Duality relations for the ASEP conditioned on a low current

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Abstract We consider the asymmetric simple exclusion process (ASEP) on a finite lattice with periodic boundary conditions, conditioned to carry an atypically low current. For an infinite discrete set of currents, parametrized by the driving strength $s_K$, $K \geq 1$, we prove duality relations which arise from the quantum algebra $U_q[\mathfrak{gl}(2)]$ symmetry of the generator of the process with reflecting boundary conditions. Using these duality relations we prove on microscopic level a travelling-wave property of the conditioned process for a family of shock-antishock measures for $N > K$ particles: If the initial measure is a member of this family with $K$ microscopic shocks at positions $(x_1, \ldots, x_K)$, then the measure at any time $t > 0$ of the process with driving strength $s_K$ is a convex combination of such measures with shocks at positions $(y_1, \ldots, y_K)$, which can be expressed in terms of $K$-particle transition probabilities of the conditioned ASEP with driving strength $s_K$.

1 Introduction

In the asymmetric simple exclusion process (ASEP) [36,26,27,34] each lattice site $k$ on a lattice $\Lambda = (1, \ldots, L)$ is occupied by at most one particle, indicated by occupation numbers $\eta(k) \in S = \{0, 1\}$. We denote by $\eta = (\eta(1), \ldots, \eta(L)) \in \Omega = S^L$ a configuration $\eta$ of the particle system. Informally speaking, in one dimension particles try to jump to the right with rate $r = wq$ and to the left with rate $\ell = wq^{-1}$. The jump attempt is successful if the target site is empty, otherwise the jump attempt is rejected. The invariant measures of the ASEP with periodic boundary conditions are well-known: For fixed particle number $N$ these are the uniform measures. From these one can construct the grandcanonical Bernoulli product measures with fugacity $z = \rho/(1 - \rho)$ where $\rho = N/L$ is the particle density on the torus. For these
measures, where each lattice site \( k \) is occupied with probability \( \rho \) independently of all other sites, one has a stationary particle current \( j^* = (r - \ell)\rho(1 - \rho) \), corresponding to an expected mean time-integrated current \( \langle J(t)/t \rangle = j^* \).

In the context of macroscopic fluctuation theory \[8\] one is interested in conditioning the process on fluctuations around some atypical mean time-integrated current \( j \neq j^* \). A question of fundamental interest is then which macroscopic density profile is most likely to realize such a large deviation of the current inside a very large (more precisely: infinite) time interval of conditioning. This large-deviation problem thus concerns an untypical ensemble of trajectories of the process. This ensemble is usually not defined by \( j \) but via Legendre transformation in terms of the canonically conjugate driving strength \( s(j) \) with \( s(j^*) = 0 \). Interestingly, for conditioning on a lower-than-typical current (i.e., for \( s < 0 \)), it was found by Bodineau and Derrida \[9\] for the weakly asymmetric simple exclusion process that there is a dynamical phase transition: For currents slightly below the typical value \( j^* \) the optimal macroscopic profile is constant as it is for \( j^* \). However, below a critical threshold \( j_c < j^* \) (corresponding to some \( s_c < 0 \)) the optimal macroscopic profile is a travelling wave with a shape resembling a smoothened shock/antishock pair.

More recently, in a similar setting, but for finite duration \( t \) of conditioning, the microscopic structure of a travelling wave in the ASEP (not weakly!) was elucidated in detail for a specific choice of negative driving strength \[5\]. One considers a certain family of inhomogeneous product measures \( \mu_k \) indexed by a lattice site \( k \), where the microscopic density profile as function of the position on the lattice has a density-jump at position \( k \) on the torus, analogous to a shock on macroscopic scale. At time \( t = 0 \) \( N \) particles (\( N \) arbitrary) are distributed according to the restricted measure \( \mu_k^N \propto \delta_{\sum \eta(k),N} \). Then at any future time \( t > 0 \) of the conditioned dynamics the measure is a convex combination \( \mu_k^N(t) = \sum_l c(l,k,t|0) \mu_k^N \) of such measures. The weights \( c(l,k,t|0) \) are the transition probabilities of a single biased random walk, thus suggesting that a shock in a macroscopic travelling wave performs a biased random walk on microscopic level.

In this work we trace back the mathematical origin of this rigorous result to certain algebraic properties of the generator of the process. Then, using these properties, we go beyond \[5\] to derive a family of duality relations that allow us to construct more complex microscopic structures corresponding to more general macroscopic optimal profiles. The starting point is the well-known fact that for reflecting boundaries, where the process is reversible, the generator of the process commutes with the generators of the quantum algebra \( U_q[gl(2)] \). This fact has been used in \[31\] to construct the canonical reversible measures and in \[33\] to derive self-duality relations for the unconditioned ASEP.\[1\]

On the torus, however, the symmetry of the generator under \( U_q[gl(2)] \) breaks down. Nevertheless, some time ago Pasquier and Saleur \[30\] found intertwining relations involving the generators of \( U_q[gl(2)] \) and the Heisenberg quantum Hamil-
tonian with a boundary twist. This quantum Hamiltonian operator became later to be known to be closely related to the generator of the conditioned ASEP [18]. Here we present a new proof for the results of [33] (and correct some typos there) for reflecting boundaries and make use of intertwining relations of [30] (correcting some typos also in that paper) to derive an infinite discrete family of duality relations for the ASEP with periodic boundary conditions. These new duality relations apply to the process conditioned on fluctuations around some untypically low mean time-integrated current.

The simplest of these duality relations proves that the homogeneous Bernoulli measure is the invariant measure for the unconditioned ASEP with periodic boundary conditions. The derivation of this well-known fact from the $U_q[\mathfrak{gl}(2)]$-symmetry of the process with reflecting boundary questions is remarkable in so far as it raises the interesting question whether one can construct the matrix product measures [20, 19] of the periodic multi-species ASEP from the $U_q[\mathfrak{gl}(n)]$-symmetry of that process with reflecting boundaries [1, 6].

From the non-trivial higher order duality relations we obtain an infinite discrete family of new microscopic “travelling waves” for the conditioned process.

2 Definitions and notation

It is convenient to work with the quantum Hamiltonian formalism [28, 34] where the generator of the process is represented by a matrix which in a judiciously chosen basis turns out to be closely related to the Hamiltonian operator of a physical quantum system. We first introduce some notation and then describe in some detail the tools required for the quantum Hamiltonian formalism for the benefit of readers not familiar with this approach.

2.1 State space and configurations

We say that a site $k \in \Lambda$ is occupied by a particle if $\eta(k) = 1$ or that it is empty if $\eta(k) = 0$. The fact that a site can be occupied by at most one particle is the exclusion principle. Occasionally we denote configurations with a fixed number of $N$ particles by $\eta_N$. The set of all configurations with $N$ particles is denoted $\Omega_N$. We also define

$$\nu(k) := 1 - \eta(k)$$

and the particle numbers

$$N(\eta) = \sum_{k=1}^{L} \eta(k), \quad V(\eta) = \sum_{k=1}^{L} \nu(k) = L - N.$$
A useful alternative way of presenting uniquely of configuration $\eta_N$ is obtained by labelling the particles consecutively from left to right (clockwise) by 1 to $N$ and their positions on $\Lambda$ by $x_i \mod L$. A configuration $\eta$ is then represented by the set $x := \{x : \eta(x) = 1\}$. We call this notation the position representation. We shall use interchangeably the arguments $\eta$, $x$, for functions of the configurations. When the argument is clear from context it may be omitted. We note the trivial, but frequently used identities $N(\eta) \equiv N(x) = |x|$ and

$$\eta(k) = \sum_{i=1}^{N(x)} \delta_{x_i,k}. \quad (3)$$

For a configuration $\eta \equiv x$ we also define the number $N_k(\eta)$ of $A$-particles to the left of a particle at site $k$

$$N_k(\eta) := \sum_{i=1}^{k-1} \eta(i) = \sum_{i=1}^{N(\eta) - 1} \sum_{l=1}^{k} \delta_{x_l,i}. \quad (4)$$

Furthermore, for $1 \leq k \leq L - 1$ we define the local permutation

$$\pi^{kk+1}(\eta) = \{\eta(1), \ldots, \eta(k-1), \eta(k+1), \eta(k), \eta(k+2), \ldots, \eta(L)\} =: \eta^{kk+1}, \quad (5)$$

and for $k = L$ we define

$$\pi^{L1}(\eta) = \{\eta(L), \ldots, \eta(k), \ldots, \eta(1)\} =: \eta^{L1}. \quad (6)$$

The space reflection is defined by

$$R(\eta) = \{\eta(L), \eta(L-1), \ldots, \eta(1)\} \quad (7)$$

corresponding to $R(\eta(k)) = \eta(L+1-k)$ for the occupation numbers.

### 2.2 Definition of the ASEP

For functions $f : \mathbb{S}^L \to \mathbb{C}$ the ASEP $\eta_t$ with periodic boundary conditions and hopping asymmetry $q$ is defined by the generator

$$\mathcal{L} f(\eta) := \sum_{\eta' \in \mathbb{S}^L} w(\eta \to \eta') [f(\eta') - f(\eta)] \quad (8)$$

where the transition rates between configurations

$$w(\eta \to \eta') = \sum_{k=1}^{L} w^{kk+1}(\eta) \delta_{\eta', \eta^{kk+1}} \quad (9)$$

are defined in terms of the local hopping rates.
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\[ w^{k+1}(\eta) = w[q\eta(k)u(k+1) + q^{-1}u(k)\eta(k+1)]. \]  \hspace{1cm} (10)

The prime at the summation symbol \( \varepsilon \) indicates the absence of the term \( \eta' = \eta \) which is omitted since \( w(\eta \to \eta) \) is not defined. The transition rates are non-zero only for a transition from a configuration \( \eta \) to a configuration \( \eta' = \eta^{k+1} \) defined by \( (5) \).

