Rough concepts

Willem Conradie a,1, Sabine Frittella b,2, Krishna Manoorkar c, Sajad Nazari b,2, Alessandra Palmigiano d,e,3, Apostolos Tzimoulis d,3, Nachoem M. Wijnberg f,g

a School of Mathematics, University of the Witwatersrand, Johannesburg, South Africa
b INSA Centre Val de Loire, Univ. Orléans, LIFO EA 4022, Bourges, France
c Technion, Haifa, Israel
d School of Business and Economics, Vrije Universiteit, Amsterdam, The Netherlands
e Department of Mathematics and Applied Mathematics, University of Johannesburg, South Africa
f Faculty of Economics and Business, University of Amsterdam, The Netherlands
g College of Business and Economics, University of Johannesburg, South Africa

Article info

Article history:
Received 29 June 2019
Received in revised form 16 May 2020
Accepted 19 May 2020
Available online 19 July 2020

Keywords:
Rough set theory
Formal concept analysis
Modal logic
Modal algebras
Rough Concept Analysis

ABSTRACT

The present paper proposes a novel way to unify Rough Set Theory and Formal Concept Analysis. Our method stems from results and insights developed in the algebraic theory of modal logic, and is based on the idea that Pawlak’s original approximation spaces can be seen as special instances of enriched formal contexts, i.e. relational structures based on formal; contexts from Formal Concept Analysis.

1. Introduction

Rough Set Theory (RST) [39] and Formal Concept Analysis (FCA) [24] are very influential foundational theories in information science, and the issues of combining [29] and unifying [5] them have received considerable attention in the literature (see [29] for a very comprehensive overview of the literature), also very recently [4,50,36].

The present paper proposes a novel way to unify RST and FCA. Our method stems from results and insights developed in the algebraic theory of modal logic, and is based on the idea that Pawlak’s original approximation spaces can be seen as special instances of enriched formal contexts (cf. Definition 2.5, see [10,11]), i.e. relational structures based on formal contexts from FCA. Mathematically, this is realized in an embedding of the class $AS$ of approximation spaces into the class $EFC$ of enriched formal contexts which—building on Drew Moshier’s category-theoretic perspective and results on formal contexts [37]—makes the following diagram commute:

1 The research of the first author was supported by a startup grant of the Faculty of Science, University of the Witwatersrand.
2 The second and fourth authors were partially funded by the grant PHC VAN GOGH 2019, project n: 42555PE and by the grant ANR JCJC 2019, project PRELAP (ANR-19-CE48-0006).
3 The research of the fifth and sixth author has been made possible by the NWO Vidi grant 016.138.314, the NWO Aspasia grant 015.008.054, and a Delft Technology Fellowship awarded in 2013.
In the above diagram, the lowermost horizontal arrow assigns each approximation space $X = (S, R)$ (seen as a Kripke frame for the modal logic S5) to its associated complex algebra $X^+ = (\mathcal{P}(S), (R), [R])$, which belongs in the class $S5\text{-BAO}^+$ of perfect S5 modal algebras (aka perfect S5 Boolean Algebras with Operators). The uppermost horizontal arrow of the diagram above assigns each enriched formal context $F = (\mathcal{P}, (R_0, R_1))$ (cf. Definition 2.5) to its associated complex algebra $F^+ = (\mathcal{P}^+, (R_0), [R_1])$, which belongs in the class $CML$ of complete modal lattices, and the leftmost vertical arrow is the natural embedding of perfect S5 modal algebras into complete modal lattices.

Working in this mathematical environment helps to highlight the division of roles between the different relations in enriched formal contexts and the different functions they perform: while the incidence relations in formal contexts is used to generate the conceptual hierarchy in the form of a concept lattice, the additional relations generate the modal operators approximating concepts in the given conceptual hierarchy. Clarifying this division of roles helps us gain a better understanding of the relationship between formal contexts and approximation spaces which, in turn, facilitates a better understanding of how FCA and RST can be integrated. Specifically, this mathematical environment also provides the background and motivation for the introduction of conceptual (co-) approximation spaces (cf. Definitions 4.1 and 4.5) as suitable structures supporting rough concepts in the same way in which approximation spaces support rough sets, i.e., via suitable relations.

As a consequence of this generalization, we obtain a novel and different set of possible interpretations for the language of the lattice-based normal modal logic discussed in [10,11]: just as S5 is motivated both as an epistemic logic and as the logic of rough sets, the logic introduced in [11] can be understood both as an epistemic logic of categories and as the logic of rough concepts. 

Structure of the paper. In Section 2, we collect preliminaries on approximation spaces and their associated complex algebras, enriched formal contexts and their associated complex algebras, and modal logic. In Section 3, we partly recall and partly develop the background theory motivating the embedding of approximation spaces into a suitable subclass of enriched formal contexts, and show that this embedding makes the diagram above commute. On the basis of these facts we introduce conceptual (co-) approximation spaces in Section 4, and prove that these structures are logically captured by a certain modal axiomatization. In Section 5, we discuss how conceptual (co-) approximation spaces can be used to model a wide variety of situations which, in their turn, allow for different interpretations of the modal operators arising from them. In Section 6, we introduce varieties of lattice-based modal algebras that capture abstract versions of the complex algebras of conceptual (co-) approximation spaces, and which provide algebraic semantics to the modal logics of conceptual (co-) approximation spaces. In Section 7, we apply the insights and constructions developed in the previous sections to extend and generalize three different (and independently developed) logical frameworks which aim at address and account for vagueness, gradedness, and uncertainty. We present conclusions and directions for further research in Section 8.

2. Preliminaries

Throughout the paper, the superscript $(\cdot)^c$ denotes the relative complement of the subset of a given set. In particular, for any binary relation $R \subseteq S \times S$, we let $R' \subseteq S \times S$ be defined by $(s, s') \in R'$ iff $(s, s') \not\in R$. For any such $R$ and any $Z \subseteq S$, we also let 

$$R[Z] := \{ z \in S \mid (z, s) \in R \text{ for some } z \in Z \}$$

and 

$$R^{-1}[Z] := \{ s \in S \mid (s, z) \in R \text{ for some } z \in Z \}$$

for any set $Z \subseteq S$.

As usual, we write $R[z]$ and $R^{-1}[z]$ instead of $R([z])$ and $R^{-1}([z])$, respectively. The relation $R$ can be associated with the following semantic modal operators: for any $Z \subseteq S$,

$$[R]Z := R^{-1}[Z] = \{ s \in S \mid (s, z) \in R \text{ for some } z \in Z \},$$

(1)

$$[R]Z := (R^{-1}[Z])^c = \{ s \in S \mid \text{ for any } z, \text{ if } (s, z) \in R \text{ then } z \in Z \},$$

(2)

and also with the following ones:

$$[R]Z := (R^{-1}[Z])^c = \{ s \in S \mid \text{ for any } z, \text{ if } (s, z) \in R \text{ then } z \notin Z \},$$

(3)

$$[R]Z := R^{-1}[Z] = \{ s \in S \mid (s, z) \in R \text{ for some } z \notin Z \}.$$  

(4)
As is well known, in the context of approximation spaces, where \( R \) is an equivalence relation encoding indistinguishability, the sets \( |R|Z, (R)Z, |R|Z, \) and \( |R|Z \) collect the elements of \( S \) that are definitely in \( Z \), possibly in \( Z \), definitely not in \( Z \), and possibly not in \( Z \). All these operations are interdefinable using relative complementation, just like their syntactic counterparts are interdefinable using Boolean negation. However, since in the setting of formal contexts this will no longer be the case, we find it useful to introduce this notation already in the classical setting.

### 2.1. Approximation spaces, their complex algebras and modal logic

**Approximation spaces** are structures \( X = (S, R) \) such that \( S \) is a set, and \( R \subseteq S \times S \) is an equivalence relation.\(^4\) For any such \( X \) and any \( Z \subseteq S \), the upper and lower approximations of \( Z \) are respectively defined as follows:

\[
\bar{Z} := \bigcup \{R(z) \mid z \in Z\} \quad \text{and} \quad \underline{Z} := \bigcup \{R(z) \mid z \in Z \text{ and } R(z) \subseteq Z\}.
\]

A rough set \( X \) is a pair \( (Z, \bar{Z}) \) for any \( Z \subseteq S \) (cf. \([1]\)). Since approximation spaces coincide with Kripke frames for the modal logic S5, notions and insights from the semantic theory of modal logic have been imported to rough set theory \([1]\). In particular, the **complex algebra** of an approximation space \( X \) (and more in general of a Kripke frame) is the Boolean algebra with operator

\[
\Box \phi \rightarrow \phi \quad \Box \phi \rightarrow \Box \Box \phi \quad \phi \rightarrow \Box \Diamond \phi.
\]

Hence, \( X^+ \) is an S5-algebra. Given a language for S5 over a set \( \text{Prop} \) of proposition variables, a **model** is a tuple \( M = (X, V) \) where \( X = (S, R) \) is a Kripke frame, and \( V: \text{Prop} \rightarrow X^+ \) on \( X \) is a valuation. The satisfaction \( M, w \vDash \phi \) of any formula \( \phi \) at states \( w \) in \( M \), and its extension \( [\phi] := \{w \in S \mid M, w \vDash \phi\} \) are defined recursively as follows:

- \( M, w \vDash p \) iff \( w \in V(p) \)
- \( M, w \vDash \phi \land \psi \) iff \( w \in [\phi] \) or \( w \in [\psi] \)
- \( M, w \vDash \neg \phi \) iff \( w \notin [\phi] \)
- \( M, w \vDash \Box \phi \) iff \( R(w) \subseteq [\phi] \)
- \( M, w \vDash \Diamond \phi \) iff \( R(w) \text{ implies } u \in [\phi] \)

Finally, as to the interpretation of sequents:

\( M \models \phi \rightarrow \psi \) iff for all \( w \in W \), if \( M, w \vDash \phi \), then \( M, w \vDash \psi \)

A sequent \( \phi \rightarrow \psi \) is valid on a Kripke frame \( X \) (in symbols: \( X \models \phi \rightarrow \psi \)) if \( M \models \phi \rightarrow \psi \) for every model \( M \) based on \( X \).

### 2.2. Rough algebras

In this subsection, we report on the definitions of several classes of algebras, collectively referred to as ‘rough algebras’, which have been introduced and studied as abstract versions of approximation spaces (cf. e.g. \([1,43,44]\)).

**Definition 2.1.** \([43]\) An algebra \( T = (L, \neg, I) \) is a **topological quasi-Boolean algebra** (tqBa) if \( L = (L, \wedge, \lor, \top, \bot) \) is a bounded distributive lattice, and for all \( a, b \in L \),

- \( \neg \neg a = a \) \quad \text{T1.} \quad \text{\textit{De Morgan}}
- \( \neg(a \lor b) = \neg a \land \neg b \) \quad \text{T2.} \quad \text{\textit{De Morgan}}
- \( 1(a \land b) = 1a \land 1b \) \quad \text{T3.} \quad \text{\textit{Idempotent}}
- \( Ia = Ia \) \quad \text{T4.} \quad \text{\textit{Idempotent}}
- \( Ia \leq a \) \quad \text{T5.} \quad \text{\textit{Idempotent}} \quad \text{\textit{Idempotent}}

An algebra \( T \) as above is a **topological quasi-Boolean algebra** \( 5 \) (tqBa5) if for all \( a \in L \),

- \( Ca = Ia \), where \( Ca := \neg Ia \).

An algebra \( 5 \) is an intermediate algebra of type 1 (1A1) iff for all \( a \in L \),

---

\(^4\) Approximation spaces form a subclass of Kripke frames, which are structures \( X = (S, R) \) such that \( S \) is a set and \( R \subseteq S \times S \) is an arbitrary relation.
A $\text{tqBa5}$ $T$ is an intermediate algebra of type 2 $(\text{IA2})$ iff for all $a \in L$,

\[\text{T9. } Ia \vee \neg Ia = T.\]

A $\text{tqBa5}$ $T$ is an intermediate algebra of type 3 $(\text{IA3})$ iff for all $a \in L$,

\[\text{T10. } Ia \vee Ib = I(a \vee b).\]

A $\text{tqBa5}$ $T$ is an intermediate algebra of type 3 $(\text{IA3})$ iff for all $a \in L$,

\[\text{T11. } Ia \leq Ib \text{ and } Ca \leq Cb \text{ imply } a \leq b.\]

A pre-rough algebra is a $\text{tqBa5}$ $T$ which satisfies T9, T10 and T11. A rough algebra is a complete and completely distributive pre-rough algebra.

2.3. Formal contexts and their concept lattices

This subsection elaborates and expands on [11,10] [Appendix]. For any relation $T \subseteq U \times V$, and any $U' \subseteq U$ and $V' \subseteq V$, let

\[T^{(1)}[U'] := \{v \in V \mid \forall u \in U'(u \Rightarrow uT v)\} \quad T^{(0)}[V'] := \{u \in U \mid \forall v \in V'(v \Rightarrow uT v)\}.\]  

(5)

Well known properties of this construction are stated in the following lemma.

**Lemma 2.2.**

1. $X_1 \subseteq X_2 \subseteq U$ implies $T^{(1)}[X_2] \subseteq T^{(1)}[X_1]$, and $Y_1 \subseteq Y_2 \subseteq V$ implies $T^{(0)}[Y_2] \subseteq T^{(0)}[Y_1]$.
2. $U' \subseteq T^{(0)}[V']$ iff $V' \subseteq T^{(1)}[U']$.
3. $U' \subseteq T^{(0)}[T^{(1)}[U']]$ and $V' \subseteq T^{(1)}[T^{(0)}[V']]$.
4. $T^{(1)}[U'] = T^{(1)}[T^{(0)}[T^{(1)}[U']]]$ and $T^{(0)}[V'] = T^{(0)}[T^{(1)}[T^{(0)}[V']]]$.
5. $T^{(0)}[\cup X] = \bigvee_{\forall v \in X} T^{(0)}[V']$ and $T^{(1)}[\cup X] = \bigvee_{\forall u \in X} T^{(1)}[U']$.

In what follows, we fix two sets $A$ and $X$, and use $a, b$ (resp. $x, y$) for elements of $A$ (resp. $X$), and $B, C, A_I$ (resp. $Y, W, X_I$) for subsets of $A$ (resp. $X$). Formal contexts, or polarities, are structures $\mathcal{P} = (A, X, I)$ such that $A$ and $X$ are sets, and $I \subseteq A \times X$ is a binary relation. Intuitively, formal contexts can be understood as abstract representations of databases [24], so that $A$ represents a collection of objects, $X$ as a collection of features, and for any object $a$ and feature $x$, the tuple $(a, x)$ belongs to $I$ exactly when object $a$ has feature $x$. As is well known, for every formal context $\mathcal{P} = (A, X, I)$, the pair of maps

\[ (\cdot)^! : \mathcal{P}(A) \rightarrow \mathcal{P}(X) \quad \text{and} \quad (\cdot)^! : \mathcal{P}(X) \rightarrow \mathcal{P}(A),\]

respectively defined by the assignments $B^! := I^{(1)}[B]$ and $Y^! := I^{(0)}[Y]$, form a Galois connection (cf. Lemma 2.2(2)), and hence induce the closure operators $(\cdot)^!$ and $(\cdot)^!$ on $\mathcal{P}(A)$ and on $\mathcal{P}(X)$ respectively. Moreover, the fixed points of these closure operators (sometimes referred to as Galois-stable sets) form complete sub-$\cap$-semilattices of $\mathcal{P}(A)$ and $\mathcal{P}(X)$ (and hence complete lattices) respectively, which are dually isomorphic to each other via the restrictions of the maps $(\cdot)^!$ and $(\cdot)^!$ (cf. Lemma 2.2(3)). This motivates the following definition:

**Definition 2.3.** For every formal context $\mathcal{P} = (A, X, I)$, a formal concept of $\mathcal{P}$ is a pair $c = (B, Y)$ such that $B \subseteq A$, $Y \subseteq X$, and $B^! = Y$ and $Y^! = B$. The set $B$ is the extension of $c$, which we will sometimes denote $[c]$, and $Y$ is the intension of $c$, sometimes denoted $\langle c \rangle$. Let $L(\mathcal{P})$ denote the set of the formal concepts of $\mathcal{P}$. Then the concept lattice of $\mathcal{P}$ is the complete lattice

\[\mathcal{P}^+ := (L(\mathcal{P}), \wedge, \vee),\]

where for every $X \subseteq L(\mathcal{P})$,

\[\bigwedge X := \left( \bigcap_{c \in X} [c] \right)^! \quad \text{and} \quad \bigvee X := \left( \bigvee_{c \in X} [c] \right)^! .\]

Then clearly, $T^+ := \bigwedge \emptyset = (A, A^!_I)$ and $\perp^+ := \bigvee \emptyset = (X^!, X)$, and the partial order underlying this lattice structure is defined as follows: for any $c, d \in L(\mathcal{P})$,

\[c \leq d \text{ iff } [c] \subseteq [d] \text{ iff } \langle d \rangle \subseteq \langle c \rangle.\]

---

\[\text{When } B = \{a\} \text{ (resp. } Y = \{x\} \text{) we write } a^! \text{ for } [a]^! \text{ (resp. } x^! \text{ for } [x]^!).\]
Theorem 2.4. (Birkhoff’s theorem, main theorem of FCA) Any complete lattice $\mathcal{L}$ is isomorphic to the concept lattice $\mathcal{P}^+$ of some formal context $\mathcal{P}$.

Proof. If $\mathcal{L} = (L, \leq)$ is a complete lattice seen as a poset, then $\mathcal{L} \cong \mathcal{P}^+$ e.g. for $\mathcal{P} := (L, L, \leq)$.

2.4. Enriched formal contexts and their complex algebras

In [11], structures similar to the following are introduced as generalizations of Kripke frames.

Definition 2.5. An enriched formal context is a tuple

$\mathcal{F} = (\mathcal{P}, R_\setminus, R_\cap)$

such that $\mathcal{P} = (A, X, I)$ is a formal context, and $R_\setminus \subseteq A \times X$ and $R_\cap \subseteq X \times A$ are $I$-compatible relations, that is, for all $x \in X$ and $a \in A$ the sets $R_\setminus^0(x)$ (resp. $R_\setminus^1(a)$) and $R_\cap^1(x)$ (resp. $R_\cap^0(a)$) are Galois-stable relative to (the incidence relation $I$ of) formal context $\mathcal{P}$:

$\left(R_\setminus^0(x)\right)^{\perp} = R_\setminus^0(x) \quad \left(R_\setminus^1(a)\right)^{\perp} = R_\setminus^1(a) \quad \left(R_\cap^0(x)\right)^{\perp} = R_\cap^0(x) \quad \left(R_\cap^1(a)\right)^{\perp} = R_\cap^1(a)$

The complex algebra of $\mathcal{F}$ is

$\mathcal{F}^+ = (\mathcal{P}^+, [R_\setminus], [R_\cap])$,

where $\mathcal{P}^+$ is the concept lattice of $\mathcal{P}$, and $[R_\setminus]$ and $[R_\cap]$ are unary operations on $\mathcal{P}^+$ defined as follows: for every $c \in \mathcal{P}^+$,

$[R_\setminus]c := \left(\left(R_\setminus_\setminus^0(c)\right)^{\perp}, \left(R_\setminus_\cap^0(c)\right)^{\perp}\right)$ and $[R_\cap]c := \left(\left(R_\cap_\setminus^0(c)\right)^{\perp}, \left(R_\cap_\setminus^0(c)\right)^{\perp}\right)$.

Several possible interpretations of the modal operators defined above are discussed below in Section 2.5 and also in Sections 5 and 7. Intuitively, the $I$-compatibility condition is intended to guarantee that the modal operators associated with $R_\setminus$ and $R_\cap$—as well as their adjoints, introduced below—are well-defined. However, as we will see in more detail below, $I$-compatibility is sufficient but not necessary for the required well-definedness. We note that the well-definedness of these operators is a second-order condition, whereas the strictly stronger $I$-compatibility is first-order. We prefer to take this elementary and more restricted class as our basic environment because, as we will see, the basic normal lattice-based modal logic introduced in the next subsection is complete w.r.t enriched formal contexts, and working with a first-order definable class of structures gives access to a host of powerful results (more on this in the last paragraph of Section 2.5). The following lemma provides equivalent reformulations of $I$-compatibility and refines [11] Lemmas 3 and 4.

Lemma 2.6.

1. The following are equivalent for every formal context $\mathcal{P} = (A, X, I)$ and every relation $R \subset A \times X$:
   (i) $R^0(x)$ is Galois-stable for every $x \in X$;
   (ii) $R^0(Y)$ is Galois-stable for every $Y \subset X$;
   (iii) $R^1(B) = R^1[B]^{\perp}$ for every $B \subset A$.

2. The following are equivalent for every formal context $\mathcal{P} = (A, X, I)$ and every relation $R \subset A \times X$:
   (i) $R^1(a)$ is Galois-stable for every $a \in A$;
   (ii) $R^1(B)$ is Galois-stable, for every $B \subset A$;
   (iii) $R^0(Y) = R^0[Y^{\perp}]$ for every $Y \subset X$.

Proof. We only prove item 1, the proof of item 2 being similar. For (i) $\Rightarrow$ (ii), since $Y = \bigcup_{x \in Y} R^0(x)$, Lemma 2.2.5 implies that $R^0(Y) = R^0(\bigcup_{x \in Y} R^0(x)) = \bigcap_{x \in Y} R^0(x)$ which is Galois-stable by (i) and the fact that Galois-stable sets are closed under arbitrary intersections. The converse direction is immediate.

(i) $\Rightarrow$ (iii). Since $(\cdot)^{\perp}$ is a closure operator, $B \subset B^{\perp}$. Hence, Lemma 2.2.1 implies that $R^1\left[B^{\perp}\right] \subset R^1[B]$. For the converse inclusion, let $x \in R^1_\setminus(B)$. By Lemma 2.2.2, this is equivalent to $B \subset R^0_\setminus(x)$. Since $R^0_\setminus(x)$ is Galois-stable by assumption, this implies that $B^{\perp} \subset R^0_\setminus(x)$, i.e., again by Lemma 2.2.2, $x \in R^1_\setminus[B^{\perp}]$. This shows that $R^1_\setminus[B] \subset R^1_\setminus[B^{\perp}]$, as required.
(iii) \(\Rightarrow\) (i). Let \(x \in X\). It is enough to show that \(\langle R^{(0)}[x] \rangle^{11} \subseteq R^{(0)}[x]\). By Lemma 2.2, \(R^{(0)}[x] \subseteq R^{(0)}[x]\) is equivalent to \(x \in R^{(1)}[R^{(0)}[x]]\). By assumption, \(R^{(1)}[R^{(0)}[x]] = R^{(1)}\left[\left(R^{(0)}[x]\right)^{11}\right]\), hence \(x \in R^{(1)}\left[\left(R^{(0)}[x]\right)^{11}\right]\). Again by Lemma 2.2, this is equivalent to \(\langle R^{(0)}[x] \rangle^{11} \subseteq R^{(0)}[x]\), as required.

For any enriched formal context \(\mathcal{F}\), let \(R_{\bullet} \subseteq X \times A\) be defined by \(x R_{\bullet} a\) iff \(a R_{\bullet} x\), and \(R_{\bullet} \subseteq A \times X\) by \(a R_{\bullet} x\) iff \(x R_{\bullet} a\). Hence, for every \(B \subseteq A\) and \(Y \subseteq X\),

\[
R^{(0)}_{\bullet}[B] = R^{(0)}[B] \quad R^{(1)}_{\bullet}[Y] = R^{(1)}[Y] \quad R^{(0)}_{\bullet}[Y] = R^{(1)}[Y]\quad R^{(1)}_{\bullet}[B] = R^{(0)}[B].
\]

By Lemma 2.6, the \(I\)-compatibility of \(R_{\bullet}\) and \(R_{\circ}\) guarantees that \(R^{(0)}_{\bullet}[Y] \supseteq R^{(0)}[Y]\) are Galois-stable sets for every \(Y \subseteq X\), and that \(R^{(0)}_{\bullet}[B] \supseteq R^{(0)}[B]\) and \(R^{(1)}_{\bullet}[B] \supseteq R^{(1)}[B]\) are Galois-stable sets for every \(B \subseteq A\), and hence implies that the maps \([R^{(0)}_{\bullet}], [R^{(0)}_{\circ}], [R^{(0)}_{\bullet}], [R^{(0)}_{\circ}]: \mathcal{P}^+ \to \mathcal{P}^+\) are well-defined. However, \(I\)-compatibility is strictly stronger than the maps \([R_{\bullet}], [R_{\circ}], [R_{\bullet}], [R_{\circ}]: \mathcal{P}^+ \to \mathcal{P}^+\) being well-defined, as is shown in the following example.

**Example 2.7.** Let \(\mathcal{P}=(A,X,I)\) with \(A=\{a,b,c\}\), \(X=\{x,y,z\}\) and \(I=\{(a,y), (a,z)\}\) be a formal context.

Let \(R=\{(a,y), (a,z), (b,x)\}\).

The concept lattice of \(\mathcal{P}\) is represented in the picture above. It is easy to verify that \(R^{(0)}[X] = \varnothing\), \(R^{(0)}[\{y,z\}] = \{a\}\) and \(R^{(0)}[\varnothing] = A\), hence \([R]: \mathcal{P}^+ \to \mathcal{P}^+\) is well defined and is the identity on \(\mathcal{P}^+\). Likewise, \(R^{(1)}[\varnothing] = \varnothing\), \(R^{(1)}[\{a\}] = \{y,z\}\) and \(R^{(1)}[X] = \mathcal{X}\), hence \([R^{-1}]: \mathcal{P}^+ \to \mathcal{P}^+\) is well defined and is the identity on \(\mathcal{P}^+\). However, \(R\) is not \(I\)-compatible. Indeed, \(R^{(0)}[x] = \{b\}\) and \(R^{(1)}[b] = \{x\}\) are not Galois-stable.

