Stationary Dilatons with Arbitrary Electromagnetic Field

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Abstract
We present two new classes of axisymmetric stationary solutions of
the Einstein-Maxwell-Dilaton equations with coupling constant \( \alpha^2 = 3 \). Both classes
are written in terms of two harmonic maps \( \lambda \) and \( \tau \). \( \lambda \) determines the
gravitational potential and \( \tau \) the electromagnetic one in such a form that we can have
an arbitrary electromagnetic field. As examples we generate two solutions with mass \( (M) \),
rotation \( (s) \) and scalar \( (\delta) \) parameters, one with electric charge \( (q) \) another one with
magnetic dipole \( (Q) \) parameter. The first solution contains the Kerr metric for \( q = \delta = 0 \).

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1 Introduction

There exist in the Universe a great amount of astrophysical objects possessing gravitational and electromagnetic fields together. Such is the case of some planets and stars possessing a magnetic dipole field like the Earth or the Sun. The Einstein-Maxwell (EM) theory predicts the existence of gravitational objects endowed with magnetic dipoles by means of complicate exact solutions [1, 2]. EM theory is actually not an unification’s theory, since here the electromagnetic field appears as energy-momentum tensor, there is in fact no explanation for its existence, it is put by hand and the electromagnetic field appears like a model. On the other hand, five-dimensional (5D) gravity is an alternative theory for understanding gravitational and electromagnetic interactions together. In this theory the electromagnetic field is a consequence of a more general unified field, it is not a model. Nevertheless, it contains an extra field not observed in nature, a scalar field called the dilaton. Scalar fields are not new in physics, they appear naturally in all the most important unified theories, like the Kaluza-Klein [3] and the super string theory, and it is put by hand in unified models like inflation or the standard model, but nobody has seen them. In [4] it is shown that there exists a class of very simple static exact solutions of the Einstein-Maxwell-Dilaton field equations where the coupling constant $\alpha$ between the scalar field and electromagnetism, remains arbitrary. It possesses a gravitational and a magnetic dipole field; its four-dimensional (4D) metric behaves very similarly than the Schwarzschild solution but coupled with a magnetic dipole. They are very simple, but they posses a scalar field interaction arising from the compactification from the 5D space, which has not yet been observed in astrophysical objects. Nevertheless, in [4] it is shown that for these solutions the dilaton interaction cannot be measured in weak gravitational fields like the Sun, even if the Sun would posses one, but it could be probably detected in stronger gravitational fields like one of a pulsar. A star like the sun is essentially static, but a pulsar is essentially not, therefore it is worth to generalize this solution for a rotating body. One should expect that the scalar field interaction does not modify the interaction of test particles for non-compact bodies like the sun, even if the rotation is taken into account. But one expects that a rotating solution gives more inside in the behavior of rotating compact bodies.

In [4], T. Matos developed a method for generating exact static solutions for the 5D Einstein equations with a $G_3$ group of motion, putting the
solutions in terms of two harmonic maps, λ and τ. The harmonic map λ determines the gravitational field and the harmonic map τ the electromagnetic one. Therefore one can choose the electromagnetic field of a monopole, dipole, quadripole, etc. In this work we generalize a set of static solutions for \( \alpha^2 = 3 \). The first class represents an electrically charged static body with mass and scalar field parameters, the second one represents a magnetic dipole with mass and scalar field parameters. We obtain their corresponding rotating solution using invariant transformations in the potential space. Unfortunately this method can be used only for \( \alpha^2 = 3 \) or \( \alpha^2 = 0 \), only in these two cases the potential space is symmetric (see ref. [4]). This work is organized as follows. In section 2 we introduce the potential space formalism for 5D-gravity. In section 3 we write the field equations in chiral form. In section 4 we generalize the first class of solutions and in section 5 the second class. We write them in terms of two harmonic maps and give some explicit solutions in section 6. In section 7 we give some conclusions and remarks.

