On Conservative and Monotone
One-dimensional Cellular Automata and
Their Particle Representation

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Abstract

Number-conserving (or conservative) cellular automata have been used in several contexts, in particular traffic models, where it is natural to think about them as systems of interacting particles. In this article we consider several issues concerning one-dimensional cellular automata which are conservative, monotone (specially “non-increasing”), or that allow a weaker kind of conservative dynamics. We introduce a formalism of “particle automata”, and discuss several properties that they may exhibit, some of which, like anticipation and momentum preservation, happen to be intrinsic to the conservative CA they represent. For monotone CA we give a characterization, and then show that they too are equivalent to the corresponding class of particle automata. Finally, we show how to determine, for a given CA and a given integer $b$, whether its states admit a $b$-neighborhood-dependent relabelling whose sum is conserved by the CA iteration; this can be used to uncover conservative principles and particle-like behavior underlying the dynamics of some CA.

Complements at http://www.dim.uchile.cl/~anmoreir/ncca

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1 Introduction

Cellular automata (CA) are discrete dynamical systems, where states taken from a finite set of possible values are assigned to each cell of some regular lattice; at each time step, the state of a cell is updated through a function whose inputs are the states of the cell and its neighbors at the previous time step. They are useful models for systems of many identical elements when the dynamics depends only on local interactions. Conservative (or “number-conserving”) cellular automata represent a special class of CA, in which the sum of all the states, that are integers, remains constant as the system is iterated. This property arises naturally when modeling phenomena such as traffic flow ([NS92]), eutectic alloys ([Koh89,Koh91], or the exchange of goods between neighboring individuals. When number-conservation is not apparent for the initial system, its detection can be interesting by itself, and may help to prove dynamical properties.

Necessary and sufficient conditions for a CA to be number-conserving are given in [BF98] for one dimension and states \{0, 1\}, and in [BF01] for one dimension and states \{0, ..., q − 1\}; a generalization for two and more dimensions is found in [DFR03]. In [Mor03] the definition—and the characterizations—are extended to allow general sets of states \(S \subset \mathbb{Z}\), and an algorithm is given to decide, for any CA, whether its states can be relabeled with integer values, so as to make it number-conserving. In [MI98] and [MTI99] the universality of reversible, number-conserving “partitioned” CA is proved for one and two dimensions, respectively. In [Mor03] the universality of usual (not partitioned) conservative CA in one dimension is proved. In fact, it is shown that any one-dimensional CA can be simulated by a conservative CA; this proves the existence of intrinsically universal conservative CA in the sense defined in [Oli01]; this notion of universality is stronger than the usual one (the ability to simulate universal Turing machines). A construction of a logically universal conservative CA in two dimensions is given in [IFIM02]; they also construct a self-reproducing model in a two-dimensional conservative CA, by embedding in it the well known Langton’s loops. Another interesting work is found in [DFR], where the CA classifications of Kůrka [Kůr97] and Braga [BCFM93,BCFV95] are intersected and the existence of conservative CA in the resulting classes is checked. A recent article by Fukš [Fuk] considers probabilistic conservative CA.

In the articles of Boccara and Fukš ([BF98,BF01]) the necessary and sufficient condition was used to list and study all the conservative CA rules with small neighborhoods and small number of states. For all the rules they study, they give a motion representation: the state of a cell is interpreted as the number of particles in it, and the CA rule is interpreted as an operator that governs the interaction of these identical, indestructible particles. Fukš [Fuk00] and
Pivato [Piv02] have independently shown that this interpretation is always possible (in the one-dimensional case). In the same spirit but with a very general definition of “particles”, Kürka [Kür03] has recently considered CA with *vanishing* particles.

For the sake of completeness and to avoid confusions, it is worth mentioning other contexts in which particles have been considered. On one hand, there are the *interacting particle systems* (IPS), with a long history in probability theory [Lig85], and the *lattice gases*, some of them with associated CA models [Boo91]; in general, they cannot be written as conservative CA. The well-known two-dimensional Margolus CA [Mar84] is number-conserving and was designed to allow rich interactions of particles; it does not fit in the definition given here, because of its alternating neighborhood. Particles have been widely used in computer graphics [HE88], sometimes using CA with the neighborhood of Margolus [TET95]. The word “particle” is also used to describe emergent particle-like structures that propagate in CA [BNR91,DMC94,HC97,HSC01]; in this last sense, it is close to the spirit of our last section, and to that in [Kür03].

In this article we consider one-dimensional cellular automata; Section 2 gives the necessary definitions and reviews (and generalizes) some relevant previous results, while Section 3 gives our definition of *particle automata* (PA) as a formalism for motion representation. Section 4 deals with several issues related to conservative CA. First we prove (again) their equivalence with the (conservative) PA; then we discuss several behaviors that PA may exhibit, showing that some of them (like anticipation and global cycles) may be intrinsic to some conservative CA. We also consider the special properties of state-conservation (where a sensible particle representation will recognize each state as a different kind of particle) and momentum preservation (which, in spite of being defined in terms of the PA, depends only on the conservative CA it represents). The main result of the paper is in Section 5, where we characterize non-increasing CA and show how to represent them with particle automata. Finally, Section 6 considers CA where the states can be relabelled, in a way that depends on the neighbors of a cell, in order to obtain a conservative dynamics (and a representation in terms of particles).

## 2 Definitions and Some Previous Results

**Cellular automata**: A *one-dimensional cellular automaton* (CA) with set of states $Q = \{0, \ldots, q - 1\}$, is any continuous function $F : Q^Z \to Q^Z$ which commutes with the shift. It is well known that cellular automata correspond to the functions $F$ that can be expressed in terms of a local function: $F(c)_i = f(c_{i+N})$, for all $c \in Q^Z$, $i \in Z$, and $N$ a fixed finite subset of $Z$, called the
neighborhood of $F$. $N$ can always be assumed to be an interval of integers which includes the origin, and we write $F(c)_{i+d} = f(c_{i+N})$, with $N = \{0, \ldots, n-1\}$ and $d \in \mathbb{Z}$ an offset; rules with the same $f$ but different $d$ will be identical up to a shift. It will be useful to define, for $n \in \mathbb{N}$ and $Q = \{0, \ldots, q-1\}$,

\[ \mathcal{A}(q, n) = \{ f : Q^{\{0, \ldots, n-1\}} \to Q \} \]

Any CA can then be expressed by an element of $\mathcal{A}(q, n)$ for some $q$ and $n$, combined with an offset $d$ which tells where the image of the neighborhood is placed. The 256 elementary CA, for instance, correspond to $\mathcal{A}(2, 3)$, usually with $d = 1$. $\mathcal{A}$ will denote the union of $\mathcal{A}(q, n)$ over all $q$ and $n$.

A common shorthand notation for cellular automata is the codification used by Wolfram [Wol86]: the code for an element $f \in \mathcal{A}(q, n)$ is given by

\[ \text{Code}(f) = \sum_{(x_1, \ldots, x_n) \in Q^n} f(x_1, \ldots, x_n) q^{\sum_{k=1}^{n} q^{n-k} x_k} \]

**Configurations:** An element in $Q^\mathbb{Z}$ is called a configuration. A configuration is said to be finite if all but a finite number of its components are 0. A configuration $c$ is said to be periodic if $c_i = c_{i+p}$, for all $i$, for some $p \in \mathbb{Z}$, $p \neq 0$; in this case, $p$ is said to be a period of $c$.

**Monotone and conserved quantities:** Consider a CA $F$ on $\mathbb{Z}$, and let $C_P$ be the set of all periodic configurations in $\mathbb{Z}$; for each $c \in C_P$ choose a period $p(c)$. Let $\phi$ be a function $\phi : Q^b \to \mathbb{R}$, where $b$ is a nonnegative integer. Then $\phi$ is said to be a non-increasing additive quantity under $F$ if and only if

\[ \sum_{k=0}^{p(c)-1} \phi(F(c)_k, \ldots, F(c)_{k+b-1}) \leq \sum_{k=0}^{p(c)-1} \phi(c_k, \ldots, c_{k+b-1}), \quad \forall c \in C_P. \]  

(1)

Similarly, $\phi$ is said to be non-decreasing additive quantity if condition (1) holds with the inequality in the other direction. It is easy to see that $\phi$ is non-increasing if and only if $-\phi$ is non-decreasing. If $\phi$ is both non-increasing and non-decreasing, it is said to be a conserved additive quantity (in this case, (1) holds with an equality sign). We say that $\phi$ is monotone if it is either non-decreasing or non-increasing. In [HT91] it is said that “an additive conserved quantity is a discrete-time analog of what we usually call a conserved quantity, such as energy, momentum and charge of a physical system”; the sentence can be rephrased for the monotone case.

**Finitary characterization:** The previous definitions consider the addition of a density function over a period of a periodic configuration. Another pos-
Possibility would be to consider the sum over finite configurations: we may say
that $\phi$ is a finitely non-increasing additive quantity if condition (1) holds for all
finite $c$ in $\mathbb{Z}$, instead of $C_P$, with the sums being taken now over the whole $\mathbb{Z}$.
(Here we are assuming that $\phi(0, \ldots , 0) = 0$; if this is not the case, we consider
$\phi = \phi - \phi(0, \ldots , 0)$ instead.) It turns out that the two notions are equivalent:

**Theorem 1 (Generalized from [DFR03])** Let $F$ be a CA and $\phi$ be a function $\phi : S^b \rightarrow \mathbb{R}$. Then $\phi$ is an additive conserved (non-increasing, non-decreasing) quantity for $F$ if and only if it is an additive finitely conserved (non-increasing, non-decreasing) quantity for $F$.

**Sketch of the proof.** In [DFR03] the equivalence is proved for conserved quantities, in dimension 1, when $\phi : Q \rightarrow Q$ is the identity; however, their proof includes both the non-increasing and the non-decreasing cases, and can be easily extended to the case of a general $\phi$. In one direction the proof is trivial: if the condition holds for all periodic configurations, and $c$ is a finite configuration, then the condition is shown to hold for $c$ by building a periodic configuration with blocks that include the non-zero part of $c$. On the other hand, if the condition is not verified by a periodic configuration with repeated word $w$, then it will be not verified for a finite configuration of the form $\ldots 000wN000\ldots$, for $N$ large enough: the surplus (or deficit) of the periodic configuration is amplified by the growing $N$, while the only terms that could reduce it (those corresponding to a neighborhood of $0w$ and $w0$) remain fixed.

Notice that the same argument can be also extended to higher dimensions: by repeating an $n$-dimensional pattern enough times, its surplus will be amplified as $N^n$, while the terms corresponding to the border, though not fixed, will grow only as $N^{n-1}$. □

The following theorem is a useful characterization of conserved quantities in one-dimensional CA.

