Multi-Particle Processes in QCD without Feynman Diagrams

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A way to efficiently compute helicity amplitudes for arbitrary tree-level scattering processes in QCD is presented. The scattering amplitude is evaluated recursively through a set of Dyson-Schwinger equations. The computational cost of this algorithm grows asymptotically as $3^n$, where $n$ is the number of particles involved in the process, compared to $n!$ in the traditional Feynman graphs approach. Unitary gauge is used and mass effects are available as well. Additionally, the color and helicity structures are appropriately transformed so the usual summation is replaced by the Monte Carlo techniques.

1. INTRODUCTION

QCD processes with many external legs are of much interest, both for testing QCD in different settings and as backgrounds for new physics processes at the Fermilab TeVatron and at the CERN LHC. However, the estimation of multi-jet production cross sections as well as their characteristic distributions is a difficult task. Perturbation theory based on Feynman graphs runs into computational problems, since the number of graphs contributing to the amplitude grows like $n!$. The counting of the graphs itself becomes a problem let alone their evaluation and the computation of the color and helicity structures which is an additional source of computational inefficiencies.

Over the last few years, new recursive algorithms based on Dyson-Schwinger equations\textsuperscript{123} or on field equations\textsuperscript{4150} have been developed in order to overcome the computational obstacles. Very recently also on shell recursive equations have been proposed\textsuperscript{89}. The amplitude calculated using Dyson-Schwinger recursive equations, which avoid Feynman diagrams, results in a computational cost growing asymptotically as $3^n$. Here, the off-shell subamplitudes are introduced which are combinations of parts of Feynman graphs. For those subamplitudes a recursion relation has been obtained which enables to express an $n$-particle amplitude in terms of all subamplitudes, starting from $1-,$ $2-,$ ... up to $(n-1)$ particles. The color and helicity structures corresponding to each subamplitude have a simpler form. Moreover, they can be appropriately transformed so the usual summation can be replaced by the Monte Carlo one.

In this article the algorithm based on Dyson-Schwinger recursive equations is presented and used in order to efficiently obtain cross sections for arbitrary multi-jet processes.

2. DESCRIPTION OF THE ALGORITHM

To illustrate how the algorithm works let us present a simple example, the gluon self-interaction:

$$g(p_1)g(p_2) \rightarrow g(p_3)g(p_4).$$

(1)

Here $p_1, p_2, p_3, p_4$ represent the external momenta involved in the scattering process, taken to be in-
coming. The subamplitude with an off-shell gluon of momentum \( P \) has contributions from three- and four-gluon vertices only. To reduce the computational complexity down to an asymptotic \( 3^n \), we replaced the four-gluon vertex with a three-gluon vertex by introducing an auxiliary field represented by the antisymmetric tensor \( H^{\mu\nu} \). The recursion for the gluons now changes only slightly. However, we have an additional equation for the auxiliary field. Diagrammatically, the full content of Dyson-Schwinger equations is presented in Fig. 1 and the corresponding equations have the following form:

\[
A_{AB}^\mu(P) = \frac{g_s}{2P^2} \sum_{p = p_1 + p_2} V^\mu_{\alpha \lambda}(P, p_1, p_2) \epsilon(p_1, p_2)
\]

\[
\{ A_{AC}^\lambda(p_1) A_{CB}^\nu(p_2) - A_{AC}^\nu(p_2) A_{CB}^\lambda(p_1) \}
\]

\[
\frac{ig_s}{2P^2} \sum_{p = p_1 + p_2} X_{\mu \lambda \rho}^\nu A_{AC}^\lambda(p_1) H_{CB}^{\lambda \rho}(p_2)
\]

\[-H_{AC}^{\lambda}(p_2) A_{CB}^{\nu}(p_1) \epsilon(p_1, p_2)\]

\[
H_{AB}^{\mu \nu}(P) = \frac{ig_s}{4} \sum_{p = p_1 + p_2} X_{\mu \lambda \rho}^{\nu \epsilon} \epsilon(p_1, p_2)
\]

