A unified approach to radial, hyperbolic, and directional distance models in Data Envelopment Analysis

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Abstract

The paper analyzes properties of a large class of “path-based” Data Envelopment Analysis models through a unifying general scheme, which includes as standard the well-known oriented radial models, the hyperbolic distance measure model, and the directional distance measure models. The scheme also accommodates variants of standard models over negative data.

Path-based models are analyzed from the point of view of nine desired properties that a well-designed model should satisfy. The paper develops mathematical tools that allow systematic investigation of these properties in the general scheme including, but not limited to, the standard path-based models. Among other results, the analysis confirms the generally accepted view that path-based models need not generate Pareto–Koopmans efficient projections, one-to-one identification, or strict monotonicity.

Keywords: Data envelopment analysis; Radial models; Hyperbolic measure model; Directional Distance Measure model; Negative data.

1 Introduction

Data Envelopment Analysis (DEA) defines a variety of models to assess the performance of Decision-Making Units (DMU). The DEA approach defines technology sets via observed input-output data in combination with certain axioms. The model links together an efficiency measure and a given technology set by means of a mathematical programming problem that seeks a benchmark dominating the assessed DMU by optimizing the value of the efficiency measure. There are two main methods of searching for the dominant unit, which give rise to two basic classes of DEA models — the slacks-based models and the path-based models. The latter are the subject of this paper.

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According to Russell and Schworm (2018), the models in the slacks-based class search for the benchmarks by “specifying the form of aggregation over the coordinate-wise [additive or multiplicative] slacks” relative to the benchmark. The main representatives of this class are the Slacks-Based Measure (SBM) model of Tone (2001), the Russell Graph Measure model of Färe, Grosskopf and Lovell (1985), the Additive Model (AM) of Charnes et al. (1985), and the Weighted Additive Models (WAM) including the Range Adjusted Measure (RAM) model (Cooper, Park and Pastor, 1999) and the Bounded Adjusted Measure (BAM) model (Cooper et al., 2011).

The models in the second class search for the benchmarks by specifying various parametric paths running from the DMU to the boundary of the technology set. In the special case of the radial BCC input or output-oriented models (Banker, Charnes and Cooper, 1984), the path is defined by a ray connecting the DMU to the origin in the space of inputs or outputs and thus represents a proportional, radial contraction of inputs or expansion of outputs. In the Directional Distance Measure (DDM) model (Chambers, Chung and Färe, 1996, 1998), the path is determined by a ray in a pre-assigned direction pointing from the assessed DMU towards the dominating part of the frontier. On this path, one then seeks the point of minimal distance to the frontier of the technology set. The hyperbolic distance measure (HDM) model (Färe, Grosskopf and Lovell, 1985) combines together the input and output oriented BCC models by using a hyperbolic path that allows for a simultaneous equiproportionate contraction of inputs and expansion of outputs. In the terminology of Russell and Schworm (2018), the three special cases above fall into the class of path-based models.

Despite a large number of research papers in DEA, there is a paucity of studies that analyze properties of either model class in a unified framework. In the context of general economic productivity theory, a series of articles Russell and Schworm (2008), Russell and Schworm (2011), Levkoff, Russell and Schworm (2011), Roshdi et al. (2018), and Hasannasab et al. (2019) provided a comprehensive analysis of efficiency measures over different types of productivity sets. The message from these papers is that the slacks-based measures, when operating in the full input-output space, identify the Pareto–Koopmans efficiency unambiguously while the path-based measures do not.

In the DEA setting, Cooper, Park and Pastor (1999) and Sueyoshi and Sekitani (2009) analyse efficiency measures and introduce a set of desirable properties that an ideal DEA model should satisfy. These properties include identification of strong efficiency; boundedness; strict monotonicity; unit invariance; and translation invariance. Selected DEA models are then classified based on these criteria.

In other work, Halická and Trnovská (2021) analyze properties of slacks-based models in a general scheme that encompasses all commonly used models in this class but also allows for construction of new models. The models in this class are characterized by projecting onto the strongly efficient frontier and therefore account for all sources of inefficiencies. Among them, the RAM model performs best when measured against eight desirable properties, satisfying seven and failing only the unique projection property due to its linearity. Recently, Aparicio and

1The terminology “non-radial” and “radial pattern models” was used for slacks-based and path-based models, respectively.
Monge (2022) proposed a convex generalization of the RAM model that falls into the general scheme of Halická and Trnovská (2021) and satisfies uniqueness of projections. The new model thus currently claims the top spot in terms of the number of desirable properties.

In contrast to slacks-based models, a comprehensive analysis of path-based models is missing in the literature. The current state of knowledge is fragmented across many articles with varying focus. This situation is exacerbated by the fact that properties of DDM models depend strongly on the choice of direction vectors and these have not been treated systematically to date. There appears to be a general consensus that path-based models do not guarantee strong efficiency of projection and hence their efficiency score is overstated; and that they are monotone but not strictly monotone. There are known difficulties with super efficiency measurement under variable returns to scale and the related measurement of productivity change over time.

In this paper we analyze path-based models in a general framework. The framework includes all standard models but also a certain generalization of the hyperbolic model and even an opportunity to go beyond hyperbolic models. We examine the properties of the general model in the light of nine desirable properties that a well-designed model ought to satisfy. In one place we thus systematically survey the properties of all existing path-based models but also offer certain guidance how to construct a model with given properties. The construction does not require non-negative data and hence permits extension of existing models designed for positive data to arbitrary (negative) data.

The paper is organized as follows. Section 2 introduces basic terminology and notation concerning, among others, the technology set and its efficient frontier over general data; desirable properties of DEA models; and standard path-based models. Section 3 proposes a general scheme for path-based models, analyzes its basic properties, and discusses its geometry. Section 4 undertakes deeper analysis of the general scheme in light of nine desirable properties. This section develops practically useful criteria for each property and illustrates them on individual standard path-based models. Section 5 extends the foregoing analysis in two directions: (a) data with negative values; (b) construction of new models with good properties. Section 6 concludes.

2 Preliminaries

Let us establish some notation. \( \mathbb{R}^d \) denotes the \( d \)-dimensional Euclidean space and \( \mathbb{R}^d_+ \) its non-negative orthant. Bold lowercase letters denote column vectors, bold uppercase letters matrices. Superscript \( T \) denotes the transpose of a column vector or a matrix. Symbol \( e \) denotes a vector of ones.

Consider a set of \( n \) observed decision making units DMU\(_j\) (\( j = 1, \ldots, n \)), each consuming \( m \) inputs \( x_{ij} \) (\( i = 1, \ldots, m \)) to produce \( s \) outputs \( y_{rj} \) (\( r = 1, \ldots, s \)). For each \( j = 1, \ldots, n \), the data of inputs and outputs of DMU\(_j\) can be arranged into the column vectors \( \mathbf{x}_j = (x_{1j}, \ldots, x_{mj})^T \in \mathbb{R}^m \) of inputs and \( \mathbf{y}_j = (y_{1j}, \ldots, y_{sj})^T \in \mathbb{R}^s \) of outputs. Finally, the input and output vectors of all DMUs form the \( m \times n \) input and \( s \times n \) output matrices \( \mathbf{X} \) and \( \mathbf{Y} \), i.e., \( \mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_n] \) and \( \mathbf{Y} = [\mathbf{y}_1, \ldots, \mathbf{y}_n] \), respectively.

In the article we will reserve the notation \( i, j \), and \( r \) for indices running the whole index.
sets \{1, \ldots, m\}, \{1, \ldots, n\}, and \{1, \ldots, s\}, respectively. We make no assumptions about the non-negativity of the data at this point. The non-negativity requirement may follow later from other assumptions and we shall alert the reader whenever that is the case.

2.1 Technology set

Based on the given data we consider the technology set

\[ \mathcal{T} = \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^s \mid X \lambda \leq x, \ Y \lambda \geq y, \ \lambda \geq 0, \ e^T \lambda = 1\}, \]  

(1)
corresponding to variable returns to scale (VRS). Note that the common non-negativity of \((x, y)\) is not imposed here. Elements of \(\mathcal{T}\) will be called units. It follows from (1) that the closed set \(\mathcal{T}\) has a non-empty interior; we shall denote its boundary by \(\partial \mathcal{T} := \mathcal{T} \setminus \text{int}\mathcal{T}\).

By \((x_o, y_o)\) we denote a unit from \(\mathcal{T}\) to be currently evaluated. For input vectors we also use the notation \(x^\text{min}, x^\text{max}, x^\text{ev}, x^\text{sd}\), where for \(i = 1, \ldots, m\) we set \(x_i^\text{min} = \min_j x_{ij}, \ x_i^\text{max} = \max_j x_{ij}, \ x_i^\text{ev} = \frac{1}{n} \sum_j x_{ij}, \) and \(x_i^\text{sd} = \sqrt{\sum_j (x_{ij} - x_i^\text{ev})^2}\). Here \(x_i^\text{sd}\) is the standard deviation of the \(i\)-th input over all DMU\(_j\), \(j = 1, \ldots, n\). The notation \(y^\text{min}, y^\text{max}, y^\text{ev}, \) and \(y^\text{sd}\) is introduced analogously for output vectors. Without loss of generality, we assume that \(x^\text{min} < x^\text{max}\) and \(y^\text{min} < y^\text{max}\). Otherwise, there would be components of inputs / outputs, where all DMUs take the same value and such components can be excluded from the analysis.

We write \((x, y) \succeq (x_o, y_o)\) if the unit \((x, y)\) dominates the unit \((x_o, y_o)\), that is, if \(x \leq x_o\) and \(y \geq y_o\). A unit \((x, y)\) strictly dominates the unit \((x_o, y_o)\) if \(x < x_o\) and \(y > y_o\). A unit \((x_o, y_o)\) \(\in \mathcal{T}\) is called strongly efficient if there is no other unit in \(\mathcal{T}\) dominating \((x_o, y_o)\), i.e., if the property that \((x, y) \in \mathcal{T}\) dominates \((x_o, y_o)\) yields \((x, y) = (x_o, y_o)\).\(^2\) A unit \((x_o, y_o)\) \(\in \mathcal{T}\) is called weakly efficient if there is no unit in \(\mathcal{T}\) that strictly dominates \((x_o, y_o)\). Evidently, any strongly efficient unit is weakly efficient, and the weakly efficient units lie on the boundary \(\partial \mathcal{T}\).

The converse is also true: every unit on the boundary \(\partial \mathcal{T}\) is weakly efficient because the definition of \(\mathcal{T}\) in (1) does not impose the non-negativity assumption on the units therein. The boundary \(\partial \mathcal{T}\) is thus partitioned into the strongly efficient frontier \(\partial^S \mathcal{T}\) containing all strongly efficient units and the remaining part \(\partial^W \mathcal{T} := \partial \mathcal{T} \setminus \partial^S \mathcal{T}\), which consists of the weakly but not strongly efficient units. In this paper we will refer to the remaining part of the boundary as the weakly efficient frontier. One thus has \(\partial \mathcal{T} = \partial^S \mathcal{T} \cup \partial^W \mathcal{T}\) and \(\partial^S \mathcal{T} \cap \partial^W \mathcal{T} = \emptyset\). The simple proof of the next lemma is omitted.

Lemma 2.1. For \((x_o, y_o) \in \mathcal{T}\), the following statements hold.

(a) \((x^M, y^M) \succeq (x_o, y_o)\);

(b) \((x_o, y_o) \in \partial \mathcal{T}\) if and only if for all \((d^x, d^y) \geq 0\) such that \((x_o - d^x, y_o + d^y) \in \mathcal{T}\), one has \((d^x, d^y) \neq 0\);

(c) \((x_o, y_o) \in \partial^S \mathcal{T}\) if and only if for all \((d^x, d^y) \geq 0\) such that \((x_o - d^x, y_o + d^y) \in \mathcal{T}\), one has that \((d^x, d^y) = 0\);

\(^2\)This is the well known Pareto–Koopmans efficiency. Some authors call such units Pareto efficient, or fully efficient — see the discussion in Cooper, Seiford and Tone (2007, p. 45).
(d) \((x_o, y_o) \in \partial W \mathcal{T}\) if and only if there exists \((d^x, d^y) \geq 0\) such that \((x_o - d^x, y_o + d^y) \in \mathcal{T}\), and if for all \((d^x, d^y) \geq 0\) such that \((x_o - d^x, y_o + d^y) \in \mathcal{T}\), one has \((d^x, d^y) \not\geq 0\).

2.2 Efficiency measures and their desired properties

On the technology set \(\mathcal{T}\) one can define various measures, which then give rise to specific DEA models. This is achieved by formulating certain mathematical programming problems that are applied separately to each evaluated unit \((x_o, y_o) \in \mathcal{T}\).

