Decidability of the theory of addition and the Frobenius map in rings of rational functions

Dimitra Chompitaki, Manos Kamarianakis† and Thanases Pheidas
University of Crete, Dept. of Mathematics & Applied Mathematics
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Abstract

We prove model completeness for the theory of addition and the Frobenius map for certain subrings of rational functions in positive characteristic. More precisely: Let \( p \) be a prime number, \( \mathbb{F}_p \) the prime field with \( p \) elements, \( F \) a field algebraic over \( \mathbb{F}_p \) and \( z \) a variable. We show that the structures of rings \( R \), which are generated over \( \mathbb{F}_p[z] \) by adjoining a finite set of inverses of irreducible polynomials of \( \mathbb{F}[z] \) (e.g., \( R = \mathbb{F}_p[z, \frac{1}{z}] \)), with addition, the Frobenius map \( x \mapsto x^p \) and the predicate \( \in F \) - together with function symbols and constants that allow building all elements of \( \mathbb{F}_p[z] \) - are model complete, i.e., each formula is equivalent to an existential formula. Further, we show that in these structures all questions, i.e., first order sentences, about the rings \( R \) may be, constructively, translated into questions about \( F \).

1 Introduction

Let \( p \) be a prime number, \( \mathbb{F}_p \) a field with \( p \) elements and \( F \) a field, algebraic over \( \mathbb{F}_p \). Let \( z \) be a variable, \( \mathbb{F}_p[z] \) and \( F[z] \) the rings of polynomials of \( z \) over \( \mathbb{F}_p \) and \( F \) accordingly, and let \( s_1, \ldots, s_\nu \) be irreducible elements of \( \mathbb{F}_p[z] \) which remain irreducible in \( F[z] \). Let \( S = \{s_1, \ldots, s_\nu\} \) and \( R = F[z, S^{-1}] \) be the ring which is generated over \( F \) by \( z \) and the inverses of the elements of \( S \). Consider \( R \) as a structure (model) of the language \( \mathcal{L}_p(z) := \{=, +, x \mapsto x^p, 0, 1, x \mapsto zx, \in F\} \) with symbols \(+\) for addition, \( x \mapsto x^p \) for the Frobenius map, constant-symbols for 0 and 1, the function symbol \( x \mapsto zx \) for the multiplication-by-\( z \) map and a symbol for belonging to \( F \). In \[\text{Rob51}\] R. Robinson proved that the ring theory of rings of polynomials such as \( \mathbb{F}_p[z] \), in the language of rings, augmented by a constant-symbol for \( z \), is undecidable. In \[\text{Den79}\]

†Corresponding Author, kamarianakis@uoc.gr
it was proved that even the positive-existential theory of a polynomial ring in this language is undecidable. The similar result for the rings $R$ was proved in [Shl93].

It is then natural to ask questions of decidability of substructures of the ring-structure of $R$. Here we prove the following. Consider $L_p := \{+,-,x \mapsto x^p, 0, 1\}$ for the restriction of the language $L_p(z)$, with interpretations as above. Also consider the extension $L_p(z)^e$ of $L_p(z)$ by predicate symbols $P_\sigma$, one for each formula $\sigma$ of $L_p$.

We interpret $P_\sigma(\alpha)$ by ‘each element of the tuple $\alpha$ is an element of $F$ and $\sigma(\alpha)$ holds true over $F$’ - we assume that all the free variables of the formula $\sigma$ are among the tuple of variables $\alpha$. We prove:

**Theorem 1.1** Let $F$ be an algebraic field extension of $\mathbb{F}_p$. Let $R$ be as above. Then the following hold:

1. The $L_p(z)^e$-theory of $R$ is effectively model-complete.

2. Every $L_p(z)^e$-sentence is equivalent in $R$ to a sentence of the form $P_\sigma$, where $\sigma$ is a sentence of $L_p$.

We prove model-completeness by constructing an algorithm which converts any existential $L_p(z)^e$-formula to an equivalent, in $R$, universal $L_p(z)^e$-formula. It is well known that model completeness of the theory of a countable model with a recursive elementary diagram implies decidability of the theory (see current developments in [CMS21]), so we obtain:

**Corollary 1.2** Assume that the $L_p$-theory of the field $F$ is decidable. Then the $L_p(z)^e$-theory of $R$ is decidable.

