Projections from surfaces of revolution in the Euclidean plane

C. Charitos¹ · P. Dospra²

Received: 23 May 2020 / Accepted: 8 October 2020 / Published online: 26 October 2020
© The Managing Editors 2020

Abstract
In this paper, we determine the class of surfaces of revolution $S$ for which there exists a smooth map $\Phi$ from a neighbourhood $U$ of $S$ to the Euclidean plane $E^2$ preserving distances infinitesimally along the meridians and the parallels of $S$ and sending the meridional arcs of $U \cap S$ to straight lines of $E^2$.

Keywords Surfaces of revolution · Meridians · Parallels · Projections

Mathematics Subject Classification 53A05 · 34A05

1 Introduction
In [4] (see [5] for the translation of [4] in English), Euler proved that there does not exist a perfect map from the sphere $S^2$ or, from a part of $S^2$, to the Euclidean plane $E^2$. Recall that a smooth map $f$ from $S^2$ (or, from a part of $S^2$) to $E^2$ is called perfect if for each $p \in S^2$ there is a neighbourhood $U(p)$ of $p$ in $S^2$ such that the restriction of $f$ on $U(p)$ preserves distances infinitesimally along the meridians and the parallels of $S^2$ and $f$ also preserves angles between meridians and parallels Charitos and Papadoperakis (2019). In modern geometric language, a perfect map is a local isometry from $S^2$ to $E^2$ and, thus, Euler’s theorem follows from the Gauss Egregium Theorem which was proved many years later. However, Euler’s method of proof is very fruitful and can be applied to similar problems, see for instance

P. Dospra
petroula.dospra@gmail.com

C. Charitos
bakis@aua.gr

¹ Department of Natural Resources Management and Agricultural Engineering, Agricultural University Athens, Iera Odos 55, 11855 Athens, Greece
² Department of Electrical and Computer Engineering, University of Western Makedonia, 50100 Kozani, Greece
Proposition 5 of Charitos and Papadoperakis (2019). Very briefly, Euler’s basic idea for the non-existence of a perfect map from $S^2$ to $E^2$, is to translate geometrical conditions to a system of differential equations and prove that this system does not have a solution. Using Euler’s method, the non-existence of a smooth map from a neighbourhood $U$ of $S^2$ to $E^2$ which preserves distances infinitesimally along the meridians and the parallels of $S^2$ and which sends the meridional arcs of $U \cap S^2$ to straight lines of $E^2$, can be shown as well Charitos and Papadoperakis (2019).

The origin of all these problems lies in the ancient problem of cartography, that is, the problem of constructing geographical maps from $S^2$ (or from a subset of $S^2$) to $E^2$ which satisfy certain specific requirements. This problem can also be considered as part of a more general question concerning the existence of coordinate transformations that preserve certain geometrical properties from one coordinate system to another. Several prominent mathematicians have studied this problem from antiquity to our days and in the course of this study, $S^2$ was replaced gradually by surfaces of revolution or by surfaces in $E^3$ in general (see Papadopoulos (2018) for an excellent historical recursion on this subject).

The goal of this work is to determine the class of surfaces of revolution $S$ for which there exists a smooth map $\Phi$ from an open neighbourhood $U$ of $S$ to $E^2$ preserving distances infinitesimally along the meridians and the parallels of $S$ and sending meridional arcs of $U \cap S$ to straight lines of $E^2$. Furthermore the map $\Phi$ is computed explicitly. For the computation of $\Phi$ we follow Euler’s ideas, that is, we convert geometrical conditions to differential equations whose solutions allow us to find $\Phi$.

As a corollary of the above result we deduce that if $p$ is a point of $S$ and the Gaussian curvature at $p$ is positive, then a map $\Phi$ as above does not exist in a neighbourhood $U$ of $p$. We also deduce that if $S_0$ is an abstract surface of constant negative curvature, such maps $\Phi$ from an open subset $U$ of $S_0$ to $E^2$ do not exist.

