The tropical discriminant in positive characteristic

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Abstract

We study singularities in tropical hypersurfaces defined by a valuation over a field of positive characteristic. We provide a method to compute the set of singular points of a tropical hypersurface in positive characteristic and the p-adic case. This computation is applied to determine all maximal cones of the tropical linear space of univariate polynomials of degree $n$ and characteristic $p$ with a fixed double root and the fan of all tropical polynomials that have 0 as a double root independently of the characteristic. We also compute, by pure tropical means, the number of vertices, edges and 2-faces of the Newton polytope of the discriminant of polynomials of degree $p$ in characteristic $p$.

1 Introduction

Given $A \subseteq \mathbb{Z}^d$ a finite subset, there is a close relation between the theory of $A$-discriminants and coherent subdivisions and the secondary polytope of $A$. We refer to [6] for a basic reference on this relation. The combinatorial nature underlying the $A$-discriminant is more apparent computing the tropical discriminant of the support $A$, [3,4,10]. However, this study is usually restricted to the case of characteristic 0. In this paper, we extend the notion of tropical singularity in a hypersurface introduced in [4] to the characteristic $p$ and the $p$-adic case, with the aim that this study will help understanding the reduction of the $A$-discriminant mod $p$.

With this idea in mind, let $\mathbb{K}$ be an algebraically closed field with a valuation $v : \mathbb{K}^* \to \mathbb{T} \subseteq \mathbb{R}$. Let $k$ be the residue field and let $p$ be a prime number. There are three possibilities for the characteristics of $\mathbb{K}$ and $k$.

- $\text{char}(\mathbb{K}) = \text{char}(k) = 0$, (equi)characteristic zero.
- $\text{char}(\mathbb{K}) = \text{char}(k) = p$, (equi)characteristic $p$.
- $\text{char}(\mathbb{K}) = 0$, $\text{char}(k) = p$, $p$-adic case.
If $V \subseteq (\mathbb{K}^\ast)^d$ is an algebraic variety of dimension $n$ in the torus, its tropicalization is the closure in $\mathbb{R}^n$ of the image of $V$ taking the valuation component-wise,

$$
trop(V) = \{(v(a_1), \ldots, v(a_d)) \in \mathbb{R}^d \mid (a_1, \ldots, a_d) \in V\}
$$

These varieties are polyhedral complexes of dimension $n$ in $\mathbb{R}^d$ and are the base of tropical geometry. In this paper, we are interested in the case that $\mathbb{K}$ is a field of characteristic $p$ and $V = \Delta_A$ is the $A$-discriminant, the set of polynomial of support $A$ having a double root in $(\mathbb{K}^\ast)^d$.

In many cases, tropical geometry do not depend in the characteristic of the underlying field. For instance, Kapranov’s theorem does not depend on the characteristic \[11, 14\]. Also, if we fix $d + 1$-supports $A_0, \ldots, A_d$ from the results of \[13, 15\] it follows that the $(A_0, \ldots, A_d)$ tropical resultant does not depend on the characteristic of the field. On the other hand, it is known that the tropical Grassmannian –understood as the image of the Grassmannian under a valuation– does depend on the characteristic of the ground field \[12\]. This is also the case of the discriminant. For instance, if $N_{p,n}$ is the Newton polytope of the discriminant of a univariate polynomial of degree $n$ in characteristic $p$, it is well known that $N_{n,0}$ is combinatorially a $(n-1)$-hypercube, while we prove (See Corollary \[4, 10\]) that the 2-faces of $N_{p,n}$ are quadrangles or triangles.

The paper is structured as follows. In Section 2, we introduce the basic notions and the notation. In Section 3, the set of singular points of a hypersurface is computed using pure tropical techniques, thus, giving a purely combinatorial method to decide if a polynomial is in the $A$-discriminant in characteristic $p$ or the $p$-adics. Section 4 is the main section, we study the (tropical) linear space of all tropical univariate polynomials of degree $n$ having a double root in characteristic $p$, some results on the tropical discriminant in characteristic $p$ and we also describe the set of tropical polynomials that are singular independently of the characteristic. In 5, we make a brief comment on the case of the $p$-adics.

## 2 Preliminaries

Let $\mathbb{K}$ be an algebraically closed field. Let $\mathbb{Q} \subseteq T \subseteq \mathbb{R}$ be a subgroup of the reals and $v : \mathbb{K}^\ast \to T$ a nontrivial valuation. We will denote by capital letters $A, B, X$ the elements and variables in $\mathbb{K}$ and by lower case $a, b, x$ the elements and variables in $T$. Given a Laurent polynomial $F = \sum_{i \in A} A_i x^i \in \mathbb{K}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$, $i = (i_1, \ldots, i_d)$, $X = X_1 \cdots X_d$ we call the tropicalization of $F$ to the formal tropical polynomial $f = \oplus_{i \in A} a_i x^i$, with $a_i = v(A_i)$, $i \in A$. The function associated to $f$ is the piecewise-affine function

$$
f(b) = \min_{i \in A} \{a_i + \langle i, b \rangle\}
$$

Different polynomials may define the same function, for instance $0 \oplus 0x \oplus 0x^2$ and $0 \oplus 1x \oplus 0x^2$. $f(b)$ is the expected valuation of $F(B)$ for $B$ an element such that $v(B) = b$. But there may be some $B$ such that $f(v(B)) \leq v(F(B))$. This can only happen if $b$ is a tropical root of $f$. That is, if the value $f(b)$ is attained
(at least) at two different monomials. We denote by \( \mathcal{T}(f) \) the set of tropical roots of \( f \).

\[
\mathcal{T}(f) = \{ b \in \mathbb{R}^n \mid \exists i, j \in A, f(b) = a_i + \langle b, i \rangle = a_j + \langle b, j \rangle \}
\]

If \( f \) is the tropicalization of \( F \) and \( V = \mathcal{V}(F) \subseteq (\mathbb{K}^*)^d \) is the hypersurface defined by \( F \) then, by the fundamental theorem of tropical geometry

\[
\mathcal{T}(f) = \{(v(B_1), \ldots, v(B_d)) \mid (B_1, \ldots, B_d) \in V \}.
\]

See for instance [7, 14].