Item 2 of Theorem 1.1 says that ‘questions’ (first-order sentences) about $R$ may be effectively translated into questions about $F$ (as a model of $L_p$). This replies positively, for the structures $R$, to a ‘program’ asked by Leonard Lipshitz: ‘Identify theories with universe a polynomial ring $F[z]$ or a field of rational functions, extending the structure of addition by commonly used operations and relations, which have the property that one can effectively translate first-order sentences of the structure into questions about $F$ and, possibly, other elementary mathematical structures, e.g., groups’. This follows the type of results of J. Ax and S. Kochen for fields of $p$-adic numbers in [AK65].

Theorem 1.1 is, in part, a generalization of the results of [PZ04], where a similar theorem was proved for rings of polynomials (i.e., when $S$ is the empty set), for any perfect field $F$ (not necessarily algebraic). A similar result (model-completeness) was proved in [Ona18] by Onay for the henselization of $\mathbb{F}_p[z]$, seen as a module over $\mathbb{F}_p[z]$ (and, more generally, over a finite field or an algebraic closure of it). The problem of whether the $L_p(z)^e$-theory of the field $\mathbb{F}_p(z)$ or $\tilde{\mathbb{F}}_p(z)$ is model-complete, or even decidable, remains open.

The structure of addition and the Frobenius map is interesting, not only for its own sake, but also because it is connected to various important algebraic and logical
problems. For example, the derivative of a function (polynomial, rational or power series) is positive-existentially definable in $L_p(z)^e$ (see the Introduction of [PZ04]). So, the structure of $R$ as a model of addition and differentiation is encodable in its $L_p(z)$-structure.

For surveys regarding the decidability properties of algebraic structures the reader may consult [PZ00], [PZ08], [Poo08] and [Koe18]. For other decidability results for polynomial rings of positive characteristic the reader may consult [Sir10] - an analogue of results of A. Semenov for the natural numbers with addition and the set of powers of a fixed prime number - and [Phe85] where it is proved that the existential theory of addition and divisibility over a ring of polynomials with coefficients in an existentially decidable field is decidable. For the algebraic and model theoretic properties of the Frobenius map see [Maz75], [CH99] and [Hru04].

Although the general strategy of proof of [PZ04] works for the rings $R$, several of its components are quite difficult to adapt. We describe briefly the main two. A polynomial $f \in \mathbb{F}_p[z]$, in $m$ variables, is additive if, for all $a$ and $b$ in $(\mathbb{F}(z))^m$, it satisfies $f(a) + f(b) = f(a+b)$. Notice that the polynomial terms of the language $L_p(z)$, with a zero constant term, are such additive polynomials. Here we will consider only additive polynomials with coefficients in $\mathbb{F}_p[z]$. Such an $f$ is called strongly normalized if its coefficients are in $\mathbb{F}_p[z]$, the degrees of $f$ with respect to each of its variables is the same, $p^s$, for some $s \in \mathbb{N} \cup \{0\}$ and the degrees of its leading coefficients are pairwise inequivalent modulo $p^s$. We develop an algorithm by which questions regarding solvability of - arbitrary - additive polynomials are reduced to similar questions for strongly normalized polynomials. The first crucial property of a strongly normalized polynomial $f$ is that, for any given $y \in R$, the inverse image $\{ x \in \mathbb{F}(z)^n : f(x) = u \}$ has a bounded height. This is relatively easy for the case that $R$ is a polynomial ring and we know this is not true if $R$ is substituted by the field of all rational functions $\mathbb{F}_p(z)$. In order to prove it for the rings $R$ we use the notion of a ‘Hasse derivative’ and recent results on the relative Algebra (see Section 3). So we prove:

**Theorem 1.3** Let $f$ be a strongly normalized additive polynomial of the variables $x = (x_1, \ldots, x_n)$. Then there is a recursive function $h$ which, to each additive polynomial $f$ of the language $L_p(z)$ and each $\ell \in \mathbb{N}$ associates a non-negative integer $h(f, \ell)$ such that the height of each element of the set $\{ x \in \mathbb{F}(z)^n : |f(x)| \leq \ell \}$ is less than or equal to $h(f, \ell)$.