2 Potential space field equations

In this section we introduce the potential formalism for 5D-gravity. This theory is characterized by the existence of a Killing vector field \( \mathbf{X} \), which generate the \( U(1) \) gauge electromagnetic group. Introducing an extra time-like Killing vector field, Neugebauer introduced the potential formalism in 5D-gravity. The field equations are then of stationary fields. This formalism consists in defining covariantly five potentials in terms of the two commuting Killing vectors \( \mathbf{X} \) and \( \mathbf{Y} \), \( \mathbf{X} \) being related to the \( U(1) \) isometry and \( \mathbf{Y} \) being related to stationarity. The five potentials are (see also ref. [5])

\[
I^2 = \kappa^4 = X^\mu X_\mu \quad f = -I Y^\mu Y_\mu + I^{-1} (X^\mu Y_\mu)^2 \\
\psi = -I^{-2} X^\mu Y_\mu \quad \epsilon_{\mu \nu} = \epsilon_{\alpha \beta \gamma \delta \mu} X^\alpha Y^\beta Y^{\gamma \delta} \\
\chi_{\nu \mu} = -\epsilon_{\alpha \beta \gamma \delta \mu} X^\alpha Y^\beta X^{\gamma \delta} 
\]

being \( \epsilon_{\alpha \beta \gamma \delta \mu} \) the Levi-Civita pseudotensor. In the adapted coordinate system where \( X = X^A \frac{\partial}{\partial x^A} = \frac{\partial}{\partial x^5} \), \( Y = Y^A \frac{\partial}{\partial x^A} = \frac{\partial}{\partial x_4} \), one finds that \( \Psi^A = (f, \epsilon, \psi, \chi, \kappa), A = 1, ..., 5 \) are the gravitational, rotational, electrostatic, magnetostatic and scalar potentials, respectively. The five dimensional field equa-
tions in terms of the potentials (1) can be derived from the Lagrangian \[6, 5\]

\[ L = \rho \frac{2}{f^2} \left[ f_{;j} f^{;i} + (\epsilon_{;i} + \psi_{;i} \chi_{;i})(e^{;i} + \psi^{;i}) \right] + \rho \frac{2}{f} \left( \kappa^2 \psi_{;i} \psi^{;i} + \frac{1}{\kappa^2} \chi_{;i} \chi^{;i} \right) + \frac{2}{3} \rho \kappa_{;i} \kappa^{;i} \]

(variation with respect to \(\Psi^A\)). Now we can define a 5D (abstract) Riemannian space \(V_5\), called the potential space, inspired on the Lagrangian (2) with metric

\[ dS^2 = \frac{1}{2f^2} \left[ df^2 + (d\epsilon + \psi d\chi)^2 \right] + \frac{1}{2f} \left( \kappa^2 d\psi^2 + \frac{1}{\kappa^2} d\chi^2 \right) + \frac{2}{3} \frac{d\kappa^2}{\kappa^2}. \]

(3)

On \(V_5\), the five potentials \(\Psi^A\) are the local coordinates which define a symmetric Riemannian space, i.e. the covariant derivative of the curvature tensor of \(V_5\) with respect to each coordinate, vanishes.

### 3 The chiral form of the field equations

Axisymmetry is represented by the existence of a third space like Killing vector field \(Z\). Then one can choose a coordinate system in which the components of the five metric depend only on two coordinates. In this case the matrix representation of the 5D Einstein field equations in the potential space \(V_5\) is \[7\]

\[ (\rho g_{;z} g^{-1})_{,z} + (\rho g_{;z} g^{-1})_{,z} = 0 \]

(4)

where Weyl’s canonical coordinates \(\rho\) and \(\zeta\) are related by \(z = \rho + i\zeta\) and its complex conjugate \(\bar{z}\). Matrix \(g\) in \(4\) is a symmetric matrix, element of the group \(SL(3, R)\), i.e.

\[ g = g^T, \quad g = \bar{g}, \quad det \ g = 1 \]

(5)

(T denotes matrix transposition). Using the invariant transformations of Lagrangian \(3\), we generate new solutions from a seed one. That means that if we have a solution of the field equation \(\Psi^A\), an invariant transformation \(3\) of the form \(\Psi^A \rightarrow \Psi'^A(\Psi^B)\) will give us a new solution (see ref. \(8\)). All invariant transformations of \(3\) were found in ref. \(7\). The group of motion of metric \(3\) is \(SL(3, R)\), which has 8 parameters. The invariant transformations of Lagrangian \(2\) can be cast into the very simple form

\[ g = Cg_0 C^T \]

