Bidirectional compression in heterogeneous settings for distributed or federated learning with partial participation: tight convergence guarantees

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Abstract

We introduce a framework – Artemis – to tackle the problem of learning in a distributed or federated setting with communication constraints and device partial participation. Several workers (randomly sampled) perform the optimization process using a central server to aggregate their computations. To alleviate the communication cost, Artemis allows to compress the information sent in both directions (from the workers to the server and conversely) combined with a memory mechanism. It improves on existing algorithms that only consider unidirectional compression (to the server), or use very strong assumptions on the compression operator, and often do not take into account devices partial participation. We provide fast rates of convergence (linear up to a threshold) under weak assumptions on the stochastic gradients (noise’s variance bounded only at optimal point) in non-i.i.d. setting, highlight the impact of memory for unidirectional and bidirectional compression, analyze Polyak-Ruppert averaging. We use convergence in distribution to obtain a lower bound of the asymptotic variance that highlights practical limits of compression. We propose two approaches to tackle the challenging case of devices partial participation and provide experimental results to demonstrate the validity of our analysis.

1 INTRODUCTION

In modern large scale machine learning applications, optimization has to be processed in a distributed fashion, using a potentially large number $N$ of workers. In the data-parallel framework, each worker only accesses a fraction of the data: new challenges have arisen, especially when communication constraints between the workers are present.

In this paper, we focus on first-order methods, especially Stochastic Gradient Descent [Bottou 1999] Robbins & Monro [1951] in a centralized framework: a central machine aggregates the computation of the $N$ workers in a synchronized way. This applies to both the distributed [e.g. Li et al. 2014] and the federated learning [introduced in Konečný et al. 2016 McMahand et al. 2017] settings.

Formally, we consider a number of features $d \in \mathbb{N}^*$, and a convex cost function $F : \mathbb{R}^d \rightarrow \mathbb{R}$. We want to solve the following convex optimization problem:

$$\min_{w \in \mathbb{R}^d} F(w) \text{ with } F(w) = \frac{1}{N} \sum_{i=1}^{N} F_i(w),$$

where $(F_i)_{i=1}^{N}$ is a local risk function for the model $w$ on the worker $i$. Especially, in the classical supervised machine learning framework, we fix a loss $\ell$ and access, on a worker $i$, $n_i$ observations $(z_k^i)_{1 \leq k \leq n_i}$ following a distribution $D_i$. In this framework, $F_i$ can be either the (weighted) local empirical risk, $w \mapsto (n_i^{-1}) \sum_{k=1}^{n_i} \ell(w, z_k^i)$ or the expected risk $w \mapsto \mathbb{E}_{z \sim D_i} [\ell(w, z)]$. At each iteration of the algorithm, each worker can get an unbiased oracle on the gradient of the function $F_i$ (typically either by choosing uniformly an observation in its dataset or in a streaming fashion, getting a new observation at each step).

Our goal is to reduce the amount of information exchanged between workers, to accelerate the learning process, limit the bandwidth usage, and reduce energy consumption. Indeed, the communication cost has been identified as an important bottleneck in the distributed settings [e.g. Strom 2015]. In their overview of the federated learning framework, [Kairouz et al. 2019] also underline in Section 3.5 two possible directions to reduce this cost: 1) compressing communication from workers to the central server (uplink) 2) compressing the downlink communication.

Most of the papers considering the problem of reducing the communication cost [Alistarh et al. 2017; Agarwal et al. 2018; Wu et al. 2018; Karimireddy et al. 2019; Mishchenko et al. 2019; Horváth et al. 2019; Li et al. 2020; Horváth & Richtarik 2020] only focus on compressing the message sent from the workers to the central node. This direction has the highest potential to reduce the total runtime given
that (i) the bandwidth for upload is generally more limited than for download, and that (ii) for some regimes with a large number of workers, the downlink communication, that corresponds to a “one-to-N” communication, may not be the bottleneck compared to the “N-to-one” uplink.

Nevertheless, there are several reasons to also consider downlink compression. First, the difference between upload and download speeds is not significant enough at all to ignore the impact of the downlink direction (see Appendix B for an analysis of bandwidth). If we consider for instance a small number $N$ of workers training a very heavy model – the size of Deep Learning models generally exceeds hundreds of MB [Dean et al., 2012; Huang et al., 2019] –, the training would not be eager to download a hundreds of MB for each update on their phone. Here again, downlink compression appears to be necessary. To encompass all situations, our framework implements compression in either or both directions with possibly different compression levels.

Bidirectional compression (i.e. compressing both uplink and downlink) raises new challenges. In the downlink step, if we compress the model, the quantity compressed does not tend to zero. Consequently the compression error significantly hinders convergence. To circumvent this problem we compress the gradient that may asymptotically approach zero. Double compression has been recently considered by [Tang et al., 2019; Zheng et al., 2019; Liu et al., 2020; Yu et al., 2019; Philippenko & Dieuleveut, 2021]. One of the most recent works, Dore, defined by Liu et al., 2020, analyzed a double compression approach, combining error compensation, a memory mechanism and model compression, with a uniform bound on the gradient variance. In this work, we provide new results on Dore-like algorithms, considering a framework without error-feedback using tighter assumptions, and quantifying precisely the impact of data heterogeneity on the convergence.

Moreover, we focus on a heterogeneous setting: the data distribution depends on each worker (thus non i.i.d.). We explicitly control the differences between distributions. In such a setting, the local gradient at the optimal point $\nabla F_i(w_*)$ may not vanish: to get a vanishing gradient error, we introduce a “memory” process [Mishchenko et al., 2019]. A very recent work by Philippenko & Dieuleveut [2021] builds upon our demonstrations precisely to handle non-i.i.d. settings: however they introduce an additional mechanism (“preserved update”) that is orthogonal to this work.

Finally, we encompass in Artemis Partial Participation (PP) settings, in which only some workers contribute to each step. Several challenges arise with PP, because of both heterogeneity and downlink compression. We propose a new algorithm in that setting, which improves on the approaches proposed by [Sattler et al., 2019; Tang et al., 2019] for bidirectional compression.

Assumptions made on the gradient oracle directly influence the convergence rate of the algorithm: in this paper, we neither assume that the gradients are uniformly bounded [as in Zheng et al., 2019] nor that their variance is uniformly bounded [as in Alistarh et al., 2017; Mishchenko et al., 2019; Liu et al., 2020; Tang et al., 2019; Horváth et al., 2019]: instead we only assume that the variance is bounded by a constant $\sigma^2_g$ at the optimal point $w_*$, and provide linear convergence rates up to a threshold proportional to $\sigma^2_g$ (as in Dieuleveut et al., 2018; Gower et al., 2019) for non distributed optimization). This is a fundamental difference as the variance bound at the optimal point can be orders of magnitude smaller than the uniform bound used in previous work: this is striking when all loss functions have the same critical point, and thus the noise at the optimal point is null! This happens for example in the interpolation regime, which has recently gained importance in the machine learning community [Belkin et al., 2019]. As the empirical risk at the optimal point is null or very close to zero, so are all the loss functions with respect to one example. This is often the case in deep learning [e.g., Zhang et al., 2017] or in large dimension regression [Mei & Montanari, 2019].

Overall, we make the following contributions:

1. We describe a framework – Artemis – that encompasses 6 algorithms (with or without up/down compression, with or without memory) which is adapted to PP. We provide and analyze in Theorem 1 a fast rate of convergence – exponential convergence up to a threshold proportional to $\sigma^2_g$, the noise at the optimal point, obtaining tighter bounds than in [Alistarh et al., 2017; Mishchenko et al., 2019].

2. We explicitly tackle heterogeneity using Assumption 4, proving that the limit variance of Artemis with memory is independent from the difference between distributions (as for SGD). This is the first theoretical guarantee for double compression that explicitly quantifies the impact of non i.i.d. data.

3. We propose two approaches to tackle the case of partial device participation when using memories. This setting is challenging due to the difficulty to synchronize memories. The second (and recommended) approach leverages the full potential of the memory to improve convergence.

4. In the non strongly-convex case, we prove the convergence using Polyak-Ruppert averaging in Theorem 2.

5. We prove convergence in distribution of the iterates, and subsequently provide a lower bounds on the asymptotic variance. This sheds light on the limits of (double) compression, which results in an increase of the algorithm’s variance, and can thus only accelerate the learning process for early iterations and up to a “moderate”
We consider the problem described in Equation (1). In the consequence, the compressed term tends in expectation to compress, and consequently the error [Mishchenko et al., 2019], we always broadcast gradients and never models. To distinguish the two compression operations we denote $C_{up}$ and $C_{dwn}$ the compression operator for downlink and uplink. At each iteration, we thus have the following steps:

1. First, each active local node sends to the central server a compression of gradient differences: $Δ_{k} = C_{up}(g_{k+1} - h_{k})$, and updates the memory term $h_{k+1} = h_{k} + αΔ_{k}$ with $α ∈ \mathbb{R}^*$. The server recovers the approximated gradients’ values by adding the received term to the memories kept on its side.

2. Then, the central server sends back the compression of the sum of compressed gradients: $Ω_{k+1} = C_{dwn} \left( \frac{1}{N} \sum_{i=1}^{N} Δ_{i} + h_{k} \right)$. No memory mechanism needs to be used, as the sum of gradients tends to zero in the absence of regularization.

The update is thus given by:

$$\egin{align*}
\forall i ∈ \{1, N\}, & Δ_{k} = C_{up}(g_{k+1} - h_{k}) \\
Ω_{k+1} = C_{dwn} \left( \frac{1}{N} \sum_{i=1}^{N} Δ_{i} + h_{k} \right) \\
w_{k+1} = w_{k} - γΩ_{k+1}
\end{align*}$$

Constants $γ, α ∈ \mathbb{R}^* × \mathbb{R}_+$ are learning rates for respectively the iterate sequence and the memory sequence. The adaptation of the framework in the case of device sampling is developed in Section 3. This is illustrated on Algorithm 1 and fig. 8 in Appendix A.

As a summary, the Artemis framework encompasses in particular these four algorithms: the variant with unidirectional compression ($ω_{dwn} = 0$) w.o. or with memory ($α = 0$ or $α ≠ 0$) recovers QSGD defined by Alistarh et al. [2017] and DIANA proposed by Mishchenko et al. [2019]. The variant using bidirectional compression ($ω_{dwn} ≠ 0$) w.o memory ($α = 0$) is called Bi-QSGD. The last and most effective variant combines bidirectional compression with memory and is the one we refer to as Artemis if no precision is given. It corresponds to a simplified version of Dore without error-feedback, but this additional mechanism did not lead to any theoretical improvement [Remark 2 in Sec. 4.1., Liu et al. 2020].

Remark 1 (Local steps). An obvious independent direction to reduce communication is to increase the number of steps performed before communication. This is the spirit of Local-SGD [Stich 2019]. It is an interesting extension to incorporate this into our framework, that we do not consider in order to focus on the compression insights.

In the following section, we present and discuss assumptions over the function $F$, the data distribution and the compression operator.

2.1 ASSUMPTIONS

We make classical assumptions on $F : \mathbb{R}^d → \mathbb{R}$.

Assumption 1 (Strong convexity). $F$ is $μ$-strongly convex, that is for all vectors $w, v ∈ \mathbb{R}^d$: $F(v) ≥ F(w) + (v - w)^T \nabla F(w) + \frac{μ}{2} \|v - w\|_2^2$. 

accuracy. Interestingly, this “moderate” accuracy has to be understood with respect to the reduced noise $σ^2$. Furthermore, we support our analysis with various experiments illustrating the behavior of our new algorithm and we provide the code to reproduce our experiments. See [this anonymized repository]. In Table [1] we highlight the main features and assumptions of Artemis compared to recent algorithms using compression.

The rest of the paper is organized as follows: in Section 2 we introduce the framework of Artemis. In Subsection 2.1 we describe the assumptions, and we review related work in Subsection 2.2. We then give the theoretical results in Section 3 we extend the result to device sampling in Section 4 we present experiments in Section 5 and finally, we conclude in Section 6.

2 PROBLEM STATEMENT

We consider the problem described in Equation (1). In the convex case, we assume that there exist at least one optimal point which we denote $w_*$, we also denote $h_i^1 = ∇F_i(w_*)$, for $i ∈ \{1, N\}$. We use $\| \cdot \|$ to denote the Euclidean norm.

A stochastic gradient $g_{k+1}^i$ is provided at iteration $k$ in $N$ to the device $i ∈ \{1, N\}$. This function is then evaluated at point $w_k$: to alleviate notation, we will use $g_{k+1}^i = g_{k+1}^i(w_k)$ and $g_{k+1}^i = g_{k+1}^i(w_*)$ to denote the stochastic gradient vectors at points $w_k$ and $w_*$ on device $i$. In the classical centralized framework (without compression), with partial participation of devices, SGD corresponds to:

$$w_{k+1} = w_k - γ \frac{1}{N} \sum_{i=1}^{N} g_{k+1}^i$$

where $γ$ is the learning rate. Here, we first consider the full participation case.

However, computing such a sequence would require the nodes to send either the gradient $g_{k+1}^i$ or the updated local model to the central server (uplink communication), and the central server to broadcast back either the averaged gradient $g_{k+1}$ or the updated global model (downlink communication). Here, in order to reduce communication cost, we perform a bidirectional compression. More precisely, we combine two main tools: 1) an unbiased compression operator $C : \mathbb{R}^d → \mathbb{R}^d$ that reduces the number of bits exchanged, and 2) a memory process that reduces the size of the signal to compress, and consequently the error [Mishchenko et al., 2012, Li et al. 2020]. That is, instead of directly compressing the gradient, we first approximate it by the memory term and, afterwards, we compress the difference. As a consequence, the compressed term tends in expectation to zero, and the error of compression is reduced. Following Tang et al. [2019], we always broadcast gradients and never models. To distinguish the two compression operations we denote $C_{up}$ and $C_{dwn}$, the compression operator for downlink and uplink. At each iteration, we thus have the following steps:

1. First, each active local node sends to the central server a compression of gradient differences: $Δ_{k} = C_{up}(g_{k+1} - h_{k})$, and updates the memory term $h_{k+1} = h_{k} + αΔ_{k}$ with $α ∈ \mathbb{R}^*$. The server recovers the approximated gradients’ values by adding the received term to the memories kept on its side.

2. Then, the central server sends back the compression of the sum of compressed gradients: $Ω_{k+1} = C_{dwn} \left( \frac{1}{N} \sum_{i=1}^{N} Δ_{i} + h_{k} \right)$. No memory mechanism needs to be used, as the sum of gradients tends to zero in the absence of regularization.

The update is thus given by:

$$\egin{align*}
\forall i ∈ \{1, N\}, & Δ_{k} = C_{up}(g_{k+1} - h_{k}) \\
Ω_{k+1} = C_{dwn} \left( \frac{1}{N} \sum_{i=1}^{N} Δ_{i} + h_{k} \right) \\
w_{k+1} = w_{k} - γΩ_{k+1}
\end{align*}$$

Constants $γ, α ∈ \mathbb{R}^* × \mathbb{R}_+$ are learning rates for respectively the iterate sequence and the memory sequence. The adaptation of the framework in the case of device sampling is developed in Section 3. This is illustrated on Algorithm 1 and fig. 8 in Appendix A.

As a summary, the Artemis framework encompasses in particular these four algorithms: the variant with unidirectional compression ($ω_{dwn} = 0$) w.o. or with memory ($α = 0$ or $α ≠ 0$) recovers QSGD defined by Alistarh et al. [2017] and DIANA proposed by Mishchenko et al. [2019]. The variant using bidirectional compression ($ω_{dwn} ≠ 0$) w.o memory ($α = 0$) is called Bi-QSGD. The last and most effective variant combines bidirectional compression with memory and is the one we refer to as Artemis if no precision is given. It corresponds to a simplified version of Dore without error-feedback, but this additional mechanism did not lead to any theoretical improvement [Remark 2 in Sec. 4.1., Liu et al. 2020].

Remark 1 (Local steps). An obvious independent direction to reduce communication is to increase the number of steps performed before communication. This is the spirit of Local-SGD [Stich 2019]. It is an interesting extension to incorporate this into our framework, that we do not consider in order to focus on the compression insights.

In the following section, we present and discuss assumptions over the function $F$, the data distribution and the compression operator.
Table 1: Comparison of frameworks for main algorithms handling (bidirectional) compression. By “non i.i.d.”, we mean that the theoretical framework encompasses and explicitly quantifies the impact of data heterogeneity on convergence (Assumption\[4\]), e.g., Dore does not assume i.i.d. workers but does not quantify differences between distributions. References: see Alistarh et al. \[2017\] for QSGD, Mishchenko et al. \[2019\] for Diana, Horváth & Richtárik \[2020\] for [HR20], Liu et al. \[2020\] for Dore, Philippenko & Dieuleveut \[2021\] for MCM and Tang et al. \[2019\] for DoubleSqueeze.

| QSGD        | Diana          | [HR20]       | Dore | Double | Dist- EF-SGD | MCM | Artemis (new) |
|-------------|----------------|-------------|------|--------|-------------|-----|---------------|
| Data        | i.i.d.         | non i.i.d.  | non i.i.d. | i.i.d. | i.i.d.       | non i.i.d. | non i.i.d.    |
| Bounded variance | Uniformly     | Uniformly   | Uniformly | Uniformly | Uniformly   | Uniformly | At optimal point |
| Compression | One-way        | One-way     | One-way   | Two-way | Two-way      | Two-way    | Two-way       |
| Error-feedback | ✓             | ✓           | ✓         | ✓      | ✓           | ✓           | ✓             |
| Memory      | ✓              | ✓           | ✓         | ✓      | ✓           | ✓           | ✓             |
| Device sampling | ✓             | ✓           | ✓         | ✓      | ✓           | ✓           | ✓             |

Note that we do not need each $F_i$ to be strongly convex, but only $F$. Also remark that we only use this inequality for $v = w_*$ in the proof of Theorems\[1\] and \[2\].

Below, we introduce cocoercivity [see Zhu & Marcotte \[1996\] for more details about this hypothesis]. This assumption implies that all $(F_i)_{i \in \mathbb{N}}$ are $L$-smooth.

**Assumption 2** (Cocoercivity of stochastic gradients (in quadratic mean)). We suppose that for all $k \in \mathbb{N}$, stochastic gradients functions $(g_k^i)_{i \in \mathbb{N}}$ are $L$-cocoercive in quadratic mean. That is, for $k \in \mathbb{N}$, $i \in \mathbb{N}$ and for all vectors $w_1, w_2 \in \mathbb{R}^d$, we have:

$$\mathbb{E}[\|g_k^i(w_1) - g_k^i(w_2)\|^2] \leq L (\|\nabla F_i(w_1) - \nabla F_i(w_2)\|_w^2).$$

E.g., this is true under the much stronger assumption that stochastic gradients functions $(g_k^i)_{i \in \mathbb{N}}$ are almost surely $L$-cocoercive, i.e.: $\mathbb{E}[\|g_k^i(w_1) - g_k^i(w_2)\|^2] \leq L (\|\nabla F_i(w_1) - \nabla F_i(w_2)\|_w^2)$.

Next, we present the assumption on the stochastic gradient’s noise. Again, we highlight that the noise is only controlled at the optimal point. To carefully control the noises process (gradient oracle, uplink and downlink compression), we introduce three filtrations $(H_k, \mathcal{G}_k, \mathcal{F}_k)_{k \geq 0}$, such that $w_k$ is $H_k$-measurable for all $k \in \mathbb{N}$. Detailed definitions are given in Appendix A.3.

**Assumption 3** (Noise over stochastic gradients computation). The noise over stochastic gradients at the global optimal point, for a mini-batch of size $b$, is bounded: there exists a constant $\sigma_* \in \mathbb{R}$, s.t. for all $k \in \mathbb{N}$, for all $i \in \mathbb{N}$, we have a.s.:

$$\mathbb{E}[\|g_{k,i} - \nabla F_i(w_i)\|^2 | H_k] \leq \sigma_*^2.$$

The constant $\sigma_*^2$ is null, e.g. if we use deterministic (batch) gradients, or in the interpolation regime for i.i.d. observations, as discussed in the Introduction. As we have also incorporated here a mini-batch parameter, this reduces the variance by a factor $b$.

Unlike Diana,\[Mishchenko et al. \[2019\], Li et al. \[2020\], Dore,\[Liu et al. \[2020\], Dist-EF-SGD, Zheng et al. \[2019\], MCM, Philippenko & Dieuleveut \[2021\] or Double-Squeeze,\[Tang et al. \[2019\], we assume that the variance of the noise is bounded only at optimal point $w_*$ and not at any point $w$ in $\mathbb{R}^d$. It results that if variance is null ($\sigma_*^2 = 0$) at optimal point, we obtain a linear convergence while previous results obtain this rate solely if the variance is null at any point (i.e. only for deterministic GD).

Also remark that Assumptions\[2\] and \[3\] both stand for the simplest Least-Square Regression (LSR) setting, while the uniform bound on the gradient’s variance does not. Next, we give the assumption that links the distributions on the different machines.

**Assumption 4** (Bounded gradient at $w_*$). There exists a constant $B \in \mathbb{R}_+$, s.t.:

$$\frac{1}{N} \sum_{i=0}^{N} \|\nabla F_i(w_*)\|^2 = B^2.$$

This assumption is used to quantify how different the distributions are on the different machines. In the streaming i.i.d. setting $-D_1 = \cdots = D_N$ and $F_1 = \cdots = F_N$ – the assumption is satisfied with $B = 0$. Combining Assumptions \[3\] and \[4\] results in an upper bound on the averaged squared norm of stochastic gradients at $w_*$: for all $k \in \mathbb{N}$, a.s.:

$$\frac{1}{N} \sum_{i=1}^{N} \mathbb{E}[\|g_{k,i} - \nabla F_i(w_i)\|^2 | H_k] \leq \frac{\sigma_*^2}{N} + B^2.$$

Finally, compression operators can be classified in two main categories: quantization [as in Alistarh et al. \[2017\], Seide et al. \[2014\], Zhou et al. \[2018\], Wen et al. \[2017\], Reisizadeh et al. \[2020\], Horváth et al. \[2019\], and sparsification [as in Stich et al. \[2018\], Aji & Heafield \[2017\], Alistarh et al. \[2018\], Khirarat et al. \[2020\]]. Theoretical guarantees provided in this paper do not rely on a particular kind of compression, as we only consider the following assumption on the compression operators $C_{up}$ and $C_{dwn}$:

**Assumption 5**. There exist constants $\omega_{up}^*, \omega_{dwn}^* \in \mathbb{R}_+$, such that the compression operators $C_{up}$ and $C_{dwn}$ verify the two following properties for all $\Delta$ in $\mathbb{R}^d$:

$$\begin{cases}
\mathbb{E}[C_{up}(\Delta)] = \Delta,
\mathbb{E}[\|C_{up}(\Delta) - \Delta\|^2] \leq \omega_{up}^* \|\Delta\|^2.
\end{cases}$$

In other words, the compression operators are unbiased and their variances are bounded. Note that Horváth & Richtárik...
2.2 RELATED WORK ON COMPRESSION

Quantization is a common method for compression and is used in various algorithms. For instance, Seide et al. [2014] are one of the first to propose to quantize each gradient component by either $-1$ or $1$. This approach has been extended in [Karimireddy et al., 2019]. Alistarh et al. [2017] define a new algorithm – QSGD – which instead of sending gradients, broadcasts their quantized version, getting robust results with this approach. On top of gradient compression, Wu et al. [2018] add an error compensation mechanism which accumulates quantization errors and corrects the gradient computation at each iteration. Diana [introduced in Mishchenko et al., 2019] introduces a “memory” term in the place of accumulating errors. Li et al. [2020] extend this algorithm and improve its convergence by using an accelerated gradient descent. Reisizadeh et al. [2020] combine unidirectional quantization with device sampling, leading to a framework closer to Federated Learning settings where devices can easily be switched off. In the same perspective, Horváth & Richtárik [2020] detail results that also consider PP. Tang et al. [2019] are the first to suggest a bidirectional compression scheme for a decentralized network. For both uplink and downlink, the method consists in sending a compression of gradients combined with an error compensation. Later, Yu et al. [2019] choose to compress models instead of compressing gradients. This approach is enhanced by Liu et al. [2020] who combine model compression with a memory mechanism and an error compensation drawing from [Mishchenko et al., 2019]. Both Tang et al. [2019] and Zheng et al. [2019] compress gradients without using a memory mechanism. However, as proved in the following section, memory is key to reducing the asymptotic variance in the heterogeneous case. A recent work written by Philippenko & Dieuleveut [2021] build upon our work and design an algorithm that is doing bidirectional compression but achieves rates of convergence identical to unidirectional compression. In their work, they take advantage of the uplink memory to handle the heterogeneous settings by reusing our demonstration’s paradigm.

We now provide theoretical results about the convergence of bidirectional compression.

3 THEORETICAL RESULTS

In this section, we present our main theoretical results on the convergence of Artemis and its variants. For the sake of clarity, the most complete and tightest versions of theorems are given in Appendices, and simplified versions are provided here. The main linear convergence rates are given in Theorem 1. In Theorem 2 we show that Artemis combined with Polyak-Ruppert averaging reaches a sub-linear convergence rate. In this section, we denote $\delta_0^2 = \|w_0 - w_*\|^2$.

**Theorem 1 (Convergence of Artemis).** Under Assumptions 7 to 5 for a step size $\gamma$ satisfying the conditions in Table 2, for a learning rate $\alpha$ and for any $k \in \mathbb{N}$, the mean squared distance to $w_*$ decreases at a linear rate up to a constant of the order of $E$: 

$$E\left[\|w_k - w_*\|^2\right] \leq (1 - \gamma \mu)^k \left(\delta_0^2 + 2\gamma^2 B^2\right) + \frac{2\gamma E}{\mu N},$$

for constants $C$ and $E$ depending on the variant (independent of $k$) given in Table 2 or in the appendix. Variants with $\alpha \neq 0$ require $\alpha \in [1/(2(\omega_{up}^2 + 1)), \alpha_{max}]$, the upper bound $\alpha_{max}$ is given in Theorem 6.