**Definition 2.1.** Let \( f = \oplus_{i \in I} a_i x^i \in \mathbb{T}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}] \) be a Laurent tropical polynomial in \( d \) variables. We say that \( f \) is a singular polynomial (with respect to a valuation \( v : \mathbb{K} \to \mathbb{T} \)) and that \( b = (b_1, \ldots, b_d) \) is a singular point of \( f \) if there exists an algebraic counterpart polynomial \( F = \sum_{i \in I} A_i X^i \in \mathbb{K}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}], B = (B_1, \ldots, B_d) \in (\mathbb{K}^*)^d \) with \( v(A_i) = a_i, v(B_i) = b_i \) and such that \( B \) is a singular point of the hypersurface defined by \( F \).

**Example 2.2.** Let \( f = 0 \oplus 0 \oplus 0 x^2 \) and \( g = 0 \oplus 1 x \oplus 0 x^2 \). Then \( \mathcal{T}(F) = \mathcal{T}(G) = \{0\} \). If we are working in the characteristic zero case, then \( f \) is singular, since it is the tropicalization of \( X^2 - 2X + 1 = (X - 1)^2 \). Moreover, \( g \) cannot be singular, since if \( G = AX^2 + BX + C \) is a polynomial with tropicalization \( g \), then \( v(A) = v(C) = 0, v(B) = 1 \) and \( v(B^2 - 4AC) = 0, \) so \( B^2 - 4AC \neq 0 \) and \( G \) does not have a double root.

On the other hand, if we now work with a \( 2 \)-adic valuation, \( v_2(2) = 1 \), then \( g \) is the tropicalization of \( X^2 - 2X + 1 \) and \( g \) is singular. \( f \) cannot be singular in this case, since if \( F = AX^2 + BX + C \) has \( v_2(A) = v_2(B) = v_2(C) = 0 \), then \( v_2(B^2 - 4AC) = \min\{0, 2\} = 0 \) and the discriminant does not vanish.

We now define the tropical Euler derivatives introduced in [8], this is the main tool we use to deal with tropical singularities.

**Definition 2.3.** Let \( F = \sum_{i \in I} A_i X^i \in \mathbb{K}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}] \). Let \( L = B_0 + B_1 X_1 + \ldots + B_d X_d \in \mathbb{Z}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}] \) be an linear polynomial defined over the integers. We define the Euler derivative with respect to \( L \) to

\[
\frac{\partial F}{\partial L} = B_0 F + \sum_{j=1}^d B_j X_j \frac{\partial F}{\partial X_j} = \sum_{i \in I} L(i) A_i X^i \in \mathbb{K}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]
\]

Analogously, let \( f \in \sum_{i \in I} a_i x^i \in \mathbb{T}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}] \). The Euler derivative with respect to \( L \) is

\[
\frac{\partial f}{\partial L} = \bigoplus_{i \in I} (v(L(i)) + a_i) x^i \in \mathbb{T}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]
\]

Note that the definition of partial Euler derivative depends on the specific valuation of the field. Note also that if \( f \) is the tropicalization of \( F \), \( v(A_i) = a_i \), then \( v(L(i)A_i) = v(L(i)) + a_i \) and \( \frac{\partial f}{\partial L} \) is the tropicalization of \( \frac{\partial F}{\partial L} \).
Remark 2.4. Let $f$ be a tropical polynomial with support $A$, let $L = b_0 + b_1 x_1 + \ldots + b_d x_d \in \mathbb{Z}[x_1^\pm 1, \ldots, x_d^\pm 1]$ and $\gcd(b_0, \ldots, b_d) = 1$. Then taking the Euler derivative in $f$ has an easy geometric interpretation.

1. Characteristic 0, we eliminate from the support of $f$ the monomials in the hyperplane $\{L = 0\}$, the coefficients remain unchanged.

2. Characteristic $p$, we eliminate from the support of $f$ all the monomials lying at lattice distance $r \equiv 0 \mod p$ from the hyperplane $\{L = 0\}$, the rest of the coefficients remain unchanged.

3. $p$-adic case, we eliminate from the support of $f$ the monomials lying in the hyperplane $L = 0$. If a monomial $i_0$ lies at lattice distance $r \equiv 0 \mod p$ from $L = 0$, we add $v_p(r)$ to the corresponding coefficient $a_{i_0}$ of $f$.

Example 2.5. Let $f = a_0 \oplus a_1 x \oplus a_2 x^2 \oplus a_3 x^3 \oplus a_4 x^4 \oplus a_5 x^5$. Let $L = x - 4$. Then, the Euler derivative of $f$ with respect to $L$ is:

- In the characteristic 0 case. In the characteristic $p$ case or $p$-adic case with $p > 3$, $\frac{\partial f}{\partial L} = a_0 \oplus a_1 x \oplus a_2 x^2 \oplus a_3 x^3 \oplus a_5 x^5$.

- If the characteristic of $\mathbb{K}$ is $p = 2$, $\frac{\partial f}{\partial L} = a_1 x \oplus a_3 x^3 \oplus a_5 x^5$.

