COARSE DIMENSION AND DEFINABLE SETS IN EXPANSIONS OF THE
ORDERED REAL VECTOR SPACE

ERIK WALSBERG

ABSTRACT. Let $E \subseteq \mathbb{R}$. Suppose there is an $s > 0$ such that $$(\{k \in \mathbb{Z}, -m \leq k \leq m - 1 : \lfloor k, k + 1 \rfloor \cap E \neq \emptyset\}) \geq m^s$$ for all sufficiently large $m \in \mathbb{N}$. Then there is an $n \in \mathbb{N}$ and a linear $T : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $T(E^n)$ is dense. It follows that if $E$ is in addition nowhere dense then $(\mathbb{R}, <, +, 0, (x \mapsto \lambda x)_{\lambda \in \mathbb{R}}, E)$ defines every bounded Borel subset of every $\mathbb{R}^n$.

1. INTRODUCTION

Let $X \subseteq \mathbb{R}^n$ be bounded and $Z \subseteq \mathbb{R}^n$. Given a positive $\delta \in \mathbb{R}$ we let $M(\delta, X)$ be the minimum number of open $\delta$-balls required to cover $X$. Equivalently $M(\delta, X)$ is the minimal cardinality of a subset $S$ of $X$ such that $x \in X$ lies within distance $\delta$ of some element of $S$. Let $B_n(p, r)$ be the open ball in $\mathbb{R}^n$ with center $p$ and radius $r > 0$ and let $B_n(r) = B_n(0, r)$. We define the coarse Minkowski dimension of $Z$ to be $$\dim_{CM}(Z) := \limsup_{r \to \infty} \frac{M(1, B_n(r) \cap Z)}{\log(r)}.$$ It is easy to see that the coarse Minkowski dimension of $Z$ is bounded above by $n$ and the coarse Minkowski dimension of a bounded set is zero. An application of the first claim of Fact 2.1 below shows that replacing one with any fixed real number $\delta > 0$ does not change the coarse Minkowski dimension. A simple computation shows that $\dim_{CM}(Z)$ is the infimum of the set of positive $s \in \mathbb{R}$ such that $M(1, B_n(r) \cap Z) < r^s$ for all sufficiently large $r > 0$.

We define $$N(X) := \left\{ (k_1, \ldots, k_n) \in \mathbb{Z}^n : X \cap \prod_{i=1}^n [k_i, k_i + 1] \neq \emptyset \right\}. $$ It is well-known and easy to see that there is a real number $K > 0$ depending only $n$ such that $$K^{-1}M(1, X) \leq N(X) \leq KM(1, X).$$ So $$\dim_{CM}(Z) = \limsup_{r \to \infty} \frac{N(B_n(r) \cap Z)}{\log(r)}.$$

Our main geometric result is Theorem 1.1.

Theorem 1.1. Suppose $E \subseteq \mathbb{R}$. If $\dim_{CM}(E) > 0$ then $T(E^n)$ is dense for some $n \in \mathbb{N}$ and linear $T : \mathbb{R}^n \rightarrow \mathbb{R}$. Equivalently, if there is a positive $s \in \mathbb{R}$ such that $N(B_n(r) \cap E) > r^s$ for all sufficiently large $r \in \mathbb{R}$ then there exist $n \in \mathbb{N}$ and linear $T : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $T(E^n)$ is dense.

The converse implication to Theorem 1.1 does not hold. Let $D = \{2^n, 2^n + n : n \in \mathbb{N}\}$. A simple computation shows that $D$ has coarse Minkowski dimension zero. Let $S : \mathbb{R}^4 \rightarrow \mathbb{R}$ be given by $S(x_1, x_2, x_3, x_4) = (x_1 - x_2) + \alpha(x_3 - x_4)$ for a fixed $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then $S(D^3)$ is dense.

