The abstract Birman–Schwinger principle and spectral stability

Marcel Hansmann\textsuperscript{a} and David Krejčířík\textsuperscript{b}

\textsuperscript{a) Fakultät für Mathematik, Technische Universität Chemnitz, Reichenhainer Strasse 41, 09107
Chemnitz, Germany; marcel.hansmann@mathematik.tu-chemnitz.de.}

\textsuperscript{b) Department of Mathematics, Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University in Prague, Trojanova 13, 12000 Prague 2, Czechia; david.krejcirik@fjfi.cvut.cz.}

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Abstract

We discuss abstract Birman–Schwinger principles to study spectra of self-adjoint operators subject to small non-self-adjoint perturbations in a factorised form. In particular, we extend and in part improve a classical result by Kato which ensures spectral stability. As an application, we revisit known results for Schrödinger and Dirac operators in Euclidean spaces and establish new results for Schrödinger operators in three-dimensional hyperbolic space.

1 Introduction

1.1 Motivations

The present paper has three purposes. The first is to develop an abstract version of the so-called Birman–Schwinger principle, which is a well known tool from the theory of Schrödinger operators. It is customarily used to transfer a differential equation to an integral equation and has been employed in many circumstances over the last half century since the pioneering works of Birman \cite{birman} and Schwinger \cite{schwinger}. In recent years, the method has been revived in the context of spectral theory of non-self-adjoint Schrödinger and Dirac operators with complex potentials as a replacement of unavailable variational techniques (see, e.g., \cite{davies, derezinski, han, hansmann, hansmann2, hansmann3, hansmann4, hansmann5} to quote just a couple of most recent works). While its usefulness is very robust, the method is usually applied to concrete problems \textit{ad hoc} and not always rigorously. Here we suggest an abstract machinery directly applicable to concrete problems. Abstract versions of the Birman–Schwinger principle have been discussed before (see Remark \ref{rem1} below), but this was usually restricted to (discrete) eigenvalues. In contrast, we also cover eigenvalues embedded in the essential spectrum as well as residual, continuous and essential spectra.

Our second goal is to use our abstract machinery to prove spectral stability given uniform bounds on the Birman–Schwinger operator. In particular, we will be able to derive such results without any smoothness assumptions (in the sense of Kato \cite{kato}) and will thus be able to extend and improve upon Kato’s classical result (Theorem \ref{thm1} below) on this topic.

Our third and final goal is to show the applicability of the abstract Birman–Schwinger principles. This will be illustrated via some known spectral enclosures for Schrödinger and Dirac operators in Euclidean spaces, which we recover, and via a completely new result, namely the spectral stability for Schrödinger operators in three-dimensional hyperbolic space.

1.2 Assumptions and notations

Throughout this paper $\mathcal{H}$ and $\mathcal{H}'$ denote complex separable Hilbert spaces and $\mathcal{B}(\mathcal{H},\mathcal{H}')$ denotes the space of bounded linear operators from $\mathcal{H}$ to $\mathcal{H}'$. As usual, we set $\mathcal{B}(\mathcal{H}) := \mathcal{B}(\mathcal{H},\mathcal{H})$, etc. We denote the inner product (which is linear in the second component) and norm in $\mathcal{H}$ as well as in $\mathcal{H}'$ by the same symbols, namely $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. The latter is also used to denote the operator norms in $\mathcal{B}(\mathcal{H},\mathcal{H}')$, $\mathcal{B}(\mathcal{H})$ and so on. The particular meaning of each symbol should always be clear from the
context. We denote the domain, kernel, range and adjoint of an operator $A$ from $\mathcal{H} \to \mathcal{H}'$ by $\mathcal{D}(A)$, $\mathcal{N}(A)$, $\mathcal{R}(A)$ and $A^*$, respectively. Recall that the spectrum $\sigma(H)$ of any closed operator $H$ in $\mathcal{H}$ is the set of those complex numbers $\lambda$ for which $H - \lambda : \mathcal{D}(H) \to \mathcal{H}$ is not bijective. The resolvent set is the complement $\rho(H) := \mathbb{C} \setminus \sigma(H)$. The point spectrum $\sigma_p(H)$ of $H$ is the set of eigenvalues of $H$ (i.e., the operator $H - \lambda$ is not injective). For the surjectivity, one says that $\lambda \in \sigma(H)$ belongs to the continuous spectrum $\sigma_c(H)$ (respectively, residual spectrum $\sigma_r(H)$) of $H$ if $\lambda \notin \sigma_p(H_V)$ and the closure of the range of $H - \lambda$ equals $\mathcal{H}$ (respectively, the closure is a proper subset of $\mathcal{H}$). Finally, we say that $\lambda \in \mathbb{C}$ belongs to the essential spectrum $\sigma_e(H)$ of $H$ if $\lambda$ is an eigenvalue of infinite geometric multiplicity or the range of $H - \lambda$ is not closed.

Our standing hypotheses are as follows.

**Assumption 1.** $H_0$ is a self-adjoint operator in $\mathcal{H}$ and $|H_0| := (H_0^2)^{1/2}$. Moreover, $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}'$ and $B : \mathcal{D}(B) \subset \mathcal{H} \to \mathcal{H}'$ are linear operators such that $\mathcal{D}(|H_0|^{1/2}) \subset \mathcal{D}(A) \cap \mathcal{D}(B)$. We assume that for some (hence all) $b > 0$

$$A(|H_0| + b)^{-1/2} \in \mathcal{B}(\mathcal{H}, \mathcal{H}'), \quad B(|H_0| + b)^{-1/2} \in \mathcal{B}(\mathcal{H}, \mathcal{H}'). \tag{1.1}$$

Next, we set $G_0 := (|H_0| + 1)$ and introduce the Birman–Schwinger operator

$$K(\lambda) := [AG_0^{-1/2}] [G_0(H_0 - \lambda)^{-1}] [BG_0^{-1/2}]^* \in \mathcal{B}(\mathcal{H}'), \quad \lambda \in \rho(H_0). \tag{1.2}$$

Our final assumption is that there exists $\lambda_0 \in \rho(H_0)$ such that

$$-1 \notin \sigma(K(\lambda_0)). \tag{1.3}$$

**Remark 1.** We note that $K(\lambda)$ is a bounded extension of the (maybe more familiar) Birman–Schwinger operator $A(H_0 - \lambda)^{-1} B^*$, defined on $\mathcal{D}(B^*)$. In particular, if $\mathcal{D}(B^*)$ is dense in $\mathcal{H}'$, then $K(\lambda) = A(H_0 - \lambda)^{-1} B^*$. For instance, the latter is true if $B$ is closable. Moreover, setting $G_\delta := (|H_0| + 1 + \delta)\delta > -1$, we note that we also have that

$$K(\lambda) = [AG_\delta^{-1/2}] [G_\delta(H_0 - \lambda)^{-1}] [BG_\delta^{-1/2}]^*, \tag{1.4}$$

as follows from the fact that functions of $H_0$ commute (taking the respective domains into account) and

$$[BG_\delta^{-1/2}]^* = [(BG_\delta^{-1/2})(G_\delta^{1/2} G_0^{-1/2})]^* = [G_\delta^{1/2} G_0^{-1/2}]^* [BG_\delta^{-1/2}]^* = [G_\delta^{1/2} G_0^{-1/2}] [BG_\delta^{-1/2}]^*.$$

**Remark 2.** There exist a variety of approaches to the Birman–Schwinger principle for factorable perturbations of a given (self-adjoint) operator $H_0$, i.e. for a suitable closed extension $H_V$ of $H_0 + B^* A$. Let us mention Kato’s pioneering work [47] and the work by Konno and Koroda [39]. As some of the more recent articles on the topic we mention works by Gesztesy et al. [30], Latushkin and Sukhtayev [42], Frank [28] and of Behrndt, ter Elst and Gesztesy [3]. The assumptions on $A, B$ and $H_0$ made in these works are not uniform but vary from paper to paper. Our own assumptions take an intermediate position. For instance, we do not assume that $A$ or $B$ are closed, which is important for some applications. An example is provided, e.g., in the remark in Appendix B of [28], which also shows that it can be advantageous to allow for the case $\mathcal{H}' \neq \mathcal{H}$.

On the other hand, we do assume that $H_0$ is self-adjoint (but not necessarily bounded below), which some of the mentioned papers don’t, and we do assume that $\mathcal{D}(|H_0|^{1/2}) \subset \mathcal{D}(A) \cap \mathcal{D}(B)$. The latter assumption allows for a quite explicit description of $H_V$ via quadratic forms (see Section 2). In contrast to this, the weaker assumption that $\mathcal{D}(H_0) \subset \mathcal{D}(A) \cap \mathcal{D}(B)$ made in some of the mentioned papers would allow to define $H_V$ only implicitly via an associated resolvent equation (see (1.1.3)).

While Assumption (1.1) is usually easy to verify in concrete applications (for instance, it is certainly true if $A$ and $B$ are closed as follows from the closed graph theorem), the direct verification of (1.3) might not be that easy. For this reason, the next lemma discusses some sufficient conditions for (1.3) which might be easier to verify.

**Lemma 1.** Assume (1.1). Then assumption (1.3) is satisfied if one of the following three conditions holds:
(i) there exists $\lambda_0 \in \rho(H_0)$ such that $\|K(\lambda_0)\| < 1$,

(ii) there exists $a \in (0,1)$ and $\delta > 0$ such that

$$\| [B(|H_0| + \delta)^{-1/2}]^* [A(|H_0| + \delta)^{-1/2}] \| \leq a,$$

(1.5)

(iii) there exists $a \in (0,1)$ and $\delta > 0$ such that

$$|(B\phi, A\psi)| \leq a \| (|H_0| + \delta)^{1/2} \| \| (|H_0| + \delta)^{1/2} \psi \|, \quad \phi, \psi \in \mathcal{D}(|H_0|^{1/2}).$$

Moreover, the assumptions (1.3) and (1.5) are both satisfied if

(iv) there exists $a \in (0,1)$ and $b > 0$ such that

$$\max (\| A\psi \|^2, \| B\psi \|^2) \leq a \| H_0 \|^{1/2} \psi \|^2 + b \| \psi \|^2, \quad \psi \in \mathcal{D}(|H_0|^{1/2}).$$

(1.7)

In addition, if $A$ and $B$ are closed then it is sufficient that (1.2) holds for $\psi \in \mathcal{D}$ where $\mathcal{D}$ is a core of $\mathcal{D}(|H_0|^{1/2})$.

Proof. (i) follows from the fact that the spectral radius is dominated by the operator norm. For (ii) we first note that in view of (1.4) for $\delta > 0$ we have

$$K(\lambda) = [A(|H_0| + \delta)^{-1/2}] [(|H_0| + \delta)(H_0 - \lambda)^{-1}] [B(|H_0| + \delta)^{-1/2}]^*, \quad \lambda \in \rho(H_0).$$

Since for two bounded operators $C, D$ we have $\sigma(CD) \setminus \{0\} = \sigma(DC) \setminus \{0\}$, we thus obtain that $-1 \notin \sigma(K(\lambda))$ if, and only if,

$$-1 \notin \sigma \left( [(|H_0| + \delta)(H_0 - \lambda)^{-1}] [B(|H_0| + \delta)^{-1/2}]^* [A(|H_0| + \delta)^{-1/2}] \right)$$

and the last condition is satisfied if the norm of the operator on the right-hand side is smaller than one. But by assumption there exist $a, \delta > 0$ such that $\| [B(|H_0| + \delta)^{-1/2}]^* [A(|H_0| + \delta)^{-1/2}] \| \leq a < 1$, hence it suffices to choose $\lambda \in \rho(H_0)$ such that $\| (|H_0| + \delta)(H_0 - \lambda)^{-1} \| \leq 1/a$. The latter is satisfied if $\lambda = i\eta$ with $\eta > 0$ sufficiently large, which concludes the proof of (ii). Continuing, we note that (iii) follows from (ii) since

$$\| [B(|H_0| + \delta)^{-1/2}]^* A(|H_0| + \delta)^{-1/2} \| \leq a \quad \Leftrightarrow \quad \| A(|H_0| + \delta)^{-1/2} B(|H_0| + \delta)^{-1/2} \| \leq a$$

$$\Leftrightarrow \forall f, g \in \mathcal{H} : \quad \| (B(|H_0| + \delta)^{-1/2} f, A(|H_0| + \delta)^{-1/2} g) \| \leq a \| f \| \| g \|$$

and

$$\Leftrightarrow \forall \phi, \psi \in \mathcal{D}(|H_0|^{1/2}) : \quad \| (B\phi, A\psi) \| \leq a \| (|H_0| + \delta)^{1/2} \phi \| \| (|H_0| + \delta)^{1/2} \psi \|.$$n

Concerning (iv) we note that given (1.7), for $\phi \in \mathcal{H}$ and $\psi = (|H_0| + \delta)^{-1/2} \phi$, where $\delta = b/a$, we obtain that

$$\| A(|H_0| + \delta)^{-1/2} \phi \|^2 \leq a \| H_0 \|^{1/2} \phi \|^2 + b \| \psi \|^2 = a \| (|H_0| + \delta)^{1/2} \psi \|^2 = a \| \phi \|^2.$$

Hence $A(|H_0| + \delta)^{-1/2}$ is bounded and $\| A(|H_0| + \delta)^{-1/2} \| \leq \sqrt{a}$ and the same is true of $B(|H_0| + \delta)^{-1/2}$ and its norm, so (1.3) is satisfied. Moreover, the validity of (1.3) follows from (ii), the submultiplicativity of the operator norm and the fact that the norm of a bounded operator and its adjoint coincide. Finally, concerning the last statement of (iv) we note that in case $A$ and $B$ are closed, the estimate (1.7) will hold for all $\phi, \psi \in \mathcal{D}(|H_0|^{1/2})$ once they hold for $\phi, \psi$ in a core of $|H_0|^{1/2}$.}

The composition $V := B^*A$ (with its natural domain) is a well defined operator in $\mathcal{H}$. However, since $H_0$ is not necessarily bounded from below, the machinery of closed sectorial forms and the customary Friedrichs extension of the operator sum $H_0 + V$ are not available to us. As a replacement, below we will introduce a unique closed extension $H_V$ of $H_0 + V$ by means of the so-called pseudo-Friedrichs extension [38, Sec. VI.3.4] (see Section 2 for more details). First, however, let us discuss our main results about this operator.