- If the characteristic is $p = 3$, $\frac{\partial f}{\partial L} = a_0 \oplus a_2 x^2 \oplus a_3 x^3 \oplus a_5 x^5$.

- In the 2-adic case, $\frac{\partial f}{\partial L} = (2 + a_0) \oplus a_1 x \oplus (1 + a_2)x^2 \oplus a_3 x^3 \oplus a_5 x^5$.

- In the 3-adic case, $\frac{\partial f}{\partial L} = a_0 \oplus (1 + a_1)x \oplus a_2 x^2 \oplus a_3 x^3 \oplus a_5 x^5$.

3 Singularities in tropical hypersurfaces

Lemma 3.1. If the tropical discriminant $\Delta_{p,A}$ of polynomials of support $A$ in characteristic $p$ is non-empty, then it is a rational polyhedral fan of pure dimension in $\mathbb{R}^{|A|}$.

Proof. Let $\Delta_{p,A}^\sim$ be the $A$-discriminant in characteristic $p$. Since it is parametrizable, $\Delta_{p,A}$ is absolutely irreducible. It follows from [2] that $\Delta_{p,A}$ is a rational polyhedral complex of the same dimension as $A$. Moreover, since $\Delta_{p,A}$ is a variety defined over the prime field $\mathbb{Z}/(p)$, then we are in a constant coefficient case, the initial ideal $in_w(I) = in_t(I)$, for every $w \in \mathbb{R}^n$ and $t \in \mathbb{R}$. It follows from [7] that the cells of $\Delta_{p,A}$ are cones and $\Delta_{p,A}$ is a rational polyhedral fan.

Theorem 3.2 ([H]). Let $f = \bigoplus_{i \in I} a_i x^i$ be a tropical polynomial with support $A$. Let $q \in T(f)$ be a point in the hypersurface defined by $f$. Then, $q$ is a singular point of $T(f)$ if and only if $q \in T(\frac{\partial f}{\partial L})$ for all $L$. 

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Thus, \( f \) defines a singular tropical hypersurface if and only if
\[
\bigcap_L \mathcal{T} \left( \frac{\partial f}{\partial L} \right) \neq \emptyset.
\]

This intersection can be given by a finite number of Euler derivatives of \( f \).

Proof. The proof given in [4] is written for the characteristic zero case, but it works in any case. The result in the general case follows from the fact that given a linear space \( V \subseteq (\mathbb{K}^*)^d \), a tropical basis of \( V \) is given by the linear polynomials vanishing in \( V \) with minimal support. These polynomials form a tropical basis independently of the characteristic of the field \( \mathbb{K} \) nor the valuation \( v \).

Example 3.3. Let \( f \) be the tropical polynomial \( f = 0 \oplus 0x^2 \oplus 0y^2 \oplus 0x^2y^2 \oplus 1x^3 \). Then, it is easy to check that \((0,0)\) is a singularity of \( f \) in characteristic different from 2, but that \( f \) is not singular in characteristic 2, because \( \frac{\partial f}{\partial x} - y \) has no tropical root.

Corollary 3.4. Let \( f = a_0 \oplus a_1 x \oplus \ldots \oplus a_n x^n \in \mathbb{T}[x] \) be a tropical univariate polynomial of degree \( n \) in characteristic \( p \neq 0 \), let \( a \in \mathbb{T}(f) \), then \( f \) is singular with double root \( a \) if and only if \( a \in \mathbb{T}(\frac{\partial f}{\partial x^i}), i = 0, \ldots, p-1 \).

Proof. If \( L = ax + b \), then \( \frac{\partial f}{\partial L} \) is, up to multiplication by a constant, equal to \( f \) or one of the Euler derivatives \( \frac{\partial f}{\partial x^i}, i = 0, \ldots, p-1 \).

The case of characteristic zero is well understood. The tropical discriminant is combinatorially dual to the Newton polytope of the discriminant, see, as a starting point, [3], [5], [6]. Here, we will follow the approach of [8], [9]. We study the univariate polynomials of degree \( n \) that have a singular point in \( 0 \in \mathbb{T} \). Any other case can be reduced to this situation.

4 The linear spaces \( H_{p,n} \)

In this section we follow the ideal of [8]. We fix the tropical point 0 and compute all polynomials with 0 as double root. The space obtained is a tropical linear space.

Definition 4.1. We define \( H_{p,n} \), with \( p \) prime or zero, the (tropical) linear space of tropical polynomials of degree \( n \) such that 0 is a tropical double root. If \( \Delta_{p,n} \) is the set of tropical singular polynomials of degree \( n \) in characteristic \( p \) then
\[
\Delta_{p,n} = \bigcup_{c \in \mathbb{T}} \{ f(cx) | f \in H_{p,n} \}
\]

However, passing from \( H_{p,n} \) to \( \Delta_{p,n} \) is not trivial. If one wants to use the techniques developed here to study \( \Delta_{p,n} \), some knowledge of tropical polynomials with two tropical roots is needed. This is an open problem even in characteristic zero.
Theorem 4.2. $H_{0,n}$ is the set of tropical polynomials $f = \oplus_{i=0}^{n} a_i x^i$ such that the minimum of $\{a_i, 0 \leq i \leq n\}$ is attained (at least) at three different monomials $\{j, k, l\}$. It is a rational fan of codimension 2 in $\mathbb{T}^n$, the maximal cones can be marked by the monomials $\{j, k, l\}$ where the minimum is attained, so it has \binom{n+1}{3} maximal cells.

