Simple multiplicative proof nets with units

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Abstract. This paper presents a simple notion of proof net for multiplicative linear logic with units. Cut elimination is direct and strongly normalising, in contrast to previous approaches which resorted to moving jumps (attachments) of par units during normalisation. Composition in the resulting category of proof nets is simply path composition: all of the dynamics happens in \text{GoI}(\text{Setp}), the geometry-of-interaction construction applied to the category of sets and partial functions.

Keywords: multiplicative linear logic, units, proof nets, geometry of interaction.

AMS subject classification: 3B47 Substructural logics, 03F52 Linear logic.

1 Introduction

Here is a passage from Girard’s *Proof Nets: the Parallel Syntax for Proof Theory* [Gir96, §A.2]1:

There are two multiplicative neutrals, 1 and \perp, and two rules, the axiom \vdash 1 and the weakening rule: from \vdash \Gamma, deduce \vdash \Gamma, \perp. Both rules are handled by means of links with one conclusion and no premise; however, \perp-links are treated like 0-ary \textit{?-}links, i.e., they must be given a default jump. Sequentialisation is immediate.

At first sight, cut elimination is unproblematic: replace a cut between conclusions 1 and \perp of zero-ary links with . . . nothing. But we notice a new problem, namely that a cut formula A can be the default jump of a \perp-link \textit{L}, and we must therefore propose another jump for \textit{L}. Usually one of the premises of the link with conclusion \textit{A} works (or the jump of \textit{L′} if \textit{A} is the conclusion of a \perp-link) works. Worse, this new jump is by no means natural (if \textit{A} is \textit{B} \otimes \textit{C}, the new jump can either be \textit{B} or \textit{C}), which is quite unpleasant. As far as we know, the only solution consists in declaring that jumps are not part of the proof-net, but rather some control structure. It is then enough to show that at least one choice of default jump is possible. This is not a very elegant solution: we are indeed working with equivalence classes of proof nets and if we want to be rigorous we shall have to endlessly check that such and such operation does not depend on the choice of default jumps.

This paper presents a very simple solution: define a multiplicative proof net with units ( neutrals) as a function from negative to positive formula leaves, satisfying the usual correctness criterion [Gir87] [DR89]. Cut elimination on binary connectives is then trivial (as usual in the unit-free setting), and we have a direct strong normalisation by standard path composition: all of the dynamics happens in \text{GoI}(\text{Setp}), the geometry-of-interaction or feedback construction [Gir89] [JSV96] [Abr96] applied to the category of sets and partial functions.

The novelty here is not the directed edges between negative and positive leaves, an idea which goes back to the origins of linear logic [Gir87] and Kelly-MacLane graphs [KM71]. The key contribution is the simply defined, strongly normalising cut elimination, over \text{GoI}(\text{Setp}).

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1Similar remarks are in the earlier Linear Logic: A Survey [Gir93, §3.6].
**The nets.** Here is a simple example of a cut-free proof net on a four-formula sequent:

\[
\begin{align*}
\bot &\xleftarrow{\neg} P \otimes (P^\bot \otimes 1) \otimes \bot \otimes \bot \\
\end{align*}
\]

The graph of the function from negative to positive leaves is shown by the directed edges. Note that all four switchings are trees. This is easier to see if we show the parse trees:

\[
\begin{align*}
\bot &\xleftarrow{\neg} P \xrightarrow{\otimes} P^\bot \xrightarrow{\otimes} P^\bot \otimes 1 \otimes \bot \otimes \bot \\
\end{align*}
\]

As with the unit-free case [Gue99, MO00], correctness can be checked in linear time (see Section 6).

**GoI dynamics.** MLL formulas and proof nets form a category with a morphism \( A \to B \) a cut-free proof net on \( \vdash A^\bot, B \). For example,

\[
((1 \otimes 1) \otimes (P \otimes P^\bot)) \otimes (1 \otimes \bot)
\]

is a morphism from the upper formula to the lower formula. (We suppress the negation on the input/upper formula, flipping polarity, so tensors are switched in the input.) The underlying GoI(Sep) morphism is:

\[
\begin{align*}
\text{An object of } \text{GoI(Sep)} \text{ is a signed set } S, \text{ whose elements we shall call } \text{leaves}, \text{ and a morphism } S \to T \text{ is a partial function from negative leaves to positive leaves (polarity flipped on the input side). Composition is standard path composition, e.g.}
\end{align*}
\]

\[
\begin{align*}
\text{which provides composition (turbo cut elimination) in the category of proof nets, e.g.}
\end{align*}
\]
is the path composition of the previous GoI diagram. This provides a simple solution to the problems articulated by Girard above.

