Non-weight modules over the super-BMS$_3$ algebra
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Abstract: In the present paper, a class of non-weight modules over the super-BMS$_3$ algebras $\mathfrak{S}^\epsilon$ ($\epsilon = 0$ or $\frac{1}{2}$) are constructed. Assume that $t = \mathbb{C}L_0 \oplus \mathbb{C}W_0 \oplus \mathbb{C}G_0$ and $\mathfrak{T} = \mathbb{C}L_0 \oplus \mathbb{C}W_0$ are the Cartan subalgebra (modulo center) of $\mathfrak{S}^0$ and $\mathfrak{S}^{\frac{1}{2}}$, respectively. These modules over $\mathfrak{S}^0$ when restricted to the $t$ are free of rank 1, while these modules over $\mathfrak{S}^{\frac{1}{2}}$ when restricted to the $\mathfrak{T}$ are free of rank 2. In particular, the $U(\mathfrak{T})$-modules of rank 1 over $\mathfrak{S}^{\frac{1}{2}}$ are isomorphic to the Virasoro modules. Then we determine the necessary and sufficient conditions for these modules being simple, as well as determining the necessary and sufficient conditions for two $\mathfrak{S}^\epsilon$-modules being isomorphic.

Key words: super-BMS$_3$, non-weight module, simple module.

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1 Introduction
Throughout this paper, we denote by $\mathbb{C}, \mathbb{C}^*, \mathbb{Z}, \mathbb{Z}_+$ and $\mathbb{N}$ the sets of complex numbers, nonzero complex numbers, integers, nonnegative integers and positive integers, respectively. All vector superspaces (resp. superalgebras, supermodules) and spaces (resp. algebras, modules) are considered to be over the field of complex numbers.

The super-BMS$_3$ algebra $\mathfrak{S}^\epsilon$ ($\epsilon = 0$ or $\frac{1}{2}$) is defined as an infinite-dimensional Lie superalgebra over $\mathbb{C}$ with basis $\{L_m, W_m, G_r, C_1, C_2 \mid m \in \mathbb{Z}, r \in \epsilon + \mathbb{Z}\}$ and satisfying the following non-trivial relations

\begin{align}
[L_m, L_n] &= (n - m)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}C_1, \\
[L_m, W_n] &= (n - m)W_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}C_2, \\
[L_m, G_r] &= (r - \frac{m}{2})G_{m+r}, \\
[G_r, G_s] &= 2W_{r+s} + \frac{1}{6}(r^2 - \frac{1}{4})\delta_{r+s,0}C_2,
\end{align}

where $m, n \in \mathbb{Z}, \epsilon = 0, \frac{1}{2}, r, s \in \epsilon + \mathbb{Z}$ and $C_1, C_2$ are central elements. In order to study the precise boundary conditions for the gauge field describing the theory, the super-BMS$_3$ algebra was introduced in [1][2]. Clearly, the $\mathbb{Z}_2$-graded of $\mathfrak{S}^\epsilon$ is defined by $\mathfrak{S}^\epsilon = \mathfrak{S}_0 \oplus \mathfrak{S}^1$, where $\mathfrak{S}_0 = \{L_m, W_m, C_1, C_2 \mid m \in \mathbb{Z}\}$ and $\mathfrak{S}^1 = \{G_r \mid r \in \epsilon + \mathbb{Z}\}$. Notice that the subspace spanned by $\{L_m, C_1 \mid m \in \mathbb{Z}\}$ is the Virasoro algebra, which is closely related to the quantum physics, conformal field theory, vertex algebras, and so on (see, e.g., [8][9][11]). The even part $\mathfrak{S}_0$ is exact the $W$-algebra $W(2,2)$ $\mathcal{W}$, which was introduced in [19] for the study of the classification of vertex operator algebras generated by weight 2 vectors. To illustrate the
difference, $\mathfrak{S}^0$ and $\mathfrak{S}^{\frac{1}{2}}$ are respectively called Ramond type super-BMS$_3$ algebra and Neveu-Schwarz type super-BMS$_3$ algebra. It is clear that $\mathfrak{S}^{\frac{1}{2}}$ is isomorphic to the subalgebra of $\mathfrak{S}^0$ spanned by $\{L_m, W_m \mid m \in 2\mathbb{Z}\} \cup \{G_r \mid r \in 2\mathbb{Z} + 1\} \cup \{C_1, C_2\}$.

As an important part of representation theory, non-weight module has drawn a lot of attention worldwide. In recent years, a class of non-weight modules on which the Cartan subalgebra acts freely were constructed and studied. This kind of non-weight modules are called free $U(\mathfrak{h})$-modules by many authors, where $U(\mathfrak{h})$ is the universal enveloping algebra of the Cartan subalgebra $\mathfrak{h}$. It is worth mentioning that the free $U(\mathfrak{h})$-modules were first constructed in [13] for the complex matrices algebra $\mathfrak{sl}_{n+1}$. In addition, these modules were introduced by a very different approach in [16]. From then on, this kind of non-weight modules were widely investigated, such as the finite-dimensional simple Lie algebras [14], the Virasoro algebra [12, 15], the Heisenberg-Virasoro algebra [5, 6], the algebra $Vir(a, b)$ [10]. Furthermore, the super version of this kind of non-weight modules were also studied. In [7], $U(\mathfrak{h})$-modules over the basic Lie superalgebras were investigated, which showed that $\mathfrak{osp}(1|2n)$ is the only basic Lie superalgebra that admits such modules. In [17], the free $U(\mathfrak{h})$-modules of rank 1 over the Ramond $N=1$ algebra and free $U(\mathfrak{h})$-modules of rank 2 over the Neveu-Schwarz $N=1$ algebra were respectively classified. Recently, a family of non-weight modules over the untwisted $N=2$ superconformal algebras were constructed in [18], which regarded as modules over the Cartan subalgebra are free of rank 2. The structure theory of $\mathfrak{S}^\epsilon$ was studied in [3, 4], but few researches involved the representation theory. In this paper, we aim to study the free $U(\mathfrak{t})$-modules of rank 1 and $U(\mathfrak{\Xi})$-modules of rank 2 respectively over the Ramond type and Neveu-Schwarz type of super-BMS$_3$ algebras.

The rest of this paper is organized as follows. In section 2, we construct a class of non-weight modules over the super-BMS$_3$ algebra $\mathfrak{S}^\epsilon$. Then the simplicity and isomorphism classes of these modules are determined. In Section 3, the free $U(\mathfrak{t})$-modules of rank 1 over the Ramond algebra $\mathfrak{S}^0$ are classified. These modules can also be regarded as free $U(\mathfrak{\Xi})$-modules of rank 2. In Section 4, the free $U(\mathfrak{\Xi})$-modules of rank 2 over the Neveu-Schwarz algebra $\mathfrak{S}^{\frac{1}{2}}$ are classified. Furthermore, we present that the category of free $U(\mathfrak{t})$-modules of rank 1 over the Ramond type $\mathfrak{S}^0$ is equivalent to the category of free $U(\mathfrak{\Xi})$-modules of rank 2 over the Neveu-Schwarz type $\mathfrak{S}^{\frac{1}{2}}$.

## 2 Non-weight modules over $\mathfrak{S}^\epsilon$

In this section, we first recall some basic definitions and results. Then we define a class of non-weight modules over $\mathfrak{S}^\epsilon$ which related to Cartan subalgebras without containing the central elements.

Let $V = V_0 \oplus V_1$ be any $\mathbb{Z}_2$-graded vector space. Then any element $v \in V_0$ (resp. $v \in V_1$) is said to be even (resp. odd). Define $|v| = 0$ if $v$ is even and $|v| = 1$ if $v$ is odd. Elements in
$V_0$ or $V_1$ are called homogeneous. Throughout this paper, all elements in superalgebras and modules are homogenous unless specified.

Assume that $G$ is a Lie superalgebra, a $G$-module is a $\mathbb{Z}_2$-graded vector space $V$ together with a bilinear map $G \times V \to V$, denoted $(x,v) \mapsto xv$ such that

$$x(yv) - (-1)^{|x||y|} y(xv) = [x,y]v$$

and

$$\mathcal{G}_i V_j \subseteq V_{i+j}$$

for all $x,y \in G, v \in V$. Thus there is a parity-change functor $\Pi$ on the category of $G$-modules to itself. In other words, for any module $V = V_0 \oplus V_1$, we have a new module $\Pi(V)$ with the same underlining space with the parity exchanged, i.e., $\Pi(V_0) = V_1$ and $\Pi(V_1) = V_0$. We use $U(\mathcal{G})$ to denote the universal enveloping algebra. All modules for Lie superalgebras considered in this paper are $\mathbb{Z}_2$-graded and all irreducible modules are non-trivial.

