A Family of Descriptive Approaches To Preferred Answer Sets

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Abstract

In logic programming under the answer set semantics, preferences on rules are used to choose which of the conflicting rules are applied. Many interesting semantics have been proposed. Brewka and Eiter’s Principle I expresses the basic intuition behind the preferences. All the approaches that satisfy Principle I introduce a rather imperative feature into otherwise declarative language. They understand preferences as the order, in which the rules of a program have to be applied. In this paper we present two purely declarative approaches for preference handling that satisfy Principle I, and work for general conflicts, including direct and indirect conflicts between rules. The first approach is based on the idea that a rule cannot be defeated by a less preferred conflicting rule. This approach is able to ignore preferences between non-conflicting rules, and, for instance, is equivalent with the answer set semantics for the subclass of stratified programs. It is suitable for the scenarios, when developers do not have full control over preferences. The second approach relaxes the requirement for ignoring conflicting rules, which ensures that it stays in the NP complexity class. It is based on the idea that a rule cannot be defeated by a rule that is less preferred or depends on a less preferred rule. The second approach can be also characterized by a transformation to logic programs without preferences. It turns out that the approaches form a hierarchy, a branch in the hierarchy of the approaches by Delgrande et al., Wang et. al., and Brewka and Eiter. Finally, we show an application for which the existing approaches are not usable, and the approaches of this paper produce expected results.

Introduction

Preferences on rules are an important knowledge representation concept. In logic programming, one usually writes general rules, and needs to express exceptions. Consider we have the following rules

\[ \begin{align*}
  r_1: & \quad \text{select(car}_1) \leftarrow \text{nice(car}_1) \\
  r_2: & \quad \neg\text{select(car}_1) \leftarrow \text{expensive(car}_1) \\
  r_3: & \quad \text{select(car}_1) \leftarrow \text{fast(car}_1) \\
\end{align*} \]

If a \text{car}_1 is both nice, expensive, and fast, the rules lead to contradiction. If we have preferences on rules, e.g., we prefer \( r_1 \) over \( r_2 \), and \( r_2 \) over \( r_3 \), we can use default negation to express exceptions between rules. Since the rules \( r_1 \) and \( r_3 \) have the same head, we have to use an auxiliary literal in order to ensure that \( r_3 \) does not defeat \( r_2 \).  

\[ \begin{align*}
  r_{1a}: & \quad \text{aux} \leftarrow \text{nice(car}_1) \\
  r_{1b}: & \quad \text{select(car}_1) \leftarrow \text{select(car}_1) \\
  r_2: & \quad \neg\text{select(car}_1) \leftarrow \text{expensive(car}_1), \text{not aux} \\
  r_3: & \quad \text{select(car}_1) \leftarrow \text{fast(car}_1), \text{not \neg select(car}_1) \\
\end{align*} \]

The hand-encoding of preferences has to use auxiliary literals, we have to split rules, and the resulting program is less readable. If the complementary literals are derived via other rules, and the program has hundreds of rules, the hand-encoding becomes even less readable.

More readable way to encode the exceptions between the rules is to make rules mutually exclusive, represent preferences using a relation on rules, and use a semantics for logic programs with preferences, in order to handle preferences.  

\[ \begin{align*}
  r_1: & \quad \text{select(car}_1) \leftarrow \text{nice(car}_1), \neg\text{select(car}_1) \\
  r_2: & \quad \neg\text{select(car}_1) \leftarrow \text{expensive(car}_1), \neg\text{select(car}_1) \\
  r_3: & \quad \text{select(car}_1) \leftarrow \text{fast(car}_1), \neg\text{select(car}_1) \\
  r_3 < r_2 < r_1 \\
\end{align*} \]

The rules \( r_1 \) and \( r_2 \) are mutually exclusive: whenever we apply the rule \( r_1 \), the rule \( r_2 \) is not applicable, and vice versa. We call this mutual exclusivity a conflict. The resulting program is much tolerant to changes. If we decide that the rule \( r_3 \) is the most preferred, and \( r_2 \) is the least preferred, only the preference relation needs to be changed, and the rules stay intact.

Several semantics for logic programs with preferences on rules have been proposed in the literature. In the first group are semantics that extend the well-founded semantics (Van Gelder, Ross, and Schlipf 1991); (Brewka 1996); (Wang, Zhou, and Lin 2000); (Schaub and Wang 2002) modify the alternating fixpoint characterization of the well-founded semantics in order to take preferences into account.

In the second group are the semantics that extend the answer set semantics (Gelfond and Lifschitz 1991). Each model of a program with preferences, called a preferred answer set, is guaranteed to be an answer set of the underlying program without preferences. (Brewka and Eiter 1999); (Wang, Zhou, and Lin 2000); (Delgrande, Schaub, and Tompits 2003) provide prescriptive (Delgrande et al. 2004) semantics, i.e. preferences are understood as the order in which the rules of a program have to be applied. A rule can be defeated only by rules that were applied before it w.r.t. to this order. Each answer set is
tested whether it can be constructed in aforementioned way. (Zhang and Foo 1997) iteratively non deterministically removes from a program less preferred rules that are defeated by the remainder of the program. (Sakama and Inoue 2000) transforms preferences on rules to preferences on literals, which leads to comparison of the sets of generating rules. Roughly speaking, answer set generated by maximal rules (w.r.t. a preference relation) are selected. (ˇSr´ anek 2008) understands preference handling as a kind of argumentation.

Brewka and Eiter have proposed Principle [1] (Brewka and Eiter 1999) that captures the intuition behind preferences on rules. If two answer sets are generated by the same rules except for two rules, and one rule is preferred over the other, an answer set generated by the less preferred rule should not be preferred.

The existing approaches to preference handling that satisfy Principle [1] (Brewka and Eiter 1999) Delgrande, Schaub, and Tompits 2003), denoted here as $\text{PAS}_{BE}$, $\text{PAS}_{WZL}$ and $\text{PAS}_{DST}$, introduce a rather imperative feature into the otherwise declarative language. They understand preferences on rules as the order in which the rules of a program have to be applied. This, on the one hand goes against declarative spirit of logic programming. On the other hand, it makes the approaches unusable in the situations when we need to automatically generate preferences.

Example 1 Consider a modified version of the scenario from (Brewka and Eiter 1999). Imagine we have a car recommender system. A program written by the developers of the system contains a database of cars and recommends them to a user.

\[
\begin{align*}
  r_1 & : \text{nice(car}_1) \leftarrow \\
  r_2 & : \text{safe(car}_2) \leftarrow \\
  r_3 & : \text{rec(car}_1) \leftarrow \text{nice(car}_1), \text{not}\text{rec(car}_1) \\
  r_4 & : \text{rec(car}_2) \leftarrow \text{nice(car}_2), \text{not}\text{rec(car}_2)
\end{align*}
\]

The system recommends nice cars to the user. We allow the user to write his/her own rules during the run time of a system. Imagine the user writes the following rules

\[
\begin{align*}
  u_1 & : \text{not}\text{rec(car}_2) \leftarrow \text{rec(car}_1) \\
  u_2 & : \text{not}\text{rec(car}_1) \leftarrow \text{rec(car}_2) \\
  u_3 & : \text{rec(car}_1) \leftarrow \text{safe(car}_1), \text{not}\text{rec(car}_1) \\
  u_4 & : \text{rec(car}_2) \leftarrow \text{safe(car}_2), \text{not}\text{rec(car}_2)
\end{align*}
\]

to say that maximally one car should be recommended, and that the user is interested in safe cars.

Due to the rules $u_1$ and $u_2$, the rule $u_3$ is conflicting with $r_4$: (i) The rule $u_1$ depends on $r_3$, and its head is in the negative body of $u_4$. (ii) The rule $u_4$ depends on $r_4$, and its head is in the negative body of $r_3$. We also have that $u_3$ is conflicting with $u_4$, and $r_3$ is conflicting with $r_4$ and $u_4$. All the conflicts are indirect – without the rules $u_1$ and $u_2$ there are no conflicts.

