text{mo}}$, i.e. the rank of $F'$ is greater than that of $C$, as $F'$ is an open $T'$-ring and $C$ is a closed $T$-ring, but since $F'$ is the outer face of $G'$, and the genus has not increased, it is immediate that $T'$ satisfies M(4) if $C$ is an internal ring of $T$. Now suppose that $C = C_*$. In this case, we have $G_0 = \text{Ext}(F)$ and, by 1) of Claim 9.8, we have $C \subseteq \text{Int}(F) \neq \emptyset$. Since $V(F \cap C_*) \neq \emptyset$, it follows from 1) of Theorem 8.16 that there is a $C' \in C \subseteq \text{Int}(F)$ with $d(C_*, w(C')) < \beta \cdot 3^{g-1} + \frac{2N_{\text{mo}}}{3}$. For each $C'' \in C \subseteq G_0$, we have $d(w(C''), C') \geq \beta \cdot 3^{g}$ by our distance conditions on $T$, as each of $C', C''$ is an internal ring.
of $T$. Since any two vertices of $C_*$ are of distance at most $\frac{N_{mo}}{6}$ apart, it follows that, for each $C'' \in \mathcal{C}_G^{G_0}$, we have
\[ d(w(C''), C_*) > (\beta \cdot 3^g - \beta \cdot 3^{g-1}) - 4N_{mo} \] and thus
\[ d(w(C''), F) \geq 2\beta \cdot 3^{g-1} - 4N_{mo} - \left( \frac{N_{mo}}{6} - 1 \right) > \beta \cdot 3^{g-1} + 4N_{mo}, \]
so $T'$ satisfies M3. 

Since $T'$ satisfies all of M0)-M5), $T'$ is indeed a mosaic. Since $|V(G')| < |V(G)|$ and $T$ is critical, we conclude that $G'$ is $L'$-colorable, so $\phi$ extends to an $L$-coloring of $G$, a contradiction.  

Combining Lemmas 9.2 and 9.7, we complete the proof of Theorem 9.1

### 10 Generalized Chords of Open Rings in Critical Mosaics

The purpose of Section 10 is to prove an analogue to Theorem 9.1 for open rings. In order to prove this result, we first prove the following lemma, which shows that if we have a short enough proper generalized chord of an open ring in a critical mosaic, then one side of the generalized chord is colorable under certain conditions.

**Lemma 10.1.** Let $\mathcal{T} = (\Sigma, G, C, L, C_*)$ be a critical mosaic and let $C \in \mathcal{C}$ be an open ring. Let $Q$ be a proper generalized chord of $C$ with $|E(Q)| \leq \frac{2N_{mo}}{3}$. Let $G = G_0 \cup G_1$ be the natural $Q$-partition of $G$ and, for each $j = 0, 1$, let $U_j$ be the unique component of $\Sigma \setminus (C \cup Q)$ such that $G \cap \text{Cl}(U_j) = G_j$. Suppose that $\text{Cl}(U_1)$ contains either a noncontractible closed curve of $\Sigma$ or an element of $\mathcal{C} \setminus \{C\}$. Then $G_0$ is $L$-colorable.

**Proof.** For each $j = 0, 1$, let $C_j := (C \cap G_j) + Q$. Since $Q$ is a proper generalized chord of $C$, each of $C_0, C_1$ is a cycle. We construct a mosaic whose underlying subgraph has strictly fewer vertices than $G$ and has $G_0 \cup C$ as a subgraph. If $\text{Cl}(U_1)$ contains a noncontractible closed curve of $\Sigma$, then it contains a noncontractible cycle of $G$, since $G$ is a 2-cell embedding. Thus, $\text{Cl}(U_1)$ either contains a noncontractible cycle of $G$ or an element of $\mathcal{C} \setminus \{C\}$. Thus, it follows from either our face-width conditions or our distance conditions on $\mathcal{T}$ that $|V(G_1 \setminus C_1)| \neq \emptyset$. Let $\Sigma^1$ be a surface obtained from $\Sigma$ by replacing $U_1$ with an open disc. Note that, if $U_1$ is not externally bounded by $C_1$, then $C \neq C_*$. Let $\Sigma^1_+ = \Sigma^1 \setminus \Sigma_+$. If $U_1$ is externally bounded by then $\Sigma^1_+ = \Sigma^1$, and if $U_1$ is internally bounded by $C_1$, then $\Sigma^1_+$ is obtained from $\Sigma^1$ by moving the north pole so that $C$ is the outer face of $G_1$.

The idea here is to apply a construction similar to that of Definition 8.3 to construct an embedding on $\Sigma^1_+$ that contains $G_0 \cup C$ as a subgraph and has fewer vertices than $G$, but because $C$ is an open ring, the path $C \cap G_1$ can be arbitrarily long, so we need to do this in two steps in order to bound the size of the new near-triangulation we construct. Let $k = \min\{|E(Q)| + 1, |E(C \cap G_1)|\}$ and let $U_2$ be the open connected component of $\Sigma \setminus (C \cup Q)$ corresponding to $U_1 \subseteq \Sigma$. Regarding $G_0 \cup C$ as an embedding on $\Sigma^1_+$, we define an embedding $H$ on $\Sigma^1_+$ which is obtained from $G_0 \cup C$ by doing the following.
1) First we add \( k \) new vertices \( w_1, \ldots, w_k \) to the open disc which replaces \( U_2 \), and we add some edges to \( \text{Cl}(U_1) \), so that \( w_1 \ldots w_k \) is a path and, for each \( 1 \leq i < i' \leq k \), the neighborhoods of \( w_i \) and \( w_{i'} \) on \( C_1 \) are subpaths of \( C_1 \) of length at least one which intersecting precisely on a common endpoint, where the respective neighborhoods of \( w_1 \) and \( w_k \) on \( C_1 \) are disjoint terminal subpaths of \( C \cap G_1 \).

2) Now, letting \( K \) be the cycle \( Q + w_1 \ldots w_k \), we add a \( K \)-web to the open disc of \( \Sigma^1_k \) externally bounded by \( K \).

