Viscoelastic properties of soft tissues in a living body measured by MR elastography

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Abstract. MRE (Magnetic resonance elastography) is a new diagnostic modality to measure the stiffness of soft tissues in a living body. It measures the displacements of waves propagating in the tissues and then this measured data is linked to the stiffness via a proper model for wave propagation in the tissues and by solving some inverse problem under this model. We will show why we can only see transverse waves inside soft tissues even if we inject longitudinal vibrations into the tissues from their surfaces by modeling the soft tissues of a living body as a nearly incompressible isotropic viscoelastic medium. We interpret the nearly incompressibility by an asymptotic analysis. As a consequence, we show that the so-called modified Stokes system is a proper model. Further, by a modified numerical integral method for solving the inverse problem under this PDE model, we can recover the viscoelasticity of soft tissues.

1. Introduction
MRE (Magnetic Resonance Elastography) is a non-destructive testing (NDT) technology which enables us to measure the displacements of attenuating shear waves propagating in soft tissues of a living body. Principally, a MRE system contains an MRI (Magnetic Resonance Imaging) machine, an external vibration system and data analysis software. By setting up a proper model which describes wave propagation in living tissues and solve some related inverse problem, we can identify the stiffness of tissues. This made MRE very useful for medical diagnostic purposes, e.g., for detection of early stage cancers which have stiffness that are five times more than those of normal tissues and cannot be seen very well by MRI because of the underdevelopment of capillary blood vessels. Another possible application of MRE is in the field of rheology. Existing devices such as rheometers can measure the stiffness and viscosity of high polymer materials only at frequencies up to 100 Hz, whereas MRE can measure those quantities at frequencies higher than 100 Hz. Also, MRE can be used for measurement of material bodies which are of much larger size than that allowed by rheometers. To develop such a kind of “high frequency rheometer” is one of the interests of our MRE research group in Hokkaido University.

The principle of MRE for medical diagnostic purposes was initiated in 1995 by R. Ehman’s team in Mayo Clinic, USA (see [17]). After that, several experimental studies and methods to solve the MRE-related inverse problem have been given (cf. [1,4–6,12–17,20–23,25,27–31]). We also gave a method to identify both the shear modulus and viscosity of a viscoelastic medium from finitely many MRE measurements (see [19]).

As mentioned above we need to have a proper model to set up the inverse problem. Since the soft tissues of a living body are usually considered to be isotropic viscoelastic and
nearly incompressible, in this paper we model the tissues as a isotropic viscoelastic media (cf. [10, 11, 26]), and we interpret the meaning of nearly incompressible by an asymptotic analysis. As a consequence of this analysis, we show that the so-called modified Stokes system approximates very well the nearly incompressible viscoelasticity model in Section 2. Further, to recover the viscoelasticity of soft tissues by MRE-measured data is an inverse problem. To solve this inverse problem, we propose a modified numerical integral method, which is considered as a stable inversion scheme, with some numerical example in Section 3.

2. Asymptotic Analysis for MRE
2.1. Linear Viscoelasticity
The MRE-measured data are two dimensional data over some cross section of a three dimensional medium and the stack of these data in one direction can give three dimensional data. If the data are uniform in some direction, we only need a two dimensional model for MRE data analysis. Otherwise, we need a three dimensional model. So, to handle both cases at the same time, let $n$ ($n = 2, 3$) be the space dimension.

Linear viscoelasticity is the usual mechanical model which describes small deformations in the soft tissues of a living body. More precisely, let

$$ U(t, x) = (U_1(t, x), \ldots, U_n(t, x)) $$

be the displacement vector of the deformation in the soft tissues of a living body at time $t \geq 0$ and point $x$ of a reference domain $\Omega \subset \mathbb{R}^n$, which is a bounded domain with Lipschitz continuous boundary $\partial \Omega$. If there is no external force acting on the tissues, then $U(t, x)$ satisfies the equations

$$ \rho(x) \partial^2_t U_i(t, x) = \sum_{l=1}^{n} \frac{\partial}{\partial x_l} \sigma_{il}(U) \quad (t > 0, x \in \Omega, 1 \leq i \leq n), $$

where $(\sigma_{il})$ is the stress tensor.

![Viscoelasticity models](image)

Figure. 1. Viscoelasticity models.

