Distributionally Robust Chance Constrained Optimal Power Flow Assuming Unimodal Distributions with Misspecified Modes

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Abstract—Chance constrained optimal power flow (CC-OPF) formulations have been proposed to minimize operational costs while controlling the risk arising from uncertainties like renewable generation and load consumption. To solve CC-OPF, we often need access to the (true) joint probability distribution of all uncertainties, which is rarely known in practice. A solution based on a biased estimate of the distribution can result in poor reliability. To overcome this challenge, recent work has explored distributionally robust chance constraints, in which the chance constraints are satisfied over a family of distributions called the ambiguity set. Commonly, ambiguity sets are only based on moment information (e.g., mean and covariance) of the random variables; however, specifying additional characteristics of the random variables reduces conservatism and cost. Here, we consider ambiguity sets that additionally incorporate unimodality information. In practice, it is difficult to estimate the mode location from the data and so we allow it to be potentially misspecified. We formulate the problem and derive a separation-based algorithm to efficiently solve it. Finally, we evaluate the performance of the proposed approach on a modified IEEE-30 bus network with wind uncertainty and compare with other distributionally robust approaches. We find that a misspecified mode significantly affects the reliability of the solution and the proposed model demonstrates a good trade-off between cost and reliability.

Index Terms—Optimal power flow, chance constraint, distributionally robust optimization, misspecified mode, \(\alpha\)-unimodality approximation \[8\], \[9\], which provides a priori guarantees on reliability, require the constraints to be satisfied over a large number of uncertainty samples. The solutions from these approaches are usually overly conservative with high costs \[7\], \[10\]. Another popular approach is to assume that the uncertainties follow a parametric distribution such as Gaussian \[4\], \[5\], \[7\]. The resulting CC-OPF is often easier to solve but the solution may have low reliability unless the assumed probability distribution happens to be close to the true one.

As an alternative, distributionally robust chance constrained (DRCC) OPF models do not depend on a single estimate of the probability distribution \[10\]–\[17\]. More specifically, DRCC models consider a family of distributions, called the ambiguity set, that share certain statistical characteristics and requires that the chance constraint holds with respect to all distributions within the ambiguity set \[18\]–\[21\]. Most existing work characterizes the ambiguity set based on moment information obtained from historical data of the uncertainty (see, e.g., \[10\], \[11\], \[13\], \[14\]). For example, a commonly adopted ambiguity set consists of all distributions whose mean and covariance agree with their corresponding sample estimates \[10\], \[11\], \[13\]. Many uncertainty distributions (e.g., those associated with wind forecast error) are unimodal and so, recently, unimodality has been incorporated to strengthen the ambiguity set and reduce the conservatism of DRCC models \[13\], \[16\], \[17\]. However, as compared to the moments, the mode location is more likely to be misspecified in sample-based estimation.

In this paper, we study a DRCC model with an ambiguity set based on moment and unimodality information with a potentially misspecified mode location. To the best of our knowledge, this paper is the first work discussing misspecification of a value related to a structural property, though others have considered misspecification of moments \[10\], \[14\], \[18\], \[19\], \[21\] and misspecification of distributions \[12\], \[22\]. Our main theoretical result shows that the distributionally robust chance constraints can be recast as a set of second-order conic (SOC) constraints. Furthermore, we derive an iterative algorithm to accelerate solving the reformulation. In this algorithm, we begin with a relaxed formulation, and in each iteration, we efficiently find the most violated SOC constraint, if any, or terminate with a globally optimal solution. We apply the theoretical results to a direct current (DC) OPF problem and conduct a case study using a modified IEEE 30-bus system with wind power. We compare our results (operational cost, reliability, computational time, and optimal
that the mode estimate can deviate from the corresponding mean estimate in all directions. This indicates the importance of considering the misspecification of mode location in DRCC models, because the mode-mean deviation shows the skewness of the uncertainty. As a result, if we misspecify the mode location (e.g., by modeling a right-skewed distribution as a left-skewed one, see Section III-D for an example), then we may mistakenly relax the chance constraint and get poor out-of-sample performance.

III. DRCC FORMULATION

A. General Formulation

In this paper, we consider the following physical constraint under uncertainty:

$$a(x)^\top \xi \leq b(x),$$  \hspace{1cm} (1)

where $x \in \mathbb{R}^l$ represents an $l$-dimensional decision variable, and $a(x) : \mathbb{R}^l \rightarrow \mathbb{R}^n$ and $b(x) : \mathbb{R}^l \rightarrow \mathbb{R}$ represent two affine functions of $x$. Uncertainty $\xi \in \mathbb{R}^n$ represents an $n$-dimensional random vector defined on probability space $(\mathbb{R}^n, \mathcal{B}^n, P_\xi)$ with Borel $\sigma$-algebra $\mathcal{B}^n$ and probability distribution $P_\xi$. The assumption that $a(x)$ and $b(x)$ are affine in $x$ is a standard assumption in existing DRCC models and consistent with the DRCC DC OPF.

To manage constraint violations due to uncertainty, one natural way is to ensure that (1) is satisfied with at least a pre-defined probability threshold $\alpha = 1 - \epsilon$, which leads to the following chance constraint [24], [25]:

$$\mathbb{P}_\xi \left( a(x)^\top \xi \leq b(x) \right) \geq 1 - \epsilon,$$  \hspace{1cm} (2)

where $1 - \epsilon$ normally takes a large value (e.g., 0.99).

B. Distributionally Robust Formulation

In reality, it may be challenging to access the (true) joint probability distribution $P_\xi$. Oftentimes we may only have a set of historical data and certain domain knowledge of $\xi$. In this case, we can consider the following distributionally robust chance constraint:

$$\inf_{P_\xi \in D_\xi} \mathbb{P}_\xi \left( a(x)^\top \xi \leq b(x) \right) \geq 1 - \epsilon.$$  \hspace{1cm} (3)

Instead of assuming that $P_\xi$ takes a specific form, we consider an ambiguity set $D_\xi$ consisting of plausible candidates of $P_\xi$. Then, we require that chance constraint (3) holds with respect to all distributions in $D_\xi$.

