Abstract

Motivated by the problem of $N$ coupled Hubbard chains, we investigate a generalisation of the Schulz-Shastry model containing two species of one-dimensional fermions interacting via a gauge field that depends on the positions of all the particles of the other species. The exact many body ground state of the model can be easily obtained through a unitary transformation of the model. The correlation functions are Luttinger-like - i.e., they decay through power laws with non-integer exponents. Through the interaction dependent correlation functions of the two-particle operators, we identify the relevant perturbations and hence, possible instabilities.
Exactly solvable models\cite{1,2,3} have always attracted a lot of interest in theoretical physics, because they serve as paradigms for more complicated systems. The fact that these models are usually in one dimension no longer make them unrealistic, however, since current technological advances have seen the advent of many semi-artificial one dimensional systems, such as quantum wires, quantum Hall bars, one-dimensional organic metals and one dimensional spin chains\cite{4}. In fact, phenomena such as one dimensional Luttinger liquids and the Haldane gap in spin chain models have actually been experimentally seen\cite{5}. Besides their role in these systems, exactly solvable models have played a very important role as a reliable test for various approximation methods and for developing qualitative understanding\cite{6}.

However, for two or more dimensions, there have been very few exact results. For instance, the large $U$ Hubbard model has been studied using several approximation schemes\cite{7}, none of which have led to completely reliable results. In recent years, there have been attempts to understand two dimensions through the coupling of one-dimensional chains. Both coupled spin chains\cite{8} and coupled Hubbard models\cite{9} with interchain hopping and interchain interactions have been studied using a variety different schemes such as weak coupling renomalisation group techniques and bosonisation\cite{10}, exact numerical diagonalisations\cite{11}, etc. Unfortunately, in the absence of any exact results, the interpretation of the results of these inter-chain coupling studies have remained difficult\cite{12}.

With the motivation of approaching two dimensional phenomena through the coupling of one-dimensional chains, in this paper, we study a generalisation of a class of models\cite{13} that can be diagonalised by a pseudo-unitary transformation and still exhibit non-trivial Luttinger liquid behaviour. Our model has two species of particles with pseudospin index $\sigma = \pm$, at the positions $x_{\sigma i}$ and with momenta $p_{\sigma i}$, with a Hamiltonian given by

$$H = \sum_{I}^{N} \sum_{\sigma i} a_I (\Pi_{\sigma i})^{2I}. \tag{1}$$

Here, $\Pi_{\sigma i} = p_{\sigma i} + \sigma A_{\sigma}(x_{\sigma i})$ is the ‘covariant momentum’ introduced in Ref.\cite{13} and $N$ is a ‘chain index’ or ‘band index’. (The nomenclature of chain or band index will be explained later.) We have chosen to have only even powers of the covariant momentum in the Hamiltonian, although positivity of energy only requires that the largest power of the covariant momentum be even. This maintains the symmetry $x \rightarrow -x$ or parity, which simplifies the presentation of the calculations, although the result goes through even when we include odd powers of the momenta. As explained in Ref.\cite{13}, particles interact via a gauge potential, given for the particle at the position $x$ by $A_{\sigma}(x) = \sum_j V(x - x_{-\sigma j}) - i.e., the potential for the particles with positive pseudospin is due to the presence of the particles with negative pseudospin and vice-versa. The potential is chosen to be an even function, vanishes at infinity and explicitly breaks time-reversal invariance, although it is invariant under a combined operation of time reversal and reversal of pseudo-spin index. We have generalised the model in Ref.\cite{13} by including a chain index (or equivalently a band index) and allowing higher powers of the covariant momentum in the Hamiltonian. Our model reduces to the Schulz-Shastry model for $I = 1$ and $a_1 = 1$. Clearly, the $a_I$ are not dimensionless, and in fact, explicitly contain a scale $\Lambda$ (except for $a_1$, which is dimensionless).
As noted by the authors in Ref. [13], the same pseudo-unitary transformation that they use to diagonalise their Hamiltonian,

\[ e^{iS(x_i, x_{-i})}p_\sigma e^{-iS(x_i, x_{-i})} = p_\sigma - \partial_{x_i} S(x_i, x_{-i}) \]  

