STATISTICS OF DARK MATTER HALOS FROM THE EXCURSION SET APPROACH

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ABSTRACT

We exploit the excursion set approach in integral formulation to derive novel, accurate analytic approximations of the unconditional and conditional first crossing distributions for random walks with uncorrelated steps and general shapes of the moving barrier; we find the corresponding approximations of the unconditional and conditional halo mass functions for cold dark matter (DM) power spectra to represent very well the outcomes of state-of-the-art cosmological N-body simulations. In addition, we apply these results to derive, and confront with simulations, other quantities of interest in halo statistics, including the rates of halo formation and creation, the average halo growth history, and the halo bias. Finally, we discuss how our approach and main results change when considering random walks with correlated instead of uncorrelated steps, and warm instead of cold DM power spectra.

Key words: cosmology: theory – dark matter – galaxies: halos – methods: analytical

Online-only material: color figures

1. INTRODUCTION

In the standard cosmological framework, the seeds of cosmic structures like quasars, galaxies, and galaxy systems are constituted by dark matter (DM) perturbations of the cosmic density field, originated by quantum effects during the early inflationary universe. The perturbations are amplified by gravitational instabilities and, as the local gravity prevails over the cosmic expansion, are enforced to collapse and virialize into bound “halos.” In turn, these tend to grow hierarchically in mass and sequentially in time, with small clumps forming first and then stochastically merging together into larger and more massive objects. The halos provide the gravitational potential wells where baryonic matter can settle in virial equilibrium, and via a number of complex astrophysical processes (cooling, star formation, feedback, etc.) originate the luminous structures that populate the visible universe.

To accurately describe and deeply understand the statistics of DM halos constitute fundamental steps toward formulating a sensible theory of galaxy formation and evolution. A milestone is constituted by the pioneering work of Press & Schechter (1974), who first provided an analytic expression for the halo mass function, i.e., the halo abundance as a function of mass and redshift; their theory envisages that a halo can collapse if it resides within a sufficiently overdense region of an initial, Gaussian perturbation field. However, the overdensity around a given spatial location depends on the scale under consideration; thus the halo abundance can be computed from the mass fraction in the density field which is above a critical threshold for collapse when conveniently smoothed on different scales.

However, a relevant issue is to consider only those perturbations that overcome the threshold on a given smoothing scale, but not on a larger one; in fact, overdense regions embedded within a larger collapsing perturbation are squeezed to smaller and smaller sizes, and care must be paid not to double-count them. To cure this “cloud-in-cloud” problem, Bond et al. (1991) developed the excursion set approach. This recognizes that the overdensity around a given spatial location executes a random walk when considered as a function of the smoothing scale (see Figure 1); moreover, if the smoothing is performed with a sharp filter in Fourier space, the steps of the random walk are uncorrelated, i.e., the walk is Markovian. In this framework the threshold for collapse plays the role of a barrier the random walk can hit; the distribution of first crossing, i.e., the probability that a walk crosses the barrier for the first time on a specific scale, can be closely related to the halo mass function, and allows any double-counting to be avoided.

The excursion set approach was soon exploited to derive the “conditional” halo mass function (see Lacey & Cole 1993), describing the mass and redshift distribution of progenitors given they belong to a descendant halo at a later time. This statistics has then been used to generate merger trees, i.e., Monte Carlo realizations of the halo merging process (see Kauffmann & White 1993; Somerville & Kolatt 1999; Cole et al. 2000; Zhang et al. 2008a). With some additional assumptions, the excursion set has also been helpful in developing models for the halo spatial clustering and bias (see Mo & White 1996; Sheth & Lemson 1999). Subsequent developments focused on the shape of the moving barrier; originally, a simple model of halo spherical collapse (Gunn & Gott 1972) was adopted, yielding a constant barrier, independent of halo size. However, Gaussian perturbations naturally possess a triaxial nature (Doroshkevich 1970), and the halo collapse is likely to be ellipsoidal rather than spherical; as a result, the threshold for collapse becomes a “moving,” mass-dependent barrier (Sheth et al. 2001; Sheth & Tormen 2002). The excursion set approach with moving barriers has proven to be rather successful in reproducing reasonably well the outcomes of cosmological N-body simulations (Sheth & Tormen 1999; Jenkins et al. 2001; Springel et al. 2005; Warren et al. 2006; Reed et al. 2007; Tinker et al. 2008; also Crocce et al. 2010; Angulo et al. 2012; Watson et al. 2013). More recent developments concern extension of the excursion set approach to non-Markovian walks with correlated steps (Maggiore & Riotto 2010a; Paranjape et al. 2012; Musso & Sheth 2012), stochastic barriers (Maggiore & Riotto 2010b; Corasaniti & Achitouv 2011), non-Gaussian initial conditions (Maggiore & Riotto 2010c; de Simone et al. 2011a, 2011b; Musso & Paranjape 2012), and application to the distribution of voids (Sheth & van de Weygaert 2004).

Despite these efforts, exact solutions to the first crossing distribution are known only for simple barriers, like the constant or linear ones, but not for the nonlinear shapes describing a realistic, ellipsoidal halo collapse. In such cases, empirical formulae have been proposed to fit the outcomes of cosmological N-body simulations (see Sheth & Tormen 2002; Parkinson et al. 2008; Neistein &
massive halo by merger. This growth history is represented by the solid colored lines connecting steps of the random walk curve; the longer the line between two steps, of first crossing. Moving toward the present, the barrier heights decrease and the points of first crossing occur at smaller S (A color version of this figure is available in the online journal.)

Figure 1. Example of a random walk executed by the overdensity contrast δ as a function of the variance S. The black solid line represents the random walk. The dotted lines illustrate the constant (green), square-root (blue), and ellipsoidal barriers (red) at two redshifts \( z = 0 \) and \( z' = 2 > z \), with the dots indicating the locations of first crossing. Moving toward the present, the barrier heights decrease and the points of first crossing occur at smaller S, corresponding to the formation of a more massive halo by merger. This growth history is represented by the solid colored lines connecting steps of the random walk curve; the longer the line between two steps, the more massive the merger with the companion halo.

(A color version of this figure is available in the online journal.)

Dekel 2008; Lam & Sheth 2009; de Simone et al. 2011a). Although such fitting formulae are inspired by the known solutions, being not derived directly from the excursion set formalism they may suffer from various problems: inconsistencies with the excursion set approach in given mass and redshift ranges; difficulties in the physical interpretation of the results; dependence on the simulation setup; poor flexibility in describing comparably well different barrier shapes; complex expressions, which make it difficult to derive analytically other statistical quantities of interest, like halo merger rates and bias. One of the main goals of this paper is to provide analytic approximations of the first crossing distribution (both unconditional and conditional) for quite general shapes of the barrier, which are derived directly from, and so are fully consistent with, the excursion set formalism.

A related important issue we address concerns the halo creation rate, i.e., the change in abundance per unit time of newly created halos as a function of mass and redshift. The seemingly simple task of deriving this quantity hides delicate conceptual problems, and is to some extent still unresolved. Naively, one can think of just taking the derivative of the halo mass function; this is plainly not correct, because such a derivative is actually the balance between the rate of halo creation by the merging of smaller ones and the rate of halo destruction by merging into larger ones (as stressed and discussed by Cavaliere et al. 1991; see also Blain & Longair 1993). Indeed, the total derivative can be positive or negative, as a consequence of the fact that small masses are dominated by the destruction and high masses by the creation term. Some authors considered the possibility of taking the “positive” term in the cosmic time derivative of the Press–Schechter mass function (e.g., Haehnelt et al. 1998); a partial justification for such an approach was provided by Sasaki (1994), who obtained the same outcome by requiring the destruction term to be scale-invariant. As a matter of fact, N-body simulations show this assumption not to hold in general, and to break down especially at small masses; in addition, Sasaki’s procedure does not provide consistent result for nonlinear barriers (see Mitra et al. 2011).

Another approach has been proposed by Sheth & Pitman (1997); they described the merging between halos in terms of the classic Smoluchowski’s (1916) binary coagulation equation (see also Silk & White 1978), whose structure naturally separates between the creation and destruction terms. In fact, as already noted by Epstein (1983), there exists a strict analogy between the excursion set approach with a white-noise power spectrum and a discrete Poisson model of binary coagulation with additive kernel. However, in the continuum limit relevant to the cosmological context, the creation and destruction terms diverge, and replacing them with the finite, discrete Poisson expressions lacks a mathematical justification. In addition, when more realistic spectra are considered, like power law or cold DM ones, the merger kernel from the excursion set is no longer symmetric, and the average number of progenitors per merger event is greater than two; thus the analogy with the binary coagulation equation no longer holds (see Benson et al. 2005; Benson 2008; Neistein & Dekel 2008; Zhang et al. 2008a).

Lacey & Cole (1993) and Kitayama & Suto (1996) attempted to derive an expression for the creation rate directly from the excursion set framework; taken at face value the result is divergent and calls for some form of regularization. They sidestepped the problem by introducing the concept of “formation” or major merger rate, according to which a halo is formed during a merger if it acquires more than half its original mass. Percival & Miller (1999), basing on Bayes’ theorem and assuming a flat prior on the distribution of the threshold for collapse at fixed smoothing scale, obtained an expression for the creation time distribution, which represents the creation rate normalized in time (the normalization being again formally infinite). All in all, these analyses failed to derive sensible, finite results for the creation rates; even the expressions provided for formation rates and creation time distribution hold only for constant barriers, and their extension to general moving barriers is non-trivial (see Mahmood & Rajesh 2005; also Moreno et al. 2009 for discussion). Another goal of this paper is to provide finite, analytic expressions for the rates of halo creation and formation with general moving barriers.
The plan of the paper is straightforward: in Section 2 we recall the basics of the excursion set formalism; in Sections 3 and 4 we derive our main results, i.e., approximated expressions of the unconditional and conditional first crossing distributions (and related mass functions) for general moving barriers; in Section 5 we apply our main results to derive other quantities of interest in halo statistics, including the rates of halo formation and creation, the average halo growth history, and the halo bias for general moving barriers; in Section 6 we summarize our main findings. Appendix A provides a brief primer on the $\xi$-regularization technique for divergent integrals, while in Appendices B and C we discuss how the excursion set approach and our main results change when considering random walks with correlated instead of uncorrelated steps, and with warm instead of cold DM power spectra.

