Comments on SUSY Exact Action in 3D Supergravity

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We consider 2+1 dimensional off-shell $\mathcal{N} = 1$ pure supergravity that is constructed from graviton, gravitino and auxiliary field. We show that the $R^2$ supersymmetric invariant and $R^{2\mu}{}_{\nu}$ supersymmetric invariant are expressed as local supersymmetric exact terms up to mass terms for gravitino. In both cases, the mass parameter is proportional to the off-shell supersymmetric cosmological constant.

I. INTRODUCTION

Calculating quantum gravity partition function in a certain reasonable way is one of the most important and fundamental questions in theoretical physics. Even in the conventional quantum field theory with spin 0, 1/2, 1 fields, the exact computation is extremely difficult in many cases, and we are often tempted to use perturbative analysis. However, if there are some supersymmetries, one can utilize these symmetries to reduce the path integral to the finite dimensional matrix models \cite{1, 2}. In this procedure, the existence of supersymmetric “exact” Lagrangian is extremely important because adding such term into the path integral would not change the final result and the WKB computation turns out to be exact by taking its coupling constant to be infinite (or zero).

By applying this technique to the gravity path integral, we would like to make the gravity path integral well-defined. In \cite{2}, the authors considered such possibility in terms of supersymmetric Chern-Simons formulation of the three-dimensional gravity. In this notes, we discuss another possibility: localization computation with local supersymmetry. We will focus on 2+1 dimensional $\mathcal{N} = 1$ supergravity, and we start with reviewing some known facts on the theory.

II. 3D $\mathcal{N} = 1$ OFF-SHELL SUPERGRAVITY

We focus on the Lorentz signature $\eta^{ab} = \text{diag}(-1, +1, +1)$, where the alphabet runs for local Lorentz indices $a, b = 0, 1, 2$. The fundamental degrees of freedom are graviton $e_\mu^a$, gravitino: $\psi_\mu$, real auxiliary field, $S$. Local supersymmetry is defined by an arbitrary Majorana spinor parameter $\epsilon$ which depends on the coordinates as follows \cite{2}:

\begin{align}
&\delta e_\mu^a = \frac{1}{2} (\bar{\epsilon} \gamma^a \psi_\mu), \quad \delta \psi_\mu = D_\mu (\hat{\omega}) \epsilon + \frac{1}{2} S \gamma_\mu \epsilon, \quad (1) \\
&\delta S = \frac{1}{4} (\bar{\epsilon} \gamma^\mu \psi_\nu (\hat{\omega}) - \frac{1}{4} (\bar{\epsilon} \gamma^a \psi_\mu) S), \quad (2)
\end{align}

where the covariant derivative is defined by $D_\mu (\omega) = \partial_\mu + \frac{1}{2} \omega_\mu^{\rho\gamma} \gamma_\rho$, and the hatted spin connection contains the contribution from torsion. And $\psi_\mu (\omega) = \frac{1}{2} (D_\mu (\hat{\omega}) \psi_\nu - D_\nu (\hat{\omega}) \psi_\mu)$. See \cite{3} for more details.

Under these transformations, the following Lagrangians are invariant up to total derivative term.

\begin{align}
L_{EH} &= e (R - \bar{\psi}_\mu \gamma^{\mu\nu} D_\nu (\hat{\omega}) \psi_\rho - 2 S^2), \quad (3) \\
L_C &= e (\bar{S} \frac{1}{8} \bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu).
\end{align}

The first one is the usual Einstein-Hilbert term. The second one corresponds to the cosmological constant term. Just by integrating out the auxiliary field, it generates the usual negative cosmological constant term, and the resultant Lagrangian turns out to be so-called $\mathcal{N} = (1, 0)$ AdS-supergravity. In addition to that, one can find supersymmetric gravitational CS term, but we omit them for simplicity. Other supersymmetric terms can be found in \cite{4} as follows.

