Affine convex body semigroups

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Abstract

In this paper we present a new kind of semigroups called convex body semigroups which are generated by convex bodies of $\mathbb{R}^k$. They generalize to arbitrary dimension the concept of proportionally modular numerical semigroup of [7]. Several properties of these semigroups are proven. Affine convex body semigroups obtained from circles and polygons of $\mathbb{R}^2$ are characterized. The algorithms for computing minimal system of generators of these semigroups are given. We provide the implementation of some of them.

Keywords: Affine semigroup, circle semigroup, convex body monoid, convex body semigroup, polygonal semigroup.

MSC-class: 20M14 (Primary), 20M05 (Secondary).

Introduction

Let $F$ be a subset of $\mathbb{R}^k$, $F = \bigcup_{i=0}^{\infty} F_i \cap \mathbb{R}^k_+$ and $\mathcal{F} = \bigcup_{i=0}^{\infty} F_i \cap \mathbb{N}^k$, where $F_i = \{iX | X \in F\}$ with $i \in \mathbb{N}$. A convex body of $\mathbb{R}^n$ is a compact convex subset with non-empty interior. If $F$ is a convex body, then the set $F$ is a monoid and $\mathcal{F}$ is a semigroup (see Proposition 1). Given a convex body $F$, we call convex body monoid (respectively semigroup) generated by $F$ to the above monoid (respectively semigroup) $F$ (respectively $\mathcal{F}$). In this work we consider the usual topology of $\mathbb{R}^k$.

In general these semigroups are not finitely generated. If $\mathcal{F}$ is a finitely generated semigroup we say that $F$ is an affine convex body semigroup. Given a convex polygon or a circle in $\mathbb{R}^2$, we study the necessary and sufficient conditions for $\mathcal{F}$ to be finitely generated. These conditions are related to the slopes of $\mathbb{R}^2$.

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the extremal rays of the minimal cone which includes $\mathcal{F}$. We give effective methods to obtain their minimal system of generators.

In [7], the authors present the numerical monoids and semigroups generated by intervals ($F = [\alpha, \beta] \subseteq \mathbb{R}_{\geq}$ with $\alpha < \beta$) called proportionally modular numerical semigroups. They prove proportionally modular numerical semigroups are characterized by a modular Diophantine inequality (see [7, Theorem 8]). We generalize this modular Diophantine inequality for the convex body monoids and semigroups (see Corollaries [3] and [20]).

The minimal system of generators of a proportionally modular numerical semigroup can be obtained by constructing a Bézout sequence connecting two reduced fractions (see [2] and [7]). In Lemma 6 it can be found an alternative method to compute this minimal system of generators.

Besides, Lemma 19 shows an easy algorithm to check if an element belongs to a circle semigroup, and Corollary 22 provides a bound for the minimal generators of these semigroups. The implementation of the algorithm to compute the minimal system of generators of a circle semigroup is available at the url [3].

The contents of this work are organized as follows. In Section 1 we give some concepts and results used during this work. We also characterize convex body semigroups in terms of Diophantine inequalities. In Section 2 some algebraic and geometrical constructions are given. Section 3 and 4 are devoted to characterize the affine semigroups generated by a polygon (polygonal semigroup) or a circle (circle semigroup). The algorithms to compute their minimal systems of generators are showed. For these cases in Section 5 we compute a bound for the minimal generators of the affine semigroup.

1 Convex semigroups

Given $\{a_1, \ldots, a_r\} \subseteq \mathbb{N}^k$, we denote by $S = \langle a_1, \ldots, a_r \rangle$ the subsemigroup of $\mathbb{N}^k$ generated by $\{a_1, \ldots, a_r\}$, that is, $\langle a_1, \ldots, a_r \rangle = \{\lambda_1 a_1 + \cdots + \lambda_r a_r | \lambda_1, \ldots, \lambda_r \in \mathbb{N}\}$. If no proper subset of $\{a_1, \ldots, a_r\}$ generates $S$, then this set is called the minimal system of generators of $S$. Every affine semigroup admits a unique minimal generating system (see [6]).

Define the cone generated by $A \subseteq \mathbb{R}^k_>$ as the set

$$L_{\mathbb{Q}_2}(A) = \left\{ \sum_{i=1}^{p} q_i a_i | p \in \mathbb{N}, q_i \in \mathbb{Q}_{\geq}, a_i \in A \right\}.$$ 

A ray is a line containing the zero element, $O$, of $\mathbb{R}^k$. A ray is defined by only one point not equal to $O$. Given $A \subseteq \mathbb{R}^k_>$, denote by $\tau_1$ and $\tau_2$ to the extremal rays of $L_{\mathbb{Q}_2}(A)$ (assume the slope of $\tau_1$ is greater than the slope of $\tau_2$), and by $\text{int}(A) = A \cap (L_{\mathbb{Q}_2}(A) \setminus \{\tau_1, \tau_2\})$. We called interior of $A$ to the set $\text{int}(A)$.

Let $F$ be a convex body of $\mathbb{R}^k$ and let

$$F = \{X \in \mathbb{R}^k_> | \text{there exists } i \in \mathbb{N} \text{ such that } \frac{X}{i} \in F \} \cup \{0\} = \bigcup_{i=0}^{\infty} F_i,$$

where $F_i = \{iX | X \in F \}$ with $i \in \mathbb{N}$.

**Proposition 1.** $F$ is a submonoid of $\mathbb{R}^k$. 

Proof. Let $P, Q \in F$. There exist $i, j \in \mathbb{N}$ and $P', Q' \in F$ such that $P = iP'$ and $Q = jQ'$. Then
\[
P + Q = iP' + jQ' = (i + j) \left( \frac{i}{i+j}P' + \left(1 - \frac{i}{i+j}\right)Q' \right).
\]
Using the convexity of $F$ we obtain $\frac{i}{i+j}P' + \left(1 - \frac{i}{i+j}\right)Q' \in F$ and so $P + Q \in F$.

We call convex body monoid of $\mathbb{R}^k$ to every submonoid $F$ of $\mathbb{R}^k$ obtained as above from a convex body of $\mathbb{R}^k$.

Denote by $d(P, Q)$ the Euclidean distance between two elements $P, Q \in \mathbb{R}^k$ and by $d(P)$ the distance $d(P, O)$. We see the convexity property is necessary to $F$ be a monoid. If $F$ is the compact and not convex set
\[
\{X \in \mathbb{R}^2_k | 3 \leq d(X) \leq 5\},
\]
the elements $(4, 0), (0, 4)$ are in $F$ but $(4, 0) + (0, 4)$ is not in $F$.