Throughout this work we adopt a standard, flat cosmology (see Komatsu et al. 2011; Hinshaw et al. 2013; Planck Collaboration 2013) with matter density parameter $\Omega_M = 0.3$, baryon density parameter $\Omega_b = 0.046$, and Hubble constant $H_0 = 100 \, h \, \text{km s}^{-1} \text{Mpc}^{-1}$ with $h = 0.70$.

2. THE EXCURSION SET APPROACH

In this section we recall the basics of the excursion set approach, highlight some delicate points useful in the sequel, and set the notation. The expert reader may jump directly to Section 3.

We start by considering a spatial location in the universe with comoving coordinate $x$ and local density contrast $\delta(x) \equiv \rho(x)/\bar{\rho} - 1$ over the background value $\bar{\rho}$. The density contrast smoothed on a scale $R$ is given by

$$\delta(x, R) = \int d^3x' W_R(|x - x'|) \delta(x'),$$

in terms of some filter function $W_R(\cdot)$. Since the behavior of $\delta(x, R)$ as a function of $R$ is the interesting quantity, without loss of generality we can pose $x = 0$ and denote $\delta(0, R)$ just as $\delta(R)$. Passing to Fourier space and indicating with a hat Fourier-transformed quantities, we can write

$$\hat{\delta}(R) = \int \frac{d^3k}{(2\pi)^3} \hat{W}_R(k) \hat{\delta}(k).$$

We consider a Gaussian fluctuation field, whose statistical properties can be completely encased in its power spectrum $P(k)$ defined as

$$\langle \hat{\delta}(k)\hat{\delta}(k') \rangle = (2\pi)^3 P(k) \delta_D(k + k'),$$

with $\delta_D(\cdot)$ being the Dirac delta function, and $\langle \cdot \rangle$ indicating the statistical average over the stochastic variables $\hat{\delta}(k)$. In terms of the power spectrum, the variance of the smoothed density field is simply

$$S(R) = \int \frac{d^3k}{(2\pi)^3} \hat{W}_R^2(k) P(k).$$

For $P(k)$ we adopt the standard cold DM shape by Bardeen et al. (1986; see their Equation (G3)) with the correction for baryons by Sugiyama (1995; see his Equation (3.9)), normalized in such a way that the rms $\sigma_R \equiv \sqrt{S(R)}$ takes on the value $\sigma_R = 0.81$ on a scale $R = 8 \, h^{-1} \text{Mpc}$ (consistent with the results by WMAP7/WMAP9 from Hinshaw et al. (2013) and by the Planck Collaboration 2013).

As to the filter function, a common and convenient choice is a simple sharp filter in Fourier space $\hat{W}_R(k) = \theta_R(k_S - k)$, where $\theta_R(\cdot)$ is the Heaviside step function and $k_S(R)$ is a cutoff wavenumber, to be related to the scale $R$. In such a case, indicating with $\hat{\delta}(k) \equiv \int d\Omega \hat{\delta}(k)/4\pi$ the polar-average in momentum space of $\hat{\delta}(k)$, one has $\delta(R) = \int_0^{k_S(R)} dk k^2 \hat{\delta}(k)/2\pi^2$ and $S(R) = \int_0^{k_S(R)} dk k^2 P(k)/2\pi^2$, hence $d\delta = d_S\delta/dk_S S = \hat{\delta}(k_S)/P(k_S)$ holds. Using these relations $dS\delta = d_S\delta/dk_S S = \hat{\delta}(k_S)/P(k_S)$ follows; moreover, polar-averaging Equation (3) yields $\langle \hat{\delta}(k_S)\hat{\delta}(k_S') \rangle = (2\pi)^3 P(k_S) \delta_D(k_S - k_S')/4\pi k_S^2 = 2\pi^2 P(k_S) |d_S| S \delta_D(S - S')/k_S^2 = P(k_S) P(k_S) \delta_D(S - S')$.

All in all, $\delta(R)$ is seen to satisfy a Langevin equation with a Dirac delta noise, in the form

$$\frac{d\delta}{dS} = \frac{\hat{\delta}(k_S)}{P(k_S)} \quad \text{where} \quad \langle \hat{\delta}(k_S) \hat{\delta}(k_S') \rangle = \frac{\hat{\delta}(k_S)}{P(k_S)} \frac{\hat{\delta}(k_S')}{P(k_S')} = \delta_D(S - S');$$

this means that $\delta(R)$ executes a Markovian random walk as a function of the smoothing scale $R$, with $S(R)$ playing the role of a time variable. In more detail, the walk starts at $\delta = 0$ when $S = 0$ corresponding to large values of $R$, and then as $R$ decreases and the pseudo-time $S(R)$ increases, $\delta(R)$ performs a stochastic motion under the influence of a Gaussian noise with variance $S(R)$. An example is illustrated in Figure 1.

A delicate point with the sharp filter in Fourier space is to associate a mass $M$ to the smoothing scale $R$ (see also discussion in Appendix B). The standard procedure, admittedly ambiguous to some extent, is to normalize the expression of the filter in real space $W_R(r) = |\sin k_S r - k_S r \cos k_S r|/2\pi^2 r^3$ to its maximum value $k_S^3/6\pi^2$, then to compute the volume $V$ enclosed in the filter, and finally to set $M = \bar{\rho} V$. This yields

$$M = 4\pi \bar{\rho} \int_0^\infty drr^2 \frac{6\pi^2}{k_S^3} W_R(r) = \frac{12\pi^2 \bar{\rho}}{k_S^3} \int_0^\infty dx \left[ \frac{\sin x}{x} - \cos x \right] = \frac{6\pi^2 \bar{\rho}}{k_S^3}.$$
Section 2 of the main text.

theory of halo collapse or by comparison with describes spherical collapse (Press & Schechter 1974; Bond et al. 1991); the triple (0 prototypical examples to illustrate our results, for simplicity referring to them as "constant," "square root," and "ellipsoidal."

In deriving the final expression we have used that \( \int_0^{\infty} dx \sin x/x = \pi/2 \), and we have computed the Borel-regularized value of the otherwise ill-defined, oscillating integral \( \int_0^{\infty} dx \cos x \); this amounts to introducing a regulator \( \Lambda \) and to computing, in place of the original integral, the well-defined quantity (see Hardy 1949 for details)

\[
\lim_{\Lambda \to 0^+} \int_0^{\infty} dx e^{-\Lambda x} \cos x = \lim_{\Lambda \to 0^+} \left[ e^{-\Lambda x} \frac{\sin x - \Lambda \cos x}{1 + \Lambda^2} \right]_0^\infty = \lim_{\Lambda \to 0^+} \frac{\Lambda}{1 + \Lambda^2} = 0. \tag{7}
\]

All in all, posing \( M = 4\pi \bar{\rho} R^3/3 \) one obtains the relation \( R = (9\pi/2)^{1/3}/k_5 \) between the smoothing scale and the cutoff wavenumber.

In the excursion set formulation a halo is formed when the walk first crosses a barrier \( B(S, t) \) with general shape

\[
B(S, t) = B_0 + B_\gamma S^\gamma = \sqrt{q} \delta_c(t) \left( 1 + \beta \left[ \frac{q \delta_c^2(t)}{S} \right]^{-\gamma} \right) = \sqrt{S} (q \nu)^{1/2} \left[ 1 + \frac{\beta}{(q \nu)^\gamma} \right], \tag{8}
\]

with \( B_0 \equiv q^{1/2} \delta_c(t) \) and \( B_\gamma \equiv \beta q^{-\gamma + 1/2} \delta_c(t)^{1-2\gamma} \). Here, \( \delta_c(t) \) is the critical threshold for collapse extrapolated from linear perturbation theory. At the current epoch \( \delta_c \approx 1.686 \) holds, although the precise value weakly depends on cosmological parameters (see Eke et al. 1996); then it evolves like \( \delta_c(t) \propto 1/D(t) \) with the cosmological time \( t \), in terms of the linear growth function \( D(t) \), see Weinberg (2008) for detailed expressions.

In addition, the dependence of the barrier on the scale \( S \) is specified by the parameters \( (q, \beta, \gamma) \) that are commonly set based on theory of halo collapse or by comparison with \( N \)-body simulations. The triple of values \((1, 0, 0)\) corresponds to a constant barrier that describes spherical collapse (Press & Schechter 1974; Bond et al. 1991); the triple \((0.707, 0.47, 0.615)\) corresponds to a nonlinear barrier that represents ellipsoidal collapse (actually a collapse with the most probable values of ellipticity and prolateness, see Sheth & Tormen 2002 for details); finally, the triple \((0.55, 0.5, 0.5)\) corresponds to a square-root barrier often considered in the literature as a simpler description of ellipsoidal collapse (Mahmood & Rajesh 2005). In fact, note that for \( \gamma > 1/2 \) not all walks are guaranteed to cross the barrier (the rms \( \delta \) of the walk scales as \( \sqrt{S} \)), and barriers corresponding to different times can intersect (an occurrence thought to represent fragmentation); thus one often adopts the square-root barrier in place of the Sheth & Tormen one, given that the two are known to reproduce comparably well the outcomes of cosmological \( N \)-body simulations.

The last equality in Equation (8) highlights that these barriers can be recast in self-similar terms with the use of the variable \( v \equiv \delta_c^2(t)/S \); this is useful to treat at the same time the first crossing distribution for different epochs and/or masses. In Figure 2, we represent the shape \( B(S) \) of these barriers (color-coded) as a function of \( v \). In the following we will use these three barriers as prototypical examples to illustrate our results, for simplicity referring to them as "constant," "square root," and "ellipsoidal."