\begin{align}
L_{R^{2\mu}{}_{\nu}} &= -\frac{1}{4} e R^{\mu\nu ab} (\Omega^+) R_{\mu\nu ab} (\Omega^+) \\
&- 2 e \bar{\psi}_ab (\Omega^-) \gamma^{\mu} D_\mu \psi_ab (\Omega^-) + \frac{1}{4} e R_{\mu\nu ab} (\Omega^+) \bar{\psi}_{\rho} \gamma^{\mu\nu} \gamma^{\rho} \psi_ab (\Omega^-) \\
&+ e S \bar{\psi}_ab (\Omega^-) \psi_ab (\Omega^-) - \frac{1}{2} e \bar{\psi}_ab (\Omega^-) \psi_ab (\Omega^-) \bar{\psi}_\mu \psi_\mu \\
&+ \frac{1}{8} e \bar{\psi}_ab (\Omega^-) \psi_ab (\Omega^-) \bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu, \quad (5)
\end{align}

where $\Omega^{\pm ab} = \bar{\omega}_\mu \gamma^{\mu} \pm S \epsilon^{\mu}{}_{ab}$ and $(\Omega^\pm)$ means that the corresponding object is defined by the covariant derivative with respect to $\Omega^\pm$. This is $R^{2\mu}{}_{\nu}$ type supersymmetric Lagrangian. A nice property is that $L_{R^{2\mu}{}_{\nu}}$ can be represented as the supersymmetric Yang-Mills action by considering the pair of indices $ab$ as the gauge index and regarding the gauge field $A^I_\mu = \Omega^+_\mu{}^{ab}$ and the gaugino $\chi^I = \psi^{ab} (\Omega^-)$:

\begin{align}
L_{\text{SYM}} &= -\frac{1}{4} e F^{\mu\nu} I F_{\mu\nu} - 2 e \bar{\chi}^I \gamma^{\mu} (D_\mu \chi^I) + \frac{1}{4} e F^{\mu\nu} I \bar{\psi}_\mu \gamma^{\mu\nu} \gamma^{\rho} \chi^I \\
&+ e S \bar{\chi}^I \chi^I - \frac{1}{2} e \bar{\chi}^I \chi^I \psi_\rho \psi_\mu + \frac{1}{8} e \bar{\chi}^I \chi^I \bar{\psi}_\rho \gamma^{\rho\nu} \psi_\nu \\
&= L_{R^{2\mu}{}_{\nu}}, \quad (6)
\end{align}

where we use the identifications $A^I_\mu = \Omega^{\pm}{}^{ab}_\mu$ and $\chi^I = \psi^{ab} (\Omega^-)$ in the final equality.
In addition, one can also find the following $R^2$ type supersymmetric term,

$$L_{R^2} = \frac{1}{16} e \hat{R}^2(\Omega^+) + \frac{1}{4} e \bar{\psi}_{\mu\nu} \gamma^{\mu\nu} \mathcal{D} \psi_{\rho\sigma} (\Omega^-) - e \partial^\mu S \partial_\mu S - \frac{1}{8} e \bar{\psi}_{\mu\nu} \gamma^{\mu\nu} \partial_{\rho\sigma} \psi_{\rho\sigma} (\Omega^-) - \frac{1}{32} e \bar{\psi}_{\mu\nu} \gamma^{\mu\nu} \partial_{\rho\sigma} \psi_{\rho\sigma} (\Omega^-) + \frac{1}{64} e \bar{\psi}_{\mu\nu} \gamma^{\mu\nu} \partial_{\rho\sigma} \psi_{\rho\sigma} (\Omega^-) \bar{\psi}_{\lambda\tau} \psi_{\tau},$$

where the hatted curvature is defined by

$$\hat{R}(\Omega^+) = R(\hat{\omega}) + 6 S^2 + 2 e \bar{\psi}_{\mu\nu} \gamma^{\mu\nu} \psi_{\nu}.$$