We shall assume partially asymmetric hopping \( q \neq 0, 1, \infty \). The constant \( w \neq 0 \) sets the time scale of the process. On the torus we identify increasing order of the lattice index with the clockwise direction. In the case of reflecting boundary conditions no jumps from site 1 to the left and no jumps from site \( L \) to the right are allowed. Increasing order of the lattice index is identified with the direction left to right. The upper summation limit \( L \) in \( (9) \) has to be replaced by \( L - 1 \), giving rise to a generator that we denote by \( \mathcal{L} \).

In order study fluctuations around some untypical integrated current, parameterized in terms of the driving strength \( s \), we define the weighted transition rates

\[ w_{\sigma}^{kk+1}(\eta) = w[q\sigma^s\eta(k)u(k+1) + q^{-1}e^{-s}u(k)\eta(k+1)], \hspace{1cm} 1 \leq k \leq L - 1 \]  \hspace{1cm} (11)

\[ w_{\sigma}^{L}(\eta) = w[q\sigma^s\eta(L)u(1) + q^{-1}e^{-s}u(L)\eta(1)], \hspace{1cm} k = L. \]  \hspace{1cm} (12)

This leads us to define the weighted generators

\[ \mathcal{L}_s f(\eta) := \sum_{k=1}^{L-1} w_{\sigma}^{kk+1}(\eta)f(\eta^{kk+1}) - w_{\sigma}^{kk+1}(\eta)f(\eta) \]  \hspace{1cm} (13)

\[ \mathcal{L}_s f(\eta) := \sum_{k=1}^{L} w_{\sigma}^{L}(\eta)f(\eta^{L}) - w_{\sigma}^{L}(\eta)f(\eta). \]  \hspace{1cm} (14)

The weighted generators give a weight \( e^s \) (\( e^{-s} \)) to each particle jump to the right (left) anywhere on the lattice and for the process with periodic boundary conditions an extra weight \( e^s \) (\( e^{-s} \)) to each particle jump to the right (left) across bond \((L, 1)\). Thus each random trajectory of the process is given a weight \( e^{tJ(t) + sL_\delta(t)} \) where \( J(t) \) is the time-integrated total current, i.e., the total number of all particle jumps to the right up to time \( t \) minus the total number of all particle jumps to the left up to time \( t \) and \( J_\delta(t) \) is the time-integrated current across bond \((L, 1)\). Notice that the diagonal part of the weighted generator does not depend on the driving strength \( s \) or \( \delta \), reflecting the fact that the random times after which jumps occur remain unchanged. For details on this construction see e.g., \([22, 23, 16] \) and specifically for the present context \([35] \).

We fix more notation and summarize some well-known basic facts from the theory of Markov processes. For a probability distribution \( P(\eta) \) we denote the expectation of a continuous function \( f(\eta) \) by \( \langle f \rangle_P := \sum_{\eta} f(\eta)P(\eta) \). The transposed generator is defined by \( \mathcal{L}^T f(\eta) := \sum_{\eta', \eta'' \in \mathcal{S}^L} f(\eta')\mathcal{L}_{\eta''} \mathcal{1}_{\eta'}(\eta) \) where \( \mathcal{1}_{\eta'}(\eta) = \delta_{\eta, \eta'} \). With this definition \( (8) \) yields for a probability distribution \( P(\eta) \) the master equation

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\( ^2 \) When the summation is over \( \Omega = \mathcal{S}^L \) we shall usually omit the set \( \mathcal{S}^L \) under the summation symbol and simply write \( \sum_{\eta}. \)
\[ \mathcal{L}^T P(\eta) = \sum_{\eta'} [w(\eta' \to \eta) P(\eta') - w(\eta \to \eta') P(\eta)]. \] (15)

An invariant measure is denoted \( \pi^*(\eta) \) and defined by

\[ \mathcal{L}^T \pi^*(\eta) = 0 \] (16)

and the normalization \( \sum_{\eta} \pi^*(\eta) = 1 \). An unnormalized measure with the property (16) is denoted \( \pi(\eta) \). The time-reversed process is defined by

\[ \mathcal{L}^{rev} f(\eta) := \sum_{\eta'} w^{rev}(\eta \to \eta') [f(\eta') - f(\eta)] \] (17)

with \( w^{rev}(\eta \to \eta') = w(\eta' \to \eta) \pi(\eta') / \pi(\eta) \). The process is reversible if \( \mathcal{L}^{rev} = \mathcal{L} \) which means that the rates satisfy the detailed balance condition \( \pi(\eta) w(\eta \to \eta') = w(\eta' \to \eta) \pi(\eta') \). A probability distribution satisfying the detailed balance condition is a reversible measure. It is easily verified that the ASEP with reflecting boundary conditions is reversible with reversible measure

\[ \pi(\eta) = q \sum_{k=1}^{\infty} \left( \frac{\mu}{2} + \mu \right)^k \eta^k = e^{\mu N(\eta)} q^2 \sum_{i=1}^{N(\eta)} x_i. \] (18)

for any \( \mu \in \mathbb{R} \).

We define the transition matrix \( H \) of the process by the matrix elements

\[ H_{\eta' \eta} = \begin{cases} -w(\eta \to \eta') & \eta \neq \eta' \\ \sum_{\eta'} w(\eta \to \eta') & \eta = \eta'. \end{cases} \] (19)

with \( w(\eta \to \eta') \) given by (9). One has

\[ \mathcal{L} f(\eta) = -\sum_{\eta'} f(\eta') H_{\eta' \eta}, \quad \mathcal{L}^T P(\eta) = -\sum_{\eta'} H_{\eta' \eta} P(\eta'). \] (20)

Notice that here the sum includes the term \( \eta' = \eta \). In slight abuse of language we shall also call \( H \) the generator of the process. Analogously we also define the weighted transition matrix where the off-diagonal elements are replaced by the weighted rates (11), (12).

For an unnormalized stationary distribution we define the diagonal matrix \( \hat{\pi} \) with the stationary weights \( \pi(\eta) \) on the diagonal. For ergodic processes with finite state space one has \( 0 < \pi(\eta) < \infty \) for all \( \eta \). In terms of this diagonal matrix we can write the generator of the reversed dynamics as \( H^{rev} = \hat{\pi} H^T \hat{\pi}^{-1} \). The reversibility condition \( H^{rev} = H \) then reads

\[ \hat{\pi}^{-1} H \hat{\pi} = H^T. \] (21)

Therefore, if one finds a diagonal matrix with the property (21) then this matrix defines a reversible measure.
2.3 Representation of the generator in the natural tensor basis

In order to write the matrix $H$ explicitly we assign to each configuration $\eta$ a canonical basis vector $|\eta\rangle$. We choose the binary ordering $i(\eta) = 1 + \sum_{k=1}^{L} \eta(k)2^{k-1}$ of the basis. Defining single-site basis vectors of dimension 2

$$|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

one then has $|\eta\rangle = |\eta(1)\rangle \otimes \ldots \otimes |\eta(L)\rangle$ where $\otimes$ denotes the tensor product. These basis vectors span the complex vector space $(\mathbb{C}^2)^{\otimes L}$ of dimension $d = 2^L$. We also define transposed basis vectors $\langle \eta | := |\eta\rangle^T$ and the inner product $\langle v | w \rangle := \sum_\eta v(\eta)w(\eta)$.

Furthermore we define the two-by-two Pauli matrices

$$\sigma^x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the two-dimensional unit matrix $\mathbb{1}$. From these we construct

$$\sigma^\pm = \frac{1}{2}(\sigma^x \pm i\sigma^y), \quad \hat{n} = \frac{1}{2}(\mathbb{1} - \sigma^z), \quad \hat{v} = \frac{1}{2}(\mathbb{1} + \sigma^z).$$

(23)

These matrices satisfy the following relations:

$$\sigma^+ \sigma^- = \hat{v}, \quad \sigma^- \sigma^+ = \hat{n}, \quad \hat{n} \hat{v} = 0, \quad \hat{v} \hat{n} = 0$$

$$\sigma^+ \hat{n} = \sigma^+, \quad \hat{n} \sigma^+ = 0, \quad \sigma^+ \hat{v} = 0, \quad \sigma^+ \hat{v} = \sigma^+, \quad \sigma^- \hat{n} = 0, \quad \hat{n} \sigma^- = \sigma^-, \quad \sigma^- \hat{v} = \sigma^-, \quad \hat{v} \sigma^- = 0.$$

(25)

With the occupation variables (1) for a single site we have the projector property

$$\hat{n}|\eta\rangle = \eta|\eta\rangle, \quad \hat{n}|\eta\rangle = \eta|\eta\rangle.$$

(26)

Having in mind the action of these operators to the right on a column vector, we call $\sigma^-$ a creation operator, and $\sigma^+$ annihilation operators. When acting to the left on a bra-vector the roles are interchanged: $\sigma^- \sigma^+$ acts as creation operator and $\sigma^- \sigma^+$ as annihilation operator.