**Laminated GoI composition for MALL nets.** Section 7 continues the GoI theme, and shows how composition (turbo cut elimination) of MALL proof nets [HG03, HG05] can be viewed as occurring in a ‘laminated’ variant of GoI(Setp).

**Related work.** Proof nets with units are in [BCST96] and [LS04]. Neither solves the problems in Girard’s quote: each suffers from the need to move \( \bot \)-jumps during elimination, so one is lumbered once again with equivalence classes. The cut-free one-sided MLL proof nets in [BCST96] are the cut-free proof nets described in Girard’s quote in a circuit/wire notation, with an additional ordering on \( \bot \)-jumps: see Section 8.1. The paper [LS04] defines a cut-free proof net on a sequent \( \Gamma \) as a separate MLL formula \( \Theta \) whose leaves from left-to-right are a permutation of those of \( \Gamma \). The \( \bot \)-jumps and axiom links are thus enveloped in an additional syntactic layer \( \Theta \): see Section 8.2. The proof nets of [MO03] for intuitionistic multiplicative linear logic with units (based on essential nets [Lam94]) involve directed edges.

Work in progress quotients the nets presented in this paper by Trimble’s *empire rewiring* [Tri94], which permits a \( \bot \)-jump target to move so long as correctness is not broken, to construct free star-autonomous categories for full coherence (cf. [BCST96, KO99, MO03, LS04]).

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### 2 Notation

By MLL we mean multiplicative linear logic with units [Gir87]. Formulas are built from literals (propositional variables \( P, Q, \ldots \) and their duals \( P^\perp, Q^\perp, \ldots \)) and units/constants/neutrals 1 and \( \perp \) by the binary connectives tensor \( \otimes \) and par \( \& \). Negation \( (\neg)^{\perp} \) extends to arbitrary formulas with \( P^{\perp\perp} = P \) on propositional variables, \( \bot^\perp = \bot \), \( \perp^\perp = 1 \), and de Morgan duality \( (A \otimes B)^{\perp} = A^{\perp} \& B^{\perp} \) and \( (A \& B)^{\perp} = A^{\perp} \otimes B^{\perp} \). An **atom** is a literal or unit. We identify a formula with its parse tree: a tree labelled with atoms at the leaves and connectives at internal vertices. A **sequent** is a non-empty disjoint union of formulas. Thus a sequent is a particular kind of labelled forest. We write comma for disjoint union. Sequents are proved using the following rules:

\[
\frac{}{P, P^\perp} \quad \frac{}{\Gamma, A \quad A^\perp, \Delta} \quad \frac{}{\Gamma, \Delta} \quad \frac{\Gamma, A \quad B, \Delta}{\Gamma, A \otimes B, \Delta} \quad \frac{\Gamma, A \quad B}{\Gamma, A \& B, \Delta}
\]

\[
\frac{}{1} \quad \frac{}{\bot} \quad \frac{}{\Gamma, \perp} \quad \frac{}{\Gamma \quad \bot}
\]

Here, and throughout this document, \( P, Q, \ldots \) range over propositional variables, \( A, B, \ldots \) over formulas, and \( \Gamma, \Delta, \ldots \) over (possibly empty) disjoint unions of formulas. Without loss of generality we restrict the axiom rule to literals [Gir87]. The propositional variables \( P, Q, \ldots \) and
the unit 1 are \textit{positive}, and their duals \(P^⊥, Q^⊥, \ldots\) and \(\bot\) are \textit{negative}. A leaf of a formula is positive/negative according to its label. A \textbf{cut pair} \(A \perp A^⊥\) is a disjoint union of complementary formulas \(A\) and \(A^⊥\) together with an undirected edge, a \textit{cut}, between their roots. A \textbf{cut sequent} is a disjoint union of a sequent and zero or more cut pairs. A \textbf{switching} of a cut sequent is any subgraph obtained by deleting one of the two argument edges of each \(\overline{\gamma}\) (see [DR89]). By an \textbf{old proof net} we mean a proof net for MLL with units as in Girard’s quote in the Introduction; see [Dan90, Reg92, GSS92, Gir93, Gir96] for history and development. (An example of an old proof net is drawn in the next section.)