**Theorem 2 (Hattori and Takesue [HT91])** Let $F$ be a one-dimensional CA with local rule $f \in \mathcal{CA}(q, n)$. Let $a$ be an arbitrary element in $Q = \{0, \ldots , q - 1\}$. Then $\phi : Q^b \rightarrow \mathbb{R}$ is an additive conserved quantity under $F$ if and only if

$$
\phi_f(x_0, \ldots , x_{b+n-2}) - \phi(x_0, \ldots , x_{b-1}) = \\
\sum_{i=1}^{b+n-2} \left\{ -\phi_f(a_i, a, x_0, \ldots , x_{b+n-2-i}) + \phi_f(a_i, a, x_1, \ldots , x_{b+n-1-i}) \right\} \\
+ \sum_{i=1}^{b-1} \left\{ \phi(a_i, a, x_0, \ldots , x_{i-1}) - \phi(a_i, a, x_1, \ldots , x_i) \right\} 
$$

(2)
for all $x_0, \ldots, x_{b+n-2} \in Q$, where
\[
\phi_f(x_0, \ldots, x_{b+n-2}) = \phi(f(x_0, \ldots, x_{n-1}), \ldots, f(x_{b-1}, \ldots, x_{b+n-2}))
\]

**Monotone and conservative CA:** A cellular automaton is said to be non-increasing (non-decreasing, conservative) if the identity of its set state is a non-increasing (non-decreasing, conservative) quantity for its dynamics. Since the condition depends only on the local rule of the CA, and not on the offset, we will define $\mathcal{CA}^-(q, n)$, $\mathcal{CA}^0(q, n)$ and $\mathcal{CA}^+(q, n)$ to be the non-increasing, conservative and non-decreasing rules in $\mathcal{CA}(q, n)$, respectively.

Theorem 2 implies that $f \in \mathcal{CA}^0(q, n)$ if and only if, for all $(x_1, \ldots, x_n) \in Q^n$,
\[
f(x_1, \ldots, x_n) = x_1 + \sum_{k=1}^{n-1} \left\{ f(0, \ldots, 0, x_2, \ldots, x_{k+1}) - f(0, \ldots, 0, x_1, \ldots, x_k) \right\}
\]  

(3)

This characterization, given in [BF01], can be generalized to higher dimensions, though its explicit form becomes hard to write (it was done by [DFR03]).

**Some more notation:** The letter $q$ will always denote the number of states, and the letter $Q$ will denote the set $\{0, \ldots, q-1\}$. With $\bar{0}$ we will denote an infinite sequence of zeroes (thus, $\bar{0}w\bar{0}$ denotes a word $w$ surrounded by infinite zeroes). For $f \in \mathcal{CA}(q, n)$, we will denote by $f(u/v)$ the block image of word $u$ when followed by word $v$:
\[
f(u/v) = f(w_0, \ldots, w_{n-1}) f(w_1, \ldots, w_n) \ldots f(w_{|u|-1}, \ldots, w_{|u|+n-2})
\]
where $w = uv$. Of course, only the $n-1$ first elements in $v$ contribute to $f(u/v)$. Furthermore, we denote $f(u) = f(u/\epsilon)$, for any $u$ with $|u| \geq n$, where $\epsilon$ is the empty word.

With this notation, the condition for $f \in \mathcal{CA}(q, n)$ to belong to $\mathcal{CA}^-(q, n)$ can be restated as
\[
\sum_{k=0}^{|w|-1} f(w/w)_k \leq \sum_{k=0}^{|w|-1} w_k \quad \forall w \in Q^*
\]

(4)

If $g(\bullet)$ is a vector-valued function, we will use the notation $[g(\bullet)]_k$ to refer to its $k$-th component. Finally, we will use the function $(\bullet)_+$, defined as $(x)_+ = x$ for $x \geq 0$, and $(x)_+ = 0$ for $x < 0$. 

6
3 Particle Automata and Motion Representations

A common way to look at conservative CA is through their representation in terms of particles: the state of each cell is interpreted as the number of particles contained in it, and a rule is given describing the motion that these particles will have, depending on the local context; we will add the possibility of vanishing, and will formalize this as particle automata.

A particle automaton (PA) with set of states \( Q = \{0, \ldots, q - 1\} \) will act on \( Q^Z \), like a CA, and like a CA it will be defined by a local rule (or rather, by a set of rules) taking as input the states of a local neighborhood, which is again some \( N \) of the form \( \{-\ell, \ldots, r\} \).

For a local configuration \( w = c_{-\ell}, \ldots, c_0, \ldots, c_r \) with \( c_0 > 0 \), a function \( g_{c_0} \) will give the new positions of the \( c_0 \) particles at the origin: we have a set \( (g_i)_{i=1,\ldots,q-1} \) of functions

\[
g_i : Q^\ell \times Q^r \rightarrow (N \cup \dagger)\]

where the dagger (\( \dagger \)) represents the “vanishing” option. Thus, for a configuration \( c \in Q^Z \); the \( k \)-th particle at position \( i \) (with \( 1 \leq k \leq c_i \)) will

\[
\begin{cases}
\text{vanish if } [g_{c_i}(c_{i-\ell}, \ldots, c_{i-1}, c_{i+1}, \ldots, c_{i+r})]_k = \dagger \\
\text{move to position } i + [g_{c_i}(c_{i-\ell}, \ldots, c_{i-1}, c_{i+1}, \ldots, c_{i+r})]_k \text{ otherwise}
\end{cases}
\]

Since the particles are undistinguishable, the order of the components of each \( g_i \) is irrelevant; only the number of components mapping to each element of \( (N \cup \dagger) \) is important.

Some examples may help to make the definition clear. Consider a PA with \( q \) states that moves all particles one position to the left. It will have \( \ell = 1, r = 0 \) (i.e., \( N = \{-1, 0\} \)), and

\[
g_\alpha(\beta) = \left(\underbrace{-1, \ldots, -1}_\alpha\right) \quad \text{for all } 0 < \alpha < q, \ 0 \leq \beta < q
\]

As a second example, consider a PA with \( Q = \{0, 1, 2\} \), \( \ell = 1, r = 2 \), and \( g_1, g_2 \) such that \( g_1(0, 2, 0) = (\dagger) \) and \( g_2(1, 0, 0) = (0, 2) \). Figure 1 shows its effect on the configuration \( 012 \bar{0} \).

Thus, a PA \( G \) is defined by a tuple \( G = (q, N, (g_i)_{i=1,\ldots,q-1}) \). The global action of \( G \) on \( c \in Q^Z \) is given by

\[
G(c)_i = \min\{ q - 1 , \# \{(j,k) : j + [g_{c_j}(c_{j-\ell}, \ldots, c_{j-1}, c_{j+1}, \ldots, c_{j+r})]_k = i \} \}
\]
In other words, $G(c)_i$ is the number of particles arriving at $i$, with a maximum of $q - 1$; this last condition prevents "overflows", but, in any case, any rule can always be fixed to avoid needing this overflow control, by sending the excess of particles to "†" (though this may require an extension of the neighborhood).

We will denote by $\mathcal{PA}(q, n)$ the set of all PA with $q$ states and neighborhood of size $|N| = \ell + r + 1 = n$, and by $\mathcal{PA}^0(q, n)$ the set of conservative PA: those members of $\mathcal{PA}(q, n)$ such that their functions $g_i$ go to $N^i$ (nothing vanishes), and avoid the overflow (i.e., the minimum in the above definition is always the right term). Notice that unlike the elements of $\mathcal{CA}(q, n)$, an element of $\mathcal{PA}(q, n)$ is not combined with an offset to define the actual PA.

**Motion Representation:** In [BF98] and [BF01] Boccara and Fuk dés give motion representations for each of the rules they study; we will follow their notation, which is an efficient and intuitive way of expressing a PA. A motion representation is a list of specific local configurations of a given a cell, with arrows indicating the motion to be performed by the particle(s) located in that cell for all these local configurations. As a simplification, if for a given local configuration, the particles do not move, then this configuration is not listed. Some examples may clarify this point. Consider

$$M_1 = \{ \text{↶10, 0011} \} \quad , \quad M_2 = \{ \text{↶20, 21} \}$$

$M_1$, for $q = 2$, is read as follows. If a particle sees an empty cell on its right, it moves to it. If that cell is occupied, but the two cells on its left are empty, it moves to the closest one. In any other case, it keeps its current position. (The neighborhood in this case is $\{-2, -1, 0, 1\}$.) Numbers may be added to the arrows when a cell may be occupied by more than one particle. This is the case in $M_2$, with $q = 3$. If two particles are in a cell, then one, two, or none of them may move to the right, depending on the space available there. Here are two more examples:

$$M_3 = \{ \text{↶k, } k = 1, \ldots, q - 1 \} \quad , \quad M_4 = \{ \text{↷10} \}$$

In $M_3$ the bullets (●) are wildcards. In this example, everything moves two steps to the left, regardless of what the states in the other cells are. If we denote by $\sigma$ the shift (the CA rule such that $f(a_0, a_1) = a_1$, with offset 0),
then $M_3$ represents the CA rule $\sigma^2 = \sigma \circ \sigma$. Rule $M_4$ (for $q = 2$) shows that a particle will move to the right if, and only if, that cell is empty; its effect on $\{0, 1\}^Z$ is the same as the elementary CA rule 184, with offset 1. The notation is easily extended to include vanishing particles, by adding a hat (') for each of the particles that vanish:

$$M_5 = \{\hat{0}, \hat{1}, 2\}$$

Here the 1’s will travel to the left until they meet a 2, and then will disappear.

Note that PA do not distinguish particles; if $k$ particles arrive at a same cell, at the next time step the rule says how many of them will move to each neighboring cell, but does not say which particles are moving. However, particles are, in a sense, distinguishable, since we know how many go from each cell to each other cell at each iteration. To say that “a particle moved from $j$ to $i$, while another moved from $i$ to $k$”, is not the same as saying “a particle moved from $j$ to $k$, and another stayed at $i$”. When arbitrarily large groups move, as for instance in the shift, we have to assume that some intermediate cells with unchanging values are changing the particles they contain; otherwise, we would need an infinite neighborhood to describe the motion. If we implement the system and want to trace the particles throughout the iterations, we need to add some criterion. A sensible choice (implicitly applied in [BF01]) is to keep the order of the particles along the line.

Both CA and PA are mappings from $S^Z$ into itself. We say that a CA $F$ is a projection of a PA $G$ if $F(c) = G(c)$, for all $c \in S^Z$. The following proposition is straightforward.

**Proposition 3** Let $G$ be a PA and $F$ be a projection of $G$. Then $G \in \mathcal{PA}$ if and only if $F \in \mathcal{CA}^-$, and $G \in \mathcal{PA}^0$ if and only if $F \in \mathcal{CA}^0$.

### 3.1 CA for a given PA

**Theorem 4** For any $G \in \mathcal{PA}(q, n)$, there is a unique CA $F$ with local rule $f \in \mathcal{CA}(q, 2n - 1)$ which is a projection of $G$.

**Proof.** The function $G$ is obviously continuous and shift-commuting. Hence, it may be written as a CA, which is uniquely defined but for the neighborhood (since the neighborhood size can always be increased). The only thing we have to check is that a neighborhood of size $2n - 1$ is enough. For this note that, in the definition of PA, the state of $c'_i$ is completely determined by the particles that will move to $i$. If we denote by $\{-\ell, \ldots, r\}$ the neighborhood of $G$ ($n = \ldots$, $r\}$ the neighborhood of $G$ (
\( \ell + r + 1 \), these particles have to be in the cells \( \{ i - r, \ldots, i + \ell \} \); their behavior, in turn, is completely determined by the values in \( \bigcup_{j=-r}^{j=\ell+r} \{ j - \ell, \ldots, j + r \} \), i.e., by \( \{ c_{-r-\ell}, \ldots, c_{r+\ell} \} \).

We will denote by \( \Pi(G) \) the smallest CA \( F \) (with respect to \( |N_F| \)) that is a projection of \( G \). If, for \( G \), we also take the minimum possible neighborhood, then, in most cases, we have \( |N_G| < |N_F| \). Take, for instance, the motion rule \( M_4 \) defined above: rule 184, a 3-input rule, is the smallest CA matching it. The reason is that an occupied cell must know if its particle will leave, i.e., must look to the left, and an empty cell must know whether a particle will arrive, i.e., it must look to the right. The relation \( |N_G| < |N_F| \) is, however, not always verified, as shown by the following examples.