C. Ambiguity Sets

In this paper, we consider three ambiguity sets, denoted as $D_\xi^i$ for $i = 1, 2, 3$, that are defined by a combination of moment and unimodality information. Precisely, we consider a generalized notion of unimodality defined as follows.

Definition 3.1: ($\alpha$-Unimodality) For any fixed $\alpha > 0$, a probability distribution $P$ on $\mathbb{R}^n$ is called $\alpha$-unimodal with mode $0$ if $t^{\alpha} P(B/t)$ is non-decreasing in $t > 0$ for every Borel set $B \in \mathcal{B}^n$. 

II. UNIMODALITY OF WIND FORECAST ERRORS & ERROR IN MEAN AND MODE ESTIMATES

In this section, we first empirically verify the unimodality of wind forecast error distributions using 10,000 data samples from [6], [7] with statistical outliers omitted (total probability < 0.1%). The samples were generated using a Markov Chain Monte Carlo mechanism [23] based on real data that includes both hourly forecast and actual wind generation in Germany. In Fig. 1, we depict the histograms of univariate and bivariate wind forecast errors with 15 bins. Both histograms empirically justify our assumption that the probability distribution of wind forecast errors is unimodal.

Next, we empirically evaluate the errors of mean and mode estimates (i.e., the peak location in the histogram). We randomly extract 100 groups of samples, each group containing 500 data points, from the wind forecast error data pool. For each group of samples, we estimate the mean by taking sample averages and estimate the mode by identifying the center of the highest bin in the 15-bin histogram. In Fig. 2, we plot all the mean and mode estimates and the differences between them. From the left subfigure, we observe that sampling errors have larger impacts on mode estimates than on mean estimates. From the right subfigure, we observe solutions) to those obtained using four alternative ambiguity sets [19], [19], [17], [20].

The remainder of this paper is organized as follows. Section II empirically verifies the (multivariate) unimodality of wind forecast errors and explores misspecification of the mode location. The proposed DRCC model and ambiguity set are introduced in Section III and the main theoretical results are presented in Section IV. Section V includes the case studies and Section VI concludes the paper.

REFERENCES
From the definition, we notice that $\alpha$ parameterizes the “degree of unimodality.” When $\alpha = n = 1$, the definition coincides with the classical univariate unimodality with mode 0. When $\alpha = n > 1$, the density function of $\xi$ (if exists) peaks at the mode and is non-increasing in any directions moving away from the mode. As $\alpha \to \infty$, the requirement of unimodality gradually relaxes and eventually vanishes. Under Definition 3.1 we define the following three ambiguity sets:

**Ambiguity set 1:** (moment only)

$$D_1^\alpha := \{P_\xi \in \mathcal{P}_n : E_{P_\xi}[\xi] = \mu_\xi, E_{P_\xi}[\xi^T] = \Sigma\},$$

**Ambiguity set 2:** (moment and $\alpha$-unimodality, fixed mode)

$$D_2^\alpha := \{P_\xi \in \mathcal{P}_n^\alpha \cap D_1^\alpha : M(\xi) = m\},$$

**Ambiguity set 3:** (moment and $\alpha$-unimodality, misspecified mode)

$$D_3^\alpha := \{P_\xi \in \mathcal{P}_n^\alpha \cap D_1^\alpha : M(\xi) \in \Xi\},$$

where $\mathcal{P}_n^\alpha$ and $\mathcal{P}_n$ denote all probability distributions on $\mathbb{R}^n$ with and without the requirement of $\alpha$-unimodality respectively; $\mu$ and $\Sigma$ denote the first and second moments of $\xi$; and $M(\xi)$ denotes a function returning the true mode location of $\xi$ with $m_i$ and $\Xi$ representing a single mode value and a connected and compact set. The compact set can be constructed using possible mode estimates calculated from samples of historical data.

Among these three ambiguity sets, we use $D_1^\alpha$ as a benchmark. Set $D_2^\alpha$ is a special case of $D_3^\alpha$, i.e., $\Xi$ only contains a single value $m_i$. In practice, since the mode estimate is influenced by sampling errors, the mode estimates from data samples are not the same single values but distribute around a certain area. The shape of this area decides the underlying structural skewness in the uncertainty distribution. Hence, we compare $D_2^\alpha$ and $D_3^\alpha$ to see how misspecified mode estimates affect the DRCC problem. In this paper, we do not additionally consider misspecified moments since this topic has been well-studied [14], [18], [19], [21] and our main results can be easily extended based on these existing works.

**D. Numerical Example**

We use a simple example to illustrate the impact of an inaccurate mode estimate. We assume random variable $\xi$ follows distribution $P_{\xi_1}$. $P_{\xi_2}$ is a biased estimate of $P_{\xi_1}$ due to sampling errors. Both distributions are illustrated in Fig. 3 where each has zero mean and unit variance. However, $P_{\xi_2}$ is right-skewed with mode at $-1$ and $P_{\xi_2}$ is left-skewed with mode at 1. Suppose that we try to reformulate $P_{\xi}(z \leq z) \geq 90\%$. Based on the given distributions, we find $z \geq 1.8$ from the correct distribution $P_{\xi_1}$ and $z \geq 0.925$ from the biased distribution $P_{\xi_2}$. In this example, we observe that a misspecified mode estimate could shrink the 90% confidence bound by almost a half and significantly decrease the reliability of the solution to the chance constraint.
B. Reformulation for $D_\xi^2$

We now present the reformulation with $D_\xi^2$, which is based on Theorem 4.2

$$\sqrt{1 - \epsilon - \tau^{-\alpha}} ||\Delta a(x)|| \leq \left( \tau - \frac{\alpha + 1}{\alpha} \right) (\mu - m)^\top a(x)$$

$$+ \tau (b(x) - \mu^\top a(x)), \ \forall \tau \geq \left( \frac{1}{1 - \epsilon} \right)^{1/\alpha}, \ \forall m \in \Xi, \ (9)$$

$$a(x)^\top m \leq b(x), \ \forall m \in \Xi, \ (10)$$

where $\Lambda := \left( \left( \frac{\alpha + 2}{\alpha} \right) (\Sigma - \mu^\top)(\frac{1}{\tau})(\mu - m)(\mu - m)^\top \right)^{1/2}$

and (10) comes from Assumption 4.2

Compared to (9), (9) is more complicated with two parameters $m$ and $\tau$ each with an infinite number of choices. To solve an optimization problem with (9), we propose an iterative solving algorithm given in Algorithm 1.