The purpose of the user’s rules is to override the default behaviour of the system in order to provide the user the best experience possible. Therefore we want the rule $u_3$ to override $r_4$, and $u_4$ to override $r_3$. Since the $u_i$ rules are only known at the run time, preferences cannot be specified beforehand by the developers of the system. Moreover, we cannot expect a user to know all the $r_i$ rules. It is reasonable to prefer each $u_i$ rule over each $r_i$ rule, and let the semantics to ignore preferences between non-conflicting rules. Hence we have the preferences:

\[
\begin{align*}
  u_1 & \text{ is preferred over } r_1 \\
  u_2 & \text{ is preferred over } r_2 \\
  \ldots \\
  u_4 & \text{ is preferred over } r_4
\end{align*}
\]

The prerequisites nice(car$_2$) and safe(car$_1$) of $r_4$ and $u_3$ cannot be derived. The only usable conflicting rules are $r_3$ and $u_3$. The rule $u_3$ being preferred, $u_4$ defines an exception to $r_3$. We expect $u_4$ to be applied, and $r_3$ defeated. The only answer set that uses $u_4$ is $S = F \cup \{\neg \text{rec(car}_1), \text{rec(car}_2)\}$, where $F = \{\text{nice(car}_1), \text{safe(car}_2)\}$. Hence $S$ is the unique expected preferred answer set.

None of the existing approaches satisfying Principle [1] works as expected. $\text{PAS}_{BE}$ does not handle indirect conflicts, and provides two preferred answer sets $S$ and $S_2 = F \cup \{\neg \text{rec(car}_1), \neg \text{rec(car}_2)\}$. $\text{PAS}_{DST}$ and $\text{PAS}_{WZL}$ provide no preferred answer set due to they imperative nature. Since $u_4$ is preferred over $r_2$, they require that $u_4$ is applied before $r_2$. It is impossible as $r_2$ is the only rule that derives $r_4$’s prerequisite.

It is not crucial for the example that the facts $r_1$ and $r_2$ are less preferred. If one feels that they should be separated from the rest of the rules, we can easily modify the program, e.g., by replacing the fact safe(car$_2$) by the fact volvo(car$_2$) and the rule safe(car$_2$) ← volvo(car$_2$).

Our goal is to develop an approach to preference handling that (i) is purely declarative, (ii) satisfies Brewka and Eiter’s Principle [1] and (iii) is usable in the above-mentioned situation.

We have already proposed such a semantics for the case of direct conflicts, and we denote it by $\text{PAS}_D$ (ˇSimko 2013). We understand this semantics as the reference semantics for the case of direct conflicts, and extend it to the case of general conflicts in this paper.

We present two approaches. The first one, denoted by $\text{PAS}_{D_G}$, is based on the intuition that a rule cannot be defeated by a less preferred (generally) conflicting rule. The approach is suitable for situations when we need to ignore preferences between non-conflicting rules, and is equivalent to the answer set semantics for the subclass of stratified programs. We consider this property to be important for the aforementioned situations as stratified programs contain no conflicts.

The second approach, denoted $\text{PAS}_{D_GNO}$, relaxes the requirement for ignoring preferences between non-conflicting rules, and stays is the NP complexity class. There are stratified programs with answer sets and no preferred answer sets according to the approach. The approach is suitable in situations when a developer has a full control over a program. The approach is based on the intuition that a rule cannot be
defeated by a less preferred rule or a rule that depends on a less preferred rule. The approach can be also characterized by a transformation from logic programs with preferences to logic programs without preferences such that the answer sets of the transformed program (modulo new special-purpose literals) are the preferred answer sets of an original one.

The two approaches of this paper and our approach for direct conflicts $PAS_D$ form a hierarchy, which in general does not collapse. Preferred answer sets of $PAS_GNO$ are preferred according to $PAS_G$, and preferred answer sets of $PAS_G$ are preferred according to $PAS_D$.

$PAS_D$ is thus the reference semantics for the case of direct conflicts. $PAS_GNO$ can be viewed as a computationally acceptable approximation of $PAS_G$. $PAS_GNO$ is sound w.r.t. $PAS_G$, but it is not complete w.r.t. $PAS_G$, meaning that each preferred answer set according to $PAS_GNO$ is a preferred answer set according to $PAS_G$, but not vice versa.

When dealing with preferences, it is always important to remember what the abstract term “preferences” stands for. Different interpretations of the term lead to different requirements on a semantics. We want to stress that we understand preferences as a mechanism for encoding exceptions between rules in this paper.

The rest of the paper is organized as follows. We first recapitulate preliminaries of logic programming, answer set semantics and our approach to preferred answer sets for direct conflicts $PAS_D$. Then we provide the two approaches to preferred answer sets for general conflicts. After that we show relation between the approaches of this paper, and also between approaches of this paper and existing approaches. Finally we show how the approaches work on the problematic program from Example 1.

Proofs not presented here can be found in the technical report [Simko 2014].

**Preliminaries**

In this section, we give preliminaries of logic programming and the answer set semantics. We recapitulate the alternative definition of answer sets based on generating sets from [Simko 2013], upon which this paper builds.

**Syntax**

Let $At$ be a set of all atoms. A literal is an atom or an expression $\neg a$, where $a$ is an atom. Literals of the form $a$ and $\neg a$ where $a$ is an atom are complementary. A rule is an expression of the form $l_0 \leftarrow l_1, \ldots, l_m$, not $l_{m+1}, \ldots, not \ l_n$, where $0 \leq m \leq n$, and each $l_i$ ($0 \leq i \leq n$) is a literal. Given a rule $r$ of the above form we use $head(r) = l_0$ to denote the head of $r$, $body(r) = \{l_1, \ldots, not \ l_m\}$ the body of $r$. Moreover, $body^+(r) = \{l_1, \ldots, l_m\}$ denotes the positive body of $r$, and $body^-(r) = \{l_{m+1}, \ldots, l_n\}$ the negative body of $r$. For a set of rules $R$, $head(R) = \{head(r) : r \in R\}$. A fact is a rule with the empty body. A logic program is a finite set of rules.

We say that a rule $r_1$ defeats a rule $r_2$ iff $head(r_1) \in body^-(r_2)$. A set of rules $R$ defeats a rule $r$ iff $head(R) \cap body^-(r) \neq \emptyset$. A set of rules $R_1$ defeats a set of rules $R_2$ iff $R$ defeats a rule $r_2 \in R_2$.

For a set of literals $S$ and a program $P$ we use $G_P(S) = \{r \in P : body^+(r) \subseteq S \land body^-(r) \cap S = \emptyset\}$.

A logic program with preferences is a pair $(P, <)$ where:
1. $(P, <)$ is a logic program, and $(ii)$ is a transitive and asymmetric relation on $P$. If $r_1 < r_2$ for $r_1, r_2 \in P$ we say that $r_2$ is preferred over $r_1$.

**Answer Set Semantics**

A set of literals $S$ is consistent iff $a \in S$ and $\neg a \in S$ holds for no atom $a$.

A set of rules $R \subseteq P$ positively satisfies a logic program $P$ iff for each rule $r \in P$ we have that: if $body^+(r) \subseteq head(R)$, then $r \in R$. We will use $Q(P)$ to denote the minimal (w.r.t. $\subseteq$) set of rules that positively satisfies $P$. It contains all the rules from $P$ that can be applied in the iterative manner: we apply a rule which positive body is derived by the rules applied before.