(6)
where the constant matrix \( C \) is also an element of \( SL(3, \mathbb{R}) \). The matrix \( g \) can be parametrized in terms of the potentials (1) as \([7]\)

\[
g = \frac{-2}{f \kappa^3} \begin{pmatrix}
    f^2 + \epsilon^2 - f \kappa^2 \psi^2 & -\epsilon & -\frac{1}{2\sqrt{2}}(\epsilon \chi + f \kappa^2 \psi) \\
    -\epsilon & 1 & \frac{1}{2\sqrt{2}} \chi \\
    -\frac{1}{2\sqrt{2}}(\epsilon \chi + f \kappa^2 \psi) & \frac{1}{2\sqrt{2}} \chi & \frac{1}{8}(\chi^2 - \kappa^2 f)
\end{pmatrix}.
\] (7)

The correct choice of the \( C \) matrix in (6) will generate new solutions. We use this method for generating the Belinsky-Ruffini solution \([10]\).

### 4 First class of solutions. Seed solution \( \Psi^A = (f, \epsilon, 0, 0, \kappa) \).

In order to obtain the new solution, we put the potentials \( \Psi^A \) in terms of the components of the matrix \( g \) and its inverse

\[
g^{-1} = -\frac{1}{2} \frac{\kappa^3}{f} \begin{pmatrix}
    1 & \frac{\epsilon + \chi \psi}{f^2 + (\epsilon + \chi \psi)^2 - f \chi^2 \kappa^{-2}} & \frac{-2\sqrt{2}\psi}{2\sqrt{2}[f \chi \kappa^{-2} - \psi(\epsilon + \chi \psi)] - 8(f \kappa^{-2} - \psi^2)} \\
    \frac{\epsilon + \chi \psi}{f^2 + (\epsilon + \chi \psi)^2 - f \chi^2 \kappa^{-2}} & f^2 + (\epsilon + \chi \psi)^2 - f \chi^2 \kappa^{-2} & \frac{-2\sqrt{2}\psi}{2\sqrt{2}[f \chi \kappa^{-2} - \psi(\epsilon + \chi \psi)] - 8(f \kappa^{-2} - \psi^2)} \\
    \frac{-2\sqrt{2}\psi}{2\sqrt{2}[f \chi \kappa^{-2} - \psi(\epsilon + \chi \psi)] - 8(f \kappa^{-2} - \psi^2)} & \frac{-2\sqrt{2}\psi}{2\sqrt{2}[f \chi \kappa^{-2} - \psi(\epsilon + \chi \psi)] - 8(f \kappa^{-2} - \psi^2)} & f^2 + (\epsilon + \chi \psi)^2 - f \chi^2 \kappa^{-2}
\end{pmatrix},
\] (8)

thus the potentials \( \Psi^A \) can be written as

\[
\kappa^3 = \frac{4g_{11}^{-1}}{g_{22}}, \quad f^2 = \frac{1}{g_{11}^{-1} g_{22}}, \quad \chi = \frac{2\sqrt{2}g_{23}}{g_{22}}, \\
\psi = \frac{1}{2\sqrt{2} g_{11}^{-1}}, \quad \epsilon = -\frac{g_{12}}{g_{22}},
\] (9)

where \( g_{ij} \) are the components of the matrix \( g \), and \( g_{ij}^{-1} \) the components of the matrix \( g^{-1} \). As seed solution we write \( f_0, \epsilon_0 \) and \( \kappa_0 \) arbitrary and \( \psi_0 = \chi_0 = 0 \) in matrix \( g_0 \) in (3). Using the invariant transformations (6) with the \( C \) matrix as

\[
C = \begin{pmatrix}
    a & b & c \\
    d & e & j \\
    i & h & k
\end{pmatrix}
\] (10)

we evaluate matrix \( g \) using the relations (3), obtaining

\[
\kappa^3 = \frac{v}{W} \kappa^3_0, \quad f^2 = \frac{f^2_0}{v W},
\]
\[ \psi = \frac{1}{2\sqrt{2V}} \left[ (u\epsilon_0 + q)(w\epsilon_0 + t) + f_0(wu f_0 - 8sz\kappa_0^{-2}) \right], \]
\[ \chi = \frac{2\sqrt{2}}{W} \left[ (de_0 - e)(i\epsilon_0 - h) + f_0\left(di f_0 - \frac{j}{8}k j \kappa_0^2\right)\right], \]
\[ \epsilon = -\frac{1}{W} \left[ (ae_0 - b)(de_0 - e) + f_0\left(ad f_0 - \frac{j}{8}c j \kappa_0^2\right)\right], \]

with
\[ V = (q + u\epsilon_0)^2 + f_0\left(u^2 f_0 - 8s^2\kappa_0^{-2}\right) \]
and
\[ W = (d\epsilon_0 - e)^2 + f_0\left(d^2 f_0 - \frac{j^2\kappa_0^2}{8}\right). \]

