Abstract
We study a family of action functionals whose critical points interpolate between frozen planet orbits for the helium atom with mean interaction between the electrons and the free fall. The rather surprising first result of this paper asserts that for the whole family, critical points are always nondegenerate. This implies that the frozen planet orbit with mean interaction is nondegenerate and gives a new proof of its uniqueness. As an application, we show that the integral count of frozen planet orbits with instantaneous interaction equals one. For this, we prove orientability of the determinant line bundle over the space of self-adjoint Fredholm operators with spectrum bounded from below, and use it to define an integer valued Euler characteristic for Fredholm sections whose linearization belongs to this class.

1 Introduction

Frozen planet orbits are periodic orbits in the helium atom in which both electrons move on a line on the same side of the nucleus. The inner electron undergoes consecutive collisions with the nucleus, while the outer electron (the actual “frozen planet”) remains almost stationary at some distance. See [14, 15] for numerical evidence for such orbits and a discussion of their role in the semiclassical treatment of the helium atom.

When trying to prove the existence of frozen planet orbits, one faces the difficulty that they cannot be obtained as perturbations of the system without interaction between the electrons. In order to deal with this problem, the second author replaced in [9] the instantaneous interaction between the two electrons by a mean interaction and proved that in this case for every negative energy there exists a unique frozen planet orbit. Building on work of Barutello, Ortega and Verzini [2], we introduced in [3] two functionals $B_{av}$ and $B_{in}$ whose critical points correspond to the Levi-Civită regularizations of frozen planet orbits for the mean and instantaneous interaction, respectively. We proved that for each $r \in [0, 1]$ the $L^2$-gradient $\nabla B_r$ of the interpolation $B_r = rB_{in} + (1 - r)B_{av}$ is
a $C^1$-Fredholm map of index 0. Now to each $C^1$-Fredholm map $F$ of index 0 with compact zero set one can associate its mod 2 Euler number $\chi(F) \in \mathbb{Z}/2\mathbb{Z}$ counting its zeroes modulo 2 after perturbation, see [3, Appendix C]. Using the compactness result from [10] and homotopy invariance of the mod 2 Euler number, we deduced

$$\chi(\nabla B_{\text{in}}) \equiv \chi(\nabla B_{av}) \equiv 1 \pmod{2}.$$ (1)

Here the functionals are considered on a suitable space of normalized simple symmetric loops. In particular, for each negative energy there exists a frozen planet orbit [3, Corollary C].

The proof that $\chi(\nabla B_{av}) \equiv 1 \pmod{2}$ in [3, Theorem D.1] was based on a further deformation of $\nabla B_{av}$ through Fredholm maps which are not gradients of functionals. Our first result in the present paper improves this to (see Corollary 4.6)

**Theorem A:** The unique normalized simple symmetric frozen planet orbit for the mean intersection functional $B_{av}$ is nondegenerate of Morse index 0.

To prove this, we introduce a family of functionals $F_r, r \in [0, \infty)$, such that for $\rho = (\sqrt{2} - 1)^2$ the critical points of $F_\rho$ and their Hessians agree with those of $B_{av}$, see Section 4.2. On the other hand, $F_0$ describes the free fall of an electron into the helium kernel (again undergoing consecutive collisions which are regularized), which is easily seen to possess a unique simple periodic orbit that is nondegenerate of Morse index 0. Thus Theorem A (as well as the uniqueness of the frozen planet orbit for $B_{av}$) will be a consequence of the following result (see Theorem 3.1):

**Theorem B:** For every $r \in [0, \infty)$, each critical point of $F_r$ is nondegenerate.

The main ingredient in the proof of Theorem B is an algebraic identity associated to critical points of $F_r$ which can be solved in terms of elliptic integrals (see Proposition 3.4 and Appendix A).

Theorem A allows us to upgrade equation (1) to an equality of integer valued Euler numbers. For this, we need to define a $\mathbb{Z}$-valued Euler number for a class of Fredholm sections including the $\nabla B_r$ above. The important feature of these Fredholm sections is that their linearizations are self-adjoint and bounded from below with respect to the $L^2$-scalar product. The main ingredient is the following abstract result which may be of independent interest.

Let $F$ be a real Hilbert space, and $E \subset F$ a dense linear subspace which is itself a Hilbert space (with a different inner product) such that the inclusion $E \hookrightarrow F$ is compact. For a real number $\Re$ consider the spaces

$$\mathcal{F}_s^{>\Re}(E, F) \subset \mathcal{F}_s(E, F) \subset \mathcal{F}(E, F)$$

where $\mathcal{F}(E, F)$ is the space of Fredholm operators $E \to F$, $\mathcal{F}_s(E, F)$ the subspace of operators that are self-adjoint as unbounded operators on $F$ with domain $E$, and $\mathcal{F}_s^{>\Re}(E, F)$ the subspace of operators whose spectrum is contained in $(\Re, \infty)$. Recall (see e.g. [13]) that the determinants $\det(D) = \Lambda^{\max}(\ker D^*) \otimes \Lambda^{\min}(\ker D)$.
\( \Lambda_{\text{max}}(\ker D) \) of \( D \in \mathcal{F}(X, Y) \) give rise to a real line bundle, the determinant line bundle \( \text{det} \to \mathcal{F}(X, Y) \). It is well-known that this line bundle is non-orientable, but we have (see Theorem 5.3):

**Theorem C:** The restriction of the determinant line bundle to \( \mathcal{F}_s(E, F) \) carries a canonical orientation.

Contrarily to our initial expectation, the restriction of the determinant line bundle to \( \mathcal{F}_s(E, F) \) is non-orientable; we construct an explicit loop over which the bundle is nontrivial in Proposition 5.7.

Consider now a Hilbert manifold \( X \) and a Hilbert space bundle \( E \to X \) with a continuous bundle inclusion \( TX \subset E \) such that \( T_xX \subset E_x \) is dense and the inclusion \( T_xX \hookrightarrow E_x \) is compact for each \( x \in X \). Denote by \( \text{Func}^{>\mathbb{R}_s}(X) \) the space of \( C^2 \)-functions \( f : X \to \mathbb{R} \) whose \( E \)-gradient \( \nabla_E f : X \to E \) is of class \( C^1 \) such that \( \nabla_E f(x) \in \mathcal{F}^{>\mathbb{R}^n}(T_xX, E_x) \) for each critical point \( x \) of \( f \). For this class of functions, Theorem C allows us to define a \( \mathbb{Z} \)-valued Euler number (see Theorem 5.12):

**Corollary A:** To each \( f \in \text{Func}^{>\mathbb{R}_s}(X) \) with compact zero set we can associate an Euler number \( \chi(\nabla_E f) \in \mathbb{Z} \) which is uniquely characterized by suitable axioms of (Transversality), (Excision), and (Homotopy).

Verifying that the \( B_r \) above belong to \( \text{Func}^{>\mathbb{R}_s}(X) \) for a suitable bundle \( E \to X \) and using the (Transversality) and (Homotopy) axioms, we deduce (see Corollary 6.2):

**Corollary B:** The integral count of normalized simple symmetric frozen planet orbits equals

\[
\chi(\nabla_{B_{\text{in}}}) = \chi(\nabla_{B_{\text{av}}}) = 1 \in \mathbb{Z}.
\]

**Remark:** Symmetric frozen planet orbits have two Morse indices: one as a symmetric frozen planet orbit, and one just as a frozen planet orbit forgetting about the symmetry. When we talk about Morse indices (which enter into the Euler number via the (Transversality) axiom in Theorem 5.12) we always mean the symmetric one. In fact, the two indices are different: the unique normalized simple symmetric periodic orbit for the regularized free fall (i.e., the functional \( \mathcal{F}_r \) for \( r = 0 \)) has index 0 as a symmetric orbit and index 1 just as a periodic orbit. As we show, the functional \( \mathcal{F}_r \) is always nondegenerate in the symmetric as well as in the just periodic sense. Therefore, the symmetric Euler characteristic equals 1 while the periodic Euler characteristic equals \(-1\). However, to our knowledge, compactness for frozen planet orbits in the homotopy from mean to instantaneous interaction has only been established in the symmetric case [11]. Therefore, it is not clear whether Corollary B has an analogue if one forgets about the symmetry. We expect that it does and that the Euler characteristic for normalized simple (not necessarily symmetric) frozen planet orbits with instantaneous interaction is \(-1\). This would fit with the findings of physicists [13] [14], who numerically detected a frozen planet orbit for instantaneous interaction which is stable and therefore has odd Conley-Zehnder index.
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2 Levi-Civita transformation

In this section we recall some background on the Levi-Civita transformation. For details we refer the reader to [3].

We abbreviate by $S^1 = \mathbb{R}/\mathbb{Z}$ the circle. We denote the $L^2$-inner product of $z_1, z_2 \in L^2(S^1, \mathbb{R})$ by

$$\langle z_1, z_2 \rangle := \int_0^1 z_1(\tau)z_2(\tau)d\tau,$$

and the $L^2$-norm of $z \in L^2(S^1, \mathbb{R})$ by

$$\|z\| := \sqrt{\langle z, z \rangle}.$$

In the sequel we will work with Sobolev spaces $H^k = W^{k,2}$, but the only relevant norms and inner products will be the ones from $L^2$.

Consider two maps

$q : S^1 \to \mathbb{R}_{\geq 0}, \quad z : S^1 \to \mathbb{R}$

related by the Levi-Civita transformation

$$q(t) = z(\tau)^2$$

for a time change $t \leftrightarrow \tau$ satisfying $0 \leftrightarrow 0$ and

$$\frac{dt}{q(t)} = \frac{d\tau}{\|z\|^2}. \quad (3)$$

This implies that the mean values of $q$ and $1/q$ are given by

$$\overline{q} := \int_0^1 q(t)dt = \int_0^1 \frac{z(\tau)^4}{\|z\|^2}d\tau = \frac{\|z\|^2}{\|z\|^2} \quad (4)$$

and

$$\int_0^1 \frac{dt}{q(t)} = \frac{1}{\|z\|^2}. \quad (5)$$

We will denote derivatives with respect to $t$ by a dot and derivatives with respect to $\tau$ by a prime. Then the first and second derivatives of $q$ and $z$ (where they are defined) are related by

$$\dot{q}(t) = 2z(\tau)z'(\tau) \frac{d\tau}{dt} = 2\|z\|^2z'(\tau) \frac{z'(\tau)}{z(\tau)} \quad (6)$$
Substituting $z^2$ and $z'^2$ by (2) and (6) this becomes
\[ \ddot{q}(t) = \frac{1}{q(t)} \left( 2\|z\|^4 \frac{z''(\tau)}{z(\tau)} - \frac{\dot{q}(t)^2}{2} \right). \] (8)

The $L^2$-norm of the derivative of $q$ is given by
\[ \|\dot{q}\|^2 = \int_0^1 \dot{q}(t)^2 dt = \int_0^1 4\|z\|^4 \left( z''(\tau) \frac{z(\tau)}{\|z\|^2} \right)^2 \frac{z(\tau)^2}{\|z\|^2} d\tau = 4\|z\|^2 \|z''\|^2. \] (9)

We can now give the precise definition of the Levi-Civita transformation. Let $z \in C^0(S^1, \mathbb{R})$ be a continuous function with finite zero set
\[ Z_z := z^{-1}(0). \]

We associate to $z$ a $C^1$-map $t_z : S^1 \to S^1$ by
\[ t_z(\tau) := \frac{1}{\|z\|^2} \int_0^\tau z(\sigma)^2 d\sigma. \] (10)

Note that $t_z(0) = 0$ and
\[ t'_z(\tau) = \frac{z(\tau)^2}{\|z\|^2}. \] (11)

Since $z$ has only finitely many zeroes, this shows that $t_z$ is strictly increasing and we conclude

**Lemma 2.1** If $z \in C^0(S^1, \mathbb{R})$ has only finitely many zeroes, then the map $t_z : S^1 \to S^1$ defined by (10) is a homeomorphism. □

It follows that $t_z : S^1 \to S^1$ has a continuous inverse
\[ \tau_z := t_z^{-1} : S^1 \to S^1. \]

Since $t_z$ is of class $C^1$, the function $\tau_z$ is also of class $C^1$ on the complement of the finite set $t_z(Z_z)$ with derivative
\[ \dot{\tau}_z(t) = \frac{\|z\|^2}{z(t)^2}, \quad t \in S^1 \setminus t_z(Z_z). \] (12)

We define a continuous map $q : S^1 \to \mathbb{R}_{\geq 0}$ by
\[ q(t) := z(\tau_z(t))^2. \] (13)

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Then the two maps \( z, q \) are related by the Levi-Civita transformation \( \tau = \tau_z \). Their zero sets

\[
Z_z = z^{-1}(0) \quad \text{and} \quad Z_q := q^{-1}(0) = t_z(Z_z)
\]

are in bijective correspondence via \( t_z \) (or equivalently \( \tau_z \)). Moreover, by (5) we have

\[
\int_0^1 \frac{ds}{q(s)} = \frac{1}{\|z\|^2} < \infty.
\]

Conversely, suppose we are given a map \( q \in C^0(S^1, \mathbb{R}_{\geq 0}) \) with finite zero set \( Z_q \) satisfying \( \int_0^1 \frac{ds}{q(s)} < \infty \). We associate to \( q \) the time reparametrization \( \tau_q : S^1 \to S^1 \),

\[
\tau_q(t) := \left( \int_0^1 \frac{ds}{q(s)} \right)^{-1} \int_0^t \frac{1}{q(s)} ds. \tag{14}
\]

Then \( \tau_q(1) = 1 \), \( \tau_q \) is of class \( C^1 \) outside the zero set \( Z_q = q^{-1}(0) \) with derivative

\[
\tau'_q(t) = \left( \int_0^1 \frac{ds}{q(s)} \right)^{-1} \frac{1}{q(t)}, \quad t \in S^1 \setminus Z_q. \tag{15}
\]

By [2, Lemma 2.1], the map \( \tau_q : S^1 \to S^1 \) is a homeomorphism whose inverse \( t_q := \tau_q^{-1} \) is of class \( C^1 \) and satisfies \( t_q(1) = \tau_q^{-1}(1) = 1 \) and

\[
t'_q(\tau) = \left( \int_0^1 \frac{ds}{q(s)} \right) q(t_q(\tau)), \quad \tau \in S^1. \tag{16}
\]

Suppose that \( z : S^1 \to \mathbb{R} \) is a continuous function satisfying

\[
z(\tau)^2 = q(t_q(\tau)). \tag{17}
\]

Then \( z \) has finite zero set \( Z_z = \tau_q(Z_q) \), so we can associate to \( z \) the homeomorphism \( t_z : S^1 \to S^1 \) defined by (10) and its inverse \( \tau_z \). We claim that

\[
\tau_q = \tau_z \quad \text{and} \quad t_q = t_z. \tag{18}
\]

It is enough to check the second equality. For this we compute

\[
\int_0^\tau z(\sigma)^2 d\sigma = \int_0^\tau q(t_q(\sigma))d\sigma \overset{(*)}{=} \left( \int_0^1 \frac{ds}{q(s)} \right)^{-1} \int_0^{t_q(\tau)} ds = \left( \int_0^1 \frac{ds}{q(s)} \right)^{-1} t_q(\tau),
\]

where \((*)\) follows from the coordinate change \( \sigma = \tau_q(s) \) and (15). Evaluating at \( \tau = 1 \) gives us

\[
\frac{1}{\|z\|^2} = \int_0^1 \frac{ds}{q(s)}. \tag{19}
\]

Therefore,

\[
t_z(\tau) = \frac{1}{\|z\|^2} \int_0^\tau z(\sigma)^2 d\sigma = t_q(\tau)
\]
and (18) is established. Hence \( q \) is the Levi-Civita transform of \( z \) defined by (13). Equation (17) does not uniquely determine \( z \) for given \( q \) because the sign of \( z \) can be arbitrarily chosen on each connected component of \( S^1 \setminus Z_z \). If \( Z_z \) consists of an even number of points, then we can determine \( z \) up to a global sign by the requirement that \( z \) changes its sign at each zero. If \( Z_z \) consists of an odd number of points, then the requirement that \( z \) changes its sign at each zero leads to \( z(\tau + 1) = -z(\tau) \), so \( z \) has period 2 rather than 1. Therefore, the preceding discussion shows

**Lemma 2.2** The Levi-Civita transformation \( z \mapsto q \) given by (13) defines for each even integer \( m \) a surjective 2-to-1 map

\[
\mathcal{L} : \{ z \in C^0(S^1, \mathbb{R}) \mid z \text{ has precisely } m \text{ zeroes and switches sign at each zero} \}
\to \{ q \in C^0(S^1, \mathbb{R}_{\geq 0}) \mid z \text{ has precisely } m \text{ zeroes and } \int_0^1 \frac{ds}{q(s)} < \infty \},
\]

and for each odd integer \( m \) a surjective 2-to-1 map

\[
\mathcal{L} : \{ z \in C^0(\mathbb{R}/2\mathbb{Z}, \mathbb{R}) \mid z \text{ has precisely } 2m \text{ zeroes and switches sign at each zero, } z(\tau + 1) = -z(\tau) \text{ for all } \tau \}
\to \{ q \in C^0(S^1, \mathbb{R}_{\geq 0}) \mid z \text{ has precisely } m \text{ zeroes and } \int_0^1 \frac{ds}{q(s)} < \infty \}.
\]

\[\square\]

### 3 The functionals \( F_r \) and their critical points

We denote by \( S^1 = \mathbb{R}/\mathbb{Z} \) the circle and abbreviate by

\[
H^1_+(S^1, \mathbb{R}) = H^1(S^1, \mathbb{R}) \setminus \{0\}, \quad H^2_+(S^1, \mathbb{R}) = H^2(S^1, \mathbb{R}) \setminus \{0\}
\]

the open subsets of the Hilbert space \( H^1(S^1, \mathbb{R}) \) respectively \( H^2(S^1, \mathbb{R}) \) where the origin is removed. For \( r \in [0, \infty) \) we consider the functional

\[
F_r : H^1_+(S^1, \mathbb{R}) \to \mathbb{R}, \quad z \mapsto 2||z||^2||z'||^2 + \frac{2}{||z||^2} + r \frac{||z||^2}{||z'||^2},
\]

where as before \( ||z|| \) is the \( L^2 \)-norm of the loop \( z \). If \( z \in H^1_+(S^1, \mathbb{R}) \) and \( \xi \in H^1(S^1, \mathbb{R}) \), then the differential of \( F_r \) at \( z \) in direction of \( \xi \) is given by

\[
DF_r(z)\xi = 4||z||^2\langle z', \xi' \rangle + 4||z'||^2\langle z, \xi \rangle - 4\langle z, \xi \rangle \frac{2\langle z, \xi \rangle}{||z||^4} + r \left( \frac{2\langle z, \xi \rangle}{||z'||^2} - \frac{4||z||^2\langle z^3, \xi \rangle}{||z'||^4} \right),
\]

where \( \langle \cdot, \cdot \rangle \) denotes the \( L^2 \)-inner product. Therefore, integration by parts shows that for \( z \in H^2_+(S^1, \mathbb{R}) \) the functional \( F_r \) possess an \( L^2 \)-gradient given by the
We see that $\nabla F_r : H^2(S^1, \mathbb{R}) \to L^2(S^1, \mathbb{R})$ is differentiable; we call its derivative the \textit{Hessian} of $F_r$. Critical points of $F_r$ are solutions of the problem

$$z'' = -bz - 2az^3.$$  \hfill (21)

A standard bootstrapping argument implies that critical points of $F_r$ are in fact smooth. If $z$ is a solution of (21), then for $n \in \mathbb{N}$ the loop $z_n$ defined as

$$z_n(\tau) = n^{-1/3} z(n\tau), \quad \tau \in S^1$$

is another solution of (21) with $a_n = n^{8/3} a$ and $b_n = n^2 b$. We say that a critical point $z$ is \textit{multiply covered} if there exists a critical point $w$ and $n > 1$ such that $z = w_n$. Otherwise we call the critical point \textit{simple}. Moreover, the functional $F_r$ is invariant under the $S^1$-action given by time shift,

$$\sigma_z(\tau) = z(\tau + \sigma), \quad \tau \in S^1,$$

where $\sigma \in S^1$ and $z \in H^1_1(S^1, \mathbb{R})$. In particular, its critical points are invariant under time shift as well. Therefore, if $z$ is a critical point of $F_r$, then $z'$ lies in the kernel of the Hessian of $F_r$ at $z$.