Fix any $\lambda \in \mathbb{C}^\ast, \alpha \in \mathbb{C}$ and $h(t) \in \mathbb{C}[t]$. For any $m \in \mathbb{Z}$, define

$$h_m(t) = mh(t) - m(m-1)\alpha \frac{h(t) - h(\alpha)}{t - \alpha}.$$

Then

$$nh_n(t) - mh_m(t) + n(t-n\alpha) \frac{\partial}{\partial t} (h_m(t)) = m(t-m\alpha) \frac{\partial}{\partial t} (h_n(t))$$

(2.1)

for any $m,n \in \mathbb{Z}$. Let $\Omega_W(\lambda,\alpha,h) = \mathbb{C}[t,s]$ as a vector space and define the action of $W$ as follows:

$$L_m(f(t,s)) = \lambda^m(s + h_m(t)) f(t,s - m) - m\lambda^m(t-m\alpha) \frac{\partial}{\partial t} (f(t,s-m)),$$

$$W_m(f(t,s)) = \lambda^m(t-m\alpha) f(t,s - m), \ C_1(f(t,s)) = C_2(f(t,s)) = 0,$$

where $m,n \in \mathbb{Z}, \alpha \in \mathbb{C}, h(t) \in \mathbb{C}[t]$.

The following results on $\Omega_W(\lambda,\alpha,h)$ were given in [5].

**Lemma 2.1.** For $\lambda \in \mathbb{C}^\ast, \alpha \in \mathbb{C}, h(t) \in \mathbb{C}[t]$, then

1. $\Omega_W(\lambda,\alpha,h)$ is a $W$-module;
2. $\Omega_W(\lambda,\alpha,h)$ is simple if and only if $\alpha \neq 0$.

Let $V = \mathbb{C}[t^2,s] \oplus t\mathbb{C}[t^2,s]$. Then $V$ is a $\mathbb{Z}_2$-graded vector space with $V_0 = \mathbb{C}[t^2,s]$ and $V_1 = t\mathbb{C}[t^2,s]$. In the following, a precise construction of an $\mathcal{G}^0$-module structure on $V$ will be presented.
Proposition 2.2. For $\lambda \in \mathbb{C}^*, \alpha \in \mathbb{C}, h(t) \in \mathbb{C}[t], f(t^2, s) \in \mathbb{C}[t^2, s]$ and $tf(t^2, s) \in t\mathbb{C}[t^2, s]$, we define the action of Ramond type super-BMS$_3$ algebra $\mathfrak{S}^0$ on $V$ as follows

\[
L_m f(t^2, s) = \lambda^m(s + h_m(t^2))f(t^2, s - m)
- m\lambda^m(t^2 - m\alpha)\frac{\partial}{\partial t^2}(f(t^2, s - m)),
\]

(2.2)

\[
L_m t f(t^2, s) = \lambda^m(s - \frac{m}{2} + h_m(t^2))tf(t^2, s - m)
- m\lambda^m(t^2 - m\alpha)t\frac{\partial}{\partial t^2}f(t^2, s - m),
\]

(2.3)

\[
W_m f(t^2, s) = \lambda^m(t^2 - m\alpha)f(t^2, s - m),
\]

(2.4)

\[
W_m t f(t^2, s) = \lambda^m(t^2 - m\alpha)tf(t^2, s - m),
\]

(2.5)

\[
G_m f(t^2, s) = \lambda^m tf(t^2, s - m),
\]

(2.6)

\[
G_m t f(t^2, s) = \lambda^m (t^2 - 2m\alpha)f(t^2, s - m),
\]

(2.7)

\[
C_i(f(t^2, s)) = C_i(tf(t^2, s)) = 0
\]

(2.8)

for $m \in \mathbb{Z}, i = 1, 2$. Then $V$ is an $\mathfrak{S}^0$-module under the action of (2.2)-(2.8), which is a free of rank 1 as a module over $U(t)$ and denoted by $\Omega_{\mathfrak{S}^0}(\lambda, \alpha, h)$.

Proof. By the trivial relation of (2.8), we omit the central elements in the following. From Lemma 2.1 (1), we obtain

\[
[L_m, L_n] f(t^2, s) = L_m L_n f(t^2, s) - L_n L_m f(t^2, s),
\]

\[
[L_m, W_n] f(t^2, s) = L_m W_n f(t^2, s) - W_n L_m f(t^2, s),
\]

\[
0 = (W_m W_n - W_n W_m) f(t^2, s),
\]

where $m, n \in \mathbb{Z}$. For any $m, n \in \mathbb{Z}$, it follows from (2.3) and (2.5) that we have

\[
L_m L_n t f(t^2, s) = L_m \left( \lambda^n(s - \frac{n}{2} + h_n(t^2))tf(t^2, s - n) \right)
- n\lambda^n(t^2 - n\alpha)t\frac{\partial}{\partial t^2}f(t^2, s - n) \right)
= \lambda^{m+n}\left( s - \frac{m}{2} + h_m(t^2) \right)\left( s - m - \frac{n}{2} + h_n(t^2) \right)f(t^2, s - m - n)
- m\lambda^{m+n}(t^2 - m\alpha)tf(t^2, s - m - n)\frac{\partial}{\partial t^2}h_n(t^2)
- m\lambda^{m+n}(t^2 - m\alpha)t\left( s - m - \frac{n}{2} + h_n(t^2) \right)\frac{\partial}{\partial t^2}f(t^2, s - m - n)
\]
\[-n\lambda^{m+n}(s - \frac{m}{2} + h_m(t^2))t(t^2 - n\alpha)\frac{\partial}{ \partial t^2} f(t^2, s - m - n)\]
\[+ mn\lambda^{m+n}(t^2 - m\alpha)t \frac{\partial}{ \partial t^2} f(t^2, s - m - n)\]
\[+ mn\lambda^{m+n}(t^2 - m\alpha)t(t^2 - n\alpha)\frac{\partial^2}{ \partial^2 (t^2)} f(t^2, s - m - n), \quad (2.9)\]

\[L_m W_n t f(t^2, s) = L_m \left( \lambda^n (t^2 - n\alpha) t f(t^2, s - n) \right)\]
\[= \lambda^{m+n}(s - \frac{m}{2} + h_m(t^2))t(t^2 - n\alpha)f(t^2, s - m - n)\]
\[-m\lambda^{m+n}(t^2 - m\alpha) f(t^2, s - m - n)\]
\[-m\lambda^{m+n}(t^2 - m\alpha)t(t^2 - n\alpha)\frac{\partial}{ \partial t^2} f(t^2, s - m - n) \quad (2.10)\]

and

\[W_n L_m t f(t^2, s) = W_n \left( \lambda^m (s - \frac{m}{2} + h_m(t^2)) t f(t^2, s - m) \right)\]
\[-m\lambda^{m}(t^2 - m\alpha)t \frac{\partial}{ \partial t^2} f(t^2, s - m)\]
\[= \lambda^{m+n}(t^2 - n\alpha)(s - n - \frac{m}{2} + h_m(t^2))t f(t^2, s - m - n)\]
\[-m\lambda^{m+n}(t^2 - n\alpha)(t^2 - m\alpha)t \frac{\partial}{ \partial t^2} f(t^2, s - m - n). \quad (2.11)\]