Let \( L^1_k \) be a list-assignment for \( H \) where each vertex of \( H \setminus G_0 \) is given arbitrary 5-list and otherwise \( L^1_k = L \). By definition, \( C \) is the outer face of \( H \). Since \( |E(K)| \geq 5 \), it follows from Proposition \( 8.4 \) that \( H \) is short-separation-free and that \( T^1_k := (\Sigma^1_k, H, L^1_k, \{C\} \cup C \subseteq G_0, C) \) is a tesselation, where \( C \) is an open \( T^1_k \)-ring with precolored path \( P^1_k := P_{T,C} \cap G_0 \). Recall that, by assumption, we have \( P_{T,C} \cap G_0 \neq \emptyset \). We claim now that \( T^1_k \) is a mosaic.

### Claim 10.2. \( T^1_k \) satisfies M5).

**Proof:** If \( \text{Cl}(U_1) \) contains a noncontractible closed curve of \( \Sigma \), then \( g(\Sigma^1_k) = 0 \) so M5) is trivially satisfied in that case. Now suppose that \( \text{Cl}(U_1) \) contains no noncontractible closed curve of \( \Sigma \). Thus, we have \( g = g(\Sigma) \).

#### Subclaim 10.2.1. \( \text{ew}(H) \geq 2.4 \beta \cdot 3^{q-2} \).

**Proof:** Suppose not. Thus, there is a noncontractible cycle \( F \) of \( H \) with \( |E(F)| < 2.4 \beta \cdot 3^{q-2} \). It follows from our construction of \( H \) that \( d_H(x,y) = d_{G_0 \cup C}(x,y) \) for any \( x,y \in V(Q) \). As \( U_2 \) is an open disc, it follows that there is a noncontractible cycle \( F' \) of \( G_0 \cup C \) which is obtained from \( F \) by replacing (if necessary) some edges of \( H \setminus (G_0 \cup C) \) with a subpath of \( C_1 \) whose endpoints lie in \( F \cap Q \), where \( |E(F')| \leq |E(F)| \). Thus, \( \text{ew}(G_0 \cup C) < 2.4 \beta \cdot 3^{q-2} \). Since \( \text{ew}(G_0 \cup C) \geq \text{ew}(G) \) and \( g = g(\Sigma) \), this contradicts the fact that \( T \) satisfies M5). ■

To finish the proof of Claim 10.2, we just need to check that the face-width conditions are satisfied. Since \( G \) is a 2-cell embedding on \( \Sigma \), it follows from our construction of \( H \) that \( H \) is a 2-cell embedding on \( T^1_k \). Combining this with our distance conditions and the fact that \( \text{fw}(G) \geq 1.1 \beta \cdot 3^{q-2} \), it follows from Fact 7.4 that \( \text{fw}(H) \geq 1.1 \beta \cdot 3^{q-2} \), so \( T^1_k \) satisfies M5). ■

Since \( T^1_k \) satisfies all of M0)-M5), we conclude that \( T^1_k \) is indeed a mosaic.

### Claim 10.3. \( |V(H)| < |V(G)| \).
Proof: Note that \(|E(K)| \leq 2|E(Q)| + 1\), so \(|E(K)| \leq 2N_{\text{mo}}\), and it follows from 3) of Proposition 8.4 that the closed disc of \(\Sigma_1\) contains at most \(36N_{\text{mo}}^2\) vertices of \(H\), and we get \(|V(H) \cap \text{Cl}(U_1)| \leq |V(C_1)| + 36N_{\text{mo}}^2\). We just need to show that \(|V(G) \cap \text{Cl}(U_1)| > |V(C_1)| + 36N_{\text{mo}}^2\). Since \(|E(Q)| \leq 2N_{\text{mo}}\), it follows from our distance conditions that every element of \(C \setminus \{C\}\) has distance at least \(\beta \cdot 3^{|\Sigma|-1} - \frac{N_{\text{mo}}}{\beta}\) from \(C_1\). If there is an element of \(C \setminus \{C\}\) in \(\text{Cl}(U_1)\), then we are done, so suppose that no such element of \(C \setminus \{C\}\) exists. Thus, every facial subgraph of \(G \cap \text{Cl}(U_1)\), except for \(C_1\), is a triangle, and \(G \cap \text{Cl}(U_1)\) contains a noncontractible cycle. In particular, \(g(\Sigma) > 0\) and \(\text{fw}(G) \geq \frac{1+\beta}{3}\).

Since \(\text{fw}(G) > 2\), \(G \cap \text{Cl}(U_1)\) contains a noncontractible cycle \(K\) such that \(K\) contains at most two chords of \(C\), and since \(\text{fw}(G) \geq \frac{1+\beta}{3}\), it follows that \((G \cap \text{Cl}(U_1)) \setminus E(C)\) contains at least \(\frac{1+\beta}{3} - 3\) edges which each have at most one endpoint in \(V(C)\). Since \(|E(C_1) \setminus E(C)| \leq 2N_{\text{mo}}\), we have \(|V(G) \cap \text{Cl}(U_1)| > |V(C_1)| + 36N_{\text{mo}}^2\). \(\blacksquare\)

Since \(\mathcal{T}_2\) is a mosaic and \(|V(H)| < |V(G)|\), it follows from the criticality of \(\mathcal{T}\) that \(H\) is \(L_2^\ast\)-colorable and so \(G_0\) is \(L\)-colorable, as desired. \(\square\)

We now state and prove the lone main result of Section 10. As in the statement of Theorem 9.1, the natural \(Q\)-partition of \(G\) in the statement below is well-defined by the face-width conditions on mosaics.

**Theorem 10.4.** Let \(\mathcal{T} = (\Sigma, G, C, L, C_\ast)\) be a critical mosaic and let \(C \in \mathcal{C}\) be an open ring. Let \(Q\) be a proper generalized chord of \(C\) with \(|E(Q)| \leq 2N_{\text{mo}}\). Let \(G = G_0 \cup G_1\) be the natural \(Q\)-partition of \(G\), and suppose that \(P_C \cap G_0\) is connected and has at least one edge. Then one of the following holds.

1) \(G_1\) is contractible and every element of \(C \setminus \{C\}\) lies in \(G_0\); OR

2) \(|E(P_C \cap G_1)| + |E(Q)| > \frac{2N_{\text{mo}}}{3}\).

