OPTIMAL CONTROL OF AN ALLEN-CAHN MODEL FOR TUMOR GROWTH THROUGH SUPPLY OF CYTOTOXIC DRUGS

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ABSTRACT. Our aim in this paper is to study an optimal control problem for a tumor growth model. The state system couples an Allen-Cahn equation and a reaction diffusion equation that models the evolution of tumor in the presence of nutrient supply. Elimination of cancer cells via cytotoxic drug is considered and the concentration of the cytotoxic drug is represented as a control variable.

2020 Mathematics Subject Classification. 35Q93, 35K20, 49K20, 49J20, 92C50.
Key words and phrases. Glioma treatment, Allen-Cahn equation, cytotoxic drugs, first order necessary optimality conditions.

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To achieve the desired tumor density with an optimal drug dosage, we consider a cost functional that depends on a free time variable representing the treatment time to be optimized.

1. Introduction.

1.1. Literature. Nowadays, various strategies are being developed in the fight against cancer, including experimental and theoretical techniques.

From the first decades of the twentieth century to the present day, and given the illustration of experimental results from various fields of cancer research, scientists have resorted to mathematical modeling. Thus, the power of mathematical modeling is used by scientists to distinguish between different mechanisms underlying important aspects of tumor development. Applied mathematics has the potential to reduce experimentation and provide scientists with information that can help control tumors by developing mathematical models that describe tumor growth (see [2, 20, 3, 13]).

The integration of mathematical investigations and experimental work have together modeled our understanding of tumor development and helped achieve some cancer treatments. So far, collaborations between biologists and clinicians, in addition to the development of mathematical models of tumor growth, may make it possible to optimize treatment for each individual (see [5, 7, 4, 6, 11]).

The Allen-Cahn equation (see [8])

\[
\frac{\partial u}{\partial t} - \Delta u + f(u) = 0, \quad \text{in } \Omega \times (0, T),
\]

(1)
is important in materials science, where \( \Omega \) represents the volume occupied by the material, \( u \) is an order parameter corresponding, for example, to the ordering of atoms per unit cell in a crystal lattice, and the function \( f \) is the derivative of a double-well potential \( F \) whose wells correspond to the phases of the material and is given by

\[
F(s) = \frac{1}{4} (s^2 - 1)^2.
\]

Furthermore, this equation has also been applied in the modeling of tumor growth (see, e.g., [17]).

The Cahn-Hilliard equation

\[
\frac{\partial u}{\partial t} = -\Delta (\Delta u - f(u)),
\]

was first introduced by Cahn and Hilliard (see [9]) to explain the phenomenon of spinodal decomposition observed in binary metal alloys.

The phase-field theory is used to derive models for problems with moving interfaces; specifically, they have been considered in the studies of tumor growth (see [24, 12, 25, 32]).

The complexity of oncology has attracted the interest of mathematicians to guide the experimental research necessary for therapy development. Mathematical models, especially those involving phase separation models, have been used to help develop therapeutic strategies for cancer (see [17, 10]). In particular, the authors in [28] have considered the following Cahn-Hilliard model for tumor growth.
\begin{align}
\begin{aligned}
\partial_t \phi &= \Delta \mu + (P \sigma - A - \alpha u) h(\varphi), \quad \text{in } \Omega \times (0, T), \\
\mu &= A \Psi'(\varphi) - B \Delta \varphi, \quad \text{in } \Omega \times (0, T), \\
\partial_t \sigma &= \Delta \sigma - C \sigma h(\varphi) + \beta (\sigma_s - \sigma), \quad \text{in } \Omega \times (0, T), \\
\partial_{\nu} \varphi = \partial_{\nu} \sigma = \partial_{\nu} \mu &= 0, \quad \text{on } \Gamma \times (0, T), \\
\varphi(0) &= \varphi_0, \quad \sigma(0) = \sigma_0, \quad \text{in } \Omega,
\end{aligned}
\end{align}

along with the relaxed cost functional

\begin{align}
\mathcal{J}_r(\varphi, u, \tau) = \frac{\beta Q}{2} \int_0^T \int_\Omega |\varphi - \varphi_Q|^2 dx dt + \frac{\beta \Omega}{2} \int_{\tau - \tau}^\tau \int_\Omega |\varphi - \varphi_{\Omega}|^2 dx dt \\
+ \frac{\beta \sigma_1}{2} \int_{\tau - \tau}^\tau \int \Omega (1 + \varphi) dx dt + \frac{\beta u}{2} \int_0^T \int_\Omega |u|^2 dx dt + \beta \tau \tau.
\end{align}

These authors have shown the existence of a solution for the problem \( \min \mathcal{J}_r(\varphi, u, \tau) \) associated to \((2)\). In addition, they have derived a simplified first-order necessary optimality condition.

Many mathematical models involving optimal control for tumor models have been studied. A distributed optimal control problem for a nonlocal convective Cahn-Hilliard equation with degenerate mobility and singular potential in three spatial dimensions is studied in [34]. A distributed optimal control of the Cahn-Hilliard system including the case of a double homogeneous energy density, where a first-order optimality condition for the original problem was derived by a boundary value process, was also studied in [29]. A distributed optimal control problem for a diffuse interface model of tumor growth was studied in [15]. We can also refer the reader to many problems dealing with tumor growth models [21, 22, 27, 18, 16, 14, 30, 26, 33, 28, 1] and the references therein.

We are interested in studying tumor growth based on a phase-field model. For this purpose, we proceed in this work and leverage from the mathematical model presented in [28].

1.2. Position of our problem. Cancer treatments include surgery, immunotherapy (boosting the immune system), radiation therapy (using radiation to kill cancer cells), and chemotherapy (using drugs to kill cancer cells). The latter three treatments are used in cycles, where a cycle is a period of treatment followed by an extended period of rest to allow the patient’s body to produce new healthy cells. The goal of these treatments is to reduce the size of the tumor until a surgery can be performed. Further treatments may be necessary to eliminate the cancer cells that remain after surgery.

Cancer drugs are known to cause the death of rapidly dividing normal cells, such as in bone marrow, hair follicles, which impairs the immune system and is fatal for the patient. In addition, a high dose of these drugs may cause resistance to treatment, so the shortest treatment time along with the optimal drug dose must be found. Thus, to optimize the growth of the tumor, the final distribution of cancer cells, the dose of drug administered to the patient, and the treatment time of a cycle, we consider a system that couples the variation of tumor density with the concentration of nutrient delivery to the tumor cells in the presence of cytotoxic drug (control). For \( T > 0 \), which is the maximum treatment time, in a bounded domain \( \Omega \subset \mathbb{R}^n, \ n = 1, 2, 3 \) with \( C^2 \) boundary \( \Gamma \), we have

\begin{align}
\partial_t \varphi - B \Delta \varphi + f(\varphi) = (P \sigma - A - \alpha u) h(\varphi), \quad \text{in } \Omega \times (0, T) =: Q,
\end{align}
\[
\frac{\partial t}{\partial t} \sigma = \Delta \sigma - C \sigma h(\varphi) + \beta(\sigma_s - \sigma), \quad \text{in} \quad Q, \\
\frac{\partial \nu}{\partial t} \varphi = \frac{\partial \nu}{\partial t} \sigma = 0, \quad \text{on} \quad \Gamma \times (0, T), \\
\varphi(0) = \varphi_0, \quad \sigma(0) = \sigma_0, \quad \text{in} \quad \Omega.
\]

Here, \(\varphi = \varphi_C - \varphi_D\), where \(\varphi_C\) and \(\varphi_D\) are the concentrations of phases \(C\) and \(D\), respectively. The double-well potential \(F(s) = 4^{-1} (s^2 - 1)^2\) of derivative \(f\) allows the coexistence of tumor and healthy cells, so that \(f(s) = s^3 - s\). The constant \(\alpha\) is positive, \(B = \lambda l^2\) is the diffusion coefficient of tumor cells, where \(\lambda\) and \(l\) denote tumor mobility and interface width, respectively. \(\sigma\) is the concentration of an unspecified chemical species that serves as a nutrient for tumor cells, while \(u\) denotes the concentration of cytotoxic drugs. The function \(h(\varphi)\) verifying \(h(-1) = 0\) and \(h(1) = 1\) is an interpolation function, and the parameters \(P, A, C, \beta\) denote the constant proliferation rate, apoptosis rate, nutrient consumption rate and nutrient supply rate, respectively. The term \(h(\varphi)P\sigma\) models the proliferation of tumor cells which is proportional to the concentration of nutrient, the term \(h(\varphi)A\) models the apoptosis of tumor cells, and \(C h(\varphi)\sigma\) models the consumption of nutrient by tumor cells only. The term \(\alpha u h(\varphi)\) models the elimination of tumor cells by the cytotoxic drug at a constant rate \(\alpha\). On the other hand, \(\sigma_s\) denotes the nutrient concentration in a pre-existing vasculature, and \(\beta(\sigma_s - \sigma)\) models the delivery of nutrients from blood vessels when \(\sigma_s > \sigma\) and the removal of nutrients from the domain \(\Omega\) when \(\sigma_s < \sigma\). In this work, the function \(u\) will act as our control. For realistic applications, the control \(u : [0, T] \rightarrow [0, 1]\), should be spatially constant, where \(u = 1\) represents full dosage and \(u = 0\) represents no dosage.

**Remark 1.**

1. When the right-hand side of (3) vanishes, we obtain the Allen-Cahn Equation (1).
2. Note that, in [28], the authors considered a Cahn-Hilliard equation coupled with a diffusion equation for tumor growth. However, in our work, we consider coupled Allen-Cahn and diffusion equations which can also be applied to tumor growth. Compared to the Cahn-Hilliard model studied in [28], this allows to simplify the mathematical analysis, while still keeping important aspects of phase separation. The numerical analysis of the model and simulations will be addressed in future works.

In Equation (3), the function \(h\) is defined over \(\mathbb{R}\), but we are only interested here in the physical domain, which is \([-1, 1]\), and since the choice of \(h\) is not unique, we can take it, for example, as in Figure 1 illustrating the phase transition.

**Figure 1.** Some examples for the interpolation function \(h\)
Now we consider the objective functional $J_r$ as in [28]. For positive constants $r, \beta_u, \beta_T$, and nonnegative constants $\beta_Q, \beta_\Omega, \beta_S$

$$J_r(\varphi, u, \tau) = \frac{\beta_Q}{2} \int_0^\tau \int_\Omega |\varphi - \varphi_Q|^2 dx dt + \frac{\beta_\Omega}{2r} \int_{\tau-r}^\tau \int_\Omega |\varphi - \varphi_\Omega|^2 dx dt$$

$$+ \frac{\beta_S}{2} \int_{\tau-r}^\tau \int_\Omega (1 + \varphi) dx dt + \frac{\beta_u}{2} \int_0^\tau \int_\Omega |u|^2 dx dt + \beta_T \tau. \tag{7}$$

Here, $\tau \in [r, T]$ represents treatment time, $\varphi_Q$ represents a desired evolution for the tumor cells, while $\varphi_\Omega$ represents a desired final distribution. The first two terms of $J_r$ are of standard tracking type often considered in the parabolic optimal control literature (see [35]) and the third term of $J_r$ measures the size of the tumor at the end of treatment. The fourth term penalizes high concentrations of the cytotoxic drug, and the fifth term of $J_r$ penalizes long treatment times (see [28]).

In this paper, we study the optimal control problem

minimize $J_r(\varphi, u, \tau)$ subject to (3) - (6), $u \in U_{ad}$, $\tau \in [r, T]$,

where the space of admissible controls $U_{ad}$ is defined by:

$$U_{ad} := \{ u \in L^\infty(Q) : 0 \leq u \leq 1, \text{a.e. in } Q \}.$$

Assumptions.

(A) The initial conditions satisfy $\varphi_0, \sigma_0 \in H^1(\Omega)$, with $0 \leq \sigma_0 \leq 1$ a.e. $x \in \Omega$. The target functions $\varphi_\Omega, \varphi_Q \in L^2(Q)$, and the vasculature nutrient concentration $\sigma_s$ satisfies $0 \leq \sigma_s \leq 1$, a.e. in $Q$.

(B) The interpolation function $h : \mathbb{R} \to [0, 1]$ is continuously differentiable and Lipschitz with Lipschitz constant $M$. In addition, the parameters $P, A, C, \beta$ are non negative constants, and $\alpha$ is a positive constant.

Throughout this work, the same letter $c$ (and, sometimes, $c', c'', c_1, c_2, c_{p,q}$) denotes a constant that may vary in the same line.

We denote by $\langle \phi \rangle$ the spatial average of a function $\phi$ in $L^1(\Omega)$,

$$\langle \phi \rangle = \frac{1}{\text{Vol}(\Omega)} \int_\Omega \phi \, dx.$$

2. Existence of solution. In this section, we study the existence of a unique weak solution of System (3)-(6). More precisely, we will prove the following theorem:

**Theorem 2.1** (Existence and uniqueness of weak solution). Assume that Assumptions (A) and (B) hold. Then, Problem (3)-(6) admits a unique weak solution $(\varphi, \sigma)$ such that $0 \leq \sigma \leq 1$ and

$$(\varphi, \sigma) \in \mathcal{Y} := (L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega)))^2.$$

Moreover, the control-to-state operator

$$S : L^2(Q) \rightarrow \mathcal{Y}$$

$$u \quad \mapsto \quad (\varphi, \sigma)$$

is continuous.
Proof. We employ the Faedo-Galerkin method (see [19]) to show the existence of solution of the parabolic System (3)-(6).

**Existence of weak solution of (3)-(6).**

The variational problems associated with (4) and (3), are given by:

\[
\int_{\Omega} (\partial_t \sigma) \xi \, dx + \int_{\Omega} \nabla \sigma \cdot \nabla \xi \, dx + \beta \int_{\Omega} \sigma \xi \, dx = -C \int_{\Omega} h(\phi) \sigma \xi \, dx + \beta \int_{\Omega} \sigma_s \xi \, dx \quad (8)
\]

and

\[
\int_{\Omega} (\partial_t \varphi) \xi \, dx + B \int_{\Omega} \nabla \varphi \cdot \nabla \xi \, dx + \int_{\Omega} f(\varphi) \xi \, dx = \int_{\Omega} (P \varphi h(\varphi) - Ah(\varphi) - \alpha uh(\varphi)) \xi \, dx,
\]

for almost every \( t \in (0, T) \) and for all \( \xi \in H^1(\Omega) \).

**A priori Estimates:**

**Estimate 1.** Formally putting \( \xi = 2\sigma \) in (8) and using Assumption (B), we get

\[
\frac{d}{dt} \| \sigma \|_{L^2(\Omega)}^2 + 2 \| \nabla \sigma \|_{L^2(\Omega)}^2 + 2\beta \| \sigma \|_{L^2(\Omega)}^2 \\
\leq C \| h(\phi) \|_{L^\infty} \| \sigma \|_{L^2(\Omega)}^2 + \beta^2 \| \sigma_s \|_{L^2(\Omega)}^2 + \| \sigma \|_{L^2(\Omega)}^2.
\]

Setting \( c_0 = \min(2, 2\beta) \) and integrating over \([0, t]\), we find

\[
\| \sigma(t) \|_{L^2(\Omega)}^2 + c_0 \left( \| \nabla \sigma \|_{L^2(0,t;L^2(\Omega))}^2 + \| \sigma \|_{L^2(0,t;L^2(\Omega))}^2 \right) \\
\leq c \| \sigma \|_{L^2(\Omega)}^2 + \beta^2 \| \sigma_s \|_{L^2(0,t;L^2(\Omega))}^2 + \| \sigma \|_{L^2(\Omega)}^2.
\]

Using Gronwall’s lemma, we have

\[
\| \sigma(t) \|_{L^2(\Omega)}^2 \leq \left( \beta^2 \| \sigma_s \|_{L^2(0,t;L^2(\Omega))}^2 + \| \sigma \|_{L^2(\Omega)}^2 \right) e^{ct}. \quad (10)
\]

Putting \( \xi = 2\varphi \) in (9), we find

\[
\frac{d}{dt} \| \varphi \|_{L^2(\Omega)}^2 + 2B \| \nabla \varphi \|_{L^2(\Omega)}^2 \leq \| h(\phi) \|_{L^\infty} \left( P \| \sigma \|_{L^2(0,t;L^2(\Omega))}^2 + \alpha^2 \| u \|_{L^2(\Omega)}^2 \right) \\
+ A^2 \| h(\phi) \|_{L^2(\Omega)}^2 + \| \varphi \|_{L^2(\Omega)}^2.
\]

From (10), we know that

\[
\| \sigma \|_{L^2(0,t;L^2(\Omega))}^2 = \int_0^t \| \sigma(s) \|_{L^2(\Omega)}^2 \, ds \leq \frac{c}{c}(e^{ct} - 1).
\]

