A QUANTUM ANALOG OF THE POINCARE–BIRKHOFF–WITT THEOREM

V. K. Kharchenko

Translated from Algebra i Logika, Vol. 38, No. 4, pp. 476-507, July-August, 1999. Original article submitted June 29, 1998.

We reduce the basis construction problem for Hopf algebras generated by skew-primitive semi-invariants to a study of special elements, called “super-letters,” which are defined by Shirshov standard words. In this way we show that above Hopf algebras always have sets of PBW-generators (“hard” super-letters). It is shown also that these Hopf algebras having not more than finitely many “hard” super-letters share some of the properties of universal enveloping algebras of finite-dimensional Lie algebras. The background for the proofs is the construction of a filtration such that the associated graded algebra is obtained by iterating the skew polynomials construction, possibly followed with factorization.

INTRODUCTION

In this article we deal with the basis construction problem for character Hopf algebras, i.e., for the Hopf algebras generated by skew primitive semi-invariants and by an Abelian group of all group-like elements. These algebras constitute an important class actively studied within the frames of the quantum group theory. The class includes all known to the date quantizations with the coalgebra structure of a Lie algebra, and probably we may think of it as an abstractly defined class of all “quantum” universal enveloping algebras. In line with this approach, “quantum” Lie algebras are couched in terms of spaces of all skew primitive elements of the character Hopf algebras endowed

\footnote{Supported by the National Society of Researchers, México (SNI, exp. 18740, 1997-2000).}
with the natural structure of an Yetter–Drinfeld module and equipped with partial quantum operations; see [1].

In the present article the basis construction problem will be reduced to treating special elements defined by Shirshov standard words, which we call “super-letters.” The main result, Theorem 2, states that the set of all monotonic restricted words in “hard” super-letters constitute a basis for a Hopf algebra. If the Hopf algebra is generated by ordinary primitive elements, the set of all “hard” super-letters constitute a basis for the Lie algebra of all primitive elements. By this token, Theorem 2 may be conceived of as one of the possible quantum analogs for the Poincare–Birkhoff–Witt theorem.

The proof and the statement of the main theorem are based on Shirshov’s combinatorial method, originally developed for Lie algebras. Using this method, Shirshov solved a number of important problems in combinatorial theory of Lie algebras. Among them are the characterization problem for a free Lie algebra over an arbitrary operator ring [3], the basis construction problem for a free Lie algebra over a field [5] (which is an outgrowth of M. Hall’s ideas in [4]), and the equality problem for Lie algebras with one defining relation or with finitely many homogeneous relations [6]. Independently, most fundamental concepts of that method were pronounced in [7] where the basis construction problem was dealt with for dual groups of the lower central series of a finitely generated free group.

A weak point in our modification of Shirshov’s method is that essential use will be made of the so-called “through” ordering of words, standard words, and super-words, for which the set of all standard words (super-letters) may be not completely ordered. For this reason, the main theorem is proved only for finitely generated Hopf algebras. In this connection, it is worth mentioning that Shirshov’s original method does not presume the use of a “through” ordering only. What it calls for is a weak restriction on the order: the end of a standard word should be less than the word itself. An example is M. Hall’s ordering from [4], which is in fact also used in the present article. However, we opt to not bring in both of the orders, to avoid (or at least minimize) misunderstanding. The reason why we do not use the M. Hall’s ordering as the main to our reasoning is because Lemma 8 becomes almost uninformative in this case.

In order to generalize the main theorem to the case of infinitely generated algebras, instead of searching suitable orders, one may apply the famous local method by Mal’tsev (cf. [8]), whereby the proof of the theorem reduces to a
logical analysis of its formulation.

The main theorem can also be used to construct bases for known quantizations of Lie algebras. For the Drinfeld–Jimbo quantizations, such were constructed in Rosso [9], Yamane [10], Lusztig [11, 12], and Kashiwara [13]. Curiously, no one of the methods by Rosso, Yamane, Lusztig, or Kashiwara presupposes that active use be made of the coalgebraic structure — instead — they all presume a detailed treatment of the algebraic. At the same time, the coproduct in Drinfeld–Jimbo quantizations, and also in every pointed Hopf algebra (cf. [14]), consists only of a “skew primitive” leading part and a linear tensor combination of a lesser degree. — This opens up unbounded prospects for inductive proofs.

An approach attempted here is aimed at a study of effects brought about by the existence of a coproduct. The Poincare–Birkhoff–Witt theorem (PBW-theorem) can also be proved in terms of a coproduct, provided that a given Lie algebra is presupposed to be embedded in a (cocommutative connected) Hopf algebra. This was in fact done in Milnor and Moore [15, Secs. 5 and 6]. The PBW-theorem in the Milnor-Moore form carries no information about primitive elements (the given Lie algebra) but gives a complete solution to the basis construction problem for a Hopf algebra modulo its solution for a given Lie algebra. The mentioned above decreasing process (unlike detailed algebraic accounts) has sharply delineated boundaries of application — it cannot give any information about the structure of a set of skew primitive elements (that is of the structure of the quantum Lie algebra itself). Therefore, it might be interesting to investigate these sets in known quantizations as quantum Lie algebras, that is together with all partial quantum operations over them [1].

In Sec. 1, we introduce basic notions and give a formulation of Shirshov’s theorem [3, Lemma 1] needed for our further constructions. All statements under this section were proved by Shirshov sometimes in a more general form. In a slightly different guise, some of them were discovered independently of Shirshov in [7].

In Sec. 2, we replace the classical commutator with a skew commutator whose “curvature” depends on the parameters of specified elements in approximately the same way as it does in color Lie super-algebras. In our case, however, the bicharacter is not assumed symmetric. Still, identity (8), which is analogous to the Jacobi identity, is valid. And so are derivative identities (9) and (11), which link the skew commutator and the basic product.
bulk of the information needed is given in Lemmas 6 and 8, in which two decreasing processes are described. One is an analog of the Hall–Shirshov construction for nonassociative words and the other is concerned with a co-product in the way mentioned above.

In Sec. 3, we pass from quantum variables to arbitrary skew primitive generators, and using the two above-mentioned decreasing processes, prove the main result, Theorem 2. On this theorem, each character Hopf algebra has the same basis as the universal enveloping algebra of a (restricted) Lie algebra. The role of a basis for the Lie algebra is played by special elements, which we call hard super-letters. The main lemmas are stated in such a way as to fit in dealing with skew primitive elements.

In Sec. 4, we derive some immediate consequences of the main theorem. In particular, it is shown that character Hopf algebras having not more than finitely many hard super-letters share some of the properties of universal enveloping algebras of finite-dimensional Lie algebras. The background for our proofs is the construction of a filtration such that the associated graded algebra is obtained by iterating the skew polynomials construction, possibly followed with factorization. Note also that the main theorem, as well as its corollaries, remain true for \((G, \lambda)\)-graded Hopf algebras and for braided bigraded Hopf algebras. In this event a group \(G\) merely defines a grading, but the algebra in question does not itself contain the \(G\). Therefore, additional restrictions on a group are unnecessary.

Finally, the quantum Serre relations can be expressed in terms of some super-letters being equal to zero. If, in these super-letters, we replace the skew commutator operation with the classical one then the original Serre relations will appear; see [1, Thm. 6.1]. Therefore, it seems absolutely realistic that all hard super-letters of the Drinfeld–Jimbo quantized enveloping algebras arise from a suitable basis for the Lie algebra by merely replacing the commutator with the skew commutator. This is likely to be true not only for the case of Drinfeld–Jimbo quantizations.

1. SHIRSHOV STANDARD WORDS

Let \(x_1, \ldots, x_n\) be a set of variables. Consider this set as an alphabet. On a set of all words in this alphabet, define the lexicographical order such that \(x_1 > x_2 > \ldots > x_n\). This means that two words \(v\) and \(w\) are compared by moving from left to right until the first distinct letter is encountered.
If not, i.e., one of the words is the beginning of the other, then a shorter word is assumed to be greater than the longer (as is common practice in dictionaries). For example, all words of length at most two in two variables respect the following order:

\[ x_1 > x_1^2 > x_1 x_2 > x_2 > x_2 x_1 > x_2^2. \]  

(1)

This order is stable under left multiplication and unstable under right. Nevertheless, if \( u > v \) and \( u \) is not the beginning of \( v \), then the inequality is preserved under right multiplication, even by different words: \( uw > vt \).

