Replica mean-field theory for Levy spin-glasses

A. Engel
Institut für Physik, Carl-von-Ossietzky-Universität, 26111 Oldenburg, Germany

Infinite-range spin-glass models with Levy-distributed interactions show a freezing transition similar to disordered spin systems on finite connectivity random graphs. It is shown that despite diverging moments of the local field distribution this transition can be analyzed within the replica approach by working at imaginary temperature and using a variant of the replica method developed for diluted systems and optimization problems. The replica-symmetric self-consistent equation for the distribution of local fields illustrates how the long tail in the distribution of coupling strengths gives rise to a significant fraction of strong bonds per spin which form a percolating backbone at the transition temperature.

PACS numbers: 02.50.-r,05.20.-y,89.75.-k

I. INTRODUCTION

Spin-glasses are model systems of statistical mechanics in which simple degrees of freedom interact via random couplings drawn from a given probability distribution [1]. The ensuing interplay between disorder and frustration gives rise to peculiar static and dynamic properties which made spin-glasses paradigms for complex systems with competing interactions. In this way the concepts and techniques developed for their theoretical understanding [2] became invaluable tools in the quantitative analysis of problems originating from such diverse fields as algorithmic complexity [3, 4, 5], game theory [6, 7], artificial neural networks [8], and cryptography [9].

In the present note we study a spin-glass for which the couplings strengths are drawn from a Levy-distribution. The main characteristic of these distributions are power-law tails resulting in diverging moments. Compared to the extensively studied spin-glass models with Gaussian [10] or other finite moment distributions Levy-distributed couplings are interesting for several reasons. On the one hand they pose new challenges to the theoretical analysis because the diverging second moments invalidate the central limit theorem which is at the bottom of many mean-field techniques. Related issues of interest include the spectral theory of random matrices with Levy-distributed entries [14, 15] and relaxation and transport on scale-free networks [16]. It is also interesting to note that the characteristic properties of the Cauchy-distribution have recently enabled progress in the mathematically rigorous analysis of matrix games with random pay-off matrices [17].

The model considered below with the help of the replica method was analyzed previously by Cizeau and Bouchaud (CB) using the cavity approach [18]. In their paper CB remark that they resorted to the cavity method because they were not able to make progress within the replica framework. This might have been caused by the fact that at that time the central quantity in the replica method was the second moment of the local field distribution, the so-called Edwards-Anderson parameter [19], which for Levy-distributed couplings is likely to diverge. Later a variant of the replica method was developed to deal with non-Gaussian local field distributions characteristic for diluted spin glasses and complex optimization problems [20]. Until now this approach was used only in situations where the local field distribution is inadequately characterized by its second moment alone and higher moments of the distribution are needed for a complete description. Here we show that the method may also be adapted to situations where the moments not even exist.

II. THE MODEL

We consider a system of $N$ Ising spins $S_i = \pm 1$, $i = 1, ..., N$ with Hamiltonian

$$H(\{S_i\}) = -\frac{1}{2N^{1/\alpha}} \sum_{(i,j)} J_{ij} S_i S_j,$$

(1)
where the sum is over all pairs of spins. The couplings $J_{ij} = J_{ji}$ are i.i.d. random variables drawn from a Levy distribution $P_\alpha(J)$ defined by its characteristic function \[22\]

$$\tilde{P}_\alpha(k) := \int dJ \, e^{-ikJ} \, P_\alpha(J) = e^{-|k|^\alpha}$$  \hspace{1cm} (2)

with the real parameter $\alpha \in (0, 2]$. The thermodynamic properties of the system are described by the ensemble averaged free energy

$$f(\beta) := -\lim_{N \to \infty} \frac{1}{\beta N} \ln Z(\beta), \hspace{1cm} (3)$$

with the partition function

$$Z(\beta) := \sum_{\{S_i\}} \exp(-\beta H(\{S_i\})). \hspace{1cm} (4)$$

Here $\beta$ denotes the inverse temperature and the overbar stands for the average over the random couplings $J_{ij}$.

### III. REPLICA THEORY

To calculate the average in (3) we employ the replica trick \[19\]

$$\ln Z = \lim_{n \to 0} \frac{Z^n - 1}{n}. \hspace{1cm} (5)$$

As usual we aim at calculating $Z^n$ for integer $n$ times, $\{S_i\} \mapsto \{S^n_i\}, a = 1, \ldots, n$, and then try to continue the results to real $n$ in order to perform the limit $n \to 0$.