Now, integrating over \([0, t]\) and since \( h \) is \( C^1, u \in U_{ad} \), we have

\[
\| \varphi(t) \|_{L^2(\Omega)}^2 + 2B \| \nabla \varphi \|_{L^2(0,t;L^2(\Omega))}^2 \leq A^2 \| h(\phi) \|_{L^2(0,t;L^2(\Omega))}^2 + \| \varphi_0 \|_{L^2(\Omega)}^2 + \| h(\phi) \|_{L^\infty} \left( P \| \sigma \|_{L^2(0,t;L^2(\Omega))}^2 + \alpha^2 \| u \|_{L^2(0,t;L^2(\Omega))}^2 \right) \\
+ \| \varphi \|_{L^2(0,t;L^2(\Omega))}^2 (1 + \| h(\phi) \|_{L^\infty}).
\]

Thanks to Gronwall’s inequality, we infer

\[
\| \varphi(t) \|_{L^2(\Omega)}^2 + 2B \| \nabla \varphi \|_{L^2(0,t;L^2(\Omega))}^2 \leq c + ce^{ct}.
\]
Estimate 2. Putting $\xi = 2\partial_t \sigma$ in (8), using Young’s inequality, and Assumption (B), we find

$$2 \|\partial_t \sigma\|^2_{L^2(\Omega)} + \frac{d}{dt} \|\nabla \sigma\|^2_{L^2(\Omega)} + \beta \frac{d}{dt} \|\sigma\|^2_{L^2(\Omega)} \leq C \|h(\varphi)\| \frac{d}{dt} \|\sigma\|^2_{L^2(\Omega)} + \beta^2 \|\sigma_s\|^2_{L^2(\Omega)} + \|\partial_t \sigma\|^2_{L^2(\Omega)}.$$

Setting $c_1 = \min(1, \beta)$, integrating over $[0, t]$ and using (10), we find

$$\|\partial_t \sigma\|^2_{L^2(0, t; L^2(\Omega))} + c_1 \|\sigma\|^2_{L^2(0, t; H^1(\Omega))} \leq c_0 \|\sigma_0\|^2_{H^1(\Omega)} + \beta^2 \|\sigma_s\|^2_{L^2(0, t; L^2(\Omega))} + e^{ct}.$$

Putting $\xi = 2\partial_t \varphi$ in (9), we obtain

$$2 \|\partial_t \varphi\|^2_{L^2(\Omega)} + B \frac{d}{dt} \|\nabla \varphi\|^2_{L^2(\Omega)} \leq \|h(\varphi)\| \frac{d}{dt} \|\varphi\|^2_{L^2(\Omega)} + \|\partial_t \varphi\|^2_{L^2(\Omega)} + \|\varphi\|^2_{L^2(\Omega)} \leq \|\varphi\|^2_{L^2(\Omega)} + \|\nabla \varphi\|^2_{L^2(\Omega)} + \|\partial_t \varphi\|^2_{L^2(\Omega)}.$$

Equivalently

$$\|\partial_t \varphi\|^2_{L^2(\Omega)} + B \frac{d}{dt} \|\nabla \varphi\|^2_{L^2(\Omega)} \leq \|h(\varphi)\| \left( \mathcal{P}^2 \|\sigma\|^2_{L^2(\Omega)} + \alpha^2 \|u\|^2_{L^2(\Omega)} + \|\partial_t \varphi\|^2_{L^2(\Omega)} \right) + \mathcal{A}^2 \|h(\varphi)\|^2_{L^2(\Omega)} + \frac{d}{dt} \|\varphi\|^2_{L^2(\Omega)}.$$

Multiplying Equation (3) by $\frac{1}{\text{Vol}(\Omega)}$ and integrating over $\Omega$ yield

$$\frac{d}{dt} \langle \varphi \rangle = -\langle f(\varphi) \rangle + \langle (\mathcal{P}\sigma - \mathcal{A}u) h(\varphi) \rangle.$$

We already know that

$$\frac{d}{dt} \langle \varphi \rangle^2 = 2\langle \varphi \rangle \frac{d}{dt} \langle \varphi \rangle \leq \langle \varphi \rangle^2 + \left( \frac{d}{dt} \langle \varphi \rangle \right)^2 \leq \langle \varphi \rangle^2 + \langle f(\varphi) \rangle^2 + \langle (\mathcal{P}\sigma - \mathcal{A}u) h(\varphi) \rangle^2.$$

Using the facts that

$$\langle \varphi \rangle^2 \leq \|\varphi\|^2_{H^1(\Omega)}$$

and

$$\langle f(\varphi) \rangle^2 \leq \|\varphi\|^2_{H^1(\Omega)}$$

we write

$$\frac{d}{dt} \langle \varphi \rangle^2 \leq \|\varphi\|^2_{L^2(\Omega)} + \|\varphi\|^2_{H^1(\Omega)} + \frac{\|h(\varphi)\|^2_{L^2(\Omega)}}{\text{Vol}(\Omega)} \left( \mathcal{P}^2 \|\sigma\|^2_{L^2(\Omega)} + \mathcal{A}^2 \text{Vol}(\Omega)^2 + \alpha^2 \|u\|^2_{L^2(\Omega)} \right).$$
Estimate 3. Equations (3) and (4) can be written in the following forms: since \((\sigma, \varphi)\) (see [23]). Therefore, we get

\[
2 \left\| \partial_t \varphi \right\|_{L^2(\Omega)}^2 + c' \left\| \varphi(t) \right\|_{H^1(\Omega)}^2 \leq c t.
\]

The last inequality yields that \(\varphi \in L^\infty(0, t; H^1(\Omega))\) and \(\partial_t \varphi \in L^2(0, t; L^2(\Omega))\).

Estimate 3. Equations (3) and (4) can be written in the following forms:

\[
\begin{align*}
-\Delta \sigma &= -\partial_t \sigma - C\sigma h(\varphi) + \beta (\sigma_s - \sigma), & \text{in } Q, \\
- B \Delta \varphi &= \partial_t \varphi + \varphi - \varphi^3 + (P\sigma - A - \alpha u) h(\varphi), & \text{in } Q.
\end{align*}
\]

Since the right hand sides are in \(L^2(\Omega)\) for a.e. \(t \in (0, T)\) and \(\partial_t \sigma = \partial_t \varphi = 0\) on \(\Gamma\), then the elliptic regularity (see [31]) yields, \((\sigma, \varphi) \in (H^2(\Omega))^2\) for a.e. \(t \in (0, T)\), and there exists \(K > 0\) such that

\[
\left\| \sigma \right\|_{H^2(\Omega)}^2 \leq K \left( \left\| \partial_t \sigma \right\|_{L^2(\Omega)}^2 + \left\| \sigma \right\|_{L^2(\Omega)}^2 + \left\| \sigma_s \right\|_{L^2(\Omega)}^2 \right)
\]

and

\[
\left\| \varphi \right\|_{H^2(\Omega)}^2 \leq K \left( \left\| \partial_t \varphi \right\|_{L^2(\Omega)}^2 + \left\| \sigma \right\|_{L^2(\Omega)}^2 + \left\| \sigma h(\varphi) \right\|_{L^2(\Omega)}^2 + \left\| u \right\|_{L^2(\Omega)}^2 + \left\| f(\varphi) \right\|_{L^2(\Omega)}^2 \right).
\]

It follows, from Estimates 1 and 2, that \((\sigma, \varphi) \in L^2(0, T; H^2(\Omega))^2\). In particular, since \((\partial_t \sigma, \partial_t \varphi) \in L^2(0, T; L^2(\Omega))^2\), then, using interpolation result, we deduce that

\[
(\sigma, \varphi) \in C^0([0, T]; H^1(\Omega))^2
\]

(see [23]). Therefore, we get

\[
(\sigma, \varphi) \in \left(L^2(0, T; H^2(\Omega)) \cap H^1(0, T; L^2(\Omega)) \cap C^0([0, T]; H^1(\Omega))\right)^2.
\]

We will use Faedo-Galerkin method (see [19]) to show the existence of weak solution of the parabolic System (3)-(6). First, the variational formulations associated with (3) and (4) are given by

\[
\int_{\Omega} \partial_t \varphi \xi dx + B \int_{\Omega} \nabla \varphi \cdot \nabla \xi dx + \int_{\Omega} f(\varphi) \xi dx = \mathcal{P} \int_{\Omega} \sigma h(\varphi) \xi dx - \mathcal{A} \int_{\Omega} h(\varphi) \xi dx - \alpha \int_{\Omega} u h(\varphi) \xi dx
\]

and

\[
\int_{\Omega} \partial_t \sigma \xi dx + \int_{\Omega} \nabla \sigma \cdot \nabla \xi dx = - \int_{\Omega} (C h(\varphi) + \beta) \sigma \xi dx + \beta \int_{\Omega} \sigma_s \xi dx
\]

for almost every \(t \in (0, T)\) and for all \(\xi \in H^1(\Omega)\).
The variational approximate problem.

Let \( \omega_j \) be an eigenfunction of the following problem
\[
\begin{aligned}
-\Delta \omega_j &= \lambda_j^2 \omega_j, & \text{in } & \Omega, \\
\partial_\nu \omega_j &= 0, & \text{on } & \Gamma,
\end{aligned}
\]
such that
\[
\|\omega_j\|_{L^2(\Omega)}^2 = \int_\Omega |\omega_j|^2 \, dx = 1.
\]
The operator \( L_N := -\Delta \) with Neumann boundary condition is self adjoint with compact resolvent from \( L^2(\Omega) = \{ u \in L^2(\Omega) : \langle u \rangle = 0 \} \) into itself, then its eigenfunctions \( \{ \omega_j \}_{j=1}^\infty \) corresponding to the eigenvalues \( \{ \lambda_j \}_{j=1}^\infty \) form an orthonormal basis in \( L^2(\Omega) \) and an orthogonal basis in
\[
D(L_N^{1/2}) = H^1_N(\Omega) = \{ \varphi \in H^1(\Omega) : \langle \varphi \rangle = 0 \text{ and } \partial_\nu \varphi = 0 \text{ on } \Gamma \}.
\]
Consider the finite dimensional eigenspaces \( V_n := \text{span} \{ \omega_j, \; j = 1, \ldots, n \} \) and the projection \( P_n \), then \( V_n \subset V_{n+1} \) and \( \cup V_n = H^1_N(\Omega) \) (see [19]). Set \( \varphi_0^n = P_n \varphi_0, \sigma_0^n = P_n \sigma_0, \sigma^n = P_n \sigma_s \), and \( u_n = P_n u \), such that \( u_n \to u \) strongly in \( L^2(0;T;L^2(\Omega)) \), \( \sigma_n \to \sigma_s \) in \( L^2(0,T;L^2(\Omega)) \), \( \sigma_0^n \to \sigma_0 \) in \( L^2(\Omega) \) and \( \varphi_0^n \to \varphi_0 \) in \( L^2(\Omega) \).

The approximated variational problem is given by
\[
\begin{aligned}
\int_\Omega \partial_t \varphi_n \omega_j \, dx + B \int_\Omega \nabla \varphi_n \cdot \nabla \omega_j \, dx + \int_\Omega f(\varphi_n) \omega_j \, dx \\
= & \mathcal{P} \int_\Omega \sigma_n h(\varphi_n) \omega_j \, dx - A \int_\Omega h(\varphi_n) \omega_j \, dx - \alpha \int_\Omega u_n h(\varphi_n) \omega_j \, dx
\end{aligned}
\]
and
\[
\begin{aligned}
\int_\Omega \partial_t \sigma_n \omega_j \, dx + \int_\Omega \nabla \sigma_n \cdot \nabla \omega_j \, dx &= - \int_\Omega (Ch(\varphi_n) + \beta) \sigma_n \omega_j \, dx + \beta \int_\Omega \sigma^n \omega_j \, dx,
\end{aligned}
\]
where \( \varphi_n = \sum_{i=1}^n a_{n,i} \omega_i \), \( \sigma = \sum_{i=1}^n b_{n,i} \omega_i \in V_n \), \( a_{n,i} = \langle \varphi_n, \omega_i \rangle \) and \( b_{n,i} = \langle \sigma_n, \omega_i \rangle \) for \( i = 1, \ldots, n \). We also have \( \| P \varphi_0 \|_{H^1(\Omega)} \leq \| \varphi_0 \|_{H^1(\Omega)} \) according to Bessel’s inequality. Equations (13) and (14) are equivalent to find \( a_n(t) \) and \( b_n(t) \) satisfying
\[
\frac{d}{dt} \sum_{i=1}^n a_{n,i} \int_\Omega \omega_i \omega_j \, dx + B \sum_{i=1}^n a_{n,i} \int_\Omega \nabla \omega_i \cdot \nabla \omega_j \, dx
\]
\[
+ \sum_{i=1}^n a_{n,i} \left( \sum_{i=1}^n a_{n,i} \omega_i - 1 \right)^2 \omega_i \omega_j \, dx = \mathcal{P} \sum_{i=1}^n b_{n,i} \int_\Omega h(\sum_{i=1}^n a_{n,i} \omega_i) \omega_i \omega_j \, dx
\]
\[
- A \int_\Omega \sum_{i=1}^n a_{n,i} \omega_i \omega_j \, dx - \alpha \int_\Omega u_n h(\sum_{i=1}^n a_{n,i} \omega_i) \omega_j \, dx + \sum_{i=1}^n a_{n,i} \int_\Omega \omega_i \omega_j \, dx
\]
and
\[
\frac{d}{dt} \sum_{i=1}^n b_{n,i} \int_\Omega \omega_i \omega_j \, dx + \sum_{i=1}^n b_{n,i} \int_\Omega \nabla \omega_i \cdot \nabla \omega_j \, dx
\]
\[
= - \sum_{i=1}^n b_{n,i} \int_\Omega \left( Ch(\sum_{i=1}^n a_{n,i} \omega_i) + \beta \right) \omega_i \omega_j \, dx + \beta \int_\Omega \sigma^n \omega_j \, dx, \quad \omega_j, \omega_i \in V_n.
\]
(15) and (16) can be written in vector form as follows

\[
\frac{da_n(t)}{dt} I_n = -Ba_n(t)W_n + \mathcal{P}b_n(t)M_n - AH_n - \alpha U_n + a_n(t)f I_n
\]

and

\[
\frac{db_n(t)}{dt} I_n = -b_n(t)W_n - b_n(t)J_n + \beta S_n,
\]

where

\[
\begin{align*}
(I_n)_{i,j} &= \int_\Omega \omega_i \omega_j dx, \quad (W_n)_{i,j} = \int_\Omega \nabla \omega_i \cdot \nabla \omega_j dx, \\
(M_n)_{i,j} &= \int_\Omega h \left( \sum_{i=1}^n a_{n,i} \omega_i \right) \omega_i \omega_j dx, \quad (H_n)_j = \int_\Omega h \left( \sum_{i=1}^n a_{n,i} \omega_i \right) \omega_j dx, \\
(J_n)_{i,j} &= \int_\Omega \left( Ch \sum_{i=1}^n a_{n,i} \omega_i \right) \omega_i \omega_j dx, \quad (S_n)_j = \int_\Omega \sigma_n \omega_j dx, \\
(U_n)_j &= \int_\Omega u_n h \left( \sum_{i=1}^n a_{n,i} \omega_i \right) \omega_j dx.
\end{align*}
\]

Recall that \(0 \leq \sigma_n \leq 1\) and \(0 \leq u_n \leq 1\), a.e. in \(Q\) and \(h\) is continuously differentiable with values between 0 and 1, so we have a system of ODEs in the following form

\[
\begin{align*}
a'_n(t) &= f(t, a_n(t), b_n(t)) \\
b'_n(t) &= g(t, a_n(t), b_n(t)).
\end{align*}
\]

It is obvious that \(F = (f, g)\) is locally Lipschitz with respect to the second variable, and the system of ODEs has a unique local solution \((a_n, b_n) \in [0, T^*]^2\) for all \(T^* \in [0, T]\). Multiplying Equation (16) by \(b_n, j\) and summing over \(j\) from 1 to \(n\), we find

\[
\|
\|
\|
\|
\]

\[
\|
\|
\|
\|
\]

It follows, that \(\varphi_n\) is bounded in \(L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))\). Therefore the solution \((a_n(t), b_n(t))\) is global in \([0, T]\). Multiply Equation (16) by \(\sigma_n, j\) and sum over \(j\) from 1 to \(n\), we get

\[
\|
\|
\|
\|
\]

We deduce that \(\varphi_n\) is bounded in \(L^2(0, T; L^2(\Omega))\). It follows from (17) that there exists a relabeled subsequence \(\{\sigma_n\}_n\) such that \(\sigma_n \rightharpoonup \sigma\) weakly in \(L^2(0, T; H^1(\Omega))\). Multiply Equation (15) by \(\frac{da_n, j(t)}{dt}\) and sum over \(j\) from 1 to \(n\), we have

\[
\|
\|
\|
\|
\]

\[
\|
\|
\|
\|
\]
Consequently, $\partial_t\varphi_n$ is bounded in $L^2(0,T;L^2(\Omega))$, so there exists a relabeled subsequence \( \{ \frac{d\varphi_n}{dt} \} \) such that \( \frac{d\varphi_n}{dt} \rightharpoonup y \) weakly in $L^2(0,T;L^2(\Omega))$. It also follows from (18) that there exists a relabeled subsequence $\{ \varphi_n \}_n$ such that $\varphi_n \rightharpoonup \varphi$ weakly in $L^2(0,T;H^1(\Omega))$.