This theorem is derived from Theorems 5 and 6, which are respectively proved in Appendices E.1 and E.2. We can make the following remarks:

1. **Linear convergence.** The convergence rate given in Theorem 1 can be decomposed into two terms: a bias term, forgotten at linear speed $(1 - \gamma \mu)^k$, and a variance residual term which corresponds to the saturation level of the algorithm. The rate of convergence $(1 - \gamma \mu)$ does not depend on the variant of the algorithm. However, the variance and initial bias do vary.

2. **Bias term.** The initial bias always depends on $\|w_0 - w_*\|^2$, and when using memory (i.e. $\alpha \neq 0$) it also depends on the difference between distributions (constant $B^2$).

3. **Variance term and memory.** On the other hand, the variance depends a) on both $\sigma^2_g/b$, and the distributions’ difference $B^2$ without memory b) only on the gradients’ variance at the optimum $\sigma^2_g/b$ with memory. Similar theorems in related literature [Liu et al., 2020; Alistarh et al., 2017; Mishchenko et al., 2019; Yu et al., 2019; Tang et al., 2019; Zheng et al., 2019] systematically had a worse bound for the variance term depending on a uniform bound of the noise variance or under much stronger conditions on the compression operator. This paper and [Liu et al., 2020] are also the first to give a linear convergence up to a threshold for bidirectional compression.

4. **Impact of memory.** To the best of our knowledge, this is the first work on double compression that explicitly
We now provide a convergence guarantee for the averaged

Theorem 2

(Convergence of \( \omega \) (resp. \( \omega^{up/down} \)), in three main asymptotic regimes:

Table 3: Upper bound on \( \gamma \) learning rate

| Memory            | \( \alpha = 0 \) | \( \alpha \neq 0 \) |
|-------------------|------------------|------------------|
| \( N \gg \omega^{up} \) | \( \frac{1}{(\omega^{down} + 1)L} \) | \( \frac{1}{2(\omega^{down} + 1)L} \) |
| \( N \approx \omega^{up} \) | \( \frac{1}{3(\omega^{down} + 1)L} \) | \( \frac{1}{5(\omega^{down} + 1)L} \) |
| \( \omega^{up} \gg N \) | \( \frac{N}{2\omega^{up}(\omega^{down} + 1)L} \) | \( \frac{N}{4\omega^{up}(\omega^{down} + 1)L} \) |

tackles the non i.i.d. case [Philippenko & Dieuleveut (2021)] also handle this setting but have mentioned that they get inspired from our work. We prove that memory makes the saturation threshold independent of \( B^2 \) for Artemis.

Variance term. The variance term increases with a factor proportional to \( \omega^{up} \) for the unidirectional compression, and proportional to \( \omega^{up} \times \omega^{down} \) for bidirectional. This is the counterpart of compression, each compression resulting in a multiplicative factor on the noise. A similar increase in the variance appears in [Mishchenko et al., 2019] and [Liu et al., 2020]. The noise level is attenuated by the number of devices \( N \), which is inversely proportional.

Link with classical SGD. For variant of Artemis with \( \alpha = 0 \), if \( \omega^{up/down} = 0 \) (i.e. no compression) we recover SGD results: convergence does not depend on \( B^2 \), but only on the noise’s variance.

Conclusion: Overall, it appears that Artemis is able to efficiently accelerate the learning during first iterations, enjoying the same linear rate as SGD with lower communication complexity, but it saturates at a higher level, proportional to \( \sigma^2 \) and independent of \( B^2 \).

The range of acceptable learning rates is an important feature of first order algorithms. In Table 3, we summarize the upper bound on \( \gamma \) for \( N \) and \( \gamma \) to guarantee a \((1 - \gamma \mu)\) convergence of Artemis. These bounds are derived from Theorems SS and S6 in three main asymptotic regimes: \( N \gg \omega^{up} \), \( N \approx \omega^{up} \) and \( \omega^{up} \gg N \). Using bidirectional compression impacts \( \gamma \) by a factor \( \omega^{down} + 1 \) in comparison to unidirectional compression. For unidirectional compression, if the number of machines is at least of the order of \( \omega^{up} \), then \( \gamma \) nearly corresponds to \( \gamma_{\text{max}} \) for vanilla (serial) SGD.

We now provide a convergence guarantee for the averaged iterate without strong convexity.

Theorem 2 (Convergence of Artemis with Polyak-Ruppert averaging). Under Assumptions 2 to 5 (convex case) with constants \( C \) and \( E \) as in Theorem 1, after running \( K \) in \( N \) iterations, for a learning rate \( \gamma = \min \left( \sqrt{\frac{\omega^{up}}{\omega^{down}}} \gamma_{\text{max}} \right) \), with \( \gamma_{\text{max}} \) as in Table 3 we have a sublinear convergence rate for the Polyak-Ruppert averaged iterate \( \bar{w}_K = \frac{1}{K} \sum_{k=0}^{K} w_k \), with \( \varepsilon_F(\bar{w}_K) = F(\bar{w}_K) - F(w) \):

\[
\varepsilon_F(w_K) \leq 2 \max \left( \sqrt{\frac{2\delta^2 E}{NK}} \gamma_{\text{max}} K \right) + \frac{2\gamma_{\text{max}} CB^2}{K}.
\]

This theorem is proved in Appendix E3. Several comments can be made on this theorem:

1. Importance of averaging This is the first theorem given for averaging for double compression. In the context of convex optimization, averaging has been shown to be optimal [Rakhlin et al., 2012].

2. Speed of convergence, if \( \sigma = 0 \), \( \sigma \neq 0 \), \( K \to \infty \). For \( \alpha \neq 0 \), \( E = 0 \), while for \( \alpha = 0 \), \( E \propto B^2 \). Memory thus accelerates the convergence from a rate \( O(K^{-1/2}) \) to \( O(K^{-1}) \).

3. Speed of convergence, general case. More generally, we always get a \( K^{-1/2} \) sublinear speed of convergence, and a faster rate \( K^{-1} \), when using memory, and if \( E \leq \delta^2_0 N/(2K\gamma_{max}^2) \) – i.e. in the context of a low noise \( \sigma^2 \), as \( E \propto \sigma^2 \). Again, it appears that bi-compression is mostly useful in low-\( \sigma^2 \) regimes or during the first iterations: intuitively, for a fixed communication budget, while bi-compression allows to perform \( \min(\omega^{up}, \omega^{down}) \)-times more iterations, this is no longer beneficial if the convergence rate is dominated by \( \sqrt{\delta^2_0 E/NK} \), as \( E \) increases by a factor \( \omega^{up} \times \omega^{down} \).

4. Memoryless case, impact of minibatch. In the variant of Artemis without memory, the asymptotic convergence rate is \( \sqrt{\delta^2_0 E/NK} \) with the constant \( E \propto \sigma^2 \). Interestingly, it appears that in the case of non i.i.d. data (\( B^2 > 0 \)), the convergence rate saturates when the size of the mini-batch increases: large mini-batches do not help. On the contrary, with memory, the variance is, as classically, reduced by a factor proportional to the size of the batch, without saturation.

3.1 CONVERGENCE IN DISTRIBUTION AND LOWER BOUND

The increase in the variance (in item 3) is not an artifact of the proof: we prove the existence of a limit distribution for the iterates of Artemis, and analyze its variance. More precisely, we show a linear rate of convergence for the distribution \( \Theta_k \) of \( w_k \) (launched from \( w_0 \)), w.r.t. the Wasserstein distance \( \mathcal{W}_2 \) [Villani, 2009]: this gives us a lower bound on the asymptotic variance. Here, we further assume that the compression operator is Stochastic sparsification [Wen et al., 2017].

Theorem 3 (Convergence in distribution and lower bound on the variance). Under Assumptions 7 to 10 (full participation setting), for \( \gamma, \alpha, E \) given in Theorem 1 and Table 3.
1. There exists a limit distribution $\pi_{\gamma,v}$ depending on the variant $v$ of the algorithm, s.t. for any $k \geq 1$, $\mathcal{W}_k(\Theta_k, \pi_{\gamma,v}) \leq (1 - \gamma \mu)^k C_0$, with $C_0$ a constant.

2. When $k$ goes to $\infty$, the second order moment $\mathbb{E}[\|w_k - w_\star\|^2]$ converges to $\mathbb{E}_{w \sim \pi_{\gamma,v}}[\|w - w_\star\|^2]$, which is lower bounded by $\Omega(\gamma E/\mu N)$ as in Theorem 1 as $\gamma \to 0$, with $E$ depending on the variant.

**Interpretation.** The second point (2.) means that the upper bound on the saturation level provided in Theorem 1 is tight w.r.t. $\sigma^2, \omega_{\text{up}}^{\text{up}}, \omega_{\text{down}}^{\text{down}}$, $\mathcal{B}^2$, $N$ and $\gamma$. Especially, it proves that there is indeed a quadratic increase in the variance w.r.t. $\omega_{\text{up}}^{\text{up}}$ and $\omega_{\text{down}}^{\text{down}}$ when using bidirectional compression (which is itself rather intuitive). Altogether, these three theorems prove that bidirectional compression can become strictly worse than usual stochastic gradient descent in high precision regimes, a fact of major importance in practice and barely (if ever) even mentioned in previous literature. To the best of our knowledge, only Mayekar & Tyagi [2020] are giving a lower bound on the asymptotic variance for algorithms using compression. There result is more general i.e., valid for any algorithm using unidirectional compression, but weaker (worst case on the oracle does not highlight the importance of noise at the optimal point and is incompatible with linear rates).

**Proof and assumptions.** This theorem also naturally requires, for the second point, Assumptions 3 to 5 to be “tight”: that is, e.g. $\operatorname{Var}(g_{k+1,i}) \geq \Omega(\sigma^2/b)$; more details and the proof are given in Appendix E. Extension to other types of compression reveals to be surprisingly non-simple, and is thus out of the scope of this paper and a promising direction.

### 4 PARTIAL PARTICIPATION

In this section we extend our work to the Partial Participation (PP) setting, considering Assumption 6.

**Assumption 6.** At each round $k$ in $\mathbb{N}$, each device has a probability $p$ of participating, independently from other workers, i.e., there exists a sequence $(B^k_i)_k$ of i.i.d. Bernoulli random variables $B(p)$, such that for any $k$ and $i$, $B^k_i$ marks if device $i$ is active at step $k$. We denote $S_k = \{i \in [1,N] \mid B^k_i = 1\}$ the set of active devices at round $k$ and $N_k = \text{card}(S_k)$ the number of active workers.

There are two approaches to extend the update rule. The first one (we will refer to it as PP1) is the most intuitive. It consists in keeping all memories $(h^k_i)_{1 \leq i \leq N}$ on the central server. This way, the central server can reconstruct at each iteration $k$ in $\mathbb{N}$ and for each device $i$ in $\{1, \cdots, N\}$ the stochastic gradient $\hat{g}^k_i$ defined as $\Delta^k_i + h^k_i$. The update equation is $w_{k+1} = w_k - \gamma C_{\text{down}} \left( \frac{1}{pN} \sum_{i \in S_k} \Delta^k_i + h^k_i \right)$, and memories are updated as usually. If both compression levels are set to 0, this approach recovers classical SGD with PP: $w_{k+1} = w_k - \frac{\gamma}{pN} \sum_{i \in S_k} g^i_w(w_k)$. It also corresponds to the proposition of Sattler et al. [2019] and Tang et al. [2019].

However, it has two important drawbacks. First, the central server has to store $N$ additional memories, which may have a huge cost. Secondly, this method saturates, even with deterministic gradients $\sigma^2_{\text{init}} = 0$ and no compression $\omega_{\text{up/down}} = 0$ (see Figure 3). Indeed, PP induces an additive noise term, i.e. the variance of the noise at the optimum point is not null because we have: $\operatorname{Var}(w_{k+1}) = \frac{1}{Np} \sum_{i \in S_k} \|\nabla F_i(w_\star)\|^2 = \frac{(1-p)\mathcal{B}^2}{Np}$. This happens for all compression regimes in Artemis, even for SGD.

We consider a novel approach (denoted PP2) that leverages the full potential of the memory, solving simultaneously the convergence issue and the need for additional memory resources. At each iteration $k$, this approach keeps a single memory $h_k$ (instead of $N$ memories) on the central server. This memory is updated at each step: $h_{k+1} = h_k + \frac{\gamma}{N} \sum_{i \in S_k} \Delta^k_i$, and the update equation becomes: $w_{k+1} = w_k - \gamma C_{\text{down}} \left( h_k + \frac{1}{pN} \sum_{i \in S_k} \Delta^k_i \right)$. This difference is far from being insignificant. Indeed in order to reconstruct the broadcast signal, we use the memory built on all devices during previous iterations, even if the device $i$ in $\{1, \cdots, N\}$ was not active! **The impact of this approach is major** as it follows that algorithms, even using bidirectional compression, can be faster than classical SGD. In this setting, SGD with memory (i.e. Artemis-PP2 with $\omega_{\text{up/down}} = 0$) will be the benchmark: see Figure 4.

**Theorem 4 (Artemis with partial participation).** Under the same assumptions and constraints on $\gamma$ and $\alpha$, when considering Assumption 5 (partial participation), Theorem 1 is still valid for PP2 with memory. We have: $E = (\omega_{\text{down}} + 1) \left( \frac{2(\omega_{\text{up}}^{\text{up}} + 1)}{p} - 1 \right) \frac{\sigma^2}{\mathcal{B}^2} + 2pC \left( 2\alpha^2(\omega_{\text{up}}^{\text{up}} + 1) - \alpha \right) \frac{\sigma^2_{\text{init}}}{\mathcal{B}^2}$.

The most important observation is that we recover again a linear convergence rate if $\sigma_{\star} = 0$. By contrast, even with memory and without compression, for PP1, there is an extra $(\mathcal{B}^2(1-p)/Np)$ term.

**Remark 2** (Impact of downlink compression). With both of these approaches, we still need to synchronize the model in order to compute the stochastic gradient on the same up-to-date model for each worker. Thus, a newly active worker must catch-up the sequence of missed updates $(\Omega_k)_k$. Of course, if a device has been inactive for too many iterations, we send the full model instead of the sequence of missed updates. Thus, this need for synchronization does not lead to additional computational resources. The mechanism is described in Algorithm 7 and does not interfere with the theoretical analysis.

### 5 EXPERIMENTS

In this section, we illustrate our theoretical guarantees on both synthetic and real datasets. The goal of this section is to confirm the theoretical findings in Theorems 1 and 2 on arxiv for the distributed case.
to \(4\) and to underline the impact of the memory. Therefore, we focus on five of the algorithms covered by our framework: Artemis with bidirectional compression (simply denoted Artemis), QSGD, Diana, Bi-QSGD, and usual SGD without any compression. In the Appendix (see Figure [S24]), we compare Artemis with other existing benchmarks: Double-Squeeze, Dore, FedSGD and FedPAAQ [see Reisizadeh et al. 2020]. We also perform experiments with optimized learning rates (see Figure S20).

In all experiments, we display the logarithm excess error \(\log_{10}(F(w_k) - F(w^*))\) w.r.t. the number of iterations \(k\) or the number of communicated bits. We use a quantization scheme (defined in Appendix A.2) with \(s = 2^0\) in full participation settings, and with \(s = 2^1\) in PP settings. Curves are averaged over 5 runs, and we plot error bars on all figures. These error bars correspond to \(\pm\) the standard deviation of the logarithm excess loss over the five runs.

We first consider two simple synthetic datasets: one for least-squares regression (with the same distribution over each machine), and one for logistic regression (with varying distributions across machines). More details are given in Appendix C on the way data is generated. We use \(N = 20\) devices, each holding 200 points of dimension \(d = 20\), and run algorithms over 100 epochs.

To illustrate theorems on real data and higher dimension, we then consider two real-world dataset: superconduct [see Hamidieh 2018 with 21 263 points and 81 features] and quantum [see Caruana et al. 2004 with 50 000 points and 65 features] with \(N = 20\) workers. To simulate non-i.i.d. and unbalanced workers, we split the dataset in heterogeneous groups, using a Gaussian mixture clustering on the TSNE representations (defined by Maaten & Hinton 2008). Thus, the distribution and number of points held by each worker largely differs between devices, see Figure S11.

**Convergence.** Figure 1a presents the convergence of each algorithm w.r.t. the number of iterations \(k\). During first iterations all algorithms make fast progress. However, because \(\sigma^2 \neq 0\), all algorithms saturate; and the saturation level is higher for double compression (Artemis, Bi-QSGD), than for simple compression (Diana, QSGD), or than for SGD. This corroborates findings in Theorem 1 and Theorem 3.

**Complexity.** On Figures 2 to 4, the loss is plotted w.r.t. the theoretical number of bits exchanged after \(k\) iterations for the quantum and superconduct dataset. This confirms that double compression should be the method of choice to achieve a reasonable precision (w.r.t. \(\sigma^2\)), whereas for high precision, a simple method like SGD results in a lower complexity.

**Linear convergence under null variance at the optimum.** To highlight the significance of our new condition on the noise, we compare \(\sigma^2 \neq 0\) and \(\sigma^2 = 0\) on Figure 1 but if we consider a situation in which \(\sigma^2 = 0\), and where the uniform bound on the gradient’s variance is not null (as opposed to experiments in Liu et al. 2020 who consider batch gradient descent), a linear convergence rate is observed. This illustrates that our new condition is sufficient to reach a linear convergence. Comparing Figure 1a with Figure S8a sheds light on the fact that the saturation level (before which double compression is indeed beneficial) is truly proportional to the noise variance at optimal point i.e. \(\sigma^2\). And when \(\sigma^2 = 0\), bidirectional compression is much more effective than the other methods (see Figure S8 in Appendix C.1.1).

**Heterogeneity and real datasets.** While in Figure 1a data is i.i.d. on machines, and Artemis is thus not expected to outperform Bi-QSGD (the difference between the two being the memory), in Figures 1b and 2 to 4 we use non-i.i.d. data. None of the previous papers on compression...
directly illustrated the impact of heterogeneity on simple examples, neither compared it with i.i.d. situations.

**Partial participation.** On Figures 3 and 4 we run experiments with only half of the devices active (randomly sampled) at each iteration with the two approaches (PP1 and PP2) described in Section 4 in a full gradient regime (same experiments in stochastic regime are given in Appendix C.2.1). This two figures emphasize the failure of the first approach PP1 compared to PP2. We observe that in this last case only, Artemis has a linear convergence. It also stresses the key role of the memory in this setting. On Figure 4 all algorithms with a (single) memory (with or without up/down compression) are better than SGD without memory. Note that Diana is defined with multiple memories, thus it cannot be compared to Artemis with PP2.

6 CONCLUSION
We propose Artemis, a framework using bidirectional compression to reduce the number of bits needed to perform distributed or federated learning. On top of compression, Artemis includes a memory mechanism which improves convergence over non-i.i.d. data. As PP is a classical setting, we designed an approach (PP2) to tackle it while leveraging the full impact of memory, outperforming existing solutions. We provide three tight theorems giving guarantees of a fast convergence (linear up to a threshold), highlighting the impact of memory, analyzing Polyak-Ruppert averaging and obtaining lower bounds by studying convergence in distribution of our algorithm. Altogether, this improves the understanding of compression combined with a memory mechanism and sheds light on challenges ahead.

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Supplementary material

In this appendix, we provide additional details to our work. In Appendix A, we give more details on Artemis, we describe the $s$-quantization scheme used in our experiments and we define the filtrations used in following demonstrations. Secondly, in Appendix B, we analyze at a finer level the bandwidth speeds across the world to get a better intuition of the state of the worldwide internet usage. Thirdly, in Appendix C, we present the detailed framework of our experiments and give further illustrations to our theorems. In Appendix D, we gather a few technical results and introduce the lemmas required in the proofs of the main results. Those proofs are finally given in Appendix E. More precisely, Theorem 1 follows from Theorems S5 and S6 which are proved in Appendices E.1 and E.2 while Theorems 2 and 3 are respectively proved in Appendices E.3 and E.4.

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A ADDITIONAL DETAILS ABOUT THE ARTEMIS FRAMEWORK

The aim of this section is threefold. First, we give the pseudo-code of Artemis. Secondly, we provide supplementary details about the quantization scheme used in our work. We also explain (based on Alistarh et al. [2017]) how quantization combined with Elias code [see Elias [1975]] reduces the required number of bits to send information. Thirdly, we define the filtrations used in our proofs and give their resulting properties.

A.1 ARTEMIS PSEUDO-CODE

We provide the pseudo-code of Artemis in Algorithm 1 and for the understanding of Artemis implementation, we give a visual illustration of the algorithm in Figure S1.

Remark 3. Remark that we have used in Algorithm 1 and for the understanding of Artemis implementation, we give a visual illustration of the algorithm in Figure S1.
Algorithm 1: Artemis - set $\alpha > 0$ to use memory.

**Input:** Mini-batch size $b$, learning rates $\alpha, \gamma > 0$, initial model $w_0 \in \mathbb{R}^d$, operators $C_{up}$ and $C_{down}, M_1$ and $M_2$ the sizes of the full/compressed gradients.

**Initialization:** Local memory: $\forall i \in [1, N]$ $h_i^0 = 0$ (kept on both central server and device $i$). Index of last participation: $k_i = 0$.

**Output:** Model $w_K$

for $k = 0, 1, 2, \ldots, K$ do

Randomly sample a set of device $S_k$

for each device $i \in S_k$ do

- **Catching up.**
  - If $k - k_i > \lceil M_1/M_2 \rceil$, send the model $w_k$
  - Else send $(\hat{\Omega}_j)_j = k_{i+1}$ and update local model: $\forall j \in [k_i + 1, k]$, $w_j = w_{j-1} - \gamma \Omega_j, s_j - 1$
  - Update index of its last participation: $k_i = k$

- **Local training.**
  - Compute stochastic gradient $g_{k+1} = g_{k+1}(w_k)$ (with mini-batch)
  - Set $\Delta_k = g_{k+1} - h_k^i$, compress it $\hat{\Delta_k} = C_{up}(\Delta_k)$
  - Update memory term: $h_{k+1}^i = h_k^i + \alpha \hat{\Delta_k}^i$
  - Send $\hat{\Delta_k}$ to central server

- Compute $\hat{g}_{k+1, s_k} = h_k + \frac{1}{pN} \sum_{i \in S_k} \hat{\Delta_k}$

- Update central memory: $h_{k+1} = h_k + \alpha \frac{1}{N} \sum_{i \in S_k} \hat{\Delta_k}$

- Back compression: $\Omega_{k+1, s_k} = C_{down}(\hat{g}_{k+1, s_k})$

- Broadcast $\Omega_{k+1}$ to all workers.

- Update model on central server: $w_{k+1} = w_k - \gamma \Omega_{k+1, s_k}$

Figure S1: Visual illustration of Artemis bidirectional compression at iteration $k \in \mathbb{N}$. $w_k$ is the model’s parameter, $C_{up}$ and $C_{down}$ are the compression operators, $\gamma$ is the step size, $\alpha$ is the learning rate which simulates a memory mechanism of past iterates and allows the compressed values to tend to zero.

A.2 QUANTIZATION SCHEME

In the following, we define the $s$-quantization operator $C_s$ which we use in our experiments. After giving its definition, we explain [based on Alistarh et al., 2017] how it helps to reduce the number of bits to broadcast.

**Definition 1** ($s$-quantization operator). Given $\Delta \in \mathbb{R}^d$, the $s$-quantization operator $C_s$ is defined by:

$$C_s(\Delta) := \text{sign}(\Delta) \times \|\Delta\|_2 \times \frac{\psi}{s}.$$  

$\psi \in \mathbb{R}^d$ is a random vector with $j$-th element defined as:

$$\psi_j := \begin{cases} 
  l + 1 & \text{with probability } s \frac{\|\Delta_j\|_2}{\|\Delta\|_2} - l \\
  l & \text{otherwise} 
\end{cases}.$$
where the level \( l \) is such that \( \frac{A_l}{\|A_l\|_2} \in \left[ \frac{l}{s}, \frac{l+1}{s} \right] \).

The \( s \)-quantization scheme verifies Assumption 5 with \( \omega_C = \min(d/s^2, \sqrt{d}/s) \). Proof can be found in [Alistarh et al., 2017].

Now, for any vector \( v \in \mathbb{R}^d \), we are in possession of the tuple \( (\|v\|^2, \phi, \psi) \), where \( \phi \) is the vector of signs of \( (v_i)_{i=1}^d \), and \( \psi \) is the vector of integer values \( (\psi_j)_{j=1}^d \). To broadcast the quantized value, we use the Elias encoding [Elias, 1975]. Using this encoding scheme, it can be shown (Theorem 3.2 of [Alistarh et al., 2017]) that:

**Proposition S1.** For any vector \( v \), the number of bits needed to communicate \( C_s(v) \) is upper bounded by:

\[
\left( 3 + \frac{3}{2} + o(1) \right) \log \left( \frac{2(s^2 + d)}{s(s + \sqrt{d})} \right) s(s + \sqrt{d}) + 32.
\]

The final goal of using memory for compression is to quantize vectors with \( s = 1 \). It means that we will employ \( O(\sqrt{d} \log d) \) bits per iteration instead of \( 32d \), which reduces by a factor \( \frac{\sqrt{d}}{\log d} \) the number of bits used by iteration. Now, in a FL setting, at each iteration we have a double communication (device to the main server, main server to the device) for each of the \( N \) devices. It means that at each iteration, we need to communicate \( 2 \times N \times 32d \) bits if compression is not used. With a single compression process like in [Mishchenko et al., 2019], [Li et al., 2020], [Wu et al., 2018], [Agarwal et al., 2018], [Alistarh et al., 2017], we need to broadcast

\[
O \left( 32N d + N \sqrt{d} \log d \right) = O \left( N d \left( 1 + \frac{\log d}{\sqrt{d}} \right) \right) = O(N d).
\]

But with a bidirectional compression, we only need to broadcast \( O \left( 2N \sqrt{d} \log d \right) \).