Then for any \(m, n \in \mathbb{Z}\), by (2.1) and (2.9), we compute that

\[[L_m, L_n] t f(t^2, s)\]
\[= \lambda^{m+n}\left( s - \frac{m}{2} + h_m(t^2) \right) t \left( s - m - \frac{n}{2} + h_n(t^2) \right) f(t^2, s - m - n)\]
\[-m\lambda^{m+n}(t^2 - m\alpha) t f(t^2, s - m - n) \frac{\partial}{ \partial t^2} h_n(t^2)\]
\[-m\lambda^{m+n}(t^2 - m\alpha) t \left( s - m - \frac{n}{2} + h_n(t^2) \right) \frac{\partial}{ \partial t^2} f(t^2, s - m - n)\]
\[-n\lambda^{m+n}\left( s - \frac{m}{2} + h_m(t^2) \right) t (t^2 - n\alpha) \frac{\partial}{ \partial t^2} f(t^2, s - m - n)\]
\[+ mn\lambda^{m+n}(t^2 - m\alpha) t \frac{\partial}{ \partial t^2} f(t^2, s - m - n)\]
\[+ mn\lambda^{m+n}(t^2 - m\alpha)(t^2 - n\alpha)\frac{\partial^2}{ \partial^2 (t^2)} f(t^2, s - m - n)\]
\[-\lambda^{m+n}\left(s - \frac{n}{2} + h_n(t^2)\right) t\left(s - n - \frac{m}{2} + h_m(t^2)\right) f(t^2, s - m - n)\]
\[+ n\lambda^{m+n}(t^2 - n\alpha) tf(t^2, s - m - n) \frac{\partial}{\partial t^2} h_m(t^2)\]
\[+ n\lambda^{m+n}(t^2 - n\alpha) t\left(s - n - \frac{m}{2} + h_m(t^2)\right) \frac{\partial}{\partial t^2} f(t^2, s - m - n)\]
\[+ m\lambda^{m+n}\left(s - \frac{n}{2} + h_n(t^2)\right) t(t^2 - m\alpha) \frac{\partial}{\partial t^2} f(t^2, s - m - n)\]
\[-mn\lambda^{m+n}(t^2 - n\alpha) t \frac{\partial}{\partial t^2} f(t^2, s - m - n)\]
\[-mn\lambda^{m+n}(t^2 - n\alpha)t(t^2 - m\alpha) \frac{\partial^2}{\partial t^2(t^2)} f(t^2, s - m - n),\]
\[= \lambda^{m+n}tf(t^2, s - m - n)\left(n(s - \frac{n}{2}) - m(s - \frac{m}{2}) + nh_n(t^2) - mh_m(t^2)\right)\]
\[+ \lambda^{m+n}tf(t^2, s - m - n)\left(n(t^2 - n\alpha) \frac{\partial}{\partial t^2} h_m(t^2) - m(t^2 - m\alpha) \frac{\partial}{\partial t^2} h_n(t^2)\right)\]
\[+ m^2\lambda^{m+n}(t^2 - m\alpha) t \frac{\partial}{\partial t^2} f(t^2, s - m - n)\]
\[-n^2\lambda^{m+n}(t^2 - n\alpha) t \frac{\partial}{\partial t^2} f(t^2, s - m - n)\]
\[+ mn\lambda^{m+n}(n - m) \alpha t \frac{\partial}{\partial t^2} f(t^2, s - m - n)\]
\[= (n - m)\lambda^{m+n}(s - \frac{m + n}{2} + h_{m+n}(t^2)) tf(t^2, s - m - n)\]
\[-(n - m)(m + n)\lambda^{m+n}(t^2 - (m + n)\alpha) t \frac{\partial}{\partial t^2} f(t^2, s - m - n)\]
\[= (n - m)L_{m+n}f(t^2, s).\]

For any \(m, n \in \mathbb{Z}\), from (2.10) and (2.11), one can check that
\[
[L_m, W_n]f(t^2, s) = (n - m)\lambda^{m+n}(t^2 - (m + n)\alpha) tf(t^2, s - m - n)
= (n - m)W_{m+n}f(t^2, s).
\]

For any \(m, n \in \mathbb{Z}\), by the similar computation, we conclude that
\[
[L_m, G_n]f(t^2, s)
= (n - \frac{m}{2})\lambda^{m+n}tf(t^2, s - m - n)
= (n - \frac{m}{2})G_{m+n}f(t^2, s),
\]
\[ [L_m, G_n]tf(t^2, s) \]
\[ = (n - \frac{m}{2})\lambda^{m+n}(t^2 - 2(m + n)\alpha)f(t^2, s - m - n) \]
\[ = (n - \frac{m}{2})G_{m+n}tf(t^2, s), \]
\[ [G_m, G_n]f(t^2, s) \]
\[ = \lambda^{m+n}(t^2 - 2m\alpha)f(t^2, s - m - n) + \lambda^{m+n}(t^2 - 2n\alpha)f(t^2, s - m - n) \]
\[ = 2W_{m+n}f(t^2, s), \]
\[ [G_m, G_n]f(t^2, s) \]
\[ = \lambda^{m+n}(t^2 - 2n\alpha)f(t^2, s - m - n) + \lambda^{m+n}(t^2 - 2m\alpha)f(t^2, s - m - n) \]
\[ = 2W_{m+n}tf(t^2, s). \]

Finally, for any \( m, n \in \mathbb{Z} \), it is easy to get that
\[ [W_m, G_n]f(t^2, s) = [W_m, G_n]tf(t^2, s) = 0. \]
This completes the proof. \( \square \)

**Theorem 2.3.** Let \( \lambda \in \mathbb{C}^*, \alpha \in \mathbb{C}, h(t) \in \mathbb{C}[t]. \) Then \( \Omega_{\mathfrak{g}^0}(\lambda, \alpha, h) \) is simple if and only if \( \alpha \neq 0. \)

**Proof.** To prove this, we consider two cases in the following.

**Case 1.** \( \alpha = 0. \)

For any \( i \in \mathbb{Z}_+ \), denote \( \pi_i := (t^2)^i\mathbb{C}[t^2, s] \oplus t\mathbb{C}[t^2, s]. \) It is clear that for any \( i \in \mathbb{Z}_+ \), \( \pi_i \) is a submodule of \( \Omega_{\mathfrak{g}^0}(\lambda, 0, h) \) for \( \lambda \in \mathbb{C}^* \) and \( h(t) \in \mathbb{C}[t]. \)

**Case 2.** \( \alpha \neq 0. \)

Let \( P = P_0 \oplus P_1 \) be a nonzero submodule of \( \Omega_{\mathfrak{g}^0}(\lambda, \alpha, h) \). It follows from Lemma 2.1 (2) that \( \Omega_{\mathfrak{g}^0}(\lambda, \alpha, h) \) is a simple \( \mathfrak{g}_0 \)-module. Then one can check that \( P_1 = \Omega_{\mathfrak{g}^0}(\lambda, \alpha, h) \) by (2.6). Consequently, \( P = \Omega_{\mathfrak{g}^0}(\lambda, \alpha, h) \), which implies that \( \Omega_{\mathfrak{g}^0}(\lambda, \alpha, h) \) is a simple \( \mathfrak{g}^0 \)-module. \( \square \)

**Remark 2.4.** For \( i \in \mathbb{Z}_+ \), the quotient module
\[ \bar{\pi}_i := (t^2)^i\mathbb{C}[t^2, s] \oplus t\mathbb{C}[t^2, s] / (t^2)^{i+1}\mathbb{C}[t^2, s] \oplus t\mathbb{C}[t^2, s] \]
is simple if and only if \( i \neq h(0) \). In fact, for any nonzero \( (t^2)^i g(s) \in \bar{\pi}_i \), we have
\[ L_m((t^2)^i g(s)) \equiv \lambda^m(s + mh(0))(t^2)^i g(s - m) - \lambda^m \bar{m}(t^2)^i g(s - m) \]
\[ \equiv \lambda^m(s + m(h(0) - i))(t^2)^i g(s - m) \mod (t^2)^{i+1}\mathbb{C}[t^2, s] \oplus t\mathbb{C}[t^2, s] \]
and
\[ W_m((t^2)^i g(s)) = 0. \]
Theorem 2.5. Let \( \lambda, \mu \in \mathbb{C}^*, \alpha, \beta \in \mathbb{C}, h(t), g(t) \in \mathbb{C}[t] \). Then

(i) \( \Pi(\Omega_{\mathfrak{g}_0}(\lambda, \alpha, h)) \neq \Omega_{\mathfrak{g}_0}(\lambda', \alpha', h') \) for any \( \lambda, \alpha, h \in \mathbb{C}, h'(t) \in \mathbb{C}[t] \);

(ii) \( \Omega_{\mathfrak{g}_0}(\lambda, \alpha, h) \cong \Omega_{\mathfrak{g}_0}(\mu, \beta, g) \) if and only if \( \lambda = \mu, \alpha = \beta \) and \( h = g \).

