Chiral fermion action
with (8,0) worldsheet supersymmetry

E. Ivanov\textsuperscript{(a)} and E. Sokatchev\textsuperscript{(b)}

\textsuperscript{(a)}Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna near Moscow, Russia
\textsuperscript{(b)}Physikalisches Institut, Universität Bonn, Nussallee 12, D-53115 Bonn, Germany

Abstract

We propose an action describing chiral fermions with an arbitrary gauge group and with manifest (8,0) worldsheet supersymmetry. The form of the action is inspired by and adapted for completing the twistor-like formulation of the $D = 10$ heterotic superstring.
1. Introduction. Chiral fermions are a necessary ingredient of the self-consistent quantum heterotic superstring theory. They serve to compensate the conformal anomaly and ensure that the critical dimension of space-time is 10 [1].

Recently various twistor-like formulations of the $D = 10$ heterotic string at the classical level have been proposed [2, 3, 4]. The basic attractive feature of twistor-like formulations of the heterotic superstring [1] is the trading of the local $\kappa$-symmetry of the $D = 10$ superstring action for $N = (8,0)$ local supersymmetry of the worldsheet. The twistor-like mechanism was first discovered by Sorokin et al [7] in the context of a $D = 3,4$ superparticle with local $N = 1,2$ worldline supersymmetry and then generalized to the $D = 6$ [8] and $D = 10$ [9] superparticle with $N = 4$ and $N = 8$ worldline supersymmetry and to heterotic superstrings in $D = 4$ [10] and $D = 6$ [12] with $N = 2,4$ worldsheet supersymmetry (see also [11]). It unveiled for the first time the transparent geometric meaning of $\kappa$-symmetry. The fact that all symmetries of the superstring action become manifest and geometrically interpretable within a twistor-like formulation gives us hope that the latter is most appropriate for covariant quantization of the heterotic superstring.

Keeping in mind this ambitious though still remote prospect, one may wonder how to consistently incorporate, to begin with at the classical level, the chiral fermions into the twistor-like formulation of the heterotic superstring while preserving all symmetries and geometric features of the latter. An early attempt in this direction was made by Tonin [2]. There the chiral fermion superfield equation of motion was introduced into the action with a Lagrange multiplier. Unfortunately, the latter turned out to propagate unwanted degrees of freedom. Another approach to this problem was proposed by Sorokin and Tonin [13]. This time the field equations of the chiral fermions are derived from a quadratic $(8,0)$ superfield constraint, once again added to the superstring action with a Lagrange multiplier. To make the derivation unambiguous and to avoid the propagation of extra degrees of freedom, they have to restrict themselves only to supersymmetric solutions of their constraint and to resort to a rather subtle positiveness argument.

A different $(8,0)$ action for chiral fermions has recently been proposed by Howe [14]. It is a standard bilinear superfield action with a specially chosen operator made out of up to 7 spinor derivatives. The action has a large abelian gauge invariance which helps to restrict the on-shell content of the theory to just chiral fermions. The superfields incorporating these fermions must satisfy a certain constraint which effectively ties up the internal symmetry index on the fermion fields to the (local) $SO(8)$ symmetry group of the worldsheet. As remarked in [14], this results in a severe restriction on the internal symmetry group (it should not be bigger than $SO(4)$).

In the present letter we propose another chiral fermion action with local $(8,0)$ worldsheet supersymmetry and with an arbitrary gauge group. It is a modification of the one of Sorokin and Tonin [13]. Like in [13], an appropriate superfield terms is included into the Wess-Zumino term of the twistor superstring action sharing the same Lagrange multiplier superfield. As shown in [3], after fixing a certain gauge this Lagrange multiplier is reduced just to the string coupling constant. Then the new term produces the standard kinetic term for chiral fermions as well as an auxiliary field term. We stress that the chiral fermion modification of the superstring Wess-Zumino term we propose is essentially different from that in [13]. It is chosen so that requiring compatibility with the standard constraints on

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1. Recently several attempts have been made to adapt the twistor-like approach to non-heterotic superstrings [6] or $p$-branes [5], but in our opinion it is still not entirely clear to what extent these formulations are self-consistent.
the background supergravity three-form does not imply equations of motion for the chiral fermions (as it does in [13]). Instead, the integrability condition results in a quadratic constraint on the chiral fermions superfields which simply reduces the off-shell component content of the superfields (quite similarly to standard linear irreducibility constraints). This constraint turns out to be also necessary for the off-shell local worldsheet supersymmetry of the chiral fermion action. The realization of off-shell supersymmetry in this action has some unusual features which we discuss in detail.