Theorem 1.1 is motivated by an application to logic that we now describe. Let $\mathbb{R}_{vec}$ be the ordered vector space $(\mathbb{R}, <, +, 0, (x \mapsto \lambda x)_{\lambda \in \mathbb{R}})$ of real numbers. For any subset $E$ of $\mathbb{R}$ let $(\mathbb{R}_{vec}, E)$ be the expansion of $\mathbb{R}_{vec}$ by a unary predicate defining $E$. When we say that a subset of $\mathbb{R}^n$ is definable in a first order expansion of $(\mathbb{R}, <, +, 0)$ such as $(\mathbb{R}_{vec}, E)$ we mean that it is first order definable possibly with parameters from $\mathbb{R}$.

Hieronymi and Tychonievich [6] show that $(\mathbb{R}_{vec}, \mathbb{Z})$ defines all bounded Borel subsets of all $\mathbb{R}^n$. In contrast, it follows from [8, 9] that every subset of $\mathbb{R}^n$ definable in $(\mathbb{R}, <, +, 0, \mathbb{Z})$ is a finite union of locally closed sets.

Date: October 17, 2021.
The theorem of Hironumy and Tyconovich is a special case of Theorem 1.2. Theorem 1.2 also follows from a more general theorem of Fornasiero, Hironumy, and Walsberg [2, Theorem 7.3, Corollary 7.5]. We let $C(E)$ be the closure of $E \subseteq \mathbb{R}$ and $Bd(E)$ be the boundary of $E$. Recall that the boundary of a subset of $\mathbb{R}$ is always closed.

**Theorem 1.2.** Suppose that $E \subseteq \mathbb{R}$ is not dense and co-dense in any nonempty open interval. Then the following are equivalent:

1. $(\mathbb{R}_{\vec{v}},E)$ does not define every bounded Borel subset of every $\mathbb{R}^n$,
2. Every subset of $\mathbb{R}$ definable in $(\mathbb{R}_{\vec{v}},E)$ either has interior or is nowhere dense,
3. $T(Bd(E)^n)$ is nowhere dense for every linear $T: \mathbb{R}^n \to \mathbb{R}$.

The implication $(3) \Rightarrow (2)$ is a corollary of a result of Friedman and Miller [3]. The implication $(1) \Rightarrow (3)$ is a corollary of the main theorem of [6]. Note that $Bd(E)$ is nowhere dense as $E$ is not dense and co-dense in any open interval. If $E$ is bounded then $(3)$ above is equivalent to a natural geometric condition on $E$. This equivalence, observed in [2, Theorem 7.3], is an easy consequence of the famous Marstrand projection theorem ([7, Chapter 9]) and the classical theorem of Steinhaus that $Z - Z := \{z - z' : z, z' \in Z\}$ has interior whenever $Z \subseteq \mathbb{R}^n$ has positive $n$-dimensional Lebesgue measure.

**Fact 1.3.** Suppose $F \subseteq \mathbb{R}$ is bounded. Then $T(F^n)$ is nowhere dense for every linear $T: \mathbb{R}^n \to \mathbb{R}$ if and only if $C(F)^n$ has Hausdorff dimension zero for all $n \in \mathbb{N}$.

Fact 1.3 does not hold for unbounded subsets of $\mathbb{R}$. The set of integers, like any countable set, has Hausdorff dimension zero, and $T(2^Z)$ is dense for any linear $T: \mathbb{R}^2 \to \mathbb{R}$ of the form $T(x,y) = x + ay$ with $a \in \mathbb{R} \setminus \mathbb{Q}$. Combining Theorem 1.1 and Theorem 1.2 we obtain the following.

**Theorem 1.4.** Suppose $E \subseteq \mathbb{R}$ is not dense and co-dense in any nonempty open interval. If $Bd(E)$ has positive Minkowski dimension then $(\mathbb{R}_{\vec{v}},E)$ defines every bounded Borel subset of every $\mathbb{R}^n$. In particular if $E$ is nowhere dense and has positive coarse Minkowski dimension then $(\mathbb{R}_{\vec{v}},E)$ defines every bounded Borel subset of every $\mathbb{R}^n$.