(2)
diagonalises any power of \( \Pi_{\sigma_i} \), as long as we choose the function \( S \) (a function of the \( 2n \) positions of the particles) to eliminate the interaction in Eq.(1). Thus, we obtain the transformed Hamiltonian given by

\[ \tilde{H} = e^{iS(x_i, x_{-i})} H e^{-iS(x_i, x_{-i})} = \sum_{\sigma_i} \sum_I a_I (p_\sigma)^{2I} = \sum_{\sigma_i} H_{\sigma_i} \]  

(3)

where the interaction pieces have been removed by the transformation. However, the eigenvalues and eigenfunctions are not the same as that for a genuinely non-interacting Hamiltonian because the boundary conditions on the wave-functions are now different.

For the single particle Hamiltonian \( H_{\sigma_i} \) in Eq.(3), the eigenvalue equation is a \( 2N^{th} \) order differential equation, and depending on the energy chosen, will have at most \( 2N \) different solutions. The general solution is given by

\[ \tilde{\psi} = \sum_I c_I e^{ik_I x} + h.c., \]  

(4)

where the \( k_I \)'s are known in terms of \( E \) and the \( N - 1 \) constants \( a_I \) (we always choose \( a_N = 1 \) without loss of generality since it only sets the overall scale). However, not all \( k_I \)'s need be integer multiples of \( 2\pi/L \), where \( L \) is the size of the system. For those that are not, the corresponding \( c_I \) vanish so as to make the wave function periodic in \( L \). Since the \( a_I \)'s are fixed, we may choose only one of the \( k_I \)'s to be independent, say \( k \), which in turn fixes the dispersion to be

\[ E(k) = \sum_I a_I (k^2)^{2I}. \]  

(5)

The Fermi points are the roots of the equation \( E(k) = E_F \). We choose an energy \( E_F \) where the \( 2N \) roots \( \{-b_I, b_I\} \) are all real and distinct (with \( b_1 < b_2 < \cdots < b_N \)). The Fermi points \( \{-F_I, F_I\} \) are given by \( F_I = 2\pi |r_I|/L \) where \( b_I = 2\pi r_I/L \) and \( |r_I| \) stands for the largest integer below \( r_I \).

We now see the justification for calling \( I \) the chain or band index. The usual identification of an \( N \)-chain model with an \( N \)-band model is made by diagonalising the kinetic energy of the \( N \)-chain model and using the \( N \) different momenta along the direction perpendicular to the chain to label the \( N \) bands or \( N \) dispersion relations [10]. The filling of these bands upto the Fermi level defines the set of \( 2N \) Fermi points \( \pm k_F^I \). Our model is slightly different from the \( N \)-chain model in that it has only a single dispersion (Eq.(3)). However, the dispersion is not quadratic and has \( N \) wells (unlike the usual quadratic dispersion which has one well per band) and \( 2N \) Fermi points. Hence, the physics it describes is similar to that of the \( N \) band model.
We are interested in the solution of the original Hamiltonian in Eq.(1) and not the transformed Hamiltonian in Eq.(3). Although the single particle energies of the two Hamiltonians are the same, their wave-functions are related by the pseudo-unitary transformation \( \psi = e^{-iS} \tilde{\psi} \), where \( S \) was chosen to cancel the interaction and is of the form

\[
S(\{x_{+i}\}, \{x_{-i}\}) = \sum_{i,j} E(x_{+i} - x_{-j}) \quad \text{where} \quad E(x) = \int_0^x dx' V(x').
\]