It follows from Proposition $3.3$ below that for each critical point $z$ of $F_r$, $r \geq 0$, there exists $\tau_0 \in S^1$ such that $z(\tau_0) = 0$. We then necessarily have $z'(\tau_0) \neq 0$, since otherwise by (21) the loop $z$ would be the constant loop at the origin which does not lie in $H^1_1(S^1, \mathbb{R})$. Thus a critical point of $F_r$ is never a fixed point of the $S^1$-action on the free loop space. In particular, its nullity, i.e., the dimension of the kernel of its Hessian, is at least one. We say that a critical point is \textit{nondegenerate} if its nullity is precisely one. Our first result asserts that nondegeneracy always holds true for nonnegative $r$. It corresponds to Theorem B from the Introduction and will be proved in Section $3.3$.

\textbf{Theorem 3.1} For every $r \in [0, \infty)$, each critical point of $F_r$ is nondegenerate.

\section*{3.1 Levi-Civita transform of critical points}

In this section we apply the Levi-Civita transformation to critical points of $F_r$. Let $z \in H^1_1(S^1, \mathbb{R})$ be a solution of (21) and $q(t) := z(\tau_z(t))^2$ its Levi-Civita
transform. We compute at points $t \in S^1 \setminus t_z(Z_z)$:

$$
\ddot{q} \overset{(A)}{=} \frac{1}{q} \left( -2\|z\|^4 (b + 2az^2) - \frac{\dot{q}^2}{2} \right)
\overset{(B)}{=} \frac{1}{q} \left( -\frac{2}{\|z\|^2} + 2\|z\|^2 ||z'||^2 + r \frac{||z||^2}{||z^2||^2} - 2r \frac{||z||^4}{||z^2||^4} q - \frac{\dot{q}^2}{2} \right)
\overset{(C)}{=} \frac{1}{q} \left( -\int_0^1 \frac{2}{q(s)} ds + \frac{||\dot{q}||^2}{2} + \frac{r}{q} q - 2r \frac{q}{\dot{q}^2} - \frac{\dot{q}^2}{2} \right).
$$

Equality (A) follows from substituting $z''$ by (21) in equation (8); equality (B) uses the expressions for $a$ and $b$ in (21); equality (C) uses $q = z^2$ as well as (3), (4) and (9). Thus $q$ satisfies the ODE

$$
\ddot{q} = \left( c - \frac{\dot{q}^2}{2} \right) \frac{1}{q} - \frac{2r}{q^2}
$$

with the constant

$$
c = \frac{||\dot{q}||^2}{2} - \int_0^1 \frac{2}{q(s)} ds + \frac{r}{q}.
$$

At the global maximum $t_{\text{max}}$ of $q$, equation (22) becomes

$$
\frac{c}{q(t_{\text{max}})} + \frac{2r}{q} = \ddot{q}(t_{\text{max}}) \leq 0,
$$

hence

$$
c \leq -\frac{2r q(t_{\text{max}})}{q^2}.
$$

Let now $t_- < t_+$ be adjacent zeroes of $q$ and consider the smooth map

$$
\beta := \frac{\ddot{q} + \frac{r}{q}}{q} : (t_-, t_+) \to \mathbb{R}.
$$

From (22) we obtain

$$
\beta q^2 = c - \frac{\dot{q}^2}{2} - \frac{r \dot{q}^2}{q^2}.
$$

With inequality (24) this implies

$$
\beta q^2 \leq -\frac{2r q(t_{\text{max}})}{q^2} - \frac{\dot{q}^2}{2} - \frac{r \dot{q}^2}{q^2} < 0,
$$

hence $\beta < 0$ on $(t_-, t_+)$. Differentiating both sides of the equation for $\beta q^2$ we get

$$
\dot{\beta} q^2 + 2\beta q \dot{q} = -\ddot{q} - \frac{r \dot{q}^2}{q^2} = -\beta q \ddot{q},
$$

and therefore

$$
\dot{\beta} q = -3\beta \dot{q}.
$$

We need the following
Lemma 3.2  Equation (25) for functions $q > 0$ and $\beta < 0$ on $(t_-, t_+)$ implies that
$$\beta = -\frac{\mu}{q^3}$$

on $(t_-, t_+)$ for some constant $\mu > 0$.

Proof:  Dividing both sides of equation (25) by $q\beta$ yields
$$\frac{d}{dt} \log(-\beta) = -3\frac{d}{dt} \log(q),$$

which by integration implies the lemma.  \qed

By Lemma 3.2 we get equation (26) on $(t_-, t_+)$ for some constant $\mu > 0$.  By definition of $\beta$, this yields the following equation for $q$:
$$\ddot{q}(t) = -\frac{\mu}{q(t)^2} - \frac{r}{q^2}$$

for $t \in (t_-, t_+)$.  It remains to compute $\mu$.  Plugging this into (22) we infer
$$\mu = -\left(c - \frac{\dot{q}(t)^2}{2}\right)q(t) + \frac{rq(t)^2}{q^2}$$

for $t \in (t_-, t_+)$.  In particular, using (6) and $q(t_\pm) = 0$ we obtain
$$\mu = \lim_{t \to t_\pm} \frac{\dot{q}(t)^2q(t)}{2} = 2\|z\|^4 z'(\tau_z(t_\pm))^2.$$  

We deduce from this that equation (27) holds on $S^1 \setminus t_\pm(Z_z)$ with a fixed $\mu$ independent of the connected component in $S^1 \setminus t_\pm(Z_z)$.  Dividing (26) by $q(t)$ and inserting $c$ from (23), we get
$$\frac{\mu}{q(t)} = -\frac{\|\dot{q}(t)^2q(t)}{2} + \int_0^1 \frac{1}{q(s)} ds - \frac{r}{q} + \frac{\dot{q}(t)^2}{2} + \frac{rq(t)}{q^2}.$$  

Integrating this equation yields
$$\mu \int_0^1 \frac{1}{q(t)} dt = 2 \int_0^1 \frac{1}{q(s)} ds,$$

and therefore
$$\mu = 2.$$  

This gives us the following statement.

Proposition 3.3  Assume that $z \in H^1_1(S^1, \mathbb{R})$ is a critical point of the frozen functional $F_r$.  Then the Levi-Civita transform $q(t) = z(\tau_z(t))^2$ of $z$ satisfies the differential equation
$$\ddot{q}(t) = -\frac{2}{q(t)^2} - \frac{r}{q^2}.$$  

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Equation (31) explains the physical meaning of critical points of $F_r$: The orbit $q(t)$ describes an electron on the line attracted by a doubly positively charged nucleus at the origin and subject to an additional force depending on its mean $q$. For $r = 0$ the mean interaction force vanishes and $q(t)$ describes the free fall of the electron into the nucleus, regularized by elastic reflection as it hits the nucleus. For $r > 0$ the mean interaction force pushes the electron towards the origin, which can be interpreted as the effect of a second electron further away and on the same side of the nucleus. Indeed, we will show in §4 that for a suitable value of $r$ the system describes frozen planet orbits in helium with mean interaction between the electrons. For $r < 0$ the mean interaction force pushes the electron away from the origin, which can be interpreted as the effect of a second electron on the other side of the nucleus. However, we will not consider the case $r < 0$ in this paper.

### 3.2 Analysis of critical points

Let $r \geq 0$ and $z \in H^1(S^1, \mathbb{R})$ be a critical point of $F_r$, i.e., a solution of (21). We denote by $\|z\|_0$ the maximum norm of $z$ and introduce the quantities

$$v := \frac{\|z\|^2}{\|z\|_0^2}, \quad w := \frac{\|z\|^2 \|z\|_0^2}{\|z\|^2_0^2}.$$  \hspace{1cm} (32)

For $n \in \mathbb{N}_0$ we consider the elliptic integral

$$I_n : (-\infty, 1) \to \mathbb{R}, \quad I_n(m) := \int_0^1 \frac{\zeta^{2n}}{(1 - \zeta^2)(1 - m\zeta^2)} d\zeta.$$  

The main result of this section is the following proposition.

**Proposition 3.4** The quantities $v$ and $w$ satisfy the equations

$$v = \frac{2}{4 + 3rw - 2rw^2} = \frac{I_1}{I_0} \left( - \frac{rw^2}{2} \right).$$  \hspace{1cm} (33)

**Proof:** After a time reparametrization, we can assume without loss of generality that $z$ attains at time 0 its global maximum

$$z_0 := \|z\|_0 = z(0).$$

Recall that there exists $\tau_0 \in S^1$ such that $z(\tau_0) = 0$. We further choose $\tau_0$ as the smallest number in $(0, 1)$ with the property that $z(\tau_0) = 0$.

We first eliminate in the formula for $b$ in (20) the variable $\|z\|$. Using (21), we obtain via integration by parts

$$\|z\| = -\langle z, z'' \rangle = b\|z\|^2 + 2a\|z\|^2.$$
Plugging this into the equation for $b$ and using the equation for $a$, we get

$$b = \frac{1}{||z||^6} - b - \frac{r}{||z||^2 \cdot ||z||^2} - \frac{r}{2||z||^2 \cdot ||z||^2},$$

implying

$$b = \frac{1}{2||z||^6} - \frac{3r}{4||z||^2 \cdot ||z||^2}. \tag{34}$$

From (21) we further infer that we have the conserved quantity

$$\frac{z'(\tau)^2}{2} + \frac{b z(\tau)^2}{2} + \frac{a z(\tau)^4}{2} = \frac{c}{2} \tag{35}$$

for some constant $c$. Since $z(\tau_0) = 0$ and $z'(\tau_0) \neq 0$ we conclude that $c$ is positive. Since $z$ attains its global maximum $z_0$ at time $\tau = 0$, we necessarily have $z'(0) = 0$ and therefore

$$b z_0^2 + a z_0^4 = c.$$

This means that $z_0^2$ is a root of the quadratic polynomial

$$p(x) = a x^2 + b x - c.$$

Moreover, positivity of $c$ yields

$$a z_0^2 + b > 0. \tag{36}$$

**Case 1: $r > 0$.**

In this case $a \neq 0$ as well and the second root of $p$ is given by $-z_0^2 - \frac{b}{a}$. In particular, the quadratic polynomial factorizes as

$$p(x) = \left(x - z_0^2\right)\left(ax + az_0^2 + b\right).$$

Plugging this into (35) we obtain

$$z'(t)^2 = -p(z(\tau)) = \left(z_0^2 - z(\tau)^2\right)\left(az(\tau)^2 + az_0^2 + b\right), \quad \tau \in S^1. \tag{37}$$

Equation (21) is invariant under reflection at the origin and time reversal. Since $z(\tau_0) = 0$, we conclude that

$$z(\tau) = -z(2\tau_0 - \tau), \quad \tau \in [\tau_0, 2\tau_0].$$

In particular, we have $z(2\tau_0) = -z(0)$ and $z'(2\tau_0) = 0$. Using once more invariance under time reversal, we conclude that

$$z(\tau) = -z(2\tau_0 + \tau), \quad \tau \in [0, 4\tau_0].$$

In particular, we have $z(4\tau_0) = z(0)$ and $z'(4\tau_0) = z'(0) = 0$. We see that $z$ is periodic of period $4\tau_0$. Since $\tau_0$ was chosen as the first positive time at which
z passes through the origin, we conclude that 4\(\tau_0\) is the minimal period of \(z\). Since \(z\) is by assumption periodic of period 1, we conclude that there exists \(n \in \mathbb{N}\) such that \(4\tau_0n = 1\), i.e.,

\[
t_0 = \frac{1}{4n}.
\]

If \(z\) is simple, then \(n = 1\); otherwise it is multiply covered. Using (37) we therefore obtain

\[
\frac{1}{4n} = \int_0^{\frac{z_0}{2}} \frac{1}{\sqrt{(z_0^2 - z^2)(az^2 + az_0^2 + b)}} dz = \int_0^1 \frac{1}{\sqrt{a z_0^2 + b}} \int_0^1 \frac{1}{\sqrt{1 - \zeta^2}(1 + \frac{a z_0^2}{a z_0^2 + b} \zeta^2)} d\zeta
\]

= \frac{1}{\sqrt{a z_0^2 + b}} f_0\left(-\frac{a z_0^2}{a z_0^2 + b}\right).

Similarly, we compute

\[
\frac{||z||^2}{4n} = \int_0^{\frac{z_0}{2}} \frac{z^2}{\sqrt{(z_0^2 - z^2)(az^2 + az_0^2 + b)}} dz = \int_0^1 \frac{\zeta^2}{\sqrt{a z_0^2 + b}} \int_0^1 \frac{\zeta^2}{\sqrt{1 - \zeta^2}(1 + \frac{a z_0^2}{a z_0^2 + b} \zeta^2)} d\zeta
\]

= \frac{z_0^2}{\sqrt{a z_0^2 + b}} I_1\left(-\frac{a z_0^2}{a z_0^2 + b}\right)

and

\[
\frac{||z^2||^2}{4n} = \int_0^{\frac{z_0}{2}} \frac{z^4}{\sqrt{(z_0^2 - z^2)(az^2 + az_0^2 + b)}} dz = \int_0^1 \frac{\zeta^4}{\sqrt{a z_0^2 + b}} \int_0^1 \frac{\zeta^4}{\sqrt{1 - \zeta^2}(1 + \frac{a z_0^2}{a z_0^2 + b} \zeta^2)} d\zeta
\]

= \frac{z_0^4}{\sqrt{a z_0^2 + b}} I_2\left(-\frac{a z_0^2}{a z_0^2 + b}\right).

The elliptic function \(I_2\) can be expressed using the elliptic functions \(I_0\) and \(I_1\) by

\[
I_2(m) = \frac{2(m+1)I_1(m) - I_0(m)}{3m}
\]

as explained in (127) in Appendix A. Hence from (38), (39), and (40) we obtain
the equality

\[
||z^2||^2 = \frac{4n ||z_0^2||}{\sqrt{az_0^2 + b}} \left( \frac{2}{3} \frac{az_0^2 + b}{az_0^2 + b} \right) I_1 \left( - \frac{az_0^2 + b}{3az_0^2} I_0 \left( - \frac{az_0^2 + b}{az_0^2 + b} \right) \right)
\]

\[
= \frac{z_0^4}{\sqrt{az_0^2 + b}} \left( \frac{az_0^2 + b}{3az_0^2} \sqrt{az_0^2 + b} - \frac{2b}{3a} \frac{\sqrt{az_0^2 + b}}{||z||^2} \right)
\]

\[
= \frac{(az_0^2 + b)z_0^2}{3a} - \frac{2b||z||^2}{3a}
\]

\[
= \frac{z_0^4}{3} + \frac{b}{3a}(z_0^2 - 2||z||^2)
\]

\[
= \frac{z_0^4}{3} + \left( \frac{||z^2||^4}{3r||z||^6} - \frac{||z^2||^2}{2||z||^2} \right) \cdot (z_0^2 - 2||z||^2)
\]

\[
= \frac{z_0^4}{3} + \frac{||z^2||^4 z_0^2}{3r||z||^6} - \frac{||z^2||^2 z_0^2}{2||z||^2} - \frac{2||z^2||^4}{3r||z||^4} + ||z^2||^2
\]

where in the second to last equality we have used the equation for \(a\) from (20) and equation (34) for \(b\). Removing \(||z^2||^2\) on both sides and multiplying the remaining terms by \(\frac{6r||z||^4}{||z^2||^2}\), we obtain the equation

\[
0 = 2r||z||^4 z_0^4 + 2z_0^2 \frac{||z^2||^2 z_0^2}{||z||^2} - 3r||z||^2 z_0^2 - 4. \tag{41}
\]

By definition of \(v\) and \(w\) we can rewrite this as

\[
0 = 2rw^2 + \frac{2}{v} - 3rw - 4
\]

or equivalently

\[
v = \frac{2}{4 + 3rw - 2rw^2}.
\]

This proves the first equation in (33). We use this together with the equation for \(a\) in (20) and equation (34) for \(b\) to compute

\[
\frac{az_0^2}{az_0^2 + b} = \frac{az_0^2}{az_0^2 + b + z_0^6}
\]

\[
= \frac{r z_0^8}{2||z||^2} + \frac{z_0^6}{2||z||^2} - \frac{3r z_0^6}{4||z||^4} \tag{41}
\]

\[
= \frac{r w^2}{2w^2} + \frac{1}{w^2} - \frac{3r w}{4w^4}
\]

\[
= \frac{r w^2}{w^2} + \frac{1}{w^2} - \frac{3r w}{w^4}
\]

\[
= \frac{r w^2}{2}. \tag{41}
\]
Dividing (39) by (38) and combining the result with (41) we get

\[ \|z\|^2 = z_0^2 \cdot \frac{I_1}{I_0} \left( -\frac{az_0^2}{a^2 + b} \right) = z_0^2 \cdot \frac{I_1}{I_0} \left( -\frac{rw^2}{2} \right). \]

Hence by definition of \( v \) this can be rephrased as

\[ v = \frac{I_1}{I_0} \left( -\frac{rw^2}{2} \right). \]

This proves the second equation in (33) and completes the proof in the case \( r \neq 0 \).

**Case 2:** \( r = 0 \).

In the case \( r = 0 \) equation (33) becomes

\[ v = \frac{1}{2} = \frac{I_1(0)}{I_0(0)}. \] (42)

By equation (123) in Appendix A we have

\[ I_n(0) = \frac{(2n - 1)!! \pi}{2^{n+1} n!}, \]

where \((2n - 1)!!\) equals \((2n - 1)(2n - 3) \cdots 1\) for \( n \geq 1 \) and 1 for \( n = 0 \). In particular,

\[ I_0(0) = \frac{\pi}{2}, \quad I_1(0) = \frac{\pi}{4} \]

and therefore the second equality in (42) follows. It remains to check the first equation in (42) which says that

\[ \|z\|^2 = \frac{z(0)^2}{2}. \] (43)

However, for \( r = 0 \) we have that \( a = 0 \) and therefore \( z \) is a solution of the ODE

\[ z'' = -bz, \]

which implies that up to scaling and time-reparametrization \( z \) is given by the cosine function. Now (43) follows from

\[ \int_0^1 \cos^2(2\pi n t) dt = \frac{1}{2}, \quad n \in \mathbb{N}. \]

This finishes the proof of (33) in the case \( r = 0 \) and the proposition follows. \( \square \)
3.3 Proof of nondegeneracy

In this section we prove Theorem 3.1. Assume first that \( r > 0 \). Let \( z \) be a critical point of \( F_r \), i.e., a solution of the problem

\[
    z'' = -bz - 2az^3, \quad (44)
\]

\[
    b = \frac{1}{2||z||^6} - \frac{3r}{4||z||^2 \cdot ||z^2||^2},
\]

\[
    a = \frac{r}{2||z^2||^4}.
\]

As in the previous section, after a time shift we may assume that \( z \) attains its maximum \( ||z||_0 = z(0) \) at \( \tau = 0 \), hence \( z'(0) = 0 \). An element \( \xi \) in the kernel of the Hessian of \( F_r \) at \( z \) is a solution of the linearized problem

\[
    \xi'' = -b\xi - 6az^2\xi - db(\xi)z - 2da(\xi)z^3. \quad (45)
\]

In order to prove nondegeneracy we need to show that \( \xi \) is a constant multiple of \( z' \). Note that

\[
    da(\xi) = -\frac{4r\langle \xi, z^3 \rangle}{||z^2||^6}, \quad db(\xi) = -\frac{3\langle \xi, z \rangle}{||z||^6} + \frac{3r\langle \xi, z \rangle}{2||z||^2 \cdot ||z^2||^2} + \frac{3r\langle \xi, z^3 \rangle}{||z||^2 \cdot ||z^2||^4}.
\]

Using (44) and (45) we obtain via integration by parts

\[
    -b\langle \xi, z \rangle - 2a\langle \xi, z^3 \rangle = \langle \xi, z'' \rangle = \langle \xi'', z \rangle = -b\langle \xi, z \rangle - 6a\langle \xi, z^3 \rangle - db(\xi)||z||^2 - 2da(\xi)||z^2||^2
\]

and therefore

\[
    4a\langle \xi, z^3 \rangle = -db(\xi)||z||^2 - 2da(\xi)||z^2||^2.
\]

Plugging into this equation the formulas for \( a \), \( da \), and \( db \), this becomes

\[
    \frac{2r\langle \xi, z^3 \rangle}{||z^2||^4} = \frac{3\langle \xi, z \rangle}{||z||^6} - \frac{3r\langle \xi, z \rangle}{2||z||^2 \cdot ||z^2||^2} + \frac{3r\langle \xi, z^3 \rangle}{||z||^2 \cdot ||z^2||^4} + \frac{8r\langle \xi, z^3 \rangle}{||z^2||^4},
\]

which simplifies to

\[
    \frac{r\langle \xi, z \rangle}{2||z||^2 \cdot ||z^2||^2} - \frac{\langle \xi, z \rangle}{||z||^6} = \frac{r\langle \xi, z^3 \rangle}{||z^2||^4}. \quad (46)
\]

In particular (recall our assumption \( r > 0 \)), we see from (46) that if \( \langle \xi, z \rangle \) vanishes the same has to hold for \( \langle \xi, z^3 \rangle \).