**Lemma 2.8.** For any enriched formal context \(\mathcal{F}=(A,X,I,R_{\bullet},R_{\circ})\), the algebra \(\mathcal{F}^+ = (\mathcal{P}^+, [R_{\bullet}], [R_{\circ}])\) is a complete normal lattice expansion such that \([R_{\bullet}]\) is completely meet-preserving and \([R_{\circ}]\) is completely join-preserving.

**Proof.** As discussed above, the \(I\)-compatibility of \(R_{\bullet}\) and \(R_{\circ}\) guarantees that the maps \([R_{\bullet}], [R_{\circ}], [R_{\bullet}], [R_{\circ}]: \mathcal{P}^+ \to \mathcal{P}^+\) are well-defined. Since \(\mathcal{P}^+\) is a complete lattice, to show that \([R_{\bullet}]\) is completely meet-preserving and \([R_{\circ}]\) is completely join-preserving, it is enough to show that \(\langle R_{\bullet} \rangle\) is the left adjoint of \([R_{\bullet}]\) and that \([R_{\circ}]\) is the right adjoint of \(\langle R_{\circ} \rangle\). For any \(c, d \in \mathcal{P}^+\),

\[
\langle R_{\bullet} \rangle c \leq d \quad \text{iff} \quad \langle d \rangle \leq R^{(0)}_{\bullet} \llbracket c \rrbracket \quad \text{ordering of concepts}
\]

\[
\text{iff} \quad \langle d \rangle \leq R^{(1)}_{\bullet} \llbracket c \rrbracket \quad \text{(6)}
\]

\[
\text{iff} \quad [c] \subseteq R^{(0)}_{\bullet} \llbracket d \rrbracket \quad \text{Lemma 2.2.2}
\]

\[
\text{iff} \quad c \leq [R_{\bullet}] d \quad \text{ordering of concepts}
\]

Likewise, one shows that \([R_{\bullet}]\) is the right adjoint of \(\langle R_{\circ} \rangle\).

The converse of the lemma above also holds, in the form of the following representation theorem of which we give only the sketch of a proof, referring to [18] for further details.

**Theorem 2.9.** (Expanded Birkhoff theorem) Any complete modal algebra\(^6\) \(\mathcal{A} = (L, \sqsubseteq, \Diamond)\) is isomorphic to the complex algebra \(\mathcal{F}^+\) of some enriched formal context \(\mathcal{F}\).

**Proof.** If \(L = (L, \leq)\) is the complete lattice underlying \(\mathcal{A}\) seen as a poset, then \(\mathcal{A} \cong \mathcal{F}^+\) where \(\mathcal{F} := (\mathcal{P}, R_{\bullet}, R_{\circ})\) is e.g. such that \(\mathcal{P} := (L, \leq)\), and for all \(a, b \in L\),

\[
a R_{\bullet} b \quad \text{iff} \quad a \leq \Box b \quad a R_{\circ} b \quad \text{iff} \quad \Diamond a \leq b.
\]

---

\(^6\) A complete modal algebra is a complete normal lattice expansion \(\mathcal{A} = (L, \sqsubseteq, \Diamond)\) such that \(L\) is a complete lattice, \(\Box\) is completely meet-preserving and \(\Diamond\) is completely join-preserving.
2.5. Basic modal logic of concepts and its enriched formal context semantics

In [10], following the general methodology for interpreting normal lattice-based logics on polarity-based (i.e. formal-context-based) relational structures discussed in [15], a lattice-based normal modal logic is introduced as the basic epistemic logic of categories, together with its semantics based on a restricted class of formal contexts. The restrictions were lifted in [11]. In what follows we report on a variation of this logic, and the semantics proposed in [11]. Intuitively, this logic is the counterpart, in the framework of formal concepts, of the basic normal modal logic $K$, and as such, it constitutes the basic logical framework in which the logic of conceptual approximation spaces embeds.

2.5.1. Basic logic and informal understanding

Let $\text{Prop}$ be a (countable or finite) set of atomic propositions. The language $\mathcal{L}$ of the basic modal logic of formal concepts is

\[ \varphi ::= \bot \mid T \mid p \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \Box \varphi \mid \Diamond \varphi \]

where $p \in \text{Prop}$. Clearly, the logical signature of this language matches the algebraic signature of the complex algebra of any enriched formal context $F = (\mathcal{P}, R_\mathcal{P}, R_\mathcal{V})$ (cf. Definition 2.5). Hence, formulas in this language can be interpreted as formal concepts of $F$. If the formal context $\mathcal{P}$ on which $F$ is based is regarded as the abstract representation of a database, atomic propositions $p \in \text{Prop}$ can be understood as atomic labels (or names) for concepts, appropriate to the nature of the database. For instance, if the database consists of music albums and their features (e.g. names of performers, types of musical instruments, number of bits per minute etc), then the atomic propositions can stand for names of music genres (e.g. jazz, rock, rap); likewise, if the database consists of movies and their features (e.g. names of directors or performers, duration, presence of special effects, presence of costumes, presence of shooting scenes, etc), then the atomic propositions can stand for movie genres (e.g. western, drama, horror); if the database consists of goods on sale in a supermarket and their features (e.g. capacity of packages, presence of additives, presence of organic certification, etc) then the conceptual labels can stand for supermarket categories (e.g. detergents, dairies, spices); if the database consists of patients in a hospital and their symptoms (e.g. fever, jaundice, vertigo, etc), then the atomic propositions can stand for diseases (e.g. pneumonia, hepatitis, diabetes). Compound formulas $\varphi \land \psi$ and $\varphi \lor \psi$ respectively denote the greatest common subconcept and the smallest common superconcept of $\varphi$ and $\psi$. In [10,11], modal operators are given an epistemic interpretation, so that, for a given agent $i \in Ag$, the formula $\Box \varphi$ was understood as “the category $\varphi$, according to agent $i$”. In the present paper we will propose different interpretations of the modal operators in connection to their ‘lower’ and ‘upper approximation’ roles (cf. Section 5, see also Section 7). The basic or minimal normal $\mathcal{L}$-logic is a set $\mathcal{L}$ of sequents $\varphi \vdash \psi$ (which intuitively read “$\varphi$ is a subconcept of $\psi$”) with $\varphi, \psi \in \mathcal{L}$, containing the following axioms:

- Sequents for propositional connectives:
  \[ p \vdash p, \quad \bot \vdash p, \quad T \vdash p, \quad p \lor q \vdash p, \quad p \lor q \vdash q. \]

- Sequents for modal operators:
  \[ T \vdash \Box T, \quad \Box p \land \Box q \vdash \Box (p \land q), \quad \Diamond \bot \vdash, \quad \Diamond (p \lor q) \vdash \Diamond p \lor \Diamond q. \]

and closed under the following inference rules:

\[
\begin{align*}
\frac{\varphi \vdash \chi \land \psi \land \psi}{\varphi \vdash \chi} & \quad \frac{\varphi \vdash \psi}{\varphi \vdash \psi} \quad \frac{\varphi \vdash \psi}{\varphi \vdash \psi} \\
\frac{\varphi \vdash \psi}{\varphi \vdash \psi} & \quad \frac{\varphi \vdash \psi}{\varphi \vdash \psi} \quad \frac{\varphi \vdash \psi}{\varphi \vdash \psi} \\
\end{align*}
\]

By an $\mathcal{L}$-logic we understand any extension of $\mathcal{L}$ with $\mathcal{L}$-axioms $\varphi \vdash \psi$. The reader can refer to [11] for more details about $\mathcal{L}$-logic such as the proof of the soundness and completeness of the logic w.r.t enriched formal contexts.

2.5.2. Interpretation in enriched formal contexts

For any enriched formal context $F = (\mathcal{P}, R_\mathcal{P}, R_\mathcal{V})$, a valuation on $F$ is a map $V : \text{Prop} \rightarrow 2^\mathcal{P}$. For every conceptual label $p \in \text{Prop}$, we let $[p] := [V(p)]$ (resp. $([p]) := ([V(p)])$) denote the extension (resp. the intension) of the interpretation of $p$ under $V$. The elements of $[p]$ are the members of concept $p$ under $V$; the elements of $([p])$ describe concept $p$ under $V$. Any valuation $V$ on $F$ extends homomorphically to an interpretation map of $\mathcal{L}$-formulas defined as follows:
above, we can equivalently rewrite it in the following recursive form:

\[
V(p) = ([p], [p])
\]

\[
V(\top) = (A, A^1)
\]

\[
V(\bot) = (X^1, X)
\]

\[
V(\phi \land \psi) = ([\phi] \cap [\psi], ([\phi] \cap [\psi])^1)
\]

\[
V(\phi \lor \psi) = (\langle [\phi] \cap ([\psi])^1, ([\phi]) \cap ([\psi]) \rangle)
\]

\[
V(\Box \phi) = \left( \left( R^0_{\Box}([\phi]) \right)^1, R^0_{\Box}([\phi]) \right)
\]

\[
V(\Diamond \phi) = \left( \left( R^0_{\Diamond}([\phi]) \right)^1, R^0_{\Diamond}([\phi]) \right).
\]

A model is a tuple \( M = (F, V) \) where \( F = (P, R, \preceq) \) is an enriched formal context and \( V \) is a valuation on \( F \). For every \( \phi \in \mathcal{L} \), we write:

\[
M, a \vdash \phi \quad \text{iff} \quad a \in [\phi]_M
\]

\[
M, x \rhd \phi \quad \text{iff} \quad x \in ([\phi]_M)
\]

and we read \( M, a \vdash \phi \) as “\( a \) is a member of category \( \phi \)”, and \( M, x \rhd \phi \) as “\( x \) describes category \( \phi \)”. Spelling out the definition above, we can equivalently rewrite it in the following recursive form:

\[
M, a \vdash p \quad \text{iff} \quad a \in [p]_M
\]

\[
M, x \rhd p \quad \text{iff} \quad x \in ([p]_M)
\]

\[
M, a \vdash \top \quad \text{always}
\]

\[
M, x \rhd \bot \quad \text{always}
\]

\[
M, a \vdash \psi \quad \text{iff} \quad M, a \vdash \phi \quad \text{and} \quad M, a \vdash \psi
\]

\[
M, x \rhd \phi \land \psi \quad \text{iff} \quad \forall a \in A, \text{ if } M, a \rhd \phi \quad \text{and} \quad M, a \rhd \psi
\]

\[
M, x \rhd \phi \lor \psi \quad \text{iff} \quad M, x \rhd \phi \quad \text{and} \quad M, x \rhd \psi
\]

\[
M, a \vdash \Box \phi \quad \text{iff} \quad \forall a \in A, \text{ if } M, a \vdash \Box \phi, \text{ then } ax
\]

\[
M, a \vdash \Diamond \phi \quad \text{iff} \quad \forall a \in A, \text{ if } M, a \vdash \Diamond \phi, \text{ then } ax
\]

Hence, in each model, \( \top \) is interpreted as the concept generated by the set \( A \) of all objects, i.e. the widest concept and hence the one with the laxest (possibly empty) description; \( \bot \) is interpreted as the category generated by the set \( X \) of all features, i.e. the smallest category and hence the one with the most restrictive description and possibly empty extension; \( \phi \land \psi \) is interpreted as the semantic category determined by the intersection of the extensions of \( \phi \) and \( \psi \) (hence, the description of \( \phi \land \psi \) certainly includes \( ([\phi]) \cup ([\psi]) \) but is possibly larger). Likewise, \( \phi \lor \psi \) is interpreted as the semantic category determined by the intersection of the intensions of \( \phi \) and \( \psi \) (hence, \( [\phi] \cup [\psi] \subseteq [\phi \lor \psi] \) but this inclusion is typically strict).

As to the interpretation of modal formulas:

\[
M, a \vdash \Box \phi \quad \text{iff} \quad \forall x \in X, \text{ if } M, x \rhd \phi, \text{ then } aR_{\Box}x
\]

\[
M, x \rhd \Box \phi \quad \text{iff} \quad \forall a \in A, \text{ if } M, a \vdash \Box \phi, \text{ then } ax.
\]

\[
M, a \vdash \Diamond \phi \quad \text{iff} \quad \forall x \in X, \text{ if } M, x \rhd \phi, \text{ then } ax
\]

\[
M, x \rhd \Diamond \phi \quad \text{iff} \quad \forall a \in A, \text{ if } M, a \vdash \Diamond \phi, \text{ then } aR_{\Diamond}a.
\]

Thus, in each model, \( \Box \phi \) is interpreted as the concept whose members are those objects which are \( R_{\Box} \)-related to every feature in the description of \( \phi \), and \( \Diamond \phi \) is interpreted as the category described by those features which are \( R_{\Diamond} \)-related to every member of \( \phi \). Finally, as to the interpretation of sequents:

\[
M \models \phi \vdash \psi \quad \text{iff} \quad \forall a \in A, \text{ if } M, a \vdash \phi, \text{ then } M, a \vdash \psi
\]

\[
M \models \phi \vdash \psi \quad \text{iff} \quad \forall x \in X, \text{ if } M, x \rhd \phi, \text{ then } M, x \rhd \phi
\]

A sequent \( \phi \vdash \psi \) is valid on an enriched formal context \( F \) (in symbols: \( F \models \phi \vdash \psi \)) if \( M \models \phi \vdash \psi \) for every model \( M \) based on \( F \).

2.5.3. Properties

The basic normal lattice-based logic \( \mathcal{L} \) pertains to the class of normal LE-logics [15] (i.e. logics algebraically captured by varieties of normal lattice expansions), for each of which, relational semantic structures based on formal contexts have been introduced (of which enriched formal contexts are an instance) and several results (e.g. soundness, completeness, Sahlqvist-type correspondence and canonicity [15], semantic cut elimination [26], a Goldblatt-Thomason theorem [20]) have been obtained in generality and uniformity. These results immediately apply to \( \mathcal{L} \) and to a wide class of axiomatic extensions.
and modal expansions of \( L \) which includes all those defined by the axioms mentioned in the present paper (see Section 2.6).

As we will see (cf. Proposition 4.3 and discussion around it), the instantiation of Sahlqvist correspondence theory for LE-logics to the modal language of \( L \) will be key to the development of the present theory, and will provide the main technical justification of the defining conditions of conceptual approximation spaces and their refinements (cf. Definition 4.1).

In conclusion, from a purely logical and algebraic perspective, it is clear that, since its propositional base is the logic of general (i.e. possibly non-distributive) lattices, the basic normal logic of formal concepts is more general, i.e. weaker, than the basic classical normal modal logic, and hence the class of algebras for the latter is a proper subclass of the class of algebras for the former. In Section 3, we will show that the class of relational models of the latter can also be embedded in the class of relational models of the former so as to preserve the natural embedding of the corresponding classes of algebras, and make the diagram discussed in Section 1 commute.

### 2.6. Axiomatic extensions and modal expansions

#### 2.6.1. Axiomatic extensions

The basic normal modal logic of formal concepts and its semantics based on enriched formal contexts provide the background system of study for several well known modal principles such as \( \Box p \vdash p, p \vdash \Diamond p, \Diamond \Diamond p \vdash p, \Box p \vdash \Box \Box p, \Diamond \Box p \vdash \Box \Diamond p \). The theory of unified correspondence \([13,19,15,14,9]\) guarantees that the validity on any given enriched formal context of each of these and other modal principles is equivalent to a first-order condition being true of the given enriched formal context, just in the same way in which the validity of e.g. \( \Box p \vdash p \) on a given Kripke frame corresponds to that Kripke frame being reflexive, and so on. In the present paper, we will restrict our attention to certain modal principles which are relevant to the development of the theory of rough concepts, and we will address their study from two different perspectives; the first one concerns lifting the first order condition which classically corresponds to the given modal principle from Kripke frames to enriched formal context. The second perspective concerns the autonomous interpretation of the first order correspondents of the modal principles on enriched formal contexts. More on this in the next sections.

#### 2.6.2. Expanding with negative modalities

In some situations (cf. Section 1) it can be useful to work with a more expressive language such as the following one:

\[
\varphi := : \top \mid t \mid p \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \Box \varphi \mid \Diamond \varphi \mid \Rightarrow \varphi \mid \leftarrow \varphi .
\]

the additional connectives \( \Rightarrow \) and \( \leftarrow \) are characterized by the following axioms and rules:

\[
\top \vdash \Rightarrow \bot \quad \Rightarrow p \land \Rightarrow q \vdash \Rightarrow (p \lor q)
\]

\[
\leftarrow \top \vdash \bot \quad \leftarrow (p \land q) \vdash \leftarrow p \land \leftarrow q
\]

By the general theory \([15,26,20]\), enriched formal contexts for this expanded language are tuples \( \mathcal{F} = (\mathcal{P}, \mathcal{R}, \mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_2) \) such that \( (\mathcal{P}, \mathcal{R}_0, \mathcal{R}_1) \) is as in Definition 2.5, and \( \mathcal{R}_0 \subseteq A \times A \text{ and } \mathcal{R}_1 \subseteq X \times X \) are \( l \)-compatible relations, that is, \( \mathcal{R}_0^{0}[b] \) (resp. \( \mathcal{R}_1^{0}[y] \)) and \( \mathcal{R}_1^{1}[a] \) (resp. \( \mathcal{R}_1^{1}[x] \)) are Galois-stable for all \( x, y \in X \) and \( a, b \in A \). The complex algebra of such an \( \mathcal{F} \) is

\[\mathcal{F}^+ = (\mathcal{P}^+, [\mathcal{R}_0], [\mathcal{R}_1], [\mathcal{R}_2]),\]

where \( (\mathcal{P}^+, [\mathcal{R}_0], [\mathcal{R}_1]) \) is as in Definition 2.5, and \( [\mathcal{R}_0] \) and \( [\mathcal{R}_1] \) are unary operations on \( \mathcal{P}^+ \) defined as follows: for every \( c \in \mathcal{P}^+ \),

\[ [\mathcal{R}_0] c := (\mathcal{R}_0^{0}[[c]] \circ (\mathcal{R}_1^{0}[[c]])^{-1}) \quad \text{and} \quad [\mathcal{R}_1] c := \left( (\mathcal{R}_1^{0}[[c]])^{-1} \circ \mathcal{R}_0^{0}[[c]] \right), \]

Valuations and models for this expanded language are defined analogously as indicated above, and for each such model \( \mathcal{M} \),

\[
\mathcal{M}, a \vdash \Rightarrow \varphi \quad \text{iff} \quad \text{for all } b \in A, \text{ if } \mathcal{M}, b \vdash \varphi, \text{ then } a \mathcal{R}_1 b
\]

\[
\mathcal{M}, x \triangleright \varphi \quad \text{iff} \quad \text{for all } a \in A, \text{ if } \mathcal{M}, a \triangleright \varphi, \text{ then } a \mathcal{R}_0 x.
\]

Thus, in each model, \( \Rightarrow \varphi \) is interpreted as the concept whose members are those objects which are \( \mathcal{R}_1 \)-related to every member of \( \varphi \), and \( \triangleright \varphi \) is interpreted as the concept described by those features which are \( \mathcal{R}_0 \)-related to every feature describing \( \varphi \). For every enriched formal context \( \mathcal{F} = (\mathcal{P}, \mathcal{R}, \mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_2) \), any valuation \( V \) on \( \mathcal{F} \) extends to an interpretation map of formulas defined as above for formulas in the \( \mathcal{L}^- \)-fragment, and for \( \Rightarrow \) and \( \triangleright \) formulas is defined as follows:
3. Approximation spaces as enriched formal contexts

In the present section, we introduce the building blocks of the methodology which will lead to the definition of conceptual approximation spaces in the next section. Specifically, we discuss how to represent any given approximation space \( X \) as an enriched formal context \( F_X \) so that the complex algebra \( F_X \) coincides with \( X \), i.e., so that the diagram of Section 1 commutes. The requirement that this representation preserves the algebra arising from each approximation space guarantees that the representation preserves the logical properties of \( X \) that can be expressed in the language of Section 2.5. This preservation is key to guaranteeing that the conceptual approximation spaces appropriately restrict to standard approximation spaces. However, and more interestingly, at the end of the present section (cf. Proposition 3.16 and ensuing discussion) we will see that the specific way in which approximation spaces are represented as enriched formal contexts also provides a way to generalize key properties from approximation spaces to enriched formal contexts; precisely this generalization guarantees the required preservation of their logically salient content, and will yield the definition of conceptual approximation spaces in Section 4.

We start by introducing typed versions of sets and relations. Based on this, we will represent sets as formal contexts. We will then move to representing Kripke frames as enriched formal contexts. Finally, we will discuss how logically salient properties of Kripke frames are lifted along this representation.

3.1. Lifting and typing relations

Throughout this section, for every set \( S \), we let \( A_S := \{ (s, s) \mid s \in S \} \), and we typically drop the subscript when it does not cause ambiguities. Hence we write e.g. \( A := \{ (s, s') \mid s, s' \in S \text{ and } s \neq s' \} \). We let \( S_A \) and \( S_X \) be copies of \( S \), representing the two domains of the polarity associated with \( S \). For every \( P \subseteq S \), we let \( P_A \subseteq S_A \) and \( P_X \subseteq S_X \) denote the corresponding copies of \( P \) in \( S_A \) and \( S_X \), respectively. Then \( P_A \) (resp. \( P_X \)) stands both for \( (P')_A \) (resp. \( (P')_X \)).

In what follows, we will rely on the notation introduced above to define typed versions of relations, sets and Kripke frames; so as to also obtain typed versions of conditions such as reflexivity and transitivity. Working with such typed versions is not essential for the sake of justifying that the notion of conceptual approximation spaces restricts appropriately to that of approximation spaces; however, as we will discuss at the end of the present section and in the next section, the typed versions are key for the more important generalization purpose of the representation.

For the sake of a more manageable notation, we will use \( a \) and \( b \) (resp. \( x \) and \( y \)) to indicate both elements of \( A \) (resp. \( X \)) and their corresponding elements in \( S_A \) (resp. \( S_X \)), relying on the types of the relations for disambiguation.

The next definition introduces notation for the four ways in which a given binary relation on a set can be lifted to its typed counterparts. The four typed versions will give rise to corresponding modal operators which, unlike in the classical setting, are not interdefinable in the setting of enriched formal contexts.

**Definition 3.1.** For every \( R \subseteq S \times S \), we let

1. \( I_R \subseteq S_A \times S_X \) such that \( a I_R x \iff a R x \);
2. \( J_R \subseteq S_X \times S_A \) such that \( x J_R a \iff x R a \);
3. \( H_R \subseteq S_A \times S_A \) such that \( a H_R b \iff a R b \);
4. \( K_R \subseteq S_X \times S_X \) such that \( x K_R y \iff x R y \).

**Lemma 3.2.** For every \( R \subseteq S \times S \),

\[
(I_R^{-1})^{-1} = I_R^{-1}, \quad (L_R^{-1})^{-1} = L_R^{-1}, \quad (H_R^{-1})^{-1} = H_R^{-1}, \quad (K_R^{-1})^{-1} = K_R^{-1}.
\]

**Proof.** As to the second identity, \( x (I_R^{-1})^{-1} a \iff a I_R x \iff a R^{-1} a \iff x J_R^{-1} a \). The remaining ones are proved similarly.

Notice that the notation \((\cdot)^{-1}\) applies less well to the setting of typed relations than to the untyped setting. Indeed, if \( R \in \mathcal{P}(S \times S) \), then \( R^{-1} = \{ (t, s) \mid (s, t) \in R \} \in \mathcal{P}(S \times S) \), and hence \((\cdot)^{-1}\) defines an operation on \( \mathcal{P}(S \times S) \). However, as the lemma above shows, this is not always so in the typed setting, where \((\cdot)^{-1}\) may transform a relation into one of another type (contrast this with the typed version of relational composition we discuss in Section 3.4).
3.2. Sets as formal contexts

**Definition 3.3.** For any set $S$, we let $P_S := (S_A, S_X, I_A)$.

**Proposition 3.4.** If $S$ is a set, then $P_S^+ \cong \mathcal{P}(S)$.

**Proof.** Consider the map $h : \mathcal{P}(S) \to P_S^+$ defined by the assignment $P \mapsto (P_A, P_X)$. To show that this map is well defined it is enough to show that $P_A = (P_X^+)^1$. Indeed,

$$
(P_X^+)^1 = \{ a \in S_A \mid \forall x [ x \in P_X \Rightarrow a I_D a] \} \\
= \{ a \in S_A \mid \forall x [ x \notin P_X \Rightarrow a = x] \} \\
= \{ a \in S_A \mid \forall x [ a = x \Rightarrow x \in P_X] \} \\
= P_A.
$$

Verifying that $h$ is a Boolean algebra isomorphism is straightforward and omitted.

The next example shows that the requirement of the preservation of the complex algebra is not met by associating $S$ with the seemingly more obvious formal context $(S_A, S_X, I_D)$. 

**Example 3.5.** Let $S = \{a, b, c\}$ and $Q := (S_A, S_X, I_A)$. Then $\mathcal{P}(S)$ and $Q^+$ are represented by the Hasse diagrams below, and are clearly non-isomorphic.