3
1.3 Our main results

The well known version of the Birman–Schwinger principle is formulated by the following equivalence.

**Theorem 1.** Suppose Assumption 1. Then
\[ \forall \lambda \in \mathbb{C} \setminus \sigma(H_0), \quad \lambda \in \sigma_p(H_V) \iff -1 \in \sigma_p(K_\lambda). \] (1.8)

We establish the validity of this equivalence in the fully abstract setting above (Theorems 3 and 4). While in slightly different settings this result has been proved a variety of times before (see, e.g., the papers cited in Remark 2), one of the main points of the present paper is that suitably adapted versions of the Birman–Schwinger principle hold also for:

- all eigenvalues \( \sigma_p(H_V) \setminus \sigma_p(H_0) \); (Theorem 5)
- residual spectrum \( \sigma_r(H_V) \setminus \sigma_r(H_0) \); (Theorems 9 and 10)
- essential spectrum \( \sigma_c(H_V) \setminus \sigma(H_0) \). (Theorem 11)

Such variants of the Birman–Schwinger principle seem to be less known. An exception is [26] in which Fanelli, Vega and one of the present authors established results of this type in the case of Schrödinger operators (see also [19, 45, 29]).

Using the Birman–Schwinger operator and the Birman–Schwinger principle, we establish stability results about the spectrum of \( H_V \), assuming that \( K_z \) is uniformly bounded in \( z \), i.e.,
\[ \sup_{z \in \rho(H_0)} \| K_z \| < \infty. \] (1.9)

The first of our main results in this direction is the following theorem.

**Theorem 2.** Suppose Assumption 1 and \( (1.9) \). Then \( \sigma(H_0) \subset \sigma(H_V) \).

**Remark 3.** It is clear that the conclusion of Theorem 2 is generally false if \( (1.9) \) is not satisfied. Just consider the case where \( A = I \) and \( B = i \cdot I \), where \( \sigma(H_V) = \sigma(H_0) + i \), i.e. the spectrum of \( H_0 \) is shifted into the complex plane.

From our point of view, the remarkable thing about Theorem 2 is that it holds without any smallness assumption on \( \sup_z \| K_z \| \). Indeed, in all applications of the Birman–Schwinger principle to spectral estimates that we are aware of one assumes that \( \sup_z \| K_z \| \) is sufficiently small and then derives information about \( \sigma(H_V) \). The fact that some information can also be obtained without assuming that the supremum is small seems to have been completely overlooked so far. A possible reason for this might be that in typical applications the spectrum of \( H_0 \) is purely essential and the assumption (1.10) usually implies that the resolvent difference of \( H_0 \) and \( H_V \) is compact, hence \( \sigma(H_0) = \sigma_c(H_0) = \sigma_c(H_V) \). In general, however, there is no reason to believe that (1.10) should imply such a compactness property.

In case that \( \| K_z \| \) is indeed uniformly small, i.e.
\[ \exists c < 1 : \sup_{z \in \rho(H_0)} \| K_z \| \leq c, \] (1.10)

one obtains much stronger information on \( \sigma(H_V) \).

**Remark 4.** Let us note that given (1.10) the Assumption 1 reduces to (1.11) since (1.3) is automatically satisfied as we discussed in Lemma 4.

**Theorem 3.** Suppose Assumption 1 and (1.10). Then the following holds:

(i) \( \sigma(H_0) = \sigma(H_V) \).

(ii) \( [\sigma_p(H_V) \cup \sigma_r(H_V)] \subset \sigma_p(H_0) \) and \( \sigma_c(H_0) \subset \sigma_c(H_V) \).

In particular, if \( \sigma(H_0) = \sigma_c(H_0) \), then \( \sigma(H_V) = \sigma_c(H_V) = \sigma_c(H_0) \).

So the spectra of \( H_V \) and \( H_0 \) coincide if the perturbation \( V \) is small in the sense of (1.10). Moreover, the spectrum of \( H_V \) is purely continuous if it is the case of \( H_0 \). As we will see, these stability properties follow directly from Theorem 2 and from Theorem 1 and its variants mentioned below it.
Remark 5. It is well known that \( \sigma(H_0) = \sigma(H_V) \) need not be true if \( \sup \|K_z\| \geq 1 \), see e.g. the proof of the \( d = 3 \) case of Theorem 2 in [27].

We do not know whether in general, given [140], the continuous-, point- and residual spectra of \( H_0 \) and \( H_V \) coincide. However, this is the case if \( A \) is relatively smooth with respect to \( H_0 \), which means that \( A: D(A) \subset \mathcal{H} \to \mathcal{H}' \) is closed with \( D(H_0) \subset D(A) \) and

\[
\sup_{z \in \mathbb{C} \setminus \mathbb{R}, \psi \in \mathcal{H} \setminus \{0\}} |\Im(z)| \cdot \|A(H_0 - z)^{-1}A\psi\|^2/\|\psi\|^2 < \infty. \tag{1.11}
\]

The notion of relative smoothness is due to Kato [37] and we should note that there exist several equivalent ways to introduce this concept.

**Corollary 1.** Suppose Assumption [1] and [1.10]. Moreover, assume that \( A \) is relatively smooth with respect to \( H_0 \). Then

\[
\sigma_c(H_V) = \sigma_c(H_0), \quad \sigma_p(H_V) = \sigma_p(H_0) \quad \text{and} \quad \sigma_r(H_V) = \sigma_r(H_0) = \emptyset.
\]

**Proof of Corollary [1]**. In view of Theorem 3 it is sufficient to show that \( \sigma_p(H_0) \subset \sigma_p(H_V) \). So suppose that for some \( \lambda \in \mathbb{R} \) and \( \psi \in D(H_0) \setminus \{0\} \) we have \( H_0\psi = \lambda\psi \). Then for \( \varepsilon > 0 \) we also have \(-i\varepsilon(H_0 - \lambda - i\varepsilon)^{-1}\psi = \psi \) and hence

\[
\|A(H_0 - \lambda - i\varepsilon)^{-1}\psi\| = \varepsilon^{-1}\|A\psi\|.
\]

Since this is true for all \( \varepsilon > 0 \), assumption (1.11) implies that \( A\psi = 0 \). We will see below that \( H_V \) is a closed extension of \( H_0 + B^*A \), so we obtain that \( \psi \in D(H_V) \) and \( H_V\psi = H_0\psi = \lambda\psi \).

**Remark 6.** Even if \( A \) and \( B \) are closed and satisfy Assumption [1] and [1.10], this does not imply that \( A \) is \( H_0 \)-smooth. For instance, the mentioned assumptions on \( A \) and \( B \) are satisfied if \( B = 0 \) and \( A \) is any closed operator from \( \mathcal{H} \to \mathcal{H}' \) with \( D(|H_0|^{1/2}) \subset D(A) \).

There is one important case where (1.10) does imply smoothness of \( A \) with respect to \( H_0 \), namely if \( A = DB \) for some \( D \in \mathcal{B}(\mathcal{H}') \). This leads to another corollary of Theorem 3.

**Corollary 2.** Suppose Assumption [1] and [1.10]. Moreover, suppose that \( A \) is closed and that \( A = DB \) for some \( D \in \mathcal{B}(\mathcal{H}') \). Then

\[
\sigma_c(H_V) = \sigma_c(H_0), \quad \sigma_p(H_V) = \sigma_p(H_0) \quad \text{and} \quad \sigma_r(H_V) = \sigma_r(H_0) = \emptyset.
\]

**Proof of Corollary [1]**. By [37] Thm. 5.1 the \( H_0 \)-smoothness of \( A \) is equivalent to the fact that

\[
\sup_{z \in \mathbb{C} \setminus \mathbb{R}, \psi \in D(A^*) \setminus \{0\}} \|((H_0 - z)^{-1} - (H_0 - \overline{z})^{-1})A^*\psi, A^*\psi\|/\|\psi\|^2 < \infty. \tag{1.12}
\]

But using our assumptions, for \( z \in \mathbb{C} \setminus \mathbb{R} \) and \( \psi \in D(A^*) \) with \( \|\psi\| = 1 \) we can estimate

\[
\|((H_0 - z)^{-1}A^*\psi, A^*\psi\| = \|(A(H_0 - z)^{-1}B^*A^*\psi, \psi)\| \leq \|K_z\| \|D\| \leq \|D\| \sup_{z \in \mathbb{C} \setminus \mathbb{R}} \|K_z\|.
\]

Using the same inequality to estimate the second term of the difference in (1.12) we see that the left-hand side of (1.12) is indeed finite. Now apply Corollary [1].

In order to put Theorem 3 and its corollaries into perspective, we need to take a closer look at Kato’s classical work [37]. We do this in the following section.

### 1.4 Kato’s results

The main result of Kato’s 1966 paper [37] is the following theorem.

**Theorem 4** ([37] Thm. 1.5]). Let \( H_0 \) be self-adjoint in \( \mathcal{H} \) and suppose that \( A, B \) are closed operators from \( \mathcal{H} \) to \( \mathcal{H}' \) with \( D(H_0) \subset D(A) \cap D(B) \) which are smooth relative to \( H_0 \). Moreover, suppose that there exists \( c < 1 \) such that

\[
\sup_{z \in \mathbb{C} \setminus \mathbb{R}} \|A(H_0 - z)^{-1}B^*\| \leq c. \tag{1.13}
\]
Then there exists a closed extension $\tilde{H}_V$ of $H_0 + B^*A$ which is similar to $H_0$ (so in particular, the continuous, point and residual spectra of $\tilde{H}_V$ and $H_0$ coincide). Moreover, the operator $\tilde{H}_V$ satisfies the generalised second resolvent equation
\begin{equation}
\forall \xi \in \mathbb{C} \setminus \mathbb{R}, \quad (\tilde{H}_V - \xi)^{-1} - (H_0 - \xi)^{-1} = -(H_0 - \xi)^{-1}B^*A(\tilde{H}_V - \xi)^{-1}.
\end{equation}

**Remark 7.** Here the similarity means that there exists an operator $W \in \mathcal{B}(\mathcal{H})$ such that $W^{-1} \in \mathcal{B}(\mathcal{H})$ and $\tilde{H}_V = WH_0W^{-1}$. In other words, $\tilde{H}_V$ is quasi-self-adjoint (cf. [11]). We note that Kato actually states his theorem for the more general case that $H_0$ is closed and densely defined with $\sigma(H_0) \subset \mathbb{R}$.

To compare Kato’s result with our results of the previous section, one first needs to check that his operator $\tilde{H}_V$ and our pseudo-Friedrichs extension $H_V$ (to be constructed below) do indeed coincide if Assumption 1 is satisfied. This will be done in the Appendix (Proposition 2) under the additional assumption that $D(A) = D(B) = D(|H_0|^{1/2})$.

Now let us start with a comparison of the assumptions of Kato and of our results above. First, we note that Kato requires the operators $A$ and $B$ to be closed, which we don’t, but that he doesn’t assume that $D(|H_0|^{1/2}) \subset D(A) \cap D(B)$, which we do. Second, we note that given Kato’s assumptions, the operator $A(H_0 - z)^{-1}B^*$ is just the closure of our Birman–Schwinger operator $K(z)$, so assumption 1.10 is the same as our assumption 1.10. In particular, let us emphasise that Kato does not provide any conclusions under the weaker assumption 1.9 as we do in Theorem 2 above. Moreover, in addition to the smallness assumption 1.13, Kato does also require that $A$ and $B$ are $H_0$-smooth (which does not follow from 1.13 as we discussed in Remark 6) so the spectral stability results we obtain in Theorem 3 and Corollary 1 are certainly not a consequence of Kato’s Theorem 4. Having made all these observations we of course also have to admit that in case that all of Kato’s (and our) assumptions are satisfied, his conclusion that $H_V$ and $H_0$ are similar is considerably stronger than our observation that their spectra coincide. In particular, using Kato’s result one can derive the following improved version of Corollary 2

**Corollary 3.** Suppose Assumption 1, 1.10 and that $D(A) = D(B) = D(|H_0|^{1/2})$. Moreover, suppose that $A$ and $B$ are closed and that $A = D_0B$ and $B = D_1A$ for some $D_0, D_1 \in \mathcal{B}(\mathcal{K}')$. Then $H_V$ and $H_0$ are similar.

**Proof.** As the proof of Corollary 2 showed, given the above assumptions $A$ and $B$ are smooth relative to $H_0$, hence Kato’s theorem applies. \hfill \Box

**Remark 8.** In particular, the previous corollary applies in case that $A = UB$, where $U \in \mathcal{B}(\mathcal{K}, \mathcal{K}')$ is a partial isometry with initial set $\mathbb{R}(B)$, since then $B = U^*A$. This example is important in applications to Schrödinger operators, see Section 7 below.

Having stated the advantages of Kato’s and our own results, let us conclude this section by noting that Kato’s proof of Theorem 1 is very different from our proof of Theorem 3. In fact, he uses the method of stationary scattering theory (and the similarity transformation $W$ he constructs has the meaning of a wave operator), while we work directly with the mentioned variants of the Birman–Schwinger principle.