Usually, there are three kinds of viscoelasticity models (see figure 4). The constitutive equations of those models are

- Voigt model:

  $$ \sigma_{il}(U) = \sum_{k,m=1}^{n} \mu_{iklm} \varepsilon_{km}(U) + \sum_{k,m=1}^{n} \eta_{iklm} \partial_t \varepsilon_{km}(U). $$

- Maxwell model:

  $$ \partial_t \varepsilon_{il}(U) = \sum_{k,m=1}^{n} \mu^{-1}_{iklm} \partial_t \sigma_{km}(U) + \sum_{k,m=1}^{n} \eta^{-1}_{iklm} \sigma_{km}(U). $$
• Zener model:

\[
\sum_{k,m=1}^{n} \mu_{ikm}^{(2)} \sigma_{km}(U) + \sum_{k,m=1}^{n} \eta_{ikm} \partial_t \sigma_{km}(U)
\]

\[
= \sum_{k,m=1}^{n} \mu_{ikm}^{(1)} \mu_{ikm}^{(2)} \epsilon_{km}(U) + \sum_{k,m=1}^{n} (\mu_{ikm}^{(1)} + \mu_{ikm}^{(2)}) \eta_{ikm} \partial_t \epsilon_{km}(U).
\]

where

\[
\varepsilon_{ij}(U) = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \quad (1 \leq i, j \leq n),
\]

\[
0 < \delta < \rho(x) \in L^\infty(\Omega) \quad \text{(usually known as about 1.0} \times 10^3 \text{ kg/m}^3 \text{ and } \delta \text{ is a constant),}
\]

\[
\mu_{ikm}(x), \mu_{ikm}^{(1)}, \mu_{ikm}^{(2)} \in L^\infty(\Omega) \quad \text{and} \quad \eta_{ikm}(x) \in L^\infty(\Omega)
\]

are the strain tensor, density, elasticity tensor and viscosity tensor, respectively. It is physically natural to assume that those viscoelasticity tensors \(0 < \delta < A_{ikm}(\mu_{ikm}^{(1)}, \mu_{ikm}^{(2)}, \eta_{ikm})\) satisfy the full symmetries:

\[
A_{ikm}(x) = A_{ikm}(x) = A_{lmk}(x) = A_{kml}(x)
\]

for any \(i,l,k,m \leq 1, i,l,k,m \leq n\), a.e. \(x \in \Omega\), and the strong convexity: there exists a constant \(C > 0\) such that

\[
\sum_{i,l,k,m=1}^{n} A_{ikm}(x) \zeta_{il} \zeta_{km} \geq C \sum_{i,l=1}^{3} \zeta_{il}^2
\]

for any symmetric matrix \((\zeta_{il})\), a.e. \(x \in \Omega\), \(i,l,k,m \leq 1, i,l,k,m \leq n\).

2.2. Generation of Time Harmonic Motions

As compared with ultrasound CT, MRI is a slow imaging modality and this is why time harmonic waves can only be used in MRE measurements. A vibration system equipped with MRI generates a time harmonic excitation on a part \(\Gamma_D\) of the surface \(\partial \Omega\) of the living body, while the remaining part \(\Gamma_N\) of \(\partial \Omega\) is traction free (cf. [31,32]). Mathematically, let \(\Gamma_D\) and \(\Gamma_N\) be open subsets of \(\partial \Omega\) with Lipschitz continuous boundaries such that \(\partial \Omega = \Gamma_D \cup \Gamma_N\), \(\Gamma_D \neq \emptyset\) and \(\Gamma_D \cap \Gamma_N = \emptyset\).

A general formulation of the boundary condition and initial condition for MRE measurements is as follows:

\[
\begin{align*}
\{ \ U(t, x) &= \chi(t) e^{i \omega t} f(x) \quad \text{on } (0, +\infty) \times \Gamma_D, \\
\partial_{\nu} U(t, x) &= 0 \quad \text{on } (0, +\infty) \times \Gamma_N
\end{align*}
\]

and

\[
U = \partial_t U = 0 \quad \text{on } \{0\} \times \Omega,
\]

respectively. Here, \(\omega\) is a given angular frequency (about 50–500 Hz) of the vibration generated by the vibration system, \(\partial_{\nu} U(t, x)\) is the conormal derivative along the outward unit normal vector \(\nu = (\nu_1, \cdots, \nu_n)\) to \(\partial \Omega\) defined by

