We identify a solvable dynamical system — interpretable to some extent as a many-body problem — and point out that — for an appropriate assignment of its parameters — it is entirely isochronous, namely all its nonsingular solutions are completely periodic (i.e., periodic in all degrees of freedom) with the same fixed period (independent of the initial data). We then identify its equilibrium configurations and investigate its behavior in their neighborhood. We thereby identify certain matrices — of arbitrary order — whose eigenvalues are all rational numbers: a Diophantine finding.

Keywords: Dynamical systems; integrable; isochronous; Diophantine; matrices; eigenvalues; conjectures; nonlinear harmonic oscillators.

1. Introduction

The strategy underlying the Diophantine findings (results and conjectures) obtained in this paper takes as starting point a dynamical system whose time evolution is demonstrably entirely isochronous, namely all its solutions are completely periodic (i.e., periodic in all its degrees of freedom) with the same fixed period (generally an integer multiple of a basic period) in its entire phase space, except possibly for a sector with vanishing dimensionality where the solutions become singular. Such systems — especially when they are characterized by equations of motion of Newtonian type — can justifiably be considered as describing nonlinear harmonic oscillators [1]. Suppose moreover that such a system possesses (at least) one, explicitly known, equilibrium configuration. It is then possible to investigate its behavior near equilibrium via the standard approach, namely by assuming that the coordinates are infinitesimally close to their values at equilibrium and thereby linearizing the equations of motion. In this manner it is generally found that the behavior of the system near (a stable) equilibrium is multiply periodic, the frequencies of the basic oscillations coinciding with the eigenvalues of a matrix constructed from the equations of motion and evaluated at the equilibrium configuration (hence, under the above assumptions, an explicitly known matrix). But if the system is entirely isochronous, its behavior near equilibrium must be as well isochronous: hence all its basic frequencies must be integer multiples of a basic frequency. The outcome of this approach is therefore to identify a matrix (of arbitrary order) all eigenvalues of which are rational numbers: a Diophantine finding!
An approach to implement this strategy begins by manufacturing a dynamical system which is
solvable. Here and throughout the term solvable is employed to identify dynamical systems whose
initial-value problem can be solved by purely algebraic techniques, typically by finding the eigen-
values of matrices explicitly known for all time in terms of the initial data, or equivalently the
zeros of explicitly time-dependent polynomials. Next, one “ω-modifies” this solvable system so that
the (still solvable) ω-modified dynamical system thereby obtained is entirely isochronous — as can
be demonstrated by exploiting the solvable character of the system. A technique to perform this
second step is by now well known, and its applicability is sufficiently wide to justify the assertion
that isochronous systems are not rare. Indeed several solvable and entirely isochronous dynamical
systems have been recently identified in this manner, which are naturally interpretable as many-
body problems, their time evolution being characterized by Newtonian equations of motion with
one-body and two-body forces (for reviews of these recent results see Refs. [2–4]). These findings
are of interest in themselves, but they also yielded — via the strategy outlined above — several
Diophantine findings and conjectures (for a review of these results see Appendix C of Ref. [4]; and
for consequential developments involving orthogonal polynomials see [5–7]).

The main technique to manufacture these solvable systems starts from a solvable matrix evolution
equation and then focuses on the evolution of its eigenvalues. Indeed — as it is by now well known, see
for instance Ref. [4] — this often leads to Newtonian equations of motion characterizing a many-body
problem in which the coordinates $z_n(t)$ of the moving particles are identified with the eigenvalues
of the evolving matrix. It is generally convenient — and we shall follow this practice hereafter — to con-
sider these coordinates as complex numbers, hence to imagine that the many-body problem thereby
manufactured describes point-like particles moving in the complex $z$-plane. (It is also generally possi-
ble — and significant — to identify this complex plane with a physical plane on which move particles
whose positions are identified by real two-vectors; we will not elaborate on this aspect in this paper).
Generally these many-body problems also feature auxiliary variables, which might be interpreted as
“time-dependent coupling constants” or equivalently as “internal parameters”, whose time evolution
is determined by additional equations of motion, nonlinearly coupled to the equations of motion satisfi-
fied by the particle coordinates $z_n(t)$. Only in special cases, via some appropriate ansatz which turns
out — as it were, “miraculously” — to be compatible with these equations of motion, one can express
in terms of the particle coordinates $z_n(t)$ (and possibly their time derivatives $\dot{z}_n(t)$) these additional
variables, getting thereby rid of them, hence obtaining a solvable many-body problem whose equa-
tions of motion only involves the coordinates $z_n(t)$ (and possibly the velocities $\dot{z}_n(t)$) of the moving
particles — and possibly in addition some arbitrary constants.