2. Preliminaries. We start by briefly recalling the structure of the heterotic superstring action in the twistor-like formulation as it has been given in [3]. The action consists of three parts, each of them given by an integral over the \((8,0)\) worldsheet superspace

\[
S = S_1 + S_2 + S_{\text{wz}}. 
\]

For our purposes it will be important to know the explicit form of \(S_{\text{wz}}\)

\[
S_{\text{wz}} = \int d^2x d^8\theta P^{MN} \left[ B_{MN} + E_\alpha^+ E_{\beta N}^- \phi_{\alpha} E_{\beta}^a - \partial_{[M} Q_{N]} \right].
\]

Here and in what follows non-underlined indices refer to the worldsheet superspace \(\{z^M\} = \{x^-, x^+, \theta^\mu\}, \mu = 1, \ldots, 8\), i.e. \(M = -, +, \mu\), while underlined ones refer to the target \(D = 10\) superspace \(z^M\). For instance, \(\underline{a} = 0, 1, \ldots, 9\) is the vector tangent space index. The symbol \([MN]\) means graded antisymmetrization. The quantities \(B_{MN}\) and \(E_\pm \) are, respectively, the worldsheet superspace pull-backs of the background \(D = 10\) supergravity two-form \(B_{MN}\) and vielbeins \(E_{\underline{M}}^A\)

\[
B_{MN} = (-)^{M(N+N)} \partial_N z^N \partial_M z^M B_{MN}, \quad E_{\underline{M}}^A = E_{\underline{M}}^A \partial_M z^M E_{\underline{A}}^M,
\]

\(E_{\underline{M}}^\pm\) are elements of the vielbein and \(E_{\underline{M}}\) of the inverse vielbein matrix on the worldsheet, \(\Phi\) is the dilaton field of background supergravity, \(Q_{N}\) and \(P^{MN}\) are worldsheet superfield Lagrange multipliers, whose rôle will be explained later on.

The term \(S_1\) in (1) produces the so called geometro-dynamical constraint

\[
E_{\underline{M}}^\alpha \equiv E_{\underline{M}}^\alpha \partial_M z^M E_{\underline{A}}^\alpha = 0
\]

on the spinor pull-back of the vector target superspace vielbeins. It has as a corollary a twistor representation for the left-handed Virasoro vector

\[
E_{\underline{\alpha}} = \frac{1}{8} E_{\underline{\alpha}}^\alpha \gamma_{\underline{\alpha} \underline{\beta}} E^\beta_{\underline{\alpha}}
\]

which provides a twistor-like solution to the left-handed Virasoro constraint \(E_{\underline{\alpha}} E_{-\underline{\alpha}} = 0\) of the superstring. The term \(S_2\) in (1) enforces an irreducibility condition on the spinor component \(e^+_{\underline{\alpha}}\) of the worldsheet inverse vielbein (see eq. (16) below); the latter covariantizes the worldsheet superspace derivatives with respect to arbitrary shifts of the right-handed coordinate \(x^+\) and thus provides the Lagrange multiplier for the right-handed Virasoro constraint. More details on these two terms can be found in [3].

The term (2) is the leading one in the twistor superstring action, since it produces the entire superstring component action. It is instructive to discuss in some detail how this works, because we will use the same mechanism for generating the chiral fermion component action.
The structure of $S_{wz}$ is entirely specified by the requirement that the result of varying with respect to the Lagrange multiplier $P_{MN}$,

$$B_{MN} = \partial_{[M}Q_{N]} - E_{[M}^aE_{N]}^\phi e^{a\phi}\partial^{\underline{a}+} E_{+}$$

(6)
is consistent with the $D=10$ supergravity constraints on the three-form field strength $H_{KMN}$ of the two-form $B_{MN}$ [15]. This means that eq. (6) implies the integrability condition

$$H_{KMN} \equiv \partial_{[K}B_{MN]} = -\partial_{[K}(E_{M}^aE_{N]}^\phi e^{a\phi}\partial^{\underline{a}+} E_{+})$$

(7)

Using the geometro-dynamical constraint (4) and its corollary (5), one can show that equation (7) is consistent with the $D=10$ background supergravity constraints and is actually equivalent to them. When (7) holds the action term $S_{wz}$ is invariant under the following gauge transformation

$$\delta P_{MN} = \partial_K \Sigma^{MNK},$$

(8)

where $\Sigma^{KMN}(z)$ is a totally (graded) antisymmetric superfield.

To see how (2) produces the standard component heterotic superstring action, one varies with respect to the Lagrange multiplier $Q_M$ and obtains

$$\partial_N P_{MN} = 0.$$  

(9)

The general solution of this equation is given by

$$P_{MN} = \partial_K \Lambda^{MNK} + \theta^8 \delta_+^{[M} \delta_-^{N]} T.$$  

(10)

The second (cohomological) term contains the constant $T$,

$$\partial_+ T = \partial_- T = 0.$$  

(11)

Then one substitutes this solution back into $S_{wz}$ and observes that the first term in (10) can be completely gauged away using the gauge freedom (8). As a result in this gauge in $P_{MN}$ there survives only one component

$$P^{+-} = \theta^8 T$$

(12)

and $S_{wz}$ (2) is reduced to the ordinary $x$-space integral of the first component of the superfield expression within the square brackets. It contains, in particular, the superstring WZ term

$$S_{wz} = T \int d^2x \partial_- z^M \partial_+ z^N B_{MN} + ...,$$  

(13)
as well as the superstring kinetic term. The constant $T$ is naturally interpreted as the superstring tension.

