Global Visibility of a Strong Curvature Singularity in Non-marginally bound Dust Collapse

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We investigate here the local versus global visibility of a space-time singularity formed due to the gravitational collapse of a spherically symmetric dust cloud having a non-zero velocity function. The conditions are investigated that ensure the global visibility of the singularity, in the sense that the outgoing null geodesics leave the boundary of the matter cloud in the future, whereas, in the past, these terminate at the singularity. Explicit examples of this effect are constructed. We require that this must be a strong curvature singularity in the sense of Tipler, to ensure the physical significance of the scenario considered. This may act as a counterexample to the weak cosmic censorship hypothesis.

key words: Cosmic Censorship Hypothesis, Gravitational Collapse, Naked Singularity, Strong Singularity.

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I. INTRODUCTION

When a sufficiently massive cloud collapses unhindered under the influence of its gravitational field, a space-time singularity is obtained. The possibility of such a singularity being non-space like has created much interest in recent years, as that would violate the cosmic censorship hypothesis. The first model of the gravitational collapse was studied by Oppenheimer and Snyder [1] in 1939, and by Datt [2] independently. The model had zero pressure, and the density distribution was homogeneous. The end state of such a collapse turns out to be a singularity from where no non-space like geodesic can escape. It was argued then that singularities were merely an artifact of the exact symmetries [3], e.g., in the case of gravitational collapse, it is the assumption of spherical symmetry that might cause the occurrence of singularities. However, in a real scenario, this would not be the case. Hence it was argued that no such singularities arise in reality. However, the cosmological evidence provided by the WMAP, COBE, and PLANCK of the cosmic microwave background radiation indicates that the precursor of our present universe is a singularity, i.e., the universe had a singular initiation. Apart from this, the singularity theorems provided by Penrose and Hawking [3, 4] prove that singularities could indeed form under very generic conditions in gravitational collapse as well as cosmology. These observations, together with the above-mentioned theorems, resolve the suspicion in the existence of singularities in the universe.

Once the existence of singularities is accepted, the next step is to comprehend the nature of the neighborhood of the singularity. One such property that needs to be investigated is the visibility of the singularity [5–9]. It is known that the big bang singularity is visible in principle because we can see the null and time-like geodesics coming from it. However, it was still unclear whether or not the singularities arising as a result of gravitational collapse are necessarily censored completely from the outside universe by an event horizon. Penrose [10], in 1969, proposed what is now known as the cosmic censorship hypothesis, which is expressed in two forms: the weak cosmic censorship hypothesis, which suggests that a singularity can never be globally naked, i.e., visible to far-away asymptotic observers. Its strong counterpart states that the singularity can never be locally naked as well.

It was, however, shown that introducing inhomogeneity in the mass profile of the collapsing cloud could change the evolution of the apparent horizon, thereby possibly allowing non-space-like geodesics to escape away from near the singularity without getting trapped [11]. As such, the assumption of a homogeneous star is not very appropriate since it is expected that a star becomes denser as we move towards its center.

Various mass distributions have been shown to give rise to visible singularities which are locally naked [12–15]. The stability of locally naked singularities due to collapsing dust cloud against some perturbation in the initial data has also been studied by Deshingkar, Joshi, and Dwivedi [16], and later by Mena, Tavakol and Joshi [17]. The local causal structure of the end state of the collapse in the presence of non-vanishing pressure have been studied wherein possibilities of locally naked singularities have been depicted [18–23]. Most of the work in literature deals with the strong cosmic censorship. Although, globally visible singularities have more observa-
tional significance. However, less work dealing with the global causal structure of such singularities is done. (see e.g. [24–29]). Deshingkar, Jhingan and Joshi [24] depicted some examples of mass functions giving rise to a globally visible singularity where the mass profile is a function of only \( r \), i.e., the fluid under consideration was dust. The collapse, in this case, is considered to be marginally bound. Later, Jhingan and Kaushik used a certain transformation of coordinates to put a restriction on the mass profile of a marginally bound collapsing dust to ensure global visibility of the singularity thus formed [27]. However, the strength of singularities formed due to the depicted mass functions and with vanishing velocity functions was not investigated in both these cases. 

