WZW model based on the extended de Sitter group

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Abstract

We study the WZW model based on the centrally extended 2D de Sitter algebra. We obtain the spacetime metric and its explicitly conformally flat expression. The symmetries of the spacetime are found by identifying the Killing vectors with the group generators. The energy-momentum tensor obtained from the affine-Sugawara construction agrees with that from the more conventional approach. The exact center charge agrees to one-loop order with the one-loop beta function equations. We have also studied the representations of the corresponding enveloping Virasoro algebra.

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There has recently been much interest in WZW models based on non-semisimple groups [1][2]. The first of such models is constructed in [2], based on the centrally extended 2D Poincaré algebra, which is used in analyzing the gauge theory of 2D gravity models. In this paper we consider the “centrally extended 2D de Sitter algebra”, which is also used in the gauge formulation of lineal gravity models [3].

The algebra has the following explicit description:

\[
[J, P_a] = \epsilon_{ab} P_b \quad [P_a, P_b] = \epsilon_{ab} (-\Lambda J + T) \quad [T, J] = [T, P_a] = 0
\]

We will call the corresponding non-compact group \( G \). The extended Poincaré algebra is given by the above with \( \Lambda = 0 \).

In general, given a Lie algebra with generators \( T^a \) (here \( T^a = P_1, P_2, J, T \)), and structure constants \( f_{cd}^{ab} \) (so \( [T^a, T^b] = if_{cd}^{ab} T^d \)), if there is a bilinear form \( \Omega_{ab} \) in the generators \( T^a \), which is symmetric (\( \Omega_{ab} = \Omega_{ba} \)), invariant

\[
f_d^{ab} \Omega_{cd} + f_d^{ac} \Omega_{bd} = 0
\]

and non-degenerate (so that there is an inverse matrix \( \Omega_{ab} \) obeying \( \Omega_{ab} \Omega_{bc} = \delta_a^c \)), then the WZW action on the surface \( \Sigma \) of a three-manifold \( B \) is

\[
S(g) = \frac{1}{4\pi} \int_\Sigma d^2 \sigma \, \Omega_{ab} A_{a\alpha} A_b^\alpha + \frac{i}{12\pi} \int_B d^3 \sigma \, \epsilon_{\alpha\beta\gamma} A_{a\alpha} A_{b\beta} A_{c\gamma} \Omega_{cd} f_d^{ab}
\]

where the \( A_{a\alpha} \)'s are defined via the left invariant one-forms \( g^{-1} \partial_\alpha g = A_{a\alpha} T^a \).

Usually for semisimple groups one can choose the bilinear form \( \Omega_{ab} = K_{ab} = f_d^{ac} f_c^{bd} \). However, for non-semi-simple groups this quadratic form is degenerate. Nevertheless, the \( G \) Lie algebra does have another non-degenerate bilinear form [3] i.e.

\[
\Omega_{ab} = k \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \frac{h}{1-b\Lambda} & \frac{1}{b\Lambda} \\
0 & 0 & \frac{1}{1-b\Lambda} & \frac{1}{b\Lambda}
\end{pmatrix}
\]

This metric on the Lie algebra has signature \((+,+,+,-\text{sign}(1-b\Lambda))\), and that will therefore be the signature of the resulting space-time metric.
In order to write (3) explicitly we need to find the $A_a$’s. To this purpose we use the following parametrization of the group manifold:

$$g = e^{a_1 P_1 + a_2 P_2} e^{u J + v T}$$

from which we obtain

$$A_a A_b^a \Omega^{ab} = \partial_a a_k \partial^a a_k + \frac{1}{1-b \Lambda} (b \partial_a u \partial^a u + 2 \partial_a v \partial^a u + \Lambda \partial_a v \partial^a v)$$

$$+ \partial^a u \frac{2(1-\cosh q)}{q^2} \epsilon_{ij} a_i \partial_a a_j + \Lambda \frac{2(\cosh q - 1) - q^2}{q^4} (\epsilon_{ij} a_i \partial_a a_j)^2$$