(11)

Solution (11) is endowed with eight new free parameters. This transformation (11) yields a solution for the potentials \( \Psi^A \) from an arbitrary seed solution without electromagnetism. If we want to obtain an asymptotically flat solution from a same one, we must fix some of the eight parameters (see ref [7]). In general, one should give an explicit seed solution in (11) and integrate equations (1) in order to obtain the corresponding metric in the space-time. In ref. [7] we gave an example. Here we provide a general integration for a special case of matrix \( C \). There exists a class of solutions which can be integrated for any seed solution. This class is derived from the matrix

\[ C_M = \begin{pmatrix} q & 0 & -s \\ 0 & 1 & 0 \\ -s & 0 & q \end{pmatrix}, \]

(12)

with \( \det C_M = q^2 - s^2 = 1 \). Substituting \( C_M \) into (11), the five potentials \( \Psi^A \) read

\[ f^2 = \frac{f_0^2}{q^2 - 8s^2 f_0\kappa_0^2}, \quad \epsilon = q\epsilon_0, \quad \chi = 2\sqrt{2}s\epsilon_0, \]
\[ \kappa^4 = \kappa_0^4 \left(q^2 - 8s^2 f_0\kappa_0^{-2}\right), \quad \psi = \frac{1}{2\sqrt{2}}\frac{q\epsilon_0\kappa_0^4}{q^2 - 8s^2 f_0\kappa_0^{-2}}. \]

(13)

Using definitions (11) we can integrate (13) to obtain the space-time metric, we arrive at

\[ dS^2 = \frac{1}{I_0 f_0} e^{2k}dz d\bar{z} + \left(g_{033} - \frac{8s^2 g_{034}^2}{T I_0^2}\right) d\phi^2 + 2\frac{g_0 g_{034}}{T} d\phi dt - \frac{f_0}{I_0 T} dt^2 + \]
\[ + f^2 \left[-\frac{2\sqrt{2}g_{034} f_0}{T} d\phi - \frac{qs}{2\sqrt{2}} \frac{1 - 8f_0\kappa_0^2}{T} dt + dx^5\right]^2, \]

(14)

where \( T = q^2 - 8s^2 f_0\kappa_0^{-2} \) (a subindex 0 means seed solution).
Stationary seed solutions of the Einstein equations in terms of harmonic maps are well-known (see ref. [8]), they are written in terms of the Ernst potential $E = f + i\epsilon$. These seed solutions can be substituted into (14) in order to obtain the metric in terms of harmonic maps. Let us give an example. We start from the Kerr-NUT solution together with a $\kappa_0$ potential as a seed solution given by

$$f_0 = \frac{\omega - 2mr - 2L_+}{\omega}, \quad \epsilon_0 = \frac{2(mL_+ - lr)}{\omega}, \quad \kappa_0 = \left(\frac{r - m + \sigma}{r - m - \sigma}\right)^\delta,$$

$$L_+ = a \cos \theta + l, \quad L_- = a \cos \theta - l, \quad \omega = r^2 + (a \cos \theta + l)^2,$$