**Proof.** By 4) of Proposition 6.8, \(P_C\) is a path of length at least one, and since \(C\) is an open ring, we have \(V(C) \setminus V(P_C) \neq \emptyset\). Given a proper generalized chord \(Q\) of \(C\), we say that \(Q\) is \(P_C\)-splitting if it has one endpoint in \(V(P_C)\) and the other endpoint in \(V(C) \setminus V(P_C)\). In particular, we have the following simple observation.

**Claim 10.5.** Let \(Q\) be a proper generalized chord of \(C\) and let \(C_0, C_1\) be the two cycles of \(C \cup Q\) which are distinct from \(C\). Then precisely one of the following holds.

1) \(Q\) is \(P_C\)-splitting; OR

2) There exists precisely one \(j \in \{0, 1\}\) such that \(P_C \cap C_j\) either has two connected components or at most one vertex.

Given Claim 10.5 above, we introduce the following notation. For any proper generalized \(Q\) of \(C\) of length at most \(\frac{2N_{\text{mo}}}{3}\) and any endpoint \(p\) of \(P_C\), we let \(G = G_0^{Q,p} \cup G_1^{Q,p}\) be the natural \(Q\)-partition of \(G\), where \(P_C \cap G_0^{Q,p}\) is a path.
of length at least one, and either $Q$ is not $P_C$-splittable or $p \in V(G_{1}^{Q,p}) \setminus V(G_{0}^{Q,p})$. It follows from Claim $10.5$ that $G_{0}^{Q,p}$ and $G_{1}^{Q,p}$ are uniquely specified by the definition above. For each endpoint $p$ of $P_C$, let $B_p$ be the set of proper generalized chords of $C$ with $E(P_C \cap G_{1}^{Q,p}) + |E(Q)| \leq \frac{2N_{C}}{3}$, where $G_{1}^{Q,p}$ either contains a noncontractible cycle or an element of $C \setminus \{C\}$. Suppose toward a contradiction that Theorem $10.4$ does not hold. Thus, letting $p, p'$ be the endpoints of $P_C$, we have $B_p \cup B_{p'} \neq \emptyset$. We now choose a $q \in \{p, p'\}$ and a $Q \in B_q$ which minimizes $|V(G_{1}^{Q,q})|$ among all the elements of $B_p \cup B_{p'}$. For each $j = 0, 1$, we let $C_j := (C \cap G_{j}^{Q,q}) + Q$, and, to avoid clutter, we also let $G_j := G_{j}^{Q,q}$.

Claim 10.6. Let $P \subseteq G_1$ be a $k$-chord of $C_1$ for some $1 \leq k \leq 2$, where $P$ has one endpoint in $\hat{Q}$ and the other endpoint in $C \cap G_1$. Then the following hold.

1) $|E(P)| = 2$; AND

2) If both endpoints of $P$ lie in $Q \cup (P_C \cap G_1)$, then the endpoints of $P$ are consecutive in $Q$.

Proof: Suppose toward a contradiction that, for some $1 \leq k \leq 2$, $G_1$ contains such a $k$-chord $P$ of $C_1$ which violates Claim $10.6$. Let $x, y$ be the endpoints of $P$, where $x \in V(\hat{Q})$ and $y \in V(C \cap G_1)$. Let $Q_0, Q_1$ be the two proper generalized chords of $C$ such that $Q_0 \cup Q_1 = Q + P$ and $Q_0 \cap Q_1 = P$. By our face-width conditions, each of the three cycles in $C_1 + P$ is contractible, so let $G_1 = H_0 \cup H_1$ be the natural $(C_1, P)$-partition of $G_1$, where, for each $i = 0, 1$, $H_i \cap (Q + P) = Q_i$.

Subclaim 10.6.1. For each $i = 0, 1$, $|E(P_C \cap H_i)| + |E(Q_i)| \leq |E(P_C \cap G_1)| + |E(Q)| \leq \frac{2N_{C}}{3}$.

Proof: By our choice of $Q$, $|E(P_C \cap G_1)| + |E(Q)| \leq \frac{2N_{C}}{3}$. Suppose that Subclaim $10.6.1$ does not hold, and suppose for the sake of definiteness that $|E(P_C \cap H_0)| + |E(Q_0)| > |E(P_C \cap G_1)| + |E(Q)|$. In particular, since $P_C \cap H_0$ is a (possibly empty) subpath of $P_C \cap G_1$, we have $|E(Q_0)| > |E(Q)|$. Thus, $x$ is an endpoint of $\hat{Q}$ and $|E(P)| = 2$. In particular, letting $x'x'$ be the unique terminal edge of $Q$ incident to $x$, we have $Q_0 = (Q - x') + P$ and $|E(Q_0)| = |E(Q)| + 1$. Furthermore, $Q_1$ has endpoints $x', y$. Since $|E(P)| = 2$, $P$ violates 2) of Claim $10.6$ so $y \in V(P_C \cap G_1)$. Since $|E(Q_0)| = |E(Q)| + 1$, and since $P_C \cap G_0$ is connected and has at least one edge, we get $P_C \cap H_0 = P_C \cap G_0$, so $y$ is an endpoint of $P_C$ and $x' \notin V(P_C)$. Furthermore, letting $x''$ be the endpoint of $Q$ which is distinct from $y$, we have $x'' \in V(P_C)$. In particular, $x''$ is an internal vertex of $P_C$, as $P_C \cap G_0$ has at least one edge. Let $D := (Q - x') + P + (C \cap H_0)$. Note that $D$ is a cyclic facial subgraph of $H_0$. Furthermore, since $x'' \in V(\hat{P}_C)$ and $y$ is an endpoint of $P_C$, and since $H_1$ contains no edges of $P_C$, it follows that $C \cap H_0$ is a terminal subpath of $P_C$. Since $|E(P_C \cap G_1)| + |E(Q)| \leq \frac{2N_{C}}{3}$, we have $|E(D)| \leq \frac{2N_{C}}{3}$ as well. Furthermore, since $x' \notin V(P_C), Q$ is $P_C$-splitting. Consider the following cases.

Case 1: Either $H_0$ is not contractible or $H_0$ contains an element of $C \setminus \{C\}$
Suppose not. Thus, there exist endpoints $C$ chord of $Q$. It follows from Subclaim 10.6.1 that each $R_k$ contains either a noncontractible cycle or an element of $Q$. Hence, we break this into two subcases.