where $q \equiv \sqrt{\Lambda (a_1^2 + a_2^2)}$. Thus we are still able to reduce the Wess-Zumino term to a surface term without introducing any singularities. By using the polar coordinates

$$a_1 \equiv r \cos \theta \quad a_2 \equiv r \sin \theta$$

and identifying the resulting action with the non-linear $\sigma$-model action of the form

$$S = \frac{k}{4\pi} \int d^2 \sigma (G_{ab} \partial_a X^a \partial^a X^b + iB_{ab} \epsilon_{a\beta} \partial^a X^a \partial^\beta X^b)$$

where $X^a = (r, \theta, u, v)$, one can read off the background space-time metric and antisymmetric tensor field. The space-time geometry is described by the metric

$$ds^2 = dr^2 + 2 \frac{\cosh \lambda r - 1}{\lambda^2} d\theta^2 - 2 \frac{\cosh \lambda r - 1}{\lambda^2} dv^2 + \frac{1}{1-b \Lambda} (b d^2 + 2du dv + \Lambda dv^2)$$

and

$$B_{12} = u \frac{\sinh \lambda r}{\lambda}$$

where $\lambda \equiv \sqrt{\Lambda}$, and the dilaton is constant because of the homogeneity of the
group manifold. In terms of \((r, \theta, u, v)\), the left invariant currents are

\[
A_{1\alpha} = \cos(u - \theta)\partial_\alpha r + \sin(u - \theta)\frac{\sinh \lambda r}{\lambda} \partial_\alpha \theta \\
A_{2\alpha} = -\sin(u - \theta)\partial_\alpha r + \cos(u - \theta)\frac{\sinh \lambda r}{\lambda} \partial_\alpha \theta \\
A_{3\alpha} = \partial_\alpha u + (\cosh \lambda r - 1)\partial_\alpha \theta \\
A_{4\alpha} = \partial_\alpha v - \frac{1}{\lambda^2}(\cosh \lambda r - 1)\partial_\alpha \theta
\]

Introducing

\[
A_{\alpha a} \equiv A_{ai}\partial_\alpha X^i \equiv (T^{-1})_{ia}\partial_\alpha X^i
\]

we have

\[
\partial_i(T^{-1})_{ja} - \partial_j(T^{-1})_{ia} = if_{a}^{bc}(T^{-1})_{ib}(T^{-1})_{jc}
\]

These are the Maurer-Cartan equations satisfied by the left invariant currents. The classical equations of motion are given by, in terms of the light-cone coordinates \(x_\pm = \frac{1}{\sqrt{2}}(\tau \pm \sigma)\),

\[
\partial_+ A_\alpha^+ = 0
\]

Thus the \(A_\alpha^+\)’s are functions of \(x_+\) only.

To check that this model is conformally invariant, we first look at the one loop beta function equations [4]

\[
R_{ab} + \frac{1}{4} H_{ab}^2 + \nabla_a \nabla_b \phi = 0 \\
\nabla^c H_{cab} + \nabla^c \phi H_{cab} = 0 \\
R + \frac{1}{12} H^2 + 2\nabla^2 \phi + (\nabla \phi)^2 - \Lambda_1 = 0
\]

where \(H_{cab} = \nabla_{[c} B_{ab]}\) and \(H_{ab}^2 = H_{acd} H_{b}^{cd}\), \(H^2 = H_{abc} H^{abc}\), and \(\Lambda_1 = 2k(c-4)/3\). One finds that the non-zero components of \(R_{ab}\) are

\[
R_{11} = \frac{1}{2}\lambda^2 \\
R_{22} = \cosh \lambda r - 1 \\
R_{23} = -\frac{1}{2}(\cosh \lambda r - 1) \\
R_{33} = -\frac{1}{2}
\]

and as the only non-zero component of \(B\) is \(B_{12} = u \frac{\sinh \lambda r}{\lambda}\), the only non-zero
component of $H$ is $H_{123} = \frac{\sinh \lambda r}{\lambda}$. Also we have $R = 3\Lambda/2$ and $H^2 = -6\Lambda$. Putting these pieces together, one verifies equations (16) with $\Lambda_1 = \Lambda$ and $c = 4 + \frac{3\Lambda}{2k}$.