Then according to the excursion set prescription, the (unconditional) halo mass function is given by

\[
\frac{dN}{dM} = \frac{\bar{\rho}}{M^2} \frac{d}{d\log M} \frac{dS}{S} f(S); \tag{9}
\]

here \( f(S) \) is the first crossing distribution, i.e., \( f(S) dS \) represents the probability that a trajectory crosses the barrier for the first time between \( S \) and \( S + dS \). This quantity may be determined as described next.
2.1. The First Crossing Distribution

It is convenient to formulate the problem in terms of the integral equation (see Zhang & Hui 2006)

$$\int_0^S dS' f(S') + \int_{-\infty}^0 d\delta P(\delta, S) = 1;$$

(10)

here $B(S)$ is the moving barrier, $f(S)$ is the first crossing distribution, and $P(\delta, S)$ is the probability for the trajectory to lie between $\delta$ and $\delta + d\delta$ at $S$. The above equation plainly states that at a given $S$ a trajectory must either have crossed the barrier at some smaller $S$ or still be below the barrier.

If no barrier were present, $P(\delta, S)$ would simply be a normal Gaussian distribution with null mean and variance $S$, to read

$$P_0(\delta, S) = \frac{e^{-\delta^2/2S}}{\sqrt{2\pi S}};$$

(11)

in the presence of the barrier, to find $P(\delta, S)$ one has to subtract from $P_0(\delta, S)$ the fraction of trajectories now at ($\delta, S$) which have crossed the barrier at some $S' < S$; this is written as

$$P(\delta, S) = P_0(\delta, S) - \int_0^S dS' f(S') P_0[\delta - B(S'), S - S'].$$

(12)

Inserting Equations (11) and (12) into Equation (10) and performing the integration over $\delta$ yields

$$\text{erfc} \left[ \frac{B(S)}{\sqrt{2}S} \right] = \int_0^S dS' f(S') \text{erfc} \left[ \frac{B(S) - B(S')}{\sqrt{2}(S - S')} \right],$$

(13)

where $\text{erfc}(\cdot)$ is the complementary error function. This constitutes a Volterra integral equation of the second kind for the unknown function $f(S)$ given the barrier shape $B(S)$. Note that the integral formulation of the excursion set approach expressed by Equation (13) holds for Markovian random walks with uncorrelated steps; in Appendix B we will discuss how the approach can be modified when some degree of correlation between the steps is considered.

For a linear barrier $B_L(S) = B_0 + B_1 S$ the solution of Equation (13) is easily derived by applying a Laplace transformation, to obtain

$$f_L(x) = e^{-B_0(B_1+\sqrt{B_1^2+2S})};$$

(14)

anti-transforming via the standard Bromwich integral (along a vertical contour $C$ in the complex plane having all the singularities of $f_L$ to the left, e.g., Arfken et al. 2013, p. 1038) leads to the “inverse Gaussian” distribution

$$f_L(S) = \frac{1}{2\pi i} \int_C dze^{-zS} f_L(z) = \frac{B_0}{\sqrt{2\pi S^3}} e^{-B_0^2(S)/2S};$$

(15)

note that the solution for a constant barrier $B(S) = B_0$ is simply recovered by placing $B_1 = 0$.

However, the barriers relevant to the collapse of DM halos are nonlinear, featuring the typical shapes $B(S) = B_0 + B_\gamma S^\gamma$ with $1/2 \leq \gamma \leq 1$ like in Equation (8); in these cases it is not possible to solve exactly Equation (13) for the first crossing distribution, and one generally resorts to numerical techniques (e.g., Zhang & Hui 2006; Benson et al. 2013). In fact, it is sufficient to discretize Equation (13) on a grid of values $S_i$ with spacing $\Delta S_i$ to obtain the solution by the recursive formula

$$f(S_i) = \frac{2}{\Delta S_{i-1}} \left\{ \text{erfc} \left[ \frac{B(S_i)}{\sqrt{2}S_i} \right] - \sum_{j=0}^{i-1} f(S_j) \text{erfc} \left[ \frac{B(S_i) - B(S_j)}{\sqrt{2}(S_i - S_j)} \right] \Delta S_{j-1} + \Delta S_j \right\},$$

(16)

where $f(S_0) \equiv 0$ by definition.

3. UNCONDITIONAL FIRST CROSSING DISTRIBUTION AND HALO MASS FUNCTION

In this section we focus on the unconditional first crossing distribution. We numerically solve Equation (16) and illustrate by the solid lines in Figure 3 the outcomes for constant, square-root, and ellipsoidal barriers (color-coded). In addition, the crosses show the outcome of state-of-the-art cosmological N-body simulations (fit to Tinker et al. 2008; see Section 1 for other references). This is in close agreement with both the square-root and the ellipsoidal distributions, actually striking an intermediate course between the two; the constant barrier result is instead substantially different.

Here our aim is to derive useful approximate expressions to the unconditional first crossing distribution in the limit $S \ll \delta^2(t)$, i.e., large masses and/or early times; in self-similar terms this corresponds to the limit $v \gg 1$. We start by considering in Equation (13) the limit for small $S$: on the left-hand side the quantity $B(S)/\sqrt{2S}$ tends to infinity so that we can exploit the asymptotic expansion of the function $\text{erfc}(x) \simeq e^{-x^2}/x\sqrt{\pi}$ for large argument; meanwhile inside the integral on the right-hand side
Equation (21), and crosses show the outcomes of cosmological

\[ S = \frac{\nu}{\ell} \]

Remarkably, the approximations perform very well over a wide range in

\( \nu \gtrsim \ell \) barriers. For example, in Appendix C we will exploit it for warm DM, and show its good agreement with recent N-body simulations.

Figure 3. Unconditional first crossing distribution, as a function of the self-similar variable \( \nu \equiv S^2/\ell^2 \), for the constant (green), square-root (blue), and ellipsoidal (red lines) barriers. Solid lines refer to the exact results on solving the integral Equation (13) via Equation (16), dashed lines illustrate our approximation given by Equation (21), and crosses show the outcomes of cosmological N-body simulations (fit to Tinker et al. 2008).

(A color version of this figure is available in the online journal.)

Differentiating both sides of Equation (13) with respect to \( S \) yields, to the lowest order,

\[ \frac{B_0 e^{-B(s)/2S}}{\sqrt{2\pi S^3}} \left[ 1 + (1 - 2\gamma) \frac{B_v}{B_0} S^\gamma \right] \simeq f(S) - \frac{B_v \gamma S^{\gamma-1}}{\sqrt{2\pi}} \int_0^S dS' \frac{f(S')}{S' - S} \left[ 1 + 2(\gamma - 1) \frac{S - S'}{S} \right]. \]  

(17)

The structure of this equation suggests to consider the following ansatz for the small \( S \) expansion of the first crossing distribution:

\[ f(S) \simeq \frac{B_0 e^{-B(s)/2S}}{\sqrt{2\pi S^3}} \left[ 1 + k_\nu B_v \frac{S^\gamma}{B_0} \right]. \]  

(18)

where \( k_\nu \) is a constant, dependent on \( \nu \), to be determined. Using it in Equation (17), and approximating the integral by the Laplace

\[ \frac{B_0 e^{-B(s)/2S}}{\sqrt{2\pi S^3}} \left[ 1 + (1 - 2\gamma) \frac{B_v}{B_0} S^\gamma \right] \simeq \frac{B_0 e^{-B(s)/2S}}{\sqrt{2\pi S^3}} \left[ 1 + (k_\nu - \gamma) \frac{B_v}{B_0} S^\gamma \right]. \]  

(19)

This implies \( k_\nu = 1 - \gamma \) and thus the approximate first crossing distribution becomes

\[ f(S) \simeq \frac{B_0 e^{-B(s)/2S}}{\sqrt{2\pi S^3}} \left[ 1 + (1 - \gamma) \frac{B_v}{B_0} S^\gamma \right]; \]  

(20)

note that for the linear (\( \gamma = 1 \)) and constant (\( B_\nu = 0 \)) barriers one recovers the exact solution Equation (15).

Specializing to the barrier shape given by Equation (8) and passing to the self-similar variable \( \nu \) we have

\[ f(\nu) \simeq \sqrt{\frac{q}{2\pi \nu}} e^{-q(1+\beta(q\nu)^{-1})^2/2} \left[ 1 + \frac{(1 - \gamma) \beta}{(q\nu)^\gamma} \right]; \]  

(21)

this approximate expression valid for \( \nu \gg 1 \) and for a generic \( \gamma \) constitutes a novel result. In Figure 3 we compare these approximations (dashed lines) of the unconditional first crossing distribution to the exact results (solid lines) for the constant, square-root, and ellipsoidal barriers. Remarkably, the approximations perform very well over a wide range in \( \nu \), and specifically better than \( \lesssim 10\% \) for \( \nu \gtrsim 0.3 \) (and better than \( \lesssim 30\% \) for \( \nu \gtrsim 0.1 \)). In terms of the characteristic mass \( M_c(z) \) defined by \( \delta_c^2(z) = S[M_c(z)] \), for a standard cold DM spectrum the range translates into \( M \gtrsim 10^{-2} M_c \) at \( z \approx 0 \), extends to \( M \gtrsim 10^{-3} M_c \) at \( z \approx 1 \) and to \( M \gtrsim 3 \times 10^{-4} M_c \) at \( z \approx 2 \), practically encompassing all masses of cosmological interest for \( z \gtrsim 2 \).

In Figure 4 we illustrate the corresponding mass function at different redshifts (coded with line styles), computed by inserting in Equation (9) our approximated expression Equation (20) for the first crossing distribution. When compared to the outcomes of cosmological N-body simulations (crosses; fit to Tinker et al. 2008) for cold DM, the constant barrier mass function considerably underestimates the abundance of simulated halos, especially at high \( z \); the square root and above all the ellipsoidal barrier mass functions perform instead substantially better.

