On Chow weight structures without projectivity and resolution of singularities

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Abstract

In this paper certain Chow weight structures on the "big" triangulated motivic categories $\mathcal{DM}^{eff}_R \subset \mathcal{DM}_R$ are defined in terms of motives of all smooth varieties over the base field. This definition allows studying basic properties of these weight structures without applying resolution of singularities; thus we don’t have to assume that the coefficient ring $R$ contains $1/p$ in the case where the characteristic $p$ of the base field is positive. Moreover, in the case where $R$ satisfies the latter assumption our weight structures are "compatible" with the weight structures that were defined in previous papers in terms of Chow motives; it follows that a motivic complex has non-negative weights if and only if its positive Nisnevich hypercohomology vanishes. The results of this article yield certain Chow-weight filtration (also) on $p$-adic cohomology of motives and smooth varieties.

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Introduction

This paper is dedicated to the study of a new definition of Chow weight structures for Voevodsky motives over an arbitrary perfect field $k$ of characteristic $p$. This definition does not depend on the existence of nice compactifications for smooth varieties (and other resolution of singularities results); this allows treating $R$-linear versions of these weight structures (on the triangulated categories $DM^\text{eff}_R \subset DM_R$) also in the case where $p$ is positive and not invertible in the coefficient ring $R$.

Now recall that Chow weight structures yield analogues of Deligne’s weights (as described for mixed Hodge structures in [Del71a] and for mixed étale sheaves in [Del71b]) for various triangulated categories of Voevodsky motives. For motives over a field certain Chow weight structures were described in [Bon10] and [Bon11] (the latter paper treated the case $p > 0$). In these articles the fact that the categories of geometric (i.e., compact) motives are generated by their subcategories of Chow motives (i.e., by $Chow^\text{eff}$ and Chow, respectively) was applied. It yields the existence of bounded weight structures on the categories $DM^\text{eff}_gm$ and $DM_gm$ of geometric motives; their hearts consist of the corresponding categories of Chow motives. Moreover, these weight structures can be extended to the corresponding "big" motivic categories (that are compactly generated by their subcategories of geometric motives; cf. Proposition 1.7 and §2.1 of [Bon18a] or §2.3 of [BoL16]). In contrast to Deligne’s weights, this gave Chow weight structures for $\mathbb{Z}[1/p]$-linear motives (in the case $p > 0$ we set $\mathbb{Z}[1/p] = \mathbb{Z}$); respectively, there exist weight filtrations and spectral sequences for all $\mathbb{Z}[1/p]$-linear (co)homology of motives. Moreover, the theory of weight structures gives non-trivial functoriality properties of these matters. Similarly to several other properties of motives, the construction of weight structures in the aforementioned papers relied on the resolution of singularities (i.e., on the Hironaka theorem in the case $p = 0$ and on the Gabber’s resolution of singularities in the setting of $\mathbb{Z}[1/p]$-linear motives for $p > 0$).

In the current paper a new construction method is applied; it gives a certain Chow weight structure $w^\text{eff}_{Chow}$ "directly" on the category $DM^\text{eff}_R = DM^\text{eff}_R(k)$ of unbounded $R$-linear motivic complexes, where $R$ is an arbitrary (commutative unital) coefficient ring; this weight structure is generated by
motives of all smooth varieties over $k$. This definition (in contrast to earlier
ones based on smooth projective varieties only) does not depend on any
resolution of singularities results; so it also "works fine" for $\mathbb{Z}$-linear motives
over any perfect field $k$ of characteristic $p > 0$; see Remark 2.1.3(4) below. A
disadvantage of this method is that it does not yield that $w_{\text{eff}}^{\text{Chow}}$ is generated
by Chow motives and restricts to the subcategory $\text{DM}^{\text{eff},R}(k)$ of geometric
motives. Still we successfully establish several other properties of $w_{\text{eff}}^{\text{Chow}}$ that
are similar to the main properties of the Chow weight structures defined
earlier. In particular, we prove that $w_{\text{eff}}^{\text{Chow}}$ can be naturally extended to a
weight structure $w_{\text{Chow}}$ on the so-called stable motivic category $\text{DM}_R$ (that
contains $\text{DM}^{\text{eff},R}_R$).

Now we describe the contents of the paper; some more information of this
sort can also be found in the beginnings of sections.

In §1 we give some categorical notation and definitions, and recall the
basics of the theory of weight structures (on compactly generated triangul-
ated categories). The most complicated part of the section is the recollection
of the properties of the category $\text{DM}^{\text{eff},R}_R$ (or $R$-linear motivic complexes) for
an arbitrary coefficient ring $R$; since the existing literature is mostly dedi-
cated to the case $R \subset \mathbb{Q}$, we are forced to compare different definitions of
$\text{DM}^{\text{eff},R}_R$. Note here that in the ("basic") cases $R = \mathbb{Z}$ and $R = \mathbb{Z}[1/p]$ those
arguments of our papers that concern "general" properties of motives can
be significantly simplified (for instance, one may apply the results of [Deg11]
and [Kel12]).

In §2 our main weight structure $w_{\text{eff}}^{\text{Chow}}$ on the category $\text{DM}^{\text{eff},R}_R$ is defined;
we prove some of its properties. In particular, we prove that $w_{\text{eff}}^{\text{Chow}}$ is generated
by $R$-linear Chow motives if either $p$ is zero or $p$ is invertible in $R$. Thus
we obtain the compatibility of the Chow weight structures defined earlier
with $w_{\text{eff}}^{\text{Chow}}$; it follows that a motivic complex has non-negative weights if
and only if its positive Nisnevich hypercohomology vanishes. Moreover, all
the scalar extension functors $- \otimes_R^{\text{mot}} R'$ (where $R'$ is a commutative unital
$R$-algebra) are weight-exact.

In §3 we prove that $w_{\text{eff}}^{\text{Chow}}$ naturally "induces" certain weight structures on
the categories of stable and birational motives (i.e., we consider a compactly
generated category $\text{DM}_R$ on which the action of the Tate twist $- \langle 1 \rangle = -(1)[2]$
is invertible, and the "birational" localization $\text{DM}^{\text{eff},R}_R/\text{DM}^{\text{eff},R}_R(1)$). These
statements are similar to the corresponding properties of Chow weight struc-

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1Weight structures generated by sets of compact objects were constructed in [Pau12];
their properties were studied in detail in [Bon16]. It is also worth noting that in several
papers (starting from [Wil09]) J. Wildeshaus has used motives of non-proper $k$-varieties
for the construction of certain motives in the heart of the Chow weight structure.
tures that were established in previous papers as well.

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1 Preliminaries

In §1.1 we introduce some definitions and notation that are mostly related to triangulated categories; we also prove an easy lemma.

In §1.2 we recall the basics of the theory of weight structures.

In §1.3 we briefly recall the basics on (unbounded) Voevodsky motivic complexes with coefficients in an arbitrary ring \( R \). We also prove some properties of the motivic extension of scalars functors.

1.1 Categorical definitions and notation

- Given a category \( \mathcal{B} \) and \( M, N \in \text{Obj} \mathcal{B} \), we say that \( M \) is a retract of \( N \) if \( \text{id}_M \) can be factored through \( N \) (recall that if \( \mathcal{B} \) is triangulated then \( M \) is a retract of \( N \) if and only if \( M \) is its direct summand).

- A subcategory \( \mathcal{D} \) of \( \mathcal{B} \) is said to be Karoubi-closed in \( \mathcal{B} \) if it contains all \( \mathcal{B} \)-retracts of its objects.

- The full subcategory \( \text{Kar}_\mathcal{B}(\mathcal{D}) \) of \( \mathcal{B} \) whose objects are all \( \mathcal{B} \)-retracts of objects of \( \mathcal{D} \) will be called the Karoubi-closure of \( \mathcal{D} \) in \( \mathcal{B} \). It is easily seen that \( \text{Kar}_\mathcal{B}(\mathcal{D}) \) is Karoubi-closed in \( \mathcal{D} \); if \( \mathcal{B} \) and \( \mathcal{D} \) are additive then \( \text{Kar}_\mathcal{B}(\mathcal{D}) \) is additive as well.

- We will say that an additive category \( \mathcal{D} \) is Karoubian if any its idempotent endomorphism is isomorphic to the composition of a retraction and a coretraction of the type \( M \oplus N \rightarrow M \rightarrow M \oplus N \).

- The symbol \( \mathcal{C} \) below will always denote some triangulated category. For a given class \( \mathcal{P} \subset \text{Obj} \mathcal{C} \) we will write \( \langle \mathcal{P} \rangle \) for the smallest full Karoubi-closed triangulated subcategory \( \mathcal{D} \) of \( \mathcal{C} \) such that \( \mathcal{P} \subset \text{Obj} \mathcal{D} \).

- For any \( A, B, C \in \text{Obj} \mathcal{C} \) we will say that \( C \) is an extension of \( B \) by \( A \) if there exists a distinguished triangle \( A \rightarrow C \rightarrow B \rightarrow A[1] \). A class \( \mathcal{P} \subset \text{Obj} \mathcal{C} \) is said to be extension-closed if it is closed with respect to extensions and contains 0.

- The smallest extension-closed Karoubi-closed class \( \mathcal{P}' \subset \text{Obj} \mathcal{C} \) containing \( \mathcal{P} \) will be called the envelope of \( \mathcal{P} \).
• For $M, N \in \text{Obj } C$ we will write $M \perp N$ if $C(M, N) = \{0\}$. For $D, E \subset \text{Obj } C$ we write $D \perp E$ if $M \perp N$ for all $M \in D, N \in E$.