4.1 The case $p = 2$

Theorem 4.3. A tropical polynomial $f$ is in $H_{2,n}$ if and only if the minimum coefficient in the even monomials of $f$ is attained twice and in the odd monomials is attained also twice. Hence, if $f$ is of degree $2n - 1$, $n \geq 2$ then there are \binom{n}{2}^2 maximal cones.

Proof. By Corollary 3.4 a tropical polynomial $f$ is singular at 0 if and only if $f_0 = \frac{\partial f}{\partial x}$ and $f_1 = \frac{\partial f}{\partial x-1}$ have $0$ as tropical root. $f_0$ is the polynomial consisting on all odd monomials of $f$ and $f_1$ is the polynomial consisting on all even monomials of $f$.

$$f_0 = \bigoplus_{i=0}^{\frac{n-1}{2}} a_{2i+1} x^{2i+1} \quad f_1 = \bigoplus_{i=0}^{\frac{n}{2}} a_{2i} x^{2i}$$

A maximal cone is characterized by the two monomials $\{i, j\}$ where $f_0$ attains its minimum and the two monomials $\{k, l\}$ where $f_1$ attains its minimum. Hence, in a polynomial of degree $2n - 1$ there are \binom{n}{2}^2 maximal cells. \qed

Theorem 4.4. The tropical discriminant $\Delta_{2,2n-1}$ in characteristic 2 of a polynomial of degree $2n - 1$ is in natural correspondence with the tropical resultant of two polynomials of degree $n - 1$ in characteristic zero. In particular, the Newton polytope of $\Delta_{2,2n-1}$ has \binom{2n-2}{n-1} extremal vertices.

These two polynomials have a common tropical root if and only if

$$g_1 = \bigoplus_{i=0}^{\frac{n}{2}} a_{2i} y^i \quad g_0 = \bigoplus_{i=0}^{\frac{n-1}{2}} a_{2i+1} y^i$$

have a common root. Hence, the polynomials in the tropical discriminant $\Delta_{2,n}$ is in bijective correspondence with the resultant of two univariate polynomials of degree $\left\lfloor \frac{n}{2} \right\rfloor$ and $\left\lfloor \frac{n-1}{2} \right\rfloor$. Since the tropical resultant of two univariate polynomials does not depend on the characteristic [13, 15], then the tropical discriminant in characteristic two is in natural bijective correspondence with the resultant of two polynomials of degree $\left\lfloor \frac{n}{2} \right\rfloor$ and $\left\lfloor \frac{n-1}{2} \right\rfloor$.

4.2 The case $p > 2$

Let us study now the case of an odd primes $p$. By Corollary 3.4 a polynomial $f$ is in $H_{p,n}$ if and only if the minimum of the coefficients of $\frac{\partial f}{\partial x-1}$ is attained twice for $i = 0, \ldots, p - 1$. Let us now describe the combinatorial types of maximal cells in $H_{p,n}$.
Theorem 4.5. The set of maximal cells in $H_{p,n}$ is exactly the following:

1. Type I. Tropical polynomials where the minimum is attained at three different monomials $i,j,k$ that produce three different residues $\mod p$. Denote them by $\{i,j,k\}$

2. Type II. Tropical polynomials where the minimum is attained at $i,j$ with the same residue $\mod p$. And, if we eliminate the monomials $r, r \equiv i \mod p$, then the minimum is attained in $k,l$ with $k \neq l \mod p$. Denote them by $\{(i,j),\{k,l\}\}$

3. Type III. Tropical polynomials where the minimum is attained at $i,j$ with the same class $\mod p$. And, if we eliminate the monomials $r, r \equiv i \mod p$, then the minimum is attained in $k,l$ with $k \equiv l \mod p$. Denote them by $\{(i,j),\{k,l\}\}$.

Proof. It is clear that any polynomial that satisfies the description of the cells is in $H_{p,n}$. For type I polynomials and any Euler derivative, there is at least two monomials in $\{i,j,k\}$ where the minimum is attained. For type II and III. For any Euler derivative $x = s$, with $s \neq r$ the minimum is attained at $\{i,j\}$, while for $\frac{\partial f}{\partial x-r}$ the minimum is attained at $\{k,l\}$. We are also describing cells of codimension 2, so we are dealing with maximal cells.

Let us now check that there is no other possibility. Let $f \in H_{p,n}$, we will show that it must belong to the closure of one of these cells. Let $L_0$ be the set of monomials where the minimum is attained. Consider $L_0 \mod p = \{[a]|a \in L_0\}$ the set of classes. If there are at least three different classes $[i],[j],[k]$ in $L_0 \mod p$, then $f$ is in the closure of the type I cell $\{i,j,k\}$. If $L_0 \mod p = \{[r]\}$ is a single class. Then $L_0$ contains at least two monomials $\{i,j\}$. Now, since $f \in H_{p,n}$, the minimum of the coefficients of $\frac{\partial f}{\partial x-r}$ has two be attained at least for two monomials $k,l$ hence $f$ is in the closure of a type II or III cell. Depending if $k \equiv l \mod p$ or not. Finally, it may happen that $L_0$ consists on exactly two different classes $\{[r],[s]\}$. Since the minimum of $\frac{\partial f}{\partial x-r}$ and $\frac{\partial f}{\partial x-s}$ is attained at least twice, there must be at least two monomials on each class $i,j \in L_0 \ {[i]=[j]=[r]}$ and $k,l \in L_0, \ {[k]=[l]=[s]}$. Thus, in this case $f$ is in the closure of a type III cell.

Finally, we have to check that we are not counting twice a maximal cell.

We now check that the notation of type III cells is well chosen.

