Optimal Discriminant Analysis in High-Dimensional Latent Factor Models

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Abstract

In high-dimensional classification problems, a commonly used approach is to first project the high-dimensional features into a lower dimensional space, and base the classification on the resulting lower dimensional projections. In this paper, we formulate a latent-variable model with a hidden low-dimensional structure to justify this two-step procedure and to guide which projection to choose. We propose a computationally efficient classifier that takes certain principal components (PCs) of the observed features as projections, with the number of retained PCs selected in a data-driven way. A general theory is established for analyzing such two-step classifiers based on any projections. We derive explicit rates of convergence of the excess risk of the proposed PC-based classifier. The obtained rates are further shown to be optimal up to logarithmic factors in the minimax sense. Our theory allows the lower-dimension to grow with the sample size and is also valid even when the feature dimension (greatly) exceeds the sample size. Extensive simulations corroborate our theoretical findings. The proposed method also performs favorably relative to other existing discriminant methods on three real data examples.

Keywords: High-dimensional classification, latent factor model, principal component regression, dimension reduction, discriminant analysis, optimal rate of convergence.

1 Introduction

In high-dimensional classification problems, a widely used technique is to first project the high-dimensional features into a lower dimensional space, and base the classification on the resulting lower dimensional projections (Ghosh, 2001; Nguyen and Rocke, 2002; Chiaromonte and Martinelli, 2002; Antoniadis et al., 2003; Biau et al., 2003; Boulesteix, 2004; Dai et al., 2006; Li, 2016; Hadef and Djebabra, 2019; Jin et al., 2021; Ma et al., 2020; Mallary et al., 2022). Despite having been widely used for years, theoretical understanding of this approach is scarce, and what kind of low-dimensional projection to choose remains unknown. In this paper we formulate a latent-variable model with a hidden low-dimensional structure to justify the two-step procedure that takes leading principal components of the observed features as projections.

Concretely, suppose our data consists of independent copies of the pair \((X, Y)\) with features \(X \in \mathbb{R}^p\) according to

\[
X = AZ + W
\]

and labels \(Y \in \{0, 1\}\). Here \(A\) is a deterministic, unknown \(p \times K\) loading matrix, \(Z \in \mathbb{R}^K\) are unobserved, latent factors and \(W\) is random noise. We assume that

(i) \(W\) is independent of both \(Z\) and \(Y\),

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(ii) \( \mathbb{E}[W] = 0_p \),

(iii) \( A \) has rank \( K \).

This mathematical framework allows for a substantial dimension reduction in classification for \( K \ll p \). Indeed, in terms of the Bayes’ misclassification errors, we prove in Lemma 1 of Section 2.1 the inequality

\[
R^*_g := \inf_g \mathbb{P}\{g(X) \neq Y\} \geq R^*_h := \inf_h \mathbb{P}\{h(Z) \neq Y\},
\]

that is, it is easier to classify in the latent space \( \mathbb{R}^K \) than in the observed feature space \( \mathbb{R}^p \). In this work, we further assume that

(iv) \( Z \) is a mixture of two Gaussians

\[
Z | Y = k \sim N_K(\alpha_k, \Sigma_{Z|Y}), \quad \mathbb{P}(Y = k) = \pi_k, \quad k \in \{0, 1\}
\]

with different means \( \alpha_0 := \mathbb{E}[Z | Y = 0] \) and \( \alpha_1 := \mathbb{E}[Z | Y = 1] \), but with the same covariance matrix

\[
\Sigma_{Z|Y} := \text{Cov}(Z | Y = 0) = \text{Cov}(Z | Y = 1),
\]

assumed to be strictly positive definite.

We emphasize that the distributions of \( X \) given \( Y \) are not necessarily Gaussian as the distribution of \( W \) could be arbitrary.

Within the above modelling framework, parameters related with the moments of \( X \) and \( Y \), such as \( \{\pi_k, \mathbb{E}[X|Y] \text{ and Cov}(X|Y)\} \), are identifiable, while \( A, \Sigma_{Z|Y}, \alpha_k, \) and \( \Sigma_W := \text{Cov}(W) \) are not. For instance, we can always replace \( Z \) by \( Z' = QU \) for any invertible \( K \times K \) matrix \( Q \) and write \( \alpha'_k = Q\alpha_k, \Sigma'_{Z|Y} = Q\Sigma_{Z|Y}Q^T \) and \( A' = AQ^{-1} \). Since we focus on classification, there is no need to impose any conditions on the latter group of parameters that render them identifiable. Although our discussion throughout this paper is based on a fixed notation of \( A, \Sigma_{Z|Y}, \Sigma_W \) and \( \alpha_k \), it should be understood that our results are valid for all possible choices of these parameters such that model (1.1) and (1.3) holds, including sub-models under which such parameters are (partially) identifiable.

Our goal is to construct a classification rule \( \hat{g}_x : \mathbb{R}^p \to \{0, 1\} \) based on the training data \( D := \{X, Y\} \) that consists of independent pairs \((X_1, Y_1), \ldots, (X_n, Y_n)\) from model (1.1) and (1.3) such that the resulting rule has small misclassification error \( \mathbb{P}\{\hat{g}_x(X) \neq Y\} \) for a new pair of \((X, Y)\) from the same model that is independent of \( D \). In this paper, we are particularly interested in \( \hat{g}_x \) that is linear in \( X \), motivated by the fact that the restriction of equal covariance in (1.4) leads to a Bayes rule that is linear in \( Z \) when we observe \( Z \) (see display (1.6) below).

Linear classifiers have been popular for decades, especially in high-dimensional classification problems, due to their interpretability and computational simplicity. One strand of the existing literature imposes sparsity on the coefficients \( \beta \in \mathbb{R}^p \) in linear classifiers \( g(x) = 1\{\beta^T x + \beta_0 \geq 0\} \) for large \( p \) (\( p \geq n \)), see, for instance, Tibshirani et al. (2002); Fan and Fan (2008); Witten and Tibshirani (2011); Shao et al. (2011); Cai and Liu (2011); Mai et al. (2012); Cai and Zhang (2019a) for sparse linear discriminant analysis (LDA) and Tarigan and Van de Geer (2006); Wegkamp and Yuan (2011) for sparse support vector machines. For instance, in the classical LDA-setting, when \( X \) itself is a mixture of Gaussians

\[
X | Y = k \sim N_p(\mu_k, \Sigma), \quad \mathbb{P}(Y = k) = \pi_k, \quad k \in \{0, 1\}
\]
with $\Sigma$ strictly positive definite, the Bayes classifier is linear with $p$-dimensional vector $\beta = \Sigma^{-1}(\mu_1 - \mu_0)$. Sparsity of $\beta$ is then a reasonable assumption when $\Sigma$ is close to diagonal, so that sparsity of $\beta$ gets translated to that of the difference between the mean vectors $\mu_1 - \mu_0$. However, in the high-dimensional regime, many features are highly correlated and any sparsity assumption on $\beta$ is no longer intuitive and becomes in fact questionable. This serves as a main motivation for this work, in which we study a class of linear classifiers that no longer requires the sparsity assumption on $\beta$, for neither construction of the classifier, nor its analysis.

### 1.1 Contributions

We summarize our contributions below.

#### 1.1.1 Minimax lower bounds of rate of convergence of the excess risk

Our first contribution in this paper is to establish minimax lower bounds of rate of convergence of the excess risk for any classifier under model (1.1) and (1.3). The excess risk is defined relative to $R^*_x$ in (1.2) which we view as a more natural benchmark than $R^*_z$ because our proposed classifier is designed to adapt to the underlying low-dimensional structure in (1.1). The relation in (1.2) suggests $R^*_x$ is also a more ambitious benchmark than $R^*_z$.

Since the gap between $R^*_z$ and $R^*_x$ quantifies the irreducible error for not observing $Z$, we start in Lemma 2 of Section 2.1 by characterizing how $R^*_x - R^*_z$ depends on $\xi^* = \lambda_K(A\Sigma_{Z|Y}A^\top)/\lambda_1(\Sigma_W)$, the signal-to-noise ratio for predicting $Z$ from $X$ (conditioned on $Y$), and $\Delta^2 = (\alpha_1 - \alpha_0)^\top\Sigma^{-1}_Z(\alpha_1 - \alpha_0)$, the Mahalanobis distance between random vectors $Z \mid Y = 1$ and $Z \mid Y = 0$. Interestingly, it turns out that $R^*_x - R^*_z$ is small when either $\xi^*$ or $\Delta$ is large, a phenomenon that is different from the setting when $Y$ is linear in $Z$. Indeed, for the latter case, the excess risk of predicting $Y$ by using the best linear predictor of $X$ relative to the risk of predicting $Y$ from $E[Y|Z]$ is small only when $\xi^*$ is large (Bing et al., 2021).

In Theorem 3 of Section 2.2, we derive the minimax lower bounds of the excess risk for any classifier with explicit dependency on the signal-to-noise ratio $\xi^*$, the separation distance $\Delta$, the dimensions $K$ and $p$ and the sample size $n$. Our results also fully capture the phase transition of the excess risk as the magnitude of $\Delta$ varies. Specifically, when $\Delta$ is of constant order, the established lower bounds are

$$
(\omega_n)^2 = \frac{K}{n} + \frac{\Delta^2}{\xi^*} + \frac{\Delta^2}{\xi^*} \frac{p}{\xi^* n}.
$$

The first term is the optimal rate of the excess risk even when $Z$ were observable; the second term corresponds to the irreducible error of not observing $Z$ in $R^*_x - R^*_z$ and the last term reflects the minimal price to pay for estimating the column space of $A$. When $\Delta \to \infty$ as $n \to \infty$, the lower bounds become $(\omega_n)^2 \exp(-\Delta^2/8)$ and get exponentially faster in $\Delta^2$. When $\Delta \to 0$ as $n \to \infty$, the lower bounds get slower as $\omega_n \min\{\omega_n/\Delta, 1\}$, implying a more difficult scenario for classification. In Section 5.3, the lower bounds are further shown to be tight in the sense that the excess risk of the proposed PC-based classifiers have a matching upper bound, up to some logarithmic factors.

To the best of our knowledge, our minimax lower bounds are both new in the literature of factor models and the classical LDA. In the factor model literature, even in linear factor regression models, there is no known minimax lower bound of the prediction risk with respect to the quadratic loss function. In the LDA literature, our results cover the minimax lower bound of the excess risk in the classical LDA as a special case and are the first to fully characterize the phase transition in $\Delta$ (see Remark 5 for details). The analysis of establishing Theorem 3 is highly non-trivial and encounters several challenges. Specifically, since the excess risk is not
a semi-distance, as required by the standard techniques of proving minimax lower bounds, the first challenge is to develop a reduction scheme based on a surrogate loss function that satisfies a local triangle inequality-type bound. The second challenge of our analysis is to allow a fully non-diagonal structure of \( \text{Cov}(X|Y) \) under model (1.1), as opposed to the existing literature on the classical LDA that assumes \( \text{Cov}(X|Y) \) to be diagonal or even proportional to the identity matrix. To characterize the effect of estimating the column space of \( A \) on the excess risk in deriving the third term of the lower bounds, our proof is based on constructing a suitable subset of the parameter space via the hypercube construction that is used for proving the optimal rates of the sparse PCA (Vu and Lei, 2013) (see the paragraph after Theorem 3 for a full discussion). Since the statistical distance (such as the KL-divergence) between thus constructed hypotheses could diverge as \( p/n \to \infty \), this leads to the third challenge of providing a meaningful and sharp lower bound that is valid for both \( p < n \) and \( p > n \).

### 1.1.2 A general two-step classification approach and the PC-based classifier

Our second contribution in this paper is to propose a computationally efficient linear classifier in Section 3.2 that uses leading principal components (PCs) of the high-dimensional feature, with the number of retained PCs selected in a data-driven way. This PC-based classifier is one instance of a general two-step classification approach proposed in Section 3.1. To be clear, it differs from naively applying standard LDA, using plug-in estimates of the Bayes rule, on the leading PCs.

To motivate our approach, suppose that the factors \( Z \) were observable. Then the optimal Bayes rule is to classify a new point \( z \in \mathbb{R}^K \) as

\[
g^*_z(z) = \mathbb{1}\{z^\top \eta + \eta_0 \geq 0\} \quad (1.6)
\]

where

\[
\eta = \Sigma^{-1}_{Z|Y}(\alpha_1 - \alpha_0), \quad \eta_0 = -\frac{1}{2}(\alpha_0 + \alpha_1)^\top \eta + \log \frac{\pi_1}{\pi_0}. \quad (1.7)
\]

This rule is optimal in the sense that it has the smallest possible misclassification error. Our approach in Section 3.1 utilizes an intimate connection between the linear discriminant analysis and regression to reformulate the Bayes rule \( g^*_z(z) \) as \( \mathbb{1}\{z^\top \beta + \beta_0 \geq 0\} \) with \( \beta = \Sigma^{-1}_{Z|Y} \text{Cov}(Z, Y) \) (and \( \beta_0 \) is given in (3.1) of Section 3). The key difference is the use of the unconditional covariance matrix \( \Sigma_Z \), as opposed to the conditional one \( \Sigma_{Z|Y} \) in (1.7). As a result, \( \beta \) can be interpreted as the coefficient of regressing \( Y \) on \( Z \), suggesting to estimate \( z^\top \beta \) by \( z^\top (Z^\top \Pi_n Z)^{-1} Z^\top \Pi_n Y \) via the method of least squares, again, in case \( Z = (Z_1, \ldots, Z_n)^\top \in \mathbb{R}^{n \times K} \) and \( z \in \mathbb{R}^K \) had been observed. Here \( Y = (Y_1, \ldots, Y_n)^\top \in \{0, 1\}^n \), \( \Pi_n = I_n - n^{-1}1_n 1_n^\top \) is the centering projection matrix and \( M^+ \) denotes the Moore-Penrose inverse of any matrix \( M \) throughout of this paper.

Since we only have access to \( x \in \mathbb{R}^p \), a realization of \( X, X = [X_1 \cdots X_n]^\top \in \mathbb{R}^{n \times p} \), and \( Y \in \{0, 1\}^n \), it is natural to estimate the span of \( z \) by \( B^\top x \) and to predict the span of \( \Pi_n Z \) by \( \Pi_n X B \), for some appropriate matrix \( B \). This motivates us to estimate the inner-product \( z^\top \beta \) by

\[
(B^\top x)^\top (B^\top X^\top \Pi_n X B)^+ B^\top X^\top \Pi_n Y := x^\top \hat{\theta}. \quad (1.8)
\]

By using a plug-in estimator \( \hat{\beta}_0 \) of \( \beta_0 \), the resulting rule \( \hat{g}_x(x) = \mathbb{1}\{x^\top \hat{\theta} + \hat{\beta}_0 \geq 0\} \) is a general two-step, regression-based classifier and the choice of \( B \) is up to the practitioner.

In this paper, we advocate the choice \( B = U_r \in \mathbb{R}^{p \times r} \) where \( U_r \) contains the first \( r \) right-singular vectors of \( \Pi_n X \), such that the projections \( \Pi_n X B \) become the first \( r \) principal components of \( X \). Intuitively, this method has promise as Stock and Watson (2002a) proves that
when \( r \) is chosen as \( K \), the projection \( \Pi_n XU_K \) accurately predicts the span of \( \Pi_n Z \) under model (1.1). Since in practice \( K \) is oftentimes unknown, we further use a data-driven selection of \( K \) in Section 3.3 to construct our final PC-based classifier. The proposed procedure is computationally efficient. Its only computational burden is that of computing the singular value decomposition (SVD) of \( X \). Guided by our theory, we also discuss a cross-fitting strategy in Section 3.2 that improves the PC-based classifier by removing the dependence from using the data twice (one for constructing \( U_r \) and one for computing \( \hat{\theta} \) in (1.8)) when \( p > n \) and the signal-to-noise ratio \( \xi^* \) is weak.

Retaining only a few principal components of the observed features and using them in subsequent regressions is known as principal component regression (PCR) (Stock and Watson, 2002a). It is a popular method for predicting \( Y \in \mathbb{R} \) from a high-dimensional feature vector \( X \in \mathbb{R}^p \) when both \( X \) and \( Y \) are generated via a low-dimensional latent factor \( Z \). Most of the existing literature analyzes the performance of PCR when both \( Y \) and \( X \) are linear in \( Z \), for instance, Stock and Watson (2002a,b); Bair et al. (2006); Bai and Ng (2008); Hahn et al. (2013); Bing et al. (2021), just to name a few. When \( Y \) is not linear in \( Z \), little is known. An exception is Fan et al. (2017), which studies the model \( Y = h(\xi_1 Z, \ldots, \xi_q Z; \varepsilon) \) and \( X = AZ + W \) for some unknown general link function \( h(\cdot) \). Their focus is only on estimation of \( \xi_1, \ldots, \xi_q \), the sufficient predictive indices of \( Y \), rather than analysis of the risk of predicting \( Y \). As \( \mathbb{E}[Y|Z] \) is not linear in \( Z \) under our model (1.1) and (1.3), to the best of our knowledge, analysis of the misclassification error under model (1.1) and (1.3) for a general linear classifier has not been studied elsewhere.

1.1.3 A general strategy of analyzing the excess risk of \( \hat{g}_x \) based on any matrix \( B \)

Our third contribution in this paper is to provide a general theory for analyzing the excess risk of the type of classifiers \( \hat{g}_x \) that uses a generic matrix \( B \) in (1.8). In Section 4 we state our result in Theorem 5, a general bound for the excess risk of the classifier \( \hat{g}_x \) based on a generic matrix \( B \). It depends on how well we estimate \( z^T \beta + \beta_0 \) and a margin condition on the conditional distributions \( Z \mid Y = k, k \in \{0, 1\} \), nearby the hyperplane \( \{z \mid z^T \beta + \beta_0 = 0\} \). This bound is different from the usual approach which bounds the excess risk \( \mathbb{P}(\hat{g}(X) \neq Y) - R^2 \) of a classifier \( \hat{g} : \mathbb{R}^p \rightarrow \{0, 1\} \) by \( 2\mathbb{E}[|\eta(Z)| - 1/2]1\{\hat{g}(X) \neq g^*_x(Z)\} \), with \( \eta(z) = \mathbb{P}(Y = 1|Z = z) \), and involves analyzing the behavior of \( \eta(Z) \) near 1/2 (see our detailed discussion in Remark 7). The analysis of Theorem 5 is powerful in that it can easily be generalized to any distribution of \( Z \mid Y \), as explained in Remark 8. Our second main result in Theorem 7 of Section 4 provides explicit rates of convergence of the excess risk of \( \hat{g}_x \) for a generic \( B \) and clearly delineates three key quantities that need to be controlled as introduced therein. The established rates of convergence reveal the same phase transition in \( \Delta \) from the lower bounds. It is worth mentioning that the analysis of Theorem 7 is more challenging under model (1.1) and (1.3) than the classical LDA setting (1.5) in which the excess risk of any linear classifier in \( X \) has a closed-form expression.

1.1.4 Optimal rates of convergence of the PC-based classifier

Our fourth contribution is to apply the general theory in Section 4 to analyze the PC-based classifiers. Consistency of our proposed estimator of \( K \) is established in Theorem 8 of Section 5.1. In Theorem 9 of Section 5.2, we derive explicit rates of convergence of the excess risk of the PC-based classifier that uses \( B = U_K \). The obtained rate of convergence exhibits an interesting interplay between the sample size \( n \) and the dimensions \( K \) and \( p \) through the quantities \( K/n \), \( \xi^* \) and \( \Delta \). Our analysis also covers the low signal setting \( \Delta = o(1) \), a regime that has not been analyzed even in the existing literature of classical LDA. Our theoretical results are valid for both fixed and growing \( K \) and are also valid even when \( p \) is much larger than \( n \). In Theorem
of Section 5.2, we also show that a PC-based LDA that uses either auxiliary data or sample splitting could surprisingly yield faster rates of convergence of the excess risk by removing the dependence between $U_K$ and $X$. These faster rates are further shown to be minimax optimal, up to a logarithmic factor, in Corollary 11 of Section 5.3. The benefit of using auxiliary data or sample splitting has also been recognized in other problems, such as the problem of estimating the optimal instrument in sparse high-dimensional instrumental variable model (Belloni et al., 2012) and the problem of inference on a low-dimensional parameter in the presence of high-dimensional nuisance parameters (Chernozhukov et al., 2018).

1.1.5 Extension to multi-class classification

Our fifth contribution is to extend the general two-step classification procedure in Section 3 to handle multi-class classification problems in Section 8. Rates of convergence of the excess risk of the proposed multi-class classifier that bases on any matrix $B$ are derived in Theorem 12. PC-based classifiers are analyzed subsequently in Corollary 13. Our theory is the first to explicitly characterize dependence of the excess risk on the number of classes, and to cover the weak separation case when $\Delta \to 0$.

We emphasize that the methodology is of its own interest. It solves a long standing issue on generalizing regression-based classification methods (Hastie et al., 2009; Izenman, 2008; Mai et al., 2012) in the classical binary LDA setting to handle multi-class classification.

The paper is organized as follows. In Section 2.1, we provide an oracle benchmark that quantifies the excess risk of the optimal classifier based on $X$. We state the minimax lower bounds of the excess risk in Section 2.2. In Section 3, we present a connection between the linear discriminant classifier by using $Z$ and regression of $Y$ onto $Z$. This key observation leads to our proposed PC-based classifier. Furthermore, we propose a data-driven selection of the number of retained principal components. A general theory is stated in Section 4 for analyzing the excess risk of the classifier $\hat{g}_0$ that uses any $B$ for the estimate $\hat{\theta}$ in (1.8). In Section 5 we apply the general result to analyze the PC-based classifiers. Simulation results are presented in Section 6 and a real data analysis is given in Section 7. Extension to multi-class classification is studied in Section 8. All the proofs are deferred to the Appendix.

**Notation:** We use the common notation $\varphi(x) = \exp(-x^2/2)/\sqrt{2\pi}$ for the standard normal density, and denote by $\Phi(x) = \int \varphi(t) 1 \{ t \leq x \} dt$ its c.d.f.. For any positive integer $d$, we write $[d] := \{1, \ldots, d\}$. For any vector $v$, we use $\|v\|_q$ to denote its $\ell_q$ norm for $0 \leq q \leq \infty$. We also write $\|v\|_Q^2 = v^\top Q^{-1} v$ for any commensurate, invertible square matrix $Q$. For any real-valued matrix $M \in \mathbb{R}^{p \times q}$, we use $M^+$ to denote the Moore-Penrose inverse of $M$, and $\sigma_1(M) \geq \sigma_2(M) \geq \cdots \geq \sigma_{\min(p,q)}(M)$ to denote the singular values of $M$ in non-increasing order. We define the operator norm $\|M\|_{\text{op}} = \sigma_1(M)$. For a symmetric positive semi-definite matrix $Q \in \mathbb{R}^{p \times p}$, we use $\lambda_1(Q) \geq \lambda_2(Q) \geq \cdots \geq \lambda_p(Q)$ to denote the eigenvalues of $Q$ in non-increasing order. We write $Q \succ 0$ if $Q$ is strictly positive definite. For any two sequences $a_n$ and $b_n$, we write $a_n \preceq b_n$ if there exists some constant $C$ such that $a_n \leq C b_n$. The notation $a_n \asymp b_n$ stands for $a_n \preceq b_n$ and $b_n \preceq a_n$. For two numbers $a$ and $b$, we write $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. We use $I_d$ to denote the $d \times d$ identity matrix and use $1_d (0_d)$ to denote the vector with all ones (zeros). For $d_1 \geq d_2$, we use $O_{d_1 \times d_2}$ to denote the set of all $d_1 \times d_2$ matrices with orthonormal columns. Lastly, we use $c, c', C, C'$ to denote positive and finite absolute constants that unless otherwise indicated can change from line to line.
2 Excess risk and its minimax optimal rates of convergence

We start in Section 2.1 by introducing the oracle benchmark relative to which the excess risk is defined. Minimax optimal rates of convergence of the excess risk are derived in Section 2.2.

2.1 Oracle benchmark

Since our goal is to predict the Bayes rule \( \mathbb{1}\{z^\top \eta_1 + \eta_0 \geq 0\} \) under model (1.3), it is natural to choose the oracle risk \( R_z^* \) in (1.2) as our benchmark, as opposed to \( R_z \). Furthermore, we always have the explicit expression

\[
R_z^* = 1 - \pi_1 \Phi \left(\frac{\Delta}{2} + \frac{\log \frac{\pi_1}{\pi_0}}{\Delta}\right) - \pi_0 \Phi \left(\frac{\Delta}{2} - \frac{\log \frac{\pi_1}{\pi_0}}{\Delta}\right),
\]  

see, for instance, Izenman (2008, Section 8.3, pp 241–244). Here,

\[
\Delta^2 := (\alpha_0 - \alpha_1)\Sigma_{Z|Y}^{-1}(\alpha_0 - \alpha_1)
\]

is the Mahalanobis distance between the conditional distributions \( Z \mid Y = 1 \sim N(\alpha_1, \Sigma_{Z|Y}) \) and \( Z \mid Y = 0 \sim N(\alpha_0, \Sigma_{Z|Y}) \). In particular, when \( \pi_0 = \pi_1 \), the expression in (2.1) simplifies to \( R_z^* = 1 - \Phi(\Delta/2) \).

Remark 1. It is immediate from (2.1) that \( \Delta \to \infty \) implies \( R_z^* \to 0 \). The case of zero Bayes error \( R_z^* \) represents the easiest classification problem and we can expect fast rates of the excess risk. If \( \Delta \to 0 \), the Bayes risk \( R_z^* \) converges to \( \min\{\pi_0, \pi_1\} \). When \( \pi_0 = \pi_1 = 1/2 \), the limit reduces to random guessing, which represents the hardest classification problem and slow rates are to be expected. When \( \pi_0 \neq \pi_1 \), we can expect fast rates, too, since the asymptotic Bayes rule always votes for the same label, to wit, the one with the largest unconditional probability. Thus, in a way, \( \Delta \approx 1 \) is the most interesting case to investigate.

The lemma below shows that \( R_z^* \geq R_z^* \), implying that \( R_z^* \) is also an ambitious benchmark.

Lemma 1. Under model (1.1) and (i) – (iii), we have

\[
R_z^* = \inf_{g : \mathbb{R}^p \to \{0,1\}} \mathbb{P}\{g(AZ + W) \neq Y\} \geq R_z^* = \inf_{h : \mathbb{R}^p \to \{0,1\}} \mathbb{P}\{h(Z) \neq Y\}.
\]

Proof. See Appendix A.1.1. \( \square \)

If \( W = 0_p \), the inequality in Lemma 1 obviously becomes an equality. More generally, if the signal for predicting \( Z \) from \( X \) under model (1.1) is large, we expect the gap between \( R_z^* \) and \( R_z^* \) to be small. To characterize such dependence, we introduce the following parameter space of \( \Theta : (A, \Sigma_{Z|Y}, \Sigma_W, \alpha_1, \alpha_0, \pi_1, \pi_0) \),

\[
\Theta(\lambda, \sigma, \Delta) = \left\{ \theta : \lambda_j(\Sigma_W) \asymp \sigma^2, \forall j \in [p], \lambda_k(\Sigma_{Z|Y} A^\top) \asymp \Delta, \forall k \in [K], \pi_0 = \pi_1 \right\}
\]  

and recall \( \Delta \) from (2.2). For any \( \theta \in \Theta(\lambda, \sigma, \Delta) \), the quantity \( \lambda/\sigma^2 \) can be treated as the signal-to-noise ratio for predicting \( Z \) from \( X \) given \( Y \) under model (1.1). The following lemma shows how the gap between \( R_z^* \) and \( R_z^* \) depends on \( \lambda/\sigma^2 \) and \( \Delta \) in the special case \( W \sim N_p(0_p, \Sigma_W) \).

Lemma 2. Under model (1.1) and (i) – (iv), suppose \( W \sim N_p(0_p, \Sigma_W) \) with \( \Sigma_W \succ 0 \). For any \( \theta \in \Theta(\lambda, \sigma, \Delta) \), we have

\[
\frac{\Delta}{1 + (\lambda/\sigma^2)} \exp \left\{ -\frac{\Delta^2}{8} \right\} \lesssim R_z^* - R_z^* \lesssim \frac{\Delta}{1 + (\lambda/\sigma^2)} \exp \left\{ -\frac{\Delta^2}{8} + \frac{\Delta^2}{8(1 + \lambda/\sigma^2)} \right\}.
\]
Remark 2. The upper bound of Lemma 2 reveals that $\lambda/\sigma^2 \to \infty$ implies $R_x^* - R_z^* \to 0$ irrespective of the magnitude of $\Delta$. Regarding to $\Delta$, we also find that $R_x^* - R_z^* \to 0$ in the following scenarios: (1) if $\Delta \to 0$, irrespective of $\lambda/\sigma^2$, (2) if $\Delta \to \infty$ and $\lambda/\sigma^2 \not\to 0$, (3) if $\Delta \approx 1$ and $\lambda/\sigma^2 \to \infty$.

The lower bound of Lemma 2, on the other hand, establishes the irreducible error for not observing $Z$. This term will naturally appear in the minimax lower bounds of the excess risk derived in the next section.

2.2 Minimax lower bounds of the excess risk

In this section, we first establish minimax lower bounds of the excess risk $R_x(\hat{g}) - R_z^*$ under model (1.1) and (1.3) for any classifier $\hat{g}$. The results are established over the parameter space $\Theta(\lambda, \sigma, \Delta)$ in (2.3) which is characterized by three quantities: $\lambda$, $\sigma^2$ and $\Delta$, all of which are allowed to grow with the sample size $n$. Our minimax lower bounds of the excess risk fully characterize the dependence on these quantities, in addition to the dimensions $K$ and $p$ and the sample size $n$.

We use $\mathbb{P}_\theta^D$ to denote the set of all distributions of $D := (X, Y)$ parametrized by $\theta \in \Theta(\lambda, \sigma, \Delta)$ under model (1.1) and (1.3). For simplicity, we drop the dependence on $\theta$ for both $R_x(\hat{g})$ and $R_z^*$. Define

$$\omega_n^* = \sqrt{\frac{K}{n} + \frac{\sigma^2}{\lambda} \Delta^2 + \frac{\sigma^2 \sigma^2 p}{\lambda n} \Delta^2}. \quad (2.4)$$

The following theorem states the minimax lower bounds of the excess risk for any classifier over the parameter space $\Theta(\lambda, \sigma, \Delta)$.

**Theorem 3.** Under model (1.1), assume (i) – (iv), $K \geq 2$, $K/(n \wedge p) \leq c_1$, $\sigma^2/\lambda \leq c_2$ and $\sigma^2 p/(\lambda n) \leq c_3$ for some sufficiently small constants $c_1, c_2, c_3 > 0$. There exists some constants $c_0 \in (0, 1)$ and $C > 0$ such that

1. If $\Delta \approx 1$, then

$$\inf_{\hat{g}} \sup_{\theta \in \Theta(\lambda, \sigma, \Delta)} \mathbb{P}_\theta^D \left\{ R_x(\hat{g}) - R_z^* \geq C \left( \omega_n^* \right)^2 \right\} \geq c_0.$$

2. If $\Delta \to \infty$ and $\sigma^2/\lambda = o(1)$ as $n \to \infty$, then

$$\inf_{\hat{g}} \sup_{\theta \in \Theta(\lambda, \sigma, \Delta)} \mathbb{P}_\theta^D \left\{ R_x(\hat{g}) - R_z^* \geq C \left( \omega_n^* \right)^2 \exp \left\{ - \left[ \frac{1}{8} + o(1) \right] \Delta^2 \right\} \right\} \geq c_0.$$

3. If $\Delta \to 0$ as $n \to \infty$, then

$$\inf_{\hat{g}} \sup_{\theta \in \Theta(\lambda, \sigma, \Delta)} \mathbb{P}_\theta^D \left\{ R_x(\hat{g}) - R_z^* \geq C \min \left\{ \frac{\omega_n^*}{\Delta}, 1 \right\} \omega_n^* \right\} \geq c_0.$$

The infima in all statements are taken over all classifiers.

**Proof.** The proof of Theorem 3 is deferred to Appendix B. \qed

The lower bounds in Theorem 3 consist of three terms: the one related with $K/n$ is the optimal rate of the excess risk even when $Z$ were observable; the second one related with $\sigma^2/\lambda$ is the irreducible error for not observing $Z$ (see, Lemma 1); the last one involving $\sigma^2 p/(\lambda n)$ is the price to pay for estimating the column space of $A$. Although the third term could get absorbed
by the second term as $\sigma^2 p/(\lambda n) \leq c_3$, we incorporate it here for transparent interpretation. The lower bounds in Theorem 3 are tight as we show in Section 5.3 that there exists a classifier whose excess risk has a matching upper bound.

**Remark 3 (Phase transition in $\Delta$).** Recall from (2.2) that $\Delta$ quantifies the separation between $N(\alpha_0, \Sigma_{Z|Y})$ and $N(\alpha_1, \Sigma_{Z|Y})$. We see in Theorem 3 a phase transition of the rates of convergence of the excess risk as $\Delta$ varies. When $\Delta$ is of constant order, the excess risk has minimax convergence rate

$$
\frac{K}{n} + \frac{\sigma^2}{\lambda} + \frac{\sigma^2 \sigma^2 p}{\lambda \lambda n}.
$$

When $\Delta \to \infty$, we see that the minimax rate of convergence of the excess risk gets faster exponentially in $\Delta^2$. For instance, if $\Delta^2 \geq C_0 \log n$ for some constant $C_0 > 0$, then the minimax rate already becomes polynomially faster in $n$ as

$$
\left[ \frac{K}{n} + \frac{\sigma^2}{\lambda} + \frac{\sigma^2 \sigma^2 p}{\lambda \lambda n} \right] \frac{1}{n^{C_1}}
$$

for some $C_1 > 0$ depending on $C_0$. Finally, when $\Delta \to 0$, a more challenging, yet important case, the minimax convergence rate of the excess risk gets slower. It is worth noting that although the oracle Bayes risk $R^*_z \to 1/2$ when $\Delta \to 0$, the minimax excess risk still converges to zero at least in $\omega^*_n$-rate. If $\omega^*_n \lesssim \Delta$, the convergence gets faster as

$$
\frac{K}{n} \frac{1}{\Delta} + \frac{\sigma^2}{\lambda} \Delta + \frac{\sigma^2 \sigma^2 p}{\lambda \lambda n} \Delta.
$$

**Remark 4 (Proof technique).** In the proof of Theorem 3, the three terms in the lower bound are derived separately in the setting where $X|Y$ is Gaussian. Since, for any classifier $\hat{g}$,

$$
R_x(\hat{g}) - R^*_x = (R_x(\hat{g}) - R^*_x) + (R^*_x - R^*_z),
$$

in view of Lemma 1, it suffices to prove the two terms related with $K/n$ and $\sigma^2 p/(\lambda n)$ constitute the lower bounds of $R_x(\hat{g}) - R^*_x$. In fact, as a byproduct of our result, we also derive minimax lower bounds of the excess risk relative to $R^*_x$. This derivation is based on constructing subsets of $\Theta(\lambda, \sigma, \Delta)$ by fixing either $A$ or $\alpha_0$ and $\alpha_1$ separately. The choice of $A$ is based on the hypercube construction for matrices with orthonormal columns (Vu and Lei, 2013, Lemma A.5). The analyses of both terms are non-standard as the excess risk is not a semi-distance, as required by standard techniques of proving minimax lower bounds. Based on a reduction scheme established in Appendix B, we show that proving Theorem 3 suffices to establish a minimax lower bound of the following loss function

$$
L_{\theta}(\hat{g}) := P_{\theta} \{ \hat{g}(X) \neq g^*_\theta(X) \}.
$$

Here $P_{\theta}$ is taken with respect to $X$ and $g^*_\theta(X)$ is the Bayes rule based on $X$ that minimizes $R_x(g)$ over $g : \mathbb{R}^p \to \{0, 1\}$. Since $L_{\theta}(\hat{g})$ is shown to satisfy a local triangle inequality-type bound such that a variant of Fano’s lemma can be applied (Azizyan et al., 2013, Proposition 2), we proved a crucial result, in Lemmas 28 and 29 of Appendix B, that

$$
\inf_{\hat{g}} \sup_{\theta \in \Theta(\lambda, \sigma, \Delta)} P_{\theta} \left\{ L_{\theta}(\hat{g}) \geq C \left( \sqrt{\frac{K}{n} \frac{1}{\Delta}} + \sqrt{\frac{\sigma^2 \sigma^2 p}{\lambda \lambda n}} \right) e^{-\Delta^2} \right\} \geq c_0
$$

for some constant $c_0 \in (0, 1)$ and $C > 0$. 
Remark 5 (Comparison with the existing literature). As mentioned above, a byproduct of our proof of Theorem 3 is the minimax lower bounds of $R_x(\hat{g}) - R^*_x$ in the setting where $X \mid Y$ is Gaussian, which have exactly the same form as Theorem 3 but without the second term related with $\sigma^2/\lambda$. It is informative to put this lower bound of $R_x(\hat{g}) - R^*_x$ in comparison to the existing literature in this special setting.

