STRONG SOLUTIONS TO A FOURTH ORDER EXPONENTIAL PDE DESCRIBING EPITAXIAL GROWTH

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Abstract. In this paper we prove the global existence of a strong solution to the initial boundary value problem for the exponential partial differential equation \( \partial_t u - \Delta e^{-\Delta u} + e^{-\Delta u} - 1 = 0 \). The equation was proposed as a continuum model for epitaxial growth of crystal surfaces on vicinal surfaces with evaporation and deposition effects [6]. Our investigations reveal that we must control the size of both \( \| e^{-\Delta u(x,0)} \|_{W^{2,2}(\Omega)} \) and \( \| e^{\Delta u(x,0)} \|_{\infty,\Omega} \) suitably to achieve our results. Related results in [8, 10] were established via the Weiner algebra framework. Here we offer a totally new approach, which seems to shed more light on the nature of exponential nonlinearity.

1. Introduction

1.1. Problem background and statement of main results. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) with \( C^2 \) boundary \( \partial \Omega \). For each \( T > 0 \), we consider the initial-boundary value problem

\[
\begin{align*}
\partial_t u &= \Delta e^{-\Delta u} - e^{-\Delta u} + 1 \quad \text{in } \Omega_T = \Omega \times (0, T), \\
\nabla u \cdot \nu &= \nabla e^{-\Delta u} \cdot \nu = 0 \quad \text{on } \Sigma_T = \partial \Omega \times (0, T), \\

u(x, 0) &= u_0(x) \quad \text{on } \Omega,
\end{align*}
\]

where \( \nu \) is the unit outward normal vector to the boundary.

Equation (1.1) can be used to describe the evolution of a crystal surface [6]. In this case, \( u \) is the surface height. The fourth order term in the equation represents the diffusion effect, while the lower order terms describe evaporation and deposition. Detailed information can be found in [6].

Epitaxial growth is an important process in forming solid films and other nano-structures. Mathematical modeling of the process has attracted wide attentions [6]. Continuum models involving exponential nonlinearity were first derived in [9] and more recently in [13, 6]. Mathematical analysis of such models in high space dimensions (\( N \geq 2 \)) is very challenging due to the lack of estimates for the exponent term. It was first observed in [11] that one had to allow the possibility that the exponent be a measure-valued function. Later, the idea of “exponential singularity” was employed in [2, 4, 5, 14, 18]. However, measure exponents do not arise in the one-dimensional case. See [3, 6].

To remove the singularity in the exponent, the authors in [8, 10] introduced a rather sophisticated critical Wiener algebra space and showed that there existed a strong (no measure) solution as long as the norm of \( u_0 \) in the Wiener algebra space was suitably small. The proof in [10] employed the Fourier transform of the power series expansion of the exponential term. A similar approach was also adopted in [8]. Here we offer a totally different perspective from which to view the problem. Our method is based upon Lemma 2.7 below, a simple result first introduced in [17]. Denote by \( \| \cdot \|_{p, \Omega} \) the norm in the space \( L^p(\Omega) \). Our investigations reveal that we can obtain global existence of a strong solution by requiring the \( W^{2,2}(\Omega) \) norm of \( e^{-\Delta u_0} \) and \( \| e^{\Delta u_0} \|_{\infty, \Omega} \) to be suitably small.

Before we state our main theorem, we give our definition of a strong solution.

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Definition: We say that a pair \((u, \rho)\) is a strong solution to (1.1)-(1.3) if the following conditions hold:

(D1) \(u, \rho \in L^\infty(0,T;W^{2,2}(\Omega)) \cap W^{1,2}(\Omega_T)\) with \(\rho \geq c_0\) for some positive number \(c_0\);
(D2) We have

\[
\frac{\partial u}{\partial t} = \Delta \rho - \rho + 1 \quad \text{a.e. on } \Omega_T,
\]

\[
-\Delta u = \ln \rho \quad \text{a.e. on } \Omega_T,
\]

\[
\nabla u \cdot \nu = \nabla \rho \cdot \nu = 0 \quad \text{a.e. on } \Sigma_T.
\]

The initial condition (1.3) is satisfied in the space \(C([0,T];L^2(\Omega))\).

Our main result is the following

Theorem 1.1 (Main Theorem). Assume:

- (H1) \(\Omega\) is a bounded domain in \(\mathbb{R}^N\) with \(C^2\) boundary;
- (H2) \(N = 2\) or \(3\);
- (H3) \(u_0 \in W^{2,2}(\Omega)\) is such that \(e^{-\Delta u_0} \in W^{2,2}(\Omega)\). Moreover, we have the consistence conditions

\[
\nabla u_0 \cdot \nu = 0, \quad \nabla e^{-\Delta u_0} \cdot \nu = 0 \quad \text{a.e. on } \partial \Omega.
\]

Then there exist two positive numbers \(s_0, s_1\) determined by \(\Omega\) only such that problem (1.1)-(1.3) has a global strong solution whenever

\[
\varepsilon_0 \equiv \|e^{-\Delta u_0}\|_{W^{2,2}(\Omega)} < s_0, \quad \|e^{\Delta u_0}\|_{\infty, \Omega} < s_1.
\]

By a global strong solution, we mean that for each \(T > 0\) there is a strong solution \(u\) to (1.1)-(1.3) on \(\Omega_T\). We can infer from (H1), (H2), and the Sobolev embedding theorem that

\[
\|u\|_{\infty, \Omega} \leq c(\Omega, p)\|u\|_{W^{2,p}(\Omega)} \quad \text{for each } u \in W^{2,p}(\Omega) \text{ whenever } p > \frac{3}{2}.
\]

Hence we also have \(e^{-\Delta u_0} \in L^\infty(\Omega)\). Note that the two inequalities in (1.5) are not contradictory. Roughly speaking, the first one controls the set where \(-\Delta u_0\) is very large, while the second one is concerned with the set where \(\Delta u_0\) is very large. In fact, our assumptions here reveal the true nature of the exponential nonlinearity. That is, the composite function \(e^{-\Delta u_0}\) can still behave well even if the exponent term \(-\Delta u_0\) displays singularity near the set \(\{-\Delta u_0 = -\infty\}\). The second inequality in (1.5) is assumed to prevent this from happening. We refer the reader to [11] for more discussions in this regard.

1.2. A priori estimates for smooth solutions. To gain some insights into our problem, we proceed to perform some formal analysis. By “formal”, we mean that the solution \(u\) to (1.1)-(1.3) is as smooth as we desire so that all the subsequent calculations in this subsection make sense. However, the essence of our approach is already demonstrated here.

To simplify our presentation, we introduce the functions

\[
\rho = e^{-\Delta u}, \quad G = \partial_t u + \rho - 1.
\]

Then (1.1) becomes

\[
-\Delta \rho + G = 0 \quad \text{in } \Omega_T.
\]

Square both sides of this equation and then integrate it with respect to \(x\) over \(\Omega\) to get

\[
\int_\Omega [(\Delta \rho)^2 + G^2] \, dx - 2 \int_\Omega G \Delta \rho \, dx = 0.
\]
Note from (1.7) that
\[
-2 \int_{\Omega} G \Delta \rho \, dx = -2 \int_{\Omega} \partial_t u \Delta \rho \, dx + 2 \int_{\Omega} |\nabla \rho|^2 \, dx
\]
\[
= -2 \int_{\Omega} e^{-\Delta u} \partial_t \Delta u \, dx + 2 \int_{\Omega} |\nabla \rho|^2 \, dx
\]
\[
= 2 \frac{d}{dt} \int_{\Omega} \rho \, dx + 2 \int_{\Omega} |\nabla \rho|^2 \, dx.
\]
Substitute this into (1.9) and integrate the resulting equation to obtain
\[
(1.10) \quad \sup_{0 \leq t \leq T} 2 \int_{\Omega} \rho(x,t)dx + \int_{\Omega_T} \left[ G^2 + (\Delta \rho)^2 + 2 |\nabla \rho|^2 \right] \, dx \, dt + \leq 4 \int_{\Omega} e^{-\Delta u_0} \, dx \leq 4\epsilon_0,
\]
where \(\epsilon_0\) is given as in (1.5).

Next we differentiate (1.8) with respect to \(t\) and then use \(G\) as a test function in the resulting equation to obtain
\[
(1.11) \quad - \int_{\Omega} \partial_t \Delta \rho \, G \, dx + \frac{1}{2} \frac{d}{dt} \int_{\Omega} G^2 \, dx = 0.
\]
Observe that
\[
\int_{\Omega} \partial_t \Delta \rho \, G \, dx = \int_{\Omega} \partial_t \Delta \rho \, \partial_t u \, dx + \int_{\Omega} \partial_t \Delta \rho \, (\rho - 1) \, dx
\]
\[
= \int_{\Omega} \partial_t e^{-\Delta u} \partial_t \Delta u \, dx - \int_{\Omega} \partial_t \nabla \rho \cdot \nabla \rho \, dx
\]
\[
= -4 \int_{\Omega} |\partial_t \sqrt{\rho}|^2 \, dx - \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla \rho|^2 \, dx.
\]
Substitute this into (1.11) to derive
\[
(1.12) \quad \sup_{0 \leq t \leq T} \int_{\Omega} \left( \frac{1}{2} |\nabla \rho|^2 + \frac{1}{2} G^2 \right) \, dx + 4 \int_{\Omega_T} |\partial_t \sqrt{\rho}|^2 \, dx
\]
\[
\leq \int_{\Omega} \left( (|\nabla \rho(x,0)|^2 + G^2(x,0)) \right) \, dx
\]
\[
= \int_{\Omega} |\nabla e^{-\Delta u_0(x)}|^2 \, dx + \int_{\Omega} |\Delta e^{-\Delta u_0(x)}|^2 \, dx \leq \epsilon_0^2.
\]
By virtue of (H2) and Lemma 2.6 below, there is a positive number \(c = c(\Omega)\) such that
\[
(1.13) \quad \|\rho(\cdot, t)\|_{\infty, \Omega} \leq c\|\rho(\cdot, t)\|_{1, \Omega} + c\|G(\cdot, t)\|_{2, \Omega} \leq c\epsilon_0.
\]

It follows from (1.7) that
\[
- \Delta u = \ln \rho \quad \text{in} \; \Omega.
\]
Integrate this equation over \(\Omega\) and use (1.2) to obtain
\[
\int_{\Omega} \ln \rho \, dx = 0.
\]
Keeping this and (1.10) in mind, we estimate
\[
\int_\Omega |\ln \rho| \, dx = \int_\Omega \ln^+ \rho \, dx + \int_\Omega \ln^- \rho \, dx \\
= 2 \int_\Omega \ln^+ \rho \, dx - \int_\Omega \ln \rho \, dx \\
\leq 2 \int_\Omega \rho \, dx \leq 2 \int_\Omega e^{-\Delta u_0(x)} \, dx \leq 2 \varepsilon_0.
\]

Fix \( L > 1 \). We have
(1.14) \(|\rho \leq \frac{1}{L}| \leq \frac{1}{\ln L} \int_{\{\rho \leq \frac{1}{L}\}} |\ln \rho| \, dx \leq \frac{2 \varepsilon_0}{\ln L}.

