How symmetric is too symmetric for large quantum speedups?

Shalev Ben-David
University of Waterloo
shalev.b@uwaterloo.ca

Supartha Podder
University of Ottawa
spodder@uottawa.ca

Abstract

Suppose a Boolean function $f$ is symmetric under a group action $G$ acting on the $n$ bits of the input. For which $G$ does this mean $f$ does not have an exponential quantum speedup? Is there a characterization of how rich $G$ must be before the function $f$ cannot have enough structure for quantum algorithms to exploit?

In this work, we make several steps towards understanding the group actions $G$ which are “quantum intolerant” in this way. We show that sufficiently transitive group actions do not allow a quantum speedup, and that a “well-shuffling” property of group actions – which happens to be preserved by several natural transformations – implies a lack of super-polynomial speedups for functions symmetric under the group action. Our techniques are motivated by a recent paper by Chailloux (2018), which deals with the case where $G = \mathbb{S}_n$.

Our main application is for graph symmetries: we show that any Boolean function $f$ defined on the adjacency matrix of a graph (and symmetric under relabeling the vertices of the graph) has a power 6 relationship between its randomized and quantum query complexities, even if $f$ is a partial function. In particular, this means no graph property testing problems can have super-polynomial quantum speedups, settling an open problem of Ambainis, Childs, and Liu (2011).

1 Introduction

One of the most fundamental questions in the field of quantum computing is the question of when quantum algorithms substantially outperform classical ones. While polynomial quantum speedups are known in many settings, super-polynomial quantum speedups are known (or even merely conjectured) for only a few select problems. An important lesson in the field has been that exponential quantum speedups only occur for certain “structured” problems: problems such as period-finding (used in Shor’s factoring algorithm [Sho97]), or Simon’s problem [Sim97], in which the input is known in advance to have a highly restricted form. In contrast, for “unstructured” problems such as blackbox search or NP-complete problems, only polynomial speedups are known (and in some models, it can be formally shown that only polynomial speedups are possible).

In this work, we are interested in formalizing and characterizing the structure necessary for fast quantum algorithms; in particular, we study the types of symmetries a Boolean function can have while still exhibiting super-polynomial quantum speedups.

1.1 Previous work

Despite the strong intuition in the field that structure is necessary for exponential quantum speedups, only a handful of works have attempted to formalize this intuition and characterize the necessary structure. All of them study the problem in the query complexity (black-box) model of quantum computation, which is a natural framework in which both period-finding and Simon’s
problem can be formally shown to give exponential quantum speedups (see [BdW02] for a survey of query complexity, or [Cle04] for a formalization of period-finding specifically).

In the query complexity model, the goal is to compute a Boolean function \( f : \Sigma^n \rightarrow \{0,1\} \) using as few queries to the bits of the input \( x \in \Sigma^n \) as possible. Here \( \Sigma \) is some finite alphabet, and each query specifies an index \( i \in [n] \) and receives the response \( x_i \in \Sigma \). A query algorithm, which may depend on \( f \) but not on \( x \), must output \( f(x) \) (to worst-case bounded error) after as few queries as possible. Quantum query algorithms are allowed to make queries in superposition; we are interested in how much advantage this gives them over randomized classical algorithms (for a formal definition of these notions, see [BdW02]).

Beals, Buhrman, Cleve, Mosca, and de Wolf [BBC+01] showed that all total Boolean functions \( f : \Sigma^n \rightarrow \{0,1\} \) have a polynomial relationship between their classical and quantum query complexities (which we denote \( \text{R}(f) \) and \( \text{Q}(f) \) respectively). This means that super-polynomial speedups are not possible in query complexity unless we impose a promise on the input: that is, unless we define \( f : P \rightarrow \{0,1\} \) with \( P \subseteq \Sigma^n \), and allow an algorithm computing \( f \) to behave arbitrarily on inputs outside of the promise set \( P \). For such promise problems (also called partial functions), provable exponential quantum speedups are known; this is the setting in which Simon’s problem and period-finding reside.

The question, then, is whether we can say anything about the structure necessary for a partial Boolean function \( f \) to exhibit a super-polynomial quantum speedup. Towards this end, Aaronson and Ambainis [AA14] showed that symmetric functions do not allow super-polynomial quantum speedups, even with a promise. Chailloux [Cha18] improved this result by improving the degree of the polynomial relationship between randomized and quantum algorithms for symmetric functions, and removing a technical requirement on the symmetry of those functions.\(^1\)

Other work attempted to characterize the structure necessary for quantum speedups in other ways. [Ben16] showed that certain types of symmetric promises do not admit any function with a super-polynomial quantum speedup, a generalization of [BBC+01] (who showed this when the promise set is \( \Sigma^n \)). [ABR16] showed that small promise sets, which contain only \( \text{poly}(n) \) inputs out of \( |\Sigma|^n \), also do not admit functions which separate quantum and classical algorithms by more than a polynomial factor.

1.2 Our contributions

In this work, we extend the results of Aaronson-Ambainis and Chailloux to other symmetry groups. To state our results, we introduce the following definition.

**Definition 1.** Let \( f : P \rightarrow \{0,1\} \) be a function with \( P \subseteq \Sigma^n \), where \( \Sigma \) is a finite alphabet and \( n \in \mathbb{N} \). We say that \( f \) is symmetric with respect to a group action \( G \) acting on domain \([n]\) if for all \( x \in P \) and all \( \pi \in G \), the string \( x \circ \pi \) defined by \((x \circ \pi)_i := x_{\pi(i)} \) satisfies \( x \circ \pi \in P \) and \( f(x \circ \pi) = f(x) \).

This definition allows us to talk about more general symmetries of a Boolean function. The case where \( G = S_n \) is the fully-symmetric group action is the one handled in Chailloux’s work [Cha18]: he showed that \( \text{R}(f) = O(\text{Q}(f)^3) \) if \( f \) is symmetric under \( S_n \). Aaronson and Ambainis [AA14] required an even stronger symmetry property. We note that when \( \Sigma \) is large, say \( |\Sigma| = n \) or larger, the class of functions symmetric under \( S_n \) is already highly nontrivial: among others, it includes functions such as Collision, an important function whose quantum query complexity was established in [AS04]; \( k \)-Sum, whose quantum query complexity required the negative-weight adversary
to establish \([BŠ13]\); and \(k\)-DISTINCTNESS, whose quantum query complexity is still open \([BKT18]\). Additionally, computational geometry functions such as \textsc{ClosestPair} (a function studied in recent work by Aaronson, Chia, Lin, Wang, and Zhang \([ACL^+19]\)) are typically symmetric under \(S_n\) as well, as the points are usually represented as alphabet symbols. If we round the alphabet to be finite, the \(G = S_n\) case already shows that no computational geometry problem of this form can have super-cubic quantum speedups.

In this work, we examine what happens when we relax the full symmetry \(S_n\) to smaller symmetry groups \(G\). We introduce some tools for showing that particular classes of group actions \(G\) do not allow super-polynomial quantum speedups; that is, we provide tools for showing that every \(f\) symmetric with respect to \(G\) satisfies \(Q(f) = R(f)^\Omega(1)\). Our primary application is the following theorem, in which \(G\) is the graph symmetry: the group action acting on strings of length \(\binom{k}{2}\) (which represent the possible edges of a graph), which includes all permutations of the edges which are induced by one of the \(k!\) relabelings of the \(k\) vertices. Functions which take in the adjacency matrix of a graph as input, and whose output depends only on the graph (and not on the labeling of its vertices), are always symmetric with respect to the graph symmetry \(G\).

**Theorem 2** (Informal). Any Boolean function \(f\) defined on the adjacency matrix of a graph (and symmetric with respect to renaming the vertices of the graph) has a polynomial (power 6) relationship between \(R(f)\) and \(Q(f)\). This holds even if \(f\) is a partial function.

(For a formal version of this theorem, see Corollary 32.)

This theorem holds even when the alphabet of \(f\) is non-Boolean. We note that this is a strict generalization of the result of Aaronson and Ambainis \([AA14]\), since any fully-symmetric function will necessarily be symmetric under the graph symmetry as well. It is also a generalization of Chailloux \([Cha18]\), except that our polynomial degree (power 6) is larger than the power 3 of Chailloux.

This theorem also settles an open problem of Ambainis, Childs, and Liu \([ACL11]\) at least for the adjacency matrix version of graph property testing. They asked whether there is any graph property testing problem with an exponential quantum speedup over the best possible classical algorithm; our theorem implies the answer is no.\(^2\) Indeed, any graph property testing problem is always symmetric under the graph symmetry group action \(G\), which means that all graph property testing problems satisfy a power 6 relationship between their quantum and classical query complexities.

