ON A CLASS OF HIGHLY SYMMETRIC MARKOV-DYCK SHIFTS

WOLFGANG KRIEGER

Abstract. A class of highly symmetric Markov-Dyck shifts is introduced. Expressions for zeta functions are derived and topological entropies are determined.

1. Introduction

Let $\Sigma$ be a finite alphabet, and let $S_{\Sigma} : \Sigma^\mathbb{Z} \to \Sigma^\mathbb{Z}$ be the left shift on $\Sigma^\mathbb{Z}$,

$$S_{\Sigma}((\sigma_i)_{i \in \mathbb{Z}}) = (\sigma_{i+1})_{i \in \mathbb{Z}}, \quad (\sigma_i)_{i \in \mathbb{Z}} \in \Sigma^\mathbb{Z}.$$  

We denote the restriction of $S_{\Sigma}$ to a $S_{\Sigma}$-invariant Borel subset $X$ of $\Sigma^\mathbb{Z}$ by $S_X$. Compact $S_{\Sigma}$-invariant subsets $X$ of $\Sigma^\mathbb{Z}$, or rather the dynamical systems $(X, S_X)$, are called subshifts. Transition matrices $A_{\sigma, \sigma'} \in \{0, 1\}, \sigma, \sigma' \in \Sigma$, that have in every row and in every column at least one entry, that is equal to 1, define Markov shifts $M(A) = \bigcup_{i \in \mathbb{Z}} \{ (\sigma_i)_{i \in \mathbb{Z}} \in \Sigma^\mathbb{Z} : A(\sigma_i, \sigma_{i+1}) = 1 \}$, that serve as prototypical examples of subshifts. We denote the Perron eigenvalue of a $\mathbb{Z}_+^+$-matrix $A$ by $\lambda(A)$. The topological entropy of the Markov shift with transition matrix $A$ is given by

$$h(M(A)) = \log \lambda,$$

(1.1)

For an introduction to the theory of subshifts see [Ki] and [LM]. A finite word in the symbols of $\Sigma$ is called admissible for the subshift $X \subset \Sigma^\mathbb{Z}$ if it appears in a point of $X$. A subshift $X \subset \Sigma^\mathbb{Z}$ is uniquely determined by its language of admissible words.

In this paper we are concerned with subshifts that are constructed from directed graphs. We denote a finite directed graph with vertex set $V$ and edge set $E$ by $G(V, E)$. The source vertex of an edge $e \in E$ we denote by $s$ and its target vertex by $t$. A strongly connected directed graph $G = G(V, E)$ has an edge shift, which is a Markov shift with alphabet $E$ and whose language of admissible words is the set of finite paths in $G$. We denote the edge shift of $G$ by $Me(G)$.

Another class of subshifts that arise from directed graphs, are the Markov-Dyck shifts. We recall their construction. For a given graph $G = G(V, E)$, let $E^- = \{ e^- : e \in E \}$ be a copy of $E$. Reverse the directions of the edges in $E^-$ to obtain the edge set $E^+ = \{ e^+ : e \in E \}$ of the reversed graph of $G(V, E^-)$. In this way one has defined a graph $G(V, E^- \cup E^+)$, that has source and target mappings, that are given by

$$s(e^-) = s(e), \quad t(e^-) = t(e),$$
$$s(e^+) = t(e), \quad t(e^+) = s(e), \quad e \in E.$$

The construction of the Markov-Dyck shift of a finite strongly connected non-circular directed graph $G(V, E^-)$ is via the graph inverse semigroup of the graph [LL Section 7.3]. With idempotents $1_V, V \in V$, we use as a generating set of the
graph inverse semigroup of \( G(V,\mathcal{E}^-) \) the set \( \mathcal{E}^- \cup \{1_V : V \in \mathcal{V}\} \cup \mathcal{E}^+ \). Besides \( 1_V^2 = 1_V, V \in \mathcal{V} \), we have the relations
\[
1_V 1_W = 0, \quad U, W \in \mathcal{V}, U \neq W,
\]
\[
f^- g^+ = \begin{cases} 1_{s(f)}, & \text{if } f = g, \\ 0, & \text{if } f \neq g, \quad f, g \in \mathcal{E}. \end{cases}
\]
\[
e^- = 1_{s(e)}e^- = e^- 1_{t(e)}, \quad 1_{t(e)}e^+ = e^+ 1_{s(e)}, \quad e \in \mathcal{E}.
\]
The alphabet of the Markov-Dyck shift of the strongly connected non-circular directed graph \( G(\mathcal{V},\mathcal{E}) \) is \( \mathcal{E}^- \cup \mathcal{E}^+ \), and its admissible words are the words
\[
(e_i)_{1 \leq i \leq I} \in (\mathcal{E}^- \cup \mathcal{E}^+)^I, \quad I \in \mathbb{N},
\]
such that
\[
\prod_{1 \leq k \leq I} \sigma_i \neq 0.
\]
We denote the Markov-Dyck shift of the directed graph \( G \) by \( Md(G) \). Markov-Motzkin shifts [KM2 Section 4.1] are versions of Markov-Dyck shifts.

For a given graph \( G = G(\mathcal{V},\mathcal{E}) \) set for \( e \in \mathcal{E} \)
\[
(e^-)^{-1} = e^+, \quad (e^+)^{-1} = e^-.
\]
The map \( e \to (e)^{-1} (e \in \mathcal{E}) \) induces an anti-automorphism of the graph inverse semigroup of \( G \), and also a time reversal \( \rho \) of the Markov-Dyck shift of \( G \), by
\[
(\rho(x))_i = (x_{-i})^{-1}, \quad x \in Md(G)
\]
We refer to \( \rho \) as the canonical time reversal of the Markov-Dyck shift of \( G \).

Denote the one-vertex directed graph with \( N > 1 \) loops by \( G(N) \). The graph inverse semigroup of \( G(N) \) is the Dyck inverse monoid (the "polycyclique" [NP]), and the subshifts \( Md(G(N)) \) are the Dyck shifts, that were introduced in [Kt1]. As shown in [Kr1] Section 4] the topological entropy of \( Md(G) \) is given by
\[
h(Md(G)) = \log(N + 1).
\]
The zeta function of the Dyck shift was obtained in [Ke Example 3, p. 79]. Also necessary and sufficient conditions for the existence of an embedding of an irreducible subshift of finite type into a Dyck shift are known [Hi1]. K-groups of the Dyck shifts were computed in [Ma1] and [KM1]. For the tail invariant measures of the Dyck shifts see [Me].

Another example of a Markov-Dyck shift is the Fibonacci-Dyck shift, that arises from the Fibonacci graph \( F \), that has two vertices \( V_1 \) and \( V_2 \), and three edges \( e_{1,2}, e_{2,1} \) and \( e_{1,1} \), and source and target mappings, that are given by
\[
V_1 = s(e_{1,2}) = t(e_{2,1}) = s(e_{1,1}) = t(e_{1,1}), \quad V_2 = s(e_{2,1}) = t(e_{1,2}).
\]
In [KM2 Section 4] the topological entropy of \( Md(\hat{F}) \) was shown to be given by
\[
h(Md(G(N))) = 3\log 2 - \log 3,
\]
and a formula for the zeta function of the Fibonacci-Dyck shift was also given. Necessary and sufficient conditions for the existence of an embedding of an irreducible subshift of finite type into the Fibonacci-Dyck shift are known [HK]. K-groups of the Fibonacci-Dyck shift were computed in [Ma4].