Set \( w = \frac{1}{\rho} \).

We easily verify that
\[
\Delta \rho = -w^{-2} \Delta w + 2w^{-3} |\nabla w|^2.
\]

Subsequently, \( w \) satisfies the boundary value problem
\[
-\Delta w + 2w = Gw^2 \text{ in } \Omega, \\
\nabla w \cdot \nu = 0 \text{ on } \partial \Omega.
\]

We can infer from Lemma 2.6 below and (H2) that there is a positive number \( c = c(\Omega) \) such that
\[
\|w(\cdot, t)\|_{\infty, \Omega} \leq c \|w(\cdot, t)\|_{1, \Omega} + c \|G(\cdot, t)w^2(\cdot, t)\|_{2, \Omega} \\
\leq c \int_{\{w \leq L\}} w \, dx + c \int_{\{w > L\}} w \, dx + c \|w(\cdot, t)\|_{\infty, \Omega} \|G(\cdot, t)\|_{2, \Omega} \\
\leq cL + c \|w(\cdot, t)\|_{\infty, \Omega} \left|\left\{ \rho \leq \frac{1}{L} \right\}\right| + c \varepsilon_0 \|w(\cdot, t)\|_{\infty, \Omega}^2 \\
\leq cL + \frac{c \varepsilon_0 \|w(\cdot, t)\|_{\infty, \Omega}}{\ln L} + c \varepsilon_0 \|w(\cdot, t)\|_{\infty, \Omega}^2.
\]

Here we have used (1.12) and (1.14). Consider the quadratic function
\[
Q(s) = c \varepsilon_0 s^2 - \left(1 - \frac{c \varepsilon_0}{\ln L}\right) s + cL \text{ on } (0, \infty).
\]

Then (1.15) says
\[
Q(\|w(\cdot, t)\|_{\infty, \Omega}) \geq 0 \text{ for each } t \in [0, T].
\]

Suppose that \( \|w(\cdot, t)\|_{\infty, \Omega} \) is a continuous function of \( t \). According to the proof of Lemma 2.7 below, if we choose \( L > 1 \) and \( \varepsilon_0 \) so that
(1.16) \[
1 - \frac{c \varepsilon_0}{\ln L} > 0, \quad \left(1 - \frac{c \varepsilon_0}{\ln L}\right)^2 > 4c^2 \varepsilon_0,
\]
then
\[
\|w(\cdot, t)\|_{\infty, \Omega} \leq \frac{1 - \frac{c \varepsilon_0}{\ln L} - \sqrt{(1 - \frac{c \varepsilon_0}{\ln L})^2 - 4c^2 \varepsilon_0 \varepsilon_0}}{2c \varepsilon_0} \equiv g(\varepsilon_0, L) \text{ for } t > 0
\]
whenever
\[
\|w(\cdot, 0)\|_{\infty, \Omega} \leq g(\varepsilon_0, L).
\]

Take the square root of the second inequality in (1.16) to derive
\[
-\frac{c}{\ln L} \varepsilon_0 - 2c \sqrt{L \varepsilon_0} + 1 > 0.
\]
Solving this inequality yields
\[(1.17)\quad \sqrt{\varepsilon} \leq \sqrt{L \ln^2 L + \frac{\ln L}{c} - \sqrt{L} \ln L} \equiv h(L).\]

By (6) in Lemma 2.4 below,
\[h(L) \leq \sqrt{\ln L}.\]
That is, (1.17) implies the first inequality in (1.16). We easily see that
\[h(1) = 0, \quad \lim_{L \to \infty} h(L) = 0.\]
Thus \(h(L)\) attains its maximum value at some point \(L_0 \in (1, \infty)\). We take
\[s_0 = h(L_0)^2.\]
To determine \(s_1\), it is easy to see that
\[g(\varepsilon_0, L) = \frac{2cL}{1 - \frac{c\varepsilon_0}{c} + \sqrt{(1 - \frac{c\varepsilon_0}{c})^2 - 4c^2 L \varepsilon_0}} - \frac{2cL}{1 - \frac{c\varepsilon_0}{c} + \sqrt{1 - \frac{c^2 L^2}{4c^2 L \varepsilon_0}} - \left(\frac{2c}{c} + 4c^2 L\right) \varepsilon_0 + 1},\]
which is an increasing function of \(\varepsilon_0\) on the interval \((0, s_0)\). Thus we take
\[s_1 = g(0, L_0) = cL_0.\]
Whenever \(\|e^{-\Delta u_0}\|_{W^{2,2}(\Omega)} < s_0, \quad \|e^{\Delta u_0}\|_{\infty, \Omega} < s_1\), we have
\[\|e^{\Delta u(t)}\|_{\infty, \Omega} \leq g(\|e^{-\Delta u_0}\|_{W^{2,2}(\Omega), L_0}) \quad \text{for all } t > 0.\]
This together with (1.13) implies
\[\Delta u \in L^\infty(\Omega).\]
In particular, the exponent term is not a measure.
A solution to (1.1)-(1.3) will be constructed as the limit of a sequence of approximate solutions. The key is to design an approximation scheme so that all the calculations in Subsection 1.2 can be justified. This is accomplished in Sections 2 and 3. To be more specific, in Section 2 we state a few preparatory lemmas and present our approximate problems. The existence of a classical solution is established for these problems. We form a sequence of approximate solutions based upon implicit discretization in the time variable. Section 3 is devoted to the proof of the discretized versions of the results in Subsection 1.2. These estimates are enough to justify passing to the limit.

2. Approximate Problems

Before we present our approximate problems, we state a few preparatory lemmas.

Lemma 2.1. Let \(\Omega\) be a bounded domain in \(\mathbb{R}^N\).
\[(i)\quad \text{If } \Omega \text{ is convex, then } \int_\Omega (\Delta u)^2 \, dx \geq \int_\Omega |\nabla^2 u|^2 \, dx \quad \text{for all } u \in W^{2,2}(\Omega) \text{ with } \nabla u \cdot \nu = 0 \text{ on } \partial \Omega.\]
\[(ii)\quad \text{If } \partial \Omega \text{ is } C^2, \text{ then there is a positive constant } c \text{ depending only on } N, \Omega \text{ and the smoothness of the boundary such that}\]
\[(2.1)\quad \int_\Omega (\Delta u)^2 \, dx + \int_\Omega |\nabla u|^2 \, dx \geq c \int_\Omega |\nabla^2 u|^2 \, dx \quad \text{for all } u \in W^{2,2}(\Omega) \text{ with } \nabla u \cdot \nu = 0 \text{ on } \partial \Omega.\]
We refer the reader to [16] for some background information on this lemma. Our existence theorem is based upon the following fixed point theorem, which is often called the Leray-Schauder Theorem ([1], p.280).

**Lemma 2.2.** Let $B$ be a map from a Banach space $B$ into itself. Assume:

(LS1) $B$ is continuous;
(LS2) the images of bounded sets of $B$ are precompact;
(LS3) there exists a constant $c$ such that

$$||z||_B \leq c$$

for all $z \in B$ and $\sigma \in [0,1]$ satisfying $z = \sigma B(z)$.

Then $B$ has a fixed point.

Relevant interpolation inequalities for Sobolev spaces are listed in the following lemma.

**Lemma 2.3.** Let $\Omega$ be a bounded domain in $\mathbb{R}^N$. Then we have:

1. $||f||_{\Omega} \leq \varepsilon ||f||_{r,\Omega} + \varepsilon^{p} ||f||_{p,\Omega}$, where $\varepsilon > 0, p \leq q < r$, and $\sigma = \left(\frac{1}{p} - \frac{1}{q}\right) / \left(\frac{1}{q} - \frac{1}{r}\right)$;
2. If $\partial \Omega$ is $C^2$, for each $\varepsilon > 0$ and each $p \in [2, 2^*)$, where $2^* = \frac{2N}{N-2}$ if $N > 2$ and any number bigger than 2 if $N = 2$, there is a positive number $c = c(\varepsilon, p)$ such that

$$||f||_{p,\Omega} \leq \varepsilon ||\nabla f||_{2,\Omega} + c||f||_{1,\Omega}$$

for all $f \in W^{1,2}(\Omega)$,

$$||\nabla g||_{p,\Omega} \leq \varepsilon ||\nabla^2 g||_{2,\Omega} + c||g||_{1,\Omega}$$

for all $g \in W^{2,2}(\Omega)$.

Finally, we collect a few frequently used elementary inequalities in the following lemma.

**Lemma 2.4.** For $x, y \in \mathbb{R}^N$, $s, t \in \mathbb{R}$, and $a, b \in (0, \infty)$, we have:

1. $x \cdot (x - y) \geq \frac{1}{2}(|x|^2 - |y|^2)$;
2. if $f$ is an increasing function on $\mathbb{R}$ and $F$ an anti-derivative of $f$, then $f(s)(s - t) \geq F(s) - F(t)$.