• Given $\mathcal{P} \subset \text{Obj } C$ we will write $\mathcal{P}^\perp$ for the class
  \[ \{ N \in \text{Obj } C : M \perp N \forall M \in \mathcal{P} \}. \]
  Dually, $^\perp \mathcal{P} = \{ M \in \text{Obj } C : M \perp N \forall N \in \mathcal{P} \}$.

• Assume that $C$ is closed with respect to (small) coproducts (we only consider small coproducts in this paper). For $D \subset C$ ($D$ is a triangulated category that may be equal to $C$) one says that $\mathcal{P}$ generates $D$ as a localizing subcategory of $C$ if $D$ is the smallest full strict triangulated subcategory of $C$ that contains $\mathcal{P}$ and is closed with respect to $C$-coproducts.

• $M \in \text{Obj } C$ is said to be compact if the functor $C(M, -) : C \to \text{Ab}$ respects coproducts.

• $C$ is said to be compactly generated if it is generated by a set of compact objects as its own localizing subcategory.

We will sometimes need the following properties of compactly generated triangulated categories.

**Lemma 1.1.1.** Assume that the category $C$ is generated as its own localizing subcategory by objects of its triangulated subcategory $C'$, the objects of $C'$ are compact in $C$, and $F : C \to D$ is an exact functor (so, the category $D$ is triangulated) that respects coproducts.

1. If $C'$ is small then there exists an exact functor $G : D \to C$ right adjoint to $F$.

2. Assume that the restriction of $F$ to $C'$ is a full embedding that sends objects of $C'$ into compact objects of $D$. Then $F$ is fully faithful.

Moreover, if the class $F(\text{Obj } C')$ generates $D$ as its own localizing subcategory then $F$ is an equivalence of categories.

3. Let $E$ be a set of objects of $\text{Obj } C'$, and denote by $\mathcal{E}$ the localizing subcategory of $C$ generated by $E$. Then the localization $C/E$ exists (i.e., the morphism classes in $C/E$ are sets), and it is closed with respect to coproducts. Moreover, the localization functor $L : C \to C/E$ respects coproducts and induces an equivalence of $\text{Kar}(Kar(C')/E')$ with the full subcategory of compact objects of $C/E$, and the class $L(\text{Obj } C')$ generates $C/E$ as its own localizing subcategory.
Proof. 1. See Theorem 8.4.4 and Lemma 5.3.6 of [Nee01].

2. Since the objects of $C'$ are compact in $C$ and their images are compact in $D$, the class $C_1$ of those $N \in \text{Obj } C$ such that the maps $C(M, N) \rightarrow D(F(M), F(N))$ are bijective for all $M \in \text{Obj } C'$, is closed with respect to coproducts. $C_1$ is also shift-invariant (since $\text{Obj } C'[1] = \text{Obj } C'$); hence it is also extension-closed. Since $C_1$ contains $\text{Obj } C'$, we obtain $C_1 = \text{Obj } C$. Next, the class $C_2$ of those $M \in \text{Obj } C$ such that the maps $C(M, N) \rightarrow D(F(M), F(N))$ are bijective for all $N \in \text{Obj } C$ is obviously closed with respect to coproducts, shifts, and extensions (here we apply the assumption that $F$ respects coproducts once again). As we have just proved, $C_2$ contains $\text{Obj } C'$; thus it coincides with $\text{Obj } C'$, i.e., $F$ is fully faithful.

The second part of the assertion easily follows from Proposition 1.1.4 of [Bon16].

3. These statements also follow from the results of [Nee01] easily; indeed, the easy arguments described in the proof of [BoS16, Proposition 4.3.1.3(III.1–2)] demonstrate that they follow from Theorem 8.3.3, Proposition 9.1.19, and Theorem 4.4.9 of [Nee01].

Now we introduce some "geometric" notation.

Our base field will be denoted by $k$; we assume that it is perfect (and fixed). We will write $p$ for its characteristic ($p$ may equal 0). Moreover, if $p = 0$ then the symbol $\mathbb{Z}[1/p]$ will denote the ring $\mathbb{Z}$.

$\text{SmVar}$ is the set of smooth (not necessarily connected) $k$-varieties.

We will use the notation $R$ for the "main" coefficient ring for motives in this paper; $R$ will always be a commutative associative unital ring.

1.2 Weight structures

Recall that $C$ will always denote some triangulated category in the current paper.

Definition 1.2.1. A couple of subclasses $C_{w \leq 0}$ and $C_{w \geq 0} \subset \text{Obj } C$ will be said to define a weight structure $w$ for a triangulated category $C$ if they satisfy the following conditions.

(i) $C_{w \leq 0}$ and $C_{w \geq 0}$ are Karoubi-closed in $C$ (i.e., contain all $C$-retracts of their elements).

(ii) Semi-invariance with respect to translations.

$C_{w \leq 0} \subset C_{w \leq 0}[1]$ and $C_{w \geq 0}[1] \subset C_{w \geq 0}$.

(iii) Orthogonality.

$C_{w \leq 0} \perp C_{w \geq 0}[1]$.

(iv) Weight decompositions.
For any $M \in \text{Obj } \mathcal{C}$ there exists a distinguished triangle

$$LM \to M \to RM \to LM[1]$$

such that $LM \in \mathcal{C}_{w\leq0}$ and $RM \in \mathcal{C}_{w\geq0}[1]$.

We will also need the following definitions.

**Definition 1.2.2.**
1. The full subcategory $H_{w} \subset \mathcal{C}$ whose object class is $\mathcal{C}_{w=0} = \mathcal{C}_{w\geq0} \cap \mathcal{C}_{w\leq0}$ is called the **heart** of $w$.

2. For $i \in \mathbb{Z}$ we will use the notation $\mathcal{C}_{w\geq i}$ (resp. $\mathcal{C}_{w\leq i}$, resp. $\mathcal{C}_{w=i}$) for the class $\mathcal{C}_{w=0}[i]$ (resp. $\mathcal{C}_{w\leq0}[i]$, $\mathcal{C}_{w=0}[i]$).

3. We will say that a weight structure $w$ is **generated** by a class $P \subset \text{Obj } \mathcal{C}$ if $\mathcal{C}_{w\leq0} = (\cup_{i>0} P[-i])^\perp$.

4. Let $\mathcal{C}'$ be a triangulated category endowed with a weight structure $w'$; let $F : \mathcal{C} \to \mathcal{C}'$ be an exact functor.

We will say that $F$ is *left weight-exact* (with respect to $w, w'$) if it maps $\mathcal{C}_{w\leq0}$ into $\mathcal{C}'_{w\leq0}$; it will be called *right weight-exact* if it sends $\mathcal{C}_{w\geq0}$ into $\mathcal{C}'_{w\geq0}$. $F$ is said to be *weight-exact* if it is both left and right weight-exact.

A collection of basic properties of weight structures can be found in §2 of [Bon16].

**Proposition 1.2.3.** I. Let $w$ be a weight structure on $\mathcal{C}$.
1. Then $\mathcal{C}_{w\leq0} = {}^\perp \mathcal{C}_{w\geq1}$ and $\mathcal{C}_{w\geq0} = \mathcal{C}_{w\leq1} {}^\perp$. Thus if $w$ is generated by a class $P$ then $P \subset \mathcal{C}_{w\geq0}$.
2. For each $i \in \mathbb{Z}$ the classes $\mathcal{C}_{w\geq i}$ and $\mathcal{C}_{w\leq i}$ are extension-closed (hence they are additive).

II. Assume that $\mathcal{C}$ is compactly generated.
1. Let $P$ be a set of compact objects of $\mathcal{C}$. Then there exists (a unique) weight structure on $\mathcal{C}$ that is generated by $P$.
2. Assume that an exact functor $F : \mathcal{C} \to \mathcal{C}'$ respects coproducts, $w$ is a weight structure on $\mathcal{C}$ that is generated by some class $P \subset \text{Obj } \mathcal{C}$, and $w'$ is a weight structure on $\mathcal{C}'$. Then $F$ is left weight-exact if and only if $F(P) \subset \mathcal{C}_{w\leq0}$.
3. Assume in addition that $F$ is surjective on objects. Then $F$ is weight-exact if and only if $w'$ is generated by $F(P)$.

These statements were actually proved in [Bon10] (cf. also Remark 1.2.3(4) of [BoST18]); yet in that paper somewhat distinct notation for weight structures was used.
We briefly recall the basics of the theory of (\(R\)-linear unbounded) Voevodsky motivic complexes. The case of an arbitrary (associative commutative unital) \(R\) was not treated in detail in the literature; yet it is not much different from the "main" case \(R = \mathbb{Z}\) (or \(R\) being a localization of \(\mathbb{Z}\); cf. §5 of [Kel12]).

\(R\)-linear unbounded motivic complexes are defined for an arbitrary \(R\); see [BeVo08 §2.3].

We start from the description of the category \(DM^\mathfrak{eff}_R\) given in loc. cit. One takes the additive category \(\text{SmCor}\) of Voevodsky smooth correspondences (the notation is taken from [Voe00]); so, \(\text{Obj SmCor} = \text{SmVar}\) and the morphisms in \(\text{SmCor}\) are algebraic analogues of multi-valued functors. \(\text{PreSh}_{\text{Nis}}(\text{SmCor}, R)\) will denote the abelian category of additive contravariant functors from \(\text{SmCor}\) into \(R\)-\text{Mod}.