We recall the quantities

\[
    v = \frac{||z||^2}{||z||_0^2}, \quad w = \frac{||z||^2 \cdot ||z||_0^3}{||z^2||^2}
\]

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\[ \hat{v} = \frac{2(\xi, z)}{||z||_0^2} - \frac{2||z||^2\xi_0}{||z||_0^4}, \quad \hat{w} = \frac{2(\xi, z)||z||_0^2 + 2||z||^2||z||_0\xi_0}{||z||^2 ||z||_0^2} - \frac{4(\xi, z^3)||z||^2||z||_0^3}{||z||^4} \]

(47)

where \( \xi_0 := \xi(0) \). From (33) we infer that

\[ \hat{v} = \frac{d}{dw} \left( \frac{2}{4 + 3rw - 2rw^2} \right) \hat{w} \]

(48)

and

\[ 0 = \frac{d}{dw} \left( \frac{4 + 3rw - 2rw^2}{I_1 I_0} \left( -\frac{rw^2}{2} \right) \right) \hat{w}. \]

(49)

As explained in Appendix A in formula (130), the quotient \( \frac{I_1}{I_0} \) of elliptic functions satisfies as a function of \( m \) the Riccati differential equation

\[ \left( \frac{I_1}{I_0} \right)' = \frac{1}{2m(1 - m)} - \frac{1}{m(1 - m)} \frac{I_1}{I_0} + \frac{1}{2(1 - m)} \left( \frac{I_1}{I_0} \right)^2. \]

Using this equation and (33) again we compute the derivative in (49) as follows:

\[
\begin{align*}
\frac{d}{dw} \left( \frac{4 + 3rw - 2rw^2}{I_1 I_0} \left( -\frac{rw^2}{2} \right) \right) &= \left(3r - 4rw\right) \frac{I_1}{I_0} \left( -\frac{rw^2}{2} \right) - rw \left(4 + 3rw - 2rw^2\right) \left( \frac{I_1}{I_0} \right)' \left( -\frac{rw^2}{2} \right) \\
&= \frac{6r - 8rw}{4 + 3rw - 2rw^2} + \frac{2(4 + 3rw - 2rw^2)}{w(2 + rw^2)} - \frac{8}{w(2 + rw^2)} \\
&= \frac{(2 + rw^2)(4 + 3rw - 2rw^2)}{w(2 + rw^2)} \\
&= \frac{36rw - 36rw^2 - 18r^2w^3 + 18r^2w^2}{w(2 + rw^2)(4 + 3rw - 2rw^2)} \\
&= \frac{18r(1 - w)(2 + rw)}{(2 + rw^2)(4 + 3rw - 2rw^2)}.
\end{align*}
\]

Since \( r \) and \( w \) are positive, we see from this formula that this derivative vanishes only if \( w = 1 \). In this case we obtain from (33) that

\[ \frac{2}{4 + r} = \frac{I_1}{I_0} \left( -\frac{r}{2} \right). \]

(50)

If we set \( m = -\frac{r}{2} \) this amounts to the equation

\[ 1 = (2 - m) \frac{I_1}{I_0}(m). \]

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By Lemma A.1 in Appendix A there are no solutions \( m < 0 \) of this equation, and therefore there are no solutions \( r > 0 \) of equation (50). Hence if \( r > 0 \) we necessarily have \( \hat{w} = 0 \), and therefore in view of (18) as well \( \hat{v} = 0 \). From (17) we infer
\[
\xi_0 = \frac{\langle \xi, z \rangle \|z\|_0}{\|z\|^2}, \quad \langle \xi, z^3 \rangle = \frac{\|z^2\|^2}{\|z\|^2} \left( \frac{\langle \xi, z \rangle}{\|z\|^2} \xi_0 \right),
\]
and therefore
\[
\langle \xi, z^3 \rangle = \frac{\|z^2\|^2}{\|z\|^2} \langle \xi, z \rangle.
\]
Plugging this into (46) we obtain
\[
\frac{r\langle \xi, z \rangle}{2\|z\|^2\|z^2\|^2} - \frac{\langle \xi, z \rangle}{\|z\|^6} = \frac{r\langle \xi, z \rangle}{\|z\|^2\|z^2\|^2},
\]
or equivalently,
\[
\frac{\langle \xi, z \rangle}{\|z\|^6} = \frac{r\langle \xi, z \rangle}{2\|z\|^2\|z^2\|^2}.
\]
Since \( r > 0 \), the two sides have opposite signs, therefore both sides have to be zero and we obtain
\[\langle \xi, z \rangle = 0.\]
Using (46) this implies
\[\langle \xi, z^3 \rangle = 0,
\]
and therefore
\[da(\xi) = 0, \quad db(\xi) = 0.\]
From \( \langle \xi, z \rangle = 0 \) and the first equation in (51) we further conclude that
\[\xi(0) = \xi_0 = 0.\]
Hence from (45) we see that \( \xi \) is a solution of the ODE
\[\xi'' = -b\xi - 6az^2\xi \quad (52)\]
with \( \xi(0) = 0 \). Applying the same reasoning to \( z' \) in place of \( \xi \), we conclude that \( z' \) also solves (52) with \( z'(0) = 0 \). From (44) and (50) we infer \( z''(0) = -b\|z\|_0 - 2a\|z\|^3 < 0 \), so we can define
\[c := \frac{\xi'(0)}{z''(0)} \in \mathbb{R}.
\]
Then \( \eta := \xi - cz' \) solves (52) with \( \eta(0) = \eta'(0) = 0 \), hence \( \eta \equiv 0 \) and \( \xi = cz' \).
This proves nondegeneracy for the case \( r > 0 \). In the case \( r = 0 \) the functional \( \mathcal{F}_0 \) is just the functional for the regularized free fall for which nondegeneracy can be checked directly, see [11, Lemma 3.6]. This finishes the proof of Theorem 3.1.
3.4 Uniqueness of symmetric critical points

In this subsection we prove a uniqueness result for critical points of \( F_r, r \geq 0 \). To formulate the result, we introduce some terminology from [3]. A symmetric critical point of \( F_r \) is a smooth map \( z : \mathbb{R}/2\mathbb{Z} \to \mathbb{R} \) satisfying the critical point equation (21) and the symmetry conditions

\[
z(1 + \tau) = -z(\tau) \quad \text{and} \quad z(\tau) = z(1 - \tau) \quad \text{for all} \ \tau.
\]  

By the discussion at the beginning of this section and Lemma 2.2, \( z \) has an odd number of zeroes in the interval \([0, 1)\) all of which are nondegenerate. Its Levi-Civitá transform \( q : S^1 \to \mathbb{R} \) has an odd number of zeroes and satisfies \( q(1 - t) = q(t) \). Note that (53) implies

\[
z(0) = z'(1/2) = 0.
\]

In particular, restriction to symmetric critical points removes the translation invariance of the functional \( F_r \).

Symmetric critical points of \( F_r \) may still be nonunique because they may be multiply covered. Therefore, we restrict to symmetric critical points \( z \) which are simple, i.e., of minimal period 2. This is equivalent to \( z \) having zeroes precisely at integer points \( \tau \in \mathbb{Z} \), and critical points at \( \tau \in 1/2 + \mathbb{Z} \). The remaining ambiguity \( z \mapsto -z \) can be removed by requiring \( z \) to be normalized by \( z(\tau) > 0 \) for all \( \tau \in (0, 1) \). Using Theorem 3.1 we will prove

Corollary 3.5 For each \( r \in [0, \infty) \), the functional \( F_r \) has a unique normalized simple symmetric critical point. This critical point is nondegenerate of index zero.

The proof is based on the following lemma.

Lemma 3.6 Let \( z \) be a simple symmetric critical point of the functional \( F_r \). Then its \( C^0 \)-norm \( ||z||_0 \) satisfies

\[
1 \leq ||z||_0 \quad (54)
\]

and

\[
||z||_0 \leq \sqrt{2 + (2r)^{1/3}}. \quad (55)
\]

Proof: After a suitable time reparametrization, the Levi-Civita transformation \( q \) of \( z \) satisfies the hypotheses of the loop \( q_2 \) in [3, Lemma D.2]. This gives us the estimates

\[
1 \leq ||q||_0 \quad (56)
\]

\[
||q||_0 \leq \min \left\{ \frac{2q}{2q + \frac{2r}{(2q)^2}} \right\}, \quad (57)
\]

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where $\bar{q} > 0$ is the average of $q$. Inequality (56) via Levi-Civita transformation implies that $1 \leq ||z||_0$, proving (54).

To get an estimate on the norm observe that for $\omega \geq 0$ and $x > 0$ we have

$$\min \left\{ x, \frac{\omega}{x^2} \right\} \leq \omega^{1/3}.$$  

We use this for $\omega := 2r$ and $x := 2\bar{q}$ together with (57) to get

$$||q||_0 \leq 2 + (2r)^{1/3}.$$  

In view of $||q||_0 = ||z||_0^2$ this proves (55) completing the proof of the lemma. □

**Proof of Corollary 3.5**: Following Sections 6.2 and 6.3 of [3], we introduce the Hilbert space of symmetric loops

$$H^2_{sym}(S^1, \mathbb{R}) := \{ z \in H^2(\mathbb{R}/2\mathbb{Z}, \mathbb{R}) \mid -z(1+\tau) = z(\tau) = z(1-\tau) \text{ for all } \tau \}$$  

and its open subset

$$X := \{ z \in H^2_{sym}(S^1, \mathbb{R}) \mid z'(0) > 0, z(\tau) > 0 \text{ for all } \tau \in (0,1) \}.$$  

We consider on $H^2_{sym}(S^1, \mathbb{R})$ the $L^2$-inner product $\langle z, w \rangle = \int_0^1 z(\tau)w(\tau)d\tau$ (which is an inner product in view of the condition $z(1+\tau) = -z(\tau)$). By the discussion at the beginning of this section, $\mathcal{F}_r : X \to \mathbb{R}$ has an $L^2$-gradient $\nabla \mathcal{F}_r$ of class $C^1$. For $R > 0$ consider the set

$$\mathcal{Z}_R := \{ (r, z) \mid \nabla \mathcal{F}_r(z) = 0 \} \subset [0, R] \times X$$  

with its induced topology. By the preceding discussion, $\mathcal{Z}_R$ consists of pairs $(r, z)$ such that $z$ is a normalized simple symmetric critical point of $\mathcal{F}_r$.

Next we show that $\mathcal{Z}_R$ is compact. Recall that for $(r, z) \in \mathcal{Z}_R$ the loop $z$ satisfies the ODE (21) with constants $a, b$ depending on $(r, z)$ given in (20). Thus the $C^3$-bound from Lemma 3.6 and equation (21) give a uniform $C^3$-bound on $z$ for $(r, z) \in \mathcal{Z}_R$, and thus compactness of $\mathcal{Z}_R$ by the Arzelà-Ascoli theorem, provided we have uniform bounds on the constants $a, b$. Note that by definition $a \geq 0$. We will use the quantities $v, w > 0$ defined in (32).

**Case 1**: $r = 0$. In this case by definition $a = 0$. Since $v = ||z||^2/||z||_0^2$, equation (33) yields $2||z||^2 = ||z||_0^2$. Now (34) and (54) imply boundedness of $b$.

**Case 2**: $r > 0$. In this case let us denote $A := rw^2 > 0$. Using this notation we can write

$$\frac{1}{w} = \sqrt{\frac{v}{A}} \leq \sqrt{\frac{R}{A}}.$$  

Equation (33) together with $v > 0$ implies

$$4 + 3\frac{1}{w}A - 2A > 0.$$
Together with the above upper bound on $\frac{1}{w}$ this gives us

$$4 + 3\sqrt{R\sqrt{A - 2A}} > 0,$$

which implies a uniform upper bound on $A$. Therefore, equation (11) implies

$$0 < I_0(-\tilde{C}) \leq I_0\left(-\frac{a||z||^2_0}{a||z||^2_0 + b}\right) \leq I_0(0) < \infty$$

for all $r \in [0, R]$. Equation (58) (with $n = 1$ since $z$ is a simple orbit) then gives us constants $c, C > 0$ with

$$c \leq a||z||^2_0 + b \leq C$$

for all $r \in [0, R]$. The second inequality in (60) together with with (58) gives us an upper bound on $a||z||^2_0$. The latter together with (54) gives an upper bound on $a$. The bounds are uniform with respect to $r \in [0, R]$. The double inequality (60) together with an upper bound on $a||z||^2_0$ gives us an upper bound on $|b|$. This concludes the proof of compactness of $Z_R$.

On the other hand, according Theorem 3.1 each critical point of $F_r$ is nondegenerate. This implies that $Z_R$ is a compact 1-dimensional submanifold transverse to the slices $\{r = \text{const}\}$. Now it is easy to see that for $r = 0$ (describing the free fall) there exists a unique normalized simple symmetric critical point $z_0$. By the proof of Proposition D.4 in [3], the Hessian of $F_0$ at $z_0$ is positive definite, so $z_0$ is nondegenerate of index zero. Therefore, $Z_R$ intersects each slice $\{r = \text{const}\}$ in a single point $z_r$, which is nondegenerate of index zero as a critical point of $F_r$. Since $R$ was arbitrary, this proves the corollary. □

4 Frozen planet orbits for mean interaction

Let us recall the variational approach for frozen planet orbits in helium from [3]. For the mean interaction between the two electrons, one considers the space

$$H_{av} = \left\{(z_1, z_2) \in H^2(S^1, \mathbb{R}^2) \mid ||z_1|| > 0, ||z_2|| > 0, \frac{||z_1^2||^2}{||z_1||^2} > \frac{||z_2^2||^2}{||z_2||^2}\right\}$$

and defines on it the functional $B_{av} : H_{av} \to \mathbb{R}$ by

$$B_{av}(z_1, z_2) := 2 \sum_{i=1}^2 \left(||z_i||^2 \cdot ||z_i'||^2 + \frac{1}{||z_i||^2}\right) - \frac{||z_1||^2 \cdot ||z_2||^2}{||z_1||^2 \cdot ||z_2||^2 - ||z_2^2||^2}.$$
Here $z_1$ and $z_2$ correspond to the Levi-Civita regularizations of the outer and inner electron, respectively. We restrict to Sobolev class $H^2$ right away because on this space the $L^2$-gradient of $B_{av}$ will be of class $C^1$. This $L^2$-gradient is given by

$$
\nabla B_{av}(z_1, z_2) = \left(-4||z_1||^2 \nu_1(z_1, z_2), -4||z_2||^2 \nu_2(z_1, z_2)\right),
$$
\(61\)

$$
\nu_1(z_1, z_2) = -z_1'' + a_1 z_1 + b_1 z_1^3, \quad \nu_2(z_1, z_2) = -z_2'' + a_2 z_2 + b_2 z_2^3,
$$



$$
a_1 = \frac{||z_1'||^2}{||z_1||^2} - \frac{1}{||z_1||^6} - \frac{||z_2||^4 \cdot ||z_1'||^2}{2||z_1||^2 \cdot (||z_2'||^2 \cdot ||z_2||^2 - ||z_2||^2 \cdot ||z_1||^2)^2},
$$

$$
b_1 = \frac{||z_2||^4}{(||z_2'||^2 \cdot ||z_2||^2 - ||z_2||^2 \cdot ||z_1||^2)^2},
$$

$$
a_2 = \frac{||z_2'||^2}{||z_2||^6} - \frac{1}{2||z_2||^2 \cdot (||z_2'||^2 \cdot ||z_2||^2 - ||z_2||^2 \cdot ||z_1||^2)^2},
$$

$$
b_2 = -\frac{||z_1||^4}{(||z_2'||^2 \cdot ||z_2||^2 - ||z_2||^2 \cdot ||z_1||^2)^2}.
$$

It was shown in [3] that $\nabla B_{av}$ is of class $C^1$; we refer to its derivative as the Hessian of $B_{av}$. According to Lemma 3.1 in [9] or Lemma D2 of [3], for each critical point $(z_1, z_2)$ of $B_{av}$, the first component $z_1$ is constant. Thus the $S^1$-action by time shift acts trivially on the first component and (by a similar argument as for $F_r$) nontrivially on the second one, leading to a 1-dimensional subspace of the kernel of the Hessian. By analogy with $F_r$, we say that a critical point of $B_{av}$ is nondegenerate if the nullity of its Hessian is precisely 1. Now we can formulate the main result of this section, whose proof will occupy the rest of this section.