In this paper we introduce a class of directed graphs, that are built from directed trees, and we study their Markov-Dyck shifts We consider directed trees all of whose edges are pointing away from the root. The height of a vertex \( V \) of the tree is the length of the path from the root to \( V \). A directed tree is said to be rotationally homogeneous, if all of its leaves have the same height, and if vertices, that have the same height, have the same out-degree. For a finite directed graph \( G = G(\mathcal{V},\mathcal{E}) \)
denote by $F_G$ the set of edges that are the only incoming edges of their target vertices. We say that the directed graph $G$ is rotationally homogeneous, if $F_G$ is a rotationally homogeneous directed tree, if the source vertex of every edge $e \in E \setminus F_G$ is a leaf of $F_G$, and if all leaves of $F_G$ have the same out-degree in $G$. The height of a rotationally homogeneous directed tree is the height of the leaves of $F_G$.

In this paper we study measures of maximal entropy, topological entropy, and zeta functions of Markov-Dyck shifts of rotationally homogeneous directed graphs. For the problem of topological conjugacy of rotationally homogeneous directed graphs see [HK]. In section 2 we consider rotationally homogeneous directed graphs of height $H(G) > 1$ and in Section 3 of height $H(G) = 1$. The directed graphs $G(N), N > 1$, can be interpreted as rotationally homogeneous directed graphs of height zero. From this point of view the results of Sections 2 and 3 extend the results of [Kr1, Section 4]. We show that the Markov-Dyck shifts of rotationally homogeneous directed graphs have two measures of maximal entropy, and we identify for a given rotationally homogeneous directed graph $G$ a companion graph, that has an edge shift with topological entropy equal to the topological entropy of the Markov-Dyck shift of $G$. As in [Kr1 Section 4] this involves constructing suitable shift invariant Borel subsets of the Markov-Dyck shift of $G$ and of the edge shift of the companion graph, that yield Borel conjugate Borel dynamical systems. By rewriting the formulas of Cardano and Vieta for the solutions of algebraic equations of degree three one obtains expressions for the topological entropy of the Markov-Dyck shifts of rotationally homogeneous directed graphs of height 2 and 5. For heights 1 and 3 solving a quadratic equation suffices. A Descartes resolvent cubic of a quartic equation offers a way of finding the topological entropy of the Markov-Dyck shifts of rotationally homogeneous directed graphs of height 7.

Given a directed graph $G = G(V,E)$, set
$$\psi(e^-) = 1, \quad \psi(e^+) = -1, \quad e \in E.$$ 

Following the terminology, that was introduced in [HI], we say that a point $x \in Md(G)$ of period $\pi$ is neutral, if there exists an $i \in \mathbb{Z}$, such that $\sum_{i \leq j < i+\pi} \psi(x_j)$ is zero, and we say that it has a negative (positive) multiplier, if there exists an $i \in \mathbb{Z}$, such that $\sum_{i \leq j < i+\pi} \psi(x_j)$ is negative (positive). General expressions for the zeta functions of Markov-Dyck shifts were derived in [KM2, Section 2] and in [Kr2, Section 2]. Following [Kr2, Section 3] these expressions were obtained by factoring the zeta function according to the classification of the periodic points of the Markov-Dyck shift as neutral, or as periodic points with negative or with positive multiplier, and by using for each case a suitable circular code. The method is based on the fact, that for circular codes $C$ in the symbols of a finite alphabet $\Sigma$, the zeta function of the set of periodic points in $\Sigma^\mathbb{Z}$, that carry a bi-infinite concatenation of code words, is related to the generating function $g_C$ of $C$ by
$$\zeta_X(C) = \frac{1}{1 - g_C}.$$ 

(see [BPR, Section 7.3]). See also [BBD1, Section E], [Ma2, Section 3], [BBD2, Section 6], [IK1, IK2, Section 2], [BH1, Section 5.2], [BH2, Section 4], and [Kr2, Section 9]. By the same method we derive in Section 4 an expression for the zeta functions of Markov-Dyck shifts specifically for the case of rotationally homogeneous directed graphs.