This last step requires that such an ansatz exist, and that someone discover it. Several successful
examples have been recently reported [8–14] (for a review see Ref. [4]), and they also led, as already
mentioned above, to Diophantine findings and conjectures. However, even when this last step does
not seem to be feasible, the interpretation of these types of solvable models as many-body problems
with additional time-dependent auxiliary variables besides the particle coordinates $z_n(t)$ and the cor-
responding velocities $\dot{z}_n(t)$ is also valid [15–18]. Moreover, the elimination of the auxiliary variables —
while important to yield a many-body problem characterized by neater equations of motion — is by
no means essential in order to apply the strategy outlined above and thereby to arrive at Diophan-
tine findings — indeed, even the fact that the equations of motion of the dynamical system under
consideration have a Newtonian look has no relevance from this point of view, although it may pro-
mote the interest per se of these dynamical systems. What is essential is to manufacture an entirely
isochronous system and then to identify explicitly a (nontrivial) equilibrium configuration of it. The
first of these two steps can be realized in many ways, since no “miracle” is now required in order to pro-
cceed; the feasibility of the second step must be investigated on a case-by-case basis. Of course whether
the entirely isochronous dynamical systems thereby obtained, and the Diophantine findings arrived
at in this manner, are deemed "interesting", is a value judgement that can just as well be issued only a posteriori and on a case-by-case basis — and is anyway in the eye of the beholder.

The results of this paper are reported in the following Sec. 2, and they are proved in Sec. 3. A terse Sec. 4 ("Outlook") completes this paper.

2. Results

In this section we introduce our solvable and entirely isochronous dynamical system, we identify its equilibria, we discuss its behavior in their neighborhood, and we report the Diophantine findings arrived at in this manner. The proofs of these results requiring any elaboration are postponed to the following Sec. 3.

2.1. A solvable, entirely isochronous, dynamical system

We take as starting point the second-order matrix ODE

$$\dot{\nu}U = 2\nu \omega U + \lambda^2 \nu^2 \dot{U}U + (\nu - 1)\dot{U}U^{-1}\dot{U}. \quad (1)$$

Here and hereafter $U \equiv U(t)$ is a time-dependent $N \times N$ matrix, with $N$ an arbitrary positive integer; superimposed dots denote time derivatives; $\iota$ denotes the imaginary unit, $\iota^2 = -1$; and $\lambda, \nu, \omega$ are three scalar constants: the third, $\omega$, is hereafter assumed to be positive, and we associated with it the basic period

$$T = \frac{2\pi}{\omega} \quad (2)$$

the other two, $\lambda$ and $\nu$, are a priori arbitrary, but we generally assume that they are rational numbers, this being sufficient — as we shall immediately see — to guarantee that all the nonsingular solutions of this matrix ODE, (1), evolve periodically with a period $\tilde{T}$ which is an integer multiple of the basic period $T$.

The solution of the initial-value problem for this matrix ODE, (1), is indeed provided by the following (explicit!) formula (see Eq. (5.1-2) of Ref. [19]; or verify by explicit computation):

$$U(t) = \exp(\lambda \omega t) \left[\cos(\omega t) + \frac{\sin(\omega t)}{p \omega} \left(U(0)U(0)^{-1} - \iota \omega \right)^{\nu} \right] U(0). \quad (3)$$

It is clear from it that, for generic initial data (of course such that $U(0)$ is invertible), the matrix $U(t)$ is periodic, $U(t + \tilde{T}) = U(t)$ with $\tilde{T} = pT$, the positive integer $p$ being simply related to the denominators of the two rational numbers $\lambda$ and $\nu$; although there are some exceptional initial data for which the time evolution of $U(t)$ runs into singularities (except when $\nu$ is an integer). It is also plain that, while the condition that both $\lambda$ and $\nu$ be rational is sufficient to guarantee that the matrix ODE (1) be entirely isochronous, the condition that (only) $\lambda$ be rational is sufficient in order that this matrix ODE be isochronous: indeed, even if the requirement that the parameter $\nu$ be rational were dropped, there clearly would still exist an open, fully dimensional, sector in the phase space of the initial data $U(0)$, $U(0)$ such that the corresponding solution (3) is periodic (with period $\tilde{T} = pT$, the positive integer $p$ being then characterized by the requirement that $p\lambda$ be integer).