**Theorem 4.1** All critical points of $B_{av}$ are nondegenerate.

### 4.1 Relating frozen planet orbits to critical points of $F_\rho$

We introduce the numerical parameters

$$
\rho := (\sqrt{2} - 1)^2, \quad \alpha := \frac{\sqrt{2} - 1}{\sqrt{2}}
$$

and observe that

$$
0 < \alpha < 1.
$$

We introduce the following constant for any $z \in H^2_+ (S^1, \mathbb{R})$:

$$
c(z) := \alpha^{-1/2} \frac{||z||^2}{||z||}
$$

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Lemma 4.2 The frozen functional $\mathcal{F}_\rho$ is related to the functional $\mathcal{B}_{av}$ by

$$\mathcal{F}_\rho(z) = \mathcal{B}_{av}(c(z), z). \quad (62)$$

Proof: This follows from the following computation:

$$\mathcal{B}_{av} \left( \frac{2^{1/4} \cdot \|z^2\|}{\sqrt{2} - 1 \cdot \|z\|}, z \right)$$

$$= 2\|z\|^2 \|z^\prime\|^2 + \frac{2}{\|z\|^2} + \frac{2(\sqrt{2} - 1) \cdot \|z\|^2}{\sqrt{2} \cdot \|z^2\|^2} - \frac{\|z\|^2}{(\sqrt{2} - 1) \|z^2\|^2}$$

$$= 2\|z\|^2 \|z^\prime\|^2 + \frac{2}{\|z\|^2} + \frac{\sqrt{2}(\sqrt{2} - 1) \cdot \|z\|^2}{\|z^2\|^2} - \frac{\|z\|^2}{(\sqrt{2} - 1) \|z^2\|^2}$$

$$= 2\|z\|^2 \|z^\prime\|^2 + \frac{2}{\|z\|^2} + \frac{(\sqrt{2} - 1)^2 \cdot \|z\|^2}{\|z^2\|^2}$$

$$= \mathcal{F}_\rho(z).$$

Let us illustrate the significance of the constant $c(z)$ from another angle. Let

$$H^2_{const} := \{ (z_1, z_2) \in H^2(S^1, \mathbb{R}^2) \mid z_1 = \text{const} \}$$

be the subspace of functions with constant first component and set

$$H^1_{av} := \mathcal{H}_{av} \cap H^2_{const} = \left\{ (z_1, z_2) \in H^2(S^1, \mathbb{R}^2) \mid z_1 = \text{const}, \|z_2\| > 0, \ z_1 > \frac{\|z_2\|^2}{\|z_2\|^2} \right\}. \quad (63)$$

Consider the following codimension 1 Hilbert submanifold of $H^2_{const}$:

$$\text{graph}(c) := \{ (c(z), z) \in H^2_{const} \mid z \in H^2_{const}(S^1, \mathbb{R}) \}.$$ 

Then the definition of $c(z)$ and $\alpha < 1$ imply

$$\text{graph}(c) \subset H^1_{av}.$$ 

Lemma 4.3 Assume that $(z_1, z_2) \in H^1_{av}$. Then the equation

$$V_1(z_1, z_2) = 0$$

is equivalent to the equation

$$z_1 = \pm c(z_2).$$

Moreover, if this is the case, then

$$a_1 = \frac{2}{z_1^2}, \quad b_1 = \frac{2}{z_1^2}. \quad (23)$$
Proof: For constant $z_1$, the functions $a_1$ and $b_1$ that enter the formula for $V_1$ simplify to
\begin{align}
 a_1 &= -\frac{1}{z_1^2} - \frac{\|z_2\|^2 z_1^2}{2(\|z_2\|^2 z_1^4 - \|z_2\|^2 z_1^2)^2}, \\
 b_1 &= \frac{\|z_2\|^4}{(\|z_2\|^2 z_1^4 - \|z_2\|^2 z_1^2)^2}.
\end{align}
(64)

Therefore, $V_1$ rewrites as follows:
\[ V_1(z_1, z_2) = a_1 z_1 + b_1 z_1^3 = -\frac{1}{z_1^2} + \frac{\|z_2\|^4 z_1^3}{2(\|z_2\|^2 z_1^4 - \|z_2\|^2 z_1^2)^2}.
\]

Thus, $V_1(z_1, z_2) = 0$ is equivalent to
\[ 2(\|z_2\|^2 z_1^4 - \|z_2\|^2 z_1^2)^2 = \|z_2\|^4 z_1^8. \] (65)

The defining inequality in (63) implies that (65) is equivalent to
\[ \sqrt{2}(\|z_2\|^2 z_1^4 - \|z_2\|^2 z_1^2) = \|z_2\|^2 z_1^2. \]

Since $z_1 \neq 0$ by (63), the latter is equivalent to
\[ \|z_2\|^2 = \frac{\sqrt{2} - 1}{\sqrt{2}} \|z_2\|^2 z_1^2, \]
in other words to
\[ z_1 = \pm \frac{2^{1/4} \cdot \|z_2\|}{\sqrt{2} - 1 \cdot \|z_2\|}. \]

This shows the first statement. The second statement follows by substituting (65) in (64). \qed

Taking the derivative of equation (62) with respect to $z$ yields
\[ DF_{\rho}(z) = D_1 B_{av}(c(z), z) Dc(z) + D_2 B_{av}(c(z), z) = D_2 B_{av}(c(z), z). \]

Here $D_1$ and $D_2$ denotes the derivative with respect to the first and second component, respectively, and $D_1 B_{av}(c(z), z) = 0$ by Lemma 4.3. As a consequence, we get the following relation between the $L^2$-gradients of $F_{\rho}(z)$ and $B_{av}$:
\[ \nabla F_{\rho}(z) = V_2(c(z), z). \] (66)

Here we have dropped the factor $-4\|z\|^2$ from $\nabla F_{\rho}(z)$ which has no relevance for the subsequent discussion. Taking another derivative with respect to $z$, we get
\[ D\nabla F_{\rho}(z) \xi = D_1 V_2(c(z), z) Dc(z) \xi + D_2 V_2(c(z), z) \xi \] (67)
for any $\xi \in H^2(S^1, \mathbb{R})$. 24
4.2 Proof of Theorem 4.1 modulo two key lemmas

Let \((z_1, z_2)\) be a critical point of \(B_{av}\). Recall from [9, 3] that \(z_1\) is constant. Therefore, Lemma 4.3 implies that \(z_1 = \pm c(z_2)\). In the following we assume \(z_1 > 0\) (the case \(z_1 < 0\) being analogous), so that

\((z_1, z_2) \in \text{graph}(c)\).

Equation (66) with \(z = z_2\) implies that \(z_2\) is a critical point of \(F_\rho\).

Let \(\xi = (\xi_1, \xi_2) \in H^2(S^1, \mathbb{R}^2)\) lie in the kernel of the Hessian at \((z_1, z_2)\), that is

\[ D_1 V_1(z_1, z_2) \xi_1 + D_2 V_1(z_1, z_2) \xi_2 = 0 \quad (68) \]

and

\[ D_1 V_2(z_1, z_2) \xi_1 + D_2 V_2(z_1, z_2) \xi_2 = 0. \quad (69) \]

We need the following two lemmas.

**Lemma 4.4** Assume that \((z_1, z_2) \in \text{graph}(c)\) and \((\xi_1, \xi_2) \in H^2(S^1, \mathbb{R}^2)\) are such that equation (68) is satisfied. Then \(\xi_1\) is constant, that is \((\xi_1, \xi_2) \in H^2_{\text{const}}\).

**Lemma 4.5** Assume that \((z_1, z_2) \in \text{graph}(c)\) and \((\xi_1, \xi_2) \in H^2_{\text{const}}\) are such that equation (68) is satisfied. Then

\[ \xi_1 = Dc(z_2) \xi_2, \]

that is \((\xi_1, \xi_2) \in T_{(z_1, z_2)} \text{graph}(c)\).

Assuming these two lemmas, set \(z = z_2\), \(c(z) = z_1\) and \(\xi = \xi_2\) in equation (67) and solve it for \(D_2 V_2(z_1, z_2) \xi_2\) to get

\[ D_2 V_2(z_1, z_2) \xi_2 = D_2 \nabla F_\rho(z_2) \xi_2 - D_1 V_2(z_1, z_2) Dc(z_2) \xi_2. \]

The latter allows us to rewrite equation (69) as

\[ D \nabla F_\rho(z_2) \xi_2 = D_1 V_2(z_1, z_2)(Dc(z_2) \xi_2 - \xi_1). \quad (70) \]

Apply Lemma 4.5 to get

\[ D \nabla F_\rho(z_2) \xi_2 = 0. \]

Since \(z_2\) is a critical point of \(F_\rho\), it is nondegenerate by Theorem 3.1, so \(\xi_2\) belongs to a subspace of dimension 1. This together with Lemma 4.5 shows that \((\xi_1, \xi_2)\) belongs to a subspace of dimension 1, giving us nondegeneracy of \((z_1, z_2)\) as a critical point of \(B_{av}\). This proves Theorem 4.1 modulo the two lemmas above.
4.3 Proof of Lemma 4.4

Observe that equation (68) is equivalent to

\[ \xi''_1 = a_1 \xi_1 + 3b_1 z_1^2 \xi_1 + \left( da_1(\xi_1, \xi_2) z_1 + db_1(\xi_1, \xi_2) z_1^3 \right). \]

Note that the term in the round brackets is a function constant in time, which we abbreviate as \textit{const}. So, we can rewrite it as

\[ \xi''_1 = (a_1 + 3b_1 z_1^2) \xi_1 + \text{const}. \]

Inserting for \(a_1, b_1\) the simplified expressions from Lemma 4.3, we see that

\[ a_1 + 3b_1 z_1^2 = \frac{2}{z_1^0} + \frac{6z_1^2}{z_1^8} = \frac{4}{z_1^0} > 0 \]

is a positive constant. Now we argue by contradiction.

Suppose that \(\xi_1\) is not constant and let \((\tau_1, \tau_2)\) be a maximal interval on which \(\xi'_2 > 0\). Then \(\xi'_1(\tau_1) = \xi'_1(\tau_2) = 0\), \(\xi''_1(\tau_1) \geq 0\), and \(\xi''_1(\tau_2) \leq 0\). Since the coefficient in front of \(\xi_1\) is constant in time and positive, we get the following chain of inequalities for \(\tau_1 \leq \tau \leq \tau_2\):

\[ 0 \leq \xi''_1(\tau_1) \leq \xi''_1(\tau) \leq \xi''_1(\tau_2) \leq 0. \]

So we must have equalities everywhere, \(\xi''(\tau) = 0\), and therefore \(\xi'(\tau)\) for all \(\tau \in (\tau_1, \tau_2)\), contradicting our hypothesis.

4.4 Proof of Lemma 4.5

\textbf{Step 1.} Let \(z = (z_1, z_2) \in \text{graph}(c)\) and \(\xi = (\xi_1, \xi_2) \in H^2_{\text{const}}\) satisfy equation (68). It follows from equation (61) that the restriction

\[ W := V_1|_{H^2_{\text{const}}} \]

lands in the space of constant functions and can, therefore, be considered as a map

\[ H^2_{\text{const}} \rightarrow \mathbb{R}. \]

Since both \(z_1\) and \(\xi_1\) are constant in \(t\), we can assume that we vary \(z_1\) within the space of constant functions when computing the derivative \(D_1 V_1\) with respect to \(z_1\). Therefore, equation (68) is equivalent to

\[ D W(z) \xi = D_1 W(z_1, z_2) \xi_1 + D_2 W(z_1, z_2) \xi_2 = 0. \] (71)

In this terminology, Lemma 4.3 is equivalent to

\[ \mathcal{H}_{av} \cap \{ W = 0 \} = \text{graph}(c). \]
Assume for the moment that

\[ DW(z) \neq 0. \]  \hfill (72)

Then

\[ T_z graph(c) = \ker DW(z), \]

since the inclusion \( T_z graph(c) \subset \ker DW(z) \) is obvious and both spaces have codimension 1 in \( H_{\text{const}}^2 \). This shows the lemma modulo (72), which is an immediate consequence of the next step.

**Step 2.** For any \( z = (z_1, z_2) \in graph(c) \), we have

\[ D_1 W(z_1, z_2) \neq 0. \]  \hfill (73)

This is achieved by a brute force computation. Since \( z_1 \) is constant in \( t \), we set \( z_1'' = 0 \) in formula (61) to get

\[ W(z_1, z_2) = a_1 z_1 + b_1 z_1^3, \]  \hfill (74)

where \( a_1, b_1 \) are given by (64) to be

\[ a_1 = -\frac{1}{z_1^6} - \frac{\left| z_2 \right|^2}{2}b_1, \quad b_1 = \frac{\left| z_2 \right|^4}{P(z_1)^2} \]

with the polynomial

\[ P(z_1) := \left| z_2 \right|^2 z_1^4 - \left| z_2 \right|^2 \left| z_1 \right|^2. \]

We rewrite (74) as

\[ W(z_1, z_2) = -\left( \frac{1}{z_1^6} + \frac{\left| z_2 \right|^2}{2}b_1 \right) z_1 + b_1 z_1^3 = -\frac{1}{z_1^6} + \frac{3}{2}b_1 \]  \hfill (75)

and compute the derivative with respect to \( z_1 \):

\[ D_1 W(z_1, z_2) = \frac{5}{z_1^6} + \frac{3}{2}z_1^2 b_1 + \frac{1}{2} z_1^3 D_1 b_1(z_1). \]

Now we differentiate \( b_1 \):

\[ D_1 b_1(z_1) = -2 \left| z_2 \right|^4 \frac{P'(z_1)}{P(z_2)^3} = -2 \frac{P'(z_1)}{P(z_1)} b_1 \]

Therefore,

\[ D_1 W(z_1, z_2) = \frac{5}{z_1^6} + \left( \frac{3}{2} z_1^2 - \frac{3}{2} \frac{P'(z_1)}{P(z_1)} \right) b_1. \]  \hfill (76)

Now we manipulate \( P(z_1) \) and \( P'(z_1) \)

\[ P(z_1) = \left| z_2 \right|^2 z_1^4 - \left| z_2 \right|^2 \left| z_1 \right|^2 \]

\[ = \left| z_2 \right|^2 z_1^4 - \alpha \left| z_2 \right|^2 \left| z_1 \right|^2 \]

\[ = (1 - \alpha) \left| z_2 \right|^2 z_1^4. \]
Here the equalities marked with (*) use \( z_1 = c(z_2) \) in the form
\[
||z_2||^2 = \alpha ||z_2||^2 z_1^2.
\]
Now we set
\[
X := P(z_1)^3 z_1^6 D_1 W(z_1, z_2)
\]
and continue from (76):
\[
X = 5P(z_1)^3 + 3 \frac{3}{2} ||z_2||^2 z_1^8 P(z_1) - ||z_2||^2 z_1^2 P'(z_1)
\]
\[
= \left( 5(1 - \alpha)^3 + \frac{3}{2}(1 - \alpha) - 2(2 - \alpha) \right) ||z_2||^6 z_1^2.
\]
We compute the numerical coefficient
\[
K := 5(1 - \alpha)^3 + \frac{3}{2}(1 - \alpha) - 2(2 - \alpha)
\]
in front of \(||z_2||^6 z_1^2\). With
\[
k := 1 - \alpha = \frac{1}{\sqrt{2}}
\]
we compute:
\[
K + 2\alpha = 5(1 - \alpha)^3 + \frac{3}{2}(1 - \alpha) - 2(2 - 2\alpha)
\]
\[
= 5k^3 + \frac{3}{2}k - 4k
\]
\[
= 5k(k^2 - \frac{1}{2}) = 0.
\]
This shows \( K = -2\alpha \neq 0 \), which implies \( X \neq 0 \) and therefore proves equation (73).

### 4.5 Uniqueness of symmetric frozen planet orbits

The discussion in the preceding subsections shows that critical points of \( B_1 = B_{av} \) (i.e., frozen planet orbits for the mean interaction) are in one-to-one correspondence with critical points of \( F_{\rho} \) for \( \rho = (\sqrt{2} - 1)^2 \). Moreover, this correspondence preserves the indices and nullities of the critical points. By Corollary 3.5, the functional \( F_{\rho} \) has a unique normalized simple symmetric critical point, which is nondegenerate of index zero. Hence the same holds for the corresponding normalized simple symmetric critical point of \( B_{av} \) and we conclude the following result, which corresponds to Theorem A in the Introduction.
Corollary 4.6 The unique normalized simple symmetric frozen planet orbit for the mean intersection functional $B_{av}$ is nondegenerate of Morse index 0. □

5 Determinant lines of self-adjoint Fredholm operators

The goal of this section is to show that under certain conditions the restriction of the determinant bundle to the space of symmetric index zero Fredholm operators bounded from below (or above) is trivial. We begin by describing the general Fredholm setting. Then we will describe the more specific Hilbert space setting and state the main result.

5.1 The determinant line bundle

For two Banach spaces $X$ and $Y$ we denote the space of continuous linear maps from $X$ to $Y$ by $\mathcal{L}(X,Y)$, the subspace of Fredholm operators from $X$ to $Y$ by $\mathcal{F}(X,Y)$, and the set of surjective Fredholm operators by $\mathcal{F}^*(X,Y)$. Recall from [13] that the determinants

$$\det(D) = \Lambda^\max(\ker D^*) \otimes \Lambda^\max(\ker D)$$

for any $D \in \mathcal{F}(X,Y)$ give rise to a real line bundle, the determinant line bundle

$$\det \to \mathcal{F}(X,Y).$$

We describe the bundle structure for $\det$ following [13]. For this recall the bundle of kernels over $\mathcal{F}^*(X,Y)$ and note that the restriction $\det|_{\mathcal{F}^*(X,Y)}$ is just the top exterior power of the kernel bundle over $\mathcal{F}^*(X,Y)$. Let now $T \in \mathcal{F}(X,Y)$ be a Fredholm operator and set $N := \dim \operatorname{coker} T$. Let

$$\Phi : \mathbb{R}^N \to Y$$

be an isomorphism onto a direct complement to $\im T$. Then the stabilized operator

$$D \oplus \Phi : X \oplus \mathbb{R}^N \to Y, \quad (x, \zeta) \mapsto D(x) + \Phi(\zeta)$$

is surjective for $D := T$ by construction. Therefore, it is surjective for any $D$ in a small open neighbourhood $U_{T,\Phi}$ of $T$. The idea is to construct a certain fiberwise linear bijection $\iota_{\Phi}$ between the restriction of the determinant bundle to $U_{T,\Phi}$ and its restriction to the image of $U_{T,\Phi}$ under stabilization. We then declare that $\iota_{\Phi}$ be a bundle isomorphism. To define $\iota_{\Phi}$ pick any $D \in U_{T,\Phi}$ and set $k := \dim \ker D$, $l := \dim \operatorname{coker} D$. Then

$$\dim \ker(D \oplus \Phi) = \operatorname{ind}(D \oplus \Phi) = (k - l) + N$$
and 

\[ \ker D \times \{0\} \subset \ker(D \oplus \Phi) \]

is a subspace of dimension \( k \). In particular,

\[ k = \dim \ker D \leq \dim \ker(D \oplus \Phi) = N + (k - l). \tag{77} \]

A complement to \( \ker D \times \{0\} \) in \( \ker(D \oplus \Phi) \) can be described as follows. Pick \( N - l \) linearly independent vectors \( \{\zeta_j\}_{j=l+1}^N \) in \( \mathbb{R}^N \) and \( N - l \) vectors \( \{\xi_j\}_{j=l+1}^N \) in \( X \) subject to

\[ D\xi_j + \Phi \zeta_j = 0, \quad j = l + 1, \ldots, N. \]

Then the collection \( \{(\xi_j, \zeta_j)\}_{j=l+1}^N \) spans the desired complement. We complete \( \{\zeta_j\}_{j=1}^N \) to a basis \( \{\zeta_j\}_{j=1}^N \) of \( \mathbb{R}^N \). Now for any

\[ \theta = (y_1^* \wedge \cdots \wedge y_1^*) \otimes (x_1 \wedge \cdots \wedge x_k) \in \det(D) \]

define

\[ \iota_{\Phi}(D, \theta) := (-1)^{kl} \frac{\det(\langle y_j^*, \Phi \zeta_i \rangle_{i,j=1, \ldots, l}(x_1, 0) \wedge \cdots \wedge (x_k, 0) \wedge (\xi_{l+1}, \zeta_{l+1}) \wedge \cdots \wedge (\xi_N, \zeta_N))}{\det(\zeta_1, \ldots, \zeta_N)}. \]

It is shown in [13] that the map \( \iota_{\Phi} \) is independent of the choices made and the collection of all maps \( \iota_{\Phi} \) does indeed define a bundle structure on the determinant bundle over \( \mathcal{F}(X, Y) \).

### 5.2 Self-adjoint Fredholm operators

Let now \( (F, \langle \cdot, \cdot \rangle_F) \) be a real Hilbert space and \( E \subset F \) a dense linear subspace. Recall that a linear map

\[ T : E \to F \]

is called **symmetric** if

\[ \langle Tx, y \rangle_F = \langle x, Ty \rangle_F \quad \text{for all } x, y \in E, \]

and **self-adjoint** if in addition for each \( y \in F \) the existence of a constant \( C_y \) such that

\[ (Tx, y)_F \leq C_y \|x\|_F \tag{78} \]

for all \( x \in E \) implies that \( y \in E \).

Assume now that \( (E, \langle \cdot, \cdot \rangle_E) \) is itself a Hilbert space, so we can talk about the space of Fredholm operators \( \mathcal{F}(E, F) \).

