ANALYTIC INTEGRABILITY OF A CLASS OF PLANAR POLYNOMIAL DIFFERENTIAL SYSTEMS

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Abstract. In this paper we find necessary and sufficient conditions in order that the differential systems of the form $\dot{x} = xf(y)$, $\dot{y} = g(y)$, with $f$ and $g$ polynomials, have a first integral which is analytic in the variable $x$ and meromorphic in the variable $y$. We also characterize their analytic first integrals in both variables $x$ and $y$.

These polynomial differential systems are important because after a convenient change of variables they contain all quasi–homogeneous polynomial differential systems in $\mathbb{R}^2$.

1. Introduction and statement of the main results. Let $\mathbb{C}$ be the set of complex numbers and $\mathbb{C}[y]$ the ring of all polynomials in the variable $y$ with coefficients in $\mathbb{C}$. In this paper we consider the polynomial differential systems of the form

$$\dot{x} = xf(y), \quad \dot{y} = g(y),$$

(1)

where $f, g \in \mathbb{C}[y]$ and are coprime. The dot denotes the derivative with respect to the independent variable $t$ real or complex. We denote by $X = (xf(y), g(y))$ the polynomial vector field associated to system (1), and we say that the degree of the system is $n = \max\{\deg xf(y), \deg g(y)\}$. For the sake of simplicity, we assume for the rest of the paper that system (1) is not linear, that is $n > 1$.

We recall that given a planar polynomial differential system (1), we say that a function $H: \mathcal{U} \subset \mathbb{C}^2 \to \mathbb{C}$ with $\mathcal{U}$ an open set, is a first integral of system (1) if $H$ is continuous, not locally constant and constant on each trajectory of the system

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contained in $\mathcal{U}$. We note that if $H$ is of class at least $C^1$ in $\mathcal{U}$, then $H$ is a first integral if it is not locally constant and

$$xf(y)\frac{\partial H}{\partial x} + g(y)\frac{\partial H}{\partial y} = 0$$

in $\mathcal{U}$. We call the integrability problem the problem of finding such a first integral and the functional class where it belongs. We say the system has an analytic first integral if it is not locally constant and $xf(y)\frac{\partial H}{\partial x} + g(y)\frac{\partial H}{\partial y} = 0$ in $\mathcal{U}$. We call the integrability problem the problem of finding such a first integral and the functional class where it belongs. We say the system has a pseudo-meromorphic first integral if there exists a first integral $H(x, y)$ which is an analytic function in the variable $x$ and a meromorphic function in the variable $y$.

The aim of this paper is to characterize the existence of first integrals of system (1) that can be described by functions that are analytic or pseudo-meromorphic.

Let $\alpha_l$ for $l = 1, \ldots, k$ be the zeros of $g$. We say that $g$ is square-free if $g(y) = \prod_{l=1}^{k}(y - \alpha_l)$ with $\alpha_l \neq \alpha_j$ for $l, j = 1, \ldots, k$ and $l \neq j$. When $g$ is square-free we define $\gamma_l = f(\alpha_l)/g'(\alpha_l)$ for $l = 1, \ldots, k$. With this notation we introduce the main result of the paper.

**Theorem 1.1.** System (1) has a pseudo-meromorphic first integral if and only if $g(y)$ is square-free. Moreover, if $\gamma_l < 0$ for all $l = 1, \ldots, k$ then the first integral is analytic, otherwise it is a pseudo-meromorphic function with poles on $y = \alpha_l$ if $\gamma_l < 0$.

The proof of Theorem 1.1 is given in section 2. Furthermore, the specific form of the first integral is given in the proof of Theorem 1.1.

**Example 1.** Consider the differential system

$$\dot{x} = xy^3, \quad \dot{y} = y + 1.$$ 

This system has the analytic first integral

$$H(x, y) = e^{-(y+1)(2y^2-5y+11)/6}x(y+1).$$

Note that $g(y) = y + 1$ is square-free, $\alpha_1 = -1$ and $\gamma_1 = -1 < 0$.