A mathematical program consist of an objective function and a set of conditions for decision variables. The values of decision variables satisfying these conditions are called feasible solutions of the program. Solving the program yields the optimal value of the objective function and a subset of feasible solutions called the optimal solutions.

The optimal value of a DEA program determines the efficiency score / value of the measure of the evaluated unit. Optimal solutions also yield projections and benchmarks of the evaluated unit — the elements of \(\partial \mathcal{T}\) that dominate \((x_o, y_o)\) and are associated with the efficiency score. Depending on the chosen measure, models and their efficiency scores have varying properties. In the first instance, one wants to know whether the efficiency score equal to one identifies strong efficiency of the evaluated unit. Some models are ambiguous in this regard and require further analysis to arrive at a conclusion.

Following the works of Färe and Lovell (1978), Pastor, Ruiz and Sirvent (1999), Sueyoshi and Sekitani (2009), Russell and Schworm (2018), Aparicio and Monge (2022), and Halická and Trnovská (2021), we highlight nine desirable properties for efficiency measurement in DEA.

(P1) Unique projection for efficiency comparison. The projection generated by the measure should be the same, regardless of the alternative optimal solution considered from the associated optimization problem.

(P2) Identification. The value of the measure equals one if and only if the evaluated unit is strongly efficient.

(P3) Strong efficiency of projections. Projections generated by the measure are strongly efficient.

(P4) Boundedness. The measure takes values between zero and one.

(P5) Units invariance. The value of the measure does not depend on the units of measurements in the input and output variables.

(P6) Translation invariance. The value of the measure is unaffected by translation in inputs or outputs.

(P7) Monotonicity. An increase in any input or decrease of any output relative to the evaluated unit, holding other inputs as well as outputs constant, reduces or maintains the value of the measure.
(P8) **Strict monotonicity.** An increase in any input or decrease of any output relative to the evaluated unit, holding other inputs as well as outputs constant, reduces the value of the measure.

(P9) **Superefficiency.** The value of the measure of a unit outside the technology set is well defined and finite.

The properties (P1) and (P3) are expressed in terms of the projections provided by the models, where (P3) specifically is essential for (P2) and (P8). The properties (P2), (P4)–(P8) focus on the value of the efficiency measure (the efficiency score) and its characteristics, while the last property (P9) is important not only for efficiency measurement of units outside $T$ but also for the frontier shift measurability (see Sueyoshi and Sekitani, 2009).

We shall now introduce some standard models that enter our analysis as special cases. We formulate these models in a way that permits certain generalizations.

### 2.3 Standard path-based models

We first present the radial oriented models introduced by Banker, Charnes and Cooper (1984). The standard formulation of these models is over non-negative data with the assumption

$$x_j \geq 0 \quad \text{and} \quad y_j \geq 0 \quad \text{for each} \quad j \in \{1, \ldots, n\}. \quad (2)$$

The input oriented BCC model can be formulated as

$$\text{(BCC-I):} \quad \min \quad \theta \quad \begin{aligned}
X \lambda & \leq \theta x_o \\
Y \lambda & \geq y_o \\
e^T \lambda & = 1, \quad \lambda \geq 0,
\end{aligned} \quad (3)$$

and the output oriented BCC model as

$$\text{(BCC-O):} \quad \min \quad \theta \quad \begin{aligned}
X \lambda & \leq x_o \\
Y \lambda & \geq \frac{1}{\beta} y_o \\
e^T \lambda & = 1, \quad \lambda \geq 0.
\end{aligned} \quad (4)$$

If we denote $\psi := \frac{1}{\beta}$ and replace $\min \theta$ by $\max \psi$, then we obtain the output BCC model in standard form, where the optimal $\psi^* \geq 1$ and the efficiency score is $\frac{1}{\psi^*}$. Among the properties (P1)–(P9), the BCC models satisfy only (P1), (P4), (P5), and (P7).

The hyperbolic distance measure model introduced by Färe, Grosskopf and Lovell (1985) combines input- and output-oriented radial measures into one measure. Commonly it is defined over positive data only, i.e.

$$x_j > 0 \quad \text{and} \quad y_j > 0 \quad \text{for each} \quad j \in \{1, \ldots, n\}. \quad (5)$$

In our formulation it reads

$$\text{(HDM):} \quad \min \quad \theta \quad \begin{aligned}
X \lambda & \leq \theta x_o \\
Y \lambda & \geq \frac{1}{\beta} y_o \\
e^T \lambda & = 1, \quad \lambda \geq 0.
\end{aligned} \quad (6)$$

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Halická and Trnovská (2019) investigated the properties of this model, proposed computational methods for its solution and derived its dual form via SDP, while Hasannasab et al. (2019) established duality via second order cone. HDM satisfies properties (P1), (P4), (P5), (P7), and (P9).

The use of the linearization $\frac{1}{\theta} - 1 \sim 1 - \theta$ around $\theta = 1$ in the HDM model leads to

\[(DDM): \min \theta \]
\[X \lambda \leq x_o - (1 - \theta)x_o \]
\[Y \lambda \geq y_o + (1 - \theta)y_o \]
\[e^T \lambda = 1, \ \lambda \geq 0.\]

If we denote $\delta = 1 - \theta$ and replace $\min \theta$ by $\max \delta$ we obtain a special version of the directional distance measure model with the directional vector $(g^x_o, g^y_o) = (x_o, y_o)$. This model, introduced by Briec (1997), is known as the proportional directional distance function. It satisfies the properties (P1), (P4), (P5), (P7), and (P9).

Now, we present the general directional distance measure model (DDM-g). Let $g^x_o \in \mathbb{R}^m_+$, $g^y_o \in \mathbb{R}^s_+$ be given directional vectors that may depend on $(x_o, y_o)$ and such that at least one component of $g^x_o$ or $g^y_o$ is positive, i.e.,

\[(g^x_o, g^y_o) \succeq 0.\]

(8)

The model admits negative data provided the nonnegativity assumption (8) on directional vectors is satisfied. The model reads

\[(DDM-g): \min \theta \]
\[X \lambda \leq x_o - (1 - \theta)g^x_o \]
\[Y \lambda \geq y_o + (1 - \theta)g^y_o \]
\[e^T \lambda = 1, \ \lambda \geq 0.\]

(9)

Once again, if we denote $\delta = 1 - \theta$ and replace $\min \theta$ by $\max \delta$, we obtain the standard form of DDM-g introduced in Chambers, Chung and Färe (1996, 1998).

Since DDM is a linearization of HDM, it is natural to ask whether there exists a generalization of HDM whose linearization corresponds to DDM-g. An affirmative answer is provided by the following model, which we refer to as the general hyperbolic distance measure (HDM-g) model,

\[(HDM-g): \min \theta \]
\[X \lambda \leq x_o - (1 - \theta)g^x_o \]
\[Y \lambda \geq y_o + \left(\frac{1}{\theta} - 1\right)g^y_o \]
\[e^T \lambda = 1, \ \lambda \geq 0.\]

(10)

This new model, too, uses pre-specified directions and admits negative data as long as the directions satisfy the non-negativity condition (8). It is easy to see that the choice $g^x_o = x_o$ and $g^y_o = y_o$ leads to the HDM model.

Without specific knowledge of the directional vector $g$ beyond (8), only the property (P1) is ensured in HDM-g and DDM-g. One needs to say more about the choice of the directional vectors to ensure validity of the remaining properties (P2)–(P9). For a discussion on the available choices of directional vectors see, for example, Färe, Grosskopf and Margaritis (2008) and Pastor, Aparicio and Zofío (2022). Table 1 presents directions $(g^x_o, g^y_o)$ that will feature in the
The subsequent analysis of HDM-g and DDM-g model families. These directions exhibit a range of properties as documented in Table 3.

| Notation | \(g_o^x\) | \(g_o^y\) | Reference |
|----------|------------|------------|-----------|
| (G1)     | \(x_o\)   | \(y_o\)   | Chambers, Fâure and Grosskopf (1996) |
| (G2)     | \(x_o - x_{\min}\) | \(y_{\max} - y_o\) | Portela, Thanassoulis and Simpson (2004) |
| (G3)     | \(x_{\max} - x_{\min}\) | \(y_{\max} - y_{\min}\) | Portela, Thanassoulis and Simpson (2004) |
| (G4)     | \(x^{ev}\) | \(y^{ev}\) | Aparicio, Pastor and Ray (2013) |
| (G5)     | \(x^{sd}\) | \(y^{sd}\) | |
| (G6)     | \(e = (1, \ldots, 1)^T\) | \(e = (1, \ldots, 1)^T\) | Chambers, Fâure and Grosskopf (1996) |

Table 1: Commonly encountered choices of directions \(g_o\). For the notation \(x_{\min}\), \(x_{\max}\), etc., see Subsection 2.1.

(G1) is perhaps the most commonly chosen direction in the family of DDM-g models, yielding the Farrell proportional distance function (see Briec and Kerstens, 2009). Directions (G2) and (G3) appear in the range directional models developed by Portela, Thanassoulis and Simpson (2004). The use of (G4) directions is discussed in Aparicio, Pastor and Ray (2013). It is also quite common to consider DDM-g with (G6) directions, which is mathematically equivalent to minimizing the \(l_{\infty}\) distance to the boundary of \(T\) (see Briec and Lesourd, 1999).

Remark 2.2. It is interesting to note that the expressions for directions (G1)–(G6) appear in the denominator of weights in specific WAM models. Indeed, direction (G6) is linked to the weights in AD model; directions (G2) and (G3) are linked to the weights in BAM and RAM models, respectively; direction (G1) corresponds to the weights in so-called Measure of Inefficiency Proportions (MIP) model by Cooper, Park and Pastor (1999); and the weights generated via (G5) appear in the WAM model of Ševčovič, Halická and Brunovský (2001).

3 A general scheme for path-based models

On comparing (9) and (10), we observe that the models DDM-g and HDM-g are qualitatively similar, differing only in the nature of their output bounds which are linear for DDM-g but non-linear for HDM-g. The latter is formalized via the function \(\psi^y(\theta) = \frac{1}{\theta}\).

We shall now generalize this approach in the direction of convex functions \(\psi^y\) on the output side and concave functions \(\psi^x\) on the input side. Our new model will encompass all the models mentioned up to this point, allowing for a unified analysis of their properties. At the same time, the generalization will permit construction of new models whose properties can be specified in advance.

By the general scheme (GS) model applied to \((x_o, y_o) \in T\) with directions \(g_o = (g_o^x, g_o^y) \geq 0\)
that may depend on \((x_0, y_0)\), we understand
\[
(GS)_o \min \theta \\
X \lambda \leq x_o + (\psi^x(\theta) - 1)g_o^x, \quad (11a)
Y \lambda \geq y_o + (\psi^y(\theta) - 1)g_o^y, \quad (11b)
e^T \lambda = 1, \quad \lambda \geq 0. \quad (11c)
\]

Here the real functions \(\psi\), their domains (dom), and images (im) satisfy the following assumptions.

(A1) \(\text{dom}(\psi^x) = (a^x, \infty)\) with \(a^x \in \{-\infty, 0\}\) and \(\text{dom}(\psi^y) = (a^y, \infty)\) with \(a^y \in \{-\infty, 0\}\);

(A2) \(\psi^x\) is concave, increasing and \(\psi^y\) is convex, decreasing;

(A3) \(\text{im}(\psi^x) = (b^x, \infty)\) with \(b^x = -\infty\) if \(g^y = 0\) and \(b^x \in \{-\infty, 0\}\) otherwise; \(\text{im}(\psi^y) = (b^y, \infty)\) with \(b^y \in \{-\infty, 0\}\). Due to monotonicity of \(\psi^x\) and \(\psi^y\) assumed in (A2), the assumption (A4) means \(\lim_{\theta \to a^x_+} \psi^x(\theta) = b^x\) and \(\lim_{\theta \to \infty} \psi^x(\theta) = \infty\) and \(\lim_{\theta \to a^y_+} \psi^y(\theta) = \infty\) and \(\lim_{\theta \to \infty} \psi^y(\theta) = b^y\).

If the vectors \(g^x_o\) and \(g^y_o\) are positive, then the GS model (11) operates in the full input-output space, hence it is a graph model in the terminology of Färe, Grosskopf and Lovell (1985).

With \(g^x_o = 0\) or \(g^y_o = 0\), one obtains output or input oriented models, respectively.