**Lemma 5.1** In the setting above, a Fredholm operator \( T \in \mathcal{F}(E, F) \) is self-adjoint if and only if it is symmetric and of index zero. Moreover, in this case we have the equalities

\[ \ker T = (\text{im} T)^\perp_{\mathcal{F}}, \quad (\ker T)^\perp_{\mathcal{F}} = \text{im} T. \tag{79} \]
Proof: Suppose first that $T$ is self-adjoint. Then it is in particular symmetric and we obtain the inclusion $\ker T \subset (\text{im } T)^{\perp F}$. On the other hand, consider $y \in (\text{im } T)^{\perp F}$. Then $(Tx, y) = 0$ for all $x \in E$, so $y$ satisfies (78) with $C_y = 0$. Since $T$ is self-adjoint, this implies $y \in E$, and symmetry of $T$ gives $(x, Ty) = 0$ for all $x \in E$. By density of $E \subset F$ this implies $Ty = 0$, so we have shown $\ker T = (\text{im } T)^{\perp F}$. This proves $\text{ind } T = 0$ and the first equality, and the second equality follows from the first one by taking orthogonal complements.

Suppose now that $T$ is symmetric and of index zero. Then the inclusion $\ker T \subset (\text{im } T)^{\perp F}$ has to be an equality because both spaces have the same finite dimension, so the equalities (79) hold. To prove self-adjointness, assume first that $T$ is surjective. Let $y \in F$ satisfy (78). By the Riesz representation theorem in $F$ for the functional $x \mapsto \langle Tx, y \rangle_F$ and surjectivity of $T$, there exists $z \in E$ with

$$\langle Tx, y \rangle_F = \langle x, Tz \rangle_F$$

for all $x \in E$. We apply symmetry of $T$ to the right hand side of the last displayed equation to get

$$\langle Tx, y - z \rangle_F = 0$$

for all $x \in E$. Since $\text{im } T = F$, this implies $y - z = 0$ and thus $y = z \in E$. This proves self-adjointness in the case that $T$ is surjective. If $T$ is not surjective, then we replace the triple $(F, E, T)$ by the triple $(F_1 := \text{im } T, E_1 := \text{im } T \cap E, T_1 := T|_{\text{im } T \cap E})$.

Since the codimension of $\text{im } T$ is finite, the linear subspace $E_1$ is dense in $F_1$ (because $\text{im } T$ possesses a complement which is contained in $E$). Equations (79) imply that $T_1$ is surjective, so by the discussion above $T_1$ is self-adjoint. Therefore, the original operator $T$ is self-adjoint. □

We denote the space of operators as in Lemma 5.1 by

$$\mathcal{F}_s(E, F) := \{ T \in \mathcal{F}(E, F) \mid \text{ind } T = 0 \text{ and } T \text{ is symmetric} \}.$$

Let $T \in \mathcal{F}_s(E, f)$. By Lemma 5.1 and its proof, $\text{im } T \cap E$ is a complement to $\ker T$ in $E$. So the restriction of $T$ to $\text{im } T \cap E$ defines an isomorphism

$$\tilde{T} := T|_{\text{im } T \cap E} : (\text{im } T \cap E, \langle \cdot, \cdot \rangle_E) \cong (\text{im } T, \langle \cdot, \cdot \rangle_F).$$

Assume now in addition that the inclusion $E \hookrightarrow F$ is compact. Then the inverse of $T$ can be viewed as a compact operator

$$\tilde{T}^{-1} : (\text{im } T, \langle \cdot, \cdot \rangle_F) \rightarrow (\text{im } T \cap E, \langle \cdot, \cdot \rangle_E) \hookrightarrow (\text{im } T, \langle \cdot, \cdot \rangle_F).$$

As such, the spectrum of $\tilde{T}^{-1}$ consists only of eigenvalues which are uniformly bounded and can accumulate only at zero, and all nonzero eigenvalues have finite multiplicity (see e.g. [5]). Since the eigenvalue zero of $T$ has only finite multiplicity by the Fredholm hypothesis, we obtain
Lemma 5.2 Consider the setting above and assume that the inclusion $E \hookrightarrow F$ is compact. Then for $T \in \mathcal{F}_s(E, F)$ the spectrum $\sigma(T) \subset \mathbb{R}$ is discrete and consists only of eigenvalues of finite multiplicity. □

For $T$ as in the preceding lemma and a measurable subset $Z \subset \mathbb{R}$ let $E^T_Z : F \to F$ denote the projection onto the sum of eigenspaces to eigenvalues in $Z$ (this is a very special case of a spectral measure [5]), so we get the spectral decomposition

$$T = \bigoplus_{\lambda \in \sigma(A)} \lambda E^T_{\lambda}.$$ 

We conclude this subsection with an important lemma about the continuity of the spectrum and of the spectral measure.

Lemma 5.3 Consider the setting as in Lemma 5.2 and $T \in \mathcal{F}_s(E, F)$.

(a) Let $I \subset \mathbb{R}$ be a compact interval such that $T$ has no eigenvalues in $I$. Then there exists an open neighbourhood $W_T$ of $T$ in $\mathcal{F}_s(E, F)$ such that the spectrum of any $T' \in W_T$ is disjoint from $I$.

(b) Let $J \subset \mathbb{R}$ be a finite interval whose boundary points are no eigenvalues of $T$. Then there exists an open neighbourhood $W_T$ of $T$ in $\mathcal{F}_s(E, F)$ such that the assignment $W_T \to \mathcal{L}(F, F), \quad T' \mapsto E^T_{J}$

is continuous.

Proof: (a) The family $\{T_t := T - t\text{Id}\}_{t \in I} \subset \mathcal{F}_s(E, F)$ consists of invertible operators, and every $T_t$ has an open neighbourhood $U_t$ which consists of invertible operators. Since the family $\{T_t\}_{t \in I}$ is compact, it is covered by finitely many $U_{ij}, j = 1, \ldots, l$ and we set

$$W_T := \bigcap_{j=1}^l (U_{ij} + t_j\text{Id}).$$

(b) By part (a), we find an open neighbourhood $W_T$ such that no $T' \in W_T$ has eigenvalues at the boundary points of $J$. The desired continuity can now be deduced e.g. from Dunford calculus [6] as follows. Let $\Gamma \subset \mathbb{C}$ be a smooth simple closed curve enclosing $J$ and disjoint from the spectrum of $T'$ for all $T' \in W_T$. We orient $\Gamma$ as the boundary of the connected component of $\mathbb{C} \setminus \Gamma$ containing $J$. Then the spectral measure in question can be expressed by the operator valued Cauchy integral formula

$$E^T_{J} = \frac{1}{2\pi i} \int_{\Gamma} (\lambda\text{Id} - T')^{-1} d\lambda, \quad (81)$$
see e.g. [6, § X.1 Formula (i)]. In loc. cit. it is stated for bounded operators, but it immediately generalizes to our case as follows. Note that the integral itself on the right hand side of (81) still makes sense. Let now $e$ be an eigenvector for an eigenvalue $\mu$ of $T'$. Then

$$(\lambda \text{Id} - T')^{-1}e = (\lambda - \mu)^{-1}e.$$  

Since the spectrum of $T'$ consists of eigenvalues, the usual Cauchy formula from complex analysis implies that the right hand side of (81) does indeed define the desired spectral measure. The continuous dependence of $E_s^{T'}$ on $T'$ now follows from the continuous dependence of the Cauchy integral on its integrand. □

5.3 The main result and strategy of proof

We introduce the following notation for any real number $R$, where $\sigma(T)$ denotes the spectrum of $T$:

$$F^>s_R(E, F) := \{ T \in F_s(E, F) \mid \sigma(T) \subset (R, +\infty) \},$$

$$F^s_R(E, F) := \{ T \in F_s(E, F) \mid \ker T = \text{coker} T = 0 \},$$

$$F^<_R(E, F) := \{ T \in F_s(E, F) \mid \sigma(T) \subset (-\infty, R) \}.$$  

Now we can state the main result of this section.

**Theorem 5.4** Consider the setting above and assume that the inclusion $\iota : E \hookrightarrow F$ is compact. Let $R$ be any real number. Then the equalities $(\text{im} T)_{1,F}^\perp = \ker T$ give rise to a canonical orientation of the restrictions

$$\det |_{F^>_R(E, F)} \quad \text{and} \quad \det |_{F^<_R(E, F)}$$

of the determinant line bundle to the subspaces of index zero symmetric Fredholm operators bounded from below or above.

The proof will occupy the rest of this section. We begin by explaining the main ideas. Observe that the involution

$$F^>_R(E, F) \rightarrow F^<_R(E, F), \quad A \mapsto -A$$

preserves the determinant line bundle. Therefore, it is enough to prove our statement for $F^>_R(E, F)$.

For $T \in F_s(E, F)$ let $v_1, \ldots, v_k$ be an orthonormal basis (with respect to $\langle \cdot, \cdot \rangle_F$) of $(\text{im} T)^{1,F} = \ker T$. We identify $\ker T$ with its dual using the Hilbert space structure. The element

$$t(T) := v_k \wedge \cdots \wedge v_1 \otimes v_1 \wedge \cdots \wedge v_k \in \Lambda^k(\ker T) \otimes \Lambda^k(\ker T) = \det(T) \quad (82)$$

does not depend on the choice of the orthonormal basis, so these elements define a canonical section $t$ of $\det |_{F_s(E, F)}$. Unfortunately, this section will turn out
to be discontinuous — it has singularities at operators that have more kernel than some of their neighbours. Observe, however, that the restriction of the determinant line bundle to the subset $\mathcal{F}_s(E, F)^*$ of invertible operators is a trivial bundle $\mathcal{F}_s(E, F)^* \times \mathbb{R}$. Moreover, the section $t$ over $\mathcal{F}_s(E, F)^*$ is the constant $1 \in \mathbb{R}$. The section $t$ can be thought of as a tautological section and its restriction to $\mathcal{F}_s(E, F)^*$ is tautologically continuous.

The idea now is to compensate the above discontinuity using a certain spectral count. Recall from Lemma 5.2 that the spectrum $\sigma(T) \subset \mathbb{R}$ of $T \in \mathcal{F}_s(E, F)$ is discrete and consists only of eigenvalues of finite multiplicity. Let

$$
\rho : \mathbb{R} \rightarrow (-\infty, 1]
$$

be a nondecreasing smooth function which equals the identity on $(-\infty, a]$ and which is constant equal to 1 on $[b, \infty)$ for some $0 < a < b$. We define our spectral count

$$
\mu : \mathcal{F}_s^>^\mathbb{R}(E, F) \rightarrow \mathbb{R}
$$

by the formula

$$
\mu(T) := \prod_{\lambda \in \sigma(T) \setminus \{0\}} \rho(\lambda),
$$

where each eigenvalue appears with its multiplicity. Note that the product above contains only finitely many factors different from 1 because all eigenvalues of $T$ are $> \mathbb{R}$ and $\rho(\lambda) = 1$ for large $\lambda$. The function $\mu$ is nowhere zero. Observe, however, that part of the spectrum may converge to 0 in a family of operators. On such a family the function $\mu$ converges to 0 and compensates the singularity in the section $t$ above. We define a modified section of the determinant line bundle over $\mathcal{F}_s^>^\mathbb{R}(E, F)$ by

$$
s(T) := \mu(T)v_k \wedge \cdots \wedge v_1 \otimes v_1 \wedge \cdots \wedge v_k.
$$

We will show that $s$ is continuous. Note that different choices of the function $\rho$ lead to positively proportional sections $s$, therefore the induced orientation is canonical.

Let $T \in \mathcal{F}_s^>^\mathbb{R}(E, F)$ be given. We specify the general Fredholm setting above to the present situation. Namely, set $N := \dim \ker T$ and pick a linear isomorphism

$$
\Phi : \mathbb{R}^N \rightarrow \ker T = (\im T)^{\perp F} \subset F.
$$

respecting scalar products. Let $U_{T, \Phi}$ be a neighbourhood of $T$ in $\mathcal{F}(E, F)$ as above and consider its restriction to $U_T := U_{T, \Phi} \cap \mathcal{F}_s^>^\mathbb{R}(E, F)$. Let $\{T_n\}_{n \in \mathbb{N}} \subset U_T$ be any sequence with

$$
\lim_{n \to \infty} T_n = T.
$$

We need to show

$$
\lim_{n \to \infty} s(T_n) = s(T). \quad (83)
$$

By a standard trick, it suffices to show that for every subsequence $(T_{n_k})$ there exists a subsequence $(T_{n_{k_j}})$ such that $\lim_{j \to \infty} s(T_{n_{k_j}}) = s(T)$. Using this, in the
proof of \( (83) \) we will repeatedly pass to subsequences of \((T_n)\), always renaming them back to \(T_n\).

Inequality \((77)\) with \(k = l\) (because of index zero) implies that after passing to a subsequence we can assume

\[
\dim \ker T_n = k \leq \dim \ker T = N
\]

for some constant \(k\). We understand the desired convergence \((83)\) in terms of the above isomorphism \(\iota \Phi\). That is, we have to prove that

\[
\lim_{n \to \infty} \iota \Phi(T_n, s(T_n)) = \iota \Phi(T, s(T)). \tag{84}
\]

Let us recall the choices we have to make in order to define the necessary objects entering the last assertion:

(i) an orthonormal basis \(v_1, \ldots, v_N\) of \(\ker T\);

(ii) an orthonormal basis \(v^n_1, \ldots, v^n_k\) of \(\ker T_n\) for each \(n \in \mathbb{N}\);

(iii) a basis \(\zeta_1, \ldots, \zeta_N\) of \(\mathbb{R}^N\) (to define \(\iota \Phi(T, \cdot)\));

(iv) a basis \(\zeta^n_1, \ldots, \zeta^n_N\) of \(\mathbb{R}^N\) for each \(n \in \mathbb{N}\) (to define \(\iota \Phi(T_n, \cdot)\));

(v) a collection \(\{\xi^n_j\}_{j=k+1}^N \subset E\) subject to\(^{35}\)

\[
T_n \xi^n_j = - \Phi \zeta^n_j, \quad j = k + 1, \ldots, N
\]

(to define \(\iota \Phi(T_n, \cdot)\)).

This allows us to write out

\[
\iota \Phi(T_n, s(T_n)) = (-1)^k C_n \mu(T_n)(v^n_1, 0) \wedge \cdots \wedge (v^n_k, 0) \wedge (\xi^n_{k+1}, \zeta^n_{k+1}) \wedge \cdots \wedge (\xi^n_N, \zeta^n_N)
\]

and

\[
\iota \Phi(T, s(T)) = (-1)^N C \mu(T)(v_1, 0) \wedge \cdots \wedge (v_k, 0) \wedge (v_{k+1}, 0) \wedge \cdots \wedge (v_N, 0)
\]

with the abbreviations

\[
C_n := \frac{\det \langle v^n_i, \Phi \zeta^n_j \rangle_{i,j=1,\ldots,k}}{\det(\zeta^n_1, \ldots, \zeta^n_k)}, \quad C := \frac{\det(\langle v_i, \Phi \zeta_j \rangle_{i,j=1,\ldots,N})}{\det(\zeta_1, \ldots, \zeta_N)}.
\]

The desired convergence \((84)\) thus follows from the following three assertions (for suitable choices (i)–(v)):

Assertion (A)

\[
\lim_{n \to \infty} v^n_1 \wedge \cdots \wedge v^n_k = v_1 \wedge \cdots \wedge v_k.
\]

Assertion (B)

\[
\lim_{n \to \infty} \mu(T_n)(-\xi^n_{k+1}, \zeta^n_{k+1}) \wedge \cdots \wedge (-\xi^n_N, \zeta^n_N) = \mu(T)(v_{k+1}, 0) \wedge \cdots \wedge (v_N, 0).
\]

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Assertion (C)

\[
\lim_{n \to \infty} \det(v^n_i, \Phi \zeta^n_j)_{i,j=1,\ldots,k} = \det(v, \Phi \zeta)_{i,j=1,\ldots,N}, \tag{86}
\]

\[
\lim_{n \to \infty} \det(\zeta^n_1, \ldots, \zeta^n_N) = \det(\zeta_1, \ldots, \zeta_N). \tag{87}
\]

Let us describe the strategy for making the choices above. We are going to show that in a certain sense \(\ker T_n\) converges to a subspace \(\ker \ker T\) of \(\ker T\) as \(n \to \infty\). Note that each \(\ker T_n\) is a member of an infinite dimensional Grassmannian. Therefore, we need an additional construction (“parametrizing the kernels”) relating \(\ker T_n\) to a subspace of \(\ker T\). This will give us the key orthogonal splitting

\[
\ker T = \ker \ker T \oplus \ker \im T,
\]

where \(\ker \im T\) is the orthogonal complement to \(\ker \ker T\) in \(\ker T\). Alternatively, \(\ker \im T\) is the limit of \(\im T_n \cap \ker T\) as \(n \to \infty\). We let \(v_1, \ldots, v_k\) be any orthonormal basis of \(\ker \ker T\) and \(v_{k+1}, \ldots, v_N\) be any orthonormal basis of \(\ker \im T\). We define \(\{\zeta_j\}_{j=1}^N\) to be the preimage of \(\{v_j\}_{j=1}^k\) under \(\Phi\) and \(\zeta_j := \zeta_j^i\) for \(j = 1, \ldots, k\). We then define \(\{v_j\}_{j=k+1}^N\) as the orthonormalization of a suitable projection of \(\{v_j\}_{j=1}^k\) to \(\ker T_n \cap \ker T\), and \(\{\Phi \zeta_j\}_{j=k+1}^N\) as the orthogonal projection of \(\{v_j\}_{j=k+1}^N\) to \(\im T_n\). This will allow us to define \(\{\xi_j\}_{j=k+1}^N\) using (85) and it remains to verify the three assertions above.

5.4 Parametrizing the kernels

Let \(X\) and \(Y\) be two Banach spaces and \(F^*(X,Y)\) be the space of surjective Fredholm operators from \(X\) to \(Y\). Pick any \(D^0 \in F^*(X,Y)\) and let \(U_{D^0} \subset F^*(X,Y)\) be a connected open neighbourhood of \(D^0\). We describe one possible trivialization of the kernel bundle over \(U_{D^0}\). Consider the direct sum \(Y \oplus \ker D^0\) and let \(C_0 \subset X\) be a closed direct complement of \(\ker D^0\) in \(X\). Let

\[
P_{\ker D^0} : X \to \ker D^0
\]

denote the projection onto \(\ker D^0\) along \(C_0\).