### 1.5 Organisation of the paper

In Section 2 we introduce the operator $H_V$ as the pseudo-Friedrichs extension of $H_0 + V$. Sections 3, 4 and 5 are devoted to establishing the aforementioned variants of the Birman–Schwinger principle for the point, residual and essential spectra, respectively. In Section 6 we provide the proofs of Theorems 2 and 3. Finally, in Section 7 we apply the abstract theorems to Schrödinger and Dirac operators; we recall some classical as well as recently established properties, and prove completely new results for Schrödinger operators in three-dimensional hyperbolic space. Finally, the appendix contains a proof that Kato’s extension $\tilde{H}_V$ and our pseudo-Friedrichs extension $H_V$ coincide given some suitable assumptions.

### 2 The pseudo-Friedrichs extension

By our standing Assumption 1, $H_0$ is a self-adjoint operator in a complex separable Hilbert space $\mathcal{K}$. Recall ([cf. 20 Sec. VI.2.7]) that the absolute value $|H_0| := (H_0^2)^{1/2}$ is also self-adjoint, $D(|H_0|) = D(H_0)$
is a core of $|H_0|^{1/2}$ and $H_0$ and $|H_0|$ commute (in the sense of their resolvents). The operator $G_0 : D(H_0) \to \mathcal{H}, G_0 = |H_0| + 1$ is bijective. We define a sesquilinear form associated with $H_0$ by

$$h_0(\phi, \psi) : = (G_0^{1/2} \phi, H_0 G_0^{-1} G_0^{1/2} \psi), \quad \phi, \psi \in D(h_0) : = D(|H_0|^{1/2}).$$

Since $H_0 G_0^{-1} \in \mathcal{B}(\mathcal{H})$ is selfadjoint, we see that $h_0$ is symmetric, i.e. $h_0(\phi, \psi) = \overline{h_0(\psi, \phi)} : = h_0^*(\phi, \psi)$ for $\phi, \psi \in D(|H_0|^{1/2})$. Moreover, $h_0(\phi, \psi) = (\phi, H_0 \psi)$ and, by symmetry, $h_0(\psi, \phi) = (H_0 \psi, \phi)$ for every $\phi \in D(|H_0|^{1/2})$ and $\psi \in D(H_0)$.

Let $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ and $B : D(B) \subset \mathcal{H} \to \mathcal{H}'$ be two operators satisfying $D(|H_0|^{1/2}) \subset D(A) \cap D(B)$ and \footnote{36 Thm. VI.3.11} \footnote{50} (see also \footnote{54} for more recent developments). It is a suitable generalisation of the Friedrichs extension in the case when $H_0$ is not necessarily bounded from below.

The idea is to replace $V$ by its sesquilinear form

$$v(\phi, \psi) : = (B \phi, A \psi), \quad \phi, \psi \in D(v) : = D(|H_0|^{1/2}).$$

Noting that, by assumption \footnote{11}, we can rewrite $v$ as

$$v(\phi, \psi) = (B G_0^{-1/2} G_0^{1/2} \phi, A G_0^{-1/2} G_0^{1/2} \psi) = (G_0^{1/2} \phi, [B G_0^{-1/2}]^* A G_0^{-1/2} G_0^{1/2} \psi), \quad \phi, \psi \in D(|H_0|^{1/2}),$$

we obtain that

$$h_V : = h_0 + v, \quad D(h_V) : = D(|H_0|^{1/2}),$$

we have

$$h_V(\phi, \psi) = (G_0^{1/2} \phi, (H_0 G_0^{-1} + [B G_0^{-1/2}]^* A G_0^{-1/2}) G_0^{1/2} \psi), \quad \phi, \psi \in D(|H_0|^{1/2}). \quad (2.1)$$

Hence, we define

$$H_V : = G_0^{1/2} (H_0 G_0^{-1} + [B G_0^{-1/2}]^* A G_0^{-1/2}) G_0^{1/2} \quad (2.2)$$

with its natural domain. Clearly, $D(H_V) \subset D(|H_0|^{1/2})$ and if $\psi \in D(D_0) \cap D(V)$, where $D(D_0) \subset D(|H_0|^{1/2})$ and $D(V) = A^{-1} D(B^*) = \{ \psi \in D(A) : A \psi \in D(B^*) \} \subset D(|H_0|^{1/2})$, then for all $\phi$ in the dense set $D(|H_0|^{1/2})$ we have $(\phi, H_V \psi) = (\phi, H_0 \psi) + (\phi, V \psi)$, so $H_V \psi = (H_0 + V) \psi$. This shows that $H_V \supset H_0 + V$ and one has the representation formula

$$\forall \phi \in D(|H_0|^{1/2}), \quad \psi \in D(H_V), \quad (\phi, H_V \psi) = h_V(\phi, \psi). \quad (2.3)$$

Now let us verify that $H_V$ is a closed operator. We will do this by showing that $\rho(H_V)$ is non-empty. For this purpose, we use assumption \footnote{13}, i.e. there exists $\lambda_0 \in \rho(H_0)$ such that $-1 \not\in \sigma(K(\lambda_0))$. Using that $I \supset G_0^{1/2} G_0^{-1} G_0^{1/2}$, with this choice of $\lambda_0$ we can write

$$H_V - \lambda_0 = G_0^{1/2} ([H_0 - \lambda_0] G_0^{-1} + [B G_0^{-1/2}]^* A G_0^{-1/2}) G_0^{1/2}. \quad (2.4)$$

In particular, we obtain that

$$(H_V - \lambda_0)^{-1} = G_0^{-1/2} ([H_0 - \lambda_0] G_0^{-1} + [B G_0^{-1/2}]^* A G_0^{-1/2})^{-1} G_0^{-1/2}, \quad (2.4)$$

provided that

$$[H_0 - \lambda_0] G_0^{-1} + [B G_0^{-1/2}]^* A G_0^{-1/2} = [H_0 - \lambda_0] G_0^{-1} \left( I + G_0 (H_0 - \lambda_0)^{-1} [B G_0^{-1/2}]^* A G_0^{-1/2} \right)$$

has a bounded inverse. But this is the case if, and only if, $-1 \not\in \sigma(G_0 (H_0 - \lambda_0)^{-1} [B G_0^{-1/2}]^* A G_0^{-1/2})$ which is the case (as we already argued in the proof of Lemma\footnote{14} (ii)) if, and only if, $-1 \not\in \sigma(K(\lambda_0))$. So we conclude that indeed $\lambda_0 \in \rho(H_V)$ and $H_V$ is closed.

Next, let us show that $D(H_V) = \mathcal{R}((H_V - \lambda_0)^{-1})$ is dense in $\mathcal{H}$. To this end, note that from \footnote{2.4} we obtain that

$$[(H_V - \lambda_0)^{-1}]^* = G_0^{-1/2} ([H_0 - \lambda_0] G_0^{-1} + [A G_0^{-1/2}]^* B G_0^{-1/2})^{-1} G_0^{-1/2}, \quad (2.5)$$
where we used that $G_0^{-1/2} \in \mathcal{B}(\mathcal{K})$ is self-adjoint and $[H_0 - \lambda_0]G_0^{-1} + [BG_0^{-1/2}]^* A G_0^{-1/2}$ is invertible in \( \mathcal{B}(\mathcal{K}) \). Now the operator on the right-hand side of (2.5) is clearly injective and hence
\[
\overline{D(H_V)} = \overline{R(H_V - \lambda_0)^{-1}} = (\mathcal{N}([H_V - \lambda_0]^{-1})^*)^\perp = \mathcal{K},
\]
so \( H_V \) is densely defined. In particular, its adjoint \( H_V^* \) exists and \( \overline{H_V} \in \rho(H_V^*) \) with
\[
(H_V^* - \overline{\lambda_0})^{-1} = G_0^{-1/2}([H_0 - \overline{\lambda_0}]G_0^{-1} + [AG_0^{-1/2}]^* BG_0^{-1/2})^{-1} G_0^{-1/2}.
\] (2.6)
It follows that \( D(H_V^*) \subset \overline{R(G_0^{-1/2})} = \overline{R(H_0)^{1/2}} \) and
\[
H_V^* = G_0^{1/2}(H_0 G_0^{-1} + [AG_0^{-1/2}]^* BG_0^{-1/2})^{-1/2} G_0^{1/2}.
\] (2.7)
Moreover, with the adjoint form
\[
v^*(\phi, \psi) := \overline{v(\psi, \phi)} = (A\phi, B\psi), \quad D(v^*) := D(v) = D([H_0]^{1/2}),
\]
we obtain the representation formula
\[
\forall \phi \in D([H_0]^{1/2}), \; \psi \in D(H_V^*), \quad (\phi, H_V^* \psi) = h_V^*(\phi, \psi),
\] (2.8)
where \( h_V^* = h_0^* + v^* \).

Let us summarise the properties of the pseudo-Friedrichs extension into the following theorem.

**Theorem 5.** Suppose Assumption 0 and set \( V := B^* A \). There exists a unique closed extension \( H_V \) of \( H_0 + V \) such that \( D(H_V) \subset D([H_0]^{1/2}) \), \( D(H_V^* ) \subset D([H_0]^{1/2}) \) and the representation formulae (2.3) and (2.8) hold.

**Proof.** It remains to verify the uniqueness claim. Let \( H_V \) be another closed extension of \( H_0 + V \) with the properties stated in the theorem. Let \( \phi \in D([H_0]^{1/2}) \) and \( \psi \in D(H_V) \subset D([H_0]^{1/2}) \). Then (2.3) and (2.1) imply that
\[
(\phi, H_V \psi) = h_0(\phi, \psi) + v(\phi, \psi) = (G_0^{1/2} \phi, H_0 G_0^{-1} G_0^{1/2} \psi) + (G_0^{1/2} \phi, C G_0^{1/2} \psi),
\]
where \( C = [BG_0^{-1/2}]^* A G_0^{-1/2} \). But this implies that \( [H_0 G_0^{-1} + C] G_0^{1/2} \psi \in D(G_0^{1/2}) = D((G_0^{1/2})^*) \) and
\[
H_V \psi = G_0^{1/2}(H_0 G_0^{-1} + C) G_0^{1/2} \psi.
\]
By (2.3), it follows that \( \psi \in D(H_V) \) and \( H_V \psi = H_V \psi \). This shows that \( H_V \subset H_V \).

Now, let \( \phi \in D([H_0]^{1/2}) \) and \( \psi \in D((H_V)^*) \subset D([H_0]^{1/2}) \). Then (2.8) implies
\[
(\phi, (H_V^*)^* \psi) = h_0^*(\phi, \psi) + v^*(\phi, \psi) = h_0(\psi, \phi) + (B \psi, A \phi) = (H_0 G_0^{-1/2} \phi, G_0^{1/2} \psi) + (C G_0^{1/2} \phi, C G_0^{1/2} \psi) = (G_0^{1/2} \phi, H_0 G_0^{-1} G_0^{1/2} \psi) + (G_0^{1/2} \phi, C^* G_0^{1/2} \psi),
\]
where the second equality employs the commutativity of \( H_0 \) and \( G_0 \). Arguing as above, this implies that
\[
(H_V^*)^* \psi = G_0^{1/2}(H_0 G_0^{-1} + C^*) G_0^{1/2} \psi
\]
and hence by (2.7) it follows that \( \psi \in D(H_V^*) \) and \( (H_V^*)^* \psi = H_V^* \psi \). This shows that \( (H_V)^* \subset H_V^* \), so \( H_V \subset H_V \).

We conclude this section about the pseudo-Friedrichs extension with the following generalised version of the second resolvent identity.

**Proposition 1.** For all \( z \in \rho(H_0) \cap \rho(H_V) \),
\[
(H_V - z)^{-1} - (H_0 - z)^{-1} = -[B(H_0 - z)^{-1}]^* A(H_V - z)^{-1}.
\] (2.9)
Proof. Given any \( f, g \in \mathcal{H} \), set \( \phi := (H_0 - \bar{z})^{-1} f \) and \( \psi := (H_V - z)^{-1} g \). Then
\[
(f, [(H_V - z)^{-1} - (H_0 - z)^{-1}] g) = ((H_0 - \bar{z}) \phi, \psi) - (\phi, (H_V - z) \psi)
= (H_0 \phi, \psi) - (\phi, H_V \psi)
= h_0(\phi, \psi) - h_V(\phi, \psi)
= (B\phi, A\psi)
= (B(H_0 - \bar{z})^{-1} f, A(H_V - z)^{-1} g)
= (f, [B(H_0 - \bar{z})^{-1}]^* A(H_V - z)^{-1} g),
\]
where the third equality holds because both \( \phi, \psi \in D(|H_0|^{1/2}). \)

3 The point spectrum

This section deals with the point spectrum of \( H_V \). As a byproduct of the following two theorems, we obtain a proof of Theorem 6. For instance, the next theorem establishes the implication \( \Rightarrow \) of Theorem 6.

**Theorem 6.** Suppose Assumption 1. Let \( H_V \psi = \lambda \psi \) with some \( \lambda \in \mathbb{C} \setminus \sigma(H_0) \) and \( \psi \in D(H_V) \setminus \{0\} \). Then \( g := A\psi \neq 0 \) and \( K_A g = -g \).

Proof. Suppose that \( g = A\psi = 0 \). Then for every \( f \in D(H_0) \) we have
\[
(H_0 f, \psi) = h_0(f, \psi) = h_V(f, \psi) = (B f, A \psi) = (f, H_V \psi) = (f, \lambda \psi).
\]
This shows that \( \psi \in D(H_0^*) = D(H_0) \) and \( H_0 \psi = H_0^* \psi = \lambda \psi \), so \( \lambda \in \sigma_p(H_0) \), a contradiction. Hence \( g \neq 0 \).