For $L > 1$ and any linear combination $u$ of these matrices we define the tensor operators $u_k := \mathbb{1}^{\otimes k-1} \otimes u \otimes \mathbb{1}^{\otimes L-k}$. By convention the zero'th tensor power of any matrix is the $c$-number 1 and $u^{\otimes 1} = u$. We note that also the tensor occupation operators $\hat{n}_k$ act as projectors

$$\hat{n}_k|\eta\rangle = \delta_{k,N(\eta)}|\eta\rangle = \sum_{i=1}^{N(\eta)} \delta_{x_i,k}|\eta\rangle,$$

(27)

with the occupation variables $\eta(k)$ or particle coordinates $x_i$ respectively understood as functions of $\eta$. The proof is trivial: The first equality is inherited from (26).
by multilinearity of the tensor product, the second equality follows from (3). Multilinearity of the tensor product also yields $u_k v_{k+1} = 1 \otimes (k-1) \otimes (u \otimes 1) (1 \otimes v) \otimes 1 \otimes (L-k-1) = 1 \otimes (k-1) \otimes (u \otimes v) \otimes 1 \otimes (L-k-1)$ and the commutator property $u_k v_j = v_j u_k$ for $k \neq l$. For $k = l$ one has relations analogous to (25).

It turns out to be convenient to introduce parameters $\alpha = q\epsilon$ and $\beta = \epsilon^2$ and express for periodic boundary conditions the weighted generator as $H(q, \alpha, \beta)$ with the convention $H(q, q, 1) = H$ for the unweighted generator. Similarly one writes $\tilde{H}(q, \alpha)$ for the weighted generator with reflecting boundary conditions with the convention $\tilde{H}(q, q) = R$. With these definitions the weighted generators $\tilde{H}(q, \alpha)$ and $H(q, \alpha, \beta)$ defined by (15) and (14) resp.

become

$$\tilde{H}(q, \alpha) = \sum_{k=1}^{L-1} h_{k,k+1}(q, \alpha)$$

$$H(q, \alpha, \beta) = \tilde{H}(q, \alpha) + h_{L,1}(q, \alpha, \beta)$$

with the hopping matrices

$$h_{k,k+1}(q, \alpha) = -w \left[ \alpha \sigma^+_k \sigma^-_{k+1} - q \hat{n}_k \hat{n}_{k+1} + \alpha^{-1} \sigma^-_k \sigma^+_{k+1} - q^{-1} \hat{n}_k \hat{n}_{k+1} \right]$$

and

$$h_{L,1}(q, \alpha, \beta) = -w \left[ \alpha \beta \sigma^+_k \sigma^-_{k+1} - q \hat{n}_k \hat{n}_{k+1} + (\alpha \beta)^{-1} \sigma^-_k \sigma^+_{k+1} - q^{-1} \hat{n}_k \hat{n}_{k+1} \right].$$

It is useful to introduce the space-reflection operator $\hat{R}$ defined by

$$\hat{R} u_k \hat{R}^{-1} = u_{L+1-k}$$

for local one-site operators $u_k$ and the diagonal transformations

$$V(\gamma) = \gamma^{\frac{1}{2}} \sum_{k=1}^{L} \sigma^z_k \gamma^{-\frac{1}{2}} \sum_{k=1}^{L} \sigma^z_k h_k$$

$$W(z) = z^{\hat{N}}$$

with the number operator $\hat{N} = \sum_{k=1}^{L} \hat{n}_k$. We note the properties

$$\hat{R} V(\gamma) \hat{R}^{-1} = V^{-1}(\gamma) = V(\gamma^{-1}),$$

$$\hat{R} W(z) \hat{R}^{-1} = W(z) = W^{-1}(z^{-1}),$$

$$\hat{R} H(q, \alpha, \beta) \hat{R}^{-1} = H(q, \alpha^{-1}, \beta^{-1})$$

$$H^T(q, \alpha, \beta) = H(q, \alpha^{-1}, \beta^{-1})$$

$$W H(q, \alpha, \beta) W^{-1} = H(q, \alpha, \beta).$$

Moreover, the transformation property

$$V(\gamma) \sigma^z_k V^{-1}(\gamma) = \gamma^{\frac{1}{2}} \sum_{k=1}^{L} \sigma^z_k$$

yields
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\[ V(\gamma) H(q, \alpha, \beta) V^{-1}(\gamma) = H(q, \alpha \gamma^{-1}, \beta \gamma^L) \]  
\[ V(\gamma) \tilde{H}(q, \alpha) V^{-1}(\gamma) = \tilde{H}(q, \alpha \gamma^{-1}). \]

Thus for periodic boundary conditions global conditioning and local conditioning are related by a similarity transformation, while for reflecting boundary conditions the conditioning can be completely absorbed into a similarity transformation. One also finds with \( \gamma = q^2, \alpha = q \) the reversibility relation

\[ V(q^2) \tilde{H}(q) V^{-1}(q^2) = \tilde{H}(q). \]  

For driving strength \( s_0 := -\ln(q) \) corresponding to \( \alpha = 1 \) one finds for \( 1 \leq k \leq L - 1 \)

\[ h_{k,k+1}(q,1) = -\frac{w}{2} \left[ \sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \Delta(\sigma_k^z \sigma_{k+1}^z - 1) + h(\sigma_k^z - \sigma_{k+1}^z) \right] \]  
with

\[ \Delta = \frac{1}{2}(q + q^{-1}), \quad h = \frac{1}{2}(q - q^{-1}) \]

and the unit-matrix \( \mathbf{1} \) of dimension \( 2^L \). Notice that for periodic boundary conditions the local divergence terms \( \sigma_k^z - \sigma_{k+1}^z \) cancel. For reflecting boundaries the local divergence term contributes opposite boundary fields \( h(\sigma_L^z - \sigma_1^z) \). With the further choice \( \bar{s}_0 = 0 \) corresponding to \( \beta = 1 \) the weighted generator \( \tilde{H} \) becomes the Hamiltonian operator \( H(q,1,1) \) of the ferromagnetic Heisenberg spin-1/2 quantum chain, while for \( \beta \neq 1 \) one has the Heisenberg chain \( H(q,1,\beta) \) with twisted boundary conditions.

Since particle number is conserved the process is trivially reducible. For each particle number \( N \) one has an irreducible process \( \eta_{N,t} \) on the state space \( \Omega_N \). We define the projector

\[ \hat{1}_N := \sum_{\eta \in \Omega_N} |\eta\rangle \langle \eta| \]

where we have used the quantum mechanical ket-bra convention \( |\eta\rangle \langle \eta| \equiv |\eta\rangle \otimes \langle \eta| \) for the tensor product of two vectors. Thus one obtains the generator

\[ H_N(q,\alpha,\beta) := \hat{1}_N H(q,\alpha,\beta) \hat{1}_N \]

for the \( N \)-particle weighted ASEP.

Notice that \( \hat{1}_N \) acts as unit matrix on the irreducible subspace corresponding to particle number \( N \). The unit matrix \( \mathbf{1} \) in the full space has the useful representation

\[ \mathbf{1} = \sum_{\eta \in \Omega} |\eta\rangle \langle \eta|. \]

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3. This is equivalent to Eq. (2.14) in \cite{33}, which, however, has a sign error and should read \( H^T = V^{-2}HV^2 \).
2.4 The quantum algebra $U_q[gl(2)]$

The quantum algebra $U_q[gl(2)]$ is the $q$-deformed universal enveloping algebra of the Lie algebra $gl(2)$. This associative algebra over $\mathbb{C}$ is generated by $L_i^\pm$, $i = 1, 2$ and $S^\pm$ with the relations

\begin{align*}
\{L_i, L_j\} &= 0 \\
L_i S^\pm &= q^{\pm(i,\pm,\pm,\pm,\pm,\pm,\pm,\pm)} S^\pm L_i \\
[S^+, S^-] &= \frac{(L_2 L_1^{-1})^2 - (L_2 L_1^{-1})^{-2}}{q - q^{-1}}
\end{align*}

(48) \hspace{1cm} (49) \hspace{1cm} (50)

Notice the replacement $q^2 \to q$ that we made in the definitions of $[11]$. It is convenient to work also with the subalgebra $U_q[sl(2)]$. We introduce the generators $N$ and $V$ via $q^{N/2} = L_1$, $q^{V/2} = L_2$ and define

$$S^z = \frac{1}{2}(N - V)$$

(51)

and the identity $I$. Then the quantum algebra $U_q[sl(2)]$ is the subalgebra generated by $q^{\pm S^z}$ and $S^\pm$ with relations

\begin{align*}
q^{S^z} q^{-S^z} &= q^{-S^z} q^{S^z} = I \\
q^{S^z} S^\pm q^{-S^z} &= q^{S^z} S^\pm \\
[S^+, S^-] &= \frac{q^{2S^z} - q^{-2S^z}}{q - q^{-1}}
\end{align*}

(52) \hspace{1cm} (53) \hspace{1cm} (54)

Observing that $N + V$ belongs to the center of $U_q[gl(2)]$, one sees that $U_q[sl(2)]$ is a subalgebra of $U_q[gl(2)]$.

It is trivial to verify that $U_q[gl(2)]$ has the two-dimensional fundamental representation $S^\pm \to \sigma^\pm$, $N \to \hat{n}$, $V \to \hat{v}$ given by the matrices. Then $\sigma^\pm$ and $\sigma^z/2$ form the two-dimensional fundamental representation of $U_q[sl(2)]$. Reducible higher-dimensional representations can be constructed using the coproduct

\begin{align*}
\Delta(S^\pm) &= S^\pm \otimes q^{-S^z} + q^{S^z} \otimes S^\pm \\
\Delta(S^z) &= S^z \otimes 1 + 1 \otimes S^z.
\end{align*}

(55) \hspace{1cm} (56)

By repeatedly applying the coproduct to the fundamental representation we obtain

$$S^\pm(k) = q^{\frac{1}{2} \sum_{j=1}^{k} \sigma^+_j} \sigma^+_k$$

$$S^z(k) = \frac{1}{2} \sigma^z_k.$$

(57) \hspace{1cm} (58)

One has
\[ S^\pm(k)S^\pm(l) = \begin{cases} q^{l-k} S^\pm(l)S^\pm(k) & k > l \\ 0 & k = l \\ q^{k-l} S^\pm(l)S^\pm(k) & k < l \end{cases} \quad (59) \]

Thus the spatial order in which particles are created (or annihilated) by applying the operators \( S^\pm(k) \) gives rise to combinatorial issues when building many-particle configurations from the reference state corresponding to the empty lattice.