For \(X \in \text{SmVar}\) we will use the notation \(R_{tr}(X)\) for the functor \(Y \mapsto \text{SmCor}(Y, X) \otimes_{\mathbb{Z}} R\); this is an object of \(\text{PreSh}_{\text{Nis}}(\text{SmCor}, R)\) that we will also consider as a complex (and so, as an object of \(D(\text{PreSh}_{\text{Nis}}(\text{SmCor}, R))\)) whose non-zero term is in degree 0. The object \(R_{tr}(X)\) is certainly compact in \(D(\text{PreSh}_{\text{Nis}}(\text{SmCor}, R))\); hence [Nee01 Theorem 8.3.3] implies that the set of all \(R_{tr}(X)\) generates \(D(\text{PreSh}_{\text{Nis}}(\text{SmCor}, R))\) as its own localizing subcategory.

\(DM^\mathfrak{eff}_R\) is defined as the Verdier quotient of \(D(\text{PreSh}_{\text{Nis}}(\text{SmCor}, R))\) by the localizing subcategory generated by the union of two sets of complexes that we will now describe. The first of these sets will be denoted by \(HI\); its elements are two-term complexes \(R_{tr}(\mathbb{A}^1 \times X) \xrightarrow{pr_X} R_{tr}(X)\) for \(X \in \text{SmVar}\) (here the morphism \(pr_X\) comes from the projection \(\mathbb{A}^1 \times X \to X\)). The second set will be denoted by Zar-MV; its elements are complexes

\[
\begin{align*}
R_{tr}(W) & \xrightarrow{\left(\begin{array}{c} -R_{tr}(k) \\ R_{tr}(g) \end{array}\right)} R_{tr}(Y) \bigoplus R_{tr}(V) & \xrightarrow{\left(\begin{array}{c} R_{tr}(f) \\ R_{tr}(j) \end{array}\right)} R_{tr}(X)
\end{align*}
\] (1.3.1)

\[\text{Recall that bounded above motivic complexes were considered in Lecture 14 of MVW06.}\]
corresponding to all Cartesian squares

$$
\begin{array}{c}
W \\
\downarrow g \\
V
\end{array} \rightarrow
\begin{array}{c}
Y \\
\downarrow f \\
X
\end{array} \quad (1.3.2)
$$

of smooth varieties such that connecting morphisms are open embeddings (hence $W = V \cap Y$) and $Y \sqcup V$ covers $X$. We will write $L'_{\mathcal{R}}$ for the corresponding localization functor and note that this functor respects coproducts and compact objects according to Lemma 1.1.1(3). Moreover, the category $\text{DM}^\text{eff}_{\mathcal{R}}$ is symmetric monoidal (see [Be V08, §2.3]).

Now let us give some alternative descriptions of $\text{DM}^\text{eff}_{\mathcal{R}}$; these are similar to certain constructions in the literature. We will need some more notation.

$\text{Sh}_{\text{Nis}}(\text{SmCor}, R)$ will denote the full subcategory of $\text{PreSh}_{\text{Nis}}(\text{SmCor}, R)$ whose objects are those functors that yield Nisnevich sheaves (on the étale site of SmVar; the objects of $\text{Sh}_{\text{Nis}}(\text{SmCor}, R)$ can be called $R$-linear sheaves with transfers). We recall that this category is abelian also (this is an easy consequence of [MVW06, Theorem 13.1]), and all $R_{\text{tr}}(X)$ are its objects (easy from Lemma 6.2 of ibid.).

We will also need the additive category $\text{SmCor}^\oplus_{\mathcal{R}}$ of all coproducts of sheaves of the type $R_{\text{tr}}(X)$ in the category $\text{Sh}_{\text{Nis}}(\text{SmCor}, R)$ (certainly, $\text{SmCor}^\oplus_{\mathcal{R}}$ is also a full subcategory of $\text{PreSh}_{\text{Nis}}(\text{SmCor}, R)$). We will write $K'(\text{SmCor}^\oplus_{\mathcal{R}})$ for the localizing subcategory of the homotopy category $K(\text{SmCor}^\oplus_{\mathcal{R}})$ that is generated by $\text{Obj} \text{SmCor}^\oplus_{\mathcal{R}}$ (here we consider objects of $\text{SmCor}^\oplus_{\mathcal{R}}$ as one-term complexes; note that $K(\text{SmCor}^\oplus_{\mathcal{R}})$ is closed with respect to coproducts).

**Proposition 1.3.1.** The following categories are canonically equivalent:

1) The localization of $K'(\text{SmCor}^\oplus_{\mathcal{R}})$ by the localizing subcategory $\mathcal{D}$ generated by $HI \cup \text{Zar-MV}$;

2) $\text{DM}^\text{eff}_{\mathcal{R}}$;

3) the localization $\text{DM}^\text{eff}_{\mathcal{R}}'$ of the category $D(\text{Sh}_{\text{Nis}}(\text{SmCor}, R))$ by the localizing subcategory generated by $HI$.

**Proof.** All the objects of the form $R_{\text{tr}}(X)$ (for $X \in \text{SmVar}$) as well as bounded complexes whose terms are of this type are compact in the categories $K'(\text{SmCor}^\oplus_{\mathcal{R}}) \subset K(\text{SmCor}^\oplus_{\mathcal{R}}) \subset K(\text{PreSh}_{\text{Nis}}(\text{SmCor}, R))$ (and also is $D(\text{PreSh}_{\text{Nis}}(\text{SmCor}, R))$). Denote the triangulated subcategory $\langle R_{\text{tr}}(X) : X \in \text{SmVar} \rangle \subset K'(\text{SmCor}^\oplus_{\mathcal{R}})$ by $K'(\text{SmCor}^\oplus_{\mathcal{R}})$. Applying Lemma 1.1.1(3) we obtain that the natural functor $K'(\text{SmCor}^\oplus_{\mathcal{R}})/\mathcal{D} \rightarrow \text{DM}^\text{eff}_{\mathcal{R}}'$ gives an isomorphism between those full subcategories whose objects come from $K'(\text{SmCor}^\oplus_{\mathcal{R}})$. 

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Applying part 2 of the lemma we obtain that the category $K'(\text{SmCor}_R^\oplus)/D$ is equivalent to $\text{DM}^{\text{eff}}_R$.

Now we prove that $\text{DM}^{\text{eff}}_R$ is equivalent to $\text{DM}^{\text{eff}}_R'$. Note that the natural functor $D(\text{PreSh}_{\text{Nis}}(\text{SmCor}, R)) \to D(\text{Sh}_{\text{Nis}}(\text{SmCor}, R))$ respects coproducts and kills all elements of Zar-MV; hence there exists an exact functor $\text{DM}^{\text{eff}}_R \to \text{DM}^{\text{eff}}_R'$ that respects coproducts. Recall also that the Nisnevich sheafifications of objects of $\text{PreSh}_{\text{Nis}}(\text{SmCor}, R)$ (considered as presheaves on the aforementioned site) are objects of $\text{Sh}_{\text{Nis}}(\text{SmCor}, R)$ (i.e., sheaves with transfers); this statement follows from [MVW06, Theorem 13.1] as well. It obviously follows that the category $D(\text{Sh}_{\text{Nis}}(\text{SmCor}, R))$ is equivalent to the localization of $D(\text{PreSh}_{\text{Nis}}(\text{SmCor}, R))$ by the localizing category generated by those presheaves whose sheafification is zero. Hence it suffices to verify that the localizing subcategory of $D(\text{PreSh}_{\text{Nis}}(\text{SmCor}, R))$ generated by $HI \cup \text{Zar-MV}$ contains all presheaves of this sort. This fact can be easily justified using an argument used in the proof of [Voe00, Theorem 3.2.6] (where it was established in the case $R = \mathbb{Z}$).

Remark 1.3.2. 1. Our list of descriptions of $\text{DM}^{\text{eff}}_R$ can be completed by means of replacing Zar-MV by a larger set of complexes. Recall that the Cartesian square (1.3.2) is called an elementary distinguished square if the morphisms $k$ and $j$ are open embeddings, $f$ and $g$ are étale, and the base change of $f$ to $X \setminus j(V)$ is an isomorphism. Denote by Nis-MV the set of complexes (1.3.1) corresponding to squares satisfying these conditions; this set obviously contains Zar-MV. Conversely, the category $\text{Kar}_{K_f(\text{SmCor}_R^\oplus)}(HI \cup \text{Zar-MV})$ contains Nis-MV; indeed, it suffices to verify this statement in the case $R = \mathbb{Z}$, and then it easily follows from the aforementioned Theorem 3.2.6 of [Voe00].

Thus in all the three descriptions in Proposition 1.3.1 the set Zar-MV may be replaced Nis-MV (and the localizations will not change). This reduces the equivalence of $\text{DM}^{\text{eff}}_R$ with $\text{DM}^{\text{eff}}_R'$ to the following well-known fact: the category $D(\text{Sh}_{\text{Nis}}(\text{SmCor}, R))$ is equivalent to the localization of $D(\text{PreSh}_{\text{Nis}}(\text{SmCor}, R))$ by the localizing subcategory generated by Nis-MV; see also §6.2 of [CiD09].

2. Let $X \in \text{SmVar}$. The aforementioned statement implies that the sheaf $R_{tr}(X)$ is compact in $D(\text{Sh}_{\text{Nis}}(\text{SmCor}, R))$.

