Numerical solutions of Korteweg de Vries and Korteweg de Vries-Burger’s equations using computer programming

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Abstract

In this paper, numerical and solitonic solutions of Korteweg de Vries (KdV) and Korteweg de Vries-Burger’s (KdVB) equations with initial and boundary conditions are calculated by sinc-collocation method. The basis of method is sinc functions. First, discretizing time derivative of KdV and KdVB’s equations using a classic finite difference formula and space derivatives by $\theta$– weighted scheme between successive two time levels is applied, then Sinc functions are used to solve these two equations. Mathematica programming is used to solve matrix representation of these equations. KdV equation describes behavior of traveling waves which is a third order non-linear partial differential equation (PDE). Maximum absolute errors are given in Tables. The figures show approximate solutions of these two equations. Three conservation laws for KdV’s equation are obtained.

Keywords: Numerical method; KdV equation; KdV-Burger equation; Sinc method; Collocation.

1 Introduction

First, In 1895, Korteweg and de Vries derived famous KdV equation that describes weakly nonlinear shallow water waves and models one directional long water wave of small amplitude, propagating in a channel. It occurs in many field of physics such as in water waves, plasma and fiber optics. Another example for this equation is pulse wave propagation in blood vessels. After its discovery scientist found solution of this equation that is called soliton. The dynamic of solitary wave is modeled by this equation.

KdVB equation was derived by Su and Gardner [1] for a wide class of nonlinear system in the weak non-linearity and long wavelength approximation. The
steady state solution of the KdVB equation has been shown to model weak plasma shocks propagation perpendicularly to a magnetic field. When diffusion dominates dispersion the steady state solutions of the KdVB equation are monotonic shocks, and when dispersion dominates, the shocks are oscillatory. The KdVB equation has been obtained when including electron inertia effects in the description of weak nonlinear plasma waves [3]. The KdVB equation has also been used in a study of wave propagation through liquid field elastic tube [4] and for a description of shallow water waves on viscous fluid.

Consider third order partial differential equations

\[ u_t + 6uu_x + u_{xxx} = 0, \quad x \in \Omega = (a, b) \subset \mathbb{R}, \quad t > 0, \quad (1) \]

as KdV equation. KdV’s equation has the analytical solution as

\[ u(x, t) = 0.5\text{sech}^2(0.5(x - t)), \quad (2) \]

and consider

\[ u_t + \varepsilon uu_x - \nu u_{xx} + \mu u_{xxx} = 0, \quad x \in \Omega = (a, b) \subset \mathbb{R}, \quad t > 0, \quad (3) \]

as KdVB’s equation that has analytical solution given as

\[ u(x, t) = \frac{-6\nu^2}{25\mu}[1 + \tanh\left(\frac{\nu}{10\mu}\left(x + \frac{6\nu^2}{25\mu}t\right)\right)] - \frac{1}{2}\text{sech}^2\left(\frac{\nu}{10\mu}\left(x + \frac{6\nu^2}{25\mu}t\right)\right), \quad (4) \]

that space variable is defined as

\[ x_i = a + (i - 1)h, \quad i = 1, \ldots, N, \quad h = \frac{|b - a|}{N - 1}, \quad (5) \]

as KdVB’s equation. In recent year numerous methods are used for solving KdV and KdVB’s equations. Wang Ju-Feng et al. obtained numerical solution of the third-order nonlinear KdV equation using the elementfree Galerkin (EFG) method which is based on the moving least-squares approximation. A variational method is used to obtain discrete equations, and the essential boundary conditions are enforced by the penalty method [5]. The dynamics of solitary waves is modeled by the Korteweg de Vries (KdV) equation. Jamrud Aminuddin and Sehah, strated by discreetizing the KdV equation using the finite difference method. The discreet form of the KdV equation is put into a matrix form. The solution the of matrix is determined using the Gauss-Jordan method [6]. Jie Shen et al. have studied the eventual periodicity of solutions to the initial and boundary value problem for the KdV equation on a half-line and with periodic boundary data. They derived a representation formula for solutions to the linearized KdV equation and rigorously establish the eventual periodicity of these solutions [7]. Julio Duarte et al. employ the Wavelet-Petrov-Galerkin method to obtain the numerical solution of the equation Korteweg-de Vries (KdV)[8]. Homotopy Perturbation Method (HPM), Variational Iteration Method (VIM) and
Homotopy Analysis Method (HAM) for the semi analytical solution of Kortweg-de Vries (KdV) type equation are applied by Foad Saadi et al [9]. Also, KdVB’s equation has been solved in recent years. M. T. Darvishi et al. have used a Numerical Solution of the KdV-Burgers Equation by Spectral Collocation Method and Darvishis Preconditionings [10]. In [11] Riccati equation expansion method is presented for constructing exact travelling wave solutions of nonlinear evolution equations. The main idea of this method is to take full advantage of the more solutions of Riccati equation to construct exact travelling wave solutions of nonlinear evolution equations. More new exact travelling wave solutions are obtained for KdVB equation. On using variable transformations and proofs of theorems, the asymptotic behaviour and the proper analytical solution of the Korteweg-de Vries-Burgers equation have been found in [12]. Anna Gao et al. studied the problem of optimal control of the viscous KdVBs equation. they develop a technique to utilize the Cole-Hopf transformation to solve an optimal control problem for the viscous KdVBs equation [13]. In [14] exact travelling wave and solitary solutions for compound KdVBs equations are obtained by using an improved sine-cosine method and the Wu elimination method. Numerical solutions of the Korteweg-deVries equation using the periodic scattering transform µ-representation is studied in [15]. Operator Splitting Methods for Generalized KortewegDe Vries Equations has been discussed in [16].