Similarly one can regard this $L_{R^2}$ matter Lagrangian as follows:

$$L_{\text{matter}} \equiv L_{R^2} = -e \partial^\mu \phi \partial_\mu \phi - \frac{1}{4} e \lambda \gamma^\mu D_{\mu} \lambda + \frac{1}{16} e f^2 + \frac{1}{8} e S \lambda \lambda + \frac{1}{32} e \lambda \gamma^\mu \lambda \partial_{\mu} \phi \lambda + \frac{1}{64} e \lambda \lambda \bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu.$$

where we use the identifications for the scalar $\phi = S$, for the spinor $\lambda = \gamma^{\mu\nu} \psi_{\mu\nu}(\Omega^-)$, and for the auxiliary scalar $f = \hat{R}(\Omega^\pm)$ in the final equality.

**III. SUSY EXACT TERMS**

For the localization calculation, the most important feature is the following point: To obtain the partition function $Z = \lim_{\epsilon \to 0} Z(t)$, we define $Z(t)$ as

$$Z(t) = \int \mathcal{D}e^\mu \mathcal{D} \psi_\mu \mathcal{D} S \ e^{i S + i t \delta V},$$

and furthermore this $Z(t)$ does not depend on the parameter $t$. Then we can take $t \to \infty$ limit to conduct the computation, and in this limit, all the contributions of the path integral are localized on the field configurations which satisfy $\delta V = 0$. In quantum field theory, this technique achieved great successes and uncovered structures of the interacting supersymmetric field theories in various dimensions. The necessary ingredients for this $t$-independence are 1: off-shell supersymmetry $\delta$, 2: supersymmetric invariant action $S = \int L$ and 3: supersymmetric exact action $\delta V$ where $V$ is a certain functional of the fields, which satisfy $\delta^2 V = 0$. Naively, we expect that its analog to the supergravity provides us an unknown structures of quantum gravity. In this notes, we try to do it.

In order to apply the above localization argument, the missing piece is the supersymmetric exact action $\delta V$, and we find that the following actions are candidates for the appropriate actions $\delta V$;

$$L_{R^2} + \text{cosm} = -\frac{1}{8} L_{R^2} - \frac{1}{4} L \bar{\psi}_{ab} \psi_{ab} + \text{cosm} \left( \begin{array}{c}
\frac{1}{8} e \text{SYM} - \frac{1}{4} L C \lambda \lambda \\
L_{\text{matter}} - \frac{1}{4} L C \lambda \lambda
\end{array} \right),$$

where $L_C$ is the supersymmetric cosmological constant given in (14). In fact, one can verify the following relations:

$$\delta \left( e [\chi \delta \chi] \right) = (\bar{e} e) L_{R^2} + \text{cosm} \left( \begin{array}{c}
(\bar{e} e) L_{R^2} + \text{cosm} \\
(\bar{e} e) L_{R^2} + \text{cosm}
\end{array} \right)$$

These relations show that above $L_{R^2} + \text{cosm}$ and $L_{R^2} + \text{cosm}$ are SUSY exact terms. Of course, the Lagrangians $L_{R^2}$ and $L_{R^2}$ preserves supersymmetry. However, in each case (11) or (12), one has a mass term for the fermion, and it is a typical supersymmetry breaking term, where the supersymmetry breaking is given by the supersymmetric cosmological constant term $L_C$.

One might wonder why these SUSY exact actions are not SUSY invariant. The reason is as follows. In rigid limit, we have $\delta^2 = 0$ in field theoretical sense, and one can show SUSY invariance just by acting additional $\delta$ to (13) or (14). However if we do not take a rigid limit ($\psi_\mu = 0$), then we have $\delta^2 \neq 0$. As a result, (13) and (14) are not SUSY invariant, even though they are SUSY exact.