According to Tipler [30], if any object hits the singularity and is crushed to zero volume, then it is called a “strong” singularity. Sufficient condition for a singularity to be “Tipler strong” was provided by Clarke and Krolak [31]. Our basic purpose here is to examine the global visibility of singularities formed due to bound dust collapse is discussed. In Sec. IV, the strength of globally visible singularity in the sense of Tipler is discussed. We end the paper with the concluding remarks and stating a few open concerns in Sec. V.

II. LEMAITRE-TOLMAN-BONDI SPACETIME

The LTB metric [32–34] is a spherically symmetric metric governing the spacetime of collapsing dust clouds. It is given by

\[
ds^2 = -dt^2 + \frac{R^2}{1 + f}dr^2 + R^2d\Omega^2\tag{1}
\]

in the comoving coordinates \( t \) and \( r \). The stress-energy tensor is of Type 1 matter field with vanishing pressure given by

\[
T^\mu_\nu = \rho \delta^\mu_\nu \delta^\nu_\mu.	ag{2}
\]

Einstein’s field equations give us the expression of density and pressure, and the information about the dynamics of the collapse as

\[
\rho = \frac{F^2}{R^2 R'},\tag{3}
\]

\[
p = -\frac{\dot{F}}{R^2 R'}.	ag{4}
\]

and

\[
\dot{R}^2 = \frac{F}{R} + f\tag{5}
\]

respectively. The superscripts dot and prime denote the partial derivative with respect to \( t \) and \( r \), respectively. Here, \( F \) and \( f \) are respectively called the Misner-Sharp mass function and the velocity function. The Misner-Sharp mass function in case of dust is a function of \( r \) only and independent of \( t \). This can be seen from Eq.(4) which tells us that \( \dot{F} = 0 \) since \( p = 0 \) in case of dust. \( F \) tells us about the mass of the collapsing cloud inside a shell of radial coordinate \( r \) at time \( t \). For zero pressure, this mass is conserved inside a fixed radial shell. The polarity of \( f \) classifies the spacetime in three different categories: bound (elliptic), marginally bound (flat) and unbound (hyperbolic) collapse, corresponding to the restrictions \( f < 0 \), \( f = 0 \) and \( f > 0 \) respectively. A collapsing solution of Einstein’s field equation is obtained by restricting the physical radius as \( R < 0 \). This means that a particular shell of fixed radial coordinate collapses to form a singularity when \( R = 0 \) for this shell. However, \( R \) vanishes also at the regular center. Both these cases can be differentiated by expressing the physical radius as

\[
R(t, r) = rv(t, r)\tag{6}
\]

Now, a shell of radial coordinate \( r \) is said to form a singularity a time \( t_s \) when \( v(t_s, r) = 0 \). Rescaling of the physical radius is done using the coordinate freedom such that

\[
R(t, r) = r,	ag{7}
\]

where \( t_i \) is the initial time. This can be rewritten as \( v(t_i, r) = 1 \). Eq.(5) can be integrated to get

\[
t - t_s(r) = -\frac{R^2 G(-fr/F)}{\sqrt{F}}.\tag{8}
\]

Here \( G(y) \) is defined as follows:

\[
G(y) = \begin{cases} 
\frac{\arcsin{\sqrt{y}}}{y^2} - \frac{\sqrt{1-y}}{y} & \text{for } 0 < y < 1, \\
\frac{2}{3} & \text{for } y = 0, \\
\frac{-\arcsinh{\sqrt{-y}}}{(-y)^{3/2}} - \frac{\sqrt{1+y}}{y} & \text{for } -\infty < y < 0.
\end{cases}\tag{9}
\]

The constant of integration in Eq.(8) can be obtained using Eq.(7) as

\[
t_s(r) = \frac{r^{3/2} G(-fr/F)}{\sqrt{F}}.\tag{10}
\]
This is called the singularity curve. It gives us the information about the time at which a shell of radial coordinate $r$ collapses to form a singularity $R = 0$.

