Generic combinatorial rigidity of periodic frameworks

Justin Malestein\textsuperscript{a}, Louis Theran\textsuperscript{b,\ast}

\textsuperscript{a} Math Department, Hebrew University, Jerusalem, Israel
\textsuperscript{b} Institut für Mathematik, Freie Universität Berlin, 14195 Berlin, Germany

Received 15 June 2010; accepted 16 October 2012
Available online 8 November 2012

Communicated by Gil Kalai

Abstract

We give a combinatorial characterization of generic minimal rigidity for planar periodic frameworks. The characterization is a true analogue of the Maxwell–Laman Theorem from rigidity theory: it is stated in terms of a finite combinatorial object and the conditions are checkable by polynomial time combinatorial algorithms.

To prove our rigidity theorem we introduce and develop periodic direction networks and $\mathbb{Z}^2$-graded-sparse colored graphs.

\copyright\ 2012 Elsevier Inc. All rights reserved.

Keywords: Combinatorial rigidity; Matroids; Periodic graphs

1. Introduction

A periodic framework is an infinite planar structure, periodic with respect to a lattice representing $\mathbb{Z}^2$, made of fixed-length bars connected by joints with full rotational degrees of freedom; the allowed continuous motions are those that preserve the lengths and connectivity of the bars, and the framework’s $\mathbb{Z}^2$-symmetry. A periodic framework is rigid if the only allowed motions are Euclidean isometries, and flexible otherwise.

The forced periodicity is a key feature of this model: there are structures that are rigid with respect to periodicity-preserving motions that are flexible if a larger class of motions is allowed. What is not required to be preserved is also noteworthy: the lattice is allowed to change as the framework moves.

\ast Corresponding author.

E-mail addresses: justinmalestein@math.huji.ac.il (J. Malestein), theran@math.fu-berlin.de (L. Theran).

0001-8708/$ - see front matter \copyright\ 2012 Elsevier Inc. All rights reserved.
doi:10.1016/j.aim.2012.10.007
Formally, a periodic framework is given by a triple $(\tilde{G}, \varphi, \tilde{\ell})$ where: $\tilde{G}$ is a simple infinite graph; $\varphi$ is a free $\mathbb{Z}^2$-action on $\tilde{G}$ by automorphisms such that the quotient is finite; and $\tilde{\ell} = (\tilde{\ell}_{ij})$ assigns a length to each edge of $\tilde{G}$.

A realization $\tilde{G}(p, L)$ of a periodic framework $(\tilde{G}, \varphi, \tilde{\ell})$ is defined to be a mapping $p$ of the vertex set $V(\tilde{G})$ into $\mathbb{R}^2$ and a representation $\mathbb{Z}^2 \rightarrow \mathbb{R}^2$ encoded by a matrix $L \in \mathbb{R}^{2 \times 2}$ (with $\mathbb{R}^2$ here viewed as translations) such that:

- the representation is equivariant with respect to the $\mathbb{Z}^2$-actions on $\tilde{G}$ and the plane; i.e., $p_{\gamma \cdot i} = p_i + L \cdot \gamma$ for all $i \in V(\tilde{G})$ and $\gamma \in \mathbb{Z}^2$.
- The specified edge lengths are preserved by $p$; i.e., $\|p_i - p_j\| = \tilde{\ell}_{ij}$ for all edges $ij \in E(\tilde{G})$.

The reader should note that together these definitions imply that, to be realizable, an abstract periodic framework must give the same length to each $\mathbb{Z}^2$-orbit of edges.

A realization $\tilde{G}(p, L)$ is rigid if the only allowed continuous motions of $p$ and $L$ that preserve the action $\varphi$ and the edge lengths are rigid motions of the plane and flexible otherwise. If $\tilde{G}(p, L)$ is rigid but ceases to be so if any $\mathbb{Z}^2$-orbit of edges in $\tilde{G}$ is removed it is minimally rigid.

These definitions of periodic frameworks and rigidity are from [2]. (See Section 16 for complete details.)

1.1. Main theorem

The topic of this paper is to determine rigidity and flexibility of periodic frameworks based only on the combinatorics of a framework—i.e., which bars are present and not their specific lengths. In general, this is not possible, and even testing rigidity of a finite framework seems to be a hard problem, with the best known algorithms relying on exponential-time Gr"obner basis computations.

However, for generic periodic frameworks, we give the following combinatorial characterization, which is analogous to the landmark Maxwell–Laman Theorem [14,8]. The colored-Laman graphs appearing in statements of theorems are defined in Section 4; the quotient graph is defined in Section 2, and genericity is defined precisely in Section 17 in terms of the coordinates of the points in a realization avoiding a nowhere-dense algebraic set. In particular, this means that the set non-generic realizations has measure zero.

**Theorem A.** Let $(\tilde{G}, \varphi, \tilde{\ell})$ be a generic periodic framework. Then a generic realization $\tilde{G}(p, L)$ of $(\tilde{G}, \varphi, \tilde{\ell})$ is minimally rigid if and only if its colored quotient graph $(G, \gamma)$ is colored-Laman.

**Theorem A** is a true combinatorial characterization of generic periodic rigidity in the plane: $(G, \gamma)$ is a finite combinatorial object and the colored-Laman condition is checkable in polynomial time. The specialization of **Theorem A** to the case where the quotient graph has only one vertex is implied by [2, Theorem 3.12].

1.2. Examples

Because infinite periodic graphs are unwieldy to work with, we will model periodic frameworks by colored graphs, which are finite directed graphs with elements of $\mathbb{Z}^2$ on the edges. These are defined in Section 2, but we show some examples here to give intuition and motivate **Theorem A**. Fig. 1(a) shows part of a periodic point set, (b) makes it more clear that it is indeed periodic by indicating the $\mathbb{Z}^2$ orbits of the points; (c) indicates the vectors representing $\mathbb{Z}^2$ by translations and shows several copies of the fundamental domain of the $\mathbb{Z}^2$-action on
the plane (often called the unit cell). Fig. 2(a) shows an example of a periodic framework, and (b) and (c) illustrate the construction from moving from a periodic framework to a colored graph, from which the entire periodic graph formed by the bars of the framework can be reconstructed. Section 2 describes the construction in detail.

The framework in Fig. 2 turns out to be generically rigid, and, in fact, over-constrained. This is because the colored graph in Fig. 2(c) has \( n = 4 \) vertices and \( m = 10 \) edges. On the other hand, there are 12 total variables defining the framework (8 from the coordinates of the points and 4 from the representation of the lattice) and 3 trivial degrees of freedom (from Euclidean motions of the plane), so there can be only 9 independent distance equations. Since the graph in Fig. 2(c) is somewhat complicated, showing that the framework from Fig. 2(a) is rigid is most easily done via Theorem A. Fig. 2(d) indicates a subgraph corresponding to a minimally rigid framework in gray; this is what we call colored-Laman.

Fig. 3 example shows an example of a very simple one degree of freedom framework (a) and its associated colored graph (c). Two things to note are: its flex (b) necessarily moves the representation of the lattice; in this rigidity model all 4-regular colored graphs are associated with frameworks that, generically, have at least one degree of freedom.

Our final example, in Fig. 4 is a simple illustration of a subtle feature of the periodic rigidity model that the colored-Laman graphs we define capture: dependencies between disconnected sub-frameworks and sub-graphs. The framework in Fig. 4(a) consists of two disconnected orbits of bars, each of which is modeled by a self-loop in the associated colored graph shown in Fig. 4(b). To be realizable at all, the lengths of the bars all need to be exactly the same: in both cases, what is being restricted is just the length of one of the lattice vectors. It follows
Fig. 2. A periodic framework: (a) part of an infinite periodic framework; (b) choosing representatives for the edges; (c) the associated colored graph; (d) a colored-Laman basis of the colored graph in (c).

Fig. 3. A flexible one degree of freedom periodic framework with a motion that changes the lattice representation (a) and (b); its associated colored graph (c).

Fig. 4. A dependent, but disconnected, periodic framework (a) and its associated colored graph (b).
that, generically, this framework is over-constrained, despite both it and its quotient being disconnected. Accounting for the interactions between sub-frameworks that affect the same parts of the lattice representation is a key aspect of the definition of colored-Laman graphs.

1.3. Periodic direction networks

A periodic direction network \((\tilde{G}, \varphi, \tilde{d})\) is an infinite multigraph \(\tilde{G}\) with a free \(\mathbb{Z}^2\)-action \(\varphi\) by automorphisms and an assignment of directions \(\tilde{d} = (\tilde{d}_{ij})_{i,j \in E(\tilde{G})}\) to the edges of \(\tilde{G}\).

A realization \(\tilde{G}(p, L)\) of a periodic direction network is a mapping \(p\) of the vertex set \(V(\tilde{G})\) into \(\mathbb{R}^2\) and a matrix \(L \in \mathbb{R}^{2 \times 2}\) representing \(\mathbb{Z}^2\) by translations of \(\mathbb{R}^2\) such that:

- The representation \(\mathbb{Z}^2 \to \mathbb{R}^2\) from \(L\) is equivariant with respect to the actions on \(\tilde{G}\) and the plane; i.e., \(p_{\gamma \cdot i} = p_i + L \cdot \gamma\) for all \(i \in V(\tilde{G})\) and \(\gamma \in \mathbb{Z}^2\).
- The specified edge directions are preserved by \(p\); i.e., \(p_j - p_i = \alpha_{ij} \tilde{d}_{ij}\) for all edges \(i, j \in E(\tilde{G})\) and some \(\alpha_{ij} \in \mathbb{R}\).

An edge \(ij\) is collapsed in a realization \(\tilde{G}(p)\) if \(p_i = p_j\); a realization in which all edges are collapsed is defined to be a collapsed realization, and a realization in which no edges are collapsed is faithful. We prove an analogue of Whiteley’s Parallel Redrawing Theorem [26,27].

**Theorem B.** Let \((\tilde{G}, \varphi, \tilde{d})\) be a generic periodic direction network. Then \((\tilde{G}, \varphi, \tilde{d})\) has a unique, up to translation and scaling, faithful realization if and only if its quotient graph \((G, \gamma)\) is colored-Laman.

1.4. Roadmap and guide to reading

Let us first briefly sketch how Theorem B implies Theorem A. All known proofs of “Maxwell–Laman-type” theorems (such as Theorem A) proceed via a linearization of the problem called infinitesimal rigidity, which is concerned with the rank of the rigidity matrix \(M_{2,3,2}(G, \gamma)\). The structure of the rigidity matrix depends on the quotient graph \((G, \gamma)\) of the periodic framework. The two main steps are to prove that for a quotient graph with \(n\) vertices and \(m\) edges:

- If \(M_{2,3,2}(G, \gamma)\) has rank \(2n + 1\) then the associated framework is rigid. (This is done in [2].)
- For almost all frameworks (these are called generic) with quotient graph \((G, \gamma)\) having \(n\) vertices and \(2n + 1\) edges, the rank of \(M_{2,3,2}(G, \gamma)\) is \(2n + 1\) if and only if \((G, \gamma)\) is colored-Laman.

The second step, where the rank of the rigidity matrix is proved from only a combinatorial assumption is the (more difficult) “Laman direction”, and it occupies most of this paper. The approach is as follows:

- We begin with a matrix \(M_{2,2,2}(G, \gamma)\), arising from a periodic direction network that has non-zero entries in same positions as the rigidity matrix, but simpler entries (vectors \((a_{ij}, b_{ij})\) instead of differences of points \(p_i - p_j\)). The rank of \(M_{2,2,2}(G, \gamma)\) is much easier to analyze directly.
- Then we apply Theorem B to a colored-Laman graph \((G, \gamma)\). For generic directions \(\mathbf{d}\) (defined in Section 12), there is a point set \(p\) and a lattice \(L\) such that \(p_i - p_j\) is in the direction of \(\mathbf{d}_{ij}\) and the two endpoints of every edge are different. Substituting in the \(p_i\) recovers the rigidity matrix \(M_{2,3,2}(G, \gamma)\) from \(M_{2,2,2}(G, \gamma)\), completing the proof.
The main task, then, is to prove Theorem B. This proceeds in three distinct steps as follows.

**The combinatorial step:** We begin by defining periodic and colored graphs (Section 2). Sections 3–7 define and develop our central combinatorial objects: colored-Laman and \((2, 2, 2)\)-colored-graphs. Our main combinatorial results are that:

- \((2, 2, 2)\)-colored-graphs give the bases of a matroid (Lemma 4.1).
- \((2, 2, 2)\)-colored graphs have an alternate characterization via a sparsity-certifying decomposition into edge-disjoint subgraphs (Lemma 4.2).
- Colored-Laman graphs are related to \((2, 2, 2)\)-colored-graphs by doubling an edge (Lemma 7.3).

**The natural representation step:** In Sections 8–10 we prove that the matroid that has \((2, 2, 2)\)-colored graphs as its bases admits a natural representation: i.e., for a periodic graph \((G, \gamma)\) with \(n\) vertices and \(2n + 2\) edges, there is a matrix \(M_{2,2,2}(G, \gamma)\) that has rank \(2n + 2\) if and only if \((G, \gamma)\) is \((2, 2, 2)\)-colored. The decomposition Lemma 4.2 lets us study \(M_{2,2,2}(G, \gamma)\) in terms of highly structured minors for which it is easier to give exact determinant formulas.

**The geometric step:** Sections 12–15 develop the geometric theory of periodic direction networks and gives the proof of Theorem B. The equations giving the edge directions are closely related to the natural representation matrix \(M_{2,2,2}(G, \gamma)\), which allows us to combinatorially predict the generic rank. To understand the geometry of a generic direction network’s realization space, we make a connection between collapsed edges in a realization and doubling of edges, bringing us back, in a natural way to colored-Laman graphs.

1.5. **Related work: generic periodic rigidity**

The question of generic rigidity and flexibility of periodic frameworks has been studied in the past. The first result of which we are aware is by Walter Whiteley [27], who showed (in our language) that there are colored graphs which lead to infinitesimally rigid periodic frameworks if the representation of the lattice is fixed. Elissa Ross [19,20] proved, using methods quite different from ours, the analogue of Theorem A when the lattice representation is fixed.

1.6. **Related work: parallel redrawing**

Our proof of Theorem B is essentially algebraic, but the geometric correspondence between parallel redrawings of finite frameworks (which are vertex displacements preserving the edge directions) and infinitesimal motions (which preserve the lengths of the bars to first order, see Section 17) has a long history as a folklore tool in engineering. The reason for this is that it is easier to see a non-trivial parallel redrawing than the associated infinitesimal motion. Whiteley developed the subject [26, Section 4] and its generalizations to higher-dimensional scene analysis [26, Section 8], and the idea of deducing the Maxwell–Laman Theorem from a direct proof of the Parallel Redrawing Theorem appears in [27].

1.7. **Related work: \((k, \ell)\)-sparse graphs**

The colored-Laman and colored-\((2, 2, 2)\) graphs we define in Section 4 are an extension of the well-studied families of \((k, \ell)\)-sparse [9] graphs: these are defined by the property that any subgraph spanning \(n'\) vertices and \(m'\) edges satisfies \(m' \leq kn' - \ell\). All known generic rigidity characterizations (e.g., [8,23,22,27]) are in terms of \((k, \ell)\)-sparse graphs with specialized...
parameters for \( k \) and \( \ell \). The combinatorial theory we develop in Section 4 is a generalization of \( (k, \ell) \)-sparsity, and runs in parallel to parts of it:

- Lemma 7.3, which relates colored-Laman graphs to colored-(2, 2, 2) graphs by edge doubling is a colored graph version of theorems of Lovász–Yemini [12] and Recski [17].
- The decomposition Lemma 4.2 is analogous to the Nash-Williams–Tutte Theorem [15,25], and we use it in a similar manner to the way [27,22] use Nash-Williams–Tutte.