Lemma 4.6. Let $[i] = [j] \neq [k] = [l] \mod p$. Then the cones $a_i = a_j \leq a_k = a_l$ and $a_k = a_l \leq a_i = a_j$ belong to the same maximal cone of type III, $\{(i,j),\{k,l\}\}$.

Proof. The two cones meet in the common face $D = \{a_i = a_j = a_k = a_l\}$. In characteristic zero, this cone is adjacent to four maximal cones of $H_{0,n}$, namely $\{i,j,k\}, \{i,j,l\}, \{i,k,l\}, \{j,k,l\}$. However, none of these cones belong to $H_{p,n}$, since $[i] = [j] \neq [k] = [l] \mod p$. The only perturbations of $D$ that still belong to $H_{p,n}$ are precisely the cones $a_i = a_j \leq a_k = a_l$ and $a_k = a_l \leq a_i = a_j$. It
follows that the maximal cone of $H_{p,n}$ containing $D$ is $a_i = a_j$, $a_k = a_l$, that is the definition of the cone $\{\{i, j\}, \{k, l\}\}$.

\begin{remark}
The only cells in common of $H_{p,n}$ and $H_{0,n}$ are type I cells. The reason for separating type II and type III cells is that if $\{\{i, j\}, \{k, l\}\}$ is a type II cell, then the cone $a_k = a_l < a_i = a_j$ is not in $H_{p,n}$. Moreover, in the proof of Theorem 4.9 we will see that if $\{\{i, j\}, \{k, l\}\}$ the set of polynomials $a_i = a_j < a_k = a_l$ and $a_k = a_l < a_i = a_j$ (and the rest of monomial higher) belong to a maximal cone of $H_{p,n}$.
\end{remark}

\begin{theorem}
In $H_{p,pn-1}$ in characteristic $p > 2$ there are:

1. $\binom{n}{3} n^3$ facets of type I
2. $pn^2 \binom{n^2}{2} \binom{p-1}{2}$ facets of type II
3. $\binom{n}{2}^2$ facets of type III

Thus, for fixed $p$ the number of facets in $H_{p,m}$ is $O(m^4)$.
\end{theorem}

\begin{proof}
If the polynomial is of degree $pn - 1$ there are exactly $n$ monomials on each class mod $p$. For type I facets, we have to choose three classes $[i], [j], [k]$ and, on each class one monomial. For type II facets $\{\{i, j\}, \{k, l\}\}$ we first choose the class of $[i], [j]$ and then the two monomials. Now, we have to choose the classes of $[k], [l]$ and one element on each class. Finally, for type III facets, $\{\{i, j\}, \{k, l\}\}$ we have just to choose the two classes $[i], [k]$ and, on each class. Two monomials.
\end{proof}

We now describe the incidence of two maximal cones of $\Delta_{p,n}$. If $C_1$ and $C_2$ are two different maximal cones of $\Delta_{p,n}$ and the common face $D = C_1 \cap C_2$ is of dimension $n - 1$ in $\mathbb{R}^{n+1}$, then any polynomial $f$ will have two double roots $a_1, a_2$. The specific values of $a_1$ and $a_2$ depend on $f \in D$, but not the fact that $a_1 = a_2$ or $a_1 \neq a_2$.

\begin{theorem}
Let $p > 2$ be a prime. Let $C_1, C_2$ be two maximal cones of $\Delta_{p,n}$ meeting on a cone dimension $n - 1$, $D = C_1 \cap C_2$. Let $a_1, a_2$ the two tropical double roots of a generic polynomial $f \in D$. Then

1. If $a_1 \neq a_2$ then $C_1$ and $C_2$ can be of any type, $D$ is a face of exactly 4 maximal cones of $\Delta_{p,n}$.
2. If $a_1 = a_2$ then, up to order:

   (a) The only obstruction for the pair of types of $C_1$ and $C_2$ is I - III.
   (b) If both $C_1 = \{i, j, k\}$ and $C_2 = \{i, j, l\}$ are of type I and the classes $[i], [j], [k], [l]$ are pairwise different, then $D$ is adjacent to 4 maximal cones of $\Delta_{p,n}$. Otherwise $D$ is adjacent to exactly 3 maximal cones of $\Delta_{p,n}$.
\end{theorem}
Proof. By abuse of notation, if \( i \in I \), the support of the polynomial, we will also write \( i = (i, a_i) \) the corresponding point in the Newton diagram of the polynomial. Let \( a_1 \neq a_2 \). We distinguish the following cases.

- Both \( C_1 \) and \( C_2 \) are of type \( I \), we are in the same situation as in characteristic 0. Since every 2-face of the Newton polytope in discriminant zero is (combinatorially) a square, \( D \) is adjacent to four different maximal cones.

- \( \big\{ i, j, k \big\} \) and \( \big\{ i, j, t \big\} \). Without loss of generality \( i \notin \big\{ l, m, r, s \big\} \) (because \( i, j, k \) are collinear). Also, \( r \notin \big\{ i, j, k \big\} \) (because \( i, j, k \) lie on an edge of the Newton polytope of \( f \) not parallel to \( \overline{rs} \)). We can either increase or decrease \( i \) or \( o \) to eliminate one of the tropical roots and passing to a maximal cone. Hence \( S \) is also adjacent to four maximal cells. The same argument holds for two cells of type \( I, III \).