The paper is organized into six sections. Section 2 outlines some of the main properties of sinc function and sinc method. In Section 3, the discretization of KdV and KdVB equations is discussed. Section 4 outlines stability analysis and section 5 introduces errors and conservation laws. Finally numerical results and the efficiency and accuracy of the proposed numerical scheme is shown by considering some numerical examples in Section 6.

2 The Sinc function

In this section the basis of sinc function is discussed[17]. The sinc function is defined on the whole real line, $-\infty < x < \infty$, by

$$\text{sinc}(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \neq 0; \\ 1, & x = 0. \end{cases}$$

(6)

For any $h > 0$, the translated sinc functions with evenly spaced nodes are given as

$$S(j, h)(z) = \text{sinc}\left(\frac{z - jh}{h}\right), \quad j = 0, \pm 1, \pm 2, \cdots.$$  

(7)

The sinc functions are cardinal for the interpolating points $z_k = kh$ in the sense that

$$S(j, h)(kh) = \delta_{jk}^{(0)} = \begin{cases} 1, & k = j; \\ 0, & k \neq j. \end{cases}$$

(8)
If \( f \) is defined on the real line, then for \( h > 0 \) the series

\[
C(f, h)(z) = \sum_{j=-\infty}^{\infty} f(jh)\text{sinc}(\frac{z-jh}{h}),
\]

is called the Whittaker cardinal expansion of \( f \) whenever this series converges. They are based in the infinite strip \( D_s \) in the complex plane

\[
D_s = \{ w = u + iv : |v| < d \leq \frac{\pi}{2} \}.
\]

Some derivatives of sinc function will be used in reduction the equation to matrix form so [13],

\[
I^{(0)}_{ji} = [S(j, h)(x)]|_{x=x_i} = \begin{cases} 
1, & j = i; \\
0, & j \neq i,
\end{cases}
\]

\[
I^{(1)}_{ji} = \frac{d}{dx}[S(j, h)(x)]|_{x=x_i} = \frac{1}{h} \begin{cases} 
0, & j = i; \\
\frac{(-1)^{(k-j)}}{(k-j)^2}, & j \neq i,
\end{cases}
\]

and

\[
I^{(2)}_{ji} = \frac{d^2}{dx^2}[S(j, h)(x)]|_{x=x_i} = \frac{1}{h^2} \begin{cases} 
0, & j = i; \\
\frac{\pi^2}{3} \frac{(-1)^{(k-j)}}{(k-j)^5} \cdot \frac{6 - \pi^2(k-j)^2}{(k-j)^2}, & j \neq i,
\end{cases}
\]

\[
I^{(3)}_{ji} = \frac{d}{dx}[S(j, h)(x)]|_{x=x_i} = \frac{1}{h^3} \begin{cases} 
0, & j = i; \\
\frac{(-1)^{(k-j)}}{(k-j)^7} \cdot [6 - \pi^2(k-j)^2], & j \neq i,
\end{cases}
\]

And so on, for even coefficients, where \( r = 1, 2, \ldots \)

\[
I^{(2r)}_{ji} = \frac{d^{2r}}{dx^{2r}}[S(j, h)(x)]|_{x=x_i} = \frac{1}{h^{2r}} \begin{cases} 
\frac{\pi^{2r}}{2^{2r+1}}, & j = i; \\
\frac{(-1)^r}{2^{2r+1}}, \sum_{l=0}^{r-1} (-1)^{l+1} \frac{2^{2l+1}}{(2l+1)!} \pi^{2l}(i-j)^{2l}, & j \neq i,
\end{cases}
\]

and for odd coefficients, where \( r = 1, 2, \ldots \)

\[
I^{(2r+1)}_{ji} = \frac{d^{2r}}{dx^{2r+1}}[S(j, h)(x)]|_{x=x_i} = \frac{1}{h^{2r+1}} \begin{cases} 
\frac{(-1)^{(j-i)}}{2^{2r+1}(j-i)^{2r+1}}, \sum_{l=0}^{r-1} (-1)^{l+1} \frac{(2r+1)l!}{(2l+1)!} \pi^{2l}(i-j)^{2l}, & j = i; \\
\frac{(-1)^{(j-i)}}{2^{2r+1}(j-i)^{2r+1}}, \sum_{l=0}^{r-1} (-1)^{l+1} \frac{(2r+1)l!}{(2l+1)!} \pi^{2l}(i-j)^{2l}, & j \neq i,
\end{cases}
\]
3 Survey of the method