2 Statement of results

Let $S$ be a surface of revolution in $E^3$. In the following we assume that all maps are of class $C^s$, $s \geq 2$. We consider a parametrization of $S$ given by

$$r(t, u) = (f(u) \cos t, f(u) \sin t, g(u)), \quad (r)$$

where $f(u) > 0$, $a < u < b$, $t \in [0, 2\pi]$, and we assume that the curve $\gamma(u) = (f(u), g(u))$ is parametrized by arc-length i.e. we have $(f')^2(u) + (g')^2(u) = 1$. Thus, the Riemannian metric on $S$ takes the form

$$ds^2 = du^2 + f^2(u) dt^2.$$

For $t$ fixed, the $u$-curves $r(t, \cdot)$ are called meridians of $S$ and they are geodesics for the metric $ds$, while for $u$ fixed, the $t$-curves $r(\cdot, u)$ are called parallels and they are not geodesics in general. Furthermore, for each point $p \in S$, there exists a unique pair
(t, u) ∈ (a, b) × [0, 2π) such that p = r(t, u), and so, we may identify p with the pair (t, u).

**Theorem 1** Assume that \( f'(u) \neq 0 \) and \( f''(u) \neq 0 \) for each \( u \in (a, b) \). Let \( U \) be an open connected subset of \( S \). Then, there exists a map \( \Phi : U \to E^2 \) satisfying properties

(C1): \( \Phi \) sends the meridional arcs of \( S \cap U \) to straight lines of \( E^2 \);
(C2): \( \Phi \) preserves distances infinitesimally along the meridians and the parallels of \( S \);

if and only if \( f^2 = cu^2 + du + k \), where \( c, d, k \) are constants with \( k > 0 \) and \( c > 0 \). Furthermore, assuming that \( \Phi(t, u) = (x(t, u), y(t, u)) \), we have that

\[
x(t, u) = u \cos(b(t)) + \int \sqrt{k} \cos(\theta_0 - b(t))dt,
\]

\[
y(t, u) = -u \sin(b(t)) + \int \sqrt{k} \sin(\theta_0 - b(t))dt,
\]

where

\[
b(t) = -\sqrt{ct} + c_0.
\]

In Figure 1, a surface of revolution \( S \) is drawn which satisfies all hypotheses of Theorem 1. In this example, by taking \( f^2 = u^2 + 1 \) we get that \( g(u) = \ln(\sqrt{u^2 + 1} + u), u > 0 \). The picture confirms that the Gaussian curvature of each point of \( S \) is negative.
Corollary 1 (1) If \( p \in S \) and the Gaussian curvature at \( p \) is positive, then for each neighbourhood \( U \) of \( p \) in \( S \) a map \( \Phi : U \to \mathbb{E}^2 \) satisfying conditions (C1) and (C2) does not exist.

(2) If \( S_0 \) is a Riemannian surface of constant negative curvature, then for each open neighbourhood \( U \subset S_0 \) a map \( \Phi : U \to \mathbb{E}^2 \) satisfying conditions (C1) and (C2) does not exist.

Condition (C1) is a natural requirement since meridians are geodesics of \( S \) and thus it is required to be sent to geodesics of \( \mathbb{E}^2 \) via \( \Phi \). Condition (C2) appears in Euler’s writings and means that the elementary length between two points \( p, q \) on a meridian (resp. two points \( p, r \) on a parallel) of \( S \) is equal to the elementary length of points \( P = \Phi(p), Q = \Phi(q) \) (resp. of points \( P, R = \Phi(r) \)). In other words, the “elements” \( pq \) and \( pr \) are equal to the “elements” \( PQ \) and \( PR \) respectively (following the terminology of [4]). In order to express (C2) rigorously, let \( p = (t, u), q = (t, u + du), r = (t + dt, u), P = \Phi(t, u), Q = \Phi(t, u + du) \) and \( R = \Phi(t + du, u) \). If we denote by \( |p_1 - p_2| \) the distance between the points \( p_1, p_2 \in S \) and by \( ||P_1 - P_2|| \) the Euclidean distance between the points \( P_1, P_2 \in \mathbb{E}^2 \), then condition (C2) means that:

