Matrix Schur Functions, Permutation Matrices, and Young Operators as Inner Product Spaces

James D. Louck
Los Alamos National Laboratory Fellow, 54 Wildflower Way, Santa Fe, NM, 87506 USA
E-mail: jimlouck@aol.com

Abstract. An inner product is defined on the space of permutation matrices and the space of matrices dual to the permutation matrices is given. The relationship of permutation matrices to the expansion of matrices of fixed line-sum is discussed. This inner product carries over to the space of linear Young operators, as does the notion of the dual space. Motivation from physics for considering such algebraic structures is also given; in particular, the real, orthogonal, irreducible representations of the symmetric group originating from the matrix Schur functions are given.

1. Background, Review and Motivation

There is, perhaps, no group that has more universal applications in quantum physics than the symmetric group $S_n$ because of the extraordinary restrictions put on the quantum states of composite systems by the Pauli principle (see Hamermesh [1]) and Wybourne [2] for such applications). The classical work of Young [3] establishing the foundations of this subject, is well-known. We begin by reviewing the definition of the symmetric group, using the one-line notation for elements of the group, which may not be familiar to all.

The symmetric group $S_n$ may be defined to be the set of all $n!$ sequences $\{\pi\}$ in the integers $1, 2, \ldots, n$, together with a multiplication rule for multiplying each pair $\rho \in S_n, \tau \in S_n$ of elements in the set $S_n$, as follows:

$$S_n = \{\pi = (\pi_1, \pi_2, \ldots, \pi_n) \mid \text{the } n \text{ parts } \pi_i, i = 1, 2, \ldots, n \text{ of } \pi \text{ are any arrangement of the integers } 1, 2, \ldots, n\},$$

$$\rho \times \tau = \rho \tau = (\rho_{\tau_1}, \rho_{\tau_2}, \ldots, \rho_{\tau_n}), \text{ all } \rho, \tau \in S_n.$$  

Technically, the symmetric group is the pair of objects $(S_n, \times)$, since one and the same set can have different rules for multiplying its elements. It is customary, however, to denote the group simply by $S_n$ once the rule for multiplying elements has been given, and write $\rho \times \tau$ as $\rho \tau$ for each pair of elements $\rho, \tau \in S_n$, as already anticipated in (1).

We use the one-line symbol $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ for the elements of $S_n$, where the meaning of this symbol is: replace 1 by $\pi_1$, replace 2 by $\pi_2$, \ldots, replace $n$ by $\pi_n$, which is written
1 \mapsto \pi_1, 2 \mapsto \pi_2, \ldots, n \mapsto \pi_n.\) This statement has, of course, no meaning on its own — it has a contextual meaning in that it must be applied in the context of some set of objects \(X = \{X_1, X_2, \ldots, X_n\},\) where the elements of \(X\) (which themselves may be sets) are enumerated by \(1, 2, \ldots, n.\) For example, the permutation \(\pi = (3, 4, 2, 1) \in S_4\) can be applied to \(x = (x_1, x_2, x_3, x_4),\) with the result that \(\pi : (x_1, x_2, x_3, x_4) \mapsto (x_3, x_4, x_2, x_1);\) or \(\pi\) can be applied to \(y = (x_2, x_1, x_4, x_3)\) with the result that \(\pi : (x_2, x_1, x_4, x_3) \mapsto (x_4, x_3, x_1, x_2).\) The instruction carried by the notation \(\pi = (3, 4, 2, 1)\) is expressed verbally by: replace 1 by 3, replace 2 by 4, replace 3 by 2, replace 4 by 1, in any collection of entities labeled by 1, 2, \ldots, n on which the permutation is defined to act. In particular, the general permutation \(\rho \in S_n\) can act on another permutation \(\tau \in S_n,\) in which case the rule gives \(\rho \times \tau = \rho \tau = (\rho_{\pi_1}, \rho_{\pi_2}, \ldots, \rho_{\pi_n}),\) all \(\rho, \tau \in S_n,\) as given in (1). Thus, property is called the closure property.

The elements of the set \(S_n\) satisfy the rules imposed on every abstract group: (i) the product of two elements in the set is again an element; (ii) the associative product rule \((\pi \rho) \tau = \pi (\rho \tau)\) holds for all elements; (iii) there is an identity element \(e = (1, 2, \ldots, n)\) that commutes with each element; (iv) there is an inverse \(\pi^{-1}\) for each element. (The unique) inverse, denoted \(\pi^{-1},\) of the sequence \(\pi = (\pi_1, \pi_2, \ldots, \pi_n) \in S_n\) is obtained from \(\pi\) by writing out the sequence \((\pi_1 = 1, \pi_2 = 2, \ldots, \pi_n = n),\) and rearranging the entries in this sequence such that the subscripts are 1, 2, \ldots, n as read left-to-right.)

This presentation of permutations as one-line symbols of sequences with a multiplication rule is quite useful, especially from the viewpoint of concise presentation.

The action of a permutation \(\pi = (\pi_1, \pi_2, \ldots, \pi_n) \in S_n\) on an arbitrary sequence of \(n\) indeterminates \(x = (x_1, x_2, \ldots, x_n)\) is defined by

\[
\pi(x) = (x_{\pi_1}, x_{\pi_2}, \ldots, x_{\pi_n}).
\]  

(2)

The action of the product \(\rho \tau\) of two elements of the group is defined by the rule: first apply \(\tau\) to the sequence \(x\) followed by applying \(\rho\) to the sequence \(\tau(x) = (x_{\tau_1}, x_{\tau_2}, \ldots, x_{\tau_n}).\) Thus, we have that

\[
(\rho \tau)(x) = \rho(\tau(x)) = (x_{(\rho \tau)_1}, x_{(\rho \tau)_2}, \ldots, x_{(\rho \tau)_n}).
\]  

(3)

The set of permutation matrices is a realization of the multiplication rules for the symmetric group \(S_n\) by \(0 - 1\) matrices of order \(n.\) The permutation matrix \(P_{\pi}, \pi \in S_n,\) is defined as the following matrix:

\[
P_{\pi} = (e_{\pi_1} e_{\pi_2} \ldots e_{\pi_n}),
\]  

(4)

where \(e_i\) is a the unit column matrix with 1 in row \(i\) and 0 in all other rows, each \(i = 1, 2, \ldots, n.\) Thus, the element in row \(i\) and column \(j\) of \(P_{\pi}\) is given by

\[
(P_{\pi})_{ij} = \delta_{i, \pi_j}.
\]  

(5)

It may be verified directly from these matrix elements that the set of \(n!\) matrices of order \(n\) defined by

\[
P_{S_n} = \{P_{\pi} \mid \pi \in S_n\}
\]  

