Relativistic levels of mesic atoms

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Abstract

We revisit the derivation of the covariant two-body scalar-fermion equation with a Coulomb interaction, presented in a previous paper. We show that it can be given the formal aspect of a Dirac equation, but for the fact that the eigenvalue is also contained in one of the coefficients and thus it is not linearly included. The discussion of the boundary value problem is therefore different, although some properties of the Dirac equation can be recovered in an approximation bringing back to the concept of reduced mass. We discuss a mixed analytic-numerical method of solution which allows to obtain very accurate results and we calculate the lowest levels, states and QED corrections for pionic and kaonic atoms.

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1. Introduction

Some years ago we began a research program on the two body covariant wave equations and we developed a relativistic wave equation for two interacting fermions [1–5]. We studied the covariance, the different limits and the spectral properties of such an equation and the first applications were made on the hyperfine levels of the Hydrogenic atoms. We then addressed to high energy physics, calculating the mass of the mesons and their radiative decays. In all cases the results were in excellent agreement with experimental results, even when dealing with the masses of light mesons for which potential models had generally failed. The essential reason for the accuracy of the results can be traced back to the complete covariance of the treatment, which includes all the relativistic and the two-body effects. In more recent times, in order to broaden the applicability of the method, we have extended our analysis by considering relativistic objects of different nature. We have thus determined the wave equations for interacting scalar-scalar and scalar-fermion particles [6], enabling thus the study of further elementary two-body systems. Again we have proved the covariance, the different limits of such equations and the general properties of their spectrum, giving them a clear physical interpretation.

In this paper we revisit and we give a further development of the relativistic scalar-fermion equation together with its application to the pionic and kaonic atoms. The physical interest for such systems is due to the fact that meson masses are two-three order of magnitude larger than the electron mass, their atomic orbits are of the order of hundred fm and their binding energies of some keV. Therefore they provide a particularly useful opportunity for studying nuclear interactions in QCD low energy regime [7,9]. The effects of the strong interactions on the spectrum is given by a shift of the purely electromagnetic levels and an absorption width, both of them experimentally measured through X-ray spectroscopy [10]. In fact the size of the atom is much larger than the range of strong interactions, so that QED itself provides almost exact values of the atomic levels and the fundamental state is actually the only one for which the nuclear force plays a role. Hence the difference between the measured transition energies from excited states to the fundamental one and the corresponding energies calculated by pure QED are essentially the same as the shift of the fundamental level. A history of the determination of the pion-nucleon coupling constant can be found in [11]. Up to the most recent papers [12, 13], the mass of the π meson has been evaluated by matching the measured values of the photon energy emitted in
the transitions between two rather high level of pionic atoms. The energy of those same decays were calculated by the Schrödinger equation with a perturbation treatment of the relativistic and QED effects [14]. Different corrections of a more phenomenological nature have also been considered, such as, for instance, the finite dimensions of the particles, usually treated as if they were uniformly charged spheres with a non vanishing radius [15–17]. A different measure involving the \( \mu \)-neutrino mass has been presented in [18]. We aim here at a more essential treatment which maintains the covariance, such to take exactly into account the relativistic effects. We therefore will add the lowest order QED corrections, due to vacuum polarization, while assuming a basic model formed by two particles which are point-like and interact by a Coulomb potential. Of course this does not mean that we believe that phenomenological corrections are negligible, but only that we defer their inclusion until we find a formulation fitting the covariant two body framework. For instance, we know that it is comparatively easy to introduce a term mimicking a finite radius of an attracting center for the case of a single particle in an external potential. For two interacting points, after the separation of the global coordinate, the relative part becomes an entangled system and the definition of a potential producing a finite radius effect in terms of the relative coordinate only seems tough. Starting with extended particles would directly lead to the formulation of a field theory producing the appropriate form factors. This is outside of our present purposes. On the other hand we notice that the final reduced form of the scalar-fermion equation develops a radial term clearly related to relativistic corrections of the classical reduced mass. Thus, besides the results, we also take the opportunity to illustrate some properties of the fermion-scalar equation, as well as to explain the ideas of the mixed analytical-numerical treatment we have used in order to obtain a very accurate solution of the spectral problem. High precision appears to be necessary, as seen by the numerical results, since the spectrum presents pairs of very close levels, whose difference becomes fainter and fainter with increasing energy. These states appear as a direct consequence of the covariant treatment which takes into account the fermionic nature of the proton. For the sake of completeness, we begin Section 2 with a sketchy derivation of the scalar-fermion equation relegating to Appendix I the summary of the kinematical variables introduced in our previous papers. We then show that the scalar-fermion equation can be cast into the form of a Dirac equation in a central potential, but for the fact that the reduction procedure produces a finite radius effect in terms of the relative coordinate only seems tough. Starting with extended particles would directly lead to the formulation of a field theory producing the appropriate form factors. In this section we recall the basic steps leading to the relativistic wave equation for a fermion and a scalar particle interacting through a Coulomb potential [6]. We then make explicit the dimensionless form of the second order system which produces e

2. The scalar-fermion equation and its spectrum

In this section we recall the basic steps leading to the relativistic wave equation for a fermion and a scalar particle interacting through a Coulomb potential [6]. We then make explicit the dimensionless form of the second order system which
defines the boundary value problem leading to the spectrum. An improvement with respect to the original derivation is the Dirac form the reduced equation can be given, which makes somewhat more familiar the treatment. The solution has been discussed analytically as far as possible in order to obtain a high precision for the spectral levels. In the next Section we will therefore develop the recurrence relations which make it possible a rapid achievement of very accurate numerical results. As we said in the Introduction, the physical systems we have considered are the pionic atom \( (H^+, \pi^-) \) and the kaonic atom \( (H^+, K^-) \), whose lowest Coulomb levels are given in Tables 1 and 2.

Let us recall some elementary properties of the Klein-Gordon (KG) and Dirac equations in order to establish the two-body relativistic equation for a scalar and a fermion. The covariance of the method has been largely discussed in the series of our previous papers [1][2][3], so that, for the sake of completeness, we just report in Appendix I the definition of the phase-space we are using and some few words to describe its physical and geometrical properties.

A Hamiltonian formulation for the Klein-Gordon equation can be given in the form of a 2-dim system:

\[
i \partial \Phi / \partial t - H_S \Phi = 0, \quad \Phi = (\phi_1, \phi_2)
\]  

(2.1)

In the standard notation \( \sigma_z, \sigma_\pm \) for the Pauli matrices, the Hamiltonian \( H_S \) has the expression

\[
H_S = -\sigma_- (\nabla^2 / 2m_S) + (\sigma_z + 2\sigma_+) m_S = \begin{pmatrix}
m_S & 2m_S \\
-\nabla^2 / 2m_S & -m_S
\end{pmatrix}
\]  

where \( m_S \) is the mass of the scalar. It is obtained from the Feshbach-Villars [21] representation of the KG equation with a transformation generated by \( \sigma_+ + \sigma_z \).