To close this introductory section, we present the worldsheet gauge group (including local $(8,0)$ supersymmetry) in the twistor formulation. The whole action (1) and in particular $S_{wz}$ are invariant under the following restricted class of diffeomorphisms of the $(8,0)$ superspace

$$\delta \theta_\alpha = -\frac{i}{2} D_\alpha \Lambda^-,$$

$$\delta x^- = \Lambda^- - \frac{1}{2} \theta_\alpha D_\alpha \Lambda^-,$$

$$\delta x^+ = \Lambda^+,$$  

(14)
with \( \Lambda^\pm (z) \) being arbitrary unconstrained superfield parameters. This group is chosen so that it leaves the “almost flat” derivatives

\[
\begin{align*}
D_\alpha &= \partial_\alpha + i \theta_\alpha \partial_- + e_\alpha^+ \partial_+
\quad \\
D_- &= \partial_- - \frac{i}{8} D_\alpha e_\alpha^+ \partial_+
\quad \\
D_+ &= \partial_+
\end{align*}
\]

(15)
covariant. In (15) the single non-trivial component of the worldsheet superspace inverse vielbeins (defined by \( D_A = E^M_A \partial_M \)) is \( e_\alpha^+ \). It is needed to ensure covariance under the shifts of \( x^+ \) and obeys the constraint (\{\}) denotes the symmetric traceless part

\[
D_{\{\alpha e_\beta^+} = 0,
\]

(16)
so that the covariant derivatives \( D_\alpha, D_- \) still form the flat algebra

\[
\{D_\alpha, D_\beta\} = 2i \delta_{\alpha\beta} D_- , \quad [D_\alpha, D_-] = 0 .
\]

(17)
The constraint (16) is produced by the term \( \mathcal{S}_2 \) in the superstring action (1).

From the explicit form of the covariant derivatives \( D_\alpha, D_- \) above one can read off the form of the worldsheet zweibein matrix. In what follows we shall make use of the vielbeins \( E^\pm_M \) entering the WZ term (2):

\[
\begin{align*}
E^+_\mu &= 1 , \quad E^+_\mu = \frac{i}{8} D_\alpha e_\alpha^+ , \quad E^+_{\mu} = - \frac{1}{8} \theta_\mu D_\alpha e_\alpha^+ ,
\quad \\
E^-_\mu &= 0 , \quad E^-_\mu = 1 , \quad E^-_{\mu} = -i \theta_\mu .
\end{align*}
\]

(18)
Finally, we give the transformation laws of the covariant derivatives and \( e_\alpha^+ \) under the restricted diffeomorphism group (14)

\[
\begin{align*}
\delta D_\alpha &= \frac{i}{2} (D_\alpha D_\beta) \Lambda^- D_\beta \\
\delta D_- &= -(D_- \Lambda^-) D_- + \frac{i}{2} (D_- D_\alpha \Lambda^-) D_\alpha \\
\delta D_+ &= - \left(D_+ \Lambda^+ + \frac{i}{2} (D_+ D_\alpha \Lambda^-) e_\alpha^+ + \frac{i}{8} (D_+ \Lambda^-) D_\alpha e_\alpha^+ \right) D_+ \\
&\quad - (D_+ \Lambda^-) D_- + \frac{i}{2} (D_+ D_\alpha \Lambda^-) D_\alpha \\
\delta e_\alpha^+ &= D_\alpha \Lambda^+ + \frac{i}{2} D_\alpha D_\beta \Lambda^- e_\beta^+ .
\end{align*}
\]

(19)

3. Chiral fermion action. Let us now turn to our basic task. We shall identify the chiral fermions with the first components of some anticommuting \((8, 0)\) superfields \( \Psi^i(z) \). The index \( i \) is in general arbitrary, but for definiteness we shall regard it as a vector \( SO(n) \) index, for instance as the vector index of \( SO(32) \). The action which we propose for this superfield will be the following addition to the superstring WZ term (14)

\[
S_f = \int d^2 x d^8 \theta \left\{ P^{MN} E^+_M (i \partial_N) \Psi^i \Psi^i + \frac{1}{8} E^{-}_N \nabla_\alpha \Psi^i \nabla_\alpha \Psi^i + P^{(\alpha\beta)} \nabla_\alpha \Psi^i \nabla_\beta \Psi^i \right\} .
\]

(20)
Here $P^{MN}$ is the same Lagrange multiplier as in the superstring action (2) and $P^{(a\beta)}$ is a new symmetric traceless Lagrange multiplier, whose meaning will be clarified later on. The covariant derivative $\nabla_{\alpha}$ is defined as follows:

$$\nabla_{\alpha} = D_{\alpha} - (\partial_+ e^+_{\alpha}) w .$$

(21)

The operator $w$ measures the weight with respect to local dilatations with the superfield parameter $D_+\Lambda^+ + \ldots$ (see the first term in the transformation law $\delta D_+$(19)). As follows from (19), the derivative $D_+$ has weight $-1$, so the vielbein $E^+_M$ present in (21) has weight $+1$. Thus, to achieve invariance of (20) we should ascribe to the superfield $\Psi^j$ the weight $-1/2$. The derivative (21) is covariant when acting on a superfield with an arbitrary weight $w$. It still satisfies the flat algebra (17) together with the modified covariant $x^-$ derivative (its own weight is zero)

$$\nabla_- = D_- + \frac{i}{8} \partial_+(D_+ e^+_{\alpha}) w .$$

(22)

Note that the term with $\partial_N\Psi$ in (20) is covariant as it stands, since the superfields $\Psi^i$ anticommute.