Denote the right-hand sides of (11b) and (11c) by \(\phi^x_o\) and \(\phi^y_o\), respectively, i.e.,
\[
\phi^x_o(\theta) := x_o + (\psi^x(\theta) - 1)g^x_o, \quad \phi^y_o(\theta) := y_o + (\psi^y(\theta) - 1)g^y_o. \quad (12)
\]
It is easy to see that the models presented in the previous section are special cases of the GS model. These models, taken in conjunction with the directions in Table 1 and the usual assumptions on the positiveness of data (i.e., condition (2) for the BCC models and (5) for the hyperbolic model) will be called the standard path-based models. The corresponding parametrizations in Table 2 indicate that only two choices of the function \(\psi^y\) are associated with the standard models: the linear, \(\psi^y(\theta) = 2 - \theta\), and the hyperbolic, \(\psi^y(\theta) = \frac{1}{\theta}\). Only one form of \(\psi^x\) appears: \(\psi^x(\theta) = \theta.\) However, other choices of \(\psi\) satisfying the assumptions (A1)–(A4) are possible, for example \(\psi^x(\theta) = \theta^p, -1 < p < 0,\) or \(\psi^x(\theta) = 1 + \ln \theta,\) and \(\psi^y(\theta) = \theta^p, p > 0,\) \(\psi^y(\theta) = 1 - \ln \theta\) with domains \((0, \infty),\) or \(\psi^y(\theta) = e^{1-\theta}\) with domain \((-\infty, \infty).\) We will be able to fine-tune the model properties by suitable choices of \(\psi^x\) and \(\psi^y\) and directions. Note that the choice of functions \(\psi^x(\theta) = \theta^{1-p}, \psi^y(\theta) = \theta^{-p}, p \in [0, 1]\) and directions \((g^x, g^y) = (x_o, y_o) > 0\) leads to the so-called generalised distance function introduced by Chavas and Cox (1999).

The next subsection shows that the \((GS)_o\) model is well defined as a convex program, i.e., for any \((x_o, y_o) \in T\) the optimal value \(\theta^*\) of \((GS)_o\) is finite and the minimum is attained. To this end, we introduce a geometric interpretation of the model.

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3 In the case \(g^x_o = 0\) or \(g^y_o = 0\) the contribution of \(\psi^x\) in (11b) or \(\psi^y\) in (11c) vanishes; in such cases we take \(\text{dom}(\psi^x) = (-\infty, \infty)\) or \(\text{dom}(\psi^y) = (-\infty, \infty),\) respectively.

4 Since \(g^x_o = 0,\) the term \(g^x_o \psi^y\) vanishes.

5 Since \(g^y_o = 0,\) the term \(g^y_o \psi^x\) vanishes.
defines a continuous path in the input-output space $\mathbb{R}^T$.

The definition of the technology set $\mathcal{T}$ is given in (1).

### 3.1 Geometric interpretation

We take the standard parameter choices $\phi_1^{\circ} = \theta x_0, \phi_0^{\circ} = y_0, \phi_1^{\circ} = \frac{1}{2} y_0$, and $\psi_x = \psi_y = \frac{1}{2}$.

It is seen from the definition of the path in (12) and (14) that each path is determined by a specific choice of $\phi_1^{\circ}, \phi_0^{\circ}, \phi_1^{\circ}$, and $\psi_x, \psi_y$ (the parameters of the path). Unless otherwise stated, we take the standard parameter choices, i.e.,

$$(x_0, y_0) \in \mathcal{T}, \quad (g_0^x, g_0^y) \geq 0, \quad \text{and} \quad \psi_x, \psi_y \quad \text{that meet (A1)--(A4)}.$$

The next theorem summarises those properties of the path $\phi_o$ that are common for all standard parameter choices.

**Theorem 3.1.** The path $\phi_o$ defined by (12) and (14) with standard parameter choices (16) has the following properties.

(a) For $\theta_1 \leq \theta_2 \in \mathcal{D}$, the point $\phi_o(\theta_1)$ on the path $\phi_o$ dominates the point $\phi_o(\theta_2)$, (i.e., $\phi_1^{\circ}(\theta_1) \geq \phi_1^{\circ}(\theta_2)$), and $\phi_o(\theta_1) \neq \phi_o(\theta_2)$, if and only if $\theta_1 < \theta_2$.

(b) $\phi_o(1) = (x_0, y_0) \in \mathcal{T}$.

(c) There exists $i$ such that $\lim_{\theta \to +\infty} \phi_i^{\circ}(\theta) = -\infty$ or $r$ such that $\lim_{\theta \to +\infty} \phi_r^{\circ}(\theta) = +\infty$ (see Figure 1 for illustration).

(d) There exists $\theta^*_o \leq 1$ such that $\phi_o(\theta^*_o) \in \mathcal{T}$ and $\phi_o(\theta) \notin \mathcal{T}$ for all $\theta < \theta^*_o$. As a consequence, the point $\phi_o(\theta^*_o)$ belongs to the boundary of $\mathcal{T}$ and $\theta^*_o$ is the finite optimal value of the (GS)$_o$ model (11).

**Proof.** (a) This follows from the assumptions placed on $g_o$ in (16), the monotonicity properties of $\psi_x, \psi_y$ and the definition of $\phi_o$ in (12).

| Model  | $g_o^x$ | $\phi_1^{\circ}(\theta)$ | $g_o^y$ | $\phi_0^{\circ}(\theta)$ | $\psi_x$ | $\psi_y$ |
|--------|---------|------------------------|---------|------------------------|---------|---------|
| BCC-I  | $x_0$   | $\theta x_0$           | 0       | $y_0$                 | $\theta$ | $\psi_y$ |
| BCC-O  | 0       | $x_0$                  | $y_0$   | $\frac{1}{2} y_0$    | $\psi_x$ | $\frac{1}{2}$ |
| DDM-g  | $g_o^x$ | $x_o + (\theta - 1) g_o^x$ | $g_o^y$ | $y_o + (1 - \theta) g_o^y$ | $\theta$ | $2 - \theta$ |
| DDM    | $x_0$   | $\theta x_o$          | $y_o$   | $(2 - \theta) y_o$   | $\theta$ | $2 - \theta$ |
| HDM    | $x_0$   | $\theta x_o$          | $y_o$   | $\frac{1}{2} y_o$    | $\theta$ | $\frac{1}{2}$ |
| HDM-g  | $g_o^x$ | $x_o + (\theta - 1) g_o^x$ | $g_o^y$ | $y_o + (\frac{1}{\beta} - 1) g_o^y$ | $\theta$ | $\frac{1}{\beta}$ |

Table 2: Parameterization of the standard path-based models.
Figure 1: Illustration of Theorem 3.1. Hyperbolic path $\phi_o(\theta)$ for direction $(10, 2)$; there exists $\bar{\theta} < 1$ such that $\phi_y(\bar{\theta}) > y^{\text{min}}$.

(b) This follows from (12) and (A3).

c) This part follows from (12), (A1), and (A4).

d) The assumptions on $g_o$ and part (c) of this theorem yield $\lim_{\theta \to a_x^+} \phi_x(\theta) = -\infty$ for some $i$ or $\lim_{\theta \to a_y^+} \phi_y(\theta) = +\infty$ for some $r$. From this, part (b) of this theorem, and continuity of $\phi_o$, it follows that there exists a finite $\bar{\theta}$ such that $\phi_y(\bar{\theta}) > y^{\text{max}}$ or $\phi_x(\bar{\theta}) < x^{\text{min}}$ (see Figure 1). Hence by Lemma 2.1(a) one obtains $\phi_o(\bar{\theta}) \notin T$. Property (a) of this theorem then yields that for any $\bar{\theta}$ such that $\phi_o(\bar{\theta}) \notin T$ one has $\phi_o(\theta) \notin T$ for all $\theta \leq \bar{\theta}$.

On the other hand, $\phi_o(1) = (x_o, y_o) \in T$ and property (a) of this theorem implies that if $\phi_o(\bar{\theta}) \in T$ for some $\bar{\theta}$, then $\phi_o(\theta) \in T$ for all $\theta \geq \bar{\theta}$. Now, by continuity of $\phi_o$, there exists a finite $\theta^*_o$ such that $\theta^*_o = \sup\{\theta : \phi_o(\theta) \notin T\} = \inf\{\theta : \phi_o(\theta) \in T\}$. We show that $\phi_o(\theta^*_o)$ belongs to the boundary $\partial T$ of the closed set $T$. Assume by contradiction that $\phi_o(\theta^*_o) \in \text{int} T$. From continuity of $\phi_o$, it follows that $\phi_o(\theta) \in \text{int} T$ for all $\theta$ close enough to $\theta^*_o$, hence also for some $\theta < \theta^*_o$ which contradicts $\theta^*_o = \inf\{\theta : \phi_o(\theta) \in T\}$. 

\[ \square \]

Thanks to Theorem 3.1, the GS model applied to $(x_o, y_o) \in T$ with directions $(g_{x_o}^*, g_{y_o}^*) \geq 0$ is well defined. The optimal value $\theta^*_o$ is called the efficiency score, or alternatively, the value of the efficiency measure for $(x_o, y_o)$. The point $(\phi_x(\theta^*_o), \phi_y(\theta^*_o))$ on the path $\phi_o$ is called the projection of $(x_o, y_o)$ in the (GS) model. Hereafter, we shall denote the projection point more compactly by

\[ (\hat{x}_o, \hat{y}_o) := (\phi_x(\theta^*_o), \phi_y(\theta^*_o)). \]
(GS)$_o$ may have multiple optimal slacks given by
\[(s^{x*}, s^{y*}) := (\hat{x}_o - X\lambda^*, Y\lambda^* - \hat{y}_o) \geq 0.\]

**Lemma 3.2.** Let $(\lambda^*_o, \theta^*_o)$ be an optimal solution of (GS)$_o$. Then at least one of the inequalities in (11b) or (11c) is tight, i.e., at least one component of the corresponding slacks is zero.

**Proof.** If all components of the slacks are positive, then thanks to the assumed continuity of the path there is $\epsilon > 0$ such that $(\lambda^*_o, \theta^*_o - \epsilon)$ is feasible, which contradicts the optimality of $\theta^*_o$. \(\square\)

### 4 Properties of the general model

We will now analyze the (GS)$_o$ model in the light of the nine desirable properties (P1)–(P9). The only property that the general scheme (GS)$_o$ satisfies regardless of the specific choice of directions is the uniqueness of projection (P1). This is a simple consequence of Theorem 3.1. The remaining properties depend on the choice of direction vectors $g_o$ and the functions $\psi^x$ and $\psi^y$. We examine them one at a time in the following subsections.

#### 4.1 Identification

(P2) (a) If the measure equals 1, then the evaluated unit is strongly efficient.

(b) If the evaluated unit is strongly efficient, then the measure equals 1.

The part (b) of this property holds universally in the (GS) scheme as indicated in the following theorem.

**Theorem 4.1.** If $(x_o, y_o)$ is strongly efficient in $\mathcal{T}$, then $\theta^*_o = 1$.

**Proof.** If $(x_o, y_o)$ is strongly efficient, then the set of points from $\mathcal{T}$ dominating $(x_o, y_o)$ contains only $(x_o, y_o)$. In view of $\phi_o(1) = (x_o, y_o)$ and Theorem 3.1(a), one has $\theta^*_o = 1$. \(\square\)

Note that $\theta^*_o = 1$ in conjunction with Theorem 3.1(d) only yield $(x_o, y_o) \in \partial W\mathcal{T}$, hence one must admit the possibility that $(x_o, y_o)$ belongs to $\partial W\mathcal{T}$ in some cases with $\theta^*_o = 1$.

**Theorem 4.2.** The GS model with positive directions violates the part (b) of (P2) for any $\mathcal{T}$.

**Proof.** Let $(x_o, y_o) \in \partial W\mathcal{T}$ and $g_o > 0$. We want to show that $\theta^*_o = 1$. Assume by contradiction that $\theta^*_o < 1$. Then $\hat{x}_o = x_o + (\psi^x(\theta^*_o) - 1)g_o^x < x_o$ and $\hat{y}_o = y_o + (\psi^y(\theta^*_o) - 1)g_o^y > y_o$, and $(\hat{x}_o, \hat{y}_o) \in \mathcal{T}$, which is, by Theorem 3.1(d), in contradiction with the weak but not strong efficiency of $(x_o, y_o)$. \(\square\)

**Remark 4.3.** By Theorem 4.2, all standard path-based models with directions (G1) and (G3)–(G6) violate the part (b) of (P2) for any $\mathcal{T}$. In the case of DDM-(G2), the fulfilment or non-fulfilment of the (b) part of (P2) depends on data configuration (the shape of $\mathcal{T}$). The fulfilment can be seen in the two-dimensional Example 4.4, where each unit $(x_o, y_o) \in \partial W\mathcal{T}$ is projected onto $\partial S\mathcal{T}$ in DDM-(G2) model, and hence $\theta^*_o < 1$. The non-fulfilment is seen in the three-dimensional Example 4.5, where the unit C belongs to $\partial W\mathcal{T}$ and the corresponding paths
Figure 2: Illustration of Example 4.4. The DDM-(G2) model projects units from $\mathcal{T}$ onto $\partial^S \mathcal{T}$ in the two dimensional technology $\mathcal{T}$.

at C point out of $\mathcal{T}$. This implies that the efficiency scores of C is equal to one. On the other hand, HDM-(G2) violates part (b) of (P2) even in Example 4.4. Indeed, since $\frac{1}{1} - 1 > 1 - \theta$ for $\theta < 1$, the hyperbolic path for C is above the directional path for C for $\theta < 1$, and hence leaves $\mathcal{T}$ at C.