\[
\lim_{du \to 0} \frac{|q - p|}{|du|} \overset{(i)}{=} \lim_{du \to 0} \frac{||P - Q||}{|du|}
\]

and

\[
\lim_{dt \to 0} \frac{|r - p|}{|du|} \overset{(ii)}{=} \lim_{dt \to 0} \frac{||R - Q||}{|du|}.
\]

Using the coordinate functions \( x(t, u) \) and \( y(t, u) \), equalities (i) and (ii) take respectively the following form:

\[
\sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2} = 1, \quad (iii)
\]

and

\[
\sqrt{\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2} = f(u). \quad (iv)
\]

Indeed, relations (iii) and (iv) correspond to the relations (I) and (II) of ([5], p. 5). In Charitos and Papadoperakis (2019), these relations are reproved using a modern mathematical language and they are labelled as relations (6) and (7) respectively. In the present work, relations (iii) and (iv) are obtained by replacing \( \cos u \) in the parametrization \((\cos u \cos t, \cos u \sin t, \sin u)\) of \( S^2 \) by the function \( f(u) \). In this way we obtain the parametrization \((f(u) \cos t, f(u) \sin t, g(u))\) of \( S \) and then we repeat the same steps. As a result, using Euler’s method, condition (C2) is translated into a system of differential equations consisting of relations (iii) and (iv).

Combining (C1) and (C2) we deduce that \( \Phi \) restricted to a meridian of \( S \) is an isometry onto its image.
Remark 2 If $f'(u) = 0$ for each $u \in (a, b)$, then, the curve $\gamma(u) = (f(u), g(u))$ is a straight line in the $(x, z)$-plane and so, the surface $S$ obtained by revolving $\gamma$ about the $z$-axis is Euclidean i.e. locally isometric to the Euclidean plane $E^2$. Furthermore, if $f''(u) = 0$ for each $u$, we deduce that $f'$ and $g'$ are constant functions since, by assumption, the curve $\gamma(u) = (f(u), g(u))$ is parametrized by arc-length. Therefore $\gamma(u)$ is a line segment and, thus, $S$ is locally isometric to $E^2$.

\section{3 Auxiliary Lemmas}

In this section we give results needed for the proof of Theorem 1.

\begin{lemma}
Assume that $f'(u) \neq 0$ and $f''(u) \neq 0$, for all $u \in (\alpha, \beta)$. Let $U$ be an open connected subset of $S$ and a map $\Phi : U \to E^2$ satisfying properties (C1) and (C2). Then, there are variables $\phi$ and $\omega$ which are functions of $t, u$ satisfying

\[
\left( \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u} \right) = (\cos \phi, \sin \phi), \quad \left( \frac{\partial x}{\partial t}, \frac{\partial y}{\partial t} \right) = (f(u) \cos \omega, f(u) \sin \omega)
\]

and

\[
f''(u) = \left( \frac{\partial \omega}{\partial u}(t, u) \right)^2 f(u).
\]

\end{lemma}

\begin{proof}
Proceeding as in the proof of Proposition 5 of Charitos and Papadoperakis (2019), we have that there are variables $\phi$ and $\omega$ which are functions of $t, u$ such that

\[
\left( \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u} \right) = (\cos \phi, \sin \phi),
\]

\[
\left( \frac{\partial x}{\partial t}, \frac{\partial y}{\partial t} \right) = (f(u) \cos \omega, f(u) \sin \omega).
\]