In order to determine the causal structure of the singularity, the time of formation of trapped surfaces should be taken into account. The boundary of all the trapped surfaces is called the apparent horizon. Null geodesics coming from the singularity gets trapped if the trapped surfaces, or more specifically the apparent horizon, forms before the formation of such singularity. This comparison of whether the apparent horizon forms before or during the singularity formation determines whether the singularity is hidden or at least locally naked. The apparent horizon is determined by equating the physical radius with the Misner-sharp mass function as $R = F(r)$. This, along with Eq.(8) and Eq.(10) gives us the time of formation of the apparent horizon as a function of radial coordinate as

\[
t_{AH}(r) = \frac{r^{3/2}G(-rf/F)}{\sqrt{F}} - FG(-f).
\]

It is also called the apparent horizon curve. Various examples of mass profiles and velocity functions have been shown to give rise to a singularity such that $t_s(0) = t_{AH}(0)$, thereby creating a possibility for non-spacelike geodesic to have a positive tangent at $r = 0$. Such singularities are at least locally naked. The geodesics may later get trapped, thereby keeping the weak cosmic censorship intact. However, it is also possible that the singular geodesic avoids getting trapped by the trapped surfaces and reaches the boundary of the collapsing cloud unhindered. Such singularities are studied in detail in the next section.

### III. GLOBAL VISIBILITY

The singularities which are only locally visible may not be of much observational significance. This is because, in such a case, an observer outside the event horizon will not be able to receive any signal escaping from the neighborhood of the singularity. For this reason, it is of extreme importance to investigate whether or not there exists a globally visible singularity.

For a singularity to be globally visible, null geodesics originating from the neighborhood of the singularity should not only avoid getting trapped by the trapped surfaces but also reach the boundary of the star before the event horizon. It turns out that the former implies the latter and vice versa. This is because, at $r = r_c$, the apparent horizon coincides with the event horizon. We know that the evolution of the event horizon of the collapsing cloud is same as the evolution of the null geodesic along with the condition that at the boundary of the cloud $r_c$, the following equality should be satisfied:

\[
F(r_c) = R(t, r_c).
\]

Now, the event horizon can not start forming after the initiation of the formation of trapped surfaces (or its boundary, i.e., the apparent horizon). This is because any null geodesic, more specifically outgoing null geodesic, forming inside the apparent horizon, will have a negative tangent and falls back into the singularity. The evolution of EH can be thought of as the evolution of the last outgoing radial null geodesic escaping the center without getting trapped. The equation of the null geodesic is given by

\[
\frac{dt}{dr} = \frac{R'}{\sqrt{1 + f}}.
\]

In the case of inhomogeneous dust, since the time of formation of AH at $r = 0$ is same as the time of formation of central singularity it can be concluded that the EH starts forming either before or during the formation of the singularity due to collapse of the central shell i.e.

\[
t_{EH}(0) \leq t_s(0).
\]

Here, $t_{EH}(r)$ is the event horizon curve which is the solution of Eq.(13) with the condition given by Eq.(12).

If we can trace a null geodesic originating from near the singularity and reaching the boundary with $F(r_c) < R(t, r_c)$, then such geodesic reaches the boundary before the event horizon and thereby makes the central region visible to the outside observer, however, the null geodesic is singular only if it escapes the center at a time which is in a particular neighborhood of time of formation of a central singularity. This neighborhood should not have size more than the Plank time. For this to be possible, first of all, the event horizon should not form way too early than the formation of the singularity at $r = 0$.

Expression of $R$ in terms of the comoving coordinates $t$ and $r$ is obtained from Eq.(8) and Eq.(10) as

\[
R = \left(\frac{r^{\frac{3}{2}}G(-fr/F) - \sqrt{Fr}}{G(-fr/F)}\right)^{\frac{2}{3}}.
\]

In the case of marginally bound collapse, this is reduced to

\[
R = \left(r^{\frac{3}{2}} - \frac{3}{2} \sqrt{Fr}\right)^{\frac{2}{3}}.
\]

However, in the case of non-marginally bound collapse, we use the Taylor expanded expression for the function $G(y)$ given by Eq.(9) around $y = 0$ for $0 < y \leq 1$ as

\[
G(y) = \frac{2}{3} + \frac{1}{5}y + \frac{3}{28}y^2 + o(y^3).
\]