Note that $Z$ has coarse Minkowski dimension one so Theorem 1.4 generalizes the result of Hieronymi and Tyconovich described above. There are subsets $E$ of $\mathbb{R}$ with coarse Minkowski dimension zero such that $(\mathbb{R}_{\vec{v}},E)$ defines every bounded Borel subset of every $\mathbb{R}^n$ such as $\{2^n, 2^n + n : n \in \mathbb{N}\}$ (see the comment after Theorem 1.1). Theorem 1.4 fails without the assumption that $E$ is not dense and co-dense in any nonempty open interval. Block-Gorman, Hironumy, and Kaplan [4] show that every closed subset of $\mathbb{R}^n$ definable in $(\mathbb{R}_{\vec{v}},\mathbb{Q})$ is already definable in $\mathbb{R}_{\vec{v}}$ and $Bd(\mathbb{Q}) = \mathbb{R}$ has coarse Minkowski dimension one.

The present paper is part of the broader study of the metric geometry of definable sets in first order structures expanding $(\mathbb{R},<,+;0)$, see [1, 2, 5]. Fornasiero, Hironumy, and Miller [1] show that if $E \subseteq \mathbb{R}$ is nowhere dense and has positive Minkowski dimension then $(\mathbb{R},<,+;0,1,E)$ defines every Borel subset of every $\mathbb{R}^n$. This statement fails over $\mathbb{R}_{\vec{v}}$, as $D = \{\frac{1}{n} : n \in \mathbb{N}, n \geq 1\}$ has Minkowski dimension one and Fact 1.3 and Theorem 1.2 together imply that every subset of $\mathbb{R}$ definable in $(\mathbb{R}_{\vec{v}},D)$ either has interior or is nowhere dense. It is shown in [2] that if $E \subseteq \mathbb{R}^n$ is closed and the topological dimension of $E$ is strictly less than the Hausdorff dimension of $E$ then $(\mathbb{R}_{\vec{v}},E)$ defines every bounded Borel subset of every $\mathbb{R}^n$.

As a closed subset of $\mathbb{R}$ has topological dimension zero if it is nowhere dense and topological dimension one if it has interior, Theorem 1.4 shows that if $E \subseteq \mathbb{R}$ is closed and the topological dimension of $E$ is strictly less than the coarse Minkowski dimension of $E$ then $(\mathbb{R}_{\vec{v}},E)$ defines every bounded Borel subset of every $\mathbb{R}^n$. It is natural to conjecture that if $E \subseteq \mathbb{R}^n$ is closed and the topological dimension of $E$ is strictly less then the coarse Minkowski dimension of $E$ then $(\mathbb{R}_{\vec{v}},E)$ defines every bounded Borel subset of every $\mathbb{R}^n$. In Theorem 4.1 we will show as a corollary to Theorem 1.4 that if $Z \subseteq \mathbb{R}^n$ is closed and has topological dimension zero and positive coarse Minkowski dimension then $(\mathbb{R}_{\vec{v}},Z)$ defines every bounded Borel subset of $\mathbb{R}^n$.

**Acknowledgements.** I thank the referee for many improvements and Philipp Hieronymi for useful discussions.

## 2. Metric Notions

We recall two useful facts about $M(\delta,X)$ and $N(X)$, both of which are easy to see. One can find more information about these invariants in Yomdin and Comte [10, Chapter 2] and many other places.
Fact 2.1. Let $n \in \mathbb{N}$. There are $K, L > 0$ such that for all bounded $X, Y \subseteq \mathbb{R}^n$ and $0 < \delta < \delta'$

$$M(\delta', X) \leq M(\delta, X) \leq K \left( \frac{\delta'}{\delta} \right)^n M(\delta', X)$$

and

$$L^{-1}M(\delta, X)M(\delta, Y) \leq M(\delta, X \times Y) \leq LM(\delta, X)M(\delta, Y)$$

In particular

$$L^{-1}M(\delta, X)^2 \leq M(\delta, X^2) \leq LM(\delta, X)^2$$

for all bounded $X \subseteq \mathbb{R}^n$.

The proof of the fact below is a straightforward computation that is essentially the same as the proof of the analogous fact for Minkowski dimension. We leave the proof to the reader.