**IV. NAIVE ATTEMPT TOWARD GRAVITY LOCALIZATION**

Let us discuss the localization argument on supergravity based on the results in previous section. As explained above, the only embarrassing term is the mass term for graviton $\psi_\mu$, or equivalently $\chi$ or $\lambda$ in (11) or (12), which prevails the SUSY invariance of SUSY exact term (13) and (14). To overcome the problem, here we try to eliminate it just by inserting the delta function $\delta(L_C)$ to the path integral in (11):

$$Z(t) = \int \mathcal{D}e^\mu \mathcal{D} \psi_\mu \mathcal{D} S \delta(L_C) e^{i \int d^3 x L_{\text{EH}} + i \int d^3 x L_C + i t \delta V},$$

where we take $S = \int (L_{\text{EH}} + L_C)$, and $\delta V$ is the one (11) or (12). It might look strange, but since the delta function can be written by introducing auxiliary field $\phi$ as an integral formula,

$$\delta(L_C) = \int \mathcal{D} \phi \ e^{i \int d^3 x L_{\text{C}} \phi},$$
we can rewrite (15) as

$$Z(t) = \int D\psi^a D\bar{\psi}_\mu DSD\varphi \ e^{i \int d^3x L_{\text{EH}} + i \int d^3x L_{C}(1+\varphi) + i\delta V}. \quad (17)$$

If the supersymmetric invariance for the deformed cosmological constant term and for the path integral measure are achieved, then this $Z(t)$ becomes $t$-independent, and we can utilize the localization technique by taking $t \to \infty$ limit. For that purpose, we require

$$\delta(\varphi L_C) = 0. \quad (18)$$

If the SUSY variation of the cosmological constant term is total divergence, say, $\delta L_C = \nabla \mu J^\mu$, then (18) implies that $\delta \varphi$ should be defined linear with respect to $\varphi$ such as

$$\delta \varphi = - \frac{\nabla \mu J^\mu}{L_C} \varphi. \quad (19)$$

However this induces quantum anomaly, i.e., Jacobian for supersymmetry variation of $\varphi$ is not one. In order to apply the conventional supersymmetric localization technique, the Jacobian for supersymmetry variation should vanish, therefore this naive method does not work, unfortunately.

V. CONCLUSION, DISCUSSION, AND FUTURE WORK

In this notes, we discussed a possibility for the application of localization technique to the quantum gravity path integral. We tried to conduct direct gravity path integral by constructing SUSY exact terms in 3D supergravity. Although SUSY exact terms are constructed, naive procedure for localization calculation fails. Our main discovery is that the $R^2_{\mu\nu}$ supersymmetric invariant, $L_{R_2}$, and $R^2$ supersymmetric invariant, $L_{R_2}$, can be represented as SUSY exact terms up to gravitino mass terms, which break supersymmetric invariance and its breaking is given by the supersymmetric cosmological constant term $L_C$. This prevents us from applying a naive localization technique to supergravity within these setups [18]. We would like to make some comments about our (rather negative) results.

First, let us comment on the difficulty of the gravity sector localization computation. In our case, as one can find the algebraic structure of local SUSY $\delta$ on 3D supergravity in [9], squared SUSY $\delta^2$ is not zero and contains SUSY $\delta$, too. This structure is coming from the existence of gravitino, and it is absent in the rigid SUSY limit $\delta_{\text{rigid}}$ [6] which guarantees the localization computation because of the nilpotent nature $\delta^2_{\text{rigid}} = 0$ in many cases. However, the possibility for localization in supergravity is not excluded even for 3D $\mathcal{N} = 1$ because what we found is just the relationship [11] - [13]. Therefore, if one can find certain better SUSY exact terms and succeed in canceling the obstructing mass term, then it should work.

Second, the mass terms in our SUSY exact Lagrangians, (11) and (12), seem to be “universal” mass terms because they are always proportional to the supersymmetric cosmological constant $L_C$ in [1]. We have no a priori reason to get such supersymmetric coefficient as the mass parameter, but there might exist certain deep reason which could be related to the algebraic structure on supergravity.

Third, it may be good to consider the same problem with extended local supersymmetries, $\mathcal{N} \geq 2$. For example, we can find off-shell formulation of $\mathcal{N} = 2$ supergravity in [7]. In 3D, conventional field theoretical localization computation is available only for $\mathcal{N} \geq 2$, therefore situation there could be better.