Example 4.4. Consider a single-input single-output example where the technology set is generated by four units; $A = (2, 2)$, $B = (4, 4)$, $C = (7, 4)$, and $D = (2, 1)$. The units A and B are strongly efficient, while C and D belong to $\partial^W \mathcal{T}$. The (G2) direction for C is $(5, 0)$ and it points from C to B which implies that $\theta^*_C < 1$. The (G2) direction for D is $(0, 1)$ and it points from D to A which implies that $\theta^*_D < 1$. Similar observation applies to all units from $\partial^W \mathcal{T}$. The situation is illustrated in Figure 2.

Example 4.5. Consider a two-input, single-output example, where the technology set is generated by two strongly efficient units, $A = (1, 0, 1)$ and $B = (0, 1, 1)$. The unit $C = (0.5, 0.5, 0.5)$ is weakly but not strongly efficient. Here $(x^{\min}, y^{\max}) = (0, 0, 1)$ and (G2) direction for C is $g_C = (0.5, 0.5, 0.5)$. Since $g_C > 0$, Theorem 4.2 yields that the part (b) of (P2) is not satisfied. This is also seen in Figure 3; the DDM-(G2) path for C with $g_C$ direction leaves $\mathcal{T}$ at C, which means that C is projected onto itself with $\theta^*_C = 1$.

The tools to determine whether a unit with the efficiency score equal to one is strongly efficient will be presented in the next subsection, where this problem will be solved in a more general setting.

4.2 Strong efficiency of projection

(P3) The projection point generated by the measure is strongly efficient.

The projection in the $(GS)_o$ model is located on the boundary of $\mathcal{T}$ but not necessarily on its strongly efficient part. Note that the lack of strong efficiency of projection is a known
Figure 3: An example of $T$, for which the DDM-(G2) model does not project $C \in \partial^W T$ onto the strongly efficient frontier $\partial^S T = |AB|$.

possibility in BCC and DDM models. We show in this subsection that this is the case of all standard path-based models.

The absence of strong efficiency of the projection has knock-on effects such as the lack of strict monotonicity of the efficiency score or the failure of one-to-one identification of efficiency. Moreover, if a unit is not projected onto the strongly efficient frontier, then its efficiency score does not capture all sources of inefficiencies in some input/output components and hence it overestimates the unit’s performance. Therefore, it is important to recognize whether the specific unit was projected onto a strongly efficient frontier or not. Unfortunately, this recognition cannot be made from the mere value of the optimal score. In connection with BCC and DDM, however, tools have been developed to detect this feature. We are now adopting them into our scheme.

The next theorem states that to identify the strong efficiency of projections one needs to know more than just the optimal value.

**Theorem 4.6.** The projection of $(x_o, y_o)$ is strongly efficient if and only if for each optimal solution $(\lambda^*_o, \theta^*_o)$ of (11), the inequality constraints (11b) and (11c) are satisfied with equality.

**Proof.** Assume by contradiction that $(x_o, y_o)$ is strongly efficient and there exists an optimal solution $(\lambda^*_o, \theta^*_o)$ of $(GS)_o$ such that at least one component of the corresponding optimal slacks is positive. Then the unit $(X\lambda^*_o, Y\lambda^*_o)$ differs from $(\tilde{x}_o, \tilde{y}_o)$, belongs to $T$ and dominates $(\check{x}_o, \check{y}_o)$, hence $(\check{x}_o, \check{y}_o)$ is not strongly efficient.
To prove the reverse implication, we assume that \((\hat{x}_o, \hat{y}_o) \in \partial W T\). According to Lemma 2.1(d) there exists \((d^x, d^y) \succeq 0\), \((d^x, d^y) \not\preceq 0\) such that \((\hat{x}_o - d^x, \hat{y}_o + d^y) \in T\). By definition of \(T\), there then exists \(\hat{\lambda} \geq 0\) such that \(e^T \hat{\lambda} = 1\), \(X\hat{\lambda} \leq \hat{x}_o - d^x\), \(Y\hat{\lambda} \geq \hat{y}_o + d^y\). This implies that \((\hat{\lambda}, \theta_o^*)\) is an optimal solution to \((GS)\) for \((x_o, y_o)\) and at least one component of inequality (11b) or (11c) is strict. \(\Box\)

Theorem 4.6 identifies projection onto the strongly efficient frontier theoretically. A practical method for the identification of strong efficiency of a weakly efficient unit \((x_o, y_o) \in T\) with \(\theta_o^* = 1\), but also of the projection of an inefficient unit \((x_o, y_o) \in T\) with \(\theta_o^* < 1\), is the so-called second phase procedure. It solves a modified program, which at the fixed value of \(\theta_o^*\) maximizes the sum of slacks. In the context of the GS scheme it reads

\[
\begin{align*}
\max & \quad e^T s^x + e^T s^y \\
\text{s.t.} & \quad X\lambda + s^x = \phi_o^*(\theta_o^*), \\
& \quad Y\lambda - s^y = \phi_o^*(\theta_o^*), \\
& \quad e^T \lambda = 1, \lambda \geq 0, s^x \geq 0, s^y \geq 0.
\end{align*}
\]  

\textbf{Theorem 4.7.} The optimal value in (18) vanishes if and only if the projection of \((x_o, y_o)\) in \((GS)\) is strongly efficient. Moreover, for each optimal solution \((\lambda^*, s^{x*}, s^{y*})\) in (18), the unit \((X\lambda^*, Y\lambda^*)\) is a strongly efficient benchmark for \((x_o, y_o)\).

\textbf{Proof.} The first part follows from Theorem 4.6. The second part claims that \((X\lambda^*, Y\lambda^*) = (\hat{x}_o - s^{x*}, \hat{y}_o + s^{y*}) \in \partial W T\). Suppose this is not the case. Then by Lemma 2.1(c), there exists a unit in \(T\) (corresponding to some \(\lambda \geq 0\) with \(e^T \lambda = 1\)) that dominates but is not equal to \((X\lambda^*, Y\lambda^*)\). The slacks of \((\hat{x}_o, \hat{y}_o)\) relative to the new unit have increased, which contradicts the optimality of the original slacks in (18). \(\Box\)

\textbf{Remark 4.8.} While in the class of slacks-based models the notions of ‘projection’ and ‘benchmark’ coincide (see Remark 2 in Halická and Trnovská, 2021), in the class of path-based models it is necessary to single out one projection from among multiple benchmarks. The projection is uniquely determined by (17) as the point (benchmark) on the path, where the path leaves \(\partial T\), which also pins down the optimal value of the objective function. However, the model is capable of generating other benchmarks, i.e., points \((X\lambda, Y\lambda)\) such that \((\lambda, \theta_o^*)\) is the optimal solution of \((GS)\), some of which may lie in \(\partial W T\). A more restricted (but still convex) subset of benchmarks in \(\partial W T\) is generated by optimal solutions of (18) but even this subset may contain multiple elements (see for example Sueyoshi and Sekitani, 2007). \(\Box\)

\textbf{Remark 4.9.} Property (P3) implies (P2). Indeed, if \(\theta_o^* = 1\), then \((x_o, y_o)\) coincides with its projection. Now, if (P3) holds, this projection is strongly efficient, which yields the (b) part of (P2). The part (a) of (P2) holds universally by Theorem 4.1.

As a consequence of the implication (P3) \(\Rightarrow\) (P2), the conclusions of Remark 4.3 regarding failure of (P2) for standard path-based models carry over to property (P3). \(\Box\)

\subsection{4.3 Boundedness}

(P4) The measure takes values between zero and one.
If \((x_o, y_o) \in T\), then automatically \(\theta_o^* \leq 1\) (since \(\theta = 1\) is a feasible solution). On the other hand, the property \(\theta_o^* \geq 0\) hinges on the domains of \(\psi^x\) and \(\psi^y\). With \(\text{dom}(\psi^x) = (0, \infty)\) or \(\text{dom}(\psi^y) = (0, \infty)\), which only occur for nonlinear \(\psi^x\) and \(\psi^y\), one has \(\theta_o^* > 0\) for any choice of \(g_o\). With linear \(\psi\) one has \(\text{dom}(\psi) = \mathbb{R}\) and \(\text{(P4)}\) may in principle fail. In the following theorem, we formulate sufficient conditions on \(g_o\) for \(\text{(P4)}\) to hold.

**Theorem 4.10.** Consider a GS model with \(\psi^x(\theta) = \theta, \psi^y(\theta) = 2 - \theta\), and directions \(g_o \geq 0\). If
\[
\max_i \left\{ \frac{x_i^{\min} + g_{io}^x - x_{io}}{g_{io}^x} : g_{io} > 0 \right\} \geq 0 \quad \text{or} \quad \max_r \left\{ \frac{y_{ro} + g_{ro}^y - y_{o}}{g_{ro}^y} : g_{ro} > 0 \right\} \geq 0,
\]
then \(\theta_o^* \geq 0\).

**Proof.** We proceed by constructing lower bounds for the optimal score \(\theta_o^*\). If \(g_{io}^x > 0\), then the \(i\)-th component of \((11b)\) yields
\[
\frac{x_i^{\min} + g_{io}^x - x_{io}}{g_{io}^x} \leq \psi^x(\theta_o^*) = \theta_o^*.
\]
Similarly, if \(g_{ro}^y > 0\), then the \(r\)-th component of \((11c)\) yields
\[
\frac{y_{ro} + g_{ro}^y - y_{o}}{g_{ro}^y} \leq \psi^y(\theta_o^*) = 2 - \theta_o^*.
\]
If the maximal lower bound is non-negative, then \(\theta_o^* \geq 0\), which completes the proof. □

Sufficient conditions for other linear \(\psi^x\) and \(\psi^y\) follow by an obvious modification of the proof.

**Remark 4.11.** Theorem 4.10 offers an easy route to verify that DDM-g with directions (G1)–(G3) satisfies boundedness \(\text{(P4)}\). A counterexample with negative \(\theta_o^*\) for directions (G4)–(G6) is provided by a single-input, single-output model with two units \(A = (1, 5)\) and \(B = (5, 1)\). Here, the average direction (G4) is \((3, 3)\) and the standard deviation direction (G5) equals \((2, 2)\). The scores for the non-efficient unit B are \(-1/3, -1, -13\), respectively, for the directions (G4)–(G6). The HDM-g family (including HDM itself) satisfies \(\text{(P4)}\) trivially thanks to the non-negative domain of \(\psi\).

The results of Theorem 4.10 are in agreement with the partial results of Sahoo, Mehdiloozad and Tone (2014) related to input oriented directional distance model. □

### 4.4 Unit invariance

(P5) The value of the measure does not depend on the units of measurements in the input and output variables.

**Theorem 4.12.** Let \(C \in \mathbb{D}^{m}_{++}\) and \(B \in \mathbb{D}^{n}_{++}\) be diagonal matrices with positive elements on the diagonal. For all \(j = 1, \ldots, n\), consider the transformed inputs \(C x_j\) and the transformed outputs \(B y_j\), respectively. Let \(g_o^x, g_o^y\) and \(g_o^C, g_o^B\) denote the directional vectors before and after the data transformation, respectively. The \((GS)_o\) model is unit invariant if and only if one has \(g_o^C = C g_o^x\) and \(g_o^B = B g_o^y\).

**Proof.** The input inequality \((11b)\) for the transformed inputs reads
\[
C \sum_{j=1}^{n} x_j \lambda_j = \sum_{j=1}^{n} (C x_j) \lambda_j \leq C x_o + (\psi^x(\theta) - 1) g_o^C \Leftrightarrow C \sum_{j=1}^{n} x_j \lambda_j \leq C (x_o + (\psi^y(\theta) - 1) g_o^y).
\]
Thanks to \(C\) being diagonal with positive entries, this is equivalent to the input inequality \((11b)\) for the original inputs. This and similar equivalence for outputs yield that the conditions...
Remark 4.13. Theorem 4.12 yields easily that GS models with (G1)–(G5) directions satisfy unit invariance (P5) but this property fails with directions (G6).