Next, we introduce the $N$ eigenvalues $\omega_n(t)$ of the $N \times N$ matrix $U(t)$ by setting

$$U(t) = R(t)Z(t)[R(t)]^{-1}, \quad (4a)$$

with

$$Z(t) = \text{diag} \{\omega_n(t)\}. \quad (4b)$$

Here $R(t)$ is of course the $N \times N$ matrix that diagonalizes the matrix $U(t)$. Here and hereafter it is understood that the index $n$ (and likewise indices such as $m, \ell$, see below) range from 1 to $N$. 
evolve according to the following $N$ eigenvalues $z_n(t)$, namely of the diagonal matrix $Z(t)$, is of course a purely algebraic task; and of course, since the matrix $U(t)$ is periodic with a period $T$ that is an integer multiple of $T$, all its eigenvalues $z_{n}(t)$ shall also be periodic functions of time with periods that are integer multiples of $T$ (these periods might themselves be integer multiples of the period $T$ due to an exchange of the eigenvalues through the time evolution; for special initial data it might happen that the time evolution of these eigenvalues runs into singularities, due to a collision of one or more of them, but this shall not happen for generic initial data). Likewise, it is a purely algebraic task to evaluate the diagonalizing $N \times N$ matrix $R(t)$, and this matrix shall also be periodic in time with a period that is an integer multiple of the basic period $T$. However this assertion is only valid modulo the property of this matrix to be defined up to multiplication from the right by an arbitrary diagonal matrix $D(t)$,

$$D(t) = \text{diag}[d_n(t)].$$

since, thanks to (4b), it makes no difference if in the right-hand side of (4a) $R(t)$ is replaced by $R(t)D(t)$ and simultaneously $[R(t)]^{-1}$ is of course replaced by $[D(t)]^{-1}[R(t)]^{-1}$.

Next, we introduce the $N \times N$ matrix

$$M(t) = [R(t)]^{-1}R(t).$$

The indeterminacy in the definition of the $N \times N$ matrix $R(t)$ due to the possibility of multiplying it from the right by an arbitrary diagonal matrix $D(t)$ entails that this matrix $M(t)$ is defined up to the "gauge" replacement

$$M(t) \Rightarrow \tilde{M}(t) = [D(t)]^{-1}M(t)D(t) + [D(t)]^{-1}D(t),$$

namely (componentwise)

$$\mu_n(t) \Rightarrow \tilde{\mu}_n(t) = \mu_n(t) + \frac{d_n(t)}{d_{\mu}(t)}$$

$$M_{mn}(t) \Rightarrow \tilde{M}_{mn}(t) = [d_n(t)]^{-1}M_{mn}(t)d_m(t), \quad n \neq m.$$ (7c)

Here and hereafter, for notational convenience, we denote as $\mu_n(t)$ (respectively $\tilde{\mu}_n(t)$) the $N$ diagonal elements of the $N \times N$ matrix $M(t)$ (respectively $\tilde{M}(t)$).

$$M_{mn}(t) \equiv \mu_n(t), \quad \tilde{M}_{mn}(t) \equiv \tilde{\mu}_n(t).$$ (8)

It is clear from (7b) that these quantities $\tilde{\mu}_n(t)$ remain undetermined (due to our freedom to assign the diagonal matrix $D(t)$), namely that we retain the privilege to assign them at our convenience — provided we simultaneously take account of the corresponding modification of the off-diagonal elements of the matrix $\tilde{M}(t)$, see (7c). And it is as well plain that, up to this indeterminacy (but hereafter we assume to restrict the selection of the diagonal matrix $D(t)$ so that its diagonal elements are periodic, $d_n(t + T) = d_n(t)$), the matrix $M(t)$ shall also be periodic with a period that is an integer multiple of the basic period $T$.