**Example 1:** Consider the PA \( G = (2, \{-3, \ldots, 2\}, g) \) with \( g \) described by the following motion representation:

\[
M_6 = \{ 0\overline{0}0, 1\overline{0}1, 1\overline{1}0, 1\overline{1}0 \}
\]

It may be checked that \( \Pi(G) = (\{0, 1\}, \{-2, \ldots, 2\}, f) \), with \( f \) given by code number 3221127170. A way to look at this situation is this: from the information given by the occupancy numbers of the neighboring cells of an occupied cell, we know that the particle will leave it and go to the left by looking at the two cells in that direction, but we do not know its precise destination. If we take the viewpoint of the destination cells, then we know it, since they do see the rest of the (PA) neighborhood of the particle. In other cases, a cell knows that it will remain occupied, but it does not know if the occupying particle will be the same.

**Example 2:** Here we show a family of PA for which the minimal CA actually requires the whole neighborhood allowed by Theorem 4. Consider the family of PA \( G_{\ell,r} \) with \( q = 3 \) described by the motion rule

\[
M_7 = \{ 2u1v00, 0u1v2 \}
\]

where \( u = 2^{\ell-1} \) and \( v = 2^{r-1} \).

If we have a local configuration \( \alpha u1v0u1v\beta \), which has size \( 2\ell + 1 + 2r \), the next state of the cell in the middle, which now contains a 0, depends both on \( \alpha \) and \( \beta \). Hence, the minimal CA has neighborhood \( \{-\ell - r, \ldots, \ell + r\} \), while \( G \) requires only \( \{-\ell, \ldots, r\} \).

If we want an example with only two different states, the PA needs to be a bit more complicated; one possibility is described by the motion rule

\[
M_8 = \{ 00u1v0, 11u1v0, 1\overline{1}0 \}
\]

where \( u = 0^{\ell-2} \) and \( v = 0^{r-1} \).
4 Particles for Conservative CA

4.1 PA for a given conservative CA

Theorem 5 For each conservative CA $F$ with local rule $f \in \mathcal{CA}^0(q, n)$, there exists $G \in \mathcal{PA}^0(q, 2n - 1)$, such that $\Pi(G) = F$.

Proof. This result was independently proved both by Fukš [Fuk00] and Pivato [Piv02]. Though very different, all these first proofs (including an unpublished one by ourselves) are rather long; the idea, however, is very simple, and we sketch it here. In addition, Theorem 15 in Section 5.4 will generalize it, providing yet another proof for the theorem.

Let $\{-\ell, \ldots, r\}$ be the neighborhood of $F$. We define $G$ with $N_G = \{-(\ell + r), \ldots, (\ell + r)\}$; for all $w = w_{-\ell-r}, \ldots, w_{\ell+r} \in Q^{2n-1}$ with $w_0 > 0$, we have to define the new positions $g_w(w_{-\ell-r}, \ldots, w_{-1}, w_1, \ldots, w_{\ell+r})$. This is done by computing the image of the configuration $c = \bar{0}w0$, with $w_0$ at the origin, and matching the particles in the image $c'$ with the particles in $c$, from left to right.

We claim that the new positions of the $w_0$ particles at the origin are in $\{-r, \ldots, \ell\}$. Suppose that one of them goes to a position to the left of $-r$: this means that the particles in $(c_i)_{i<0}$ did not match all the particles in $(c'_i)_{i<-r}$. Since these $c'_i$ depend only on the values of $(c_i)_{i<0}$, this would contradict $f \in \mathcal{CA}^0$ for the configuration $\bar{0}w_{-\ell-r}w_{-1}0$. The symmetric argument shows that no particle from the origin moves to the right of position $\ell$.

Now take any $u \in Q^*$, and define $c = \bar{0}uw0$ and $c' = F(c)$. As before, we match the particles in $c$ and $c'$, from left to right. Notice that we have added $\sum_{i=0}^{\lfloor |u|/2 \rfloor} u_i$ particles in the preimage; since $f \in \mathcal{CA}^0$, we have also added the same number of particles in the image, and they must be to the left of $-r$ (the rest of the image has not changed). Hence, the movement of the $w_0$ particles at the origin is the same as before, and we see that the effect of applying $G$ on any finite configuration $c$ is the same as the effect of matching the particles with those from the image of $c$ through $F$. We conclude that $F$ and $G$ have the same effect on $c$, and $F = \Pi(G)$. □

Corollary 6 In Theorem 5, if the CA $F$ has offset $\ell$ (and hence neighborhood $\{-\ell, \ldots, r\}$, with $r = n - 1 - \ell$), then $G$ has neighborhood $\{-(\ell + r), \ldots, (\ell + r)\}$, and for each $i = 1, \ldots, q - 1$, $g_i(Q^{2\ell+2r}) \subset \{-r, \ldots, \ell\}^i$ (i.e., the particles move only to $\{-r, \ldots, \ell\}$).

Example 3: If we take $f \in \mathcal{CA}^0(2, 5)$ with code #288146448 (see Table 1)
and offset 2, we obtain a PA with a motion representation given by
\[ M_9 = \{ \circ 00, \circ 01, \circ 10, \circ 11, \circ 101 \} \]
If we set the offset to 0, the result is
\[ M_{10} = \{ \circ 101, \circ 101, \circ 111 \} \]

**Example 4:** If we take \( f \in \mathcal{C}_0^0(2, 4) \) with code \#49024 (see Table 2) and use offset 1, the resulting PA has motion representation
\[ M_{11} = \{ \circ 0, \circ 00 \} \]

\[ \square \]

|     | 0000 | 01000 | 10000 | 11000 | 11001 |
|-----|------|-------|-------|-------|-------|
| 00000 | 0 | 0 | 1 | 1 | 1 |
| 00001 | 0 | 0 | 1 | 1 | 1 |
| 00010 | 0 | 1 | 0 | 1 | 0 |
| 00011 | 0 | 1 | 0 | 1 | 1 |
| 00100 | 0 | 0 | 1 | 0 | 0 |
| 00101 | 0 | 1 | 0 | 1 | 1 |
| 00110 | 0 | 0 | 1 | 0 | 0 |
| 00111 | 1 | 0 | 1 | 1 | 0 |
| 00111 | 1 | 0 | 1 | 1 | 1 |

Table 1
Lookup table for rule \#2881464448 in \( \mathcal{C}_0^0(2, 5) \)

|     | 0000 | 0100 | 1000 | 1100 | 1110 |
|-----|------|------|------|------|------|
| 00000 | 0 | 0 | 0 | 0 | 1 |
| 00001 | 0 | 0 | 0 | 0 | 1 |
| 00010 | 0 | 0 | 1 | 1 | 0 |
| 00011 | 0 | 0 | 1 | 1 | 1 |
| 00100 | 0 | 0 | 1 | 1 | 1 |
| 00101 | 0 | 1 | 1 | 1 | 1 |
| 00110 | 0 | 0 | 1 | 1 | 1 |
| 00111 | 1 | 1 | 1 | 1 | 1 |

Table 2
Lookup table for rule \#49024 in \( \mathcal{C}_0^0(2, 4) \)

### 4.2 On Some Behaviors of PA

The theorem in the preceding section shows how to construct, for a given conservative CA \( F \), a conservative PA \( G \) such that \( F = \Pi(G) \). We will call this the *canonical* PA for \( F \), since it is the only one that preserves the order of the particles along the line. However, it is not the only PA that matches \( F \) (in fact, there are infinite PA matching any given CA). For this reason, we shall discuss some behaviors that a PA may exhibit, and whether or not they can be intrinsic to certain CA; this may be relevant in the applications.

**Local cycles:** For a PA and a given configuration, we say that there is a *local m-cycle* in the iteration if there is a chain of particles \( p_0, p_1, \ldots, p_m \), with
\( p_0 = p_m, \) all located in different cells, such that, for \( i = 0, \ldots, m - 1, \) each \( p_i \) moves to the cell occupied previously by particle \( p_{i+1}. \) This behavior may be unwanted if the rule is supposed to express the motion of indistinguishable elements. The following motion representation has a 3-cycle; less trivial local cycles can be found in other PA (and it is even possible to construct PA with local cycles of arbitrary length).

\[ M_{12} = \{ \text{0110, 0110, 0110} \} \]

**Order preservation:** We say that a PA preserves the order if there is no configuration in which a particle moves from position \( i_0 \) to position \( i_1, \) while another moves from \( j_0 \) to \( j_1, \) with \( i_0 < j_0 \leq j_1 < i_1, \) or \( i_1 < j_1 \leq j_0 < i_0. \) This behavior may be wanted, for instance, when modeling cars moving on a one-lane road. Order-preserving PA do not admit local cycles.

**Anticipation:** We say that a PA with states \( \{0, \ldots, q-1\} \) exhibits anticipation, if for some configuration a cell in a state \( s \) receives, from other cells, a number of particles \( t \) such that \( t + s > q - 1 \) (as if, as a result of the rule, the neighboring cells “knew” that some particles will leave the cell). For \( q = 2, \) anticipation is what happens when a particle moves to a currently occupied cell. In the case of highway traffic, for instance, most drivers usually move anticipating the motion of the cars ahead (assuming that they are not going to stop), and this is considered in the models. An important case in which anticipation may be unwanted is the situation in which movement is optional: the particles may decide to stay in their place or to move, but if they move, they must obey the PA rule. In this situation, an anticipating rule would cause collisions (or overflows, if \( q > 2 \)).

**Global cycles:** We say that there is a global cycle if there is a chain of particles \( \{p_i\}_{i \in \mathbb{N}}, \) located at different positions, such that, for all \( i, \) particle \( p_i \) moves to the cell previously occupied by particle \( p_{i+1}. \) The reason to call this a “cycle” is the following. By removing local cycles, the chain may be assumed to approach \( \infty \) (or \(-\infty\)) as \( m \to \infty. \) If the configuration is—spatially—periodic of period \( p, \) then we may identify it with the torus \( \mathbb{Z}_p, \) and the chain is in fact a cycle. If the configuration is not periodic, as we follow the chain we will at some point find a repetition (due to the finite number of possibilities) of a block of length larger than \( |N|. \) At this point we may cut the part of the configuration starting with the block and ending with it, and repeat it to produce a periodic configuration with a cyclic replacement. A trivial example of global cycles is the PA that just shifts the configuration, as motion rule \( M_3 \) given above.

A special case are global cycles of anticipatory motion (which is always the case for global cycles when \( q = 2 \)). Such cycles may be unwanted if we are modeling agents with local information, and we do not want them to use the
information “I am on a torus”, or “I am in an infinite queue”.

**Theorem 7** For any conservative CA \( F \) there exists a \( G \in \mathcal{PA}^0 \) such that \( \Pi(G) = F \) and which preserves the order (and hence, has no local cycles). On the other hand, there do exist rules in \( \mathcal{CA}^0 \) for which anticipation and global cycles are intrinsic: with any offset, they are not the projection of any PA without these features.

**Proof.** The first part follows from the construction in Theorem 5: the canonical PA preserves the order. For the second, we just need to exhibit a CA for which the claimed property is true.