1.8. Related work: matroid constructions

Another way to view our \( \mathbb{Z}^2 \)-graded-sparse graphs is in terms of constructions on matroids. To make the connection clear, we give an outline of our proof in these terms:

- We argue directly that \((1, 1, 2)\)-graphs are the bases of a matroid with a specific rank function (Lemma 5.1).
- That \((2, 2, 2)\)-graphs give the bases of a matroid (Lemma 4.1) and the associated decomposition follows from the Nash-Williams matroid construction.
- The colored-Laman matroid can then be obtained via Dilworth Truncation [3, Section 7.7], which, in terms of submodular functions means going from \( f(\cdot) \) to \( f(\cdot) - 1 \) as the function generating the matroid.

For linearly representable matroids, Matroid Union and Dilworth Truncation both have counterparts in terms of the representation. These are:

- Adding a new block to the representing matrix with the same filling pattern but different generic variables, for Matroid Union [3, Proposition 7.16.4].
- Confining the rows of the representing matrix to lie in a generic hyperplane, for Dilworth Truncation (see [10]).

The subtlety, for us, is that we are after a specific representation for the colored-Laman graph matroid, namely the rigidity matrix \( M_{2,3,2}(G) \). This means that we need an argument specific to our geometric setting, and not just general results on Dilworth Truncation.

1.9. Notations

In what follows \((\hat{G},\varphi)\) denotes an (infinite) periodic graph with \( \mathbb{Z}^2 \)-action \( \varphi \). Colored graphs are denoted \((G,\gamma)\). \( G \) always refers to a finite graph with vertex set \( V = V(G) \) and edge set \( E(G) \). For finite graphs, the parameters \( n, m \) and \( c \) refer to the number of vertices, edges, and connected components respectively. Graphs may have multiple edges (and so are multigraphs) as well as self-loops; edges and loops are oriented and denoted \( i j \), where \( i \) is the tail and \( j \) is the head. Since edges and self-loops play the same role in the combinatorial and linear theories appearing here, for economy of language, we do not distinguish between them. Self-loops are treated as edges with the same vertex playing the role of head and tail; self-loops are also treated as cycles with only one vertex. Multiple copies of the same edge are treated as distinguished; where there is a source of confusion in the indexing, we explicitly note it. The coloring \( \gamma = (\gamma_{ij})_{i,j \in E} \) is a vector mapping edges \( ij \) to elements \( \gamma_{ij} = (\gamma_{ij}^1, \gamma_{ij}^2) \) of \( \mathbb{Z}^2 \).

Subgraphs \((G', \gamma)\) of a colored graph \((G, \gamma)\) are taken to be edge-induced, with the induced coloring from \( \gamma \), and have \( n' \) vertices, \( m' \) edges, and \( c' \) connected components.

A matrix \( M \) is denoted in bold; if \( M \) is \( m \times n \), then \( M[A, B] \) denotes the submatrix induced by the row indices \( A \subset [m] \) and column indices \( B \subset [n] \). Vectors \( v \) are denoted in bold. Point sets \( p = (p_i)_{i=1}^n \subset \mathbb{R}^2 \) of \( n \) points in the plane are taken both as indexed sets of points and flattened vectors \( p \in \mathbb{R}^{2n} \); each point \( p_i = (x_i, y_i) \).
2. Colored and periodic graphs

In this section we introduce periodic graphs and colored graphs, which provide the combinatorial setting for this paper. What we call colored graphs are also known as “gain graphs” or “voltage graphs” [28]; the terminology of colored graphs originates with Igor Rivin’s work on hypothetical zeolites [18].

2.1. Periodic graphs

A periodic graph \((\tilde{G}, \varphi)\) is defined to be a simple infinite graph with an associated action \(\varphi : \mathbb{Z}^2 \to \text{Aut}(\tilde{G})\) that is free on edges and vertices and such that the quotient graph \(\tilde{G}/\mathbb{Z}^2\) is finite. We denote the action of \(\gamma \in \mathbb{Z}^2\) on a vertex \(i\) or edge \(ij\) by \(\gamma \cdot i\) and \(\gamma \cdot (ij)\) respectively.

2.2. \(\mathbb{Z}^2\)-colored graphs

We now define \(\mathbb{Z}^2\)-colored graphs (shortly, colored graphs, since all the graphs in this paper have \(\mathbb{Z}^2\)-colors), which are finite objects that capture the essential information in a periodic graph. A \(\mathbb{Z}^2\)-colored graph \((G, \gamma)\) is a directed graph \(G\) together with a vector \(\gamma_{ij} = (\gamma_{i j})\) assigning each edge of \(G\) a group element in \(\mathbb{Z}^2\), which we call the color.

Given a colored graph \((G, \gamma)\), we define its development \((\tilde{G}, \varphi)\) as follows: set

\[
V(\tilde{G}) = \{(i, \gamma) \mid i \in V(G), \gamma \in \mathbb{Z}^2\}.
\]

Similarly, define the edge set as \(E(\tilde{G}) = E(G) \times \mathbb{Z}^2\). Set the tail of \((ij, \gamma)\) to be \((i, \gamma)\) and the head of \((ij, \gamma)\) to be \((j, \gamma_{i j} + \gamma)\). These definitions induce a \(\mathbb{Z}^2\)-action \(\varphi\) via

\[
\gamma' \cdot (i, \gamma) = (i, \gamma' + \gamma) \quad \text{and} \quad \varphi'(ij, \gamma) = (ij, \gamma' + \gamma).
\]

We can also define a colored graph, called the colored quotient, for each periodic graph \((\tilde{G}, \varphi)\). Let the quotient graph \(G = \tilde{G}/\mathbb{Z}^2\) have \(n\) vertices and \(m\) edges. Select one representative from each \(\mathbb{Z}^2\)-orbit of vertices in \(V(\tilde{G})\) (there will be \(n\) of these) to represent \(V(G)\). We define a coloring \(\gamma\) of \(G\) as follows:

- Orient \(G\) in some way.
- Let \(ij \in E(G)\) be a directed edge of \(G\) and let \(\tilde{ij}\) be the unique lift of \(ij\) with tail \(\tilde{i}\) being the chosen representative for \(i \in V(\tilde{G})\).
- The head of \(\tilde{ij}\) is \(\gamma \cdot j\) where \(\tilde{j}\) is the chosen representative of \(j\) in \(V(\tilde{G})\) and \(\gamma\) is uniquely defined by the choices of \(\tilde{i}\) and \(\tilde{j}\). We define \(\gamma\) to be the color \(\gamma_{i j}\) on \(ij \in E(G)\).

If \((\tilde{G}, \varphi)\) is already realized on a periodic point set, a convenient way to pick the representatives is by the points in a fundamental domain of the \(\mathbb{Z}^2\)-action on the plane, as we did in Fig. 2(b). The following lemma shows that the associated colored graph encodes all the data of the periodic graph.

**Lemma 2.1.** The development of every colored graph is a periodic graph, and, in particular, every periodic graph is the development of its colored quotient, for any choice of representatives.

**Proof.** Let \((G, \gamma)\) be a colored graph. It is clear from the definition that the \(\mathbb{Z}^2\)-action \(\varphi\) on the development \(\tilde{G}\) acts freely on the edges and vertices, so \((\tilde{G}, \varphi)\) is periodic. Moreover, by construction, \(G\), as an undirected graph, is the quotient \(\tilde{G}/\mathbb{Z}^2\). To see that \((G, \gamma)\) is the colored quotient of \(\tilde{G}\), let \(G\) retain its original orientation and select as representatives for \(V(G)\) the
vertices \((i, (0, 0)) \in V(\tilde{G})\), and for edges \(ij \in E(G)\) the unique lift of \(ij\) with tail \((i, (0, 0))\) this lift then has head \((j, \gamma_{ij})\), as desired. \(\square\)

In a slightly different language, Lemma 2.1 says that the colors in a colored graph \((G, \gamma)\) encode the covering map to \(G\) from its development \(\tilde{G}\) induced by \(\varphi\).

2.3. The \(\mathbb{Z}^2\)-rank of a colored graph

Different choices of vertex representatives of \(V(G)\) will yield different colorings. The “right” finite graph counterpart to \((\tilde{G}, \varphi)\) is the finite graph \(G = \tilde{G}/\mathbb{Z}^2\) along with the data of a homomorphism \(\rho : H_1(G, \mathbb{Z}) \to \mathbb{Z}^2\), where \(H_1\) is the first homology group. One can show that there is a natural bijective correspondence between pairs \((\tilde{G}, \varphi)\) and pairs \((G, \rho)\). For our purposes, it will be technically simpler to use colored graphs.

A fundamental notion used in our paper is that of the \(\mathbb{Z}^2\)-rank of a colored graph. Let \(C\) be a simple cycle in a colored graph \((G, \gamma)\) and fix a traversal order of \(C\). We define the function \(\rho(C)\) to be:

\[
\rho(C) = \left( \sum_{\gamma_{ij} \text{ traversed from } i \text{ to } j} \gamma_{ij} \right) - \left( \sum_{\gamma_{ij} \text{ traversed from } j \text{ to } i} \gamma_{ij} \right).
\]

With a slight abuse of notation, we denote the induced homomorphism \(H_1(G, \mathbb{Z}) \to \mathbb{Z}^2\) by \(\rho\) as well. The \(\mathbb{Z}^2\)-rank of \((G, \gamma)\) is then defined to be the rank of the subgroup of \(\mathbb{Z}^2\) generated by the image \(\rho(G, \gamma)\) of \(\rho\). Equivalently, this is the number of linearly independent vectors in \(\rho(G, \gamma)\).

Note that the \(\mathbb{Z}^2\)-rank is invariant under the choice of representatives when taking the quotient of a periodic graph to obtain a colored graph.

2.4. Facts about the \(\mathbb{Z}^2\)-rank

The following basic lemmas about the \(\mathbb{Z}^2\)-rank will be useful later.

**Lemma 2.2.** Let \((G, \gamma)\) be a colored graph with \(\mathbb{Z}^2\)-rank \(k\). Then for any choice of cycle basis \(B\) for \(H_1(G, \mathbb{Z})\), there are \(k\) cycles in \(B\) with independent images in \(\mathbb{Z}^2\).

**Proof.** If the map \(\rho\) extends to a well-defined linear map on \(H_1(G, \mathbb{Z})\), then we are done. This follows from the fact that if the cycle \(C_3 = C_1 \Delta C_2\), then if edges in \(C_1 \cap C_3\) are traversed forwards, edges in \(C_2 \cap C_3\) are traversed backwards. Thus \(\rho(C_3) = \rho(C_1) + \rho(C_2)\), since the contributions of edges in \(C_1 \cap C_2\) cancel on the r.h.s. \(\square\)

**Lemma 2.3.** Let \((G, \gamma)\) be a connected colored graph, and let \(ij\) be colored edge not in \(E(G)\). Then the \(\mathbb{Z}^2\)-rank of \((G + ij, \gamma)\) is at most one more than the \(\mathbb{Z}^2\)-rank of \((G, \gamma)\).

**Proof.** Pick a cycle basis for \(H_1(G, \mathbb{Z})\) and then extend it to a basis of \(H_1(G + ij, \mathbb{Z})\). This process adds at most one cycle \(C\), and, thus, the dimension of \(\rho(G + ij, \gamma)\) is at most one more than that of \(\rho(G, \gamma)\). \(\square\)

**Lemma 2.4.** Let \((G, \gamma)\) be a colored graph and let \((G', \gamma)\) be a subgraph of \((G, \gamma)\). Suppose that adding the (colored) edge \(ij\) causes the \(\mathbb{Z}^2\)-rank of \((G, \gamma)\) to increase, and that \(ij\) is spanned by a connected component of \(G'\). Then adding \(ij\) to \(G'\) causes its \(\mathbb{Z}^2\)-rank to increase.

**Proof.** Pick a spanning forest \(F'\) in \(G'\) and extend it to a spanning forest \(F\) of \(G\). The collection of cycles formed by each edge of \(E(G) - F\) and \(F\) gives a basis of \(H_1(G, \mathbb{Z})\). In particular, since
adding the edge $ij$ to $G$ causes an increase in the $\mathbb{Z}^2$-rank. Lemma 2.2 implies that the $\rho$-image of the cycle formed by $ij$ and $F$ is not in the span of $\rho(G, \gamma)$. By hypothesis, this cycle uses only $ij$ and edges from $F'$, and since $\rho(G', \gamma)$ is contained in $\rho(G, \gamma)$, the $\mathbb{Z}^2$-rank of $(G', \gamma)$ increases as well. \hfill \Box

2.5. The $\mathbb{Z}^2$-image and the development

Although none of our proofs explicitly depend on it, we want to refer to the following fact about how the $\mathbb{Z}^2$-image of a colored graph relates to the connectivity of its development. In the interest of space, we skip the proof, which is straightforward, but requires a case analysis.

Lemma 2.5. Let $(G, \gamma)$ be a colored graph with $G$ connected and $\mathbb{Z}^2$-rank $k$. Let $\Gamma(G) < \mathbb{Z}^2$ be the subgroup of $\mathbb{Z}^2$ generated by $\rho(G, \gamma)$, and let $\ell$ be the index of $\Gamma(G)$ in $\mathbb{Z}^2$. Let $(\tilde{G}, \varphi)$ be the development. Then:

- If $k = 2$, then $\ell$ is finite, and the development has $\ell$ infinite connected components.
- If $k = 1$, then $\ell$ is infinite, and the development has infinitely many infinite connected components. These components can be indexed by $\mathbb{Z} \times [\ell']$, where $\ell'$ is the index of a subgroup of $\mathbb{Z}$.
- If $k = 0$, then $\ell$ is infinite and the development has infinitely many finite connected components, which can be indexed by $\mathbb{Z} \times \mathbb{Z}$.

Fig. 4(a) shows an example of the $k = 1$ case of Lemma 2.5.

2.6. Doubling edges

In the sequel, we make use of the operation of doubling an edge $ij$ in colored graphs. Let $(G, \gamma)$ be a colored graph, and let $ij \in E(G)$ be an edge with color $\gamma_{ij}$. Then, the graph with $ij$ doubled, $(G + (ij)_c, \gamma)$ is the colored graph $(G, \gamma)$ with a new edge $(ij)_c$ with the same color $\gamma_{ij}$ as $ij$.

3. Colored-Laman graphs

Let $(G, \gamma)$ be a colored graph. We define $(G, \gamma)$ to be $(2, 3, 2)$-graded-sparse (shortly, colored-Laman-sparse) if for all edge-induced subgraphs $(G', \gamma')$ of $(G, \gamma)$ on $n'$ vertices, $m'$ edges, $c'$ connected components, and $\mathbb{Z}^2$-rank $k'$,

$$m' \leq 2n' - 3 + 2k' - 2(c' - 1).$$

If, in addition, $(G, \gamma)$ has $m = 2n + 1$ edges, we define $(G, \gamma)$ to be a colored-Laman graph.

Because of the key role they play, the rest of this section gives examples of colored graphs that are and are not colored-Laman, with some discussion of the aspects of periodic rigidity each example captures. Section 7 proves the key properties of colored-Laman graphs that we need; these are stated in terms of the related family of $(2, 2, 2)$-graphs, which are developed in Section 4.

3.1. Connection between colored-Laman graphs and Laman graphs

If the rank in $\mathbb{Z}^2$ is zero for all subgraphs, then for a graph to be colored-Laman-sparse, we have

$$m' \leq 2n' - 3 - 2(c' - 1).$$
The Laman graphs that characterize minimal rigidity of finite planar frameworks are defined by the counting condition “\(m' \leq 2n' - 3\)”, but it is not hard to show, using [9, Lemma 4], that minimal violations of (uncolored) Laman-sparsity happen on connected subgraphs, so the two conditions give the same family of graphs. Thus Laman-sparse graphs, with any coloring, are always colored-Laman-sparse.

3.2. The smallest colored-Laman graph

Fig. 5 shows an example of the smallest colored-Laman graphs: a single vertex with three self-loops with colors such that every two colors are linearly independent. Rigidity of this example, and its higher-dimensional generalizations were shown in [2]. Note that if any two of the colors are linearly dependent, the subgraph containing just these loops is not colored-Laman-sparse.

3.3. A larger example

Fig. 6 is a larger example of a colored-Laman graph.

3.4. Colored-Laman graphs need not be Laman-spanning

Fig. 7(c) shows that colored-Laman graphs need not have any spanning subgraph that, after forgetting the colors, corresponds to a minimally rigid generic finite framework.