Every noncommutative polynomial \( f \) in \( x_1, \ldots, x_n \) is a linear combination of words \( f = \sum \alpha_i u_i \). By \( \overline{f} \) we denote a leading word which occurs in this decomposition with a nonzero coefficient. In the general case the leading word of a product does not equal the product of leading words of the factors. For example, if \( f = x_1 + x_1 x_2 \) and \( g = x_3 + x_3 x_2 \) then \( \overline{fg} = x_1 x_2 x_3 \neq x_1 x_3 = \overline{f} \overline{g} \). But if the leading word of \( f \) is not the beginning of any other word in \( f \), then

\[ \overline{fg} = \overline{f} \overline{g}. \]  

(2)

Indeed, the inequalities \( \overline{f} > u_j \) can be multiplied from the right by (possibly distinct) elements \( \overline{fv}_k > u_j v_s \). In particular, if \( f \) is an homogeneous polynomial, i.e., all words \( u_i \) have the same length, then formula (2) is true.

The set of all words is not completely ordered since there exist infinite decreasing chains — for instance,

\[ x_1 > x_1^2 > x_1^3 > \ldots > x_1^n > \ldots. \]  

(3)

Yet, all of its finite subsets are completely ordered. This will allow us to use induction on the leading word, provided that bounds are set on the lengths of words, \( l(v) \), or on degrees of the polynomials under consideration.

**Definition 1.** A word \( u \) is called standard (in the sense of Shirshov) if, for each representation \( u = u_1 u_2 \), where \( u_1 \) and \( u_2 \) are nonempty words, the inequality \( u > u_2 u_1 \) holds. For example, in (3), there is only one standard word, namely, \( x_1 \), and in (1), there are three: \( x_1, x_1 x_2, \) and \( x_2 \).

**Lemma 1.** If \( u = sv \) is a standard word and \( s \) is nonempty then \( v \) is not the beginning of \( u \).

**Proof.** Suppose, to the contrary, that \( u = vs' \). By the definition of a standard word, we then have \( sv = vs' > s'v \), i.e., \( s > s' \). Similarly, \( vs' = sv > vs \), whence \( s' > s \), a contradiction.
LEMMA 2. A word $u$ is standard if and only if it is greater than any one of its endings.

Proof. If the word $u$ is standard and $u = vv_1$ then $vv_1 > v_1 v$ by definition. By the previous lemma, $v_1$ is not the beginning of $vv_1$, and hence $u = vv_1$ and $v_1 v$ differ already in their first $l(v_1)$ letters. Therefore, $u > v_1$. Conversely, if $u = u_1 u_2$ and $u > u_2$ then $u$ is not the beginning of $u_2$, and so the above inequality remains true under right multiplication of the right hand side by $u_1$.

LEMMA 3. If $u$ and $v$, $u > v$, are standard words then $u^h > v$.

Proof. If $u$ is not the beginning of $v$ then $u > v$ can be multiplied from the right by different words. Suppose that $v = u^k v'$ and that $v'$ does not begin with $u$. If $k \geq h$ then $u^k > v$ as a beginning. If $k < h$ then $v'$ is nonempty and $v' < v < u$. It follows that $v = u^k \cdot v' < u^k \cdot u \cdot u^{h-k-1} = u^h$.

LEMMA 4. Let $u$ and $u_1$ be standard words such that $u = u_3 u_2$ and $u_2 > u_1$. Then

$$uu_1 > u_3 u_1, \quad uu_1 > u_2 u_1. \tag{4}$$

Proof. First we show that $u_2 u_1 > u_1$. If $u_1$ does not begin with $u_2$, the inequality follows immediately from $u_2 > u_1$. Assume that $u_1 = u_2^k \cdot u_1'$ and $u_2$ is not the beginning of $u_1'$. Since $u_1$ is standard, we have $u_2^k u_1' > u_2^k u_1'$, i.e., $u_2 u_1' > u_1'$. Hence $u_2 u_1 = u_2^k \cdot u_2 u_1' > u_2^k \cdot u_1' = u_1$. Multiplying this inequality from the left by $u_3$ yields the first inequality required. Consider the second. Since $u$ is a standard word, $u_3 u_2 > u_2$ by Lemma 2, and $u_3 u_2$ is of course not the beginning of $u_2$. We can therefore multiply the latter inequality from the right by $u_1$.

Recall that a nonassociative word is one where $[,]$ are somehow arranged to show how multiplication applies. The set of nonassociative words can be defined inductively by the following axioms:

1. all letters are nonassociative words;
2. if $[v]$ and $[w]$ are nonassociative words then $[[v][w]]$ is a nonassociative word;
3. there are no other nonassociative words.

Definition 2. A nonassociative word $[u]$ is said to be standard (in the sense of Shirshov) if:

1. an (associative) word $u$ obtained from this word by removing the brackets is standard;
2. if $[u] = [[v][w]]$ then $[v]$ and $[w]$ are standard nonassociative words;
(3) if \([u = [[[v_1][v_2]][w]]]\) then \(v_2 \leq w\).

The Shirshov theorem (cf. [3, Lemma 1]). Each standard word can be uniquely bracketed so that the resulting nonassociative word is standard.

This theorem, combined with the inductive definition of a set of all nonassociative words, immediately implies that every standard (associative) word \(u\) has a decomposition \(u = vw\), where \(v > w\) and \(v\) and \(w\) are standard. Yet, for the associative decomposition (as distinct from nonassociative one), the words \(v\) and \(w\) are not defined uniquely. The factors \(v\) and \(w\) in the nonassociative decomposition \([u = [[v][w]]]\), we note, can be defined to be standard words such that \(u = vw\), where \(v\) has a least possible length; see [16].

2. DECOMPOSITION OF QUANTUM POLYNOMIALS INTO LINEAR COMBINATIONS OF MONOTONIC SUPER-WORDS

Let \(x_1, \ldots, x_n\) be quantum variables, i.e., associated with each letter \(x_i\) are an element \(g_i\) of a fixed Abelian group \(G\) and a character \(\chi^i : G \to k^*\). For every word \(u\), denote by \(g_u\) an element of the group \(G\) which results from \(u\) by replacing each occurrence of the letter \(x_i\) with \(g_i\). This group-like element is denoted also by \(G(u)\), provided that \(u\) is an unwieldy expression. Likewise, by \(\chi^u\) we denote a character which results from \(u\) by replacing all \(x_i\) with \(\chi^i\). For a pair of words \(u\) and \(v\), put

\[
p_{uv} = \chi^u(g_v).
\]

(5)

Obviously, the following equalities hold:

\[
p_{uu_1} v = p_u p_{u_1} v, \quad p_{uv} v = p_{uv} p_{u_1},
\]

(6)

that is the operator \(p\) is a bicharacter defined on a semigroup of all words. Sometimes we denote this operator by \(p(u, v)\). Define a bilinear termal operation, a skew commutator, on a set of all quantum polynomials by setting

\[
[u, v] = uv - p_{uv} vu.
\]

(7)

This satisfies the following identity:

\[
[[u, v], w] = [u, [v, w]] + p_{uv}^{-1}[[u, w], v] + (p_{vw} - p_{uv}^{-1})[u, w] \cdot v,
\]

(8)
which is similar to the Jacobi identity, where \( \cdot \) stands for usual multiplication in a free algebra, and which can be easily verified by direct computations using (7). Likewise, the following formulas of skew derivations, by which the skew commutator is linked to multiplication, are valid:

\[
[u, v \cdot w] = [uv] \cdot w + p_{uw} v \cdot [uw],
\]

\[ (9) \]

\[
[u \cdot v, w] = p_{vw} [uw] \cdot v + u \cdot [vw].
\]

\[ (10) \]

**Definition 3.** A *super-letter* is a polynomial equal to a standard nonassociative word with brackets defined as in operation (7).

By the Shirshov theorem, every standard word \( u \) is associated with a super-letter \([u]\). If we remove the brackets in \([u]\) as is done in definition (7) we obtain an homogeneous polynomial whose leading word is equal to \( u \), and this leading word occurs in the decomposition of \([u]\) with coefficient 1. This is easily verified by induction on the degree. Indeed, if \([u] = [[v][w]]\) then the super-letter \([u]\) is equal to \([v][w] - p_{uw} [w][v]\). By the induction hypothesis, \([v]\) and \([w]\) are homogeneous polynomials with the leading words \( v \) and \( w \), respectively. Therefore, the leading word of the first summand equals \( vw \) and has coefficient 1; the leading word of the second equals \( wv \) and is less than \( vw \) by definition.