We also give another proof of the latter fact. According to Exercise 13.5 of [MVW06] (see also Lemma 13.4 of loc. cit.), for each $i \in \mathbb{Z}$ and a bounded above complex $C$ of objects of $\text{Sh}_{\text{Nis}}(\text{SmCor}, R)$ the group $D(\text{Sh}_{\text{Nis}}(\text{SmCor}, R))(R_{tr}(X), C[i])$ is naturally isomorphic
to $H_{\text{Nis}}^i(X, C)$ (i.e., to the $i$th Nisnevich hypercohomology group of $X$ with coefficients in $C$). This fact immediately extends to all objects $D(\text{Sh}_{\text{Nis}}(\text{SmCor}, R))$. Since for any family $(C_j)$ of objects of $D(\text{Sh}_{\text{Nis}}(\text{SmCor}, R))$ we have $H^0(X, \bigoplus_j C_j) \cong \bigoplus_j H^0(X, C_j)$ (see Corollary 1.1.11 and §1.1.12 of [CiD12]), we obtain the compactness statement in question.

3. Hence the category $D(\text{Sh}_{\text{Nis}}(\text{SmCor}, R))$ is compactly generated. Since the localization functor $L_R : D(\text{Sh}_{\text{Nis}}(\text{SmCor}, R)) \to \text{DM}_{\text{eff}}^R$ respects coproducts (see Lemma 1.1.13), part 1 of the lemma gives the existence of an adjoint functor $i_R : \text{DM}_{\text{eff}}^R \to D(\text{Sh}_{\text{Nis}}(\text{SmCor}, R))$ that is certainly a full embedding. We will often identify the categories $\text{DM}_{\text{eff}}^R$ and $\text{DM}_{\text{eff}}^R$ with the essential image of $i_R$ (that is called the category of motivic complexes). We will use the notation $\mathcal{M}_R(X)$ for the image of $R_{\text{tr}}(X)$ (for $X \in \text{SmVar}$) in all these categories.

It is easily seen (see Theorem 9.1.16 of [Nee01]) that motivic complexes are characterized inside the category $D(\text{Sh}_{\text{Nis}}(\text{SmCor}, R))$ by the following homotopy invariance conditions for the presheaves $H_{\text{Nis}}^i(\cdot, C)$: for all $i \in \mathbb{Z}$ and $X \in \text{SmVar}$ we have $H_{\text{Nis}}^i(X, C) \cong H_{\text{Nis}}^i(X \times \mathbb{A}^1, C)$. The functor $i_R \circ L_R$ can be described by an explicit formula (see [MVW06] Remark 14.7); yet we will not need this fact below.

We note also that the adjunction between $L_R$ and $i_R$ combined with the isomorphism mentioned in part 2 of this remark implies that the group $\text{DM}_{\text{eff}}^R(\mathcal{M}_R(X), C[i])$ is naturally isomorphic to $H_{\text{Nis}}^i(X, C)$ for each motivic complex $C$ and $i \in \mathbb{Z}$.

4. One of the advantages of our first description of $\text{DM}_{\text{eff}}^R$ is that it simplifies checking the compactness of all $\mathcal{M}_R(X)$ in this category. Moreover, we will use some more properties of $\text{DM}_{\text{eff}}^R$ established in §6 of [BeV08]. Still we note that in the case $R = \mathbb{Z}$ all these statements were proved in the papers of V. Voevodsky and F. Déglise; to generalize them to the case of an arbitrary $R$ one may apply Proposition 1.3.3 below.

In particular, below we will apply the following property of $\text{DM}_{\text{eff}}^R$: for each smooth $Y/k$ and smooth proper $X/k$ all of whose connected components are of dimension $n$ and $i \geq 0$ the group $\text{DM}_{\text{eff}}^R(R_{\text{tr}}(Y), R_{\text{tr}}(X)[i])$ vanishes if $i > 0$ and equals $CH^n(X \times Y) \otimes \mathbb{Z} R$ if $i = 0$; see [BeV08, Corollary 6.7.3]. Recalling also that the composition of morphisms in the full subcategory $\text{Corr}^{\text{rat}}_R$ of $\text{DM}_{\text{eff}}^R$ whose objects are all $R_{\text{tr}}(X)$, is compatible with the composition of morphisms in the category of effective Chow motives (note that it suffices to prove this statement in the
case $R = \mathbb{Z}$; see Proposition [1.3.3 below] we obtain the following: the additive category $\text{Chow}_{\text{eff}}^R = \text{Kar}_{\text{DM}_{\text{eff}}^R}^R \text{Corr}_{\text{rat}}^R$ is the natural $R$-linear version of effective Chow motives.

We will also need some properties of the "extension of scalars" for motivic complexes.

**Proposition 1.3.3.** Let $R'$ be an associative commutative unital $R$-algebra. Then the natural functor $\otimes_R R' : D(\text{Sh}_{\text{Nis}}(\text{SmCor}, R)) \to D(\text{Sh}_{\text{Nis}}(\text{SmCor}, R'))$ yields a commutative diagram

$$
\begin{array}{cccc}
\text{SmCor} & \overset{M_R}{\longrightarrow} & \text{DM}_{\text{eff}}^R & \overset{i_R}{\longrightarrow} & D(\text{Sh}_{\text{Nis}}(\text{SmCor}, R)) & \overset{L_R}{\longrightarrow} & \text{DM}_{\text{eff}}^{R'} \\
\downarrow & & \downarrow \otimes_R \text{mot } R' & & \downarrow \otimes_R R' & & \downarrow \otimes_R \text{mot } R' \\
\text{SmCor} & \overset{M_{R'}}{\longrightarrow} & \text{DM}_{\text{eff}}^{R'} & \overset{i_{R'}}{\longrightarrow} & D(\text{Sh}_{\text{Nis}}(\text{SmCor}, R')) & \overset{L_{R'}}{\longrightarrow} & \text{DM}_{\text{eff}}^{R'}
\end{array}
$$

(1.3.3)
of functors.

*Proof.* The existence (and commutativity) of the right hand square in our diagram is obvious.

Next, recall that motivic complexes are characterized by the homotopy invariance of the presheaves $H^i_{\text{Nis}}(\_, C)$ (see Remark [1.3.2(3)]. Combining this fact with Theorem 22.3 of [MVW06] (that says that the Nisnevich sheafification respects the homotopy invariance of presheaves with transfers) we obtain that the functor $\otimes_R R'$ sends $R$-linear motivic complexes into $R'$-linear ones. Combining the latter statement with [Bon16, Proposition 1.1.1(III)] (that treats Bousfield localizations of triangulated categories following [Nee01, §9]; cf. also Remark 1.3.3(3) of ibid.) we easily obtain the commutativity of the middle square in the diagram.

It remains to note that the commutativity of the left hand square in the diagram follows immediately from the commutativity of the diagram obtained from our one by means of deleting the second column (certainly, the horizontal arrows passing through it should be composed pairwisely).

*Remark 1.3.4.* 1. Obviously the functor $\otimes_R R'$ respects coproducts. Since $L_R$ is surjective on objects, and both $L_R$ and $L_{R'}$ respect coproducts (see Remark [1.3.2(3)]), we obtain that the functor $\otimes_R \text{mot } R'$ respects coproducts also.

The functors $i_R$ and $i_{R'}$ respect coproducts as well (see Remark [1.3.2(3) once again), but we will not need this fact.

4Note that the category $\text{DM}_{\text{eff}}^R$ is Karoubian according to Proposition 1.6.8 of [Nee01]; hence $\text{Chow}_{\text{eff}}^R$ is Karoubian also.
2. The cases $R = \mathbb{Z}$ and $R = \mathbb{Z}[1/p]$ appear to be the most interesting in the context of this paper. If one restricts to these cases then the corresponding motivic extension of scalars functor can be described as a certain Verdier localization; see appendix A of [Kel12].

2 On the Chow weight structure for effective motives

In §2.1 we give the definition of our "effective" Chow weight structure $w_{Chow}^{eff}$ and discuss its relation to the Chow weight structures defined in previous papers. Our description of $w_{Chow}^{eff}$ implies that a motivic complex has non-negative weights if and only if its positive Nisnevich hypercohomology vanishes.

In §2.2 the relation of our weight structure $w_{Chow}^{eff}$ to motivic twists is studied; this gives bounds on weights of certain motives. We also prove that the twist functor $-\langle 1 \rangle = -1[2]$ is weight-exact; this statement is important for the next section.

Since the distinctions between $DM_R^{eff}$, $DM_R^{eff'}$, and the category of motivic complexes (see Remark [1.3.2][3]) are irrelevant for our arguments below, we will use the notation $DM_R^{eff}$ to the category of motivic complexes.

2.1 The definition of $w_{Chow}^{eff}$ and its comparison with the Chow weight structures defined earlier

We will use the notation $w_{Chow}^{eff}$ for the weight structures generated by the set $\mathcal{M}_R(\text{SmVar})$ in the category $DM_R^{eff}$ (see Proposition [1.2.3][1.1]). We will write just $w_{Chow}^{eff}$ for it and call it the Chow weight structure when this will cause no ambiguity.

Remark 2.1.1. The definition of $w_{Chow}^{eff}$ along with the Remark [1.3.2][3] obviously imply that the motivic complex $C'$ belongs to $DM_R^{eff}_{w_{Chow}^{eff}} \geq 0$ if and only if for any $X \in \text{SmVar}$ and $i > 0$ we have $H^i_{Nis}(X, C) = \{0\}$.

Applying Proposition [1.2.3][1.1] we also obtain that $\mathcal{M}_R(\text{SmVar}) \subset DM_R^{eff}_{w_{Chow}^{eff}} \leq 0$.

We will use the notation SmPrVar for the set of smooth projective $k$-varieties.

