The Exponential Decay of Gluing Maps for $J$-Holomorphic map Moduli Spaces

An-Min Li and Li Sheng

Department of Mathematics, Sichuan University Chengdu, PRC

Abstract

We prove the exponential decay of the derivative of the gluing maps with respect to the gluing parameter.

1 Introduction and Preliminary

The gluing analysis plays an important role in Gromov-Witten Invariants theory. In this paper we study the gluing estimates, in particular the estimates of the derivatives of the gluing maps with respect to the gluing parameter $r$. We describe now the problem and state our main result. We only consider the case of gluing one nodal, for general cases the estimates are the same.

1.1 $J$-holomorphic maps from Riemann surface with one nodal point

Let $(M, \omega, J)$ be a closed $C^\infty$ symplectic manifold of dimension $2m$ with $\omega$-tame almost complex structure $J$, where $\omega$ is a symplectic form. Then there is a Riemannian metric

$$G_J(v, w) := \langle v, w \rangle_J := \frac{1}{2} (\omega(v, Jw) + \omega(w, Jv))$$

(1.1)

for any $v, w \in TM$. Following [7] we choose the complex linear connection

$$\tilde{\nabla}_X Y = \nabla_X Y - \frac{1}{2} J (\nabla_X J) Y$$

induced by the Levi-Civita connection $\nabla$ of the metric $G_J$.

Let $(\Sigma, j, y, q)$ be a marked nodal Riemann surface of genus $g$ with $n$ marked points $y = (y_1, \ldots, y_n)$ and one nodal point $q$. We write the marked nodal Riemann surface as

$$\Sigma = \Sigma_1 \cup \Sigma_2, j = (j_1, j_2), y = (y_1, y_2), q = (p_1, p_2),$$

(1.3)

where $(\Sigma_i, j_i, y_i, q_i), i = 1, 2$, are smooth Riemann surfaces of genus $g_i$ with $n_i$ marked points $y_i$ and puncture $q_i$. We say that $q_1, q_2$ are paired to form $q$. We assume that $(\Sigma_i, j_i, y_i, q_i)$ is stable, i.e., $n_i + 1 > 2 - 2g_i$, $i = 1, 2$. Let $z_i$ be a local complex coordinate around $q_i$, $z_i(q_i) = 0$, $i = 1, 2$. Let

$$z_1 = e^{-s_1 - 2\pi \sqrt{-1} t_1}, \quad z_2 = e^{s_2 + 2\pi \sqrt{-1} t_2},$$

(1.4)

$(s_i, t_i)$ are called the holomorphic cylindrical coordinates near $q_i$. In terms of the holomorphic cylindrical coordinates we write

$$\Sigma_1 := \Sigma_1 \setminus \{q_1\} \cong \Sigma_{10} \cup \{[0, \infty) \times S^1\},$$

$$\Sigma_2 := \Sigma_2 \setminus \{q_2\} \cong \Sigma_{20} \cup \{(-\infty, 0) \times S^1\}.$$

(1.5)

1 partially supported by a NSFC grant
2 anminliscu@126.com, lshengscu@gmail.com
Here \( \Sigma_0 \subset \Sigma_i \), \( i = 1, 2 \), are compact surfaces with boundary. Put \( \tilde{\Sigma} = \Sigma \setminus \{q_1, q_2\} = \tilde{\Sigma}_1 \cup \tilde{\Sigma}_2 \). We introduce the notations

\[
\Sigma_i(R_0) = \Sigma_0 \cup \{(s_i, t_i) \mid |s_i| \leq R_0\}, \quad \Sigma(R_0) = \Sigma_1(R_0) \cup \Sigma_2(R_0).
\]

We choose a local coordinate system \((a_1, a_2) \in A_1 \times A_2\) for complex structure on \( \tilde{\Sigma}_1 \cup \tilde{\Sigma}_2 \), where \( A_i \subset \mathbb{R}^{6n_i - 6 + 2m_i} \) is an open set, and \( a_i = (j_i, y_i) \).

For any gluing parameter \((r, \tau)\) with \( r \geq R_0 \) and \( \tau \in S^1 \) we construct a surface \( \Sigma(r) = \Sigma_1 \#(r) \Sigma_2 \) as follows, where and later we use \((r)\) to denote gluing parameters. We cut off the part of \( \Sigma_i \) with cylindrical coordinate \(|s_i| > \frac{r}{2}\) and glue the remainders along the collars of length \( r \) of the cylinders with the gluing formulas:

\[
s_1 = s_2 + 2r, \quad t_1 = t_2 + \tau.
\]

\( \Sigma(R_0) \) can naturally equate to the subset of \( \Sigma(r) \). Then \( (a_1, a_2, r, \tau) \) is a local coordinate system near \((\Sigma, j, y, q)\) in the Teichmüller space \( T_{g,n} \). For any \( a = (a_1, a_2) \in A_1 \times A_2 \) with \( a_i = (j_i, y_i) \), let \( j_{(r),a} \) be the complex structure on \( \Sigma(r) \) satisfying

\[
\dot{j}_{(r),a}|_{\Sigma(R_0)} = j_i, \quad \dot{j}_{(r),a} \frac{\partial}{\partial s_1} |_{\Sigma(R) \setminus \Sigma(R_0)} = \frac{\partial}{\partial t_1} |_{\Sigma(R) \setminus \Sigma(R_0)}
\]

where \( z = e^{-r - 2\pi \sqrt{-1} \tau} \). If no danger of confusion we denote \( j_{(r),a} \) by \( j_a \).

We may choose a family of metrics \( g_i \) on \( \tilde{\Sigma}_i \) in the given conformal class \( j_i \), depending on \( a_i \in A_i \) smoothly, such that, restricting to \( \Sigma \setminus \Sigma(R_0) \),

\[
g_i = (ds_i)^2 + (dt_i)^2,
\]

the standard cylinder metric. Then we define a metric \( g \) on \( \tilde{\Sigma} \) as \( g = g_1 \oplus g_2 \).

Let \( u = (u_1, u_2) \), where \( u_i : \Sigma \to M \) is \((j_i, J)\)-holomorphic map such that \( u_1(q_1) = u_2(q_2) \). We choose the local normal coordinates \((x^1, \ldots, x^{2m})\) in a neighborhood \( O_u(q) \) of \( u(q) \) such that

\[
(x^1, \ldots, x^{2m})(u(q)) = 0, \quad J(0) \frac{\partial}{\partial x^i} |_{0} = \frac{\partial}{\partial x^{m+i}} |_{0}, \quad J(0) \frac{\partial}{\partial x^{m+i}} |_{0} = - \frac{\partial}{\partial x^i} |_{0}, \quad i \leq m.
\]

In terms of the holomorphic cylindrical coordinates \((s_i, t_i)\) over each tube the linearized operator \( D_{u_i} \) takes the following form (see Appendix 6.1)

\[
D_{u_i} = \frac{\partial}{\partial s_i} + J_0 \frac{\partial}{\partial t_i} + F^1_{u_i} \frac{\partial}{\partial t_i} + \cdots + F^k_{u_i} \frac{\partial}{\partial t_i},
\]

where \( \sum_{p+q=m} \left| \frac{\partial^p F^q}{\partial s_i^p \partial t_i^q} \right| \to 0 \), for \( l, i = 1, 2, \forall d \geq 0 \), exponentially and uniformly in \( t_i \) as \(|s_i| \to \infty \). Here \( J_0 \) is the standard complex structure in \( O_u(q) \) such that \( J_0 \frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^{m+i}} \), \( J_0 \frac{\partial}{\partial x^{m+i}} = - \frac{\partial}{\partial x^i} \). Therefore, the operator \( H_{s_i} = J_0 \frac{\partial}{\partial t_i} + F^1_{u_i} \frac{\partial}{\partial t_i} \) converges to \( H_\infty = J_0 \frac{d}{dt} \). Obviously, the operator \( D_{u_i} \) is not Fredholm operator because over nodal end the operator \( H_\infty = J_0 \frac{d}{dt} \) has zero eigenvalue. The \( \ker H_\infty \) consists of constant vectors in \( T_{u(q)} M \). To recover a Fredholm theory we use weighted spaces \( W^{k,2,\alpha}(u^*TM) \). In this paper we take \( k \geq 2 \). Fix a positive function \( W \) on \( \Sigma \) which has order equal to \( e^{\alpha|s|} \) on each end, where \( \alpha \) is a small constant such that \( 0 < \alpha < 1 \) and over each end \( H_\infty - \alpha = J_0 \frac{d}{dt} - \alpha \) is invertible. We will write the weight function simply as \( e^{\alpha|s|} \). For any section \( h \in C_\infty(\tilde{\Sigma}; u^*TM) \) and section \( \eta \in C_\infty(\tilde{\Sigma}; u^*TM \otimes \Lambda^{0,1}_T \tilde{\Sigma}) \) we define
the norms
\[ \| h \|_{k,2,\alpha} = \left( \int_{\Sigma} e^{2\alpha|s|} \sum_{i=0}^{k} |\nabla_i h|^2 \text{dvol}_\Sigma \right)^{1/2}, \tag{1.9} \]
\[ \| \eta \|_{k-1,2,\alpha} = \left( \int_{\Sigma} e^{2\alpha|s|} \sum_{i=0}^{k-1} |\nabla_i \eta|^2 \text{dvol}_\Sigma \right)^{1/2}. \tag{1.10} \]

Denote by \( W^{k,2,\alpha}(\Sigma; u^*TM) \) and \( W^{k-1,2,\alpha}(\Sigma; u^*TM \otimes \Lambda_{j_0}^{0,1} T^* \Sigma) \) the complete spaces with respect to the norms (1.9) and (1.10) respectively. The operator
\[ D_u : W^{k,2,\alpha}(\Sigma; u^*TM) \to W^{k-1,2,\alpha}(\Sigma; u^*TM \otimes \Lambda_{j_0}^{0,1} T^* \Sigma) \]
is a Fredholm operator as long as \( \alpha \) does not lie in the spectrum of the operator \( H_\infty \).

We choose \( R_0 \) so large that \( u(\{|s| \geq \frac{R}{2}\}) \) lie in \( O_{u(q)} \) for any \( r > R_0 \). In this coordinate system we identify \( T_x M \) with \( T_{u(q)} M \) for all \( x \in O_{u(q)} \). Any \( h_0 \in T_{u(q)} M \) may be considered as a vector field in the coordinate neighborhood. We fix a smooth cutoff function \( \varphi \):
\[ \varphi(s) = \begin{cases} 1, & \text{if } |s| \geq \tilde{d}, \\ 0, & \text{if } |s| \leq \frac{\tilde{d}}{2} \end{cases} \]
where \( \tilde{d} \) is a large positive number. Put
\[ \tilde{h}_0 = \varphi h_0. \]

Then for \( \tilde{d} \) large enough \( \tilde{h}_0 \) is a section in \( \mathcal{W}^{k,2,\alpha}(\Sigma; u^*TM) \) supported in the tube \( \{(s,t)||s| \geq \frac{\tilde{d}}{2}, t \in S^1\} \).

Denote
\[ \mathcal{W}^{k,2,\alpha}(\Sigma; u^*TM) = \{ h + \tilde{h}_0 | h \in W^{k,2,\alpha}(\Sigma; u^*TM), h_0 \in \ker H_\infty \}. \]

We define weighted Sobolev norm on \( \mathcal{W}^{k,2,\alpha} \) by
\[ \| h + \tilde{h}_0 \|_{\mathcal{W}^{k,2,\alpha}} = \| h \|_{k,2,\alpha} + |h_0|, \]
where \( |h_0| = [G_f(h_0, h_0)_{u(q)}]^{\frac{1}{2}} \). If no danger of confusion, later we will denote
\[ W^{k,2,\alpha}_u = W^{k,2,\alpha}(\Sigma; u^*TM), \quad W^{k,2,\alpha}_u = W^{k,2,\alpha}(\Sigma; u^*TM), \quad L^{k-1,2,\alpha}_a = W^{k-1,2,\alpha}(\Sigma, u^*TM \otimes \Lambda_{j_0}^{0,1} T^* \Sigma). \]

Let
\[ B_1 = \{ v_i = \exp_{u_i}(h_i) | h_i \in W^{k,2,\alpha}_u \}, \]
\[ B_1 \times_q B_2 := \{ v = (v_1, v_2) \in B_1 \times B_2 | v_1(q_1) = v_2(q_2) \}. \]

For any \( \rho > 0 \) set
\[ O_{h_0}(\rho) := \{ v \in B_1 \times_q B_2 | v = \exp_u(h + \tilde{h}_0), \| h + \tilde{h}_0 \|_{W^{k,2,\alpha}} < \rho \}. \tag{1.11} \]

Let \( \mathcal{E}_i \) be the infinite dimensional Banach bundle over \( A_i \times B_i \), whose fiber at \( (a_i, v_i) \) is \( W^{k-1,2,\alpha}(\Sigma, v_i^*TM \otimes \Lambda_{j_0}^{0,1} T^* \Sigma) \). We have a Fredholm system \( (A_i \times B_i, \mathcal{E}_i, \mathcal{O}) \). We will fix a complex structure \( a_o = (a_{o1}, a_{o2}) \).
1.2 Preglueing

Let \( b_{oi} = (a_{oi}, u_i), i = 1, 2, \) where \( a_{oi} \in i, u_i : \Sigma_i \to M \) are \((j_{oi}, J)\)-holomorphic maps with \( u_1(q_1) = u_2(q_2) \). Denote \( b_o = (b_{o1}, b_{o2}) = (a_o, u) = (j_o, y_o, u) \). Let \( r > 4R_0 \). We glue the map \((u_1, u_2)\) to get a preglueing maps \( u_{(r)} \), a family of approximate \((j_o, J)\)-holomorphic maps as follows. Set

\[
\begin{align*}
  u_{(r)} = \begin{cases} 
    u_1 & \text{on } \Sigma_{10} \cup \{(s_1, t_1)|0 \leq s_1 \leq \frac{r}{2}, t_1 \in S^1\} \\
    u_1(q) = u_2(q) & \text{on } \{(s_1, t_1)|\frac{3r}{4} \leq s_1 \leq \frac{5r}{4}, t_1 \in S^1\} \\
    u_2 & \text{on } \Sigma_{20} \cup \{(s_2, t_2)|0 \geq s_2 \geq -\frac{r}{2}, t_2 \in S^1\}.
  \end{cases}
\end{align*}
\]

To define the map \( u_{(r)} \) in the remaining part we fix a smooth cutoff function \( \beta : \mathbb{R} \to [0, 1] \) such that

\[
\beta(s) = \begin{cases} 
  1 & \text{if } s \geq 1 \\
  0 & \text{if } s \leq 0
\end{cases}
\]

(1.12) and \( \sqrt{1 - \beta^2} \) is a smooth function, \( 0 \leq \beta'(s) \leq 4 \) and \( \beta^2(\frac{1}{2}) = \frac{1}{2} \). We define

\[
u_{(r)} = u_1(q) + \beta \left(3 - \frac{4s_1}{r}\right)(u_1(s_1, t_1) - u_1(q)) + \beta \left(\frac{4s_1}{r} - 5\right)(u_2(s_1 - 2r, t_1 - \tau) - u_2(q)).
\]

We introduce a notation \( \beta_{i;R}(s_i) \). For any \( R > 0 \) denote

\[
\begin{align*}
  \beta_{1;R}(s_1) &= \beta \left(\frac{1}{2} + \frac{r - s_1}{R}\right), \quad \beta_{2;R}(s_2) = \sqrt{1 - \beta^2 \left(\frac{1}{2} - \frac{s_2 + r}{R}\right)},
\end{align*}
\]

(1.13) where \( \beta \) is the cut-off function defined in (1.12). Then we have

\[
\beta_{2;R}^2(s_1 - 2r) = \left(1 - \beta^2 \left(\frac{1}{2} - \frac{s_1 - r}{R}\right)\right) = 1 - \beta_{1;R}^2(s_1).
\]

(1.14) For any \( \eta \in C^\infty(\Sigma_{(r)}; u_{(r)}^* TM \otimes \wedge_{j_o}^{0,1} T\Sigma_{(r)}) \), let

\[
\eta_i(p) = \begin{cases} 
  \eta & \text{if } p \in \Sigma_i(R_0) \\
  \beta_{i;2}(s_i)(\eta_{1}(s_i, t_i) & \text{if } p \in \Sigma_i(r + 1) \setminus \Sigma_i(R_0) \\
  0 & \text{if } p \in \Sigma_i \setminus \Sigma_i(r + 1)
\end{cases}
\]

(1.15) Then \( \eta_i \) can be considered as a section over \( \Sigma_i \), i.e., \( \eta_i \in C^\infty(\Sigma_i; u_i^* TM \otimes \wedge_{j_o}^{0,1} T\Sigma_i) \). If no danger of confusion we will denote (1.15) by \( \eta_i = \beta_{i;2} \eta \). Define

\[
\|\eta\|_{k-1,2,\alpha,x} = \|\eta_1\|_{\Sigma_{1,k-1,2,\alpha}} + \|\eta_2\|_{\Sigma_{2,k-1,2,\alpha}}.
\]

(1.16) We now define a norm \( \| \cdot \|_{k,2,\alpha,x} \) on \( C^\infty(\Sigma_{(r)}; u_{(r)}^* TM) \). For any section \( h \in C^\infty(\Sigma_{(r)}; u_{(r)}^* TM) \) denote

\[
h_0 = \int_{S^1} h(r, t)dt,
\]

(1.17) \( h_1(s_1, t_1) = (h - h_0)(s_1, t_1)\beta_{1;2}(s_1), \quad h_2(s_2, t_2) = (h - h_0)(s_2, t_2)\beta_{2;2}(s_2). \)

(1.18) We define

\[
\|h\|_{k,2,\alpha,x} = \|h_1\|_{\Sigma_{1,k,2,\alpha}} + \|h_2\|_{\Sigma_{2,k,2,\alpha}} + |h_0|.
\]

(1.19)
Denote the resulting completed spaces by

\[ W^{k-1,2,\alpha}_r(\Sigma(x); u_r^*)TM \otimes \Lambda^{0,1}_o T\Sigma(x) \]  
and \[ W^{k,2,\alpha}_r(\Sigma(x); u_r^*)TM \]

respectively. To simplify notations we will denote

\[ W^{k,2,\alpha}_r; u_r^*; TM = W^{k,2,\alpha}_r(\Sigma(x), u_r^*; TM), \quad L^{k-1,2,\alpha}_r; u_r^*; TM = W^{k-1,2,\alpha}_r(\Sigma(x), u_r^*; TM \otimes \Lambda^{0,1}_o T\Sigma(x)). \]

Set \( \mathbb{D} = \{ z = e^{-2r-\sqrt{-2}T\pi} | R_0 < r \leq \infty, \quad 0 \leq \tau \leq 1 \} \) and for \( (r) \in \mathbb{D} \) denote

\[ \mathcal{B}_r = \left\{ v_r : \Sigma(x) \to M \mid v_r = \exp_{u_r^*} h_r, \quad h_r \in W^{k,2,\alpha}_r \right\}. \]

For any \( R > R_0, \rho > 0 \) denote

\[ O_b(R; \rho) := \bigcup_{r \geq R, \tau \in \mathbb{S}^1} \left\{ (e^{-2r-2\sqrt{-2}\tau}, v_r) \mid v_r = \exp_{u_r^*} h_r \in \mathcal{B}_r, \|h_r\|_{k,2,\alpha, r < \rho} \right\}. \]

### 1.3 Local regularization

We want to use the implicit function theorem to get \((j, J)\)-holomorphic maps from \( \Sigma(r) \to M \). When the transversality fails we need to take the "regularization". We explain this now. Fix \( a_o = (a_{o1}, a_{o2}) \), where \( a_o = (j_o, y_o) \).