Thus, in correspondence with distinct standard words \( u \) and \( v \) are distinct super-letters \([u]\) and \([v]\), and the order on a set of super-letters can be defined as follows:

\[
[u] > [v] \iff u > v.
\]

\[ (11) \]

**Definition 4.** A word in super-letters is called a *super-word*. A super-word is said to be *monotonic* if it has the form

\[
W = [u_1]^{k_1} [u_2]^{k_2} \ldots [u_m]^{k_m},
\]

\[ (12) \]

where \( u_1 < u_2 < \ldots < u_m \).

We recall that the *constitution* of \( u \) is a sequence of integers \( (m_1, m_2, \ldots, m_n) \) such that \( u \) has degree \( m_1 \) in \( x_1 \), degree \( m_2 \) in \( x_2 \), etc. Since super-letters and super-words are homogeneous in each of the variables, their constitutions can be defined in the obvious manner. Because \( G \) is commutative, the elements \( g_u \) and the characters \( \chi^u \) are the same for all words of a same constitution. For super-letters and super-words, therefore, \( G(W) = g_w \) and \( p(U, V) = p_{uv} \) are defined uniquely.
On the set of all super-words, consider a lexicographic order defined by
the ordering of super-letters in (11).

**Lemma 5.** A monotonic super-word \( W = [w_1]^{k_1}[w_2]^{k_2} \cdots [w_m]^{k_m} \) is
greater than a monotonic super-word \( V = [v_1]^{m_1}[v_2]^{m_2} \cdots [v_k]^{m_k} \) if and only
if the word \( w = w_1^{k_1}w_2^{k_2} \cdots w_m^{k_m} \) is greater than the word \( v = v_1^{m_1}v_2^{m_2} \cdots v_k^{m_k} \).
Moreover, the leading word of the polynomial \( W \), when decomposed into a
sum of monomials, equals \( w \) and has coefficient 1.

**Proof.** Let \( W > V \). Then \( w_1 \geq v_1 \) in view of the ordering of super-
letters. If \( w_1 = v_1 \), we can remove one factor from the left of both \( V \) and
\( W \), and then proceed by induction. Therefore, we will put \( w_1 > v_1 \). If \( w_1 \)
is not the beginning of \( v_1 \), then the latter inequality can be multiplied from
the right by suitable distinct elements, which yields \( w > v \), as required. Let
\( v_1 = (w_1^{k_1}w_2^{k_2} \cdots w_s^{k_s-1})w_s^{l_1} \cdot v_1' \), where \( 0 \leq l < k_s \). Note that the term between
the parentheses may be missing (in which case \( s = 1, l > 0 \)), and \( w_s \) is not the
beginning of \( v_1' \). If \( v_1' \) is a nonempty word, then \( v_1' < v_1 < w_1 \leq w_s \), since \( v_1 \)
is standard. To obtain \( v < w \), the inequality \( v_1' < w_s \) will be multiplied from
the left by one element \((w_1^{k_1}w_2^{k_2} \cdots w_s^{k_s-1})w_s^{l_1} \cdot v_1' \), and from the right by (possibly)
different elements. If \( v_1' \) is the empty word, again we arrive at a contradiction
with \( v_1 \) being standard. Indeed, if \( l > 0 \), then the word \( v_1 \) should be greater
than its end \( w_s \); therefore, \( w_1 > v_1 > w_s \), which contradicts the fact that
\( w_1 \leq w_s \) is valid for all \( s \geq 1 \). If \( l = 0 \), then \( s > 1 \), since \( v_1 \) begins with \( w_1 \).
It follows that \( v_1 \) is greater than its end \( w_{s-1} \), which is again a contradiction
with \( w_1 > v_1 > w_{s-1} \).

The second part of the lemma follows from the fact that the leading word
of a product of homogeneous polynomials equals the product of leading words
of the factors.

The lemma cannot be extended to the case of nonmonotonic super-words,
for example, \([x_1] \cdot [x_3] \geq [x_1x_2] \) and \( x_1x_3 < x_1x_2 \).

**Lemma 6.** Let \( u \) and \( u_1 \) be standard words and \( u > u_1 \). Then the
polynomial \([u][u_1]\) is a linear combination of super-words in the super-letters
\([u]\) which lie properly between \([u]\) and \([u_1]\) and are such that \( w \leq uu_1 \). In
this case the degree of every summand in each of the variables \( x_1, \ldots, x_n \) is
equal to a respective degree of \( uu_1 \).

**Proof.** If the nonassociative word \([u][u_1]\) is standard then it defines a
super-letter \([w]\). In this case \( u > w \) since \( u \) is the beginning of \( w \), and \( w > u_1 \)
by Lemma 2. In particular, the lemma is valid if the degrees of \( u \) and \( u_1 \) are
equal to 1. And we can therefore proceed by induction on the length of \( uu_1 \).

Suppose that our lemma is true if the length of \( uu_1 \) is less than \( m \). Choose a pair \( u, u_1 \) with a greatest word \( [u] \), so that the polynomial \([u][u_1]\) does not enjoy the required decomposition and the length of \( uu_1 \) equals \( m \). Then the word \([u][u_1]\) is not standard, i.e., \([u] = [u_3][u_2]\) with \( u_2 > u_1 \). We introduce the notation for super-letters \( U_i = [u_i], i = 1, 2, 3 \). By Jacobi identity (8), we can write

\[
[U_3 U_2 U_1] = [U_3 [U_2 U_1]] + p_{u_1 u_2}^{-1} [U_3 U_1 U_2] + (p_{u_2 u_1} - p_{u_1 u_2}^{-1}) [U_3 U_1] \cdot U_2. \tag{13}
\]

It follows that \( u_3 > u > u_2 > u_1 \). By the induction hypothesis, \([U_3 U_1]\) can be represented as \( \sum \alpha_i \prod_k [w_{ik}] \), where \( u_3 > u_3 u_1 \geq w_{ik} > u_1 \). Using Lemma 4, we obtain \( u > uu_1 > u_3 u_1 \geq w_{ik} \), i.e., all super-letters \([w_{ik}]\) satisfy the requirements of the present lemma. Furthermore, the word \( u \) cannot be the beginning of \( u_2 \), and so \( u > u_2 \) implies \( uu_1 > u_2 \). Thus the super-letter \( U_2 \), too, satisfies the requirements. Consequently, the second [in view of (7)] and third summands of (13) have the required decomposition.

Using the induction hypothesis, for the first summand we obtain

\[
[U_2 U_1] = \sum_i \beta_i \prod_k [v_{ik}], \tag{14}
\]

where \( u_2 > u_2 u_1 \geq v_{ik} > u_1 \). By Lemma 4, \( uu_1 > u_2 u_1 \geq v_{ik} \), i.e., the super-letters \([v_{ik}]\) satisfy the conditions of the lemma. Rewrite the first summand using skew-derivation formula (9), with the first factor replaced by (14). With this, the first summand turns into a linear combination of words in the super-letters \([v_{ik}]\) and skew commutators \([u_3][v_{ik}]\). Since \( u_3 > u > u_2 > v_{ik} \) and the length of \( v_{ik} \) does not exceed that of \( u_2 u_1 \), the induction hypothesis applies to yield

\[
[u_3][v_{ik}] = \sum_j \gamma_j \prod_l [w_{jt}], \tag{15}
\]

where \( u_3 > u_3 v_{ik} \geq w_{jt} > v_{ik} \). In this case \( u_2 u_1 \geq v_{ik} \) implies \( uu_1 = u_3 u_2 u_1 \geq u_3 v_{ik} \); in addition, \( w_{jt} > v_{ik} > u_1 \), i.e., the super-letters \([w_{jt}]\) also satisfy the conditions.