Now for every \( \phi \in \mathcal{H} \), one has
\[
(\phi, K_A g) = \langle [AG_0^{-1/2}]^* \phi, [G_0(H_0 - \lambda)^{-1}]^* BG_0^{-1/2} \phi \rangle
g = \langle [BG_0^{-1/2}]^* G_0(H_0 - \lambda)^{-1}^* [AG_0^{-1/2}]^* \phi, A \psi \rangle
= \langle (B \eta, \psi) = v(\eta, \psi) \rangle
\]
with \( \eta := G_0^{-1/2} G_0(H_0 - \lambda)^{-1}^* [AG_0^{-1/2}]^* \phi \in D(|H_0|^{1/2}) \). Using (2), it follows that
\[
(\phi, K_A g) = (\eta, H_V \psi) = h_0(\eta, \psi)
g = \lambda (\eta, \psi) - h_0(\eta, \psi)
g = \lambda (G_0^{1/2} \eta, G_0^{-1} G_0^{1/2} \psi) - (G_0^{1/2} \eta, H_0 G_0^{-1} G_0^{1/2} \psi)
g = - (G_0^{1/2} \eta, H_0 - \lambda) G_0^{-1} G_0^{1/2} \psi
= - (\lambda G_0^{1/2} \phi, G_0(H_0 - \lambda)^{-1} \psi)
= - (\lambda G_0^{1/2} \phi, G_0^{1/2} \psi)
= - (\phi, A \psi)
g = - (\phi, g).
\]
Since this is true for every \( \phi \in \mathcal{H} \), it follows that \( K_A g = -g \).

The following theorem establishes the opposite implication \( \Leftarrow \) of Theorem 6.

**Theorem 7.** Suppose Assumption 1. Let \( K_A g = -g \) with some \( \lambda \in \mathbb{C} \setminus \sigma(H_0) \) and \( g \in \mathcal{H} \setminus \{0\} \). Then \( \psi := G_0^{1/2} (H_0 - \lambda)^{-1} [B G_0^{-1/2}]^* \psi \in D(H_V) \), \( \psi \neq 0 \) and \( H_V \psi = \lambda \psi \).

Proof. Since \( \psi \in D(|H_0|^{1/2}) \) we see that if \( \psi = 0 \), then \( 0 = AG_0^{-1/2} G_0^{1/2} \psi = K_A g = -g \), leading to a contradiction. Hence \( \psi \neq 0 \). Now for every \( \phi \in D(|H_0|^{1/2}) \)
\[
h_V(\phi, \psi) = h_0(\phi, \psi) + v(\phi, \psi) = (G_0^{1/2} \phi, H_0 G_0^{-1} G_0^{1/2} \psi) + (B \phi, A \psi)
= (G_0^{1/2} \phi, H_0 (H_0 - \lambda)^{-1} [B G_0^{-1/2}]^* \psi) + (B \phi, A \psi)
= (G_0^{1/2} \phi, [B G_0^{-1/2}]^* \psi) + \lambda (G_0^{1/2} \phi, (H_0 - \lambda)^{-1} [B G_0^{-1/2}]^* \psi) + (B \phi, K_A g)
= \lambda (\phi, \psi).
\]
At the same time, by (2.4)

\[ h_V(\phi, \psi) = (G_0^{1/2} \phi, (H_0 G_0^{-1} + [B G_0^{-1/2}]^* A G_0^{-1/2}) G_0^{1/2} \psi) \]

for every \( \phi \in D([H_0]^{1/2}) \). But this implies that

\[ (H_0 G_0^{-1} + [B G_0^{-1/2}]^* A G_0^{-1/2}) G_0^{1/2} \psi \in D(G_0^{1/2}), \]

hence by (2.2) we obtain that \( \psi \in D(H_V) \) and

\[ H_V \psi = G_0^{1/2} [H_0 G_0^{-1} + [B G_0^{-1/2}]^* A G_0^{-1/2}] G_0^{1/2} \psi = \lambda \psi. \]

We continue with a theorem extending the implication \( \implies \) of Theorem 4 to suitable points \( \lambda \in \sigma(H_0) \).

**Theorem 8.** Suppose Assumption 1. Let \( g \) denote the spectral measure of \( H_0 \) where \( h \neq 0 \) and \( \lambda_\pm \in \sigma(H_0) \). Then \( B \lambda \in \sigma(H_0) \) and \( B \lambda \in \sigma(H_0) \).

**Proof.** As in the proof of Theorem 4 we see that \( g \neq 0 \).

Now we note that \( \lambda \) is real and so \( \lambda + i\epsilon \notin \sigma(H_0) \) for all \( \epsilon \in \mathbb{R} \setminus \{0\} \). As in the proof of Theorem 4 for every \( \lambda \in \mathbb{R} \), we have

\[
(\phi, K_{\lambda + i\epsilon} g) = -([A G_0^{-1/2}]^* \phi, G_0 (H_0 - \lambda - i\epsilon)^{-1} (H_0 - \lambda) G_0^{-1} G_0^{1/2} \psi)
\]

where

\[
I(\epsilon) := \epsilon ([A G_0^{-1/2}]^* \phi, G_0 (H_0 - \lambda - i\epsilon)^{-1} G_0^{-1} G_0^{1/2} \psi). \quad (3.3)
\]

It remains to show that \( I(\epsilon) \) vanishes as \( \epsilon \to 0 \). Using the spectral theorem, we have

\[
I(\epsilon) = \int_{\delta(H_0)} f(\epsilon) d([A G_0^{-1/2}]^* \phi, E_0(r) G_0^{1/2} \psi)
\]

with \( f(\epsilon) := \epsilon \frac{r - \lambda - i\epsilon}{r - \lambda} \), where \( E_0 \) denotes the spectral measure of \( H_0 \). First, one has

\[
f(\epsilon) \xrightarrow{\epsilon \to 0} \begin{cases} 0 & \text{if } r \neq \lambda, \\ i & \text{if } r = \lambda. \end{cases}
\]

In any case, however, \( E_0(\{\lambda\}) = 0 \) because \( \lambda \notin \sigma_p(H_0) \). Hence, \( f(\epsilon) \to 0 \) as \( \epsilon \to 0 \) almost everywhere with respect to the spectral measure. Second, neglecting the real part of \( r - \lambda - i\epsilon \), one has

\[
|f(\epsilon)| \leq \begin{cases} 1 & \text{if } 3\lambda = 0, \\ \frac{|\epsilon|}{|3\lambda + \epsilon|} & \text{if } 3\lambda \neq 0, \end{cases}
\]

where the last inequality holds for all \( \epsilon \) with sufficiently small \( |\epsilon| \). Hence \( |f(\epsilon)| \) is bounded by an \( \epsilon \)-independent constant and

\[
\int_{\delta(H_0)} d\|([A G_0^{-1/2}]^* \varphi, E_0(r) G_0^{1/2} \psi)\| = \|([A G_0^{-1/2}]^* \varphi)\| G_0^{1/2} \psi \| < \infty.
\]

The dominated convergence theorem implies that \( I(\epsilon) \to 0 \) as \( \epsilon \to 0 \).

**Remark 9.** Until (3.3), the proof of Theorem 8 follows the lines of [26] proof of Lem. 2] or [24] proof of Lem. 3] dealing with Schrödinger or Dirac operators, respectively. To show that \( I(\epsilon) \to 0 \) as \( \epsilon \to 0 \) in the abstract case, here we have developed a completely new approach.

**Corollary 4.** Suppose Assumption 4. Let \( \lambda \in \sigma_p(H_V) \).
(i) If \( \lambda \not\in \sigma(H_0) \), then \( \|K_\lambda\| \geq 1 \).

(ii) If \( \lambda \in \sigma_c(H_0) \), then \( \liminf_{\varepsilon \to 0^\pm} \|K_{\lambda+i\varepsilon}\| \geq 1 \).

Proof. Let \( \lambda \in \sigma_p(H_V) \), let \( \psi \neq 0 \) be a corresponding eigenvector and set \( \phi = A\psi \neq 0 \).

If \( \lambda \not\in \sigma(H_0) \), then Theorem 6 implies \( \phi \neq 0 \)

\[ \|\phi\|^2 \|K_\lambda\| \geq |\langle \phi, K_\lambda \phi \rangle| = \|\phi\|^2, \]

from which the claim (i) immediately follows.

If \( \lambda \in \sigma_c(H_0) \), we similarly write

\[ \|\phi\|^2 \|K_{\lambda+i\varepsilon}\| \geq |\langle \phi, K_{\lambda+i\varepsilon} \phi \rangle| \]

Taking the limit \( \varepsilon \to 0^\pm \), Theorem 8 implies

\[ \|\phi\|^2 \liminf_{\varepsilon \to 0^\pm} \|K_{\lambda+i\varepsilon}\| \geq \|\phi\|^2, \]

from which the desired claim (ii) immediately follows since, again, \( \phi \neq 0 \).

4 The residual spectrum

In view of the general characterisation (see, e.g., [41, Prop.5.2.2])

\[ \sigma_r(H_V) = \{ \lambda \not\in \sigma_p(H_V) : \bar{\lambda} \in \sigma_p(H_V^\ast) \}, \] (4.1)

the analysis of the residual spectrum of \( H_V \) can be reduced to the analysis of the point spectrum of the adjoint \( H_V^\ast \).

From the construction of the pseudo-Friedrichs extension in Section 2, it is clear that the roles of \( A \) and \( B \) are just interchanged when considering \( H_V^\ast \). It leads one to consider the adjoint Birman–Schwinger operator

\[ K_\ast^\varepsilon = \left[ BG_0^{-1/2} \right] \left[ G_0(H_0 - z)^{-1} \right] \left[ AG_0^{-1/2} \right]^\ast. \] (4.2)

In view of the above considerations, Theorems 4, 6, 7 and 8 remain true if, in their statements, we simultaneously replace \( H_V \) by \( H_V^\ast \), \( A \) by \( B \), \( B \) by \( A \) and \( K_\lambda \) by \( K_\lambda^\ast \) (notice the complex conjugate of \( \lambda \) in the latter). As a consequence of (4.1), we therefore get the following theorem extending Theorem 1 to the residual spectrum.

**Theorem 9.** Suppose Assumption 4. Then

\[ \forall \lambda \in \mathbb{C} \setminus \sigma(H_0), \quad \lambda \in \sigma_r(H_V) \iff -1 \in \sigma_r(K_\lambda^\ast). \]

Similarly, we get the following theorem extending Theorem 8 to the residual spectrum.

**Theorem 10.** Suppose Assumption 4. Let \( H_V^\ast \psi = \bar{\lambda} \psi \) with some \( \lambda \in \sigma_r(H_V) \cap \sigma_c(H_0) \) and \( \psi \in \mathcal{D}(H_V^\ast) \setminus \{0\} \). Then \( g := B\psi \neq 0 \) and \( K_{\lambda+i\varepsilon}^\ast g \xrightarrow{\varepsilon \to 0^\pm} -g \).

As consequence, we also get the following analogue of Corollary 4.

**Corollary 5.** Suppose Assumption 4. Let \( \lambda \in \sigma_r(H_V) \).

(i) If \( \lambda \not\in \sigma(H_0) \), then \( \|K_\lambda^\ast\| \geq 1 \).

(ii) If \( \lambda \in \sigma_c(H_0) \), then \( \liminf_{\varepsilon \to 0^\pm} \|K_{\lambda+i\varepsilon}\| \geq 1 \).
5 The essential spectrum

As mentioned in the introduction, among the variety of definitions of essential spectra for non-self-adjoint operators, here we choose that of Wolf (denoted by $\sigma_W$ in [21, Chap. IX.1]). That is, $\lambda \in C$ belongs to the essential spectrum $\sigma_e(H)$ of a closed operator $H$ in $\mathcal{K}$ if $\lambda$ is an eigenvalue of infinite geometric multiplicity or the range of $H - \lambda$ is not closed. This is equivalent to the existence of a sequence $\{\psi_n\}_{n \in \mathbb{N}} \subset D(H)$ weakly convergent to zero such that $\|\psi_n\| = 1$ for every $n \in \mathbb{N}$ and $(H - \lambda)\psi_n \to 0$ as $n \to \infty$.

The following theorem is a modification of Theorem 8 to deal with the essential spectrum. Note, however, that we do not require that the sequence is weakly converging to zero in this theorem. The admissible points therefore satisfy $\lambda \in \sigma_e(H_V) \cup \sigma_p(H_V)$. However, better results of Theorems 6 and 8 are available for eigenvalues.

**Theorem 11.** Suppose Assumption 3. Let $(H_V - \lambda)v_n \to 0$ as $n \to \infty$ with some $\lambda \in \mathbb{C} \setminus \sigma(H_0)$ and $\{v_n\}_{n \in \mathbb{N}} \subset D(H_V)$ such that $\|v_n\| = 1$ for all $n \in \mathbb{N}$. Then $\phi_n := Av_n \neq 0$ for all sufficiently large $n$ and

$$\lim_{n \to \infty} \frac{(\phi_n, K\lambda\phi_n)}{\|\phi_n\|^2} = -1.$$  

(5.1)

**Proof.** First of all, let us show that $\phi_n \neq 0$ for all sufficiently large $n$. In fact, we establish the stronger fact that

$$\liminf_{n \to \infty} \|\phi_n\| > 0.$$  

(5.2)

By contradiction, let us assume that there exists a subsequence $\{\phi_{n_j}\}_{j \in \mathbb{N}}$ such that $n_j \to \infty$ and $\phi_{n_j} = Av_{n_j} \to 0$ as $j \to \infty$. From the identity (5.2) and the hypothesis, we deduce that for $f_j := (H_0 - \lambda)^{-1}v_{n_j}$, we have

$$|h_0(f_j, \psi_{n_j}) - \lambda(f_j, \psi_{n_j})| \leq |h_V(f_j, \psi_{n_j}) - \lambda(f_j, \psi_{n_j})| + |(Bf_j, Av_{n_j})|$$

$$= |(f_j, (H_V - \lambda)v_{n_j})| + |(B(H_0 - \lambda)^{-1}v_{n_j}, \phi_{n_j})|$$

$$\leq \|f_j\|\|(H_V - \lambda)v_{n_j}\| + \|B(H_0 - \lambda)^{-1}\|\|\phi_{n_j}\|$$

$$\leq \|(H_0 - \lambda)^{-1}\|\|(H_V - \lambda)v_{n_j}\| + \|B(H_0 - \lambda)^{-1}\|\|\phi_{n_j}\|.$$  

Here we used that $B(H_0 - \lambda)^{-1} = (BG_0^{-1/2}/(G_0[H_0 - \lambda])^{-1}) \in \mathcal{B}(\mathcal{K}, \mathcal{K}')$. In particular, we see that $|h_0(f_j, \psi_{n_j}) - \lambda(f_j, \psi_{n_j})| \to 0$ for $j \to \infty$. On the other hand, since $f_j \in D(H_0)$ we also have

$$h_0(f_j, \psi_{n_j}) - \lambda(f_j, \psi_{n_j}) = (\langle H_0 - \lambda \rangle f_j, \psi_{n_j}) = \|\psi_{n_j}\|^2 = 1$$

for every $j \in \mathbb{N}$, which leads to a contradiction. Hence $\phi_n \neq 0$ for all sufficiently large $n$ and (5.2) holds true.