In particular, there hold the inequalities

$$a(\ln a - \ln b) \geq a - b \quad \text{and}$$

$$(a - b) \ln a \geq a \ln a - b \ln b - (\ln a - \ln b);$$

we have

$$a(\ln a - \ln b) \geq a - b \quad \text{and}$$

$$(a - b)(\ln a - \ln b) \geq 2\left(\sqrt{a} - \sqrt{b}\right)^2;$$

there hold

$$(a + b)^{\alpha} \leq a^\alpha + b^\alpha$$

if $0 < \alpha \leq 1$,

$$(a + b)^{\alpha} \leq 2^{\alpha-1}(a^\alpha + b^\alpha)$$

if $\alpha > 1$,

$$ab \leq \varepsilon a^p + \frac{1}{\varepsilon q/p} b^q$$

if $\varepsilon > 0, p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

The proof of the lemma is also rather elementary. We refer the reader to [11] for details.

**Lemma 2.5.** Let $\{y_n\}, n = 0, 1, 2, \ldots$, be a sequence of positive numbers satisfying the recursive inequalities

$$y_{n+1} \leq cb^ny_n^{1+\alpha} \quad \text{for some } b > 1, c, \alpha \in (0, \infty).$$

If

$$y_0 \leq c^{-\frac{1}{\alpha}}b^{-\frac{1}{\alpha}},$$

then $\lim_{n \to \infty} y_n = 0$. 
We will only focus on the case

(2.8) \[ \int_\Omega |\nabla (w - k_n + 1)|^2 dx = \int_\Omega f [(w - k_n + 1) - y_{n+1}] dx. \]

For each \( s > 1 \) we obtain from the Sobolev inequality that

\[ \int_\Omega f [(w - k_n + 1) - y_{n+1}] dx \leq \|f\|_{\frac{2s}{s+2},\Omega} \|(w - k_n + 1) - y_{n+1}\|_{s,\Omega} \]

\[ \leq cK \{w \geq k_n + 1\}^{\frac{s+2}{s}} \|\nabla (w - k_n + 1)\|_{2,\Omega} \]

\[ \leq cK \{w \geq k_n + 1\}^{1 - \frac{1}{p}} \|\nabla (w - k_n + 1)\|_{2,\Omega}. \]

Use this in (2.8) to get

\[ \|\nabla (w - k_n + 1)^+\|_{2,\Omega} \leq cK \{w \geq k_n + 1\}^{1 - \frac{1}{p}}. \]

With this and the Sobolev inequality in mind, we deduce that

\[ y_{n+1} \leq \left( \int_\Omega [(w - k_n + 1)^+]^s dx \right)^{\frac{1}{s}} \{w \geq k_n + 1\}^{1 - \frac{1}{s}} \]

\[ \leq c \left( \left( \int_\Omega |\nabla (w - k_n + 1)^+|^{\frac{2s}{s+2}} dx \right)^{\frac{s+2}{2s}} + \int_\Omega (w - k_n + 1)^+ dx \right) \{w \geq k_n + 1\}^{\frac{s+2}{s}} \]

\[ \leq c \left( \|\nabla (w - k_n + 1)^+\|_{2,\Omega} \{w \geq k_n + 1\}^{\frac{1}{s}} + \int_\Omega (w - k_n + 1)^+ dx \right) \{w \geq k_n + 1\}^{\frac{1}{s+1}} \]

\[ \leq c \left( K \{w \geq k_n + 1\}^{1 + \frac{1}{2} - \frac{1}{p} + y_n} \right) \{w \geq k_n + 1\}^{\frac{1}{s+1}}. \]

We easily see that

\[ y_n \geq \frac{1}{|\Omega|} \int_{\{w \geq k_n + 1\}} (w - k_n)^+ dx \geq \frac{K}{2^{n+1}|\Omega|} \{w \geq k_n + 1\}. \]
Take
\[ \alpha = \min \left\{ 1 - \frac{1}{p} s - \frac{1}{s} \right\}. \]

Our assumption on \( p \) implies \( \alpha > 0 \).

We can obtain from (2.9) that
\[
y_{n+1} \leq cK \left| \{ w \geq k_{n+1} \} \right|^{1+\alpha} + cy_n \left| \{ w \geq k_{n+1} \} \right|^\alpha \\
\leq \frac{c_2^{(1+\alpha)n}}{K^\alpha} y_n^{1+\alpha}.
\]

According to Lemma 2.5, if we choose \( K \) so large that
\[
y_0 = \frac{1}{|\Omega|} \int_{\Omega} w^+ dx \leq cK,
\]
then \( w \leq K \). In view of (2.7), we conclude
\[
w \leq c\|w\|_{1,\Omega} + c\|f\|_{p,\Omega}.
\]

Lemma 2.7. Let \( h(\tau) \) be a continuous non-negative function defined on \([0, T_0]\) for some \( T_0 > 0 \). Suppose that there exist three positive numbers \( \varepsilon, \delta, b \) such that
\begin{equation}
(2.10) \quad h(\tau) \leq \varepsilon h^{1+\delta}(\tau) + b \text{ for each } \tau \in [0, T_0].
\end{equation}

Then
\begin{equation}
(2.11) \quad h(\tau) \leq \frac{1}{[\varepsilon(1+\delta)]^{\frac{1}{\delta}}} \equiv s_0 \text{ for each } \tau \in [0, T_0],
\end{equation}

provided that
\begin{equation}
(2.12) \quad \varepsilon \leq \frac{\delta^\delta}{(b + \delta)^\delta (1 + \delta)^{1+\delta}} \text{ and } h(0) \leq s_0.
\end{equation}

The proof is given in [17]. For the convenience of the reader, we will reproduce it here.

Proof. Consider the function \( f(s) = \varepsilon s^{1+\delta} - s + b \) on \([0, \infty)\). Then condition (2.10) simply says
\begin{equation}
(2.13) \quad f(h(\tau)) \geq 0 \text{ for each } \tau \in [0, T_0].
\end{equation}

It is easy to check that the function \( f \) achieves its minimum value at \( s_0 = \frac{1}{[\varepsilon(1+\delta)]^{\frac{1}{\delta}}} \). The minimum value
\[
f(s_0) = \frac{\varepsilon}{[\varepsilon(1+\delta)]^{\frac{1+\delta}{\delta}}} - \frac{1}{[\varepsilon(1+\delta)]^{\frac{1}{\delta}}} + b
\]
\[
= b - \frac{\delta}{\varepsilon^{\frac{1}{\delta}} (1 + \delta)^{\frac{1}{\delta}}}.\]

By the first inequality in (2.12), \( f(s_0) \leq -\delta \). Consequently, the equation \( f(s) = 0 \) has exactly two solutions \( 0 < s_1 < s_2 \) with \( s_0 \) lying in between. Evidently, \( f \) is positive on \([0, s_1)\), negative on \((s_1, s_2)\), and positive again on \((s_2, \infty)\). The range of \( h \) is a closed interval because of its continuity, and this interval is either contained in \([0, s_1]\) or \((s_2, \infty)\) due to (2.13). The latter cannot occur due to the second inequality in (2.12). Thus the lemma follows. \( \square \)
We largely follow [14] for the construction of approximate problems. For this purpose, let
\( \tau > 0 \) and \( v \in L^2(\Omega) \).

Consider the boundary value problem
\[
\begin{align*}
-\Delta \rho + \rho + \tau \ln \rho &= -\frac{u - v}{\tau} + 1 \quad \text{in } \Omega,
-\Delta u + \tau u &= \ln \rho \quad \text{in } \Omega,
\nabla u \cdot \nu = \nabla \rho \cdot \nu &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]
This problem will serve as a basis for our approximation. To obtain an existence assertion for this problem, we first need to study
\[
\begin{align*}
-\Delta \rho + \rho + \tau \ln \rho &= f \quad \text{in } \Omega,
\nabla \rho \cdot \nu &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]
where \( f \) is a given function in \( L^2(\Omega) \). A weak solution to this problem is a function \( \rho \in W^{1,2}(\Omega) \) such that
\[
\int_{\Omega} \nabla \rho \nabla \varphi dx + \int_{\Omega} \rho \varphi + \tau \int_{\Omega} \ln \rho \varphi dx = \int_{\Omega} f \varphi dx \quad \text{for each } \varphi \in W^{1,2}(\Omega).
\]
Of course, (2.20) implies \( \rho > 0 \) a.e. on \( \Omega \).

**Lemma 2.8.** For each \( f \in L^2(\Omega) \) there is a unique weak solution to (2.18)-(2.19).

**Proof.** For the existence part, we consider the approximate problem
\[
\begin{align*}
-\Delta \rho_\delta + \rho_\delta + \tau \psi_\delta(\rho_\delta) &= f \quad \text{in } \Omega,
\nabla \rho_\delta \cdot \nu &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]
where \( \delta \in (0,1) \) and
\[
\psi_\delta(s) = \begin{cases} 
\ln(s + \delta) & \text{if } s > 0, \\
\ln \delta & \text{if } s \leq 0.
\end{cases}
\]
Existence of a weak solution to this problem is standard, we will omit its proof. Next, we proceed to show that we can take \( \delta \to 0 \) in (2.21)-(2.22). To this end, let
\[
s_\delta \equiv 1 - \delta \in (0,1).
\]
Then we have
\[
\psi_\delta(s_\delta) = 0.
\]
Subtract \( s_\delta \) from both sides of (2.21) and use \( \rho_\delta - s_\delta \) as a test function in the resulting equation to get
\[
\int_{\Omega} |\nabla \rho_\delta|^2 dx + \int_{\Omega} (\rho_\delta - s_\delta)^2 dx + \tau \int_{\Omega} \psi_\delta(\rho_\delta)(\rho_\delta - s_\delta) dx
\]
\[
= \int_{\Omega} (f - s_\delta)(\rho_\delta - s_\delta) dx
\]
\[
\leq \frac{1}{2} \int_{\Omega} (\rho_\delta - s_\delta)^2 dx + \frac{1}{2} \int_{\Omega} (f - s_\delta)^2 dx.
\]
Thus,
\[
\int_{\Omega} |\nabla \rho_\delta|^2 dx + \int_{\Omega} (\rho_\delta - s_\delta)^2 dx + \tau \int_{\Omega} \psi_\delta(\rho_\delta)(\rho_\delta - s_\delta) dx \leq c.
\]
Here and in what follows the letter $c$ denotes a positive number independent of $\delta$. Note that 
\[ \psi_\delta(\rho_\delta)(\rho_\delta - s_z) \geq 0 \text{ a.e. on } \Omega. \]
This together with (2.23) implies that \{\rho_\delta\} is bounded in $W^{1,2}(\Omega)$. We may assume that 
$\rho_\delta \to \rho$ weakly in $W^{1,2}(\Omega)$, strongly in $L^2(\Omega)$, and a.e. on $\Omega$.

