Coarse grained and fine dynamics in trapped ion Raman schemes

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Abstract

A novel result concerning Raman coupling schemes implemented using trapped ions is obtained. By means of an operator perturbative approach, it is shown that the complete time evolution of these systems can be expressed, with a high degree of accuracy, as the product of two unitary evolutions. The first one describes the time evolution related to an effective coarse grained dynamics. The second is a suitable correction restoring the fine dynamics suppressed by the coarse graining performed to adiabatically eliminate the nonresonantly coupled atomic level.

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1 Introduction

Trapped ions provide an effective platform for observing interesting aspects of quantum mechanics and for realizing useful applications in the context of quantum computation [1, 2, 3].

In these traps, a time-dependent quadrupolar electromagnetic field is responsible for a charged particle motion which may be kinematically assimilated to the motion of a massive spot subjected to a quadratic potential. Such a circumstance provides the possibility of describing the center of mass of an ion confined into a rf Paul trap as a quantum harmonic oscillator [4, 5, 6]. In addition the ion possesses atomic degrees of freedom related to the electronic motion around the nucleus [2, 4, 5, 6, 7].

Acting upon the system via laser fields, it is possible to induce vibronic transitions described by Jaynes-Cummings-like Hamiltonians [7, 8]. Such interactions are characterized by nonlinearities governed by the so called Lamb-Dicke parameter which, in a spherically symmetric trap, is nothing but the ratio between the width of the ion vibrational ground wave function and the laser wavelength. Controlling the Lamb-Dicke parameter leaving unchanged the laser frequency would then provide the possibility of implementing a wider variety of Hamiltonian models. Unfortunately, these two parameters are strictly related, the Lamb-Dicke parameter being proportional to the inverse of the wavelength, hence proportional to the laser frequency. Therefore, they cannot be independently adjusted. Nevertheless, such a possibility can be achieved exploiting a fundamental property of Raman coupling schemes. In these couplings a three-level system is subjected to a far detuned Λ-scheme. Under some assumptions and within approximations concerning the time scale used to observe the system, the dynamics may be well enough described via an effective Hamiltonian thinkable as a Jaynes-Cummings-like Hamiltonian related to an effective laser field, with frequency and wave-vector given by the differences between the corresponding parameters of the two real Λ-scheme lasers. Then, changing the angle between the two real laser propagation directions leads to the possibility of obtaining effective lasers such that the product of their wavelength and frequency is not the velocity of light. The price to pay is the complete ignorance about the detailed system dynamics at a fine time scale.

In this paper we will try to overcome such a limit by means of a new approach. More precisely, we will analyze the dynamics of a three-level trapped ion subjected to a Raman Λ coupling scheme using a very convenient perturbative decomposition of the evolution operator of the system. We will show that the Raman scheme time evolution can be factorized, at the second perturbative order (hence with a high degree of accuracy), into two unitary evolutions. The first one can be interpreted as the effective time evolution which may be obtained adiabatically eliminating one of the three atomic levels involved in the coupling scheme [9]. Such a dynamics concerns the coarse grained variables and is in accordance with already well known results [7]. The second unitary evolution introduces the correction necessary to take into account the fine deviation from the coarse grained time evolution.
The physical system

The physical system on which we focus is a three-level harmonically trapped ion subjected to a Raman coupling scheme involving atomic transitions. The relevant Schrödinger picture Hamiltonian of the Raman Λ-scheme is then given by

\[ \hat{H}_\Lambda(t) = \sum_{l=1,2,3} \hbar \omega_l \hat{\sigma}_l + \hbar \nu \sum_{\alpha=x,y,z} \hat{a}^\dagger_{\alpha} \hat{a}_\alpha 
+ \left[ \hbar g_{13} e^{-i(\vec{k}_{13} \cdot \vec{r} - \omega_{13} t)} \hat{\sigma}_{13} + h.c. \right] 
+ \left[ \hbar g_{23} e^{-i(\vec{k}_{23} \cdot \vec{r} - \omega_{23} t)} \hat{\sigma}_{23} + h.c. \right], \tag{1} \]

where \( \hat{\sigma}_{kl} \equiv |k\rangle \langle l| \) (with \( k, l = 1, 2, 3 \)), \( \{ |l\rangle \} \) being the considered three atomic levels and \( \{ \hbar \omega_l \} \) the corresponding energies; \( \hat{a}_\alpha \) (\( \alpha = x, y, z \)) is the annihilation operator related to the center of mass harmonic motion along the direction \( \alpha \) (we will denote the associated Fock basis by \( \{ \psi_\alpha^n \} \)). For the sake of simplicity (but without loss of generality), we have assumed to deal with a 3D degenerate parabolic trap with single frequency \( \nu \). The two laser fields responsible for the coupling terms are characterized by complex strengths (proportional to the laser amplitude and to the atomic dipole operator, and including the laser phases), wave vectors and frequencies \( g_{13}, \vec{k}_{13}, \omega_{13} \) and \( g_{23}, \vec{k}_{23}, \omega_{23} \) respectively.