The free Dirac Hamiltonian \( H_F \) in spherical coordinates reads

\[
H_F = \begin{pmatrix}
m_F & 0 & q_0 & \sqrt{2} q_- \\
0 & m_F & -\sqrt{2} q_+ & q_0 \\
q_0 & \sqrt{2} q_+ & -m_F & 0 \\
-\sqrt{2} q_- & -q_0 & 0 & -m_F
\end{pmatrix}
\]  

(2.3)

where \( m_F \) is the mass of the fermion and the spherical derivatives \( q_+, q_- \) and \( q_0 \) are defined by

\[
q_\pm = (\pm q_+ + i q_\mp) / \sqrt{2}, \quad q_0 = q_z, \quad q_k \rightarrow -i \partial / \partial r_k
\]  

(2.4)

Denoting the fermion radial coordinate by \( \xi = (x^2 + y^2 + z^2)^{1/2} \), the use of the spherical spinors

\[
\Omega_{\ell', \ell, m}(\theta, \phi) = \begin{pmatrix}
(j + m)^{1/2} (2j)^{-1/2} Y_{\ell', m-\ell}(\theta, \phi) \\
(j - m)^{1/2} (2j)^{-1/2} Y_{\ell, m+\ell}(\theta, \phi)
\end{pmatrix}, \quad \Omega_{\ell, \ell, m}(\theta, \phi) = \begin{pmatrix}
-jm + 1)^{1/2} (2j + 2)^{-1/2} Y_{\ell, m-\ell}(\theta, \phi) \\
(j + m + 1)^{1/2} (2j + 2)^{-1/2} Y_{\ell, m+\ell}(\theta, \phi)
\end{pmatrix}
\]  

(2.5)

allows for the construction of a Dirac state where angular momentum and parity are diagonal:

\[
\psi(\xi, \theta, \phi) = \begin{pmatrix}
\varphi(\xi, \theta, \phi) \\
\chi(\xi, \theta, \phi)
\end{pmatrix} = \begin{pmatrix}
a(\alpha) \Omega_{\ell m}(\theta, \phi) \\
b(\alpha) \Omega_{\ell' m}(\theta, \phi)
\end{pmatrix}, \quad \ell = j - 1/2, \quad \ell' = j + 1/2
\]  

(2.6)

We now write the relativistic Hamiltonian for a scalar and a fermion interacting through a Coulomb potential expressed in the coordinates introduced in Appendix I:

\[
H = H_S \otimes I_4 + I_2 \otimes H_F + I_4 V(r), \quad V(r) = -\alpha / r
\]  

(2.7)

It generates the stationary eigenvalue equation

\[
H \Phi = M \Phi, \quad M = m_F + m_S + E
\]  

(2.8)

where the states of opposite parity have the form

\[
\Phi_1(r) = \begin{pmatrix}
a_1(\alpha) \Omega_{\ell m}(\theta, \phi) \\
a_2(\alpha) \Omega_{\ell' m}(\theta, \phi) \\
a_3(\alpha) \Omega_{\ell m}(\theta, \phi) \\
a_4(\alpha) \Omega_{\ell' m}(\theta, \phi)
\end{pmatrix}, \quad \Phi_2(r) = \begin{pmatrix}
a_1(\beta) \Omega_{\ell m}(\theta, \phi) \\
a_2(\alpha) \Omega_{\ell' m}(\theta, \phi) \\
a_3(\alpha) \Omega_{\ell m}(\theta, \phi) \\
a_4(\beta) \Omega_{\ell' m}(\theta, \phi)
\end{pmatrix}
\]  

(2.9)
and $a_i(r)$ are coefficients depending upon the relative radial variable only. We redefine two of the unknown functions by letting $a_2(r) \to -ia_2(r)$, $a_4(r) \to -ia_4(r)$ and we introduce the dimensionless variables

$$s = m_F r, \quad \sigma = m_3/m_F, \quad \epsilon = E/m_F, \quad \lambda = M/m_F = 1 + \sigma + \epsilon,$$  (2.10)

By substituting (2.9) into (2.7) we get a system of eight equations equal in pairs, and the final system to be solved is

$$\begin{align*}
\left( \frac{d}{ds} - \frac{j - 1}{2} \right) a_1(s) + \left( \lambda + \frac{\alpha}{s} - \sigma + P \right) a_2(s) - 2\sigma a_4(s) &= 0 \\
\left( \frac{d}{ds} + \frac{j + 3}{2} \right) a_2(s) - \left( \lambda + \frac{\alpha}{s} - \sigma - P \right) a_1(s) + 2\sigma a_3(s) &= 0 \\
\left( \frac{d}{ds} + \frac{j + 3}{2} \right) a_4(s) - \left( \lambda + \frac{\alpha}{s} + \sigma - P \right) a_3(s) - \frac{\nabla^2}{2\sigma} a_1(s) &= 0 \\
\left( \frac{d}{ds} - \frac{j - 1}{2} \right) a_3(s) + \left( \lambda + \frac{\alpha}{s} + \sigma + P \right) a_4(s) + \frac{\nabla^2}{2\sigma} a_2(s) &= 0 \\
\end{align*}$$

(2.11)

where $P = \pm 1$, the positive sign referring to the state $\Phi_1(s)$, the negative sign to $\Phi_0(s)$. Notice that the Laplace operator of the third equation contains an angular momentum contribution $r^2 (j - 1/2)(j + 1/2)$, while in the last equation the angular momentum contribution is $r^2 (j + 1/2)(j + 3/2)$. Actually, from (2.11) it turns out that $a_3(r)$ and $a_4(r)$ can be expressed in terms of $a_1(r)$ and $a_2(r)$ by the algebraic relations, equal for both parities:

$$a_3(s) = \frac{\left((\lambda - \sigma)s + a_2(s) - s^2\right)a_1(s) - a_2(s)}{4s(\lambda s + \alpha)\sigma}, \quad a_4(s) = \frac{\alpha a_1(s) + \left((\lambda - \sigma)s + a_2(s) - s^2\right)a_2(s)}{4s(\lambda s + \alpha)\sigma}. \quad (2.12)$$