2. The case $H(G) > 1$

Let $H \in \mathbb{N}$, and let there be given data
$$N = (N_h)_{1 \leq h \leq H+1} \in \mathbb{N}^H \times (\mathbb{N} \setminus \{1\}).$$
The data \( \mathbf{N} \) specify a rotationally homogeneous directed graph \( \hat{G}(\mathbf{N}) \) with vertex set
\[
V(\mathbf{N}) = \{V(0)\} \cup \bigcup_{1 \leq h \leq H} \{V((n_{h_1})_{1 \leq h_1 \leq h}) : (n_{h_2})_{1 \leq h_2 \leq h} \in \prod_{1 \leq h_3 \leq h} [1, N_{h_3}]\},
\]
and edge set \( \mathcal{F}(\mathbf{N}) \cup \mathcal{E}(\mathbf{N}) \), where
\[
\mathcal{F}(\mathbf{N}) = \bigcup_{1 \leq h \leq H} \{f((n_{h_1})_{1 \leq h_1 \leq h}) : (n_{h_2})_{1 \leq h_2 \leq h} \in \prod_{1 \leq h_3 \leq h} [1, N_{h_3}]\},
\]
\[
\mathcal{E}(\mathbf{N}) = \{e((n_{h_1})_{1 \leq h \leq H+1}) : (n_{h_2})_{1 \leq h \leq H+1} \in \prod_{1 \leq h \leq H+1} [1, N_{h}]\},
\]
and with source and target mappings, that are given by
\[
s(f(n_1)) = V(0), \quad 1 \leq n_1 \leq N_1,
\]
and
\[
s(f((n_{h_2})_{1 \leq h_2 \leq h})) = V(n_{h_2})_{1 \leq h_2 \leq h},
\]
\[
t(f((n_{h_2})_{1 \leq h_2 \leq h})) = V((n_{h_2})_{1 \leq h_2 \leq h}), \quad (n_{h_2})_{1 \leq h_2 \leq h} \in \prod_{1 \leq h_3 \leq h} [1, N_{h_3}]\), 1 < h \leq H,
\]
\[
s(e((n_{h_2})_{1 \leq h \leq H+1})) = V((n_{h_2})_{1 \leq h \leq H}),
\]
\[
t(e((n_{h_2})_{1 \leq h \leq H+1})) = V(0), \quad (n_{h_2})_{1 \leq h \leq H+1} \in \prod_{1 \leq h \leq H+1} [1, N_{h}],
\]
For given data \( \mathbf{N} = (N_{h})_{1 \leq h \leq H+1} \in \mathbb{N}^H \times (\mathbb{N} \setminus \{1\}) \), we denote for \( J \in \mathbb{N} \) by \( Y_J(\text{Md}(\hat{G}(\mathbf{N}))) \) the set of \( y \in \text{Md}(\hat{G}(\mathbf{N})) \), such that
\[
\sum_{-J_0 < J_0 < 0} \psi(y_{J_0}) \geq 0, \quad 0 < J_0 < J,
\]
and
\[
\sum_{-J_0 < J_0 < 0} \psi(y_{J_0}) = -1.
\]
We set
\[
(2.1) \quad Y(\text{Md}(\hat{G}(\mathbf{N}))) = \bigcap_{J \in \mathbb{Z}} S_{\text{Md}(\hat{G}(\mathbf{N}))}^J \left( \bigcup_{J \in \mathbb{N}} Y_J(\text{Md}(\hat{G}(\mathbf{N}))) \right).
\]
For the given data \( \mathbf{N} \) we also define an \( \mathbb{N} \)-matrix \( A_{h,h'}(\mathbf{N})_{1 \leq h,h' \leq H+1} \) by
\[
A_{h,h+1}(\mathbf{N}) = N_{h+1}, \quad A_{h+1,1}(\mathbf{N}) = 1, \quad 1 \leq h \leq H, A_{H+1,1}(\mathbf{N}) = N_1, \quad A_{1,H+1}(\mathbf{N}) = 1.
\]
We also set \( C^{-}(\mathbf{N}) = \{c_{h}^{-}(n_h) : 1 \leq n_h \leq N_h, 1 \leq h \leq H+1\} \), \( C^{+}(\mathbf{N}) = \{c_{h}^{+} : 1 \leq h \leq H+1\} \), and we use as a directed graph with adjacency matrix \( A(\mathbf{N}) \) a directed graph \( \overline{G}(\mathbf{N}) \) with vertex set \([1, H+1]\), with edge set \( C^{-}(\mathbf{N}) \cup C^{+}(\mathbf{N}) \), and with source and target mappings, that are given by
\[
s(c^{-}(h,n_h)) = h - 1, \quad 1 \leq n_h \leq N_h, 1 < h \leq H + 1,
\]
\[
s(c^{-}(1,n_1)) = H + 1, \quad 1 \leq n_1 \leq N_1,
\]
\[
s(c^{-}(h,n_2)) = h, \quad 1 \leq n_h \leq N_h, 1 < h \leq H + 1,
\]
\[
s(c^{+}(h)) = h, \quad 1 < h \leq H + 1,
\]
\[
t(c^{+}(h+1)) = h, \quad 1 \leq h \leq H,
\]
\[
t(c^{+}(1)) = H + 1.
\]
We set
\[ \varphi(c^+_h(n_h)) = 1, \quad \varphi(f^+) = -1, \quad 1 \leq n_h \leq N_h, \]
\[ \varphi(c^-_h) = -1, \quad \quad 1 \leq h \leq H + 1, \]
and we denote for \( I \in \mathbb{N} \) by \( X_I(ME(\overline{G}(\mathbb{N}))) \) the set of \( x \in ME(\overline{G}(\mathbb{N})) \), such that
\[ \sum_{-I_0 < i < 0} \varphi(y_{i_k}) \geq 0, \quad 0 < I_0 < I, \]
and
\[ \sum_{-I < i < 0} \varphi(y_j) = -1. \]

We set
\[ (2.2) \quad X(ME(\overline{G}(\mathbb{N}))) = \bigcap \bigcup_{i \in \mathbb{Z}} S^i_{ME(\overline{G}(\mathbb{N}))} ( \bigcup_{I \in \mathbb{N}} X_I(ME(\overline{G}(\mathbb{N}))) ). \]

**Lemma 2.1.** For \( H > 1 \), and for data
\[ \mathbf{N} = (N_h)_{1 \leq h \leq H+1} \in \mathbb{N}^H \times (\mathbb{N} \setminus \{1\}) \]
the Borel dynamical system
\[ (Y(MD(\hat{G}(\mathbb{N}))), S_{Y(MD(\hat{G}(\mathbb{N})))} \upharpoonright Y(MD(\hat{G}(\mathbb{N})))) \]
is Borel conjugate to the Borel dynamical system
\[ (Y(MD(\hat{G}(\mathbb{N}))), S_{Y(MD(\hat{G}(\mathbb{N})))} \upharpoonright Y(MD(\hat{G}(\mathbb{N})))) \]

**Proof.** We set
\[ \Omega(f^-((n_h)_1 \leq h \leq n_h)) = c^-_h(n_h), \]
\[ \Omega(f^+((n_h)_1 \leq h \leq n_h)) = c^+_h(n_h), \quad 1 \leq h \leq H, \]
and
\[ \Omega(e^-((n_h)_1 \leq h \leq n_h+1)) = c^-_{h+1}(n_h+1), \quad \Omega(e^+((n_h)_1 \leq h \leq n_h+1)) = c^+_{h+1}. \]
The 1-block map \( \Omega \) yields by
\[ (\omega(y))_0 = \Omega(y_0), \quad y \in Y(MD(\hat{G}(\mathbb{N}))), \]
a homomorphism
\[ \omega : (Y(MD(\hat{G}(\mathbb{N}))), S_{Y(MD(\hat{G}(\mathbb{N})))} \upharpoonright Y(MD(\hat{G}(\mathbb{N})))) \rightarrow \]
\[ (X(ME(\overline{G}(\mathbb{N}))), S_{ME(\overline{G}(\mathbb{N})))} \upharpoonright X(ME(\overline{G}(\mathbb{N})))) \]
The map \( \omega \) is in fact a Borel conjugacy. We indicate how for \( y \in Y(MD(\hat{G}(\mathbb{N}))) \), \( y_0 \) is reconstructed from \( x = \omega(y) \), or, more generally, how for \( x \in X(ME(\overline{G}(\mathbb{N}))) \), one can find \( \omega^{-1}(x)_0 \).

For this we set inductively
\[ I_1(x) = I, \quad x \in X_I \in ME(\overline{G}(\mathbb{N})), \quad I \in \mathbb{N}, \]
\[ I_{k+1}(x) = I_k(x) + I_1(S^{-I_k(x)}_{ME(\overline{G}(\mathbb{N}))}(x)), \quad x \in ME(\overline{G}(\mathbb{N})), \quad k \in \mathbb{N}. \]

We note that
\[ \varphi(x_{-I_k(x)}) = 1, \quad \sum_{-I_k(x) < i < 0} \varphi(x_i) = k - 1, \quad k \in \mathbb{N}. \]
We distinguish six cases. In the case that
\[ x_0 \in \{c^+_1(n_1) : 1 \leq n_1 \leq N_1\}, \]
one has $n_1 \in [1, N_1]$ determined by
\[ x_0 = c_1^-(n_1), \]
and one has
\[ y_0 = f^-(n_1). \]

In the case that
\[ x_0 = c_{H+1}^+, \]
one has $(n_h)_{1 \leq h \leq H+1} \in \prod_{1 \leq h \leq H+1} [1, N_h]$ determined by
\[ x_{-I_h} = c_h^-(n_h), \quad 1 \leq h \leq H+1, \]
and one has
\[ y_0 = e^+((n_h)_{1 \leq h \leq H+1}). \]