Since $\psi_\delta$ is a Lipschitz function, we can use $\psi_\delta(\rho_\delta)$ as a test function in (2.21) to deduce
\[ \int_\Omega \psi_\delta'(\rho_\delta)|\nabla \rho_\delta|^2 \, dx + \int_\Omega \rho_\delta \psi_\delta(\rho_\delta) + \int_\Omega \psi_\delta^2(\rho_\delta) \, dx = \int_\Omega f \psi_\delta(\rho_\delta) \, dx. \]
Remember that $\psi_\delta'(\rho_\delta) \geq 0$. Thus, we can conclude from (2.24) that
\[ \int_\Omega \psi_\delta^2(\rho_\delta) \, dx \leq c(\tau). \]

In view of Fatou's lemma and (2.25), we must have
\[ |\\{\rho \leq 0\}| = 0 \]
and
\[ \int_\Omega \ln^2 \rho \, dx = \int_{\{\rho > 0\}} \ln^2 \rho \, dx \leq \lim_{\delta \to 0} \int_\Omega \psi_\delta^2(\rho_\delta) \, dx \leq c. \]

We are ready to pass to the limit in (2.21).

The uniqueness of a weak solution to (2.18)-(2.19) is trivial because $\rho + \tau \ln \rho$ is strictly increasing.

The proof is complete. \hfill \Box

**Lemma 2.9.** Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary, and assume that (2.14) hold. Then there is a weak solution to (2.15)-(2.17). If, in addition, $v \in L^\infty(\Omega)$, then we have
\[ \ln \rho \in L^\infty(\Omega). \]

**Proof.** We essentially follows the argument in Section 4, [14]. To proceed, we define an operator $B$ from $W^{1,2}(\Omega)$ into itself as follows: For each $w \in W^{1,2}(\Omega)$ we first solve the problem
\begin{align*}
-\Delta \rho + \rho + \tau \ln \rho &= -\frac{w - v}{\tau} + 1 \text{ in } \Omega, \\
\nabla \rho \cdot \nu &= 0 \text{ on } \partial \Omega.
\end{align*}
By Lemma 2.8 there is a unique weak solution $\rho \in W^{1,2}(\Omega)$ with $\ln \rho \in L^2(\Omega)$ to the above problem. We use the function $\rho$ so obtained to form the problem
\begin{align*}
-\Delta u + \tau u &= \ln \rho \text{ in } \Omega, \\
\nabla u \cdot \nu &= 0 \text{ on } \partial \Omega.
\end{align*}
The classical existence theory asserts that there is a unique weak solution $u \in W^{1,2}(\Omega)$ to (2.29)-(2.30). We define
\[ B(w) = u. \]
Clearly, $B$ is well-defined. As in Section 4, [14], we can conclude that $B$ is continuous and maps bounded sets into precompact ones. Next, we show that there is a positive number $c$ such that
\[ \|u\|_{W^{1,2}(\Omega)} \leq c \]
for all $u \in W^{1,2}(\Omega)$ and $\sigma \in [0,1]$ satisfying
\[ u = \sigma B(u). \]
This equation is equivalent to the boundary value problem

\begin{align}
-\Delta \rho + \rho + \tau \ln \rho &= -\frac{u - v}{\tau} + 1 \quad \text{in } \Omega, \\
-\Delta u + \tau u &= \sigma \ln \rho \quad \text{in } \Omega, \\
\nabla u \cdot \nu = \nabla \rho \cdot \nu &= 0 \quad \text{on } \partial \Omega.
\end{align}

In view of \((2.26)\), we may assume that \(\rho\) is bounded away from 0 below. Thus we can use \(\ln \rho\) as a test function in \((2.32)\) to get

\begin{align}
\int _{\Omega} \frac{|\nabla \rho|^2}{\rho} \, dx + \int _{\Omega} (\rho - 1) \ln \rho \, dx + \tau \int _{\Omega} \ln^2 \rho \, dx &\leq -\frac{1}{\tau} \int _{\Omega} (u - v) \ln \rho \, dx.
\end{align}

Use \(u\) as a test function in \((2.33)\) to deduce

\begin{align}
\sigma \int _{\Omega} u \ln \rho \, dx = \int _{\Omega} |\nabla u|^2 \, dx + \tau \int _{\Omega} u^2 \, dx \geq 0.
\end{align}

Drop the first term in \((2.35)\) and then use the above equation to get It immediately follows that

\begin{align}
\int _{\Omega} (\rho - 1) \ln \rho \, dx + \tau \int _{\Omega} \ln^2 \rho \, dx \leq \frac{1}{\tau} \int _{\Omega} v \ln \rho \, dx.
\end{align}

Subsequently,

\begin{align}
\int _{\Omega} (\rho - 1) \ln \rho \, dx + \int _{\Omega} \ln^2 \rho \, dx \leq c(\tau) \int _{\Omega} v^2 \, dx.
\end{align}

This together with \((2.33)\) implies \((2.31)\).

To see \((2.26)\), we first establish the estimate

\begin{align}
\tau \| \ln \rho \|_{p, \Omega} \leq \left\| \frac{u - v}{\tau} \right\|_{p, \Omega} \quad \text{for each } p \geq 2.
\end{align}

For this purpose, we introduce the function

\begin{align}
h_\varepsilon(s) = \begin{cases} 
1 & \text{if } s > \varepsilon, \\
s & \text{if } |s| \leq \varepsilon, \\
-1 & \text{if } s < -\varepsilon, \quad \varepsilon > 0.
\end{cases}
\end{align}

Use \(|\ln \rho|^{p-1}h_\varepsilon(\rho - 1)\) as a test function in \((2.27)\) to derive

\begin{align}
\int _{\Omega} |\ln \rho|^{p-1}h_\varepsilon(\rho - 1)(\rho - 1) \, dx + \tau \int _{\Omega} \ln \rho |\ln \rho|^{p-1}h_\varepsilon(\rho - 1) \, dx \leq -\int _{\Omega} \frac{u - v}{\tau} |\ln \rho|^{p-1}h_\varepsilon(\rho - 1) \, dx.
\end{align}

Here we have used the fact that \(|\ln \rho|^{p-1}h_\varepsilon(\rho - 1)\) is an increasing function of \(\rho\). Taking \(\varepsilon \to 0\) yields

\begin{align}
\tau \int _{\Omega} |\ln \rho|^p \leq \int _{\Omega} \frac{u - v}{\tau} \left| \frac{u - v}{\tau} |\ln \rho|^{p-1}h_\varepsilon(\rho - 1) \right| dx \leq \left\| \frac{u - v}{\tau} \right\|_{p, \Omega} \| \ln \rho \|_{p, \Omega}^{p-1}.
\end{align}

The estimate \((2.37)\) follows. Now take \(p \to \infty\) in \((2.37)\) to get

\begin{align}
\tau \| \ln \rho \|_{\infty, \Omega} \leq \left\| \frac{u - v}{\tau} \right\|_{\infty, \Omega}.
\end{align}

Lemma \(2.6\) asserts that for each \(q > \frac{N}{2}\) there is a positive number \(c = c(N, \Omega, \tau)\) such that

\begin{align}
\| u \|_{\infty, \Omega} \leq c \| u \|_{1, \Omega} + c \| \ln \rho \|_{q, \Omega} \leq c \| \ln \rho \|_{q, \Omega}.
\end{align}
This combined with (2.38) implies
\[ \| \ln \rho \|_{\infty, \Omega} \leq c \| u \|_{\infty, \Omega} + c \| v \|_{\infty, \Omega} \leq c \| \ln \rho \|_{q, \Omega} + c \| v \|_{\infty, \Omega} \leq \varepsilon \| \ln \rho \|_{\infty, \Omega} + c(\varepsilon) \| \ln \rho \|_{1, \Omega} + c \| v \|_{\infty, \Omega}, \quad \varepsilon > 0. \]

The last step is due to the interpolation inequality (1) in Lemma 2.3. Taking \( \varepsilon \) suitably small yields (2.26). The proof is complete. \( \square \)

Note that for Lemma 2.9 we do not have to assume (H2).

To conclude this section, we would like to make some remarks about the possible non-negativity of \( u \). Since \( u \) represents the surface height in our model, it is natural for us to expect \( u \geq 0 \).

In this regard, one is tempted to consider the following approximation
\[ \begin{align*}
-\Delta \rho + \rho + \tau \ln \rho &= \frac{w - v}{\tau} + 1 \quad \text{in } \Omega, \\
-\Delta u + \tau \ln u &= \ln \rho \quad \text{in } \Omega, \\
\nabla u \cdot \nu &= \nabla \rho \cdot \nu = 0 \quad \text{on } \partial \Omega.
\end{align*} \]

It turns out that this problem does have a solution. The proof is only a slight modification of that for Lemma 2.9. To see this, we define an operator \( B \) from \( W^{1,2}(\Omega) \) into itself as follows: For each \( \psi \in W^{1,2}(\Omega) \) we first solve the problem
\[ \begin{align*}
-\Delta \rho + \rho + \tau \ln \rho &= \frac{w - v}{\tau} + 1 \quad \text{in } \Omega, \\
\nabla \rho \cdot \nu &= 0 \quad \text{on } \partial \Omega.
\end{align*} \]

We use the function \( \rho \) so obtained to form the problem
\[ \begin{align*}
-\Delta u + \tau \ln u &= \ln \rho \quad \text{in } \Omega, \\
\nabla u \cdot \nu &= 0 \quad \text{on } \partial \Omega.
\end{align*} \]

By replacing the second term in (2.18) by \( \delta \rho \) and then taking \( \delta \to 0 \) in the resulting problem, we can also conclude that there is a unique weak solution to the preceding problem. See [14] for details. Define
\[ B(\psi) = \psi. \]

Clearly, \( B \) is well-defined. Once again, we can infer from Section 4 in [14] that \( B \) is continuous and maps bounded sets into precompact ones.