Since

\[
\frac{\partial^2}{\partial u \partial t} = \frac{\partial^2}{\partial t \partial u},
\]

we have

\[
- \sin \phi \cdot \frac{\partial \phi}{\partial t} = f' \cdot \cos \omega - f \cdot \sin \omega \cdot \frac{\partial \omega}{\partial u},
\]

\[
\cos \phi \cdot \frac{\partial \phi}{\partial t} = f' \cdot \sin \omega + f \cdot \cos \omega \cdot \frac{\partial \omega}{\partial u}.
\]

Multiplying the first of the above equality by $\cos \omega$, the second by $\sin \omega$ and adding, we deduce that
\begin{align*}
(- \sin \phi \cdot \cos \omega + \cos \phi \cdot \sin \omega) \frac{\partial \phi}{\partial t} \\
= f' \cdot \cos^2 \omega - f \cdot \sin \omega \cdot \cos \omega \cdot \frac{\partial \omega}{\partial u} + f' \cdot \sin^2 \omega + f \cdot \cos \omega \cdot \sin \omega \cdot \frac{\partial \omega}{\partial u}
\end{align*}

if and only if

\[ \sin(\phi - \omega) \cdot \frac{\partial \phi}{\partial t} = f'. \]

Similarly, multiplying the first equality by \( \cos \phi \), the second by \( \sin \phi \) and adding, we obtain

\[ \sin u \cdot \cos \omega \cdot \cos \phi - \\
\cos u \cdot \sin \omega \cdot \frac{\partial \omega}{\partial u} \cdot \cos \phi - \sin u \cdot \sin \omega \cdot \phi + \cos u \cdot \cos \omega \cdot \frac{\partial \omega}{\partial u} \cdot \sin \phi = 0 \]

which implies that

\[ f \cdot \sin(\phi - \omega) \frac{\partial \omega}{\partial u} = -f' \cdot \cos(\phi - \omega). \]

On the other hand, condition (C1) implies that the meridians are mapped to straight lines, and so, we have

\[ \frac{\partial \phi}{\partial u} = 0. \]

Thus, differentiating

\[ \sin(\phi - \omega) \frac{\partial \phi}{\partial t} = f' \]

with respect to \( u \) and using the previous equality, we obtain

\[ -\cos(\phi - \omega) \cdot \frac{\partial \omega}{\partial u} \cdot \frac{\partial \phi}{\partial t} = f''. \]

Multiplying

\[ \sin(\phi - \omega) \frac{\partial \phi}{\partial t} = f' \]

by

\[ \frac{\partial \omega}{\partial u} \cdot \frac{\partial \phi}{\partial t}, \]

\( \odot \) Springer
we have
\[ f \cdot \sin(\phi - \omega) \cdot \left( \frac{\partial \omega}{\partial u} \right)^2 \cdot \frac{\partial \phi}{\partial t} = -f' \cdot \cos(\phi - \omega) \cdot \frac{\partial \omega}{\partial u} \cdot \frac{\partial \phi}{\partial t}. \]

Hence, combining the above equalities, we deduce
\[ f \cdot f' \cdot \left( \frac{\partial \omega}{\partial u} \right)^2 = f' \cdot f'' \]
which implies that
\[ f'(u)(f''(u) - \left( \frac{\partial \omega}{\partial u} \right)^2 f(u)) = 0. \]

Since we have \( f'(u) \neq 0 \), we obtain the result. \( \square \)

**Lemma 2** Assume that \( f'(u) \neq 0 \) and \( f''(u) \neq 0 \), for all \( u \in (\alpha, \beta) \) and let \( a(u) \), \( u \in (\alpha, \beta) \) be a function satisfying
\[ 2f'a' + fa'' = 0 \text{ and } (a')^2 = f''/f \]
for all \( u \in (\alpha, \beta) \). Then we have \((f(u))^2 = cu^2 + du + k\) and
\[ a(u) = \arctan \left( \frac{2c}{\sqrt{-\Delta}} \left( u + \frac{d}{2c} \right) \right), \]
where \( \Delta = d^2 - 4ck \).