The level \( |3\rangle \) is assumed to be dipole-coupled to both the levels \( |1\rangle \) and \( |2\rangle \) via far detuned lasers. Precisely, the two laser frequencies are chosen in such a way that

\[ \Delta \equiv \omega_3 - \omega_1 - \omega_{13} = \omega_3 - \omega_2 - \omega_{23}, \tag{2} \]

where the detuning \( \Delta \) satisfies the condition

\[ |\Delta| >> |g_{13}|, |g_{23}|, \nu. \tag{3} \]

The analysis of such an Hamiltonian model has been already carried out, for instance in ref. [9], providing the adiabatic elimination of the nonresonantly coupled atomic level \( |3\rangle \) following the path pointed out in refs. [10, 11] and including the motional degrees of freedom. Indeed, due to the large detuning, the transitions for instance from the level \( |1\rangle \) to the level \( |3\rangle \) are very fast and immediately followed by decays on the atomic level \( |2\rangle \). Therefore, considering only coarse grained observables, meaning that the system is observed at a “rough enough time scale”, effectively eliminates the far detuned level; namely, at such a time scale, the only observables and hence meaningful dynamical behaviors, involve levels \( |1\rangle \) and \( |2\rangle \) as a result of time averaging second order processes having \( |3\rangle \) as an intermediate virtual level. This procedure hence suppresses the fine dynamics, that is it sacrifices any information concerning the fast dynamics the third level is involved in.

In the following we present a perturbative approach to the solution of the dynamical problem related to \( \hat{H}_\Lambda \) overcoming the limit of the coarse graining, making it possible to study also the fine dynamics discarded by the adiabatic
elimination. The first step consists in passing to a rotating frame, meaning that the time-dependent Hamiltonian \( \hat{H}_\Lambda \) is canonically transformed via the operator

\[
\hat{R}(t) = e^{-i\hat{A}t},
\]

where

\[
\hat{A} = (\omega_3 - \Delta) (\hat{\sigma}_{11} + \hat{\sigma}_{22} + \hat{\sigma}_{33}) - \omega_{13} \hat{\sigma}_{11} - \omega_{23} \hat{\sigma}_{22}
\]

\[
= \omega_1 \hat{\sigma}_{11} + \omega_2 \hat{\sigma}_{22} + (\omega_3 - \Delta) \hat{\sigma}_{33},
\]

into the following time-independent rotating frame Hamiltonian:

\[
\hat{H} := \hat{R}^\dagger (\hat{H}_\Lambda(t) - \hat{A}) \hat{R}(t) = \hbar \Delta \left( \hat{H}_0 + \hat{H}_B + \hat{H}_\updownarrow \right),
\]

where \( \hat{H} \) is a dimensionless Hamiltonian which is the sum of the three hermitian operators \( \hat{H}_0 \), \( \hat{H}_B \) and \( \hat{H}_\updownarrow \) defined as

\[
\hat{H}_0 := \hat{\sigma}_{33},
\]

\[
\hat{H}_B := \frac{\nu}{\Delta} \sum_{\alpha=x,y,z} \hat{a}_\alpha^\dagger \hat{a}_\alpha,
\]

\[
\hat{H}_\updownarrow := \left[ \frac{g_{13}}{\Delta} e^{-i\vec{k}_{13} \cdot \vec{r}} \hat{\sigma}_{13} + h.c. \right] + \left[ \frac{g_{23}}{\Delta} e^{-i\vec{k}_{23} \cdot \vec{r}} \hat{\sigma}_{23} + h.c. \right].
\]

Considering the assumption given by the inequality (3), both \( \hat{H}_B \) and \( \hat{H}_\updownarrow \) may be thought of as perturbations with respect to \( \hat{H}_0 \). In fact, introducing the dimensionless perturbative parameter

\[
\lambda := \frac{g}{\Delta}, \quad g \equiv \max\{\nu, |g_{13}|, |g_{23}|\},
\]

both \( \hat{H}_B \) and \( \hat{H}_\updownarrow \) are first order perturbations in \( \lambda \):