**Time complexity analysis of simple vs double compression for the 1-quantization schema.** Using quantization with \( s = 1 \), and then the Elias code [defined in Elias, 1975] to communicate between servers, leads to reduce from \( O(N d) \) to \( O( \sqrt{d} \log(d) ) \) the number of bits to send, for each direction. Getting an estimation of the total time complexity is difficult and inevitably dependant of the considered application. Indeed, as highlighted by Figure S3, download and upload speed are always different. The biggest measured difference between upload and download is found in Europa for mobile broadband; their ratio is around 3.5.

Denoting \( v_d \) and \( v_u \) the speed of download and upload (in bits per second), we typically have \( v_d = \rho v_u \), \( 3.5 > \rho > 1 \).

Then for unidirectional compression, each iteration takes \( O \left( \frac{N d}{v_d} + \frac{N \sqrt{d} \log(d)}{v_u} \right) \approx O \left( \frac{N d}{\rho v_u} \right) \) seconds, while for a bidirectional one it takes only \( O \left( \frac{N \sqrt{d} \log(d)}{v_d} + \frac{N \sqrt{d} \log(d)}{v_u} \right) \approx O \left( \frac{N \sqrt{d} \log(d)}{v_u} \right) \) seconds.

In other words, unless \( \rho \) is really large (which is not the case in practice as stressed by Figure S3), double compression reduces by several orders of magnitude the global time complexity, and bidirectional compression is by far superior to unidirectional.

### A.3 FILTRATIONS

In this section we provide some explanations about filtrations - especially a rigorous definition - and how it is used in the proofs of Theorems 2, 3, S5 and S6. We recall that we denoted by \( \omega_C^{\text{up}} \) and \( \omega_C^{\text{down}} \) the variance factors for respectively uplink and downlink compression.

Let a probability space \( (\Omega, \mathcal{A}, \mathbb{P}) \) with \( \Omega \) a sample space, \( \mathcal{A} \) an event space, and \( \mathbb{P} \) a probability function. We recall that the \( \sigma \)-algebra generated by a random variable \( X : \Omega \to \mathbb{R}^m \) is

\[
\sigma(X) = \{ X^{-1}(A) : A \in \mathcal{B}(\mathbb{R}^m) \},
\]

where \( \mathcal{B}(\mathbb{R}^m) \) is the Borel set of \( \mathbb{R}^m \).
We can make the following observations for all $i$.

1. Stochastic gradients. It corresponds to the noise associated with the stochastic gradients computation on device $i$ at epoch $k$. We have:
   $$
   \forall k \in \mathbb{N}, \forall i \in [0, \ldots, N], \quad g^i_{k+1} = \nabla F_i(w_k) + \xi^i_{k+1}(w_k), \text{ with } \mathbb{V}(\xi^i_{k+1}) \text{ bounded}.
   $$

2. Uplink compression: this noise corresponds to the uplink compression when local gradients are compressed. Let $k \in \mathbb{N}$ and $i \in [0, \ldots, N]$, suppose we want to compress $\Delta^i_k \in \mathbb{R}^d$, then the associated noise is $\epsilon^i_{k+1}$ with $\mathbb{V}(\epsilon^i_{k+1}(\Delta^i_k)) \leq \omega^\text{up} \|\Delta^i_k\|^2$, where $\omega^\text{up} \in \mathbb{R}^+$ is defined by the uplink compression schema (see Assumption 5). And it follows that:
   $$
   \forall k \in \mathbb{N}, \forall i \in [0, \ldots, N], \quad \tilde{\Delta}^i_k = \Delta^i_k + \epsilon^i_{k+1}(\Delta^i_k) \iff \tilde{g}^i_{k+1} = g^i_{k+1} + \epsilon^i_{k+1}(\tilde{g}^i_{k+1}, S_k).
   $$

3. Downlink compression. This noise corresponds to the downlink compression, when the global model parameter is compressed. Let $k \in \mathbb{N}$, suppose we want to compress $\tilde{g}^i_{k+1}, S_k \in \mathbb{R}^d$, then the associated noise is $\epsilon_{k+1}(\tilde{g}^i_{k+1}, S_k)$ with $\mathbb{V}(\epsilon_{k+1}) \leq \omega^\text{down} \|\tilde{g}^i_{k+1}, S_k\|^2$. There is:
   $$
   \forall k \in \mathbb{N}, \quad \Omega_{k+1,S_k} = C_{\epsilon}(\tilde{g}^i_{k+1}, S_k) = \tilde{g}^i_{k+1}, S_k + \epsilon_{k+1}(\tilde{g}^i_{k+1}, S_k).
   $$

4. Random sampling. This randomness corresponds to the partial participation of each device. We recall that according to Assumption 6, each device has a probability $p$ of being active. For $k \in \mathbb{N}$, for $i \in [1, N]$, we note $B^i_k \sim \mathcal{B}(p)$ the Bernoulli random variable that marks if a device is active or not at step $k$.

This “succession of noises” in the algorithm is illustrated in Figure [S2]. In order to handle these four sources of randomness, we define five sequences of nested $\sigma$-algebras.

**Definition 2.** We note $\mathcal{F}_k$ the filtration associated to the stochastic gradient computation noise, $\mathcal{G}_k$ the filtration associated to the uplink compression noise and $\mathcal{H}_k$ the filtration associated to the downlink compression noise, $\mathcal{B}_k$ the filtration associated to the random device participation randomness. For $k \in \mathbb{N}^*$, we define:

\[
\begin{align*}
\mathcal{F}_k &= \sigma(\Gamma_{k-1}, (\xi^i_{k})_{i=1}^N) \\
\mathcal{G}_k &= \sigma(\Gamma_{k-1}, (\epsilon^i_{k})_{i=1}^N) \\
\mathcal{H}_k &= \sigma(\Gamma_{k-1}, (\xi^i_{k})_{i=1}^N, (\epsilon^i_{k})_{i=1}^N) \\
\mathcal{B}_{k-1} &= \sigma((B^i_{k-1})_{i=1}^N) \\
\mathcal{I}_k &= \sigma(\mathcal{H}_k \cup \mathcal{B}_{k-1})
\end{align*}
\]

with

\[
\Gamma_k = \{(\xi^i_{t})_{t \in [0, N]}, (\epsilon^i_{t})_{t \in [0, N], \in [1, N], \epsilon_t, (B^i_{k-1})_{i=1}^N}_{t \in [1, k]} \text{ and } \Gamma_0 = \{\emptyset\}.
\]

We can make the following observations for all $k \geq 1$:

- From these three definitions, it follows that our sequences are nested.
   $$
   \mathcal{F}_1 \subset \mathcal{G}_1 \subset \mathcal{H}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{H}_K.
   $$

However, $\mathcal{B}_k$ is independent of the other filtrations.

\[
\begin{align*}
w_k \xrightarrow{\xi^i_{k+1}} g^i_{k+1} \xrightarrow{\epsilon^i_{k+1}} B^i_k \rightarrow \tilde{g}^i_{k+1,S_k} = \sum_{i \in S_k} g^i_{k+1} \xrightarrow{\epsilon_{k+1}} \Omega_{k+1,S_k} = C(\tilde{g}^i_{k+1,S_k})
\end{align*}
\]

Figure S2: The sequence of successive additive noises in the algorithm.
• \( I_k = \sigma (H_k \cup B_{k-1}) = \sigma (\Gamma_k) \), and the aim is to express the expectation w.r.t. all randomness i.e \( I_k \).
• \( w_k \) is \( I_k \)-measurable.
• \( g_{k+1}(w_k) \) is \( F_{k+1} \)-measurable.
• \( \hat{g}_{k+1}(w_k) \) is \( G_{k+1} \)-measurable.
• \( B_{k-1}^i \) is \( B_{k-1} \)-measurable.
• \( g_{k+1,S_k} \) and \( \Omega_{k+1,S_k} \) are respectively \( \sigma (F_{k+1} \cup B_k) \)-measurable, \( \sigma (G_{k+1} \cup B_k) \)-measurable and \( \sigma (H_{k+1} \cup B_k) \)-measurable. Note that \( F_{k+1} \) contains \( \Gamma_k \), and thus all \( (B_{k-1}^i)_{i=1}^N \), but does not contain all the \( (B_k^i)_{i=1}^N \).

As a consequence, we have Propositions S2 to S8. Please, take notice that for sake of clarity Propositions S2 to S6 are stated without taking into account the random participation \( S_k \). All this proposition remains identical when adding partial participation, the results only have to be expressed w.r.t. active nodes \( S_k \).

Below Proposition S2 gives the expectation over stochastic gradients conditionally to \( \sigma \)-algebras \( H_k \) and \( F_{k+1} \).

**Proposition S2 (Stochastic Expectation).** Let \( k \in \mathbb{N} \) and \( i \in [1, N] \). Then on each local device \( i \in [1, N] \) we have almost surely (a.s.):

\[
\begin{align*}
\mathbb{E} [ g^i_{k+1} | F_{k+1} ] &= g^i_{k+1} \\
\mathbb{E} [ g^i_{k+1} | H_k ] &= \nabla F^i_i(w_k),
\end{align*}
\]

which leads to:

\[
\begin{align*}
\mathbb{E} [ g^i_{k+1} | F_{k+1} ] &= g^i_{k+1} \\
\mathbb{E} [ g^i_{k+1} | H_k ] &= \nabla F(w_k).
\end{align*}
\]

Proposition S3 gives expectation of uplink compression (information sent from remote devices to central server) conditionally to \( \sigma \)-algebras \( F_{k+1} \) and \( G_{k+1} \).

**Proposition S3 (Uplink Compression Expectation).** Let \( k \in \mathbb{N} \) and \( i \in [1, N] \). Recall that \( \hat{g}^i_{k+1} = g^i_{k+1} + \epsilon^i_k \), then on each local device \( i \in [1, N] \), we have a.s:

\[
\begin{align*}
\mathbb{E} [ \hat{g}^i_{k+1} | F_{k+1} ] &= \hat{g}^i_{k+1} \\
\mathbb{E} [ \hat{g}^i_{k+1} | G_{k+1} ] &= \hat{g}^i_{k+1} \\
\mathbb{E} [ \hat{g}^i_{k+1} | H_k ] &= \nabla F(w_k).
\end{align*}
\]

which leads to

\[
\begin{align*}
\mathbb{E} [ \hat{g}^i_{k+1} | F_{k+1} ] &= \hat{g}^i_{k+1} \\
\mathbb{E} [ \hat{g}^i_{k+1} | G_{k+1} ] &= \hat{g}^i_{k+1} \\
\mathbb{E} [ \hat{g}^i_{k+1} | H_k ] &= \nabla F(w_k).
\end{align*}
\]

From Assumption 5 it follows that variance over uplink compression can be bounded as expressed in Proposition S4.

**Proposition S4 (Uplink Compression Variance).** Let \( k \in \mathbb{N} \) and \( i \in [1, N] \). Recall that \( \Delta^i_k = \hat{g}^i_{k+1} + h^i_k \), using Assumption 5 following hold a.s:

\[
\begin{align*}
\mathbb{E} \left[ \| \dot{\Delta}^i_k - \Delta^i_{k+1} \|^2 \Big| F_{k+1} \right] &\leq \omega_{up}^i \| \Delta^i_{k+1} \|^2 \\
(\iff)\quad \mathbb{E} \left[ \| g^i_{k+1} - g^i_{k+1} \|^2 \Big| F_{k+1} \right] &\leq \omega_{up}^i \| g^i_{k+1} \|^2 when no memory \right).
\end{align*}
\]

Concerning downlink compression (information sent from central server to each node), Proposition S5 gives its expectation w.r.t \( \sigma \)-algebras \( G_{k+1} \) and \( H_{k+1} \).

**Proposition S5 (Downlink Compression Expectation).** Let \( k \in \mathbb{N} \), recall that \( \Omega_{k+1} = C_{dwn}(\hat{g}^i_{k+1}) = \hat{g}^i_{k+1} + \epsilon_k \), then a.s.

\[
\begin{align*}
\mathbb{E} [ \Omega_{k+1} | G_{k+1} ] &= \hat{g}^i_{k+1} \\
\mathbb{E} [ \Omega_{k+1} | H_{k+1} ] &= \Omega_{k+1} \\
\mathbb{E} [ \Omega_{k+1} | G_{k+1} ] &= \hat{g}^i_{k+1}.
\end{align*}
\]

Next proposition states that downlink compression can be bounded as for Proposition S4.

**Proposition S6 (Downlink Compression Variance).** Let \( k \in \mathbb{N} \), using Assumption 5 following holds a.s:

\[
\begin{align*}
\mathbb{E} \left[ \| \Omega_{k+1} - \hat{g}^i_{k+1} \|^2 \Big| G_{k+1} \right] &\leq \omega_{dwn}^i \| \hat{g}^i_{k+1} \|^2.
\end{align*}
\]
Now, we give in Propositions \([S7]\) and \([S8]\) the expectation and the variance w.r.t. devices random sampling noise.

**Proposition S7** (Expectation of device sampling). *Let \( k \in \mathbb{N} \), let’s note \( a_{k+1} = \frac{1}{N} \sum_{i=1}^{N} a_{k+1}^{i} \) and \( a_{k+1,S_k} = \frac{1}{pN} \sum_{i \in S_k} a_{k+1}^{i} \), where \( (a_{k+1}^{i})_{i=0}^{N} \in (\mathbb{R}^{d})^{N} \) are \( N \) random variables independent of each other and \( \mathcal{J}_{k+1} \)-measurable, for a \( \sigma \)-field \( \mathcal{J}_{k+1} \) s.t. \((B_{k}^{i})_{i=1}^{N} \) are independent of \( \mathcal{J}_{k+1} \). We have a.s:*

\[
\mathbb{E}[a_{k+1,S_k} | \mathcal{J}_{k+1}] = a_{k+1}.
\]

The vector \( a_{k+1} \) (resp. the \( \sigma \)-field \( \mathcal{J}_{k+1} \)) may represent various objects, for instance: \( g_{k+1}, \hat{g}_{k+1}, \Omega_{k+1} \) (resp. \( \mathcal{F}_{k+1}, \mathcal{G}_{k+1}, \mathcal{H}_{k+1} \)).

**Proof.** For any \( k \in \mathbb{N}^* \), we have that:

\[
\mathbb{E}[a_{k+1,S_k} | \mathcal{J}_{k+1}] = \mathbb{E}\left[ \frac{1}{pN} \sum_{i \in S_k} a_{k+1}^{i} | \mathcal{J}_{k+1} \right] = \mathbb{E}\left[ \frac{1}{pN} \sum_{i=0}^{N} a_{k+1}^{i} B_{k}^{i} | \mathcal{J}_{k+1} \right]
\]

by linearity of the expectation,

\[
= \frac{1}{pN} \sum_{i=0}^{N} \mathbb{E}\left[ a_{k+1}^{i} B_{k}^{i} | \mathcal{J}_{k+1} \right] \text{ because } (a_{k+1}^{i})_{i=1}^{N} \text{ are } \mathcal{J}_{k+1} \text{-measurable},
\]

\[
= \frac{1}{N} \sum_{i=0}^{N} a_{k+1}^{i} = a_{k+1} \text{ because } (B_{k}^{i})_{i=1}^{N} \text{ are independent of } \mathcal{J}_{k+1},
\]

which allows to conclude. \( \square \)

**Proposition S8** (Variance of device sampling). *Let \( k \in \mathbb{N}^* \), with the same notation as Proposition \([S7]\) we have a.s:*

\[
\mathbb{V}[a_{k+1,S_k} | \mathcal{J}_{k+1}] = \frac{1-p}{pN^2} \sum_{i=0}^{N} \|a_{k+1}^{i}\|^2.
\]

**Proof.** Let \( k \in \mathbb{N}^* \),

\[
\mathbb{V}[a_{k+1,S_k} | \mathcal{J}_{k+1}] = \mathbb{V}\left[ \frac{1}{pN} \sum_{i=0}^{N} a_{k+1}^{i} B_{k}^{i} | \mathcal{J}_{k+1} \right]
\]

because \( (B_{k}^{i})_{i=1}^{N} \) are independent,

\[
= \frac{1}{p^2N^2} \sum_{i=0}^{N} \mathbb{V}\left[ a_{k+1}^{i} B_{k}^{i} | \mathcal{J}_{k+1} \right] \text{ because } (a_{k+1}^{i})_{i=1}^{N} \text{ are } \mathcal{J}_{k+1} \text{-measurable},
\]

\[
= \frac{1}{p^2N^2} \sum_{i=0}^{N} \|a_{k+1}^{i}\|^2 \mathbb{V}[B_{k}^{i} | \mathcal{J}_{k+1}] \text{ because } (B_{k}^{i})_{i=1}^{N} \text{ are independent of } \mathcal{J}_{k+1}.
\]

\( \square \)

**B BANDWIDTH SPEED**

In a network configuration where download would be much faster than upload, bidirectional compression would present no benefit over unidirectional, as downlink communications would have a negligible cost. However, this is not the case in practice: to assess this point, we gathered broadband speeds, for both download and upload communications, for fixed broadband (cable, T1, DSL ...) or mobile (cellphones, smartphones, tablets, laptops ...) from studies carried out in 2020 over the 6 continents by [Speedtest.net](https://www.speedtest.net) [see noa]. Results are provided in Figure \([S3]\) comparing download and upload speeds. The
The dataset is picked from a study carried out by Speedtest.net [see noa]. This study has measured the bandwidth speeds in 2020 across the six continents. In order to get a better understanding of this dataset, we illustrate the speeds distribution on Figures S3 to S5. In Figures S4a, S4b, and S5, unlike Figure S3, we do not aggregate data by countries of a same continents. This allows to analyse the speeds ratio between upload and download with the proper value of each countries. Looking at Figures S4a, S4b, and S5, it is noticeable that in the world, the ratio between upload and download speed is between 1 and 5, and not between 1 and 3.5 as Figure S3 was suggesting since we were aggregating data by continents. There are only nine countries in the world having a ratio higher than 5. In Europe: Malta, Belgium and Montenegro. In Asia: South Korea. In North America: Canada, Saint Vincent and the Grenadines, Panama and Costa Rica. In Africa: Western Sahara. The highest ratio is 7.7 observed in Malta.
We build two different synthetic dataset for i.i.d. or non-i.i.d. cases. We use linear regression to tackle the i.i.d case. For non-i.i.d. setting, we generate two different datasets based on a logistic model with two different parameters: $z_j^i = x_i^j y_j^i 1 \leq j \leq n_i = (X^i, Y^i)$ following a distribution $D_i$. We use $N = 10$ devices, each holding 200 points of dimension $d = 20$ for least-square regression and $d = 2$ for logistic regression. We ran algorithms over 100 epochs.

**Choice of the step size for synthetic dataset.** For stochastic descent, we use a step size $\gamma = \frac{1}{L \sqrt{\kappa}}$ with $k$ the number of iteration, and for the batch descent we choose $\gamma = \frac{1}{L}$. For i.i.d. setting, we use a linear regression model without bias. For each worker $i$, data points are generated from a normal distribution $\mathcal{N}(0, \Sigma)$. And then, for all $j$ in $[1, n_i]$, we have: $y_j^i = \langle w, x_j^i \rangle + \epsilon_i$ with $\epsilon_i \sim \mathcal{N}(0, \lambda^2)$ and $w$ the true model.

To obtain $\sigma_* = 0$, it is enough to remove the noise $\epsilon_i$ by setting the variance $\lambda^2$ of the dataset distribution to 0. Indeed, using a least-square regression, for all $i$ in $[1, N]$, the cost function evaluated at point $w$ is $F_i(w) = \frac{1}{2} \| X_i^T w - Y_i \|^2$. Thus the stochastic gradient $g_j^i(w)$ on device $i$ in $[1, N]$ is $g_j^i(w) = (X_j^i)^T w - Y_j^i) X_j^i$. On the other hand, the true gradient is $\nabla F_i(w) = \mathbb{E} X_i^T X_i^T (w - w^*)$. Computing the difference, we have for all device $i$ in $[1, N]$ and all $j$ in $[1, n_i]$:

$$g_j^i(w) - F_i(w) = (X_j^i)^T w - \mathbb{E} X_i^T X_i^T (w - w_*) + (X_j^i)^T (w_* - Y_j^i) X_j^i \sim \mathcal{N}(0, \lambda^2)$$

This is why, if we set $\lambda = 0$ and evaluate eq. (S3) at $w_*$, we get back Assumption 3 with $\sigma_* = 0$, and as a consequence, the stochastic noise at the optimum is removed. Remark that it remains a stochastic gradient descent, and the uniform bound on the gradients noise is not 0. We set $\lambda^2 = 0 (\Rightarrow \sigma_*^2 = 0)$ in Figure S8. Otherwise, we set $\lambda^2 = 0.4$.

For non-i.i.d., we generate two different datasets based on a logistic model with two different parameters: $w_1 = \ldots = \ldots$
Figure S6: Data distribution for logistic regression to simulate non-i.i.d. data. Half of the device hold first dataset, and the other half the second one.

Figure S7: Synthetic dataset, Least-Square Regression with noise ($\sigma_* \neq 0$). In a situation where data is i.i.d., the memory does not present much interest, and has no impact on the convergence. Because $\sigma_*^2 \neq 0$, all algorithms saturate; and saturation level is higher for double compression (Artemis, Bi-QSGD), than for simple compression (Diana, QSGD) or than for SGD. This corroborates findings in Theorem 1 and Theorem 3.

(10,10) and $w_2 = (10,-10)$. Thus the model is expected to converge to $w_* = (10,0)$. We have two different data distributions $x_1 \sim \mathcal{N}(0, \Sigma_1)$ and $x_2 \sim \mathcal{N}(0, \Sigma_2)$, and for all $i$ in $[1, N]$, for all $k$ in $[1, n_i]$, $y_{ik} = R \left\{ \text{Sigm} \left( \left( w_{(i \mod 2)+1} \left( x_{(i \mod 2)+1} \right) \right) \right) \right\} \in \{-1, +1\}$. That is, half the machines use the first distribution $\mathcal{N}(0, \Sigma_1)$ for inputs and model $w_1$ and the other half the second distribution for inputs and model $w_2$. Here, $R$ is the Rademacher distribution and Sigm is the sigmoid function defined as $\text{Sigm}: x \mapsto \frac{e^x}{1 + e^x}$. These two distributions are presented on Figure S6.

C.1.1 Least-square regression

In this section, we present all figures generated using Least-Square regression. Figure S7 corresponds to Figure 1a.

As explained in the main of the paper, in the case of $\sigma_* \neq 0$ (Figure S7), algorithm using memory (i.e Diana and Artemis) are not expected to outperform those without (i.e QSGD and Bi-QSGD). On the contrary, they saturate at a higher level. However, as soon as the noise at the optimum is 0 (Figure S8), all algorithms (regardless of memory), converge at a linear rate exactly as classical SGD.
Figure S8: **Synthetic dataset, Least-Square Regression without noise** ($\sigma_* = 0$). Without surprise, with i.i.d data and $\sigma_* = 0$, the convergence of each algorithm is linear. Thus, in i.i.d. settings, the impact of the memory is negligible, but this will not be the case in the non-i.i.d. settings as underlined by Figure S9.

Figure S9: **Synthetic dataset, Logistic Regression on non-i.i.d. data** using a batch gradient descent (to get $\sigma_* = 0$). The benefit of memory is obvious, it makes the algorithm to converge linearly, while algorithms without are saturating at a high level. This stress on the importance of using the memory in non-i.i.d. settings.

### C.1.2 Logistic regression

In this section, we present all figures generated using a logistic regression model. Figure S9 corresponds to Figure 1b. Data is non-i.i.d. and we use a full batch gradient descent to get $\sigma_* = 0$ to shed into light the impact of memory over convergence. Figure S10 is using same data and configuration as Figure S9, except that it is combined with a Polyak-Ruppert averaging. Note that in the absence of memory the variance increases compared to algorithms using memory. To generate these figures, we didn’t take the optimal step size. But if we took it, the trade-off between variance and bias would be worse and algorithms using memory would outperform those without.

### C.2 REAL DATASETS: QUANTUM AND SUPERCONDUCT

In this section, we present details about experiments conducted on real-life datasets: superconduct (from Caruana et al. [2004]) where we use a least-square regression, and quantum (from Hamidieh [2018]) with a logistic regression. All figures can be found in the notebooks provided in supplementary materials.

In this following, we present results on superconduct and quantum in the setting of full device participation. We detail experiments in the PP setting in Appendix C.2.1. Next, we address the issue of the optimal step size in Appendix C.2.2. In Appendix C.2.3 we compare Artemis to other existing algorithm doing compression in a distributed learning framework. Finally, we estimate in Appendix C.3 the carbon footprint of our work.
Figure S10: **Polyak-Ruppert averaging, synthetic dataset.** Logistic Regression on non-i.i.d. data using a batch gradient descent (to get $\sigma_* = 0$) and a Polyak-Ruppert averaging. The convergence is linear as predicted by Theorem 2 because $\sigma_* = 0$. Best seen in colors.

Figure S11: TSNE representations. Best seen in colors.

In order to simulate non-i.i.d. data and to make the experiments closer to real-life usage, we split the dataset in heterogeneous groups using a Gaussian mixture clustering on TSNE representations (defined by Maaten & Hinton 2008). Thus, the data are highly non-i.i.d. and unbalanced over devices. We plot on Figure S11 the TSNE representation of the two real datasets.

There are $N = 20$ devices for superconduct and quantum datasets. For superconduct, there are between 250 and 3900 points by worker, with a median at 750; and for quantum, there are between 900 and 10500 points, with a median at 2300. On each figure, we indicate which step size $\gamma$ has been used.