Under the classical LDA model (1.5), Cai and Zhang (2019b) derives the minimax lower bounds of $R_x(\hat{g}) - R^*_x$ over a suitable parameter space for $\Delta \gtrsim 1$, which have the same form as ours with $K/n + \sigma^4p\Delta^2/(\lambda^2n)$ replaced by $s/n$ for $s := \|\Sigma^{-1}(\mu_1 - \mu_0)\|_0$. In contrast, our lower bounds reflect the benefit of considering an approximate lower-dimensional structure of $X \mid Y$ under (1.1) and (1.5) instead of directly assuming sparsity on $\Sigma^{-1}(\mu_1 - \mu_0)$. These two lower bounds coincide in the low-dimensional setting ($p < n$) when there is no sparsity in $\Sigma^{-1}(\mu_1 - \mu_0)$, that is $s = p$, and when there is no low-dimensional hidden factor model (that is, $X = Z$ with $K = p$, $A = I_p$ and $W = 0_p$). On the other hand, Cai and Zhang (2019a) only established the phase transition between $\Delta \approx 1$ and $\Delta \to \infty$ whereas we are able to derive the minimax lower bound for $\Delta \to 0$, a case that has not even been analyzed in the classical LDA literature.

Technically, it is also worth mentioning that the latent model structure on $X$ via (1.1) brings considerable additional difficulties for establishing the lower bounds of $R_x(\hat{g}) - R^*_x$. Indeed, for any $\theta \in \Theta(\lambda, \sigma, \Delta)$, the covariance matrix of $X \mid Y$ is $\Sigma(\theta) = A\Sigma_{Z \mid Y}A^\top + \Sigma_W$ which cannot be chosen as a diagonal matrix to simplify the analysis as done by Cai and Zhang (2019b). Furthermore, to derive the term $\sigma^4p\Delta^2/(\lambda^2n)$ in the lower bound for quantifying the error of estimating the column space of $A$, we need to carefully choose the subset of $\Theta(\lambda, \sigma, \Delta)$ via the hypercube construction (Vu and Lei, 2013, Lemma A.5) that has been used for proving the optimal rates of the sparse PCA. Since the statistical distance (such as KL-divergence) between any two of thus constructed hypotheses of $\Theta(\lambda, \sigma, \Delta)$ is diverging whenever $p/n \to \infty$ (see, Lemma 27 in Appendix B), a different analysis than the standard one (for instance, in Azizyan et al. (2013)) has to be used to allow $p > n$ and a large amount of work is devoted to provide a meaningful and sharp lower bound that is valid for both $p < n$ and $p > n$ (see Lemma 28 for details).

3 Methodology

In this section, we describe our classification method based on $n$ i.i.d. observations from model (1.1) and (1.3). We first state a general method in Section 3.1 which is motivated by the optimal oracle rule $g^*_z$ in (1.6) and (1.7), and is based on prediction of the unobserved factors $Z_1, \ldots, Z_n, Z$ in the features $X_1, \ldots, X_n, X$ by projections. In Section 3.2 we state our proposed methods via principal component projections as well as a cross-fitting strategy for high-dimensional scenarios. Selection of the number of principal components is further discussed in Section 3.3.

3.1 General approach

The first idea is to change the classification problem into a regression problem, at the population level. The close relationship between LDA and regression has been observed before, see, for instance, Section 8.3.3 in Izenman (2008), Hastie et al. (2009) and Mai et al. (2012). Let
\[ \Sigma_Z = \text{Cov}(Z) \] be the unconditional covariance matrix of \( Z \). Define
\[ \beta = \pi_0 \pi_1 \Sigma_Z^{-1} (\alpha_1 - \alpha_0), \quad (3.1) \]
\[ \beta_0 = -\frac{1}{2} (\alpha_0 + \alpha_1)^\top \beta + \pi_0 \pi_1 \left[ 1 - (\alpha_1 - \alpha_0)^\top \beta \right] \log \frac{\pi_1}{\pi_0}. \]

**Proposition 4.** Let \( \eta, \eta_0 \) and \( \beta, \beta_0 \) be defined in (1.7) and (3.1), respectively. Under model (1.3) and assumption (iv), we have
\[ z^\top \eta + \eta_0 \geq 0 \iff z^\top \beta + \beta_0 \geq 0. \]

Furthermore,
\[ \beta = \Sigma_Z^{-1} \text{Cov}(Z, Y). \]

**Proof.** The proof of Proposition 4 can be found in Appendix A.2. \( \square \)

**Remark 6.** In fact, our proof shows that the first statement of Proposition 4 still holds if we replace \( \pi_0 \pi_1 \) in the definition of \( \beta \) by any positive value coupled with corresponding modification of \( \beta_0 \) (see Lemma 14 in Appendix A.2 for the precise statement). The advantage of using \( \pi_0 \pi_1 \) in (3.1) is that \( \beta \) can be obtained by simply regressing \( Y \) on \( Z \). For this choice of \( \beta \), our proof also reveals
\[ z^\top \eta + \eta_0 = \frac{1}{\pi_0 \pi_1} \left[ 1 - (\alpha_1 - \alpha_0)^\top \beta \right] \left( z^\top \beta + \beta_0 \right) = \frac{1 + \pi_0 \pi_1 \Delta^2}{\pi_0 \pi_1} (z^\top \beta + \beta_0), \quad (3.2) \]
a key identity that will used later in Section 8 to extend our approach for handling multi-class classification problems.

Proposition 4 implies the equivalence between the linear rules \( g^*_z(z) \) in (1.7) and
\[ g_z(z) := \mathbb{1} \{ z^\top \beta + \beta_0 \geq 0 \} \quad (3.3) \]
based on, respectively, the halfspaces \( \{ z \mid z^\top \eta + \eta_0 \geq 0 \} \) and \( \{ z \mid z^\top \beta + \beta_0 \geq 0 \} \). According to Proposition 4, if \( Z = (Z_1^\top, \ldots, Z_n^\top)^\top \in \mathbb{R}^{n \times K} \) were observed, it is natural to use the least squares estimator \( (Z^\top \Pi_n Z)^+ Z^\top \Pi_n Y \) to estimate \( \beta \). Recall that \( \Pi_n = I_n - n^{-1} 1_n 1_n^\top \) is the centering matrix and \( M^+ \) is the Moore-Penrose inverse of any matrix \( M \). Since in practice only \( X = (X_1^\top, \ldots, X_n^\top)^\top \in \mathbb{R}^{n \times p} \) is observed, we propose to estimate \( z^\top \beta \) by
\[ x^\top \hat{\beta} := x^\top B (\Pi_n X B)^+ Y = x^\top B (B^\top X^\top \Pi_n X B)^+ B^\top X^\top \Pi_n Y \quad (3.4) \]
with \( x \in \mathbb{R}^p \) being one realization of \( X \) from model (1.1). Here in principal \( B \in \mathbb{R}^{p \times q} \) could be any matrix with any \( q \in \{1, \ldots, p\} \). Furthermore, we estimate \( \beta_0 \) by
\[ \hat{\beta}_0 := -\frac{1}{2} (\hat{\mu}_0 + \hat{\mu}_1)^\top \hat{\theta} + \hat{\pi}_0 \hat{\pi}_1 \left[ 1 - (\hat{\mu}_1 - \hat{\mu}_0)^\top \hat{\theta} \right] \log \frac{\hat{\pi}_1}{\hat{\pi}_0} \quad (3.5) \]
based on standard non-parametric estimates
\[ n_k = \sum_{i=1}^n \mathbb{1} \{ Y_i = k \}, \quad \hat{n}_k = \frac{n_k}{n}, \quad \hat{\mu}_k = \frac{1}{n_k} \sum_{i=1}^n X_i \mathbb{1} \{ Y_i = k \}, \quad k \in \{0, 1\}. \quad (3.6) \]
Our final classifier is
\[ g^*_z(x) := \mathbb{1} \{ x^\top \hat{\theta} + \hat{\beta}_0 \geq 0 \}. \quad (3.7) \]
Notice that \( \hat{\theta}, \hat{\beta}_0 \) and \( g^*_z(x) \) all depend on \( B \) implicitly.
3.2 Principal component (PC) based classifiers

Though the classifier in (3.7) can use any matrix $B$, in this paper we mainly consider the choice $B = U_r \in \mathbb{R}^{p \times r}$, for some $r \in \{1, \ldots, p\}$, where the matrix $U_r$ consists of the first $r$ right-singular vectors of $\Pi_n X$, the centered $X$. In this case, $x^\top \hat{\theta}$ is the famous principal component regression (PCR) predictor by using $r$ principal components (Hotelling, 1957). The optimal choice of $r$ would be $K$, the number of latent factors, when it is known in advance. We analyze the classifier with $B = U_K$ in Theorem 9 of Section 5.2.

Suggested by our theory, in the high-dimensional setting $p > n$, performance of the PC-based classifiers can be improved either by using an additional dataset or via data-splitting.

In several applications, such as semi-supervised learning, researchers also have access to an additional set of unlabelled data. Given an additional data matrix $\tilde{X} \in \mathbb{R}^{n' \times p}$ with i.i.d. (unlabelled) observations from model (1.1) with $n' \asymp n$ and independent of $X$ in (3.4), it is often beneficial to use $B = \tilde{U}_K$ based on the first $K$ right singular vectors of $\Pi_{n'} \tilde{X}$. This classifier is analyzed in Theorem 10 of Section 5.2.

When additional data is not available, we advocate to use a sample splitting technique called $k$-fold cross-fitting (Chernozhukov et al., 2018). First, we randomly split the data into $k$ folds, and for each fold, we use it as $X$ to construct $\hat{U}_r$ and use the remaining data as $X$ and $Y$ to obtain $\hat{\theta}$ and $\hat{\beta}_0$ from (3.4) and (3.5), respectively. In the end, the final classifier is constructed via (3.7) based on the averaged $k$ pairs of $\hat{\theta}$ and $\hat{\beta}_0$. Theoretically, it is straightforward to show that the resulting classifiers share the same conclusions as Theorem 10 for $k = \mathcal{O}(1)$.

Empirically, since this cross-fitting strategy ultimately uses all data points, it might mitigate the efficiency loss due to sample splitting. Standard choices of $k$ include $k = 2$ and $k = 5$ while the latter is reported to have smaller standard errors (Chernozhukov et al., 2018).

3.3 Estimation of the number of retained PCs

When $K$ is unknown, we propose to estimate it by

$$\hat{K} := \arg \min_{0 \leq k \leq \bar{K}} \frac{\sum_{j > k} \sigma_j^2}{np - c_0(n + p)k}, \quad \text{with} \quad \bar{K} := \left\lfloor \frac{\nu}{2c_0(1 + \nu)(n \wedge p)} \right\rfloor,$$

for absolute constants $c_0$ and $\nu > 1$. The latter is introduced to avoid division by zero and can be set arbitrarily large. The choice of $c_0 = 2.1$ is used in all of our simulations and has overall good performance. The sum $\sum_j \sigma_j u_j v_j^\top$, with non-increasing $\sigma_j$, is the singular-value-decomposition (SVD) of $\Pi_n X$ or $\Pi_{n'} \tilde{X}$.

Criterion (3.8) was originally proposed in Bing and Wegkamp (2019) for selecting the rank of the coefficient of a multivariate response regression model and is further adopted by Bing et al. (2021) for selecting the number of retained principal components under the framework of factor regression models. It also has close connection to the well-known elbow method. The main computation of solving (3.8) is to compute the SVD of $\Pi_n X$ once. In Section 5.1 we show the consistency of $\hat{K}$, ensuring that the classifier with $B = \hat{U}_{\hat{K}}$ shares the same theoretical properties as the one with $B = U_K$.

4 A general strategy of bounding the excess classification error

In this section, we establish a general theory for analyzing the excess risk of the classifier $\hat{g}_x$ in (3.7) that uses any matrix $B$ for the estimate $\hat{\theta}$ in (3.4). The main purpose is to establish high-level conditions that yield a consistent classifier constructed in Section 3 in the strong sense

$$R_x(\hat{g}_x) := \mathbb{P}\{\hat{g}_x(X) \neq Y\} \rightarrow R_x^*,$$
and further to provide its rate of convergence. We recall that $\mathbb{P}$ is taken with respect to $(X,Y)$.

For convenience, we introduce the notation

$$
\hat{G}_x(x) := x^T \hat{\theta} + \hat{\beta}_0, \quad G_z(z) := z^T \beta + \beta_0
$$

(4.1)

such that $\hat{g}_x(x) = 1\{\hat{G}_x(x) \geq 0\}$ from (3.7) and, using the equivalence in Proposition 4,

$$
g^*_z(z) = 1\{G_z(z) \geq 0\}.
$$

(4.2)

Recall that $\hat{g}_x$ depends on the choice of $B$ via $\hat{\theta}$ and $\hat{\beta}_0$.

The following theorem provides a general bound for the excess risk of $\hat{g}_x$ that uses any $B$ in (3.4). Its proof can be found in Appendix A.3.1.

**Theorem 5.** Under model (1.1), assume (i) – (iv). For all $t > 0$, we have

$$
R_x(\hat{g}_x) - R^*_x \leq \mathbb{P}\{|\hat{G}_x(X) - G_z(Z)| > t\} + c_x t \mathbb{P}(t)
$$

(4.3)

where $c_x = \Delta^2 + (\pi_0 \pi_1)^{-1}$ and

$$
P(t) = \pi_0 \left[ \Phi(R) - \Phi(R - t \ c_x / \Delta) \right] + \pi_1 \left[ \Phi(L + t \ c_x / \Delta) - \Phi(L) \right]
$$

(4.4)

with

$$
L = -\frac{\Delta}{2} - \frac{\log \frac{\pi_1}{\pi_0}}{\Delta}, \quad R = -\frac{\Delta}{2} - \frac{\log \frac{\pi_0}{\pi_1}}{\Delta}.
$$

**Remark 7.** The quantity $P(t)$ in (4.4) is in fact

$$
\pi_0 \mathbb{P}\{-t < G_z(Z) < 0 \mid Y = 0\} + \pi_1 \mathbb{P}\{0 < G_z(Z) < t \mid Y = 1\}
$$

which describes the probabilistic behavior of the margin of the hyperplane $\{z : G_z(z) = 0\}$ that separates the distributions $Z \mid Y = 0$ and $Z \mid Y = 1$. Conditions that control the margin between $Z \mid Y = 0$ and $Z \mid Y = 1$ are more suitable in our current setting and have a different perspective than the usual margin condition in Tsybakov (2004) that controls the probability $\mathbb{P}\{|\eta(Z) - 1/2| < \delta\}$ for any $0 < \delta \leq 1/2$, with $\eta(z) := \mathbb{P}(Y = 1 \mid Z = z)$.

**Remark 8** (Extension to non-linear classifiers). The proof of Theorem 5 also allows us to analyze more complex classifiers. Indeed, let $\Lambda_z(z)$ be the logarithm of the ratio between $\mathbb{P}(Z = z, Y = 1)$ and $\mathbb{P}(Z = z, Y = 0)$, and let $\hat{\Lambda}_z(x)$ be an arbitrary estimate of $\Lambda_z(z)$. We can easily derive from our proof of Theorem 5 the following excess risk bound for the classifier $\hat{g}_x(x) = 1\{\hat{\Lambda}_z(x) \geq 0\}$,

$$
R_x(\hat{g}_x) - R^*_x \leq \mathbb{P}\{|\hat{\Lambda}_z(X) - \Lambda_z(Z)| > t\}
$$

$$
+ t \pi_0 \mathbb{P}\{-t < \Lambda_z(Z) < 0 \mid Y = 0\} + t \pi_1 \mathbb{P}\{0 < \Lambda_z(Z) < t \mid Y = 1\},
$$

(4.5)

for any $t > 0$. Therefore, bound in (4.5) can be used as an initial step for analyzing any classification problems, particularly suitable for situations where conditional distributions $Z \mid Y$ are specified. The remaining difficulty is to find a good estimator $\hat{\Lambda}_z(x)$ and to control $|\hat{\Lambda}_z(X) - \Lambda_z(Z)|$. For instance, when $Z \mid Y = k$, for $k \in \{0, 1\}$, have Gaussian distributions with different means and different covariances, the Bayes rule of using $Z$ (equivalently, $\Lambda_z(Z)$) becomes quadratic, leading to an estimator $\hat{\Lambda}_z(x)$ that is quadratic in $x$ as well. Since both the procedure and the analysis are different, we will study this setting in a separate paper.
From (4.1), we find the identity
\[ \hat{G}_x(X) - G_z(Z) = Z^\top (A^\top \hat{\theta} - \beta) + W^\top \hat{\theta} + \hat{\beta}_0 - \beta_0. \] (4.6)

To establish its deviation inequalities, our analysis uses the following distributional assumption on \( W \) from (1.1). We assume that
\( W = \Sigma_{W}^{1/2} \bar{W} \) and \( \bar{W} \) is a mean-zero \( \gamma \)-subGaussian random vector with \( \mathbb{E}[\bar{W}\bar{W}^\top] = I_p \) and \( \mathbb{E}[\exp(u^\top \bar{W})] \leq \exp(\gamma^2/2) \), for all \( ||u||_2 = 1 \).

We stress that the distributions of \( X \mid Y \) need not be Gaussian. In addition, we require that
\( (vi) \ \pi_0 \) and \( \pi_1 \) are fixed and bounded from below by some constant \( c \in (0, 1/2] \).

The following proposition states a deviation inequality of \( |\hat{G}_x(X) - G_z(Z)| \) which holds with high probability under the law \( \mathbb{P}^D \). It depends on three quantities:
\[ \hat{r}_1 := ||\Sigma_{Z}^{1/2} (A^\top \hat{\theta} - \beta)||_2, \quad \hat{r}_2 := ||\hat{\theta}||_2, \quad \hat{r}_3 := \frac{1}{\sqrt{n}} ||W(P_B - P_A)||_{\text{op}}. \] (4.7)

For any matrix \( M \), let \( P_M \) denote the projection onto its column space. From (4.6), appearance of the first two quantities in (4.7) is natural since \( Z \) and \( W \) are independent of \( \theta \) and \( \hat{\beta}_0 \), and \( Z^\top w \) and \( W^\top v \) have subGaussian tails for any \( v \in \mathbb{R}^p \) and \( w \in \mathbb{R}^K \) under the distributional assumptions (iv) and (v). The third quantity \( ||W(P_B - P_A)||_{\text{op}} \) in (4.7) originates from \( \hat{\beta}_0 - \beta_0 \) and reflects the benefit of using a matrix \( B \) that estimates the column space of \( A \) well.

**Proposition 6.** Under model (1.1), assume (i) – (vi) and \( K \log n \leq cn \) for some constant \( c > 0 \). For any \( a \geq 1 \), we have
\[ \mathbb{P}^D \left\{ \mathbb{P} \left\{ |\hat{G}_x(X) - G_z(Z)| \geq \omega_n(a) \right\} \leq n^{-a} \right\} = 1 - \mathcal{O}(n^{-1}). \] (4.8)

Here, for some constant \( C > 0 \) depending on \( \gamma \) only,
\[ \omega_n(a) = C \left\{ \sqrt{a \log n} \left( \hat{r}_1 + ||\Sigma_{W}||_{\text{op}}^{1/2} \hat{r}_2 \right) + \hat{r}_2 \hat{r}_3 + \sqrt{\log n} \right\}. \] (4.9)

**Proof.** See Appendix A.3.2.

**Theorem 7.** Under model (1.1), assume (i) – (vi) and \( K \log n \leq cn \) for some constant \( c > 0 \). For any \( a \geq 1 \) and any sequence \( \omega_n > 0 \), on the event \{\( \omega_n(a) \leq \omega_n \}\), the following holds with probability \( 1 - \mathcal{O}(n^{-1}) \) under the law \( \mathbb{P}^D \),
\[ R_x(\hat{g}_x) - R_x^* \lesssim n^{-a} + \begin{cases} \omega_n^a, & \text{if } \Delta \asymp 1; \\ \omega_n^2 \exp \left\{ -[c_\pi + o(1)]\Delta^2 \right\}, & \text{if } \Delta \to \infty \text{ and } \omega_n = o(1); \\ \omega_n^2 \exp \left\{ -[c' + o(1)]\Delta^2 \right\}, & \text{if } \Delta \to 0, \pi_0 \neq \pi_1 \text{ and } \omega_n = o(1); \\ \omega_n \min\{1, \omega_n/\Delta\}, & \text{if } \Delta \to 0 \text{ and } \pi_0 = \pi_1. \end{cases} \]

Here \( c_\pi \) and \( c' \) are some absolute positive constants and \( c_\pi = 1/8 \) if \( \pi_0 = \pi_1 \).
Hence, it remains to find a deterministic sequence $\omega_n \to 0$ such that $P\{\hat{\omega}_n(a) \leq \omega_n\} \to 1$ as $n \to \infty$. Further, in view of (4.9), all we need is to find deterministic upper bounds of $\hat{r}_1, \hat{r}_2$ and $\hat{r}_3$. In such way Theorem 7 serves as a general tool for analyzing the excess risk of the classifier constructed via (3.4) – (3.7) by using any matrix $B$.

Later in Section 5 we apply Theorem 7 to analyze several classifiers, including the principal components based classifier by choosing $B = U_K$ and $B = \hat{U}_K$ as well as their counterparts based on the data-dependent choice $\hat{K}$. For these PC-based classifiers, we will find a sequence $\omega_n$ that closely matches the sequence $\omega^*_n$ in (2.4) under suitable conditions, up to logarithmic factors in $n$, for our procedure. In view of Theorem 3, this rate turns out to be minimax-optimal over a subset of the parameter space considered in Theorem 3, up to $\log(n)$ factors.

Although not pursued in this paper, it is worth mentioning some other reasonable choices of $B$ including, for instance, the identity matrix $I_p$ which leads to the generalized least squares based classifier, the estimator of $A$ in Bing et al. (2020), the projection matrix from supervised PCA (Bair et al., 2006; Barshan et al., 2011) and the projection matrix obtained via partial least squares regression (Nguyen and Rocke, 2002; Barker and Rayens, 2003).

Remark 9. We observe the same phase transition in Theorem 7 for $\Delta \approx 1$ amd $\Delta \to \infty$ as discussed in Remark 3. When $\Delta \to 0$, it is interesting to see that the rate of convergence depends on whether or not $\pi_1$ and $\pi_0$ are distinct, as explained in Remark 1.

To the best of our knowledge, upper bounds of the excess risk in the regime $\Delta = o(1)$ are not known in the existing literature. Our result in this regime relies on a careful analysis which does not require any condition on $\Delta$, in contrast to the existing analysis of the classical high-dimensional LDA problems. For instance, under model (1.5), Cai and Zhang (2019a) assumes $\Delta^2_x := (\mu_1 - \mu_0)^\top \Sigma^{-1}(\mu_1 - \mu_0) \gtrsim 1$ and $\Delta^2_x(s \log n/n) = o(1)$ to derive the convergence rate of their estimator of $\Sigma^{-1}(\mu_1 - \mu_0)$ with $s = \|\Sigma^{-1}(\mu_1 - \mu_0)\|_0$. As a result, their results of excess misclassification risk only hold for $\Delta_x \gtrsim 1$.

5 Rates of convergence of the PC-based classifier

We apply our general theory in Section 4 to several classifiers corresponding to different choices of $B = U_K, B = \hat{U}_K, B = U_{\hat{K}}$ and $B = \hat{U}_{\hat{K}}$ in (3.4). Since our analysis is beyond the parameter space $\Theta(\lambda, \sigma, \Delta)$ in (2.3), we first generalize the signal-to-noise ratio $\lambda/\sigma^2$ of predicting $Z$ from $X$ given $Y$ by introducing

$$\xi^* := \frac{\lambda_K(A\Sigma_{Z|Y}A^\top)}{\lambda_1(\Sigma_W)}.$$ (5.1)

We also need the related quantity

$$\xi := \frac{\lambda_K(A\Sigma_{Z|Y}A^\top)}{\delta_W},$$ (5.2)

that characterizes the signal-to-noise ratio of predicting $Z$ from $X = ZA^\top + W$. Indeed, note that we replaced $\lambda_1(\Sigma_W)$ by

$$\delta_W = \lambda_1(\Sigma_W) + \frac{\text{tr}(\Sigma_W)}{n}$$ (5.3)

and the largest eigenvalue of the random matrix $W^\top W/n$ is of order $O_p(\delta_W)$ under assumption (v) (see, for instance, Bing et al. (2021, Lemma 22)).

5.1 Consistent estimation of the latent dimension $K$

Since in practice the true $K$ is often unknown, we analyze the estimated rank $\hat{K}$ selected from (3.8).
Consistency of $\hat{K}$ under the factor model (1.1) when $Z$ is a zero-mean subGaussian random vector has been established in Bing et al. (2021, Proposition 8). Here we establish such property of $\hat{K}$ under (1.1) where $Z$ follows a mixture of two Gaussian distributions. Let $r_e(\Sigma_W) = \text{tr}(\Sigma_W)/\lambda_1(\Sigma_W)$ denote the effective rank of $\Sigma_W$.

**Theorem 7.** Let $\hat{K}$ be defined in (3.8) for some absolute constant $c_0 > 0$. Under model (1.1), assume (i) – (vi), and, in addition,

$$K \leq \hat{K}, \quad \log p \leq Cn, \quad \xi \geq C', \quad \text{and} \quad r_e(\Sigma_W) \geq C''(n \wedge p)$$

for some constants $C, C', C'' > 0$. Then,

$$\mathbb{P}^D\{\hat{K} = K\} = 1 - \mathcal{O}(n^{-1}).$$

**Proof.** The proof is deferred to Appendix A.5.1

The condition $K \leq \hat{K}$ holds, for instance, if $K \leq c'(n \wedge p)$ with $c' \leq \nu/(2c_0(1+\nu))$. Condition $r_e(\Sigma_W) \geq C'(n \wedge p)$ holds, for instance, in the commonly considered setting

$$0 < c \leq \lambda_p(\Sigma_W) \leq \lambda_1(\Sigma_W) \leq C < \infty$$

while being more general.

Note that we require $\xi$ to be sufficiently large. This condition is also needed in deriving the rates for both our PC-based classification procedures in the next section.

Theorem 8 implies that the classifier that uses $B = U_{\hat{K}}$ ($B = \tilde{U}_K$) has the same excess risk bound as that uses $B = U_K$ ($B = \tilde{U}_K$). For this reason, we restrict our analysis in the remaining of this section to $B$ based on the first $K$ principal components of $U$ and $\tilde{U}$.

### 5.2 PC-based LDA by using the true dimension $K$

The following theorem states the excess risk bounds of $\tilde{g}_x$ that uses $B = U_K$. Its proof can be found in Appendix A.5.2. We use the notation $\kappa$ for the condition number $\lambda_1(A\Sigma_ZA^\top)/\lambda_K(A\Sigma_ZA^\top)$ of the matrix $A\Sigma_ZA^\top$.

**Theorem 9.** Under model (1.1), assume (i) – (vi). If $K \log n \leq cn$ and $\xi \geq C\kappa^2$ for some constants $c, C > 0$, then for any $a \geq 1$ and

$$\omega_n(a) = \left(\sqrt{\frac{K \log n}{n}} + \min\{1, \Delta\} \sqrt{\frac{1}{\xi^*}} + \sqrt{\frac{\kappa}{\xi^2}}\right) \sqrt{a \log n}, \quad (5.4)$$

we have $\mathbb{P}^D\{\tilde{\omega}_n(a) \leq \omega_n(a)\} = 1 - \mathcal{O}(n^{-1})$. Hence, with this probability, the conclusion of Theorem 7 holds for the classifier that uses $B = U_K$ for $\omega_n(a)$ in (5.4).

**Remark 10.** Theorem 9 requires $\xi \geq C\kappa^2$. As shown in the proof, this condition can be relaxed to $\xi \geq C$ in which case Theorem 9 holds with $\omega_n(a)$ in (5.4) replaced by

$$\left(\sqrt{\frac{K \log n}{n}} + \sqrt{\frac{1}{\xi^*}} + \sqrt{\frac{\kappa}{\xi^2}} \frac{1}{\xi}\right) \sqrt{a \log n}. \quad (5.5)$$

The above rate is slower than (5.4) when $\Delta \to 0$ as $n \to \infty$.

Similarly, the classifier that uses $B = \tilde{U}_K$ also has the same guarantees as stated in Remark 10 when $\xi \geq C$. However, for larger $\xi$ such as $\xi \geq C\kappa^2$, the following theorem states a smaller excess risk bound of $\tilde{g}_x$ that uses $B = \tilde{U}_K$ comparing to Theorem 9. Its proof can be found in Appendix A.5.3.
Theorem 10. Under the same conditions of Theorem 9, for any $a > 0$ and

$$\omega_n(a) = \left(\sqrt{\frac{K \log n}{n}} + \min\{1, \Delta\}\sqrt{\frac{1}{\xi^*}}\right) \sqrt{a \log n},$$

(5.6)

we have $\mathbb{P}^D\{\tilde{\omega}_n(a) \lesssim \omega_n(a)\} = 1 - \mathcal{O}(n^{-1}).$ Hence, with this probability, the conclusion of Theorem 7 holds for the classifier that uses $B = \tilde{U}_K$ for $\omega_n(a)$ in (5.6).

Remark 11 (Polynomially fast rates). In view of Theorems 9 & 10, fast rates (of the order $\mathcal{O}(n^{-a})$ for arbitrary $a \geq 1$) are obtained for both PC-based procedures, provided that (a) $\Delta^2 \gg \log n$ or (b) $1/\Delta^2 \gg \log n$ and $\pi_0 \neq \pi_1$.

Remark 12 (Advantage of using an independent dataset or data splitting). Comparing to (5.4) in Theorem 9, the convergence rate of the excess risk of the classifier that uses $B = \tilde{U}_K$ does not have the third term $\sqrt{\kappa/\xi^2}$. This advantage only becomes evident when $p > n$ and $\xi^*$ is not sufficiently large. We refer to Remark 13 below for detailed explanation.

To understand why using $\tilde{U}_K$ that is independent of $X$ yields smaller excess risk, recall that the third term in (5.4) originates from predicting $Z$ from $X$ and its derivation involves controlling $\|W(P_{U_K} - P_{\lambda})\|_{\text{op}}$. Since $U_K$ is constructed from $X$, hence also depends on $W$, the dependence between $W$ and $U_K$ renders a slow rate for $\|W(P_{U_K} - P_{\lambda})\|_{\text{op}}$. The fact that auxiliary data can bring improvements (in terms of either smaller prediction / estimation error or weaker conditions) is a phenomenon that has been observed in other problems, such as the problem of estimating the optimal instrument in sparse high-dimensional instrumental variable model (Belloni et al., 2012) and the problem of inference on a low dimensional parameter in the presence of high-dimensional nuisance parameters (Chernozhukov et al., 2018).

Remark 13 (Simplified rates within $\Theta(\lambda, \sigma, \Delta)$). To obtain more insight from the results of Theorems 9 & 10, consider $\theta \in \Theta(\lambda, \sigma, \Delta)$ in (2.3) with $\Delta \approx 1$ such that $\pi_0 = \pi_1$, $1/\xi^* \asymp \sigma^2/\lambda$, $1/\xi \asymp (\sigma^2/\lambda)(1 + p/n)$ and $\kappa \asymp 1$. In this case, combining Theorems 7, 9 and 10 reveals that, with probability $1 - \mathcal{O}(n^{-1}),$

$$R_x(\hat{g}_x) - R^*_z \lesssim \left[\frac{K \log n}{n} + \frac{\sigma^2}{\lambda} + \left(\frac{p \sigma^2}{n \lambda}\right)^2\right] \log n,$$

(5.7)

$$R_x(\hat{g}_x) - R^*_z \lesssim \left[\frac{K \log n}{n} + \frac{\sigma^2}{\lambda}\right] \log n,$$

(5.8)

We have the following conclusions.

(1) If $p < n$, the two rates above coincide and equal (5.8), whence consistency of both PC-based classifiers requires that $K \log^2 n/n \rightarrow 0$ and $\sigma^2 \log n/\lambda \rightarrow 0$.

(2) If $p > n$, it depends on the signal-to-noise ratio (SNR) $\lambda/\sigma^2$ whether or not consistency of the classifier with $B = U_K$ requires additional condition.

(a) If the SNR is large such that

$$\frac{\lambda}{\sigma^2} \gtrsim \min \left\{\left(\frac{p}{n}\right)^2, \frac{p}{\sqrt{nK \log n}}\right\},$$

(5.9)

the two rates in (5.7) and (5.8) also coincide and equal (5.8). In this case, there is no apparent benefit of using an auxiliary data set.

(b) For relatively smaller values of SNR that fail (5.9), the effect of using $B = \tilde{U}_K$ based on an independent data set $X$ is real as evidenced in Figure 1 below where we keep $\lambda/\sigma^2$, $n$ and $K$ fixed but let $p$ grow.
Figure 1: Illustration of the advantage of constructing $\tilde{U}_K$ from an independent dataset: PCLDA represents the PC-based classifier based on $B = U_K$ while PCLDA-split uses $B = \tilde{U}_K$ that is constructed from an independent $\tilde{X}$. Oracle-LS is the oracle benchmark that uses both $Z$ and $\tilde{Z}$ while Bayes represents the risk of using the oracle Bayes rule. The y-axis represents the misclassification errors in percentage. We fix $n = 100$ and $K = 5$ and keep $\lambda/\sigma^2$ fixed, while we let $p$ grow. We refer to Section 6 for detailed data generating mechanism.

(c) It is worth mentioning that if the SNR is sufficiently large such that

$$\frac{\lambda}{\sigma^2} \gtrsim \max \left\{ \left( \frac{p}{n} \right)^2, \frac{p}{\sqrt{nK \log n}} \right\},$$

both errors due to not observing $Z$ and estimation of the column space of the matrix $A$ are negligible compared to the parametric rate $K/n$, to wit, both rates in (5.7) and (5.8) reduce to $K \log^2 n/n$.

Conditions $\lambda \gtrsim p$ and $\sigma^2 = O(1)$ are common in the analysis of factor models with a diverging number of features (Stock and Watson, 2002a; Bai and Li, 2012; Fan et al., 2013). For instance, $\lambda \gtrsim p$ holds when eigenvalues of $\Sigma_{Z\mid Y}$ are bounded and a fixed proportion of rows of $A$ are i.i.d. realizations of a sub-Gaussian random vector with covariance matrix having bounded eigenvalues as well. Nevertheless, consistency of the PC-based classifiers only requires $\lambda/(\sigma^2 \log n) \rightarrow \infty$ for $B = U_K$ and $\lambda/(\sigma^2 \log n) \rightarrow \infty$ for $B = \tilde{U}_K$, both of which are much milder conditions.