Consider third order partial differential equations

\[ u_t + 6uu_x + u_{xxx} = 0, \quad x \in \Omega = (a, b) \subset \mathbb{R}, \quad t > 0, \tag{17} \]

as KdV equation and

\[ u_t + \varepsilon uu_x - \nu u_{xx} + \mu u_{xxx} = 0, \quad x \in \Omega = (a, b) \subset \mathbb{R}, \quad t > 0, \tag{18} \]

as KdVB’s equation, with the initial condition

\[ u(x, 0) = f(x), \quad x \in \Omega, \tag{19} \]

and the boundary conditions

\[ u(a, t) = g_a(t), \quad u(b, t) = g_b(t), \quad t \geq 0, \tag{20} \]

where \( \varepsilon, \mu \) and \( \nu \) are constants. By discrdeting time derivative of KdV’s equation using a classic finite difference formula and space derivatives by \( \theta \)-weighted scheme we have

\[
\frac{u^{n+1} - u^n}{\delta t} + \theta(\varepsilon(uu_x)^{n+1} + \mu(u_{xxx})^{n+1}) \\
+ (1 - \theta)(\varepsilon(uu_x)^n + \mu(u_{xxx})^n) = 0, \tag{21}
\]

so using Taylor expansion for the term \( uu_x \) and considering \( \varepsilon = 6 \) and \( \mu = 1 \) we have

\[
u^{n+1} + \delta t \theta(6[u^n u_x^{n+1} + u_x^n u_x^{n+1}] + u_{xxx}^{n+1}) \
= u^n + 6(2 \theta - 1)(uu_x)^n + \delta t(1 - \theta)u_{xxx}^n \tag{22} \]

for KdV equation and with same calculation for KdVB equation we have

\[
u^{n+1} + \delta t \theta(\varepsilon[u^n u_x^{n+1} + u_x^n u_x^{n+1}] - \nu u_{xx}^{n+1} + \mu u_{xxx}^{n+1}) \
= u^n + \delta t \theta u^n u_x^n - \delta t(1 - \theta)[\varepsilon uu_x^n - \nu u_{xx}^n + \mu u_{xxx}^n] \tag{23} \]

Now we use approximate solution as

\[ u(x, t^n) = u^n(x) \simeq \sum_{j=1}^N u_j^n S_j(x). \tag{24} \]

where

\[ S_j(x) = \sin\left(\frac{x - (j - 1)h - a}{h}\right). \tag{25} \]

By substituting above approximate solution in Eq. (22) a matrix representation is obtained for KdV equation as

\[ Mu^{n+1} = R \tag{26} \]
where

\[
\begin{align*}
A_d &= \begin{bmatrix} I_{ij}^{(0)} \end{bmatrix} : i = 2, ..., N - 1, j = 1, ..., N, \ 0 \text{ elsewhere} \end{bmatrix}_{N \times N}, \\
A_b &= \begin{bmatrix} I_{ij}^{(0)} \end{bmatrix} : i = 1, N, j = 1, \ldots, N, \ 0 \text{ elsewhere} \end{bmatrix}_{N \times N}, \\
B &= -\begin{bmatrix} I_{ij}^{(1)} \end{bmatrix} : i = 2, \ldots, N - 1, j = 1, \ldots, N, \ 0 \text{ elsewhere} \end{bmatrix}_{N \times N}, \\
C &= \begin{bmatrix} I_{ij}^{(2)} \end{bmatrix} : i = 2, \ldots, N - 1, j = 1, \ldots, N, \ 0 \text{ elsewhere} \end{bmatrix}_{N \times N}, \\
G &= -\begin{bmatrix} I_{ij}^{(3)} \end{bmatrix} : i = 2, \ldots, N - 1, j = 1, \ldots, N, \ 0 \text{ elsewhere} \end{bmatrix}_{N \times N},
\end{align*}
\]

so with these definition for KdV equation we have

\[
\begin{align*}
M &= [A_d + A_b + \theta \delta t (6(E + D) + G)], \\
R &= [A_d - \delta t \{6(2\theta - 1)E - (1 - \theta)G\}]u^n + F^{n+1},
\end{align*}
\]

and with same substitution for KdVB’s equation

\[
\begin{align*}
M &= [A_d + A_b + \theta \delta t(\mu (E + D) + \nu G - \nu C)], \\
R &= [A_d - \delta t \{\varepsilon(2\theta - 1)E - (1 - \theta)(\mu G - \nu C)\}]u^n + F^{n+1},
\end{align*}
\]

4 Stability Analysis

In this section stability analysis of approximate solution for linearized equation is discussed[18]. The error at nth time level is

\[
e^n = u^n_{\text{exact}} - u^n_{\text{approximate}}.
\]