Let \( \tilde{E} \) be the infinite dimensional Banach bundle over \((B_1 \times_q B_2) \mid O_b_{o}(\rho) \) whose fiber at \( b \in O_b_{o}(\rho) \) is

\[ \tilde{E}_b := \left\{ \beta((R_0 - s_i)\eta(s_i, t_i))\eta \in \tilde{E}_1 \times \tilde{E}_2 \right\}. \]

\( \tilde{E} \) can be viewed as a infinite dimensional Banach bundle over \( \tilde{E}_r \mid O_b_{o}(R, \rho) \) for \( r > R > R_0 \). Denote by \( \text{inj}_M \) the injective radius of \((M, G_J)\). Given \( \xi \in W^{k,2,\alpha}_r \) with \( \|\xi\|_{L_{\infty}} < \text{inj}_M \), let

\[ \Phi_{u_r^*}(\xi) : u_r^*; TM \to (\exp_{u_r^*} \xi)^*TM \]

(1.20) denote the complex bundle isomorphism, given by parallel transport along the geodesics \( s \to \exp_{u_r^*} (s\xi) \) with respect to the connection \( \tilde{\nabla} \). There is a neighborhood \( O_b_{o}(R, \rho) \), over which \( \tilde{E} \) is trivialized. Since \( u_r^*|_{\Sigma(x)} = u|_{\Sigma(x)} \), there is an isomorphism \( \Phi_{b_o, b} : \tilde{E}_b \to \tilde{E}_b \) for any \( b \in O_b_{o}(R, \rho) \), where \( \Phi_{b_o, b} \) is induced by \( \Phi \). We can choose a finite dimensional subspace \( K_{b_o} = (K_{b_o, 1}, K_{b_o, 2}) \subset \tilde{E}|_{b_o} = (\tilde{E}_{b_o, 1}, \tilde{E}_{b_o, 2}) \) such that

\[ K_{b_o} \text{ and } \text{im} D_{u_1} \tilde{E}|_{b_o, 1}, \quad K_{b_o} \text{ and } \text{im} D_{u_2} \tilde{E}|_{b_o, 2}. \]

By a small perturbation we may assume that every member of \( K_{b_o} \) is \( C^\infty \) and supports in the compact subset \( \Sigma(x) \) for some large number \( R_0 \). Then \( K_{b_o} \) can be considered a subspace of \( L^{k-1, -2, \alpha}_r; u_r^*; TM \) in a natural way. We define a thickened Fredholm system \((K_{b_o} \times O_b_{o}(R, \rho), K_{b_o} \times \tilde{E}|_{O_b_{o}(R, \rho)}, \tilde{S})\) with

\[ \tilde{S}(\kappa, b) = \tilde{\partial}_{j_o, J} v + i(\kappa, b), \quad \kappa \in K_{b_o}, \]

(1.21) where \( i(\kappa, b) = \Phi_{b_o, b} \kappa \) and \( b = (a_o, v) \). Denote by \( DS_{(\kappa, b)} \) the linearized operator of \( \tilde{S} \) at \((\kappa, b)\). Then

\[ DS(0, b_0)|_{K_{b_o} \times W^{k,2,\alpha}_u : K_{b_o} \times W^{k,2,\alpha}_u \to L^{k-1, -2, \alpha}_u} \]

is surjective. Let \( Q(0, b_0) \) be a right inverse of \( DS_{(0, b_0)} \). We call \((\kappa, b)\) a perturbed \((j_o, J)\)-holomorphic map, if \((\kappa, b)\) satisfies \( S(\kappa, b) = 0 \). If no confusion, we denote \( K_{b_o} \) by \( K \). Let \((\kappa_o, b_o)\) be a perturbed \((j_o, J)\)-holomorphic map. Denote by \( DS_{(\kappa_o, b_o)} \) the linearized operator of \( S \) at \((\kappa_o, b_o)\), by \( Q(\kappa_o, b_o) \) a right inverse of \( DS_{(\kappa_o, b_o)} \). Denote by \( DS_{(\kappa_o, b_o)} \) the linearized operator of \( S \) at \((\kappa_o, b_o)\).
1.4 Some operators

Given \( \eta \in L^{k-1,2,\alpha}_{r,u(r)} \) denote
\[
(\eta_1(s_1,t_1), \eta_2(s_2,t_2)) = (\beta_{1,2}(s_1)\eta(s_1,t_1), \beta_{2,2}(s_2)\eta(s_2,t_2)) ,
\]
(1.22)

\[
Q(\kappa_0, b_0)(\eta_1, \eta_2) = (\kappa, h) = (\kappa(h_1, h_2)) , \ h_i \in W^{k,2,\alpha}(\Sigma_i; u_i^*TM),
\]
(1.23)

where \( \eta(s_1,t_1) \) denote the expression of \( \eta \) in terms of the coordinates \( (s_i, t_i) \). We define \( h(r) \in W^{k,2,\alpha}_{r,u(r)} \) by
\[
h(r) = \beta_{1,r}(s_1)h_1(s_1,t_1) + \beta_{2,r}(s_1 - 2r)h_2(s_1 - 2r, t_1 - \tau).
\]
(1.24)

Note that, in the part \( \{|s_i| \geq \frac{\xi}{2}\} \), \( h_1 \) and \( h_2 \) are identified as vectors in \( T_u(q)M \), so the meaning of definition (1.24) is clear. Since \( i(\kappa, b) \) supports in \( \Sigma(R_0) \) for all \( \kappa \in K \) and
\[
u(r)|_{\{s_1 \leq \frac{\xi}{2}\}} = u_1|_{\{s_1 \leq \frac{\xi}{2}\}}, \quad u(r)|_{\{s_2 \geq \frac{\xi}{2}\}} = u_2|_{\{s_2 \leq \frac{\xi}{2}\}},
\]
we have \( i(\kappa, b(r)) = i(\kappa, b) \) along \( u(r) \). Then we define an approximate right inverse
\[
Q'(\kappa_0, b_0)\eta = (\kappa, h(r)).
\]
(1.25)

It is easy to show that \( DS(\kappa_0, b_0) \circ Q'(\kappa_0, b_0) \) is invertible when \( r \) big enough (cf. the proof of Lemma 3.2). Then a right inverse \( Q(\kappa_0, h(r)) \) of \( DS(\kappa_0, b_0) \) is given by
\[
Q(\kappa_0, b_0) = Q'(\kappa_0, b_0)(DS(\kappa_0, b_0) \circ Q'(\kappa_0, b_0))^{-1}.
\]
(1.26)

For a fixed gluing parameter \( (r) = (r, \tau) \) we define a map
\[
I_r : \ker DS(\kappa_0, b_0) \rightarrow \ker DS(\kappa_0, b_0)
\]
as follows. For any \( (\kappa, h + \hat{h}_0) \in \ker DS(\kappa_0, b_0) \), where \( h = (h_1, h_2) \in W^{k,2,\alpha}_u \), we set
\[
h(r) = \beta_{1,r}(s_1)h_1(s_1,t_1) + \beta_{2,r}(s_1 - 2r)h_2(s_1 - 2r, t_1 - \tau) + \hat{h}_0,
\]
(1.27)

and define
\[
I_r(\kappa, h + \hat{h}_0) = (\kappa, h(r)) - Q(\kappa_0, b_0) \circ DS(\kappa_0, b_0)(\kappa, h(r)).
\]
(1.28)

It is easy to see that \( I_r(\kappa, h + \hat{h}_0) \in \ker DS(\kappa_0, b_0) \). We can prove that when \( r \) large enough \( I_r \) is an isomorphism (cf. 3.2).

It is important to estimate the derivative of the gluing map with respect to \( r \). To this end we need to take the derivative \( \frac{\partial}{\partial r} \) for \( Q(\kappa_0, b_0) \) and other operators. Note that both \( Q(\kappa_0, b_0) \) and \( f_r \) are global operators, so we need a global estimate. On the other hand, since the domain \( \Sigma_r \) depends on \( r \), in order to make the meaning of the derivative \( \frac{\partial}{\partial r} \) for these operators clear we need transfer all operators defined over \( \Sigma_r \) into a family of operators defined over \( \Sigma_1 \cup \Sigma_2 \), depending on \( (r) \). We first define three maps
\[
H_r : L^{k-1,2,\alpha}_{r,u(r)} \rightarrow L^{k-1,2,\alpha}_u, \quad P_r : L^{k-1,2,\alpha}_u \rightarrow L^{k-1,2,\alpha}_{r,u(r)}, \quad \phi_r : W^{k,2,\alpha}_u \rightarrow W^{k,2,\alpha}_{r,u(r)}
\]
as following. Given \( \eta \in L^{k-1,2,\alpha}_{r,u(r)} \) define
\[
H_r\eta = (\beta_{1,2}(s_1)\eta(s_1,t_1), \beta_{2,2}(s_2)\eta(s_2,t_2)),
\]
where \( \eta(s_1, t_1) \) is the expression of \( \eta \) in terms the coordinates \((s_1, t_1)\). Given \((\eta_1, \eta_2) \in L^{k-1,2,\alpha}_u \) define

\[
P_r(\eta_1, \eta_2) = \begin{cases} 
\eta_i & \text{if } p \in \Sigma(r/2) \\
b_1,2(s_1)\eta_1(s_1, t_1) + \beta_2,2(s_1 - 2r)\eta_2(s_1 - 2r, t_1 - \tau) & \text{if } p \in \Sigma(r) \setminus \Sigma(r/2) 
\end{cases} 
\quad (1.29)
\]

If no danger of confusion we will denote \((1.29)\) by \( P_r(\eta_1, \eta_2) = \sum \beta_i,2\eta_i \).

Given \((h_1 + \hat{h}_0, h_2 + \hat{h}_0) \in W^{k,2,\alpha}_u \) with \( \text{supp} \ h_1 \subset \Sigma(3r/2) \), define

\[
\phi_r \left( h_1 + \hat{h}_0, h_2 + \hat{h}_0 \right)_{\Sigma(r/2)} = \left( h_1 + \hat{h}_0 \right)_{\Sigma(r/2)}, \\
\phi_r \left( h_1 + \hat{h}_0, h_2 + \hat{h}_0 \right)_{\frac{r}{2} \leq s_1 \leq \frac{3r}{2}} = \left( h_1(s_1, t_1) + h_2(s_1 - 2r, t_1 - \tau) + \hat{h}_0 \right)_{\frac{r}{2} \leq s_1 \leq \frac{3r}{2}}.
\]

By \((1.14)\) one can check that

\[
P_r H_r = Id, \quad H_r P_r(\eta_1, \eta_2) = (\tilde{\xi}_1, \tilde{\xi}_2).
\]

where

\[
\tilde{\xi}_1 = \beta_1,2(\beta_1,2\eta_1(s_1, t_1) + \beta_2,2\eta_2(s_1 - 2r, t_1 - \tau)), \quad \tilde{\xi}_2 = \beta_2,2(\beta_1,2\eta_1(s_2 + 2r, t_2 + \tau) + \beta_2,2\eta_2(s_2, t_2)).
\]

In particular, \( H_r \) is injective and \( P_r \) is surjective.

Next we introduce the following three operators

\[
(Q'^*)_r(\kappa, \eta) : L^{k-1,2,\alpha}_{u_1}(r, u_{2(r)}) \to W^{k,2,\alpha}_u, \quad Q'^*_{\kappa}(\kappa, \eta) : L^{k-1,2,\alpha}_{u_1}(r, u_{2(r)}) \to W^{k,2,\alpha}_u, \quad I^*_r : \ker DS_{(\kappa, \eta)} \to K_{b_0} \times W^{k,2,\alpha}_u.
\]

Given \( \eta \in L^{k-1,2,\alpha}_{u_1}(r) \), denote

\[
(\kappa, (h_1, h_2)) = Q_{(\kappa, \eta)} H_r \eta.
\]

Set

\[
h^* = (\beta_1,2(s_1)h_1(s_1, t_1), \beta_2,2(s_2)h_2(s_2, t_2)) \in W^{k,2,\alpha}_u.
\]

Define

\[
(Q'^*)_{(\kappa, \eta)} \eta = (\kappa, h^*), \quad Q'^*_{(\kappa, \eta)} = (Q'^*)_{(\kappa, \eta)} (DS_{(\kappa, \eta)} Q'^*_{(\kappa, \eta)})^{-1}.
\]

Then we have maps

\[
(Q'^*)_{(\kappa, \eta)} P_r : L^{k-1,2,\alpha}_{u_1} \to W^{k,2,\alpha}_u, \quad Q'^*_{(\kappa, \eta)} P_r : L^{k-1,2,\alpha}_{u_1} \to W^{k,2,\alpha}_u.
\]

For any \((\kappa, \zeta + \hat{\zeta}_0) \in \ker DS_{(\kappa, \eta)} \), where \( \zeta = (\zeta_1, \zeta_2) \in W^{k,2,\alpha}_u \), we set

\[
\zeta = (\zeta_1 \beta_1,2 + \hat{\zeta}_0, \zeta_2 \beta_2,2 + \hat{\zeta}_0).
\]

Define

\[
I^*_r(\kappa, \zeta + \hat{\zeta}_0) = (\kappa, \zeta^* - Q'^*_{(\kappa, \eta)} \circ DS_{(\kappa, \eta)} (Id, \phi_r)(\kappa, \zeta^*)).
\]

By the definition we have

\[
I_r = (Id, \phi_r) \circ I^*_r, \quad Q_{(\kappa, \eta)}(\kappa, \phi_r) = (Id, \phi_r) \circ Q'^*_{(\kappa, \eta)}.
\]
Theorem 1.1. The main result of this paper is the following estimate.

1.5 Main result

By implicit function theorem, there exists a small neighborhood \( O_r \) of \( 0 \in \ker DS_{(\kappa, b)} \) and a unique smooth map

\[ f_r : O_r \rightarrow L_{r, u(r)}^{k-1,2.\alpha} \]

such that for any \((\kappa, \zeta) \in O_r\),

\[ dv + J \cdot dv \cdot j_{\alpha} + i (\kappa_{\alpha} + \kappa_{\beta}, b) = 0, \]

where \( b = (r, \tau, a_{\alpha}, v), \) \( v = \exp_{u(r)}(h) \) and

\[ P_{b,b(r)} = \Phi_{u(r)}(h)^{-1} : W^{k-1,2.\alpha}_r (\Sigma(r), v^* TM \otimes \Lambda_{j_\alpha}^1 T^* \Sigma(r)) \rightarrow L^{k-1,2.\alpha}_r. \]

Corollary 1.2. As a consequence we have

\[ I_r + Q_{(\kappa, b(r))} \circ f_r \circ I_r \]

from \( O \) into the space of perturbed \((j_{\alpha}, J)\)-holomorphic maps, where \( O \) is a neighborhood of \( 0 \) in \( \ker DS_{(\kappa, b)} \).

We consider the operator

\[ I_r^* + Q_{(\kappa, b(r))}^* f_r I_r : \ker DS_{(\kappa, b)} \rightarrow K \times W^{k,2.\alpha}_u. \]

It is easy to see that, restricting to \( \Sigma(R_0) \), we have

\[ I_r^* (\kappa, \zeta) + Q_{(\kappa, b(r))}^* f_r (I_r (\kappa, \zeta)) = I_r (\kappa, \zeta) + Q_{(\kappa, b(r))} f_r (I_r (\kappa, \zeta)). \]

1.5 Main result

The main result of this paper is the following estimate.

Theorem 1.1. Let \( \Sigma \) be a marked nodal Riemann surface with one nodal point \( q \). Let \((\kappa, b) = (\kappa_{\alpha}, a_{\alpha}, u) \in K \times O_{b_0}(\rho)\) be a perturbed \((j_{\alpha}, J)\)-holomorphic map from \( \Sigma \) to \( M \), where \( u = (u_1, u_2) : \Sigma_1 \cup \Sigma_2 \rightarrow M \) with \( u_1(q) = u_2(q) \) and

\[ DS_{(\kappa, b)} \mid_{K \times W^{k,2.\alpha}_u} : K \times W^{k,2.\alpha}_u \rightarrow L^{k-1,2.\alpha}_u \]

is surjective. Denote by \( Q_{(\kappa, b)} : L^{k-1,2.\alpha}_u \rightarrow K \times W^{k,2.\alpha}_u \) a right inverse of \( DS_{(\kappa, b)} \). Let \( Q_{(\kappa, b)} \) be the right inverse of \( DS_{(\kappa, b)} \) defined in \((1.26)\). Then the following hold.

Let \( c \in (0, 1) \) be a fixed constant. For any \( 0 < \alpha < \frac{1}{100c} \), there exists two positive constants \( C_1, d \) such that for any \((\kappa, \zeta) \in \ker DS_{(\kappa, b)} \) satisfying \( \| (\kappa, \zeta) \|_{W^{k,2.\alpha}} \leq d \), we have the following estimate

\[ \left\| \frac{\partial}{\partial r} \left( I_r^* (\kappa, \zeta) + Q_{(\kappa, b(r))}^* f_r (I_r (\kappa, \zeta)) \right) \right\|_{k-1,2.\alpha} \leq C_1 e^{-(\kappa-5\alpha)^c} (d + 1), \]

when \( r \) large enough, where \( \| \cdot \|_{k,2.\alpha} \) is defined in \((5.1)\).

As a consequence we have

Corollary 1.2. Let \( l \in \mathbb{Z}^+ \) be a fixed integer. There exists positive constants \( C_{\alpha, l}, d, R_0 \) such that for any \((\kappa, \zeta) \in \ker DS_{(\kappa, b)} \) with \( \| (\kappa, \zeta) \|_{W^{k,2.\alpha}} \leq d \), restricting to the compact set \( \Sigma(R_0) \), the following estimate hold

\[ \left\| \frac{\partial}{\partial r} \left( I_r (\kappa, \zeta) + Q_{(\kappa, b(r))} f_r (I_r (\kappa, \zeta)) \right) \right\|_{C^l (\Sigma(R_0))} \leq C_{\alpha, l} e^{-(\kappa-5\alpha)^c} (d + 1). \]
In order to apply our result to the study of $J$-holomorphic map moduli spaces we should let $a = (a_1, a_2)$ vary in $A_1 \times A_2$ and need to take a sum of several $K_h$. In the section [5] we extend the Theorem [1.1] to more general setting and to a neighborhood of $a_0$. In our next paper [5] we use the theorem to show that the Gromov-Witten invariants can be defined as an integral over top strata of virtual neighborhood. Furthermore, we prove that such invariants satisfy all the Gromov-Witten axioms of Kontsevich and Manin.

2 Some important estimates

In this section we give some important estimates which will be used in this paper.

2.1 Exponential decay theorem for $J$-holomorphic maps

Denote $B_r(0) = \{ z \in \mathbb{C} | |z| \leq r \}, A(r, R) := \{ z \in \mathbb{C} | r \leq |z| \leq R \}$. Take the standard Euclidean metric on $B_r(0)$ and $A(r, R)$.

The following two theorems in [7] (see Lemma 4.3.1 and Lemma 4.7.3) are fundamental results:

Theorem 2.1. Let $(M, J)$ be a compact almost complex manifold. Suppose that $M$ is equipped with any Riemannian metric. Then there exists a constant $\hbar > 0$ such that the following holds. If $r > 0$ and $u : B_r(0) \to M$ is a $J$-holomorphic curve then

$$\int_{B_r(0)} |du|^2 < \hbar \Rightarrow |du(0)|^2 < \frac{8}{\pi r^2} \int_{B_r(0)} |du|^2.$$ 

Theorem 2.2. Let $(M, \omega)$ be a compact symplectic manifold and $J$ be an $\omega$-tame almost complex structure. Then, for every $c < 1$, there exist positive constants $\hbar = \hbar(M, \omega, J, \nu)$ and $C = C(c)$ such that every $J$-holomorphic curve $u : A(r, R) \to M$ with

$$E(u, A(r, R)) < \hbar,$$

satisfies for any $\log 2 \leq T \leq \log \sqrt{R/r}$,

$$E(u, A(e^T r, e^{-T} R)) \leq C e^{-2cT} E(u, A(r, R)).$$

Now we use the cylinder coordinates $(s, t)$. Fix a constant $R_0 \in (0, r/4)$. Take the standard complex structure $j$ and a smooth metric $g$ on $(R_0, 2r - R_0) \times S^1$ such that

$$g = ds^2 + dt^2, \quad \text{in} \quad R_0 \leq s \leq 2r - R_0.$$ 

We can restate Theorems 2.1 and 2.2 as following.