(6)

satisfies all the group rules for the symmetric group \(S_n\) in which now the rule of multiplication is ordinary matrix multiplication. Thus, \(P_{\pi} P_{\tau} = P_{\pi \tau},\) each pair \(\rho, \tau \in S_n.\) In particular, \(P_{\pi}^{-1} = P_{\pi}^T,\) where \(T\) denotes matrix transposition. Thus, we have a representation of \(S_n\) by a set of real, orthogonal matrices of order \(n.\) A unit row matrix presentation can also be used, but we will always use the column matrix form.
The action $\pi(x)$ of a permutation on a sequence $x = (x_1, x_2, \ldots, x_n)$ can presented in matrix form as ordinary row-on-column matrix multiplication:

$$P_\pi x = x_\pi,$$

where now $x$ is the column matrix $x = \text{col}(x_1, x_2, \ldots, x_n)$ and $x_\pi$ is the column matrix $x_\pi = \text{col}(x_{\pi_1}, x_{\pi_2}, \ldots, x_{\pi_n})$. In particular, we have that

$$P_\pi e_j = e_{\pi_j} = \sum_{i=1}^{n} \delta_{i,\pi_j} e_i.$$  \hfill (8)

Permutation matrices are of interest on their own. This is because there is a class of matrices \{A\} over the real numbers $\mathbb{R}$, or the complex numbers $\mathbb{C}$, that have an expansion in terms of permutation matrices:

$$A = \sum_{\pi \in S_n} a_\pi P_\pi, \text{ each } a_\pi \in \mathbb{C}, \hfill (9)$$

Before developing the properties of this class of matrices further in Sect. 2, we illustrate by three principal examples their occurrence in problems in physics and chemistry, hence, their intrinsic interest.

Each of the following sets of matrices has an expansion in terms of permutation matrices: doubly stochastic, magic square, and alternating sign, as we next briefly discuss:

(i) A doubly stochastic matrix of order $n$ has line-sum $r = 1$ (see (18) below) with elements $a_{ij} \in \mathbb{N}$, the nonnegative real numbers. We denote the set of all doubly stochastic matrices of order $n$ by $\mathcal{A}_n$. The existence of the expansion (9) for each $A \in \mathcal{A}_n$ is known as Birkhoff’s theorem. A simple proof of property (9) (over the positive integers) is found in Brualdi and Ryser [4]. Our interest stems from the role of doubly stochastic matrices in the quantum theory of angular momentum of composite systems (see Ref. [5,6]).

This theory and its relation to combinatorics has been developed extensively in Ref. [5]. It suffices here to remark that all $3n - j$ coefficients can be realized as the elements of a real orthogonal matrix, called a recoupling matrix, each of which is a doubly stochastic matrix.

(ii) A magic square matrix of order $n$ has line-sum equal to arbitrary $r \in \mathbb{R}_+$, the positive integers, with elements $a_{ij} \in \mathbb{N}$. We denote the set of all magic square matrices of order $n$ by $\mathcal{M}_n(r)$. Magic square matrices are the simplest example of the more general class of matrices of order $n$, denoted by $\mathcal{M}^p_{n \times n}(\alpha, \beta)$ in Ref. [5] and defined below in Sect. 4, for which the row and column sums $r_i$ and $c_j$ are not all equal, but are equal to the parts $r_i = \alpha_i$ and $c_j = \alpha_j$ of the weights (also called contents) $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \vdash p$ and $\alpha' = (\alpha_1', \alpha_2', \ldots, \alpha_n') \vdash p$ of the pair of Gelfand-Tsetlin patterns that enumerate the irreducible integer representations of the general linear group $GL(n, \mathbb{C})$ and the unitary group $U(n)$. (The notation $\alpha \vdash p$, etc. denotes that the parts of $\alpha$ sum to a prescribed positive integer $p$.) The sets $\mathcal{M}^p_{n \times n}(\alpha, \beta)$ occur in many contexts; significantly, they are basic to the definition of what are called matrix Schur functions (see Sect. 4), all of which is discussed in great detail in Ref. [5].

Little is known about the cardinality of the set $\mathcal{M}^p_{n \times n}(\alpha, \beta)$. A magic square is the special case: $\mathcal{M}_n(r) = \mathcal{M}_{n \times n}^{nr}(\alpha, \beta)$, for which $\alpha_i = \beta_i = r$, each $i = 1, 2, \ldots, n$. Closed formulas for the cardinality of the set $\mathcal{M}_n(r)$ are known for $n = 1, 2, 3, 4, 5$, for which $|\mathcal{M}_1(r)| = 1$, $|\mathcal{M}_2(r)| = r + 1$, $N_3(r) = \binom{r+4}{5} - \binom{r+3}{5}$, and the explicit formulas for $n = 4, 5$ have been noted recently in Ref. [7].

(iii) An alternating sign matrix of order $n$ has line-sum $r = 1$ with elements $a_{ij} \in A = \{-1, 0, 1\}$, such that when a given row (column) is read from left (top)-to-right (bottom) and all zeros
in that row (column) are ignored, the row (column) contains either a single 1, or is an alternating series 1, −1, 1, −1, . . . , 1, −1, 1. We denote the set of all alternating sign matrices of order \( n \) by \( \mathcal{A}_S n \). The discovery and proof of the formula for the number of alternating sign matrices of order \( n \) is a fascinating story of the interrelations between topics in mathematics, as reviewed in the essay by Bressoud and Propp [8], the book by Bressoud [9], and the article by Propp [10]. The formula is

\[
|\mathcal{A}_S n| = d_n = \frac{1}{(n+1)!} \prod_{j=0}^{n-1} (3j + 1)! 
\]

This relation was first proved by Zeilberger [11] and augmented by a closely related, and, perhaps, more significant formula in Zeilberger [12], since (10) follows quite easily from the second result (see Ref.[5]). The integer \( d_n \) also gives the cardinality of several other sets of ”combinatorial” entities. A physical interpretation of the number \( d_n \) is in counting the number of states of a two-dimensional ”6-vertex ” model of ice, known as square ice. Kupenberg [13] uses this 6-vertex model of statistical mechanics to give another derivation of the relation (10).

It is interesting to note that the collection of \( n! \) permutation matrices of order \( n \) is a subset of each of the classes of matrices: doubly stochastic, magic square, and alternating sign, and the subset of permutation matrices is, at the same time, a basis for expanding all such matrices of order \( n \).

We next turn to the mathematical framework for this develop of the properties of permutation matrices.

