Systematic proof of the existence of Yangian symmetry in chiral Gross–Neveu models

Tamas Hauer∗†

Institute for Theoretical Physics
Lorand Eötvös University
H-1088 Budapest, Puskin u. 5-7, Hungary

Abstract

The existence of non-local charges, generating a Yangian symmetry is discussed in generalized chiral Gross-Neveu models. Their conservation can be proven by a finite-loop perturbative computation, the order of which is determined from group theoretic constants and is independent of the number of flavors. Examples, where the 1-loop calculation is sufficient, include the $SO(n)$-models and other more exotic groups and representations.

∗E-mail address: hauer@mit.edu
†Address: Center for Theoretical Physics, MIT, Cambridge, MA 02139
**Introduction.** In the theory of two dimensional integrable systems a fundamental role is played by those conserved quantities which guarantee the solvability of the models. While in many cases classical integrability is manifest, the proof of conservation and even the proper definition of these charges may be quite subtle in the corresponding quantum theory. In certain models where the symmetry algebra is generated by nonlocal charges \cite{1,2} the approach originally proposed by Lüscher \cite{3} proved to be fruitful. In these (asymptotically free) field theories, the properties of nonlocal expressions of the local currents can be traced back to the short distance singularities of the current algebra. The existence of Lüscher's nonlocal conserved charge is the consequence of the fact that the operator product expansion (OPE) of the currents close on themselves and their derivatives, which replaces, in the quantum theory, the zero-curvature condition of the classical currents. (See \cite{4} for a summary and \cite{5} for a complete review). The question about the closure of the current algebra is a delicate one, and the answer varies from one model to another. In the simplest case (like the O(n) non-linear sigma model \cite{3}, a large class of generalized sigma models \cite{3} or the chiral SU(n) Gross-Neveu (GN)-model \cite{3}) there are too few degrees of freedom to form operators which may ruin the conservation, while in other theories, extra fields may be constructed and it is the dynamics of the model which determines their ultimate presence or absence. For example, in the $\mathbb{C}P^{N-1}$-model, the conservation is ruined by the extra term \cite{7}, while in its supersymmetric partner this quantum anomaly disappears \cite{8}. In a previous paper \cite{4} we studied the SU$(n)$ (multiflavor), chiral Gross-Neveu model ((M)CGN)\cite{9} where - thanks to renormalization group invariance - a one-loop perturbative computation proved to be decisive and saved the desired form of the current algebra. The aim of the present letter is to extend this argument to a general class of chiral GN-type models, which are defined as Lagrangian field theories with current-current interaction, and where the symmetry algebra generated by the currents is an *arbitrary simple Lie-algebra*. This family of theories (without flavor multiplicity) was studied in \cite{10} where, the non-local charges generating
The classical Yangian were constructed. Our goal is to investigate whether the conservation of the non-local generator of the algebra survives the quantization using the above strategy. We will explicitly calculate the leading exponent of the perturbative coupling constant of the OPE-coefficients in terms of group-theoretic constants and obtain a simple formula (eq. (14)), expressing the order of the needed perturbative calculation in terms of quadratic Casimirs of certain representations. We also give a large class of examples where zero- or one-loop results yield conclusion and show that the multiplicity (flavor) of the fermion field does not affect the question. These theories include the $SO(n)$ models\footnote{For the (one-flavor) $O(n)$-symmetric $GN$-models, non-local Ward identities were proven in leading order of the $1/n$-expansion in \cite{11}} and a bunch of previously uninvestigated\footnote{This refers to the quantum case, the study of the classical conservation laws is similar for the whole family \cite{10}} models characterized by different groups, representations and flavor multiplicity.

The plan of the paper is as follows. First, we summarize the important points in the connection between the non-local conservation laws and the OPE of the current, then shortly review the RG-argument developed in \cite{4} showing how the anomalous dimension of certain operators play fundamental role in the analysis. When generalizing to arbitrary groups, we then express these quantities in terms of quadratic Casimirs of certain representations, deriving the formula (14), which determines the order of the needed perturbative calculation. Finally, we present examples where no more then a 1-loop computation is sufficient to reach conclusion.