4.1 KdV’s equation

By considering the obtained matrix we have

\[
\begin{align*}
[H + \delta t \theta K]e^{n+1} &= [H - \delta t(1 - \theta)K]e^n, \\
\end{align*}
\]

where \(H = [A_d + A_b]A^{-1}\) and \(K = [6E + G]A^{-1}\) so,

\[
e^{n+1} = Pe^n
\]

where \(P = [H + \delta t \theta K]^{-1}[H - \delta t(1 - \theta)K]\). This method is stable if \(\| P \|_2 \leq 1\) or \(\rho(P) \leq 1\) which is spectral radius of the matrix \(P\). The stability is assured if all the eigenvalues of the matrix \([H + \delta t \theta K]^{-1}[H - \delta t(1 - \theta)K]\) satisfy the following condition

\[
\left| \frac{\lambda_H - \delta t(1 - \theta)\lambda_K}{\lambda_H + \delta t \theta \lambda_K} \right| \leq 1
\]
where \( \lambda_H \) and \( \lambda_K \) are eigenvalues of the matrices \( H \) and \( K \) respectively. When \( \theta = 0.5 \), the inequality (28) becomes

\[
\left| \frac{\lambda_H - 0.5\delta t\lambda_K}{\lambda_H + 0.5\delta t\lambda_K} \right| \leq 1. \tag{29}
\]

In the case of complex eigenvalues \( \lambda_H = a_h + ib_h \) and \( \lambda_K = a_k + ib_k \), where \( a_h, a_k, b_h \) and \( b_k \) are any real numbers, the inequality (29) takes the following form,

\[
\left| \frac{(a_h - 0.5\delta ta_k) + i(b_h - 0.5\delta tb_k)}{(a_h - 0.5\delta ta_k) + i(b_h - 0.5\delta tb_k)} \right| \leq 1. \tag{30}
\]

The inequality (30) is satisfied if \( a_ha_k + b_hb_k \geq 0 \). For real eigenvalues, the inequality (29) holds true if either \( (\lambda_H \geq 0 \text{ and } \lambda_K \geq 0) \) or \( (\lambda_H \leq 0 \text{ and } \lambda_K \leq 0) \).

This shows that the scheme is unconditionally stable if \( a_ha_k + b_hb_k \geq 0 \), for complex eigenvalues and if either \( (\lambda_H \geq 0 \text{ and } \lambda_K \geq 0) \) or \( (\lambda_H \leq 0 \text{ and } \lambda_K \leq 0) \), for real eigenvalues. When \( \theta = 0 \), the inequality (29) becomes

\[
\left| 1 - \frac{\delta t\lambda_k}{\lambda_h} \right| \leq 1,
\]

i.e.,

\[
\delta t \leq \frac{2\lambda_H}{\lambda_K} \text{ and } \frac{\lambda_H}{\lambda_K} \geq 0
\]

Thus for \( \theta = 0 \), the scheme is conditionally stable. The stability of scheme for the other values of \( \theta \) can be investigate in a similar manner. The stability of the scheme and conditioning of the component matrices \( H, K \) of the matrix \( P \) depend on the weight parameter \( \theta \) and the minimum distance between any two collocation points \( h \) in the domain set \([a, b]\).

### 4.2 KdVB’s equation

By considering the obtained matrix we have

\[
[H + \delta t\theta K]e^{n+1} = [H - \delta t(1 - \theta)K]e^n, \tag{31}
\]

where \( H = [A_d + A_b]A^{-1} \) and \( K = [\varepsilon E + \mu G - \nu C]A^{-1} \) so,

\[
e^{n+1} = Pe^n
\]

where \( P = [H + \delta t\theta K]^{-1}[H - \delta t(1 - \theta)K] \). This method is stable if \( \| P \|_2 \leq 1 \) or \( \rho(P) \leq 1 \) which is spectral radius of the matrix \( P \). The stability is assured if all the eigenvalues of the matrix \([H + \delta t\theta K]^{-1}[H - \delta t(1 - \theta)K] \) satisfy the following condition

\[
\left| \frac{\lambda_H - \delta t(1 - \theta)\lambda_K}{\lambda_H + \delta t\theta\lambda_K} \right| \leq 1 \tag{32}
\]
where $\lambda_H$ and $\lambda_K$ are eigenvalues of the matrices $H$ and $K$ respectively. When $\theta = 0.5$, the inequality (32) becomes
\[
\left| \frac{\lambda_H - 0.5\delta t \lambda_K}{\lambda_H + 0.5\delta t \lambda_K} \right| \leq 1. \tag{33}
\]
In the case of complex eigenvalues $\lambda_H = a_h + ib_h$ and $\lambda_K = a_k + ib_k$, where $a_h, a_k, b_h$ and $b_k$ are any real numbers, the inequality (33) takes the following form,
\[
\left| \frac{(a_h - 0.5\delta ta_k) + i(b_h - 0.5\delta tb_k)}{(a_h - 0.5\delta ta_k) + i(b_h - 0.5\delta tb_k)} \right| \leq 1. \tag{34}
\]
The inequality (34) is satisfied if $a_h a_k + b_h b_k \geq 0$. For real eigenvalues, the inequality (33) holds true if either ($\lambda_h \geq 0$ and $\lambda_k \geq 0$) or ($\lambda_h \leq 0$ and $\lambda_k \leq 0$).
This shows that the scheme is unconditionally stable if $a_h a_k + b_h b_k \geq 0$, for complex eigenvalues and if either ($\lambda_h \geq 0$ and $\lambda_k \geq 0$) or ($\lambda_h \leq 0$ and $\lambda_k \leq 0$), for real eigenvalues. When $\theta = 0$, the inequality (33) becomes
\[
\left| 1 - \frac{\delta t \lambda_k}{\lambda_h} \right| \leq 1,
\]
i.e.,
\[
\delta t \leq 2\frac{\lambda_H}{\lambda_K} \quad \text{and} \quad \frac{\lambda_H}{\lambda_K} \geq 0
\]
Thus for $\theta = 0$, the scheme is conditionally stable. The stability of scheme for the other values of $\theta$ can be investigated in a similar manner. The stability of the scheme and conditioning of the component matrices $H, K$ of the matrix $P$ depend on the weight parameter and the minimum distance between any two collocation points $h$ in the domain set $[a, b]$.