Fact 2.2. For any $X \subseteq \mathbb{R}^n, Y \subseteq \mathbb{R}^n$ and $k \in \mathbb{N}$ we have

$$\dim_{CM}(X \times Y) \leq \dim_{CM}(X) + \dim_{CM}(Y)$$

and

$$\dim_{CM}(X^k) = k \dim_{CM}(X).$$

Suppose that $X \subseteq \mathbb{R}^n, Y \subseteq \mathbb{R}^n$, $f$ is a map $X \rightarrow Y$, and $\lambda, \delta > 0$. Then $f$ is a $(\lambda, \delta)$-quasi-isometry if

$$\frac{1}{\lambda} \|x - x'\| - \delta \leq \|f(x) - f(x')\| \leq \lambda \|x - x'\| + \delta \quad \text{for all} \quad x, x' \in X,$$

and if for every $y \in Y$ we have $\|f(x) - y\| < \delta$ for some $x \in X$. We say that $f$ is a quasi-isometry if it is a $(\lambda, \delta)$-quasi-isometry for some $\lambda, \delta > 0$. It is well-known and easy to see that if there is a quasi-isometry $X \rightarrow Y$ then there is also a quasi-isometry $Y \rightarrow X$. A map $g : X \rightarrow \mathbb{R}^n$ is a quasi-isometric embedding if it yields a quasi-isometry $X \rightarrow g(X)$.

Lemma 2.3. Suppose $X \subseteq \mathbb{R}^n, Y \subseteq \mathbb{R}^n, 0 \in X, 0 \in Y$, and $f : X \rightarrow Y$ is a quasi-isometry such that $f(0) = 0$. Then $X$ and $Y$ have the same coarse Minkowski dimension.

Lemma 2.3 holds without the assumptions that $0 \in X, 0 \in Y$, and $f(0) = 0$. We do not prove this more general result to avoid technicalities.

Proof. We show that $\dim_{CM}(Y) \leq \dim_{CM}(X)$. As there is a quasi-isometry $Y \rightarrow X$ that also maps 0 to 0 the same argument yields the other inequality. Fix $\lambda, \delta > 0$ such that $f$ is a $(\lambda, \delta)$-quasi-isometry.

Fix $r > 0$. Let $X(r) = B_r(0) \cap X$ and $Y(r) = B_r(0) \cap Y$. Let $\{B_{m}(p_i, 1)\}_{i=1}^k$ be a minimal covering of $X(r)$ by balls with radius 1. Then $\{f(B_{m}(p_i, 1))\}_{i=1}^k$ covers $f(X(r))$. Let $q_i = f(p_i)$ for all $i$. As $f$ is a $(\lambda, \delta)$-quasi-isometry we see that $f(B_{m}(p_i, 1))$ is contained in $B_{m}(q_i, \lambda + \delta)$ for all $i$. So $\{B_{m}(q_i, \lambda + \delta)\}_{i=1}^k$ covers $f(X(r))$.

We now show that every point in $Y(r\lambda^{-1} - 2\delta)$ lies within distance $\delta$ of $f(X(r))$. Fix $y \in Y(r\lambda^{-1} - 2\delta)$. As $f$ is a $(\lambda, \delta)$-quasi-isometry there is $x \in X$ such that $\|f(x) - y\| < \delta$. Suppose $\|x\| > r$. Then as $f(0) = 0$ we have

$$\|f(x)\| \geq \frac{1}{\lambda} \|x\| - \delta \geq r\lambda^{-1} - \delta.$$

As $\|f(x) - y\| < \delta$ the triangle inequality yields $\|y\| > r\lambda^{-1} - 2\delta$. Contradiction.