5.3 Optimality of the PC-based LDA by sample splitting

We now show that the PC-based LDA by sample splitting achieves the minimax lower bounds in Theorem 3, up to multiplicative logarithmic factors of $n$. Recalling that (2.3), for any $\theta \in \Theta(\lambda, \sigma, \Delta)$, one has $\pi_0 = \pi_1$, $1/\xi^* \asymp \sigma^2/\lambda$, $1/\xi \asymp (\sigma^2/\lambda)(1 + p/n)$ and $1 \lesssim \kappa \lesssim 1 + \Delta^2$. Based on Theorem 10, we have the following corollary for the classifier that uses $B = \tilde{U}_K$. Its proof can be found in Appendix A.5.4. We use the notation $\lesssim$ for inequalities that hold up to a multiplicative logarithmic factor of $n$. Recall $\omega^\ast_n$ from (2.4).

**Corollary 11.** Under model (1.1), assume (i) – (v), $K \log n \leq cn$, $\kappa^2 \sigma^2/\lambda \leq c'$ and $\kappa^2 \sigma^2 p/(\lambda n) \leq c''$ for some constants $c, c', c'' > 0$. For any $\theta \in \Theta(\lambda, \sigma, \Delta)$, with probability $1 - O(n^{-1})$, the classifier that uses $B = \tilde{U}_K$ satisfies the following statements.
(1) If $\Delta \asymp 1$, then
\[ R_x(\hat{g}_x) - R^*_x \preceq (\omega_n^*)^2. \]

(2) If $\Delta \to \infty$, and additionally, $(\log n + \Delta^2)K \log n/n \to 0$ and $(\log n + \Delta^2)\sigma^2/\lambda \to 0$ as $n \to \infty$, then
\[ R_x(\hat{g}_x) - R^*_x \preceq (\omega_n^*)^2 \exp\left\{ - \left[ \frac{1}{8} + o(1) \right] \Delta^2 \right\}. \]

(3) If $\Delta \to 0$ as $n \to \infty$, then
\[ R_x(\hat{g}_x) - R^*_x \preceq \min\left\{ \frac{\omega_n^*}{\Delta}, 1 \right\} \omega_n^*. \]

In view of Theorem 3 and Corollary 11, we conclude the optimality of PC-based procedure that uses $B = \tilde{U}_K$ over $\Theta(\lambda, \sigma, \Delta)$. For $\Delta \to \infty$, if conditions in (2) are not met such as $\Delta^2 \gtrsim n/K$ or $\Delta^2 \gtrsim \lambda/\sigma^2$, the PC-based procedure still has $n^{-a}$ convergence rate of its excess risk, for arbitrary large $a \geq 1$, as commented in Remark 11.

Regarding the PC-based classifier that does not resort to sample splitting, according to Theorems 3 & 9, its excess risk also achieves optimal rates of convergence when $\lambda/\sigma^2$ is large in the precise sense that
\[ \frac{\lambda}{\sigma^2} \gtrsim \min\left\{ \frac{1}{\min\{1, \Delta\}}, \left(\frac{p}{\sqrt{nK\log n}}\right)^2, \frac{p}{\sqrt{nK\log n}} \right\}, \]
holds.

6 Simulation study

We conduct various simulation studies in this section to compare the performance of our proposed algorithm with other competitors. For our proposed algorithm, we call it PCLDA standing for the Principal Components based LDA. The name PCLDA-$K$ is reserved when the true $K$ is used as input. When $K$ is estimated by $\hat{K}$, we use PCLDA-$\hat{K}$ instead. We call PCLDA-CF-$k$ the PCLDA with $k$-fold cross-fitting. We consider $k = 5$ in our simulation as suggested by Chernozhukov et al. (2018). To set a benchmark for PCLDA-CF-$k$, we use PCLDA-split that uses an independent copy of $X$ to compute $\tilde{U}_K$. On the other hand, we compare with the nearest shrunken centroids classifier (PAMR) (Tibshirani et al., 2002), the $\ell_1$-penalized linear discriminant (PenalizedLDA) (Witten and Tibshirani, 2011) and the direct sparse discriminant analysis (DSDA) (Mai et al., 2012). Finally, we choose the performance of the oracle procedure (Oracle-LS) as benchmark in which Oracle-LS uses both $Z$ and $Z$ to estimate $\beta$, $\beta_0$ and the classification rule $g_z(Z)$ in (3.3).

We generate the data as follows. First, we set $\pi_0 = \pi_1 = 0.5$, $\alpha_0 = 0_K$ and $\alpha_1 = 1_K\sqrt{\eta/K}$. The parameter $\eta$ controls the signal strength $\Delta$ in (2.2). We generate $\Sigma_{Z|Y}$ by independently sampling its diagonal elements $[\Sigma_{Z|Y}]_{ii}$ from Unif(1,3) and set its off-diagonal elements as
\[ [\Sigma_{Z|Y}]_{i,j} = \sqrt{[\Sigma_{Z|Y}]_{ii} [\Sigma_{Z|Y}]_{jj}} (-1)^{i+j} (0.5)^{|i-j|}, \quad \text{for each } i \neq j. \]

The covariance matrix $\Sigma_W$ is generated in the same way, except we set $\text{diag}(\Sigma_W) = 1_p$. The rows of $W \in \mathbb{R}^{n \times p}$ are generated independently from $N_p(0, \Sigma_W)$. Entries of $A$ are generated

\(^{1}\text{PAMR, PenalizedLDA and DSDA are implemented in the R packages pamr, penalizedLDA and TULIP, respectively.}\)
independently from \( N(0, 0.3^2) \). The training data \( Z, X \) and \( Y \) are generated according to model (1.1) and (1.3). In the same way, we generate 100 data points that serve as test data for calculating the (out-of-sample) misclassification error for each algorithm.

In the sequel, we vary the dimensions \( n \) and \( p \) as well as the signal strength \( \Delta \) in (2.2), one at a time. For each setting, we repeat the entire procedure 100 times and averaged misclassification errors for each algorithm are reported.

6.1 Vary the sample size \( n \)
We set \( \eta = 5, K = 10, p = 300 \) and vary \( n \) within \( \{50, 100, 300, 500, 700\} \). The left-panel in Figure 2 shows the averaged misclassification error (in percentage) of each algorithm on the test data sets. Since \( \hat{K} \) consistently estimates \( K \), we only report the performance of PCLDA-\( K \). We also exclude the performance of PCLDA-split and PCLDA-CF-5 since they all have similar performance as PCLDA-\( K \). The blue line represents the optimal Bayes error. All algorithms perform better as the sample size \( n \) increases. As expected, Oracle-LS is the best because it uses the true \( Z \) and \( Z \). Among the other algorithms, PCLDA-\( K \) has the closest performance to Oracle-LS in all settings. The gap between PCLDA-\( K \) and Oracle-LS does not close as \( n \) increases. According to Theorem 9, this is because such a gap mainly depends on \( 1/\xi \) which does not vary with \( n \).

![Figure 2: The averaged misclassification errors of each algorithm. We vary \( n \) in the left panel while vary \( \Delta \) in the right one. The y-axis represents the percentage (%) of misclassified points.](image)

6.2 Vary the signal strength \( \Delta^2 \)
We fix \( K = 5, n = 100, p = 300 \) and vary \( \eta \) within \( \{2, 4, 6, 8, 10\} \). As a consequence, the signal strength \( \Delta^2 \) varies within \( \{3.1, 6.3, 9.4, 12.6, 15.7\} \). The right-panel of Figure 2 depicts the averaged misclassification errors of each algorithm. For the same reasoning as before, we exclude PCLDA-\( \hat{K} \), PCLDA-CF-5 and PCLDA-split. It is evident that all algorithms have

\[\text{This is as expected since our data generating mechanism ensures } \xi^* \approx p \text{ in which case PCLDA-split has no clear advantage comparing to PCLDA-}\( K \) (see, discussions after Theorem 10).\]
better performance as the signal strength $\Delta$ increases. Among them, PCLDA-K has the closest performance to Oracle-LS and Bayes in all settings.

6.3 Vary the feature dimension $p$

We examine the performance of each algorithm when the feature dimension $p$ varies across a wide range. Specifically, we fix $K = 5$, $\eta = 5$, $n = 100$ and vary $p$ within $\{100, 300, 500, 700, 900\}$. Figure 3 shows the misclassification errors of each algorithm. The performance of PCLDA-K improves and gets closer to that of Oracle-LS as $p$ increases, in line with Theorem 9. The gap between Oracle-LS and Bayes is due to the fact that both $n$ and $\Delta$ are held fixed.

![Figure 3: The averaged misclassification errors of each algorithm for various choices of $p$. The $y$-axis represents the percentage (%) of misclassified points.](image)

7 Real data analysis

To further illustrate the effectiveness of our proposed method, we analyze three popular gene expression datasets (leukemia data, colon data and lung cancer data)\(^3\), which have been widely used to test classification methods, see, for instance, Alon et al. (1999); Singh et al. (2002); Nguyen and Rocke (2002); Dettling (2004) and also, the more recent literature, Fan and Fan (2008); Mai et al. (2012); Cai and Zhang (2019a). These datasets contain thousands or even over ten-thousand features with around one hundred samples (see Table 1 for the summary). In such challenging settings, LDA-based classifiers that are designed for high-dimensional data are not only easy to interpret but also have competing and even superior performance than other, highly complex classifiers such as classifiers based on kernel support vector machines, random forests and boosting (Dettling, 2004; Mai et al., 2012).

\(^3\)Leukemia data is available at [www.broad.mit.edu/cgi-bin/cancer/datasets.cgi](http://www.broad.mit.edu/cgi-bin/cancer/datasets.cgi). Colon data is available from the R package [plsgenomics](http://plsgenomics). Lung cancer data is available at [www.chestsurg.org](http://www.chestsurg.org).
Table 1: Summary of three data sets.

| Data name   | p   | n   | n0 (category)                  | n1 (category)                        |
|-------------|-----|-----|---------------------------------|--------------------------------------|
| Leukemia    | 7129| 72  | 47 (acute lymphoblastic leukemia) | 25 (acute myeloid leukemia)          |
| Colon       | 2000| 62  | 22 (normal)                     | 40 (tumor)                           |
| Lung cancer | 12533| 181 | 150 (adenocarcinoma)            | 31 (malignant pleural mesothelioma)  |

Since the goal is to predict a dichotomous response, for instance, whether one sample is a tumor or normal tissue, we compare the classification performance of each algorithm. For all three data sets, the features are standardized to zero mean and unit standard deviation. For each dataset, we randomly split the data, within each category, into 70% training set and 30% test set. Different classifiers are fitted on the training set and their misclassification errors are computed on the test set. This whole procedure is repeated 100 times. The averaged misclassification errors (in percentage) as well as their standard deviations of each algorithm are reported in Table 2. Our proposed PC-based LDA classifiers have the smallest misclassification errors over all datasets. Although PCLDA-CF-5 only has the second best performance in Colon and Lung cancer data sets, its performance is very close to that of PCLDA-$\hat{K}$.

Table 2: The averaged misclassification errors (in percentage). The numbers in parentheses are the standard deviations over 100 repetitions.

|            | PCLDA-$\hat{K}$ | PCLDA-CF-5 | DSDA    | PenalizedLDA | PAMR    |
|------------|-----------------|------------|---------|--------------|---------|
| Leukemia   | 3.57 (0.036)    | 3.04 (0.032)| 5.52 (0.044)| 3.91 (0.043) | 4.61 (0.039) |
| Colon      | 16.37 (0.077)   | 18.11 (0.082)| 18.11 (0.07)  | 33.95 (0.086) | 19.00 (0.089) |
| Lung cancer| 0.55 (0.008)    | 0.60 (0.009)| 1.69 (0.017) | 1.80 (0.026) | 0.91 (0.011)  |

8 Extension to multi-class classification

In this section, we discuss how to extend the previously discussed procedure to multi-class classification problems in which $Y$ has $L$ classes, $\mathcal{L} := \{0, 1, \ldots, L - 1\}$, for some positive integer $L \geq 2$, and model (1.3) holds, that is,

$$Z \mid Y = k \sim N_K(\alpha_k, \Sigma_{Z \mid Y}), \quad P(Y = k) = \pi_k, \quad k \in \mathcal{L}. \quad (8.1)$$

In particular, the covariance matrices for the $L$ classes are the same.

For a new point $z \in \mathbb{R}^K$, the oracle Bayes rule assigns it to $k \in \mathcal{L}$ if and only if

$$k = \arg \max_{\ell \in \mathcal{L}} P(Y = \ell \mid Z = z) = \arg \max_{\ell \in \mathcal{L}} \log \frac{P(Z = z, Y = \ell)}{P(Z = z, Y = 0)}$$

$$= \arg \max_{\ell \in \mathcal{L}} \left( z^\top \eta^{(\ell)} + \eta^{(0)}_{\ell} \right) = \arg \max_{\ell \in \mathcal{L}} G_z^{(\ell)(0)}(z) \quad (8.2)$$

where

$$\eta^{(\ell)} = \Sigma_{Z \mid Y}^{-1}(\alpha_\ell - \alpha_0), \quad \eta^{(0)}_{\ell} = -\frac{1}{2}(\alpha_0 + \alpha_\ell)^\top \eta^{(\ell)} + \log \frac{\pi_\ell}{\pi_0}, \quad \forall \ell \in \mathcal{L}. \quad (8.3)$$

Notice that $G_z^{(0)(0)}(z) = 0$ and, for any $\ell \in \mathcal{L} \setminus \{0\}$, the proof of (3.2) reveals that,

$$G_z^{(\ell)(0)}(z) = z^\top \eta^{(\ell)} + \eta^{(0)}_{\ell} = \frac{1}{\pi_0 \pi_\ell [1 - (\alpha_\ell - \alpha_0)^\top \beta^{(\ell)}]} \left( z^\top \beta^{(\ell)} + \beta^{(0)} \right) \quad (8.4)$$
\[ \hat{\beta}_{0}^{(\ell)} = -\frac{1}{2}(\alpha_0 + \alpha_\ell)\top \beta^{(\ell)} + \hat{\pi}_\ell \left( 1 - (\alpha_\ell - \alpha_0)\top \beta^{(\ell)} \right) \log \frac{\hat{\pi}_\ell}{\hat{\pi}_0}. \]

In view of (8.2) and (8.4), for a new point \( x \in \mathbb{R}^p \) and any matrix \( B \in \mathbb{R}^{p \times q} \) with \( 1 \leq q \leq p \), we propose the following multi-class classifier
\[ \hat{g}_x^*(x) = \arg \max_{\ell \in \mathcal{L}} \hat{G}_x^{(\ell)(0)}(x) \]
where \( \hat{G}_x^{(\ell)(0)}(x) = 0 \) and, for any \( \ell \in \mathcal{L} \setminus \{0\} \),
\[ \hat{G}_x^{(\ell)(0)}(x) = \frac{1}{\hat{\pi}_0 \hat{\pi}_\ell [1 - (\hat{\mu}_\ell - \hat{\mu}_0)\top \hat{\theta}^{(\ell)}]} \left( x\top \hat{\theta}^{(\ell)} + \hat{\beta}_0^{(\ell)} \right) \]
with
\[ \hat{\pi}_\ell = \frac{n_\ell}{n_0 + n_\ell}, \]
\[ \hat{\theta}^{(\ell)} = B \left( \Pi_{(n_0 + n_\ell)} X^{(\ell)} B \right)^+ Y^{(\ell)}, \]
\[ \hat{\beta}_0^{(\ell)} = -\frac{1}{2}(\hat{\mu}_0 + \hat{\mu}_\ell)\top \hat{\theta}^{(\ell)} + \hat{\pi}_\ell \left( 1 - (\hat{\mu}_\ell - \hat{\mu}_0)\top \hat{\theta}^{(\ell)} \right) \log \frac{\hat{\pi}_\ell}{\hat{\pi}_0}. \]

Here \( n_\ell \) and \( \hat{\mu}_\ell \) are the non-parametric estimates as (3.6) and both the submatrix \( X^{(\ell)} \in \mathbb{R}^{(n_0 + n_\ell) \times p} \) of \( X \) and the response vector \( Y^{(\ell)} = \{0, 1\}^{(n_0 + n_\ell)} \) correspond to samples with label in \( \{0, \ell\} \). Note that \( Y^{(\ell)} \) is encoded as 1 for observations with label \( \ell \) and 0 otherwise.

To analyze the classifier \( \hat{g}_x^* \) in (8.6), its excess risk depends on
\[ \tilde{r}_1 = \max_{\ell \in \mathcal{L} \setminus \{0\}} \| [\Sigma_Z^{(\ell)}]^{1/2} (A\top \hat{\theta}^{(\ell)} - \beta^{(\ell)}) \|_2, \quad \tilde{r}_2 = \max_{\ell \in \mathcal{L} \setminus \{0\}} \| \hat{\theta}^{(\ell)} \|_2 \]
as well as \( \tilde{r}_3 \) as defined in (4.7). Here \( \Sigma_Z^{(\ell)} := \text{Cov}(Z \mid Y \in \{0, \ell\}) \). Analogous to (4.9), for some constant \( C = C(\gamma) > 0 \), define
\[ \hat{\omega}_n = C \sqrt{\log n} \left( \tilde{r}_1 + \| \Sigma_W \|_{op}^{1/2} \tilde{r}_2 + \tilde{r}_2 \tilde{r}_3 + \sqrt{\frac{L}{n}} \right). \]

For ease of presentation, we also assume there exists some sequence \( \Delta > 0 \) and some absolute constants \( C > c > 0 \) such that
\[ c \frac{\Delta}{\min_{k, \ell \in \mathcal{L}, k \neq \ell} \| \alpha_\ell - \alpha_k \|_{\Sigma_Z}} \leq \max_{k, \ell \in \mathcal{L}, k \neq \ell} \| \alpha_\ell - \alpha_k \|_{\Sigma_Z} \leq C \Delta. \]

The following theorem extends Theorem 7 to multi-class classification by establishing rates of convergence of the excess risk of \( \hat{g}_x^* \) in (8.6) for a general \( B \in \mathbb{R}^{p \times q} \).

**Theorem 12.** Under model (1.1) and (8.1), assume (i) – (iii) and (8.10). Further assume \( c/L \leq \min_{k \in \mathcal{L}} \pi_k \leq \max_{k \in \mathcal{L}} \pi_k \leq C/L \) and \( L \log n \leq c'n \) for some constants \( c, c', C > 0 \). Then, for any sequence \( \omega_n > 0 \) satisfying \((1 + \Delta^2)\omega_n = o(1)\) as \( n \to \infty \), on the event \( \{ \hat{\omega}_n \leq \omega_n \} \), the following holds with probability at least \( 1 - \mathcal{O}(n^{-1}) \) under the law \( \mathbb{P}^D \).
(1) If $\Delta \asymp 1$, then
$$
R_x(\hat{g}_x^*) - R_z^* \lesssim L \omega_n^2.
$$

(2) If $\Delta \to \infty$, then, for some constant $c'' > 0$,
$$
R_x(\hat{g}_x^*) - R_z^* \lesssim L \omega_n^2 \exp \left\{ - \left[ c'' + o(1) \right] \Delta^2 \right\}
$$

(3) If $\Delta = o(1)$, then,
$$
R_x(\hat{g}_x^*) - R_z^* \lesssim L \omega_n \min \left\{ \frac{\omega_n}{\Delta}, 1 \right\}.
$$

Proof. The proof can be found in Appendix A.6.

Condition (8.10) is only assumed to simplify the presentation. It is straightforward to derive results based on our analysis when the separation $\|\alpha_\ell - \alpha_k\|_{\Sigma_x Y}$ is not of the same order for all $\ell, k \in \mathcal{L}$. For the third case, $\Delta = o(1)$, our proof also allows to establish different convergence rates depending on whether or not $\pi_k$ and $\pi_\ell$ are distinct for each $k \neq \ell$, analogous to the last two cases of Theorem 7. However, we opt for the current presentation for succinctness.

Theorem 12 immediately leads to the following corollary for the PC-based classifier that use $B = U_K$ and $B = \tilde{U}_K$. Furthermore, Theorem 8 also ensures that similar guarantees can be obtained for the classifiers in (8.6) that use $B = \hat{U}_K$ and $B = \tilde{U}_K$.

**Corollary 13.** Grant the conditions in Theorem 12. Further assume $\xi \geq C \kappa^2$ for some constant $C > 0$. The conclusion of Theorem 12 holds for the classifier in (8.6) that uses

(1) $B = U_K$ with
$$
\omega_n = \left( \sqrt{\frac{LK \log n}{n}} + \min \{ 1, \Delta \} \sqrt{\frac{T}{\xi^*}} + \sqrt{\frac{\kappa}{\xi^*}} \right) \sqrt{\log n},
$$

(2) $B = \tilde{U}_K$ with
$$
\omega_n = \left( \sqrt{\frac{LK \log n}{n}} + \min \{ 1, \Delta \} \sqrt{\frac{T}{\xi^*}} \right) \sqrt{\log n}.
$$

Proof. See Appendix A.6.3.

**Remark 14.** Multi-class classification problems based on discriminant analysis have been studied, for instance, by Witten and Tibshirani (2011); Clemmensen et al. (2011); Mai et al. (2019); Cai and Zhang (2019b). Theoretical guarantees are only provided in Mai et al. (2019) and Cai and Zhang (2019b) under the classical LDA setting for moderate / large separation scenarios, $\Delta \gtrsim 1$, and for fixed $L$, the number of classes. Our results fully characterize dependence of the excess risk on $L$ and also cover the weak separation case, $\Delta \to 0$. On the other hand, our proposed procedure solves a long standing issue on generalizing regression-based classification methods (Hastie et al., 2009; Izenman, 2008; Mai et al., 2012) in the classical binary LDA setting to handle multi-class classification.

**Remark 15.** The classifier in (8.6) chooses $Y = 0$ as the baseline. In practice, we recommend taking each class as the baseline one at the time and averaging the predicted probabilities. Specifically, it is easy to see that, for any baseline choice $k \in \mathcal{L}$ and for any $\ell \in \mathcal{L}$,
$$
P(Y = \ell \mid Z = z) = \frac{P(Z = z, Y = \ell)}{\sum_{k' \in \mathcal{L}} P(Z = z, Y = k')} = \frac{\exp \left\{ G_z^{(\ell k)}(z) \right\}}{\sum_{k' \in \mathcal{L}} \exp \left\{ G_z^{(k k')}(z) \right\}}
$$
where $G^{(l|k)}(z)$ is defined analogous to (8.2) with $k$ in lieu of 0. Therefore, for any new data point $x \in \mathbb{R}^p$, the averaged version of the classifier in (8.6) is

$$\arg \max_{\ell \in \mathcal{L}} \frac{1}{L} \sum_{k \in \mathcal{L}} \sum_{k' \in \mathcal{L}} \exp \left\{ \hat{G}^{(l|k)}(x) \right\} \sum_{k' \in \mathcal{L}} \exp \left\{ \hat{G}^{(l'|k)}(x) \right\}$$

with $\hat{G}^{(l|k)}(x)$ defined analogous to (8.7). The empirical finite sample performance and theoretical analysis of this classifier will be studied elsewhere.

Appendix

We first provide in Appendix A section-by-section main proofs for the results in Sections 2 – 5 and 8 except Theorem 3. The proof of minimax lower bounds in Theorem 3 is stated separately in Appendix B. Technical lemmas and auxiliary lemmas are collected in Appendices C and D, respectively.

A Main proofs

A.1 Proofs of Section 2

A.1.1 Proof of Lemma 1

We observe that

$$R^*_x := \inf_{g} \mathbb{P}\{g(AZ + W) \neq Y\} \geq \mathbb{E}_W \inf_{g} \mathbb{P}\{g(AZ + W) \neq Y \mid W\} \geq \mathbb{E}_W \inf_{h} \mathbb{P}\{h(Z) \neq Y\} = \inf_{h} \mathbb{P}\{h(Z) \neq Y\} := R^*_z.$$  \hspace{1cm} (A.1)

In the derivation (A.1) above, the infima are taken over all measurable functions $g : \mathbb{R}^p \to \{0, 1\}$ and $h : \mathbb{R}^K \to \{0, 1\}$, and note that the second inequality uses the independence between $W$ and $(Y, Z)$.

A.1.2 Proof of Lemma 2

We define

$$\Delta^2_x := (\alpha_1 - \alpha_0)^\top A^\top (A\Sigma_{Z|Y}A^\top + \Sigma_W)^{-1}A(\alpha_1 - \alpha_0).$$  \hspace{1cm} (A.2)

From standard LDA theory (Izenman, 2008, pp 241-244),

$$R^*_x = 1 - \pi_1 \Phi \left( \frac{\Delta_x}{2} + \log \frac{\pi_1}{\pi_0} \right) - \pi_0 \Phi \left( \frac{\Delta_x}{2} - \log \frac{\pi_1}{\pi_0} \right)$$

which simplifies for $\pi_0 = \pi_1$ to $R^*_x = 1 - \Phi (\Delta_x/2)$. Hence, we have

$$R^*_x - R^*_z = \Phi \left( \frac{\Delta}{2} \right) - \Phi \left( \frac{\Delta_x}{2} \right).$$
Since, by an application of the Woodbury identity,

\[ \Delta^2 - \Delta^2_x = (\alpha_1 - \alpha_0)^\top \left[ \Sigma^{-1/2}_{Z|Y} - A^\top (A\Sigma_{Z|Y} A^\top + \Sigma_{W})^{-1} A \right] (\alpha_1 - \alpha_0) \]

\[ = (\alpha_1 - \alpha_0)^\top \Sigma^{-1/2}_{Z|Y} \left( I_K + \Sigma_{Z|Y} A^\top \Sigma_{W}^{-1/2} A \right) \Sigma^{-1/2}_{Z|Y} (\alpha_1 - \alpha_0) \]  

(A.3)

we have

\[ \Delta \geq \Delta_x, \quad \Delta^2 - \Delta^2_x \leq \frac{\Delta^2}{1 + \lambda_K(H)} \]  

(A.4)

with \( H = \Sigma_{Z|Y} A^\top \Sigma_{W}^{-1/2} A \). Since

\[ \lambda_K(H) \geq \lambda_K(A \Sigma_{Z|Y} A^\top) \]

(5.1)

= \( \xi^* \),

and the function \( x \mapsto x/(1 + x) \) is increasing for \( x > 0 \), we further find that

\[ \Delta^2 \geq \Delta^2_x \geq \Delta^2 \frac{\lambda_K(H)}{1 + \lambda_K(H)} \geq \Delta^2 \frac{\xi^*}{1 + \xi^*}. \]  

(A.5)

Finally, using the mean value theorem, we find

\[ R^*_x - R^*_y \leq \frac{1}{2} (\Delta - \Delta_x) \varphi \left( \frac{\Delta_x}{2} \right) = \frac{1}{2} \Delta^2 - \Delta^2_x \varphi \left( \frac{\Delta_x}{2} \right) \]

\[ \leq \frac{1}{2\sqrt{2\pi}} \cdot \frac{\Delta}{1 + \lambda_K(H)} \exp \left\{ -\Delta^2_x/8 \right\} \]

\[ \leq \frac{1}{2\sqrt{2\pi}} \cdot \frac{\Delta}{1 + \xi^*} \exp \left\{ -\frac{\xi^*}{8(1 + \xi^*)} \Delta^2 \right\}. \]

Our claim of the upper bound thus follows from \( \xi^* \approx \lambda/\sigma^2 \) for any \( \theta \in \Theta(\lambda, \sigma, \Delta) \).

To prove the lower bound of \( R^*_x - R^*_y \), note that

\[ \Delta^2 - \Delta^2_x \geq \frac{\|\alpha_1 - \alpha_0\|_{\Sigma_{Z|Y}}}{} \]

\[ = \frac{\Delta^2}{1 + \lambda_1(H)}. \]

This implies

\[ \Delta^2_x \leq \frac{\lambda_1(H)}{1 + \lambda_1(H)} \Delta^2. \]

Similarly, by the mean value theorem and \( \Delta \geq \Delta_x \) from (A.4),

\[ R^*_x - R^*_y = \Phi \left( \frac{\Delta}{2} \right) - \Phi \left( \frac{\Delta_x}{2} \right) \]

\[ \geq \frac{1}{2} (\Delta - \Delta_x) \varphi \left( \frac{\Delta}{2} \right) = \frac{1}{2} \Delta^2 - \Delta^2_x \varphi \left( \frac{\Delta}{2} \right) \]

\[ \geq \frac{1}{2\sqrt{2\pi}} \cdot \frac{\Delta}{1 + \lambda_1(H)} \exp \left\{ -\Delta^2/8 \right\} \]

\[ \geq \frac{1}{4\sqrt{2\pi}} \cdot \frac{\Delta}{1 + \lambda_1(H)} \exp \left\{ -\Delta^2/8 \right\}. \]

The result follows from this inequality and \( \lambda_1(H) \approx \lambda/\sigma^2 \) for any \( \theta \in \Theta(\lambda, \sigma, \Delta) \).
A.2 Proof of Proposition 4

We prove Proposition 4 by proving the following more general result. Define, for any scalar \( a > 0 \),

\[
\beta^a = a \Sigma_Z^{-1}(\alpha_1 - \alpha_0), \quad (\text{A.6})
\]

\[
\beta_0^a = -\frac{1}{2}(\alpha_0 + \alpha_1)^\top \beta^a + \left[a - \pi_0 \pi_1 (\alpha_1 - \alpha_0)^\top \beta^a\right] \log \frac{\pi_1}{\pi_0}.
\]

**Lemma 14.** Let \( \eta, \eta_0 \) and \( \beta^a, \beta_0^a \) be defined in (1.7) and (A.6), respectively. Under model (1.1) and (1.3) and Assumption (iv), for any \( a > 0 \), we have

\[
z^\top \eta + \eta_0 \geq 0 \iff z^\top \beta^a + \beta_0^a \geq 0.
\]

Furthermore, the parameters \( \beta := \beta^a \) and \( \beta_0 := \beta_0^a \) defined in (A.6) with \( a = \pi_0 \pi_1 \) satisfies

\[
\beta = \Sigma_Z^{-1} \text{Cov}(Z, Y)
\]

and

\[
z^\top \eta + \eta_0 = \frac{1}{\pi_0 \pi_1} \left[1 - (\alpha_1 - \alpha_0)^\top \beta\right] (z^\top \beta + \beta_0). \quad (\text{A.7})
\]

**Proof.** To prove the first statement, write

\[
G^*_z(z) := z^\top \eta + \eta_0 = z^\top \eta - \frac{1}{2}(\alpha_0 + \alpha_1)^\top \eta + \log \frac{\pi_1}{\pi_0}.
\]

It suffices to show that, for any \( a > 0 \),

\[
\eta = a - \pi_0 \pi_1 (\alpha_1 - \alpha_0)^\top \beta^a \quad (\text{A.8})
\]

and

\[
a - \pi_0 \pi_1 (\alpha_1 - \alpha_0)^\top \beta^a > 0. \quad (\text{A.9})
\]

To show (A.9), observe that (see, Fact 1)

\[
\Sigma_Z = \Sigma_{Z|Y} + \pi_0 \pi_1 (\alpha_1 - \alpha_0)(\alpha_1 - \alpha_0)^\top. \quad (\text{A.10})
\]

By the Woodbury formula,

\[
\Sigma_Z^{-1}(\alpha_1 - \alpha_0) = \Sigma_{Z|Y}^{-1}(\alpha_1 - \alpha_0) - \frac{\pi_0 \pi_1 \|\alpha_1 - \alpha_0\|^2_{\Sigma_{Z|Y}}}{1 + \pi_0 \pi_1 \|\alpha_1 - \alpha_0\|^2_{\Sigma_{Z|Y}}} \Sigma_{Z|Y}^{-1}(\alpha_1 - \alpha_0)
\]

\[
= \frac{1}{1 + \pi_0 \pi_1 \Delta^2} \Sigma_{Z|Y}^{-1}(\alpha_1 - \alpha_0). \quad (\text{1.2})
\]

This gives

\[
\|\alpha_1 - \alpha_0\|^2_{\Sigma_Z} = \frac{\Delta^2}{1 + \pi_0 \pi_1 \Delta^2} \quad (\text{A.11})
\]

which implies

\[
1 - \pi_0 \pi_1 \|\alpha_1 - \alpha_0\|^2_{\Sigma_Z} = \frac{1}{1 + \pi_0 \pi_1 \Delta^2} > 0. \quad (\text{A.12})
\]

Hence (A.9) follows as

\[
a - \pi_0 \pi_1 (\alpha_1 - \alpha_0)^\top \beta^a = a \left(1 - \pi_0 \pi_1 \|\alpha_1 - \alpha_0\|^2_{\Sigma_Z}\right) = \frac{a}{1 + \pi_0 \pi_1 \Delta^2}.
\]
We proceed to show \((A.8)\). By using \((A.10)\) and the Woodbury formula again,

\[
\eta = \Sigma_Z^{-1}(\alpha_1 - \alpha_0)
\]

\[
= \Sigma_Z^{-1}(\alpha_1 - \alpha_0) + \frac{\pi_0 \pi_1 \|\alpha_1 - \alpha_0\|^2_{E_Z}}{1 - \pi_0 \pi_1 \|\alpha_1 - \alpha_0\|^2_{E_Z}} \Sigma_Z^{-1}(\alpha_1 - \alpha_0)
\]

\[
= \left[1 + \frac{\pi_0 \pi_1 \|\alpha_1 - \alpha_0\|^2_{E_Z}}{1 - \pi_0 \pi_1 \|\alpha_1 - \alpha_0\|^2_{E_Z}}\right] \frac{\beta^a}{a}
\]

\[
= \frac{1}{1 - \pi_0 \pi_1 \|\alpha_1 - \alpha_0\|^2_{E_Z}} \frac{\beta^a}{a}.
\]

This proves \((A.8)\) and completes the proof of the first statement.

To prove the second statement, by definition and the choice of \(a = \pi_0 \pi_1\),

\[
\beta = a \Sigma_Z^{-1}(\alpha_1 - \alpha_0) = \Sigma_Z^{-1}(\alpha_1 - \alpha_0) \pi_0 \pi_1.
\]

On the other hand,

\[
[Cov(Z)]^{-1}Cov(Z, Y) = \Sigma_Z^{-1}(E[ZY] - E[Z]E[Y]) = \Sigma_Z^{-1}\pi_1(\alpha_1 - \pi_0 \alpha_0 + \pi_1 \alpha_1) = \Sigma_Z^{-1}\pi_0 \pi_1(\alpha_1 - \alpha_0),
\]

proving our claim.