This can then be used in Eq.(15) to write $R$ explicitly as

\[
R(t, r) = \frac{5F}{2f} \left(1 - \sqrt{1 - \frac{4f}{5F} \left(r^{\frac{3}{2}} - \frac{3f}{10F}r^{\frac{3}{2}} - \frac{3}{2} \sqrt{Fr}\right)^{\frac{2}{3}}}\right),
\]
for non-vanishing velocity function, i.e. \( f \neq 0 \). Here, we have considered the expansion of \( G \) from Eq.(17) in Eq.(18) only up to first order. Hence, large value of the ratio \( \frac{F}{f} \) may not give a good approximation. Therefore, in our investigation, we make sure to keep this ratio small by considering positive velocity function having small deviation from zero.

Deshingkar, Jhingan, and Joshi [24] studied the global causal structure of the end state of marginally bound collapse, wherein three different mass distributions were considered. These mass distribution had first-order, second-order, and third-order inhomogeneity terms. The general result obtained was that a higher magnitude of the inhomogeneity term corresponded to the end state as a globally visible singularity. Here, we analyze the global behavior of the singularity formed by bound collapse and for a mass function and the velocity function given by

\[
F = F_0 r^3 + F_2 r^5, \quad f = b_00 r^2
\]

The boundary of the cloud is found such that the density smoothly matches to zero there. Hence, the boundary is given by

\[
r_c = \sqrt{-\frac{3F_0}{5F_2}}.
\]

This is a second-order inhomogeneity in the mass function. As seen in Fig. (1) the singularity is at least locally naked for chosen values of \( F_0 \) and \( F_2 \). However, in Fig. (1a) the event horizon starts forming before the formation of the central singularity, thereby making the singularity globally hidden. The singular geodesic can escape the singularity but later gets trapped and falls back. Now, increasing the magnitude of the inhomogeneity term \( F_2 \), as seen in Fig. (1b) affects the evolution of

\[
t(r, v) = t(0, v) + r\chi_1(v) + r^2\chi_2(v) + r^3\chi_3(v) + O(r^4),
\]

\[
\lim_{\lambda \to 0} \lambda^2 R_{ij} K^i K^j > 0.
\]

Here, \( \lambda \) is the affine parameter along the null geodesic with \( \lambda = 0 \) at the singularity. We can use this criterion to put a restriction on a particular parameter signifying the non-linear relation between the physical radius and the tangent of the outgoing radial null geodesic at the singular center. The time curve can be Taylor expanded around the center \( r = 0 \) as follows:
FIG. 2: Causal structure of a Tipler strong singularity formed as an end state of a bound (elliptic) collapsing dust cloud. Apparent horizon, event horizon, and singular null geodesics are represented by dashed black curves, solid black curves, and solid blue curves, respectively. $\chi_1 = \chi_2 = 0$ and $\chi_3 > 0$. (a) The evolution of the event horizon starts from the center before the formation of the central singularity. Singular null geodesics, if at all, can escape the singularity gets trapped later and falls back in, making the singularity only locally naked. (b) The evolution of the event horizon starts during the formation of the central singularity. Singular null geodesics can escape and reach the faraway observer. Here, $\frac{dR}{dt} \sim 10^{-3}$ initially, and reduces in magnitude thereafter, in both these cases.

FIG. 3: Causal structure of a Tipler strong singularity formed as an end state of an unbound (hyperbolic) collapsing dust cloud. Apparent horizon, event horizon, and singular null geodesics are represented by dashed black curves, solid black curves, and solid blue curves, respectively. $\chi_1 = \chi_2 = 0$ and $\chi_3 > 0$. The mass profile, which ends as a globally visible singularity in bound case (see Fig. (2)), ends as a globally hidden singularity in unbound case. Here, $\frac{dR}{dt} \sim 10^{-3}$ initially, and reduces in magnitude thereafter.