In the superstring action the internal symmetry group of the chiral fermions, e.g., $SO(32)$ is gauged. Therefore one needs to add the pull-back of the $D = 10$ super-Yang-Mills connection $E^M_A(z)$ to the covariant derivative (21). However, this does not affect the algebra of the derivatives, provided the connection $A^M(z)$ satisfies the constraints of $D = 10$ super-Yang-Mills theory (14). Therefore, for simplicity in what follows we shall omit the gauge field term in $\nabla_{\alpha}$.

Now we are ready to explain why we propose the chiral fermion superfield action just in the form (20). First of all, we wish to produce the standard component kinetic term of the chiral fermions

$$\sim \bar{\psi}^j \partial_- \psi^j , \quad \bar{\psi}^j = \Psi^j|_{\theta = 0} ,$$

using the same mechanism as in the case of the superstring WZ term, i.e. by passing to the gauge (12). This accounts for the term $\partial_N\Psi^j\Psi^i$ in (20). It is of the same form as the one proposed in (13). However, the second $\Psi$ term in (20) was not included in the action in (13). The fact that it should be there follows from two requirements:

(i) the integrability condition (7), which is equivalent to the $D = 10$ supergravity constraints, should not be affected by adding the action (20) to the original superspace WZ term (2);

(ii) the constraint on the superfields $\Psi^i$ following from requirement (i) should not put the theory on shell. Precisely this point accounts for the main difference between our action and that of (13).

Let us first explain how requirement (i) is satisfied. In the presence of the new action term (20) varying with respect to $P^{MN}$ leads to modifications of the two- and three-forms in eq. (13) and its consequence (17). We denote them by $\hat{B}_{MN}$ and $\hat{H}_{KMN}$, respectively:

$$B_{MN} \Rightarrow B_{MN} + \hat{B}_{MN} , \quad H_{KMN} \Rightarrow H_{KMN} + \hat{H}_{KMN} , \quad \hat{H}_{KMN} = \partial_{[K} \hat{B}_{MN]} .$$

(23)

So, we require

$$\hat{H}_{[KMN]} = 0 .$$

(24)
To find the constraint on $\Psi$ following from (24), we replace for the moment the expression $\nabla^\alpha \Psi^i \nabla_\alpha \Psi^i$ in (20) by an unknown $X$ and then write down

$$\hat{B}_{MN} = E_M^\dagger (i \partial_N \Psi^i \bar{\Psi}^i + \frac{1}{8} E_N^- X) - (-)^{MN} (M \leftrightarrow N).$$

(25)

Using the explicit form of the zweibein matrix (18) it is straightforward though a bit tedious to compute all the components of $\hat{H}_{KMN}$ and to see that the necessary and sufficient condition for them to vanish is

$$\nabla^\alpha \Psi^i \nabla_\beta \Psi^i - \frac{1}{8} \delta_{\alpha\beta} X = 0 \Rightarrow X = \nabla^\alpha \Psi^i \nabla_\alpha \Psi^i.$$

(26)

Let us give explicitly several components of $\hat{H}_{KMN}$

$$\hat{H}_{\alpha\beta\gamma} = -2i E_\alpha^\dagger \left[ (\nabla^\beta \Psi^i \nabla_\gamma \Psi^i - \frac{1}{8} \delta_{\beta\gamma} X) - \frac{1}{16} \theta_\beta (16i \nabla_\gamma \Psi^i \bar{\Psi}^i - \nabla_\gamma X) \right] + \text{symmetrization}$$

(27)

$$\hat{H}_{+\alpha\beta} = -2i (\nabla^\alpha \Psi^i \nabla_\beta \Psi^i - \frac{1}{8} \delta_{\alpha\beta} X) + i \frac{1}{8} \theta_\alpha (16i \nabla_\beta \Psi^i \bar{\Psi}^i - \nabla_\beta X) + (\alpha \leftrightarrow \beta)$$

(28)

$$\hat{H}_{\alpha+-} = \frac{1}{8} (\nabla_\alpha X - 16i \nabla_- \Psi^i \nabla_\alpha \Psi^i).$$

(29)

From the vanishing of the last component it follows that

$$16i \nabla_- \Psi^i \nabla_\beta \Psi^i - \nabla_\beta X = 0,$$

(30)

then from the vanishing of the second one follows the constraint (26). Eq. (30) is in fact a corollary of (24).