\[
\hat{H} = \hat{H}(\lambda) = \hat{H}_0 + \lambda \sum_{\alpha=x,y,z} \hat{a}_\alpha^\dagger \hat{a}_\alpha + \lambda \sum_{j=1,2} \left[ \chi_{j,3} e^{-i\vec{k}_{j3} \cdot \vec{r}} \hat{\sigma}_{j3} + h.c. \right],
\]

where \( \chi \equiv \nu / g \leq 1 \), \( \chi_{j,3} \equiv g_{j,3} / g \), \( |\chi_{j,3}| \leq 1 \), and we notice that, due to condition (3), \( \lambda \ll 1 \). It is worth noting that the circumstance that \( \hat{H}_B \) is treated as a perturbation leads to the eccentric situation of an unperturbed Hamiltonian, \( \hat{H}_0 \), wherein the bosonic degrees of freedom are absent. Nevertheless, as we shall see, such a mathematical artifice reveals fruitful in order to succeed in factorizing the coarse grained dynamics and its fine correction.

Our solving procedure relies on a suitable canonical transformation \( e^{i\hat{Z}(\lambda)} \) of the rotating frame Hamiltonian such that

\[
e^{i\hat{Z}(\lambda)} \hat{R} e^{-i\hat{Z}(\lambda)} = \hbar \Delta \left( \hat{H}_0 + \hat{C}(\lambda) \right),
\]

where \( \hat{C}(\lambda), \hat{Z}(\lambda) \) depend analytically on the perturbative parameter \( \lambda \) and \( \hat{C}(\lambda) \) is a constant of motion with respect to the unperturbed dynamics, i.e.
\[ [\hat{H}_0, \hat{C}(\lambda)] = 0. \] This transformation allows to give a very convenient decomposition of the evolution operator associated with the rotating frame Hamiltonian, namely
\[
\exp\left(-\frac{i}{\hbar} t\right) = e^{-i\hat{Z}(\lambda)} \exp(i\Delta \hat{H}_0 t) \exp(i\Delta \hat{C}(\lambda) t) e^{i\hat{Z}(\lambda)}. \tag{11}
\]

At this point, truncating the power expansions
\[
\hat{C}(\lambda) = \lambda \hat{C}_1 + \lambda^2 \hat{C}_2 + \cdots + \lambda^n \hat{C}_n + \cdots, \quad \hat{Z}(\lambda) = \lambda \hat{Z}_1 + \lambda^2 \hat{Z}_2 + \cdots + \lambda^n \hat{Z}_n + \cdots
\]
at a given perturbative order, one obtains by formula (11) useful expressions of the evolution operator. This procedure has been developed in a general setting in refs. [12, 13]. In the next section, we want to recall briefly the mathematical background and to show how the operators \( \hat{C}_1, \hat{C}_2, \ldots, \hat{Z}_1, \hat{Z}_2, \ldots \) can be computed by a suitable iterative process. Then, we will give the explicit solutions for our case up to the second perturbative order.

3 Perturbative analysis of the rotating frame Hamiltonian

Let \( \hat{H}_u, \hat{H}_p \) be hermitian operators and assume that \( \hat{H}_u \) has a purely discrete spectrum. Denote by \( E_0 < E_1 < E_2 < \ldots \) the (possibly degenerate) eigenvalues of \( \hat{H}_u \) and by \( \hat{P}_0, \hat{P}_1, \hat{P}_2, \ldots \) the associated eigenprojectors. Now, consider the operator \( \hat{H}(\lambda) = \hat{H}_u + \lambda \hat{H}_p, \lambda \in \mathbb{C} \), which is hermitian if \( \lambda \) is real. It is possible to show that, under certain conditions [14], there exist positive constants \( r_0, r_1, r_2, \ldots \) and a simply connected neighbourhood \( \mathcal{I} \) of zero in \( \mathbb{C} \) such that the following contour integral on the complex plane
\[
\hat{P}_m(\lambda) = \frac{i}{2\pi} \oint_{|E-E_m|=r_m} \left( \hat{H}(\lambda) - E \right)^{-1} dE, \quad \lambda \in \mathcal{I},
\]
defines a projection \( \hat{P}_m(\lambda)^2 = \hat{P}_m(\lambda) \), which is an orthogonal projection for real \( \lambda \), with \( \hat{P}_m(0) = \hat{P}_m \), and \( \mathcal{I} \ni \lambda \mapsto \hat{P}_m(\lambda) \) is an analytic operator-valued function. Moreover, the range of \( \hat{P}_m(\lambda) \) is an invariant subspace for \( \hat{H}(\lambda) \), hence
\[
\hat{H}(\lambda) \hat{P}_m(\lambda) = \hat{P}_m(\lambda) \hat{H}(\lambda) \hat{P}_m(\lambda), \tag{12}
\]
and there exists an analytic family \( \hat{U}(\lambda) \) of invertible operators such that
\[
\hat{P}_m = \hat{U}(\lambda) \hat{P}_m(\lambda) \hat{U}(\lambda)^{-1}, \quad \hat{U}(0) = \text{Id}, \tag{13}
\]
and \( \hat{U}(\lambda) = e^{i\hat{Z}(\lambda)} \), \( \lambda \in \mathcal{I} \), with \( \hat{Z}(\lambda^*) = \hat{Z}(\lambda)^\dagger \) (hence, for real \( \lambda \), \( \hat{Z}(\lambda) \) is hermitian and \( \hat{U}(\lambda) \) is unitary), where \( \mathcal{I} \ni \lambda \mapsto \hat{Z}(\lambda) \) is analytic. One can show easily that the function \( \lambda \mapsto \hat{U}(\lambda) \) is not defined uniquely by condition (13) even in the simplest case when \( \hat{H}_u \) has a nondegenerate spectrum. Anyway, the nonuniqueness in the definition of \( \hat{U}(\lambda) \) is not relevant if one is only interested in