3.5. Example: checking edge-induced subgraphs is essential

That the colored-Laman counts are defined on subsets of edges and not subsets of vertices is essential: the colored graph in Fig. 8 has \(2n + 1\) edges, and meets the colored-Laman counts on all vertex-induced subgraphs, but it is not colored-Laman. The subgraph in Fig. 8 that fails the colored-Laman condition is the colored graph in Fig. 4(b).
Fig. 7. A colored-Laman graph that does not have a Laman graph as a spanning subgraph.

Fig. 8. A colored graph that is not colored-Laman, but meets the sparsity counts on all vertex induced subgraphs. Edges are shown with their orientation and colors. Sparsity is violated on the disconnected subgraph including the two self-loops with the same color.

Fig. 9. A colored-Laman graph with $\mathbb{Z}^2$-image generating a finite-index subgroup of $\mathbb{Z}^2$ (a) and its development (b) with connected components indicated by color.

3.6. Finite-index $\mathbb{Z}^2$-image

Fig. 9(a) shows a slight variation of the example in Fig. 5: the underlying colored graph is still just three self-loops, but the second coordinates of all the colors have been multiplied by two. Lemma 2.5 implies that developing this colored graph to a periodic graph, as in Fig. 9 (shown just as a periodic graph, since the bars of an associated framework would, necessarily overlap), yields two connected components. This example illustrates that forcing all the motions to preserve periodicity with respect to the $\mathbb{Z}^2$-action means that there are rigid periodic frameworks that are not connected, even though their quotient graph is.

4. (2, 2, 2)-colored graphs

Let $(G, \gamma)$ be a colored graph with $n$ vertices and $m$ edges. We define $(G, \gamma)$ to be $(2, 2, 2)$-$\mathbb{Z}^2$-graded-sparse (shortly, $(2, 2, 2)$-sparse) if for all edge-induced subgraphs $(G', \gamma')$ of $(G, \gamma)$
on \( n' \) vertices, \( m' \) edges, \( c' \) connected components, and \( \mathbb{Z}^2 \)-rank \( k' \),

\[
m' \leq 2n' - 2 + 2k' - 2(c' - 1).\]

If, in addition, \((G, \gamma)\) has \( m = 2n + 2 \) edges, we define \((G, \gamma)\) to be a \((2, 2, 2)\)-colored-graph (shortly, a \((2, 2, 2)\)-graph). More generally, if \((G, \gamma)\) is \((2, 2, 2)\)-sparse, has \( \mathbb{Z}^2 \)-rank \( k \), and has \( m = 2n - 2 + 2k \) edges, we define it to be a \((2, 2, k)\)-colored-graph, or a \((2, 2, k)\)-graph, for short.

4.1. Relationship to \((k, \ell)\)-sparsity

Our definition of colored-Laman-sparsity and \((2, 2, 2)\)-sparsity has two features that distinguish it from the traditional \((k, \ell)\)-sparsity counts:

- The number of edges allowed in a subgraph is controlled by the \( \mathbb{Z}^2 \)-rank.
- The number of edges in a colored-Laman graph on \( n \) vertices is \( 2n + 1 \); similarly, a \((2, 2, 2)\)-graph has \( 2n + 2 \) edges. This is outside of the “matroidal range” [9, Theorem 2] for \((k, \ell)\)-sparse graphs.

The next several sections develop the combinatorial and matroidal properties of \((2, 2, k)\)-graphs that we need to prove Theorem B. We start with some preliminaries from matroid theory.

4.2. Matroid preliminaries

A matroid \( M \) on a ground set \( E \) is a combinatorial structure that captures properties of linear independence. Matroids have many equivalent definitions, which may be found in a monograph such as [16]. For our purposes, the most convenient formulation is in terms of bases: a matroid \( M \) on a finite ground set \( E \) is presented by its bases \( B \subset 2^E \), which satisfy the following properties:

- The set of bases \( B \) is not empty.
- All elements \( B \subset B \) have the same cardinality, which is the rank of \( M \).
- For any two distinct bases \( B_1, B_2 \subset B \), there are elements \( e_1 \in B_1 - B_2 \) and \( e_2 \in B_2 \) such that \( B_2 + \{e_1\} - \{e_2\} \subset B \).

In this paper, the ground set is the colored graph \((K_6^{6,4}, \gamma)\), where \( K_6^{6,4} \) is the complete graph on \( n \) vertices with 6 distinguished copies of each edge and 4 distinguished self-loops on each vertex. The coloring \( \gamma \) can vary, but will always be fixed in the statements of theorems; i.e., we are defining a family of matroids indexed by coloring.

The reason we define the ground set this way is because at most 6 parallel edges or 4 self-loops can appear in any \((2, 2, 2)\)-sparse colored graph. Thus there is no loss of generality in making this restriction.

In addition, we need the following two fundamental results of matroid theory.

**Proposition 1** ([6]). Let \( f \) be a non-negative, increasing, integer-valued, submodular function on the power set of a finite set \( E \). Then the collection of subsets

\[
\mathcal{B}_f = \{ E' \subset E : f(E') = f(E) \text{ and for all } E'' \subset E', f(E'') \leq |E''| \}
\]

gives the bases of a matroid defined to be \( M_f \).
Proposition 2 ([6]). Let \( f \) be a non-negative, increasing, integer-valued, submodular function on the power set of a finite set \( E \). Then the matroid \( M_{2f} \) (in the sense of Proposition 1) has as its bases the collection of subsets

\[
B_{2f} = \{ E' \subset E : E \text{ is the disjoint union of 2 elements of } B_f \}.
\]

4.3. Main combinatorial lemmas

We have two key combinatorial results. The first one shows that the sparsity counts give rise to a matroidal family of graphs. We define the resulting matroids to be the \((2, 2, k)\)-matroids.

Lemma 4.1. Let \( \gamma \) be a coloring of \( K_{6,4n}^6 \) with \( \mathbb{Z}^2 \)-rank \( k \). Then, the family of \((2, 2, k)\)-colored-graphs, if non-empty, forms the bases of a matroid on \((K_{6,4n}^6, \gamma)\).

The second is the equivalent of the Nash-Williams–Tutte Theorem [15,25] for \((2, 2, \cdot)\)-colored graphs. Instead of decompositions into spanning trees, we seek decompositions into \((1, 1, k)\)-graphs, which we now define. (See the figures in Section 5 for some examples of \((1, 1, k)\)-graphs.)

Let \((G, \gamma)\) be a colored graph with \( \mathbb{Z}^2 \)-rank \( k \). We define \((G, \gamma)\) to be a \((1, 1, k)\)-graph if \( G \) is a spanning tree plus \( k \) additional edges. For the purposes of this definition, we allow “empty” spanning trees when there is only one vertex.

Lemma 4.2. A graph \((G, \gamma)\) is a \((2, 2, k)\)-graph if and only if it is the edge-disjoint union of two spanning \((1, 1, k)\)-graphs.

Fig. 10 shows an example of a \((2, 2, 2)\)-graph and a certifying decomposition into two \((1, 1, 2)\)-graphs.

4.4. Proof strategy for Lemmas 4.1 and 4.2

We will study the function

\[
g(G, \gamma) = 2n + 2k - 2c
\]

where \( n, c \) and \( k \) have their usual meaning of the number of vertices, connected components, and the \( \mathbb{Z}^2 \)-rank. To show \( g \) is submodular, we study the function \( f = \frac{1}{2} g \), which turns out to be the
rank function of a matroid. We then infer that \( g \) satisfies the hypotheses of Proposition 1, from which Lemmas 4.1 and 4.2 are immediate. We carry out this strategy in Sections 5 and 6.

Before moving on with the proof, we want to point out a subtlety in the transition from \((2,2,2)\)-graphs to the union of two \((1,1,2)\)-graphs. The definition of \((2,2,2)\)-graphs makes clear that they have \(\mathbb{Z}_2\)-rank 2. However, Lemma 4.2 says more: in fact, any \((2,2,2)\)-graph decomposes into two disjoint, connected, spanning subgraphs that each have \(\mathbb{Z}_2\)-rank 2 in isolation.

5. The \((1,1,k)\)-matroids

Let \((G, \gamma)\) be a colored graph with \(n\) vertices, \(c\) connected components, and \(\mathbb{Z}_2\)-rank \(k\). Recall that we have defined the function

\[
f(G, \gamma) = n + k - c
\]

which, after fixing a coloring of the ground set \((K_{6}^{4}, \gamma)\), is a function from subsets of the edges of \((K_{n}^{6,4})\) to the non-negative integers. We are going to prove the following.

**Lemma 5.1.** Let \(\gamma\) be a coloring of \(K_{n}^{6,4}\), and suppose that \((K_{n}^{6,4}, \gamma)\) has \(\mathbb{Z}_2\)-rank \(k\). Then:

- \(f\) is non-negative, monotone, and submodular (i.e., Proposition 1 applies to it).
- The matroid \(M_f\) from Proposition 1 has \((1,1,k)\)-graphs as its bases.

We start with some examples and simple structural results about \((1,1,k)\)-graphs.

5.1. Structure of \((1,1,k)\)-graphs

It is immediate from the definition of \((1,1,k)\)-graphs and the \(\mathbb{Z}_2\)-rank that:

**Lemma 5.2.** Let \((G, \gamma)\) be a \((1,1,k)\)-graph. Then,

- If \(k = 0\), then \(G\) is a tree.
- If \(k = 1\), then \(G\) is a tree plus one additional edge, and the unique cycle in \(G\) has non-trivial \(\mathbb{Z}_2\)-image.
- If \(k = 2\), then \(G\) is a tree plus two additional edges, and there are two cycles with linearly independent \(\mathbb{Z}_2\)-images.

Fig. 11 shows an example of a \((1,1,1)\)-graph. Fig. 12 shows an example of two types of \((1,1,2)\)-graphs. It is not hard to see that every \((1,1,2)\)-graph contains a subdivision of either a vertex with two-self loops, three copies of a single edge (e.g., Fig. 12(a)), or an edge with a self-loop on each endpoint (e.g., Fig. 12(b)).
5.2. An analogy to spanning trees

If we consider the developments of $(1, 1, k)$-graphs to periodic graphs, we can make an analogy to connectivity in finite graphs. We do not rely on Lemma 5.3, so we simply state it. However, readers who are familiar with the role played by trees in proofs of combinatorial rigidity characterizations may well find it instructive.

Lemma 5.3. Let $(G, \gamma)$ be a $(1, 1, k)$-graph, and let $(\tilde{G}, \varphi)$ be the development of $(G, \gamma)$. Then removing any edge of $G$, disconnects every connected component of $\tilde{G}$. Moreover, the development is connected if and only if $k = 2$ and $\rho(G, \gamma)$ is all of $\mathbb{Z}^2$.

Lemma 5.3 makes the connection between spanning trees (which are bases of the graphic matroid) and $(1, 1, 2)$-graphs (which are bases of a matroid on colored graphs): trees are minimally connected finite graphs and $(1, 1, 2)$-graphs have minimally connected periodic developments if $\rho(G, \gamma)$ generates $\mathbb{Z}^2$.

5.3. Maximizers of $f$

We now begin our study of the function $f$ in more detail, first considering the graphs maximizing the function $f$. These will turn out to be $(1, 1, k)$-graphs for a $\mathbb{Z}^2$-rank $k$ coloring of $K_n^{6,4}$.

Lemma 5.4. Let $\gamma$ be a coloring of $K_n^{6,4}$, and suppose that $(K_n^{6,4}, \gamma)$ has $\mathbb{Z}^2$-rank $k$. The maximum value of $f(G', \gamma)$, over any colored sub-graph $(G', \gamma)$ of $(K_n^{6,4}, \gamma)$ is $n + k - 1$.

Proof. Immediate from the definition.

Lemma 5.5. Let $\gamma$ be a coloring of $K_n^{6,4}$, and suppose that $(K_n^{6,4}, \gamma)$ has $\mathbb{Z}^2$-rank $k$. Suppose that $(G', \gamma)$ is a colored subgraph of $(K_n^{6,4}, \gamma)$ and $f(G', \gamma) = n + k - 1$. Then $G'$ is connected.

Proof. $G'$ spans at most $n$ vertices and has $\mathbb{Z}^2$-rank at most $k$, and thus the contribution of the positive terms to $f(G', \gamma)$ is at most $n + k$. For $f(G', \gamma)$ to be $n + k - 1$, we must then have the number of connected components $c'$ equal to one.

Lemma 5.6. Let $\gamma$ be a coloring of $K_n^{6,4}$, and suppose that $(K_n^{6,4}, \gamma)$ has $\mathbb{Z}^2$-rank $k$. Let $(G', \gamma)$ be any colored sub-graph of $(K_n^{6,4}, \gamma)$ with this coloring and $n - 1 + k$ edges. Then $f(G', \gamma) = n + k - 1$ if and only if $(G', \gamma)$ is a $(1, 1, k)$-graph.

Proof. First, we suppose that $f(G', \gamma) = n + k - 1$. This means that Lemma 5.5 applies, so $G'$ is connected, and thus some $n - 1$ of the edges of $G'$ form a spanning tree $T$ of $G'$. If $k = 0$,
then we are done. If \( k = 1 \), then there is one more edge \( ij \), which creates a unique cycle \( C \) composed of \( ij \) and the path from \( i \) to \( j \) in \( T \). Since \( f(G', \gamma) = n, \rho(C) \neq (0, 0) \), which implies that \((G', \gamma)\) is a \((1,1,1)\)-graph. If \( k = 2 \), then \( G' \) has two additional edges in addition to \( T \), and \( f(G', \gamma) = n + 1 \) implies that there are two cycles with linearly independent \( \mathbb{Z}^2 \)-image in \((G', \gamma)\), so it is a \((1,1,2)\)-graph.

In the other direction, it is immediate from the definitions that if \((G', \gamma)\) is a \((1,1,k)\)-graph, then \( f(G', \gamma) = n + k - 1 \). \( \square \)

5.4. Submodularity of \( f \)

Next we show that \( f \) meets the hypotheses of Proposition 1. We will verify the following form of the submodular inequality: let \((G', \gamma)\) be a colored graph and let \((G'', \gamma)\) be a subgraph of \((G', \gamma)\). To show that \( f \) is submodular, it is enough to prove that for any colored edge \( ij \notin E(G') \):

\[
f(G'' + ij, \gamma) - f(G'', \gamma) \geq f(G' + ij, \gamma) - f(G', \gamma).
\]

Before proving that \( f \) is submodular, we need the following lemma.

**Lemma 5.7.** Let \( \gamma \) be a coloring of \( K_n^{6,4} \). Let \((G', \gamma)\) be any colored sub-graph of \((K_n^{6,4}, \gamma)\), and let \( ij \) be any colored edge of the ground set not in \( E(G') \). Then \( f(G' + ij, \gamma) - f(G', \gamma) \) is either zero or one.

**Proof.** The proof is a case analysis based on how the new edge \( ij \) interacts with the connected components of \( G' \). (We remind the reader at this point that \( ij \) may be a self-loop; i.e., \( i \) may equal \( j \).)

Let \( n', c' \) and \( k' \) be the number of vertices, number of connected components and \( \mathbb{Z}^2 \)-rank of \((G', \gamma)\). Similarly let \( n'', c'', k'' \) be the same quantities for \((G' + ij, \gamma)\).

**Case 1:** \( ij \) is disjoint from all connected components of \( G \) (Fig. 13(a)).

If \( ij \) is disjoint from \( G' \) and \( ij \) is an edge, then \( n'' = n' + 2, c'' = c' + 1 \), and, since adding \( ij \) cannot create any new cycles in \( G', k' = k'' \). Thus we have \( f(G' + ij, \gamma) = n'' + k'' - c'' = n' + 2 + k' - (c' + 1) = f(G', \gamma) + 1 \).

If \( ij \) is a self-loop, then \( n'' = n' + 1 \) and \( c'' = c' + 1 \), so the only possible change can come from \( k'' = k' \). This is either zero or one, depending on whether the color \( \gamma_{ij} \) is in the span of \( \rho(G', \gamma) \).