Our tools apply to other group actions as well. We show that highly transitive group actions also are not consistent with exponential quantum speedups.

**Theorem 3** (informal). Let \(G\) be a \(n^{O(1)}\)-transitive group action on \([n]\), and let \(f\) be a (possibly partial) Boolean function on strings of length \(n\) which is symmetric under \(G\). Then \(Q(f) = R(f)^\Omega(1)\).

(A formal version of this theorem can be found in Corollary 23.)

We note that this theorem does not subsume the previous one, as graph symmetry group actions are not even 2-transitive.

Further, we are able to generalize our results to directed graph symmetries, hypergraph symmetries, and bipartite graph symmetries. We also provide a reasonably clean framework in which we prove these results, and show that various natural operations on group actions preserve the lack-of-exponential-quantum-speedup property.

\(^2\)An alternative version of graph property testing is called the adjacency list model, in which it is possible to directly query the list of neighbors of each vertex. Query functions on adjacency list graphs can achieve large quantum speedups – for example, the glued trees problem \([CCD^+03]\) – but it is still open whether property testing graph problems in the adjacency list model can achieve exponential quantum speedups. We conjecture that such exponential speedups for graph property testing do exist – perhaps by a modification of the glued trees problem.
Finally, we examine the other direction, and exhibit some classes of group actions whose symmetries do allow exponential quantum speedups. In particular, we show the following.

**Theorem 4** (informal). The order of the group action does not characterize whether it allows exponential quantum speedups. In particular, there is a group action $G_n$ on $[n]$ and a different group action $H_n$ on $[n]$ such that $|G_n| = |H_n| = n^{\Omega(n)}$, but every function $f$ that’s symmetric under $G$ has $R(f) = O(Q(f)^3)$ while there exists a function $f$ that’s symmetric under $H$ and has $Q(f) = O(1)$ and $R(f) = n^{\Omega(1)}$.

(A formal version of this theorem can be found in Theorem 38.)

This theorem says that even very large group actions may still be consistent with exponential quantum speedups; to characterize the group actions which do not allow super-polynomial quantum advantage, we must use some richness measure other than the order of the group action (and also other than transitivity, as some 1-transitive group actions allow exponential quantum speedups and some don’t). We leave such a characterization as an intriguing open problem for future work.

**Open Problem 1.** Is there a clean combinatorial characterization of the classes of group actions $G$ that allow super-polynomial quantum speedups, and the ones that don’t? For example, is there a combinatorial measure $M(G)$ such that any function $f$ symmetric under $G$ satisfies something like $R(f) = Q(f)^{O(M(G))}$, and also such that there always exists a function $g$ symmetric under $G$ for which $R(g) = Q(g)^{\Omega(M(G))}$?

We view Open Problem 1 as an important direction for understanding the nature of quantum speedups.

### 1.3 Our techniques

Our main tool is a simple observation from [Cha18]. Suppose that $f$ is symmetric under the full symmetric group action $S_n$. Zhandry [Zha13] showed that distinguishing a random permutation from $S_n$ from a random small-range function $\alpha : [n] \to [n]$ with $|\alpha([n])| = r$ requires $\Omega(r^{1/3})$ quantum queries$^3$. Now, if $Q$ was a quantum algorithm solving $f$ using $T$ queries, then $Q$ also outputs $f(x)$ on input $x \circ \pi$ (the input $x$ with bits shuffled according to $\pi$) for any $\pi \in S_n$, since $f$ is symmetric under $S_n$. In particular, $Q(x \circ \pi)$ for a random $\pi \in S_n$ still outputs $f(x)$. However, the $T$-query algorithm $Q$ cannot distinguish a random $\pi \in S_n$ from a random function $\alpha : [n] \to [n]$ with range $r \approx T^3$; hence $Q$ must output $f(x)$ to constant error even when run on $x \circ \alpha$ for a random small-range function $\alpha$. This property can be used to simulate $Q$ classically: a classical algorithm $R$ will simply sample a small-range function $\alpha$, explicitly query the entire string $x \circ \alpha$ (possible to do using $O(T^3)$ queries since $\alpha$ has range only $O(T^3)$), and then simulate $Q$ on the string $x \circ \alpha$. This is an $O(T^3)$-query classical algorithm for computing $f$, created out of a $T$-query quantum algorithm for $f$.

The above trick can be generalized from the fully-symmetric group action $S_n$ to any other group action $G$, so long as we can show that it is hard for a $T$-query quantum algorithm to distinguish $G$ from a small-range function with range $O(\text{poly}(T))$. The question of whether there exists an arbitrary symmetric function $f$ with a quantum speedup is therefore reduced to the question of whether the concrete task of distinguishing $G$ from the set of all small-range functions can be done quickly using a quantum algorithm. That is, if $D_{n,r}$ is the set of all strings in $[n]^n$ which use only $r$ unique symbols, then we care about the quantum query cost of distinguishing $D_{n,r}$ from $G$; if this

---

$^3$Actually, we will show that a version of Zhandry’s result that is sufficient for our purposes follows easily from the collision lower bound, so his techniques are not necessary for our results.
cost is \( r^{\Omega(1)} \), then no function which is symmetric under \( G \) can exhibit a super-polynomial quantum speedup. In this case, we call \( G \) well shuffling. We show that the well-shuffling property is preserved under various operations one might perform on a group action, and that these operations allow us to prove many group actions are well-shuffling simply by reduction to \( S_n \).

2 Preliminaries

2.1 Query complexity

We start with introducing some standard notation from query complexity. A Boolean function will be a \( \{0,1\} \)-valued function \( f \) on strings of length \( n \), with \( n \in \mathbb{N} \). We will use \( \text{Dom}(f) \) to denote the domain of \( f \), and we will always have \( \text{Dom}(f) \subseteq \Sigma^n \) where \( \Sigma \) is a finite alphabet. The function \( f \) is called total if \( \text{Dom}(f) = \Sigma^n \), and otherwise it is called partial.

For a (possibly partial) Boolean function \( f \), we use \( R_\varepsilon(f) \) to denote its randomized query complexity to error \( \varepsilon \), as defined in [BdW02]. This is the minimum number of queries required in the worst case by a randomized algorithm which computes \( f \) to worst-case error \( \varepsilon \). We use \( Q_\varepsilon(f) \) to denote the quantum query complexity to error \( \varepsilon \) of \( f \), also defined in [BdW02]. This is the minimum number of queries required in the worst case by a randomized algorithm which computes \( f \) to worst-case error \( \varepsilon \). When \( \varepsilon = 1/3 \), we omit it and simply write \( R(f) \) and \( Q(f) \).

An important tool for lower bounding query complexity is the minimax theorem, the original version of which was given by Yao for zero-error (Las Vegas) randomized algorithms [Yao77]. Here we will need a bounded-error, quantum version of the minimax theorem. Bounded-error versions of the minimax theorem can be shown using linear programming duality (see also [Ver98] who proved a minimax theorem in the setting where both the error and the expected query complexity are measured against the same hard distribution). A similar technique works for quantum query complexity; this result is folklore, and we prove it in Appendix A.

Lemma 5 (Minimax for bounded error quantum algorithms). Let \( f \) be a (possibly partial) Boolean function with \( \text{Dom}(f) \subseteq \Sigma^n \), and let \( \varepsilon \in (0,1/2) \). Then there is a distribution \( \mu \) supported on \( \text{Dom}(f) \) which is hard for \( f \) in the following sense: any quantum algorithm using fewer than \( Q_\varepsilon(f) \) quantum queries for computing \( f \) must have average error \( > \varepsilon \) on inputs sampled from \( \mu \).

Note that achieving average error \( \varepsilon \) against a known distribution \( \mu \) is always easier than achieving worst-case error \( \varepsilon \); the minimax theorem says that there is a hard distribution against which achieving average error \( \varepsilon \) is just as hard as achieving worst-case error \( \varepsilon \).

2.2 Group actions

We review some basic definitions about group actions.

Definition 6 (Group action). A group action is a pair \( (D,G) \) where \( D \) is a set and \( G \) is a set of bijections \( \pi : D \rightarrow D \), such that \( G \) forms a group under composition (i.e. \( G \) contains the identity function and is closed under composition and inverse of the bijections). We will often denote a group action simply by \( G \), with the domain \( D \) being implicit.

In other words, a group action is simply a set of permutations of a domain \( D \) which is closed under composition and inverse. In this work we will generally take \( D = [n] \), where \( [n] \) denotes the set \( \{1,2,\ldots,n\} \) for \( n \in \mathbb{N} \). The set \( [n] \) will represent the indices of an input string, or equivalently, the queries an algorithm is allowed to make.