**Remark 3.6.** The construction of **Definition 3.3** is perhaps the simplest way of associating a polarity $P_S$ with a set $S$ so that $P_S^+ \cong \mathcal{P}(S)$. However, there are others, which are equivalent to the one above precisely in the sense that **Proposition 3.4** holds for each of them. Hence, in the category of formal contexts, all these constructions will give rise to isomorphic formal contexts. We mention two more alternative constructions, since they will become relevant in Section 7.2. The first one consists in defining $P_S$ as

$$P_S := (S_A, 2 \times S_X, I_A),$$

where $2$ is the two-element Boolean algebra, and $I_A$, represented as characteristic function $I_A : S_A \times (2 \times S_X) \to 2$, is defined by the assignment $(a, (\alpha, x)) \mapsto \Delta(a, \alpha, x)$, i.e. $I_A(a, (\alpha, x))$ iff $\alpha = 1$, or $\alpha = 0$ and $a \sim x$. Since $2 \times S_X \cong S_{X_0} \cup S_{X_1}$, consider the map $h : \mathcal{P}(S) \to P_S^+$ defined by the assignment $P \mapsto (P_A, P_X^+ \cup S_{X_1})$. To show that this map is well defined it is enough to show that

$$P_A = (P_X^+ \cup S_{X_1})^1.$$ 

Indeed,

$$
(P_X^+ \cup S_{X_1})^1 = \{ a \in S_A \mid \forall (\alpha, x) [ (\alpha, x) \in P_X^+ \cup S_{X_1} \Rightarrow I_A(a, (\alpha, x))] \} \\
= \{ a \in S_A \mid \forall x [ x \notin P_X \Rightarrow a \sim x] \} \\
= \{ a \in S_A \mid \forall x [ a = x \Rightarrow x \in P_X] \} \\
= P_A
$$

The second construction consists in defining $P_S$ as

$$P_S := (S_A, 2^{S_X}, I_A).$$
where $I_{\Delta}$, represented as characteristic function $I_{\Delta} : S_{\Delta} \times 2^{S_{\Delta}} \to 2$, is defined by the assignment $(a, f) \mapsto f(a) - 0$, i.e. $I_{\Delta}(a, f)$ if $a \neq f$ iff $\forall x(f(x) = a - x)$ iff $\forall x(I_{\Delta}(a, x) \Rightarrow f(x) = 0)$. Since $2^{S_{\Delta}} \equiv \mathcal{P}(S_{\Delta})$, consider the map $h : \mathcal{P}(S_{\Delta}) \to \mathcal{P}_{\Sigma}^+$ defined by the assignment $P \mapsto (P_{A}, \{Q_{X} \mid Q \subseteq P\})$. To show that this map is well defined it is enough to show that $P_{A} = \{Q_{X} \mid Q \subseteq P\}$. Indeed,

$$
\{Q_{X} \mid Q \subseteq P\} = \{a \in S_{A} \mid \forall Q[Q \subseteq P \Rightarrow I_{\Delta}(a, Q_{X})]\} = \{a \in S_{A} \mid \forall Q[Q \cap P = \emptyset \Rightarrow a \neq Q]\} = \{a \in S_{A} \mid \forall Q[a \in Q \Rightarrow Q \cap P = \emptyset]\} = P_{A}.
$$

### 3.3. Kripke frames as enriched formal contexts

In the present subsection, we extend the construction of the previous subsection from sets to Kripke frames. For any Kripke frame $\mathcal{K} = (S, R)$, we let $F_{\mathcal{K}} := (P_{\mathcal{K}}, I_{\mathcal{K}}, J_{\mathcal{K}})$ where $P_{\mathcal{K}} = (S, S_{\mathcal{K}}, I_{\mathcal{K}})$ is defined as in the previous subsection. Since the concept lattice of $P_{\mathcal{K}}$ is isomorphic to $\mathcal{P}(S)$, the relations $I_{\mathcal{K}} \subseteq S_{\mathcal{K}} \times S_{\mathcal{K}}$ and $J_{\mathcal{K}} \subseteq S_{\mathcal{K}} \times S_{\mathcal{K}}$ are trivially $I_{\mathcal{K}}$-compatible, hence $F_{\mathcal{K}}$ is an enriched formal context (cf. Definition 2.5).

Recall that the complex algebra of $\mathcal{K}$ is the Boolean algebra with operator $\mathcal{K}^{+} = (\mathcal{P}(S), [R], [\not\exists])$ (cf. Section 2.1). The next proposition verifies that the embedding of Kripke frames into enriched formal contexts defined by the assignment $\mathcal{K} \mapsto F_{\mathcal{K}}$ makes the diagram of Section 1 commute.

**Proposition 3.7.** If $\mathcal{K}$ is a Kripke frame, then $F_{\mathcal{K}}^{+} \equiv \mathcal{K}^{+}$.

**Proof.** By Proposition 3.4, the complete lattice underlying $F_{\mathcal{K}}^{+}$ is $\mathcal{P}(S)$, so it is enough to show that for every $P \subseteq S$,

$$
(\langle R \rangle P_{A})_{A} = (\langle R '_{\mathcal{K}} \rangle P)_{A} \quad \text{and} \quad (\langle R \rangle P)_{A} = (\langle R '_{\mathcal{K}} \rangle P)_{A} = (\langle R '_{\mathcal{K}} \rangle P)_{A}.
$$

Similarly, one can show that

$$
(\langle R \rangle P)_{A} = (\langle R '_{\mathcal{K}} \rangle P)_{A} = (\langle R '_{\mathcal{K}} \rangle P)_{A} = (\langle R '_{\mathcal{K}} \rangle P)_{A}.
$$

Analogously to what we have observed in the previous subsection, representing $R$ by means of e.g. the relation $I_{\mathcal{K}}$ rather than $I_{\mathcal{K}}$ would fail to preserve the complex algebras, as shown by the following example.

**Example 3.8.** Consider the Kripke frame $\mathcal{K} = (S, \Delta)$ and the enriched formal context $F = (S, S_{\mathcal{K}}, I_{\mathcal{K}}, H_{\mathcal{K}})$.

By Proposition 3.4, $\mathcal{K}^{+}$ and $F^{+}$ are both based on $\mathcal{P}(S)$. Moreover, the operations $\langle A \rangle$ and $\langle \not\exists \rangle$ on $\mathcal{K}^{+}$ (i.e. the impossibility and skepticism operators on $\mathcal{P}(S)$ arising from $\Delta$) coincide with the Boolean negation. However, none of the operations $\langle A \rangle$, $\langle \not\exists \rangle$, $\langle \not\exists \rangle$ is a Boolean negation on $\mathcal{P}(S)$. Indeed, if $P$ is a subset of $S$ and $h : \mathcal{P}(S) \to \mathcal{P}_{\Sigma}^{+}$ is defined as in the proof of Proposition 3.4, then the extension of $\langle A \rangle$ is $S_{A}$ if $P = S$, is $P_{A}$ if $P$ is a coatom (i.e., the relative complement of a singleton), and is $\emptyset$ in any other case; the extension of $\langle \not\exists \rangle$ is $\emptyset$ if $P = \emptyset$, is $P_{A}$ if $P$ is a singleton, and is $\emptyset$ in any other case; the extension of $\langle \not\exists \rangle$ is $\emptyset$ if $P = S$, is $P_{A}$ if $P$ is a coatom, and is $S_{A}$ in any other case.

**Remark 3.9.** The two alternative lifting constructions discussed in Remark 3.6 can be expanded as follows, so as to accommodate the lifting of Kripke frames: as to the first one, for any Kripke frame $\mathcal{K} = (S, R)$, we let $F_{\mathcal{K}} := (P_{\mathcal{K}}, I_{\mathcal{K}}, J_{\mathcal{K}})$ where $P_{\mathcal{K}} := (S, 2^{S_{\mathcal{K}}}, I_{\mathcal{K}})$ and $I_{\mathcal{K}} : S_{\mathcal{K}} \times 2^{S_{\mathcal{K}}} \to 2$ is defined as $(a, Q) \mapsto x \Rrightarrow R(a, x) \Rightarrow x \neq Q$, and $J_{\mathcal{K}} : 2^{S_{\mathcal{K}}}) \times S_{\mathcal{K}} \to 2$ is defined as $(Q, a) \mapsto \forall x[R(a, x) \Rightarrow x \neq Q]$. As to the second one, for any Kripke frame $\mathcal{K} = (S, R)$, we let $F_{\mathcal{K}} := (P_{\mathcal{K}}, I_{\mathcal{K}}, J_{\mathcal{K}})$ where $P_{\mathcal{K}} := (S, 2^{S_{\mathcal{K}}}, I_{\mathcal{K}})$, and $I_{\mathcal{K}} : S_{\mathcal{K}} \times 2^{S_{\mathcal{K}}} \to 2$ is defined as $(a, Q) \mapsto \forall x[R(a, x) \Rightarrow x \neq Q]$, and $J_{\mathcal{K}} : 2^{S_{\mathcal{K}}} \times S_{\mathcal{K}} \to 2$ is defined as $(Q, a) \mapsto \forall x[R(a, x) \Rightarrow x \neq Q]$. 

---

7 Notice that, for any polarity $P = (A, X, I)$, if $\Delta_{P}$ (resp. $\Delta'_{P}$) is $J$-compatible, then every subset of $A$ (resp. $X$) is Galois-stable, and hence the concept lattice $P^{+}$ is a Boolean algebra.
\[
\left( J^0_P \right)_{P[A]}^1
\]
\[
\left( J^0_Q \right)_{Q[A]}^1
\]
\[
\left( \{ (a, x) \mid \forall a \in P \Rightarrow (x, x) \in \mathcal{R} \} \right)^1
\]
\[
\left( \{ Q \in 2^X \mid \forall a \in P \Rightarrow \mathcal{Q}(a) \} \right)^1
\]
\[
\left( \{ (a, x) \mid x = 1 \text{ or } x \neq R^{-1}[P] \} \right)^1
\]
\[
\left( \{ (Q \mid \exists x \in P \Rightarrow xRa) \} \right)^1
\]
\[
\left( \{ (R, P)_A \} \right)^1
\]

### 3.4. Lifting properties of relations

Based on the construction introduced in Section 3.3, in the present subsection we discuss how properties of accessibility relations of Kripke frames can be characterized as properties of their corresponding liftings. Towards this goal, let us recall that the usual composition of relations $R, T \subseteq X \times X$ is $R \circ T := \{ (u, v) \mid (u, v) \in R \text{ and } (v, w) \in T \text{ for some } w \in X \}$. The following definition slightly modifies those in [5,37]. As will be clear from Lemma 3.15, this definition can be understood as the ‘typed counterpart’ of the usual composition of relations in the setting of formal contexts. In Section 5, we will also discuss how this composition plays out in concrete contexts.

**Definition 3.10.** For any formal context $P = (A, X, I)$,

1. for all relations $R, T \subseteq X \times A$, the $I$-composition $R \circ T \subseteq X \times A$ is such that, for any $a \in A$ and $x \in X$,
   \[
   (R \circ T)(a) = R(0)[T(0)[a]], \quad \text{i.e. } x(R \circ T)a \iff x \in R(0)[T(0)[a]].
   \]
2. for all relations $R, T \subseteq A \times X$, the $I$-composition $R \circ T \subseteq A \times X$ is such that, for any $a \in A$ and $x \in X$,
   \[
   (R \circ T)(a) = R(1)[T(1)[a]], \quad \text{i.e. } a(R \circ T)x \iff a \in R(1)[T(1)[a]].
   \]

When the context is fixed and clear, we will simplify notation and write e.g. $R \circ T$ in stead of $R \circ T$. In these cases, we will also refer to $I$-composition as `composition`.

Notice that $I$-composition preserves the types of the input relations, and hence defines a binary operation on each algebra of relations of the same type. Under the assumption of $I$-compatibility, equivalent, alternative formulations of the $I$-composition of relations are available, as the following lemma shows.

**Lemma 3.11.** If $P = (A, X, I)$ is a formal context, then

1. for any $I$-compatible relations $R, T \subseteq X \times A$ and any $a \in A$ and $x \in X$,
   \[
   x(R \circ T)a \iff a \in T(1)[R(1)[a]].
   \]
2. for all $I$-compatible relations $R, T \subseteq A \times X$, and any $a \in A$ and $x \in X$,
   \[
   a(R \circ T)x \iff x \in T(1)[R(1)[a]].
   \]

**Proof.** The proofs of the two items are similar, so we will only prove item 1. By Definition 3.10, $x(R \circ T)a$ iff $x \in R(0)[T(0)[a]]$. By Lemma 2.6(ii) and the $I$-compatibility of $R$, the set $R(0)[T(0)[a]]$ is Galois-stable, so $x \in R(0)[T(0)[a]]$ iff $x \in R(0)[T(0)[a]]$. By Lemma 2.6(ii) the latter is the case iff $T(1)[R(1)[a]] \subseteq T(1)[x][a]$, which, by Lemma 2.6(2(iii)) is the case iff $T(0)[a] \subseteq T(1)[x][a]$. By Lemma 2.2(i), this implies that $T(1)[R(1)[a]] \subseteq T(1)[T(0)[a]]$, which is equivalent to $T(1)[R(1)[a]] \subseteq T(0)[a]$ (since $T(0)[a]$ is Galois stable). Once again applying Lemmas 2.2(ii) and 2.6(ii) we find that this is equivalent to $a \in T(1)[R(1)[a]]$. The argument for the converse is symmetric.
Lemma 3.12. If \( R, T \subseteq X \times A \) (resp. \( R, T \subseteq A \times X \)) and \( R \) is \( I \)-compatible, then so is \( R, T \).

Proof. We only prove the statement for \( R, T \subseteq X \times A \). Let \( a \in A \). By definition, \( (R; T)^{(0)}[a] = R^{(0)}[\left( T^{(0)}[a] \right)] \); since \( R \) is \( I \)-compatible, by Lemma 2.6.1, \( R^{(0)}[\left( T^{(0)}[a] \right)] \) is Galois-stable. By a similar argument one shows that \( (R; T)^{(1)}[x] \) is Galois-stable for any \( x \in X \).

The following lemma is a variant of [Lemmas 6 and 7] [5].

Lemma 3.13.

1. If \( R, T \subseteq A \times X \) are \( I \)-compatible, then for any \( B \subseteq A \) and \( Y \subseteq X \),
\[
(R; T)^{(0)}[Y] = R^{(0)}[\left( I^{(1)}(T^{(0)}[Y]) \right)] = (R; T)^{(1)}[B] = R^{(1)}[\left( I^{(0)}(T^{(1)}[B]) \right)].
\]

2. If \( R, T \subseteq X \times A \) are \( I \)-compatible, then for any \( B \subseteq A \) and \( Y \subseteq X \),
\[
(R; T)^{(1)}[Y] = R^{(1)}[\left( I^{(1)}(T^{(1)}[Y]) \right)] = (R; T)^{(0)}[B] = R^{(0)}[\left( I^{(0)}(T^{(0)}[Y]) \right)].
\]

Proof. We only prove the first identity of item 1, the remaining identities being proved similarly.

\[
R^{(0)}[\left( I^{(1)}(T^{(0)}[Y]) \right)] = R^{(0)}[\left( I^{(1)}(\bigcup_{x \in Y}[X]) \right)] = R^{(0)}[\left( I^{(1)}(\bigcap_{x \in Y}[X]) \right)] \quad \text{Lemma 2.2.5}
\]

\[
= R^{(0)}[\left( I^{(1)}(T^{(0)}[X]) \right)] \quad \text{T}^{(0)}[X] \text{ Galois – stable}
\]

\[
= R^{(0)}[\left( I^{(1)}(I^{(0)}[T^{(0)}[X]]) \right)] \quad \text{Lemma 2.2.5}
\]

\[
= R^{(0)}[\left( I^{(0)}(I^{(1)}[T^{(0)}[X]]) \right)] \quad \text{Lemma 2.6}
\]

\[
= \bigcap_{x \in Y} R^{(0)}[\left( I^{(1)}[T^{(0)}[X]] \right)] \quad \text{Lemma 2.2.5}
\]

\[
= \bigcap_{x \in Y} (R; T)^{(0)}[x] \quad \text{Definition of } R; T
\]

\[
= (R; T)^{(0)}[\bigcup_{x \in Y}[X]] \quad \text{Lemma 2.2.5}
\]

\[
= (R; T)^{(0)}[Y].
\]

Lemma 3.14.

1. If \( R, T, U \subseteq X \times A \) are \( I \)-compatible, then \( (R; T; U) = R; (T; U) \).
2. If \( R, T, U \subseteq A \times X \) are \( I \)-compatible, then \( (R; T; U) = R; (T; U) \).

Proof. We only show item 2. For every \( x \in X \),
\[
(R; (T; U))^{(0)}[x] = R^{(0)}[\left( I^{(1)}(T; U)^{(0)}[x] \right)]
\]

\[
= R^{(0)}[\left( I^{(1)}(T^{(0)}[U^{(0)}[x]]) \right)]
\]

\[
= (R; T)^{(0)}[\left( I^{(1)}[U^{(0)}[x]] \right)]
\]

\[
= ((R; T); U)^{(0)}[x].
\]
Lemma 3.15. For all $R, T \subseteq S \times S$,

1. $I_{(R; T)} = I^T_R : I^T_T$.
2. $J_{(R; T)} = J^T_R : J^T_T$.

Proof. We only prove item 2, the proof of item 1 being similar. For any $a \in S$,

$$
J_{(R; T)}^{0}[a] = \left( \{ x \in S \mid \forall b [ xRb \Rightarrow bT^c a] \} \right)_x
= \left( \{ x \in S \mid \forall b [ xTb \Rightarrow bR b] \} \right)_x
= \left( \{ x \in S \mid \forall b [ b \in T^{0}[a] \Rightarrow xRb] \} \right)_x
= \left\{ x \in S_x \mid \forall b [ b \in (T^{0}[a])_A \Rightarrow xRb] \right\}_x
= J_R^{[0]}(\{ x \in S \mid xTa \})_A
= J_R^{[0]}(\{ x \in S \mid xTa \})_A
= J_R^{[0]}(\{ x \in S \mid xT^c a \})
= J_R^{[0]}(\{ x \in S \mid xT^c a \})
$$

A relation $R \subseteq S \times S$ is sub-delta if $R = \{(z, z) \mid z \in Z\}$ for some $Z \subseteq S$, and is dense if $\forall x \forall y (sRt \Rightarrow \exists u (sRu \& uRt))$.

Proposition 3.16. For any Kripke frame $\mathcal{X} = (S, R)$,

1. $R$ is reflexive iff $I^c_R \subseteq I^c_{R'}$ iff $I^c_{R'} \subseteq I^c_{R}$.
2. $R$ is transitive iff $I^c_R \subseteq I^c_{R'}$ iff $I^c_{R'} \subseteq I^c_{R}$.
3. $R$ is symmetric iff $I^c_R = I^c_{R'}$ iff $I^c_{R'} = I^c_{R}$.
4. $R$ is sub-delta iff $I^c_{R'} \subseteq I^c_{R}$ iff $I^c_{R'} \subseteq I^c_{R}$.
5. $R$ is dense iff $I^c_R : I^c_R \subseteq I^c_{R'} : I^c_{R'}$.

Proof.

1. The reflexivity of $R$ is encoded in the inclusion $\Delta \subseteq R$, which can be equivalently rewritten as $R^c \subseteq \Delta^c$, which hence lifts as $I^c_{R} \subseteq I^c_{\Delta}$ and $J^c_{R} \subseteq J^c_{\Delta}$, with the latter inclusion being equivalent to $J^c_{\Delta} \subseteq J^c_{\Delta}$ (cf. Lemma 3.2), given that $\Delta^c$ is symmetric.

2. The transitivity of $R$ is encoded in the inclusion $R \circ R \subseteq R$, which can be equivalently rewritten as $R^c \subseteq (R \circ R)^c$, which hence lifts as $I^c_R \subseteq I^c_{R \circ R}$, and by Lemma 3.15 can be equivalently rewritten as $I^c_R \subseteq I^c_{R} : I^c_R$. The second inclusion is proved similarly.

3. The symmetry of $R$ is encoded in the identity $R = R^{-1}$, which can be equivalently rewritten as $R^{-1} = R^c$, which hence lifts as $I^c_R = I^c_{R^{-1}} = J^c_{R^{-1}}$ or equivalently as $J^c_{R} = I^c_{R}$.

4. $R$ is sub-delta if $R \subseteq \Delta$, which can be equivalently rewritten as $\Delta^c \subseteq R^c$, which hence lifts as $I^c_R \subseteq I^c_{\Delta}$ and $J^c_R \subseteq J^c_{\Delta}$, with the latter inclusion being equivalent to $I^c_{\Delta} \subseteq J^c_{\Delta}$, given that $\Delta^c$ is symmetric.

5. The denseness of $R$ is encoded in the inclusion $R \subseteq R \circ R$, which can be equivalently rewritten as $(R \circ R)^c \subseteq R^c$, which hence lifts as $I^c_{R \circ R} \subseteq I^c_{R^c}$, and by Lemma 3.15 can be equivalently rewritten as $I^c_{R} : I^c_{R} \subseteq I^c_{R^c}$. The second inclusion is proved similarly.

The proposition above characterizes well known untyped properties of binary relations on a given set in terms of typed properties of their liftings. As discussed at the beginning of the present section, this characterization provides the basis for generalizing the typed versions of these lifted properties to arbitrary formal contexts, e.g. by adopting the following terminology:

Definition 3.17. For any polarity $P = (A, X, I)$, a relation $R \subseteq A \times X$ (resp. $T \subseteq X \times A$) is

- reflexive if $R \subseteq I$ (resp. if $T^{-1} \subseteq I$)
- transitive if $R \subseteq R; R$ (resp. if $T \subseteq T; T$)
- subdelta if $I \subseteq R$ (resp. if $I \subseteq T^{-1}$)
- dense if $R; R \subseteq R$ (resp. if $T; T \subseteq T$).
4. Conceptual (co-) approximation spaces

In the present section we discuss the main contribution of this paper, namely the definition of conceptual (co-) approximation spaces. This definition is both a generalization and a modularization of Pawlak’s approximation spaces, aimed at modelling indiscernibility with relations which are not necessarily equivalence relations, and at exploring the possibility of modelling the interior and closure of sets via different relations.

Proposition 3.16 suggests a way for identifying, among the enriched formal contexts, the subclass of those which properly generalize approximation spaces. Recall that for any enriched formal context $\mathbf{F} = (\mathcal{P}, R_I, R_C)$, the relations $R_I$ and $R_C$ are defined as in Section 2.4.

**Definition 4.1.** A conceptual approximation space is an enriched formal context $\mathbf{F} = (\mathcal{P}, R_I, R_C)$ verifying the following condition:

$$R_I; R_C \subseteq I.$$ (8)

Such an $\mathbf{F}$ is reflexive if $R_I \subseteq I$ and $R_C \subseteq I$, is symmetric if $R_I = R_C$ or equivalently if $R_I = R_C$, and is transitive if $R_I \subseteq R_C; R_C$ and $R_C \subseteq R_I; R_C$ (cf. Definition 3.17).

The definition above identifies subclasses of enriched formal contexts defined by first-order conditions which characterize the required behaviour of the modal operators arising from these structures; specifically, that the modal operators $[R_I]$ and $[R_C]$ associated with a reflexive, symmetric and transitive conceptual approximation space are an interior and a closure operator on the lattice of concepts respectively, and are hence suitable to serve as lower and upper approximations of concepts (cf. Proposition 4.3). The following proposition accounts for the fact that, when restricted to formal contexts that correspond to sets, the notion of a reflexive, symmetric and transitive conceptual approximation space (cf. Definition 4.1) exactly captures the classical notion of an approximation space. This proposition is an immediate consequence of Propositions 3.16 and 3.7.

**Proposition 4.2.** If $\mathcal{X} = (S, R)$ is an approximation space, then $\mathbf{F}_\mathcal{X} = (\mathcal{P}_\mathcal{X}, I_{\mathcal{X}}, J_{\mathcal{X}})$ is a reflexive, symmetric and transitive conceptual approximation space such that $\mathbf{F}_\mathcal{X} \equiv \mathcal{X}^+$.

The following proposition shows that, from the perspective of logic, Definition 4.1 is the appropriate generalization of the notion of approximation space, since it characterizes in a modular way the axioms of the lattice-based counterpart of the modal logic of classical approximation spaces. In particular, by item 1, condition (8) characterizes the minimal condition for $\diamond$ and $\Box$ to convey the upper and lower approximations of given concepts; moreover, items 2 and 4 (resp. 3 and 5) exactly characterize the conditions under which the semantic $\Box(\phi)$ (resp. $\diamond$) is an interior (resp. closure) operator.

**Proposition 4.3.** For any enriched formal context $\mathbf{F} = (\mathcal{P}, R_I, R_C)$:

1. $\mathbf{F} \models \Box \phi \iff R_C; R_C \subseteq I.$
2. $\mathbf{F} \models \Box \phi \iff R_I \subseteq I.$
3. $\mathbf{F} \models \phi \iff R_I \subseteq I.$
4. $\mathbf{F} \models \Box \phi \iff R_C \subseteq R_I; R_C.$
5. $\mathbf{F} \models \diamond \phi \iff R_C \subseteq R_I; R_C.$
6. $\mathbf{F} \models \phi \iff R_C \subseteq R_C.$
7. $\mathbf{F} \models \Box \phi \iff I \subseteq R_C.$
8. $\mathbf{F} \models \phi \iff I \subseteq R_C.$
9. $\mathbf{F} \models \Box \phi \iff I \subseteq R_C.$
10. $\mathbf{F} \models \Box \phi \iff I \subseteq R_C.$
11. $\mathbf{F} \models \diamond \phi \iff R_I; R_C \subseteq R_C.$
12. $\mathbf{F} \models \diamond \phi \iff I \subseteq R_C.$

**Proof.** Notice that the modal principles in all items of the lemma are Sahlqvist (cf. [15] Definition 3.5)). Hence, they all have first order correspondents, both on Kripke frames and on enriched formal contexts, which can be effectively computed e.g. by running the algorithm ALBA (cf. [15] Section 4)) on each of them. In the remainder of this proof, we will use ALBA to compute these first-order correspondents on enriched formal contexts. In what follows, the variables $j$ are interpreted as elements of the set $J := \{(a^4, a^3) | a \in A\}$ which completely join-generates $I^+$, and the variables $m$ as elements of $M := \{(x^3, x^4) | x \in X\}$ which completely meet-generates $I^+$. 