This can also be studied nonperturbatively by generalizing the Sugawara construction to non-semi-simple algebras. The Lie algebra of Eq.(1) admits a $2 \times 2$ representation

$$
P_1 = \frac{\lambda}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P_2 = \frac{\lambda}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad J = \begin{pmatrix} b' - \frac{i}{2} & 0 \\ 0 & b' + \frac{i}{2} \end{pmatrix}, \quad T = b' \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^2 \end{pmatrix}
$$

One finds that (here $t^a = P_1, P_2, J, T$)

$$
\text{Tr} \ t^a t^b = \frac{\Lambda}{2k} \Omega^{ab}
$$

provided $4b'^2 = 1/(1 - b\Lambda)$. Thus the Polyakov-Wiegmann composition rule still holds, and one can obtain the Kac-Moody symmetry of the model [5][6]. Introducing

$$
J^a(x_+)(t) \equiv 2\Omega^{ab}A_{b+}(x_+)
$$

we have the canonical commutators

$$
[J^a(x_+), J^b(y_+)] = -2\pi \delta(x_+-y_+)f^{ab}J^c(x_+) - 4\pi i \delta'(x_+-y_+)\Omega^{ab}
$$

These currents correspond to fermionic currents in a free fermion theory under the non-abelian bosonization. Introducing $z = e^{ix_+}$ (hence $|z|=1$) and expanding $J^a(z)$ in Laurent series

$$
J^a(z) = \sum_{n=-\infty}^{\infty} J^a_n z^{-n-1}
$$

one obtains

$$
[J^a_m, J^b_n] = i f^{ab} J^c_{m+n} + 2\Omega^{ab}m\delta_{m+n,0}
$$

which can be viewed as a representation of the Kac-Moody algebra in terms of the currents. Following [7] we take the moments of the energy-momentum tensor to
be

\[ L_m = L_{ab} \sum_k : J^a_{m+k} J^b_{-k} : \equiv L_{ab} \left[ \sum_{k \leq -m/2} J^a_{m+k} J^b_{-k} + \sum_{k > -m/2} J^b_{-k} J^a_{m+k} \right] \] (24)

where \( L_{ab} = L_{ba} \). Namely \( T(z) = \sum_{n=-\infty}^{\infty} L_n z^{-n-2} \) is the energy-momentum tensor operator. Our normal ordering convention is the same as that used in [8]. Hermiticity of these generators implies \( J^a_n = J^a_{-n} \) and \( L_n = L_{-n} \). Using Eq.(23), we calculate

\[
[L_m, J^a_n] = L_{cd} \sum_k \left[ J^c_{m+k} J^d_{-k}, J^a_n \right] \psi(\epsilon k) \\
= -n \left( 4L_{cd}\Omega^{ac} + L_{bc}f^{ac}f^{eb} \right) J^d_{m+n} + iL_{cd}f^{ac} \left\{ \sum_k \left[ J^d_{m+k} J^e_{-k} : \right.ight. \\
+ \sum_k : J^e_{m+k} J^d_{-k} : + 2\Omega^{de}\delta_{m+n,0} \left[ \sum_{k > -m} (k + m) + \sum_{k > 0} k \right] \psi(\epsilon k) \right\} 
\] (25)

where the symmetric cut-off function [8]

\[
\psi(x) = \begin{cases} 
1 & |x| \leq 1 \\
0 & |x| > 1 
\end{cases} 
\] (26)

is to regularize possible ambiguities under shifting indices. We have suppressed the function \( \psi(x) \) for those terms that do not have such ambiguities. Requiring that the currents be primary fields of the energy-momentum tensor with conformal weight one