where \( X_{\mu \nu}^{\lambda \rho} \) is the new auxiliary field-gluon-gluon vertex:

\[
X_{\mu \nu}^{\lambda \rho} = g_{\mu \lambda} g_{\nu \rho} - g_{\nu \lambda} g_{\mu \rho}
\]

and \( A, B, C = 1, 2, 3 \), and \( \epsilon(p_1, p_2) \) is the Fermi sign factor. In order to be more transparent we will write explicitly the three-gluon vertex part as well as the auxiliary one, suppressing the color indices and using the light-cone representation:

\[
A_{AB}^{\mu}(P) \sim (A(p_1) \cdot A(p_2))(p_2 - p_1)^{\mu} + (p_1 \cdot A(p_2) + P \cdot A(p_2)) A_{AB}^{\mu}(p_1) - (p_2 \cdot A(p_1) + P \cdot A(p_1)) A_{AB}^{\mu}(p_2)
\]

\[+ A_{\nu}(p_1) H_{\mu \nu}^{\rho}(p_2) - A_{\nu}(p_1) H_{\mu \nu}^{\rho}(p_2) \]

\[
H_{\mu \nu}^{\rho}(P) \sim A_{\nu}(p_1) A_{\nu}^{\mu}(p_2) - A_{\nu}^{\mu}(p_1) A_{\nu}(p_2)
\]

where all momenta are taken to be incoming. These equations, the off-shell fields, are the main building blocks of the gluon self-interaction, which is to be constructed iteratively. After \( n - 1 \) steps, where \( n \) is the number of particles under consideration, one can get the total amplitude

\[
A(p_1, p_2, ..., p_n) = \hat{A}_{AB}(P_i) \cdot A_{AB}(p_i),
\]

where

\[
P_i = \sum_{j \neq i} p_j
\]

so that \( P_i + p_i = 0 \). The hat denotes functions given by the expression from the previous step except for the propagator term. This is because the outgoing momentum \( P_i \) must be on shell and the propagator term is removed by the amputation procedure. The iteration begins with the initial condition for the external particles. For gluons we have

\[
A_{AB}^{\mu}(p_i) = \varepsilon_{\lambda}^{\mu}(p_i) \delta_{AC} \delta_{DB},
\]

where \( i \) enumerates the external particles, \( i = 1, ..., n \), \( \varepsilon \) denotes the polarization vector and \( \lambda = \pm 1 \).

In order to label and systematically control the momenta of external particles and their relevant intermediate combination, we give all momenta in the binary representation, see e.g.
For the process under consideration, and the external particles with momenta \( p_i^\mu \), \( i = 1, 2, 3, 4 \) we assign to the momentum \( P^\mu \) a binary vector \( \hat{m} = (m_1, m_2, m_3, m_4) \) with components which are either 0 or 1 as follows:

\[
P^\mu = \sum_{i=1}^{4} m_i p_i^\mu. \tag{7}
\]

The binary vector can now be uniquely represented by the integer

\[
m = \sum_{i=1}^{4} 2^{i-1} m_i. \tag{8}
\]

In particular, one can write:

\[
g(1)g(2) \to g(4)g(8).
\]

All subamplitudes can now be replaced by the corresponding integers

\[
A^\mu(P) \to A^\mu(m),
\]

\[
H^{\mu\nu}(P) \to H^{\mu\nu}(m).
\]

This representation allows us to establish a natural ordering of the momenta based on the notion of level, defined simply as

\[
l = \sum_{i=1}^{4} m_i. \tag{9}
\]

All external momenta are at the level 1, whereas the total amplitude corresponds to the unique level \( n = 4 \). With levels we can see the natural iteration path of the equations. We start from the subamplitudes at the level 1 which are the external momenta together with the initial conditions, then go to the subamplitudes at the level 2 and so on until we reach level \( n = 4 \). The Fermi sign factor is also expressed in the integer number language

\[
\epsilon(P_1, P_2) \to \epsilon(m_1, m_2) \tag{10}
\]

and we use the following formula

\[
\epsilon(m_1, m_2) = (-1)^{\chi(m_1, m_2)} \tag{11}
\]

with

\[
\chi(m_1, m_2) = \sum_{i=1}^{2} \sum_{j=1}^{i-1} \hat{m}_{1i} \hat{m}_{2j}. \tag{12}
\]

where the hat means that this particular component is equal to 0 if the corresponding external particle is a boson. This sign factor takes into account the sign change when two identical fermions are interchanged.