Consider a CA \( F \) with the local rule \( f \in \mathcal{CA}^0(2,5) \) of Example 3. We will show now that any \( G \in \mathcal{PA}^0 \) such that \( \Pi(G) = F \) must have global cycles (and, in particular, anticipation), for any offset. Let \( G \in \mathcal{PA}^0(q,n) \) be such that \( \Pi(G) = F \). Consider a configuration

\[
\ldots 00000.11111\ldots 1111111,000000\ldots
\]

where the sequence of 1’s is longer than \( 2n \); the dot and the comma are there for reference. The image of this configuration, assuming an offset 0, is

\[
\ldots 00001.11111\ldots 1110011,000000\ldots
\]

If the particles in the middle of the configuration *are moving*, then they are moving without seeing any 0’s; hence, the particles in the configuration \( \bar{1} \) (where all states are 1) would also be moving, and that would be a global cycle. On the other hand, if the particles in the middle are *not moving*, then there are two particles that are moving somehow from one end of the region of 1’s to the other, which is a contradiction, since the region is larger than the neighborhood of the PA.

For any other choice of the offset, we obtain the same situation, except for an offset of 2. But in that case, we can consider the configuration

\[
000000.1010101010\ldots 010101,000000
\]

whose image is

\[
000000.0010101010\ldots 010101,01000
\]

and produces the same result as above: for any PA, there is an anticipation to the right, and it allows a global cycle. \( \square \)

If anticipation and global anticipatory cycles may be intrinsic to a rule in \( \mathcal{CA}^0 \), then it is natural to ask about the decidability of these properties: given
a rule \( f \in \mathcal{CA}_0 \), can we decide if, for some offset, there does exist a PA without anticipation (and/or without global anticipatory cycles) from which the CA is the projection? The answer is not trivial. It is easy to check these properties on a given PA; however, to check them on a given CA rule, we must consider all the possible PA for which the CA is the projection. Consider the CA of Example 4: the canonical PA given by Theorem 5 shows anticipation, but a non-anticipating PA with the same projection does exist, and is described by the motion representation

\[ M_{13} = \{ 110 \ 010 \} \]

Hence, to find a non-anticipating PA we may need to drop the condition of order preservation. In fact, the situation is even worse. The canonical PA of Theorem 5 has a nice feature, stated in Corollary 6: a particle moves always to a cell that “sees” it in its CA neighborhood. As we shall see in the next sections, we can ask for the same feature when obtaining PA for monotone CA and for state-conserving CA; it seems to be “natural”, and some authors have taken for granted that only motion rules with this property need to be considered. But to avoid anticipation, we may need to drop this condition too. It is violated in the PA described by \( M_{13} \), and it can be shown that anticipation cannot be avoided without violating it for the local rule of Example 4, combined with any offset. Thus, we must consider a rather large set of PA for a given CA rule in order to decide if anticipation can be avoided or not.

**Theorem 8** For a given \( f \in \mathcal{CA}_0 \), it may be decided whether an offset \( \ell \) exists for which the CA defined by \( f \) and \( \ell \) is the projection of some PA without anticipation. If the answer is positive, that PA can be found.

**Proof.** First we must notice that for a given \( f \in \mathcal{CA}_0 \), there is at most one offset \( \ell \) for which a non-anticipating PA can exist. This was already seen in the proof of Theorem 7 for the rule of Example 3: when we evaluated a configuration of the form \( \overline{01^m0} \), with an arbitrarily large \( m \), an offset had to be imposed to prevent the movement of particles from one extremity of the 1’s to the other. For the general case, let us suppose that there is a non-anticipating PA \( G \) such that its projection is \( f \) with offset \( \ell \) (and hence neighborhood \( \{-\ell, \ldots, r\} \), with \( r = n-1-\ell \)). Then we may assume that \( G \) has a neighborhood \( \{-L, \ldots, R\} \) with \( L \geq \ell \) and \( R \geq r \). Consider a configuration \( c = \overline{0(q-1)^M0} \), with \( M \) arbitrarily large. We have the situation depicted in Table 3.

| \( c \) | \( \overline{0} \) | \( \overline{0}^r(q-1)^\ell \) | \( (q-1)^{L-\ell} \) | \( (q-1)^{M-(L+R)} \) | \( (q-1)^{R-r} \) | \( (q-1)^{r} \) | \( 0 \) | \( c' \) | \( \overline{0} \) | \( \overline{0}^r(q-1)^{\ell+r} \) | \( (q-1)^{L-\ell} \) | \( (q-1)^{M-(L+R)} \) | \( (q-1)^{R-r} \) | \( f((q-1)^{\ell+r}0^{\ell+r}) \) | \( 0 \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( e \) | 0 | \( \overline{0}^r(q-1)^\ell \) | \( (q-1)^{L-\ell} \) | \( (q-1)^{M-(L+R)} \) | \( (q-1)^{R-r} \) | \( (q-1)^{r} \) | \( 0 \) | \( e' \) | 0 | \( \overline{0}^r(q-1)^{\ell+r} \) | \( (q-1)^{L-\ell} \) | \( (q-1)^{M-(L+R)} \) | \( (q-1)^{R-r} \) | \( f((q-1)^{\ell+r}0^{\ell+r}) \) | \( 0 \) |

Table 3
Iteration on \( \overline{0}(q-1)^M0 \)
The PA must keep the particles in region D fixed: they only see cells in state \( q - 1 \) around them. If they move, then we would have anticipation (and a global anticipatory cycle) for the configuration where all cells are in state \( q - 1 \). Furthermore, the particles in regions B and C cannot move to region D (this would imply anticipation), nor to regions E, F and G (since \( M \) is arbitrarily large). Hence, \( G \) must match the \( L(q - 1) \) particles from regions B and C with the particles that \( c' \) has in regions B and C, and this requires

\[
L(q - 1) = (L - \ell)(q - 1) + \sum_{i=1}^{n-1} f(0...0(q-1)...(q-1))
\]

which implies

\[
\ell = \frac{1}{q - 1} \sum_{i=1}^{n-1} f(0...0(q-1)...(q-1))
\] (5)

Thus an offset is imposed for the existence of a non-anticipating PA. If the right side of (5) is not an integer, then we are done with \( f \): any PA will have to show anticipation, and, moreover, a global anticipatory cycle (since the anticipation takes place through the arbitrarily large region D, and hence, will also take place in the configuration with all cells in state \( q - 1 \)).

We will attempt to define a non-anticipating PA \( G \) with neighborhood \( \{-L, R\} \), where \( L = L_1 + \ell, R = R_1 + r \), and \( L_1, R_1 \) are “large” numbers that will be precised below. If the attempt fails, we will show that a global anticipatory cycles must exist.

The construction of \( G \) is inspired by the proof of Theorem 5. For every sequence \( w = w_{-L}, \ldots, w_R \in Q^{L+R+1} \), we will consider the configuration \( c = \bar{0}w\bar{0} \) with \( w_0 \) at the origin and will use it to define the new positions of the \( w_0 \) particles. As before, we write \( c' = F(c) \), where \( F \) is the CA defined by \( f \) and the (now fixed) offset \( \ell \).

If \( G \) is to be non-anticipating, then there are certain cells were we already know that some particles cannot move. Without anticipation, the cell at position \( i \) can receive at most \( q - 1 - c_i \) particles from other cells. If \( c'_i > q - 1 - c_i \), then at least \( c'_i - (q - 1 - c_i) \) particles in position \( i \) must stay there, without moving. For every \( i \in \mathbb{Z} \), we define

\[
b_i = (c'_i + c_i - q + 1)_+ \quad a_i = c_i - b_i \quad d_i = c'_i - b_i
\]

Thus, \( (b_i)_{i \in \mathbb{Z}} \) are the numbers of particles fixed at the different positions, \( (a_i)_{i \in \mathbb{Z}} \) contains the numbers of particles in the preimage which need to be associated to some particles in the image, and \( (d_i)_{i \in \mathbb{Z}} \) has the numbers of particles in the image which need to be associated with some in the preimage. Clearly, \( a_i = b_i = c_i = 0 \) for \( i \notin \{-L_1 - \ell, R_1 + r\} \), and \( d_i = 0 \) for \( i \notin \{-L_1 - \ell - r, R_1 + r + \ell\} \). As in Theorem 5, we associate the two sets of particles, \( (a_i)_{i \in \mathbb{Z}} \) and \( (d_i)_{i \in \mathbb{Z}} \).
from left to right. The PA is thus defined: the particles at the origin have found their new locations, possibly fixing some of them \((b_0)\), and associating the rest to the available particles in the image. Notice that in general this PA does not preserve the order of the particles.

Suppose now that (for at least some \(w\)), a particle from the origin is sent to a position outside of \(\{-L_1, R_1\}\). We consider the case in which it is associated to a position to the left of \(-L_1\) (the other case is symmetric); this means that the non-fixed particles to the left of 0 in \(c\) were not enough to match all the non-fixed particles to the left of \(-L_1\) in \(c'\), i.e.

\[
\sum_{i=-L_1-\ell}^{1} a_i < \sum_{i=-L_1-\ell-r}^{1} d_i \\
\iff \sum_{i=-L_1}^{1} a_i < \sum_{i=-L_1-\ell-r}^{1} d_i - \sum_{i=-L_1-\ell}^{1} a_i \\
\iff \sum_{i=-L_1}^{1} a_i < \sum_{i=-L_1-\ell-r}^{1} d_i + \sum_{i=-L_1-\ell}^{1} b_i - \sum_{i=-L_1-\ell}^{1} b_i - \sum_{i=-L_1-\ell}^{1} a_i \\
\iff \sum_{i=-L_1}^{1} a_i < -L_1-\ell \sum_{i=-L_1-\ell}^{1} c_i 
\]

(6)

On the other hand,

\[
-\sum_{-L_1-\ell-r}^{1} c_i' \leq -L_1-1+r \sum_{i=-L_1-\ell}^{1} c_i \leq r(q-1) + \sum_{i=-L_1-\ell-r}^{1} c_i 
\]

(7)

(otherwise, the configuration \(0c_{-L_1-\ell}, \ldots , c_{-L_1-1+r}, 0\) would contradict \(f \in CA^0\)). Combining (6) and (7) we obtain \(\sum_{i=-L_1}^{1} a_i < r(q-1)\); thus, by choosing \(L_1 \geq r(q-1)L_2\) (where \(L_2\) has still to be chosen) we can guarantee that there will be an interval \(I = \{s, \ldots , s+L_2-1\} \subset \{-L_1, \ldots , -1\}\), such that \(a_i = 0\) for all \(i \in I\), i.e., all particles in this region are fixed.