The last statement follows immediately from \((A.8)\) with \(a = \pi_0 \pi_1\).

\[\square\]

A.3 Proofs of Section 4

A.3.1 Proof of Theorem 5

Since \(D = \{X, Y\}\) is independent of \((X, Z, W, Y)\), we treat quantities that are only related with \(D\) fixed throughout the proof. Recall the definitions of \(\hat{G}_x\) and \(G_x\) in \((4.1)\). By definition,

\[
R_x(\hat{g}_z) = \pi_0 P \{\hat{G}_x(X) \geq 0 \mid Y = 0\} + \pi_1 P \{\hat{G}_x(X) < 0 \mid Y = 1\}
\]

and

\[
R^*_x = \pi_0 P \{G_x(Z) \geq 0 \mid Y = 0\} + \pi_1 P \{G_x(Z) < 0 \mid Y = 1\}.
\]

Recall that \(X = AZ + W\) and write \(f_{Z|k}(z)\) for the p.d.f. of \(N_K(\alpha_k, \Sigma_Z|Y)\) at the point \(z \in \mathbb{R}^K\) for \(k \in \{0, 1\}\). We have

\[
R_x(\hat{g}_z) - R^*_x = \pi_0 \mathbb{E}_W \mathbb{E}_Z \left[1 \{\hat{G}_x(AZ + W) \geq 0\} - 1 \{G_x(Z) \geq 0\} \mid Y = 0, W = w\right]
\]

\[
+ \pi_1 \mathbb{E}_W \mathbb{E}_Z \left[1 \{\hat{G}_x(AZ + W) < 0\} - 1 \{G_x(Z) < 0\} \mid Y = 1, W = w\right]
\]

\[
= \mathbb{E}_W \int \left(1 \{\hat{G}_x(Az + w) \geq 0\} - 1 \{G_x(z) \geq 0\}\right) \left(\pi_0 f_{Z|0}(z) - \pi_1 f_{Z|1}(z)\right) dz
\]

\[
= \mathbb{E}_W \int_{G_z \geq 0, G_x < 0} \left(\pi_0 f_{Z|0}(z) - \pi_1 f_{Z|1}(z)\right) dz + \mathbb{E}_W \int_{G_z < 0, G_x \geq 0} \left(\pi_1 f_{Z|1}(z) - \pi_0 f_{Z|0}(z)\right) dz.
\]
The penultimate step uses the assumption that $W$ is independent of both $Z$ and $Y$. Notice that

$$
\pi_0f_{Z|0}(z) - \pi_1f_{Z|1}(z) = \pi_0f_{Z|0}(z) \left[ 1 - \frac{\pi_1 f_{Z|1}(z)}{\pi_0 f_{Z|0}(z)} \right] = \pi_0f_{Z|0}(z) \left( 1 - \exp\{G^*_z(z)\} \right)
$$

with

$$
G^*_z(z) = \log \frac{\pi_1 f_{Z|1}(z)}{\pi_0 f_{Z|0}(z)} = z^\top \eta + \eta_0 = \frac{1 + \pi_0 \pi_1 \Delta^2}{a} G_z(z) := c_s G_z(z)
$$

from Lemma 14 and (A.7). This implies the identity

$$(I) = \pi_0 E_W E_Z \left[ 1 \left\{ \hat{G}_x(AZ + w) \geq 0, G_z(Z) < 0 \right\} \left( 1 - \exp\{G^*_z(z)\} \right) | Y = 0, W = w \right].$$

Define, for any $t \geq 0$, the event

$$E_t := \left\{ |\hat{G}_x(AZ + W) - G_z(Z)| \leq t \right\}.$$ (A.13)

We obtain

$$(I) = \pi_0 E_W E_Z \left[ 1 \left\{ \hat{G}_x(AZ + w) \geq 0, G_z(Z) < 0 \right\} \left( 1 - e^{G^*_z(Z)} \right) 1\{E_t\} | Y = 0, W = w \right]$$

$$+ \pi_0 E_W E_Z \left[ 1 \left\{ \hat{G}_x(AZ + w) \geq 0, G_z(Z) < 0 \right\} \left( 1 - e^{G^*_z(Z)} \right) 1\{E_t^c\} | Y = 0, W = w \right]$$

$$\leq \pi_0 c_s t E_Z \left[ 1 \left\{ -t \leq G_z(Z) < 0 \right\} | Y = 0 \right] + \pi_0 \mathbb{P}(E_t^c | Y = 0).$$

In the last step we use the basic inequality $1 + x \leq \exp(x)$ for all $x \in \mathbb{R}$ together with $-t \leq G_z(Z) < 0$ and $-G^*_z(Z) \leq c_s t$ on the event $\{G_z \geq 0, G_z < 0\} \cap E_t$.

By using analogous arguments and from the identity

$$\pi_1 f_{Z|1}(z) - \pi_0 f_{Z|0}(z) = \pi_1 f_{Z|1}(z) \left( 1 - \exp\{-G^*_z(z)\} \right),$$

we find (II) is equal to

$$\pi_1 E_W E_Z \left[ 1 \left\{ \hat{G}_x(AZ + w) < 0, G_z(Z) \geq 0 \right\} \left( 1 - \exp\{-G^*_z(z)\} \right) 1\{E_t\} | Y = 1, W = w \right]$$

$$+ \pi_1 E_W E_Z \left[ 1 \left\{ \hat{G}_x(AZ + w) < 0, G_z(Z) \geq 0 \right\} \left( 1 - \exp\{-G^*_z(z)\} \right) 1\{E_t^c\} | Y = 1, W = w \right]$$

$$\leq \pi_1 c_s t E_Z \left[ 1 \left\{ -t \leq G_z(Z) < 0 \right\} | Y = 0 \right] + \pi_1 \mathbb{P}(E_t^c | Y = 1).$$

Combining the bounds for (I) and (II) and using $G^*_z(z) = c_s G_z(z)$, we conclude that

$$R_x(g_x) - R^*_x \leq \mathbb{P}(E_t^c) + \pi_0 c_s t \mathbb{P}\{-c_s t < G^*_z(Z) < 0 | Y = 0\}$$

$$\quad + \pi_1 c_s t \mathbb{P}\{0 < G^*_z(Z) < c_s t | Y = 1\}.$$

Using the fact that

$$G^*_z(Z) | Y = 1 \sim N\left( \frac{1}{2} \Delta^2 + \log \frac{\pi_1}{\pi_0}, \Delta^2 \right),$$

$$G^*_z(Z) | Y = 0 \sim N\left( -\frac{1}{2} \Delta^2 + \log \frac{\pi_1}{\pi_0}, \Delta^2 \right),$$

the proof easily follows. □

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A.3.2 Proof of Proposition 6

Recall that

\[ \hat{r}_1 := \| \Sigma_Z^{1/2} (A^\top \hat{\theta} - \beta) \|_2, \quad \hat{r}_2 := \| \hat{\theta} \|_2, \quad \hat{r}_3 := \frac{1}{\sqrt{n}} \| W (P_B - P_A) \|_{\text{op}}. \]

The proof of Proposition 6 consists of two parts: we first show that, for any \( a \geq 1 \), with probability at most \( O(n^{-a}) \),

\[
|\hat{G}_{\hat{\beta}}(X) - G_{\hat{\beta}}(Z)| \geq C \sqrt{a \log n} \left( \hat{r}_1 + \| \Sigma_W \|_{\text{op}}^{1/2} \hat{r}_2 \right) + \left| \hat{\beta}_0 - \beta_0 + \frac{1}{2} (\alpha_1 + \alpha_0)^\top (A^\top \hat{\theta} - \beta) \right|, \tag{A.14}
\]

Notice that randomness of the right-hand side only depends on \( D \). We then prove in Lemma 15 that with probability \( 1 - O(n^{-1}) \),

\[
\left| \hat{\beta}_0 - \beta_0 + \frac{1}{2} (\alpha_1 + \alpha_0)^\top (A^\top \hat{\theta} - \beta) \right| \leq C \left( \hat{r}_1 + \| \Sigma_W \|_{\text{op}}^{1/2} \hat{r}_2 + \hat{r}_2 \hat{r}_3 + \sqrt{\frac{\log n}{n}} \right), \tag{A.15}
\]

which together with (A.14) yields the claim.

To prove (A.14), starting with

\[
\hat{G}_{\hat{\beta}}(X) - G_{\hat{\beta}}(Z) = \left( Z - \frac{\alpha_1 + \alpha_0}{2} \right)^\top (A^\top \hat{\theta} - \beta) + W^\top \hat{\theta} + \hat{\beta}_0 - \beta_0 + \frac{1}{2} (\alpha_1 + \alpha_0)^\top (A^\top \hat{\theta} - \beta),
\]

we observe that \( \hat{\theta} \) and \( \hat{\beta}_0 \) are independent of \( W \) and \( Z \). Since \( W^\top \hat{\theta} \) given \( \hat{\theta} \) is subGaussian with parameter \( \gamma \sqrt{\Sigma_W \hat{\theta} \Sigma_W^\top \hat{\theta}} \leq \gamma \| \Sigma_W \|_{\text{op}}^{1/2} \hat{r}_2 \), we find that, for any \( \alpha > 0 \),

\[
P \left\{ |W^\top \hat{\theta}| \geq \gamma \sqrt{2 \alpha \log n} \| \Sigma_W \|_{\text{op}}^{1/2} \hat{r}_2 \right\} \leq 2n^{-\alpha}. \tag{A.16}
\]

To bound \( (Z - (\alpha_1 + \alpha_0)/2)^\top (A^\top \hat{\theta} - \beta) \), by conditioning on \( Y = 0 \) and \( \hat{\theta} \), and by recalling that \( Z \mid Y = 0 \sim N_K(\alpha_0, \Sigma_Z | Y) \), we have

\[
P \left\{ \left| \left( Z - \frac{1}{2} (\alpha_1 + \alpha_0) \right)^\top (A^\top \hat{\theta} - \beta) \right| \geq M + t \sqrt{V} \mid Y = 0, \hat{\theta} \right\} \leq 2e^{-t^2/2}
\]

for all \( t \geq 0 \), where

\[
M = \frac{1}{2} |(\alpha_1 - \alpha_0)^\top (A^\top \hat{\theta} - \beta)|, \quad V = (A^\top \hat{\theta} - \beta)^\top \Sigma_Z | Y (A^\top \hat{\theta} - \beta).
\]

By the Cauchy-Schwarz inequality and (A.10),

\[
V \leq \| \Sigma_Z^{1/2} \Sigma_Z | Y \Sigma_Z^{1/2} \|_{\text{op}} \| \Sigma_Z^{1/2} (A^\top \hat{\theta} - \beta) \|_2^2 \leq \| \Sigma_Z^{1/2} (A^\top \hat{\theta} - \beta) \|_2^2 = \hat{r}_1^2.
\]

Furthermore, by (A.11), we have

\[
M \leq \frac{1}{2} \| \alpha_1 - \alpha_0 \|_{\Sigma_Z} \| \Sigma_Z^{1/2} (A^\top \hat{\theta} - \beta) \|_2 \leq \| \Sigma_Z^{1/2} (A^\top \hat{\theta} - \beta) \|_2 = \hat{r}_1.
\]
These bounds of $V$ and $M$ yield that, for any $\alpha > 0$,

$$
\mathbb{P}\left\{ \left| \left( Z - \frac{\alpha_1 + \alpha_0}{2} \right)^\top (A^\top \hat{\theta} - \beta) \right| \geq \left( \sqrt{\alpha \log n} + 1 \right) \hat{r}_1 \mid Y = 0 \right\} \leq 2n^{-\alpha}.
$$

By the same arguments, the above also holds by conditioning on $Y = 1$, hence further holds by unconditioning on $Y$. Together with (A.16), the proof of (A.14) is complete by taking $\alpha \geq 1$, concluding the proof of Proposition 6.

\textbf{Lemma 15.} Under conditions of Proposition 6, with probability $1 - \mathcal{O}(n^{-1})$,

$$
\left| \hat{\beta}_0 - \beta_0 + \frac{1}{2}(\alpha_1 + \alpha_0)^\top (A^\top \hat{\theta} - \beta) \right| \leq C \left( \hat{r}_1 + \| \Sigma \|_{op}^{1/2} \hat{r}_2 + \hat{r}_3 \sqrt{\frac{\log n}{n}} \right)
$$

for some constant $C = C(\gamma) > 0$.

\textbf{Proof.} By definition,

$$
\left| \hat{\beta}_0 - \beta_0 + \frac{1}{2}(\alpha_1 + \alpha_0)^\top (A^\top \hat{\theta} - \beta) \right| \leq \frac{1}{2} \left( A\alpha_0 + A\alpha_1 - (\hat{\mu}_0 - \hat{\mu}_1)^\top \hat{\theta} \right)
$$

$$
+ \left[ \hat{\pi}_0 \hat{\pi}_1 \left[ 1 - (\hat{\mu}_1 - \hat{\mu}_0)^\top \hat{\theta} \right] \log \frac{\hat{\pi}_1}{\pi_0} - \pi_0 \hat{\pi}_1 \left[ 1 - (\alpha_1 - \alpha_0)^\top \beta \right] \log \frac{\pi_1}{\pi_0} \right].
$$

We proceed to bound $R_1$ and $R_2$ separately.

\textbf{Bounding $R_1$.} By recalling that, for any $k \in \{0, 1\}$,

$$
\hat{\mu}_k = \frac{1}{n_k} \sum_{i=1}^{n} X_i \mathbb{I}\{Y_i = k\} = \frac{1}{n_k} \sum_{i=1}^{n} (AZ_i + W_i) \mathbb{I}\{Y_i = k\} := A\hat{\alpha}_k + \hat{W}_k,
$$

we have

$$
\left| \alpha_k^\top A^\top \hat{\theta} - \hat{\mu}_k^\top \hat{\theta} \right| \leq \left| (\alpha_k - \hat{\alpha}_k)^\top A^\top \hat{\theta} \right| + \left| \hat{W}_k^\top \hat{\theta} \right|
$$

$$
\leq \left| (\alpha_k - \hat{\alpha}_k)^\top \beta \right| + \left| (\alpha_k - \hat{\alpha}_k)^\top (\beta - A^\top \hat{\theta}) \right| + \left| \hat{W}_k^\top \hat{\theta} \right|
$$

$$
\leq \left| (\alpha_k - \hat{\alpha}_k)^\top \beta \right| + \| \Sigma_Z^{-1/2} (\alpha_k - \hat{\alpha}_k) \|_2 \| \Sigma_Z^{1/2} (\beta - A^\top \hat{\theta}) \|_2
$$

$$
+ \| PA \hat{W}_k \|_2 \| \hat{\theta} \|_2 + \| (P_B - P_A) \hat{W}_k \|_2 \| \hat{\theta} \|_2.
$$

The last step uses

$$
\hat{W}_k^\top \hat{\theta} = \hat{W}_k P_B B (\Pi_n X B)^\top Y = \hat{W}_k (P_A + P_B - P_A) \hat{\theta}
$$

and Cauchy-Schwarz inequality. By invoking Lemma 31 and using

$$
\| \Sigma_Z^{1/2} \beta \|_2 = \pi_0 \pi_1 \| \alpha_1 - \alpha_0 \|_{\Sigma_Z} \overset{(A.11)}{=} \pi_0 \pi_1 \sqrt{\frac{\Delta^2}{\pi_0 \pi_1 \Delta^2}} \leq 1,
$$

(A.18)
from (vi), we further have
\[
\left| (\alpha_k - \hat{\alpha}_k) \right| + \| \Sigma_Z^{-1/2} (\alpha_k - \hat{\alpha}_k) \|_2 \| \Sigma_Z^{1/2} (\beta - A^\top \hat{\theta}) \|_2 \lesssim \sqrt{\frac{\log n}{n}} + \sqrt{\frac{K \log n}{n}} \hat{r}_1
\]
with probability \( 1 - \mathcal{O}(1/n) \). Lemma 30 yields
\[
\mathbb{P}_D \left\{ \frac{p_{\min}}{n} \geq c (\pi_0 \wedge \pi_1) \geq c \pi_0 \pi_1 \right\} \geq 1 - 2n^{-1}. \tag{A.19}
\]
After collecting the above terms and using Fact 1 and \( K \log n \lesssim n \), we obtain
\[
\left| \alpha_k^\top A^\top \hat{\theta} - \hat{\mu}_k^\top \hat{\theta} \right| \lesssim \hat{r}_1 \sqrt{\frac{K \log n}{n}} + \sqrt{\frac{\log n}{n}} \leq (\| P_A \hat{W} \|_2 + \| (P_B - P_A) \hat{W} \|_2) \hat{r}_2
\]
with probability \( 1 - \mathcal{O}(1/n) \). Notice that
\[
\| (P_B - P_A) \hat{W} \|_2^2 = \frac{1}{n_1} \| (P_B - P_A) W Y \|_2^2 \\ \leq \frac{1}{n_1} \| W (P_B - P_A) \|_{\text{op}} \| Y \|_2 \sqrt{n} \\ \lesssim \hat{r}_3
\]
by (A.19) and, similarly,
\[
\| (P_B - P_A) \hat{W} \|_2 \leq \frac{1}{\sqrt{n}} \| W (P_B - P_A) \|_{\text{op}} \| \mathbb{I}(Y = 0) \|_2 \sqrt{n} \lesssim \hat{r}_3
\]
Then use Lemma 32 to obtain
\[
\left( \| P_A \hat{W} \|_2 + \| (P_B - P_A) \hat{W} \|_2 \right) \hat{r}_2 \lesssim \hat{r}_2 \sqrt{\| \Sigma_W \|_{\text{op}} \left( \frac{K \log n}{n} + \hat{r}_2 \hat{r}_3 \right)}
\]
which further implies
\[
R_1 \lesssim \hat{r}_1 \sqrt{\frac{K \log n}{n}} + \sqrt{\frac{\log n}{n}} + \hat{r}_2 \left( \sqrt{\| \Sigma_W \|_{\text{op}} \left( \frac{K \log n}{n} + \hat{r}_3 \right)} \right),
\]
with probability \( 1 - \mathcal{O}(1/n) \). Therefore, with the same probability, we have
\[
\left| (\alpha_0 - \alpha_1) \right| \left( \beta - (\hat{\mu}_0 - \hat{\mu}_1)^\top \hat{\theta} \right) \\ \leq \left| (\alpha_0 - \alpha_1) \right| \left( \beta - A^\top \hat{\theta} \right) + \left| (\alpha_0 - \alpha_1) \right| A^\top \hat{\theta} - (\hat{\mu}_0 - \hat{\mu}_1)^\top \hat{\theta} \\ \leq \| (\alpha_0 - \alpha_1) \|_{\Sigma_Z} \left| \Sigma_Z^{1/2} (\beta - A^\top \hat{\theta}) \right|_2 + \sum_{k \in \{0,1\}} \left| \alpha_k A^\top \hat{\theta} - \hat{\mu}_k^\top \hat{\theta} \right| \\ \lesssim \hat{r}_1 + \sqrt{\frac{\log n}{n}} + \hat{r}_2 \left( \sqrt{\| \Sigma_W \|_{\text{op}} \left( \frac{K \log n}{n} + \hat{r}_3 \right)} \right). \tag{A.20}
\]
In the last step, we also use \( \| \alpha_1 - \alpha_0 \|_{\Sigma_Z} \lesssim 1 \) from Fact 1 and \( K \log n \lesssim n \) to collect terms.

**Bounding \( R_2 \).** We bound from above the following two terms separately:
\[
R_{21} := \left| \hat{\pi}_0 \hat{\pi}_1 (\hat{\mu}_1 - \hat{\mu}_0)^\top \hat{\theta} - \pi_0 \pi_1 (\alpha_1 - \alpha_0)^\top \beta + \pi_0 \pi_1 - \hat{\pi}_0 \hat{\pi}_1 \right| \left| \log \frac{\hat{\pi}_1}{\pi_0} \right|,
\]
\[
R_{22} := \left| \pi_0 \pi_1 - \pi_0 \pi_1 (\alpha_1 - \alpha_0)^\top \beta \right| \left| \log \frac{\hat{\pi}_1}{\pi_0} - \log \frac{\pi_1}{\pi_0} \right|.
\]
We start with
\[ R_{21} \leq \hat{\pi}_0 \hat{\pi}_1 \left| (\hat{\mu}_1 - \hat{\mu}_0)\hat{\theta} - (\alpha_1 - \alpha_0)\beta \right| \cdot \left| \log \frac{\hat{\pi}_1}{\pi_0} \right| \]
\[ + \left| \hat{\pi}_0 \hat{\pi}_1 - \pi_0 \pi_1 \right| \pi_0 \pi_1 \left| \alpha_1 - \alpha_0 \right| \Sigma_{Z} \cdot \left| \log \frac{\hat{\pi}_1}{\pi_0} \right| + \left| \hat{\pi}_0 \hat{\pi}_1 - \pi_0 \pi_1 \right| \cdot \left| \log \frac{\hat{\pi}_1}{\pi_0} \right| \]
\[ \leq \hat{\pi}_0 \hat{\pi}_1 \left| (\hat{\mu}_1 - \hat{\mu}_0)\hat{\theta} - (\alpha_1 - \alpha_0)\beta \right| \cdot \left| \log \frac{\hat{\pi}_1}{\pi_0} \right| \]
\[ + \left| \hat{\pi}_0 - \pi_0 \right| \cdot \left| \log \frac{\hat{\pi}_1}{\pi_0} \right| \pi_0 \pi_1 \left| \alpha_1 - \alpha_0 \right| \Sigma_{Z} + \left| \hat{\pi}_0 - \pi_0 \right| \cdot \left| \log \frac{\hat{\pi}_1}{\pi_0} \right| \]
by using
\[ |\hat{\pi}_0 \hat{\pi}_1 - \pi_0 \pi_1| = |(\hat{\pi}_0 - \pi_0)\hat{\pi}_1 + (\hat{\pi}_1 - \pi_1)\pi_0| = |(\hat{\pi}_0 - \pi_0)(\hat{\pi}_1 - \pi_0)| \leq |\hat{\pi}_0 - \pi_0| \] (A.21)
in the last line. The concavity of \( x \mapsto \log(x) \) implies
\[ \left| \log \frac{\hat{\pi}_1}{\pi_0} \right| \leq \left| \frac{\hat{\pi}_1 - \pi_0}{\pi_1 \wedge \pi_0} \right| \]
and \( \pi_0 \pi_1 \left| \alpha_1 - \alpha_0 \right| \Sigma_{Z} \leq 1 \) follows from (A.11). We invoke the bound (A.20) on \( R_1 \), use Lemma 30, inequality (C.2) and condition (vi) to obtain
\[ \mathbb{P}^D \left\{ R_{21} \leq \tilde{r}_1 + \sqrt{\frac{\log n}{n} + \tilde{r}_2 \| \Sigma W \|_{\text{op}}^2} \sqrt{\frac{K \log n}{n} + \tilde{r}_2 \tilde{r}_3} \right\} \geq 1 - cn^{-1}. \]

To bound \( R_{22} \), notice from (A.12) that
\[ \pi_0 \pi_1 - \pi_0 \pi_1 (\alpha_1 - \alpha_0)\beta = \pi_0 \pi_1 \left[ 1 - \pi_0 \pi_1 \left| \alpha_1 - \alpha_0 \right| \Sigma_{Z} \right] = \frac{\pi_0 \pi_1}{1 + \pi_0 \pi_1 \Delta^2}. \]
Use
\[ \left| \log \frac{\hat{\pi}_1}{\pi_0} - \log \frac{\pi_1}{\pi_0} \right| \leq \left| \frac{\hat{\pi}_1 - \pi_1}{\pi_0} \right| \cdot \frac{\pi_0 \sqrt{\hat{\pi}_0 \hat{\pi}_1}}{\pi_0 \pi_1} \]
\[ \leq \max \left\{ \left| \frac{\hat{\pi}_1 \pi_0 - \pi_1 \hat{\pi}_0}{\pi_0 \pi_1} \right|, \left| \frac{\hat{\pi}_1 \pi_0 - \pi_1 \hat{\pi}_0}{\pi_0 \hat{\pi}_1} \right| \right\} \]
and
\[ |\hat{\pi}_1 \pi_0 - \hat{\pi}_1 \hat{\pi}_0| \leq |\hat{\pi}_1 - \pi_1| \pi_0 + |\pi_1| \hat{\pi}_0 - \hat{\pi}_0| \]

\[ R_{22} \leq \frac{\pi_0 \pi_1}{1 + \pi_0 \pi_1 \Delta^2} \left( \sqrt{\frac{\pi_0}{\pi_1} + \sqrt{\frac{\pi_1}{\pi_0}}} \right) \sqrt{\frac{\log n}{n}} \leq \sqrt{\frac{\log n}{n}} \]
with probability \( 1 - O(1/n) \). Combining the bounds of \( R_1, R_{21} \) and \( R_{22} \) yields the desired result. \qed
A.3.3 Proof of Theorem 7

On the event \( \{ \hat{w}_n(a) \leq \omega_n \} \), we use the result of Theorem 5 with \( t = \omega_n \). It remains to bound from above

\[
T := \pi_0 c_s \omega_n [\Phi(R) - \Phi(R - c_s \omega_n / \Delta)] + \pi_1 c_s \omega_n [\Phi(L + c_s \omega_n / \Delta) - \Phi(L)]
\]

(A.22)

with

\[
c_s = \frac{1}{\pi_0 \pi_1} + \Delta^2, \quad L = -\frac{1}{2} \Delta - \frac{\log \pi_1}{\Delta}, \quad R = \frac{1}{2} \Delta - \frac{\log \pi_0}{\Delta}.
\]

By the mean-value theorem, we have the bound

\[
T \leq \frac{c_s^2 \omega_n^2}{\Delta} \exp(-m^2/2) \quad \text{with} \quad m \in \left[ L, L + \frac{c_s \omega_n}{\Delta} \right] \cup \left[ R - \frac{c_s \omega_n}{\Delta}, R \right].
\]

We consider three scenarios:

1. \( \Delta \asymp 1 \). In this case, \( c_s \asymp 1 \) and \( m \asymp 1 \), so that

\[
T \lesssim \omega_n^2.
\]

2. \( \Delta \rightarrow \infty \). In this case, \( c_s \asymp \Delta^2 \), \( c_s \omega_n / \Delta \asymp \omega_n \Delta = o(\Delta) \), whence \( m^2 = c_s \Delta^2 + o(\Delta^2) \) with \( c_s = 1/8 \) if \( \pi_0 = \pi_1 \), and

\[
T \lesssim \omega_n^2 \Delta^3 \exp \left[ -c_s \Delta^2 + o(\Delta^2) \right] = \omega_n^2 \Delta \exp \left[ -c_s \Delta^2 + o(\Delta^2) \right].
\]

(3a) \( \Delta \rightarrow 0 \) and \( \pi_1 \) and \( \pi_0 \) are distinct. In this case \( c_s \asymp 1 \), \( L = -\log(\pi_1/\pi_0)/\Delta + o(1) \), \( R = -\log(\pi_1/\pi_0)/\Delta + o(1) \), \( c_s \omega_n / \Delta \asymp \omega_n / \Delta = o(1/\Delta) \), whence \( m = -\log(\pi_1/\pi_0)/\Delta + o(1/\Delta) \) and

\[
T \lesssim \frac{\omega_n^2}{\Delta} \exp \left[ -\frac{\log(\pi_1/\pi_0)}{\Delta^2} + o \left( \frac{1}{\Delta^2} \right) \right] = \omega_n^2 \exp \left[ -\frac{\log(\pi_1/\pi_0)}{\Delta^2} + o \left( \frac{1}{\Delta^2} \right) \right].
\]

(3b) \( \Delta \rightarrow 0 \) and \( \pi_0 = \pi_1 \). In this case, \( c_s \asymp 1 \), \( L = -\Delta/2 = -R \). Thus

\[
T \lesssim \frac{\omega_n^2}{\Delta}.
\]

The second bound \( T \lesssim \omega_n \) follows directly from (A.22).

In view of the above three cases, on the event \( \{ \hat{w}_n(a) \leq \omega_n \} \), the proof is complete by invoking Theorem 5.

\[ \square \]

A.4 A general tool of bounding \( \hat{r}_1 \) and \( \hat{r}_2 \) for a generic choice \( B = \hat{A} \)

In this section, we provide a general result of establishing \( \hat{r}_1, \hat{r}_2 \) and \( \hat{r}_3 \) for \( \hat{B} \) in (3.4) with \( B = \hat{A} \). Here \( \hat{A} \in \mathbb{R}^{p \times q} \) is any matrix and its dimension \( q \) is also allowed to be random. Recall that \( P_{\hat{A}} \) is the projection matrix onto the column space of \( \hat{A} \) and \( P_{\hat{A}} = I_p - P_{\hat{A}} \). Define

\[
\hat{\psi} = \frac{1}{n} \sigma^2 (\Pi_n X P_{\hat{A}}), \quad \hat{\eta} = \frac{1}{n} \sigma^2 (\Pi_n X P_{\hat{A}}).
\]

For any \( \hat{A} \), the following theorem bounds \( \hat{r}_1 = \| \Sigma_{2}^{1/2} (A^T \hat{B} - \beta) \|_2 \) and \( \hat{r}_2 = \| \hat{B} \|_2 \) in terms of the two quantities above as well as \( \hat{r}_3 \) and

\[
\hat{\zeta} := \left( \frac{1}{n} \left\| (P_{\hat{A}} - P_{A}) W^T \Pi_n Y \right\|_2 \right)^2.
\]

(A.23)
**Theorem 16.** Under model (1.1) and (1.3), assume (i) – (vi) and \( K \log n \leq cn \) for some sufficiently small constant \( c > 0 \). With probability \( 1 - cn^{-1} \), we have

\[
\hat{r}_1 \lesssim \sqrt{\frac{K \log n}{n} + \sqrt{\frac{\|\Sigma W\|_{op}}{\lambda K} + \sqrt{\frac{\|\Sigma W\|_{op} + \hat{\tau}_3^2}{\bar{\eta}}} + \| P_{\hat{A}} - P_A \|_{op} \sqrt{\hat{\psi}}}},
\]

and

\[
\hat{r}_2 \lesssim \sqrt{\frac{\tau}{\eta}} \min \left\{ 1, \left( \sqrt{\frac{K \log n}{n}} + \min\{1, \Delta\} \right) \hat{\phi} + \sqrt{\frac{\hat{\tau}}{\eta}} \right\}
\]

where

\[
(\hat{\phi})^2 = 1 + \frac{\|\Sigma W\|_{op} + \hat{\tau}_3^2}{\eta} + \sqrt{\frac{\lambda_1(A \Sigma Z A^\top)}{\eta}} \left( \sqrt{\frac{\|\Sigma W\|_{op}}{\eta} \frac{K \log n}{n}} + \hat{r}_3 \sqrt{\kappa} + \frac{\|\Sigma W\|_{op} + \hat{\tau}_3^2}{\sqrt{\lambda K}} \right) \hat{r}_2.
\]  

(A.24)

The following theorem states an alternative bound of \( \hat{r}_1 \) which is potentially smaller than that in Theorem 16. It requires the following event,

\[
E_A = \left\{ \sqrt{\kappa} \left\| P_{\hat{A}} - P_A \right\|_{op} \leq e'' \right\}
\]

where \( e'' > 0 \) is some sufficiently small constant.

**Theorem 17.** Under conditions of Theorem 16, assume \( \|\Sigma W\|_{op} \leq c' \lambda K \) for some \( c' > 0 \). Then, on the event \( E_A \), the following holds with probability \( 1 - cn^{-1} \),

\[
\hat{r}_1 \lesssim \sqrt{\frac{K \log n}{n}} + \sqrt{\frac{\hat{\tau}}{\lambda K}} + \left( \sqrt{\kappa} \frac{\|\Sigma W\|_{op}}{\eta} \frac{K \log n}{n} + \hat{r}_3 \sqrt{\kappa} + \frac{\|\Sigma W\|_{op} + \hat{\tau}_3^2}{\sqrt{\lambda K}} \right) \hat{r}_2.
\]

We prove Theorem 16 below and the proof of Theorem 17 follows from the same argument in conjunction with Lemma 20.