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obtaining an expression of the evolution operator associated with \( \hat{H}(\lambda) \) of the general form \( \text{11} \). We will see soon that there is a natural condition which fixes a unique solution for \( \hat{U}(\lambda) \).

Now, let us define the operator

\[
\hat{K}(\lambda) := \hat{U}(\lambda) \hat{H}(\lambda) \hat{U}(\lambda)^{-1},
\]

(14)

which, for real \( \lambda \), is unitarily equivalent to \( \hat{H}(\lambda) \). Using relations \( \text{12} \) and \( \text{13} \), we find

\[
\hat{K}(\lambda) \hat{P}_m = \hat{U}(\lambda) \hat{H}(\lambda) \hat{P}_m \hat{U}(\lambda)^{-1} = \hat{U}(\lambda) \hat{P}_m (\lambda) \hat{H}(\lambda) \hat{P}_m (\lambda) \hat{U}(\lambda)^{-1}
\]

and hence: \( \hat{K}(\lambda) \hat{P}_m = \hat{P}_m \hat{K}(\lambda) \hat{P}_m \). It follows that \( \hat{K}(\lambda) = \hat{K}(\lambda) \) and then we obtain the following important decomposition formula:

\[
\hat{U}(\lambda) \hat{H}(\lambda) \hat{U}(\lambda)^{-1} = \hat{H}_u + \hat{C}(\lambda),
\]

(15)

where \( [\hat{C}(\lambda), \hat{H}_u] = 0 \), i.e. \( \hat{C}(\lambda) \) is a constant of the motion with respect to the time evolution generated by \( \hat{H}_u \). At this point, we can obtain perturbative expressions of the unknown operators \( \hat{C}(\lambda), \hat{U}(\lambda) \) by means of a recursive algebraic procedure.

Indeed, since the functions \( \lambda \mapsto \hat{C}(\lambda) \) and \( \lambda \mapsto \hat{Z}(\lambda) \) are analytic in \( \mathcal{I} \) and \( \hat{C}(0) = \hat{Z}(0) = 0 \), we can write:

\[
\hat{C}(\lambda) = \sum_{n=1}^{\infty} \lambda^n \hat{C}_n, \quad \hat{Z}(\lambda) = \sum_{n=1}^{\infty} \lambda^n \hat{Z}_n \quad \lambda \in \mathcal{I}.
\]

In order to determine the operators \( \{\hat{C}_n\} \) and \( \{\hat{Z}_n\} \), we substitute the exponential form \( e^{i\hat{Z}(\lambda)} \) of \( \hat{U}(\lambda) \) in formula \( \text{15} \) thus getting

\[
\hat{H}(\lambda) + \sum_{n=1}^{\infty} \frac{i^n}{n!} \text{ad}_{\hat{Z}(\lambda)}^n \hat{H}(\lambda) = \hat{H}_u + \hat{C}(\lambda),
\]

where we recall that \( \text{ad}_{\hat{Z}(\lambda)} \hat{H}(\lambda) := [\hat{Z}(\lambda), \hat{H}(\lambda)] \).

Next, inserting the power expansions \( \text{16} \) in this equation, in correspondence to the various perturbative orders, we obtain the following set of conditions:

\[
\hat{C}_1 - i [\hat{Z}_1, \hat{H}_u] - \hat{H}_p = 0, \quad [\hat{C}_1, \hat{H}_u] = 0
\]

\[
\hat{C}_2 - i [\hat{Z}_2, \hat{H}_u] + \frac{1}{2} [\hat{Z}_1, [\hat{Z}_1, \hat{H}_u]] - i [\hat{Z}_1, \hat{H}_p] = 0, \quad [\hat{C}_2, \hat{H}_u] = 0
\]

\[
\vdots
\]

where we have taken into account also the additional constraint \( [\hat{C}(\lambda), \hat{H}_u] = 0 \).