**Example 2** Consider the following program $P$:

$$
\begin{align*}
\text{r}_1: & \ a \leftarrow \\
\text{r}_2: & \ b \leftarrow a \\
\text{r}_3: & \ d \leftarrow c
\end{align*}
$$

We have that $R_1 = \{r_1, r_2\}$ and $R_2 = \{r_1, r_2, r_3\}$ positively satisfy $P$. On the other hand $R_3 = \{r_1\}$ do not positively satisfy $P$ as $body^+(r_2) \subseteq head(R_3)$ and $r_2 \notin R_3$.

We also have that $Q(P) = R_1$.

The reduct $P^R$ of a logic program $P$ w.r.t. a set of rules $R \subseteq P$ is obtained from $P$ by removing each rule $r$ with $head(R) \cap body^-(r) \neq \emptyset$.

A set of rules $R \subseteq P$ is a generating set of a logic program $P$ iff $R = Q(P^R)$.

**Definition 1 (Answer set)** A consistent set of literals $S$ is an answer set of a logic program $P$ iff there is a generating set $R$ such that $head(R) = S$.

**Example 3** Consider the following program $P$

$$
\begin{align*}
\text{r}_1: & \ a \leftarrow not \ b \\
\text{r}_2: & \ c \leftarrow d, not \ b \\
\text{r}_3: & \ b \leftarrow not \ a
\end{align*}
$$

Let $R = \{r_1\}$. When constructing $P^R$ we remove $r_3$ as $body^-(r_3) \cap head(R) \neq \emptyset$. We get that $P^R = \{r_1, r_2\}$, and $Q(P^R) = \{r_1\}$. The rule $r_2$ is not included as $d \in body^+(r_2)$ cannot be derived. We have that $Q(P^R) = R$. Therefore $R$ is a generating set of $P$ and $\{a\} = head(R)$ is an answer set of $P$.

It holds that: if a set of rules $R$ is a generating set of a logic program $P$, and $S = head(R)$ is consistent, then $R = G_P(S)$.

**Conflicts**

Informally, two rules are conflicting, if their applicability is mutually exclusive: if the application of one rule causes the other rule to be inapplicable, and vice versa. We divide general conflicts into two disjunctive categories:

- direct conflicts, and
- indirect conflicts.
In case of a direct conflict, application of a conflicting rule causes immediately the other rule to be inapplicable.

**Definition 2 (Directly Conflicting Rules)** We say that rules \( r_1 \) and \( r_2 \) are directly conflicting iff: (i) \( r_1 \) defeats \( r_2 \), and (ii) \( r_2 \) defeats \( r_1 \).

**Example 4** Consider the following program
\[
\begin{align*}
r_1 &: a \leftarrow \neg b \\
r_2 &: b \leftarrow \neg a
\end{align*}
\]
The rules \( r_1 \) and \( r_2 \) are directly conflicting. If \( r_1 \) is used, then \( r_2 \) is not applicable, and vice versa.

In case of an indirect conflict, another, intermediate rule, has to be used. The following example illustrated the idea.

**Example 5** Consider the following program
\[
\begin{align*}
r_1 &: x \leftarrow \neg b \\
r_2 &: b \leftarrow \neg a \\
r_3 &: a \leftarrow x, \neg y \\
r_4 &: y \leftarrow
\end{align*}
\]
Now, the rule \( r_1 \) is not able to make \( r_2 \) inapplicable on its own. The rule \( r_3 \) is also needed. Therefore we say that \( r_1 \) and \( r_2 \) are indirectly conflicting, and the conflict is formed via the rules \( r_3 \).

When trying to provide a formal definition of a general conflict, one has to address several difficulties.

First, an indirect conflict is not always effectual. The following example illustrates what we mean by that.

**Example 6** Consider the following program
\[
\begin{align*}
r_1 &: x \leftarrow \neg b \\
r_2 &: b \leftarrow \neg a \\
r_3 &: a \leftarrow x, \neg y \\
r_4 &: y \leftarrow
\end{align*}
\]
When the rule \( r_3 \) is used, the rule \( r_1 \) cannot be used. However, if we use \( r_1 \), the rule \( r_2 \) is still applicable as the rule \( r_3 \) that depends on \( r_1 \) and defeats \( r_2 \) is defeated by the fact \( r_1 \). Note that this cannot happen in the case of direct conflicts.

Second, we need to define that an indirect conflict is formed via rules that are somehow related to a conflicting rule.

**Example 7** Consider the following program:
\[
\begin{align*}
r_1 &: a \leftarrow \neg b \\
r_2 &: x \leftarrow \neg a \\
r_3 &: b \leftarrow
\end{align*}
\]
If we fail to see that \( r_3 \) does not depend on \( r_2 \), we come to wrong conviction that \( r_1 \) and \( r_2 \) are conflicting via \( r_3 \) as (i) \( r_1 \) defeats \( r_2 \), and (ii) \( r_3 \) defeats \( r_1 \).

Third, in general, the rules depending on a rule are conflicting, thus creating alternatives, in which the rule is/is not conflicting. The following example illustrates this.

**Example 8** Consider the following program:
\[
\begin{align*}
r_1 &: x \leftarrow \neg c \\
r_2 &: a \leftarrow x, \neg b \\
r_3 &: b \leftarrow x, \neg a \\
r_4 &: c \leftarrow \neg a
\end{align*}
\]
Since the rules \( r_2 \) and \( r_3 \) are directly conflicting, they cannot be used at the same time. If \( r_2 \) is used, \( r_1 \) and \( r_4 \) are conflicting via \( r_2 \). If \( r_3 \) is used, \( r_1 \) and \( r_4 \) are not conflicting.

In this paper we are going to address these issues from a different angle. Instead of defining a general conflict between two rules, we will move to sets of rules and define conflicts between sets of rules in the later sections.

**Approach to Direct Conflicts**

In this section we recapitulate our semantics for direct conflicts (Simko 2013), which we generalize in this paper for the case of general conflicts.

We say that a rule \( r_1 \) directly overrides a rule \( r_2 \) w.r.t. a preference relation \( \prec \) iff (i) \( r_1 \) and \( r_2 \) are directly conflicting, and (ii) \( r_2 \prec r_1 \).

The reduce \( P^R \) of a logic program with preferences \( P = (P, \prec) \) w.r.t. a set of rules \( R \subseteq P \) is obtained from \( P \) by removing each rule \( r_1 \in P \), for which there is a rule \( r_2 \in R \) such that:

- \( r_2 \) defeats \( r_1 \), and
- \( r_1 \) does not directly override \( r_2 \) w.r.t. \( \prec \).

A set of rules \( R \subseteq P \) is a preferred generating set of a logic program with preferences \( P = (P, \prec) \) iff \( R = Q(P^R) \).

A consistent set of literals \( S \) is a preferred answer set of a logic program with preferences \( P \) iff there is a preferred generating set \( R \) of \( P \) such that \( \text{head}(R) = S \).

We will use \( P, AS_D(P) \) to denote the set of all the preferred answer sets of \( P \) according to this definition.

It holds that each preferred generating set of \( P = (P, \prec) \) is a generating set of \( P \).

**Principles**

An important direction in preference handling research is the study of principles that a reasonable semantics should satisfy. Brewka and Eiter have proposed first two principles (Brewka and Eiter 1999).

Principle I tries to capture the meaning of preferences. If two answer sets are generated by the same rules except for two rules, the one generated by a less preferred rule is not preferred.

**Principle I (Brewka and Eiter 1999)** Let \( P = (P, \prec) \) be a logic program with preferences. \( S_1, S_2 \) be two answer sets of \( P \). Let \( G_P(S_1) = R \cup \{r_1\} \) and \( G_P(S_2) = R \cup \{r_2\} \) for \( R \subseteq P \). Let \( r_2 \prec r_1 \). Then \( S_2 \) is not a preferred answer set of \( P \).

Principle II says that the preferences specified on a rule with an unsatisfied positive body are irrelevant.