From the coproduct one obtains the tensor representations of \( U_q[\mathfrak{sl}(2)] \), denoted by capital letters,

\[ S^\pm = \sum_{k=1}^L S^\pm(k), \quad S^\pm = \sum_{k=1}^L S^\mp(k). \quad (60) \]

For the full quantum algebra \( U_q[\mathfrak{gl}(2)] \) the tensor generators are \( S^\pm \) and \( \hat{N} = \sum_{k=1}^L \hat{r}_k + \hat{\nu}_k. \) The unit \( I \) is represented by the 2\( L \)-dimensional unit matrix \( I := 1^\otimes L \).

For reflecting boundary conditions the Heisenberg Hamiltonian \( \hat{H}(q, 1) \) is symmetric under the action of \( \hat{H}(q, 1) \) \( \otimes \) \( \mathfrak{sl}(2) \). This symmetry property is the origin of the duality relations derived in (53) and will also be used extensively below. In fact, for \( 1 \leq k \leq L - 1 \) one has \([h_{k,k+1}(q, 1), S^\pm] = [h_{k,k+1}(q, \alpha), S^\mp] = 0\), which imply

\[ [\hat{H}(q, 1), S^\pm] = 0 \quad (61) \]

and, equivalently to (59), the diagonal symmetries \([\hat{H}(q, \alpha), \hat{N}] = [\hat{H}(q, \alpha), \hat{V}] = 0\), thus giving rise to the \( U_q[\mathfrak{sl}(2)] \) symmetry of \( \hat{H}(q, 1) \).

We stress that \([h_{L,1}(q, 1), S^\pm] \neq 0\). One the other hand \([h_{L,1}(q, \alpha, \beta), S^\mp] = 0\).

Hence for periodic boundary conditions the symmetry breaks down to only a residual \( U(1) \) symmetry \([H_{(q, \alpha, \beta), S^\mp}] = 0\) generated by \( S^\mp \), which corresponds to particle number conservation since the \( z \)-component of the total spin \( S^z \) is related to the particle number operator \( \hat{N} \) through \( S^z = L/2 - \hat{N} \).

We also define

\[ S^\pm(q, \alpha) = \sum_{k=1}^L S^\pm_k(q, \alpha) \quad (62) \]

where

\[ S^\pm_k(q, \alpha) = \alpha^{\pm \frac{(L+1-2k)}{2}} q^{\pm \frac{1}{2} \sum_{i=1}^{L-k+1} \sigma_i^+ \sum_{i=1}^{L-k+1} \sigma_i^-}. \quad (63) \]

The diagonal transformation (40) and the defining relation (53) yield

\[ V(\gamma)S^\pm(q, \alpha)V^{-1}(\gamma) = S^\pm(q, \alpha \gamma^{-1}) \quad (64) \]

\[ W(z)S^\pm(q, \alpha)W^{-1}(z) = z^{\pm 1} S^\pm(q, \alpha). \quad (65) \]

Notice that \( S^\pm(q, 1) = S^\pm \) as defined in (60). Hence \( S^\pm(q, \alpha) \) and \( S^\pm \) also form a representation of \( U_q[\mathfrak{sl}(2)] \). Since according to (61) \( \hat{H}(q, 1) \) commutes with the generators \( S^\pm = S^\pm(q, 1) \) we conclude from (42) that \( \hat{H}(q, \alpha) \) commutes with \( S^\pm(q, \alpha) \), which together with \( S^\mp \) form an equivalent representation of \( U_q[\mathfrak{sl}(2)] \). In particular, the generator of the ASEP with reflecting boundary conditions \( \hat{H} = \hat{H}(q, q) \) commutes with \( S^\pm := S^\pm(q, q) \).

We note that
\[
(S^\pm (q, \alpha))^T = S^\pm (q, \alpha^{-1}) \\
\hat{R}S^\pm (q, \alpha)\hat{R}^{-1} = S^\pm (q^{-1}, \alpha^{-1}).
\] (66)

To prove the second equality one uses \(\hat{R}S^\pm (q, \alpha)\hat{R}^{-1} = S^\pm_{L+1-k}(q^{-1}, \alpha^{-1})\) which comes from (66).

Finally we introduce the symmetric \(q\)-number
\[
[x]_q := \frac{q^x - q^{-x}}{q - q^{-1}}
\] (68)

for \(q, q^{-1} \neq 0\) and \(x \in \mathbb{C}\). This definition can be applied straightforwardly to finite-dimensional matrices through the Taylor expansion of the exponential. For integers we also define the \(q\)-factorial
\[
[n]_q! := \begin{cases} 
1 & n = 0 \\
\prod_{k=1}^{n} [k]_q & n \geq 1.
\end{cases}
\] (69)

### 2.5 Duality in the quantum Hamiltonian formalism

For self-containedness we briefly review how to express expectation values using the matrix representation of the generator which allows to state the notion of duality in a neat matrix form \([37, 21]\).

A probability measure \(P(\eta)\) is represented by the column vector
\[
| P \rangle = \sum_\eta P(\eta) | \eta \rangle.
\] (70)

Next we define the summation vector
\[
| s \rangle := \sum_\eta | \eta \rangle
\] (71)

which is the row vector where all components are equal to 1. The expectation \(\langle f \rangle_P\) of a function \(f(\eta)\) with respect to a probability distribution \(P(\eta)\) is the inner product
\[
\langle f \rangle_P = \langle f | P \rangle = \langle s | \hat{f} | P \rangle
\] (72)

where
\[
\hat{f} := \sum_\eta f(\eta) | \eta \rangle \langle \eta |
\] (73)

is a diagonal matrix with diagonal elements \(f(\eta)\). Notice that
\[
f(\eta) = \langle \eta | \hat{f} | \eta \rangle = \langle s | \hat{f} | \eta \rangle.
\] (74)
One obtains the diagonal matrix \( \hat{f} \) corresponding to a function \( f(\eta) \) by substituting in \( f(\eta) \) the variable \( \eta(k) \) by the diagonal matrix \( \hat{n}_k \).

For a Markov process \( \eta_t \) the master equation (15) for a probability measure \( P(\eta_t) := \text{Prob}\{ \eta_t = \eta \} \) reads
\[
\frac{d}{dt} |P(t)\rangle = -H|P(t)\rangle
\]
which implies
\[
|P(t)\rangle = e^{-Ht}|P_0\rangle
\]
for an initial probability measure \( P_0(\eta) \equiv P(\eta_0) \) at time \( t = 0 \). We write the expectation of a function \( f(\eta_t) \) as
\[
\langle f(t) \rangle := \sum_{\eta} f(\eta)P(\eta_t) = \sum_{\eta} f(\eta)|\eta\rangle e^{-Ht}|P_0\rangle = \langle s|\hat{f}e^{-Ht}|P_0\rangle.
\]

If the initial distribution needs to be specified we use an upper index \( \langle f(t) \rangle^{P_0} \). Normalization implies \( \langle s|P(t)\rangle = 1 \) for all \( t \geq 0 \) and therefore \( \langle s|H = 0 \). A stationary distribution, denoted by \( \langle \pi^* \rangle \), is a right eigenvector of \( H \) with eigenvalue 0, i.e., \( H|\pi^*\rangle = 0 \) and normalization \( \langle s|\pi^*\rangle = 1 \). For the ergodic subspaces with fixed particle number \( N \) it is unique.

In order to introduce duality we consider a process \( \xi_t \) with generator \( H \) and a process \( x_t \) with generator \( G \) which may have different countable state spaces \( \Omega_A \) and \( \Omega_B \). Consider also a family of functions \( f^x : \Omega_A \to \mathbb{C} \) indexed by \( x \in \Omega_B \) and a family of functions \( g^\xi : \Omega_B \to \mathbb{C} \) indexed by \( \xi \in \Omega_A \) such that \( f^x(\xi) = g^\xi(x) =: D(x, \xi) \). Let the process \( \xi_t \) start at some fixed \( \xi \in \Omega_A \) and let \( x_t \) start at some fixed \( x \in \Omega_B \). Then the two processes are said to be dual with respect to the duality function \( D(x, \xi) \) if
\[
\langle f^x(t) \rangle = \langle g^\xi(t) \rangle^x.
\]
As pointed out in [21] this property can be stated neatly in terms of the generators as
\[
DH = G^TD
\]
where the duality matrix \( D \) is defined by
\[
D = \sum_{\xi \in \Omega_A} \sum_{x \in \Omega_B} D(x, \xi)|x\rangle\langle \xi |
\]
By construction one has \( D(x, \xi) = \langle x|D|\xi \rangle \).

### 2.6 Shock/Antishock measures

In vector notation a product measure with marginals \( \rho_k \) is a tensor vector \( |\{\rho_k\}\rangle = |\rho_1\rangle \otimes \ldots \otimes |\rho_L\rangle \) with the single-site column vectors \( |\rho_k\rangle = (1 - \rho_k, \rho_k)^T \). It is con-
venient to introduce the local fugacity
\[ z_k = \frac{\rho_k}{1 - \rho_k} \] (81)

and write the product measure in the form \( \{|\zeta_i\} = |z_1 \cdots z_L / Z_L \) with \( z_k = (1, z_k)^T \) and normalization \( Z_L = \prod_{k=1}^L [z_k / (1 + z_k)] \).