This infinite set of equations can be solved recursively. The first equation, together with the first constraint, determines \( \hat{Z}_1 \) up to an operator commuting
with \( \hat{H}_n \) and \( \hat{C}_1 \) uniquely and so on. It is convenient to eliminate the arbitrariness in the determination of the operators \( \{ \hat{Z}_n \} \) choosing the minimal solution characterized by the additional condition \( \sum_m \hat{P}_m \hat{Z}_n \hat{P}_m = 0 \), \( n = 1, 2, \ldots \).

In our particular case, we have the following identifications:

\[
\begin{aligned}
\begin{cases}
\hat{H}_n = \hat{H}_0, \\
\hat{H}_B = \hat{H}_B + \hat{H}_2.
\end{cases}
\end{aligned}
\] (17)

Notice that the two (infinitely degenerate) eigenspaces of the unperturbed Hamiltonian \( \hat{H}_0 \) are associated with the eigenprojectors

\[
\{ \hat{P}_m \}_{m=g,c} = \{ \hat{P}_g = 1_B \otimes (\hat{\sigma}_{11} + \hat{\sigma}_{22}) \, , \, \hat{P}_c = 1_B \otimes (\hat{\sigma}_{33}) \},
\] (18)

where

\[
1_B = \sum_{n_x,n_y,n_z} (|\psi_{n_x}^x\rangle \langle \psi_{n_x}^x|) \otimes \cdots \otimes (|\psi_{n_z}^z\rangle \langle \psi_{n_z}^z|)
\] (19)

is the identity in the vibrational Hilbert space. Accordingly, for the operators \( \{ \hat{C}_1, \hat{Z}_1, \hat{C}_2, \hat{Z}_2, \ldots \} \) forming the minimal solution, we get at the first perturbative order the following expressions:

\[
\begin{aligned}
\begin{cases}
\lambda \hat{C}_1 = \sum_{m=g,c} \hat{P}_m \left( \hat{H}_B + \hat{H}_2 \right) \hat{P}_m, \\
\lambda \hat{Z}_1 = i \sum_{j \neq k} \hat{P}_j \left( \hat{H}_B + \hat{H}_2 \right) \hat{P}_k.
\end{cases}
\end{aligned}
\] (20)

Similarly, at the second order, we have:

\[
\begin{aligned}
\begin{cases}
\lambda^2 \hat{C}_2 = \sum_{m=g,c} \hat{P}_m \left\{ i \hat{Z}_1 \hat{H}_B + \hat{H}_1 - \frac{1}{2} \left[ \hat{Z}_1, \hat{H}_1 \right] \right\} \hat{P}_m, \\
\lambda^2 \hat{Z}_2 = i \sum_{j \neq k} \hat{P}_j \left\{ i \hat{Z}_1 \hat{H}_B + \hat{H}_1 - \frac{1}{2} \left[ \hat{Z}_1, \hat{H}_1 \right] \right\} \hat{P}_k.
\end{cases}
\end{aligned}
\] (21)

Eventually, performing explicit calculations, we find that

\[
\lambda \hat{C}_1 = \hat{H}_B,
\] (22)

\[
\lambda^2 \hat{C}_2 = - \frac{|g_{13}|^2}{\Delta^2} \hat{\sigma}_{11} - \frac{|g_{23}|^2}{\Delta^2} \hat{\sigma}_{22} + \frac{|g_{13}|^2 + |g_{23}|^2}{\Delta^2} \hat{\sigma}_{33}
\]

\[
- \left( \frac{g_{13}g_{32}}{\Delta^2} e^{-i\vec{k}_{13} \cdot \vec{r}} e^{i\vec{k}_{23} \cdot \vec{r}} \hat{\sigma}_{12} + h.c. \right),
\] (23)

where we have set \( g_{3j} = g_{j3} \), and

\[
\lambda \hat{Z}_1 = i \left( \frac{g_{13}}{\Delta} e^{-i\vec{k}_{13} \cdot \vec{r}} \hat{\sigma}_{13} - h.c. \right) + i \left( \frac{g_{23}}{\Delta} e^{-i\vec{k}_{23} \cdot \vec{r}} \hat{\sigma}_{23} - h.c. \right),
\] (24)