Theorem 2.1.2. 1. $\text{Chow}_R^{eff} \subset H_{w_{Chow}^{eff}}$

2. Assume that $R$ is a $\mathbb{Z}[1/p]$-algebra. Then $w_{Chow}^{eff}_R$ is also generated by the set $\mathcal{M}_R(\text{SmPrVar})$. 

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3. The functor $- \otimes_{\mathbb{R}}^{\text{mot}} \mathbb{R}'$ (as defined in Proposition 1.3.3) is weight-exact (with respect to the weight structures $w_{\text{Chow}_{\text{eff}}}^R$ and $w_{\text{Chow}_{\text{eff}}}^R'$).

Proof. 1. Since the category $\text{Chow}_{\text{eff}}^R$ is Karoubi-closed in $\text{DM}_{\text{eff}}^R$, it suffices to verify that $\mathcal{M}_R(\text{SmPrVar}) \subset \text{DM}_{\text{eff}}^R w_{\text{Chow}_{\text{eff}}}^R = 0$.

We have already noted that $\mathcal{M}_R(\text{SmVar}) \subset \text{DM}_{\text{eff}}^R w_{\text{Chow}_{\text{eff}}}^R \leq 0$. Hence it remains to prove that $\mathcal{M}_R(\text{SmVar}) \perp \mathcal{M}_R(\text{SmPrVar})[i]$ for all $i > 0$. The latter fact follows from Remark 1.3.2(4) immediately.

2. According to Proposition 1.2.3(I.1) it suffices to verify that $(\bigcup_{i<0} \mathcal{M}_R(\text{SmPrVar})[i])^{\perp} = (\bigcup_{i<0} \mathcal{M}_R(\text{SmVar})[i])^{\perp}$.

Obviously, the second of these classes is contained in the first one. To check the inverse inclusion it suffices to prove that $\mathcal{M}_R(X)$ belongs to the envelope of the set $\bigcup_{i \leq 0} \mathcal{M}_R(\text{SmPrVar})[i]$ for each $X \in \text{SmVar}$.

To prove $\bigcup_{i<0} \mathcal{M}_R(\text{SmPrVar})[i]^{\perp} = (\bigcup_{i<0} \mathcal{M}_R(\text{SmVar})[i])^{\perp}$.

3. Proposition 1.2.3(II.2) obviously implies that the functor $- \otimes_{\mathbb{R}}^{\text{mot}} \mathbb{R}'$ is left weight-exact.

To prove that $- \otimes_{\mathbb{R}}^{\text{mot}} \mathbb{R}'$ is right weight-exact we should verify the following: if for a motivic complex $C$ we have $H^i_{\text{Nis}}(X, C) = \{0\}$ for all $X \in \text{SmVar}$ and $i < 0$, then the same condition is also fulfilled for $C \otimes_{\mathbb{R}}^{\text{mot}} \mathbb{R}' \in \text{Obj} D(\text{Sh}_{\text{Nis}}(\text{SmCor}, \mathbb{R}'))$ (see Proposition 1.3.3). Certainly, to check this vanishing statement one can consider $C \otimes_{\mathbb{R}}^{\text{mot}} \mathbb{R}'$ as an object of $D(\text{Sh}_{\text{Nis}}(\text{SmCor}, R'))$ (so here we apply to $C$ the forgetful functor $F : D(\text{Sh}_{\text{Nis}}(\text{SmCor}, R')) \to D(\text{Sh}_{\text{Nis}}(\text{SmCor}, R))$, that is right adjoint to the functor $- \otimes_{\mathbb{R}} \mathbb{R}'$). Next, the complex $F(C \otimes_{\mathbb{R}} \mathbb{R}')$ can be obtained by tensoring $C$ by a flat $\mathbb{R}$-module resolution of $\mathbb{R}'$; since the latter is concentrated in non-positive degrees, this gives the result in question (cf. [MVW06, Definition 14.2, §8]).

Remark 2.1.3. 1. Since $\text{Chow}_{\text{eff}}^R \subset Hw_{\text{Chow}_{\text{eff}}}^R$, the subcategory $\text{Chow}_{\text{eff}}^R$ is negative in $\text{DM}_{\text{eff}}^R$, i.e., $\text{Obj} \text{Chow}_{\text{eff}}^R \perp \text{Obj} \text{Chow}_{\text{eff}}^R[i]$ for all $i > 0$. Since $\text{Chow}_{\text{eff}}^R$ is Karoubian, Corollary 2.1.2 of [BoS18] gives a unique weight structure $w$ on $\langle \text{Obj} \text{Chow}_{\text{eff}}^R \rangle$ such that $\text{Chow}_{\text{eff}}^R \subset Hw$. Moreover, $\text{Chow}_{\text{eff}}^R = \ldots$

\footnote{It was assumed in ibid. that $p > 0$; still all the arguments of that paper can be applied in the case $p = 0$ also. Moreover, if $p = 0$ then one can apply Theorem 6.2.1(1) of [Bon09].}

\footnote{We certainly can assume that the complex $C$ has zero terms in positive degrees; this allows us to apply the results of ibid. Moreover, this calculation can also be easily made using other descriptions of $\text{DM}_{\text{eff}}^R$ given above.}
$Hw$, the class $\langle \text{Obj Chow}^{\text{eff}}_{R} \rangle_{w \leq 0}$ coincides with the envelope of $\bigcup_{i \leq 0} \text{Obj Chow}^{\text{eff}}_{R}[i]$, and the class $\langle \text{Obj Chow}^{\text{eff}}_{R} \rangle_{w \geq 0}$ equals the envelope of $\bigcup_{i \geq 0} \text{Obj Chow}^{\text{eff}}_{R}[i]$.

In the settings considered in [Bon10] and [Bon11] it is known that $\langle \text{Obj Chow}^{\text{eff}}_{R} \rangle$ coincides with the subcategory $\text{DM}^{\text{eff}}_{\text{gm}, R}$ of compact objects of $\text{DM}^{\text{eff}}_{R}$ (DM$^{\text{eff}}_{\text{gm}, R}$ is called the category of effective geometric motives). Respectively, this weight structure $w$ was called the Chow one.

2. Now, for any weight structure $(\mathcal{C}, w)$ the class $\mathcal{C}_{w \leq 0}$ is closed with respect to all $\mathcal{C}$-coproducts (of its objects). Since $w^{\text{eff}}_{\text{Chow}}$ is generated by a set of compact objects of $\text{DM}^{\text{eff}}_{R}$, the class $\text{DM}^{\text{eff}}_{R} w^{\text{eff}}_{\text{Chow} \geq 0}$ is also closed with respect to $\text{DM}^{\text{eff}}_{R}$-coproducts (weight structures of this type are called smashing ones).

Next, take the localizing subcategory $\text{DM}^{\text{eff}}_{R} \text{Chow}^{\text{eff}}_{R}$ generated by $\text{Obj Chow}^{\text{eff}}_{R}$ in $\text{DM}^{\text{eff}}_{R}$, and consider the weight structure $w^{\prime}_{\text{Chow}^{\text{eff}}_{R}}$ generated by $\text{Obj Chow}^{\text{eff}}_{R}$ in $\text{DM}^{\text{eff}}_{R} \text{Chow}^{\text{eff}}_{R}$ (note that we can assume that $\text{Obj Chow}^{\text{eff}}_{R}$ is a set, so that we can apply Proposition 1.2.3(I.1)). We immediately obtain that the embedding $\langle \text{Obj Chow}^{\text{eff}}_{R} \rangle \to \text{DM}^{\text{eff}}_{R} \text{Chow}^{\text{eff}}_{R}$ is weight-exact with respect to the weight structures $w$ and $w^{\prime}_{\text{Chow}^{\text{eff}}_{R}}$.

Moreover, Corollary 2.3.1(1) of [BoS17] implies that the embedding $\text{DM}^{\text{eff}}_{R} \text{Chow}^{\text{eff}}_{R} \to \text{DM}^{\text{eff}}_{R}$ is weight-exact (with respect to $w^{\prime}_{\text{Chow}^{\text{eff}}_{R}}$ and $w^{\text{eff}}_{\text{Chow}}$) as well. Applying Proposition 1.2.3(I.1) one can easily deduce that $\langle \text{Obj Chow}^{\text{eff}}_{R} \rangle_{w \leq 0} = \text{DM}^{\text{eff}}_{R} w^{\text{eff}}_{\text{Chow} \leq 0} \cap \text{Obj} \langle \text{Obj Chow}^{\text{eff}}_{R} \rangle$, $\langle \text{Obj Chow}^{\text{eff}}_{R} \rangle_{w \geq 0} = \text{DM}^{\text{eff}}_{R} w^{\text{eff}}_{\text{Chow} \geq 0} \cap \text{Obj} \langle \text{Obj Chow}^{\text{eff}}_{R} \rangle$, $\text{DM}^{\text{eff}}_{R} w^{\prime}_{\text{Chow}^{\text{eff}}_{R} \leq 0} = \text{DM}^{\text{eff}}_{R} w^{\prime}_{\text{Chow}^{\text{eff}}_{R} \leq 0} \cap \text{Obj} \text{DM}^{\text{eff}}_{R} \text{Chow}^{\text{eff}}_{R}$, and $\text{DM}^{\text{eff}}_{R} w^{\prime}_{\text{Chow}^{\text{eff}}_{R} \geq 0} = \text{DM}^{\text{eff}}_{R} w^{\text{eff}}_{\text{Chow} \geq 0} \cap \text{Obj} \text{DM}^{\text{eff}}_{R} \text{Chow}^{\text{eff}}_{R}$ (see Proposition 1.2.5(1) of ibid.).