386
\[ \forall p \exists p \leq \Diamond p \]

\[ \forall p \exists j \forall m (j \leq \Box p \land p \leq j \leq m) \Rightarrow j \leq m \]  
first approximation

\[ \forall p \forall j \forall m (j \leq \Box p \land p \leq j \leq m) \Rightarrow j \leq m \]  
adjunction

\[ \forall j \exists m (j \leq \Box m \Rightarrow j \leq m) \]  
Ackermann’s Lemma

\[ \forall m \exists j (j \leq \Box m \leq j \leq m) \]  
J c. join – generates \( \mathcal{F}^+ \)

Translating the universally quantified algebraic inequality above into its concrete representation in \( \mathcal{F}^+ \) requires using the interpretation of \( m \) as ranging in \( M \) and the definition of \( \mathcal{R}_\Delta \) and \( \mathcal{R}_\bullet \), as follows:

\[ \forall x \in X \ R_0^{(1)} \left( \mathcal{R}_0^{(0)} [x^1] \right) \subseteq x^1 \]

iff  
\[ \forall x \in X \ R_0^{(1)} \left( \mathcal{R}_0^{(0)} [x] \right) \subseteq F_0 [x] \]  
Lemma 2.6 since \( \mathcal{R}_\bullet \) is \( I \) – compatible

iff  
\[ \mathcal{R}_\bullet \subseteq \mathcal{R}_\Delta \]  
By definition

The proofs of the remaining items are collected in Appendix A.

The following are immediate consequences of the proposition above.

**Corollary 4.4.** For any enriched formal context \( \mathcal{F} = (\mathcal{P}, \mathcal{R}_\Delta, \mathcal{R}_\bullet) \),

1. If \( \mathcal{R}_\Delta \subseteq I \) and \( \mathcal{R}_\bullet \subseteq I \), then \( \mathcal{R}_\bullet \subseteq \mathcal{I} \), hence \( \mathcal{F} \) is a conceptual approximation space.
2. If \( I \subseteq \mathcal{R}_\Delta \) and \( I \subseteq \mathcal{R}_\bullet \), then \( I \subseteq \mathcal{R}_\bullet \subseteq \mathcal{R}_\Delta \), hence \( \mathcal{F} \) is a conceptual co-approximation space (cf. Definition 4.5 below).
3. If \( \mathcal{I} \subseteq \mathcal{R}_\Delta \), then \( \mathcal{R}_\Delta \subseteq \mathcal{R}_\bullet \subseteq \mathcal{I} \).
4. If \( I \subseteq \mathcal{R}_\Delta \), then \( \mathcal{R}_\Delta \subseteq \mathcal{R}_\bullet \).
5. If \( \mathcal{R}_\Delta \subseteq I \), then \( \mathcal{R}_\Delta \subseteq \mathcal{R}_\bullet \subseteq \mathcal{R}_\Delta \).
6. If \( \mathcal{R}_\bullet \subseteq I \), then \( \mathcal{R}_\bullet \subseteq \mathcal{R}_\Delta \subseteq \mathcal{R}_\bullet \).

**Proof.** 1. If \( \mathcal{R}_\Delta \subseteq I \) and \( \mathcal{R}_\bullet \subseteq I \), then by items 2 and 3 of Proposition 4.3, \( \mathcal{F} \models \Box \phi \vdash \phi \) and \( \mathcal{F} \models \phi \vdash \Diamond \phi \), hence \( \mathcal{F} \models \Box \phi \vdash \Diamond \phi \), which, by item 1 of Proposition 4.3, is equivalent to \( \mathcal{R}_\Delta \subseteq \mathcal{R}_\bullet \subseteq \mathcal{I} \), as required. The proof of item 2 is similar.

3. If \( I \subseteq \mathcal{R}_\Delta \) then by item 7 of Proposition 4.3, \( \mathcal{F} \models \phi \vdash \Box \phi \), hence \( \mathcal{F} \models \Box \phi \vdash \Diamond \phi \), which, by item 4 of Proposition 4.3, is equivalent to \( \mathcal{R}_\Delta \subseteq \mathcal{R}_\bullet \subseteq \mathcal{I} \), as required. The proof of the remaining items are similar.

In the classical setting, the ‘sub-delta’ condition (cf. Section 3.4, discussion before Proposition 3.16) identifies a very restricted class of relations, namely those corresponding to tests, which are not much mentioned in the literature of modal logic besides the context of PDL. Consequently, their corresponding modal axiomatic principles \( p \vdash \Box p \) and \( \Diamond p \vdash p \) do not commonly occur in the literature either. However, as stated in Proposition 4.3(8) and (9), in the present setting these principles characterize a natural and meaningful class of relations (which we refer to as ‘sub-delta’, by extension), which, as noticed in [27], has already cropped up in the literature on rough formal concept analysis [30], and is potentially useful for applications (see SubSections 5.4 and 5.5 below).

**Definition 4.5.** A conceptual co-approximation space is an enriched formal context \( \mathcal{F} = (\mathcal{P}, \mathcal{R}_\Delta, \mathcal{R}_\bullet) \) verifying the following condition:

\[ I \subseteq \mathcal{R}_\Delta \mathcal{R}_\bullet \]  
(9)

Such an \( \mathcal{F} \) is sub-delta if \( I \subseteq \mathcal{R}_\Delta \) and \( I \subseteq \mathcal{R}_\bullet \), is symmetric if \( \mathcal{R}_\bullet = \mathcal{R}_\bullet \), or equivalently if \( \mathcal{R}_\bullet = \mathcal{R}_\Delta \), and is dense if \( \mathcal{R}_\Delta \subseteq \mathcal{R}_\Delta \) and \( \mathcal{R}_\Delta \subseteq \mathcal{R}_\bullet \).

Recalling that the operators \( \Box \) and \( \Diamond \) are monotone (cf. 2.5), by items 8–11 of Proposition 4.3, sub-delta and dense conceptual co-approximation spaces are exactly those such that \( \mathcal{R}_\Delta \) is a closure operator and \( \mathcal{R}_\bullet \) is an interior operator. Hence, these operations are suitable to provide the upper and lower approximations of concepts, but in the reverse roles than those of Definition 4.1. Proposition 4.3.12 characterizes condition (9) of the definition above precisely in terms of these reverse roles.

| Conceptual approximation spaces | Conceptual co-approximation spaces |
|--------------------------------|-----------------------------------|
| reflexive \( \Box \phi \vdash \phi \) | \( \phi \vdash \Box \phi \) |
| transitive \( \Diamond \phi \vdash \Box \Diamond \phi \) | \( \Diamond \Diamond \phi \vdash \Diamond \phi \) |

387
5. Examples of conceptual (co-) approximation spaces

In this section, we illustrate, by way of examples, how conceptual (co-) approximation spaces can be used to model a variety of situations, and how, under different interpretations, the modal logic of formal concepts can be used to capture each of these situations.

5.1. Reflexive and symmetric conceptual approximation space

Consider a database, represented as a formal context \( P = (A, X, I) \). As usual, the intuitive understanding of \( a x \) is “object \( a \) has feature \( x \).” Consider an \( I \)-compatible relation \( R \subseteq A \times X \), intuitively understood as “there is evidence that object \( a \) has feature \( x \),” or “object \( a \) demonstrably has feature \( x \).” Under this intuitive understanding, it is reasonable to assume that \( R \subseteq I \).

Then, letting \( R_c := R \) and \( R_s := R^{-1} \), for any concept \( c \in P^+ \),

\[
[[R_c]c = R^0[[c]] = \{ a \in A \mid \forall x(x \in \{c\} \Rightarrow aRx) \}.
\]

That is, the members of \([R_c]c\) are exactly those objects that demonstrably have all the features in the description of \( c \), and hence \([R_c]c\) can be understood as the category of the ‘certified members’ of \( c \). The assumption that \( R \subseteq I \) implies that \( [[R_c]c = R^0[[c]] \subseteq I^0[[c]] = [c] \), hence \([R_c]c\) is a sub-concept of \( c \). Moreover,

\[
[[R_s]c] = R^{-1}[[c]] = \{ a \in A \mid \forall x(x \in \{c\} \Rightarrow aRx) \}.
\]

That is, \((R_s)c\) is the concept described by the set of features that each member of \( c \) demonstrably has, and hence \((R_s)c\) can be understood as the category of the ‘potential members’ of \( c \). Indeed, if \( a \in A \) lacks some feature that each member of \( c \) demonstrably has, then it is impossible for \( a \) to be a member of \( c \). The assumption that \( R \subseteq I \) implies that \( [[(R_s)c] = R^{-1}[[c]] \subseteq I^{-1}[[c]] = [c] \), and hence \((R_s)c\) is a super-concept of \( c \). Hence, \([R_c]c\) and \((R_s)c\) can be taken as the lower and upper approximations of the concept \( c \), respectively. Notice that, while in approximation spaces the relation \( R \) relates indiscernible sets, and thus encodes the information we do not have, in the present setting we take the opposite perspective and let \( R \) encode our (possibly partial) knowledge or information. This perspective is consistent with the fact that, when an approximation space \( X \) is lifted to \( F_X \), the indiscernibility relation \( R \) is encoded into the liftings of \( R \). By construction, the enriched formal context \( F = (P, R_c, R_s) \), with \( R_c \) and \( R_s \) as above, is a symmetric and reflexive, hence dense (cf. Corollary 4.4.3 and 4.4) conceptual approximation space.

5.2. Reflexivity and transitivity used normatively

Besides having a descriptive function, conditions such as reflexivity and transitivity can also be used normatively. To illustrate this point, let \( P = (A, X, I) \) represent the information of an undergraduate course, where \( A \) is the set of definitions/ notions the course is about, \( X \) is the set of properties, facts and results that can be attributed to these notions, and \( a x \) iff ‘notion \( a \) has property \( x \).’ Concepts arising from this representation are clusters of notions, intentionally described by Galois-stable sets of properties. Let \( R \subseteq A \times X \) be an \( I \)-compatible relation representing the amount of information about this course that a student remembers. Then \( R \) being reflexive corresponds to the requirement that the student’s recollections be correct, and \( R \) being transitive corresponds to the requirement that if the student remembers that notion \( a \) has property \( x \), then the student should also remember that notion \( a \) has every (other) property which is shared by all the notions which the student remembers to have property \( x \). So, under this interpretation, transitivity captures a desirable requirement that well organized and rational memories ought to satisfy (but might not satisfy), and hence in certain contexts, conditions such as reflexivity and transitivity can be used e.g. to discriminate between the ‘good’ and the ‘bad’ approximations of \( I \). By items 2–5 of Proposition 4.3, these ‘good’ approximations can be identified not only by first-order conditions on conceptual approximation spaces, but also by means of ‘modal axioms’, i.e. sequents in the language of the modal logic of concepts (cf. Section 2.5).

5.3. Relativizations

A natural large class of reflexive and transitive conceptual approximation spaces consists of those enriched formal contexts \( F = (A, X, I, R) \) such that \( R = I \cap (B \times Y) \) for some \( B \subseteq A \) and \( Y \subseteq X \), in which case we say that \( R \) relativizes \( I \) to \( B \subseteq A \) and \( Y \subseteq X \). Recall that a polarity is separated if for all \( a, b \in A \) if \( a \sim b \) then \( (a, x) \in I \) and \( (b, x) \notin I \) for some \( x \in X \), and for all \( x, y \in X \) if \( x \sim y \) then \( (a, x) \in I \) and \( (a, y) \notin I \) for some \( a \in A \). In separated polarities, \( \emptyset = X^1 = A^1 \) is Galois-stable.

**Lemma 5.1.** For any separated polarity \( P = (A, X, I) \) and any \( B \subseteq A \) and \( Y \subseteq X \), the structure \( F = (A, X, I, R_c, R_s) \) such that \( R_c = R = I \cap (B \times Y) \) and \( R_s = R^{-1} \) is a reflexive, symmetric and transitive conceptual approximation space.

\footnote{When \( Y = X \) then we say that \( R \) relativizes \( I \) to \( B \subseteq A \), and when \( B = A \) then we say that \( R \) relativizes \( I \) to \( Y \subseteq X \).}
Proof. For any \( x \in X \), either \( R^0(0)[x] = I^0(0)[x] \) if \( x \in Y \), or \( R^0(0)[x] = \emptyset \) if \( x \notin Y \). In both cases, \( R^0(0)[x] \) is Galois-stable. Likewise, one shows that \( R^1(0)[a] \) is Galois-stable for any \( a \in A \), which completes the proof that \( \mathcal{F} \) is an enriched formal context. By construction, \( \mathcal{F} \) is symmetric; moreover \( R \subseteq I \), hence \( \mathcal{F} \) is also reflexive. To show that \( \mathcal{F} \) is transitive, it is enough to show that \( R^0(0)[x] \subseteq R^0(0)[I^1(0)[R^0(0)[x]]] \) for every \( x \in X \). We proceed by cases. If \( x \notin Y \), then \( R^0(0)[x] = \emptyset \) and hence the inclusion holds. If \( x \in Y \), then \( R^0(0)[x] = I^0(0)[x] \subseteq B \), hence \( I^1(0)[R^0(0)[x]] = R^1(0)[R^0(0)[x]] \subseteq Y \). For any \( x \in X \), the set of patients, \( A \) shows that \( R^0(0)[x] \) is symmetric, and \( R^1(0)[x] \) is transitive, that is, \( R^0(0)[x] = R^0(0)[R^1(0)[R^0(0)[x]]] \), as required.

Enriched formal contexts \( \mathcal{F} = (A, X, I, R) \) such that \( R = I \cap (B \times Y) \) for some \( B \subseteq A \) and \( Y \subseteq X \) naturally arise in connection with databases endowed with designated sets of objects and/or features. Concrete examples of these abound: for instance, in databases of market products, subsets of features that are relevant for the decision-making of certain classes of consumers can naturally be modelled as designated subsets; in databases of words (as objects) and Twitter ‘tweets’ (as features), hashtags can be grouped together so as to form designated subsets of objects; in longitudinal datasets, objects and features first appearing after a certain point in time can form designated sets. Under this intuitive understanding, letting \( R_0 := R \) and \( R_0 := R^{-1} \), for any concept \( c \in P^+ \),

\[
[[R_0]c] = R^0(0)[[[c]c]] = \{a \in A \mid \forall x(x \in [[c]c]) \Rightarrow aRx\}.
\]

That is, \( [R_0]c \) is the empty category if \( [[c]c] \not\subseteq Y \) and if \( [[c]c] \subseteq Y \), then the members \( [R_0]c \) are exactly those designated objects that have all the features in the description of \( c \), and hence \( [R_0]c \) can be understood as the designated restriction of \( c \). Moreover,

\[
[[R_0]c] = R^1(0)[[[c]c]] = \{x \in X \mid \forall a(a \in [[c]c]) \Rightarrow aRx\}
\]

That is, \( [R_0]c \) is the universal category if \( [[c]c] \subseteq B \), and if \( [[c]c] \subseteq B \), then \( [R_0]c \) is the concept described by the set of designated features that are common to all members of \( c \), and hence, \( [R_0]c \) can be understood as the designated enlargement of \( c \).

5.4. Sub-delta and symmetric conceptual co-approximation space

Text databases can be modelled as formal contexts \( P = (A, X, I) \) such that \( A \) is a set of documents, \( X \) is a set of words, and \( aRx \) is understood as “document \( a \) has word \( x \) as its keyword”. Formal concepts arising from such a database can be understood as themes or topics, intentionally described by Galois-stable sets of words. Consider an \( I \)-compatible relation \( R \subseteq A \times X \), intuitively understood as \( aRx \) iff “document \( a \) has word \( x \) or one of its synonyms as its keyword”. This understanding makes it reasonable to assume that \( I \subseteq R \). Then, letting \( R_1 := R \) and \( R_0 := R^{-1} \), for any concept (theme) \( c \in P^+ \),

\[
[[R_1]c] = R^0(0)[[[c]c]] = \{a \in A \mid \forall x(x \in [[c]c]) \Rightarrow aRx\}.
\]

That is, the members of \( [R_1]c \) are those documents the keywords of which include all the words describing topic \( c \) or their synonyms. The assumption that \( I \subseteq R \) implies that \( [c] = I^0(0)[[[c]c]] \subseteq R^0(0)[[[c]c]] = [[R_1]c] \), hence \( [R_1]c \) is a super-concept of \( c \). Moreover,

\[
[[R_0]c] = R^1(0)[[[c]c]] = \{x \in X \mid \forall a(a \in [[c]c]) \Rightarrow aRx\}.
\]

That is, \( [R_0]c \) is the theme by the set of common keywords of all documents in \( c \) and their synonyms. The assumption that \( I \subseteq R \) implies that \( [[c]c] = I^1(0)[[[c]c]] \subseteq R^1(0)[[[c]c]] = [[R_0]c] \), and hence \( [R_0]c \) is a sub-concept of \( c \). By construction, the enriched formal context \( \mathcal{F} = (P, R_1, R_0, c) \), with \( R_1 \) and \( R_0 \) as above is a symmetric and sub-delta, hence transitive (cf. Corollary 4.4.5 and .6) conceptual co-approximation space.

5.5. Sub-delta and denseness used normatively

To illustrate how sub-delta and denseness can also be used normatively, let \( P = (A, X, I) \) represent a hospital, where \( A \) is the set of patients, \( X \) is the set of symptoms, and \( aRx \) iff “patient \( a \) has symptom \( x \)”. Concepts arising from this representation are syndromes, intentionally described by Galois-stable sets of symptoms. Let \( R \subseteq A \times X \) be an \( I \)-compatible relation the intuitive interpretation of which is \( aRx \) iff “according to the doctor, patient \( a \) has symptom \( x \)” or “the doctor attributes symptom \( x \) to patient \( a \)”. Under this interpretation, \( R \) being sub-delta corresponds to the requirement that if a patient has a symptom, then the doctor correctly attributes this symptom to the patient. The relation \( R \) being dense corresponds to the requirement that for any given patient \( a \) and symptom \( x \), if, according to the doctor, \( a \) has all the symptoms shared by all patients to whom he/she attributes symptom \( x \), then the doctor will also attribute \( x \) to \( a \). So, under this interpretation, the denseness condition corresponds to a principled and grounded way of making attributions by default. In conclusion, the first-order conditions of being sub-delta and dense capture desirable requirements that doctors’ judgments ought to have, and hence, as discussed in previous examples, in certain situations they can be useful to discriminate between ‘good’ and ‘bad’ approximations of \( I \). By items 8–11 of Proposition 4.3, the ‘good’ approximations identified by these conditions can be identified also in the language of the modal logic of concepts (cf. Section 2.5).
5.6. Conceptual bi-approximation spaces

In the previous subsections, we have discussed situations which are naturally formalized either by conceptual approximation spaces or by conceptual co-approximation spaces. However, in certain cases we might want to work with enriched formal contexts \((A,X,I,R,S)\) such that the \(I\)-compatible relations \(R\) and \(S\) give rise to a co-approximation space and to an approximation space respectively. We will refer to these structures as conceptual bi-approximation spaces. Examples of conceptual bi-approximation spaces naturally arise in connection with Kent’s rough formal contexts [30], as is discussed in what follows [a more detailed presentation can be found in [27] [Section 3]].

Rough formal contexts, introduced by Kent in [30] as structures synthesizing approximation spaces and formal contexts, are tuples \(\mathbb{G} = (\mathbb{P}, E)\) such that \(\mathbb{P} = (A,X,I)\) is a polarity, and \(E \subseteq A \times A\) is an equivalence relation. For every \(a \in A\) we let \((a)_E := \{b \in A \mid aEb\}\). The relation \(E\) induces two relations \(R, S \subseteq A \times X\) approximating \(I\), defined as follows: for every \(a \in A\) and \(x \in X\),

\[
aRxiff b \in (a)_E; \quad aSxiff b \in (a)_E.
\]

By definition, \(R\) and \(S\) are \(E\)-definable (i.e. \(R(a) = \bigcup_{x \in X} (a)_E\) and \(S(a) = \bigcup_{y \in Y} (a)_E\) for any \(x \in X\), and \(E\) being reflexive immediately implies that \(S \subseteq I\) and \(I \subseteq R\). Intuitively, \(R\) can be understood as the lax version of \(I\) determined by \(E\), and \(S\) as its strict version determined by \(E\). Let us assume that \(R\) and \(S\) are \(I\)-compatible. This assumption does not imply that \(E\) is \(I\)-compatible: indeed, let \(\mathbb{G} = (\mathbb{P}, \Delta)\) for any polarity \(\mathbb{P}\) such that not all singleton sets of objects are Galois-stable. Hence, \(E = \Delta\) is not \(I\)-compatible, but \(E = \Delta\) implies that \(R = S = \Pi\), hence \(R\) and \(S\) are \(I\)-compatible. However, under the assumption that \(E\) is also \(I\)-compatible, the following inclusions hold (cf. [27] [Lemma 3])

\[
R; R \subseteq R \quad \text{and} \quad S \subseteq S; S.
\]

Hence, rough formal contexts \(\mathbb{G} = (\mathbb{P}, E)\) such that \(E\), \(R\) and \(S\) are \(I\)-compatible give naturally rise to conceptual bi-approximation spaces the ‘approximation reduct’ of which is reflexive and transitive, and ‘co-approximation reduct’ of which is sub-delta and dense.

5.7. Expanded modal language of Kent’s rough formal contexts

Related to what was discussed in the previous subsection and in Section 2.6, in [27], a modal logic is introduced for rough formal contexts \(\mathbb{G} = (\mathbb{P}, E)\) such that \(E\), \(R\) and \(S\) are \(I\)-compatible. The modal signature of this logic is \(\{\Box, \Diamond, \Box, \Diamond\}\), where \(\Box\) and \(\Diamond\) (resp. \(\Box\) and \(\Box\)) are interpreted using \(S\) and \(S^{-1}\) (resp. \(R\) and \(R^{-1}\)). This language can be expanded with negative modal connectives \(\neg\) interpreted on rough formal contexts via the completely join-reversing operators \(|E|\) and \(|E^{-1}|\) on \(\mathbb{P}^+\) arising from \(E \subseteq A \times A\) (cf. Section 2.6), and such that for all \(c, d \in \mathbb{P}^+\),

\[
c \leq |E|d \iff d \leq |E^{-1}|c.
\]

Also for \(\neg\) and \(\neg\), correspondence-type results similar to those of Proposition 4.3 hold, such as the following:

**Proposition 5.2.** For any enriched formal context \(\mathbb{F} = (\mathbb{P}, \mathbb{R})\):

1. \(\mathbb{F} \models \phi \vdash \Box \phi \iff R = R\).
2. \(\mathbb{F} \models \Box \neg \phi \vdash \neg \phi \iff R = A \times A\).
3. \(\mathbb{F} \models \phi \vdash \Diamond \phi \vdash \bot \iff \forall a \in R_0(a) \Rightarrow a \in X^\perp\).
4. \(\mathbb{F} \models \neg \phi \vdash \Diamond \phi \vdash \bot \iff \forall a \in R_0(a) \Rightarrow A \subseteq R_0(a)\).

The proof of the proposition above can be found in Appendix A. The lemma below shows that, on rough formal contexts such that \(E\), \(R\) and \(S\) are \(I\)-compatible, the condition expressing the reflexivity (resp. transitivity) of \(E\) is equivalent to the condition expressing the reflexivity (resp. transitivity) of \(S\). Since the latter conditions are modally definable as \(\leq_{c, \Diamond, \Box}\) inequalities, these results are preliminary to completely axiomatize the modal logic of rough formal contexts (more on this in Section 8).

**Lemma 5.3.** For any enriched formal context \(\mathbb{G} = (\mathbb{P}, \mathbb{R})\), let \(S \subseteq A \times X\) denote the lower approximation of \(I\) induced by \(R, S \subseteq A \times A\). If \(S\) is \(I\)-compatible, then

1. \(\Delta \subseteq R\) \iff \(S \subseteq I\).
2. \(R \circ R, S \subseteq R\) \iff \(S \subseteq S; S\).

\(^9\) In [27], rough formal contexts \(\mathbb{G}\) as above such that \(R, S\) and \(E\) are \(I\)-compatible are referred to as amenable rough formal contexts, cf. Definition 4 therein.
The proof of the lemma above can be found in Appendix A.

5.8. Expanded modal language with negative modalities

In the present section, we discuss a situation which can be modelled by polarity-based structures supporting the semantics of negative modalities, similarly to the structures discussed in the previous subsection. Let \( P = (A, X, I) \) where \( A \) is the set of producers (companies), \( X \) is the set of products, and \( ax \) iff "company \( a \) produces \( x \)." Then, if \( a \) and \( b \) are companies, \( b \) is a total competitor of \( a \) (notation: \( a/b \)) iff \( a \subseteq b \), and are relative competitors (notation: \( aRb \)) if \( b \cap a \sim \emptyset \). By definition, \( T \) is an equivalence relation, while \( R \) is symmetric. When \( T, R \subseteq A \times A \) are \( I \)-compatible relations, they give rise to the completely join-reversing operators \( [T] \) and \( [R] \) on \( P^+ \) such that, for every concept \( c \in P^+ \),

\[
[T]c = T^0[[c]] = \{ a \in A \mid \forall b (b \in [c] \Rightarrow a/b) \}
\]

and

\[
[R]c = R^0[[c]] = \{ a \in A \mid \forall b (b \in [c] \Rightarrow aRb) \}.
\]

That is, the members of \( [T]c \) (resp. \( [R]c \)) are those companies which are total (resp. relative) competitors of each company in \( c \). It is interesting to notice that, while in Kent’s structures the relation \( E \subseteq A \times A \) gives rise to the two relations \( R, S \subseteq A \times X \) approximating \( I \), in the present case it is \( I \subseteq A \times X \) which gives rise to two relations \( R, S \subseteq A \times A \).