Algorithm 1: Iterative solving algorithm

Initialization: $i = 1$, $\tau_0 = \left( \frac{1}{\tau^*} \right)^{1/\alpha}$, and $m_0 = \{ \text{any singular point in } \Xi \}$

Iteration $i$:

Step 1: Solve the reformulated optimization problem with (9) using $\tau_j$ and $m_j$ for all $j = 0, \ldots, i - 1$ and obtain optimal solution $x^*_i$. All $\tau_j$ and $m_j$ values are collected from previous iterations;

Step 2: Find worst case $\tau^*$ and $m^*$ that result in the largest violation of (9) under $x^*_i$: IF $m^*$ and $\tau^*$ does not exist. STOP and RETURN $x^*_i$ as optimal solution; ELSE GOTO Step 3;

Step 3: Set $\tau_i = \tau^*$, $m_i = m^*$, and $i = i + 1$;

Note that the reformulated optimization problem in Step 1 contains only SOC constraints.

C. Step 2 of Algorithm 1

The challenge is how to efficiently perform Step 2 of Algorithm 1. In the following, we assume $a(x^*_i) \neq 0$, otherwise (9) is satisfied with $x^*_i$ regardless of the values of $\tau$ and $m$.

Next, we define the following terms

$$h = a(x^*_i)^\top (\mu - m)/\alpha, \ \tilde{c} = b(x^*_i) - \mu^\top a(x^*_i),$$

$$\tilde{R} = \sqrt{a(x^*_i)^\top \left( \frac{\alpha + 2}{\alpha} \right) (\Sigma - \mu^\top)a(x^*_i)},$$

$$g(\tau) = \sqrt{\frac{1 - \epsilon - \tau^{-\alpha}}{\epsilon}} f(\tau) = -(\alpha \tau - \alpha - 1).$$

Since $m \in \Xi$, we have $h \in [\tilde{h}, \tilde{h}]$ where

$$\tilde{h} = \max_{m \in \Xi} a(x^*_i)^\top (\mu - m)/\alpha, \ \tilde{h} = \min_{m \in \Xi} a(x^*_i)^\top (\mu - m)/\alpha.$$

From Assumption 4.1, we have $[\tilde{h}, \tilde{h}] \in (-\tilde{R}, \tilde{R})$ and transform (9) into

$$\left[ g(\tau) \sqrt{\tilde{R}^2 - \tilde{h}^2} + f(\tau) \tilde{h} \right] - \tilde{c} \tau \leq 0, \ \forall \tau \in [\tilde{h}, \tilde{h}], \ \forall \tau \geq \tau_0. \ \ (12)$$

Since the left side of (12) is not jointly convex or concave in $h$ and $\tau$ (see a proof in Appendix A), we can not find the global maximum value for the left side by simply checking the boundary values or stationary points. Therefore, we propose the following algorithm to efficiently find the global maximum.

We notice that for given a $\tau$ and if $h \in [-\tilde{R}, \tilde{R}]$, the maximum value of $g(\tau) \sqrt{\tilde{R}^2 - \tilde{h}^2} + f(\tau) \tilde{h}$ equals $\tilde{R} \sqrt{g(\tau)^2 + f(\tau)^2}$ with maximizer $\hat{h}(\tau) = \frac{f(\tau)}{\sqrt{g(\tau)^2 + f(\tau)^2}} \tilde{R}$. Next, by taking the derivative of $\hat{h}$, we observe that $\hat{h}$ is a strictly decreasing function of $\tau$. Hence, we can compute $\tau$ and $\tau$ that cause $\hat{h}$ to reach its boundary values by solving $\hat{h}(\tau) = \tilde{h}$ and $\hat{h}(\tau) = \tilde{h}$. Since $\epsilon < 1/2$ and $\alpha \geq 1$, we have $\tau_0 < (\alpha + 1)/\alpha$ and hence $\hat{h}(\tau_0) = \tilde{R}$. Then we know $[\tilde{h}, \tilde{h}] \geq \tau_0$ as $[\tilde{h}, \tilde{h}] < \tilde{R}$.

To efficiently solve these two equalities, we will use a golden section search by first solving for $\tau$ on $[\tau_0, \infty]$ and then for $\tau$ on $[\tau_0, \tilde{h}]$. To efficiently apply a golden section search on $\tau$, we need to find a finite upper bound instead of $\infty$. The following lemma describes the selection of the finite upper bound $\tau_1$ and the best region to conduct the golden section search.

Lemma 4.1: If $\frac{1}{\alpha} h_0 \geq 0$, $\tau_0 = \frac{\alpha + 1}{\alpha}$. The golden section search of $\tau$ can be conducted on $[\tau_0, \frac{\alpha + 1}{\alpha}]$. If $\frac{1}{\alpha} h_0 < 0$, $\tau_1 = -\frac{1}{\alpha} h_0 (\frac{1}{\alpha} h_0 - (\alpha + 1))/\alpha$. The search can be conducted on $[\frac{\alpha + 1}{\alpha}, \tau_1]$. The proof is given in Appendix B.

Furthermore, from Assumption 4.2 we have $\tilde{c} \geq -\alpha \tilde{h} \geq -\alpha \tilde{h}$.

Based on the threshold values $\tau$ and $\tau$, we divide our discussion into three cases.