Denoting for simplicity $f(s) = a_1(s)$, $g(s) = -a_2(s)$ and introducing the parameters

$$\Lambda = \frac{\lambda}{2} + \frac{1 - \sigma^2}{2\lambda}, \quad B = \frac{\alpha(1 - \sigma^2)}{2\lambda^2}, \quad \rho = \frac{\alpha}{\lambda} \quad (2.13)$$

the differential problem to be solved reduces to the second order boundary condition problem with eigenvalue $\epsilon$ (or $\lambda$)

$$\begin{align*}
\frac{d}{ds} f(s) + \left( \frac{j}{s} + \frac{1}{2(s + \rho)} \right) f(s) - \left( \Lambda + 1 + \frac{\alpha}{2s} - \frac{B}{s + \rho} \right) g(s) &= 0 \\
\frac{d}{ds} g(s) + \left( \frac{j + 1}{s} + \frac{1}{2(s + \rho)} \right) g(s) + \left( \Lambda - 1 + \frac{\alpha}{2s} - \frac{B}{s + \rho} \right) f(s) &= 0 \\
\end{align*}$$

(2.14)

The system (2.14) can be given a more convenient form by substituting the angular momentum $j$ with $\kappa = -j - 1/2$ as in [19] and by changing the unknown functions $f(s)$, $g(s)$ with $\phi(s)$, $\chi(s)$ according to

$$f(s) = \frac{\phi(s)}{\sqrt{s(s + \rho)}}, \quad g(s) = \frac{\chi(s)}{\sqrt{s(s + \rho)}}. \quad (2.15)$$

We get

$$\begin{align*}
\frac{d}{ds} \phi(s) + \frac{\kappa}{s} \phi(s) - \left( \Lambda + 1 + \frac{\alpha}{2s} - \frac{B}{s + \rho} \right) \chi(s) &= 0 \\
\frac{d}{ds} \chi(s) - \frac{\kappa}{s} \chi(s) + \left( \Lambda - 1 + \frac{\alpha}{2s} - \frac{B}{s + \rho} \right) \phi(s) &= 0 \\
\end{align*}$$

(2.16)

The system (2.16) is formally identical to a Dirac equation in a central potential [19]. However, besides the fact that for a given value of the parameter $\Lambda$ there are two determinations of the eigenvalue $\lambda$ we are concerned with, i.e. $\lambda = \Lambda \pm (\Lambda^2 + \sigma^2 - 1)^{1/2}$, the major point that forbids the system (2.16) to be treated as a genuine Dirac equation is the...
dependence of the parameter $B$ on $\lambda$ and hence on $\Lambda$. In particular, for (2.16) the spectrum is no more dependent only upon the absolute value of $\kappa$ as for the Dirac equation in a central potential \cite{19}. Thus the level degeneracy is removed and the states with $\kappa$ appear separately, although their difference is really small, as we can see from the numerical results. Finally notice that an actual Dirac equation in a Coulomb potential with coupling constant $\alpha/2$ is obtained for equal fermion and scalar masses: in this case $\sigma = 1$, $B = 0$ and $\Lambda = \alpha/2$.

For general masses, the solution of (2.16) cannot be completely analytical. It is a boundary value problem with singularities at the origin and at infinity and we use a double shooting framework for obtaining the eigenvalues and the points. Since the system is regular in $(0, \infty)$, we calculate two more holomorphic series solutions, each of which centered sufficiently close to one of the singular points. We then match them to the approximate solution coming from the corresponding singular point. The continuity of components of these two series at an arbitrarily chosen crossing point gives, as usual, the spectral condition.

We conclude this section with an observation on the limits of (2.11) for large values of the mass of each one of the two components. Since $\lambda + \alpha/s - \sigma = 1 + \epsilon + \alpha/s$, when the mass of the scalar tends to infinity, namely $\sigma \rightarrow \infty$, from (2.10) we see that the last two equations of (2.11) are identically vanishing and the two first ones give the Dirac equation \cite{19}

$$
\begin{align*}
\left( \frac{d}{ds} + \frac{1 + \kappa}{s} \right) a_1(s) + \left( 1 + \epsilon + \frac{\alpha}{s} + 1 \right) a_2(s) &= 0 \\
\left( \frac{d}{ds} + \frac{1 - \kappa}{s} \right) a_2(s) - \left( 1 + \epsilon + \frac{\alpha}{s} - 1 \right) a_1(s) &= 0
\end{align*}
$$

(2.17)

When the mass of the fermion tends to infinity we recover the KG equation for the scalar. This is easily seen by first making the change of variable $s = (1/\sigma) u$ and taking the mass of the lighter particle as mass scale. In order to normalize the physical parameters to the mass of the scalar, we define $\eta = \epsilon/\sigma$. Developing $a_j(u) = (1/\sigma) a_j^{(1)}(u) + a_j^{(0)}(u)$, $j = 1, 4$, at the order $\sigma^{-2}$ we find the vanishing of $a_2^{(1)}(u)$ and $a_4^{(1)}(u)$. At the order $\sigma^{-1}$ we then see that $a_2^{(0)}(u)$, $a_3^{(1)}(u)$ and $a_4^{(0)}(u)$ can be expressed in terms of $a_1^{(1)}(u)$, which, in turn, satisfies

$$
\nabla^2 a_1^{(1)}(u) + \left( 1 + \eta + \frac{\alpha}{\eta} \right)^2 a_1^{(1)}(u) = 0
$$

(2.18)

namely the KG equation in a Coulomb field.

3. The solution of the boundary value problem.

In this section we give the details of our method for calculating the spectrum of the system (2.14), together with the results of the pure Coulomb levels of the $(H^s, \pi^s)$ and $(H^s, K^s)$ atoms. A global exact analytical solution is not available. Therefore we look for a result in the form of piecewise approximate analytical solutions expressed by series and, in order to accelerate the convergence, by Padé approximants. This allows to obtain any required precision and to have an effective control of it. Clearly, any improvement of the accuracy occurs to the detriment of the computation time. However, numerical values of very large numbers of coefficients of the series involved are efficiently and rapidly found by recurrence procedures. Only the calculations of the Padé are somewhat longer, still completely remaining within acceptable time limits even for a normal workstation. Here below we give, thus, the explicit form of the recurrence relations of (2.16) for the series solution at the origin, the asymptotic solution at infinity and the holomorphic solution at any regular point of the open positive line $(0, \infty)$.

The solution at the origin.