Theorem 2.3. Fix a constant $c \in (0, 1)$. Let $(M, \omega)$ be a compact symplectic manifold and $J$ be an $\omega$-tame almost complex structure. Then there exist positive constants $\hbar = \hbar(M, \omega, J, \nu)$ and $C_1 = C_1(c)$ such that every $(j, J)$-holomorphic map $u : [R_0, 2r - R_0] \times S^1 \to M$ with

$$E(u, R_0 \leq s \leq 2r - R_0) := \int_{[R_0, 2r - R_0] \times S^1} |du|^2 < \hbar, \quad (2.1)$$
and any \( R_0 + \log 2 \leq R \leq r \), we have
\[ E(u, R \leq s \leq 2r - R) \leq C_1 E(u, R_0 \leq s \leq 2r - R_0) e^{-2\epsilon(R - R_0)}, \]
\[ \left| \frac{\partial u}{\partial s}(s, t) \right| + \left| \frac{\partial u}{\partial t}(s, t) \right| \leq C_1 \sqrt{E(u, R_0 \leq s \leq 2r - R_0) e^{-\epsilon(R - R_0)}}, \quad \forall \ R + \frac{1}{2} \leq s \leq 2r - R - \frac{1}{2}. \]

If we take \( \mathbb{R}^{2m} \) instead of \( M \), a similar result also hold. Let \( \omega_\alpha \) be the standard symplectic form in \( \mathbb{R}^{2m} \). Let \( u : [R_0, 2r - R_0] \times S^1 \to \mathbb{R}^{2m} \) be a map satisfying \( du + J_0 \cdot du \cdot j = 0 \). Denote
\[ e(R_0) := E(u, R_0 \leq s \leq 2r - R_0) := \int_{[R_0, 2r - R_0] \times S^1} u^* \omega_\alpha = \int_{[R_0, 2r - R_0] \times S^1} |\nabla u|^2 \, ds \, dt, \]
where \( |\nabla u|^2 = \sum_{i=1}^{2m} \left( \frac{\partial u^i}{\partial x^i} \right)^2 + \sum_{i=1}^{2m} \left( \frac{\partial u^i}{\partial x^i} \right)^2 \). We have

**Lemma 2.4.** There is a constant \( C \) depending only on \( \epsilon \) such that for all solution \( u : [R_0, 2r - R_0] \times S^1 \to \mathbb{R}^{2m} \) of the equations \( du + J_0 \cdot du \cdot j = 0 \), and any \( R_0 + \log 2 \leq R \leq r \), we have,
\[ e(R) \leq e^{-2\epsilon(R - R_0)} e(R_0), \]  \[ \left| \frac{\partial u}{\partial s}(s, t) \right| + \left| \frac{\partial u}{\partial t}(s, t) \right| \leq C \sqrt{e(R_0) e^{-\epsilon(R - R_0)}}, \quad \forall \ R + \frac{1}{2} \leq s \leq 2r - R - \frac{1}{2}. \]  

**Proof.** We give a sketch of the proof. For any loop \( \gamma : S^1 \to \mathbb{R}^{2m} \) and any smooth map \( W : D_1(0) \to \mathbb{R}^{2m} \) satisfying \( W(\partial D_1(0)) = \gamma \), we define an action functional by
\[ \mathcal{A}(\gamma) = - \int_{D_1(0)} W^* \omega_\alpha. \]
Since \( \omega_\alpha \) is exact in \( \mathbb{R}^{2m} \), the action functional \( \mathcal{A}(\gamma) \) is well defined. Denote by \( L(\gamma) \) the length of \( \gamma \). Then isoperimetric inequality can be written as (see Page 85 of \([7]\))
\[ |\mathcal{A}(\gamma)| \leq \frac{1}{4\pi} L(\gamma)^2. \]  \[ \text{(2.4)} \]
Denote \( u_s : S^1 \to \mathbb{R}^{2m} \) given by \( u_s(t) = u(s, t) \). Then
\[ e(R) = |\mathcal{A}(u_R) - \mathcal{A}(u_{2r-R})|. \]
By the same argument of \([7]\) (see Page 105 of \([7]\)), using inequalities \text{(2.4)} one can prove \text{(2.2)}. It is well known that \( |\nabla u|^2 \) is subharmonic function. Then
\[ |\nabla u|^2(p) \leq \frac{4}{\pi} \int_{D_1(p)} |\nabla u|^2, \quad \forall \ p \in [R_0 + 1, 2r - R_0 - 1] \times S^1. \]  \[ \text{(2.5)} \]
Then \text{(2.3)} follows from \text{(2.5)}. \hfill \Box

**Remark 2.5.** Theorem 2.3 and Lemma 2.4 also hold for \( r = \infty \).

### 2.2 Estimates for the equation \( \tilde{\partial}_j J_0 \zeta = \chi \)

Fix \( \alpha \in (0, \frac{1}{100}) \). The multiplication by \( e^{\alpha s} \) gives an isomorphism from \( W^{k,2,\alpha}(\mathbb{R} \times S^1; \mathbb{R}^{2m}) \) to \( W^{k,2}(\mathbb{R} \times S^1; \mathbb{R}^{2m}) \) and
\[ C^{-1} \| e^{\alpha s} f \|_{W^{k,2}(\mathbb{R} \times S^1; \mathbb{R}^{2m})} \leq \| f \|_{W^{k,2,\alpha}(\mathbb{R} \times S^1; \mathbb{R}^{2m})} \leq C \| e^{\alpha s} f \|_{W^{k,2}(\mathbb{R} \times S^1; \mathbb{R}^{2m})}, \]  \[ \text{(2.6)} \]
for some constant $C > 0$ depending only on $k$ and $\alpha$. It is easy to check that

$$\bar{\partial}_j J_0 h = \eta,$$

if and only if $(\bar{\partial}_j J_0 - \alpha)(e^{\alpha s} h) = e^{\alpha s} \eta$. (2.7)

Obviously, $L := J_0 \partial_x^2 - \alpha$ is an invertible elliptic operator on $H^1(S^1)$. It is well known that the operator $\bar{\partial}_j J_0 - \alpha : W^{k,2}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \rightarrow W^{k-1,2}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ has a right inverse

$$Q_\alpha : W^{k-1,2}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \rightarrow W^{k,2}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$$

with

$$\|Q_\alpha \rho\|_{k,2} \leq C(k, \alpha)\|ho\|_{k-1,2}, \quad \forall \rho \in W^{k-1,2}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$$

(see Proposition 3.4 in [2]). Denote

$$\chi \in \mathbb{R} \t \alpha,$$

First we construct Lemma 2.6.

Let $\eta \in L^k_{\infty} \mathbb{R} \times S^1, \mathbb{R}^{2n}$ and $h + \hat{h}_0 \in W^{k,2}_{\infty}$ be a solution of $D_u(h + \hat{h}_0) = \eta$ over $\Sigma \setminus \Sigma(R_0)$. Suppose that, for any $p, q \geq 0$,

$$\left| \frac{\partial^{p+q} F^1_u}{\partial s^q_0 \partial t^1} \right| \leq C_{p,q} e^{-\xi |s_i|}, \quad \forall \ |s_i| \geq R_0, \ l = 1, 2$$

(2.9)

for some constant $C_{p,q} > 0$. Then for any $0 < \alpha < \frac{7}{8}$, there exists a constant $C > 0$ such that for any $R > \max\{R_0, d\}$ and $R' > 2 + R$

$$\|h\|_{s_i \geq R} \leq C \left( e^{-\xi (R' - R)} + \frac{e^{-\xi (R') R}}{R} \right) \left| \hat{h}_0 \right|_{W^{k,2}_{\infty}} + \| \eta\|_{s_i \geq R} \|k-1,2,0\|$$

(2.10)

for some constant $C' > 0$ independent of $r$ such that

$$\|h\|_{s_i \geq R} \leq C' \left[ \left( e^{-\xi (R' - R)} + \frac{e^{-\xi (R') R}}{R} \right) \|\eta\|_{k-1,2,0} + \| \eta\|_{s_i \geq R} \|k-1,2,0\| \right].$$

(2.11)

**Proof.** First we construct $\zeta$ such that, restricting to $s \geq R + 1$,

$$\bar{\partial}_j J_0 (h - \zeta) = 0.$$  (2.12)

Denote $\chi = \beta(s_1 - R) \left( \eta - F^1_u(h + \hat{h}_0) - F^2_u \frac{\partial h}{\partial t} \right)$. Obviously, supp $\chi \subset \{s \geq R\}$ and

$$\chi = \eta - F^1_u(h + \hat{h}_0) - F^2_u \frac{\partial h}{\partial t}, \quad \text{for any } s \geq R + 1.$$
Let $\zeta = Q_{J_0,\alpha}(\chi)$. Then $\zeta$ satisfies (2.12) in $s \geq R + 1$. By (2.8) and using the the exponential decay of $F^i_u, i = 1, 2$ we get

$$\|\zeta\|_{k,2,\alpha} \leq C \|\chi\|_{k-1,2,\alpha} \leq C \left( e^{-\epsilon(R)} \|h + \hat{h}_0\|_{k,2,\alpha} + \|\eta\|_{s \geq R} \|k-1,2,\alpha\rangle \right). \quad (2.13)$$

Since for any nonnegative integers $p, q$,

$$\partial_j \partial_j \left( \frac{\partial^{p+q}}{\partial^p \partial^q \partial \eta} (h - \zeta) \right) = 0, \quad \text{in } s \geq R + 1 \quad (2.14)$$

using Lemma 2.4 with $r = \infty$ we conclude that for any $l \leq k$

$$|\partial^l (h - \zeta)(s, t)| \leq C e^{-\epsilon(s-R)}\|h - \zeta\|_{l,2} \leq C e^{-\epsilon(s-R)}\|h - \zeta\|_{k,2,\alpha}, \quad \forall s \geq R + 2.$$

By $\lim_{s \to \infty} (h - \zeta) = 0$ and the integration with respect to $s$, we have

$$\|\zeta\|_{s \geq R} \|k,2,\alpha \leq C e^{-(\epsilon-R)(s-R)}\|h - \zeta\|_{k,2,\alpha}. \quad (2.15)$$

Then by (2.13) and (2.15) we get

$$\|h\|_{s \geq R} \|k,2,\alpha \leq \|\zeta\|_{s \geq R} \|k,2,\alpha \leq C e^{-\epsilon(R-R)}\|h - \zeta\|_{k,2,\alpha} + C \left( e^{-\epsilon(R)}\|h + \hat{h}_0\|_{k,2,\alpha} + \|\eta\|_{s \geq R} \|k-1,2,\alpha\rangle \right) \leq C \left( e^{-\epsilon(R)}\|h - \zeta\|_{k,2,\alpha} + C\|\eta\|_{s \geq R} \|k-1,2,\alpha\rangle \right).$$

We obtain the estimate (2.10). \hfill \Box

### 3 Gluing estimates

Let $(\kappa_0, \beta_0) = (\kappa_0, \alpha_0, u) = (\kappa_0, J_0, y_0, u)$ be a perturbed $(j_0, J)$-holomorphic map from $\Sigma$ to $M$. For any $(\kappa, h + \hat{h}_0) \in K \times W^{k,2,\alpha},$ where $h = (h_1, h_2) \in W^{k,2,\alpha}_u$, we define

$$\|\kappa, h\|_{k,2,\alpha} = |\kappa| + |h_1|_{k,2,\alpha} + |h_2|_{k,2,\alpha}, \quad \|\kappa, h + \hat{h}_0\|_{W^{k,2,\alpha}} = \|\kappa, h\|_{k,2,\alpha} + |\hat{h}_0|.$$ \quad (3.1)

For any $(\kappa, h(\tau)) \in K \times W^{k,2,\alpha}_{\tau}$ we define

$$\|\kappa, h(\tau)\|_{k,2,\alpha,\tau} = |\kappa| + |h(\tau)|_{k,2,\alpha,\tau}. \quad (3.2)$$

In this section and the next section we derive gluing estimates. To simplify notations we let $C$ denote a generic constant whose value may change from line-to-line, but is independent of $(\tau)$.

#### 3.1 Estimates of right inverse

First we recall the definition of $Q'_{(\kappa_0, \beta_0)}$. Given $\eta \in L^{k-1,2,\alpha}_{\tau, u(\tau)}$ denote

$$Q_{(\kappa_0, \beta_0)}(\eta_1, \eta_2) = (\beta_1; \beta_2) \eta(s_1, t_1), \beta_2; \beta_2(\eta_2(s_2, t_2)),$$

Then

$$Q'_{(\kappa_0, \beta_0)}\eta := (\kappa, h(\tau)) = (\kappa, \beta_1; \beta_2 h_1(s_1, t_1) + \beta_2; \beta_2(\eta_2(s_2, t_2)(s_2 - 2\tau))h_2(s_1 - 2\tau, t_1 - \tau)).$$
Lemma 3.1. For any $\eta \in L^{k-1,2,\alpha}_R$, we have

$$DS_{(\kappa, b(r))} \circ Q'_{(\kappa, b(r))} \eta - \eta = \frac{1}{2} \sum (\partial_1^2)_{(\kappa, b(r))} h_i + \sum \beta_i r^2 (F_{u(r)}^1 - F_{u_i}^1) h_i$$

(3.3)

Proof: Since $DS_{(\kappa, b_0)}(\kappa, h) = d_h(\kappa, b_0) (\kappa, h) + D_\alpha h = (\eta_1, \eta_2)$ we have

$$DS_{(\kappa, b(r))} \circ Q'_{(\kappa, b(r))} \eta = \eta \quad \text{for } |s| \leq \frac{r}{2}.$$  

(3.4)

It suffices to calculate the left hand side in the annulus $\{ \frac{r}{2} \leq |s| \leq \frac{3r}{2} \}$. By choosing $r$ large enough we may assume that $\{ \frac{r}{2} \leq |s| \leq \frac{3r}{2} \} \subset \Sigma \setminus \Sigma(R_0)$. Note that in this annulus

$$d_h(\kappa, b_0) = 0, \quad DS_{(\kappa, b_0)} h_i = D_{u_i} h_i = \eta_i,$$

and the proof is complete.

Lemma 3.2. Suppose that $DS_{(\kappa, b_0)}|_{K \times W^{k,2,\alpha}} : K \times W^{k,2,\alpha}_R \to L^{k-1,2,\alpha}_R$ is surjective. Denote by $Q_{(\kappa, b_0)} : L^{k-1,2,\alpha}_R \to K \times W^{k,2,\alpha}_R$ a bounded right inverse of $DS_{(\kappa, b_0)}$. Then $DS_{(\kappa, b(r))}$ is surjective for $r$ large enough.

Moreover, there is a right inverse $Q_{(\kappa, b(r))}$ such that

$$||Q_{(\kappa, b(r))}|| \leq C$$

(3.6)

for some constant $C > 0$ independent of $r$.

Proof: We first show that

$$||Q_{(\kappa, b(r))}|| \leq C$$

(3.7)

$$||DS_{(\kappa, b(r))} \circ Q'_{(\kappa, b(r))} - I|| \leq \frac{2}{3}$$

(3.8)

for some constant $C > 0$ independent of $r$. By the definition (1.17) and $0 \leq \beta_i r \leq 1$ we have

$$|(\tilde{h}_i(r))| \leq e^{-\alpha r} \max_{t \in S^1} |e^{\alpha r} h_i(r, t)| \leq e^{-\alpha r} \max_{t \in S^1} \sum |e^{\alpha r} h_i(r, t)|$$

(3.9)

$$\leq C e^{-\alpha r} \sum_{i=1}^3 \|h_i(s_1, t_i)\|_{r-1 \leq s_1 \leq r+1} \|k, 2, \alpha \leq C e^{-\alpha r} \sum_{i=1}^3 \||h_i\|_{k, 2, \alpha}$$

where we used the Sobolev embedding theorem in the second inequality. By $||Q_{(\kappa, b_0)}|| \leq C$ and the definition of $|| \cdot ||_{k, 2, \alpha, r}$ we have

$$||(\kappa, h_i)||_{k, 2, \alpha, r} = |\kappa| + \sum \||\beta_i^r h_i - (\hat{h}_i(r))||_{k, 2, \alpha} + |(\hat{h}_i(r))|$$

$$\leq |\kappa| + \sum \||\beta_i^r h_i - (\hat{h}_i(r))||_{k, 2, \alpha} + C \sum ||h_i||_{k, 2, \alpha}$$

$$\leq 2(C + 1) ||(\kappa, h_1, h_2)||_{k, 2, \alpha} \leq C ||(\eta_1, \eta_2)||_{k-1, 2, \alpha} \leq C ||\eta||_{k-1, 2, \alpha, r}.$$
where we used (3.9) in the second inequality. Then (3.7) follows.

We prove (3.8). It follows from (3.3) that

\[
\left\| DS_{(\kappa, b_{(r)})} \circ Q_{(\kappa, b_{(r)})} r - \eta \right\|_{k-1,2,\alpha,r} \leq \frac{1}{2}\|\eta\|_{k-1,2,\alpha,r} + \frac{C}{r} \sum\|h_i\|_{k,2,\alpha} \leq \left( \frac{C}{r} + \frac{1}{2}\right) \|\eta\|_{k-1,2,\alpha,r},
\]

where we used \( \frac{1}{4} \leq \sum \beta_{i,r} \beta_{i,2} \leq \sqrt{2}, \sum\|F_t(i)| \leq C e^{-t} \) in the first inequality, and used \( \|Q_{(\kappa, b_{(r)})}\| \leq C \) in the last inequality. Then (3.8) follows when \( r \) large enough.

The estimate (3.3) implies that \( DS_{(\kappa, b_{(r)})} \circ Q_{(\kappa, b_{(r)})} \) is invertible, and a right inverse \( Q_{(\kappa, b_{(r)})} \) of \( DS_{(\kappa, b_{(r)})} \) is given by

\[
Q_{(\kappa, b_{(r)})} = Q_{(\kappa, b_{(r)})} (DS_{(\kappa, b_{(r)})})^{-1} \circ Q_{(\kappa, b_{(r)})}.
\]

Then the Lemma follows. \( \square \)

### 3.2 Isomorphism between ker \( DS_{(\kappa, b_{(r)})} \) and ker \( DS_{(\kappa, b_{(r)})} \)

For any \( (\kappa, h + \hat{h}_0) \in \text{ker} DS_{(\kappa, b_{(r)})} \), where \( h = (h_1, h_2) \in W^{k,2,\alpha}_u \), we set

\[
h_{(r)} = \beta_{1,r} (s_1) h_1 (s_1, t_1) + \beta_{2,r} (s_1 - 2r) h_2 (s_1 - 2r, t_1 - \tau) + \hat{h}_0,
\]

Recall that \( I_r : \text{ker} DS_{(\kappa, b_{(r)})} \rightarrow \text{ker} DS_{(\kappa, b_{(r)})} \) defined by

\[
I_r (\kappa, h + \hat{h}_0) = (\kappa, h_{(r)}) - Q_{(\kappa, b_{(r)})} \circ DS_{(\kappa, b_{(r)})} (\kappa, h_{(r)}).
\]

**Lemma 3.3.** \( I_r \) is an isomorphisms for \( r \) large enough, and

\[
\|I_r\| \leq C,
\]

for some constant \( C > 0 \) independent of \( r \).

**Proof.** The proof is basically a similar gluing argument as in [2]. The proof is divided into 2 steps.

**Step 1.** We define a map \( I'_r : \text{ker} DS_{(\kappa, b_{(r)})} \rightarrow \text{ker} DS_{(\kappa, b_{(r)})} \) and show that \( I'_r \) is injective for \( r \) large enough. For any \( (\kappa, h) \in \text{ker} DS_{(\kappa, b_{(r)})} \) we denote by \( h_i \) the restriction of \( h \) to the part \( |s_i| \leq r + 1 \), we get a pair \( (h_1, h_2) \). Set

\[
h_0 = \int_{S^1} h (r, t) dt.
\]

We denote

\[
\hat{h} = ((h_1 - h_0) \hat{\beta}_{1,2} + \hat{h}_0, (h_2 - h_0) \hat{\beta}_{2,2} + \hat{h}_0)
\]

and define \( I'_r : \text{ker} DS_{(\kappa, b_{(r)})} \rightarrow \text{ker} DS_{(\kappa, b_{(r)})} \) by

\[
I'_r (\kappa, h) = (\kappa, \hat{h}) - Q_{(\kappa, b_{(r)})} \circ DS_{(\kappa, b_{(r)})} (\kappa, \hat{h}),
\]

where \( Q_{(\kappa, b_{(r)})} \) denotes the right inverse of \( DS_{(\kappa, b_{(r)})} \) from \( W^{k,2,\alpha}_u \) to \( L^{k-1,2,\alpha}_u \). Since \( DS_{(\kappa, b_{(r)})} \circ Q_{(\kappa, b_{(r)})} = 1 \), we have \( I'_r (\text{ker} DS_{(\kappa, b_{(r)})}) \subset \text{ker} DS_{(\kappa, b_{(r)})} \).