Next, we show that there is a positive number \( c \) such that
\[ \| u \|_{W^{1,2}(\Omega)} \leq c \]
for all \( \psi \in W^{1,2}(\Omega) \) and \( \sigma \in [0, 1] \) satisfying
\[ u = \sigma B(u). \]

This equation is equivalent to the boundary value problem
\[ \begin{align*}
-\Delta \rho + \rho + \tau \ln \rho &= \frac{w - v}{\tau} + 1 \quad \text{in } \Omega, \\
-\Delta u + \tau \sigma (\ln u - \ln \sigma) &= \sigma \ln \rho \quad \text{in } \Omega, \\
\nabla u \cdot \nu &= \nabla \rho \cdot \nu = 0 \quad \text{on } \partial \Omega.
\end{align*} \]

Integrate (2.43) over \( \Omega \) to get
\[ \int_{\Omega} u \, dx = -\tau \int_{\Omega} \rho \, dx - \tau^2 \int_{\Omega} \ln \rho \, dx + \int_{\Omega} v \, dx + \tau |\Omega|. \]
Obviously, (2.36) is still valid. Use it in the above equation to get
\[ \left| \int_{\Omega} u \, dx \right| \leq c. \]

Use \( u - 1 \) as a test function in (2.44) to get
\[ \int_{\Omega} |\nabla u|^2 \, dx \leq \int_{\Omega} \left( \sigma \ln \rho + \tau \sigma \ln \sigma \right) u \, dx \leq \varepsilon \int_{\Omega} u^2 \, dx + c(\varepsilon). \]

We deduce from Poincaré’s inequality that
\[ \int_{\Omega} u^2 \, dx \leq \frac{2}{\varepsilon} \left( u - \frac{1}{|\Omega|} \int_{\Omega} u \, dx \right)^2 + \frac{2}{|\Omega|} \left( \int_{\Omega} u \, dx \right)^2 \leq c \int_{\Omega} |\nabla u|^2 \, dx + c. \]

By taking \( \varepsilon \) suitably small, we obtain (2.42).

Unfortunately, when we try to pass to the limit in the system (2.39)-(2.41), we run into an insurmountable problem. That is, the existence of a non-negative solution to (1.1) remains open. This is probably not a surprise because it is well known that the bi-harmonic heat equation does not satisfy the maximum principle, i.e., solutions change sign no matter how one prescribes the initial boundary conditions. However, certain nonlinearities in fourth-order equations can allow the existence of non-negative solutions [12, 15].

3. Proof of Theorem 1.1

The proof of Theorem 1.1 will be divided into several lemmas. First, we present our approximation scheme. This is based upon Lemma 2.9. Then we proceed to derive estimates similar to those in Subsection 1.2 for our approximate problems. These estimates are shown to be sufficient to justify passing to the limit.

Let \( T > 0 \) be given. For each \( j \in \{1, 2, \cdots, \} \) we divide the time interval \([0, T]\) into \( j \) equal sub-intervals. Set
\[ \tau = \frac{T}{j}. \]

We assume that \( j \) is so large that
\[ \tau < \min \left\{ 1, \frac{1}{\| u_0 \|_{W^{2,2}(\Omega)}}, \frac{1}{8T} \right\}. \]

Let \( u_0 \) be given as in (H3). For \( k = 1, \cdots, j \), we solve recursively the system
\[ \frac{u_k - u_{k-1}}{\tau} + \rho_k - \Delta \rho_k + \tau \ln \rho_k = 1 \quad \text{in } \Omega, \]
\[ -\Delta u_k + \tau u_k = \ln \rho_k \quad \text{in } \Omega, \]
\[ \nabla \rho_k \cdot \nu = \nabla u_k = 0 \quad \text{on } \partial \Omega. \]

Set
\[ t_k = k\tau. \]
We can form the following functions on $\Omega_T$ by setting

$$
\tilde{u}_j(x,t) = \frac{t-t_{k-1}}{\tau} u_k(x) + \left(1 - \frac{t-t_{k-1}}{\tau}\right) u_{k-1}(x),
$$

$$
\bar{u}_j(x,t) = u_k(x),
$$

$$
\overline{\rho}_j(x,t) = \rho_k(x),
$$

$$
\hat{\rho}_j(x,t) = \frac{t-t_{k-1}}{\tau} \rho_k(x) + \left(1 - \frac{t-t_{k-1}}{\tau}\right) \rho_{k-1}(x),
$$

$$
\overline{G}_j(x,t) = \frac{u_k - u_{k-1}}{\tau} + \rho_k - 1 \equiv G_k,
$$

$$
\tilde{\sigma}_j(x,t) = \frac{t-t_{k-1}}{\tau} \sqrt{\rho_k(x)} + \left(1 - \frac{t-t_{k-1}}{\tau}\right) \sqrt{\rho_{k-1}(x)}
$$

whenever $x \in \Omega$, $t \in (t_{k-1}, t_k]$. In the last equation, we take

$$
(3.11) \quad \rho_0 = e^{-\Delta u_0 + \tau u_0}.
$$

Subsequently, we can rewrite (3.2)-(3.4) as

$$
(3.12) \quad \partial_t \tilde{u}_j - \Delta \overline{\rho}_j + \bar{\sigma}_j + \tau \ln \overline{\rho}_j = 1 \quad \text{in } \Omega_T,
$$

$$
(3.13) \quad -\Delta \overline{\sigma}_j + \tau \overline{\rho}_j = \ln \overline{\rho}_j \quad \text{in } \Omega_T.
$$

We proceed to derive a priori estimates for the sequences $\{\tilde{u}_j, \bar{u}_j, \overline{\rho}_j, \hat{\rho}_j, \tilde{\sigma}_j, \overline{G}_j, \ln \overline{\rho}_j\}$. The discretized version of (1.10) is the following

Lemma 3.1. We have

$$
\int_{\Omega_T} \left( (\Delta \overline{\rho}_j)^2 + (\overline{G}_j + \tau \ln \overline{\rho}_j)^2 + 2 | \nabla \overline{\rho}_j|^2 + 8\tau | \nabla \sqrt{\overline{\rho}_j}|^2 \right) \, dx dt + 2 \sup_{0 \leq t \leq T} \int_{\Omega} (\overline{\rho}_j - \ln \overline{\rho}_j) \, dx
$$

$$
+ 2\tau \int_{\Omega_T} | \nabla \overline{\rho}_j|^2 \, dx dt + 2\tau \int_{\Omega_T} (\overline{\rho}_j - 1)^2 \, dx dt + 2\tau \int_{\Omega_T} (\overline{\rho}_j - 1) \ln \overline{\rho}_j \, dx dt
$$

$$
\leq c(\Omega, N) \| e^{-\Delta u_0(x)} \|_{1, \Omega} + 4\tau \| u_0 \|_{1, \Omega}.
$$

Here $c$ depends only on $N, \Omega$.

Proof. Using (3.9), we can write (3.2) as

$$
(3.15) \quad G_k + \tau \ln \rho_k - \Delta \rho_k = 0 \quad \text{in } \Omega.
$$

Square both sides of this equation and integrate the resulting equation over $\Omega$ to derive

$$
(3.16) \quad \int_{\Omega} \left[ (G_k + \tau \ln \rho_k)^2 + (\Delta \rho_k)^2 \right] \, dx - 2 \int_{\Omega} (G_k + \tau \ln \rho_k) \Delta \rho_k \, dx = 0.
$$

We easily see that

$$
(3.17) \quad -2 \int_{\Omega} (G_k + \tau \ln \rho_k) \Delta \rho_k \, dx = -2 \int_{\Omega} \frac{u_k - u_{k-1}}{\tau} \Delta \rho_k \, dx + 2 \int_{\Omega} | \nabla \rho_k|^2 \, dx + 8\tau \int_{\Omega} | \nabla \sqrt{\rho_k}|^2 \, dx.
$$

Thus we only need to be concerned with the second integral in the above equation. For this purpose, we use $\tau(\rho_k - 1)$ as a test function in (3.2) to yield

$$
\int_{\Omega} (\rho_k - 1)(u_k - u_{k-1}) \, dx = -\tau \int_{\Omega} | \nabla \rho_k|^2 \, dx - \tau \int_{\Omega} (\rho_k - 1)^2 \, dx
$$

$$
- \tau^2 \int_{\Omega} (\rho_k - 1) \ln \rho_k \, dx.
$$
On the other hand, we can conclude from (3.3) and (3.4) that

\[
-\Delta(u_k - u_{k-1}) + \tau(u_k - u_{k-1}) = \ln \rho_k - \ln \rho_{k-1} \quad \text{in } \Omega, \tag{3.19}
\]

\[
\nabla(u_k - u_{k-1}) \cdot \nu = 0 \quad \text{on } \partial \Omega. \tag{3.20}
\]

Note that the above system also holds for \( k = 1 \) due to (3.11) and (1.4). With these in mind, we estimate

\[
-2 \int_\Omega \frac{u_k - u_{k-1}}{\tau} \Delta \rho_k \, dx = -\frac{2}{\tau} \int_\Omega (\rho_k - 1) \Delta(u_k - u_{k-1}) \, dx
\]

\[
= \frac{2}{\tau} \int_\Omega (\rho_k - 1) (\ln \rho_k - \ln \rho_{k-1}) \, dx - 2 \int_\Omega (\rho_k - 1) (u_k - u_{k-1}) \, dx
\]

\[
\geq \frac{2}{\tau} \int_\Omega [(\rho_k - \ln \rho_k) - (\rho_{k-1} - \ln \rho_{k-1})] \, dx + 2\tau \int_\Omega |\nabla \rho_k|^2 \, dx
\]

\[
+ 2\tau \int_\Omega (\rho_k - 1)^2 \, dx + 2\tau \int_\Omega (\rho_k - 1) \ln \rho_k \, dx. \tag{3.21}
\]