\[
\lambda^2 \hat{Z}_2 = \nu \left\{ \left( \frac{g_{13}}{\Delta} \hat{X}_{13} \hat{\sigma}_{13} + \frac{g_{31}}{\Delta} \hat{X}_{31} \hat{\sigma}_{31} \right) + \left( \frac{g_{23}}{\Delta} \hat{X}_{23} \hat{\sigma}_{23} + \frac{g_{32}}{\Delta} \hat{X}_{32} \hat{\sigma}_{32} \right) \right\},
\] (25)
where:

\[
\begin{align*}
\hat{X}_j^3 &:= i[e^{-i\vec{k}_j \cdot \vec{r}}, \sum_{\alpha=x,y,z} \hat{a}_\alpha \hat{a}^\dagger_\alpha], \\
\hat{X}_3^j &:= i[e^{i\vec{k}_j \cdot \vec{r}}, \sum_{\alpha=x,y,z} \hat{a}_\alpha \hat{a}^\dagger_\alpha] = \hat{X}_j^3,
\end{align*}
\]

with \(j = 1, 2\).

The interpretation of this result leads to a very interesting fact. Indeed, it turns out that once the unitary transformation \(e^{i\hat{Z}(\lambda)}\) has been applied to the rotating frame Hamiltonian \(\hat{\mathcal{H}}\) (recall eq. (10)), the time evolution of the system is described, at the second order in the parameter \(\lambda\), by the Hamiltonian

\[
\hat{\mathcal{H}}^{12} = \hat{\mathcal{H}}_1^{12} + \hat{\mathcal{H}}_3,
\]

(26)

where, in order to display a more transparent formula, we set

\[
\begin{align*}
\hat{\mathcal{H}}_1^{12} &:= \hbar \nu \sum_{\alpha=x,y,z} (\hat{a}_\alpha \hat{a}^\dagger_\alpha) \otimes (\hat{\sigma}_{11} + \hat{\sigma}_{22}) + \hbar \tilde{\omega}_1 \hat{\sigma}_{11} + \hbar \tilde{\omega}_2 \hat{\sigma}_{22} \\
&\quad + \left[ \hbar g_{12} e^{-i\vec{k}_{12} \cdot \vec{r}} \hat{\sigma}_{12} + h.c. \right], \\
\hat{\mathcal{H}}_3 &:= \hbar \nu \sum_{\alpha=x,y,z} (\hat{a}_\alpha \hat{a}^\dagger_\alpha) \otimes \hat{\sigma}_{33} + \hbar (\Delta + \tilde{\omega}_3) \hat{\sigma}_{33},
\end{align*}
\]

(27)

(28)

with:

\[
\begin{align*}
\tilde{\omega}_j &= -\frac{|g_{j3}|^2}{\Delta}, \quad j = 1, 2, \\
\tilde{\omega}_3 &= \frac{|g_{13}|^2 + |g_{23}|^2}{\Delta}, \quad g_{12} = \frac{g_{13} g_{32}}{\Delta}, \quad \vec{k}_{12} = \vec{k}_{13} - \vec{k}_{23}.
\end{align*}
\]

Thus, the transformed Hamiltonian is the sum of two decoupled Hamiltonians \(\hat{\mathcal{H}}_{12}\) and \(\hat{\mathcal{H}}_3\), \([\hat{\mathcal{H}}_{12}, \hat{\mathcal{H}}_3] = 0\), ‘living’ respectively in the ranges of the orthogonal projectors \(\hat{P}_g\) and \(\hat{P}_e\). This is a consequence of the fact that \([\hat{P}_m, \hat{C}_n] = 0\), \(m = g, e\), \(n = 1, 2, \ldots\). It is worth noting that the Hamiltonian \(\hat{\mathcal{H}}_{12}\) can be regarded as the rotating frame Hamiltonian of a trapped two-level ion in interaction with a laser field characterized by the following parameters:

\[
\begin{align*}
\omega_{12} &\equiv \omega_{13} - \omega_{32} = \omega_2 - \omega_1, \\
\vec{k}_{12} &\equiv \vec{k}_{13} - \vec{k}_{23}.
\end{align*}
\]

(29)

This effective coupling can be compared with the result found performing the adiabatic elimination of the level \(|3\rangle\) (see ref. [9]). We will come back to this point in the next section.