2. Algebra of Permutation Matrices

The set of permutation matrices \( \mathcal{P}_S n \) defined by (6) can be taken as the basis elements in a real or complex linear vector space \( \mathcal{P}_n \) with an inner product: Thus, if we define the matrix \( A \) by

\[
A = \sum_{\pi \in S_n} a_\pi P_\pi, \text{ each } a_\pi \in \mathbb{C}, \tag{11}
\]

then it is well-known that the set of all such matrices, obtained by letting the expansion coefficients take on all real (resp., complex) values, is a real (resp., complex) linear vector space. Indeed, if we take a second such matrix,

\[
B = \sum_{\pi \in S_n} b_\pi P_\pi, \text{ each } b_\pi \in \mathbb{C}, \tag{12}
\]

then the number defined by

\[
(A \mid B) = \text{Tr}(A^\dagger B) \tag{13}
\]

is an inner product on the vector space \( \mathcal{P}_n \), where \( ^\dagger \) denotes the complex conjugated transpose of \( A \), and \( \text{Tr} \) denotes the trace of a matrix. Then:

\[
(P_\pi \mid P_{\pi'}) = \sum_{i=1}^{n} \delta_{\pi_i, \pi'_i} \text{ and } (A \mid B) = \sum_{i=1}^{n} \sum_{\pi, \pi' \in S_n} \delta_{\pi_i, \pi'_i} a_\pi^* b_{\pi'}. \tag{14}
\]

where \( ^* \) denotes complex conjugation.
The element $a_{ij}$ in row $i$ and column $j$ of a matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ of the form (11) is given in terms of the expansion coefficients $a_\pi$ by
\[ a_{ij} = \sum_{\pi \in S_n^i} a_\pi, \quad (15) \]
where $S_n^i$ is the subset of permutations in $S_n$ defined by
\[ S_n^i = \{ \pi \in S_n | \pi_j = i \}. \quad (16) \]
These subsets of $S_n$ then satisfy the following union relations:
\[ \bigcup_{k=1}^{n} S_n^i = S_n, \quad \bigcup_{k=1}^{n} S_n^k = S_n, \quad i, j = 1, 2, \ldots, n. \quad (17) \]
It follows, in turn, from (15) that each row-sum $r_i$ and each column-sum $c_j$ of the elements of $A$ is given, respectively, by
\[ r_i = \sum_{k=1}^{n} a_{ik} = c_j = \sum_{k=1}^{n} a_{kj} = \sum_{\pi \in S_n} a_\pi. \quad (18) \]
Thus, all row and column sums are equal. We refer to this common sum as the line-sum of the matrix $A$ and denote it by $l_A$:
\[ l_A = \sum_{\pi \in S_n} a_\pi. \quad (19) \]
Each matrix $A \in \mathcal{P}_n$ is allowed to have its own line-sum $l_A$; that is, we do not require all such matrices to have the same line-sum.

Next, we take the product of two matrices $A, B \in \mathcal{P}_n$ and obtain:
\[ AB = C = \sum_{\pi \in S_n} c_\pi P_\pi \in \mathcal{P}_n, \quad (20) \]
where the group multiplication rule for permutation matrices gives the expansion coefficients as
\[ c_\pi = \sum_{\rho \in S_n} a_\rho b_{\rho^{-1}\pi}. \quad (21) \]
Thus, we find that the line-sum of the product matrix $AB$ satisfies the product rule:
\[ l_{AB} = l_A l_B. \quad (22) \]

A linear vector space that also has a product defined for all pairs of vectors such that the product of each pair is again a vector in the space is called an algebra — here, the algebra of permutation matrices, which we denote by $\mathcal{A} \mathcal{P}_n$.

Thus far we have given no hint as to how to determine the expansion coefficients $a_\pi$ for a matrix $A \in \mathcal{A} \mathcal{P}_n$. This, in turn, raises the question of the linear independence of the set $\mathcal{P}_n$ of permutation matrices; that is, the dimensionality of the vector space we have denoted by $\mathcal{P}_n$. 


As is the case for any linear vector space, a set of permutation matrices \( \{ P_\pi \mid \pi \in \Sigma_n \subset S_n \} \) is linearly independent over the real (complex) numbers if and only if the relation

\[
\sum_{\pi \in \Sigma_n} a_\pi P_\pi = 0_n \quad \text{(zero matrix of order } n) \tag{23}
\]

implies that \( a_\pi = 0 \), each \( \pi \in \Sigma_n \). To my knowledge, my postdoctoral student, John H. Carter [14] was the first to prove: The number \( b_n \) of linearly independent permutation matrices of order \( n \), hence, also the dimensionality of the space \( \mathcal{P}_n \) is given by

\[
\dim \mathcal{P}_n = b_n = (n - 1)^2 + 1. \tag{24}
\]

Curiously, we could not find this elementary result in the published literature, although probably known to many. The derivation of this number is repeated in Ref. [7], together with the proof of several sets \( \Sigma_n \) of permutations giving a basis set of linearly independent permutations matrices:

\[
\mathbb{P}_\Sigma_n = \{ P_\pi \mid \pi \in \Sigma_n \subset S_n \}. \tag{25}
\]

Then, each matrix \( A \) given by (11) can now be rewritten in the form

\[
A = \sum_{\pi \in \Sigma_n} a_\pi P_\pi, \text{ each } a_\pi \in \mathbb{C}. \tag{26}
\]

The expansion coefficients, or coordinates \( a_\pi \) relative to the basis \( \mathbb{P}_\Sigma_n \) are now unique.

The unique coordinates \( a_\pi \) in the expansion (26) relative to a basis \( \Sigma_n \) can be determined by using the dual basis to \( \mathbb{P}_\Sigma_n \), which we denote by \( \mathbb{P}_\Sigma^* \). This is carried out as follows: The dual matrices are defined by the orthogonality requirements:

\[
(\mathbb{P}_\pi \mid P_{\pi'}) = \delta_{\pi,\pi'}, \text{ each pair } \pi, \pi' \in \Sigma_n. \tag{27}
\]

Thus, the coordinates in relation (26) are given uniquely by

\[
a_\pi = (\mathbb{P}_\pi | A). \tag{28}
\]

It is, however, difficult to determine the basis set of matrices in the dual basis:

\[
\mathbb{P}_\Sigma^* = \{ \mathbb{P}_\pi \mid \pi \in \Sigma_n \}. \tag{29}
\]

We have determined the general set \( \mathbb{P}_\Sigma^* \) in Ref. [7] for a specific basis set \( \Sigma_n \). Here, we just note the example for \( n = 3 \), where the orthogonality property (27) can be verified explicitly.