It can then be shown (see the following Sec. 3) that the matrix evolution equation (1) implies that the $N$ quantities $z_{n}(t)$, see (4b), and the $N(N-1)$ quantities

$$Y_{mn}(t) = |z_{m}(t) - z_{n}(t)|M_{mn}(t), \quad m \neq n$$ (9)

evolve according to the following $N + N(N-1) = N^2$ equations of motion:

$$\nu z_n = 2\hbar \omega z_n + (\hbar^2 \omega^2 z_n + (\nu - 1) \frac{\hbar^2}{\omega^2}$$

$$+ \sum_{\ell=1,\ell \neq n}^{N} \frac{Y_{\ell n}Y_{n\ell}}{z_n - z_{\ell}} \left( \nu + 1 + (\nu - 1) \frac{\omega^2}{\hbar^2} \right),$$ (10a)
Clearly the equilibrium configurations of the system (10) are provided by the formulae

\[ Y_{nm} = \left[ \frac{\dot{z}_x - \dot{z}_m}{z_x - z_m} + \frac{2\lambda \omega}{\nu} \frac{\nu - 1}{\nu} \left( \frac{\dot{z}_x}{z_x} + \frac{\dot{z}_m}{z_m} \right) - \rho_n + \mu_m \right] Y_{nm} \]

\[ - \sum_{\ell \neq 1, \ell \neq m} Y_{nm} \left( \frac{1}{\gamma_{n} - x_{\ell}} - \frac{1}{\gamma_{m} - x_{\ell}} \right), \quad n \neq m. \]  \hspace{1cm} (10b)

Note that the first set of these ODEs, (10a), look like \( N \) Newtonian equations of motion for \( N \) “particle coordinates” \( z_{nm} \). except that they also feature the \( N(N - 1) \) “auxiliary variables” \( Y_{nm} \)—play the role of “time-dependent coupling constants”— whose time evolution is then specified by the \( N(N - 1) \) (first-order) ODEs (10b). Of course the previous discussion entails that — remarkably — the generic solution of this system of \( N^2 \) autonomous ODEs is completely periodic with a period that is an integer multiple of the basic period \( T \).

2.2. Equilibria

Clearly the equilibrium configurations of the system (10) are provided by the formulæ

\[ z_{nm}(t) = \frac{x_{nm}}{\omega}, \quad Y_{nm}(t) = y_{nm}, \quad \mu_n(t) = i\omega y_{nm}. \]  \hspace{1cm} (11)

with the \( N + N(N - 1) = N^2 \) (time-independent) quantities \( x_{nm}, y_{nm} \) solutions of the system of \( N^2 \) algebraic equations

\[ \left( \lambda^2 - \nu^2 \right) X_{nm} = \sum_{\ell = 1, \ell \neq m}^{N} \left\{ \frac{\dot{y}_{\ell} t_{\ell m}}{x_{\ell} - x_{m}} \left( \nu + 1 - (\nu - 1)\frac{\dot{z}_x}{x_{\ell}} \right) \right\}, \]  \hspace{1cm} (12a)

\[ \frac{2\lambda}{\nu} = \gamma_n - \gamma_m + \sum_{\ell = 1, \ell \neq m}^{N} \left\{ \frac{\dot{y}_{\ell} t_{\ell m}}{y_{nm}} \left[ \frac{1}{\gamma_{n} - x_{\ell}} - \frac{1}{\gamma_{m} - x_{\ell}} \right] \right\}, \quad n \neq m. \]  \hspace{1cm} (12b)

where we retain the privilege to assign at our convenience the \( N \) constants \( \gamma_n \).

As shown in the following section, a solution of these equations (hereafter identified as Case 1) reads

\[ x_n = \exp \left( \frac{2\pi i n}{N} \right), \quad y_{nm} = \frac{2\nu}{\pi} \exp \left( \frac{\pi (n + m)}{N} \right), \quad \gamma_n = 0. \]  \hspace{1cm} (13a)

provided

\[ \lambda = \pm \frac{\nu}{N} (2 - N). \]  \hspace{1cm} (13b)

There is no restriction on the parameter \( \nu \), except the requirement that it be rational, implying of course that \( \lambda \) is as well rational.

In the following section it is also shown that a second class of solutions (hereafter identified as Case 2) exists provided

\[ \lambda = 0, \quad \nu = 1, \]  \hspace{1cm} (14a)

as detailed by the following formulæ:

\[ y_{nm} = \frac{1}{x_{nm}} \quad \gamma_n = \frac{\pi^2}{3} + c, \]  \hspace{1cm} (14b)

where \( c \) is an irrelevant arbitrary parameter and the numbers \( x_n \) are now the \( N \) zeros of the Hermite polynomial of order \( N \):

\[ H_N(x_n) = 0. \]  \hspace{1cm} (14c)
2.3. Behavior near equilibria

To investigate the behavior of the system (10) in the neighborhood of its equilibria one sets (see (11))

\[ z_\nu(t) = \frac{x_\nu + \epsilon y_\nu(t)}{\epsilon^2}, \quad Y_\nu(t) = y_{\nu m} + \epsilon w_{\nu m}(t), \quad \mu_\nu(t) = \omega_\nu \gamma_\nu, \]

(15)
treating \( \epsilon \) as an (infinitesimally) small parameter. Note that, for simplicity, we keep fixed the variables \( \mu_\nu(t) \) at their equilibrium values.