**Proof** Putting \( y = a' \), we have the differential equation
\[ y' + \frac{2f'}{f} y = 0. \]
Its solution is
\[ y = a' = Ce^{-2 \int \frac{f'}{f} du}. \]
It follows that
\[ (a')^2 = C^2 e^{-4 \int \frac{f'}{f} du}, \]
and hence
\[ \frac{f''}{f} = C^2 e^{-4 \int \frac{f'}{f} du}. \]
and so, we obtain

$$\ln f'' - \ln f = K - 4 \int \frac{f'}{f} du.$$  

Differentiating the above equality, we get

$$\frac{f^{(3)}}{f''} - \frac{f'}{f} = -4 \frac{f'}{f},$$

and therefore we deduce

$$f^{(3)} f + 3 f'' f' = 0.$$

On the other hand, we have

$$(ff')'' = ((ff')')' = (f' f' + ff'')' = 2 f'' f + f'' f + ff^{(3)} = f^{(3)} f + 3 f'' f',$$

hence we get

$$(ff')'' = 0.$$  

It follows that $(ff')' = c,$ hence we have $ff' = c_1 u + d_1,$ and so, we get $(f^2)' = cu + d.$ Thus, we obtain

$$f^2 = cu^2 + du + k.$$  

Taking the first and the second derivative, we have

$$f' = \frac{1}{2} \frac{2cu + d}{\sqrt{cu^2 + du + k}} \quad \text{and} \quad f'' = \frac{4ck - d^2}{4(cu^2 + du + k)^{3/2}}.$$  

Thus

$$\frac{f''}{f} = \frac{4ck - d^2}{4(cu^2 + du + k)^2} \quad \text{and} \quad \frac{f'}{f} = \frac{2cu + d}{2(cu^2 + du + k)}.$$  

Since

$$\frac{f''}{f} = (a')^2 = C^2 e^{-4 \int \frac{f'}{f} du},$$

we have

$$\frac{f''}{f} = \frac{C^2}{f^4}.$$  

Springer
Thus, we obtain
\[
\frac{4ck - d^2}{4(cu^2 + du + k)^2} = \frac{C^2}{(cu^2 + du + k)^2},
\]
and therefore
\[
C^2 = \frac{4ck - d^2}{4}.
\]
Let \(\Delta = d^2 - 4ck\) be the discriminant of \(cu^2 + du + k\). Thus, we get
\[
C = \frac{\sqrt{-\Delta}}{2}.
\]
Furthermore, we have
\[
a' = Ce^{-2\int \frac{f'}{f} du} = \frac{\sqrt{-\Delta/2}}{cu^2 + du + k}
\]
and thus
\[
a = \int a' du = \int \frac{\sqrt{-\Delta/2}}{cu^2 + du + k} du = \int \frac{\sqrt{-\Delta/2}}{c(u + \frac{d}{2c})^2 + (\frac{\sqrt{-\Delta}}{2c})^2} du.
\]
Hence, we obtain
\[
a(u) = \arctan \left( \frac{2c}{\sqrt{-\Delta}} \left( u + \frac{d}{2c} \right) \right).
\]

\[\Box\]

4 Proof of Theorem 1

Suppose that there exists a map \(\Phi : U \to E^2\) satisfying properties (C1) and (C2). By Lemma 1, there are variables \(\phi\) and \(\omega\) which are functions of \(t, u\) satisfying
\[
\left( \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u} \right) = (\cos \phi, \sin \phi), \quad (1)
\]
\[
\left( \frac{\partial x}{\partial t}, \frac{\partial y}{\partial t} \right) = (f(u) \cos \omega, f(u) \sin \omega). \quad (2)
\]
and
\[
f''(u) = \left( \frac{\partial \omega}{\partial u} (t, u) \right)^2 f(u). \quad (3)
\]
By \((C1)\), the meridians are mapped to straight lines, and so, we have
\[
\frac{\partial \phi}{\partial u} = 0.
\]