**Lemma 7.** Every nonmonotonic super-word is a linear combination of lesser monotonic super-words of a same constitution, whose super-letters all lie (not strictly) between the greatest and the least super-letters of a super-word given.
Proof. We proceed by induction on the degree. Whenever super-letters of a given super-word are rearranged, the degree of a polynomial remains fixed; therefore, the least super-word of degree \( \leq m \) will be monotonic. Assume that the lemma is true for super-words of degree \( < m \), letting \( W = UU_1 \cdots U_t \) be a least super-word of degree \( m \) for which our lemma fails. If the super-word \( U_1 \cdots U_t \) is not monotonic, by the induction hypothesis, then, it is a linear combination of lesser monotonic super-words \( W_i \). And we can now apply the induction hypothesis to \( UW_i \). Let

\[
W = UU_1^{k_1} \cdots U_t^{k_t}, \quad U_1 < U_2 < \ldots < U_t. \tag{16}
\]

If \( U \leq U_1 \) then \( W \) is monotonic, and there is nothing to prove. Let \( U > U_1 \). Then

\[
W = [UU_1]U_1^{k_1-1} \cdots U_t^{k_t} + p_{uu} U_1 UU_1^{k_1-1} \cdots U_t^{k_t}. \tag{17}
\]

The second summand being a super-word is less than \( W \), and so we can write it in the required form. By Lemma 6, the factor \([UU_1]\) in the first term can be represented as \( \sum_i \alpha_i \prod_j [w_{ij}] \), where the super-letters \([w_{ij}]\) are less than \( U \). Consequently, the super-letters \( \prod_j [w_{ij}]U_1^{k_1-1} \cdots U_t^{k_t} \) are less than \( W \), i.e., the first term, and hence also \( W \), will have the required representation.

**THEOREM 1.** The set of all monotonic super-words constitute a basis for a free algebra \( \mathbb{k}\{x_1, \ldots, x_n\} \).

Proof. Since the letters \( x_1, \ldots, x_n \) are super-letters, every polynomial is a linear combination of monotonic super-words by Lemma 7. Our present goal is to prove that the set of all monotonic super-words is linearly independent. Let

\[
\sum_i \alpha_i W_i = 0 \tag{18}
\]

and assume that \( W = [w_1]^{k_1}[w_2]^{k_2} \cdots [w_m]^{k_m} \) is a leading super-word in (18). By Lemma 5, the leading word of \( W \) equals \( w = w_1^{k_1}w_2^{k_2} \cdots w_m^{k_m} \). Note that this word occurs exactly once in (18). Suppose, to the contrary, that \( W \) does also occur in the decomposition \( V = [v_1]^{m_1}[v_2]^{m_2} \cdots [v_k]^{m_k} \). Then the word \( w \) is less than or equal to the leading word \( v = v_1^{m_1}v_2^{m_2} \cdots v_k^{m_k} \) in the decomposition of \( V \), which contradicts the fact that \( W > V \) by Lemma 5.

Consider a free enveloping algebra in a given set of quantum variables \( H\{x_1, \ldots, x_n\} = G \ast \mathbb{k}\{x_1, \ldots, x_n\} \), on which the coproduct is defined by
setting
\[ \Delta(x_i) = x_i \otimes 1 + g_{x_i} \otimes x_i, \quad \Delta(g) = g \otimes g, \]
and group-like elements commute with variables via \( xg = \chi_x(g)x \). It follows that \( G \otimes k\langle x_1, \ldots, x_n \rangle \) turns into a Hopf algebra; for details, see [1, Sec. 3].

**Lemma 8.** The coproduct at a super-letter \( W = [w] \) is represented thus:
\[ \Delta([w]) = [w] \otimes 1 + g_w \otimes [w] + \sum \alpha_i G(W''_i)W'_i \otimes W''_i, \]
where \( W'_i \) are nonempty words in less super-letters than is \( [w] \). Moreover, the sum of degrees of super-words \( W'_i \) and \( W''_i \) in each variable \( x_j \) equals the degree of \( W \) in that variable, i.e., the sum of structural elements of \( W'_i \) and \( W''_i \) is equal to the constitution of \( W \).

**Proof.** We use induction on the length of a word \( w \). For letters, by (19), there is nothing to prove. Let \( W = [U, V], U = [u], \) and \( V = [v] \). Assume that the decompositions
\[ \Delta(U) = U \otimes 1 + g_u \otimes U + \sum \alpha_i G(U''_i)U'_i \otimes U''_i, \]
and
\[ \Delta(V) = V \otimes 1 + g_v \otimes V + \sum \beta_j G(V''_j)V'_j \otimes V''_j \]
satisfy the requirements of the lemma. Using (7) and properties of a bicharacter \( p \), we can write
\[ \Delta(W) = \Delta(U)\Delta(V) - p_{uv}\Delta(V)\Delta(U) = W \otimes 1 + g_w \otimes W + \]
\[ (1 - p_{uv}p_{vu})g_u V \otimes U + \sum \beta_j p(U, V''_j)G(V''_j)[UV'_j] \otimes V''_j + \]
\[ \sum \beta_j g_u G(V''_j)V'_j \otimes (UV''_j - p_{uv}p(V'_j, U)V''_j U) + \]
\[ \sum \alpha_i G(U''_i)(U'_i \cdot V - p_{uv}p(V, U''_i)U'_i \otimes U''_i + \sum \alpha_i p(U'_i, V)g_v G(U''_i)U'_i \otimes [U''_i V] + \]
\[ \sum \alpha_i \beta_j G(U''_i V''_j)(p(U'_i, V''_j)U'_j \otimes U''_j V''_j - p_{uv}p(V'_j, U''_i)U'_i V'_j \otimes V''_j U''_i). \]

Collecting similar terms in this formula will result in the canceling of terms of the form \( g_u U \otimes V \) only. We claim that all left parts of the remaining tensors in (23) admit the required decomposition. First, in view of the induction hypothesis, all super-letters of all super-words \( V'_j \) are less than
V, which are in turn less than W because v is the end of a standard word w. Moreover, by the induction hypothesis again, u cannot be the beginning of any word u’ such that the super-letter [u’] would occur in super-words U_i’. Therefore, u > u’ implies uv > u’ or W > [u’]. Thus all but the first and fourth super-words on the left-hand sides of all tensors depend only on super-letters which are less than W.

We want to apply Lemma 6 to the fourth tensor. Let V_j' = ∏ k V_{ik}, where V_{ik} = [v_{ik}] are less than V. By formula (9) the polynomial [U, V_j'] is a linear combination of words in the super-letters V_{ik} and skew commutators [U, V_{ik}]. By Lemma 6, each of these commutators is a linear combination of words in the super-letters [v’] such that v’ ≤ uw_{ik}. In view of v_{ik} < v, we obtain v’ < uv = w.

The statement concerning the constitution follows immediately from formula (23) and the induction hypothesis.

**LEMMA 9.** The coproduct at a super-word W is represented thus:

\[ \Delta(W) = W \otimes 1 + G(W) \otimes W + \sum_i \alpha_i G(W_i'') W_i' \otimes W_i'' , \]

where the sum of constitutions of W_i' and W_i'' equals the constitution of W.

**Proof.** It suffices to observe that \( \Delta \) is an homomorphism of algebras. Here, we can no longer assert that W_i' < W.

### 3. BASIS FOR A CHARACTER HOPF ALGEBRA

Consider a Hopf algebra H generated by a set of skew primitive semi-invariants a_1, . . . , a_n and by an Abelian group G of all group-like elements. Denote by H_a a subalgebra generated by a_1, . . . , a_n. Then H = GH_a since by definition, semi-invariants obey the following commutation rule:

\[ a g = \chi^a(g) \cdot g a. \]

Let x_1, . . . , x_n be quantum variables with the same parameters as a_1, . . . , a_n, respectively, that is \( \chi^{x_i} = \chi^{a_i} \) and \( g_{x_i} = g_{a_i} \). Then there exists an homomorphism

\[ \varphi : k(x_1, . . . , x_n) \rightarrow H_a \]
which maps $x_i$ to $a_i$. This allows us to extend all the combinatorial notions applied to the words in $x_1, \ldots, x_n$ in the above sections to the words in $a_1, \ldots, a_n$.

With $a_1, \ldots, a_n$ we associate the respective natural degrees $d_1, \ldots, d_n$. In this way, every word, super-letter, and super-word of a constitution $(m_1, \ldots, m_n)$ have degree $m_1d_1 + \ldots + m_nd_n$.

**Definition 5.** A $G$-super-word is a product of the form $gW$, where $g \in G$ and $W$ is a super-word. The degree, constitution, length, and other concepts which apply with $G$-super-words are defined by the super-word $W$. Alternatively, we assume that the degree and the constitution of $g \in G$ are equal to zero. In view of (25), every product of super-letters and group-like elements equals a linear combination of $G$-super-words of the same constitution.

**Definition 6.** A super-letter $[u]$ is said to be hard if it is not a linear combination of words of the same degree in less super-letters than is $[u]$ and of $G$-super-words of a lesser degree.