**Proof of Theorem 16.** By definition,

\[
A^\top \hat{\theta}_{\hat{A}} = A^\top \hat{A}(\Pi_n X \hat{A})^\top Y.
\]

Let \( \hat{Z} = Z \Sigma Z^{-1/2} \). Define the event

\[
E_z := \left\{ c \leq \frac{1}{n} \lambda_K(\hat{Z}^\top \Pi_n \hat{Z}) \leq \frac{1}{n} \lambda_1(\hat{Z}^\top \Pi_n \hat{Z}) \leq C \right\}.
\]  

(A.26)

According to part (vi) of Lemma 31, \( \mathbb{P}(E_z) \geq 1 - c'n^{-1} \). Using \( (\Pi_n Z)^\top Z = (Z^\top \Pi_n Z)^{-1} Z^\top \Pi_n Z = I_K \) and \( X = Z A^\top + W \), we find

\[
A^\top \hat{\theta}_{\hat{A}} = (\Pi_n Z)^\top Z A^\top \hat{A}(\Pi_n X \hat{A})^\top Y
\]

\[
= (\Pi_n Z)^\top X \hat{A}(\Pi_n X \hat{A})^\top Y - (\Pi_n Z)^\top W \hat{A}(\Pi_n X \hat{A})^\top Y
\]

\[
= (\Pi_n Z)^\top Y - (\Pi_n Z)^\top P_{\Pi_n X \hat{A}}^\perp Y - (\Pi_n Z)^\top W \hat{A}(\Pi_n X \hat{A})^\top Y.
\]

It follows that

\[
\Sigma_{\hat{Z}}^{1/2}(A^\top \hat{\theta}_{\hat{A}} - \beta) = (\Pi_n \hat{Z})^\top Y - \Sigma_{\hat{Z}}^{1/2} \beta - (\Pi_n \hat{Z})^\top P_{\Pi_n X \hat{A}}^\perp Y - (\Pi_n \hat{Z})^\top W \hat{A}(\Pi_n X \hat{A})^\top Y
\]

where we also used \( \Sigma_{\hat{Z}}^{-1/2}(\Pi_n \hat{Z})^\top = (\Pi_n \hat{Z})^\top \). The first result now follows after invoking Lemmas 18 and 19 and the basic inequality \((a + b + c)^2 \leq 3(a^2 + b^2 + c^2)\). The second result is proved in Lemma 19. \( \square \)
A.4.1 Key lemmas used in the proof of Theorems 16 & 17

Lemma 18. Under conditions of Theorem 16, with probability $1 - cn^{-1}$,

$$\left\| (\Pi_n \tilde{Z})^\top Y - \Sigma_Z^{1/2} \beta \right\|_2 \lesssim \sqrt{\frac{K \log n}{n}}.$$  

Proof. Write $\tilde{\alpha} := E[Z]$ and 

$$\tilde{\beta} := (\Pi_n \tilde{Z})^\top Y$$

such that 

$$\frac{1}{n} \tilde{Z}^\top \Pi_n \tilde{Z} = \frac{1}{n} \tilde{Z}^\top \Pi_n Y = \frac{1}{n} \sum_{i=1}^n \left( \tilde{Z}_i - \tilde{\alpha} \right) \mathbb{1}\{-Y_i = 1\} = \frac{1}{n} \sum_{i=1}^n \left( \tilde{Z}_i - \Sigma_Z^{-1/2} \tilde{\alpha} \right) \mathbb{1}\{-Y_i = 1\} + \frac{1}{n} \sum_{i=1}^n \left( \tilde{Z}_i - \Sigma_Z^{-1/2} \tilde{\alpha} \right) \mathbb{1}\{Y_i = 1\}.$$  

Define 

$$\tilde{\alpha}_k := \frac{1}{n_k} \sum_{i=1}^n \mathbb{1}\{Y_i = k\} Z_i, \quad k \in \{0, 1\}. \quad \text{(A.27)}$$

By the identity 

$$\Sigma_Z^{-1/2} (\tilde{\alpha}_1 - \tilde{\alpha}_0) = \frac{1}{n_1} \sum_{i=1}^n \mathbb{1}\{Y_i = 1\} \tilde{Z}_i - \frac{1}{n_0} \sum_{i=1}^n \mathbb{1}\{Y_i = 0\} \tilde{Z}_i = \frac{1}{n_0 n_1} \sum_{i=1}^n \mathbb{1}\{Y_i = 1\} \tilde{Z}_i - \frac{1}{n_0} \sum_{i=1}^n \tilde{Z}_i = \frac{1}{n_0 n_1} \sum_{i=1}^n \mathbb{1}\{Y_i = 1\} (\tilde{Z}_i - \Sigma_Z^{-1/2} \tilde{\alpha}_0) - \frac{1}{n_0} \sum_{i=1}^n (\tilde{Z}_i - \Sigma_Z^{-1/2} \tilde{\alpha}),$$

we obtain 

$$\frac{1}{n} \tilde{Z}^\top \Pi_n \tilde{Z} = n_0 n_1 \Sigma_Z^{-1/2} (\tilde{\alpha}_1 - \tilde{\alpha}_0) + \frac{2n_1}{n^2} \sum_{i=1}^n (\tilde{Z}_i - \Sigma_Z^{-1/2} \tilde{\alpha}).$$

Recall that $\beta^a$ is defined in (A.6). Since 

$$\tilde{\alpha} = \tilde{\pi}_0 \tilde{\pi}_1 = \frac{n_0 n_1}{n^2}, \quad a = \pi_0 \pi_1,$$

we have 

$$\frac{1}{n} Z^\top \Pi_n Z \left( \tilde{\beta} - \Sigma_Z^{1/2} \beta \right) = \tilde{\alpha} \Sigma_Z^{-1/2} (\tilde{\alpha}_1 - \tilde{\alpha}_0) - \frac{1}{n} \tilde{Z}^\top \Pi_n \tilde{Z} \Sigma_Z^{1/2} \beta \tilde{\alpha} + \frac{1}{n} \tilde{Z}^\top \Pi_n \tilde{Z} \Sigma_Z^{1/2} (\tilde{\beta} - \beta) + \frac{2n_1}{n^2} \sum_{i=1}^n (\tilde{Z}_i - \Sigma_Z^{-1/2} \tilde{\alpha})$$

$$= \tilde{\alpha} \Sigma_Z^{-1/2} (\tilde{\alpha}_1 - \alpha_1 - \tilde{\alpha}_0 + \alpha_0) + \tilde{\alpha} \left( I_K - \frac{1}{n} \tilde{Z}^\top \Pi_n \tilde{Z} \right) \Sigma_Z^{-1/2} (\alpha_1 - \alpha_0) + (\tilde{\alpha} - a) \frac{1}{n} \tilde{Z}^\top \Pi_n \tilde{Z} \Sigma_Z^{-1/2} (\alpha_1 - \alpha_0)$$

$$+ \frac{2n_1}{n^2} \sum_{i=1}^n (\tilde{Z}_i - \Sigma_Z^{-1/2} \tilde{\alpha}).$$
On the event $\mathcal{E}_z$, we find
\[
\begin{align*}
&c \left\| \bar{\beta} - \bar{\Sigma}_Z^{1/2} \beta \right\|_2 \\
&\leq \hat{a} \left( \left\| \Sigma_Z^{-1/2}(\alpha_1 - \alpha_1) \right\|_2 + \left\| \Sigma_Z^{-1/2}(\alpha_0 - \alpha_0) \right\|_2 \right) + C|\hat{a} - a|\left\| \alpha_1 - \alpha_0 \right\| \Sigma_Z \\
&+ \hat{a} \left\| \Pi_K \frac{1}{n} \tilde{Z}^\top \Pi_n \tilde{Z} \right\|_{op} \left\| \alpha_1 - \alpha_0 \right\| \Sigma_Z + \frac{n_1}{n^2} \left\| \sum_{i=1}^n (\tilde{Z}_i - \Sigma_Z^{-1/2} \tilde{\alpha}) \right\|_2.
\end{align*}
\]

By invoking Lemmas 30 and 31 together with (A.21), we have
\[
\begin{align*}
c \left\| \bar{\beta} - \bar{\Sigma}_Z^{1/2} \beta \right\|_2 &\leq a \sqrt{\frac{K \log n}{n_{\min}}} + \pi_1 \sqrt{\frac{K \log n}{n}} + \left\| \alpha_1 - \alpha_0 \right\| \Sigma_Z \sqrt{\frac{\pi_0 \pi_1 \log n}{n}} \\
&+ a \left\| \alpha_1 - \alpha_0 \right\| \Sigma_Z \left( \sqrt{\frac{K \log n}{n}} + \sqrt{\frac{\log n}{n}} \left\| \alpha_1 - \alpha_0 \right\| \Sigma_Z \right),
\end{align*}
\]
with probability $1 - c'n^{-1}$. The result follows after using (A.18) and (A.19). \qed

Lemma 19. Suppose $K \log n \leq cn$ for some constant $c > 0$. On the event $\mathcal{E}_z$, the following inequalities hold with probability $1 - c'n^{-1}$ for any $\hat{A}$,
\[
\begin{align*}
\left\| (\Pi_n \tilde{Z})^+ P_{\Pi_n, \hat{A}} Y \right\|_2 &\leq \left\| P_{\hat{A}, \Pi_n} Y \right\|_{op} \sqrt{\frac{\psi}{\lambda K}} + \sqrt{\frac{\Sigma W_{\Pi_n, \hat{A}} Y}{\lambda K}}, \\
\left\| (\Pi_n \tilde{Z})^+ W \hat{A} (\Pi_n, \hat{A})^+ Y \right\|_2 &\leq \sqrt{\frac{\Sigma W_{\Pi_n, \hat{A}} Y}{\lambda K} + \frac{\bar{s}_2^2}{\eta}}, \\
\left\| \theta_{\hat{A}} \right\|_2 &\leq \sqrt{\frac{1}{\eta}} \min \left\{ 1, \left( \sqrt{\frac{K \log n}{n}} + \min\{1, \Delta\} \right) \hat{\phi} + \sqrt{\frac{\zeta}{\eta}} \right\}.
\end{align*}
\]

Proof. We start by proving the first three claims. By definition, for any $Q \in \mathbb{R}^{q \times K}$,
\[
(\Pi_n \tilde{Z})^+ P_{\Pi_n, \hat{A}}^\perp Y = \Sigma_Z^{1/2} (Z^\top \Pi_n Z)^+ (Z^\top \Pi_n \hat{A})^+ Y
\]
\[
= \Sigma_Z^{1/2} (Z^\top \Pi_n Z)^+ (Z - X \hat{A} Q)^+ \Pi_n P_{\Pi_n, \hat{A}}^\perp Y
\]
\[
= \Sigma_Z^{1/2} (Z^\top \Pi_n Z)^+ (Z A A^\top - X \hat{A} Q)^+ \Pi_n P_{\Pi_n, \hat{A}}^\perp Y
\]
\[
= \Sigma_Z^{1/2} (Z^\top \Pi_n Z)^+ (X A^\top - X \hat{A} Q)^\top \Pi_n P_{\Pi_n, \hat{A}}^\perp Y
\]
where the last equality uses $X = Z A^\top + W$. Take $Q = \hat{A}^+ A^\top$ to obtain
\[
(\Pi_n \tilde{Z})^+ P_{\Pi_n, \hat{A}}^\perp Y = I + \Pi
\]
(A.28)
where
\[
I := \Sigma_Z^{1/2} (Z^\top \Pi_n Z)^+ A^+ (\Pi_n X P_{\hat{A}}^\perp)\top P_{\Pi_n, \hat{A}}^\perp Y,
\]
\[
\Pi := -\Sigma_Z^{1/2} (Z^\top \Pi_n Z)^+ A^+ W^\top \Pi_n P_{\Pi_n, \hat{A}}^\perp Y.
\]
On the event $\mathcal{E}_z$, by using $A^+ P_{\tilde{A}} \perp (A^T A)^{-1} A^T (P_{\tilde{A}}^\perp - P_{\hat{A}}^\perp)$, we further have

$$I \leq c^{-2} \frac{1}{n} \left\| \Sigma_{Z}^{-1/2} A^+ (P_{\tilde{A}}^\perp - P_{\hat{A}}^\perp) \right\|_{\text{op}} \left( \frac{1}{\sqrt{n}} \left\| \Pi_n X P_{\hat{A}}^\perp \right\|_{\text{op}} \right)^2 \leq c^{-2} \|\Sigma_{Z}^{-1/2} A^+\|_{\text{op}} \left\| P_{\tilde{A}} - P_A \right\|_{\text{op}} \frac{1}{\sqrt{n}} \left\| \Pi_n X P_{\hat{A}}^\perp \right\|_{\text{op}} \left( \frac{1}{\sqrt{n}} \left\| \Pi_n X \hat{A} Y \right\|_{2} \right)^2.$$

Note that

$$\|\Sigma_{Z}^{-1/2} A^+\|_{\text{op}}^2 = \|\Sigma_{Z}^{-1/2} A^+ A \Sigma_{Z}^{-1/2}\|_{\text{op}} = \frac{1}{\lambda_K (A \Sigma_{Z} A^T)} \leq \frac{1}{\lambda_K (A \Sigma_{Z} Y A^T)}$$

and

$$\frac{1}{\sqrt{n}} \left\| \Pi_n X \hat{A} Y \right\|_{2} \leq \frac{1}{\sqrt{n}} \left\| Y \right\|_{2} = \sqrt{\frac{m_1}{n}}. \tag{A.29}$$

We conclude

$$I \leq c^{-2} \frac{1}{n} \|P_{\tilde{A}} - P_A\|_{\text{op}}.$$

Regarding $\Pi$, one similarly has

$$\Pi \leq c^{-2} \frac{1}{n} \|\Sigma_{Z}^{-1/2} A^+ \Pi_n \|_{\text{op}} \frac{1}{\sqrt{n}} \left\| A^\top \Sigma_{Z}^{-1/2}\right\|_{\text{op}} \leq \sqrt{\frac{\|\Sigma_{W}\|_{\text{op}}}{\lambda_K}},$$

with probability $1 - \mathcal{O}(n^{-1})$. The last step invokes Lemma 32. The first claim follows after we combine these bounds of $I$ and $\Pi$.

To prove the second claim. From $(A.29)$ and $n_1 \leq n$, we find

$$\mathbb{P}^D \left\{ \left\| \Pi_n \hat{Z} \right\|_{\text{op}} + \left\| W A^\top \Pi_n X \hat{A} + Y \right\|_{2} \leq \|M\|_{\text{op}} \right\} \cap \mathcal{E}_z \geq 1 - cn^{-1},$$

where $M = (\Pi_n X \hat{A} )^\top \hat{W} \hat{A} \Pi_n X \hat{A}$. Adding and subtracting $P_A$ gives

$$\|W \hat{A} \Pi_n X \hat{A}\|_{\text{op}} \leq \|WP_A \hat{A} \Pi_n X \hat{A}\|_{\text{op}} + \|W (P_{\tilde{A}} - P_A) \hat{A} \Pi_n X \hat{A}\|_{\text{op}} \leq (\|WP_A\|_{\text{op}} + \|W (P_{\tilde{A}} - P_A)\|_{\text{op}}) \|\hat{A} \Pi_n X \hat{A}\|_{\text{op}}.$$

To bound from above $\|\hat{A} \Pi_n X \hat{A}\|_{\text{op}}$, write the SVD of $\hat{A}$ as $\hat{A} = UDV^\top$ where $U \in \mathbb{R}^{p \times r_0}$ and $V \in \mathbb{R}^{r \times r_0}$ are orthogonal matrices with $r_0 = \text{rank}(\hat{A})$. We have

$$\|\hat{A} \Pi_n X \hat{A}\|_{\text{op}}^2 \leq \left\| \hat{A} \right\|_{\text{op}} \left\| \Pi_n X \right\|_{\text{op}} \leq \left\| \hat{A} \right\|_{\text{op}} \left\| \Pi_n X \hat{A}\right\|_{\text{op}} \leq \left\| \hat{A} \right\|_{\text{op}} \left\| \Pi_n X \right\|_{\text{op}} \leq \sigma_F^{-2} \left\| \Pi_n X \right\|_{\text{op}} \leq (\tilde{m})^{-1} \tag{A.30}$$
where we used $\|FF^\top\|_{\text{op}} = \|F^\top F\|_{\text{op}}$ for any matrix $F$ in (i), $\text{rank}(\Pi_n X U) = \text{rank}(\Pi_n X P_\tilde{A}) = \tilde{r}$ in (ii) and

$$\sigma_2^2(\Pi_n X U) = \lambda_{\text{F}}(\Pi_n X U U^\top X \Pi_n) = \lambda_{\text{F}}(\Pi_n X P_\tilde{A}^2 X \Pi_n) = \sigma_2^2(\Pi_n X P_\tilde{A})$$

in (iii). By collecting terms, the second result follows from (A.23) and Lemma 32.

Regarding the third result, the bound $\sqrt{1/\eta}$ follows immediately from

$$\|\hat{A}(\Pi_n X \hat{A})^+ Y\|_2 \leq \|\hat{A}(\Pi_n X \hat{A})^+\|_{\text{op}} \|Y\|_2 \leq \sqrt{n} \|\hat{A}(\Pi_n X \hat{A})^+\|_{\text{op}}$$

and (A.30). To prove the other bound, start with

$$\left\| \hat{A}(\Pi_n X \hat{A})^+ Y \right\|_2 \leq \left\| \hat{A}(\hat{A}^\top X^\top \Pi_n X \hat{A})^+ \hat{A}^\top A \Sigma Z^1/2 \right\|_{\text{op}} \left\| \Sigma Z^{-1/2} \hat{A}^\top \Pi_n Y \right\|_2$$

$$+ \left( \left\| \hat{A}(\hat{A}^\top X^\top \Pi_n X \hat{A})^+ \hat{A}^\top \right\|_{\text{op}} \left\| P_\tilde{A} W^\top \Pi_n Y \right\|_2 \right).$$

Display (A.30) and Lemma 33 ensure that, with probability $1 - O(n^{-1})$, the second term on the right hand side is bounded by

$$\frac{1}{\eta} \left( \sqrt{\|\Sigma W\|_{\text{op}}} \sqrt{\frac{K \log n}{n}} + \sqrt{\tilde{\zeta}} \right). \quad \text{(A.31)}$$

For the first term, we have, on $\mathcal{E}_z$, with probability $1 - O(n^{-1})$,

$$\frac{1}{n} \left\| \Sigma Z^{-1/2} \Pi_n Y \right\|_2 \leq \left\| (\Pi_n \tilde{Z})^+ Y \right\|_2$$

$$\leq \left\| (\Pi_n \tilde{Z})^+ Y - \Sigma Z^{-1/2} \beta \right\|_2 + \left\| \Sigma Z^{-1/2} \beta \right\|_2$$

$$\leq \sqrt{\frac{K \log n}{n}} + \min\{1, \Delta\} \quad \text{(A.32)}$$

where the last step uses Lemma 18 and $\|\Sigma Z^{-1/2} \beta\|^2 = \pi_0 \pi_1 \|\alpha_1 - \alpha_0\|^2_{\Sigma Z}$ together with (A.11). It remains to bound $n^2 \|\hat{A}(\hat{A}^\top X^\top \Pi_n X \hat{A})^+ \hat{A}^\top A \Sigma Z^1/2\|_{\text{op}}$ which is less than

$$n^2 \left\| \hat{A}(\hat{A}^\top X^\top \Pi_n X \hat{A})^+ \hat{A}^\top \right\|_{\text{op}} \left\| \Sigma Z^{1/2} \hat{A}(\hat{A}^\top X^\top \Pi_n X \hat{A})^+ \hat{A}^\top A \Sigma Z^{1/2} \right\|_{\text{op}}$$

$$\leq \frac{n}{\eta} \left\| \Sigma Z^{1/2} \hat{A}(\hat{A}^\top X^\top \Pi_n X \hat{A})^+ \hat{A}^\top A \Sigma Z^{1/2} \right\|_{\text{op}}$$

by (A.30).

Observe that, on the event $\mathcal{E}_z$,

$$n \left\| \Sigma Z^{1/2} \hat{A}(\hat{A}^\top X^\top \Pi_n X \hat{A})^+ \hat{A}^\top A \Sigma Z^{1/2} \right\|_{\text{op}}$$

$$\leq \left\| (\hat{Z}^\top \Pi_n \tilde{Z})^{1/2} \Sigma Z^{1/2} \hat{A}(\hat{A}^\top X^\top \Pi_n X \hat{A})^+ \hat{A}^\top A \Sigma Z^{1/2} (\hat{Z}^\top \Pi_n \tilde{Z})^{1/2} \right\|_{\text{op}}$$

$$= \left\| (\hat{A}^\top X^\top \Pi_n X \hat{A})^{1/2} \hat{A}^\top \Pi_n \tilde{Z} A \hat{A}^\top \left[ (\hat{A}^\top X^\top \Pi_n X \hat{A})^{1/2} \right] \right\|_{\text{op}}$$

$$+ 2 \left\| (\hat{A}^\top X^\top \Pi_n X \hat{A})^{1/2} \hat{A}^\top W^\top \Pi_n Z A \hat{A}^\top \left[ (\hat{A}^\top X^\top \Pi_n X \hat{A})^{1/2} \right] \right\|_{\text{op}}$$

$$\leq 1 + \left( \|W P_\tilde{A}\|^2_{\text{op}} + 2 \sqrt{\lambda_1(A \Sigma Z A^\top)} \|P_\tilde{A} W^\top \Pi_n \tilde{Z}\|_{\text{op}} \right) \|\hat{A}(\hat{A}^\top X^\top \Pi_n X \hat{A})^+ \hat{A}^\top \|_{\text{op}}$$

$$\leq 1 + \frac{1}{\eta} \left[ \|\Sigma W\|_{\text{op}} + n \tilde{r}_3^2 + \sqrt{\lambda_1(A \Sigma Z A^\top)} \sqrt{\frac{K \log n}{n}} + \tilde{r}_3 \right] \quad \text{(A.33)}$$
The last step invokes Lemma 32 together with (A.30). Combining (A.31) with (A.32) and (A.33) yields the desired result.

On the event \( \mathcal{E}_A \) defined in (A.25), the following lemma states potentially faster rates of the quantities analyzed in Lemma 19. Recall that (A.23).

**Lemma 20.** Suppose \( K \log n \leq cn \) for some constant \( c > 0 \). On the event \( \mathcal{E}_z \cap \mathcal{E}_A \), the following inequalities hold with probability \( 1 - c'n^{-1} \) for any \( \hat{A} \)

\[
\left\| (\Pi_n \hat{Z})^+ P_{\Pi_n, X \hat{A}}^\perp Y \right\|_2 \lesssim \sqrt{\frac{\|\Sigma_W\|_{\text{op}} K \log n}{n}} + \sqrt{\frac{\zeta}{\lambda_K}} + \left( \sqrt{\frac{\kappa \|\Sigma_W\|_{\text{op}} K \log n}{n}} + \hat{r}_3 \sqrt{\kappa} + \frac{\|\Sigma_W\|_{\text{op}} + \hat{r}_3^2}{\sqrt{\lambda_K}} \right) \hat{r}_2,
\]

\[
\left\| (\Pi_n \hat{Z})^+ W \hat{A}(\Pi_n X \hat{A})^+ Y \right\|_2 \lesssim \left( \sqrt{\frac{\|\Sigma_W\|_{\text{op}} K \log n}{n}} + \hat{r}_3 \right) \hat{r}_2.
\]

**Proof.** To show the first result, using the decomposition (A.28) yields

\[
(\Pi_n \hat{Z})^+ P_{\Pi_n, X \hat{A}}^\perp Y = \Sigma_{Z}^{1/2} (Z^\top \Pi_n Z)^+ A^+ \left[ P_{\hat{A}}^\perp X^\top - W^\top \right] \Pi_n P_{\Pi_n, X \hat{A}}^\perp Y
\]

\[
= \Sigma_{Z}^{1/2} (Z^\top \Pi_n Z)^+ A^+ P_{\hat{A}}^\perp A Z^\top \Pi_n P_{\Pi_n, X \hat{A}}^\perp Y
\]

\[-\Sigma_{Z}^{1/2} (Z^\top \Pi_n Z)^+ A^+ P_{\hat{A}}^\perp W^\top \Pi_n P_{\Pi_n, X \hat{A}}^\perp Y.
\]

By invoking \( \mathcal{E}_z \), we find

\[
\left\| \Sigma_{Z}^{1/2} (Z^\top \Pi_n Z)^+ A^+ P_{\hat{A}}^\perp A Z^\top \Pi_n P_{\Pi_n, X \hat{A}}^\perp Y \right\|_2
\]

\[
\lesssim \frac{1}{n} \left\| \Sigma_{Z}^{-1/2} A^+ (P_{\hat{A}}^\perp - P_{\hat{A}}^\perp) A \Sigma_{Z}^{1/2} \right\|_{\text{op}} \left\| Z^\top \Pi_n P_{\Pi_n, X \hat{A}}^\perp Y \right\|_2
\]

\[
\leq \sqrt{\kappa} \left\| P_{\hat{A}} - P_{\hat{A}}^\perp \right\|_{\text{op}} \frac{1}{n} \left\| (\Pi_n \hat{Z})^+ P_{\Pi_n, X \hat{A}}^\perp Y \right\|_2
\]

by \( \mathcal{E}_z \).

It follows that, on the event \( \mathcal{E}_A \cap \mathcal{E}_z \),

\[
\left\| (\Pi_n \hat{Z})^+ P_{\Pi_n, X \hat{A}}^\perp Y \right\|_2 \lesssim \left\| \Sigma_{Z}^{1/2} (Z^\top \Pi_n Z)^+ A^+ P_{\hat{A}}^\perp W^\top \Pi_n P_{\Pi_n, X \hat{A}}^\perp Y \right\|_2
\]

\[
\lesssim \frac{1}{n} \left\| \Sigma_{Z}^{-1/2} A^+ P_{\hat{A}}^\perp W^\top \Pi_n P_{\Pi_n, X \hat{A}}^\perp Y \right\|_2
\]

\[
\leq \frac{1}{\sqrt{\lambda_K}} \left( \frac{1}{n} \left\| P_{\hat{A}}^\perp W^\top \Pi_n Y \right\|_2 + \frac{1}{n} \left\| P_{\hat{A}}^\perp W^\top \Pi_n P_{\Pi_n, X \hat{A}}^\perp Y \right\|_2 \right).
\]

Since

\[
\frac{1}{n} \left\| P_{\hat{A}}^\perp W^\top \Pi_n P_{\Pi_n, X \hat{A}}^\perp Y \right\|_2
\]

\[
= \frac{1}{n} \left\| P_{\hat{A}}^\perp W^\top \Pi_n X \hat{A}(\Pi_n X \hat{A})^+ Y \right\|_2
\]

\[
\leq \frac{1}{n} \left\| P_{\hat{A}}^\perp W^\top \Pi_n Z A^+ \hat{A}(\Pi_n X \hat{A})^+ Y \right\|_2 + \frac{1}{n} \left\| P_{\hat{A}}^\perp W^\top \Pi_n W \hat{A}(\Pi_n X \hat{A})^+ Y \right\|_2
\]

\[
\leq \left( \frac{1}{n} \left\| P_{\hat{A}}^\perp W^\top \Pi_n \hat{Z} \right\|_{\text{op}} \sqrt{\lambda_1 (A \Sigma Z A^\top)} + \frac{1}{n} \left\| W P_{\hat{A}}^\perp \right\|_{\text{op}}^2 \right) \left\| \hat{A}(\Pi_n X \hat{A})^+ Y \right\|_2,
\]

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the first result follows by invoking Lemma 33.

To prove the second claim, Following the proof of the second claim in Lemma 19, we find

\[
\left\| (\Pi_n \hat{Z})^+ W \hat{A} (\Pi_n X \hat{A})^+ Y \right\|_2 \leq \frac{1}{n} \left\| \hat{Z}^T \Pi_n W \hat{A} \right\|_{\text{op}} \left\| \hat{A} (\Pi_n X \hat{A})^+ Y \right\|_2
\]

The result then follows by invoking the first statement in Lemma 33 to bound \( \| \hat{Z}^T \Pi_n W \hat{A} \|_{\text{op}} \).

\[ \square \]

A.5 Proofs of Section 5

A.5.1 Proof of Theorem 8

We show \( \hat{K} = K \) with probability tending to one. Let \( \mu_n = c_0(n + p) \). Under conditions of Theorem 8, Proposition 8 in Bing et al. (2021) shows that \( \hat{K} \leq K \) holds on the event

\[
\mathcal{E}_W = \{ \sigma_1^2(W) \leq n \delta W \} \cap \left\{ c \text{ tr}(\Sigma_W) \leq \frac{1}{n} \| W \|_F^2 \leq C \text{ tr}(\Sigma_W) \right\}
\]

and \( \mathbb{P}(C) \geq 1 - O(n^{-1}) \). Thus it remains to prove \( \hat{K} \geq K \) on the event \( \mathcal{E}_z \cap \mathcal{E}_W \). From Corollary 10 of Bing and Wegkamp (2019), we need to verify

\[
\sigma_2^2(\Pi_n Z A^T) \geq \mu_n \|\Pi_n W\|_F^2 \left[ \frac{\sqrt{2}^2 + \sqrt{\frac{np}{np - \mu_n K}}} 2 \right]^2.
\]

For the left-hand-side, invoking \( \mathcal{E}_z \) in (A.26) gives

\[
\sigma_2^2(\Pi_n Z A^T) \geq c n \lambda_K (A \Sigma Z A^T) \geq c n \lambda_K.
\]

Regarding the right-hand-side, by invoking \( \mathcal{E}_W \) and using

\[
K \leq \hat{K} \leq \frac{\nu \frac{np}{1 + \nu \mu_n}}
\]

from (3.8), it can be bounded from above by

\[
\mu_n \| W \|_F^2 \left[ \frac{\sqrt{2}^2 + \sqrt{1 + \nu}} 2 \right]^2 \leq c \text{ tr}(\Sigma_W)^2 \frac{n + p} p.
\]

The proof is then completed by observing that

\[
\frac{\text{ tr}(\Sigma_W) n + p} \lambda_K \frac{n_p}{np} \leq \frac{\lambda_1(\Sigma_W)} {n \lambda_K} \frac{\lambda_1(\Sigma_W)} \lambda_K = \frac{\delta W} \lambda_K = \frac{1} \xi
\]

is sufficiently small.

\[ \square \]

A.5.2 Proof of Theorem 9

We first prove the result when \( \xi \geq C \) as stated in Remark 10. According to Theorem 7, it suffices to invoke Theorem 16 with \( \tilde{\eta} = \sigma_2^2(X)/n, \tilde{\psi} = \sigma_2^2(X)/n \) and \( \tilde{\gamma}_3 \leq \| P_{U_K} - P_A \|_{\text{op}} \sqrt{\delta W} \) together with a bound for \( \| P_{U_K} - P_A \|_{\text{op}} \). First, by using

\[
\mathbb{P}(D) \left\{ \frac{1}{n} \sigma_1^2(W) \leq c \delta W \right\} \geq 1 - e^{-n},
\]

(A.34)
for some constant $c > 0$, and by using Weyl’s inequality, we find that
\[ \hat{\psi} = \frac{1}{n} \sigma^2_k (\Pi_n X) \leq \frac{1}{n} \sigma^2_1 (\Pi_n W) \leq \frac{1}{n} \sigma^2_1 (W) \leq c \delta_W, \tag{A.35} \]
with probability $1 - \mathcal{O}(1/n)$. Second, with the same probability, we find that
\[ \hat{\eta} = \frac{1}{n} \sigma^2_k (\Pi_n X) \geq \frac{1}{n} \left\{ \sigma_k (\Pi_n Z A^\top) - \sigma_1 (\Pi_n W) \right\}^2 \geq \frac{1}{n} \left\{ \sigma_k (\Pi_n \hat{Z}) \sigma_k (\Sigma_Z^{1/2} A^\top) - \sigma_1 (\Pi_n W) \right\}^2 \geq \lambda_K (A \Sigma_Z A^\top) \geq \lambda_K, \tag{A.36} \]
using Weyl’s inequality, inequality (A.35), our assumption $\xi = \lambda_K / \delta_W \geq C$ and the event $E_z$ together with $\sigma^2_k (\Sigma_Z^{1/2} A^\top) = \lambda_K (A \Sigma_Z A^\top) \geq \lambda_K$. Finally, invoke Lemma 21 to identify
\[ \hat{\zeta} \leq \frac{1}{n} \|\Pi_n Y\|^2 \leq \kappa \delta_W, \]
which together with $\xi \geq \kappa^2$ and $K \log n \leq n$ imply that $\hat{\zeta} \leq \hat{\eta}$ and $\hat{\phi} \leq 1$. Then invoking Theorem 17 gives that, with probability $1 - \mathcal{O}(n^{-1}),$
\[ \hat{r}_1 \leq \sqrt{\frac{K \log n}{n}} + \sqrt{\frac{\kappa^2 \delta_W}{\lambda_K}} + \left\{ \sqrt{\frac{\kappa}{\xi^*}} \frac{K \log n}{n} + \sqrt{\frac{\kappa}{\xi^*}} \frac{1}{\xi^*} \right\} \left( \frac{\sqrt{\kappa \log n}}{n} + \min\{1, \Delta\} + \sqrt{\frac{\kappa}{\xi^2}} \right) \]
\[ \approx \sqrt{\frac{K \log n}{n}} + \sqrt{\frac{\kappa}{\xi^2}} + \kappa \min\{1, \Delta\}, \]
\[ \hat{r}_2 \leq \sqrt{\frac{1}{\lambda_K}} \left( \frac{\sqrt{\kappa \log n}}{n} + \min\{1, \Delta\} + \sqrt{\frac{\kappa}{\xi^2}} \right). \]
We conclude (5.4) after collecting terms. \qed
Lemma 21. Under conditions of Theorem 9, with probability $1 - O(n^{-1})$, one has

$$\|P_U - P_A\|_{op} \lesssim \sqrt{\kappa \delta W / \lambda_K} \wedge 1.$$  

Proof. By the variant (Yu et al., 2014) of Davis-Kahan theorem,

$$\|P_U - P_A\|_{op} \leq \frac{2 \|X^\top \Pi_n X - AZ^\top \Pi_n Z A^\top\|_{op}}{\lambda_K (AZ^\top \Pi_n Z A^\top)} \leq \frac{2 \|W^\top \Pi_n W\|_{op}}{\lambda_K (AZ^\top \Pi_n Z A^\top)} \lesssim \sqrt{\kappa \delta W / \lambda_K} \wedge 1.$$  

On the event $\mathcal{E}_z$, by also recalling (A.36) and (A.35), we have

$$\|P_U - P_A\|_{op} \lesssim \delta W / \lambda_K (A \Sigma Z A^\top) + \sqrt{\delta W / \lambda_K (A \Sigma Z A^\top)} \sqrt{\kappa \delta W / \lambda_K (A \Sigma Z A^\top)} \wedge 1,$$

with probability at least $1 - cn^{-1}$. Using $\xi = \lambda_K / \delta W \geq C$ to simplify terms completes the proof. 