Due to the algebraic decay $P_\alpha(J) \sim |J|^{-\alpha - 1}$ of the distribution $P_\alpha(J)$ for large $|J|$ the average $Z^n(\beta)$ does not exist for real $\beta$. On the other hand, for a purely imaginary temperature, $\beta = -ik, k \in \mathbb{R}$, we find from the very definition of $P_\alpha(J)$, cf. \[2\]

$$Z^n(-ik) = \sum_{\{S^n_i\}} \exp \left(- \frac{|k|^\alpha}{2N} \sum_{i,j} \sum_a S^n_i S^n_j \alpha + O(1) \right). \hspace{1cm} (6)$$

Note that the replica Hamiltonian is extensive which justifies a-posteriori the scaling of the interaction strengths with $N$ used in \[1\]. The determination of $Z^n$ can be reduced to an effective single site problem by introducing the distributions

$$c(\vec{S}) = \frac{1}{N} \sum_i \delta(\vec{S}_i, \vec{S}), \hspace{1cm} (7)$$

where $\vec{S} = \{S^n_i\}$ stands for a spin vector with $n$ components. We find after standard manipulations \[20\]

$$Z^n(-ik) = \int_{\vec{S}} d\vec{S} \delta(\sum S(\vec{S}) - 1) \exp \left(- N \left[ \sum_{\vec{S}} c(\vec{S}) \ln c(\vec{S}) + \frac{|k|^\alpha}{2} \sum_{\vec{S}, \vec{S}^\prime} c(\vec{S}) c(\vec{S}^\prime) |\vec{S} \cdot \vec{S}^\prime|^\alpha \right] \right). \hspace{1cm} (8)$$

In the thermodynamic limit, $N \to \infty$, the integral in \[8\] can be calculated by the saddle-point method. The corresponding self-consistent equation for $c(\vec{\sigma})$ has the form

$$c(\vec{\sigma}) = \Lambda(n) \exp \left(- |k|^\alpha \sum_{\vec{S}} c(\vec{S}) |\vec{S} \cdot \vec{\sigma}|^\alpha \right), \hspace{1cm} (9)$$

where the Lagrange parameter $\Lambda(n)$ enforces the constraint $\sum_{\vec{S}} c(\vec{S}) = 1$ resulting from \[7\].
IV. REPLICA SYMMETRY

Within the replica symmetric approximation one assumes that the solution of (9) is symmetric under permutations of the replica indices implying that the saddle-point value of \(c(\vec{S})\) depends only on the sum, \(s := \sum_a S^a\), of the components of the vector \(\vec{S}\). It is then convenient to determine the distribution of local magnetic fields \(P(h)\) from its relation to \(c(s)\) as given by \(20\)

\[
c(s) = \int dh \, P(h) \, e^{-ikh_s} \quad \quad \quad \quad \quad \quad P(h) = \int \frac{ds}{2\pi} \, e^{ish} \, c\left(\frac{s}{k}\right).
\]

Note that the \(P(h)\) defined in this way is normalized only after the limit \(n \to 0\) is taken. The distribution of local magnetic fields is equivalent to the free energy \(f(\beta)\) since all thermodynamic properties may be derived from suitable averages with \(P(h)\) \(21\).

To get an equation for \(P(h)\) from (9) we need to calculate

\[
\sum_S e^{-ikh_s} |\vec{S} \cdot \vec{\sigma}|^\alpha = \int \frac{dr d\vec{r}}{2\pi} |r|^\alpha e^{ir \vec{r}} \sum_S \exp\left(-ikh_s - ir \vec{S} \cdot \vec{\sigma}\right) = \int \frac{dr d\vec{r}}{2\pi} |r|^\alpha e^{ir \vec{r}} \sum_S \prod_a \exp\left(-iS^a(kh + \hat{r} \sigma^a)\right)
\]

\[
= \int \frac{dr d\vec{r}}{2\pi} |r|^\alpha e^{ir \vec{r}} \left[2 \cos(kh + \hat{r})\right]^{\frac{\alpha}{2}} \left[2 \cos(kh - \hat{r})\right]^{\frac{\alpha}{2}}
\]

\[
\to \int \frac{dr d\vec{r}}{2\pi} |r|^\alpha e^{ir \vec{r}} \frac{\cos(kh + \hat{r})}{\cos(kh - \hat{r})},
\]

where the limit \(n \to 0\) was performed in the last line and \(\sigma := \sum_a \sigma^a\). Using \(\Lambda(n) \to 1\) for \(n \to 0\) \(20\) we therefore find from (9) in the replica symmetric approximation

\[
c(\sigma) = \exp\left(-|k|^\alpha \int dhP(h) \int \frac{dr d\vec{r}}{2\pi} |r|^\alpha \exp\left(i\vec{r} \cdot \frac{\sigma}{2} \ln\frac{\cos(kh + \hat{r})}{\cos(kh - \hat{r})}\right)\right).
\]