We note that our approximation to the halo mass function based on Equations (9) and (20) is extremely flexible with respect to changes of the power spectrum. For example, in Appendix C we will exploit it for warm DM, and show its good agreement with recent N-body simulations.
Figure 4. Unconditional mass function as a function of halo mass at different redshifts, for the constant (green), square-root (blue), and ellipsoidal (red) barriers. Crosses illustrate the outcomes of N-body simulations (fit to Tinker et al. 2008). A standard cold DM power spectrum has been adopted. (A color version of this figure is available in the online journal.)

Figure 5. Conditional first crossing distribution, as a function of the self-similar variable \( \nu_c \equiv (\Delta \delta_c)^2 / \Delta S \), for the constant (green), square-root (blue), and ellipsoidal (red lines) barriers. Solid lines refer to the exact results by solving numerically Equation (16), as a function of \( \nu_c \), for a descendant mass \( M = 10^{13} M_\odot \) (close to the characteristic mass at \( z = 0 \)) and to a difference \( \Delta z = 1 \) between the redshift of the progenitors and of the descendant. A standard cold DM power spectrum has been adopted. (A color version of this figure is available in the online journal.)

4. CONDITIONAL FIRST CROSSING DISTRIBUTION AND HALO MASS FUNCTION

In this section we focus on the conditional first crossing distribution, i.e., the probability of an object to feature a mass in the range between \( M' \) and \( M' + dM' \) at time \( t' \) provided that it becomes incorporated into a larger object of mass \( M > M' \) at a later time \( t > t' \). As can be easily understood on the basis of Figure 1, in the excursion set approach this two-barrier problem is equivalent to determining the first crossing distribution of a random walk with a single barrier of the form \( B(\Delta S, t, t') = B(S', t') - B(S, t) \), starting point \( \delta_c(t) \), and pseudo-time variable \( \Delta S = S' - S \). Note that if the barrier \( B(S, t) = \delta_c(t) \) were constant, then \( B(\Delta S) = \delta_c(t') - \delta_c(t) = \Delta \delta_c \) would apply, and the result would entail just rescaling the unconditional distribution from the self-similar variable \( \nu \equiv \delta_c^2 / S \) to \( \nu_c \equiv (\Delta \delta_c)^2 / \Delta S \).

In Figure 5 we illustrate as solid lines the exact solutions by solving numerically Equation (16), as a function of \( \nu_c \), for a descendant mass \( M = 10^{13} M_\odot \) and redshift interval \( \Delta z = 1 \). In fact, note that the conditional distributions, except for constant barrier case, do not depend solely on \( \nu_c \) but also slightly on the descendant halo mass and redshift; the rescaled unconditional mass functions are plotted for reference in Figure 5 as dotted lines.

Here, our aim is to derive useful approximated expressions for the conditional first crossing distributions. Since \( B(\Delta S, t, t') \) is a weakly nonlinear barrier, we can approximate it as \( B(\Delta S) \simeq C_0 + C_1 \Delta S + C_2 (\Delta S)^2 \), and the corresponding first crossing distribution as \( f(\Delta S) \simeq f_L(\Delta S) + C_2 \delta_{C_2} f|_{C_2=0} \), where \( f_L(\Delta S) \) is the inverse Gaussian distribution of Equation (15). The unknown quantity...
\[ \partial_{c_2} f_{|c_2=0} \] may be derived from the following steps. First, we apply the operator \( \partial_{c_2} (\cdot)_{|c_2=0} \) to Equation (13) and obtain

\[
-\sqrt{2} (\Delta S)^2 \frac{e^{-(c_0+c_1\Delta S)/2\Delta S}}{\sqrt{\pi \Delta S}} = \int_0^{\Delta S} d\Delta S' \partial_{c_2} f_{c_2=0}(\Delta S') \text{erfc} \left( C_1 \sqrt{\frac{\Delta S - \Delta S'}{2}} \right)
\]

Then we perform a Laplace transformation term by term. To this purpose, we use the following Laplace rules (see, e.g., Abramowitz & Stegun 1972, p. 1020): \( \int_0^t dx' F(x-x')G(x') = \hat{F}(s) \hat{G}(s), e^{x t} F(x) = \hat{F}(s-k), (-1)^n x^n F(x) = d^n \hat{F}(s)/ds^n \), holding for generic functions \( F(x), G(x) \) and their transforms \( \hat{F}(s), G(s) \), for any real number \( k \) and for any integer \( n \). We also use the following Laplace transforms of elementary functions: \( e^{k\sqrt{t}/\sqrt{\pi x}} = e^{-k^2t/\sqrt{\pi}}, e^{x^2} \text{erfc}(k, \sqrt{x}) = 1/\sqrt{x} (\sqrt{x} + k), k e^{-k^2/4x}/2\sqrt{\pi x^3} \rightarrow e^{-k^2/2}, 2\sqrt{x}/\pi \rightarrow s^{-3/2} \), holding for any real number \( k > 0 \).

After some tedious algebra, we get

\[
-\sqrt{2} e^{-c_0 c_1} \frac{d^2}{d(\Delta S')^2} \left[ \frac{e^{-(c_0+c_1\sqrt{\Delta S})/2\Delta S}}{\sqrt{\Delta S'}} \right]_{|\Delta S' = \Delta S + c_1/2} = \frac{\partial_{c_2} \hat{f}_{c_2=0}(\Delta S)}{[C_0 + C_1 \Delta S + C_1 C_1 + 2\Delta S + (C_1 + 2\Delta S)^3/2]}.
\]

Rearranging the various terms yields

\[
\partial_{c_2} \hat{f}_{c_2=0}(\Delta S) = -C_0 e^{-c_0(c_1+\sqrt{c_1^2+2\Delta S})} \left[ \frac{C_0}{(C_1^2 + 2\Delta S)^{1/2}} + \frac{1 + C_0 C_1}{C_1^2 + 2\Delta S + (C_1 + 2\Delta S)^3/2} \right].
\]

Then we anti-transform term by term. To this purpose, we use the following inverse Laplace transforms of elementary functions:

\( e^{-k\sqrt{t}/\sqrt{\pi x}} \rightarrow e^{-k^2/\sqrt{\pi}} x, e^{-k^2/2} \rightarrow \text{erfc}(k, \sqrt{x}) e^{-k^2/4x} - k \text{erfc}(k, \sqrt{x}) \), holding for any real number \( k > 0 \). This leads to

\[
\partial_{c_2} f_{c_2=0}(\Delta S) = -C_0 e^{-(c_0+c_1\Delta S^2)/2\Delta S} \left[ \frac{e^{c_1^2/2\Delta S}}{2} \text{erfc} \left( \frac{C_0}{\sqrt{2\Delta S}} \right) + \frac{C_0 + C_1 \Delta S}{\sqrt{2\pi \Delta S}} \right].
\]

finally, we find

\[
f(\Delta S) \simeq \frac{C_0 e^{-(c_0+c_1\Delta S^2)/2\Delta S}}{\sqrt{2\pi \Delta S^3}} \left[ 1 - C_2 (\Delta S)^{3/2} \left[ \frac{\sqrt{\pi}}{2} e^{c_1^2/2\Delta S} \text{erfc} \left( \frac{C_0}{\sqrt{2\Delta S}} \right) + \frac{C_0 + C_1 \Delta S}{\sqrt{2\pi \Delta S}} \right] \right].
\]

We now specialize our computation to the popular family of barrier shapes of Equation (8), in terms of the normalized coefficients

\[
\tilde{C}_0 \equiv \frac{C_0}{\sqrt{S^3}} = \left[ q \frac{\delta^2(t)}{S} \right]_{t=0}^{1/2} + \beta \left[ q S \delta^2(t)^3 \right]_{t=0}^{1/2} - \beta \left[ q S \delta^2(t)^3 \right]_{t=0}^{-1/2},
\]

\[
\tilde{C}_1 \equiv C_1 S^{1/2} \beta \gamma \left[ q S \delta^2(t)^3 \right]_{t=0}^{-y+1/2},
\]

\[
\tilde{C}_2 \equiv C_2 S^{3/2} = -\frac{\beta \gamma (1 - y)}{2} \left[ q S \delta^2(t)^3 \right]_{t=0}^{-y+1/2}.
\]

Using these expressions, the (approximate) conditional first crossing distribution takes the form

\[
f(\Delta S) \simeq \tilde{C}_0 e^{-(c_0+c_1\Delta S^2)/2\Delta S} \left[ 1 - \tilde{C}_2 \left( \frac{\Delta S}{S} \right)^{3/2} \left[ \frac{\sqrt{\pi}}{2} e^{c_1^2/2\Delta S} \text{erfc} \left( \frac{\tilde{C}_0}{\sqrt{2\Delta S}} \right) + \tilde{C}_0 \sqrt{\Delta S} + \tilde{C}_1 \sqrt{\Delta S} \right] \right].
\]

This is a novel result that generalizes to finite time difference the expression of Zhang et al. (2008b).

In Figure 5 we compare this approximation (dashed lines) for the conditional first crossing distribution to the exact results (solid line) for the constant, square-root, and ellipsoidal barriers. Remarkably, the approximation performs very well over a wide range in \( v_e \), and specifically better than \( \lesssim 10\% \) for \( v_e \geq 0.3 \); in terms of halo masses, for a standard cold DM spectrum the range translates into \( M'/M \gtrsim 10^{-3} \) with \( \Delta z = 1 \). Note that the mass range where the approximation works to this accuracy level becomes larger for higher descendant mass and smaller redshift difference; actually, for \( \Delta z \lesssim 0.1 \), the approximation works very well for all the masses of cosmological interest.
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Figure 6. Conditional mass function as a function of progenitor halo mass, for a descendent mass of $10^{13} M_\odot$ and various differences between redshift of the progenitors and the descendant. Color code refers to the constant (green), square-root (blue), and ellipsoidal (red) barriers. A standard cold DM power spectrum has been adopted. (A color version of this figure is available in the online journal.)