Let \( (\kappa, h) \in \text{ker} DS_{(\kappa, b_{(r)})} \) such that \( I'_r (\kappa, h) = 0 \). First we prove \( h_0 = 0 \). Since \( d_{(\kappa, b_{(r)})} (\kappa, h) \) and \( D_{u_{(i,2}} (\hat{\beta}_{i,2} (h_i - \hat{h}_0)) \) have compact support and \( F_t (i) \in W^{k,2,\alpha}_u, t = 1,2 \), we have \( DS_{(\kappa, b_{(r)})} (\kappa, \hat{h}) \subset L^{k-1,2,\alpha}_u \). Then \( Q_{(\kappa, b_{(r)})} \circ DS_{(\kappa, b_{(r)})} (\kappa, \hat{h}) \subset K \times W^{k,2,\alpha}_u \). By (3.14) we have \( h_0 = 0 \).
Next we estimate $\| (\kappa, \tilde{h}) \|_{k,2, \alpha}$. From (3.14), by $I'_r(\kappa, h) = 0$, we have
\[
\| (\kappa, \tilde{h}) \|_{k,2, \alpha} \leq C \| d_i(\kappa, b_0) (\kappa, \tilde{h}) + D_u(h) \|_{k-1, 2, \alpha}
\]
\[
= \left\| d_i(\kappa, b_0) (\kappa, \tilde{h}) + D_u(h) - \left( \beta_{1,2} \left( D_u(h) + d_i(\kappa, b_i(r) (\kappa, h) \right), \beta_{2,2} \left( D_u(h) + d_i(\kappa, b_i(r) (\kappa, h) \right) \right) \right\|_{k-1, 2, \alpha}
\]
for some constant $C > 0$, where we used $(\kappa, h) \in \ker DS_{b_0(r)}$ in the last inequality. We choose $r > 4R_0$. As
\[
d_i(\kappa, b_0(r)) (\kappa, h), \ d_i(\kappa, b_0(r)) (\kappa, h) \|_{s_i \geq R_0} = d_i(\kappa, b_0(r)) (\kappa, \tilde{h}) \|_{s_i \geq R_0} = 0
\]
and $\beta_{2,2} |s_i| \leq r - 1 = 1$ we have $(\beta_{1,2} d_i(\kappa, b_0(r)) (\kappa, h), \beta_{2,2} d_i(\kappa, b_0(r)) (\kappa, h)) = d_i(\kappa, b_0(r)) (\kappa, \tilde{h})$. Set $(\tilde{\beta}) h = (\tilde{\beta}_{1,2} h_1, \tilde{\beta}_{2,2} h_2)$. Therefore
\[
\| (\kappa, \tilde{h}) \|_{k,2, \alpha} \leq C \| (\tilde{\beta}) h \|_{k-1, 2, \alpha} + C \sum_{i=1}^{2} \| \beta_{i,2} (F_u^1 - F_u^{1(r)}) h \|_{k-1, 2, \alpha}
\]
\[
+ C \sum_{i=1}^{2} \| \beta_{i,2} (F_u^2 - F_u^{2(r)}) \tilde{\beta} h \|_{k-1, 2, \alpha}.
\]
Note that $F_u^l = F_u^{l(r)}$, $l = 1, 2$ in $\{ |s_i| \leq \frac{r}{2} \}$. By exponential decay of $F_u^l, F_u^{l(r)}, l = 1, 2$, in $\{ \frac{r}{2} \leq s_1 \leq \frac{3r}{2} \}$, there exists a constant $C > 0$ such that
\[
\sum_{i=1}^{2} \| \beta_{i,2} (F_u^1 - F_u^{1(r)}) h \|_{k-1, 2, \alpha} + \sum_{i=1}^{2} \| \beta_{i,2} (F_u^2 - F_u^{2(r)}) \tilde{\beta} h \|_{k-1, 2, \alpha} \leq C e^{-\frac{r}{2}} \| \beta h \|_{k,2, \alpha}.
\]
Since $(\tilde{\beta}_{1,2}) h_1$ supports in $r - 1 \leq s_1 \leq r + 1$, and over this part
\[
|\tilde{\beta}_{1,2}| \leq 4, \ r - 1 \leq |s_2| \leq r + 1, \ e^{2\alpha |s_1|} \leq e^{2\alpha |s_2|}, \ \beta_{1,2} + \beta_{2,2} \geq 1, \ h_1 = h_2,
\]
we obtain
\[
\| (\tilde{\beta}_{1,2}) h_1 \|_{k-1, 2, \alpha} \leq C \| h_1 \|_{r-1 \leq s_1 \leq r+1} \| \Sigma_{1,k,1,2, \alpha} \leq C \| \sum_{i=1}^{2} \beta_i h_i \|_{r-1 \leq s_i \leq r+1} \| \Sigma_{1,k,1,2, \alpha}
\]
\[
\leq C \sum_{i=1}^{2} \| \beta_{i,2} h_i \|_{r-1 \leq s_i \leq r+1} \| k-1, 2, \alpha \leq C \| \Sigma_{1,k,1,2, \alpha, r} \| \| h \|_{k-1, 2, \alpha, r} \leq C e^{-(\kappa \alpha) \frac{r}{2}} \| h \|_{k-1, 2, \alpha, r}
\]
where we have used Corollary 2.6 with $R' = r - 1, R = \frac{r-1}{\alpha}$ and $\eta = 0$ in the last inequality. Similar inequality for $(\tilde{\beta}_{2,2}) h_2$ also holds. So we have
\[
\| (\tilde{\beta}) h \|_{k-1, 2, \alpha} \leq C e^{-(\kappa \alpha) \frac{r}{2}} \| h \|_{k-1, 2, \alpha, r} = C e^{-(\kappa \alpha) \frac{r}{2}} \| \tilde{h} \|_{k,2, \alpha}.
\]
Hence
\[
\| (\kappa, \tilde{h}) \|_{k,2, \alpha} \leq C e^{-(\kappa \alpha) \frac{r}{2}} \| \tilde{h} \|_{k,2, \alpha} \leq 1/2 \| \tilde{h} \|_{k,2, \alpha} = 1/2 \| \tilde{h} \|_{k,2, \alpha} \leq 1/2 \| \tilde{h} \|_{k,2, \alpha} \leq 1/2 \| \tilde{h} \|_{k,2, \alpha}
\]
(3.15) gives us
\[
|\kappa| = 0, \ \| \tilde{h} \|_{k,2, \alpha} = 0.
\]
Note that $\beta_{i,2} h_i |s_i| \leq r = h |s_i| \leq r$. It follows that $\kappa = 0, \ h = 0$. So $I'_r$ is injective.

**Step 2.** Let $(\kappa, h + \tilde{h}_0) \in \ker DS_{b_0}$ with $I'_r(\kappa, h + \tilde{h}_0) = 0$. Since $\| Q(\kappa, b_0) \|$ is uniformly bounded, from (1.28) and (3.6), we have
\[
\| (\kappa, h(r)) \|_{k,2, \alpha, r} = \| I'_r(\kappa, h + \tilde{h}_0) - (\kappa, h(r)) \|_{k,2, \alpha, r} \leq C \| DS_{b_0(r)} (\kappa, h(r)) \|
\]
for some constant $C > 0$. By a similar calculation as in (3.5) we obtain

\[ DS(\kappa, h, \beta, r(\kappa, h)) = D_{\kappa, \beta, r}(\kappa, h) + \frac{1}{r} \sum_{i=1}^{2} \partial \beta_{i, r}(u, h) \leq C \left( \| h \|_{k, 2, \alpha} + |h_0| \right) \]

(3.16)

where we used $D S(\kappa, h, \beta, r(\kappa, h)) = 0$. Then we have

\[ \| (\kappa, h, r(\kappa, h)) \|_{k, 2, \alpha} \leq \frac{C}{r} \left( \| h \|_{k, 2, \alpha} + |h_0| \right) \]

(3.17)

for some constant $C > 0$. Since $d_i(\kappa, u)|_{s_i \geq R_0} = d_i(\kappa, u)|_{s_i \geq R_0} = 0$, for any $(\kappa, h) \in \ker D S(\kappa, b_{r(\kappa, h)})$, restricting in $\{ \{ s_i \} \geq R_0 \}$, we have

\[ \partial_{j_i}(h + \hat{h}_0) + F_{u_i}^1(h + \hat{h}_0) + F_{u_i}^2 \partial_i(h + \hat{h}_0) = D S(\kappa, h, \beta, r(\kappa, h)) = 0. \]

Let $\epsilon' \in (0, 1)$ be a constant. Applying Corollary 2.6 with $R = \max\left( \frac{\ln \epsilon' + \ln 2}{\alpha}, R_0 + 2 \right)$ and $R' = 2R$, we conclude that the restriction of $h$ to $|s_i| \geq 2R$ satisfies

\[ \| h \|_{s_i \geq 2R} \leq 2C e^{-\alpha R} (\| h \|_{k, 2, \alpha} + |h_0|) \leq \epsilon' (\| h \|_{k, 2, \alpha} + |h_0|), \]

therefore

\[ \| h \|_{k, 2, \alpha} \geq \| h \|_{s_i \leq 2R} \| k_{2, \alpha} + |h_0| \geq (1 - \epsilon') (\| h \|_{k, 2, \alpha} + |h_0|), \]

(3.18)

for $r > 4R$. Then (3.17) and (3.18) give us $h = 0$ and $h_0 = 0$, and so $\kappa = 0$. Hence $I_r$ is injective.

The step 1 and step 2 together show that both $I_r$ and $I'_r$ are isomorphisms for $r$ large enough. \qed

3.3 Gluing maps

Choose $R_0$ large such that

\[ \sum_{i=1}^{2} E(u_i; |s_i| \geq \frac{R_0}{r}) \leq \frac{\bar{h}}{8}. \]

(3.19)

where $h$ is the constant in Theorem 2.3

Lemma 3.4. Suppose that $S(\kappa, b_o) = 0$. Then there exists a constant $C > 0$ independent of $r$ such that for $r > R_0$

\[ \| S(\kappa, b_r) \|_{k-1, 2, \alpha, r} \leq C e^{-\epsilon \alpha \frac{r}{2}}. \]

(3.20)

Proof. Since $u_r)|_{s_i \leq \frac{r}{2}} = u|_{s_i \leq \frac{r}{2}}$, we have $S(\kappa, b_r)|_{s_i \leq \frac{r}{2}} = 0$. Note that $i(\kappa, b_r) = 0$ in $\{ s_i \leq \frac{r}{2}, s_1 \leq \frac{3r}{2} \}$. So we get

\[ S(\kappa, b_r) = \beta \left( 3 - \frac{4s_1}{r} \right) \overline{\partial_{j_i}}(u_1(s_1, t_1)) + \beta \left( \frac{4s_1}{r} - 5 \right) \overline{\partial_{j_i}}(u_2(s_1 - 2r, t_1 - \tau)) + \frac{\partial}{\partial s_1} \left( \beta \left( 3 - \frac{4s_1}{r} \right) \right) \overline{u_1(s_1, t_1) - u_1(q)} + \frac{\partial}{\partial s_1} \left( \beta \left( \frac{4s_1}{r} - 5 \right) \right) (u_2(s_1 - 2r, t_1 - \tau) - u_2(q)). \]

(3.21)

By (3.19) and Theorem 2.3 we can obtain

\[ |du(s_i, t_i)| \leq C e^{-\epsilon |s_i|} \quad \text{for any } \frac{r}{2} \leq |s_i| \leq \frac{3r}{2} \]

(3.22)

Then (3.20) follows from the exponential decay of $|du|$.

\[ \qed \]
For fixed \((r)\) we consider the family of maps:
\[
F_{(r)} : K \times W_{r,\delta_{(r)}}^{k,2,\alpha} \to L_{r,\delta_{(r)}}^{k-1,2,\alpha}, \quad F_{(r)}(\kappa, h) = P_{b_r(b_{(r)})} \left( \bar{\partial}_{2}(v + i(\kappa, b)) \right),
\]
where \(b = (r, \tau, a_0, v), \quad v = \exp_{u_{(r)}} h\) and
\[
P_{b_{(r)}} = \Phi_{u_{(r)}}(h)^{-1} : W_{r,\delta_{(r)}}^{k-1,2,\alpha}(\Sigma_{(r)}, v^*TM \otimes T^*\Sigma_{(r)}) \to L_{r,\delta_{(r)}}^{k-1,2,\alpha}.
\]
Let \(g_o\) be the metrics on \((\Sigma, j_o)\) as in Section 3.1. We denote by \(c_{g_o}\) the norm of the Sobolev embedding \(W^{k,2}(\Sigma, g_o) \to C^{k-\lambda}(\Sigma, g_o)\).

For every \((r)\), \(F_{(r)}(\kappa, h)\) is a smooth function of \((\kappa, h)\). Consider the path \(\mathbb{R} \to K \times W_{r,\delta_{(r)}}^{k,2,\alpha} : \lambda \to (\lambda \kappa, \lambda \zeta)\).

**Lemma 3.5. 1.** \(\frac{d}{d\lambda} F_{(r)}(\lambda \kappa, \lambda \zeta)\big|_{\lambda=0} = dF_{(r)}(0)(\kappa, \zeta) = DS_{(\kappa_0, b_{(r)})}(\kappa, \zeta)\).

**Lemma 3.5. 2.** For every constant \(d_1 > 0\) there exists a constant \(C > 0\) such that the following holds for every metric \(g_{o,(r)}\) on \(\Sigma_{(r)}\) with \(c_{g_{o,(r)}} < d_1\): if \(u_{(r)} \in W_{r,\delta_{(r)}}^{k,2,\alpha}(\Sigma_{(r)}, M)\) and \((\kappa, h) \in K \times W_{r,\delta_{(r)}}^{k,2,\alpha}\) satisfying
\[
\|du_{(r)}\|_{k-2,\alpha,r} \leq d_1, \quad \|\kappa, h\|_{k,2,\alpha,r} \leq d_1
\]
then
\[
\|dF_{(r)}(\kappa, h) - dF_{(r)}(0)\| \leq C\|\kappa, h\|_{k,2,\alpha,r}.
\]

Here \(\| \cdot \|\) denotes the operator norm on \(L(K \times W_{r,\delta_{(r)}}^{k,2,\alpha}, L_{r,\delta_{(r)}}^{k-1,2,\alpha})\).

The proof is similar to the proofs of Proposition 3.1.1 and Proposition 3.5.3 in [7], we omit it here.

Now we check that \(F_{(r)}\) satisfies the assumption of Theorem 6.1. There exists two constants \(\epsilon_o > 0\) and \(C_o > 0\) such that \(g_o\) is a complete Riemannian metric with injectivity radius \(\text{inj}(\Sigma, g_o) > \epsilon_o\) and sectional curvature \(|\text{Rm}(\Sigma, g_o)| < C_o\). Then there exists a constant \(C > 0\) depending only on \(\epsilon_o\) and \(C_o\) such that \(c_{g_o} < C\) (for the Sobolev embedding theorem see [14]). Then we have
\[
\|h\|_{C^{k-2}(\Sigma_{(r)})} \leq \sum \|\beta_i\|_{C^{k-2}(\Sigma_{(r)})} \leq \sum c_{g_o} \|\beta_i\|_{W^{k-2}(\Sigma_{(r)})} \leq 2c_{g_o} \|h\|_{W^{k-2}(\Sigma_{(r)})}.
\]
It follows that \(c_{g_{o,(r)}} < C\). On the other hand, by Theorem 2.3 we have
\[
\|du_{(r)}\|_{k-1,2,\alpha,r} \leq \sum \|\beta_i\|_{W^{k-1,2,\alpha}} + Ce^{-(\epsilon-o)^{\frac{r}{2}}} \leq C.
\]
Choosing \(d_1\) large enough, by Lemma 3.5 we have
\[
\|dF_{(r)}(\kappa, h) - DS_{(\kappa_0, b_{(r)})}\| \leq \|\kappa, h\|_{k,2,\alpha,r}.
\]
By Lemma 3.4 we have
\[
\|F_{(r)}(0)\|_{k-1,2,\alpha,r} = \|S_{(\kappa_0, b_{(r)})}\|_{k-1,2,\alpha,r} \leq Ce^{-(\epsilon-o)^{\frac{r}{2}}},
\]
Then \(F_{(r)}\) satisfies the conditions in Implicit function theorem 6.1 when \(r\) large enough and \(\|\kappa, h\|_{k,2,\alpha,r}\) small enough. Hence the zero set of \(F_{(r)}\) is locally the form \((\kappa_r, \zeta_r) + Q_{(\kappa_0, b_{(r)})} \circ f_{(r)}(\kappa_r, \zeta_r)\), i.e.
\[
F_{(r)} \left( (\kappa_r, \zeta_r) + Q_{(\kappa_0, b_{(r)})} \circ f_{(r)}(\kappa_r, \zeta_r) \right) = 0
\]
where \((\kappa_r, \zeta_r) \in \text{ker } DS_{(\kappa_0, b_{(r)})}\) with \(\|\kappa_r, \zeta_r\|\) small.

Since \(I_r\) is an isomorphism for \(r\) large enough we have a gluing map
\[
I_r + Q_{(\kappa_0, b_{(r)})} \circ f_{(r)} \circ I_r
\]
from \(O\) into the space of perturbed \((j_o, J)\)-holomorphic maps, where \(O\) is a neighborhood of 0 in \(\ker DS_{(\kappa_0, b_{(r)})}\).
4 Estimates of derivatives with respect to gluing parameters $r$

In this section we prove Theorem 1.1. We fix an arbitrary $a_0 = (a_{o1}, a_{o2}) \in A_1 \times A_2$.

**Assumption (⋆).** Let $\Sigma$ be a marked nodal Riemann surface with one nodal point $q$. Let $(\kappa_o, b_o) = (\kappa_o, a_o, u) \in K \times O(\rho)$ be a perturbed $(j_o, J)$-holomorphic map from $\Sigma$ to $M$, where $u = (u_1, u_2) : \Sigma_1 \cup \Sigma_2 \rightarrow M$ with $u_1(q) = u_2(q)$ and

$$DS_{(\kappa_o, b_o)}|_{K \times W_u^{k,2,\alpha}} : K \times W_u^{k,2,\alpha} \rightarrow L_u^{k-1,2,\alpha}$$

is surjective. Denote by $Q_{(\kappa_o, b_o)} : L_u^{k-1,2,\alpha} \rightarrow K \times W_u^{k,2,\alpha}$ a right inverse of $DS_{(\kappa_o, b_o)}$. Let $Q_{(\kappa_o, b_o)(r)}$ be the right inverse of $DS_{(\kappa_o, b_o)}(r)$ defined in (4.26).

To simplify notations we denote

$$D := DS_{(\kappa_o, b_o)(r)}, \quad Q := Q_{(\kappa_o, b_o)(r)};$$

$$Q' := Q'_{(\kappa_o, b_o)(r)}; \quad (Q')^* := (Q')^*_{(\kappa_o, b_o)(r)}; \quad Q^* := Q^*_{(\kappa_o, b_o)(r)}.$$  

The main result of this section is to prove Theorem 1.1. We first prove some lemmas.

### 4.1 Estimates for $\frac{\partial}{\partial r}((Q')^* P_r)$

We define an operator $X : L_u^{k-1,2,\alpha} \rightarrow L_r^{k-1,2,\alpha}$ as

$$X(\eta_1, \eta_2) = DQ^* P_r(\eta_1, \eta_2) - P_r(\eta_1, \eta_2).$$  

(4.1)

By Lemma 3.1, we have

$$X(\eta_1, \eta_2) = \sum \left( \partial^i \beta_i (r) \right) h_i + \sum \left( \beta_i (r) (F^1_{u(r)} - F^1_{u(r)}) h_i + \sum \left( \beta_i (r) (F^2_{u(r)} - F^2_{u(r)}) \partial h_iight.\right) + \left( \sum \beta_i (r) h_i \right) (4.2)$$

where $(\kappa, h) = Q_{(\kappa_o, b_o)} H_r P_r (\eta_1, \eta_2)$. Then $\text{supp} \ X(\eta_1, \eta_2) \subset \{ \frac{r}{2} \leq |s| \leq \frac{r+1}{2} \}$.

**Lemma 4.1.** For any $(\eta_1, \eta_2) \in L_u^{k-1,2,\alpha}$ the following estimates hold:

(a) $\| (H_r P_r)(\eta_1, \eta_2) \|_{k-1,2,\alpha} \leq C \sum_{i=1}^{2} \| \eta_i \|_{|s_i| \leq r+1} \| \Sigma_i, k-1,2,\alpha \}$

(b) $\| (H_r P_r)(\eta_1, \eta_2) \|_{|s| \geq R} \| k-1,2,\alpha \leq C \sum_{i=1}^{2} \| \eta_i \|_{|s_i| \leq R} \| k-1,2,\alpha \}$

(c) $\| (Q')^* P_r(\eta_1, \eta_2) \|_{k,2,\alpha} \leq C \sum_{i=1}^{2} \| H_r P_r(\eta_1, \eta_2) \|_{|s_i| \leq r+1} \| \Sigma_i, k-1,2,\alpha \}$

(d) $\| \frac{\partial}{\partial r}((Q')^* P_r)(\eta_1, \eta_2) \|_{k-1,2,\alpha} \leq C \left( e^{-(c-o)\frac{r}{2}} \sum \| \eta_i \|_{|s_i| \leq r+1} \| k-1,2,\alpha \} + \sum_{i=1}^{2} \| \eta_i \|_{|s_i| \leq r+1} \| k-1,2,\alpha \} \right)$

(e) $\| (Q')^* P_r(\eta_1, \eta_2) \|_{\frac{r}{2} \leq |s| \leq \frac{r+1}{2}} \| k,2,\alpha} \leq C \left[ e^{-(c-o)\frac{r}{2}} \| H_r P_r(\eta_1, \eta_2) \|_{k-1,2,\alpha} + \| H_r P_r(\eta_1, \eta_2) \|_{|s_i| \geq \frac{r}{2}} \| k-1,2,\alpha \} \right]$.

**Proof.** By definition

$$\left( H_r P_r)(\eta_1, \eta_2) = (\tilde{\eta}_1, \tilde{\eta}_2), \right. \quad (4.3)$$
By (1.13) we have
\[ \frac{\partial \tilde{\eta}_1}{\partial r} = \beta_{1;2} \left( \frac{\partial}{\partial r} \eta_1(s_1, t_1) \right) + \beta_{2;2} \left( \frac{\partial}{\partial r} \eta_2(s_1, t_1) \right). \]
Then (a) follows from (4.4) and the definition of \( \beta_{1;2} \).

By (1.13) we have
\[ \frac{\partial \tilde{\eta}_1}{\partial r} = \beta_{1;2} \left( \frac{\partial}{\partial r} \eta_1(s_1, t_1) \right) + \beta_{2;2} \left( \frac{\partial}{\partial r} \eta_2(s_1, t_1) \right). \]
A same estimate for \( \frac{\partial \tilde{\eta}_2}{\partial r} \) also holds. Then (b) follows from \( \frac{\partial}{\partial r} (H_r P_r)(\eta_1, \eta_2) = \left( \frac{\partial}{\partial r} \tilde{\eta}_1, \frac{\partial}{\partial r} \tilde{\eta}_2 \right) \).

Denote \((\kappa, h_1, h_2) = Q_{(\kappa, h_0)} H_r P_r(\eta_1, \eta_2)\). Recall that (cf. (1.33))
\[ (Q')^* P_r(\eta_1, \eta_2) = (\kappa, \beta_{1;r}, h_1, \beta_{2;r}, h_2). \]
Then (c) follows from (a), \( |\beta_{1;r}| \leq 1 \) and \( ||Q_{(\kappa, h_0)}|| \leq C \).