Now, we define \(t = 2 \max\{\ell, r\}\), and choose \(L_2 = (t+1)q^t\). In this way, we guarantee the existence of a word \(u\) of length \(t\) that occurs at least twice, without overlap, in \(c_s, \ldots , c_{s+L_2-1}\). We can write it as \(u = u_1u_2\), with \(|u_1| = |u_2| = \max\{\ell, r\}\). Let \(-T\) and \(-T'\) be the positions to the left of the first occurrence of \(u\), and to the right of its second occurrence, respectively. If we write \(x = c_{-L_1-\ell}, \ldots , c_{-T}\), \(x' = c_{-L_1-\ell-r}, \ldots , c'_{-T}, y = c_{-T'}, \ldots , c_{R_1+r}\), and \(y' = c'_{-T'}, \ldots , c'_{R_1+r+t}\), we can rewrite \(c\) as \(c = 0xu_1u_2u_3u_1u_2y\) for some \(u_3\), and we have that

\[
F(c) = F(0xu_1u_2u_3u_1u_2y) = 0x'\bar{u}_1'\bar{u}_2'u_3'u_1\bar{u}_2'y'\bar{0} 
\]
If we define $c^M = \bar{0}x_1(u_2u_3u_1)^M u_2y\bar{0}$, for an arbitrary positive integer $M$, we obtain that

$$F(c^M) = F(\bar{0}x_1(u_2u_3u_1)^M u_2y\bar{0}) = \bar{0}x_1'(u_2' u_3'u_1')^M u_2'y\bar{0}$$

The situation is similar to the first paragraphs of this proof. Since $M$ is arbitrarily large, we have an arbitrarily large region were particles cannot be moved by any non-anticipating PA. If we choose $M$ such that $|uu_3| M > L' + R' + 1$, where $\{-L', ..., R'\}$ is the neighborhood of any candidate non-anticipating PA, we see that the PA will have to match the particles of $xu_1$ with those of $x'u_1$. This requires an equal number of particles in both of them, i.e.,

$$\sum_{i=-L_1-\ell}^{-T+t} c_i = \sum_{i=-L_1-\ell-r}^{-T+t} c_i'$$

$$\iff \sum_{i=-L_1-\ell}^{-1} c_i + \sum_{i=-L_1-\ell-r}^{-T+t} a_i + \sum_{i=-L_1-\ell}^{-T+t} b_i = \sum_{i=-L_1-\ell}^{-T+t} c_i' + \sum_{i=-L_1-\ell-r}^{-T+t} a_i$$

$$\iff \sum_{i=-L_1-\ell}^{-T+t} b_i - \sum_{i=-L_1-\ell}^{-1} c_i' = \sum_{i=-L_1-\ell-r}^{-T+t} c_i' - \sum_{i=-L_1-\ell}^{-T+t} c_i - \sum_{i=-L_1-\ell-r}^{-1} a_i > 0$$

where the last inequality uses (6). We have arrived to a contradiction, since $b_i \leq c_i'$, for all $i$.

From all the previous discussion, we see that our construction of $G$, with neighborhood $\{-L_1-\ell, ..., R_1+r\}$, $L_1 = r(q-1)(t+1)q^4$, $R_1 = \ell(q-1)(t+1)q^4$, will move the particles from a position $i$ to $\{i - L_1, ..., i + R_1\}$; if not, then $f$ does not admit a non-anticipating PA, and it exhibits global anticipatory cycles, with any offset.

We still have to show that $G$ is really non-anticipating, and that $F$ is its projection. For the first fact, we have to notice that the cell at position $i$ is receiving at most $d_i$ particles (it may receive less than $d_i$, since the assignment of non-fixed particles may fix some of them). For $G$ to be non-anticipating we need

$$d_i \leq q - 1 - c_i \iff c_i' - b_i \leq q - 1 - c_i \iff c_i' + c_i - q + 1 \leq b_i$$

which follows from the definition of $b_i$.

To show that $F$ is a projection of $G$, we follow again the scheme of Theorem 5, and consider the effect of adding an arbitrary word $u$ to the left of $w$. Instead of $c = \bar{0}w\bar{0}$, we take now $\tilde{c} = \bar{0}uw\bar{0}$, and proceed as before: we define $\tilde{b}_i$, $\tilde{a}_i$, $\tilde{d}_i$,
fix $\tilde{b}_i$ particles at position $i$, and associate the “free” particles from left to right. We will show that the destination of the $w_0$ particles at the origin is exactly the same as before; this implies that applying $G$ to a finite configuration $c$ has the same effect of matching the particles with those from $F(c)$, and we conclude that $\Pi(G) = F$.

In order to show that the destination of the particles at the origin has not changed, we will show that the number of “free particles” sent from $\{-L_1, \ldots, -1\}$ to positions to the left of $-L_1$ is not changed. Since for $i > -L_1$ we have not only $\tilde{c}_i = c_i$, but also $\tilde{c}'_i = c'_i$ (and hence $\tilde{b}_i = b_i$, $\tilde{a}_i = a_i$, $\tilde{d}_i = d_i$), this implies that the destinations of the particles at the origin remain the same.

The number of “free particles” sent from positions $\{-L_1, \ldots, -1\}$ to positions to the left of $-L_1$ is equal to the difference between the number of particles to the left of $-L_1$ in the image, and the number of particles to the left of $-L_1$ in the preimage. Thus, we have to show that

$$\sum_{l=-L_1-\ell-r}^{u-1} c'_i - \sum_{l=-L_1-\ell}^{u-1} c_i = \sum_{l=-L_1-\ell-[u]-r}^{u-1} \tilde{c}'_i - \sum_{l=-L_1-\ell-[u]}^{u-1} \tilde{c}_i$$

$$\iff \sum_{l=-L_1-\ell-r}^{u-1} c'_i = \sum_{l=-L_1-\ell-[u]-r}^{u-1} \tilde{c}'_i - \sum_{i=0}^{u-1} u_i$$

(8)

Since $F$ is conservative, and since $\tilde{c}'_i = c'_i$ for $i > -L_1$, we have

$$\sum_{l=-L_1-\ell}^{R_1+r} w_i = \sum_{l=-L_1-\ell-[u]-r}^{u-1} c'_i + \sum_{l=-L_1}^{R_1+r+l} c'_i$$

(9)

and

$$\sum_{i=0}^{u-1} u_i + \sum_{l=-L_1-\ell}^{R_1+r} w_i = \sum_{l=-L_1-\ell-[u]-r}^{u-1} \tilde{c}'_i + \sum_{i=-L_1}^{R_1+r+l} \tilde{c}'_i = \sum_{l=-L_1-\ell-[u]-r}^{u-1} \tilde{c}'_i + \sum_{i=-L_1}^{R_1+r+l} \tilde{c}'_i$$

(10)

We obtain (8) from (9) and (10).

**Corollary 9** The intrinsic presence of anticipation and the intrinsic presence of global anticipatory cycles are equivalent properties for conservative CA rules.

**Proof.** One direction is trivial, since global anticipatory cycles include anticipation. The other follows from Theorem 8: if the procedure fails to give a non-anticipating PA, then a global anticipatory cycle occurs. 

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A comment about the feasibility of finding a non-anticipating PA is due. Even if Theorem 8 asks, in principle, for the evaluation of a huge number of configurations \( ((q-2)(q-1)^{(n-1)(q-1)^{(t+1)t^q+n-1}} \), with \( t = 2 \max\{\ell, r\} \), more practical implementations are possible: if we apply the procedure of the theorem to a configuration \( \bar{w}_\alpha, \ldots, \bar{w}_0, \ldots, \bar{w}_\beta, \bar{0} \), with \( \alpha \geq \ell \) and \( \beta \geq r \), and we find that the \( w_0 \) particles at the origin move to positions in \( \{-\alpha+\ell, \ldots, \beta-r\} \), then the argument contained in the last part of the proof holds, and we do not need to evaluate the configurations of the form \( uw'v \): the movement of these \( w_0 \) will be the same in all cases.

4.3 State-conserving CA

Some CA satisfy a condition stronger that number-conservation: they conserve the number of cells in each state, throughout the iterations. We say that \( f \in \mathcal{CA}(q, n) \) is state-conserving if it verifies

\[
|w|^{-1} \sum_{k=0}^{t} \delta_\alpha(f\{(w/w)_k\}) = |w|^{-1} \sum_{k=0}^{t} \delta_\alpha(w_k) \quad \forall w \in Q^*, \alpha \in Q
\]

where \( \delta_\alpha(x) \) is 1 for \( x = \alpha \) and 0 otherwise. It is plain to see that state-conserving CA are conservative, and that all elements in \( \mathcal{CA}^0(2, n) \) are state-conserving. State-conserving CA may be characterized as follows:

**Proposition 10** A necessary and sufficient condition for \( f \in \mathcal{CA}(q, n) \) to be state-conserving is that, for all \( \alpha, x_1, \ldots, x_n \in Q \),

\[
\delta_\alpha(f(x_1, \ldots, x_n)) = \delta_\alpha(x_1) + \sum_{k=1}^{n-1} \delta_\alpha(f(0, \ldots, 0, x_2, \ldots, x_{k+1})) - \delta_\alpha(f(0, \ldots, 0, x_1, \ldots, x_k))
\]

**Proof.** For each \( \alpha \in Q \), we apply Theorem 2 to the density function \( \delta_\alpha \). \( \square \)

Notice that \( \delta_\alpha \) is a non-linear function. Thus, unlike the characterization of \( \mathcal{CA}^0 \) in 3, this characterization cannot be used to create a linear system whose solutions would give all the CA satisfying the condition; however, it may be used as a test to look for state conservation in \( \mathcal{CA}^0 \).

**State conservation is local:** If there is a state \( \alpha \) in a configuration, then there must be a state \( \alpha \) close to it in the image. Suppose this is not true. Then there are configurations where no \( \alpha \) appears in the image of a window
of length \( L \) around the original \( \alpha \), for arbitrarily large \( L \). We may choose \( L \) large enough to assure that there is a word of length \( n \) which is repeated in the window; we cut the configuration between the repetitions (keeping one of them), and obtain a periodic configuration without a state \( \alpha \) in its image, which is a contradiction.

**The right PA for a state-conserving CA:** If state-conservation is local, then the most reasonable way to look at the rule in terms of particles is to consider each state as a single particle, with different states corresponding to different types of particles (and, in fact, the numerical values of the states turn out to be irrelevant, and could be replaced by colors or letters). But then we would like to have a kind of particle automaton that takes a particle of type \( \alpha \) and a surrounding configuration, and moves it to the position of a particle of type \( \alpha \) in the image; it would be defined by a tuple \((q, \{-\ell, \ldots, r\}, g)\), with \( g : Q^\ell \times Q^r \to \mathbb{N} \), and \( g \) would determine the new position of the particle currently located at the origin in \( \mathbb{N} \). If we want to keep the image of the states as occupancy numbers, then what we would want is a particle automaton that moves all the particles of a cell together: instead of evaluating to an arbitrary vector in \( \mathbb{N}^3 \), each \( g_i \) (in the notation for PA) would evaluate to a vector \((j, j, \ldots, j)\), for some \( j \in \mathbb{N} \).

**Constructing the right PA:** Unfortunately, this is not the PA that will result from the application of Theorem 5 (unless we have \( q = 2 \), or a particular case like the shift). However, the construction given in the proof of Theorem 5 may be fixed for the case of state-conserving CA: for each configuration, we assign now to each particle (i.e., to each state in each cell) the position of its correlative particle (from left to right) in the image of the configuration (this can be done, thanks to the state-conservation). The proof then proceeds exactly in the same way, and the resulting PA will have the same neighborhood (and the particles will arrive in the same zone inside it) as stated in Corollary 6. In general, order will not be preserved.

**Example 5:** Consider \( f \in \mathcal{CA}(3, 3) \) with code \#6768185473053, and offset 1. Using Theorem 5 we obtain the motion representation

\[
M_{14} = \{ \hat{1} \to 21 \}
\]

However, this CA is state-conserving: a ‘2’ will travel to the right as long as it is immersed in a background of 1’s. Depending on the application, it may be therefore more appropriate to use the special version of the construction (as described above), and obtain the motion representation

\[
M_{15} = \{ \hat{2} \to 21, \hat{2} \to 21 \}
\]
where the numbers in the arrows might be dropped, since they will always represent the motion of the complete “particle”, 1 or 2. □

4.4 Momentum Conservation

So far we have considered one additive conserved quantity, the mass. It is natural to ask about other quantities that frequently follow conservation laws, as, for instance, momentum. Notice that this question does not apply to CA, but is natural for a PA.