3 Proof nets

A \textbf{leaf function} on a cut sequent is a function from its negative leaves to its positive leaves. A \textbf{proof net} on a cut sequent \(\Gamma\) is a leaf function \(f\) on \(\Gamma\) satisfying:

- \textbf{Matching}. For any propositional variable \(P\), the restriction of \(f\) to \(P\)-labelled leaves is a bijection between the \(P\)-labelled leaves of \(\Gamma\) and the \(P^⊥\)-labelled leaves of \(\Gamma\).

- \textbf{Switching}. For any switching \(\Gamma'\) of \(\Gamma\), the undirected graph obtained by adding the edges of \(f\) to \(\Gamma'\) is a tree (acyclic and connected).

See page 2 for an example. This definition amounts to a restricted case of an old proof net: restrict \(\perp\)-jumps to target positive leaves and reject unit axiom links (use \(\perp \to 1\) jumps instead). In addition, we orient all axiom links from negative to positive. Stating this the other way round, the above definition relaxes to the old definition thus: (a) on \(\perp\)-labelled leaves allow \(f\) to target any vertex (equivalently subformula) of \(\Gamma\), not just a positive leaf, (b) distinguish between two kinds of edges from \(\perp\) to 1 (jump \textit{versus} axiom link), and (c) draw axiom links unoriented. Here is an example of an old proof net:

\[
\begin{align*}
\perp & \quad P & \quad P^⊥ & \quad 1 & \quad \perp \\
& \quad \otimes & & \quad \otimes & \quad \gamma \\
\end{align*}
\]

which in original proof net notation is:

\[
\begin{align*}
\perp & \quad P & \quad P^⊥ & \quad 1 & \quad \perp \\
& \quad \otimes & & \quad \otimes & \quad \gamma \\
\end{align*}
\]

Axiom links are shown as three-segment straight edges, and jumps from \(\perp\)-links \(\overline{\perp}\) are shown curved and directed.

Translation from a proof to a proof net is as usual, with a \(\perp\)-jump added at each \(\perp\)-rule, but now with choice of target restricted to positive atoms only. Note that well-definedness relies on the observation that every provable MLL sequent contains a positive atom. The translation becomes deterministic upon marking a positive leaf in the conclusion of every \(\perp\)-rule. For example, each of the following marked proofs translates (deterministically) into the proof net on page 2.
Marks are shown by underlining; when a sequent has just one positive atom, we leave the mark implicit. (Downward tracking of $\bot$’s is vertical, except through the tensor rule.)

**THEOREM 1 (SEQUENTIALISATION)** Every proof net is a translation of a proof.

This is simply a restriction of the theorem for old proof nets. Correctness is verifiable in linear time (a simple corollary of the unit-free case [Gue99, MO00]): see Section 6.

4 Cut elimination

Let $f$ be a proof net on the cut sequent $\Gamma, A \vdash A^\bot$. The result $f'$ of eliminating the cut in $A \vdash A^\bot$ is:

- **Atom.** Suppose $A$ is an atom. Without loss of generality, $A$ is positive. Delete $A \vdash A^\bot$ and reset any $f$-edge to $A$ to target $f(A^\bot)$ instead.

- **Compound.** Suppose $A$ is a compound formula. Without loss of generality $A = B \otimes C$ and $A^\bot = B^\bot \otimes C^\bot$. Replace $A \vdash A^\bot$ by $B \vdash B^\bot, C \vdash C^\bot$. The leaves, and $f$, remain unchanged.

Schematically:

**THEOREM 2** Cut elimination is well-defined: eliminating a cut from a proof net yields a proof net.

**Proof.** The atomic case is trivial, since switchings and cycles correspond before and after the elimination. The compound case is the same as the usual unit-free elimination [Gir87, DR89, Gir93].
**Proposition 1** Cut elimination is locally confluent.

*Proof.* The only non-trivial case is a pair of atomic eliminations. This case is clear from the following schematic involving two interacting atomic cut redexes $A \perp A$ and $B \perp B$.

![Diagram](image)

**Theorem 3** Cut elimination is strongly normalising.

*Proof.* It is locally confluent, and eliminating a cut reduces the number of vertices of the cut sequent.