**Definition 7.** We say that the height of a super-letter $[u]$ of degree $d$ equals a natural number $h$ if $h$ is least with the following properties:

1. $p_u$ is a primitive root of unity of degree $t \geq 1$, and either $h = t$ or $h = tl^k$, where $l$ is the characteristic of the ground field;

2. a super-word $[u]^{h}$ is a linear combination of super-words of degree $hd$ in less super-letters than is $[u]$ and of $G$-super-words of a lesser degree.

If, for the super-letter $[u]$, the number $h$ with the above properties does not exist then we say that the height of $[u]$ is infinite.

**Definition 8.** The monotonic $G$-super-word

$$g[u_1]^{n_1}[u_2]^{n_2} \cdots [u_k]^{n_k}$$

is said to be restricted if each of the numbers $n_i$ is less than the height of the super-letter $u_i$.

**Theorem 2.** If a Hopf algebra $H$ is generated by a set skew-primitive semi-invariants $a_1, \ldots, a_n$ and by an Abelian group $G$ of all group-like elements, then the set of all monotonic restricted $G$-super-words in hard super-letters constitute a basis for $H$.

---

\[2^*\] Note that further argument will remain true for the case where $d_1, \ldots, d_n$ are arbitrary positive elements of a linearly ordered additive Abelian group.
The proof will proceed through a number of lemmas. For brevity, we call a super-word (a $G$-super-word) admissible if it is monotonic restricted and is a word in hard super-letters only.

**LEMA 10.** Every nonadmissible super-word of degree $d$ is a linear combination of lesser admissible super-words of degree $d$ and of admissible $G$-super-words of a lesser degree. Also, all super-letters occurring in super-words of degree $d$ of this linear combination are less than or equal to a greatest super-letter of the super-word given.

The proof is by induction on the degree. Assume that the lemma is valid for super-words of degree $< m$. Let $W$ be a least super-word of degree $m$ for which the required representation fails. By Lemma 7, the super-word $W$ is monotonic. If it has a nonhard super-letter, by definition, we can replace it with a linear combination of $G$-super-words of a lesser degree and of words in less super-letters of the same degree. Removing the parentheses turns $W$ into a linear combination of $G$-super-words of a lesser degree and of lesser super-words of the same degree, a contradiction with the choice of $W$. If $W$ contains a subword $[u]^k$, where $k$ equals the height of $[u]$, then we can replace it as is specified above, which gives us a contradiction again. Thus the $W$ is itself monotonic restricted and is a word in hard super-letters only.

In order to prove Theorem 2, it remains to show that admissible $G$-super-words are linearly independent. Consider an arbitrary linear combination $T$ of admissible $G$-super-words and let $U = U_1^{n_1}U_2^{n_2} \cdots U_k^{n_k}$ be its leading super-word of degree $m$. Multiplying, if necessary, that combination by a group-like element, we can assume that $U$ occurs once without a group-like element:

$$T = U + \sum_{j=1}^{r} \alpha_j g_j U + \sum_{i} \alpha_i g_i V_i^{n_{i1}}V_i^{n_{i2}} \cdots V_i^{n_{is}}. \quad (27)$$

In the next three lemmas, we accept the following inductive assumption on $m$ and on $r$:

the set of all admissible $G$-super-words of degree $m$ which are less than $U$, of admissible $G$-super-words of degree $< m$, and of $G$-super-words $g_j U$, $1 \leq j \leq r$, is linearly independent [we can assume $r = 0$ in (27)].

In view of this assumption and Lemma 10, every super-word of degree $m$ which is less than $U$, and every super-word of degree $< m$, can be uniquely decomposed into a linear combination of admissible $G$-super-words. For brevity, such will be referred to as a basis decomposition.
LEMMA 11. If $T$ is a skew primitive element then $r = 0$ and all $G$-super-words of degree $m$ in (27) are super-words.

Proof. Rewrite the linear combination $T$ as follows:

$$T = U + \sum_{i=1}^{k} \gamma_i g_i W_i + W', \quad (28)$$

where $g_i W_i$ are distinct $G$-super-words of degree $m$ in (27) and $W'$ a linear combination of $G$-super-words of degree $< m$. In the expression

$$\Delta(T) - T \otimes 1 - g_t \otimes T, \quad (29)$$

consider all tensors of the form $gW \otimes \ldots$, where $W$ is of degree $m$. By Lemma 9, the sum of all such tensors equals

$$\sum_{i=1}^{r} \gamma_i g_i W_i \otimes g_i - \sum_{i=1}^{r} \gamma_i g_i W_i \otimes 1 = \sum_{i=1}^{r} \gamma_i g_i W_i \otimes (g_i - 1). \quad (30)$$

By inductive assumption (*), the elements $g_i W_i$ are linearly independent modulo all left parts of tensors of degree $< m$ in (29). Therefore, if (29) vanishes then either $\gamma_i = 0$ or $g_i = 1$ for every $i$, as required.

LEMMA 12. If $T$ is a skew primitive element then $U = U_1^{n_1}$ and all super-words of degree $m$ except $U$ are words in less super-letters than is $U_1$.

Proof. By the preceding lemma, we can assume that

$$T = \sum_{i} \alpha_i g_i V_{i1}^{n_{i1}} V_{i2}^{n_{i2}} \cdots V_{is}^{n_{is}}, \quad (31)$$

where $V_{ij} = [v_{ij}]$ are hard super-letters, $\alpha_i$ are nonzero coefficients, and $g_i = 1$ if $V_i$ is of degree $m$. We apply coproduct to (31). By (8), then, the right-hand side assumes the form

$$\sum_{i} \alpha_i (g_i \otimes g_i) \prod_{j=1}^{s} (V_{ij} \otimes 1 + g_{ij} \otimes V_{ij} + \sum_{\theta} g_{ij\theta} V'_{ij\theta} \otimes V''_{ij\theta})^{n_{ij}}, \quad (32)$$

where $V'_{ij\theta} < V_{ij}$ and $\deg V'_{ij\theta} + \deg V''_{ij\theta} = \deg V_{ij}$.

Let $[v]$ be the largest super-letter occurring in super-words of degree $m$ in (31). Since all super-words of (31) are monotonic, this super-letter stands at the end of some super-words $V_i$, i.e., $[v] = V_{is}$. If one of these super-words
depends only on \([v]\), i.e., \(V_i = [v]^k\), then \(V_i\) is a leading term, as required. Therefore, we assume that every super-word of degree \(m\) ending with \([v]\) is a word in more than one super-letter.

Let \(k\) be the largest exponent \(n_{is}\) of \([v]\) in \(T\). Consider all tensors of the form \(g[v]^k \otimes \ldots\) obtained in (32) by removing the parentheses and applying the basis decomposition to all left parts of tensors in all terms except \(T \otimes 1\) (all of these terms are of degree \(< m\)).

All left parts of tensors which appear in

\[
\Delta(V_i) = (g_i \otimes g_i) \prod_{j=1}^{s} (V_{ij} \otimes 1 + g_{ij} \otimes V_{ij} + \sum_{\theta} g_{ij\theta} V'_{ij\theta} \otimes V''_{ij\theta})^{n_{ij}}
\]

by removing the parentheses arise from the word \(V_i = \alpha_i g_{i1} V_{i1}^{n_{i1}} V_{i2}^{n_{i2}} \cdots V_{is}^{n_{is}}\) by replacing some of the super-letters \(V_{ij}\) either with group-like elements \(g_{ij}\) or with \(G\)-super-words \(g_{ij\theta} V'_{ij\theta}\) of a lesser degree in less super-letters. The right parts are, respectively, products obtained by replacing super-letters \(V_{ij}\) or super-words \(V''_{ij\theta}\) multiplied from the left by \(g_i\).

If, under the replacements above, a new super-word is greater in degree than \([v]^k\), then its basis decomposition will give rise to terms of the form \(g[v]^k \otimes \ldots\). In this case, however, the right parts of those terms are of degree less than \(m - k\deg([v])\) since the sum of degrees of both parts of the tensors either remains equal to \(m\) or decreases.

If a new super-word is of degree less than the degree of \([v]^k\), or the super-word is itself less than \([v]^k\) then its basis decomposition will be freed of terms of the form \(g[v]^k \otimes \ldots\); see Lemma 10.

If a new super-word is of degree equal to that of \([v]^k\) and \(V_i\) is of degree less than \(m\) then the new super-word can be greater than or equal to \([v]^k\). In this case the right-hand sides of the new tensors are of degree less than \(m - k\deg([v])\) because the sum of degrees of the left- and right-hand sides of the tensors is less than \(m\).

