Complexity of Oscillatory Integration
for Univariate Sobolev Spaces

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Abstract

We analyze univariate oscillatory integrals for the standard Sobolev spaces $H^s$ of periodic and non-periodic functions with an arbitrary integer $s \geq 1$. We find matching lower and upper bounds on the minimal worst case error of algorithms that use $n$ function or derivative values. We also find sharp bounds on the information complexity which is the minimal $n$ for which the absolute or normalized error is at most $\varepsilon$. We show surprising relations between the information complexity and the oscillatory weight. We also briefly consider the case of $s = \infty$. 

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1 Introduction

We study the approximate computation of univariate oscillatory integrals (Fourier coefficients)
\[ I_k(f) = \int_0^1 f(x) e^{-2\pi i k x} \, dx, \quad i = \sqrt{-1}, \tag{1} \]
where \( k \in \mathbb{Z} \) and \( f \in H^s \). We improve the known upper bounds and also prove matching lower bounds, i.e., we study the complexity of this computational problem. By \( H^s \) we mean the standard Sobolev (Hilbert) space; we study spaces of periodic and non-periodic functions defined on \([0, 1]\) with an arbitrary integer \( s \geq 1 \). We usually consider a finite \( s \) but we also briefly consider the case of \( s = \infty \). Although we consider arbitrary integers \( k \), our emphasis is for large \( |k| \) and we explain our results here only for such \( k \).

We compute the initial error (the norm of \( I_k \)) as well as the worst case error of our algorithms exactly. This is possible since we assume that \( k \) is an integer. For the periodic case the initial error is of order \( |k|^{-s} \), whereas for the non-periodic case it is independent of \( s \) and is roughly \( |k|^{-1} \). This means that the initial error for the periodic case is much smaller for large \( s \). For \( s = \infty \), the periodic case leads to the space of only constant functions and the problem becomes trivial since the initial error is zero for all \( k \neq 0 \). The non-periodic case is still reasonable with the initial error roughly \( |k|^{-1} \).

For a finite \( s \) and the periodic case, we prove that an algorithm that uses \( n \) function values at equally spaced points is nearly optimal, and its worst case error is bounded by \( C_s (n+|k|)^{-s} \) with an exponentially small \( C_s \) in \( s \). For the non-periodic case, we first compute successive derivatives up to order \( s - 1 \) at the end-points \( x = 0 \) and \( x = 1 \). These derivatives values are used to periodize the function and this allows us to obtain similar error bounds like for the periodic case. Asymptotically in \( n \), the worst case error of the algorithm is of order \( n^{-s} \) independent of \( k \) for both periodic and non-periodic cases.

Near optimality of this algorithm is shown by proving a lower bound of order \( (n+|k|)^{-s} \) which holds for all algorithms that use the values of function and derivatives up to order \( s - 1 \) at \( n \) arbitrarily chosen points from \([0, 1]\). We establish the lower bound by constructing a periodic function that vanishes with all its derivatives up to order \( s - 1 \) at the points sampled by a given algorithm, belongs to the unit ball of the space \( H^s \), and its oscillatory integral is of order \( (n+|k|)^{-s} \).

For \( s = \infty \), we provide two algorithms which compute successive derivatives and/or function values at equally spaced points. The worst case error of one of these algorithms is super exponentially small in \( n \). For \( s = \infty \), we do not have a matching lower bound.

We consider the absolute and normalized error criteria. For the absolute error criterion, we want to find the information complexity which is defined as the smallest \( n \) for which
the $n$th minimal error is at most $\varepsilon \in (0, 1)$, whereas for the normalized error criterion, the information complexity is the smallest $n$ for which the $n$th minimal error reduces the initial error by a factor $\varepsilon$. For a finite $s$ we obtain the following results.

- For the absolute error criterion and the periodic case, the information complexity is zero if $\varepsilon > 1/(2\pi|k|)^s$ and otherwise is roughly $\varepsilon^{-1/s} - |k|$. This means that in this case the problem becomes easier for large $|k|$.

- For the normalized error criterion and for the periodic case, the information complexity is of order $|k|\varepsilon^{-1/s}$. Hence, in this case the problem becomes harder for larger $|k|$.

- For the absolute error criterion and the non-periodic case, the information complexity is zero if $\varepsilon \geq 1.026/(2\pi|k|)$ and otherwise is roughly lower bounded by $\varepsilon^{-1/s} - |k|$ and upper bounded by $\varepsilon^{-1/s} + 2s - 1 - |k|$. As for the periodic case, the problem becomes easier for large $k$.

- For the normalized error criterion and the non-periodic case, the information complexity is of order $|k|^{1/s} \varepsilon^{-1/s}$ for very small $\varepsilon$. In this case, the dependence on $|k|$ is more lenient than for the periodic case especially if $s$ is large.

The dependence on $|k|$ is quite intriguing if $|k|$ goes to infinity. For $s = 1$ and fixed $\varepsilon$, the information complexity goes to infinity linearly with $|k|$. However, the situation is quite different for $s \geq 2$. Then for large $|k|$ the information complexity is bounded by $2s$ if $\varepsilon$ is fixed or if $\varepsilon$ tends to zero like $|k|^{-\eta}$ with $\eta \in (0, s - 1)$.

For $s = \infty$, we obtain only upper bounds on the information complexity. For $\varepsilon$ tending to zero they are roughly $\ln(\varepsilon^{-1})/\ln(\ln(\varepsilon^{-1}))$ independent of $|k|$.

There are several recent papers about the approximate computation of highly oscillatory univariate integrals with the weight $\exp(2\pi i kx)$, where $x \in [0, 1]$ and $k$ is an integer (or $k \in \mathbb{R}$) which is assumed to be large in the absolute sense, see Domínguez, Graham and Smyshlyaev [4], Iserles and Nørsett [6], Melenk [8], Chapter 3 of Olver [11], and Huybrechs and Olver [5] for a survey. Some authors mainly present asymptotic error bounds as $k$ goes to infinity for algorithms that use $n$ function or derivative values. It is usually done for $C^\infty$ or even analytic functions. There are not too many papers that contain explicit error bounds depending on $k$ and $n$. Examples include [4, 8, 11]. All these papers also contain pointers to the further relevant literature.

There is a discussion in the literature concerning the question whether “high oscillation”, i.e., large $|k|$, means that the problem is “easy” or “difficult”. For this question it is useful to distinguish between the absolute and normalized error criteria and, in addition, it is important to know the initial errors.
The absolute error criteria means that the error is at most $\varepsilon$, whereas the normalized error criteria means that the error is at most $\varepsilon$ times the initial error. The initial error is the error of the zero algorithm and only depends on the formulation of the problem. It turns out that in the setting of our paper the initial error is small for large $|k|$ which makes the absolute error criterion easier than the normalized error criterion. We show that the answer to the question whether the problem is easy or difficult for large $|k|$ depends on the error criterion we choose as well as on the relation between $|k|$, $\varepsilon$ and the assumed smoothness of integrands.

We did not find a computation of the initial error in the literature and we did not find lower bounds on the error of algorithms that use $n$ function or derivatives values. In this paper, we present the formulas for the initial error as well as matching lower and upper bounds on the minimal errors of algorithms.

2 Preliminaries

We study the Sobolev space $H^s$ for a finite $s \in \mathbb{N}$, i.e.,

$$H^s = \{ f : [0, 1] \to \mathbb{C} \mid f^{(s-1)} \text{ is abs. cont., } f^{(s)} \in L^2 \}$$

with the inner product

$$\langle f, g \rangle_s = \sum_{\ell=0}^{s-1} \int_0^1 f^{(\ell)}(x) \, dx \int_0^1 g^{(\ell)}(x) \, dx + \int_0^1 f^{(s)}(x) \overline{g^{(s)}(x)} \, dx$$

$$= \sum_{\ell=0}^{s-1} \langle f^{(\ell)}, 1 \rangle_0 \langle g^{(\ell)}, 1 \rangle_0 + \langle f^{(s)}, g^{(s)} \rangle_0,$$

where $\langle f, g \rangle_0 = \int_0^1 f(x) \overline{g(x)} \, dx$, and norm $\| f \|_{H^s} = \langle f, f \rangle_s^{1/2}$. We later comment on the space $H^\infty$ for $s = \infty$.

Remark 1. Probably the most standard inner product on the Sobolev space $H^s$ is

$$\langle f, g \rangle_{s,*} = \sum_{\ell=0}^{s} \langle f^{(\ell)}, g^{(\ell)} \rangle_0 .$$

Obviously, the norms $\| \cdot \|_{H^s}$ and $\| \cdot \|_{H^s_{\ast}} = \langle f, f \rangle_{s,*}^{1/2}$ are equivalent. What is more surprising, the bounds on the embedding constants are independent of $s$ and close to one. More precisely, we have

$$\frac{12}{13} \| f \|_{H^s_{\ast}} \leq \| f \|_{H^s} \leq \| f \|_{H^s_{\ast}} \quad \text{for all } f \in H^s \text{ and } s \in \mathbb{N}.$$
The second inequality is trivial, whereas the first inequality seems to be new and its proof is given in the appendix.

From (5) it clearly follows that all results presented in this paper for the space $H^s$ equipped with $\langle \cdot, \cdot \rangle_s$ are practically the same as for the space $H^s$ equipped with $\langle \cdot, \cdot \rangle_{s,*}$. We choose to work with the inner product $\langle \cdot, \cdot \rangle_s$ since the analysis in this case is easier and more straightforward.

We want to solve the following problem:

- What is the complexity of the approximate computation of oscillatory integrals of the form

$$I_k(f) = \int_0^1 f(x) e^{-2\pi i k x} \, dx, \quad i = \sqrt{-1},$$

where $k \in \mathbb{Z}$ and $f \in H^s$? Our emphasize is on large $|k|$. We improve the known upper bounds and also prove matching lower bounds.

We ask (and answer) the same question also for the periodic case, i.e., for the subspace of $H^s$ given by

$$\widetilde{H}^s = \{ f \in H^s \mid f^{(\ell)}(0) = f^{(\ell)}(1) \text{ for } \ell = 0, 1, \ldots, s - 1 \}$$

equipped with the same inner product as for the space $H^s$. Note that for $f, g \in \widetilde{H}^s$ this inner product simplifies to

$$\langle f, g \rangle_s = \langle f, 1 \rangle_0 \langle g, 1 \rangle_0 + \langle f^{(s)}, g^{(s)} \rangle_0.$$

The results are presented in the following order. We first consider the integration problem for periodic functions, i.e., for functions from $\widetilde{H}^s$, and then, using this knowledge, we analyze the integration problem for the space $H^s$.

The results of this paper could be stated also for real-valued functions, where $I_k(f)$ can be written, for example, as

$$I_k(f) = \int_0^1 f(x) \cos(2\pi k x) \, dx$$

for $k \in \mathbb{Z}$, losing only some negligible constants. We decided to work with complex-valued functions to ease the notation.
3 The periodic case

As already indicated, we first analyze oscillatory integration over $\tilde{H}^s$. That is, we want to approximate the integral

$$I_k(f) := \int_0^1 f(x) e^{-2\pi ikx} \, dx = \int_0^1 f(x) \cos(2\pi k x) \, dx - i \int_0^1 f(x) \sin(2\pi k x) \, dx,$$

where $k \in \mathbb{Z}$ and $f \in \tilde{H}^s$ with $s \in \mathbb{N}$. Although $k$ can be any integer, the emphasis of this paper is for large $|k|$. In this case the weight functions $\cos(2\pi k x)$ and $\sin(2\pi k x)$ highly oscillate and therefore the approximation of $I_k$ is called an (highly) oscillatory integration problem.

We consider the worst case error on the unit ball of $\tilde{H}^s$ for algorithms that use function values or, more generally, function and derivatives (up to order $s - 1$) values. Note that for $f \in H^s$, the values $f^{(j)}(x)$ are well defined for all $j = 0, 1, \ldots, s - 1$ and $x \in [0, 1]$.

It is well known that adaption does not help, see Bakhvalov [1], and linear algorithms are optimal, see Smolyak [12]. These results can be also found in e.g., [9, 10, 13]. This means that without loss of generality we may consider linear algorithms of the form

$$A_n(f) = \sum_{j=1}^n a_j f^{(\ell_j)}(x_j)$$

for some $a_j \in \mathbb{C}$, $\ell_j \in [0, s - 1]$ and $x_j \in [0, 1]$. Observe that we allow the use of derivatives $f^{(\ell_j)}(x_j)$ as in [6]. An important special case are linear algorithms that use only function values, i.e.,

$$A_n(f) = \sum_{j=1}^n a_j f(x_j).$$

To summarize the problem: We want to compute a single Fourier coefficient $I_k(f)$ for $f \in H^s$ by algorithms of the form (8) or (9). The problem is described by $k$ and $s$, while $n$ describes the amount of resources of an algorithm $A_n$. The algorithm (the knots $x_j$ and the weights $a_j$) may depend in an arbitrary way on $k$ and $s$ and $n$. For the upper bounds we work (in the periodic case) with equidistant knots. We do not know whether these knots are optimal but our lower bounds (valid for all algorithms) show that they are at least almost optimal.

Of course, for $s = 1$ there is no difference between (8) and (9). We will see that for all $s$ the complexity results are similar for both classes of algorithms (8) and (9). 

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The worst case error of \( A_n \) is defined as

\[
\tilde{e}(A_n) = \sup_{f \in \tilde{H}^s, \|f\|_{\tilde{H}^s} \leq 1} |I_k(f) - A_n(f)|,
\]

whereas the \( n \)th minimal worst case error is

\[
\tilde{e}(n, k, s) := \inf_{A_n} \tilde{e}(A_n).
\]  \((10)\)

We use the tilde to indicate that we consider the periodic case, i.e., the class \( \tilde{H}^s \). The particular case \( n = 0 \) corresponds to the zero algorithm \( A_0 = 0 \) and gives the so-called initial error

\[
\tilde{e}(0, k, s) := \sup_{f \in \tilde{H}^s, \|f\|_{\tilde{H}^s} \leq 1} |I_k(f)| = \|I_k\|_{\tilde{H}^s \to \mathbb{C}}.
\]  \((11)\)

**Remark 2.** We believe that this is a simple but already interesting model problem for approximating highly oscillatory integrals. Later we plan to study the multivariate case and tractability and we believe that, from a practical point of view, the integrals

\[
S_k(f) = \int_{\mathbb{R}^d} f(x) e^{ikx_1} \exp(-\|x\|^2) \, dx
\]

for smooth integrands \( f : \mathbb{R}^d \to \mathbb{C} \) are more interesting. Here \( x_1 \) is the first coordinate of a vector \( x \in \mathbb{R}^d \). We start, however, with the integral \((1)\) since it seems to be the simplest interesting case of oscillatory integrals.