A.5.3 Proof of Theorem 10

Similar as the proof of Theorem 9, we aim to invoke Theorem 17 with $\hat{A} = \tilde{U}_K$ and $\hat{\eta} = \sigma_K^2 (\Pi_n \tilde{X} \tilde{U}_K) / n$. By invoking Lemma 21 with $U_K$ and $X$ replaced by $\tilde{U}_K$ and $\tilde{X}$, respectively, we have

$$\|P_{\tilde{U}_K} - P_A\|_{op} \lesssim \sqrt{\kappa / \xi}.$$  

with probability $1 - O(n^{-1})$. The event $\mathcal{E}_A$ in (A.25) of Theorem 17 thus holds under $\xi \gtrsim \kappa^2$. Furthermore, since $\hat{A} = \tilde{U}_K$ is independent of $X$, hence independent of $W$, an application of Lemma 35 yields

$$P \left\{ \frac{1}{n} \|W (P_A - P_A)^2 \|_{op} \lesssim \|H\|_{op} + \frac{\text{tr}(H)}{n} \right\} \geq 1 - e^{-n},$$

where $H = W^{1/2} (P_A - P_A)^2 W^{1/2}$ satisfies $\text{tr}(H) \leq 2K \|H\|_{op} \leq 2K \|P_A - P_A\|_{op} \|\Sigma W\|_{op}$. It then follows that, with the same probability,

$$\hat{r}_3^2 \lesssim \|\Sigma W\|_{op} \|P_A - P_A\|_{op} \lesssim \frac{\kappa \|\Sigma W\|_{op}}{\xi}.$$  

(A.38)

Similarly, the same proof for the last result of Lemma 32 with $P_A$ replaced by $P_{\hat{A}} - P_A$ yields that, with probability $1 - O(n^{-1})$,

$$\frac{1}{n} \| (P_{\hat{A}} - P_A) W^\top \Pi_n Y \|_2 \lesssim \|P_{\hat{A}} - P_A\|_{op} \sqrt{\|\Sigma W\|_{op} K \log n / n},$$

implying

$$\hat{\xi} \lesssim \frac{\kappa \|\Sigma W\|_{op} K \log n}{\xi}. $$

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To bound from below \( \widehat{\eta} \), we first find that

\[
\sqrt{n \widehat{\eta}} = \sigma_K(\Pi_n X \hat{U}_K) \\
\geq \sigma_K(\Pi_n Z A^\top \hat{U}_K) - \sigma_1(\Pi_n W \hat{U}_K) \\
\geq \sigma_K(\Pi_n \tilde{Z}) \sigma_K(\Sigma_{Z}^{1/2} A^\top \hat{U}_K) - \sigma_1(W \hat{U}_K).
\]

Since, with probability \( 1 - O(n^{-1}) \),

\[
\sigma_K(\Sigma_{Z}^{1/2} A^\top \hat{U}_K) = \sigma_K(\Sigma_{Z}^{1/2} A^\top) - \sigma_1(\Sigma_{Z}^{1/2} (P_A - P_{\hat{U}_K}^{\top})) \\
\geq \sqrt{\lambda_K(A \Sigma Z A^\top) - \sigma_1(\Sigma_{Z}^{1/2} (P_A - P_{\hat{U}_K}^{\top})))} \\
\geq \sqrt{\lambda_K(A \Sigma Z A^\top) - \lambda_1(A \Sigma Z A^\top) \sqrt{\kappa}} \\
\geq \sqrt{\lambda_K(A \Sigma Z A^\top)} \geq \sqrt{\lambda_K},
\]

where the last step uses (A.10) and \( \xi \gtrsim \kappa^2 \), by also using \( \sigma_1(W \hat{U}_K) \leq \sigma_1(W) \) and (A.34) together with \( \mathcal{E}_z \), we conclude

\[
P_{\mathcal{D}}\{ \widehat{\eta} \gtrsim \lambda_K \} = 1 - O(n^{-1}). \tag{A.39}
\]

Then invoking Theorem 17 gives that, with probability \( 1 - O(n^{-1}) \),

\[
\hat{r}_1 \lesssim \sqrt{\frac{K \log n}{n}} + \sqrt{\frac{\kappa \xi^* \xi}{\xi^* \xi}} \left( \sqrt{\frac{1}{\xi^* \xi}} \frac{K \log n}{n} + \sqrt{\frac{1}{\xi^* \xi}} \frac{K \log n}{n} \right) \min\{1, \Delta\}
\]

\[
\hat{r}_2 \gtrsim \sqrt{\frac{1}{\lambda_K} \left( \sqrt{\frac{K \log n}{n}} + \min\{1, \Delta\} \right)},
\]

implying that

\[
\omega_n(a) \asymp \left[ \sqrt{\frac{K \log n}{n}} + \frac{\kappa \xi^* \xi}{\xi^* \xi} \min\{1, \Delta\} + \sqrt{\frac{1}{\xi^* \xi}} \min\{1, \Delta\} \right] \sqrt{a \log n}
\]

\[
\asymp \left( \sqrt{\frac{K \log n}{n}} + \min\{1, \Delta\} \sqrt{\frac{1}{\xi^* \xi}} \right) \sqrt{a \log n}.
\]

We used \( \xi \gtrsim \kappa^2 \) in the last step. The proof is complete. \( \square \)

### A.5.4 Proof of Corollary 11

Since \( \sigma^2(1 + p/n) \leq c'\lambda \) implies \( \xi \geq C \) for some constant \( C(c') > 0 \), the proof follows from Theorem 10 by choosing \( a = \Delta^2/\log n + 1 \) for \( \omega_n(a) \) in (5.6) and by noting that

\[
\omega_n(a) \asymp \left( \sqrt{\frac{K \log n}{n}} + \min\{1, \Delta\} \right) \sqrt{\log n + \Delta^2}.
\]

\( \square \)
A.6 Proofs of Section 8

For notational convenience, define
\[ G_z^{(l|k)}(z) := \left( z - \frac{\alpha_l + \alpha_k}{2} \right) ^ \top \Sigma_{Z|Y}^{-1}(\alpha_l - \alpha_k) + \log \frac{\pi_l}{\pi_k}, \quad \forall \ell, k \in \mathcal{L}. \] (A.40)

In particular, for any \( \ell \in \mathcal{L} \), we have
\[ G_z^{(l|0)}(z) = \left( z - \frac{\alpha_l + \alpha_0}{2} \right) ^ \top \Sigma_{Z|Y}^{-1}(\alpha_l - \alpha_0) + \log \frac{\pi_l}{\pi_0} \]
\[ = z^\top \eta^{(l)} + \eta_0^{(l)} \]
\[ = \frac{1}{\pi_0 \pi_\ell} [1 - (\alpha_l - \alpha_0)^\top \beta^{(l)}] \left( z^\top \beta^{(l)} + \beta_0^{(l)} \right). \]

Further recall that
\[ \hat{G}_z^{(l|0)}(x) := \frac{1}{\pi_0 \pi_\ell} [1 - (\hat{\mu}_l - \hat{\mu}_0)^\top \hat{\eta}^{(l)}] \left( x^\top \hat{\eta}^{(l)} + \hat{\beta}_0^{(l)} \right), \quad \forall \ell \in \mathcal{L}. \]

For any \( t \geq 0 \), define the event
\[ \mathcal{E}_t = \bigcap_{\ell \in \mathcal{L}} \left\{ \left| \hat{G}_z^{(l|0)}(X) - G_z^{(l|0)}(Z) \right| \leq t \right\}. \] (A.41)

Finally, we write for simplicity
\[ \Delta_{(l|k)} = \| \pi_l - \pi_k \|_{\Sigma_{Z|Y}}, \quad \forall \ell, k \in \mathcal{L}. \] (A.42)

A.6.1 Proof of Theorem 12

By definition, we start with
\[ R_x(\hat{g}_z^*) - R_x^* \]
\[ = \sum_{k \in \mathcal{L}} \pi_k \left\{ \mathbb{E} \left[ \mathbb{1} \left\{ \hat{g}_z^*(X) \neq k \right\} \mid Y = k \right] - \mathbb{E} \left[ \mathbb{1} \left\{ g_z^*(Z) \neq k \right\} \mid Y = k \right] \right\} \]
\[ = \sum_{k \in \mathcal{L}} \pi_k \mathbb{E} \left[ \mathbb{1} \left\{ \hat{g}_z^*(X) \neq k, g_z^*(Z) = k \right\} \mid Y = k \right] - \sum_{k \in \mathcal{L}} \pi_k \mathbb{E} \left[ \mathbb{1} \left\{ \hat{g}_z^*(X) = k, g_z^*(Z) \neq k \right\} \mid Y = k \right] \]
\[ = \sum_{k,l \in \mathcal{L}, k \neq l} \pi_k \mathbb{E} \left[ \mathbb{1} \left\{ \hat{g}_z^*(X) = \ell, g_z^*(Z) = k \right\} \mid Y = k \right] - \sum_{k,l \in \mathcal{L}, k \neq l} \pi_k \mathbb{E} \left[ \mathbb{1} \left\{ \hat{g}_z^*(X) = k, g_z^*(Z) = \ell \right\} \mid Y = k \right] \]
\[ = \sum_{k,l \in \mathcal{L}, k \neq l} \left\{ \pi_k \mathbb{E} \left[ \mathbb{1} \left\{ \hat{g}_z^*(X) = \ell, g_z^*(Z) = k \right\} \mid Y = k \right] - \pi_l \mathbb{E} \left[ \mathbb{1} \left\{ \hat{g}_z^*(X) = \ell, g_z^*(Z) = k \right\} \mid Y = \ell \right] \right\}. \]

Recall that \( f_{Z|k}(z) \) is the p.d.f. of \( Z = z \mid Y = k \) for each \( k \in \mathcal{L} \). Repeating arguments in the proof of Theorem 7 gives
\[ R_x(\hat{g}_z) - R_x^* = \sum_{k,l \in \mathcal{L}, k \neq l} \mathbb{E}_W \int_{\hat{g}_z^* = \ell, g_z^* = k} \left( \pi_k f_{Z|k}(z) - \pi_l f_{Z|\ell}(z) \right) dz \]
\[ = \sum_{k,l \in \mathcal{L}, k \neq l} \mathbb{E}_W \int_{\hat{g}_z^* = \ell, g_z^* = k} \pi_k f_{Z|k}(z) \left( 1 - \exp \left\{ G_z^{(l|k)}(z) \right\} \right) dz \]
with $G_z^{(l,k)}(z)$ defined in (A.40). Since
\[
G_z^{(l,k)}(z) = G_z^{(l,0)}(z) - G_z^{(k,0)}(z),
\]
the event \( \{ \hat{g}_x (X) = l, g_x^* (Z) = k \} \cap E_t \) implies
\[
0 > G_z^{(l,k)}(z) \geq \hat{G}_z^{(l,0)}(X) - \hat{G}_z^{(k,0)}(X) - 2t \geq -2t, \quad \forall \ t > 0.
\]
By repeating the arguments of analyzing term (I) in the proof of Theorem 7, we obtain that, for any \( t > 0 \),
\[
R_x (\hat{g}_x) - R_x^* \leq \sum_{k, l \in \mathcal{L}, k \neq l} \left\{ 2 \pi k \mathbb{E} Z \left[ \mathbb{I} \{ -2t \leq G_z^{(l,k)}(Z) \leq 0 \mid Y = k \} \right] + \pi k \mathbb{P} (E_t^c \mid Y = k) \right\} \\
\leq (L - 1) \sum_{k \in \mathcal{L}} 2 \pi k t \max_{\ell \in \mathcal{L} \setminus \{ k \}} \left[ \Phi \left( \frac{R^{(l,k)}(k)}{2} \right) - \Phi \left( \frac{R^{(l,k)}(k) - \frac{2t}{\Delta^{(l,k)}}}{2} \right) \right] + (L - 1) \mathbb{P} (E_t^c) \quad (A.44)
\]
\[
\leq (L - 1) \sum_{k \in \mathcal{L}} 4 \pi k t^2 \max_{\ell \in \mathcal{L} \setminus \{ k \}} \frac{1}{\Delta^{(l,k)}} \exp \left( - \frac{m^{(l,k)}_{(2)}}{2} \right) + (L - 1) \mathbb{P} (E_t^c)
\]
where
\[
R^{(l,k)}(k) = \frac{\Delta^{(l,k)}}{2} - \log \frac{m^{(l,k)}_{(2)}}{\Delta^{(l,k)}}, \quad m^{(l,k)}_{(2)} \in \left[ R^{(l,k)}(k) - \frac{2t}{\Delta^{(l,k)}}, R^{(l,k)}(k) \right].
\]
The penultimate step uses the fact that
\[
G_z^{(l,k)}(Z) \mid Y = k \sim N \left( -\Delta^{(l,k)} R^{(l,k)}, \Delta^2 \right)
\]
while the last step applies the mean-value theorem. By choosing
\[
t^* = (1 + \Delta^4) \omega_n
\]
and invoking condition (8.10) and \((1 + \Delta^2) \omega_n = o(1)\), we find that:

(a) If \( \Delta \rightarrow 1 \), then
\[
R_x (\hat{g}_x) - R_x^* \lesssim L \omega_n^2 + L \mathbb{P} (E_t^c). \]

(b) If \( \Delta \rightarrow \infty \), then \( \Delta^2 \omega_n = o(1) \) ensures that \( m^{(l,k)}_{(2)} \asymp \Delta \) hence
\[
R_x (\hat{g}_x) - R_x^* \lesssim L \omega_n^2 e^{-c \Delta^2 + o(\Delta^2)} + L \mathbb{P} (E_t^c). \]

(c) If \( \Delta \rightarrow 0 \), then \( t^* \asymp \omega_n \) and
\[
R_x (\hat{g}_x) - R_x^* \lesssim L \omega_n^2 \Delta + L \mathbb{P} (E_t^c). \]

For \( \Delta \rightarrow 0 \), by (A.44), we also have
\[
R_x (\hat{g}_x) - R_x^* \lesssim L \min \left\{ \frac{\omega_n^2}{\Delta}, \omega_n \right\} + L \mathbb{P} (E_t^c). \]

In view of cases (a) – (c), since the event \( \{ \hat{\omega}_n \leq \omega_n \} \) implies
\[
\mathbb{P} (E_t^c) \leq \mathbb{P} \left\{ \max_{\ell \in \mathcal{L}} \left| \hat{G}_x^{(l,0)}(X) - G_x^{(l,0)}(Z) \right| \geq (1 + \Delta^4) \hat{\omega}_n \right\},
\]
It remains to prove that, with probability \( 1 - O(n^{-1}) \), the right-hand side of the above display is no greater than \( n^{-1} e^{-\Delta^2} \). This is proved by combining Lemmas 22 and 23. \( \square \)
A.6.2 Lemmas used in the proof of Theorem 12

The following lemma establishes the probability tail of the event \( E_t \) defined in (A.41) for \( t = \tilde{\omega}_n \), a random sequence defined below whose randomness only depends on \( D \). Recall \( \hat{r}_1 \) and \( \hat{r}_2 \) from (8.8). Set

\[
\tilde{\omega}_n = \max_{\ell \in \mathcal{L}} C \left\{ \frac{\hat{r}_1 + \| \Sigma_W \|_{op}^{1/2} \hat{r}_2}{\| \tilde{\pi}_0 \tilde{\pi}_\ell [1 - (\hat{\mu}_\ell - \hat{\mu}_0)^\top \hat{\theta}(t)] \|} \left( \sqrt{\log n + \Delta} \right) + \left( \tilde{\beta}_0(t) - \hat{\beta}_0(t) \right) + \frac{1}{2} \left( \alpha_1 + \alpha_0 \right)^\top (A^\top \hat{\theta}(t) - \beta(t)) \right\},
\]

(A.45)

\[
\begin{align*}
\text{Lemma 22.} \quad & \text{Under conditions of Theorem 12, we have,} \\
& \mathbb{P} \left\{ \max_{\ell \in \mathcal{L}} \left| \tilde{G}_\ell(t)(X) - G_\ell(t)(Z) \right| \geq \tilde{\omega}_n \right\} \leq n^{-1} e^{-\Delta^2}.
\end{align*}
\]

Proof. Pick any \( \ell \in \mathcal{L} \). By definition,

\[
\left| \tilde{G}_\ell(t)(X) - G_\ell(t)(Z) \right| \leq I + II + III
\]

where

\[
I = \left( X^\top \hat{\theta}(t) - Z^\top \beta(t) - \frac{1}{2} \left( \alpha_1 + \alpha_0 \right)^\top (A^\top \hat{\theta}(t) - \beta(t)) \right) \left\| \tilde{\pi}_0 \tilde{\pi}_\ell [1 - (\hat{\mu}_\ell - \hat{\mu}_0)^\top \hat{\theta}(t)] \right\|
\]

\[
II = \frac{\left( \tilde{\beta}_0(t) - \beta(t) \right) + \frac{1}{2} \left( \alpha_1 + \alpha_0 \right)^\top (A^\top \hat{\theta}(t) - \beta(t))}{\tilde{\pi}_0 \tilde{\pi}_\ell [1 - (\alpha_\ell - \alpha_0)^\top \beta(t)]},
\]

\[
III = \left( \frac{1}{\tilde{\pi}_0 \tilde{\pi}_\ell [1 - (\hat{\mu}_\ell - \hat{\mu}_0)^\top \hat{\theta}(t)]} - \frac{1}{\tilde{\pi}_0 \tilde{\pi}_\ell [1 - (\alpha_\ell - \alpha_0)^\top \beta(t)]} \right) \left\| Z^\top \beta(t) + \tilde{\beta}_0(t) \right\|
\]

(8.4)

First, notice that the numerator of I is bounded from above by

\[
W^\top \hat{\theta}(t) + \left( Z - \frac{1}{2} (\alpha_\ell + \alpha_0) \right)^\top (A^\top \hat{\theta}(t) - \beta(t)),
\]

which, by the arguments of proving Proposition 6 and by conditioning on \( Y = k \) for any \( k \in \mathcal{L} \), with probability \( 1 - \mathcal{O}(n^{-a}) \) for any \( a > 0 \), is no greater than

\[
C \left( \sqrt{a \log n} + \left\| \alpha_k - \frac{1}{2} (\alpha_\ell + \alpha_0) \right\|_{\Sigma_{\ell}^{(t)}} \right) \left\| \left[ \Sigma_Z(t) \right]^{1/2} (A^\top \hat{\theta}(t) - \beta(t)) \right\|_2
\]

\[
+ C \sqrt{a \log n} \| \hat{\theta}(t) \|_2 \| \Sigma_W \|_{op}^{1/2}
\]

\[
\lesssim \left( \sqrt{a \log n} + \max_{k \in \mathcal{L}} \Delta (k_0) + 1 \right) \left\| \left[ \Sigma_Z(t) \right]^{1/2} (A^\top \hat{\theta}(t) - \beta(t)) \right\|_2 + \sqrt{a \log n} \| \hat{\theta}(t) \|_2 \| \Sigma_W \|_{op}^{1/2}
\]

\[
\lesssim \left( \sqrt{a \log n} + \Delta + 1 \right) \left( \left\| \left[ \Sigma_Z(t) \right]^{1/2} (A^\top \hat{\theta}(t) - \beta(t)) \right\|_2 + \| \hat{\theta}(t) \|_2 \| \Sigma_W \|_{op}^{1/2} \right).
\]

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In the second step, we also used
\[ \|\alpha_k - \alpha_0\|^2_{Z^2} \leq \|\alpha_k - \alpha_0\|^2_{Z^2Z|Y} \left\| \Sigma_{Z|Y}^{1/2} (\Sigma_{Z}^{\ell})^{-1}\Sigma_{Z|Y}^{1/2} \right\|_{op} \leq \Delta_k, \quad \forall k \in \mathcal{L}. \]

Again, by the arguments of proving Proposition 6, with probability \(1 - O(n^{-a})\) for any \(a > 0\),
\[ Z^T \eta^{(2)} + \eta_0^{(2)} \lesssim \|\alpha_\ell - \alpha_0\|_{Z^2} \sqrt{a \log n} + \left| \left( \alpha_k - \frac{\alpha_\ell + \alpha_0}{2} \right) \Sigma_{Z|Y}^{-1} (\alpha_\ell - \alpha_0) \right| \lesssim \Delta_{(\ell|0)} \left( \sqrt{a \log n} + \Delta_{(\ell|0)} + \Delta_{(k|0)} \right) \lesssim \Delta \left( \sqrt{a \log n} + \Delta \right). \]

Taking \(a = C + \Delta^2/\log n\) for some positive constant \(C\) in these two bounds yields the claim. \(\square\)

We proceed to bound from above \(\hat{\omega}_n\) defined in (A.45) by \(\hat{\omega}_n\) in (8.9). Recall that
\[ \hat{\omega}_n = C \sqrt{\log n} \left( \hat{\rho}_1 + \|\Sigma_W\|_o^2 \hat{\rho}_2 + \hat{\rho}_3 + \sqrt{\frac{L}{n}} \right). \]

**Lemma 23.** Under conditions of Theorem 12, we have
\[ \mathbb{P} \{ \hat{\omega}_n \lesssim (1 + \Delta^4)\hat{\omega}_n \} = 1 - O(n^{-1}). \]

**Proof.** We first bound from above the numerators of the last two terms in \(\hat{\omega}_n\) defined in (A.45). By Lemma 30 and \(\pi_k \asymp 1/L\) for all \(k \in \mathcal{L}\), we have
\[ \mathbb{P} \left\{ \max_{\ell \in \mathcal{L}} |\hat{\pi}_\ell - \pi_\ell| \lesssim \sqrt{\log n} \right\} = 1 - O(Ln^{-C}). \]
for some constant \(C > 1\). With the same probability, using \(L \log n \lesssim n\) further yields that, for any \(\ell \in \mathcal{L}\),
\[ \hat{\pi}_\ell \asymp \frac{1}{L}, \quad n_\ell \asymp \frac{n}{L} \]
as well as
\[ |\hat{\pi}_\ell - \pi_\ell| = \left| \frac{\hat{\pi}_\ell - \pi_\ell}{\hat{\pi}_\ell + \pi_\ell} \right| + \left| \frac{\pi_\ell (\hat{\pi}_\ell - \pi_\ell + \pi_0 - \pi_0)}{(\hat{\pi}_\ell + \pi_0)(\pi_\ell + \pi_0)} \right| \lesssim \sqrt{\frac{L \log n}{n}}, \quad \hat{\pi}_\ell \asymp 1. \]

Pick any \(\ell \in \mathcal{L}\). By following the same arguments of proving Lemma 15 and using the condition \(KL \log n \lesssim n\), we have, with probability \(1 - O(n^{-C})\),
\[ \max \left\{ \left| \beta_0^{(\ell)} - \beta_0^{(\ell)} + \frac{1}{2} (\alpha_1 + \alpha_0)^{T} (A^{T} \hat{\beta}(\ell) - \beta(\ell)) \right|, \quad \left| \hat{\pi}_0 \pi_0 [1 - (\hat{\mu}_\ell - \hat{\mu}_0)^{T} \hat{\beta}(\ell)] - \pi_0 \hat{\pi}_0 [1 - (\alpha_\ell - \alpha_0)^{T} \beta(\ell)] \right| \right\} \lesssim \hat{\rho}_1 + \|\Sigma_W\|_o^2 \hat{\rho}_2 + \hat{\rho}_3 + \sqrt{\frac{L \log n}{n}} \leq \omega_n. \quad (A.46) \]

By taking the union bounds over \(\ell \in \mathcal{L}\), the above bound also holds for all \(\ell \in \mathcal{L}\) with probability \(1 - O(Ln^{-C})\).
It remains to bound from below \(|\tilde{\pi}_0\tilde{\pi}_\ell [1 - (\hat{\mu}_\ell - \hat{\mu}_0)^\top \hat{\theta}(\ell)]|\). To this end, repeating arguments of proving Lemma 14 gives

\[
\text{Cov}(Z, \mathbb{1}\{Y = \ell\} | Y \in \{0, \ell\}) = \tilde{\pi}_0\tilde{\pi}_\ell (\alpha_\ell - \alpha_0),
\]

and, by recalling that \(\Sigma_\ell = \text{Cov}(Z | Y \in \{0, \ell\})\),

\[
\|\alpha_\ell - \alpha_0\|^2_{\Sigma(\ell)} = \|\alpha_\ell - \alpha_0\|^2_{\Sigma Z} = (A.42) \frac{\Delta^2(\ell)}{1 + \tilde{\pi}_0\tilde{\pi}_\ell \Delta^2(\ell)}. \tag{A.42}
\]

It then follows that

\[
\tilde{\pi}_0\tilde{\pi}_\ell [1 - (\alpha_\ell - \alpha_0)^\top \beta(\ell)] = \tilde{\pi}_0\tilde{\pi}_\ell \left[1 - \tilde{\pi}_0\tilde{\pi}_\ell \|\alpha_\ell - \alpha_0\|^2_{\Sigma(\ell)}\right] = \frac{\tilde{\pi}_0\tilde{\pi}_\ell}{1 + \tilde{\pi}_0\tilde{\pi}_\ell \Delta^2(\ell)}. \tag{A.46}
\]

Thus, by (A.46), condition (8.10) and condition \((1 + \Delta^2)\omega_n = o(1)\), we find that, with probability \(1 - \mathcal{O}(Ln^{-C})\),

\[
\left|\tilde{\pi}_0\tilde{\pi}_\ell [1 - (\hat{\mu}_\ell - \hat{\mu}_0)^\top \hat{\theta}(\ell)]\right| \geq \frac{\tilde{\pi}_0\tilde{\pi}_\ell}{1 + \tilde{\pi}_0\tilde{\pi}_\ell \Delta^2(\ell)} - \omega_n \geq \frac{\tilde{\pi}_0\tilde{\pi}_\ell}{1 + \tilde{\pi}_0\tilde{\pi}_\ell \Delta^2(\ell)}. \tag{A.46}
\]

Combining the last display with (A.46) gives that, with probability \(1 - \mathcal{O}(n^{-1})\),

\[
\tilde{\omega}_n \lesssim \max_{\ell \in \mathcal{L}} \left\{ (1 + \Delta^2) \left\{ (\sqrt{\log n} + \Delta) \left(\hat{r}_1 + \hat{r}_2\|\Sigma_W\|_{\text{op}}^{1/2}\right) + \left(\hat{r}_1 + \|\Sigma_W\|_{\text{op}}^{1/2}\hat{r}_2 + \hat{r}_3 \sqrt{\frac{L \log n}{n}} \right) \left(1 + \Delta \sqrt{\log n + \Delta^2}\right) \right\} \right\}
\]

\[
\lesssim (1 + \Delta^4)\omega_n,
\]

completing the proof. \(\square\)

**A.6.3 Proof of Corollary 13**

In view of Theorem 12, we only need to bound from above \(\hat{r}_1, \hat{r}_2\) and \(\hat{r}_3\) for each choice of \(B\). Inspecting the proofs of Theorems 16 & 17 reveals that the same conclusions in Theorems 16 & 17 hold with \(K\) replaced by \(KL\). Consequently, repeating the steps in the proofs of Theorems 9 & 10 yields the desired result. \(\square\)

**B Proof of the minimax lower bounds of the excess risk**

**Proof of Theorem 3.** Recall that \(\pi_0 = \pi_1 = 1/2\). It suffices to consider \(\alpha_1 = -\alpha_0 = \alpha\). Further recall that \(K/(n \vee p) \leq c_1, \sigma^2/\lambda \leq c_2\) and \(\sigma^2 p/(\lambda n) \leq c_3\) for sufficiently small positive constants \(c_1, c_2\) and \(c_3\).

To prove Theorem 3, it suffices to consider the Gaussian case. Specifically, for any \(\theta = (A, \Sigma_Z\mid Y, \Sigma_W, \alpha, -\alpha, 1/2, 1/2)\), consider

\[
X \mid Y = 1 \sim N_p(\mu_\theta, \Sigma_\theta) \quad \text{and} \quad X \mid Y = 0 \sim N_p(-\mu_\theta, \Sigma_\theta) \tag{B.1}
\]

with \(\mu_\theta = A\alpha, \quad \Sigma_\theta = A\Sigma_Z\mid Y A^\top + \Sigma_W\).
In this case, the Bayes rule of using $X$ is
\[
g_0^\star(x) = \mathbb{I}\{G_0^\star(x) \geq 0\} = \mathbb{I}\left\{2x^\top \Sigma_\theta^{-1} \mu_\theta \geq 0\right\}.
\] (B.2)

For any classifier $\hat{g}: \mathbb{R}^p \to \{0,1\}$, one has
\[
R_x(\hat{g}) - R_x^\star = R_x(\hat{g}) - R_x(g_0^\star) + R_x(g_0^\star) - R_x^\star.
\]
Lemma 2 together with $\sigma^2/\lambda \leq c_2$ ensures that, for any $\theta \in \Theta(\lambda, \sigma, \Delta)$,
\[
R_x(g_0^\star) - R_x^\star \geq \frac{\sigma^2}{\lambda} \Delta \exp\left(-\frac{\Delta^2}{8}\right).
\] (B.3)

Note that $g_0^\star$ has the smallest risk over all measurable functions $\hat{g}: \mathbb{R}^p \to \{0,1\}$. We proceed to bound from below $R_x(\hat{g}) - R_x(g_0^\star)$ by splitting into two scenarios depending on the magnitude of $\Delta$.

**Case 1:** $\Delta \gtrless 1$. We may assume $\Delta \geq 2$ for simplicity. It suffices to show
\[
\inf_{\hat{g}} \sup_{\theta \in \Theta(\lambda, \sigma, \Delta)} \mathbb{P}^{D}_{\theta}\left\{R_x(\hat{g}) - R_x(g_0^\star) \geq \eta \frac{\Delta^2}{8} \right\} \geq c_0,
\] (B.4)
where
\[
\delta = \frac{\sigma^2}{\sigma^2 + \lambda} \frac{\Delta^2}{8}
\] (B.5)
and
\[
\eta = C \left[\frac{K}{n} + \frac{\sigma^4(3-K)}{\lambda^2 n}\right].
\] (B.6)

We take the leading constant $C > 0$ in $\eta$ small enough such that

(a) $C < 3(c_1 + c_2c_3)$, where $c_1, c_2, c_3$ are defined in Theorem 3.

(b) $C < \min(C_1, C_2)/6$, where $C_1$ and $C_2$ are defined in (B.14) and (B.15).

These two requirements will become apparent soon.

To prove (B.4), we first introduce another loss function
\[
L_\theta(\hat{g}) = \mathbb{P}_{\theta}\{\hat{g}(X) \neq g_0^\star(X)\}.
\] (B.7)

We proceed to bound $R_x(\hat{g}) - R_x(g_0^\star)$ from below by using $L_\theta(\hat{g})$. By following the same arguments in the proof of Theorem 5 with $G_z(Z)$ replaced by $G_\theta^\star(X)$, one can deduce that
\[
R_x(\hat{g}) - R_x(g_0^\star) = \mathbb{P}_{\theta}\{\hat{g}(X) \neq Y\} - \mathbb{P}_{\theta}\{g_\theta^\star(X) \neq Y\} := I + II
\]
where
\[
I = \pi_0 \mathbb{E}_{\theta}\left[\mathbb{I}\{\hat{g}(X) = 1, G_\theta^\star(X) < 0\} (1 - \exp(G_\theta^\star(X)) \mid Y = 0\}\right],
\]
\[
II = \pi_1 \mathbb{E}_{\theta}\left[\mathbb{I}\{\hat{g}(X) = 0, G_\theta^\star(X) \geq 0\} (1 - \exp(-G_\theta^\star(X)) \mid Y = 1\}\right].
\]

For any $t > 0$,
\[
I \geq \pi_0 \mathbb{E}_{\theta}\left[\mathbb{I}\{\hat{g}(X) = 1, G_\theta^\star(X) \leq -t\} (1 - \exp(G_\theta^\star(X)) \mid Y = 0\}\right] \\
\geq \pi_0 (1 - e^{-t}) \mathbb{E}_{\theta}\left[\mathbb{I}\{\hat{g}(X) = 1, G_\theta^\star(X) \leq -t\} \mid Y = 0\right] \\
\geq \pi_0 (1 - e^{-t}) \left\{\mathbb{E}_{\theta}\left[\mathbb{I}\{\hat{g}(X) = 1, G_\theta^\star(X) < 0\} \mid Y = 0\right] - \mathbb{P}_{\theta}\left(-t \leq G_\theta^\star(X) < 0 \mid Y = 0\right)\right\} \\
= \pi_0 (1 - e^{-t}) \left\{\mathbb{E}_{\theta}\left[\mathbb{I}\{\hat{g}(X) = 1, g_\theta^\star(X) = 0\} \mid Y = 0\right] - \mathbb{P}_{\theta}\left(-t \leq G_\theta^\star(X) < 0 \mid Y = 0\right)\right\}.
\]
An application of the mean value theorem yields 

$$
\eta
$$

and, similarly,

$$
\eta
$$

Next, choose

$$
\eta
$$

for any

$$
0 < t < \frac{1}{2}
$$

Then, for

$$
0 < t < \frac{1}{2}
$$

for any

$$
0 < t < \frac{1}{2}
$$

Similarly,

$$
\Pi \geq \pi_1 \left(1 - e^{-t}\right) \left\{ \mathbb{E}_{\theta} \left[ 1 \{ \hat{g}(X) = 0, g_\theta^*(X) = 1 \} \mid Y = 1 \right] - \mathbb{P}_\theta \left( 0 \leq G_\theta^*(X) \leq t \mid Y = 1 \right) \right\}.
$$

Combine these two lower bounds, the identity \( \pi_0 = \pi_1 = 1/2 \) and the inequality \( 1 - \exp(-t) \geq t/2 \) for \( 0 < t < 1 \) to obtain,

$$
R_x(\hat{g}) - R_x(g_\theta^*) \geq \frac{t}{2} \left\{ L_\theta(\hat{g}) - \frac{t}{2\Delta_x} \varphi(R_t) - \frac{t}{2\Delta_x} \varphi(L_t) \right\},
$$

for any \( 0 < t < 1 \). From (A.2), we see that \( \Delta_x^2 = 4\mu_{\theta} \Sigma_{\theta}^{-1} \mu_{\theta} \), and we easily find

$$
\left( G_\theta^*(X) \mid Y = 0 \right) = \left( 2X^\top \Sigma_{\theta}^{-1} \mu_{\theta} \mid Y = 0 \right) \sim N \left( -\frac{1}{2} \Delta_x^2, \Delta_x^2 \right),
$$

and, similarly,

$$
G_\theta^*(X) \mid Y = 1 \sim N \left( \frac{1}{2} \Delta_x^2, \Delta_x^2 \right).
$$

An application of the mean value theorem yields

$$
R_x(\hat{g}) - R_x(g_\theta^*) \geq \frac{t}{2} \left\{ L_\theta(\hat{g}) - \frac{t}{2\Delta_x} \varphi(R_t) - \frac{t}{2\Delta_x} \varphi(L_t) \right\},
$$

for

$$
R_t \in \left[ \frac{1}{2} \Delta_x - \frac{t}{\Delta_x}, \frac{1}{2} \Delta_x \right], \quad L_t \in \left[ -\frac{1}{2} \Delta_x, -\frac{1}{2} \Delta_x + \frac{t}{\Delta_x} \right], \quad 0 < t < 1.
$$

Then, for \( 0 < t < \min(1, \Delta_x^2/2) \), we easily find from (B.8) that

$$
\frac{t}{2\Delta_x} \left\{ \varphi(R_t) + \varphi(L_t) \right\} \leq \frac{t}{\Delta_x} \sqrt{\frac{e}{2\pi}} \exp \left( -\frac{\Delta_x^2}{8} \right).
$$

Hence, for any \( 0 < t \leq \min(1, \Delta_x^2/2) \), we proved that

$$
\inf_{\theta} \sup_{\hat{g}} \mathbb{P}_\theta \left\{ R_x(\hat{g}) - R_x(g_\theta^*) \geq \frac{\eta}{\Delta} \exp \left( -\frac{\Delta_x^2}{8} + \delta \right) \right\},
$$

which holds by using \( K/n \leq c_1, \sigma^2/\lambda \leq c_2, \sigma^2 \eta/(\lambda n) \leq c_3 \) and requirement (a) of the constant \( C \) in the definition (B.6) of \( \eta \).

In the proof of the lower bounds (B.14) and (B.15) below, we consider subsets of \( \Theta(\lambda, \sigma, \Delta) \) such that, for any \( \theta \in \Theta(\lambda, \sigma, \Delta) \),

$$
\Delta_x^2 = \frac{\lambda}{\sigma^2 + \lambda} \Delta_x^2.
$$

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This implies
\[ \frac{\Delta^2}{2} \leq \Delta_x^2 \leq \Delta^2, \] (B.11)
provided that \( \sigma^2/\lambda \leq c_2 \leq 1 \), and, using (B.5),
\[ -\frac{\Delta^2}{8} + \delta = -\frac{\Delta_x^2}{8}. \] (B.12)
Note that (B.10) further implies \( t^* \leq 1 \leq \Delta^2/4 \leq \Delta_x^2/2 \). Then, by plugging \( t^* \) into (B.9) and using (B.11) and (B.12), we find
\[
\begin{align*}
\inf \sup_{\hat{g}} \mathbb{P}_\theta \left\{ R_x(\hat{g}) - R_x(g_0^*) \geq \frac{\eta}{\Delta} \exp \left( -\frac{\Delta_x^2}{8} + \delta \right) \right\} \\
\geq \inf \sup_{\hat{g}} \mathbb{P}_\theta \left\{ L_\theta(\hat{g}) \geq \left( \frac{e}{\pi} \right)^{1/4} \sqrt{\frac{\eta}{\Delta}} \exp \left( -\frac{\Delta_x^2}{8} + \delta \right) + \left( \frac{e}{\pi} \right)^{1/4} \sqrt{\frac{\eta}{\Delta}} \sqrt{\Delta} \exp \left(-\frac{\Delta_x^2}{8} \right) \right\} \\
= \inf \sup_{\hat{g}} \mathbb{P}_\theta \left\{ L_\theta(\hat{g}) \geq 2 \left( \frac{e}{\pi} \right)^{1/4} \sqrt{\frac{\eta}{\Delta}} \exp \left(-\frac{\Delta_x^2}{8} \right) \right\}.
\end{align*}
\]
In the next two sections we prove the inequalities
\[
\begin{align*}
\inf \sup_{\hat{g}} \mathbb{P}_\theta \left\{ L_\theta(\hat{g}) \geq C_1 \sqrt{\frac{K}{n}} \frac{1}{\Delta} \exp \left( -\frac{\Delta_x^2}{8} \right) \right\} &\geq (1 + c_0)/2, \tag{B.14} \\
\inf \sup_{\hat{g}} \mathbb{P}_\theta \left\{ L_\theta(\hat{g}) \geq C_2 \sqrt{\frac{\sigma^4(p-K)}{\lambda^2 n}} \exp \left(-\frac{\Delta_x^2}{8} \right) \right\} &\geq (1 + c_0)/2, \tag{B.15}
\end{align*}
\]
for some positive constants \( C_1 \) and \( C_2 \). By using requirement (b) for the leading constant \( C \) in the definition (B.6) of \( \eta \), we can conclude from the final lower bound (B.13) the proof of Theorem 3 for \( \Delta \gtrsim 1 \).

**Case 2:** \( \Delta = o(1) \). We further consider two cases and recall that
\[ \omega^*_n = \sqrt{\frac{K}{n} + \frac{\sigma^2}{\lambda} \Delta^2 + \frac{\sigma^2 \sigma^2}{\lambda^2} \Delta^2}. \]
When \( \omega^*_n = o(\Delta) \), we now prove the lower bound \( (\omega^*_n)^2/\Delta \). By choosing
\[ t_1 = c_t \sqrt{\frac{K}{n} + \frac{\sigma^4(p-K)}{\lambda^2 n} \Delta^2} \leq 1 \] (B.16)
in (B.8) for some constant \( c_t > 0 \) and by using \( \varphi(R_{t_1}) \leq 1 \), \( \varphi(L_{t_1}) \leq 1 \) and \( \Delta_x \leq \Delta \), we find
\[ R_x(\hat{g}) - R_x(g_0^*) \geq \frac{c_t}{2} L_\theta(\hat{g}) \sqrt{\frac{K}{n} + \frac{\sigma^4(p-K)}{\lambda^2 n} \Delta^2} - \frac{c_t^2}{2} \left[ \frac{K}{n} + \frac{\sigma^4(p-K)}{\lambda^2 n} \Delta^2 \right]. \]
From (B.14) and (B.15), it follows that
\[
\begin{align*}
\inf \sup_{\hat{g}} \mathbb{P}_\theta \left\{ R_x(\hat{g}) - R_x(g_0^*) \geq \frac{c_t C_3}{2} \left[ \frac{K}{n} \Delta + \frac{\sigma^4(p-K)}{\lambda^2 n} \Delta \right] \exp \left(-\frac{\Delta_x^2}{8} \right) \\
- \frac{c_t^2}{2} \left[ \frac{K}{n} \Delta + \frac{\sigma^4(p-K)}{\lambda^2 n} \Delta \right] \right\} &\geq c_0
\end{align*}
\]
for some constant \( C_3 > 0 \) depending on \( C_1 \) and \( C_2 \). Therefore, by using \( \Delta_x \geq \Delta / 2 \) and \( \Delta = o(1) \) and taking \( c_t \) sufficiently small, we conclude

\[
\inf \sup \mathbb{P}_\theta^D \left\{ R_x(\hat{g}) - R_x(g_\theta^*) \geq c_1 C_3 \left[ \frac{K}{n} \frac{1}{\Delta} + \frac{\sigma^4(p - K)}{\lambda^2 n} \Delta \right] \right\} \geq c_0.
\]

The above display together with (B.3) proves the lower bound \((\omega_n^*)^2 / \Delta\).