The most essential difference from the approach of Sorokin and Tonin becomes clear at this point. They do not include the $X$ term in the superstring WZ action and so from the same consistency condition (24) they derive the constraint

$$\nabla^\alpha \Psi^i \nabla_\beta \Psi^i = 0.$$

(31)

It is much stronger than (26) in that it implies the equations of motion for $\Psi^i$. Below we will show that our constraint (26) is purely kinematical and serves to reduce the off-shell field content of $\Psi^i$.

We would also like to comment on the approach of Howe [14]. His action is based on a linear constraint on the chiral fermion superfield $\Psi^i q$:

$$D_\alpha \Psi^i q = (\gamma^i)_{\alpha a} P^{\dot{\alpha}} q.$$

(32)

Here the $SO(32)$ index is split into an $SO(8)$ index $i'$ and an $SO(4)$ index $q$. The indices $\alpha$ and $\dot{\alpha}$ are $s$ and $c$ spinor indices of $SO(8)$ and $\gamma^i$ is the $SO(8)$ gamma matrix. It should be pointed out that eq. (32) provides a solution (but not the general one) to our constraint (26) for this particular arrangement of the indices. The advantage of the constraint (32) is its linearity, but its serious drawback (noted in [14] as well) is the strong restriction on

6
the chiral fermion gauge group, which is at most $SO(4)$ (or a tensor product of several $SO(4)$’s).

Coming back to our action (20), it remains to explain the rôle of the second Lagrange multiplier term in it. On the one hand, it serves to impose the constraint (26) on shell. On the other hand, its presence helps to maintain the gauge freedom (8) which allows one to bring $P_{MN}$ to the form (12). The gauge variation of the $P_{MN}$ term in (20) is proportional to the constraint (26), so it can be compensated by an appropriate variation of the new Lagrange multiplier $P^{(\alpha\beta)}$. The explicit form of this variation is not too enlightening, so we do not give it here. Note that $P^{(\alpha\beta)}$ has its own gauge freedom

$$\delta P^{(\alpha\beta)} = \nabla_\gamma \Lambda^{(\gamma\alpha\beta)},$$

which allows one to gauge away some components of this superfield.

The meaning of the constraint (26) and how it helps to make the action (20) supersymmetric in the gauge (12) will be discussed in the next Section. In this gauge (20) takes the form

$$S_f = T \int d^2 x d^8 \theta \left\{ \theta^8 \left( i \nabla_\gamma \Psi^i \Psi^i + \frac{1}{8} \nabla_\alpha \Psi^i \nabla_\alpha \Psi^i + P^{(\alpha\beta)} \nabla_\alpha \Psi^i \nabla_\beta \Psi^i \right) \right\}. \quad (34)$$

4. Peculiarities of the chiral fermion action. Here we discuss some unusual features of the action (34) as it stands, leaving aside its superstring-inspired appearance. It is worth mentioning that though we have introduced it in the framework of the $D = 10$ heterotic superstring and of an $(8,0)$ worldsheet superspace, its basic features actually do not depend on the Grassmann dimension of the worldsheet superspace. So we may consider a general $(N,0)$ superspace, which corresponds to the change $8 \rightarrow N$ in (34). We assume that the $\theta$’s are always real and transform, in general, according to the vector representation of the automorphism group $SO(N)$ \footnote{In the case $N = 8$ (as well as $N = 4, 2, 1$), due to the triality property of $SO(8)$, the $\theta$’s can equally well be placed in either the $s$ or $c$ spinor representations of $SO(8)$. This is in fact necessary if one wants to fix a light-cone gauge in which the worldsheet $\theta$’s are identified with one half of the target superspace ones (see \cite{3} for details).}. We also assume that the derivatives $\nabla_\alpha, \nabla_-$ in (34) are flat, i.e. we put $e^+ = 0$. Actually, one can always choose a gauge in which all covariant derivatives are almost flat \cite{12}, \cite{3},

$$e^+_\alpha = i \theta_\alpha g_{--}, \quad D_- = \partial_- + g_{--} \partial_+, \quad (35)$$

where $g_{--}$ is the component of the worldsheet metric responsible for the second Virasoro constraint. For simplicity we put $g_{--}$ equal to zero (which is only possible locally), although this is not essential for what follows.

First we demonstrate that the action

$$S = T \int d^2 x d^N \theta \left\{ \theta^N \left( i \partial_- \Psi^i \Psi^i + \frac{1}{N} D_\alpha \Psi^i D_\alpha \Psi^i \right) + P^{(\alpha\beta)} D_\alpha \Psi^i D_\beta \Psi^i \right\} \quad (36)$$

is supersymmetric despite the presence of explicit $\theta$’s. The easiest way to see this is to observe that

$$i \partial_- \Psi^i \Psi^i + \frac{1}{N} D_\alpha \Psi^i D_\alpha \Psi^i = \frac{1}{N} D_\alpha (\Psi^i D_\alpha \Psi^i) \quad (37)$$
then to integrate in both terms of the action by parts and, finally, to rewrite it in a very simple form resembling Chern-Simons type actions \[17\]

\[
S = T \int d^2 xd^N \theta P^\alpha \Psi^i D_\alpha \Psi^i
\]  

(38)

with

\[
P^\alpha = \theta^{N-1}_\alpha + D_\beta P^{(\beta \alpha)} , \quad \theta^{N-1}_\alpha \equiv D_\alpha \theta^N
\]  

(39)

(actually, when passing from (36) to (38), there appears a minus sign in the case of an odd \(N\), but it can be absorbed into the constant \(T\)).