Insertion of this ansatz in the equations of motion (10) yields the following (constant-coefficient, linearized) system of \( N + N(N - 1) = N^2 \) ODEs for the \( N + N(N - 1) = N^2 \) dependent variables \( w_\nu(t) \) and \( w_{\nu m}(t) \) (with \( n \neq m \)):

\[
\nu \ddot{w}_\nu - 2i \omega_\nu \dot{w}_\nu - \omega^2 \left[ \sum_{j=1}^{N} A_{\nu,j} \nu w_j + \sum_{j=1,j \neq \nu}^{N} B_{\nu,j} \nu w_j \right] = 0, \quad (16a)
\]

\[
\nu \ddot{w}_{\nu m} + 2 \sum_{j=1}^{N} C_{\nu,j} \nu w_j + 2 \sum_{j=1,j \neq \nu}^{N} E_{\nu,j} \nu w_j = 0, \quad n \neq m, \quad (16b)
\]

where the \( N \times N \) matrix \( A \), the \( N \times [N(N - 1)] \) matrix \( B \), the two \( [N(N - 1)] \times N \) matrices \( C \) and \( D \), and the \( [N(N - 1)] \times [N(N - 1)] \) matrix \( E \) are defined componentwise in terms of the equilibrium data as follows:

\[
A_{\nu,j} = \delta_{\nu j} \left[ \frac{1}{x_\nu - x_j} \right]^{N^2} - \left( 1 - \delta_{\nu j} \right) \frac{y_{\nu j} y_{\nu m}}{(x_\nu - x_j)^2} \left[ \nu - 1 + 2(\nu - 1) \frac{x_j - (x_\nu - x_j)}{x_\nu - x_j} \right]^2, \quad (17a)
\]

\[
B_{\nu,j} = \frac{\delta_{\nu j} y_{\nu j}}{x_\nu - x_j} \left( \nu + 1 + (\nu - 1) \frac{x_j - (x_\nu - x_j)}{x_\nu - x_j} \right), \quad \ell \neq j, \quad (17b)
\]

\[
C_{\nu,j} = \frac{\delta_{\nu j} y_{\nu j}}{x_\nu - x_j} \left( \nu - 1 \right) \left( \frac{\delta_{\nu j}}{x_\nu - x_j} + \frac{\delta_{\nu m}}{x_m - x_j} \right) y_{\nu m}, \quad n \neq m, \quad (17c)
\]

\[
D_{\nu,m} = -\sum_{j=1,j \neq \nu,m}^{N} \left( \frac{y_{\nu j} y_{\nu m}}{(x_\nu - x_j)^2} \left[ \frac{\delta_{\nu j}}{x_\nu - x_j} + \frac{\delta_{\nu m}}{x_m - x_j} \right] \right) + (1 - \delta_{\nu m})(1 - \delta_{\nu m}) \frac{y_{\nu m} y_{\nu m}}{(x_\nu - x_j)^2} \left( \frac{1}{x_\nu - x_j} \right)^2 + \frac{\nu - 1}{\nu x_j}, \quad n \neq m, \quad (17d)
\]

\[
E_{\nu,j} = \delta_{\nu j} \frac{2 \Delta_\nu}{\nu} + \gamma_\nu + \gamma_m + \delta_{\nu j}(1 - \delta_{\nu j}) y_{\nu j} \left( \frac{1}{x_\nu - x_j} + \frac{1}{x_m - x_j} \right) + \frac{\nu - 1}{\nu x_j}, \quad n \neq m, \quad \ell \neq j, \quad (17e)
\]

Here and hereafter \( \delta_{\nu m} \) denotes the Kronecker symbol, \( \delta_{\nu m} = 1 \) if \( n = m \), \( \delta_{\nu m} = 0 \) if \( n \neq m \).