A second point where the strategy of [PZ04] needs significant adaptations is where we need to prove that the image of a strongly normalized polynomial, whose leading coefficients form a basis of $\mathbb{F}(z)$ over $\mathbb{F}(z^{p^s})$ (where $p^s$ is the degree of the polynomial), is ‘almost all of $R$’ (in the sense of Lemma 2.4). Our method works only for algebraic fields $F$ - not for general perfect fields. Item 2 of Theorem 1.1 even for the polynomial case, is obviously stronger than the results of [PZ04] and its proof requires some improvements in the logical treatment of the subject - see the proof of Section 4.

In the rest of this section, we fix our notation and give a sketch of the proof.
Section 2 contains some necessary algebraic results. We prove Theorem 1.3 in Section 3 and Theorem 1.1 in Section 4. Some tedious elementary proofs are gathered in the Appendix.

1.1 Notation, Definitions and some elementary algebraic Facts

We use the following notations and definitions.

1. $\mathbb{N}$ is the set of natural numbers.

2. $p$ is a prime number, $\mathbb{F}_p$ is a field with $p$ elements, $\overline{\mathbb{F}}_p$ is an algebraic closure of $\mathbb{F}_p$.

3. $F$ is a field of characteristic $p$ such that $\mathbb{F}_p \subseteq F \subseteq \overline{\mathbb{F}}_p$.

4. $z$ is a variable, $F[z]$ is the ring of polynomials of $z$ with coefficients in $F$ and $F(z)$ is its field of quotients, i.e., rational functions of $z$ with coefficients in $F$.

5. $S$ is a finite set $S = \{s_1, \ldots, s_\nu\}$ of irreducible elements of $\mathbb{F}_p[z]$, which we assume to remain irreducible as elements of $F[z]$.

6. $R$ is the ring $F[z, s_1^{-1}, \ldots, s_\nu^{-1}]$, i.e., the ring generated over $F[z]$ by the inverses of the elements of $S$.

7. The language $L_p$ is defined as $L_p := \{+, x \mapsto x^p, =, 0, 1\}$. Its symbols are interpreted as follows: $+$ denotes addition, $x \mapsto x^p$ denotes the Frobenius map, $=$ denotes equality, 0 and 1 are constant symbols for the obvious elements of $\mathbb{F}_p$.

8. The language $L_p(z)$ is defined by $L_p(z) = L_p \cup \{\in \in F, x \mapsto zx\}$ where $x \mapsto zx$ is the multiplication-by-$z$ map, and $\in F$ is a unary predicate denoting belonging to $F$.

9. If $\alpha = (\alpha_1, \ldots, \alpha_m)$ is a tuple of the variables $\alpha_i$, then $\alpha \in F$ denotes the relation $\wedge_{i=1}^{m} \alpha_i \in F$.

10. The language $L_p(z)^e$ is the extension of $L_p(z)$ by the predicates $P_\sigma$, where the index $\sigma$ ranges over the set of formulas $\sigma(a_1, \ldots, a_n)$ of $L_p$. We assume that all the free variables of $\sigma$ are among $a_1, \ldots, a_n$. The predicate $P_\sigma$ associated to $\sigma(a_1, \ldots, a_n)$ is interpreted as the $n$-ary relation “$a_1, \ldots, a_n \in F$ and $\sigma(a_1, \ldots, a_n)$ is true in $F$”.

11. $R$ denotes the model of the language $L_p(z)^e$, with universe $R$, with symbols interpreted as above.

12. A variable ranging only over $F$ will be called an $F$-variable.
13. For \( x \in F(z) \), if \( Q \) is an irreducible element of \( F[z] \), we write \( \text{ord}_Q(x) \) for the order of \( x \) at \( Q \). Recall that by convention \( \text{ord}_Q(0) = \infty \). If \( Q = z - \rho \), with \( \rho \in F \), then we write \( \text{ord}_\rho \) instead of \( \text{ord}_Q \). The order at infinity of \( x \), denoted \( \text{ord}_\infty(x) \), is the degree of the denominator of \( x \) minus the degree of its numerator.