The last step is due to (2.2). Collecting (3.11) and (3.21) in (3.16) gives

\[
\int_\Omega \left( (\Delta \rho_k)^2 + (G_k + \tau \ln \rho_k)^2 + 2 |\nabla \rho_k|^2 + 8\tau |\nabla \sqrt{\rho_k}|^2 \right) \, dx
\]

\[
+ \frac{2}{\tau} \int_\Omega [(\rho_k - \ln \rho_k) - (\rho_{k-1} - \ln \rho_{k-1})] \, dx + 2\tau \int_\Omega |\nabla \rho_k|^2 \, dx
\]

\[
+ 2\tau \int_\Omega (\rho_k - 1)^2 \, dx + 2\tau \int_\Omega (\rho_k - 1) \ln \rho_k \, dx \leq 0.
\]

Multiplying through the inequality by \( \tau \) and summing up the resulting one over \( k \), we obtain

\[
\int_{\Omega_T} \left( (\Delta \rho_j)^2 + (G_j + \tau \ln \rho_j)^2 + 2 |\nabla \rho_j|^2 + 8\tau |\nabla \sqrt{\rho_j}|^2 \right) \, dx dt + 2 \sup_{0 \leq t \leq T} \int_\Omega (\rho_j - \ln \rho_j) \, dx
\]

\[
+ 2\tau \int_{\Omega_T} |\nabla \rho_j|^2 \, dx dt + 2\tau \int_{\Omega_T} (\rho_j - 1)^2 \, dx dt + 2\tau \int_{\Omega_T} (\rho_j - 1) \ln \rho_j \, dx dt
\]

\[
\leq 4 \int_{\Omega_T} \left( e^{-\Delta u_0(x)} + \tau u_0(x) + \Delta u_0 - \tau u_0 \right) \, dx \leq 4e^{\tau \|u_0\|_{\infty, \Omega}} \int_{\Omega} e^{-\Delta u_0(x)} \, dx - 4\tau \int_{\Omega} u_0 \, dx
\]

\[
\leq c(\Omega, N) \|e^{-\Delta u_0(x)}\|_{1, \Omega} + 4\tau \|u_0\|_{1, \Omega}.
\]

The last step is due to (3.1) and (1.6). This finishes the proof. \( \square \)

An immediate consequence of this lemma is

\[
\sup_{0 \leq t \leq T} \int_\Omega (\rho_j + |\ln \rho_j|) \, dx \leq c(\Omega, N) \varepsilon_0 + (16\|u_0\|_{1, \Omega} + 4|\Omega|T) \tau. \tag{3.22}
\]

To see this, we first integrate (3.13) over \( \Omega \) to obtain

\[
\tau \int_\Omega \bar{u}_j \, dx = \int_\Omega \ln \rho_j \, dx.
\]

Then we calculate

\[
\int_\Omega \rho_j \, dx = \int_\Omega (\rho_j - \ln \rho_j) \, dx + \int_\Omega \ln \rho_j \, dx
\]

\[
\leq c(\Omega, N) \varepsilon_0 + 2\|u_0\|_{1, \Omega} \tau + \tau \int_\Omega \bar{u}_j \, dx. \tag{3.23}
\]
On the other hand,
\[
\int_{\Omega} |\ln \overline{p_j}| \, dx = \int_{\Omega} \left( \ln^+ \overline{p_j} + \ln^- \overline{p_j} \right) \, dx \\
= -2 \int_{\{\overline{p_j} < 1\}} \ln \overline{p_j} \, dx \quad + \quad \int_{\Omega} \ln \overline{p_j} \, dx \\
\leq 2 \int_{\Omega} (\overline{p_j} - \ln \overline{p_j}) \, dx + \tau \int_{\Omega} \overline{u_j} \, dx \\
\leq c(\Omega, N) \varepsilon_0 + 4\|u_0\|_{1,\Omega} + \tau \int_{\Omega} \overline{u_j} \, dx.
\]

Here we have used the fact that \( \overline{p_j} - \ln \overline{p_j} > 0 \). Adding this inequality to (3.23) gives
\[
(3.24) \quad \int_{\Omega} \overline{p_j} \, dx + \int_{\Omega} |\ln \overline{p_j}| \, dx \leq c(\Omega, N) \varepsilon_0 + 6\|u_0\|_{1,\Omega} + 2\tau \int_{\Omega} \overline{u_j} \, dx.
\]

We integrate (3.2) over \( \Omega \) to derive
\[
\int_{\Omega} \frac{u_k - u_{k-1}}{\tau} \, dx = - \int_{\Omega} \rho_k \, dx - \tau \int_{\Omega} \ln \rho_k \, dx + |\Omega|.
\]

Multiply through this equation by \( \tau \) and sum up the resulting equation over \( k \) to get
\[
\sup_{0 \leq t \leq T} \int_{\Omega} \overline{u_j} \, dx \leq \int_{\Omega} u_0 \, dx + \int_{\Omega} \overline{p_j} \, dx + \tau \int_{\Omega} |\ln \overline{p_j}| \, dx dt + |\Omega|T
\]
\[
\leq \|u_0\|_{1,\Omega} + T \sup_{0 \leq t \leq T} \int_{\Omega} \overline{p_j} \, dx + \tau T \sup_{0 \leq t \leq T} \int_{\Omega} |\ln \overline{p_j}| \, dx + |\Omega|T.
\]

(3.25) Keeping this in mind, we deduce from (3.24) that
\[
\sup_{0 \leq t \leq T} \int_{\Omega} (\overline{p_j} + |\ln \overline{p_j}|) \, dx \leq c(\Omega, N) \varepsilon_0 + 6\|u_0\|_{1,\Omega} + 2\sup_{0 \leq t \leq T} \int_{\Omega} \overline{u_j} \, dx \\
\leq c(\Omega, N) \varepsilon_0 + 6\|u_0\|_{1,\Omega} + 4\varepsilon_0 + 4\sup_{0 \leq t \leq T} \int_{\Omega} (\overline{p_j} + |\ln \overline{p_j}|) \, dx
\]
\[
+ 2|\Omega|T\tau.
\]

(3.26) According to (3.1), \( 4\tau T < \frac{1}{2} \). Use this in the above inequality to get (3.22).

Now we are ready to obtain a discretized version of (1.12).

Lemma 3.2. We have
\[
\sup_{0 \leq t \leq T} \int_{\Omega} \left( \frac{1}{2} \overline{G_j}^2 + \left( \frac{1}{2} + \frac{\tau}{2} \right) |\nabla \overline{p_j}|^2 + \frac{\tau}{2} (\overline{p_j} - 1)^2 \right) \, dx + 2 \int_{\Omega_T} (\partial_\tau \overline{\sigma_j})^2 \, dx dt \\
+ \sup_{0 \leq t \leq T} \tau^2 \int_{\{\overline{p_j} > 1\}} \overline{p_j} \ln \overline{p_j} \, dx + \sup_{0 \leq t \leq T} \tau \int_{\Omega} (\overline{p_j} - \ln \overline{p_j}) \, dx
\]
\[
\leq c\varepsilon_0 + (c + cT)\tau.
\]

Here \( c \) depends only on \( \Omega, N \).

Proof. Define
\[
(3.28) \quad G_0 = \Delta \rho_0 - \tau \ln \rho_0.
\]

This combined with (3.15) implies that
\[
(3.29) \quad \frac{G_k - G_{k-1}}{\tau} - \Delta \left( \frac{\rho_k - \rho_{k-1}}{\tau} \right) + \ln \rho_k - \ln \rho_{k-1} = 0 \quad \text{in} \quad \Omega \quad \text{for each} \quad k \in \{1, 2, 3, \cdots, j\}.
\]
We can easily derive from (1.4) that
\[ \nabla \rho_0 \cdot \nu = 0 \] on \( \partial \Omega \).
Thus, we can use \( G_k \) as a test function in (3.29) (even for \( k = 1 \)) to get
\[ \frac{1}{\tau} \int_{\Omega} (G_k - G_{k-1}) G_k \, dx \]
\[ + \frac{1}{\tau} \int_{\Omega} \nabla G_k \cdot \nabla (\rho_k - \rho_{k-1}) \, dx + \int_{\Omega} (\ln \rho_k - \ln \rho_{k-1}) G_k \, dx = 0. \]
(3.30)

Once again, we can use (3) in Lemma 2.4 to handle the first term. The second integral in (3.30) can be evaluated as follows:
\[ \frac{1}{\tau} \int_{\Omega} \nabla G_k \cdot \nabla (\rho_k - \rho_{k-1}) \, dx \]
\[ = -\frac{1}{\tau} \int_{\Omega} (\rho_k - \rho_{k-1}) \Delta \left( \frac{u_k - u_{k-1}}{\tau} \right) \, dx + \frac{1}{\tau} \int_{\Omega} \nabla \rho_k \cdot \nabla (\rho_k - \rho_{k-1}) \, dx \]
\[ \geq \frac{1}{\tau^2} \int_{\Omega} (\rho_k - \rho_{k-1}) (\ln \rho_k - \ln \rho_{k-1}) \, dx - \frac{1}{\tau} \int_{\Omega} (\rho_k - \rho_{k-1})(u_k - u_{k-1}) \, dx \]
\[ + \frac{1}{2\tau} \int_{\Omega} (|\nabla \rho_k|^2 - |\nabla \rho_{k-1}|^2) \, dx \]
\[ \geq \frac{2}{\tau^2} \int_{\Omega} (\sqrt{\rho_k} - \sqrt{\rho_{k-1}})^2 \, dx - \frac{1}{\tau} \int_{\Omega} (\rho_k - \rho_{k-1})(u_k - u_{k-1}) \, dx \]
\[ + \frac{1}{2\tau} \int_{\Omega} (|\nabla \rho_k|^2 - |\nabla \rho_{k-1}|^2) \, dx. \]
(3.31)