**Case 2:** \( i \) is in some connected component of \( G' \) and \( j \) is not (Fig. 13(b)).

In this case, \( j \) becomes a leaf (\( ij \) cannot be a self-loop in this case). We have \( n'' = n' + 1, c' = c'' \), and, since no cycle is created, \( k'' = k' \). It follows that \( f(G' + ij, \gamma) = f(G', \gamma) + 1 \).
Fig. 14. Cases in the proof of Lemma 5.7: (a) Case 3; (b) Case 4. The edge $ij$ to be added is indicated by a dashed line.

**Case 3:** $i$ and $j$ are in different connected components of $G'$ (Fig. 14(a)).

In other words, $ij$ is a bridge in $G + ij$ (and so $ij$ cannot be a self-loop). No new cycles are created, so $k' = k''$. Since $c'' = c' - 1$ and $n' = n'' f(G' + ij, \gamma) = f(G', \gamma) + 1$.

**Case 4:** $ij$ is in the span of some connected component of $G$ (Fig. 14(b)).

The number of vertices and connected components is fixed, and the proof follows from Lemma 2.3. (The treatment of self-loops is uniform in this case.) □

**Lemma 5.8.** Let $\gamma$ be a coloring of $K_n^{6,4}$. The function $f$ from subgraphs $(G', \gamma)$ of $(K_n^{6,4}, \gamma)$ is submodular.

**Proof.** We check the submodular inequality. To do this, we let $(G', \gamma)$ be a colored subgraph of the ground set, $(G'', \gamma)$ be a subgraph of $(G', \gamma)$, and $ij$ a colored edge not in $(G', \gamma)$. We need to show that

$$f(G'' + ij, \gamma) - f(G'', \gamma) \geq f(G' + ij, \gamma) - f(G', \gamma).$$

By Lemma 5.7, both sides of the inequality to prove are either zero or one. It follows that when the r.h.s. is zero, we are done. What is left is to assume that the r.h.s. is one, and show that this implies that the l.h.s. is as well.

Since $f(G' + ij, \gamma) - f(G', \gamma) = 1$, we know from the proof of Lemma 5.7 that this was due either to:

- One of Cases 1, 2, or 3.
- Case 4, where the $\mathbb{Z}_2$-rank increased. (This is the only possibility if $ij$ is a self-loop.)

Because $G''$ is a subgraph of $G'$, if the increase $f(G' + ij, \gamma) - f(G', \gamma) = 1$ is due to Cases 1, 2, or 3, then $ij$ is not in the span of any connected component of $G''$ either. This means that adding $ij$ to $G''$ will force one of these cases as well, and the desired inequality follows.

To complete the proof, we suppose that the increase $f(G' + ij, \gamma) - f(G', \gamma) = 1$ came from Case 4. Again, if $ij$ was not in the span of any connected component of $G''$, this forces one of the first three cases analyzed in the proof of Lemma 5.7, and we are done. Otherwise, all the hypotheses of Lemma 2.4 are met, and the lemma follows. □

We remark that we have shown more about $f$ than was strictly necessary to apply Proposition 1. As readers familiar with matroid theory will have noticed, Lemmas 5.7 and 5.8 together imply that $f$ is actually the rank function of a matroid.

**5.5. Proof of Lemma 5.1**

The non-negativity and monotonicity of the function $f$ follow from Lemma 5.7 and submodularity was checked in Lemma 5.8. Thus Proposition 1 applies to $f$, and we have the first statement.
For the second statement, Lemma 5.6 implies that the bases of the resulting matroid \( M_f \) are \((1, 1, k)\)-graphs. □

6. The \((2, 2, k)\)-matroids

We have the tools in place to prove our main combinatorial results on \((2, 2, k)\)-graphs.

6.1. Proof of Lemmas 4.1 and 4.2

Fix a coloring \( \gamma \). On the one hand, since Lemma 5.8 implies that \( f \) meets the conditions of Proposition 1, \( 2f \) does too, so \( M_{2f} \) has as its bases \((2, 2, k)\)-colored graphs if the \( \mathbb{Z}^2 \)-rank of \((K_n^{6,4}, \gamma)\) is \( k \), since the right hand side of the defining sparsity condition is just \( 2f(G', \gamma) \). This proves Lemma 4.1.

On the other hand, Proposition 2 says that the bases of \( M_{2f} \) are exactly the graphs that decompose into two disjoint bases of \( M_f \). By Lemma 5.1, these are the decompositions required by Lemma 4.2. □

7. The colored-Laman matroid

Although it is a corollary of Theorem A, we can give a proof of the following lemma directly.

**Lemma 7.1.** Let \((K_n^{6,4}, \gamma)\) have \( \mathbb{Z}^2 \)-rank 2. Then the family of colored-Laman graphs forms the bases of a matroid.

**Proof.** This is an application of Proposition 1 to the function

\[
h(G', \gamma) = 2n' - 3 + 2k' - 2(c' - 1)
\]

where \( n' , c' , \) and \( k' \) have their usual meanings. Since the function \( h \) is \( 2f(G', \gamma) - 1 \), and \( f \) is submodular and monotone, \( h \) is as well. Non-negativity of \( h \) is not hard to check. □

As discussed in the introduction, Lemma 7.1 amounts to saying that the colored-Laman matroid is the (combinatorial) Dilworth Truncation of the \((2, 2, 2)\)-matroid.

7.1. Circuits in the colored-Laman matroid

If a colored graph \((G, \gamma)\) is not colored-Laman-sparse, then it must have some subgraph \((G', \gamma)\) with \( m' \) edges and \( m' > h(G', \gamma) \geq 2f(G, \gamma) \), where \( g \) is defined above and \( f \) is the function defined in Section 4.

We define a colored graph \((G, \gamma)\) to be a colored-Laman circuit if:

- \( G \) has \( m = 2f(G, \gamma) \) edges.
- Removing any edge \( ij \) from \( G \) results in a colored-Laman-sparse graph \((G - ij, \gamma)\).

We note that \((G - ij, \gamma)\) may not be a colored-Laman graph. For example Fig. 4 shows a colored-Laman circuit, but it has no spanning subgraph that is colored-Laman.

The colored-Laman circuits are the minimal obstructions to colored-Laman-sparsity.

**Lemma 7.2.** Let \((G, \gamma)\) be a colored graph, and suppose that \((G, \gamma)\) is not colored-Laman-sparse. Then \((G, \gamma)\) has as an edge-induced subgraph \((G', \gamma)\) a colored-Laman circuit.
Proof. We begin by extracting a maximal subgraph \((B, \gamma)\) of \((G, \gamma)\) that is colored-Laman-sparse. The matroidal property (Lemma 7.1) implies that these all have the same size, and since \((G, \gamma)\) is not colored-Laman-sparse, \((B, \gamma)\) is not all of \((G, \gamma)\). Let \(ij\) be an edge not in \(B\), and consider the subgraph \((B + ij, \gamma)\).

Since each subgraph of \((B, \gamma)\) gained at most one more edge, \((B + ij, \gamma)\) is colored-(2, 2, 2) sparse and since \((B + ij, \gamma)\) is not colored-Laman-sparse, some subgraph \((C, \gamma)\) of it on \(n'\) vertices, \(m'\) edges, \(c'\) connected components, and \(\mathbb{Z}^2\)-rank \(k'\) must have

\[
m' = 2n' + 2k' - 2c' = 2f(G').
\]

A final appeal to the matroidal property of colored-Laman graphs shows that there is a unique minimal such \((C, \gamma)\), which will be a colored-Laman circuit as desired. \(\square\)

7.2. Characterization of colored-Laman graphs by edge doubling

The following characterization of colored-Laman graphs is very similar in spirit to the Lovász–Yemini [12] and Recski [17] characterizations of Laman graphs, and the proof is similar as well.

Lemma 7.3. Let \((G, \gamma)\) be a colored graph with \(n\) vertices and \(2n + 1\) edges. Then \((G, \gamma)\) is colored-Laman if and only if doubling any edge \(ij\) results in a \((2, 2, 2)\)-colored-graph \((G + (ij)_c, \gamma)\).

Proof. First suppose that \((G, \gamma)\) is colored-Laman. Any subgraph of \(G + (ij)_c\) that is a subgraph of \(G\) already satisfies the sparsity counts. Suppose then that a subgraph \(G'\) of \(G + (ij)_c\) contains \((ij)_c\). If \(G'\) also contains the edge \(ij\), then \(G'\) is \(G' - (ij)_c\) plus one edge (which adds no new vertices or components to \(G' - (ij)_c\)). Since \(G' - (ij)_c\) is a subgraph of \(G\), if \(G' - (ij)_c\) has \(m''\) edges, \(n''\) vertices, \(c''\) components and \(\mathbb{Z}^2\)-rank \(k''\), then

\[
m'' \leq 2n'' - 3 + 2k'' - 2(c'' - 1).
\]

Observe that \(G'\) has \(m' = m'' + 1\) edges, \(n' = n''\) vertices, \(c' = c''\) components and \(\mathbb{Z}^2\)-rank \(k' \geq k''\) and hence \(m' \leq 2n' - 2 + 2k' - 2(c' - 1)\). If \(G'\) does not contain the edge \(ij\), then \(G' - (ij)_c + ij\), a subgraph of \(G\), has the same rank and number of vertices, edges, and components as \(G'\).

On the other hand, if \((G, \gamma)\) is not colored-Laman, then there is some subgraph \((G', \gamma)\) with \(n'\) vertices, \(\mathbb{Z}^2\)-rank \(k'\), and at least \(2n' - 2 + 2k' - 2(c' - 1)\) edges. Then doubling any edge in \(G'\) results in a graph that is not \((2, 2, 2)\)-colored. \(\square\)

8. Natural representations

In Sections 6 and 7, we proved that the \((2, 2, k)\)-colored graphs and colored-Laman graphs on \(n\) vertices each give the bases of a matroid. In matroidal terms, the rigidity Theorem A states that the rigidity matrix (defined in Section 17) for generic periodic bar-joint frameworks represents the colored-Laman matroid: linear independence among the rows of the matrix corresponds bijectively to independence in the associated combinatorial matroid.

The next step in the program set out in the introduction is to give linear representations of the \((2, 2, k)\)-colored matroids which are natural in the sense that the matrices obtained have the same dimensions as the corresponding rigidity matrices and non-zero entries at the same positions.
We now give the detailed definitions and state the main representation results on \((2, 2, k)\)-matroids.

8.1. The generic rank of a matrix

A generic matrix has as its non-zero entries generic variables, or formal polynomials over \(\mathbb{R}\) in generic variables. Its generic rank is given by the largest number \(r\) for which \(M\) has an \(r \times r\) matrix minor with a determinant that is formally non-zero.

Let \(M\) be a generic matrix in \(m\) generic variables \(x_1, \ldots, x_m\), and let \(v = (v_i) \in \mathbb{R}^m\). We define a realization \(M(v)\) of \(M\) to be the matrix obtained by replacing the variable \(x_i\) with the corresponding number \(v_i\). A vector \(v\) is defined to be a generic point if the rank of \(M(v)\) is equal to the generic rank of \(M\); otherwise \(v\) is defined to be a non-generic point.

We will make extensive use of the following well-known facts from algebraic geometry (see, e.g., [5]):

- The rank of a generic matrix \(M\) in \(m\) variables is equal to the maximum over \(v \in \mathbb{R}^m\) of the rank of all realizations \(M(v)\).
- The set of non-generic points of a generic matrix \(M\) is an algebraic subset of \(\mathbb{R}^m\).
- The rank of a generic matrix \(M\) in \(m\) variables is at least as large as the rank of any specific realization \(M(v)\); i.e., generic rank can be established by a single example.

8.2. Generic representations of matroids

Let \(M\) be a matroid on ground set \(E\). We define a generic matrix \(M\) to be a generic representation of \(M\) if:

- There is a bijection between the rows of \(M\) and the ground set \(E\).
- A subset of rows of \(M\) attains the rank of the matrix \(M\) if and only if the corresponding subset of \(E\) is a basis of \(M\).

8.3. The natural representation of the \((2, 2, k)\)-matroids

Let \((G, \gamma)\) be a colored graph with \(n\) vertices and \(m\) edges. We define the matrix \(M_{2,2,2}(G, \gamma)\) to be the \(m \times 2n + 4\) matrix with the filling pattern indicated below:

\[
\begin{pmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \gamma_{ij}^1a_{ij} & \gamma_{ij}^2b_{ij} \\
0 \ldots 0 & -a_{ij} & -b_{ij} & 0 \ldots 0 & a_{ij} & b_{ij} & 0 \ldots 0 & \gamma_{ij}^1a_{ij} & \gamma_{ij}^2b_{ij} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]

The rows of \(M_{2,2,2}(G, \gamma)\) are indexed by the edges \(ij \in E(G)\). The columns are indexed as follows: the first \(2n\) columns are indexed by the vertices \(V(G)\), with two columns for every vertex; the last 4 are associated with the coordinate projections of \(\gamma\) onto \(\mathbb{Z}\), with each getting two columns \(L_1\) and two columns \(L_2\). The entries \(a_{ij}\) and \(b_{ij}\) are generic variables, with different copies of an edge getting distinct variables. Notice that the sign pattern of the matrix encodes the underlying orientation of \(G\).

We now state our representation result for the \((2, 2, k)\)-matroid.

**Lemma 8.1.** Let \((G, \gamma)\) be a colored graph with \(2n - 2 + 2k\) edges and \(\mathbb{Z}^2\)-rank \(k\). Then \((G, \gamma)\) is \((2, 2, k)\)-colored if and only if \(M_{2,2,2}(G, \gamma)\) has generic rank \(2n - 2 + 2k\).
From this we have the immediate corollary.

**Corollary 8.2.** If \((K_n^{6,4}, \gamma)\) has \(\mathbb{Z}^2\)-rank 2, then the matrix \(M_{2,2,2}(K_n^{6,4}, \gamma)\) represents the \((2, 2, 2)\)-colored matroid.

To prove Lemma 8.1, we first establish analogous results for \((1, 1, k)\)-graphs in Section 9. This is done using determinant formulas similar to standard ones for the graphic matroid. The deduction, in Section 10, of Lemma 8.1 from the results of Section 9 and Lemma 4.2 is then a standard argument that is nearly the same as that for the Matroid Union Theorem for representable matroids [3, Proposition 7.6.14] or the specialization to the graphic matroid [27, Theorem 1].

**9. Natural representations of \((1, 1, k)\)-graphs**

Let \((G, \gamma)\) be a colored graph, and define the matrix \(M_{1,1,2}(G, \gamma)\) to have the filling pattern indicated below:

\[
\begin{pmatrix}
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & -a_{ij} & \cdots & a_{ij} & \cdots & 0 \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
\gamma_{ij}^1a_{ij} & \gamma_{ij}^2a_{ij} & \gamma_{ij}^1a_{ij} & \cdots & \cdots & 0 & \cdots \\
\end{pmatrix}
\]

The row and column indexing is similar to that for \(M_{2,2,2}(G, \gamma)\): there are \(m\) rows, one for each edge, and \(n + 2\) columns, one for each vertex and two associated with the coordinate projections of the coloring.

**Lemma 9.1.** Let \((G, \gamma)\) be a colored graph with \(n - 1 + k\) edges. Then \((G, \gamma)\) is \((1, 1, k)\)-colored if and only if \(M_{1,1,2}(G, \gamma)\) has generic rank \(n - 1 + k\).

**Proof.** Lemmas 9.4–9.6 prove the lemma for each rank. \(\square\)

The rest of the section contains the proofs.

**9.1. Natural representation of the graphic matroid**

The sub-matrix induced by the \(n\) columns is standard in matroid theory: it is the usual generic representation of the graphic matroid. We will denote this sub-matrix \(M_{1,1}(G)\) (there is no dependence on the coloring, so the notation suppresses it). The following lemma is standard.

**Lemma 9.2 ([11, Section 4]).** Let \(G\) be a graph with \(n\) vertices and \(n - 1\) edges. Let \(M^*_{1,1}(G)\) be a (necessarily square) matrix minor of \(M_{1,1}(G)\) obtained by dropping any one column. Then:

- \(\det(M^*_{1,1}(G)) = \pm \prod_{ij \in E(G)} a_{ij}\) if \(G\) is a tree.
- \(\det(M^*_{1,1}(G)) = 0\) otherwise.