14. An additive polynomial is a polynomial of the form
\[
f(x) = \sum_{i=1}^{n} f_i(x_i),
\]
where \( x = (x_1, \ldots, x_n) \) and, for each \( i \),
\[
f_i(x_i) = b_i x_i^{s(i)} + \sum_{j=1}^{s(i)} c_{i,j} x_i^{p(i)-j},
\]
with \( b_i, c_{i,j} \in \mathbb{F}_p[z] \). The degree of \( f \) is \( \text{deg}(f) := \max_i \{ p^{s(i)} \} \).

15. We say that the additive polynomial \( f(x) \) is a polynomial of all the variables of the tuple \( x = (x_1, \ldots, x_m) \) whenever the degree of \( f \) in each of the variables \( x_i \) is positive, for \( i = 1, \ldots, m \).

16. For \( s \in \mathbb{N} \), let \( \mathcal{V}_s(\mathbb{F}_p) \) be \( \mathbb{F}_p(z)^s \) considered as a vector space over the field \( \mathbb{F}_p(z^{p^s}) \). Respectively, let \( \mathcal{V}_s(F) \) be \( F(z) \) considered as a vector space over the field \( F(z^{p^s}) \).

17. An additive polynomial \( f \) is called normalized if all degrees \( s(i) \) are equal and the set of leading coefficients \( \{ b_i : i = 1, \ldots, n \} \) is linearly independent over \( \mathcal{V}_s(\mathbb{F}_p) \). An additive polynomial is called \( p \)-basic if it is normalized and the set \( \{ b_1, \ldots, b_n \} \) forms a basis for \( \mathcal{V}_s(\mathbb{F}_p) \). Moreover, \( f \) is strongly normalized if \( f \) is normalized and the degrees of the \( b_i \) are pair-wise inequivalent modulo \( p^s \), where \( p^s \) is the degree of \( f \).

18. **Definition 1.4** A proper transformation is a tuple \( \xi = (\xi_1, \ldots, \xi_n) \) such that each \( \xi_i(X, \beta) \) is an additive polynomial of the variables \( x_1, \ldots, x_m, \beta_1, \ldots, \beta_\mu \) and such that the map \( \xi \) defined by
\[
\xi : \quad R^m \times F^\mu \rightarrow R^n \quad \begin{cases} (x_1, \ldots, x_m, \beta_1, \ldots, \beta_\mu) \mapsto (\xi_1(x_1, \ldots, x_m, \beta_1, \ldots, \beta_\mu), \ldots, \\
\xi_n(x_1, \ldots, x_m, \beta_1, \ldots, \beta_\mu)) \end{cases}
\]
is surjective. Also note that the composition of proper transformations is again a proper transformation.

We will use proper transformations in order to change variables.

19. Let \( x \) be an tuple of variables and \( f \) be an additive polynomial of the variables of \( x \). When all the variables of the additive polynomial \( H \) are \( F \)-variables then we write \( \text{Im}_F(H) := \{ y \in R \mid \exists \alpha \in F \ H(\alpha) = y \} \). We write \( \text{Im}(f) := \{ y \in R \mid \exists x \in R \ f(x) = y \} \).
20. A **bounded term** is any expression of the form $\frac{1}{e}G(\alpha)$, where $e \in \mathbb{F}_p[z]$ and $G(\alpha)$ is an additive polynomial of the tuple of $F$-variables $\alpha$. Notice that a bounded term is not always a term of the language $\mathcal{L}_p(z)^e$ but we will be writing them with the understanding that ‘clearing of the denominators’ is performed immediately - this makes sense because the only relation symbol of $\mathcal{L}_p(z)^e$, apart from $\in \mathbb{F}$, is equality. We write $\text{Im}_F(\frac{1}{e}G)$ for $\frac{1}{e}\text{Im}_F(G)$.