Define a convex body semigroup as the intersection of a convex body monoid with $\mathbb{N}^k$. In general, these semigroups are not full affine semigroup, that is, they can not be expressed using linear Diophantine equations (see [6]). To see this, consider a convex body $F$ of $\mathbb{R}^k$ fulfilling that it has at least an element $P$ satisfying that $P + e_1 \in F$, where $e_1$ is the first element of the canonical basis of $\mathbb{R}^k$, and $e_1 \not\in F$. This implies the elements $P, P + e_1 \in F$ but $(P + e_1) - P = e_1 \not\in F$.

The following result is a generalization of Theorem 8 of [7] and it provides an inequality which characterizes the elements of a convex body monoid of $\mathbb{R}^k$.

Observe that if a ray intersects with $F_1$ in only a point (respectively a segment), then the intersection of the ray with any other $F_i$ with $i > 1$ is also a point (respectively a segment). Denote by $\overrightarrow{PQ}$ the segment joining $P$ and $Q$.

**Proposition 2.** Let $\tau$ be a non-negative slope ray. Then, for all $X \in F \cap \tau$ there exist $a, b \in \mathbb{R}_+$ with $1 < a < b$, such that
\[
a \cdot d(X) \mod b \leq d(X).
\]

**Proof.** If $X \in F \cap \tau$, then there exists $i \in \mathbb{N}$ such that $X \in F_i$. If $i = 0$, then $X = 0$ and there exist $a, b \in \mathbb{R}_+$ such that the inequality is clearly satisfied.

Assume that $X \in F_i$, with $i > 0$. Observe the intersection $\tau \cap F_i$, can be only a point or a segment. If $\tau \cap F_i = \{X\}$ then there exists $P \in F$ such that $X = iP$ and $d(X) = id(P)$. Taking now a number $a \in (1, \infty)$ we obtain $a < ai$ and $ad(X) \mod ai(\tau) = 0 \leq d(X)$. If $\tau \cap F_i = \overrightarrow{PQ}$ (assume $d(P) < d(Q)$), then $X \in i\overrightarrow{PQ}$ and $d(X)$ belongs to a submonoid of $\mathbb{R}_+$ generated by $\{d(P), d(Q)\}$. By [7] Theorem 8], we conclude there exist $a, b \in (1, \infty)$ with $b > a$ such that $a \cdot d(X) \mod b \leq d(X)$.

From the above proposition it can be concluded that $a$ and $b$ depend only of the vector $\overrightarrow{OX}$. This fact allows us to characterize the elements of a convex body semigroup from an inequality. Denote by $\tau$ the ray containing the point $X$.

**Corollary 3.** An element $X \in \mathbb{N}^k$ belongs to $\text{int}(F)$ if and only if the following conditions are fulfilled:
1. $\tau \cap F$ is a segment $\overline{PQ}$ with $P, Q \in \text{int}(F)$.

2. \[
\frac{d(Q)}{d(Q) - d(P)} \mod \frac{d(P)d(Q)}{d(Q) - d(P)} \leq d(X).
\]

Proof. It is straightforward from Proposition 2 and the proof of Theorem 8 in \cite{7}.

2 Tools

Let $F$ be a convex body of $\mathbb{R}_2^2$ and $\tau_1, \tau_2$ the extremal rays of $L_{Q_2}(F)$ (assume the slope of $\tau_1$ is greater than the slope of $\tau_2$). Observe that $F$ is contained in the cone $L_{Q_2}(F)$. The subsemigroup $L_{Q_2}(F) \cap \mathbb{N}^2$ is denoted by $C$.

In general for every semigroup equal to the set of non-negative integer solutions of a system of inequalities (for instance $C$), its minimal system of generators can be determined by obtaining the minimal solutions of a system of Diophantine equations (see \cite{11} and \cite{14}).

Lemma 4. Let $\tau$ be a rational slope ray, $g, s \in \tau \cap \mathbb{N}^2$ and $\overrightarrow{u} \in \mathbb{R}^2$. Define $R_i$ the parallelogram determined by the elements $g+(i-1)s$, $g+is$ and $g+(i-1)s+\overrightarrow{u}$ with $i \in \mathbb{N}$. If $R_i \subset \mathbb{R}_2^2$, then $R_i \cap \mathbb{N}^2 = (R_i \cap \mathbb{N}^2) + (i-1)s$.

Proof. By construction $R_i = R_1 + (i-1)s$ for every $i \in \mathbb{N}$. Since $s \in \mathbb{N}^2$, then $R_i \cap \mathbb{Z}^2 = (R_i \cap \mathbb{Z}^2) + (i-1)s$. In case $R_1 \subset \mathbb{R}_2^2$, we obtain that $R_i \cap \mathbb{N}^2 = (R_i \cap \mathbb{N}^2) + (i-1)s$.

Lemma 5. Let $P, Q \in \mathbb{Q}_{\geq}$ (respectively $P, Q \in \mathbb{Q}_2$). The semigroup $\mathcal{I} = (\bigcup_{i \in \mathbb{N}} i\overrightarrow{PQ}) \cap \mathbb{N}$ (respectively $\mathcal{I} = (\bigcup_{i \in \mathbb{N}} i\overrightarrow{PQ}) \cap \mathbb{N}^2$) is finitely generated and there exists an algorithm to determine its minimal system of generators.

Proof. Assume that $\overrightarrow{PQ} \subset \mathbb{R}_2$. The elements $P' = (P, 1)$ and $Q' = (Q, 1)$ belong to $\mathbb{Q}_2$. Denote by $C'$ the semigroup $L_{Q_2}(\{P', Q'\}) \cap \mathbb{N}^2$. The set $C'$ is determined by the rational systems of inequalities given by the two rays containing the points $P'$ and $Q'$. Thus $C'$ is finitely generated. The semigroup $\mathcal{I}$ is the projection onto the first coordinate of the elements of $C'$ and therefore it is finitely generated.

Let consider now the case $\overrightarrow{PQ} \subset \mathbb{R}_2^2$. Define again $P' = (P, 1)$ and $Q' = (Q, 1)$ elements of $\mathbb{Q}_3$. Take now $\overrightarrow{u}$ a normal vector to the subspace $\langle \overrightarrow{OP'}, \overrightarrow{OQ'} \rangle$ and two vectorial planes $\pi_1$ and $\pi_2$ generated by $\{\overrightarrow{OP'}, \overrightarrow{u}\}$ and $\{\overrightarrow{OQ'}, \overrightarrow{u}\}$ respectively.

Let $C'$ be the semigroup finitely generated by the minimal solutions of the system of rational inequalities determined by the plane containing the points $\{O, P', Q'\}$, and the cone delimited by $\pi_1$ and $\pi_2$. Since $\mathcal{I}$ is the projection onto the first and second coordinate of $C'$, it is finitely generated.