**Example.** The dual permutation matrices \( \mathbb{P}_\pi \) for \( \pi \in \Sigma_3 = \{(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 2, 1)\} \) (the permutation \( (3, 1, 2) \) is missing):

\[
\begin{align*}
\mathbb{P}_{1,2,3} &= (2P_{1,2,3} - P_{2,1,3} + P_{2,3,1} - P_{3,1,2} - P_{3,2,1})/3, \\
\mathbb{P}_{2,1,3} &= (2P_{2,1,3} - P_{2,1,3} + 3P_{2,3,1} - 3P_{3,2,1} + 2P_{1,3,2} + 2P_{3,2,1})/9, \\
\mathbb{P}_{2,3,1} &= (2P_{2,3,1} - P_{2,1,3} + 2P_{2,3,1} - P_{1,3,2} - P_{3,2,1})/3, \\
\mathbb{P}_{1,3,2} &= (2P_{1,3,2} - 2P_{2,1,3} + 2P_{2,3,1} - 3P_{1,3,2} + 5P_{3,2,1})/9, \\
\mathbb{P}_{3,2,1} &= (2P_{3,2,1} - 2P_{2,1,3} + 2P_{2,3,1} - 3P_{1,3,2} + 2P_{3,2,1})/9. \quad \Box
\end{align*}
\]
We point out that it is never the case that the subset $\Sigma_n \subset S_n$ of $b_n = (n-1)^2 + 1$ linearly independent permutation matrices is a subgroup of $S_n$. Nonetheless, it is still true that

$$AB = \sum_{\rho, \tau \in \Sigma_n} a_{\rho} b_{\tau} P_{\rho \tau} = \sum_{\pi \in \Sigma_n} d_{\pi} P_{\pi}, \quad (31)$$

for two matrices $A$ and $B$ written in the form (11) and (12) with coordinates $a_{\rho}$, $b_{\tau}$. But now the coordinates of the product $AB$ are not given by the rule (21), since $\Sigma_n$ is not a group. The first sum in (31) splits into a sum of two terms: those for which $\rho \tau \in \Sigma_n$ and those for which $\rho \tau \notin \Sigma_n$. The permutation matrices $P_{\rho \tau}$ for $\rho \tau$ in the second split sum must be re-expressed in terms of the $P_{\pi}, \pi \in \Sigma_n$ to obtain the formula for the coordinates $d_{\pi}$ of the product $AB$. However, the product rule $l_{AB} = l_A l_B$ for line-sums given by (22) still holds — it is a basis independent relation.

3. An Algebra of Young Operators

We recall that a linear operator is a rule for mapping vectors in a linear vector space $V$ into new vectors in the same space. Let $V_n$ denote a linear vector space of finite-dimension $\dim V_n = n$ over the complex numbers $\mathbb{C}$. Also, let $V_n$ be an inner product space with an orthonormal basis set of vectors $B_n = \{|1\rangle, |2\rangle, \ldots, |n\rangle\}$ (Dirac bra-ket notation). Then, a linear operator $Y_\pi$ can be defined with the property

$$Y_\pi |i\rangle = |\pi_i\rangle, \text{ each } i = 1, 2, \ldots, n. \quad (32)$$

The linear property then means that the action of $Y_\pi, \pi \in S_n$, on each vector $|\alpha\rangle = \sum_{i=1}^{n} \alpha_i |i\rangle \in V_n$, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^n$, is given by

$$Y_\pi \sum_{i=1}^{n} \alpha_i |i\rangle = \sum_{i=1}^{n} \alpha_i Y_\pi |i\rangle = \sum_{i=1}^{n} \alpha_i |\pi_i\rangle = \sum_{i=1}^{n} \alpha_{\pi_i^{-1}} |i\rangle = |\alpha_{\pi^{-1}}\rangle, \quad (33)$$

where $\alpha_{\pi^{-1}} = (\alpha_{\pi_1^{-1}}, \alpha_{\pi_2^{-1}}, \ldots, \alpha_{\pi_n^{-1}})$. The linear operator $Y_\pi$ is the simplest example of a Young operator associated with the symmetric group $S_n$. The product $Y_\pi Y_\tau$ of two such Young operators is then defined by $(Y_\pi Y_\tau)(\alpha) = Y_\pi(Y_\tau(\alpha))$, which must hold for each vector $|\alpha\rangle \in V_n$. This relation gives the operator identity on $V_n$ expressed by $Y_\pi Y_\tau = Y_\tau Y_\pi$, as required of a group action.

We can now extend the definition of a Young operator to arbitrary sums (over $\mathbb{C}$) of the elementary Young operators $Y_\pi$, as given by

$$Y(x) = \sum_{\pi \in S_n} x_\pi Y_\pi, \text{ each } x_\pi \in \mathbb{C}. \quad (34)$$

It is a straightforward exercise to prove the well-known result: The set of all linear operators $\{Y(x) | x \in \mathbb{C}^n\}$ is itself a linear vector space $V_n^{\text{op}}$ over the complex numbers. We also refer to $Y(x)$ as a Young operator. But now we have the additional property that, for each pair of Young operators $Y(x) \in V_n^{\text{op}}$ and $Y(y) \in V_n^{\text{op}}$, it also the case that the product $Y(x)Y(y)$ is a Young operator; that is, we have

$$Y(x)Y(y) = Y(z) \in V_n^{\text{op}}; \quad z_\pi = \sum_{\rho \in S_n} x_\rho y_{\rho^{-1} \pi}. \quad (35)$$
Thus, we have an algebra of Young operators. See Hamermesh [1, pp. 239-246] for a more general discussion of this algebra.)

The matrix elements of the elementary Young operator $Y_\pi$ on the orthonormal basis $B_n$ of the vector space $V_n$ are given by

$$\langle i | Y_\pi | j \rangle = \delta_{i,\pi_j} = (P_\pi)_{ij}. \quad (36)$$

Thus, the matrix representation of the elementary Young operator $Y_\pi$ on the vector space $V_n$ is by the permutation matrix $P_\pi$. This result implies:

*Every relation in Sect. 2 that holds between permutation matrices $P_\pi$ and their linear combinations $A, B, \ldots$ over $\mathbb{C}$ also holds between the elementary Young operators $Y_\pi$ and their linear combinations $Y(a), Y(b), \ldots$ over $\mathbb{C}$."