where

$$\chi_i(v) = \left. \frac{d^2 t}{d r^2} \right|_{r=0}. \tag{23}$$

For a singularity to be at least locally visible, the tangent of the future directed radial null geodesic from the singularity at $r \to 0$ should be positive. In the $(R, u)$ frame, where $u = r^\alpha$ with $\alpha > 1$, this tangent is written as $X_0 = \lim_{r \to 0} \frac{dR}{du}$. It can be shown that

$$X_0^3 = \lim_{r \to 0} \frac{1}{\alpha - 1} \left( \chi_1(0) + 2r\chi_2(0) + 3r^2\chi_3(0) \right) + 4r^3\chi_4(0) + o(r^4) \sqrt{M_0(0)} r^{\frac{4-\alpha}{2}}. \tag{24}$$

Here, the relation between the tangent of ORNG at the singularity and the components $\chi_i$ of the Taylor expansion of the time curve at $v = 0$ is depicted. To ensure the positivity of $X_0$, the first non-zero $\chi_i$ should be positive.

Now, it is known that Eq. (21) can be satisfied only if $\alpha \geq 3$. Also, the necessary criterion for the singularity

| $b_{00}$ | $F_3$ | $t_{EH}(0)$ | $b_{00}$ | $F_3$ | $t_{EH}(0)$ |
|--------|--------|-------------|--------|--------|-------------|
| $10^{-1}$ | -5 | 0.586922 | $10^{-3}$ | -5 | 0.611443 |
| $10^{-1}$ | -20 | 0.646240 | $10^{-3}$ | -20 | 0.666319 |
| $10^{-1}$ | -50 | 0.646667 | $10^{-3}$ | -50 | 0.666467 |
| $10^{-1}$ | -100 | 0.646667 | $10^{-3}$ | -100 | 0.666467 |
| $10^{-2}$ | -200 | 0.646667 | $10^{-3}$ | -200 | 0.666468 |
| $10^{-2}$ | -5 | 0.609225 | $10^{-4}$ | -5 | 0.611663 |
| $10^{-2}$ | -20 | 0.664501 | $10^{-4}$ | -20 | 0.666500 |
| $10^{-2}$ | -50 | 0.664667 | $10^{-4}$ | -50 | 0.666467 |
| $10^{-2}$ | -100 | 0.664667 | $10^{-4}$ | -100 | 0.666468 |
| $10^{-2}$ | -200 | 0.664668 | $10^{-4}$ | -200 | 0.666467 |
to be at least locally naked is given by $\alpha \leq 3$. Hence, the necessary criterion for a singularity to be strong and locally naked is given by [11]

$$\alpha = 3. \tag{25}$$

However, for $\alpha = 3$, if at all $\chi_1$ or $\chi_2$ is/are non-zero, then $X_0$ blows up. Hence, we will have to make sure that $\chi_1$ and $\chi_2$ should be zero. More specifically, $\chi_1$ and $\chi_2$ should be of order at least $r^3$ and $r^2$ respectively to avoid blowing up of $X_0$. The integral expression of $\chi_1$, $\chi_2$ and $\chi_3$ are as follows [13, 35]:

$$\chi_1(v) = -\frac{1}{2} \int_v^1 \left( \frac{M_i}{v} + b_{01} \right)^2 dv, \tag{26}$$

$$\chi_3 = \int_v^1 \left( \frac{b_{01}}{\left( \frac{M_i}{v} + b_{00} \right)^2} \right)^2 \left( -\frac{5}{16} \left( \frac{b_{01}}{\frac{M_i}{v} + b_{00}} \right) \right)^2 + 3 \left( \frac{M_i}{v} + b_{02} \right) - \frac{1}{2} \left( \frac{M_i}{v} + b_{03} \right)^2 dv. \tag{27}$$