Case 1: If $\tau \in [\tau_0, \tilde{h}]$, $h^* = \tilde{h}$. Then from (12), we find

$$g(\tau) \sqrt{\tilde{R}^2 - \tilde{h}^2} + f(\tau) \tilde{h} - \tilde{c} \tau \leq 0.$$

Then, we transform the above constraint into the following equivalent form:

$$F_1(\tau) = C_1 g(\tau) - (\tilde{c} + \alpha \tilde{h}) \tau + (\alpha + 1) \tilde{h} \leq 0, \ (13)$$

where $C_1 = \sqrt{\tilde{R}^2 - \tilde{h}^2} \geq 0$. The left side of (13) is concave on $\tau$. Define the derivative of the left side as $F'_1(\tau) = C_1 g'(\tau) - (\tilde{c} + \alpha \tilde{h})$. We observe that $F'_1(\tau_0) > 0$ as $g'(\tau_0) \rightarrow \infty$ and

1) if $F'_1(\tilde{\tau}) \leq 0$, $\tau^*$ is the unique solution of $F'_1(\tau) = 0$ within the domain $[\tau_0, \tilde{\tau}]$;

2) else if $F'_1(\tilde{\tau}) > 0$, $\tau^* = \tilde{\tau}$.

Case 2: If $\tau \in [\tilde{\tau}, \tau_1]$, $h^* = \tilde{h}(\tau)$. Then from (12), we find

$$\tilde{R} \sqrt{g(\tau)^2 + f(\tau)^2} - \tilde{c} \tau \leq 0.$$

The above problem is a one-dimensional problem on $\tau$. We transform it into the following form:

$$F_2(\tau) = \tilde{R} \left( g(\tau)^2 + f(\tau)^2 \right) - \tilde{c} \tau \leq 0. \ (14)$$

We observe that $F_2(\tau)$ is differentiable on $[\tilde{\tau}, \tau_1]$. Then, we know that the extreme value of $F_2(\tau)$ happens at the critical
points (boundary points \( \tau, \tau_1 \) or \( \tau_2 \) such that that \( F_2' (\tau_1) = 0 \)). In the following numerical analysis, we present efficient ways to find \( \tau^* \) which maximize the left side of (14).

The first and second derivative of the left side of (14) are

\[
F_2' (\tau) = \hat{R}^2 \left( \frac{\alpha}{\epsilon} \tau^{-\alpha-1} + 2\alpha (\alpha \tau - \alpha - 1) \right) - 2\hat{c}^2 \tau
\]

\[
= \frac{\alpha \hat{R}^2}{\epsilon} \tau^{-\alpha-1} + (2\alpha^2 \hat{R}^2 - 2\hat{c}^2) \tau - 2\hat{R}^2 \alpha (\alpha + 1),
\]

\[
F_2''(\tau) = -\frac{\alpha \hat{R}^2 (\alpha + 1)}{\epsilon} \tau^{-\alpha-2} + (2\alpha^2 \hat{R}^2 - 2\hat{c}^2).
\]

Given this, there are two conditions.

**Condition 1:** If \( 2\alpha^2 \hat{R}^2 - 2\hat{c}^2 \leq 0 \), \( F_2'' (\tau) \) is monotonically decreasing on \( \tau \) and \( F_2 (\tau) \) is concave on \( \tau \). Then,

1) if \( F_2' (\xi) \leq 0 \), \( \tau^* = \xi \);  
2) else if \( F_2' (\xi) > 0 \) and \( F_2' (\xi) \leq 0 \), \( \tau^* = \xi \) or the unique solution of \( F_2' (\tau) = 0 \) within the domain \([\tau, \xi]\).  
3) else if \( F_2' (\xi) > 0 \), \( \tau^* = \xi \).

**Condition 2:** If \( 2\alpha^2 \hat{R}^2 - 2\hat{c}^2 > 0 \), \( F_2'' (\tau) \) is monotonically increasing on \( \tau \) and \( F_2 (\tau) \) is convex on \( \tau \). Then,

1) if \( F_2' (\tau) \leq 0 \), \( F_2 (\tau) \) is decreasing within the domain. To find \( \tau^* \), we follow the same discussions as in Condition 1;  
2) else if \( F_2'' (\tau) > 0 \) and \( F_2'' (\tau) \leq 0 \), \( F_2'' (\tau) \) is first decreasing and then increasing. Define \( F_2 = (\tau, \tau) \) where \( F_2' (\tau) = 0 \) within the domain \([\tau, \tau] \), \( F_2 = (\tau, \tau) \), and \( F_2'' (\tau) \).

a) if \( 0 \leq F_2 (\tau) \), \( \tau^* = \tau \);  
b) if \( F_2 (\tau) \leq 0 \) or \( F_2 (\tau) \leq 0 \), \( \tau^* = \tau \) or the unique solution of \( F_2' (\tau) = 0 \) within the domain \([\tau, \tau] \);  
c) if \( F_2 (\tau) \leq 0 \), \( \tau^* = \tau \) or \( \tau \) that maximizes \( F_2 (\tau) \);  
d) if \( F_2 (\tau) \leq 0 \), \( \tau^* = \tau \) or \( \tau \) that maximizes \( F_2 (\tau) \);  
e) if \( F_2 (\tau) \leq 0 \), \( \tau^* = \tau \) or \( \tau \) that maximizes \( F_2 (\tau) \).

3) else if \( F_2' (\tau) > 0 \), \( F_2 (\tau) \) is convex on \( \tau \). \( \tau^* = \tau \) or \( \tau \) that maximizes \( F_2 (\tau) \).