**OPE and Non-local Charge** In the field theories under consideration we have a set of conserved, local currents: $\partial_{\mu} j^{a\mu}(x) = 0$, transforming under the adjoint representation of a simple Lie-group, $G$, with charges satisfying\footnote{$Q^a \equiv \int dx j^{a\mu}(x, t); \quad f^{abc} f^{bcd} = -C_{adj} \delta^{ad}$.}

\begin{equation}
[Q^a, Q^b] = f^{abc} Q^c. \tag{1}
\end{equation}

In addition we require the QFT be renormalizable and asymptotically free and that all operators in the adjoint representation of $G$ have higher canonical di-
mension than the current. If these conditions hold then, the general form of the current-current OPE (up to vanishing terms as \( x \to 0 \)) is the following:

\[
f^{abc}j^b_\mu(x)j^c_\nu(0) = C^{\rho}_{\mu\nu}(x)j^\rho_\sigma(0) + D^{\sigma\rho}_{\mu\nu}(x)\partial_\sigma j^\rho_\mu(0) + \sum_i E_i O^{a}_{i[\mu\nu]}(0),
\]

where \( C^{\rho}_{\mu\nu}(x) \) and \( D^{\sigma\rho}_{\mu\nu}(x) \) are functions with leading singularities \( \mathcal{O}(|x|^{-1-0}) \) and \( \mathcal{O}(|x|^{-0}) \) \((-0 \text{ stands for logarithmic-like singularities}) \) which are explicitly given in terms of one model-dependent scalar function, \( \xi(x^2) \), while \( E_i \) are constant OPE-coefficients multiplying antisymmetric tensor operators, \( O^{a}_{i[\mu\nu]}(x) \). Using the OPE (2) one can prove that, the quantum analog of the classical charge

\[
Q_1^a = \frac{1}{4} \int_{-\infty}^{\infty} dy_1 dy_2 \epsilon(y_1 - y_2) f^{abc}j^b_0(t,y_1)j^c_0(t,y_2) + \int_{-\infty}^{\infty} dy j^a_1(t,y),
\]

can always be consistently defined, but it is conserved only if all the \( E_i \)'s are zero (both statements are independent of the concrete form of \( \xi(x^2) \)). Phrasing the condition in the above form immediately yields the straightforward strategy for proving the existence of the quantum charge: one assumes the presence of every antisymmetric tensor operator allowed by the symmetries, and computes the corresponding OPE-coefficients in some way; if all of them are zero then the conservation is proved. This program was successfully carried out in the \( CP^{N-1} \)-model with fermions, where supersymmetry prevented the classical value from receiving quantum corrections; in our case renormalization group invariance will be of great help.

**OPE and Renormalization Group** As we showed in [4], the calculation of the OPE-coefficients can be done using perturbation theory in models without dimensionful parameters, since in this case renormalization group invariance highly restricts the possible form of \( E_i \). In the \( SU(n) \) MCGN we faced one extra operator and we expressed the leading exponent of its perturbative expansion in terms of its 1-loop anomalous dimension, however – as we will see – the argument is neither specific to \( SU(n) \) nor is it restricted to the extra operator standing alone.
For simplicity, assume that the extra operators under consideration renormalize multiplicatively (do not mix with others) and the anomalous dimension of $O_{i[\mu \nu]}^{a}$ is given by

$$D \log(Z_i) = \eta_{i,1} g^2 + \eta_{i,2} g^4 + \ldots,$$  \hspace{1cm} (4)

where the lhs. is the renormalization-group derivative of the renormalization constant $Z_i$, corresponding to $O_{i[\mu \nu]}^{a}$ and $g$ is the perturbative coupling of the model. For the OPE-coefficient,

$$E_i = E_{i,0} g^{2\alpha_i} (1 + E_{i,1} g^2 + E_{i,2} g^4 + \ldots),$$  \hspace{1cm} (5)

the renormalization group equation yields the following relation between $\alpha_i$, $\eta_{i,1}$, and the one-loop beta function coefficient, $\beta_0$:

$$\alpha_i = -\frac{\eta_{i,1}}{2\beta_0},$$  \hspace{1cm} (6)

This equation is the key point in the argument since $\alpha_i$ is either a positive integer, in which case it determines the order of the perturbative calculation needed to decide whether $E_i$ vanishes or not, or if it is negative or non-integer then it does not allow $O_{i[\mu \nu]}^{a}$ to be present in the OPE.