**Principle II (Brewka and Eiter 1999)** Let \( S \) be a preferred answer set of a logic program with preferences \( P = (P, \prec) \), and \( r \) be a rule such that \( \text{body}^+(r) \not\subseteq S \). Then \( S \) is a preferred answer set of a logic program with preferences \( P' = (P', \prec') \), where \( P' = P \cup \{r\} \) and \( \prec' \cap (P \times P) = \prec \).
We have an indirect conflict between the rules referred. Brewka and Eiter did not consider preferences should not cause a consistent program to be inconsistent.

**Principle III** Let \( \mathcal{P} = (P, \prec) \) be a logic program with preferences. If \( P \) has an answer set, then \( \mathcal{P} \) has a preferred answer set.

Before we proceed, we remind that our approach to preference handling is for general conflicts, and understands preferences on rules as a mechanism for expressing exception between rules. Using this view, we show that Principle II and Principle III should be violated by a semantics, and hence are not relevant under this understanding of preferences.

**Example 9** Consider the following program \( \mathcal{P} = (P, \prec) \)

\[
\begin{align*}
\text{r}_1 & : \text{select}(a) \leftarrow \text{not } \text{select}(a) \\
\text{r}_2 & : \text{select}(b) \leftarrow \text{not } \text{select}(b) \\
\text{r}_3 & : \text{select}(a) \leftarrow \text{not } \text{select}(b)
\end{align*}
\]

\( \text{r}_2 < \text{r}_1 \)

The program is stratified, and has the unique answer set \( S = \{ \text{select}(a), \text{select}(b) \} \). Since there are no conflicts between the rules, the unique answer set should be preferred. We construct \( \mathcal{P}' = (P', \prec) \), \( P' = P \cup \{ \text{r}_4 \} \), by adding the rule

\[
\text{r}_4 : \text{select}(b) \leftarrow \text{select}(a)
\]

We have an indirect conflict between the rules \( \text{r}_1 \) and \( \text{r}_2 \) via \( \text{r}_3 \) and \( \text{r}_4 \). The rule \( \text{r}_1 \) being preferred, \( S \) should not be a preferred answer set of \( \mathcal{P}' \). Hence Principle II is violated: \( \text{body}^+(\text{r}_4) = \{ \text{select}(a) \} \subseteq S \), but \( S \) is not a preferred answer set of \( \mathcal{P}' \).

**Example 10** Consider the following program \( \mathcal{P} = (P, \prec) \)

\[
\begin{align*}
\text{r}_1 & : \text{select}(a) \leftarrow \text{not } \text{select}(a) \\
\text{r}_2 & : \text{select}(a) \leftarrow \text{not } \text{select}(a)
\end{align*}
\]

\( \text{r}_2 < \text{r}_1 \)

When we interpret preference \( \text{r}_1 \prec \text{r}_2 \) as a way of saying that \( \text{r}_1 \) defines an exception to \( \text{r}_2 \) and not vice versa, the program has the following meaning:

\[
\begin{align*}
\text{r}_1 & : \text{select}(a) \\
\text{r}_2 & : \text{select}(a) \leftarrow \text{not } \text{select}(a)
\end{align*}
\]

Hence \( S = \{ \text{select}(a) \} \) is the unique preferred answer set of \( \mathcal{P} \).

We construct \( \mathcal{P}' = (P', \prec) \), \( P' = P \cup \{ \text{r}_3 \} \), by adding the rule

\[
\text{r}_3 : \text{inc} \leftarrow \text{select}(a), \text{not } \text{inc}
\]

The program \( \mathcal{P}' \) has the following meaning:

\[
\begin{align*}
\text{r}_1 & : \text{select}(a) \\
\text{r}_2 & : \text{not } \text{select}(a) \\
\text{r}_3 & : \text{inc} \leftarrow \text{select}(a), \text{not } \text{inc}
\end{align*}
\]

The program has no answer set, and hence \( \mathcal{P}' \) has no preferred answer set.

Hence Principle III is violated: The program \( \mathcal{P}' \) has an answer set, but \( \mathcal{P}' \) has no preferred answer set.

**Approach One to General Conflicts**

In this section we generalize our approach to direct conflicts to the case of general conflicts. As we have already noted, we deliberately avoid defining what a general conflict between two rules is. We will define when two sets of rules are conflicting instead. For this reason we develop an alternative definition of an answer set as a set of sets of rules, upon which the semantics for preferred answer sets will be defined.

**Alternative Definition of Answer Sets**

A building block of the alternative definition of answer sets is a fragment. The intuition behind a fragment is that it is a set of rules that can form the one hand side of a conflict. The positive bodies of the rules must be supported in a non-cyclic way.

**Definition 3 (Fragment)** A set of rules \( R \subseteq P \) is a fragment of a logic program \( P \) iff \( \mathcal{Q}(R) = R \).

**Example 11** Consider the following program \( P \) that we will use to illustrate the definitions of this paper.

\[
\begin{align*}
\text{r}_1 & : a \leftarrow x \\
\text{r}_2 & : x \leftarrow \text{not } b \\
\text{r}_3 & : b \leftarrow \text{not } a
\end{align*}
\]

The sets \( F_1 = \emptyset, F_2 = \{ \text{r}_2 \}, F_3 = \{ \text{r}_3 \}, F_4 = \{ \text{r}_2, \text{r}_1 \}, F_5 = \{ \text{r}_2, \text{r}_3 \}, F_6 = \{ \text{r}_1, \text{r}_2, \text{r}_3 \} \) are all the fragments of the program. For example, \( \{ \text{r}_1 \} \) is not a fragment as \( \mathcal{Q}(\{ \text{r}_1 \}) = \emptyset \).

**Notation 1** We will denote by \( F(P) \) the set of all the fragments of a program \( P \).

**Notation 2** Let \( P \) be a logic program and \( E \subseteq F(P) \).

We will denote \( R(E) = \bigcup_{X \in E} X \), and \( \text{head}(E) = \text{head}(R(E)) \).

Given a guess of fragments, we define the reduce. Since fragments are sets of rules, we can speak about defeating between fragments.

**Definition 4 (Reduce)** Let \( P \) be a logic program and \( E \subseteq F(P) \).

The reduce \( P^E \) of \( P \) w.r.t. \( E \) is obtained from \( P \) by removing each fragment \( X \in F(P) \) for which there is \( Y \in E \) that defeats \( X \).

**Example 12** (Example III continued) Let \( E_1 = \{ F_1, F_2, F_3 \} \). We have that \( P^{E_1} = \{ F_1, F_2, F_3 \} \). The fragments \( F_3, F_5 \), and \( F_6 \) are removed as they contain the rule \( \text{r}_3 \) which is defeated by \( F_4 \in E_1 \).

Let \( E_2 = \{ F_2 \} \). We have that \( P^{E_2} = \{ F_1, F_2, F_3, F_4, F_5, F_6 \} \). Since no rule has \( x \) in its negative body, no fragment is removed.
A stable fragment set, an alternative notion to the notion of answer set, is a set of fragments that is stable w.r.t. to the reduction.

**Definition 5 (Stable fragment set)** A set \( E \subseteq F(P) \) is a stable fragment set of a program \( P \) iff \( P^E = E \).

**Example 13 (Example 12 continued)** We have that \( P^{E_1} = E_1 \), so \( E_1 \) is a stable fragment set. On the other hand, \( E_3 \) is not a stable fragment set as \( P^{E_2} \neq E_2 \).

**Proposition 1** Let \( P \) be a logic program, and \( E \subseteq F(P) \). 
\( E \) is a stable fragment set of \( P \) iff \( R(E) \) is a generating set of \( P \) and \( E = \{ T : T = Q(T) \text{ and } T \subseteq R(E) \} \).

From Proposition 1 we directly have that the following is an alternative definition of answer sets.

**Proposition 2** Let \( P \) be a logic program and \( S \) a consistent set of literals.
\( S \) is an answer set of \( P \) iff there is a stable fragment set \( E \) of \( P \) such that \( \text{head}(E) = S \).