We define orbits and transitivity of group actions, both of which are standard definitions.
Definition 7 (Orbit). Let \( G \) be a group action on domain \( D \), and let \( i \in D \). Then the orbit of \( i \) is the set \( \{\pi(i) : \pi \in G\} \). A subset of \( D \) is an orbit of \( G \) if it is the orbit of some \( i \in D \) with respect to \( G \).

Definition 8 (Transitivity). We say that a group action \( G \) on domain \( D \) is \( k \)-transitive if for all distinct \( i_1, i_2, \ldots, i_k \in D \) and distinct \( j_1, j_2, \ldots, j_k \in D \), there exists some \( \pi \in G \) such that \( \pi(i_t) = j_t \) for all \( t = 1, 2, \ldots, k \).

2.3 Symmetric functions

We introduce some notation that will be used throughout this paper to talk about symmetric functions.

Definition 9 (Notation for permuting strings). Let \( \pi \) be a permutation on \([n]\), and let \( x \in \{0,1\}^n \). We use \( x \circ \pi \) to denote the string whose characters have been permuted by \( \pi \); that is, \( (x \circ \pi)_i := x_{\pi(i)} \). More generally, \( x \circ \pi \) is similarly defined when \( \pi \) is merely a function \([n] \rightarrow [n]\) rather than a permutation.

Note that if we view a string \( x \in \Sigma^n \) as a function \([n] \rightarrow \Sigma \) with \( x(i) := x_i \), then \( x \circ \pi \) is simply the usual function composition of \( x \) and \( \pi \). This notation allows us to easily define symmetric functions.

Definition 10 (Symmetric function). Let \( G \) be a group action on \([n]\), and let \( f \) be a (possibly partial) Boolean function with \( \text{Dom}(f) \subseteq \Sigma^n \). We say \( f \) is symmetric under \( G \) if for all \( x \in \text{Dom}(f) \) and all \( \pi \in G \) we have \( x \circ \pi \in \text{Dom}(f) \) and \( f(x \circ \pi) = f(x) \).

In order for asymptotic bounds such as \( Q(f) = R(f)^{\Omega(1)} \) to be well-defined, we actually need to talk about classes of functions rather than individual functions. To do that, we will need to talk about classes of group actions. We introduce the following definition, which defines, for a class of group actions \( \mathcal{G} \), the set of all functions symmetric under some group action in \( \mathcal{G} \). We denote this set by \( F(\mathcal{G}) \).

Definition 11 (Class of symmetric functions). Let \( \mathcal{G} = \{G_i\}_{i \in I} \) be a (possibly infinite) set of finite group actions, with \( G_i \) acting on \([n_i] \) for each \( i \in I \). Here \( I \) is an arbitrary index set and \( n_i \in \mathbb{N} \) for all \( i \in I \). Then define \( F(\mathcal{G}) \) to be the set of all (possibly partial) Boolean functions that are symmetric under some \( G_i \). That is, we have \( f \in F(\mathcal{G}) \) if and only if \( f : \text{Dom}(f) \rightarrow \{0,1\} \) is a function with \( \text{Dom}(f) \subseteq [m]^n \) for some \( n, m \in \mathbb{N} \), and \( f \) is symmetric under \( G_i \) for some \( i \in I \) such that \( n_i = n \).

(In the above definition, \([m] \) represents the alphabet \( \Sigma \).)

3 Well-shuffling group actions

In this section we first define the notion of a well-shuffling class of group actions, which will be a class \( \mathcal{G} \) of group actions \( G \) that are hard to distinguish from the set of small-range functions via a quantum query algorithm. We will then show that a well-shuffling class of group actions does not allow super-polynomial quantum speedups. This result (Theorem 15) converts the task of showing group actions do not allow quantum speedups into the task of showing those group actions are well-shuffling, a much simpler objective.

We start by defining the set of small-range strings \( D_{n,r} \).
Lemma 5

Definition 12 (Small-range strings). For $n, r \in \mathbb{N}$, let $D_{n,r}$ be the set of all strings $\alpha$ in $[n]^n$ for which the number of unique alphabet symbols in $\alpha$ is at most $r$.

We identify a string $\alpha \in [n]^n$ with a function $[n] \to [n]$. Then $D_{n,r}$ is the set of all functions $[n] \to [n]$ with range size at most $r$. Next, we define $\text{cost}(G, r)$ as the quantum query complexity of distinguishing $G$ from $D_{n,r}$ (where $G$ is a group action acting on $[n]$).

Definition 13 (Cost). Identify a permutation on $[n]$ with a string in $[n]^n$ in which each alphabet symbol occurs exactly once. Then a group action $G$ on $[n]$ corresponds to a subset of $[n]^n$. For $r < n$, let $\text{cost}_r(G, r)$ be the minimum number of quantum queries needed to distinguish $G$ from $D_{n,r}$ to worst-case error $\epsilon$; that is, $\text{cost}_r(G, r) := Q_\epsilon(f)$, where $f$ has domain $G \cup D_{n,r} \subseteq [n]^n$ and is defined by $f(x) = 1$ if $x \in G$ and $f(x) = 0$ if $x \in D_{n,r}$. When $r \geq n$, we set $\text{cost}_r(G, r) := \infty$. When $\epsilon = 1/3$, we omit it and write $\text{cost}(G, r)$.

We note that since $\text{cost}_r(G, r)$ is defined as the worst-case quantum query complexity of a Boolean function, it satisfies amplification, meaning that the precise value of $\epsilon$ does not matter so long as it is a constant in $(0,1/2)$ and so long as we do not care about constant factors.

We define a well-shuffling class of group actions as follows.

Definition 14 (Well-shuffling group actions). Let $G$ be a collection of group actions. We say $G$ is well-shuffling if $\text{cost}(G, r) = \Omega(1)$ for all $G \in G$ and $r \in \mathbb{N}$. More explicitly, we say $G$ is well-shuffling with power $a \in \mathbb{N}$ if there exists $b \in \mathbb{N}$ such that $\text{cost}(G, r) \geq r^{1/a}/b$ for all $G \in G$ and all $r \in \mathbb{N}$.

We note that $\text{cost}(G, r) \geq r^{1/a}/b$ is always satisfied when $r$ is greater than or equal to the domain size of $G$, since in that case $\text{cost}(G, r) = \infty$. Hence to show well-shuffling we only need to worry about $r$ smaller than $n$, the domain size of the group action $G$.

The following theorem will play a central role in this work: it shows that a well-shuffling collection of group actions does not allow super-polynomial quantum speedups.

Theorem 15. Let $f : \text{Dom}(f) \to \{0,1\}$ be a partial Boolean function on $n \in \mathbb{N}$ bits, with $\text{Dom}(f) \subseteq \Sigma^n$ (where $\Sigma$ is a finite alphabet). Let $G$ be a group action on $[n]$, and suppose that $f$ is symmetric under $G$. Then there is a universal constant $c \in \mathbb{N}$ such that

$$R(f) \leq \min\{ r \in \mathbb{N} : \text{cost}(G, r) \geq c Q(f) \}.$$ 

Consequently, if $G$ is a well-shuffling collection of group actions with power $a$, then for all $f \in F(G)$ we have $R(f) = O(Q(f)^a)$.

In order to prove this theorem, we will need the following minimax theorem for the cost measure.

Lemma 16 (Minimax for cost). Let $r, n \in \mathbb{N}$ satisfy $r < n$, let $\epsilon \in [0,1/2)$, and let $G$ be a group action on $[n]$. Then there is a distribution $\mu$ on $D_{n,r}$ that is hard in the following sense. Let $\mu'$ be the uniform distribution on $G \subseteq [n]^n$. Then any quantum algorithm for distinguishing $G$ from $D_{n,r}$ which uses fewer than $\text{cost}_r(G, r)$ queries must either make error $> \epsilon$ on average against $\mu$, or else make error $> \epsilon$ against $\mu'$ (i.e. it fails to distinguish $\mu$ from the uniform distribution on $G$).