In a PA $G = (q, N, (g_i)_{i=1,...,q-1})$ with $N = \{-\ell, ..., r\}$, we define the velocity of a particle at a given time step as the difference between its position at the next time step and its current position; equivalently, as the value that $g_i$ assigns to it. Thus, the sum of the velocities of the particles at a cell $i$ of a configuration $c \in S^Z$ is

$$V_i(c) = \sum_{k=1}^{c_i} [g_{c_i}(c_{i-\ell}, ..., c_{i-1}, c_{i+1}, ..., c_{i+r})]_k$$

We will say that a PA preserves the momentum if, and only if,

$$\sum_{i=0}^{p(c)-1} V_i(c) = \sum_{i=0}^{p(c)-1} V_i(c') \quad \forall c \in C_P,$$

where $C_P$ are the periodic configurations of $Q^Z$ and $c' = G(c)$. In other words, the function $\phi : Q^{\ell+r+1} \rightarrow Z$ defined by

$$\phi(x_0, ..., x_{\ell+r}) = \sum_{k=1}^{x_i} [g_{x_i}(x_0, ..., x_{\ell-1}, x_{\ell+1}, ..., x_{\ell+r})]_k$$

is asked to be the density of an additive preserved quantity for $\Pi(G)$. Direct application of Theorem 2 yields the following proposition.

**Proposition 11** Momentum preservation is a decidable property of PA.

In fact, in spite of being defined in terms of the particle representation, momentum preservation depends only on the conservative CA we are representing, as shown by the next theorem. Notice that this is not true for non-conservative CA.

**Theorem 12** Let $F$ be a conservative CA and let $G$ and $G'$ be PA such that $\Pi(G) = \Pi(G') = F$. Then the following three are equivalent:

(i) $G$ preserves momentum
(ii) $G'$ preserves momentum

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(iii) $F$ verifies
\[
\sum_{i \in \mathbb{Z}} \sum_{j \leq i} c_j - c'_j = \sum_{i \in \mathbb{Z}} \sum_{j \leq i} c'_j - c''_j \quad \forall c \in C_F, \text{ where } c' = F(c), c'' = F^2(c)
\]

**Proof.** Since momentum is an additive quantity, Theorem 1 implies that condition (12) can be tested on the configurations of $C_F$ instead of $C_P$. Therefore, (i)⇒(ii) follows from (i)⇒(iii) together with the fact that $\Pi(G) = \Pi(G')$, and we just need to show (i)⇒(iii).

If $F = \Pi(G)$, then $F(c) = G(c)$ for all $c \in Q^\mathbb{Z}$, and we can forget $F$. Let $c$ be a finite configuration, $c' = G(c)$ and $c'' = G(c')$. Then what we must show is that
\[
\sum_{i \in \mathbb{Z}} V_i(c) = \sum_{i \in \mathbb{Z}} V_i(c') \iff \sum_{i \in \mathbb{Z}} \sum_{j \leq i} c_j - c'_j = \sum_{i \in \mathbb{Z}} \sum_{j \leq i} c'_j - c''_j
\]
In fact, what we have is that
\[
\sum_{i \in \mathbb{Z}} \sum_{j \leq i} c_j - c'_j = \sum_{i \in \mathbb{Z}} V_i(c) = \sum_{i \in \mathbb{Z}} \sum_{j=1}^{c_i} [g_{c_i}(c_i-\ell, ..., c_{i-1}, c_{i+1}, ..., c_{i+r})]_j \quad (13)
\]
To see this, consider the contribution of each particle in $c$ to each of the sums. On the left side, the $j$-th particle at $c_k$ contributes its displacement, $v = [g_{c_k}(c_k-\ell, ..., c_{k-1}, c_{k+1}, ..., c_{k+r})]_j$. Without loss of generality, suppose $v \geq 0$. The left side of (13) is the addition, over $i \in \mathbb{Z}$, of $\sum_{j \leq i} c_j - c'_j$, the accumulated difference between $c$ and $c'$. The particle is moving from $k$ to $k + v$; thus, it contributes $+1$ to this sum, for the $v$ terms corresponding to $i = k, ..., k + v - 1$, i.e., its movement contributes with $v$ to the total sum. \[ \square \]

**Corollary 13** Momentum preservation is a decidable property in $\mathcal{CA}^0$.

**Example 6:** Most of the momentum preserving NCPA with small neighborhoods are trivial (identity, shifts); the rest of the cases consist of rules that only allow movements with zero sum, as in the following motion representations.
\[
M_{16} = \{ \text{00110}, \text{00110} \}, \quad M_{17} = \{ \text{120} \}
\]
Of course, more sophisticated CA with momentum preservation can be constructed for larger neighborhoods. \[ \square \]

5 **The Monotone Case**

In this section we deal with the characterization and particle representation of monotone one-dimensional CA. In fact, we will talk almost exclusively about
non-increasing CA (\(\mathcal{CA}^-\)), but it must be noticed that this is equivalent to talking about non-decreasing CA (or, at least, each result about the former translates into a result about the latter). In fact, there is a one-to-one correspondence between the elements of \(\mathcal{CA}^-\) and those of \(\mathcal{CA}^+\), by replacing “particles” with “non-particles” and vice-versa: for each \(f \in \mathcal{CA}^-\) we have a \(\tilde{f} \in \mathcal{CA}^+\) defined by

\[
\tilde{f}(x_1, \ldots, x_n) = q - 1 - f(q - 1 - x_1, \ldots, q - 1 - x_n)
\]

5.1 An Only Sufficient Condition

A first idea for a characterization of monotone CA would be the replacement of the equality for an inequality in condition (3), i.e., we would like to say that \(f\) is in \(\mathcal{CA}^-(q, n)\) if and only if, for all \((x_1, \ldots, x_n) \in \mathbb{Q}^n\),

\[
f(x_1, \ldots, x_n) \leq x_1 + \sum_{k=1}^{n-1} \left\{ f(0, \ldots, 0, x_{2k}, \ldots, x_{2k+1}) - f(0, \ldots, 0, x_{2k}, \ldots, x_k) \right\}
\]

However, this condition is only sufficient. To see that it is sufficient, consider any periodic configuration, and add both sides of the inequality along a whole period: all the terms in the sums of the right side will cancel, and we obtain (1). On the other hand, the condition is not necessary, as shown by the following example.

Example 7: Let \(F\) be the elementary CA 72. Here \(q = 2\), the offset is 1, and

\[
f(x_1, x_2, x_3) = \begin{cases} 
1 & \text{if } x_2 = 1 \text{ and } x_1 + x_3 = 1 \\
0 & \text{otherwise}
\end{cases}
\]

It is easy to see that \(f \in \mathcal{CA}^-\): the only way in which a cell can get a 1 in the image is to already have one in the preimage. On the other hand, condition (14) is not verified:

\[
1 = f(1, 1, 0) > 1 + f(0, 1, 0) + f(0, 0, 1) - f(0, 1, 1) - f(0, 0, 1) = 0
\]
5.2 Domination and maximal elements in \( \mathcal{C}A^- \)

We will say that \( h \in \mathcal{C}A(q, n) \) dominates \( f \in \mathcal{C}A(q, n) \) if \( f(w) \leq h(w) \) for all \( w \in Q^n \). It is easy to see that any rule \( f \in \mathcal{C}A \) which is dominated by a rule \( h \in \mathcal{C}A^0 \) belongs to \( \mathcal{C}A^- \). This arises a natural question: are all elements of \( \mathcal{C}A^-(q, n) \) dominated by elements of \( \mathcal{C}A^0(q, n) \)? In fact, this is true for \( q = 2, n = 3 \) (the elementary CA). However, the general answer is negative:

**Example 8:** Consider \( f \in \mathcal{C}A(3, 2) \) defined by

\[
\begin{align*}
f(x_1, x_2) &= \begin{cases} 
2 & \text{if } x_1 = 2 \\
1 & \text{if } x_1 \in \{0, 1\} \text{ and } x_2 = 1 \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\]

It is easy to see that \( f \in \mathcal{C}A^- \), since \( f(x_1, x_2) \leq x_1 \). Now suppose that there is \( h \in \mathcal{C}A^0(3, 2) \) such that \( f \leq h \). Since \( h \) is conservative, it must verify \( h(0, 0) = 0, h(1, 1) = 1 \) and \( h(2, 2) = 2 \); since it dominates \( f \), it must also verify \( h(2, 0) = 2 \) and \( h(2, 1) = 2 \). From (3) we have

\[
2 = h(2, 0) = 2 + h(0, 0) - h(0, 2) \quad \text{and} \quad 2 = h(2, 1) = 2 + h(0, 1) - h(0, 2)
\]

Since \( h(0, 0) = 0 \), we get \( h(0, 2) = 0 \) and hence \( h(0, 1) = 0 \), which is less than \( f(0, 1) = 1 \).

The CA defined by \( f \) (with offset 0) keeps the 2’s untouched, while the 1’s travel to the left until they hit a 2, and disappear (it is the projection of the PA described by \( M_5 \) in Section 3). Most small examples of non-dominated rules in \( \mathcal{C}A^- \) follow the same pattern: a wall of some kind, and particles that move until they meet it and disappear. Intuitively, a CA dominating them would have to preserve everything which is being destroyed (and it cannot do this only “next to the wall”), at the same time that it follows the particles in their movement; therefore, it would have to increase the total mass. \( \square \)

We will say that \( f \in \mathcal{C}A^-(q, n) \) is maximal if it is not dominated by another element of \( \mathcal{C}A^-(q, n) \).

5.3 A Characterization of \( \mathcal{C}A^- \)

In the following proposition we show that monotony is decidable. Unfortunately, our characterization is computationally useful only for small values of
Let \( f \) be a local rule in \( CA(q, n) \). Then \( f \in CA^-(q, n) \) if and only if

\[
\sum_{k=0}^{w-1} f(w/w)_k \leq \sum_{k=0}^{w-1} w_k, \quad \forall w \in L(q, n)
\]

(15)

where \( L(q, n) \) is the set of all words in \( Q^* \) such that

\[
(w_i \mod |w|, \ldots, w(i+n-2) \mod |w|) \neq (w_j \mod |w|, \ldots, w(j+n-2) \mod |w|) \quad \text{for } i \neq j
\]

In other words, \( L(q, n) \) are the words that do not repeat a subword of length \( n-1 \), when considered as circular words. They correspond to all the cycles in the de Bruijn graph \( B(n-2, q) \) that do not repeat edges; their maximum length is that of the Eulerian paths in \( B(n-2, q) \), which is \( q^{n-1} \).

**Proof.** Since (15) is the restriction of (4) to some configurations, the condition is obviously necessary. To show its sufficiency, we have to consider a word \( w \in S^* \setminus L(q, n) \). Then \( w \) must have a subword \( \alpha \) of length \( n-1 \) which occurs twice in \( w \). There are two cases: the occurrences of \( \alpha \) overlap, or they do not.