Combining the previous paragraphs we see that $\{B_{m}(q_i, \lambda + 2\delta)\}_{i=1}^k$ covers $Y(r\lambda^{-1} - 2\delta)$. Thus

$$M(\lambda + 2\delta, Y(r\lambda^{-1} - 2\delta)) \leq M(1, X(r)) \quad \text{for all} \quad r > 0.$$

Applying the first claim of Fact 2.1 we obtain a constant $L > 0$ depending only on $m$ such that

$$LM(1, Y(r\lambda^{-1} - 2\delta)) \leq M(\lambda + 2\delta, Y(r\lambda^{-1} - 2\delta))$$

hence

$$LM(1, Y(r\lambda^{-1} - 2\delta)) \leq M(1, X(r)).$$

Taking logarithms of of both sides of the expression above, dividing both sides by $\log(r)$, and taking the limit as $r \rightarrow \infty$ we see that $\dim_{CM}(Y) \leq \dim_{CM}(X)$. □
3. Proof of Theorem 1.1

Let $S$ be the unit circle in $\mathbb{R}^2$. Given $u \in S$ we let $T_u : \mathbb{R}^2 \to \mathbb{R}$ be the orthogonal projection parallel to $u$, i.e., $T_u$ is the orthogonal projection such that $T_u(x) = T_u(y)$ if and only if $x - y = tu$ for some $t \in \mathbb{R}$. For our purposes a double wedge around $u \in S$ is a subset of $\mathbb{R}^2$ of the form

$$C_{s,\varepsilon}^u := \{tv : t \in \mathbb{R}, |t| > s, v \in S, \|v - u\| < \varepsilon \}$$

for some $s, \varepsilon > 0$.

**Lemma 3.1.** Let $F$ be a nonempty subset of $\mathbb{R}^2$ and $u \in S$. If $F \cup \{x - y : x, y \in F \}$ is disjoint from some double wedge around $u$ then the restriction of $T_u$ to $F$ is a quasi-isometric embedding $F \to \mathbb{R}$.

Lemma 3.1 is a quasi-isometric version of a well-known fact from geometric measure theory: if $F$ is a nonempty subset of $\mathbb{R}^2$ such that $F \cup \{x - y : x, y \in F \}$ is disjoint from a double wedge of the form $C_{s,\varepsilon}^u$, then the restriction of $T_u$ to $F$ is a bilipschitz embedding $F \to \mathbb{R}$. This fact is applied in [1, 5].

**Proof.** Suppose that $F \cup \{x - y : x, y \in F \}$ is disjoint from $C_{s,\varepsilon}^u$. As $T_u$ is an orthogonal projection we have $\|T_u(x) - T_u(x')\| \leq \|x - x'\|$ for all $x, x' \in \mathbb{R}^2$, so it suffices to obtain a lower bound on $\|T_u(x) - T_u(x')\|$ of the appropriate form.

After making a change of coordinates if necessary we suppose $u = (0,1)$ so that $T_u(x,y) = x$ for all $(x,y) \in \mathbb{R}^2$. Then we have

$$C_{s,\varepsilon}^u = \{(x, y) \in \mathbb{R}^2 : |y| > \lambda |x| \quad \text{and} \quad \|(x, y)\| > s \}$$

for some $\lambda > 0$ depending only on $\varepsilon$. Thus, if $(x, y) \in F - F$ then either $\|(x, y)\| < s$ or $|y| \leq \lambda |x|$. Equivalently, for all $(x, y), (x', y') \in F$ we either have

$$\|(x, y) - (x', y')\| < s \quad \text{or} \quad |y-y'| \leq \lambda |x-x'|.$$

In the latter case we have

$$\|(x, y) - (x', y')\| \leq |x-x'| + |y-y'| \leq (1+\lambda) |x-x'|$$

hence

$$\frac{1}{1+\lambda} \|(x, y) - (x', y')\| \leq |x-x'|.$$

In the first case we have

$$\|(x, y) - (x', y')\| - s < |x-x'|.$$

So for all $(x, y), (x', x') \in F$ we have

$$\frac{1}{1+\lambda} \|(x, x') - (y, y')\| - s \leq |x-x'|.$$

So the restriction of $T_u$ to $F$ is a quasi-isometric embedding $F \to \mathbb{R}^2$. □

We let $\mathbb{H}$ be the upper half plane $\{(x, y) \in \mathbb{R}^2 : y > 0 \}$ and let $S^+ = S \cap \mathbb{H}$. A wedge in $\mathbb{H}$ around $u \in S^+$ is a set of the form

$$C_{s,\varepsilon}^{u,+} := \{tv : t \in \mathbb{R}, t > s, v \in S, \|v - u\| < \varepsilon \}$$

such that $C_{s,\varepsilon}^{u,+} \subseteq \mathbb{H}$.