- Both \( C_1 \) and \( C_2 \) are of type \( II \) or \( III \). Assume that \( C_1 = \big\{ \{i, j\}, \{k, l\} \big\}, C_2 = \big\{ \{r, s\}, \{t, u\} \big\}. \) Note that it is impossible that \( \{i, j\} \subseteq \{r, s, t, u\} \). First, \( \{r, s\} \neq \{i, j\} \neq \{t, u\} \). Hence \( i, j \) is not parallel to the lines \( \overline{rs} \) and \( \overline{tu} \). Second, it can not happen that \( i \notin \{r, s\} \) and \( j \notin \{t, u\} \), because in that case \( \{r\} = [i] = [j] = [t] \) mod \( p \) and this is impossible. Hence we may assume that \( i \notin \{r, s, t, u\} \) and \( r \notin \{i, j, k, l\} \) and we still have four possibilities to perturb this configuration into maximal cones of \( \Delta_{p,n} \).

Now, assume that \( a_1 = a_2 \), by dehomogenization, we may assume that \( a_1 = a_2 = 0 \). We also distinguish different cases:

- Both \( C_1 \) and \( C_2 \) are of type \( I \). They are of the form \( C_1 = \big\{ \{i, j\}, \{k, l\} \big\}, C_2 = \big\{ \{r, s\}, \{t, u\} \big\}. \) If \( [k] \neq [t] \), then the case is like the characteristic zero case, \( D \) is adjacent to the four cells of type \( I \), \( \big\{i, j, k\big\}, \big\{i, j, t\big\}, \big\{i, k, t\big\}, \big\{j, k, t\big\}. \) But, if \( [k] = [t] \), then \( D \) is a face of three maximal cones \( \big\{ \{i, j, k\}, \{i, j, t\} \big\} \) and \( \big\{ \{k, t\}, \{i, j\} \big\}. \) Note that \( [i] \neq [j] \), then this cell is never of type \( III \).

- Types \( I - II \). In this case, the two cells are \( \big\{ \{i, j\}, \{k, l\} \big\}, \big\{ \{i, j\}, \{k, l\} \big\} \), so again, \( D \) is a face of three maximal cells. \( \big\{ \{i, j, k\}, \{i, j, l\}, \{\{k, l\}, \{i, j\} \big\} \).

- Types \( I - III \). If \( \big\{ \{i, j, k\}, \{j, r\}, \{k, s\} \big\} \) meet in \( D \), we have five monomials with the same value at the tropical root, so \( D \) is at least of codimension 3 in \( \mathbb{R}^n \).

- Types \( II - II \). Two cells of type \( II \) can meet in two different ways.

1. \( \big\{ \{i, j\}, \{k, l\} \big\} \) and \( \big\{ \{i, j\}, \{k, m\} \big\} \), \( D \) is a face of these two type \( II \) cells and the other may be the cell of type \( II \) or \( III \) \( \big\{ \{i, j\}, \{l, m\} \big\} \) or \( \big\{ \{i, j\}, \{l, m\} \big\} \) depending if \( [l] \neq [m] \) or not.

2. \( \big\{ \{i, j\}, \{k, l\} \big\} \) and \( \big\{ \{i, m\}, \{k, l\} \big\} \), \( D \) is a face of three maximal cells. The previous two plus \( \big\{ \{j, m\}, \{k, l\} \big\} \). Note that, in this case \( [i] = [j] = [m] \), so it is not important if \( a_l = a_j < a_m < a_k = a_l \) to decide if the polynomial is singular or not, because the only important derivative \( \frac{\partial f}{\partial a_{i-1}} \) erases monomials \( i, j, m \).

- Types \( II - III \). The two cells are of the form \( \big\{ \{i, j\}, \{k, l\} \big\} \) and \( \big\{ \{i, j\}, \{l, m\} \big\} \). This is one of the cases studied above. The codimension one cell is adjacent to these two facets plus \( \big\{ \{i, j\}, \{k, m\} \big\} \).
• Type III - III. Two different cells of type III meet, they must be of one of the following cases:

1. \{\{i, j\}, \{k, l\}\} and \{\{i, j\}, \{k, m\}\} so there is again only another facet adjacent to this cell, \{\{i, j\}, \{l, m\}\}.

2. \{\{i, j\}, \{k, l\}\} and \{\{i, j\}, \{n, o\}\}, with \(|k| \neq |o| \neq |i|\). The other adjacent cell is \{\{k, l\}, \{n, o\}\}.

\[\square\]

**Corollary 4.10.** Let \(p > 2\) be a prime. If \(n < p\) then \(\Delta_{p,n} = \Delta_{0,n}\). If \(n \geq p\) then the 2-faces of the Newton polytope of \(\Delta_{p,n}\) are quadrangles or triangles.

**Proof.** If \(n < p\), then the Euler derivative \(\frac{\partial f}{\partial x_i}\) is the same in characteristics 0 and \(p\), hence \(\Delta_{p,n} = \Delta_{p,n}\). If \(n \geq p\) the 2-faces of \(\Delta_{p,n}\) are combinatorially dual to the cones \(D\) in \(\Delta_{p,n}\) of codimension 2. By Theorem 4.9 \(D\) is a face of four or three maximal cones. Hence the 2-faces can only be quadrangles and triangles.

\[\square\]

**Theorem 4.11.** The Newton polytope of \(\Delta_{p,p}\) has \(2^{p-1} - 1\) vertices, \((p - 1)(2^{p-2} + p/2 - 2)\) edges, \((p-1)^2(2^{p-3} - 1)\) quadrangles and \((p^3)\) triangles.

