Crossover from Percolation to Self-Organized Criticality

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(December 17, 2021)

We include immunity against fire as a new parameter into the self-organized critical forest-fire model. When the immunity assumes a critical value, clusters of burnt trees are identical to percolation clusters of random bond percolation. As long as the immunity is below its critical value, the asymptotic critical exponents are those of the original self-organized critical model, i.e. the system performs a crossover from percolation to self-organized criticality. We present a scaling theory and computer simulation results.

PACS numbers: 05.40.+j, 05.70.Jk, 05.70.Ln

Several extended nonequilibrium systems show scaling behavior over a wide range of parameter values and independently of the initial conditions. They are called self-organized critical (SOC) and might explain the ubiquity of fractal structures in nature. By analogy to equilibrium critical phenomena, the question arises if SOC phenomena are universal, i.e. if the critical behavior depends only on few properties of the model as dimension, conservation laws, or number of components. So far, examples for both universal and nonuniversal behavior have been found. While the critical behavior of the sandpile model seems to be robust with respect to various changes, the critical exponents of an earthquake model change continuously as function of a parameter which characterizes the degree of conservation. In equilibrium critical phenomena, such a continuous change of critical exponents occurs only in exceptional cases, e.g. the eight vertex model. Instead, critical behavior is characterized by a crossover between two fixed points, when certain parameters are changed. This crossover is described by scaling functions and a crossover exponent \( \phi \). In this letter, we report for the first time a crossover phenomenon in SOC systems by including a new parameter, the immunity \( g \), in the SOC forest-fire model. \( g \) is the probability that fire cannot pass from one tree to a neighboring tree. When the immunity assumes a critical value \( g_c \), the forest becomes very dense, and clusters that are burnt by a lightning stroke are identical to percolation clusters. As long as the immunity is below its critical value, the asymptotic critical exponents of the model are those of the original SOC forest-fire model, i.e. the system performs a crossover from percolation to SOC.

The forest-fire model is defined on a \( d \)-dimensional hypercubic lattice with \( L^d \) sites. Each site is either occupied by a tree, a burning tree, or it is empty. The state of the system is parallely updated according to the following rules:

1. Burning tree \( \rightarrow \) empty site.
2. Tree \( \rightarrow \) burning tree with probability \( 1 - g^n \) if \( n \geq 1 \) nearest neighbors are burning.
3. Tree \( \rightarrow \) burning tree with probability \( f \) if no nearest neighbor is burning.
4. Empty site \( \rightarrow \) tree with probability \( p \).

Starting with arbitrary initial conditions, after some time the system approaches a steady state the properties of which depend on the parameter values but not on the initial state. We always chose the lattice size so large that the steady state is also independent of the boundary conditions. Let \( \rho_t \) be the density of trees, \( \rho_e \) the density of empty sites, and \( \rho_f \) the density of burning trees in the steady state. Large forests and therefore large fires occur only if the lightning probability \( f \) is much smaller than the tree growth probability \( p \). If additionally large and small fires look alike, the system becomes scale invariant. Small fires live only for few time steps and are extinguished before new trees grow in their neighborhood. Scale invariance therefore can only be observed if tree growth is so slow that large fires also are extinguished before new trees grow at the edge of the burning forests. We conclude that SOC behavior occurs in the forest-fire model if the following conditions are satisfied: Lightning occurs seldom compared to tree growth, and tree growth is much slower than the lifetime of a fire, i.e.

\[
f \ll p \ll (f/p)^{\nu'}
\]

with an appropriate exponent \( \nu' \). Eq. (1) represents a double separation of timescales. Since in the steady state the mean number of burning trees equals the mean number of growing trees, the mean number of trees destroyed by a lightning stroke is

\[
\bar{s} = \frac{\rho_e}{\rho_t} p \frac{p}{f} \approx \frac{1 - \rho_t}{\rho_t} p \frac{p}{f}.
\]

In the last step, we have neglected the fire density \( \rho_f \) which vanishes in the limit of perfect time scale separation. In \( d \geq 2 \) dimensions and for values of the immunity below its critical value, the critical forest density \( \rho^*_t = \lim_{f/p \to 0} \rho_t \) is smaller than 1, and Eq. (2) represents a power law \( \bar{s} \propto (f/p)^{-\gamma} \) with \( \gamma = 1 \), indicating a
critical point at $f/p = 0$. Close to this critical point, i.e. if $f \ll p$, there is scaling over many orders of magnitude.

Several critical exponents characterizing this scaling behavior have been defined \[3\]. The forest density satisfies for small $f/p$ a power law

$$\rho_c^f - \rho_c \propto (f/p)^{1/\delta}$$ \hspace{1cm} (3)

as first stated in \[4\] and \[5\].