**Example 2.** Consider the differential system

$$\dot{x} = xy^3, \quad \dot{y} = y - 1.$$ 

This system has the pseudo-meromorphic first integral

$$H(x, y) = e^{(1-y)(2y^2+5y+11)/6}x(y-1).$$

Note that $g(y) = y - 1$ is square-free, $\alpha_1 = 1$ and $\gamma_1 = 1 > 0$.

System (1) is of separate variables and appears in many situations. In Lemma 2.2 of [1] it is proved that there exists a blow-up change of variables that transforms any quasi-homogeneous polynomial differential system into a differential system (1). However we point out that not all the planar polynomial differential systems (1) come from quasi-homogenous polynomial differential systems. We recall that a polynomial differential system

$$\dot{x} = P(x, y) \quad \dot{y} = Q(x, y)$$

is quasi-homogeneous if there exists $s_1, s_2, d \in \mathbb{N}$ (here $\mathbb{N}$ denotes the set of positive integers) such that for arbitrary $\alpha \in \mathbb{C}$,

$$P(\alpha^{s_1}x, \alpha^{s_2}y) = \alpha^{s_1-1+d}P(x, y), \quad Q(\alpha^{s_1}x, \alpha^{s_2}y) = \alpha^{s_2-1+d}Q(x, y).$$ (2)
From Theorem 3.1b) of [1] and Proposition 1 of [3] it follows the next result.

**Theorem 1.2.** The quasi–homogeneous polynomial differential system (2) has an analytic first integral if and only if \( g(y) \) is square–free, \( \deg f < \deg g \) and \( \gamma_i \in \mathbb{Q} \) for \( i = 1, 2, \ldots, k \) and \( 1 + \gamma_1 + \gamma_2 + \cdots + \gamma_k \geq 0 \).

Note that Theorem 1.1 extends the result of Theorem 1.2 only valid for the quasi–homogeneous polynomial differential systems. Recall that quasi–homogeneous polynomial differential systems can be written as a subclass of the polynomial differential systems (1) using the Lemma 2.2 of [1].

2. **Proof of Theorem 1.1.** Assume that system (1) has a pseudo–meromorphic first integral. Then it can be written as a power series in \( x \) in the form

\[
H(x, y) = \sum_{l \geq 0} a_l(y)x^l, \tag{3}
\]

where \( a_l(y) \) is a meromorphic function in the variable \( y \). Then, it must satisfy

\[
xf(y) \frac{\partial H}{\partial x} + g(y) \frac{\partial H}{\partial y} = 0,
\]

that is

\[
0 = \sum_{l \geq 0} lf(y)a_l(y)x^l + \sum_{l \geq 0} g(y)a'_l(y)x^l = \sum_{l \geq 0} (lf(y)a_l(y) + g(y)a'_l(y))x^l.
\]

Hence,

\[
a'_0(y) = 0 \quad \text{that is} \quad a_0(y) = \text{constant}
\]

and for \( l \geq 1 \),

\[
lf(y)a_l(y) + g(y)a'_l(y) = 0 \quad \text{that is} \quad \frac{a'_l(y)}{a_l(y)} = \frac{-lf(y)}{g(y)}. \tag{4}
\]

If \( \deg f \geq \deg g \) and we consider the division of \(-lf(y)\) by \( g(y)\) we can write

\[
lf(y) = q(y)g(y) + r(y),
\]

where \( r(y) \) cannot be zero taking into account that \( f \) and \( g \) are coprime and \( \deg r < \deg g \). Hence equation (4) takes the form

\[
\frac{a'_l(y)}{a_l(y)} = -\frac{q(y) - r(y)}{g(y)}.
\]