**9.2. Cycle elimination lemma**

The defining property of \((1, 1, k)\)-graphs is the presence of \(k\) cycles with non-trivial, independent \(\rho\)-images. The following lemma shows how to make the connection explicit.

**Lemma 9.3.** Let \((G, \gamma)\) be a colored graph on \(n\) vertices with a cycle \(C\) and an edge \(i_0j_0\) on \(C\). Then \(M_{1,1,2}(G, \gamma)\) can be put into a form by elementary operations such that:
• The row \( r_{i_0 j_0} \) corresponding to \( i_0 j_0 \) has all zeros in the first \( n \) columns and \( \rho(C) \) in the last two columns.

• All other rows have entries \( \pm 1 \) or 0 in the first \( n \) columns and \( \gamma_{ij} \) in the last two columns.

**Proof.** First scale each row by \( 1/a_{ij} \). It is easy to check (and is a standard result, e.g., [4, Problem 16-3]) that after scaling every row by \( 1/a_{ij} \):

\[
  r_{i_0 j_0} = \sum_{ij \neq i_0 j_0 \in C} r_{ij} + \sum_{ij \neq i_0 j_0 \in C} r_{ij}
\]

equals a row vector with zeros in the first \( n \) columns and \( \rho(C) \) in the last two columns. \( \square \)

The next several lemmas establish natural representation of \((1, 1, k)\)-colored graphs.

### 9.3. Natural representation for \((1, 1, 0)\)-graphs

**Lemma 9.4.** Let \((G, \gamma)\) have \(\mathbb{Z}^2\) rank 0, \(n\) vertices and \(m = n - 1\) edges. Then:

• \( M_{1,1,2}(G, \gamma) \) has generic rank \( n - 1 \) if and only if \((G, \gamma)\) is \((1, 1, 0)\)-colored (i.e., \( G \) is a tree).

• The minor \( M^*_{1,1,2}(G, \gamma) \) obtained by dropping both columns in the \( L \) block and any of the first \( n \) columns indexed by vertices has determinant:

\[
  \det (M^*_{1,1,2}(G, \gamma)) = \pm \prod_{ij \in E} a_{ij}
\]

if \( G \) is a tree and 0 otherwise.

**Proof.** First we suppose that \( G \), the underlying graph is a tree. In this case, dropping in the \( L \) block of \( M_{1,1,2}(G, \gamma) \) and one other column leaves the matrix \( M^*_{1,1,1}(G) \), defined above. **Lemma 9.2** then implies both the desired rank and determinant formula.

Now suppose that \( G \) is not tree. We check that any \((n - 1) \times (n - 1)\) minor of \( M_{1,1,2}(G, \gamma) \) has a determinant that is formally zero. Since \( G \) is not a tree, it must contain a cycle \( C \). After applying **Lemma 9.3** to \((G, \gamma)\) we obtain a matrix with a row of all zeros, since \( \rho(C) = (0, 0) \) by hypothesis. Thus \( M_{1,1,2}(G, \gamma) \) is rank deficient as desired. \( \square \)

### 9.4. Natural representation for \((1, 1, 1)\)-graphs

**Lemma 9.5.** Let \((G, \gamma)\) have \(\mathbb{Z}^2\) rank 1, \(n\) vertices and \(m = n\) edges. Then:

• \( M_{1,1,2}(G, \gamma) \) has generic rank \( n \) if and only if \((G, \gamma)\) is a \((1, 1, 1)\)-graph.

• The minor \( M^*_{1,1,2}(G, \gamma) \) obtained by dropping some column in the \( L \) block and any of the first \( n \) columns indexed by vertices has determinant:

\[
  \det (M^*_{1,1,2}(G, \gamma)) = \pm t_q \prod_{ij \in E} a_{ij}
\]

if \( G \) is \((1, 1, 1)\)-colored and \( C \) is its unique cycle with \( \rho(C) = (t_1, t_2) \) \((q \in \{1, 2\})\), and 0 otherwise.

**Proof.** Suppose that \((G, \gamma)\) is a \((1,1,1)\)-graph with cycle \( C \) spanning an edge \( i_0 j_0 \). Applying **Lemma 9.3** gives \( M_{1,1,2}(G, \gamma) \) a form where: \( r_{i_0 j_0} \) has zeros in the first \( n \) columns and \( \rho(C) \) in the \( L \) columns; and all other entries in the first \( n \) columns are zero or \( \pm 1 \). Since \( \rho(C) = (t_1, t_2) \neq (0, 0) \) assume, w.l.o.g., that \( \rho_1 \neq 0 \). Consider the minor formed by dropping any of the first \( n \) columns and the second column from the block \( L \).
The determinant of this minor (by expanding along the remaining column from \( L \)) is 
\[ \rho_1 \cdot \det(\mathbf{A}), \] where \( \mathbf{A} \), the complementary cofactor, is \( M^*_{1,1}(G - i_0j_0) \). Because \( (G, \gamma) \) is a 
\((1, 1, 1)\)-graph, Lemma 5.2 implies that \( G - i_0j_0 \) is a tree, and so \( \det(\mathbf{A}) = \pm 1 \) by Lemma 9.2.

The desired determinant formula then follows from noting that the effect on the determinant
by the scaling in Lemma 9.3 is to multiply it by \( \prod_{ij \in E(G)} a_{ij} \).

If \( (G, \gamma) \) is not \((1,1,1)\)-colored, it has more than one cycle. Let \( C_1 \) and \( C_2 \) be two such cycles with 
\( \rho(C_r) = (t_r^1, t_r^2) \) \((r = 1, 2)\). Since there are \( ij \in C_1 \) with \( ij \not\in C_2 \) and \( i'j' \in C_2 \) with
\( i'j' \not\in C_1 \), we can apply Lemma 9.3 two times to put it in a form where the \( ij \) row has all zeros
in the first \( n \) columns and \( \rho(C_1) \) in the last two and the \( i'j' \) row has zeros in the first
\( n \) columns and \( \rho(C_2) \) in the last two.

There are two types of \( n \times n \) minors to consider:
- Minors with both columns from \( L \) have determinant \( \pm(t_1^2 - t_1^1) \), but this is zero, since the
\( \mathbb{Z}^2 \)-rank 1 hypothesis implies that \( \rho(C_1) \) and \( \rho(C_2) \) are linearly dependent.
- Minors with at most one column from \( L \). The determinant is zero, since either the minor has
a row of all zeros or the cofactor in the expansion on the remaining column of \( L \) always
does. \( \square \)

9.5. Natural representation for \((1, 1, 2)\)-graphs

**Lemma 9.6.** Let \((G, \gamma)\) have \( \mathbb{Z}^2 \)-rank 2, \( n \) vertices and \( m = n + 1 \) edges. Then:
- \( M^*_{1,1,2}(G, \gamma) \) has generic rank \( n \) if and only if \((G, \gamma)\) is a \((1, 1, 2)\)-graph.
- The minor \( M^*_{1,1,2}(G, \gamma) \) obtained by dropping any of the first \( n \) columns indexed by vertices
has determinant:
\[
\det (M^*_{1,1,2}(G, \gamma)) = \pm(t_1^2 - t_1^1) \prod_{ij \in E} a_{ij}
\]
if \( G \) is \((1, 1, 1)\)-colored, with cycles \( C_1 \) and \( C_2 \), \( \rho(C_q) = (t_q^1, t_q^2) \) \((q \in \{1, 2\})\) and 0
otherwise.

**Proof.** If \((G, \gamma)\) is \((1, 1, 2)\)-colored then it has two cycles \( C_1 \) and \( C_2 \), with linearly independent
\( \rho \) images. The structural Lemma 5.2 for \((1, 1, 2)\)-graphs implies that we can find edge \( i_1j_1 \) and
\( i_2j_2 \) on cycle \( C_1 \) but not \( C_2 \) and \( C_2 \) but not \( C_1 \), respectively.

Adopting the arguments and notation from the proof of Lemma 9.5, we get the desired
Determinant formula, since the \( \mathbb{Z}^2 \)-rank 2 hypothesis implies that \( (t_1^2 - t_1^1) \neq 0 \).

On the other hand, if \((G, \gamma)\) fails to be \((1, 1, 2)\)-colored, then we can iteratively apply
Lemma 9.3 three times to put it \( M_{1,1,2}(G, \gamma) \) in a form where there are three rows with all
zeros in the first \( n \) columns, which shows it to be rank deficient. \( \square \)

The proof of Lemma 9.1 is now complete.

9.6. Maximum rank lemma

We conclude with a small result about the maximum rank of \( M_{1,1,2}(G, \gamma) \).

**Lemma 9.7.** Let \((G, \gamma)\) have \( \mathbb{Z}^2 \)-rank \( k \) and \( m > n - 1 + k \) edges. Then \( M_{1,1,2}(G, \gamma) \) has a row
dependency.

**Proof.** If \( m > n + 2 \) this follows from the shape. The other case is where \( m \leq n + 2 \). The edge
counts imply that Lemma 9.3 can be applied \( k + 1 \) times to leave \( k + 1 \) rows with all zeros in the
first \( n \) columns. Using the same arguments as above, the determinant of any \( m \times m \) submatrix is
zero. \( \square \)
10. Natural representations of \((2, 2, k)\)-graphs

The main step in the proof of \textbf{Lemma 8.1} is to determine the rank of \(M_{2,2,2}(G, \gamma)\) when \((G, \gamma)\) decomposes into two \((1, 1, 2)\)-graphs.

\textbf{Lemma 10.1.} Let \((G, \gamma)\) be a colored graph with \(2n - 2 + 2k\) edges and \(\mathbb{Z}^2\)-rank \(k\). Then \((G, \gamma)\) is the edge-disjoint union of two \((1, 1, k)\)-graphs if and only if \(M_{2,2,2}(G, \gamma)\) has generic rank \(2n - 2 + 2k\).

The proof is quite similar to that from \cite[Proposition 7.16.4]{3} for Matroid Union for linearly-representable matroids.

\textbf{Proof.} Let \((G, \gamma)\) be a \(\mathbb{Z}^2\)-rank \(k\) colored graph with \(m = 2n - 2 + 2k\) edges. Let \(M^*\) be any \(m \times m\) submatrix of \(M_{2,2,2}(G, \gamma)\), and let \(A\) be the set of columns of \(M^*\) with \(a_{ij}\) and \(B\) be the set of columns with \(b_{ij}\). We compute the determinant using the Laplace expansion:

\[
\det (M^*) = \sum_{\substack{X \subseteq [m] \\ |X| = |A|}} \pm \det (M^*[X, A]) \cdot \det (M^*[m - X, B]).
\]

The key observation is that each of the sub-determinants in the sum has the form of a minor of \(M_{1,1,2}(G', \gamma)\) for an edge-induced subgraph of \((G, \gamma)\), and the sub-determinants correspond to disjoint subgraphs. First note that, \textbf{Lemmas 9.1 and 9.7} imply that unless \(M^*\) was obtained by dropping: one column from the first \(n\) in each of \(A\) and \(B\) and \(2 - k\) columns from \(L_1\) and \(L_2\) in each \(A\) and \(B\); at least one of the sub-determinants is zero in every term.

Now consider an \(M^*\) of the form described. By \textbf{Lemma 9.1}, unless both \(X\) and \([m] - X\) correspond to \((1, 1, k)\)-colored subgraphs of \((G, \gamma)\), every term in the determinant expansion has a zero factor, so the whole determinant is zero. On the other hand, if there is such a decomposition, then the whole determinant cannot cancel, since combinatorially different decompositions give rise to combinatorially different monomials in the \(a_{ij}\) and \(b_{ij}\). \(\Box\)

10.1. Proof of \textbf{Lemma 8.1}

The lemma is immediate from \textbf{Lemma 10.1} and the \((2, 2, k)\)-graph decomposition \textbf{Lemma 4.2}. \(\Box\)

11. Periodic rigidity on the line

As a warm up result, we will give a combinatorial characterization of periodic rigidity on the Euclidean line \(\mathbb{R}\). The definitions of frameworks and their associated colored quotient graphs are specializations of those for the planar case:

- An abstract 1d-periodic framework \((\tilde{G}, \varphi, \tilde{\ell})\) is given by an infinite graph with a free \(\mathbb{Z}\)-action that has finite quotient, and an assignment \(\tilde{\ell} = (\tilde{\ell}_{ij})_{i,j \in E(\tilde{G})}\) of edge-lengths that respects the action \(\varphi\).
- A realization \(\tilde{G}(p, L)\) of the abstract framework is a mapping of \(V(\tilde{G})\) onto a periodic point set \(p = (x_i)_{i \in V(\tilde{G})}\) such that the edge lengths are respected.

The relationship between a 1d-periodic framework and the associated quotient graph (which will have colors in \(\mathbb{Z}\)) is also similar to the planar case. \textbf{Fig. 15} shows an example.

Rigidity and flexibility are also defined (on realizations) in a similar way: a framework is rigid if the only allowed continuous motions are translations of the line and otherwise flexible.
In this section, we will show the following.

**Theorem C.** Let $(\tilde{G}, \varphi, \tilde{\ell})$ be a generic 1d-periodic framework. Then a generic realization $\tilde{G}(p, L)$ of $(\tilde{G}, \varphi, \tilde{\ell})$ is minimally rigid if and only if its quotient graph $(G, \gamma)$ is a $(1, 1, 1)$-graph.

The analogous result for finite frameworks on the line is that a framework is minimally rigid if and only if the graph formed by the bars is a tree (see, e.g., [7, Section 2.5]). We will give two arguments. The first is geometric and does not generalize to the plane. The second uses (as we need to in the plane) infinitesimal rigidity and relies on the natural representations of $(1, 1, k)$-graphs from Section 9. As is the case for finite frameworks, periodic direction networks are trivial objects in dimension one, so we do not develop them.

Because this is a “warmup” to indicate intuition, we will elide some details in the interest of brevity. Readers who are familiar with rigidity theory may wish to skip to Section 12.

11.1. Geometric proof

Let $(\tilde{G}, \varphi, \tilde{\ell})$ be an abstract 1d-periodic framework. In principle, to specify a realization, we have to specify infinitely many points $x_i$: one for each vertex of $\tilde{G}$. However, the assumption that the $\mathbb{Z}$-action $\varphi$ has finite quotient means that there is really only finite information present. In particular, once we know:

- the location of one point in each $\mathbb{Z}$-orbit of vertices;
- the real number $L$ representing $\mathbb{Z}$ by translations;

we can reconstruct the entire realization.

It is not hard to see that the continuity of the distance function implies that any connected component of $\tilde{G}$ is rigid ([7, Section 2.5] contains the details). Thus any connected component containing two vertices in the same $\mathbb{Z}$-orbit fixes $L$. Such a connected component is necessarily infinite, since $\tilde{G}$ is periodic. Lemma 2.5 then implies that the colored quotient must contain a cycle with $\mathbb{Z}$-rank one. (Fig. 16 shows what happens when this fails.) We also observe that if no infinite connected component of $\tilde{G}$ hits a vertex in each $\mathbb{Z}$-orbit, then there are two orbits that can move independently of each other, leading to a flexible framework. Thus any rigid periodic framework on the line must contain an infinite connected component that hits every $\mathbb{Z}$-orbit of vertices. Sufficiency of the same condition is clear, so we have shown that a periodic framework on the line is minimally rigid if and only if its colored quotient is a $(1, 1, 1)$-graph. □
11.2. Proof via natural representations

We now give a second proof of Theorem C that follows the general approach we use to prove Theorem A.

The continuous theory: Rigidity and flexibility are determined by the solution space to the infinite set of length equations:

- $|x_j - x_i| = \tilde{\ell}_{ij}$, for all edges $ij \in \tilde{G}$
- $x_{\gamma,i} = x_i + \gamma \cdot L$, for all $i \in V(\tilde{G})$ and $\gamma \in \mathbb{Z}$

where the unknowns are the points $x_i$ and the lattice representation $L$.