Proof. (i) Let \( \psi : \Pi(\Omega_{\mathfrak{g}_0}(\lambda, \alpha, h)) \rightarrow \Omega_{\mathfrak{g}_0}(\lambda', \alpha', h') \) be an isomorphism. Let \( \mathbf{1} \) and \( \mathbf{1}' \) be the generators of the free \( \mathbb{C}[L_0, W_0, G_0] \)-modules \( \Pi(\Omega_{\mathfrak{g}_0}(\lambda, \alpha, h)) \) and \( \Omega_{\mathfrak{g}_0}(\lambda', \alpha', h') \), respectively. We see that there exists some \( c, d \in \mathbb{C}^* \) such that \( \psi(\mathbf{1}) = ct\mathbf{1}' \) and \( \psi(\mathbf{1}) = d\mathbf{1}' \). For \( r \in \mathbb{Z} \), we have

\[
G_r \psi(\mathbf{1}) = G_r(c t\mathbf{1}') = c \lambda' (t^2 - 2r \alpha') \mathbf{1}' \quad \text{and} \quad \psi(G_r \mathbf{1}) = d \lambda' \mathbf{1}'.
\]

Clearly, \( G_r \psi(\mathbf{1}) \neq \psi(G_r \mathbf{1}) \), which yields a contradiction.

(ii) The sufficiency is obvious. Suppose \( \Omega_{\mathfrak{g}_0}(\lambda, \alpha, h) \cong \Omega_{\mathfrak{g}_0}(\mu, \beta, g) \) as \( \mathfrak{g}_0 \)-module. Then \( \Omega_{\mathfrak{g}_0}(\lambda, \alpha, h) \cong \Omega_{\mathfrak{g}_0}(\mu, \beta, g) \) as \( \mathfrak{g}_0 \)-modules. By Theorem 3.2 in [5], we have \( \lambda = \mu, \alpha = \beta \) and \( h = g \). \( \square \)

Assume that \( M = \mathbb{C}[t, s] \oplus \mathbb{C}[y, x] \). Then \( M \) is a \( \mathbb{Z}_2 \)-graded vector space with \( M_0 = \mathbb{C}[t, s] \) and \( M_1 = \mathbb{C}[y, x] \). The \( \mathfrak{g}_{\frac{1}{2}} \)-module structure on \( M \) will be characterized in the following.

Proposition 2.6. For \( \lambda \in \mathbb{C}^*, \alpha \in \mathbb{C}, h(t) \in \mathbb{C}[t], h(y) \in \mathbb{C}[y], f(t, s) \in \mathbb{C}[t, s] \) and \( k(y, x) \in \mathbb{C}[y, x] \), we define the action of Neveu-Schwarz type super-BMS\(_3\) algebra \( \mathfrak{g}_{\frac{1}{2}} \) on \( M \) as follows

\[
L_m f(t, s) = \lambda^m (s + h_m(t)) f(t, s - m) - m \lambda^m (t - m \alpha) \frac{\partial}{\partial t} (f(t, s - m)),
\]

(2.12)

\[
L_m k(y, x) = \lambda^m (x - \frac{m}{2} + h_m(y)) k(y, x - m) - m \lambda^m (y - m \alpha) \frac{\partial}{\partial y} k(y, x - m),
\]

(2.13)

\[
W_m f(t, s) = \lambda^m (t - m \alpha) f(t, s - m),
\]

(2.14)

\[
W_m k(y, x) = \lambda^m (y - m \alpha) k(y, x - m),
\]

(2.15)

\[
G_r f(t, s) = \lambda^{- \frac{r}{2}} f(y, x - r),
\]

(2.16)

\[
G_r k(y, x) = \lambda^{\frac{r}{2}} (t - 2r \alpha) k(t, s - r),
\]

(2.17)

\[
C_i \{ f(t, s) \} = C_i (k(y, x)) = 0,
\]

(2.18)

where \( m \in \mathbb{Z}, i = 1, 2, r \in \frac{1}{2} + \mathbb{Z} \). Then \( M \) is an \( \mathfrak{g}_{\frac{1}{2}} \)-module under the action of \( (2.12) - (2.18) \), which is a free of rank 2 as a module over \( U(\mathfrak{s}) \) and denoted by \( \Omega_{\mathfrak{g}_{\frac{1}{2}}}(\lambda, \alpha, h) \).
Proof. By the similar calculations appeared in Lemma 2.1 (1) and Proposition 2.2, we have

\[
[L_m, L_n]f(t, s) = L_m L_n f(t, s) - L_n L_m f(t, s),
\]
\[
[L_m, L_n]k(y, x) = L_m L_n k(y, x) - L_n L_m k(y, x),
\]
\[
[L_m, W_n]f(t, s) = L_m W_n f(t, s) - W_n L_m f(t, s),
\]
\[
[L_m, W_n]k(y, x) = L_m W_n k(y, x) - W_n L_m k(y, x),
\]
\[
(W_m W_n - W_n W_m) f(t, s) = (W_m W_n - W_n W_m) k(y, x) = 0,
\]

where \( m, n \in \mathbb{Z} \). For any \( m \in \mathbb{Z}, r \in \frac{1}{2} + \mathbb{Z} \), we deduce that

\[
[L_m, G_r]f(t, s) = L_m \left( \lambda^{r - \frac{1}{2}} f(y, x - r) \right)
\]
\[
- G_r \left( \lambda^m (s + h_m(t)) f(t, s - m) - m \lambda^m (t - m \alpha) \frac{\partial}{\partial t} (f(t, s - m)) \right)
\]
\[
= (r - \frac{m}{2}) \lambda^{m + r - \frac{1}{2}} f(y, x - m - r)
\]
\[
= (r - \frac{m}{2}) G_{m+r} f(t, s).
\]

For any \( m \in \mathbb{Z}, r, q \in \frac{1}{2} + \mathbb{Z} \), we obtain that

\[
[L_m, G_r]k(y, x) = (r - \frac{m}{2}) \lambda^{m + r + \frac{1}{2}} (t - 2(m + r) \alpha) f(t, s - m - r)
\]
\[
= (r - \frac{m}{2}) G_{m+r} k(y, x),
\]
\[
[G_r, G_p]f(t, s) = 2 \lambda^{r+p} (t - (r + p) \alpha) f(t, s - r - p)
\]
\[
= 2 W_{r+p} f(t, s),
\]
\[
[G_r, G_p]k(y, x) = 2 \lambda^{r+p} (y - (r + p) \alpha) k(y, x - r - p)
\]
\[
= 2 W_{r+p} k(y, x).
\]

At last, for any \( m \in \mathbb{Z}, r \in \frac{1}{2} + \mathbb{Z} \), it is easy to get

\[
[W_m, G_r]f(t, s) = [W_m, G_r] k(y, x) = 0.
\]

This proposition holds. \( \square \)
Assume that $\sigma : \mathfrak{g}^{1 \frac{1}{2}} \to \mathfrak{g}^0$ is a linear map defined by

$$L_m \mapsto \frac{1}{2} L_{2m} + \frac{1}{16} \delta_{m,0} C_1,$$

$$W_m \mapsto \frac{1}{2} W_{2m} + \frac{1}{16} \delta_{m,0} C_2,$$

$$G_r \mapsto \frac{1}{\sqrt{2}} G_{2r},$$

$$C_1 \mapsto 2C_1,$$

$$C_2 \mapsto 2C_2$$

for $m \in \mathbb{Z}, r \in \mathbb{Z} + \frac{1}{2}$. It is straightforward to verify that $\sigma$ is an injective Lie superalgebra homomorphism. Clearly, $\mathfrak{g}^{1 \frac{1}{2}}$ can be regarded as a subalgebra of $\mathfrak{g}^0$. Then $\Omega_{\mathfrak{g}^0}(\lambda, \alpha, h)$ also can be regarded as an $\mathfrak{g}^{1 \frac{1}{2}}$-module. It is straightforward to verify that $\sigma$ is an injective Lie superalgebra homomorphism. Clearly, $\mathfrak{g}^{1 \frac{1}{2}}$ can be regarded as a subalgebra of $\mathfrak{g}^0$. Then $\Omega_{\mathfrak{g}^0}(\lambda, \alpha, h)$ also can be regarded as an $\mathfrak{g}^{1 \frac{1}{2}}$-module.

**Proposition 2.7.** $\Omega_{\mathfrak{g}^0}(\lambda, \alpha, h)$ is a simple $\mathfrak{g}^{1 \frac{1}{2}}$-module if and only if it is a simple $\mathfrak{g}^0$-module.