**Passing to the limit.**

Assume that $\psi \in C_0^\infty(0,T;H^1(\Omega))$, we have

\[
\int_0^T \int_\Omega \frac{d\varphi_n}{dt} \psi(t) dx dt = - \int_0^T \int_\Omega \varphi_n(t) \psi'(t) dx dt
\]

as $n \to \infty$, 

\[
- \int_0^T \int_\Omega \varphi(t) \psi'(t) dx dt = \int_0^T \int_\Omega \frac{d\varphi}{dt} \psi(t) dx dt.
\]

So, we deduce that $y = \frac{d\varphi}{dt}$. On the other hand

\[
\int_0^T \int_\Omega \nabla \varphi_n \cdot \psi dx dt = - \int_0^T \int_\Omega \varphi_n \nabla \psi dx dt
\]

as $n \to \infty$, 

\[
- \int_0^T \int_\Omega \varphi \nabla \psi dx dt = \int_0^T \int_\Omega \nabla \varphi \cdot \psi dx dt.
\]

Furthermore, we have that $\varphi_n$ is bounded in $L^2(0,T;H^1(\Omega))$ compactly embedded in $L^2(0,T;L^2(\Omega))$, then $\varphi_n \rightharpoonup \varphi$ strongly in $L^2(0,T;L^2(\Omega))$, and $h$ is Lipschitz, we infer

\[
h(\varphi_n) \rightharpoonup h(\varphi) \quad \text{strongly in } L^2(0,T;L^2(\Omega))
\]

and

\[
f(\varphi_n) \rightharpoonup f(\varphi) \quad \text{strongly in } L^2(0,T;L^2(\Omega)).
\]

Therefore, (13) converges weakly to

\[
\int_0^T \int \frac{d\varphi}{dt} \psi dx dt + B \int_0^T \int \nabla \varphi \cdot \nabla \psi dx dt + \int_0^T \int f(\varphi) \psi dx dt
\]

\[
= \int_0^T \int (P\sigma - A - \alpha u) h(\varphi) \psi dx dt.
\]

Since $\varphi \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$, then $\varphi \in C([0,T];H^1(\Omega))$. Now choose a function $\psi \in C^1([0,T];H^1(\Omega))$ with $\psi(T) = 0$ in the above equation, we get

\[
\int_0^T \int \varphi \psi' dx dt + B \int_0^T \int \nabla \varphi \cdot \nabla \psi dx dt + \int_0^T \int f(\varphi) \psi dx dt
\]

\[
= \int_0^T \int (P\sigma - A - \alpha u) h(\varphi) \psi dx dt.
\]

Thus, we have

\[
- \int_0^T \int \varphi \psi' dx dt + B \int_0^T \int \nabla \varphi \cdot \nabla \psi dx dt + \int_0^T \int f(\varphi) \psi dx dt
\]

\[
= \int_0^T \int (P\sigma - A - \alpha u) h(\varphi) \psi dx dt + \int \varphi(0) \psi(0) dx.
\]
Similarly, we have in (13) that
\[-\int_0^T \int_\Omega \varphi_n \psi' \, dx \, dt + B \int_0^T \int_\Omega \nabla \varphi_n \cdot \nabla \psi \, dx \, dt + \int_0^T \int_\Omega f(\varphi_n) \psi \, dx \, dt = \int_0^T \int_\Omega (P\sigma_n - A - \alpha u_n) h(\varphi_n) \psi \, dx \, dt + \int_\Omega \varphi_n(0) \psi(0) \, dx,\]
so that by passing to the limit in this equation, we obtain
\[-\int_0^T \int_\Omega \varphi' \psi \, dx \, dt + B \int_0^T \int_\Omega \nabla \varphi \cdot \nabla \psi \, dx \, dt + \int_0^T \int_\Omega f(\varphi) \psi \, dx \, dt = \int_0^T \int_\Omega (P\sigma - A - \alpha u) h(\varphi) \psi \, dx \, dt + \int_\Omega \varphi(0) \psi(0) \, dx.\]
Note that after subtracting (19) and the above equation, we obtain \(\int_\Omega (\varphi(0) - \varphi_0) \psi(0) \, dx = 0\). Then, we deduce that \(\varphi(0) = \varphi_0\) a.e. in \(L^2(\Omega)\). Similarly (14) converges weakly to
\[-\int_0^T \int_\Omega \sigma' \psi \, dx \, dt + \int_0^T \int_\Omega \nabla \sigma \cdot \nabla \psi \, dx \, dt = \int_0^T \int_\Omega \sigma h(\varphi) \psi \, dx \, dt - A \int_0^T \int_\Omega h(\varphi) \psi \, dx \, dt - \int_\Omega \sigma(0) \psi(0) \, dx.\]
Which is equivalent to
\[-\int_0^T \int_\Omega \sigma' \psi \, dx \, dt + \int_0^T \int_\Omega \nabla \sigma \cdot \nabla \psi \, dx \, dt = \int_0^T \int_\Omega \sigma h(\varphi) \psi \, dx \, dt - A \int_0^T \int_\Omega h(\varphi) \psi \, dx \, dt - \int_\Omega \sigma(0) \psi(0) \, dx.\]
Back to (14), we have
\[-\int_0^T \int_\Omega \sigma_n' \psi \, dx \, dt + \int_0^T \int_\Omega \nabla \sigma_n \cdot \nabla \psi \, dx \, dt = \int_0^T \int_\Omega \sigma_n h(\varphi_n) \psi \, dx \, dt - A \int_0^T \int_\Omega h(\varphi_n) \psi \, dx \, dt - \int_\Omega \sigma_n(0) \psi(0) \, dx.\]
Finally, by passing to the limit in this equation, we obtain
\[-\int_0^T \int_\Omega \sigma' \psi \, dx \, dt + \int_0^T \int_\Omega \nabla \sigma \cdot \nabla \psi \, dx \, dt = \int_0^T \int_\Omega \sigma h(\varphi) \psi \, dx \, dt - A \int_0^T \int_\Omega h(\varphi) \psi \, dx \, dt - \int_\Omega \sigma(0) \psi(0) \, dx.\]
Hence, by subtracting (20) and the above equation, we find that \(\sigma(0) = \sigma_0\) a.e. in \(L^2(\Omega)\).

**Boundedness property of \(\sigma\).**

Recall that, \(0 \leq \sigma_0 \leq 1\). Let \(\sigma^- = \min(0, -\sigma)\) and substitute \(\xi\) in Equation (12) by \(-\sigma^- \in H^1(\Omega)\), we obtain
\[
\frac{1}{2} \frac{d}{dt} \|\sigma^-\|^2_{L^2(\Omega)} = -\|\nabla \sigma^-\|^2_{L^2(\Omega)} - C \int_\Omega h(\varphi) |\sigma^-|^2 \, dx - \beta \int_\Omega \sigma \sigma^- \, dx - \beta \int_\Omega |\sigma^-|^2 \, dx \leq 0.
\]
It follows from the positivity property of $C$, $h$, $\sigma_s$ and $\beta$, that
\[
\sup_{t \in [0, T]} \left\| \sigma^-(t) \right\|_{L^2(\Omega)}^2 \leq \left\| \sigma^-_0 \right\|_{L^2(\Omega)}^2.
\]
On the other hand, since $\sigma_0 \geq 0$ a.e. in $\Omega$, then, we get $\sigma^-(0) = 0$ a.e. in $\Omega$ which leads to $\sigma^- = 0$ a.e. in $Q$. Hence $\sigma \geq 0$ a.e. in $Q$.

Similarly, let $(\sigma - 1)^+ = \max(0, \sigma - 1)$, and substitute $\xi$ in (12) by $(\sigma - 1)^+$, we obtain
\[
\frac{1}{2} \frac{d}{dt} \left\| (\sigma - 1)^+ \right\|^2 + \left\| \nabla (\sigma - 1)^+ \right\|_{L^2(\Omega)}^2 + \int_{\Omega} (Ch(\varphi) + \beta) \left| (\sigma - 1)^+ \right|^2 \, dx
\]
\[
+ C \int_{\Omega} h(\varphi) (\sigma - 1)^+ \, dx + \beta \int_{\Omega} (1 - \sigma_s) (\sigma - 1)^+ \, dx = 0.
\]
It follows that
\[
\sup_{t \in [0, T]} \left\| (\sigma - 1)^+(t) \right\|^2 \leq \left\| (\sigma - 1)^+(0) \right\|_{L^2(\Omega)}^2.
\]
However, $\sigma_0 \leq 1$ a.e. in $\Omega$, so that $\sigma_0 - 1 \leq 0$ a.e. in $\Omega$, and $(\sigma_0 - 1)^+ = 0$ a.e. in $\Omega$.

We infer that $(\sigma - 1)^+ = 0$ a.e. in $Q$, and $\sigma \leq 1$ a.e. in $Q$.

**Continuous dependence on the control.**

This section is devoted to study the continuous dependence of the control-to-state operator on the control $u$, for this purpose, let $u_1, u_2 \in \mathcal{U}_{ad}$ be given, along with the corresponding solutions $(\varphi_1, \sigma_1)$ and $(\varphi_2, \sigma_2)$ of (3)-(6) with same initial data $\varphi_0$ and $\sigma_0$. Set $u = u_1 - u_2$, $\varphi = \varphi_1 - \varphi_2$, and $\sigma = \sigma_1 - \sigma_2$ satisfying for all $\xi \in H^1(\Omega)$
\[
\int_{\Omega} \partial_t \sigma \xi \, dx + \int_{\Omega} \nabla \sigma \cdot \nabla \xi \, dx = - \int_{\Omega} (Ch(\varphi_1) + \beta) \sigma \xi \, dx - C \int_{\Omega} \sigma_2 (h(\varphi_1) - h(\varphi_2)) \, \xi \, dx
\]
and
\[
\int_{\Omega} \partial_t \varphi \xi \, dx + B \int_{\Omega} \nabla \varphi \cdot \nabla \xi \, dx + \int_{\Omega} (f(\varphi_1) - f(\varphi_2)) \xi \, dx = \mathcal{P} \int_{\Omega} h(\varphi_1) \sigma \xi \, dx
\]
\[
+ \int_{\Omega} (\mathcal{P} \sigma_2 - A - \alpha u_1) (h(\varphi_1) - h(\varphi_2)) \, \xi \, dx - \alpha \int_{\Omega} h(\varphi_2) u \xi \, dx.
\]
Some calculations yield
\[
\frac{d}{dt} \| \varphi \|^2_{L^2(\Omega)} + 2B \| \nabla \varphi \|^2_{L^2(\Omega)} \leq \mathcal{P}^2 \| \sigma \|^2_{L^2(\Omega)} + c \| \varphi \|^2_{L^2(\Omega)} + \alpha^2 \| u \|^2_{L^2(\Omega)}
\]
and
\[
\frac{d}{dt} \| \sigma \|^2_{L^2(\Omega)} + 2 \| \sigma \|^2_{L^2(\Omega)} \leq C^2 M^2 \| \varphi \|^2_{L^2(\Omega)} + \| \sigma \|^2_{L^2(\Omega)}.
\]
Combining the last inequalities, we have
\[
\frac{d}{dt} \left( \| \varphi \|^2_{L^2(\Omega)} + \| \sigma \|^2_{L^2(\Omega)} \right) + 2B \| \nabla \varphi \|^2_{L^2(\Omega)} + 2 \| \nabla \sigma \|^2_{L^2(\Omega)}
\]
\[
\leq (1 + \mathcal{P}^2) \| \sigma \|^2_{L^2(\Omega)} + c \| \varphi \|^2_{L^2(\Omega)} + \alpha^2 \| u \|^2_{L^2(\Omega)}.
\]
Putting $c' = \max(1 + \mathcal{P}^2; c)$, integrating over $[0, t]$ and thanks to Gronwall’s inequality, we find
\[
\| \sigma(t) \|^2_{L^2(\Omega)} + \| \varphi(t) \|^2_{L^2(\Omega)} + 2 \| \nabla \sigma \|^2_{L^2(0, t; L^2(\Omega))} + 2B \| \nabla \varphi \|^2_{L^2(0, t; L^2(\Omega))}
\]
\[
\leq \alpha^2 \| u \|^2_{L^2(0, t; L^2(\Omega))} e^{c't}.
\]
Moreover, putting $\xi = \partial_t \phi$ in (21), and $\xi = \partial_t \sigma$, we find
\[
\|\partial_t \varphi\|_{L^2(\Omega)}^2 + B \frac{d}{dt} \|\nabla \varphi\|_{L^2(\Omega)}^2 \leq \frac{d}{dt} \|\varphi\|_{L^2(\Omega)}^2 + \mathcal{P}^2 \|\sigma\|_{L^2(\Omega)}^2 + \alpha^2 \|u\|_{L^2(\Omega)}^2 + cM^2 \|\varphi\|_{L^2(\Omega)}^2 + \mathcal{P} \sigma_2 - A - \alpha u_1 (h(\varphi_1) - h(\varphi_2)) - (ah(\varphi_2)u).
\]
Using Assumption (B), in addition to the following inequality
\[
\|f(\varphi_1) - f(\varphi_2)\|_{L^2(\Omega)}^2 \leq \|f'(\varphi)\|_{L^2(\Omega)}^2 ,
\]
it follows that
\[
\left( \frac{d}{dt} \langle \varphi \rangle \right)^2 \leq \|f'(\varphi)\|_{L^2(\Omega)}^2 + \mathcal{P}^2 \|h(\varphi_1)\|_{L^\infty(\Omega)}^2 \|\sigma\|_{L^2(\Omega)}^2 + M^2 \|\mathcal{P} \sigma_2 - A - \alpha u_1 \|_{L^\infty(\Omega)}^2 \|\varphi\|_{L^2(\Omega)}^2 + \alpha^2 \|h(\varphi_2)\|_{L^\infty(\Omega)}^2 \|\sigma\|_{L^2(\Omega)}^2 + \alpha^2 \|u\|_{L^2(\Omega)}^2 .
\]
Now, we can write
\[
\left( \frac{d}{dt} \langle \varphi \rangle \right)^2 + \langle \varphi \rangle^2 \leq \|\varphi\|_{L^2(\Omega)}^2 + \mathcal{P}^2 \|\sigma\|_{L^2(\Omega)}^2 + \alpha^2 \|u\|_{L^2(\Omega)}^2 .
\]
Adding the last inequality with (22), we find
\[
\|\partial_t \varphi\|_{L^2(\Omega)}^2 + \frac{d}{dt} \|\varphi\|_{H^1(\Omega)}^2 \leq \frac{d}{dt} \|\varphi\|_{L^2(\Omega)}^2 + \frac{d}{dt} \|\varphi\|_{H^1(\Omega)}^2 + 2\mathcal{P}^2 \|\sigma\|_{L^2(\Omega)}^2 + 2\alpha^2 \|u\|_{L^2(\Omega)}^2 + \alpha^2 \|\varphi\|_{L^2(\Omega)}^2 .
\]
Combining (23) and the above equation, and integrating over $[0, t]$, we have
\[
\|\partial_t \varphi\|_{L^2(0, t; L^2(\Omega))}^2 + \|\partial_t \sigma\|_{L^2(0, t; L^2(\Omega))}^2 + \|\nabla \varphi(t)\|_{H^1(\Omega)}^2 + \|\nabla \sigma(t)\|_{L^2(\Omega)}^2 \leq \alpha^2 \|u\|_{L^2(0, t; L^2(\Omega))}^2 \left( ce^c t + c' \right) .
\]
In addition, we know that
\[
\|\varphi\|_{L^2(0, t; H^2(\Omega))}^2 \leq k \left( \|\partial_t \varphi\|_{L^2(0, t; L^2(\Omega))}^2 + \alpha^2 \|u\|_{L^2(0, t; L^2(\Omega))}^2 + \|\varphi\|_{L^2(0, t; L^2(\Omega))}^2 + \|\varphi\|_{L^2(0, t; H^2(\Omega))}^2 \right) .
\]
and
\[
\|\sigma\|_{L^2(0, t; H^2(\Omega))}^2 \leq k \left( \|\partial_t \sigma\|_{L^2(0, t; L^2(\Omega))}^2 + \|\sigma\|_{L^2(0, t; L^2(\Omega))}^2 \right) .
\]
Finally, we have
\[
\|\varphi\|_{L^2(0, t; H^2(\Omega))}^2 + \|\sigma\|_{L^2(0, t; H^2(\Omega))}^2 \leq c \|u\|_{L^2(0, t; L^2(\Omega))}^2 .
\]
3. Existence of a minimizer.