6. Conceptual rough algebras

Proposition 4.3 provides us with a modular link between logical axioms and subclasses of enriched formal contexts. This link can be extended to classes of algebras. In the present section, we introduce the classes of algebras that are the ‘conceptual’ counterparts of the classes of rough algebras listed in Section 2.2, in the sense that they can be understood as abstract representations of conceptual approximation and co-approximation spaces, in the same way in which rough algebras are to approximation spaces. Algebras based on ideas developed in the present section have been introduced in [27] motivated by the algebraic and proof-theoretic development of the logic of Kent’s rough formal contexts (cf. Section 5.7).

Definition 6.1. A conceptual (co-) rough algebra is a structure \( A = (\mathbb{L}, \Box, \Diamond) \) such that \( \mathbb{L} \) is a complete lattice, and \( \Box \) and \( \Diamond \) are unary operations on \( \mathbb{L} \) such that \( \Box \) is completely meet-preserving, \( \Diamond \) is completely join-preserving, and for any \( a \in \mathbb{L} \),

\[
\Box a \leq \Diamond a \quad \text{(resp. } \Diamond a \leq \Box a \text{)}.
\]

A conceptual rough algebra \( A \) as above is reflexive if for any \( a \in \mathbb{L} \),

\[
\Box a \leq a \quad \text{and} \quad a \leq \Diamond a.
\]

and is transitive if for any \( a \in \mathbb{L} \),

\[
\Box a \leq \Box \Box a \quad \text{and} \quad \Diamond a \leq \Diamond \Diamond a.
\]

A conceptual co-rough algebra \( A \) as above is sub-delta if for any \( a \in \mathbb{L} \),

\[
a \leq \Box a \quad \text{and} \quad \Diamond a \leq a.
\]

and is dense if for any \( a \in \mathbb{L} \),

\[
\Box \Box a \leq a \quad \text{and} \quad \Diamond \Diamond a \leq a.
\]

Finally, A conceptual (co-) rough algebra \( A \) as above is symmetric if for any \( a \in \mathbb{L} \),

\[
a \leq \Box a \quad \text{and} \quad \Diamond a \leq a.
\]

We let \( RA^+ \) (resp. \( CA^+ \)) denote the class of conceptual rough algebras (resp. co-rough algebras).

Proposition 6.2. If \( A = (\mathbb{L}, \Box, \Diamond) \in RA^+ \cup CA^+ \), then \( A \cong \mathbb{F} \) for some conceptual (co-) approximation space \( \mathbb{F} \) such that \( A \) validates any of the inequalities (11)–(15) iff \( \mathbb{F} \) does.

Proof. By assumption, \( \mathbb{L} \) is a complete lattice, hence \( \mathbb{L} \cong P^+ \) for some polarity \( P = (A, X, I) \) (cf. Theorem 2.4). For any \( a \in A \) and \( x \in X \), let \( a := (a^+, a^-) \in \mathbb{P}^+ \cong \mathbb{L} \) and \( x := (x^+, x^-) \in \mathbb{P}^+ \cong \mathbb{L} \). Let \( \mathbb{F} := (P, R_\Box, R_\Diamond) \), where \( R_\Box \subseteq A \times X \) and \( R_\Diamond \subseteq X \times A \) are defined as follows: for every \( a \in A \) and \( x \in X \),

\[
R_\Box^0[a] := \{ b \in A \mid b \leq \Box x \} \cong [\Box x] \quad \text{and} \quad R_\Diamond^0[a] := \{ y \in X \mid \Diamond a \leq y \} \cong (\Diamond a).
\]

Then, recalling that \( A \) join-generates \( P^+ \) identified with \( \mathbb{L} \), and \( X \) meet-generates \( P^+ \) identified with \( \mathbb{L} \), the definition above immediately implies that \( R_\Box \) and \( R_\Diamond \) are \( I \)-compatible. Moreover, for every \( x \in X \),

\[
[R_\Box x] = R_\Box^0[[x]] = R_\Box^0[x^+] = R_\Box^0[x] \cong [\Box x]
\]

and

\[
[R_\Diamond x] = R_\Diamond^0[[x]] = R_\Diamond^0[x^-] = R_\Diamond^0[x] \cong [\Diamond x].
\]
which is enough to prove that \([R_c]\) \(\cong \Box\), since both operations are completely meet-preserving and \(X\) meet-generates \(P^+\). Analogously, one shows that \([R_c]\) \(\cong \Diamond\), which completes the proof of the first part of the statement. The second part of the statement is an immediate consequence of the definition of satisfaction and validity of enriched formal contexts.

The following definition introduces the abstract versions of the algebras of Definition 6.1.

**Definition 6.3.** An abstract conceptual (co-) rough algebra (acronyms acra and accra, respectively) is a structure \(A = (\mathbb{L}, \Box, \Diamond)\) such that \(\mathbb{L}\) is a bounded lattice, and \(\Box\) and \(\Diamond\) are unary operations on \(\mathbb{L}\) such that \(\Box\) is finitely meet-preserving, \(\Diamond\) is finitely join-preserving, and for any \(a \in \mathbb{L}\),

\[
\Box a \leq \Diamond a \quad \text{(resp. } \Diamond a \leq \Box a). \]

An \(A\) as above is **reflexive** if for any \(a \in \mathbb{L}\),

\[
\Box a \leq a \quad \text{and} \quad a \leq \Diamond a.
\]

is **transitive** if for any \(a \in \mathbb{L}\),

\[
\Box a \leq \Box \Diamond a \quad \text{and} \quad \Diamond \Diamond a \leq \Diamond a.
\]

is **symmetric** if for any \(a \in \mathbb{L}\),

\[
a \leq \Box \Diamond a \quad \text{and} \quad \Diamond \Box a \leq a.
\]

is **sub-delta** if for any \(a \in \mathbb{L}\),

\[
a \leq \Box a \quad \text{and} \quad \Diamond a \leq a.
\]

and is **dense** if for any \(a \in \mathbb{L}\),

\[
\Box a \leq a \quad \text{and} \quad \Diamond a \leq \Diamond a.
\]

We let \(RA\) (resp. \(CA\)) denote the class of abstract conceptual (co-) rough algebras.

The classes of algebras defined above form varieties of lattice expansions (LEs) for which several duality-theoretic, universal algebraic and proof-theoretic results are available in generality and uniformity (cf. discussion at the end of Section 2.5). In particular, the inequalities in the definition above are all Sahlqvist inequalities (cf. [15] Definition 3.5), and hence the expanded Birkhoff’s representation theorem for complete modal algebras via enriched formal contexts (cf. Theorem 2.9) specializes to conceptual (co-) approximation spaces thanks to the fact that the relevant axioms are Sahlqvist and hence canonical (see [18] for an expanded treatment). Notice that reflexive and transitive accras are the ‘nondistributive’ (i.e. general lattice-based, hence conceptual) counterparts of topological quasi Boolean algebras (tqBa) (cf. [1]). In the next definition we introduce them, together with their mirror-image version (cf. discussion in Section 4).

**Definition 6.4.** A conceptual tqBa is a reflexive and transitive acra \(A\). A conceptual co-tqBa is a sub-delta and dense accra \(A\).

In the next definition we introduce the ‘nondistributive’ (i.e. general lattice-based) counterparts of the topological quasi Boolean algebras 5 (tqBa5), intermediate algebras of types 2 and 3 (IA2 and IA3), and pre-rough algebras (pra).

**Definition 6.5.** A conceptual tqBa (resp. co-tqBa) \(A\) as above is a conceptual tqBa5 (resp. co-tqBa5) if for any \(a \in \mathbb{L}\),

\[
\Diamond \Box a \leq \Box a \quad \text{and} \quad \Diamond a \leq \Box \Diamond a. \tag{17}
\]

A conceptual tqBa5 (resp. co-tqBa5) \(A\) as above is a conceptual IA2 (resp. co-IA2) if for any \(a, b \in \mathbb{L}\),

\[
\Box (a \lor b) \leq \Box a \lor \Box b \quad \text{and} \quad \Diamond a \lor \Diamond b \leq \Diamond (a \lor b), \tag{18}
\]

and is a conceptual IA3 (resp. co-IA3) if for any \(a, b \in \mathbb{L}\),

\[
\Box a \leq \Box b \quad \text{and} \quad \Diamond a \leq \Diamond b \quad \text{imply} \quad a \leq b. \tag{19}
\]

A conceptual prerough algebra (resp. conceptual co-prerough algebra) is a conceptual tqBa (resp. co-tqBa) \(A\) verifying (17)–(19).

**7. Applications**

In the present section, we apply suitably adapted versions of the methodology developed in the previous sections to generalize – from predicates to concepts – three very different and mutually independent semantic frameworks, respectively aimed at accounting for vagueness (Section 7.1), gradedness (Section 7.2), and uncertainty (Section 7.3).
7.1. Vague concepts

The vague vs discrete nature of categories is a central issue in the foundations of categorization theory, since it concerns the limits of applicability of linguistic, perceptual, cognitive and informational categories. Vague concepts such as ‘red’, ‘tall’, ‘heap’ or ‘house’ admit borderline cases, namely cases for which we are uncertain as to whether the concept should apply or not. Closely related to this, vague predicates give rise to paradoxes such as the sorites paradox, which in its best known formulation involves a heap of sand, from which grains are individually removed. Under the assumption that removing a single grain does not turn a heap into a non-heap, the paradox arises when considering what happens when the process is repeated enough times: is a single remaining grain still a heap? If not, when did it change from a heap to a non-heap? The assumption that a single remaining grain does not turn a heap into a non-heap, the paradox arises when considering what happens when the process is repeated enough times: is a single remaining grain still a heap? If not, when did it change from a heap to a non-heap? The assumption of the paradox can be formulated more abstractly as the following tolerance principle (cf. [6]): if a predicate \( P \) applies to an object \( a \), and \( a \) and \( b \) differ very little in respects relevant to the application of the predicate \( P \), then \( P \) also applies to \( b \). In [6], vague predicates are defined as those for which the tolerance principle holds, and a logical framework for the treatment of vague predicates is introduced which allows to validate the tolerance principle while preserving modus ponens. The logical framework of [6] hinges on the interplay among three notions of truth: the tolerant, the classical and the strict. Below we briefly report on the main definitions,\(^{10}\) and then we apply the insights developed in Section 3 to generalize this approach from vague predicates to vague concepts.

A (propositional) T-model over a set \( \text{AtProp} \) of proposition letters (cf. [6] Definition 4]) is a structure \( M = (D, \{ \sim_p \mid p \in \text{AtProp} \}, V) \), such that \( D \) is a nonempty set, \( \sim_p \subseteq D \times D \) is reflexive and symmetric for every \( p \in \text{AtProp} \), and \( V : \text{AtProp} \to \mathcal{P}(D) \) is a map. Propositional formulas \( \phi \) are satisfied classically at states \( a \in D \) on T-models \( M \) in the usual way (in symbols: \( M, a \vDash \phi \)); in addition, the strict and tolerant satisfaction relations (in symbols: \( M, a \vartriangleright \phi \) and \( M, a \triangleright \phi \)) are defined recursively as follows:

\[
\begin{align*}
M, a \triangleright p & \iff M, b \vDash p \text{ for some } b \in D \text{ such that } a \sim_p b; \\
M, a \triangleright p & \iff M, b \vDash p \text{ for every } b \in D \text{ such that } a \sim_p b; \\
M, a \triangleright \bot & \iff \text{never}; \\
M, a \triangleright \top & \iff \text{always}; \\
M, a \triangleright \neg \phi & \iff M, a \vartriangleright \phi; \\
M, a \triangleright \phi \lor \psi & \iff M, a \triangleright \phi \text{ or } M, a \triangleright \psi; \\
M, a \triangleright \phi \land \psi & \iff M, a \triangleright \phi \text{ and } M, a \triangleright \psi; \\
M, a \vartriangleright \psi & \iff M, a \triangleright \psi.
\end{align*}
\]

Letting \( V^1(\phi) := \{ a \in D \mid M, a \triangleright \phi \} \) and \( V^2(\phi) := \{ a \in D \mid M, a \vartriangleright \phi \} \), the following inclusions readily follow from the definitions above:

\[
V^2(\phi) \subseteq V(\phi) \subseteq V^1(\phi).
\]

Notice that the first two clauses in the definition of \( \equiv^1 \) and \( \equiv^2 \) can be rewritten as follows:

\[
M, a \equiv^1 p \iff M, a \equiv^2 (\sim_p) p; \\
M, a \equiv^2 p \iff M, a \equiv^1 (\sim_p) p.
\]

Clearly, T-models are generalizations of approximation spaces in which indiscernibility relations do not need to be transitive,\(^{11}\) and are specific for each predicate. Hence, the insights and the lifting construction of Section 3.3 can be applied to T-models, analogously to what is done in Section 4. Doing so, we readily arrive at the following.

**Definition 7.1.** A conceptual T-model over a set \( \text{AtProp} \) of atomic category labels is a structure \( M = (P, \{ R_p \mid p \in \text{AtProp} \}, V) \) such that \( P = (A, X, I) \) is a polarity, \( R_p \subseteq A \times X \) is \( I \)-compatible and reflexive for every \( p \in \text{AtProp} \), and \( V : \text{AtProp} \to P^{+} \) is a map.

In each conceptual T-model \( M \), the standard membership and description relations for concept-terms \( \phi \) of the propositional lattice language \( \mathcal{L} \) over \( \text{AtProp} \) (in symbols: \( M, a \vDash \phi \) and \( M, x \vDash \phi \)) are defined as indicated in Section 2.5.

\(^{10}\) For simplicity of presentation, here we only report on the propositional fragment of the framework of [6]. Also, given that the setting of [6] is classical, only a minimal functionally complete set of propositional connectives are considered in [6]. However, for the sake of an easier comparison with the setting of conceptual T-models which will be introduced in what follows, we explicitly report the satisfaction clauses for the whole signature of classical propositional logic.

\(^{11}\) Dropping the transitivity requirement has been independently explored in the Rough Set Theory literature, see e.g., [49].
**Definition 7.2.** For any conceptual T-model $\mathcal{M}$, the following strict and tolerant membership and description relations:

- $\mathcal{M}, a \vDash \phi$ which reads: object $a$ is definitely a member of category $\phi$
- $\mathcal{M}, a \vDash^1 \phi$ which reads: feature $x$ definitely describes category $\phi$
- $\mathcal{M}, a \vDash^1 \phi$ which reads: object $a$ is loosely a member of category $\phi$
- $\mathcal{M}, a \vDash^1 \phi$ which reads: feature $x$ loosely describes category $\phi$

are defined recursively as follows:

- $\mathcal{M}, a \vDash [R_p]p$ iff $a \in [R_p]p$.
- $\mathcal{M}, a \vDash^1 [R_p]p$ iff $a \in [R_p]p$.
- $\mathcal{M}, a \vDash^1 [R_p] \top$ iff $a \in [R_p] \top$.
- $\mathcal{M}, a \vDash^1 \bot$ iff $a \in \bot$.
- $\mathcal{M}, a \vDash^1 \phi \land \psi$ iff $a \in [\phi] \cap [\psi]$.
- $\mathcal{M}, a \vDash^1 \phi \lor \psi$ iff $a \in [\phi] \cup [\psi]$.
- $\mathcal{M}, a \vDash^1 \phi \rightarrow \psi$ iff $a \in [\phi] \rightarrow [\psi]$.
- $\mathcal{M}, a \vDash^1 \phi \leftrightarrow \psi$ iff $a \in [\phi] \leftrightarrow [\psi]$.

Hence, any conceptual T-model $M = (p, \{R_p | p \in \text{AtProp}\}, V)$ induces the tolerant and strict interpretations $V^t, V^s : \mathcal{L} \rightarrow \{0, 1\}$ of $\mathcal{L}$-terms, defined as follows:

$$V^t(p) = \left( R^1 p[\{p\}], R^1 p[\{p\}] \right)$$

$$V^s(p) = \left( R^0 p[\{p\}], R^0 p[\{p\}] \right)$$

$$V^t(\top) = \left( A, A \right)$$

$$V^s(\top) = \left( A, A \right)$$

$$V^t(\bot) = \left( X, X \right)$$

$$V^s(\bot) = \left( X, X \right)$$

$$V^t(\phi \land \psi) = \left( [\phi] \cap [\psi], [\phi] \cap [\psi] \right)$$

$$V^s(\phi \land \psi) = \left( [\phi] \cap [\psi], [\phi] \cap [\psi] \right)$$

$$V^t(\phi \lor \psi) = \left( [\phi] \cup [\psi], [\phi] \cup [\psi] \right)$$

$$V^s(\phi \lor \psi) = \left( [\phi] \cup [\psi], [\phi] \cup [\psi] \right)$$

$$V^t(\phi \rightarrow \psi) = \left( [\phi] \rightarrow [\psi], [\phi] \rightarrow [\psi] \right)$$

$$V^s(\phi \rightarrow \psi) = \left( [\phi] \rightarrow [\psi], [\phi] \rightarrow [\psi] \right)$$

$$V^t(\phi \leftrightarrow \psi) = \left( [\phi] \leftrightarrow [\psi], [\phi] \leftrightarrow [\psi] \right)$$

$$V^s(\phi \leftrightarrow \psi) = \left( [\phi] \leftrightarrow [\psi], [\phi] \leftrightarrow [\psi] \right)$$

**Lemma 7.3.** For any conceptual T-model $\mathcal{M}$ and any $\mathcal{L}$-term $\phi$, $V^s(\phi) \leq V^t(\phi)$.

**Proof.** By induction on $\phi$. As to the base cases, $R_p \subseteq I$ for every $p \in \text{AtProp}$ implies that $[p] = R^0 p[\{p\}] \subseteq l^0 p[\{p\}] = [p]$ and $[p]^t = R^1 p[\{p\}] \subseteq l^1 p[\{p\}] = [p]$. and moreover, $[\bot] = R^0 X \subseteq l^0 X = [\bot]$ and $[\top] = R^1 A \subseteq l^1 A = [\top]$. The induction steps are straightforward.
**Definition 7.4.** For any conceptual T-model \( \mathcal{M} \) and any \( p \in \text{AtProp} \), the relation \( R_p \) induces *similarity relations* on objects and on features defined as follows: for any \( a, b \in A \) and \( x, y \in X \),

\[
a \sim_{R_p} b \quad \text{iff} \quad R_p^{(3)}[a] \subseteq b, \quad x \sim_{R_p} y \quad \text{iff} \quad R_p^{(3)}[x] \subseteq y.
\]

We use the same symbol to denote both relations and rely on the input arguments \( (a, b \text{ for objects, } x, y \text{ for features}) \) for disambiguation. Clearly, the reflexivity of \( R_p \) implies that each \( \sim_{R_p} \) is reflexive. However, in general these relations are neither transitive nor symmetric. The next lemma shows that when \( R_p \) is the lifting of some (classical) similarity relation \( \sim_p \), the similarity relations \( \sim_{R_p} \) induced by \( R_p \) are isomorphic copies of \( \sim_p \).

**Lemma 7.5.** For every T-model \( \mathcal{M} = (D, \{ \sim_p \mid p \in \text{AtProp} \}, V) \), its associated conceptual T-model \( \mathcal{F}_\mathcal{M} := \left( (D_A, D_X, I_A), \{ I_p \mid p \in \text{AtProp} \}, V \right) \) (cf. Section 3.3) is such that for every \( p \in \text{AtProp} \) and all \( d, d' \in D \),

\[
d \sim_p d' \quad \text{iff} \quad d \sim_{I_p} d'.
\]

**Proof.** In the displayed equivalence above, \( d \sim_{I_p} d' \) refers both to \( d, d' \in D_A \) and to \( d, d' \in D_X \). For \( d, d' \in D_A \),

\[
d \sim_{I_p} d' \quad \text{iff} \quad I_p^{(3)}[d] \subseteq (d')^{*}
\]

\[
\text{iff} \quad \{ e \in D_X \mid d \sim_p e \} \subseteq \{ e \in D_X \mid d' \sim_p e \}
\]

\[
\text{iff} \quad \{ e \in D_X \mid d \sim_p e \}^c \subseteq \{ e \in D_X \mid d' \sim_p e \}^c
\]

\[
\text{iff} \quad d' \in \{ e \in D_X \mid d \sim_p e \}^c \quad \text{and} \quad \{ e \in D_X \mid d' \sim_p e \}^c = \{ d' \}
\]

\[
\text{iff} \quad (d, d') \notin I_p
\]

\[
\text{iff} \quad d \sim_p d'.
\]

The proof in the case in which \( d, d' \in D_X \) is similar, and is omitted.

**Lemma 7.6.** For any conceptual T-model \( \mathcal{M} \) and any \( p \in \text{AtProp} \),

1. If \( \mathcal{M}, a \Vdash p \) and \( a \sim_{R_p} b \), then \( \mathcal{M}, b \Vdash p \);
2. If \( \mathcal{M}, x \sim_p y \) and \( x \sim_{R_p} y \), then \( \mathcal{M}, y \Vdash p \).

**Proof.** 1. The assumption \( \mathcal{M}, a \Vdash p \) can be rewritten as \( a \in [p]^A = R_p^{(3)}([p]), \) i.e. \( ([p]) \subseteq R_p^{(3)}[a] \). Hence, \( R_p^{(3)}([p]) \subseteq I_p^{(3)}([p]) = ([p]) \subseteq R_p^{(3)}[a] \). We need to show that \( \mathcal{M}, b \Vdash p \), i.e. \( b \in \left( R_p^{(3)}([p]) \right)^c \), i.e. \( R_p^{(3)}([p]) \subseteq b^c \). The latter inclusion immediately follows from \( R_p^{(3)}([p]) \subseteq R_p^{(3)}[a] \) and the assumption that \( a \sim_{R_p} b \), i.e. \( R_p^{(3)}[a] \subseteq b^c \). The proof of the second item is similar, and is omitted.

As discussed in [6] Section 6], the semantics of T-models successfully handles e.g. the following version of the sorites paradox, formulated in terms of the strict and the tolerant notions of truth:

\[
\text{If } \mathcal{M}, a \Vdash p \text{ and } a \sim_{R_p} a_{i+1} \text{ for every } 1 \leq i \leq n \text{ then } \mathcal{M}, a \Vdash p.
\]

Indeed, since \( \sim_p \) does not need to be transitive, T-models exist which falsify (20), while for every T-model \( \mathcal{M} \),

\[
\text{if } \mathcal{M}, a \Vdash p \text{ and } a \sim_p b \text{ then } \mathcal{M}, b \Vdash p.
\]

The following versions of the sorites paradox, formulated both w.r.t. objects and w.r.t. features, can be handled equally well by conceptual T-models:

\[
\text{If } \mathcal{M}, a \Vdash p \text{ and } a \sim_{R_p} a_{i+1} \text{ for every } 1 \leq i \leq n \text{ then } \mathcal{M}, a \Vdash p
\]

\[
\text{If } \mathcal{M}, x \Vdash p \text{ and } x \sim_{R_p} x_{i+1} \text{ for every } 1 \leq i \leq n \text{ then } \mathcal{M}, x \Vdash p
\]

Indeed, by Lemma 7.6, each single step is valid on every conceptual T-model; however, if \( \mathcal{M} \) is a T-model falsifying (20), then by Lemma 7.5, its associated conceptual T-model \( \mathcal{F}_\mathcal{M} \) will falsify (22) and (23).
7.2. A many-valued semantics for the modal logic of concepts

In [22] several kinds of many-valued Kripke frame semantics are proposed for the language of normal modal logic. Based on this, [3] introduces complete axiomatizations for the many-valued counterparts of the classical normal modal logic \( K \) corresponding to some of the semantic settings introduced there, while simultaneously generalizing the choice of truth value space from Heyting algebras to residuated lattices, subject to certain constraints. In the present subsection, we illustrate how the insights and results developed in the previous sections can be suitably adapted to generalize the logical framework of [22,3] from (many-valued) Kripke frames to (many-valued) enriched formal contexts. Below, we will first provide a brief account of the framework from [22,3] we are going to generalize. Since it is intended only as an illustration, our account focuses on a restricted many-valued setting in which the algebra of truth-values is a Heyting algebra. Moreover, in presenting the classical many-valued modal logic framework, we will cover only the portion that is directly involved in the subsequent generalization, and we leave a more complete and general treatment for a separate paper (cf. [17]).