**Case A: No overlap.** Without loss of generality, we can assume that \( w \) begins with \( \alpha \), and write it as \( w = \alpha u \alpha v \). In addition, we can assume that both \( \alpha u \in L(q, n) \) and \( \alpha v \in L(q, n) \): if not, we apply the whole argument to them (recursively). Thus, they verify (15), and we have

\[
\sum_{k=0}^{w-1} f(w/w)_k = \sum_{k=0}^{w-1} (\alpha u \alpha v/\alpha u \alpha v)_k
\]

\[
= \sum_{k=0}^{\alpha u - 1} (\alpha u/\alpha)_k + \sum_{k=0}^{\alpha v - 1} (\alpha v/\alpha)_k
\]

\[
\leq \sum_{k=0}^{\alpha u - 1} (\alpha u)_k + \sum_{k=0}^{\alpha v - 1} (\alpha v)_k = \sum_{k=0}^{w-1} w_k
\]

**Case B: With overlap.** In this case, two occurrences of \( \alpha \) overlap. It follows that \( \alpha \) has a prefix \( \beta \) such that \( \alpha_i = \beta_i \mod |\beta| \). Note that

\[
f(w/w) = f(w/\alpha) = f(\beta \alpha u/\alpha) = f(\beta/\alpha) f(\alpha u/\alpha)
\]

As before, we can assume that inequality (15) is satisfied for the words \( \alpha u \) and \( \beta \) (in fact, since \( |\beta| < n-1 \), we apply it to \( \beta^m \), with \( m|\beta| \geq n-1 \), and divide by \( m \) to obtain it for \( \beta \)). Thus we get
As we said before, this test is not very practical, since it involves checking the condition for a large number of words, which grows as \( q^n \). A careful listing of the cycles in de Bruijn graph \( B(n - 2, q) \) could make this a bit lower, but not much: there are at least \( (q!)^{q^{n-3}} q^{(n-2)} \) Eulerian cycles in \( B(n - 2, q) \).

5.4 Particle Representation for CA−

Once we have characterized one-dimensional monotone CA, and since we know that one-dimensional conservative CA can be represented through particle automata, a natural problem to consider is the representation of non-increasing CA in terms of the movement of particles. Clearly, the projection of any PA is a non-increasing CA. For the converse, we will show how to construct a PA for a given non-increasing CA.

More precisely, we will show how to associate to any cell in the image of a configuration, the location that its particles had in that configuration (this is a particular case of the particle identifications defined in [Kur03]). Consider a word \( w \in Q^n \), with image \( f(w) > 0 \). Since the CA is non-increasing, we have that \( f(w) \leq \sum_{i=0}^{n-1} w_i \). We will impose that the \( f(w) \) particles in the image cell must come from the cells in which \( w_0, \ldots, w_{n-1} \) were located. Moreover, the particles will be assumed to be contiguous in the preimage. Then their location in \( w \) is defined by a single value \( s(w), s(w) \in \{0, \ldots, \sum_{i=0}^{n-1} w_i - f(w)\} \), as shown in the following scheme:

\[
\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\text{\( s(w) \)} & \text{\( f(w) \)} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

Denote by \( p(w,i) \) the number of particles contributed by each \( w_i \) to the cell that holds \( f(w) \) in the image; this can be better understood in the scheme above, by seeing how many of the \( f(w) \) selected particles fall in \( i \), but its precise value can be written as

\[
p(w,i) = \left( \min \left\{ s(w) + f(w), \sum_{j \leq i} w_j \right\} - \max \left\{ s(w), \sum_{j < i} w_j \right\} \right)_{+}
\]
and we have \( f(w) = \sum_{i=0}^{n-1} p(w, i) \).

**Theorem 15** Let \( f \in \mathcal{CA}(q, n) \). Then a necessary and sufficient condition for \( f \) to be in \( \mathcal{CA}^- \) is that there exists \( s : \mathbb{Z} \to \mathbb{Z} \), with \( s(w) \in \{0, \ldots, \sum_{i=0}^{n-1} w_i - f(w)\} \) for all \( w = w_0, \ldots, w_{n-1} \) and \( s(w) = 0 \) for all \( w \) with \( f(w) = 0 \), such that \( p(w, i) \) defined by (17) verifies

\[
\sum_{i=0}^{n-1} p((w_{n-1-i}, \ldots, w_{2n-2-i}), i) \leq w_{n-1} \quad \text{for all } w_0, \ldots, w_{2n-2} \in \mathbb{Z} \tag{18}
\]

**Proof.**

**Sufficient condition.** Suppose that there exists \( s \) with these properties, and consider any finite configuration \( c \in \mathbb{Z} \). Then

\[
\sum_{j \in \mathbb{Z}} f(c_j, \ldots, c_{j+n-1}) = \sum_{j \in \mathbb{Z}} \sum_{i=0}^{n-1} p((c_j, \ldots, c_{j+n-1}), i) = \sum_{j \in \mathbb{Z}} \sum_{i=0}^{n-1} p((c_{j-i}, \ldots, c_{j-i+n-1}), i) \leq \sum_{j \in \mathbb{Z}} c_j
\]

**Necessary condition.** Suppose that \( f \in \mathcal{CA}^- \). For each \( w = w_0, \ldots, w_{n-1} \) in \( \mathbb{Z} \), define \( s(w) = 0 \) if \( f(w) = 0 \), and

\[
s(w) = \sum_{i=0}^{n-1} w_i + \min_{v \in \mathbb{Z}} \left\{ |v|^{-1} \sum_{i=0}^{n+|v|-1} v_i - \sum_{i=0}^{n} f(wv/0^{n-1}), i \right\} \leq \sum_{i=0}^{n-1} c_j
\]

(19)

otherwise. It verifies

\[
0 \leq s(w) \leq \sum_{i=0}^{n-1} w_i - \sum_{i=0}^{n-1} f(w/0^{n-1}), i \leq \sum_{i=0}^{n-1} w_i - f(w)
\]

where the lower bound follows from \( f \in \mathcal{CA}^- \), and the upper one is obtained by taking \( v \) as the empty word (or as “0”) in the definition of \( s(w) \).

It is important to notice that \( s \) can be determined with a finite computation: if \( v \) is a word of minimal length where the minimum is reached, then \( |v| \leq q^{n-1} \).

The situation is similar to the proof of Theorem 14; since we are not taking circular words, we need a word of length greater \( q^{n-1} + n - 1 \) to assure the existence of a repeated subword \( \alpha \) of length \( n - 1 \), and this length is reached by \( wv \). Then \( wv \) can be written as \( x\alpha y\alpha z \), and we can define \( v' \) such that
\[ wv' = x\alpha z, \text{ which verifies} \]

\[
\sum_{i=0}^{n-1} w_i + \sum_{i=0}^{|v'|-1} v'_i - \sum_{i=0}^{n+|v'|-1} f(wv'/0^n) \leq s(w)
\]

where the last inequality is implied by the non-decreasing property: if it is false, then the periodic configuration with periodic pattern \( \alpha y \) contradicts the monotony. Here we assumed that the two occurrences of \( \alpha \) do not overlap; if they do, the reasoning is similar, akin to that in case B of Theorem 14. In both cases, what we get is a violation of the minimality of \( |v| \).

We still have to show that \( s(w) \) verifies (18). This follows from the fact that different particles in the image take different particles as their preimages.

Consider a configuration \( c \in Q^n \) and two positive coordinates in \( F(c) \); we may assume that the offset is 0 (i.e., \( F(c)_0 = f(c_0, \ldots, c_{n-1}) \)), and that the two positions are 0 and some \( k > 0 \). Consider a scheme like (16), with the \( \sum_{i=0}^{k+n-1} c_i \) particles of the positions \( 0, \ldots, c_{k+n-1} \) aligned in a row, and number them from left to right. Then the \( f(c_0, \ldots, c_{n-1}) \) particles landing at position 0 are taken as the particles labeled with numbers

\[
s(c_0, \ldots, c_{n-1}) + 1, \ldots, s(c_0, \ldots, c_{n-1}) + f(c_0, \ldots, c_{n-1})
\]

while the \( f(c_k, \ldots, c_{k+n-1}) \) particles landing at \( k \) are taken as the particles labeled with the numbers

\[
\sum_{i=0}^{k-1} c_k + s(c_k, \ldots, c_{k+n-1}) + 1, \ldots, \sum_{i=0}^{k-1} c_k + s(c_k, \ldots, c_{k+n-1}) + f(c_k, \ldots, c_k+n-1)
\]

All we need to show is that they do not overlap, i.e.,

\[
s(c_0, \ldots, c_{n-1}) + f(c_0, \ldots, c_{n-1}) \leq \sum_{i=0}^{k-1} c_k + s(c_k, \ldots, c_k+n-1)
\]

Consider any word \( v \in Q^n \), and put the word \( v' = (c_n, \ldots, c_{k+n-1}, v_0, \ldots, v_{|v|}) \) in (19) for the definition of \( s(c_0, \ldots, c_{n-1}) \). We obtain that

\[
s(c_0, \ldots, c_{n-1}) \leq \sum_{i=0}^{n-1} c_i + \sum_{i=0}^{|v'|-1} v'_i - \sum_{i=0}^{n+|v'|-1} f(c_0, \ldots, c_{n-1}, v'/0^n)_i
\]

\[
= \sum_{i=0}^{k-1} c_i + \sum_{i=k}^{k+n-1} c_i + \sum_{i=0}^{|v|-1} v_i - \sum_{i=0}^{k-1} f(c_i, \ldots, c_i+n-1) - \sum_{i=0}^{|v|+n-1} f(c_k, \ldots, c_k+n-1, v/0^n)_i
\]
Thus, any \( v \in Q^* \) verifies
\[
\sum_{i=k}^{k+n-1} c_i + \sum_{i=0}^{\lvert v \rvert - 1} v_i - \sum_{i=0}^{\lvert v \rvert + n-1} f(c_k, \ldots, c_{k+n-1}, v/0^{n-1})_i \\
\geq s(c_0, \ldots, c_{n-1}) - \sum_{i=0}^{k-1} c_i + \sum_{i=0}^{k-1} f(c_i, \ldots, c_{i+n-1})
\]
and therefore, taking the minimum over all \( v \),
\[
s(c_k, \ldots, c_{k+n-1}) \\
\geq s(c_0, \ldots, c_{n-1}) - \sum_{i=0}^{k-1} c_i + \sum_{i=0}^{k-1} f(c_i, \ldots, c_{i+n-1}) \\
\geq s(c_0, \ldots, c_{n-1}) - \sum_{i=0}^{n-1} c_i + f(c_0, \ldots, c_{n-1})
\]
\( \Box \)

**Corollary 16** Let \( F \) be a CA with local rule \( f \in CA^{-}(q, n) \). Then there exists \( G \in PA(q, 2n - 1) \) such that \( \Pi(G) = F \).

**Proof.** The functions \( s \) and \( p \) as in Theorem 15 determine the particle automaton. For any configuration \( w_0, \ldots, w_{n-1}, \ldots, w_{2n-2} \) with \( w_{n-1} > 0 \), we define the motion of the \( w_{n-1} \) particles as follows: the number of particles moving from \( n - 1 \) to \( i \), \( i = 0, \ldots, n - 1 \), is given by \( p_i = p((w_{n-1-i}, \ldots, w_{2n-2-i}), i) \), and the number of particles that die (move to \( \dagger \)) is given by \( w_{n-1} - \sum_{i=0}^{n-1} p_i \). In other words, we define a \( g_{w_{n-1}} \) such that
\[
\#\{k : [g_{w_{n-1}}(w_0, \ldots, w_{n-2}, w_n, \ldots, w_{2n-2})]_k = i\} = p_i
\]
and \( \#\{k : [g_{w_{n-1}}(w_0, \ldots, w_{n-2}, w_n, \ldots, w_{2n-2})]_k = i\} = w_{n-1} - \sum_{i=0}^{n-1} p_i \)
\( \Box \)

**Corollary 17 (Another test for CA monotony)** The previous theorems establish the equivalence between (one-dimensional) non-increasing CA and particle automata. This provides an alternative characterization of \( CA^{-} \): to see if a certain CA belongs to this class, check the existence of a function \( s \) with the properties of Theorem 15. Finding it in the way described in the proof would need (in definition 19) as much work as the characterization in Theorem 14; it is usually easier to check for its existence by testing the possible functions \( s \). Since \( s : Q^n \to \mathbb{Z} \), and \( 0 \leq s(w) \leq \sum w_i - \sum f(w) \), a number of possible \( s \) is \( (qn)^n \), which is more than the work in Theorem 14, but using that \( s(w) = 0, p(w, i) = 0 \) for \( f(w) = 0 \), and applying the necessary conditions of Theorem 15, the possibilities for \( s \) can be drastically reduced in a practical implementation.
Remark: There is no canonical PA. As seen in Section 4, we can always take a “canonical” particle representation for a (one-dimensional) conservative CA, which is unique, and is characterized by the preservation of the order of the particles. In the monotone case, there is no canonical form: for instance, when two particles are close to each other and one disappears, there is an arbitrary decision favouring the survival of one of them.

| q | n | \( CA^0 \) | state-cons. CA | \( CA^- \) | maximal \( CA^- \) |
|---|---|---|---|---|---|
| 2 | 2 | 2 | 2 | 6 | 2 |
| 2 | 3 | 5 | 5 | 46 | 5 |
| 2 | 4 | 22 | 22 | 2756 | 38 |
| 2 | 5 | 428 | 428 | ? | ? |
| 2 | 6 | 133184 | 133184 | ? | ? |
| 3 | 2 | 4 | 2 | 708 | 6 |
| 3 | 3 | 144 | 15 | ? | ? |
| 3 | 4 | 5448642 | ? | ? | ? |
| 4 | 2 | 10 | 2 | 3732576 | ? |
| 4 | 3 | 89588 | 89 | ? | ? |

Table 4
Some demographics: Number of one-dimensional conservative, state-conserving, non-increasing, and maximal non-increasing rules, for some \( q \) and \( n \).