Let $s$ be the number of trees destroyed by a fire. The size distribution of forest-fire model

$$sn(s) \propto s^{1-\tau}C(s/s_{\text{max}})$$ \hspace{1cm} (4)

with

$$s_{\text{max}} \propto (f/p)^{-\lambda}$$. \hspace{1cm} (5)

Inserting Eq. (3) and Eq. (4) into Eq. (2), we obtain

Together with Eqs. (4), (5) and (7), this gives

$$\tau(g_{c}) \equiv \tau_{c} = \tau_{\text{perc}} = 187/91 \approx 2.05$$ \hspace{1cm} (11)

and

$$\mu(g_{c}) \equiv \mu_{c} = \mu_{\text{perc}} = 91/48 \approx 1.90.$$ \hspace{1cm} (12)

Bonds that are immune during one time step might be non immune during the next one, and consequently lightning strokes at the same site burn down different clusters at different times. When $f/p$ is finite, there is a cutoff in cluster size, since large fires are stopped by empty sites that have been left from earlier fires. The mean forest density is no longer 1. We determined the critical exponents $\lambda$, $\delta$, and $\nu$ at $g = g_c$ by computer simulations in $d = 2$ dimensions and obtained

$$\lambda_{c} = 0.92(3), \delta_{c} = 0.15(1), \nu_{c} = 0.484(2).$$ \hspace{1cm} (13)

The scaling relations Eq. (6) and Eq. (8) still hold at $g = g_c$ and are confirmed by our simulations. Using Eqs. (8) and (9), we obtain

$$\tilde{s} \propto (f/p)^{-\gamma_{c}}$$ \hspace{1cm} (14)

with $\gamma_{c} = 1 - 1/\delta_{c}$.

Our simulations yield $\gamma_{c} = 0.84(2)$, in agreement with Eq. (14).

When the immunity is just below its critical value $(g_{c} - g) \ll 1)$, the situation becomes more complicated. On small length scales, a system close to the percolation threshold cannot be distinguished from a system exactly at the percolation threshold. On large length scales, however, the difference can be seen. If the initial state were a completely dense forest, a finite fraction of all trees were connected by non immune bonds. With increasing immunity, the forest density increases, since fewer trees are burnt. At the critical immunity $g_{c}$, the critical forest density is $\rho_{c}^f = 1$. Then we have the following situation: The forest is completely dense in the limit $f/p \to 0$, and clusters that are destroyed by fire are percolation clusters of bond percolation. Consequently the exponents $\tau$ and $\mu$ are given by percolation theory

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or non immune (in reality, rule 2. says that they might be immune during one time step and non immune during the next one). Then the dynamics on the sites which initially belonged to the infinite cluster are completely decoupled from dynamics on the remaining sites. We can consider the sites which belonged to the infinite cluster as an independent subsystem where no immunity exists. They form a two-dimensional lattice with another symmetry than the original hypercubic one. Since we already know from earlier simulations that the exponents of the SOC forest-fire model do not change when lattice symmetry is changed, we expect that the critical exponents on our subsystem are just the ones which we obtained for $g = 0$ and which have been assigned an index "soc". The remaining sites are connected by non immune bonds to form finite clusters of bond percolation. On each of these clusters, the dynamics are independent of the other clusters and of the infinite subsystem. The largest of these clusters have a radius of the order of the percolation correlation length $\xi_{perc} \propto (g_c - g)^{-\nu_{perc}}$. When $f/p$ becomes very small, these finite clusters are very rarely struck by lightning, and consequently the mean tree density on these clusters approaches the value 1 in the limit $f/p \to 0$. Each time lightning strikes such a cluster, there is a fire which has the size and the fractal dimension of a finite percolation cluster. As long as $f/p$ is so small that the correlation length $\xi \propto (f/p)^{-\nu}$ is much larger than the percolation correlation length $\xi_{perc}$, all large fires occur on the infinite subsystem, and the exponents $\nu$, $\delta$, and $\lambda$ are those of the subsystem, i.e. the SOC ones. The exponents $\tau$ and $\mu$ of the large fires are by the same reasoning also the SOC ones. But for fires with a radius smaller than the percolation correlation length, $\tau$ and $\mu$ assume their percolation values since the infinite subsystem contains only a small portion of all sites and consequently most of the small fires occur on the finite clusters. When $f/p$ becomes so large that the forest clusters on our infinite subsystem are no more larger than the percolation correlation length, our system is dominated by the dynamics on the finite clusters and becomes indistinguishable from a system at $g = g_c$. The exponents $\nu$, $\delta$, and $\lambda$ then are identical to those at $g = g_c$, and the exponents $\tau$ and $\mu$ are those of percolation theory on all length scales up to the correlation length. So far, we considered the case of small $g_c - g$. When the immunity is far away from its critical value, the infinite subsystem contains a large portion of all sites, and the percolation-dominated behavior cannot occur any more.