**Example 14 (Example 13 continued)** \( E_1 = \{ F_1, F_2, F_4 \} \) and \( E_3 = \{ F_1, F_3 \} \) are the only stable fragment sets of the program. The sets \( \{ a, x \} = \text{head}(E_1) \) and \( \{ b \} = \text{head}(E_3) \) are the only answer sets of the program.

**Preferred Answer Sets**

In this subsection we develop our first definition of preferred answer sets for general conflicts from the alternative definition of answer sets based on stable fragment sets.

The basic intuition behind the approach is that a rule cannot be defeated by a less preferred conflicting rule. This intuition is realized by modifying the definition of reduct. We do not allow a fragment \( X \) to be removed because of a fragment \( Y \) if \( Y \) uses less preferred conflicting rules. For this purpose we use the term “override”.

**Definition 6 (Conflicting Fragments)** Fragments \( X \) and \( Y \) are conflicting iff (i) \( X \) defeats \( Y \), and (ii) \( Y \) defeats \( X \).

**Example 15 (Example 11 continued)** Let us recall the fragments: \( F_2 = \{ r_2 \} \), \( F_3 = \{ r_3 \} \), and \( F_4 = \{ r_2, r_1 \} \). The fragments \( F_3 \) and \( F_4 \) are conflicting as \( \text{head}(r_3) \in \text{body}^-(r_2) \) and \( \text{head}(r_1) \in \text{body}^-(r_3) \). On the other hand, \( F_2 \) and \( F_3 \) are not conflicting. The fragment \( F_3 \) defeats \( F_2 \), but not the other way around as \( \text{head}(r_2) \not\in \text{body}^-(r_3) \).

**Definition 7 (Override)** Let \( X \) and \( Y \) be conflicting fragments. We say that \( X \) overrides \( Y \) w.r.t. a preference relation \(<\) iff for each \( r_1 \in X \) that is defeated by \( Y \), there is \( r_2 \in Y \) defeated by \( X \), and \( r_2 < r_1 \).

**Example 16 (Example 15 continued)** Let us continue with preference \( r_2 < r_3 \). We have that \( F_3 \) overrides \( F_4 \) and \( F_3 \) overrides \( F_0 \). On the other hand \( F_3 \) does not override \( F_2 \) because \( F_2 \) does not defeat \( F_3 \). From the following Proposition 3 we also have that \( F_3 \) does not override \( F_6 \).

**Proposition 3** Let \( P = (P, <) \) be a logic program with preferences, \( X \) and \( Y \) be fragments of \( P \).
If \( X \) overrides \( Y \) w.r.t. \(<\), then \( Y \) does not override \( X \) w.r.t. \(<\).

When constructing the reduct w.r.t. a guess, a fragment \( X \) cannot be removed because of a fragment \( Y \) which is overridden by \( X \).

**Definition 8 (Reduct)** Let \( P = (P, <) \) be a logic program with preferences, and \( E \subseteq F(P) \).

The reduct \( P^E \) of \( P \) w.r.t. \( E \) is obtained from \( F(P) \) by removing each \( X \in F(P) \) such that there is \( Y \in E \) that:
- \( Y \) defeats \( X \), and
- \( X \) does not override \( Y \) w.r.t. \(<\).

**Example 17 (Example 16 continued)** Let \( E_1 = \{ F_1, F_2, F_4 \} \). We have that \( P^{E_1} = \{ F_1, F_2, F_3, F_4 \} \). Now, the fragment \( F_3 \) is not removed as the only fragment from \( E_1 \) that defeats it is \( F_4 \), but \( F_3 \) overrides \( F_4 \).

**Definition 9 (Preferred stable fragment set)** Let \( P = (P, <) \) be a logic program with preferences, and \( E \subseteq F(P) \).

We say that \( E \) is a preferred stable fragment set of \( P \) iff \( P^E = E \).

**Example 18 (Example 16 continued)** Now we have that \( P^{E_1} \neq E_1 \), so \( E_1 \) is not a preferred stable fragment set. On the other hand, \( E_3 = \{ F_1, F_3 \} \) is a preferred stable fragment set as \( P^{E_3} = E_3 \).

**Definition 10 (Preferred answer set)** Let \( P = (P, <) \) be a logic program with preferences, and \( S \) be a consistent set of literals.

\( S \) is a preferred answer set of \( P \) iff there is a preferred stable fragment set \( E \) of \( P \) such that \( \text{head}(E) = S \).

We will use \( \mathcal{PAS}_G(P) \) to denote the set of all the preferred answer sets of \( P \) according to this definition.

**Example 19 (Example 18 continued)** The set \( E_3 = \{ F_1, F_3 \} \) is the only preferred stable fragment set, and \( \{ b \} = \text{head}(E_3) \) is the only preferred answer set of the program.

**Proposition 4** Let \( P = (P, <) \) be a logic program with preferences, and \( E \subseteq F(P) \).
If \( E \) is a preferred stable fragment set of \( P \), then \( E \) is a stable fragment set of \( P \).

**Properties**

Preferred answer sets as defined in Definition 10 enjoy the following nice properties.

**Proposition 5** Let \( P = (P, <) \) be a logic program with preferences. Then \( \mathcal{PAS}_G(P) \subseteq \mathcal{AS}(P) \).

**Proposition 6** Let \( P = (P, \emptyset) \) be a logic program with preferences. Then \( \mathcal{PAS}_G(P) = \mathcal{AS}(P) \).

**Proposition 7** Preferred answer sets as defined in Definition 10 satisfy Principle 7.

**Proposition 8** Let \( P_1 = (P, <_1) \), \( P_2 = (P, <_2) \) be logic programs with preferences such that \( <_1 \sqsubseteq <_2 \).

Then \( \mathcal{PAS}_G(P_2) \subseteq \mathcal{PAS}_G(P_1) \).

On the subclass of stratified programs, the semantics is equivalent to the answer set semantics. We consider this property to be an important one as stratified programs contain no conflicts.
Proposition 9 Let $\mathcal{P} = (P, \prec)$ be a logic program with preferences such that $P$ is stratified. Then $\mathcal{P} AS_G(P) = AS(P)$.

The following example illustrates how the approach works on stratified programs.

Example 20 Consider a problematic program from [Brewka and Eiter 1999]:

\begin{align*}
r_1: & \ a \leftarrow \neg b \\
r_2: & \ b \leftarrow \\
\end{align*}

$r_2 \prec r_1$

The program is stratified and has a unique answer set $S = \{b\}$.

The program has the following fragments $F_0 = \emptyset$, $F_1 = \{r_1\}$, $F_2 = \{r_2\}$, $F_3 = \{r_1, r_2\}$. The set $E = \{F_0, F_2\}$ is a unique stable fragment set.

We have that $F_2$ defeats both $F_1$, and $F_3$. Neither $F_1$, nor $F_3$ override $F_2$ as they are not conflicting with $F_2$. This is the reason why preference $r_2 \prec r_1$ is ignored here, and both $F_1$ and $F_3$ are removed during the reduction: $\mathcal{P}^E = \{F_0, F_2\} = E$. Therefore $S$ is a unique preferred answer set.

From the computational complexity point of view, so far, we have established only the upper bound. Establishing the lower bound remains among open problems for future work.

Proposition 10 Given a logic program with preferences $\mathcal{P}$, deciding whether $\mathcal{P}$ has a preferred answer set is in $\Sigma_3^P$.

Approach Two to General Conflicts

If we have an application domain, where we can relax the requirements for preference handling in a sense that we no longer require preferences between non-conflicting rules to be ignored, we can ensure that the semantics stays in the NP complexity class.

In this section we simplify our first approach by using the following intuition for preference handling: a rule cannot be defeated by a less preferred rule or a rule depending on a less preferred rule.