Contrary to original HELAC, the computational part consists of only one step, where couplings allowed by the lagrangian defined by fusion rules are only explored, see Ref. [10] for technical details. Subsequently, the helicity configurations are set up. There are two possibilities, either exact summation over all \( 2^{n_1} \times 2^{n_2} \times 2^{n_3} \) helicity configurations, where \( n_1 \) is the total number of quarks, \( n_2 \), the total number of antiquarks and \( n_3 \), the total number of gluons; or Monte Carlo summation. For example for a gluon the second option is achieved by introducing the polarization vector

\[
\hat{\epsilon}^\mu_\phi(p) = e^{i\phi} \hat{\epsilon}^\mu_+(p) + e^{-i\phi} \hat{\epsilon}^\mu_-(p), \tag{13}
\]

where \( \phi \in (0, 2\pi) \) is a random number. By integrating over \( \phi \) we can obtain the sum over helicities

\[
\frac{1}{2\pi} \int_0^{2\pi} d\phi \ \hat{\epsilon}^\mu_\phi(p)(\hat{\epsilon}^\mu_\phi(p))^* = \sum_{\lambda = \pm} \epsilon^\lambda_\phi(p)(\epsilon^\lambda_\phi(p))^*.
\]

The same idea can be applied to the helicity of quarks and antiquarks.

Finally, the color factor is evaluated iteratively. Once again, we have two options. Either we proceed by computing all \( 3^{n_u} \times 3^{n_d} \) color configurations, where the gluon is treated as a quark-antiquark pair and \( n_u, n_d \) is the number of quarks and antiquarks respectively, or particular configurations are chosen by the Monte Carlo method (for more details see Ref. [10]).

For the process \( g(P_1)g(P_2) \to g(P_4)g(P_8) \) which we are using through this section as an example the momenta involved in the various levels of calculation are the following:

1st Level:

\[
P_1 = (0001) = p_1, \quad P_2 = (0010) = p_2
\]

\[
P_4 = (0100) = p_3, \quad P_8 = (1000) = p_4
\]

2nd Level:
\[ P_6 = (0110) = p_2 + p_3 \]
\[ P_{10} = (1010) = p_2 + p_4 \]
\[ P_{12} = (1100) = p_3 + p_4 \]

3th Level: \( P_{14} = (1110) = p_2 + p_3 + p_4 \)

4th Level: \( P_{15} = (1111) = p_1 + p_2 + p_3 + p_4 \)

Note that we have chosen the particle number 1 as our ending point and, therefore, we have only computed all the relevant momentum combinations where the momentum \( p_1 \) does not appear. This excludes all odd integers between 1 and \( 2^n - 2 \). The momentum \( p_1 \) is combined only once, in the last \( n \)-th level. Starting from the second level we get:

\[ A^\mu(6) \sim V^{\nu\rho\sigma}(6, 2, 4) A_\rho(2) A_\sigma(4) \]
\[ H^{\mu\nu}(6) \sim X^{\mu\nu\rho\sigma} A_\rho(2) A_\sigma(4) \]
\[ A^\mu(10) \sim V^{\nu\rho\sigma}(10, 2, 8) A_\rho(2) A_\sigma(8) \]
\[ H^{\mu\nu}(10) \sim X^{\mu\nu\rho\sigma} A_\rho(2) A_\sigma(8) \]
\[ A^\mu(12) \sim V^{\nu\rho\sigma}(12, 4, 8) A_\rho(4) A_\sigma(8) \]
\[ H^{\mu\nu}(12) \sim X^{\mu\nu\rho\sigma} A_\rho(4) A_\sigma(8) \]