When \( \omega_n^* / \Delta \gtrsim 1 \), we proceed to prove the lower bound \( \omega_n^* \). Notice that \( \omega_n^* \geq \Delta \) implies \( \sqrt{K/n} \geq \Delta \), which, in view of (B.14) and by \(-\Delta_x \leq -\Delta / 2 = o(1)\), further implies

\[
\inf \sup \mathbb{P}_\theta^D \left\{ L_\theta(\hat{g}) \geq C_L \right\} \geq c_0
\]

for some \( C_L \in (0, 1] \). By choosing \( t_1 \) as (B.16) in (B.8), we have \( t_1 \simeq \sqrt{K/n}, t_1 / \Delta \gtrsim 1 \) and

\[
\max \{ \varphi(R_{t_1}), \varphi(L_{t_1}) \} \lesssim \exp \left( -\frac{c t_1^2}{\Delta^2} \right),
\]

hence

\[
\inf \sup \mathbb{P}_\theta^D \left\{ R_x(\hat{g}) - R_x(g_\theta^*) \geq \frac{C_L t_1}{2} - \frac{t_1^2}{2 \Delta} \exp \left( -\frac{c t_1^2}{\Delta^2} \right) \right\} \geq c_0.
\]

By choosing \( c_t \) to be sufficiently large and \( t_1 / \Delta \gtrsim 1 \), we have

\[
\frac{t_1}{\Delta} \exp \left( -\frac{c t_1^2}{\Delta^2} \right) \leq \frac{C_L}{2},
\]

such that

\[
\inf \sup \mathbb{P}_\theta^D \left\{ R_x(\hat{g}) - R_x(g_\theta^*) \geq \frac{C_L t_1}{4} \right\} \geq c_0.
\]

The claim then follows from

\[
\frac{t_1}{4} + \frac{\sigma^2}{\lambda} \Delta^2 \simeq \sqrt{\frac{K}{n}} + \frac{\sigma^2 p}{\lambda^2 n} \Delta^2 + \frac{\sigma^2}{\lambda} \Delta^2 \simeq \sqrt{\frac{K}{n}} + \omega_n^*
\]

by using \( \Delta \lesssim 1 \), \( \sqrt{K/n} \gtrsim \Delta \), \( \sigma^2 \lesssim \lambda \) and \( p \sigma^2 \lesssim n \lambda \).

\[\square\]

**B.1 Proof of (B.15)**

**Proof.** We aim to invoke the following lemma to obtain the desired lower bound. The lemma below follows immediately from the proof of Proposition 1 in Azizyan et al. (2013) together with Theorem 2.5 in Tsybakov (2009).

**Lemma 24.** Let \( M \geq 2 \) and \( \theta_0, \ldots, \theta_M \in \Theta \). For some constant \( c_0 \in (0, 1/8] \), \( \gamma > 0 \) and any classifier \( \hat{g} \), if \( \KL(\mathbb{P}_\theta^D, \mathbb{P}_\theta^D) \leq c_0 \log M \) for all \( 1 \leq i \leq M \), and \( L_{\theta_i}(\hat{g}) < \gamma \) implies \( L_{\theta_j}(\hat{g}) \geq \gamma \) for all \( 0 \leq i \neq j \leq M \), then

\[
\inf \sup_{\hat{g}} \sup_{i \in \{1, \ldots, M\}} \mathbb{P}_\theta^D \{ L_{\theta_i}(\hat{g}) \geq \gamma \} \geq \frac{\sqrt{M}}{\sqrt{M} + 1} \left[ 1 - 2c_0 - \sqrt{\frac{2c_0}{\log M}} \right].
\]
To this end, we start by describing our construction of hypotheses of \(\theta \in \Theta(\lambda, \sigma, \Delta)\) defined in (2.3). Without loss of generality, we assume \(\sigma = 1\) and \(\Sigma_{Z|Y} = I_K\). We consider a subspace of \(\Theta(\lambda, \sigma, \Delta)\) where \(\lambda_1(\Sigma_{Z|Y}A^\top) = \lambda_K(\Sigma_{Z|Y}A^\top) = \lambda\). By further writing \(\Sigma_{Z|Y}A^\top = AA^\top = \lambda BB^\top\) with \(B \in O_{p\times K}\), we consider

\[
\theta^{(j)} = \left(\sqrt{\lambda} B^{(j)}, I_K, I_p, \alpha, -\alpha, 1, \frac{1}{2}, \frac{1}{2}\right), \quad \text{for } j = 1, \ldots, M, \tag{B.17}
\]

where

\[
\alpha = \begin{bmatrix} \Delta/2 \\ 0_{K-1} \end{bmatrix}, \quad B^{(j)} = \begin{bmatrix} \sqrt{1 - \varepsilon^2} & 0 \\ 0_{K-1} & I_K \\ \varepsilon J^{(j)} & 0_{p-K} \end{bmatrix}, \tag{B.18}
\]

with

\[
\varepsilon^2 = c_0 c_1 \frac{(p - K)}{\lambda n} \frac{1}{2\lambda + \Delta^2} \tag{B.19}
\]

for some constants \(c_0 \in (0, 1/8]\) and \(c_1 > 0\). Here \(J^{(1)}, \ldots, J^{(M)} \in O_{(p-K)\times 1}\) are chosen according to the hypercube construction in Lemma 25 with \(m = p - K\). It is easy to see that \(\theta^{(j)} \in \Theta(\lambda, \sigma = 1, \Delta)\) for all \(1 \leq j \leq M\). Lemma 26 below collects several useful properties of \(\theta^{(j)}\).

Next, to apply Lemma 24, it suffices to verify

1. \(\text{KL}(P_{\theta^{(i)}}, P_{\theta^{(j)}}) \leq c_0 \log(M - 1)\) for all \(1 \leq i \leq M\);
2. \(L_{\theta^{(i)}}(\hat{g}) + L_{\theta^{(j)}}(\hat{g}) \geq 2\gamma\), for all \(1 \leq i \neq j \leq M\) and any measurable \(\hat{g}\), with

\[
\gamma \asymp e^{-\Delta^2/8} \sqrt{\frac{\varepsilon^2}{\lambda}}, \quad \Delta^2 = \frac{\lambda}{1 + \lambda} \Delta^2.
\]

The first claim is proved by invoking Lemmas 25 and 27 together with the choice of \(\varepsilon\) in (B.19) while the second claim is proved in Lemma 28. The result then follows by noting that

\[
\varepsilon^2 \asymp \frac{p - K}{n\lambda(1 + \Delta^2)} \asymp \frac{p - K}{n\lambda\Delta^2}.
\]

\[\square\]

**B.1.1 Lemmas used in the proof of (B.15)**

The following lemma is adapted from Vu and Lei (2013, Lemma A.5).

**Lemma 25** (Hypercube construction). Let \(m \geq 1\) be an integer. There exist \(J^{(1)}, \ldots, J^{(M)} \in O_{m\times 1}\) with the following properties:

1. \(\|J^{(i)} - J^{(j)}\|_2^2 \geq 1/4\) for all \(i \neq j\), and
2. \(\log M \geq \max\{cm, \log m\}\), where \(c > 1/30\) is an absolute constant.

**Proof.** The case for \(m \geq c\) is proved in Vu and Lei (2013, Lemma A.5) by taking \(m = s\). For \(m = 2\), one can choose \(J^{(i)} = (-1)^i e_1\), for \(i = 1, 2\), and \(J^{(i)} = (-1)^i e_2\), for \(i = 3, 4\), such that \(M = 4\) and \(\|J^{(i)} - J^{(j)}\|_2^2 = 4\). Here \(\{e_1, e_2\}\) represents the set of canonical vectors in \(\mathbb{R}^2\). For \(m = 1\), one can simply take \(J^{(i)} = (-1)^i\) for \(i = 1, 2\). \(\square\)
The following lemma collects some useful identities, under the choices of $\theta^{(i)}$ in (B.17) – (B.18).

**Lemma 26.** Fix any $i \in \{1, \ldots, M\}$. Let $B^{(i)}$ and $\alpha$ defined in (B.18). Further let

$$\Sigma^{(i)} = \lambda B^{(i)}(B^{(i)})^\top + I_p, \quad \mu^{(i)} = \sqrt{\lambda} B^{(i)} \alpha.\] $$

(i) $|\Sigma^{(i)}| = (\lambda + 1)^K$ and

$$\begin{align*}
(Sigma^{(i)})^{-1} &= \frac{1}{\lambda + 1} B^{(i)}(B^{(i)})^\top + I_p - B^{(i)}(B^{(i)})^\top \\
&= I_p - \frac{\lambda}{\lambda + 1} B^{(i)}(B^{(i)})^\top .
\end{align*}\] $$

(ii) $$(\Sigma^{(i)})^{-1} \mu^{(i)} = \frac{\sqrt{\lambda}}{1 + \lambda} B^{(i)} \alpha - \frac{\sqrt{\lambda}}{1 + \lambda^2} B^{(i)} \alpha.$$

(iii) $$(\mu^{(i)})^\top (\Sigma^{(i)})^{-1} \mu^{(i)} = \frac{\lambda}{1 + \lambda} \alpha^\top (B^{(i)})^\top B^{(i)} \alpha = \frac{\lambda}{1 + \lambda} \frac{\Delta^2}{4}.$$ 

**Proof.** Notice that $B^{(i)} \in O_{p \times K}$. Then part (i) is easy to verify. Parts (ii) and (iii) follow immediately from (B.18) and (B.20). \qed

Let $\mathbb{P}_{\theta^{(i)}}$, for $2 \leq i \leq M$, denote the distribution of $(X, Y)$ parametrized by $\theta^{(i)}$. The following lemma provides upper bounds of the KL-divergence between $\mathbb{P}_{\theta^{(1)}}$ and $\mathbb{P}_{\theta^{(i)}}$.

**Lemma 27 (KL-divergence).** For any $\theta^{(i)}$, let

$$(X \mid Y = 1) \sim N_p(\mu^{(i)}, \Sigma^{(i)}), \quad (X \mid Y = 0) \sim N_p(-\mu^{(i)}, \Sigma^{(i)})$$

with $\mu^{(i)} = \sqrt{\lambda} B^{(i)} \alpha$, $\Sigma^{(i)} = \lambda B^{(i)}(B^{(i)})^\top + I_p$ and $B^{(i)} \in O_{p \times K}$. Then

$$\text{KL}(\mathbb{P}_{\theta^{(1)}}, \mathbb{P}_{\theta^{(i)}}) \leq n \left( \frac{2\lambda}{1 + \lambda} + \frac{\Delta^2}{2} \right) \lambda \varepsilon^2.$$ 

**Proof.** Since $(X, Y)$ contains $n$ i.i.d. copies of $(X, Y)$, it suffices to prove

$$\text{KL}(\mathbb{P}_{\theta^{(1)}}, \mathbb{P}_{\theta^{(i)}}) = \text{KL} \left( N_p(\mu^{(1)}, \Sigma^{(1)}), N_p(\mu^{(i)}, \Sigma^{(i)}) \right) \leq \left( \frac{2\lambda}{1 + \lambda} + \frac{\Delta^2}{2} \right) \lambda \varepsilon^2.$$ 

By the formula of KL-divergence between two multivariate normal distributions,

$$\text{KL}(\mathbb{P}_{\theta^{(1)}}, \mathbb{P}_{\theta^{(i)}}) \leq \frac{1}{2} \left\{ \text{tr} \left[ (\Sigma^{(i)})^{-1} (\Sigma^{(1)} - \Sigma^{(i)}) \right] + \log \left( \frac{|\Sigma^{(i)}|}{|\Sigma^{(1)}|} \right) \right\}$$

$$\quad + \frac{1}{2} \left( \mu^{(i)} - \mu^{(1)} \right)^\top (\Sigma^{(i)})^{-1} \left( \mu^{(i)} - \mu^{(1)} \right) := I_1 + I_2.$$
Combining the bounds of \( I \)

\[ I_1 = \frac{\lambda^2}{1 + \lambda} \cdot \frac{1}{2} \| B^{(i)} (B^{(i)})^T - B^{(1)} (B^{(1)})^T \|_F^2 \leq \frac{\lambda^2}{1 + \lambda} \frac{\varepsilon^2}{2} \| J^{(i)} - J^{(1)} \|_2^2. \]

For \( I_2 \), by using part (i) of Lemma 26 together with

\[ \mu^{(i)} - \mu^{(1)} = \sqrt{\lambda} (B^{(i)} - B^{(1)}) \alpha = \frac{\Delta \sqrt{\lambda}}{2} \varepsilon (J^{(i)} - J^{(1)}), \]

from (B.18), we find

\[ I_2 = \frac{\lambda \Delta^2}{8} \varepsilon^2 (J^{(i)} - J^{(1)})^T \left( I_p - \frac{\lambda}{\lambda + 1} B^{(i)} (B^{(i)})^T \right) (J^{(i)} - J^{(1)}) \]

\[ \leq \frac{\lambda \Delta^2}{8} \varepsilon^2 \| J^{(i)} - J^{(1)} \|_2^2. \]

Combining the bounds of \( I_1 \) and \( I_2 \) and using \( \| J^{(i)} - J^{(1)} \|_2^2 \leq 4 \) complete the proof. \( \square \)

Recall that \( L_{\theta(\cdot)} \) is defined in (B.7). The following lemma establishes lower bounds of

\[ L_{\theta^{(i)}(\cdot)} (\hat{g}) + L_{\theta^{(j)}(\cdot)} (\hat{g}) \]

for any measurable \( \hat{g} \).

**Lemma 28.** Let \( \theta^{(i)} \) for \( 1 \leq i \leq M \) be constructed as (B.17) – (B.18). Under conditions of Theorem 3, for any measurable \( \hat{g} \), one has

\[ L_{\theta^{(i)}(\cdot)} (\hat{g}) + L_{\theta^{(j)}(\cdot)} (\hat{g}) \gtrsim e^{-\Delta_x^2/8} \sqrt{\frac{\varepsilon^2}{\lambda}} \]

with \( \Delta_x^2 = \lambda \Delta^2 / (1 + \lambda) \).

**Proof.** Pick any \( i \neq j \in \{1, \ldots, M\} \) and any \( \hat{g} \). For simplicity, we write \( \theta = \theta^{(i)} \) and \( \theta' = \theta^{(j)} \) with corresponding \( B = B^{(i)} \) and \( B' = B^{(j)} \). We also write \( L_\theta = L_\theta(\hat{g}) \) and \( L_{\theta'} = L_{\theta'}(\hat{g}) \). The proof consists of three steps:

(a) Bound \( L_\theta + L_{\theta'} \) from below by a \( p \)-dimensional integral;

(b) Reduce the \( p \)-dimensional integral to a 2-dimensional integral;

(c) Bound from below the 2-dimensional integral.

**Step (a)** By definition in (B.7),

\[ L_\theta + L_{\theta'} = \int_{\hat{g} \neq g_\theta^*} dP_\theta(x) + \int_{\hat{g} \neq g_{\theta'}^*} dP_{\theta'}(x) \]

\[ \geq \min \{ dP_\theta(x), dP_{\theta'}(x) \} \]

\[ \geq \min \{ dP_\theta(x), dP_{\theta'}(x) \} . \]

In the last step we used

\[ \{ g_\theta^* \neq g_{\theta'}^* \} = \{ \hat{g} = g_\theta^*, \hat{g} \neq g_{\theta'}^* \} \cup \{ \hat{g} \neq g_\theta^*, \hat{g} = g_{\theta'}^* \} \]

\[ \subseteq \{ \hat{g} \neq g_\theta^* \} \cup \{ \hat{g} \neq g_{\theta'}^* \} . \]
Since

\[ P_\theta = \frac{1}{2} N_p(\mu_\theta, \Sigma_\theta) + \frac{1}{2} N_p(-\mu_\theta, \Sigma_\theta) \]

and \( g^*_\theta(x) = 1 \{ x^T \Sigma_\theta^{-1} \mu_\theta \geq 0 \} \) from (B.2), we obtain

\[ L_\theta + L_{\theta'} \]

\[ \geq \frac{1}{2} \int_{x^T \Sigma_\theta^{-1} \mu_\theta \geq 0} \frac{1}{(2\pi)^{p/2}} \min \left\{ |\Sigma_\theta|^{-1/2} \left[ \exp \left( -\frac{1}{2} \| x - \mu_\theta \|^2_{\Sigma_\theta} \right) + \exp \left( -\frac{1}{2} \| x + \mu_\theta \|^2_{\Sigma_\theta} \right) \right], \right\} \]

\[ \geq \frac{1}{2} \int_{x^T \Sigma_\theta^{-1} \mu_\theta \geq 0} \frac{1}{(2\pi)^{p/2}} \min \left\{ |\Sigma_{\theta'}|^{-1/2} \left[ \exp \left( -\frac{1}{2} \| x - \mu_{\theta'} \|^2_{\Sigma_{\theta'}} \right) + \exp \left( -\frac{1}{2} \| x + \mu_{\theta'} \|^2_{\Sigma_{\theta'}} \right) \right], \right\} \]

\[ \geq \int_{x^T \Sigma_\theta^{-1} \mu_\theta \geq 0} |\Sigma_{\theta'}|^{-1/2} \min \left\{ \exp \left( -\frac{1}{2} \| x - \mu_{\theta'} \|^2_{\Sigma_{\theta'}} \right), \exp \left( -\frac{1}{2} \| x + \mu_{\theta'} \|^2_{\Sigma_{\theta'}} \right) \right\} \]

\[ \geq e^{-\frac{\Delta^2}{4}} \int_{x^T \Sigma_\theta^{-1} \mu_\theta \geq 0} \frac{|\Sigma_{\theta'}|^{-1/2}}{(2\pi)^{p/2}} \min \left\{ \exp \left( -\frac{1}{2} x^T \Sigma_{\theta'}^{-1} x \right), \exp \left( -\frac{1}{2} x^T \Sigma_{\theta'}^{-1} x \right) \right\} \]

\[ \geq e^{-\frac{\Delta^2}{4}} \int_{x^T \Sigma_\theta^{-1} \mu_\theta \geq 0} \frac{|\Sigma_{\theta'}|^{-1/2}}{(2\pi)^{p/2}} \min \left\{ \exp \left( -\frac{1}{2} x^T \Sigma_{\theta'}^{-1} x \right), \exp \left( -\frac{1}{2} x^T \Sigma_{\theta'}^{-1} x \right) \right\} \]

The equality uses the fact that \( X \) has the same distribution as \(-X\) and the identity

\[ |\Sigma_\theta| = |\Sigma_{\theta'}| = (\lambda + 1)^K \]

from part (i) of Lemma 26. The last step uses the fact that

\[ \frac{\Delta^2}{4} = \frac{(\lambda + 2) \mu_{\theta'}^T \Sigma_{\theta'}^{-1} \mu_{\theta}}{\lambda} = \frac{\Delta^2}{4} = \frac{\mu_{\theta'}^T \Sigma_{\theta'}^{-1} \mu_{\theta'}}{\lambda} \]

from part (iii) of Lemma 26.

**Step (b)** In the following, we provide a lower bound for

\[ T := \int_{x^T \Sigma_\theta^{-1} \mu_\theta \geq 0} \frac{|\Sigma_{\theta'}|^{-1/2}}{(2\pi)^{p/2}} \min \left\{ \exp \left( -\frac{1}{2} x^T \Sigma_{\theta'}^{-1} x \right), \exp \left( -\frac{1}{2} x^T \Sigma_{\theta'}^{-1} x \right) \right\} \]

Recall from (B.18) and (B.21) that

\[ \Sigma_{\theta}^{-1} = I_p - \frac{\lambda}{1 + \lambda} B_{-1} B_{-1}^T = \frac{\lambda}{1 + \lambda} B_{1} B_{1}^T, \]

\[ \Sigma_{\theta'}^{-1} = I_p - \frac{\lambda}{1 + \lambda} B_{-1} B_{-1}^T = \frac{\lambda}{1 + \lambda} B_{1} B_{1}^T. \]
Further note from part (ii) of Lemma 26 that

\[ \Sigma^{-1}_\theta \mu_\theta = \frac{\sqrt{\lambda}}{1 + \lambda} \frac{\Delta}{2} B_1, \quad \Sigma^{-1}_{\theta'} \mu_{\theta'} = \frac{\sqrt{\lambda}}{1 + \lambda} \frac{\Delta}{2} B'_1. \]

Plugging these expressions in \( T \) yields

\[
T = \int_{\mathbf{x}^\top B_1 \geq 0} \frac{|\Sigma_\theta|^{-1/2}}{(2\pi)^{p/2}} \exp \left( -\frac{1}{2} \mathbf{x}^\top \left( \mathbf{I}_p - \frac{\lambda}{\lambda + 1} B_{-1} B_{-1}^\top \right) \mathbf{x} \right) \\
\min \left\{ \exp \left( \frac{1}{2} \mathbf{x}^\top \frac{\lambda}{\lambda + 1} B_1 B_1^\top \mathbf{x} \right), \quad \exp \left( \frac{1}{2} \mathbf{x}^\top \frac{\lambda}{\lambda + 1} B'_1 B'_1^\top \mathbf{x} \right) \right\} \, d\mathbf{x}.
\]

Let \( H \in \mathcal{O}_{p \times p} \) such that

\[
HB_1 = \begin{bmatrix} a & b \\ b & 0 \end{bmatrix} := \begin{bmatrix} u \\ 0_{p-2} \end{bmatrix}, \quad HB'_1 = \begin{bmatrix} a \\ -b \end{bmatrix} := \begin{bmatrix} v \\ 0_{p-2} \end{bmatrix}, \quad a > 0. \quad (B.24)
\]

Such an \( H \) exists since \( [B_1, B'_1] \in \mathbb{R}^{p \times 2} \) has rank 2 and \( \|B_1\|_2 = \|B'_1\|_2 = 1 \). By changing variables \( y = Hx \) and by writing \( y_1 = (y_1, y_2) \), we obtain

\[
T = \int_{y_1, u \geq 0} \frac{|\Sigma_\theta|^{-1/2}}{(2\pi)^{p/2}} \exp \left( -\frac{1}{2} y_1^\top H \left( \mathbf{I}_p - \frac{\lambda}{\lambda + 1} B_{-1} B_{-1}^\top \right) H^\top y \right) \\
\min \left\{ \exp \left( \frac{\lambda (y_1^\top u)^2}{2(1 + \lambda)} \right), \quad \exp \left( \frac{\lambda (y_1^\top v)^2}{2(1 + \lambda)} \right) \right\} \, dy.
\]

Denote

\[
Q := H \left( \mathbf{I}_p - \frac{\lambda}{\lambda + 1} B_{-1} B_{-1}^\top \right)^{-1} H^\top = H(\lambda B_{-1} B_{-1}^\top + \mathbf{I}_p)H^\top. \quad (B.25)
\]

Notice that \( |Q| = (\lambda + 1)^{K-1} = |\Sigma_\theta|/(\lambda + 1) \) by (B.23). We further have

\[
T = \frac{1}{\sqrt{\lambda + 1}} \int_{y_1, u \geq 0} \frac{|Q|^{-1/2}}{(2\pi)^{p/2}} \exp \left( -\frac{1}{2} y_1^\top Q^{-1} y \right) \\
\min \left\{ \exp \left( \frac{\lambda (y_1^\top u)^2}{2(1 + \lambda)} \right), \quad \exp \left( \frac{\lambda (y_1^\top v)^2}{2(1 + \lambda)} \right) \right\} \, dy
\]

\[
= \frac{1}{\sqrt{\lambda + 1}} \int_{a y_1 + b y_2 \geq 0} \frac{|Q_{11}|^{-1/2}}{2\pi} \exp \left( -\frac{1}{2} y_1^\top (Q_{11})^{-1} y_1 \right) \\
\min \left\{ \exp \left( \frac{\lambda (a y_1 + b y_2)^2}{2(1 + \lambda)} \right), \quad \exp \left( \frac{\lambda (a y_1 - b y_2)^2}{2(1 + \lambda)} \right) \right\} \, dy_1
\]

where \( Q_{11} \) is the first \( 2 \times 2 \) submatrix of \( Q \). Recall that \( a > 0 \) and on the area of integration \( \{a y_1 + b y_2 \geq 0, ay_1 - by_2 < 0\} \) we have

\[
\exp \left( \frac{\lambda (a y_1 + b y_2)^2}{2(1 + \lambda)} \right) \geq \exp \left( \frac{\lambda (a y_1 - b y_2)^2}{2(1 + \lambda)} \right) \iff y_1 \geq 0.
\]

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Splitting $T$ into two parts further gives

$$
T = \frac{1}{\sqrt{\lambda + 1}} \int_{a y_1 + b y_2 \geq 0} \frac{|Q_{II}|^{-1/2}}{2\pi} \exp \left[ -\frac{1}{2} y_I^\top \left( Q_{II}^{-1} - \frac{\lambda}{1 + \lambda} v v^\top \right) y_I \right] dy_I \\
+ \frac{1}{\sqrt{\lambda + 1}} \int_{a y_1 - b y_2 \geq 0} \frac{|Q_{II}|^{-1/2}}{2\pi} \exp \left[ -\frac{1}{2} y_I^\top \left( Q_{II}^{-1} - \frac{\lambda}{1 + \lambda} u u^\top \right) y_I \right] dy_I \\
:= T_1 + T_2.
$$

**Step (c)** We bound from below $T_1$ first. Denote

$$
G = \left( Q_{II}^{-1} - \frac{\lambda}{1 + \lambda} v v^\top \right)^{-1} = Q_{II} + \frac{\lambda}{1 + \lambda} Q_{II} v v^\top Q_{II} = Q_{II} + \lambda Q_{II} v v^\top Q_{II}
$$

where the second equality uses the Sherman-Morrison formula and the third equality is due to the fact that

$$
v^\top Q_{II} v = B_1^\top H^\top H (\lambda B_{-1} B_{-1}^\top + I_p) H^\top H B_1' \quad \text{by (B.24) and (B.25)}
$$

$$
= \lambda B_1^\top B_{-1} B_{-1}^\top B_1' + 1 \quad \text{by } H \in \mathcal{O}_{p \times p}
$$

$$
= 1 \quad \text{by (B.18).} \quad (B.27)
$$

Further observe that

$$
|G| = |Q_{II}| \left| I_2 + \lambda Q_{II}^{1/2} v v^\top Q_{II} \right| = |Q_{II}| \left| (1 + \lambda v^\top Q_{II} v) \right| = |Q_{II}|(1 + \lambda).
$$

We obtain

$$
T_1 = \int_{a y_1 + b y_2 \geq 0} \frac{|G|^{-1/2}}{2\pi} \exp \left[ -\frac{1}{2} y_I^\top G^{-1} y_I \right] dy_I \\
= \int_{a y_1 - b y_2 \geq 0} \frac{|G|^{-1/2}}{2\pi} \exp \left[ -\frac{1}{2} y_I^\top G^{-1} y_I \right] dy_I.
$$

By changing of variables $z = G^{-1/2} y_I$ again and writing

$$
\zeta_1 = G^{1/2} v, \quad \zeta_2 = G^{1/2} \left[ \begin{array}{c} a \\ 0 \end{array} \right]
$$

for simplicity, one has

$$
T_1 = \int_{z^T \zeta_1 < 0} \frac{1}{2\pi} e^{-\frac{1}{2} z^T z} dz = \int_{\zeta_1 \cos \theta + \zeta_2 \sin \theta < 0} \frac{1}{\pi} d\theta.
$$

Note that, the integral is simply the area within the half unit circle $\{(x, y) : x^2 + y^2 \leq 1, y \geq 0\}$ intersected by vectors $\zeta_1$ and $\zeta_2$. We thus conclude

$$
T_1 = \frac{1}{2\pi} \arccos(\zeta_1, \zeta_2) \geq \frac{1}{2\pi} \left\| \zeta_1 - \zeta_2 \right\|_2
$$

where $\zeta_1 = \|\zeta_1\|_2, \zeta_2 = \|\zeta_2\|_2$ and $\arccos(\zeta_1, \zeta_2)$ denotes the length of the arc between $\zeta_1$ and $\zeta_2$.  

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We proceed to calculate $\|\tilde{\zeta}_1 - \tilde{\zeta}_2\|_2$. First note that
\[
\|\zeta_1\|_2^2 = v^\top G v \overset{(B.26)}{=} v^\top \left( Q_{11} + \lambda Q_{11} v v^\top Q_{11} \right) v \overset{(B.27)}{=} 1 + \lambda.
\] Since
\[
Q_{11} v \overset{(B.25)}{=} H_I(\lambda B_{-1} B_{-1}^\top + I_p) H_I v \overset{(B.24)}{=} H_I(\lambda B_{-1} B_{-1}^\top + I_p) H_I H_B' = H_I H_B',
\] we obtain
\[
\|\zeta_2\|_2^2 = \frac{1}{4}(u + v)^\top G(u + v)
= \frac{1}{4}(B_1 + B_1')^\top H_I^\top \left( Q_{11} + \lambda Q_{11} v v^\top Q_{11} \right) H_I (B_1 + B_1')
= \frac{1}{4}(B_1 + B_1')^\top H_I^\top \left[ H_I(\lambda B_{-1} B_{-1}^\top + I_p) H_I^\top + \lambda H_I B_1' B_1'^\top H_I^\top \right] H_I (B_1 + B_1')
= \frac{1}{4}(B_1 + B_1')^\top \left( I_p + \lambda B_1' B_1'^\top \right) (B_1 + B_1')
= \frac{1}{4}\left[ \lambda + 2 + 2(\lambda + 1) B_1^\top B_1' + \lambda(B_1^\top B_1')^2 \right].
\] The penultimate step uses the orthogonality between $B_{-1}$ and $B_1 + B_1'$. Since
\[
1 - B_1^{-1} B_1' = \frac{1}{2}\|B_1 - B_1'\|_2^2 = \frac{\epsilon^2}{2}\|J^{(i)} - J^{(j)}\|_2^2 \leq 2\epsilon^2
\] which can be bounded by a sufficiently small constant, we have $B_1^\top B_1' \asymp 1$ hence $\|\zeta_2\|_2^2 \asymp \lambda + 1$.

Finally, similar arguments yield
\[
\zeta_1^\top \zeta_2 = \frac{1}{2} v^\top G(u + v)
= \frac{1}{2}(B_1')^\top \left( I_p + \lambda B_1' B_1'^\top \right) (B_1 + B_1')
= \frac{1}{2}(1 + \lambda)(1 + B_1^\top B_1')
\asymp 1 + \lambda.
\] We thus have, after a bit algebra,
\[
\|\zeta_1\|_2^2 \|\zeta_2\|_2^2 - (\zeta_1^\top \zeta_2)^2 = \frac{1}{4}(1 + \lambda)(1 + B_1^\top B_1')(1 - B_1^\top B_1') \asymp (1 + \lambda)\epsilon^2,
\] hence
\[
\frac{1}{2}\|\tilde{\zeta}_1 - \tilde{\zeta}_2\|_2^2 = \frac{\|\zeta_1\|_2^2 \|\zeta_2\|_2^2 - \zeta_1^\top \zeta_2}{\|\zeta_1\|_2 \|\zeta_2\|_2}
= \frac{\|\zeta_1\|_2^2 \|\zeta_2\|_2^2 - (\zeta_1^\top \zeta_2)^2}{\|\zeta_1\|_2 \|\zeta_2\|_2} \frac{1}{\|\zeta_1\|_2 \|\zeta_2\|_2}
\asymp \frac{\epsilon^2}{1 + \lambda}
\] implying that
\[
T_1 \gtrsim \sqrt{\frac{\epsilon^2}{\lambda}}.
\] Following the same line of reasoning, we can derive the same lower bound for $T_2$. We conclude that
\[
L_\theta + L_\theta' \gtrsim e^{-\Delta^2 / 8} \sqrt{\frac{\epsilon^2}{\lambda}},
\] which completes the proof. \qed
B.2 Proof of (B.14)

The proof of (B.14) follows the same lines of reasoning as the proof of (B.15). To construct hypotheses of $\Theta(1, \sigma = 1, \Delta)$, we consider

$$\theta^{(j)} = \left( \sqrt{\lambda} B, I_K, I_p, \alpha^{(j)}, \alpha^{(j)}, \frac{1}{2}, \frac{1}{2} \right), \quad \text{for } j = 1, \ldots, M',$$

with $B \in \mathcal{O}_{p \times K}$ and

$$\alpha^{(j)} = \frac{\Delta}{2} \left[ \sqrt{1 - (\varepsilon')^2} \varepsilon' J^{(j)} \right].$$

(B.29)

Here $J^{(j)}$ for $j = 1, \ldots, M'$ are again chosen according to Lemma 25 with $m = K - 1$ and

$$(\varepsilon')^2 = c_0 c_1 (K - 1) \frac{n \Delta^2}{4}.$$  

(B.30)

for some constant $c_0 \in (0, 1/8]$ and $c_1 > 0$. Notice that $\|\alpha^{(j)}\|_2^2 = \Delta^2 / 4$ for all $j \in \{0, 1, \ldots, M'\}$, so that $\theta^{(j)} \in \Theta(1, \sigma = 1, \Delta)$. From part (iii) of Lemma 26, we also have

$$\frac{\Delta^2}{4} \frac{(\lambda \Sigma^{1/2}) \mu^{((j))} \Sigma^{1/2} \mu^{((j))}}{1 + \lambda} \geq \frac{\lambda}{1 + \lambda} \|\alpha^{(j)}\|_2^2 = \frac{\Delta^2}{4}, \quad \forall j \in \{1, \ldots, M'\}.$$  



Next, to invoke Lemma 24, it remains to verify

1. $\text{KL}(\mathbb{P}_{\theta^{(i)}}, \mathbb{P}_{\theta^{(i)}}) \leq c_0 \log M'$ for all $1 \leq i \leq M'$;
2. $L_{\theta^{(i)}}(\hat{g}) + L_{\theta^{(i)}}(\hat{g}) \geq 2\gamma$, for all $1 \leq i \neq j \leq M'$ and any $\hat{g}$, with

$$\gamma \geq \frac{1}{\Delta^2} e^{-\Delta^2/4} \sqrt{\frac{K}{n}}, \quad \Delta^2 = \frac{\lambda}{1 + \lambda} \Delta^2.$$  

To prove (1), note that the distribution of $(Y, X)$ parametrized by $\theta^{(i)}$ is

$$\mathbb{P}_{\theta^{(i)}} = \frac{1}{2} N_p (\mu_{\theta^{(i)}}, \Sigma_{\theta^{(i)}}) + \frac{1}{2} N_p (-\mu_{\theta^{(i)}}, \Sigma_{\theta^{(i)}})$$

with $\mu_{\theta^{(i)}} = \sqrt{\lambda} B \alpha^{(i)}$ and $\Sigma_{\theta^{(i)}} = \lambda BB^T + I_p$. Following the arguments in the proof of Lemma 27 yields

$$\text{KL}(\mathbb{P}_{\theta^{(i)}}, \mathbb{P}_{\theta^{(i)}}) = \frac{1}{2} (\mu_{\theta^{(i)}}, \mu_{\theta^{(i)}})^T (\lambda BB^T + I_p)^{-1} (\mu_{\theta^{(i)}}, \mu_{\theta^{(i)}})$$

$$= \frac{\lambda}{2} (\alpha^{(i)} - \alpha^{(1)})^T B^T \frac{1}{\lambda + 1} BB^T B (\alpha^{(i)} - \alpha^{(1)}) \quad \text{by (B.20)},$$

$$= \frac{\lambda \Delta^2}{8(1 + \lambda)} (\varepsilon')^2 8 \|J^{(i)} - J^{(1)}\|_2^2 \quad \text{by (B.31)}$$

Claim (1) then follows from $\log M' \geq cK$ by using Lemma 25 and the additivity of KL divergence among independent distributions. Since claim (2) is proved in Lemma 29, the proof is complete.  