We begin with the computation of the initial errors \( \tilde{e}(0, k, s) = \|I_k\|_{\tilde{H}^s \to \mathbb{C}} \). Since \( I_k \) is a continuous linear functional defined on the Hilbert space \( \tilde{H}^s \), Riesz’s theorem implies that for each \( k \in \mathbb{Z} \) and \( s \in \mathbb{N} \), there exists a function \( \tilde{h}_{k,s} \in \tilde{H}^s \) such that

\[
I_k(f) = \left\langle f, \tilde{h}_{k,s} \right\rangle_s \quad \text{for all} \quad f \in \tilde{H}^s.
\]

The function \( \tilde{h}_{k,s} \) is called the representer of \( I_k \) for the space \( \tilde{H}^s \). It is well known and easy to show that \( \|\tilde{h}_{k,s}\|_{\tilde{H}^s} = \|I_k\|_{\tilde{H}^s \to \mathbb{C}} \).

To find \( \tilde{h}_{k,s} \) consider the particular function \( e_k(x) = e^{2\pi ikx} \). Clearly, \( e_k \in \tilde{H}^s \). Using integration by parts, we obtain for \( k \neq 0 \)

\[
\left\langle f, e_k \right\rangle_s = \left\langle f, 1 \right\rangle_0 \left\langle e_k, 1 \right\rangle_0 + \left\langle f^{(s)}, e_k^{(s)} \right\rangle_0 = \left\langle f^{(s)}, e_k^{(s)} \right\rangle_0 = (-1)^s \left\langle f, e_k^{(2s)} \right\rangle_0 = (2\pi k)^{2s} \left\langle f, e_k \right\rangle_0 = (2\pi k)^{2s} I_k(f).
\]  \((12)\)

For \( k = 0 \), we have \( \left\langle f, e_k \right\rangle_s = \left\langle f, 1 \right\rangle_0 = I_0(f) \). Hence we obtain
Proposition 3. Let $k \neq 0$. The representer of $I_k$ for the space $\tilde{H}^s$ is

$$\tilde{h}_{k,s}(x) = (2\pi k)^{-2s} e^{2\pi i k x}$$

and the initial error is

$$\tilde{e}(0, k, s) = \|\tilde{h}_{k,s}\|_{H^s} = \frac{1}{(2\pi |k|)^s}.$$ 

Additionally, for $k = 0$ we have $\tilde{h}_{0,s}(x) = 1$ and $\tilde{e}(0, 0, s) = 1$.

We now present a few linear algorithms whose worst case errors are of order $(n + |k|)^{-s}$. We will prove later that this is the best possible order.

For $n \geq 1$, we first define the linear algorithm

$$A_{n}^{\text{QMC}}(f) = \frac{1}{n} \sum_{j=1}^{n} f(j/n) e^{-2\pi i k j/n} \quad \text{for all } f \in \tilde{H}^s.$$ 

We use the superscript QMC to stress that the algorithm uses equal weights $1/n$ for the function $f(\cdot) \exp(-2\pi i k \cdot)$. This means that this is a QMC (quasi Monte Carlo) algorithm. This is the standard way to compute Fourier coefficients for periodic functions, with computation of integrals for a range of $k$ facilitated by the FFT. Observe, however, that our problem is to compute $I_k(f)$ for a single $k$.

As we shall see, the worst case error of $A_{n}^{\text{QMC}}$ is small only if $n$ is sufficiently large with respect to $|k|$. Later, we will modify the algorithm $A_{n}^{\text{QMC}}$ to have a good error bound for all $n$. First we prove the following theorem.

Theorem 4.

(i) The worst case error of $A_{n}^{\text{QMC}}, n \geq 1$, is

$$\tilde{e}(A_{n}^{\text{QMC}}) = \left(\sum_{j=1}^{\infty} \left( \frac{1}{\max\{1, (2\pi (jn + k))^{2s}\}} + \frac{1}{\max\{1, (2\pi (jn - k))^{2s}\}} \right) \right)^{1/2}.$$ 

(ii) For any $1 \leq n \leq |k|$ we have

$$\tilde{e}(A_{n}^{\text{QMC}}) > \tilde{e}(0, k, s).$$ 

(iii) For any $n > |k|$ we have

$$\tilde{e}(A_{n}^{\text{QMC}}) = \left(\sum_{j=1}^{\infty} \left( \frac{1}{(2\pi (jn + k))^{2s}} + \frac{1}{(2\pi (jn - k))^{2s}} \right) \right)^{1/2} \leq \frac{2}{(2\pi)^{s}} \frac{1}{(n - |k|)^{s}}.$$
(iv) Let $\alpha \in (0,1)$. Then for $n > [(1 + \alpha)/(1 - \alpha)] |k|$ we have

$$\tilde{e}(A_n^{QMC}) < \frac{2}{(2\pi \alpha)^s} \frac{1}{n + |k|^s}.$$ 

\hfill \Box

**Proof.** For $h \in \mathbb{Z}$, let $e_h(x) = e^{2\pi i hx}$ for $x \in [0,1]$. Since $f$ is a periodic function from $\tilde{H}^s$ we can write

$$f(x) = \sum_{h \in \mathbb{Z}} \hat{f}_h e_h(x),$$

with the Fourier coefficients $\hat{f}_h = \int_0^1 f(x) e^{-2\pi i hx} \, dx$. Since $f$ is smooth the last series is also pointwise convergent. Then

$$A_n^{QMC}(f) = \frac{1}{n} \sum_{h \in \mathbb{Z}} \hat{f}_h \sum_{j=1}^n [e^{2\pi i (h-k)/n}]^j.$$ 

Note that the sum with respect to $j$ is zero if $h - k \neq 0 \mod n$, and is equal to $n$ if $h - k = 0 \mod n$. Therefore we can restrict $h$ to $h = k + jn$ with $j \in \mathbb{Z}$, and

$$A_n^{QMC}(f) = \sum_{j \in \mathbb{Z}} \hat{f}_{k+jn}.$$ 

We have $\hat{f}_k = I_k(f)$ which yields

$$I_k(f) - A_n^{QMC}(f) = - \sum_{j \in \mathbb{Z} \setminus 0} \hat{f}_{k+jn}.$$ 

Let $a_h = \max\{1, (2\pi h)^{2s}\}$. Clearly $a_h = a_{-h}$. Since $e_h$'s are orthogonal in $\tilde{H}^s$ and $\|e_h\|_{\tilde{H}^s} = a_h$, for $f \in \tilde{H}^s$ we have

$$\|f\|_{\tilde{H}^s}^2 = \sum_{h \in \mathbb{Z}} |\hat{f}_h|^2 a_h < \infty.$$ 

Hence,

$$|I_k(f) - A_n^{QMC}(f)| = \left| \sum_{j \in \mathbb{Z} \setminus 0} \hat{f}_{k+jn} a_{k+jn}^{1/2} a_{k+jn}^{-1/2} \right| \leq \left( \sum_{j \in \mathbb{Z} \setminus 0} |\hat{f}_{k+jn}|^2 a_{k+jn} \right)^{1/2} \left( \sum_{j \in \mathbb{Z} \setminus 0} a_{k+jn}^{-1} \right)^{1/2}$$ 

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and
\[ |I_k(f) - A_{n}^{\text{QMC}}(f)| \leq \|f\|_{H^s} \left( \sum_{j=1}^{\infty} \left( \frac{1}{\max\{1, (2\pi(jn + k))^{2s}\}} + \frac{1}{\max\{1, (2\pi(jn - k))^{2s}\}} \right) \right)^{1/2}. \]
The last inequality becomes an equality if we take
\[ f = c \sum_{j \in \mathbb{Z}, 0} \hat{f}_{k+jn} \] with \( c \neq 0 \), and \( \hat{f}_{k+jn} = a_{k+jn}^{-1}. \)
We can choose \( c \) such that \( \|f\|_{H^s} = 1 \). This yields the formula for \( \tilde{e}(A_{n}^{\text{QMC}}) \) and proves (i).

Using (i), we obtain for \( n \in [1, |k|] \) that
\[ \tilde{e}(A_{n}^{\text{QMC}}) > \left( \frac{1}{\max\{1, (2\pi(|k| - n))^{2s}\}} \right)^{1/2} > \left( \frac{1}{\max\{1, (2\pi k)^{2s}\}} \right)^{1/2} = \tilde{e}(0, k, s). \]
This proves (ii).

We now estimate \( \tilde{e}(A_{n}^{\text{QMC}}) \) for \( n > |k| \). For such an integer \( n \) and all \( j \in \mathbb{N} \) we have
\[ \frac{1}{\max\{1, (2\pi(jn + k))^{2s}\}} + \frac{1}{\max\{1, (2\pi(jn - k))^{2s}\}} \leq \frac{2}{(2\pi(jn - |k|))^{2s}}. \]
This yields
\[ \tilde{e}^2(A_{n}^{\text{QMC}}) \leq \frac{2}{(2\pi)^{2s}} \sum_{j=1}^{\infty} \frac{1}{(jn - |k|)^{2s}}. \]
We have
\[
\begin{align*}
\sum_{j=1}^{\infty} \frac{1}{(jn - |k|)^{2s}} &= \frac{1}{(n - |k|)^{2s}} + \sum_{j=2}^{\infty} \frac{1}{(jn - |k|)^{2s}} \\
&\leq \frac{1}{(n - |k|)^{2s}} + \int_{1}^{\infty} \frac{dx}{(nx - |k|)^{2s}} \\
&= \frac{1}{(n - |k|)^{2s}} - \frac{1}{(2s - 1)n} \left. (nx - |k|)^{-(2s-1)} \right|_{1}^{\infty} \\
&= \frac{1}{(n - |k|)^{2s}} + \frac{1}{(2s - 1)n (n - |k|)^{2s-1}} \\
&= \left( 1 + \frac{n - |k|}{(2s - 1)n} \right) \frac{1}{(n - |k|)^{2s}} \leq \frac{2s}{2s - 1} \frac{1}{(n - |k|)^{2s}} \\
&\leq \frac{2}{(n - |k|)^{2s}}.
\end{align*}
\]
This completes the estimate of \( \tilde{e}(A_{n}^{\text{QMC}}) \) for \( n > |k| \), and proves (iii).

If \( n > \left( (1+\alpha)/(1-\alpha) \right) |k| \) we have \( n - |k| > \alpha (n + |k|) \) and \( (n - |k|)^{-s} < \alpha^{-s} (n + |k|)^{-s} \). Then (iii) easily yields (iv) and completes the proof. \( \square \)

We comment on Theorem 4. Note that for \( k = 0 \), the point (ii) cannot happen and the assumptions of (iii) and (iv) always hold. We now discuss this theorem for \( k \neq 0 \). We start with (iv). Obviously, if \( \alpha \) is close to zero then the condition on \( n \) is relaxed. However, the upper bound on \( \tilde{e}(A_{n}^{\text{QMC}}) \) is weaker since the factor \( (2\pi \alpha)^{-s} \) goes to infinity. On the other hand, if \( \alpha \) goes to one then the condition on \( n \) is more severe but the upper bound (in terms of \( (n + |k|)^{-s} \)) on \( \tilde{e}(A_{n}^{\text{QMC}}) \) is better. This means that there is a tradeoff between the condition on \( n \) and the quality of the upper bound on \( \tilde{e}(A_{n}^{\text{QMC}}) \). This problem disappears if \( n \) goes to infinity. Then we can take \( \alpha \) close to one. In fact, the formula for \( \tilde{e}(A_{n}^{\text{QMC}}) \) for \( n \) tending to infinity yields

\[
\lim_{n \to \infty} \tilde{e}(A_{n}^{\text{QMC}})^{n^{s}} = \left( \frac{2\zeta(2)}{(2\pi)^{s}} \right)^{1/2},
\]  

where \( \zeta \) is the Riemann zeta function, \( \zeta(x) = \sum_{j=1}^{\infty} \frac{1}{j^{x}} \) for \( x > 1 \).

Remark 5. It is interesting that the right hand side of (13) appears for other problems. First of all, it is the norm of the embedding of \( \tilde{H}^{s} \cap \{ f \in \tilde{H}^{s} | \int_{\Omega} f(x) \, dx = 0 \} \), equipped with the norm \( \| f(s) \|_{L_{2}} \), into \( C([0,1]) \), see [7, Theorem 1.2]. This was proven by calculating the diagonal values of the corresponding reproducing kernel. Moreover, the right hand side of (13) equals \( \frac{1}{s!} \| B_{s} \|_{L_{2}} \), where \( B_{s} \) is the Bernoulli polynomial of degree \( s \), see [7, Lemma 2.16].