To every operator \(D \in U_{D^0}\) we associate an operator \(\tilde{D} \in L(X,Y \oplus \ker D^0)\) defined by \(\tilde{D}(x) := (D(x), P_{\ker D^0}(x))\). By construction the operator \(\tilde{D}\) is bijective. Therefore, by shrinking \(U_{D^0}\) if necessary, we may assume that \(\tilde{D}\) is bijective for every \(D \in U_{D^0}\). As a consequence, each \(\tilde{D}\) admits continuous inverse depending continuously on \(D\). Given any \(x \in X\), let \(\tilde{x} \in X\) be the unique solution of the equation

\[
\tilde{D}(\tilde{x}) = (D(x), 0). \tag{88}
\]

Writing out equation (88) in components gives us

\[
D(\tilde{x}) = D(x), \quad P_{\ker D^0}(\tilde{x}) = 0.
\]
In other words,
\[ D(x - \tilde{x}) = 0, \quad \tilde{x} \in \mathcal{C}_0. \]
Observe that \( \tilde{x} \) depends continuously on \( (D, x) \in U_{D^0} \times X \), and \( \tilde{x} = 0 \) for \( D = D^0 \) and any \( x \in \ker D^0 \). So the isomorphisms
\[ Q_D : \ker D^0 \xrightarrow{\cong} \ker D, \quad Q_D(x) := x - \tilde{x}. \]
depend continuously on \( D \in U_{D^0} \) and
\[ Q_{D^0} = \text{Id}_{\ker D^0}. \]
To apply this in our situation set \( X := E \oplus \mathbb{R}^N, Y := F, D^0 := T \oplus \Phi, D := T_n \oplus \Phi \). Furthermore, in our case \( \ker T \cong \ker T \times \{0\} = \ker(T \oplus \Phi) \) and we can take \( \mathcal{C}_0 := \text{im } T \cap E \). Then the preceding construction gives us isomorphisms
\[ Q_n := Q_{T_n \oplus \Phi} : \ker T \xrightarrow{\cong} \ker(T_n \oplus \Phi) \]
such that
\[ \lim_{n \to \infty} Q_n = \text{Id}_{\ker T}. \quad (89) \]
Equation (89) implies in particular
\[ \| (Q_n)_1 - \text{Id}_{\ker T} \| \to 0, \quad \| Q_n^{-1} |_{\ker T \times \{0\}} - \text{Id}_{\ker T_n \times \{0\}} \| \to 0 \quad (90) \]
as \( n \to \infty \), where \( (Q_n)_1 \) denotes the first component of \( Q_n \).
Using the isomorphisms \( Q_n \) we can now establish the orthogonal splitting \( \ker T = \ker T \oplus \ker \im T \). For two subspaces \( H_1, H_2 \) of a finite dimensional Hilbert space \( (H, \langle \cdot, \cdot \rangle_H) \) we define
\[ \langle H_1, H_2 \rangle := \sup \{ \| u \| \cdot \| v \| : \langle u, v \rangle_H \in H_1 \times H_2, \| u \| = \| v \| = 1 \}. \]
Note that \( \langle H_1, H_2 \rangle = 0 \) if and only if \( H_1 \) and \( H_2 \) are orthogonal to each other. We consider \( \ker T \) and \( \ker(T_n \oplus \Phi) \) as finite dimensional Hilbert spaces with the scalar products induced from \( F \). Let \( \perp \) denote “orthogonal complement in \( \ker(T_n \oplus \Phi) \)” for the discussion below. For a subspace \( K \) of \( \ker(T_n \oplus \Phi) \) let \( (K)_2 \subset \mathbb{R}^N \) denote the projection of \( K \) on the second component. Observe that
\[ (\ker T_n \times \{0\})_{1 \perp} = \{ (\xi, \zeta) \in (\im T_n \cap E) \times \mathbb{R}^N | T_n \xi = -\Phi \zeta \}. \]
Since \( T_n \) restricts to an isomorphism from \( \im T_n \cap E \) to \( \im T_n \), we get
\[ \Phi((\ker T_n \times \{0\})_{1 \perp})_2 = \im T_n \cap \ker T. \]
Since the Grassmannian of \( k \)-dimensional subspaces of \( \ker T \) is compact, we get (after passing to a subsequence if necessary) the following limits:
\[ \ker T := \lim_{n \to \infty} Q_n^{-1}(\ker T_n \times \{0\}), \quad \ker \im T := \lim_{n \to \infty} (\im T_n \cap \ker T). \quad (91) \]
For a closed subspace $C \subset F$ we denote by 

$$P_C : F \rightarrow F$$

the orthogonal projection onto $C$. In terms of orthogonal projections, the preceding equations mean 

$$P_{\ker T_n} = \lim_{n \rightarrow \infty} P_{Q_n^{-1} \left( \ker T_n \times \{0\} \right)}, \quad P_{\text{im} T_n} = \lim_{n \rightarrow \infty} P_{\text{im} T_n \cap \ker T}. \tag{92}$$

Our next goal is to show orthogonality of $\ker T_n$ and $\text{im} T_n$. Note that $
 \ker T_n \times \{\{0\}\} \text{ and } \text{im} T_n \cap \ker T$ are orthogonal to each other as subspaces of $F$. This together with equation (90) implies that 

$$\langle \text{im} T_n \cap \ker T, Q_n^{-1} \left( \ker T_n \times \{0\} \right) \rangle \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{91}$$

The desired orthogonality now follows from equation (91). Since 

$$\dim \ker T = k \quad \text{and} \quad \dim \text{im} T = N - k,$$

we get the orthogonal splitting 

$$\ker T = \ker \ker T \oplus \text{im} T. \tag{93}$$

### 5.5 Proof of Theorem 5.4

The choices (i)-(v). Now we are ready to make the choices (i)-(v) in Section 5.3. For (i) let $v_1, \ldots, v_N$ be an orthonormal basis of $\ker T$ such that $v_1, \ldots, v_k$ form a basis of $\ker \ker T$, and $v_{k+1}, \ldots, v_N$ form a basis of $\text{im} T$. For (ii), let $GS_n$ denote the Gram-Schmidt retraction from the space of all bases of $\ker T_n$ to the space of orthonormal bases of $\ker T_n$ and set 

$$\{v_j^k\}_{j=1}^k := GS_n \left( P_{\ker T_n \times \{0\}} Q_n \langle \{v_j^k\}_{j=1}^k \rangle \right). \tag{94}$$

For (iii) we define 

$$\zeta_j := \Phi^{-1} v_j, \quad j = 1, \ldots, N. \tag{95}$$

For (iv) we define 

$$\zeta_j^n := \begin{cases} \zeta_j & 1 \leq j \leq k, \\ \Phi^{-1} P_{\text{im} T_n \cap \ker T} v_j & k + 1 \leq j \leq N. \end{cases} \tag{96}$$

The second part of this definition rewrites as 

$$\Phi \zeta_j^n = P_{\text{im} T_n \cap \ker T} v_j, \quad j = k + 1, \ldots, N. \tag{97}$$

In view of the second equation in (92) and $v_j \in \ker \ker T$ for $j = k + 1, \ldots, N$ this implies 

$$\Phi \zeta_j^n \rightarrow P_{\ker \ker T} v_j = v_j \quad \text{as } n \rightarrow \infty, \quad j = k + 1, \ldots, N. \tag{98}$$
Applying $\Phi^{-1}$ to both sides of the last equation, we get
\[ \zeta^n_j \to \zeta_j \quad \text{as} \quad n \to \infty, \quad j = k + 1, \ldots, N. \tag{99} \]
In particular, we see that $\{\zeta^n_j\}_{j=1}^N$ form a basis of $\mathbb{R}^N$ for large $n$. For (v), we use the isomorphism
\[ \tilde{T}_n := T_n|_{\text{im}T_n \cap E} : \text{im}T_n \cap E \xrightarrow{\cong} \text{im}T_n \tag{100} \]
to define
\[ \xi^n_j := -\tilde{T}_n^{-1}\Phi\zeta^n_j, \quad j = 1, \ldots, N \tag{101} \]
With these choices, we will now verify assertions (A)–(C) from Section 5.3.

**Proof of Assertion (A).** We will use the following simple lemma.

**Lemma 5.5** Let $\{A_n\}_{n \in \mathbb{N}}$ and $\{B_n\}_{n \in \mathbb{N}}$ be two bounded sequences of operators $A_n \in \mathcal{L}(Y, Z)$, $B_n \in \mathcal{L}(X, Y)$ between Banach spaces. Assume convergence $B := \lim_{n \to \infty} B_n$ and the equality $A_nB_n = 0$ for all $n \in \mathbb{N}$. Then $\lim_{n \to \infty} A_nB = 0$.

**Proof:** Since the sequence $\{A_n\}_{n \in \mathbb{N}}$ is bounded, we get
\[ \|A_n(B - B_n)\| \to 0 \quad \text{as} \quad n \to \infty \]
uniformly with respect to $m$. Set $m := n$ and use $A_nB_n = 0$ to conclude.

We apply the lemma with $X = Y = Z = \ker T$ and the projections
\[ A_n := Q_n^{-1}P_{(\ker T_n \times \{0\})^\perp}Q_n, \quad B_n := P_{Q_n^{-1}(\ker T_n \times \{0\})^\perp}. \]
Since obviously $A_nB_n = 0$, and $\lim_{n \to \infty} B_n = P_{\ker T_n}T$ by equation (92), Lemma 5.5 yields
\[ \lim_{n \to \infty} Q_n^{-1}P_{\ker T_n \times \{0\}}^\perp Q_nP_{\ker T_n}T = 0. \tag{102} \]
In view of $P_{\ker T_n \times \{0\}}^\perp + P_{\ker T_n \times \{0\}} = \text{Id}_{\ker T}$ and equation (89), this implies
\[ \lim_{n \to \infty} P_{\ker T_n \times \{0\}}Q_nP_{\ker T_n}T = P_{\ker T_n}T. \tag{103} \]
This implies
\[ P_{\ker T_n \times \{0\}}Q_n\{v_j\}_{j=1}^k \to \{v_j\}_{j=1}^k \quad \text{as} \quad n \to \infty, \tag{104} \]
and since $\{v_j\}_{j=1}^k$ is orthonormal we get
\[ \{v^n_j\}_{j=1}^k \to \{v_j\}_{j=1}^k \quad \text{as} \quad n \to \infty. \tag{105} \]
This proves Assertion (A).
Proof of Assertion (C). The choice (95) of \( \{ \zeta_j \}_{j=1,...,N} \) implies that the right hand side of (86) equals 1. The choice (96) of \( \{ \zeta_n^j \}_{j=1,...,k} \) and equation (105) implies that the left hand side of (86) equals 1. Equation (87) follows from equations (96) (for \( j \leq k \)) and (99) (for \( j > k \)). □

Proof of Assertion (B). First, we prepare several convergence statements. Let \( \varepsilon > 0 \) be such that 0 is the only eigenvalue of \( T \) in \((-2\varepsilon, 2\varepsilon)\) and the function \( \rho \) equals the identity on \((-\varepsilon, \varepsilon)\). Then by continuity (80) of the spectral measure we get

\[
\lim_{n \to \infty} E_{T_n}(-\varepsilon, \varepsilon) = E_{T} = P_{\ker T}, \quad \lim_{n \to \infty} E_{T_n}^{\mathbb{R}\setminus(-\varepsilon, \varepsilon)} = \text{Id} - P_{\ker T}.
\]

(106)

Let \( k + 1 \leq j \leq N \). We apply the operations in (106) to both sides of equation (98) to get

\[
\lim_{n \to \infty} E_{T_n}(-\varepsilon, \varepsilon) \Phi \zeta_n^j = v_j, \quad \lim_{n \to \infty} E_{T_n}^{\mathbb{R}\setminus(-\varepsilon, \varepsilon)} \Phi \zeta_n^j = 0.
\]

(107)

Observe that \( P_{\ker T_n} \Phi \zeta_j^n = 0 \) and define

\[
z_n^j := E_{T_n}(-\varepsilon, \varepsilon) \Phi \zeta_n^j = E_{T_n}(-\varepsilon, \varepsilon) \zeta_n^j, \quad \tilde{z}_j^n := E_{T_n}^{\mathbb{R}\setminus(-\varepsilon, \varepsilon)} \Phi \zeta_j^n.
\]

Note that \( \Phi \zeta_j^n = z_j^n + \tilde{z}_j^n \).

Recall the isomorphism \( \tilde{T}_n \) from (100). The absolute values of the eigenvalues of \( \tilde{T}_n^{-1} := \lim_{n \to \infty} E_{T_n}^{\mathbb{R}\setminus(-\varepsilon, \varepsilon)} \) are bounded above by \( \varepsilon^{-1} \). Since the spectral radius of a compact self-adjoint operator equals its norm, this implies

\[
\| \tilde{T}_n^{-1} \tilde{z}_j^n \| = \| \tilde{T}_n^{-1} z_j^n \| \leq \| \tilde{T}_n^{-1} \| \| z_j^n \| \leq \varepsilon^{-1} \| \tilde{z}_j^n \|.
\]

(108)

The preceding discussion can be summarized as follows for \( j = k + 1, \ldots, N \):

\[
\Phi \zeta_j^n = z_j^n + \tilde{z}_j^n, \quad \lim_{n \to \infty} z_j^n = v_j, \quad \lim_{n \to \infty} \tilde{z}_j^n = \lim_{n \to \infty} \tilde{T}_n^{-1} z_j^n = 0.
\]

(109)

Second, we manipulate the spectral count \( \mu \) and determinants of certain finite dimensional operators. Set

\[
\mu_n := \prod_{\lambda \in \sigma(T_n) \cap (\mathbb{R}\setminus(-\varepsilon, \varepsilon))} \rho(\lambda).
\]

Since \( \rho(\lambda) = \lambda \) for \( \lambda \in (-\varepsilon, \varepsilon) \), we get

\[
\mu(T_n) = \mu_n \prod_{\lambda \in \sigma(T_n) \cap ((-\varepsilon, +\varepsilon) \setminus \{0\})} \lambda = \mu_n \det(\tilde{T}_n |\lim_{n \to \infty} E_{T_n}^{(-\varepsilon, +\varepsilon) \setminus \{0\}})
\]

(110)

and

\[
\lim_{n \to \infty} \mu_n = \mu(T).
\]

(111)
Now we attend to the main part of the argument. Using the notation $\Lambda^b_j w_j = \mu(T_n) \Lambda^N_j = k + 1 (-\xi_1^n, \xi_2^n) = \mu(T) \Lambda^N_j = k + 1 (v_j, 0)$. Identifying vectors $\xi \in E$ and $\zeta \in \mathbb{R}^N$ with their images $(\xi, 0)$ and $(0, \zeta)$ in $E \oplus \mathbb{R}^N$ and using equations (109) and (110), this reads
\[
\lim_{n \to \infty} \mu(T_n) \Lambda^N_j = k + 1 (T_n - 1 z_j^n + T_n - 1 z_j^n + \zeta_j^n) = \mu(T) \Lambda^N_j = k + 1 v_j.
\] We split the wedge product on the left hand side as the leading term plus the rest,
\[
\Lambda^N_j = k + 1 (T_n - 1 z_j^n + T_n - 1 z_j^n + \zeta_j^n) = \Lambda^N_j = k + 1 T_n - 1 z_j^n + \tilde{R}_n.
\] We first discuss the rest term $\tilde{R}_n$. Modulo signs a typical summand is
\[
\left(\Lambda^l_{i=k+1} T_n - 1 z_j^n\right) \wedge \left(\Lambda^m_{i=l+1} T_n - 1 z_j^n\right) \wedge \left(\Lambda^N_{i=m+1} \zeta_j^n\right)
\]
for some $k \leq l \leq m \leq N$ with $l < N$. Equation (112) and the third equation in (109) imply that the second and third factors in the last displayed equation remain bounded as $n \to \infty$. For the first factor, we pick orthonormal bases $e_{j+1}^i, \ldots, e_N^j$ of the spaces in $E^T_{n, \varepsilon + \varepsilon}(0)$ consisting of eigenvectors of $T_n$ and converging to an orthonormal system $e_{j+1}, \ldots, e_N$ as $n \to \infty$. We write $z_j^n = \sum_{i=k+1}^N c_j^i e_i^j$ in these bases, with coefficients $c_j^i \in \mathbb{R}$. Since $z_j^n \to v_j$ by the second part of (109), the sequence $\{c_j^i\}_{n \in \mathbb{N}}$ converges and thus in particular remains bounded as $n \to \infty$, for all $i, j = k + 1, \ldots, N$. This way the first factor $\tilde{T}_n - 1 z_j^n \wedge \cdots \wedge \tilde{T}_n - 1 z_j^n$ can be viewed as a homogeneous polynomial in $\lambda_{k+1}, \ldots, \lambda_N$ of degree $l - k$ strictly less than $N - k$. Moreover, this polynomial has bounded coefficients in $\Lambda^l - k \text{im } E^T_{\varepsilon + \varepsilon}(0)$, and each variable $\lambda_j - 1$ for $j = k + 1, \ldots, N$ enters every monomial with power 0 or 1. This implies
\[
\lim_{n \to \infty} \mu(T_n) \left(\Lambda^l_{i=k+1} T_n - 1 z_j^n\right) = 0,
\]
and therefore
\[
\lim_{n \to \infty} \mu(T_n) \tilde{R}_n = 0.
\]
Now we consider the leading term in equation (113) which we rewrite as
\[
\mu(T_n) \Lambda^N_j = k + 1 T_n - 1 z_j^n = \mu(T_n) \det(T_n - 1 \text{im } E^T_{\varepsilon + \varepsilon}(0)) \Lambda^N_j = k + 1 z_j^n = \mu_n \Lambda^N_j = k + 1 z_j^n.
\]
where the first equality follows from the definition of the determinant of a finite dimensional operator and the second equality follows from equation (110). In view of equation (111) and the second equation in (109), this implies
\[
\lim_{n \to \infty} \mu(T_n) \Lambda^N_j = k + 1 T_n - 1 z_j^n = \mu(T) \Lambda^N_j = k + 1 v_j,
\]
which together with (115) proves Assertion (B) and thus Theorem 5.4. □
Remark 5.6 Recall that on the subset $\mathcal{F}_s^{>\mathbb{R}}(E, F)^*$ consisting of invertible operators we have another orientation of the determinant bundle given by the section $t$, see (82) and the discussion following it. Observe that for $T \in \mathcal{F}_s^{>\mathbb{R}}(E, F)^*$ the values $s(T)$ and $t(T)$ differ by $\mu(T)$, where $\mu(T)$ is the product of all negative eigenvalues of $T$ and some positive factor resulting from the restriction $\rho|_{(0, +\infty)}$. We rephrase this in terms of the respective orientations. Let us call the orientation of $\det|_{\mathcal{F}_s^{>\mathbb{R}}(E, F)^*}$ induced by $t$ the tautological orientation. Any other orientation of $\det|_{\mathcal{F}_s^{>\mathbb{R}}(E, F)^*}$ differs from the tautological one by a sign. For $T \in \mathcal{F}_s^{>\mathbb{R}}(E, F)$ let $i(T)$ denote the (finite) number of negative eigenvalues of $T$. Then the restriction of the orientation $s$ to $\mathcal{F}_s^{>\mathbb{R}}(E, F)^*$ is related to the tautological orientation by

$$s(T) = (-1)^{i(T)} t(T) \quad \text{for } T \in \mathcal{F}_s^{>\mathbb{R}}(E, F)^*.$$

5.6 A counterexample

In this subsection we construct a loop of self-adjoint Fredholm operators with unbounded spectrum and nonorientable determinant bundle, thus showing that the boundedness of the spectrum from below or above in Theorem 5.4 is essential.

We consider $l^2 := \{x = (x_n)_{n \in \mathbb{Z}} \mid x_n \in \mathbb{R}, \sum_{n \in \mathbb{Z}} x_n^2 < \infty\}$ as a real Hilbert space with scalar product

$$\langle x, y \rangle := \sum_{n \in \mathbb{Z}} x_n y_n.$$

Similarly, we consider $h^1 := \{x = (x_n)_{n \in \mathbb{Z}} \mid x_n \in \mathbb{R}, \sum_{n \in \mathbb{Z}} n^2 x_n^2 < \infty\}$ as a real Hilbert space with scalar product

$$\langle x, y \rangle_{h^1} := \sum_{n \in \mathbb{Z}} x_n y_n + \sum_{n \in \mathbb{Z}} n^2 x_n y_n.$$

Since $h^1 \subset l^2$ is dense and the inclusion $h^1 \hookrightarrow l^2$ is compact, the pair $(h^1, l^2)$ satisfies the hypotheses of Theorem 5.4. Our goal is to prove

**Proposition 5.7** The restriction $\det|_{\mathcal{F}_s(h^1, l^2)}$ of the determinant line bundle to the space of self-adjoint Fredholm operators $l^2 \supset h^1 \hookrightarrow l^2$ is non-orientable.

The proof uses the following lemma. Recall the standard Hilbert space basis $\{e_n := (\ldots 0, 1, 0, \ldots)\}_{n \in \mathbb{Z}}$ of $l^2$, where the 1 occupies position $n$. 

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Lemma 5.8 There exists a continuous path \( \{ U_\tau \} \) of unitary isomorphisms of \( l^2 \) that restrict to (non-unitary) isomorphisms of \( h^1 \) such that

(i) \( U_1 \) maps \( e_n \) to \( e_{n-1} \) for all \( n \in \mathbb{Z} \);
(ii) \( U_2 = \text{Id} \);
(iii) \( U^{-1}_\tau(e_0) \) is a positive linear combination of \( e_0 \) and \( e_1 \) for all \( \tau \in (1, 2) \).