\[
\partial_{\nu} U(t, x) := \sum_{i=1}^{n} \sigma_{ii}(U) \nu_i \quad (1 \leq i \leq n),
\]

and \(\chi(t) \in C^\infty([0, \infty))\) is a cut off function such that

\[
\chi(t) = \begin{cases} 
0 & (0 \leq t \leq \frac{1}{2}), \\
1 & (t \geq 1).
\end{cases}
\]
To prove the well-posedness of the initial-boundary-value problem (1) (4), (5), we need to assume that all the coefficients are of class $C^1$ near $\partial \Omega$. More precisely, we make the following assumption: there exists a compact set $F \subset \Omega$ such that $F \cap \partial \Omega = \emptyset$ and those viscoelasticity tensors $A$’s are all of class $C^1$ in $\Omega \setminus F$.

Since the time harmonic vibrations generated by the vibration system take some time to reach the stationary state, we should ascertain how fast the vibrations become stationary. In this section we will show that the solution of the initial-boundary-value problem (1), (4), (5) converges exponentially to a stationary solution. This result is comforting, because it shows that we can quickly have time harmonic vibrations when we vary the angular frequencies of the vibrations so that it is possible to successively conduct MRE measurements with different frequencies.

To begin with we introduce the Sobolev spaces $\dot{H}^s(\Gamma_D’)$ of fractional order $s = 1/2$ or $3/2$ to which the data of the displacement boundary condition $f$ belongs. For an open subset $\Gamma_D’$ of $\Gamma_D$ with a boundary away from $\partial \Gamma_D$ and of class $C^2$, we denote by $\dot{H}^s(\Gamma_D’)$ the set of distributions in the usual fractional Sobolev space $H^s(\Gamma_D’)$ compactly supported in $\Gamma_D’$ (cf. [7]). This can be naturally imbedded into $H^s(\Gamma_D)$. Further, we introduce the spaces

$$V := \{ u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_D \}, \quad V_0 := \{ u \in V : \nabla \cdot u = 0 \text{ in } \Omega \}.$$ 

Associated with these, we define $V^*$ and $V_0^*$ as the dual spaces of $V$ and $V_0$, respectively, via the continuous extension of the $L^2(\Omega)$ inner product. The norms of $V$, $V_0$, $V^*$, $V_0^*$ are denoted by $\| \cdot \|_V$, $\| \cdot \|_{V_0}$, $\| \cdot \|_{V^*}$, $\| \cdot \|_{V_0^*}$, and for some Sobolev space $X$, we define $\| f \|_{L^2((0,t);X)} := \int_0^t \| f \|_X dt$ ($t > 0$).

By applying the argument in [24], we know that the displacement $U(t,x)$ of the wave generated by the vibration system converges exponentially to a stationary state.

**Theorem 2.1.** 1) For any given Dirichlet input $f(x) \in \dot{H}^{3/2}(\Gamma_D)$, there exists a unique solution $U(t,x) \in C^1([0, +\infty); H^1(\Omega)) \cap C^2([0, +\infty); L^2(\Omega))$ to the mixed problem (1), (4) and (5).

Moreover, this solution $U(t,x)$ satisfies

$$\| U(t) \|_{H^1(\Omega)} + \| \partial_t U(t) \|_{L^2(\Omega)} \leq C \| f \|_{\dot{H}^{3/2}(\Gamma_D)},$$

where $C$ is a positive constant independent of $f$.

2) There exist constants $M, \beta > 0$, such that

$$\| U(t,x) - e^{i\omega t} u(x) \|_{H^1(\Omega)} \leq M e^{-\beta t},$$
where \( u(x) := (u_1(x), \ldots, u_n(x)) \in H^1(\Omega) \) is the unique solution to the mixed boundary value problem for the stationary viscoelasticity equation (we call it stationary viscoelasticity model for simplicity):

\[
\begin{aligned}
\sum_{l=1}^{n} \frac{\partial}{\partial x_l} \sigma'_{il}(u) + \rho \omega^2 u &= 0 \quad \text{on } \Omega, \\
u &= f \quad \text{on } \Gamma_D, \quad (1 \leq i \leq n). \\
\partial \nu u := \sum_{l=1}^{n} \sigma'_{il}(u) \nu_l &= 0 \quad \text{on } \Gamma_N
\end{aligned}
\]