(15)

where $r$ and $\theta$ are the Boyer-Lindquist coordinates; constants $a$, $m$ and $l$ are the rotation, mass and NUT parameters, respectively. Solution (12) with seed solution (15) is an axisymmetric stationary exact solution of 5D gravity. The resulting metric is

$$dS^2 = \frac{\omega}{\omega - 2mr - 2L_+} \left(\frac{r - m + \sigma}{r - m - \sigma}\right)^\delta \left(r^2 - 2mr + L_+L_-\right) e^{2k_s} \left(\frac{dr^2}{\Delta} + d\Omega^2\right) + \frac{1}{D} \left(\frac{r - m + \sigma}{r - m - \sigma}\right)^\delta \times \left\{-(\omega - 2mr + 2lL_+)dt^2 + (4qa(mr + l)\sin^2 \theta + 2l \cos \theta \Delta)\frac{\omega - 2mr + 2L_+}{r^2 - 2mr + L_+L_-} dt d\phi + \left[q^2(\omega - 2mr - 2L_+)\left(\frac{4a^2 \sin^2 \theta (mr + l) + 2l \cos \theta \Delta}{r^2 - 2mr + L_+L_-}\right)^2\right] d\phi^2\right\} + \left(\frac{r - m + \sigma}{r - m - \sigma}\right)^\delta D \left(A_3 d\phi + A_4 dt + dx^5\right)^2$$

(16)

where

$$A_3 = -2\sqrt{2} \left(\frac{r - m + \sigma}{r - m - \sigma}\right)^{2\delta} \frac{4a^2 \sin^2 \theta (mr + l) + 2l \cos \theta \Delta}{r^2 - 2mr + L_+L_-} D,$$

$$A_4 = -\frac{q}{2} \left(\frac{r - m + \sigma}{r - m - \sigma}\right)^{2\delta} q^2 (\omega - 2mr + 2IL_+),$$

$$D = \omega \left(\frac{r - m + \sigma}{r - m - \sigma}\right)^{2\delta} q^2 - 8s^2(\omega - 2mr + 2IL_+), \quad \Delta = r^2 - 2mr + a^2 - l^2$$

(17)

$$e^{2k_s} = 6 \left[\sqrt{\rho^2 + (\zeta - m)^2} + \sqrt{\rho^2 + (\zeta + m)^2}\right] - 4m^2$$

$$4\sqrt{\rho^2 + (\zeta - m)^2}\left[\rho^2 + (\zeta + m)^2\right]$$

7
with \( \rho = \sqrt{r^2 - 2mr + a^2 - \ell^2}\) \( \sin \theta \) and \( \zeta = (r - m) \cos \theta \), \( (18) \)

Here \( m, a, q \) and \( \sigma \) are constants related by the restrictions \( \sigma^2 = m^2 + l^2 - a^2, q^2 - s^2 = 1 \). Metric \((16)\) reduces to the Kramer metric \([11]\) for \( C = \text{diag}(1, 1, 1) \) and \( l = 0 \), to the Kerr metric for \( C = \text{diag}(1, 1, 1), l = 0, \delta = 0 \), and to the Belinsky-Ruffini \([10]\) solution by setting \( \delta = \frac{1}{2}, l = 0 \). The study of singularities of metric \((16)\) will be given elsewhere \([12]\).

## 5 Second class of solutions. Seed solution

\( \Psi^A = (f, 0, 0, \chi, \kappa) \).

From the astrophysical point of view, magnetic dipoles are more interesting. Now we start from a magnetized static seed solution, with \( f, \chi \) and \( \kappa \) arbitrary and \( \epsilon = \psi = 0 \). In order to obtain the new solution, we proceed in the same way as in the case before. Thus, we find the expressions for the potentials

\[
\begin{align*}
\kappa^4 &= \frac{4B}{\kappa_0^4} \kappa_0^4 \\
f^2 &= \frac{\kappa_0^4}{A} f_0^2 \\
\chi &= -\frac{1}{\sqrt{2} A} \left\{ 8 id f_0^2 + 8d(h + \frac{1}{2}\sqrt{2} k \chi_0) + j[2\sqrt{2} k_0 + k(h_0 + k_0)] \right\} \\
\epsilon &= \frac{1}{4 A} \left\{ 8ad f_0^2 + 8ae + 2\sqrt{2} \chi_0 (bj + ce) + c j(\chi_0^2 - f_0 \kappa_0^2) \right\} \\
\psi &= -\frac{1}{4\sqrt{2} B} \left\{ t q \kappa_0^2 + u \omega (f_0^2 \kappa_0^2 - f_0 \chi_0^2) + 2\sqrt{2} f_0 \chi_0 (ws + uz) - 8zs f_0 \right\}
\end{align*}
\]

\( (19) \)

where

\[
\begin{align*}
A &= -\frac{1}{4} \left\{ 8d^2 f_0^2 + e(8e + 4\sqrt{2} j \chi_0) + j^2(\chi_0^2 - f_0 \kappa_0^2) \right\} \\
B &= -\frac{1}{2} \left\{ q^2 \kappa_0^2 + u^2 (f_0^2 \kappa_0^2 + f_0 \chi_0^2) + 4s f_0 (\sqrt{2} u \chi_0 - 2s) \right\}.
\end{align*}
\]

\( (20) \)

and again, we integrate solution \((20)\) for any seed solution using matrix \((15)\).