Proof. Let $f$ be the function which asks to distinguish $G$ from $D_{n,r}$ in the worst case. Then by the minimax theorem (Lemma 5), there is a hard distribution $\nu$ for $f$, such that any quantum algorithm using fewer than $Q_\epsilon(f) = \text{cost}_r(G, r)$ queries must make more than $\epsilon$ error against $\nu$. Let $\nu'$ be the distribution we get by applying a uniformly random permutation from $G$ to a sample from $\nu$. Then $\nu'$ is still a hard distribution for $f$. Indeed, if it were not a hard distribution, there would be some quantum algorithm $Q$ solving $f$ against $\nu'$ using too few queries; but in that case,
Theorem 15

Theorem 15 is that we can show a class of group actions \( \mathcal{G} \) does not allow super-polynomial quantum speedups simply by showing that it is well-shuffling – that is, by showing that \( G \in \mathcal{G} \) is hard to distinguish from the set of small-range functions \( D_{n,r} \) using a quantum query algorithm.

Proof. (Of Theorem 15.) Let \( Q \) be a quantum algorithm for \( f \) which uses \( Q(f) \) queries. Amplify it to \( Q' \) by repeating 3 times and taking the majority vote; then it uses \( 3Q(f) \) queries and makes worst-case error \( 7/27 \) instead of \( 1/3 \). Using Lemma 16, let \( \mu \) be the hard distribution on \( D_{n,r} \) which is hard to distinguish from \( G \) to error \( \epsilon \), where we pick \( r \) later and pick \( \epsilon \) to be a constant close to \( 1/2 \). Sample \( \alpha \) from \( \mu \), and consider the string \( x \circ \alpha \) with \( (x \circ \alpha)_i = x_{\alpha(i)} \).

Now, \( Q' \) succeeds on \( f \) to error \( 7/27 \), and \( f \) is invariant under \( G \), so \( Q' \) outputs \( f(x) \) to error \( 7/27 \) when run on \( x \circ \pi \) for each \( \pi \in G \). In particular, consider picking \( \pi \) from \( G \) uniformly at random, and running \( Q' \) on \( x \circ \pi \) where the string \( x \in Dom(f) \) is fixed. Compare this to the behavior of \( Q' \) on \( x \circ \alpha \), where \( \alpha \) is sampled from the hard distribution \( \mu \) on \( D_{n,r} \).

If \( Q' \) did not output \( f(x) \) on \( x \circ \alpha \) to error at most \( 1/3 \), then we could convert \( Q' \) to an algorithm distinguishing \( \pi \) from \( \alpha \) with constant error. This is because \( Q' \) outputs \( f(x) \) to error at most \( 7/27 < 1/3 \) on input \( x \circ \pi \); hence \( Q' \) behaves differently when run on \( x \circ \alpha \) and on \( x \circ \pi \). We can convert \( Q' \) to an algorithm \( Q'' \) which hard codes the input \( x \), and receives either a random \( \pi \) from \( G \) or a random \( \alpha \) from \( \mu \) as input. This algorithm \( Q'' \) will only make \( 3Q(f) \) queries to \( \pi \) or \( \alpha \), but its acceptance probability differs by a constant gap between the two distributions, which (using some standard re-balancing) we can use to distinguish \( G \) from \( \mu \) to a constant error.

Now, assuming the distribution \( \mu \) was picked to be hard enough (i.e. \( \epsilon \) was chosen sufficiently close to \( 1/2 \)), this means that \( 3Q(f) \), the query cost of \( Q'' \), is at least \( \text{cost}_c(G, r) \). Since \( \text{cost}_c(G, r) \) is the worst-case quantum query complexity of a Boolean function, it can be amplified. We conclude that if \( Q' \) failed to output \( f(x) \) on input \( x \circ \alpha \) (with \( \alpha \leftarrow \mu \)) to error at most \( 1/3 \), then we have \( Q(f) = \Omega(\text{cost}(G, r)) \), that is, \( Q(f) > \text{cost}(G, r)/c \) for some universal constant \( c \) (from amplification).

Now assume that \( Q(f) \leq \text{cost}(G, r)/c \). Then \( Q' \) has error at most \( 1/3 \) for computing \( f(x) \) when run on \( \alpha(x) \), with \( \alpha \) chosen from \( \mu \). Since \( \alpha \in D_{n,r} \) uses at most \( r \) alphabet symbols, a randomized algorithm can simulate \( Q' \) simply by picking \( \alpha \) from \( \mu \) and querying all the \( r \) bits of \( x \) used in the string \( x \circ \alpha \), fully determining that string. This algorithm \( R \) uses \( r \) queries, and makes at most \( 1/3 \) error, so we conclude that \( R(f) \leq r \).

By correctly picking \( r \), we conclude that \( R(f) \leq \min \{ r \in \mathbb{N} : \text{cost}(G, r) \geq cQ(f) \} \), as desired. Finally, note that if \( \text{cost}(G, r) \geq r^{1/a} / b \), then by picking \( r = (bcQ(f))^a \) we get \( \text{cost}(G, r) \geq cQ(f) \). From this it follows that \( R(f) \leq (bcQ(f))^a = \mathcal{O}(Q(f)^a) \), as desired.

The upshot of Theorem 15 is that we can show a class of group actions \( \mathcal{G} \) does not allow super-polynomial quantum speedups simply by showing that it is well-shuffling – that is, by showing that \( G \in \mathcal{G} \) is hard to distinguish from the set of small-range functions \( D_{n,r} \) using a quantum query algorithm.
4 Showing group actions are well-shuffling

In this section, we introduce some tools for showing that a collection of group actions is well-shuffling. Due to Theorem 15, a well-shuffling collection of group actions does not allow any super-polynomial quantum speedups for the class of functions symmetric under it, so these tools can be directly used to show that certain symmetries are not consistent with large quantum speedups.

4.1 The symmetric group action

The first fundamental result is that the class of full symmetric group actions $S_n$ is well-shuffling. This was shown by Zhandry [Zha13] in a different context, though we also provide a simpler proof by a reduction from the collision problem.

**Theorem 17.** There is a universal constant $C$ such that any quantum algorithm distinguishing a permutation in $S_n$ from a string in $D_{n,r}$ must make at least $r^{1/3}/C$ queries.

This theorem says that $S_n$ is hard to distinguish from $D_{n,r}$ (moreover, Zhandry [Zha13] showed that the hard distribution over $D_{n,r}$ is uniform, but we do not need this fact).

**Proof.** When $n$ is a multiple of $r$, then each $(n/r)$-to-1 function has range $r$ and each 1-to-1 function is a permutation; hence distinguishing $(n/r)$-to-1 from 1-to-1 functions is a sub-problem of distinguishing $D_{n,r}$ from $S_n$. This sub-problem is the collision problem, from which an $\Omega(r^{1/3})$ lower bound directly follows [AS04, Amb05, Kut05]. When $n$ is not a multiple of $r$ but $r \leq n/2$, we can just set $n' = r\lceil n/r \rceil$, and then distinguishing $(n'/r)$-to-1 from 1-to-1 functions with domain size $n'$ still reduces to distinguishing $D_{n,r}$ from $S_n$.  

From Theorem 17, the following two corollaries immediately follow (in light of Theorem 15).

**Corollary 18.** The set of symmetric group actions $S = \{S_n\}_{n \in \mathbb{N}}$ is well-shuffling with power 3.

**Corollary 19.** All (possibly partial) Boolean functions $f$ that are symmetric under the full symmetric group action $S_n$ satisfy $R(f) = O(Q(f)^3)$.

Apart from Theorem 17, the main tools we use to prove the well-shuffling property are transformations on group actions which approximately preserve cost$(G, r)$. We outline several such transformations and invariances. Since we prove Theorem 17 by a reduction from collision, and since our main tools from here on out are additional reductions, it’s effectively the case that all lower bounds in this paper work by reductions from collision.

4.2 The case of highly-transitive group actions

We next show that a collection of highly-transitive group actions is always well-shuffling; we define the notion of highly-transitive collections below.

**Definition 20** (Highly transitive). We say a collection $\mathcal{G}$ of group actions is highly transitive if each group action $G \in \mathcal{G}$ is $r^{\Omega(1)}$-transitive, where $n$ is the domain size of $G$. In other words, $\mathcal{G}$ is highly transitive if there exist some constants $a, b \in \mathbb{N}$ such that each $G \in \mathcal{G}$ is $(n^{1/a}/b)$-transitive, where $n$ is the domain size of $G$.

To show that highly transitive collections of group actions are well-shuffling, we show that group actions with high transitivity look nearly indistinguishable from $S_n$ to any quantum algorithm, and that they therefore share the well-shuffling property of the group actions $S_n$. More formally, we have the following theorem.
Theorem 21 (Similar-looking group actions have similar costs). Suppose $G$ and $H$ are group actions on $[n]$ and $k \leq n$ is a positive integer such that for each $i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_k \in [n]$, it holds that
\[
\left| \Pr_{\pi \leftarrow G} [\forall \ell \pi(i_\ell) = j_\ell] - \Pr_{\pi \leftarrow H} [\forall \ell \pi(i_\ell) = j_\ell] \right| \leq n^{-10k}.
\]
Then $\text{cost}(H, r) \geq \Omega(\min\{k, \text{cost}(G, r)\})$. In particular, if $k \geq n^{\Omega(1)}$ and if $\text{cost}(G, r) \geq r^{\Omega(1)}$, then we have $\text{cost}(H, r) \geq r^{\Omega(1)}$.

