Hyers-Ulam Stability of Pompeiu’s Point

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Abstract. In this paper, we investigate the stability of Pompeiu’s points in the sense of Hyers-Ulam.

1. Introduction

In 1946, Pompeiu [8] derived a variant of Lagrange’s mean value theorem, now known as Pompeiu’s mean value theorem.

Definition 1.1. For every real valued function $f$ differentiable on an interval $[a,b]$ not containing 0 and for all pairs $x_1 \neq x_2$ in $[a,b]$, there exists a point $\xi$ in $(x_1, x_2)$ such that
\[ \frac{x_1f(x_2) - x_2f(x_1)}{x_1 - x_2} = f(\xi) - \xi f'(\xi). \]

Such an intermediate point $\xi$ will be called Pompeiu’s point of the function $f$. The geometric meaning of this is that the tangent at the point $(\xi, f(\xi))$ intersects on the $y$-axis at the same point as the secant line connecting the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

In 1954, Hyers and Ulam [4] considered the stability of differential expressions and proved the following theorem, by which many mathematicians have obtained some interesting theorems.

Theorem 1.2. Let $f : \mathbb{R} \to \mathbb{R}$ be $n$-times differentiable in a neighborhood $N$ of the point $\eta$. Suppose that $f^{(n)}(\eta) = 0$ and $f^{(n)}(x)$ changes sign at $\eta$. Then, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for each function $h : \mathbb{R} \to \mathbb{R}$ which is $n$-times differentiable in $N$ and satisfies $|h(x) - f(x)| < \delta$ in $N$, there exists a point $\xi$ in $N$ such that $h^{(n)}(\xi) = 0$ and $|\xi - \eta| < \varepsilon$.

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Let \([a, b] \subset \mathbb{R}\) be a closed interval and
\[
\phi = \{ f : [a, b] \to \mathbb{R} \mid \text{f is continuously differentiable, } f'(a) = f'(b) \}.
\]

In 2003, M. Das, T. Riedel and P. K. Sahoo [1] gave a Hyers-Ulam type stability result for Flett’s points.

**Theorem 1.3.** Let \(f \in \phi\) and \(\eta\) be a Flett’s point of \(f\) in \((a, b)\). Assume that there is a neighborhood \(N\) of \(\eta\) in \((a, b)\) such that \(\eta\) is the unique Flett’s point of \(f\) in \(N\). Then for each \(\varepsilon > 0\), there exists a \(\delta > 0\) such that for every \(h \in \phi\) satisfying \(h(a) = f(a)\) and \(\left| h(x) - f(x) \right| < \delta\) for all \(x\) in \(N\), there exists a point \(\xi \in N\) such that \(\xi\) is a Flett’s point of \(h\) and \(\left| \xi - \eta \right| < \delta\).

Unfortunately, there are some errors in the proof of M. Das et al.. In 2009, W. Lee, S. Xu and F. Ye [6] constructed a counter example to show that theorem is incorrect, then they proved the Hyers-Ulam stability of the Sahoo-Riedel’s point, and as a corollary they got the correct theorem of the stability of Flett’s point.

**Theorem 1.4.** Let \(f, h : [a, b] \to \mathbb{R}\) be differentiable and \(\eta\) be a Sahoo-Riedel’s point of \(f\) in \((a, b)\). If \(f\) has 2nd derivative at \(\eta\) and \(f''(\eta)(\eta - a) - 2f'(\eta) + \frac{2(f(\eta) - f(a))}{\eta - a} \neq 0\), then corresponding to any \(\varepsilon > 0\) and any neighborhood \(N \subset (a, b)\) of \(\eta\), there exists a \(\delta > 0\) such that for every \(h \in \phi\) satisfying \(|h(x) - f(x)| < \delta\) for all \(x\) in \(N\) and \(h'(b) - h'(a) = f'(b) - f'(a)\), there exists a point \(\xi \in N\) such that \(\xi\) is a Sahoo-Riedel’s point of \(h\) and \(\left| \xi - \eta \right| < \varepsilon\).

In 2010, P. Găvruţă, S.-M. Jung and Y. Li [3] investigated the stability of the Lagrange’s mean value points.

**Theorem 1.5.** Let \(a, b, \eta\) be real numbers satisfying \(a < \eta < b\). Assume that \(f : \mathbb{R} \to \mathbb{R}\) is a twice continuously differentiable function and \(\eta\) is the unique Lagrange’s mean value point of \(f\) in an open interval \((a, b)\) and moreover that \(f''(\eta) \neq 0\). Suppose \(g : \mathbb{R} \to \mathbb{R}\) is a differentiable function. Then, for a given \(\varepsilon > 0\), there exists a \(\eta > 0\) such that if \(|f(x) - g(x)| < \eta\) for all \(x \in [a, b]\), then there is a Lagrange’s mean value point \(\xi \in (a, b)\) of \(g\) with \(\left| \xi - \eta \right| < \varepsilon\).