21. The **height** of a rational function $u = \frac{a}{b} \in F(z)$, where $a, b \in F[z]$ are coprime, is

$$|u| := \max\{\deg(a), \deg(b)\}.$$  

The height of 0 is not defined.

22. The **partial fraction decomposition** of a rational function, from elementary algebra, is given by:

**Fact 1.5** Let $x \in F(z)$ and let $Q_1, \ldots, Q_r$ be all the monic, irreducible polynomials of $F[z]$ at which $x$ has a pole. Then $x$ can be written in a unique way as

$$x = g(z) + \sum_{i,j} \frac{d_{i,j}(z)}{Q_i^j},$$  \hspace{1cm} (3)

where the index $i$ ranges in the set $\{1, \ldots, r\}$ and, for each $i$, the index $j$ ranges in a non-empty subset of $\mathbb{N}$. Each of $g(z)$ and $d_{i,j}(z)$ is in $F[z]$ and for each $(i, j)$ the degree of the polynomial $d_{i,j}(z)$ is less than the degree of $Q_i$.

**Fact 1.6** (a) Let $u \in R$ and $c \in F[z]$ with $\deg(c) = d > 0$. Then there is a $v \in R$ and a polynomial $r$, of degree less than $d$, such that $u = vc + r$. If no irreducible factor (over $F[z]$) of $c$ is invertible in $R$, then $v$ and $r$ are unique.

(b) Given $u \in R, c \in F[z] \setminus \{0\}$ and $N \in \mathbb{N}$, there are $v \in R$ and $r_i \in F[z]$, for $i = 0, \ldots, N$, with $\deg(r_i) < \deg(c)$ such that

$$u = r_0 + r_1 c + \cdots + r_N c^N + v c^{N+1}.$$

(c) Let $Q \in \mathbb{F}_p[z]$ which is not divisible by any polynomial that is irreducible in $F[z]$ and invertible in $R$, of degree $d \geq 1$. Then, for any $x \in R$, the formula $x \notin F$ is equivalent in $R$ to

$$\exists y \exists \alpha_0 \ldots \exists \alpha_{d-1}$$

$$[\alpha_0, \ldots, \alpha_{d-1} \in F \land x = y Q + \alpha_{d-1} z^{d-1} + \cdots + \alpha_1 z + \alpha_0 \land (y \neq 0 \lor \bigvee_{i=1}^{d-1} \alpha_i \neq 0)]$$

The proof is elementary. For completeness, we include one in Section A.1.

**Fact 1.7** A system of equations may substituted by one equation due to the equivalence

$$(x = 0 \land y = 0) \iff x^p + z y^p = 0.$$
Fact 1.8 Let $F$ be a perfect field. Let $s \in \mathbb{N} \cup \{0\}$. Then:

- The set $\{ z^i | 0 \leq i \leq p^s - 1 \}$ is a basis of both $V_s(F_p)$ and $V_s(F)$.
- Let $B$ be a subset of $F_p(z)$ which is linearly independent in $V_s(F_p)$. Then it is also linearly independent in $V_s(F)$. Consequently, any basis of $V_s(F_p)$ is also a basis of $V_s(F)$.

1.2 Existential Formulas

Let $u$ be a term of $L_p(z)^e$. From Fact 1.6, the formula $u \notin F$ can be substituted by the equivalent $\exists \alpha, x [\alpha \in F \land u = \alpha_0 + \alpha_1 z + \cdots + \alpha_{r-1} z^{r-1} + gx \land x \neq 0]$, where $g$ is an irreducible polynomial in $F[z]$, of degree $r$, which is not in $S$. If $u$ is a term but not a variable, we can replace the formula $u \in F$ by the equivalent $\exists \alpha [\alpha \in F \land u = \alpha]$. Therefore, any formula is equivalent in $\mathcal{R}$ to a formula in which the negation of the predicate-symbol $\in F$ does not occur and in which $\in F$ is applied only to variables. Such a formula of $L_p(z)^e$, where all quantified variables range over $F$ will be called a bounded formula (cf. [PZ04], p. 1021).

Fact 1.9 The set $\{ x \in R \setminus \{0\} : |x| \leq k \} \cup \{0\}$ is definable in $L_p(z)$ by a bounded existential formula, i.e., one which is existential and its quantified variables are all $F$-variables.