However, as noted above, since there is only finite information, we can identify this space with the more tractable:

$$(x_j + \gamma_{ij} \cdot L - x_i)^2 = \ell_{ij}^2$$

for all colored edges $ij$ of the quotient graph $(G, \gamma)$.

We define the set of solutions to these equations to be the realization space $\mathcal{R}(G, \gamma)$ of the colored framework $(G, \gamma, \ell)$. We note that since $\tilde{\ell}$ had to assign the same length to each $\mathbb{Z}$-orbit of edges, the colored framework is well-defined. The configuration space $\mathcal{C}(G, \gamma)$ of the colored framework is then defined to be the quotient $\mathcal{R}(G, \gamma)/\text{Euc}(1)$ of the realization space by isometries of the line.

This formalism allows us to define rigidity: a realization of a 1d-periodic framework is rigid when it is isolated in the configuration space.

The infinitesimal theory: The rigidity question, then, turns out to be one about the dimension of the configuration space near a realization. In the most general setting, this is a difficult question, but at a smooth point, an adaptation of the arguments of Asimow and Roth [1] show that a realization is rigid if and only if the tangent space of the realization space is one-dimensional.

Taking the formal differential of the equations defining the realization space and dividing by two, we obtain the system

\begin{align*}
(x_j + \gamma_{ij} \cdot L - x_i)^2 &= \ell_{ij}^2 \\
\end{align*}
where \( \eta_{ij} = x_j + \gamma_{ij} L - x_i \), which we define to be the 1d-rigidity matrix. The kernel of this 1d-rigidity matrix is identified with the tangent space \( T_p L(\mathcal{R}(G, \gamma)) \) of the realization space at the point \((p, L)\).

**Genericity and the combinatorial theory:** Provided that the \( \eta_{ij} \) are all non-zero, this matrix is just \( M_{1,1,1}(G, \gamma) \) with the last column discarded. The condition for any of the \( \eta_{ij} \) being zero is a measure-zero algebraic subset of \( \mathbb{R}^{n+1} \), which we define to be the non-generic set of realizations. If the \( x_i \) and \( L \) avoid the non-generic set, then Lemma 9.5 implies that the 1d-rigidity matrix has corank one if and only if \((G, \gamma)\) contains a spanning \((1, 1, 1)\)-graph, completing the proof. \( \square \)

12. Periodic and colored direction networks

We recall the definitions from the introduction. A periodic direction network \((\tilde{G}, \varphi, \tilde{d})\) is an infinite multigraph \( \tilde{G} \) with a free \( \mathbb{Z}^2 \)-action \( \varphi \) by automorphisms and an assignment of directions \( \tilde{d} = (d_{ij})_{ij \in E(\tilde{G})} \) to the edges of \( \tilde{G} \).

A realization \( \tilde{G}(p, L) \) of a periodic direction network is a mapping \( p \) of the vertex set \( V(\tilde{G}) \) into \( \mathbb{R}^2 \) and a matrix \( L \in \mathbb{R}^{2 \times 2} \) representing \( \mathbb{Z}^2 \) by translations of \( \mathbb{R}^2 \) such that:

- The representation \( \mathbb{Z}^2 \to \mathbb{R}^2 \) from \( L \) is equivariant with respect to the actions on \( \tilde{G} \) and the plane; i.e., \( p_{\varphi \cdot i} = p_i + L \cdot \gamma \) for all \( i \in V(\tilde{G}) \) and \( \gamma \in \mathbb{Z}^2 \).
- The specified edge directions are preserved by \( p \); i.e., \( p_j - p_i = \alpha_{ij} \tilde{d}_{ij} \) for all edges \( ij \in E(\tilde{G}) \) and some \( \alpha_{ij} \in \mathbb{R} \).

An edge \( ij \) is collapsed in a realization \( \tilde{G}(p) \) if \( p_i = p_j \); a realization in which all edges are collapsed is defined to be a collapsed realization, and a realization in which no edges are collapsed is faithful. Our main result on periodic direction networks is the following.

**Theorem B.** Let \((\tilde{G}, \varphi, \tilde{d})\) be a generic periodic direction network. Then \((\tilde{G}, \varphi, \tilde{d})\) has a unique, up to translation and scaling, faithful realization if and only if its quotient graph \((G, \gamma)\) is colored-Laman.

In the next two sections we develop the tools we need, and then give the proof in Section 15.

12.1. Colored direction networks

To study realizations of periodic direction networks we first reduce the problem to a finite linear system. A colored direction network \((G, \gamma, d)\) is defined to be a colored graph along with an assignment of directions to the edges. A realization \( G(p, L) \) of the colored direction network \((G, \gamma, d)\) is a mapping \( p = (p_i)_{i \in V(G)} \) into \( \mathbb{R}^2 \) such that:

\[
p_j + L\gamma_{ij} - p_i = \alpha_{ij} d_{ij}
\]

for some real number \( \alpha_{ij} \).

An edge \( ij \in E(G) \) is collapsed in a realization \( G(p, L) \) if \( p_i = p_j + L\gamma_{ij} \); a realization with no collapsed edges is defined to be faithful. A realization is collapsed if all edges are collapsed.

The problems of periodic direction network realization and colored direction network realization are equivalent.
Lemma 12.1. Let $(\tilde{G}, \varphi, d)$ be a periodic direction network. Then the realizations of $(\tilde{G}, \varphi, d)$ are in bijective correspondence with the realizations of the colored direction network $(G, \gamma, d)$ on the quotient graph $(G, \gamma)$. Furthermore, a realization of $(\tilde{G}, \varphi, d)$ is collapsed if and only if the corresponding realization of $(G, \gamma, d)$ is.

Proof. The proof is very similar to that of Lemma 2.1. Any realization $G(p, L)$ of $(G, \gamma, d)$ can be extended to a $\tilde{G}(p, L)$ realization of $(\tilde{G}, \varphi, d)$ via the $\mathbb{Z}^2$-action induced by $\gamma$; in the other direction, a periodic realization $\tilde{G}(p, L)$ induces a colored realization of $(G, \gamma, d)$ via the vertex representatives in $\tilde{G}$ of the vertices of the quotient graph $G$. □

We define the colored direction network realization system $P(G, \gamma, d)$ to be given by:

$$\left\{ p_j + L \gamma_{ij} - p_i, d_{ij} \right\} = 0 \quad \text{for all edges } i \in E(G).$$

The unknowns are the points $p_i$ and the matrix $L$; the given data are the edge directions $d_{ij}$.

13. Properties of colored direction networks

We develop the properties of the system $P(G, \gamma, d)$ that we will need.

13.1. Collapsed realizations of colored-Laman graphs

Collapsed realizations of colored direction networks on colored-Laman graphs have a simple form: they force the lattice representation to be trivial and put all the points on top of each other.

Lemma 13.1. Let $(G, \gamma)$ be colored-Laman. Then, a realization $G(p, L)$ of $(G, \gamma, d)$ is collapsed if and only if $L = \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right)$ and $p_i = p_j$ for all $i, j \in V(G)$.

Proof. Summing the relations $p_j + L \gamma_{ij} - p_i = 0$ over a cycle $C$ yields the equation $L \rho(C) = 0$. Since there are two cycles with linearly independent $\rho(C)$, this implies that $L = \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right)$. The fact that $p_i = p_j$ for all $i, j \in V(G)$ then follows from the connectedness of colored-Laman graphs (which follows, for instance, by Lemmas 4.2 and 7.3). The converse is clear. □

13.2. Translation invariance

This lemma formalizes the geometric observation that translating any realization of a colored direction network results in another realization.

Lemma 13.2. The set of solutions $(p, L)$ to $P(G, \gamma, d)$ is invariant under translation of the points $p_i$ and scaling of $(p, L)$.

Proof. Let $t$ be a vector in $\mathbb{R}^2$ and $\lambda$ a scalar in $\mathbb{R}$. Then

$$\left( \lambda p_j + t \right) + \lambda L \gamma_{ij} - (\lambda p_i - t), d_{ij} = \lambda \left( p_j + L \gamma_{ij} - p_i, d_{ij} \right).$$

□

13.3. Relationship to the $(2, 2, 2)$-matroid

Our main tool for moving back and forth between (geometric) colored direction networks and (combinatorial) colored graphs is that the system $P(G, \gamma, d)$ is closely related to the generic representation of the $(2, 2, 2)$-matroid.
Lemma 13.3. The solutions $p, L$ to the system $P(G, \gamma, d)$ are the $(p, L)$ satisfying
$$M_{2,2,2}(G, \gamma)(p, L)^T = 0.$$

Proof. Using the bilinearity of the inner product we get
$$\langle p_j + L\gamma_{ij} - p_i, d_{ij}^\perp \rangle = \langle p_j - p_i, d_{ij}^\perp \rangle + \langle L_1, \gamma_{ij}^1 d_{ij}^\perp \rangle + \langle L_2, \gamma_{ij}^2 d_{ij}^\perp \rangle$$
where the $L_i$ are the columns of the matrix $L$. In matrix form this is $M_{2,2,2}(G, \gamma)$. □

Lemma 13.3 implies that we can determine the dimension of generic colored direction network realization spaces using our results on natural representations of the $(2, 2, 2)$-matroid.

Lemma 13.4. Let $(G, \gamma)$ be a colored graph with $n$ vertices, and $m$ edges. The generic rank of the system $P(G, \gamma, d)$ (with the coordinates of $p_1, \ldots, p_n$ and entries of $L$ as the unknowns) is $m$ if and only if $(G, \gamma)$ is $(2, 2, 2)$-sparse. In particular, it is $2n + 2$ if and only if $(G, \gamma)$ is a $(2, 2, 2)$-graph.

Proof. Apply Lemma 13.3 and then Corollary 8.2. □

13.4. Genericity for colored direction networks

Combining Lemmas 13.3 and 13.4 we see that the set of directions for which the rank of $P(G, \gamma, d)$ is not predicted combinatorially by $(2, 2, 2)$-sparsity is a measure-zero algebraic subset of $\mathbb{R}^{2m}$.

Lemma 13.5. Let $(G, \gamma)$ be a $\mathbb{Z}^2$-rank $k(2, 2, 2)$-$\mathbb{Z}^2$-graded-sparse colored graph with $n$ vertices, $c$ connected components, and $m \leq 2n + 2k - 2c$ edges. The set of edge directions $d$ such that the rank of $P(G, \gamma, d)$ is $m$ is the (open, dense) complement of an algebraic subset of $\mathbb{R}^{2m}$.

Proof. By Lemmas 13.3 and 13.4, the rank is $m$ unless $d$ is a common zero of all the $m \times m$ minors of the matrix $M_{2,2,2}(G, \gamma)$, which is a nowhere-dense closed algebraic subset of $\mathbb{R}^{2m}$. □

14. Collapse of colored-Laman circuits

In this short section we prove the main technical lemmas we need for Theorem B.

14.1. Generic direction networks on colored-Laman circuits

Generic colored direction networks on colored-Laman circuits (defined in Section 7) have very simple realization spaces: all realizations are collapsed.

Lemma 14.1. Let $(G, \gamma)$ be a colored-Laman circuit with $n$ vertices, $c$ connected components, $\mathbb{Z}^2$-rank $k$, and $m = 2n + 2k - 2c$ edges. Furthermore, assume that $P(G, \gamma, d)$ has rank $2n + 2k - 2c$ (this is possible by Lemma 13.5). Then all solutions of $P(G, \gamma, d)$ are collapsed.

Lemma 14.1 will follow from the following general fact about the subspace of collapsed realizations of any colored direction network.
Fig. 17. Constructing a collapsed realization of a tree: (a) the underlying colored graph; (b) the location of the points in a colored realization; (c) in the development, we do not see any of the edges, since all the vertex orbits are just translates of the same point set, reflecting the fact that the direction condition is trivially met for a collapsed edge.

**Lemma 14.2.** Let $(G, \gamma)$ be a colored graph with $n$ vertices, $c$ connected components, $\mathbb{Z}^2$-rank $k$. Any direction network on the graph $(G, \gamma)$ has a $(4 - 2k + 2c)$-dimensional space of collapsed realizations.

The intuition behind this lemma is that we can freely select the position of one vertex in each connected component which then determines the location of the rest of the vertices in that component, accounting for the $2c$ term. Additionally, when the $\mathbb{Z}^2$-rank is non-zero, the representation $L$ of the lattice is restricted if we want to get a collapsed realization, giving the $4 - 2k$.

**Proof of Lemma 14.2.** We first account for the 4 parameters in the lattice representation $L$. Select the matrix $L$ such that:

$$L \cdot \rho(C) = 0$$

for every cycle $C$ in $G'$. Under this condition, the number of free parameters in $L$ is $4 - 2k$.

For now, assume that $G$ is connected, let $T$ be a spanning tree of $G$. For distinct vertices $v$ and $w$ in $G$, define $P_{vw}$ to be the path from $v$ to $w$ in $T$, and define $\sigma_{vw} \in \mathbb{Z}^2$ to be:

$$\sigma_{vw} = \left( \sum_{ij \in P_{vw}} \gamma_{ij} \right) - \left( \sum_{ij \in P_{vw}} \gamma_{ij} \right) \text{traversed from } i \text{ to } j.$$

Select a root vertex $r$ and set $p_r = (x_r, y_r)$ arbitrarily, and then set $p_i = p_r - L \cdot \sigma_{ri}$. We check that all edges $ij$ in $G$ are collapsed. If $ij$ is in the tree $T$, then $\gamma_{ij} = \sigma_{rj} - \sigma_{ri}$. It then follows that

$$p_j - p_i = L \cdot \sigma_{ri} - L \cdot \sigma_{rj} = -L \cdot \gamma_{ij}$$

so all the tree edges are collapsed. (See Fig. 17 for an example of the construction.)

For non-tree edges $ij$, let $C_{ij}$ be the fundamental cycle of $ij$ with respect to $T$. Using the identity $\rho(C_{ij}) - \gamma_{ij} = \sigma_{ri} - \sigma_{rj}$ we compute

$$p_j - p_i = L \cdot \sigma_{ri} - L \cdot \sigma_{rj} = L \rho(C_{ij}) - L \cdot \gamma_{ij}$$

and, since $L \cdot \rho(C_{ij}) = 0$ (by construction), the edge $ij$ is collapsed as well. (Fig. 18 shows an example.)

The general case of the lemma follows from considering the connected components one by one. □
Lemma 14.1 follows nearly immediately from Lemma 14.2.

**Proof of Lemma 14.1.** The hypothesis of the lemma is that the realization space is $(4-2k+2c)$-dimensional. By Lemma 14.2 the space of collapsed solutions has at least this dimension, so the two coincide.  

14.2. Collapsed edges and doubling an edge

We now turn to the case in which the underlying colored graph of the colored direction network is not $(2,2,2)$-colored. In this case, collapsed edges can be given a combinatorial interpretation.

**Lemma 14.3.** Let $(G, \gamma, d)$ be a generic colored direction network, and let $ij$ be an edge of $E(G)$. Suppose every solution $(p, L)$ of $P(G, \gamma, d)$ has $p_i = p_j + L\gamma_{ij}$ (i.e., $ij$ is collapsed). Then, $(p, L)$ is a solution to $P(G, \gamma, d)$ if and only if it is a solution to $P(G + (ij)_c, \gamma, d')$ for any extension $d'$ of the assignment $d$ to $G + (ij)_c$.

**Proof.** Since every solution of $P(G, \gamma, d)$ has $p_j + L\gamma_{ij} - p_i = 0$, we can add a new constraint of the form $(p_j + L\gamma_{ij} - p_i, (a, b)) = 0$ without changing its solution set. This is equivalent to a system of the form $P(G + (ij)_c, \gamma, d')$ where $d'$ is an extension of the assignment $d$ to $G + (ij)_c$.

15. Genericity and Proof of Theorem B

We are nearly ready to prove Theorem B.

15.1. Genericity for colored-Laman direction networks

The last technical tool we need is a description of the set of generic directions for direction networks on colored-Laman graphs.