For the solution at the origin we make the usual expansion of the unknown functions $\phi(s)$ and $\chi(s)$:

$$
\phi(s) = s^\nu \sum_{n=0}^{\infty} \phi_n s^n, \quad \chi(s) = s^\nu \sum_{n=0}^{\infty} \chi_n s^n
$$

(3.1)
The index \( n \) is obtained by the relations with \( n = 0 \):

\[
\begin{align*}
(v + \kappa) \phi_0 - (\alpha/2) \chi_0 &= 0 \\
(\alpha/2) \phi_0 + (v - \kappa) \chi_0 &= 0
\end{align*}
\]

which actually are an eigenvalue system in \( v \), yielding

\[
v = \sqrt{k^2 - \alpha^2/4}, \quad \phi_0 = 1, \quad \chi_0 = (2/\alpha)(v + \kappa)
\]

By substituting (3.1) in the system (2.16), expanded in the neighborhood of \( s = 0 \), and letting

\[
\Sigma_{\phi,n} = \sum_{\rho=0}^{n} (\rho)^{-\rho} \phi_{n-\rho}, \quad \Sigma_{\chi,n} = \sum_{\rho=0}^{n} (-\rho)^{-\rho} \chi_{n-\rho}
\]

we obtain the recurrence relations for \( n \geq 1 \). They read

\[
\begin{pmatrix}
(v + \kappa + n) & -\alpha/2 \\
\alpha/2 & v - \kappa + n
\end{pmatrix}
\begin{pmatrix}
\phi_n \\
\chi_n
\end{pmatrix} =
\begin{pmatrix}
(\Lambda + 1) \chi_{n-1} - (B/\rho) \Sigma_{\chi,n-1} \\
-(\Lambda - 1) \phi_{n-1} + (B/\rho) \Sigma_{\phi,n-1}
\end{pmatrix}
\]

Solving in \( \phi_n \) and \( \chi_n \), we get the expressions to be implemented by a numerical code:

\[
\begin{align*}
\phi_n &= \frac{1}{2n (n + 2 \nu)} \left( 2 (v + n - \kappa) \left( (\Lambda + 1) \chi_{n-1} - (B/\rho) \Sigma_{\chi,n-1} \right) - \alpha \left( (\Lambda - 1) \phi_{n-1} - (B/\rho) \Sigma_{\phi,n-1} \right) \right) \\
\chi_n &= \frac{1}{2n (n + 2 \nu)} \left( \alpha \left( (\Lambda + 1) \chi_{n-1} - (B/\rho) \Sigma_{\chi,n-1} \right) - (v + \kappa + n) \left( (\Lambda - 1) \phi_{n-1} - (B/\rho) \Sigma_{\phi,n-1} \right) \right)
\end{align*}
\]

The corresponding series have a finite radius of convergence \( s_0 \). A point in \((0, s_0)\) will be chosen to match this solution with the corresponding series described here below.

**The solution at a regular point.**

We first shift the system (2.16) at a point \( u_0 > 0 \) by letting \( s = u + u_0 \), so that \( u = 0 \) is a regular point. We have

\[
\begin{align*}
\frac{d}{du} \phi(u) + \frac{\kappa}{u + u_0} \phi(u) - \frac{\Lambda + (\alpha/2)}{2(u + u_0)} \phi(u) - \frac{B}{u + u_0 + \rho} + 1 \chi(u) &= 0 \\
\frac{d}{du} \chi(u) - \frac{\kappa}{u + u_0} \chi(u) + \left( \Lambda + \frac{\alpha}{2(u + u_0)} - \frac{B}{u + u_0 + \rho} - 1 \right) \phi(u) &= 0
\end{align*}
\]

The expansion of the solutions at the regular point \( u = 0 \) are now the holomorphic series

\[
\phi(u) = \sum_{n=0}^{\infty} \phi_n u^n, \quad \chi(u) = \sum_{n=0}^{\infty} \chi_n u^n
\]

The coefficients \((\phi_0, \chi_0)\) are arbitrary integration constants that will be chosen to be the pairs \((1, 0)\) and \((0, 1)\) in order to obtain a set of fundamental solutions, necessary for the matching with the expansions at the origin and at infinity. For \( n \geq 1 \), assuming that the coefficients with negative index are vanishing, a substitution into (5.7) yields the following expressions of \( \phi_n \) and \( \chi_n \):

\[
\begin{align*}
\phi_n &= \frac{1}{n (u_0 + \rho) u_0} \left( -((\rho + 2 u_0) (n - 1) + (\rho + u_0) \kappa) \phi_{n-1} + \left( (\Lambda + 1) u_0^2 + \left( (\Lambda + 1) \rho - B + \alpha/2 \right) u_0 + \rho \alpha/2 \right) \chi_{n-1} - \left( \kappa + n - 2 \right) \phi_{n-2} + \left( (\rho + 2 u_0) (\Lambda + 1) - B + \alpha/2 \right) \chi_{n-2} + \left( (\Lambda + 1) \chi_{n-1} \right) \right) \\
\chi_n &= \frac{1}{n (u_0 + \rho) u_0} \left( -((\Lambda - 1) u_0^2 + \left( (\Lambda - 1) \rho - B + \alpha/2 \right) u_0 + \rho \alpha/2 \right) \phi_{n-1} - \left( (\rho + 2 u_0) (n - 1) - (\rho + u_0) \kappa \right) \chi_{n-1} - \left( (\Lambda - 1) (\rho + 2 u_0) - B + \alpha/2 \right) \phi_{n-2} + \left( (k - n + 2) \chi_{n-2} - (\Lambda - 1) \phi_{n-1} \right)
\end{align*}
\]

by which accurate numerical values of the solutions are rapidly obtained.
The solution at infinity.