**Proof.** We only prove sufficiency. From

$$\sigma(L_m)f(t^2, s) = \frac{1}{2} \lambda^{2m} (s + h_{2m}(t^2)) f(t^2, s - 2m) - m \lambda^{2m} (t^2 - 2m\alpha) \frac{\partial}{\partial t^2} (f(t^2, s - 2m)),$$

$$\sigma(W_m)f(t^2, s) = \frac{1}{2} \lambda^{2m} (t^2 - 2m\alpha) f(t^2, s - 2m)$$

and $\alpha \neq 0$, we see that $\Omega_{\mathfrak{g}^0}(\lambda, \alpha, h)_{\bar{0}}$ is a simple $\sigma(W)$-module. Then by $\sigma(G_r)f(t^2, s) = \frac{1}{\sqrt{2}} \lambda^{2r} f(t^2, s - 2r)$, one has $t1 \in \Omega_{\mathfrak{g}^0}(\lambda, \alpha, h)_{\bar{1}}$, which yields that $\Omega_{\mathfrak{g}^0}(\lambda, \alpha, h)$ is a simple $\mathfrak{g}^{1 \frac{1}{2}}$-module. \qed

**Proposition 2.8.** Let $\lambda \in \mathbb{C}^*, \alpha \in \mathbb{C}$ and $g(t), h(t) \in \mathbb{C}[t]$. Suppose that for any $m \in \mathbb{Z}$ $g_m(\frac{1}{2} t) = \frac{1}{2} h_{2m}(t)$. Then as $\mathfrak{g}^{1 \frac{1}{2}}$-modules, we have $\Omega_{\mathfrak{g}^{1 \frac{1}{2}}}(\lambda, \alpha, g) \cong \Omega_{\mathfrak{g}^0}(\sqrt{\lambda}, \alpha, h)$.

**Proof.** Let

$$\Psi : \Omega_{\mathfrak{g}^{1 \frac{1}{2}}}(\lambda, \alpha, g) \longrightarrow \Omega_{\mathfrak{g}^0}(\sqrt{\lambda}, \alpha, h)$$

$$f(t, s) \longmapsto f(\frac{1}{2} t^2, \frac{1}{2} s)$$

$$k(y, x) \longmapsto \sqrt{\frac{\lambda}{2}} k(\frac{1}{2} t^2, \frac{1}{2} s)$$
be the linear map from $\Omega^1_{\mathfrak{S}^1_2}(\lambda, \alpha, g)$ to $\Omega^0_{\mathfrak{S}^0}(\sqrt{\lambda}, \alpha, h)$. Clearly, $\Psi$ is bijective. Now we show that it is an $\mathfrak{S}^1_2$-module isomorphism.

For any $m \in \mathbb{Z}$, $r \in \frac{1}{2} + \mathbb{Z}$, $g(t), h(t) \in \mathbb{C}[t]$, $f(t, s) \in \mathbb{C}[t, s]$, $k(y, x) \in \mathbb{C}[y, x]$, we check that

$$\Psi(L_m f(t, s)) = \Psi(\lambda^m (s + g_m(t)) f(t, s - m)$$

$$- m \lambda^m (t - m\alpha) \frac{\partial}{\partial t}(f(t, s - m))$$

$$= \lambda^m \left( \frac{s}{2} + g_m(\frac{1}{2}t^2) \right) f(\frac{1}{2}t^2, \frac{1}{2}s - m)$$

$$- m \lambda^m (\frac{1}{2}t^2 - m\alpha) \frac{\partial}{\partial (\frac{1}{2}t^2)}(f(\frac{1}{2}t^2, \frac{1}{2}s - m)),$$

$$\Psi(L_m k(y, x)) = \Psi(\lambda^m (x - \frac{m}{2} + g_m(y)) k(y, x - m)$$

$$- m \lambda^m (y - m\alpha) \frac{\partial}{\partial y} k(y, x - m)$$

$$= \sqrt{\frac{\lambda}{2}} \lambda^m \left( \frac{1}{2}s - \frac{m}{2} + g_m(\frac{1}{2}t^2) \right) tk(\frac{1}{2}t^2, \frac{1}{2}s - m)$$

$$- m \sqrt{\frac{\lambda}{2}} \lambda^m (\frac{1}{2}t^2 - m\alpha)t \frac{\partial}{\partial (\frac{1}{2}t^2)} k(\frac{1}{2}t^2, \frac{1}{2}s - m),$$

$$\Psi(W_m f(t, s)) = \Psi(\lambda^m (t - m\alpha) f(t, s - m) = \lambda^m (\frac{1}{2}t^2 - m\alpha) f(\frac{1}{2}t^2, \frac{1}{2}s - m),$$

$$\Psi(W_m k(y, x)) = \sqrt{\frac{\lambda}{2}} \lambda^m (\frac{1}{2}t^2 - m\alpha) tk(\frac{1}{2}t^2, \frac{1}{2}s - m),$$

$$\Psi(G_r f(t, s)) = \Psi(\lambda^{r - \frac{1}{2}} f(y, x - r)) = \frac{\lambda^r}{\sqrt{2}} tf(\frac{1}{2}t^2, \frac{1}{2}s - r),$$

$$\Psi(G_r k(y, x)) = \Psi(\lambda^{r + \frac{1}{2}} (t - 2r\alpha) k(t, s - r)) = \lambda^{r + \frac{1}{2}} (\frac{1}{2}t^2 - 2r\alpha) k(\frac{1}{2}t^2, \frac{1}{2}s - r),$$

$$\Psi(C_i f(t, s))) = \Psi(C_i k(y, x))) = 0$$
for \( i = 1, 2 \). Then by the following calculation

\[
\sigma(L_m)\Psi(f(t, s)) = \frac{1}{2} L_{2m} f\left(\frac{1}{2} t^2, \frac{1}{2} s \right)
\]

\[
= \frac{1}{2} \lambda^m (s + h_{2m}(t^2)) f\left(\frac{1}{2} t^2, \frac{1}{2} (s - 2m) \right)
- m \lambda^m (t^2 - 2m \alpha) \frac{\partial}{\partial(t^2)} f\left(\frac{1}{2} t^2, \frac{1}{2} (s - 2m) \right),
\]

\[
\sigma(L_m)\Psi(k(y, x)) = \frac{1}{2} L_{2m} \sqrt{\frac{\lambda}{2}} tk\left(\frac{1}{2} t^2, \frac{1}{2} s \right)
\]

\[
= \sqrt{\frac{\lambda}{2}} \lambda^m (s - \frac{m}{2} + \frac{1}{2} h_{2m}(t^2)) tk\left(\frac{1}{2} t^2, \frac{1}{2} (s - m) \right)
- m \sqrt{\frac{\lambda}{2}} \lambda^m (t^2 - m \alpha) t \frac{\partial}{\partial(t^2)} k\left(\frac{1}{2} t^2, \frac{1}{2} (s - m) \right),
\]

\[
\sigma(W_m)\Psi(f(t, s)) = \frac{1}{2} W_{2m} f\left(\frac{1}{2} t^2, \frac{1}{2} s \right)
\]

\[
= \lambda^m (\frac{1}{2} t^2 - m \alpha) f\left(\frac{1}{2} t^2, \frac{1}{2} s - m \right),
\]

\[
\sigma(W_m)\Psi(k(y, x)) = \frac{1}{2} W_{2m} \sqrt{\frac{\lambda}{2}} tk\left(\frac{1}{2} t^2, \frac{1}{2} s \right)
\]

\[
= \sqrt{\frac{\lambda}{2}} \lambda^m (\frac{1}{2} t^2 - m \alpha) tk\left(\frac{1}{2} t^2, \frac{1}{2} (s - m) \right),
\]

\[
\sigma(G_r)\Psi(f(t, s)) = \frac{1}{\sqrt{2}} G_{2r} f\left(\frac{1}{2} t^2, \frac{1}{2} s \right)
\]

\[
= \frac{1}{\sqrt{2}} \lambda^r f\left(\frac{1}{2} t^2, \frac{1}{2} s - r \right),
\]

\[
\sigma(G_r)\Psi(k(y, x)) = \frac{1}{\sqrt{2}} G_{2r} \sqrt{\frac{\lambda}{2}} tk\left(\frac{1}{2} t^2, \frac{1}{2} s \right)
\]

\[
= \lambda^r + \frac{1}{4} (\frac{1}{2} t^2 - 2r \alpha) k\left(\frac{1}{2} t^2, \frac{1}{2} s - r \right),
\]

\[
\sigma(C_i(f(t, s))) = \sigma(C_i(k(y, x))) = 0,
\]

one has

\[
\Psi(L_m f(t, s)) = \sigma(L_m)\Psi(f(t, s)), \Psi(L_m k(y, x)) = \sigma(L_m)\Psi(k(y, x)),
\]

\[
\Psi(W_m f(t, s)) = \sigma(W_m)\Psi(f(t, s)), \Psi(W_m k(y, x)) = \sigma(W_m)\Psi(k(y, x)),
\]

\[
\Psi(G_r f(t, s)) = \sigma(G_r)\Psi(f(t, s)), \Psi(G_r k(y, x)) = \sigma(G_r)\Psi(k(y, x)),
\]

\[
\Psi(C_i(f(t, s))) = \sigma(C_i(f(t, s))) = \Psi(C_i(k(y, x))) = \sigma(C_i(k(y, x))) = 0.
\]
for \(i = 1, 2\). Thus, we conclude that \(\Psi\) is an \(\mathfrak{S}^\frac{1}{2}\)-module isomorphism. \(\square\)

Now we give an example of the conditions in Proposition 2.8

**Example 2.9.** Let \(h(t) = t, g(t) = t - \alpha\). For any \(m \in \mathbb{Z}\), we can check

\[
g_m(t) = m(t + \alpha) - 2m\alpha = \frac{1}{2}h_{2m}(t).
\]

According to Propositions 2.7, 2.8 and Theorem 2.3, we see that the following results.