**Case 3:** If \( \tau \in [\tau, \infty) \), \( h^* = \hat{h} \). Then from (12), we find

\[
g(\tau) \sqrt{\hat{R}^2 - \hat{h}^2} + f(\tau) \hat{h} - \hat{c} \tau \leq 0,
\]

which we transform into the following equivalent form

\[
F_3 (\tau) = C_3 g(\tau) - (\hat{c} + \alpha \hat{h}) \tau + (\alpha + 1) |\hat{h}| = 0
\]

where \( C_3 = \sqrt{\hat{R}^2 - \hat{h}^2} \). Define the derivative of the left hand side of (15) as \( F_3' (\tau) = C_3 g'(\tau) - (\hat{c} + \alpha \hat{h}) \). Then \( F_3 (\tau) \) is concave on \( \tau \) and as \( \tau \to \infty \), \( F_3 (\tau) \leq 0 \). Then,

1) if \( \hat{c} + \alpha \hat{h} = 0 \), \( F_3 (\tau) \) is an increasing function and \( \tau^* = \infty \);  
2) else if \( \hat{c} + \alpha \hat{h} > 0 \), as \( \tau \to \infty \), \( F_3 (\tau) < 0 \). Based on the concavity of \( F_3 (\tau) \), we find

a) if \( F_3' (\tau) \leq 0 \), \( \tau^* = \tau \);  
b) else if \( F_3' (\tau) > 0 \), \( \tau^* = \tau \) or the unique solution of \( F_3' (\tau) = 0 \) within the domain \([\tau, \infty) \). To efficiently apply the golden section search, we determine an effective finite upper bound instead of \( \infty \). Let the effective upper bound be \( \tau_2 \), we have

\[
F_3 (\tau_2) = C_3 g' (\tau_2) - (\hat{c} + \alpha \hat{h}) \leq 0.
\]

**Lemma 4.2:** A feasible selection of \( \tau_2 \) is

\[
\tau_2 = \left[ -1 + \sqrt{1 + 4(1 - \epsilon) C_2} \right]^{-\frac{1}{2}},
\]

where \( C_2 = \frac{\alpha^2 C_3}{\epsilon (\alpha + 1)^2} \). The proof is given in Appendix C.

Then, instead of a search on \([\tau, \infty) \), we only need to search on \([\tau, \tau_2] \).

Combining all three cases, we can find the overall worst \( \tau^* \) and \( h^* \) given \( x_i \). If (12) is satisfied with these parameters, then there is no violated constraint in Step 2 of Algorithm 1. If (12) is not satisfied, we need to use the worst case \( \tau^* \) and \( m^* \) in Step 3 and the iteration continues. Depending on how we define \( \Xi \), \( m^* \) are different functions of \( h^* \).

**D. Candidates of \( \Xi \)**

In this section, we demonstrate how the selection of \( \Xi \) affects the determination of \( h^* \), \( \Xi \), and \( m^* \). Specifically, we give two examples of \( \Xi \) and show how to exactly reformulate (10) (i.e., Assumption 4.2) and how to calculate \( h^* \) and \( \Xi \), given \( x_i \). Furthermore, we show how to find the worst case \( m^* \) from \( h^* \).

**Rectangular Support:** We assume that \( \Xi = [k, k] \) and hence we can reformulate (10) as

\[
a(x)^\top \left( \frac{k + K}{2} \right) + |a(x)|^\top \left( \frac{k - K}{2} \right) \leq b(x).
\]

Furthermore, given \( x_i \), we have the following relationships due to (17).

\[
h = \left[ a(x_i)^\top \left( \mu - \frac{k + K}{2} \right) - |a(x_i)|^\top \left( \frac{k - K}{2} \right) \right] / \alpha,
\]

\[
\Xi = \left[ a(x_i)^\top \left( \mu - \frac{k + K}{2} \right) + |a(x_i)|^\top \left( \frac{k - K}{2} \right) \right] / \alpha.
\]

Based on (17) and (18), if we have the worst case \( h^* \), we find the worst case \( m^* \) by solving (19) for \( \lambda_r \) and substituting in (20):

\[
h^* = \left[ a(x_i)^\top \left( \mu - \frac{k + K}{2} \right) + \lambda_r |a(x_i)|^\top \left( \frac{k - K}{2} \right) \right] / \alpha,
\]

\[
m^* = \left( \frac{k + K}{2} \right) - \lambda_r \text{sign} (a(x_i)) \left( \frac{k - K}{2} \right),
\]

where \( \text{sign} (a(x)) \) returns a diagonal matrix whose diagonal elements equal the sign of each element in \( a(x) \).
Ellipsoidal Support: We assume that \( \Xi = \{ m : m = m_{c} + P^{1/2}u, \|u\|_{2} \leq 1 \} \), where \( P > 0 \). Then we can reformulate (10) as
\[
a(x)^{T} m_{c} + \left\| P^{1/2} a(x) \right\|_{2}^{2} \leq b(x). \tag{21}\]
Furthermore, due to (11), we have the following relationships:
\[
\begin{align*}
\underline{h} &= \left( a(x^{*})^{T} (\mu - m_{c}) - \left\| P^{1/2} a(x^{*}) \right\|_{2} \right) / \alpha, \tag{22}
\bar{h} &= \left( a(x^{*})^{T} (\mu - m_{c}) + \left\| P^{1/2} a(x^{*}) \right\|_{2} \right) / \alpha. \tag{23}
\end{align*}
\]
Next, if we have the worst case \( h^{*} \), we find the worst case \( m^{*} \) directly by solving (24) for \( \lambda_{c} \) and substituting in (25).
\[
\begin{align*}
h^{*} &= \left[ a(x^{*})^{T} (\mu - m_{c}) - \lambda_{c} \left\| P^{1/2} a(x^{*}) \right\|_{2} \right] / \alpha, \tag{24}
m^{*} &= m_{c} + \lambda_{c} \left\| P^{1/2} a(x^{*}) \right\|_{2}. \tag{25}
\end{align*}
\]

V. CASE STUDY

A. Simulation Setup

We consider the DC OPF problem from [16]. We assume that the system has two wind power plants with wind forecast error \( \tilde{w} = [\tilde{w}_{1}, \tilde{w}_{2}]^{T} \). With \( N_{G} \) generators and \( N_{B} \) buses, the design variables are generation \( P_{G} \in \mathbb{R}^{N_{G}} \), up and down reserve capacities \( R^{up}_{G}, R^{dn}_{G} \in \mathbb{R}^{N_{G}} \), and a distribution vector \( d_{G} \in \mathbb{R}^{N_{G}} \), which determines the real-time reserve provision from each generator used to balance the wind forecast error. The full problem formulation is as follows.