Proof. Let $Q$ be a quantum algorithm for distinguishing $H$ from $D_{n,r}$ which uses $\text{cost}(H, r)$ and achieves worst-case error $1/3$. If $\text{cost}(H, r) \geq k$, we are done, so assume $\text{cost}(H, r) < k$. Now, $Q$ can be converted into a polynomial of degree at most $2\text{cost}(H, r)$ in the variables $z_{ij}$, where $z_{ij} = 1$ if the input $x$ satisfies $x_i = j$ and otherwise $z_{ij} = 0$ (see [AS04]). This polynomial $p$ satisfies $p(x) \in [0, 1/3]$ if $x \in D_{n,r}$ and $p(x) \in [2/3, 1]$ if $x \in H$. It has $n^2$ variables and degree $d = 2\text{cost}(H, r)$. We assume it has no monomials that always evaluate to 0 (for example, $z_{11}z_{12}$, which is always 0 as $x_1$ cannot be both 1 and 2), because if it had such monomials we could just delete them.

We claim that the sum of absolute values of coefficients of $p$ is at most $n^{3d}$, where $d = 2\text{cost}(H, r)$ is its degree. To see this, first note that there are at most $\binom{n^2}{d}$ monomials of $p$ of degree $d$; for each such monomial $m$, let $p_m$ be the polynomial consisting of all terms in $p$ that use a subset of the variables in $m$. Then the sum of the absolute values of the coefficients of $p$ is at most $\binom{n^2}{d}$ times the maximum sum of absolute values of the coefficients in one of the polynomials $p_m$; since $\binom{n^2}{d} \leq n^{3d}$, it suffices to upper bound the sum of absolute values of coefficients of $p_m$ for arbitrary $m$. Now, $m$ consists of $d$ variables $z_{itj}$, for $t = 1, 2, \ldots, d$, which equal 1 when $x_{it} = j_t$ and equal 0 otherwise. Consider feeding into the quantum algorithm an input string where $x_i = \ast$ when $i \notin \{i_1, i_2, \ldots, i_d\}$, and $x_{it}$ is either $j_t$ or $\ast$ for $t = 1, 2, \ldots, d$. The quantum algorithm will accept the string with some probability between 0 and 1, which means the polynomial $p$ computing the acceptance probability of $Q$ will evaluate to something between 0 and 1. But such inputs “zero out” all terms that use variables outside of $m$, and hence turn $p$ into $p_m$. From this we can conclude that $p_m$ is bounded in $[0, 1]$ for all inputs it receives in $\{0, 1\}^d$. But polynomials bounded in $[0, 1]$ on the Boolean hypercube can have sum of coefficients at most $5^d$ (one way to analyze would be to recall that a bounded polynomial in the $\{-1, 1\}$ basis has its sum of squares of coefficients equal to at most 1, and has at most $2^d$ coefficients, so by Cauchy-Schwartz, the sum of absolute values of coefficients is at most $2^d/2$; converting the $\{-1, 1\}$ basis to the $\{0, 1\}$ basis requires plugging in $(2z - 1)$ terms into the variables, which can increase the sum of absolute values by a factor of at most $3^d$, for a total of at most $(3\sqrt{2})^d \leq 5^d$). Assuming $n \geq 5$, we get an upper bound of $n^{3d}$ on the sum of absolute values of coefficients of $p$.

We have $d \leq 2k$, so this sum is also at most $n^{6k}$. Now, on each input $x$, the expected output of $p(\pi(x))$ when $\pi$ is sampled uniformly from $H$ is a linear combination of the expectations of the monomials of $p$. For each monomial, this expectation is just the probability that the monomial is satisfied, which by the condition on $G$ and $H$ is within $n^{-10k}$ of the expectation under $\pi \leftarrow G$. It follows that the expectation of $p(\pi(x))$ when $\pi \leftarrow H$ is within $n^{6k}n^{-10k} = n^{-4k}$ of the expectation of $p(\pi(x))$ when $\pi \leftarrow G$. But this expectation is simply the acceptance probability of $Q$. Hence the acceptance probability of $Q$ on the uniform distribution on $H$ is within $n^{-4k}$ of the acceptance probability of $Q$ on the uniform distribution on $G$.

Since $Q$ distinguishes $H$ from $D_{n,r}$, it distinguishes the uniform distribution on $H$ from any string in $D_{n,r}$. Since it does not distinguish the uniform distribution on $H$ from the uniform distribution on $G$, $Q$ must also distinguish the uniform distribution on $G$ from any input in $D_{n,r}$.
to error $1/3 + n^{-4k}$. By amplifying, we can get this down to error $1/3$, meaning that $\cost(G, r) = O(\cost(H, r))$, as desired.

To show that highly-transitive group actions are well-shuffling, we now only need to show that a $k$-transitive group action $G$ looks like $S_n$ when examining any $k$ bits. This directly follows from the definition of transitivity.

**Corollary 22.** If $G$ is $k$-transitive, then $\cost(G, r) = \Omega(\min\{k, r^{1/3}\})$, where the constant in the big-$\Omega$ is universal.

**Proof.** This follows directly from Theorem 21, setting $H$ to be the $k$-transitive group action we care about and setting $G = S_n$. To see this, observe that $k$-transitivity completely determines $\Pr_{\pi \sim \mathcal{G}}[\forall \ell \in [k] \; \pi(i_\ell) = j_\ell]$, and that both $G$ and $H$ are $k$-transitive; hence this expression is the same for both $G$ and $H$, and the difference between the two expressions is exactly 0 (certainly less than $n^{-10k}$).

From this, the formal version of Theorem 3 follows.

**Corollary 23.** If $\mathcal{G}$ is a highly transitive collection of group actions, then it is well-shuffling, and hence $R(f) = O(\poly(Q(f)))$ for $f \in F(\mathcal{G})$.

### 4.3 Transformations for graph symmetries

Next, we introduce some additional transformations on group actions which approximately preserve the cost; the transformations in this section will allow us to show that graph property group actions (and several variants of them) are well-shuffling.

#### 4.3.1 Transformation for directed graphs

We start by defining an extension of a group action $G$ on $[n]$ to an action on $[n]^\ell$. The notation in the definition below comes from [Ker13].

**Definition 24.** Let $G$ be a group action on domain $D$, and let $\ell \in \mathbb{N}$. Define $G^{(\ell)}$ to be the group action which acts on domain $D^\ell$ by $\pi(i_1, i_2, \ldots, i_\ell) = (\pi(i_1), \pi(i_2), \ldots, \pi(i_\ell))$ for each $\pi \in G$ (so the number of permutations in $G^{(\ell)}$ is the same as the number of permutations in $G$).

Define $G^{<\ell>}$ to be the group action $G^{(\ell)}$ with domain restricted to the subset $D^{<\ell>} \subseteq D^\ell$ consisting of all distinct $\ell$-tuples of elements of $D$.

We show that these transformations both preserve the cost, at least when $\ell$ is constant. We start with $G^{(\ell)}$.

**Theorem 25.** Let $G$ be a group action on $[n]$, and let $H$ be the group action $G^{(\ell)}$. Then we have $\cost(H, r^\ell) \geq \cost(G, r)/\ell$.

**Proof.** Let $Q$ be an algorithm distinguishing $H$ from $D_{n, r, \ell}$. Let $\mu$ be the hard distribution for $G$, such that no algorithm using fewer than $\cost(G, r)$ can distinguish $\mu$ from the uniform distribution on $G$. Then $\mu$ is a distribution on $D_{n, r}$. Let $\mu'$ be the distribution on $D_{n, r, \ell}$ that we get by sampling $\alpha \leftarrow \mu$, and returning $\alpha'$ defined by $\alpha'(z) = (\alpha(z_1), \alpha(z_2), \ldots, \alpha(z_\ell))$ for each $z \in [n]^\ell$ (here we identify $[n]^\ell$ with $[n]^\ell$). Note that if $\alpha$ has range $r$, then $\alpha'$ has range at most $r^\ell$.