If a new super-word is of degree equal to the degree of \([v]^k\), but \(V_i\) does not end with \([v]^k\), i.e., \(V_i = W_i[v]^s\), \(0 \leq s < k\), then the new super-word is less than \([v]^k\) since its first super-letter is less than \([v]\). (All super-letters of \(W_i\) cannot be replaced with group-like elements, since otherwise the new word would be of degree less than or equal to the degree of \([v]^s\).)

Finally, if \(V_i = W_i[v]^k\) then a super-word of degree \(k\deg([v])\), which is greater than or equal to \([v]^k\), may appear only if all super-letters of the
super-words $W_i$ are replaced with group-like elements, but $[v]$ is not. Here, the resulting tensor is of the form $g(W_i)[v]^k \otimes \alpha_i W_i$.

We fix an index $t$ such that $V_i$ ends with $[v]^k$, letting $t = 1$. Then the sum of all tensors of the form $G(W_1)[v]^k \otimes \ldots$ in $\Delta(T) - T \otimes 1$ is equal to

$$G(W_1)[v]^k \otimes \left( \sum_j \alpha_j W_j + W' \right),$$

(33)

where $W'$ is a linear combination of basis elements of degree less than $m - k \deg([v])$, and $j$ runs through the set of all indices $i$ such that $V_i = W_i[v]^k$, $G(W_i) = G(W_1)$, and the degree of $W_i$ equals $m - k \deg([v])$. Since $W_i$ are distinct nonempty basis super-words of degree less than $m$, tensor (33) is nonzero.

**Lemma 13.** Under the conditions of Lemma 12, either $n_1 = 1$ or $p(U_1, U_1)$ is a primitive root of unity of degree $t \geq 1$, in which case $n_1 = t$ or the characteristic of a base field equals $l > 0$, and $n_1 = tl^k$.

**Proof.** By the previous lemma, the linear combination $T$ can be written in the form

$$T = U^k + \sum_i \alpha_i g_i V_i^{n_i_1} V_i^{n_i_2} \ldots V_i^{n_i_s},$$

(34)

where $U = [u]$ is greater than all super-letters $V_i$ of degree $m$. First let $\xi = 1 + p_u + p_u^2 + \ldots + p_u^{k-1} \neq 0$ and assume $k > 1$.

In the basis decomposition of $\Delta(T) - T \otimes 1$, consider tensors of the form $U^{k-1} \otimes \ldots$. All super-letters $V_{ij}$ in super-words of degree $m$ are less than $[u]$; therefore, tensors of this form may appear under the basis decomposition of a tensor of $\Delta(V_i) - V_i \otimes 1$ only if either the left part of that tensor is of degree greater than $(k - 1)\deg([u])$ or $V_i$ is of degree less than $m$. In either case the right part is of less degree than is $[u]$. As above, if we remove the parentheses in

$$\Delta(U^k) = (U \otimes 1 + g_u \otimes U + \sum_{\tau} U'_\tau \otimes U''_\tau)^k,$$

(35)

we see that the left parts of the resulting tensors arise from the super-word $U^k$ by replacing some super-letters $U$ either with $g_u$ or with super-words $U'_\tau$ of a lesser degree in less super-letters than is $U$. It follows that a super-word of degree $(k - 1)\deg(U)$ which is greater than or equal to $U^{k-1}$ appears only if exactly one super-letter is replaced with a group element. Using the
commutation rule $U^s g_u = p_{uu}^s g_u U^s$, we see that the sum of all tensors of the form $g_u U^{k-1} \otimes \ldots$ equals
\[ g_u U^{k-1} \otimes (\xi U + W), \]
where $W$ is a linear combination of basis $G$-super-words of degree less than $\text{deg}(U)$. Consequently, (29) is nonzero for $k \neq 1$.

Now let $\xi = 0$. Then $p_{uu}^k = 1$. Therefore, $p_{uu}$ is a primitive root of unity of some degree $t$ (we put $t = 1$ if $p = 1$) and the number $k$ is divisible by $t$. We can write $k$ in one of the forms $t \cdot q$ or $t l^k \cdot q$, where $l$ is the characteristic of a base field, in which $q \cdot 1 \neq 0$. Put $h = t$ or $h = t l^k$, respectively. Since $(U \otimes 1) \cdot (g_u \otimes U) = p_{uu}(g_u \otimes U) \cdot (U \otimes 1)$, use will be made of the quantum Newton binomial formula
\[ (U \otimes 1 + g_u \otimes U)^h = U^h \otimes 1 + g_{uu}^h \otimes U^h. \] (36)
This implies that if we remove the parentheses in
\[ \Delta(U^h) = ((U \otimes 1 + g_{uu} \otimes U) + \sum_i G(U''_i) U'_i \otimes U''_i)^h, \] (37)
then Lemma 8 gives
\[ \Delta(U^h) = U^h \otimes 1 + g_{uu}^h \otimes U^h + \sum_q G(U''_q) U'_q \otimes U''_q, \] (38)
where all super-words $U'_q$ are less than $U^h$ and are of less degree than is $U^h$.
In this formula, we note, all terms $U^r \otimes \ldots, r < h$, whose left parts are greater than $U^h$, are banished. This allows us to treat $U^h$ in (34) as a single block, or as a new formal super-letter $\{U^h\}$ such that $\{U^h\} < U$, and $\{U^h\} > [v_{ij}]$ if $u^h > v_{ij}$ (which is equivalent to $u > v_{ij}$ by Lemma 3), i.e.,
\[ T = \{U^h\}^q + \sum_i \alpha_i g_i V^{n_{i1}}_1 V^{n_{i2}}_2 \ldots V^{n_{is}}_s. \] (39)
Since $p(U^h, U^h) = p_{hh}^h = 1$, we have
\[ \xi_1 = 1 + p(U^h, U^h) + \ldots + p(U^h, U^h)^{q-1} = q \neq 0. \]
As in the case above, assuming that $\{U^h\}$ is a single block, we can compute the sum of all tensors of the form $g_{uu}^h \{U^h\}^{q-1} \otimes \ldots$ in the basis decomposition of $\Delta(T) - T \otimes 1$ (provided that $q > 1$):
\[ g_{uu}^h \{U^h\}^{q-1} \otimes (q \cdot \{U^h\} + W), \] (40)

where $W$ is a linear combination of basis $G$-super-words of less degree than is $U^h$. By the induction hypothesis, tensor (40) is nonzero, and so therefore is (29).

The equality $T = 0$ does not hold. Indeed, if it did, then $T$ would be a skew primitive element, which is nonzero in view of Lemma 13 and definitions of hard super-letters and their heights. Inductive assumption ($\ast$) for $r = 0$ is obviously valid if $U$ is smallest among generators $a_i$, since group-like elements, i.e., $G$-super-words of degree zero, are always linearly independent. Theorem 2 is proved.

4. SOME COROLLARIES

In this section, again we write $H$ for a Hopf algebra generated by an Abelian group $G$ of all group-like elements and by skew primitive semi-invariants $a_1, \ldots, a_n$ with which degrees $d_1, \ldots, d_n$ are associated.

**COROLLARY 1.** The set of all $G$-words in $a_1, \ldots, a_n$, obtained by dropping all brackets from monotonic restricted $G$-super-words in hard super-letters, constitute a basis for $H$.

**Proof.** Decompose an arbitrary word $v$ in $a_1, \ldots, a_n$ as is specified in Theorem 1, namely, $v = \sum \alpha_j V_j$, where $V_j = [v_{j1}]^{n_1} \cdots [v_{jk}]^{n_k}$ are monotonic super-words of the same constitution. By Lemma 5, the leading word appearing in $\sum \alpha_j V_j$ under decomposition (7) equals $v_s = v_{s1}^{n_1} \cdots v_{sk}^{n_k}$, where $V_s$ is the leading super-word among all $V_j$. Therefore, $v = v_s$, $\alpha_s = 1$ — this is still a decomposition in the free algebra.

We use induction on the degree. Let $w$ be a minimal word of degree $d$ which is not a linear combination of the $G$-words specified in the statement. The word, as in the preceding paragraph, is decomposed thus: $w = \sum \alpha_j W_j$.

If the leading super-word $W_s$ is admissible, then $w$ arises from $W_s$ by dropping the brackets, and so there is nothing to prove. If $W_s$ is not admissible, $W_s$ is the required linear combination by Lemma 10 and inductive assumption ($\ast$). We have $w = (\sum_{j \neq s} \alpha_j W_j) + W_s$, where the first summand is a linear combination of words which are less than $w$, and again the inductive assumption applies.