Taking the derivative \( \frac{\partial}{\partial r} \) of (4.5) we obtain
\[ \frac{\partial}{\partial r} ((Q')^* P_r)(\eta_1, \eta_2) = \left( h_1 \frac{\partial}{\partial r} \beta_{1;r}, \beta_{2;r}, h_2 \frac{\partial}{\partial r} \beta_{2;r} \right) \]
On the other hand, since \( \frac{\partial}{\partial r} (\kappa, h_1, h_2) = Q_{(\kappa, h_0)} \left( \frac{\partial}{\partial r} (H_r P_r)(\eta_1, \eta_2) \right) \), we have
\[ \left\| \frac{\partial}{\partial r} \kappa(s_1, t_1) \right\|_{k-2,2,\alpha} \leq C \left\| (\eta_1, \eta_2) \right\|_{k-1,2,\alpha}. \]
where we have used the bound of right inverse and (b) in the inequality. Since \( h \frac{\partial}{\partial r} \kappa(s_1, t_1) \subset \{ \frac{3}{2} \leq |s_1| \leq \frac{3\varepsilon}{2} \} \), we have
\[ \left\| (0, h_1 \frac{\partial}{\partial r} \beta_{1;r}, h_2 \frac{\partial}{\partial r} \beta_{2;r}) \right\|_{k-2,2,\alpha} \leq C \left\| (h_1, h_2) \right\|_{k-1,2,\alpha}. \]
Since \( d_{(\kappa, h_0)} |R_0 \leq |s_1| \leq 2r - R_0 = 0 \) we have
\[ D_u(h_1, h_2) = H_r P_r(\eta_1, \eta_2), \quad \text{for } R_0 \leq |s_1| \leq 2r - R_0. \]
By \((\kappa, h_1, h_2) = Q_{(\kappa, h_0)} H_r P_r(\eta_1, \eta_2) \) and \( \kappa |R_0 \leq |s_1| \leq 2r - R_0 = 0 \), applying (2.11) of Corollary 2.6 with \( R' = \frac{\varepsilon}{2} \), \( R = \frac{\varepsilon}{2} \) and \( \eta = H_r P_r(\eta_1, \eta_2) \), we conclude that
\[ \left\| (h_1, h_2) \right\|_{k-1,2,\alpha} \leq C \left( e^{-(k-\alpha)\frac{\varepsilon}{2}} \left\| H_r P_r(\eta_1, \eta_2) \right\|_{k-2,2,\alpha} + \left\| H_r P_r(\eta_1, \eta_2) \right\|_{k-1,2,\alpha} \right) \]
where we have used (a) in the last inequality. By (4.6), (4.7), (4.8) and (4.10) we get (d). (e) follows from (4.10) and \( \left\| (\beta_{1;r}, h_1, \beta_{2;r}, h_2) \right\|_{k-2,2,\alpha} \leq \left\| (h_1, h_2) \right\|_{k-1,2,\alpha} \) □
**Lemma 4.2.** There exists a constant $C > 0$, such that for any $(\kappa, h + \hat{h}_0) \in K \times \mathcal{W}_{u}^{k,2,\alpha}$ with supp $h_i \subset \{|s_i| \leq \frac{Mr}{2}\}$, we have

$$||H, D(Id, \phi_r)(\kappa, (h + \hat{h}_0, h_2 + \hat{h}_0))||_{k-1,2,\alpha} \leq C||(\kappa, h + \hat{h}_0, h_2 + \hat{h}_0)||_{k,2,\alpha}.$$

In particular, if $(\kappa, h + \hat{h}_0)$ satisfies $D(\kappa, h + \hat{h}_0)||_{|s_i| \leq \frac{Mr}{2}} = 0$, we have

$$||H, D(Id, \phi_r)(\kappa, h_1 \beta_1, \hat{h}_0, h_2; \beta_2, \hat{h}_0)||_{k-1,2,\alpha} \leq C||(h_1, h_2)||_{\frac{Mr}{2}} ||h_2, \hat{h}_0||_{k,2,\alpha} + e^{(\alpha-\gamma)\frac{Mr}{2}}||h_0||.$$

**Proof.** By the same calculation as in (3.5) we have, in $\{|s_i| \leq r + 1\}$,

$$H_r, D(Id, \phi_r)(\kappa, (h_1 + \hat{h}_0, h_2 + \hat{h}_0)) = H_r, D(\kappa, \sum h_i + \hat{h}_0) := (\tilde{\eta}_1, \tilde{\eta}_2)$$

where

$$\tilde{\eta}_i = \beta_{i,2} \left( \sum_{i=1}^{2} D_{u_i} h_i + d\nu_{(\kappa, h_0)}(\kappa, h + \hat{h}_0) + \sum_{i=1}^{2} (F_{u_i}^1 - F_{u_i}^1) h_i + F_{u_i}^1 \hat{h}_0 + \sum_{i=1}^{2} (F_{u_i}^1 - F_{u_i}^1) \partial_i h_i \right).$$

Then the first inequality follows. Note that $(\kappa, h_{(r)}) = (Id, \phi_r)(\kappa, h_1 \beta_1, \hat{h}_0, h_2; \beta_2, \hat{h}_0)$. Applying (3.16), the exponential decay of $F_{u_i}^1 \partial_i h_i, k = 1, 2$, and

$$\beta_{i,2}||s_i|| \leq \frac{Mr}{2} = 1, \quad \beta_{i,2}||s_i|| \leq \frac{Mr}{2} = 0, \quad D(h + \hat{h}_0)||_{s_i} \leq \frac{Mr}{2} = 0, \quad F_{u_i}^1 - F_{u_i}^1||_{s_i} \leq \frac{Mr}{2} = 0, \quad i, l = 1, 2,$

we can prove the second inequality. □

**Lemma 4.3.** There exists a constant $C > 0$, such that for any $(\eta_1, \eta_2) \in L_{u}^{k-1,2,\alpha}$ the following estimates hold:

1. $||H_r, X(\eta_1, \eta_2)||_{k-1,2,\alpha} \leq C\left[ e^{-(\alpha-\gamma)\frac{Mr}{2}}||\eta_1, \eta_2||_{|s_i| \leq r + 1||k-1,2,\alpha} + ||\eta_1, \eta_2||_{\frac{Mr}{2}} ||\eta_1, \eta_2||_{|s_i| \leq r + 1||k,2,\alpha} \right]$.

2. $||\frac{\partial}{\partial r} (H_r, X)(\eta_1, \eta_2)||_{k-1,2,\alpha} \leq C\left[ e^{-(\alpha-\gamma)\frac{Mr}{2}}||\eta_1, \eta_2||_{|s_i| \leq r + 1||k-1,2,\alpha} + ||\eta_1, \eta_2||_{\frac{Mr}{2}} ||\eta_1, \eta_2||_{|s_i| \leq r + 1||k,2,\alpha} \right]$.

**Proof.** Denote $(\kappa, h_1, h_2) = Q_{(\kappa, h_0)} H_r, P_r(\eta_1, \eta_2)$. Note that supp $H_r, X \subset \{|s_i| \leq r + 1\}$. From the definition of $X$ (see (4.2)) we have

$$||H_r, X(\eta_1, \eta_2)||_{k-1,2,\alpha} \leq C\left( (h_1, h_2)||_{\frac{Mr}{2}} ||h_2, \eta_2||_{k,2,\alpha} + C ||(\eta_1, \eta_2)||_{\frac{Mr}{2}} ||(\eta_1, \eta_2)||_{|s_i| \leq r + 1||k,2,\alpha} \right).$$

Then (1) follows from (4.10) and (a) in Lemma 4.1. Taking derivative $\frac{\partial}{\partial r}$ to $H_r, X(\eta_1, \eta_2)$ we get

$$\frac{\partial}{\partial r} (H_r, X)(\eta_1, \eta_2) = \sum \xi_1^i h_i + \sum \xi_2^i h_i + \sum \xi_3^i \hat{h}_i \left( \sum \rho_{1,2} \frac{\partial h_i}{\partial t} + \rho_{1,2} \frac{\partial h_i}{\partial t} \right)$$

$$+ \left( \sum \lambda_1^{i,2} \frac{\partial^2 h_i}{\partial t^2} + \lambda_1^{i,2} \frac{\partial^2 h_i}{\partial t^2} \right),$$

where

$$\xi_1^i = \frac{\partial}{\partial r} \left( \beta_{i,2} (\beta_{i,2} h_i + h_i) \right), \quad \rho_1^i = \beta_{i,2} (F_{u_i}^1 - F_{u_i}^1), \quad \lambda_1^i = \beta_{i,2} (F_{u_i}^1 - F_{u_i}^1) - F_{u_i}^1.$$}

By the same proof of (d) in Lemma 4.1 we have

$$||\frac{\partial}{\partial r} (H_r, X)(\eta_1, \eta_2)||_{k-2,\alpha} \leq C\left( ||(\eta_1, h_2)||_{\frac{Mr}{2}} ||(h_1, h_2)||_{\frac{Mr}{2}} ||(h_1, h_2)||_{\frac{Mr}{2}} \right)$$

$$\leq C\left[ ||(\eta_1, \eta_2)||_{\frac{Mr}{2}} ||(\eta_1, \eta_2)||_{\frac{Mr}{2}} + e^{-(\alpha-\gamma)\frac{Mr}{2}} ||(\eta_1, \eta_2)||_{|s_i| \leq r + 1||k-1,2,\alpha} \right],$$

where we have used (4.7) and (4.10) in the last inequality. Then (2) is proved. □
4.2 Estimates of $\left\| \frac{\partial}{\partial r} \left( H_r \circ (DS_{(\kappa_0,b_0)}) \circ Q'_{\kappa_0,b_0}) \right) \right\|_{k-2,2,\alpha}$

Lemma 4.4. There exists a constant $C > 0$, such that for any $(\eta_1, \eta_2) \in L^2_{\alpha}$ the following estimates hold:

(A) $H_r(DQ')^{-1}P_r(\eta_1, \eta_2)|_{|s| \leq \frac{\tau}{2}} = (\eta_1, \eta_2)|_{|s| \leq \frac{\tau}{2}}$.

(B) $\left\| H_r(DQ')^{-1}P_r(\eta_1, \eta_2)|_{|s| \leq \frac{\tau}{2}} \right\|_{k-1,2,\alpha}$

$$\leq C \left[ e^{-(\tau-\alpha) \frac{\tau}{2}} \left( \| \eta_1, \eta_2 \|_{|s| \leq r+1} \right) \left\| k-1,2,\alpha \right\| + \left\| \eta_1, \eta_2 \right\|_{|s| \leq r+1} \left\| k-1,2,\alpha \right\| \right].$$

(C) $\left\| \frac{\partial}{\partial r} (H_r(DQ')^{-1}P_r) (\eta_1, \eta_2) \right\|_{k-2,2,\alpha}$

$$\leq C \left[ e^{-(\tau-\alpha) \frac{\tau}{2}} \left( \| \eta_1, \eta_2 \|_{|s| \leq r+1} \right) \left\| k-1,2,\alpha \right\| + \left\| \eta_1, \eta_2 \right\|_{|s| \leq r+1} \left\| k-1,2,\alpha \right\| \right].$$

Proof. For any $(\eta_1, \eta_2) \in L^2_{\alpha}$, let $\eta_r := P_r(\eta_1, \eta_2) = \sum_{i=1}^{2} \beta_i \eta_i$. Denote

$$Q_{(\kappa_0,b_0)}(H_r \eta_r) = (\kappa, h_1, h_2).$$

Then $Q' \eta_r = (\kappa, \sum_{i=1}^{2} \beta_i \eta_i)$. Let $\tilde{\eta}_r = (DQ')^{-1} \eta_r$. By the definition of $X$ and the invertibility of $DQ'$, we have

$$\eta_r - \tilde{\eta}_r = (DQ') \tilde{\eta}_r - \tilde{\eta}_r = X(H_r \tilde{\eta}_r), \quad \eta_r - \tilde{\eta}_r = (DQ')^{-1} (DQ' - I) \eta_r = (DQ')^{-1} X(\eta_1, \eta_2).$$

(4.12)

It follows that $\tilde{\eta}_r |_{|s| \leq \frac{\tau}{2}} = \eta_r |_{|s| \leq \frac{\tau}{2}}$. Then (A) follows.

By (4.12) and $\| H_r \eta \|_{k-1,2,\alpha} = \| \eta \|_{k-1,2,\alpha, r}$, we have

$$\left\| H_r \tilde{\eta}_r |_{|s| \leq \frac{\tau}{2}} \right\|_{k-1,2,\alpha} \leq \| \eta_r \|_{|s| \leq \frac{\tau}{2}} \left\| k-1,2,\alpha \right\| + \left\| (DQ')^{-1} X(\eta_1, \eta_2) \right\|_{k-1,2,\alpha, r}$$

$$\leq C \left[ \| \eta_r \|_{|s| \leq r+1} \left\| k-1,2,\alpha \right\| + \| X(\eta_1, \eta_2) \|_{k,2,\alpha, r} \right]$$

$$= C \left[ \| \eta_r \|_{|s| \leq r+1} \left\| k-1,2,\alpha \right\| + \| H_r X(\eta_1, \eta_2) \|_{k,2,\alpha} \right]$$

$$\leq C \left[ e^{-(\tau-\alpha) \frac{\tau}{2}} \left( \| \eta_1, \eta_2 \|_{|s| \leq r+1} \right) \left\| k-1,2,\alpha \right\| + \left\| \eta_1, \eta_2 \right\|_{|s| \leq r+1} \left\| k-1,2,\alpha \right\| \right],$$

where we have used (a) of Lemma 4.1 and the bound of $\| (DQ')^{-1} \|$ in the second inequality. We used (1) of Lemma 4.3 in the last inequality. Then (B) follows.

We prove (C). Multiplying $H_r$ on both sides of (4.12) and taking the derivative $\frac{\partial}{\partial r}$ we have

$$\frac{\partial}{\partial r} (H_r(\tilde{\eta}_r)) = \frac{\partial}{\partial r} (H_r(\eta_r)) - \frac{\partial}{\partial r} \left[ H_r(DQ')^{-1} P_r \circ H_r X(\eta_1, \eta_2) \right].$$

(4.13)

On the other hand, by $H_r(DQ') P \circ H_r(DQ')^{-1} X(\eta_1, \eta_2) = H_r X(\eta_1, \eta_2)$, we get

$$H_r P_r \circ \frac{\partial}{\partial r} [H_r(DQ')^{-1} X(\eta_1, \eta_2)]$$

(4.14)

$$= H_r(DQ')^{-1} P_r \circ \frac{\partial}{\partial r} [H_r X(\eta_1, \eta_2)] - H_r(DQ')^{-1} P_r \frac{\partial}{\partial r} [H_r(DQ') P_r] \circ H_r(DQ')^{-1} X(\eta_1, \eta_2).$$

Note that

$$\frac{\partial}{\partial r} [H_r(DQ')^{-1} X(\eta_1, \eta_2)] = H_r P_r \frac{\partial}{\partial r} [H_r(DQ')^{-1} X(\eta_1, \eta_2)] + \frac{\partial}{\partial r} (H_r P_r) H_r(DQ')^{-1} X(\eta_1, \eta_2).$$
Then inserting (4.14) into (4.13) we get

\[
\frac{\partial}{\partial r}(H_r\tilde{\eta}) = \frac{\partial}{\partial r}(H_rP_r)(\eta_1, \eta_2) + (I) + (II) + (III),
\]

where

\[
(I) = -\frac{\partial (H_rP_r)}{\partial r} \circ H_r(DQ')^{-1}P_r \circ H_rX(\eta_1, \eta_2),
\]

\[
(II) = -H_r(DQ')^{-1}P_r \circ \frac{\partial (H_rX)}{\partial r}(\eta_1, \eta_2)
\]

\[
(III) = H_r(DQ')^{-1}P_r \circ \frac{\partial}{\partial r}(H_r(DQ')P_r) \circ H_r(DQ')^{-1}P_r \circ H_rX(\eta_1, \eta_2).
\]

By (b) of Lemma 4.1 we have,

\[
\left\| \frac{\partial}{\partial r}(H_rP_r)(\eta_1, \eta_2) \right\|_{k-2,2,\alpha} \leq C \|(\eta_1, \eta_2)\|_{k-1,2,\alpha} \leq 1\|s_i\|_{k-1,2,\alpha}.
\] (4.15)

Next we calculate (I), (II) and (III).

**Calculation for (I).** Using (4.15) with \((\eta_1, \eta_2)\) replaced by \((H_r(DQ')^{-1}P_r \circ H_rX(\eta_1, \eta_2))\), we obtain that

\[
\|H_r(DQ')^{-1}P_r \circ H_rX(\eta_1, \eta_2)\|_{k-1,2,\alpha} \leq C\|H_rX(\eta_1, \eta_2)\|_{k-1,2,\alpha} \leq \|s_i\|_{k-1,2,\alpha}.
\] (4.16)

where we have used (A), (B) with \((\eta_1, \eta_2)\) replaced by \((H_rX(\eta_1, \eta_2))\) and \(\text{supp } H_rX(\eta_1, \eta_2) \subset \left\{ \frac{r}{2} \leq |s_i| \leq \frac{3r}{2} \right\}\). Then

\[
\|I\|_{k-2,2,\alpha} \leq C\|H_rX(\eta_1, \eta_2)\|_{k-1,2,\alpha} \leq \|s_i\|_{k-1,2,\alpha}.
\] (4.17)

**Calculation for (II).** Applying (A) and (B) with \((\eta_1, \eta_2)\) replaced by \(\frac{\partial (H_rX)}{\partial r}(\eta_1, \eta_2)\), we have

\[
\|II\|_{k-2,2,\alpha} \leq C\left\| \frac{\partial (H_rX)}{\partial r}(\eta_1, \eta_2) \right\|_{k-2,2,\alpha}.
\] (4.18)

**Calculation for (III).** Set \(\xi = H_r(DQ')^{-1}P_r \circ H_rX(\eta_1, \eta_2)\). Multiplying \(H_r\) on both sides of (4.1) and taking the derivative \(\frac{\partial}{\partial r}\) we have

\[
\frac{\partial (H_rX)}{\partial r} \xi + \frac{\partial (H_rP_r)}{\partial r} \xi = \frac{\partial}{\partial r}(H_r(DQ')P_r)\xi,
\]

where \((\eta_1, \eta_2)\) is replaced by \(\xi\). Since \(\text{supp } \frac{\partial}{\partial r}(H_r(DQ')P_r)(\eta_1, \eta_2) \subset \left\{ \frac{r}{2} \leq |s_i| \leq r + 1 \right\}\), using (4.19), (b) of Lemma 4.1 and (2) of Lemma 4.3 we get

\[
\left\| \frac{\partial}{\partial r}(H_r(DQ')P_r)\xi \right\|_{k-2,2,\alpha} \leq C \left[ \left\| \xi \right\|_{k-2,2,\alpha} + e^{-\epsilon \alpha \frac{r}{2}} \left\| \xi \right\|_{k-1,2,\alpha} \right].
\] (4.20)

Applying (A) and (B) with \((\eta_1, \eta_2)\) replaced by \(\frac{\partial}{\partial r}(H_r(DQ')P_r)\xi\), we have

\[
\|III\|_{k-2,2,\alpha} \leq C\left\| \frac{\partial}{\partial r}(H_r(DQ')P_r)\xi \right\|_{k-2,2,\alpha} \leq C \left[ \|H_rX(\eta_1, \eta_2)\|_{k-1,2,\alpha} + e^{-\epsilon \alpha \frac{r}{2}} \|H_rX(\eta_1, \eta_2)\|_{k-1,2,\alpha} \right].
\] (4.21)

where we have used (4.20) and (4.16) in the last inequality.

Combining the estimates (4.15), (4.17), (4.18) and (4.21), we obtain

\[
\left\| \frac{\partial}{\partial r}(H_r(DQ')^{-1}P_r)(\eta_1, \eta_2) \right\|_{k-2,2,\alpha} \leq C\left[ \left\| \eta_1, \eta_2 \right\|_{\frac{r}{2} \leq |s_i| \leq r + 1} + \left\| \frac{\partial (H_rX)}{\partial r}(\eta_1, \eta_2) \right\|_{k-2,2,\alpha} \right] + C\|H_rX(\eta_1, \eta_2)\|_{k-1,2,\alpha}.
\]

where we have used \(\text{supp } H_rX(\eta_1, \eta_2) \subset \frac{r}{2} \leq |s_i| \leq r + 1\). The lemma follows from (1) and (2) of Lemma 4.3.
4.3 Estimates of $\frac{\partial r}{\partial r}$

Lemma 4.5. There exists a constant $C > 0$, independent of $r$, such that for any $(\kappa, h + \hat{h}_0) \in \ker DS_{(\kappa, h_0)}$,

$$\left\| \frac{\partial}{\partial r} I^*_{r}(\kappa, h + \hat{h}_0) \right\|_{k-2, \alpha, r} \leq C \| h_i \|_{\frac{r}{2}} \leq \frac{r}{2} \| k, 2, \alpha, r \| + C_{e^{-c} \| h \|}.$$

(4.22)

Proof. Recall that $I^*_{r}(\kappa, h + h_0) = (\kappa, h^*_r) - Q^* D(\kappa, h(r))$. By $P_r H_r = 1d$ we have

$$\frac{\partial}{\partial r} I^*_{r}(\kappa, h, h_0) = \begin{pmatrix} 0, \frac{\partial^2 \beta}{\partial r^2} h_1, \frac{\partial^2 \beta}{\partial r^2} h_2 \end{pmatrix} + \frac{\partial}{\partial r} (Q^* P_r) \circ H_r D(\kappa, h(r)) + Q^* P_r \circ \frac{\partial}{\partial r} (H_r D(\kappa, h(r))).$$

Note that $d_i(\kappa, h_r) = d_i(\kappa, h_0)$. By (3.16), equalities $F^k_{u_r} | s_i | \leq \frac{r}{2}, i, k = 1, 2$ and the definition of $\beta_i$ we have supp $D(\kappa, h(r)) \subset \{ \frac{r}{2} \leq | s_i | \leq \frac{r}{2} \}$. Then

$$\left\| [H_r \circ D(\kappa, h(r))] \right\|_{k-2, \alpha, r} \leq C \| h_i \|_{\frac{r}{2}} \leq \frac{r}{2} \| k, 2, \alpha, r \| + C e^{c \| h \|}.$$

(4.23)

(4.24)

Note that

$$Q^* P_r = (Q')^* P_r \circ H_r (DQ')^{-1} P_r$$

(4.25)

$$\frac{\partial}{\partial r} (Q^* P_r) = \frac{\partial}{\partial r} ((Q')^* P_r) \circ H_r (DQ')^{-1} P_r + (Q')^* P_r \frac{\partial}{\partial r} (H_r (DQ')^{-1} P_r).$$

(4.26)

Then lemma follows from (4.23), (4.24) and Lemma 4.4.