So the solution in the potential space takes the form

\[
\begin{align*}
\kappa^4 &= \frac{\kappa_0^4}{q^2 - 8s^2 f_0 \kappa_0^2} (q^2 - 8s^2 f_0 \kappa_0^2) \\
f &= \frac{f_0}{\sqrt{q^2 - 8s^2 f_0 \kappa_0^2}}, \\
\chi &= q \kappa_0, \\
\epsilon &= \frac{1}{2\sqrt{2}} s \chi_0
\end{align*}
\]

\( (19) \)
In the space-time, solution (21) can be integrated to obtain

\[ dS^2 = \frac{1}{T} \left\{ T^{\frac{1}{2}} e^{\kappa_0} f_0 dz d\bar{z} + \left[ T^{\frac{1}{2}} e^{\kappa_0} - \frac{8 s^2 A_0 f_0}{T^{\frac{1}{2}}} \right] d\phi^2 - \frac{2 \sqrt{2} s A_0 f_0}{T^{\frac{1}{2}}} d\phi dt - \frac{f_0}{T^{\frac{1}{2}}} dt^2 \right\} + I^2 \left( \frac{q A_3}{T} d\phi - \frac{q s}{2 \sqrt{2}} \frac{1 - 8 s f_0 \kappa_0^{-2}}{T} dt + dx^5 \right)^2. \]

(22)

where \( T = q^2 - 8 s^2 f_0 \kappa_0^{-2} \) and the electromagnetic four potential is given by

\[ A_3 = \frac{q A_{03}}{q^2 - 8 s^2 f_0 \kappa_0^{-2}} \quad \text{and} \quad A_4 = \frac{-q s (1 - 8 s^2 f_0 \kappa_0^{-2})}{2 \sqrt{2} (q^2 - 8 s^2 f_0 \kappa_0^{-2})}, \]

(23)

with \( A_1 = A_2 = 0 \), and the scalar field fulfills the identity \( I^3 = \kappa^2 \). We can combine solution (22) with the seed solutions in terms of harmonic maps, in order to get rotating solutions with arbitrary electromagnetic field. This is done in the next section.

6 Explicit solutions

In this section we apply the results given in ref. [13], were static solutions of 5D gravity were written in terms of two harmonics maps \( \lambda \) and \( \tau \). In [13], many classes of solutions in terms of one and two harmonic maps were found. Here we give only one example. There exist two subclasses of static solutions of five-dimensional gravity which are very similar to the Schwarzschild space-time (see ref. [14]). These subclasses were generalized for any dilaton theory in [13]. Here we use their five-dimensional version. The static metric reads

\[ ds^2 = \frac{1}{T} \left\{ e^{2(k_2 + k_4)} g_{22}^\gamma \left\{ d\tau^2 + \frac{d\kappa_0}{g_{22}^\gamma} \left( e^{2(k_2 + k_4)} d\theta^2 + \sin^2 \theta d\phi^2 \right) \right\} + \frac{1 - 2m}{g_{22}^\gamma} dt^2 \right\} + I^2 (A_3 \ d\phi + dx^5)^2 \]

\[ A_{3,\zeta} = Q \rho \tau, \quad A_{3,\xi} = -Q \rho \tau, \quad \kappa^2 = I^3 = \frac{k^3 e^{\gamma_0 \theta}}{1 - 2m / g_{22}^\beta} \]