All these results demonstrate that $w^{\text{eff}}_{\text{Chow}}$ is "closely related" to weight structures "generated" (in the corresponding senses) by Chow motives; this is why we call $w^{\text{eff}}_{\text{Chow}}$ a Chow weight structure. Note also that to prove that $w^{\text{eff}}_{\text{Chow}} = w^{\prime}_{\text{Chow}^{\text{eff}}_{R}}$ it suffices to verify that $\text{DM}^{\text{eff}}_{R} \text{Chow}^{\text{eff}}_{R} = \text{DM}^{\text{eff}}_{R}$; the latter equality is equivalent to $\langle \text{Obj Chow}^{\text{eff}}_{R} \rangle = \text{DM}^{\text{eff}}_{\text{gm}, R}$.

3. The main disadvantage of $w^{\text{eff}}_{\text{Chow}}$ (for a general $R$ and $p > 0$) is that we cannot describe its heart explicitly, and do not know whether this weight structure restricts to $\text{DM}^{\text{eff}}_{\text{gm}, R}$.

Recall also that for any functor $H$ from $\text{DM}^{\text{eff}}_{R}$ into an abelian category the weight structure $w^{\text{eff}}_{\text{Chow}}$ gives a certain filtration on the values of $H$ (see [Bon10] Proposition 2.1.2)), that can be called the Chow-weight one (in
Remark 2.4.3 and Remark 2.4.5 it is explained that Chow-weight filtrations vastly generalized Deligne’s weight filtrations on singular and étale cohomology of varieties); these filtrations are $DM^{eff}_R$-functorial. Thus our Chow weight structure gives interesting weight filtrations for any $R$ (in particular, we can consider them for "$p$-adic" functors).

If $H$ is homological or cohomological then the theory of weight structures developed in §2.3–2.4 of [Bon10] yields a relation of cohomology of arbitrary motivic complexes to that of objects of the heart of $w^{eff}_{Chow}$. The disadvantages of $w^{eff}_{Chow}$ described above make the application of this result rather difficult; yet cf. Proposition 2.2.2(1) below.

4. Theorem 2.1.2(2) is the only statement in this paper whose proof relies on certain resolution of singularities results for $k$-varieties; note that these statements were crucial for the corresponding results of [Bon10] and [Bon11]. This hints that the methods of the current paper can be applied to categories of relative motives (over a base scheme distinct from the spectrum of a field). Recall also that a certain compactly generated Chow weight structure has found interesting applications to the so-called relative $K$-motives in §2.3 of [BoL16].

2.2 On twists and their weight-exactness

Now let us recall the notion of (Tate) twists $-\langle n \rangle$ and $-\langle n \rangle$, and Gysin distinguished triangles; these are important for motives.

Since the identity of the point $\text{Spec } k$ factors through $\mathbb{P}^1$, we have $\mathcal{M}_R(\mathbb{P}^1) \cong R \bigoplus R(1)$, where $R$ is the motif of the point and $R(1)$ is the motif that Voevodsky called the Tate one (though calling it the Lefschetz motif would may be more appropriated) and denoted by $R(1)[2]$. Certainly $R(1) \in \text{Obj Chow}^{eff}_R$.

Since $\mathcal{M}_R(X) \otimes \mathcal{M}_R(Y) \cong \mathcal{M}_R(X \times Y)$ for any $X, Y \in \text{SmVar} (\text{here we use the tensor product on } DM^{eff}_R)$, the subcategory $\text{Chow}^{eff}_R \subset DM^{eff}_R$ is closed with respect to the tensor product. Hence the $n$th tensor power $R(n)$ of $R(1)$ (that can also be denoted by $R(n)[2n]$) is an object of $\text{Chow}^{eff}_R$ for each $n \geq 0$. We will use the notation $M \langle n \rangle$ for $M \otimes R(n)$ for any $M \in \text{Obj } DM^{eff}_R$.

The properties of the tensor product of $DM^{eff}_R$ easily imply that the functor $-\langle n \rangle$ respects coproducts and the compactness of objects.

We will need the following facts.

\footnote{Since the category $DM^{eff}_R$ is compactly generated by the set $\mathcal{M}_R(\text{SmVar})$, its subcategory $DM^{eff}_{gm,R}$ of compact objects coincides with $\langle \mathcal{M}_R(\text{SmVar}) \rangle$. It follows immediately that the functor $-\langle n \rangle$ respects compactness. However, below we will only apply the compactness of the elements of $\mathcal{M}_R(\text{SmVar})(n)$.}
Lemma 2.2.1. Assume that $n \geq 0$.

1. Let $Z \in \text{SmVar}$ be an equicodimensional closed subvariety of codimension $n$ in a smooth variety $X$ (i.e., the connected components of $Z$ are of codimension $n$ in $X$). Then there exists a Gysin distinguished triangle

$$
\mathcal{M}_R(X \setminus Z) \to \mathcal{M}_R(X) \to \mathcal{M}_R(Z)\langle n \rangle \to \mathcal{M}_R(X \setminus Z)[1]
$$
in $\text{DM}^{eff}_R$.

2. The endo-functor $\langle n \rangle$ is left weight-exact; in particular, $\mathcal{M}_R(\text{SmVar})\langle n \rangle \subset \text{DM}^{eff}_R$.

Proof. 1. This assertion is a re-formulation of [BeV08 Proposition 6.3.1] (see also Proposition 4.3 of [Deg08a]).

2. Since the functor $\langle n \rangle$ respects coproducts, Proposition 2.2.2. says that it suffices to verify whether $\mathcal{M}_R(\text{SmVar})\langle n \rangle \subset \text{DM}^{eff}_R$.

To illustrate these properties of motives and twists we prove two easy statements on "weight bounds".

Proposition 2.2.2. 1. Assume that $U$ is a variety of the form $X \setminus \cup_{i=1}^n Z_i$, where $X$, all of $Z_i$, and all the intersections of subfamilies of $Z_i$ are smooth proper $k$-varieties; suppose moreover that the intersections of any $m$ of distinct $Z_i$’s is empty (for some $m \leq n \in \mathbb{Z}$).

Then the motif $\mathcal{M}_R(U)$ belongs to the envelope of $\cup_{i=0}^m \text{Obj Chow}^{eff}_R[i]$, and so, to $\text{Obj Chow}^{eff}_R) \cap \text{DM}^{eff}_R$.

2. Let $f : U \to V$ be an open dense embedding, where $U, V \in \text{SmVar}$. Then $\text{Cone}(\mathcal{M}_R(f)) \in \text{DM}^{eff}_R$.

Proof. 1. Since $\text{Chow}^{eff}_R \subset H^{\text{Chow}}^{eff}_R$ and the class $\text{DM}^{eff}_R \cap \text{DM}^{eff}_R$ is closed with respect to extensions and retractions, it suffices to verify that $\mathcal{M}_R(U)$ belongs to the envelope of $\cup_{i=0}^m \text{Obj Chow}^{eff}_R[i]$. We prove this statement by induction on $n$. In the case $n = 1$ it is obvious.

Assume now that the statement is fulfilled for all $n' < n$. We present $U$ as $(X \setminus \cup_{i=2}^n Z_i) \setminus (Z_1 \setminus \cup_{i=2}^n Z_i)$. Denote the variety $X \setminus \cup_{i=2}^n Z_i$ by $X'$, and the connected components of $Z_1 \setminus \cup_{i=2}^n Z_i$ by $Y_j$ (here $1 \leq j \leq l$ for some $l \in \mathbb{Z}$).

According to our inductive assumption, the motif $\mathcal{M}_R(X')$ belongs to the envelope of $\cup_{i=0}^n \text{Obj Chow}^{eff}_R[i]$, and all $\mathcal{M}_R(Y_j)$ belong to the envelope of $\cup_{i=1}^{m-l} \text{Obj Chow}^{eff}_R[i]$. Denote the codimensions of $Y_j$ in $X$ by $c_j$. Since $\text{Obj Chow}^{eff}_R \subset \text{Obj Chow}^{eff}_R$, all $\mathcal{M}_R(Y_j)\langle c_j \rangle$ also belong to the envelope of $\cup_{i=0}^{m-l} \text{Obj Chow}^{eff}_R[i]$. 

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Next, the Gysin distinguished triangle (see Lemma 2.2.1(1)) gives distinguished triangles

$$\mathcal{M}_R(Y_{r+1})\langle c_{r+1} \rangle[-1] \to \mathcal{M}_R(X'\setminus(\sqcup_{j=1}^l Y_j)) \to \mathcal{M}_R(X'\setminus(\sqcup_{j=1}^l Y_j)) \to \mathcal{M}_R(Y_{r+1})\langle c_{r+1} \rangle$$

for all $r$ between 0 and $l - 1$. Since $U = X' \setminus (\sqcup_{j=1}^l Y_j)$, these triangles yield our assertion.

2. There obviously exists a sequence of open dense embeddings $U_0 = U \subset U_1 \subset \ldots \subset U_m = V$ (for some $m > 0$) such that the varieties $U_{i+1} \setminus U_i$ are smooth and equicodimensional in $U_{i+1}$ for all $i$ between 0 and $m - 1$.