In both cases the minimal system of generators of $\mathcal{I}$ is obtained by an effective way from the set given by the projection of a system of generators of $C'$. A minimal system of generators of $C'$ can be computed from the solutions of a system of Diophantine inequalities.

Lemma 6. Let $\tau$ be a ray and $\overrightarrow{PQ}$ a segment over $\tau$ with $P, Q \in \mathbb{R}^2 \setminus \mathbb{Q}_2$ (assume $d(P) < d(Q)$). Then the semigroup $\mathcal{I} = (\bigcup_{i \in \mathbb{N}} i\overrightarrow{PQ}) \cap \mathbb{N}^2$ is finitely generated and there exists an algorithm for computing its minimal system of generators.
Proof. If \( \tau \) has negative or irrational slope then \((\bigcup_{i \in \mathbb{N}} iPQ) \cap \mathbb{N}^2 = \emptyset\), and therefore the result is straightforward.

Assume the slope of \( \tau \) is not negative and rational. Let \( k \) be the smallest positive integer fulfilling that \( kQ - (k + 1)P \in \mathbb{R}^2_\geq \). By construction the integer \( k \) exists and it can be determined, then the ray with vertex \((k + 1)P\) and determined by \( PQ \) is included in the monoid \( \bigcup_{i \in \mathbb{N}} iPQ \).

Let \( T \) be the finite set \( O((k + 1)P) \cap \mathbb{N}^2 \) and let

\[
\begin{align*}
    d_1 & = \min \left( \bigcup_{i=1}^{k+1} \{d(H,iP) | H \in T \} \right) / (k + 1), \\
    d_2 & = \min \left( \bigcup_{i=1}^{k+1} \{d(H,iQ) | H \in T \} \right) / k.
\end{align*}
\]

Consider the segment \( PQ' \) with \( P' = P - d_1 \vec{u} \) and \( Q' = Q + d_2 \vec{u} \) where \( \vec{u} \) is the unitary direction vector of \( \tau \).

The segment \( PQ' \) verifies that \( PQ' \cap \mathbb{N}^2 \supset (\bigcup_{i \in \mathbb{N}} iPQ') \cap \mathbb{N}^2 \geq \).

Since \( PQ' \cap \mathbb{N}^2 \) are rational, by Lemma 5 we conclude that \( \mathcal{I} \) is finitely generated.

The minimal system of generators of \( \mathcal{I} \) can be computed in an effective way following the steps of this proof:

- Compute the smallest \( k \in \mathbb{N} \) such that \( kQ - (k + 1)P \in \mathbb{R}^2_\geq \).
- Compute the set \( T \) and the values \( d_1 \) and \( d_2 \).
- Compute the vector \( \vec{u} \) and take the rational points \( P' \) and \( Q' \).
- Apply Lemma 5.

In particular, the above result can be used to obtain a system of generators of a proportionally modular semigroup. This is an alternative method to the one presented in [7].

The following results are used to find system of generators of convex body semigroups.

Lemma 7. Let \( \{g_1, \ldots, g_p\} \subset \mathbb{N}^2 \) be the minimal system of generators of a semigroup \( F \) and \( \tau = g_1Q \) an extremal ray of \( F \). Assume that \( g_1 \) generates \( \mathbb{N}^2 \cap \tau \) and consider \( \{s_1, \ldots, s_t\} \) the minimal system of generators of a subsemigroup of \( \mathbb{N}^2 \cap \tau \). Let \( F' \) be the semigroup generated by \( B = B_1 \cup B_2 \) with

\[
B_1 = \{s_1, \ldots, s_t, g_2, \ldots, g_p\}, \; B_2 = \bigcup_{i=2}^{p} \{g_i + g_1, \ldots, g_i + (\lambda_t - 1)g_1\},
\]

where \( 0 < \lambda_1 < \cdots < \lambda_t \) are the integers such that \( s_i = \lambda_i g_1 \). Then the semigroup \( F' \) verifies:

- \( F' \cap \tau = \langle s_1, \ldots, s_t \rangle \).
• $F' \setminus \tau = F \setminus \tau$.

Proof. Clearly $F' \cap \tau = \langle s_1, \ldots, s_t \rangle$.

On the other hand, let $g \in F \setminus \tau$. There exist $\mu_1, \ldots, \mu_p \in \mathbb{N}$ with $\sum_{i=2}^{p} \mu_i \neq 0$, such that $g = \sum_{i=1}^{p} \mu_i g_i$. Without lost of generality we can assume that $\mu_2 > 1$. There are three possibilities:

- If $\mu_1 = 0$, then it is trivial that $g \in F' \setminus \tau$.
- If $\lambda > \mu_1 > 0$, then $g = g_2 + \mu_1 g_1 + (\mu_2 - 1) \sum_{i=3}^{p} \mu_i g_i$.
- If $\mu_1 \geq \lambda > 0$, then there exist $u, v \in \mathbb{N}$ such that $\mu_1 = u \lambda + v$, with $\lambda > v$. Thus, $g = u (\lambda g_1) + g_2 + v g_1 + (\mu_2 - 1) \sum_{i=3}^{p} \mu_i g_i$.

In any of the above cases we obtain that $g \in F' \setminus \tau$ and we can conclude that $F' \setminus \tau = F \setminus \tau$ (trivially $F' \setminus \tau \subset F \setminus \tau$).

Lemma 8. Let $F \subset \mathbb{N}^2$ be a finitely generated semigroup and $a \in F$. The set $F \setminus \{a\}$ is a semigroup if and only if $a$ is a minimal generator of $F$. Besides if $B = \{a, f_2, \ldots, f_t\}$ is the minimal system of generators of $F$, then the semigroup $F \setminus \{a\}$ is generated by

$$\{f_2, \ldots, f_t, f_2 + a, \ldots, f_t + a, 2a, 3a\}.$$

Proof. Assume that $F \setminus \{a\}$ is a semigroup and that $a$ is not a minimal generator of $F$. Then there exist $a_1, a_2 \in F \setminus \{a\}$ such that $a = a_1 + a_2$, which contradicts the fact that $F \setminus \{a\}$ is a semigroup.

Conversely, assume that $a$ is a minimal generator of $F$ (remind the semigroup $F$ has a unique system of generators). To prove that $F \setminus \{a\}$ is a semigroup it is only necessary to show that the addition is an operation on this set. Let $x, y \in F \setminus \{a\}$, then $x + y \in F \setminus \{a\}$ (if not we have that $x + y = a$, which is impossible because $a$ is a minimal generator of $F$).