\[
[L_m, J^a_n] = -n J^a_{m+n} 
\] (27)

one finds [7][9] that \( L_{ab} \) is invariant and that

\[
L_{ab}^{-1} = 4\Omega^{ab} + K^{ab} 
\] (28)
Next one can use Eq.(27) to verify the Virasoro algebra

\[
[L_m, L_n] = L_{cd} \sum_k [L_m, J^c_{n+k} J^d_{-k}] \psi(\epsilon k)
\]

\[= (m - n)L_{m+n} + \frac{1}{3}(m^3 - m)L_{cd} \Omega^{cd} \delta_{m+n,0} \quad (29)\]

Therefore \( c = 4L_{ab} \Omega^{ab} \). In our case, the inverse inertia tensor is given by

\[
L_{ab} = \frac{1}{2(2k - \Lambda)} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -\Lambda & 1 \\
0 & 0 & 1 & -b - \frac{1 - b\Lambda}{2k}
\end{pmatrix} \quad (30)
\]

This is the expected result since our model is equivalent to an \( SU(1, 1) \otimes \mathbb{R} \) WZW model. The center charge is

\[
c = 4 + \frac{3\Lambda}{2k - \Lambda} = \frac{6k}{2k - \Lambda} + 1 = 4 + \frac{3\Lambda}{2k} + O\left(\frac{1}{k^2}\right) \quad (31)
\]

where the contribution \( 6k/(2k - \Lambda) \) is from the \( SU(1, 1) \) factor. A similar analysis has been carried out for a larger class of non-semi-simple groups [10].

Conformally invariant theories with plane wave metrics were studied in [11]. It was shown that the beta function equations are satisfied to all loop orders due to the vanishing of curvature invariants. The study was extended to cases where the metrics are of the form of Brinkmann’s generalized plane-fronted waves with parallel rays in Ref.[12]. This method does not apply here, since all curvature invariants do not vanish. (And hence that rank two tensors such as \( R^2 R_{ab} \) do not vanish.)

It remains interesting to study the construction of these curvature invariants. One finds that the metric of Eq.(10) is conformally flat, and that

\[
\nabla_f R_{abcd} = 0 \quad \nabla_f H_{abc} = 0 \quad (32)
\]

Hence the beta functions must all be covariantly constant to all loop orders. In particular, \( \beta^\Phi \) must be a constant to all orders in \((1/k)\). It was conjectured that
the conditions Eqs.(16) and (32) imply that the Wess-Zumino term is never renormalized to all loop orders [13]. Now let us consider a curvature invariant of order \( k \)

\[
Q = Q(g_{ab}, R_{abcd}, \ldots, \nabla f_1 \ldots \nabla f_k R_{abcd})
\]  

(33)

We see that only 0-th order invariants can be constructed. Moreover, the 14 0-th order curvature invariants of Eq.(10) are all constants, due to

\[
\partial_f Q = \nabla_f Q = 0
\]

(34)

Explicitly, these invariants are the 10 vanishing quantities constructed with the use of the Weyl tensor and the 4 invariants formed out of the Ricci tensor [14]

\[
Q_n \equiv \text{Tr} \left( R^b_a \right)^n = \frac{3 \Lambda^n}{2^n} \quad n = 1, 2, 3, 4
\]

(35)

The model describes a homogeneous space-time; the left and right action of \( G \) on itself gives rise to a seven dimensional symmetry group of the space-time, as the left and right actions of the central generator \( T \) coincide. By a shift in \( v \) and then a rotation with angle \( u \) in the \((a_1, a_2)\) plane, we can re-write Eq.(10) as

\[
ds^2 = d\hat{s}^2 + dv^2
\]

with

\[
d\hat{s}^2 = \frac{4}{\Lambda} (dr^2 + \sinh^2 r \ d\theta^2 - \cosh^2 r \ du^2)
\]