The last step is due to (2.4). To estimate the second to last integral in (3.31), we use \( \rho_k - \rho_{k-1} \) as a test function in (3.2) and then apply (3) and (2.3) in Lemma 2.4 to obtain
\[ -\frac{1}{\tau} \int_{\Omega} (u_k - u_{k-1})(\rho_k - \rho_{k-1}) \, dx \]
\[ = \int_{\Omega} \nabla \rho_k \nabla (\rho_k - \rho_{k-1}) \, dx + \int_{\Omega} (\rho_k - 1)(\rho_k - \rho_{k-1}) \, dx \]
\[ + \tau \int_{\Omega} \ln \rho_k(\rho_k - \rho_{k-1}) \, dx \]
\[ \geq \frac{1}{2} \int_{\Omega} (|\nabla \rho_k|^2 - |\nabla \rho_{k-1}|^2) \, dx + \frac{1}{2} \int_{\Omega} ((\rho_k - 1)^2 - (\rho_{k-1} - 1)^2) \, dx \]
\[ + \tau \int_{\Omega} (\rho_k \ln \rho_k - \rho_{k-1} \ln \rho_{k-1}) \, dx - \tau \int_{\Omega} (\rho_k - \rho_{k-1}) \, dx. \]

Calculating the third integral in (3.30), we invoke (2.2) to obtain
\[ \int_{\Omega} (\ln \rho_k - \ln \rho_{k-1}) G_k \, dx \]
\[ = \frac{1}{\tau} \int_{\Omega} (\ln \rho_k - \ln \rho_{k-1})(u_k - u_{k-1}) \, dx \]
\[ + \int_{\Omega} (\ln \rho_k - \ln \rho_{k-1})(\rho_k - 1) \, dx \]
\[ \geq \int_{\Omega} (-\Delta (u_k - u_{k-1}) + \tau(u_k - u_{k-1}))(u_k - u_{k-1}) \, dx \]
\[ + \int_{\Omega} (\rho_k - \rho_{k-1}) \, dx - \int_{\Omega} (\ln \rho_k - \ln \rho_{k-1}) \, dx \]
\[ \geq \int_{\Omega} [(\rho_k - \ln \rho_k) - (\rho_{k-1} - \ln \rho_{k-1})] \, dx. \]
Using the preceding results in \(3.30\) yields
\[
\frac{1}{2\tau} \int_{\Omega} (G_k^2 - G_{k-1}^2) \, dx + \left( \frac{1}{2\tau} + \frac{1}{2} \right) \int_{\Omega} (|\nabla \rho_k|^2 - |\nabla \rho_{k-1}|^2) \, dx \\
+ 2 \int_{\Omega} \left( \frac{\sqrt{\rho_k} - \sqrt{\rho_{k-1}}}{\tau} \right)^2 \, dx + \frac{1}{2} \int_{\Omega} ((\rho_k - 1)^2 - (\rho_{k-1} - 1)^2) \, dx \\
+ \tau \int_{\Omega} (\rho_k \ln \rho_k - \rho_{k-1} \ln \rho_{k-1}) \, dx - \tau \int_{\Omega} (\rho_k - \rho_{k-1}) \, dx \\
+ \int_{\Omega} [(\rho_k - \ln \rho_k) - (\rho_{k-1} - \ln \rho_{k-1})] \, dx \leq 0.
\]
Multiply through the inequality by \(\tau\), sum the resulting inequality over \(k\), and thereby obtain
\[
\sup_{0 \leq t \leq T} \int_{\Omega} \left( \frac{1}{2} G_j^2 + \left( \frac{1}{2} + \frac{1}{2} \right) |\nabla \rho_j|^2 + \frac{\tau}{2} (\rho_j - 1)^2 \right) \, dx + 2 \int_{\Omega_T} (\partial_t \sigma_j)^2 \, dxdt \\
+ \sup_{0 \leq t \leq T} \tau^2 \int_{\{\rho_j > 1\}} \rho_j \ln \rho_j \, dx + \sup_{0 \leq t \leq T} \tau \int_{\Omega} (\rho_j - \ln \rho_j) \, dx \\
\leq \int_{\Omega} G_0^2 \, dx + \int_{\Omega} |\nabla \rho_0|^2 \, dx + \tau \int_{\Omega} (\rho_0 - 1)^2 \, dx - 2 \sup_{0 \leq t \leq T} \tau^2 \int_{\{\rho_j > 1\}} \rho_j \ln \rho_j \, dx \\
+ 2\tau^2 \sup_{0 \leq t \leq T} \int_{\Omega} \rho_j \, dx + \tau \int_{\Omega} (\rho_0 - \ln \rho_0) \, dx.
\]
(3.32)
We calculate from \(3.11\) and \(3.28\) that
\[
\nabla \rho_0 = e^{\tau u_0} \nabla e^{-\Delta u_0} + \tau e^{-\Delta u_0} e^{\tau u_0} \nabla u_0,
\]
\[
\Delta \rho_0 = e^{\tau u_0} \Delta e^{-\Delta u_0} + 2\tau \nabla e^{-\Delta u_0} \cdot e^{\tau u_0} \nabla u_0 + \tau e^{-\Delta u_0 + \tau u_0} (\tau |\nabla u_0|^2 + \Delta u_0).
\]
We can derive from \(1.1\) and \(3.1\) that
\[
\int_{\Omega} |\nabla \rho_0|^2 \, dx \leq c \varepsilon_0^2, \quad \int_{\Omega} G_0^2 \, dx \leq c \varepsilon_0^2.
\]
Here \(c\) depends only on \(\Omega, N\). Moreover,
\[
\int_{\Omega} \rho_0^2 \, dx = \int_{\Omega} e^{-2\Delta u_0 + 2\tau u_0} \, dx \leq c \varepsilon_0^2,
\]
\[
\int_{\Omega} (\rho_0 - \ln \rho_0) \, dx = \int_{\Omega} e^{-\Delta u_0 + \tau u_0} \, dx - \tau \int_{\Omega} u_0 \, dx \leq c \varepsilon_0 + 1,
\]
\[
\tau^2 \sup_{0 \leq t \leq T} \int_{\Omega} \rho_j \, dx \leq c \varepsilon_0 + c \tau + c T \tau^3.
\]
The last inequality is due to Lemma 3.1. Collecting these estimates in (3.32) gives the lemma. 
\(\square\)

**Lemma 3.3.** The sequence \(\{\rho_j\}\) is bounded \(W^{1,2}(\Omega_T)\).

**Proof.** Since \(u_0 \in L^\infty(\Omega)\) we can infer from Lemma 2.9 that
\[
\ln \rho_j(\cdot, t) \in L^\infty(\Omega) \quad \text{for each} \ j \ \text{and each} \ t \in [0, T].
\]
Thus we can use \(\tau \ln \rho_k\) as a test function in (3.15) to get
\[
\int_{\Omega} \tau^2 \ln^2 \rho_k \, dx \leq - \int_{\Omega} G_k \tau \ln \rho_k \, dx \leq \frac{1}{2} \int_{\Omega} G_k^2 \, dx + \frac{1}{2} \int_{\Omega} \tau^2 \ln^2 \rho_k \, dx,
\]
from whence follows
\[
\int_{\Omega} \tau^2 \ln^2 \rho_j \, dx \leq \int_{\Omega} G_j^2 \, dx.
\]
By virtue of (H2) and Lemma 2.6, there is a positive number $c = c(\Omega)$ such that
\[
\|\rho_j(-, t)\|_{\infty, \Omega} \leq c\|\rho_j(-, t)\|_{1, \Omega} + c\|G_j + \tau \ln \rho_j\|_{2, \Omega} 
\]  
(3.34)
\[
\leq c\|\rho_j(-, t)\|_{1, \Omega} + c\|G_j\|_{2, \Omega} \leq c. 
\]
We are ready to estimate, with the aid of Lemma 3.2, that
\[
\int_{\Omega_T} (\partial_t \tilde{\rho}_j)^2 \, dx \, dt = \sum_{k=1}^{j} \int_{t_{k-1}}^{t_k} \int_{\Omega} \left( \frac{\rho_k - \rho_{k-1}}{\tau} \right)^2 \, dx \, dt 
\]
\[
= \sum_{k=1}^{j} \int_{t_{k-1}}^{t_k} \int_{\Omega} \left( \sqrt{\rho_k} + \sqrt{\rho_{k-1}} \right)^2 \left( \frac{\sqrt{\rho_k} - \sqrt{\rho_{k-1}}}{\tau} \right)^2 \, dx \, dt 
\]
(3.35)
\[
\leq 4\|\rho_j\|_{\infty, \Omega_T} \int_{\Omega_T} (\partial_t \tilde{\rho}_j)^2 \, dx \, dt \leq c. 
\]
As for the gradient with respect to the space variables, we have
\[
\int_{\Omega_T} |\nabla \tilde{\rho}_j|^2 \, dx \, dt = \sum_{k=1}^{j} \int_{t_{k-1}}^{t_k} \int_{\Omega} \left| \frac{\tau t - \tau t_{k-1}}{\tau} \nabla \rho_k + \left( 1 - \frac{\tau t - \tau t_{k-1}}{\tau} \right) \nabla \rho_{k-1} \right|^2 \, dx \, dt 
\]
\[
\leq \sum_{k=1}^{j} \int_{t_{k-1}}^{t_k} \left[ \frac{\tau t - \tau t_{k-1}}{\tau} \int_{\Omega} |\nabla \rho_k|^2 \, dx + \left( 1 - \frac{\tau t - \tau t_{k-1}}{\tau} \right) \int_{\Omega} |\nabla \rho_{k-1}|^2 \, dx \right] \, dt 
\]
\[
= \sum_{k=1}^{j} \tau \left( \int_{\Omega} |\nabla \rho_k|^2 \, dx + \int_{\Omega} |\nabla \rho_{k-1}|^2 \, dx \right) \int_{\Omega_T} |\nabla \tilde{\rho}_j|^2 \, dx \, dt \leq c. 
\]
(3.36)
The last step is due to Lemma 3.1. The proof is complete. \(\square\)

It follows that \(\{\tilde{\rho}_j\}\) is precompact in \(L^2(\Omega_T)\). Note that
\[
\int_{\Omega_T} |\tilde{\rho}_j - \rho_j|^2 \, dx \, dt = \sum_{k=1}^{j} \int_{t_{k-1}}^{t_k} \int_{\Omega} \left( \frac{\rho_k - \rho_{k-1}}{\tau} \right)^2 \, dx \, dt 
\]
\[
= \sum_{k=1}^{j} \frac{\tau^3}{2} \int_{\Omega} (\partial_t \tilde{\rho}_j)^2 \, dx 
\]
\[
= \frac{\tau^2}{2} \int_{\Omega_T} (\partial_t \tilde{\rho}_j)^2 \, dx \, dt \leq c\tau^2. 
\]
Subsequently, \(\{\rho_j\}\) is also precompact in \(L^2(\Omega_T)\). As a result, we can select a subsequence of \(\{\rho_j\}\), still denoted by \(\{\rho_j\}\), such that
\[
\rho_j \text{ converges a.e. on } \Omega_T. 
\]