Once the conditional first crossing distribution is known, the conditional mass function can be computed as

$$dN/dM' = \left( \frac{M}{M'} \right)^{3/2} \Delta S f(\Delta S) \left| \frac{d\log \Delta S}{d\log M'} \right|.$$  \hspace{1cm} (29)

In Figure 6 we illustrate the outcomes at different redshifts (line style-coded), computed by inserting our approximation Equation (28) into Equation (29). Note that for small redshift difference $\Delta z \lesssim 0.1$ our result concurs with that by Zhang et al. (2008b).

5. APPLICATIONS TO HALO STATISTICS

In this section we apply our main results given by Equations (20) and (28) to study various quantities of interest in halo statistics, and compare them with the outcomes of $N$-body simulations.

5.1. Rates of Halo Creation

The rates at which DM halos of given mass are created by merging of smaller ones is an essential ingredient in galaxy formation and evolution models. To recall some examples, these rates have been used to investigate active galactic nucleus/quasar activity (e.g., Wyithe & Loeb 2003; Mahmood et al. 2005; Lapi et al. 2006), FIR/submillimeter (submm) selected galaxies (e.g., Lapi et al. 2011; Cai et al. 2013), Lyman break galaxies and cosmic reionization (e.g., Kolatt et al. 1999; Mao et al. 2007), abundances of binary supermassive black holes and related coalescence rates (e.g., Milosavljevic & Merritt 2001; Volonteri et al. 2002), astrophysics of first stars (e.g., Santos et al. 2002; Scannapieco et al. 2003), galaxy clustering (e.g., Percival et al. 2003; Xia et al. 2012), and statistics of galaxy–galaxy gravitational lensing (e.g., Lapi et al. 2012).

As discussed in the references above (see also Cavaliere et al. 1991; Blain & Longair 1993; Sasaki 1994; Haehnelt et al. 1998), to address these issues one cannot rely on the total derivative $dN/dt$ of the halo mass function, because it includes a balance between the rate of halo creation by merging of smaller ones, and the rate of halo destruction by merging into larger ones. Indeed, the total derivative can be positive or negative, as a consequence of the fact that small masses are dominated by the destruction and high masses by the creation term. Extracting from $dN/dt$ the rates of halo creation is actually a non-trivial task.

Such a delicate issue can be attacked with the excursion set approach (see Lacey & Cole 1993; Kitayama & Suto 1996; Mahmood & Rajesh 2005; Moreno et al. 2009). The problem is that in such a framework all halos continuously merge with other ones, and therefore are newly born to some extent; as shown below, pursuing this viewpoint leads to divergent expressions for the creation rates that need to be regularized. On the other hand, to obtain a finite answer without the need for a regularization, some of the aforementioned authors have envisaged that a new halo is born only when its change in mass during a merger has been substantial, i.e., the halo has undergone a major merger. In the above literature such major merger rates are usually dubbed “formation rates,” to distinguish them from the “creation rates” that instead are computed irrespective of the mass change during the merger event. Given that both these quantities are widely used in the literature and extremely relevant for galaxy formation models, our purpose here is to provide novel analytic expressions of the creation and formation rates for general moving barriers.

We start by computing the rate $d^2p_{M'\to M}/dM'dt$ at which a halo of mass $M$ originates from a progenitor halo with mass in the range between $M'$ and $M' + dM'$ over an infinitesimal time step $dt$ (e.g., Kitayama & Suto 1996, their Equation (8)); in other words, this quantity represents the progenitor distribution for infinitesimal look-back times. It can be derived from the above expression Equation (28) on taking the limit $t' \to t$ and changing variable from $\Delta S$ to $M'$, i.e., $d^2p_{M'\to M}/dM'dt \equiv \lim_{t'\to t} f(\Delta S)|d\Delta S/dM'|/\Delta t$. 


In such a limit \( \bar{C}_0 \simeq C_0 \), \[ \frac{d^2 P_{M \rightarrow M}}{dM \, dt} \rightarrow 0, \] and the result reads

\[
\frac{d^2 P_{M \rightarrow M}}{dM \, dt} = \frac{C_0 \, |\dot{\xi}(t)| e^{-C_1 \Delta S / 2S}}{\sqrt{2\pi(\Delta S)^3}} \left| \frac{dS}{dM} \right| \left\{ 1 - \bar{C}_2 \left( \frac{\Delta S}{S} \right)^{3/2} \left[ \frac{\sqrt{\pi}}{2} + \bar{C}_1 \left( \frac{\Delta S}{S} \right)^{1/2} \right] \right\}, \tag{30}
\]

in terms of the quantities

\[
\bar{C}_0 \equiv \sqrt{q} \left[ 1 - \beta (2\gamma - 1) (qv)^{-\gamma} \right],
\]

\[
\bar{C}_1 \equiv \beta \gamma (qv)^{-\gamma+1/2},
\]

\[
\bar{C}_2 \equiv -\frac{\beta \gamma(1 - \gamma)}{2} (qv)^{-\gamma+1/2}
\]

that depend only on the self-similar variable \( v = \delta^2(t)/S \). The above expression coincides with that obtained by Zhang et al. (2008b).

Then the halo formation rates are given by (see, e.g., Kitayama & Suto 1996, their Equations (8) and (13))

\[
\frac{\partial_t \tilde{N}^{\text{form}}}{\partial t} = N \int_0^{M_f} dM' \frac{d^2 P_{M \rightarrow M}}{dM \, dt} \tag{32}
\]

where \( M_f \) is a suitable fraction of the final mass \( M \), typically \( M_f \approx M/2 \); as mentioned at the beginning of this section, the rationale of the above expression is that a new halo is formed if more than half of its final mass is acquired during the merger, i.e., if the merger has been a “major” one. Note that to simplify the notation we have indicated the unconditional mass function with \( N \) in place of \( dN/dM \).

Using Equation (30) and changing variable from \( dN/dM \) to \( x = \sqrt{S/\Delta S} \), yields

\[
\frac{\partial_t \tilde{N}^{\text{form}}}{\partial t} = \frac{2\sqrt{2\pi \Delta S}}{\sqrt{S}} \int_0^{M_f} dM' \frac{d^2 P_{M \rightarrow M}}{dM \, dt} \left[ 1 - \sqrt{\frac{\pi}{2}} \frac{\bar{C}_2}{C_1} \right]. \tag{33}
\]

Remarkably, the integrals can be carried out analytically and we obtain

\[
\frac{\partial_t \tilde{N}^{\text{form}}}{\partial t} = 2\sqrt{2\pi \Delta S} \int_0^{M_f} dM' \frac{d^2 P_{M \rightarrow M}}{dM \, dt} \left[ 1 - \sqrt{\frac{\pi}{2}} \frac{\bar{C}_2}{C_1} \right]. \tag{34}
\]

with \( s_f \equiv S_f / \Delta S - 1 \). This expression for a general moving barrier constitutes a novel result. In Figure 7 we illustrate the formation rates as a function of halo mass at different redshifts, for the constant, square-root, and ellipsoidal barriers.

On the other hand, the halo “creation” rates are given by (see, e.g., Kitayama & Suto 1996, their Equations (8) and (10))

\[
\frac{\partial_t \tilde{N}^{\text{crea}}}{\partial t} = N \int_0^{M_f} dM' \frac{d^2 P_{M \rightarrow M}}{dM \, dt}; \tag{35}
\]

this constitutes the limit for \( M_f \rightarrow M \) in the expression for the formation rates; as mentioned at the beginning of this section, the rationale of the above expression is that a new halo is continuously created by merging of smaller ones, irrespective of the mass change during the merger. Plainly, the integral in Equation (35) does not converge, because the term proportional to \( s_f^{-1/2} \) in Equation (34) blows up when \( s_f \rightarrow 0 \) (and since we are considering infinitesimal time steps \( dt \), even the expression of the integrand in Equation (35) would be ill defined around \( M' \rightarrow M \), cf. the footnote concerning Equation (30)). However, this divergence (or ill definiteness) is easily recognized to be unphysical, being related to the continuum nature of this treatment where infinitesimal mass changes can take place. In fact, one is actually counting the transition of an object into itself; this has been pointed out for a constant barrier by Kitayama & Suto (1996), by showing that the same divergence is present in the destruction rate (that can be obtained from the creation rate and the unconditional distribution using Bayes’ theorem), and that the two exactly cancel out in the difference of the rates to yield the total derivative of the mass function.

Here, we derive a finite result for the creation rate on exploiting the \( \xi \)-regularization technique for divergent integrals; a brief primer on the latter is provided in Appendix A. First of all, we introduce in Equation (35) a cutoff \( \Lambda \), and rewrite conveniently the upper integration limit as \( M (1 + 1/\Lambda^2) \). Then we change variable from \( M' \) to \( x = \sqrt{S/\Delta S} \); this turns the upper limit of integration into \( \left[ S_{M(1+1/\Lambda^2)/S} - 1 \right]^{-1/2} = \Lambda |d log \, M / d log \, S|^{1/2} \), and we obtain

\[
\frac{\partial_t \tilde{N}^{\text{crea}}}{\partial t} = 2\sqrt{2\pi \Delta S} \lim_{\Lambda \rightarrow \infty} \int_0^{\Lambda |d \log \, M / d \log \, S|^{1/2}} dx e^{-C_1^2/2x^2} \left[ 1 - \sqrt{\frac{\pi}{2}} \frac{\bar{C}_2}{C_1} \right]. \tag{36}
\]

Note that the terms proportional to \( \bar{C}_0 \sqrt{S/\Delta S} \rightarrow C_0 \, |\dot{\xi}(t)| \sqrt{S/\Delta S} \) appearing for \( t' \rightarrow t \) in the exponential and in the square bracket of Equation (28) are subdominant since \( dt \rightarrow 0 \), for any finite value of \( \Delta S \). On the other hand, if \( \Delta S \) were also to tend to zero (corresponding to \( M' \rightarrow M \)), then the limit would become ill defined and some form of regularization would be needed. We will come back to the issue later in this section when dealing with the creation rates.
Formation rates as a function of halo mass at different redshifts, for the constant (green), square-root (blue), and ellipsoidal (red) barriers. A formation mass $M_{\text{form}} = M/2$ and a standard cold DM power spectrum have been adopted.