We leave to the reader to verify that an existential formula of $L_p(z)^e$ is equivalent in $\mathcal{R}$ to a disjunction of formulas of the form:

$$\phi(u, \{v_j\}_{j \in J}) : \exists x, \alpha [\alpha \in F \land \psi(x, \alpha)]$$

where

$$\psi(x, \alpha) : f(x) + H(\alpha) = u \land \bigwedge_{j \in J} e_j(x) + G_j(\alpha) \neq v_j \land P(\alpha)$$

under the conventions:

- $x = (x_1, \ldots, x_m)$ is a tuple of the variables $x_i$.
- $\alpha$ is a tuple of $F$-variables, each of them distinct from each variable of $x$.
- $f$ is an additive polynomial of all the variables of $x$ (hence, by convention, we have $\deg_x(f) > 0$ for every $i$), with coefficients in $F_p[z]$.
- Each $e_j$ is an additive polynomial of some of the variables of $x$.
- $H$ is an additive polynomial in some of the variables of $\alpha$.
- Each $G_j$ is an additive polynomial in some of the variables of $\alpha$.
- $u$ and the $v_j$ are terms of $L_p(z)$. 

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• No variables among those of \( x \) or \( \alpha \) occurs in \( u \) or any of the \( v_j \).

• The predicate symbol \( P_\sigma(\alpha) \) may have more variables than those of \( \alpha \) occurring in it.

1.3 Overview of this article.

We start with a formula \( \phi \) as above. In Section 2, we show that, through a certain type of change of variables (see proper transformation in Definition 1.4), one may assume that the polynomial \( f \) is strongly normalized or the zero polynomial. The way to do this is similar to that of [PZ04], but several details have to be adjusted to the new rings \( R \). In particular, Lemma 2.4 constitutes a new approach. In Section 3, we prove Theorem 1.3. This allows us to substitute sets of the form \( \{ x \in F(z) : |f(x)| \leq k \} \) by sets of the form \( \{ x \in F(z) : |x| \leq h \} \), where \( k, h \in \mathbb{N} \). Finally, in Section 4, we use the previous steps in order to show that \( \phi \) is equivalent to a universal formula.

2 Strongly normalized polynomials and properties

The following lemmas will be used in the proof of Theorem 1.3. In this Section, we prove Theorem 1.3.

Lemma 2.1 (counterpart of [PZ04], Lemmas 3.3 and 3.4) Let \( f \) be an additive polynomial in \( m_0 \) variables, with coefficients in \( \mathbb{F}_p[z] \). Then, there is a proper transformation \( \xi : R^m \times F^k \rightarrow R^{m_0} \), a strongly normalized additive polynomial \( \tilde{f} \) in \( m \) variables, with coefficients in \( \mathbb{F}_p[z] \) and an additive polynomial \( G \) in only \( F \)-variables, each one of them distinct from the variables of \( \tilde{f} \), such that:

• \( f \circ \xi = \tilde{f} + G \),

• \( m \leq m_0 \),

• \( \deg(\tilde{f}) \leq \deg(f) \),

• \( \text{Im}(f) = \text{Im}(\tilde{f}) + \text{Im}_F(G) \),

• \( \xi \) and \( \tilde{f} \) are effectively computable from \( f \).

The proof is similar to those of Lemma 3.3 and 3.4 of [PZ04]. For completeness, we provide outlines in Section A.4.

Lemma 2.2 For any strongly normalized additive polynomial \( \tilde{f} \) in the variables of \( x = (x_1, \ldots, x_m) \), ranging over \( R \), there is an additive polynomial \( h \), in the variables of \( v = (v_1, \ldots, v_{p^s-m}) \), also ranging over \( R \), each one distinct from the variables of \( x \), such that \( \tilde{f} + h \) is \( p \)-basic and strongly normalized.
For a proof, see Lemma 3.5(a) of [PZ04].