## 5 Errors and Conservation Laws

In this section, two error norms is defined that will be used for showing the accuracy of the method as follows
\[
L_2 = \|u - \tilde{u}\|_2 = \sqrt{\sum_{j=1}^{N} |u_j - \tilde{u}_j|^2},
\]
\[
L_\infty = \|u - \tilde{u}\|_\infty = \max |u_j - \tilde{u}_j|, \quad 1 \leq j \leq N
\]
where $u, \tilde{u}$ are exact and approximate solution respectively.
KdV equation has three conservation laws as follows
\[
I_1 = \int_a^b u(x,t)dx,
\]
\[
I_2 = \int_a^b u(x,t)^2dx,
\]
\[
I_3 = \int_a^b [u(x,t)^2 - \frac{1}{3}u(x,t)^3]dx,
\]
Where $I_1$, $I_2$, $I_3$ represents mass, momentum and energy that show conservation properties of collocation method by,

$$I_1 \simeq h \sum_{j=1}^{N} u_j^n;$$

$$I_2 \simeq h \sum_{j=1}^{N} (u_j^n)^2;$$

$$I_3 \simeq h \sum_{j=1}^{N} [(u_j^n)^2 - \frac{1}{3}(u_j^n)^3].$$

Sinc-collocation scheme satisfies the properties $I_1$ and $I_2$ in the conservative case.

6 Numerical Solution

In this section $L_2$ and $L_\infty$ are obtained and shown in Tables and approximate solution of KdV and KdVB’s equations is shown in Figures.

6.1 KdV equation

By considering $\mu = 1$, $\varepsilon = 6$, $\theta = 0.5$, in Table 1, two kinds of error is calculated for $n = 100$, $a = -15$, $b = 15$, $\delta t = 0.1$, $T = 0.1, \ldots, 0.9$. Three invariants of conservative laws is obtained. Table 2 indicates errors for $\delta t = 0.01$ and Table 3, shows error for $deltat = 0.001$. Table 4, errors are given for different time levels $T = 0.001, \ldots, 0.009$. By Tables, Errors reduce when time step decreases. Figure 1, indicates solitonic solutions of KdV equation in different time level of $T = 1, \ldots, 9$ for $n = 100$, $a = -10$, $b = 20$, $\delta t = 0.01$. Figure 2, indicates solitonic solutions of KdV equation in different time level of $T = 1, \ldots, 9$ for $n = 100$, $a = -15$, $b = 15$, $\delta t = 0.1$.

| Time | $L_\infty$ | $L_2$ | $I_1$ | $I_2$ | $I_3$ |
|------|------------|-------|-------|-------|-------|
| 0.1  | 4.55798 x 10^{-4} | 7.31340 x 10^{-5} | 2 | 0.666667 | -0.088889 |
| 0.2  | 9.28126 x 10^{-5} | 1.34140 x 10^{-4} | 2 | 0.666667 | -0.088853 |
| 0.3  | 1.28469 x 10^{-4} | 1.81748 x 10^{-4} | 2 | 0.666667 | -0.088865 |
| 0.4  | 1.34961 x 10^{-4} | 2.20735 x 10^{-4} | 2 | 0.666667 | -0.088767 |
| 0.5  | 1.39977 x 10^{-4} | 2.54695 x 10^{-4} | 2 | 0.666667 | -0.088857 |
| 0.6  | 1.58066 x 10^{-4} | 2.85701 x 10^{-4} | 2 | 0.666667 | -0.088847 |
| 0.7  | 1.77771 x 10^{-4} | 3.15070 x 10^{-4} | 2 | 0.666667 | -0.088839 |
| 0.8  | 1.98187 x 10^{-4} | 3.43619 x 10^{-4} | 2 | 0.666667 | -0.088832 |
| 0.9  | 2.20096 x 10^{-4} | 3.71849 x 10^{-4} | 2 | 0.666667 | -0.088825 |