(24)
The functions $g_{22}$, $k_s$ and $k_e$ for the subclass a) are

\[ g_{22} = a_1 e^{q_1 \tau} + a_2 e^{q_2 \tau}, \]
\[ k_{s,\zeta} = \frac{\rho}{2\alpha^2} (\lambda_{,\zeta} - \tau_0 \tau_{,\zeta})^2, \quad k_{e,\zeta} = -\rho \gamma q_1 q_2 (\tau_{,\zeta})^2, \quad \tau_0 = q_1 + q_2, \]

and for the subclass b) are

\[ g_{22} = a_1 \tau + 1, \quad k_{s,\zeta} = \frac{\rho}{2\alpha^2} (\lambda_{,\zeta})^2, \quad k_e = 0, \quad \tau_0 = 0, \]

where $\zeta = \rho + i z = \sqrt{r^2 - 2mr} \sin \theta + i (r - m) \cos \theta$. $A = A_i dx^i$, $i = 1, ..., 4$ is the electromagnetic four potential, $m$ the mass parameter, $\gamma = \frac{1}{2}$, $\beta = \frac{3}{2}$; $Q$, $a_1 + a_2 = 1$, $q_1$ and $q_2$ are constants subjected to the restrictions

\[ 2\gamma a_1 a_2 (q_1 - q_2)^2 + \kappa_0^2 Q^2 = 0 \]

for the subclass a), and

\[ 2\gamma a_1^2 - \kappa_0^2 Q^2 = 0 \]

for the subclass b). Metric (24) is convenient because we can interpret $m$ as the mass parameter, $g_{22}$ as the magnetic field contribution to the metric and the expression between brackets {...} as the four-dimensional space-time metric. Hence metric (24) can be interpreted as a 5D magnetized Schwarzschild solution. Reference [16] enlists a set of solutions of the harmonic map equation $(\rho \tau_{,\zeta})_{,\zeta} + (\rho \tau_{,\zeta})_{,\zeta} = 0$ and their corresponding magnetic potential $A_3$. The harmonic map $\lambda$ determines the gravitational potential, while the harmonic map $\tau$ determines the electromagnetic one. For a magnetic dipole, the harmonic map $\tau$ and its corresponding magnetic field read

\[ \tau = \frac{\cos \theta}{(r - m)^2 - m^2 \cos^2 \theta} \quad A_3 = \frac{Q (r - m) \sin^2 \theta}{(r - m)^2 - m^2 \cos^2 \theta}. \]  

(25)

In general, $\tau$ can be chosen in order to obtain monopoles, dipoles, quadripoles, etc.

### 6.1 Mass and angular momentum

Now we substitute the values of $g_{033}$, $g_{044}$ and $A_{03}$ from metric (24) for an arbitrary electromagnetic field, i.e., we substitute (24) into the metric (23) to obtain
\[ dS^2 = \frac{1}{T^2} \left\{ T^2 e^{2(k_s + k_e)} g^2_{22}(\frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\theta^2) + \\
[T^2 g^2_{22} r^2 e^{2(k_s + k_e)} \sin^2 \theta - \frac{8s^2 q}{T^2} A^2_{03} \frac{1 - \frac{2m}{r}}{g^2_{22}}] d\phi^2 + \\
\frac{-2\sqrt{2} T A_{03}}{T^2} \frac{1 - \frac{2m}{r}}{g^2_{22}} d\phi dt - \frac{1}{T^2} \frac{1 - \frac{2m}{r}}{g^2_{22}} dt^2 \right\} + \\
I^2 \left( \frac{q}{T} A_{03} d\phi - \frac{q s}{T} (1 - 8\frac{1}{T^2} e^{-\tau_0} (1 - 2\frac{m}{r})^2 g_{22}) dt + dx^5 \right)^2, \]  
(26)

where \( T = q^2 - 8s^2 \frac{1}{T^2} e^{-\tau_0} (1 - 2\frac{m}{r})^2 g_{22} \) and \( q^2 - s^2 = 1 \). Metric (26) is an exact solution of 5D gravity. Setting \( s = 0 \) we recover the seed metric (24).

For the magnetic field (25), we can obtain the mass and the electromagnetic parameters. For both subclasses a) and b), metric (26) contains the mass parameter

\[ M = m \left[ 1 + \frac{8s^2}{h^2} \right] = qm. \]  
(27)

and an angular momentum per unit of mass given by

\[ a = \frac{s Q}{4M}, \]  
(28)

provided that \( h = 2 \).