**Definition 2.3** Let $f$ an additive polynomial over $F(z)$, in $m$ variables. We define the relation $\sim_f$ over $F(z)$ by

$$u \sim_f u' \text{ if and only if } u' - u = f(\tilde{x}), \text{ for some } \tilde{x} \in R^m.$$  \hspace{1cm} (7)

**Lemma 2.4** Assume that $F \subset \tilde{F}_p$. Then there is a primitive recursive function $E_{ord}$ from the set of $p$-basic polynomials of $L_p(z)$ to $\mathbb{N}$ such that, for any $p$-basic polynomial $f$ over $F_p[z]$ the following hold:

1. For any $u \in R$, there exists $u' \in R$ with the following properties:
   - $u \sim_f u'$.
   - Each pole of $u'$ is a pole of $u$ - the word ‘pole’ includes the pole at infinity.
   - For all $\rho \in \tilde{F}_p \cup \{\infty\}$, we have $|\text{ord}_\rho(u')| \leq E_{ord}(f)$.

2. Assume that $f$ has degree $p^s$, with $s \geq 0$. Write $N = E_{ord}(f)$, let $\alpha = (\alpha_0, \ldots, \alpha_{N-1})$ be a tuple of distinct $F$-variables and $G(\alpha) := \sum_{i=0}^{N-1} \alpha_i z^i$. Then,

$$R = \text{Im}(f) + \text{Im}_F \left( \frac{1}{e N} G \right),$$  \hspace{1cm} (8)

where $e$ is the product of all elements of $S$.

**Proof:** Consider an arbitrary $s \geq 0$, write $q = p^s$ and consider a $p$-basic polynomial $f(x) = \sum_{i=1}^{q} b_i x_i + g(x)$, of degree $q$, with coefficients in $F_p[z]$, where $x = (x_1, \ldots, x_q)$ is the tuple of variables of $f$ and $g$ is an additive polynomial in $x$, of degree less than $q$, hence at most $q/p$.

Since $f$ is $p$-basic, both $\{z^i : 0 \leq i \leq q-1\}$ and $\{b_1, \ldots, b_q\}$ are bases of $V_s(F_p)$. Therefore, there is a non-singular matrix $A$ such that $B = AZ$, where $Z$ and $B$ are the vectors of the $z^i$ and the $b_i$ respectively - in some order. Obviously $A$ has entries in $F_p[z^q]$. Write $\Delta$ for the determinant of $A$. Using Cramer’s Rule to solve the system $B = AZ$ for the variable $z^i$, we obtain that, for any $i \in \{0, \ldots, q-1\}$, there are $e_{i,j} \in F_p[z]$ such that

$$\Delta z^i = \sum_{j=1}^{q} e_{i,j}^q b_j.$$  \hspace{1cm} (9)

We will now prove this auxiliary claim.

**Claim 2.5** Let $v = \frac{a}{Q}$, where $a, Q \in F[z]$, $\ell \in \mathbb{N}$ and $Q$ is a monic, irreducible polynomial over $F[z]$. If $\Delta$ divides $a$ then there exists a $v' \in F[z]$ such that

1. $v \sim_f v'$,

2. the poles of $v'$ are roots of $Q$ and,
3. $|\text{ord}_Q(v')| \leq \frac{t+r}{p} = \frac{1}{p}(|\text{ord}_Q(v)| + r)$, where $r$ is the least non-negative integer such that $p \ell + r$.

Proof: Let $r$ be the least non-negative integer such that $\ell + r = kq$ for some $k \in \mathbb{N}$. Clearly $0 \leq r < q$. Since $\Delta$ divides $a$, we write $aQ^r = \Delta a'$ for some $a' \in F[z]$. By Fact 1.8 there are $a_i \in F[z]$ so that

$$a' = \sum_{i=0}^{p^r-1} a_i^q z^i.$$  

Then, using Fact 2 there are $e_{i,j} \in F[z]$ such that, for each $i = 0, \ldots q - 1$ we have $\Delta z^i = \sum_{j=1}^q e_{i,j} b_j$, hence

$$v = \frac{\sum_{i=0}^{q-1} a_i^q \Delta z^i}{Q^{kq}} = \frac{\sum_{i=0}^{q-1} \sum_{j=1}^q a_i^q e_{i,j} b_j}{Q^{kq}} = \sum_{j=1}^q b_j \tilde{x}_j^q,$$