Here, the constitutive equations for this stationary case are as follows:

- **Voigt model:**
  \[
  \sigma'_{il}(u) = \sum_{k,m=1}^{n} \mu_{iklm} \varepsilon_{km}(u) + i \omega \sum_{k,m=1}^{n} \eta_{iklm} \varepsilon_{km}(u).
  \]

- **Maxwell model:**
  \[
  \partial \varepsilon_{il}(u) = i \omega \sum_{k,m=1}^{n} \mu_{iklm}^{-1} \sigma'_{km}(u) + \sum_{k,m=1}^{n} \eta_{iklm}^{-1} \sigma'_{km}(u).
  \]

- **Zener model:**
  \[
  \sum_{k,m=1}^{n} \mu_{iklm}^{(2)} \sigma'_{km}(u) + i \omega \sum_{k,m=1}^{n} \eta_{iklm} \sigma'_{km}(u) = \sum_{k,m=1}^{n} \mu_{iklm}^{(1)} \mu_{iklm}^{(2)} \varepsilon_{km}(u) + i \omega \sum_{k,m=1}^{n} (\mu_{iklm}^{(1)} + \mu_{iklm}^{(2)}) \eta_{iklm} \varepsilon_{km}(u).
  \]

**Remark.**

(i) By Theorem 2.1, \( U(t,x) \) converges to \( e^{i \omega t} u(x) \) exponentially in time. Hence, it is reasonable to say that we know \( u(x) \) in \( \overline{\Omega} \) if we know \( U(t,x) \) on \( [0,T] \times \overline{\Omega} \) for some sufficiently large \( T > 0 \).

(ii) The mixed boundary value problem of (6) is well-posed, i.e., for any given \( f \in H^{1/2}(\Gamma_D) \), there exists a unique solution \( u(x) \in H^1(\Gamma_D) \) to (6) (cf. [8]).

### 2.3. Asymptotic Interpretation of Isotropic “Nearly Incompressible”

For simplicity, we assume that the soft tissues of a living body are isotropic, i.e., the viscoelasticity tensors \( \mu_{iklm}, \eta_{iklm} \) are given as

\[
\begin{aligned}
\mu_{iklm} := &\lambda(x) \delta_{il} \delta_{km} + \mu(x) (\delta_{ik} \delta_{lm} + \delta_{im} \delta_{lk}), \\
\eta_{iklm} := &\zeta(x) \delta_{il} \delta_{km} + \eta(x) (\delta_{ik} \delta_{lm} + \delta_{im} \delta_{lk})
\end{aligned}
\]

with the Kronecker delta \( \delta_{ij} \) for any \( x \in \overline{\Omega} \). \( i, l, k, m \) (\( 1 \leq i, l, k, m \leq 3 \)). Here \( \lambda(x) \) and \( \mu(x) \) are the Lamé modulus, while \( \zeta(x) \) and \( \eta(x) \) are the viscosity coefficients. Especially, \( \mu(x) \) and \( \eta(x) \) are called shear modulus and shear viscosity respectively. Physically well known Possion’s ratio \( \nu \) is given by \( \nu = \lambda/2(\lambda + \mu) \). For the isotropic medium, the strong convexity (3) becomes

\[
\mu > \delta, \quad \eta > \delta, \quad 3\lambda + 2\mu > \delta, \quad 3\zeta + 2\eta > \delta \quad \text{a.e. in } \Omega
\]

for some constant \( \delta > 0 \).
Based on the discussions above, the displacement \( u(x) \) of the wave measured by MRE is governed by the stationary viscoelasticity model (6). The Poisson ratio \( \nu \) of soft tissues of living bodies has values close to 0.5. In other words, for soft tissues \( \lambda \) is much larger than \( \mu \). In this case, we call the stationary viscoelastic model nearly incompressible. The typical order of the value for \( \lambda \) is GPa while the order of \( \mu \) is kPa, which implies \( \nu = 0.4999999 < 0.5 \). What does this mean in terms of the wave speed and wave length of the shear wave and compressional wave? Take the Voigt type viscoelasticity medium for example, the speed of shear wave \( v_s \) and that of compressional wave \( v_c \) are given by

\[
v_s = \sqrt{\frac{2(\mu^2 + \omega^2\eta^2)}{\rho(\mu + \sqrt{\mu^2 + \omega^2\eta^2})}}, \quad v_c = \sqrt{\frac{2(\lambda + 2\mu)^2 + \omega^2(\zeta + 2\eta)^2}{\rho(\lambda + 2\mu) + \sqrt{(\lambda + 2\mu)^2 + \omega^2(\zeta + 2\eta)^2}}}
\]