Let $B = \{a, f_2, \ldots, f_t\}$ the minimal system of generators of $F$ (without lost of generality we assume that $a$ is the first element of $B$). Trivially, $\{f_2, \ldots, f_t, f_2 + a, \ldots, f_t + a, 2a, 3a\} \subset F \setminus \{a\}$. Let $f \in F \setminus \{a\} \subset F$, therefore $\exists \lambda, \lambda_2, \ldots, \lambda_t \in \mathbb{N}$ such that $f = \lambda a + \sum_{i=2}^{t} \lambda_i f_i$. If $\lambda \neq 1$, there exist $\alpha, \beta \in \mathbb{N}$ verifying that $\lambda = 2 \alpha + 3 \beta$, thus

$$f = \lambda a + \sum_{i=2}^{t} \lambda_i f_i = \alpha (2a) + \beta (3a) + \sum_{i=2}^{t} \lambda_i f_i.$$ 

If $\lambda = 1$, since $a \notin F \setminus \{a\}$, there exists $\lambda_{i_0} \geq 1$, such that

$$f = a + \sum_{i=2}^{t} \lambda_i f_i = (f_{i_0} + a) + (\lambda_{i_0} - 1)f_{i_0} + \sum_{i=2, i \neq i_0}^{t} \lambda_i f_i.$$ 

In any case, $\{f_2, \ldots, f_t, f_2 + a, \ldots, f_t + a, 2a, 3a\}$ is a system of generators of $F \setminus \{a\}$. 

\qed
Corollary 9. Let $\mathcal{F}$ be a finitely generated semigroup and $A \subset \mathcal{F}$ be a finite subset. If $\mathcal{F} \setminus A$ is a semigroup, then $\mathcal{F} \setminus A$ is a finitely generated semigroup. Furthermore, there exists an algorithm to compute a system of generators of $\mathcal{F} \setminus A$.

Proof. Assume that $A = \{a_1, \ldots, a_n\} \subset \mathcal{F}$ and assume that $B$ is the minimal system of generators of $\mathcal{F}$. Using the proof of Lemma 8 at least an element of $A$ must be an element of $B$. Assume that $a_1 \in B$, then by Lemma 8 we obtain that $\mathcal{F}_1 = \mathcal{F} \setminus \{a_1\}$ is a subsemigroup of $\mathbb{N}^2$. Denote by $B_1$ to the minimal system of generators of the semigroup $\mathcal{F}_1$ which is obtained from the system of generators of $\mathcal{F}_1$ constructed as in Lemma 8. Using again the above reasoning with the sets $A_1 = A \setminus \{a_1\}$, $\mathcal{F}_1$ and $B_1$, we obtain a new semigroup $\mathcal{F}_2 = \mathcal{F}_1 \setminus \{a_1\}$, where $a_i \in A_1 \cap B_1$ with $i \in \{2, \ldots, n\}$. Since $A$ is finite, this method stops after a finite number of steps and we obtain a finite system of generators $B_n$ of the semigroup $\mathcal{F}_n = \mathcal{F} \setminus A$. 

3 Convex polygonal semigroups

In general the semigroup generated by a convex body of $\mathbb{R}^2$ is not finitely generated. In this section partial results on semigroups generated by convex polygons are presented and the affine convex polygonal semigroups are characterized.

Denote by $P_i = (p_{i1}, p_{i2})$ with $i = 1, \ldots, n$ the vertices of a compact convex polygon $F \subset \mathbb{R}_2^3$ ordered in the clockwise direction. We denote this set by $\mathcal{P}$ and by $\mathcal{P}$ the associated semigroup.

Proposition 10. If $\mathcal{P} \subset \mathbb{Q}_2$, then $\mathcal{P}$ is finitely generated. Furthermore, there exists an algorithm which determines its minimal system of generators.

Proof. Let $\mathcal{P} = \{P_1, \ldots, P_n\}$ the set of vertices of $F$ and consider the set of points $\mathcal{P}' = \{(P_1, 1), \ldots, (P_n, 1)\} \subset \mathbb{Q}_2^3$. Take now the cone $\mathcal{C} \subseteq \mathbb{N}^3$ delimited by the planes that contain the origin and two consecutive points of $\mathcal{P}'$. Since this cone is defined by rational inequalities, it is finitely generated.

A system of generators of $\mathcal{P}$ is the set formed by the projection onto the first two coordinates of a system of generators of $\mathcal{C}$. From this set of generators of $\mathcal{P}$ one can compute its minimal system of generators.

Suppose now the extremal ray $\tau_1$ of $L_{\mathbb{Q}_2}(F)$ intersects $F$ in only one point $P_1$, denote by $V_i$ the intersection of $(iP_1)(iP_2)$ and $((i+1)P_n)((i+1)P_1)$ for every $i \in \mathbb{N}$. Note that for the initial values of $i$ it is possible that these points does not exist (see Figure 7).

Lemma 11. Every point $V_i$ belongs to a parallel line to $\tau_1$.

Proof. Clearly $(iP_1)(iP_2)$ and $((i+1)P_n)((i+1)P_1)$ are not parallel, their lengths increase with no limit and keep one of their vertices in the ray $\tau_1$. They intersect in only one point $V_i$ for $i \gg 0$.

After some basic computations the reader can check that the distance between $V_i$ and $\tau_1$ is constant and equal to

$$\frac{p_{12}^2p_{11}p_{n1} - p_{12}p_{21}p_{11}p_{n2} + a_1^2p_{n2}p_{22} - p_{11}p_{22}p_{12}p_{n1}}{(-p_{22}p_{n1} + p_{11}p_{22} + p_{12}p_{n1} + p_{n2}p_{21} - p_{n2}p_{11} - b_1p_{22}) \sqrt{p_{12}^2 + p_{11}^2}}.$$

Thus, the points $V_i$ are in a line parallel to $\tau_1$. 

7
In this case there exists $i_0$ such that
\[
\int(P) \setminus \bigcup_{i \geq i_0} P_i \subset \int(C) \setminus \bigcup_{i \geq i_0} \text{triangle}\{iP_1, (i + 1)P_1, V_i\}.
\]
We illustrate this property in Figure 1 (in this figure $i_0 = 6$).

![Figure 1: Image of a convex polygonal semigroup.](image)

For the sake of simplicity we have used the points $P_1, P_2$ and $P_n$ in the above results, but the result can be extended to the intersection of $F$ and an extremal ray when this intersection is only a point.

We focus now our attention when $F$ is a particular triangle.