Integrating this equation we have

\[
a_l(y) = C_l e^{-Q(y)} e^{\int \frac{r(v)}{g(v)} dv} \tag{5}
\]

where \( C_l \) is a constant of integration and \( Q'(y) = q(y) \). Note that inserting \( a_l(y) \) in the function \( H \) given in (3) we get

\[
H(x, y) = e^{-Q(y)} e^{\int \frac{r(v)}{g(v)} dv} \sum_{l \geq 0} C_l x^l.
\]

Since we are assuming that \( H \) is a pseudo-meromorphic first integral, it must be a meromorphic function in the variable \( y \), i.e.,

\[
e^{-Q(y)} e^{\int \frac{r(v)}{g(v)} dv} \tag{6}
\]

must be a meromorphic function in the variable \( y \). Therefore, since the first factor of (6) is an analytic function, we must study the second factor in (6).
Assume that \( g \) is not square free. Using an affine transformation of the form \( z = y + \alpha \) with \( \alpha \in \mathbb{C} \) if it is necessary, we can assume that \( z \) is a multiple of \( g \), that is, \( \hat{g}(z) = z^m R(z) \), where \( \hat{g}(z) = g(z - \alpha) \) with \( m > 1 \) an integer and \( R(0) \neq 0 \). Since \( f \) and \( g \) are coprime we also have \( \hat{r}(0) \neq 0 \), where \( \hat{r}(z) = r(z - \alpha) \). Now we develop \( \hat{r}(z)/\hat{g}(z) \) in simple fractions of \( z \), that is,

\[
\frac{\hat{r}(z)}{\hat{g}(z)} = \frac{c_m}{z^m} + \frac{c_m-1}{z^{m-1}} + \cdots + \frac{c_1}{z} + \frac{\alpha(z)}{R(z)},
\]

where \( \alpha(z) \) is a polynomial with \( \deg \alpha(z) < \deg R(z) \), and \( c_i \in \mathbb{C} \) for \( i = 1, \ldots, m \). Note that \( c_m \neq 0 \). Therefore integrating this last expression we have

\[
\exp \left( \int \frac{\hat{r}(z)}{\hat{g}(z)} \, dz \right) = \exp \left( \frac{c_m}{(1-m)z^{m-1}} \right) \cdot \exp \left( \int \left( \frac{c_m-1}{z^{m-1}} + \cdots + \frac{c_1}{z} + \frac{\alpha(z)}{R(z)} \right) \, dz \right).
\]

Note that the first exponential factor cannot be simplified by any part of the second exponential factor. Moreover \( c_m \neq 0 \) and we get a contradiction with the fact that the left hand side must be a meromorphic function in the variable \( y \) while \( \exp(c_m/(1-m)z^{m-1}) \) has an essential singularity at \( z = 0 \), and this it is not meromorphic in \( z \). Therefore, we conclude that \( g(y) \) is square-free. Hence we write

\[
\frac{r(z)}{g(z)} = \frac{\gamma_1}{z - \alpha_1} + \cdots + \frac{\gamma_k}{z - \alpha_k}.
\]

Then,

\[
\int \frac{r(z)}{g(z)} \, dz = \sum_{j=0}^{k} \int \frac{\gamma_j}{z - \alpha_j} \, dz = \sum_{j=0}^{k} \gamma_j \log(z - \alpha_j)
\]

and, consequently,

\[
e^{\int \frac{r(z)}{g(z)} \, dz} = \prod_{j=0}^{k} (z - \alpha_j)^{\gamma_j}.
\] (7)

Note that this expression is always a meromorphic function. If \( \gamma_j > 0 \) for all \( j = 1, \ldots, k \) then it is an analytic function in the variable \( y \), otherwise it is meromorphic with poles on the \( \alpha_j \) such that \( \gamma_j < 0 \). Hence \( a_k(y) \) is an analytic function in \( y \) if \( \gamma_j < 0 \) for \( j = 1, \ldots, k \), and it is meromorphic with poles on the \( \alpha_j \) with \( \gamma_j > 0 \).