(cf. [3, 10]). Hence, a large \( \lambda \) implies that the wavelength of the compressional wave is much longer than the one of the shear wave. This explains why we cannot observe any compressional wave in MRE measurements visually, but there still remains a question how can we explain it mathematically. The answer is that we can find an approximate model which no longer contains a large \( \lambda \) as a parameter but its solution approximate very well the solution of the nearly incompressible model.

By applying an asymptotic analysis, after taking \( \lambda \gg \mu \) into account, we have the following so-called modified Stokes model:

\[
\begin{cases}
\nabla \cdot (2(G' + iG'')\varepsilon(u)) - \nabla p + \rho \omega^2 u = 0 \quad \text{in } \Omega, \\
\n\nabla \cdot u = 0 \quad \text{in } \Omega, \\
\n\nabla \cdot u = f \quad \text{on } \Gamma_D, \\
\\partial_n u := [2(G' + iG'')\varepsilon(u) - p] \nu = 0 \quad \text{on } \Gamma_N.
\end{cases}
\]

(7)

Here, \( p \) is usually called pressure. The storage modulus and loss modulus are defined as follows:

- Voigt model:
  \[
  G' = \mu, \quad G'' = \omega\eta.
  \]

- Maxwell model:
  \[
  G' = \frac{\mu(\omega\eta)^2}{\mu^2 + (\omega\eta)^2}, \quad G'' = \frac{\mu^2(\omega\eta)}{\mu^2 + (\omega\eta)^2}.
  \]

- Zener model:
  \[
  G' = \mu^{(1)} + \frac{(\mu^{(2)})^2(\omega\eta)^2}{(\mu^{(2)})^2 + (\omega\eta)^2}, \quad G'' = \frac{(\mu^{(2)})^2(\omega\eta)}{(\mu^{(2)})^2 + (\omega\eta)^2}.
  \]

Here, we must remark that the storage modulus and loss modulus are usually the viscoelastic quantities measured by using rheometer.

Further, we can prove the well-posedness of the mixed boundary value problems for modified Stokes systems (7), i.e. for any given \( f \in H^{1/2}(\Gamma^D_D) \), there exists a unique \((u, p) \in H^1(\Omega) \times L^2(\Omega)\) which is the solution to (7) (cf. [8]).

2.4. Numerical Examinations
In what follows we will examine via numerical simulations how well this modified Stokes model could recover our experimental data obtained by using the Micro-MRE system we have now in Hokkaido University, Japan. We used the software FreeFem++ (http://www.freefem.org/ff++/) for the finite element scheme (cf. [2]) to solve the mixed boundary value problem. Since
MRE can measure the displacement of stationary-state wave motion in any specified plane, say in the plane plane \(x_3 = 0\), and our phantom is uniform in the \(x_3\) direction, we used the plain strain assumption to simulate the wave in the plane. Of course there may be a modeling error in assuming this, because the injected vibration may not be uniform in the \(x_3\) direction. More details on the plane strain assumption is as follows.

**Plane Strain Assumption.** Consider a relatively long cylindrical body with the \(x_3\) axis as its central axis. If the body forces and traction on the lateral boundaries are independent of \(x_3\) and have no \(x_3\) component, then the wave displacement field in the body can be put in the reduced form \(u := (u_1(x_1, x_2), u_2(x_1, x_2), 0)\). This deformation is referred to as a state of plane strain in the \(x_1, x_2\)-plane, and thus the three dimensional problem is reduced to a two dimensional formulation in the region \(\Omega\) in the \(x_1, x_2\)-plane. Using the strain-displacement relations, the strains corresponding to this plane problem become \(\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0\).