Specifically, we define for a set \( x \) of lattice sites with cardinality \( K = |x| \) the following family \( |\nu_x\rangle = |z_1 \cdots z_L / Z_L \) of shock/antishock measures, or SAM for short, in terms of the fugacities

\[ z_k = \begin{cases} 
q^{2l} & \text{for } x_l < k < x_{l+1}, \quad l \in \{0, 1, \ldots, K+1\} \\
\infty & \text{for } k \in x
\end{cases} \] (82)

with \( x_0 = 0 \) and \( x_{K+1} = L + 1 \). On coarse-grained scale with \( \xi = k/L \) and \( \xi^r_l = x_l / L \) the macroscopic density profile \( \rho(\xi) \) corresponding to the fugacities \( z_k \) has discontinuities at \( \xi = \xi^r_l \) with constant fugacity ratios \( z^+_l/z^-_l = q^2 \) where \( z^+_l = z_{\xi^r_l + 1} \).

In forward (clockwise) direction and for \( q > 1 \) this is an upward step, corresponding to a shock profile for the ASEP (with positive bias \( q > 1 \)). Between site \( L \) and site 1 there is downward jump with fugacity ratio \( q^{-2K} \). On macroscopic scale this constitutes an antishock at position \( \xi^a = \xi^r_l - (1+\kappa) \) mod 1, hence the term SAM. These shock measures are closely related to the shock measures defined in [4] and also to the infinite-volume shock measures studied in [3] where the shock positions \( x_i \) are occupied by second-class particles.

With a different normalization factor the general SAM (82) with constant fugacity jumps \( q^2 \) can be written as

\[ |\bar{\mu}_x\rangle := \prod_{j=1}^K z^{-1} q^{\sum_{j=1}^{x_j-1} \hat{n}_i + \sum_{j=x_j+1}^{L} \hat{\bar{n}}_i} |z\rangle \propto |\nu_x\rangle. \] (83)

Here
\[ |z\rangle := |z\rangle^\otimes L \] (84)

for \( K = 0 \) is the unnormalized homogeneous product measure corresponding to the Bernoulli product measure \( |\rho\rangle = |z\rangle / (1 + z)^L \) where \( \rho = z / (1 + z) \).

From the SAM defined by (83) and (84) we construct a second type of SAM’S using the transformations (83) and (84)

\[ |\mu_x\rangle := z^{-K V(q^{2K})} \prod_{j=1}^K W(q^{x_j - 1}) |\bar{\mu}_x\rangle \] (85)

\[ = \prod_{j=1}^K z^{-1} q^{\sum_{j=1}^{x_j-1} \hat{n}_i - \sum_{j=x_j+1}^{L} \hat{\bar{n}}_i} |z\rangle. \] (86)

We illustrate the definition for \( K = 1 \) and \( K = 2 \).

For \( K = 1 \) the SAM (83) reduces to
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\[ |\mu_x\rangle := z^{-1} q^2 \sum_{i=1}^{x} (x-i) \hat{h}_i - \sum_{i=x+1}^{L} \hat{h}_i + \sum_{i=x+1}^{L} \hat{p}_i |\hat{z}\rangle \]  

(87)

with \(1 \leq x \leq L\). This corresponds to local fugacities

\[ z_k = \begin{cases} 
q^2 \frac{2(2k-I+L)}{L} & \text{for } 1 \leq k < x \\
\infty & \text{for } k = x \\
q^2 \frac{2(2k-I-L)}{L} & \text{for } x < k \leq L 
\end{cases} \]

(88)

and therefore to densities

\[ \rho_k = \begin{cases} 
\frac{1}{2} \left[ 1 - \tanh \left( \frac{E}{L} (2k-x-y+L \frac{\kappa+2}{2}) \right) \right] & \text{for } 1 \leq k < x \\
\frac{1}{2} \left[ 1 - \tanh \left( \frac{E}{L} (2k-y-x+L \frac{\kappa+2}{2}) \right) \right] & \text{for } k = x \\
\frac{1}{2} \left[ 1 - \tanh \left( \frac{E}{L} (2k-x-y+L \frac{\kappa-2}{2}) \right) \right] & \text{for } x < k < y \\
\frac{1}{2} \left[ 1 - \tanh \left( \frac{E}{L} (2k-y-x+L \frac{\kappa-2}{2}) \right) \right] & \text{for } k = y \\
\frac{1}{2} \left[ 1 - \tanh \left( \frac{E}{L} (2k-y-x+L \frac{\kappa-2}{2}) \right) \right] & \text{for } y < k \leq L 
\end{cases} \]

(90)

where \(E = \ln q\) and \(\kappa = \ln z / E\) corresponding to \(z = q^\kappa\). These measures are closely related to the type-II shock measures defined in [5]. On coarse-grained scale with \(\xi = k / L\) and \(\xi^t = x / L\) the macroscopic density profile \(\rho(\xi)\) has a discontinuity at \(\xi = \xi^t\) with amplitude \(A^t := \rho^+ - \rho^- = \tanh (E(\kappa+1)/2) - \tanh (E(\kappa+1)/2)\), where \(\rho^+ = \lim_{\epsilon \to 0} \rho(\xi^t \pm \epsilon)\). In forward (clockwise) direction and for \(E > 0\) this is an upward step with fugacity ratio \(z^+ / z^- = q^2\), corresponding to a shock profile for the ASEP (with positive bias \(E > 0\)). For strong asymmetry \(E = \epsilon L\) one has near \(k = x - L(1 + \kappa)/2\) a smoothened downward “step” with an intrinsic width \(\approx 1 / \epsilon\) on lattice scale. On macroscopic scale this constitutes an antishock at position \(\xi^a = \xi^t - (1 + \kappa)/2 \mod 1\).

For \(K = 2\) the SAM \(|\mu_{x,y}\rangle\) with \(1 \leq x < y \leq L\) has local fugacities

\[ z_k = \begin{cases} 
q^2 \frac{2(2k-x-y+L)}{L} & \text{for } 1 \leq k < x \\
\infty & \text{for } k = x \\
q^2 \frac{2(2k-x-y-L)}{L} & \text{for } x < k \leq y \\
\infty & \text{for } k = y \\
q^2 \frac{2(2k-y-x-L)}{L} & \text{for } y < k \leq L 
\end{cases} \]

(90)

corresponding to densities

\[ \rho_k = \begin{cases} 
\frac{1}{2} \left[ 1 - \tanh \left( \frac{E}{L} (2k-x-y+L \frac{\kappa+2}{2}) \right) \right] & \text{for } 1 \leq k < x \\
\frac{1}{2} \left[ 1 - \tanh \left( \frac{E}{L} (2k-y-x+L \frac{\kappa+2}{2}) \right) \right] & \text{for } k = x \\
\frac{1}{2} \left[ 1 - \tanh \left( \frac{E}{L} (2k-x-y+L \frac{\kappa-2}{2}) \right) \right] & \text{for } x < k < y \\
\frac{1}{2} \left[ 1 - \tanh \left( \frac{E}{L} (2k-y-x+L \frac{\kappa-2}{2}) \right) \right] & \text{for } k = y \\
\frac{1}{2} \left[ 1 - \tanh \left( \frac{E}{L} (2k-y-x+L \frac{\kappa-2}{2}) \right) \right] & \text{for } y < k \leq L 
\end{cases} \]

(91)

On macroscopic scale this density profile has two shock discontinuities at \(\xi^1 = x / L \mod 1\) and \(\xi^2 = y / L \mod 1\). Both fugacity ratios are of magnitude \(q^2\). For strong asymmetry \(E = \epsilon L\) there are two antishocks at \(\xi^1 = (\xi^1 + \xi^2)/2 - (\kappa+2)/4 \mod 1\) and \(\xi^2 = (\xi^1 + \xi^2)/2 - (\kappa+2)/4 \mod 1\).
3 Results

Before stating the new results we recall the duality relation for the ASEP with reflecting boundary conditions derived in [33], Eq. (3.12). We reformulate this duality relation slightly and correct a sign error in Eq. (3.12) of [33]. We also give a new proof, parts of which are then used to prove the new results given below. We also present a generalized and slightly reformulated version of the intertwiner relation Eq. (2.62b) of [30] for periodic boundary conditions, also with a correction of some sign errors in that formula.

Theorem 1. (Schütz, [33]) The ASEP with reflecting boundary conditions and asymmetry parameter \( q \) is self-dual w.r.t. the duality function

\[
D(x, \eta) = \prod_{j=1}^{\lfloor x \rfloor} q^{-2x_j} Q_{x_j}(\eta)
\]

where

\[
Q_{x_j}(\eta) = q^{x_j-1} \eta(i) - q^{x_j+1} \eta(i) \eta(x_j).
\]

Remark 1. Because of particle number conservation also

\[
\tilde{D}(x, \eta) = q^{\lfloor x \rfloor(N(\eta) - 1)} D(x, \eta) = \prod_{j=1}^{\lfloor x \rfloor} q^{2N_{x_j}(\eta) - 2x_j} \eta(x_j)
\]

is a duality function with the particle numbers \( N(\eta) \) (2) and \( N_{x}(\eta) \) (4). This is the duality function (3.12) of [33].

Proposition 1. Let \( H(\cdot, \cdot, \cdot) \) be the conditioned generator (29) of the ASEP with periodic boundary conditions and let \( \eta_K \in \Omega_K \) be any configuration with \( K \) particles. Then for \( 0 \leq n \leq L - K \) one has the intertwining relation

\[
\left[ (S^{\pm}(q, \alpha))^n H(q, \alpha, q^{2n} \beta^{\pm}) - H(q, \alpha, \beta^{\pm})(S^{\pm}(q, \alpha))^n \right] | \eta_K \rangle = 0
\]

with

\[
\beta^{\pm} = q^{\pm(L-2K)} \alpha^{-L}
\]

and the generators \( S^{\pm}(q, \alpha) \) (62) of \( U_q[\mathfrak{gl}(2)] \).