In this paper, we prove the Hyers-Ulam stability of Pompeiu’s point by employing the ideas of theorem 1.3, 1.4 and 1.5.

2. Hyers-Ulam Stability of Pompeiu’s Point

In this section, we investigate the stability of the Pompeiu’s point.

**Theorem 2.1.** Let \(f, h : [a, b] \to \mathbb{R}\) be differentiable and \(\eta\) be a Pompeiu’s point of \(f\). If \(f\) has 2nd derivative at \(\eta\) with
\(f''(\eta) \neq 0\),
then corresponding to any $\varepsilon > 0$, there exists a $\delta > 0$ such that for every $h$ satisfying $|h(t) - f(t)| < \delta$ for all $t \in [a, b]$, there exists a point $\xi \in (a, b)$ such that $\xi$ is a Pompeiu’s point of $h$ with $|\xi - \eta| < \varepsilon$.

**Proof.** Without loss of generality, we shall assume that $a, b > 0$. Define a real valued function $F$ on the interval $[\frac{1}{b}, \frac{1}{a}]$ by

$$F(t) = tf\left(\frac{1}{t}\right).$$

Since $f$ is differentiable on $[a, b]$, and $0$ is not in $[a, b]$, we see that $F$ is differentiable on $\left(\frac{1}{b}, \frac{1}{a}\right)$ and

$$F'(t) = f\left(\frac{1}{t}\right) - \frac{1}{t}f'\left(\frac{1}{t}\right).$$

Consider the auxiliary function $G_F(t) : [\frac{1}{b}, \frac{1}{a}] \to \mathbb{R}$ corresponding to $F$ defined by

$$G_F(t) = F(t) - \frac{F\left(\frac{1}{b}\right) - F\left(\frac{1}{a}\right)}{\frac{1}{b} - \frac{1}{a}}(t - \frac{1}{a}).$$

Since $\eta$ is a Pompeiu’s point, we have

$$G_F'(\frac{1}{\eta}) = F'(\frac{1}{\eta}) - \frac{F\left(\frac{1}{b}\right) - F\left(\frac{1}{a}\right)}{\frac{1}{b} - \frac{1}{a}} = f(\eta) - \eta f'(\eta) - \frac{af(b) - bf(a)}{a - b} = 0.$$ 

Moreover, by the assumption that $f''(\eta) \neq 0$, we obtain that

$$G_F''(\frac{1}{\eta}) = F''(\frac{1}{\eta}) = \eta^3 f''(\eta) \neq 0,$$

which implies $G_F'(t)$ changes sign at $\frac{1}{\eta}$.

According to theorem 1.2, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for each function $\phi : [\frac{1}{b}, \frac{1}{a}] \to \mathbb{R}$ which is differentiable in $\left(\frac{1}{b}, \frac{1}{a}\right)$ and satisfies $|\phi(t) - G_F(t)| < \frac{\delta}{a}$ in $[\frac{1}{b}, \frac{1}{a}]$, there exists a point $\xi_0$ in $\left(\frac{1}{b}, \frac{1}{a}\right)$ such that $\phi'(\xi_0) = 0$ and $|\xi_0 - \frac{1}{a}| < \frac{\delta}{a^2}.$

Now, let us define differentiable functions $H$ and $G_H$ by

$$H(t) = th\left(\frac{1}{t}\right)$$

and

$$G_H(t) = H(t) - \frac{H\left(\frac{1}{b}\right) - H\left(\frac{1}{a}\right)}{\frac{1}{b} - \frac{1}{a}}(t - \frac{1}{a}).$$

Recall $F(t) = tf\left(\frac{1}{t}\right)$, we have

$$|H(t) - F(t)| = \left|th\left(\frac{1}{t}\right) - tf\left(\frac{1}{t}\right)\right| \leq \frac{1}{a}|h\left(\frac{1}{t}\right) - f\left(\frac{1}{t}\right)|.$$
for all $t \in \left[ \frac{1}{b}, \frac{1}{a} \right]$. On the other hand,

$$
|G_H(t) - G_F(t)| \leq \left| H(t) - \frac{H(\frac{1}{b}) - H(\frac{1}{a})}{\frac{1}{b} - \frac{1}{a}}(t - \frac{1}{a}) - \left( F(t) - \frac{F(\frac{1}{b}) - F(\frac{1}{a})}{\frac{1}{b} - \frac{1}{a}}(t - \frac{1}{a}) \right) \right|
$$