**Subcase 1.1:** $H_0 = \text{Int}(D)$

In this case, either $D$ is not inward contractible or $\text{Int}(D)$ contains an element of $C \setminus \{C\}$. Let $g = g(S)$ and $g' = g(S_D)$. Since $d(D, C) = 0$, and since $\text{Rk}(C) = 2N_{\text{mo}}$ and $g \geq \max\{g', g - g'\}$, we contradict Corollary 8.23.

**Subcase 1.2:** $H_0 = \text{Ext}(D)$

In this case, $C \neq C^*$ and $D$ separates $C^*$ from $C$, so $C \subseteq \text{Int}(D)$. Since $C$ is an open ring and $V(C \cap D) \neq \emptyset$, this contradicts 2) of Corollary 8.24.

**Case 2:** $H_0$ is contractible and $H_0$ contains no element of $C \setminus \{C\}$

In this case, by our assumption on $Q$, either $H_1$ is not contractible or $H_1$ contains an element of $C \setminus \{C\}$. Let $R = x'x + P$. Note that $R$ is a proper 3-chord of $C$, neither endpoint of which is an internal vertex of $P$. Thus, for each endpoint $P$, we have $H_1 = G_{1, p}$. Since $|V(H_1)| < |V(G_1)|$, this contradicts the minimality of $|V(G_1)|$.

It follows from Subclaim 10.6.1 that each $Q_0$ and $Q_1$ has length at most $\frac{2N_{\text{mo}}}{3}$. Each of $Q_0, Q_1$ is a proper generalized chord of $C$, so, for each $i \in \{0, 1\}$ and endpoint $p_i$ of $P_C$, each of $G_{0, p_i}$ and $G_{1, p_i}$ is well-defined.

**Subclaim 10.6.2.** There exists an $i \in \{0, 1\}$ such that, for each endpoint $p_i$ of $P_C$, we have $H_i = G_{0, p_i}$.

**Proof:** Suppose not. Thus, there exist endpoints $p_0, p_1$ of $P$ (where possibly $p_0 = p_1$) such that $H_i = G_{0, p_i}$ for each $i = 0, 1$. Since $Q \in B_q$ and $H_0 \cup H_1 = G_1$, it follows from (ii) that there is at least one $i \in \{0, 1\}$ such that $H_i$ contains either a noncontractible cycle or an element of $C \setminus \{C\}$. Since $|V(H_i)| < |V(G_1)|$ for each $i = 0, 1$, we contradict the minimality of $|V(G_1)|$.

**Subclaim 10.6.3.** At least one endpoint of $Q$ lies in $P_C$, and furthermore, $Q$ is not $P_C$-splitting.

**Proof:** If neither endpoint of $Q$ lies in $P_C$, then, for each $i = 0, 1$, the path $P_C \cap H_i$ has at most one vertex and thus $H_i = G_{1, q}^\ast$, contradicting Subclaim 10.6.2. Thus, at least one endpoint of $Q$ lies in $P_C$. Now suppose toward a contradiction that $Q$ is $P_C$-splitting. Thus, $P_C \cap G_1$ is a path of length at least one which contains $q$. If $y \in V(C \cap G_1 \setminus P_C$, then, for each $i = 0, 1$, we have $H_i = G_{i, q}^\ast$, contradicting Subclaim 10.6.2. Thus, we have $y \in V(C \cap G_1 \cap P_C$. It follows that precisely one of $Q_0, Q_1$ has both endpoints in $P_C$, and the other one is $P_C$-splitting, so suppose without loss of generality that $Q_0$ has both endpoints in $P_C$ and $Q_1$ is $P_C$-splitting. Thus, we have $q \in V(H_1) \setminus V(Q_1)$ and furthermore, $P_C \cap H_0$ has one connected component and $P_C \setminus (H_0 \setminus Q_0)$ has two connected components. We conclude that $H_0 = G_{0, q}^\ast$ and $H_1 = G_{1, q}^\ast$. Since $|V(H_1)| < |V(G_1)|$, it
follows from the minimality of $G_1$ that $Q_1 \notin B_q$, so it follows from (†) that $H_0$ either contains a noncontractible cycle or an element of $C \setminus \{C\}$. Since $|E(P_C \cap G_1)| + |E(Q)| \leq \frac{2N_{mo}}{3}$, it follows that $G$ contains a cycle $F$ with $|E(F)| \leq N_{mo}$, where $F$ has nonempty intersection with $P_C$ and $F$ separates $C$ from either a noncontractible cycle of $G$ or an element of $C \setminus \{C\}$. Since $R_k(C) - \frac{3N_{mo}}{2} \geq \frac{N_{mo}}{2}$ and each vertex of $C$ has distance at most $\frac{N_{mo}}{2}$ from $w(C)$, we contradict Corollary 8.23. ■

It follows from Subclaim 10.6.3 that both endpoints of $Q$ lie in $P_C$ and at least one of them is an internal vertex of $P_C$. Thus, $P_C \cap G_1$ has two connected components.

**Subclaim 10.6.4.** $y \in V(G \cap C_1) \cap V(P_C)$.

**Proof:** Suppose not. Thus, for each $i = 0, 1$, $P_C \cap H_i$ is nonempty and connected. Let $q'$ be the endpoint of $P_C$ which is distinct from $q$. Since $y \in V(G \cap C_1) \setminus V(P_C)$, there exists an $i \in \{0, 1\}$ such that $H_i = G_1^q Q_i$ and $H_{1-i} = G_1^{q', Q_{1-i}}$, contradicting Subclaim 10.6.2. ■

Since $y \in V(G \cap C_1) \cap V(P_C)$, and both endpoints of $Q$ lie in $P_C$, there exists an $i \in \{0, 1\}$ such that $P_C \cap H_i$ has one connected component and $P_C \cap H_{1-i}$ has two connected components. Thus, $H_{1-i}$ contains no noncontractible cycles or elements of $C \setminus \{C\}$, or else we contradict the minimality of $|V(G_1)|$. Since $G_1 = H_0 \cup H_1$, it follows that $H_i$ contains either an element of $C \setminus \{C\}$ or a noncontractible cycle, so $(P_C \cap H_i) + Q_i$ is a cycle of length at most $N_{mo}$ which separates $C$ from either a noncontractible cycle of $G$ or an element of $C \setminus \{C\}$, contradicting Corollary 8.23. This completes the proof of Claim 10.6. ■

We now have the following.