**Proof.** We give a pure tropical proof of this theorem. First, note that in \(\Delta_{p,p}\) there are only cells of type I and II. Let \(N_0\) and \(N_p\) be the Newton polytopes of \(\Delta_{0,p}\) and \(\Delta_{p,p}\) respectively. The vertices of \(N_q\) correspond to the cones in the complement of \(\Delta_{q,p}\), that is, combinatorial types of polynomials without double roots. Edges correspond to codimension 1 cones in \(\Delta_{q,p}\) and 2-faces of \(N_q\) with codimension 2 cones in \(\Delta_{q,p}\). The Newton polytope \(N_0\) of \(\Delta_{0,p}\) is the secondary polytope of \(\{0, \ldots, p\}\), combinatorially a hypercube of dimension \(p - 1\). Its vertices correspond to the triangulations of the set \([p] = \{0, \ldots, p\}\). Note that, reducing \(\mod p\), the only modifications appear around the vertex \(V_0 = (A_0A_p)^{p-1}\), corresponding to the triangulation \([0, p]\). This vertex is not in \(N_p\), since \(f \in C_0\) is not enough information to decide if \(f \in \Delta_{p,p}\) or not. The common cones of type I in \(\Delta_{0,p}\) and \(\Delta_{p,p}\) are those of type \(\{i, j, k\}\) except \(\{0, i, p\}\), \(1 \leq i \leq p - 1\) that are not cones in \(\Delta_{p,p}\). These are the edges in \(N_0\) linking \(V_0\) with the vertices \([0, i, p]\) of \(N_0\). Hence, \(p - 1\) edges disappear reducing \(\mod p\). On the other hand, the new edges corresponding to type II cells \(\{\{0, p\}, \{i, j\}\}\) appear in \(N_p\). Thus we have \((p - 1)2^{p-2} - (p - 1) + (p-1)^2 = (p - 1)(2^{p-2} + p/2 - 2)\) edges in \(N_p\). In \(N_0\) there are \((p-1)^2(2^{p-3})\) quadrangles, from these \((p-1)^2\) are adjacent to \(V_0\) and no new quadrangle appear. Hence \(N_p\) has \((p-1)^2(2^{p-3} - 1)\) quadrangles. Finally, the number of triangles correspond to codimension 2 cones in \(\Delta_{p,p}\) that are faces of type II cones. There are two possibilities, according to Theorem 4.9. Either two edges of the triangle are type I cones, the edges correspond to the cones \(\{i, j, p\}, \{i, j, 0\}, \{\{0, p\}, \{i, j\}\}\), and the vertices of the triangle to the subdivisions \([0, i, p], [0, j, p], [0, i, j, p]\). These are \((p-1)^3\) triangles that correspond to the quadrangles of \(N_0\) that disappear. The other triangles are those that consist on only type II cells. Their edges are of
the form $[\{0, p\}, \{i, j\}], [\{0, p\}, \{i, k\}], [\{0, p\}, \{j, k\}]$ and the vertices correspond to triangulations $[0, i, p], [0, j, p]$ and $[0, k, p]$. These are completely new triangles and the corresponding faces in $\Delta_{p, p}$ that lie in the cell dual to $V_0$ in $N_0$. There are $\binom{n-1}{3}$ such triangles. Hence, there are $\binom{n-1}{2} + \binom{n-1}{3} = \binom{n-1}{2}$ triangles in $N_p$.

Note that there are no new vertices in $N_p$, by the analysis of the edges of the triangle. The vertex of $N_p$ corresponding to the triangulation $[0, i, p]$ correspond to the (dehomogenized, closed) cone $a_0 = a_p \geq a_i \geq a_j, j \notin \{i, 0, p\}$.

**Remark 4.12.** The Theorem suggests that reducing the discriminant mod $p$, in terms of the Newton polytope, consists in just erasing the monomials corresponding to triangulations with a segment of length multiple of $p$. This is not the case. For instance, let $n = 5$, and compare $N_{0,5}$ and $N_{3,5}$ the Newton polytopes of the discriminant of degree 5 in characteristics 0 and 3. Then $N_{0,5} - N_{3,5} = \{(0, 2, 4, 0, 0, 2), (2, 0, 0, 4, 2, 0), (2, 0, 0, 5, 0, 1), (0, 4, 0, 0, 4, 0), (1, 0, 5, 0, 0, 2)\}$ which are precisely the triangulations with a segment of length 3. But $N_{3,5} - N_{0,5} = \{(1, 3, 1, 0, 0, 3), (0, 3, 0, 0, 1, 3, 1)\}$ are new vertices.

### 4.3 Universally singular polynomials

In this section, we analyze which polynomials have a multiple tropical root at 0 independently on the characteristic. That is, $\Delta(0) = \cap p H_{p, n} \cap H_{0, n}$. We have the following result.

**Theorem 4.13.** $\Delta(0)$ is a rational polyhedral fan of codimension 3 in $\mathbb{R}^n$. The cones of codimension 3 consist on polynomials such that the minimum is attained at three different monomials $i, j, k$ such that

- $d = \text{rad}(j - i) = \text{rad}(k - i) = \text{rad}(k - j)$.
- If we eliminate all the monomials $l \equiv i \mod d$ in $f$, then the minimum is attained in two different monomials $r, s$ such that $|r - i|, |s - i| \in \mathbb{Z}/(d)^*$ are units $\mod d$.