Remark 2. Defining the duality matrix \( D_K^{\pm,n} = 1_{K+n}(S^{\pm}(q, \alpha))^n 1_K \) with the projector (45) and using (38) the intertwiner relation (95) can be expressed as the duality relation

\[
D_K^{\pm,n} H_K(q, \alpha, q^{2n} \beta^{\pm}) = (H_{K+n}(q, \alpha^{-1}, \beta^{\pm}))^T D_K^{\pm,n}
\]

with the projected generator (46). We shall focus on the formulation (95) of this duality.

\[\text{Notice a sign error in front of the term } 2k_i \text{ in Eq. (3.12) of [33] and pay attention to the different convention } q \leftrightarrow q^{-1}.\]
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Remark 3. For $\alpha = 1$ this is the result (2.62b) of \[30\]. The proof of Proposition \[1\] is entirely analogous to the derivation given in \[30\] since the generalized form (2.55) follows trivially from the result of \[30\] for $\alpha = 1$ through the similarity transformation (3.3). Some ingredients of the proof, with sign errors in \[30\] corrected, are presented in the appendix.

We focus now on global conditioning ($\alpha \neq q, \beta = 1$) and local conditioning $\alpha = q, \beta \neq 1$. The main results of this work are the following theorems.

Theorem 2. Let $H_N^K := 1_N H(q, q^{1 - \frac{2K}{N}}, 1) 1_N$ be the generator (4.2) of the globally conditioned ASEP with $N$ particles and periodic boundary conditions and driving strength $s = -2K/L \ln q$. Furthermore, let $|\mu_N^x(\cdot)| := 1_N |\mu_x^x(\cdot)|$ be the unnormalized shock-antishock measure (5.5) restricted to $N$ particles and

$$|\mu_N^x(t)| := e^{-H_N^K t} |\mu_N^x(\cdot)|$$

with $K = |x|$. Then

$$|\mu_N^x(t)| = \sum_{y \in \Omega_K} P_N^x(y, t|x, 0) |\mu_N^y(\cdot)|$$

where $P_N^x(y, t|x, 0) := \langle y | e^{-H_N^K t} | x \rangle$ is the conditioned $K$-particle transition probability from $x$ to $y$ at time $t$ with driving strength $s' = -2N/L \ln q$.

Remark 4. The significance of this result lies in the fact that the conditioned evolution of an $N$-particle SAM is fully determined by the conditioned transition probability of only $K$ particles, in analogy to the evolution of shocks in the infinite lattice explored in \[4, 5\].

Remark 5. For $K = 1$ a related result was obtained in \[5\] for a normalized and slightly different definition of the shock measures. The proof of \[5\] is by explicit computations relying on the presence of a single shock. The present proof for the generalized $K \geq 1$ shows that the mathematical origin of the conditioned shock motion is the duality relation (2.55).

Theorem 3. Let $\tilde{H}_N^K := 1_N H(q, q, q^{2K}) 1_N$ be the generator (4.2) of the locally conditioned ASEP with $N$ particles and periodic boundary conditions and boundary driving strength $\bar{s} = -2K \ln q$. Furthermore, let $|\tilde{\mu}_N^x(\cdot)| := 1_N |\tilde{\mu}_x^x(\cdot)|$ be the unnormalized shock-antishock measure (5.2) restricted to $N$ particles and

$$|\tilde{\mu}_N^x(t)| := e^{-\tilde{H}_N^K t} |\tilde{\mu}_N^x(\cdot)|$$

with $K = |x|$. Then

$$|\tilde{\mu}_N^x(t)| = \sum_{y \in \Omega_K} \tilde{P}_N^x(y, t|x, 0) |\tilde{\mu}_N^y(\cdot)|$$

where $\tilde{P}_N^x(y, t|x, 0) := \langle y | e^{-\tilde{H}_N^K t} | x \rangle$ is the boundary-conditioned $K$-particle transition probability from $x$ to $y$ at time $t$ with driving strength $\bar{s}' = -2N \ln q$.

5 Eqs. (2.62a) and (2.62b) of \[30\] have some sign errors which are corrected in Proposition \[1\].
4 Proofs

4.1 Proof of Theorem

Proof. We first note

Lemma 1. Let

\[ \tilde{S} = \sum_{n=0}^{L} \tilde{S}_n, \quad \tilde{Q}_x = q^{\sum_{i=1}^{L} n_i - \sum_{i=1}^{L} r_i}. \]  \hspace{1cm} (102)

Then for a configuration \( x \in \Omega_N \) with \( N = |x| \) particles one has

\[ \langle x | \tilde{S} \rangle = \langle s | \prod_{i=1}^{N} \tilde{Q}_x \rangle. \] \hspace{1cm} (103)

The proof is completely analogous to the proof in [7] of (164) with \( y = \emptyset \).

Now we observe that with the reversible measure (18) and with (74) we can write

\[ D(x, \eta) = \pi^{-1}(x) \langle | s | \prod_{i=1}^{N} \tilde{Q}_x | \eta \rangle = f^\eta(x) = g^\eta(x). \] \hspace{1cm} (104)

Then the following chain of equalities holds and proves the theorem:

\[ \langle f^\eta(t) \rangle^\eta := \sum_{\xi} f^\eta(\xi) \langle | s | \prod_{i=1}^{N} \tilde{Q}_x | \xi \rangle \] \hspace{1cm} (105)

\[ = \sum_{\xi} \pi^{-1}(x) \langle | s | \prod_{i=1}^{N} \tilde{Q}_x | \xi \rangle \langle | s | \prod_{i=1}^{N} \tilde{Q}_x | \eta \rangle \] \hspace{1cm} (106)

\[ = \pi^{-1}(x) \langle | s | \prod_{i=1}^{N} \tilde{Q}_x e^{-\tilde{H}t} | \eta \rangle \] \hspace{1cm} (107)

\[ = \pi^{-1}(x) \langle | s | \tilde{S} e^{-\tilde{H}t} | \eta \rangle \] \hspace{1cm} (108)

\[ = \pi^{-1}(x) \langle | s | e^{-\tilde{H}t} \tilde{S} | \eta \rangle \] \hspace{1cm} (109)

\[ = \pi^{-1}(x) \sum_{y} \langle | s | e^{-\tilde{H}t} | y \rangle \langle | s | \tilde{Q}_x | \eta \rangle \] \hspace{1cm} (110)

\[ = \sum_{y \in \Omega_N} \langle | s | \pi^{-1} e^{-\tilde{H}t} | y \rangle \pi^{-1}(y) \langle | s | \prod_{i=1}^{N} \tilde{Q}_x | \eta \rangle \] \hspace{1cm} (111)

\[ = \sum_{y \in \Omega_N} \langle | s | \pi^{-1} e^{-\tilde{H}t} | y \rangle \pi^{-1}(y) \langle | s | \prod_{i=1}^{N} \tilde{Q}_x | \eta \rangle \] \hspace{1cm} (112)

\[ = \sum_{y \in \Omega_N} \pi^{-1}(y) \langle | s | e^{-\tilde{H}t} | y \rangle \] \hspace{1cm} (113)

\[ =: \langle g^\eta(t) \rangle^x \] \hspace{1cm} (114)
The following ingredients were used: Eqs. (105) and (113): The expressions (77) for expectations; Eqs. (106) and (118): The expression (112) for the duality function; Eq. (107): The expressions (77) and the representation (47) of the unit matrix of dimension $2^L$; Eq. (108): The expression (103) of part of the duality function in terms of the symmetry operator $\hat{S}$; Eq. (109). The $U_q[\mathfrak{sl}(2)]$ symmetry (61). Eq. (110). Particle number conservation and the representation (33) of the unit matrix in the subspace of $N$ particles; Eq. (111). The diagonal matrix representation of the reversible measure (13). Eq. (112). Reversibility (21) \hfill $\square$

### 4.2 Proofs of Theorems 2 and 3

Before we set out to prove Theorem 2 and Theorem 3, we show that the SAM (85) can be generated by the action of the particle creation operator $U_q[\mathfrak{sl}(2)]$.

**Lemma 2.** Let $\mathbf{x}$ be a configuration of $K = |\mathbf{x}|$ particles and let

$$\bar{\mu}^N := \bar{\mu}_k \delta_{z, n, 1} \eta(k, N), \quad \mu^N := \mu_k \delta_{z, n, 1} \eta(k, N) \quad (115)$$

be the SAM's defined by (82), (85) restricted to $N \geq K$ particles. Then the vector representations $|\bar{\mu}^N \rangle := 1_N |\bar{\mu}_k \rangle$ and $|\mu^N \rangle := 1_N |\mu_k \rangle$ can be written as

$$|\bar{\mu}^N \rangle = z^{N-K} \left( S^{-1}(q^{-1}, q) \right)^{N-K} |\mathbf{x} \rangle \quad (116)$$

$$|\mu^N \rangle = z^{N-K} V\left( q^{-1} \frac{\gamma \eta(q^{-1})}{q} \right)^{N-K} |\mathbf{x} \rangle \quad (117)$$

in terms of the generators (62) of $U_q[\mathfrak{sl}(2)]$ and the transformation (63).