Conversely, assume that \( g \) is square-free and that \( f(y) = q(y)g(y) + r(y) \). We will show that

\[
H(x, y) = xe^{-\int q(y) \, dy} (y - \alpha_1)^{\gamma_1} \cdots (y - \alpha_k)^{\gamma_k},
\] (8)

with \( \gamma_i = r(\alpha_i)/g'(\alpha_i) \) for \( i = 1, \ldots, k \) is a pseudo-meromorphic function, and it is analytic if all \( \gamma_j < 0 \) for \( j = 1, \ldots, k \). Note that the function \( H \) defined in (8) is just the function \( H \) in (3) with only a term different from zero when \( l = 1 \), and thus satisfying (4) with \( l = 1 \) and (5) with \( Q(y) = \int q(y) \, dy \), and with \( e^{\int \frac{r(z)}{g(z)} \, dz} \) as in (7). Now we show that indeed it is a first integral of system (1). We set \( \phi(y) = (y - \alpha_1)^{\gamma_1} \cdots (y - \alpha_k)^{\gamma_k} \). Note that

\[
0 = xf(y)\frac{\partial H}{\partial x} + g(y)\frac{\partial H}{\partial y}
= xf(y)e^{-\int q(y) \, dy} \phi(y) + xg(y)(-q(y)\phi(y) - \phi'(y))e^{-\int q(y) \, dy}
= xe^{-\int q(y) \, dy}(f(y)\phi(y) - g(y)q(y)\phi(y) - g(y)\phi'(y))
= xe^{-\int q(y) \, dy}(r(y)\phi(y) - g(y)\phi'(y)).
\]
To see that this last expression is identically zero it is equivalent to see that
\[ \frac{\phi'(y)}{\phi(y)} = \frac{r(y)}{g(y)}. \]

Recalling the expression of \( \phi(y) \) we have
\[ \frac{\phi'(y)}{\phi(y)} = \frac{\gamma_1}{y - \alpha_1} + \frac{\gamma_2}{y - \alpha_2} + \cdots + \frac{\gamma_k}{y - \alpha_k}. \]
Taking common denominator and recalling that \( g(y) = c(y - \alpha_1)(y - \alpha_2) \cdots (y - \alpha_k) \)
we obtain
\[ \frac{\phi'(y)}{\phi(y)} = \frac{c}{g(y)} \sum_{i=1}^{k} \gamma_i \prod_{j=1, j \neq i}^{k} (y - \alpha_j). \]
Now substituting the values of \( \gamma_i = r(\alpha_i)/g'(\alpha_i) \) and taking into account that
\[ g'(\alpha_i) = c \prod_{j=1, j \neq i}^{k} (\alpha_i - \alpha_j), \]
we get
\[ \frac{\phi'(y)}{\phi(y)} = \frac{1}{g(y)} \sum_{i=1}^{k} r(\alpha_i) \prod_{j=1, j \neq i}^{k} \frac{y - \alpha_j}{\alpha_i - \alpha_j} = \frac{r(y)}{g(y)}. \] (9)
The last expression in the sum, recalling that \( \deg r < \deg g \), is the expression of the Lagrange polynomial which interpolates \( r(y) \) in the \( k \) points \((\alpha_i, r(\alpha_i))\), for \( i = 1, 2, \ldots, k \), see for more details [2]. Therefore this polynomial is \( r(y) \), and we conclude that the expression (9) is satisfied. This completes the proof of the theorem.

REFERENCES
[1] J. Giné, M. Grau and J. Llibre, Polynomial and rational first integrals for planar quasi-homogeneous polynomial differential systems, *Discrete and Continuous Dynamical Systems, Series A*, 33 (2013), 4531–4547.
[2] E. Isaacson and H. B. Keller, *Analysis of Numerical Methods*, Dover Publications, Inc., New York, 1994.
[3] J. Llibre and X. Zhang, Polynomial first integrals for quasi-homogeneous polynomial differential systems, *Nonlinearity*, 15 (2002), 1269–1280.

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