**Proof.** First, note that if $p > n + 1$ is a prime, then the Euler derivative in characteristic $p$ coincides with one Euler derivative in characteristic zero, so $H_{0, n} = H_{p, n}$. This means that $\Delta(0)$ is the intersection of finitely many rational polyhedral fans in $\mathbb{R}^{n+1}$. So it is a rational polyhedral fan. Now, note that $H_{0, n}$ only contains facets of type $I$, while $H_{2, n}$ contains only facets of type $III$. This means that $\Delta(0)$ does not contain any cell of codimension 2. Consider a polynomial $f$ satisfying the hypothesis. It is clear that the corresponding cell is of codimension three, since we have only three conditions $a_i = a_j = a_k = a_s$. It is also clear that $f \in H_{0, n}$. Now, $(k - i) = (k - j) + (j - i)$ and have the same radical $d$. In this case, $d$ must be even and $i, j, k$ must be of the same parity and $r - i, s - i$ must be odd. This means that $f \in H_{2, n}$. Next, let $p$ be a prime not dividing $d$. Taking an Euler derivative in characteristic $p$ can only erase at most one of the monomials $\{i, j, k\}$ in the support and the minimum in the derivative will be attained in the other two monomials. It follows that $f \in H_{p, n}$. Finally, let $q$ be a prime dividing $d$. If we take a derivative $\mod q$, then either...
Theorem 5.1. Let \( p \not| d \) and we have the first condition. For the second condition, note that the primes taking the derivative \( \partial f \) show that the classes \( \mod p \) is singular in characteristic zero, the minimum of the coefficients is attained corresponding polynomial \( f \) and the tropical discriminants are polyhedral complexes of codimension 1 in fan. Because non-zero constants appear while taking Euler derivatives. These \( q \) and every class \( i, j, k \) in three different monomials \( n \) by construction that the polynomial belongs to \( \Delta(0) \) \( \mod p \) do not pose any problem and that, for any prime \( q \) dividing \( d \) the minimum is attained in \( \partial f \mod i \). For any prime \( q \) dividing \( d \) there are two monomials \( r_q, s_q \) where the minimum is attained in \( \partial f \mod q \). But if these two monomial depend on \( q \), then the codimension of the cell must be at least 4 and we are out of hypothesis. Hence, the same monomials \( r_q, s_q \) are valid for every \( q \). Since \( \partial f \not|_i \mod q \) for any \( q \) dividing \( d \), then \( \partial f \not|_i \mod q \) if \( q \not| d \) and \( |l| = |i| \mod q \). So, this is a cell of codimension \( k + 2 \). It follows that we can always construct maximal cells of codimension \( O(\log(n)) \).

Theorem 4.14. If \( n \) is big enough, then \( \Delta(0)_n \) is not a pure rational polyhedral fan in codimension three, there are maximal faces of codimension \( \Omega(\log(n)) \).

Proof. Take \( n = 2 \cdot 4^k \) and consider the first \( k \) primes \( p_1, \ldots, p_k \). Let \( d = \prod_{i=1}^k p_i < 4^k < n \). Take the polynomial \( f \) such that takes the value 0 at 0, \( d, 2d \). And, for each \( 1 \leq i \leq k \) takes the value \( i \) at the monomials \( d/p_i, d/p_i + d \). And, for every other monomial takes arbitrary values bigger than \( k \). It is clear by construction that the polynomial belongs to \( \Delta(0)_n \). Because, for every prime \( q \) and every class \( l \), the minimum of \( \partial f \mod q \) is attained in (at least) two of the monomials in \( \{i, j, k\} \) or the minimum is attained in \( d/q, d/q + d \) if \( q \not| d \) and \( \lfloor t \rfloor = \lceil i \rceil \mod q \). So, this is a cell of codimension \( k + 2 \). It follows that we can always construct maximal cells of codimension \( \Omega(\log(n)) \).

5 A note on the \( p \)-adic case

So far we have studied only the equicharacteristic case in which the field \( \mathbb{K} \) has the same characteristic as the residue field \( k \). The \( p \)-adic case has also interest on its own, however, under a \( p \)-adic valuation, the discriminant is no longer a fan. Because non-zero constants appear while taking Euler derivatives. These tropical discriminants are polyhedral complexes of codimension 1 in \( \mathbb{R}^{n+1} \). We show that the \( p \)-adic discriminant performs a kind of interpolation between the discriminant in characteristic zero and characteristic \( p \).

If we take a generic point in a maximal cell of the discriminant, the corresponding polynomial \( f \) will have only one double root \( a \). Let \( L_0 \) be the set of monomials where the minimum is attained at \( a \) and \( L_1 \) the set of monomials where the second minimum is attained at \( a \).
• On a small ball centered at the origin, the $p$-adic discriminant equals the discriminant in characteristic $p$, $\Delta_p \cap B(0, \epsilon) = \Delta_{p,n} \cap B(0, \epsilon)$ for $0 < \epsilon << 1$.

• In the subset of $\mathbb{R}^n$ where $L_0 << L_1$, the maximal cells in common of the $p$-adic discriminant and the characteristic $p$ tropical discriminant are precisely the cones of type I.

Proof. Taking the Euler derivative with respect to $x - i$ of a polynomial in $p$-adics increases the value of the monomials $[j] = [i]$ by the $p$-adic valuation of $i - j$. If the coefficients are small enough, taking this derivative has the same effect as deleting the monomials for the matter of finding the minimum. Hence the first item follows.

For the second item, Note that if we have a type $I$ cone in the discriminant of characteristic $p$ then the minimum is attained at three monomials $i, j, k$ defining three different classes $[i], [j], [k]$ mod $p$. Taking Euler derivatives $\frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j}, \frac{\partial f}{\partial x_k}$ in the $p$-adic case do not modify the other two monomials there the minimum is attained, so these cells are in common between the two discriminants. Now, if we have a type $II$ cone in the characteristic $p$ discriminant, then the minimum is attained at two different monomials $i, j$ with $[i], [j]$. Since $L_0 << L_1$, it follows that the minimum in the $p$-adic Euler derivative $\frac{\partial f}{\partial x_i}$ is attained in the monomial $j$ alone. So, it is not a cell in the $p$-adic discriminant.

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