**Theorem 3.1.** Assume that Assumptions (A) and (B) hold and let \( \mathcal{J}_r \) be defined by (7). Then there exist \((u^*, \tau^*) \in \mathcal{U}_{ad} \times [r, T]\) such that
\[
\mathcal{J}_r(u^*, \tau^*) \leq \mathcal{J}_r(u, \tau), \quad \text{for every } (u, \tau) \in \mathcal{U}_{ad} \times [r, T].
\]

**Proof.** The cost functional \( \mathcal{J}_r \) is bounded from below, and therefore it has a finite infimum. Consider a minimizing sequence \( \{(u_n, \tau_n)\}_{n \in \mathbb{N}} \) with \( u_n \in \mathcal{U}_{ad} \) and \( \tau_n \in [r, T] \) and the corresponding weak solutions \((\varphi_n, \sigma_n)_{n \in \mathbb{N}} \) on the interval \([0, T]\) with \( \varphi_n(0) = \varphi_0 \) and \( \sigma_n(0) = \sigma_0 \), \( n \in \mathbb{N} \), such that
\[
\lim_{n \to \infty} \mathcal{J}_r(\varphi_n, \tau_n, u_n) = \inf_{(\phi, w, s)} \mathcal{J}_r(\phi, w, s).
\]

We have, \( u_n \in \mathcal{U}_{ad} \), then \( 0 \leq u_n \leq 1 \), a.e in \( Q \), for all \( n \in \mathbb{N} \). Since \(\{\tau_n\}_n \) is a bounded sequence, then there exists a relabeled subsequence \( \tau_n \) satisfying
\[
\lim_{n \to \infty} \tau_n = \tau_* \in [r, T].
\]

We also have
\[
\begin{align*}
& u_n \to u_* \quad \text{weakly in } \quad L^2(0, T; L^2(\Omega)), \\
& \varphi_n \to \varphi_* \quad \text{strongly in } \quad C([0, T]; L^2(\Omega)) \cap L^2(0, T; L^2(\Omega)), \\
& \sigma_n \to \sigma_* \quad \text{strongly in } \quad C([0, T]; L^2(\Omega)) \cap L^2(0, T; L^2(\Omega)),
\end{align*}
\]
where the couple \((\varphi_*, \sigma_*)\) satisfy (3)-(6), with \( 0 \leq u_*, \sigma_* \leq 1 \), a.e. in \( Q \).

Applying Lebesgue dominated convergence theorem, we obtain
\[
\begin{align*}
\chi_{[0, \tau_n]}(t) \to \chi_{[0, \tau_*]}(t) & \quad \text{strongly in } \quad L^p(0, T) \quad p \in [1, \infty) \\
\chi_{[\tau_n - r, \tau_n]}(t) \to \chi_{[\tau_* - r, \tau_*]}(t) & \quad \text{strongly in } \quad L^p(0, T) \quad p \in [1, \infty).
\end{align*}
\]

Passing to the limit, we get
\[
\int_0^{\tau_n} \int_{\Omega} |\varphi_n - \varphi_Q|^2 dx \, dt = \int_0^T \|\varphi_n - \varphi_Q\|_{L^2(\Omega)}^2 \chi_{[0, \tau_n]}(t) \, dt
\]
\[
\to \int_0^T \|\varphi_* - \varphi_Q\|_{L^2(\Omega)}^2 \chi_{[0, \tau_*]}(t) \, dt
\]
\[
= \int_0^{\tau_*} \|\varphi_* - \varphi_Q\|_{L^2(\Omega)} \, dt \quad \text{as } n \to \infty \quad \text{in } \quad L^2(0, T; L^2(\Omega))
\]
and
\[
\frac{1}{r} \int_{\tau_n - r}^{\tau_n} \left( \frac{\beta_0}{2} \|\varphi_n - \varphi_Q\|_{L^2(\Omega)}^2 + \frac{\beta_S}{2} \int_{\Omega} (1 + \varphi_n) \, dx \right) \, dt
\]
\[
\to \frac{1}{r} \int_{\tau_* - r}^{\tau_*} \left( \frac{\beta_0}{2} \|\varphi_* - \varphi_Q\|_{L^2(\Omega)}^2 + \frac{\beta_S}{2} \int_{\Omega} (1 + \varphi_*) \, dx \right) \, dt
\]
as \( n \to \infty \) in \( L^2(0, T; L^2(\Omega)) \).

Furthermore, using the weak lower semicontinuity of the \( L^2(Q) \) norm, and the fact that \( \tau_n \to \tau_* \) as \( n \to \infty \), we find
\[
\liminf_{n \to \infty} \int_0^{\tau_n} \|u_n\|_{L^2(\Omega)}^2 \, dt - \int_0^{\tau_*} \|u_*\|_{L^2(\Omega)}^2 \, dt \geq 0.
\]

After passing to the limit in \( \mathcal{J}_r(\varphi_n, u_n, \tau_n) \), we obtain
\[
\inf_{(\phi, w, s)} \mathcal{J}_r(\phi, w, s) = \lim_{n \to \infty} \mathcal{J}_r(\varphi_n, u_n, \tau_n) \geq \mathcal{J}_r(\varphi_*, u_*, \tau_*),
\]
which implies that \((u_*, \tau_*)\) is a minimizer of the problem. \( \square \)
4. Well-posedness of the linearized system. In order to establish the existence of the Fréchet derivative of the control-to-state operator with respect to the control, we consider the linearized system at \( u^* \), for \( w \in L^2(0,T;L^2(\Omega)) \).

\[
\begin{aligned}
\frac{\partial}{\partial t} \Phi - B\Delta \Phi + \Phi f'(\varphi^*) = (P\Sigma - \alpha w)h(\varphi^*) + (P\sigma^* - A - \alpha u^*)\Phi h'(\varphi^*), & \quad \text{in } Q, \\
\frac{\partial}{\partial t} \Sigma = \Delta \Sigma - C\Sigma h(\varphi^*) - C\sigma^* \Phi h'(\varphi^*) - \beta \Sigma, & \quad \text{in } Q, \\
\Sigma(0) = \Phi(0) = 0, & \quad \text{on } \Gamma \times (0,T), \\
\end{aligned}
\]

(24)

satisfying the following variational formulations, for \( \xi \in H^1(\Omega) \),

\[
\int_{\Omega} \frac{\partial}{\partial t} \Phi \xi dx + B \int_{\Omega} \nabla \Phi \cdot \nabla \xi dx + \int_{\Omega} f'(\varphi^*) \Phi \xi dx = \int_{\Omega} (P\Sigma - \alpha w)h(\varphi^*) \xi dx + \int_{\Omega} (P\sigma^* - A - \alpha u^*)\Phi h'(\varphi^*) \xi dx
\]

(25)

and

\[
\int_{\Omega} \frac{\partial}{\partial t} \Sigma \xi dx + \int_{\Omega} \nabla \Sigma \cdot \nabla \xi dx = -\int_{\Omega} ((Ch(\varphi^*) + \beta) \Sigma + C\sigma^* \Phi h'(\varphi^*) \xi) dx.
\]

(26)

**Theorem 4.1.** Let \( u^* \in U_{ad} \). Then, System (24) admits a unique solution \((\Phi, \Sigma)\) in \((L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))) \cap C([0,T];L^2(\Omega)))^2\).

**Proof.** As we did before, we will use Galerkin method. Reconsider the finite dimensional space spanned by the first \( n \) eigenfunctions \( \{\omega_i\}_i \) associated to the first \( n \) eigenvalues of \( -\Delta \) operator. We are looking for the functions of the form

\[
\Phi_n(x,t) = \sum_{i=1}^n a_{n,i}(t)\omega_i(x) \quad \text{and} \quad \Sigma_n(x,t) = \sum_{i=1}^n b_{n,i}(t)\omega_i(x)
\]

satisfying

\[
\int_{\Omega} \left( \frac{\partial}{\partial t} \Phi_n v + B\nabla \Phi_n \cdot \nabla v - h(\varphi^*)(P\Sigma_n - \alpha w)v \right) dx
\]

\[-\int_{\Omega} h'(\varphi^*)(P\sigma^* - A - \alpha u^*)\Phi_n - f'(\varphi^*)\Phi_n v \right) dx = 0
\]

and

\[
\int_{\Omega} \left( \frac{\partial}{\partial t} \Sigma_n v + \nabla \Sigma_n \cdot \nabla v + Ch(\varphi^*)\Sigma_n v + C\sigma^* h'(\sigma)\Phi_n v + \beta \Sigma_n v \right) dx = 0.
\]

Let \( \Phi_{n,0} = P_{V_n} \Phi_0 \) and \( \Sigma_{n,0} = P_{V_n} \Sigma_0 \) such that \( \Phi_{n,0} \to \Phi_0 \) and \( \Sigma_{n,0} \to \Sigma_0 \) in \( L^2(\Omega) \). Substituting \( \omega_j \in V_n, \ j = 1, \ldots, n \), for \( v \) in both equations to have

\[
\int_{\Omega} \left( \frac{\partial}{\partial t} \Phi_n \omega_j + B\nabla \Phi_n \cdot \nabla \omega_j - h(\varphi^*)(P\Sigma_n - \alpha w)\omega_j \right) dx
\]

\[-\int_{\Omega} h'(\varphi^*)(P\sigma^* - A - \alpha u^*)\Phi_n - f'(\varphi^*)\Phi_n \omega_j dx = 0
\]

and

\[
\int_{\Omega} \left( \frac{\partial}{\partial t} \Sigma_n \omega_j + \nabla \Sigma_n \cdot \nabla \omega_j + Ch(\varphi^*)\Sigma_n \omega_j + C\sigma^* h'(\varphi^*)\Phi_n \omega_j + \beta \Sigma_n \omega_j \right) dx = 0.
\]
Consequently,

\[
\int_{\Omega} \frac{d}{dt} \sum_{i=1}^{n} a_{n,i}(t) \omega_i \omega_j \, dx + \int_{\Omega} B \sum_{i=1}^{n} a_{n,i}(t) \nabla \omega_i \cdot \nabla \omega_j \, dx \\
- \int_{\Omega} h(\varphi^*) \left( P \int_{\Omega} \sum_{i=1}^{n} b_{n,i}(t) \omega_i - \alpha \nu \right) \omega_j \, dx \\
- \int_{\Omega} h'(\varphi^*) (P \sigma^* - A - \alpha u^*) \sum_{i=1}^{n} a_{n,i}(t) \omega_i \omega_j \, dx - \int_{\Omega} f'(\varphi^*) \sum_{i=1}^{n} a_{n,i}(t) \omega_i \omega_j \, dx = 0
\]

and

\[
\int_{\Omega} \left( \frac{d}{dt} \sum_{i=1}^{n} b_{n,i}(t) \omega_i \omega_j + \sum_{i=1}^{n} b_{n,i}(t) \nabla \omega_i \cdot \nabla \omega_j + Ch(\varphi^*) \sum_{i=1}^{n} b_{n,i}(t) \omega_i \omega_j \right) \, dx \\
+ \int_{\Omega} \left( C \sigma^* h'(\varphi^*) \sum_{i=1}^{n} a_{n,i}(t) \omega_i \omega_j + \beta \sum_{i=1}^{n} b_{n,i}(t) \omega_i \omega_j \right) \, dx = 0
\]

Its vector form is

\[
\frac{d}{dt} a_n(t) I_n + B a_n(t) J_n = h(\varphi^*) P b_n(t) I_n + h(\varphi^*) \alpha W_n + h'(\varphi^*) (P \sigma^* - A - \alpha u^*) a_n(t) I_n - f'(\varphi^*) a_n(t) I_n
\]

and

\[
\frac{d}{dt} b_n(t) I_n + b_n(t) J_n = -Ch(\varphi^*) b_n(t) I_n - C \sigma^* h'(\varphi^*) a_n(t) I_n - \beta b_n(t) I_n,
\]

where

\[(J_n)_{i,j} = \int \nabla \omega_i \cdot \nabla \omega_j \, dx \quad \text{and} \quad (W_n)_{i,j} = \int w \omega_j \, dx.\]

Equivalently

\[
\left\{ \begin{array}{l}
\dot{a}_n(t) = (h(\varphi^*) P b_n(t) + h'(\varphi^*) (P \sigma^* - A - \alpha u^*) a_n(t) - f'(\varphi^*) a_n(t)) I_n \\
+ h(\varphi^*) \alpha W_n - B a_n(t) J_n = f(t, a_n, b_n) \\
\dot{b}_n(t) = -b_n(t) J_n - (Ch(\varphi^*) + \beta) b_n(t) I_n - C \sigma^* h'(\varphi^*) a_n(t) I_n = g(t, a_n, b_n).
\end{array} \right.
\]

It is easy to see that the function \( F = (f(t, a_n, b_n), g(t, a_n, b_n)) \) is locally Lipschitz with respect to \( a_n \) and \( b_n \), so the ODE system admits a unique local in time solution. Multiplying Equation (27) by \( a_{n,j}(t) \) and (28) by \( b_{n,j}(t) \), and summing over \( j \) from 1 to \( n \), moreover, due to the fact that \( \alpha \) is positive, \( \beta, C, P, A \) are non negative, we obtain

\[
\frac{1}{2} \frac{d}{dt} \| \Phi_n \|_{L^2(\Omega)}^2 + B \| \nabla \Phi_n \|_{L^2(\Omega)}^2 \leq \frac{1}{2} \| h(\varphi^*) \|_{\infty} P^2 \| \Sigma_n \|_{L^2(\Omega)}^2 + \| \Phi_n \|_{L^2(\Omega)}^2
\]

\[
+ \frac{1}{2} \| h(\varphi^*) \|_{\infty} \alpha^2 \| w \|_{L^2(\Omega)}^2 + \| h'(\varphi^*) P \sigma^* \| \| \Phi_n \|_{L^2(\Omega)}^2 + \| f'(\varphi^*) \| \| \Phi_n \|_{L^2(\Omega)}^2
\]