(36)

being a metric for the 3D de Sitter space, which can be realized as an embedding in the 4D flat space

\[
ds^2 = -dt^2 + dx^2 + dy^2 - dw^2
\]

(37)

with the transformations [15]

\[
t = \frac{2}{\lambda} \sin u \cosh r \quad x = \frac{2}{\lambda} \cos \theta \sinh r \quad y = \frac{2}{\lambda} \sin \theta \sinh r \quad w = \frac{2}{\lambda} \cos u \cosh r
\]

(38)

We see that the seven symmetries of Eq.(10) are associated with the six symmetries of the 3D de Sitter space and a translation in the flat dimension. Using a different
set of embedding coordinates

\[ \begin{align*}
    t &= \cos \chi \\
    x &= \sin \chi \sinh \eta \cos \phi \\
    y &= \sin \chi \sinh \eta \sin \phi \\
    w &= \sin \chi \cosh \eta
\end{align*} \]

one can bring the metric

\[ ds^2 = dv^2 + dr^2 + \sinh^2 r \, d\theta^2 - \cosh^2 r \, du^2 \]  

(40)

to the Robertson-Walker form

\[ ds^2 = dv^2 - d\chi^2 + \sin^2 \chi \left( d\eta^2 + \sinh^2 \eta \, d\phi^2 \right) \]  

(41)

whose explicitly conformally flat expression is known. We thus find

\[ ds^2 = \cos^2 \frac{1}{2}(v + \chi) \cos^2 \frac{1}{2}(v - \chi) \left[ 4d\alpha d\beta + (\alpha - \beta)^2 (d\eta^2 + \sinh^2 \eta \, d\phi^2) \right] \]  

(42)

where

\[ \begin{align*}
    \alpha &= \tan \frac{1}{2}(v + \chi) \\
    \beta &= \tan \frac{1}{2}(v - \chi)
\end{align*} \]  

(43)

The spacetime inside the bracket in Eq.(42) is manifestly flat. The result can be viewed as a generalization of the fact that the exact plane wave studied in [2]

\[ ds^2 = dx^2 - 2du \left[ dv + \frac{1}{2} \epsilon_{ij} x^i dx^j \right] + bdu^2 \]  

(44)

is conformally flat with conformal factor \( \cos^{-2}(u/2) \). The above space-time has non-vanishing Ricci tensor, a covariantly constant null vector and vanishing curvature invariants. It also has seven symmetries.

The explicit forms of the symmetry of the metric Eq.(10) can be found as follows. The group generators, when acting on group parameter space, can be
represented as differential operators [16]. With the parametrization of Eq.(5), we have

\[
\begin{align*}
P_1 &= \cos(u - \theta) \frac{\partial}{\partial r} + \frac{\lambda \sin(u - \theta)}{\sinh \lambda r} \frac{\partial}{\partial \theta} - \sin(u - \theta) \tanh \frac{\lambda}{2} r \left( \frac{\lambda}{\partial u} - \frac{1}{\lambda} \frac{\partial}{\partial v} \right) \\
P_2 &= -\sin(u - \theta) \frac{\partial}{\partial r} + \frac{\lambda \cos(u - \theta)}{\sinh \lambda r} \frac{\partial}{\partial \theta} - \cos(u - \theta) \tanh \frac{\lambda}{2} r \left( \frac{\lambda}{\partial u} - \frac{1}{\lambda} \frac{\partial}{\partial v} \right) \\
J &= \frac{\partial}{\partial u} \\
T &= \frac{\partial}{\partial v}
\end{align*}
\]

(45)

They correspond to the right action of the group, i.e., with infinitesimal elements acting from the right. We can write the above \(T^a\)'s (here \(T^a = P_1, P_2, J, T\)) as

\[
T^a = (T^a)^i \frac{\partial}{\partial X^i} = T^a_i \frac{\partial}{\partial X^i}
\]

(46)

where the \(T^a_i\)'s are introduced in Eq.(13). Indeed, from the structure equation Eq.(14) and \(\partial(TT^{-1}) = 0\) one obtains

\[
T^a_i \partial_i T^{bj} - T^{bi} \partial_i T^a_j = i f^{ab}_{c} T^{cj}
\]

(47)

which are the desired group algebras.