Using this result in (10) and performing the transformations \(r \to r/k, \hat{r} \to \hat{r} k\) we get

\[
P(h) = \int \frac{ds}{2\pi} \exp\left(i\sigma \cos\left(2\cos(kh + \hat{r})\right)\right).
\]

We are now in the position to continue this result back to real values of the temperature by simply setting \(k = i\beta\). In this way we find the following self-consistent equation for the replica symmetric field distribution \(P(h)\) of a Levy spin-glass at inverse temperature \(\beta\)

\[
P(h) = \int \frac{ds}{2\pi} \exp\left(i\sigma \cos\left(2\cos(kh + \hat{r})\right)\right).
\]

V. SPIN GLASS TRANSITION

From (17) we infer that the paramagnetic field distribution, \(P(h) = \delta(h)\), is always a solution. To test its stability we plug into the r.h.s. of (17) a distribution \(P_0(h)\) with a small second moment, \(\epsilon_0 := \int dh P_0(h) h^2 \ll 1\), calculate the l.h.s. (to be denoted by \(P_1(h)\)) by linearizing in \(\epsilon_0\) and compare the new second moment, \(\epsilon_1 := \int dh P_1(h) h^2\), with \(\epsilon_0\). We find \(\epsilon_1 > \epsilon_0\), i.e. instability of the paramagnetic state, if the temperature \(T\) is smaller than a critical value \(T_{f,\alpha}\) determined by

\[
(T_{f,\alpha})^\alpha = - \int \frac{dr d\vec{r}}{2\pi} |r|^\alpha e^{ir \vec{r}} \tanh^2 \hat{r} = - \frac{\Gamma(\alpha + 1)}{\pi} \cos\left(\frac{\alpha + 1}{2}\right) \int \frac{d\vec{r}}{|r|^\alpha + 1} \tanh^2 \hat{r}.
\]

This result for the freezing temperature is essentially the same as the one obtained by CB using the cavity method \(18\). Our somewhat more detailed prefactor ensures that the limit \(\alpha \to 2\) correctly reproduces the value \(T_{f}^{SK} = \sqrt{2}\) of the SK-model \(10\). The dependence of \(T_{f,\alpha}\) on \(\alpha\) is shown in fig. 11.
The peculiarities of the spin-glass transition in the present system are apparent from the similarity between (17) and analogous results for strongly diluted spin glasses and disordered spin systems on random graphs [11, 20, 21]. To make this analogy more explicit we rewrite (17) in a form that allows to perform the s-integration to obtain

\[
P(h) = \int \frac{ds}{2\pi} e^{ish} \sum_{d=0}^{\infty} \frac{(-1)^d}{d!} \int \prod_{i=1}^{d} \left( dh_i P(h_i) \frac{dr_i \hat{r}_i}{2\pi} |r_i|^\alpha e^{ir_i \hat{r}_i} \right) \exp \left( -i \frac{s}{2\beta} \sum_{i=0}^{d} \ln \frac{\cosh \beta(h_i + \hat{r}_i)}{\cosh \beta(h_i - \hat{r}_i)} \right) \\
= \sum_{d=0}^{\infty} \frac{(-1)^d}{d!} \int \prod_{i=1}^{d} \left( dh_i P(h_i) \frac{dr_i \hat{r}_i}{2\pi} |r_i|^\alpha e^{ir_i \hat{r}_i} \right) \delta \left( h - \frac{1}{\beta} \sum_{i=0}^{d} \tanh^{-1}(\tanh \beta h_i \tanh \beta \hat{r}_i) \right). 
\]

This form of the self-consistent equation is similar to those derived within the cavity approach for systems with locally tree-like topology [5, 11, 20] and may also form a suitable starting point for a numerical determination of \( P(h) \) using a population-dynamical algorithm [21].

VI. DISCUSSION

Infinite-range spin-glasses with Levy-distributed couplings are interesting examples of classical disordered systems. The broad variations in coupling strengths brought about by the power-law tails in the Levy-distribution violate the Lindeberg condition for the application of the central limit theorem and give rise to non-Gaussian cavity field distributions with diverging moments. We have shown that it is nevertheless possible to derive the replica symmetric properties of the system in a compact way by using the replica method as developed for the treatment of strongly diluted spin glasses and optimization problems [20] which focuses from the start on the complete distribution of fields rather than on its moments.