**Proposition 12.** Let $F$ be a triangle delimited by $\{P_1, P_2, P_3\}$ with $P_1 \in \mathbb{Q}^2_2$ and $P_2, P_3 \in \mathbb{R}^2_2 \setminus \mathbb{Q}^2_2$, such that $P_1 \in \tau_1$ and $P_2 P_3 \subset \tau_2$, where $\tau_1$ and $\tau_2$ are the extremal rays of $L_{\mathbb{Q}^2_2}(F)$. Then $P$ is finitely generated and there exists an algorithm to compute its minimal system of generators.

**Proof.** By Lemma 11, for all integer $i \gg 0$ the distance between the point $\overline{iP_1 P_2} \cap (i + 1)P_1 P_3$ and the line $\tau_1$ is constant. Let $j_0$ be the smallest integer such that $j_0 P_1 \cap (j_0 + 1)P_1 P_3 \neq \emptyset$ and $j_0 P_1 \in \mathbb{N}^2$. Denote by $s_1$ the element of $P$ which generates $P \cap \tau_1$, by $V$ the point $\overline{j_0 P_1 P_2} \cap (j_0 + 1)P_1 P_3$, and let $j_1$ be the smallest integer such that $j_1 P_1 = j_0 P_1 + s_1$.

Denote by $T_1$ the finite set of integer points belonging to the parallelogram $G$ with edges the segment $\overline{j_0 P_1}(j_1 P_1)$ and the segment determined by the points $j_0 P_1$ and $j_0 P_1 + (j_0 P_1) V$, but they are not in $P$. By Lemma 4, the integer points of $G$ obtained applying the translations defined by $i s_1$ with $i \in \mathbb{N}$ are the translated of $T_1$. Furthermore, we clearly have the distances of the points of $T_1 + i s_1$ to the edges of the triangles contained in the parallelogram $G + i s_1$ are constant for all $i \in \mathbb{N}$.

Denote by $T_2$ the finite set of integer points of the region delimited by $\tau_1, \tau_2$ and $j_0 P_1 P_3$ which does not belong to $P$, and let $T = T_1 \cup T_2$ (see Figure 2).

Consider
\[
d_1 = \min \left( \bigcup_{i=1}^{j_1} \{d(H, iP_1 P_2) | H \in T\} \right),
\]
and
\[
d_2 = \min \left( \bigcup_{i=0}^{j_1} \{d(H, (i + 1)P_1 P_3) | H \in T\} \right).
\]
Once we know the distances $d_1$ and $d_2$ we can move in $\tau_2$ the vertices $P_2$ and $P_3$ until we reach two rational points $P'_2$ and $P'_3$ (since the slope of $\tau_2$ is rational, there are an infinite number of possibilities to take these points into segments that including $P_2P_3$) to form a new triangle $F'$ with rational vertices \{$P_1, P'_2, P'_3$\} such that

\[
\mathcal{P} = \left( \bigcup_{i \in \mathbb{N}} iF' \right) \cap \mathbb{N}^2,
\]

as shown in Figure 3 where dotted lines correspond to the new rational triangle with rational vertices.

As the vertices of $F'$ are rational, the semigroup $\mathcal{P}$ is finitely generated and its minimal system of generators can be computed (see Proposition 10).
Proposition 13. Let $F \subset \mathbb{R}_2^2$ be a convex polygon fulfilling that $\tau_1$ and $\tau_2$ have rational slopes and $\tau_1 \cap F$ and $\tau_2 \cap F$ are segments. Then $\mathcal{P}$ is finitely generated and there exists an algorithm which determines its minimal system of generators.

Proof. Let $\tau_1 \cap F = \overline{P_1P_2}$ and $\tau_2 \cap F = \overline{P_{j_0+1}P_i}$. By construction there exists the least integer $j_0$, such that the region $G$ bounded by $\tau_1$, $\tau_2$ and the segment $j_0\overline{P_iP_{j_0+1}}$ verifies $C \setminus G \subset \left( \bigcup_{i \geq j_0} iF \right) \cap \mathbb{N}^2$. Define the finite set $T = G \setminus \mathcal{P}$.

Since $\mathcal{P}$ is the set $C \setminus T$, we conclude that $\mathcal{P}$ is finitely generated (see Corollary 0).

An algorithm to determine a system of generators of $\mathcal{P}$ is the following:

1. Compute the generators of $\mathcal{P} \cap \tau_1$ and $\mathcal{P} \cap \tau_2$ (use Lemma 6).
2. Construct a semigroup $F'$ verifying $F' \cap \tau_1 = \mathcal{P} \cap \tau_1$, $F' \cap \tau_2 = \mathcal{P} \cap \tau_2$ and $F' \setminus \{\tau_1, \tau_2\} = C \setminus \{\tau_1, \tau_2\}$ (use Lemma 7). This semigroup is obtained using the system of generators of $C$ and the generators set of the preceding step.
3. Eliminate from $F'$ all the points of $T$ (use Lemma 8).

This process ends after a finite number of steps obtaining a system of generators of $\mathcal{P}$ which can used to get its minimal system of generators. \hfill \Box

Theorem 14. The semigroup $\mathcal{P}$ is finitely generated if and only if $F \cap \tau_1$ and $F \cap \tau_2$ contain rational points. Furthermore, in such case there exists an algorithm to compute the minimal system of generators of $\mathcal{P}$.

Proof. Assume $F \cap \tau_1 \subseteq \mathbb{R}_2^2 \setminus \mathbb{Q}^2$ and let $G = \{s_1, s_2, \ldots, s_r\}$ be a system of generators of $\mathcal{P}$. This implies that $\mathcal{P} \cap \tau_1 = \emptyset$.

Consider $s_k \in G$ such that the vector $\overrightarrow{O s_k}$ has maximum slope respect to the points of $G$. Since $\mathcal{P} \cap \tau_1 = \emptyset$, there exists at least an element $Q \in \mathbb{Q}^2$ in the interior of the cone delimited by $\tau_1$ and the ray defined by $s_k$.

There exists $u \in \mathbb{N}$ such that $uQ$ belongs to a polygon $F_u$, but $uQ$ is not generated by $G$. Thus, $\mathcal{P}$ is not finitely generated which is a contradiction.

If $F \cap \tau_2$ has not rational points, the proof that $\mathcal{P}$ is not finitely generated is similar than above.

Conversely, assume the intersections of $F$ with $\tau_1$ and $\tau_2$ contain rational points. There are several cases:

1. If $\tau_1 \cap F$ and $\tau_2 \cap F$ are segments, this case is already solved in Proposition 13.
2. If $\tau_1 \cap F$ has only a point and $\tau_2 \cap F$ is a segment, then take $\tau'_1$ a ray with rational slope such that the intersection of the polygon $F$ with the region delimited by $\tau_1$ and $\tau'_1$ is a triangle $F'_2$. The set $F'_2 = F \setminus F'_1$ verifies the conditions of Proposition 13.