Thus, \((1)\) yields
\[
\frac{\partial}{\partial u} \left( \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u} \right) = \frac{\partial}{\partial u} \left( \cos \phi, \sin \phi \right) = \left( -\sin \phi \frac{\partial \phi}{\partial u}, \cos \phi \frac{\partial \phi}{\partial u} \right) = (0, 0).
\]

It follows
\[
x(t, u) = ug_1(t) + g_2(t), \quad y(t, u) = uh_1(t) + h_2(t). \tag{4}
\]

Therefore, the function \(\frac{\partial \omega}{\partial u}\) is a function depending only on the variable \(u\), and, hence, there exist functions \(a(u)\) and \(b(t)\) such that
\[
\omega(t, u) = a(u) + b(t). \tag{5}
\]

Combining \((2)\) and \((5)\), we deduce
\[
\left( \frac{\partial x}{\partial t}, \frac{\partial y}{\partial t} \right) = (f \cos(a(u) + b(t)), f \sin(a(u) + b(t))), \tag{6}
\]
and using that
\[
\frac{\partial^2 x}{\partial u \partial t} = \frac{\partial^2 x}{\partial t \partial u},
\]
\((4)\) and \((6)\) imply that
\[
\frac{\partial}{\partial u} f \cos(a(u) + b(t)) = \frac{\partial}{\partial t} g_1(t).
\]

Therefore, for each \(t\) and \(u\), we deduce
\[
f'(u) \cos((a(u) + b(t)) - f(u) \sin(a(u) + b(t))a'(u) = g_1'(t). \tag{7}
\]

Similarly, from
\[
\frac{\partial^2 y}{\partial u \partial t} = \frac{\partial^2 y}{\partial t \partial u},
\]
we get
\[
f'(u) \sin((a(u) + b(t)) + f(u) \cos(a(u) + b(t))a'(u) = h_1'(t), \tag{8}
\]
for each \(t\) and \(u\).
By taking the derivative of (7) with respect to \( u \), we have

\[
    f'' \cos \omega - 2f'a' \sin \omega - f(a')^2 \cos \omega - fa'' \sin \omega = 0
\]

and so, we get

\[
    \sin \omega (2f'a' + fa'') - (f'' - f(a')^2) \cos \omega = 0
\]

Assuming that \( \sin \omega \neq 0 \), we obtain

\[
    2f'a' + fa'' = 0. \tag{9}
\]

If \( \sin \omega = 0 \), then \( \cos \omega \neq 0 \). Thus, by taking the derivative of (8) we can derive the same differential equation (9), restricting, if necessary, the domain where the functions \( f \) and \( a \) are defined. Lemma 2 implies that

\[
    f^2 = cu^2 + du + k
\]

and

\[
    a(u) = \arctan \left( \frac{2c}{\sqrt{-\Delta}} \left( u + \frac{d}{2c} \right) \right), \tag{10}
\]

where \( \Delta = d^2 - 4ck \).

Using (9) and (10), we get

\[
    \frac{f'}{f} = -\frac{a''}{2a'} = a' \tan a,
\]

which is equivalent to

\[
    f' \cos a - fa' \sin a = 0. \tag{11}
\]

If \( f \cos a = 0 \), then the above equality implies that \( fa' \sin a = 0 \). Since \( f(u)a'(u) \neq 0 \), for every \( u \), we have \( \sin a = 0 \) which is a contradiction. Thus, dividing the above equality by \( f \cos a \), we obtain \( \frac{f'}{f} \tan a + a' = 0 \). Substituting \( f'/f \) by \( a' \tan a \) we deduce \( a'(\tan a)^2 + a' = 0 \), whence \( (\tan a)^2 = -1 \) which is a contradiction. Hence, we have

\[
    f' \sin a + fa' \cos a \neq 0.
\]