**Convex settings** are given in Table S1. Experiments have been performed with 150 epochs in the stochastic regime, and 800 epochs in the full batch regime. We use quantization [defined in Alistarh et al., 2017] with $s = 2^{10}$ for all experiments, except in the case of partial participation where we used $s = 2^{11}$.

Figures S12 and S14 correspond to Figure 2. We observe on these figures the benefit of the memory. The level of saturation of algorithms using memory is much lower than those without memory. Additionally, Theorem 1 highlights that the level of saturation (see constant $E$ of Table 2) is proportional to the level of compression $\omega^{up/dwn}_C$. This is indeed observed on Figures S12 to S15.

In the case of the quantum dataset (see Figure S12), Artemis is not only better than Bi-QSGD, but in fact, as good as QSGD. That is to say, we achieve to make an algorithm using a bidirectional compression, as good as an algorithm handling unidirectional compression.
Table S1: Settings of experiments.

| Setting                  | quantum references | superconduct references |
|--------------------------|--------------------|-------------------------|
| model                    | Caruana et al. 2004| Hamidi [2018]           |
| dimension $d$            | 66                 | 82                      |
| training dataset size    | 50,000             | 21,200                  |
| batch size $b$           | 256                | 64                      |
| compression rate $s$     | $2^0$ (i.e. two levels) |                        |
| norm quantization        | $\| \cdot \|_2$    |                         |
| momentum $m$             | no momentum        |                         |
| step size $\gamma$      | $1/L$              |                         |

C.2.1 Partial participation

In this section we provide additional experiments on partial participation in the stochastic regime. Only half of the devices (randomly sampled) participate at each round, we use $2^1$-quantization.

Figure S16 presents the first naive approach (PP1) to handle partial participation. This naive solution fails to properly converge. In the other hand, algorithms using PP2 - SGD with memory i.e $\omega_c^{up/dwn} = 0$, Artemis with unidirectional compression i.e. $\omega_c^{down} = 0$ and Artemis with $\omega_c^{up/dwn} \neq 0$ - presents much better convergence, see Figure S17. As an example, SGD with memory matches the results of SGD in the case of full participation (Figures S12 and S14). However, the convergence of QSGD and Bi-QSGD is unchanged as there is no difference between the two approaches in the absence of memory.
Figure S13: **Quantum.** least-square regression, $\sigma_* = 0$, $\gamma = 1/L$, $b = 256$, non-i.i.d.. Best seen in colors.

Figure S14: **Superconduct.** least-square regression, $\sigma_* \neq 0$, $\gamma = 1/L$, $b = 64$, non-i.i.d.. Best seen in colors.

Figure S15: **Superconduct.** least-square regression, $\sigma_* = 0$, $\gamma = 1/L$, $b = 64$, non-i.i.d.. Best seen in colors.
Figure S16: Partial participation, stochastic regime - PP1. $\sigma_* \neq 0$, $N = 20$ workers, $p = 0.5$, $b > 1$ (150 iter.) With this variant, all algorithms are saturating at a high level.

Figure S17: Partial participation, stochastic regime - PP2. $\sigma_* \neq 0$, $N = 20$ workers, $p = 0.5$, $b > 1$ (150 iter.).

The result in the full gradient regime is given in Section 5. On Figures 3 and 4, we can observe that our new algorithm PP2 has a linear convergence unlike PP1.

As a conclusion on partial participation: with PP2, we observe the significant impact of memory when using non-i.i.d. data. Comparing Figure S17 to Figure S16, the saturation of algorithms with $\alpha$ different to zero (i.e. using memory) is much lower; and classical SGD is outperformed by its variant using the memory mechanism.

C.2.2 Optimized step size

In this section, we want to address the issue of the optimal step size. On Figure S18 we plot the minimal loss after 150 iterations for each of the 5 algorithms. We can see that algorithms with memory clearly outperform those without. Then, on Figure S19 we present the loss of Artemis after 150 iterations for various step size: $N = 20$, $\frac{1}{2L}$, $\frac{1}{L}$, $\frac{1}{2L}$, $\frac{1}{4L}$, $\frac{1}{8L}$ and $\frac{1}{16L}$. This helps to understand which step size should be taken to obtain the best accuracy after $k$ in [1, 150] iterations. Finally, on Figure S20, we plot the loss obtained with the optimal step size $\gamma_{opt}$ of each algorithms (found with Figure S18) w.r.t the number of communicated bits.

On Figure S18, it is interesting to note that the memory allows to increase the maximal step size. So, the optimal step size is $\gamma_{opt} = \frac{1}{L}$ for Artemis, but is $\gamma_{opt} = \frac{1}{2L}$ for BiQSGD.

We plot the loss of Artemis after 150 iterations for different step size on Figure S19. As stressed by Figure S18, after 150 iterations, the best accuracy for both datasets is indeed obtained with $\gamma_{opt} = \frac{1}{L}$. And we observe that (as for Vanilla SGD), the optimal step size of Artemis decreases with the number of iterations (e.g., for quantum, it is $1/L$ before 50 iterations and $1/2L$ after). This is consistent with Theorem I.
Figure S18: Searching for the optimal step size $\gamma_{opt}$ for each algorithm. X-axis - value on step size, Y-axis - minimal loss after running 250 iterations.

Figure S19: Loss w.r.t. step size $\gamma$.

In conclusion of this subsection, Figures S18 to S20 allow to conclude on the significant impact of memory in a non-i.i.d. settings, and to claim that bidirectional compression with memory is by far superior (up to a threshold) to the four other algorithm: SGD, QSGD, Diana and BiQSGD.

### C.2.3 Comparing Artemis with other existing algorithms

On Figure S21 we compare Artemis with other existing algorithms: FedSGD, FedPAQ, Diana, Dore and Double-Squeeze. We take $\gamma = 1/(2^L)$ because otherwise FedSGD and FedPAQ diverge. These two algorithms present worse performance because they have not been designed for non-i.i.d. datasets.

We can observe that Double-Squeeze (which only uses error-feedback) is outperformed by Artemis. Besides, we observe that Dore (which combines this mechanism with memory) has identical rate of convergence than Artemis. It underlines that for unbiased operators of compression, the enhancement comes from the memory and not from the error-feedback.

FedPAQ (unidirectional compression) has a very fast convergence during first iterations, but then saturates at a level higher than for Artemis-like algorithms. FedSGD (no compression) presents a convergence’s rate worse that vanilla SGD because it does not correctly handle heterogeneous datasets.
As part as a community effort to report the amount of experiments that were performed, we estimated that overall our experiments ran for 220 to 270 hours end to end. We used an Intel(R) Xeon(R) CPU E5-2667 processor with 16 cores.

The carbon emissions caused by this work were subsequently evaluated with Green Algorithm built by [Lannelongue et al., 2020]. It estimates our computations to generate 30 to 35 kg of CO2, requiring 100 to 125 kWh. To compare, it corresponds to about 160 to 200km by car. This is a relatively moderate impact, matching the goal to keep the experiments for an illustrative purpose.

C.3 CPU USAGE AND CARBON FOOTPRINT

In this section, we introduce a few technical lemmas that will be used in the proofs. In Appendix [D.1] we give four simple lemmas, while in Appendix [D.2] we present a lemma which will be invoked in Appendix [E] to demonstrate Theorems [S5] to [S7].

**Notation.** Let $k^* \in \mathbb{N}$ and $(a_{k+1})_{i=0}^N \in (\mathbb{R}^d)^N$ random variables independent of each other and $\mathcal{J}_{k+1}$-measurable, for a $\sigma$-field $\mathcal{J}_{k+1}$ s.t. $(B^i_{k+1})_{i=1}^N$ are independent of $\mathcal{J}_{k+1}$. Then in all the following demonstration, we note $a_{k+1} = \frac{1}{N} \sum_{i=1}^N a^i_{k+1}$ and $a_{k+1, S_k} = \frac{1}{pN} \sum_{i \in S_k} a^i_{k+1}$.

The vector $a_{k+1}$ (resp. the $\sigma$-field $\mathcal{J}_{k+1}$) may represent various objects, for instance: $g_{k+1}, \hat{g}_{k+1}, G_{k+1}$ (resp. $F_{k+1}, G_{k+1}$).

**Remark 4.** We can add the following remarks on the assumptions.

- Assumption [4] can be extended to probabilities depending on the worker $(p_i)_{i \in [1, N]}$. 

---

**Figure S20:** Optimal step size for each of the algorithms. X-axis in # bits.

**Figure S21:** Artemis compared to other existing algorithms. $\gamma = 1/(2L)$. X-axis in # epoch or in # bits.
• Assumption 3 requires in fact to access a sequence of i.i.d. compression operators \( C_{\text{up/dwn},k} \) for \( k \in \mathbb{N} \) – but for simplicity, we generally omit the \( k \) index.

• Assumption 4 in fact only requires that for any \( i \in [1, N] \), \( \mathbb{E} \left[ \| g_{k+1,i} - \nabla F_i(w_i) \| \right] \leq \frac{(\sigma_i)^2}{k} \), and the results then hold for \( \sigma_i = \frac{1}{N} \sum_{i=1}^{N} (\sigma_i)^2 \). In other words, the bounds does not need to be uniform over workers, only the average truly matters.

### D.1 USEFUL IDENTITIES AND INEQUALITIES

**Lemma S1.** Let \( N \in \mathbb{N} \) and \( d \in \mathbb{N} \). For any sequence of vector \( (a_i)_{i=1}^{N} \in \mathbb{R}^d \), we have the following inequalities:

\[
\left\| \sum_{i=1}^{N} a_i \right\|^2 \leq \left( \sum_{i=1}^{N} \| a_i \|^2 \right) \leq N \sum_{i=1}^{N} \| a_i \|^2 .
\]

The first part of the inequality corresponds to the triangular inequality, while the second part is Cauchy’s inequality.

**Lemma S2.** Let \( \alpha \in [0, 1] \) and \( x, y \in (\mathbb{R}^d)^2 \), then:

\[
\| \alpha x + (1 - \alpha) y \|^2 = \alpha \| x \|^2 + (1 - \alpha) \| y \|^2 - \alpha(1 - \alpha) \| x - y \|^2 .
\]

This is a norm’s decomposition of a convex combination.

**Lemma S3.** Let \( X \) be a random vector of \( \mathbb{R}^d \), then for any vector \( x \in \mathbb{R}^d \):

\[
\mathbb{E} \| X - \mathbb{E}X \|^2 = \mathbb{E} \| X - x \|^2 - \| \mathbb{E}X - x \|^2 .
\]

This equality is a generalization of the well known decomposition of the variance (with \( x = 0 \)).

**Lemma S4.** If \( F : \mathcal{X} \subset \mathbb{R}^d \to \mathbb{R} \) is strongly convex, then the following inequality holds:

\[ \forall (x, y) \in \mathbb{R}^d, \langle \nabla F(x) - \nabla F(y) \mid x - y \rangle \geq \mu \| x - y \|^2 . \]

This inequality is a consequence of strong convexity and can be found in [Nesterov 2004] equation 2.1.22.

### D.2 LEMMAS FOR PROOF OF CONVERGENCE

Below are presented technical lemmas needed to prove the contraction of the Lyapunov function for Theorems S5 and S6. In this section we assume that Assumptions I to 6 are verified. In Appendices D.2.1 and D.2.2, we separate lemmas that required only for the case with memory or without.

The first lemma is very simple and straightforward from the definition of \( \Delta_k^i \). We remind that \( \Delta_k^i \) is the difference between the computed gradient and the memory hold on device \( i \). It corresponds to the information which will be compressed and sent from device \( i \) to the central server.

**Lemma S5** (Bounding the compressed term). The squared norm of the compressed term sent by each node to the central server can be bounded as following:

\[ \forall k \in \mathbb{N} , \forall i \in [1, N] , \quad \| \Delta_k^i \|^2 \leq 2 \left( \| g_{k+1,i}^* - h_i^* \|^2 + \| h_k^i - h_i^* \|^2 \right) . \]

**Proof.** Let \( k \in \mathbb{N} \) and \( i \in [1, N] \), we have by definition:

\[ \| \Delta_k^i \|^2 = \| g_{k+1,i}^* - h_i^* \|^2 = \| (g_{k+1,i} - h_i^*) + (h_i^* - h_k^i) \|^2 . \]

Applying Lemma S1 gives the expected result. \( \square \)
Below, we show up a recursion over the memory term $h_{k+1}^i$ involving the stochastic gradients. This recursion will be used in Lemma S13. The existence of recursion has been first shed into light by [Mishchenko et al., 2019].

**Lemma S6** (Expectation of memory term). The memory term $h_{k+1}^i$ can be expressed using a recursion involving the stochastic gradient $g_{k+1}$:

$$
\forall k \in \mathbb{N}, \forall i \in [1, N], \quad \mathbb{E} \left[ h_{k+1}^i \mid \mathcal{F}_{k+1} \right] = (1 - \alpha)h_k^i + \alpha g_{k+1}.
$$

**Proof.** Let $k \in \mathbb{N}$ and $i \in [1, N]$. We just need to decompose $h_k^i$ using its definition:

$$
h_k^i = h_k^i + \alpha \Delta_k^i = h_k^i + \alpha (\hat{g}_{k+1}^i - h_k^i),$$

and considering that $\mathbb{E} \left[ \hat{g}_{k+1}^i \mid \mathcal{F}_{k+1} \right] = g_{k+1}$ (Proposition S3), the proof is completed.

In Lemma S7, we rewrite $\|g_{k+1}\|^2$ and $\|g_{k+1} - h_k^i\|^2$ to make appears:

1. the noise over stochasticity
2. $\|g_{k+1} - g_{k+1,*}\|^2$ which is the term on which will later be applied cocoercivity (see Assumption 2).

Lemma S7 is required to correctly apply cocoercivity in Lemma S14.

**Lemma S7** (Before using co-coercivity). Let $k \in [0, K]$ and $i \in [1, N]$. The noise in the stochastic gradients as defined in Assumptions 3 and 4 can be controlled as following:

$$
\frac{1}{N} \sum_{i=1}^{N} \mathbb{E} \left[ \|g_{k+1}^i\|^2 \mid \mathcal{H}_k \right] \leq \frac{2}{N} \sum_{i=1}^{N} \left( \mathbb{E} \left[ \|g_{k+1}^i - g_{k+1,*}^i\|^2 \mid \mathcal{H}_k \right] + \frac{\sigma_b^2}{b} + B^2 \right),
$$

(S4)

$$
\frac{1}{N} \sum_{i=1}^{N} \mathbb{E} \left[ \|g_{k+1}^i - h_k^i\|^2 \mid \mathcal{H}_k \right] \leq \frac{2}{N} \sum_{i=1}^{N} \left( \mathbb{E} \left[ \|g_{k+1}^i - g_{k+1,*}^i\|^2 \mid \mathcal{H}_k \right] + \frac{\sigma_b^2}{b} \right).
$$

(S5)

**Proof.** Let $k \in \mathbb{N}$. For eq. (S4):

$$
\|g_{k+1}^i\|^2 = \|g_{k+1}^i - g_{k+1,*}^i + g_{k+1,*}^i\|^2 \\
\leq 2 \left( \|g_{k+1}^i - g_{k+1,*}^i\|^2 + \|g_{k+1,*}^i\|^2 \right) \text{ using inequality of Lemma S1}
$$

Taking expectation with regards to filtration $\mathcal{H}_k$ and using Assumptions 3 and 4 gives the first result.

For eq. (S5), we use Lemma S1 and we write:

$$
\|g_{k+1}^i - h_k^i\|^2 = \|(g_{k+1}^i - g_{k+1,*}^i) + (g_{k+1,*}^i - \nabla F_i(w_*))\|^2 \\
\leq 2(\|g_{k+1}^i - g_{k+1,*}^i\|^2 + \|g_{k+1,*}^i - \nabla F_i(w_*)\|^2).
$$

Taking expectation, we have:

$$
\mathbb{E} \left[ \|g_{k+1}^i - h_k^i\|^2 \mid \mathcal{H}_k \right] \leq 2 \left( \mathbb{E} \left[ \|g_{k+1}^i - g_{k+1,*}^i\|^2 \mid \mathcal{H}_k \right] + \mathbb{E} \left[ \|g_{k+1,*}^i - \nabla F_i(w_*)\|^2 \mid \mathcal{H}_k \right] \right) \\
\leq 2 \left( \mathbb{E} \left[ \|g_{k+1}^i - g_{k+1,*}^i\|^2 \mid \mathcal{H}_k \right] + \frac{\sigma_b^2}{b} \right) \text{ using Assumption 3}
$$

□
Demonstrating that the Lyapunov function is a contraction requires to bound \( \|g_{k+1,S_k}\|^2 \) which needs to control each term \( (\|g_{k+1,S_k}\|^2)_{i=1}^{N} \) of the sum. This leads to invoke smoothness of \( F \) (consequence of Assumption 2).

**Lemma S8.** Regardless if we use memory, we have the following bound on the squared norm of the gradient, for all \( k \) in \( \mathbb{N} \):

\[
\mathbb{E} \left[ \|g_{k+1,S_k}\|^2 \mid I_k \right] \leq \frac{1}{pN^2} \sum_{i=1}^{N} \mathbb{E} \left[ \|g_{k+1}^i - h_i^*\|^2 \mid H_k \right] + L \langle \nabla F(w_k) \mid w_k - w_* \rangle .
\]

**Proof.** Let \( k \in \mathbb{N} \),

\[
\|g_{k+1,S_k}\|^2 = \left\| \frac{1}{pN} \sum_{i \in S_k} g_{k+1}^i \right\|^2 = \left\| \frac{1}{pN} \sum_{i \in S_k} (g_{k+1}^i - \nabla F_i(w_k)) + \frac{1}{pN} \sum_{i \in S_k} \nabla F_i(w_k) \right\|^2.
\]

Now taking conditional expectation w.r.t \( \sigma(I_k \cup B_k) \) (including \( B_k \) in the \( \sigma \)-field allows to not consider the randomness associated to the device sampling):

\[
\mathbb{E} \left[ \|g_{k+1,S_k}\|^2 \mid \sigma(I_k \cup B_k) \right] = \mathbb{E} \left[ \left\| \frac{1}{pN} \sum_{i \in S_k} g_{k+1}^i - \nabla F_i(w_k) + \frac{1}{pN} \sum_{i \in S_k} \nabla F_i(w_k) \right\|^2 \mid \sigma(I_k \cup B_k) \right].
\]

Expanding this squared norm:

\[
\mathbb{E} \left[ \|g_{k+1,S_k}\|^2 \mid \sigma(I_k \cup B_k) \right] = \mathbb{E} \left[ \left\| \frac{1}{pN} \sum_{i \in S_k} g_{k+1}^i - \nabla F_i(w_k) \right\|^2 \mid \sigma(I_k \cup B_k) \right] + 2 \mathbb{E} \left[ \left( \frac{1}{pN} \sum_{i \in S_k} g_{k+1}^i - \nabla F_i(w_k) \right) \left( \frac{1}{pN} \sum_{i \in S_k} \nabla F_i(w_k) \right) \right] \mid \sigma(I_k \cup B_k) \right] + \mathbb{E} \left[ \left\| \frac{1}{pN} \sum_{i \in S_k} \nabla F_i(w_k) \right\|^2 \mid \sigma(I_k \cup B_k) \right].
\]

Moreover, \( \forall i,j \in \{1, N\} \), \( \mathbb{E} \left[ \langle g_{k+1}^i - \nabla F_i(w_k) \mid \nabla F_j(w_k) \rangle \mid \sigma(I_k \cup B_k) \right] = 0 \) and \( \nabla F(w_k) \) is \( \sigma(I_k \cup B_k) \)-measurable:

\[
\mathbb{E} \left[ \|g_{k+1,S_k}\|^2 \mid \sigma(I_k \cup B_k) \right] \leq \mathbb{E} \left[ \left\| \frac{1}{pN} \sum_{i \in S_k} g_{k+1}^i - \nabla F_i(w_k) \right\|^2 \mid \sigma(I_k \cup B_k) \right] + \mathbb{E} \left[ \left\| \nabla F_{s_k}(w_k) \right\|^2 \mid \sigma(I_k \cup B_k) \right].
\]

To compute \( \|\nabla F_{s_k}(w_k)\|^2 \), we apply cocoercivity (see Assumption 2) and next we take expectation w.r.t \( \sigma \)-algebra \( I_k \):

\[
\mathbb{E} \left[ \|\nabla F_{s_k}(w_k)\|^2 \mid I_k \right] = L \mathbb{E} \left[ \|\nabla F_{s_k}(w_k)\mid I_k \mid w_k - w_* \right] = L \langle \nabla F(w_k) \mid w_k - w_* \rangle.
\]

Now, for sake of clarity we note \( \Pi = \left\| \frac{1}{pN} \sum_{i \in S_k} g_{k+1}^i - \nabla F_i(w_k) \right\|^2 \), then:

\[
\mathbb{E} \left[ \Pi \mid \sigma(I_k \cup B_k) \right] = \frac{1}{p^2N^2} \sum_{i \in S_k} \mathbb{E} \left[ \|g_{k+1}^i - \nabla F_i(w_k)\|^2 \mid \sigma(I_k \cup B_k) \right] + \frac{1}{p^2N^2} \sum_{i,j \in S_k, i \neq j} \mathbb{E} \left[ \langle g_{k+1}^i - \nabla F_i(w_k) \mid g_{k+1}^j - \nabla F_j(w_k) \rangle \mid \sigma(I_k \cup B_k) \right] \mid = 0 \text{ by independence of } (g_{k+1}^i)_{i=0}^N \\right] + \mathbb{E} \left[ \left\| \frac{1}{pN} \sum_{i \in S_k} (g_{k+1}^i - \nabla F_i(w_*)) + (\nabla F_i(w_* - \nabla F_i(w_k)) \right\|^2 \mid \sigma(I_k \cup B_k) \right].
\]
Developing the squared norm a second time:

\[
\mathbb{E} \left[ \Pi \mid \sigma(\mathcal{I}_k \cup B_k) \right] = \frac{1}{p^2N^2} \sum_{i \in S_k} \mathbb{E} \left[ \left\| g_{k+1}^i - \nabla F_i(w_*) \right\|^2 \mid \sigma(\mathcal{I}_k \cup B_k) \right]
\]

\[
+ \frac{2}{p^2N^2} \sum_{i \in S_k} \mathbb{E} \left[ \langle g_{k+1}^i - \nabla F_i(w_*) \mid \nabla F_i(w_*) - \nabla F_i(w_k) \rangle \mid \sigma(\mathcal{I}_k \cup B_k) \right]
\]

\[
+ \frac{1}{p^2N^2} \sum_{i \in S_k} \left\| \nabla F_i(w_k) - \nabla F_i(w_*) \right\|^2.
\]

Then,

\[
\mathbb{E} \left[ \Pi \mid \sigma(\mathcal{I}_k \cup B_k) \right] = \frac{1}{p^2N^2} \sum_{i \in S_k} \mathbb{E} \left[ \left\| g_{k+1}^i - \nabla F_i(w_*) \right\|^2 \mid \sigma(\mathcal{I}_k \cup B_k) \right]
\]

\[
- \frac{2}{p^2N^2} \sum_{i \in S_k} \langle \nabla F_i(w_k) - \nabla F_i(w_*) \mid \nabla F_i(w_*) - \nabla F_i(w_k) \rangle
\]

\[
+ \frac{1}{p^2N^2} \sum_{i \in S_k} \left\| \nabla F_i(w_k) - \nabla F_i(w_*) \right\|^2
\]

\[
= \frac{1}{p^2N^2} \sum_{i \in S_k} \mathbb{E} \left[ \left\| g_{k+1}^i - \nabla F_i(w_*) \right\|^2 \mid \sigma(\mathcal{I}_k \cup B_k) \right] - \left\| \nabla F_i(w_k) - \nabla F_i(w_*) \right\|^2
\]

applying cocoercivity (Assumption 3):

\[
\mathbb{E} \left[ \Pi \mid \sigma(\mathcal{I}_k \cup B_k) \right] \leq \frac{1}{p^2N^2} \sum_{i \in S_k} \mathbb{E} \left[ \left\| g_{k+1}^i - \nabla F_i(w_*) \right\|^2 \mid \sigma(\mathcal{I}_k \cup B_k) \right] - L \langle \nabla F_i(w_k) - \nabla F_i(w_*) \mid w_k - w_* \rangle. \quad (S7)
\]

Now we consider the randomness associated to device sampling. Remember that because \( \mathcal{I}_k \subset \sigma(\mathcal{I}_k \cup B_k) \), we have \( \mathbb{E} \left[ \Pi \mid \mathcal{I}_k \right] = \mathbb{E} \left[ \mathbb{E} \left[ \Pi \mid \sigma(\mathcal{I}_k \cup B_k) \right] \mid \mathcal{I}_k \right] \). Thus, we consider now \( \Pi \) w.r.t. the \( \sigma \)-field \( \mathcal{I}_k \).