At this point we can forget the particular structure of \(P^\alpha\) (39) coming from the original definition of the chiral fermion action. Instead, we can define the action by eq. (38) with \(P^\alpha\) satisfying the constraint

\[
D_\alpha P^\alpha = 0 .
\]  

(40)

It is easy to check that (39) provides the general solution to this constraint up to an arbitrary right-moving function \(a(x^+)\) in front of the first (cohomological) term in (39). Note, however, that the action (38) and the constraint (40) are invariant under \(x^+\) dependent scale transformations

\[
\Psi^i \rightarrow \lambda^{1/2}(x^+) \Psi^i , \quad P^\alpha \rightarrow \lambda^{-1}(x^+) P^\alpha
\]  

(41)

which allow us to gauge \(a(x^+)\) into any non-zero constant. So, the action (38) with the additional constraint (40) describe the general situation.

In this Chern-Simons-like representation the chiral fermion action does not include explicit \(\theta\)'s (they are hidden in the solution of the constraint (40)) and is invariant (together with the constraint) with respect to the local transformations (14) with \(\Lambda^+ = \Lambda^+(x^+)\), provided \(P^\alpha\) transforms according to the law

\[
\delta P^\alpha = -\frac{i}{2} (D_\beta D^\alpha \Lambda^-) P^\beta + \frac{1}{2} (N-2)(\partial_- \Lambda^-) P^\alpha.
\]  

(42)

The second term in (42) cancels the variation of the superspace integration measure in (38).

Now let us discuss some peculiar features of the realization of rigid supersymmetry in the above action. To this end we will go to components by varying in (36) with respect to the Lagrange multiplier \(P^{(\alpha \beta)}\), substituting the constraint back into the action (this will be justified below) and finally integrating over the \(\theta\)'s. The \(\theta\) integration is now trivial because of the presence of the factor \(\theta^N\). In this way we obtain that for an arbitrary \(N\) the chiral fermions are described off shell by the action

\[
S_f = T \int d^2 x (i \partial_- \bar{\Psi}^i \Psi^i + \frac{1}{N} b_i^\alpha b_i^\alpha) ; \quad b_i^\alpha \equiv D_\alpha \Psi^i |_{\theta=0} ,
\]  

(43)

supplemented by the nonlinear superfield constraint

\[
D_\alpha \Psi^i D_\beta \Psi^i - \frac{1}{N} \delta_{\alpha \beta} D_\gamma \Psi^i D_\gamma \Psi^i = 0 .
\]  

(44)

The first unusual feature of the action (43) is that, irrespective of the value of \(N\), it involves only the two first fields from the \(\theta\) expansion of \(\Psi^i\). Nevertheless, it is off-shell supersymmetric! To see this, we first write down explicitly the first two component
constraints following from the superfield one (44)

\[ b^i_{\{\alpha} b^i_{\beta\}} = 0 , \]

\[ \phi^i_{[\gamma\{\alpha]} b^i_{\beta]} + i\gamma\{\alpha} \partial_- \psi^i b^i_{\beta]} = 0 \]

with

\[ \phi^i_{[\alpha\beta]} \equiv \frac{1}{2} D_{[\alpha} D_{\beta]} \Psi^i|_{\theta=0} . \]

The \( \gamma/\beta \) trace of (46) yields the important relation

\[ \phi^i_{[\alpha\beta]} b^i_{\gamma} - i(N-1) \partial_- \psi^i b^i_{\gamma} = 0 , \]

while the part with mixed symmetry results in some further restriction on \( \phi^i_{[\alpha\beta]} \) which does not involve the physical field \( \psi^i \) (its explicit form is not needed for our purposes).

Now, let us make a rigid supersymmetry transformation in (43) (the transformation laws for the component fields follow from the expansion of the superfield \( \Psi^i \)),

\[ \delta\psi^i = \epsilon_\alpha b^i_\alpha , \quad \delta b^i_\alpha = \epsilon_\gamma (\phi^i_{[\gamma\alpha]} + i\delta_{\gamma\alpha} \partial_- \psi^i) . \]

The variation of the action is

\[ \delta S_f = T \int d^2x \frac{2}{N} \epsilon_\gamma [\phi^i_{[\gamma\alpha]} b^i_\alpha - i(N-1) \partial_- \psi^i b^i_\gamma] \]

and it vanishes in virtue of (47). Thus, the action (43) is supersymmetric due to the superfield constraint (44). As we have just seen, in fact only the component (47) of this constraint is involved in achieving the supersymmetry of the action; the other component constraints are needed to ensure that the whole set of them are supersymmetric in their own right.