4.5 Translation invariance

(P6) The value of the measure is unaffected by translation in inputs or outputs.

Theorem 4.14. The \((GS)\) model is translation invariant if and only if the directional vectors \(g_x^o\) and \(g_y^o\) are invariant to the translation of inputs and outputs, respectively.

Proof. Let \(c \in \mathbb{R}^m\) and \(b \in \mathbb{R}^s\) are the translation vectors for inputs and outputs, and hence the translated inputs and outputs are \(x_j + c\) and \(y_j + b\), for all \(j = 1, \ldots, n\) respectively. The condition \(\sum_{j=1}^{n} \lambda_j = 1\) and the invariance of \(g_x^o\) on the translation of inputs imply that the input inequality (11b) of \((GS)\) for translated inputs is equivalent to the input inequality (11b) of \((GS)\) for original inputs:

\[
\sum_{j=1}^{n} (x_j + c)\lambda_j = \sum_{j=1}^{n} x_j\lambda_j + c \leq x_o + c + (\psi^x(\theta) - 1)g_x^o \iff \sum_{j=1}^{n} x_j\lambda_j \leq x_o + (\psi^x(\theta) - 1)g_x^o.
\]

Similar equivalence in outputs proves the theorem.

Translation invariance of DDM-g models is analyzed in detail by Aparicio, Pastor and Vidal (2016). This subsection is consistent with their results and extends them to the whole class of GS models.

Remark 4.15. Using Theorem 4.14, one easily verifies that GS models with directions (G2), (G3), (G5), and (G6) satisfy translation invariance (P6) but those with directions (G1) and (G4) do not.

4.6 Monotonicity

(P7) An increase in any input or decrease of any output relative to the evaluated unit, holding other inputs as well as outputs constant, reduces or maintains the value of the measure.

For any fixed \(\bar{\theta} \in \mathcal{D}\), we consider a function \(\phi(\bar{\theta}): (x_o, y_o) \mapsto \phi_o(\bar{\theta})\) mapping each point \((x_o, y_o) \in \mathcal{T}\) to the point \((\phi^x_o(\bar{\theta}), \phi^y_o(\bar{\theta}))\) on the path \(\phi_o(\theta)\) at \(\theta = \bar{\theta}\). The monotonicity property of the models states that for any \((x_o, y_o)\) and \((x_q, y_q)\) in \(\mathcal{T}\) one has

\[
(x_o, y_o) \succ (x_q, y_q) \Rightarrow \theta^*_o \geq \theta^*_q.
\]

The monotonicity of a model is linked with the monotonicity of \(\phi(\bar{\theta})\), which we define as follows.

Definition 4.16. We say that \(\phi(\bar{\theta})\) is monotone on \(\mathcal{T}\) at \(\bar{\theta} \in \mathcal{D}\) if for any two units \((x_o, y_o)\) and \((x_q, y_q)\) in \(\mathcal{T}\), one has

\[
(x_o, y_o) \succ (x_q, y_q) \Rightarrow \phi_q(\bar{\theta}) \succ \phi_o(\bar{\theta}).
\]
For function $\phi(\theta)$ such that $[\phi^x(\theta)]_i$ depends only on $x_{io}$ and $[\phi^y(\theta)]_r$ depends only on $y_{ro}$, the monotonicity property in Definition 19 simply means that $[\phi^x(\theta)]_i$ and $[\phi^y(\theta)]_r$ are nonincreasing in $x_{io}$ and $y_{ro}$, respectively.

**Lemma 4.17.** Let $(x_o, y_o)$ and $(x_i, y_i)$ be two units in $T$ with the corresponding optimal values $\theta^*_o, \theta^*_i$. If $f_o(\theta^*_o) \geq f_i(\theta^*_i)$, then each optimal solution $(\theta^*_o, \lambda^*_o)$ of $(GS)_o$ is a feasible solution of $(GS)_i$. Furthermore, $\theta^*_o \leq \theta^*_i$.

**Proof.** The assumptions of the lemma and feasibility of $(\theta^*_o, \lambda^*_o)$ for $(GS)_o$ yield

$$X\lambda^*_o \leq \phi^x_o(\theta^*_o) \leq \phi^x_i(\theta^*_i), \quad Y\lambda^*_o \geq \phi^y_o(\theta^*_o) \geq \phi^y_i(\theta^*_i).$$

Therefore $(\theta^*_o, \lambda^*_o)$ is also feasible for $(GS)_i$. Since $\theta^*_o$ is the optimal value for the minimization problem $(GS)_q$, we get $\theta^*_q \leq \theta^*_o$.

The next theorem is a consequence of Lemma 4.17. Here we assume that the assumption of monotonicity is satisfied only at $\bar{\theta} \in D$, where $D$ is a set of those $\bar{\theta}$ for which there exists $(x_o, y_o) \in T$ such that $\theta^*_o = \bar{\theta}$.

**Theorem 4.18.** Suppose that $\phi(\theta)$ is monotone at any $\bar{\theta} \in D$. Then the GS model satisfies the property of monotonicity (P7).

**Remark 4.19.** Note that if $g_o$ does not depend on $(x_o, y_o)$, then $\phi(\bar{\theta})$ is monotone for any choice of $\psi^x$ and $\psi^y$ at any $\theta \in D$. Hence, by Theorem 4.18, (GS) models with directions (G3)–(G6) satisfy the property (P7). For (G1) direction with arbitrary (standard) $\psi^x$ and $\psi^y$, the monotonicity assumption of Theorem 4.18 is satisfied at $\bar{\theta} \in D$, provided $\psi^x(\bar{\theta}) > 0$ and $\psi^y(\bar{\theta}) > 0$ hold. These conditions are satisfied by the DDM-(G1) and HDM-(G1) models since by Remark 4.11, in these models $\theta^*$ attains only positive values on $T$. Hence, both models meet the property (P7). For (G2) direction with arbitrary (standard) $\psi^x$ and $\psi^y$, the monotonicity holds at $\bar{\theta} \in D$ provided $\psi^x(\bar{\theta}) > 0$ and $\psi^y(\bar{\theta}) < 0$ hold. These conditions are again satisfied by both DDM-(G2) and HDM-(G2). Indeed, according to Remark 4.11, DDM-(G2) meets (P4), hence $\theta^*_o > 0$. Furthermore, $\theta^*_o \geq \frac{1}{2}$ on $T$ in HDM-(G2) by an easy computation.

In conclusion, all standard path-based models have the property of monotonicity (P7).

**4.7 Strict monotonicity**

(P8) An increase in any input or decrease of any output relative to the evaluated unit, holding other inputs as well as outputs constant, reduces the value of the measure.

We commence with a necessary condition for strict monotonicity (P8).

**Theorem 4.20.** If there exists $(x_o, y_o) \in \partial^W T$ such that $\theta^*_o = 1$, then the GS model does not meet the property of strict monotonicity.

**Proof.** Since $(x_o, y_o)$ is not a strongly efficient unit, by Theorem 4.7 there exists an optimal solution $(\lambda^*, s^{x*}, s^{y*})$, of (18) such that $(s^{x*}, s^{y*}) \neq 0$ and $(X\lambda^*, Y\lambda^*) \in \partial^S T$. Since $(x_p, y_p) = (X\lambda^*, Y\lambda^*)$ is strongly efficient, Theorem 4.1 yields $\theta^*_p = 1$. Strict monotonicity is violated because $(x_p, y_p)$ dominates and is different from $(x_o, y_o)$ but $\theta^*_o = \theta^*_p = 1$. 

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We have shown in Theorem 4.2 that positive directions rule out validity of (P2). As a consequence of Theorem 4.20, an analogous statement applies to (P8).

**Theorem 4.21.** The GS model with positive directions fails in (P8) for any $T$ defined by (1).

**Remark 4.22.** As shown in Examples 4.4 and 4.5, all standard path-based models exhibit $(x_o, y_o) \in \partial^W T$ such that $\theta_o = 1$ for some data configurations and hence by Theorem 4.20 do not satisfy (P8).

### 4.8 Superefficiency

(P9) The value of the measure of a unit outside the technology set is well defined and finite.

It is well understood that for oriented models superefficiency may fail for some $(x_o, y_o)$. Specifically, if $(x_o, y_o) \notin T$, some of the inequalities in (11b) or (11c) are not satisfied for $\theta = 1$. To remedy the situation, one must increase the value of $\theta = 1$ and so to increase $\phi^x_o(\theta)$ on the input side and decrease $\phi^y_o(\theta)$ on the output side. Should one component of $\phi^x_o(\theta)$ or $\phi^y_o(\theta)$ not depend on $\theta$, this will not be possible, and the corresponding inequalities remain unfulfilled for some choices of $(x_o, y_o) \notin T$. It follows that if one wishes to ensure (P9) for arbitrary $(x_o, y_o) \notin T$, one must consider only the graph models where $g^y_o > 0$ and $g^x_o > 0$.

**Theorem 4.23.** For $(x_o, y_o) \notin T$ and $(g^x_o, g^y_o) \geq 0$, the following statements hold.

(a) If $(GS)_o$ is feasible, i.e., if there exists $\hat{\theta} \in \text{dom}(\psi)$ such that $(\phi^x_o(\hat{\theta}), \phi^y_o(\hat{\theta})) \in T$, then $(GS)_o$ admits an optimal solution and the optimal value satisfies $\theta^*_o > 1$.

(b) Let $g_o > 0$. If $y^{\text{min}} - y_o + g^y_o > 0$ or $\text{im}(\psi^y) = (-\infty, \infty)$, then $(GS)_o$ admits an optimal solution and the optimal value satisfies $\theta^*_o > 1$.

(c) Assume $\text{im}(\psi^y) = (0, \infty)$ and let $g_o > 0$. If (i) there does not exist $\lambda \geq 0$ with $e^T \lambda = 1$ such that the inequality $Y \lambda \geq y_o - g^y_o$ is satisfied or (ii) $y^{\text{max}}_o - y^r_o + g^y_o < 0$ for some $r$, then $(GS)_o$ is infeasible.

**Proof.** (a) The proof runs along the lines of the proof of Theorem 3.1(d). The fact that $\theta^*_o > 1$ is a simple consequence of strict monotonicity of $\phi^x_o(\theta)$, $\phi^y_o(\theta)$, $(x_o, y_o) \notin T$, and $T$ being closed.

(b) Since $X \lambda \leq x^{\text{max}}$ and $Y \lambda \geq y^{\text{min}}$, a sufficient condition for the feasibility of $(GS)_o$ is the existence of $\bar{\theta} \in D$, such that

$$x^{\text{max}} - x_o + g^x_o \leq \psi^x(\bar{\theta})g^x_o,$$

$$y^{\text{min}} - y_o + g^y_o \geq \psi^y(\bar{\theta})g^y_o. \tag{20}$$

Since $g^x_o > 0$, $\psi^x$ is increasing and $\lim_{\theta \to +\infty} \psi^x(\theta) = +\infty$, the constraint (20) is satisfied for $\bar{\theta}$ sufficiently large. Since $\psi^y$ is decreasing, the case $\lim_{\theta \to +\infty} \psi^y(\theta) = +\infty$ implies that the constraint (21) is satisfied for $\bar{\theta}$ sufficiently large. On the other hand, if $\lim_{\theta \to +\infty} \psi^x(\theta) = 0$, then the assumptions $y^{\text{min}} - y_o + g^y_o > 0$ and $g^y_o > 0$ guarantee that (21) is satisfied for $\bar{\theta}$ sufficiently large.

(c) Let us first consider the case (i). By assumption, for each feasible $\lambda$ ($\lambda \geq 0, e^T \lambda = 1$),
there exists $r$ such that $(Y \lambda)_r < y_{ro} - g^y_{ro}$. The assumption on $\psi^y$ implies $(Y \lambda)_r < y_{ro} - g^y_{ro} < y_{ro} - g^y_{ro} + \psi^y(\theta)g^y_{ro}$ for all $\theta \in \text{dom}(\psi^y)$. Hence we have shown that $Y \lambda \geq y_o - g^y_{ro} + \psi^y(\theta)g^y_{ro}$ is violated for all feasible $\lambda$ and $\theta \in (1, \infty)$ and therefore $(GS)_o$ is infeasible. Consider now the case (ii). Since $(Y \lambda)_r \leq y^\text{max}_r$ for all feasible $\lambda$, we have $(Y \lambda)_r < y_{ro} - g^y_{ro}$ and the claim follows by part (i).