General theorems ~\[22\] state that asymptotic solutions at infinity of \(2.16\) are of the form
\[
\phi(s) = e^{-\omega s} s^\delta \Phi(s), \quad \chi(s) = e^{-\omega s} s^\delta X(s)
\]  
(3.10)
The unknown functions \((\Phi(s), X(s))\) are solutions of the system
\[
\frac{d}{ds} \Phi(s) + \left(-\omega + \frac{\delta + \kappa}{s}\right) \Phi(s) - \left(\Lambda + \frac{\alpha}{2s} - \frac{B}{s + \rho} + 1\right) X(s) = 0
\]
\[
\frac{d}{ds} X(s) + \left(-\omega + \frac{\delta - \kappa}{s}\right) X(s) + \left(\Lambda + \frac{\alpha}{2s} - \frac{B}{s + \rho} - 1\right) \Phi(s) = 0
\]  
(3.11)
By substituting into \(3.11\) the expansions
\[
\Phi(s) = \sum_{n=0}^{\infty} \Phi_n s^{-n}, \quad X(s) = \sum_{n=0}^{\infty} X_n s^{-n} \quad (s + \rho)^{-1} = \sum_{n=0}^{\infty} (-\rho)^n s^{-n+1}
\]  
(3.12)
at the zero order we find the system
\[
\begin{pmatrix}
  -\omega & -\Lambda - 1 \\
  \Lambda - 1 & -\omega
\end{pmatrix}
\begin{pmatrix}
  \Phi_0 \\
  X_0
\end{pmatrix} = 0
\]  
(3.13)
which yields
\[
\omega = \sqrt{1 - \Lambda^2}, \quad \Phi_0 = 1, \quad X_0 = -\omega/(1 + \Lambda)
\]  
(3.14)
Letting
\[
\Sigma_{\Phi,n} = \sum_{p=0}^{n} (-\rho)^p \Phi_{n-p}, \quad \Sigma_{\chi,n} = \sum_{p=0}^{n} (-\rho)^p X_{n-p}
\]  
(3.15)
at order \(-n\), with \(n \geq 1\) we find the relations
\[
\begin{pmatrix}
  -\omega & -\Lambda - 1 \\
  \Lambda - 1 & -\omega
\end{pmatrix}
\begin{pmatrix}
  \Phi_n \\
  X_n
\end{pmatrix} = \begin{pmatrix}
  -(\delta + \kappa - n + 1) \Phi_{n-1} + (\alpha/2) X_{n-1} - B \Sigma_{\chi,n-1} \\
  -(\delta - \kappa - n + 1) X_{n-1} - (\alpha/2) \Phi_{n-1} + B \Sigma_{\Phi,n-1}
\end{pmatrix}
\]  
(3.16)
The matrix in \(3.16\) is singular in view of \(3.13\), with eigenvalues \((0, -2\omega)\). Therefore we transform the system to the basis where the matrix is diagonal. Obviously, in such a basis, one of the two relations obtained from \(3.16\) is homogeneous. For \(n = 1\) this homogeneous equation gives the value of \(\delta\), namely
\[
\delta = -(2\omega)^{-1} (\Lambda (2B - \alpha))
\]  
(3.17)
In order to obtain a solvable set of recurrence equations allowing to determine the coefficients \(\Phi_n, X_n\) for \(n \geq 1\), we have considered the first equation by subtracting the homogeneous relation from the inhomogeneous one, and a second equation obtained by rescaling the homogeneous equation from \(n\) to \(n + 1\), which thus becomes an inhomogeneous equation in \(\Phi_n\) and \(X_n\). The result is
\[
\Phi_n = \Delta^{-1} \left( -A_X C_\Phi \Phi_{n-1} - A_X C_X X_{n-1} + \gamma \left(2 \gamma \omega B \rho - A_X B\right) \Sigma_{\Phi,n-1} + \left(2 \gamma \omega B \rho + A_X B\right) \Sigma_{\chi,n-1}\right)
\]
\[
X_n = \Delta^{-1} \left( A_\Phi C_\Phi \Phi_{n-1} + A_\Phi C_X X_{n-1} + \gamma \left(-2 \omega B \rho + A_\Phi B\right) \Sigma_{\Phi,n-1} - (2 \omega B \rho + A_\Phi B) \Sigma_{\chi,n-1}\right)
\]  
(3.18)
where, for simplicity in writing, we have defined

\[ A_\Phi = n - \delta - \kappa - (2B - \alpha)(\Lambda + 1)/(2\omega), \]
\[ C_\Phi = n - 1 - \kappa - \delta - (\Lambda + 1)\alpha/(2\omega), \]
\[ A_\chi = (1/2)\alpha - (\Lambda + 1)(n - \delta + \kappa)/\omega - B, \]
\[ C_\chi = (1/2)\alpha + (\Lambda + 1)(n - 1 - \delta + \kappa)/\omega, \]
\[ T = (\Lambda + 1)/\omega, \]
\[ \Delta = (-2A_\Phi T + 2A_\chi)\omega \]

(3.19)

It is well known that the expansion of the solution at infinity produces an asymptotic series. Thus, for convergence, in the numerical treatment we have applied to this series the Padé approximant technique.

The results for the pure Coulomb levels of the pionic and kaonic atoms are reported in the following Tables 1 and 2.

| State | \( \kappa \) | \( n = 1 \) | \( n = 2 \) | \( n = 3 \) | \( n = 4 \) | \( n = 5 \) |
|-------|------|------|------|------|------|------|
| \( s_{1/2} \) | \(-1\) | -3235.0931859 | -808.7611897 | -359.4465684 | -202.1877820 | -129.3998072 |
| \( p_{1/2} \) | \(+1\) | -808.7426715 | -359.4410815 | -202.1854647 | -129.3986221 |
| \( p_{3/2} \) | \(-2\) | -808.7424909 | -359.4410280 | -202.1854447 | -129.3986105 |
| \( d_{3/2} \) | \(+2\) | -359.4399305 | -202.1849817 | -129.3983734 |
| \( d_{5/2} \) | \(-3\) | -359.4399127 | -202.1849742 | -129.3983696 |
| \( f_{3/2} \) | \(+3\) | -359.4399127 | -202.1849742 | -129.3983696 |
| \( f_{5/2} \) | \(-4\) | -359.4399305 | -202.1849817 | -129.3986221 |
| \( g_{7/2} \) | \(+4\) | -359.4399305 | -202.1849817 | -129.3986221 |
| \( g_{9/2} \) | \(-5\) | -359.4399305 | -202.1849817 | -129.3986221 |

Table 1: The pure Coulomb spectrum of the pionic atom \((H^+, \pi^-)\) expressed in eV. We have taken the proton mass \( m_p = 938.2720813 \text{ MeV} \), the pion mass \( m_\pi = 139.57061 \text{ MeV} \), the fine structure constant \( \alpha = 0.0072973525698163 \) [23]. In the first row \( n = n_\ell + |\kappa| \) is the principal quantum number, where \( n_\ell \) is the radial quantum number [19].

| State | \( \kappa \) | \( n = 1 \) | \( n = 2 \) | \( n = 3 \) | \( n = 4 \) | \( n = 5 \) |
|-------|------|------|------|------|------|------|
| \( s_{1/2} \) | \(-1\) | -8612.9384350 | -2153.205298 | -956.9862506 | -538.3038471 | -344.5140798 |
| \( p_{1/2} \) | \(+1\) | -2153.205298 | -956.9862506 | -538.3038471 | -344.5140798 |
| \( p_{3/2} \) | \(-2\) | -2153.205298 | -956.9862506 | -538.3038471 | -344.5140798 |
| \( d_{3/2} \) | \(+2\) | -956.9797133 | -538.3019039 | -344.5130848 |
| \( d_{5/2} \) | \(-3\) | -956.9793768 | -538.3019039 | -344.5130848 |
| \( f_{3/2} \) | \(+3\) | -956.9793768 | -538.3019039 | -344.5130848 |
| \( f_{5/2} \) | \(-4\) | -956.9797133 | -538.3019039 | -344.5130848 |
| \( g_{7/2} \) | \(+4\) | -956.9797133 | -538.3019039 | -344.5130848 |
| \( g_{9/2} \) | \(-5\) | -956.9797133 | -538.3019039 | -344.5130848 |

Table 2: The pure Coulomb spectrum of the kaonic atom \((H^+, K^-)\) expressed in eV. \( m_p \) and \( \alpha \) are as in Table 1. We have taken the mass of the \( K \) meson \( m_K = 493.677 \text{ MeV} \) [23]. In the first row \( n = n_\ell + |\kappa| \) is the principal quantum number, where \( n_\ell \) is the radial quantum number [19].