The definition of the approach follows the structure of our approach for direct conflicts. The presented intuition is realized using a set $\mathcal{P}^R$ in the definition of reduct.

Definition 11 (Reduct) Let $\mathcal{P} = (P, \prec)$ be a logic program with preferences, and $R \subseteq P$ be a set of rules.

The reduct $\mathcal{P}^R$ of $\mathcal{P}$ w.r.t. $R$ is obtained from $\mathcal{P}$ by removing each rule $r \in P$ such that body$^-(r) \cap$ head($T^R_r$) $\neq \emptyset$, where $T^R_r = \mathcal{Q}(\{p \in R : p \prec r\})$.

Example 21 (Example 16 continued) Let us recall the program:

\begin{align*}
r_1: & \ a \leftarrow x \\
r_2: & \ x \leftarrow \neg b \\
r_3: & \ b \leftarrow \neg a \\
r_2 & < r_3
\end{align*}

Let $R_3 = \{r_1, r_2\}$. We have that $T^R_1 = R_3$, $T^R_2 = R_3$. On the other hand $T^R_3 = \emptyset$ as $r_2 \prec r_3$ and $r_1$ depends on $r_2$. No rule less preferred, and no rule that depends on a rule less preferred than $r_3$ can be used to defeat $r_3$. In this case no rule can defeat $r_3$.

Hence $\mathcal{P}^R_3 = \{r_1, r_2, r_3\}$.

Definition 12 (Preferred generating set) Let $\mathcal{P} = (P, \prec)$ be a logic program with preferences, and $R$ be a generating set of $\mathcal{P}$.

We say that $R$ is a preferred generating set of $\mathcal{P}$ iff $R = \mathcal{Q}(\mathcal{P}^R)$.

Example 22 (Example 21 continued) We have that $\mathcal{Q}(\mathcal{P}^R_3) = P \neq R_3$. Hence $R_3$ is not a preferred generating set.

Definition 13 (Preferred answer set) Let $\mathcal{P} = (P, \prec)$ be a logic program with preferences, and $S$ be a consistent set of literals.

$S$ is a preferred answer set of $\mathcal{P}$ iff there is a preferred generating set $R$ such that $S = \text{head}(R)$.

We will use $\mathcal{P} AS_G(P)$ to denote the set of all the preferred answer sets of $\mathcal{P}$ according to this definition.

Example 23 (Example 22 continued) The set $R_2 = \{r_3\}$ is the only preferred generating set, and $\{b\} = \text{head}(R_2)$ is the only preferred answer set.

Transformation

It turns out that the second approach can be characterized by a transformation from programs with preferences to programs without preferences in a way that the answer sets of the transformed program correspond (modulo new special-purpose literals) to the preferred answer sets of an original program.

The idea of the transformation is to use special-purpose literals and auxiliary rules in order to allow a rule to be defeated only by $T^R_r$ where $R$ is a preferred generating set guess. We first present the definition of the transformation and then explain each rule.

Notation 3 If $r$ is a rule of a program $P$, then $n_r$ denotes a new literal not occurring in $P$.

If $r$ is a rule of a program $P$, and $x$ is a literal of $P$, then $x^r$ denotes a new literal not occurring in $P$ and different from $n_q$ for each $q \in P$. For a set of literals $S$, $S^r$ denotes $\{x^r : x \in S\}$.

We will also use $\text{inc}$ to denote a literal not occurring in $P$ and different from all previously mentioned literals.

Definition 14 (Transformation) Let $\mathcal{P} = (P, \prec)$ be a logic program with preferences.

Let $r$ be a rule. Then $P^r_r$ is the set of the rules

\begin{align*}
\text{head}(r) & \leftarrow n_r & (1) \\
n_r & \leftarrow \text{body}^+(r), \neg \text{body}^-(r)^r & (2)
\end{align*}

and the rule

\begin{align*}
\text{head}(p)^r & \leftarrow \text{body}^+(p)^r, n_p & (3)
\end{align*}

for each $p \in P$ such that $p \not\prec r$, and the rule

\begin{align*}
\text{inc} & \leftarrow n_r, x, \neg \text{inc} & (4)
\end{align*}
for each \( x \in \text{body}^-(r) \).
\[
t(P) = \bigcup_{r \in P} t_P(r).
\]

A preferred generating set guess \( R \) is encoded using \( n_r \) literals. The meaning of a literal \( n_r \) is that a rule \( r \) was applied. In order to derive \( n_r \) literals, we split each rule \( r \) of a program into two rules: The rule \( 2 \) derives literal \( n_r \), and the rule \( 1 \) derives the head of the original rule \( r \).

The special-purpose literals \( x^r \) are used in the negative body of the rule \( 2 \) in order to ensure that only \( T^R_r \) can defeat a rule \( r \). The \( x^r \) literals are derived using the rules of the form \( 4 \).

The rules of the form \( 4 \) ensure that no answer set of \( t(P) \) contains both \( n_r \) and \( x \). This condition is needed in order to ensure that \( R \) is also a generating set.

Example 24 Consider again our running program \( P \):
\[
\begin{align*}
r_1: & \quad a \leftarrow x \\
r_2: & \quad x \leftarrow \text{not } b \\
r_3: & \quad b \leftarrow \text{not } a
\end{align*}
\]

\( r_2 < r_3 \)
\[
t(P) \text{ is as follows:}
\begin{align*}
a & \leftarrow n_{r_1} \\
x & \leftarrow n_{r_2} \\
b & \leftarrow n_{r_3} \\
n_{r_1} & \leftarrow x \\
n_{r_2} & \leftarrow \text{not } b \\
n_{r_3} & \leftarrow \text{not } a^r \\
a^r & \leftarrow x^{r_1}, n_{r_1} \\
a^r & \leftarrow x^{r_2}, n_{r_2} \\
a^r & \leftarrow x^{r_3}, n_{r_3} \\
x^r & \leftarrow n_{r_1} \\
x^r & \leftarrow n_{r_2} \\
x^r & \leftarrow n_{r_3} \\
b^r & \leftarrow n_{r_3} \\
b^r & \leftarrow n_{r_2} \\
b^r & \leftarrow n_{r_3} \\
\text{inc} & \leftarrow n_{r_2}, b, \text{not inc} \\
\text{inc} & \leftarrow n_{r_3}, a, \text{not inc}
\end{align*}
\]

Now, as \( r_2 < r_3 \), a transformed rule deriving \( x^{r_3} \) coming from \( r_2 \) is not included.

The transformation captures the semantics of preferred answer sets as defined in Definition 13.

Proposition 11 Let \( P = (P, <) \) be a logic program with preferences. Let \( S \) be a set of all the literals constructed from the atoms of \( P \), and \( N_P(S) = \{ n_r : r \in G_P(S) \} \), and \( \text{Aux}(S) = \bigcup_{r \in P} \text{head}(T^P_r) \), where \( R = G_P(S) \).

If \( S \) is a preferred answer set of \( P \), then \( A = S \cup N_P(S) \cup \text{Aux}(S) \) is an answer set of \( t(P) \).

If \( A \) is an answer set of \( t(P) \), then \( S = A \cap \text{Lit} \) is a preferred answer set of \( P \), and \( A = S \cup N_P(S) \cup \text{Aux}(S) \).

Properties
Preferred answer sets as defined in Definition 13 enjoy several nice properties.

Proposition 12 Let \( P = (P, <) \) be a logic program with preferences. Then \( \mathcal{PAS}_{GNO}(P) \subseteq \mathcal{AS}(P) \).

Proposition 13 Let \( P = (P, \emptyset) \) be a logic program with preferences. Then \( \mathcal{PAS}_{GNO}(P) = \mathcal{AS}(P) \).

Proposition 14 Preferred answer sets as defined in Definition 13 satisfy Principle 9.