We can compute the difference between \( S(x_{-i} = L) \) and \( S(x_{-i} = 0) \) for any particular negative pseudospin coordinate \( x_{-i} \) as

\[
S(x_{-i} = L) - S(x_{-i} = 0) = \sum_j [E(x_{+j} - L) - E(x_{+j})]
= n_+^T \int_0^L V(x) dx \equiv n_+^T \delta.
\]

in terms of a phase shift \( \delta \) and \( n_+^T \) which is the total number of positive pseudospin particles. One gets a similar result if we choose the reference particle to be a positive pseudospin particle, with the only difference that \( n_+^T \) gets replaced by \( n_-^T \) and \( \delta \) by \( -\delta \). Hence the quantisation condition on the wave-numbers of the particles becomes

\[
Lk_{\pm i} \mp n_{\mp i}^T \delta = 2\pi n_{\pm i}
\]

where the \( n_{\pm i} \) are integer quantum numbers analogous to those used in the non-interacting case. Since, in general, \( n_{\pm i}^T \delta \not\equiv \text{integral multiple of } 2\pi \), the free Hamiltonian and the interacting Hamiltonian are in different Hilbert spaces.

So far, all the arguments used by Schulz and Shastry have gone through for our model as well. The differences begin when we try to construct the many body ground state and the spectrum of low energy excitations. For ease of presentation, we will now specialise to the two band case, explicitly perform the calculations leading to the low energy effective Hamiltonian, and then generalise to the case of \( N \) bands.

For the two-band case, the single particle dispersion is given by

\[
\tilde{H} = e^{iS} H e^{-iS} = \sum_{i\sigma} a_4 P_{\sigma i}^4 + a_2 P_{\sigma i}^2
\]

where \( a_4 = \Lambda^2 \) has length dimension two and \( a_2 = -1 \) is dimensionless. As mentioned before, we restrict the Fermi level to lie within the double well - i.e., we have four distinct Fermi points \((-F_2, -F_1)\) on the left and \((F_1, F_2)\) on the right. For each energy, the degeneracy is either four or two , depending on whether or not both \( k_1 \) and \( k_2 \) satisfies the boundary condition given in Eq.(10). However, all that really matters is that energy levels in both the wells below the Fermi level are filled. Let us assume that in band 1, there are \( n_1 \) states below the Fermi level and in band 2, there are \( n_2 \) states below the Fermi level for both pseudospins. The ground state energy is then given by

\[
E_{gs} = 2 \sum_{\sigma = \pm} \sum_{i = -n_1}^{n_2} [h_{1i}^4 - h_{2i}^2]
\]
where \( h_1 = \Lambda^2 (\frac{2\pi}{L})^4 \) and \( h_2 = (\frac{2\pi}{L})^2 \). We have assumed that \( n_{T \delta}^T / 2\pi \) is an integer and the factor of two takes care of the contributions from both the right and left moving sectors. (Note that the states on the right branches of both the wells are right movers, whereas the states on the left branches of both wells are left movers. See figure below.)

\[ E^{(2)} = \left[ \sum_{\pm} \sum_{-(n_1-n_{R\pm1})}^{(n_2+n_{R\pm2})} \left[ h_1 (i + c_{\pm})^4 - h_2 (i + c_{\pm})^2 \right] + R \rightarrow L \right]_2 \]  

**Fig 1.** Dispersion for the two band model. For the Fermi energy \( E_F \), the two bands are denoted by dotted indices (band 1) and full lines (band 2). \((-F_2, F_1)\) are clearly left-mover Fermi points and \((-F_1, F_2)\) are right mover Fermi points. The inset shows the dispersion of the three band model, with left-mover branches denoted by full lines and right-mover branches denoted by dotted lines.