**Lemma 15.1.** Let $(G, \gamma)$ be a colored-Laman graph on $n$ vertices (and thus $m = 2n + 1$ edges). The set of directions $d \in \mathbb{R}^{4n+2}$ such that:

- $P(G, \gamma, d)$ has rank $2n + 1$
- for all edges $ij \in E(G)$, $P(G + (ij)_c, \gamma, d')$ has rank $2n + 2$ for some $d'$ extending $d$

is the open, dense complement of an algebraic subset of $\mathbb{R}^{4n+2}$

**Proof.** Applying Lemma 13.5 to $(G, \gamma)$ and each graph $(G + (ij)_c, \gamma)$ yields a finite set of nowhere dense algebraic subsets of $\mathbb{R}^{4n+2}$ for which the statement of the lemma does not hold. The union of these is algebraic and nowhere dense, as required.
15.2. Remark on genericity

We remark that non-generic sets of directions come in two types:

- Those for which $P(G, \gamma, d)$ has rank less than $2n + 1$
- Those for which some $P(G + (ij)_c, \gamma, d')$ has rank less than $2n + 2$.

Both of these conditions are necessary for the proof of Theorem B, and thus the genericity assumption given here cannot be weakened too much. They also have slightly different geometric interpretations:

- If the rank of $P(G, \gamma, d)$ is not maximum, then there is a larger than expected space of non-collapsed realizations preserving the given directions. These additional degrees of freedom translate to non-trivial infinitesimal motions of periodic frameworks via a standard trick from parallel redrawing.

- The rank of $P(G + (ij)_c, \gamma, d')$ not increasing means that the given directions are not realizable as part of the difference set of points in the plane, which implies collapsed edges even before doubling. Intuitively, the rank of the colored direction network system does not rise when doubling a collapsed edge because there is no new constraint on its direction.

15.3. Proof of Theorem B

Let $(G, \gamma)$ be a colored-Laman graph, and select $d$ as in Lemma 15.1. By Lemma 13.4, $P(G, \gamma, d)$ has a 3-dimensional solution space. The set of collapsed solutions is two-dimensional by Lemma 13.1. Hence, there is a solution $p = \hat{p}$ and $L = \hat{L}$ that is not collapsed, and by Lemma 13.2 we can assume that $\hat{p}_1 = (0, 0)$. Any other solution with $p_1 = (0, 0)$ is $(\hat{p}, \hat{L})$ up to scaling by some real number $\lambda$.

We suppose, for a contradiction, that some edge $ij$ is collapsed in $(\hat{p}, \hat{L})$. Because all the realizations are scalings of $(\hat{p}, \hat{L})$, $ij$ must be collapsed in all realizations. It follows from Lemma 14.3 that $P(G, \gamma, d)$ has the same solution space as $P(G + (ij)_c, \gamma, d')$ where $d'$ is chosen as in Lemma 15.1.

The combinatorial Lemma 7.3 implies that $(G + (ij)_c, \gamma)$ is $(2, 2, 2)$-colored. By the hypothesis on $d'$, from Lemma 15.1, $P(G + (ij)_c, \gamma, d)$ has full rank, and then Lemma 14.1 implies that all solutions of $P(G + (ij)_c, \gamma, d')$ are collapsed. This contradicts our assumption that $(\hat{p}, \hat{L})$ is not collapsed, proving that, if $(G, \gamma)$ is colored-Laman and $d$ is chosen generically as in Lemma 15.1, then all realizations with at least one non-collapsed edge are faithful.

As noted above, the realization space is three dimensional. Lemma 13.2 shows that there is a 2-dimensional subspace of translations and, since the system is homogeneous, scaling provides an independent dimension of realizations. This proves that the faithful realization is unique up to translation and scale.

In the other direction, if $(G, \gamma)$ is not colored-Laman, then Corollary 8.2 and Lemma 14.1, applied to colored-Laman circuit supplied by Lemma 7.2 imply that some edge collapses.

16. Periodic and colored rigidity

With Theorem B proved, we return from the setting of direction networks to that of bar-joint rigidity. Sections 16–18 follow the same three-step outline used for the 1d-periodic case in Section 11.2, going from the continuous rigidity theory to the combinatorics of colored-Laman graphs and then (generically), back again. We start by recalling the definition of periodic frameworks from the introduction.
16.1. Periodic frameworks

A periodic framework is defined by a triple $(\tilde{G}, \varphi, \tilde{\ell})$ where: $\tilde{G}$ is a simple infinite graph; $\varphi$ is a free $\mathbb{Z}^2$-action on $\tilde{G}$ by automorphisms such that the quotient is finite; and $\tilde{\ell} = (\ell_{ij})$ assigns a length to each edge of $\tilde{G}$.

A realization $\tilde{G}(p, L)$ of a periodic framework $(\tilde{G}, \varphi, \tilde{\ell})$ is defined to be a mapping $p$ of the vertex set $V(\tilde{G})$ into $\mathbb{R}^2$ and a representation $\mathbb{Z}^2 \to \mathbb{R}^2$ encoded by a matrix $L \in \mathbb{R}^{2 \times 2}$ (with $\mathbb{R}^2$ here viewed as translations) such that:

- the representation is equivariant with respect to the $\mathbb{Z}^2$-actions on $\tilde{G}$ and the plane; i.e., $p_{\gamma \cdot i} = p_i + L \cdot \gamma$ for all $i \in V(\tilde{G})$ and $\gamma \in \mathbb{Z}^2$;
- the specified edge lengths are preserved by $p$: i.e., $\|p_i - p_j\| = \tilde{\ell}_{ij}$ for all edges $ij \in E(\tilde{G})$.

To be realizable, a periodic framework needs to assign the same length to edges in the same $\mathbb{Z}^2$-orbit, and from now on we make this assumption, since we are interested in analyzing generic realizations.

16.2. Periodic rigidity and flexibility

The realization space of a periodic framework is defined to be the algebraic set $\mathcal{R}(\tilde{G}, \varphi, \tilde{\ell})$ of all realizations. The group of 2-dimensional Euclidean isometries, $\text{Euc}(2)$, acts naturally on $\mathcal{R}(\tilde{G}, \varphi, \tilde{\ell})$; for $\phi \in \text{Euc}(2)$ with rotational part $\phi_0 \in \text{Euc}(2)$, the action is given by

$$\phi(\tilde{G}(p, L)) = \tilde{G}(\phi(p), \phi_0 \circ L).$$

The configuration space $\mathcal{C}(\tilde{G}, \varphi, \tilde{\ell}) = \mathcal{R}(\tilde{G}, \varphi, \tilde{\ell})/\text{Euc}(2)$ is defined to be the quotient of the realization space by Euclidean motions. A realization $\tilde{G}(p, L)$ is rigid if $\tilde{G}(p, L)$ is isolated in the configuration space and minimally rigid if it is rigid but ceases to be so when the $\mathbb{Z}^2$-orbit of any edge $ij \in E(\tilde{G})$ is removed. Since $\mathcal{R}(\tilde{G}, \varphi, \tilde{\ell})$ is a subset of an infinite-dimensional space, its topology merits some discussion. The interested reader can refer to [13, Appendix A].

16.3. Main theorem

We can now state our main theorem.

**Theorem A.** Let $(\tilde{G}, \varphi, \tilde{\ell})$ be a generic periodic framework. Then a generic realization $\tilde{G}(p, L)$ of $(\tilde{G}, \varphi, \tilde{\ell})$ is minimally rigid if and only if its colored quotient graph $(G, \gamma)$ is colored-Laman.

The proof will make use of (technically simpler) colored frameworks, which we now define.

16.4. Colored frameworks

A priori, the realization space $\mathcal{R}(\tilde{G}, \varphi, \tilde{\ell})$ could be an unwieldy infinite dimensional object. However, since $\tilde{G}/\mathbb{Z}^2$ is finite, the realization space is really finite dimensional. We now make this precise via the following definition.

A $\mathbb{Z}^2$-colored framework is defined as a triple $(G, \gamma, \ell)$ where $(G, \gamma)$ is a $\mathbb{Z}^2$-colored graph and $\ell = (\ell_{ij})_{ij \in E(G)}$ is an assignment of lengths to the edges of $G$.

A realization $G(p, L)$ of a $\mathbb{Z}^2$-colored framework $(G, \gamma, \ell)$ is an assignment $p = (p_i)_{i \in V(G)}$ of points to the vertices of $G$ and a choice of matrix $L \in \mathbb{R}^{2 \times 2}$ such that for all $ij \in E(G)$

\[ \tilde{\ell}_{ij} = \ell_{ij}. \]

\[ \|p_i - p_j\| = \ell_{ij} \]

\[ p_i + L \cdot \gamma = p_{\gamma \cdot i} \]

\[ \phi(p) = \phi(p) \]

\[ \phi(\tilde{G}(p, L)) = \tilde{G}(\phi(p), \phi_0 \circ L). \]

\[ \phi(\tilde{G}(p, L)) = \tilde{G}(\phi(p), \phi_0 \circ L). \]

The reference [13] is a previous version of the present paper.
we have
\[ \| \mathbf{p}_j + \mathbf{L} \cdot \gamma_{ij} - \mathbf{p}_i \|^2 = \ell_{ij}^2. \] (1)

It is clear from the definition that the realization space \( \mathcal{R}(G, \gamma, \ell) \) is naturally identified with a subvariety of \( \mathbb{R}^{2n+4} = (\mathbb{R}^2)^n \times \mathbb{R}^{2 \times 2} \) where \( n = |V(G)| \). As with Lemma 2.1, there is a dictionary between triples \( (G, \gamma, \ell) \) and triples \( (\tilde{G}, \phi, \tilde{\ell}) \) where \( \tilde{G} \) is the development of \( (G, \gamma) \) and \( \tilde{\ell} \) is obtained by assigning \( \ell_{ij} \) to every edge in the fiber over \( ij \in E(G) \).

16.5. Continuous rigidity of colored frameworks

As in the theory of finite (uncolored) frameworks in Euclidean space, if no vertex is “pinned down”, then there are always trivial motions of a realization that arise from Euclidean isometries. A realization is then rigid if these are the only motions. We now make the various notions of rigidity precise in the colored framework setting.

The isometry group \( \text{Euc}(2) \) of the Euclidean plane acts naturally on \( \mathcal{R}(G, \gamma, \ell) \). For any \( \phi \in \text{Euc}(2) \), let \( \phi_0 \in \mathbb{R}^{2 \times 2} \) be the rotational part. Then the action
\[ \phi \cdot (\mathbf{p}_1, \ldots, \mathbf{p}_n, \mathbf{L}) = (\phi(\mathbf{p}_1), \ldots, \phi(\mathbf{p}_n), \phi_0 \cdot \mathbf{L}) \]
preserves \( \mathcal{R}(G, \gamma, \ell) \). We define motions given by \( \text{Euc}(2) \) to be trivial, and we define the configuration space \( \mathcal{C}(G, \gamma, \ell) \) to be \( \mathcal{R}(G, \gamma, \ell)/\text{Euc}(2) \).

Let \( (G, \gamma, \ell) \) be a \( \mathbb{Z}^2 \)-colored framework. A realization \( G(\mathbf{p}, \mathbf{L}) \) of \( (G, \gamma, \ell) \) is rigid if the corresponding point in \( \mathcal{C}(G, \gamma, \ell) \) is isolated. Otherwise, it is flexible. If \( G(\mathbf{p}, \mathbf{L}) \) is rigid and is flexible after the removal of any edge, we say \( G(\mathbf{p}, \mathbf{L}) \) is minimally rigid.

16.6. Equivalence of periodic and colored frameworks

The following proposition can be obtained from [2] by translating the arguments into the setting of colored frameworks.

**Proposition 3** ([2, Theorem 3.1]). Let \( (\tilde{G}, \phi, \tilde{\ell}) \) be a periodic framework and \( (G, \gamma, \ell) \) an associated \( \mathbb{Z}^2 \)-colored graph. There is a natural homeomorphism \( \Psi : \mathcal{R}(\tilde{G}, \phi, \tilde{\ell}) \rightarrow \mathcal{R}(G, \gamma, \ell) \) respecting the action of \( \text{Euc}(2) \). In particular, \( \tilde{G}(\tilde{\mathbf{p}}, \tilde{\mathbf{L}}) \) is rigid if and only if \( \Psi(\tilde{G}(\tilde{\mathbf{p}}, \tilde{\mathbf{L}})) \) is rigid.

Fig. 19 shows the correspondence between periodic and colored frameworks associated with the same colored graph.
17. Infinitesimal colored rigidity

We now introduce infinitesimal rigidity, a linearization of the rigidity problem that is more tractable than the quadratic system of length equations. The rigidity matrix \( M_{2,3,2}(G, \gamma, p, L) \) of a colored framework is defined by the differential of the system (1):

\[
\begin{bmatrix}
  i & j & \ldots & \ldots & \ldots & \ldots & L_1 & L_2 \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}
\]

where \( \eta_{ij} = p_j + L \cdot \gamma_{ij} - p_i \). This matrix was first computed in [2].

The kernel of the rigidity matrix is defined to be the space of infinitesimal motions, which spans the tangent space \( T(p, L) \mathbb{R}(G, \gamma, \ell) \) of the realization space at the point \( (p, L) \).

It is shown in [2] (and easy to check via direct computation) that the Lie algebra of \( \text{Euc}(2) \) always induces a 3-dimensional subspace of infinitesimal motions. A realization \( G(p, L) \) is defined to be infinitesimally rigid if the space of infinitesimal motions is 3-dimensional and infinitesimally flexible otherwise. Infinitesimal rigidity is equivalent to the rigidity matrix having corank 3. Infinitesimal rigidity always implies rigidity, but the converse holds only up to a nowhere dense set of non-generic realizations, which we define below.

17.1. Genericity for colored frameworks

A realization \( G(p, L) \) is defined to be generic if the rank of the rigidity matrix is maximized over all choices of \( p \) and \( L \); i.e., \( M_{2,3,2}(G, \gamma, p, L) \) achieves its generic rank at \( G(p, L) \). The important thing, for our purposes, is that the generic rank of the rigidity matrix depends only on the underlying colored graph \( (G, \gamma) \).

Thus, we define \( (G, \gamma) \) to be generically rigid (resp. flexible) if generic \( G(p, L) \in \mathbb{R}^{2n+4} \) are rigid (resp. flexible). Similarly define generic infinitesimal rigidity (resp. flexibility) of \( (G, \gamma) \). We define \( (G, \gamma) \) to be generically minimally rigid if generic \( G(p, L) \) are minimally rigid.

The analogue of the following lemma for the non-periodic setting follows from the main theorem of [1]. This result says intuitively that for generic realizations, continuous and infinitesimal rigidity have the same behavior. With minor modifications, the proofs of [1] carry over to our setting [13, Appendix A].

Lemma 17.1. A colored graph \( (G, \gamma) \) is:

- Generically rigid if and only if it is generically infinitesimally rigid.
- Generically flexible if and only if it is generically infinitesimally flexible.

18. Generic periodic rigidity: proof of the Main Theorem A

This completes the required background, and we are ready to prove our main result.

18.1. Proof of Theorem A

Let \( (G, \gamma) \) be a colored graph with \( n \) vertices and \( m = 2n + 1 \) edges. We may reduce to the case \( m = 2n + 1 \) since if \( m \neq 2n + 1 \), the colored graph \( (G, \gamma) \) is neither colored-Laman nor generically minimally rigid. By Lemma 17.1, it suffices to verify that the generic rank of
$M_{2,3,2}(G, \gamma, p, L)$ is $2n + 1$ if and only if $(G, \gamma)$ is colored-Laman, since removing any edge will lead to a rigidity matrix with corank at least 4.

First, suppose that $(G, \gamma)$ is not colored-Laman. Then by Lemma 7.2, it contains a colored-Laman circuit $(G', \gamma')$ on $n' > 2$ vertices, $c'$ components, rank $k'$ and $m' = 2n' + 2k' - 2c'$ edges. This subgraph induces a submatrix $M'$ of the same form as the rigidity matrix with $2n' + 2k' - 2c'$ rows and $2n' + 4$ columns with non-zero entries.