**Handling first term of eq. (S7):**

\[
\frac{1}{p^2N^2} \sum_{i \in S_k} \mathbb{E} \left[ \left\| g_{k+1}^i - \nabla F_i(w_*) \right\|^2 \mid \mathcal{I}_k \right] = \frac{1}{p^2N^2} \sum_{i=1}^{N} B_i^k \mathbb{E} \left[ \left\| g_{k+1}^i - \nabla F_i(w_*) \right\|^2 \mid \mathcal{I}_k \right]
\]

\[
= \frac{1}{pN^2} \sum_{i=1}^{N} \mathbb{E} \left[ \left\| g_{k+1}^i - \nabla F_i(w_*) \right\|^2 \mid \mathcal{I}_k \right].
\]

**Handling second term of eq. (S7):**

\[
L \left( \frac{1}{p^2N^2} \sum_{i \in S_k} \nabla F_i(w_k) - \nabla F_i(w_*) \mid w_k - w_* \right) = L \left( \frac{1}{p^2N^2} \sum_{i=0}^{N} \left( \nabla F_i(w_k) - \nabla F_i(w_*) \right) B_i^k \mid w_k - w_* \right).
\]

Taking expectation w.r.t \( \sigma \)-algebra \( \mathcal{I}_k \):

\[
\mathbb{E} \left[ L \left( \frac{1}{p^2N^2} \sum_{i \in S_k} \nabla F_i(w_k) - \nabla F_i(w_*) \mid w_k - w_* \right) \mid \mathcal{I}_k \right]
\]

\[
= L \left( \frac{1}{pN^2} \sum_{i=0}^{N} \nabla F_i(w_k) - \nabla F_i(w_*) \mid w_k - w_* \right)
\]

\[
= \frac{L}{pN} \left( \nabla F(w_k) - \nabla F(w_*) \mid w_k - w_* \right).
\]
Injecting this in eq. (S7):

\[
\mathbb{E} [\Pi | \mathcal{I}_k] \leq \frac{1}{pN^2} \sum_{i=1}^{N} \mathbb{E} \left[ \|g_{k+1}^i - \nabla F_i(w_*)\|^2 | \mathcal{I}_k \right] - \frac{L}{pN} \langle \nabla F(w_k) | w_k - w_* \rangle .
\]

Recall that we note \( h_* = \nabla F_i(w_*) \), returning to eq. (S6) and invoking again cocoercivity:

\[
\mathbb{E} \left[ \|g_{k+1}\|^2 | \mathcal{I}_k \right] \leq \frac{1}{pN^2} \sum_{i=1}^{N} \mathbb{E} \left[ \|g_{k+1}^i - h_*^i\|^2 | \mathcal{I}_k \right] + (1 - \frac{1}{pN})L \langle \nabla F(w_k) | w_k - w_* \rangle ,
\]

which we simplify by considering that:

\[
\mathbb{E} \left[ \|g_{k+1}\|^2 | \mathcal{I}_k \right] \leq \frac{1}{pN^2} \sum_{i=1}^{N} \mathbb{E} \left[ \|g_{k+1}^i - h_*^i\|^2 | \mathcal{I}_k \right] + L \langle \nabla F(w_k) | w_k - w_* \rangle .
\]

\[\square\]

### D.2.1 Lemmas for the case without memory

In this subsection, we give lemmas that are used only to demonstrate Theorem S5 (i.e. without memory).

Lemma S9 helps to pass for \( k \in \mathbb{N} \) from \( \hat{g}_{k+1,S_k} \) to \( (\hat{g}_{k+1})_{i=1}^N \), this lemma will allow to invoke Lemma S10.

**Lemma S9.** In the case without memory, we have the following bound on the squared norm of the compressed gradient (randomly sampled), for all \( k \in \mathbb{N} \):

\[
\mathbb{E} \left[ \|\hat{g}_{k+1,S_k}\|^2 | \mathcal{G}_{k+1} \right] = \frac{(1-p)}{pN^2} \sum_{i=1}^{N} \|\tilde{g}_{k+1}^i\|^2 + \|\hat{g}_{k+1}\|^2 .
\]

**Proof.** The proof is quite straightforward using the expectation and the variance of \( \hat{g}_{k+1,S_k} \) computed in Propositions S7 and S8. We just need to decompose as following to easily obtain the result:

\[
\mathbb{E} \left[ \|\hat{g}_{k+1,S_k}\|^2 | \mathcal{G}_{k+1} \right] = \mathbb{V} \left[ \hat{g}_{k+1,S_k} | \mathcal{G}_{k+1} \right] + \mathbb{E} \left[ \|\hat{g}_{k+1,S_k} | \mathcal{G}_{k+1} \|^2 \right] .
\]

\[\square\]

Lemma S10 is used to remove the uplink compression noise straight after Lemma S9 has been applied.

**Lemma S10** (Expectation of the squared norm of the compressed gradient when no memory). In the case without memory, we have the following bound on the squared norm of the compressed gradient (randomly sampled), for all \( k \in \mathbb{N} \):

\[
\mathbb{E} \left[ \|\hat{g}_{k+1}\|^2 | \mathcal{H}_k \right] \leq \frac{\alpha_{up}}{N^2} \sum_{i=0}^{N} \mathbb{E} \left[ \|\hat{g}_{k+1}^i\|^2 | \mathcal{H}_k \right] + \frac{1}{N^2} \sum_{i=0}^{N} \mathbb{E} \left[ \|\tilde{g}_{k+1}^i - h_*^i\|^2 | \mathcal{H}_k \right] + L \langle \nabla F(w_k) | w_k - w_* \rangle .
\]

**Proof.** Let \( k \in \mathbb{N} \), first, we write as following:

\[
\|\hat{g}_{k+1}\|^2 = \|\hat{g}_{k+1} - g_{k+1} + g_{k+1}\|^2 = \|\hat{g}_{k+1} - g_{k+1}\|^2 + 2 \langle \hat{g}_{k+1} - g_{k+1} | g_{k+1} \rangle + \|g_{k+1}\|^2 .
\]

Taking stochastic expectation (recall that \( g_{k+1} \) is \( \mathcal{F}_{k+1} \)-measurable and that \( \mathcal{H}_k \subset \mathcal{F}_{k+1} \)):

\[
\mathbb{E} \left[ \|\hat{g}_{k+1}\|^2 | \mathcal{F}_{k+1} \right] | \mathcal{H}_k = \mathbb{E} \left[ \|\hat{g}_{k+1} - g_{k+1}\|^2 | \mathcal{F}_{k+1} \right] | \mathcal{H}_k + 2 \times \mathbb{E} \left[ \langle \hat{g}_{k+1} - g_{k+1} | g_{k+1} \rangle | \mathcal{F}_{k+1} \right] | \mathcal{H}_k + \mathbb{E} \left[ \|g_{k+1}\|^2 | \mathcal{H}_k \right] .
\]
We need to find a bound for each of the terms of above eq. (S8). The last term is handled in Lemma S8, narrowing it down to the case $p = 1$.

It follows that we just need to bound $\|\hat{g}_{k+1} - g_{k+1}\|^2$:

$$
E \left[ \left\| \hat{g}_{k+1} - g_{k+1} \right\|^2 \left| \mathcal{F}_{k+1} \right. \right] = E \left[ \left\| \hat{g}_{k+1} - E \left[ \hat{g}_{k+1} \left| \mathcal{F}_{k+1} \right. \right] \right\|^2 \left| \mathcal{F}_{k+1} \right. \right] \\
= E \left[ \left\| \frac{1}{N} \sum_{i=1}^N \hat{g}_{k+1}^i - E \left[ \hat{g}_{k+1}^i \left| \mathcal{F}_{k+1} \right. \right] \right\|^2 \left| \mathcal{F}_{k+1} \right. \right] \\
= \frac{1}{N^2} \sum_{i=0}^N E \left[ \left\| \hat{g}_{k+1}^i - g_{k+1}^i \right\|^2 \left| \mathcal{F}_{k+1} \right. \right] \\
+ \frac{1}{N} \sum_{i \neq j} E \left[ \left( \hat{g}_{k+1}^i - g_{k+1}^i \right) \left( \hat{g}_{k+1}^j - g_{k+1}^j \right) \left| \mathcal{F}_{k+1} \right. \right] \\
= 0 \text{ because } (\hat{g}_{k+1}^i)_{i=1}^N \text{ are independents} \\
= \frac{1}{N^2} \sum_{i=1}^N E \left[ \left\| \hat{g}_{k+1}^i - g_{k+1}^i \right\|^2 \left| \mathcal{F}_{k+1} \right. \right].
$$

Combining with Proposition S4 we hold that:

$$
E \left[ \left\| \hat{g}_{k+1} - g_{k+1} \right\|^2 \left| \mathcal{F}_{k+1} \right. \right] \leq \frac{\omega_{\text{up}}^2}{N^2} \sum_{i=1}^N \left\| g_{k+1}^i \right\|^2.
$$

Now, we proved that:

$$
\begin{align*}
E \left[ \left\| \hat{g}_{k+1} - g_{k+1} \right\|^2 \left| \mathcal{F}_{k+1} \right. \right] &\leq \frac{\omega_{\text{up}}^2}{N^2} \sum_{i=1}^N E \left[ \left\| g_{k+1}^i \right\|^2 \left| \mathcal{F}_{k+1} \right. \right] \\
E \left[ \left( \hat{g}_{k+1} - g_{k+1} \right) \left( g_{k+1} \right) \left| \mathcal{F}_{k+1} \right. \right] &\leq 0 \quad \text{(Proposition S3)} \\
E \left[ \left\| g_{k+1} \right\|^2 \left| \mathcal{H}_k \right. \right] &\leq \frac{1}{N^2} \sum_{i=0}^N E \left[ \left\| g_{k+1}^i - h_k^i \right\|^2 \left| \mathcal{H}_k \right. \right] \\
&+ L \left\langle \nabla F(w_k), w_k - w_0 \right\rangle \quad \text{(Lemma S8 with } p = 1). \\
\end{align*}
$$

Thus, we obtain from eq. (S8):

$$
E \left[ \left\| \hat{g}_{k+1} \right\|^2 \left| \mathcal{H}_k \right. \right] \leq \frac{\omega_{\text{up}}^2}{N^2} \sum_{i=1}^N E \left[ \left\| g_{k+1}^i \right\|^2 \left| \mathcal{H}_k \right. \right] + \frac{1}{N^2} \sum_{i=1}^N E \left[ \left\| g_{k+1}^i - h_k^i \right\|^2 \left| \mathcal{H}_k \right. \right] + L \left\langle \nabla F(w_k), w_k - w_0 \right\rangle.
$$

\(\square\)

**Lemma S11.** In the case without memory, we have the following bound on the squared norm of the local compressed gradient, for all $k \in \mathbb{N}$, for all $i \in [1, N]$: $E \left[ \left\| \hat{g}_{k+1} \right\|^2 \left| \mathcal{F}_{k+1} \right. \right] \leq (\omega_{\text{up}}^2 + 1) \left\| g_{k+1} \right\|^2$

**Proof.** Let $k \in \mathbb{N}$ and $i \in [1, N]$:

$$
E \left[ \left\| \hat{g}_{k+1} \right\|^2 \left| \mathcal{F}_{k+1} \right. \right] = E \left[ \left\| \hat{g}_{k+1} - g_{k+1} \right\|^2 \left| \mathcal{F}_{k+1} \right. \right] \\
= E \left[ \left\| \hat{g}_{k+1} - g_{k+1} \right\|^2 \left| \mathcal{F}_{k+1} \right. \right] + 2 E \left[ \left\langle \hat{g}_{k+1} - g_{k+1}, g_{k+1} \right\rangle \left| \mathcal{F}_{k+1} \right. \right] + E \left[ \left\| g_{k+1} \right\|^2 \left| \mathcal{F}_{k+1} \right. \right]
$$

To obtain the result, we need to recall that $\left\| g_{k+1} \right\|^2$ is $\mathcal{F}_{k+1}$-measurable, and then to use Proposition S6

\(\square\)
D.2.2 Lemmas for the case with memory

In this subsection, we give lemmas that are used only to demonstrate Theorems S6 and S7 (i.e. with memory).

In order to derive an upper bound on the squared norm of \( \| w_{k+1} - w_s \| \), for \( k \in \mathbb{N} \), we need to control \( \| \tilde{g}_{k+1,S_k} \| ^2 \). This term is decomposed as a sum of three terms depending on:

1. the recursion over the memory term \( (\hat{h}_k^i) \)
2. the difference between the stochastic gradient at the current point and at the optimal point (later controlled by co-coercivity)
3. the noise over stochasticity.

**Lemma S12.** In the case with memory, we have the following upper bound on the squared norm of the compressed gradient, for all \( k \in \mathbb{N} \):

\[
\mathbb{E} \left[ \| \tilde{g}_{k+1,S_k} \|^2 \bigg| \mathcal{I}_k \right] \leq 2 \left( \frac{2(\omega_C^{up} + 1)}{p} - 1 \right) \frac{1}{N^2} \sum_{i=1}^N \mathbb{E} \left[ \| g_{k+1}^i - g_{k+1,s}^i \|^2 \bigg| \mathcal{I}_k \right] \\
+ \left( \frac{2(\omega_C^{up} + 1)}{p} - 2 \right) \frac{1}{N^2} \sum_{i=1}^N \mathbb{E} \left[ \| h_k^i - h_s^i \|^2 \bigg| \mathcal{I}_k \right] \\
+ L \langle \nabla F(w_k) \big| w_k - w_s \rangle \\
+ 2\sigma_s \times \frac{2(\omega_C^{up} + 1)}{p} - 1 \right).
\]

**Proof.** We take the expectation w.r.t. the \( \sigma \)-algebra \( \mathcal{I}_{k+1} \): Doing a bias-variance decomposition:

\[
\mathbb{E} \left[ \| \tilde{g}_{k+1,S_k} \|^2 \bigg| \mathcal{I}_{k+1} \right] = \mathbb{E} \left[ \| \tilde{g}_{k+1,S_k} - \tilde{g}_{k+1} \|^2 \bigg| \mathcal{I}_{k+1} \right] + \mathbb{E} \left[ \| \tilde{g}_{k+1} \|^2 \bigg| \mathcal{I}_{k+1} \right].
\]

**First term.**

\[
\mathbb{E} \left[ \| \tilde{g}_{k+1,S_k} - \tilde{g}_{k+1} \|^2 \bigg| \mathcal{I}_{k+1} \right] = \mathbb{E} \left[ \left\| \frac{1}{Np} \sum_{i \in S_k} \tilde{\Delta}_k^i + h_k - \frac{1}{N} \sum_{i=1}^N \tilde{\Delta}_k^i + h_k \right\|^2 \bigg| \mathcal{I}_{k+1} \right]
\]

\[
= \mathbb{E} \left[ \left\| \frac{1}{Np} \sum_{i=1}^N \tilde{\Delta}_k^i (B_i - p) \right\|^2 \bigg| \mathcal{I}_{k+1} \right]
\]

\[
= \frac{1}{N^2 p^2} \sum_{i=1}^N \mathbb{E} \left[ (B_i - p)^2 \bigg| \mathcal{I}_{k+1} \right] \left\| \tilde{\Delta}_k^i \right\|^2,
\]

by independence of device sampling and because \( \{\Delta_k^i\}_{i=1}^N \) are \( \mathcal{I}_{k+1} \)-measurable. Next:

\[
\mathbb{E} \left[ \| \tilde{g}_{k+1,S_k} - \tilde{g}_{k+1} \|^2 \bigg| \mathcal{I}_{k+1} \right] = \frac{1}{N^2 p^2} \sum_{i=1}^N p(1-p) \mathbb{E} \left[ \left\| \tilde{\Delta}_k^i \right\|^2 \bigg| \mathcal{I}_{k+1} \right].
\]

Now we take expectation w.r.t. the \( \sigma \)-algebra \( \mathcal{I}_k \). Because \( \mathcal{I}_k \subset \mathcal{F}_{k+1} \), we have for all \( i \in \{1, \cdots, N\} \),

\[
\mathbb{E} \left[ \left\| \Delta_k^i \right\|^2 \bigg| \mathcal{F}_{k+1} \bigg| \mathcal{I}_k \right] = \mathbb{E} \left[ \left\| \Delta_k^i \right\|^2 \bigg| \mathcal{I}_k \right]
\]

and we use Proposition S4

\[
\mathbb{E} \left[ \| \tilde{g}_{k+1,S_k} - \tilde{g}_{k+1} \|^2 \bigg| \mathcal{I}_k \right] = \frac{(\omega_C^{up} + 1)(1-p)}{N^2 p} \sum_{i=1}^N \mathbb{E} \left[ \left\| \Delta_k^i \right\|^2 \bigg| \mathcal{I}_k \right], \text{ with Lemma S5}
\]

\[
= \frac{2(\omega_C^{up} + 1)(1-p)}{N^2 p} \sum_{i=1}^N \mathbb{E} \left[ \| g_{k+1}^i - h_s^i \|^2 \bigg| \mathcal{I}_k \right] + \mathbb{E} \left[ \| h_k^i - h_s^i \|^2 \bigg| \mathcal{I}_k \right].
\]
We take expectation w.r.t. the $\sigma$-algebra $I_k$, thus the first term is handled with Lemma $S8$:

$$
\mathbb{E} \left[ \|g_{k+1} - g_k\|^2 \mid I_k \right] = \mathbb{E} \left[ \|g_k\|^2 \mid I_k \right] + \mathbb{E} \left[ \|\tilde{g}_{k+1} - g_{k+1}\|^2 \mid I_k \right].
$$

We take expectation w.r.t. the $\sigma$-algebra $I_k$, thus the first term is handled with Lemma $S8$:

$$
\mathbb{E} \left[ \|g_{k+1}\|^2 \mid I_k \right] \leq \frac{1}{N^2} \sum_{i=1}^N \mathbb{E} \left[ \|g_{k+1} - h_i\|^2 \mid I_k \right] + L \left\langle \nabla F(w_k) \mid w_k - w_* \right\rangle.
$$

Considering the second term, by independence of the “$N$” compressions and using as previously Proposition $S4$ (because $I_k \subset \mathcal{F}_{k+1}$), we have:

$$
\mathbb{E} \left[ \|\tilde{g}_{k+1} - g_{k+1}\|^2 \mid I_k \right] \leq \frac{2 \omega_{up,i}^i}{N^2} \sum_{i=1}^N \mathbb{E} \left[ \|g_{k+1} - h_i\|^2 \mid I_k \right] + \frac{2 \omega_{up}^i}{N^2} \sum_{i=1}^N \mathbb{E} \left[ \|h_i - h_i^*\|^2 \mid I_k \right].
$$

At the end:

$$
\mathbb{E} \left[ \|\tilde{g}_{k+1}\|^2 \mid I_k \right] = \frac{2 \omega_{up,i}^i}{N^2} + \frac{1}{N^2} \sum_{i=1}^N \mathbb{E} \left[ \|g_{k+1} - h_i\|^2 \mid I_k \right] + \frac{2 \omega_{up}^i}{N^2} \sum_{i=1}^N \mathbb{E} \left[ \|h_i^* - h_i^*\|^2 \mid I_k \right] + L \left\langle \nabla F(w_k) \mid w_k - w_* \right\rangle.
$$

We can combine the first and second term, and it follows that:

$$
\mathbb{E} \left[ \|\tilde{g}_{k+1}, S_k\|^2 \mid I_k \right] \leq \left( \frac{2(\omega_{up,i}^i + 1)(1 - p)}{p} + 2 \omega_{up}^i + 1 \right) \frac{1}{N^2} \sum_{i=1}^N \mathbb{E} \left[ \|g_{k+1} - h_i\|^2 \mid I_k \right]
$$

$$
+ \left( \frac{2(\omega_{up,i}^i + 1)(1 - p)}{p} + 2 \omega_{up}^i \right) \frac{1}{N^2} \sum_{i=1}^N \mathbb{E} \left[ \|h_i - h_i^*\|^2 \mid I_k \right]
$$

$$
+ L \left\langle \nabla F(w_k) \mid w_k - w_* \right\rangle
$$

we can now apply Lemma $S7$ to conclude the proof:

$$
\mathbb{E} \left[ \|\tilde{g}_{k+1}, S_k\|^2 \mid I_k \right] \leq 2 \left( \frac{2(\omega_{up,i}^i + 1)(1 - p)}{p} + 2 \omega_{up}^i + 1 \right) \frac{1}{N^2} \sum_{i=1}^N \mathbb{E} \left[ \|g_{k+1} - g_{k+1,*}\|^2 \mid I_k \right]
$$

$$
+ \left( \frac{2(\omega_{up,i}^i + 1)(1 - p)}{p} + 2 \omega_{up}^i \right) \frac{1}{N^2} \sum_{i=1}^N \mathbb{E} \left[ \|h_i - h_i^*\|^2 \mid I_k \right]
$$

$$
+ L \left\langle \nabla F(w_k) \mid w_k - w_* \right\rangle
$$

$$
+ \frac{2 \sigma_x}{N b} \times \left( \frac{2(\omega_{up,i}^i + 1)(1 - p)}{p} + 2 \omega_{up}^i + 1 \right),
$$

and simplifying each coefficient gives the result.

To show that the Lyapunov function is a contraction, we need to find a bound for each terms. Bounding $\|w_{k+1} - w_*\|^2$, for $k \in \mathbb{N}$, flows from update schema (see eq. $[3]$) decomposition. However the memory term $\|h_{k+1}^i - h_i^*\|^2$ involved in the Lyapunov function doesn’t show up naturally.

The aim of Lemma $S13$ is precisely to provide a recursive bound over the memory term to highlight the contraction. Like Lemma $S6$, the following lemma comes from [Mishchenko et al. 2019].
Lemma S13 (Recursive inequalities over memory term). Let $k \in \mathbb{N}$ and let $i \in [1, N]$. The memory term used in the uplink broadcasting can be bounded using a recursion:

$$
\mathbb{E} \left[ \|h_{k+1}^i - h_*^i\|^2 \mid I_k \right] \leq (1 + p(2\alpha^2 \omega_{up} + 2\alpha^2 - 3\alpha)) \mathbb{E} \left[ \|h_k^i - h_*^i\|^2 \mid I_k \right] + 2p(2\alpha^2 \omega_{up} + 2\alpha^2 - \alpha) \mathbb{E} \left[ \|g_{k+1} - g_{k+1,*}\|^2 \mid I_k \right] + 2p \frac{\sigma^2}{b} (2\alpha^2 (\omega_{up}^2 + 1) - \alpha).
$$

Proof. The proof is done in two steps:

1. First, we provide a recursive bound on memory when the device is used for the update (i.e such that for $k \in \mathbb{N}$, for $i$ in $\{1, \ldots, N\}$, $B^i_k = 1$).

2. Then, we generalize to the case with all $i$ in $[1, N]$ regardless to if they are used at the round $k$.

First part. Let $k \in \mathbb{N}$ and let $i \in [1, N]$ such that $B^i_k = 1$

$$
\mathbb{E} \left[ \|h_{k+1}^i - h_*^i\|^2 \mid F_{k+1} \right] = \mathbb{E} \left[ h_{k+1}^i \mid F_{k+1} \right] - h_*^i \| \mathbb{E} \left[ \|h_{k+1}^i - h_*^i\|^2 \mid F_{k+1} \right]
$$

and now with Lemma S6

$$
\mathbb{E} \left[ \|h_{k+1}^i - h_*^i\|^2 \mid F_{k+1} \right] = (1 - \alpha) h_k^i + \alpha g_{k+1}^i - h_*^i \|^2 + \mathbb{E} \left[ \|h_{k+1}^i - \mathbb{E} h_{k+1}^i \mid F_{k+1} \| \mathbb{E} \right] F_{k+1}^i.
$$

Now recall that $h_{k+1}^i = h_k^i + \alpha \hat{\Delta}_k^i$ and $\mathbb{E} \left[ \hat{\Delta}_k^i \mid F_{k+1} \right] = \hat{\Delta}_k^i$,

$$
\mathbb{E} \left[ \|h_{k+1}^i - h_*^i\|^2 \mid F_{k+1} \right] = (1 - \alpha) (h_k^i - h_*^i) + \alpha (g_{k+1}^i - h_*^i) \|^2 + \alpha^2 \mathbb{E} \left[ \|\hat{\Delta}_k - \hat{\Delta}_k\|^2 \mid F_{k+1} \right].
$$

Using Lemma S2 of Appendix D.1 and Proposition S4

$$
\mathbb{E} \left[ \|h_{k+1}^i - h_*^i\|^2 \mid F_{k+1} \right] \leq (1 - \alpha) \|h_k^i - h_*^i\|^2 + \alpha \|g_{k+1}^i - h_*^i\|^2 - \alpha (1 - \alpha) \|h_k^i - g_{k+1}^i\|^2 + \alpha^2 \|\omega_{up}\|^2 \|\Delta_k\|^2.
$$

Because $h_k^i - g_{k+1}^i = \Delta_k$:

$$
\mathbb{E} \left[ \|h_{k+1}^i - h_*^i\|^2 \mid F_{k+1} \right] \leq (1 - \alpha) \|h_k^i - h_*^i\|^2 + \alpha \|g_{k+1}^i - h_*^i\|^2 + \alpha (\alpha \omega_{up} + 1) \|\Delta_k\|^2,
$$

and using Lemma S3

$$
\|h_k^i - g_{k+1}^i\| \leq (1 - \alpha) \|h_k^i - h_*^i\|^2 + \alpha \|g_{k+1}^i - h_*^i\|^2 + 2\alpha \alpha \omega_{up}(\alpha \omega_{up} + 1) \|\Delta_k\|^2.
$$

Finally using eq. S5 of Lemma S7 and writing that:

$$
\mathbb{E} \left[ \|g_{k+1} - h_*^i\|^2 \mid H_k \right] = \mathbb{E} \left[ \|g_{k+1} - h_*^i\|^2 \mid F_{k+1} \right] \mid H_k \ (\text{because } H_k \in F_{k+1})
$$


we have:
\[
\mathbb{E} \left[ \| h_{k+1}^i - h^*_i \|^2 \mid \mathcal{H}_k \right] \leq (1 + 2\alpha^2 \omega_{\text{up}}^2 + 2\alpha^2 - 3\alpha) \mathbb{E} \left[ \| h_k^i - h^*_i \|^2 \mid \mathcal{H}_k \right] + 2(2\alpha^2 \omega_{\text{up}}^2 + 2\alpha^2 - \alpha) \mathbb{E} \left[ \| g_{k+1} - g_{k+1,*} \|^2 \mid \mathcal{H}_k \right] + 2 \frac{\sigma^2}{k} (2\alpha^2 (\omega_{\text{up}}^2 + 1) - \alpha),
\]
which conclude the first part of the proof. Now we take the general case with \( \forall i \in [1, N], B^i_k = 0 \) or 1.