We argue for linear independence. Let $\sum_{d} \beta_d g_d w_i = 0$. Then $w_i = \sum \alpha_{ij} W_{ij}$, where $w_i$ is obtained by dropping all brackets from the leading
super-word $W_{is}$, and $\alpha_{is} = 1$. Therefore, $W_{is}$ are admissible super-words. Now the equality $\sum \beta_{it} \alpha_{ij} g_{it} W_{ij} = 0$ leads us to a contradiction. Indeed, by Lemma 10, the nonleading super-words $W_{ij}$ decrease under the basis decomposition, either in degree or in ordering.

Sometimes we find it useful to apply the following criterion which allows us to forget about skew commutators in computing hard super-letters.

**COROLLARY 2.** A super-letter $[u]$ is hard if and only if the standard word $u$ is not a linear combination of lesser words of degree $\text{deg}(u)$ and of $G$-words of a lesser degree.

**Proof.** Let $u = \sum \alpha_i w_i + u_0$, where $w_i < u$ and $\text{deg}(u_0) < \text{deg}(u)$. Decompose the words $u$ and $w_i$ as was done at the beginning of Corollary 1. We obtain $u = [u] + \sum \beta_j U_j$ and $w_i = \sum \beta_t W_{it}$, where the super-words $U_j$ are less than $[u]$, and $w_i$ equals the leading word of a polynomial defined by the leading super-word $W_{is}$. Since $u > w_i$, we have $[u] > W_{is}$ by Lemma 5, and hence $[u]$ is greater than all $W_{it}$. Consequently, the basis decompositions of $U_i$ and $W_{it}$ have only super-words which either are less than $[u]$ or of a lesser degree. For hard super-letters $[u]$, therefore, the equality $[u] + \sum \beta_j U_j - \sum_i \alpha_i \sum_j \beta_j W_{it} - u_0 = 0$ is an impossibility.

Conversely, if $[u] = \sum \alpha_i W_i + U_0$, where $W_i$ depends on super-letters less than $[u]$, then

$$u = [u] + (u - [u]) = \sum \alpha_i W_i + U_0 + (u - [u]),$$

and the polynomial in the right part has no monomials whose degree equals the degree of $u$ and which are greater than or equal to $u$.

**COROLLARY 3.** Lemmas 11-13 are valid without assumption ($\ast$).

**COROLLARY 4.** A Hopf algebra $H$ is finite-dimensional if and only if the group $G$ and the set of all hard super-letters are finite, and each hard super-letter has finite height.

**COROLLARY 5.** If $H$ has only a finite number of hard super-letters and $G$ is finitely generated, then $H$ is (left and right) Noetherian.

**COROLLARY 6.** Let the group algebra $k[G]$ has no zero divisors. If $H$ has only a finite number of hard super-letters, of which each has infinite height, then $H$ has no zero divisors and has a classical skew field of quotients.
As in the case of classical Lie algebras, in order to prove the last two corollaries, we need only construct on $H$ a filtration $H_0 \subseteq H_1 \subseteq \ldots \subseteq H_k \subseteq \ldots$ such that the associated graded algebra $D(H)$ satisfies the required properties; see, e.g., [17, Ch. V, Sec. 3, Thms. 4, 5].

**Construction of the filtration.** Assume that $H$ has finitely many hard super-letters. Consider a set $R$ of all words in $a_1, \ldots, a_n$, whose degree does not exceed the maximal degree of a hard super-letter multiplied by a maximal finite height, or by 2 if all heights are infinite. In this case $R$ is composed of all standard words defining hard super-letters and of all products $uv, u^h$, where $[u]$ and $[v]$ are hard super-letters and $h$ is the height of $u$. Let words of $R$ all respect the lexicographical ordering described at the beginning of Sec. 1 and $n(u)$ be the number of words in $R$ which are less than or equal to $u$. The largest word $a_1$ is defined by the number $L = n(a_1)$. Denote by $M$ an arbitrary natural number which is greater than the length of any word in $R$. Define the filtration degree on hard super-letters using the formula

$$
\text{Deg}([u]) = M^{L+1}\text{deg}(u) + M^{n(u)},
$$

where $\text{deg}(u)$ is specified by the constitution of $u$, and $\text{deg}(u) = d_1m_1 + \ldots + d_nm_n$. The filtration degree of a basis element $gW$ equals the sum of filtration degrees of all of its super-letters. The filtration degree of an arbitrary element $T \in H$ equals the maximal filtration degree of the basis elements occurring in its basis decomposition.

**Lemma 14.** The function $\text{Deg}$ defines a filtration on $H$, so that $H_0 = k[G]$, $H_k = \{T \in H \mid \text{Deg}(T) \leq k\}$.

**Proof.** We have to show that $H_kH_s \subseteq H_{k+s}$, i.e., $\text{Deg}(T_1 \cdot T_2) \leq \text{Deg}(T_1) + \text{Deg}(T_2)$. To do this, we construct an additional degree function $D'$ on a set of all linear combinations of (not necessarily admissible) super-words in the super-letters defined by all standard words of the vocabulary $R$. The $D'$-degree of a super-letter is defined by formula (41). The $D'$-degree of a product of super-letters equals the sum of degrees of its factors. Accordingly, the $D'$-degree of a linear combination equals the maximum of $D'$-degrees of its summands. Of course we do not claim that the various linear combinations defining equal elements of $H$ have the same $D'$-degrees.

If we assume that $T_1 \cdot T_2$ is obtained from $T_1$ and $T_2$ by merely removing the parentheses, then $\text{Deg}(T_1) + \text{Deg}(T_2) = D'(T_1 \cdot T_2)$. Therefore, it suffices
to specify how a basis decomposition of super-words proceeds in a way that $D'$ is kept unincreased. Our plan is as follows. First, we replace nonhard super-letters via Definition 6, next replace all subwords $[u]^h$, where $h$ is the height of a hard super-letter $[u]$, then apply the decreasing decomposition of Lemma 7, and again replace nonhard super-letters, etc.

Let $[u]$ be a nonhard super-letter defined by $u$ in $R$ as follows:

$$[u] = \sum_i \alpha_i \prod_j [w_{ij}] + \sum_s \alpha_s g_s \prod_t [v_{st}],$$

(42)

where $[w_{ij}]$ are less than $[u]$, $n(w_{ij}) \leq n(u) - 1$, of the combined ordinary degree $\text{deg}(u)$, and $\sum_t \text{deg}(v_{st}) \leq \text{deg}(u) - 1$. By the definition of $R$, all words $w_{ij}$ and $v_{st}$, as well as the words which result from the summands of (42) by dropping all brackets and multiplication signs, belong to the vocabulary $R$. In particular, their lengths are less than $M$. Therefore, the number of factors in every summand of (42) is less than $M$. Thus we may write

$$D'(\prod_j [w_{ij}]) = \sum_j M^{L+1} \text{deg}(w_{ij}) + \sum_j M^{n(w_{ij})} < M^{L+1} \text{deg}(u) + M \cdot M^{n(u)-1} = D'(u).$$

(43)

Similarly,

$$D'(\prod_t [v_{st}]) = \sum_t M^{L+1} \text{deg}(v_{st}) + \sum_t M^{n(v_{st})} < M^{L+1} (\text{deg}(u) - 1) + M \cdot M^L = D'(u) - M^{n(u)} - M^{L+1} + M^{L+1} < D'(u).$$

(44)

The argument is the same if $[u]^h$ is changed with $\sum_i \prod_j [w_{ij}] + \sum_s g_s \prod_t [v_{st}]$, where $[u]$ is a hard super-letter of height $h$. We have

$$D'(\sum_i \prod_j [w_{ij}] + \sum_s g_s \prod_t [v_{st}]) < D'( [u]^h).$$

(45)

We proceed to the decreasing process of Lemma 7. The second summand in (17) has the same $D'$-degree as $W$. By Lemma 6, the factor $[UU_1]$ in the first summand can be represented as $\sum_i \alpha_i \prod_j [w_{ij}]$, where the super-letters $[w_{ij}]$ are less than $U$, and they all are in $R$ since their ordinary degrees are less than or equal to the ordinary degree of $UU_1$. Therefore, $n(w_{ij}) \leq n(u) - 1$. And the constitutions being equal indicates that the number of factors in $\prod_j$
does not exceed $M$. We have

$$
\begin{align*}
D'(\prod_j [w_{ij}]) &= \sum_j M^{L+1} \deg(w_{ij}) + \sum M^n(w_{ij}) < \\
M^{L+1} \deg(UU_1) + M \cdot M^n(u)^{-1} < M^{L+1} \deg(UU_1) + M^n(u) + M^n(u_1) = D'(UU_1).
\end{align*}
$$

(46)

The lemma is proved.