**Corollary 2.10.** Let \(\lambda \in \mathbb{C}^*, \alpha \in \mathbb{C}, h(t) \in \mathbb{C}[t]\). Then \(\Omega_{\mathfrak{S}^\frac{1}{2}}(\lambda, \alpha, h)\) is simple if and only if \(\alpha \neq 0\).

By the similar arguments in the proof of Theorem 2.3, it yields the results as follows.

**Theorem 2.11.** Let \(\lambda, \mu \in \mathbb{C}^*, \alpha, \beta \in \mathbb{C}, h(t), g(t) \in \mathbb{C}[t]\). Then

(i) \(\Pi(\Omega_{\mathfrak{S}^\frac{1}{2}}(\lambda, \alpha, h)) \not\sim \Omega_{\mathfrak{S}^\frac{1}{2}}(\lambda', \alpha', h')\) for any \(\lambda, \lambda' \in \mathbb{C}, h'(t) \in \mathbb{C}[t]\);

(ii) \(\Omega_{\mathfrak{S}^\frac{1}{2}}(\lambda, \alpha, h) \sim \Omega_{\mathfrak{S}^\frac{1}{2}}(\mu, \beta, g)\) if and only if \(\lambda = \mu, \alpha = \beta\) and \(h = g\).

### 3 Classification of free \(U(t)\)-modules of rank 1 over \(\mathfrak{S}^0\)

Assume that \(V = V_0 \oplus V_1\) is an \(\mathfrak{S}^0\)-module such that it is free of rank 1 as a \(U(t)\)-module, where \(t = CL_0 \oplus CW_0 \oplus CG_0\). It follows from the algebra structure of \(\mathfrak{S}^0\) in \(\mathbb{L}1\) that we have

\[
L_0W_0 = W_0L_0, \quad L_0G_0 = G_0L_0, \quad W_0G_0 = G_0W_0, \quad G_0^2 = W_0.
\]

Thus \(U(t) = \mathbb{C}[L_0, W_0] \oplus G_0\mathbb{C}[L_0, W_0]\). Choose a homogeneous basis element \(1\) in \(V\). Without loss of generality, up to a parity, we may assume \(1 \in V_0\) and

\[
V = U(t)1 = \mathbb{C}[L_0, W_0]1 \oplus G_0\mathbb{C}[L_0, W_0]1
\]

with \(V_0 = \mathbb{C}[L_0, W_0]1\) and \(V_1 = G_0\mathbb{C}[L_0, W_0]1\). Then we can suppose that \(L_01 = s1, W_01 = t^21, G_01 = t1\). In the following, \(V = V_0 \oplus V_1\) is equal to \(\mathbb{C}[t^2, s]1 \oplus t\mathbb{C}[t^2, s]1\) with \(V_0 = \mathbb{C}[t^2, s]1\) and \(V_1 = t\mathbb{C}[t^2, s]1\).

**Remark 3.1.** The free \(U(t)\)-module \(V\) of rank 1 over \(\mathfrak{S}^0\) can be viewed as a free \(\mathbb{C}[L_0, W_0]\)-module of rank 2 by defining \(1_0 := 1, 1_1 := G_01\).

According to \(\mathfrak{S}_0 \cong \mathcal{W}\), it is clear that \(V_0\) can be regarded as a \(\mathcal{W}\)-module which is free of rank 1 as a \(\mathbb{C}[L_0, W_0]\)-module. From Lemma 2.11, for any \(m \in \mathbb{Z}\), \(f(t^2, s) \in \mathbb{C}[t^2, s]\), there exist \(\lambda \in \mathbb{C}^*, \alpha \in \mathbb{C}, h(t) \in \mathbb{C}[t]\) such that

\[
L_m(f(t^2, s)) = \lambda^m(s + h_m(t^2))f(t^2, s - m) - m\lambda^m(t^2 - m\alpha)\frac{\partial}{\partial t^2}(f(t^2, s - m)), \quad (3.1)
\]

\[
W_m(f(t^2, s)) = \lambda^m(t^2 - m\alpha)f(t^2, s - m), \quad C_1(f(t^2, s)) = C_2(f(t^2, s)) = 0. \quad (3.2)
\]
Firstly, we give the following results.

**Lemma 3.2.** For \( m \in \mathbb{Z}, f(t^2, s) \in \mathbb{C}[t^2, s], tf(t^2, s) \in t\mathbb{C}[t^2, s], \) we obtain

(i) \( G_m tf(t^2, s)\mathbf{1} = f(t^2, s - m)G_m \mathbf{1}; \)

(ii) \( G_m f(t^2, s)\mathbf{1} = f(t^2, s - m)G_m \mathbf{1}; \)

(iii) \( C_i tf(t^2, s)\mathbf{1} = 0 \) for \( i = 1, 2. \)

**Proof.** (i) It is easy to get that \( G_m L_0 \mathbf{1} = (L_0 - m)G_m \mathbf{1} \) by the relations of \( \mathcal{G}^0. \) Recursively, we conclude that \( G_m L_0^n \mathbf{1} = (L_0 - m)^n G_m \mathbf{1} \) for \( n \in \mathbb{Z}_+. \) Hence, \( G_m tf(t^2, s)\mathbf{1} = f(t^2, s - m)G_m \mathbf{1}. \) Similarly, we obtain (ii).

(iii) By \( (3.2), \) we know that \( C_i f(t^2, s)\mathbf{1} = 0, \) which gives \( C_i \mathbf{1} = 0 \) for \( i = 1, 2. \) Thus \( C_i tf(t^2, s)\mathbf{1} = tf(t^2, s)C_i \mathbf{1} = 0 \) for \( i = 1, 2. \)

**Lemma 3.3.** For \( m \in \mathbb{Z}, \) we get \( G_m \mathbf{1} = \lambda^m \mathbf{1}. \)

**Proof.** To prove this, we suppose \( G_m \mathbf{1} = tf_m(t^2, s)\mathbf{1} \in t\mathbb{C}[t^2, s] \mathbf{1} \) for \( m \in \mathbb{Z}. \) For any \( m \in \mathbb{Z}, \) by \( C_m^2 \mathbf{1} = W_{2m} \mathbf{1}, \) we have

\[
f_m(t^2, s - m)(2\lambda^m(t^2 - m\alpha) - t^2 f_m(t^2, s)) = \lambda^{2m}(t^2 - 2m\alpha). \tag{3.3}
\]

Considering the degree of \( s \) and \( t^2 \) in above equation, we conclude that \( f_m(t^2, s - m) = f_m \in \mathbb{C}. \) Now we rewrite (3.3) as

\[
f_m((2\lambda^m - f_m)t^2 - 2m\lambda^m\alpha) = \lambda^{2m}(t^2 - 2m\alpha),
\]

which implies \( f_m = \lambda^m. \)

Now we present the main result of this section, which gives a complete classification of free \( U(t) \)-modules of rank 1 over \( \mathcal{G}^0. \)

**Theorem 3.4.** Assume that \( V \) is an \( \mathcal{G}^0 \)-module such that the restriction of \( V \) as a \( U(t) \)-module is free of rank 1. Then up to a parity, \( V \cong \Omega_{\mathcal{G}^0}(\lambda, \alpha, h) \) for \( \lambda \in \mathbb{C}^*, \alpha \in \mathbb{C}, h(t) \in \mathbb{C}[t] \) with the \( \mathcal{G}^0 \)-module structure defined as in (2.2) - (2.8).