7.2.1. Many-valued modal logic

For an arbitrary but fixed complete Heyting algebra \( A \) (understood as the algebra of truth-values), an \( A \)-Kripke frame (cf. [22] Definition 4.1) is a structure \( \mathcal{X} = (W, R) \) such that \( W \) is a nonempty set and \( R \) is an \( A \)-valued relation, i.e. it is a map \( R : W \times W \to A \) which we will equivalently represent as \( R : W \to A^W \), where \( A^W \) denotes the set of maps \( f : W \to A \), so \( R[w] : W \to A \) for every \( w \in W \). As is well known, the algebraic structure of \( A \) lifts to \( A^W \) by defining the operations pointwise; moreover, any \( A \)-valued relation \( R \) induces operations \( [R], (R) : A^W \to A^W \) such that, for every \( f : W \to A \),

\[
[R]f : W \to A \quad \quad \quad (R)f : W \to A
\]

\[
w \mapsto \bigwedge_{w \in W} (R[w](v) \to f(v)) \quad \quad \quad w \mapsto \bigvee_{w \in W} (R[w](v) \land f(v)).
\]

An \( A \)-model over a set AtProp of atomic propositions is a tuple \( \mathcal{M} = (\mathcal{E}, V) \) such that \( \mathcal{E} \) is an \( A \)-Kripke frame and \( V : \text{AtProp} \to A^W \). Every such \( V \) has a unique homomorphic extension, also denoted \( V : \mathcal{L} \to A^W \), where \( \mathcal{L} \) denotes the \( \{\Box, \Diamond\} \) modal language over AtProp, which in its turn induces \( \alpha \)-satisfaction relations for each \( \alpha \in A \) (in symbols: \( \mathcal{M}, w \models^\alpha \phi \)), such that for every \( \phi \in \mathcal{L} \),

\[
\mathcal{M}, w \models^\alpha \phi \iff \alpha \leq (V(\phi))(w).
\]

This can be equivalently expressed by means of the following recursive definition:

\[
\begin{align*}
\mathcal{M}, w \models^\alpha p & \iff \alpha \leq (V(p))(w); \\
\mathcal{M}, w \models^\alpha \top & \iff \alpha \leq (V(\top))(w) \text{ i.e. always}; \\
\mathcal{M}, w \models^\alpha \bot & \iff \alpha \leq (V(\bot))(w) \text{ i.e. iff } \alpha = \bot; \\
\mathcal{M}, w \models^\alpha \phi \land \psi & \iff \mathcal{M}, w \models^\alpha \phi \text{ and } \mathcal{M}, w \models^\alpha \psi; \\
\mathcal{M}, w \models^\alpha \phi \lor \psi & \iff \mathcal{M}, w \models^\alpha \phi \text{ or } \mathcal{M}, w \models^\alpha \psi; \\
\mathcal{M}, w \models^\alpha \phi \to \psi & \iff (V(\phi))(w) \land \alpha \leq (V(\psi))(w); \\
\mathcal{M}, w \models^\alpha \Box \phi & \iff \alpha \leq ([R](V(\phi)))(w); \\
\mathcal{M}, w \models^\alpha \Diamond \phi & \iff \alpha \leq ([R](V(\phi)))(w).
\end{align*}
\]

When \( A \) is the Boolean algebra \( 2 \), these definitions coincide with the usual ones in classical modal logic. For every \( \alpha \in A \), let \( \{\alpha\} : W \to A \) be defined by \( \nu \mapsto \alpha \) if \( \nu = w \) and \( \nu \mapsto \bot \) if \( \nu \neq w \). Then, for every \( f \in A^W \),

\[
f = \bigvee_{w \in W} \{f(w)\}. \tag{24}
\]

For any set \( W \), the \( A \)-subsethood relation between elements of \( A^W \) is the map \( S_W : A^W \times A^W \to A \) defined as \( S_W(f, g) := \bigwedge_{w \in W} (f(w) \to g(w)) \). If \( S_W(f, g) = 1 \) we also write \( f \subseteq g \).

7.2.2. Many-valued FCA

Any \( A \)-valued relation \( R : U \times W \to A \) induces maps \( R^{(0)}[-] : A^W \to A^U \) and \( R^{(1)}[-] : A^U \to A^W \) given by the following assignments: for every \( f : U \to A \) and every \( u \in W \),

\[
R^{(1)}[f] : W \to A \quad \quad \quad R^{(0)}[u] : U \to A
\]

\[
x \mapsto \bigwedge_{a \in U} (f(a) \to R(a, x)) \quad \quad \quad a \mapsto \bigvee_{x \in W} (u(x) \to R(a, x))
\]

These maps are such that, for every \( f \in A^A \) and every \( u \in A^X \),

\[
S_X(f, R^{(0)}[u]) = S_X(u, R^{(1)}[f]). \tag{25}
\]

396
that is, the pair of maps \( R^0 [] \) and \( R^1 [] \) form an \( A \)-Galois connection.

A formal \( A \)-context or \( A \)-polarity (cf. [2]) is a structure \( \mathbb{P} = (A, X, I) \) such that \( A \) and \( X \) are sets and \( I : A \times X \rightarrow A. \) Any formal \( A \)-context induces maps \( (\cdot)^*: A^* \rightarrow A^* \) and \( (\cdot)^*: A^* \rightarrow A^* \) given by \( (\cdot)^* = f^0 \) and \( (\cdot)^* = f^1 \).

In [2], it is shown that every \( A \)-Galois connection arises from some formal \( A \)-context. A formal \( A \)-context is a pair \((f, u) \in A^* \times A^* \) such that \( f^1 = u \) and \( u^1 = f \). It follows immediately from this definition that if \((f, u) \) is a formal \( A \)-context, then \( f^1 = f \) and \( u^1 = u \), that is, \( f \) and \( u \) are stable. The set of formal \( A \)-contexts can be partially ordered as follows:

\[
(f, u) \leq (g, v) \quad \text{iff} \quad f \subseteq g \quad \text{iff} \quad v \subseteq u.
\]

Ordered in this way, the set of the formal \( A \)-contexts of \( \mathbb{P} \) is a complete lattice, which we denote \( \mathbb{P}^+ \).

### 7.2.3. Many-valued enriched formal contexts

Building on [2], we can generalize the definition of enriched formal contexts to the many-valued setting as follows:

**Definition 7.7.** An enriched formal \( A \)-context is a structure \( \mathcal{E} = (\mathbb{P}, R_C, R_\circ) \) such that \( \mathbb{P} = (A, X, I) \) is a formal \( A \)-context and \( R_C : A \times X \rightarrow A \) and \( R_\circ : X \times A \rightarrow A \) are \( I \)-compatible, i.e. \( R_C^0 (\{ \underline{a} \}), R_C^1 (\{ \underline{x} \}), R_\circ^0 (\{ \underline{a} \}) \) and \( R_\circ^1 (\{ \underline{x} \}) \) are stable for every \( \underline{x} \in A, a \in A \) and \( x \in X \).

**Definition 7.8.** The complex algebra of an enriched formal \( A \)-context \( \mathcal{E} = (\mathbb{P}, R_C, R_\circ) \) is the algebra \( \mathcal{E}^+ = (\mathbb{P}^+, [R_C], [R_\circ]) \) where \([R_C], [R_\circ] : \mathbb{P}^+ \rightarrow \mathbb{P}^+ \) are defined by the following assignments: for every \( c = ([\underline{a}], [\underline{x}]) \in \mathbb{P}^+ \),

\[
[R_C]c = \left( R_C^0 ([\underline{a}]), R_C^1 ([\underline{x}]) \right) \quad \text{and} \quad [R_\circ]c = \left( R_\circ^0 ([\underline{a}]), R_\circ^1 ([\underline{x}]) \right).
\]

As in the crisp setting, the \( I \)-compatibility of \( R_C \) and \( R_\circ \) guarantees that the operations \([R_C], [R_\circ] \) (and in fact also their adjoints) are well-defined.

**Lemma 7.9.**

1. The following are equivalent for every formal \( A \)-context \( \mathbb{P} = (A, X, I) \) and every \( A \)-relation \( R : A \times X \rightarrow A. \)
   (i) \( R^0 (\{ \underline{x} \}) \) is Galois-stable for every \( x \in X \) and \( \underline{x} \in A; \)
   (ii) \( R^0 [u] \) is Galois-stable for every \( u : X \rightarrow A; \)
   (iii) \( R^1 [f] = R^1 [f^1] \) for every \( f : A \rightarrow A. \)

2. The following are equivalent for every formal \( A \)-context \( \mathbb{P} = (A, X, I) \) and every \( A \)-relation \( R : A \times X \rightarrow A. \)

   (i) \( R^1 (\{ \underline{a} \}) \) is Galois-stable for every \( a \in A \) and \( \underline{a} \in A; \)
   (ii) \( R^1 [f] \) is Galois-stable for every \( f : A \rightarrow A; \)
   (iii) \( R^0 [u] = R^0 [u^1] \) for every \( u : X \rightarrow A. \)

**Proof.** Since \( R^0 \left( \bigvee_{j \neq i} u_j \right) = \bigwedge_{j \neq i} R^0 [u_j] \), it is immediate that (i) and (ii) are equivalent. Therefore let us show that (ii) and (iii) is equivalent. By (25) it follows that

\[
f \subseteq R^0 [u] \iff u \subseteq R^1 [f]. \tag{26}
\]

Let us assume that \( R^0 [u] \) is stable for every \( u : X \rightarrow A \) and show that \( R^1 [f] \subseteq R^1 [f^1] \), the converse inclusion following from the antitonicity of \( R^1 \):

\[
R^1 [f] \subseteq R^1 [f^1] \iff f \subseteq R^0 \left( R^1 [f^1] \right) \tag{by 26}
\]

\[
\iff f^1 \subseteq R^0 \left( R^1 [f^1] \right) \tag{by assumption}
\]

\[
\iff R^1 [f] \subseteq R^1 [f^1]. \tag{by 26}
\]

Now assume that \( R^1 [f] = R^1 [f^1] \) for every \( f : A \rightarrow A. \) We want to show that \( R^0 [u]^1 \subseteq R^0 [u]: \)
For any set \( \mathcal{I} \)

\[
R^0[\mathcal{I}] \subseteq R^0[\mathcal{I}]
\]

\[
\iff \ u \subseteq R^0[\mathcal{I}]
\]

\[
\iff \ u \subseteq \left( R^0[\mathcal{I}] \right)^{\downarrow}_1
\]

\[
\iff \ \left( R^0[\mathcal{I}] \right)^{\downarrow}_1 \subseteq R^0[\mathcal{I}]
\]

(by 26).

The proof of 2. follows verbatim.

**Lemma 7.10.** For any enriched formal \( A \)-context \( \mathcal{F} = (\mathcal{P}, R^+, R^-) \), the algebra \( \mathcal{F}^+ = (\mathcal{P}^+, \mathcal{R}^+, (R^+_0)) \) is a complete normal lattice expansion such that \( R^+_0 \) is completely meet-preserving and \( (R^+_0) \) is completely join-preserving.

**Proof.** Let us show that \( R^+_0 \) is completely meet-preserving. Let \( f_j : A \to A \) be stable concepts. Notice that

\[
\left( \bigvee_{j \in J} f_j \right)^{\downarrow}_1 = \left( \bigwedge_{j \in J} f_j \right)^{\downarrow}_1
\]

We have

\[
R^0\left[ \left( \bigvee_{j \in J} f_j \right)^{\downarrow}_1 \right] = R^0\left[ \bigwedge_{j \in J} f_j \right] = \bigwedge_{j \in J} R^0\left[ f_j \right]
\]

the first equality following from Lemma 7.9.2(iii). The proof for \( R^-_0 \) goes similarly using Lemma 7.9.1(iii).

Applying the methodology developed in the previous sections, in what follows, we embed (and thereby represent) sets into \( A \)-polarities and \( A \)-Kripke frames into enriched formal \( A \)-contexts so as to preserve their complex algebras.

**Definition 7.11.** For any set \( W \), the formal \( A \)-context associated with \( W \) is

\[
P_W := (W, A \times W, I_A),
\]

where \( I_A : W \times (A \times W) \to A \) is defined by \( I_A(w, (x, y)) = \Delta(w, x) \to y \). That is, \( I_A(w, (x, y)) = \top \) if \( w \sim y \) and \( I_A(w, (x, y)) = x \) if \( w = y \). Alternatively, the formal \( A \)-context associated with \( W \) is

\[
P_W := (W, A^W, I_A),
\]

where \( I_A : W \times A^W \to A \) is defined by \( I_A(w, f) = \bigwedge_{z \in Z} (f(z) \to \Delta(w, z) \to 0) = f(w) \to 0 \).

**Lemma 7.12.** Let \( W \) be a set. Then \( P^+_W \cong A^W \).

**Proof.** It is enough to show that \( f = f^{\downarrow}_1 \) for any \( f \in A^W \). Using the first definition, for any \( f : W \to A \), the map \( f^{\downarrow}_1 : A \times W \to A \) is defined as follows: for every \( (\beta, x) \in A \times W \),

\[
f^{\downarrow}_1(\beta, x) = \bigwedge_{w \in W} (f(w) \to I_A(w, (x, x)))
\]

\[
= (f(v) \to \beta).
\]

The map \( f^{\downarrow}_1 : W \to A \) is defined as follows: for every \( w \in W \),

\[
f^{\downarrow}_1(w) = \bigwedge_{(\beta, x) \in A \times W} \left( f^{\downarrow}_1(\beta, x) \to I_A(w, (\beta, x)) \right)
\]

\[
= \bigwedge_{\beta \in A} \left( f^{\downarrow}_1(\beta, w) \to \beta \right)
\]

\[
= \bigwedge_{\beta \in A} \left( (f(w) \to \beta) \to \beta \right)
\]

\[
= f(w).
\]

The identity marked with \((*)\) is an instance of \( a = \bigwedge_{b \in A} (a \to b) \to b \) which is valid in any complete Heyting algebra \( A \); indeed, \( \bigwedge_{b \in A} (a \to b) \to b \leq (a \to a) \to a = \top \to a = a \); conversely, \( a \leq \bigwedge_{b \in A} (a \to b) \to b \) iff \( a \leq (a \to b) \to b \) for each \( b \), iff \( a \wedge (a \to b) \leq b \) for each \( b \), which is always true.

Using the second definition, for any \( f : W \to A \), the map \( f^{\downarrow}_1 : A^W \to A \) is defined as follows: for every \( g : W \to A \),

\[
f^{\downarrow}_1(g) = \bigwedge_{w \in W} (f^{\downarrow}_1(w) \to g(w))
\]

\[
= \bigwedge_{\beta \in A} \left( f^{\downarrow}_1(\beta) \to g \right)
\]

\[
= \bigwedge_{\beta \in A} \left( (f(w) \to \beta) \to g \right)
\]

\[
= f(g).
\]
\[ f^1(g) = \bigwedge_{w \in W} (f(w) \rightarrow I_\lambda(w, g)) \]
\[ = \bigwedge_{w \in W} \left( f(w) \rightarrow (g(w) \rightarrow 0) = \bigwedge_{w \in W} ((f(w) \land g(w)) \rightarrow 0) = \left( \bigvee_{w \in W} f(w) \land g(w) \right) \rightarrow 0 = \neg \left( \bigvee_{w \in W} f(w) \land g(w) \right). \]

Hence, \( f^1(f) = \neg \left( \bigvee_{w \in W} f(w) \land f(w) \right) = \neg \left( \bigwedge_{w \in W} f(w) \right) \). The map \( f^{11} : W \rightarrow A \) is defined as follows: for every \( w \in W, \)
\[ f^{11}(w) = \bigwedge_{g \in A^W} \left( f^1(g) \rightarrow I_\lambda(w, g) \right) \]
\[ = \bigwedge_{g \in A^W} \left( f^1(g) \rightarrow (g(w) \rightarrow 0) \right) \]
\[ = \bigwedge_{g \in A^W} \left( \neg \left( \bigvee_{v \in W} f(v) \land g(v) \right) \land g(w) \right) \rightarrow 0 \]
\[ = \neg \left( \bigvee_{g \in A^W} \left( \neg \left( \bigvee_{v \in W} f(v) \land g(v) \right) \land g(w) \right) \right) \]
\[ = f(w). \]

for \( g = f, \neg \left( \bigvee_{w \in W} f(v) \land f(w) \right) \leq \neg \left( \bigvee_{w \in W} f(v) \land f(v) \right) \land f(w) \leq \left( \bigvee_{g \in A^W} \left( \neg \left( \bigvee_{w \in W} f(v) \land g(v) \right) \land g(w) \right) \right). \)

**Definition 7.13.** For any \( A \)-Kripke frame \( \mathcal{X} = (W, R) \), the enriched formal \( A \)-context associated with \( \mathcal{X} \) is
\[ E_\mathcal{X} := (P_W, I_K, J_K), \]
where \( P_W \) is as in **Definition 7.11**, and \( I_K : W \times (A \times W) \rightarrow A \) is defined by \( I_K(w, (x, v)) = R(w, v) \rightarrow x \) and \( J_K : (A \times W) \times W \rightarrow A \) is defined by \( J_K((x, w), v) = R(w, v) \rightarrow x \).

**Lemma 7.14.** Let \( \mathcal{X} = (W, R) \) be an \( A \)-Kripke frame. Then \( E_\mathcal{X} \cong \left( A^W, [R], \langle R \rangle \right) \).

**Proof.** By **Lemma 7.12**, it is enough to show that, for every \( f \in A^W, \)
\[ |Rf| = I_0^0 \left[ f \right] \quad \text{and} \quad \langle Rf \rangle = \left( I_0^0 \left[ f \right] \right)^1. \quad (27) \]

To prove the first identity, recall that \( |Rf| : W \rightarrow A \) is defined as
\[ (|Rf|)(w) = \bigwedge_{v \in W} (R(w)(v) \rightarrow f(v)), \]

and \( I_0^0 \left[ f \right] : W \rightarrow A \) is defined as
\[ \left( I_0^0 \left[ f \right] \right)(w) = \bigwedge_{(\beta, v) \in A \times A} \left( f(\beta, v) \rightarrow I_K(w, (\beta, v)) \right) = \bigwedge_{(\beta, v) \in A \times A} ((f(\beta) \rightarrow f(v)) \rightarrow (R(w, v) \rightarrow \beta)). \]

Hence,
\[ \left( I_0^0 \left[ f \right] \right)(w) = \bigwedge_{(\beta, v) \in A \times A} ((f(\beta) \rightarrow f(v)) \rightarrow (R(w, v) \rightarrow \beta)) \]
\[ \leq \bigwedge_{v \in W} (f(v) \rightarrow f(v)) \rightarrow (R(w, v) \rightarrow f(v))) \quad \text{for} \ \beta := f(v) \]
\[ = \bigwedge_{v \in W} (R(w, v) \rightarrow f(v)) \]
\[ = \bigwedge_{v \in W} (R(w, v) \rightarrow f(v)) \]
\[ = (|Rf|)(w). \]

To show that
\[ \bigwedge_{v \in W} (R(w, v) \rightarrow f(v))) \leq \bigwedge_{(\beta, v) \in A \times A} ((f(\beta) \rightarrow f(v)) \rightarrow (R(w, v) \rightarrow \beta)), \]
it is enough to show that for each \((\beta, \upsilon) \in A \times W\),
\[
R(w, \upsilon) \rightarrow f(\upsilon) \leq (f(\upsilon) \rightarrow \beta) \rightarrow (R(w, \upsilon) \rightarrow \beta).
\]
The inequality above is an instance of \(b \rightarrow a \leq (a \rightarrow c) \rightarrow (b \rightarrow c)\), which is valid in every Heyting algebra. To see this,
\[
\begin{align*}
& b \rightarrow a \leq (a \rightarrow c) \rightarrow (b \rightarrow c) \\
& \text{iff } (b \rightarrow a) \wedge (a \rightarrow c) \leq b \rightarrow c \\
& \text{iff } b \wedge (b \rightarrow a) \wedge (a \rightarrow c) \leq c
\end{align*}
\]
and indeed, \(b \wedge (b \rightarrow a) \wedge (a \rightarrow c) \leq a \wedge (a \rightarrow c) \leq c\). To prove that \(\langle R \rangle f = \left( f^0 \right) ^1 \), recall that \(\langle R \rangle f : W \rightarrow A\) is defined as
\[
\langle R \rangle f(w) = \bigvee_{v \in W} (\langle R \rangle[w](v) \wedge f(v)),
\]
and \(\left( f^0 \right) ^1 : W \rightarrow A\) is defined as
\[
\left( f^0 \right) ^1 (w) = \bigwedge_{(\beta, \upsilon) \in A \times W} \left( f^0(\beta, \upsilon) \rightarrow I_\delta(w, (\beta, \upsilon)) \right) = \bigwedge_{b \in A} \left( f^0(\beta) \rightarrow \beta \right).
\]
Hence,
\[
\left( f^0 \right) ^1 (w) = \bigwedge_{(\beta, \upsilon) \in A \times W} \left( f^0(\beta, \upsilon) \rightarrow I_\delta(w, (\beta, \upsilon)) \right) \\
= \bigwedge_{b \in A} \left( f^0(\beta) \rightarrow \bigwedge_{v \in W} (f(v) \rightarrow (\langle R \rangle[w] \rightarrow \beta)) \rightarrow \beta \right) \\
= \bigwedge_{b \in A} \left( f^0(\beta) \rightarrow \bigwedge_{v \in W} ((f(v) \wedge \langle R \rangle[w]) \rightarrow \beta) \rightarrow \beta \right) \\
= \bigwedge_{b \in A} \left( \left( \bigvee_{v \in W} (f(v) \wedge \langle R \rangle[w]) \rightarrow \beta \right) \rightarrow \beta \right) \\
= \bigwedge_{b \in A} \left( (\langle R \rangle f(w) \rightarrow \beta) \rightarrow \beta \right) \\
= \left( \langle R \rangle f \right)(w).
\]

**Lifting reflexivity.** In the crisp setting of the previous sections, we discussed how properties of Kripke frames can be lifted to properties of their corresponding enriched formal contexts, and we used these property-lifting results to motivate the definition of conceptually approximation spaces (cf. Sections 3.4 and 4). Below, we illustrate, by way of an example, that this ‘lifting method’ works also in the many-valued setting. Specifically, we show how the property of reflexivity of \(A\)-Kripke frames can be lifted to enriched formal \(A\)-contexts. An \(A\)-Kripke frame \(X = (W, R)\) is reflexive if \(\Delta(w, w) = T^A\) for all \(w \in W\), or equivalently, if \(\Delta(w, v) \leq R(w, v)\) for all \(w, v \in W\), where \(\Delta(w, v) = T\) if \(w = v\) and \(\Delta(w, v) = \bot\) if \(w \neq v\).

**Proposition 7.15.** If \(X = (W, R)\) is an \(A\)-Kripke frame, \(X\) is reflexive iff \(I_\Delta(w, (x, v)) \leq I_\Delta(w, (x, v))\) for all \(w, v \in W\) and \(x \in A\).

**Proof.** If \(R\) is reflexive, i.e. if \(\Delta(w, v) \leq R(w, v)\) for all \(w, v \in W\), then \(I_\Delta(w, (x, v)) = R(w, v) \rightarrow x \leq \Delta(w, v) \rightarrow x = I_\Delta(w, (x, v))\) for any \(x \in A\). Conversely, suppose that \(I_\Delta(w, (x, v)) \leq I_\Delta(w, (x, v))\) for all \(w, v \in W\) and \(x \in A\), i.e. \(R(w, v) \rightarrow x \leq \Delta(w, v) \rightarrow x\) for all \(w, v\) and \(x\). Then in particular, setting \(x = R(w, v)\), we have \(T^A = R(w, v) \rightarrow R(w, v) \leq \Delta(w, v) \rightarrow R(w, v)\), and hence, by residuation, \(\Delta(w, v) \leq R(w, v)\).

Analogously to Proposition 3.16 eliciting Definition 3.17, Proposition 7.15 elicits the following.