4. The lowest order radiative corrections

The lowest order QED corrections to the spectrum is given by the one-loop electron vacuum polarization or Uehling potential. It takes into account the vacuum polarization produced by the creation of electron-positron pairs, since the contributions due to other particles, such as the lepton \( \mu \), are much smaller. Its form, in terms of the dimensionless variable \( s \) and expressed in proton mass units, is

\[
V_\nu(s) = -\frac{2\alpha^2}{3\pi s} \int_1^\infty e^{-\frac{2m_e}{m_p} sx} \sqrt{x^2 - 1} \left( \frac{2x^2 + 1}{2x^4} \right) dx
\]

(4.1)

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where $m_p$ is the proton mass and $m_e$ the electron mass. It can be expressed in terms of special functions

$$V_U(s) = \frac{2\alpha^2}{3\pi s} \left( \frac{m_e s}{4m_p} G_{30}^{13} \left( 0, -1/2, -1/2 \left( \frac{m_e}{m_p} s \right)^2 \right) + \left( \frac{m_e s}{4m_p} \right)^3 G_{13}^{20} \left( -1/2, -1, -3/2 \right) \left( \frac{m_e}{m_p} s \right)^2 \right)$$  \hspace{1cm} (4.2)$$

where $G_{pq}^{rs}(\zeta_1,...,\zeta_q | \zeta_r)$ is the Meijer G function [24].

The next order corrections are given by the two-loop irreducible part of the vacuum polarization, also known as Källen-Sabry potential

$$V_{KS}(s) = -\frac{\alpha^3}{3\pi^2 s} \int_0^1 x e^{-\frac{2m_e}{m_p} \frac{s}{\sqrt{1-x^2}}} \left[ x(3-x^2) (3-2 \ln \left( \frac{1-x}{4} \right) - 2 \ln(x)) + \ln \left( \frac{1+x}{1-x} \right) \left( \frac{11}{16} (3-x^2)(1+x^2) + \frac{x^4}{4} \right) \right] x^2 - (3-x^2)(1+x^2) \left( \text{Li}_2 \left( \frac{1-x}{1+x} \right) + 2\text{Li}_2 \left( \frac{1-x}{1+x} \right) + \text{Li}_2 \left( \frac{1+x}{1-x} \right) \left( \frac{3}{2} \ln \left( \frac{1+x}{2} \right) - \ln(x) \right) \right) + \frac{3}{8} (5-3x^2) dx$$  \hspace{1cm} (4.3)$$

where $\text{Li}_2(x)$ is the Dilogarithm function.

We finally add the contribution of the reducible part of the two-loop vacuum polarization potential, namely

$$V_R(s) = -\frac{\alpha^3}{9\pi^2 s} \int_0^1 x^2 e^{-\frac{2m_e}{m_p} \frac{s}{\sqrt{1-x^2}}} \left( 1 - x^2 \right) \left( 16 - 6x^2 + 3x(3-x^2) \ln \left( \frac{1-x}{1+x} \right) \right)$$  \hspace{1cm} (4.4)$$

The detailed results of each one of the corrections on each atomic level are shown in Appendix II, together with some comments. Here below we give the QED-corrected transitions from the excited to the ground state for $(H^+, \pi^-)$ and $(H^+, K^-)$.

| Transition     | $n = 2$ | $n = 3$ | $n = 4$ | $n = 5$ |
|----------------|---------|---------|---------|---------|
| $^s s_{1/2} \rightarrow ^d s_{1/2}$ | 2429.2202909 | 2878.7969532 | 3036.1184511 | 3108.9286213 |
| $^p p_{1/2} \rightarrow ^d s_{1/2}$ | 2429.5729174 | 2878.8990487 | 3036.1611897 | 3108.9504262 |
| $^d d_{1/2} \rightarrow ^s s_{1/2}$ | - | 2878.9112441 | 3036.1663831 | 3108.9530963 |
| $^p f_{1/2} \rightarrow ^s s_{1/2}$ | - | - | 3036.1668317 | 3108.9533385 |
| $^g g_{1/2} \rightarrow ^s s_{1/2}$ | - | - | - | 3108.9533999 |

Table 3: The values in eV of the transitions from the excited to the ground state of $(H^+, \pi^-)$. The QED corrected value of the ground level is $E_{0, (H^+, \pi^-)} = -3238.3591996$ eV.

| Transition     | $n = 2$ | $n = 3$ | $n = 4$ | $n = 5$ |
|----------------|---------|---------|---------|---------|
| $^s s_{1/2} \rightarrow ^s s_{1/2}$ | 6479.1535386 | 7677.1203701 | 8096.2083183 | 8290.1408666 |
| $^p p_{1/2} \rightarrow ^s s_{1/2}$ | 6480.8162040 | 7677.6072875 | 8096.4213611 | 8290.2449542 |
| $^d d_{1/2} \rightarrow ^s s_{1/2}$ | - | 7677.7851662 | 8096.4842761 | 8290.2811709 |
| $^p f_{1/2} \rightarrow ^s s_{1/2}$ | - | - | 8096.5009525 | 8290.2897706 |
| $^g g_{1/2} \rightarrow ^s s_{1/2}$ | - | - | - | 8290.2904556 |

Table 4: The values in eV of the transitions from the excited to the ground state of $(H^+, K^-)$. The QED corrected value of the ground level is $E_{0, (H^+, K^-)} = -8634.8934971$ eV.

We have done a comparison with experimental evaluations of the transitions between higher levels of the pionic atom and we have found complete agreement within the experimental errors [25][26]. The experimental results for kaonic atom are more rare and affected by larger errors [27]. There is however agreement between our numerical values and those reported in [27] taken from [28].
5. Conclusions.

In this paper we have calculated levels and states, up to \( n = 5 \), of the pionic and kaonic Hydrogen atoms in a two body quantum relativistic framework for a scalar and a fermion. The original four dimensional system is reduced to a pair of first order differential equations formally identical to a Dirac equation in a central potential, but for the eigenvalue. This fact, together with the not purely Coulomb interaction, implies the non conservation of the Johnson-Lippmann operator \([29]\) and thus the splitting of the levels having the same \( n \) and \( \kappa \) of equal absolute value but opposite sign. This is quite evident for \( n = 2 \), much less for increasing values of \( n \) and \(|\kappa|\), as shown in Tables 1 and 2. The radiative corrections, Table 5, share the same behavior, with the hints concerning the contributions due to \( V_K \). The numerical calculations have been carried on with high accuracy, in order to give a clearcut distinction and enumeration of the levels although, nowadays, they still cannot be experimentally observed. In any case the great physical relevance of the mesic atoms is well known: for instance many decades ago Dalitz underlined the great importance of the “definitive determination of the energy level shifts in the K-Hydrogen and Deuterium, because of their direct connection with the physics of K-nucleon interaction and their complete independence from all other kinds of measurements which bear on this interaction” (quoted by \([9]\)). This justifies the efforts in order to obtain the electromagnetic spectrum as precise as possible. The experiments now in progress mainly in kaonic atoms \([9, 30]\) will hopefully give a breakthrough in this direction.