**Chiral Gross-Neveu models.** Now we turn to the CGN models, they are defined by the following Minkowskian action:

$$S = \int d^2 x \left( \bar{\psi} i \partial \psi - \frac{g^2}{2} j^a_\mu j^a_\mu \right).$$  \hspace{1cm} (7)

The fermionic field, $\psi$ transforms under the irreducible representation, $R$ (whereas $\bar{\psi}$ transforms under $\bar{R}$) of the simple (color) Lie-group, $G$ and in case of the multiflavor models it also carries a multiplicity (flavor) index. The current in the interaction Lagrangian is defined as

$$j^a_\mu(x) \equiv \bar{\psi}(x) T^a \gamma_\mu \psi(x),$$  \hspace{1cm} (8)

where $T^a$ are generator matrices in representation $R$ (in case of the multiflavor models summation over the multiplicity indices is understood). In order not to
get in conflict with our assumption that the color current is the operator in the adjoint representation of $G$ of the lowest canonical dimension, we require that the decomposition of $\bar{R} \otimes R$ into irreducible representations does not contain the adjoint more than once.

The first steps in determining the operator content of the OPE (2) are: collecting the set of operators allowed by symmetry and canonical dimensional analysis; and then – following our method – calculating their 1-loop renormalization. Apart from the current, there is one bilinear operator in the adjoint representation, $i(\partial_{\mu}\bar{\psi}T^{a}\gamma_{\nu}\psi - \bar{\psi}T^{a}\gamma_{\nu}\partial_{\mu}\psi)$ which, however, has opposite $C$-parity to the current’s and is excluded. Therefore canonical dimensional analysis allows only for operators which are quadrilinear expressions of the fermionic field. In order to identify these fields we search for operators in the adjoint representation composed of the direct product $\bar{R} \otimes R \otimes \bar{R} \otimes R$ that is, we have to decompose this quadratic product into irreducibles. The following sequence will prove to be useful: decompose first $\bar{R} \otimes R$ (form bilinears) and then the pairwise products (form bilinear of bilinears) from the two direct sums and look for the adjoint representations:

$$(\bar{R} \otimes R) \otimes (\bar{R} \otimes R) = (\ldots \oplus R_{1} \oplus \ldots) \otimes (\ldots \oplus R_{2} \oplus \ldots) = \ldots \oplus R_{\text{adj}} \oplus \ldots \quad (9)$$

It turns out that this decomposition will guarantee the multiplicative renormalization and it is the Casimir of $R_{1}$ and $R_{2}$ “defined” above that enters the anomalous dimension. To write down explicitly the operator we have in mind, denote the projectors on the basis elements of $R_{1}$ and $R_{2}$ with $C_{1}^{b} \alpha$ and $C_{2}^{b} \rho$, respectively and the one on the basis elements in the adjoint representation by $h_{a_{\alpha \rho}}^{\alpha}$:

$$O_{[\mu \nu]}^{a} = h_{a_{\alpha \rho}}^{\alpha}(\bar{\psi}\gamma_{\mu}\gamma_{\nu}C^{1 \alpha} \psi)(\bar{\psi}\gamma_{\mu}C^{2 \rho} \psi). \quad (10)$$