**Turbo cut elimination.** As with standard unit-free MLL proof nets, normalisation can be completed in a single step. For $l$ the $i$th leaf of a formula $A$ in a cut pair $A \perp A$, let $l \perp$ denote the $i$th leaf of $A \perp$. The normal form of a cut sequent $\Gamma$ is the sequent $|\Gamma|$ obtained by deleting all cut pairs. Given a proof net $f$ on $\Gamma$, its normal form $|f|$ is the proof net on $|\Gamma|$ obtained by replacing every set of edges $(l_0, l_1), (l_1 \perp, l_2), (l_2 \perp, l_3), \ldots, (l_{n-1} \perp, l_n)$ in $f$ in which only $l_0$ and $l_n$ occur in $|\Gamma|$ by the single edge $(l_0, l_n)$. By a simple induction on the number of vertices of cut sequents, $|f|$ is precisely the normal form of $f$ under one-step cut elimination. (In particular, this implies $|f|$ is indeed a proof net.)

**5 GoI dynamics**

Cut elimination yields a category $\mathcal{N}$ of MLL proof nets. Objects are MLL formulas, and a morphism $A \to B$ is a proof net on the (cut-free) sequent $A \perp, B$ (cf. [HG03], for example). The composite of $f : A \to B$ and $g : B \to C$ is the normal form of the proof net $f \cup g$ on $A \perp, B \perp, C$. Composition is associative because cut elimination is strongly normalising. The identity $A \to A$, a leaf function on $A \perp, A$, has an edge between the $i$th leaf of $A \perp$ and the $i$th leaf of $A$ for each $i$, oriented from negative to positive.
We generally draw \( f : A \rightarrow B \) with \( A \) above \( B \), and suppress the negation on \( A \). For example, the identity \( \bot \otimes P \rightarrow \bot \otimes P \)

\[
\begin{array}{c}
\begin{array}{c}
1 \otimes P^\perp \\
\otimes P
\end{array}
\end{array}
\]

becomes

\[
\begin{array}{c}
\begin{array}{c}
\bot \otimes P
\end{array}
\end{array}
\]

Similarly, a composition such as

\[
\begin{array}{c}
\begin{array}{c}
(P^\perp \otimes Q^\perp) \otimes Q^\perp \\
\bot \otimes P
\end{array}
\end{array}
\]

involving the aforementioned identity \( \bot \otimes P \rightarrow \bot \otimes P \) becomes:

\[
\begin{array}{c}
\begin{array}{c}
(P \otimes Q^\perp) \otimes Q
\end{array}
\end{array}
\]

A more interesting example of composition is on page 3 of the Introduction.

**Underlying GoI category.** The category \( \text{GoI}(\text{Setp}) \), the result of applying the geometry-of-interaction or feedback construction \( \text{GoI} \) \cite{Gir89,JSV96,Abr96} to the category \( \text{Setp} \) of sets and partial functions, has the following structure. An object is a signed set \( S \), whose elements we shall call *leaves* (each signed either *positive* or *negative*), and a morphism \( S \rightarrow T \) is a *leaf function*: a partial function from \( S^+ + T^- \) to \( S^- + T^+ \), where \((-)^+\) (resp. \((-)^-)\) restricts to positive (resp. negative) leaves. For example,

\[
\begin{array}{c}
\begin{array}{c}
\bullet
\end{array}
\end{array}
\]

is a (total) morphism from the upper signed set (4 positive \( \bullet \) and 2 negative \( \circ \) leaves) to the lower one (2 positive and 3 negative leaves). Composition is simply (finite) path composition: for an example, see page 2 of the Introduction. Turbo cut elimination is the very same path composition, hence there is a forgetful (faithful) functor from the category \( \mathcal{N} \) of MLL proof nets to \( \text{GoI}(\text{Setp}) \), extracting the leaves from a formula. Again, see the Introduction for examples.
6 Linear complexity of proof net correctness

**Theorem 4 (Linear Complexity)** Verification of proof net correctness is linear in the number of leaves: if $f$ is a leaf function on a cut sequent $\Gamma$, then determining whether $f$ is a proof net can be done in linear time in the number of leaves of $\Gamma$.