(A color version of this figure is available in the online journal.)

Creation rates as a function of halo mass at different redshifts, for the constant (green), square-root (blue), and ellipsoidal (red) barriers. A standard cold DM power spectrum has been adopted.

(A color version of this figure is available in the online journal.)

Only the first integral diverges, and according to the procedure outlined in Appendix A, we find the $\zeta$-regularized value

$$\lim_{\Lambda \to \infty} \int_0^\Lambda dx e^{-\frac{C_1}{2}x^2} \approx \frac{1}{2} \left| \frac{d \log M}{d \log S} \right|^{1/2} - \sqrt{\frac{\pi}{2}} \bar{C}_1.$$  (37)

Finally, we obtain for the creation rates

$$\partial_t N_{\text{crea}} = \frac{\bar{C}_0 |\delta_c| N}{\sqrt{2\pi S}} \left( \left| \frac{d \log M}{d \log S} \right|^{1/2} - \sqrt{2\pi} \left( \bar{C}_1 + 2 \frac{\bar{C}_2}{\bar{C}_1} \right) \right) = \frac{\sqrt{q} |\delta_c| N}{\sqrt{2\pi S}} g(\nu);$$  (38)

here we have highlighted for compactness the factor $g(\nu)$, which depends only on the self-similar variable $\nu = \delta_c^2/S$ at given power spectrum. The expression of Equation (38) constitutes a novel result. In Figure 8 we illustrate the creation rates as a function of halo mass at different redshifts, for the constant, square-root, and ellipsoidal barriers.

Now to compare our results with $N$-body simulations (see Moreno et al. 2009), we compute the creation time distribution $c(t|m)$; this is defined as the time-normalized creation rate

$$c(t|m) = \frac{\partial_t N_{\text{crea}}}{\int_0^\infty dt \partial_t N_{\text{crea}}}. $$  (39)
Using our expressions Equations (21) and (38) we can write
\[ c(\nu) = \frac{c(t|m)}{\left| \frac{\delta_c}{\nu} \right|} \frac{\sqrt{\nu} \ f(\nu) \ g(\nu)}{\int_0^\infty d\nu \sqrt{\nu} \ f(\nu) \ g(\nu)}; \]

this shows that the creation time distribution can be put in self-similar form. In Figure 9, we illustrate the outcomes for different barriers with the solid lines. Note that our result for non-constant barriers is somewhat different from that obtained by Percival & Miller (1999) on using Bayes’ theorem and assuming a uniform \( \delta_c \) distribution at given \( S \); their outcome is still given by Equation (40) but with \( g(\nu) = 1 \), and is illustrated by the dotted lines in Figure 9; the crosses refer to the results of cosmological\textsuperscript{N}-body simulations (see Moreno et al. 2009). We find a very good agreement between the \textsuperscript{N}-body data and our result for the ellipsoidal barrier.

Finally, for the sake of completeness, we also compare the formation and creation rates with two of their proxies often adopted in the literature; it is beyond the scope of the present paper to discuss their underlying theoretical background (but see Section 1 for a quick overview). The first is obtained by taking the “positive” term in the cosmic time derivatives of the mass function (see Haehnelt et al. 1998; Blain & Longair 1993; Sasaki 1994); using our approximated expression in Equation (21) one obtains
\[ \frac{\partial t N_{\text{pos}}}{\partial t} = \left| \frac{\delta_c}{\nu} \right| N \ \delta_c \ \nu, \]

an expression strictly holding in the limit \( \nu \gg 1 \), i.e., large masses. For brevity we will refer to this prescription as “coagulation” rate. The second has been proposed by Sheth & Pitman (1997), based on the analogy between the excursion set approach with a white-noise power spectrum and a discrete Poisson model of binary coagulation with additive kernel. Their prescription is written as
\[ \frac{\partial t N_{\text{coag}}}{\partial t} = \left| \frac{\delta_c}{\nu} \right| N \ \delta_c \ \nu, \]

an expression strictly holding in the limit \( \nu \gg 1 \), i.e., large masses. For brevity we will refer to this prescription as “coagulation” rate. In Figure 10 we illustrate the creation, positive, formation, and coagulation rates (color-coded) as a function of halo mass at different redshifts, for an ellipsoidal barrier; the former three rates are very similar for masses \( M \gg 10^{10} M_\odot \) and redshift \( z \gg 1 \), with somewhat larger differences at lower redshift and smaller masses. The coagulation rate is substantially lower at small masses, although it converges to the other prescriptions at large masses (where actually Equation (42) is supposed to hold).

5.2. Halo Growth History

The halo conditional mass function can be exploited to construct a merger tree, i.e., a numerical realization of the halo merger process. This procedure with its delicate issues has been extensively discussed in the literature (see Kauffmann & White 1993; Somerville & Kolatt 1999; Cole et al. 2000; Parkinson et al. 2008; Neistein & Dekel 2008; Zhang et al. 2008a), and is beyond the scope of the present paper; however, from Equation (30) we can derive the average growth history of DM halos that would obtain after averaging over many realizations of a full merger tree. To this purpose, we first compute the rate of accretion onto a mass \( M \) at time \( t \) as
\[ \dot{M}(M, t) = \int_{M_{\text{min}}}^M dM' (M - M') \frac{d^2 P_{M \rightarrow M}}{dM'dt}, \]
To derive the “mean” evolutionary track of an individual halo we set the lower integration limit \( M_{\text{min}} \) in Equation (43) to 0 (see Miller et al. 2006, their Equation (4)); practically, a lower limit is provided by the “mass resolution” of the tree. Note that, as shown by Neistein & Dekel (2008), there is no double-counting issue associated to the range of integration since in the excursion set framework the typical merger event is not binary but involves multiple progenitors even for infinitesimal time steps; this is mirrored in the integrand of Equation (43), which is not symmetric with respect to \( M' \) and \( M - M' \).

Following Neistein et al. (2006, see their Appendix and Equations (31) and (33)) we can also compute a first-order, analytical estimate of the “main progenitor” evolutionary track by setting \( M_{\text{min}} = M/2 \); this corresponds to considering at each branch of a merger tree only the most massive progenitor. The “mean” and the “main progenitor” growth histories for different final masses at \( z = 0 \) are illustrated by the blue and red lines in the top panel of Figure 11. We stress that for the first time these outcomes have been obtained on adopting the ellipsoidal barrier, and as such extend previous analysis based on the constant barrier (e.g., Miller et al. 2006; Neistein et al. 2006) and constitute novel results. The main progenitor history is plainly biased toward early times relative to the mean one, and can be considered the typical history of halos residing in rich environments or on peaks of the initial density perturbation fields.

It is worth noticing that we recover the typical two-stage evolution of a halo concurrently found in many state-of-the-art \( N \)-body simulations and semi-analytic studies (see Zhao et al. 2003; Fakhouri et al. 2010; Wang et al. 2011; Lapi & Cavaliere 2011); such evolution comprises an early fast collapse and a late slow accretion, roughly separated by the epoch when the halo accretion timescale becomes comparable to the Hubble time. This separation epoch is highlighted by the dots on each evolutionary track. In Figure 12, we compare our results with the growth histories extracted from \( N \)-body simulations by McBride et al. (2009); see also earlier works by Wechsler et al. (2002; van den Bosch 2002). These authors have shown that the halo mass growth from simulations can be parameterized as \( M(z)/M(0) = (1 + z)^{\beta} e^{\gamma z} \) in terms of two shape parameters \( \beta \) and \( \gamma \); for galactic halos with current mass \( M(0) \approx 2-3 \times 10^{12} \, M_\odot \) the average values \( \beta = 0.1 \) and \( \gamma = 0.69 \) apply. The corresponding evolution is illustrated in the figure by crosses. Our reference result for the growth history of a halo with mass \( 2.5 \times 10^{12} \, M_\odot \) is reported as a cyan line. However, for fair comparison with McBride et al. (2009), we must redo the computation after applying two changes: first, we include in the minimum mass \( M_{\text{min}} \) entering Equation (43) their mass resolution of \( 10^9 \, M_\odot \) (green line); second, we modify our normalization of the mass variance \( \sigma_8 = 0.81 \) to match their value 0.9 (orange line). Our overall outcome is in striking agreement with the \( N \)-body result.

The bottom panel of Figure 11 is focused on evolutionary tracks starting at \( z = 2 \) instead of \( z = 0 \). Comparing the two panels, it is evident that the histories ending at \( z = 2 \) feature a more rapid evolution; for example, halos of \( 10^{13} \, M_\odot \) at \( z = 2 \) are still in, or very close to, their fast collapse stage, while the same halos at \( z = 0 \) are well within the slow accretion regime. This implies that the modeling of the baryonic processes occurring inside a given halo should be quite different, depending on whether its formation occurred at low or high redshift.