**Second part.** Let \( k \in \mathbb{N} \) and let \( i \in [1, N] \).

To resume, if the device participate to the iteration \( k \), we have
\[
\mathbb{E} \left[ \| h_{k+1}^i - h^*_i \|^2 \mid \mathcal{H}_k \right] \leq (1 + T_1) \mathbb{E} \left[ \| h_k^i - h^*_i \|^2 \mid \mathcal{H}_k \right] + 2T_2 \mathbb{E} \left[ \| g_{k+1} - g_{k+1,*} \|^2 \mid \mathcal{H}_k \right] + 2T_3,
\]
ontherwise:
\[
\mathbb{E} \left[ \| h_{k+1}^i - h^*_i \|^2 \mid \mathcal{H}_k \right] = \mathbb{E} \left[ \| h_k^i - h^*_i \|^2 \mid \mathcal{H}_k \right].
\]
In other words, for all \( i \in [1, N] \):
\[
\mathbb{E} \left[ \| h_{k+1}^i - h^*_i \|^2 \mid \mathcal{H}_k \right] \leq (1 + T_1) B_k^i \mathbb{E} \left[ \| h_k^i - h^*_i \|^2 \mid \mathcal{H}_k \right] + 2T_2 B_k^i \mathbb{E} \left[ \| g_{k+1} - g_{k+1,*} \|^2 \mid \mathcal{H}_k \right] + 2T_3 B_k^i + (1 - B_k^i) \mathbb{E} \left[ \| h_k^i - h^*_i \|^2 \mid \mathcal{H}_k \right]
\]
\[
\leq (1 + T_1 B_k^i) \mathbb{E} \left[ \| h_k^i - h^*_i \|^2 \mid \mathcal{H}_k \right] + 2T_2 B_k^i \mathbb{E} \left[ \| g_{k+1} - g_{k+1,*} \|^2 \mid \mathcal{H}_k \right] + 2T_3 B_k^i,
\]
Taking expectation w.r.t \( \sigma \)-algebra \( \mathcal{I}_k \) gives the result.

After successfully invoking all previous lemmas, we will finally be able to use co-coercivity. Lemma [S14](#) shows how Assumption 2 is used to do it. After this stage, proof will be continued by applying strong-convexity of \( F \).

**Lemma S14 (Applying co-coercivity).** This lemma shows how to apply co-coercivity on stochastic gradients.
\[
\forall k \in \mathbb{N}, \quad \frac{1}{N} \sum_{i=1}^{N} \mathbb{E} \left[ \| g_{k+1}^i - g_{k+1,*} \|^2 \mid \mathcal{H}_k \right] \leq L \langle \nabla F(w_k) \mid w_k - w_* \rangle.
\]

**Proof.** Let \( k \in \mathbb{N} \).
\[
\frac{1}{N} \sum_{i=1}^{N} \mathbb{E} \left[ \| g_{k+1}^i - g_{k+1,*} \|^2 \mid \mathcal{H}_k \right] \leq \frac{1}{N} \sum_{i=1}^{N} L \langle \mathbb{E} \left[ g_{k+1}^i - g_{k+1,*} \mid \mathcal{H}_k \right] \mid w_k - w_* \rangle \text{ using Assumption 2}
\]
\[
\leq L \left( \frac{1}{N} \sum_{i=1}^{N} \mathbb{E} \left[ g_{k+1}^i - g_{k+1,*} \mid \mathcal{H}_k \right] \mid w_k - w_* \right)
\]
\[
\leq L \left( \frac{1}{N} \sum_{i=1}^{N} \nabla F_i(w_k) - \nabla F_i(w_* \mid w_k - w_* \right).
\]
E PROOFS OF THEOREMS

In this section we give demonstrations of all our theorems, that is to say, first the proofs of Theorems S5 and S6 from which flow Theorem 1. Their demonstration sketch is drawn from Mishchenko et al. [2019]. And in a second time, we give a complete demonstration of theorems stated in the main paper: Theorems 2 and 3.

For the sake of demonstration, we define a Lyapunov function $V_k$ as in Mishchenko et al. [2019], Liu et al. [2020], with $k$ in $[1, K]$:

$$V_k = \|w_k - w_*\|^2 + 2\gamma^2 2C_1 \frac{1}{N} \sum_{i=1}^{N} \|h^i_k - h^i_*\|^2 .$$

The Lyapunov function is defined combining two terms:

1. the distance from parameter $w_k$ to optimal parameter $w_*$
2. the memory term, the distance between the next element prediction $h^i_k$ and the true gradient $h^i_* = \nabla F_i(w_*)$.

The aim is to proof that this function is a $(1 - \gamma \mu)$ contraction for each variant of Artemis, and also when using Polyak-Ruppert averaging. To show that it’s a contraction, we need three stages:

1. we develop the update schema defined in eq. $[3]$ to get a first bound on $\|w_k - w_*\|^2$
2. we find a recurrence over the memory term $\|h^i_k - h^i_*\|^2$
3. and finally we combines the two equations to obtain the expected contraction using co-coercivity and strong convexity.

E.1 PROOF OF MAIN THEOREM FOR ARTEMIS - VARIANT WITHOUT MEMORY

Theorem S5 (Unidirectional or bidirectional compression without memory). Considering that Assumptions 1 to 6 hold. Taking $\gamma$ such that

$$ \gamma \leq \frac{pN}{L(\omega_c^{down} + 1) (pN + 2(\omega_c^{up} + 1))} ,$$

then running Artemis with $\alpha = 0$ (i.e without memory), we have for all $k$ in $\mathbb{N}$:

$$E \|w_{k+1} - w_*\|^2 \leq (1 - \gamma \mu)^{k+1} \|w_0 - w_*\|^2 + 2\gamma^{k+1} E \frac{p}{\mu pN} ,$$

with $E = (\omega_c^{down} + 1) \left((\omega_c^{up} + 1) \frac{\sigma^2}{\beta} + (\omega_c^{up} + 1 - p)B^2 \right)$. In the case of unidirectional compression (resp. no compression), we have $\omega_c^{down} = 0$ (resp. $\omega_c^{up/down} = 0$).

Proof. In the case of variant of Artemis with $\alpha = 0$, we don’t have any memory term, thus $p = 0$ and we don’t need to use the Lyapunov function.

Let $k$ in $\mathbb{N}$, we start by writing that by definition of eq. $[3]$:

$$\|w_{k+1} - w_*\|^2 = \|w_k - \gamma \Omega_{k+1,S_k} - w_*\|^2 = \|w_k - w_*\|^2 - 2\gamma \langle \Omega_{k+1,S_k} | w_k - w_* \rangle + \gamma^2 \|\Omega_{k+1,S_k}\|^2 ,$$

with $\Omega_{k+1,S_k} = C_{down} \left(\frac{1}{N} \sum_{i=1}^{N} \hat{g}^i_{k+1}\right)$. First, we have $E \left[\Omega_{k+1,S_k} | \sigma(\mathcal{G}_{k+1} \cup B_k)\right] = \tilde{g}_{k+1,S_k}$; secondly considering that:

$$E \left[\|\Omega_{k+1,S_k}\|^2 | \sigma(\mathcal{G}_{k+1} \cup B_k)\right] = V(\Omega_{k+1,S_k}) + E \left[\Omega_{k+1,S_k} | \sigma(\mathcal{G}_{k+1} \cup B_k)\right] = (\omega_c^{down} + 1) \|\tilde{g}_{k+1,S_k}\|^2 ,$$

it leads to:

$$E \left[\|w_{k+1} - w_*\|^2 | \sigma(\mathcal{G}_{k+1} \cup B_k)\right] = E \left[\|w_k - w_*\|^2 | \sigma(\mathcal{G}_{k+1} \cup B_k)\right] - 2\gamma \langle \tilde{g}_{k+1,S_k} | w_k - w_* \rangle + \gamma^2 (\omega_c^{down} + 1) \|\tilde{g}_{k+1,S_k}\|^2 .$$
Now, if we take expectation w.r.t $\sigma$-algebra $\mathfrak{I}_k \subset \sigma(\mathcal{G}_{k+1} \cup \mathcal{B}_k)$ (the inclusion is true because $\mathcal{G}_{k+1}$ contains $\mathcal{B}_{k-1}$); with use of Propositions S2, S3 and S7 and Lemma S9 we obtain:

\[
\mathbb{E} \left[ \|w_{k+1} - w^*\|^2 \mid \mathfrak{I}_k \right] = \mathbb{E} \left[ \|w_k - w^*\|^2 \mid \mathfrak{I}_k \right] - 2\gamma \langle \nabla F(w_k) \mid w_k - w^* \rangle + \gamma^2(\omega_{c}^{up} + 1) \left( \frac{(1 - p)}{pN^2} \sum_{i=0}^{N} \mathbb{E} \left[ \|\hat{g}_{k+1}^i\|^2 \mid \mathfrak{I}_k \right] \right) + \gamma^2(\omega_{c}^{down} + 1) \mathbb{E} \left[ \|\hat{g}_{k+1}\|^2 \mid \mathfrak{I}_k \right].
\]

For sake of clarity we temporarily note: $\Pi = \frac{(1 - p)}{pN^2} \sum_{i=0}^{N} \mathbb{E} \left[ \|\hat{g}_{k+1}^i\|^2 \mid \mathfrak{I}_k \right] + \mathbb{E} \left[ \|\hat{g}_{k+1}\|^2 \mid \mathfrak{I}_k \right]$. Recall that in fact, $\Pi$ is equal to $\mathbb{E} \left[ \|\hat{g}_{k+1, s_k}\|^2 \mid \mathfrak{I}_k \right]$.

Using Lemmas S10 and S11 we have:

\[
\mathbb{E} [\Pi \mid \mathfrak{I}_k] \leq \frac{(\omega_{c}^{up} + 1 - p)}{pN^2} \sum_{i=0}^{N} \mathbb{E} \left[ \|\hat{g}_{k+1}^i\|^2 \mid \mathfrak{I}_k \right] + \frac{1}{N} \sum_{i=0}^{N} \mathbb{E} \left[ \|\hat{g}_{k+1}^i - h_i^*\|^2 \mid \mathfrak{I}_k \right] + L \langle \nabla F(w_k) \mid w_k - w^* \rangle .
\]

Let's introducing the noise at optimal point $w^*$ with the two equations of Lemma S7:

\[
\mathbb{E} [\Pi \mid \mathfrak{I}_k] \leq \frac{(\omega_{c}^{up} + 1 - p)}{pN^2} \sum_{i=1}^{N} 2 \left( \mathbb{E} \left[ \|g_{k+1}^i - g_{k+1, i, s}\|^2 \mid \mathfrak{I}_k \right] + \frac{\sigma^2}{b} + B^2 \right) + \frac{1}{N^2} \sum_{i=1}^{N} 2 \left( \mathbb{E} \left[ \|g_{k+1}^i - g_{k+1, i, s}\|^2 \mid \mathfrak{I}_k \right] + \frac{\sigma^2}{b} \right) + L \langle \nabla F(w_k) \mid w_k - w^* \rangle .
\]

\[
\leq \frac{2(\omega_{c}^{up} + 1)}{pN^2} \sum_{i=1}^{N} \mathbb{E} \left[ \|\hat{g}_{k+1} - \hat{g}_{k+1, i, s}\|^2 \mid \mathfrak{I}_k \right] + L \langle \nabla F(w_k) \mid w_k - w^* \rangle + \frac{2 \times \left( (\omega_{c}^{up} + 1) \frac{\sigma^2}{b} + \omega_{c}^{up} + 1 - p \right) B^2}{pN} .
\]
Invoking cocoercivity (Assumption 2):

\[
E\left[\Pi \mid I_k\right] \leq \frac{2(\omega_c^{up} + 1)}{pN^2} \sum_{i=1}^{N} E\left[ L \left( g_{k+1} - g_{k+1,*} \mid w_k - w_* \right) \mid I_k \right] + L \left( \nabla F(w_k) \mid w_k - w_* \right) + \frac{2 \times (\omega_c^{up} + 1) \sigma_b^2 + (\omega_c^{up} + 1 - p)B^2}{pN} \leq \frac{2(\omega_c^{up} + 1)L}{pN} \left( \nabla F(w_k) \mid w_k - w_* \right) + L \left( \nabla F(w_k) \mid w_k - w_* \right) + \frac{2 \times (\omega_c^{up} + 1) \sigma_b^2 + (\omega_c^{up} + 1 - p)B^2}{pN}.
\]

Finally, we can inject eq. (S10) in eq. (S9) to obtain:

\[
E \left[ \|w_{k+1} - w_*\|^2 \mid I_k\right] \leq \|w_k - w_*\|^2 - 2\gamma \left( 1 - \gamma L(\omega_c^{dwn} + 1)(\omega_c^{up} + 1) - \gamma L(\omega_c^{dwn} + 1) \right) \|w_k - w_*\|^2 + 2 \times (\omega_c^{dwn} + 1) \left( \omega_c^{up} + 1 \sigma_b^2 + (\omega_c^{up} + 1 - p)B^2 \right) \leq \frac{2 \times (\omega_c^{dwn} + 1) \left( \omega_c^{up} + 1 \sigma_b^2 + (\omega_c^{up} + 1 - p)B^2 \right)}{pN}.
\]

We need \(1 - \gamma L(\omega_c^{dwn} + 1)(\omega_c^{up} + 1) - \gamma L(\omega_c^{dwn} + 1) \geq 0\) in order to further apply strong convexity. This condition is equivalent to:

\[
\gamma \leq \frac{2pN}{L(\omega_c^{dwn} + 1) (pN + 2(\omega_c^{up} + 1))}.
\]

Finally, using strong convexity of \(F\) (Assumption 1), we rewrite Equation (S11):

\[
E \left[ \|w_{k+1} - w_*\|^2 \mid I_k\right] \leq \|w_k - w_*\|^2 - 2\mu \left( 1 - \gamma L(\omega_c^{dwn} + 1)(\omega_c^{up} + 1) - \gamma L(\omega_c^{dwn} + 1) \right) \|w_k - w_*\|^2 + 2 \gamma^2 \frac{E}{pN} \leq \left( 1 - 2\mu \left( 1 - \gamma L(\omega_c^{dwn} + 1)(\omega_c^{up} + 1) - \gamma L(\omega_c^{dwn} + 1) \right) \right) \|w_k - w_*\|^2 + 2 \gamma^2 \frac{E}{pN}.
\]

To guarantee a convergence in \((1 - \gamma\mu)\), we need:

\[
\frac{1}{2} \geq \gamma \frac{L(\omega_c^{dwn} + 1)(\omega_c^{up} + 1)}{pN} + \frac{L(\omega_c^{dwn} + 1)}{2} = \gamma \leq \frac{pN}{L(\omega_c^{dwn} + 1) (pN + 2(\omega_c^{up} + 1))},
\]

which is stronger than the condition obtained to correctly apply strong convexity. Then we are allowed to write:

\[
E \left[ \|w_{k+1} - w_*\|^2 \leq (1 - \gamma\mu)E \|w_k - w_*\|^2 + 2 \gamma^2 \frac{E}{pN} \right. \longleftrightarrow \left. E \left[ \|w_{k+1} - w_*\|^2 \leq (1 - \gamma\mu)^{k+1}E \|w_0 - w_*\|^2 + 2 \gamma^2 \frac{E}{pN} \times \frac{1 - (1 - \gamma\mu)^{k+1}}{\gamma\mu} \right. \longleftrightarrow \left. E \left[ \|w_{k+1} - w_*\|^2 \leq (1 - \gamma\mu)^{k+1} \|w_0 - w_*\|^2 + 2 \gamma \frac{E}{\mu pN} \right.,
\]

\]
and the proof is complete. 

\[\square\]

E.2 PROOF OF MAIN THEOREM FOR ARTEMIS - VARIANT WITH MEMORY

Theorem S6 (Unidirectional or bidirectional compression with memory). Considering that Assumptions 1 to 6 hold. We use \(w_\star\) to indicate the optimal parameter such that \(\nabla F(w_\star) = 0\), and we note \(h_\star^i = \nabla F_i(w_\star)\). We define the Lyapunov function:

\[V_k = \|w_k - w_\star\|^2 + 2\gamma^2 C \frac{1}{N} \sum_{i=1}^{N} \|h_k^i - h_\star^i\|^2.\]

We defined \(C \in \mathbb{R}^+\), such that:

\[
\frac{(\omega_c^\text{dwn} + 1) (\omega_c^\text{up} + 1) - 1}{\alpha p (3 - 2\alpha (\omega_c^\text{up} + 1))} \leq C \leq \frac{N - \gamma L (\omega_c^\text{dwn} + 1) \left( N + \frac{4(\omega_c^\text{up} + 1)}{p} - 2 \right)}{4\gamma L p \alpha (2\alpha (\omega_c^\text{up} + 1) - 1)}. \tag{S12}
\]

Then, using Artemis with a memory mechanism (\(\alpha \neq 0\)), the convergence of the algorithm is guaranteed if:

\[
\begin{align*}
\frac{1}{2(\omega_c^\text{up} + 1)} & \leq \alpha < \min \left\{ \frac{3}{2(\omega_c^\text{up} + 1)}, \frac{3N - \gamma L (\omega_c^\text{dwn} + 1) \left( 3N + \frac{8(\omega_c^\text{up} + 1)}{p} - 2 \right)}{2(\omega_c^\text{up} + 1)(N - \gamma L (\omega_c^\text{dwn} + 1)(N + 2))} \right\} \\
\gamma & < \min \left\{ \frac{(\omega_c^\text{dwn} + 1) \left( 1 + \frac{2}{Np} \right) L}{(\omega_c^\text{dwn} + 1) \left( N + \frac{4(\omega_c^\text{up} + 1)}{p} - 2 \right) L}, \frac{3}{(\omega_c^\text{dwn} + 1) \left( 3 + \frac{8(\omega_c^\text{up} - 1) - 2p}{Np} \right) L} \right\}. \tag{S13}
\end{align*}
\]

And we have a bound for the Lyapunov function:

\[\mathbb{E} V_{k+1} \leq (1 - \gamma p)^{k+1} \left( \|w_0 - w_\star\|^2 + 2C\gamma^2 B^2 \right) + 2\gamma \frac{E}{\mu N}, \]

with

\[E = \frac{\sigma^2}{b} \left( (\omega_c^\text{dwn} + 1) \left( \frac{2(\omega_c^\text{up} + 1)}{p} - 1 \right) + 2p C (2\alpha^2 (\omega_c^\text{up} + 1) - \alpha) \right).\]

In the case of unidirectional compression (resp. no compression), we have \(\omega_c^\text{dwn} = 0\) (resp. \(\omega_c^\text{up/dwn} = 0\)).

Proof. Let \(k \in [1, K]\), by definition of the update schema in the partial-participation setting: \(w_{k+1} = w_k - \gamma \Omega_{k+1, S_k}\), with \(\Omega_{k+1, S_k} = C_{\text{dwn}} \left( \frac{1}{p N} \sum_{i \in S_k} (\Delta_k^i + h_k^i) \right)\), thus:

\[
\|w_{k+1} - w_\star\|^2 = \|w_k - w_\star + \gamma \Omega_{k+1, S_k}\|^2 \\
= \|w_k - w_\star\|^2 - 2\gamma \langle \Omega_{k+1, S_k} \mid w_k - w_\star \rangle + \gamma^2 \|\Omega_{k+1, S_k}\|^2.
\]

First, we have \(\mathbb{E} [\Omega_{k+1, S_k} \mid \sigma(\mathcal{G}_{k+1} \cup \mathcal{B}_k)] = \hat{g}_{k+1, S_k}\); secondly considering that:

\[
\mathbb{E} \left[ \|\Omega_{k+1, S_k}\|^2 \mid \sigma(\mathcal{G}_{k+1} \cup \mathcal{B}_k) \right] = \mathbb{V}(\Omega_{k+1, S_k}) + \mathbb{E} [\Omega_{k+1, S_k} \mid \sigma(\mathcal{G}_{k+1} \cup \mathcal{B}_k)]^2 = (\omega_c^\text{dwn} + 1) \|\hat{g}_{k+1, S_k}\|^2,
\]

In the case of unidirectional compression (resp. no compression), we have \(\omega_c^\text{dwn} = 0\) (resp. \(\omega_c^\text{up/dwn} = 0\)).
it leads to:

\[
\mathbb{E} \left[ \|w_{k+1} - w_*\|^2 \mid \sigma(G_{k+1} \cup B_k) \right] = \mathbb{E} \left[ \|w_k - w_*\|^2 \mid \sigma(G_{k+1} \cup B_k) \right] - 2\gamma \langle \hat{g}_{k+1} \mid w_k - w_* \rangle \\
+ \gamma^2 (\omega_c^{\text{down}} + 1) \|\hat{g}_{k+1} \mid w_k - w_* \|^2.
\]

we can invoke Lemma S12 with the \(\sigma\)-algebras \(I_k\):

\[
\mathbb{E} \left[ \|w_{k+1} - w_*\|^2 \mid I_k \right] \leq \|w_k - w_*\|^2 - 2\gamma \mathbb{E} \left[ \langle \hat{g}_{k+1} \mid w_k - w_* \rangle \mid I_k \right] \\
+ 2\gamma^2 (\omega_c^{\text{down}} + 1) \left( \frac{2(\omega_c^{\text{up}} + 1)}{p} - 1 \right) \frac{1}{N^2} \sum_{i=1}^N \mathbb{E} \left[ \|g_i^{k+1} - g_i^{k_1,s} \|^2 \mid I_k \right] \\
+ \gamma^2 (\omega_c^{\text{down}} + 1) \left( \frac{2(\omega_c^{\text{up}} + 1)}{p} - 2 \right) \frac{1}{N^2} \sum_{i=1}^N \mathbb{E} \left[ \|h_i^{k} - h_i^{s} \|^2 \mid I_k \right] \\
+ \gamma^2 (\omega_c^{\text{down}} + 1) L \langle \nabla F(w_k) \mid w_k - w_* \rangle \\
+ \frac{2\sigma_s}{Nb} \times \gamma^2 (\omega_c^{\text{down}} + 1) \left( \frac{2(\omega_c^{\text{up}} + 1)}{p} - 1 \right). 
\]  

(S14)

Note that in the case of unidirectional compression, we have \(\Omega_{k+1} = \hat{g}_{k+1}\), and the steps above are more straightforward. Recall that according to Lemma S13 (and taking the sum), we have:

\[
\frac{1}{N^2} \sum_{i=1}^N \mathbb{E} \left[ \|h_i^{k+1} - h_i^{s} \|^2 \mid I_k \right] \\
\leq (1 + p(2\alpha^2 \omega_c^{\text{up}} + 2\alpha^2 - 3\alpha)) \frac{1}{N^2} \sum_{i=1}^N \mathbb{E} \left[ \|h_i^k - h_i^s \|^2 \mid I_k \right] \\
+ 2p(2\alpha^2 \omega_c^{\text{up}} + 2\alpha^2 - \alpha) \frac{1}{N^2} \sum_{i=1}^N \mathbb{E} \left[ \|g_i^k - g_i^{k_1,s} \|^2 \mid I_k \right] \\
+ \frac{2p \sigma_s^2}{Nb} (2\alpha^2 \omega_c^{\text{up}} + 1 - \alpha) 
\]  

(S15)

With a linear combination S14 + \(2\gamma^2 C\) S15:

\[
\mathbb{E} \left[ \|w_{k+1} - w_*\|^2 \mid I_k \right] + \gamma^2 C \frac{1}{N^2} \sum_{i=1}^N \mathbb{E} \left[ \|h_i^{k+1} - h_i^s \|^2 \mid I_k \right] \\
\leq \|w_k - w_*\|^2 - 2\gamma \mathbb{E} \left[ \langle \hat{g}_{k+1} \mid w_k - w_* \rangle \mid I_k \right] \\
+ 2\gamma^2 \left( \omega_c^{\text{down}} + 1 \right) \left( \frac{2(\omega_c^{\text{up}} + 1)}{p} - 1 \right) + 2pC(2\alpha^2 \omega_c^{\text{up}} + 2\alpha^2 - \alpha) \\
\times \frac{1}{N^2} \sum_{i=1}^N \mathbb{E} \left[ \|g_i^{k+1} - g_i^{k_1,s} \|^2 \mid I_k \right] \\
+ 2\gamma^2 C \left( \frac{\omega_c^{\text{down}} + 1}{2C} \left( \frac{2(\omega_c^{\text{up}} + 1)}{p} - 2 \right) + 1 + p(2\alpha^2 \omega_c^{\text{up}} + 2\alpha^2 - 3\alpha) \right) \frac{1}{N^2} \sum_{i=1}^N \mathbb{E} \left[ \|h_i^k - h_i^s \|^2 \mid I_k \right] \\
+ \gamma^2 (\omega_c^{\text{down}} + 1) L \langle \nabla F(w_k) \mid w_k - w_* \rangle \\
+ \frac{2\gamma^2}{N} \left( \frac{\sigma_s^2}{b} \left( \omega_c^{\text{down}} + 1 \right) \left( \frac{2(\omega_c^{\text{up}} + 1)}{p} - 1 \right) + 2pC (2\alpha^2 (\omega_c^{\text{up}} + 1) - \alpha) \right). 
\]  

(S15)
Because $\mathcal{H}_k \subset \mathcal{F}_{k+1}$, because $\tilde{g}_{k+1}$ is independent of $B_k$, and with Propositions S2 and S3

$$E[\tilde{g}_{k+1} | \mathcal{I}_k] = E[E[\tilde{g}_{k+1} | \mathcal{F}_{k+1}] | \mathcal{H}_k] = E[g_{k+1} | \mathcal{H}_k] = \nabla F(w_k).$$

We transform $\|g_{k+1} - \hat{g}_{k+1, *}\|$ applying co-coercivity (Lemma S14):

$$E V_{k+1} \leq \|w_k - w_*\|^2 - 2\gamma \left(1 - \gamma L \left(\frac{\omega_{c, \text{dnw}} + 1}{2} + \frac{A_C}{N}\right)\right) \langle \nabla F(w_k) | w_k - w_*\rangle$$

$$+ 2\gamma^2 C D C - \frac{1}{N^2} \sum_{i=1}^N E \left[\left\|h^i_k - h^i_*\right\|^2 \mathcal{I}_k\right] + \frac{2\gamma^2 E}{N}.$$ (S16)