**Proof.** Verifying the Matching condition is clearly linear time. The Switching condition is a simple corollary of the unit-free theorem [Gue99, MO00]: the function $f$ determines a standard unit-free proof structure $\hat{f}$ on $\hat{\Gamma}$, as follows. First, replace every cut pair $A \perp A$ by $A \otimes A$. We may assume every positive leaf has an incoming $f$-edge: every literal does, by Matching; if the $1$ of a subformula $A \otimes 1$ doesn’t, replace $A \otimes 1$ by $A$; if the $1$ of $A \forall 1$ doesn’t, Switching fails. Re-label each positive literal to $1$ and each negative literal to $\perp$. Replace each $1$ by $1_n$ where $n \geq 1$ is the number of $f$-edges targeting the $1$, and $1^n$ denotes the tensor product of $n$ copies of $1$ (bracketed arbitrarily); re-target the $n$ edges to the old $1$ to each target a distinct new $1$ of $1^n$. Finally, view the symbols $\perp$ and $1$ as complementary literals, so we have formed a standard proof structure $\hat{f}$ on a cut-free, unit-free MLL sequent $\hat{\Gamma}$. To clarify, here is $\hat{f}$ for $f$ the proof net on page 2:

$$\perp \quad (1 \otimes 1 \otimes 1) \quad \exists (\perp \otimes (1 \otimes 1)) \quad \perp \quad \perp$$

By construction the original $f$ on $\Gamma$ is correct iff $\hat{f}$ on $\hat{\Gamma}$ is correct in the usual unit-free sense. The construction of $\hat{f}$ is linear time in the number of leaves. □

**Corollary 1** The theorem above extends to old proof nets (i.e., when $f$ is a function from negative leaves to vertices of $\Gamma$, optionally with a differentiation between axiom links $\perp 1$ and jumps $\perp 1$).

**Proof.** First, if differentiating, replace every axiom link $\perp 1$ by a jump $\perp 1$. Rewrite every compound subformula or negative leaf $A$ targeted by a $\perp$-jump to $A \otimes 1$, and shift any $\perp$-jumps which targeted $A$ to target the new $1$ instead. This yields a function $\tilde{f}$ from negative leaves to positive leaves which is correct iff $f$ is correct; apply the above theorem to $\tilde{f}$. To clarify, here is $\tilde{f}$ for the old proof net $f$ drawn on page 4:

```
    \perp
   / \  /
  /   \ /  \\
 /     /   \\
\   / \   \\
 \ \ \ \ \
  \ / \ / \\
 /   /   \\
/   /   \\
/  / \  \\
/ /   /  \\
\ \ \ \ \
  \ / \ / \\
   \ / \ /
    \perp
```

The construction $f \mapsto \tilde{f}$ is linear time in the number of leaves. □
7 Laminated GoI composition for MALL nets

Continuing the GoI theme, we observe that composition (turbo cut elimination) of MALL proof nets \cite{HG03, HG05} can be viewed as occurring in a ‘laminated’ variant of GoI(\text{Setp}).\footnote{Since the MALL proof nets are unit-free, we could work with the category of sets and partial injections instead of \text{Setp}.}

Leaf functions \( f : S \to T \) and \( g : T \to U \) synchronise in \( T \), denoted \( f \bowtie_T g \) or simply \( f \bowtie g \), if

\[
\begin{align*}
(\text{im } f) \cap T &= (\text{dom } g) \cap T \\
(\text{dom } f) \cap T &= (\text{im } g) \cap T
\end{align*}
\]

In other words, for every (signed) leaf \( t \in T \) there is an edge into \( t \) iff there is an edge out of \( t \) (in \( f \cup g \), viewed as a directed graph on \( S + T + U \)).

Let \( \text{fSetp} \) be the category of finite sets and partial functions (a full subcategory of \( \text{Setp} \)). Thus \( \text{GoI}(\text{fSetp}) \) is the full subcategory of \( \text{GoI}(\text{Setp}) \) with finite objects (signed sets). Finiteness will be critical for associativity of composition in the construction below.

Define the category \( \text{Lam}(\text{GoI}(\text{fSetp})) \) as follows. Objects are inherited from \( \text{GoI}(\text{fSetp}) \) (finite signed sets), and a morphism \( S \to T \) is a set \( l \) of leaf functions \( S \to T \), which we call a laminated leaf function from \( S \) to \( T \). Given laminated leaf functions \( l : S \to T \) and \( m : T \to U \), their composite \( l;m : S \to U \) is by synchronised composition:

\[
l;m = \{ f;g \text{ such that } f \in l, \ g \in m, \text{ and } f \bowtie g \},
\]