### 4 Dynamics of the Raman scheme

The question of what the complete dynamics of the system is now arises. First, it will be convenient to adopt the following notation. Given a couple of functions...
Next, let us denote by $\hat{T}_\Lambda$ the evolution operator associated with the Raman scheme:

$$i\hbar \left( \frac{d}{dt} \hat{T}_\Lambda \right) (t) = \hat{H}_\Lambda(t) \hat{T}_\Lambda(t), \quad \hat{T}_\Lambda(0) = \text{Id.} \quad (30)$$

Expressing $\hat{T}_\Lambda$ in terms of the evolution operator associated with the rotating frame Hamiltonian yields:

$$\hat{T}_\Lambda(t) = \hat{R}(t) \exp\left(-\frac{i}{\hbar} \hat{\mathcal{H}}_\Lambda t \right). \quad (31)$$

Now, according to what we have shown in the previous section, we have:

$$\hat{T}(t) := \exp\left(-\frac{i}{\hbar} \hat{\mathcal{H}}_\Lambda t \right) \approx e^{-i\hat{\mathcal{Z}}(\lambda)} \exp\left(-i\Delta e^{i\hat{\mathcal{Z}}(\lambda)} \hat{H}(\lambda) e^{-i\hat{\mathcal{Z}}(\lambda)} t \right) e^{i\hat{\mathcal{Z}}(\lambda)} 
\approx e^{-i(\lambda \hat{Z}_1 + \lambda^2 \hat{Z}_2)} e^{-i\Delta (\hat{H}_0 + \lambda \hat{C}_1 + \lambda^2 \hat{C}_2) t} e^{i(\lambda \hat{Z}_1 + \lambda^2 \hat{Z}_2)}, \quad (32)$$

where we have truncated the power expansions of $\hat{Z}(\lambda)$ and $\hat{C}(\lambda)$ at the second order in $\lambda$. Formula (32) provides an approximate expression of the evolution operator in the remarkable form of a one-parameter group of unitary transformations. Nevertheless, in order to achieve an approximate expression allowing a direct comparison with the coarse grained dynamics, we still need to perform some manipulation. To this aim, observe that, since the commutators $[\lambda^2 \hat{C}_2, \lambda \hat{Z}_1 + \lambda^2 \hat{Z}_2]$ and $[\lambda^2 \hat{C}_2, \hat{H}_0 + \lambda \hat{C}_1] = [\lambda^2 \hat{C}_2, \lambda \hat{C}_1]$ are of the third order in $\lambda$, maintaining our degree of approximation we can write

$$e^{-i(\lambda \hat{Z}_1 + \lambda^2 \hat{Z}_2)} e^{-i\Delta (\hat{H}_0 + \lambda \hat{C}_1 + \lambda^2 \hat{C}_2) t} \approx \lambda^2 e^{-i\Delta \lambda \hat{C}_2 t} e^{-i\Delta (\hat{H}_0 + \lambda \hat{C}_1) t}, \quad (33)$$

Therefore, we can manipulate the second order expression (32) of $\hat{T}(t)$ as follows:

$$\hat{T}(t) \approx e^{-i\Delta \lambda \hat{C}_2 t} e^{-i(\lambda \hat{Z}_1 + \lambda^2 \hat{Z}_2)} e^{-i\Delta (\hat{H}_0 + \lambda \hat{C}_1) t} e^{i(\lambda \hat{Z}_1 + \lambda^2 \hat{Z}_2)}$$

Finally, we find a remarkable decomposition of $\hat{T}$:

$$\hat{T}(t) \approx \hat{T}_\Lambda(t) \hat{T}_1(t), \quad (33)$$
where we have set

$$T_e(t) := \exp\left(-i \Delta \left( \hat{H}_0 + \lambda \hat{C}_1 + \lambda^2 \hat{C}_2 \right) t \right),$$

$$\hat{\Omega}_l(t) := \exp\left(-i \left( \lambda \hat{Z}_1(t) + \lambda^2 \hat{Z}_2(t) \right) \right) \exp\left(i \left( \lambda \hat{Z}_1 + \lambda^2 \hat{Z}_2 \right) \right),$$

with \( \hat{Z}_k(t) \equiv e^{i \Delta (\hat{H}_0 + \lambda \hat{C}_1) t} \hat{Z}_k e^{-i \Delta (\hat{H}_0 + \lambda \hat{C}_1) t}, k = 1, 2. \) It is worth emphasizing that, by a completely analogous procedure\(^1\), we get also

$$\hat{T}(t) \overset{\lambda^2}{\cong} \hat{T}_l(t) \hat{T}_e(t), \quad \hat{T}_l(t) := \hat{T}_l(-t)^\dagger.$$  

Notice that, due to the specific dependence of \( \hat{T}_l \) on \( t \), one has that \( \hat{\Omega}_l(t) \neq \hat{T}_l(t). \)