**Proof.** For any \( f(t^2, s) \in \mathbb{C}[t^2, s], \) by Lemma 3.2 (ii) and Lemma 3.3 we obtain

\[
G_m f(t^2, s)\mathbf{1} = f(t^2, s - m)G_m \mathbf{1} = \lambda^m f(t^2, s - m) t\mathbf{1}. \tag{3.4}
\]
Moreover, we have

\[ L_m tf(t^2, s) \mathbf{1} = L_m G_0 f(t^2, s) \mathbf{1} = G_0 L_m f(t^2, s) \mathbf{1} - \frac{m}{2} G_m f(t^2, s) \mathbf{1} = \lambda^m (s - \frac{m}{2} + h_m(t^2)) f(t^2, s - m) \mathbf{1} - m \lambda^m (t^2 - m \alpha) \frac{\partial}{\partial t^2} (f(t^2, s - m)) \mathbf{1} \]

and

\[ W_m tf(t^2, s) \mathbf{1} = W_m G_0 f(t^2, s) \mathbf{1} = G_0 W_m f(t^2, s) \mathbf{1} = \lambda^m (t^2 - m \alpha) f(t^2, s - m) \mathbf{1}. \]

Suppose \( G_m \mathbf{1} = g_m(t^2, s) \mathbf{1} \in \mathbb{C}[t^2, s] \mathbf{1} \) for \( m \in \mathbb{Z} \). From \( G_m G_m \mathbf{1} = W_{2m} \mathbf{1} \), it is easy to check that \( g_m(t^2, s) = \lambda^m(t^2 - 2m \alpha) \). Thus \( G_m \mathbf{1} = \lambda^m(t^2 - 2m \alpha) \mathbf{1} \) for \( m \in \mathbb{Z} \). Then by Lemma 3.2 (i), one has

\[ G_m tf(t^2, s) \mathbf{1} = \lambda^m(t^2 - 2m \alpha) f(t^2, s - m) \mathbf{1}. \]

\[ \square \]

4 Classification of free \( U(\mathfrak{T}) \)-modules of rank 2 over \( \mathfrak{S}_2 \)

By the definition of Neveu-Schwarz super-BMS\(_3\) algebra \( \mathfrak{S}_2 \), we see that it has a 2-dimensional Cartan subalgebra \( \mathfrak{T} = \mathbb{C} L_0 \oplus \mathbb{C} W_0 \) without containing the center. Then we have \( U(\mathfrak{T}) = \mathbb{C}[L_0, W_0] \). At the same time, we see that \( \{ W_m | m \in \mathbb{Z} \} \) can be generated by odd elements \( G_{\frac{1}{2} + r} \) for \( r \in \mathbb{Z} \). Then the pure even or pure odd non-trivial \( \mathfrak{S}_2 \)-modules are isomorphic to Virasoro modules, namely, the free \( U(\mathfrak{T}) \)-modules of rank 1 are isomorphic to Virasoro modules. Hence, we will classify the free \( U(\mathfrak{T}) \)-modules of rank 2 over the \( \mathfrak{S}_2 \) in the following.

Assume that \( M = M_0 \oplus M_1 \) is an \( \mathfrak{S}_2 \)-module such that it is free of rank 2 as a \( U(\mathfrak{T}) \)-module with two homogeneous basis elements \( v \) and \( w \). If the parities of \( v \) and \( w \) are the same, for any \( r \in \mathbb{Z} + \frac{1}{2} \) then \( G_{\pm r} v = G_{\pm r} w = 0 \). Therefore,

\[ W_0 v = \frac{1}{2} [G_r, G_{-r}] v = 0, W_0 w = \frac{1}{2} [G_r, G_{-r}] w = 0, \]

which gives a contradiction. Then we conclude that \( v \) and \( w \) have different parities. Set \( v = \mathbf{1}_0 \in M_0 \) and \( w = \mathbf{1}_1 \in M_1 \). As a vector space, we get \( M_0 = \mathbb{C}[t, s] \mathbf{1}_0 \) and \( M_1 = \mathbb{C}[y, x] \mathbf{1}_1 \).
By $\mathcal{S}_0 \cong \mathcal{W}$, we know that $M_0$ and $M_1$ are both can be viewed as $\mathcal{W}$-modules. According to Lemma 2.1 there exist $\lambda, \mu \in \mathbb{C}^*$, $\alpha, \beta \in \mathbb{C}$, $h(t) \in \mathbb{C}[t]$ and $g(y) \in \mathbb{C}[y]$ such that

\[ L_m f(t, s) = \lambda^m(s + h_m(t))f(t, s - m) \]
\[ -m\lambda^m(t - m\alpha)\frac{\partial}{\partial t}(f(t, s - m)), \quad (4.1) \]
\[ W_m f(t, s) = \lambda^m(t - m\alpha)f(t, s - m), C_i(f(t, s)) = 0 \quad (4.2) \]

for $i = 1, 2$, $f(t, s) \in \mathbb{C}[t, s]$ and

\[ L_m f(y, x) = \mu^m(x + g_m(y))f(y, x - m) \]
\[ -m\mu^m(y - m\beta)\frac{\partial}{\partial y}(f(y, x - m)), \quad (4.3) \]
\[ W_m f(y, x) = \mu^m(y - m\beta)f(y, x - m), C_i(f(y, x)) = 0 \quad (4.4) \]

for $i = 1, 2$, $f(y, x) \in \mathbb{C}[y, x]$.

Now we give the following two preliminary results for later use.

**Lemma 4.1.** Let $\lambda, \mu \in \mathbb{C}^*$, $\alpha, \beta \in \mathbb{C}$, $f(t, s) \in \mathbb{C}[t, s], k(y, x) \in \mathbb{C}[y, x]$. Then $\lambda = \mu, \alpha = \beta$ and there exists $c \in \mathbb{C}^*$ such that one of the following two cases occurs.

(i) $G_{\frac{1}{2}}1_{\bar{0}} = c1_{1_{\bar{1}}}, G_{\frac{1}{2}}1_{\bar{1}} = \frac{1}{c}\lambda(t - \alpha)1_{\bar{0}}$;

(ii) $G_{\frac{1}{2}}1_{\bar{0}} = \frac{1}{c}\lambda(y - \alpha)1_{1_{\bar{1}}}, G_{\frac{1}{2}}1_{\bar{1}} = c1_{\bar{0}}$.

**Proof.** Assume that $G_{\frac{1}{2}}1_{\bar{0}} = f(y, x)1_{\bar{1}}$ and $G_{\frac{1}{2}}1_{\bar{1}} = g(t, s)1_{\bar{0}}$. According to $[G_{\frac{1}{2}}, G_{\frac{1}{2}}]1_{\bar{0}} = 2W_11_{\bar{0}}$, we get

\[ G_{\frac{1}{2}}^21_{\bar{0}} = f(W_0, L_0 - \frac{1}{2})G_{\frac{1}{2}}1_{\bar{1}} = f(t, s - \frac{1}{2})g(t, s)1_{\bar{0}} \]

and $G_{\frac{1}{2}}^21_{\bar{1}} = W_11_{\bar{0}} = \lambda(t - \alpha)$, which imply $f(t, s - \frac{1}{2})g(t, s) = \lambda(t - \alpha)$. Therefore,

\[ f(t, s - \frac{1}{2}) = c, g(t, s) = \frac{1}{c}\lambda(t - \alpha) \quad \text{or} \quad f(t, s - \frac{1}{2}) = \frac{1}{c}\lambda(t - \alpha), g(t, s) = c \quad (4.5) \]

for $c \in \mathbb{C}^*$. Similarly, by $G_{\frac{1}{2}}^21_{\bar{1}} = W_11_{\bar{1}}$, we see that $g(y, x - \frac{1}{2})f(y, x) = \mu(y - \beta)$. Then by (4.5), we always obtain $g(y, x - \frac{1}{2})f(y, x) = \lambda(y - \alpha)$, which gives $\lambda = \mu, \alpha = \beta$.  

Due to Lemma 4.1 up to a parity, we can assume $\lambda = \mu, \alpha = \beta, G_{\frac{1}{2}}1_{\bar{0}} = 1_{\bar{1}}$ and $G_{\frac{1}{2}}1_{\bar{1}} = \lambda(t - \alpha)1_{\bar{0}}$ without loss of generality.