Remark 4.24. With $g_o > 0$, DDM-g models are feasible for all units outside of the technology set by Theorem 4.23(b) $\text{dom}(\psi) = (-\infty, \infty)$. As a result, the entire family of the DDM-g models meets the property $(P9)$ provided the directional vectors are positive. Note that the positivity of $(G2)$ directions for $(x_o, y_o) \notin T$ is satisfied only if $x_o > x^\text{min}$ and $y_o < y^\text{max}$.

In the class of HDM-g models over positive data, only the HDM model with positive $(G1)$ directions is superefficient since it satisfies condition (ii) of Theorem 4.23(b).

The HDM-g models over positive data with $(G3)$–$(G6)$ directions do not garantee $(P9)$. Namely, one can always choose $(x_o, y_o) \notin T$ with $y_{ro} > y^{\text{max}}_r + g_{ro}$ for some $r$. The infeasibility of the corresponding $(GS)_o$ then follows from (ii) of Theorem 4.23(c).

The condition (ii) is too coarse for HDM-(G2) because the inequality $y_{ro} > y^{\text{max}}_r + g_{ro}$ is incompatible with positive $(G2)$ directions. We shall instead provide an example that meets condition (i) of Theorem 4.23(c). Consider a one-input two-output production technology with the technology set generated by two units, $A = (1, 1, 10)$ and $B = (1, 10, 1)$. One can easily show that the unit $(x_o, y_o) = (2, 8, 8)$ is outside $T$. The corresponding $(G2)$ direction equals $g_o = (2, 2, 2)$. An easy calculation now shows that condition (i) of Theorem 4.23(c) is satisfied, hence HDM-g with $(G2)$ directions does not guarantee superefficiency $(P9)$.

Remark 4.25. The definition of the technology set $T$ in (1), even when applied to non-negative data, does not presuppose non-negative values of outputs. Consequently, for the purposes of superefficiency measurement, we allow projections of units $0 \leq (x_o, y_o) \notin T$ onto elements of the frontier with negative outputs. Our results thus differ from the studies of other authors, where projections onto the negative part of the frontier are treated by definition as infeasible.

For example, in Briec and Kerstens (2009) (in the context of general economic productivity theory) DDM-g is classified as not having property $(P9)$ only because some units project onto the negative part of the frontier. To rule out such source of infeasibility, authors separately consider models, where the technology set is extended by the so-called free disposal cone, which then matches our approach. Johnson and McGinnis (2009) praise HDM, which in contrast to DDM-g is always feasible for positive data. This can also be seen in our $(GS)$ scheme since the HDM projection with $(G1)$ directions reads $(\theta x_o, \frac{1}{\theta} y_o)$ and it is thus positive for positive data.

5 Extensions

Section 4 has introduced some theoretical tools helpful in analysing the nine desirable properties $(P1)$–$(P9)$ over arbitrary data. In remarks of Section 4, these tools were applied to the standard path-based models over non-negative data. Next, we shall expand the analysis to examine the nine properties for standard models over arbitrary data (Subsection 5.1). Furthermore, we will
Table 3: Properties of the DDM-g model for $\psi^x(\theta) = \theta$, $\psi^y(\theta) = 2 - \theta$ and the HDM-g model for $\psi^x(\theta) = \theta$, $\psi^y(\theta) = \frac{1}{\theta}$ (in brackets) with respect to different choices of vector pairs $g_o^x, g_o^y$. $\checkmark$ – the property is satisfied for positive directions; $\checkmark^{**}$ – the property is satisfied for positive data but not for general data; $\times^*$ – the property holds only for some very specific $T$.

| Direction notation | [G1] | [G2] | (G3) | [G4] | (G5) | (G6) |
|--------------------|------|------|------|------|------|------|
| $g_o^x$            | $|x_o|$ | $|x_o - x_{\min}^o|$ | $x_{\max}^o - x_{\min}^o$ | $|x_{\max}^\psi|$ | $x_{sd}$ | $e$ |
| $g_o^y$            | $|y_o|$ | $|y_{\max}^o - y_{\min}^o|$ | $y_{\max}^o - y_{\min}^o$ | $|y_{\max}^\psi|$ | $y_{sd}$ | $e$ |

(P2) identification | $\times(\times)$ | $\times^*(\times)$ | $\times(\times)$ | $\times(\times)$ | $\times(\times)$ | $\times(\times)$ |
(P3) strong efficiency | $\times(\times)$ | $\times^*(\times)$ | $\times(\times)$ | $\times(\times)$ | $\times(\times)$ | $\times(\times)$ |
(P4) $\theta_o^* \in [0, 1]$ | $\checkmark^{**}(\checkmark)$ | $\checkmark(\checkmark)$ | $\checkmark(\checkmark)$ | $\checkmark(\checkmark)$ | $\checkmark(\checkmark)$ | $\checkmark(\checkmark)$ |
(P5) unit invariance | $\checkmark(\checkmark)$ | $\checkmark(\checkmark)$ | $\checkmark(\checkmark)$ | $\checkmark(\checkmark)$ | $\checkmark(\checkmark)$ | $\checkmark(\checkmark)$ |
(P6) translation invariance | $\times(\times)$ | $\checkmark(\checkmark)$ | $\checkmark(\checkmark)$ | $\checkmark(\checkmark)$ | $\checkmark(\checkmark)$ | $\checkmark(\checkmark)$ |
(P7) monotonicity | $\checkmark^{**}(\checkmark)$ | $\checkmark(\checkmark)$ | $\checkmark(\checkmark)$ | $\checkmark(\checkmark)$ | $\checkmark(\checkmark)$ | $\checkmark(\checkmark)$ |
(P8) strict monotonicity | $\times(\times)$ | $\times^*(\times)$ | $\times(\times)$ | $\times(\times)$ | $\times(\times)$ | $\times(\times)$ |
(P9) superefficiency | $\checkmark^*(\checkmark^{**})$ | $\checkmark^*(\checkmark^{**})$ | $\checkmark^*(\checkmark^{**})$ | $\checkmark^*(\checkmark^{**})$ | $\checkmark^*(\checkmark^{**})$ | $\checkmark^*(\checkmark^{**})$ |

5.1 Properties of standard path-based models over arbitrary data

The particular choices of vectors $g_o$ in Table 3 are based on the list found in Table 1. Since we now allow for negative data, the non-negativity of directions is ensured by taking absolute values where necessary. This is the case of expressions for (G1) a (G4), but also for (G2), where values of units outside $T$ might harm the assumption $g_o \geq 0$. The modified directions with absolute values are denoted $|G1|$, $|G2|$, and $|G4|$. The list of desirable properties in Table 3 omits property (P1), which in the (GS) scheme is satisfied regardless of the choice of $\psi^x$, $\psi^y$, or $g_o$. Observe that all models of Table 3 fail in common properties (P2), (P3), and (P8). Since (P2) is not satisfied in any of the considered models, the failure of (P3) is inevitable in view of the implication (P3) $\Rightarrow$ (P2) shown in Remark 4.9.

- **Strong efficiency of projections (P3).** According to Remark 4.9, all models fail (P3) with non-negative data and therefore also for arbitrary data.

- **Boundedness (P4):** For $\psi^x(\theta) = \theta$, $\psi^y(\theta) = \frac{1}{\theta}$, allowing arbitrary data produces no change; for all directions $g_o \geq 0$ one again obtains $\theta_o^* > 0$. Similarly, for $\psi^x(\theta) = \theta$, $\psi^y(\theta) = 2 - \theta$, and directions $|G2|=|G2|$ and (G3) one has $\theta_o^* \geq 0$ by Theorem 4.10. However, for directions $|G1|$ the status changes; negative data may lead to negative score as shown in the one-input

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It may happen that DDM-(G2) meets (P2), (P3), and (P8) over some very specific $T$ — see Figure 2.
one-output example with two units \( A = (-1, 3), B = (3, 1) \), where the directional vector for \( B \) is \((3, 1)\) and its score equals \(-1/3\).

By Remark 4.11, DDM-g models with directions (P4)–(P6) do not guarantee non-negative efficiency scores for positive data, hence the same is true for general data.

- **Unit and translation invariance** (P5) and (P6). There is no change compared to the positive data case and the conclusions of Remarks 4.13, 4.15 remain valid.

- **Monotonicity** (P7). Allowing for negative data makes no difference to the status of (P7) for those directions that do not explicitly depend on \((x_o, y_o)\). Therefore, models with (G3), (G4), (G5), and (G6) directions satisfy (P7). Likewise, directions \(|G2|=G2\) meet the assumptions of Theorem 4.18, hence here, too, there is no change.

The situation is different for directions \(|G1|\), which do not satisfy the assumptions of Theorem 4.18. To see that (P7) may fail, consider a one-input one-output counterexample with three units: efficient unit \( A = (-1, 4) \) and two inefficient units \( B = (0.1), C = (1, 0) \). Although \( B \) dominates \( C \), one has \( \theta_B^* = -10 < \theta_C^* = -1 \). We remark that directions \(|x_o|\), resp. \(|y_o|\) appear in Cheng, Zervopoulos and Qian (2013) in connection with radial oriented models. There it is observed that monotonicity may fail if zero is an internal point of the data.

- **Strict monotonicity** (P8). According to Remark 4.22, all models fail (P8) with non-negative data and therefore also for arbitrary data.

- **Superefficiency** (P9). DDM-g models with directions \(|G1|, |G2|, (G3), (G4), (G5), \) and (G6) retain property (P9) for arbitrary data (as long as the directions are positive). Likewise, negative result for HDM-g models with \(|G2|, (G3), (G4), (G5), \) and (G6) directions carry over to arbitrary data. Theorem 4.23 does not resolve the status of HDM-g models with \(|G1|\) directions. However, one can find an example, where the model is infeasible. Consider a single-input single-output case, where the technology \( T \) is generated by a single unit \( A = (2, 0) \). For \( B = (1, 10) \not\in T \), whose \(|G1|\) direction reads \( g_B = (1, 10) \), the output inequality \( 0 \geq 10 \theta^{-1} \) is not satisfied for any \( \theta > 0 \).

### 5.2 Non-standard directions with good properties

We will focus our attention on the (G2) directions, which we have already shown yield the invariance properties (P5) and (P6) for arbitrary \( \psi^x \) and \( \psi^y \) satisfying (A1)–(A4) (see Remarks 4.13 and 4.15). These directions allow for a small modification that will also ensure the validity of (P5) and (P7). The idea is to choose \( g_o \) so that the corresponding path \( \phi_o \) crossing \((x_o, y_o)\) at \( \theta = 1 \) passes through the point \((x^\text{min}, y^\text{max})\) at \( \theta = \theta^\text{min} \) (see Figure 3). Then \( \theta^\text{min} \) serves as the lower bound for the efficiency score as indicated in the next lemma. Its proof follows by simple calculation and is therefore omitted.

**Lemma 5.1.** Let \( \theta^\text{min} \in [0, 1) \cap D \) and \((x_o, y_o) \in T \setminus \{(x^\text{min}, y^\text{max})\}\). The path \( \phi_o(\theta) \) runs through \((x^\text{min}, y^\text{max})\) at \( \theta = \theta^\text{min} \), i.e., \( \phi_o(\theta^\text{min}) = (x^\text{min}, y^\text{max}) \) if and only if the directions
Figure 4: All paths with directions (G2.0) intersect at the point \((x_{\min}, y_{\max})\) (Lemma 5.1). Furthermore, (G2.0) projects all units in two-dimensional technology \(T\) onto the strongly efficient frontier.

\[ g^x_0, g^y_0 \text{ satisfy} \]
\[ g^x_0 = \frac{x_o - x_{\min}}{1 - \psi^x(\theta_{\min})}, \quad g^y_0 = \frac{y_{\max} - y_o}{\psi^y(\theta_{\min}) - 1}. \]  

(G2.0)

Remark 5.2. The GS models with directions (G2.0) satisfy (P5) and (P6) for arbitrary \(\psi^x\) and \(\psi^y\) with properties (A1)–(A4) by Theorems 4.12 and 4.14. By Lemma 5.1, the same holds for the boundedness property (P4). Finally, after substituting directions (G2.0) into the right-hand sides in (12) and performing simple calculations, the monotonicity of \(\phi(\theta)\) obtains for each \(\theta \geq \theta_{\min}\). Hence, the monotonicity property (P7), too, holds universally with directions (G2.0).