where \( f;g : S \to U \) is the (sequential) composite of leaf functions \( f : S \to T \) and \( g : T \to U \) in \( \text{GoI}(\text{fSetp}) \). Thus \( l;m \) collects the pairwise composition of all synchronising leaf functions \( f \in l \) and \( g \in m \). The identity laminated leaf function \( \text{id}_S : S \to S \) comprises every leaf function \( S \to S \) contained in the identity leaf function \( S \to S \) in \( \text{GoI}(\text{fSetp}) \). The identity law \( \text{id}_S ; l = l = l ; \text{id}_S \) holds since every leaf function \( f \in l \) synchronises with a unique leaf function in \( \text{id}_S \). Composition is associative since given leaf functions \( f : S \to T \) and \( g : T \to U \),

\[
f \bowtie g \implies (\text{dom } f ; g ) \cap S = (\text{dom } f ) \cap S \text{ and } (\text{im } f ; g ) \cap U = (\text{im } g ) \cap U.
\]

This stable domain-image property depends critically on finiteness: if infinite signed sets are available, the property fails, and so does associativity of composition. Let \( f : S \to T, g : T \to U \) and \( h : U \to V \) be the following composable leaf functions

\[
\begin{array}{c}
S \\
T \\
U \\
V
\end{array}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet
\end{array}
\begin{array}{c}
f \\\
g \\\
h
\end{array}
\begin{array}{c}
\cdots
\end{array}
\]
so $S,T,V$ are singletons, $U$ is countably infinite, and $g$ and $h$ continue as suggested by the ellipsis. Note that $f \circ g \sim h$. The stable domain property fails for $g$ and $h$: $(\text{dom } g) \cap T = T$ but $(\text{dom } g; h) \cap T$ is empty (since $g; h$ is empty). Thus

$$f \neq g; h \quad \text{but} \quad f; g \sim h$$

so $\{f\}; (\{g\}; \{h\}) \neq (\{f\}; \{g\}); \{h\}$ for the corresponding singleton laminated leaf functions. (Note, however, that $f; (g; h) = (f; g); h$, so lamination is important.)

The canonical embedding $\text{GoI}(\text{fSetp}) \hookrightarrow \text{Lam}(\text{GoI}(\text{fSetp}))$ is more interesting than one might have anticipated. It is of course the identity on objects (finite signed sets), but not the ‘identity’ on leaf functions in the naive sense of mapping a leaf function $f : S \to T$ in $\text{GoI}(\text{fSetp})$ to the singleton laminated leaf function $\{f\} : S \to T$ in $\text{Lam}(\text{GoI}(\text{fSetp}))$. Rather, the image of $f$ comprises every leaf function $S \to T$ which is contained in $f$. Thus the embedding takes the downward closure of $f$. Observe how this preserves identities.

**Forgetful functor from MALL nets.** There is a forgetful functor from the category of MALL proof nets $[\text{HG05}, \S 5.2]$ to $\text{Lam}(\text{GoI}(\text{fSetp}))$, extracting the (signed) leaves of a MALL formula: the synchronisation property $\circ$ is the matching property $[\text{HG05}, \S 5.11]$, and synchronised composition corresponds to the definition of the normal form of a set of MALL linkings, by turbo cut elimination $[\text{HG05}, \S 5.11]$ (the path composition reduction of $[\text{HG05}, \S 5.11]$ being composition in $\text{GoI}(\text{fSetp})$).

One would hope to be able to find relationships between MALL proof nets, $\text{Lam}(\text{GoI}(\text{fSetp}))$ and the geometry of interaction for additives as in $[\text{Gir95, AJ94}]$. Since MLL units are the focus of the present paper, this is best left for another occasion.

**The lamination construction.** The construction of $\text{Lam}(\text{GoI}(\text{fSetp}))$ from $\text{GoI}(\text{fSetp})$ abstracts to a general construction $\text{Lam}(\mathcal{C})$ on a category $\mathcal{C}$ equipped with a suitable synchronisation relation $\circ$ between homsets: enumerate the properties of $\circ$ used above to ensure that $\text{Lam}(\mathcal{C})$ is a category. However, whether or not the lamination construction leads anywhere interesting remains to be seen. It may be useful in constructing models for the additives, perhaps in concert with double glueing $[\text{Tan97, HS03}]$.

### 8 Previous approaches

Girard’s passage quoted on the first page of the Introduction gives a convenient summary of old proof nets. Normalisation is hampered by having to move targets of $\bot$-jumps.