Due to the long tails in the distribution of coupling strengths Levy spin-glasses interpolate between systems with many, i.e. \( O(N) \), weak couplings per spin as the Sherrington-Kirkpatrick model and systems with few, i.e. \( O(1) \), strong couplings per spin as the Viana-Bray model. The majority of the \( N - 1 \) random interactions coupled to each spin are very weak (of order \( N^{-1/\alpha} \)). These weak couplings will influence only the very low temperature behaviour which may be expected to be similar to that of the SK-model. On the other hand the largest of \( N \) random numbers drawn independently from the distribution (2) is of order \( N^{1/\alpha} \) [23] and hence every spin also shares a fraction of
strong bonds, $J_{ij} = \mathcal{O}(1)$, which are for $|J_{ij}| > 1/\beta$ practically frozen. With decreasing temperature a growing backbone of frozen bonds builds up that percolates at the transition temperature $T_{f,\alpha}$ \cite{18}. The mechanism for the freezing transition is hence rather different from that operating in the Sherrington-Kirkpatrick model and resembles the one taking place in disordered spin systems on random graphs with local tree-structure.

Acknowledgments

I would like to thank Daniel Grieser, Rémi Monasson and Martin Weigt for clarifying discussions.

\begin{thebibliography}{23}
\bibitem{1} Binder K. and Young, A. P., Rev. Mod. Phys. \textbf{58}, 801 (1986)
\bibitem{2} Mezard M., Parisi G., and Virasoro M. A., \textit{Spin-glass Theory and Beyond} (World Scientific, Singapore, 1987)
\bibitem{3} Monasson R., Zecchina R., Kirkpatrick S., Selman B., and Troyanski L., Nature \textbf{400}, 133 (1999)
\bibitem{4} Mezard M., Parisi G., and Zecchina R., Science \textbf{297}, 812 (2002)
\bibitem{5} Hartmann A. K. and Weigt M., \textit{Phase Transitions in Combinatorial Optimization Problems} (Wiley VCH, Weinheim, 2005)
\bibitem{6} Diederich S. and Opper M., Phys. Rev. \textbf{A39}, 4333 (1989)
\bibitem{7} Berg J. and Engel A., Phys. Rev. Lett. \textbf{81}, 4999 (1998)
\bibitem{8} Engel A. and Van den Broeck C., \textit{Statistical Mechanics of Learning}, (Cambridge University Press, Cambridge, 2001)
\bibitem{9} Mislovaty R., Klein E., Kanter I., and Kinzel W., Phys. Rev. Lett. \textbf{91}, 118701 (2003)
\bibitem{10} Sherrington D. and Kirkpatrick S., Phys. Rev. Lett. \textbf{35}, 1972 (1975)
\bibitem{11} Viana L. and Bray A. J., J. Phys. \textbf{C18}, 3037 (1985)
\bibitem{12} Kanter I. and Sompolinski H., Phys. Rev. Lett. \textbf{58}, 164 (1987)
\bibitem{13} Mezard M. and Parisi G., Europhys. Lett. \textbf{3}, 1067 (1987)
\bibitem{14} Burda Z., Jurkiewicz J., Nowak M. A., Papp G., and Zahed I., cond-mat/0602087
\bibitem{15} Birol G., Bouchaud J.-P., and Potters M., cond-mat/0609070
\bibitem{16} Albert R. and Barabasi A.-L., Rev. Mod. Phys. \textbf{74}, 47 (2002)
\bibitem{17} Roberts D. P. Int. J. Game Theory \textbf{34}, 167 (2006)
\bibitem{18} Cizeau P. and Bouchaud J.-P., J. Phys. \textbf{A26}, L187 (1993)
\bibitem{19} Edwards S. F. and Anderson P. W., J. Phys. \textbf{F5}, 965 (1975)
\bibitem{20} Monasson R., J. Phys. \textbf{A31}, 513 (1998)
\bibitem{21} Mezard M. and Parisi G., Eur. Phys. J. B \textbf{20}, 217 (2001)
\bibitem{22} Gnedenko B. V. and Kolmogorov A. N., \textit{Limit Distributions for Sums of Independent Random Variables} (Addison-Wesley, Reading MA, 1954)
\bibitem{23} Bouchaud J.-P. and Georges A., Phys. Rep. \textbf{195}, 127 (1990), App. B
\end{thebibliography}