and

\[
\frac{1}{2} \frac{d}{dt} \| \Sigma_n \|_{L^2(\Omega)}^2 + \| \Sigma_n \|_{H^1(\Omega)}^2 \leq \frac{1}{2} \left( \| C \sigma^* h'(\varphi^*) \| \| \Sigma_n \|_{L^2(\Omega)}^2 + \| \Phi_n \|_{L^2(\Omega)}^2 \right).
\]
Adding (29) to the last inequality, we get
\[
\frac{1}{2} \frac{d}{dt} \left( \|\Phi_n\|^2_{L^2(\Omega)} + \|\Sigma_n\|^2_{L^2(\Omega)} \right) + B \|\nabla \Phi_n\|^2_{L^2(\Omega)} + \|\Sigma_n\|^2_{H^1(\Omega)} \\
\leq c \|\Sigma_n\|^2_{L^2(\Omega)} + c' \|\Phi_n\|^2_{L^2(\Omega)} + c'' \|w\|^2_{L^2(\Omega)}.
\]
Integrating the above equation with respect to time leads to
\[
\|\Phi_n(s)\|^2_{L^2(\Omega)} + \|\Sigma_n(s)\|^2_{L^2(\Omega)} + \int_0^s 2 \left( B \|\nabla \Phi_n(t)\|^2_{L^2(\Omega)} + \|\Sigma_n(t)\|^2_{H^1(\Omega)} \right) dt \\
\leq \|\Phi_n(0)\|^2_{L^2(\Omega)} + \|\Sigma_n(0)\|^2_{L^2(\Omega)} + C_3 \|w\|^2_{L^2(0,s;L^2(\Omega))} \\
+ \int_0^s \left( C_1 \|\Sigma_n(t)\|^2_{L^2(\Omega)} + C_2 \|\Phi_n(t)\|^2_{L^2(\Omega)} \right) dt,
\]
for \( s \in (0,T) \). By virtue of Gronwall’s lemma, we obtain
\[
\|\Phi_n(s)\|^2_{L^2(\Omega)} + \|\Sigma_n(s)\|^2_{L^2(\Omega)} + \int_0^s \left( B \|\nabla \Phi_n(t)\|^2_{L^2(\Omega)} + \|\Sigma_n(t)\|^2_{L^2(\Omega)} \right) dt \\
\leq \left( c + c \|w\|^2_{L^2(0,s;L^2(\Omega))} \right) e^{cs}.
\]
Putting \( \xi = 1 \) in (26), and multiplying it by \( \frac{1}{Vol(\Omega)} \), we find
\[
\frac{d}{dt} \langle \Phi \rangle = -(f'(\varphi)\Phi) + \langle \mathcal{P} h(\varphi^*) \Sigma - \alpha \langle wh(\varphi^*) \rangle + ((\mathcal{P} \sigma^* - A - \alpha u^*) h'(\varphi^*)\Phi),
\]
so that
\[
\left( \frac{d}{dt} \langle \Phi \rangle \right)^2 \leq c \|\Phi\|^2_{L^2(\Omega)} + \mathcal{P}^2 \|\Sigma\|^2_{L^2(\Omega)} + \alpha^2 \|w\|^2_{L^2(\Omega)}.
\]
Consequently,
\[
2 \langle \Phi \rangle \frac{d}{dt} \langle \Phi \rangle \leq c \|\Phi\|^2_{L^2(\Omega)} + \mathcal{P}^2 \|\Sigma\|^2_{L^2(\Omega)} + \alpha^2 \|w\|^2_{L^2(\Omega)}.
\]
Multiplying (27) and (28) by \( a'_{n,j} \) and \( b'_{n,j} \), respectively, and summing over \( j \) from 1 to \( n \), we find
\[
\left\| \frac{d\Phi_n}{dt} \right\|^2_{L^2(\Omega)} + \frac{B}{2} \frac{d}{dt} \|\nabla \Phi_n\|^2_{L^2(\Omega)} \\
\leq \frac{1}{2} \|h(\varphi^*)\|_{L^\infty(\Omega)} \left( \mathcal{P}^2 \|\Sigma_n\|^2_{L^2(\Omega)} + \alpha^2 \|w\|^2_{L^2(\Omega)} + \left\| \frac{d\Phi_n}{dt} \right\|^2_{L^2(\Omega)} \right) \\
+ \frac{1}{2} \|h'(\varphi^*)\|_{L^\infty(\Omega)} \|\mathcal{P}\| \|\sigma^* - A - \alpha u^*\|_{L^\infty(\Omega)} \frac{d}{dt} \|\Phi_n\|^2_{L^2(\Omega)} + \frac{1}{2} \|f'(\varphi^*)\|_{L^\infty(\Omega)} \frac{d}{dt} \|\Phi_n\|^2_{L^2(\Omega)}.
\]
Adding (30) to the last inequality, we find
\[
\left\| \frac{d\Phi_n}{dt} \right\|^2_{L^2(\Omega)} + c \frac{d}{dt} \|\Phi_n\|^2_{H^1(\Omega)} \leq 2 \mathcal{P}^2 \|\Sigma\|^2_{L^2(\Omega)} + c \frac{d}{dt} \|\Phi_n\|^2_{L^2(\Omega)} + 2 \alpha^2 \|w\|^2_{L^2(\Omega)}.
\]
On the other hand, we have
\[
\left\| \frac{d\Sigma_n}{dt} \right\|^2_{L^2(\Omega)} + \frac{d}{dt} \|\nabla \Sigma_n\|^2_{L^2(\Omega)} \\
\leq \frac{\|C h(\varphi^*) + \beta\|_{L^\infty(\Omega)}}{2} \|\Sigma_n\|^2_{L^2(\Omega)} + \frac{1}{2} \|C \sigma^* h'(\varphi^*)\|^2_{L^2(\Omega)} \|\Phi_n\|^2_{L^2(\Omega)}.
\]
Combining (31) and the last inequality, and integrating over \([0, t]\), we find
\[
\left\| \frac{d\Phi_n}{dt} \right\|^2_{L^2(0,t;L^2(\Omega))} + \left\| \frac{d\Sigma_n}{dt} \right\|^2_{L^2(0,t;L^2(\Omega))} + c \left\| \Phi_n(t) \right\|^2_{H^1(\Omega)} + \left\| \nabla \Sigma_n(t) \right\|^2_{L^2(\Omega)} \\
\leq 2\alpha^2 \left\| w \right\|^2_{L^2(0,t;L^2(\Omega))} + c \left\| \Phi_n(0) \right\|^2_{H^1(\Omega)} + \left\| \nabla \Sigma_n(0) \right\|^2_{L^2(\Omega)} \\
+ c' \left\| \Sigma_n \right\|^2_{L^2(0,t;L^2(\Omega))} + c'' \left\| \Phi_n \right\|^2_{L^2(0,t;L^2(\Omega))}.
\]
Consequently, \( \{\Phi_n\} \) and \( \{\Sigma_n\} \) are uniformly bounded in
\[
L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega)),
\]
therefore, the solution is global in time. Moreover, there exists a relabeled subsequence such that
\[
\Phi_n \xrightarrow{\ast} \Phi \text{ weakly star in } L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))
\]
and
\[
\Sigma \xrightarrow{\ast} \Sigma \text{ weakly star in } L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega)).
\]
Since this system is linear, it is easy to pass to the limit. Let \( \{\Phi_i, \Sigma_i\}_{i=1,2} \) be two weak solutions to (25)-(26). Setting \( \Phi := \Phi_1 - \Phi_2 \) and \( \Sigma := \Sigma_1 - \Sigma_2 \) with \( w = 0 \), then \( \Sigma \) and \( \Phi \) satisfy (25) and (26), respectively. Therefore, regularity estimates still hold and it implies
\[
\left\| \Phi \right\|_{L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))} + \left\| \Sigma \right\|_{L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))} \leq 0,
\]
hence \( \Phi = \Sigma = 0 \). \( \square \)

5. Fréchet differentiability of control-to-state operator with respect to the control. Let \( w \in L^2(Q) \), set \( (\varphi^w, \sigma^w) := S(u^* + w), (\varphi^*, \sigma^*) := S(u^*) \) and \( u^w := u^* + w \). We express the remainders \( \rho \) and \( \theta \) as \( \theta = \varphi^w - \varphi^* - \Phi \), and \( \rho = \sigma^w - \sigma^* - \Sigma \).

**Theorem 5.1.** Let \((\Phi, \Sigma)\) be the solution of the linearized system at \( u^* \), then the remainders \( \theta \) and \( \rho \) satisfy
\[
\left\| (\theta, \rho) \right\|^2_{\mathcal{Y}} \leq c \left\| w \right\|^4_{L^2(Q)},
\]
with
\[
\mathcal{Y} := L^2(0,s;H^2(\Omega)) \cap H^1(0,s;L^2(\Omega)) \cap L^\infty(0,s;H^1(\Omega)), \ s \in (0,T).
\]
Then the control-to-state operator is Fréchet differentiable with respect to the control.

**Proof.** For all \( \xi \in H^1(\Omega) \), we have
\[
\int_{\Omega} \partial_t \varphi^w \xi dx + B \int_{\Omega} \nabla \varphi^w \cdot \nabla \xi dx + \int_{\Omega} f(\varphi^w) \xi dx = \int_{\Omega} (P\sigma^w - A - \alpha u^w)h(\varphi^w) \xi dx \\
\int_{\Omega} \partial_t \varphi^* \xi dx + B \int_{\Omega} \nabla \varphi^* \cdot \nabla \xi dx + \int_{\Omega} f(\varphi^*) \xi dx = \int_{\Omega} (P\sigma^* - A - \alpha u^*)h(\varphi^*) \xi dx
\]
and
\[
\int_{\Omega} \partial_t \Phi \xi dx + B \int_{\Omega} \nabla \Phi \cdot \nabla \xi dx + \int_{\Omega} f'(\varphi^*) \Phi h(\varphi^*) \xi dx \\
= \int_{\Omega} (P\Sigma - \alpha u^w)h(\varphi^*) \xi dx + \int_{\Omega} (P\sigma^* - A - \alpha u^*) \Phi h'(\varphi^*) \xi dx.
\]
Respectively,
\[
\begin{align*}
\int_{\Omega} \frac{\partial_t \sigma^w}{\partial t} \xi dx + \int_{\Omega} \nabla \sigma^w \cdot \nabla \xi dx &= - C \int_{\Omega} \sigma^w h(\varphi^w) \xi dx + \beta \int_{\Omega} (\sigma_s - \sigma^w) \xi dx, \\
\int_{\Omega} \frac{\partial_t \sigma^*}{\partial t} \xi dx + \int_{\Omega} \nabla \sigma^* \cdot \nabla \xi dx &= - C \int_{\Omega} \sigma^* h(\varphi^*) \xi dx + \beta \int_{\Omega} (\sigma_s - \sigma^*) \xi dx, \\
\int_{\Omega} \frac{\partial_t \Sigma}{\partial t} \xi dx + \int_{\Omega} \nabla \Sigma \cdot \nabla \xi dx &= - C \int_{\Omega} \Sigma h(\varphi^*) \xi dx - C \int_{\Omega} \sigma^* \Phi h'(\varphi^*) \xi dx - \beta \int_{\Omega} \Sigma \xi dx.
\end{align*}
\]

Then the remainder \( \theta \) satisfies, for all \( \xi \in H^1(\Omega) \)
\[
\begin{align*}
\int_{\Omega} \partial_t \theta \xi dx + B \int_{\Omega} \nabla \theta \cdot \nabla \xi dx + \int_{\Omega} (f(\varphi^w) - f(\varphi^*)) \xi dx &= \int_{\Omega} (P\sigma^w - A - \alpha u^w) h(\varphi^w) \xi dx - \int_{\Omega} (P\sigma^* - A - \alpha u^*) h(\varphi^*) \xi dx \\
&\quad - \int_{\Omega} (P\Sigma - \alpha w) h(\varphi^*) \xi dx - \int_{\Omega} (P\sigma^* - A - \alpha u^*) \Phi h'(\varphi^*) \xi dx.
\end{align*}
\]

Besides, we have that
\[
\begin{align*}
\sigma^w h(\varphi^w) - \sigma^* h(\varphi^*) + \sigma^* h(\varphi^w) - \sigma^w h(\varphi^*) + \sigma^w h(\varphi^w) + \sigma^* h(\varphi^*) - \sigma^* h(\varphi^*) \\
+ \sigma^* h(\varphi^*) - \Sigma h(\varphi^*) - (\Phi h')'(\varphi^*) \\
= \sigma^w (h(\varphi^w) - h(\varphi^*)) - \sigma^* (h(\varphi^w) - h(\varphi^*)) + \sigma^w h(\varphi^w) - \sigma^* h(\varphi^w) \\
- \Sigma h(\varphi^*) + \sigma^* h(\varphi^w) - \sigma^* h(\varphi^*) - (\Phi h')'(\varphi^*) \\
= (h(\varphi^w) - h(\varphi^*)) (\sigma^w - \sigma^* + h(\varphi^*) (\sigma^w - \sigma^* - \Sigma) \\
+ \sigma^* (h(\varphi^w) - h(\varphi^*) - (\Phi h')'(\varphi^*)).
\end{align*}
\]

Thanks to Taylor expansion with integral remainder, we have
\[
\begin{align*}
\int_{\Omega} \partial_t \xi dx + B \int_{\Omega} \nabla \theta \cdot \nabla \xi dx + \int_{\Omega} (f(\varphi^w) - f(\varphi^*)) \xi dx &= \int_{\Omega} (P\sigma^w - A - \alpha u^w) h(\varphi^w) \xi dx - \int_{\Omega} (P\sigma^* - A - \alpha u^*) h(\varphi^*) \xi dx \\
&\quad - \int_{\Omega} (P\Sigma - \alpha w) h(\varphi^*) \xi dx - \int_{\Omega} (P\sigma^* - A - \alpha u^*) \Phi h'(\varphi^*) \xi dx.
\end{align*}
\]

The remainder
\[
R = \int_{0}^{1} h''(z(\varphi^w) + (1 - z)\varphi^*) (1 - z) dz.
\]

is bounded, with
\[
\|R\|_{\infty} \leq c_R.
\]

We then deduce that
\[
(h(\varphi^w) - h(\varphi^*)) (\sigma^w - \sigma^*) + h(\varphi^*) (\sigma^w - \sigma^* - \Sigma) + \sigma^* (h(\varphi^w) - h(\varphi^*) - (\Phi h')'(\varphi^*)) \\
= (h(\varphi^w) - h(\varphi^*)) (\sigma^w - \sigma^*) + h(\varphi^*) (\sigma^w - \sigma^* - \Sigma) + \sigma^* (h(\varphi^w) - h(\varphi^*) - (\Phi h')'(\varphi^*)).
\]

Therefore the variational formulation, for \( \xi \in H^1(\Omega) \) is given by
\[
\begin{align*}
\int_{\Omega} \partial_t \xi dx + B \int_{\Omega} \nabla \theta \cdot \nabla \xi dx + \int_{\Omega} (f(\varphi^w) - f(\varphi^*)) \xi dx &= \int_{\Omega} (P\sigma^w - A - \alpha u^w) h(\varphi^w) \xi dx - \int_{\Omega} (P\sigma^* - A - \alpha u^*) h(\varphi^*) \xi dx \\
&\quad + \int_{\Omega} (P\Sigma - \alpha w) h(\varphi^*) \xi dx - \int_{\Omega} (P\sigma^* - A - \alpha u^*) \Phi h'(\varphi^*) \xi dx. \tag{32}
\end{align*}
\]
Now, put \( \xi = \theta \) in (32), we get

\[
\frac{1}{2} \frac{d}{dt} \| \theta \|_{L^2(\Omega)}^2 + B \| \nabla \theta \|_{L^2(\Omega)}^2 + \int_{\Omega} \int_{\Omega} f'(\varphi^*) \theta^2 \, dx + \int_{\Omega} (\varphi^w - \varphi^*)^2 R \theta \, dx
\]

\[
= \mathcal{P} \int_{\Omega} (h(\varphi^w) - h(\varphi^*)) (\sigma^w - \sigma^*) \theta \, dx + \int_{\Omega} \mathcal{P} h(\varphi^*) \rho \theta \, dx
\]

\[
+ \int_{\Omega} (\mathcal{P} \sigma^* - A - \alpha u^*) h'(\varphi^*) \theta^2 \, dx + \int_{\Omega} (\mathcal{P} \sigma^* - A - \alpha u^*) (\varphi^w - \varphi^*)^2 R \theta \, dx
\]

\[
- \alpha \int_{\Omega} w (h(\varphi^w) - h(\varphi^*)) \theta \, dx.
\]

Applying holder’s inequality, and as well as Young’s, and knowing that \( h \) is a Lipschitz function with constant \( M \), and \( R \) is bounded by some \( c_R \), we obtain

\[
\frac{d}{dt} \| \theta \|_{L^2(\Omega)}^2 + 2B \| \nabla \theta \|_{L^2(\Omega)}^2 \leq 2P \| \varphi^w - \varphi^* \|_{L^2(\Omega)} \| \sigma^w - \sigma^* \|_{L^\infty(\Omega)} \| \theta \|_{L^2(\Omega)}
\]

\[
+ \| h(\varphi^*) \|_{L^\infty} \left( \mathcal{P}^2 \| \rho \|_{L^2(\Omega)}^2 + \| \theta \|_{L^2(\Omega)}^2 \right) + 2c \| h'(\varphi^*) \|_{L^\infty(\Omega)} \| \theta \|_{L^2(\Omega)}
\]

\[
+ 2c_R \| \varphi^w - \varphi^* \|_{L^2(\Omega)} \| \varphi^w - \varphi^* \|_{L^2(\Omega)} \| \theta \|_{L^2(\Omega)}
\]

\[
+ 2M \| \varphi^w - \varphi^* \|_{L^\infty(\Omega)} \| w \|_{L^2(\Omega)} \| \theta \|_{L^2(\Omega)}.
\]

Using the boundedness of \( h'(\varphi^*), h(\varphi^*) \), and integrating with respect to time, we get

\[
\| \theta(s) \|_{L^2(\Omega)}^2 + 2B \| \nabla \theta \|_{L^2(\Omega)}^2 \leq 2P \| \varphi^w - \varphi^* \|_{L^2(\Omega)} \| \sigma^w - \sigma^* \|_{L^\infty(\Omega)} \| \theta \|_{L^2(\Omega)}
\]

\[
+ \| h(\varphi^*) \|_{L^\infty} \left( \mathcal{P}^2 \| \rho \|_{L^2(\Omega)}^2 + \| \theta \|_{L^2(\Omega)}^2 \right) + 2c \| h'(\varphi^*) \|_{L^\infty(\Omega)} \| \theta \|_{L^2(\Omega)}
\]

\[
+ 2c_R \| \varphi^w - \varphi^* \|_{L^2(\Omega)} \| \varphi^w - \varphi^* \|_{L^2(\Omega)} \| \theta \|_{L^2(\Omega)}
\]

\[
+ 2M \| \varphi^w - \varphi^* \|_{L^\infty(\Omega)} \| w \|_{L^2(\Omega)} \| \theta \|_{L^2(\Omega)}.
\]

Using the embedding \( L^2(0, s; H^2(\Omega)) \hookrightarrow L^2(0, s; L^\infty(\Omega)) \), and applying Young’s inequality, we obtain

\[
\| \theta(s) \|_{L^2(\Omega)}^2 + 2B \| \nabla \theta \|_{L^2(\Omega)}^2 \leq (\mathcal{P}M)^2 \| \varphi^w - \varphi^* \|_{L^2(0, s; L^2(\Omega))}^2 \| \sigma^w - \sigma^* \|_{L^2(0, s; L^2(\Omega))}^2
\]

\[
+ \| \theta \|_{L^\infty(0, s; L^2(\Omega))}^2 + \| \rho \|_{L^\infty(0, s; L^2(\Omega))}^2 \| \theta \|_{L^2(0, s; L^2(\Omega))}^2 + c \| \theta \|_{L^\infty(0, s; L^2(\Omega))}^2
\]

\[
+ c_R^2 \| \varphi^w - \varphi^* \|_{L^\infty(0, s; H^2(\Omega))}^2 \| \varphi^w - \varphi^* \|_{L^\infty(0, s; L^2(\Omega))}^2 + \| \theta \|_{L^\infty(0, s; L^2(\Omega))}^2
\]

\[
+ M^2 \| \varphi^w - \varphi^* \|_{L^\infty(0, s; H^2(\Omega))}^2 \| w \|_{L^2(0, s; L^2(\Omega))}^2 + \frac{1}{2} \| \theta \|_{L^\infty(0, s; L^2(\Omega))}^2.
\]