Now we discuss the meaning of the component constraints. Splitting \( b^i_\alpha \) into a “radial” and an “angular” parts,

\[ b^i_\alpha = m \hat{b}^i_\alpha , \]

we can rewrite (45) as

\[ m (\hat{b}^i_\alpha \hat{b}^i_\beta - \frac{1}{N} \delta_{\alpha\beta}) = 0 . \]

If one assumes \( m \) to be non-singular off-shell (this is not so on-shell, see below), then (51) implies that \( \hat{b}^i_\alpha \) is an orthogonal \( SO(N) \) matrix if \( n = N \) or it represents the coset \( SO(n)/SO(n-N) \) if \( n > N \). In order for this to make sense we have to take \( n \geq N \). With the help of \( \hat{b}^i_\alpha \) one can covariantly split any \( SO(n) \) vector into an \( SO(n-N) \) projection and an orthogonal \( N \)-dimensional one which is inert under \( SO(n) \), but is transformed in a proper way by the automorphism \( SO(N) \). So \( \hat{b}^i_\alpha \) is a sort of a “bridge” relating these two groups. The meaning of the constraint (47) becomes clear now: it states that such an \( SO(N) \) projection of the component \( \phi^i_{[\gamma\alpha]} \) for some irreducible combination of its \( SO(N) \) indices is not independent, but is expressed in terms of \( \partial_- \psi^i \). As was already mentioned, (46) implies more constraints on this projection which amount to the

\[^{3}\text{Note that for the supersymmetry of (43) it is enough to impose the weaker constraint } D_{\alpha} D_{[\alpha} \Psi^i D_{\beta]} \Psi^i = 0, \text{ the first component of which is just eq. (47). The reason for choosing (43) is the requirement of consistency with the twistor-like formulation of the superstring, as explained earlier.} \]
vanishing of some other of its $SO(N)$ irreducible components. The rest of $\phi^i_{[\gamma\delta]}$ remains arbitrary. For instance, in the case $N = 2$ we have only the constraint (47) on the $n$ fields $\phi^i = \epsilon_{\alpha\beta}\phi^i_{\alpha\beta}$, so in the latter there remain $n - 2$ independent components; in the case $N = 3$ we have five more constraints of the second type alongside (47), which together leave $3n - 8$ independent components in the $3n$ fields $\phi_{[\alpha\beta]}$, etc. It can be shown that all of the subsequent component constraints in (44) have the typical structure of (46): they mean that some $b$ projections of the corresponding higher components of $\Psi^i$ either vanish or are expressed in terms of the lower ones.

It should be pointed out that a priori it is not so evident that the constraint (44) is purely kinematical and does not produce differential conditions of the kind of equations of motion for some higher components of $\Psi^i$ and for certain values of $N$. If it were the case, we would not be allowed to substitute the constraint back into the action. Fortunately, simple arguments show that for any $N$ this constraint remains purely kinematical off shell. To convince ourselves, let us pull out a constant part from $\Psi^i$,

$$\Psi^i = \theta_\alpha \delta_{ai} + \Psi_i$$

and split the index $i$ into $i = (\alpha, i')$, where $\alpha$ and $i'$ run, respectively, from 1 to $N$ and from $N + 1$ to $n$. Then $\Psi_i$ splits into a pair of superfields,

$$\Psi_i = (\Psi_\alpha, \Psi_{i'})$$.

In terms of these superfields the original constraint (44) can be rewritten as a linear one,

$$D_\alpha \Phi_\beta + D_\beta \Phi_\alpha - \delta_{\alpha\beta} \frac{2}{N} D_\gamma \Phi_\gamma = 0,$$

where

$$\Phi_\alpha \equiv -\Psi_\alpha + \frac{1}{2} \Psi_\beta D_\alpha \Psi_\beta + \frac{1}{2} \Psi_{i'} D_\alpha \Psi_{i'} .$$

Thus, we have a purely kinematical constraint for the superfield $\Phi_\beta$ which has a simple general solution:

$$\Phi_\alpha = D_\alpha G(z) + \theta_\alpha F(x^+, x^-) .$$

On the other hand, the relation (55) is just a canonical redefinition of $\Psi_\alpha$: one can reexpress it in terms of $\Phi_\alpha$ and $\Psi_{i'}$ from (55) by means of iterations. The superfield $\Psi_{i'}$ remains entirely unconstrained. So, it becomes clear that for any $N$ the constraint (44) does not contain any dynamics off shell and merely expresses some components of $\Psi_i$ in terms of others (or puts them equal to zero). The corresponding Lagrange multiplier $P^{[\alpha\beta]}$ does not contain propagating degrees of freedom. We note that just because the constraint (44) is kinematical, there is no local symmetry of Siegel’s type (18) associated with $P^{[\alpha\beta]}$ in the actions (20), (34), (36), (38). Once again, this is in contrast to ref. (13), where the constraint (44) is dynamical and its Lagrange multiplier needs some sort of Siegel’s invariance in order not to propagate.