The second order fluctuation in the energy due to the addition of \( n_{R\pm1} \) and \( n_{L\pm1} \) particles for the right and left movers can also be computed. It is given by
\[\begin{align*}
\sum_{I} \sum_{\pm} [h_{1} f_{1}(n_{I}) + h_{2} f_{2}(n_{I})] (n_{R\pm I} - c_{\mp})^{2} \\
= \sum_{I} \sum_{\pm} g(n_{I}) [J_{\pm I} \pm n_{\mp} \delta/2\pi]^{2} + (n_{\pm I})^{2}
\end{align*}\]

where \(c_{\sigma} = n_{T}^{\sigma} \delta/2\pi\) in the first row and the subscript 2 is to indicate that we keep only terms up to quadratic order in the fluctuations. In the second row, \(f_{1}(n_{I}) = n_{I}(n_{I} + 1)(2n_{I} + 1)\) and \(f_{2} = n_{I} + 1/2\), and in the third row, we have defined the current \(J_{\pm I} = n_{R\pm I} - n_{L\pm I}\), the charge \(n_{\pm I} = n_{R\pm I} + n_{L\pm I}\) and the ‘density’ \(g(n_{I}) = h_{1} f_{1}(n_{I}) + h_{2} f_{2}(n_{I})\). The total charge is clearly \(n_{T}^{\pm} = n_{\pm 1} + n_{\pm 2}\). Note however, that unlike the Schulz-Shastry model, here fourth order fluctuations do exist, which we neglect because we are only interested in low energy fluctuations.

We bosonise as in the single band case by introducing boson fields \(\phi_{\pm I}\) with their conjugate momenta \(\Pi_{\pm I}\). These are related to the currents and charge densities as

\[\begin{align*}
n_{\pm I} &= \frac{L}{\sqrt{\pi}} \partial_{x} \phi_{\pm I} \\
J_{\pm I} &= -\frac{L}{\sqrt{\pi}} \Pi_{\pm I}.
\end{align*}\]

(We use the notation of Ref.\[14\].) To rewrite the effective Hamiltonian for the low energy fluctuations in terms of the boson fields, we have to identify the function \(g(n_{I})L/2\pi = \rho_{I}\) as an effective density after which we obtain

\[H = \sum_{I} \sum_{\pm} \int dx \rho_{I} \left\{ [-\Pi_{I\pm} \pm \frac{\delta}{\pi} (\partial_{x} \phi_{1\mp} + \partial_{x} \phi_{2\mp})]^{2} + (\partial_{x} \phi_{\pm I})^{2} \right\}.\]

But interestingly, although \(\rho_{I}\) contains information about the scale, the low energy effective Hamiltonian is scale invariant - there are no mass terms (or cosine terms leading to mass terms) for the boson fields. A similar redefinition of variables as in the one band case,

\[\tilde{\phi}_{\pm I} = \phi_{\pm I} , \quad \tilde{\Pi}_{\pm I} = \Pi_{\pm I} \mp \frac{\delta}{\pi} (\partial_{x} \phi_{x1} + \partial_{x} \phi_{x2})\]

leads to a non-interacting form of the Hamiltonian given by

\[H = \sum_{I} \sum_{\pm} \int dx \rho_{I} \left\{ [(\tilde{\Pi}_{\pm I})^{2} + (\partial_{x} \tilde{\phi}_{\pm I})^{2} \right\}.\]

Thus the correlators of the tilde fields are just free field correlators. In terms of the non-tilde bosonic variables or equivalently in terms of the fermion fields, the Hamiltonian is not non-interacting. However, since they are explicitly known in terms of the free fields, their correlators can also be explicitly calculated.

In fact, at this stage, the generalisation to \(N\) chains is obvious. The single particle dispersion of the \(N\)-band model has \(N\) wells and \(2N\) Fermi points. The Hamiltonian for
quadratic fluctuations about the Fermi points is precisely the same as that in Eq.(17) with the replacement
\[ (\partial_x \phi_{\mp 1} + \partial_x \phi_{\mp 2}) \rightarrow \sum_J \partial_x \phi_{\mp J} \]  
(20)

As before, the redefinition of \( \phi_I \) and \( \Pi_I \) in terms of the tilde fields leads to the non-interacting form of the Hamiltonian in Eq.(19) with the sum going over all \( N \) bands.