Therefore, it follows that

\[
\| \theta(s) \|_{L^2(\Omega)}^2 + 2B \| \nabla \theta \|_{L^2(\Omega)}^2 \leq c_1 \| w \|_{L^2(0, s; L^2(\Omega))}^4 + c_2 \| \theta \|_{L^2(0, s; L^2(\Omega))}^4 + \mathcal{P} \| \rho \|_{L^2(0, s; L^2(\Omega))}^2.
\]
On the other hand, $\rho$ satisfies the following variational inequality, for $\xi \in H^1(\Omega)$

\[
\int_{\Omega} \partial_t \rho \xi dx + \int_{\Omega} \nabla \rho \cdot \nabla \xi dx = -C \int_{\Omega} (\sigma - \sigma^*) (h(\varphi^w) - h(\varphi^*)) \xi dx
\]

Furthermore, putting $\xi = \rho$ in (34), we get

\[
\frac{d}{dt} \frac{1}{2} \|\rho\|^2_{L^2(\Omega)} + \|\nabla \rho\|^2_{L^2(\Omega)} + \beta \|\rho\|^2_{L^2(\Omega)} \
\leq C \|\sigma - \sigma^*\|_{L^\infty(\Omega)} \|h(\varphi^w) - h(\varphi^*)\|_{L^2(\Omega)} \|\rho\|_{L^2(\Omega)} + C \|h(\varphi^*)\|_{L^\infty(\Omega)} \|\rho\|_{L^2(\Omega)}^2
\]

Putting $c = \min(1; \beta)$ and using the boundedness of $h(\varphi^*), h'(\varphi^*), \sigma^*$, in addition to the fact that $h$ is Lipschitz with constant $M$, and the embedding $L^2(0,s; H^2(\Omega)) \hookrightarrow L^2(0,s; L^\infty(\Omega))$, then integrating on $[0,s]$ for $s \in [0,T]$, we get

\[
\|\rho(s)\|^2_{L^2(\Omega)} + 2c \|\rho\|^2_{L^2(0,s; H^1(\Omega))} \
\leq (CM)^2 \|\sigma - \sigma^*\|^2_{L^2(0,s; H^2(\Omega))} \|\varphi^w - \varphi^*\|^2_{L^\infty(0,s; L^2(\Omega))} + \|\rho\|^2_{L^2(0,s; L^2(\Omega))} + C \|\nabla \theta\|^2_{L^2(0,s; L^2(\Omega))} + \|\rho\|^2_{L^2(0,s; L^2(\Omega))}
\]

Combining (33) and the above inequality, we find

\[
\|\rho(s)\|^2_{L^2(\Omega)} + \|\theta(s)\|^2_{L^2(\Omega)} + 2c \|\rho\|^2_{L^2(0,s; H^1(\Omega))} + 2B \|\nabla \theta\|^2_{L^2(0,s; L^2(\Omega))}
\]

Applying Gronwall’s inequality, we get

\[
\|\rho(s)\|^2_{L^2(\Omega)} + \|\theta(s)\|^2_{L^2(\Omega)} + 2c \|\rho\|^2_{L^2(0,s; H^1(\Omega))} + 2B \|\nabla \theta\|^2_{L^2(0,s; L^2(\Omega))}
\]

Furthermore, putting $\xi = \partial_t \theta$ in (32), we obtain

\[
\|\partial_t \theta\|^2_{L^2(\Omega)} + \frac{B}{2} \frac{d}{dt} \|\nabla \theta\|^2_{L^2(\Omega)} \
\leq \frac{1}{2} \frac{d}{dt} \|\theta\|^2_{L^2(\Omega)} + CC_R \|\varphi^w - \varphi^*\|_{L^\infty(\Omega)} \|\varphi^w - \varphi^*\|_{L^2(\Omega)} \|\partial_t \theta\|_{L^2(\Omega)}
\]

using the Lipschitz property of the function $h$, in addition to the boundedness of $\sigma^*, h'(\varphi^*), h(\varphi^*)$, and $R$ in addition to Young’s inequality, and the continuous
embedding $H^2(\Omega) \rightarrow L^\infty(\Omega)$, we get
\[
\|\partial_t \theta\|_{L^2(\Omega)}^2 + B \frac{d}{dt} \|\nabla \theta\|_{L^2(\Omega)}^2 \leq \frac{d}{dt} \|\theta\|_{L^2(\Omega)}^2 + M^2 \alpha^2 \|w\|_{L^2(\Omega)}^2 \|\varphi^w - \varphi^*\|_{H^2(\Omega)}^2 \\
+ 2c_E \|\varphi^w - \varphi^*\|_{L^\infty(\Omega)}^2 \|\varphi^w - \varphi^*\|_{L^2(\Omega)} \|\partial_t \theta\|_{L^2(\Omega)} \\
+ 2PM \|\varphi^w - \varphi^*\|_{L^\infty(\Omega)} \|\sigma^w - \sigma^*\|_{L^2(\Omega)} \|\partial_t \rho\|_{L^2(\Omega)} + P \|\rho\|_{L^2(\Omega)}^2.
\]
Integrating over $[0, s]$, and applying Young’s and Schwarz inequalities, as well as the embedding
\[
L^2(0, s; H^2(\Omega)) \hookrightarrow L^2(0, s; L^\infty(\Omega)),
\]
we obtain
\[
\|\partial_t \theta\|_{L^2(0, s; L^2(\Omega))}^2 + B \|\nabla \theta(s)\|_{L^2(\Omega)}^2 \leq c \|\theta(s)\|_{L^2(\Omega)}^2 + \|\partial_t \rho\|_{L^2(0, s; L^2(\Omega))}^2 \\
+ cE \|\varphi^w - \varphi^*\|_{L^2(0, s; H^2(\Omega))}^2 \|\varphi^w - \varphi^*\|_{L^\infty(0, s; L^2(\Omega))}^2 + \|\sigma^w - \sigma^*\|_{L^\infty(0, s; L^2(\Omega))}^2 \\
+ (M \alpha)^2 \|w\|_{L^2(0, s; L^2(\Omega))}^2 \|\varphi^w - \varphi^*\|_{L^2(0, s; H^2(\Omega))}^2 \\
+ (PM)^2 \|\varphi^w - \varphi^*\|_{L^2(0, s; H^2(\Omega))}^2 \|\sigma^w - \sigma^*\|_{L^\infty(0, s; L^2(\Omega))}^2.
\]
On the other hand, putting $\xi = \partial_t \rho$ in (34), and proceeding in the same way like above, we get
\[
\|\partial_t \rho\|_{L^2(0, s; L^2(\Omega))}^2 + B \|\nabla \rho(s)\|_{L^2(\Omega)}^2 \leq \|\partial_t \theta\|_{L^2(0, s; L^2(\Omega))}^2 \\
+ (CM)^2 \|\sigma^w - \sigma^*\|_{L^\infty(0, s; L^2(\Omega))}^2 \|\varphi^w - \varphi^*\|_{L^2(0, s; L^2(\Omega))}^2 \\
+ (MC)^2 \|\varphi^w - \varphi^*\|_{L^\infty(0, s; L^2(\Omega))}^2 \|\varphi^w - \varphi^*\|_{L^2(0, s; L^2(\Omega))}^2 \\
+ c \|\partial_t \rho\|_{L^2(0, s; L^2(\Omega))}^2.
\]
Integrating with respect to time, using the embedding
\[
L^2(0, s; H^2(\Omega)) \hookrightarrow L^2(0, s; L^\infty(\Omega)),
\]
and the continuous dependence on control, we obtain
\[
\|\partial_t \rho\|_{L^2(0, s; L^2(\Omega))}^2 + \|\nabla \rho(s)\|_{L^2(\Omega)}^2 \leq c \|w\|_{L^2(0, s; L^2(\Omega))}^4 + c' \|\theta\|_{L^2(0, s; L^2(\Omega))}^2 \\
+ c \|\rho\|_{L^2(0, s; L^2(\Omega))}^2.
\]
Combining (35) and the above inequality, we obtain
\[
\|\partial_t \theta\|_{L^2(0, s; L^2(\Omega))}^2 + \|\partial_t \rho\|_{L^2(0, s; L^2(\Omega))}^2 + B \|\nabla \theta(s)\|_{L^2(\Omega)}^2 + \|\nabla \rho(s)\|_{L^2(\Omega)}^2 \\
\leq c \|w\|_{L^2(0, s; L^2(\Omega))}^4 + c_1 \|\rho\|_{L^2(0, s; L^2(\Omega))}^2 + c_2 \|\theta\|_{L^2(\Omega)}^2 + c_3 \|\theta\|_{L^2(0, s; L^2(\Omega))}^2.
\]
Noting that, there exist a constant $c_4$ such that $c_4 \geq c_1$ and $c_4 \geq c_3$, we have
\[
c_1 \|\rho\|_{L^2(0, s; L^2(\Omega))}^2 + c_3 \|\theta\|_{L^2(0, s; L^2(\Omega))}^2 \leq c_4 \left( \|\rho\|_{L^2(0, s; L^2(\Omega))}^2 + \|\theta\|_{L^2(0, s; L^2(\Omega))}^2 \right)
\leq c_4 \left( \int_0^s \left( \|\rho(t)\|_{L^2(\Omega)} + \|\theta(t)\|_{L^2(\Omega)}^2 \right) dt \right) \leq c_4 (e^{c_0 s} - 1) \|w\|_{L^2(0, s; L^2(\Omega))}^4.
\]
and a constant $c_5$ such that $c_5 \geq c_2$ and $c_5 \geq C$, we also have
\[
c_2 \|\theta(s)\|_{L^2(\Omega)}^2 + c_5 \|\rho(s)\|_{L^2(\Omega)}^2 \leq c_5 \left( \|\theta(s)\|_{L^2(\Omega)}^2 + \|\rho(s)\|_{L^2(\Omega)}^2 \right)
\leq c_5 \left( e^{c_0 s} \|w\|_{L^2(0, s; L^2(\Omega))}^4 \right).
Finally, we obtain
\[
\|\partial_t \theta\|_{L^2(0,s;L^2(\Omega))}^2 + \|\partial_t \rho\|_{L^2(0,s;L^2(\Omega))}^2 + B \|\nabla \theta(s)\|_{L^2(\Omega)}^2 + \|\nabla \rho(s)\|_{L^2(\Omega)}^2 \leq c \|w\|_{L^2(0,s;L^2(\Omega))}^4.
\]

One can view (32) and (34) as the weak formulations of the following problem
\[
-B\Delta \theta = -\partial_t \theta
\]
\[
+ \mathcal{P} (h(\varphi^w) - h(\varphi^s)) (\sigma^w - \sigma^s) + h(\varphi^s)\rho + \sigma^s h'(\varphi^s) \theta + (\varphi^w - \varphi^s)^2 R
\]
and
\[
-\Delta \rho = -\partial_t \rho - C (h(\varphi^w) - h(\varphi^s)) (\sigma^w - \sigma^s) - C h(\varphi^s)\rho
\]
\[
- C \sigma^s h'(\varphi^s) \theta - C \sigma^s (\varphi^w - \varphi^s)^2 R + \beta \rho.
\]
Thus, using elliptic regularity, we get
\[
\|\theta\|_{L^2(0,s;H^2(\Omega))}^2 + \|\rho\|_{L^2(0,s;H^2(\Omega))}^2 \leq c \|w\|_{L^2(0,s;L^2(\Omega))}^4.
\]
\[
\square
\]

6. Differentiability of the cost functional.

6.1. Differentiability of the cost functional with respect to time. The Fréchet derivative of the reduced cost functional at \((u^*, \tau^*)\) with respect to time is given as follows:
\[
D_{\tau} \mathcal{F}_r(u^*, \tau^*) = \frac{\beta_Q}{2} \int_{\Omega} (|\varphi(\tau^*) - \varphi_Q(\tau^*)|^2 - |\varphi(0) - \varphi_Q(0)|^2) \, dx
\]
\[
+ \frac{\beta\Omega}{2r} \int_{\Omega} (|\varphi(\tau^*) - \varphi_Q(\tau^*)|^2 - |\varphi(\tau^* - r) - \varphi_Q(\tau^* - r)|^2) \, dx
\]
\[
+ \frac{\beta_u}{2r} \int_{\Omega} (\varphi(\tau^*) - \varphi(\tau^* - r)) \, dx + \frac{\beta_u}{2} \int_{\Omega} |u^*(T)|^2 \, dx + \beta_T,
\]

which is equivalent to
\[
D_{\tau} \mathcal{F}_r(u^*, \tau^*)
\]
\[
= \frac{\beta_Q}{2} \|\varphi(\tau^*) - \varphi_Q(\tau^*)\|_{L^2(\Omega)}^2 + \frac{\beta\Omega}{2r} \int_{\Omega} (\varphi(\tau^*) - \varphi(\tau^* - r)) \, dx + \beta_T
\]
\[
+ \frac{\beta_u}{2} \|u(T)\|_{L^2(\Omega)}^2
\]
\[
+ \frac{\beta\Omega}{2r} \left( \|\varphi(\tau^*) - \varphi_Q(\tau^*)\|_{L^2(\Omega)}^2 - \|\varphi(\tau^* - r) - \varphi_Q(\tau^* - r)\|_{L^2(\Omega)}^2 \right),
\]

the proof of the differentiability of the cost functional with respect to time follows as in [28].

6.2. Fréchet differentiability of the cost functional with respect to the control. The Fréchet derivative of the cost functional at \((u^*, \tau^*)\) with respect to the control is given as follows:
\[
D_u \mathcal{F}_r(u^*, \tau^*) w = \beta_Q \int_{\Omega} \int_0^{\tau} \left| \varphi^w - \varphi_Q \right| \Phi^w \, dx \, dt + \beta\Omega \int_{\tau^*-r}^{\tau} \int_{\Omega} \left| \varphi^s - \varphi_Q \right| \Phi^w \, dx \, dt
\]
\[
+ \frac{\beta_u}{r} \int_{\tau^*-r}^{\tau} \int_{\Omega} \Phi^w \, dx \, dt + \beta_u \int_0^{T} \int_{\Omega} u \, w \, dx \, dt.
\]
In order to eliminate the term $\Phi^w$ from the above equation, we apply Lagrangian principle, to this end, we define the Lagrangian function with respective Lagrangian multipliers $p$ and $q$ by

$$L(\varphi, \sigma, u, p, q) = J_r(\varphi, u) - \int_Q (\partial_t \sigma - \Delta \sigma + C \sigma h(\varphi) - \beta(\sigma_s - \sigma)) q \, dx \, dt - \int_Q (\partial_t \varphi - B \Delta \varphi - (\mathcal{P} \sigma - \mathcal{A} - \alpha u) h(\varphi) + f(\varphi)) p \, dx \, dt.$$ 

6.3. **Adjoint system.** The adjoint system is given as follows

$$D_\varphi L(\varphi^*, \sigma^*, u^*, p, q) \varphi = 0$$

and

$$D_\sigma L(\varphi^*, \sigma^*, u^*, p, q) \sigma = 0,$$

where

$$D_\varphi L(\varphi^*, \sigma^*, u^*, p, q) \varphi = \frac{1}{2r} \int_0^{\tau^*} \int_\Omega \chi_{[\tau^*-r, \tau]}(t) (2\beta \Omega (\varphi^* - \varphi_\Omega) + \beta_s) \varphi \, dx \, dt$$

$$+ \int_0^{\tau^*} \int_\Omega (\mathcal{P} \sigma^* - \mathcal{A} - \alpha u^*) h'(\varphi^*) p \varphi \, dx \, dt - \int_0^{\tau^*} \int_\Omega f'(\varphi^*) p \varphi \, dx \, dt - \int_0^{\tau^*} \int_\Omega C \sigma^* h'(\varphi^*) q \varphi \, dx \, dt,$$

and

$$D_\sigma L(\varphi^*, \sigma^*, u^*, p, q) \sigma = \int_0^{\tau^*} \int_\Omega \partial_t q \sigma \, dx \, dt + \int_0^{\tau^*} \int_\Omega \Delta q \sigma \, dx \, dt$$

$$+ \int_0^{\tau^*} \int_\Omega \mathcal{P} h(\varphi^*) p \sigma \, dx \, dt - C \int_0^{\tau^*} \int_\Omega h(\varphi^*) q \sigma \, dx \, dt - \beta \int_0^{\tau^*} \int_\Omega q \sigma \, dx \, dt.$$ 

The adjoint system is then written as

$$-\partial_t p - B \Delta p = (\mathcal{P} \sigma^* - \mathcal{A} - \alpha u^*) h'(\varphi^*) p + \beta Q (\varphi^* - \varphi_Q)$$

$$+ \frac{1}{2r} \chi_{[\tau^*-r, \tau]}(t) (2\beta \Omega (\varphi^* - \varphi_\Omega) + \beta_s) - f'(\varphi^*) p - C \sigma^* h'(\varphi^*) q,$$

in $[0, T] \times \Omega$,

$$-\partial_t q - \Delta q = \mathcal{P} h(\varphi^*) p - (\mathcal{C} h(\varphi^*) + \beta) q,$$

in $[0, T] \times \Omega$,

$$\partial_p p = \partial_q q = 0,$$

on $[0, T] \times \Gamma$,

$$p(\tau^*) = q(\tau^*) = 0,$$

in $\Omega$.