Here $M_i$ are the components non-minimally coupled to $r^i$ in the Taylor expansion of $M$ around $r = 0$. $M$ is called the mass profile, having relation with the misner sharp mass function as $F(r) = r^3 M(r)$. This relation is dictated by the regularity conditions. Also $b_{00}$ in Eq.(26-28) are the components non-minimally coupled with $r^i$ in the Taylor expansion of the velocity profile $b_0(r)$ around the center $r = 0$. Regularity condition dictates that $f(r) = r^2 b_0(r)$ The mass profile and the velocity profile together determine the polarity of $\chi_3$. For positive $\chi_3$, we have a strong at least locally naked singularity provided $\chi_1$ and $\chi_2$ vanish at $v = 0$. Such analysis was not done in [24] in the case of marginally bound collapse for various mass functions considered therein. One such example of mass function and velocity function for which $\chi_1$ and $\chi_2$ vanish is given as follows:

$$F = F_0 r^3 + F_3 r^6, \quad f = b_{00} r^2. \tag{29}$$

The boundary of the cloud is found such that the density smoothly matches to zero there. Hence, the boundary is given by

$$r_c = \left( -\frac{F_0}{2F_3} \right)^{\frac{2}{3}}. \tag{30}$$

Similar to the previous mass function, this mass function, along with a positive velocity, also gives at least a locally naked singularity for chosen values of $F_0$ and $F_3$. $\chi_3 > 0$ in this case. However, in Fig. (2a), outgoing singular radial null geodesics having positive tangent at the center later gets trapped and falls back to the singularity. Increasing the magnitude of the inhomogeneity term, $F_3$, alters the evolution of the event horizon in such a way that its initiation now coincides with the time of formation of singularity due to collapsing central shell, thereby allowing singular null geodesics to escape and reach the faraway observer, as observed in Fig. (2b).

In the case of a marginally bound collapse of dust, third-order inhomogeneity in the mass profile can give globally naked singularity for a wide range of $F_3 < 0$ [16]. It can be seen from Eq.(26-28) that such singularity is Tipler strong.

In the case of unbound collapse, we consider a velocity function to have a positive value. It is found in Fig. (3) that the mass function giving rise to the globally naked singularity as the end state of bound collapse, gives a globally hidden singularity as the end state of unbound collapse having velocity function with the same magnitude but opposite polarity. Furthermore, it is observed in Tab. (1) that at least so long as the mass function and the velocity function is of the form Eq.(29) along with $b_{00} > 0$, a wide range of coefficients in such mass and velocity function gives a globally hidden singularity as the end state.

In Fig. (4), dynamics of the collapse of the fluid are shown for a particular mass function such that the outgoing radial null geodesics get trapped, and there is no causal connection between the singular region and the outside observer. The singularity thus obtained is, however, locally naked, as seen in Fig. (5). Fig. (6) depicts the evolution of the density profile, event horizon, and singular geodesics escaping the boundary of the cloud without getting trapped by any trapped surfaces. A different value of the mass function is considered here. The magnitude of the inhomogeneity term in the Misner-Sharp mass function is more in this case. An asymptotic observer may observe the wavefronts of the escaped sig-
FIG. 4: Evolution of the collapsing star and the global causal structure is depicted here. $F_0 = 1$, $F_3 = -15$ and $F_i = 0$ for $i \not= 1, 3$. $b_{00} = -0.001$ and $b_{0j} = 0$ for $j \not= 0$. $\frac{dR}{dt} \sim 10^{-3}$ initially and reduces in magnitude thereafter. The singularity is Tipler strong with $\chi_1 = \chi_2 = 0$ and $\chi_3 \neq 0$. The solid black disk represents the event horizon which increases in size with time. No singular geodesic can escape and reach the boundary.

FIG. 5: Local causal structure is depicted here. $F_0 = 1$, $F_3 = -15$ and $F_i = 0$ for $i \not= 1, 3$. $b_{00} = -0.001$ and $b_{0j} = 0$ for $j \not= 0$. $\frac{dR}{dt} \sim 10^{-3}$ initially and reduces in magnitude thereafter. The singularity is Tipler strong with $\chi_1 = \chi_2 = 0$ and $\chi_3 \neq 0$. Behavior of singular outgoing radial null geodesic wave front is represented by blue color. Event horizon is represented by black colored circle.
FIG. 6: Evolution of the collapsing star and the global causal structure is depicted here. $F_0 = 1$, $F_3 = -20$ and $F_i = 0$ for $i \neq 1, 3$. $b_{00} = -0.001$ and $b_{0j} = 0$ for $j \neq 0$. $\frac{\kappa R}{f} \sim 10^{-3}$ initially and reduces in magnitude thereafter. The singularity is Tipler strong with $\chi_1 = \chi_2 = 0$ and $\chi_3 \neq 0$. The solid black disk represents the event horizon which increases in size with time. Escaping singular null geodesic wave fronts are represented by red and blue circles which increases with time.