In the case, that
\[ x_0 \in \{c_h^-(n_h) : 1 \leq h \leq N_h, 1 < h \leq H\}, \]
one has $(n_h)_{1 \leq h < h_0} \in \prod_{1 \leq h < h_0} [1, N_h]$ determined by
\[ x_{-I_{h_0-1}} = c_{h_0}^-(n_{h_0}), \quad 1 \leq h_0 \leq h, \]
and $n_h \in [1, N_h]$ determined by
\[ x_0 = c_h^-(n_h), \]
and one has
\[ y_0 = f^+((n_h)_{1 \leq h \leq h_0}). \]

In the case that
\[ x_0 \in \{c_{H+1}^+ : 1 \leq n_{H+1} \leq N_{H+1}\}, \]
one has
\[ (n_h)_{1 \leq h \leq H} \in \prod_{1 \leq h \leq H} [1, M_h] \]
determined by
\[ x_{-I_h} = c_h^-(n_h), \quad 1 \leq h \leq H, \]
and $n_{H+1} \in [1, N_{H+1}]$ determined by
\[ x_0 = c_{H+1}^-, \]
and one has
\[ y_0 = e^-((n_h) : 1 \leq h \leq H+1). \]

In the case, that
\[ x_0 \in \{c_h^+ : 1 \leq h \leq H\}, \]
one has
\[ (n_h)_{1 \leq h \leq h_0} \in \prod_{1 \leq h \leq h_0} [1, N_{h_0}] \]
determined by
\[ x_{I_{h_0+1}} = c_h^-(n_{h_0}), \quad 1 \leq h_0 \leq h, \]
and one has
\[ y_0 = f^-((n_h)_{1 \leq h \leq h_0}). \]

In the case, that
\[ x_0 = c_1^+, \]
one has $n_1 \in [1, N_1]$ determined by
\[ x_{-1}(x) = c_1^-(n_1), \]
and one has
\[ y_0^- = f^-((n_1). \]
Theorem 2.2. For $H > 1$ and for data
\[ \mathbf{N} = (N_h)_{1 \leq h \leq H+1} \in \mathbb{N}^H \times (\mathbb{N} \setminus \{1\}), \]
the Markov-Dyck shift of $\hat{G}(\mathbf{N})$ has two measures of maximal entropy, and its topological entropy is equal to the topological entropy of the edge shift of $\overline{G}(\mathbf{N})$.

Proof. Let $\mu_\mathbf{N}$ denote the measure of maximal entropy of $\text{Me}(\hat{G}(\mathbf{N}))$. It follows from $N_{H+1} > 1$, that
\[ \mu_\mathbf{N}(\{x \in \text{Me}(\overline{G}(\mathbf{N})) : \varphi(x_0 = 1)\}) > \mu_\mathbf{N}(\{x \in \text{Me}(\overline{G}(\mathbf{N})) : \varphi(x_0 = -1)\}), \]
and therefore
\[ \mu_\mathbf{N}(X(\text{Me}(\overline{G}(\mathbf{N})))) \geq \mu_\mathbf{N}(\{x \in \text{Me}(\overline{G}(\mathbf{N}))) : \lim_{i \to \infty} \sum_{1 \leq i \leq I} \varphi(x_j) = \infty\}) = 1. \]
The canonical time reversal of $\text{Me}(\hat{G}(\mathbf{N}))$ carries the Borel set $Y^{+}(\text{Me}(\hat{G}(\mathbf{N})))$ into a Borel set $Y^{-}(\text{Me}(\hat{G}(\mathbf{N})))$, that is symmetric to $Y^{-}(\text{Me}(\hat{G}(\mathbf{N})))$. As a consequence of the Poincaré recurrence theorem every ergodic shift invariant probability measure on $\text{Me}(\hat{G}(\mathbf{N}))$ assigns measure one to $Y^{-}(\text{Me}(\hat{G}(\mathbf{N}))) \cup Y^{+}(\text{Me}(\hat{G}(\mathbf{N})))$ (see [KT1 Section 4]. Apply Lemma 2.1 to obtain a measure of maximal entropy $\text{Me}(\overline{G}(\mathbf{N}))$ on $\text{Me}(\overline{G}(\mathbf{N}))$. The image of this measure under the canonical time reversal of $\text{Me}(\hat{G}(\mathbf{N}))$ is a measure of maximal entropy of $\text{Me}(\hat{G}(\mathbf{N}))$, that assigns measure one to $Y^{+}(\text{Me}(\hat{G}(\mathbf{N})))$. By Lemma 2.1 there are no other measures of maximal entropy. \qed

For $H > 1$, and for data
\[ \mathbf{N} = (N_h)_{1 \leq h \leq H+1} \in \mathbb{N}^H \times (\mathbb{N} \setminus \{1\}) \]
set
\[ \Sigma(\mathbf{N}) = \sum_{1 \leq h \leq H+1} N_h, \quad \Pi(\mathbf{N}) = \prod_{1 \leq h \leq H+1} N_h. \]
We list the Perron eigenvalues of $A(\mathbf{N})$ for heights 2, 3, 5 and 7. The topological entropy of $\text{Me}(\overline{G}(\mathbf{N}))$ can then be found by Theorem (2.2) (see (1.1)). We follow in this case procedures for solving algebraic equations of degree less than or equal to four by radicals.