We now comment on Theorem 4 when \( n \in [1,|k|] \). In this case we know that the algorithm \( A_{n}^{\text{QMC}} \) is even worse than the zero algorithm. For instance, take \( n = |k| \). Then it is easy to conclude from (i) that

\[
\tilde{e}(A_{n}^{\text{QMC}})^{2} > 1 + \frac{1}{(4\pi |k|)^{2s}}.
\]  

Note that (14) is almost worst possible since every quadrature rule \( A_{n} \) with positive weights which sum up to one satisfies

\[
\tilde{e}(A_{n}) \leq \sup_{f \in \tilde{H}^{s}, \| f \|_{\tilde{H}^{s}} \leq 1} (|I_{k}(f)| + |A_{n}(f)|) \leq \| I_{k} \|_{\tilde{H}^{s} \to C} + \sup_{f \in \tilde{H}^{s}, \| f \|_{\tilde{H}^{s}} \leq 1} \sup_{x \in [0,1]} |f(x)| = \frac{1}{(2\pi |k|)^{s}} + \| \text{Id} \|.
\]
Here, $\text{Id} : \tilde{H}^s \to C([0,1])$ is the embedding operator, i.e., $\text{Id}f = f$ for all $f \in H^s$. We can estimate its norm as follows. We know that $\tilde{H}^s$ is a reproducing kernel Hilbert space with the kernel

$$\tilde{K}_s(x,t) = 1 + (-1)^{s-1} B_{2s}(\{x-t\}) = 1 + \sum_{h \in \mathbb{Z} \setminus 0} e^{2\pi i h(x-t)} \frac{(2\pi h)^{2s}}{(2\pi)^{2s}}$$

where $B_k^* = B_k/k!$ is the $k$th normalized Bernoulli polynomial, see [22], and $\{x-t\}$ is the fractional part of $x-t$. This implies for $f$ with $\|f\|_{H^s} \leq 1$ that

$$f^2(x) = \left< f, \tilde{K}_s(\cdot, x) \right>_{\tilde{H}^s} \leq \|f\|_{H^s}^2 \tilde{K}_s(x,x) \leq 1 + \frac{2}{(2\pi)^{2s}} \sum_{j=1}^{\infty} \frac{1}{j^{2s}}.$$ 

Hence, $\|\text{Id}\|^2 \leq 1 + 2\zeta(2s)/(2\pi)^{2s}$ and

$$\tilde{e}(A_n) \leq 1 + \frac{1}{(2\pi |k|)^s} + \frac{(2\zeta(2s))^{1/2}}{(2\pi)^s}$$

which for large $s$ is close to one as the right hand-side of (14).

We now show how to modify the algorithm $A_n^{\text{QMC}}$ such that its worst case error is smaller than the initial error $\tilde{e}(0,k,s)$ with no condition on $n$. It turns out that the weight $n^{-1}$ used by the algorithm $A_n^{\text{QMC}}$ is too large.

**Theorem 6.** For $a \in \mathbb{R}$, consider the algorithm of the form

$$A_{n,a}(f) = \frac{a}{n} \sum_{j=1}^{n} f(j/n) \exp^{-2\pi i k j/n} \quad \text{for all} \quad f \in \tilde{H}^s.$$ 

The worst case error of $A_{n,a}$ is minimized with respect to $a$ for

$$a = a_n^* = \frac{[\tilde{e}(0,k,s)]^2}{[\tilde{e}(0,k,s)]^2 + [\tilde{e}(A_n^{\text{QMC}})]^2},$$

and

$$\tilde{e}(A_{n,a_n^*}) = \frac{\tilde{e}(0,k,s) \tilde{e}(A_n^{\text{QMC}})}{\sqrt{[\tilde{e}(0,k,s)]^2 + [\tilde{e}(A_n^{\text{QMC}})]^2}}.$$ 

\footnote{The formula of the reproducing kernel of $H^s$ given as (10.2.4) on page 130 in [14] and as Example 21 on page 320 in [2] has a typo. The term $B_k^* B_k^*$ should be replaced by $\sum_{j=1}^{s} B_j^* B_j^*$, as is correctly stated in the original paper [3], where this result is proved.}
Clearly, 
\[ a^*_n < 1 \quad \text{and} \quad \tilde{e}(A_{n,a^*_n}) < \min \{ \tilde{e}(0, k, s), \tilde{e}(A_{n}^\text{QMC}) \}. \]

**Proof.** Repeating the analysis of the first part of the proof of Theorem 4, we obtain
\[ I_k(f) - A_{n,a}(f) = (1-a)\tilde{f}_k - a \sum_{j \in \mathbb{Z} \setminus \{0\}} \tilde{f}_{k+jn}. \]

Similarly as before we use \( a_h = \max \{ 1, (2\pi h)^{2s} \} \) and conclude that
\[ \tilde{e}(A_{n,a}) = \left( \frac{(1-a)^2}{a_k} + a^2 \sum_{j \in \mathbb{Z} \setminus \{0\}} \frac{1}{a_{k+jn}} \right)^{1/2} = \left( (1-a)^2[\tilde{e}(0, k, s)]^2 + a^2[\tilde{e}(A_{n}^\text{QMC})]^2 \right)^{1/2}. \]

Clearly, the last expression is minimized with respect to \( a \) for \( a = a^*_n \) from which we obtain the form of \( \tilde{e}(A_{n,a^*_n}) \). This completes the proof.

We discuss \( a^*_n \) which decreases the weight \( n^{-1} \) in the algorithm \( A_{n,a^*_n} \). For \( n \in [1, k] \), the point (iii) of Theorem 4 yields that \( a^*_n < 1/2 \). For \( n = |k| \geq 1 \) we know from (14) that \( \tilde{e}(A_{n}^\text{QMC}) > 1 \), and therefore \( a^*_n \leq [\tilde{e}(0, k, s)]^2 = (2\pi |k|)^{-2s} \) which is polynomially small in \( |k| \) and exponentially small in \( s \). On the other hand, if \( k \) is fixed and \( n \) goes to infinity then \( a^*_n \) goes to one and the algorithm \( A_{n,a^*_n} \) becomes the same as the algorithm \( A_{n}^\text{QMC} \).

The algorithm \( A_{n,a^*_n} \) has a (small) computational drawback since it requires the exact value of \( a^*_n \) which is given by the infinite series describing the worst case error of \( \tilde{e}(A_{n}^\text{QMC}) \). Of course, it can be precomputed to an arbitrary accuracy.

There is another simple idea how to modify the algorithm \( A_{n}^\text{QMC} \) without computing \( a^*_n \). Namely, for small \( n \) we use the zero algorithm whereas for large \( n \) we use the algorithm \( A_{n}^\text{QMC} \). More precisely, for \( n = 0, 1, \ldots \), we define the algorithm
\[ A^*_n(f) = \begin{cases} 0 & \text{if } n = 0 \text{ or } n < 2|k|, \\ A_{n}^\text{QMC}(f) & \text{if } n \geq \max(1, 2|k|). \end{cases} \quad (15) \]

The algorithm \( A^*_n \) uses no information on \( f \) if \( n = 0 \) or \( n < 2|k| \), and \( n \) function values otherwise. Based on Theorem 4 and the discussion after its proof it is easy to show

**Corollary 7.** We have
\[ \tilde{e}(A^*_n) \begin{cases} = 1 & \text{for } k = 0 \text{ and } n = 0, \\ = \frac{1}{(2\pi |k|)^s} & \text{for } k \neq 0 \text{ and } n \in [0, 2|k|], \\ \leq \frac{1}{(2\pi)^s (n-|k|)^s} & \text{for } n \geq \max(1, 2|k|). \end{cases} \]
Furthermore,

\[ \tilde{e}(A_n^*) \leq \left( \frac{3}{2\pi} \right)^s \frac{2}{(n + |k|)^s} \quad \text{for all} \quad n \geq 1. \tag{16} \]

**Proof.** Assume first that \( k = 0 \). Then for \( n = 0 \) we have \( A_0^* = 0 \) and \( \tilde{e}(A_0^*) = 1 \). For \( n \geq 1 \) we have \( A_n^* = A_n^{QMC} \) and we use Theorem 4(iii) to get the third estimate on \( \tilde{e}(A_n^*) \).

Assume now that \( k \neq 0 \). For \( n \in [0, 2|k|] \) the error of \( A_n^* = 0 \) is the initial error which is \((2\pi|k|)^{-s}\). For \( n \geq 2|k| \) we have \( A_n^* = A_n^{QMC} \) and the estimate on \( \tilde{e}(A_n^*) \) follows from Theorem 4(iii).

We now prove the estimate (16). Again assume first that \( k = 0 \). Consider first the case \( n \geq \max(1, 2|k|) \). We can now apply Theorem 4(iv) with \( \alpha = 1/3 \) and then

\[ \tilde{e}(A_n^*) = \tilde{e}(A_n^{QMC}) \leq \left( \frac{3}{2\pi} \right)^s \frac{2}{(n + |k|)^s}, \]

as claimed. It remains to consider the case \( n \in [1, 2|k|] \) for \( k \neq 0 \). Then \( |k| > (n + |k|)/3 \) and

\[ \tilde{e}(A_n^*) = \frac{1}{(2\pi|k|)^s} \leq \left( \frac{3}{2\pi(n + |k|)} \right)^s = \left( \frac{3}{2\pi} \right)^s \frac{1}{(n + |k|)^s}, \]

as claimed. This completes the proof. \( \square \)

**Remark 8.** Another possible modification of the algorithm \( A_n^{QMC} \) for small \( n \) is the “Filon-type” approach, see e.g. [1, 5, 6, 8]. For such an algorithm one assumes to have a given set of functions \( \{b_1, \ldots, b_N\} \), e.g. \( b_\ell \) could be polynomials, and that we know the values \( I_k(b_\ell) \) for all \( \ell = 1, \ldots, N \). Using these functions and the function values \( y_j = f(x_j), j = 1, \ldots, n \), we compute an approximation of the input \( f \) of the form \( F_n(x) = \sum_{k=1}^N c_k(y_1, \ldots, y_n) b_k(x) \). The “Filon-type” algorithm \( A_n^{**} \) for \( I_k \) is then given by \( A_n^{**}(f) = \sum_{k=1}^N c_k(y_1, \ldots, y_n) I_k(b_\ell) \). In Theorem 9 we prove a lower bound valid for all algorithms that use function and derivatives values. This lower bound shows that our algorithm \( A_n^* \) is almost optimal but does not exclude the possibility that a suitably chosen Filon-type algorithm is even slightly better than \( A_n^* \), i.e., \( \tilde{e}(A_n^{**}) < \tilde{e}(A_n^*) \) although \( \tilde{e}(A_n^{**}) \) must be also of order \((n + |k|)^{-s}\).

We stress that all algorithms considered so far use only function values although we allow also computation of derivatives up to order \( s - 1 \). Furthermore, they use function values at equally spaced points and use the same weights \( n^{-1} \) or \( a_n^* n^{-1} \) for large \( n \). Although algorithms that minimize the worst case error are probably not of this form, we now prove a lower bound on the order of convergence of an arbitrary algorithm, and show that this order is \((n + |k|)^s\). Hence the algorithm \( A_n^* \) enjoys the best possible order of convergence. Additionally, the algorithm \( A_n^* \) is easy to implement.
Theorem 9. Consider the integration problem $I_k$ defined over the space $\tilde{H}^s$ of periodic functions with $s \in \mathbb{N}$. Let $\tilde{e}(n, k, s)$ be the $n$th minimal worst case error of all algorithms that use at most $n$ function or derivatives (up to order $s-1$) values, see (10). There is a number $c_s > 0$ such that

$$\frac{c_s}{(n + |k|)^s} \leq \tilde{e}(n, k, s) \leq \left(\frac{3}{2\pi}\right)^s \frac{2}{(n + |k|)^s}$$

for all $k \in \mathbb{Z}$ and $n \in \mathbb{N}$.

Proof. The upper bound has been already shown for the algorithm $A^*_n$. Hence, we only need to prove the lower bound.

Let $A_n$ be an arbitrary algorithm of the form (8) that uses $f^{(\ell)}(x_j)$ for some $\ell \in [0, s-1]$ and $x_j \in [0, 1]$ for $j = 1, 2, \ldots, n$. Suppose that for $f \in \tilde{H}^s$ we get $f^{(\ell)}(x_j) = 0$ for all $j = 1, 2, \ldots, n$. Since $-f$ also belongs to $\tilde{H}^s$, the algorithm $A_n$ cannot distinguish between $I_k(f)$ and $I_k(-f) = -I_k(f)$. Therefore $|I_k(f)|$ is a lower bound on the worst case error of $A_n$. This leads to a well-known inequality

$$\tilde{e}(A_n) \geq \sup\{|I_k(f)| : f \in \tilde{H}^s, \|f\|_{H^s} \leq 1, N(f) = 0\},$$

where

$$N(f) = |f^{(\ell)}(x_j), j = 1, 2, \ldots, n|$$

Below we will construct a function $f$ with large $|I_k(f)|$ and all of the $s \cdot n$ values $f^{(\ell)}(x_j), j = 1, \ldots, n, \ell = 0, \ldots, s-1$, are equal to zero. Obviously, such a function $f$ satisfies $N(f) = 0$.

We consider a real-valued $f$ and the real part of $I_k(f)$ which is

$$\tilde{I}_k(f) = \int_0^1 f(x) \cos(2\pi k x) \, dx$$

of $I_k(f)$. Define the disjoint subintervals $T_{i,k} \subset [0, 1]$ such that for all $x \in T_{i,k}$ we have $|\cos(2\pi k x)| \geq 1/\sqrt{2}$. There are $2|k| + 1$ such subintervals. For $k = 0$ we have $T_{1,0} = [0, 1]$, whereas for $k \neq 0$ the lengths of $T_{i,k}$’s for $i = 1, 2, \ldots, 2|k| + 1$ are $\frac{1}{4|k|}, \frac{1}{8|k|}, \ldots, \frac{1}{4|k|}, \frac{1}{8|k|}$ with the total length 1/2. The points $x_1, \ldots, x_n$ used by $A_n$ may divide the $T_{i,k}$ further and altogether we obtain $m \in [2|k| + 1, 2|k| + 1 + n]$ intervals $\tilde{T}_{1,k}, \ldots, \tilde{T}_{m,k}$; all the endpoints of the $\tilde{T}_{i,k}$ coincide with an endpoint of one of the $T_{i,k}$ or are one of the $x_j$. Again, the sum of the lengths of the $\tilde{T}_{i,k}$ is 1/2.

We define $\Phi(x) = d_s(\cos^2(\pi x/2))^s$ for $|x| \leq 1$ and $\Phi(x) = 0$ otherwise. Then $\Phi \in C^s(\mathbb{R})$ and we can choose $d_s > 0$ in such a way that $\|\Phi\|_{H^s([-1,1])} = 1$. 