At level 3:

\[ A^\mu(14) \sim V^{\nu\rho\sigma}(14, 2, 12) A_\rho(2) A_\sigma(12) - \]
\[ X^{\mu\nu\rho\sigma} A_\rho(2) H_{\rho\sigma}(12) + V^{\mu\nu\rho}(14, 4, 10) A_\rho(4) A_\sigma(10) \]
\[ -X^{\mu\nu\rho\sigma} A_\rho(4) H_{\rho\sigma}(10) + V^{\mu\nu\sigma}(14, 8, 6) A_\rho(8) A_\sigma(6) \]
\[ -X^{\mu\nu\rho\sigma} A_\rho(8) H_{\rho\sigma}(6) \]

At the last, 4-th level, the total amplitude is computed by combining this subamplitude with the remaining one, which describes the momentum \( p_1 \) and is simply given by

\[ A(15) \sim A_\mu(1) \cdot A^\mu(14). \]

### 3. NUMERICAL RESULTS

As an example the algorithm has been used to compute total cross sections for multiple jets production. The CMS energy was chosen \( \sqrt{s} = 14 \) TeV and the following cuts were applied

\[ p_T > 60 \text{ GeV}, \quad |\eta| < 2.5, \quad \Delta R > 1.0 \quad (14) \]

where \( p_T = \sqrt{p_x^2 + p_y^2} \) is the transverse momentum of a jet, \( \eta = -\ln\tan(\theta/2) \) is the pseudorapidity. Many methods can be used to define what is meant by a jet of hadron. One commonly used is the ’cone’ description of a jet which is the transverse energy, \( E_T \), concentration in a cone of radius

\[ \Delta R = \sqrt{\Delta \phi^2 + \Delta \eta^2} \]

with

\[ \Delta \phi_{ij} = \arccos \left( \frac{p_{x_j} p_{x_i} + p_{y_j} p_{y_i}}{p_T p_T} \right). \]

All results are obtained with a fixed strong coupling constant calculated at the \( M_Z \) scale. There are several parameterizations for the parton structure functions, we used CTEQ6 PDF’s parametrization [11,12]. For the phase space generation we used either PHEGAS [13] or a flat phase-space generator RAMBO [14].

The results for the total cross section from 4 up to 8 gluons are listed above in Table 1. We give

| Process | \( \sigma_T \pm \text{error (nb)} \) | \( \sigma_T^{MC} \pm \text{error (nb)} \) |
|---------|---------------------------------|---------------------------------|
| \( gg \to 2g \) | 4611.55 ± 38.13 | 4627.18 ± 33.28 |
| \( gg \to 3g \) | 152.444 ± 2.490 | 152.137 ± 2.822 |
| \( gg \to 4g \) | 12.9072 ± 0.4070 | 12.6137 ± 0.4619 |
| \( gg \to 5g \) | 1.04254 ± 0.05300 | 1.04446 ± 0.10390 |
| \( gg \to 6g \) | 0.07577 ± 0.00597 | 0.07261 ± 0.00516 |
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Figure 2. Distribution in $z = |M_I|^2 / \sum_i |M_i|^2$, where $|M_I|^2$ is the square matrix element for one particular color configuration normalized to the sum of all possible is plotted. The upper plot corresponds to the $gg \rightarrow gg$ process, the lower one to the $gg \rightarrow ggg$. Solid line crosses denote summation over all color configurations whereas dashed, the Monte Carlo summation.

The next result we present is an example of summation over all color configurations in comparison to Monte Carlo summation. In Fig. 2 the X-axis variable is

$$z = |M_I|^2 / \sum_i |M_i|^2$$

where $|M_I|^2$ is the square matrix element for one particular color configuration normalized to the sum of all possible is plotted. The agreement is easy visible.

4. SUMMARY

An efficient tool for automatic computation of helicity amplitudes and cross sections for multi-particle final states in QCD has been presented.

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