**Lemma 3.2.** Suppose $F \subseteq \mathbb{H}$ intersects every wedge in $\mathbb{H}$. Then there is a $u \in S^+$ such that $T_u(F)$ is dense.

The reader may find that drawing a few pictures greatly assists in comprehending the proof of Lemma 3.2. We let $p = (-1,0)$ and $o = (0,0)$. Note that if $z \in \mathbb{H}$, $q$ is a positive real number, and $u \in S^+$, then $T_u(z) = q$ if and only if $\angle pouch = \angle pqz$.

**Proof.** We show that the set of $u \in S^+$ such that $T_u(F)$ is dense in $\mathbb{R}$ is comeager in $S^+$. It suffices to show that

$$\{u \in S^+ : T_u(F) \cap I \neq \emptyset \}$$

is open and dense in $S^+$ for every nonempty open interval $I$ with rational endpoints. Fix a nonempty open interval $I = (q_1, q_2)$ with rational endpoints. We suppose that $q_1, q_2 > 0$ for the sake of simplicity, the more general case follows by trivial modifications of our argument. The map $T : S^+ \times \mathbb{R}^2 \to \mathbb{R}$ given by $T(u, x) = T_u(x)$ is continuous. Thus if $T_u(x) \in I$ then $T_u(x) \in I$ for all $v \in S^+$ sufficiently close to $u$. It follows that the set of $u$ such that $T_u(F) \cap I \neq \emptyset$ is open in $S^+$.

It now suffices to show that the set of $w \in S^+$ such that $T_w(F) \cap I \neq \emptyset$ is dense in $S^+$. Fix $u, v \in S^+$ such that $\angle pouch < \angle pov$ and let $J$ be the set of $w \in S^+$ such that $\angle pou < \angle pov < \angle pov$. We show there
is a $w \in J$ such that $T_w(F) \cap I \neq \emptyset$. Let $r_1, r_2 \in \mathbb{H}$ be such that $\angle pq_1r_1 = \angle pou$ and $\angle pq_2r_2 = \angle pov$. Let $D$ be the set of points in $\mathbb{H}$ that lie in between the rays $\overline{q_1r_1}$ and $\overline{q_2r_2}$. It is easy to see that

$$D = \bigcup_{q \in I} \{ r \in \mathbb{H} : \angle pou < \angle qrr < \angle pov \} = \bigcup_{q \in I} \bigcup_{w \in J} T_w^{-1}(\{q\}) = \bigcup_{w \in J} T_w^{-1}(I).$$

It therefore suffices to show that $D$ intersects $F$. Let $z_1, z_2 \in \mathbb{S}$ be such that

$$\angle pou < \angle poz_1 < \angle poz_2 < \angle pov.$$

As $\angle pq_1r_1 < \angle poz_1 < \angle poz_2 < \angle pq_2r_2$, we see that every element of $\overline{oz_1}$ or $\overline{oz_2}$ sufficiently far from the origin lies in $D$. It follows that there is a $t > 0$ such that

$$W := \{ z \in \mathbb{H} : \|z\| \geq t, \angle poz_1 < \angle poz < \angle poz_2 \} \subseteq D.$$

Then $W$ is a wedge in $\mathbb{H}$ and so contains an element of $F$. Thus $D$ contains an element of $F$. \qed

Lemma 3.3. Suppose $E \subseteq \mathbb{R}$. Then one of the following holds:

1. there is a $u \in \mathbb{S}$ such that the restriction of $T_u$ to $E^2$ is a quasi-isometric embedding $E^2 \to \mathbb{R}$,
2. there is a linear $S : \mathbb{R}^4 \to \mathbb{R}$ such that $S(E^4)$ is dense.