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Let the length of the interval $\hat{T}_{i,k}$ be $1/n_i$ and let $y_i$ be its midpoint. For $i = 1, 2, \ldots, m$, we define a scaled version of $\Phi$ by

$$\Phi_i(x) = \frac{\text{sgn}(\cos(2\pi ky_i))}{(2n_i)^s} \Phi(2n_i x - 2n_i y_i) \quad \text{for all} \quad x \in \mathbb{R}.$$ 

Note that the support of $\Phi_i$ is $\hat{T}_{i,k}$ and $\|\Phi_i\|_{H^s([-1,1])} \leq 1$. Furthermore,

$$\hat{I}_k(\Phi_i) = \frac{1}{(2n_i)^s} \int_{\hat{T}_{i,k}} |\cos(2\pi k x)| \Phi(2n_i(x - y_i)) \, dx \geq \frac{1}{2^{s+1/2} n_i^s} \int_{\hat{T}_{i,k}} \Phi(2n_i(x - y_i)) \, dx = \frac{d_s}{2^{s+3/2} n_i^{s+1}} \int_{-1}^{1} (\cos^2(\pi t/2))^s \, dt.$$

Finally we define our “fooling function” by

$$f = \sum_{i=1}^{m} \Phi_i.$$ 

It is easy to check that $f \in \widetilde{H}^s$ with $N(f) = 0$ and $\|f\|_{H^s} \leq 1$. We can also estimate the integral and obtain

$$|I_k(f)| \geq |\hat{I}_k(f)| = \sum_{i=1}^{m} \hat{I}_k(\Phi_i) \geq \tilde{c}_s \sum_{i=1}^{m} n_i^{-s-1}$$

with

$$\tilde{c}_s = \frac{d_s}{2^{s+3/2}} \int_{-1}^{1} (\cos^2(\pi t/2))^s \, dt > 0.$$ 

It is easy to check by standard means that

$$\min_{n_i: \sum_{i=1}^{m} n_i^{-1} = 1/2} \sum_{i=1}^{m} n_i^{-s-1} = \frac{1}{2^{s+1}} \frac{1}{m^s} \geq \frac{1}{2^{s+1} (2|k| + 1 + n)^s} \geq \frac{1}{2 \cdot 4^s (n + |k|)^s}.$$ 

This proves the lower bound with $c_s = \tilde{c}_s/(2 \cdot 4^s)$. $\square$

We stress that the lower bound in Theorem 9 holds for a larger class of algorithms than the class (8) for $s > 1$. Namely it holds for algorithms

$$A_n(f) = \sum_{j=1}^{n} \sum_{t=1}^{s-1} a_{j,t} f^{(t)}(x_j)$$
for arbitrary \( a_j, \ell \in \mathbb{C} \) and \( x_j \in [0, 1] \). That is, we now use \( n \cdot s \) values of \( f \) and its derivatives instead of \( n \), however, we still have “only” \( n \) sample points to choose.

Theorem 9 states that both lower and upper bounds on the \( n \)th minimal error decay with \( |k| \). Does it really mean that high oscillation makes the problem easy? The answer to this question depends on whether we consider the absolute or normalized error criterion.

For the absolute error criterion, the information complexity \( \tilde{n}_{\text{abs}}(\varepsilon, k, s) \) is defined as the minimal \( n \) for which the error is at most \( \varepsilon \in (0, 1) \). That is,

\[
\tilde{n}_{\text{abs}}(\varepsilon, k, s) = \min \{ n \mid \tilde{e}(n, k, s) \leq \varepsilon \}.
\]

Clearly, \( \tilde{n}_{\text{abs}}(\varepsilon, k, s) = 0 \) for \( \varepsilon \geq \tilde{e}(0, k, s) \) since we can solve the problem by the zero algorithm. For \( \varepsilon < \tilde{e}(0, k, s) \) we can bound \( \tilde{n}_{\text{abs}}(\varepsilon, k, s) \) by Theorem 9. This implies the following corollary.

**Corollary 10.** Consider the absolute error criterion for the integration problem \( I_k \) defined over the periodic space \( \tilde{H}^s \). Let \( c_s \) be from Theorem 9

- For \( k = 0 \) and all \( \varepsilon \in (0, 1) \) we have

  \[
  c_s^{1/s} \left( \frac{1}{\varepsilon} \right)^{1/s} \leq \tilde{n}_{\text{abs}}(\varepsilon, 0, s) \leq \left[ \frac{3}{2\pi} \left( \frac{\sqrt{2}}{\varepsilon} \right)^{1/s} \right].
  \]

- For \( k \neq 0 \) and \( \varepsilon \in [1/(2\pi |k|)^s, 1) \) we have

  \[
  \tilde{n}_{\text{abs}}(\varepsilon, k, s) = 0,
  \]

whereas for \( \varepsilon \in (0, 1/(2\pi |k|)^s) \) we have

\[
 c_s^{1/s} \left( \frac{1}{\varepsilon} \right)^{1/s} - |k| \leq \tilde{n}_{\text{abs}}(\varepsilon, k, s) \leq \left[ \frac{3}{2\pi} \left( \frac{\sqrt{2}}{\varepsilon} \right)^{1/s} \right] - |k|.
\]

This means that for the absolute error criterion the problem becomes easier for large \( |k| \), but the asymptotic behavior of \( \tilde{n}_{\text{abs}}(\varepsilon, k, s) \), as \( \varepsilon \to 0 \), does not depend on \( k \).

We now turn to the normalized error criterion in which we want to reduce the initial error \( \tilde{e}(0, k, s) \) by a factor \( \varepsilon \in (0, 1) \). That is, the information complexity \( \tilde{n}_{\text{nor}}(\varepsilon, k, s) \) is defined as

\[
\tilde{n}_{\text{nor}}(\varepsilon, k, s) = \min \{ n \mid \tilde{e}(n, k, s) \leq \varepsilon \tilde{e}(0, k, s) \}. 
\]
In this case we always have \( \tilde{n}_{\text{nor}}(\varepsilon, k, s) \geq 1 \). Note that for \( k = 0 \) we have \( \tilde{e}(0,0,s) = 1 \) and there is no difference between the normalized and absolute error criteria.

For \( k \neq 0 \) the situation is quite different. From Theorem 4, Theorem 9 and Proposition 3 it is easy to prove the following corollary.

**Corollary 11.** Consider the normalized error criterion for the integration problem \( I_k \) defined over the periodic space \( \tilde{H}^s \). Let \( c_s \) be from Theorem 9.

For all \( k \neq 0 \) and all \( \varepsilon \in (0,1) \) we have

\[
|k| \left( 2\pi \left( \frac{c_s}{\varepsilon} \right)^{1/s} - 1 \right) \leq \tilde{n}_{\text{nor}}(\varepsilon, k, s) \leq |k| \left[ 3 \left( \frac{\sqrt{2}}{\varepsilon} \right)^{1/s} - 1 \right],
\]

which can be written as

\[
\tilde{n}_{\text{nor}}(\varepsilon, k, s) = \Theta \left( \frac{|k|}{\varepsilon^{1/s}} \right) \quad \text{as} \quad \varepsilon \to 0.
\]

Hence, for the normalized error criterion the problem becomes harder for large \(|k|\). It is interesting that the dependence on \(|k|\) is linear and does not depend on \( s \). In particular, for fixed \( s \) and fixed \( \varepsilon < (2\pi)^s c_s \) we have

\[
\lim_{|k| \to \infty} \tilde{n}_{\text{nor}}(\varepsilon, k, s) = \infty. \tag{17}
\]

**4 The non-periodic case**

We now turn to the case of non-periodic functions, i.e., we consider the Sobolev space

\[
H^s = \{ f : [0, 1] \to \mathbb{C} \mid f^{(s-1)} \text{ is abs. cont.}, f^{(s)} \in L_2 \} \tag{18}
\]

for a finite \( s \in \mathbb{N} \). The inner product \( \langle \cdot, \cdot \rangle_s \) in \( H^s \) is again defined by (3).

Clearly, for all \( j = 1, 2, \ldots, s \) we have \( H^s \subset H^j \) and \( \|f\|_{H^j} \leq \|f\|_{H^s} \) for all \( f \in H^s \). This follows from the inequality

\[
\int_0^1 |f''(x)|^2 \, dx \geq \int_0^1 |f(x)|^2 \, dx - \left( \int_0^1 f(x) \, dx \right)^2
\]

for differentiable functions \( f \) and implies that the unit ball of \( H^s \) is a subset of the unit ball of \( H^j \).
Again we want to approximate the integral

\[ I_k(f) := \int_0^1 f(x) e^{-2\pi i kx} \, dx, \]  

where \( k \in \mathbb{Z} \) and \( f \in H^s \) with \( s \in \mathbb{N} \). Without loss of generality we consider linear algorithms \( A_n \) of the form (8). Similarly as before, we define the worst case error of \( A_n \) as

\[ e(A_n) := \sup_{f \in H^s, \|f\|_{H^s} \leq 1} \|I_k(f) - A_n(f)\|, \]

and the \( n \)-th minimal worst case error as

\[ e(n, k, s) := \inf_{A_n} e(A_n). \]

In particular, the initial error is given by

\[ e(0, k, s) := \sup_{f \in H^s, \|f\|_{H^s} \leq 1} \|I_k(f)\| = \|I_k\|_{H^s \to \mathbb{C}}, \]

compare with (10) and (11). We do not now use the tilde to stress the non-periodic case.

Note that \( H^s \) is obviously a superset of \( \tilde{H}^s \) and hence, lower bounds that were proved in Section 3 for \( \tilde{H}^s \) also hold for \( H^s \), i.e.,

\[ e(n, k, s) \geq \tilde{e}(n, k, s). \]  

We start with the computation of the initial error. As we shall see, for large \( s \) and \( |k| \), it is now much larger than for the periodic case. In particular, the initial error for \( k \neq 0 \) does not tend to zero if \( s \) tends to infinity.

Similarly to (12) we want to compute the representer \( h_{k,s} \) of \( I_k \) in \( H^s \). Using the same functions \( e_k(x) = e^{2\pi i kx} \), which satisfy \( \|e_k\|_{H^s} = (2\pi |k|)^s \), we obtain

\[ \langle f, e_k \rangle_s = \left\langle f^{(s)}, e_k^{(s)} \right\rangle_0 = (-1)^s \left\langle f, e_k^{(2s)} \right\rangle_0 + \sum_{\ell=0}^{s-1} (-1)^\ell \left[ f^{(s-\ell-1)} e_k^{(s+\ell)} \right]_0 = (2\pi k)^{2s} I_k(f) + (-1)^s \sum_{\ell=1}^s (2\pi i k)^{2s-\ell} \left( f^{(\ell-1)}(1) - f^{(\ell-1)}(0) \right). \]  

Here we use the fact that \( k \) is an integer. Surprisingly, the functionals \( f^{(\ell-1)}(1) - f^{(\ell-1)}(0) \), \( \ell = 1, \ldots, s \), or more precisely their representers in \( H^s \), have some nice properties that will
be useful in the following analysis. These representers are given by the normalized Bernoulli polynomials

\[ B^*_\ell(x) = \frac{1}{\ell!} B_\ell(x), \]

(22)

where the Bernoulli polynomials \( B_\ell, \ell \geq 0, \) are the unique polynomials that are given by

\[ \int_t^{t+1} B_\ell(x) \, dx = t^\ell \quad \text{for all} \quad t \in \mathbb{R} \quad \text{with} \quad 0^0 = 1. \]

To see this, note that \( [B^*_\ell]' = B^*_{\ell-1} \) as well as \( \int_0^1 B^*_\ell(x) \, dx = 0, \ell \geq 1, \) and \( \int_0^1 B^*_0(x) \, dx = 1. \)

In particular, this implies for \( f \in H^s \) and \( \ell \leq s \) that

\[ \langle f, B^*_\ell \rangle_s = \langle f(\ell), 1 \rangle_0 = f^{(\ell-1)}(1) - f^{(\ell-1)}(0), \]

which proves the claim. Additionally, this shows

\[ \| B^*_\ell \|_{H^s} = 1 \quad \text{and} \quad \langle B^*_\ell, B^*_m \rangle_s = 0 \]

(23)

for \( \ell, m \in \{0, 1, \ldots, s\} \) with \( \ell \neq m \) and, consequently,

\[ H^s = \tilde{H}^s \oplus \{ B^*_1 \} \oplus \cdots \oplus \{ B^*_s \}, \]

see e.g. [14, Section 10.2].

Using (21) we obtain the following proposition.

**Proposition 12.** Let \( k \neq 0. \) The representer of \( I_k \) in \( H^s \) is

\[ h_{k,s}(x) = (2\pi k)^{-2s} e^{2\pi i k x} - \sum_{\ell=1}^{s} (-1)^\ell (2\pi i k)^{-\ell} B^*_\ell(x) \]

and the initial error is

\[ e(0, k, s) = \| h_{k,s} \|_{H^s} = \sqrt{\frac{2}{(2\pi k)^{2s}} + \sum_{\ell=1}^{s-1} \frac{1}{(2\pi k)^{2\ell}}} = \frac{\beta_{k,s}}{2\pi |k|} \]

with \( \beta_{k,1} = \sqrt{2} \) and

\[ 1 \leq \beta_{k,s} = \sqrt{\frac{(2\pi k)^{2s} + (2\pi k)^2 - 2}{(2\pi k)^{2s} - (2\pi k)^2(s-1)}} \leq \sqrt{1 + \frac{2}{(2\pi k)^2 - 1}} \leq 1.02566 \]

for \( s > 1. \) Note that \( \lim_{k \to \infty} \beta_{k,s} = 1. \)

For \( k = 0, \) the representer is \( h_{0,s} = 1 \) and the initial error is one, \( e(0, 0, s) = 1. \)
We are ready to discuss algorithms for the non-periodic case. One of the ideas to get such algorithms is first to periodize functions $f$ from $H^s$ by computing $f^{(0)}(0), \ldots, f^{(s-1)}(0)$ and $f^{(0)}(1), \ldots, f^{(s-1)}(1)$, and then apply the algorithm $A^*_{n-2s}$ from Section 3. Of course, this requires to assume that $n \geq 2s$ which is a bad assumption if $s$ is large or even impossible to satisfy if $s = \infty$. Therefore for $n < 2s$ we need to proceed differently. As already discussed $f \in H^s$ implies that $f \in H^j$ for all $j \leq s$. Therefore we can use periodization for $H^j$ by computing $f^{(0)}(0), \ldots, f^{(j-1)}(0)$ and $f^{(0)}(1), \ldots, f^{(j-1)}(1)$ as long as $n \geq 2j$. Then we can again apply the algorithm $A^*_{n-2j}$ from Section 3. Formally, this algorithm was studied only for $H^s$ but it is obvious that its error can be also analyzed for $H^j$ with the change of $s$ to $j$.