The corresponding Gysin distinguished triangles along with the octahedral axiom of triangulated categories give distinguished triangles $\mathcal{M}_R(U_{i+1}\setminus U_i)\langle c_i \rangle \to \text{Cone}(\mathcal{M}_R(U_i) \to \mathcal{M}_R(V)) \to \text{Cone}(\mathcal{M}_R(U_{i+1} \to \mathcal{M}_R(V)) \to \mathcal{M}_R(U_{i+1}\setminus U_i)\langle c_i \rangle[1]$ for all $i$ between 0 and $m - 1$, where $c_i$ is the codimension of $U_{i+1} \setminus U_i$ in $U_{i+1}$. Hence $\text{Cone}(\mathcal{M}_R(f))$ belongs to the envelope of (all) $\mathcal{M}_R(U_{i+1}\setminus U_i)\langle c_i \rangle$. Since the class $\text{DM}_{\text{eff}}^{\text{eff}}_{\mathcal{M}_R}^r\langle c_{\text{Chow}} \leq 0 \rangle$ is extension-closed, it remains to apply Lemma 2.2.1(2).

Now let us prove a theorem that is crucial for the next section.

**Theorem 2.2.3.** The endo-functor $-\langle n \rangle$ is weight-exact for all $n \geq 0$.

*Proof.* Since this functor is left weight-exact (see Lemma 1.1.1(2)), it remains to verify that it is right weight-exact. Obviously it suffices to verify the latter statement in the case $n = 1$.

Now we argue somewhat similarly to the proof of Theorem 2.1.1(3).

For a motivic complex $C$ such that $H_{\text{Nis}}^i(X, C) = \{0\}$ for all $X \in \text{SmVar}$ and $i > 0$ we should check that these conditions are also fulfilled for $C(1)$.

We note that the natural forgetful functor $\text{For} : \text{DM}_{\text{eff}}^r_{\mathcal{M}_R} \to \text{DM}_{\text{eff}}^r$ commutes with the twist $-\langle 1 \rangle$ (this can be easily checked using the first of the descriptions of $\text{DM}_{\text{eff}}^r$ listed in Proposition 1.3.1). Moreover, the group $H_{\text{Nis}}^i(X, C)$ is certainly isomorphic to $H_{\text{Nis}}^i(X, \text{For}(C))$ for all $X \in \text{SmVar}$ and $i \in \mathbb{Z}$. Thus we can assume $R = \mathbb{Z}$; we fix some $C \in \text{DM}_{\text{eff}}^r_{\mathcal{M}_R}$. For $\text{DM}_{\text{eff}}^r_{\mathcal{M}_R}$.

We consider the following cohomology theories on smooth $k$-varieties: for each $r \geq 0$, $q \in \mathbb{Z}$, and $X \in \text{SmVar}$ we will use the notation $H_r^q(x)$ for the group $\text{DM}_{\text{eff}}^r(\mathcal{M}_Z(X)\langle r \rangle [r - i], C(1))$. We should verify that $H_0^i(X) = \{0\}$ for all $i > 0$ and $X \in \text{SmVar}$. For this purposes we consider the following converging (coniveau) spectral sequence:

$$E_1^{r,q} = \prod_{x \in X^{(r)}} H_r^q(x) \Rightarrow H_r^{r+q}(X),$$

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where $X^{(r)}$ is the set of points of $X$ of codimension $r$, and for a presentation of $x \in X^{(r)}$ as $\lim_j X_j$ for $X_j \in \text{SmVar}$ we define $H^*_j(x)$ as $\lim_j H^*_j(X_j)$. This spectral sequence and its convergence is given by Proposition 4.3.1(I.3) of [Bon18b] (see Remark 4.3.2(2) of loc. cit.; see also [Deg08b] §6.3.2).

Now denote by $t$ the unbounded version of the Voevodsky homotopy $t$-structure (see [Deg11, §5.1]). According to Lemma 2.2.4 below, we have $C \in \text{DM}_{\mathbb{Z}}^{eff} t_{\leq 0}$; hence $C(1) \in \text{DM}_{\mathbb{Z}}^{eff} t_{\leq -1}$. Since the spectra of function fields are the henselizations of the corresponding smooth varieties at their generic points, we obtain that $E^{0,q}_i = \{0\}$ for $q \geq 0$. Next, for a function field $K/k$, a presentation of $\text{Spec } K$ as an inverse limit of smooth varieties $X_j$, and any $r > 0$ we have

$$H^*_r(\text{Spec } K) = \lim_j \text{DM}_{\mathbb{Z}}^{eff} (\mathcal{M}_{\mathbb{Z}}(X_j) \langle r \rangle, C(1) \langle r + q \rangle)$$

$$\cong \lim_j \text{DM}_{\mathbb{Z}}^{eff} (\mathcal{M}_{\mathbb{Z}}(X_j) \langle r - 1 \rangle, C \langle r + q \rangle);$$

here we apply the Cancellation theorem (this is Corollary 4.10 of [Voe10]). Since $\mathcal{M}_{\mathbb{Z}}(X_j) \langle r - 1 \rangle \in \text{DM}_{\mathbb{Z}}^{eff} w_{\text{Chow}}^{\geq 0}$ (see Lemma 2.2.1(2)) and $\text{DM}_{\mathbb{Z}}^{eff} w_{\text{Chow}}^{\leq -1} \perp C$, we obtain that $E^{0,q}_i = \{0\}$ if $r + q > 0$ (also in the case $r > 0$). It obviously follows that $H^{r+q}_0(X) = \{0\}$ if $r + q > 0$, as desired.

It remains to prove the following properties of $t$.

**Lemma 2.2.4.** 1. A motivic complex $C$ belongs to $\text{DM}_{\mathbb{Z}}^{eff} t_{\leq 0}$ if and only if for each scheme $S$ that is the henselization of a variety $X \in \text{SmVar}$ in some point and any presentation of $S$ as $\lim_j S_j$, where $S_j \in \text{SmVar}$, the group $\lim_j H^*_j(\text{Nis}(S_j, C))$ vanishes if $i > 0$.

2. $\text{DM}_{\mathbb{Z}}^{eff} t_{\leq 0}(1) \subset \text{DM}_{\mathbb{Z}}^{eff} t_{\leq -1}$.

3. $\text{DM}_{\mathbb{Z}}^{eff} w_{\text{Chow}}^{\geq 0} \subset \text{DM}_{\mathbb{Z}}^{eff} t_{\leq 0}$.

**Proof.** 1. According to Corollary 5.2 of [Deg11], $C$ belongs to $\text{DM}_{\mathbb{Z}}^{eff} t_{\leq 0}$ if and only if for each $i > 0$ the Nisnevich sheafification of the presheaf $H^i(-, C)$ is zero. Hence it suffices to recall that the Nisnevich sheafification of a presheaf of abelian groups $P$ (on the étale site of $\text{SmVar}$) vanishes if and only if $\lim_j P(S_j) = \{0\}$ for all projective systems $(S_j)$ as in the assertion.

2. Easy from Theorem 5.3 of loc. cit.

3. If $C$ belongs to $\text{DM}_{\mathbb{Z}}^{eff} w_{\text{Chow}}^{\leq 0}$, $i > 0$, and $S_j \in \text{SmVar}$ for all $j$, then the group $\lim_j H^*_j(\text{Nis}(S_j, C))$ vanishes. Hence the direct limits in part 1 of this lemma vanish also.
Remark 2.2.5. 1. Most probably all parts of the lemma are fulfilled for the natural $R$-linear version of $t$ (for any $R$). This statement is especially easy to verify in the case where $R$ is a localization of $\mathbb{Z}$ (i.e., $R \subset \mathbb{Q}$); see Remark 1.3.4(2) and [Bon16 Proposition 5.6.2(II.3)].

2. Now recall that the functor $-\langle 1 \rangle$ is fully faithful (the $R$-linear version of this statement was established in [BeV08 §6.1]). We will assume that this functor is a full embedding and denote its essential image by $\text{DM}^{eff}_R\langle 1 \rangle$.

Combining our theorem with Proposition 1.2.3(I.1) one can easily prove the following "cancellation theorem" for $w^{eff}_{\text{Chow}}$:

$$\text{DM}^{eff}_R w^{eff}_{\text{Chow}} \leq 0 \cap \text{Obj}(\text{DM}^{eff}_R\langle 1 \rangle) = \text{DM}^{eff}_R w^{eff}_{\text{Chow}} = 0 \langle 1 \rangle$$

and

$$\text{DM}^{eff}_R w^{eff}_{\text{Chow}} \geq 0 \cap \text{Obj}(\text{DM}^{eff}_R\langle 1 \rangle) = \text{DM}^{eff}_R w^{eff}_{\text{Chow}} = 0 \langle 1 \rangle.$$ 

3 Chow weight structures for stable and birational motives

Now we study the naturally defined Chow weight structures on the categories obtained from $\text{DM}^{eff}_R$ by means of the twist $\langle 1 \rangle$.

In §3.1 we "extend" $w^{eff}_{\text{Chow}}$ to the "stable" motivic category $\text{DM}_R$ (on which the functor $\langle 1 \rangle$ is invertible); respectively, the embedding $\text{DM}^{eff}_R \to \text{DM}_R$ is weight-exact.

In §3.2 we prove that $w^{eff}_{\text{Chow}}$ induces a weight structure $w^{bar}$ on the category $\text{DM}^{eff}_R/\text{DM}^{eff}_R\langle 1 \rangle$ of birational motives; this weight structure can be easily described.