Proposition 15 Let \( P_1 = (P_1, <_1) \) and \( P_2 = (P_2, <_2) \) be logic programs with preferences such that \( <_1 \subseteq <_2 \). Then \( \mathcal{PAS}_{GNO}(P_2) \subseteq \mathcal{PAS}_{GNO}(P_1) \).

The approach two is not equivalent to the answer set semantics for the subclass of stratified programs.

Proposition 16 There is a logic program with preferences \( P = (P, <) \) where \( P \) is stratified and \( \mathcal{PAS}_{GNO}(P) = \emptyset \).

Example 25 shows such a program. Example 20 and 25 illustrate the main difference between the two approaches. While \( \mathcal{PAS}_{G} \) ignores preferences between non-conflicting rules, \( \mathcal{PAS}_{GNO} \) is not always able to do so.

Example 25 Consider again the program from Example 20:
\[
r_1: a \leftarrow \text{not } b \\
r_2: b \leftarrow \\
r_2 < r_1
\]

The program is stratified and has a unique answer set \( S = \{ b \} \). A unique generating set \( R = \{ r_2 \} \) corresponds to the answer set \( S \).

We have that \( T^R_{r_1} = \emptyset \). The rule \( r_2 \) is not included as \( r_2 < r_1 \). Due to a simplicity of the approach, preference \( r_2 < r_1 \) is not ignored. Hence \( \text{head}(T^R_{r_1}) \cup \text{body}^-(r_1) = \emptyset \), and \( r_1 \in P^R \). From that \( \mathcal{Q}(P^R) \neq R \), and \( S \) is not a preferred answer set.

On the other hand the approach stays in the NP complexity class.

Proposition 17 Deciding whether \( \mathcal{PAS}_{GNO}(P) \neq \emptyset \) for a logic program with preferences \( P \) is NP-complete.

Proof: Membership: Using Proposition 11 we can reduce the decision problem \( \mathcal{PAS}_{GNO}(P) \neq \emptyset \) to the problem \( \mathcal{AS}(t(P)) \neq \emptyset \) (in polynomial time), which is in NP.

Hardness: Deciding \( \mathcal{AS}(P) \neq \emptyset \) for a program \( P \) is NP-complete. Using Proposition 13 we can reduce it to the decision \( \mathcal{PAS}_{GNO}((P, \emptyset)) \neq \emptyset \).

Relation between the Approaches of this Paper
It turns out that the approaches of this paper form a hierarchy, which does not collapse.

Notation 4 Let \( A \) and \( B \) be names of semantics.

We write \( A \subseteq B \) iff each preferred answer set according to \( A \) is a preferred answer set according to \( B \).

We write \( A = B \) iff \( A \subseteq B \) and \( B \subseteq A \).

Proposition 18 \( \mathcal{PAS}_{GNO} \subseteq \mathcal{PAS}_{G} \subseteq \mathcal{PAS}_{D} \)

Proposition 19 \( \mathcal{PAS}_{D} \not\subseteq \mathcal{PAS}_{G} \)

Proposition 20 \( \mathcal{PAS}_{G} \not\subseteq \mathcal{PAS}_{GNO} \)

We interpret the results as follows. The semantics \( \mathcal{PAS}_{D} \) is the reference semantics for the case of direct conflicts. The semantics \( \mathcal{PAS}_{GNO} \) and \( \mathcal{PAS}_{G} \) extend the semantics to the case of indirect conflicts. The semantics \( \mathcal{PAS}_{G} \) ignores preferences between non-conflicting rules, e.g., it is equivalent to the answer set semantics for the subclass
of stratified programs (Stratified programs contain no conflicts). If an application domain allows it, we can drop the requirement for ignoring preferences between non-conflicting rules and use the semantics $\mathcal{PAS}_{GNO}$ that stays in the NP complexity class. The semantics $\mathcal{PAS}_{GNO}$ is sound w.r.t. $\mathcal{PAS}_G$ but it is not complete w.r.t. $\mathcal{PAS}_G$. Some preferred answer sets according to $\mathcal{PAS}_G$ are not preferred according to $\mathcal{PAS}_{GNO}$ due to preferences between non-conflicting rules.

### Relation to Existing Approaches

Schaub and Wang [Schaub and Wang 2003] have shown that the approaches [Delgrande, Schaub, and Tompits 2003; Wang, Zhou, and Lin 2000; Brewka and Eiter 1999], referred here as $\mathcal{PAS}_{DST}$, $\mathcal{PAS}_{WZL}$, $\mathcal{PAS}_{BE}$ form a hierarchy.

**Proposition 21** ([Schaub and Wang 2003]) $\mathcal{PAS}_{DST} \subseteq \mathcal{PAS}_{WZL} \subseteq \mathcal{PAS}_{BE}$

We have shown that our approach for direct conflicts continues in this hierarchy (Simko 2013).

**Proposition 22** (Simko 2013) $\mathcal{PAS}_{BE} \subseteq \mathcal{PAS}_G$

The relations $\mathcal{PAS}_{DST} \subseteq \mathcal{PAS}_{GNO}$ and $\mathcal{PAS}_{WZL} \subseteq \mathcal{PAS}_G$ are the only subset relation between our semantics for general conflicts $\mathcal{PAS}_{GNO}$, $\mathcal{PAS}_G$ and $\mathcal{PAS}_{DST}$, $\mathcal{PAS}_{WZL}$ and $\mathcal{PAS}_{BE}$.

**Proposition 23** $\mathcal{PAS}_{DST} \subseteq \mathcal{PAS}_{GNO}$.

**Proposition 24** $\mathcal{PAS}_{WZL} \subseteq \mathcal{PAS}_G$.

**Proposition 25** $\mathcal{PAS}_{GNO} \not\subseteq \mathcal{PAS}_{BE}$.

### Corollary 1

- $\mathcal{PAS}_{GNO} \not\subseteq \mathcal{PAS}_{WZL}$, $\mathcal{PAS}_{GNO} \not\subseteq \mathcal{PAS}_{DST}$,
- $\mathcal{PAS}_G \not\subseteq \mathcal{PAS}_{BE}$, $\mathcal{PAS}_G \not\subseteq \mathcal{PAS}_{WZL}$, $\mathcal{PAS}_G \not\subseteq \mathcal{PAS}_{DST}$.

**Proposition 26** $\mathcal{PAS}_{WZL} \not\subseteq \mathcal{PAS}_{GNO}$

The overall hierarchy of the approaches is depicted in Figure 1.

![Figure 1: The hierarchy of the approaches.](image)

### Example 26

We recall the program:

$$ r_1: \text{nice(car}_1) \leftarrow $$

$$ r_2: \text{safe(car}_2) \leftarrow $$

$$ r_3: \text{rec(car}_1) \leftarrow \text{nice(car}_1), \neg \text{rec(car}_1) $$

$$ r_4: \text{rec(car}_2) \leftarrow \text{nice(car}_2), \neg \text{rec(car}_2) $$

$$ u_1: \neg \text{rec(car}_2) \leftarrow \text{rec(car}_1) $$

$$ u_2: \neg \text{rec(car}_1) \leftarrow \text{rec(car}_2) $$

$$ u_3: \text{rec(car}_1) \leftarrow \text{safe(car}_1), \neg \text{rec(car}_1) $$

$$ u_4: \text{rec(car}_2) \leftarrow \text{safe(car}_2), \neg \text{rec(car}_2) $$

$$ r_i < u_j \text{ for each } i \text{ and } j. $$

The program has two answer sets $S_1 = \{\text{rec(car}_1), \neg \text{rec(car}_2)\} \cup \emptyset$ and $S_2 = \{\text{rec(car}_1), \neg \text{rec(car}_2\} \cup \emptyset$ where $F = \{\text{nice(car}_1), \text{safe(car}_2\}$. As we mentioned in Introduction, $S_2$ is the intended unique preferred answer set.