(36)

**Theorem 6.1.** Let $u^*$ be an optimal control, and $(\varphi^*, \sigma^*) = S(u^*)$ be the corresponding state. Then the adjoint Problem (36) has a unique solution $(p, q)$

$$\in (H^1(0, \tau^*; L^2(\Omega)) \cap L^\infty(0, \tau^*; H^1(\Omega))) \cap L^2(0, \tau^*; H^2(\Omega)) \cap C([0, \tau^*]; H^1(\Omega)))^2.$$
Proof. The variational formulations of the adjoint system, for \( \xi \in H^1(\Omega) \), are given by:

\[
- \int_{\Omega} \partial_t p \xi \, dx + B \int_{\Omega} \nabla p \cdot \nabla \xi \, dx = \int_{\Omega} (P\sigma^* - A - \alpha\sigma^*) \, h'(\varphi^*) q \xi \, dx \\
+ \int_{\Omega} \beta Q (\varphi^* - \varphi_Q) \, \xi \, dx + \frac{1}{2r} \int_{\Omega} \chi_{[r,2r]}(t) (2\beta \Omega (\varphi^* - \varphi_\Omega) + \beta_s) \xi \, dx \\
- \int_{\Omega} f'(\varphi^*) p \xi \, dx - \int_{\Omega} C \sigma^* h'(\varphi^*) q \xi \, dx
\]

and

\[
- \int_{\Omega} \partial_t q \xi \, dx + \int_{\Omega} \nabla q \cdot \nabla \xi \, dx = \int_{\Omega} \mathcal{P} h(\varphi^*) p \xi \, dx - \int_{\Omega} (Ch(\varphi^*) + \beta) q \xi \, dx.
\]

Now, we will establish the a priori estimates for (36), for this purpose, we put \( \xi = p \) in (37) and \( \xi = q \) in (38) in addition to using \( f'(\varphi^*)p^2 \geq -p^2 \), to obtain

\[
\frac{d}{dt} \|p\|^2_{L^2(\Omega)} + 2B \|\nabla p\|^2_{L^2(\Omega)} \leq c \|p\|^2_{L^2(\Omega)} + \|\beta Q (\varphi^* - \varphi_Q)\|^2_{L^2(\Omega)} \\
+ \left| \frac{1}{2r} (2\beta \Omega (\varphi^* - \varphi_\Omega) + \beta_s) \right|^2_{L^2(\Omega)} + 2 \|p\|^2_{L^2(\Omega)} + c' \left( \|p\|^2_{L^2(\Omega)} + \|q\|^2_{L^2(\Omega)} \right) \\
\leq c \|p\|^2_{L^2(\Omega)} + c' \|q\|^2_{L^2(\Omega)} + c''.
\]

On the other hand, we have

\[
\frac{d}{dt} \|q\|^2_{L^2(\Omega)} + 2 \|\nabla q\|^2_{L^2(\Omega)} + 2\beta \|q\|^2_{L^2(\Omega)} = -2 \int_{\Omega} Ch(\varphi^*) q^2 \, dx + \int_{\Omega} \mathcal{P} h(\varphi^*) pq \, dx.
\]

Therefore

\[
\frac{d}{dt} \|q\|^2_{L^2(\Omega)} + 2 \|\nabla q\|^2_{L^2(\Omega)} \leq \mathcal{P}^2 \|p\|^2_{L^2(\Omega)} + (2C + 1) \|q\|^2_{L^2(\Omega)}.
\]

Combining (39) and the above inequality, and putting \( c_0 = \min(2; 2\beta) \), we obtain

\[
\frac{d}{dt} \left( \|p\|^2_{L^2(\Omega)} + \|q\|^2_{L^2(\Omega)} \right) + 2B \|\nabla p\|^2_{L^2(\Omega)} + \|q\|^2_{H^1(\Omega)} \\
\leq c \|p\|^2_{L^2(\Omega)} + c' \|q\|^2_{L^2(\Omega)} + c'' \\
\leq c_1 \left( \|p\|^2_{L^2(\Omega)} + \|q\|^2_{L^2(\Omega)} \right) + c' + c''
\]

where \( c_1 \) is a constant depending on \( \mathcal{P}, C, A, \alpha, \beta \), so that \( c \leq c_1 \) and \( c' \leq c_1 \).

Integrating over time and applying Gronwall’s lemma, we find

\[
\|p(s)\|^2_{L^2(\Omega)} + \|q(s)\|^2_{L^2(\Omega)} + 2B \|\nabla p\|^2_{L^2(0,s,L^2(\Omega))} + \|q\|^2_{L^2(0,s,H^1(\Omega))} \leq c''e^{c_1s}, \quad s \in (0, \sigma^*],
\]

where \( c_2 \) depends on \( c'' \), \( \|p_0\|^2_{L^2(\Omega)} \) and \( \|q_0\|^2_{L^2(\Omega)} \). Furthermore, putting \( \xi = \partial_t p \) in (37) and \( \xi = \partial_t q \) in (38), we obtain

\[
\|\partial_t p\|^2_{L^2(\Omega)} + B \frac{d}{dt} \|\nabla p\|^2_{L^2(\Omega)} + \|\partial_t q\|^2_{L^2(\Omega)} + \frac{d}{dt} \|\nabla q\|^2_{L^2(\Omega)} \\
\leq c \|p\|^2_{L^2(\Omega)} + c' \|\partial_t p\|^2_{L^2(\Omega)} + c_1 \|q\|^2_{L^2(\Omega)} + c_2 \|\partial_t q\|^2_{L^2(\Omega)} + c_3 \frac{d}{dt} \|p\|^2_{L^2(\Omega)} + c' + c''
\]
Consequently,
\[
\|\partial_t p\|^2_{L^2(0,s;L^2(\Omega))} + B \|\nabla p(s)\|^2_{L^2(\Omega)} + \|\nabla q(s)\|^2_{L^2(\Omega)} + \|\partial_t q\|^2_{L^2(0,s;L^2(\Omega))} \\
\leq c \|p\|^2_{L^2(0,s;L^2(\Omega))} + c_1 \|q\|^2_{L^2(0,s;L^2(\Omega))} + c \|p(s)\|^2_{L^2(\Omega)} + c''s \\
+ B \|\nabla p(0)\|^2_{L^2(\Omega)} + B \|\nabla q(0)\|^2_{L^2(\Omega)}.
\] (41)

However, due to (40), we have
\[
\|p\|^2_{L^2(0,s;L^2(\Omega))} + \|q\|^2_{L^2(0,s;L^2(\Omega))} = \int_0^s \|p(t)\|^2_{L^2(\Omega)} dt + \int_0^s \|q(t)\|^2_{L^2(\Omega)} dt \\
\leq \int_0^s c''e^{c't} dt \\
\leq c (e^{c's} - 1).
\]

As well, we know that
\[
\left( \left( \frac{d}{dt} (p)^2 \right) \right)^2 + (p)^2 \leq c \|p\|^2_{L^2(\Omega)} + \|\beta Q (\varphi^* - \varphi_Q)\|^2_{L^2(\Omega)} + c \|q\|^2_{L^2(\Omega)} \\
+ \left\| \frac{1}{2r} (2\beta_1 (\varphi^* - \varphi_1) + \beta_s) \right\|^2_{L^2(\Omega)}.
\]

Adding the above inequality to (41), and integrating over time, \(s \in (0, \tau^*]\), we find
\[
\|\partial_t p\|^2_{L^2(0,s;L^2(\Omega))} + B \|\nabla p(s)\|^2_{L^2(\Omega)} + \|\nabla q(s)\|^2_{H^1(\Omega)} + \|\partial_t q\|^2_{L^2(0,s;L^2(\Omega))} \leq c.
\]

Therefore, we deduce that
\[
(p, q) \in (H^1(0, \tau^*; L^2(\Omega)) \cap L^\infty(0, \tau^*; H^1(\Omega)) \cap C([0, \tau^*]; L^2(\Omega)))^2.
\]

Rewriting the adjoint equations as
\[
-B \Delta p = \partial_t p + (\mathcal{P} \sigma^* - A - \alpha u^*) h'(\varphi)p + \beta Q (\varphi^* - \varphi_Q) \\
+ \frac{1}{2r} \chi_{[\tau^* - r, \tau^*]}(t) (2\beta_1 (\varphi^* - \varphi_1) + \beta_s) - C_\sigma h'(\varphi^*)q - f'(\varphi^*)p, \quad \text{in } [0, T] \times \Omega, \\
-\Delta q = \partial_t q + \mathcal{P} h(\varphi^*)p - (Ch(\varphi^*) + \beta) q, \quad \text{in } [0, T] \times \Omega, \\
\partial_n p = \partial_n q = 0, \quad \text{on } [0, T] \times \Gamma, \\
p(\tau^*) = q(\tau^*) = 0, \quad \text{in } \Omega,
\]

and thanks to elliptic regularity, we deduce that \((p, q) \in (L^2(0, \tau^*, H^2(\Omega)))^2\).

**Existence of solution for the adjoint system.** We apply a Galerkin approximation and consider a basis \(\{\omega_i\}_{i \in \mathbb{N}}\) of \(H^1(\Omega)\) that is orthonormal in \(L^2(\Omega)\). Look for the functions
\[
p_{n,i}(x, t) = \sum_{i=1}^n P_{n,i}(t) \omega_i(x) \quad \text{and} \quad q_{n,i}(t, x) = \sum_{i=1}^n Q_{n,i}(t) \omega_i(x)
\]
such that
\[
p_n(0) = p_0^n, \quad q_n(0) = q_0^n, \quad p_n(\tau^*) = q_n(\tau^*) = 0.
\]
which satisfy the following approximate problem:

\[ -\int_\Omega \partial_t p_n v dx + B \int_\Omega \nabla p_n \cdot \nabla v dx = \int_\Omega (\mathcal{P} \sigma^* - \mathcal{A} - \alpha u^*) h'(\varphi^*) p_n v dx \\
+ \int_\Omega \left( \beta Q (\varphi^* - \varphi Q) + \frac{1}{2r} \chi_{[r^*, r^*, r^*, r^*]}(t) (2\beta \Omega (\varphi^* - \varphi \Omega) + \beta_s) \right) v dx \\
- \int_\Omega C \sigma^* h'(\varphi^*) q_n v dx - \int_\Omega f'(\varphi^*) p_n v dx \]

and

\[ -\int_\Omega \partial_t q_n v dx + \int_\Omega \nabla q_n \cdot \nabla v dx = \int_\Omega \mathcal{P} h(\varphi^*) p_n v dx - \int_\Omega (\mathcal{C} h(\varphi^*) + \beta) q_n v dx, \quad (42) \]

for all \( v \in V_n = \text{span} \{ \omega_i, i = 1, \ldots, n \} \). In particular, put \( v = \omega_j \) in (42) and (43) to get, \( \forall i, j = 1, \ldots, n \),

\[ \frac{d}{dt} \sum_{i=1}^{n} P_{n,i}(t) \int_\Omega \omega_i(x) \omega_j(x) dx + B \sum_{i=1}^{n} P_{n,i}(t) \int_\Omega \nabla \omega_i(x) \cdot \nabla \omega_i(x) dx \\
= \sum_{i=1}^{n} P_{n,i}(t) \int_\Omega (\mathcal{P} \sigma^* - \mathcal{A} - \alpha^*) h'(\varphi^*) \omega_i(x) \omega_j(x) dx \\
+ \int_\Omega \beta Q (\varphi^* - \varphi Q) \omega_j(x) dx + \int \frac{1}{2r} \chi_{[r^*, r^*, r^*, r^*]}(t) (2\beta \Omega (\varphi^* - \varphi \Omega) + \beta_s) \omega_j(x) dx \\
- \sum_{i=1}^{n} Q_{n,i}(t) \int_\Omega C \sigma^* h'(\varphi^*) \omega_i(x) \omega_j(x) dx - \sum_{i=1}^{n} P_{n,i}(t) \int_\Omega f'(\varphi^*) \omega_i(x) \omega_j(x) dx \]

and

\[ \frac{d}{dt} \sum_{i=1}^{n} Q_{n,i}(t) \int_\Omega \omega_i \omega_j dx + \sum_{i=1}^{n} Q_{n,i}(t) \int_\Omega \nabla \omega_i(x) \cdot \nabla \omega_j(x) dx \\
= \sum_{i=1}^{n} P_{n,i}(t) \int_\Omega \mathcal{P} h(\varphi^*) \omega_i(x) \omega_j(x) dx - \sum_{i=1}^{n} Q_{n,i}(t) \int_\Omega (\mathcal{C} h(\varphi^*) + \beta) \omega_i(x) \omega_j(x) dx. \]

Consequently, the above equation and (44) are ODEs of the following forms:

\[ -P_n(t)' I_n + BP_n(t) J + L + \chi_{[r^*, r^*, r^*, r^*]}(t) M - Q_n(t) H - P_n(t) F \quad (45) \]

and

\[ -Q_n(t)' I_n + Q_n(t) J = P_n(t) N - Q_n(t) R, \quad (46) \]

with the conditions \( p_n(x, \tau^*) = q_n(x, \tau^*) = 0 \). Here

\[ \left\{ \begin{array}{ll}
J_{i,j} = \int_\Omega \nabla \omega_i(x) \cdot \nabla \omega_j(x) dx, \\
K_{i,j} = \int_\Omega (\mathcal{P} \sigma^* - \mathcal{A} - \alpha^*) h'(\varphi^*) \omega_i(x) \omega_j(x) dx, \\
N_{i,j} = \int_\Omega \mathcal{P} h(\varphi^*) \omega_i(x) \omega_j(x) dx, \\
R_{i,j} = \int_\Omega (\mathcal{C} h(\varphi^*) + \beta) \omega_i(x) \omega_j(x) dx, \\
H_{i,j} = \int_\Omega C \sigma^* h'(\varphi^*) \omega_i(x) \omega_j(x) dx, \\
F_{i,j} = \int_\Omega f'(\varphi^*) \omega_i(x) \omega_j(x) dx, \\
L_j = \int_\Omega \beta Q (\varphi^* - \varphi Q) \omega_j(x) dx, \\
M_j = \int \frac{1}{2r} (2\beta \Omega (\varphi^* - \varphi \Omega) + \beta_s) \omega_j(x) dx.
\right. \]
First, we will consider the Cauchy Problem (45)-(46) on the interval \((\tau^*-r, \tau^*)\), so that it has the form
\[
P_n(t)'I_n = BP_n(t)J - P_n(t)K - L - M - Q_n(t)H + P_n(t)F
\]
and
\[
Q_n(t)'I_n = Q_n(t)J - P_n(t)N + Q_n(t)R.
\]
In other words, we have
\[
(x'(t), y'(t)) = (f(t, x(t), y(t)), g(t, x(t), y(t)))
\]
The right hand side of the above equation is locally Lipschitz with respect to \((x, y)\), hence according to Cauchy Lipschitz theorem, the Problem (45)-(46) has a unique solution on \((\tau^*-r, \tau^*)\), where \(s \in (\tau^*-r, \tau^*)\).

As well, we consider the System (45)-(46) on the interval \((0, \tau^*-r)\), we have
\[
P_n(t)'I_n = BP_n(t)J - P_n(t)K - L + Q_n(t)H
\]
and
\[
Q_n(t)'I_n = Q_n(t)J - P_n(t)N + Q_n(t)R.
\]
We then rewrite System (47)-(48) as
\[
(x'(t), y'(t)) = (f(t, x(t), y(t)), g(t, x(t), y(t)))
\]
where \(f(t, x(t), y(t)) = BxJ - xK - L + yH\) and \(g(t, x(t), y(t)) = yJ - xN + yR\).
The function \((f(t, x(t), y(t)), g(t, x(t), y(t)))\) is locally Lipschitz with respect to \((x, y)\), so Cauchy Lipschitz theorem implies that the System (47)-(48) has a unique solution on the interval \((0, s)\) where \(s \in (0, \tau^*-r)\).

It follows from the a priori estimates derived formally in the previous section, but for the approximated solution \((p_n, q_n)\), that the solution is global and is defined on the whole interval \([0, \tau^*]\). Then the a priori estimates yields that up to a relabeled subsequence,

\[
(p_n, q_n) \rightharpoonup (p, q), \ \text{weakly star in } L^\infty(0, \tau^*, L^2(\Omega))
\]
and

\[
(p_n, q_n) \rightarrow (p, q), \ \text{weakly in } L^2(0, \tau^*, H^1(\Omega)).
\]
Moreover \((p_n, q_n) \in C([0, \tau^*], L^2(\Omega))^2\).