**Claim 10.7.** $G_1$ contains no chord of $C_1$ with both endpoints in $Q$. Furthermore, for any 2-chord $P$ of $C_1$ with $P \subseteq G_1$ and both endpoints in $Q$, the endpoints of $P$ are consecutive vertices of $Q$.

**Proof:** Suppose toward a contradiction that there is a path $P$ of length at most two which violates Claim 10.7. Since $|E(Q)| + |E(P \cap G_1)| \leq \frac{2N_{mo}}{3}$ by assumption, and since the endpoints of $P$ are not consecutive in $Q$, we immediately have the following.

**Subclaim 10.7.1.** Each vertex of $C_1 \cup P$ has distance at most $\frac{N_{mo}}{2}$ from $w(C)$.

Possibly we have $P = C \cap G_1$ and the endpoints of $P$ are also the endpoints of $Q$. In that case, $C_1$ is a cycle of length at most $N_{mo}$, and since $|V(C)| \geq 5$ by 4) of Proposition 6.8, we have $|V(C \cap G_0)| > 2$ and $C_1$ is a separating cycle of $G$. On the other hand, if $P \neq C \cap G_1$, then, letting $Q'$ be the proper generalized chord of $C$ obtained from $Q$ by replacing $xQy$ with $P$, we have $|E(Q')| \leq |E(Q)|$ by our assumption on $P$, and it follows from the minimality of $|V(G_1)|$ that $C$ lies on the opposite side of the cycle $(xQy) + xy$ from either a noncontractible cycle of $G$ or an...
element of \( C \setminus \{ C \} \). In any case, since \(|E(Q)| \leq \frac{2N_m}{3}\) and \( G \) intersects with \( P_C \) at most on the endpoints of \( P_C \), it follows from Subclaim [10.7.1] that there is a separating cycle \( D \) of \( G \) with \( d(D, C \setminus P_C) \leq \frac{N_m}{3} \), where \(|E(D)| \leq N_m\) and \( D \) separates \( C \) from either a noncontractible cycle of \( G \) or an element of \( C \setminus \{ C \} \). Since \( \text{Rk}(C) - \frac{3N_m}{2} \geq \frac{N_m}{2} \) we contradict Corollary [8.23].

Let \( P_1 := (P_C \cap G_1) + Q \). Possibly \( P_C \cap G_1 = \emptyset \), but in any case, since \( G_0 \) contains at least one edge of \( P_C \), it follows that \( P_1 \) is a subpath of \( C_1 \), and it follows from Claim [10.7] that any edge of \( G_1 \) with both endpoints in \( Q \) is an edge of \( Q \). It follows from Claim [10.6] that \( G_1 \) has no chord of \( C_1 \) with an endpoint in \( V(Q) \) and the other endpoint in \( V(C \cap G_1) \), so any \( L \)-coloring of \( G_0 \) is also a proper \( L \)-coloring of the subgraph of \( G \) induced by \( V(G_0) \) and extends to an \( L \)-coloring of the subgraph of \( G \) induced by \( V(G_0 \cup P_C) \), even if \( G_1 \) contains edges of \( P_C \). Thus, by Lemma [10.1] \( V(G_0) \) admits an \( L \)-coloring \( \psi \), where \( V(P_1) \) is \( L_Q^\psi \)-colorable.

Let \( C' := \{ C_1 \} \cup \{ F \in C : F \subseteq G_1 \} \) and let \( F_* \) be the outer face of \( G_1 \). That is, \( F_* = C_1 \) if \( C_1 \neq C_* \) and otherwise \( F_* = C_1 \). Finally, let \( T' := (\Sigma, G_1, C', L_Q^{\psi}, F_*) \). Now, \( T' \) is a tessellation in which \( C_1 \) is an open ring with precolored path \( P_1 \). In particular, we have \( w_{T'}(C_1) = V(C_1) \setminus V(P_1) \), and thus \( w_{T'}(C_1) = w_T(C) \cap V(G_1) \).

Claim 10.8. \( T' \) is a mosaic.

**Proof:** Since \( w_{T'}(C_1) = w_T(C) \cap V(G_1) \), it is immediate that \( T' \) satisfies the distance conditions M3)-M4). Since \(|E(P_1)| \leq \frac{2N_m}{3} \), M0) is satisfied and M2) is immediate. It follows from Claims [10.6 and 10.7] that M1) is still satisfied, so now we just need to check M5). We have \( \text{ew}(G_1^{Q,p}) \geq \text{ew}(G) \geq 2.4 \cdot 3^{\theta(\Sigma)-2} \) by the monotonicity of edge-with, and since \( \text{fw}(G) \geq 1.1 \cdot 3^{\theta(\Sigma)-2} \), it follows from our distance conditions, together with Fact [7.4] that \( \text{fw}(G_1) \geq 1.1 \beta \cdot 3^{\theta(\Sigma)-2} \), so \( T' \) satisfies M5) as well.

We claim now that \( G_1 \) is \( L_Q^\psi \)-colorable. If \( |V(G_1)| < |V(G)| \), then this just follows immediately from the criticality of \( T \) so suppose that \( |V(G_1)| = |V(G)| \). Thus, \( V(G_0) = V(Q) \) and \( |V(Q)| \geq 3 \) (indeed, \( G_0 \) is a triangle), so \( Q \) has a vertex with an \( L \)-list of size five. Thus, \( \sum_{v \in V(G_1)} |L_Q^{\psi}(v)| \leq \sum_{v \in V(G)} |L(v)| \). It again follows from the criticality of \( T \) that \( G_1 \) is \( L_Q^\psi \)-colorable. In any case, \( \psi \) extends to an \( L \)-coloring of \( G \), a contradiction.

To conclude Section [10] we prove the following consequence of Theorem [10.4]

**Proposition 10.9.** Let \( T = (\Sigma, G, C, L, C_*) \) be a critical mosaic. Then the following hold.

1) Each ring of \( C \) is an induced cycle; AND

2) \(|C| + g(\Sigma) > 1|.|\)

**Proof.** We first prove 1). By Proposition [9.6] each closed ring is an induced cycle, so let \( C \) be an open ring and
suppose toward a contradiction that there is a chord \( xy \) of \( C \). By M1), we have \( x, y \notin V(P_C) \). Let \( G = G_0 \cup G_1 \) be the natural \( xy \)-partition. By 4) of Proposition \( 6.8 \) \( |E(P_C)| \geq 1 \), so there is a unique \( j \in \{0,1\} \) with \( P_C \subseteq G_j \), say \( j = 0 \) without loss of generality. By Theorem \( 10.4 \) \( G_1 \) is contractible, and each element of \( C \setminus \{C\} \) lies in \( G_0 \). For each \( j = 0,1 \), let \( C_j := (G \cap C_j) + xy \).