The minimal system of generators of the semigroup generated by $F'_1$ can be computed (use Proposition 12). Analogously, the minimal system of generators of the semigroup generated by $F'_2$ can be computed (use Proposition 13). Since $\mathcal{P}$ is the union of the semigroups generated by $F'_1$ and $F'_2$, the semigroup $\mathcal{P}$ is finitely generated by the union of the above systems of generators.
3. If $\tau_1 \cap F$ and $\tau_2 \cap F$ are two points, we proceed as follows. Take $\tau'_1$ and $\tau'_2$ two rays with rational slopes such that the polygons obtained from the intersection of $F$ and the region delimited by $\tau_1$ and $\tau'_1$, and by $\tau_2$ and $\tau'_2$, are two triangles. The intersection of the polygon $F$ and the region delimited by $\tau'_1$ and $\tau'_2$ verifies the condition of Proposition 13 (see Figure 4).

![Figure 4: Polygon with only a vertex in each extremal rays.](image)

Once again, a system of generators of $P$ can be obtained by applying Proposition 12 and Proposition 13 to the above regions.

In any case the semigroup $P$ is finitely generated and its minimal system of generated can be computed algorithmically.

4 Circle semigroups

This section is devoted to semigroups generated by circles. The reason of this section is that most of the results of Section 3 are not valid for this kind of semigroups.

Let $C$ be the circle (a convex body) with center $(a, b)$ and radius $r$. Denote by $C_i$ the circle with center $(ia, ib)$ and radius $ir$, and by $S = \bigcup_{i=0}^{\infty} C_i \cap \mathbb{N}^2$ the semigroup generated by $C$. As in the preceding sections, denote by $\tau_1$ and $\tau_2$ the extremal rays of $L_{\mathbb{Q}_2}(C \cap \mathbb{R}_2^2)$ where the slope of $\tau_1$ is greater than the slope of $\tau_2$, and by $\mathcal{C}$ the positive integer cone $L_{\mathbb{Q}_2}(C \cap \mathbb{R}_2^2) \cap \mathbb{N}^2$. In such case, $\text{int}(C) = C \setminus \{\tau_1, \tau_2\}$.

**Lemma 15.** Suppose that $C \cap \tau_2$ is a point. If $P_i$ is the closest point to $\tau_2$ belonging to $C_i \cap C_{i+1}$ then $\lim_{i \to \infty} d(P_i, \tau_2) = 0$.

**Proof.** Denote by $h_i$ the distance $d(P_i, \tau_2)$. Without lost of generality, assume that $\tau_2$ is the line $\{y = 0\}$. This is possible because the distances between the points of our construction are invariant by turn centered in the origin. Graphically the situation is as shown in Figure 5.

\[1\] For the initial values of $i$ it is possible to obtain that $C_i \cap C_{i+1} = \emptyset$, see Figure 6.
Since the slope of $\tau_2$ is zero, the circles have radius $ib$ and therefore $h_i = d(P_i, \tau_2)$ is equal to the second coordinate of $P_i$.

With these hypothesis, the point $P_i$ is the solution of the following system of equations closest to the axis $OX$

$$
\begin{align*}
C_i & \equiv (x - ai)^2 + (y - bi)^2 = (bi)^2 \\
C_{i+1} & \equiv (x - a(i+1))^2 + (y - b(i+1))^2 = b^2(i+1)^2.
\end{align*}
$$

That is,

$$
\begin{align*}
x &= \frac{a^4(1+2i) + b\sqrt{-a^4(a^2 - 4b^2i(1+i))}}{2a(a^2 + b^2)}, \\
y &= \frac{a^2(b + 2bi) - \sqrt{-a^4(a^2 - 4b^2i(1+i))}}{2(a^2 + b^2)}.
\end{align*}
$$

Then the distance is

$$
h_i = d(P_i, \tau_2) = \frac{a^2b + 2a^2bi - \sqrt{-a^6 + 4a^4b^2i + 4a^4b^2i^2}}{2(a^2 + b^2)}.
$$

(2)

It is straightforward to prove that $\lim_{i \to \infty} h_i = 0$.

**Remark 16.** If $C \cap \tau_1$ has only a point, denote by $P_i'$ the point of $C_i \cap C_{i+1}$ closest to $\tau_1$. Using the symmetry of $\bigcup_{i=0}^{\infty} C_i$ with respect to the line joining the centers of the circles, we obtain that $d(P_i', \tau_1) = d(P_i, \tau_2)$.

The following Lemma asserts that $\text{int}(C) \backslash \text{int}(S)$ has a finite number of points if $C \subset \mathbb{R}_0^2$.

**Lemma 17.** Let $C \subset \mathbb{R}_0^2$ be a circle. There exists $d \in \mathbb{R}_0^2$ such that

$$
\{ P \in \text{int}(C) | d(P) > d \} \subset S.
$$

Furthermore, $d$ can be computed algorithmically.
Proof. Consider two rectangles in $C$ whose bases are segments determined by two consecutive points of the semigroup in $\tau_1$ for the firsts rectangle and in $\tau_2$ for the second and with height (the same for both) a sufficiently small value to obtain no points of $\mathbb{N}^2$ into them (excepting in their bases). Denote by $d'$ this height.

For the sake of simplicity we consider that $\tau_2$ is the line $\{y = 0\}$. In this case the rectangles are as in Figure 6.

Figure 6: Construction 1.

Denote by $T_1, T_2 \in S$ the vertices of the base of the rectangle over the line $\tau_2$.

Consider now the region of the cone obtained applying to the above rectangle all the translations defined by the vector $\overrightarrow{OT_1}$ and all its positive multiples. This construction is done over $\tau_1$ and over $\tau_2$ (see Figure 7). In this region there are not integer points (Lemma 4).

Figure 7: Construction 2.

Let $i_0 \in \mathbb{N}$ the first term of the sequence of heights $\{h_i\}$ (defined in (2)).

Note the point $T_1$ is a natural multiple of the point $\tau_2 \cap C$ and that $T_2 = 2T_1$. 

\[\text{(2)}\]
such that $b_i < d'$. Lemma \ref{lem:existence} asserts the existence of $i_0$.

Then there exists $d \in \mathbb{R}_+ \geq d_0$ determined by the circle $C_{i_0}$ such that $\{P \in \text{int}(C)|d(P) > d\} \subset \bigcup_{i \geq i_0} C_i \cap \mathbb{N}^2 \subset S$. In Figure \ref{fig:construction3} observe that $i_0 = 6$. □

![Figure 8: Construction 3.](image)

The region delimited by $\tau_1$, $\tau_2$ and the circle with center the origin and radius $d$ of the above lemma (Figure \ref{fig:construction3}) can be replaced by the triangle delimited by the $\tau_1$, $\tau_2$ and the line joining the points of the intersection of such lines with the circle $C_{i_0}$. This simplifies the computation of the integer points of the region.