The rest of the proof is similar to that of Theorem 8. Since $\lambda \notin \sigma(H_0)$, (5.1) implies $\langle \phi_n, K\lambda\phi_n \rangle = v(\eta_n, \psi_n)$, where $\eta_n := G_0^{-1/2}[G_0[H_0 - \lambda]^{-1}]^{-1}G_0^{-1/2}\phi_n$ belongs to $D([H_0]^{1/2})$ and $\|\eta_n\| \leq C_0\|\phi_n\|$ with some constant $C_0$ independent of $n$. In analogy with (3.2), we have

$$v(\eta_n, \psi_n) = h_V(\eta_n, \psi_n) - h_0(\eta_n, \psi_n)$$

$$= (\eta_n, (H_V - \lambda)\psi_n) + \lambda(\eta_n, \psi_n) - h_0(\eta_n, \psi_n)$$

$$= (\eta_n, (H_V + \lambda)\psi_n) - \|\phi_n\|^2.$$  

Consequently,

$$\frac{|\langle \phi_n, K\lambda\phi_n \rangle|}{\|\phi_n\|^2} + 1 = \frac{|\langle \eta_n, (H_V - \lambda)\psi_n \rangle|}{\|\phi_n\|^2} \leq \frac{\|\eta_n\|}{\|\phi_n\|^2} \|\psi_n\| \leq \frac{C_0}{\|\phi_n\|^2} \|\psi_n\| \leq C_0 \|\psi_n\|.$$  

Using (5.2) and the hypothesis, we get the desired claim. \hfill \Box

**Remark 10.** Theorem 11 is inspired by [26, Lem. 3] proved for Schrödinger operators with $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Here we have developed an abstract approach and included real points $\lambda \notin \sigma(H_0)$ as well.
Corollary 6. Suppose Assumption 1. If \( \lambda \in [\sigma_c(H_V) \cup \sigma_p(H_V)] \setminus \sigma(H_0) \), then \( \|K_\lambda\| \geq 1 \).

Proof. Let \( \lambda \notin \sigma(H_0) \). If \( \lambda \in \sigma_p(H_V) \), then the claim follows from part (i) of Corollary 4. However, the following alternative argument applies as well. Given any \( \lambda \in \sigma_c(H_V) \cup \sigma_p(H_V) \), let \( \{\psi_n\}_{n \in \mathbb{N}} \subset D(H_V) \) be a corresponding sequence satisfying \( \|\psi_n\| = 1 \) for every \( n \in \mathbb{N} \) and \( H_V \psi_n - \lambda \psi_n \to 0 \) as \( n \to \infty \). By Theorem 11 the sequence \( \{\psi_n\}_{n \in \mathbb{N}} \) defined by \( \phi_n = A \psi_n \) has non-zero elements for all sufficiently large \( n \) and

\[
\|K_\lambda\| \geq \lim_{n \to \infty} \frac{|(\phi_n, K_\lambda \phi_n)|}{\|\phi_n\|^2} = 1,
\]

where the estimate is due to the Schwarz inequality. \( \square \)

6 The remaining proofs

Proof of Theorem 3

First, let us note that given 1.6, Corollary 4 implies that \( \sigma_p(H_V) \subset \sigma_p(H_0) \) and, noting that \( \|K_z\| = \|K_z^*\| \) for every \( z \in \rho(H_0) \), Corollary 5 implies that \( \sigma_c(H_V) \subset \sigma_c(H_0) \). Here we used that the residual spectrum of a self-adjoint operator is empty. Taken together we thus showed that

\[
[\sigma_p(H_V) \cup \sigma_c(H_V)] \subset \sigma_p(H_0),
\]

which is the first statement of part (ii) of Theorem 3. Now let us note that in general, \( \sigma_c(H_V) \subset \sigma_c(H_0) \), so by Corollary 4 we obtain that

\[
\sigma_c(H_V) \subset \sigma(H_0).
\]

The inclusions 1.6 and 1.7 ensure that \( \sigma(H_V) \subset \sigma(H_0) \). Since the reverse inclusion will be shown in the proof of Theorem 2 (which, to be sure, does not rely in any way on the results of Theorem 3), we obtain that \( \sigma(H_V) = \sigma(H_0) \), which is part (i) of Theorem 3. In particular, this implies that \( \sigma_c(H_0) \subset \sigma(H_V) \) and since \( \sigma_p(H_0) \cap \sigma_c(H_0) = \emptyset \), the inclusion 1.6 implies that \( \sigma_c(H_0) \subset \sigma_c(H_V) \), which is the second statement of part (ii) of Theorem 3. This concludes the proof. \( \square \)

Proof of Theorem 2

Interestingly enough, and in contrast to the proof of Theorem 3 given above, the proof of this theorem does not rely on the Birman–Schwinger principles. We will prove the theorem by contradiction.

So assume that \( 1.6 \) holds and set \( C_0 := \sup_{z \in \mathbb{C} \setminus \mathbb{R}} \|K_z\| < \infty \). Moreover, let us suppose that there exists \( \lambda_0 \in \sigma(H_0) \cap \rho(H_V) \). We will derive a contradiction in four steps:

Step 1. Since \( \lambda_0 \in \sigma(H_0) \) and \( H_0 \) is selfadjoint there exists a sequence \( \{f_n\} \) in \( D(H_0) \) such that \( \|f_n\| = 1, n \in \mathbb{N} \), and \( (H_0 - \lambda_0)f_n \to 0 \) for \( n \to \infty \). In particular, since \( \lambda_0 \in \mathbb{R} \) for every \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) we obtain that

\[
A[(H_0 - \lambda)^{-1} - (\lambda_0 - \lambda)^{-1}]f_n = (\lambda_0 - \lambda)^{-1}A(H_0 - \lambda)^{-1}(\lambda_0 - H_0)f_n \to 0 \quad (n \to \infty).
\]

Here we used that \( A(H_0 - \lambda)^{-1} = (AG_0^{-1/2})(G_0^{1/2}(H_0 - \lambda)^{-1}) \in \mathcal{B}(\mathcal{H}, \mathcal{H}') \) by assumption 1.6.

Step 2 (compare the proof of Theorem 11). We have \( L := \liminf_{n \to \infty} \|Af_n\| > 0 \). Indeed, suppose to the contrary. Then there would exist a subsequence \( \{f_{n_j}\} \) of \( \{f_n\} \) such that \( Af_{n_j} \to 0 \) for \( j \to \infty \). Since \( \lambda_0 = \lim_{n \to \infty} \rho(H_V) \), we could then estimate

\[
|b_{Vj}(f_{n_j}, (H_V^* - \lambda_0)^{-1}f_{n_j})| \\
= |b_{j}(f_{n_j}, (H_V^* - \lambda_0)^{-1}f_{n_j}) + v_{n_j}(f_{n_j}, (H_V^* - \lambda_0)^{-1}f_{n_j}) - \lambda_0(f_{n_j}, (H_V^* - \lambda_0)^{-1}f_{n_j})| \\
= |b_{j}(f_{n_j}, (H_V^* - \lambda_0)^{-1}f_{n_j}) + ( Af_{n_j}, B(H_V^* - \lambda_0)^{-1}f_{n_j}) - \lambda_0(f_{n_j}, (H_V^* - \lambda_0)^{-1}f_{n_j})| \\
\leq \|(H_0 - \lambda_0)f_{n_j}\| \|\|((H_V^* - \lambda_0)^{-1}f_{n_j}) + ( Af_{n_j}, B(H_V^* - \lambda_0)^{-1}f_{n_j})| \\
+ \|Af_{n_j}\|\|B(H_V^* - \lambda_0)^{-1}\|.
\]
Here we used that $B(H_V^* - \lambda_0)^{-1} \in \mathcal{B}(\mathcal{H}, \mathcal{K}')$ as follows from (2.3) and assumption (1.1). In particular, we see that

$$h_V^*(f_{n_j}, (H_V^* - \lambda_0)^{-1} f_{n_j}) = \lambda_0(\lambda, (H_V^* - \lambda_0)^{-1} f_{n_j}) \to 0 \quad (j \to \infty).$$

On the other hand, since $(H_V^* - \lambda_0)^{-1} f_{n_j} \in D(H_V^*)$, we also obtain from (2.3) that

$$h_V^*(f_{n_j}, (H_V^* - \lambda_0)^{-1} f_{n_j}) - \lambda_0(\lambda, (H_V^* - \lambda_0)^{-1} f_{n_j}) = (f_{n_j}, (H_V^* - \lambda_0)(H_V^* - \lambda_0)^{-1} f_{n_j}) = \|f_{n_j}\|^2 = 1$$

for all $j \in \mathbb{N}$, which leads to a contradiction. Hence $L = \liminf_{n \to \infty} \|A f_n\| > 0$.

**Step 3.** Now let $\varepsilon_0 > 0$ such that $\lambda_0 + i\varepsilon \in \rho(H_0) \cap \rho(H_V)$ for all $\varepsilon \in (0, \varepsilon_0)$. Then using the resolvent identity (2.9), the triangle inequality and the fact that

$$A[B(H_0 - \lambda_0 + i\varepsilon)^{-1}] = A[BG_0^{-1/2}G_0(H_0 - \lambda_0 + i\varepsilon)^{-1}G_0^{-1/2}] = K(\lambda_0 + i\varepsilon),$$

for all $\varepsilon \in (0, \varepsilon_0)$ we obtain that

$$A(H_V - \lambda_0 - i\varepsilon)^{-1} = A(\lambda_0 - i\varepsilon)^{-1} - A[B(H_0 - \lambda_0 + i\varepsilon)^{-1}]A(H_V - \lambda_0 - i\varepsilon)^{-1}$$

$$\geq (A(H_0 - \lambda_0 - i\varepsilon)^{-1} - A[B(H_0 - \lambda_0 + i\varepsilon)^{-1}]A(H_V - \lambda_0 - i\varepsilon)^{-1})$$

$$\geq (A(H_0 - \lambda_0 - i\varepsilon)^{-1} - C_0||A(H_V - \lambda_0 - i\varepsilon)^{-1}||).$$

Hence for all $\varepsilon \in (0, \varepsilon_0)$ and $n \in \mathbb{N}$ we obtain (with the $f_n$’s as in Step 1) that

$$\|A(H_V - \lambda_0 - i\varepsilon)^{-1}\| \geq (1 + C_0^{-1})\|A(H_0 - \lambda_0 - i\varepsilon)^{-1}\|$$

$$\geq (1 + C_0^{-1})\|A(H_0 - \lambda_0 - i\varepsilon)^{-1}f_n\|. \quad (6.4)$$

**Step 4.** Now fix some $\varepsilon \in (0, \varepsilon_0)$ and choose $n(\lambda_0, \varepsilon) \in \mathbb{N}$ such that, using (6.3) with $\lambda = \lambda_0 + i\varepsilon$, we have

$$\|A([H_0 - \lambda_0 - i\varepsilon]^{-1} - (-i\varepsilon)^{-1})f_n\| \leq 1 \quad (n \geq n(\lambda_0, \varepsilon)).$$

The triangle inequality implies that for $n \geq n(\lambda_0, \varepsilon)$

$$\|A([H_0 - \lambda_0 - i\varepsilon]^{-1})f_n\| \geq \frac{1}{\varepsilon}\|Af_n\| - 1$$

and hence using (6.4) we obtain that

$$\|A(H_V - \lambda_0 - i\varepsilon)^{-1}\| \geq (1 + C_0^{-1})\left(\frac{1}{\varepsilon}\|Af_n\| - 1\right), \quad n \geq n(\varepsilon, \lambda_0). \quad (6.5)$$

Now consider the lines inferior of both sides of (6.5) with respect to $n \to \infty$ and use Step 2 to obtain that

$$\|A(H_V - \lambda_0 - i\varepsilon)^{-1}\| \geq (1 + C_0^{-1})\left(\frac{L}{\varepsilon} - 1\right).$$

But since $L > 0$ and $\varepsilon \in (0, \varepsilon_0)$ was arbitrary, this implies that

$$\limsup_{\varepsilon \to 0} \|A(H_V - \lambda_0 - i\varepsilon)^{-1}\| = \infty. \quad (6.6)$$

But $\lambda_0 \in \rho(H_V)$ and the function

$$\lambda \mapsto A(H_V - \lambda)^{-1} = A(H_V - \lambda_0)^{-1} + (\lambda - \lambda_0)A(H_V - \lambda_0)^{-1}(H_V - \lambda)^{-1}$$

is analytic (hence continuous) in a neighbourhood of $\lambda_0$, so

$$\lim_{\varepsilon \to 0} \|A(H_V - \lambda_0 - i\varepsilon)^{-1}\| = \|A(H_V - \lambda_0)^{-1}\| < \infty$$

(that $A(H_V - \lambda_0)^{-1} \in \mathcal{B}(\mathcal{H}, \mathcal{K}')$ can be seen by writing the operator as $[AG_0^{-1/2}[G_0^{-1/2}H_V - \lambda_0)^{-1}][G_0^{-1/2}]$, which is okay since $D(H_V) \subset D([H_0]^{-1/2})$, and noting that here the first operator is bounded by (1.1) and the second is bounded by the closed graph theorem). This contradicts (6.6) and hence $\sigma(H_0) \subset \rho(H_V)$.