### 6.2 Electromagnetic field components

For the magnetic dipole, the non-vanishing components of the electromagnetic four potential read

\[ A_3 = \frac{qQ(r - m) \sin^2 \theta}{T[(r - m)^2 - m^2 \cos^2 \theta]}, \quad A_4 = -\frac{qs(1 - 8\frac{1}{T^2} e^{-\tau_0} (1 - 2\frac{m}{r})^2 g_{22})}{2\sqrt{2} T} \]  
(29)

which correspond to a dipole magnetic field with magnetic charge

\[ Q_M = qQ, \]  
(30)

for both subclasses a) and b).
7 Final Remarks

We have found two classes of solutions of five-dimensional gravity. The first one can be generated from any vacuum solution of the Einstein equations, plus a scalar field, the result is a charged solution of five-dimensional gravity. In fact, if we write the seed metric in Papapetrou form

$$dS^2 = \frac{1}{I_0} \left\{ \frac{1}{f_0} \left[ e^{2k_0} dz \, \bar{d}z + \rho^2 \, d\phi^2 \right] - f_0 (dt + A_0 \, d\phi)^2 \right\} + I_0^2 \, dx^5,$$

the resulting metric reads

$$dS^2 = \frac{1}{I} \left\{ \frac{1}{f} \left[ e^{2k_0} dz \, \bar{d}z + \rho^2 \, d\phi^2 \right] - f (dt + q A_0 \, d\phi)^2 \right\} + I^2 (A_3 \, d\phi + A_4 \, dt + dx^5)^2,$$

with the functions $f, I, A_3$ and $A_4$ given by

$$f = \frac{f_0}{T^{1/2}}, \quad I = I_0 T^{1/2},$$

$$A_3 = -\frac{2\sqrt{2} f_0 A_0}{T}, \quad A_4 = -\frac{q s}{2\sqrt{2}} \left( \frac{1 - 8 s^2 f_0 I_0^{-3}}{T} \right).$$

The second class can be generated from a seed static solution of five-dimensional gravity which posses magnetic field, i.e.

$$dS^2 = \frac{1}{I_0} \left\{ \frac{1}{f_0} \left[ e^{2k_0} dz \, \bar{d}z + \rho^2 \, d\phi^2 \right] - f_0 \, dt^2 \right\} + I_0^2 (A_{03} \, d\phi + dx^5)^2,$$

giving as result a rotating solution

$$dS^2 = \frac{1}{I} \left\{ \frac{1}{f} \left[ e^{2k_0} dz \, \bar{d}z + \rho^2 \, d\phi^2 \right] - f (dt - 2\sqrt{2} s A_{03} \, d\phi)^2 \right\} + I^2 (A_3 \, d\phi + A_4 \, dt + dx^5)^2,$$

with the functions $f, I, A_3$ and $A_4$ given by

$$f = \frac{f_0}{T^{1/2}}, \quad I = I_0 T^{1/2},$$

$$A_3 = \frac{q A_{03}}{T}, \quad A_4 = -\frac{q s}{2\sqrt{2}} \left( \frac{1 - 8 s^2 f_0 I_0^{-3}}{T} \right).$$
With the first class we generated some well-known solutions as the Belinsky-Rufini space-time. This latter solution is not a Black Hole since the scalar field forms a naked singularity because the scalar field parameter $\delta$ is fixed. We suspect that our solution (16) contains Black Holes for different values of $\delta$. At last the Kerr Black Hole is contained here for $l = \delta = 0$, but we do not know at the moment if other Black Holes are contained in metric (16). The behavior of this metric is in some sense similar to the Kerr-Newman Black Hole. It contains a magnetic dipole moment which vanishes for the rotation parameter $a = 0$. This means that the magnetic dipole is provoked by induction from the electric field. (16) represents a rotating electrically charged mass, with an induced magnetic dipole. The second class of solutions is quite different. If the rotation parameter $s$ vanishes, the solution becomes static and the magnetic field is just of a magnetically charged sphere, but the electric charge vanishes as well. This means that here the electric charge is induced by the rotation of the magnetic dipole. This solution represents a rotating mass with a magnetic dipole charge. Quite surprising is that the magnetic field does not alter during the rotation, which can be seen from relations (21). Nevertheless, there is no way to conserve the rotation dropping out the magnetic field, because if $Q = 0$, then the rotation parameter $a$ vanishes as well.

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