It follows that the \(T^a\)'s are Killing vectors of the metric [17], which can be expressed as

\[
G^{ij} = \Omega_{ab} T^{ai} T^{bj}
\]

(48)

In fact, by using the commutators Eq.(47) and the invariant condition Eq.(2), one can verify

\[
T^{ak} \partial_k G^{ij} - G^{kj} \partial_k T^{ai} - G^{ik} \partial_k T^a_j = 0
\]

(49)

which are equivalent to the Killing equations. The group invariant measure is
found to be proportional to the volume element of the metric Eq.(10):

\[ |\det T^{ai}|^{-1} = |\det \Omega_{ab}|^{1/2} \sqrt{G} = \frac{\sinh \lambda r}{\lambda} \]  

(50)

Furthermore, the quadratic Casimir operator is equal to the Laplacian associated with the metric:

\[ \Delta \equiv \Omega^{ab} T^{a} T^{b} = \frac{1}{\sqrt{G}} \partial_{i} \left( \sqrt{G} G^{ij} \partial_{j} \right) \]  

(51)

To see this, we use the formula, which is true for an arbitrary non-singular matrix \( X \),

\[ \frac{\partial \det X}{\det X} = \text{Tr} \left( X^{-1} \partial X \right) \]  

(52)

and the commutation relations Eq.(47), to obtain

\[ \Omega^{ab} T^{a} T^{b} - \frac{1}{\sqrt{G}} \partial_{i} \left( \sqrt{G} G^{ij} \partial_{j} \right) = -\Omega_{ab} f^{ac}_{\phantom{ac}c} T^{c} \partial_{j} \]  

(53)

But this vanishes due to the invariant condition Eq.(2):

\[ -\Omega_{ab} f^{ac}_{\phantom{ac}c} T^{b} \partial_{j} = \Omega_{ac} f^{ac}_{\phantom{ac}b} T^{b} \partial_{j} = 0 \]  

(54)

The group representation functions, which we denote as \( |lmp> \), are simultaneous eigenfunctions of \( \Delta, J \) and \( T \). They form the regular representation of the group.

The remaining three Killing vectors correspond to the left action of the group. The corresponding group generators are given by

\[ P_{1} = \cos \theta \frac{\partial}{\partial r} - \lambda \sin \theta \coth \lambda r \frac{\partial}{\partial \theta} - \sin \theta \tanh \lambda r \frac{\lambda}{2} \left( \lambda \frac{\partial}{\partial u} - \frac{1}{\lambda} \frac{\partial}{\partial v} \right) \]  

\[ P_{2} = \sin \theta \frac{\partial}{\partial r} + \lambda \cos \theta \coth \lambda r \frac{\partial}{\partial \theta} + \cos \theta \tanh \lambda r \frac{\lambda}{2} \left( \lambda \frac{\partial}{\partial u} - \frac{1}{\lambda} \frac{\partial}{\partial v} \right) \]  

\[ J = \frac{\partial}{\partial \theta} + \frac{\partial}{\partial u} \]  

\[ T = \frac{\partial}{\partial v} \]  

(55)

From the above one obtains the left invariant measure and the Casimir operator. In this case they are the same as the right invariant ones. The two sets of generators
are independent \((T \text{ and } \bar{T} \text{ are the same})\). Together they form the anti-de Sitter algebra \(so(2, 2)\) plus a center element. Since the metrics constructed via the left or right invariant one-forms are the same, the left and right invariant measures will always be the same whenever such metrics can be constructed [18]. Note that at any given point, we can always find four Killing vectors \(\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right)\), as required by the homogeneity of the spacetime.