$\mathcal{PAS}_G$ : We start by listing fragments of the program. We denote by $F_i$ fragments formed by the facts. Let $F_0 = \emptyset$, $F_1 = \{r_1\}$, $F_2 = \{r_2\}$, $F_3 = \{r_1, r_2\}$.

The rules $r_3$ and $u_4$ are conflicting. We denote by $A_i$ fragments containing the rule $r_i$: $A_1 = \{r_1\}$, $A_2 = \{r_1, r_3, u_1\}$, $A_3 = \{r_1, r_2, r_3\}$, $A_4 = \{r_1, r_2, r_3, u_1\}$.

We denote by $B_i$ fragments containing the rule $u_4$. Let $B_1 = \{r_2, u_4\}$, $B_2 = \{r_2, u_4, u_2\}$, $B_3 = \{r_1, r_2, u_4\}$, $B_4 = \{r_1, r_2, u_4, u_2\}$.

A stable fragment set $E_1 = \{F_0, F_1, F_2, F_3, A_1, A_2, A_3, A_4\}$ corresponds to the answer set $S_1$ and a stable fragment set $E_2 = \{F_0, F_1, F_2, F_3, B_1, B_2, B_3, B_4\}$ corresponds to the answer set $S_2$.

We have that $B_3$ overrides both $A_2$ and $A_4$. Hence $B_3 \in \mathcal{P}_{E_1}$, and $\mathcal{P}_{E_1} = \emptyset$. Hence $S_1$ is not a preferred answer set.

On the other hand $E_2 = \mathcal{P}_{E_1}$, and $S_2$ is a preferred answer set.

$\mathcal{PAS}_{GNO}$ : A generating set $R_1 = \{r_1, r_2, r_3, u_1\}$ corresponds to the answer set $S_1$, and $R_2 = \{r_1, r_2, u_4, u_2\}$ corresponds to the answer set $S_2$.

We have that $T_{R_1}^{u_4} = \{u_1\}$. The rules $r_1, r_2, r_3$ are not included as they are less preferred that $u_4$. Hence body$_\neg (u_4) \cap \text{head}(T_{R_1}^{u_4}) = \emptyset$. Therefore $u_4$ cannot be defeated, i.e., $u_4 \in \mathcal{P}_{R_1}$. Hence $R_1 \notin Q(\mathcal{P}_{R_1})$, and the answer set $S_1$ is not a preferred answer set.

On the other hand $R_2 = Q(\mathcal{P}_{R_2})$, and the answer set $S_2$ is a preferred answer set.

### Conclusions

When dealing with preferences it is always important to remember what the abstract term “preferences” represents. In this paper we understand preferences as a mechanism for encoding exceptions. In case of conflicting rules, the preferred
rules define exceptions to less preferred ones, and not the other way around. For this interpretation of preferences, it is important that a semantics for preferred answer sets satisfies Brewka and Eiter’s Principle [2] All the existing approaches for logic programming with preferences on rules that satisfy the principle introduce an imperative feature into the language. Preferences are understood as the order in which the rules of a program are applied.

The goal of this paper was to develop a purely declarative approach to preference handling satisfying Principle [2]. We have developed two approaches \( \mathcal{PAS}_G \) and \( \mathcal{PAS}_{GNO} \). The first one is able to ignore preferences between non-conflicting rules. For example, it is equivalent with the answer set semantics on stratified programs. It is designed for situations, where developer does not have full control over preferences. An example is a situation where a user is able to write his/her own rules in order to override developer’s rules. If the user’s rules are not known until run-time of the system, we have to prefer all the user’s rules over the developer’s rules. To the best of our knowledge, no existing approach for logic programming with preferences satisfying Principle [2] is usable in this situation. On the other hand, in situations where we can drop the requirement for ignoring preferences between non-conflicting rules, e.g. if a developer has full control over the program, we can use \( \mathcal{PAS}_{GNO} \) which is in the \( NP \) complexity class. Naturally, since the requirement for ignoring preferences between non-conflicting rules was dropped, there are stratified programs with answer sets and no preferred answer sets according to \( \mathcal{PAS}_{GNO} \).

The two presented approaches are not independent. They form a hierarchy, a branch in the hierarchy of the approaches \( \mathcal{PAS}_{DST}, \mathcal{PAS}_{WZL}, \mathcal{PAS}_{BE} \) and \( \mathcal{PAS}_D \).

One of our future goals is to better understand the complexity of the decision problem \( \mathcal{PAS}_G(\mathcal{P}) \neq \emptyset \). So far, we have \( \Sigma^p_2 \) membership result. It is not immediately clear whether the problem is also \( \Sigma^p_2 \) hard.

We also plan to investigate relation between \( \mathcal{PAS}_G \) and argumentation, and to implement a prototype solver for the semantics using a meta-interpretation technique of (Eiter et al. 2003).

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References

[Brewka and Eiter 1999] Brewka, G., and Eiter, T. 1999. Preferred Answer Sets for Extended Logic Programs. Artificial Intelligence 109(1-2):297–356.

[Brewka 1996] Brewka, G. 1996. Well-Founded Semantics for Extended Logic Programs with Dynamic Preferences. Journal of Artificial Intelligence Research 4:19–36.

[Delgrande et al. 2004] Delgrande, J. P.; Schaub, T.; Tompits, H.; and Wang, K. 2004. A classification and survey of preference handling approaches in nonmonotonic reasoning. Computational Intelligence 20(2):308–334.

Delgrande, Schaub, and Tompits 2003] Delgrande, J. P.; Schaub, T.; and Tompits, H. 2003. A Framework for Compiling Preferences in Logic Programs. Theoretical Computer Science 3(2):129–187.

[Eiter et al. 2003] Eiter, T.; Faber, W.; Leone, N.; and Pfeifer, G. 2003. Computing Preferred Answer Sets by Meta-Interpretation in Answer Set Programming. Theoretical Computer Science 3(4-5):463–498.

[Gelfond and Lifschitz 1991] Gelfond, M., and Lifschitz, V. 1991. Classical Negation in Logic Programs and Disjunctive Databases. New Generation Computing 9(3-4):365–386.

[Sakama and Inoue 2000] Sakama, C., and Inoue, K. 2000. Prioritized logic programming and its application to commonsense reasoning. Artificial Intelligence 123(1-2):185–222.

[Schaub and Wang 2002] Schaub, T., and Wang, K. 2002. Preferred well-founded semantics for logic programming by alternating fixpoints: Preliminary Report. In 9th International Workshop on Non-Monotonic Reasoning, 238–246.

[Schaub and Wang 2003] Schaub, T., and Wang, K. 2003. A semantic framework for preference handling in answer set programming. Theoretical Computer Science 3(4-5):569–607.

[Šefránek 2008] Šefránek, J. 2008. Preferred answer sets supported by arguments. In Proceedings of Twelfth International Workshop on Non-Monotonic Reasoning.

[Šimko 2013] Šimko, A. 2013. Extension of Gelfond-Lifschitz Reduction for Preferred Answer Sets : Preliminary Report. In Proceedings of 27th Workshop on Logic Programming (WLP2013), 2–16.

[Šimko 2014] Šimko, A. 2014. Proofs for the Approaches to Preferred Answer Sets with General Conflicts. Technical report, Department of Applied Informatics, Comenius University in Bratislava.

[Van Gelder, Ross, and Schlipf 1991] Van Gelder, A.; Ross, K. A.; and Schlipf, J. S. 1991. The Well-founded Semantics for General Logic Programs. Journal of the ACM.

[Wang, Zhou, and Lin 2000] Wang, K.; Zhou, L.; and Lin, F. 2000. Alternating Fixpoint Theory for Logic Programs with Priority. In Proceedings of the First International Conference on Computational Logic, 164–178.

[Zhang and Foo 1997] Zhang, Y., and Foo, N. Y. 1997. Answer Sets for Prioritized Logic Programs. In Proceedings of the 1998 International Logic Programming Symposium, 69–83.