Another idea to obtain algorithms for small $n$ relative to $s$ is to use the integration of Taylor’s expansion of $f \in H^s$ at $\frac{1}{2}$. As we shall see this approach is appropriate if $|k|$ is relatively small with respect to $n$. To explain these ideas more precisely we need some preparations.

### 4.1 Periodization

Let $1 \leq j \leq s$ be given. For $f \in H^s$, we compute

$$f^{(0)}(0), \ldots, f^{(j-1)}(0)$$

and

$$f^{(0)}(1), \ldots, f^{(j-1)}(1).$$

With this information we define a polynomial $p_{f,j}$ of degree at most $j$ such that $\tilde{f}_j = f - p_{f,j}$ is a periodic function from $\tilde{H}^j$. To obtain the polynomial $p_{f,j}$, we use the normalized Bernoulli polynomials from (22). In particular, $B^*_0(x) = 1$ and $B^*_1(x) = x - \frac{1}{2}$. For $m \geq 1$, we have $[B^*_m] = B^*_{m-1}$ which yields

$$[B^*_m]^\ell = B^*_{m-\ell} \quad \text{for all} \quad \ell = 0, 1, \ldots, m. \quad (24)$$

Furthermore,

$$B^*_1(1) - B^*_1(0) = 1 \quad \text{and} \quad B^*_m(1) - B^*_m(0) = 0 \quad \text{for all} \quad m \neq 1. \quad (25)$$

For $f \in H^s \subset H^j$, we define the polynomials $p_{f,j}$ by

$$p_{f,j}(x) := \sum_{m=0}^{j-1} (f^{(m)}(1) - f^{(m)}(0)) B^*_m(x). \quad (26)$$

We stress that the computation of the value $p_{f,j}(x)$ requires the $2j$ values of $f^{(m)}(1)$ and $f^{(m)}(0)$ for $m = 0, 1, \ldots, j - 1$. 

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For \( \ell = 0, 1, \ldots, j - 1 \), we conclude from (24) that

\[
p_{f,j}^{(\ell)}(x) = \sum_{m=\max(0,\ell-1)}^{j-1} \left( f^{(m)}(1) - f^{(m)}(0) \right) B_{m+1-\ell}^*(x).
\]

Using (25) we obtain

\[
p_{f,j}^{(\ell)}(1) - p_{f,j}^{(\ell)}(0) = f^{(\ell)}(1) - f^{(\ell)}(0) \quad \text{for all} \quad \ell = 0, 1, \ldots, j - 1.
\]

This implies that \( f - p_{f,j} \in \tilde{H}^j \) for all \( f \in H^s \).

Since \( f - p_{f,j} \in \tilde{H}^j \) and the norm of \( I_k \) restricted to the space \( \tilde{H}^j \) is given by Proposition 3 with \( s \) replaced by \( j \), we know that

\[
|I_k(f) - I_k(p_{f,j})| \leq \|I_k\|_{\tilde{H}^j} \|f - p_{f,j}\|_{H^j} = \frac{\|f - p_{f,j}\|_{H^j}}{\max\{1, (2\pi |k|)^j\}}
\]

and, by Parseval’s identity (in \( \tilde{H}^j \)), that

\[
\|f - p_{f,j}\|_{H^j}^2 = \left| \langle f - p_{f,j}, 1 \rangle \right|^2 + \sum_{\ell \in \mathbb{Z} \setminus 0} \left| \langle f - p_{f,j}, e_{\ell} \rangle \right|^2 (2\pi \ell)^{-2j}
= \left| \langle f, 1 \rangle \right|^2 + \sum_{\ell \in \mathbb{Z} \setminus 0} \left| \langle f, e_{\ell} \rangle \right|^2 (2\pi \ell)^{-2j} \leq \|f\|_{H^j}^2 \leq \|f\|_{H^s}^2.
\]

Here we used the fact that \( \langle p_{f,j}, e_k \rangle_j = 0 \) for all \( k \in \mathbb{Z} \). This proves that

\[
|I_k(f) - I_k(p_{f,j})| \leq \frac{\|f\|_{H^s}}{\max\{1, (2\pi |k|)^j\}}.
\]

Note that the last upper bound is not small for \( k = 0 \). However, if \( k \neq 0 \) then for \( j \in [1, s] \) for all \( f \in H^s \) we have

\[
|I_k(f) - I_k(p_{f,j})| \leq \frac{\|f\|_{H^s}}{(2\pi |k|)^j},
\]

which is exponentially small in \( j \).

We now show how to compute \( I_k(p_{f,j}) \) exactly. Indeed,

\[
I_k(p_{f,j}) = \sum_{m=0}^{j-1} \left( f^{(m)}(1) - f^{(m)}(0) \right) I_k(B_{m+1}^*).
\]
and it is enough to compute $I_k(B_{m+1}^*)$. For $k = 0$ we have $I_0(B_{m+1}^*) = \int_0^1 B_{m+1}^*(x) \, dx = 0$ for all $m = 0, 1, \ldots, j - 1$. Hence, $I_0(p_{f,j}) = 0$.

For $k \neq 0$, we use the Fourier expansion of the normalized Bernoulli polynomials $B_{m+1}^*$,

$$B_{m+1}^*(x) = \frac{1}{(2\pi i)^{m+1}} \sum_{\ell \in \mathbb{Z} \setminus 0} \frac{e^{2\pi i \ell x}}{\ell^{j+1}} \quad \text{for all} \quad x \in [0, 1]. \quad (29)$$

This yields

$$I_k(B_{m+1}^*) = -\frac{1}{(2\pi i)^{m+1}} \sum_{\ell \in \mathbb{Z} \setminus 0} \frac{\int_0^1 e^{2\pi i (\ell-k) x} \, dx}{\ell^{j+1}} = -\frac{1}{(2\pi ik)^{m+1}}.$$

Hence,

$$I_k(p_{f,j}) = \begin{cases} 0 & \text{for } k = 0 \\ \frac{1}{(2\pi ik)^{m+1}} & \text{for } k \neq 0. \end{cases}$$

For $k \neq 0$, the computation of $I_k(p_{f,j})$ requires the $2j$ values of $f^{(\ell)}(1)$ and $f^{(\ell)}(0)$ for $\ell = 0, 1, \ldots, j - 1$ which are also needed for the computation of $p_{f,j}(x)$.

### 4.2 Taylor’s Expansion

For $n \in [1, s]$, we use Taylor’s expansion of $f \in H^s$ at $\frac{1}{2}$. Let

$$T_{f,n}(x) = f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)(x - \frac{1}{2}) + \cdots + \frac{f^{(n-1)}\left(\frac{1}{2}\right)}{(n-1)!} (x - \frac{1}{2})^{n-1} \quad \text{for all} \quad x \in [0, 1].$$

Then

$$f(x) - T_{f,n}(x) = \left(x - \frac{1}{2}\right)^n \frac{1}{(n-1)!} \int_0^1 (1-t)^{n-1} f^{(n)}\left(\frac{1}{2} + t\left(x - \frac{1}{2}\right)\right) \, dt \quad \text{for all} \quad x \in [0, 1].$$

This allows us to estimate $I_k(f - T_{f,n})$ since

$$I_k(f - T_{f,n}) = \frac{1}{(n-1)!} \int_0^1 \int_0^1 e^{-2\pi i k x} \left(x - \frac{1}{2}\right)^n (1-t)^{n-1} f^{(n)}\left(\frac{1}{2} + t\left(x - \frac{1}{2}\right)\right) \, dt \, dx,$$

and

$$|I_k(f - T_{f,n})| \leq \frac{1}{(n-1)!} \int_0^1 \int_0^1 |x - \frac{1}{2}|^n |f^{(n)}\left(\frac{1}{2} + t\left(x - \frac{1}{2}\right)\right)| \, dt \, dx.$$
We now change variables \( y = \frac{1}{2} + t(x - \frac{1}{2}) \in [0, 1] \) so that \( dy = (x - \frac{1}{2}) dt \) and then

\[
|I_k(f) - I_k(T_{f,n})| \leq \frac{1}{(n-1)!} \int_0^1 \int_0^1 |x - \frac{1}{2}|^{n-1} |f^{(n)}(y)| \, dx \, dy \leq \frac{1}{2^{n-1} n!} \|f^{(n)}\|_{L^2}.
\]

This proves that for \( n \in [1, s] \) and all \( f \in H^s \) we have

\[
|I_k(f) - I_k(T_{f,n})| \leq \frac{1}{2^{n-1} n!} \|f\|_{H^s}. \tag{30}
\]

Furthermore, we can compute \( I_k(T_{f,n}) \) exactly if we know \( f(\frac{1}{2}), f'(\frac{1}{2}), \ldots, f^{(n-1)}(\frac{1}{2}) \). Indeed,

\[
I_k(T_{f,n}) = \sum_{\ell=0}^{n-1} f^{(\ell)}(\frac{1}{2}) \frac{1}{\ell!} \int_0^1 e^{-2\pi i k x} (x - \frac{1}{2})^\ell \, dx.
\]

For \( k = 0 \), we have

\[
I_0(T_{f,n}) = \sum_{\ell=0}^{n-1} f^{(\ell)}(\frac{1}{2}) \frac{1 + (-1)^\ell}{2^\ell+1 (\ell + 1)!} \tag{31}
\]

For \( k \neq 0 \), we use integration by parts and show that

\[
\frac{1}{\ell!} \int_0^1 e^{-2\pi i k x} (x - \frac{1}{2})^\ell \, dx = \frac{1}{(2\pi i k)^{\ell+1}} \sum_{m=0}^\ell \frac{i^m (k\pi)^m ((-1)^m - 1)}{m!}.
\]

Hence for \( k \neq 0 \), we have

\[
I_k(T_{f,n}) = \sum_{\ell=0}^{n-1} \frac{f^{(\ell)}(\frac{1}{2})}{(2\pi i k)^{\ell+1}} \sum_{m=0}^\ell \frac{i^m (k\pi)^m ((-1)^m - 1)}{m!}. \tag{32}
\]

### 4.3 Algorithms

With the preparations done in the previous two subsections, we are ready to define algorithms for the non-periodic case.

- Assume first that \( k \in \mathbb{Z} \setminus \{0\} \).

We discuss algorithms based on periodization for \( f \in H^s \). We define the algorithm \( A_n^{\text{Per}} \) for all even \( n \in [2, 2s] \) and for \( n = 2s + \ell \) with \( \ell \in \mathbb{N}_0 \).
For even \( n \in [2, 2s) \) we compute \( f^{(0)}(0), \ldots, f^{(n-2)/2}(0), f^{(0)}(1), \ldots, f^{(n-2)/2}(1) \), and define

\[
A_{n}^{\text{Per}}(f) = I_{k}(p_{f,n/2})
\]

with \( p_{f,n/2} \) given by (26) for \( j = n/2 \leq s \).

For \( n = 2s + \ell \) with \( \ell \in \mathbb{N}_0 \), we compute \( f^{(0)}(0), \ldots, f^{(s-1)}(0), f^{(0)}(1), \ldots, f^{(s-1)}(1) \) to obtain the polynomial \( p_{f,s} \). Then we define

\[
A_{n}^{\text{Per}}(f) = I_{k}(p_{f,s}) + A_{\ell+1}^{*}(f - p_{f,s})
\]

with the algorithm \( A_{\ell+1}^{*} \) from Section 5 defined by (13). The algorithm \( A_{\ell+1}^{*} \) uses no extra information on \( f \) if \( \ell < 2|k| - 1 \). For \( \ell \geq 2|k| - 1 \), the algorithm \( A_{\ell+1}^{*} \) uses extra \( \ell \) function values at \( j/(\ell + 1) \) for \( j = 1, 2, \ldots, \ell \). Note that we have already computed the function value at \( j/(\ell + 1) \) for \( j = \ell + 1 \).

We stress that the algorithm \( A_{n}^{\text{Per}} \) is well defined for \( n = 2s + \ell \) since \( f - p_{f} \in \tilde{H}^{s} \) and \( \tilde{H}^{s} \) is the domain of the algorithm \( A_{\ell+1}^{*} \). For \( f \in \tilde{H}^{s} \) we have \( p_{f,j} = 0 \) for all \( j \in [1, s] \), and therefore \( A_{n}^{\text{Per}}(f) = 0 \) for all even \( n \in [2, 2s) \) and \( A_{n}^{\text{Per}}(f) = A_{\ell+1}^{*}(f) \) for \( n = 2s + \ell \).

The algorithm \( A_{n}^{\text{Per}} \) uses at most \( n \) evaluations of \( f \). Indeed, for even \( n \in [2, 2s) \) it uses \( n/2 \) evaluations at the endpoint points \( x = 0 \) and \( x = 1 \), so that the total number is \( n \). For \( n = 2s + \ell \), the algorithm \( A_{n}^{\text{Per}} \) uses two function values and \( 2(s - 1) \) values of derivatives of \( f \) at \( x = 0 \) and \( x = 1 \), as well as at most \( \ell \) functions values at \( j/(\ell + 1) \) for \( j = 1, \ldots, \ell \), which is \( 2 + 2(s - 1) + \ell = 2s + \ell = n \), as claimed.