Claim 10.10. \( G_0 \) is \( L \)-colorable.

Proof: Let \( C'_* \) be the outer face of \( G_0 \). That is, \( C'_* = C_* \) if \( C \) is an internal ring, and otherwise \( C'_* = C_0 \). Let \( T' := (\Sigma, G_0, (C \setminus \{C\}) \cup \{C_0\}, L, C'_* \). We claim now that \( T' \) is a mosaic. Firstly, \( T' \) is a tessellation, and, since \( xy \) has one endpoint outside of \( P_C \), it follows that \( C'_xy \) is an open \( T' \)-ring with precolored path \( V(P_C) \), so \( T' \) satisfies M0), and it is clear that \( T' \) satisfies M1) as well. Since \( C_0 \) is an open ring of \( T' \) with the same precoloring as \( C \), and since \( V(C_0) \subseteq V(C) \), it is immediate that M2)-M4) are also satisfied, so we just need to check M5). Since edge-width is monotone, we have \( \text{ew}(G_0) \geq 2.4\beta \cdot 3^{\beta(\Sigma)-2} \), so it follows from Fact \( 7.4 \) together with our distance conditions, that \( \text{fw}(G_0) \geq 1.1\beta \cdot 3^{\beta(\Sigma)-2} \). □

Applying Claim \( 10.10 \) let \( \psi \) be an \( L \)-coloring of \( G_0 \). Since every element of \( C \setminus \{C\} \) lies in \( G_0 \), and \( G_1 \) is contractible, it follows from Theorem \( 1.1 \) that \( G_1 \) is \( L_\psi \)-colorable, so \( \psi \) extends to an \( L \)-coloring of \( G \), contradicting the fact that \( T \) is critical. This proves 1). Now we prove 2). Suppose toward a contradiction that \( |C| + g(\Sigma) \leq 1 \). Since the outer face of \( G \) is an element of \( C \), we have \( \Sigma = S^2 \). Let \( \psi \) be the unique \( L \)-coloring of \( V(P_{C_*}) \) and let \( F \) be the outer face of \( G \setminus P_{C_*} \). Note that every vertex of \( G \setminus P_{C_*} \) has an \( L_\psi \)-list of size at least five. Since \( G \) is not \( L \)-colorable, \( G \setminus P_{C_*} \) is not \( L_\psi \)-colorable. By 1), \( C_* \) is an induced cycle of \( G \). If \( C_* \) is a closed ring, then, since \( C_* \) is \( L \)-predictable, it follows from Theorem \( 1.1 \) that \( G \setminus P_{C_*} \) is \( L_\psi \)-colorable, a contradiction. Thus, \( C_* \) is an open ring. Let \( p, p' \) be the endpoints of the path \( C_* \setminus P_{C_*} \). Possibly \( p = p' \). Since \( C_* \) has no chords, it follows from M1) that every vertex of \( V(F) \setminus \{p, p'\} \) has an \( L_\psi \)-list of size three, and furthermore, \( \{p, p'\} \) either consists of one vertex with an \( L_\psi \)-list of size at least one, or two vertices with \( L_\psi \)-lists of size at most two. In the first case, it follows from Theorem \( 1.1 \) that \( G \setminus P_{C_*} \) is \( L_\psi \)-colorable, and in the second case, it follows from Theorem \( 1.2 \) that \( G \setminus P_{C_*} \) is \( L_\psi \)-colorable. In either case, we have a contradiction. □

11 Bands of Open Rings in Critical Mosaics

This section consists of a main result and two useful corollaries. Intuitively, this main result states that in a critical mosaic \( T = (\Sigma, G, C, L, C_*), \) if \( C \in C \) is an open \( T \)-ring, then \( G \) does not contain any “shortcuts” of the precoloring path, which is made precise below. We begin with the following definitions.
Definition 11.1. Let $\mathcal{T} = (\Sigma, G, \mathcal{C}, L, C, s)$ be a critical mosaic and let $C \in \mathcal{C}$ be an open ring. Given a proper generalized chord $Q$ of $P_C$ with $|E(Q)| \leq N_{mo}$, we have the following.

1) We let $Q_{\text{aug}}$ be the unique cycle of $Q \cup P_C$ and we let $U_Q$ be the unique connected component of $\Sigma \setminus (C \cup Q)$ such that $\partial(U) = Q_{\text{aug}}$ and $\text{Cl}(U_Q) \cap V(G) \not\subseteq V(C)$.

2) We say that $Q$ is a $C$-band if there is either an element of $C \setminus \{C\}$ or a noncontractible closed curve of $\Sigma$ in $U_Q$, and we say that $Q$ is a short $C$-band if $Q$ is a $C$-band and $|E(Q_{\text{aug}})| \leq N_{mo}$.

In the setting above, note that $Q_{\text{aug}}$ is not necessarily a generalized $C$-chord, as it possibly intersects with $C$ on many vertices. Indeed, if the path $C \setminus P_C$ is sufficiently short, then there is a $C$-band $Q$ with $\hat{Q} = C \setminus P_C$. In any case, the open set $U_Q$ from 1) of Definition 11.1 is uniquely specified. Our main result for Section 11 is the following.

Theorem 11.2. Let $\mathcal{T} = (\Sigma, G, \mathcal{C}, L, C, s)$ be a critical mosaic and let $C \in \mathcal{C}$ be an open $\mathcal{T}$-ring. Then,

1) $G$ does not contain a short $C$-band; AND

2) For any proper generalized chord $Q$ of $C$ with $|E(Q)| \leq N_{mo}$, where the endpoints of $Q$ lie in $V(P_C)$, we have $|E(P_C)| + |E(Q)| > |E(Q_{\text{aug}} \cap P_C)| + \frac{2N_{mo}}{3};$ AND

3) $|E(P_C)| = \frac{2N_{mo}}{3}$ and furthermore, for each $v \in D_1(C)$ with a neighbor in $P_C$, the graph $G[N(v) \cap V(P_C)]$ is a path of length at most one.