Proof: The following proof was found by Bernd Schmidt. The idea is to concatenate the following two rotational isotopies defined in terms of the standard basis of \( l^2 \). For \( \tau \in [0, 1/2] \) we define a unitary operator \( V^1_\tau \) of \( l^2 \) by requiring that it sends
\[
\begin{align*}
  e_{-n} &\mapsto \cos(\pi \tau)e_{-n} - \sin(\pi \tau)e_{n+1}, \\
  e_{n+1} &\mapsto \sin(\pi \tau)e_{n} + \cos(\pi \tau)e_{n+1}
\end{align*}
\]
for all \( n \geq 0 \). Similarly, for \( \tau \in [0, 1/2] \) we define a unitary operator \( V^2_\tau \) of \( l^2 \) by requiring that it sends
\[
\begin{align*}
  e_0 &\mapsto e_0, \\
  e_{-n} &\mapsto \cos(\pi \tau)e_{-n} + \sin(\pi \tau)e_{n}, \\
  e_n &\mapsto -\sin(\pi \tau)e_{n} + \cos(\pi \tau)e_n
\end{align*}
\]
for all \( n > 0 \). It is clear that both operators \( V^1_\tau \) and \( V^2_\tau \) restrict to isomorphisms of \( h^1 \). Define
\[
V_\tau := \begin{cases} V^1_\tau & \tau \in [0, 1/2], \\
V^2_{\tau-1/2} \circ V^1_{1/2} & \tau \in [1/2, 1]. \end{cases}
\]
For \( \tau = 0 \) we have \( V_0 = \text{Id} \). For \( \tau = 1 \) and \( n \geq 0 \):
\[
\begin{align*}
V_1 e_{-n} &= V^2_{1/2} V^1_{1/2} e_{-n} = -V^2_{1/2} e_{n+1} = e_{-(n+1)}, \\
V_1 e_{n+1} &= V^2_{1/2} V^1_{1/2} e_{n+1} = V^2_{1/2} e_{-n} = e_n.
\end{align*}
\]
To prove the statement about the preimage of \( e_0 \), we compute for \( \tau \in [0, 1/2] \):
\[
V_\tau(\cos(\pi \tau)e_0 + \sin(\pi \tau)e_1) = V^1_\tau(\cos(\pi \tau)e_0 + \sin(\pi \tau)e_1) \\
= \cos(\pi \tau)(\cos(\pi \tau)e_0 - \sin(\pi \tau)e_1) + \sin(\pi \tau)(\sin(\pi \tau)e_0 + \cos(\pi \tau)e_1) = e_0.
\]
Since \( V^2_\tau \) preserves \( e_0 \) for all \( \tau \in [0, 1/2] \), it follows that \( V^{-1}_\tau(e_0) \) is a positive linear combination of \( e_0 \) and \( e_1 \) for all \( \tau \in (0, 1) \). Therefore, backward reparametrized family \( \tau := V_{2-\tau} \) for \( \tau \in [1, 2] \) has the desired properties. □

Proof of Proposition 5.7: To construct our counterexample, we consider \( L^2([0, 1], \mathbb{C}) \) as a real Hilbert space with scalar product
\[
\langle f, g \rangle := \text{Re} \int_0^1 f(t)\bar{g}(t)dt.
\]
Similarly, we consider $H^1([0,1], \mathbb{C})$ as a real Hilbert space with scalar product
\[
\langle f, g \rangle_{H^1} := \text{Re} \int_0^1 f(t) \bar{g}(t) \, dt + \text{Re} \int_0^1 f'(t) \bar{g}'(t) \, dt.
\]

We abbreviate from now on
\[
L^2 := L^2([0,1], \mathbb{C}) \quad \text{and} \quad H^1 := H^1([0,1], \mathbb{C}).
\]

Consider the following densely defined operator in $L^2$:
\[
L := -i \frac{d}{dt} : L^2 \supset H^1 \rightarrow L^2
\]

and the following loop of dense subspaces of $L^2$:
\[
E_\tau := \{ f \in H^1 \mid f(0) \in \mathbb{R}, \ e^{i\pi \tau} f(1) \in \mathbb{R} \}, \quad \tau \in [0,1].
\]

Note that $E_0 = H^1$ and the inclusion $E_\tau \hookrightarrow L^2$ is compact for all $\tau \in [0,1]$. We define the restrictions
\[
L_\tau := L|_{E_\tau} : L^2 \supset E_\tau \rightarrow L^2.
\]

We claim that $L_\tau$ is symmetric. Indeed,
\[
\langle L_\tau f, g \rangle = \text{Re} \int_0^1 -if'(t) \bar{g}(t) \, dt = \text{Re} \left( -i \bar{f}_0 \bar{g}_0^1 \right) - \text{Re} \int_0^1 -if'(t) \bar{g}(t) \, dt.
\]

Now $f(0) \bar{g}(0) \in \mathbb{R}$ and $f(1) \bar{g}(1) = e^{i\pi \tau} f(1) e^{i\pi \tau} \bar{g}(1) \in \mathbb{R}$ implies $\text{Re} \left( -i \bar{f}_0 \bar{g}_0^1 \right) = 0$, which in view of $if \bar{g}' = -f \bar{g}''$ yields
\[
\langle L_\tau f, g \rangle = \langle f, L_\tau g \rangle.
\]

The spectrum of $L_\tau$ consists of eigenvalues
\[
\lambda_n^\tau = \pi (n - \tau), \quad n \in \mathbb{Z}
\]

with 1-dimensional eigenspaces spanned by the eigenvectors
\[
e_n^\tau(t) = e^{i\pi(n-\tau)t}, \quad n \in \mathbb{Z}.
\]

In particular, the spectrum of $L_\tau$ is unbounded from both sides. Note that $\ker L_\tau = 0$ for $\tau \in (0,1)$ and $\ker L_0 = \mathbb{R} e_0^0$. It is easy to see that $(\text{im} L_\tau)^\perp = \ker L_\tau$, so the operator $L_\tau$ is Fredholm of index 0, and therefore self-adjoint by Lemma 5.1 for all $\tau \in [0,1]$. Since the inclusion $E_\tau \hookrightarrow L^2$ is compact, the eigenvectors $e_n^\tau$ of $L_\tau$ form a Hilbert space basis of $L^2$ for all $\tau \in [0,1]$. In particular, for $\tau = 0$ we get a unitary isomorphism
\[
l^2 \xrightarrow{\cong} L^2, \quad e_n \mapsto e_n^0
\]
that restricts to an isomorphism $h^1 \xrightarrow{\sim} E_0$. In the sequel we will use this isomorphism to identify $l^2$ with $L^2$ and $e_n$ with $e_n^0$, so we can view the $U_\tau$ from Lemma 5.8 as unitary isomorphisms of $L^2$ restricting to isomorphisms of $E_0$.

The loop $\{L_\tau\}_{\tau \in [0,1]}$ of self-adjoint Fredholm operators is the heart of the counterexample. Unfortunately, these operators have varying domains of definition. Therefore, we will conjugate $\{L_\tau\}_{\tau \in [0,1]}$ with a suitable loop of unitary operators to bring all the operators from the desired loop on the same footing. Let $\{U_\tau\}_{\tau \in [1,2]}$ be the path of unitary operators in $L^2$ obtained from the one in Lemma 5.8 via identification (116).

We define a modified loop $\{\tilde{E}_\tau\}_{\tau \in [0,2]}$ of subspaces of $L^2$ by

$$\tilde{E}_\tau := \begin{cases} E_\tau & \tau \in [0,1], \\ E_0 = E_1 & \tau \in [1,2]. \end{cases}$$

We define a loop $\{\Phi_\tau\}_{\tau \in [0,2]}$ of isomorphisms of $L^2$ that restrict to isomorphisms $\Phi_\tau|_{E_0} : E_0 \to \tilde{E}_\tau$ by

$$\Phi_\tau f := \begin{cases} e^{-i\pi \tau} f & \tau \in [0,1], \\ U_\tau f & \tau \in [1,2]. \end{cases}$$

Note that $e^{-i\pi \tau} e_n = e^{-i\pi \tau} e_n^0 = e_{n-1} = U_1 e_n$ and $U_2 = \text{Id}$, so $\{\Phi_\tau\}_{\tau \in [0,2]}$ is indeed a continuous loop. Finally, we define

$$\tilde{L}_\tau := L|_{\tilde{E}_\tau}$$

and

$$T_\tau := \Phi_\tau^{-1} \tilde{L}_\tau \Phi_\tau$$

for $\tau \in [0,2]$. The definition of $\Phi_\tau$ and Lemma 5.8 imply that $\{T_\tau\}_{\tau \in [0,2]}$ is a loop of self-adjoint operators in $L^2$ with domain of definition $E_0$. This puts us in the setting of Section 5 with $F := L^2$, $E := E_0$ and $\{T_\tau\}_{\tau \in [0,2]} \subset \mathcal{F}(E, F)$.

Note that

$$\ker T_\tau = \begin{cases} \mathbb{R} e_0 & \tau = 0, \\ 0 & \tau \in (0,1), \\ \mathbb{R} U_\tau^{-1} e_0 & \tau \in [1,2]. \end{cases}$$

To stabilize the loop $\{T_\tau\}_{\tau \in [0,2]}$, we fix some $a, b > 0$ and consider the vector

$$G := ae_0 + be_1.$$

The positivity in Lemma 5.8 (iii) and the positivity of $a$ and $b$ imply that

$$\langle G, U_\tau^{-1} E_0 \rangle > 0.$$
for all $\tau \in [1, 2]$. Therefore, the vector $G$ is transverse to $\text{im} T_\tau = (\ker T_\tau) ^\perp$ for all $\tau \in [1, 2]$, so the stabilized operator

$$\hat{T}_\tau : E_0 \oplus \mathbb{R} \to L^2, \quad (f, \zeta) \mapsto T_\tau f + \zeta G$$

is surjective for all $\tau \in [0, 2]$. The discussion at the beginning of Section 5 implies that the determinant bundle of $\{T_\tau\}_{\tau \in [0, 2]}$ is isomorphic to the kernel bundle of $\{\hat{T}_\tau\}_{\tau \in [0, 2]}$. This loop splits naturally into two parts.

The part $\tau \in [0, 1]$:

Using $-i\frac{d}{d\tau} e_\tau = \pi ne_\tau$, we compute for $f = \sum_{n \in \mathbb{Z}} c_n e_n \in E_0$:

$$T_\tau f = e^{i\pi n \tau t} \left( -\frac{d}{d\tau} \right) e^{-i\pi n \tau t} f = e^{i\pi n \tau t} \left( -\pi \tau e^{-i\pi \tau t} f - e^{-i\pi \tau t} i \frac{d}{dt} f \right)$$

$$= -\pi \tau f - i \frac{d}{dt} f = \sum_{n \in \mathbb{Z}} \pi(n - \tau) c_n e_n.$$

Hence the equation

$$\hat{T}_\tau(f, \zeta) = \sum_{n \in \mathbb{Z}} \pi(n - \tau) c_n e_n + \zeta(ae_0 + be_1) = 0$$

is equivalent to $c_n = 0$ for all $n \neq 0, 1$ and

$$\pi(-\tau) c_0 + \zeta a = \pi(1 - \tau) c_1 + \zeta b = 0.$$

Thus $\ker \hat{T}_\tau$ is described by the equations

$$\zeta = \frac{1}{a} \pi \tau c_0 = \frac{1}{b} \pi (\tau - 1) c_1$$

and

$$(c_0, c_1) \in \mathbb{R}(a(\tau - 1), b\tau).$$

The part $\tau \in [1, 2]$:

Here we write $U_\tau f = \sum_{n \in \mathbb{Z}} c_n e_n$. A computation similar to the one above shows that

$$\hat{T}_\tau(f, \zeta) = \sum_{n \in \mathbb{Z}} \pi n c_n e_n + \zeta(ae_0 + be_1) = 0$$

is equivalent to $c_n = 0$ for all $n \neq 0, 1$ and

$$0 + \zeta a = \pi c_1 + \zeta b = 0.$$

The last equations mean that $\zeta = c_1 = 0$, and therefore

$$\ker \hat{T}_\tau = \mathbb{R}U_\tau^{-1} e_0 \oplus 0 = \mathbb{R}(c_0(\tau)e_0 + c_1(\tau)e_1) \oplus 0$$

with coefficients $c_i(\tau) \in \mathbb{R}$ which according to Lemma 5.8 (iii) satisfy $c_i(\tau) > 0$ for $\tau \in (1, 2)$ and $i = 0, 1$. 

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After rescaling we may assume \( c_1(1) = b \) and \( c_0(2) = a \). Then the kernels \( \{ \ker \hat{T}_\tau \}_{\tau \in [0,2]} \) are spanned by the continuous section \( (f(\tau), \zeta(\tau)) \) with

\[
f(\tau) = \begin{cases} 
  a(\tau - 1)c_0 + b\tau e_1 & \tau \in [0,1], \\
  c_0(\tau)e_0 + c_1(\tau)e_1 & \tau \in [1,2],
\end{cases}
\]

\[
\zeta(\tau) = \begin{cases} 
  \pi \tau(\tau - 1) & \tau \in [0,1], \\
  0 & \tau \in [1,2].
\end{cases}
\]

Since \( f(0) = -ae_0 \) and \( f(2) = ae_0 \), this shows nontriviality of the kernel bundle \( \{ \ker \hat{T}_\tau \}_{\tau \in [0,2]} \) over the circle, and thus of the determinant bundle of \( \{ T_\tau \}_{\tau \in [0,2]} \). In view of isomorphism (116), this concludes the proof of Proposition 5.7. \( \square \)

**Remark 5.9** For \( E, F \) as in Theorem 5.4 consider the spaces of operators

\[
\mathcal{F}_s^{>\mathcal{N}}(E, F)^* \subset \mathcal{F}_s^{>\mathcal{N}}(E, F) \subset \mathcal{F}_s(E, F) \subset \mathcal{F}_0(E, F),
\]

where \( \mathcal{F}_0(E, F) \) denotes the space of Fredholm operators of index zero. The space \( \mathcal{F}_0(E, F) \) is connected and \( \pi_1 \mathcal{F}_0(E, F) = \mathbb{Z}/2\mathbb{Z} \), see e.g. Proposition 1.3.5 and Corollary 1.5.10. (here and in the following \( \pi_1 \) is the fundamental group based at the identity). Proposition 5.7 shows that \( \pi_1 \mathcal{F}_s(E, F) \) and the map \( \pi_1 \mathcal{F}_s(E, F) \to \pi_1 \mathcal{F}_0(E, F) \) induced by the inclusion are nontrivial. On the other hand, \( \mathcal{F}_s^{>\mathcal{N}}(E, F)^* \) has infinitely many connected components given by the numbers \( i(T) \) of negative eigenvalues, cf. Remark 5.6. It would be interesting to determine other homotopy groups of the spaces in (117). For example, if one could show that \( \mathcal{F}_s^{>\mathcal{N}}(E, F) \) is simply connected, then this would provide an alternative proof of the orientability of the determinant bundles in Theorem 5.4.

**Remark 5.10** The non-orientability phenomenon described in Proposition 5.7 arises for example in the algebraic count of intersections of nonorientable Lagrangians, or of Reeb chords between nonorientable Legendrians.

### 5.7 The Euler number of gradient vector fields

Consider now a Hilbert manifold \( X \) (an open subset of a Hilbert space will suffice for our purposes) and a Hilbert space bundle \( E \to X \) with a continuous bundle inclusion \( TX \subset E \) such that \( T_x X \subset E_x \) is dense and the inclusion \( T_x X \hookrightarrow E_x \) is compact for each \( x \in X \). Let \( E \) be equipped with a fibrewise Hilbert space inner product \( \langle , \rangle_E \). In our application, \( X \) and \( E \) will be maps of Sobolev class \( H^2 \) and \( L^2 \), respectively.

Let \( f : X \to \mathbb{R} \) be a \( C^1 \)-map which has an \( E \)-gradient of class \( C^1 \), i.e., a \( C^1 \)-section \( \nabla_E f : X \to E \) satisfying

\[
Df(x)w = \langle \nabla_E f(x), w \rangle_E \quad \text{for all } w \in T_x X, x \in X.
\]

Note that this condition uniquely determines \( \nabla_E f \). Taking a derivative of this equation at \( x \in \text{Crit}(f) \) we obtain

\[
D^2 f(x)(v, w) = \langle D\nabla_E f(x)v, w \rangle_E \quad \text{for all } v, w \in T_x X, x \in \text{Crit}(f),
\]
where $D\nabla_E f(x) : T_x X \to E_x$ is the linearization of the section $\nabla_E f$ at $x$. Hence the second derivative $D^2 f(x)$ is the composition of the continuous linear maps

$$T_x X \otimes T_x X \xrightarrow{T_x X \otimes \text{id}} T_x X \otimes E_x \xrightarrow{\langle \cdot, \cdot \rangle_{E_x}} \mathbb{R}.$$ 

This implies that $f : X \to \mathbb{R}$ is of class $C^2$, so its second derivative $D^2 f(x)(v, w)$ is symmetric in $v, w$ (this follows from the usual finite dimensional result applied to the map $(s, t) \mapsto f \circ \exp_x(sv + tw)$). Therefore, the linear map

$$D\nabla_E f(x) : T_x X \to E_x$$

is symmetric in the sense of Section 5. For $\mathfrak{R} \in \mathbb{R}$ we denote by

$$\text{Func}^{>\mathfrak{R}}_s(X) \subset C^1(X, \mathbb{R})$$

the space of $C^1$-functions $f : X \to \mathbb{R}$ with $E$-gradient of class $C^1$ such that $D\nabla_E f(x) \in \mathcal{F}_s^{>\mathfrak{R}}(T_x X, E_x)$ for each $x \in \text{Crit}(f)$. Then Theorem 5.4 implies

**Corollary 5.11** The real line bundle

$$\mathcal{L} \to \{(f, x) \in \text{Func}^{>\mathfrak{R}}_s(X) \times X \mid Df(x) = 0\}$$

associating to $(f, x)$ the determinant line $\det(D\nabla_E f(x))$ has a canonical orientation. □

For a critical point $x$ of $f \in \text{Func}^{>\mathfrak{R}}_s(X)$ we denote by $\text{ind}(x) \in \mathbb{N}_0$ the maximal dimension of a subspace of $T_x X$ on which $D^2 f(x)$ is negative definite, or equivalently, the number of negative eigenvalues (with multiplicity) of $D\nabla_E f(x)$. A critical point $x$ is called nondegenerate if $D\nabla_E f(x) : T_x X \to E_x$ is invertible.

**Theorem 5.12** To each $f \in \text{Func}^{>\mathfrak{R}}_s(X)$ with compact critical point set we can associate its Euler number $\chi(\nabla_E f) \in \mathbb{Z}$ which is uniquely characterized by the following axioms:

**(Transversality)** If all critical points of $f$ are nondegenerate, then there are only finitely many critical points and

$$\chi(\nabla_E f) = \sum_{x \in \text{Crit}(f)} (-1)^{\text{ind}(x)}.$$

**(Excision)** For any open neighbourhood $\tilde{X} \subset X$ of $\text{Crit}(f)$ we have $\chi(\nabla_E f) = \chi(\nabla_E f|_{\tilde{X}})$.

**(Homotopy)** If $F : [0, 1] \times X \to \mathbb{R}$ is a $C^1$-map with $E$-gradient of class $C^1$ and compact critical point set such that $f_t = F(t, \cdot) \in \text{Func}^{>\mathfrak{R}}_s(X)$ for each $t \in [0, 1]$, then $\chi(\nabla_E f_0) = \chi(\nabla_E f_1)$.}

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Proof: The proof consists in applying the proof of [3] Theorem C.1 to the $E$-gradients, and upgrading it to integer coefficients using the orientations of the determinant bundles. Here are the details.