From the formulas of Sections 3 and 4.1 we find the explicit form of \( A_{n}^{\text{Per}} \). For even \( n \in [2, 2s) \) and \( n = 2s + \ell \) with \( \ell < 2|k| - 1 \) we have

\[
A_{n}^{\text{Per}}(f) = \sum_{j=0}^{(n-2)/2} \frac{f^{(j)}(0) - f^{(j)}(1)}{(2\pi i k)^{j+1}},
\]

whereas for \( n = 2s + \ell \) with \( \ell \geq 2|k| - 1 \), we have

\[
A_{n}^{\text{Per}}(f) = \sum_{j=0}^{s-1} \frac{f^{(j)}(0) - f^{(j)}(1)}{(2\pi i k)^{j+1}} + \frac{1}{\ell + 1} \sum_{j=1}^{\ell+1} \left( f\left(\frac{j}{\ell + 1}\right) - p_{f,s}\left(\frac{j}{\ell + 1}\right) \right) \exp^{-2\pi i k j/(\ell+1)}.
\]

Note that for \( s = 1 \) the algorithm \( A_{n}^{\text{Per}} \) uses only function values since \( p_{f,1}(x) = (f(1) - f(0))(x - \frac{1}{2}) \), whereas for \( s \geq 2 \) it also uses derivatives of \( f \). The weights used by the
algorithm $A_{n}^{\text{Per}}$ are complex. However, the sum of their absolute values is bounded by an absolute constant independent of $n$ since it is known that the values of the normalized Bernoulli polynomials $B_{j}^{*}$, which are present in $p_{f,s}$, are exponentially small in $j$. This implies numerical stability of the algorithm $A_{n}$.

Obviously, the derivatives $f^{(j)}(0)$ and $f^{(j)}(1)$ for $j = 1, 2, \ldots, s - 1$ can be approximated by computing $2(s - 1)$ extra function values; the worst case error of this approximation for functions from the unit ball of $H^{s}$ can be made arbitrarily small. Hence, there are algorithms that use only $n$ function values and they have a worst case error arbitrarily close to the worst case error of $A_{n}^{\text{Per}}$. However, stability of such algorithms is not clear since then we must use huge coefficients. We leave it as an open problem for $s \geq 2$ if there are stable algorithms that use only $n$ function values and whose worst case error are comparable to the algorithm $A_{n}^{\text{Per}}$.

We are ready to bound the worst case error of $A_{n}^{\text{Per}}$.

**Theorem 13.** For $k \neq 0$, we have

- for even $n \in [2, 2s],
  \[ e(A_{n}^{\text{Per}}) \leq \frac{1}{(2\pi|k|)^{n/2}}, \]

- for $n > 2s,
  \[ e(A_{n}^{\text{Per}}) \leq \left(\frac{3}{2\pi}\right)^{s} \frac{2}{(n - 2s + 1 + |k|)^{s}}. \]

**Proof.** For even $n \in [2, 2s]$, we have

\[ I_{k}(f) - A_{n}^{\text{Per}}(f) = I_{k}(f) - I_{k}(p_{f,n/2}) \]

and (28) implies the bound on $e(A_{n})$.

For $n = 2s + \ell$, we clearly have $f = f - p_{f,s} + p_{f,s}$ for all $f \in H^{s}$. By definition of $A_{n}^{\text{Per}}$ and linearity of $I_{k}$ we obtain

\[ I_{k}(f) - A_{n}^{\text{Per}}(f) = I_{k}(f - p_{f,s}) - A_{\ell+1}^{*}(f - p_{f,s}). \]

From (16) we know that

\[ |I_{k}(f - p_{f,s}) - A_{\ell+1}^{*}(f - p_{f,s})| \leq \left(\frac{3}{2\pi}\right)^{s} \frac{2}{(n - 2s + 1 + |k|)^{s}} \|f - p_{f,s}\|_{H^{s}}. \]

Then (27) with $j = s$ yields $\|f - p_{f,s}\|_{H^{s}} \leq \|f\|_{H^{s}}$, which implies the bound on $e_{n}(A_{n}^{\text{Per}})$. This completes the proof. \qed
Assume now that \( k \in \mathbb{Z} \).

Although \( k \) is now an arbitrary integer, our emphasis will be later on \( k = 0 \) or, more generally, on \(|k|\) small relative to \( n \). We discuss algorithms based on Taylor’s expansion and periodization for \( f \in H^s \). We define the algorithm \( A^{\text{Tay-Per}}_n \) for all \( n \in [1, s] \) and for \( n = 2s + \ell \) with \( \ell \in \mathbb{N}_0 \).

For \( n \in [1, s] \), we compute \( f^{(0)}(\frac{1}{2}), \ldots, f^{(n-1)}(\frac{1}{2}) \) and define

\[
A^{\text{Tay-Per}}_n(f) = I_k(T_{f,n}),
\]

where \( T_{f,n} \) is Taylor’s expansion of \( f \) at \( \frac{1}{2} \) up to the \((n-1)\)st derivative and \( I_k(T_{f,n}) \) is explicitly given by (31) for \( k = 0 \) and by (32) for \( k \neq 0 \). For \( n = 2s + \ell \) with \( \ell \in \mathbb{N}_0 \), we define

\[
A^{\text{Tay-Per}}_n(f) = A^{\text{Per}}_n(f),
\]

where \( A^{\text{Per}}_n \) is from the previous subsection.

Clearly, the algorithm \( A^{\text{Tay-Per}}_n \) uses at most \( n \) evaluations of \( f \). For \( s = 1 \) it uses only function values and it is defined for all \( n \). For \( s \geq 2 \), it also uses derivatives of \( f \) and it is not defined for \( n \in [s + 1, 2s - 1] \). Its weights are complex and the sum of their absolute values is uniformly bounded in \( n \). Hence, \( A^{\text{Tay-Per}}_n \) is stable, and a similar remark on the approximation of derivatives by function values can be made as for the algorithm \( A^{\text{Per}}_n \).

The worst case error of \( A^{\text{Tay-Per}}_n \) can be easily bounded by (30) and Theorem 13. We summarize these bounds in the following theorem.

**Theorem 14.** For an arbitrary integer \( k \), we have

- for \( n \in [1, s] \),
  \[
e(A^{\text{Tay-Per}}_n) \leq \frac{1}{2^{n-1} n!},\]
  - for \( n \geq 2s \)
  \[
e(A^{\text{Tay-Per}}_n) \leq \left( \frac{3}{2\pi} \right)^s \frac{2}{(n - 2s + 1 + |k|)^s}.\]

We now comment on Theorems 13 and 14 for a finite \( s \). For \( k = 0 \) and initial \( n \), i.e., even \( n \leq 2s \) or \( n \leq s \), we can only apply Theorem 14. It tells us that for \( n \in [1, s] \) the error bound of \( A^{\text{Tay-Per}}_n \) is exponentially small in \( n \). Note that for non-zero \( k \) we can use both theorems. For the initial \( n \) and \(|k|\) small relative to \( n \), the first bound of Theorem 14 is smaller than the first bound in Theorem 13. On the other hand, for large \(|k|\) relative to \( n \),
the opposite is true. Obviously for \( n > 2s \), both theorems coincide and the error bound of \( A_n^{\text{Tay-Per}} = A_n^{\text{Per}} \) is of the form

\[
e(A_n^{\text{Per}}) \leq \left( \frac{3}{2\pi} \right)^s \frac{2}{(n - 2s + 1 + |k|)^s}.
\]

(33)

The last bound yields an upper bound on the \( n \)th minimal error \( e(n, k, s) \) for \( n \geq 2s \). Combining this with (20) and Theorem 9 we obtain sharp lower and upper bounds on the minimal errors \( e(n, k, s) \).

**Theorem 15.** Consider the integration problem \( I_k \) defined over the space \( H^s \) of non-periodic functions with \( s \in \mathbb{N} \). Then

\[
\frac{c_s}{(n + |k|)^s} \leq e(n, k, s) \leq \left( \frac{3}{2\pi} \right)^s \frac{2}{(n + |k| - 2s + 1)^s},
\]

for all \( k \in \mathbb{Z} \) and \( n \geq 2s \). The positive number \( c_s \) is from Theorem 9.

We stress that Theorem 15 presents an upper bound on the minimal errors \( e(n, k, s) \) only for \( n \geq 2s \), the lower bound holds for all \( n \). The reason is that we need \( 2s \) function and derivatives values to periodize the function \( f \) which enables us to use the algorithm \( A_n^{\ast} \). We do not know sharp bounds on \( e(n, k, s) \) for \( n \in [1, 2s) \). However, we know that \( e(n, k, s) \) is at most \( 1/(2^{n-1}n!) \) for \( n \leq s \) and all \( k \), see Theorem 14 and at most \( (2\pi|k|)^{-n/2} \) for \( n \leq 2s \) and all \( k \neq 0 \), see Theorem 13. Of course, the problem of the minimal errors \( e(n, k, s) \) for initial \( n \) it is not very important as long as \( s \) is not too large.

The minimal errors \( e(n, k, s) \) for the non-periodic case have a peculiar property for \( s \geq 2 \) and large \( k \). Namely, for \( n = 0 \) we obtain the initial error which is of order \( |k|^{-1} \), whereas for \( n \geq 2s \) it becomes of order \( |k|^{-s} \). Hence, the dependence on \( |k|^{-1} \) is short-lived and disappears quite quickly. For instance, take \( s = 2 \). Then \( e(n, k, s) \) is of order \( |k|^{-1} \) only for \( n = 0 \) and maybe for \( n = 1, 2, 3 \), and then becomes of order \( |k|^{-2} \).

We now briefly discuss the absolute and normalized error criteria for the non-periodic case. For the **absolute error criterion**, the information complexity \( n^{\text{abs}}(\varepsilon, k, s) \) for \( \varepsilon \in (0, 1) \) is defined as

\[
n^{\text{abs}}(\varepsilon, k, s) = \min \{ n \mid e(n, k, s) \leq \varepsilon \}.
\]

Clearly, \( n^{\text{abs}}(\varepsilon, k, s) = 0 \) for \( \varepsilon \geq e(0, k, s) \). For \( \varepsilon < e(0, k, s) \) we can bound \( n^{\text{abs}}(\varepsilon, k, s) \) by Theorem 15. This implies the following corollary.

**Corollary 16.** Consider the absolute error criterion for the integration problem \( I_k \) defined over the space \( H^s \). Let \( c_s \) be from Theorem 9.
• For \( k = 0 \) and all \( \varepsilon \in (0, 1) \) we have
\[
c_s^{1/s} \left( \frac{1}{\varepsilon} \right)^{1/s} \leq n^{\text{abs}}(\varepsilon, 0, s) \leq \left\lfloor \left( \frac{3}{2\pi} \right) \left( \frac{2}{\varepsilon} \right)^{1/s} \right\rfloor + 2s - 1.
\]

• For \( k \neq 0 \) and \( \varepsilon \in [\beta_{k,s}/(2\pi|k|), 1) \), with \( \beta_{k,s} \) from Proposition 12, we have
\[
n^{\text{abs}}(\varepsilon, k, s) = 0,
\]
whereas for \( \varepsilon \in (0, \beta_{k,s}/(2\pi|k|)) \) we have
\[
c_s^{1/s} \left( \frac{1}{\varepsilon} \right)^{1/s} - |k| \leq n^{\text{abs}}(\varepsilon, k, s) \leq 2s + \max \left\{ 0, \left\lfloor \left( \frac{3}{2\pi} \right) \left( \frac{2}{\varepsilon} \right)^{1/s} \right\rfloor - 1 - |k| \right\}.
\]

Similarly as for the periodic case, this means that for the absolute error criterion the problem for the non-periodic case becomes easier for large \( |k| \). However, for \( k \neq 0 \), the condition on \( \varepsilon \) is now quite different for \( s \geq 2 \) as compared to the periodic case, see Corollary 10. We also stress that the asymptotic behaviors of \( \tilde{n}^{\text{abs}}(\varepsilon, k, s) \) and \( n^{\text{abs}}(\varepsilon, k, s) \) are of order \( \varepsilon^{-1/s} \) and do not depend on \( k \).

We now turn to the \textit{normalized error criterion} for which the information complexity \( n^{\text{nor}}(\varepsilon, k, s) \) for \( \varepsilon \in (0, 1) \) is defined as
\[
\tilde{n}^{\text{nor}}(\varepsilon, k, s) = \min \left\{ n \mid e(n, k, s) \leq \varepsilon e(0, k, s) \right\}.
\]

We always have \( n^{\text{nor}}(\varepsilon, k, s) \geq 1 \). For \( k = 0 \) we have \( e(0, 0, s) = 1 \) and there is no difference between the normalized and absolute error criteria also for the non-periodic case.

For \( k \neq 0 \) the situation is quite different. From Theorem 15, Proposition 3 as well as the estimates of \( \beta_{k,s} \), it is easy to obtain the following corollary.

Corollary 17. Consider the normalized error criterion for the integration problem \( I_k \) defined over the space \( H^s \). Let \( c_s \) be from Theorem 9.

For all \( k \neq 0 \) and all \( \varepsilon \in (0, 1) \) we have
\[
c_s^{1/s} \left( \frac{\sqrt{2\pi|k|}}{\varepsilon} \right)^{1/s} - |k| \leq n^{\text{nor}}(\varepsilon, k, s) \leq 2s + \max \left\{ 0, \left\lfloor \left( \frac{3}{2\pi} \right) \left( \frac{4\pi|k|}{\varepsilon} \right)^{1/s} \right\rfloor - 1 - |k| \right\},
\]
which can be written as
\[
n^{\text{nor}}(\varepsilon, k, s) = \Theta \left( \frac{|k|^{1/s}}{\varepsilon^{1/s}} \right) \quad \text{as} \quad \varepsilon \to 0.
\]

(34)
The asymptotic expression (34) shows that for the normalized error criterion the problem becomes harder for large $|k|$ and small $\varepsilon$. The dependence on $k$ is through $|k|^{1/s}$ and decreases with $s$. This should be compared with the periodic case, where the dependence on $|k|$ is linear. Hence, the dependence on $k$ is the same for $s = 1$, and the periodic case is harder than the non-periodic case for $s \geq 2$ and small $\varepsilon$.