Now, the goal is to apply strong convexity of $F$ (Assumption 1) using the inequality presented in Lemma S4. But then we must have:

$$1 - \gamma L \left(\frac{\omega_{c, \text{dnw}} + 1}{2} + \frac{A_C}{N}\right) \geq 0,$$

However, in order to later obtain a convergence in $(1 - \gamma \mu)$, we will use a stronger condition and, instead, state that we need:

$$\gamma L \left(\frac{\omega_{c, \text{dnw}} + 1}{2} + \frac{A_C}{N}\right) \leq \frac{1}{2}$$

$$\iff A_C \leq \frac{(1 - \gamma L (\omega_{c, \text{dnw}} + 1))N}{2\gamma L}$$

$$\iff \left(\omega_{c, \text{dnw}} + 1\right) \left(\frac{2(\omega_{c, \text{up}} + 1)}{p} - 1\right) + 2pC(2\alpha^2 \omega_{c, \text{up}} + 2\alpha^2 - \alpha) \leq \frac{(1 - \gamma L (\omega_{c, \text{dnw}} + 1))N}{2\gamma L}$$

$$\iff C \leq \frac{N - \gamma L (\omega_{c, \text{dnw}} + 1) \left(N + \frac{4(\omega_{c, \text{up}} + 1)}{p} - 2\right)}{4\gamma L p \alpha (2\alpha (\omega_{c, \text{up}} + 1) - 1)}.$$ (S17)

This holds only if the numerator and the denominator are positive:

$$\begin{cases}
N - \gamma L (\omega_{c, \text{dnw}} + 1) \left(N + \frac{4(\omega_{c, \text{up}} + 1)}{p} - 2\right) > 0 \iff \gamma < \frac{N}{(\omega_{c, \text{dnw}} + 1) \left(N + \frac{4(\omega_{c, \text{up}} + 1)}{p} - 2\right) L}
\end{cases}
$$

Strong convexity is applied, and we obtain:

$$E V_{k+1} \leq \left(1 - 2\gamma \mu \left(1 - \gamma L (\omega_{c, \text{dnw}} + 1) - \frac{\gamma L A_C}{N}\right)\right) \|w_k - w_*\|^2$$

$$+ 2\gamma^2 C D C - \frac{1}{N} \sum_{i=1}^N E \left[\left\|h^i_k - h^i_*\right\|^2 \mathcal{I}_k\right] + \frac{2\gamma^2 E}{N}.$$ (S17)

To guarantee a $(1 - \gamma \mu)$ convergence, constants must verify:

$$\begin{cases}
\gamma L (\omega_{c, \text{dnw}} + 1) - \frac{\gamma L A_C}{N} \leq \frac{1}{2} \\
D_C \leq 1 - \gamma \mu \iff \frac{\omega_{c, \text{dnw}} + 1}{C} \leq p(3\alpha - 2\alpha^2 \omega_{c, \text{up}} - 2\alpha) - \gamma \mu$$
\end{cases}$$

$$\iff C \geq \frac{(\omega_{c, \text{dnw}} + 1) \left(\frac{\omega_{c, \text{up}} + 1}{p} - 1\right)}{\alpha p (3 - 2\alpha (\omega_{c, \text{up}} + 1)) - \gamma \mu}.$$
In the following we will consider that $\frac{\gamma_{\mu}}{\alpha} = o(1)$ which is possible because $\alpha$ is independent of $\omega^{\text{up}}_C$ and $\omega^{\text{down}}_C$ and it result to:

$$\alpha p (3 - 2 \alpha (\omega^{\text{up}}_C + 1)) - \gamma_{\mu} \sim o(1)$$

Thus, the condition on $C$ becomes:

$$\frac{(\omega^{\text{down}}_C + 1) \left( \frac{\omega^{\text{up}}_C + 1}{p} - 1 \right)}{\alpha p (3 - 2 \alpha (\omega^{\text{up}}_C + 1))} \leq C,$$

which is correct only if $\alpha \leq \frac{3}{2(\omega^{\text{up}}_C + 1)}$.

And we obtain the following conditions on $C$:

$$\frac{(\omega^{\text{down}}_C + 1) \left( \frac{\omega^{\text{up}}_C + 1}{p} - 1 \right)}{\alpha p (3 - 2 \alpha (\omega^{\text{up}}_C + 1))} \leq C \leq \frac{N - \gamma_L (\omega^{\text{down}}_C + 1) \left( N + \frac{4(\omega^{\text{up}}_C + 1)}{p} - 2 \right)}{4\gamma L p \alpha (2 \alpha (\omega^{\text{up}}_C + 1) - 1)}.$$ 

It follows, that the above interval is not empty if:

$$\frac{(\omega^{\text{down}}_C + 1) \left( \frac{\omega^{\text{up}}_C + 1}{p} - 1 \right)}{\alpha p (3 - 2 \alpha (\omega^{\text{up}}_C + 1))} \leq \frac{N - \gamma_L (\omega^{\text{down}}_C + 1) \left( N + \frac{4(\omega^{\text{up}}_C + 1)}{p} - 2 \right)}{4\gamma L p \alpha (2 \alpha (\omega^{\text{up}}_C + 1) - 1)}.$$ 

For sake of clarity we denote momentarily $\tilde{\gamma} = (\omega^{\text{down}}_C + 1)\gamma_L$ and $\Pi = \frac{\omega^{\text{up}}_C + 1}{p}$, hence the below condition becomes:

$$8\alpha (\omega^{\text{up}}_C + 1)(\Pi - 1)\tilde{\gamma} - 4(\Pi - 1)\tilde{\gamma} \leq 3N - 3\tilde{\gamma} (N + 2 + 4\Pi - 2)$$

$$- 2\alpha (\omega^{\text{up}}_C + 1)N + 2\tilde{\gamma} (\omega^{\text{up}}_C + 1)(N + 4\Pi - 2)$$

$$\Leftrightarrow 2\alpha (\omega^{\text{up}}_C + 1)(N - \tilde{\gamma} (N + 2)) \leq 3N - \tilde{\gamma} (3N + 8\Pi - 2)$$

And at the end, we obtain:

$$\alpha \leq \frac{3N - \gamma_L (\omega^{\text{down}}_C + 1) \left( 3N + \frac{8(\omega^{\text{up}}_C + 1)}{p} - 2 \right)}{2(\omega^{\text{up}}_C + 1)(N - \gamma_L (\omega^{\text{down}}_C + 1)(N + 2))}.$$

Again, this implies two conditions on gamma:

$$\begin{cases}
3N - \gamma_L (\omega^{\text{down}}_C + 1) \left( 3N + \frac{8(\omega^{\text{up}}_C + 1)}{p} - 2 \right) > 0 \iff \gamma < \frac{3}{(\omega^{\text{down}}_C + 1) \left( 3 + \frac{8(\omega^{\text{up}}_C + 1) - 2p}{Np} \right) L} \\
N - \gamma_L (\omega^{\text{down}}_C + 1)(N + 2) > 0 \iff \gamma < \frac{1}{(\omega^{\text{down}}_C + 1) \left( 1 + \frac{2}{N} \right) L}
\end{cases}$$

The constant $C$ exists, and from eq. (S17) we are allowed to write:

$$EV_{k+1} \leq (1 - \gamma_{\mu})EV_k + 2\gamma^2 E \frac{N}{N}$$

$$\Leftrightarrow EV_{k+1} \leq (1 - \gamma_{\mu})^k EV_0 + 2\gamma^2 E \frac{N}{N} \times \frac{1 - (1 - \gamma_{\mu})^{k+1}}{\gamma_{\mu}}$$

$$\Rightarrow EV_{k+1} \leq (1 - \gamma_{\mu})^{k+1} EV_0 + 2\gamma E \frac{\mu N}{\mu N}.$$
Because $V_0 = \mathbb{E} \| w_0 - w_* \|^2 + 2\gamma^2 C \frac{1}{N} \sum_{i=0}^{N} \| h_i^* \|^2 \leq \| w_0 - w_* \|^2 + 2C\gamma^2 B^2$ (using Assumption 4), we can write:

$$
\mathbb{E} V_{k+1} = (1 - \gamma \mu)^{k+1} \left( \| w_0 - w_* \|^2 + 2C\gamma^2 B^2 \right) + 2\gamma \frac{E}{\mu N}.
$$

Thus, we highlighted that the Lyapunov function

$$
V_k = \| w_k - w_* \|^2 + 2\gamma^2 C \frac{1}{N} \sum_{i=1}^{N} \| h_k^i - h_*^i \|^2
$$

is a $(1 - \gamma \mu)$ contraction if $C$ is taken in a given interval, with $\gamma$ and $\alpha$ satisfying some conditions. This guarantee the convergence of the Artemis using version 1 or 2 with $\alpha \neq 0$ (algorithm with uni-compression or bi-compression combined with a memory mechanism).

\[\square\]

### E.3 PROOF OF THEOREM - POLYAK-RUPPERT AVERAGING

**Theorem S7** (Unidirectional or bidirectional compression using memory and averaging). We suppose now that $F$ is convex, thus $\mu = 0$ and we consider that Assumptions 2 to 6 hold. We use $w_*$ to indicate the optimal parameter such that $\nabla F(w_*) = 0$, and we note $h_*^i = \nabla F_i(w_*)$. We define the Lyapunov function:

$$
V_k = \| w_k - w_* \|^2 + 2\gamma^2 C \frac{1}{N} \sum_{i=1}^{N} \| h_k^i - h_*^i \|^2.
$$

We defined $C \in \mathbb{R}^+$, such that:

$$(\omega_c^{\text{down}} + 1) \left( \frac{\omega_c^{\text{up}} + 1}{p} - 1 \right) \leq C L \left( \omega_c^{\text{down}} + 1 \right) \left( \frac{N + 4(\omega_c^{\text{up}} + 1)}{p} - 2 \right).
$$

(S18)

Then running variant of Artemis with $\alpha \neq 0$, hence with a memory mechanism, and using Polyak-Ruppert averaging, the convergence of the algorithm is guaranteed if:

$$
\begin{align*}
\frac{1}{2(\omega_c^{\text{up}} + 1)} \leq \alpha &< \min \left\{ \frac{3}{2(\omega_c^{\text{up}} + 1)}, \frac{3N - \gamma L (\omega_c^{\text{down}} + 1) \left( 3N + \frac{8(\omega_c^{\text{up}} + 1)}{p} - 2 \right)}{2(\omega_c^{\text{up}} + 1) (N - \gamma L (\omega_c^{\text{down}} + 1)(N + 2))} \right\} \\
\gamma &< \min \left\{ \frac{1}{(\omega_c^{\text{down}} + 1) \left( 1 + \frac{2}{np} \right) L}, \frac{3}{(\omega_c^{\text{down}} + 1) \left( 3 + \frac{8(\omega_c^{\text{up}} - 1) - 2p}{np} \right) L}, \frac{N + 4(\omega_c^{\text{up}} + 1)}{p} - 2 \right\}.
\end{align*}
$$

(S19)

And we have the following bound:

$$
F \left( \frac{1}{K} \sum_{k=0}^{K} w_k \right) - F(w_*) \leq \frac{\| w_0 - w_* \|^2 + 2C\gamma^2 B^2}{\gamma K} + \frac{2E}{N},
$$

(S20)

with $E = (\omega_c^{\text{down}} + 1) \left( \frac{2(\omega_c^{\text{up}} + 1)}{p} - 1 \right) \frac{\sigma^2}{\gamma} + 2pC (2\alpha^2 (\omega_c^{\text{up}} + 1) - \alpha) \frac{\sigma^2}{\gamma}$.

Equation (S20) can be written as in Theorem 2 if we take $\gamma = \min \left( \sqrt{\frac{N\sigma^2}{2E}}, \gamma_{\text{max}} \right)$, where $\gamma_{\text{max}}$ is the maximal possible value of $\gamma$ as precised by Equation (S19):

$$
F \left( \frac{1}{K} \sum_{k=0}^{K} w_k \right) - F(w_*) \leq 2 \max \left( \sqrt{\frac{2\delta^2 E}{NK}}, \frac{\delta_0^2}{\gamma_{\text{max}} K} \right) + \frac{2\gamma_{\text{max}} C B^2}{K}.
$$
Proof. Starting from eq. (S16) from the proof of Theorem S6:

\[
\mathbb{E}V_{k+1} \leq \|w_k - w_*\|^2 - 2\gamma \left( 1 - \gamma L \left( \frac{\omega_c^{\text{dwn}} + 1}{2} + \frac{A_C}{N} \right) \right) \langle \nabla F(w_k) \mid w_k - w_* \rangle \\
+ 2\gamma^2 CD_C \frac{1}{N^2} \sum_{i=1}^{N} \mathbb{E} \left[ \|h^i_k - h^i_*\|^2 \mid I_k \right] + \frac{2\gamma^2}{N} E.
\]

But this time, instead of applying strong convexity of \( F \), we apply convexity (Assumption 1 but with \( \mu = 0 \)):

\[
\mathbb{E}V_{k+1} \leq \|w_k - w_*\|^2 - 2\gamma \left( 1 - \gamma L \left( \frac{\omega_c^{\text{dwn}} + 1}{2} + \frac{A_C}{N} \right) \right) (F(w_k) - F(w_*)) \\
+ 2\gamma^2 CD_C \frac{1}{N^2} \sum_{i=1}^{N} \mathbb{E} \left[ \|h^i_k - h^i_*\|^2 \mid I_k \right] + \frac{2\gamma^2}{N} E.
\] (S21)

As in Theorem S6, we want:

\[
\gamma L \left( \frac{\omega_c^{\text{dwn}} + 1}{2} + \frac{A_C}{N} \right) \leq \frac{1}{2} \\
\iff C \leq \frac{N - \gamma L(\omega_c^{\text{dwn}} + 1) \left( N + \frac{4(\omega_c^{\text{up}} + 1)}{p} - 2 \right)}{4\gamma L \alpha (2\alpha(\omega_c^{\text{up}} + 1) - 1)}.
\] (S22)

which holds only if the numerator and the denominator are positive:

\[
\begin{cases}
N - \gamma L(\omega_c^{\text{dwn}} + 1) \left( N + \frac{4(\omega_c^{\text{up}} + 1)}{p} - 2 \right) > 0 \iff \gamma < \frac{N}{(\omega_c^{\text{dwn}} + 1) \left( N + \frac{4(\omega_c^{\text{up}} + 1)}{p} - 2 \right)} L \\
2\alpha(\omega_c^{\text{up}} + 1) - 1 \leq 0 \iff \alpha \geq \frac{1}{2(\omega_c^{\text{up}} + 1)}.
\end{cases}
\]

Returning to eq. (S21), taking benefit of eq. (S22) and passing \( F(w_k) - F(w_*) \) on the left side gives:

\[
\gamma (F(w_k) - F(w_*)) \leq \|w_k - w_*\|^2 + 2\gamma^2 CD_C \frac{1}{N^2} \sum_{i=1}^{N} \mathbb{E} \left[ \|h^i_k - h^i_*\|^2 \mid I_k \right] - \mathbb{E}V_{k+1} + \frac{2\gamma^2}{N} E.
\]

If \( D_C \leq 1 \), then:

\[
\gamma (F(w_k) - F(w_*)) \leq \mathbb{E}V_k - \mathbb{E}V_{k+1} + \frac{2\gamma^2}{N} E,
\]

summing over all \( K \) iterations:

\[
\gamma \left( \frac{1}{K} \sum_{k=0}^{K} F(w_k) - F(w_*) \right) \leq \frac{1}{K} \sum_{k=0}^{K} \left( \mathbb{E}V_k - \mathbb{E}V_{k+1} + 2\gamma^2 \frac{E}{N} \right) \\
\leq \frac{\mathbb{E}V_0 - \mathbb{E}V_{K+1}}{K} + 2\gamma^2 \frac{E}{N} \quad \text{because } E \text{ is independent of } K.
\]

Thus, by convexity:

\[
F \left( \frac{1}{K} \sum_{k=0}^{K} w_k \right) - F(w_*) \leq \frac{1}{K} \sum_{k=0}^{K} F(w_k) - F(w_*) \leq \frac{V_0}{\gamma K} + 2\gamma \frac{E}{N}.
\]
Last step is to extract conditions over $\gamma$ and $\alpha$ from requirement $D_C \leq 1$:

$$D_C < 1 \iff \frac{\omega_{\text{dwn}} + 1}{2C} \left( \frac{2(\omega_{\text{up}} + 1)}{p} - 2 \right) < 3\alpha - 2\alpha^2 \omega_{\text{up}} - 2\alpha \iff C > \frac{\omega_{\text{dwn}} + 1}{\alpha p (3 - 2\alpha(\omega_{\text{up}} + 1))}.$$

and the second inequality is correct only if $\alpha \leq \frac{3}{2(\omega_{\text{dwn}} + 1)}$.

From this development follows the following conditions on $p$, which are equivalent to those obtain in Theorem S6:

$$\frac{(\omega_{\text{dwn}} + 1)(\omega_{\text{up}} + 1)}{\alpha p (3 - 2\alpha(\omega_{\text{up}} + 1))} \leq C \leq \frac{N - \gamma L(\omega_{\text{dwn}} + 1) \left( N + \frac{4(\omega_{\text{up}} + 1)}{p} - 2 \right)}{4\gamma L p \alpha (2\alpha(\omega_{\text{up}} + 1) - 1)}.$$

This interval is not empty:

$$\frac{(\omega_{\text{dwn}} + 1)(\omega_{\text{up}} + 1)}{\alpha p (3 - 2\alpha(\omega_{\text{up}} + 1))} \leq C \iff 3N - \gamma L(\omega_{\text{dwn}} + 1) \left( 3N + \frac{8(\omega_{\text{up}} + 1)}{p} - 2 \right) \leq \frac{N - \gamma L(\omega_{\text{dwn}} + 1) \left( N + \frac{4(\omega_{\text{up}} + 1)}{p} - 2 \right)}{4\gamma L p \alpha (2\alpha(\omega_{\text{up}} + 1) - 1)}.$$

Again, this implies two conditions on gamma:

$$\begin{cases}
3N - \gamma L(\omega_{\text{dwn}} + 1) \left( 3N + \frac{8(\omega_{\text{up}} + 1)}{p} - 2 \right) > 0 \iff \gamma < \frac{3}{(\omega_{\text{dwn}} + 1) \left( 3 + \frac{8(\omega_{\text{up}} + 1) - 2p}{Np} \right)} L \\
N - \gamma L(\omega_{\text{dwn}} + 1)(N + 2) > 0 \iff \gamma < \frac{1}{(\omega_{\text{dwn}} + 1) \left( 1 + \frac{2}{N} \right)} L.
\end{cases}$$

which guarantees the existence of $C$ and thus the validity of the above development.

As a conclusion:

$$F \left( \frac{1}{K} \sum_{k=0}^{K} w_k \right) - F(w_*) \leq \frac{V_0}{\gamma K} + 2\gamma \frac{E}{N} \leq \frac{\|w_0 - w_*\|^2 + 2\gamma CB^2}{\gamma K} + 2\gamma \frac{E}{N}.$$

Next, our goal is to define the optimal step size $\gamma_{opt}$. With this aim, we bound $2\gamma \frac{CB^2}{K}$ by $2\gamma_{max} \frac{CB^2}{K}$. This leads to ignore this term when optimizing the step size and thus to obtain a simpler expression of $\gamma_{opt}$. This approximation is relevant, because $\frac{B^2}{K}$ is “small”. And we obtain:

$$F \left( \frac{1}{K} \sum_{k=0}^{K} w_k \right) - F(w_*) \leq \frac{\|w_0 - w_*\|^2}{\gamma K} + 2\gamma \frac{E}{N} + 2\gamma_{max} \frac{CB^2}{K}.$$

This is valid for all variants of Artemis, with step-size in table and $E, p$ in Theorem. Subsequently, the “optimal” step size (at least the one minimizing the upper bound) is

$$\gamma_{opt} = \sqrt{\frac{\|w_0 - w_*\|^2 N}{2EK}}.$$
resulting in a convergence rate as $2\sqrt{\frac{\|w_0 - w_*\|^2}{NK} + \frac{2\gamma_{max}CB^2}{K}}$, if this step size is allowed. If $\sqrt{\frac{\|w_0 - w_*\|^2}{NK}} + \frac{2\gamma_{max}CB^2}{K} \geq \gamma_{max}$

(= $\gamma_{max}B^2 \leq \frac{\|w_0 - w_*\|^2}{\gamma_{max}K}$), then the bias term dominates and the upper bound is $\sqrt{\frac{\|w_0 - w_*\|^2}{NK}} + \frac{2\gamma_{max}CB^2}{K}$. Overall, the convergence rate is given by:

$$F\left(\frac{1}{K} \sum_{k=0}^{K} w_k\right) - F(w_*) \leq 2\max\left(\sqrt{\frac{2\|w_0 - w_*\|^2}{NK} + \frac{\|w_0 - w_*\|^2}{\gamma_{max}K}}\right) + 2\gamma_{max}CB^2.$$ 

\[\square\]

### E.4 PROOF OF THEOREM S4 - CONVERGENCE IN DISTRIBUTION

In this section, we give the proof of Theorem S3 in a full participation setting. The theorem is decomposed into two main points, that are respectively derived from Propositions S9 and S10, given in Appendices E.4.2 and E.4.3. We first introduce a few notations in Appendix E.4.1.

We consider in this section the Stochastic Sparsification compression operator $C_q$, which is defined as follows: for any $x \in \mathbb{R}^d$, $C_q(x) \overset{\text{def}}{=} \frac{1}{q}(x_1 B_1, \ldots, x_d B_d)$, with $(B_1, \ldots, B_d) \sim \mathcal{B}(q)^\otimes n$ i.i.d. Bernoullis with mean $q$. That is, each coordinate is independently assigned to $0$ with probability $1-q$ or rescaled by a factor $q^{-1}$ in order to get an unbiased operator.

**Lemma S15.** This compression operator satisfies Assumption S with $\omega_C = q^{-1} - 1$.

Moreover, if I consider a random variable $(B_1, \ldots, B_d) \sim \mathcal{B}(q)^\otimes n$ and define almost surely $C_q(x) \overset{a.s.}{=} \frac{1}{q}(x_1 B_1, \ldots, x_d B_d)$, then we also have that for any $x, y \in \mathbb{R}^d$, $C_q(x) - C_q(y) = C_q(x - y)$.

#### E.4.1 Background on distributions and Markov Chains

We consider Artemis iterates $(w_k, (h^i_k))_{k \in \mathbb{N}} \in \mathbb{R}^{d(1+N)}$ with the following update equation:

$$\begin{align*}
  w_{k+1} = w_k - \gamma C_{\text{down}} \left( \frac{1}{N} \sum_{i=1}^{N} C_{\text{up}} \left( g^{i+1}_k - h^i_k \right) + h^i_k \right) \\
  h^{i+1}_k = h^i_k + \alpha C_{\text{up}} \left( g^{i+1}_k - h^i_k \right)
\end{align*}$$

(S23)

We see the iterates, for a constant step size $\gamma$, as a homogeneous Markov chain, and denote $R_{\gamma,v}$ the Markov kernel, which is the equivalent for continuous spaces of the transition matrix in finite state spaces. Let $R_{\gamma,v}$ be the Markov kernel on $(\mathbb{R}^{d(1+N)}, \mathcal{B}(\mathbb{R}^{d(1+N)}))$ associated with the SGD iterates $(w_k, (h^i_k))_{k \in \mathbb{N}} \geq 0$ for a variant $v$ of Artemis, as defined in Algorithm 1 and with $\tau$ a constant specified afterwards, where $\mathcal{B}(\mathbb{R}^{d(1+N)})$ is the Borel $\sigma$-field of $\mathbb{R}^{d(1+N)}$. [Meyn & Tweedie 2009] provide an introduction to Markov chain theory. For readability, we now denote $(h^i_k)$ for $(h^i_k)_{i \in [1,N]}$.

**Definition 3.** For any initial distribution $\nu_0$ on $\mathcal{B}(\mathbb{R}^{d(1+N)})$ and $k \in \mathbb{N}, \nu_0 R_{\gamma,v}^k$ denotes the distribution of $(w_k, (h^i_k))$ starting at $(w_0, (h^i_0))$ distributed according to $\nu_0$.

We can make the following comments:

1. **Initial distribution.** We consider deterministic initial points, i.e., $(w_0, (\tau(h^i_0)))$ follows a Dirac at point $(w_0, (\tau(h^i_0)))$. We denote this Dirac $\delta_{w_0} \otimes \otimes_{i=1}^{N} \delta_{\tau h^i_0} \overset{\text{def}}{=} \delta_{w_0} \otimes \delta_{\tau h^i_0} \otimes \cdots \otimes \delta_{\tau h^i_0}$.

2. **Notation in the main text:** In the main text, for simplicity, we used $\Theta_k$ to denote the distribution of $w_k$ when launched from $(w_0, (\tau(h^i_0)))$. Thus $\Theta_k$ corresponds to the distribution of the projection on first $d$ coordinates of $(\delta_{w_0} \otimes \otimes_{i=1}^{N} \delta_{\tau h^i_0} R_{\gamma,v}^k)$.

3. **Case without memory:** In the memory-less case, we have $(h^i_k)_{k \in \mathbb{N}} \equiv 0$, and could restrict ourselves to a Markov kernel on $(\mathbb{R}^d, \mathcal{B}^d)$.

For any variant $v$ of Artemis, we prove that $(w_k, (h^i_k))_{k \geq 0}$ admits a limit stationary distribution

$$\Pi_{\gamma,v} = \pi_{\gamma,v,w} \otimes \pi_{\gamma,v,(h)}$$

(S24)

and quantify the convergence of $((\delta_{w_0} \otimes \otimes_{i=1}^{N} \delta_{\tau h^i_0} R_{\gamma,v}^k))_{k \geq 0}$ to $\Pi_{\gamma,v}$, in terms of Wasserstein metric $\mathcal{W}_2$. 