**Definition 7.16.** Let \(P = (A, X, I)\) be an \(A\)-polarity. An \(A\)-relation \(R : A \times X \rightarrow A\) is reflexive iff \(R(a, x) \leq I(a, x)\) for all \(a \in A\) and \(x \in X\).
7.2.4. Many-valued semantics for the logic of concepts

The discussion and results above justify the introduction of the following many-valued semantic framework for the modal logic of concepts:

**Definition 7.17.** A conceptual A-model over a set AtProp of atomic propositions is a tuple \( M = (I, V) \) such that \( I = (A, X, I, R_1, R_2) \) is an enriched formal A-context and \( V : \text{AtProp} \to \mathbb{F}^* \). For every \( p \in \text{AtProp} \), let \( V(p) := ([p], ([p])] \), where \([p] : A \to A \) and \((p)[X \to A] \) and \([p])^0 = ([p]) \) and \((p)])^1 = ([p]) \). Letting \( \mathcal{L} \) denote the \( \{\Box, \Diamond\} \) modal language over AtProp, every \( V \) as above has a unique homomorphic extension, also denoted \( V : \mathcal{L} \to \mathbb{F}^* \), defined as follows:

\[
\begin{align*}
V(p) & = ([p], ([p])], \\
V(T) & = \left( \top^A, (\top^A)^1 \right), \\
V(\bot) & = \left( (\top^A)^1, \top^A \right), \\
V(\phi \land \psi) & = \left( [\phi] \land [\psi], ([\phi] \land [\psi])^1 \right), \\
V(\phi \lor \psi) & = \left( ([\phi] \lor ([\psi])^1, ([\phi] \lor ([\psi]))^1 \right), \\
V(\Box \phi) & = \left( R^0_0([\phi]), (R^0_0([\phi])^1 \right), \\
V(\Diamond \phi) & = \left( (R^0_0([\phi])^1, R^0_0([\phi]))^1 \right)
\end{align*}
\]

which in its turn induces \( \alpha \)-membership relations for each \( \alpha \in A \) (in symbols: \( M, \alpha \vdash \phi \)), and \( \alpha \)-description relations for each \( \alpha \in A \) (in symbols: \( M, \alpha \vdash \phi \))—cf. discussion in Section 2.5—such that for every \( \phi \in \mathcal{L} 

\[
\begin{align*}
M, \alpha \vdash \phi & \iff \alpha \leq [\phi](a), \\
M, \alpha \vdash \phi & \iff \alpha \leq ([\phi](x)).
\end{align*}
\]

This can be equivalently expressed by means of the following recursive definition:

\[
\begin{align*}
M, \alpha \vdash \phi & \iff \alpha \leq [\phi](a); \\
M, \alpha \vdash T & \iff \alpha \leq (\top^A)^1(a) \text{ i.e. always}; \\
M, \alpha \vdash \bot & \iff \alpha \leq (\top^A)^1(a) = \bigwedge_{x \in X} (\top^A)(x) \rightarrow I(a, x) = \bigwedge_{x \in X} (I(a, x)); \\
M, \alpha \vdash \phi \land \psi & \iff \alpha \leq (([\phi] \land ([\psi])^1(a) = \bigwedge_{x \in X} ([\phi](x) \land ([\psi])(x)) \rightarrow I(a, x)); \\
M, \alpha \vdash \Box \phi & \iff \alpha \leq (R^0_0([\phi])^1(a) = \bigwedge_{x \in X} ((R^0_0([\phi])(x)) \rightarrow I(a, x)) \\
M, \alpha \vdash \Diamond \phi & \iff \alpha \leq ((R^0_0([\phi])^1(a) = \bigwedge_{x \in X} ((R^0_0([\phi])(x)) \rightarrow I(a, x)) \\
M, \alpha \vdash \phi \iff \alpha \leq ([\phi](x)); \\
M, \alpha \vdash \bot & \iff \alpha \leq (\top^A)^1(x) \text{ i.e. always}; \\
M, \alpha \vdash T & \iff \alpha \leq (\top^A)^1(x) = \bigwedge_{a \in A} (\top^A)(a) \rightarrow I(a, x) = \bigwedge_{a \in A} (I(a, x)); \\
M, \alpha \vdash \phi \land \psi & \iff M, \alpha \vdash \phi \text{ and } M, \alpha \vdash \psi; \\
M, \alpha \vdash \phi \lor \psi & \iff M, \alpha \vdash \phi \text{ and } M, \alpha \vdash \psi; \\
M, \alpha \vdash \Box \phi & \iff \alpha \leq (R^0_0([\phi])^1(x) = \bigwedge_{a \in A} ([\phi](a) \rightarrow \Box^0(A, a)) \\
M, \alpha \vdash \Diamond \phi & \iff \alpha \leq (R^0_0([\phi])^1(x) = \bigwedge_{a \in A} ([\phi](a) \rightarrow \Diamond^0(A, a)) \\
M, \alpha \vdash \Box \phi & \iff \alpha \leq (R^0_0([\phi])^1(x) = \bigwedge_{a \in A} ((R^0_0([\phi])(a)) \rightarrow I(a, x)) \\
M, \alpha \vdash \Diamond \phi & \iff \alpha \leq (R^0_0([\phi])^1(x) = \bigwedge_{a \in A} ((R^0_0([\phi])(a)) \rightarrow I(a, x))
\end{align*}
\]
7.2.5. Axiomatic characterization of reflexive enriched formal A-contexts

With the definition above in place, we are now in a position to show, as an illustration, that the characterization of reflexivity of Proposition 4.3 extends to the many-valued setting.

**Proposition 7.18.** For any enriched formal A-context \( F = (\mathcal{P}, R_\subseteq, R_\subseteq) \),

\[
F \models \Box \phi \iff \phi \quad \text{iff} \quad R_\subseteq \leq I.
\]

**Proof.** It is sufficient to establish that the following are equivalent in \( F^+ \):

\[
\forall p[\Box p \leq p] \quad \text{(ALBA algorithm [15])}
\]

\[
\forall x \in X [R_0([x/x]^{1}) \leq [x/x]^{1}] \quad (m := [x/x]^{1}, [x/x]^{1})
\]

\[
\forall x \in X [R_0([x/x]^{1}) \leq [x/x]^{1}] \quad (I - \text{compatibility of } R_\subseteq)
\]

\[
\forall x \in X [a_x \rightarrow aR_\subseteq x \leq a \rightarrow ax] \quad (+)
\]

\[
R_\subseteq \leq I \quad (**)
\]

To justify the equivalence to (+) we note that \( R_0([x/x]^{1})(a) = \bigwedge_{y \in X}([x/x](y) \rightarrow aR_\subseteq y) = [x/x](x) \rightarrow aR_\subseteq x = x \rightarrow aR_\subseteq x \) and that \( [x/x]^{1}(a) = \bigwedge_{y \in X}([x/x](y) \rightarrow aly) = [x/x](x) \rightarrow ax = x \rightarrow ax \). For the equivalence to (**) note that choosing \( x = aR_\subseteq x \) in (+) yields \( \bigwedge \leq aR_\subseteq x \rightarrow ax \) which, by residuation, is equivalent to (**) of \( \bigwedge \). The converse direction is immediate by the monotonicity of \( \rightarrow \) in the second coordinate.

Lifting other relevant properties such as transitivity, in the way just illustrated by the example of reflexivity, leads to the definition of many-valued conceptual approximation spaces. We will develop this theory in a dedicated paper.

7.3. Dempster-Shafer theory on conceptual approximation spaces

Dempster-Shafer theory [45] is a mathematical framework for decision-making under uncertainty in situations in which some predicates cannot be assigned subjective probabilities. In such cases, Dempster-Shafer theory proposes to replace the missing value with a range of values, the lower and upper bounds of which are assigned by belief and plausibility functions (the definitions of which are reported below). The affinity between Dempster-Shafer theory and rough set theory has been noticed very early on [38]; systematic connections are established in [46,49], and a logical account of Dempster-Shafer's theory was developed in [25]. In this section, we lay the groundwork for the possibility of using conceptual approximation spaces as the basic structures for developing a Dempster-Shafer theory of concepts. Towards this goal, we first show how (probabilistic) approximation spaces arise in connection with finite probability spaces, and then use a suitably modified version of the lifting construction on probabilistic approximation spaces to propose a polarity-based generalization of probability spaces in which the basic notions of Dempster-Shafer's theory can be generalized from predicates to concepts.

7.3.1. Belief and plausibility functions

Recall that a belief function (cf. [45] [Chapter 1,page 5]) on a set \( S \) is a map \( \text{bel}: \mathcal{P}(S) \rightarrow [0,1] \) such that \( \text{bel}(S) = 1 \), and for all \( n \in \mathbb{N}_0 \),

\[
\text{bel}(A_1 \cup \ldots \cup A_n) \geq \sum_{S \subseteq \{1, \ldots, n\}} (-1)^{|S|} \text{bel}(\bigcap_{i \in S} A_i),
\]

and a plausibility function on \( S \) is a map \( \text{pl}: \mathcal{P}(S) \rightarrow [0,1] \) such that \( \text{pl}(S) = 1 \), and for all \( n \in \mathbb{N}_0 \),

\[
\text{pl}(A_1 \cup A_2 \cup \ldots \cup A_n) \leq \sum_{S \subseteq \{1, \ldots, n\}} (-1)^{|S|} \text{bel}(\bigcap_{i \in S} A_i).
\]

For every belief function \( \text{bel} \) as above, the assignment \( X \mapsto 1 - \text{bel}(X) \) defines a plausibility function on \( S \), and for every plausibility function \( \text{pl} \) as above, the assignment \( X \mapsto 1 - \text{pl}(X) \) defines a belief function on \( S \).

7.3.2. Probability spaces

Prime examples of belief and plausibility functions arise from probability spaces. A probability space (cf. [21] [Section 2, page 3]) is a structure \( (S, \mathcal{A}, \mu) \) where \( S \) is a nonempty (finite) set, \( \mathcal{A} \) is a \( \sigma \)-algebra of subsets of \( S \), and \( \mu: \mathcal{A} \rightarrow [0,1] \) is a countably additive probability measure. Let \( e: \mathcal{A} \rightarrow \mathcal{P}(S) \) denote the natural embedding of \( \mathcal{A} \) into the powerset algebra of \( S \). Any \( \mu \) as above induces the inner and outer measures (cf. [21] [Section 2, page 4]) \( \mu_i, \mu_o: \mathcal{P}(S) \rightarrow [0,1] \), respectively defined as

\[
\mu_i(Z) := \sup \{ \mu(b) \mid b \in \mathcal{A} \text{ and } e(b) \subseteq Z \} \quad \text{and} \quad \mu_o(Z) := \inf \{ \mu(b) \mid b \in \mathcal{A} \text{ and } Z \subseteq e(b) \}.
\]

By construction, \( \mu_i(e(b)) = \mu(b) = \mu_o(e(b)) \) for every \( b \in \mathcal{A} \) and \( \mu_o(Z) = 1 - \mu_i(Z) \) for every \( Z \subseteq S \). Moreover, for every probability space \( (S, \mathcal{A}, \mu) \), the inner (resp. outer) measure induced by \( \mu \) is a belief (resp. plausibility) function on \( S \) (cf. [21] [Proposition 3.1]).
Notice that in a finite probability space $\mathcal{X} = (S, \mathcal{A}, \mu)$, the natural embedding $e : \mathcal{A} \rightarrow \mathcal{P}(S)$ is a complete lattice homomorphism (in fact it is a complete Boolean algebra homomorphism, but in the context of Boolean algebras, these two notions collapse). Hence, the right and left adjoints of $e$ exist, denoted $i, \gamma : \mathcal{P}(S) \rightarrow \mathcal{A}$ respectively, and defined as $i(Y) := \bigcup \{ a \in \mathcal{A} \mid e(a) \subseteq Y \}$ and $\gamma(Y) := \bigcap \{ a \in \mathcal{A} \mid Y \subseteq e(a) \}$. Notice that by construction and the fact that $e$ is injective, $i(\emptyset) := \bigcup \{ a \in \mathcal{A} \mid e(a) \subseteq \emptyset \} = \emptyset$, and $\gamma(S) := \bigcap \{ a \in \mathcal{A} \mid S \subseteq e(a) \} = \bigcap \{ T \} = \top$.

**Lemma 7.19.** For every finite probability space $\mathcal{X} = (S, \mathcal{A}, \mu)$, and every $Y \in \mathcal{P}(S)$,

$$\mu_i(Y) = \mu(i(Y)) \quad \text{and} \quad \mu^*(Y) = \mu(\gamma(Y)).$$

**Proof.** We only show the first identity. By definition,

$$\begin{align*}
\mu_i(Y) &= \bigvee \{ \mu(a) \mid a \in \mathcal{A} \text{ and } e(a) \subseteq Y \} \\
&= \mu\left( \bigcup \{ a \in \mathcal{A} \mid a \in \mathcal{A} \text{ and } e(a) \subseteq Y \} \right) \quad \text{(\mu is additive)} \\
&= \mu(\gamma(Y)).
\end{align*}$$

**From probability spaces to approximation spaces.** Consider the operations $\square, \Diamond : \mathcal{P}(S) \rightarrow \mathcal{P}(S)$, defined as $\square Y := e(i(Y))$ and $\Diamond Y := e(\gamma(Y))$. The next lemma shows how finite approximation spaces arise from finite probability spaces.

**Lemma 7.20.** For every finite probability space $\mathcal{X} = (S, \mathcal{A}, \mu)$, the operations $\square$ and $\Diamond$ defined above:

1. are complete normal modal operators;
2. are S4 operators (i.e. $\square$ is an interior operator and $\Diamond$ is a closure operator on $\mathcal{P}(S)$);
3. are adjoint to each other, i.e. $\Diamond Y \subseteq Z$ iff $Y \subseteq \square Z$ for all $Y, Z \in \mathcal{P}(S)$;
4. are S5 operators;
5. are dual to each other, i.e. $\square Y = \neg \Diamond \neg Y$ for all $Y \in \mathcal{P}(X)$;
6. can be respectively identified with the semantic box and diamond operators arising from the relation $R \subseteq S \times S$ defined as $R(x, y)$ iff $x \in \Diamond(y)$.
7. the relation $R$ defined above is an equivalence relation.

**Proof.** 1. Immediately follows from the definitions, and $e$ preserving both complete joins and meets; 2. $\square Y \subseteq Y$ (resp. $Y \subseteq \Diamond Y$) readily follows from the definition of $i$ (resp. $\gamma$); $\square Y \subseteq \square \square Y$ (resp. $\Diamond \Diamond Y \subseteq \Diamond Y$) readily follows from the definitions of $\square$ and $\Diamond$, and $e e = e$ (resp. $e e = e$); 3. follows from the definitions of $\square$ and $\Diamond$, and $\gamma \circ e \circ i$; 4. by adjunction, $\Diamond Y \subseteq Y$ iff $Y \subseteq \square \Diamond Y$ and $\square Y \subseteq \square \square Y$ iff $\Diamond \Diamond Y \subseteq Y$, as required; 5. since $e$ is an injective Boolean homomorphism, to prove $e i(Y) = \neg e \gamma(\neg Y)$ it is enough to show $i(Y) = \neg \gamma(\neg Y)$.

$$
\begin{align*}
\neg \gamma(\neg Y) &= \neg \bigcap \{ a \in \mathcal{A} \mid \neg Y \subseteq e(a) \} \quad \text{(def. of } \gamma) \\
&= \bigcup \{ a \in \mathcal{A} \mid \neg Y \subseteq e(a) \} \quad \text{(De Morgan)} \\
&= \bigcup \{ a \in \mathcal{A} \mid e(-a) \subseteq Y \} \quad \text{(\neg self adj.)} \\
&= \bigcup \{ b \in \mathcal{A} \mid e(b) \subseteq Y \} \quad \text{(e BA – hom.)} \\
&= i(Y). \quad \text{(def. of } i)
\end{align*}
$$

6. By 5, it is enough to show that $\langle R \rangle Y = \Diamond Y$ for every $Y \in \mathcal{P}(S)$. Recall that $\Diamond = e \gamma$ is completely $\cup$-preserving, cf. item 1 of this lemma. Hence:

$$
\begin{align*}
\langle R \rangle Y &= R^{-1}[Y] \\
&= \{ x \in X \mid x \Diamond y \text{ for some } y \in Y \} \\
&= \{ x \in X \mid x \in \Diamond(y) \text{ for some } y \in Y \} \\
&= \bigcup \{ i(Y) \mid y \in Y \} \\
&= \Diamond \bigcup \{ i(Y) \mid y \in Y \} \\
&= \Diamond Y.
\end{align*}
$$

7. By item 2, the inclusions $Y \subseteq \Diamond Y$ and $\Diamond \Diamond Y \subseteq Y$ hold for every $Y \in \mathcal{P}(S)$, and by item 5, $\Diamond = \langle R \rangle$ and $\square = [R]$. Hence, the reflexivity and transitivity of $R$, i.e. $x \in R^{-1}[x]$ and $R^{-1}[R^{-1}[x]] \subseteq R^{-1}[x]$ for every $x \in S$, can be rewritten as $\{ x \} \subseteq \Diamond \{ x \}$ and $\Diamond \Diamond \{ x \} \subseteq \Diamond \{ x \}$ respectively, and immediately follow by instantiating the inclusions above to $Y := \{ x \}$; symmetry, i.e.

---

12 Equivalently, $y$ is an $R$-successor of $x$ iff $x$ is in the closure of $y$. 

---
\[ R^{-1}[x] = R[x] \] for every \( x \in S \), follows from \( \square \) and \( \Diamond \) being adjacent to each other (cf. item 3) and \( \Box = [R] \) (cf. item 6), which imply that \( \Diamond = \langle R^{-1} \rangle \). Hence, \( R^{-1}[x] = \Diamond \{x\} = \langle R^{-1} \rangle \{x\} = R[x] \), as required.

**Definition 7.21.** (cf. e.g. [25] [Section 2]) A (finite) probabilistic S5-Kripke frame is a triple \( \mathcal{K} = (S, A, R, \mu) \) such that

\((S, \mu)\) is a (finite) probability space, and \( R \subseteq S \times S \) is an equivalence relation which is compatible with \( \mu \), i.e. letting \( \langle R \rangle : \mathcal{P}(S) \to \mathcal{P}(S) \) denote the semantic diamond induced by \( R \), then \( \langle R \rangle = e \circ \gamma' \) for some \( \gamma' : \mathcal{P}(S) \to A \). A (finite) probabilistic S5-Kripke model is a tuple \( M = (\mathcal{K}, V) \) such that \( \mathcal{K} \) is a (finite) probabilistic S5-Kripke frame, and \( V : \text{Prop} \to \mathcal{P}(S) \) is a valuation.

**Definition 7.22.** If \( \mathcal{X} = (S, A, \mu) \) is a finite probability space, let \( \mathcal{K}_{\mathcal{X}} := (S, A, R, \mu) \), where \( R \subseteq S \times S \) is as in Lemma 7.20(6).

**Lemma 7.23.** For any finite probability space \( \mathcal{X} = (S, A, \mu) \), the relation \( R \) defined as in Lemma 7.20(6) is the finest equivalence relation compatible with \( A \).

**Proof.** Let \( R' \subseteq S \times S \) be compatible with \( A \). By assumption, \( \Diamond' = \langle R' \rangle = e \circ \gamma' \) for some \( \gamma' : \mathcal{P}(S) \to A \). Since \( R' \) is reflexive, \( Y \subseteq \Diamond' \diamond R = e \circ \gamma'(Y) \). Hence by adjunction, \( \gamma(Y) \subseteq \gamma'(Y) \) for every \( Y \in A \). Hence, if \( Y := \{y\} \), this yields \( xRy \) iff \( x \Diamond y \) for every \( x \in S \), which shows that \( R \subseteq R' \), as required.

Summing up, every finite probability space \( \mathcal{X} = (S, A, \mu) \) can be endowed with a structure of approximation space by means of an indiscernibility relation on \( S \) which is ‘canonical’, in the sense that it is the most informative equivalence relation which is compatible with \( A \). Conversely, as is well known, for every (finite) approximation space \( \mathcal{X} = (S, R) \), the Boolean sub-algebra \( A \) of \( \mathcal{P}(S) \) generated by taking unions of the equivalence blocks of the equivalence relation \( R \) is a \( \sigma \)-algebra of subsets of \( S \), and hence approximation spaces can be regarded as the (purely qualitative) bases of finite probability spaces.

**Generalizing to conceptual probability spaces.** The lifting methodology discussed in the previous sections provides the motivation for the following

**Definition 7.24.** A conceptual probability space is a structure \( \mathcal{S} = (P, A, \mu) \) where \( P = (A, X, I) \) is a finite polarity,\(^\text{14}\) \( A \) is a \( \sigma \)-algebra of formal concepts of \( P \), i.e. a lattice embedding \( e : A \to P^+ \) exists, and \( \mu : A \to [0, 1] \) is a (countably additive) probability measure.

Similarly to the case of probability spaces, which can be endowed with the structure of approximation spaces, conceptual probability spaces can be endowed with the structure of conceptual approximation spaces. Since this section is only intended as a showcase of possibilities, we do not provide an explicit definition of it, but a precise definition can be straightforwardly extracted from the following discussion and Lemma. Let \( \mathcal{S} = (P, A, \mu) \) be a conceptual probability space. Since \( A \) and \( P^+ \) are finite lattices, the lattice embedding \( e : A \to P^+ \) has both its right and left adjoint \( I, \gamma : P^+ \to A \) which are defined as

\[ I(c) := \bigvee\{ a \in A \mid e(a) \leq c \} \quad \text{and} \quad \gamma(c) := \bigwedge\{ a \in A \mid c \leq e(a) \}, \]

Notice that by construction and the fact that \( e \) is injective, \( I(\bot^+) := \bigvee\{ a \in A \mid e(a) \leq \bot^+ \} = \bigvee\{ \bot^+ \} = \bot^+ \), and \( \gamma(\top^+) := \bigwedge\{ a \in A \mid \top^+ \leq e(a) \} = \bigwedge\{ \top^+ \} = \top^+ \). Let \( \Box, \Diamond : P^+ \to P^+ \) be defined as \( \Box c := e(I(c)) \) and \( \Diamond c := e(\gamma(c)) \). In what follows, we let \( a = (a^+, a^-) \) and \( x = (x^+, x^-) \) for any \( a \in A \) and \( x \in X \).

**Lemma 7.25.** For any finite conceptual probability space \( \mathcal{S} = (P, A, \mu) \), the operations \( \Box \) and \( \Diamond \) defined above:

1. are complete normal modal operators;
2. are \( S4 \) operators (i.e. \( \Box \) is an interior operator and \( \Diamond \) is a closure operator on \( P^+ \));
3. are adjoint to each other, i.e. \( \varnothing c \leq d \iff c \leq \Box d \) for all \( c, d \in P^+ \);
4. are \( S5 \) operators;
5. can be respectively identified with \( [R] \) and \( \langle R^{-1} \rangle \) for the \( I \)-compatible relation \( R \subseteq A \times X \) defined as \( R(a, x) \) iff \( a \leq \Box x \);
6. the relation \( R \) defined above is such that \( R \subseteq I \) and \( R \subseteq R \).

\( ^\text{13} \) Probabilistic S5-Kripke models are also known in the literature as total \( \Box \)-probabilistic Kripke models (cf. [25] [Section 2]). In order to be coherent with the literature in modal logic, we refer to semantic structures without valuations of propositional formulas as ‘frames’ and to those with valuations as ‘models’. Hence, we will also refer to the probability structures of [21] (i.e. tuples \( M := (X, V) \) such that \( X \) is a probability space and \( V : \text{Prop} \to \mathcal{P}(S) \) a valuation) as probability models.

\( ^\text{14} \) A polarity \( P \) is finite if its associated concept lattice \( P^+ \) is finite.
Proof. Items 1–4 are verified analogously to the corresponding items of Lemma 7.20; their proofs are omitted. By construction, $R^0[|] = \{ a \mid aRx \} = \{ a \mid a \leq \square x \} = \square [x]$ which is Galois-stable for every $x \in X$. Likewise, by item 3, $R^1[|] = \{ x \mid aRx \} = \{ x \mid a \leq \square x \} = \{ x \mid \diamond a \leq x \} = \{ x \mid \square a \}$ which is Galois-stable for every $a \in A$. This finishes the proof that $R$ is $I$-compatible. As to the remaining part of the statement, by item 3, and since adjoints uniquely determine each other, it is enough to show that $\square c = R[c]$ for every $c \in P^+$, and since they are both finitely (i.e. in this case, completely) meet-preserving and $P^+$ is meet-generated by the set $\{ x \mid x \in X \}$, it is enough to show that $\square [x] = [R[x]]$ for any $x \in X$.

\[
\begin{align*}
\square [x] &= R^0[|]\{ x \mid x \in X \} \\
&= \{ a \in A \mid \forall y (y \leq x \Rightarrow aRy) \} \\
&= \{ a \in A \mid \forall y ( (\square y) \subseteq (\{x\}) \Rightarrow a \leq \square y) \} \\
&= \{ a \in A \mid \forall y (x \leq y \Rightarrow a \leq \square y) \} \\
&= \{ a \in A \mid a \leq \square x \} \\
&= \square [x]
\end{align*}
\]

6. By adjunction, $\iota(c) \leq \iota(c)$ implies that $\square c = e\iota(c) \leq c$ for any $c \in P^+$, and hence if $aRx$ then $a \leq \square x \leq x$, i.e. $alx$, which proves that $R \subseteq I$. By adjunction, $e\iota(a) = e(a)$ for every $a \in A$, hence $\square c = e\iota(c) = e\iota(e\iota(c)) = e\square c$ for any $c \in P^+$, therefore $aRx$ iff $a \leq \square x = \square a$, i.e. $a \leq \square y$ for every $y \in X$ such that $\square x \leq y$, i.e. $aRx$ for every $y \in (\square x)$. By definition, $a(R: R)x$ iff $aRx$ for every $y \in (R^0[|]) = (\square x)$. Indeed,

\[
\begin{align*}
(R^0[|]) &= \{ y \mid \forall b ( b \in R^0[|] \Rightarrow bly) \} \\
&= \{ y \mid \forall b ( bRx \Rightarrow bly) \} \\
&= \{ y \mid \forall b ( b \leq \square x \Rightarrow b \leq y) \} \\
&= \{ y \mid \square x \leq y \} \\
&= (\square x).
\end{align*}
\]

The facts shown in this section motivate the idea that the systematic connections between rough set theory and Dempster-Shafer theory can be extended to the formal environment of rough concepts. Some steps in this direction have already been taken in [23], but even more can be shown (see discussion in Section 8).

8. Conclusions and further directions

8.1. The lifting methodology as a defining tool

We have considered a construction for lifting Kripke frames to enriched formal contexts. This was formulated as the solution to the category-theoretic requirement that the diagram in Section 1 should commute. The latter arises naturally from the duality-theoretic insight that every Kripke frame and enriched formal context has an algebraic representation in the form of its associated complex algebra. We have shown that the lifting construction ‘restricts well’ (i.e. modularly) to subclasses of Kripke frames and enriched formal contexts defined by a number of conditions—including reflexivity and transitivity—which are both natural and very relevant to Rough Set Theory; in particular, each of these conditions can be considered independently of the others, and the commutativity of the diagram is preserved for each resulting restricted class. The modularity of these restrictions is not only mathematically elegant but has also a logical import: via an application of unified correspondence theory, the conditions we have considered can be characterized in terms of the validity of certain logical axioms (cf. Proposition 4.3), and the commutativity of the diagram in each restricted subclass implies that the logical axioms are preserved by the lifting construction from Kripke frames to enriched formal contexts.

The lifting construction and its mathematical and logical properties have provided us with a principled basis from which we proposed the notion of conceptual approximation spaces (cf. Definition 4.1), as the FCA counterparts of approximation spaces, thus obtaining a unifying formal environment for RST and FCA in which to formalize and reason about rough concepts.