We now compute correlation functions using the representation for the fermion operators in terms of the non-interacting boson fields given by
\[ \psi_{R\pm I} = \exp(\tilde{\phi}_{R\pm I} \mp \frac{\delta}{2\pi} \sum_I \tilde{\phi}_{\mp I}), \]
(21)
\[ \psi_{L\pm I} = \exp(\tilde{\phi}_{L\pm I} \pm \frac{\delta}{2\pi} \sum_I \tilde{\phi}_{\mp I}). \]
(22)

(We follow the notation in Ref.[14] and define \( \phi_{R\pm I} = 1/2(\phi_{\pm I} - \int_{-\infty}^{\infty} \Pi_{\pm I}(x) dx) \) and \( \phi_{L\pm I} = 1/2(\phi_{\pm I} + \int_{-\infty}^{\infty} \Pi_{\pm I}(x) dx) \).) The one-particle correlation function is given by
\[ G_{Z\pm I}(x) = \langle \psi_{Z\pm I}(x) \psi_{Z\pm I}^\dagger(0) \rangle \sim x^{-\eta} \]
(23)
with \( \eta = 1 + N(\delta^2/2\pi^2) \) for both right and left movers (\( Z = R/L \)), for both pseudospins and for all \( I \). As in the one-band case, the fermion has an anomalous dimension given by \( \eta \neq \text{integer} \). This is the indication that the system is a Luttinger liquid and not a Fermi liquid. The interesting point to note here is the dependence of the anomalous dimension on the number of chains. The model is not just a collection of one-band Luttinger liquids - there exists a genuine dependence on the number of bands.

We can also compute exponents of the two-particle operators(TPO) in order to identify the relevant perturbations and hence potential singularities. In the one-band case, the only non-trivial two-particle correlations involved excitations at both the right and left Fermi points, because these were the only two Fermi points. Here, however, we can have non-trivial two-particle correlations involving excitations at two right Fermi points and two left Fermi points as well. These exponents for the two particle correlations are tabulated below.

### TWO PARTICLE CORRELATIONS

| TPO | \( \eta \) | TPO\((I \neq J) \) | \( \eta \) |
|-----|-----|------|-----|
| \( \psi_{R \pm I}^\dagger \psi_{L \pm J} \) | 2 | \( \psi_{Z \pm I}^\dagger \psi_{Z \pm J} \) | \( 2(1 + N\frac{\delta^2}{\pi^2}) \) |
| \( \psi_{R \pm I}^\dagger \psi_{L \pm J} \) | \( 2(1 \mp \frac{\delta}{2\pi}) + N\frac{\delta^2}{\pi^2} \) | \( \psi_{Z \pm I}^\dagger \psi_{Z \pm J} \) | \( 2(1 + N\frac{\delta^2}{\pi^2}) \) |
| \( \psi_{R \pm I} \psi_{L \pm J} \) | \( 2(1 + N\frac{\delta^2}{\pi^2}) \) | \( \psi_{Z \pm I} \psi_{Z \pm J} \) | 2 |
| \( \psi_{R \pm I} \psi_{L \pm J} \) | \( 2(1 \pm \frac{\delta}{\pi}) + N\frac{\delta^2}{\pi^2} \) | \( \psi_{Z \pm I} \psi_{Z \pm J} \) | \( 2(1 + N\frac{\delta^2}{\pi^2}) \) |

Fortunately, they are independent of the band index and only depend on whether they involve both right and left Fermi points or right (left) movers at both Fermi points. Interestingly, none of the RR or LL exponents lead to relevant perturbations. This is in agreement
with the weak coupling RG approach[15], where there is a non-zero contribution to the four-point vertex only when there is momentum transfer between left and right Fermi points. In the single chain case considered in Ref.[15], there was no possibility of momentum transfers between two Fermi points on the left or two on the right, since the model only had one on each side. However, even in the more general case of \(N\) left-moving Fermi points and \(N\) right moving Fermi points[16], graphs involving loop momenta in two left-moving shells or two right moving shells are always zero, because the energies have the same sign and the contour integral for the energy vanishes.