\[
\begin{align*}
\min P_{G}^{T} [C_{1} P_{G} + C_{2}^{T} P_{G} + C_{R}^{T} (R^{up}_{G} + R^{dn}_{G})] \tag{26a}
s.t.: \quad P_{G} &\leq P_{\text{inj}} \leq P_{I} \tag{26b}
R_{G} &\leq -d_{G} (\tilde{w}_{1}, \tilde{w}_{2}) \tag{26c}
P_{\text{inj}} &\leq C_{G} (P_{G} + R_{G}) + C_{W} (P_{W}^{f} + \tilde{w}) - C_{L} P_{L} \tag{26d}
-P_{\text{inj}} &\leq P_{G} + R_{G} \leq P_{G} \tag{26e}
-R^{dn}_{G} &\leq R_{G} \leq R^{up}_{G} \tag{26f}
1_{1 \times N_{G}} d_{G} &\leq 1 \tag{26g}
C_{1} P_{G} + C_{W} (P_{W}^{f} - C_{L} P_{L}) &\geq 0 \tag{26h}
R^{up} &\geq 0_{N_{G} \times 1}, R^{dn} \geq 0_{N_{G} \times 1} \tag{26i}
\end{align*}
\]
where \( [C_{1}] \in \mathbb{R}^{N_{G} \times N_{G}}, C_{2} \in \mathbb{R}^{N_{G}}, \) and \( C_{R} \in \mathbb{R}^{N_{G}} \) are cost parameters. Constraint (26b) bounds the power flow, which is calculated from the power injections \( P_{\text{inj}} \) defined in (26d) and the parameter matrix \( A \), by the line limits \( P_{I} \). Constraint (26c) computes the real-time reserve usage \( R_{G} \) for each generator. In (26d) \( P_{W}^{f} \) is the wind forecast, \( P_{L} \) is the load, and \( C_{G}, C_{W}, \) and \( C_{L} \) are matrices that map generators, wind power plants, and loads to buses: (26f) restricts the reserve capacity; (26g) enforces balance with and without wind forecast error; and (26h) ensures all decision variables are non-negative.

We next analyze how the data size of each sample \( N_{\text{data}} \) and the number of bins within the histogram \( N_{\text{bin}} \) affect the estimate of the mode support. Figure 4 shows that if we change \( N_{\text{bin}} \) from 15 to 30 the histograms no longer show a unimodal distribution, as compared to Fig. 1. The problem is exacerbated as \( N_{\text{bin}} \) grows.
We next explore the impact of the size of the data pool. We first use the entire data pool to select 100 samples with

### Fig. 4. Histogram of univariate and bivariate wind forecast errors (30 bins).
Plant 2 — wind forecast error (MW)

Plant 2 — wind forecast error (MW)

Fig. 5. Mode values from samples with different $N_{\text{data}}$ and $N_{\text{bin}}$. Data is sampled from the full data pool.

TABLE I
FULL POOL: $k$ AND $\overline{k}$ (MW) OF FOUR RECTANGULAR SETS $\Xi$

| $N_{\text{bin}}$ | $N_{\text{data}} = 100$ | $N_{\text{data}} = 1000$ |
|------------------|-------------------------|--------------------------|
| 15               | $-4.44$ 0.10            | $-3.45$ 0.17             |
| Plant 1          | $-4.44$ 0.24            | $-3.69$ -0.11            |
| Plant 2          | $-4.36$ 0.19            | $-3.02$ -0.93            |
| 30               | $-4.22$ 0.22            | $-3.06$ -0.39            |

TABLE II
PARTIAL POOL: $k$ AND $\overline{k}$ (MW) OF FOUR RECTANGULAR SETS $\Xi$

| $N_{\text{bin}}$ | $N_{\text{data}} = 50$ | $N_{\text{data}} = 200$ |
|------------------|-------------------------|--------------------------|
| 10               | $-5.77$ 0.58            | $-3.52$ 0.09             |
| Plant 1          | $-5.05$ 0.44            | $-4.43$ 0.06             |
| Plant 2          | $-5.82$ 0.06            | $-4.36$ -0.09            |
| 20               | $-5.76$ 0.04            | $-3.86$ 0.19             |

different data sizes $N_{\text{data}}$ (100 and 1000) and number of bins $N_{\text{bin}}$ (15 and 30) and show scatter plots of the mode values in Fig. 5. As $N_{\text{data}}$ gets larger, the mode values are more condensed and hence more accurate. When $N_{\text{bin}} = 30$ and $N_{\text{data}} = 100$ mode values appear in several disjoint regions, but this disjointness is mitigated as $N_{\text{data}}$ increases to 1000. Based on the scatter plots, we determine the parameters $k$, $\overline{k}$ of the four rectangular sets $\Xi$ used in $D^2_\xi$. The results are given in Table I.

We repeated the analysis using only a partial data pool, specifically, we randomly selected 1000 data from the full pool to comprise the partial pool. We also use different choices of $N_{\text{data}}$ and $N_{\text{bin}}$. The scatter plots are shown in Fig. 6 and parameter values for $\Xi$ are given in Table II.

2) Objective Costs: We next analyze the objective costs and the optimal reserve capacities using different ambiguity sets. The results are summarized in Table III. In all case studies, since we focus on mode misspecification not moment misspecification, moments are calculated using the full or partial data pool and all ambiguity sets use the same moments.

For ambiguity set $D^2_\xi$, we perform tests with the following six fixed mode estimates.

- M1: mode determined using the full (partial) data pool with histogram of 15 (10) bins. This case demonstrates the performance of $D^2_\xi$ with an accurate mode estimate.
- M2: mode determined using the full (partial) data pool with histogram of 30 (20) bins. This case shows how $N_{\text{bin}}$ affects the result.
- M3-6: combinations of the largest $\overline{k}$ and the smallest $k$ of both plants from Table I (full pool) and Table II (partial pool). These cases demonstrate the affect of outlying data samples.