**Passing to the limit.** Recall that, the variational formulations of the approximate system are given by
\[
-\frac{d}{dt}(p_n, v)_{L^2(\Omega)} + B(\nabla p_n, \nabla v)_{L^2(\Omega)} = ((P\sigma^* - A - \alpha u^*) h'(\varphi^*)p_n, v)_{L^2(\Omega)}
\]
\[
\left. + \left(\beta_Q(\varphi^* - \varphi) + \frac{1}{2r} \chi_{[\tau^* - r, \tau^*]}(t) (2\beta_Q(\varphi^* - \varphi^\Omega) + \beta_s)\right), v \right)_{L^2(\Omega)}
\]
\[-((C\sigma^* h'(\varphi^*)q_n, v)_{L^2(\Omega)} - (f'(\varphi^*)p, v)
\]
and
\[
-\frac{d}{dt}(q_n, v)_{L^2(\Omega)} + (\nabla q_n, \nabla v)_{L^2(\Omega)} = (PH(\varphi^*)p, v)_{L^2(\Omega)} - ((Ch(\varphi^*) + \beta) q_n, v)_{L^2(\Omega)}.
\]
Integrating by parts over \([0, \tau^*]\), \(\forall \psi \in D([0, \tau^*])\), we obtain

\[
\int_0^{\tau^*} (p_n(t), v\psi'(t))_{L^2(\Omega)} \, dt + B \int_0^{\tau^*} (\nabla p_n(t), \nabla v\psi(t))_{L^2(\Omega)} \, dt
\]

\[
= \int_0^{\tau^*} ((\mathcal{P}\sigma^* - A - \alpha u^*) h'(\varphi^*) p_n(t), v\psi(t))_{L^2(\Omega)} \, dt + (p_0^n, v\psi(0))_{L^2(\Omega)}
\]

\[
+ \int_0^{\tau^*} \left( \beta_P (\varphi^* - \varphi_Q) + \frac{1}{2r} \chi_{[\tau^*-r, \tau^*]}(t) (2\beta_Q (\varphi^* - \varphi_Q) + \beta_s), v\psi(t) \right)_{L^2(\Omega)} \, dt
\]

\[- \int_0^{\tau^*} (\mathcal{C}\sigma^* h'(\varphi^*) q_n, v\psi(t))_{L^2(\Omega)} \, dt - \int_0^{\tau^*} \left( f'(\varphi^*) p_n, v\psi(t) \right)_{L^2(\Omega)} \, dt
\]

and

\[
\int_0^{\tau^*} (q_n(t), v\psi'(t))_{L^2(\Omega)} \, dt + \int_0^{\tau^*} (\nabla q_n(t), \nabla v\psi(t))_{L^2(\Omega)} \, dt = (q_0^n, v\psi(0))_{L^2(\Omega)}
\]

\[+ \int_0^{\tau^*} (\mathcal{P} h(\varphi^*) p_n(t), v\psi(t))_{L^2(\Omega)} \, dt - \int_0^{\tau^*} \left( (\mathcal{C} h(\varphi^*) + \beta) q_n(t), v\psi(t) \right)_{L^2(\Omega)} \, dt.
\]

Noting that

\[
\int_0^{\tau^*} (p_n(t), v\psi'(t))_{L^2(\Omega)} \, dt + B \int_0^{\tau^*} (\nabla p_n(t), \nabla v\psi(t))_{L^2(\Omega)} \, dt
\]

\[\rightarrow \int_0^{\tau^*} (p(t), v\psi'(t))_{L^2(\Omega)} \, dt + B \int_0^{\tau^*} (\nabla p(t), \nabla v\psi(t))_{L^2(\Omega)} \, dt
\]

and

\[
\int_0^{\tau^*} ((\mathcal{P}\sigma^* - A - \alpha u^*) h'(\varphi^*) p_n(t), v\psi(t))_{L^2(\Omega)} \, dt + (p_0^n, v\psi(0))_{L^2(\Omega)}
\]

\[\rightarrow \int_0^{\tau^*} ((\mathcal{P}\sigma^* - A - \alpha u^*) h'(\varphi^*) p(t), v\psi(t))_{L^2(\Omega)} \, dt + (p_0, v\psi(0))_{L^2(\Omega)}.
\]

On the other hand

\[
\int_0^{\tau^*} (q_n(t), v\psi'(t))_{L^2(\Omega)} \, dt + \int_0^{\tau^*} (\nabla q_n(t), \nabla v\psi(t))_{L^2(\Omega)} \, dt - (q_0^n, v\psi(0))_{L^2(\Omega)}
\]

\[\rightarrow \int_0^{\tau^*} (q(t), v\psi'(t))_{L^2(\Omega)} \, dt + \int_0^{\tau^*} (\nabla q(t), \nabla v\psi(t))_{L^2(\Omega)} \, dt - (q_0, v\psi(0))_{L^2(\Omega)}
\]

and

\[
\int_0^{\tau^*} (\mathcal{P} h(\varphi^*) p_n(t), v\psi(t))_{L^2(\Omega)} \, dt - \int_0^{\tau^*} \left( (\mathcal{C} h(\varphi^*) + \beta) q_n(t), v\psi(t) \right)_{L^2(\Omega)} \, dt
\]

\[\rightarrow \int_0^{\tau^*} (\mathcal{P} h(\varphi^*) p(t), v\psi(t))_{L^2(\Omega)} \, dt - \int_0^{\tau^*} \left( (\mathcal{C} h(\varphi^*) + \beta) q(t), v\psi(t) \right)_{L^2(\Omega)} \, dt.
\]
Using (37) and (38), as well as the fact that $\bigcup V_n$ is dense in $H^1(\Omega)$, we find, for all $v \in V_n$, $\psi \in D([0, \tau^*])$

$$
\int_0^{\tau^*} (p(t), v\psi(t))_{L^2(\Omega)} dt + B \int_0^{\tau^*} (\nabla p(t), \nabla v\psi(t))_{L^2(\Omega)} dt = (p(0), v\psi(0))_{L^2(\Omega)}
$$

$$
+ \int_0^{\tau^*} ((P\sigma^* - A - \alpha u^*) h'(\varphi^*)p(t), v\psi(t))_{L^2(\Omega)} dt
$$

$$
+ \int_0^{\tau^*} (\beta_Q (\varphi^* - \varphi_\Omega), v\psi(t))_{L^2(\Omega)} dt
$$

$$
+ \frac{1}{2r} \int_0^{\tau^*} (\chi_{[r-r^*, r^*]}(t) (2\beta_\Omega (\varphi^* - \varphi_\Omega) + \beta_\alpha), v\psi(t))_{L^2(\Omega)} dt
$$

$$
- \int_0^{\tau^*} (C\sigma^* h'(\varphi^*)q, v\psi(t))_{L^2(\Omega)} dt - \int_0^{\tau^*} (f'(\varphi^*)p, v\psi(t))_{L^2(\Omega)} dt
$$

and

$$
\int_0^{\tau^*} (q(t), v\psi'(t))_{L^2(\Omega)} dt + \int_0^{\tau^*} (\nabla q(t), \nabla v\psi(t))_{L^2(\Omega)} dt = (q(0), v\psi(0))_{L^2(\Omega)}
$$

$$
+ \int_0^{\tau^*} (P\varphi^*) p(t), v\psi(t))_{L^2(\Omega)} dt - \int_0^{\tau^*} ((C\varphi^*) + \beta) q(t), v\psi(t))_{L^2(\Omega)} dt.
$$

Thus, it follows that $p(0) = p_0$, and $q(0) = q_0$ a.e. in $L^2(\Omega)$. \qed

7. Simplification of the first-order necessary optimality condition. Let $(u^*, \tau^*)$ be a minimizer of the problem with corresponding state variables $(\varphi^*, \sigma^*) = S(u^*)$, and adjoint variables $(p, q)$ associated to $(\varphi^*, \sigma^*)$. Let $w := u - u^* \in L^2(Q)$ for any $u \in U_{ad}$, and let $(\Phi, \Sigma)$ be the linearized state variables associated to $w$.

**Proposition 1.** The optimal control $u^*$ and the optimal treatment time $\tau^*$ satisfy the following simplified first-order necessary optimality conditions

$$
(D_u J_r(u^*, \tau^*))(u - u^*) = \beta_u \int_0^\tau \int_\Omega u^*(u - u^*) dx dt + \alpha \int_0^\tau \int_\Omega h(\varphi^*)(u - u^*)p dx dt \geq 0
$$

and

$$
D_{\tau} J_r(u^*, \tau^*)(s - \tau^*) \geq 0, \forall s \in [r, T].
$$

**Proof.** Setting $\varphi(t) = \varphi_0$ for $t \leq 0$, and $\varphi_Q \in H^1(0, T; L^2(\Omega)), \varphi^*, \varphi_\Omega \in H^1(-r, T; L^2(\Omega))$, then the cost functional (7) can be expressed as

$$
J_r(\varphi, u, \tau) = \frac{\beta_u}{2} \|w\|^2_{L^2(Q)} + \beta_T \tau + \frac{\beta_Q}{2} \int_0^\tau \|\varphi - \varphi_Q\|^2_{L^2(\Omega)} dt
$$

$$
+ \frac{\beta_\Omega}{2r} \int_0^\tau \int_\Omega |(\varphi - \varphi_\Omega)(t)|^2 dx dt
$$

$$
+ \frac{\beta_S}{2r} \int_0^\tau \int_\Omega |(\varphi(t) - \varphi(t-r))| dx dt + \frac{\beta_\Omega}{2r} \int_{-r}^0 \int_\Omega (\varphi_0 - \varphi_\Omega(t)) dx dt
$$

$$
+ \frac{\beta_S}{2r} \int_{-r}^0 \int_\Omega (1 + \varphi_0) dx dt.
$$
We have
\[
D_r J_r(u^*, \tau^*) = \beta r + \frac{\beta Q}{2} \| \varphi^*(\tau^*) - \varphi Q(\tau^*) \|^2_{L^2(\Omega)} + \frac{\beta S}{2r} \int_{\Omega} (\varphi^*(\tau^*) - \varphi^*(\tau^* - r)) \, dx \\
+ \frac{\beta Q}{2r} \left( \| (\varphi^* - \varphi_Q)(\tau^*) \|^2_{L^2(\Omega)} - \| (\varphi^* - \varphi_Q)(\tau^* - r) \|^2_{L^2(\Omega)} \right).
\]

So, the optimal control and time \((u^*, \tau^*)\) satisfy the following first order necessary optimality condition with respect to time
\[
D_r J_r(u^*, \tau^*)(s - \tau^*) \geq 0, \quad \forall s \in [r, T].
\]

This condition can be simplified by taking the following arguments on \(s\). If \(\tau^* \in (r, T)\), take \(s = \tau^* + h\) or \(s = \tau^* - h\) for \(h > 0\), then we obtain \(D_r J_r(u^*, \tau^*) = 0\).

If \(\tau^* = r\), then \(s - \tau^* \geq 0\) for any \(s \in [r, T]\), so we have \(D_r J_r(u^*, \tau^*) \geq 0\). Finally, if \(\tau^* = T\), we deduce that \(D_r J_r(u^*, \tau^*) \leq 0\).

The optimal control and time \((u^*, \tau^*)\) satisfy the following first-order necessary optimality condition for the control
\[
(D_u J_r(u^*, \tau^*))(u - u^*) = (D_u J_r(u^*, \tau^*))w = \beta Q \int_0^T \int_{\Omega} (\varphi^* - \varphi_Q) \Phi dx \, dt \\
+ \frac{\beta Q}{r} \int_0^{\tau^*} \int_{\Omega} (\varphi^* - \varphi_Q) \Phi dx \, dt + \frac{\beta S}{2r} \int_{\tau^* - r}^{\tau^*} \int_{\Omega} \Phi dx \, dt + \beta_0 \int_0^T \int_{\Omega} u^* w dx \, dt \geq 0.
\]

In order to simplify this condition, put \(\xi = \Phi\) in (37), \(\xi = \Sigma\) in (38) and integrate over \([0, \tau^*]\), which yields
\[
-\int_0^{\tau^*} \int_{\Omega} \partial_t p \Phi \, dx \, dt = -B \int_0^{\tau^*} \int_{\Omega} \nabla p \cdot \nabla \Phi \, dx \, dt \\
+ \int_0^{\tau^*} \int_{\Omega} (\mathcal{P} \sigma^* - \mathcal{A} - \alpha u^*) h'(\varphi^*) p \Phi \, dx \, dt \\
+ \int_0^{\tau^*} \int_{\Omega} \beta_0 (\varphi^* - \varphi_Q) \Phi \, dx \, dt + \int_{\tau^* - r}^{\tau^*} \int_{\Omega} \frac{1}{2r} (2\beta_0 (\varphi^* - \varphi_Q) + \beta_s) \Phi \, dx \, dt \\
- \int_0^{\tau^*} \int_{\Omega} C \sigma^* h'(\varphi^*) q \Phi \, dx \, dt - \int_0^{\tau^*} \int_{\Omega} f'(\varphi^*) p \Phi \, dx \, dt
\]
and
\[
-\int_0^{\tau^*} \int_{\Omega} \partial_t q \Sigma \, dx \, dt = -\int_0^{\tau^*} \int_{\Omega} \nabla q \cdot \nabla \Sigma \, dx \, dt \\
+ \int_0^{\tau^*} \int_{\Omega} \mathcal{P} h(\varphi^*) p \Sigma \, dx \, dt - \int_0^{\tau^*} \int_{\Omega} (\mathcal{C}(\varphi^*) + \beta) q \Sigma \, dx \, dt.
\]
On the other hand, substituting \(\xi = p\) in (25) and \(q = \Sigma\) in (26) leads to
\[
\int_0^{\tau^*} \int_{\Omega} \partial_t \Phi p \, dx \, dt + B \int_0^{\tau^*} \int_{\Omega} \nabla \Phi \cdot \nabla p \, dx \, dt = \int_0^{\tau^*} \int_{\Omega} (\mathcal{P} \Sigma - \alpha w) h(\varphi^*) p \, dx \, dt \\
+ \int_0^{\tau^*} \int_{\Omega} (\mathcal{P} \sigma^* - \mathcal{A} - \alpha u^*) h'(\varphi^*) \Phi p \, dx \, dt - \int_0^{\tau^*} \int_{\Omega} f'(\varphi^*) \Phi \, dx \, dt
\]
and
\[
\int_{0}^{\tau^*} \int_{\Omega} \partial_t \Sigma q \, dx \, dt + \int_{0}^{\tau^*} \int_{\Omega} \nabla \Sigma \cdot \nabla q \, dx \, dt
= - \int_{0}^{\tau^*} \int_{\Omega} (Ch(\varphi^*) + \beta) \Sigma q \, dx \, dt - \int_{0}^{\tau^*} \int_{\Omega} C\sigma^* h'(\varphi^*) \Phi q \, dx \, dt.
\]

Due to the facts that $p(\tau^*) = q(\tau^*) = 0$, and $\Phi(0) = \Sigma(0) = 0$, and upon integrating by parts with respect to time, we obtain
\[
- \int_{0}^{\tau^*} \int_{\Omega} \partial_t p \Phi \, dx \, dt = \int_{0}^{\tau^*} \int_{\Omega} p \partial_t \Phi \, dx \, dt.
\]
Consequently, substituting (50) into (52), we obtain
\[
\int_{0}^{\tau^*} \int_{\Omega} \beta Q(\varphi^* - \varphi Q) \Phi \, dx \, dt + \int_{0}^{\tau^*} \int_{\Omega} \frac{1}{2r} (2\beta_\Omega (\varphi^* - \varphi \Omega) + \beta_s) \Phi \, dx \, dt
= \int_{0}^{\tau^*} \int_{\Omega} (P \Sigma - \alpha w) h(\varphi^*) p \, dx \, dt + \int_{0}^{\tau^*} \int_{\Omega} C\sigma^* h'(\varphi^*) q \Phi \, dx \, dt.
\]

As well, upon substituting (51) into (53), we obtain
\[
\int_{0}^{\tau^*} \int_{\Omega} \beta Q(\varphi^* - \varphi Q) \Phi \, dx \, dt + \int_{0}^{\tau^*} \int_{\Omega} \frac{1}{2r} (2\beta_\Omega (\varphi^* - \varphi \Omega) + \beta_s) \Phi \, dx \, dt
+ \alpha \int_{0}^{\tau^*} \int_{\Omega} h(\varphi^*) wp \, dx \, dt = 0.
\]

Substituting the above equation into (49) to obtain
\[
(D_u J_r(u^*, \tau^*))(u - u^*)
= \beta_u \int_{0}^{\tau^*} \int_{\Omega} u^*(u - u^*) \, dx \, dt + \alpha \int_{0}^{\tau^*} \int_{\Omega} h(\varphi^*)(u - u^*) p \, dx \, dt \geq 0.
\]

\[\square\]

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Received August 2021; revised November 2021; early access January 2022.

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