Being more general, this environment allows for a wider range of variations. In particular, the notions of conceptual co-approximation spaces and bi-approximation spaces naturally arise from the lifting methodology, and interestingly subsume established approaches in the literature on the unification of RST and FCA, such as Kent’s Rough Concept Analysis [30].

Finally, in Section 7, we applied the lifting methodology to three theories which proceed from various motivations and starting points, but are all related to rough set theory in that they aim to account for the vagueness of predicates, their grad-enedness, and reasoning under uncertainty in situations in which not all events can be assigned a probability. Precisely the fact that these theories are so different and differently motivated is significant evidence of the robustness of the lifting methodology.

From the viewpoint of logic, these results contribute a novel intuitive understanding of the modal logic of categories and concepts discussed in Section 2.5, and specifically show that the epistemic interpretation proposed in [10,11] is one among...
many possible interpretations. Hence, these results witness the potential of this logical framework as a flexible and powerful tool to address categorical reasoning in a wide range of fields intersecting and involving information sciences. Mathematically, these results pave the way to several research directions, including those we discuss below.

8.2. The lifting construction and Sahlqvist theory

Proposition 4.3 characterizes the validity of well known modal axioms, such as those classically corresponding to reflexivity and transitivity, on enriched formal contexts in terms of the ‘lifted versions’ of these conditions (cf. Proposition 3.16 and Definition 3.17). We think of this phenomenon as stability under lifting. This suggests that this phenomenon might apply to wider classes of modal axioms, such as the Sahlqvist and inductive inequalities (cf. [[15] Definitions 3.4 and 3.5]). Formulating and proving such a general result would provide a more systematic perspective on this phenomenon, and would illuminate the relationship between Sahlqvist correspondence theory in the classical (i.e. Boolean) setting and Sahlqvist correspondence theory in the lattice-based setting from yet another angle.

8.3. The lifting construction and the Goldblatt-Thomason theorem on enriched formal contexts

Closely related to the previous question is the question concerning the nature of the relation between stability under lifting and the characterization of modal definability on enriched formal contexts given in [20]. This is the object of ongoing investigation for some of the authors.

8.4. Complete axiomatization for the logic of rough formal contexts

An open issue concerning the class of Kent’s rough formal contexts (cf. Sections 5.6 and 5.7) is to find a complete axiomatization for its naturally associated modal logic. A key step towards this goal is to verify, by means of the characterization of [20], whether the first-order conditions expressing the relationship between $E$ and the approximations $R$ and $S$ arising from $E$ are modally definable. This issue is addressed in [20].

8.5. Conceptual T-models

In Section 7.1, we have used the lifting methodology to illustrate how the semantic framework of [6] can be generalized from vague predicates to vague concepts, and have shown how conceptual counterparts of the sorites paradox can be accounted for in this framework. These results set the stage for a systematic investigation of vague concepts, making use of insights and results from formal philosophy and semantics of natural language.

8.6. Many-valued conceptual approximation spaces

In Section 7.2, we have adapted the lifting methodology to the setting of many-valued Kripke frames, and used it to define a semantic framework for the many-valued logic of categories and concepts, which is suitable to describe and reason about e.g. ‘to which extent’ a given object is a member of a certain category and ‘to which extent’ a given feature describes a category. This setting is especially suited to provide formal models of core situations and phenomena studied in areas of business science such as strategy, marketing and entrepreneurship and innovation. One such phenomenon is category-spanning (i.e. the extent to which a market-product belongs to more than one product-category), which impacts the perceptions and evaluations of consumers and which has thus been recognized as a key predictor of the success of products in markets [40]. Another key predictor is the extent to which certain core features can be attributed to market-products, which functions as a value-anchor for objects evaluated as members of new emerging categories [31]. Future research also concerns using the many-valued setting for understanding the nature and sources of graded membership, for instance by modelling situations in which graded membership can be considered to result from the presence of multiple competing classification systems in one domain [48] or different audiences attaching different category labels to the same objects [32].

8.7. Towards a Dempster-Shafer theory of concepts

In Section 7.3, we showed that the systematic connections between the mathematical environments of Dempster-Shafer theory and of rough set theory can be extended, via lifting, from set-based structures to polarity-based structures. This paves the way for the development of a Dempster-Shafer theory of concepts. Preliminary steps in this direction have been taken in [23], where, however, the connection with a modal/epistemic logic approach to Dempster-Shafer theory is not yet developed. The results of Section 7.3 set the stage for the development of an epistemic-logical approach to the Dempster-Shafer theory of concepts generalizing the approach of [25,42].
8.8. Conceptual rough algebras and proof calculi

The links between rough set theory and proof theory are mainly mediated by the theory of the varieties of algebras introduced in connection with approximation spaces. In [44], sequent calculi which are sound and complete but without cut elimination are defined for the logics naturally associated with the classes of algebras discussed in [43]. Sequent calculi with cut elimination and a non-standard version of subformula property have been introduced in [34] for some but not all of these logics. These difficulties are overcome in [28], where a family of calculi with standard cut elimination and subformula property are introduced for the logics of the classes of algebras discussed in [43], and with the same methodology, in [27], a family of calculi endowed with comparable properties is introduced for varieties of lattice-based modal algebras which can be thought of as the counterparts of the algebras of [43] for the setting of Kent’s rough formal concepts [30]. Further directions in this line of research will include developing complete axiomatizations and proof calculi for the logics discussed in the previous paragraphs as well as for the dynamic and many-valued logics discussed below.

8.9. Polarity-based semantics and graph-based semantics

In the first paragraph of this section we remarked that the formal environment of rough concepts provides new sets of interpretations for the lattice-based modal logic of Section 2.5. Several other interpretations for the same logic are proposed in [8,7,16], based on a different but related semantics (referred to as graph-based semantics) which is grounded on Ploščica’s duality and representation theory for bounded lattices [41]. Inspired by the rough set theory approach, the relation $E$ in the graphs $(Z,E)$ on which the graph-based models are based is interpreted as an indiscernibility relation. However, instead of using $E$ to generate modal operators, $E$ is used to generate a complete lattice via the polarity $(Z,A,E)$ (in the many-valued setting, via the $A$-polarity $(Z,A \times Z, I)$). An interesting feature of $E$ is that it is assumed to be reflexive but neither transitive nor symmetric. This agrees with what many researchers in the rough set theory community have advocated [49,47].

8.10. Towards modelling categorical dynamics

Conceptual approximation spaces are a natural starting point to model how categorization systems change. This is a core issue in a wide range of disciplines which include computational linguistics and information retrieval (for tracking the change in the meaning of lexical terms), AI (e.g. for modelling how clusters are generated by machine learning clustering algorithms), social and cognitive sciences (e.g. for modelling cognitive and social development of individuals), and management science (to analyze market dynamics in terms of the evolution of categories of products and producers). This is presently ongoing work, based on a suitable generalization of the techniques and results on dynamic updates on algebras developed in [35,33,12].

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Proofs

A.1. Proof of Proposition 4.3

\[
\forall p (\square p \leq p) \iff \\
\forall p \forall m ((j \leq p \land p \leq m) \Rightarrow j \leq m) \quad \text{first approximation} \\
\forall j \forall m (j \leq m \Rightarrow j \leq m) \quad \text{Ackermann’s Lemma} \\
\forall m (\square m \leq m) \quad \text{J c. join – generates } F^+ \\
i.e. \quad \forall x \in X \quad R[0]^0[x]\subseteq I[0]^0[x] \quad \text{translation} \\
\forall x \in X \quad R[0]^0[x]\subseteq I[0]^0[x] \quad \text{Lemma 2.6 since } R[0] \text{ is } I \text{ – compatible} \\
\iff R[0] \subseteq I. \quad \text{By definition}
\]
3. 
\[ \forall p [p \leq \diamond p] \]
iff \[ \forall \overline{p} \forall \overline{m} [j \leq p \land \dot{i}p \leq m] \Rightarrow j \leq m \]  
first approximation
iff \[ \forall \overline{p} \forall \overline{m} [j \leq p \land \dot{i}p \leq \overline{m}] \Rightarrow j \leq m \]  
adjunction
iff \[ \forall \overline{m} [j \leq \overline{m} \Rightarrow j \leq m] \]  
Ackermann’s Lemma
iff \[ \forall \overline{m} \overline{m} \leq m \]  
J c. join – generates \( F^+ \)
i.e. \[ \forall x \in X \ R_{\overline{m}}^0[x] \subseteq \overline{f}^0[x] \]
iff \[ \forall x \in X \ R_{\overline{m}}^0[x] \subseteq \overline{f}^0[x] \]
Lemma 2.6 since \( R_{\overline{m}} \) is \( I \) – compatible
iff \[ R_{\overline{m}} \leq I. \]
By definition

4. 
\[ \forall p [\boxdot p \leq \diamond p] \]
iff \[ \forall \overline{p} \forall \overline{m} [j \leq \boxdot p \land \dot{i}p \leq m] \Rightarrow \diamond j \leq m \]  
first approximation
iff \[ \forall \overline{p} \forall \overline{m} [\dot{i}j \leq m \Rightarrow \diamond j \leq m] \]  
Ackermann’s Lemma
iff \[ \forall \overline{p} [\dot{i}j \leq \overline{m}] \]  
Ackermann’s Lemma
iff \[ \forall a \in A \ R_{\overline{m}}^0[a] \subseteq \overline{f}^0[a] \]
translation
iff \[ \forall a \in A \ R_{\overline{m}}^0[a] \subseteq \overline{f}^0[a] \]
Lemma 2.6 since \( R_{\overline{m}} \) is \( I \) – compatible
iff \[ R_{\overline{m}} \leq R_{\overline{m}} : R_{\overline{m}}. \]
By definition

5. 
\[ \forall p [\diamond p \leq \diamond p] \]
iff \[ \forall \overline{p} \forall \overline{m} [j \leq p \land \dot{i}p \leq m] \Rightarrow \diamond j \leq m \]  
first approximation
iff \[ \forall \overline{p} \forall \overline{m} [\dot{i}j \leq m \Rightarrow \diamond j \leq m] \]  
Ackermann’s Lemma
iff \[ \forall \overline{p} [\dot{i}j \leq \overline{m}] \]  
Mc. meet – generates \( F^+ \)
i.e. \[ \forall a \in A \ R_{\overline{m}}^0[a] \subseteq \overline{f}^0[a] \]
translation
iff \[ \forall a \in A \ R_{\overline{m}}^0[a] \subseteq \overline{f}^0[a] \]
Lemma 2.6 since \( R_{\overline{m}} \) is \( I \) – compatible
iff \[ R_{\overline{m}} \leq R_{\overline{m}} : R_{\overline{m}}. \]
By definition

6. 
\[ \forall p [\boxdot p \leq \boxdot p] \]
iff \[ \forall \overline{p} \forall \overline{m} [j \leq \boxdot p \land \dot{i}p \leq m] \Rightarrow \boxdot j \leq m \]  
first approximation
iff \[ \forall \overline{p} \forall \overline{m} [\dot{i}j \leq m \Rightarrow \boxdot j \leq m] \]  
Ackermann’s Lemma
iff \[ \forall \overline{p} [\dot{i}j \leq \overline{m}] \]  
adjunction
iff \[ \forall a \in A \ R_{\overline{m}}^0[a] \subseteq \overline{f}^0[a] \]
translation
iff \[ \forall a \in A \ R_{\overline{m}}^0[a] \subseteq \overline{f}^0[a] \]
Lemma 2.6 since \( R_{\overline{m}} \) and \( R_{\overline{m}} : R_{\overline{m}} \) – compatible
iff \[ R_{\overline{m}} \leq R_{\overline{m}}. \]
By definition

7. 
\[ \forall p [\overline{\boxdot p} \leq p] \]
iff \[ \forall \overline{p} \forall \overline{m} [j \leq \overline{p} \land p \leq m] \Rightarrow \overline{p} \leq m \]  
first approximation
iff \[ \forall \overline{p} [\overline{p} \leq \overline{m}] \]  
Ackermann’s Lemma
iff \[ \forall \overline{p} [\dot{i}j \leq m \Rightarrow \overline{p} \leq m] \]  
adjunction
iff \[ \forall a \in A \ R_{\overline{m}}^0[a] \subseteq \overline{f}^0[a] \]
translation
iff \[ \forall a \in A \ R_{\overline{m}}^0[a] \subseteq \overline{f}^0[a] \]
Lemma 2.6 since \( R_{\overline{m}} \) and \( R_{\overline{m}} : R_{\overline{m}} \) – compatible
iff \[ R_{\overline{m}} \leq R_{\overline{m}}. \]
By definition

408
8.
\[ \forall p [p \leq \Box p] \]
\[ \text{iff} \quad \forall x [x \leq \Box p \land p \leq m] \Rightarrow \Box x \leq m \]  
first approximation
\[ \text{iff} \quad \forall x [x \leq m \Rightarrow \Box x \leq m] \quad \text{Ackermann's Lemma} \]
\[ \text{iff} \quad \forall m [m \leq \Box m] \quad J \text{ c. join – generates } F^+ \]
\[ \text{i.e. } \forall x \in X \quad I^0 [x] \subseteq R^0 [\Box x^{-1}] \quad \text{translation} \]
\[ \text{iff} \quad \forall x \in X \quad I^0 [x] \subseteq R^0 [\Box x] \quad \text{Lemma 2.6 since } R_\Box \text{ is } I - \text{ compatible} \]
\[ I \subseteq R_\Box. \quad \text{By definition} \]

9.
\[ \forall p [\Box p \leq \Box p] \]
\[ \text{iff} \quad \forall x [x \leq \Box p \land p \leq m] \Rightarrow \Box x \leq m \]  
first approximation
\[ \text{iff} \quad \forall x [x \leq m \Rightarrow \Box x \leq m] \quad \text{Ackermann's Lemma} \]
\[ \text{iff} \quad \forall m [m \leq \Box m] \quad \text{adjunction} \]
\[ \text{iff} \quad \forall m [m \leq \Box m] \quad J \text{ c. join – generates } F^+ \]
\[ \text{i.e. } \forall x \in X \quad I^0 [x] \subseteq R^0 [\Box x^{-1}] \quad \text{translation} \]
\[ \text{iff} \quad \forall x \in X \quad I^0 [x] \subseteq R^0 [\Box x] \quad \text{Lemma 2.6 since } R_\Box \text{ is } I - \text{ compatible} \]
\[ I \subseteq R_\Box. \quad \text{By definition} \]

10.
\[ \forall p [\Box \Box p \leq \Box p] \]
\[ \text{iff} \quad \forall x [x \leq \Box \Box p \land p \leq m] \Rightarrow \Box x \leq m \]  
first approximation
\[ \text{iff} \quad \forall x [x \leq \Box \Box m \Rightarrow \Box x \leq m] \quad \text{Ackermann's Lemma} \]
\[ \text{iff} \quad \forall m [m \leq \Box \Box m] \quad J \text{ c. join – generates } F^+ \]
\[ \text{i.e. } \forall x \in X \quad R^0 [I^1 [R^0 [\Box x^{-1}]]] \subseteq R^0 [\Box x^{-1}] \quad \text{translation} \]
\[ \text{iff} \quad \forall x \in X \quad R^0 [I^1 [R^0 [\Box x^{-1}]]] \subseteq R^0 [\Box x] \quad \text{Lemma 2.6 since } R_\Box \text{ is } I - \text{ compatible} \]
\[ R_{\Box}; R_\Box \subseteq R_{\Box}. \quad \text{By definition} \]

11.
\[ \forall p [\Box p \leq \Box \Box p] \]
\[ \text{iff} \quad \forall x [x \leq \Box p \land p \leq m] \Rightarrow \Box x \leq m \]  
first approximation
\[ \text{iff} \quad \forall x [x \leq \Box \Box m \Rightarrow \Box x \leq m] \quad \text{Ackermann's Lemma} \]
\[ \text{iff} \quad \forall m [m \leq \Box \Box m] \quad \text{Mc.meet – generates } F^+ \]
\[ \text{i.e. } \forall a \in A \quad R^0 [I^1 [R^0 [a^{-1}]]] \subseteq R^0 [a^{-1}] \quad \text{translation} \]
\[ \text{iff} \quad \forall a \in A \quad R^0 [I^1 [R^0 [a^{-1}]]] \subseteq R^0 [a] \quad \text{Lemma 2.6 since } R_\Box \text{ is } I - \text{ compatible} \]
\[ R_{\Box}; R_\Box \subseteq R_{\Box}. \quad \text{By definition} \]

12.
\[ \forall p [\Box p \leq \Box \Box p] \]
\[ \text{iff} \quad \forall x [x \leq \Box p \land p \leq m] \Rightarrow \Box x \leq m \]  
first approximation
\[ \text{iff} \quad \forall x [x \leq \Box \Box m \Rightarrow \Box x \leq m] \quad \text{Ackermann's Lemma} \]
\[ \text{iff} \quad \forall m [m \leq \Box \Box m] \quad \text{adjunction} \]
\[ \text{iff} \quad \forall m [m \leq \Box \Box m] \quad J \text{ c. join – generates } F^+ \]
\[ \text{i.e. } \forall x \in X \quad I^0 [x] \subseteq R^0 [I^1 [R^0 [\Box x^{-1}]]] \quad \text{translation} \]
\[ \text{iff} \quad \forall x \in X \quad I^0 [x] \subseteq R^0 [I^1 [R^0 [\Box x^{-1}]]] \quad \text{Lemma 2.6 since } R_\Box \text{ is } I - \text{ compatible} \]
\[ I \subseteq R_{\Box}; R_\Box. \quad \text{By definition} \]
A.2. Proof of Proposition 5.2

1. \[
\forall p[p \leq \rhd p] \iff \forall p \forall i(j \leq p \land i \leq \rhd p) \Rightarrow j \leq \rhd i \]
   first approximation

   \[
   \forall p \forall j(i \leq p \land \rhd j) \iff \forall p \forall i(j \leq p \land j \leq \rhd i) \]
   adjunction

   \[
   \forall j(j \leq \rhd i) \iff \forall i(j \leq \rhd i) \]
   Ackermann’s Lemma

   \[
   \forall i \in [\mathbb{J} 
   \iff \forall i \in [\mathbb{J} \leq \rhd \mathbb{T}] \]
   \[
   \forall i \leq \rhd \forall j \leq \rhd \]
   \[
   \forall i \leq \rhd \left( \forall j \leq \rhd \right) \]
   \[
   \rhd \text{completely join – reversing} \]

   \[
   \forall i \leq \rhd T \]

2. \[
R_\rhd = A \times A \]

   \[
   \forall a \forall b[a \in R_0[b]] \]

   \[
   \forall a \forall b[a \in R_0[b]] \]

   \[
   \forall a \forall b[a \subseteq R_0[b]] \]

   \[
   \forall i \leq \bigvee_{j \in \mathbb{J}} j \]

   \[
   \forall i \leq \bigvee_{j \in \mathbb{J}} j \]

   \[
   \forall i \leq \left( \bigvee_{j \in \mathbb{J}} j \right) \]

   \[
   \rhd \text{completely join – reversing} \]

   \[
   \forall i \leq \rhd T \]

3. \[
\forall p[p \land \rhd p \leq \bot] \iff \forall p \forall j(j \leq p \land \rhd p) \Rightarrow j \leq \bot \]
   first approximation

   \[
   \forall p \forall j(j \leq p \land j \leq \rhd p) \Rightarrow j \leq \bot \]
   adjunction

   \[
   \forall j(j \leq \rhd j) \Rightarrow j \leq \bot \]
   Ackermann’s Lemma

   \[
\forall a \in A \left[ a \subseteq R_0[a] \Rightarrow a \subseteq X^i \right] \]
   translation

   \[
\forall a \in A \left[ a \in R_0[a] \Rightarrow a \in X^i \right] \]
   Lemma 2.6 since \( R_\rhd \) is \( I \) – compatible

4. \[
\forall p[T \leq \rhd (p \land \rhd p)] \]

   \[
\forall p \forall j(j \leq p \land \rhd p) \Rightarrow T \leq \rhd j \]
   first approximation

   \[
\forall p \forall j(j \leq p \land j \leq \rhd p) \Rightarrow T \leq \rhd j \]
   adjunction

   \[
\forall j(j \leq \rhd j) \Rightarrow T \leq \rhd j \]
   Ackermann’s Lemma

   \[
\forall a \in A \left[ a \subseteq R_0[a] \Rightarrow a \subseteq X^i \right] \]
   translation

   \[
\forall a \in A \left[ a \in R_0[a] \Rightarrow a \subseteq R_0[a] \subseteq X^i \right] \]
   Lemma 2.6 since \( R_\rhd \) is \( I \) – compatible
A.3. Proof of Lemma 5.3

1. 
\[ S \subseteq I \]
iff \( \forall x \in X \ [S^{(0)}[x] \subseteq I^{(0)}[x]] \)
iff \( \forall a \in A \forall x \in X \ [a \in S^{(0)}[x] \Rightarrow a \in I^{(0)}[x]] \)
iff \( \forall a \in A \forall x \in X \ [R^{(1)}[a] \subseteq I^{(0)}[x] \Rightarrow a \in I^{(0)}[x]] \)
\[ \text{definition of } S \]
iff \( \forall a \in A \forall x \in X \ [R^{(1)}[a] \subseteq I^{(0)}[x] \Rightarrow a \in I^{(0)}[x]] \)
iff \( \forall j \in m \Rightarrow j \leq m \)
iff \( \forall j \leq \mathbb{I} \)
iff \( \forall a \in A \ [a^{(1)} \subseteq R^{(0)}[a^{(1)}]] \)
iff \( \forall a \in A \ [a \in R^{(0)}[a]] \)
iff \( \Delta \subseteq R_0 \).

2. 
\[ R_0 \circ R_0 \subseteq R_0 \]
iff \( \forall c \in A \ [R_0 \circ R_0^{(0)}[c] \subseteq R^{(0)}[c]] \)
iff \( \forall a \forall c \in A \ [a \in \bigcup_{b \in R^{(0)}[c]} R^{(0)}[b] \Rightarrow a \in R^{(0)}[c]] \)
iff \( \forall a \forall b \forall c \in A \ [(a \in R^{(0)}[b] \land b \in R^{(0)}[c]) \Rightarrow a \in R^{(0)}[c]] \)
iff \( \forall a \forall b \forall c \in A \ [(a^{(1)} \subseteq R^{(0)}[b^{(1)}] \land b^{(1)} \subseteq R^{(0)}[c^{(1)}]) \Rightarrow a^{(1)} \subseteq R^{(0)}[c^{(1)}]] \)
iff \( \forall j \in \mathbb{I} \land j \leq \mathbb{I} \Rightarrow j \leq \mathbb{I} \)
iff \( \forall i \leq j \Rightarrow \forall k(j \leq k \Rightarrow i \leq k) \)
iff \( \forall i \leq j \Rightarrow \forall k(k \leq j \Rightarrow k \leq i) \)
iff \( \forall i \leq j \Rightarrow \forall j \leq i \)
iff \( \forall i \leq j \Rightarrow i \leq j \)
iff \( \forall \mathbb{I} \leq \mathbb{I} \).

\[ S \subseteq S; S \]
iff \( \forall x \in X \ [S^{(0)}[x] \subseteq S^{(0)}[I^{(1)}[S^{(0)}[x]]] \]
iff \( \forall a \in A \forall x \in X \ [a \in S^{(0)}[x] \Rightarrow a \in S^{(0)}[I^{(1)}[S^{(0)}[x]]] \]
iff \( \forall a \in A \forall x \in X \ [R^{(1)}[a] \subseteq I^{(0)}[x] \Rightarrow R^{(1)}[a] \subseteq I^{(0)}[I^{(1)}[S^{(0)}[x]]]] \)
\[ \text{definition of } S \]
iff \( \forall a \in A \forall x \in X \ [R^{(1)}[a] \subseteq I^{(0)}[x] \Rightarrow \forall b \in R^{(1)}[a] \Rightarrow b \in S^{(0)}[x]] \]
iff \( \forall a \in A \forall x \in X \ [R^{(1)}[a] \subseteq I^{(0)}[x] \Rightarrow \forall b \in R^{(1)}[a] \Rightarrow R^{(1)}[b] \subseteq I^{(0)}[x]] \]
iff \( \forall a \in A \forall x \in X \ [R^{(1)}[a^{(1)}] \subseteq I^{(0)}[x] \Rightarrow \forall b \in R^{(1)}[a^{(1)}] \Rightarrow R^{(1)}[b^{(1)}] \subseteq I^{(0)}[x]] \)
\[ \text{Galois – stable} \]
R.I – compatible
iff \( \forall j \in m \Rightarrow j \leq m \)
iff \( \forall j \leq \mathbb{I} \)
iff \( \forall i \leq j \Rightarrow \forall m(i \leq m \Rightarrow j \leq m) \)
iff \( \forall i \leq j \Rightarrow j \leq m \)
iff \( \forall i \leq j \Rightarrow j \leq m \)
iff \( \forall i \leq j \Rightarrow i \leq j \)
iff \( \forall \mathbb{I} \leq \mathbb{I} \).

411
[48] N.M. Wijnberg, Classification systems and selection systems: the risks of radical innovation and category spanning, Scand. J. Manage. 27 (3) (2011) 297–306.

[49] Y.Y. Yao, P.J. Lingras, Interpretations of belief functions in the theory of rough sets, Inf. Sci. 104 (1–2) (1998) 81–106.

[50] Yiyu Yao, Rough-set concept analysis: interpreting rs-definable concepts based on ideas from formal concept analysis, Inf. Sci. 346–347 (2016) 442–462.