We repeat the significant relations: (i) dimension of the vector space $V_n^{OP}$ of Young operators:

$$\dim V_n^{OP} = b_n = (n - 1)^2 + 2;$$

(ii) inner product of elementary Young operators:

$$\langle Y_\pi | Y_{\pi'} \rangle = \sum_{i=1}^{n} \delta_{\pi_i, \pi'_i};$$

(iii) inner product of Young operators $Y(a)$ and $Y(b)$:

$$\langle Y_a | Y_b \rangle = \sum_{i=1} b_i \sum_{\pi, \pi' \in S_n} \delta_{\pi_i, \pi'_i} a_{\pi}. \quad (36)$$

(iv) elementary dual Young operators defined by the orthogonality requirements:

$$\langle Y_\pi | Y_{\pi'} \rangle = \langle P_\pi | P_{\pi'} \rangle = \delta_{\pi, \pi'}, \ \text{each pair } \pi, \pi' \in S_n;$$

(v) coordinates in a basis:

$$Y(a) = \sum_{\pi \in S_n} a_{\pi} Y_\pi, \text{ each } a_{\pi} \in \mathbb{C}, \text{ where } a_{\pi} = \langle Y_\pi | Y(a) \rangle = \langle P_\pi | P(a) \rangle.$$

**Example 1.**

$$P_{1,2,3} + P_{2,3,1} + P_{3,1,2} - P_{1,3,2} - P_{3,2,1} - P_{2,1,3} = 0 \text{ (zero matrix)}$$

if and only if

$$Y_{1,2,3} + Y_{2,3,1} + Y_{3,1,2} - Y_{1,3,2} - Y_{3,2,1} - Y_{2,1,3} = 0 \text{ (zero operator)}.$$ \quad (37)

**Example 2.** The five relations (30) for permutation matrices and their duals carry over exactly to elementary Young operators and their duals simply by making the substitutions $P_\pi \mapsto V_\pi$ and $P_\pi \mapsto V_{\pi'}$.

We hasten to point out that Young operators can be defined on other vectors spaces that give rise to a more general theory of their algebra (see Hamermesh [1]). It is nonetheless the case that the inner product $\langle Y_\pi | Y_{\pi'} \rangle = \sum_{i=1} b_i \sum_{\pi, \pi' \in S_n} \delta_{\pi_i, \pi'_i}$ can always be defined, since it depends only on the properties of the permutations $\pi, \pi' \in S_n$.

4. **Matrix Schur Functions**

A real, orthogonal, irreducible representation of the symmetric group is obtained directly from the specialization of a class of many-variable polynomials known as matrix Schur functions. Whereas an ordinary Schur function is based on a single partition, a matrix Schur function is based on a number of stacked partitions that satisfy certain "betweenness" relations. The definition and properties of the matrix Schur functions is the subject of considerable length in Ref. [5]. Here, we list in Items 1-15 some of the properties of matrix Schur functions for the present application to the irreducible representations of $S_n$. The matrix Schur functions are completely defined by various of these relations, as discussed and proved in Ref. 5.

(i) Notation for matrix Schur functions: $D \left( \begin{array}{c} m' \\ \lambda \\ m \end{array} \right) (Z)$ (also called $D^\lambda_m$ polynomials and Gelfand-Tsetlin polynomials in Ref. [5])
(ii) Matrix Schur functions are homogeneous polynomials in the indeterminates (variables) $Z^A = \prod_{1 \leq i,j \leq n} z_{ij}^{a_{ij}}$, each $a_{ij} \in \mathbb{N}$:

$$D \left( \begin{array}{c} m' \\ \lambda \\ m \end{array} \right) (Z) = \sum_{A \in \mathcal{M}_{n \times n}(\alpha, \alpha')} C \left( \begin{array}{c} m' \\ \lambda \\ m \end{array} \right) (A) \frac{Z^A}{A^!} \quad (38)$$

(iii) Definitions of symbols:

(a) $\lambda$ is a partition with $n$ parts: $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, each $\lambda_i \in \mathbb{N}$, with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$.

(b) $\left( \begin{array}{c} \lambda \\ m \end{array} \right)$ is a stacked array of partitions:

$$\left( \begin{array}{c} \lambda \\ m \end{array} \right) = \begin{pmatrix}
\lambda_1 & \lambda_2 & \cdots & \lambda_j & \cdots & \lambda_{n-1} & \lambda_n \\
m_{1,n-1} & m_{2,n-1} & \cdots & m_{j,n-1} & \cdots & m_{n-1,n-1} \\
 & & & & & \\
 & & & & & \\
 & & & & & \\
 & & & & & \\
m_{1,j} & m_{2,j} & \cdots & m_{j,j} \\
 & & & & & \\
 & & & & & \\
 & & & & & \\
m_{1,2} & m_{2,2} \\
m_{1,1}
\end{pmatrix} \quad (39)$$

in which row $j$ is a partition with $j$ parts, and adjacent rows $(m_{j-1}, m_j), j = 2, 3, \ldots, n$, satisfy the "betweenness conditions:"

$$m_j = (m_{1,j}, m_{2,j}, \ldots, m_{j,j}), \quad m_{i,j-1} \in [m_{i,j}, m_{i+1,j}], \quad \text{each } i = 1, 2, \ldots, j - 1, \quad (40)$$

$$[a, b], a \leq b, \quad \text{denotes the closed interval of nonnegative integers: } [a, b] = \{a, a + 1, \ldots, b\}.$$

(c) Lexical patterns (patterns they satisfy betweenness rule):

$$\mathcal{G}_\lambda = \left\{ \begin{pmatrix} \lambda \\ m \end{pmatrix} \middle| m \text{ is a lexical pattern} \right\} \quad (41)$$

(iv) The notation $\left( \begin{array}{c} m' \\ \lambda \\ m \end{array} \right)$ denotes a pair of lexical GT patterns that share the same partition $\lambda$, where the pattern $\left( \begin{array}{c} m' \\ \lambda \end{array} \right)$ denotes the inversion of the pattern $\left( \begin{array}{c} \lambda \\ m \end{array} \right)$, and the shared partition $\lambda$ is written only once. Thus, it is these double GT patterns that enumerate the matrix Schur functions $D \left( \begin{array}{c} m' \\ \lambda \\ m \end{array} \right) (Z)$, in the $n^2$ indeterminates $Z = (z_{ij})_{1 \leq i,j \leq n}$.

(v) The C-coefficients in the right-hand side in the definition of the matrix Schur functions that multiply the polynomial factor $Z^A/A^!$ are, up to multiplying factors, the elements of a real orthogonal matrix. The details are not important here (see Ref. [5] for details).

(vi) The set $\mathcal{M}_{n \times n}(\alpha, \alpha')$:
(a) Weights of the GT patterns:

\[ \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \text{ of } \binom{\lambda}{m} ; \quad \alpha' = (\alpha'_1, \alpha'_2, \ldots, \alpha'_n) \text{ of } \binom{\lambda'}{m'} . \tag{42} \]

(b) Homogeneity properties of the polynomials:

degree \( \alpha_i \) in row \( z_i = (z_{i1}, z_{i2}, \ldots, z_{in}) \) of \( Z \),

degree \( \alpha'_j \) in column \( z^j = (z_{i1}, z_{i2}, \ldots, z_{in}) \) of \( Z \),

\[ p = \sum_i \lambda_i = \sum_i \alpha_i = \sum_i \alpha'_i . \tag{43} \]

(c) \( \mathbb{M}^p_{n \times n}(\alpha, \alpha') \) denotes the set of all matrix arrays \( A \) of order \( n \) such that the sum of all elements in row \( i \) of \( A \) is \( \lambda_i \) and the sum of all elements in column \( j \) of \( A \) is \( \alpha_j' \), where \( \lambda \vdash p, \alpha \vdash p, \alpha' \vdash p. \)

(vii) The summation is over all \( A \in \mathbb{M}^p_{n \times n}(\alpha, \alpha') \) : The matrix Schur functions are homogeneous polynomials of total degree \( p \) in the indeterminates \( Z \).