$\square$
7 Applications

In this section, we apply the abstract theorems to concrete problems.

7.1 Schrödinger operators in the Euclidean spaces

Given any positive integer \( d \), let \( H_0 := -\Delta \) in \( \mathcal{H} := L^2(\mathbb{R}^d) \) with \( \mathcal{D}(H_0) := H^2(\mathbb{R}^d) \). One has \( \sigma(H_0) = [0, +\infty) \) and the spectrum is purely absolutely continuous. The absolute value \( |H_0| \) satisfies \( \|H_0\|^{1/2} \psi\| = \|\nabla \psi\| \) for every \( \psi \in \mathcal{D}(|H_0|^{1/2}) = H^1(\mathbb{R}^d) \).

Given any \( V \in L^1_{\text{loc}}(\mathbb{R}^d) \), we use the decomposition

\[
V(x) = \text{sgn} V(x) |V(x)| = \text{sgn} V(x) |V(x)|^{1/2} |V(x)|^{1/2} \tag{7.1}
\]

for almost every \( x \in \mathbb{R}^d \). Here \( \text{sgn} z := z/|z| \) if \( z \neq 0 \) and \( \text{sgn} z := 0 \) if \( z = 0 \). We choose \( A(x) := |V(x)|^{1/2} \) and \( B(x) := \text{sgn} V(x) |V(x)|^{1/2} \). We use the same symbols \( A, B \) for the associated operators of multiplication with \( \mathcal{D}(A) = \mathcal{D}(B) = \mathcal{D}(|H_0|^{1/2}) \). Note that by the Sobolev inequality, a sufficient condition to satisfy (7.1) is \( V = V_1 + V_2 \) with \( V_1 \in L^p(\mathbb{R}^d) \) and \( V_2 \in L^\infty(\mathbb{R}^d) \), where

\[
p = 1 \quad \text{if} \quad d = 1, \\
p > 1 \quad \text{if} \quad d = 2, \quad \text{and} \quad \|V_1\|_{L^p(\mathbb{R}^d)} < C_{p,d}. \tag{7.2}
\]

Here \( C_{1,1} := \infty \) (the largeness of the norm \( \|V_1\|_{L^1(\mathbb{R}^3)} \) is unrestricted if \( d = 1 \)) and \( C_{p,d} := d(d - 2)|S^d|/4 \) if \( d \geq 3 \), where \( |S^d| \) denotes the volume of the \( d \)-dimensional unit sphere (cf. [44, Thm. 8.3]). If \( d = 2 \), an estimate on the constant \( C_{p,2} \) is also known (cf. [44, Thm. 8.5(ii)]), but we shall not need it. In summary, \( V \) falls within the class of perturbations considered in Assumption 1 and the pseudo-Friedrichs extension \( H_V \) is well defined.

Remark 11. Since \( H_0 \) is bounded from below, the associated form \( h_0[\psi] = \||H_0|^{1/2}\psi\|^2 = \|\nabla \psi\|^2, \mathcal{D}(h_0) = H^1(\mathbb{R}^d) \), is closed and bounded from below. The form of the perturbation \( V \) reads \( \psi[\psi] = \int_{\mathbb{R}^d} V|\psi|^2, \mathcal{D}(V) = H^1(\mathbb{R}^d) \). Under our assumption (7.2), the perturbed form \( h_V \) is closed and sectorial with \( \mathcal{D}(h_V) = \mathcal{D}(h_0) = H^1(\mathbb{R}^d) \). Since the Friedrichs extension of the operator \( H_0 + V \) initially defined on \( \mathcal{D} := C_0^\infty(\mathbb{R}^d) \) is the only \( m \)-sectorial extension with domain contained in \( \mathcal{D}(h_V) \) (cf. [39, Thm. VI.2.11]), it follows that the pseudo-Friedrichs extension \( H_V \) defined by Theorem 12 is actually the usual Friedrichs extension.

Spectral properties of \( H_V \) substantially differ in high dimensions \( d \geq 3 \) and low dimensions \( d = 1, 2 \).

7.1.1 High dimensions

Applying the abstract results of Theorems 3 and 4, we get the following spectral stability.

Theorem 12 ([37, Thm. 6.4] [27, Thm. 2] & [29, Thm. 3.2]). Let \( d \geq 3 \) and \( V \in L^{d/2}(\mathbb{R}^d) \). There exists a positive dimensional constant \( c_d \) such that if

\[
\|V\|_{L^{d/2}(\mathbb{R}^d)} < c_d, \tag{7.3}
\]

then

\[
\sigma(H_V) = \sigma_c(H_V) = [0, +\infty). 
\]

Moreover, \( H_V \) and \( H_0 \) are similar to each other.

Proof. The idea of the proof in all dimensions \( d \geq 3 \) is due to Frank [27]. Based on a uniform Sobolev inequality due to [35], Frank established the resolvent estimate (cf. [27, Eq. (8)])

\[
(\mathcal{H}_0 - z)^{-1}_{L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)} \leq k_{p,d} |z|^{-(d+2)/2+d/p},
\]

for all \( z \in \mathbb{C} \setminus [0, +\infty) \).
where \(2d/(d+2) \leq p \leq 2(d+1)/(d+3)\), \(1/p + 1/p' = 1\) and \(k_{p,d}\) is a positive constant. For every \(\phi, \psi \in H^1(\mathbb{R}^d)\) and \(z \in [0, +\infty)\) we obtain, taking Remark 4 into account,

\[
|\langle \phi, K_z \psi \rangle| = \left| \left| \left| V \right| \right|^{1/2} \phi, (H_0 - z)^{-1} \left| \left| V \right| \right|^{1/2} \tilde{\psi} \right|
\leq k_{p,d} |z|^{-(d+2)/2 + d/p} \left\| \left| V \right| \right|^{1/2} \tilde{\phi} \left\| \left| V \right| \right|^{1/2} \tilde{\psi} \right\|_{L^p(\mathbb{R}^d)}
\leq k_{p,d} |z|^{-(d+2)/2 + d/p} \left\| \tilde{\phi} \right\|_{L^p(\mathbb{R}^d)} \left\| \tilde{\psi} \right\|_{L^p(\mathbb{R}^d)}
\]

where \(\tilde{\psi} := (\text{sgn } V) \psi\), so \(\left\| \tilde{\psi} \right\| = \left\| \psi \right\|\). Since \(H^1(\mathbb{R}^d)\) is dense in \(L^2(\mathbb{R}^d)\), this inequality extends to the whole Hilbert space and we get

\[
\left\| K_z \right\| \leq k_{p,d} |z|^{-(d+2)/2 + d/p} \left\| V \right\|_{L^p(\mathbb{R}^d)}\left\| \psi \right\|_{L^p(\mathbb{R}^d)}.
\]

Choosing \(p := 2d/(d+2)\), we get the uniform (i.e. \(z\)-independent) bound

\[
\left\| K_z \right\| \leq k_{p,d} \left\| V \right\|_{L^{d/2}(\mathbb{R}^d)}. \tag{7.4}
\]

By assuming \(1 \leq p \leq d\) with \(c_d := k_{p,d}\), we get the validity of (1.10). It follows by Theorem 3 that the spectrum of \(H_V\) is purely continuous and equal to \([0, +\infty)\). Furthermore, the same estimates as above ensure that the supremum in (1.12) is bounded (also for \(A\) being replaced by \(B\)) from above by \(2k_{p,d} \left\| V \right\|_{L^{2/3}(\mathbb{R}^d)}\). Consequently, \(A, B\) are \(H_0\)-smooth and hence similarity of \(H_0\) and \(H_V\) follows by Corollary 4.

**Remark 12.** Assuming smallness of \(V\) in different scales of Lebesgue spaces, Theorem 12 comes back to Kato [37] Thm. 6.4. The identification of the optimal Lebesgue space \(L^{d/2}(\mathbb{R}^d)\) (thanks to the availability of the uniform Sobolev inequality [38]) and the present proof is due to Frank [27, Thm. 2], who established the absence of (discrete) eigenvalues of \(H_V\) outside \([0, +\infty)\). In [29] Thm. 3.2], Frank and Simon excluded (embedded) eigenvalues inside \([0, +\infty)\) as well. The novelty of our statement here is that we additionally show that Frank’s condition actually implies the stability of the continuous and residual spectra, too, and even the similarity of \(H_V\) to \(H_0\).

For physical applications in dimension \(d = 3\), the space \(L^{3/2}(\mathbb{R}^3)\) is too restrictive, for it excludes potentials with critical singularities \(V(x) \sim |x|^{-2}\) as \(x \to 0\). To include the singular potentials, Frank [27 Thm. 3] showed that the \(L^{3/2}\)-norm can be replaced by the Morrey–Campanato norm. Alternatively, one can use the following old observation of Kato.

**Theorem 13** ([37] Thm. 6.1). Let \(d = 3\) and \(V \in L^1_{\text{loc}}(\mathbb{R}^d)\). Let \(L\) be the integral operator in \(L^2(\mathbb{R}^3)\) with the kernel

\[
\frac{|V(x)|^{1/2} |V(y)|^{1/2}}{4\pi |x - y|}.
\]

If \(L\) is bounded and there exists a constant \(c < 1\) such that

\[
\left\| L \right\| \leq c,
\]

then the conclusions of Theorem 13 hold true.

**Proof.** The idea of the proof is based on the explicit knowledge of the integral kernel of \((H_0 - z)^{-1}\) in \(\mathbb{R}^3\):

\[
G_z(x,y) := \frac{e^{-\sqrt{\pi} |x-y|}}{4\pi |x-y|},
\]

where \(z \in \mathbb{C} \setminus (0, +\infty)\) and \(x, y \in \mathbb{R}^3\) with \(x \neq y\). We use the branch of the square root on \(\mathbb{C} \setminus (-\infty, 0]\) with positive real part. The peculiarity of dimension \(d = 3\) is that one has the uniform pointwise bound

\[
\forall z \in \mathbb{C} \setminus (0, +\infty), \ x, y \in \mathbb{R}^3, \ x \neq y, \quad |G_z(x,y)| \leq G_0(x,y)\tag{7.6}
\]

Consequently, for every \(\phi, \psi \in C_0^\infty(\mathbb{R}^3)\), one has

\[
|\langle \phi, K_z \psi \rangle| \leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (|V|^{1/2} |\phi|)(x) G_z(x,y) (|V|^{1/2} |\psi|)(y) \, dx \, dy
\leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (|V|^{1/2} |\phi|)(x) G_0(x,y) (|V|^{1/2} |\psi|)(y) \, dx \, dy. \tag{7.7}
\]

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Note that the last integral is well defined because the functions \( \phi, \psi \) are assumed to have a compact support. Using the definition of \( L \) and the fact that the space \( C_0^\infty(\mathbb{R}^3) \) is dense in \( L^2(\mathbb{R}^3) \), one gets
\[
|\langle \phi, K_z \psi \rangle| \leq (|\phi|, L |\psi|) \leq c||\phi||||\psi||
\]
for every \( \phi, \psi \in L^2(\mathbb{R}^3) \). Consequently, \( ||K_z|| \leq c \) uniformly in \( z \in \mathbb{C} \setminus [0, +\infty) \), so (1.10) holds true. Furthermore, the same estimates as above ensure that the supremum in (1.12) is bounded from above by \( 2c \). Hence, the sufficient conditions of the abstract Theorem 5 and Corollary 3 are satisfied.

It is desirable to obtain sufficient conditions which guarantee the validity of (7.4). An obvious choice is to bound the operator norm of \( L \) by its Hilbert–Schmidt norm leading to the sufficient condition
\[
||V||_{R(\mathbb{R}^3)} := \sqrt{\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |V(x)||V(y)| \frac{dx dy}{|x-y|^2}} < 4\pi,
\]
where \( ||\cdot||_{R(\mathbb{R}^3)} \) is the Rollnik norm. This weaker version of Theorem 13 is mentioned in [37, Rem. 6.2] (see also [49, Thm. III.12] and [46, Thm. XIII.21] for partial results). Note that \( R(\mathbb{R}^3) \supset L^{3/2}(\mathbb{R}^3) \) by the Sobolev inequality.

An alternative approach was followed by Fanelli, Vega and one of the present authors in [20].

**Theorem 14** ([37, Thm. 6.1], [20, Thm. 1]). Let \( d = 3 \) and \( V \in L^1_{\text{loc}}(\mathbb{R}^d) \). If there exists a constant \( c < 1 \) such that
\[
\forall \psi \in H^1(\mathbb{R}^3), \quad \int_{\mathbb{R}^3} |V||\psi|^2 \leq c \int_{\mathbb{R}^3} |\nabla \psi|^2,
\]
then the conclusions of Theorem 13 hold true.