Table1: Errors and invariants for $n = 100$, $a = -15$, $b = 15$, $\delta t = 0.1$, $T = 0.1, \ldots, 0.9$. |
| Time | $L_\infty$ | $L_2$ | $I_1$ | $I_2$ | $I_3$ |
|------|------------|-------|-------|-------|-------|
| 0.1  | $1.74342 \times 10^{-6}$ | $1.91251 \times 10^{-6}$ | 2 | 0.666667 | -0.0888889 |
| 0.2  | $2.31194 \times 10^{-6}$ | $2.87693 \times 10^{-6}$ | 2 | 0.666667 | -0.0888888 |
| 0.3  | $2.73488 \times 10^{-6}$ | $3.64966 \times 10^{-6}$ | 2 | 0.666667 | -0.0888887 |
| 0.4  | $3.04988 \times 10^{-6}$ | $4.32025 \times 10^{-6}$ | 2 | 0.666667 | -0.0888886 |
| 0.5  | $3.32879 \times 10^{-6}$ | $4.95178 \times 10^{-6}$ | 2 | 0.666667 | -0.0888885 |
| 0.6  | $3.49763 \times 10^{-6}$ | $5.53733 \times 10^{-6}$ | 2 | 0.666667 | -0.0888884 |
| 0.7  | $3.93719 \times 10^{-6}$ | $6.16498 \times 10^{-6}$ | 2 | 0.666667 | -0.0888883 |
| 0.8  | $1.98187 \times 10^{-6}$ | $6.75605 \times 10^{-6}$ | 2.0001 | 0.666667 | -0.0888883 |
| 0.9  | $4.69034 \times 10^{-6}$ | $7.26574 \times 10^{-6}$ | 2.0001 | 0.666667 | -0.0888882 |

Table 2: Errors and invariants for $n = 100$, $a = -15$, $b = 15$, $\delta t = 0.01$, $T = 0.1, \ldots, 0.9$.

| Time | $L_\infty$ | $L_2$ | $I_1$ | $I_2$ | $I_3$ |
|------|------------|-------|-------|-------|-------|
| 0.1  | $1.74199 \times 10^{-6}$ | $1.75980 \times 10^{-6}$ | 2 | 0.666667 | -0.0888889 |
| 0.2  | $2.30437 \times 10^{-6}$ | $2.54006 \times 10^{-6}$ | 2 | 0.666667 | -0.0888888 |
| 0.3  | $2.71876 \times 10^{-6}$ | $3.16315 \times 10^{-6}$ | 2 | 0.666667 | -0.0888887 |
| 0.4  | $3.07014 \times 10^{-6}$ | $3.70810 \times 10^{-6}$ | 2 | 0.666667 | -0.0888886 |
| 0.5  | $3.27326 \times 10^{-6}$ | $4.23733 \times 10^{-6}$ | 2 | 0.666667 | -0.0888885 |
| 0.6  | $3.69103 \times 10^{-6}$ | $4.73099 \times 10^{-6}$ | 2 | 0.666667 | -0.0888884 |
| 0.7  | $4.03713 \times 10^{-6}$ | $5.23450 \times 10^{-6}$ | 2 | 0.666667 | -0.0888883 |
| 0.8  | $4.20408 \times 10^{-6}$ | $5.77575 \times 10^{-6}$ | 2 | 0.666667 | -0.0888883 |
| 0.9  | $4.59416 \times 10^{-6}$ | $6.26060 \times 10^{-6}$ | 2 | 0.666667 | -0.0888883 |

Table 3: Errors and invariants for $n = 100$, $a = -15$, $b = 15$, $\delta t = 0.001$, $T = 0.1, \ldots, 0.9$.

| Time | $L_\infty$ | $L_2$ | $I_1$ | $I_2$ | $I_3$ |
|------|------------|-------|-------|-------|-------|
| 0.01 | $7.90227 \times 10^{-7}$ | $4.72378 \times 10^{-7}$ | 2 | 0.666667 | -0.0888889 |
| 0.02 | $1.03034 \times 10^{-6}$ | $7.25023 \times 10^{-7}$ | 2 | 0.666667 | -0.0888889 |
| 0.03 | $1.14705 \times 10^{-6}$ | $9.14997 \times 10^{-7}$ | 2 | 0.666667 | -0.0888889 |
| 0.04 | $1.21740 \times 10^{-6}$ | $1.07375 \times 10^{-6}$ | 2 | 0.666667 | -0.0888889 |
| 0.05 | $1.28565 \times 10^{-6}$ | $1.21327 \times 10^{-6}$ | 2 | 0.666667 | -0.0888889 |
| 0.06 | $1.41361 \times 10^{-6}$ | $1.33912 \times 10^{-6}$ | 2 | 0.666667 | -0.0888889 |
| 0.07 | $1.51656 \times 10^{-6}$ | $1.45490 \times 10^{-6}$ | 2 | 0.666667 | -0.0888889 |
| 0.08 | $1.60373 \times 10^{-6}$ | $1.56261 \times 10^{-6}$ | 2 | 0.666667 | -0.0888889 |
| 0.09 | $1.67755 \times 10^{-6}$ | $1.66392 \times 10^{-6}$ | 2 | 0.666667 | -0.0888889 |

Table 4: Errors and invariants for $n = 100$, $a = -15$, $b = 15$, $\delta t = 0.001$, $T = 0.1, \ldots, 0.9$ and $T = 0.001, \ldots, 0.009$. 