The following Theorem characterizes affine circle semigroups and provides an algorithm to compute their minimal system of generators.

\textbf{Theorem 18.} The semigroup $S$ is finitely generated if and only if $C \cap \tau_1$ and $C \cap \tau_2$ have rational points. Furthermore, in such case the minimal system of generators of $S$ can be computed algorithmically.

\textbf{Proof.} If $S$ is finitely generated proceed as in Theorem \ref{thm:finitelygenerated}.

For the reciprocal we consider several cases. If $C \cap \mathbb{R}^2_+ = \emptyset$, then $S = \{0\}$ and therefore it is finitely generated. In other case, compare the semigroups $S$ and $C$. The relationship between the sets $\text{int}(C)$ and $\text{int}(S)$ is the following: if $P \in \text{int}(C) \setminus \text{int}(S)$ then $d(P) \leq d$, where $d$ is the distance determined by Lemma \ref{lem:distance}. Therefore $\text{int}(C) \setminus \text{int}(S)$ is finite. In addition, given $P \in \mathbb{N}^2$ with $d(P) > d$, $P \in \text{int}(C)$ if and only if $P \in \text{int}(S)$.

To study the relationship between $C \cap \tau_1$ and $S \cap \tau_1$, and $C \cap \tau_2$ and $S \cap \tau_2$, we must consider four cases:

1. Assume that $C \cap \tau_1$ and $C \cap \tau_2$ have only one point (this situation is similar to that shown in Figure \ref{fig:construction3}). In this case, if $C \cap \tau_1 = \langle g_1 \rangle$ and $C \cap \tau_2 = \langle g_2 \rangle$, then all the elements of $S \cap \tau_1$ and $S \cap \tau_2$ are natural multiples of $g_1$ or $g_2$.

2. Assume that $C \cap \tau_1$ is a point and $C \cap \tau_2$ is a segment. In this case $\tau_2$ is the line $\{y = 0\}$ (see Figure \ref{fig:construction3}). We compare again the semigroups $S$ and $C$:

   - Note that if $C \cap \tau_1 = \langle g_1 \rangle$ then all the elements of $S \cap \tau_1$ are natural multiples of $g_1$. 

14
• The set \((C \cap \tau_2) \setminus (S \cap \tau_2)\) is finite. Besides, \(S \cap \tau_2\) is a finitely generated semigroup and its minimal system of generators can be computed algorithmically (see Lemma 6).

3. Assume that \(C \cap \tau_1\) is a segment and \(C \cap \tau_2\) is a point. This case is similar to the above case.

4. Assume that \(C \cap \tau_1\) and \(C \cap \tau_2\) are segments. In this case \(\tau_1\) is the line \(\{x = 0\}\) and \(\tau_2\) is the line \(\{y = 0\}\). Then the sets \((C \cap \tau_1) \setminus (S \cap \tau_1)\) and \((C \cap \tau_2) \setminus (S \cap \tau_2)\) are finite. Besides, \(S \cap \tau_1\) and \(S \cap \tau_2\) are two finitely generated semigroups and their minimal systems of generators can be computed algorithmically (see Lemma 6).

We have obtained that in any case \(S\) is the set obtained after eliminate from \(C\) a finite number of points of its interior and some points of its extremal rays.

See now how a system of generators of \(S\) can be built. We construct explicitly a set of generators of the semigroup \(S'\) such that \(S' \cap \tau_1 = S \cap \tau_1, S' \cap \tau_2 = S \cap \tau_2,\) and \(\text{int}(S') = \text{int}(C)\). This set will be used in Corollary 22.

Denote by \(\{g_1, \ldots, g_p\}\) the minimal system of generators of \(C\) where \(g_1 \in \tau_1\) and \(g_2 \in \tau_2\). If we consider the first case and assume that \(s_1\) and \(s_2\) are the minimal elements of \(S\) in \(\tau_1\) and \(\tau_2\), then there exist \(k_1, k_2 \in \mathbb{N}\) such that \(s_1 = k_1 g_1\) and \(s_2 = k_2 g_2\). By using Lemma 7 on \(s_1\) and after on \(s_2\), the semigroup \(S'\) is generated by

\[
\{s_1, s_2, g_3, \ldots, g_p\} \cup \left( \bigcup_{i=2}^{p} \{g_i + g_1, \ldots, g_i + (k_1 - 1)g_1\} \right) \cup \{s_1 + g_2, \ldots, s_1 + (k_2 - 1)g_2\} \cup \\
\bigcup_{j=1}^{k_2-1} \left( \bigcup_{i=2}^{p} \{g_i + g_1 + jg_2, \ldots, g_i + (k_1 - 1)g_1 + jg_2\} \right) \cup \bigcup_{i=1}^{p} \{g_i + g_2, \ldots, g_i + (k_2 - 1)g_2\}.
\]

(3)

Consider now the second case (analogously for the third case). There exists \(k_1 \in \mathbb{N}\) such that \(s_1 = k_1 g_1 \in \tau_1\), and there exist \(\lambda_1, \ldots, \lambda_t \in \mathbb{N}\) such that \(\lambda_1 < \cdots < \lambda_t\) and \(S \cap \tau_2\) is generated minimally by \(\{(\lambda_i, 0) = \lambda_i (1, 0) | i = 1, \ldots, t\}\) \((g_2 = (1, 0))\). By using Lemma 7, one obtain a system of generators of the
the origin with the distance to the center of \( C \) where \( k \) is satisfied by the elements of \( S \).

\[
S \subseteq S'
\]

At the end of this process the minimal system of generators of \( S \) can be computed from a system of generators of \( S' \). The idea of the algorithm is to eliminate from the minimal system of generators of \( S' \) the finite set of element \( S' \setminus S \) by using the algorithm shown in Corollary \( \ref{corollary:algorithm} \).

The following Lemma allows to check if an element belongs to the semigroup \( S \) by using its distance to the origin.

**Lemma 19.** Let \( (x, y) \in \mathbb{N}^2 \). The element \( (x, y) \in S \) if and only if \( (x, y) \in C_k \cup C_{k+1} \) with \( k = \lfloor \sqrt{x^2 + y^2} \rfloor \in \mathbb{N} \).

**Proof.** Given \( (x, y) \in S \), the following inequalities holds

\[
kd((a, b)) \leq d((x, y)) \leq (k + 1)d((a, b)),
\]

where \( k = \lfloor \sqrt{x^2 + y^2} \rfloor \). Then \( (x, y) \) belongs to \( C_k \) and/or to \( C_{k+1} \).