$$
\leq |H(t) - F(t)| + \left| \left( \frac{1}{b} - \frac{1}{a} \right) \left( \frac{H(\frac{1}{b}) - F(\frac{1}{b})}{\frac{1}{b} - \frac{1}{a}} \right) \right| + \left| \left( \frac{1}{b} - \frac{1}{a} \right) \left( \frac{H(\frac{1}{a}) - F(\frac{1}{a})}{\frac{1}{b} - \frac{1}{a}} \right) \right|
$$

$$
\leq |H(t) - F(t)| + \left| \left( \frac{1}{b} - \frac{1}{a} \right) \left( \frac{H(\frac{1}{b}) - F(\frac{1}{b})}{\frac{1}{b} - \frac{1}{a}} \right) \right| + \left| \left( \frac{1}{b} - \frac{1}{a} \right) \left( \frac{H(\frac{1}{a}) - F(\frac{1}{a})}{\frac{1}{b} - \frac{1}{a}} \right) \right|
$$

for all $t \in \left[ \frac{1}{b}, \frac{1}{a} \right]$. Let $|h(t) - f(t)| < \delta$ for all $t \in [a, b]$, we have $|G_H(t) - G_F(t)| \leq \frac{3\delta}{a}$ for all $t \in \left[ \frac{1}{b}, \frac{1}{a} \right]$, which implies that there exists a point $\xi_0$ in $(\frac{1}{b}, \frac{1}{a})$ such that $G_H(\xi_0) = 0$ and $|\xi_0 - \frac{1}{2}| \leq \frac{1}{2\beta}.$

Define $\xi = \frac{1}{\xi_0}$. Recall $G'_H(\xi_0) = 0$, which implies $H'(\xi_0) = \frac{H(\frac{1}{b}) - H(\frac{1}{a})}{\frac{1}{b} - \frac{1}{a}}$, we obtain that

$$
b(\xi) - \xi h'(\xi) = \frac{\frac{1}{b}h(b) - \frac{1}{a}h(a)}{\frac{1}{b} - \frac{1}{a}} = \frac{ah(b) - bh(a)}{a-b},
$$

from which it follows that $\xi$ is a Pompeiu’s point of $h$. Moreover,

$$
|\xi - \eta| = \left| \frac{1}{\xi_0} - \eta \right| = \left| \frac{\xi_0 - \frac{1}{\eta}}{\xi_0 \cdot \frac{1}{\eta}} \right| \leq b^2 \left| \frac{\xi_0 - \frac{1}{\eta}}{\xi_0 \cdot \frac{1}{\eta}} \right| \leq \varepsilon.
$$

The proof is completed. 

\textbf{Corollary 2.2.} Let $f, h : [a, b] \rightarrow \mathbb{R}$ be differentiable and $\eta$ be a Pompeiu’s point of $f$. If $f$ has 2nd derivative at $\eta$ and

$$
f''(\eta) \neq 0,
$$

then corresponding to any $\varepsilon > 0$, there exists a $\delta > 0$ such that for every $h$ satisfying $|h(t) - f(t) - c| < \delta$ for $t$ in $[a, b]$, where $c$ is a constant, there exists a point $\xi \in (a, b)$ such that $\xi$ is a Pompeiu’s point of $h$ and $|\xi - \eta| < \varepsilon$.

The following counter example shows that Theorem 2.1 will be incorrect if we remove the condition $f''(\eta) \neq 0$.

\textbf{Example 2.3.} Let $[a, b] = [1, 2]$,

$$
f(x) = 0.
$$

Then, we can see that $f(x)$ is twice differentiable on $(1, 2)$ and $f''(x) = 0$ for all $x \in (1, 2)$. What’s more, every $x \in (1, 2)$ is a Pompeiu’s point of $f(x)$. Let $\eta = \frac{7}{4}$, which is a Pompeiu’s point of $f$. 
For sufficiently small $\delta > 0$, define

$$h(x) = \delta[4(x - \frac{3}{2})^2 - 1]$$

for $x \in [1, 2]$. Then, we can know from the geometric meaning of Pompeiu’s point that the Pompeiu’s point $\xi$ of $h(x)$ is in $(1, \frac{3}{2})$, or rather, $\xi = \sqrt{2}$ is the unique Pompeiu’s point of $h$ in $[1, 2]$.

Finally, we have

$$|\xi - \eta| = |\sqrt{2} - \frac{7}{4}| > \frac{1}{4}.$$ 

In other words, for all $\delta > 0$, there exists a twice differentiable function $h$ satisfying $|f - h| < \delta$, but there is no Pompeiu’s point of $h$ in the neighborhood of $\eta$ which is a Pompeiu’s point of $f$.

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