Appendix I. Variables, kinematics, dynamics.

In previous papers we have explained in detail the kinematic setting and the covariance of the treatment. Referring to those papers for details, we report here, for the sake of completeness, a brief synthesis useful to contextualize the definitions of dimensional and dimensionless variables which appear in our equations. We are given the phase space of two relativistic points, spanned by the Minkowski coordinates \( x^\mu_1, x^\mu_2 \) and conjugate momenta \( p^\mu_1, p^\mu_2 \), the Lorentz metric tensor being \( \eta_\mu^\nu = \text{diag}(1, -1, -1, -1) \). It is first instrumental to introduce a canonical transformation to ‘relative’ and ‘global’ coordinates \( \tilde{\mu} = x^\mu_1 - x^\mu_2, \ X^\mu = (1/2) (x^\mu_1 + x^\mu_2) \), with conjugate momenta \( \tilde{\nu} = (1/2) (p^\mu_1 - p^\mu_2) \). \( P^\mu = p^\mu_1 + p^\mu_2 \).

We then make a second canonical transformation defined by its action on the momenta: it acts as the identity on \( P^\mu \) and transforms \( \tilde{\nu} \) by boosting it to the \( P = 0 \) frame, thus defining \( q_\alpha = \tilde{e}_\alpha^\mu (P) \bar{q}_\alpha \), where \( \alpha = (0, A), A = 1, 2, 3 \) and \( \tilde{e}_\alpha^\mu (P) = L^{-1}(P) e_\alpha^\nu \) is the matrix of the Lorentz boost. This is a point-like transformation easily completed by the generating function yielding the corresponding coordinates conjugate to \( q_\alpha \) and \( P^\mu \), namely: \( r_\alpha = \tilde{e}_\alpha^\nu (P) \bar{r}_\alpha \) and \( Z^\nu = X^\nu + P^{-2} (P^\mu \tilde{q}_\nu - P^\nu \tilde{q}_\mu) \bar{q}_\mu + (P_0 + P)^{-1} W^{\nu} \), where \( W^{\nu} \) is the Pauli-Lubanski tensor. It is then obvious that \( q_0 \) and \( r_0 \) are Lorentz invariant such as \( q = (q_\alpha q_\alpha)^{1/2} \), \( r = (r_\alpha r_\alpha)^{1/2} \), since \( q = (q_\alpha) \) and \( r = (r_\alpha) \) are Wigner vectors. Moreover \( Z \) is a position vector of the Newton-Wigner type and \( Z^0 \) has the covariance of a Lorentz time. The essential point is that the mass shell conditions \( p_1^2 = m_1^2 \) and \( p_2^2 = m_2^2 \) expressed in these coordinates, amount to the elimination of \( q_0 \) from the resulting energy (and Hamiltonian) of the relative system: \( (m_1^2 + q^2)^{1/2} + (m_2^2 + q^2)^{1/2} \) which plays the role of the invariant mass. We then see that \( Z^0 \) remains the only one time in the system, since the relative time \( r_0 \) is a cyclic variable and disappears: we have therefore realized a canonical reduction of the phase space where \( m_1 \) and \( m_2 \) appear as fixed parameters, overcoming the problem of the relative energy which becomes an irrelevant quantity. The cyclic time \( r_0 \) assumes the role of a gauge-type variable, arbitrarily fixed in order to recover separate descriptions of the two particles world lines. The physical mass spectrum is determined by the invariant Hamiltonian completed with invariant interaction terms depending on \( q \) and \( r \). Thus the action of the Poincaré group is well defined and the dynamics is covariant. Finally, the quantization is done in the tensor product of the particle Hilbert spaces, using gamma matrices for fermions and an adapted Feshbach-Villars representation for the scalars, so that the square roots are eliminated from \( H \) in any case.
Appendix II. The lowest order radiative correction.

We report here the detailed list of the QED corrections to the levels of Tables 1 and 2.

| State | $\kappa$ | $V_H(H^+,\pi^-)$ | $V_{KS}(H^+,\pi^-)$ | $V_K(H^+,K^-)$ | $V_{KS}(H^+,K^-)$ | $V_K(H^+,K^-)$ |
|-------|--------|-----------------|-----------------|----------------|----------------|----------------|
| $1s_{1/2}$ | $-1$ | $-3.2413174$ | $-2.0900666 \times 10^{-2}$ | $3.7955745 \times 10^{-3}$ | $-21.7953398$ | $-1.1473759 \times 10^{-3}$ |
| $2s_{1/2}$ | $-1$ | $-3.6831740 \times 10^{-1}$ | $-2.2911091 \times 10^{-1}$ | $4.8071080 \times 10^{-1}$ | $-2.4146932$ | $-1.2917109 \times 10^{-1}$ |
| $2p_{1/2}$ | $+1$ | $-3.5796665 \times 10^{-2}$ | $-3.7779853 \times 10^{-2}$ | $-2.5718395 \times 10^{-5}$ | $-7.7448229 \times 10^{-1}$ | $-6.5377131 \times 10^{-1}$ |
| $2p_{3/2}$ | $-2$ | $-3.5796506 \times 10^{-2}$ | $-3.7779729 \times 10^{-2}$ | $-2.5718666 \times 10^{-5}$ | $-7.7447115 \times 10^{-1}$ | $-6.5376474 \times 10^{-1}$ |
| $3s_{1/2}$ | $-1$ | $-1.0756349 \times 10^{-1}$ | $-6.6578532 \times 10^{-2}$ | $1.4242775 \times 10^{-4}$ | $-6.9476605 \times 10^{-1}$ | $-3.6514009 \times 10^{-1}$ |
| $3p_{1/2}$ | $+1$ | $-1.1407182 \times 10^{-2}$ | $-1.1722417 \times 10^{-4}$ | $-7.3772433 \times 10^{-6}$ | $-2.1392561 \times 10^{-1}$ | $-1.7166863 \times 10^{-1}$ |
| $3p_{3/2}$ | $-2$ | $-1.1407110 \times 10^{-2}$ | $-1.1722378 \times 10^{-4}$ | $-7.3772720 \times 10^{-6}$ | $-2.1392247 \times 10^{-1}$ | $-1.7166496 \times 10^{-1}$ |
| $3d_{5/2}$ | $+2$ | $4.4463586 \times 10^{-4}$ | $6.3140024 \times 10^{-6}$ | $7.0773072 \times 10^{-7}$ | $3.8501657 \times 10^{-2}$ | $-4.4371663 \times 10^{-4}$ |
| $3d_{3/2}$ | $-3$ | $-4.4463503 \times 10^{-4}$ | $6.3139928 \times 10^{-6}$ | $7.0772989 \times 10^{-7}$ | $3.8501463 \times 10^{-2}$ | $-4.4371479 \times 10^{-4}$ |
| $4s_{1/2}$ | $-1$ | $-4.5156315 \times 10^{-6}$ | $-2.7904635 \times 10^{-6}$ | $6.0079123 \times 10^{-8}$ | $-2.9047328 \times 10^{-1}$ | $-1.5204619 \times 10^{-1}$ |
| $4p_{1/2}$ | $+1$ | $-4.9207863 \times 10^{-3}$ | $-5.0167795 \times 10^{-3}$ | $3.0743111 \times 10^{-6}$ | $-8.9047505 \times 10^{-2}$ | $-7.0538006 \times 10^{-2}$ |
| $4p_{3/2}$ | $-2$ | $-4.9207642 \times 10^{-3}$ | $-5.0167631 \times 10^{-3}$ | $3.0743244 \times 10^{-6}$ | $-8.9046201 \times 10^{-2}$ | $-7.0537249 \times 10^{-2}$ |
| $4d_{5/2}$ | $+2$ | $-2.4729903 \times 10^{-4}$ | $-3.4598119 \times 10^{-4}$ | $-3.8489862 \times 10^{-7}$ | $-1.8088916 \times 10^{-2}$ | $-1.9925191 \times 10^{-2}$ |
| $4d_{3/2}$ | $-3$ | $-2.4729857 \times 10^{-4}$ | $-3.4598067 \times 10^{-4}$ | $-3.8489856 \times 10^{-7}$ | $-1.8088823 \times 10^{-2}$ | $-1.9925089 \times 10^{-2}$ |
| $5s_{1/2}$ | $+1$ | $-5.0484980 \times 10^{-6}$ | $-6.4383418 \times 10^{-6}$ | $9.7052084 \times 10^{-8}$ | $-1.8361008 \times 10^{-3}$ | $-2.6515938 \times 10^{-3}$ |
| $5p_{1/2}$ | $+4$ | $-5.0484980 \times 10^{-6}$ | $-6.4383351 \times 10^{-6}$ | $9.7051988 \times 10^{-8}$ | $-1.8360956 \times 10^{-3}$ | $-2.6515873 \times 10^{-3}$ |
| $5p_{3/2}$ | $+4$ | $-5.0484980 \times 10^{-6}$ | $-6.4383418 \times 10^{-6}$ | $9.7052084 \times 10^{-8}$ | $-1.8361008 \times 10^{-3}$ | $-2.6515938 \times 10^{-3}$ |