On the other hand, by taking the derivative of \( f' \sin a + fa' \cos a \), we have

\[
    (f' \sin a + fa' \cos a)' = f'' \sin a + f'a' \cos a \\
    + f' \cos a + fa'' \cos a - f(a')^2 \sin a.
\]
In order to prove that this expression is zero, it suffices to show that
\[
\frac{f''}{f} \tan a + 2\frac{f'}{f}a' + a'' - (a')^2 \tan a = 0
\]
and one can verify, by substitution, that this relation holds. Furthermore, we have
\[
a'(0) = \frac{\sqrt{-\Delta}}{2k}, \quad f'(0) = \frac{d}{2\sqrt{k}}, \quad f(0) = \sqrt{k}, \quad \tan a(0) = \frac{d}{\sqrt{-\Delta}}.
\]
Then, we obtain
\[
f' \sin a + f a' \cos a = \sqrt{c}. \quad (12)
\]
By expanding relation (7), we obtain
\[
f'(\cos a(u) \cos b(t) - \sin a(u) \sin b(t))
- f a'(\sin a(u) \cos b(t) + \sin b(t) \cos a(u)) = g'_1(t),
\]
and from (11), (12) the relation \(g'_1(t) = \sqrt{c} \sin b(t)\) follows.
Similarly, from (9) we have:
\[
f'(\sin a \cos b + \sin b \cos a) + f a' (\cos a \cos b - \sin a \sin b) = h'_1(t),
\]
hence
\[
\cos b(f' \sin a + f a' \cos a) + \sin b(f' \cos a - f a' \sin a) = h'_1(t),
\]
and so, we obtain \(h'_1(t) = \sqrt{c} \cos b(t)\). Therefore, we get
\[
g'_1(t) = \sqrt{c} \sin b(t) \quad \text{and} \quad h'_1(t) = \sqrt{c} \cos b(t).
\]
We will proceed now with the computation of the projection \(\Phi\). By hypothesis, we have, that \((\partial \phi/\partial u) = 0\). Hence \(\phi\) is a function only of \(t\). From (1), (2) and (4) we have that
\[
\left( \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u} \right) = (\cos \phi(t), \sin \phi(t)) = (g_1, h_1) \quad (13)
\]
and
\[
\left( \frac{\partial x}{\partial t}, \frac{\partial y}{\partial t} \right) = (f(u) \cos(a(u) + b(t)), f(u) \sin(a(u) + b(t)))
= (ug'_1 + g'_2, uh'_1 + h'_2). \quad (14)
\]
Consequently, (13) and (14) imply
\[(g_1)^2 + (h_1)^2 = 1\]
and
\[u^2((g'_1)^2 + (h'_1)^2) + 2u(g'_1g'_2 + h'_1h'_2) + (g'_2)^2 + (h'_2)^2 = cu^2 + du + k,\]
respectively. Therefore, we have:
\[(g'_2)^2 + (h'_2)^2 = k\]
\[2(g'_1g'_2 + h'_1h'_2) = d.\]

From the first of the previous relations we deduce that there exists a function \(r(t)\) such that
\[(g'_2, h'_2) = (\sqrt{k} \cos r(t), \sqrt{k} \sin r(t))\]  \hspace{1cm} (15)
while from the second, in combination with (13) and (12), we deduce that \(2\sqrt{ck} \sin(b + r) = d\) and so, we get
\[\sin(b + r) = \frac{d}{2\sqrt{ck}}.\]

Therefore, there exists real number \(\theta_0\) such that
\[r(t) = \theta_0 - b(t).\]

Furthermore, from (13) we have that
\[(g'_1, h'_1) = (-\phi' \sin \phi, \phi' \cos \phi) = (\sqrt{c} \sin b, \sqrt{c} \cos b),\]
and so, we have the following two cases:

a) \(\phi' = \sqrt{c}\) and \(\phi(t) = -b(t)\). Thus, we have
\[b'(t) = -\phi' = -\sqrt{c},\]
whence
\[b(t) = -\sqrt{ct} + c_0.\]