Proof. We first prove 1). We begin with the following intermediate result.

Claim 11.3. For any short $C$-band $Q$, every vertex of $C$ lies in $\partial(U_Q)$.

Proof: Suppose toward a contradiction that there is a $C$-band $Q$ with $V(C) \not\subseteq \partial(U_Q)$. Thus, $Q_{\text{aug}}$ separates either an element of $C \setminus \{C\}$ or a noncontractible cycle of $G$ from at least one vertex of $C$. We have $\text{Rk}(C) - \frac{3|E(Q_{\text{aug}})|}{2} \geq \frac{N_{mo}}{2}$, and since each vertex of $P_C$ has distance at most $\frac{N_{mo}}{3}$ from $w(C)$, we contradict Corollary 8.23.

Now we have the following

Claim 11.4. For any short $C$-band $Q$ and any proper generalized chord $P$ of $Q_{\text{aug}}$ with $P \subseteq \text{Cl}(U_Q)$ and $|E(P)| \leq 2$, the endpoints of $P$ are consecutive in $Q$. In particular, $|E(P)| = 2$.

Proof: Suppose there is a path $P \subseteq \text{Cl}(U_Q)$ which violates Claim 11.4. By M1), $P$ has at most one endpoint in $P_C$. If $P$ has one endpoint in $P_C$, and one endpoint in $Q_{\text{aug}} \setminus$, then, since $\text{Cl}(U_Q)$ contains either a noncontractible cycle or an element of $C \setminus \{C\}$, it follows that there exists a short $C$-band $Q'$ such that $V(C) \not\subseteq \partial(U_Q')$, contradicting Claim 11.3. Thus, both endpoints of $P$ lie in $Q$. By our face-width conditions, each cycle of $Q_{\text{aug}} \cup P$ is contractible.
so let \( H_0 \cup H_1 \) be the natural \((Q_{\text{aug}}, P)-\text{partition}\) of \( G \cap \text{Cl}(U_Q) \), where \( P_C \cap Q_{\text{aug}} \subseteq H_0 \). Since \( P_C \cap Q_{\text{aug}} \) has at least one edge, \( H_0 \) and \( H_1 \) are uniquely specified. Furthermore, since the endpoints of \( P \) are not consecutive in \( Q_{\text{aug}} \), we get that \( H_0 \) contains no noncontractible cycle of \( G \) or element of \( C \setminus \{C\} \), or else there is a short \( C \)-band \( Q' \) with \( P \subseteq Q' \), where \( Q' \) violates Claim 11.3. For each \( i = 0, 1 \), let \( F_i \) be the cycle \((H_i \cap Q_{\text{aug}}) + P\). Since the endpoints of \( P \) are not consecutive in \( Q_{\text{aug}} \) we have \(|E(F_i)| \leq |E(Q)| \leq N_{\text{mo}} \) for each \( i = 0, 1 \), and \( F_1 \) separates at least one vertex of \( C \) from a noncontractible cycle of \( G \) or an element of \( C \setminus \{C\} \). It follows from Corollary 8.23 that \( d(F_1, C \setminus P_C) > 2N_{\text{mo}} - \frac{3|E(F_1)|}{2} \). Now, at least one endpoint of \( P \) has distance at most \( \frac{|E(F_0)| - |E(P)| + |E(P_C)|}{2} - |E(F_0 \cap P_C)| \) from \( V(C \setminus P_C) \), so we get

\[
\frac{|E(F_0)| - |E(P)| + |E(P_C)|}{2} - |E(F_0 \cap P_C)| + \frac{3|E(F_1)|}{2} > 2N_{\text{mo}}
\]

Since \( |E(F_0)| + |E(F_1)| = |E(F_0 \cap P_C)| + |E(Q)| + 2|E(P)| \), we have the following inequality

\[
|E(F_1)| + |E(P)| + \frac{|E(Q)| + (|E(P_C)| - |E(F_0 \cap P_C)|)}{2} > 2N_{\text{mo}}
\]

Since \( |E(P_C)| \leq \frac{2N_{\text{mo}}}{3} \) and \( |E(Q)| \leq |E(Q_{\text{aug}})| \leq N_{\text{mo}} \), the fractional term on the left-hand side of the inequality above is bounded from above by \( \frac{2N_{\text{mo}}}{6} \). Since \( |E(F_1)| \leq N_{\text{mo}} \), we have \( \frac{11N_{\text{mo}}}{6} + 2 > 2N_{\text{mo}} \), which is false. \( \blacksquare \)

Now we have enough to finish the proof of 1) of Theorem 11.2. Suppose toward a contradiction that \( Q' \) is a mosaic. Since \( |E(Q_{\text{aug}})| \leq N_{\text{mo}} \), M0 is satisfied, and M1) is immediate, since \( Q_{\text{aug}} \) is a closed \( T' \)-ring. It follows from Claim 11.4 that \( Q_{\text{aug}} \) is an expectable, and thus \( L_{Q_{\text{aug}}}^{\psi} \)-predictable, facial subgraph of \( G' \), so 2) is satisfied as well. Since \( V(Q_{\text{aug}}) = V(C) \), we get that each vertex of \( Q_{\text{aug}} \) has distance at most \( \frac{N_{\text{mo}}}{9} \) from \( \psi(L)(C) \). Since \( \text{Rk}(T'|Q_{\text{aug}}) \leq \text{Rk}(T|C) - N_{\text{mo}} \), the distance conditions M3)-M4) are still satisfied. Now we just need to check M5). Since edge-width is monotone, we have ew(#G) ≥ ew(G) ≥ 2.4β · 3^{p(3)}-2, and since \( |E(Q_{\text{aug}})| \leq N_{\text{mo}} \) and \( \text{fw}(G) \geq 1.1β \cdot 3^{p(3)}-2 \), it follows from our distance conditions, together with Fact 7.3 that \( \text{fw}(G') \geq 1.1β \cdot 3^{p(3)}-2 \). Thus, M5) is satisfied as well, and \( T' \) is indeed a mosaic. Since \( V(G) = V(G') \) and \( C \) contains at least one vertex with an \( L \)-list of size at least three, we have \( \sum_{v \in V(G')} |L_{Q_{\text{aug}}}^{\psi}(v)| < \sum_{v \in V(G)} |L(v)| \), so it follows from the criticality of \( T \) that \( G' \) is \( L' \)-colorable, and thus \( G \) is \( L \)-colorable, a contradiction. This proves 1).