For ambiguity set $D^3_\xi$, we perform tests with different $\Xi$, specifically, $\Xi_1 : 100 \times 15$, $\Xi_2 : 1000 \times 15$, $\Xi_3 : 100 \times 30$, $\Xi_4 : 1000 \times 30$, $\Xi_5 : 50 \times 10$, $\Xi_6 : 200 \times 10$, $\Xi_7 : 50 \times 20$, and $\Xi_8 : 200 \times 20$, where the first number refers to $N_{\text{data}}$ and the second number refers to $N_{\text{bin}}$. In each case we use the parameters $k$, $\overline{k}$ from Tables I and II.

As shown in Table III $D^3_\xi$ has the highest objective cost since it does not include the assumption of unimodality. The cost of $D^3_\xi$ varies with the mode estimate. We observe opposite variations on the total up and down reserve capacities since different mode estimates lead to different estimates of the skewness of the uncertainty distribution. Comparing M1 and M2 to M3-6 we see that inaccurate estimation of the mode could lead to either higher or lower costs. Furthermore, results for M1 and M2 are significantly different demonstrating the effect of different choices of $N_{\text{bin}}$. 
TABLE III
OBJECTIVE COSTS AND RESERVE CAPACITIES

|                | $D_1^1$ M1 | $D_2^1$ M2 | $D_3^1$ M3 | $D_4^1$ M4 | $D_5^1$ M5 | $D_6^1$ M6 | $D_1^2$ M1 | $D_2^2$ M2 | $D_3^2$ M3 | $D_4^2$ M4 | $D_5^2$ M5 | $D_6^2$ M6 | $D_1^3$ M1 | $D_2^3$ M2 | $D_3^3$ M3 | $D_4^3$ M4 | $D_5^3$ M5 | $D_6^3$ M6 |
|----------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| Full pool      |             |             |             |             |             |             |             |             |             |             |             |             |             |             |             |             |             |             |
| Total Cost     | 26160       | 19440       | 19546       | 18993       | 19547       | 19526       | 19542       | 19949       | 19818       | 19982       | 19896       | 19818       | 19982       | 19818       | 19982       | 19896       | 19818       | 19982       |
| Generation Cost| 13032       | 11515       | 11504       | 11506       | 11481       | 11491       | 11506       | 11522       | 11522       | 11522       | 11522       | 11514       | 11522       | 8304        | 8460        | 8373        | 8304        | 8460        |
| Reserve Cost   | 13129       | 7925        | 8042        | 7488        | 8065        | 8035        | 8036        | 8427        | 8296        | 8460        | 8460        | 8460        | 8460        | 8460        | 8460        | 8460        | 8460        | 8460        |
| Up Reserve (MW)| 38.8        | 26.8        | 26.2        | 26.3        | 25.1        | 26.1        | 26.1        | 27.1        | 26.8        | 27.1        | 27.0        | 26.7        | 27.1        | 26.7        | 27.1        | 26.7        | 27.1        |
| Down Reserve (MW)| 26.9      | 12.9        | 14.0        | 11.1        | 15.2        | 14.1        | 14.1        | 15.1        | 14.7        | 15.2        | 14.9        | 14.8        | 15.2        | 14.8        | 15.2        | 14.8        | 15.2        |
| Partial pool   |             |             |             |             |             |             |             |             |             |             |             |             |             |             |             |             |             |             |             |
| Total Cost     | 21845       | 16759       | 16740       | 15250       | 15915       | 16735       | 16718       | 17883       | 17909       | 17876       | 17909       | 16791       | 17949       | 11514       | 11522       | 8304        | 8460        | 8373        |
| Generation Cost| 11713       | 11376       | 11376       | 11324       | 11295       | 11336       | 11367       | 11420       | 11420       | 11420       | 11420       | 11371       | 11420       | 8304        | 8460        | 8373        | 8304        | 8460        |
| Reserve Cost   | 10132       | 5383        | 5364        | 3926        | 4620        | 5399        | 5351        | 6463        | 6489        | 6456        | 6489        | 5420        | 6529        | 8304        | 8460        | 8373        | 8304        | 8460        |
| Up Reserve (MW)| 31.1        | 20.1        | 20.1        | 17.5        | 16.1        | 19.4        | 19.1        | 22.0        | 22.0        | 21.9        | 22.0        | 19.2        | 22.1        | 8304        | 8460        | 8373        | 8304        | 8460        |
| Down Reserve (MW)| 19.6      | 6.8         | 6.7         | 2.1         | 7.0         | 7.6         | 7.7         | 10.3        | 10.4        | 10.4        | 10.4        | 7.9         | 10.6        | 8304        | 8460        | 8373        | 8304        | 8460        |

The costs of $D_1^1$ are higher than those of $D_2^1$ since the solution is designed to cope with mode misspecification. The costs do not vary significantly as a function of $N_{bin}$ and $N_{data}$. For a given $N_{bin}$, as $N_{data}$ increases, the costs decrease since the mode estimates are more closely clustered.

The cost of $D_2^1$ is higher than the costs of $D_2^2$ with $M_1$, demonstrating the benefit in allowing the mode to be different than the mean. The cost of $D_2^2$ is close to that of $D_2^3$ with $M_3$ since the mode estimates are widely distributed in this case; however, the cost of all other $D_2^3$ is below that of $D_2^5$. As expected, $D_2^5$ is lower bounded by the fixed mode ambiguity sets $D_2^4$ and $D_2^6$, and upper bounded by $D_3^5$.

3) Reliability: Using the solutions we generated, we run out-of-sample test with 20 samples of 5000 wind forecast errors to evaluate the joint reliability of each optimal solution. We define the joint reliability as the percentage of wind forecast errors for which all chance constraints are satisfied. Then, we compare the reliability results with our pre-defined probability level $(1 - \epsilon = 95\%)$. The results are summarized in Table IV.

We observe that reliability ranking almost always matches the cost ranking. Ambiguity sets $D_3^1$ and $D_3^2$ have the most conservative solutions and hence higher reliability and costs. The reliability of $D_3^3$ is lower bounded by the reliability of $D_2^4$ and $D_2^6$, and upper bounded by the reliability of $D_3^5$. It also shows robustness against the selection of $N_{data}$ and $N_{bin}$. For the full pool, all ambiguity sets achieve constraint satisfaction above 95%. For the partial pool, $D_2^4$ and $D_2^5$ fail to meet the threshold, while ambiguity sets with misspecified modes $D_3^4$, arbitrary modes $D_3^5$, or no unimodality assumptions $D_3^6$ achieve constraint satisfaction above 95%.