**Proof.** From Lemma 1 and (66) one finds

$$\sum_{n=0}^{L-K} \frac{(S^{-1}(q^{-1}, q))^n}{[n]_q!} |\mathbf{x} \rangle = \prod_{j=1}^{K} \left( q^{z_{i+1}^{-1}} \gamma_1 + \gamma_{y_i} \right) |\mathbf{x} \rangle \quad (118)$$

Notice that for $z = 1$ one has $|z = 1 \rangle = |s \rangle$.

The transformation (63) yields $V(\gamma) |\mathbf{x} \rangle = \gamma^{z \Sigma_{j=1}^{K} (2x_j - L - 1)} |\mathbf{x} \rangle$ and (64) gives $S^{-1}(q^{-1}, q) = V^{-1}(\lambda) S^{-1}(q^{-1}, q\lambda^{-1}) V(\lambda)$. Putting this together and using (65) turns (118) into

$$V(\gamma) \sum_{n=0}^{L-K} \frac{(S^{-1}(q^{-1}, q\lambda^{-1}))^n}{[n]_q!} |\mathbf{x} \rangle = V(\gamma \lambda) \prod_{j=1}^{K} \left( \lambda^{z^{-1} \Sigma_{j=1}^{K} (2x_j - L - 1) q - \Sigma_{j=1}^{K} \hat{\gamma}_j + \Sigma_{j=1}^{K} \hat{\lambda} \hat{\gamma}_j \right) |\mathbf{z} \rangle \quad (119)$$
Now we choose $\lambda = q^{2N}$ and $\gamma = q^{2(N-K)}$ to obtain

$$V(q^{2(N-K)}) \sum_{n=0}^{t-K} z^n \left( S^{-}(q^{-1}, q^{1-2N}) \right)^n |x\rangle.$$ 

$$= \prod_{j=1}^{K} \frac{q^{2N} (2x_j - L - 1) - 1}{2} \sum_{n=0}^{t-K} z^n \left( S^{-}(q^{-1}, q^{1-2N}) \right)^n |x\rangle.$$ 

Finally one applies the projector $1_N$ on both sides of the equation. On the l.h.s. this projects out the term with $n = N - K$, corresponding to the r.h.s. of (117). On the r.h.s. the projection allows us to substitute the number $N$ in the first power of $q$ by the number operator $\hat{N}$ (34). Thus the terms proportional to $L + 1$ cancel and the expression (86) remains under the projection operator. Therefore the r.h.s. is equal to $|\mu^N\rangle$. Similarly one chooses $\gamma = \lambda = 1$ to obtain (116). \(\square\)

### 4.2.1 Proof of Theorem 2

Now we are in a position to prove (99). Consider a $K$-particle configuration $x$ and the duality relation (95) with $n = N - K$ and $\beta = 1$:

$$H(q, q^{2N}) (S^{-}(q, q^{2N-1}))^{N-K} |x\rangle = H(q, q^{2N-1}, 1) \left( S^{-}(q, q^{2N-1}) \right)^{N-K} |x\rangle.$$ 

With the transformation (33) with $\gamma^L = q^{2K-2N}$ one uses (41) to cast this in the form

$$H(q, q^{2N-1}, 1) \left( S^{-}(q, q^{2N-1}) \right)^{N-K} |x\rangle = H(q, q^{2N-1}, 1) \left( S^{-}(q, q^{2N-1}) \right)^{N-K} |x\rangle.$$ 

or, alternatively,

$$H(q, q^{2N-1}, 1) V^{-1}(\gamma) \left( S^{-}(q, q^{2N-1}) \right)^{N-K} |x\rangle = V^{-1}(\gamma) \left( S^{-}(q, q^{2N-1}) \right)^{N-K} H(q, q^{2N-1}, 1) |x\rangle.$$ 

Applying (37), (67), (35) this turns into
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\[ H(q,q^{1-2\tilde{K}},1)V(\gamma)\frac{(S^{-}(q^{-1},q^{1-2\tilde{K}}))^N-H}{[N-K]q!}|\bar{x}\rangle \]

\[ = V(\gamma)\frac{(S^{-}(q^{-1},q^{1-2\tilde{K}}))^N-H}{[N-K]q!}H(q,q^{1-2\tilde{K}},1)|\bar{x}\rangle. \] (124)

where \(|\bar{x}\rangle = V^{-1}(\gamma)|x\rangle\) is an arbitrary \(K\)-particle configuration.

Since \(H\) conserves particle number, this relation remains valid for any power of \(H\). Thus we find

\[ e^{-H_N^\alpha}\frac{V(\gamma)(S^{-}(q^{-1},q^{1-2\tilde{K}}))^N-H}{[N-K]q!}|\bar{x}\rangle \]

\[ = V(\gamma)\frac{(S^{-}(q^{-1},q^{1-2\tilde{K}}))^N-H}{[N-K]q!}e^{-H_N^\alpha}|\bar{x}\rangle \] (125)

\[ = \sum_{\eta_K} V(\gamma)\frac{(S^{-}(q^{-1},q^{1-2\tilde{K}}))^N-H}{[N-K]q!}|\eta_K\rangle\langle \eta_K|e^{-H_N^\alpha}|\bar{x}\rangle \] (126)

where in the last equality we have inserted the unit operator restricted to \(K\)-particle states. Using (117) of Lemma 2 then proves (99). \(\square\)

4.2.2 Proof of Theorem 3

The proof of Theorem 3 is similar. For \(\alpha = q^{-1}\) where \(\beta = q^{2K}\) one has

\[ \frac{(S^{-}(q,q^{-1}))^N-H}{[N-K]q!}H(q,q^{-1},q^{2N})|x\rangle = H(q,q^{-1},q^{2K})\frac{(S^{-}(q,q^{-1}))^N-H}{[N-K]q!}|x\rangle. \] (127)

Applying space reflection (37), (67) this becomes

\[ \frac{(S^{-}(q^{-1},q))^N-H}{[N-K]q!}H(q,q,q^{-2N})|x\rangle = H(q,q,q^{-2K})\frac{(S^{-}(q^{-1},q))^N-H}{[N-K]q!}|x\rangle. \] (128)

Here we dropped the tilde over the configuration \(x\) since it is arbitrary.

Projecting on \(N\) particles and iterating this duality over powers of \(H_N^K\) yields

\[ e^{-H_N^\alpha}\frac{(S^{-}(q^{-1},q))^N-H}{[N-K]q!}|x\rangle \]

\[ = \frac{(S^{-}(q^{-1},q))^N-H}{[N-K]q!}e^{-H_N^\alpha}|x\rangle. \] (129)

Inserting the unit operator restricted to \(K\)-particle states and Using (116) of Lemma 2 then proves (101). \(\square\)

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Appendix

We present some details of the proof of Proposition (1) which are not shown in [30] and from which Proposition (1) follows by the similarity transformation (33).

We define \( e_L(\cdot, \cdot) := h_{L,1}(\cdot, \cdot) \), see (31). By explicit matrix multiplications one finds from the relations (25) for the bulk operators

\[
S^+_k (q, \alpha) e_L(\alpha', q', \beta q^{-2}) S^-_k (q, \alpha) = e_L(\alpha', q', \beta q^{-2}) S^+_k (q, \alpha) \quad 2 \leq k \leq L - 1
\]

(130)

\[
e_L(\alpha', q', \beta) S^+_k (q, \alpha) = S^+_k (q, \alpha) e_L(\alpha', q', \beta q^2) \quad 2 \leq k \leq L - 1
\]

(131)

and for the boundary operators

\[
S^+_1 (q, \alpha) e_L(\alpha', q', \beta) = q^{-1/2} \alpha^{1/2(L-1)} [(q')^{-1} \sigma_1^+ \hat{v}_L - \alpha' \beta \hat{v}_1 \sigma_1^+] q^{-S^+_L}
\]

(132)

\[
S^-_1 (q, \alpha) e_L(\alpha', q', \beta) = q^{1/2} \alpha^{-1/2(L-1)} [q' \sigma_1^- \hat{h}_L - (\alpha' \beta)^{-1} \hat{h}_1 \sigma_1^-] q^{-S^-_L}
\]

(133)

\[
S^+_1 (q, \alpha) e_L(\alpha', q', \beta) = q^{1/2} \alpha^{-1/2(L-1)} [q' \sigma_1^- \hat{v}_L - (\alpha' \beta)^{-1} \sigma_1^+ \hat{v}_1] q^{-S^+_L}
\]

(134)

\[
S^-_1 (q, \alpha) e_L(\alpha', q', \beta) = q^{-1/2} \alpha^{1/2(L-1)} [(q')^{-1} \sigma_1^+ \hat{h}_L - \alpha' \beta \sigma_1^- \hat{h}_1] q^{-S^-_L}\]

(135)

and

\[
e_L(\alpha', q', \beta) S^+_1 (q, \alpha) = q^{-1/2} \alpha^{1/2(L-1)} [q' \sigma_1^+ \hat{h}_L - \alpha' \beta \sigma_1^+] q^{-S^-_L} \]

(136)

\[
e_L(\alpha', q', \beta) S^-_1 (q, \alpha) = q^{1/2} \alpha^{-1/2(L-1)} [(q')^{-1} \sigma_1^- \hat{v}_L - (\alpha' \beta)^{-1} \sigma_1^+ \hat{v}_1] q^{-S^-_L} \]

(137)

\[
e_L(\alpha', q', \beta) S^+_1 (q, \alpha) = q^{1/2} \alpha^{-1/2(L-1)} [q' \sigma_1^- \hat{h}_L - (\alpha' \beta)^{-1} \sigma_1^+ \hat{h}_1] q^{-S^-_L} \]