In the following we consider highest weight representations of the enveloping Virasoro algebra of Eq.(29). The states \(|lmp\rangle\) provide representations for the zero mode generators \(J^a_0\). Assuming there exists a highest weight state \(|lmp, N\rangle\), where \(N\) denotes additional degrees of freedom, satisfying [19]

\[
J^a_n|lmp, N\rangle = 0 \quad L_n|lmp, N\rangle = 0 \quad (n > 0) \quad (56)
\]

The Virasoro operator \(L_0\), when acting on this highest weight state, can be represented as

\[
L_0 = L_{ab}J^a_0 J^b_0 = L_{ab}T^a T^b \quad (57)
\]

Since \(L_{ab}\) is invariant, \(L_0\) can be identified with the Laplace operator associated with the metric

\[
\tilde{G}^{ij} = L_{ab} T^{ai} T^{bj} \quad (58)
\]

Note that \(\tilde{G}^{ij}\) is proportional to \(G^{ij}\) up to a redefinition of \(b\). We then obtain

\[
L_0 = \frac{k}{2(2k - \Lambda)} \Delta - \frac{1 - b\Lambda}{4k(2k - \Lambda)} \frac{\partial^2}{\partial v^2} = \frac{k}{2(2k - \Lambda)} \Delta - \frac{1 - b\Lambda}{4k(2k - \Lambda)} \frac{\partial^2}{\partial v^2} \quad (59)
\]

This gives the conformal weight \(h_{lmp}\) of the highest weight state in terms of the eigenvalues of \(\Delta\) and \(T\). In the geometric formulation of the general affine-Virasoro construction, the center charge is given by [20]

\[
c = \dim G + 4\tilde{R} \quad (60)
\]

where \(\tilde{R}\) is the curvature scalar associated with the metric \(\tilde{G}^{ij}\). One easily verifies that this reproduces our previous result for the center charge.
Descendant states are formed by the action of a series of $L_{-n}$ ($n > 0$) on $|lmp, N>$. A linear combination of states that vanishes is known as a null state. The representation of the Virasoro algebra is constructed from the Verma module consisting of the highest weight state and the set of states

$$L_{-k_1}L_{-k_2} \ldots L_{-k_n}|lmp, N> \quad 1 \leq k_1 \leq k_2 \leq \ldots \leq k_n$$

by removing all null states and their descendants. Unitarity of the representation requires the non-existence of negative norm states. (If $O^\dagger O|\alpha> = -|\alpha>$, then $O|\alpha>$ is a “negative norm” state.) This in turn requires that the Kac determinant be non-negative, which results in the conditions $c \geq 1$ and $h_{lmp} \geq 0$, in addition to the discrete series with $0 < c < 1$. If our parameters are such that $c > 1$, so that there are no null descendant states except at $h_{lmp} = 0$, we can write down the Virasoro characters as

$$\chi_{lmp}(q, \theta_1, \theta_2) = q^{-c/24} \Tr_{lmp} q^L e^{i\theta_1 J + i\theta_2 T} = \frac{q^{-(c/24) + h_{lmp}} e^{i(m+p)\theta_1 + i\Lambda p\theta_2}}{\prod_{n=1}^\infty (1 - q^n)}$$

for $h_{lmp} \neq 0$, and

$$\chi_{lmp}(q, \theta_1, \theta_2) = \frac{q^{-c/24} e^{i(m+p)\theta_1 + i\Lambda p\theta_2} (1 - q)}{\prod_{n=1}^\infty (1 - q^n)}$$

for $h_{lmp} = 0$.

In conclusion, we have studied the WZW model based on the “centrally extended 2D de Sitter algebra”. We found the explicitly conformally flat expression of the spacetime metric, described by Eqs.(42) and (43). The energy-momentum tensor obtained from the affine-Sugawara construction agrees with that from the more conventional approach. The exact center charge agrees to one-loop order with the one-loop beta function equations. We have also studied the representations of the corresponding Virasoro algebra.
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