Consider the case $H(G) = 2$. Let
\[ \mathbf{N}_3 = (N_h)_{1 \leq h \leq 3} \in \mathbb{N}^2 \times (\mathbb{N} \setminus \{1\}). \]
The characteristic polynomial of $A(\mathbf{N}_3)$ is given by
\[ \text{Det}(z \mathbf{1} - A(\mathbf{N}_3)) = -z^3 - \Sigma(\mathbf{N}_3)z + 1 + \Pi(\mathbf{N}_3). \]
Set
\[ \Delta(\mathbf{N}_3) = (1 + \Pi(\mathbf{N}_3))^2 - \frac{4}{27} \Sigma(\mathbf{N}_3)^3. \]
In the case that $\Delta(\mathbf{N}_3) > 0$ the Perron eigenvalue of $\lambda(A(\mathbf{N}_3))$ is given by
\[ \lambda(A(\mathbf{N}_3)) = \sqrt[3]{1 + \Pi(\mathbf{N}_3) + \sqrt{D(\mathbf{N}_3)}} + \sqrt[3]{1 + \Pi(\mathbf{N}_3) - \sqrt{D(\mathbf{N}_3)}}. \]
In the case that $D(\mathbf{N}_3) < 0$, the Perron eigenvalue of $\lambda(A(\mathbf{N}_3))$ is given by
\[ \Phi(\mathbf{N}_3) = \arccos \left( \frac{3(1 + \Pi(\mathbf{N}_3))}{2\Sigma(\mathbf{N}_3)\sqrt{\Sigma(\mathbf{N}_3)}} \right), \quad 0 \leq \Phi < \frac{\pi}{2}. \]
\[ \lambda(A(\mathbf{N}_3)) = \cos \left( \frac{\Phi(\mathbf{N}_3)}{2} \right). \]
Consider the case $H(G) = 3$. Let
\[
\mathbf{N}_4 = (N_h)_{1 \leq h \leq 4} \in \mathbb{N}^4 \times (\mathbb{N} \setminus \{1\}).
\]
Set
\[
\Gamma_0(\mathbf{N}_4) = N_1N_3 + N_2N_4.
\]
The characteristic polynomial of $A(\mathbf{N}_4)$ is given by
\[
\det(zI - A(\mathbf{N}_4)) = z^4 - S(\mathbf{N}_4)z^2 - 1 + \Gamma_0(\mathbf{N}_4) - \Pi(\mathbf{N}_4).
\]
Set
\[
\Delta(\mathbf{N}_4) = S(\mathbf{N}_4)^2 + \Pi(\mathbf{N}_4) + 1 - \Gamma_0(\mathbf{N}_4).
\]
The Perron eigenvalue of $A(\mathbf{N}_4)$ is given by
\[
\lambda(A(\mathbf{N}_4)) = \sqrt{\frac{1}{2}(\Sigma(\mathbf{N}_4) + \sqrt{\Delta(\mathbf{N}_4)})}.
\]
Consider the case $H(G) = 6$. Let
\[
\mathbf{N}_6 = (N_h)_{1 \leq h \leq 6} \in \mathbb{N}^5 \times (\mathbb{N} \setminus \{1\}).
\]
Set
\[
\Gamma_0(\mathbf{N}_6) = N_1N_3N_5 + N_2N_4N_6,
\]
\[
\Gamma_1(\mathbf{N}_6) = N_1N_3 + N_1N_4 + N_1N_5 + N_2N_4 + N_2N_5 + N_2N_6 +
N_3N_5 + N_5N_4 + N_4N_6.
\]
The characteristic polynomial of $A(\mathbf{N}_6)$ is given by
\[
\det(zI - A(\mathbf{N}_6)) = z^6 - \Sigma(\mathbf{N}_6) + \Gamma_1(\mathbf{N}_6)z + \Pi(\mathbf{N}_6) - 1 - \Gamma_0(\mathbf{N}_6).
\]
Set
\[
p(\mathbf{N}_6) = \frac{1}{3}(3\Gamma_1(\mathbf{N}_6) - \Sigma(\mathbf{N}_6))^2,
\]
\[
q(\mathbf{N}_6) = \frac{1}{27}(-2\Sigma(\mathbf{N}_6)^3 + 9\Sigma(\mathbf{N}_6)\Gamma_1(\mathbf{N}_6) - 27\Pi(\mathbf{N}_6) - 27\Gamma_0(\mathbf{N}_6) - 27),
\]
and
\[
\Delta(\mathbf{N}_6) = q(\mathbf{N}_6)^2 + \frac{4}{27}p(\mathbf{N}_6)^3.
\]
In the case that $\Delta(\mathbf{N}_6) \geq 0$, the Perron eigenvalue of $A(\mathbf{N}_6)$ is given by
\[
\lambda(A(\mathbf{N}_6)) = \sqrt{\frac{1}{2}\Sigma(\mathbf{N}_6) + \sqrt{\frac{1}{2}(-q(\mathbf{N}_6) + \sqrt{\Delta(\mathbf{N}_6)})} + \sqrt{\frac{1}{2}(-q(\mathbf{N}_6) - \sqrt{\Delta(\mathbf{N}_6)})}}.
\]
In the case that $\Delta(\mathbf{N}_6) < 0$, the Perron eigenvalue of $A(\mathbf{N}_6)$ is given by
\[
\Phi(\mathbf{N}_6) = \arccos\left(\frac{3q(\mathbf{N}_6)}{2p(\mathbf{N}_6)}\sqrt{-\frac{4}{27}p(\mathbf{N}_6)}\right),
\]
\[
\lambda(A(\mathbf{N}_6)) = \sqrt{\frac{1}{2}\Sigma(\mathbf{N}_6) + 2\sqrt{-\frac{4}{27}p(\mathbf{N}_6)\cos(\Phi(\mathbf{N}_6))}}.
\]
We consider the case $H(G) = 7$. Let
\[
\mathbf{N}_8 = (N_h)_{1 \leq h \leq 8} \in \mathbb{N}^7 \times (\mathbb{N} \setminus \{1\}).
\]
Set
\[
\begin{align*}
\Gamma_0(N_8) &= N_1 N_3 N_5 N_7 + N_2 N_4 N_6 N_8, \\
\Gamma_2(N_8) &= N_1 N_3 N_5 + N_1 N_3 N_6 + N_1 N_5 N_7 + \\
&N_1 N_4 N_6 + N_1 N_4 N_7 + N_1 N_5 N_7 + \\
&N_2 N_4 N_6 + N_2 N_4 N_7 + N_2 N_5 N_8 + \\
&N_2 N_5 N_7 + N_2 N_5 N_8 + N_2 N_6 N_8 + \\
&N_3 N_5 N_7 + N_3 N_5 N_8 + N_3 N_6 N_8 + \\
&N_4 N_6 N_8 + N_3 N_6 N_8 + N_4 N_6 N_8, \\
\Gamma_4(N_8) &= N_1 N_3 + N_1 N_4 + N_1 N_5 + N_1 N_6 + N_1 N_7 + \\
&N_1 N_3 + N_1 N_4 + N_1 N_5 + N_1 N_6 + N_1 N_7 + \\
&N_2 N_4 + N_2 N_5 + N_2 N_6 + N_2 N_7 + N_2 N_8 + \\
&N_3 N_5 + N_3 N_6 + N_3 N_7 + N_3 N_8 + N_4 N_6 + \\
&N_4 N_7 + N_4 N_8 + N_5 N_7 + N_5 N_8 + N_6 N_8.
\end{align*}
\]