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Fig1: Solitonic solutions of KdV equation in different time level of $T = 1, \ldots, 9$ for $n = 100$, $a = -10$, $b = 20$, $\delta t = 0.01$

Fig2: Solitonic solutions of KdV equation in different time level of $T = 1, \ldots, 9$ for $n = 100$, $a = -15$, $b = 15$, $\delta t = 0.1$
6.2 KdVB equation

By considering $\mu = 0.1, \varepsilon = 2, \nu = 0.005, \theta = 0.5$, in Table 5, two kinds of error is calculated for $n = 100, a = -100, b = 100, \delta t = 0.02, T = 1, \ldots, 9$.

Table 6 indicates errors for $\mu = 0.001, \varepsilon = 1, \nu = 0.001, \theta = 0.5$ and $n = 100, a = -40, b = 100, \delta t = 0.05, T = 1, \ldots, 9$.

In Table 7, error is obtained for $n = 16, a = 0, b = 100, \delta t = 0.00001, T = 0.0001, \ldots, 0.0009$ and $\mu = 0.001, \varepsilon = 1, \nu = 0.001, \theta = 0.5$.

Table 8 shows errors for $n = 16, a = -40, b = 40, \delta t = 0.02, T = 1, \ldots, 9$ and $\mu = 0.1, \varepsilon = 2, \nu = 0.005, \theta = 0.5$.

In Table 9, errors are calculated for $n = 16, a = 8, b = 99, \delta t = 0.05, T = 1, \ldots, 9$ and $\mu = 0.001, \varepsilon = 1, \nu = 0.001, \theta = 0.5$. Errors reduces by decreasing of time steps.

Figure 3, indicates solutions of KdVB equation for $n = 100, a = -100, b = 100, \delta t = 0.02, T = 1$ and $\mu = 0.1, \varepsilon = 2, \nu = 0.005, \theta = 0.5$.

Figure 4, indicates solutions of KdVB equation for $n = 100, a = -40, b = 100, \delta t = 0.05, T = 1$ and $\mu = 0.1, \varepsilon = 1, \nu = 0.1, \theta = 0.5$.

Figure 5 shows solutions of KdVB equation for $n = 100, a = -40, b = 100, \delta t = 0.05, T = 1$ and $\mu = 0.01, \varepsilon = 1, \nu = 0.01, \theta = 0.5$.

In Figure 6, numerical solution of KdVB equation is shown for $n = 100, a = -40, b = 100, \delta t = 0.05, T = 1$ and $\mu = 0.001, \varepsilon = 1, \nu = 0.001, \theta = 0.5$.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
Time & $\ell_\infty$ & $\ell_2$ \\
\hline
1 & 2.06127x10^{-6} & 4.62479x10^{-6} \\
2 & 1.03034x10^{-6} & 7.25023x10^{-7} \\
3 & 1.14705x10^{-6} & 9.14997x10^{-7} \\
4 & 1.21740x10^{-6} & 1.07375x10^{-6} \\
5 & 1.28565x10^{-6} & 1.21327x10^{-6} \\
6 & 1.41361x10^{-6} & 1.33912x10^{-6} \\
7 & 1.51656x10^{-6} & 1.45490x10^{-6} \\
8 & 1.60373x10^{-6} & 1.56261x10^{-6} \\
9 & 1.67755x10^{-6} & 1.66392x10^{-6} \\
\hline
\end{tabular}
\caption{Errors for $\mu = 0.1, \varepsilon = 2, \nu = 0.005, \theta = 0.5$ and $n = 100, a = -100, b = 100, \delta t = 0.02, T = 1, \ldots, 9$.}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
Time & $\ell_\infty$ & $\ell_2$ \\
\hline
1 & 5.49372x10^{-7} & 1.00098x10^{-6} \\
2 & 1.09951x10^{-6} & 1.99828x10^{-6} \\
3 & 1.65039x10^{-6} & 2.99190x10^{-6} \\
4 & 2.20200x10^{-6} & 3.98187x10^{-6} \\
5 & 2.75434x10^{-6} & 4.96819x10^{-6} \\
6 & 3.30738x10^{-6} & 5.95089x10^{-6} \\
7 & 3.86111x10^{-6} & 6.92997x10^{-6} \\
8 & 4.41551x10^{-6} & 7.90547x10^{-6} \\
9 & 4.97059x10^{-6} & 8.87738x10^{-6} \\
\hline
\end{tabular}
\caption{Errors for $\mu = 0.1, \varepsilon = 2, \nu = 0.005, \theta = 0.5$ and $n = 100, a = -40, b = 100, \delta t = 0.05, T = 1, \ldots, 9$.}
\end{table}
Table 6: Errors for $\mu = 0.001$, $\varepsilon = 1$, $\nu = 0.001$, $\theta = 0.5$ and $n = 100$, $a = -40$, $b = 100$, $\delta t = 0.05$, $T = 1, \ldots, 9$.