Unfortunately, all these considerations are based on the assumption that bonds are permanently immune or non immune which in reality is not the case. Consequently, there exists no subsystem which is decoupled from the rest of the system. Nevertheless, the main conclusions should remain valid: On length scales smaller than the percolation correlation length, the system cannot be distinguished from a system at $g = g_c$. When $f/p$ becomes very small, there are fires which spread further than the percolation correlation length. These fires are stopped by empty sites that were created by earlier fires. This is again the same mechanism as in the limit $g = 0$ or on the infinite subsystem in the case of fixed immune bonds: fires that would spread indefinitely if there were no empty sites are stopped by empty sites. We conclude that these large fires lead again to the critical exponents $\lambda_{soc}$, $\nu_{soc}$, and $\delta_{soc}$.

Led by these considerations, we make the following scaling ansatz for the correlation length:

$$\xi = (f/p)^{-\nu} F \left( \frac{g_c - g}{(f/p)^\phi} \right). \quad (15)$$

It is plausible that the crossover from percolation-like to SOC behavior takes place when $f/p$ becomes so small that the correlation length exceeds the percolation correlation length, which suggests that the crossover exponent $\phi$ is

$$\phi = \nu_c/\nu_{perc}. \quad (16)$$

The scaling function $F(x)$ is constant for small $x$ and is $\propto x^{(\nu_{soc} - \nu_c)/\phi}$ for large $x$. Analogous scaling laws hold for $s_{max}$ and $\rho_c^x - \rho_c$. We already mentioned above that the critical forest density is $\rho_c^x = 1$ at $g_c$. We therefore expect an additional power law

$$1 - \rho_c^x(g) \propto (g_c - g)^y. \quad (17)$$

The exponent $y$ is obtained from the scaling ansatz

$$1 - \rho_c = (f/p)^{1/\delta} G \left( \frac{g_c - g}{(f/p)^\phi} \right). \quad (18)$$

In the limit $f/p \to 0$, the forest density becomes independent of $f/p$ and assumes a value $\rho_c^x \neq 1$. Therefore $G(x) \propto x^{1/\rho_c^x}$ for large $x$, yielding

$$y = \nu_{perc}/\nu_c\delta_c. \quad (19)$$

Our simulations confirm all these results. They were performed using the same method as in for lattices of up to 8192$^2$ sites and values of $f/p$ down to $10^{-6}$. The immunity varied between $(g_c - g) = 0.02$ and $(g_c - g) = 0.001$. The range of $f/p$ is limited by finite-size effects for small $f/p$ and noncritical behavior for large $f/p$, the range of $g$ was selected from the condition that at least part of the crossover region is covered. Fig. 4 shows the scaling function for the correlation length $F(x)$ for different values of $g_c - g$. The scaling ansatz Eq. (15) is well confirmed since all curves coincide. The dashed line represents $F(0)$ as obtained from the simulations at $g_c$. We also checked the scaling relation Eq. (19). Fig. 2 shows the critical forest density as function of $g_c$. We obtain $y = 0.43(2)$, in agreement with Eq. (18) (remember that $\nu_{perc} = 4/3$). Since the difference between $\tau_{soc}$ and $\tau_{perc}$, as well as between $\mu_{soc}$ and $\mu_{perc}$ is very small, the crossover behavior of $n(s)$ and $R(s)$ could not be evaluated.
In [5], we also defined exponents $\nu', \mu'$ and $\alpha$ describing the temporal behavior of the fires. The crossover in $\mu'$ and $\nu'$, which enters the condition for time scale separation Eq. (1), is analogous to the crossover in $\mu$ and $\nu$ [13]. The change in $\alpha$ is too small for any crossover to be observable.

To conclude, we have shown by analytic arguments and by computer simulations that the forest-fire model performs a crossover from percolation to SOC when the immunity is close to its critical value. This crossover is characterized by scaling functions which are defined in the same way as in crossover phenomena at equilibrium phase transitions.

Although all simulations were performed in $d = 2$ dimensions, we expect that this crossover behavior can also be observed in higher dimensions. In $d = 1$, the critical immunity is $g_c = 0$, and no crossover can take place. For $d \geq 6$, simulations suggest that the critical exponents assume their mean-field values which are identical to those of percolation [5]. Consequently there is no crossover in $d \geq 6$ dimensions.

ACKNOWLEDGMENTS

We thank E. Frey for useful discussions.

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\[(1 - \rho_t)^{-1}\]

vs.

\[g_c - g\]