The characteristic polynomial of \( A(N_8) \) is given by
\[
\text{Det}(zI - A(N_8)) = z^8 - \Sigma(N_8) z^6 + \Gamma_4(N_8) z^4 - \Gamma_2(N_8) z^2 + \Gamma_0(N_8) + II(N_8) - 1.
\tag{2.8}
\]
We set
\[
\begin{align*}
a_3(N_8) &= -\Sigma(N_8), \quad a_2(N_8) = \Gamma_4(N_8), \\
a_1(N_8) &= -\Gamma_2(N_8), \quad a_0(N_8) = \Gamma_0(N_8) + II(N_8) - 1.
\end{align*}
\]
With the variable \( y = z^2 \), we obtain from (2.8) the quartic equation
\[
y^4 + a_3(N_8)y^2 + a_2(N_8)y + a_1(N_8)y + a_0(N_8) = 0.
\tag{2.9}
\]
With the coefficients
\[
\begin{align*}
p(N_8) &= \frac{1}{8}(8a_2(N_8) - 3a_3(N_8)^2), \\
q(N_8) &= \frac{1}{27}(a_3(N_8)^3 - 4a_3(N_8)a_2(N_8) + 8a_1(N_8)), \\
r(N_8) &= \frac{1}{256}(-4a_3(N_8)^4 - 64a_3(N_8)a_1(N_8) + 16a_3(N_8)^2a_2(N_8)) + a_0(N_8),
\end{align*}
\]
and have with the variable \( x = y - \frac{1}{4}a_3(N_8) \), the depressed version
\[
x^4 + p(N_8)x^2 + q(N_8)x + r(N_8) = 0.
\tag{2.10}
\]
Set
\[
A_0(N_8) = -q(N_8)^2, \quad A_1(N_8) = p(N_8)^2 - 4r(N_8), \quad A_2(N_8) = 2p(N_8).
\]
A Descartes resolvent (see, for instance, [11 Section 6.2], also see [14 Section 59]) of the depressed version (2.10) is given by
\[
u^3 + A_2(N_8)u^2 + A_1(N_8)u + A_0(N_8) = 0.
\tag{2.11}
\]
Set
\[
\begin{align*}
P(N_8) &= \frac{1}{4}(3A_1(N_8) - A_2(N_8)^2), \\
Q(N_8) &= \frac{1}{27}(2A_2(N_8)^3 - 9A_1(N_8)A_1(N_8)) + A_0(N_8).
\end{align*}
\]
The depressed version of the resolvent (2.11) is given by
\[
v = u - \frac{1}{3}A_2(N_8),
\]
\[ v^3 + P(N_8)v + Q(N_8) = 0. \]

Set
\[ \Delta(N_8) = Q(N_8)^2 + \frac{1}{2}P(N_8)^2. \]

In the case that \( \Delta(N_8) > 0 \),

set
\[ u(N_8) = \frac{1}{3}A_2(N_8) + \frac{1}{2}(Q(N_8) + \sqrt{\Delta(N_8)}) + \frac{1}{2}(Q(N_8) - \sqrt{\Delta(N_8)}), \]

and set
\[ \delta(N_8) = p(N_8)u(N_8)^4 + q(N_8)u(N_8)^3 - \frac{p(N_8)^2}{4} - 4r(N_8)u(N_8)^2 - 3q(N_8)^2; \]

The Perron eigenvalue of \( A(N_8) \) is given by
\[ (2.12) \]
\[ \lambda(A(N_8)) = \begin{cases} \sqrt{\frac{1}{2}(Q(N_8) + \sqrt{\Delta(N_8)}) + \frac{1}{2}(Q(N_8) - \sqrt{\Delta(N_8)})}, & \text{if } \delta(N_8) > 0, \\ \sqrt{\frac{1}{2}(Q(N_8) + \sqrt{\Delta(N_8)}) + \frac{1}{2}(Q(N_8) - \sqrt{\Delta(N_8)})}, & \text{if } \delta(N_8) < 0. \end{cases} \]

In the case that \( \Delta(N_8) > 0 \),

set
\[ u(N_8) = 2\sqrt{-\frac{P(N_8)}{3}} \cos\left(\frac{1}{3}\Phi(N_8)\right). \]

The Perron eigenvalue of \( A(N_8) \) is given by
\[ (2.13a) \]
\[ \Phi(N_8) = \arccos\left(-\frac{3Q(N_8)}{2P(N_8)\sqrt{-\frac{P(N_8)}{3}}}\right), \]

\[ (2.13b) \]
\[ \lambda(A(N_8)) = \begin{cases} \sqrt{\frac{1}{2}(Q(N_8) + \sqrt{\Delta(N_8)}) + \frac{1}{2}(Q(N_8) - \sqrt{\Delta(N_8)})}, & \text{if } q(N_8) > 0, \\ \sqrt{\frac{1}{2}(Q(N_8) + \sqrt{\Delta(N_8)}) + \frac{1}{2}(Q(N_8) - \sqrt{\Delta(N_8)})}, & \text{if } q(N_8) < 0. \end{cases} \]

**Proposition 2.3.** Let \( H > 1, L > 1 \), let
\[ N = (N_h)_{1 \leq h \leq H + 1} \in \mathbb{N}^H \times (\mathbb{N} \setminus \{1\}), \]

and let
\[ \tilde{N} = ((N_h))_{1 \leq h \leq H + 1} \in \tilde{\mathbb{N}} \times (\mathbb{N} \setminus \{1\}). \]

The edge shift of \( (G(N)) \) is the image of the edge shift of \( (G(\tilde{N})) \) under a 1-bi-resolving homomorphism.

**Proof.** We introduce notation by setting
\[ (\tilde{c}_h)^-_{1 \leq h \leq L(H + 1)} = ((N_h)_{1 \leq h \leq H + 1})_{1 \leq \ell \leq L} \]
\[ (\tilde{c}_h)^+_{1 \leq h \leq L(H + 1)} = ((N_h)_{1 \leq h \leq H + 1})_{1 \leq \ell \leq L} \]

and
\[ \Theta(c^+_{h+\ell(H+1)}(nh)) = c^+_{h}(nh), \]
\[ \Theta(c^-_{h+\ell(H+1)}(nh)) = c^+_{h}(nh), \quad 1 \leq h \leq H + 1, 0 \leq \ell < L. \]
From the one-block map $\Theta$ we have a homomorphism
$$\vartheta : ME(\tilde{G}(\tilde{N})) \to ME(G(N))$$
by
$$\vartheta(\tilde{x}) = (\Theta(\tilde{x}_i))_{i \in \mathbb{Z}}$$
One shows, that $\vartheta$ is 1-right-resolving. Let
$$\tilde{x} \in ME(G(N)),$$
and for $h \in [1, H + 1], \ell \in [0, L)$, let
$$t(x_0) = h, \quad t(\tilde{x}_0) = h - \ell(H + 1).$$
If
$$x_1 = c^+_{h+1}(n_{h+1})$$
then necessarily
$$\tilde{x}_1 = \tilde{c}_{h+1+\ell(H+1)} \mod (L(H+1)(n_{h+1}),$$
and if
$$x_1 = c^+_{h},$$
then necessarily
$$\tilde{x}_1 = \tilde{c}^{+}_{h+\ell(H+1)}.$$
With the directed graphs $\tilde{G}(N,M)$ and $\overline{G}(N,M)$ and the mapping $\varphi$ at hand one can define Borel sets $X(M \in \tilde{G}(N,M))$ and $M \in (\overline{G}(N,M))$ that are analogous to the sets defined in (2.1) and (2.2). Then one proceeds exactly as in the case $H(G) > 1$. We list the corresponding statements.