It will be also interesting to consider the analog of our argument with Euclidean supergravity. (For relevant works on 3D Euclidean pure gravity with negative cosmological constant, see for example, [2, 8, 9, 10].) Crucial difference is that in Euclidean signature, modular invariance is strong enough to determine (some of) non-perturbative effects. It would be great if we can derive the summation over Modular group discussed in [10, 12] in direct supergravity localization calculation without relying on the power of modular invariance. This should be done along the line of [9], where the sum over modular group appears naturally as the sum over all of the localization locus, $J_{\mu\nu} = 0$, which are solutions of all the complex Einstein equation.

Before we end, let us discuss the physical meaning of conducting gravity path integral, $Z = \int [\mathcal{D}g_{\mu\nu}] e^{iS[g_{\mu\nu}]}$. Even if we succeed in conducting the metric path integral $[\mathcal{D}g_{\mu\nu}]$ exactly, whether it gives an exact partition function for quantum gravity or not, depends on whether the metric $g_{\mu\nu}$ is a fundamental degree of freedom in quantum gravity. We have learned from holography that bulk gravity is an effective theory, which is valid and emerging typically in the large $N$ limit of QCD-like $SU(N)$ gauge theory as a dual effective description. Furthermore, a metric, which is dual to gauge-singlet stress-tensor, is a dominating degree of freedom only in low temperature phase [13, 14]. In fact, in high temperature phase, rather than metric, black hole microstates are the dominating degrees of freedom [14]. Given these, how much is the bulk metric path integral meaningful calculation?

To answer this, the analogy to QCD helps; gravity in low temperature phase is like chiral Lagrangian in QCD, where the dynamical degrees of freedom are pion field $\pi$’s, instead of quarks and gluons. Then conducting gravity path integral $\int [\mathcal{D}g_{\mu\nu}] e^{iS[g_{\mu\nu}]}$ corresponds to conducting pion field path integral $\int [\mathcal{D}\pi] e^{iS_{\text{chiral}}}$ in the chiral lagrangian. Of course we know the fundamental theory behind chiral lagrangian is QCD, and the exact answer for the partition function for QCD can be obtained only after by conducting the path integral for quark-gluon fields, rather than pion fields. Pion field path integral of the chiral lagrangian never gives the right
answer for QCD, due to its lack of quark and gluon degree of freedom which are dominating in high temperature phase [20]. As one cannot describe quark-gluon plasma by multi-pion fields, we expect that black hole microstates are not describable by multi-gravitons (see [13] for a nice overview) [21]. In this way, we expect that the naive bulk metric path integral, $Z = \int [Dg_{\mu\nu}] e^{S[g_{\mu\nu}]}$, is not non-perturbatively-defined quantity, at least in bulk where we have space-time dimensions larger than three. (Note however in three-dimension, modular invariance of the partition function is powerful enough to determine the contributions of BTZ black hole microstates, see [3, 8, 9, 10].) To obtain an exact partition function for full quantum gravity, we have to rely on the dual non-perturbatively defined boundary theory path integral [22].

However what we try to calculate in this paper is not this quantity (partition function), but rather supersymmetric black hole microstates. This is because of supersymmetry, significant reduction of degrees of freedom occurs. Therefore, the SUSY index calculation from the bulk metric by conducting $\int [Dg_{\mu\nu}]$ is still meaningful even in bulk.

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Appendix A: Spinor notations and formulas

The clifford algebra is generated by the following two by two matrices.