As for the property (P9), the situation is more nuanced for two reasons. First, superefficiency considers all points \((x_o, y_o) \notin T\), of which only a small subset with \(x_o \geq x_{\min}\) and \(y_o \leq y_{\max}\) yields non-negative directions (G2.0). This leads us to consider absolute values of expressions in (G2.0) as explained in Subsection 5.1. Second, if the range of \(\psi^y\) is bounded from below, then feasibility is compromised on the output side for the units (outside \(T\)) with very high outputs. Thus, for \(\psi^y\) with bounded range superefficiency fails. In contrast, if the range of \(\psi^y\) is unbounded, hence by Theorem 4.23(b) a GS model with function \(\psi^y\) satisfies superefficiency for those directions (G2.0) that are positive, i.e., superefficiency holds for all units outside \(T\) except for a negligible set where some direction components are zero.

Regarding the properties (P2), (P3), and (P8), the situation is similar to the case of the DDM-(G2) model. The existence of a point \((x_o, y_o) \in \partial W T\) with \(\theta^*_o = 1\) will cause all three properties to fail simultaneously (for (P2) and (P3) this is immediate; for (P8) see Theorem 4.20). Example 4.5 together with Figure 3 show that such points can be found also for directions (G2.0) in technological sets of dimension greater than 2. In the case of two-dimensional \(T\), the properties can be satisfied as illustrated in Figure 2 for DDM-(G2) and in Figure 4 for linear \(\psi^x = \theta\) and the hyperbolic \(\psi^y = \frac{1}{\theta}\).
\[
\psi^y = 2 - \theta - \theta^p, p > 0 \quad 1 - \ln \theta \quad e^{1-\theta}
\]

| \psi^y | 2 - \theta | \theta^p, p > 0 | 1 - \ln \theta | e^{1-\theta} |
| domain | (\(-\infty, \infty\)) | (0, \infty) | (0, \infty) | (\(-\infty, \infty\)) |
| image | (\(-\infty, \infty\)) | (0, \infty) | (0, \infty) | (0, \infty) |
| \theta_{\min} | 0 | 1/2 | \e^{-1} | 0 |
| \g^x_o | | \|x_o - x^{min}\| | \|x_o - x^{min}\| | \|x_o - x^{min}\| |
| \g^y_o | | \|y^{max} - y_o\| | \|y^{max} - y_o\| | \|y^{max} - y_o\| |
| \(P_1\), \(P_4\)--\(P_7\) | \(\checkmark\) | \(\checkmark\) | \(\checkmark\) | \(\checkmark\) |
| \(P_2\), \(P_3\), \(P_8\) | \(\times^*\) | \(\times^*\) | \(\times^*\) | \(\times^*\) |
| \(P_9\) | \(\checkmark^*\) | \(\times\) | \(\checkmark^*\) | \(\times\) |

Table 4: Values and properties of directions \((G2.0)\) taken in absolute value for \(\psi^x(\theta) = \theta\) and three choices of \(\psi^y\) and \(\theta_{\min}\). \(\checkmark^*\) - the property is satisfied for positive directions; \(\times^*\) – the property holds only for some very specific \(T\).

We now examine the \((GS)\) scheme for linear \(\psi^x(\theta) = \theta\) and four choices of \(\psi^y\) that satisfy assumptions \((A1)\)--\((A4)\): \(\psi^y_1(\theta) = 2 - \theta\); \(\psi^y_2(\theta) = \theta^p, p > 0\); \(\psi^y_3(\theta) = 1 - \ln \theta\); and \(\psi^y_4(\theta) = e^{1-\theta}\). These functions offer a variety of combinations of domains and ranges as shown in Table 4. Observe that \(\psi^y_1\) and \(\psi^y_2\) with \(p = 1\), have already appeared in the context of DDM-g and HDM-g models, respectively. To evaluate the directions \((G2.0)\), one has to specify the value of \(\theta_{\min} \in [0, 1)\). Only for \(\psi^y_1\) and \(\psi^y_4\) is one able to choose the ideal value \(\theta_{\min} = 0\). The remaining two cases do not have zero in their domain and for those we select ad-hoc positive values of \(\theta_{\min}\) yielding simple expressions for directions. Intuitively, the choice of \(\theta_{\min}\) affects the “scaling” of optimal scores for non-efficient units. The resulting values are summarized in Table 4. Observe that the directions \((G2.0)\) for \(\psi^y_1\) with the choice of \(\theta_{\min} = 0\) lead to \((G2)\). In contrast, \(\psi^y_2\) with \(p = 1\) and \(\theta_{\min} = 1/2\) generate directions whose \(x\) component is multiplied by a factor of 2 compared to \((G2)\).

The statuses regarding \((P1)\)--\((P9)\) follow from Remark 5.2, where they were analyzed for general \(\psi^x\) and \(\psi^y\). Note that the models with \(\psi^y_2\) and \(\psi^y_4\) fail \((P9)\) due to the boundedness of their ranges.

### 5.3 Numerical illustration with mixed data

We present a numerical example that examines directions \((G1)\) and \((G2.0)\) for models appearing in Table 4. To evaluate the efficiency score, as well as to obtain the second phase result, we use the CVX modelling system (see Grant and Boyd, 2008, 2014) for solving convex programs, implemented in Matlab. Data of 30 Taiwanese electrical machinery producers are taken from Table 5 of Tone, Chang and Wu (2020) and consist of two inputs and two outputs. The outputs contain negative data. The results are presented in Tables 5 and 6.

The choice of particular function \(\psi^y\) has only minimal influence on the ranking of the best placed units but the effect is more pronounced for very inefficient units, where the change in ranking may be as high as 10 places out of 30 companies. On the other hand, the choice of...
Function $\psi^y(\theta) = 2 - \theta$

| Directions | $\psi^y(\theta) = 2 - \theta$ | $\psi^y(\theta) = 1/\theta$ |
|------------|---------------------------------|---------------------------------|
| DMU        | Score | Rank | Score | Rank | Score | Rank | Score | Rank |
| 1          | 0.66* | 15   | 0.759 | 28   | 0.713* | 14   | 0.401 | 14   |
| 2          | 1*    | 1-9  | 1*    | 1-9  | 1*    | 1-9  | 1*    | 1-9  |
| 3          | 0.683 | 14   | 0.888 | 12   | 0.691 | 15   | 0.578* | 11   |
| 4          | 1*    | 1-9  | 1*    | 1-9  | 1*    | 1-9  | 1*    | 1-9  |
| 5          | 1*    | 1-9  | 1*    | 1-9  | 1*    | 1-9  | 1*    | 1-9  |
| 6          | 0.175 | 29   | 0.784* | 26   | 0.256 | 29   | 0.145* | 28   |
| 7          | 0.69  | 13   | 0.86*  | 17   | 0.763 | 13   | 0.254* | 19   |
| 8          | 0.283 | 26   | 0.801  | 24   | 0.3   | 28   | 0.167* | 24   |
| 9          | 0.126* | 30   | 0.744*  | 29   | 0.194* | 30   | 0.112* | 30   |
| 10         | 0.372 | 22   | 0.777  | 27   | 0.526 | 19   | 0.156* | 27   |
| 11         | 0.37  | 23   | 0.707*  | 30   | 0.512 | 20   | 0.169* | 23   |
| 12         | 1*    | 1-9  | 1*    | 1-9  | 1*    | 1-9  | 1*    | 1-9  |
| 13         | 0.927 | 10   | 0.968* | 10   | 0.929 | 10   | 0.876  | 10   |
| 14         | 1*    | 1-9  | 1*    | 1-9  | 1*    | 1-9  | 1*    | 1-9  |
| 15         | 0.217 | 28   | 0.793* | 25   | 0.339 | 25   | 0.114  | 29   |
| 16         | 0.658 | 16   | 0.881  | 15   | 0.68  | 16   | 0.309  | 16   |
| 17         | 0.22  | 27   | 0.807* | 22   | 0.319 | 27   | 0.165* | 25   |
| 18         | 0.438 | 19   | 0.861  | 16   | 0.494 | 21   | 0.233* | 21   |
| 19         | 1*    | 1-9  | 1*    | 1-9  | 1*    | 1-9  | 1*    | 1-9  |
| 20         | 0.415 | 21   | 0.841  | 18   | 0.448 | 22   | 0.165  | 26   |
| 21         | 0.335 | 24   | 0.804  | 23   | 0.408 | 24   | 0.27   | 17   |
| 22         | 1*    | 1-9  | 1*    | 1-9  | 1*    | 1-9  | 1*    | 1-9  |
| 23         | 1*    | 1-9  | 1*    | 1-9  | 1*    | 1-9  | 1*    | 1-9  |
| 24         | 0.765 | 12   | 0.882  | 14   | 0.765 | 12   | 0.547  | 13   |
| 25         | 0.431 | 20   | 0.814* | 20   | 0.442 | 23   | 0.372  | 15   |
| 26         | 1*    | 1-9  | 1*    | 1-9  | 1*    | 1-9  | 1*    | 1-9  |
| 27         | 0.305 | 25   | 0.815* | 19   | 0.33  | 26   | 0.255  | 18   |
| 28         | 0.447 | 18   | 0.812* | 21   | 0.581 | 18   | 0.176* | 22   |
| 29         | 0.835* | 11   | 0.933* | 11   | 0.85*  | 11   | 0.576* | 12   |
| 30         | 0.452 | 17   | 0.883* | 13   | 0.598* | 17   | 0.239* | 20   |

**Average** 0.627 0.880 0.671 0.509

**Minimum** 0.126 0.707 0.194 0.112

**Correct** 12 21 13 21

**Table 5:** Efficiency scores for models with $\psi^x = \theta$, two choices of $\psi^y$, directions (G1) and (G2.0). An asterisk next to the efficiency value indicates projection onto the strongly efficient frontier. The last row indicates number of units projected onto the strongly efficient frontier.

$\psi^y$ has little impact on the number of units projected onto the strongly efficient frontier, where the main determinant is the choice of directions. The number of strongly efficient projections fluctuates between 12 and 13 for the (G1) directions, while it is between 20 and 23 units for the (G2.0) directions. The choice of the scaled (G2) direction (G2.0) not only leads to a larger number of correct efficiency evaluations, but in the case of hyperbolic and logarithmic model, it also improves the discriminatory power of the model.
Table 6: Efficiency scores for models with $\psi^x = \theta$, two choices of $\psi^y$, directions (G1) and (G2.0). An asterisk next to the efficiency value indicates projection onto the strongly efficient frontier. The last row indicates number of units projected onto the strongly efficient frontier.

| Function $\psi^y(\theta) = \exp(1 - \theta)$ | $\psi^y(\theta) = 1 - \ln \theta$ |
|---------------------------------------------|---------------------------------------------|
| Directions $\theta_{\text{min}} = 0$ | Directions $\theta_{\text{min}} = 0.01$ |
| DMU | Score | Rank | Score | Rank | Score | Rank | Score | Rank |
|---------------------------------------------|---------------------------------------------|
| 1 | 0.686* | 14 | 0.697 | 28 | 0.6898* | 14 | 0.565* | 18 |
| 2 | 1* | 1-9 | 1* | 1 | 1* | 1 | 1* | 1 |
| 3 | 0.686 | 15 | 0.833 | 12 | 0.687 | 15 | 0.686* | 12 |
| 4 | 1* | 1-9 | 1* | 2 | 1* | 2 | 1* | 2 |
| 5 | 1* | 1-9 | 1* | 3 | 1* | 3 | 1* | 3 |
| 6 | 0.19 | 29 | 0.698* | 26 | 0.203 | 29 | 0.454* | 27 |
| 7 | 0.73 | 13 | 0.799* | 16 | 0.733 | 13 | 0.627* | 15 |
| 8 | 0.283 | 26 | 0.711 | 23 | 0.283 | 26 | 0.437 | 28 |
| 9 | 0.131* | 30 | 0.652* | 29 | 0.137* | 30 | 0.407* | 30 |
| 10 | 0.439 | 20 | 0.698 | 27 | 0.457 | 20 | 0.5* | 23 |
| 11 | 0.429 | 22 | 0.626* | 30 | 0.446 | 21 | 0.422 | 29 |
| 12 | 1* | 1-9 | 1* | 4 | 1* | 4 | 1* | 4 |
| 13 | 0.928 | 10 | 0.953 | 10 | 0.928 | 10 | 0.922 | 10 |
| 14 | 1* | 1-9 | 1* | 5 | 1* | 5 | 1* | 5 |
| 15 | 0.232 | 28 | 0.708* | 25 | 0.246 | 28 | 0.464* | 26 |
| 16 | 0.668 | 16 | 0.827 | 13 | 0.669 | 16 | 0.678 | 13 |
| 17 | 0.238 | 27 | 0.726* | 22 | 0.251 | 27 | 0.488* | 24 |
| 18 | 0.457 | 19 | 0.79 | 17 | 0.463 | 19 | 0.566 | 17 |
| 19 | 1* | 1-9 | 1* | 6 | 1* | 6 | 1* | 6 |
| 20 | 0.425 | 23 | 0.763 | 18 | 0.428 | 23 | 0.512 | 22 |
| 21 | 0.357 | 24 | 0.711 | 24 | 0.366 | 24 | 0.527* | 21 |
| 22 | 1* | 1-9 | 1* | 7 | 1* | 7 | 1* | 7 |
| 23 | 1* | 1-9 | 1* | 8 | 1* | 8 | 1* | 8 |
| 24 | 0.765 | 12 | 0.82 | 15 | 0.765 | 12 | 0.618 | 16 |
| 25 | 0.432 | 21 | 0.729* | 21 | 0.434 | 22 | 0.543* | 20 |
| 26 | 1* | 1-9 | 1* | 9 | 1* | 9 | 1* | 9 |
| 27 | 0.308 | 25 | 0.733* | 20 | 0.31 | 25 | 0.484* | 25 |
| 28 | 0.51 | 18 | 0.741* | 19 | 0.523 | 18 | 0.551* | 19 |
| 29 | 0.842* | 11 | 0.899* | 11 | 0.843* | 11 | 0.808* | 11 |
| 30 | 0.524 | 17 | 0.823* | 14 | 0.537 | 17 | 0.64* | 14 |
| Average | 0.642 | 0.831 | | 0.647 | 0.697 | | |
| Minimum | 0.131 | 0.626 | 0.137 | 0.407 |
| Correct | 12 | 20 | 12 | 23 |

6 Conclusions

The paper formulates a general scheme for path-based models that includes BCC, DDM, and HDM as special cases. Models belonging to the scheme are analyzed in light of nine desirable properties. Only the uniqueness of projection (P1) is universal; the remaining properties must be judged individually and typically depend on the choice of directions, functions $\psi$, and possibly also on the configuration of data that defines the technology set $\mathcal{T}$.