For positive \(\delta\), the only relevant perturbations are
\[
\psi_R^{\dagger} \psi_L - J \eta = 1 + (1 - \delta/\pi)^2 + (N - 1)\delta^2/\pi^2, \\
\psi_R \psi_L + J \eta = 1 + (1 - \delta/\pi)^2 + (N - 1)\delta^2/\pi^2.
\]

(24)

Clearly as the number of chains increases, the exponent increases until at some critical value of \(N = N_c \propto 1/\delta\), both the operators above cease to be relevant. Similarly, for negative \(\delta\), the relevant perturbations are
\[
\psi_R^{\dagger} \psi_L - J \eta = 1 + (1 + \delta/\pi)^2 + (N - 1)\delta^2/\pi^2, \\
\psi_R \psi_L + J \eta = 1 + (1 + \delta/\pi)^2 + (N - 1)\delta^2/\pi^2.
\]

(25)

which cease to be relevant beyond \(N_c\). Hence, for \(N > N_c\) chains, there are no relevant perturbations at all. The system is always a Luttinger liquid. For \(N < N_c\), the system, depending on which instability grows, (which perturbation will be added), which in turn, will be dictated by the realistic model that we wish to study, will be in different ground states.

Let us compare our results with the results obtained by giving additional internal degrees of freedom to the \(\sigma = \pm\) particles[13]. If we assume that they occur in \(m\)-flavours, then the Hamiltonian is just
\[
H = \sum_{I=1}^{m} \sum_{\sigma\bar{\sigma}} a_I (\Pi_{I\sigma\bar{\sigma}})^2.
\]

(26)

Surprisingly, an analogous calculation leads precisely to the same exponents as in Eqs.(24) and (25) with \(N\) replaced by \(m\). However, in this case, there genuinely exist \(2m\) degrees of freedom, and the various two particle correlators have physical meaning. \(I = J\) give two particle correlators of the same particle, whereas there is no analogue of \(I \neq J\) correlators. The exponents are actually independent of the particle index because of the internal symmetry. For our Hamiltonian in Eq.(1), however, there are only two degrees of freedom corresponding to the \(\sigma = \pm\) particles. It is only after linearising around the different Fermi points and assuming that each of the linearised fermions can be bosonised independently that we have \(N\) independent R/L moving fermions or bosons, whose correlators can be computed independently. For the original fermions, the only relevant charges are \(n_{I\pm}^T\) and the relevant currents are \(J_{I\pm} = \sum_I (n_{R\pm I} - n_{L\pm I})\).

Many of the issues in coupled chain models, however, remain unaddressed in this rather simple model, which is perhaps better thought of as a single chain model with a more
complicated band structure. To really apply this model to $N$-chains, one would have to modify the model, so that there is some analog of the interplay between interchain hopping and intra-chain interactions. However, note that even as an $N$-band model, it is not trivial that the correlation functions are identical to those of the $m$- flavour model.

In conclusion, we have studied a general model with $2N$ Fermi points, (an $N$-band or $N$ chain model), which is exactly solvable and has non-trivial Luttinger liquid behaviour. We computed the exponents of the various two-particle operators and found the possible relevant perturbations. Interestingly, we found that the exponents have non-trivial dependence on the number of chains - they are not merely additive. Furthermore, beyond a certain number of chains $N_c$, which is inversely proportional to the strength of the interaction $\delta$, all perturbations are irrelevant and the Luttinger liquid ground state is robust. Thus, this model, if it can be effectively generalised to higher dimensions would be a good starting point to study possible Luttinger liquid ground states in higher dimensions.

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