Table 5: QED corrections to the $(H^+,\pi^-)$ and $(H^+,K^-)$ levels. In the first two columns we indicate the state and the corresponding $\kappa$. In the last six columns of the first line we specify the atomic systems and the potentials responsible for the energy corrections. The results are expressed in eV.

Appendix III. How the reduced mass can be brought to bear in a two-body relativistic framework.

The notion of ‘reduced mass’ comes from the reduction of the Newtonian two-body dynamics and is indeed artificial in a relativistic formulation of a two-body problem. Despite this, however, this quantity may produce some benefit even in a relativistic treatment, giving an approximation of the levels to the order $a$. The accuracy could possibly be increased by carrying on series expansions to higher orders. Indeed we are able to single out the interaction term of (2.14) leading to the reduced mass and we give some arguments to show the connection.

We recall that the levels of the Dirac equation in a Coulomb field with coupling constant $a$, in units of the fermion mass have the well known analytic expression

$$\epsilon_0(a,\kappa,n_r) = \left(1 + \frac{a^2}{n_r + \sqrt{\kappa^2 - a^2}}\right)^{-1/2} - 1$$

(III.1)

where $n_r$ is the radial quantum number of the target. For future convenience we observe that, for constant $c$,

$$\epsilon_0(c,a,\kappa,n_r) = c^2 \epsilon_0(a,\kappa,n_r) + O(a^4)$$

(III.2)
Taking into account the definitions of the parameters, if we develop the two equations (2.16) to the first order in $\alpha$ and the subsequent result to the first order in $\epsilon$ we find a new Dirac equation in the Coulomb field with interaction constant $\mu_R \alpha$ and eigenvalue $\mu_R \epsilon$, namely

$$\frac{d}{ds}\phi(s) + \frac{\kappa}{s}\phi(s) - \left(\frac{\mu_R \alpha}{s} + \mu_R \epsilon + 2\right)\chi(s) = 0$$

$$\frac{d}{ds}\chi(s) - \frac{\kappa}{s}\chi(s) + \left(\frac{\mu_R \alpha}{s} + \mu_R \epsilon\right)\phi(s) = 0$$

(III.3)

where $\mu_R = \sigma/(1 + \sigma)$. Therefore, using the relation (III.2) with $c = \mu_R$, the solution to the first order in $\alpha$ of the original system (2.16) is given by

$$\epsilon = \mu_R \epsilon_0(\alpha, \kappa, n)$$

(III.4)

Multiplying both sides by the fermion mass $m_F$, we finally see that

$$E = m_F \epsilon = m_R \epsilon_0(\alpha, \kappa, n)$$

(III.5)

where $m_R = m_S m_F/(m_S + m_F)$ is the classical reduced mass. Notice that for $\sigma = 1$, namely when the scalar and fermion masses are equal, the coefficient $B$ is vanishing and (2.16) is exactly the Dirac equation (III.3) with $\mu_R = 1/2$. In this case the Coulomb coupling constant is $\alpha/2$ and $\Lambda = \lambda/2$. The perturbation treatment of (2.16) could start from the eigenvalues (II.1) with $\alpha$ substituted by $\mu_R \alpha$. The corresponding eigenfunctions of (III.3)

$$\phi = e^{-a s} (as)^{-1} + \frac{\gamma (1 + \mu_R \epsilon)^{1/2}}{N_1 L_{n-1}^{2y}(2as) + N_2 L_n^{2y}(2as)}$$

$$\chi = e^{-a s} (as)^{-1} - \frac{\gamma (1 - \mu_R \epsilon)^{1/2}}{N_1 L_{n-1}^{2y}(2as) + N_2 L_n^{2y}(2as)}$$

(III.6)

where $L_n^m$ are the associated Legendre polynomials $\gamma = (\kappa^2 + \mu_R^2 \alpha^2)^{1/2}$, $\alpha = \mu_R \alpha (\kappa^2 + n^2 + 2n \gamma)^{-1/2}$ and the normalization constants $N_1, N_2$ can be recovered from [19]. We finally observe that, being a genuine Dirac equation, (III.3) conserves the Johnson-Lippmann operator which is not conserved in (2.16). This suggests that indeed the breaking has to be very small, as confirmed by the numerical results.

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