To calculate the anomalous dimension of this operator one has to compute the one-loop four-particle correlation function. This contains four divergent Feynman-diagrams the sum of which is proportional to the following expression:

$$\rho = h_{a_{\alpha \rho}}^{a} ((C^{1 \alpha} T_{ab})_{ij} (C^{2 \rho} T_{ab})_{kl} - (C^{1 \alpha} T_{ab})_{ij} (T_{ab} C^{2 \rho})_{kl} +$$
\[(T^b C^{1\alpha})_{ij} (T^b C^{2\rho})_{kl} - (T^b C^{1\alpha})_{ij} (C^{2\rho} T^b)_{kl}\),

(11)

where \(i..l\) are color indices. If one treats \(C^1\) and \(C^2\) as tensor operators acting on \(R\) and recalls their commutation relation with \(T^b\) this can be rewritten in terms of the generators \(\tau^{1b}\) and \(\tau^{2b}\) on representations \(R_1\) and \(R_2\), respectively:

\[
\rho = h^a_{\alpha\rho} (\tau^{1b})^\alpha \tau^{2b})^\rho (C^{1\alpha'})_{ij} (C^{2\rho'})_{kl} = -\frac{1}{2} (C_{R_1} + C_{R_2} - C_{adj}) h^a_{\alpha\rho} (C^{1\alpha'})_{ij} (C^{2\rho'})_{kl},
\]

(12)

where we also used the tensor transformation properties of \(h^a\). Thus the operator renormalizes multiplicatively at one-loop order and its anomalous dimension is determined by the quadratic Casimirs \(C_{R_1}, C_{R_2}, C_{adj}\), of representations \(R_1\), \(R_2\) and the adjoint:

\[
\eta_1 = \frac{1}{2\pi} (C_{R_1} + C_{R_2} - C_{adj}),
\]

(13)

and this together with the one-loop \(\beta\)-function, \(\beta_0 = -\frac{C_{adj}}{4\pi}\), yields our magic number, \(\alpha\):

\[
\alpha = \frac{C_{R_1} + C_{R_2} - C_{adj}}{C_{adj}}
\]

(14)

Equation (14) is the main technical result of this paper. The power of this simple formula resides in that, \(\alpha\) - which determines the order at which the needed perturbative calculation becomes “exact” - may be such a small integer that this calculation can be done in finite amount of time (as in \[4\], where it was 1) or hopefully non-integer, in which case simply no real computation is needed. Notice furthermore that, since (11) contains the flavor indices in a trivial way, the multiplicative renormalization is also true for operators with nontrivial flavor-structure (in the multiflavor models) and the same formula applies to their anomalous dimension.

Let us summarize the above in the following recipe. Take a CGN-model, which is defined by \(G\), \(R\) and the number of flavors. Decompose the product of representations \((\bar{R} \otimes R) \otimes (\bar{R} \otimes R)\) into irreducibles and find the adjoints; following the
proposed decomposition, to every copy of the adjoint representation correspond two other ones, \( R_1 \) and \( R_2 \). Calculate \( \alpha_i \) using (14) for every case obtained and take the highest nonnegative integer, \( \alpha \) out of them. Calculate the OPE (2) up to \( \alpha \) loops perturbatively and see whether it closes on the color currents or not; the fact you obtain is exact.

**Examples.** We now consider applications of (14) to various CGN models and look for the ones where no real computation is needed. In ref. [4] we calculated the OPE-coefficients up to \( g^2 \) in perturbation theory. Though we kept the \( SU(n) \)-models in mind, the computation is identical for any group and representation, and the statement that the OPE closes on the currents themselves up to 1-loop order is valid in all (M)CGN models. Therefore in the models under consideration, the conservation of the non-local charge is proved whenever among the \( \alpha_i \)'s there is no integer greater than one.