Proof nets for MLL with units are given in $[\text{BCST96}]$ and $[\text{LS04}]$. Neither solves the problems in Girard’s quote: each suffers from the need to move $\bot$-jumps during elimination, so one is lumbered once again with equivalence classes.

#### 8.1 Circuit nets

The cut-free one-sided MLL proof nets in $[\text{BCST96}]$ are $^3$ cut-free old proof nets (as described in Girard’s quote, page 10 in circuit/wire notation, with an additional ordering on $\bot$-jumps. For example, the old proof net on page 10 is drawn thus:

---

$^3$See the introduction to Section 2 of $[\text{BCST96}]$. 

---
Links are drawn as circular nodes, formulas are drawn as (labelled) wires, and \( \bot \)-jumps are drawn dotted. By an \textit{MLL proof net} in the [BCST96] setting we mean the special case when the base is a set of propositional variables, and \((-)^\bot\) is restricted to propositional variables (as usual with MLL formulas). The primary net definition in [BCST96] is two-sided; a one-sided net is simply a two-sided net with the tensor unit 1 on the input side (see the paragraph following Corollary 5.3 of [BCST96]). In drawing the one-sided net above, we omitted this input unit and its jump. The minor difference with old proof nets is that when multiple \( \bot \)-jumps target the same wire, they are ordered along the wire; in an old proof net there is no such ordering on \( \bot \)-jumps targeting the same subformula.

The problem with normalisation (see Girard’s passage on page 11) remains. For example, if we cut against the \( P \gamma( P^\bot \otimes 1) \) wire above, we do not have a cut redex: first we must re-wire the incoming \( \bot \)-jump to elsewhere in the empire of the \( \bot \); we’re once again resorting to equivalence classes for normalisation.

A key feature of the approach in [BCST96] is the modularity over negation and planarity. Circuit nets modulo equivalence describe the free linearly distributive and star-autonomous categories over a polygraph (e.g., over a category), yielding full coherence. For an internal language presentation of free star-autonomous categories, with full coherence, see [KO99] (again modulo an equivalence/congruence).

### 8.2 Syntactic nets

The paper [LS04] defines a proof net on a cut sequent \( \Gamma \) as a separate MLL formula \( \Theta \) whose leaves from left-to-right are a permutation of those of \( \Gamma \). The formula \( \Theta \) is shown upside down above the sequent, and the permutation is represented by permitting argument edges to cross in the upper half. The \( \bot \)-attachments and axiom links are thus enveloped in an additional syntactic layer \( \Theta \), with \( \bot \)-attachments as \( \sim_{\bot} \) and axiom links as \( A^\otimes_{A^\bot} \). Here is an example of a proof net on the three-formula sequent \( \bot, 1 \otimes P, \bot \otimes ((P^\bot \otimes P^\bot) \gamma P) \), essentially Figure 2 of [LS04]:

![Syntactic net example](image-url)
As with \[\text{BCST96}\] nets, the problem with normalisation (see Girard’s passage on page II) remains. For example⁴, if \(\Gamma\) is the cut sequent \(P\perp, P P\perp, Q, Q\perp, \perp\) and \(\Theta\) is the proof net given by the MLL formula \((P\perp \otimes P)\otimes (((P\perp \otimes P)\otimes (Q \otimes Q\perp)) \otimes \perp)\) (with identity permutation on leaves) then the cut cannot be reduced immediately. First one must apply invertible linear distributivity / commutativity / associativity to \(\Theta\), subject to the constraint of not breaking the correctness criterion (i.e., a form of empire-rewiring [Tri94] [BCST96]). Thus one is again resorting to equivalence classes for normalisation (see Theorem 4.3 of [LS04]). Syntactic nets modulo equivalence describe the free star-autonomous category with strict double involution \(A = A^\perp\perp\) generated by a set.

References

[Abr96] S. Abramsky. Retracing some paths in process algebra. In Proc. CONCUR ’96, pages 1–17, 1996.

[AJ94] S. Abramsky and R. Jagadeesan. New foundations for the geometry of interaction. Information and Computation, 111(1):53–119, 1994. Extended version of the paper published in the Proceedings of LICS’92.

[BCST96] R. F. Blute, J. R. B. Cockett, R. A. G. Seely, and T. H. Trimble. Natural deduction and coherence for weakly distributive categories. Journal of Pure and Applied Algebra, 113:229–296, 1996.