In the vein of a link to observations, we remark that such massive halos at \( z \sim 2 \) are the host of bright submm selected galaxies, as demonstrated by studies of the clustering properties and of the cosmic infrared background autocorrelation function (see Cooray et al. 2010; Xia et al. 2012). Specifically, Lapi et al. (2011) have shown that these galaxies selected with limiting flux \( S \gtrsim 35 \, mJy \) at 250 \( \mu \)m exhibit star formation rates in the range \( 300-3000 \, M_\odot \, \text{yr}^{-1} \). This implies that the fast collapse of massive halos at these substantial redshifts \( z \gtrsim 1.5 \) originates physical conditions in the associated baryons leading to rapid cooling and condensation into stars. As expected from basic physical arguments and shown by numerical simulations (e.g., Zhao et al. 2003; Wang et al. 2011), the halo fast collapse includes the rapid merging of large clumps, losing most of their angular momentum by relaxation processes over a few dynamical times (see Lapi & Cavaliere 2011). This concurs with high-resolution photometric and spectroscopic observations of \( z \sim 2 \) galaxies, showing that the star formation occurs in clumps located within the central \( kpc \)-sized regions, and that any residual
rotational motion is unstable and expected to be dissipated over a few hundred million years (e.g., Genzel et al. 2011). On the other hand, during the slow accretion stage at low redshift, a quite different baryonic evolution is expected.

5.3. Halo Bias

The excursion set approach may be exploited to evaluate the correlation between halo abundances and their environment. This is usually done by computing the (Eulerian) halo bias (see Mo & White 1996; Sheth & Tormen 1999):

$$b(M, z) = 1 + \frac{1}{\delta_{c, 0}} \left[ \frac{N(M, \delta_c \to M_0, \delta_{c, 0})}{N(M, \delta_c) V} - 1 \right];$$

(44)

in the expression above what matters is the ratio between the number of halos at redshift $z$ with density contrast $\delta_c$ that will end up in an environment with volume $V$, mass $M_0 \simeq \bar{\rho} V \gg M$, and density contrast $\delta_{c, 0} \ll \delta_c$; we have denoted $dN/dM$ by $N$ to simplify the notation.

Since conditions where $\Delta S \simeq S \gg S_0$ are relevant in this context, we cannot exploit our approximate expression for the conditional mass function derived in Section 4; however, it is rather easy to single out that, when written in terms of the appropriate scaling variable $\nu_c \equiv (\Delta \delta_c)^2 / \Delta S$, its shape should be close to the unconditional distribution; this is because the barrier of the conditional problem $B(\Delta S, \delta_c, \delta_{c, 0}) = B(S, \delta_c) - B(S_0, \delta_{c, 0}) \simeq B(S, \Delta \delta_c)$ tends to the unconditional one as $S \gg S_0$. Using Equations (21) and (44)
of halo mass at different redshifts, for the constant, square-root, and ellipsoidal barriers (color-coded); top panel is for \( z = 1 \).

Comparison between our computation based on Equation (43) (colored lines) and the outcomes of N-body simulations (see McBride et al. 2009; crosses) for the growth history of a DM halo with current mass \( M(0) = 2.5 \times 10^9 M_\odot \). Cyan line is our result for \( M_{\text{min}} = 0 \) in Equation (43) corresponding to an infinite mass resolution, and for the value \( \sigma_8 = 0.81 \) of the mass variance normalization; green line is for \( M_{\text{min}} = 10^9 M_\odot \) but still \( \sigma_8 = 0.81 \); orange line is for \( M_{\text{min}} = 10^9 M_\odot \) and \( \sigma_8 = 0.9 \), to match the values adopted in the N-body simulations analyzed by McBride et al. (2009). (A color version of this figure is available in the online journal.)

(A color version of this figure is available in the online journal.)

one obtains

\[
b(M, \delta) = 1 + \frac{1}{\delta_0} \left[ \frac{v_c f(v_c)}{v f(v)} - 1 \right] \approx 1 - \frac{2}{\delta_c} \left[ 1 + \frac{d \log f}{d \log v} \right],
\]

where \( f \) is the unconditional distribution; the second equality follows from considering that \( v_c \simeq v(1 - 2\delta_0/\delta_c) \) and \( f(v_c) \approx f(v) - 2(\delta_0/\delta_c) v f'(v) \) in the relevant limits \( S \gg S_0 \) and \( \delta_c \gg \delta_0 \). Note that in this way we obtain the bias function for halos observed at redshift \( z_{\text{obs}} = 0 \) but formed at redshift \( z > z_{\text{obs}} \); on the other hand, the bias for halos observed as soon they are formed at \( z_{\text{obs}} = z \) can be obtained by replacing \( \delta_c(z) \rightarrow \delta_c(z_{\text{obs}}) \) in Equation (45).

Our results of using the explicit expression of \( f(v) \) from Equation (21) in Equation (45) are illustrated in Figure 13 as a function of halo mass at different redshifts, for the constant, square-root, and ellipsoidal barriers (color-coded); top panel is for \( z_{\text{obs}} = z \) and bottom panel is for \( z_{\text{obs}} = 0 \). Our results for the ellipsoidal barrier compare well to the outcomes of cosmological N-body simulations (crosses; fit to Tinker et al. 2005), although they tend to overpredict the asymptotic bias of low-mass halos and slightly underpredict that of high-mass halos (see also discussion by Tinker et al. 2010). We stress that, as explicitly shown by Ma et al. (2011, see their Table 1 and Figure 1) for the case of a constant barrier, the inclusion of non-Markovian effects and of stochasticity in the barrier can be relevant in obtaining a better agreement between the bias function computed from the excursion set approach and that measured from N-body simulations.

6. SUMMARY AND CONCLUSIONS

The formation and evolution of DM halos constitutes a highly complex problem, whose attack ultimately requires cosmological N-body simulations on supercomputers. However, some analytic grasp is most welcome to better interpret their outcomes to provide approximated yet flexible analytic representations of the results, to develop strategies for future simulation setups, and to quickly explore the effects of modifying the excursion set assumptions or the cosmological framework (Section 1).

To this purpose, we have exploited the excursion set approach in integral formulation (Section 2) to derive novel, accurate approximated expressions of the unconditional (Section 3) and conditional (Section 4) first crossing distributions, for random walks with uncorrelated steps and general shapes of the barrier. We find the corresponding approximation of the halo mass functions for cold DM power spectra to represent very well the outcomes of state-of-the-art cosmological N-body simulations. In addition, we have applied these results to derive and confront with simulations (Section 5) various quantities of interest in halo statistics, including the rates of halo formation and creation, the average halo growth history, and the halo bias for general moving barriers.

In Table 1 we list our main results, recall their location in the paper, and cross-reference to the corresponding equations and figures. A library of IDL-based routines to easily compute the quantities derived in this paper is available at the URL http://people.sissa.it/~lapi/rates (currently under construction).

Finally, we have discussed how our results are affected when considering random walks with correlated instead of uncorrelated steps (Appendix B) and warm instead of cold DM power spectra (Appendix C). We stress that above and around the warm DM free-streaming mass scale, our approximations to the halo mass function agree very well with the outcomes of current N-body simulations.
Figure 13. Halo bias as a function of halo mass at different formation redshift $z_f$, for the constant (green), square-root (blue), and ellipsoidal (red) barriers. Top panel is for observation redshift $z_{\text{obs}}=z$ and bottom one for $z_{\text{obs}}=0$. Crosses are the results of $N$-body cosmological simulations (fit to Tinker et al. 2005). A standard cold DM power spectrum has been adopted.

(A color version of this figure is available in the online journal.)

Table 1

| Results                              | Sections | Equations | Figures |
|--------------------------------------|----------|-----------|---------|
| Unconditional first crossing distribution$^a$ | 3        | 20, 21    | 3       |
| Unconditional halo mass function$^b$  | 2, 3     | 9, 20     | 4       |
| Conditional first crossing distribution | 4        | 28        | 5       |
| Conditional halo mass function       | 4        | 28, 29    | 6       |
| Halo formation rates                 | 5.1      | 32, 34    | 7       |
| Halo creation rates                  | 5.1      | 35, 38    | 8, 10   |
| Halo creation time distribution      | 5.1      | 39, 40    | 9       |
| Halo growth history                  | 5.2      | 43        | 11, 12  |
| Halo bias                            | 5.3      | 44, 45    | 13      |

Notes.

$^a$ Results for correlated steps are discussed in Appendix B and illustrated in Figure 14.

$^b$ Results for warm DM power spectra are discussed in Appendix C and illustrated in Figure 15.

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APPENDIX A

\( \zeta \)-REGULARIZATION TECHNIQUE FOR DIVERGENT INTEGERS

Here we provide a brief primer on the \( \zeta \)-regularization technique for divergent series and integrals, helpful in Section 5.1; the interested reader can find more mathematical details and physical applications in classic textbooks like Hardy (1949), Birrell & Davies (1984), and Elizalde et al. (1994).

The \( \zeta \)-function was originally defined by Euler as

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s};
\]

(A1)

this series is convergent for any complex number \( s \) with \( \Re(s) > 1 \), i.e., real part exceeding 1.

Riemann suggested to extend the definition of \( \zeta(s) \) to the whole complex plane by analytic continuation. For example, one can use the expression involving the Dirichlet alternating series

\[
\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}
\]

(A2)

which coincides with the standard definition above for \( \Re(s) > 1 \) but is well defined for any \( s \) with \( \Re(s) > 0 \) except the point \( s = 1 \) which is a single pole. For \( s \approx 1 \) it can be shown that \( \zeta(s) \) features the approximate behavior

\[
\zeta(s) \simeq \frac{1}{s-1} + \gamma_{\text{em}},
\]

(A3)

where \( \gamma_{\text{em}} = 0.5772\ldots \) is the Euler–Mascheroni constant.