For fixed $\varepsilon$ and varying $|k|$, the difference in the behavior of the information complexity in $|k|$ is even more dramatic and depends on $s$. Consider first $s = 1$. Then Corollary 17 yields for $\varepsilon < \sqrt{2\pi c_k}$ that
\[
\lim_{|k| \to \infty} n^\text{nor}(\varepsilon, k, s) = \infty,
\]
as for the periodic case, see (17).

Assume now that $s \geq 2$. In this case, the information complexity for the non-periodic problem does not go to infinity with $|k|$ in contrast to the periodic case, see again (17). This simply follows from Corollary 17 since the second term of the maximum behaves like $O(|k|^{1/s}) - |k|$ and goes to $-\infty$. Hence
\[
\limsup_{|k| \to \infty} n^\text{nor}(\varepsilon, k, s) \leq 2s.
\] (35)
This is even true if we choose $\varepsilon$ slowly decreasing with $|k|$, say $\varepsilon_k = |k|^{-\eta}$ for some $\eta \in (0, s - 1)$. Indeed, then $|k|/\varepsilon_k = |k|^{1+\eta}$ and $O(|k|^{(1+\eta)/s}) - |k|$ still goes to $-\infty$ and (35) is again valid. This discussion can be summarized as follows.

**Corollary 18.** For the non-periodic case and the normalized error criterion

- for $s \geq 1$, oscillatory integration becomes harder in $|k|$ asymptotically in $\varepsilon$,
- for $s = 1$ and fixed small $\varepsilon$, oscillatory integration becomes harder in $|k|$,
- for $s \geq 2$ and fixed $\varepsilon$ or even for $\varepsilon^{-1} = O(|k|^{\eta})$ with $\eta \in (0, s - 1)$, oscillatory integration becomes easy since $n^\text{nor}(\varepsilon, k, s)$ is at most $2s$ for large $|k|$.

## 5 The case of $s = \infty$

We briefly discuss the space $H^\infty$ which is defined as
\[
H^\infty = \{ f \in C^\infty([0, 1]) \mid \sum_{\ell=0}^\infty \| f^{(\ell)} \|_{L_2}^2 < \infty \},
\]
where $\|f^{(\ell)}\|_{L_2}$ denotes the $L_2 = L_2([0, 1])$ norm of $f^{(\ell)}$. Note that $H^\infty$ consists of infinitely many times differentiable functions. In particular, all polynomials belong to $H^\infty$ but $e_i(x) = \exp(2\pi i h x)$ belongs to $H^\infty$ iff $h = 0$.

We equip the space $H^\infty$ with the two inner products

$$
\langle f, g \rangle_\infty = \sum_{\ell=0}^{\infty} \langle f^{(\ell)}, 1 \rangle_0 \langle g^{(\ell)}, 1 \rangle_0,
$$

$$
\langle f, g \rangle_{\infty, *} = \sum_{\ell=0}^{\infty} \langle f^{(\ell)}, g^{(\ell)} \rangle_0.
$$

As for a finite $s$, the norms generated by these inner products are closely related since we have

$$
\frac{12}{11} \|f\|_{H^\infty} \leq \|f\|_{H^s} \leq \|f\|_{H^\infty} \quad \text{for all } f \in H^\infty,
$$

see the appendix. This means that it is enough to consider only one of these inner product and, as before, we choose $\langle \cdot, \cdot \rangle_\infty$ for simplicity of the analysis.

**Proposition 19.** Polynomials are dense in $H^\infty$, i.e., for any $f \in H^\infty$ and any positive $\varepsilon$ there is a polynomial $p$ such that

$$
\|f - p\|_{H^\infty} \leq \varepsilon.
$$

**Proof.** We begin by showing that for an absolutely continuous function $g$ for which $g' \in L_2([0, 1])$ we have

$$
\|g - g(\frac{1}{2})\|_{L_2} \leq \frac{1}{2} \|g'\|_{L_2}. \quad (36)
$$

Indeed, $g(x) - g(\frac{1}{2}) = \int_{1/2}^{x} g'(t) \, dt$ and

$$
|g(x) - g(\frac{1}{2})| \leq \int_{1/2}^{x} |g'(t)| \, dt \leq \left( \int_{1/2}^{x} dt \right)^{1/2} \left( \int_{1/2}^{x} |g'(t)|^2 \, dt \right)^{1/2} \leq x - \frac{1}{2} \|g'\|_{L_2}.
$$

Hence,

$$
\|g - g(\frac{1}{2})\|_{L_2} \leq \left( \int_{0}^{1} |x - \frac{1}{2}| \, dx \right)^{1/2} \|g'\|_{L_2} = \frac{1}{2} \|g'\|_{L_2},
$$

as claimed.

Take now an arbitrary $f \in H^\infty$. For any positive $\delta$ there exists $\ell^* = \ell^*(f, \delta) \in \mathbb{N}$ such that

$$
\sum_{\ell=\ell^*}^{\infty} \|f^{(\ell)}\|_{L_2}^2 \leq \delta^2.
$$

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In particular, \( \|f^{(\ell^*)}\|_{L^2} \leq \delta \). Taking \( g' = f^{(\ell^*)} \) we conclude from (36) that
\[
\|f^{(\ell^*-1)} - f^{(\ell^*)}(\frac{1}{2})\|_{L^2} \leq \frac{1}{2} \delta.
\]
For \( \ell^* \geq 2 \), we take \( g' = f^{(\ell^*-1)} - f^{(\ell^*)}(\frac{1}{2}) \) and we have again from (36)
\[
\|f^{(\ell^*-2)} - f^{(\ell^*)}(\frac{1}{2}) \cdot -\frac{1}{2} - f^{(\ell^*-2)}(\frac{1}{2})\|_{L^2} \leq \frac{1}{4} \delta.
\]
Repeating this procedure we conclude that for
\[
p(x) = f^{(1/2)} + f^{(1/2)}(x - 1/2) + \cdots + \frac{f^{(\ell^*-1)}(1/2)}{(\ell^* - 1)!}(x - 1/2)^{\ell^*-1}
\]
we have
\[
\|f^{(\ell)} - p^{(\ell)}\|_{L^2} \leq 2^{\ell^*-\ell} \delta \quad \text{for all } j = 0, 1, \ldots, \ell^* - 1.
\]
Hence,
\[
\|f - p\|_{H^\infty}^2 \leq \sum_{\ell=0}^{\infty} \|f^{(\ell)} - p^{(\ell)}\|_{L^2}^2 = \sum_{\ell=0}^{\ell^*-1} \|f^{(\ell)} - p^{(\ell)}\|_{L^2}^2 + \sum_{\ell=\ell^*}^{\infty} \|f^{(\ell)} - p^{(\ell)}\|_{L^2}^2
\]
\[
\leq \delta^2 \sum_{\ell=0}^{\ell^*-1} 4^{\ell^* - \ell} + \delta^2 = \frac{4}{3} \delta^2.
\]
Taking \( \delta = \sqrt{3/4} \varepsilon \), Proposition 19 is proved.

It is easy to see that that the periodic subspace
\[
\tilde{H}^\infty = \{ f \in H^\infty \mid f^{(\ell)}(0) = f^{(\ell)}(1) \text{ for } \ell \in \mathbb{N}_0 \}
\]
consists only of constant functions. Indeed, since \( \|I_k\|_{\tilde{H}^\infty \rightarrow \mathbb{C}} \leq \|I_k\|_{\tilde{H}^s \rightarrow \mathbb{C}} \) for all \( s \in \mathbb{N} \) and for \( k \neq 0 \) we have \( \|I_k\|_{\tilde{H}^s \rightarrow \mathbb{C}} = (2\pi|k|)^{-s} \), we conclude that \( I_k = 0 \) for all \( k \neq 0 \). This means that \( f \in \tilde{H}^\infty \) implies that \( f = \text{constant} \), as claimed. It is also easy to check that the reproducing kernel of \( \tilde{H}^\infty \) is \( \tilde{K}_\infty(x,t) = 1 \).

Let \( \tilde{\epsilon}(n,k,\infty) \) be the minimal errors for \( \tilde{H}^\infty \). Then \( \tilde{\epsilon}(0,0,\infty) = 1 \) and \( \tilde{\epsilon}(n,0,\infty) = 0 \) for all \( n \geq 1 \), whereas \( \tilde{\epsilon}(n,k,\infty) = 0 \) for all \( n \geq 0 \) and \( k \neq 0 \).

This means that the periodic case is trivial and cannot be used as a tool for the non-periodic case. That is why our lower bound on \( \tilde{\epsilon}(n,k,s) \) which was quite useful for a finite \( s \) is meaningless for \( s = \infty \). In fact, the problem of non-trivial lower bounds for \( H^\infty \) is open.
Proposition 19 together with (23) shows that the set of normalized Bernoulli polynomials \( \{ B^*_j \}_{j=0,1,...} \) is a complete orthonormal basis of \( H^\infty \) and therefore the reproducing kernel \( K_\infty \) is given by
\[
K_\infty(x, t) = \sum_{j=0}^{\infty} B^*_j(x) B^*_j(t) \quad \text{for all} \quad x, t \in [0, 1].
\]

We now present some upper error bounds on the minimal errors \( e(n, k, \infty) \) for \( H^\infty \). In fact, we derived the upper bounds in Theorems 13 and 14 in such a way that they can be used even for \( s = \infty \).

We start with the initial error. For \( k = 0 \), the representer of \( I_0 \) is 1 and
\[
e(0, 0, \infty) = 1,
\]
whereas for \( k \neq 0 \), the proof of Proposition 12 can be modified for \( s = \infty \) and yields that the representer of \( I_k \) is
\[
h_{k,\infty}(x) = -\sum_{\ell=1}^{\infty} (-1)^\ell (2\pi ik)^{-\ell} B^*_\ell(x)
\]
and
\[
e(0, k, \infty) = \frac{\beta_{k,\infty}}{2\pi|k|}
\]
with
\[
\beta_{k,\infty} = \left( \frac{4\pi^2k^2}{4\pi^2k^2 - 1} \right)^{1/2} \in [1, 1.013].
\]

For \( k = 0 \) and all \( n \geq 1 \), we can apply the first error bound in Theorem 14 which states that
\[
e(n, 0, \infty) \leq \frac{1}{2^{n-1}n!},
\]
which is super exponentially small in \( n \).

For \( k \neq 0 \) and all even \( n \), we apply the first error bounds in Theorems 13 and 14 which state that
\[
e(n, k, \infty) \leq \min\left( \frac{1}{2^{n-1}n!}, \frac{1}{(2\pi|k|)^{n/2}} \right).
\]

Note that by Stirling’s approximation we have
\[
\frac{1}{2^{n-1}n!} \leq 2 \left( \frac{e}{2n} \right)^n.
\]
It is easy to check that the right hand side is smaller than \( \varepsilon \) iff
\[
n(\ln(2n) - 1) \geq \ln(2/\varepsilon)
\]
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which holds, in particular, if \( n \geq 2 \ln(\varepsilon^{-1})/\ln \ln(\varepsilon^{-1}) \) and \( \varepsilon < e^{-e} = 0.135 \ldots \)

These upper error bounds can be used to estimate the information complexities \( n_{\text{abs}} \) and \( n_{\text{nor}} \) for \( \varepsilon < e^{-e} \). For \( k = 0 \), we have

\[
n_{\text{abs}}(\varepsilon, 0, \infty) = n_{\text{nor}}(\varepsilon, 0, \infty) \leq \left\lceil 2 \frac{\ln \varepsilon^{-1}}{\ln \ln \varepsilon^{-1}} \right\rceil.
\]

For \( k \neq 0 \), we have

\[
n_{\text{abs}}(\varepsilon, k, \infty) \leq \left\lceil 2 \min \left\{ \frac{\ln \varepsilon^{-1}}{\ln \ln \varepsilon^{-1}}, \frac{\ln \varepsilon^{-1}}{\ln (2\pi|k|)} \right\} \right\rceil,
\]

and

\[
n_{\text{nor}}(\varepsilon, k, \infty) \leq \left\lceil 2 \min \left\{ \frac{\ln \varepsilon^{-1} + \ln(2\pi|k|)}{\ln (\ln \varepsilon^{-1} + \ln(2\pi|k|))}, \frac{\ln \varepsilon^{-1}}{\ln (2\pi|k|)} + 1 \right\} \right\rceil.
\]

These estimates are valid for all \( \varepsilon < e^{-e} \). Note that asymptotically, when \( \varepsilon \) tends to zero, all information complexity are upper bounded by roughly \( \ln(\varepsilon^{-1})/\ln(\ln(\varepsilon^{-1})) \) independent of \( k \).

6 Appendix

We now prove (5) which shows the embeddings constants between the space \( H^s \) equipped with the norm \( \| \cdot \|_{H^s} \) given by (3) and \( \| \cdot \|_{H^s}^* \) given by (4). Since \( |\langle f^{(\ell)}, 1 \rangle_0|^2 \leq \|f^{(\ell)}\|_{L^2}^2 \) we clearly have \( \|f\|_{H^s} \leq \|f\|_{H^s}^* \) for all \( f \in H^s \).