In the light of formula (33) (or (36)), the time evolution in the rotating frame, given by \( \hat{T}_l \), may be thought of as a process consisting of two fundamental components. One of these is an effective time evolution, described by \( \hat{T}_e \), in which the levels \(|1\rangle, |2\rangle \) are decoupled from the level \(|3\rangle \). The other component, the one \( \hat{T}_l \) (or \( \hat{T}_l') \) is responsible for, is a correction to \( \hat{T}_e \) and involves fast transitions (the operators \( \hat{Z}_1(t) \) and \( \hat{Z}_2(t) \) oscillate at the detuning frequency \( \Delta \)) from and to the third atomic level. Considering the complete time evolution (31), observe that the unitary evolution described by \( \hat{R} \hat{T}_e \) corresponds to the effective dynamics obtained in (22) restricting the analysis to the coarse grained observables. In fact, \( \hat{R} \hat{T}_e \) is the evolution operator associated with the time-dependent effective Hamiltonian

$$\hat{H}_e = \hat{H}^{(12)}_e + \hat{H}^{(3)}_e,$$

where

$$\hat{H}^{(12)}_e := \hbar \nu \sum_{\alpha=x,y,z} \left( \hat{a}_\alpha^\dagger \hat{a}_\alpha \right) \otimes \left( \hat{\sigma}_{11} + \hat{\sigma}_{22} \right) + \hbar (\omega_1 + \tilde{\omega}_1) \hat{\sigma}_{11} + \hbar (\omega_2 + \tilde{\omega}_2) \hat{\sigma}_{22}$$

$$+ \left[ \hbar g_{12} e^{-i(\hat{\sigma}_{12} - \omega_{12} t) \hat{H}_e + h.c.} \right],$$

$$\hat{H}^{(3)}_e := \hbar \nu \sum_{\alpha=x,y,z} \left( \hat{a}_\alpha^\dagger \hat{a}_\alpha \right) \otimes \hat{\sigma}_{33} + \hbar (\omega_3 + \tilde{\omega}_3) \hat{\sigma}_{33}.$$  

We remark that we have deduced this result analytically; no adiabatic approximation has been performed. We also stress that a relevant difference between \( \hat{T}_e \) and \( \hat{T}_l \) consists in the kind of time dependence. Indeed, on one hand, the unitary evolution \( \hat{T}_e \) forms a one-parameter group, hence it is expressible as the exponential of a generator multiplied by \( t \). It follows that a truncated power expansion of the exponential retains its validity only on a finite time span. On

\(^1\)The calculation may be carried on directly, i.e. step by step as in the previous case but changing the reordering of the exponentials in (22) and subsequent formulae, or from (36) exploiting the fact that \( \hat{T}(t) = \hat{T}(-t)^\dagger \).
the other hand, $\hat{T}_f$ can be expressed as the exponential of an operator whose time dependence involves only sinusoidal factors. In fact, we have:

\[
\hat{T}_f \approx e^{-i(\lambda(\hat{Z}_1(t) - \hat{Z}_1) + \lambda^2(\hat{Z}_2(t) - \hat{Z}_2))} e^{\frac{1}{2}[\lambda\hat{Z}_1(t) + \lambda^2\hat{Z}_2(t), \lambda\hat{Z}_1 + \lambda^2\hat{Z}_2]}
\]

where $[\hat{Z}_1(t), \hat{Z}_1] = 0$ has been used. It then follows that the truncated expansion

\[
\hat{T}_f \approx 1 - i\lambda\left(\hat{Z}_1(t) - \hat{Z}_1\right) - i\lambda^2\left(\hat{Z}_2(t) - \hat{Z}_2\right) - \frac{1}{2}\lambda^2\left(\hat{Z}_1(t) - \hat{Z}_1\right)^2
\]

is legitimated independently on time.

Summarizing, we have shown that the standard adiabatic elimination technique provides an effective dynamics, described by $\hat{R}\hat{T}_e$, that differs from the complete second order dynamics of the Raman scheme for the presence of another unitary evolution which can be cast in the form of the exponential of a rapidly oscillating operator function of time. Therefore, the factorization into a coarse grained and a fine dynamics given by eq. (33) makes the correction to the adiabatic approximation solution very readable and easy to be calculated, in view of the expression (40). It is worth noting that the corrections due to $\hat{T}_f$ are small in amplitude, since the operators $\lambda\hat{Z}_1, \lambda\hat{Z}_1(t)$ and $\lambda^2\hat{Z}_2, \lambda^2\hat{Z}_2(t)$ are respectively of the first and second order in the perturbative parameter. Moreover, they provide terms oscillating at the detuning frequency. Hence, as expected, the fine dynamics is small and fast. As a conclusive remark, we wish to emphasize that this micro-fast behavior brought to the light with the help of the method applied here should become of practical interest in connection with time resolution improvements of experiments.

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