**Lemma 4.2.** For any $m \in \mathbb{Z}, r \in \frac{1}{2} + \mathbb{Z}$, we get
(i) $g_m(y) - h_m(y) = -\frac{m}{2};$

(ii) $G_r \mathbf{1}_0 = \lambda^{r - \frac{1}{2}} \mathbf{1}_1;$

(iii) $G_r \mathbf{1}_1 = \lambda^{r + \frac{1}{2}} (t - 2r \alpha) \mathbf{1}_0.$

**Proof.** (i) Fix $r \in \frac{1}{2} + \mathbb{Z}$ and $r \neq \frac{3}{2}.$ Based on Lemma 4.1, we get

\[
\left(\frac{3}{4} - \frac{1}{2}r\right)G_r \mathbf{1}_0 = [L_{r - \frac{1}{2}}, G_{r \frac{1}{2}}] \mathbf{1}_0
\]

\[
= L_{r - \frac{1}{2}} G_{r \frac{1}{2}} \mathbf{1}_0 - G_{r \frac{1}{2}} L_{r - \frac{1}{2}} \mathbf{1}_0
\]

\[
= L_{r - \frac{1}{2}} \mathbf{1}_1 - G_{r \frac{1}{2}} \left(\lambda^{r - \frac{1}{2}} (s + h_{r - \frac{1}{2}}(t))\right) \mathbf{1}_0
\]

\[
= \lambda^{r - \frac{1}{2}} \left(g_{r - \frac{1}{2}}(y) + \frac{1}{2} - h_{r - \frac{1}{2}}(y)\right) \mathbf{1}_1
\]

and

\[
\left(\frac{3}{4} - \frac{1}{2}r\right)G_r \mathbf{1}_1 = [L_{r - \frac{1}{2}}, G_{r \frac{1}{2}}] \mathbf{1}_1
\]

\[
= L_{r - \frac{1}{2}} G_{r \frac{1}{2}} \mathbf{1}_1 - G_{r \frac{1}{2}} L_{r - \frac{1}{2}} \mathbf{1}_1
\]

\[
= L_{r - \frac{1}{2}} \left(\lambda(t - \alpha)\right) \mathbf{1}_0 - G_{r \frac{1}{2}} \left(\lambda^{r - \frac{1}{2}} (x + g_{r - \frac{1}{2}}(y))\right) \mathbf{1}_1
\]

\[
= \lambda^{r + \frac{1}{2}} \left(h_{r - \frac{1}{2}}(t) - g_{r - \frac{1}{2}}(t) + \frac{1}{2}\right)(t - \alpha) - (r - \frac{1}{2})(t - (r - \frac{1}{2})\alpha) \mathbf{1}_0.
\]

(4.6)

By (4.6) and (4.7), we check

\[
\lambda^{2r} \left(g_{r - \frac{1}{2}}(t) + \frac{1}{2} - h_{r - \frac{1}{2}}(t)\right) \left(h_{r - \frac{1}{2}}(t) - g_{r - \frac{1}{2}}(t) + \frac{1}{2}\right)(t - \alpha)
\]

\[-(r - \frac{1}{2})(t - (r - \frac{1}{2})\alpha))\right) \mathbf{1}_0
\]

\[
= \left(\frac{3}{4} - \frac{1}{2}r\right) G_r \mathbf{1}_0 = \left(\frac{3}{4} - \frac{1}{2}r\right)^2 W_{2r} \mathbf{1}_0
\]

\[
= \lambda^{2r} \left(\frac{3}{4} - \frac{1}{2}r\right)^2 \left(t - 2r \alpha\right) \mathbf{1}_0,
\]

which implies

\[
\left(g_{r - \frac{1}{2}}(t) + \frac{1}{2} - h_{r - \frac{1}{2}}(t)\right) \left(h_{r - \frac{1}{2}}(t) - g_{r - \frac{1}{2}}(t) + \frac{1}{2}\right)(t - \alpha)
\]

\[-(r - \frac{1}{2})(t - (r - \frac{1}{2})\alpha))\right) = \left(\frac{3}{4} - \frac{1}{2}r\right)^2 \left(t - 2r \alpha\right).
\]

(4.8)
Set \( \tau_r(t) := g_{r,\frac{1}{2}}(t) - h_{r,\frac{1}{2}}(t) \). We rewrite (4.8) as

\[
(\tau_r(t) + \frac{1}{2}) \left( \left( \frac{1}{2} - \tau_r(t) \right)(t - \alpha) - (r - \frac{1}{2})(t - (r - \frac{1}{2})\alpha) \right) = \left( \frac{3}{4} - \frac{1}{2}r \right)^2(t - 2r\alpha).
\]

Comparing the coefficients of \( t \) in above equation, we immediately get

\[
(\tau_r(t) + \frac{1}{2})^2 - \frac{3 - 2r}{2}(\tau_r(t) + \frac{1}{2}) + \left( \frac{3 - 2r}{4} \right)^2 = 0.
\]

Thus \( \tau_r(t) + \frac{1}{2} = \frac{3 - 2r}{4} \) for \( r \neq \frac{3}{2} \). From \([L_1, G_{\frac{1}{2}}]1_0 = 0\), it is easy to get \( g_1(y) - h_1(y) = -\frac{1}{2} \).

Then \( g_m(y) - h_m(y) = -\frac{m}{2} \) for \( m \in \mathbb{Z} \).

(ii) Using (i) in (4.6), we obtain \( G_r1_0 = \lambda^{-\frac{r}{2}}1_1 \) for any \( r \in \frac{1}{2} + \mathbb{Z} \) and \( r \neq \frac{3}{2} \). Moreover, we check that

\[
-\frac{3}{2}G_{\frac{1}{2}}1_0 = [L_2, G_{\frac{1}{2}}]1_0 = L_2G_{\frac{1}{2}}1_0 - G_{\frac{1}{2}}L_21_0 = \frac{3}{2}\lambda1_1.
\]

Hence, part (ii) also hold for \( r = \frac{3}{2} \).

By the similar discussion as (ii), the (iii) clears. \( \square \)

**Lemma 4.3.** For any \( r \in \frac{1}{2} + \mathbb{Z} \), \( h(t) \in \mathbb{C}[t], f(t, s) \in \mathbb{C}[t, s], g(y) \in \mathbb{C}[y], f(y, x) \in \mathbb{C}[y, x] \), we have

\[
G_r f(t, s)1_0 = \lambda^{-\frac{r}{2}}f(y, x - r)1_1, \quad G_r f(y, x)1_1 = \lambda^{r + \frac{1}{2}}(t - 2r\alpha)f(t, s - r)1_0.
\]

**Proof.** According to \( G_rL_0 = (L_0 - r)G_r \) and Lemma 4.2 we obtain that

\[
G_r f(t, s)1_0 = G_r f(W_0, L_0)1_1 = f(W_0, L_0 - r)G_r1_0 = \lambda^{-\frac{r}{2}}f(y, x - r)1_1,
\]

\[
G_r f(y, x)1_1 = G_{\frac{1}{2}} f(W_0, L_0)1_1 = f(W_0, L_0 - r)G_r1_1 = \lambda^{r + \frac{1}{2}}(t - \alpha)f(t, s - r)1_0.
\]

The lemma holds. \( \square \)

We present the main result of this section, which gives a complete classification of free \( U(\mathfrak{g}) \)-modules of rank 2 over \( \mathfrak{g}_{\frac{1}{2}} \).

From (4.1), (4.2), (4.3), (4.4) and Lemmas 4.1, 4.3 we conclude that the results as follows.

**Theorem 4.4.** Assume that \( M \) is an \( \mathfrak{g}_{\frac{1}{2}} \)-module such that the restriction of \( M \) as a \( U(\mathfrak{g}) \)-module is free of rank 2. Then up to a parity, we have \( M \cong \Omega_{\mathfrak{g}_{\frac{1}{2}}} \lambda, \alpha, h \) for \( \lambda \in \mathbb{C}^*, \alpha \in \mathbb{C}, h(t) \in \mathbb{C}[t] \) with the \( \mathfrak{g}_{\frac{1}{2}} \)-module structure defined in (2.12)-(2.18).

It follows from Proposition 2.7, Theorem 3.4 and Theorem 4.4 that gives

**Corollary 4.5.** The category of free \( U(\mathfrak{g}) \)-modules of rank 1 over \( \mathfrak{g}^0 \) is equivalent to the category of free \( U(\mathfrak{g}) \)-modules of rank 2 over \( \mathfrak{g}_{\frac{1}{2}} \).

**Remark 4.6.** We note that these modules are new classes of simple modules over \( \mathfrak{g}^\epsilon \).
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