(138)

\[
e_L(\alpha', q', \beta) S^-_1 (q, \alpha) = q^{-1/2} \alpha^{1/2(L-1)} [q' \sigma_1^+ \hat{v}_L - \alpha' \beta \sigma_1^- \hat{v}_1] q^{-S^-_L} \]

(139)

Consider now \( q = q' \) and \( \alpha = \alpha' \). From the quantum algebra symmetry and from the previous relations one obtains (omitting the \( q, \alpha \)-dependence)

\[
S^\pm H(\beta) - H(\beta') S^\pm = S^\pm e_L(\beta') - e_L(\beta') S^\pm
\]

(140)

\[
= \left[ e_L(q^{-2} \beta) - e_L(\beta') \right] \sum_{k=2}^{L-1} S^\pm_k + (S^\pm_1 + S^\pm_2) e_L(\beta) - e_L(\beta') \left( S^\pm_1 + S^\pm_2 \right).
\]

(141)

Observe that

\[
S^+_1 = q^{-1/2} \alpha^{1/2(L-1)} \sigma_1^+ q^{-S^-_1}, \quad S^-_1 = q^{1/2} \alpha^{-1/2(L-1)} \sigma_1^- q^{-S^-_1},
\]

(142)

\[
S^+_2 = q^{1/2} \alpha^{-1/2(L-1)} \sigma_1^+ q^{-S^-_2}, \quad S^-_2 = q^{-1/2} \alpha^{1/2(L-1)} \sigma_1^- q^{-S^-_2}
\]

(143)

and the auxiliary relations

\[
\sigma_1^+ e_L(\beta) = q^{-1} \sigma_1^+ \hat{v}_L - \alpha \beta \hat{v}_1 \sigma_1^+ \]

(144)

\[
e_L(\beta) \sigma_1^+ = q \sigma_1^+ \hat{h}_L - \alpha \beta \hat{h}_1 \sigma_1^+ \]

(145)
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\begin{align}
\sigma_L^+ e_L(\beta) &= q\hat{v}_1 \sigma_L^+ - (\alpha \beta)^{-1} \sigma_L^+ \hat{v}_L, \\
e_L(\beta) \sigma_L^+ &= q^{-1} \hat{h}_1 \sigma_L^+ - (\alpha \beta)^{-1} \sigma_L^+ \hat{h}_L,
\end{align}

and

\begin{align}
\sigma_L^- e_L(\beta) &= q\sigma_L^- \hat{h}_L - (\alpha \beta)^{-1} \hat{h}_1 \sigma_L^- , \\
e_L(\beta) \sigma_L^- &= q^{-1} \sigma_L^- \hat{v}_L - (\alpha \beta)^{-1} \hat{v}_1 \sigma_L^- , \\
\sigma_L^- e_L(\beta) &= q^{-1} \hat{h}_1 \sigma_L^- - \alpha \beta \sigma_L^- \hat{h}_L, \\
e_L(\beta) \sigma_L^- &= q \hat{v}_1 \sigma_L^- - \alpha \beta \sigma_L^- \hat{v}_L.
\end{align}

Thus one obtains

\begin{align}
(S_L^+ + S_L^+)^e_L(\beta) - e_L(\beta') (S_L^+ + S_L^+)
&= A^+(\beta, \beta') \beta^{1/2} \alpha^{L/2} q^{-S-1} + B^+(\beta, \beta') \beta^{-1/2} \alpha^{-L/2} q^{S+1}
\end{align}

with

\begin{align}
A^+(\beta, \beta') &= \frac{q^{1/2}}{(\alpha \beta)^{1/2}} \sigma_1^+ \left[ q^{-1} \hat{v}_L - q \hat{n}_L \right] - \frac{(\alpha \beta)^{1/2}}{q^{1/2}} \sigma_L^+ \left[ q \hat{v}_1 - q \frac{\beta'}{\beta} \hat{n}_1 \right] \\
B^+(\beta, \beta') &= \frac{q^{1/2}}{(\alpha \beta)^{1/2}} \sigma_1^+ \left[ q^{-1} \frac{\beta}{\beta'} \hat{n}_L - q^{-1} \hat{v}_L \right] - \frac{(\alpha \beta)^{1/2}}{q^{1/2}} \sigma_L^+ \left[ q^{-1} \hat{n}_1 - q \hat{v}_1 \right].
\end{align}

and

\begin{align}
(S_L^- + S_L^-)^e_L(\beta) - e_L(\beta') (S_L^- + S_L^-)
&= A^- (\beta, \beta') \beta^{-1/2} \alpha^{-L/2} q^{-S-1} + B^- (\beta, \beta') \beta^{1/2} \alpha^{L/2} q^{S+1}
\end{align}

with

\begin{align}
A^- (\beta, \beta') &= \frac{(\alpha \beta)^{1/2}}{q^{1/2}} \sigma_1^+ \left[ q \hat{n}_L - q^{-1} \hat{v}_L \right] - \frac{q^{1/2}}{(\alpha \beta)^{1/2}} \sigma_L^- \left[ q^{-1} \hat{n}_1 - q^{-1} \frac{\beta'}{\beta} \hat{v}_1 \right] \\
B^- (\beta, \beta') &= \frac{(\alpha \beta)^{1/2}}{q^{1/2}} \sigma_1^+ \left[ q \frac{\beta'}{\beta} \hat{v}_L - q \hat{n}_L \right] - \frac{q^{1/2}}{(\alpha \beta)^{1/2}} \sigma_L^- \left[ q \hat{v}_1 - q^{-1} \hat{n}_1 \right].
\end{align}

With the choice $\beta' = q^{-2} \beta$ (141) reduces to

\begin{equation}
S^\pm H(\beta) - H(q^{-2} \beta) S^\pm = (S_L^+ + S_L^+)^e_L(\beta) - e_L(q^{-2} \beta) (S_L^+ + S_L^+).
\end{equation}

For $S^+$ the r.h.s. reduces to

\begin{align}
&\left\{ \frac{q^{1/2}}{(\alpha \beta)^{1/2}} \sigma_1^+ \left[ q^{-1} \hat{v}_L - q \hat{n}_L \right] - \frac{(\alpha \beta)^{1/2}}{q^{1/2}} \sigma_L^+ \left[ q \hat{v}_1 - q^{-1} \hat{n}_1 \right] \right\} \\
&\times \left[ B^{1/2} \alpha^{-L/2} q^{-S-1} - B^{-1/2} \alpha^{L/2} q^{S+1} \right]
\end{align}
With (142), (143) one thus arrives at

\[ S^+ H(\beta) - H(q^{-2}\beta)S^+ = [q^{-1}\hat{\sigma}_L - q\hat{\sigma}_L] S^+_1 \left[ 1 - \beta^{-1}\alpha^{L-2} q^{2S^+} \right] \]

\[ + [q\hat{\sigma}_1 - q^{-1}\hat{\sigma}_1] S^+_L \left[ 1 - \beta \alpha^L q^{2S^+} \right]. \quad (159) \]

Notice that the action of the pseudo commutator on states with particle number satisfying

\[ q^{L-2N+2} = \beta^{\alpha^L} \quad (160) \]

vanishes.

Similarly one obtains for \( S^- \) the r.h.s. of (158)

\[ \left\{ \frac{(\alpha\beta)^{1/2}}{q^{1/2}} (\alpha\beta)^{1/2} [q^{-1}\hat{\sigma}_L - q\hat{\sigma}_L] - \frac{q^{1/2}}{(\alpha\beta)^{1/2}} [q\hat{\sigma}_1 - q^{-1}\hat{\sigma}_1] \right\} \]

\[ \times \left[ \beta^{1/2}q^{L/2} q^{S^- - 1} - \beta^{-1/2} \alpha^{-L/2} q^{-S^- + 1} \right] \]

which yields

\[ S^- H(\beta) - H(q^{-2}\beta)S^- = - [q^{-1}\hat{\sigma}_L - q\hat{\sigma}_L] S^-_1 \left[ 1 - \beta \alpha^{L-2S^-} \right] \]

\[ - [q\hat{\sigma}_1 - q^{-1}\hat{\sigma}_1] S^-_L \left[ 1 - \beta^{-1}\alpha^{L-2S^-} \right]. \quad (161) \]

Notice that the action of the pseudo commutator on states with particle number satisfying

\[ q^{-L+2N+2} = \beta^{\alpha^L} \quad (162) \]

vanishes.

In compact form (158) can thus be written

\[ S^\pm H(\beta) - H(q^{-2}\beta)S^\pm = \pm [q^{-1}\hat{\sigma}_L - q\hat{\sigma}_L] S^\pm_1 \left[ 1 - \beta^{\pm1}\alpha^{\pm L} q^{2S^\pm} \right] \]

\[ \pm [q\hat{\sigma}_1 - q^{-1}\hat{\sigma}_1] S^\pm_1 \left[ 1 - \beta^{-1}\alpha^{\pm L} q^{2S^\pm} \right]. \quad (163) \]

One can iterate. E.g. for \( (S^-)^2 \) one obtains

\[ \langle S^- \rangle^2 H(\beta) - H(q^{-2}\beta)\langle S^- \rangle^2 \]

\[ = (1 + q^{-2}) [q\hat{\sigma}_L - q^{-1}\hat{\sigma}_1] S^-_1 \left( \sum_{k=2}^{L-1} S^-_k \right) \left[ 1 - \beta \alpha^L q^{2S^-} \right] \]

\[ + (1 + q^{-2}) [q^{-1}\hat{\sigma}_1 - q\hat{\sigma}_1] \left( \sum_{k=2}^{L-1} S^-_k \right) S^-_L \left[ 1 - \beta^{-1}\alpha^L q^{2S^-} \right]. \quad (164) \]

Iterating further as in (30) one arrives at Proposition 1.
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