$$
\gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

Charge conjugation matrix is

$$
C = i\gamma^0.
$$

We always use Majorana fermion throughout this notes. It satisfies

$$
\bar{\psi} = \psi^\dagger i\gamma^0 = \psi^T C
$$

and it is equivalent to $\psi^* = \psi$ and it means real fermion. For Majorana fermions $\psi, \chi, \epsilon$, we have the following formulas:

$$
\bar{\psi} \chi = \bar{\chi} \psi, \quad (A4)
$$

$$
(\gamma^\mu \psi) \chi = -\bar{\psi} \gamma^\mu \chi, \quad (A5)
$$

$$
\bar{\chi} \gamma^\mu \psi = -\bar{\psi} \gamma^\mu \chi, \quad (A6)
$$

$$
\bar{\chi} \gamma^\mu \gamma^\nu \psi = \bar{\psi} \gamma^\nu \gamma^\mu \chi, \quad (A7)
$$

$$
\bar{\chi} \gamma^\mu \gamma^\nu \gamma^\rho \psi = -\bar{\psi} \gamma^\rho \gamma^\nu \gamma^\mu \chi, \quad (A8)
$$

$$
\epsilon(\bar{\chi} \psi) + 2(\bar{\psi} \epsilon) + \gamma^\mu \epsilon(\bar{\chi} \gamma^\mu \psi) = 0 \quad (A9)
$$

$$
(\bar{\psi} \epsilon)(\bar{\epsilon} \chi) = -\frac{1}{2}(\bar{\epsilon} \epsilon)(\bar{\psi} \chi). \quad (A10)
$$

The formula in (A10) is useful in the calculation of (B3) and (B4).

Appendix B: Proof of (B3)

As commented in the main part of this notes, if we define $A_I^\mu = \Omega^{ab}_{\mu}, \chi^I = \psi^{ab}(\Omega^-)$, these redefined multiplet satisfy

$$
\delta A_I^\mu = - (\bar{\epsilon} \gamma^\mu \chi), \quad \delta \chi = \frac{1}{8} \gamma^{\mu\nu}(F_{\mu\nu} + 2 \bar{\psi}[\gamma_{\mu\nu}] \chi)\epsilon, \quad (B1)
$$

where we omit the index $I$ from now on. Just using these SUSY transform, we calculate

$$
\delta \left[ e(\bar{\chi} \delta \chi) \right] = \delta e \cdot (\bar{\chi} \delta \chi) + e(\bar{\delta \chi} \delta \chi) + e(\bar{\chi} \delta^2 \chi) \quad (B2)
$$

as follows. We use (A10) many times.

$$
\delta e \cdot (\bar{\chi} \delta \chi) = (\bar{\epsilon} \epsilon) e \left\{ -\frac{1}{32} (\bar{\chi} \gamma^{\mu\nu} \gamma^\rho \psi) F_{\mu\nu} - \frac{1}{16} (\bar{\chi} \chi)(\bar{\psi}_\mu \gamma^{\rho\nu} \psi) \\
- \frac{1}{16} (\bar{\chi} \chi)(\bar{\psi}_\mu \gamma^\rho \psi) \right\} \quad (B3)
$$

$$
\epsilon(\bar{\delta \chi} \delta \chi) = (\bar{\epsilon} \epsilon) e \left\{ \frac{1}{32} F_{\mu\nu} F^{\mu\nu} + \frac{1}{8} (\bar{\psi} \gamma^\rho \chi) F_{\mu\nu} \\
+ \frac{1}{16} (\bar{\chi} \chi)(\bar{\psi}_\mu \psi) + \frac{1}{16} (\bar{\chi} \chi)(\bar{\psi}_\mu \gamma^{\mu\nu} \psi) \right\} \quad (B4)
$$

$$
\epsilon(\bar{\chi} \delta^2 \chi) = (\bar{\epsilon} \epsilon) e \left\{ (\frac{1}{16} (\bar{\chi} \gamma^{\mu\nu} \gamma^\rho \psi) + \frac{1}{32} (\bar{\psi}_\mu \gamma^{\mu\nu} \psi) + \frac{1}{16} (\bar{\psi}_\mu \gamma^{\mu\nu} \psi)) F_{\nu\mu} \\
+ \frac{1}{16} (\bar{\chi} \chi)(\bar{\psi}_\mu \psi) - \frac{1}{64} (\bar{\chi} \chi)(\bar{\psi}_\mu \gamma^{\mu\nu} \psi) \\
+ \frac{1}{4} (\bar{\chi} \gamma^\rho D_{\mu} \chi) - \frac{6}{16} S(\bar{\chi} \chi) \right\}. \quad (B5)
$$