Then $Q$ distinguishes $\mu'$ from the uniform distribution on $H$. The latter distribution is the same as what you get when sampling $\pi$ uniformly from $G$, and returning $\pi'$ defined by $\pi'(z) = (\pi(z_1), \pi(z_2), \ldots, \pi(z_\ell))$ for $z$ in the domain of $H$. This means that $Q$ can be used to distinguish $\mu'$
from the uniform distribution on $G$: all we need is to simulate every query of $Q$ using $\ell$ queries to the input $\alpha$. The desired result follows. \hfill \Box

To handle $G^{<\ell>}$, we first observe that restricting the domain of a group action to some union of its orbits does not decrease its cost.

**Lemma 26.** Let $G$ be a group action on $[n]$, and let $S \subseteq [n]$ be a union of orbits of $G$. Let $G'$ be the group action $G$ acting only on $S$. Then $\text{cost}(G') \geq \text{cost}(G)$.  

**Proof.** We identify $S$ with $|S|$ without loss of generality. If $Q$ distinguishes $G'$ from $D_{|S|,r}$, then we can turn it into $Q'$ distinguishing $G$ from $D_{n,r}$ by having $Q'$ run $Q$ and make queries only from $|S|$. \hfill \Box

The fact that $G^{<\ell>}$ does not decrease the cost of $G$ too much then follows as a corollary of Theorem 25 and Lemma 26.

**Corollary 27.** Let $G$ be a group action on $[n]$, and let $H$ be the group action $G^{<\ell>}$. Then $\text{cost}(H, r^\ell) \geq \text{cost}(G, r)/\ell$.  

**Proof.** All we need is to note that $G^{<\ell>}$ is the group action $G^{(\ell)}$ with domain restricted to $[n]^{<\ell>}$, which is a union of orbits because $\pi \in G^{(\ell)}$ always sends a tuple with unique entries to another tuple with unique entries (since a permutation on $[n]$ is applied to each entry). The desired result then follows from Theorem 25 and Lemma 26. \hfill \Box

We now observe that the transformation $G^{<\ell>}$ immediately allows us to show that directed graph symmetries are well-shuffling.

**Corollary 28** (Directed graph symmetries). The set $G = \{G_k\}_{k \in \mathbb{N}}$ of all directed graph symmetries is well-shuffling with power 6. Here the group action $G_k$ acts on a domain of size $n = k(k - 1)$ representing the possible arcs of a $k$-vertex directed graph, and $G_k$ consists of all $k!$ permutations on these arcs that act by relabeling the vertices.

**Proof.** This immediately follows by observing that $G_k = S_k^{<2>}$. To see this, note that the domain of $G_k$ is the set of all ordered pairs $(x, y) \in [k]$ with $x \neq y$, which is precisely $[k]^{<2>}$, and the permutations in $G_k$ are just those in $S_k$ applied to both coordinates, which is precisely relabeling the vertices. Corollary 27 then gives $\text{cost}(G_k, r) \geq \text{cost}(S_k, \sqrt{r})/2$, which is at least $\Omega(r^{1/6})$ by Corollary 17. \hfill \Box

The collection of directed hypergraph symmetries is similarly well-shuffling.

**Corollary 29** (Directed hypergraph symmetries). The set $G_p = \{G_k\}_{k \in \mathbb{N}}$ consisting of all $p$-uniform directed hypergraph symmetries is well-shuffling with power $3p$.  

**Proof.** This follows from the same argument as Corollary 28. \hfill \Box

### 4.3.2 Transformation for undirected graphs

To handle undirected graphs, we introduce yet another operation on group actions which approximately preserves the cost.

**Theorem 30.** Let $G$ be a group action acting on $[n]$, and let $S_1, S_2, \ldots, S_k \subseteq [n]$ be a partition of $[n]$ into $k$ equal parts, such that for all $\pi \in G$, all $t \in [k]$, and all $i, j \in S_t$, the outputs $\pi(i)$ and $\pi(j)$ lie in the same set $S_t$. Let $H$ be the group action on $[k]$ induced by $G$, where for each $\pi \in G$ we have $\pi' \in H$ such that $\pi'(t) = t'$ if $i \in S_t$ and $\pi(i) \in S_{t'}$. Then $\text{cost}(H, r) \geq \text{cost}(G, r)$.  

Proof. Let $Q$ be a quantum algorithm distinguishing $H$ from $D_{k,r}$ using $\text{cost}(H,r)$ queries. We construct an algorithm $Q'$ for distinguishing $G$ from $D_{n,r}$. Let $\mu$ be the hard distribution on $D_{n,r}$ that is hard to distinguish from the uniform distribution on $G$. The algorithm $Q'$ fixes a unique $i_t \in S_t$ for each $t = 1, 2, \ldots, k$. On input $\alpha$ from $G \cup D_{n,r}$, the algorithm $Q'$ will run $Q$ in the following way: each query $t \in [k]$ that $Q$ makes will be turned into the query $i_t \in [n]$ for $\alpha$, and the output $\alpha(i_t)$ will be converted into the symbol $t'$ such that $\alpha(i_t) \in S_{t'}$ and returned to $Q$. In this way, the algorithm $Q'$ effectively runs $Q$ on the mapped string $\phi(\alpha) \in [k]^k$, where $\phi(\alpha)_t$ is the symbol $t'$ such that $\alpha(i_t) \in S_{t'}$.

Now, if $\alpha \in D_{n,r}$, then $\phi(\alpha) \in D_{k,r}$, while if $\alpha \in G$, we have $\phi(\alpha) \in H$. Since $Q$ distinguishes $H$ from $D_{k,r}$, it follows that $Q'$ distinguishes $G$ from $D_{n,r}$ using the same number of queries, as desired. 

We are now finally ready to prove the formal version of Theorem 2, showing that the collection of (undirected) graph symmetries is well-shuffling.

**Definition 31** (Graph Symmetries). The collection of graph symmetries is the set $G = \{G_k\}_{k \in \mathbb{N}}$ of group actions with $G_k$ acting on $[n]$ with $n = k(k-1)/2$, such that the domain $[n]$ represents the set of all possible edges in a $k$-vertex graph, and $G_k$ acts on these edges and permutes them in a way that corresponds to relabelling the vertices of the underlying graph.

**Corollary 32.** The set of all graph symmetries is well-shuffling with power 6. Hence $R(f) = O(Q(f)^6)$ for functions $f$ symmetric under a graph symmetry.

**Proof.** Let $G$ be a directed graph symmetry on domain size $k(k-1)$, and partition this domain into $k(k-1)/2$ sets of size 2 of the form $\{(x,y), (y,x)\}$ for $x, y \in [k]$. Then the induced group action $H$ on these sets (from Theorem 30) is precisely the undirected graph symmetry on graphs of size $k$. Since $\text{cost}(H,r) \geq \text{cost}(G,r)$, and since the directed graph symmetries are well-shuffling with power 6, it follows that the undirected graph symmetries are also well-shuffling with power 6. 

Using similar arguments, we can show a similar result for hypergraphs.

**Corollary 33.** For every constant $p \in \mathbb{N}$, the collection of all $p$-uniform hypergraph symmetries is well-shuffling.

### 4.3.3 Transformations for bipartite graphs

We introduce yet more operations on group actions for the case of bipartite graph symmetries.

**Definition 34** (Product of group actions). Let $G_1$, and $G_2$ be two group actions acting on $[n_1]$ and $[n_2]$ respectively. Then the group product action $G_1 \times G_2$ is a group action acting on $[n_1 n_2]$ such that for any $(\pi_1, \pi_2) \in G_1 \times G_2$, and any $k \in [n_1]$ and $\ell \in [n_2]$ we have $(\pi_1, \pi_2)(k, \ell) = (\pi_1(k), \pi_2(\ell))$.

(In the above, we identify $[n_1] \times [n_2]$ with $[n_1 n_2]$.)

**Theorem 35.** For all $G_1, G_2$ acting on $[n_1]$ and $[n_2]$ respectively and for all $r$, $\text{cost}(G_1 \times G_2, r^2) \geq \min\{\text{cost}(G_1, r), \text{cost}(G_2, r)\}$.