Lemma 3.1. For data $(N,M) \in \mathbb{N} \times (\mathbb{N} \setminus \{1\})$ the Borel dynamical system
\[
(X(M \in \tilde{G}(N,M), S_{M \in (\overline{G}(N,M))} \upharpoonright X(M \in (\overline{G}(N,M))))
\]
is Borel conjugate to the Borel dynamical system
\[
Y(M \in (\tilde{G}(N,M))), S_{Y(M \in (\overline{G}(N,M)))} \upharpoonright Y(M \in (\tilde{G}(N,M))).
\]

Theorem 3.2. For data $(N,M) \in \mathbb{N} \times (\mathbb{N} \setminus \{1\})$ the Markov-Dyck shift of $\hat{G}(N,M)$ has two measures of maximal entropy, and its topological entropy is given by
\[
h(M \in (\tilde{G}(N,M))) = \frac{1}{2}(\log(N + 1) + \log(M + 1)).
\]

Proposition 3.3. Let $H > 1, L > 1$, let
\[
N = (N_h)_{1 \leq h \leq H+1} \in \mathbb{N}^H \times (\mathbb{N} \setminus \{1\}),
\]
and let
\[
\tilde{N} = ((N_h)_{1 \leq h \leq H+1})_{0 \leq \ell < L}.
\]
The edge shift of $(\tilde{G}(N))$ is the image of the edge shift of $(\tilde{G}(\tilde{N}))$ under a 1-bi-resolving homomorphism.

Proposition 3.4. Let $H > 1$ and let
\[
N = (N_h)_{1 \leq h \leq H+1} \in \mathbb{N}^H \times (\mathbb{N} \setminus \{1\}).
\]
The topological entropy of the Markov Dyck shift of $\tilde{G}(N)$ is equal to the topological entropy of the edge shift of $\tilde{G}(N)$.

For Proposition 3.3 and Proposition 3.4 compare (2.5) and (3.1), and (2.3) and (1.2).

For $N > 1$ let $\hat{G}(N)$ be the one-vertex graph with $N$ loops, and let $\overline{G}(N)$ be the one-vertex graph with $N + 1$ loops. The graphs $\hat{G}(N)$ and $\overline{G}(N)$ can be viewed (for the case $H(G) = 0$) as analogous to the graphs $\hat{G}((N)_1 \leq h \leq H+1)$ and $\overline{G}((N)_1 \leq h \leq H+1)$ respectively (for the case $H(G) \in \mathbb{N}$). This is the case $H(G) = 0$, that was considered in [Kr1]. As the edge shift of the graph $\hat{G}(N)$, which is the full shift on $N + 1$ symbols, is the image of the edge shift of the graph $\hat{G}((N)_1 \leq h \leq H+1), H \in \mathbb{N}$, under a 1-bi-resolving homomorphism, it follows from (1.2) that
\[
h(M \in (\tilde{G}((N)_1 \leq h \leq H+1))) = \log(N + 1), \quad N > 1, H \in \mathbb{Z}_+.
\]

Comparing (3.2) and (2.3), and (3.2) and (2.5).

4. Zeta functions

As shown in [Ke], the zeta function of the Dyck shift is given by
\[
\zeta_{M \in (\tilde{G}((N)))}(z) = \frac{2(1 + \sqrt{1 - 4Nz^2})}{(1 - 2N + \sqrt{1 - 4Nz^2})^2}, \quad N > 1.
\]

We consider the case $H(G) \in \mathbb{N}$. Let there be given data
\[
N = (N_h)_{1 \leq h \leq H+1} \in \mathbb{N}^H \times (\mathbb{N} \setminus \{1\}), \quad H \in \mathbb{N}.
\]
We associate to each vertex $V \in \mathcal{V}(N)$ of the graph $\tilde{G}(N)$ the circular code $C(V)$ that contains the paths
\[
b = (b_i)_{1 \leq i \leq 2L}.
\]
in $\hat{G}(N)$ such that
\[ s(b_1) = t(b_{2I}) = V, \]
and
\[ \sum_{1 \leq j \leq J} \psi(b_j) > 0, \quad 1 < J < 2I, \]
and
\[ \sum_{1 \leq j \leq 2I} \psi(b_j) = 0. \]

As a consequence of rotational homogeneity one has that for $h \in [1, H]$ the generating functions of the codes $C(V), V \in \prod_{1 \leq h \leq h}[1, N_h]$, have a common value, that we denote by $g_h = g_h(N)$. For polynomials
\[ P_h(0), P_h(1), Q_h(0), Q_h(1), \quad 0 \leq h \leq H, \]
that are defined inductively by
\[ P_h(0)(z) = N_{h+1}z^2, \quad P_h(1)(z) = 0, \quad Q_h(0)(z) = 1, \quad Q_h(1)(z) = 1, \]
and
\[ P_h(0)(z) = N_{h+1}z^2Q_{h+1}^{(0)}, \quad P_h(1)(z) = N_{h+1}z^2Q_{h+1}^{(1)}, \]
\[ Q_h(0)(z) = Q_{h+1}^{(0)}(z) - P_h(0)(z), \quad Q_h(1)(z) = Q_{h+1}^{(1)}(z) - P_h(1)(z), \quad 0 \leq h < H, \]
it holds that
\[ g_h = \frac{P_h(0) - P_h(1) - g_{h+1}}{Q_h(0) - Q_h(1) - g_{h+1}}, \quad 0 \leq h < H, \]
\[ g_H = \frac{P_H(0) - P_H(1) - g_0}{Q_H(0) - Q_H(1) - g_0}. \]

It is seen, that the generating functions $g_h, 0 \leq h \leq H$, are given by periodic Jacobi continued fractions (see [Ka, Section 2]). It follows that
\[ g_0 = \frac{1}{2Q_1^{(0)}} \left( P_0(0) + Q_0(0) - \sqrt{(P_0(0) + Q_0(0))^2 - 4P_0(0)(Q_0(1))^2} \right). \]
Corresponding formulae for the generating functions $g_h, 0 < h \leq H$, can be obtained from (4.2) or by cyclically permuting the data.

**Proposition 4.1.** For data
\[ N = (N_b)_{1 \leq h \leq H+1} \in N^h \times (N \setminus \{1\}), \quad H \in N, \]
the zeta function of the Markov-Dyck shift of $\hat{G}(N)$ is given by
\[ \zeta_{MD(\hat{G}(N))} = \prod_{0 \leq h \leq H} (1 - g_h(N)) - \Pi(N)z^{2(H+1)} - \prod_{0 \leq h \leq H} [(1 - g_h(N))^2 - \Pi_{1 \leq h \leq h(N_h)}]. \]

**Proof.** The zeta function of the set of neutral periodic points of $MD(\hat{G}(N))$ is
\[ \prod_{0 \leq h \leq H} [(1 - g_h)^{-\Pi_{1 \leq h \leq h(N_h)}}]. \]
(see [KM4]). For the periodic points of $MD(\hat{G}(N))$ with negative multiplier we use the circular code $C^-$ of cycles
\[ b = (h_i)_{1 \leq i \leq 2I + H + 1}, \quad I \in \mathbb{Z}_+, \]
in $E^- \cup E^-$ such that
\[ s(b_1) = t(b_{2I + H + 1}) = V(0), \]
and such that
\[ \sum_{1 \leq i \leq J} \psi(b_j) \geq 0, \quad 1 \leq i \leq 2I + H + 1, \]
and
\[ \sum_{1 \leq i \leq 2I + H + 1} \psi(b_i) = H + 1. \]

The generating function of the code \( C^- \) is given by
\[(4.4) \quad g_C = z^{2(H+1)} \prod_{0 \leq h \leq H} N_{H+1}(1-g_h)^{-1}.\]

The canonical time reversal of \( M_\text{d}(\hat{G}(N)) \) carries periodic points with a negative multiplier into periodic points with a positive multiplier and vice versa. Apply (4.3) and (4.4).