Proof. Consider $E^2 - E^2 \subseteq \mathbb{R}^2$. If $E^2 - E^2$ is disjoint from a double wedge in $\mathbb{R}^2$ then Lemma 3.2 shows that some $T_u$ quasi-isometrically embeds $E^2$ into $\mathbb{R}$.

Suppose $E^2 - E^2$ intersects every double wedge in $\mathbb{R}^2$. Note that if $(x, y) \in E^2 - E^2$ then $(-x, -y)$ is also an element of $E^2 - E^2$. It is easy to see that this implies that $E^2 - E^2$ intersects every wedge in $\mathbb{H}$. Applying Lemma 3.3 we fix a $u \in \mathbb{S}$ such that $T_u(E^2 - E^2)$ is dense. Let $S : \mathbb{R}^4 \to \mathbb{R}$ be the linear function given by

$$S(x, y, x', y') = T_u(x - x', y - y') \quad \text{for all} \quad x, y, x', y' \in \mathbb{R}.$$

Then $S(E^4)$ is dense. \qed

We now prove Theorem 1.1.

Proof. Suppose towards a contradiction that $E \subseteq \mathbb{R}$ has positive coarse Minkowski dimension and $T(E^n)$ is not dense for every $n \in \mathbb{N}$ and linear $T : \mathbb{R}^n \to \mathbb{R}$. We may suppose that $0 \in E$. Let $S$ be the collection of sets of the form $T(E^n)$ for linear $T : \mathbb{R}^n \to \mathbb{R}$. It is easy to see that if $F \in S$ and $T : \mathbb{R}^n \to \mathbb{R}$ is linear then $T(F^n)$ is also in $S$. We let $s$ be the supremum of the coarse Minkowski dimensions of members of $S$. Every element of $S$ has coarse Minkowski dimension $\leq 1$, so $s$ exists and $s \leq 1$. As $\dim_{CM}(E) > 0$ we have $s > 0$. Let $F \in S$ be such that $\dim_{CM}(F) > \frac{1}{2}s$. An application of Lemma 3.3 yields a linear $T : \mathbb{R}^2 \to \mathbb{R}$ such that the restriction of $T$ to $F^2$ is a quasi-isometric embedding $F^2 \to \mathbb{R}$. Lemma 2.3 and Fact 2.2 together show that

$$\dim_{CM}T(F^2) = \dim_{CM}(F^2) = 2 \dim_{CM}(F) > s.$$

But $T(F^2) \in S$, contradiction. \qed

4. A COROLLARY IN $\mathbb{R}^n$

We prove a higher dimensional version of the second claim of Theorem 1.4. (Recall that a closed subset of $\mathbb{R}^n$ has topological dimension zero if and only if it is nowhere dense.)

Theorem 4.1. Suppose $Z$ is a closed subset of $\mathbb{R}^n$ with topological dimension zero. If $Z$ has positive coarse Minkowski dimension then $(R_{\text{vec}}, Z)$ defines all bounded Borel subsets of all $\mathbb{R}^n$.

Proof. We suppose that $(R_{\text{vec}}, Z)$ does not define all bounded Borel subsets of all $\mathbb{R}^n$ and show that $\dim_{CM}(Z) = 0$. Given $1 \leq k \leq n$ we let $\pi_k : \mathbb{R}^n \to \mathbb{R}$ be given by

$$\pi_k(x_1, \ldots, x_n) = x_k \quad \text{for all} \quad (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

An application of [2, Theorem D, Theorem E] shows that $\pi_k(Z)$ is nowhere dense for all $1 \leq k \leq n$. Theorem 1.4 shows that $\dim_{CM} \pi_k(Z) = 0$ for all $1 \leq k \leq n$. As $Z$ is a subset of $\pi_1(Z) \times \ldots \times \pi_n(Z)$ repeated application of Fact 2.2 shows that $\dim_{CM}(Z) = 0$. \qed
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Department of Mathematics, Statistics, and Computer Science, Department of Mathematics, University of California, Irvine, 340 Rowland Hall (Bldg. # 400), Irvine, CA 92697-3875
E-mail address: ewalsber@uci.edu
URL: http://www.math.illinois.edu/~erikw