4.4 Estimates of $\frac{\partial}{\partial r} \left[ I^*_{r}(\kappa, \zeta) + Q^*_{(\kappa, b_r)} f_r (I_r (\kappa, \zeta)) \right]$

Let $d > 0$ be a small constant such that

$$\| (\kappa, \zeta) \|_{k, 2, \alpha} \leq d.$$

We set

$$(\kappa^*_r, \xi^*_r) := I^*_{r}(\kappa, \zeta) + Q^*_{(\kappa, b_r)} \circ f_r (I_r (\kappa, \zeta)),$$

(4.27)

$$(\kappa_r, \xi_r) := I_r (\kappa, \zeta) + Q_{(\kappa, b_r)} \circ f_r (I_r (\kappa, \zeta)).$$

(4.28)

where $\kappa_r \in K$, $\xi^*_r \in W^{k, 2, \alpha}_a$. Denote $v_r = \exp u_r \xi_r$ and $b_r = (r, \tau, a_\alpha, v_r)$. We have

$$\frac{\partial}{\partial \tau} j \cdot v_r + i(\kappa_r + \kappa_r, v_r) = 0.$$

(4.29)

From the Implicit Function Theorem (see Theorem 6.1, Lemma 3.3, Lemma 3.5, and Lemma 3.4), we have

$$\| f_r (I_r (\kappa, \zeta)) \|_{k-2, \alpha, r} \leq \| DQ f_r (I_r (\kappa, \zeta)) \|_{k-2, \alpha, r} \leq C \| Q f_r (I_r (\kappa, \zeta)) \|_{k-2, \alpha, r}.$$

(4.30)

$$\leq C \left\| (I_r (\kappa, \zeta)) \right\|_{k-2, \alpha, r} \leq C \| f_r (0) \|_{k-2, \alpha, r} + C \| d f_r \circ (\theta f_r (\kappa, \zeta)) \|_{k-2, \alpha, r} \leq C \left( \| \kappa, \zeta \|_{k, 2, \alpha} + e^{-\frac{r}{2}} \right).$$

where we use the intermediate value theorem in the third inequality and $\theta \in (0, 1)$. It follows that $\| (\kappa_r, \xi_r) \| \leq C d$ as $r$ large enough.

For any small $(\kappa, \zeta) \in \ker DS_{(\kappa, h_0)}$, $\exp u_r \zeta$ converges to a point as $| s_i | \to \infty$ (see Lemma 2.6). It follows that $F^l_{\exp u_i} \zeta, l = 1, 2$, converges to zero exponentially. By Theorem 2.3 and the definition of $u_r$ we have

$$| d u_r (r) | \leq C e^{-c | s_i |}, \quad \text{for any } \frac{r}{4} \leq | s_i | \leq r.$$

(4.31)
By choosing $\delta$ small and $R_0$ large we have

$$E(v_{(r)}; |s|_{i} \geq R_0) \leq \delta,$$

where $\delta$ is the constant in Theorem 4.3. Then we have for any $\frac{r}{2} \leq |s_{i}| \leq r$

$$|dE_{(r)}(v)| \leq Ce^{-\delta |s_{i}|}.$$  \hfill (4.32)

By \((4.31), (4.32), (6.2)\) and \((6.3)\) we conclude that in the part $C > 0$, when $r > 4R_0$. \hfill (4.33)

for some constant $C > 0$, when $r > 4R_0$.

Next we recall a fact about the exponential map on a compact Riemannian manifold $M$ (see \cite{7}, Page 362, Remark 10.5.5). There are two smooth families of endomorphisms

$$E_{i}(p, \xi): T_{\pi}M \to T_{\exp_{p}\xi}M, \quad i = 1, 2,$$

that are characterized by the following property. Let $\gamma: \mathbb{R} \to M$ be any smooth path in $M$ and $v(t) \in T_{\gamma(t)}M$ be any smooth vector field along this path then the derivative of the path $t \to \exp_{\gamma(t)}(v(t))$ is given by the formula

$$\frac{d}{dt} \exp_{\gamma}(v) = E_{1}(\gamma, v)\dot{\gamma} + E_{2}(\gamma, v)\dot{\gamma},$$

where $\dot{\gamma} = \frac{dv}{dt}$. We have

$$E_{1}(p, 0) = E_{2}(p, 0) = Id : T_{p}M \to T_{p}M, \quad \forall p \in M,$$

and $E_{i}(p, \xi)$ are uniformly invertible for sufficiently small $\xi$. Since $M$ is compact, there exists a constant $\epsilon$ such that for any $p \in M$ and $\xi \in T_{p}M$ with $|\xi|_{T_{p}M} \leq \epsilon$, $E_{i}(p, \xi)$ are uniformly invertible.

**Lemma 4.6.** There exist two constant $C > 0$ such that for any $(\kappa, \zeta) \in \text{ker}DS_{(\rho, \lambda, 0)}$ we have

$$\left\|H_{p}f_{\rho} \circ I_{\kappa}(\rho, \lambda)\right\|_{s_{i} = \frac{r}{2}} \leq Ce^{-(\epsilon - \alpha)\frac{r}{4}}(1 + \|\kappa, \zeta\|), \quad \forall r \geq 8R_0. \hfill (4.34)$$

**Proof.** Let $DS_{(\rho, \lambda, 0)}$ act on $x_{(r)}$. Since $di_{(\rho, \lambda, 0)}R_{0} \leq |s_{i}| \leq 2r - R_{0} = 0$ we get, in $\{R_{0} \leq |s_{i}| \leq 2r - R_{0}\}$

$$f_{(r)}(I_{\rho}(\kappa, \zeta)) = \partial_{\gamma_{0}, \lambda_{0}}\xi_{(r)} + F_{1, u(\rho)}\partial_{\xi_{(r)}} + F_{2, u(\rho)}\partial_{t}\xi_{(r)}, \quad \partial_{\gamma_{0}, \lambda_{0}}v_{(r)} = 0, \hfill (4.35)$$

Since

$$dv_{(r)} = E_{1}(u_{(r)}, \xi_{(r)})(du_{(r)}) + E_{2}(u_{(r)}, \xi_{(r)})(d\nabla\xi_{(r)}),$$

by \((4.31), (4.32)\) and the elliptic estimate we have

$$\left\|E_{2}(u_{(r)}, \xi_{(r)})(d\nabla\xi_{(r)})\right\|_{k-1, 2, \alpha} \leq Ce^{-\epsilon |s_{i}|}, \quad \frac{r}{4} \leq |s_{i}| \leq r. \hfill (4.36)$$

Note that $[E_{2}(u_{(r)}, \xi_{(r)})]^{-1}$ is uniformly bounded as $\|\xi_{(r)}\|_{k, \alpha, \rho}$ is small. Then

$$\left\|\nabla\xi_{(r)}\right\|_{s_{i} = \frac{r}{4}} \leq Ce^{-(\epsilon - \alpha)\frac{r}{4}}, \hfill (4.37)$$

for some constant $C > 0$. Note that

$$\nabla_{s}x_{(r)} = \frac{\partial x_{(r)}(r)}{\partial s} - \sum_{i, j} \Gamma_{ij}^{l} \frac{\partial x_{(r)}(r)}{\partial s} \frac{\partial x_{(r)}}{\partial s},$$

$$\nabla_{\alpha}x_{(r)} = \sum_{i, j} \Gamma_{ij}^{l} \frac{\partial x_{(r)}(r)}{\partial s} \frac{\partial x_{(r)}}{\partial s}.$$
Lemma 4.7. There exists a constant $C > 0$ such that for any $(\kappa, \zeta) \in \ker DS_{s_0, b_0}$ we have

$$\left\| \frac{\partial}{\partial s_i} (\xi^*_{(r)})_i \right\|_{k \leq |s_i| \leq \frac{R}{\gamma}} \leq C e^{-(\kappa - 5\alpha) \frac{R}{\gamma}} \|\xi\|_{k, 2, \alpha} + 1).$$

Proof. From the definition of $I^*_r$ (see (1.35) and (4.27)) we have, in $\{|s_i| \geq \frac{R}{\gamma}\}$,

$$(\kappa^*_r, \zeta^*_r) = (\kappa, \zeta^*_r) - Q^* P_r \circ H_r D(\mathbf{I}_d, \phi_r)(\kappa, \zeta^*_r) + Q^* P_r \circ H_r f_r I_r(\kappa, \zeta) = (\kappa, \zeta^*_r) - (Q^*)^* P_r \circ H_r (DQ)^{-1} P_r \circ H_r D(\mathbf{I}_d, \phi_r)(\kappa, \zeta^*_r) + (Q^*)^* P_r \circ H_r (DQ')^{-1} P_r \circ H_r f_r I_r(\kappa, \zeta).$$

Note that

$$\left\| \xi^*_{(r)} \right\|_{k \leq |s_i| \leq \frac{R}{\gamma}} \leq C e^{\alpha r} \left\| \xi_{(r)} \right\|_{k \leq |s_i| \leq \frac{R}{\gamma}} \leq C e^{-(\kappa - 5\alpha) \frac{R}{\gamma}} \|\xi\|_{k, 2, \alpha}, \quad (4.38)$$

where we have applied Lemma 2.6 with $R' = \frac{R}{\gamma}$, $R = \frac{R}{\gamma}$ and $\eta = 0$ in the last inequality.

By (e) of Lemma 4.1, Lemma 4.4, and Lemma 4.6 we have

$$\left\| (Q^*)^* P_r \circ H_r (DQ)^{-1} P_r \circ H_r f_r I_r(\kappa, \zeta) \right\|_{k \leq |s_i| \leq \frac{R}{\gamma}} \leq C e^{-(\kappa - \alpha) \frac{R}{\gamma}} \|\xi\|_{k, 2, \alpha} + 1). \quad (4.39)$$

Similar, by (e) of Lemma 4.1 and (4.23) we have

$$\left\| (Q^*)^* P_r \circ H_r (DQ')^{-1} P_r \circ H_r D(\mathbf{I}_d, \phi_r)(\kappa, \zeta^*_r) \right\|_{k \leq |s_i| \leq \frac{R}{\gamma}} \leq C e^{-(\kappa - \alpha) \frac{R}{\gamma}} \|\xi\|_{k, 2, \alpha} + 1).$$

Using Lemma 4.2 and Lemma 4.6 we have

$$\left\| \frac{\partial}{\partial s_i} (\xi^*_{(r)})_i \right\|_{k \leq |s_i| \leq \frac{R}{\gamma}} \leq C \left( \left\| \xi^*_{(r)} \right\|_{k \leq |s_i| \leq \frac{R}{\gamma}} + e^{-(\kappa - \alpha) \frac{R}{\gamma}} \|\xi\|_{k, 2, \alpha} + 1 \right) \leq C e^{-(\kappa - 5\alpha) \frac{R}{\gamma}} \|\xi\|_{k, 2, \alpha} + 1)$$

where we have used (4.38) in the last inequality.

Lemma 4.8. For any $\epsilon > 0$, there are two constants $C > 0$ and $R_1 > R_0$ depending only on $\epsilon, k, \alpha$ and the geometry of $M$, such that for any $r > R_1$ and $(\kappa, \zeta) \in \ker DS_{s_0, b_0}$ with $\|\kappa, \zeta\| \leq d$, we have

$$\left\| H_r \circ D(\mathbf{I}_d, \phi_r) \left( \frac{\partial}{\partial r} (\kappa^*_r, \zeta^*_r) \right) \right\|_{k, 2, \alpha} \leq C \left( d \left\| \frac{\partial}{\partial r} (\kappa^*_r, \zeta^*_r) \right\|_{k, 2, \alpha} + e^{-(\kappa - 5\alpha) \frac{R}{\gamma}} \right). \quad (4.40)$$

Proof. We estimate $\|\beta_1 D(\mathbf{I}_d, \phi_r) \left( \frac{\partial}{\partial r} (\kappa^*_r, \zeta^*_r) \right) \|_{k, 2, \alpha}$. The estimates of $\|\beta_2 D(\mathbf{I}_d, \phi_r) \left( \frac{\partial}{\partial r} (\kappa^*_r, \zeta^*_r) \right) \|_{k, 2, \alpha}$ is the same. Let $b = (r, \tau, j_0, v_{(r)_0})$. First we construct $\tilde{u}_{(r)}$ and $\tilde{v}_{(r)}$ defined over $\Sigma_1$ as follows:

$$\tilde{u}_{(r)} = \begin{cases} u_{(r)}, & \text{if } s_1 \in \Sigma_1(r + 1), \\ u_{(r)}(q) + \beta (r + 2 - s_1)(u_{(r)}(s_1, t_1) - u_{(r)}), & \text{if } s_1 \geq r + 1 \end{cases} \quad (4.41)$$

$$\tilde{v}_{(r)} = \begin{cases} \xi_{(r)}, & \text{if } s_1 \in \Sigma_1(r + 1), \\ \beta (r + 2 - s_1) \xi_{(r)}(s_1, t_1), & \text{if } s_1 \geq r + 1 \end{cases} \quad (4.42)$$

Define $\tilde{v}_{(r)} = \exp_{\tilde{u}_{(r)}} \tilde{v}_{(r)}$. So the meaning of $\frac{\partial}{\partial r} (\tilde{u}_{(r)}, \tilde{v}_{(r)})$ and $\nabla \tilde{v}_{(r)}$ is clear. Set

$$\Lambda_r := P_{b, h_{(r)}(\tilde{v}_{(r)} + i(\kappa_\alpha + \kappa_r, b_r))}, \quad \tilde{\Lambda}_r := P_{b, h_{(r)}(\tilde{v}_{(r)} + i(\kappa_\alpha + \kappa_r, \tilde{v}_{(r)}))}.$$
where \( \tilde{b} = (z, a_\alpha, \tilde{v}_r(r)) \) and \( \tilde{b}_r(r) = (z, a_\alpha, \tilde{u}_r(r)) \). We calculate \( \frac{\partial}{\partial r} \beta_{1:2} \Lambda_r \). By (4.29) we have \( \Lambda_r = 0 \) and \( \beta_{1:2} \Lambda_r = 0 \). Then

\[
\frac{\partial}{\partial r} (\beta_{1:2} \Lambda_r) = \frac{\partial}{\partial r} (\beta_{1:2} \Lambda_r) = \beta_{1:2} P_{b_b(r)} \left( D_{\tilde{v}_r(r)} \left( \frac{\partial \tilde{v}_r(r)}{\partial r} \right) + d_{i_{(\kappa_\alpha + \kappa_r, \tilde{b}_r)}} \left( \frac{\partial \tilde{v}_r(r)}{\partial r} \right) \right) = \beta_{1:2} P_{b_b(r)} \left( \left( E_1 (\tilde{u}_r(r), \tilde{\xi}_r(r)) \right) \frac{\partial \tilde{u}_r(r)}{\partial r} + E_2 (\tilde{u}_r(r), \tilde{\xi}_r(r)) \tilde{\nabla}_r \tilde{\xi}_r(r) \right) + d_{i_{(\kappa_\alpha + \kappa_r, \tilde{b}_r)}} \left( \frac{\partial \tilde{v}_r(r)}{\partial r} \right)
\]

Since

\[
d_{i_{(\kappa_\alpha + \kappa_r, \tilde{b}_r)}} \left( \frac{\partial \tilde{v}_r(r)}{\partial r} \right) \bigg|_{|s| \geq R_0} = 0, \quad \tilde{\nabla}_r \tilde{\xi}_r(r) \bigg|_{\Sigma(R_0)} = \phi_r \frac{\partial \xi^*_r(r)}{\partial r} \bigg|_{\Sigma(R_0)}
\]

we can conclude that

\[
d_{i_{(\kappa_\alpha + \kappa_r, \tilde{b}_r)}} \left( \frac{\partial \tilde{v}_r(r)}{\partial r} \right) = d_{i_{(\kappa_\alpha + \kappa_r, \phi_r)}} \left( \frac{\partial \tilde{v}_r(r)}{\partial r} \right),
\]

Note that in \( \{ s_1 \leq r + 1 \} \)

\[
\tilde{\xi}_r(r) = \xi_r(r) = \xi^*_1(s_1, t_1) + \xi^*_2(s_1 - 2r, t_1 - r).
\]

(4.43)

Taking derivative \( \tilde{\nabla}_r \) of (4.43) we get, in \( \{ s_1 \leq r + 1 \} \)

\[
\tilde{\nabla}_r \tilde{\xi}_r(r) = \phi_r \tilde{\nabla}_r \xi^*_r - 2 \tilde{\nabla}_r \xi^*_r 2.
\]

Then by \( v_r(r) |_{s \leq r+1} = \tilde{v}_r(r) |_{s \leq r+1} \) and \( u_r(r) |_{s \leq r+1} = \tilde{u}_r(r) |_{s \leq r+1} \) we get,

\[
\frac{\partial}{\partial r} (\beta_{1:2} \Lambda_r) = P_{b_b(r)} \beta_{1:2} (s_1, t_1) ((E) + (G) + (H) + (F)),
\]

(4.44)

where

\[
(E) = D_{v_r(r)} \left( E_2 (u_r(r), \xi_r(r)) \phi_r \frac{\partial \xi^*_r(r)}{\partial r} \right), \quad (F) = d_{i_{(\kappa_\alpha + \kappa_r, \phi_r)}} \left( \frac{\partial \xi^*_r(r)}{\partial r} \right),
\]

\[
(G) = -2 D_{v_r(r)} \left( E_2 (u_r(r), \xi_r(r)) \tilde{\nabla}_r \xi^*_r \right),
\]

\[
(H) = D_{v_r(r)} \left( E_1 (u_r(r), \xi_r(r)) \frac{\partial (u_r(r))}{\partial r} + E_2 (u_r(r), \xi_r(r)) \phi_r \left( \tilde{\nabla}_r (\xi^*_r - \frac{\partial \xi^*_r(r)}{\partial r}) \right) \right).
\]

By \( \frac{\partial}{\partial r} (\beta_{1:2} \Lambda_r) = 0 \) and (4.44) we have

\[
\left\| \beta_{1:2} D (I, \phi_r) \left( \frac{\partial}{\partial r} \left( \kappa^*_r, \xi^*_r(r) \right) \right) \right\|_{k-2,2,\alpha} \leq (I) + (II) + (III) + (IV),
\]

(4.45)

where

\[
(I) = \left\| \beta_{1:2} \left( P_{b_b(r)} D_{v_r(r)} \left( E_2 (u_r(r), \xi_r(r)) \phi_r \frac{\partial \xi^*_r(r)}{\partial r} \right) - D_{u(r)} \left( \phi_r \frac{\partial \xi^*_r(r)}{\partial r} \right) \right) \right\|_{k-2,2,\alpha},
\]

(II) = \( \left\| \beta_{1:2} P_{b_b(r)} (H) \right\|_{k-2,2,\alpha} \),

(III) = \( \left\| \beta_{1:2} P_{b_b(r)} (G) \right\|_{k-2,2,\alpha} \),

(IV) = \( \left\| \beta_{1:2} P_{b_b(r)} (F) - \beta_{1:2} d_{i_{(\kappa_\alpha + \phi_r)}} \left( \frac{\partial \kappa^*_r, \phi_r}{\partial r} \right) \right\|_{k-2,2,\alpha} \).

(4.46)
We calculate (I). There is a constant $C > 0$ depending only on the geometry of $M$ such that

\[
(I) \leq C\|\xi(r)\|_{k,2,\alpha} \leq \beta_{1;2}\phi_r \frac{\partial}{\partial r} \xi^* \|_{k-1,2,\alpha} \leq C\|\xi(r)\|_{k,2,\alpha} \left\| \frac{\partial}{\partial r} \xi^* \right\|_{k-1,2,\alpha}.
\]

Essentially, this estimate has been obtained by McDuff and D. Salamon (see (3.5.5), P68, [7]) we omit the proof here.