In this example, $D_3^5$ can be use to approximate $D_3^3$ since they have similar reliability. However, $D_3^2$ is less conservative than $D_3^5$ if $\Xi$ does not include the global worst case mode. Set $D_3^1$ is also more applicable to multivariate unimodality as $D_3^5$ is only defined for $\alpha = 1$.

4) Computational Effort: Table IV shows the iteration count and computational time for $D_2^2$ and $D_3^1$. The problems can be solved within 10 iterations and the computational time grows linearly with the number of iterations. Set $D_2^6$ requires more iterations than $D_2^2$. Problems using ambiguity sets $D_3^1$, $D_3^2$, and $D_3^5$ can each be solved in a single run, and each takes less than one second.

VI. CONCLUSION

In this paper, we proposed a distributionally robust chance constrained optimal power flow formulation considering uncertainty distributions with known moments and generalized unimodality with misspecified modes. We derived an efficient solving algorithm using the separation approach. In each iteration of the algorithm, the problem contains only SOC constraints and hence can be solved with commercial solvers. Using wind forecast errors, we found that the distribution of mode estimates are highly dependent on the data pool size, the data size of each sample, and the number of bins used in the histogram. We tested our approach on a modified IEEE 30-bus system and compared our results to those generated with other ambiguity sets. Without the assumption of unimodality, we obtain overly conservative results as unrealistic distributions are included in the ambiguity set. Considering unimodality, but with fixed mode, the results are highly dependent on the quality of the mode estimate. Considering unimodality with misspecified mode, the results are relatively consistent across different mode supports and the performance is bounded by that of the fixed-mode model and that of the arbitrary-mode model. With univariate unimodality and large mode deviations, the misspecified-mode model can be well approximated by the arbitrary-mode model.

Future work will extend the current results by considering more accurate descriptions of the mode support. For example, we could represent the mode support as a union of disjoint sets that matches the mode profile. Other directions include evaluating the approach on a more realistic system and studying how the current approach works in the cases with other misspecified information such as moments.

APPENDIX A

CONVEXITY AND CONCAVITY OF (12)

Here we prove the left side of (12) is neither jointly convex nor concave in $h$ and $\tau$ through counter examples. We first pick $\alpha = R = 1$, $\epsilon = 0.05$, and $c = 0$ without loss of generality. Then we select two groups of points and calculate...
the left-side values $v$. Group 1: $[h, \tau, v] = (0.1, 2, 2.985)$ and $(0.3, 3, 3.05)$, then the midpoint $(0.2, 2.5, 3.15)$ has a value higher than line segment value $3.0175$ (concave). Group 2: $[h, \tau, v] = (0.4, 11, 0.1900)$ and $(0.6, 10, -1.5015)$, then the midpoint $(0.5, 10.5, -0.6093)$ has a value lower than line segment value $-0.65125$ (convex).

**APPENDIX B**

**PROOF OF LEMMA 4.1**

We first check if $\hat{h} \geq 0$. If so, we know $\tau_1 \in [\tau_0, \frac{\alpha+1}{\alpha}]$ as $\hat{h}(\frac{\alpha+1}{\alpha}) = 0$ and $\hat{h}$ is decreasing. Hence, we can conduct the golden section search on $[\tau_0, \frac{\alpha+1}{\alpha}]$.

Next, if $\hat{h} < 0$, we know $\tau_1 > \frac{\alpha+1}{\alpha}$ and we have

$$\hat{h}(\tau_1) < h_2(\tau_1) = \frac{\frac{f(\tau_1)}{1-\epsilon} + f(\tau_1)^2}{1 + \frac{f(\tau_1)^2}{\epsilon} R}.$$

If we further force $h_2(\tau_1) = h$, we have $\hat{h}(\tau_1) < h_2(\tau_1) = h$ and $\tau \in [\frac{\alpha+1}{\alpha}, 1]$. The equality $h_2(\tau_1) = h$ will always have a solution on $[\frac{\alpha+1}{\alpha}, \infty)$ as $h_2(\frac{\alpha+1}{\alpha}) = 0$ and as $\tau \to \infty$, $h_2(\tau) = -R$.

Next we solve the equality and find

$$f(\tau_1) = \hat{h} \sqrt{\frac{1-\epsilon}{\epsilon(R^2 - h^2)}} \Rightarrow$$

$$\tau_1 = -\left( \frac{\hat{h} \sqrt{\frac{1-\epsilon}{\epsilon(R^2 - h^2)}} - (\alpha+1)}{\alpha} \right).$$

**APPENDIX C**

**PROOF OF LEMMA 4.2**

We have the following relationship because $\tau \geq \tau_0 > 1$.

$$g'(\tau) = \frac{\alpha(\tau-1)}{2g(\tau)} \leq g_2(\tau) = \frac{\alpha(\tau-1)}{2g(\tau)}.$$

Then, we have the following relationship where $\tau_2$ is the effective upper bound.

$$F_2'(\tau_2) \leq F_4(\tau_2) = C_3g_2(\tau_2) - (\hat{c} + \alpha h_2) = 0.$$

The last equality will always have solution on $[\bar{\tau}, \infty]$ since $F_4(\tau) \geq F_2(\tau) > 0$ and as $\tau \to \infty$, $F_4(\tau) < 0$. By solving the equality, we obtain

$$C_2(\tau_2-\alpha)^2 + \tau_2^{-\alpha} - (1-\epsilon) = 0,$$

where $C_2 = \frac{\alpha^2 C_2^2}{\pi^2 (\epsilon + a_2)^2}$. This is a quadratic equation of $\tau_2^{-\alpha}$ and we find

$$\tau_2 = \left[ -1 + \sqrt{1 + 4(1-\epsilon)C_2} \right]^{-\frac{1}{\alpha}}.$$

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