(viii) Orthogonality relations in an inner product denoted \( (, ) \) (isomorphic to boson inner product):

\[ \left( Z^{A'}, Z^{A} \right) = A! \delta_{A',A}, \]

\[ \left( D \left( \begin{array}{c} m' \\ \lambda \end{array} \right), D \left( \begin{array}{c} m'' \\ \lambda' \end{array} \right) \right) = \delta_{m,m'} \delta_{m',m''} \delta_{\lambda,\lambda'} M(\lambda), \]

\[ M(\lambda) = \prod_{i=1}^{n} (\lambda_i + n - i)! / 1!2! \cdot (n-1)! \text{Dim} \lambda, \tag{44} \]

\[ \text{Dim} \lambda = \prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j + j - i)! / 1!2! \cdot (n-1)! \text{(Weyl)}, \]

\[ D^\lambda(I_n) = I_{\text{Dim} \lambda}. \]

(ix) Invertibility:

\[ \frac{Z^A}{A!} = \sum_{\lambda \vdash p} \sum_{m,m' \in G_\lambda(\alpha,\alpha')} \frac{1}{M(\lambda) A! C} \binom{m'}{m} D \left( \begin{array}{c} m' \\ \lambda \end{array} \right) (A) D \left( \begin{array}{c} m' \\ \lambda \end{array} \right) (Z), \tag{45} \]

each \( A \in \mathbb{M}^p_{n \times n}(\alpha, \alpha') \), and \( G_\lambda(\alpha,\alpha') \) denotes the set of double GT patterns of weight \( (\alpha, \alpha') \) and shape \( \lambda \).

(x) Robinson, Schensted, Knuth (RSK) identity (expresses invertibility):

\[ \sum_{\lambda \vdash p} K(\lambda, \alpha) K(\lambda, \alpha') = |\mathbb{M}^p_{n \times n}(\alpha, \alpha')|. \tag{46} \]

(xi) Multiplication property: Fundamental multiplication rule:

\[ D^\lambda(X)D^\lambda(Y) = D^\lambda(XY), \text{ for arbitrary } X, Y. \tag{47} \]

(xii) Transpositional symmetry: \( Z \) replaced by the transpose \( Z^T \):

\[ D^\lambda(Z^T) = (D^\lambda(Z))^T. \tag{48} \]
(xiii) Group property: If $Z$ is specialized to be a member of any matrix group, continuous or finite, representations of that group are obtained. Indeed, this applies as well to any multiplicative matrix algebra. In particular, irreducible representations of the general linear group $GL(n, \mathbb{C})$ and the unitary group $U(n)$ are obtained:

(xiv) Diagonal property: $Z$ restricted to the diagonal matrix denoted $\text{diag}(Z) = \text{diag}(z_1, z_2, \ldots, z_n)$:

$$D\left(\begin{array}{c} m' \\ \lambda \\ m \end{array}\right)(\text{diag}(Z)) = \delta_{m,m'} z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n},$$  \hspace{1cm} (49)

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is the weight of the GT pattern $\binom{\lambda}{m}$.

(xv) Schur functions. The trace of the above relation gives the ordinary Schur functions:

$$s_{\lambda}(z) = \text{Tr} D^\lambda(\text{diag}(Z)) = \sum_{\alpha} K(\lambda, \alpha) x^\alpha,$$  \hspace{1cm} (50)

where the sum is over the distinct weights $\alpha$ of $\binom{\lambda}{m}$.

(xvi) Generalized MacMahon’s master theorem:

$$\frac{1}{\det(I_n^2 - tX \otimes Y)} = \sum_{k \geq 0} t^k \sum_{\lambda \vdash k} \text{Tr} D^\lambda(X) \text{Tr} D^\lambda(Y),$$  \hspace{1cm} (51)

in which $X$ and $Y$ are arbitrary matrices of order $n$ of commuting indeterminates, and $X \otimes Y$ is their Kronecker product (see Méndez [18] and Ref. [5]).

5. Real Orthogonal Irreducible Representations of $S_n$

5.1. Matrix Schur Function Real Orthogonal Irreducible Representations of $S_n$

The irreducible representations of the symmetric group are obtained from the matrix Schur functions defined by (38) in two steps:

(i). Impose conditions $\lambda \vdash n$; $\alpha, \alpha' \in \mathbb{G}_\lambda(1^n)$:

$$D\left(\begin{array}{c} m' \\ \lambda \\ m \end{array}\right)(Z) = \sum_{\pi \in S_n} C\left(\begin{array}{c} m' \\ \lambda \\ m \end{array}\right)(P_{\pi}) z_{\pi_1,1} z_{\pi_2,2} \cdots z_{\pi_n,n},$$  \hspace{1cm} (52)

in consequence of $M^n_{n \times n}(\alpha, \alpha') = \{P_{\pi} \mid \pi \in S_n\}$ and $(P_{\pi})_{ij} = \delta_{i,\pi_j}$. This result is of interest on its own because it gives a whole class of matrices $D^\lambda(Z), \lambda \vdash n$, such that the determinantal multiplication rule holds:

$$D^\lambda(X) D^\lambda(Y) = D^\lambda(XY).$$  \hspace{1cm} (53)

(ii). Set $Z = P_{\pi'}$ in (52) (and then rename $\pi'$ to be $\pi$):

$$D\left(\begin{array}{c} m' \\ \lambda \\ m \end{array}\right)(P_{\pi}) = \begin{cases} C\left(\begin{array}{c} m' \\ \lambda \\ m \end{array}\right)(P_{\pi}), & \text{for } \alpha, \alpha' \in \mathbb{G}_\lambda(1^n), \\ 0, & \text{all other weights} \end{cases}$$  \hspace{1cm} (54)