We will show by contradiction that $M'$ has rank less than $2n' + 2k' - 2c'$. Suppose $M'$ has full rank. Consider the direction network on $(G', \gamma')$ with directions $d$ given by the edge directions $\eta_{ij}$ of $G(p, L)$. Since $G(p, L)$ is itself a realization of the direction network, not all realizations are collapsed. However, the matrix for the system $P(G, \gamma, d)$ can be obtained from $M'$ by swapping and negating some columns. Hence, the system $P(G, \gamma, d)$ has full rank, and by Lemma 14.1, all solutions are collapsed, a contradiction. Since $M'$ has the same rank as the corresponding $2n' + 2k' - 2c'$ rows in the rigidity matrix, $M_{2,3,2}(G, \gamma, p, L)$ must have a row dependency, and thus rank strictly less than $2n + 1$.

Now we suppose that $(G, \gamma)$ is colored-Laman. We will show it has full rank by an example. Construct a generic (in the sense of Lemma 15.1) direction network $(G, \gamma, d)$ on $(G, \gamma)$. By Theorem B, for generic $d$, this direction network has a unique, up to translation and scaling, faithful realization $G(p, L)$. Thus, for all $ij \in E(G)$, there is $\alpha_{ij} \neq 0$ such that $p_j + L \gamma_{ij} - p_i = \alpha_{ij}d_{ij}$. By replacing $d_{ij}$ with $d_{ij}^{-1}/\alpha_{ij}$ and swapping and negating some columns in $M_{2,2,2}(G, \gamma)$, we obtain the rigidity matrix $M_{2,3,2}(G, \gamma, p, L)$. Since all such operations do not affect the rank, $G(p, L)$ is infinitesimally rigid. \hfill $\Box$

18.2. Remarks

Although we proved the rigidity Theorem A from the direction network Theorem B algebraically, using matrix manipulations, there is a more geometric way to view the argument.

Let $G(p, L)$ be a realization of a colored framework with underlying colored graph $(G, \gamma)$. This realization induces a colored direction network $(G, \gamma, d)$, where the direction $d_{ij} = p_j + L \gamma_{ij} - p_i$. Now let $G(p', L')$ be another realization of $(G, \gamma, d)$. By construction, we know that, for every edge $ij$ in the colored graph $(G, \gamma)$,

$$\left( p_j + L \gamma_{ij} - p_i, (p_j' - p_j + (L - L') \gamma_{ij} - (p_i' - p_i)))^{-1} \right) = 0. $$

In other words, the difference between $(p, L)$ and another realization of the colored direction network $(G, \gamma, d)$ turned by $90^\circ$ gives an infinitesimal motion of the colored framework $G(p, L)$. The same fact for planar finite frameworks is classical.

19. Conclusions and further directions

We considered the question of generic combinatorial periodic rigidity in the plane, and, with Theorem A, gave a complete answer. To conclude we indicate some consequences and potential further directions.

19.1. Fixed-lattice frameworks

Elissa Ross considered a specialization of the planar periodic rigidity problem in which the lattice representation $L$ is fixed. She proved, in our language the following proposition.
**Proposition 4** ([19,20]). Let \((\tilde{G}, \varphi, \tilde{\ell})\) be a generic periodic framework and further suppose that the lattice representation \(L\) is fixed, with \(L\) non-singular. Then a generic realization of \((\tilde{G}, \varphi, \tilde{\ell})\) is minimally rigid if and only if its quotient graph \((G, \gamma)\):

- Has \(n\) vertices and \(m = 2n - 2\) edges.
- Every subgraph \(G'\) on \(n'\) vertices and \(m'\) edges with \(\mathbb{Z}^2\)-rank zero satisfies \(m' \leq 2n' - 3\).
- Every subgraph \(G'\) on \(n'\) vertices and \(m'\) edges satisfies \(m' \leq 2n' - 2\).

We define a colored graph satisfying the properties of Proposition 4 to be a **Ross graph**. Ross graphs are related to colored-Laman graphs via the following combinatorial equivalence. (The colored-Laman graph in Fig. 7 arises from the construction in Lemma 19.1.)

**Lemma 19.1.** Let \((G, \gamma)\) be a colored graph with \(n\) vertices and \(m\) edges. Then \((G, \gamma)\) is a Ross graph if and only if for any vertex \(i \in V(G)\) adding three self-loops at vertex \(i\) with colors \((1, 0), (0, 1), \) and \((1, 1)\) yields a colored-Laman graph \((G', \gamma)\).

Proposition 4 can then be obtained from Theorem A and the observation that the rigidity matrix of the augmented graph \((G', \gamma)\) has the form

\[
M_{2,3,2}(G', \gamma', p, L) = \begin{pmatrix}
i & j & L_1 & L_2 \\
M_{2,3,2}(G, \gamma, p, L) \\
0 \ldots 0 & L_1 & L_1 \\
0 \ldots 0 & L_2 & L_2 \\
0 \ldots 0 & L_1 + L_2 & L_1 + L_2
\end{pmatrix}
\]

which implies that any infinitesimal motion acts trivially on the lattice representation \(L\).

### 19.2. Crystallographic rigidity: other symmetry groups

Our Main Theorem A does not close the field of Maxwell–Laman-type Theorems for planar frameworks with forced symmetry. Perhaps the most natural question raised by the present work is whether similar results are possible when \(\mathbb{Z}^2\) is replaced by another crystallographic group.

### 19.3. Periodic parallel redrawing and scene analysis

We introduced periodic direction networks with the goal of proving a characterization of generic infinitesimal periodic rigidity, and thus have focused narrowly on the properties needed for that purpose. However, as discussed in the introduction, there is a more general theory of *parallel redrawing* and *scene analysis*, which relate finite direction networks and frameworks to projections of polyhedral scenes [26, Sections 4 and 8]. Determining the extent to which these theories generalize to the periodic case would be very interesting.

### 19.4. Group-graded sparsity and algorithmic periodic rigidity

We introduced and studied two families of colored graphs: colored-Laman graphs and \((2, 2, 2)\)-graphs. These are matroidal and, via general augmenting path algorithms for matroid union, recognizable in polynomial time.
Fig. 20. Restricting the $\mathbb{Z}^2$-action on the periodic graph from Fig. 9(b) to the sub-lattice generated by (1, 0) and (0, 2): (a) the resulting colored graph; (b) the development, with connected components indicated by color. Since the black points can translate independently of the gray ones, any associated framework has at least these non-trivial degrees of freedom.

Two combinatorial questions that arise are:

- Is there a more general theory of matroidal hereditary sparsity for $\mathbb{Z}^d$-colored graphs?
- Are there cleaner, more efficient algorithms for recognizing colored-Laman and $(2, 2, 2)$-graphs?

For finite frameworks, the answers to both of these questions are affirmative [9].

19.5. Passing to sub-lattices

Elissa Ross mentions the following conjecture, which relates to the example from Section 3.6.

**Conjecture 19.2** ([20, Conjecture 8.2.8]). Let $\tilde{G}(\mathbf{p}, \mathbf{L})$ be an infinitesimally rigid periodic framework with periodic graph $(\tilde{G}, \varphi)$. Let $\Lambda < \mathbb{Z}^2$ be any sub-lattice, and define $(\tilde{G}, \varphi')$ to be the periodic graph obtained by replacing the $\mathbb{Z}^2$-action $\varphi$ with the induced $\Lambda$-action $\varphi'$. Then $\tilde{G}(\mathbf{p}, \mathbf{L})$ is an infinitesimally rigid realization of the induced abstract periodic framework on $(\tilde{G}, \varphi')$.

Informally, what this conjecture says is that a generic, rigid periodic framework remains so even if we enlarge the class of allowed motions by relaxing the periodicity constraint to hold only on a sub-lattice. Geometrically, this means just expanding the fundamental domain of the $\mathbb{Z}^2$-action on the plane induced by $\mathbf{L}$. The example in Fig. 9 shows that Conjecture 19.2 is false, even in a, much weaker, combinatorial version: if we take the sub-lattice to be the one generated by (1, 0) and (0, 2), we get the periodic framework and associated colored graph in Fig. 20. It is easy to see that the two connected components can translate independently of each other, and that a maximal colored-Laman-sparse subgraph is simply one of the connected components.

This counterexample generalizes. Suppose that $(G, \gamma)$ is a colored-Laman graph with $n$ vertices. The operation of passing to a sub-lattice $\Lambda$ corresponding to an index $\ell$ subgroup of $\mathbb{Z}^2$ in the associated periodic framework means, in combinatorial terms, passing to an $\ell$-sheeted cover $(G^*, \gamma^*)$ of the colored graph $(G, \gamma)$. Thus $G^*$ has $\ell n$ vertices and $2\ell n + \ell$ edges.

On the other hand, if $\rho(G, \gamma)$ generates a finite index subgroup $\Gamma < \mathbb{Z}^2$, and we take the corresponding sub-lattice $\Lambda$, then, by Lemma 2.5, $G^*$ has at least two connected components, and thus any colored-Laman-sparse subgraph of $(G^*, \gamma^*)$ can have at most $2\ell n + 1 - 2$ edges which is too few to be a colored-Laman graph. Repeating the same construction, but with $(G, \gamma)$ a subgraph of a colored-Laman graph $(H, \gamma)$ with $\rho(H, \gamma) = \mathbb{Z}^2$, we see that the colored graph cover $(H^*, \gamma^*)$ corresponding to the bad sub-lattice $\Lambda$ need not be disconnected.
It would be interesting to resolve the following combinatorial question about colored graphs, which is a kind of “doubly generic” version of Conjecture 19.2.

**Question 19.3.** Let $(G, \gamma)$ be a colored-Laman graph. Does a generic finite-sheeted cover of $(G, \gamma)$ that arises from passing to a sub-lattice in the development have a spanning subgraph that is colored-Laman?

We leave the meaning of generic intentionally vague, but it seems plausible that there are a finite number of maximal “bad” sub-lattices to avoid.

**Acknowledgments**

We would like to thank Igor Rivin for his encouragement and many productive discussions on this topic. The anonymous referee’s comments helped us greatly improve our exposition, and we thank them for their careful reading. We became interested in this problem as part of a larger project to study the rigidity and flexibility of zeolites [21,18,24], which is supported by NSF CDI-I grant DMR 0835586 to Rivin and M. M. J. Treacy. LT’s final preparation of this paper was supported by the European Research Council under the European Union’s Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement no 247029-SDModels.

**References**

[1] L. Asimow, B. Roth, The rigidity of graphs, Trans. Amer. Math. Soc. (ISSN: 0002-9947) 245 (1978) 279–289. http://dx.doi.org/10.2307/1998867.

[2] Ciprian S. Borcea, Ileana Streinu, Periodic frameworks and flexibility, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. (ISSN: 1364-5021) 466 (2121) (2010) 2633–2649. http://dx.doi.org/10.1098/rspa.2009.0676.

[3] Thomas Brylawski, Constructions, in: Theory of Matroids, in: Encyclopedia Math. Appl., vol. 26, Cambridge Univ. Press, Cambridge, 1986, pp. 127–223. http://dx.doi.org/10.1017/CBO9780511629563.010.

[4] Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, Clifford Stein, Introduction to Algorithms, third ed., MIT Press, Cambridge, MA, ISBN: 978-0-262-03384-8, 2009.

[5] David Cox, John Little, Donal O’Shea, Ideals, varieties, and algorithms, in: Undergraduate Texts in Mathematics, third ed., Springer, New York, 2007. http://dx.doi.org/10.1007/978-0-387-35651-8. ISBN: 978-0-387-35650-9.

[6] Jack Edmonds, Gian-Carlo Rota, Submodular set functions (abstract), in: Waterloo Combinatorics Conference, University of Waterloo, Ontario, 1966.

[7] Jack Graver, Counting on frameworks, in: Mathematics to Aid the Design of Rigid Structures, in: The Dolciani Mathematical Expositions, vol. 25, Mathematical Association of America, Washington, DC, ISBN: 0-88385-331-0, 2001.

[8] G. Laman, On graphs and rigidity of plane skeletal structures, J. Engrg. Math. (ISSN: 0022-0833) 34 (1970) 331–340.

[9] Audrey Lee, Ileana Streinu, Pebble game algorithms and sparse graphs, Discrete Math. (ISSN: 0012-365X) 308 (8) (2008) 1425–1437. http://dx.doi.org/10.1016/j.disc.2007.07.104.

[10] L. Lovász, Flats in matroids and geometric graphs, in: Combinatorial Surveys (Proc. Sixth British Combinatorial Conf., Royal Holloway Coll., Egham, 1977), Academic Press, London, 1977, pp. 45–86.

[11] László Lovász, Combinatorial Problems and Exercises, second ed., AMS Chelsea Publishing, Providence, RI, ISBN: 978-0-8218-4262-1, 2007.

[12] L. Lovász, Y. Yemini, On generic rigidity in the plane, SIAM J. Algebr. Discrete Methods (ISSN: 0196-5212) 3 (1) (1982) 91–98. http://dx.doi.org/10.1137/0603009.

[13] Justin Malestein, Louis Theran, Generic combinatorial rigidity of periodic frameworks, 2010. Preprint, arXiv:1008.1837v2.

[14] J. Maxwell, On the calculation of the equilibrium and stiffness of frames, Phil. Mag. (1864) Series 4.

[15] C.St.J.A. Nash-Williams, Edge-disjoint spanning trees of finite graphs, J. Lond. Math. Soc. (2) (ISSN: 0024-6107) 36 (1961) 445–450.
[16] James G. Oxley, Matroid Theory, Oxford Science Publications, The Clarendon Press Oxford University Press, New York, ISBN: 0-19-853563-5, 1992.

[17] András Recski, A network theory approach to the rigidity of skeletal structures. II. Laman’s theorem and topological formulae, Discrete Appl. Math. (ISSN: 0166-218X) 8 (1) (1984) 63–68. http://dx.doi.org/10.1016/0166-218X(84)90079-9.

[18] Igor Rivin, Geometric simulations: a lesson from virtual zeolites, Nat. Mater. 5 (2006) 931–932. http://dx.doi.org/10.1038/nmat1792.

[19] Elissa Ross, Periodic rigidity, in: Talk at the Spring AMS Sectional Meeting, 2009. http://www.ams.org/meetings/sectional/1050-52-71.pdf.

[20] Elissa Ross, The rigidity of periodic frameworks as graphs on a torus. Ph.D. Thesis, York University, 2011. http://www.math.yorku.ca/~ejross/RossThesis.pdf.

[21] A. Sartbaeva, S. Wells, M. Treacy, M. Thorpe, The flexibility window in zeolites, Nat. Mater. (Jan) (2006).

[22] Ileana Streinu, Louis Theran, Slider-pinning rigidity: a Maxwell–Laman-type theorem, Discrete Comput. Geom. (ISSN: 0179-5376) 44 (4) (2010) 812–837. http://dx.doi.org/10.1007/s00454-010-9283-y.

[23] Tiong-Seng Tay, Rigidity of multigraphs. I. Linking rigid bodies in n-space, J. Combin. Theory Ser. B (ISSN: 0095-8956) 36 (1) (1984) 95–112. http://dx.doi.org/10.1016/0095-8956(84)90016-9.

[24] M. Treacy, I. Rivin, E. Balkovsky, K. Randall, Enumeration of periodic tetrahedral frameworks. II. polynodal graphs, Microporous Mesoporous Mater. 74 (1–3) (2004) 121–132.

[25] W.T. Tutte, On the problem of decomposing a graph into n connected factors, J. London Math. Soc. (ISSN: 0024-6107) 36 (1961) 221–230.

[26] Walter Whiteley, Some matroids from discrete applied geometry, in: J. Bonin, James G. Oxley, B. Servatius (Eds.), Matroid Theory, in: Contemporary Mathematics, vol. 197, American Mathematical Society, 1996, pp. 171–311.

[27] Walter Whiteley, The union of matroids and the rigidity of frameworks, SIAM J. Discrete Math. (ISSN: 0895-4801) 1 (2) (1988) 237–255. http://dx.doi.org/10.1137/0401025.

[28] Thomas Zaslavsky, A mathematical bibliography of signed and gain graphs and allied areas, in: Dynamic Surveys 8, Electron. J. Combin. (ISSN: 1077-8926) 5 (1998) 124 (electronic). http://www.combinatorics.org/Surveys/index.html. Manuscript prepared with Marge Pratt.