It is therefore clear that around its equilibria our system oscillates with the \( N^2 + N - S \) basic eigenfrequencies \( \omega_{\nu m} \),

\[
w_\nu(t) = \sum_{s=1}^{S} a_{\nu s} \exp(i \nu_\nu \omega_{\nu s} t), \quad w_{\nu m}(t) = \sum_{s=1}^{S} b_{\nu m s} \exp(i \nu_{\nu m} \omega_{\nu m} t), \quad (18a)
\]
where the $S$ numbers $\eta_s$ are the $S$ roots of the following polynomial equation, of degree $S$ in the variable $\varphi$:

$$
\det \begin{pmatrix}
(v_1^2 - 2\eta_1)I + A & B \\
\eta C + D & \eta E
\end{pmatrix} = 0.
$$

(18b)

The (block) structure of the square matrix of order $N^2$ whose determinant constitutes the left-hand side of this equation is implied by the definitions of the matrices $A, B, C, D$ and $E$ as specified above, while of course $I$ respectively $N$ denote the unit matrices of order $N$ respectively $N(N-1)$.

### 2.4. Diophantine findings

It is plain from the above treatment — in particular, from the isochronous character of the system under consideration, implying of course that also its behavior around its equilibria must be completely periodic — that the $S = N^2 + N$ roots of the Eq. (18b) must all be rational numbers: a Diophantine finding.

Let us now report the explicit form — whose derivation is outlined in the following Sec. 3 — of the matrices $A, B, C, D, E$ characterizing this Diophantine finding, corresponding to the two equilibria reported above.

In Case 1:

$$
A_{\alpha,\ell} = \left(\frac{2}{3}\right)\alpha^2(N-1)(\alpha N + \alpha N^2 + 6)i\sigma
$$

$$
+ (1 - \delta_{\alpha,\ell})\alpha^2 \begin{pmatrix}
\sin \frac{(n - \ell)\pi}{N} \end{pmatrix} - 2 \begin{pmatrix}
\alpha N + 1 + 2(\alpha N - 1)\exp \left[\frac{2\pi i(n - \ell)}{N}\right]
\end{pmatrix},
$$

$$
B_{\alpha,\ell,\ell} = i\alpha \delta_{\alpha,\ell} \begin{pmatrix}
\sin \frac{(n - \ell)\pi}{N} \end{pmatrix} + \begin{pmatrix}
\exp \left[\frac{2\pi i(n - \ell)/N}{N}\right]
\end{pmatrix}, \quad \ell \neq j,
$$

$$
C_{\alpha,\ell,\ell} = -i\alpha \delta_{\alpha,\ell} \begin{pmatrix}
\sin \frac{(n - \ell)\pi}{N} \end{pmatrix} + \begin{pmatrix}
\exp \left[\frac{2\pi i(n - \ell)/N}{N}\right]
\end{pmatrix}, \quad \ell \neq j,
$$

$$
D_{\alpha,\ell,\ell} = \delta_{\alpha,\ell} \begin{pmatrix}
\sin \frac{(n - \ell)\pi}{N} \end{pmatrix} + \begin{pmatrix}
\exp \left[\frac{2\pi i(n - \ell)/N}{N}\right]
\end{pmatrix}, \quad \ell \neq j,
$$

$$
E_{\alpha,\ell,\ell} = \delta_{\alpha,\ell} \begin{pmatrix}
\sin \frac{(n - \ell)\pi}{N} \end{pmatrix} + \begin{pmatrix}
\exp \left[\frac{2\pi i(n - \ell)/N}{N}\right]
\end{pmatrix}, \quad \ell \neq j.
$$
where for notational convenience we set

$$\alpha = \frac{\nu}{N}.$$  \hspace{1cm} (19f)

For $N = 2, N = 3$ respectively $N = 4$ the explicit evaluation of the determinants (see (18b) with (19)) yields

$$\det \begin{pmatrix} (\nu \eta - 2\lambda \eta)A + B & B \\ \eta C + D & \eta D \end{pmatrix} = \det \begin{pmatrix} -\nu (\nu + \frac{\eta}{2}) & \nu^2 (1 - \frac{\eta}{2}) \\
\nu^2 (1 - \frac{\eta}{2}) & -\nu (\nu + \frac{\eta}{2}) \end{pmatrix} \begin{pmatrix} -\nu - \nu & \nu \\
\nu & 0 \end{pmatrix}$$

$$= (\nu \eta)^2 (\eta - 2)(\eta - \nu)(\nu + \nu),$$ \hspace{1cm} (20a)