Consider $f \in \text{Func}_{>\mathbb{R}}^1(X)$ with compact critical point set. Its $E$-gradient defines a $C^1$-Fredholm section $\nabla_E f : X \to E$ of index zero with compact zero set $(\nabla_E f)^{-1}(0) = \text{Crit}(f)$. By Kuiper’s theorem, we can trivialize the Hilbert space bundle $E \cong X \times Y$ and thus view $\nabla_E f$ as a $C^1$-Fredholm map $X \to Y$ to a Hilbert space $Y$. This map satisfies the hypotheses of [3] Theorem C.1, so it has a mod 2 Euler number uniquely characterized by the analogous of the (Transversality), (Excision) and (Homotopy) axioms. By Corollary 5.11, the determinant line bundle $\text{det}(D\nabla_E f) \to \text{Crit}(f)$ is canonically oriented. This implies that the transversely cut out 0- and 1-dimensional submanifolds in the proof of [3] Theorem C.1 inherit canonical orientations and we get a well-defined integer valued Euler number $\chi(\nabla_E f) \in \mathbb{Z}$ (see [4]). The formula in the (Transversality) axiom follows from Remark 5.6. □

Remark 5.13 We have stated Corollary 5.11 and Theorem 5.12 only for gradient vector fields because this covers their applications in this paper. Their proofs actually give more general statements for $C^1$-sections $S : X \to E$ such that $DS(x) \in F^\infty_\mathbb{R}(T_x X, E_x)$ for each $x \in S^{-1}(0)$. These should be compared to results on orientability of Fredholm maps and their mapping degree in the literature, see e.g. [5, 7] and the references therein.

6 Frozen planet orbits with instantaneous interaction

In this section we consider the real helium atom with the instantaneous interaction between the electrons according to their Coulomb repulsion. Frozen planet orbits of this system are smooth maps $q = (q_1, q_2) : S^1 \to \mathbb{R}^2$ satisfying

\[
\begin{align*}
\ddot{q}_1(t) &= -\frac{2}{q_1(t)^2} + \frac{1}{(q_1(t) - q_2(t))^2}, \\
\ddot{q}_2(t) &= -\frac{2}{q_2(t)^2} + \frac{1}{(q_1(t) - q_2(t))^2}
\end{align*}
\] (118)

as well as the condition

\[
q_1(t) > q_2(t) \geq 0 \quad \text{for all } t \in S^1.
\] (119)

Thus $q_1$ describes the outer electron and $q_2$ the inner electron, where the latter undergoes collisions with the nucleus at the origin. To regularize these collisions, the following setup was introduced in [3]. One considers the space

\[
\mathcal{H}_{in} := \left\{ z = (z_1, z_2) \in H^2(S^1, \mathbb{R}^2) \left| \left\| z_1 \right\| > 0, \left\| z_2 \right\| > 0, z_1^2(\tau_{z_1}(t)) - z_2^2(\tau_{z_2}(t)) > 0 \text{ for all } t \in S^1 \right. \right\},
\] (120)
where \( z_i \) corresponds to the Levi-Civita regularization of \( q_i \) and \( \tau_{z_i} = t_{z_i}^{-1} \) are the time reparametrizations defined in Section 2. Note that \( \mathcal{H}_{in} \) is an open subset of the Hilbert space \( H^2(S^1, \mathbb{R}^2) \) and the last condition in its definition corresponds to condition (119). Integrating this condition we see that \( \mathcal{H}_{in} \subset \mathcal{H}_{av} \). The instantaneous interaction functional \( B_{in} : \mathcal{H}_{in} \to \mathbb{R} \) is defined by

\[
B_{in}(z_1, z_2) := \sum_{i=1}^{2} \left( 2\|z_i\|^2\|z'_i\|^2 + \frac{2}{\|z_i\|^2} - \int_0^1 \frac{1}{z_1^2(\tau_{z_1}(t)) - z_2^2(\tau_{z_2}(t))} \, dt \right). \tag{121}
\]

It is proved in [3] that under the Levi-Civita transformations \( q_i(t) = z_i(\tau_i(t)) \) critical points of \( B_{in} \) correspond to solutions of (118) and (119).

We interpolate between the mean and instantaneous interaction functionals by

\[
B_r := (1 - r)B_{in} + B_{av} : \mathcal{H}_{in} \to \mathbb{R}, \quad r \in [0, 1].
\]

It is proved in [3] that for each \( r \in [0, 1] \) the \( L^2 \)-gradient

\[
\nabla B_r = (1 - r)\nabla B_{in} + \nabla B_{av} : \mathcal{H}_{in} \to L^2(S^1, \mathbb{R}^2)
\]

is a \( C^1 \)-Fredholm map of index zero.

As in [3] §6 and §3.4 above, we remove the symmetries of \( B_r \) by restricting it to a suitable subspace. For \( k \in \mathbb{N}_0 \) we introduce the Hilbert space of symmetric loops

\[
H^k_{sym}(S^1, \mathbb{R}^2) := \{ z = (z_1, z_2) \in H^k(\mathbb{R}/2\mathbb{Z}, \mathbb{R}^2) \mid z_1(1 + \tau) = z_1(\tau) = z_1(1 - \tau) \text{ and } -z_2(1 + \tau) = z_2(\tau) = z_2(1 - \tau) \text{ for all } \tau \}.
\]

We consider on \( H^2_{sym}(S^1, \mathbb{R}^2) \) the \( L^2 \)-inner product \( \langle z, w \rangle = \sum_{i=1}^{2} \int_0^1 z_i(\tau)w_i(\tau) \, d\tau \) and define the open subset

\[
X := \{ z = (z_1, z_2) \in H^2_{sym}(S^1, \mathbb{R}^2) \mid \|z_1\| > 0, \|z_2\| > 0, \\
z'_1(\tau_1(t)) - z'_2(\tau_2(t)) > 0 \text{ for all } t, \\
z'_2(0) > 0, \ z_1(\tau) > 0 \text{ for all } \tau \in (0, 1) \text{ and } i = 1, 2 \}.
\]

Note that the first two lines in the definition of \( X \) correspond to the conditions for \( \mathcal{H}_{in} \), and the third line implies that \( z_2 \) is simple and normalized in the sense of §3.4 We will refer to critical points of \( B_r \) on \( X \) and their Levi-Civita transforms as normalized simple symmetric frozen planet orbits for \( B_r \).

It is proved in [2] that for each \( r \in [0, 1] \) the \( L^2 \)-gradient \( \nabla B_r : X \to H^0_{sym}(S^1, \mathbb{R}^2) \) is a \( C^1 \)-Fredholm map of index zero, and the critical point set \( Z \) of the \( C^2 \)-map

\[
[0, 1] \times X \to \mathbb{R}, \quad (r, z) \mapsto B_r(z) \tag{122}
\]

is compact. We wish to apply Theorem 5.12 to this map. For this, we need the following lemma.
Lemma 6.1 There exists a constant $R > 0$ such that for each $(r, z) \in \mathcal{Z}$ the spectrum of the Hessian $D\nabla B_r(z)$ is contained in $(R, \infty)$.

Proof: According to [3] the Hessian at $(r, z) \in \mathcal{Z}$ has the form

$$D\nabla B_r(z) = \mathcal{P}_z + \mathcal{R}_{r,z} : H^2_{\text{sym}}(S^1, \mathbb{R}^2) \to H^0_{\text{sym}}(S^1, \mathbb{R}^2),$$

where the leading order term is given on $v = (v_1, v_2) \in H^2_{\text{sym}}(S^1, \mathbb{R}^2)$ by

$$\mathcal{P}_z v = (-4\|z_1\|^2 v_1', -4\|z_2\|^2 v_2'),$$

and the lower order term $\mathcal{R}_{r,z}$ extends to a bounded operator $H^1_{\text{sym}}(S^1, \mathbb{R}^2) \to H^0_{\text{sym}}(S^1, \mathbb{R}^2)$ depending continuously on $(r, z)$. Compactness of $\mathcal{Z}$ implies the existence of uniform constants $\delta, C > 0$ (independent of $r, z$) such that

$$\langle \mathcal{P}_z v, v \rangle = 2\sum_{i=1}^2 (-4\|z_i\|^2 v_i'', v_i) = 2\sum_{i=1}^2 4\|z_i\|^2 \|v_i''\|^2 \geq \delta \|v''\|^2$$

and

$$\|\mathcal{R}_{r,z} v\| \leq C\|v\|_{H^1} = C(\|v\| + \|v''\|).$$

It follows that

$$\langle D\nabla B_r(z) v, v \rangle \geq \delta \|v''\|^2 - C(\|v\| + \|v''\|)\|v\|$$

$$= \delta \left(\|v''\| - \frac{C}{2\delta}\|v\|\right)^2 - \left(C + \frac{C^2}{4\delta}\right)\|v\|^2$$

$$\geq -\left(C + \frac{C^2}{4\delta}\right)\|v\|^2,$$

and thus

$$\langle \left(D\nabla B_r(z) + C + \frac{C^2}{4\delta}\right) v, v \rangle \geq 0.$$

This shows that the spectrum of $D\nabla B_r(z)$ is contained in $(R, \infty)$ for any $R < -C - \frac{C^2}{4\delta}$. □

By the preceding discussion and Lemma 6.1 the map (122) satisfies the hypotheses of Theorem 5.12. It follows that the integral Euler number $\chi(\nabla B_r) \in \mathbb{Z}$ is defined for each $r \in [0, 1]$ and independent of $r$. Now by Corollary 4.6 the functional $B_1 = B_{av}$ has a unique normalized simple symmetric critical point, which is nondegenerate of index zero. Therefore, Theorem 5.12 implies the following result which corresponds to Corollary B from the Introduction.

Corollary 6.2 The integral count of normalized simple symmetric frozen planet orbits equals

$$\chi(\nabla B_{in}) = \chi(\nabla B_{av}) = 1 \in \mathbb{Z}.$$ □
A Elliptic integrals

For \( n \in \mathbb{N}_0 \) we consider the elliptic integrals

\[ I_n : (-\infty, 1) \to \mathbb{R}, \quad m \mapsto \int_0^1 \frac{\zeta^{2n}}{\sqrt{(1 - \zeta^2)(1 - m\zeta^2)}} d\zeta. \]

At \( m = 0 \) these integrals can be computed in terms of Euler’s beta function and evaluated elementarily, namely

\[
I_n(0) = \int_0^1 \frac{\zeta^{2n}}{\sqrt{1 - \zeta^2}} d\zeta = \frac{1}{2} \int_0^1 \zeta^{n - \frac{1}{2}} (1 - \zeta)^{-\frac{1}{2}} d\zeta = \frac{B(n + \frac{1}{2}, \frac{1}{2})}{2} = \frac{\Gamma(n + \frac{1}{2})\Gamma(\frac{1}{2})}{2\Gamma(n + 1)} = \frac{(2n - 1)!!\Gamma(\frac{1}{2})^2}{2^{n+1}n!} = \frac{(2n - 1)!!\pi}{2^{n+1}n!}.
\]

Here \((2n - 1)!!\) equals \((2n - 1)(2n - 3) \cdots 1\) for \( n \geq 1 \) and 1 for \( n = 0 \). For \( m \) different from zero these elliptic integrals can be expressed via elliptic integrals of the first and second kind. These are defined for \( m \in (-\infty, 1) \) by

\[
K(m) = \int_0^1 \frac{1}{\sqrt{(1 - \zeta^2)(1 - m\zeta^2)}} d\zeta, \quad E(m) = \int_0^1 \frac{\sqrt{1 - m\zeta^2}}{\sqrt{1 - \zeta^2}} d\zeta.
\]

For \( n \) equal to zero or one this is obvious. Indeed, we have just

\[ I_0(m) = K(m) \quad \text{(124)} \]

for any \( m \in (-\infty, 1) \) and if \( m \neq 0 \), then

\[ I_1(m) = \frac{K(m) - E(m)}{m}. \quad \text{(125)} \]

For larger \( n \) this follows from the recursion formula

\[ I_{n+2}(m) = \frac{2(n + 1)(m + 1)I_{n+1}(m)}{(2n + 3)m} - \frac{(2n + 1)I_n(m)}{(2n + 3)m} \quad \text{(126)} \]
which allows to express $I_n$ with the help of $I_0$ and $I_1$ and consequently in terms of $K$ and $E$ using (124) and (125). The recursion formula (126) follows from

$$0 = \int_0^1 \frac{d}{d\xi} \left( \xi^{n+\frac{1}{2}} \sqrt{1-(m+1)\xi + m\xi^2} \right) d\xi$$

$$= \int_0^1 \left( (n+\frac{1}{2})\xi^{n-\frac{1}{2}} \sqrt{1-(m+1)\xi + m\xi^2} + \frac{\xi^{n+\frac{1}{2}}(2m\xi - m - 1)}{2\sqrt{1-(m+1)\xi + m\xi^2}} \right) d\xi$$

$$= \int_0^1 \left( (2n+1)\xi^{n-\frac{1}{2}}(1-(m+1)\xi + m\xi^2) + \xi^{n+\frac{1}{2}}(2m\xi - m - 1) \right) d\xi$$

$$= \int_0^1 \frac{(2n+3)m\xi^{n+\frac{3}{2}} - (2n+2)(m+1)\xi^{n+\frac{1}{2}} + (2n+1)\xi^{n-\frac{1}{2}}}{2\sqrt{(1-\xi)(1-m\xi)}} d\xi$$

$$= \int_0^1 \frac{(2n+3)m\xi^{2n+3} - (2n+2)(m+1)\xi^{2n+1} + (2n+1)\xi^{2n-1}}{(1-\xi^2)(1-m\xi^2)} d\xi$$

$$= (2n+3)mI_{n+2}(m) - 2(n+1)(m+1)I_{n+1}(m) + (2n+1)I_n(m).$$

In particular, we obtain

$$I_2(m) = \frac{2(m+1)I_1(m)}{3m} - \frac{I_0(m)}{3m},$$

which with the help of (124) and (125) we can alternatively write as

$$I_2(m) = \frac{(m+2)K(m) - 2(m+1)E(m)}{3m^2}.\tag{127}$$

The derivatives of the elliptic integrals of the first and second kind can be expressed as a linear combination of them with $m$-dependent coefficients as follows

$$K'(m) = \frac{E(m) - (1-m)K(m)}{2m(1-m)}, \quad E'(m) = \frac{E(m) - K(m)}{2m}.\tag{128}$$

For $E$ this is a straightforward application of (126). Indeed,

$$E'(m) = -\frac{1}{2} \int_0^1 \frac{\zeta^2}{\sqrt{(1-\zeta^2)(1-m\zeta^2)}} d\zeta$$

$$= -\frac{1}{2} \frac{I_1(m)}{m}$$

$$= \frac{E(m) - K(m)}{2m}.\tag{128}$$
For $K$ this is more involved and follows from the following computation

\[
0 = -m \int_0^1 \frac{\zeta \sqrt{1 - \zeta^2}}{\sqrt{1 - m\zeta^2}} d\zeta
= -m \int_0^1 \frac{(1 - \zeta^2)(1 - m\zeta^2) - \zeta^2(1 - m\zeta^2) + m\zeta^2(1 - \zeta^2)}{\sqrt{(1 - \zeta^2)(1 - m\zeta^2)^3}} d\zeta
= -m \int_0^1 \frac{1 - 2\zeta^2 + m\zeta^4}{\sqrt{(1 - \zeta^2)(1 - m\zeta^2)^3}} d\zeta
= \int_0^1 \frac{m(1 - m)\zeta^2 - (1 - m\zeta^2)^2 + (1 - m)(1 - m\zeta^2)}{\sqrt{(1 - \zeta^2)(1 - m\zeta^2)^3}} d\zeta
= 2m(1 - m)K'(m) - E(m) + (1 - m)K(m).
\]

It follows from (128) that the quotient of the elliptic integrals $E$ and $K$ satisfies the Riccati differential equation

\[
\left( \frac{E}{K} \right)' = -\frac{1}{2m} + \frac{1}{m} \frac{E}{K} - \frac{1}{2m(1 - m)} \left( \frac{E}{K} \right)^2,
\]  

(129)

as the following computation shows:

\[
\left( \frac{E}{K} \right)' = \frac{E'K - EK'}{K^2}
= \frac{1}{2m} \frac{E}{K} - \frac{1}{2m} \frac{1}{2m(1 - m)} \left( \frac{E}{K} \right)^2 + \frac{1}{2m} \frac{E}{K}
= -\frac{1}{2m} + \frac{1}{m} \frac{E}{K} - \frac{1}{2m(1 - m)} \left( \frac{E}{K} \right)^2,
\]

Combining (124) and (125) we get

\[
\frac{I_1}{I_0} = \frac{1}{m} \left( 1 - \frac{E}{K} \right).
\]

Differentiating this expression and using (129) we compute

\[
\left( \frac{I_1}{I_0} \right)' = \frac{1}{m^2} \frac{E}{K} - \frac{1}{m^2} \frac{E}{K}'
= \frac{1}{m^2} \frac{E}{K} - \frac{1}{m^2} \frac{1}{2m} \frac{E}{K} + \frac{1}{m^2} \frac{E}{K} + \frac{1}{2m} \frac{I_1}{I_0} \left( \frac{E}{K} \right)^2
= -\frac{1}{2m^2} + \frac{1}{2m} \frac{I_1}{I_0} \left( 1 - \frac{I_1}{I_0} \right)^2
\]

so that we obtain for the quotient of $I_1$ and $I_0$ the Riccati differential equation

\[
\left( \frac{I_1}{I_0} \right)' = \frac{1}{2m(1 - m)} - \frac{1}{m(1 - m)} \frac{I_1}{I_0} + \frac{1}{2(1 - m)} \left( \frac{I_1}{I_0} \right)^2.
\]  

(130)
We end this appendix with a technical lemma about the function

\[ F: (-\infty, 1) \to \mathbb{R}, \quad m \mapsto (2 - m) \frac{I_1}{I_0}(m) \]

which we need to prove our nondegeneracy of critical points of collision type for the frozen functional for positive parameters \( r \). From (123) we have

\[ I_0(0) = \frac{\pi}{2}, \quad I_1(0) = \frac{\pi}{4}, \]

and therefore

\[ F(0) = 1. \]

Our technical lemma is the following.

**Lemma A.1** For \( m < 0 \) we have \( F(m) > 1 \).

**Proof:** We consider the function

\[ G: (-\infty, 1) \to \mathbb{R}, \quad m \mapsto (2 - m)I_1(m) \]

and show that it is strictly decreasing for negative \( m \). For that purpose we compute its derivative

\[
\begin{align*}
G'(m) &= -\int_0^1 \frac{\zeta^2}{\sqrt{(1 - \zeta^2)(1 - m\zeta^2)}} d\zeta \\
&\quad + \frac{2 - m}{2} \int_0^1 \frac{\zeta^4}{\sqrt{(1 - \zeta^2)(1 - m\zeta^2)^3}} d\zeta \\
&= \frac{1}{2} \int_0^1 \frac{(2 - m)\zeta^4 - 2\zeta^2(1 - m\zeta^2)}{(1 - \zeta^2)(1 - m\zeta^2)^3} d\zeta \\
&= \frac{1}{2} \int_0^1 \frac{\zeta^2((2 + m)\zeta^2 - 2)}{(1 - \zeta^2)(1 - m\zeta^2)^3} d\zeta.
\end{align*}
\]

If \( m \) is negative the enumerator is nonpositive, and strictly negative for \( \zeta \in (0, 1) \). This shows that

\[ G'(m) < 0 \text{ for } m < 0. \]

Since \( I_0 = K \) is strictly increasing, we see that

\[ F = \frac{G}{I_0} \]

as the quotient of a positive strictly decreasing function and a positive strictly increasing function is strictly decreasing for negative values of \( m \). Since \( F(0) = 1 \), we conclude that

\[ F(m) > 1 \text{ for } m < 0, \]

which proves the lemma. \( \square \)
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