3.1 The weight structure $w^{\text{Chow}}$ on $\text{DM}_R$

The full faithfulness of the functor $-\langle 1 \rangle$ yields the existence of a triangulated category $\text{DM}_R$ that is closed with respect to coproducts, is equipped with a full embedding $i : \text{DM}^{eff}_R \to \text{DM}_R$ that respects coproducts and the compactness of objects, and such that the functor $-\langle 1 \rangle_{\text{DM}_R} = - \otimes_{\text{DM}_R} i(R(1))$ is an auto-equivalence of $\text{DM}_R$ (see §4.2 of [Deg11] and §1.1 of [CiD13]).
We assume that the functor \(-\otimes_{\text{DM}_R} i(R(1))\) is invertible on \(\text{DM}_R\), and for each \(n \in \mathbb{Z}\) we will denote its \(n\)th composition power by \(-\langle n \rangle\). Certainly, these functors respect coproducts and the compactness of objects. We will assume that \(\text{DM}_R^{eff}\) is a full subcategory of \(\text{DM}_R\); so we will write \(\mathcal{M}_R(X)\) instead of \(i(\mathcal{M}_R(X))\). Recall also that the category \(\text{DM}_R\) is generated by the set \(\bigcup_{j \in \mathbb{Z}} (\mathcal{M}_R(\text{SmVar}) \langle j \rangle)\) as its own localizing subcategory.

We define \(w_{\text{Chow}} = w_{\text{Chow},R}\) as the weight structure generated by the set \(\bigcup_{j \in \mathbb{Z}} (\mathcal{M}_R(\text{SmVar}) \langle j \rangle)\). Applying Theorem 2.2.3 we easily obtain the following statements.

**Theorem 3.1.1.** 1. The auto-equivalences \(-\langle n \rangle\) are weight-exact with respect to \(w_{\text{Chow}}\) (for all \(n \in \mathbb{Z}\)).

2. The embedding \(i : \text{DM}_R^{eff} \to \text{DM}_R\) is weight-exact (with respect to \(w_{\text{eff} \text{Chow}}\) and \(w_{\text{Chow}}\)).

**Proof.** 1. Since the functor \(-\langle n \rangle\) is an auto-equivalence of \(\text{DM}_R\) that induces a bijection on the set of all \(\text{DM}_R\)-retracts of elements of \(\bigcup_{j \in \mathbb{Z}} (\mathcal{M}_R(\text{SmVar}) \langle j \rangle)\), and this set generates \(w_{\text{Chow}}\) also, the functor \(-\langle n \rangle\) is left weight-exact. Applying Proposition 1.2.3(I.1) we also obtain that \(-\langle n \rangle\) is right weight-exact.

2. Since \(i\) respects coproducts and sends "generating objects" of \(w_{\text{eff} \text{Chow}}\) into that of \(w_{\text{Chow}}\), applying Proposition 1.2.3(II.2) we obtain that \(i\) is left weight-exact.

To prove that \(i\) is also right weight-exact we should check for any \(C \in \text{Obj DM}_R^{eff}\) that if \(\mathcal{M}_R(\text{SmVar})[s] \perp C\) for all \(s < 0\) then \(\mathcal{M}_R(\text{SmVar})[s] \langle r \rangle \perp C\) for all \(s < 0\) and \(r \in \mathbb{Z}\). If \(r \geq 0\) then the statement follows from Lemma 2.2.1(2). If \(r < 0\) then for each \(X \in \text{SmVar}\) we have \(\text{DM}_R(\mathcal{M}_R(X)[s] \langle r \rangle, C) \cong \text{DM}_R(\mathcal{M}_R(X)[s], C \langle -r \rangle) = \text{DM}_R^{eff}(\mathcal{M}_R(X)[s], C \langle -r \rangle)\). Since \(C \langle -r \rangle \in \text{DM}_R^{eff}\) according to Theorem 2.2.3 and the weight structure \(w_{\text{eff} \text{Chow}}\) is generated by \(\mathcal{M}_R(\text{SmVar})\), we obtain that \(\mathcal{M}_R(X)[s] \perp C \langle -r \rangle\) indeed.

**Remark 3.1.2.** Denote by \(\text{Chow}_R\) the full subcategory of \(\text{DM}_R\) whose object class equals \(\bigcup_{i \in \mathbb{Z}} \text{Obj DM}_R^{eff}(i)\); this category is obviously equivalent to the category of \(R\)-linear Chow motives. Note also that Theorem 3.1.1 implies that \(\text{Chow}_R \subset Hw_{\text{Chow}}\).

Similarly to Remark 2.1.3 we obtain that on the localizing subcategory \(\text{DM}_R^{\text{Chow} R}\) of \(\text{DM}_R\) generated by \(\text{Obj Chow}_R\) there exists a weight structure generated by \(\text{Obj Chow}_R\), and the embedding \(\text{DM}_R^{\text{Chow} R} \to \text{DM}_R\) is weight-exact.

\(\text{DM}_R^{eff} \to \text{DM}_R^{eff}[\langle -1 \rangle] \to \text{DM}_R\) are weight-exact (cf. Theorem 3.1.1).
3.2 The relation of $w_{\text{Chow}}^{\text{eff}}$ to the birational weight structure

Applying Remark 2.2.5(2) we consider the full triangulated subcategory $\text{DM}^{\text{eff}}_R \langle 1 \rangle$ of $\text{DM}^{\text{eff}}_R$. In [KaS17] the Verdier localization $\text{DM}_R^{\text{bir}} = \text{DM}^{\text{eff}}_R / \text{DM}^{\text{eff}}_R \langle 1 \rangle$ was considered (in the cases $R = \mathbb{Z}$ and $R = \mathbb{Z}[1/p]$). This category is called the category of birational motivic complexes; the reason for this is that the localization functor $\pi : \text{DM}^{\text{eff}}_R \to \text{DM}_R^{\text{bir}}$ sends the motives of birationally equivalent smooth varieties into isomorphic objects (see Lemma 2.2.1(1)). The composition $\pi \circ M_R$ will be denoted by $M^{\text{bir}}_R$.

**Proposition 3.2.1.** 1. The elements of $M^{\text{bir}}_R(\text{SmVar})$ are compact in $\text{DM}_R^{\text{bir}}$.

2. The functor $\pi$ is weight-exact with respect to the weight structures $w_{\text{eff}}^{\text{Chow}}$ and $w_{\text{bir}}^{\text{R}}$, where $w_{\text{bir}}^{\text{R}}$ is generated by the set $M^{\text{bir}}_R(\text{SmVar})$.

**Proof.** 1. See Lemma 1.1.1(3).

2. According to [Bon10, Proposition 8.1.1(1)] there exists a weight structure on $\text{DM}_R^{\text{bir}}$ such that the functor $\pi$ is weight-exact. Applying Proposition 1.2.3(II.3) we obtain that this weight structure is generated by $M^{\text{bir}}_R(\text{SmVar})$.

**Remark 3.2.2.** 1. A significant distinction of $w_{\text{bir}}^{\text{R}}$ from $w_{\text{eff}}^{\text{Chow}}$ in the context of this paper is that we can prove that $w_{\text{bir}}^{\text{R}}$ restricts to the subcategory $\text{DM}^{\text{bir}}_{gm,R} = \langle M^{\text{bir}}_R(\text{SmVar}) \rangle$ of compact objects of $\text{DM}^{\text{bir}}_R$. Indeed, Lemma 1.1.1(3) implies that $\text{DM}^{\text{bir}}_{gm,R}$ is equivalent to $\text{Kar}(\text{DM}^{\text{eff}}_{gm,R} / \text{DM}^{\text{eff}}_{gm,R} \langle 1 \rangle)$, and the existence on the latter category of a weight structure generated by the subcategory $\text{Chow}^{\text{bir}}_R$ (whose objects are retracts of elements of $M^{\text{bir}}_R(\text{SmVar})$) was established in [BoS16, §5.2]. The heart of this restricted weight structure equals $\text{Chow}^{\text{bir}}_R$; thus [BoS17, Corollary 2.3.1(1)] (as well as [Bon10, Theorem 4.5.2]) implies that the heart of $w_{\text{bir}}^{\text{R}}$ consists of all retracts of coproducts of elements of $M^{\text{bir}}_R(\text{SmVar})$.

2. Arguing similarly to [BoS16, §5] one can prove that the category $\text{DM}^{\text{bir}}_R$ is equivalent to the localization of $K'(\text{SmCor}_R)$ (see point 1 in Proposition 1.3.1) by the localizing subcategory, generated by cones of all $M(f)$, where $f : U \to X$ is a dense open embedding of smooth $k$-varieties. On the category $K'(\text{SmCor}_R)$ we take the "stupid" weight structure, generated by $R_{tr}(\text{SmVar})$; cf. [BoS17, Remark 2.3.2(1)] and [BoS18, Remark 1.2.3(1)]. Applying [BoS16, Theorem 4.3.1.4] (or [BoS17, Theorem 3.1.3(3(ii))]) we obtain that there exists a weight structure on $\text{DM}^{\text{bir}}_R$ such that the localization $K'(\text{SmCor}_R) \to \text{DM}^{\text{bir}}_R$ is weight-exact. Hence Proposition 1.2.3(II.3) implies that this weight structure coincides with $w_{\text{bir}}^{\text{R}}$ also. Thus any element of $\text{DM}^{\text{bir}}_R w_{\text{bir}}^{\text{R}} \leq 0$ (resp. of $\text{DM}^{\text{bir}}_R w_{\text{bir}}^{\text{R}} \geq 0$) is a retract of a object that has a
pre-image in $K'(\text{SmCor}_R^p)$ that is presented by a $\text{SmCor}_R^p$-complex concentrated in non-negative (resp. non-positive) degrees (see Proposition 3.1.1(1) of [BoS17]).

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