Note that the strict inequality signs in (46) show that the inequality

$$
\deg([v] \cdot [u] - p_{uv}[v] \cdot [u]) < \deg(u) + \deg(v)
$$

is valid for all hard super-letters $U = [u], U_1 = [v], u > v$. Similarly, (45) gives us

$$
\deg([u]^h) < h\deg(u),
$$

where $h$ is the height of a hard super-letter $[u]$.

**Associated graded algebra.** With each hard super-letter $[u]$ we associate a new variable $x_u$. Denote by $H^*$ an algebra generated by $x_u$ and defined by the relations $x_u x_v = p_{uv} x_v x_u$, where $u > v$. This algebra can be constructed by iterating the skew polynomials construction. Namely, let $[u_1] < [u_2] < \ldots < [u_s]$ all be hard super-letters. Denote by $H^*_k$ a subalgebra generated by $x_{u_1}, \ldots, x_{u_k}$. Then $H^*_k$ is isomorphic to an algebra of polynomials in one variable. The map $\varphi : x_v \rightarrow p_{uv} x_v$, where $u = u_{k+1}, v = u_i, i \leq k$, determines an automorphism of $H^*_k$. The commutation rule $x_u x_v = p_{uv} x_v x_u$ can be written in the form $x_u x_v = x_v^\varphi x_u$. Therefore, the algebra $H^*_k$ is isomorphic to an algebra of skew polynomials over $H^*_k$. In particular, the algebra $H^*$ is Noetherian and has no zero divisors.

Define the action of $G$ on $H^*$ by the formula $x_u^g = \chi^u(g) \cdot x_u$. Let $H^*[G]$ be a skew group algebra of $G$ with coefficients from $H^*$.

**THEOREM 3.** The associated graded algebra $D(H)$ is isomorphic to the quotient algebra of $H^*[G]$ with respect to relations $x_u^h = 0$, where $[u]$ runs through the set of all hard super-letters of finite height and $h$ equals the height of $[u]$.

**Proof.** Denote by $x_u$ an element of $D(H)$ defined by the coset $[u] + H_{m-1}$, where $[u]$ is a hard super-letter and $m = \deg(u)$. Then the zero component $k[G]$ and the elements $x_u$ generate $D(H)$. Formulas (47) and (48) show that $x_u x_v = p_{uv} x_v x_u$ and $x_u^h = 0$ hold if $u > v$ and $h$ is the height of $[u]$. All the monotonic restricted $G$-words in $x_u$ are linearly independent in $D(H)$ since
the filtration degree of every admissible word equals the sum of filtration degrees of all super-letters of that word.

**Braided bigraded Hopf algebras.** The above results can be easily extended to the case of \((G, \lambda)\)-graded Hopf algebras (see, e.g., [18, p. 206]) and to the case of braided bigraded Hopf algebras (cf. [1]). These objects are not Hopf algebras in the ordinary sense, but still they are Hopf algebras in some categories. In view of Radford’s results, these algebras have embeddings in the ordinary Hopf algebras; see [19, Thm. 1 and Prop. 2]. The embeddings are obtained by adding the elements of \(G\) treated as group-like elements. In this case primitive elements correspond to skew primitive ones. If a given \((G, \lambda)\)-graded Hopf algebra \(H\) is generated by primitive elements then the enveloping Hopf algebra is character and separated, i.e., \(H_a \cap k[G] = 0\).

All the concepts of this article can be easily extended to the case of \((G, \lambda)\)-graded Hopf algebras by vanishing group-like elements and by replacing the bicharacter \(\rho(u, v)\) with \(\lambda^{-1}(g_u, g_v)\). We are now in a position to formulate the following corollaries.

**COROLLARY 7.** If a \((G, \lambda)\)-graded Hopf algebra \(H\) is generated by a set of primitive elements \(a_1, \ldots, a_n\) then the set of all monotonic restricted words in hard super-letters constitutes a basis for \(H\).

**COROLLARY 8.** Let \(H\) be a \((G, \lambda)\)-graded Hopf algebra generated by primitive elements. \(H\) is finite-dimensional if and only if the set of all hard super-letters is finite and each hard super-letter has finite height.

**COROLLARY 9.** Let \(H\) be a \((G, \lambda)\)-graded Hopf algebra generated by primitive elements. If \(H\) has only a finite number of hard super-letters then it is Noetherian.

**COROLLARY 10.** Let \(H\) be a \((G, \lambda)\)-graded Hopf algebra which is generated by primitive elements and has only a finite number of hard super-letters. If all these hard super-letters have infinite height then \(H\) has no zero divisors and has a classical field of quotients.

In the corollaries above, we note, restrictions on a group are unnecessary since the group merely defines a bigrading, but the algebra in question would not contain it.

**Acknowledgement.** I want to thank Dr. Juan Antonio Montaraz, Director of FES-C, Dr. Suemi Rodriguez-Romo, and Virginia Lara Sagahon for providing beautiful facilities for my research work at FES-C UNAM, México. Thanks also are due to participants of Shirshov Seminar on Ring
Theory (Institute of Mathematics of RAS) for interesting comments on the subject matter.

REFERENCES

1. V. K. Kharchenko, “An algebra of skew primitive elements,” Algebra and Logic, 37, No. 2, 101–126 (1998).

2. Y. Ju. Reshetikhin, L. A. Takhtadzhyan, and L. D. Faddeev, “Quantizations of Lie groups and Lie algebras,” Leningrad Mat. Zh., 1, No. 1, 193-225 (1990).

3. A. I. Shirshov, “On free Lie rings,” Mat. Sb., 45(87), No. 2, 113-122 (1958).

4. M. Hall, “A basis for free Lie rings and higher commutators in free groups,” Proc. Am. Math. Soc., 1, 575-581 (1950).

5. A. I. Shirshov, “On bases for free Lie algebra,” Algebra Logika, 1, No. 1, 14-19 (1962).

6. A. I. Shirshov, “Some algorithmic problems for Lie algebras,” Sib. Mat. Zh., 3, No. 2, 292-296 (1962).

7. K. T. Chen, R. H. Fox, and R. C. Lyndon, “Free differential calculus IV, the quotient groups of the lower central series,” Ann. Math., 68, 81-95 (1958).

8. A. I. Mal’tsev, “On representations of models,” Dokl. Akad. Nauk SSSR, 108, No. 1, 27-29 (1956).

9. M. Rosso, “An analogue of the Poincare–Birkhoff–Witt theorem and the universal $R$-matrix of $U_q(sl(N + 1))$,” Comm. Math. Phys., 124, 307-318 (1989).

10. H. Yamane, “A Poincaré-Birkhoff-Witt theorem for quantized universal enveloping algebras of type $A_N$,” Publ., RIMS. Kyoto Univ., 25, 503-520 (1989).
11. G. Lusztig, “Canonical bases arising from quantized enveloping algebras,” *J. Am. Math. Soc.*, 3, No. 2, 447-498 (1990).

12. G. Lusztig, “Quivers, perverse sheaves, and quantized enveloping algebras,” *J. Am. Math. Soc.*, 4, No. 2, 365-421 (1991).

13. M. Kashiwara, “On crystal bases of the q-analog of universal enveloping algebras,” *Duke Math. J.*, 63, No. 2, 465-516 (1991).

14. E. J. Taft and R. L. Wilson, “On antipodes in pointed Hopf algebras,” *J. Alg.*, 29, 27-32 (1974).

15. J. W. Milnor and J. C. Moore, “On the structure of Hopf algebras,” *Ann. Math. (2)*, 81, 211-264 (1965).

16. A. I. Shirshov, “Some problems in the theory of rings which are close to associative,” *Usp. Mat. Nauk*, 13, No. 6(84), 3-20 (1958).

17. N. Jacobson, *Lie Algebras*, Interscience, New York (1962).

18. S. Montgomery, *Hopf Algebras and Their Actions on Rings*, Reg. Conf. Ser. Math., 82, Am. Math. Soc., Providence, RI (1993).

19. D. E. Radford, “The structure of Hopf algebras with a projection,” *J. Alg.*, 92, 322-347 (1985).