Summing up (B3), (B4) and (B5), we get the result in (B3).
Appendix C: Proof of \[14\]

If we define $\phi = S$, $\lambda = \gamma^\mu\psi_{\mu}(\Omega^-)$, $f = \hat{R}(\Omega^\pm)$, then these fields satisfy

\[
\delta \phi = \frac{1}{4} \epsilon \lambda, \quad \delta \lambda = \gamma^\nu \epsilon [\partial_\nu \phi - \frac{1}{4} \psi_{\nu}] - \frac{1}{4} \epsilon f
\]

\[
\delta f = -\epsilon \gamma^\mu [D_\mu (\hat{\omega}) \lambda - \gamma^\nu \psi_{\mu} (\partial_\nu \phi - \frac{1}{4} \psi_{\nu})] + \frac{1}{2} S (\epsilon \lambda).
\]

By using these SUSY transformations, we calculate

\[
\delta \left[ e (\lambda^2 \delta \lambda) \right] = \delta e \cdot (\lambda^2 \lambda) + e (\delta \lambda^2 \lambda) + e (\lambda^2 \cdot \delta \lambda)
\]

and each term is given by as follows:

\[
\delta e \cdot (\lambda^2 \lambda) = (\epsilon e) e \left\{ -\frac{1}{4} \left( \bar{\psi}_{\mu} \gamma^\nu \gamma^\lambda \lambda \right) \partial_\nu \phi - \frac{1}{32} \left( \bar{\lambda} \lambda \right) (\bar{\psi}_{\mu} \psi_{\mu}) \right\},
\]

\[
(\epsilon e) e \left\{ -\partial_\mu \psi \partial_\mu \phi + \frac{1}{32} (\bar{\lambda} \lambda) (\bar{\psi}_{\mu} \psi_{\mu}) + \frac{1}{16} f^2 \right\},
\]

\[
(\lambda^2 \delta \lambda) = (\epsilon e) e \left\{ -\partial_\mu \psi \partial_\mu \phi + \frac{1}{32} (\bar{\lambda} \lambda) (\bar{\psi}_{\mu} \psi_{\mu}) \right\},
\]

\[
(\lambda^2 (\delta \lambda)) = (\epsilon e) e \left\{ -\partial_\mu \psi \partial_\mu \phi + \frac{1}{32} (\bar{\lambda} \lambda) (\bar{\psi}_{\mu} \psi_{\mu}) \right\}.
\]

Combining \[C3\], \[C4\] and \[C5\], we arrive at \[14\].

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[17] Here, we consider the partition function with Lorentz signature. So it is plausible to take the weight as $e^{i\phi}$. In the $t \to \infty$ limit, the configuration which satisfy $\delta V = 0$ is dominant thanks to the Riemann-Lesbesgue lemma.

[18] However, as commented in \[3\], the bosonic part of the equation $L_{R^2} = 0$ is exactly equivalent to the integrability condition for the Killing spinor equation which is equivalent to the condition $\delta \psi = 0$. This fact might illuminate the possibility of localizing calculous with $L_{R^2}$ action, but of course, we should overcome the graviton mass term problem in \[13\].

[19] These can easily seen from comparison of the $N$-dependence of the entropy between thermal-gas/black holes in bulk and confinement/deconfinement phase in the boundary \[13, 14\].

[20] If we pay attention to physics only in the confinement phase, namely, only low temperature phase and neglect all non-perturbative effects, then the results of pion path integral of chiral lagrangian is still meaningful as an effective theory.

[21] Note also that the argument of \[13, 14\] works in bulk where we have space-time dimensions larger than three. Even in three dimensional space-time case, BTZ black hole microstates are regarded as a different primary’s conformal family \[3, 6, 8, 10\].

[22] It gives, at most, approximately valid and meaningful quantity only in low temperature phase, i.e., bulk phase without black holes.