[Dan90] V. Danos. La logique linéaire appliquée à l’étude de divers processus de normalisation et principalement du lambda calcul. PhD thesis, Univ. de Paris, 1990.

[DR89] V. Danos and L. Regnier. The structure of multiplicatives. Archive for Mathematical Logic, 28:181–203, 1989.

[Gir87] J.-Y. Girard. Linear logic. Theoretical Computer Science, 50:1–102, 1987.

[Gir89] J.-Y. Girard. Towards a geometry of interaction. In Categories in Computer Science and Logic, volume 92 of Contemporary Mathematics, pages 69–108, 1989. Proc. of June 1987 meeting in Boulder, Colorado.

[Gir93] J.-Y. Girard. Linear logic: A survey. Proceedings of the International Summer School of Marktoberdorf, NATO Advanced Science Institutes, series F94, pages 63–112, 1993. Also in P. De Groote editor, The Curry-Howard Isomorphism, 193–255, Département de Philosophie, Université Catholique de Louvain, Cahiers du Centre de Logique 8, Academia Press.

[Gir95] J.-Y. Girard. Geometry of interaction III: Accommodating the additives. In Advances in Linear Logic, volume 222 of London Math. Soc. LNS, pages 329–389. Cambridge University Press, 1995.

[Gir96] J.-Y. Girard. Proof-nets: the parallel syntax for proof theory. In Logic and Algebra, volume 180 of Lecture Notes In Pure and Applied Mathematics. Marcel Dekker, New York, 1996.

[GSS92] J.-Y. Girard, A. Scedrov, and P. J. Scott. Bounded linear logic: A modular approach to polynomial time computability. Theor. Comp. Sci., 97:1–66, 1992.

[Gue99] S. Guerrini. Correctness of multiplicative proof nets is linear. In Proc. Logic in Computer Science ’99, 1999.

[HG03] D. J. D. Hughes and R. J. van Glabbeek. Proof nets for unit-free multiplicative additive linear logic (extended abstract). Proc. Logic in Computer Science ’03, 2003.

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⁴This example is drawn just after Lemma 4.2 in [LS04], with different literal labels.
[HG05] D. J. D. Hughes and R. J. van Glabbeek. Proof nets for unit-free multiplicative-additive linear logic. *ACM Transactions of Computational Logic*, 2005. To appear. Invited submission November 2003, revised January 2005. Full version of the LICS’03 paper [HG03]. Available at http://boole.stanford.edu/~dominic/papers.

[HS03] J. M. E. Hyland and A. Schalk. Glueing and orthogonality for models of linear logic. *Theoretical Computer Science*, 294:183–231, 2003.

[JSV96] A. Joyal, R. Street, and D. Verity. Traced monoidal categories. *Math. Proc. Camb. Phil. Soc.*, 119:447–468, 1996.

[KM71] G. M. Kelly and S. Mac Lane. Coherence in closed categories. *J. Pure and Applied Algebra*, 1:97–140, 1971.

[KO99] T. W. Koh and C.-H. L. Ong. Explicit substitution internal languages for autonomous and *-autonomous categories (preliminary version). In Proceedings of the 8th Conference on Category Theory and Computer Science, Edinburgh, September 1999, Electronic Notes in Theoretical Computer Science, volume 29, 1999.

[Lam94] F. Lamarche. Proof nets for intuitionistic linear logic I: Essential nets. Unpublished note, 1994.

[LS04] F. Lamarche and L. Straßburger. On proof nets for multiplicative linear logic with units. In *Proc. CSL’04*, volume 3210 of *Lecture Notes in Computer Science*, pages 145–159, 2004.

[MO00] A. S. Murawski and C.-H. L. Ong. Dominator trees and fast verification of proof nets. In *Proc. Logic in Computer Science ’00*, 2000.

[MO03] A. S. Murawski and C.-H. L. Ong. Exhausting strategies, joker games and full completeness for IMLL with unit. *Theor. Comp. Sci.*, 294:269–305, 2003.

[Reg92] L. Regnier. *Lambda-Calcul et Réseaux*. PhD thesis, Univ. Paris VII, 1992.

[Tan97] A. Tan. *Full Completeness Results for Models of Linear Logic*. PhD thesis, Cambridge, Oct 1997.

[Tri94] T. H. Trimble. *Linear logic, bimodules, and full coherence for autonomous categories*. PhD thesis, Rutgers University, 1994.