Thus, to detect if an element is in \( S \), it is enough to compare its distance to the origin with the distance to the center of \( C \). After that, it only remains to check if the point belongs to two circles of \( S \).

In the following result, Proposition \( \ref{proposition:distance} \) is used to obtain several inequalities satisfied by the elements of \( S \).
Corollary 20. Every $X = (x, y) \in S \setminus \{\tau_1, \tau_2\}$ satisfies

$$\frac{1}{2} \left( \frac{(a, b) \cdot (x, y)}{\sqrt{(d(X)r)^2 - [(b, -a) \cdot (x, y)]^2}} + 1 \right) d(X) \mod \frac{d(X)(d((a, b))^2 - r^2)}{2\sqrt{(d(X)r)^2 - [(b, -a) \cdot (x, y)]^2}} \leq d(X).$$

Proof. Repeating the reasonings of Proposition 2 and Corollary 3, the coefficients of the inequality (1) are determined by the points of the intersection of $C$ and the ray given by $X$.

In this case, the points are

$$P = \left( x \left( \frac{ax + by - \sqrt{-(bx - ay)^2 + (x^2 + y^2)r^2}}{x^2 + y^2} \right) \right),$$

$$Q = \left( \frac{ax + by + \sqrt{-(bx - ay)^2 + (x^2 + y^2)r^2}}{x^2 + y^2} \right),$$

and

$$d(P) = \frac{ax + by - \sqrt{-(bx - ay)^2 + (x^2 + y^2)r^2}}{\sqrt{x^2 + y^2}},$$

$$d(Q) = \frac{ax + by + \sqrt{-(bx - ay)^2 + (x^2 + y^2)r^2}}{\sqrt{x^2 + y^2}}.$$

By Corollary 3, $d(X)$ verifies the inequality

$$\frac{d(Q)}{d(Q) - d(P)} d(X) \mod \frac{d(Q)d(P)}{d(Q) - d(P)} \leq d(X),$$

where

$$\frac{d(Q)}{d(Q) - d(P)} = \frac{1}{2} \left( \frac{(a, b) \cdot (x, y)}{\sqrt{(d(X)r)^2 - [(b, -a) \cdot (x, y)]^2}} + 1 \right)$$

and

$$\frac{d(Q)d(P)}{d(Q) - d(P)} = \frac{d(X)(d((a, b))^2 - r^2)}{2\sqrt{(d(X)r)^2 - [(b, -a) \cdot (x, y)]^2}}.$$

If the intersection of an extremal ray $\tau$ with the initial circle is a segment, the above result is also fulfilled by all points of $S \cap \tau$. When the above mentioned intersection is only one point, the inequality we get is the inequality that appears in the proof of Proposition 2.
Example 21. Consider the circle $C$ with center $(7/3, 4/3)$ and radius $1/3$. We are going to apply the algorithm shown in Theorem 18 to the semigroup $S$ generated by $C$.

We compute the integer cone $C$ delimited by the extremal rays of $L_{\mathbb{Q}\mathbb{Z}}(C)$. This cone is minimally generated by

$$\{(4,3), (12,5), (2,1), (3,2), (7,3)\}.$$  

With the notation of Theorem 18, $g_1 = (4,3), g_2 = (12,5), s_1 = (32, 24) = 8g_1$ and $s_2 = (96, 40) = 8g_2$.

Applying the construction of the system of generators of $S'$ of (4), the semigroup $S'$ is minimally generated by

$$\{(2,1), (3,2), (7,3), (7,5), (11,8), (15,11), (19,14), (23,17), (27,20), (31,23), (32,24), (96,40), (19,8), (31,13), (43,18), (55,23), (67,28), (79,33), (91,38)\}.$$  

This semigroup is equal to $S$ in their extreme rays and equal to $C$ in their interiors.

The finite set $S' \setminus S$ has 13 points. By using Corollary 9, we eliminate recurrently from $S'$ the points of $S' \setminus S$ obtaining the minimal system of generators of $S$ (see Figure 10):

$$\{(5,3), (6,4), (7,3), (7,4), (7,5), (8,4), (9,5), (9,6), (10,5), (11,6), (11,8), (13,6), (15,11), (18,8), (19,14), (23,10), (23,17), (27,20), (31,23), (32,24), (33,14), (35,26), (38,16), (50,21), (55,23), (67,28), (79,33), (91,38), (96,40), (115,48), (127,53), (139,58)\}.$$  

Figure 10: The minimal generators set of the semigroup generated by the circle with center $(7/3, 4/3)$ and radius $1/3$.

This example has been computed by using our program CircleSG available in [3] (this program requires Wolfram Mathematica 7 to run).
5 Bounding the minimal system of generators

Assume that \( S \) is an affine semigroup obtained from a circle and consider the norm \( ||(x_1, \ldots, x_n)||_1 = \sum_{i=1}^{n} |x_i| \). Denote by \( M \) the maximum of the norms of the elements of the minimal system of generators of the cone \( C \). One can find several bounds for this value (see [5] and [8]).

Following the notation given in the proof of Theorem 18, denote by \( l \) the cardinality of the finite set \( \text{int}(S') \setminus \text{int}(S) \). Furthermore, the minimal elements of \( S \) in \( \tau_1 \) and \( \tau_2 \) are integer multiples of \( g_1 \) or \( g_2 \). Denote by \( k \) the maximum of such integers.

**Corollary 22.** Every element \( s \) of the minimal system of generators of \( S \) fulfills that

\[
||s||_1 \leq 3^l(2k - 1)M.
\]

**Proof.** The minimal system of generators of \( S' \) can be obtained from (4), (4) or (4). Thus, the norm of their elements can bounded by the value

\[
(2k-1)M = \max\{kM, M, (k-1)M + M, kM + (k-1)M, (k-1)M + (k-1)M + M\},
\]

where every value \( \{kM, M, (k-1)M + M, kM + (k-1)M, (k-1)M + (k-1)M + M\} \) is a bound for the elements of the subsets obtained in (4), (4) and (4).

To obtain a system of generators of \( S \), we apply sequentially to the elements of \( \text{int}(S') \setminus \text{int}(S) \) the algorithm described in Corollary 9. For the first iteration one has the bound is the maximum of \( \{(2k - 1)M, 2(2k - 1)M, 3(2k - 1)M\} \).

Since the above method is applied as many times as elements has the set \( \text{int}(S') \setminus \text{int}(S) \), a bound for the elements of the minimal system of generators of \( S \) is \( 3^l(2k - 1)M \).

\[\square\]

**Remark 23.** Analogously, a bound for the minimal generators of a convex polygonal semigroup can be obtained.

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