Then, we get
\[(g_1, h_1) = (\cos(-b(t)), \sin(-b(t))) = (\cos b(t), -\sin b(t)).\]  \hspace{1cm} (16)
Thus, combining (4), (15) and (16) we deduce

\[ x(t, u) = u \cos(b(t)) + \int \sqrt{k} \cos(\theta_0 - b(t)) \, dt \]
\[ y(t, u) = -u \sin(b(t)) + \int \sqrt{k} \sin(\theta_0 - b(t)) \, dt. \]

b) \( \phi' = -\sqrt{c} \) and \( \phi(t) = \pi - b(t) \). Proceeding as above, we deduce the result. Furthermore, substituting \( b(t) \) in the integrals above we may calculate them and thus we may find explicit formulas for the map \( \Phi \).

Conversely, by substituting the above expressions of \( x(t, u) \) and \( y(t, u) \) into (iii) and (iv), and assuming that \( f'^2 = cu^2 + du + k \), we see that condition (\( C_1 \)) is easily verified. Also, condition (\( C_2 \)) is satisfied, since \( \frac{\partial x}{\partial u} = \cos \phi \) implies that

\[ \phi = \arccos \left( \frac{\partial x}{\partial u} \right) \]

and so, by taking the derivative with respect to \( u \), we obtain that \( \frac{\partial \phi}{\partial u} = 0 \). Hence, Theorem 1 is proven.

5 Proof of Corollary 1

(1) The Gaussian curvature of each point of \( S \) is given by the formula

\[ K = -\frac{f''}{f} \]

(see Formula (9), p. 162, in the Example 4 of do Carmo (1976)). On the other hand, in the proof of Lemma 2 we have shown that \( f''/f > 0 \). Therefore, \( K < 0 \) at every point of \( S \) and thus statement (1) is proven.

(2) The surfaces of revolution of constant negative curvature are well known. A description of them can be found for example in (Gray (2006), Theorem 15.22, p. 477). Obviously these surfaces of revolution \( R \) does not have the form of the surface \( S \) given in Theorem 1. Therefore, for any point \( p \in R \) and for any neighborhood \( U \subset R \) of \( p \) there does not exist a map \( \Phi : U \rightarrow E^2 \) satisfying the conditions (\( C_1 \)) and (\( C_2 \)). On the other hand, if \( S_0 \) is a Riemannian surface of constant negative curvature \( k < 0 \), it is well known that \( S_0 \) is locally isometric to surface of revolution \( R \) of constant curvature \( k \). Therefore our statement follows.

References

Charitos, C., Papadoperakis, I.: On the existence of a perfect map from the 2-sphere to the Euclidean plane, Eighteen Essays in Non-Euclidean Geometry, IRMA Lectures in Mathematics and Theoretical Physics 29, Eds V. A. Papadopoulos, EMS, Alberge (2019)
do Carmo, M. P.: Differential Geometry of Curves and Surfaces, Prentice-Hall (1976)
Gray, A.: Modern Differential Geometry of Curves and Surfaces, 3rd edn. Chapman and Hall/RCR, (2006)
Euler, Leonhard: *De reprezentatione superficiei sphaericae super plano*, Acta Academiae Scientarum
Impertialis Petropolitanae 1777, 1778, pp. 107-132 Opera Omnia: Series 1, Volume 28, pp. 248-275
Euler, Leonard: (translation by G. Heine), *On the mapping of Spherical Surfaces onto the Plane*, http://
eulerarchive.maa.org/docs/translations/E490en.pdf
Papadopoulos, A.: *Quasiconformal mappings, from Ptolemy’s geography to the work of Teichmüller*, in Uni-
formization, Riemann–Hilbert correspondence, Calabi–Yau manifolds, and Picard–Fuchs equations
(L. Ji and S.-T. Yau, eds.), Advanced Lectures in Mathematics 42, Higher Education Press Beijing,
and International Press, Boston, 237–315 (2018)

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps
and institutional affiliations.