**Proof.** Let $H = G_1 \times G_2$ and $m = n_1 n_2$. Let $Q$ be an algorithm distinguishing $H$ from $D_{m,r^2}$. Let $\mu_1$ be the hard distribution for $G_1$, and let $\mu_2$ be the hard distribution for $G_2$. Then $\mu_1$ is a distribution on $D_{n_1,r}$ and $\mu_2$ is a distribution on $D_{n_2,r}$. Let $\mu'$ be the distribution on $D_{m,r}$ that we get by sampling $\alpha_1 \leftarrow \mu_1$, and $\alpha_2 \leftarrow \mu_2$ independently, and returning $\alpha' = (\alpha(z_1), \alpha(z_2))$. Note that if $\alpha_1$ and $\alpha_2$ have range $r$, then $\alpha'$ has range at most $r^2$. Now, since $Q$ distinguishes $D_{m,r^2}$
from $G_1 \times G_2$, it must also distinguish $\mu'$ from the uniform distribution over $G_1 \times G_2$, which itself is the product of the uniform distribution on $G_1$ and the uniform distribution on $G_2$. Let $\nu_1$ be the uniform distribution on $G_1$, and let $\nu_2$ be the uniform distribution on $G_2$. Consider the behavior of $Q$ on $\mu_1 \times \nu_2$. It must either distinguish this distribution from $\mu_1 \times \mu_2$, or else from $\nu_1 \times \nu_2$ (since it distinguishes $\mu_1 \times \mu_2$ and $\nu_1 \times \nu_2$ from each other). In the first case, we can construct $Q'$ which artificially generates the sample from $\mu_1$ and uses $Q$ to distinguish $\mu_2$ from $\nu_2$. In the second case, we can construct $Q'$ which artificially generates the sample from $\nu_2$ and uses $Q$ to distinguish $\mu_1$ from $\nu_1$. Hence $\text{cost}(G_1 \times G_2, r^2) \geq \min\{\text{cost}(G_1, r), \text{cost}(G_2, r)\}$, as desired.

Corollary 36. The collection $\mathcal{G}$ of all bipartite graph symmetries with equal parts is well-shuffling.

Proof. This immediately follow by observing that bipartite graph symmetries are the symmetries $S_k \times S_k$. Then Theorem 35 and Theorem 17 give the desired result.

4.3.4 Other transformations

We introduce one final transformation, which merges two group actions into one. This transformation also does not decrease the cost. While we have no direct application for it, we will mention this transformation in some discussion in the next section.

Lemma 37 (Merger). Let $G$ and $H$ be two group actions on $[n]$, and let $F = \langle G, H \rangle$ be the group action on $[n]$ which is the closure of $G \cup H$ under composition. Then $\text{cost}(F, r) \geq \text{cost}(G, r)$.

Proof. Since $G$ is a subset of $F$, distinguishing $G$ from $D_{n,r}$ is strictly easier than distinguishing $F$ from $D_{n,r}$.

5 Group actions with exponential quantum speedups

In this section, we exhibit some group actions that do allow super-polynomial quantum speedups. These serve as a barrier to proving that certain natural classes of group actions are well-shuffling.

To start, note that some of the most well-known examples of exponential quantum speedups in query complexity already have some mild symmetries.

- Period finding (the query task behind Shor’s algorithm) gives a periodic string and asks for the period; see [Sho94, Cle04] for a full definition. This function is symmetric under the cyclic group action $Z_n$.

- Simon’s problem [Sim97] promises that the input string $x$ represents a function with a hidden shift $s \in \{0, 1\}^{\log n}$, such that $x_i = x_j$ if and only if $i = j$ or $i \oplus j = s$, and asks to find the hidden shift $s$. It is not hard to convert this to a decision problem by requiring the function to output only one bit of information about $s$; in this form, Simon’s problem is symmetric under the group action which permutes $[n]$ by flipping some bits of the binary representation of each $i \in [n]$, i.e. the group action $Z_2^{\log n}$.

- In Forrelation, the input takes the form of two strings $x$ and $y$ of length $n/2$ each, and the task is to estimate the sum $\sum_{i,j \in [n/2]} (-1)^{(i,j)} x_i y_j$; for a full definition, see [AA15]. Forrelation is symmetric under the group action which permutes the bits in the binary representation of each $i$ (the group $S_{\log n}$).
Another way to convert Simon’s problem to a decision problem is to define a Boolean function $f$ which outputs 1 on strings that satisfy the Simon promise and outputs 0 on strings that are far from satisfying the promise; see [BFNR08] for a full definition. This version of Simon’s problem is symmetric under the group action that can both flip the individual bits of $i$ and permute them, a group action of order $n \cdot (\log n)!$ which is the merger of the group actions $S_{\log n}$ and $Z_2^{\log n}$ above.

While these examples are not exhaustive, other functions with exponential quantum speedups tend to have a similar flavor, being symmetric under group actions which contain only poly($n$) or maybe $n^{O(\log n)}$ permutations instead of the maximum of $n^{O(n)}$.

This might suggest that in order to get an exponential quantum speedup we always need mild symmetries, with the order of the group action being small (compared to the maximum of $n! = n^{\Theta(n)}$). However, this turns out not to be true. Theorem 38 demonstrates that even very large group actions may still be consistent with exponential quantum speedups. This means that a characterization of the group actions which do not allow super-polynomial quantum advantage must use some richness measure other than the order of the group action.

**Theorem 38** (Exponential quantum speedup with high symmetry). For infinitely many $n \in \mathbb{N}$, there is a group action $H_n$ acting on $[n]$ such that $|H_n| = n^{\Omega(n)}$, and yet there exists a function $f$ that’s symmetric under $H_n$ and has $Q(f) = O(1)$ and $R(f) = n^{\Omega(1)}$.

**Proof.** Consider the function $f = \text{For}_{\sqrt{n}} \circ \text{Triv}_{\sqrt{n}}$ that we obtain by composing the Forrelation function $\text{For}$ (as defined in [AA14]) of input size $\sqrt{n}$ with $\sqrt{n}$ copies of the trivial function $\text{Triv}$ of size $\sqrt{n}$. The function $\text{Triv}_m$ is a promise problem that only takes two inputs, $0^m$ and $1^m$. It outputs 0 on $0^m$ and 1 on $1^m$.

Observe that $R(\text{Triv}) = Q(\text{Triv}) = 1$ (even for computing this function exactly). From [AA14], we have $Q(\text{For}_m) = 1$ and $R(\text{For}_m) = \tilde{\Omega}(\sqrt{m})$. By composing the quantum algorithm for $\text{For}_{\sqrt{n}}$ with the exact quantum algorithm for $\text{Triv}$, we get that $Q(f) = 1$. On the other hand, by taking the hard distribution for $R(\text{For}_{\sqrt{n}})$ and replacing each 0 of the input with $0^{\sqrt{n}}$ and each 1 with $1^{\sqrt{n}}$, we clearly get a distribution over inputs to $f$ that is hard for randomized algorithms; it follows that $R(f) = \tilde{\Omega}(n^{1/4})$.

On the other hand, we claim that $f$ is highly symmetric. Indeed, each copy of $\text{Triv}_{\sqrt{n}}$ is symmetric under the group action $S_{\sqrt{n}}$, which has size $\sqrt{n}^{\Theta(\sqrt{n})} = n^{\Theta(\sqrt{n})}$. Since there are $\sqrt{n}$ copies of $\text{Triv}_{\sqrt{n}}$, they are together symmetric under $S_{\sqrt{n}} \times S_{\sqrt{n}} \times \cdots \times S_{\sqrt{n}} = \left(S_{\sqrt{n}}\right)^{\sqrt{n}}$, a group action of order $\left(n^{\Theta(\sqrt{n})}\right)^{\sqrt{n}} = n^{\Theta(n)}$. \qed

This theorem tells us that while we would like to say that sufficiently “rich” group actions do not allow exponential quantum speedups, such a richness notion cannot simply be the order of the group. We note that such a richness notion also cannot be transitivity: some 1-transitive group actions allow exponential quantum speedups and some don’t. For example, as we have seen, graph properties are 1-transitive and yet functions symmetric under graph property group actions do not exhibit exponential quantum speedups. On the other hand, cyclic group actions are 1-transitive and they are consistent with exponential quantum speedups (e.g. period finding). Characterizing richness necessary for a group action to disallow exponential quantum speedups remains a fascinating open problem.
Acknowledgements

We thank Scott Aaronson for many helpful discussions.

A Proof of the quantum minimax lemma

We prove Lemma 5, which we restate below.

**Lemma 5** (Minimax for bounded error quantum algorithms). Let $f$ be a (possibly partial) Boolean function with $\text{Dom}(f) \subseteq \Sigma^n$, and let $\epsilon \in (0, 1/2)$. Then there is a distribution $\mu$ supported on $\text{Dom}(f)$ which is hard for $f$ in the following sense: any quantum algorithm using fewer than $Q_\epsilon(f)$ quantum queries for computing $f$ must have average error $> \epsilon$ on inputs sampled from $\mu$.