The paper generates mathematical tools (sufficient and/or necessary conditions for the va-
Validity of individual properties) allowing analysis of models that fall under the GS scheme. The usefulness of these tools has been demonstrated on the standard path-based models, their modifications to general (negative) data, and models with non-linear, non-hyperbolic functions $\psi$.

The general scheme is flexible enough to allow modification of existing models designed for positive data to arbitrary (negative) data. For this modification it is important to rewrite models such as BCC-I (3), BCC-O (4), or HDM (6), in the form of GS scheme (Table 2), where the expressions for directions are specified. In the presence of negative data, these expressions are taken in absolute value. We have illustrated this approach on a numerical example with mixed data in Subsection 5.2.

The paper provides complete analysis of standard path-based models and their modifications to negative data regarding the nine properties (Table 3). Along with the uniqueness of projection (P1), all standard models satisfy monotonicity (P7). On the other hand, all models fail (P2), (P3), and (P8) simultaneously for some $T$. Overall, the models with (G2) directions satisfy or partially satisfy the largest number of properties, followed by models with (G3) directions.

Directions have a strong influence on model properties. Models with positive directions fail (P2), (P3), and (P8) for every technology set (Theorems 4.2 and 4.21 and Remark 4.9). This is especially true for positive data and all directions apart from (G2). For models with (G2) directions, the status of properties (P2), (P3), and (P8) is ambiguous insofar as it depends on data configuration. There are examples where all three properties fail but also data configurations (in particular, single input, single output technology sets), where all three are met.

The lack of universal validity of (P2), (P3), and (P8) can be considered a characteristic feature of path-based models. In contrast, Halická and Trnovská (2021) show that slacks-based models do satisfy these properties. This observation provides deeper rationale for distinguishing the two categories (not yet fully recognized in the literature) of path-based and slacks-based models. Some authors (e.g., Sahoo, Mehdiloozad and Tone, 2014) find that models with adjective ‘directional’ satisfy at least one of the properties (P2), (P3), and (P8). In each case, the model in question is in substance, despite its name, slacks-based. Classification of a given model into a path-based or slacks-based scheme is thus more indicative of model properties.

The paper also opens up several new research directions.

- Simultaneous failure of (P2), (P3), and (P8) raises the question whether these properties are linked. It is fairly immediate that (P3) implies (P2); such implication is valid also outside our scheme. However, the connection between (P3) and (P8) is not known or obvious. We aim to address it in future work.

- The general scheme (introduced in envelopment form) allows derivation of a dual multiplicative model, which can be related to the shadow profit inefficiency. This opens way to obtain relationships between the technical and overall profit inefficiency similar to that known for DDM (Chambers, Chung and Färe, 1998).

- To simplify exposition, we have only considered technologies with variable returns to scale. The VRS assumption is essential to allow for negative data (see, e.g., Portela, Thanassoulis and Simpson, 2004). On the other hand, many results in the paper can be
adapted to technology sets with other types of returns to scale, at the cost of considering only non-negative data.

- The numerical results in Subsection 5.3 indicate that directions (G2.0) may be helpful in practical applications in the sense that a higher number of DMUs are projected onto the strongly efficient frontier compared to other directions. It is natural to ask whether this finding holds up for other data sets.

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References

Aparicio, J. and J. F. Monge (2022). The generalized range adjusted measure in data envelopment analysis: Properties, computational aspects and duality. *European Journal of Operational Research* 302(2), 621–632.

Aparicio, J., J. T. Pastor, and S. C. Ray (2013). An overall measure of technical inefficiency at the firm and at the industry level: The 'lost profit on outlay'. *European Journal of Operational Research* 226(1), 154–162.

Aparicio, J., J. T. Pastor, and F. Vidal (2016). The directional distance function and the translation invariance property. *Omega* 58, 1–3.

Banker, R. D., A. Charnes, and W. W. Cooper (1984). Some models for estimating technical and scale inefficiencies in data envelopment analysis. *Management Science* 30(9), 1078–1092.

Briec, W. (1997). A graph-type extension of Farrell technical efficiency measure. *Journal of Productivity Analysis* 8(1), 95–110.

Briec, W. and K. Kerstens (2009). Infeasibility and directional distance functions with application to the determinateness of the Luenberger productivity indicator. *Journal of Optimization Theory and Applications* 141(1), 55–73.

Briec, W. and J. Lesourd (1999). Metric distance function and profit: Some duality results. *Journal of Optimization Theory and Applications* 101(1), 15–33.

Chambers, R. G., Y. Chung, and R. Färe (1996). Benefit and distance functions. *Journal of Economic Theory* 70(2), 407–419.

Chambers, R. G., Y. Chung, and R. Färe (1998). Profit, directional distance functions, and Nerlovian efficiency. *Journal of Optimization Theory and Applications* 98(2), 351–364.

Chambers, R. G., R. Färe, and S. Grosskopf (1996). Productivity growth in APEC countries. *Pacific Economic Review* 1(3), 181–190.

Charnes, A., W. W. Cooper, B. Golany, L. Seiford, and J. Stutz (1985). Foundations of data envelopment analysis for Pareto-Koopmans efficient empirical production functions. *Journal of Econometrics* 30(1-2), 91–107.
Chavas, J.-P. and T. L. Cox (1999). A generalized distance function and the analysis of production efficiency. *Southern Economic Journal* 66(2), 294–318.

Cheng, G., P. Zervopoulos, and Z. Qian (2013). A variant of radial measure capable of dealing with negative inputs and outputs in data envelopment analysis. *European Journal of Operational Research* 225(1), 100–105.

Cooper, W. W., K. S. Park, and J. T. Pastor (1999). RAM: A range adjusted measure of inefficiency for use with additive models, and relations to other models and measures in DEA. *Journal of Productivity Analysis* 11(1), 5–42.

Cooper, W. W., J. T. Pastor, F. Borras, J. Aparicio, and D. Pastor (2011). BAM: A bounded adjusted measure of efficiency for use with bounded additive models. *Journal of Productivity Analysis* 35(2), 85–94.

Cooper, W. W., L. M. Seiford, and K. Tone (2007, December). *Data Envelopment Analysis: A Comprehensive Text with Models, Applications, References and DEA-Solver Software*. Springer US.

Färe, R., S. Grosskopf, and C. A. K. Lovell (1985). *The Measurement of Efficiency of Production*. Springer Netherlands.

Färe, R., S. Grosskopf, and D. Margaritis (2008, February). Efficiency and productivity: Malmquist and more. In H. O. Fried, C. A. Knox Lovell, and S. S. Schmidt (Eds.), *The Measurement of Productive Efficiency and Productivity Growth*, pp. 522–622. Oxford University Press.

Färe, R. and C. A. K. Lovell (1978). Measuring the technical efficiency of production. *Journal of Economic Theory* 19(1), 150–162.

Grant, M. and S. Boyd (2008). Graph implementations for nonsmooth convex programs. In V. Blondel, S. Boyd, and H. Kimura (Eds.), *Recent Advances in Learning and Control*, Lecture Notes in Control and Information Sciences, pp. 95–110. Springer-Verlag Limited. [http://stanford.edu/~boyd/graph_dcp.html](http://stanford.edu/~boyd/graph_dcp.html).

Grant, M. and S. Boyd (2014, March). CVX: Matlab software for disciplined convex programming, version 2.1. [http://cvxr.com/cvx](http://cvxr.com/cvx).

Halická, M. and M. Trnovská (2019). Duality and profit efficiency for the hyperbolic measure model. *European Journal of Operational Research* 278(2), 410–421.

Halická, M. and M. Trnovská (2021). A unified approach to non-radial graph models in data envelopment analysis: Common features, geometry, and duality. *European Journal of Operational Research* 289(2), 611–627.

Hasannasab, M., D. Margaritis, I. Roshdi, and P. Rouse (2019). Hyperbolic efficiency measurement: A conic programming approach. *European Journal of Operational Research* 278(2), 401–409.

Johnson, A. L. and L. F. McGinnis (2009). The hyperbolic-oriented efficiency measure as a remedy to infeasibility of super efficiency models. *Journal of the Operational Research Society* 60(11), 1511–1517.

Levkoff, S. B., R. R. Russell, and W. Schworm (2011, September). Boundary problems with the “Russell” graph measure of technical efficiency: A refinement. *Journal of Productivity Analysis* 37(3), 239–248.

Pastor, J. T., J. Aparicio, and J. L. Zofío (2022). *Benchmarking Economic Efficiency: Technical and Allocative Fundamentals*. International Series in Operations Research and Management Science. Springer Cham.
Pastor, J. T., J. L. Ruiz, and I. Sirvent (1999). An enhanced DEA Russell graph efficiency measure. *European Journal of Operational Research* 115(3), 596–607.

Portela, M. C. A. S., E. Thanassoulis, and G. Simpson (2004, October). Negative data in DEA: A directional distance approach applied to bank branches. *Journal of the Operational Research Society* 55(10), 1111–1121.

Roshdi, I., M. Hasannasab, D. Margaritis, and P. Rouse (2018). Generalised weak disposability and efficiency measurement in environmental technologies. *European Journal of Operational Research* 266(3), 1000–1012.

Russell, R. R. and W. Schworm (2008, November). Axiomatic foundations of efficiency measurement on data-generated technologies. *Journal of Productivity Analysis* 31(2), 77–86.

Russell, R. R. and W. Schworm (2011, March). Properties of inefficiency indexes on (input, output) space. *Journal of Productivity Analysis* 36(2), 143–156.

Russell, R. R. and W. Schworm (2018, October). Technological inefficiency indexes: A binary taxonomy and a generic theorem. *Journal of Productivity Analysis* 49(1), 17–23.

Sahoo, B. K., M. Mehdiloozad, and K. Tone (2014). Cost, revenue and profit efficiency measurement in DEA: A directional distance function approach. *European Journal of Operational Research* 237(3), 921–931.

ˇSevˇcovíˇc, D., M. Halická, and P. Brunovský (2001). DEA analysis for a large structured bank branch network. *Central European Journal of Operations Research* 9(4), 329–342.

Sueyoshi, T. and K. Sekitani (2007). Measurement of returns to scale using a non-radial dea model: A range-adjusted measure approach. *European Journal of Operational Research* 176(3), 1918–1946.

Sueyoshi, T. and K. Sekitani (2009). An occurrence of multiple projections in DEA-based measurement of technical efficiency: Theoretical comparison among DEA models from desirable properties. *European Journal of Operational Research* 196(2), 764–794.

Tone, K. (2001). A slacks-based measure of efficiency in data envelopment analysis. *European Journal of Operational Research* 130(3), 498–509.

Tone, K., T.-S. Chang, and C.-H. Wu (2020). Handling negative data in slacks-based measure data envelopment analysis models. *European Journal of Operational Research* 282(3), 926–935.