To obtain the other estimate, we consider \( f \in H^s \) which can be written as

\[
f = \sum_{j=0}^n \langle f, B_j^* \rangle_s B_j^* + \sum_{h \in \mathbb{Z} \setminus \{0\}} \langle f, e_h \rangle_s e_h
\]

since the normalized Bernoulli polynomials \( B_j^* \) and \( e_h^* = \exp(2\pi i h \cdot)/(2\pi|h|)^s \) are an orthonormal basis of \( H^s \) with respect to \( \langle \cdot, \cdot \rangle_s \). Let \( \{b_j\}_{j \in \mathbb{N}} \) be some ordering of this orthonormal
basis. Then, clearly, we have

$$\| f \|_{H^s}^2 = \left\| \sum_{j \in \mathbb{N}} \langle f, b_j \rangle_s b_j \right\|_{H^s}^2 = \sum_{j, m \in \mathbb{N}} \langle f, b_j \rangle_s \langle f, b_m \rangle_s \langle b_j, b_m \rangle_{s,*}$$

$$\leq \sum_{j, m \in \mathbb{N}} \left| \langle f, b_j \rangle_s \right|^2 \left| \langle b_j, b_m \rangle_{s,*} \right|$$

$$\leq \left( \max_{m \in \mathbb{N}} \left\{ \sum_{j \in \mathbb{N}} \left| \langle b_j, b_m \rangle_{s,*} \right| \right\} \right) \left( \sum_{j \in \mathbb{N}} \left| \langle f, b_j \rangle_s \right|^2 \right)$$

$$= \left( \max_{m \in \mathbb{N}} \left\{ \sum_{j \in \mathbb{N}} \left| \langle b_j, b_m \rangle_{s,*} \right| \right\} \right) \| f \|_{H^s}^2$$

$$=: M_s \| f \|_{H^s}^2.$$ (37)

To bound $M_s$ we estimate $| \langle b_j, b_m \rangle_{s,*} |$ for all possible $b_j, b_m \in \{ B^*_j, e^*_h \}_{j=0,1,...,s,h \in \mathbb{Z}\setminus\{0\}}$.

We start with the case where both $b_j, b_m$ are in $\{ e^*_h \}_{h \in \mathbb{Z}\setminus\{0\}}$. This case is easy since $\{ e^*_h \}_{h \in \mathbb{Z}\setminus\{0\}}$ is also an orthogonal basis in $\tilde{H}^s$ with the inner product $\langle \cdot, \cdot \rangle_{s,*}$. We have for $h \in \mathbb{Z}\setminus\{0\}$ and $\ell \neq h$ that

$$\langle e^*_h, e^*_\ell \rangle_{s,*} = 0 \quad \text{and} \quad \| e^*_h \|_{H^s} = \frac{1}{(2\pi|h|)^s} \left( \sum_{\ell=0}^{s} (2\pi|h|)^{2\ell} \right)^{1/2}.$$ 

Hence

$$\left| \langle e^*_h, e^*_\ell \rangle_{s,*} \right| = \| e^*_h \|_{H^s}^2 \leq \frac{1}{1 - (2\pi|h|)^{-2}} \leq \frac{4\pi^2}{4\pi^2 - 1}. \quad (38)$$

To treat the case where $b_j, b_m$ are in $\{ B^*_j \}_{j=0,1,...,s}$, we need the following known properties of the normalized Bernoulli polynomials

$$\langle B^*_m, B^*_0 \rangle_0 = 1 \quad \text{for } m = 0 \text{ and } 0 \quad \text{for } m \geq 1,$$

$$\langle B^*_m, B^*_j \rangle_0 = (-1)^{\min(m,j)-1} B^*_m+j(0), \quad \text{for all } m, j \geq 1$$

$$B^*_0(0) = 1,$$

$$B^*_{2m}(0) = \frac{2(-1)^{m+1}}{(2\pi)^{2m}} \zeta(2m) \quad \text{for all } m \geq 1,$$

$$B^*_{2m+1}(0) = 0 \quad \text{for all } m \geq 1.$$ 

Here, as always, $\zeta$ is the Riemann zeta function.
From these properties for \( m \in [0, s] \) we conclude

\[
\| B_m^* \|_{H^s}^2 = \sum_{\ell=0}^{s} \| [B_m^*]^{(\ell)} \|_{L^2}^2 = \sum_{\ell=0}^{m} \| B_{m-\ell}^* \|_{L^2}^2 = 1 + 2 \sum_{\ell=1}^{m} \zeta(2\ell) \frac{(2\pi)^{2\ell}}{(2\pi)^{2\ell}}.
\]

Hence

\[
\| B_m^* \|_{H^s}^2 \leq 1 + \frac{2\zeta(2)}{(2\pi)^{2}} + \frac{2\zeta(4)}{4\pi^2(4\pi^2 - 1)}.
\]

Since \( \zeta(2) = \pi^2/6 \) and \( \zeta(4) = \pi^4/90 \), we conclude that

\[
\| B_m^* \|_{H^s}^2 \leq 1 + \frac{1}{12} + \frac{\pi^2}{180(4\pi^2 - 1)}.
\] (39)

We now consider \( \langle B_m^*, B_j^* \rangle_{s, s} \) for all \( m, j \in [0, s] \) and \( m \neq j \). Let \( m' = \max\{m, j\} \) and \( j' = \min\{m, j\} \). Furthermore, let \( \kappa_{m'-j'} = 0 \) for odd \( m' - j' \) and \( \kappa_{m'-j'} = 1 \) for even \( m' - j' \). Then

\[
\langle B_m^*, B_j^* \rangle_{s, s} = \langle B_m^*, B_0^* \rangle_0 = \delta_{m,0},
\]

whereas for \( m, j \in [1, s] \) and \( m \neq j \) we have

\[
\langle B_m^*, B_j^* \rangle_{s, s} = \sum_{\ell=0}^{j'-1} \langle B_{m-\ell}^*, B_{j-\ell}^* \rangle_0 = \sum_{\ell=0}^{j'-1} (-1)^{j'-\ell-1} B_{m+j-2\ell}(0)
\]

\[
= 2\kappa_{m'-j'} (-1)^{(m'-j')/2} \sum_{\ell=0}^{j'-1} \zeta(m+j-2\ell) \frac{(2\pi)^{m+j-2\ell}}{(2\pi)^{m+j-2\ell}}
\]

\[
= 2\kappa_{m'-j'} (-1)^{(m'-j')/2} \sum_{\ell=1}^{j'} \zeta(m'-j'+2\ell) \frac{(2\pi)^{m'-j'+2\ell}}{(2\pi)^{m'-j'+2\ell}}.
\]

Note that the smallest argument of \( \zeta \) for even \( m' - j' \) is 4. Therefore

\[
| \langle B_m^*, B_j^* \rangle_{s, s} | \leq 2\zeta(4) \frac{\kappa_{m'-j'}}{(2\pi)^{m'-j'}} \sum_{\ell=1}^{\infty} \frac{1}{(2\pi)^{2\ell}} \leq \frac{2\zeta(4)}{4\pi^2 - 1} \frac{\kappa_{m'-j'}}{(2\pi)^{m'-j'}}.
\]

From this we have

\[
\sum_{m=0}^{s} | \langle B_m^*, B_j^* \rangle_{s, s} | \leq \| B_j^* \|_{H^s}^2 + 2\zeta(4) \frac{(2\pi)^{m'-j'}}{4\pi^2 - 1} \sum_{m=0}^{s} \frac{\kappa_{m'-j'}}{(2\pi)^{m'-j'}}.
\]
The non-zero terms correspond to even $m' - j'$ and each non-zero term $\kappa_{m' - j'}/(2\pi)^{m' - j'}$ may appear at most twice. Therefore

$$
\sum_{m=0}^{s} |\langle B_m^*, B_j^* \rangle_{s,*}| \leq \|B_j^*\|_{H^s}^2 + \frac{4\zeta(4)}{4\pi^2 - 1} \sum_{\ell=0}^{\infty} \frac{1}{(2\pi)^{2\ell}} = \|B_j^*\|_{H^s}^2 + \frac{4\pi^4}{90(4\pi^2 - 1)^2}.
$$

(40)

We now consider the coefficients $\langle B_j^*, e_h^* \rangle$ for $j = 0, 1, \ldots, s$ and $h \in \mathbb{Z} \setminus \{0\}$. We start by showing that we have

$$
\int_0^1 B_j^*(x) e^{-2\pi i hx} \, dx = \begin{cases} 
0 & \text{for } j = 0, \\
\frac{-1}{(2\pi i h)^j} & \text{for } j \geq 1.
\end{cases}
$$

For $j = 0$, it is zero since the integral of $e_h^*$ is zero for $h \neq 0$.

We now use induction on $j$. For $j = 1$, we have $B_1^*(x) = x - \frac{1}{2}$ and using integration by parts we get

$$
\int_0^1 (x - \frac{1}{2}) e^{-2\pi i hx} \, dx = \frac{-1}{2\pi i h} \int_0^1 (x - \frac{1}{2}) \, d e^{-2\pi i hx} = \frac{-1}{2\pi i h} \left( 1 - \int_0^1 e^{-2\pi i hx} \, dx \right) = \frac{-1}{2\pi i h},
$$

as claimed.

For $j > 1$, we again use integration by parts, and the property of Bernoulli polynomials (24) and (25) to obtain

$$
\int_0^1 B_j^*(x) e^{-2\pi i hx} \, dx = \frac{-1}{2\pi i h} \int_0^1 B_j^*(x) \, d e^{-2\pi i hx} = \frac{1}{2\pi i h} \int_0^1 [B_j^*]'(x) e^{-2\pi i hx} \, dx = \frac{1}{2\pi i h} \int_0^1 B_{j-1}^*(x) e^{-2\pi i hx} \, dx = \frac{-1}{(2\pi i h)^j},
$$

as claimed.

From this we conclude that for $\ell \leq j$ and $j \neq 0$,

$$
\langle [B_j^*]^{(\ell)}, [e^{2\pi i h \cdot}]^{(\ell)} \rangle_0 = (-2\pi i h)\ell \langle B_{j-\ell}^*, e^{2\pi i h \cdot} \rangle_0 = -(2\pi h)^2\ell - j i^{-j}.
$$

Clearly, for $j = 0$ we have $\langle [B_j^*]^{(\ell)}, [e^{2\pi i h \cdot}]^{(\ell)} \rangle_0 = 0$.

Then for $h \neq 0$ we have

$$
\langle B_j^*, e_h^* \rangle_{s,*} = \sum_{\ell=0}^{s} \langle [B_j^*]^{(\ell)}, [e_h^*]^{(\ell)} \rangle_0 = \frac{1}{(2\pi |h|)^s} \sum_{\ell=0}^{j-1} \langle [B_j^*]^{(\ell)}, [e^{2\pi i h \cdot}]^{(\ell)} \rangle_0
$$

$$
= \begin{cases} 
0 & \text{for } j = 0, \\
\frac{-1}{(2\pi |h|)^s (2\pi i h)^j} \sum_{\ell=0}^{j-1} (2\pi h)^{2\ell} & \text{for } j \geq 1.
\end{cases}
$$
This allows us to compute $\left| \langle B_j^*, e_h^* \rangle_{s,*} \right|$ for $h \neq 0$ and $j = 0, \ldots, s$. For $j = 0$ it is zero, and for $j \geq 1$ we have

$$
\left| \langle B_j^*, e_h^* \rangle_{s,*} \right| = \frac{1}{(2\pi|h|)^{s+j}} \frac{(2\pi h)^{2j} - 1}{(2\pi h)^2 - 1} < \frac{(2\pi h)^j - 1}{(2\pi h)^{2} - 1} \leq \frac{4\pi^2}{2\pi^2} - 1 \leq \frac{1}{(2\pi|h|)^{s-j+2}}. \quad (41)
$$

We now are ready to bound $M_s$ from (37). For this we define

$$
M_{s,1} := \max_{h \in \mathbb{Z}\setminus\{0\}} \left\{ \left| \langle e_h^*, e_h^* \rangle_{s,*} \right| + \sum_{j=0}^{s} \left| \langle B_j^*, e_h^* \rangle_{s,*} \right| \right\}
$$

and

$$
M_{s,2} := \max_{j=0,\ldots,s} \left\{ \sum_{h \in \mathbb{Z}\setminus\{0\}} \left| \langle B_j^*, e_h^* \rangle_{s,*} \right| + \sum_{m=0}^{s} \left| \langle B_m^*, B_j^* \rangle_{s,*} \right| \right\}
$$

such that $M_s = \max\{M_{s,1}, M_{s,2}\}$.

Clearly, $M_s$ is strictly larger than one since the term inside of the maximum of $M_{s,2}$ corresponding to $j = 0$ equals one. We now show that $M_s$ is close to one.

From (38) and (41) we get

$$
M_{s,1} \leq \frac{4\pi^2}{4\pi^2 - 1} \left( 1 + \sum_{j=1}^{s} \frac{1}{(2\pi)^{s-j+2}} \right) \leq \frac{4\pi^2}{4\pi^2 - 1} \left( 1 + \frac{1}{(2\pi)^2 - 2\pi} \right) \leq 1.057.
$$

We now bound $M_{s,2}$. We use (41) to bound the first sum in $M_{s,2}$,

$$
\sum_{h \in \mathbb{Z}\setminus\{0\}} \left| \langle B_j^*, e_h^* \rangle_{s,*} \right| \leq \frac{4\pi^2}{4\pi^2 - 1} \sum_{h \in \mathbb{Z}\setminus\{0\}} \frac{1}{(2\pi|h|)^{s-j+2}} = \frac{8\pi^2}{4\pi^2 - 1} \frac{\zeta(s-j+2)}{(2\pi)^{s-j+2}} \leq \frac{\zeta(2)}{(2\pi)^2} = \frac{\pi^2}{3(4\pi^2 - 1)} \leq 0.0855.
$$

Using (39) and (40) we can bound the second sum in $M_{s,2}$ by

$$
\sum_{m=0}^{s} \left| \langle B_m^*, B_j^* \rangle_{s,*} \right| \leq \frac{13}{12} + \frac{\pi^2}{180(4\pi^2 - 1)} + \frac{4\pi^4}{90(4\pi^2 - 1)^2} \leq 1.0877.
$$

This shows that $M_{s,2} \leq 1.1732$ and, consequently, $M_s \leq 1.1732$ and $\sqrt{M_s} \leq 1.0832 \leq \frac{13}{12}$.

From (37) we finally obtain

$$
\|f\|_{H^s} \leq \sqrt{M_s} \|f\|_{H^s} \leq \frac{13}{12} \|f\|_{H^s}
$$

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for all $f \in H^s$, as claimed

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