Thus we have seen that there are no problems with the off-shell supersymmetry of the chiral fermion action. However, going on shell in this action is rather subtle. As follows from (13), on shell

$$b^i_{\alpha} = 0$$

and we are no longer allowed to divide by $m$ in the constraints (45), (46) and the subsequent ones. At first sight this does not lead to difficulties as (45), (46) are satisfied
identically if \( b_\alpha^i \) satisfies (57). However, beginning with \( N = 3 \), in the higher-order constraints there appear terms which are not multiplied by \( b_\alpha^i \) and so do not vanish on shell. For instance, in the \( N = 3 \) case on shell there remains the following quadratic constraint on the field \( \phi_{[\gamma\alpha]}^i \)

\[
\phi_{[\gamma\alpha]}^i \phi_{[\gamma\beta]}^i \epsilon_{\alpha\beta\sigma} = 0 .
\]  

(58)

For higher \( N \) there appear quadratic constraints of this type including derivatives of fields. This would seem to create difficulties if these fields were somehow present in the action. However, the only auxiliary field entering the component action (43) is in fact the “radial part” \( m \) of \( b_\alpha^i \). The others are needed only to supersymmetrize the constraint (47) (itself required for the supersymmetry of the action). This indicates that the appearance of such strange constraints on shell is harmless. When we vary with respect to the fields \( \psi^i \) and \( m \) in the action to obtain the equations of motion, we should vary the constraints as well. The latter simply serve to partially express the variations of the higher-order components of \( \Psi^i \) through \( \delta \psi^i \) and \( \delta m \). The presence of the terms of the type (58) in the component constraints results in factors of \( m^{-1} \) in the expressions for the variations of the higher-order fields. The more we approach the minimum, the bigger the variations of these fields become, tending to infinity at the minimum itself. But we should not care about these variations as they do not contribute to the action functional, i.e. do not generate any equations of motion. Another argument is that on shell \((N,0)\) supersymmetry is trivially realized. Indeed, there the supersymmetry variation (48) of \( \psi^i \) vanishes, but this does not contradict the algebra of supersymmetry as on shell

\[
\partial_- \psi^i = 0 .
\]  

(59)

In conclusion we can say that the form of the chiral fermion action with manifest (local) worldsheet supersymmetry presented here completes the twistor-like formulation of the heterotic string in \( D = 10 \) and can serve as a starting point for a new attempt to covariantly quantize the theory.

Acknowledgements The authors have profited from discussions with F. Delduc, A. Galperin and V. Fateev. E.I. would like to thank Prof. V. Rittenberg for hospitality at the Physics Institute of the University of Bonn, where this work has been done. E.I. is also grateful to the Russian Foundation of Fundamental Research, grant 93-02-3821, and to the International Science Foundation, grant M9T000, for financial support.

References

[1] M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory*, CUP, 1987.

[2] M. Tonin, *Phys. Lett.* 283B (1992) 213; *Int. J. Mod. Phys.* A7 (1992) 6013.

[3] F. Delduc, A. Galperin, P. Howe and E. Sokatchev, *Phys. Rev.* D47 (1992) 578.

[4] S. Aoyama, P. Pasti and M. Tonin, *Phys. Lett.* 283B (1992) 213;
I. Bandos, D. Sorokin, M. Tonin and D. Volkov, *Phys. Lett.* 319B (1993) 445.

[5] A. Galperin and E. Sokatchev, *Phys. Rev.* D48 (1993) 4810;
V. Chikalov and A. Pashnev, *Mod. Phys. Lett.* A8 (1993) 285;
P. Pasti and M. Tonin, preprint DFPD/94/TH/05, Padova (1994);
I. Bandos, M. Cederwall, D. Sorokin and D. Volkov, preprint ITP-94-10, Göteborg (1994).

[6] P. Pasti and M. Tonin, *Nucl. Phys.* B418 (1994) 337;
E. Bergshoeff and E. Sezgin, preprint CTP TAMO-67/93.

[7] D. P. Sorokin, V. I. Tkach and D. V. Volkov, *Mod. Phys. Lett.* A4 (1989) 901;
D. P. Sorokin, V. I. Tkach, D. V. Volkov and A. A. Zheltukhin, *Phys. Lett.* 216B (1989) 302.

[8] F. Delduc and E. Sokatchev, *Class. Quantum Grav.* 9 (1991) 361.

[9] A. Galperin and E. Sokatchev, *Phys. Rev.* D46 (1992) 714.

[10] E. A. Ivanov and A. A. Kapustnikov, *Phys. Lett.* 267B (1991) 175.

[11] N. Berkovits, *Phys. Lett.* 232B (1989)184; 241B (1990) 497; *Nucl. Phys.* B350 (1991) 193; B358 (1991) 169.

[12] F. Delduc, E. Ivanov and E. Sokatchev, *Nucl. Phys.* B384 (1992) 334.

[13] D. Sorokin and M. Tonin, *Phys. Lett.* 326B (1994) 84.

[14] P. Howe, A note on chiral fermions and heterotic strings, King’s College preprint, 1994.

[15] B. E. W. Nilsson, *Nucl. Phys.* B188 (1981) 176.

[16] E. Witten, *Nucl. Phys.* B266 (1986) 245.

[17] P. Howe and P. Townsend, Phys. Lett. 259B (1991) 285.

[18] W. Siegel, *Nucl. Phys.* B238 (1984) 307.