We prove the following proposition:

**Definition 4.** For all probability measures $\nu$ and $\lambda$ on $\mathcal{B}(\mathbb{R}^d)$, such that $\int_{\mathbb{R}^d} \|w\|^2 \, d\nu(w) < +\infty$ and $\int_{\mathbb{R}^d} \|w\|^2 \, d\lambda(w) \leq +\infty$, define the squared Wasserstein distance of order 2 between $\lambda$ and $\nu$ by

$$W_2^2(\lambda, \nu) := \inf_{\xi \in \Gamma(\lambda, \nu)} \int \|x - y\|^2 \xi(dx, dy),$$

where $\Gamma(\lambda, \nu)$ is the set of probability measures $\xi$ on $\mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d)$ satisfying for all $A \in \mathcal{B}(\mathbb{R}^d)$, $\xi(A \times \mathbb{R}^d) = \nu(A)$, $\xi(\mathbb{R}^d \times A) = \lambda(A)$.

### E.4.2 Proof of the first point in Theorem 3

We prove the following proposition:

**Proposition S9.** Under Assumptions [I to S] for $C_a$ the Stochastic Sparsification compression operator, for any variant $v$ of the algorithm, there exists a limit distribution $\Pi_{\gamma,v}$, which is stationary, such that for any $k$ in $\mathbb{N}$, for any $\gamma$ satisfying conditions given in Theorems S5 and S6

$$W_2^2((\delta_{w_0} \otimes \otimes_{i=1}^{N} \delta_{\tau_h^1_i}) R_{k,\gamma,v}^k, \Pi_{\gamma,v}) \leq (1 - \gamma \mu)^k \left( \int_{(w',h') \in \mathbb{R}^d(1+N)} \| (w_0, \tau(h^1_i)) - (w', \tau(h^1_i)) \|^2 \, d\Pi_{\gamma,v}(w', (h^1_i)) \right).$$

**Proof:** We consider two initial distributions $\nu_0^a$ and $\nu_0^b$ for $(w_0, \tau(h^1_i))$ with finite second moment and $\gamma > 0$. Let $(w_0^a, \tau(h^{1,a}_i))$ and $(w_0^b, \tau(h^{1,b}_i))$ be respectively distributed according to $\nu_0^a$ and $\nu_0^b$. Let $(w_k^a, \tau(h^{1,a}_i))_{k \geq 0}$ and $(w_k^b, \tau(h^{1,b}_i))_{k \geq 0}$ the Artemis iterates, respectively starting from $(w_0^a, \tau(h^{1,a}_i))$ and $(w_0^b, \tau(h^{1,b}_i))$, and sharing the same sequence of noises, i.e.,

- built with the same gradient oracles $g_{k+1}^{i,a} = g_{k+1}^{i,b}$ for all $k \in \mathbb{N}, i \in [1, N]$.
- the compression operator used for both recursions is almost surely the same, for any iteration $k$, and both uplink and downlink compression. We denote these operators $C_{\text{down},k}$ and $C_{\text{up},k}$ the compression operators at iteration $k$ for respectively the uplink compression and downlink compression.

We thus have the following updates, for any $u \in \{a, b\}$:

$$\begin{cases} w_{k+1}^u &= w_k^u - \gamma C_{\text{down},k} \left( \frac{1}{N} \sum_{i=1}^{N} C_{\text{up},k} \left( g_{k+1}^{i} - h_{k}^{i,u} \right) + h_{k}^{i,a} \right) \\ h_{k+1}^{i,u} &= h_k^{i,u} + \alpha C_{\text{up},k} \left( g_{k+1}^{i} - h_{k}^{i,u} \right) \end{cases}$$

(S26)
The proof is obtained by induction. For a $k \in \mathbb{N}$, let $\left( (w_k^a, \tau(h_k^{i,a})), (w_k^b, \tau(h_k^{i,b})) \right)$ be a coupling of random variable in $\Gamma(\nu_0^a R_{\gamma,v}^k, \nu_0^b R_{\gamma,v}^k)$ as in Definition 4, that achieve the equality in the definition, i.e.,

$$\mathcal{W}_2^2(\nu_0^a R_{\gamma,v}^k, \nu_0^b R_{\gamma,v}^k) = \mathbb{E} \left[ \left( w_k^a, \tau(h_k^{i,a}) \right) - \left( w_k^b, \tau(h_k^{i,b}) \right) \right]^2.$$

(S27)

Existence of such a couple is given by Villani (2009, theorem 4.1). Then $\left( (w_{k+1}^a, \tau(h_{k+1}^{i,a})), (w_{k+1}^b, \tau(h_{k+1}^{i,b})) \right)$ obtained after one update from Equation (S26) belongs to $\Gamma(\nu_0^a R_{\gamma,v}^{k+1}, \nu_0^b R_{\gamma,v}^{k+1})$, and as a consequence:

$$\mathcal{W}_2^2(\nu_0^a R_{\gamma,v}^{k+1}, \nu_0^b R_{\gamma,v}^{k+1}) \leq \mathbb{E} \left[ \left( w_{k+1}^a, \tau(h_{k+1}^{i,a}) \right) - \left( w_{k+1}^b, \tau(h_{k+1}^{i,b}) \right) \right]^2$$

$$= \mathbb{E} \left[ \|w_{k+1}^a - w_{k+1}^b\|^2 \right] + \tau^2 \sum_{i=1}^{N} \mathbb{E} \left[ \|h_{k+1}^{i,a} - h_{k+1}^{i,b}\|^2 \right]$$

$$= \mathbb{E} \left[ \|w_{k+1}^a - w_{k+1}^b\|^2 \right] + 2\gamma^2 \frac{C}{N} \sum_{i=1}^{N} \mathbb{E} \left[ \|h_{k+1}^{i,a} - h_{k+1}^{i,b}\|^2 \right],$$

with $\tau^2 = 2\gamma^2 \frac{C}{N}$, where $C$ depends on the variant as in Theorem 1.

We now follow the proof of the previous theorems to control respectively $\mathbb{E} \left[ \|w_{k+1}^a - w_{k+1}^b\|^2 \right]$ and $\mathbb{E} \left[ \|h_{k+1}^{i,a} - h_{k+1}^{i,b}\|^2 \right]$.

First, following the proof of Equation (S14), we get, using the fact that the compression operator is random sparsification, thus that $\mathcal{C}(x) - \mathcal{C}(y) = \mathcal{C}(x - y)$:

$$\mathbb{E} \left[ \|w_{k+1}^a - w_{k+1}^b\|^2 \right] \leq \mathbb{E} \left[ \|w_k^a - w_k^b\|^2 - 2\gamma \langle \nabla \mathcal{F}(w_k^a) - \nabla \mathcal{F}(w_k^b) \mid w_k^a - w_k^b \rangle \right]$$

$$+ \frac{2(\omega_{up}^a)(\omega_{dwn}^a + 1)}{N^2} \sum_{i=1}^{N} \mathbb{E} \left[ \|g_{k+1}^i(w_k^a) - g_{k+1}^i(w_k^b)\|^2 \right]$$

$$+ \frac{2\omega_{up}^b(\omega_{dwn}^b + 1)}{N^2} \sum_{i=1}^{N} \mathbb{E} \left[ \|h_{k+1}^{i,a} - h_{k+1}^{i,b}\|^2 \right]$$

$$+ \gamma^2 \omega_{dwn}^b \mathbb{E} \left[ \langle \nabla \mathcal{F}(w_k^a) - \nabla \mathcal{F}(w_k^b) \mid w_k^a - w_k^b \rangle \right].$$

This expression is nearly the same as in Equation (S14), apart from the constant term depending on $\sigma^2$ that disappears.

Note that with a more general compression operator, for example for quantization, it is not possible to derive such a result.

Similarly, we control $\mathbb{E} \left[ \|h_{k+1}^{i,a} - h_{k+1}^{i,b}\|^2 \right]$ using the same line of proof as for Equation (S15), resulting in:

$$\frac{1}{N^2} \sum_{i=0}^{N} \mathbb{E} \left[ \|h_{k+1}^{i,a} - h_{k+1}^{i,b}\|^2 \right] \leq (1 + p(2\alpha^2 \omega_{up}^a + 2\alpha^2 - 3\alpha)) \frac{1}{N^2} \sum_{i=0}^{N} \mathbb{E} \left[ \|h_{k+1}^{i,a} - h_{k+1}^{i,b}\|^2 \right]$$

$$+ 2(2\alpha^2 \omega_{up}^a + 2\alpha^2 - 3\alpha) \frac{1}{N^2} \sum_{i=0}^{N} \mathbb{E} \left[ \|g_{k+1}^i(w_k^a) - g_{k+1}^i(w_k^b)\|^2 \right].$$

Combining both equations, and using Assumptions 1 and 2 and Equation (S27) we get, under conditions on the learning rates $\alpha, \gamma$ similar to the ones in Theorems 3, 5, and 6, that

$$\mathcal{W}_2^2(\nu_0^a R_{\gamma,v}^{k+1}, \nu_0^b R_{\gamma,v}^{k+1}) \leq (1 - \gamma \mu) \mathcal{W}_2^2(\nu_0^a R_{\gamma,v}^k, \nu_0^b R_{\gamma,v}^k).$$

And by induction:

$$\mathcal{W}_2^2(\nu_0^a R_{\gamma,v}^{k+1}, \nu_0^b R_{\gamma,v}^{k+1}) \leq (1 - \gamma \mu)^k \mathcal{W}_2^2(\nu_0^a, \nu_0^b).$$

\]
From the contraction above, it is easy to derive the existence of a unique stationnary limit distribution: we use Picard fixed point theorem, as in [Dieuleveut et al. 2018]. This concludes the proof of Proposition S9.

**E.4.3 Proof of the second point of Theorem 3**

To prove the second point, we first detail the complementary assumptions mentioned in the text, then show the convergence to the mean squared distance under the limit distribution, and finally give a lower bound on this quantity.

**Complementary assumptions.**

To prove the lower bound given by the second point, we need to assume that the constants given in the assumptions are tight, in other words, that corresponding lower bounds exist in Assumptions 3 to 5.

**Assumption 7** (Lower bound on noise over stochastic gradients computation). The noise over stochastic gradients at optimal global point for a mini-batch of size \( b \) is lower bounded. In other words, there exists a constant \( \sigma_\ast \in \mathbb{R} \), such that for all \( k \) in \( \mathbb{N} \), for all \( i \) in \([1, N]\), we have a.s.:

\[
\mathbb{E} \left[ \| g_{k+1,i}^0 - \nabla F_i (w_\ast) \|^2 \mid \mathcal{H}_k \right] \geq \frac{\sigma_\ast^2}{b}.
\]

**Assumption 8** (Lower bound on local gradient at \( w_\ast \)). There exists a constant \( B \in \mathbb{R} \), s.t.:

\[
\frac{1}{N} \sum_{i=1}^{N} \| \nabla F_i (w_\ast) \|^2 \geq B^2.
\]

**Assumption 9** (Lower bound on the compression operator’s variance). There exists a constant \( \omega_C \in \mathbb{R}^* \) such that the compression operators \( C_{up} \) and \( C_{down} \) verify the following property:

\[
\forall \Delta \in \mathbb{R}^d, \mathbb{E}[\| C_{up, down} (\Delta) - \Delta \|^2] = \omega_C \| \Delta \|^2.
\]

This last assumption is valid for Stochastic Sparsification.

Moreover, we also assume some extra regularity on the function. This restricts the regularity of the function beyond Assumption 2 and is a purely technical assumption in order to conduct the detailed asymptotic analysis. It is valid in practice for Least Squares or Logistic regression.

**Assumption 10** (Regularity of the functions). The function \( F \) is also times continuously differentiable with second to fifth uniformly bounded derivatives: for all \( k \in \{2, \ldots, 5\} \), \( \sup_{w \in \mathbb{R}^d} \| F^{(k)} (w) \| < \infty \).

**Convergence of moments.**

We first prove that \( \mathbb{E}[\| w_k - w_\ast \|^2] \) converges to \( \mathbb{E}_{w \sim \pi_{\gamma,v}} [\| w - w_\ast \|^2] \) as \( k \) increases to \( \infty \).

We have that the difference satisfies, for random variables \( w_k \) and \( w \) following distributions \( \delta_{w_0} R_{\gamma,v}^k \) and \( \pi_{\gamma,v} \), and coupled such that they achieve the equality in Equation (S25):

\[
\Delta_{g,k} := \mathbb{E}[\| w_k - w_\ast \|^2 - \mathbb{E}_{w \sim \pi_{\gamma,v}} [\| w - w_\ast \|^2]]
\]

\[
= \mathbb{E}_{w_k, w \sim \pi_{\gamma,v}} [\| w_k - w_\ast \|^2 - \| w - w_\ast \|^2]
\]

\[
= \mathbb{E}_{w_k, w \sim \pi_{\gamma,v}} [(\| w_k - w_\ast \| - \| w - w_\ast \|)(\| w_k - w_\ast \| + \| w - w_\ast \|)]
\]

\[
\overset{C.S.}{\leq} \mathbb{E}_{w_k, w \sim \pi_{\gamma,v}} [(\| w_k - w_\ast \| - \| w - w_\ast \|)^2] \mathbb{E}_{w_k, w} [(\| w_k - w_\ast \| + \| w - w_\ast \|)^2]^{1/2}
\]

\[
\overset{T.1}{\leq} \mathbb{E}_{w_k, w \sim \pi_{\gamma,v}} [(\| w_k - w \|)^2] \mathbb{E}_{w_k, w \sim \pi_{\gamma,v}} [(\| w_k - w_\ast \| + \| w - w_\ast \|)^2]^{1/2}
\]

\[
\overset{(i)}{=} \mathbb{E}_{w_k, w \sim \pi_{\gamma,v}} [(\| w_k - w \|)^2] (2L)^{1/2}
\]

\[
\overset{(ii)}{=} (\mathcal{W}_2^2(\delta_{w_0} R_{\gamma,v}^k, \pi_{\gamma,v})2L)^{1/2}
\]

\[
\overset{(iii)}{=} 0.
\]
Where we have used Cauchy-Schwarz inequality at line C.S., triangular inequality at line T.I., the fact that the moments are bounded by a constant \( L \) at line (i), the fact that the distributions are coupled such that they achieve the equality in Equation \( \text{(S25)} \) at line (ii), and finally Proposition \( \text{S9} \) for the conclusion at line (iii).

Overall, this shows that the mean squared distance (i.e., saturation level) converges to the mean squared distance under the limit distribution.

**Evaluation of \( E_{w \sim \pi_{\gamma,v}}[\| w - w_* \|^2] \).**

In this section, we denote \( \Xi_{k+1}(w_k, h_k) \) the global noise, defined by

\[
\Xi_{k+1}(w_k, h_k) = \nabla F(w_k) - C_{\text{down}} \left( \frac{1}{N} \sum_{i=1}^{N} C_{\text{up}}(g^i_{k+1}(w_k) - h^i_k) + h^i_k \right),
\]

such that \( w_{k+1} = w_k - \gamma \nabla F(w_k) + \gamma \Xi_{k+1}(w_k, h_k) \).

In the following, we denote \( a \odot 2 := a a^T \) the second order moment of \( a \). We define \( \text{Tr} \) the trace operator and \( \text{Cov} \) the covariance operator such that \( \text{Cov}(\Xi(w, h)) = E[(\Xi(w, h))^\odot 2] \), where the expectation is taken on the randomness of both compressions and the gradient oracle. We make a final technical assumption on the regularity of the covariance matrix.

**Assumption 11.** We assume that:

1. \( \text{Cov}(\Xi(w, h)) \) is continuously differentiable, and there exists constants \( C \) and \( C' \) such that for all \( w, h \in \mathbb{R}^{d(1+N)} \),
   \[
   \max_{a=1,2,3} \text{Cov}(a)(w, h) \leq C + C' \| (w, h) - (w_*, h_*) \|^2.
   \]
2. \( \langle \Xi(w_*, h_*) \rangle \) has finite order moments up to order 8.

**Remark:** with the Stochastic Sparsification operator, this assumption can directly be translated into an assumption on the moments and regularity of \( g^i_k \). Note that Point 2 in Assumption 11 is an extension of Assumption 3 to higher order moments, but still at the optimal point. Under this assumption, we have the following lemma:

**Lemma S16.** Under Assumptions 7 to 10 and 11 we have that

\[
E_{\pi_{\gamma,v}}[\| w - w_* \|^2] = \gamma \text{Tr}(A \text{Cov}(\Xi(w_*, h_*))) + O(\gamma^2),
\]

with \( A := (F''(w_*) \odot I + I \odot F''(w_*))^{-1} \).

The intuition of the proof is natural: using the stability of the limit distribution, we have that if we start from the stationary distribution, i.e., \( (w_0, h_0) \sim \Pi_{\gamma,v} \), then \( (w_1, h_1) \sim \Pi_{\gamma,v} \).

We can thus write:

\[
E_{\pi_{\gamma,v}}[(w - w_*)^\odot 2] = E[(w_1 - w_*)^\odot 2] = E[(w_0 - w_* - \gamma \nabla F(w_0) + \gamma \Xi(w_0, h_0))^\odot 2].
\]

Then, expanding the right hand side and using the fact that \( E[\Xi(w_0, h_0)|\mathcal{H}_0] = 0 \), then the fact that \( E[(w_1 - w_*)^\odot 2] = E[(w_0 - w_*)^\odot 2] \), and expanding the derivative of \( F \) around \( w_* \) (this is where we require the regularity assumption Assumption 10), we get that:

\[
\gamma (F''(w_*) \odot I + I \odot F''(w_*)) + O(\gamma) \]

\[
E_{\pi_{\gamma,v}}[(w - w_*)^\odot 2] = \gamma^2 E[\Xi(w, h) \sim \Pi_{\gamma,v}][\Xi(w, h)^\odot 2].
\]

Thus:

\[
E_{\pi_{\gamma,v}}[(w - w_*)^\odot 2] = \gamma A E[\Xi(w, h) \sim \Pi_{\gamma,v}][\Xi(w, h)^\odot 2] + O(\gamma^2).
\]

\[
\Rightarrow E_{\pi_{\gamma,v}}[\| (w - w_*) \|^2] = \gamma \text{Tr}(AE[\Xi(w, h) \sim \Pi_{\gamma,v}][\Xi(w, h)^\odot 2]) + O(\gamma^2).
\]

Finally, we use that \( E[\Xi(w, h) \sim \Pi_{\gamma,v}] \) to get Lemma S16.
More formally, we can rely on Theorem 4 in [Dieuleveut et al., 2018]: under Assumptions 1 to 5, 10 and 11 all assumptions required for the application of the theorem are verified and the result follows.

To conclude the proof, it only remains to control \( \text{Cov}(\Xi(w_*, h_*)) \). We have the following Lemma:

**Lemma S17.** Under Assumptions 7 to 9 we have that, for any variant \( v \) of the algorithm, with the constant \( E \) given in Theorem 7 depending on the variant:

\[
\text{Tr} ( \text{Cov}(\Xi(w_*, h_*))) = \Omega \left( \frac{\gamma E}{\mu N} \right). \tag{S29}
\]

Combining Lemmas S16 and S17 and using the observation that \( A \) is lower bounded by \( \frac{1}{2E} \) independently of \( \gamma, N, \sigma_*, B \), we have proved the following proposition:

**Proposition S10.** Under Assumptions 7 to 9 and 7 to 11 we have that

\[
\mathbb{E}[\|w_k - w_*\|^2] \to \mathbb{E}_{\pi_*, v} \left[ \|w - w_*\|^2 \right] = \Omega \left( \frac{\gamma E}{\mu N} \right) + O(\gamma^2), \tag{S30}
\]

where the constant in the \( \Omega \) is independent of \( N, \sigma_*, \gamma, B \) (it depends only on the regularity of the operator \( A \)).

Before giving the proof, we make a couple of observations:

1. This shows that the upper bound on the limit mean squared error given in Theorem 7 is tight with respect to \( N, \sigma_*, \gamma, B \). This underlines that the conditions on the problem that we have used are the correct ones to understand convergence.
2. The upper bound is possibly not tight with respect to \( \mu \), as is clear from the proof: the tight bound is actually \( \text{Tr} (A \text{Cov}(\Xi(w_*, h_*))) \). Getting a tight upper bound involving the eigenvalue decomposition of \( A \) instead of only \( \mu \) is an open direction.
3. In the memory-less case, \( h \equiv 0 \) and all the proof can be carried out analyzing only the distribution of the iterates \( (w_k)_k \) and not necessarily the couple \( (w_k, (h_k^i)_i)_k \).

We now give the proof of Lemma S17.

**Proof.** With memory, we have the following:

\[
\text{Tr} ( \text{Cov}(\Xi(w_*, h_*))) = \mathbb{E} \left[ \left\| C_{\text{dwn}} \left( \frac{1}{N} \sum_{i=1}^N C_{\text{up}}(g_i^1(w_*) - h_i^*) + h_i^* \right) \right\|^2 \right]
\]

\[
\overset{(i)}{=} \left( 1 + \omega_{\text{dwn}}^c \right) \mathbb{E} \left[ \left\| \frac{1}{N} \sum_{i=1}^N C_{\text{up}}(g_i^1(w_*) - h_i^*) + h_i^* \right\|^2 \right]
\]

\[
\overset{(ii)}{=} \left( 1 + \omega_{\text{dwn}}^c \right) \frac{1}{N^2} \sum_{i=1}^N \mathbb{E} \left[ \left\| C_{\text{up}}(g_i^1(w_*) - h_i^*) \right\|^2 \right]
\]

\[
\overset{(iii)}{=} \left( 1 + \omega_{\text{dwn}}^c \right) \frac{1}{N^2} \sum_{i=1}^N \mathbb{E} \left[ \left\| B_k^i C_{\text{up}}(g_i^1(w_*) - h_i^*) \right\|^2 \right]
\]

\[
\overset{(iv)}{=} \left( 1 + \omega_{\text{dwn}}^c \right) \frac{1}{N} \sum_{i=1}^N \left( 1 + \omega_{\text{up}}^c \right) \mathbb{E} \left[ \left\| g_i^1(w_*) - h_i^* \right\|^2 \right]
\]

\[
\overset{(v)}{\geq} \left( 1 + \omega_{\text{dwn}}^c \right) \frac{1}{N} \left( 1 + \omega_{\text{up}}^c \right) \frac{\sigma_1^2}{b}
\]

At line (i) we use Assumption 9 for the downlink compression operator with constant \( \omega_{\text{dwn}}^c \). At line (ii) we use the fact that \( \sum_{i=1}^N h_i^* = \nabla F(w_*) = 0 \), the independence of the random variables \( C_{\text{up}}(g_i^1(w_*) - h_i^*) \), \( C_{\text{up}}(g_i^1(w_*) - h_j^*) \) for \( i \neq j \) and the fact that they have 0 mean. Line (iii) makes appear the bernoulli variable \( B_j \) that mark if a worker is activate or not at round \( k \). We use Assumption 9 for the uplink compression operator with constant \( \omega_{\text{up}}^c \) in line (iv); and finally
Assumption 7 at line (v) to lower bound the variance of the gradients at the optimum. This proof applies to both simple and double compression with $ω_{\text{dwn}} = 0$ or not.

Remark that for the variant 2 of Artemis, the constant $E$ given in Theorem 1 has a factor $α^2C(ω_\text{C} + 1)$: combining with the value of $C$, this term is indeed of the order of $(1 + ω_{\text{dwn}})(1 + ω_{\text{up}})$.

Without memory, we have the following computation:

$$\text{Tr} \left( \text{Cov}(Ξ(w_*, 0)) \right) = E \left[ \left\| C_{\text{dwn}} \left( \frac{1}{N} \sum_{i=1}^{N} C_{\text{up}}(g_1^i(w_*)) \right) \right\|^2 \right]$$

(i) $$(1 + ω_{\text{dwn}})E \left[ \left\| \frac{1}{N} \sum_{i=1}^{N} C_{\text{up}}(g_1^i(w_*)) - h_*^i \right\|^2 \right]$$

(ii) $$\frac{(1 + ω_{\text{dwn}})}{N^2} \sum_{i=1}^{N} E \left[ \left\| C_{\text{up}}(g_1^i(w_*)) - h_*^i \right\|^2 \right]$$

(iii) $$\frac{(1 + ω_{\text{dwn}})}{N^2} \sum_{i=1}^{N} E \left[ \left\| C_{\text{up}}(g_1^i(w_*)) - g_1^i(w_*) \right\|^2 + \left\| g_1^i(w_* - h_*^i \right\|^2 \right]$$

At line (i) we use Assumption 9 for the downlink compression operator with constant $ω_{\text{dwn}}$ and the fact that $\sum_{i=1}^{N} h_*^i = \nabla F(w_*) = 0$, then at line (ii) the independence of the random variables $C_{\text{up}}(g_1^i(w_*)) - h_*^i$ with mean 0, then a Bias Variance decomposition at line (iii).

$$\text{Tr} \left( \text{Cov}(Ξ(w_*, 0)) \right) \overset{(iv)}{=} \frac{(1 + ω_{\text{dwn}})}{N^2} \sum_{i=1}^{N} E \left[ \omega_{\text{up}} \left\| (g_1^i(w_*)) \right\|^2 + \left\| g_1^i(w_* - h_*^i \right\|^2 \right]$$

(v) $$\frac{(1 + ω_{\text{dwn}})}{N^2} \sum_{i=1}^{N} E \left[ \omega_{\text{up}} \left( \left\| g_1^i(w_* - h_*^i \right\|^2 + \left\| h_*^i \right\|^2 \right) + \left\| g_1^i(w_* - h_*^i \right\|^2 \right]$$

Next we use Assumption 9 for the uplink compression operator with constant $ω_{\text{up}}$ at line (iv). Line (v) is another Bias-Variance decomposition and we finally conclude by using Assumptions 7 and 8 at line (vi) and reorganizing terms.

We have showed the lower bound both with or without memory, which concludes the proof.