As a warm-up we repeat the result in the \( SU(n) \) theories with the fermions being in the fundamental representation. The decomposition of the direct product is:

\[
(\bar{R} \otimes R) \otimes (\bar{R} \otimes R) = (1 \otimes R_{\text{adj}}) \oplus (R_{\text{adj}} \otimes 1) \oplus (R_{\text{adj}} \otimes R_{\text{adj}}) \oplus (1 \otimes 1). \tag{15}
\]

The corresponding Casimirs are \( C_1 = 0; C_{\text{adj}} = 1 \), which yields \( \alpha_{1,\text{adj}} = 0 \) and \( \alpha_{\text{adj},\text{adj}} = 1 \). The largest integer is 1, this was why we performed the one-loop computation in [4] and found that the quantum charge is conserved in the multiflavor \( SU(n) \)-models. (In [4] other arguments, like C-parity were also used to rule out quadrilinear operators, which is not needed here since their \( \alpha \) are smaller than the largest allowed one.)

Now consider the \( SO(n) \) models with the fermions being in the vector representation. As we expect, here we face more operators than in the \( SU(n) \)-case. We repeat the decomposition for \( SO(n) \):

\[
(\bar{R} \otimes R) \otimes (\bar{R} \otimes R) = (1 \otimes R_{\text{adj}}) \oplus (R_{\text{adj}} \otimes 1) \oplus (R_{\text{adj}} \otimes R_{\text{adj}}) \oplus \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \oplus (R_S \otimes R_{\text{adj}}) \oplus (R_{\text{adj}} \otimes R_S) \oplus (R_S \otimes R_S) \oplus \ldots \tag{16}
\]

In this case \( R = \bar{R} \) and \( R_S \) stands for the symmetric tensor representation, and
we did not list the representations not containing the adjoint here. Furthermore
the Casimirs, $C_1 = 0; C_{adj} = n - 2; C_S = n$ give the following $\alpha$’s:

$$\alpha_{1,adj} = 0, \quad \alpha_{adj,adj} = 1, \quad \alpha_{S,adj} = \frac{n}{n - 2}, \quad \alpha_{S,S} = \frac{n + 2}{n - 2}.$$  \hspace{1cm} (17)

The largest integer is 5, 3 and 2 for $n = 3, 4$ and 6, respectively and 1 in all other
cases, which proves the vanishing of the OPE-coefficient of extra operators for
almost every $n$; moreover under the mildest assumption about the continuity of
the crucial coefficient in terms of $n$ one conjectures its vanishing for all $n$. This
proves the conjecture that the $SO(n)$ CGN-models also possess the Yangian
algebra $[13]$ (which can be used to prove their integrability) and this is equally
true for the multiflavor models as well. We can also consider other representa-
tions: as an example take the exotic $SO(7)$ model with the fields in the 8 spinor
representation. One finds that all the $\alpha$’s are non-integers except a 1, thus this
model also possesses Yangian symmetry.

Let us now see how the procedure works for other groups taking $F_4$ first as an
example. Let the fermions be in representation 26 which is self-conjugate and
the bilinears decompose according to $26 \otimes 26 = 1 \oplus 26 \oplus 52 \oplus 273 \oplus 324$. The
quadratic Casimirs are 0, 12, 18, 24, 26, respectively and the corresponding $\alpha$’s
are summarized in the following table:

| $C_1 \otimes C_2$ | $1 \otimes 52$ | $26 \otimes 26$ | $26 \otimes 273$ | $52 \otimes 52$ | $52 \otimes 324$ |
|------------------|--------------|----------------|----------------|----------------|----------------|
| $\alpha$         | 0            | $\frac{1}{3}$  | 1              | 1              | $\frac{13}{3}$ |

| $C_1 \otimes C_2$ | $273 \otimes 273$ | $273 \otimes 324$ | $324 \otimes 324$ |
|------------------|--------------------|------------------|--------------------|
| $\alpha$         | $\frac{5}{3}$      | $\frac{20}{3}$   | $\frac{17}{3}$    |

We can see that the biggest integer is 1 from which one concludes the absence
of extra operators in the OPE and the conservation of the non-local charge in
the $F_4$-model. It is straightforward to repeat the argument for other groups
and show that the same conclusion can be drawn for the $G = E_7, R = 56$-model
however in case of $G = E_6, R = 27$ and $G = E_8, R = 248$ we obtain $\alpha_{650 \otimes 650} = 2$
and $\alpha_{30380 \otimes 30380} = 3$, respectively, that is, a two and three-loop perturbative
calculation is needed in these models. It is clear that one can go on and play
around with other groups and representations using a table of group dimensions and indices without difficulty. Note however that, to prove the non-conservation of the non-local charge in a specific model one can not avoid going beyond one-loop order in perturbation theory.