**Proof.** By [BSS03], there is a finite bound $B$ expressible in terms of $n$ and $|\Sigma|$ on the necessary size of the work space register for a quantum algorithm solving $f$ to error $\epsilon$. This means the quantum query algorithms we deal with can be assumed without loss of generality to have work space size $B$. A quantum algorithm making $T$ queries can be represented as a sequence of $T$ unitary matrices of size upper bounded by $B$; this can be arranged as a finite vector of complex numbers. It is not hard to see that the set of all such valid quantum algorithms is a compact set.

For a quantum algorithm $Q$, let $\text{err}(Q, x)$ denote the error $Q$ makes when run on input $x \in \text{Dom}(x)$; this is $\Pr[Q(x) \neq f(x)]$, where $Q(x)$ is the random variable for the measured output of $Q$ when run on $x$. We note that $\text{err}(Q, x)$ is a continuous function of $Q$. Let $v_Q$ be the vector in $\mathbb{R}^{|\text{Dom}(f)|}$ defined by $v_Q[x] := \text{err}(Q, x)$. Then $v_Q$ is a continuous function of $Q$. Further, let $V$ be the set of all such vectors $v_Q$ for valid quantum algorithms $Q$ which make at most $Q_\epsilon(f) - 1$ queries. Since the set of such valid quantum algorithms is compact and since $v_Q$ is continuous in $Q$, we conclude that $V$ is compact. Furthermore, we claim that $V$ is convex: this is because for any two quantum algorithms $Q$ and $Q'$, there is a quantum algorithm $Q''$ which behaves like their mixture (in terms of its error on each input $x$).

Next, let $\Delta \subseteq \mathbb{R}^{||\text{Dom}(f)||}$ be the set of all probability distributions over $\text{Dom}(f)$. Then $\Delta$ is also convex and compact. Finally, define $\alpha : V \times \Delta \to \mathbb{R}$ by $\alpha(v, \mu) := \mathbb{E}_{x \leftarrow \mu} v[x] = \sum_{x \in \text{Dom}(f)} \mu[x] v[x]$. Then $\alpha$ is continuous in each coordinate, and is *saddle*: that is, $\alpha(\cdot, \mu)$ is convex for each $\mu \in \Delta$ (indeed, it is linear), and $\alpha(v, \cdot)$ is concave for each $v \in V$ (indeed, it is also linear). A standard minimax theorem (e.g. [Sio58]) then gives us

$$\min_{v \in V} \max_{\mu \in \Delta} \alpha(v, \mu) = \max_{\mu \in \Delta} \min_{v \in V} \alpha(v, \mu).$$

For the left hand side, it is clear that the maximum over $\mu$ (once the vector $v$ has been chosen) is the same as the maximum over $x \in \text{Dom}(f)$ of $v[x]$. This makes the left hand side the minimum over $v \in V$ of $||v||_\infty$, or equivalently, the minimum worst-case error of quantum algorithms making at most $Q_\epsilon(f) - 1$ queries. By the definition of $Q_\epsilon(f)$, this minimum must be strictly greater than $\epsilon$ (or else $Q_\epsilon(f)$ would be smaller). Hence the left hand side is strictly greater than $\epsilon$.

Looking at the right hand side, we see that we get a single distribution $\mu$ such that every quantum algorithm $Q$ making at most $Q_\epsilon(f) - 1$ queries must make error against $\mu$ which is greater than $\epsilon$, as desired. \qed
References

[AA14] Scott Aaronson and Andris Ambainis. The need for structure in quantum speedups. *Theory of Computing*, 10:133–166, 2014. URL: http://theoryofcomputing.org/articles/v010a006/.

[AA15] Scott Aaronson and Andris Ambainis. Forrelation: A problem that optimally separates quantum from classical computing. In *Proceedings of the Forty-Seventh Annual ACM on Symposium on Theory of Computing*, pages 307–316. ACM, 2015.

[AB16] Scott Aaronson and Shalev Ben-David. Sculpting quantum speedups. In *31st Conference on Computational Complexity (CCC)*, pages 26:1–26:28, 2016. doi:10.4230/LIPIcs.CCC.2016.26.

[ACL11] Andris Ambainis, Andrew M Childs, and Yi-Kai Liu. Quantum property testing for bounded-degree graphs. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques*, pages 365–376. Springer, 2011.

[ACL+19] Scott Aaronson, Nai-Hui Chia, Han-Hsuan Lin, Chunhao Wang, and Ruizhe Zhang. On the quantum complexity of closest pair and related problems. *arXiv preprint arXiv:1911.01973*, 2019.

[Amb05] Andris Ambainis. Polynomial degree and lower bounds in quantum complexity: Collision and element distinctness with small range. *Theory of Computing*, 1(1):37–46, 2005.

[AS04] Scott Aaronson and Yaoyun Shi. Quantum lower bounds for the collision and the element distinctness problems. *Journal of the ACM*, 51(4):595–605, July 2004. URL: http://doi.acm.org/10.1145/1008731.1008735, doi:10.1145/1008731.1008735.

[BBC+01] Robert Beals, Harry Buhrman, Richard Cleve, Michele Mosca, and Ronald De Wolf. Quantum lower bounds by polynomials. *Journal of the ACM (JACM)*, 48(4):778–797, 2001. arXiv:quant-ph/9802049, doi:10.1145/502090.502097.

[BdB02] Harry Buhrman and Ronald de Wolf. Complexity measures and decision tree complexity: a survey. *Theoretical Computer Science*, 288(1):21–43, 2002. doi:10.1016/S0304-3975(01)00144-X.

[Ben16] Shalev Ben-David. The structure of promises in quantum speedups. In *11th Conference on the Theory of Quantum Computation, Communication and Cryptography (TQC)*, pages 7:1–7:14, 2016. doi:10.4230/LIPIcs.TQC.2016.7.

[BFNR08] Harry Buhrman, Lance Fortnow, Ilan Newman, and Hein Röhrig. Quantum property testing. *SIAM Journal on Computing*, 37(5):1387–1400, 2008.

[BKT18] Mark Bun, Robin Kothari, and Justin Thaler. The polynomial method strikes back: Tight quantum query bounds via dual polynomials. In *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing*, pages 297–310. ACM, 2018.

[BŠ13] Aleksandrs Belovs and Robert Špalek. Adversary lower bound for the k-sum problem. In *Proceedings of the 4th Conference on Innovations in Theoretical Computer Science, ITCS ’13*, pages 323–328, 2013. URL: http://doi.acm.org/10.1145/2422436.2422474, doi:10.1145/2422436.2422474.
Howard Barnum, Michael Saks, and Mario Szegedy. Quantum query complexity and semi-definite programming. In 18th Conference on Computational Complexity (CCC 2003), pages 179–193, 2003. doi:10.1109/CCC.2003.1214419.

Andrew M Childs, Richard Cleve, Enrico Deotto, Edward Farhi, Sam Gutmann, and Daniel A Spielman. Exponential algorithmic speedup by a quantum walk. In Proceedings of the thirty-fifth annual ACM symposium on Theory of computing, pages 59–68. ACM, 2003.

André Chailloux. A note on the quantum query complexity of permutation symmetric functions. 10th Innovations in Theoretical Computer Science Conference (ITCS 2019), 2018.

Richard Cleve. The query complexity of order-finding. Information and Computation, 192(2):162–171, 2004.

Adalbert Kerber. Applied finite group actions, volume 19. Springer Science & Business Media, 2013.

Samuel Kutin. Quantum lower bound for the collision problem with small range. Theory of Computing, 1(1):29–36, 2005.

Peter W Shor. Algorithms for quantum computation: Discrete logarithms and factoring. In Proceedings 35th annual symposium on foundations of computer science, pages 124–134. Ieee, 1994.

Peter W. Shor. Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer. SIAM Journal on Computing, 26(5):1484–1509, 1997. arXiv:quant-ph/9508027.

Daniel R Simon. On the power of quantum computation. SIAM journal on computing, 26(5):1474–1483, 1997.

Maurice Sion. On general minimax theorems. Pacific Journal of Mathematics, 8(1):171–176, 1958. doi:10.2140/pjm.1958.8.171.

Nikolai K. Vereshchagin. Randomized boolean decision trees: Several remarks. Theoretical Computer Science, 207(2):329 – 342, 1998. doi:10.1016/S0304-3975(98)00071-1.

A. Yao. Probabilistic computations: Toward a unified measure of complexity. Proceedings of the 18th IEEE Symposium on Foundations of Computer Science (FOCS), pages 222–227, 1977. doi:10.1109/SFCS.1977.24.

Mark Zhandry. A note on the quantum collision and set equality problems. arXiv preprint arXiv:1312.1027, 2013. arXiv:1312.1027.