Now we calculate (II). Since

\[
\nabla \xi^* = \frac{\partial}{\partial r} \xi^* = \sum \Gamma_{ij}^k \frac{\partial u_i}{\partial r} (\xi^*)^j \frac{\partial}{\partial s^k},
\]

by the definition of $u(r)$ we conclude that

\[
\text{supp} \beta_{1;2} P_{b,b_1}(\xi) \subset \{ \xi \leq |s_i| \leq r + 1 \}.
\]

Then by (4.31) and (4.33) we have

\[
(II) \leq Ce^{-(\epsilon-\alpha)\xi^2} \quad (4.48)
\]

for a constant $C > 0$.

It follows from Lemma 4.7 that

\[
(III) \leq C\|\frac{\partial}{\partial x^2} (\xi^* r_2) \|_{s_2 \leq r+1} \leq C\|\xi(r)\|_{k,2,\alpha} \leq Ce^{-(\epsilon-5\alpha)\xi^2}(\|\xi\|_{k,2,\alpha} + 1).
\]

Finally we estimate (IV). Let $f(\lambda) = \lambda + \kappa, v(\lambda) = \exp u(r)(\lambda \xi(r))$ and $b(\lambda) = (\kappa, v(\lambda)), \lambda \in [0,1]$. In the following we omit the restriction $\{|s_i| \leq r + 1\}$. Since $i$ and parallel translation are smooth with respect to $(\kappa, u)$, we have

\[
(IV) \leq \beta_{1;2} P_{b,b_1}(F) - \beta_{1;2} \frac{\partial}{\partial x^2} (\kappa, \xi(r)) \left\| \xi^* \right\|_{k,2,\alpha, r} + \beta_{1;2} \frac{\partial}{\partial s^2} (\kappa, \xi(r)) \right\|_{k,2,\alpha, r} \leq C\|\xi(r)\|_{k,2,\alpha} \left\| \frac{\partial}{\partial x^2} (\xi^* r_2) \|_{k,2,\alpha}.
\]

Then the lemma follows from the estimates of (I), (II), (III), (IV) and $\|\kappa, \xi(r)\|_{k,2,\alpha} \leq C d$. □

**Proof of Theorem 1.1** From (4.27) we have

\[
\frac{\partial}{\partial r}(\kappa^*_r, \xi^*_r) = \frac{\partial}{\partial r} (Q^* P_r) H_r f_r (I_r (\kappa, \xi)) + Q^* P_r \frac{\partial}{\partial r} (H_r f_r (I_r (\kappa, \xi))). \quad (4.49)
\]

Then multiplying $H_r D(Id, \phi_r)$ on both sides of (4.49) we get

\[
H_r D(Id, \phi_r) \left( \frac{\partial}{\partial r}(\kappa^*_r, \xi^*_r) \right) = H_r D(Id, \phi_r) \left( \frac{\partial}{\partial r} (Q^* P_r) H_r f_r (I_r (\kappa, \xi)) \right) + H_r P_r \frac{\partial}{\partial r} (H_r f_r (I_r (\kappa, \xi))).
\]

In the above calculation we have used $D(Id, \phi_r) \circ Q^* = DQ = Id$. It follows together with (4.40) that

\[
\left\| H_r P_r \frac{\partial}{\partial r} (H_r f_r (I_r (\kappa, \xi))) \right\|_{k-2,2,\alpha} \leq Cd \left\| \frac{\partial}{\partial r}(\kappa^*_r, \xi^*_r) \right\|_{k-1,2,\alpha} + C e^{-(\epsilon-5\alpha)\xi^2} (A) + (B). \quad (4.50)
\]
where
\[
(A) = \left\| H_r D(I_d, \phi_r) \left( \frac{\partial}{\partial \tau} (I_r^*(\kappa, \zeta)) \right) \right\|_{k-2,2,\alpha}, \quad (B) = \left\| H_r D(I_d, \phi_r) \left( \frac{\partial}{\partial \tau} (Q^* P_r \circ H_r f_r)(I_r(\kappa, \zeta)) \right) \right\|_{k-2,2,\alpha}.
\]

By Lemma 4.2, Lemma 4.3, taking \( R = \frac{\tau}{2}, R' = \frac{\tau}{2} \) in Lemma 2.6, we conclude that
\[
(A) \leq C \left\| \frac{\partial}{\partial \tau} I_r^*(\kappa, \zeta) \right\|_{k-1,2,\alpha} \leq C \| \zeta_i \|_{\frac{1}{2} \leq |\zeta_i| \leq \frac{\tau}{2}} \| k,2,\alpha + C e^{-(\epsilon - \alpha) \frac{\tau}{4}} |\zeta_0|
\]
\[
\leq C e^{-(\epsilon - \alpha) \frac{\tau}{4}} (\| \zeta_i \|_{\frac{1}{2} \leq |\zeta_i| \leq \frac{\tau}{2}} \| k,2,\alpha + |\zeta_0|)
\]
where \( \zeta = (\zeta_1 + \zeta_0, \zeta_2 + \zeta_0) \). By (4.26), Lemma 4.1, Lemma 4.4, Lemma 4.6 and (4.30) we get
\[
\left\| \frac{\partial}{\partial \tau} (Q^* P_r \circ H_r f_r)(I_r(\kappa, \zeta)) \right\|_{k-1,2,\alpha} \leq C e^{-(\epsilon - \alpha) \frac{\tau}{4}} (\| (\kappa, \zeta) \|_{k,2,\alpha} + 1).
\] (4.51)

By Lemma 4.2 we have
\[
(B) \leq \left\| \frac{\partial}{\partial \tau} (Q^* P_r \circ H_r f_r)(I_r(\kappa, \zeta)) \right\|_{k-1,2,\alpha} \leq C e^{-(\epsilon - \alpha) \frac{\tau}{4}} (\| (\kappa, \zeta) \|_{k,2,\alpha} + 1)
\]
Inserting the estimates of (A) and (B) into (4.50) we have
\[
\left\| H_r P_r \frac{\partial}{\partial \tau} (H_r f_r)(I_r(\kappa, \zeta)) \right\|_{k-1,2,\alpha} \leq C e^{-(\epsilon - 5\alpha) \frac{\tau}{4}} + C d \left\| \frac{\partial}{\partial \tau} \left( \kappa^*_r, \xi^*_r \right) \right\|_{k-1,2,\alpha}.
\] (4.52)

By (4.25), Lemma 4.1 and Lemma 4.4 we get
\[
\left\| Q^* P_r \frac{\partial}{\partial \tau} (H_r f_r)(I_r(\kappa, \zeta)) \right\|_{k-1,2,\alpha} = \left\| Q^* P_r \left( H_r P_r \frac{\partial}{\partial \tau} (H_r f_r)(I_r(\kappa, \zeta)) \right) \right\|_{k-1,2,\alpha}
\]
\[
\leq C \left\| H_r P_r \frac{\partial}{\partial \tau} (H_r f_r)(I_r(\kappa, \zeta)) \right\|_{k-1,2,\alpha}.
\]

Using (4.51), (4.52) and Lemma 4.6 Lemma 4.5 we get
\[
\left\| \frac{\partial}{\partial \tau} \left( \kappa^*_r, \xi^*_r \right) \right\|_{k-1,2,\alpha} \leq C e^{-(\epsilon - 5\alpha) \frac{\tau}{4}} + C d \left\| \frac{\partial}{\partial \tau} \left( \kappa^*_r, \xi^*_r \right) \right\|_{k-1,2,\alpha}.
\] (4.53)

Choose \( d \) small such that \( 4 C d < 1 \). Then Theorem 1.1 is proved. \( \Box \)

**Proof of Corollary 1.2** It is easy to see that, restricting to \( \Sigma(R_0) \), we have
\[
I_r^*(\kappa, \zeta) + Q^*_{(\kappa_0, b_r)} f_r(I_r(\kappa, \zeta)) = I_r(\kappa, \zeta) + Q_{(\kappa_0, b_r)} f_r(I_r(\kappa, \zeta)).
\]

So we have an estimate for \( \left\| \frac{\partial}{\partial \tau} \left[ I_r(\kappa, \zeta) + Q_{(\kappa_0, b_r)} f_r(I_r(\kappa, \zeta)) \right] \right\|_{\Sigma(R_0)} \). By Sobolev embedding theorem and the standard elliptic estimates we get Theorem 1.2 \( \Box \)

**Remark 4.9.** Repeating the all arguments in this section, one can prove that there exists a constant \( C > 0 \) such that
\[
\left\| \frac{\partial}{\partial \tau} \left[ I_r^*(\kappa, \zeta) + Q^*_{(\kappa_0, b_r)} f_r(I_r(\kappa, \zeta)) \right] \right\|_{k-1,2,\alpha} \leq C e^{-(\epsilon - 5\alpha) \frac{\tau}{4}} (d + 1)
\] (4.54)
for any \( (\kappa, \zeta) \in \ker D S_{(\kappa_0, b_r)} \). Since we need only a bound for \( \frac{\partial}{\partial \tau} (\cdot) \), the calculations are much simpler. For example, consider (b) in Lemma 4.1, we have
\[
\left\| \frac{\partial}{\partial \tau} (H_r P_r)(\eta_1, \eta_2) \right\|_{k-2,2,\alpha} \leq C \sum_{i=1}^{2} \left\| \eta_i \right\|_{|r-1| \leq |s_i| \leq r+1} \right\|_{\Sigma, k-1,2,\alpha}.
\] (4.55)

In fact, by (1.13) we have
\[
\frac{\partial \eta_1}{\partial \tau} = -\beta_1 \frac{\sqrt{1 - \beta_2^2}}{2} \frac{\partial \eta_1}{\partial \tau} (s_1 - 2r, t_1 - \tau).
\] (4.56)

Then (4.55) follows from \( H_r P_r(\eta_1, \eta_2) = (\tilde{\eta}_1, \tilde{\eta}_2) \) and (4.56).
5 Extension

In this section we extend the Theorem 1.2 to more general setting.

5.1 Gluing several nodes

Let $(\Sigma, j, y, q)$ be a marked nodal Riemann surface of genus $g$ with $n$ marked points $y = (y_1, \ldots, y_n)$ and $e$ nodal points $q = (q_1, \ldots, q_e)$. Suppose that $\Sigma$ has $e$ smooth components $\Sigma_i$. We assume that every component $(\Sigma_{i}, j_{i}, y_{i}, q_{i})$ is stable. Let $A = A_1 \times A_2 \times \ldots \times A_e$ be the space of complex structures (including marked points). Let $u = (u_1, \ldots, u_e)$, where $u_{i} : \Sigma_{i} \to M$ be $(j_{i}, J_{i})$-holomorphic map.

For every node $q_i$ we choose the holomorphic cylindrical coordinates near the node $q_i$. We glue $\Sigma$ and $u$ at each node $q_i$ with parameter $(r_i, \tau_i)$ as in sections 1.2 to get $\Sigma_{(r)}$ and the pregluing map $u_{(r)}$. Denote $z_i = e^{-r_i - 2\sqrt{-1} \tau_i}$ and $z = (z_1, \ldots, z_e)$. Set

$$b_o = (a_o, 0, u), \quad b_r := (a_o, u_{(r)}).$$

We can define $B_{(r)}$, $W_{r,u_{(r)}}^{k,2,\alpha}$ and $L_{r,u_{(r)}}^{k-1,2,\alpha}$ as in section 1.2. The Weil-Petersson metric induces a distance $d_{\Delta}(a_o, a)$ on $A$. Set

$$O_{b_o}(R, \delta, \rho) := \{(a, z, v_{(r)})|(a, z) \in A \times D_r, \ v_{(r)} \in B_{(r)}, \ |r_i| < e^{-2R}, \ d_{\Delta}(a_o, a) < \delta, \ |h_{(r)}|s_{2,\alpha,r} < \rho \},$$

where $v_{(r)} = \exp_{u_{(r)}}(h_{(r)})$. Denote by $g_o$ the metric on $(\Sigma, j_o)$, and $|r| := \min\{r_1, \ldots, r_e\}$.

Lemma 5.1. For $|r| > R_0$ there is an isomorphism

$$I_{(r)} : \ker DS_{(\kappa, b_o)} \to \ker DS_{(\kappa, b_{(r)})}.$$ 

In order to get a global regularization we need to take a sum of several $K_{b_o}$. So we consider the following setting. Let $K$ be a $N$-dimensional linear space. Let

$$i : K \times A \times W_{r,u_{(r)}}^{k,2,\alpha}(\Sigma(R_0)\times (u_{(r)}|_{\Sigma(R_0)})^{*}TM) \to W_{r,u_{(r)}}^{k-1,2,\alpha}(\Sigma(R_0)\times (u_{(r)}|_{\Sigma(R_0)})^{*}TM \otimes \wedge_{jo}^{0,1}T_{\Sigma(R_0)}$$

be a smooth map such that $D_{v} + d\iota(\kappa, a, v|_{\Sigma(R_0)})$ is surjective for any $(\kappa, b) \in K \times O_{b_o}(R, \delta, \rho)$, where $b = (a, z, v), \ v = \exp_{u_{(r)}} h$.

Define a thickened Fredholm system $(K \times O_{b_o}(R, \delta, \rho), K \times E|_{O_{b_o}(R, \delta, \rho)} , S)$ with

$$S(\kappa, b) = \partial_{ja, j_{r}} v + i(\kappa, b).$$

(5.2)

For fixed $(r)$ we consider the family of maps:

$$F_{(r)} : K \times A \times W_{r,u_{(r)}}^{k,2,\alpha}(\Sigma_{(r)} \times (u_{(r)}|_{\Sigma(R_0)})^{*}TM) \to W_{r,u_{(r)}}^{k-1,2,\alpha}(\Sigma_{(r)} \times (u_{(r)}|_{\Sigma(R_0)})^{*}TM \otimes \wedge_{jo}^{0,1}T_{\Sigma_{(r)}}) ,$$

$$F_{(r)}(\kappa, a, h) = \Psi_{ja, j_{r}} \Phi_{u(r)}(h)^{-1}(\partial_{ja, j_{r}} v + i(\kappa, b)),$$

where $b = (a, z, v), \ v = \exp_{u_{(r)}} h$ and $\Psi_{ja, j_{r}}$ is defined in section 6.3.

By implicit function theorem (Theorem 6.1, Theorem 6.2), there exist $R > 0$, a small neighborhood of $O_{a_o}(\delta) \subset A$ and a small neighborhood $O_{(r)}$ of 0 $\in \ker DS_{b_{(r)}}$ and a unique smooth map

$$f_{(r)} : O_{a_o}(\delta) \times O_{(r)} \to W_{r,u_{(r)}}^{k-1,2,\alpha}(\Sigma_{(r)} \times (u_{(r)}|_{\Sigma(R_0)})^{*}TM \otimes \wedge_{jo}^{0,1}T_{\Sigma_{(r)}})$$

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such that for any \((a, (\kappa, \zeta)) \in O_{a_0}(\delta) \times O(\varepsilon)\) and \(|r| > R\),
\[
\mathcal{F}(r) \left( a, I_R(\kappa, \zeta) + Q_{(\kappa, b_1)} \circ f_{a}(r) \circ I_R(\kappa, \zeta) \right) = 0. \tag{5.3}
\]

Denote by \(Q_{(\kappa, b_1)}\) the right inverse of \(DS_{(\kappa, b_1)}\). Then Theorem 1.2 can be directly extended as

**Theorem 5.2.** Let \(l \in \mathbb{Z}^+\) be a fixed integer. There exists positive constants \(C_{2, l}, h, R_0\) such that for any \((\kappa, \zeta) \in \ker DS_{(\kappa_0, b_0)}\) with \(|(\kappa, \zeta)| < d\), restricting to the compact set \(\Sigma(R_0)\), for any \(a \in \mathcal{O}_i\), the following estimate holds
\[
\left\| \frac{\partial}{\partial r_i} \left( I_R(\kappa, \zeta) + Q_{(\kappa_0, b_0)} \circ f_{a}(r) \circ I_R(\kappa, \zeta) \right) \right\|_{C^l(\Sigma(R_0))} + \left\| \frac{\partial^2}{\partial r_i \partial r_j} \left( I_R(\kappa, \zeta) + Q_{(\kappa_0, b_0)} \circ f_{a}(r) \circ I_R(\kappa, \zeta) \right) \right\|_{C^l(\Sigma(R_0))} \leq C_{2, l} e^{-(5\alpha)\frac{r_i + r_j}{4}},
\]
\(i = 1, \ldots, c\).

### 5.2 Estimates of higher derivatives

Let \((s_i^l, t_i^l), l = 1, 2\) be the cylinder coordinates near the node \(q_i\). Set
\[
V_i := \bigcup_{l=1}^2 \{ (s_i^l, t_i^l) \in \Sigma \mid \frac{s_i^l}{2} \leq |s_i^l| \leq \frac{3s_i^l}{2} \}.
\]

Denote
\[
\begin{align*}
Glu_{a_0}(r)(\kappa, \zeta) &= I_R(\kappa, \zeta) + Q_{(\kappa_0, b_0)} \circ f_{a_0}(r) \circ I_R(\kappa, \zeta), \\
Glu_{a_0}^*(r)(\kappa, \zeta) &= I_R^*(\kappa, \zeta) + Q_{(\kappa_0, b_0)}^* \circ f_{a_0}(r) \circ I_R(\kappa, \zeta).
\end{align*}
\]

In this subsection we prove

**Theorem 5.3.** There exists positive constants \(C, d, R_0\) such that for any \((\kappa, \zeta) \in \ker DS_{(\kappa_0, b_0)}\) with \(|(\kappa, \zeta)| < d\), for any \(X_i \in \{ \frac{\partial}{\partial r_i}, \frac{\partial}{\partial t_i} \}, i = 1, \ldots, c\), the following estimate holds
\[
\left\| X_i X_j \left( Glu_{a_0}(r)(\kappa, \zeta) \right) \right\|_{k-2, 2, \alpha} + \left\| X_i \left( Glu_{a_0}^*(r)(\kappa, \zeta) \right) \right\|_{V_i} \leq C e^{-(5\alpha)\frac{r_i + r_j}{4}},
\]
\(1 \leq i \neq j \leq c\), for any \(a \in \mathcal{O}_i\). In particular, restricting to the compact set \(\Sigma(R_0)\) and for any \(l \in \mathbb{Z}^+\),
\[
\left\| X_i X_j \left( Glu_{a_0}(r)(\kappa, \zeta) \right) \right\|_{C^l(\Sigma(R_0))} \leq C_l e^{-(5\alpha)\frac{r_i + r_j}{4}},
\]

for some constant \(C_l\).