The corresponding matrices then have the multiplication property:

$$D^\lambda(P_{\pi}) D^\lambda(P_{\pi'}) = D^\lambda(P_{\pi} P_{\pi'}).$$  \hspace{1cm} (55)
The set of matrices \( \{ D^\lambda(P_\pi) \mid \pi \in S_n \} \) constitute a real, orthogonal, irreducible representation of \( S_n \) for each partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \vdash n \), hence, all real, orthogonal, irreducible representations, up to orthogonal similarity. The order of the matrices \( D^\lambda(P_\pi) \) is now denoted by \( \dim \lambda \); it is equal to the number of GT patterns \( \lambda \vdash n \) such that the weight of \( \binom{\lambda}{m} \) is \( \alpha = (1^n) \). The RSK relation (46) reduces now to the sum-of-squares identity:

\[
 n! = \sum_{\lambda \vdash n} K(\lambda, 1^n)K(\lambda, 1^n). \tag{56}
\]

5.2. Jucys-Murphy Real Orthogonal Representations of \( S_n \)

The set of matrices \( \{ M_\pi \} \) is defined to be \( \pi = (\pi_1, \pi_2, \ldots, \pi_n) \), in which all parts \( \pi_k = k \), except for parts \( i \) and \( j \), for which \( \pi_i = j \) and \( \pi_j = i \) for \( i \neq j \). The permutation \( (i, j) \) is called a transposition, since it interchanges \( i \) and \( j \) in its action, and leaves all other integers unchanged. It has the properties \( (i, j) = (j, i)^{-1} \). Thus, there are in all \( n(n-1)/2 \) distinct transpositions in \( S_n \), which can be chosen to be \( (i, j) \), \( 1 \leq i < j \leq n \).

It is well-known at this conference from the lectures by Lulek [19], and others (see, for example, Lulek et al. [20], Jakubczyk et al. [21]), that the \( n-1 \) Jucys-Murphy operators (Young operators) defined by \( M_2 = Y_{(1,2)}, M_3 = Y_{(1,3)} + Y_{(2,3)}, M_4 = Y_{(1,4)} + Y_{(2,4)} + Y_{(3,4)}, \ldots, M_n = Y_{(1,n)} + Y_{(2,n)} + \cdots + Y_{(n-1,n)} \) mutually commute. In our presentation, we choose to replace the Jucys-Murphy (JM) operators by the modified set \( M_2, M_2 + M_3, M_2 + M_3 + M_4, \ldots \); that is, we use the modified Jucys-Murphy operators \( M_k^{(n)} \) defined by

\[
 M_k^{(n)} = \sum_{1 \leq i < j \leq k} Y_{(i,j)}, \quad k = 2, 3, \ldots, n. \tag{57}
\]

This set is, of course, still an independent set of \( n-1 \) mutually commuting Young operators belonging to the group algebra of \( S_n \).

We have defined in Sect. 3 the action of the elementary Young operators on a very special vector space \( V_n \) of dimension \( \dim V_n = n \) with basis \( B_n \), and with the action of each elementary Young operator \( Y_{(i,j)} \) given explicitly by relation (32). This action determines completely the properties of the set of JM operators defined by relations (57), without further assumptions. But on this space \( V_n \), each elementary Young operator \( Y_{(i,j)} \) is represented by the permutation matrix \( P_{(i,j)} \). Thus, it must be the case that the spectrum of the JM operators \( M_k^{(n)} \) is exactly that obtained by diagonalizing the associated matrices, which are real symmetric; that is,

\[
P_k^{(n)} = \sum_{1 \leq i < j \leq n} P_{(i,j)} = \begin{pmatrix}
 J_k + \frac{1}{2} k(k-3)I_k & 0_{n-k} \\
 0_{n-k} & (k)I_{n-k}
\end{pmatrix}, \tag{58}
\]

where \( J_k \) denotes the matrix of order \( k \) with elements all equal to 1, and \( 0_{n-k} \) denotes the zero matrix of order \( n-k \). For \( k = n \), the last row and column are to be omitted in this relation; that is, \( P_n^{(n)} = J_n + \frac{1}{2} k(k-3)I_n \).

The characteristic equation of the matrix \( P_k^{(n)} \) is given by

\[
 \left( \lambda - \binom{k-1}{2} \right) + 1 \left( \lambda - \binom{k}{2} \right)^{n-k+1} = 0, \tag{59}
\]
where \( \binom{n}{m} \) denotes a binomial coefficient. Thus, we find that the real symmetric matrix \( P_k^{(n)} \) of order \( n \) has two distinct eigenvalues:

\[
\lambda_k^{(n)}(1) = \binom{k-1}{2} - 1 \quad \text{with multiplicity} \quad k - 1,
\]

\[
\lambda_k^{(n)}(2) = \binom{k}{2} \quad \text{with multiplicity} \quad n - k + 1,
\]

where this result holds for each of the matrices for \( k = 2, 3, \ldots, n \), and the binomial coefficient \( \binom{k}{2} \) is defined to be 0.

There always exists a real orthogonal matrix \( R \) of order \( n \) that simultaneously diagonalizes a set of commuting symmetric matrices. Thus, there exists an \( R \) such that

\[
R^T P_k^{(n)} R = D_k^{(n)} = \begin{pmatrix}
(\binom{k-1}{2} - 1) I_{k-1} & 0_{n-k+1} \\
0_{n-k+1} & \binom{k}{2} I_{n-k+1}
\end{pmatrix},
\]

where \( R \) is the same for all \( k = 2, 3, \ldots, n \). For the case at hand, the matrix \( R = (R_1 R_2 \cdots R_n) \) is unique up to \( \pm \) sign of its columns \( R_i, i = 1, 2, \ldots, n \), a result that can be proved as follows: Relation (61) implies \( DR = RD \), where \( D = \sum_{k=2}^{n} D_k^{(n)} \) is a diagonal matrix with **distinct** elements. But then it follows trivially that \( R \) is unique up to choice of signs of its columns. This is just the expression in matrix form that the set of \( n - 1 \) mutually commuting symmetric matrices \( P_k^{(n)}, k = 2, 3, \ldots, n \) is a complete set. Equivalently, this is the expression of the well-known fact (Lulek [19]) that the set of \( n - 1 \) PM operators (57) is a complete set of mutually commuting Hermitian operators on the space \( \mathcal{V}_n \). (The Hermitian property is expressed by

\[
\langle a | M_k^{(n)} | b \rangle = \langle b | M_k^{(n)} | a \rangle^* = \sum_{i,j=1}^{n} a_i b_j^*(P_k^{(n)})_{ij} \quad \text{for each pair of vectors} \quad |a\rangle, |b\rangle \in \mathcal{V}_n.
\]

We thus obtain the JM **reducible representation** \( J_\pi \) given by orthogonal similarity to a permutation matrix:

\[
J_\pi = R^T P_\pi R, \quad \pi \in S_n,
\]

where this representation is characterized as the unique diagonalization of the complete set of real symmetric matrices in relation (58) by the real, orthogonal matrix \( R \), which is unique up to \( \pm \) signs of its columns.