$$\det \begin{pmatrix} (\nu \eta - 2\lambda \eta)A + B & B \\ \eta C + D & \eta D \end{pmatrix} = (\nu \eta)^3 (\eta - 2)(\eta + 2)^2 \left( \eta - \frac{2\nu}{3} \right) \left( \eta + \frac{2\nu}{3} \right)$$

$$\cdot \left( \eta + \frac{4\nu}{3} \right) \left( \eta + \frac{4\nu}{3} \right) \left( \eta - \frac{2\nu}{3} + 2 \right) \left( \eta + \frac{2\nu}{3} + 2 \right),$$ \hspace{1cm} (20b)

respectively

$$\det \begin{pmatrix} (\nu \eta - 2\lambda \eta)A + B & B \\ \eta C + D & \eta D \end{pmatrix} = (\nu \eta)^4 (\eta - 2)(\eta + 2)^3 \left( \eta - \frac{2\nu}{3} \right) \left( \eta + \frac{2\nu}{3} \right) \left( \eta - \nu \right) \left( \eta + \nu \right) \left( \eta - \frac{2\nu}{3} \right)$$

$$\cdot \left( \eta + \frac{3\nu}{3} \right) \left( \eta + \frac{3\nu}{3} \right) \left( \eta - \frac{2\nu}{3} + 2 \right) \left( \eta + \frac{2\nu}{3} + 2 \right)^2 (\eta - \nu + 2)(\eta + \nu + 2).$$ \hspace{1cm} (20c)

The $9 \times 9$ respectively $16 \times 16$ matrices in the left-hand side of the last two, (20b) respectively (20c), of these $3$ formulae are too large to be displayed.

These findings suggest the following:

**Conjecture 2.4.1.** For arbitrary $N$

$$\det \begin{pmatrix} (\nu \eta - 2\lambda \eta)A + B & B \\ \eta C + D & \eta D \end{pmatrix} = (\nu \eta)^N (\eta - 2)(\eta + 2)^N \prod_{k=1}^{N-1} \left( \eta - \frac{2\nu}{N} \right) \left( \eta + \frac{2\nu}{N} \right)$$

$$\cdot \prod_{k=1}^{N-2} \left( \eta - \frac{2\nu}{N} + 2 \right)^{N-1-k} \left( \eta + \frac{2\nu}{N} + 2 \right)^{N-1-k}. \hspace{1cm} \blacksquare$$

Let us recall that the left-hand side of this formula is the determinant of a specific $N^2 \times N^2$ block matrix, see (19), while the right-hand side is a factorized polynomial of order $N^2 + N$ in the variable $\eta$. To arrive at this result the assumption that $\nu$ be a rational number played a role, but obviously this conjecture is applicable even if this requirement does not hold.

In Case 2:

$$A_{n,\ell} = -2(\eta - x_n)^2 (\eta - x_n)^2 (\eta - x_n)$$

$$B_{n,\ell} = -2(\eta - x_n)^2 (\eta - x_n)^2 (\eta - x_n)^2, \hspace{1cm} \ell \neq j,$$ \hspace{1cm} (22b)

$$C_{n,m} = (\delta_{nm} - \delta_{n\ell}) (x_n - x_m)^2, \hspace{1cm} n \neq m,$$ \hspace{1cm} (22c)
Conjecture 2.4.2. For arbitrary \(N\)

\[
\det \begin{bmatrix} (\nu q^2 - 2\lambda_0)1 + A & B \\ (\nu q^2 + D) & (\nu A + E) \end{bmatrix} = q^{(N-1)}(q - N)(q + N) \prod_{k=1}^{N-1} [(q - k)(q + k)]^{N-k}.
\]  

For \(N = 2, N = 3\) respectively \(N = 4\) the explicit evaluation of the determinants (see (18b) with (22)) yields

\[
\det \begin{bmatrix} (\nu q^2 - 2\lambda_0)1 + A & B \\ (\nu q^2 + D) & (\nu A + E) \end{bmatrix} = \begin{pmatrix} q^2 - \frac{2}{3} & \frac{1}{3} & 1 & -1 \\ \frac{1}{3} & q^2 - \frac{2}{3} & -1 & 1 \\ \frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

\[
= \begin{pmatrix} \frac{1}{2}q + \frac{1}{2}q & -\frac{1}{2}q + \frac{1}{2}q & \frac{1}{2}q + \frac{1}{2}q & 0 \\ -\frac{1}{2}q + \frac{1}{2}q & \frac{1}{2}q + \frac{1}{2}q & 0 & 0 \\ \frac{1}{2}q + \frac{1}{2}q & 0 & \frac{1}{2}q + \frac{1}{2}q & -\frac{1}{2}q \\ 0 & 0 & 0 & \frac{1}{2}q + \frac{1}{2}q \end{pmatrix}
\]

\[
= q^4(q - 1)^2(q + 1)^2(q - 2)(q + 2)(q - 3)(q + 3).
\]  