By the plane strain assumption, we can have the reduced modified Stokes model for \(u(x) = (u_1(x), u_2(x))\) at \(x \in \Omega \subset \mathbb{R}^2\). We will notice from our following simulation that this assumption seems to be appropriate in our case. Moreover, by looking at the simulation shown in figure 3, we conclude that our modified Stokes model is a proper PDE model for our MRE problem.

![Figure 3](image_url)

**Figure. 3.** Numerical simulation (left: experimental data, right: simulated data).

3. **Modified Numerical Integral Method**

In this section, we will present a modified numerical integral method to recover the viscoelasticity \(G'\) and \(G''\) by knowing the interior stationary wave displacement \(u(x)\) in \(\Omega\) which generated by the boundary input \(f\). For creating standard for MRE, a phantom is used as a sample instead of a living body. In this case we can assume that \(G', G''\) are piecewise constants. Our inversion scheme presented here for is only for such a case. For non piecewise constant inhomogeneous \(G', G''\) we developed a least square method in [9].

3.1. **Curl Operator**

To filter the pressure \(p\) in (7) which cannot be measured, we apply the curl operator \(w = \nabla \times u\) and our PDE model becomes as follows:

\[(G' + iG'')\Delta w + \rho(2\pi f)^2 w = 0 \quad \text{in} \ \Omega' \subset \subset \Omega\]
3.2. Mollification
To apply the above curl operator and further inversion procedure to the measured data, we need to denoise the data. In order to have the enough smoothness, we employ the mollifier as follows:
\[(S_\epsilon f)(x) = \int_{\mathbb{R}^2} s_\epsilon(y - x) \hat{f}(y) dy\]
for bounded measurable function \(f\) defined on a bounded domain \(\Omega\), where the bounded measurable function \(\hat{f}\) is an zero extension of \(f\) to a bounded domain \(\Omega_\epsilon\) such that \(\Omega \subset \Omega_\epsilon\), \(0 < \text{dist}(\Omega, \partial \Omega_\epsilon) \leq \epsilon\). Here, \(s_\epsilon(x) = \epsilon^{-2} s((\epsilon^{-1} x) (\epsilon > 0)\). The function \(s\) is a nonnegative \(C^1\) function over \(\mathbb{R}^2\) such that \(\text{supp} s \subset \Omega\) and
\[\int_{\mathbb{R}^2} s_\epsilon(x) dx = 1.\]
Then we have \(S_\epsilon f \in C^3(\overline{\Omega})\) (cf. [18]).

3.3. Modified Numerical Integral Method
Now, we shall go to the real inversion procedure. We only focus on recovering \(G'\), because the major interest here is to diagnose early stage cancer. Hence, we only give the formula for recovering \(G'\) as following:
\[G' = -\rho \omega^2 \text{Re} \left( \frac{\int_D |w|^2 dx}{\int_D w \Delta w dx} \right),\]
where the “testing function” is \(w = \nabla \times S_\epsilon u\) after applying the curl operator to the mollified data \(S_\epsilon u\), \(D \subset \subset \Omega' \subset \subset \Omega\) is the test region whose size is approximately equal to one local wavelength. Because we can prove that \(\text{Re} \int_D w \Delta w dx\) is negative by applying the unique continuation principle, we can say our computation is stable.

3.4. Numerical Result
By applying the above recovery scheme, we can recover the storage modulus \(G'\) of the PAAm gel phantom from the 0.3 T MRE-measured data obtained by the Micro-MRE in Hokkaido University (see figure 3.4).

We must remark that because of the size of test region, we cannot have a good recovery near the original boundary. In the interior region, the mean values of recovered storage modulus \(G'\) in the dashed circles are shown in table 3.4. The real values below are measured by a rheometer (ARES-2KFRT, TA Instruments, frequency: 0.1 ~ 10 Hz, strain mode: 5%).

| Table 1. Mean values of recovered storage modulus \(G'\) |
|----------------------------------|-----------------|-----------------|
| mean value (kPa)                | 31.100          | 10.762          |
| standard deviation              | 0.535           | 0.201           |
| real value (kPa)                | 32.5456         | 9.2472          |
| relative error                  | 0.0444          | 0.1638          |
Figure. 4. Recovery of Storage modulus (above: MRE-measured data, below: storage modulus of PAAm phantom).

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