**Proof.** First of all, notice that (7.9) is equivalent to \( ||V^{1/2}H_0^{-1/2}g||^2 \leq c ||g||^2 \) for every \( g \in R(H_0^{1/2}) \). Since \( 0 \in \sigma_c(H_0) \) (in fact, the spectrum of \( H_0 \) is purely continuous), the range \( R(H_0^{1/2}) \) is dense in \( L^2(\mathbb{R}^3) \). Consequently, \( |V^{1/2}H_0^{-1/2}| \) extends to a bounded operator in \( L^2(\mathbb{R}^3) \) with
\[
|||V^{1/2}H_0^{-1/2}|| \leq \sqrt{c}.
\]
By taking the adjoint, \( H_0^{-1/2}|V|^{1/2} \) also extends to a bounded operator in \( L^2(\mathbb{R}^3) \) with
\[
||H_0^{-1/2}|V|^{1/2}|| \leq \sqrt{c}.
\]

We come back to the inequality (7.4) valid for every \( \phi, \psi \in C_0^\infty(\mathbb{R}^3) \). Using the dominated convergence theorem, we write
\[
|\langle \phi, K_z \psi \rangle| \leq \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( |V^{1/2}| \phi \right)(x) G_{-\varepsilon^2}(x, y) \left( |V^{1/2}| \psi \right)(y) \, dx \, dy
\]
\[
= \lim_{\varepsilon \to 0^+} \left( \left( |V^{1/2}| \phi \right), (H_0 + \varepsilon^2)^{-1} |V^{1/2}| \psi \right)
\]
\[
= \lim_{\varepsilon \to 0^+} \left( (H_0 + \varepsilon^2)^{-1} |V^{1/2}| \phi, (H_0 + \varepsilon^2)^{-1} |V^{1/2}| \psi \right)
\]
\[
= (H_0^{-1/2} |V|^{1/2} |\phi|, H_0^{-1/2} |V|^{1/2} |\psi|)
\]
\[
\leq ||H_0^{-1/2} |V|^{1/2}|| \langle |\phi||\psi| \rangle
\]
\[
\leq c ||\phi|| ||\psi||.
\]
Here the last equality employs that \( |V^{1/2} |\phi||V^{1/2} |\psi| \in R(H_0^{1/2}) \). Since \( C_0^\infty(\mathbb{R}^3) \) is dense in \( L^2(\mathbb{R}^3) \), we get \( ||K_z|| \leq c \) uniformly in \( z \in \mathbb{C} \setminus [0, +\infty) \), so (1.10) holds true. Furthermore, the same estimates as above ensure that the supremum in (1.12) is bounded from above by the constant \( 2c \). Hence, the sufficient conditions of the abstract Theorem 5 and Corollary 3 are satisfied.

**Remark 13.** Except for the similarity of \( H_V \) and \( H_0 \), Theorem 14 was derived in [20] without the knowledge of Kato’s Theorem 4 from [37]. Unaware of Theorem 2, the inclusion \( \sigma_c(H_V) \subset \sigma(H_0) \) was derived by explicitly constructing a singular sequence of \( H_V \) corresponding to all points of \([0, +\infty)\).

It turns out that the hypotheses (7.4) and (7.5) are equivalent. The fact that (7.5) implies (7.4) is clear from the proof of Theorem 14. Conversely, \( L = TT^* \) with \( T := |V|^{1/2}H_0^{-1/2} \), so (36) implies \( ||T|| \leq \sqrt{c} \), which is (7.10) equivalent to (7.9).
By the Sobolev inequality, \( L^{3/2} \) holds provided that \( V \in L^{3/2}(\mathbb{R}^3) \) and (cf. \( L^2 \))
\[
\|V\|_{L^{3/2}(\mathbb{R}^3)} < C_{3/2,3} = 3^{3/2} \pi^2 / 4.
\]
This gives an estimate to the constant \( c_3 \) of Theorem 12. It turns out that this value is optimal as demonstrated by Frank [27, Thm. 2]. Outside of the range of the Lebesgue as well as Rollnik classes, sufficient conditions ensuring (7.3) follow by the Hardy inequality \(-\Delta \geq (1/4)|x|^{-2}\), see [26] Eq. (7). To conclude, let us compare the smallness sufficient conditions which ensure that the operators \( H_V \) and \( H_0 \) are similar to each other in the three-dimensional situation:

Lebesgue \( L^{3/2} \) \quad Rollnik \( R \) \quad form-subordination \quad Kato

We expect that Theorem 14 extends to higher dimensions.

**Conjecture 1.** Let \( d > 3 \) and \( V \in L^1_{\text{loc}}(\mathbb{R}^d) \). If there exists a constant \( c < 1 \) such that
\[
\forall \psi \in H^1(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} |V||\psi|^2 \leq c \int_{\mathbb{R}^d} |\nabla \psi|^2,
\]
then the conclusions of Theorem 12 hold true.

### 7.1.2 Low dimensions

The spectral stability does not hold in low dimensions \( d = 1, 2 \), because of the criticality of the Laplacian when \( d < 3 \). Indeed, it is well known (see, e.g., [16, Thm. XIII.11]) that \( H_V \) possesses at least one (discrete) negative eigenvalue whenever \( V \in C_0^\infty(\mathbb{R}^d) \) is real-valued, non-positive and non-trivial and \( d = 1, 2 \). In dimension \( d = 2 \), however, the spectral stability can be achieved by adding a magnetic field to \( H_0 \), see [25].

In any case, the Birman–Schwinger principle can be used to obtain sharp estimates for the eigenvalues, even when \( V \) is complex-valued. Here we focus on dimension \( d = 1 \), where a simple formula for the integral kernel of the resolvent of \( H_0 \) is available.

**Theorem 15** ([11, Thm. 4] & [19, Corol. 2.16]). Let \( d = 1 \) and \( V \in L^1(\mathbb{R}) \).

(i) \( \sigma_+(H_V) = \emptyset \).

(ii) \( \sigma_0(H_V) = [0, +\infty) \).

(iii) \( \sigma_p(H_V) \subset \{ \lambda \in \mathbb{C} : |\lambda| \leq \frac{1}{4}\|V\|_{L^1(\mathbb{R})}^2 \} \).

**Proof.** Property (i) is a general fact for Schrödinger operators because of the \( \mathcal{T} \)-self-adjointness property \( H_V^* = \mathcal{T} H_V \mathcal{T} \), where \( \mathcal{T} \psi := \psi \) is the complex conjugation (time-reversal operator in quantum mechanics). Consequently, if \( \lambda \) is an eigenvalue of \( H_V^* \), then necessarily \( \lambda \) is an eigenvalue of \( H_V \), so (i) follows by the general criterion (14).

The other properties employ the fact that the unperturbed resolvent \( (H_0 - z)^{-1} \) is an integral operator in \( L^2(\mathbb{R}) \) with the kernel
\[
G_z(x, y) := \frac{e^{-\sqrt{-1}|x-y|}}{2\sqrt{-z}},
\]
where \( z \in \mathbb{C} \setminus [0, +\infty) \). Consequently,
\[
\forall z \in \mathbb{C} \setminus [0, +\infty), \quad x, y \in \mathbb{R}, \quad |G_z(x, y)| = \frac{e^{-\sqrt{-1}|x-y|}}{2\sqrt{-z}} \leq \frac{1}{2\sqrt{|z|}}.
\]

Property (ii) follows because of the compactness of \( K_z \). Under the hypotheses \( V \in L^1(\mathbb{R}) \), the operator \( H_V \) is m-sectorial (cf. Remark 11). Hence, there exists a negative \( z \) with sufficiently large \( |z| \) such that \( z \in \rho(H_0) \cap \rho(H_V) \) and \( (H_0 - z)^{-1} \) is m-accretive. Then
\[
\|K_z\|_{18}^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} |V(x)||G_z(x, y)|^2 |V(y)| \, dx \, dy \leq \frac{\|V\|^2_{L^1(\mathbb{R})}}{4|z|},
\]
where $\| \cdot \|_{HS}$ denotes the Hilbert–Schmidt norm, so $K_z$ is compact. By Proposition [1]

$$(HV - z)^{-1} - (H_0 - z)^{-1} = -[\text{sgn } V |V|^{1/2}(H_0 - z)^{-1}]^*|V|^{1/2}(HV - z)^{-1}.$$ 

Since $|V|^{1/2}(HV - z)^{-1}$, $\text{sgn } V$ and $(H_0 - z)^{-1/2}$ are bounded operators, the difference of the resolvents is compact if the operator $T := |V|^{1/2}(H_0 - z)^{-1/2}$ is compact. This is the case if, and only if, $TT^*$ is compact. It remains to notice that $\|TT^*\|_{HS} = \|K_z\|_{HS}$ and recall the general stability theorem [21 Thm. IX.2.4].

Property (iii) is the main part of the theorem. Similarly as above, we have

$$\|K_z\|^2 \leq \|K_z\|_{HS}^2 \leq \frac{\|V\|^2_{L^1(\mathbb{R})}}{4|z|}$$

for every $z \in \mathbb{C} \setminus [0, \infty)$. Consequently, $\|K_z\| > 1$ if $|z| > \frac{1}{4}\|V\|^2_{L^1(\mathbb{R})}$. This proves the desired inclusion (including the embedded eigenvalues) by virtue of Corollary [1]

The same machinery has been recently applied to possibly non-self-adjoint biharmonic Schrödinger operators in [33] and the wave operator with complex-valued damped in [30]. The Birman–Schwinger principle is not limited to continuous spaces; see [31][3] for Schrödinger operators on lattices.

### 7.2 Dirac operators in the three-dimensional Euclidean space

Let $H_0 := -i \alpha \cdot \nabla + m \alpha_4$ in $\mathcal{H} := L^2(\mathbb{R}^3; \mathbb{C}^4)$ with $D(H_0) := H^1(\mathbb{R}^3; \mathbb{C}^4)$, where $m > 0$ is a constant and $\alpha := (\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_\mu$ being the usual $4 \times 4$ Hermitian Dirac matrices satisfying the anticommutation rules $\alpha_\mu \alpha_\nu + \alpha_\nu \alpha_\mu = 2\delta_{\mu\nu} I_{\mathbb{C}^4}$ for $\mu, \nu \in \{1, \ldots, 4\}$ and the dot denotes the scalar product in $\mathbb{R}^3$. One has $\sigma(H_0) = (-\infty, -m] \cup [+m, +\infty)$ and the spectrum is purely absolutely continuous.

Notice that $H_0^2 = (-\Delta + m^2) I_{\mathbb{C}^4}$, where $-\Delta + m^2$ is the self-adjoint Schrödinger operator in $L^2(\mathbb{R}^3)$ with the usual domain $H^2(\mathbb{R}^3)$. The absolute value of $H_0$ thus equals $|H_0| = \sqrt{-\Delta + m^2} I_{\mathbb{C}^4}$, which is again a self-adjoint operator when considered on the domain $H^1(\mathbb{R}^3; \mathbb{C}^4)$. The form domain of $\sqrt{-\Delta + m^2}$ equals the fractional Sobolev space $H^{1/2}(\mathbb{R}^3)$, cf. [44] Sec. 7.11. Notice that $C_0^\infty(\mathbb{R}^3)$ is dense in $H^{1/2}(\mathbb{R}^3)$, cf. [44] Sec. 7.14.

Given any $V \in L^1_{\text{loc}}(\mathbb{R}^3; \mathbb{C}^{4 \times 4})$, we use the matrix polar decomposition

$$V(x) = U(x) |V(x)| = U(x) |V(x)| |V(x)|^{1/2}$$

for almost every $x \in \mathbb{R}^3$. Here $U(x)$ is unitary and $|V(x)| = \sqrt{V(x)^* V(x)}$ as before. We set $A(x) := |V(x)|^{1/2}$ and $B(x) := |V(x)|^{1/2} U(x)^*$ as in the case of Schrödinger operators. Now, however, we have $A(x) U(x)^* \neq U(x)^* A(x)$ in general, which somewhat complicates the analysis. We use the same symbols $A, B$ for the extended operators of matrix multiplication initially defined on $D := C_0^\infty(\mathbb{R}^3; \mathbb{C}^4)$. Notice that $D$ is dense in $H^{1/2}(\mathbb{R}^3; \mathbb{C}^4) = D(|H_0|^{1/2})$.

To minimise conditions imposed on the matrix-valued potential $V$, we follow [24] and consider the matrix norm $v(x) := \| V(x) \|_{\mathbb{C}^4 \to \mathbb{C}^4}$ for almost every $x \in \mathbb{R}^3$. The non-negative scalar function $v$ belongs to $L^1_{\text{loc}}(\mathbb{R}^3)$. Note that $v(x) = \| V(x) \|_{\mathbb{C}^4 \to \mathbb{C}^4} = \| V(x) \|_{L^2(\mathbb{R}^3; \mathbb{C}^4)}$. We assume that there exist numbers $a \in (0, 1)$ and $b \in \mathbb{R}$ such that

$$\forall f \in C_0^\infty(\mathbb{R}^3), \quad \int_{\mathbb{R}^3} v(x)|f(x)|^2 \, dx \leq a \int_{\mathbb{R}^3} |\sqrt{-\Delta} f(x)|^2 \, dx + b \int_{\mathbb{R}^3} |f(x)|^2 \, dx . \tag{7.12}$$

Then Assumption [1] holds true. A sufficient condition to satisfy (7.12) is $v = v_1 + v_2$ with $v_1 \in L^3(\mathbb{R}^3)$ and $v_2 \in L^6(\mathbb{R}^3)$, where $\|v_1\|_{L^3(\mathbb{R}^3)} < (2\pi)^{3/2}$. This can be shown with help of the Hölder inequality and a quantified version of the Sobolev-type embedding $H^{1/2}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$, see [24] Prop. 1. Alternative sufficient conditions can be obtained by means of Kato’s inequality $\sqrt{-\Delta} \geq (2/\pi) |x|^{-1}$, see [24] Rem. 3.

In summary, $V$ falls within the class of perturbations considered in Assumption [1] and the pseudo-Friedrichs extension $H_V$ is well defined. Contrary to Schrödinger operators, the Dirac operators cannot be introduced via the Friedrichs extension because of the unboundedness from below of the latter.