The 16 \times 16 matrix in the left-hand side of the last, (23c), of these 3 formulae is too large to be displayed.
Let us recall that the left-hand side of this formula is the determinant of a specific \( N^2 \times N^2 \) block matrix, see (22) where the numbers \( x_n \) are the \( N \) zeros of the Hermite polynomial of order \( N \) (see (14c)), while the right-hand side is a factorized polynomial of order \( N^2 + N \) in the variable \( \eta \).

3. Proofs

The starting point to obtain the equations of motion (10) is the observation that, via the definition (6), time-differentiation of (4a) yields

\[
\dot{U} = \dot{R}(\dot{Z} + [M, Z])R^{-1},
\]

(25a)

\[
\ddot{U} = R(\dot{Z} + [M, Z] + [M, [M, Z]])R^{-1}.
\]

(25b)

Here and hereafter the notation \([A, B] = AB - BA\):

\[
[A, B] = AB - BA.
\]

(26)

Via these formulae the \( N \times N \) matrix ODE yield the equations (11) is reduced to a trivial computation via the identity [20]

\[
\sum_{\ell=1}^{N} \exp\left(\frac{i\eta\pi\ell}{N}\right) \exp\left(-\frac{i\eta\pi\ell}{N}\right) = N - 1,
\]

(28a)

\[
\sum_{\ell=1}^{N} \exp\left(i\eta\pi\ell\right) \exp\left(-i\eta\pi\ell\right) = N - 1.
\]

(28b)

And the corresponding matrices (17), see (19), are immediately yielded by (13) via the trigonometric identity [21,22]

\[
\sum_{\ell=1}^{N-1} \left[ \sin\left(\frac{\pi\eta\ell}{N}\right) \right]^{-2} = \frac{N^2 - 1}{3}.
\]

(28c)

Likewise, the verification that the formulae (13) of Case 1 provide a solution of the system of \( N^2 \) algebraic equations (11) is a matter of trivial algebra using the identity

\[
\frac{x_n - x_m}{(x_n - x_m)(x_m - x_2)} = -\frac{1}{x_n - x_2} - \frac{1}{x_m - x_2},
\]

(29)

(to be conveniently inserted in (12b) with (14)) and the second and third of the following 4 formulae (see Appendix C of Ref. [19])

\[
\sum_{\ell=1}^{N} \frac{1}{x_n - x_\ell} = x_n,
\]

(30a)

\[
\sum_{\ell=1}^{N} \frac{1}{(x_n - x_\ell)^2} = 2(N - 1) - \frac{2x_n^2}{3},
\]

(30b)

\[
\sum_{\ell=1}^{N} \frac{1}{(x_n - x_\ell)^3} = \frac{x_n}{2}.
\]

(30c)
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\[ \sum_{\ell \neq n, \ell = 1}^{N} \frac{1}{x_\ell - x_n} = \frac{2(N + 2) - x_n^2}{2(N - 1) - x_n^2} \quad (30d) \]

satisfied by the \( N \) zeros of the Hermite polynomial of degree \( N \). And also the derivation (via the 4 formulae (30)) of the expressions (22) from (17) with (14) is a matter of trivial if tedious algebra, using repeatedly (in particular to get (22d)) the identity (29).

4. Outlook

As explained in the introductory Sec. 1, the technique described and utilized in this paper can be used relatively widely (although its application is not quite trivial), yielding Diophantine findings (results and conjectures). This has provided the motivation to append the Roman numeral I to this paper: indeed a second paper following the same pattern and arriving thereby at additional Diophantine findings is in the pipeline [20]. As already mentioned in the introductory Sec. 1, the interest of these findings is in the eye of the beholder: we feel this kind of findings deserve to be ascertained and to be eventually recorded in standard compilations of mathematical formulae, as done for instance in [22].

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