Now we prove 2) of Theorem 11.2. Suppose toward a contradiction that \( |E(P_C)| + |E(Q)| \leq |E(Q_{\text{aug}} \cap P_C)| + \frac{2N_{\text{mo}}}{3} \).

Let \( U \) be the unique connected component of \( \Sigma \setminus (C \cup Q) \) with \( U \neq U_Q \) and \( Q \subseteq \partial(U) \). Since \( C \) is an open ring, \( P_C \cap \text{Cl}(U) \) has two connected components. By Theorem 10.4 \( \text{Cl}(U) \) contains no noncontractible cycles of \( G \) or
elements of $C \setminus \{C\}$. By 2) of Proposition[10.9] $g(\Sigma) + |C \setminus \{C\}| > 1$, and since $G$ is a 2-cell embedding, it follows that that $\text{Cl}(U_Q)$ contains either an element of $C \setminus \{C\}$ or a noncontractible cycle, contradicting 1). Now we prove 3).

Suppose that $|E(P_C)| \neq \frac{2N_{mo}}{3}$. Since $|E(P_C)| \leq \frac{2N_{mo}}{3}$ and $N_{mo}$ is a multiple of three, we have $|E(P_C)| < \frac{2N_{mo}}{3}$.

By 1) of Proposition[10.9] $C$ is an induced cycle of $G$, so it follows from our triangulation conditions that there is a 2-chord of $C$ whose endpoints are consecutive vertices of $P_C$. Applying 2) to this 2-chord of $C$, we have $2 + (|E(P_C)| - 1) > \frac{2N_{mo}}{3}$, contradicting our assumption that $|E(P_C)| < \frac{2N_{mo}}{3}$. Now let $v \in D_1(C)$, where $v$ has a neighbor in $P_C$. Suppose toward a contradiction that $G[N(v) \cap V(P_C)]$ is not a path of length at most one. Then $G$ contains a 2-chord $Q$ of $C$ whose endpoints lie in $P_C$ but are not consecutive in $P_C$. In particular, we have $|E(Q_{aug} \cap P_C)| \geq 2$. Since $|E(P_C)| = \frac{2N_{mo}}{3}$, we contradict 2). □

The result above, together with the results of Section[9] and[10] yields the following consequence.

**Theorem 11.5.** Let $T = (\Sigma, G, C, L, C_*)$, let $C \in C$, and let $Q$ be a generalized chord of $Q$ with $|E(Q)| \leq \frac{N_{mo}}{3}$. Let $G = G_0 \cup G_1$ be the natural $Q$-partition of $G$. Then there exists a unique $j \in \{0, 1\}$ such that the following hold.

A) $G_{1-j}$ is contractible and each $C' \in C \setminus \{C\}$ lies in $G_j$.

B) If $C$ is an open ring, then $P_C \cap G_j$ is connected and has at least one edge.

**Proof.** Given a $j \in \{0, 1\}$, we call $j$ ideal if it satisfies A) and, if $C$ is an open ring, then it also satisfies B). Let $U$ be the unique connected component of $\Sigma \setminus C$ such that $Q \not\subseteq \text{Cl}(U)$, i.e the unique component of $\Sigma \setminus C$ containing no vertices of $G$. Since $G$ is a 2-cell embedding, $U$ is an open disc. By 2) of Proposition[10.9] we have $g(\Sigma) + |C \setminus \{C\}| > 0$. It follows that at most one $j \in \{0, 1\}$ can satisfy A), so we just need to show that there exists an ideal $j \in \{0, 1\}$. If $C$ is a closed ring, it it follows from Theorem[9.1] that there is an ideal $j \in \{0, 1\}$. Now suppose that $C$ is an open ring. Consider the following cases.

**Case 1:** $Q$ is an improper generalized chord of $C$

In this case, let $j \in \{0, 1\}$ be the unique index such that $C \subseteq G_j$. Thus, $P_C \subseteq G_j$ and $V(P_C \cap G_j)$ consists of at most one vertex. If $C \neq C_*$, then it follows from Corollary[8.24] that $j$ is ideal, so suppose that $C = C_*$. In that case, we have $G_j = \text{Ext}(Q)$, and since $\text{Rk}(C) = 2N_{mo}$, it follows from a) of Corollary[8.23] that $j$ is ideal.

**Case 2:** $Q$ is a proper generalized chord of $C$

If neither endpoint of $Q$ lies in $P_C$, then it follows from Theorem[10.4] that there is a ideal $j \in \{0, 1\}$. Now suppose that precisely one endpoint of $Q$ lies in $P_C$. Since $|E(Q)| \leq \frac{N_{mo}}{3}$ and $|E(P_C)| \leq \frac{2N_{mo}}{3}$, there is at least one $k \in \{0, 1\}$ such that $|E(Q)| + |E(P_C \cap G_k)| \leq \frac{2N_{mo}}{3}$, so it again follows from Theorem[10.4] that there is an ideal $j \in \{0, 1\}$. Now
suppose that both endpoints of $Q$ lie in $\bar{P}_C$ and let $j \in \{0, 1\}$ be the unique index such that $G_{1-j} \cap P_C = Q_{\text{aug}} \cap P_C$. By 1) of Theorem 11.2, $j$ is ideal, so we are done. \hfill \square

As indicated in Section 4, we need to show that the underlying graph of a critical mosaic $(\Sigma, G, C, L, C_*)$ has high-enough representativity near the special faces to guarantee that the graph obtained by cutting $G$ along a path between two of the rings of $C$ still satisfies M5. In Paper II, we use the results of this paper to prove the following.

**Theorem 11.6.** Let $\mathcal{T} = (\Sigma, G, C, L, C_*)$ be a critical mosaic, let $C \in C$, and let $P$ be a generalized chord of $C$. Suppose that each cycle of $C \cup P$ which is distinct from $C$ is noncontractible. Then $|E(P)| \geq 2.3\beta \cdot 3g(\Sigma)^{-2}$. 

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49