gular null geodesic highly redshifted. Null geodesic escaping from closer to the singularity will be more redshifted. The light traveling from more close to the singularity is also traveling closer to the event horizon. One could deduce that more redshifted the light is, more significant it is, in respect of holding traces of the quantum gravity. All the evolutions are in the comoving frame. Fig. (4-6) helps in visualizing the evolution of the collapsing cloud along with the evolution of the event horizon and null trajectories. They also depict the dynamics of density variation of the collapsing cloud due to inhomogeneous mass distribution, bright light indicating denser.

V. CONCLUDING REMARKS

Some concluding remarks and open concerns are mentioned below:

1. End state of a marginally bound collapse has been studied in [16]. However, such a scenario is a very particular case. Here we consider a non-marginally bound collapse of the inhomogeneous dust cloud and study the causal structure of the singularity formed as the end state. Unless the globally naked singularity is not Tipler strong, it should not be taken as a serious counter-example to the weak cosmic censorship. Here, we have proved the existence of such strong, globally visible singularity formed due to bound collapse.

2. In deriving the explicit expression of the physical radius in terms of $t$ and $r$ in Eq.(18) for non-vanishing velocity function, only the first component of the Taylor expansion of $G$ is used from Eq.(17). Hence the accuracy of our further analysis will get affected for large values of the term $\frac{\kappa R}{f}$. To minimize the error, small values of the magnitude of the velocity function is considered. For larger values, higher-order terms in the expansion of $G$ from Eq.(17) will have to be taken into account. Once the explicit expression of the physical radius is achieved, one can study the dynamics of the event horizon, apparent horizon, and singular radial null geodesics to investigate the global causal structure of the singularity.

3. It is the event horizon, which evolves like an outgoing radial null geodesic, which starts from the singularity satisfying the equality of the physical radius and the Misner-sharp mass function at the boundary of the collapsing fluid. Hence, any outgoing radial null geodesic with the property that $F < R$ at $r = r_c$ has to start from the center at a time before the formation of the singularity. However, this time difference between the escape of the
light and the formation of the singularity can be reduced as much as desired. For such a null geodesic to be singular, it should escape from the region, which is in a small neighborhood of the singularity. This small neighborhood should have a measure of the order of Planck length. Only then will such untrapped null geodesic be considered significant and will be expected to contain traces relevant to deepen our understanding of how gravity works in the quantum regime.

4. In terms of observational significance, if at all there exists a globally visible singularity, it may be difficult to distinguish between singular and non-singular geodesics escaping such singularity and received by a telescope. However, light waveform, which is more redshifted, is expected to come from the region, which is more close to the singularity as compared to the waveform, which is less redshifted.

5. Consider Eq. (29) with negative $F_3$ and positive $b_{00}$. This corresponds to the unbound collapse of fluid with third-order inhomogeneity in mass profile. It is found that as far as such mass and velocity functions are considered, we may have $t_{EH}(0) < t_s(0)$, which means that globally visible singularity may not be achieved. This argument is supported by data in Tab. (1). So far, no concrete statement about the global visibility of a strong singularity formed due to unbound collapse of dust can be made, and further investigation is needed. It may be possible that for some other combination of mass function and velocity function (unbound), the collapse ends in a globally visible singularity. This will be investigated in more detail in our future work.

6. A very important concern is that our analysis is restricted to the end state of a collapsing dust cloud, i.e., the pressure of the collapsing fluid is considered to be zero. The effect on the global causal structure of the singularity in the presence of pressure is unknown. To understand the behavior of singular null geodesics numerically requires information about the explicit expression of the physical radius. However, this is difficult to obtain when the Misner-Sharp mass function varies with time, which is the case when there is non-zero pressure. Investigating the global visibility of a Tipler strong singularity formed due to the collapse of a cloud having such time-varying Misner-Sharp mass function will be a significant step towards understanding the cosmic censorship.

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