6 Conservation Associated to Blocks

In [Mor03], Theorem 6, a method was described to decide, for a given CA, whether its states can be relabelled with integer numbers so as to make the CA number-conserving; in Section 4, we see that this can be used, in turn, to see if the dynamics of the original CA can be understood in terms of an operator acting on a system of indestructible particles.

In this section we consider a generalization of that theorem, giving values to the words of a given length, instead of the single states; we want this function of the blocks to be preserved by the iteration of the CA. In other words, we are looking for an additive conserved quantity; however, we will impose a further condition: it must separate the images of the blocks with different values at the origin.

We want to view the function as an assignment of a value to the state in a cell, but a value which depends on the states of its neighbors: the same state
a may represent 2 or 3 particles, depending on whether or not it is followed by, say, another state a. Thus we can detect if the given CA may be seen as the projection, to fewer states, of a conservative CA which in turn is seen as an interaction of particles, restricted to certain configurations. In particular, this may automatically detect particle-like behavior in some non-conservative CA.

**Theorem 18** Let F be a one dimensional CA with state set $S \subset \mathbb{Z}$, and let $b$ be a positive integer. Then it may be decided whether or not there is a $b$-neighborhood-dependent relabelling of the states of $S$, which distinguishes the states, and whose sum is preserved by $F$.

**Proof.** Consider a cellular automaton $F$ with states $S \subset \mathbb{Z}$, neighborhood size $n$, and local rule $f \in CA(q, n)$. Without loss of generality, the offset can be assumed to be 0. Let $\phi$ be a relabelling map, $\phi : S^b \to \mathbb{Z}$. For $c \in S^\mathbb{Z}$, we define $\Phi(c)$ as $\Phi(c)_i = \phi(c_i, \ldots, c_{i+b-1})$. Let $F_\phi$ be the induced CA that makes the diagram

$$
\begin{align*}
S^\mathbb{Z} & \xrightarrow{\Phi} \Phi(S^\mathbb{Z}) \\
F \downarrow & \quad F_\phi \downarrow \\
S^\mathbb{Z} & \xrightarrow{\Phi} \Phi(S^\mathbb{Z})
\end{align*}
$$

commute. $F_\phi$ acts on the subshift $\Phi(S^\mathbb{Z})$, and has the same neighborhood as $F$. We want to determine possible mappings $\phi$ that would make $F_\phi$ number-conserving. In other words, we want to find $\phi$ such that

$$
\sum_{k=0}^{p(c)-1} \phi(c_1, \ldots, c_{i+b-1}) = \sum_{k=0}^{p(c)-1} \phi(f(c_i \ldots c_{i+n-1}), \ldots, f(c_{i+b-1} \ldots c_{i+b+n-2})) \quad (20)
$$

for all $c \in C_P$. Fix an arbitrary $\alpha \in S$. From Theorem 2 we have that a necessary and sufficient condition for (20) is that

$$
\phi_f(a_0, \ldots, a_{b+n-2}) - \phi(a_0, \ldots, a_{b-1}) = \\
\sum_{i=1}^{n+b-2} \left\{ - \phi_f(\alpha, \ldots, \alpha, a_0, \ldots, a_{n+b-2-i}) + \phi_f(\alpha, \ldots, \alpha, a_1, \ldots, a_{n+b-1-i}) \right\} + \\
\sum_{i=1}^{b-1} \phi(\alpha, \ldots, \alpha, a_0, \ldots, a_{b-i-1}) - \phi(\alpha, \ldots, \alpha, a_1, \ldots, a_{b-i}) \quad (21)
$$

holds for all $a_0, \ldots, a_{n+b-2} \in S$; here we use the notation

$$
\phi_f(x_0, \ldots, x_{b+n-2}) = \phi(f(x_0, \ldots, x_{n-1}), \ldots, f(x_{b-1}, \ldots, x_{b+n-2})).
$$

This condition can be used to get an homogeneous linear system, where the $|S|^b$ unknown values are $\{\phi(w), w \in S^b\}$. The solution will be a linear subspace.
\( V \subset \mathbb{R}^{s^b} \); if \( V = \{0\} \), then the algorithm gives a negative answer. If \( V \neq \{0\} \), we are not ready yet: we demand the solution to distinguish between different states in the first position (we are interpreting the image of the block \( a_0, \ldots, a_{b-1} \) as the value assigned to \( a_0 \)). Thus we ask that

\[
\phi(\{\alpha\} \times S^{b-1}) \cap \phi(\{\beta\} \times S^{b-1}) = \emptyset \quad \forall \alpha \neq \beta.
\]
or, in other words, \( \phi \) cannot belong to the hyperplane \( \phi(u) = \phi(v) \), for \( u, v \in S^b \), \( u_0 \neq v_0 \). Let \( \{E_{u,v}\}_{(u,v) \in I} \) be the collection of these hyperplanes, where \( I = \{(u,v) : u, v \in S^b, u_0 \neq v_0\} \). Then we want to see if \( V \setminus (\bigcup_{(u,v) \in I} E_{u,v}) \neq \emptyset \). Since each \( V \cap E_{u,v} \) is a subspace of \( V \), and linear spaces cannot be finite unions of proper subspaces, we have:

\[
V \setminus \left( \bigcup_{(u,v) \in I} E_{u,v} \right) = \emptyset \iff V = \bigcup_{(u,v) \in I} V \cap E_{u,v}
\]

\[
\iff \exists (u,v) \in I : V = V \cap E_{u,v} \iff \exists (u,v) \in I : V \subseteq E_{u,v}
\]

and this last condition can be checked by adding the hyperplane equations to the linear system.

If solutions do exist, then there are integer solutions and they can be found in finite time. Any such solution will be the desired assignment of values to the states in the original CA, that depend on their contexts, and whose sum is preserved by the dynamics of the CA. \( \square \)

**The range of values for \( b \).** There is not always a value of \( b \) that allows us to find a number-conserving relabelling of the states: this is the case, for instance, of the trivial CA with states \( \{0, 1\} \), neighborhood \( \{0\} \), and \( f(0) = f(1) = 0 \). A natural question is: For a given CA, how many values of \( b \) must we check before we conclude that no \( b \) will work? This is an open question at the moment of this writing, since we do not currently have a bound for \( b \); intuitively, we can expect that such a bound exists, since there is no reason why the considered neighborhood should be arbitrarily large with respect to the neighborhood of the CA. The problem is akin to that stated (and left open) first in [Tak89] and then in [HT91] of bounding the range of the additive conserved quantities for the elementary reversible CA considered there; both are particular cases of the general problem of bounding the range of the additive conserved quantities of a given CA.

**Particle representation for the relabelled CA.** The construction given in Theorem 5 can be trivially restricted to any subset of the possible configurations of states; in particular, it can be restricted to the subshift \( \Phi(S^Z) \), yielding a particle representation of the operation of our CA on the states and its neighbors.

**Example 9:** Consider \( f \in CA(3,4) \) with code 64056. It is not conservative,
and it can be checked that there is no reassignment of values to its states that makes it number-conserving. However, there are solutions $\phi$ of the system given by equation (21), for $b = 3$. The smallest such $\phi$ (in norm) is

$$\phi(0**) = 0$$

$$\phi(100) = 2$$

$$\phi(101) = \phi(110) = \phi(111) = 1$$

We obtain for it the motion rule

$$\begin{array}{cccccccc}
2000 & \sim & \sim & \sim & \sim & 1\hat{1} & 1\hat{2} \\
\end{array}$$

restricted to the configurations where no 1 is followed by two zeros, and all 2 are. This is a representation of the original CA in terms of the interaction of indestructible particles, and highlights one of the propagating defects in the evolution of this CA. □

7 Conclusion

This paper presented particle automata as simple systems of interacting particles, that give a complementary view on the behavior of conservative and monotone cellular automata. They are defined in terms of a global operator induced by a local rule with some finite neighborhood (in analogy to the CA formalism), and may be easily extended to higher dimensions or different topologies, as we expect to do in a future article.

The special class of conservative PA is equivalent to the class of conservative CA; we discussed several properties that the conservative PA may exhibit,
some of which depend only on their CA projections. These intrinsic properties of the particle representation include *anticipation*, the existence of global anticipatory cycles, and *momentum conservation*, in a natural sense defined here. Theorem 8 shows that the intrinsic presence of the first two is equivalent, and that this presence is a decidable property. Some properties that are not intrinsic are the preservation of order and the existence of local cycles, both of which can be always avoided by taking the “canonical” PA given by the equivalence result. Momentum conservation is an interesting case, since it turns out to depend only of the conservative CA, in spite of being defined (and making sense) in terms of a PA.

The conservative PA given by the construction algorithm may fail to be the most useful one for a given CA. This is the case with state-conserving CA; for them, the natural particle representation is one where the different states represent different particles (instead of occupancy numbers). We gave a characterization of state-conserving CA, and explained the construction of appropriate PA for them.

We have extended the work on conservative CA in two directions: on one hand, we have considered CA that are not conservative, but nearly so, since we can see them as the projection of particular configurations of a conservative CA with more states; their dynamics can be then understood in terms of indestructible particles.

On the other hand, we have considered monotone CA, and we have shown that they can be characterized, and, moreover, that they can also be represented by means of a particle system. This equivalence is less obvious than in the conservative case; one of the difficulties is that no “canonical” PA exists for monotone CA.

One open problem was already mentioned in the previous section: in Theorem 18, to find a bound for the block size $b$ that needs to be considered before the existence of a solution $\phi$ is discarded. Another natural problem, which arises from the text, is the following: to extend Theorem 18 to allow $\phi$ to be monotone. This would be very helpful, since it would help to detect particle-like behavior that includes decay or annihilation. Such an extension is already possible with the current results, but it would be computationally hard; the challenge is to make it in an efficient way.

*Please notice that some additional information, like rule lists, as well as some software, is available at our website:*

http://www.dim.uchile.cl/~anmoreir/ncca/

*Other software (like the C++ routines used to determine and solve the linear systems for listing the CA rules) is available, by request, from A. Moreira.*
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