**Conclusion.** Once a theory possesses “non-local” charges besides the usual “local” ones, it can be shown that, a Yangian symmetry algebra is generated. Due to the non-local nature of the generators however, this is not a direct multiplier of the two-dimensional Poincare group \[12, 13\]. This fact reduces the dynamical question about the mass spectrum of a field theory to one on the classification of representations of the underlying algebra (see e.g. \[14, 15\]); and after identifying the spectrum, the S-matrix is also highly constrained being proportional to the R-matrix in the given representation. Another approach to the determination of the S-matrix, (which was originally developed by Lüscher) uses the fact that the action of the non-local charge on asymptotic one-particle states is straightforwardly computed, in terms of which all the asymptotic matrix elements can be obtained \[12, 16\]. These results lead to the absence of particle production and factorization \[3, 17\].

In this letter, various chiral Gross-Neveu models – which are identified by their symmetry group, representation and multiplicity of the fields – have been considered, and the existence of the non-local generator of the expected Yangian symmetry algebra was discussed. It has long been known that the \(SU(n)\) one-flavor model possess the Yangian symmetry but one expects that there may be other theories with this property, too. In a previous article we proved this for the multiflavor \(SU(n)\) model and the method developed there was extended to the whole family in the present paper. The intricate question related to the non-perturbative definition of the model could be answered using finite order perturbation theory (thanks to asymptotic freedom) – often a one-loop calculation is sufficient. Examples for the latter case were presented proving among others that in the \(SO(n)\) models the non-local charge can also be defined and
that, more exotic theories possess it as well; moreover that, this property is independent of the number of flavors. We also have found CGN-models where – within this framework – only a higher-loop calculation could decide which class they belong to; and it is an open question whether there exist generalized CGN quantum theories without the non-local charge generating Yangian symmetry at all.

**Acknowledgment.** I am grateful to J. Balog and P. Forgács for helpful discussions. This work was partly supported by the Hungarian National Science Fund OTKA, grant No. T19917.

**References**

[1] H.J. de Vega, H. Eichenherr and J.M. Maillet *Comm.Math.Phys.* 92 (1984) 507.

[2] H.J. de Vega, H. Eichenherr and J.M. Maillet *Nucl. Phys.* B240 (1984) 377.

[3] M. Lüscher, *Nucl. Phys.* B135 (1978) 1.

[4] T. Hauer, hep-th/9702016, to appear in *Nucl. Phys* B

[5] E. Abdalla, M.C.B. Abdalla and K.D. Rothe *Nonperturbative methods in two-dimensional quantum field theory* Singapore, World Scientific (1991)

[6] E. Abdalla et al, *Nucl. Phys.* B210 (1982) 181.

[7] E. Abdalla et al, *Phys Rev.* D23 (1981) 1800.

[8] E. Abdalla et al, *Phys Rev.* D27 (1983) 825.

[9] D.J. Gross and A. Neveu, *Phys. Rev.* D10 (1974) 3235.

[10] H.J. de Vega, H. Eichenherr and J.M. Maillet *Phys. Lett.* 132B (1983) 337.

[11] T.L. Curtright and C. Zachos *Phys. Rev.* D24 (1981) 2661.
[12] M. Lüscher, *Unpublished notes*

[13] D. Bernard, *Comm.Math.Phys.* 137 (1991) 191.

[14] A. Belavin, *Phys. Lett.* B283 (1992) 67.

[15] T. Nakanishi, *Nucl. Phys.* B439 (1995) 441.

[16] D. Buchholz and J.T. Lopuszanski, *Lett. Math. Phys.* 3 (1979) 175.

[17] E. Abdalla and A. Lima-Santos, *Rev.Bras.Fis* 12 (1982) 293.