**Proof.** We give a sketch of the proof. Denote \(\eta = (\eta_1, \cdots, \eta_c)\). Set
\[
D_i^j(R_0) = \{ (s_i^l, t_i^l) \in \Sigma \mid |s_i^l| \geq R_0 \}, \quad D^j(R_0) = \bigcup_{i=1}^2 D_i^j(R_0).
\]

Denote
\[
\beta_{1, i, R}(s_i^l) = \beta \left( \frac{1}{2} + \frac{r_i - s_i^l}{R} \right), \quad \beta_{2, i, R}(s_i^l) = \sqrt{1 - \beta^2 \left( \frac{1}{2} - \frac{s_i^l}{R} \right)}.
\]

We can define \(h_1, h_2, \tilde{h}_1, \tilde{h}_2, H_R, P_R\) as before. Let \(\eta_i^l = \eta_i^l|_{D_i^j(R_0)}, l = 1, 2\). Obviously \(H_R P_R(\eta)|_{D^j(R_0)} = (\beta_{1, i, R}(\sum_{l=1}^2 \beta_{1, l, 2} \eta_i^l), \beta_{2, i, R}(\sum_{l=1}^2 \beta_{1, l, 2} \eta_i^l))\). It is easy to see that for any \(1 \leq i \neq j, \ell \leq c\) and \(l = 1, 2\),
\[
\frac{\partial (H_R P_R)(\eta)|_{V_j}}{\partial r_i} = 0, \quad \frac{\partial^2 (H_R P_R)(\eta)}{\partial r_i \partial r_j}|_{V_j} = 0, \quad \frac{\partial^2 \beta_{1, i, \ell}}{\partial r_i \partial r_j} = 0, \quad \frac{\partial^2 \beta_{2, i, \ell}}{\partial r_i \partial r_j} = 0,
\]

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\let\hline\relax
\begin{align*}
\frac{\partial \beta_{i,j,r_i}}{\partial r_i} &= \frac{\partial \beta_{i,j,r_i}}{\partial r_i} = 0, \quad \text{supp} \frac{\partial \beta_{i,j,r_i}}{\partial r_i} \subset V_i, \quad \text{supp} \frac{\partial \beta_{i,j,r_i}}{\partial r_i} \subset V_i.
\end{align*}

In the following we assume that $1 \leq i \neq j \leq e$. Let $(\kappa, h) = (\kappa, h_1, \cdots, h_i) = Q_{(\kappa, h_0)}(H_T P_T)(\eta)$. Then we have \( \frac{\partial^2 \kappa}{\partial r_i \partial r_j} = 0 \) and \( \frac{\partial^2 h_i}{\partial r_i \partial r_j} = 0 \). It follows that

\[
\begin{align*}
\frac{\partial^2 h^*_i}{\partial r_i \partial r_j} &= 0, \\
\frac{\partial^2 (H_T(D(\kappa, h_i)))}{\partial r_i \partial r_j} &= \frac{\partial^2 (H_T(D(\kappa, h_i)))}{\partial r_i \partial r_j} = 0.
\end{align*}
\]

Let \( h^*_i = h_i|_{D^j(h_0)}, l = 1, 2 \). Then \( (h^*_1, h^*_2) \) is the restriction of \( h \) near the node \( q_i \). Obviously, \( (Q')^* P_T(\eta)_{D^c} = (\kappa, \beta_{i,j,r}, h^*_1, \beta_{2,i,r}, h^*_2) \). Taking the derivative \( \frac{\partial^2}{\partial r_i \partial r_j} \) of \( (Q')^* P_T \) we obtain

\[
\begin{align*}
\frac{\partial}{\partial r_j}((Q')^* P_T)(\eta) \bigg|_{V_i} &= \left( 0, \beta_{1,i,r}, \frac{\partial h^*_1}{\partial r_i} \right) \frac{\partial}{\partial r_j}((Q')^* P_T)(\eta) \bigg|_{V_i} = \delta_{i,j} \left( 0, \frac{\partial h^*_1}{\partial r_i} \right), \\
\frac{\partial^2}{\partial r_i \partial r_j}((Q')^* P_T)(\eta) \bigg|_{D^c} &= \delta_{i,j} \left( 0, \frac{\partial h^*_1}{\partial r_i} \right).
\end{align*}
\]

Applying (2.11) of Lemma 2.6 we have

\[
\begin{align*}
\left\| \frac{\partial}{\partial r_j}((Q')^* P_T)(\eta) \bigg|_{V_i} \right\|_{k-1,2,\alpha} + \left\| \frac{\partial^2}{\partial r_i \partial r_j}((Q')^* P_T)(\eta) \bigg|_{D^c} \right\|_{k-2,2,\alpha} 
\leq C e^{-\frac{(\epsilon-\alpha)\epsilon}{4}} \|\eta\|_{k-1,2,\alpha} + C e^{-\frac{(\epsilon-\alpha)\epsilon}{4}} \|\eta\|_{k-1,2,\alpha}.
\end{align*}
\]

Similar we obtain that

\[
\begin{align*}
\left\| \frac{\partial}{\partial r_j} (H_T(DQ')^{-1} P_T)(\eta) \bigg|_{V_i} \right\|_{k-1,2,\alpha} + \left\| \frac{\partial^2}{\partial r_i \partial r_j} (H_T(DQ')^{-1} P_T)(\eta) \bigg|_{k-2,2,\alpha} 
\leq C e^{-\frac{(\epsilon-\alpha)\epsilon}{4}} \|\eta\|_{k-1,2,\alpha} + C e^{-\frac{(\epsilon-\alpha)\epsilon}{4}} \|\eta\|_{k-1,2,\alpha}.
\end{align*}
\]

and

\[
\begin{align*}
\left\| \frac{\partial}{\partial r_j} I^* (\kappa, h + \tilde{h}_0) \bigg|_{V_i} \right\|_{k-1,2,\alpha} + \left\| \frac{\partial^2}{\partial r_i \partial r_j} I^* (\kappa, h + \tilde{h}_0) \bigg|_{k-2,2,\alpha} 
\leq C \left( e^{-\frac{(\epsilon-\alpha)\epsilon}{4}} \|h\|_{V_i} + e^{-\frac{(\epsilon-\alpha)\epsilon}{4}} \|\eta\|_{k,2,\alpha} \right) + C e^{(\epsilon-\alpha)\epsilon\epsilon} \|\tilde{h}_0\|.
\end{align*}
\]

Note that, restricting in \( V_i \),

\[
\nabla \frac{\partial}{\partial r_j} \tilde{\xi}(\tau) = \phi_r \nabla \frac{\partial}{\partial r_j} \xi^*(\tau), \quad \frac{\partial u(\tau)}{\partial r_j} = 0.
\]

Then by the same calculation of Lemma 4.8 we have

\[
\left\| H_T \circ D(Id, \phi_r) \frac{\partial}{\partial r_j} \left( \kappa^*_r, \xi^*_r \right) \bigg|_{V_i} \right\|_{k-1,2,\alpha} \leq Cd \left\| \frac{\partial}{\partial r_j} \left( \kappa^*_r, \xi^*_r \right) \bigg|_{V_i} \right\|_{k-1,2,\alpha}.
\]

Using (5.4), (5.5), (5.6) and the same proof of Theorem 1.1 word by word, we have

\[
\left\| \frac{\partial}{\partial r} \left( \kappa^*_r, \xi^*_r \right) \bigg|_{V_i} \right\|_{k-1,2,\alpha} + \left\| \frac{\partial}{\partial r (H_T f_T I^* (\kappa, \xi))} \bigg|_{V_i} \right\|_{k-2,2,\alpha} \leq C e^{-5\alpha + \frac{\epsilon\epsilon}{4}}.
\]

Using (5.7), Theorem 1.1 and the Cauchy-Schwarz inequality, by the same argument of Lemma 4.8, we have

\[
\left\| H_T D(Id, \phi_r) \frac{\partial^2}{\partial r_i \partial r_j} \left( \kappa^*_r, \xi^*_r \right) \right\|_{k-2,2,\alpha} \leq Cd \left\| \frac{\partial^2}{\partial r_i \partial r_j} \left( \kappa^*_r, \xi^*_r \right) \right\|_{k-2,2,\alpha} \leq C e^{-5\alpha + \frac{\epsilon\epsilon}{4}}.
\]
Taking the derivative \( \frac{\partial^2}{\partial r_i \partial r_j} \) of (4.27) and multiplying \( H_T D(Id, \phi_r) \) on both sides we get

\[
H_T D(Id, \phi_r) \circ \frac{\partial^2 (\kappa^*_r, \xi^*_r)}{\partial r_i \partial r_j} = H_T D(Id, \phi_r) \circ \frac{\partial^2 (Q^*_r P_r)}{\partial r_i \partial r_j} + H_T f_{(r)} I_r(\kappa, \zeta) + H_T P_r \frac{\partial^2 (H_T f_{(r)} I_r(\kappa, \zeta))}{\partial r_i \partial r_j} + H_T D(Id, \phi_r) \circ \frac{\partial (Q^*_r P_r)}{\partial r_i} \frac{\partial (H_T f_{(r)} I_r(\kappa, \zeta))}{\partial r_j}.
\]

Using (4.50), (5.3), \( \frac{\partial (H_T P_r)}{\partial r_j} \subset V_j, \) Lemma 4.6 and

\[
\frac{\partial}{\partial r_j} (H_T f_{(r)} I_r(\kappa, \zeta)) = \frac{\partial (H_T P_r)}{\partial r_j} \circ H_T f_{(r)} I_r(\kappa, \zeta) + H_T P_r \frac{\partial}{\partial r_j} (H_T f_{(r)} I_r(\kappa, \zeta)),
\]

by the same argument of (4.51) we get

\[
\left\| \frac{\partial (Q^*_r P_r)}{\partial r_i} \frac{\partial (H_T f_{(r)} I_r(\kappa, \zeta))}{\partial r_j} \right\|_{k-2,2,\alpha} \leq C e^{-\frac{\gamma r+\zeta_r}{4}}.
\]

Then repeating the proof of (4.52) we have

\[
\left\| H_T P_r \frac{\partial^2 (H_T f_{(r)} I_r(\kappa, \zeta))}{\partial r_i \partial r_j} \right\|_{k-2,2,\alpha} \leq C e^{-\frac{\gamma r+\zeta_r}{4}} + C d \left\| \frac{\partial^2}{\partial r_i \partial r_j} \left( \kappa^*_r, \xi^*_r \right) \right\|_{k-2,2,\alpha}.
\]

Then as in the proof of (4.53) we conclude that

\[
\left\| \frac{\partial^2}{\partial r_i \partial r_j} \left( \kappa^*_r, \xi^*_r \right) \right\|_{k-2,2,\alpha} \leq C e^{-\frac{\gamma r+\zeta_r}{4}} + C d \left\| \frac{\partial^2}{\partial r_i \partial r_j} \left( \kappa^*_r, \xi^*_r \right) \right\|_{k-2,2,\alpha}.
\]

Choose \( d \) small such that \( 4Cd < 1 \). Since

\[
I^*_r(\kappa, \zeta) + Q^*_r(\kappa, \zeta, u) f_{(r)} I_r(\kappa, \zeta) = I_r(\kappa, \zeta) + Q(\kappa, \zeta, u) f_{(r)} I_r(\kappa, \zeta) \quad \text{on} \quad \Sigma(R_0)
\]

Theorem 5.3 holds. \( \square \)

6 Appendix

6.1 Linearized operator

Choose local normal coordinates \((x^1, \ldots, x^{2m})\) in a neighborhood \( O_{u(q)} \) of \( u(q) \) such that

\[
(x^1, \ldots, x^{2m})(u(q)) = 0, \quad J \frac{\partial}{\partial x^i} \bigg|_0 = \frac{\partial}{\partial x^{m+i+1}} \bigg|_0, \quad J \frac{\partial}{\partial x^{2m+i}} \bigg|_0 = -\frac{\partial}{\partial x^i} \bigg|_0, \quad i \leq m.
\]

For any \( h \in W^{k,2}(\Sigma, u^* TM) \) we can write \( h = \sum_{i=1}^{2m} h^i \frac{\partial}{\partial x^i} \), with \( h^i \in W^{k,2}(\Sigma, \mathbb{R}) \). For fixed \( j \), denote by \( D_u^{(j)} \) the linearized operator of \( \partial_{j,j} \) at \( u \). Let \((s, t)\) be the local coordinates on \( \Sigma \) with \( j \frac{\partial}{\partial s} = \frac{\partial}{\partial t} \). Since

\[
D_u^{(j)} h = \frac{1}{2} (\nabla h + J(u) \nabla h \circ j) - \frac{1}{4} J(u) \nabla h J (du - J(u) du \circ j)
\]

we have

\[
D_u^{(j)} h \left( \frac{\partial}{\partial s} \right) = \frac{1}{2} \sum_{i=1}^{2m} h^i \frac{\partial^2 h^i}{\partial s \partial x^i} + J_0 \frac{\partial h^i}{\partial s} \frac{\partial}{\partial x^i} - \frac{1}{4} J(u) \nabla h J \left( \frac{\partial u^i}{\partial s} - J(u) \frac{\partial u^i}{\partial t} \right) \frac{\partial}{\partial x^i} + \sum_{i=1}^{2m} \left( J(u(q)) - J_0 \right) \frac{\partial h^i}{\partial s} \frac{\partial}{\partial x^i}.
\]

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Let \( (J^k_l) \) be the matrix such that \( \sum_{k=1}^{m} J^k_l \frac{\partial}{\partial x^l} := J(\frac{\partial}{\partial x^l}) \). Let \( \Gamma^k_l \) be the Christoffel symbol of a connection \( \nabla \) with respect to a local frame \( (\frac{\partial}{\partial x^l}) \), i.e., \( \nabla \frac{\partial}{\partial x^l} = \sum_{l=1}^{m} \Gamma^k_l \frac{\partial}{\partial x^l} \). Then we can write \( D_u h \left( \frac{\partial}{\partial x^l} \right) \) as

\[
2D_u^j h \left( \frac{\partial}{\partial s} \right) = 2 \sum_{i=1}^{m} \left( \frac{\partial h^j_i}{\partial s} + J_0 \frac{\partial h^j_i}{\partial t} \right) \frac{\partial}{\partial x^i} - \frac{1}{2} \sum_{i, k, l, a=1}^{m} h^j_i J^k_l(u) \left( \nabla \frac{\partial}{\partial x^l} J^k_a \right)_a \left( \frac{\partial u^a}{\partial s} - \sum_{e=1}^{m} J^a_e(u) \frac{\partial u^e}{\partial t} \right) \frac{\partial}{\partial x^i} + \sum_{i, k, l, a=1}^{m} h^j_i \left( \frac{\partial u^k}{\partial s} \Gamma^j_{kl} + \sum_{a=1}^{m} J^j_l(u) \frac{\partial u^k}{\partial t} \Gamma^a_{kl} \right) \frac{\partial}{\partial x^i} + \sum_{i, k=1}^{m} (J^j_i(u) - (J_0)^i_j) \frac{\partial h^j_i}{\partial t} \frac{\partial}{\partial x^i}.
\]

\( 2D_a^j h \left( \frac{\partial}{\partial s} \right) \) may simply be written as \( D^a_u h \) when no ambiguity can arise. In the matrix form, \( D_u^j \) can be written as

\[
D_u^j \left( \begin{array}{c} h^1 \\ \vdots \\ h^{2m} \end{array} \right) = \frac{\partial}{\partial s} \left( \begin{array}{c} h^1 \\ \vdots \\ h^{2m} \end{array} \right) + J_0 \frac{\partial}{\partial t} \left( \begin{array}{c} h^1 \\ \vdots \\ h^{2m} \end{array} \right) + F_u^1 \left( \begin{array}{c} h^1 \\ \vdots \\ h^{2m} \end{array} \right) + F_u^2 \left( \begin{array}{c} h^1 \\ \vdots \\ h^{2m} \end{array} \right)
\]

where \( F_u^1, F_u^2 \) are matrices given by

\[
(F_u^1)^i_j = 2 \sum_{k=1}^{m} \left( \frac{\partial u^k}{\partial s} \Gamma^i_{kl} + \sum_{a=1}^{m} J^j_l(u) \frac{\partial u^k}{\partial t} \Gamma^a_{kl} \right) - \frac{1}{2} \sum_{k, a=1}^{m} J^j_l(u) \left( \nabla \frac{\partial}{\partial x^l} J^k_a \right)_a \left( \frac{\partial u^a}{\partial s} - \sum_{e=1}^{m} J^a_e(u) \frac{\partial u^e}{\partial t} \right) - \sum_{i, k, l, a=1}^{m} h^j_i \left( \frac{\partial u^k}{\partial s} \Gamma^j_{kl} + \sum_{a=1}^{m} J^j_l(u) \frac{\partial u^k}{\partial t} \Gamma^a_{kl} \right) \frac{\partial}{\partial x^i} + \sum_{i, k=1}^{m} (J^j_i(u) - (J_0)^i_j) \frac{\partial h^j_i}{\partial t} \frac{\partial}{\partial x^i}.
\]

\( (F_u^2)^i_j = J^j_i(u) - (J_0)^i_j \)

6.2 Implicit function theorem

The following theorem is a restatement of Theorem A.3.3 and Proposition A.3.4 in [7].

**Theorem 6.1.** Let \( (A, \| \cdot \|_A), (X, \| \cdot \|_X) \) and \( (Y, \| \cdot \|_Y) \) be Banach spaces, \( U \subset X \) be open sets and \( V \subset A \), \( U \subset X \) be open sets and \( F : V \times U \rightarrow Y \) be a continuously differentiable map. For any \( (a, x) \in V \times U \) define

\[
D_a F(a, x)(g) = \frac{d}{dt} F(a + tg, x)|_{t=0}, \quad D_x F(a, x)(h) = \frac{d}{dt} F(a, x + th)|_{t=0}, \quad \forall \ g \in A, \ h \in X.
\]

Suppose that \( D_x F(a_0, x_0) \) is surjective and has a bounded linear right inverse \( Q_{(a_0, x_0)} : Y \rightarrow X \) with \( \| Q_{(a_0, x_0)} \| \leq C \) for some constant \( C > 0 \). Choose a positive constant \( \delta > 0 \) such that

\[
\| D_x F(a, x) - D_x F(a_0, x_0) \| \leq \frac{1}{2C}, \quad \forall \ x \in B_\delta(x_0, X), \ a \in B_\delta(a_0, A).
\]

where \( B_\delta(a_0, A) = \{ a \in A \mid \| a - a_0 \|_A \leq \delta \}, B_\delta(x_0, X) = \{ x \in X \mid \| x - x_0 \|_X \leq \delta \}. \) Suppose that \( x_1 \in X \) and \( a \in B_\delta(a_0, A) \) satisfies

\[
\| F(a, x_1) \|_Y < \frac{\delta}{4C}, \quad \| x_1 - x_0 \|_X \leq \frac{\delta}{8}.
\]

Then there exists a unique \( x \in X \) such that

\[
F(a, x) = 0, \quad x - x_1 \in im \ Q, \quad \| x - x_0 \|_X \leq \delta, \quad \| x - x_1 \|_X \leq 2C \| F(a, x_1) \|_Y.
\]

Moreover, if \( \| F(a_0, x_0) \|_Y \leq \frac{\delta}{4C} \), there exist a constant \( \delta' > 0 \) and a unique family differential map \( f_a : ker D_x F(a_0, x_0) \rightarrow Y \) such that for any \( (a, x) \in F^{-1}(0) \cap (B_\delta'(a_0, A) \times B_\delta'(x_0, X)) \), we have

\[
F(a, x) = 0 \iff x = x_0 + \zeta + Q_{(a_0, x_0)} \circ f_a(\zeta), \quad \zeta \in ker D_x F(a_0, x_0).
\]
The following is a version of the implicit function theorem with parameters. For the proof please see [6].

**Theorem 6.2.** $F$ satisfies the assumption of Theorem 6.1. If $F : V \times U \rightarrow Y$ is of class $C^\ell$, where $\ell$ is a positive integer, then there exists a constant $\delta' > 0$ such that $F^{-1}(0) |_{B_{\delta'}(a_o,A) \times B_{\delta'}(x_o,X)}$ is $C^\ell$ manifold, and $\xi \rightarrow x_o + \xi + Q \circ f_o(\xi)$ is a $C^\ell$-chart of $F^{-1}(0) |_{B_{\delta'}(a_o,A) \times B_{\delta'}(x_o,X)}$. In particular,

$$\|D_a (x_o + \xi + Q(a_o,x_o) \circ f_o(\xi)) \| \leq C,$$

where $C > 0$ is a constant depending only on $C_1$, $C$, $\delta'$, $\|f_o\|$ and $\|D^2_{a,x} F(a,x_o)\|$.

### 6.3 An isomorphism between $u^*TM \otimes \wedge^{0,1}_j T^* \Sigma$ and $u^*TM \otimes \wedge^{0,1}_j T^* \Sigma$

Let $J(\Sigma) \subset End(T \Sigma)$ denote the manifold of complex structures on $\Sigma$ and $j_o \in J(\Sigma)$. For any $j \in J(\Sigma)$ near $j_o$ we can write $j = (I + H)j_o(I + H)^{-1}$ where $H \in T_{j_o} J(\Sigma)$. We define two maps

$$\Psi_{j_o,j} : u^*TM \otimes \wedge^{0,1}_j T^* \Sigma \rightarrow u^*TM \otimes \wedge^{0,1}_j T^* \Sigma$$

and

$$\Psi_{j,j_o} : u^*TM \otimes \wedge^{0,1}_j T^* \Sigma \rightarrow u^*TM \otimes \wedge^{0,1}_j T^* \Sigma$$

by

$$\Psi_{j_o,j}(\eta) = \frac{1}{2}(\eta - \eta \cdot j_o j), \quad \Psi_{j,j_o}(\varpi) = \frac{1}{2}(\varpi - \varpi \cdot j j_o).$$

Since $J\eta = -\eta j_o$ and $J\varpi = -\varpi j_o$, one can check that $J\Psi_{j_o,j}(\eta) = -\Psi_{j_o,j}(\eta) j$ and $J\Psi_{j,j_o}(\varpi) = -\Psi_{j,j_o}(\varpi) j_o$. Then $\Psi_{j_o,j}$ and $\Psi_{j,j_o}$ are well defined.

**Lemma 6.3.** $\Psi_{j_o,j}$ is an isomorphism when $|H|$ small enough.

**Proof.** By the definition we have

$$\Psi_{j,j_o} \Psi_{j_o,j}(\eta) = \frac{1}{4}(2\eta - \eta \cdot (jj_o + joj))).$$

A direct calculation gives us

$$1 - C|H| \leq \|\Psi_{j,j_o}\| \leq 1 + C|H|$$

where $|H| = \sup_{p \in \Sigma, X \in T_{j_o} \Sigma} \left\{ |(HX,X)_{g_o(p)}| \right\} \left\{ |(X,X)_{g_o(p)}| = 1 \right\}$. Then $\Psi_{j,j_o} \Psi_{j_o,j}$ is an isomorphism as $|H|$ small enough. In particular, $\Psi_{j_o,j}$ is injective and $\Psi_{j,j_o}$ is surjective. Similarly $\Psi_{j_o,j} \Psi_{j,j_o}$ is also an isomorphism. Hence $\Psi_{j_o,j}$ and $\Psi_{j,j_o}$ are isomorphisms. $\Box$

$\Psi_{j_o,j}$ induces an isomorphism

$$\Psi_{j_o,j} : W^{k-1,2,\alpha}(\Sigma, u^*TM \otimes \wedge^{0,1}_j T^* \Sigma) \rightarrow W^{k-1,2,\alpha}(\Sigma, u^*TM \otimes \wedge^{0,1}_j T^* \Sigma)$$

in a natural way.

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