The irreducible real orthogonal JM representations can now be obtained from the matrix Schur functions in exactly the same manner as the GT representation \( D^\lambda(P_\pi), \pi \in S_n \), given by (55). We simply replace \( P_\pi \) in relation (55) by \( J_\pi \) given by (62) and use the multiplication property (47) to obtain:

\[
D^\lambda(J_\pi) = D^\lambda(R^T) D^\lambda(P_\pi) D^\lambda(R), \quad \text{each} \ \pi \in S_n,
\]

where the matrix \( D^\lambda(R) \) of order \( \dim \lambda \) is a real orthogonal matrix obtained from the matrix Schur function (38) by setting \( Z = R \), choosing \( \lambda \vdash n \), and restricting the weights of the patterns \( \binom{\lambda}{m} \) and \( \binom{\lambda}{m'} \) to be \( \alpha = \alpha' = (1^n) \). As expected, each real orthogonal irreducible JM representation of \( S_n \) is orthogonally similar to the real orthogonal irreducible GT representation corresponding to the same partition \( \lambda \vdash n \).

**Example.** It is useful to illustrate the above theory for \( n = 3 \):

\[
\begin{align*}
\lambda_k^{(3)}(1) &= \binom{k-1}{2} - 1 \quad \text{with multiplicity} \quad k - 1, \\
\lambda_k^{(3)}(2) &= \binom{k}{2} \quad \text{with multiplicity} \quad n - k + 1,
\end{align*}
\]
(i) Matrices of the commuting JM Young operators $M_{k}^{(3)}$, $k = 2, 3$:

$$P_{2}^{(3)} = P_{2,1,3} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$P_{3}^{(3)} = P_{2,1,3} + P_{3,2,1} + P_{1,3,2} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \quad (64)$$

(ii) Simultaneous eigenvalue-eigenvector relations:

columns of $R = (R_1 R_2 R_3)$:

$$R_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad R_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \quad R_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (65)$$

simultaneous eigenvectors:

$$P_{2}^{(3)} R_1 = -R_1, \quad P_{2}^{(3)} R_2 = R_2, \quad P_{2}^{(3)} R_3 = R_3;$$

$$P_{3}^{(3)} R_1 = 0 R_1, \quad P_{2}^{(3)} R_2 = 0 R_2, \quad P_{2}^{(3)} R_3 = 3 R_3. \quad (66)$$

Thus, we have the simultaneous diagonalization:

$$R^T P_{2}^{(3)} R = \text{diag}(-1, 1, 1), \quad R^T P_{3}^{(3)} R = \text{diag}(0, 0, 3), \quad (67)$$

where $R$ is unique up to signs of its columns because the sum of the two diagonal matrices in (67) has unique eigenvalues $\text{diag}(-1, 1, 4)$. This gives the reducible JM real orthogonal representation $J_\pi = R^T P_\pi R$ of $S_3$ by real orthogonal matrices of order 3. The three irreducible JM real orthogonal representations of $S_3$ are then given by

$$D^\lambda(J_\pi) = D^\lambda(R^T) D^\lambda(P_\pi) D^\lambda(R), \quad (68)$$

where $\lambda = (1, 1, 1), (2, 1, 0), (3, 0, 0)$. These are the identity representation, the real orthogonal representation of order 2, and the alternating representation.

6. Conclusions

The basic role of permutations matrices has been illustrated through their occurrence in the expansion of three types of matrices of fixed line-sum: doubly stochastic, magic squares, and alternating sign, all of which are of interest in physical applications. It has been shown that the algebra of permutation matrices stands on its own and has implications for the algebra of Young operators. The occurrence of a natural inner product for permutations allows for the unique expansion of arbitrary fixed line-sum matrices and arbitrary Young operators in terms of any basis set $\Sigma_n \subset S_n$ of $(n-1)^2 + 1$ linearly independent matrices (Young operators) by use of the dual space. Finally, the elemental role of permutation matrices has been outlined in the determination of the real, orthogonal, irreducible representations of the symmetric group by application of the matrix Schur functions, both in the Gelfand-Tsetlin basis and in the Jucys-Murphy basis.
References

[1] Hamermesh M Group Theory and Its Application to Physical Problems 1962 Reading: Addison-Wesley.
[2] Wybourne B G Proc. Conf. Symmetry and Structural Properties of Condensed Matter ed. T Lulek, W Florek and B Lulek 1990 Singapore: World Scientific p. 253.
[3] Young A The Collected Papers of Alfred Young 1871-1940, 1977 Toronto: University of Toronto Press.
[4] Brualdi R A and Ryser H J Combinatorial Matrix Theory 39 Cambridge: Cambridge University Press.
[5] Louck J D Unitary Symmetry and Combinatorics 2009 Singapore: World Scientific.
[6] Louck J D Foundations of Physics 27 1085.
[7] Louck J D Unitary Symmetry and Combinatorics: Addendum (In preparation)
[8] Bressoud D and Propp J 1999 Notices of the AMS 46 637.
[9] Bressoud D M Proofs and Confirmations: The Story of the Alternating Sign Matrix Conjecture 1999 Cambridge: Cambridge University Press.
[10] Propp J 2001 Discrete mathematics and Theoretical Computer Proceedings AA (DM-CCG) 43.
[11] Zeilberger D 1996 Electronic J. of Combinatorics 3 R13.
[12] Zeilberger D 1996 J. Math. 2 59.
[13] Kupenberg G 1996 Inter. Math. Res. Notes 1996 139.
[14] Carter J H Permutation Matrices and the Representation of Matrices with Fixed Line-Sum 2004 (LAUR 04-5399, Los Alamos National Laboratory, Los Alamos, NM 87545 unpublished).
[15] Macdonald I Symmetric Functions and Hall Polynomials 1979 (second edition 1995) Oxford: Oxford University Press.
[16] Knuth D E 1970 Pac. J. Math. 34 709.
[17] Stanley R P Enumerative Combinatorics (vols. 1 and 2) Cambridge: Cambridge University Press.
[18] Méndez M Proc. Conf. Symmetry and Structural Properties of Condensed Matter ed T Lulek, A Wal, and B Lulek 2003 Singapore: World Scientific p 265.
[19] Lulek T Proc. Conf. Symmetry and Structural Properties of Condensed Matter ed T Lulek, B Lulek, and A. Wal 2002 Singapore: World Scientific p. 279.
[20] Lulek T Lulek B Jakubczyk D and Jakubczyk P 2006 Physica B: Condensed Matter 382 162.
[21] Jakubczyk P Topolewicz S Wal A and Lulek T 2009 Open Systems & Information Dynamics 19 221.