To apply the Birman–Schwinger principle to $H_V$, one customarily uses the identity $(H_0 - z)^{-1} = (H_0 + z)(H_0^2 - z^2)^{-1}$ to get an explicit formula for the unperturbed resolvent. More specifically, $(H_0 - z)^{-1}$
is an integral operator in $\mathcal{H}$ with the kernel obtained by applying $H_0 + z$ to the Green function $G_0$, at energy $m^2 - z^2$. Estimating the norm of $K_0$ by the Hilbert–Schmidt norm and applying Corollary 4 one obtains various enclosures for the eigenvalues of $H_V$. This strategy was followed by Fanelli and one of the present authors in [24]. As an example, we mention the following result.

**Theorem 16** ([24] Thm. 2). Assume $v \in L^3(\mathbb{R}^3) \cap L^{3/2}(\mathbb{R}^3)$. If
\[
C_1 \|v\|_{L^3(\mathbb{R}^3)} + C_2 |\Re| \|v\|_{L^{3/2}(\mathbb{R}^3)} < 1,
\] where
\[
C_1 := \left( \frac{\pi}{2} \right)^{1/3} \sqrt{1 + e^{-1} + 2e^{-2}} \quad \text{and} \quad C_2 := \frac{217/6}{3e^{2/3}},
\] then $\lambda \notin \sigma_p(H_V)$.

Given a potential $V$ with sufficiently small norm $\|v\|_{L^3(\mathbb{R}^3)}$, the hypothesis excludes the existence of eigenvalues in thin tubular neighbourhoods of the imaginary axis, with the thinness determined by the norm $\|v\|_{L^{3/2}(\mathbb{R}^3)}$. Note that eigenvalues embedded in the essential spectrum $(-\infty, -m] \cup [+m, +\infty)$ are excluded as well. As an alternative result, [24, Thm. 1] provides a quantitative enclosure for more general potentials satisfying merely $V$ is a core of $H_0$. The same machinery has been recently applied to non-self-adjoint Dirac operators on lattices.

### 7.3 Schrödinger operators in three-dimensional hyperbolic space

In order to derive completely new results with the help of the Birman–Schwinger principle, we eventually consider Schrödinger operators in hyperbolic spaces. This class of operators does not seem to have been considered previously in the non-self-adjoint context except for the recent works [11, 31]. However, the study of spectral properties of self-adjoint realisations is enormous, see, e.g., [32, 33, 35, 4, 9, 6] and references therein. Here we restrict ourselves to the three-dimensional case and refer to [41] and [11] for the hyperbolic plane and higher dimensions, respectively.

Let $\mathbb{H}^3$ be the three-dimensional hyperbolic space, i.e. a complete, simply connected Riemannian manifold with all sectional curvatures equal to $-1$. There are three (isometric) standard realisations of $\mathbb{H}^3$ given by the half-space, ball and hyperboloid models (cf. [32, Sec. 1]), but we shall not need them. We denote by $H_0$ the self-adjoint Laplacian in $\mathcal{H} := L^2(\mathbb{H}^3)$, introduced in a standard way as the Friedrichs extension of the Laplace–Beltrami operator initially defined on $\mathcal{D} := C_0^\infty(\mathbb{H}^3)$. More specifically, $H_0$ is the operator associated with the closed form $h_0(\psi) := \int_{\mathbb{H}^3} |\nabla \psi|^2$ with $D(h_0) := H^1(\mathbb{H}^3)$ being the usual Sobolev space. The absolute value $|h_0|$ satisfies $\|h_0\|^{1/2} = \|\nabla \psi\|$ for every $\psi \in D(|h_0|^{1/2}) = H^1(\mathbb{H}^3)$. Note that $C_0^\infty(\mathbb{H}^3)$ is a core of $|h_0|^{1/2}$. It is well known [32, Sec. 2] that
\[
\sigma(H_0) = [1, +\infty),
\] and that the spectrum is purely absolutely continuous. The shifted operator $H_0 - 1$ is subcritical, meaning that it satisfies a Hardy-type inequality (see [2, 5] for original proofs and [3] for recent improvements)
\[
\int_{\mathbb{H}^3} |\nabla \psi|^2 - \int_{\mathbb{H}^3} |\psi|^2 \geq \frac{1}{4} \int_{\mathbb{H}^3} \frac{|\psi(x)|^2}{\rho(x, x_0)^2} \, dx ,
\] where $\rho(x, x_0)$ denotes the Riemannian distance between the points $x, x_0 \in \mathbb{H}^3$ and $x_0$ is fixed.

Now let $V \in L_{loc}^1(\mathbb{H}^3)$ and make the same decomposition (7.11) as in the Euclidean case. The operators $A, B$ are defined analogously. We assume the subordination condition
\[
\exists C < 1, \quad \forall \psi \in H^1(\mathbb{H}^3), \quad \int_{\mathbb{H}^3} |V||\psi|^2 \leq C \left( \int_{\mathbb{H}^3} |\nabla \psi|^2 - \int_{\mathbb{H}^3} |\psi|^2 \right) .
\] Then Assumption 1 holds true and the pseudo-Friedrichs extension $H_V$ is well defined. It coincides with the usual $m$-sectorial Friedrichs extension in this case, because (7.16) ensures that $V$ is relatively
form-bounded with respect to $H_0$ with the relative bound less than 1. In view of (7.15), a sufficient condition to satisfy (7.10) is given by the pointwise inequality $|V(x)| ≤ (c/4)ρ(x,x_0)^{-2}$ for almost every $x \in \mathbb{H}^3$.

We note that (7.10) implies that the shifted operator $H_V - 1$ is m-accretive and hence, in particular, the spectrum of $H_V$ is contained in the complex half-plane $\{ \lambda : \Re \lambda ≥ 1 \}$. Actually, a much stronger statement is true.

**Theorem 17.** If (7.10) holds, then

$$\sigma(H_V) = \sigma_c(H_V) = [1, +\infty).$$

Moreover, $H_V$ and $H_0$ are similar to each other.

**Proof.** The proof is similar to the proof of Theorem 14. We start with an equivalent formulation of (7.16).

Writing $g := (H_0 - 1)^{1/2}\psi$ in (7.16), we have

$$\|V^{1/2}(H_0 - 1)^{-1/2}g\|^2 ≤ c\left(\|\nabla(H_0 - 1)^{-1/2}g\|^2 - \|(H_0 - 1)^{-1/2}g\|^2\right) = c\|g\|^2.$$

Since $1 \in \sigma_c(H_0)$ (in fact, the spectrum of $H_0$ is purely continuous), the range of $(H_0 - 1)^{-1/2}$ is dense in $L^2(\mathbb{H}^3)$ and we see that (7.17) is equivalent to

$$\|V^{1/2}(H_0 - 1)^{-1/2}\|^2 ≤ c.$$

It follows (by taking the adjoint) that also

$$\|(H_0 - 1)^{-1/2}|V|^{1/2}\|^2 ≤ c.\tag{7.18}$$

The main ingredient of the proof is the explicit form of the integral kernel $G_z(x,y)$ of the unperturbed resolvent $(H_0 - z)^{-1}$ which is given by

$$G_z(x,y) := \frac{e^{-\sqrt{(z-1)}\rho(x,y)}}{4\pi \sinh \rho(x,y)},\tag{7.19}$$

where $z \in \mathbb{C} \setminus (1, +\infty)$ and $x,y \in \mathbb{H}^3$ with $x \neq y$. To get (7.19), one may integrate the formula for the heat kernel [18, p. 179] over positive times. As in the Euclidean case (cf. (7.6)), one has the uniform pointwise bound

$$\forall z \notin (1, +\infty), \ \forall x,y \in \mathbb{H}^3, \ x \neq y, \ \ |G_z(x,y)| \leq G_1(x,y).$$

Consequently, for every $\phi, \psi \in C_0^\infty(\mathbb{H}^3)$, one has

$$\langle (\phi, K_z\psi) \rangle \leq \int_{\mathbb{H}^3} \int_{\mathbb{H}^3} (|V|^{1/2}\phi)(x) G_z(x,y) (|V|^{1/2}\psi)(y) \, dx \, dy$$

$$\leq \int_{\mathbb{H}^3} \int_{\mathbb{H}^3} (|V|^{1/2}\phi)(x) G_1(x,y) (|V|^{1/2}\psi)(y) \, dx \, dy$$

$$= \lim_{\epsilon \to 0^+} \int_{\mathbb{H}^3} \int_{\mathbb{H}^3} (|V|^{1/2}\phi)(x) G_1-e^{-\epsilon^2}(x,y) (|V|^{1/2}\psi)(y) \, dx \, dy$$

$$= \lim_{\epsilon \to 0^+} (\|V|^{1/2}\phi, (H_0 - 1 + \epsilon^2)^{-1}|V|^{1/2}\psi)$$

$$= \lim_{\epsilon \to 0^+} ((H_0 - 1 + \epsilon^2)^{-1/2}|V|^{1/2}\phi, (H_0 - 1)^{-1/2}|V|^{1/2}\psi)$$

$$= (\|H_0 - 1\|^{-1/2}|V|^{1/2}\phi, (H_0 - 1)^{-1/2}|V|^{1/2}\psi)$$

$$\leq c\|\phi\|\|\psi\|.$$

Here the limits are justified with help of the dominated convergence theorem and the last inequality follows by (7.18). Since $C_0^\infty(\mathbb{H}^3)$ is dense in $L^2(\mathbb{H}^3)$, we get $\|K_z\| ≤ c$ uniformly in $z \in \mathbb{C} \setminus [1, +\infty)$, so (7.10) holds true. Furthermore, the same estimates as above ensure that the supremum in (7.12) is bounded from above by the constant $2c$. Hence, the sufficient conditions of the abstract Theorem 3 and Corollary 3 are satisfied.
A Kato’s and pseudo-Friedrichs extensions coincide

Suppose that $H_0, A, B$ satisfy Assumption\cite{22} and that in addition $A, B$ are closed and smooth relative to $H_0$. Moreover, suppose that

$$D(A) = D(B) = D(|H_0|^{1/2}).$$

Let $H_V$ denote the pseudo-Friedrichs extension constructed in Section\cite{22} and let $\tilde{H}_V$ denote the closed extension of $H_0 + B^*A$ provided by Kato’s Theorem\cite{24}.

**Proposition 2.** Given the above assumptions we have $H_V = \tilde{H}_V$.

**Proof.** By [37] Theorem 1.5], $A$ is smooth relative to $H_V$ and $B$ is smooth relative to $\tilde{H}_V$, hence $D(\tilde{H}_V) \subset D(A) = D(|H_0|^{1/2})$ and $D(\tilde{H}_V) \subset D(B) = D(|H_0|^{1/2})$, which establishes two of the uniqueness requirements of Theorem\cite{25} It remains to verify (2.3) and (2.8). Let $\phi \in D(|H_0|^{1/2})$ and $\psi \in D(\tilde{H}_V)$. Given $\xi \in \mathbb{C} \setminus \mathbb{R}$, let $g \in \mathfrak{F}$ be the unique vector satisfying $\psi = (\tilde{H}_V - \xi)\iota g$. Then, using (1.14),

$$h_0(\phi, \psi) = (G_0^{1/2} \phi, H_0 G_0^{-1/2} \psi) = (G_0^{1/2} \phi, H_0 G_0^{-1/2} (\tilde{H}_V - \xi)^{-1} g)
= (G_0^{1/2} \phi, H_0 G_0^{-1/2} (H_0 - \xi)^{-1} g) - (G_0^{1/2} \phi, H_0 G_0^{-1/2} (H_0 - \xi)^{-1} B^* A (\tilde{H}_V - \xi)^{-1} g)
= (\phi, g) + \xi (\phi, (H_0 - \xi)^{-1} g) - (G_0^{1/2} \phi, H_0 G_0^{-1/2} (H_0 - \xi)^{-1} B^* A \psi).
$$

If $\phi \in D(H_0)$, then

$$(G_0^{1/2} \phi, H_0 G_0^{-1/2} (H_0 - \xi)^{-1} B^* A \psi) = (H_0 \phi, (H_0 - \xi)^{-1} B^* A \psi)
= [(H_0 - \xi)^{-1} B^*]^* H_0 \phi, A \psi)
= (B(H_0 - \xi)^{-1} H_0 \phi, A \psi)
= (B \phi, A \psi) + \xi (B(H_0 - \xi)^{-1} \phi, A \psi)
= \nu(\phi, \psi) + \xi (\phi, (H_0 - \xi)^{-1} B^* A \psi),$$

where we used that $(H_0 - \xi)^{-1} B^* = [(H_0 - \xi)^{-1} B^*]^* = [B(H_0 - \xi)^{-1}]^*$ which follows from the fact that $B^*$ is densely defined (since $B$ is closed) and $[(H_0 - \xi)^{-1} B^*] = B(H_0 - \xi)^{-1} \in \mathcal{B}(\mathfrak{F}, \mathfrak{F}')$ is densely defined as well. The obtained identity extends to all $\phi \in D(|H_0|^{1/2})$, since $D(H_0)$ is a core of $D(|H_0|^{1/2})$. Therefore, using (1.14) again,

$$h_V(\phi, \psi) = h_0(\phi, \psi) + \nu(\phi, \psi)
= (\phi, g) + \xi (\phi, (H_0 - \xi)^{-1} g) - \xi (\phi, (H_0 - \xi)^{-1} B^* A \psi)
= (\phi, g) + \xi (\phi, (\tilde{H}_V - \xi)^{-1} g)
= (\phi, (\tilde{H}_V - \xi) \psi) + \xi (\phi, \psi) = (\phi, \tilde{H}_V \psi),$$

for every $\phi \in D(|H_0|^{1/2})$ and $\psi \in D(\tilde{H}_V)$. This establishes (2.3). The validity of (2.8) can be proved in the same manner. The uniqueness of the pseudo-Friedrichs extension ensures that necessarily $H_V = \tilde{H}_V$ as desired.

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