In addition, Riemann discovered that the \( \zeta \)-function satisfies the functional equation

\[
\zeta(s) = 2^s \pi^{s-1} \sin \left( \frac{\pi s}{2} \right) \Gamma(1-s) \zeta(1-s),
\]

(A4)

where \( \Gamma(s) \equiv \int_0^\infty dt \, t^{s-1} e^{-t} \) is the Euler \( \Gamma \)-function. This can be used to extend the definition of \( \zeta(s) \) even to values of \( s \) with \( \Re(s) \leq 0 \). In particular, for \( s \to 0 \) the functional equation yields

\[
\zeta(0) \simeq \frac{1}{\pi} \frac{\pi s}{2} \frac{1}{(1-s) - 1} = -\frac{1}{2}.
\]

This is the origin of the weird-looking expression

\[
\zeta(0) = \sum_{n=1}^{\infty} \frac{1}{n^0} = 1 + 1 + 1 + \cdots = -\frac{1}{2};
\]

(A6)

in fact, the reader should keep in mind that \( \zeta(s) \) for \( s = 0 \) is only formally related to the infinite sum at the second and third member.

The value \( \zeta(0) \) derived above can be effectively used to regularize the divergent improper integral

\[
\lim_{\Lambda \to \infty} \int_0^\Lambda dx = \lim_{\Lambda \to \infty} \Lambda = \infty;
\]

(A7)

this can be done by replacing the integral with a series, which turns out to be closely related to \( \zeta(0) \), in the form

\[
\lim_{\Lambda \to \infty} \int_0^\Lambda dx = \lim_{\Lambda \to \infty} \sum_{n=0}^{\Lambda} 1 = 1 + \lim_{\Lambda \to \infty} \sum_{n=1}^{\Lambda} \frac{1}{n^0} = 1 + \zeta(0) = 1 - \frac{1}{2} = \frac{1}{2}.
\]

(A8)

Integrals involving more general integrands but with the same diverging behavior can also be easily regularized by adding and subtracting convenient quantities and using the result of Equation (A8). For example, we show below how to regularize the divergent integral appearing in Section 5.2 (\( \eta \) is a positive constant):

\[
\lim_{\Lambda \to \infty} \int_0^{\eta\Lambda} dx e^{-k^2/2x^2} = \lim_{\Lambda \to \infty} \left\{ \int_0^{\eta\Lambda} dx e^{-k^2/2x^2} - \eta\Lambda + \eta \int_0^{\Lambda} dx \right\} = \lim_{\Lambda \to \infty} \left[ x e^{-k^2/2x^2} + \sqrt{\frac{\pi}{2}} k \text{ erf} \left( \frac{k}{\sqrt{2x}} \right) \right]_0^{\eta\Lambda} - \eta\Lambda + \frac{\eta}{2}
\]

(A9)

\[
= \eta\Lambda - \sqrt{\frac{\pi}{2}} k - \eta\Lambda + \frac{\eta}{2} = -\sqrt{\frac{\pi}{2}} k + \frac{\eta}{2}.
\]
APPENDIX B

EXCURSION SET APPROACH WITH CORRELATED STEPS

In the main text we have been concerned with the first crossing problem when the steps of the random walk are uncorrelated. In this Appendix, we discuss how the integral formulation of the excursion set approach introduced in Section 2 can be modified to account for some degree of correlation between the steps. Recently, correlated random walks have received considerable interest (see Paranjape et al. 2012; Musso & Sheth 2012); in fact, it may well be that such models are more realistic. This is because the random walk $\delta(\mathbf{S})$ is truly Brownian, i.e., with uncorrelated steps, only if a sharp filter function in Fourier space is adopted. However, as pointed out in Section 2 in this case there is an ambiguity of a sort in associating a mass to the filtering scale. On the other hand, using a real-space top-hat window function gives a well-defined relation between mass and smoothing scale, but necessarily introduces some degree of correlations among the steps of the random walk. Note that for the slowly varying cold DM power spectra, this problem is somewhat academic, since the shape of $S(M)$ depends very weakly on the choice of the filter function; however, it is not so for, e.g., warm DM spectra (see discussion by Benson et al. 2013). In addition, correlations between the steps of the walk can also be introduced by non-Gaussian features in the initial spectrum of perturbations (see Maggiore & Riotto 2010c).

For the sake of definiteness, here we mainly consider the extreme instance according to which the steps of the random walk are completely correlated. Then one expects that the walk $\delta(\mathbf{S})$ does not proceed through stochastic zigzags, but instead grows almost monotonically with $S$. This means that if $\delta$ first exceeded the barrier $B(\mathbf{S})$ at $S$, it was certainly below that at all $S' < S$. Then the constraint imposed by Equation (12) is superfluous and $P(\delta, S) \sim P_0(\delta, S)$ holds for $\delta < B(\mathbf{S})$ to a very good approximation. Simple algebra yields

\[ \int_0^S dS' f(S') = \frac{1}{2} \text{erfc} \left[ \frac{B(S)}{\sqrt{2S}} \right], \]

which constitutes the counterpart of Equation (13) for completely correlated steps.

Differentiating with respect to $S$ leads to the first crossing distribution

\[ f(S) = -e^{-B^2(S)/2S} \frac{d}{dS} \left[ \frac{B(S)}{\sqrt{S}} \right]. \]

Specializing to the barrier shape $B(S) = B_0 + B_\gamma S' \gamma$ yields

\[ f(S) = \frac{B_0}{2} e^{-B^2(S)/2S} \left[ 1 + (1 - 2 \gamma) \frac{B_\gamma}{B_0} S' \right]. \]

Note that in the constant barrier case ($B_\gamma = 0$), this is exactly half of the result for uncorrelated steps, and actually coincides with the expression found by Press & Schechter (1974).

In Figure 14, we compare the unconditional first crossing distribution for uncorrelated (solid lines) and completely correlated (dashed lines) steps. It is apparent that the outcome of cosmological $N$-body simulations (crosses) lies much closer to the excursion set result for uncorrelated steps. This is also the case if one consider strongly but not completely correlated steps like in the framework developed by Musso & Sheth (2012), whose results are illustrated in Figure 14 by the dotted lines.
Figure 15. Unconditional mass function as a function of halo mass at different redshifts, for the ellipsoidal barrier. Black line refers to a standard cold DM power spectrum, red line to a warm DM with Fermi–Dirac mass $m_{FD} = 1.5$ keV, blue line to $m_{FD} = 0.75$ keV, and green line to $m_{FD} = 0.25$ keV. The latter is compared at $z = 0$ with the outcome of the $N$-body simulation by Schneider et al. (2012; green crosses) for the same mass 0.25 keV. Lines are thick above, and thin below, the free-streaming mass of the warm DM particle.

(A color version of this figure is available in the online journal.)

All in all, models with correlated steps need much of an improvement before reproducing the halo mass function at a level comparable to models with uncorrelated steps; this improvement, although only on an effective/empirical basis, may come from considering a stochastic in place of a deterministic barrier possibly including some drift (see Robertson et al. 2009; Maggiore & Riotto 2010b; Corasaniti & Achitouv 2011).

APPENDIX C
EXCURSION SET APPROACH WITH WARM DARK MATTER

In the main text, we have performed computations for cold DM as a reference. In this Appendix, we show that our formulation of the excursion set approach is extremely flexible with respect to changes of the power spectrum. Specifically, here we focus on warm DM (e.g., Dodelson & Widrow 1994), and test our approximation to the halo mass function against recent $N$-body simulations.

To this purpose, we follow Bode et al. (2001) and Barkana et al. (2001) in modifying the cold DM power spectrum to impose a cutoff below the warm DM free-streaming length scale, and in including the mass-dependent behavior of the linear threshold for collapse $\delta_c(M, z)$ enforced by the warm DM particle’s residual thermal velocities. Specifically, for the resulting power spectrum we take Equations (3) and (4) in Barkana et al. (2001), while for the threshold we take Equations (7)–(10) in Benson et al. (2013).

Practically, in computing the mass function after Equation (9) we exploit our results on the first crossing distribution for an ellipsoidal barrier Equation (20), then connect variance $S$ and mass $M$ with the warm DM power spectrum, and finally introduce the mass-dependent $\delta_c(M, z)$ in the result a posteriori. This procedure provides reasonable results above the free-streaming mass scale. In fact, in Figure 15 we compare our result on the mass function at $z = 0$ for $m_{FD} = 0.25$ keV with the outcomes from the recent $N$-body simulation by Schneider et al. (2012), finding an excellent agreement. However, below the free-streaming mass scale the computations based on the ellipsoidal barrier are quite uncertain, because the barrier $B(S, t)$ itself is poorly understood. In fact, there is some debate as to whether halos with such small masses can form at all (see Bode et al. 2001; Wang & White 2007). If this is the case, the overall result would be a more pronounced cutoff of the mass function just below the free-streaming scale (see discussion by Smith & Markovic 2011; Menci et al. 2012; Benson et al. 2013). To highlight this uncertainty, in Figure 15 we have indicated with a thick line our results above, and with a thin line below, the free-streaming scale.

In Figure 15, we also present the outcomes of the unconditional mass function at different redshifts for values of the warm DM particle mass (color-coded) $m_{FD} = 0.25, 0.75$ and 1.5 keV, compared with the standard cold DM outcome. As is customary we quote here the Fermi–Dirac mass $m_{FD}$, i.e., the mass that the particles would have if they were thermal relics (decoupled in thermal equilibrium); this is convenient because the masses of warm DM particles produced in different microscopic scenarios can be easily related to $m_{FD}$, see de Vega & Sanchez (2012).

All in all, the warm DM results are practically indistinguishable from the cold DM ones at the high-mass end, then on approach to the free-streaming mass progressively deviate downward (and then are practically cut off). The masses where the deviation sets in are smaller for lower Fermi–Dirac mass and higher redshift. In particular, we stress that for $m_{FD} \lesssim 1.5$ keV, at redshift $z \gtrsim 6$ and masses $M \lesssim 10^{11} M_\odot$ the strongly flattening behavior of the halo mass function should be mirrored by the faint end in the luminosity function of Lyman break galaxies up to $z \sim 10$ (e.g., Bouwens et al. 2011); there dust obscuration should not constitute a relevant issue, and the flattening could be so prominent as to not be easily ascribed to supernova feedback (see Mao et al. 2007; Z.-Y. Cai et al. 2013, in preparation).