$\square$
Lemma 29. Let $\theta^{(i)}$ for $1 \leq i \leq M'$ be constructed as (B.28) – (B.29). Under $K/n \leq c_1$ and $1/\lambda \leq c_2$, for any measurable $\hat{g}$, one has

$$L_{\theta^{(i)}}(\hat{g}) + L_{\theta^{(j)}}(\hat{g}) \geq \frac{1}{\Delta_x} e^{-\Delta^2_x / 8} \sqrt{\frac{K}{n}},$$

with $\Delta^2_x = \lambda \Delta^2/(1 + \lambda)$.

Proof. The proof uses the same reasoning for proving Lemma 28. Pick any $i \neq j \in \{0, \ldots, M'\}$ and write $L_\theta = L_{\theta^{(i)}}(\hat{g})$ and $L_{\theta'} = L_{\theta^{(j)}}(\hat{g})$. From (B.22), one has

$$L_{\theta^{(i)}} + L_{\theta^{(j)}} \geq e^{-\Delta^2_y / 8} \int_{x \in \Sigma} \left| \frac{\Sigma^{-1/2} \exp \left( -\frac{1}{2} x^\top \Sigma^{-1} x \right) }{(2\pi)^{p/2}} \right| dx$$

where $\Sigma := \Sigma_\theta = \Sigma_{\theta'} = \lambda BB^\top + I_p$. Let $H \in \mathcal{O}_{p \times p}$ such that

$$H \Sigma^{-1} \mu_\theta = \begin{bmatrix} a \\ b \\ 0_{p-2} \end{bmatrix}, \quad H \Sigma^{-1} \mu_{\theta'} = \begin{bmatrix} a \\ -b \\ 0_{p-2} \end{bmatrix}, \quad a > 0.$$

By changing variable $y = Hx$ and writing $y_i^\top = (y_1, y_2)$, we find

$$L_{\theta^{(i)}} + L_{\theta^{(j)}} \geq e^{-\Delta^2_y / 8} \int_{y_i \geq 0} \frac{H \Sigma \Sigma^\top}{2\pi} \left| \int_{y_i^\top v < 0} \frac{\Sigma^{-1/2} \exp \left( -\frac{1}{2} y_i^\top H \Sigma^{-1} H^\top y \right) }{(2\pi)^{p/2}} dy_i \right| dy_i,$$

where $Q_{11}$ is the first $2 \times 2$ matrix of $Q = H \Sigma H^\top$.

By another change of variable and the same reasoning in the proof of Lemma 28,

$$L_{\theta^{(i)}} + L_{\theta^{(j)}} \geq e^{-\Delta^2_y / 8} \int_{z \in Q_{11}^{1/2} u \geq 0} \frac{1}{2\pi} \left| \int_{z \in Q_{11}^{1/2} v < 0} \frac{\Sigma^{-1/2} \exp \left( -\frac{1}{2} z^\top z \right) }{(2\pi)^{p/2}} dv \right| dz,$$

$$\geq e^{-\Delta^2_y / 8} \frac{1}{2\pi} \| \tilde{c}_1 - \tilde{c}_2 \|_2,$$

where

$$\tilde{c}_1 = \frac{Q_{11}^{1/2} u}{u^\top Q_{11} u}, \quad \tilde{c}_2 = \frac{Q_{11}^{1/2} v}{v^\top Q_{11} v}.$$
we conclude
\[ L_{q(i)} + L_{q(j)} \gtrsim e^{-\Delta^2/8} \|Q_{1/2}(u-v)\|_2 \geq \frac{1}{\Delta} e^{-\Delta^2/8} \sqrt{\lambda} \sqrt{K/n}. \]
Using \( \lambda \geq c \) completes the proof.

\section*{C Technical lemmas}

Consider \( \pi_0 + \pi_1 = 1 \). This section contains some basic relations between \( \alpha_0 \) and \( \alpha_1 \), collected in Fact 1, as well as some useful technical lemmas. For simplicity, we write \( \mathbb{P} \) for \( \mathbb{P}^D \) from now on.

\textbf{Fact 1.} Let \( \bar{\alpha} := \pi_0 \alpha_0 + \pi_1 \alpha_1 \). One has
\[ \pi_0 \alpha_0 \alpha_0^\top + \pi_1 \alpha_1 \alpha_1^\top - \bar{\alpha} \bar{\alpha}^\top = \pi_0 \pi_1 (\alpha_1 - \alpha_0)(\alpha_1 - \alpha_0)^\top. \]
Additionally, for any \( M \in \mathbb{R}^{K \times K} \), one has
\[ \pi_0 \alpha_0^\top M \alpha_0 + \pi_1 \alpha_1^\top M \alpha_1 - \bar{\alpha}^\top M \bar{\alpha} = \pi_0 \pi_1 (\alpha_1 - \alpha_0)^\top M (\alpha_1 - \alpha_0). \]
As a result,
\[ \alpha_0^\top M \alpha_0 + \alpha_1^\top M \alpha_1 - \bar{\alpha}^\top M \bar{\alpha} \leq \max\{\pi_0, \pi_1\} \cdot (\alpha_1 - \alpha_0)^\top M (\alpha_1 - \alpha_0). \]

The following lemma provides concentration inequalities of \( \hat{\pi}_k - \pi_k \).

\textbf{Lemma 30.} For any \( k \in \{0,1\} \) and all \( t > 0 \),
\[ \mathbb{P}\left(|\hat{\pi}_k - \pi_k| > \sqrt{\pi_k(1 - \pi_k)t/n} + \frac{t}{n}\right) \leq 2e^{-t/2}. \]

In particular, if \( \pi_0 \pi_1 \geq 2 \log n/n \), then for any \( k \in \{0,1\} \),
\[ \mathbb{P}\left(|\hat{\pi}_k - \pi_k| < \sqrt{8 \pi_0 \pi_1 \log n/n}\right) \geq 1 - 2n^{-1}. \]

Furthermore, if \( \pi_0 \pi_1 \geq C \log n/n \) for some sufficiently large constant \( C \), then
\[ \mathbb{P}\left\{c \pi_k \leq \hat{\pi}_k \leq c' \pi_k\right\} \geq 1 - 2n^{-1}. \]

\textbf{Proof.} The first result follows from an application of the Bernstein inequality for bounded random variables. The second one follows by choosing \( t = 2 \log n \) and the last one can be readily seen from the second display.

\section*{C.1 Deviation inequalities of quantities related with \( Z \)}

Recall that \( \bar{\alpha} = \mathbb{E}[Z] \), \( \Sigma_Z = \text{Cov}(Z) \) and \( \tilde{Z} = Z \Sigma_Z^{-1/2} \). Let the centered \( \bar{Z} \) as
\[ R = (R_1, \ldots, R_n)^\top, \quad \text{with} \quad R_i = \tilde{Z}_i - \Sigma_Z^{-1/2} \bar{\alpha}. \]

The following lemma provides concentration inequalities of \( \hat{\alpha}_k - \alpha_k \) and some useful bounds related with the random matrices \( R \) and \( \tilde{Z}^\top \Pi_n \tilde{Z} \).
Lemma 31. Suppose that model (1.3) holds.

(i) For any deterministic vector $u \in \mathbb{R}^K$, for all $t > 0$,

$$
\mathbb{P} \left\{ \left| u^\top (\hat{\alpha}_k - \alpha_k) \right| \geq t \sqrt{\frac{u^\top \Sigma_{Z|Y} u}{n_k}} \right\} \leq 2 e^{-t^2/2}.
$$

(ii)

$$
\mathbb{P} \left\{ \left\| \Sigma_{Z}^{-1/2} (\hat{\alpha}_k - \alpha_k) \right\|_2 \leq 2 \sqrt{K \log \frac{n}{n_k}} \right\} \geq 1 - 2K/n^2.
$$

(iii) With probability $1 - 4Kn^{-2} - 4n^{-1}$,

$$
\frac{1}{n} \left\| \sum_{i=1}^{n} R_i \right\|_2 \leq 2(2 + \sqrt{2}) \sqrt{\frac{K \log n}{n}}.
$$

(iv) For any deterministic vector $u, v \in \mathbb{R}^K$, with probability $1 - 4n^{-e'} - 4n^{-1} - 8Kn^{-2}$,

$$
\left| u^\top \left( \frac{1}{n} \sum_{i=1}^{n} R_i R_i^\top - I_K \right) v \right| \lesssim \|u\|_2 \|v\|_2 \sqrt{\frac{\log n}{n}} (1 + \|\alpha_1 - \alpha_0\| \Sigma_{Z})
$$

(v) With probability $1 - 4n^{-e''K} - 4n^{-1} - 8Kn^{-2}$,

$$
\left\| \frac{1}{n} R^\top R - I_K \right\|_{op} \lesssim \sqrt{\frac{K \log n}{n}} + \frac{\log n}{n} + \|\alpha_1 - \alpha_0\| \Sigma_{Z} \sqrt{\frac{\log n}{n}}.
$$

(vi) Assume $K \log n \leq c_0 n$ for some sufficiently small constant $c_0 > 0$. With probability $1 - 4n^{-e''K} - 4n^{-1} - 8Kn^{-2}$,

$$
c \leq \frac{1}{n} \lambda_K (R^\top R) \leq \frac{1}{n} \lambda_1 (R^\top R) \leq C
$$

holds for some constants $0 < c \leq C < \infty$ depending on $c_0$ only.

(vii) Under conditions of (vi), there exists some absolute constants $c, C, C' > 0$ such that, with probability $1 - C'n$, one has

$$
\left\| \frac{1}{n} \tilde{Z}^\top \Pi_n \tilde{Z} - I_K \right\|_{op} \leq C' \sqrt{\frac{K \log n}{n}}
$$

and

$$
c \leq \frac{1}{n} \lambda_K (\tilde{Z}^\top \Pi_n \tilde{Z}) \leq \frac{1}{n} \lambda_1 (\tilde{Z}^\top \Pi_n \tilde{Z}) \leq C.
$$

Proof. Without loss of generality, we assume $\tilde{\alpha} = 0$ such that $\tilde{Z} = R$.

To prove (i), by conditioning on $Y_i$, the fact that $Z_i \mid Y_i = k$ are i.i.d. $N(\alpha_k, \Sigma_{Z|Y})$ implies that, for all $t > 0$ and any deterministic $u \in \mathbb{R}^K$,

$$
\mathbb{P} \left\{ \left| u^\top (\hat{\alpha}_k - \alpha_k) \right| \geq t \sqrt{\frac{u^\top \Sigma_{Z|Y} u}{n_k}} \mid Y \right\} \leq 2 \exp \left( -\frac{t^2}{2} \right).
$$

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The statement above yields (i) by unconditioning on $Y$.

To show part (ii), by taking $t = 2\sqrt{\log n}$ and $u = \Sigma_{Z}^{-1/2} e_k$ in (i) together with the union bounds over $k \in [K]$, we conclude

$$
\|\Sigma_{Z}^{-1/2} (\hat{\alpha}_k - \alpha_k)\|_2 \leq \sqrt{K} \max_k |e_k^T \Sigma_{Z}^{-1/2} (\hat{\alpha}_k - \alpha_k)|
$$

$$
\leq 2 \sqrt{e_k^T \Sigma_{Z}^{-1/2} \Sigma_{Z|Y} \Sigma_{Z}^{-1/2} e_k \sqrt{K \log n}}
$$

$$
\leq 2 \sqrt{\frac{K \log n}{n_k}}
$$

with probability $1 - 2K/n^2$. The last inequality also uses $\|\Sigma_{Z}^{-1/2} \Sigma_{Z|Y} \Sigma_{Z}^{-1/2}\|_{op} \leq 1$, deduced from (A.10).

To prove part (iii), without loss of generality, assume $\bar{\alpha} = 0$. By adding and subtracting terms and using

$$
E[Z] = \bar{\alpha} = 0 = \pi_1\alpha_1 + \pi_0\alpha_0,
$$

we obtain the identity

$$
\sum_{i=1}^{n} Z_i = \sum_{i: Y_i = 1} Z_i + \sum_{i: Y_i = 0} Z_i
$$

$$
= \sum_{i: Y_i = 1} (Z_i - \alpha_1) + \sum_{i: Y_i = 0} (Z_i - \alpha_0) + (n_1 - n\pi_1)\alpha_1 + (n_0 - n\pi_0)\alpha_0
$$

$$
= \sum_{i: Y_i = 1} (Z_i - \alpha_1) + \sum_{i: Y_i = 0} (Z_i - \alpha_0) + (n\pi_0 - n_0)\alpha_1 + (n - n\pi_0)\alpha_0
$$

$$
= \sum_{i: Y_i = 1} (Z_i - \alpha_1) + \sum_{i: Y_i = 0} (Z_i - \alpha_0) + (n\pi_0 - n_0)\alpha_1 + (n_0 - n\pi_0)(\alpha_1 - \alpha_0)
$$

where in the third line we used $n_0 + n_1 = n$ and $\pi_0 + \pi_1 = 1$. Therefore, by recalling that (A.27) and (3.6),

$$
\frac{1}{n} \left\| \sum_{i=1}^{n} \tilde{Z}_i \right\|_2 \leq \sqrt{\frac{n_1}{n}} \left\| \Sigma_{Z}^{-1/2} (\hat{\alpha}_1 - \alpha_1) \right\|_2 + \sqrt{\frac{n_0}{n}} \left\| \Sigma_{Z}^{-1/2} (\hat{\alpha}_0 - \alpha_0) \right\|_2
$$

$$
|\tilde{\pi}_0 - \pi_0| \cdot \|\alpha_1 - \alpha_0\|_{\Sigma_{Z}}.
$$

Invoking part (ii) and Lemma 30, together with

$$
\pi_0\pi_1\|\alpha_1 - \alpha_0\|_{\Sigma_{Z}}^2 \leq 1
$$

deduced from (A.11), completes the proof of (iii).
To prove (iv), notice that

$$\sum_{i=1}^{n} Z_i Z_i^\top = \sum_{i:Y_i=1} Z_i Z_i^\top + \sum_{i:Y_i=0} Z_i Z_i^\top$$

$$= \sum_{k\in\{0,1\}} \left[ \sum_{i:Y_i=k} (Z_i - \alpha_k)(Z_i - \alpha_k)^\top + n_k (\hat{\alpha}_k \alpha_k^\top + \alpha_k \hat{\alpha}_k^\top) - n_k \alpha_k \alpha_k^\top \right]$$

$$= \sum_{k\in\{0,1\}} \left[ \sum_{i:Y_i=k} (Z_i - \alpha_k)(Z_i - \alpha_k)^\top + n_k (\hat{\alpha}_k - \alpha_k)\alpha_k^\top + n_k \alpha_k (\hat{\alpha}_k - \alpha_k)^\top \right]$$

$$+ \sum_{k\in\{0,1\}} n_k \alpha_k \alpha_k^\top.$$ 

Since (A.10), (C.1) and Fact 1 imply

$$\Sigma_Z = \Sigma_{Z|Y} + \sum_{k\in\{0,1\}} \pi_k \alpha_k \alpha_k^\top,$$

we obtain, for any $u, v \in \mathbb{R}^K$,

$$u^\top \left( \frac{1}{n} \sum_{i=1}^{n} Z_i Z_i^\top - \Sigma_Z \right) v = \sum_{k\in\{0,1\}} \frac{n_k}{n} u^\top \left[ \frac{1}{n} \sum_{i:Y_i=k} (Z_i - \alpha_k)(Z_i - \alpha_k)^\top - \Sigma_{Z|Y} \right] v^\top$$

$$+ \sum_{k\in\{0,1\}} \frac{n_k}{n} v^\top (\hat{\alpha}_k - \alpha_k)\alpha_k^\top u + \sum_{k\in\{0,1\}} \frac{n_k}{n} u^\top (\hat{\alpha}_k - \alpha_k)\alpha_k^\top v$$

$$+ \sum_{k\in\{0,1\}} (\hat{\pi}_k - \pi_k) u^\top \alpha_k \alpha_k^\top v.$$ (C.3)

Notice that

$$u^\top \left( \frac{1}{n} \sum_{i=1}^{n} \tilde{Z}_i \tilde{Z}_i^\top - I_K \right) v = \tilde{u}^\top \left( \frac{1}{n} \sum_{i=1}^{n} Z_i Z_i^\top - \Sigma_Z \right) \tilde{v}$$

with $\tilde{u} = \Sigma_Z^{-1/2} u$ and $\tilde{v} = \Sigma_Z^{-1/2} v$. An application of Lemma 36 yields

$$\left| \tilde{u}^\top \left( \frac{1}{n} \sum_{i:Y_i=k} (Z_i - \alpha_k)(Z_i - \alpha_k)^\top - \Sigma_{Z|Y} \right) \tilde{v} \right| \leq c' \sqrt{\tilde{u}^\top \Sigma_{Z|Y} \tilde{u} \tilde{v}^\top \Sigma_{Z|Y} \tilde{v} \left( \sqrt{\frac{\log n}{n_k}} + \frac{\log n}{n_k} \right)}$$

with probability $1 - 2n^{-c''}$. By further invoking Lemma 30 and part (i), we conclude

$$\left| \tilde{u}^\top \left( \frac{1}{n} \sum_{i=1}^{n} Z_i Z_i^\top - \Sigma_Z \right) \tilde{v} \right|$$

$$\leq \sqrt{\tilde{u}^\top \Sigma_{Z|Y} \tilde{u} \tilde{v}^\top \Sigma_{Z|Y} \tilde{v}} \sum_{k\in\{0,1\}} \frac{n_k}{n} \left( \sqrt{\frac{\log n}{n_k}} + \frac{\log n}{n_k} \right) + \sqrt{\tilde{u}^\top \Sigma_{Z|Y} \tilde{v} \sum_{k\in\{0,1\}} \frac{n_k \log n}{n^2} |\tilde{u}^\top \alpha_k|}$$

$$+ \sqrt{\tilde{u}^\top \Sigma_{Z|Y} \tilde{u} \sum_{k\in\{0,1\}} \frac{n_k \log n}{n^2} |\tilde{v}^\top \alpha_k|} + \sqrt{\frac{\pi_0 \pi_1 \log n}{n} \sum_{k\in\{0,1\}} |\tilde{u}^\top \alpha_k|^2}.$$
with probability $1 - 4n^{-c''} - 4n^{-1} - 8Kn^{-2}$. Since
\[
|\tilde{u}^\top \alpha_k| \leq \|u\|_2 \|\alpha_k\|_{\Sigma_Z}
\]
from the Cauchy-Schwarz inequality, by noting that
\[
\tilde{u}^\top \Sigma_{Z|Y} \tilde{u} \leq \|u\|_2^2 \|\Sigma_{Z|Y}^{-1/2} \Sigma_{Z}^{-1/2}\|_{\text{op}} \leq \|u\|_2^2
\]
and invoking Fact 1 for
\[
\sum_{k \in \{0, 1\}} \|\alpha_k\|_{\Sigma_Z} \leq \sqrt{2} \|\alpha_1 - \alpha_0\|_{\Sigma_Z}, \quad \sum_{k \in \{0, 1\}} \|\alpha_k\|_{\Sigma_Z}^2 \leq \|\alpha_1 - \alpha_0\|_{\Sigma_Z}^2,
\]
we conclude, with the same probability,
\[
|\tilde{u}^\top \left( \frac{1}{n} \sum_{i=1}^n Z_i Z_i^\top - \Sigma_Z \right) \tilde{v}| \lesssim \|u\|_2 \|v\|_2 \sqrt{\log n} \left(1 + \|\alpha_1 - \alpha_0\|_{\Sigma_Z} + \sqrt{\alpha_0 \pi_1} \|\alpha_1 - \alpha_0\|_{\Sigma_Z} \right)
\]
where we used (C.2) in the last line.

Next, we prove (v) by bounding from above
\[
\sup_{u \in \mathbb{R}^K} \left| \frac{1}{n} \sum_{i=1}^n Z_i Z_i^\top - \Sigma_Z \right|_{\text{op}} \leq c'
\]
with probability $1 - 2n^{-c''} K$. The result follows by the same arguments of proving (iv) and also by noting that the other terms are bounded uniformly over $u \in \mathbb{R}^K$.

As a result of (v), part (vi) follows from the bound (A.18) and Weyl’s inequality.

Finally, to prove (vii), observe that
\[
\frac{1}{n} \tilde{Z}^\top \Pi_n \tilde{Z} = \frac{1}{n} \sum_{i=1}^n \tilde{Z}_i \tilde{Z}_i^\top - \Sigma_Z^{-1/2} \tilde{Z} \tilde{Z}^\top \Sigma_Z^{-1/2}
\]
with $\tilde{Z} = \sum_{i=1}^n Z_i/n$. Consequently,
\[
\left\| \frac{1}{n} \tilde{Z}^\top \Pi_n \tilde{Z} - I_K \right\|_{\text{op}} \leq \left\| \frac{1}{n} \tilde{Z}^\top \tilde{Z} - I_K \right\|_{\text{op}} + \left\| \frac{1}{n} \sum_{i=1}^n \tilde{Z}_i \right\|_2^2.
\]
Invoking (iii) and (v) gives the desired result. The bounds on eigenvalues of $\tilde{Z}^\top \Pi_n \tilde{Z}$ follow from Weyl’s inequality.
C.2 Deviation inequalities of quantities related with $W$

The following lemma provides deviation inequalities for various quantities related with $W$. Recall that

$$W_{(k)} = \frac{1}{n_k} \sum_{i=1}^{n} W_i \mathbb{1}\{Y_i = k\}, \quad \forall \ k \in \{0, 1\}.$$  

Further recall that $\mathcal{E}_z$ is defined in (A.26).

**Lemma 32.** Grant assumption (i) – (vi) and $K \log n \lesssim n$. We have the following results.

\[
\mathbb{P}\left\{ \frac{1}{\sqrt{n}} \left\| W_A^+ \Sigma_Z^{-1/2} \right\|_{\text{op}} \leq \sqrt{\frac{\| \Sigma_W \|_{\text{op}}}{\lambda_K}} \right\} \geq 1 - O(n^{-1}),
\]

\[
\mathbb{P}\left\{ \frac{1}{\sqrt{n}} \| WP_A \|_{\text{op}} \lesssim \sqrt{\| \Sigma_W \|_{\text{op}}} \right\} \geq 1 - e^{-n},
\]

\[
\mathbb{P}\left\{ \max_{k \in \{0, 1\}} \left\| P_A \bar{W}_{(k)} \right\|_2 \lesssim \sqrt{\| \Sigma_W \|_{\text{op}}} \sqrt{\frac{K \log n}{n}} \right\} \geq 1 - O(n^{-1}),
\]

\[
\mathbb{P}\left\{ \frac{1}{n} \left\| Z^T \Pi_n WP_A \right\|_{\text{op}} \lesssim \sqrt{\| \Sigma_W \|_{\text{op}}} \sqrt{\frac{K \log n}{n}} \right\} \cap \mathcal{E}_z \geq 1 - n^{-cK},
\]

\[
\mathbb{P}\left\{ \frac{1}{n} \left\| P_A W^T \Pi_n Y \right\|_2 \lesssim \sqrt{\| \Sigma_W \|_{\text{op}}} \sqrt{\frac{K \log n}{n}} \right\} \geq 1 - O(n^{-1}).
\]

Here $c > 0$ is some absolute constant.

**Proof.** Recall that $W = \bar{W} \Sigma_W^{1/2}$. Invoke Lemma 35 with $G = \bar{W}$ and $H = \Sigma_W^{1/2} A^+ \Sigma_Z^{-1} A^+ \Sigma_W^{1/2}$ together with $\text{tr}(H) \leq K\|H\|_{\text{op}}, \|H\|_{\text{op}} \leq \| \Sigma_W \|_{\text{op}}/\lambda_K$ and $K \lesssim n$ to obtain

$$\mathbb{P}\left\{ \frac{1}{\sqrt{n}} \left\| W_A^+ \Sigma_Z^{-1/2} \right\|_{\text{op}} \leq \sqrt{\frac{\| \Sigma_W \|_{\text{op}}}{\lambda_K}} \right\} \geq 1 - 2n^{-cK}.$$  

Similarly, by invoking Lemma 35 and using $K \log n \leq cn$, the second result follows from

\[
\frac{1}{n} \| WP_A \|_{\text{op}} \lesssim \| P_A \Sigma_W P_A \|_{\text{op}} + \frac{1}{n} \text{tr}(P_A \Sigma_W P_A) \leq \| \Sigma_W \|_{\text{op}} \left( 1 + \frac{K}{n} \right)
\]

with probability at least $1 - e^{-n}$.

Regarding the third result, since $\Sigma_W^{-1/2} \bar{W}_{(k)}$ given $Y$ is $\sqrt{\gamma^2/n_k}$-subGaussian, Lemma 34 gives

\[
\left\| P_A \bar{W}_{(k)} \right\|_2 \lesssim \sqrt{\frac{1}{n} \left[ \text{tr}(P_A \Sigma_W P_A) + \| P_A \Sigma_W P_A \|_{\text{op}} \log n \right]} \leq \sqrt{\frac{K + \log n}{n}} \| \Sigma_W \|_{\text{op}},
\]

with probability $1 - O(1/n)$. The last inequality in (C.5) uses $\text{tr}(P_A \Sigma_W P_A) \leq K \| \Sigma_W \|_{\text{op}}$.  

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To prove the fourth result, let $P_A = U_A U_A^\top$ with $U_A \in \mathcal{O}_{p \times K}$. Further let $\mathcal{N}_K(1/4)$ be the $(1/4)$-net of $\mathcal{S}^K$. By the properties of $\mathcal{N}_K(1/4)$, we have

\[
\frac{1}{n} \| \hat{Z}^\top \Pi_n W P_A \|_{op} = \frac{1}{n} \| \hat{Z}^\top \Pi_n W U_A \|_{op} = \sup_{u \in \mathcal{S}^K, v \in \mathcal{S}^K} u^\top \hat{Z}^\top \Pi_n W U_A v \\
\leq 2 \max_{u \in \mathcal{N}_K(1/4), v \in \mathcal{N}_K(1/4)} u^\top \hat{Z}^\top \Pi_n W U_A v.
\]

Furthermore,

\[
u^\top \hat{Z}^\top \Pi_n W U_A v = \frac{1}{n} \sum_{i=1}^n u^\top \left( \frac{1}{n} \sum_{i=1}^n \tilde{Z}_i \right) (W_i - \bar{W})^\top U_A v
= \frac{1}{n} \sum_{i=1}^n u^\top \left( \tilde{Z}_i - \Sigma_Z^{-1/2} \tilde{\alpha} \right) (W_i - \bar{W})^\top U_A v
= \frac{1}{n} \sum_{i=1}^n u^\top \left( \tilde{Z}_i - \Sigma_Z^{-1/2} \tilde{\alpha} \right) W_i^\top U_A v - \frac{1}{n} \sum_{i=1}^n \left( \tilde{Z}_i - \Sigma_Z^{-1/2} \tilde{\alpha} \right) W_i^\top U_A v.
\]

By (iii) of Lemma 31 and (C.5), the second term can be bounded from above, uniformly over $u, v \in \mathcal{N}_K(1/4)$, as

\[
\left\| \frac{1}{n} \sum_{i=1}^n \left( \tilde{Z}_i - \Sigma_Z^{-1/2} \tilde{\alpha} \right) \right\|_2 \left\| U_A v \right\|_2 \lesssim \sqrt{\| \Sigma_W \|_{op} \frac{K \log n}{n}}
\]

with probability $1 - cn^{-1}$.

It remains to show that the same bound holds for the first term in (C.6). Since $Z$ and $W$ are independent, conditioning on $\tilde{Z}$, we know $u^\top (\tilde{Z}_i - \Sigma_Z^{-1/2} \tilde{\alpha}) W_i^\top U_A v$ is sub-Gaussian with sub-Gaussian constant equal to

\[
\sqrt{\| U_A^\top \Sigma_W U_A \|} \sqrt{u^\top (\tilde{Z}_i - \Sigma_Z^{-1/2} \tilde{\alpha}) (\tilde{Z}_i - \Sigma_Z^{-1/2} \tilde{\alpha})^\top u} \leq \sqrt{\| \Sigma_W \|_{op} \frac{1}{n} R_i R_i^\top u},
\]

recalling that $R_i = \tilde{Z}_i - \Sigma_Z^{-1/2} \tilde{\alpha}$. Thus, $n^{-1} \sum_{i=1}^n u^\top (\tilde{Z}_i - \Sigma_Z^{-1/2} \tilde{\alpha}) W_i^\top U_A v$ is sub-Gaussian with sub-Gaussian constant equal to

\[
\frac{1}{n} \sqrt{\| \Sigma_W \|_{op} \sum_{i=1}^n u^\top R_i R_i^\top u} \leq \frac{1}{n} \frac{\| \Sigma_W \|_{op} \| \frac{1}{n} R^\top R \|_{op}}{n}.
\]

We conclude that, for each $u, v \in \mathcal{N}_K(1/4)$,

\[
P \left\{ \frac{1}{n} \sum_{i=1}^n u^\top (\tilde{Z}_i - \Sigma_Z^{-1/2} \tilde{\alpha}) W_i^\top U_A v \geq t \sqrt{\| \Sigma_W \|_{op} \| \frac{1}{n} R^\top R \|_{op}} \right\} \leq e^{-t^2/2}.
\]

The result follows by choosing $t = C \sqrt{K \log n}$ for some sufficiently large constant $C > 0$, taking a union bounds over $\mathcal{N}_K(1/4)$ together with $|\mathcal{N}_K(1/4)| \leq 9^K$, and invoking (v) of Lemma 31.

Finally, to prove the last claim, recall from (A.17) that

\[
W^\top \Pi_n Y = W^\top Y - \frac{1}{n} W^\top 1_n 1_n^\top Y = n_1 (W_{(1)} - \bar{W}),
\]

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with $W = \sum_{i=1}^{n} W_i$. We thus find that, with probability $1 - O(n^{-1})$,

$$\frac{1}{n} \left\| P_A W^\top \Pi_n Y \right\|_2 \leq \left\| P_A W_{(1)} \right\|_2 + \left\| P_A \bar{W} \right\|_2 \lesssim \sqrt{K \log n} + \sqrt{\|\Sigma_W\|_{\text{op}}} \quad (C.7)$$

where the last step uses the bound in (C.5).

The following lemma analyzes some of the quantities in Lemma 32 with $A$ replaced by any estimator $\hat{A} \in \mathbb{R}^{p \times q}$.

**Lemma 33.** Under conditions of Lemma 32, for any $\hat{A} \in \mathbb{R}^{p \times q}$, we have

$$\Pr \left\{ \frac{1}{n} \left\| W P_{\hat{A}} \right\|_{\text{op}} \lesssim \sqrt{\|\Sigma_W\|_{\text{op}}} + \frac{1}{\sqrt{n}} \left\| W (P_{\hat{A}} - P_A) \right\|_{\text{op}} \right\} \geq 1 - e^{-n},$$

$$\Pr \left\{ \frac{1}{n} \left\| \tilde{Z}^\top \Pi_n W P_{\hat{A}} \right\|_{\text{op}} \lesssim \sqrt{\|\Sigma_W\|_{\text{op}}} \sqrt{\frac{K \log n}{n}} + \frac{1}{\sqrt{n}} \left\| W (P_{\hat{A}} - P_A) \right\|_{\text{op}} \right\} \cap \mathcal{E}_z \geq 1 - n^{-cK},$$

$$\Pr \left\{ \frac{1}{n} \left\| P_A W^\top \Pi_n Y \right\|_2 \lesssim \sqrt{\|\Sigma_W\|_{\text{op}}} \sqrt{\frac{K \log n}{n}} + \frac{1}{n} \left\| (P_{\hat{A}} - P_A) W^\top \Pi_n Y \right\|_2 \right\} \geq 1 - O(n^{-1}).$$

**Proof.** In view of Lemma 32, the first and third results follow from the triangle inequality. For the second claim, it suffices to bound from above

$$\frac{1}{n} \left\| \tilde{Z}^\top \Pi_n W (P_{\hat{A}} - P_A) \right\|_{\text{op}} \leq \frac{1}{\sqrt{n}} \left\| \Pi_n \tilde{Z} \right\|_{\text{op}} \frac{1}{\sqrt{n}} \left\| W (P_{\hat{A}} - P_A) \right\|_{\text{op}}.$$ 

Invoking $\mathcal{E}_z$ gives the claim. \qed

### D Auxiliary lemmas

The following lemma is the tail inequality for a quadratic form of sub-Gaussian random vectors. We refer to Bing et al. (2021, Lemma 16) for its proof, also see, Lemma 30 in Hsu et al. (2014).

**Lemma 34.** Let $\xi \in \mathbb{R}^d$ be a $\gamma_\xi$ sub-Gaussian random vector. For all symmetric positive semi-definite matrices $H$, and all $t \geq 0$,

$$\Pr \left\{ \xi^\top H \xi > \gamma_\xi^2 \left( \sqrt{\text{tr}(H)} + \sqrt{2t\|H\|_{\text{op}}} \right)^2 \right\} \leq e^{-t}.$$

The following lemma provides an upper bound on the operator norm of $GHG^\top$ where $G \in \mathbb{R}^{n \times d}$ is a random matrix and its rows are independent sub-Gaussian random vectors. It is proved in Lemma 22 of Bing et al. (2021).

**Lemma 35.** Let $G$ be a $n \times d$ matrix with rows that are independent $\gamma$ sub-Gaussian random vectors with identity covariance matrix. Then for all symmetric positive semi-definite matrices $H$,

$$\Pr \left\{ \frac{1}{n} \|GHG^\top\|_{\text{op}} \leq \gamma^2 \left( \sqrt{\frac{\text{tr}(H)}{n}} + \sqrt{6\|H\|_{\text{op}}} \right)^2 \right\} \geq 1 - e^{-n}$$

Another useful concentration inequality of the operator norm of the random matrices with i.i.d. sub-Gaussian rows is stated in the following lemma. This is an immediate result of Vershynin (2012, Remark 5.40).
Lemma 36. [Bing et al. (2021, Lemma 16)] Let $G$ be an $n \times d$ matrix whose rows are i.i.d. $\gamma$ sub-Gaussian random vectors with covariance matrix $\Sigma_Y$. Then for every $t \geq 0$, with probability at least $1 - 2e^{-ct^2}$,

$$\left\| \frac{1}{n} G^\top G - \Sigma_Y \right\|_{op} \leq \max \left\{ \delta, \delta^2 \right\} \| \Sigma_Y \|_{op},$$

with $\delta = C\sqrt{d/n} + t/\sqrt{n}$ where $c = c(\gamma)$ and $C = C(\gamma)$ are positive constants depending on $\gamma$.

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