We define a function \(\tilde{F}_j(x, t)\) on \(\Omega_T\) as follows: For each \((x, t) \in \Omega_T\) there is a \(k \in \{1, 2, \cdots, j\}\) such that \(t \in (t_{k-1}, t_k]\). Subsequently, we set
\[
\tilde{F}_j(x, t) = \frac{t - t_{k-1}}{\tau} (G_k(x) + \tau \ln \rho_k(x)) + \left( 1 - \frac{t - t_{k-1}}{\tau} \right) (G_{k-1}(x) + \tau \ln \rho_{k-1}(x)). 
\]
We can write (3.15) as
\[
(3.37) \quad -\Delta \tilde{\rho}_j = -\tilde{F}_j \text{ in } \Omega_T. 
\]
Of course, we also have the boundary condition
\begin{equation}
\nabla \tilde{\rho}_j \cdot \nu = 0 \quad \text{on } \partial \Omega \times (0, T).
\end{equation}

Set
\begin{equation}
\tilde{w}_j = \frac{1}{\tilde{\rho}_j}.
\end{equation}

We can infer from (2.23) that $\tilde{\rho}_j$ is bounded away from 0 below for each fixed $j$. Thus $\tilde{w}_j$ is well-defined. An elementary calculation from (3.37) and (3.38) shows that $\tilde{w}_j$ satisfies the problem
\begin{align*}
-\Delta \tilde{w}_j + 2\tilde{w}_j^{-1}|\nabla \tilde{w}_j|^2 &= \tilde{F}_j \tilde{w}_j^2 \quad \text{in } \Omega, \\
\nabla \tilde{w}_j \cdot \nu &= 0 \quad \text{on } \partial \Omega
\end{align*}
for each $t \in [0, T].$

**Lemma 3.4.** The sequence $\{\tilde{w}_j\}$ is bounded in $L^\infty(\Omega_T)$.

**Proof.** The proof largely mimics what we did in Subsection 1.2. First, we can deduce from (3.33) and Lemmas 3.1 and 3.2 that
\begin{align*}
\sup_{0 \leq t \leq T} \| \ln \tilde{\rho}(-,t) \|_{1,\Omega} + \sup_{0 \leq t \leq T} \| \tilde{F}_j (-,t) \|_{2,\Omega} \\
\quad \leq \sup_{0 \leq t \leq T} \| \ln \tilde{\rho}(-,t) \|_{1,\Omega} + 2 \sup_{0 \leq t \leq T} \| \tilde{G}_j (-,t) \|_{2,\Omega} \\
\quad \leq c(\varepsilon_0 + c_1 \tau).
\end{align*}

Here $c$ depends on $\Omega$ only, while $c_1$ depends on both $\|u_0\|_{1,\Omega}$ and $\Omega_T$. Now pick a number $L$ from $(1, \infty)$. Note that $-\ln s$ is convex on $(0, 1)$. Also, for each $t \in [0, T]$ there is a $k \in \{1, 2, \cdots, j\}$ such that $t \in (t_{k-1}, t_k]$. With these in mind, we derive that
\begin{align*}
|\{\tilde{w}_j > L\}| &= \left| \left\{ \tilde{\rho}_j < \frac{1}{L} \right\} \right| \\
&\leq \frac{1}{\ln L} \int_{\{\tilde{\rho}_j \leq \frac{1}{L}\}} |\ln \tilde{\rho}_j| dx \\
&\leq \frac{1}{\ln L} \int_{\{\tilde{\rho}_j \leq \frac{1}{L}\}} \left( \frac{t - t_{k-1}}{\tau} |\ln \rho_k(x)| + \left(1 - \frac{t - t_{k-1}}{\tau}\right) |\ln \rho_{k-1}(x)| \right) dx \\
&\leq \frac{\sup_{0 \leq t \leq T} \| \ln \tilde{\rho}(-,t) \|_{1,\Omega}}{\ln L} \leq c(\varepsilon_0 + c_1 \tau).
\end{align*}

This is the new version of (1.14). For the remainder, all we need to do is to substitute $\{w, \varepsilon_0, G\}$ in Subsection 1.2 for $\{\tilde{w}_j, \varepsilon_0 + c_1 \tau, \tilde{F}_j\}$ here. Thus, the new version of (1.15) is: Lemma 2.6 asserts that there is a positive number $c = c(\Omega)$ such that
\begin{align*}
\|\tilde{w}_j\|_{\infty, \Omega} &\leq c \|\tilde{w}_j\|_{1,\Omega} + c \|\tilde{F}_j \tilde{w}_j^2\|_{2,\Omega} \\
&\leq c \int_{\{\tilde{w}_j > L\}} \tilde{w}_j dx + c \int_{\{\tilde{w}_j \leq L\}} \tilde{w}_j dx + c(\varepsilon_0 + c_1 \tau) \|\tilde{w}_j\|_{\infty, \Omega}^2 \\
&\leq c \|\tilde{w}_j\|_{\infty, \Omega} \left|\{\tilde{w}_j > L\}\right| + cL + c(\varepsilon_0 + c_1 \tau) \|\tilde{w}_j\|_{\infty, \Omega}^2 \\
&\leq \frac{c(\varepsilon_0 + c_1 \tau) \|\tilde{w}_j\|_{\infty, \Omega}}{\ln L} + cL + c(\varepsilon_0 + c_1 \tau) \|\tilde{w}_j\|_{\infty, \Omega}^2.
\end{align*}

Recall that $\tilde{\rho}_j$ is piece-wise linear in the time variable $t$. Thus $\|\tilde{\rho}_j(-,t)\|_{\infty, \Omega}$ is a continuous function of $t$ for each fixed $j$. As we mentioned earlier, $\tilde{\rho}_j$ is bounded away from 0 below for each fixed $j$. We can conclude that $\|\tilde{w}_j(-,t)\|_{\infty, \Omega}$ is also a continuous function of $t$ for each fixed $j$. This enables us to apply the proof of Lemma 2.7. Let $L_0$, $s_0$, $s_1$, $g$ be determined as in Subsection 1.2. If
\begin{equation}
\varepsilon_0 + c_1 \tau < s_0, \quad c\Delta u_0 < s_1
\end{equation}
then
\[ \|\tilde{w}_j(\cdot, t)\|_{\infty, \Omega} \leq g(\varepsilon_0 + c_1 \tau, L_0) \leq g(\varepsilon_0 + c_1, L_0) \] for all \( t > 0 \).

On account of (1.5), (3.39) holds for \( j \) sufficiently large. Hence (3.40) also holds for \( j \) sufficiently large. This completes the proof of the lemma.

\[ \square \]

**Lemma 3.5.** The sequence \( \{\tilde{u}_j\} \) is bounded in \( W^{1,2}(\Omega_T) \).

**Proof.** First, we derive from Lemma 3.2 and (3.34) that
\[ \int_{\Omega} (\partial_t \tilde{u}_j)^2 \, dx \leq 2 \int_{\Omega} G_j^2 \, dx + 2 \int_{\Omega} (\tilde{p}_j - 1)^2 \, dx \leq c. \]

Use \( \bar{u}_j \) as a test function in (3.13) to get
\[ (3.41) \int_{\Omega} |\nabla \bar{u}_j|^2 \, dx + \tau \int_{\Omega} \bar{u}_j^2 \, dx = \int_{\Omega} \bar{u}_j \ln \tilde{p}_j \, dx. \]

By (3.25) and Poincaré’s inequality, we have
\[ \int_{\Omega} \bar{u}_j \ln \tilde{p}_j \, dx = \int_{\Omega} \left( \bar{u}_j - \frac{1}{|\Omega|} \int_{\Omega} \bar{u}_j \, dx \right) \ln \tilde{p}_j \, dx + \frac{1}{|\Omega|} \int_{\Omega} \bar{u}_j \, dx \int_{\Omega} \ln \tilde{p}_j \, dx \]
\[ \leq \varepsilon \int_{\Omega} \left( \bar{u}_j - \frac{1}{|\Omega|} \int_{\Omega} \bar{u}_j \, dx \right)^2 \, dx + c(\varepsilon) \]
\[ \leq c \varepsilon \int_{\Omega} |\nabla \bar{u}_j|^2 \, dx + c. \]

Use this in (3.41) to get
\[ \sup_{0 \leq t \leq T} \int_{\Omega} |\nabla \bar{u}_j|^2 \, dx \leq c. \]

Use Poincaré’s inequality again to derive
\[ \int_{\Omega} \bar{u}_j^2 \, dx \leq 2 \int_{\Omega} \left( \bar{u}_j - \frac{1}{|\Omega|} \int_{\Omega} \bar{u}_j \, dx \right)^2 \, dx + \frac{2}{|\Omega|} \left( \int_{\Omega} \bar{u}_j \, dx \right)^2 \leq c. \]

By a calculation similar to (3.36), we obtain
\[ \|\tilde{u}_j\|_{W^{1,2}(\Omega)} \leq c. \]

The proof is complete.

To finish the proof Theorem 1.1, we square both sides of (3.13) and integrate to get
\[ \int_{\Omega} (\Delta \bar{p}_j)^2 \, dx + 2\tau \int_{\Omega} |\nabla \bar{u}_j|^2 \, dx + \tau^2 \int_{\Omega} \bar{u}_j^2 \, dx = \int_{\Omega} \ln^2 \tilde{p}_j \, dx \leq c. \]

This together with (2.1) implies
\[ \sup_{0 \leq t \leq T} \|\bar{u}_j\|_{W^{2,2}(\Omega)} \leq c. \]

Similarly,
\[ \sup_{0 \leq t \leq T} \|\bar{p}_j\|_{W^{2,2}(\Omega)} \leq c. \]

We are ready to pass to the limit in the system (3.12)–(3.13). This completes the proof of Theorem 1.1.
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