**Corollary 4.2.** For data \((N,M) \in \mathbb{N} \times (\mathbb{N} \setminus \{1\})\) set
\[ F(N,M)(z) = \frac{1}{2} \left( 1 - (N - M)z^2 - \sqrt{(1 + (N - M)z^2)^2 - 4Nz^2} \right). \]

The zeta function of \( M_\text{d}(\hat{G}(N,M)) \) is given by
\[ \zeta_{M_\text{d}(\hat{G}(N,M))}(z) = \frac{F(N,M)F(M,N)}{F(N,M)^N(F(N,M)F(M,N) - NM^2)}. \]

**Proof.** The corollary follows from Proposition (4.1).

As is seen from (4.2), in the case of periodic data \(((N_h)_{1 \leq h \leq H+1})_{0 \leq \ell < L}\), the sequence of generating functions \((g_{h+\ell(h+1)})_{1 \leq h \leq H+1, 0 \leq \ell < L}\) has the same period \( L \) as the data.

**Proposition 4.3.** For data
\[ N = ((N_h)_{1 \leq h \leq H+1})_{0 \leq \ell < L} \in \mathbb{N}^L \times (\mathbb{N} \setminus \{1\}), \quad H \in \mathbb{N}, \]
the zeta function of the Markov-Dyck shift of \( \hat{G}(N) \) is given by
\[ \zeta_{M_\text{d}(\hat{G}(N))} = \left( \prod_{0 \leq h \leq H} (1 - g_h(N))^L - \Pi(N^L z^{2(H+1)})^{-2} \prod_{0 \leq h \leq H} [(1 - g_h(N))^2 - \Pi_{1 \leq h \leq H} (N_h)] \right). \]

Proposition 4.3 together with (4.1) gives
\[ \zeta_{M_\text{d}(\hat{G}(N)_{1 \leq n \leq H+1})}(z) = 2^{-\frac{N+1}{2}}(1 + \sqrt{1 - 4Nz^2})^{2H+2} \frac{N+1}{N-1}((1 + \sqrt{1 - 4Nz^2})^{H+1} - (2Nz)^{H+1})^{-2}, \quad H \in \mathbb{Z}_+. \]

**References**

[BBD1] M.-P. Béal, M. Blockelet, C. Dima, *Zeta functions of finite-type-Dyck shifts are \( N \)-algebraic*, DOI: 10.1109/ITA.2014.6804286

[BBD2] M.-P. Béal, M. Blockelet, C. Dima, *Sofic-Dyck shifts*, Theoretical Computer Science 609, (2016), 226 – 244

[BH1] M.-P. Béal, P. Heller, *Generalized Dyck shifts*, CSR 2017, LNCS 10304 (2017), 99 – 111

[BH2] M.-P. Béal, P. Heller *Shifts of k-nested sequences*, Theoretical Computer Science 658 (2017), 18 – 26

[BPR] J. Berstel, D. Perrin, Ch. Reutenauer, *Codes and automata*, Encyclopedia of Mathematics and its Applications 129, Cambridge University Press (2010)

[H] T. Hamachi, K. Isoue, *Embeddings of shifts of finite type into the Dyck shift*, Monatsh. Math. 145 (2005), 107 – 129
[HK] T. Hamachi, W. Krieger, Families of directed graphs and topological conjugacy of the associated Markov-Dyck shifts, in preparation

[I] K. Inoue, The zeta function, periodic points and entropies of the Motzkin shift, arXiv: math.DS/0602100 (2006).

[IK1] K. Inoue, W. Krieger, Subshifts from sofic shifts and Dyck shifts, zeta functions and topological entropy, arXiv: 1001.1839 [math.DS] (2010).

[IK2] K. Inoue, W. Krieger, Excluding words from Dyck shifts, arXiv: 1305.4720 [math.DS] (2013).

[Ka] Y. Kato, Mixed periodic Jacobi continued fractions, Nagoya Math. J. 104 (1986), 129 – 148

[Ke] G. Keller, Circular codes, loop counting, and zeta-functions, J. Combinatorial Theory 56 (1991) 75–83.

[Ki] B. P. Kitchens, Symbolic dynamics, Springer, Berlin, Heidelberg, New York (1998)

[Kr1] W. Krieger, On the uniqueness of the equilibrium state, Math. Systems Theory 8 (1974/75), 97 – 104

[Kr2] W. Krieger, On subshift presentations, Ergod. Th. & Dynam. Sys. 37 (2017), 1253 – 1290

[KM1] W. Krieger, K. Matsumoto, A lambda-graph system for the Dyck shift and its K-groups, Doc. Math. 8 (2003), 79 – 96.

[KM2] W. Krieger, K. Matsumoto, Zeta functions and topological entropy of the Markov-Dyck shifts, Münster J. Math. 4 (2011), 171–184

[KM3] W. Krieger, K. Matsumoto, A notion of synchronization of symbolic dynamics and a class of C*-algebras, Acta Appl. Math. 126 (2013), 263 – 275.

[KM4] W. Krieger, K. Matsumoto, Markov-Dyck shifts, neutral periodic points and topological conjugacy, [arXiv:1511.03025] [math.DS] (2018)

[L] M. V. Lawson, Inverse semigroups, World Scientific, Singapure, New Jersey, London and Hong Kong (1998).

[LM] D. Lind and B. Marcus, An introduction to symbolic dynamics and coding, Cambridge University Press, Cambridge (1995)

[Ma1] K. Matsumoto, K-theoretic invariants and conformal measures of the Dyck shifts, International J. of Mathematics 16 (2005), 213 – 248

[Ma2] K. Matsumoto, C*-algebras arising from Dyck systems of topological Markov chains, Math. Scand. 109 (2011), 31 – 54

[Ma3] K. Matsumoto, On the Markov-Dyck shifts of vertex type, Discrete and Continuous Dynam. Sys. 36 (2016), 403 – 422

[Ma4] K. Matsumoto, K-theory for the simple C*- algebra of the Fibonacci-Dyck shift, Acta Sci. Math. (Szeged) 83 (2017), 177 – 200

[Me] T. Meyerovitch, Tail invariant measures of the Dyck shift, Israel J. of Math. 163 (2008), 61 – 83

[NP] M. Nivat and J.-F. Perrot, Une généralisation du monoïde bicyclique, C. R. Acad. Sc. Paris, 271 (1970), 824–827.

[T] J.-P. Tignol, Galois’ Theory of Algebraic Equations, World Scientific, New Jersey, London, Singapure (2016)

[V] B. L. van der Waerden, Algebra I, Die Grundlagen der mathematischen Wissenschaften in Einzeldarstellungen, Vol. XXXIII, Springer, Berlin, Heidelberg, New York (1955)