| Time | $L_\infty$ | $L_2$ |
|------|------------|-------|
| 1    | $1.84693 \times 10^{-12}$ | $6.66561 \times 10^{-12}$ |
| 2    | $3.69387 \times 10^{-12}$ | $1.33287 \times 10^{-11}$ |
| 3    | $5.54080 \times 10^{-12}$ | $1.99924 \times 10^{-11}$ |
| 4    | $7.38774 \times 10^{-12}$ | $2.66562 \times 10^{-11}$ |
| 5    | $9.23467 \times 10^{-12}$ | $3.33201 \times 10^{-11}$ |
| 6    | $1.10816 \times 10^{-11}$ | $3.99840 \times 10^{-11}$ |
| 7    | $1.29285 \times 10^{-11}$ | $4.66479 \times 10^{-11}$ |
| 8    | $1.47755 \times 10^{-11}$ | $5.33118 \times 10^{-11}$ |
| 9    | $1.66224 \times 10^{-11}$ | $5.99758 \times 10^{-11}$ |

Table 7: Errors for $\mu = 0.001$, $\varepsilon = 1$, $\nu = 0.001$, $\theta = 0.5$ and $n = 16$, $a = 0$, $b = 100$, $\delta t = 0.00001$, $T = 0.0001, \ldots, 0.0009$.

| Time | $L_\infty$ | $L_2$ |
|------|------------|-------|
| 1    | $7.82718 \times 10^{-8}$ | $3.31435 \times 10^{-7}$ |
| 2    | $1.57051 \times 10^{-7}$ | $6.62416 \times 10^{-7}$ |
| 3    | $2.36335 \times 10^{-7}$ | $9.92942 \times 10^{-7}$ |
| 4    | $3.16123 \times 10^{-7}$ | $1.32301 \times 10^{-6}$ |
| 5    | $3.23467 \times 10^{-7}$ | $1.99924 \times 10^{-11}$ |
| 6    | $4.77201 \times 10^{-7}$ | $2.63863 \times 10^{-6}$ |
| 7    | $5.58486 \times 10^{-7}$ | $2.96637 \times 10^{-6}$ |
| 8    | $6.40266 \times 10^{-7}$ | $3.20135 \times 10^{-6}$ |
| 9    | $7.22539 \times 10^{-7}$ | $3.43837 \times 10^{-6}$ |

Table 8: Errors for $\mu = 0.1$, $\varepsilon = 2$, $\nu = 0.005$, $\theta = 0.5$ and $n = 16$, $a = -40$, $b = 40$, $\delta t = 0.02$, $T = 1, \ldots, 9$.

| Time | $L_\infty$ | $L_2$ |
|------|------------|-------|
| 1    | $2.26009 \times 10^{-8}$ | $8.83534 \times 10^{-8}$ |
| 2    | $4.52028 \times 10^{-8}$ | $1.76087 \times 10^{-7}$ |
| 3    | $6.78055 \times 10^{-8}$ | $2.65005 \times 10^{-7}$ |
| 4    | $9.04092 \times 10^{-8}$ | $3.53006 \times 10^{-7}$ |
| 5    | $1.13014 \times 10^{-7}$ | $4.41592 \times 10^{-7}$ |
| 6    | $1.35619 \times 10^{-7}$ | $5.29861 \times 10^{-7}$ |
| 7    | $1.58226 \times 10^{-7}$ | $6.18113 \times 10^{-7}$ |
| 8    | $1.80833 \times 10^{-7}$ | $7.06350 \times 10^{-7}$ |
| 9    | $2.03441 \times 10^{-7}$ | $7.94570 \times 10^{-7}$ |

Table 9: Errors for $\mu = 0.001$, $\varepsilon = 1$, $\nu = 0.001$, $\theta = 0.5$ and $n = 16$, $a = 8$, $b = 99$, $\delta t = 0.05$, $T = 1, \ldots, 9$.  

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Fig3: Solutions of KdVB equation for $n = 100, a = -100, b = 100, \delta t = 0.02, T = 1$ and $\mu = 0.1, \epsilon = 2, \nu = 0.005, \theta = 0.5$.

Fig4: Solutions of KdVB equation for $n = 100, a = -40, b = 100, \delta t = 0.05, T = 1$ and $\mu = 0.1, \epsilon = 1, \nu = 0.1, \theta = 0.5$. 
Fig5: solutions of KdVB equation for $n = 100, a = -40, b = 100, \delta t = 0.05, T = 1$ and $\mu = 0.01, \varepsilon = 1, \nu = 0.01, \theta = 0.5$.

Fig6: Solution of KdVB equation for $n = 100, a = -40, b = 100, \delta t = 0.05, T = 1$ and $\mu = 0.001, \varepsilon = 1, \nu = 0.001, \theta = 0.5$
Conclusion

The collocation method using sinc basis is applied for solving KdV and KdVB's equations. Three invariant of conservation laws are calculated for KdV equation. The method is computationally attractive and results are shown through tables and figures.

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