where $\tilde{x}_j := \frac{1}{Q^{kq}} \sum_{i=0}^{q-1} a_i e_{i,j}$. Setting $\tilde{x} := (\tilde{x}_1, \ldots, \tilde{x}_q)$, we have $v = f(\tilde{x}) - g(\tilde{x})$. Clearly, every $\tilde{x}_j$ is in $R$, since $Q$ is an element of $S$ hence invertible in $R$. Furthermore, observe that the order of $g(\tilde{x})$ at $Q$ is at least $-kq$, since the coefficients of $g$ are in $\mathbb{F}_p[z]$. So we set $v' = -g(\tilde{x})$ and the proof is complete. \hfill \hfill \hfill \hfill \hfill \diamondsuit

We now proceed in the proof of Item 1 of the Lemma. Consider any $u \in F(z)$. Write it as partial fractions, i.e., as sum of terms of the form $u_{Q,\ell} = \frac{a}{Q^\ell}$ or $h$, where $a, h, Q \in F[z]$, $\ell \in \mathbb{N}$ and $Q$ is a monic, irreducible polynomial over $F[z]$. For each term $u_{Q,\ell}$, we will find $u'_{Q,\ell} \in F(z)$, whose poles (except the one at infinity) are roots of $Q$, with order bounded by $E_{\text{ord}}$, to be determined, such that

$$u_{Q,\ell} \sim_f u'_{Q,\ell}.$$  

Obviously $\sim_f$ is an equivalence and additive relation, so $u$ will be related by $\sim_f$ to the sum $\hat{u} = \sum_{Q,\ell} u'_{Q,\ell} + h$. Finally, we will treat the polynomial part $h$ of $\hat{u}$ in a similar way, to obtain a $u'$ with the required properties.

Since $\Delta$ is in $\mathbb{F}_p[z]$, there is a finite extension $\mathbb{F}_q^{m}$ of $\mathbb{F}_p$ in which $\Delta$, as a polynomial in $z$, splits into linear factors (and $m \geq 1$). Then, the product of the distinct irreducible (over $\mathbb{F}_p[z]$) factors of $\Delta$ divides $z^{p^m} - z$, in $\mathbb{F}_p[z]$. Taking some high enough power $p^{m_0}$ of $p$, such that the multiplicity of each irreducible factors of $\Delta$ is $\leq p^{m_0}$ we have that $\Delta$ divides $(z^{p^m} - z)^{p^{m_0}} = z^{p^{m+m_0}} - z^{p^{m_0}}$.

We rewrite the term $u_{Q,\ell}$ as

$$u_{Q,\ell} = \frac{a}{Q^\ell} = \frac{-a(Q^{p^{m+m_0}} - Q^{p^{m_0}})}{Q^{\ell + p^{m_0}}} + \frac{aQ^m}{Q^\ell}. \quad (10)$$

Let $u_1 := -\frac{a(Q^{p^{m+m_0}} - Q^{p^{m_0}})}{Q^{\ell + p^{m_0}}} \quad \text{and} \quad u_2 := \frac{aQ^m}{Q^\ell} = Q^m u_{Q,\ell}$. Since $m > 0$, we have that the order of $u_2$ at $Q$ is greater than the order of $u_{Q,\ell}$. Moreover, the numerator of $u_1$ is
divisible by $\Delta$, based on the remarks of the previous paragraph. Hence, by Claim 2.5 there is a $u_1' \in F[z, \frac{1}{Q}]$ such that $u_1 \sim f u_1'$ such that $|\text{ord}_Q(u_1')| \leq \frac{1}{p}(\ell + p^{m_0} + r)$, where $r$ is the least non-negative integer such that $p\ell + p^{m_0} + r$.

Note that if $\frac{1}{p}(\ell + p^{m_0} + r) \geq \ell$ then $p^{m_0} + r \geq (p - 1)\ell$ and therefore, $\ell < \frac{p^{m_0} + q}{p - 1}$, since $r < q$. Consequently, if $|\text{ord}_Q(u_1)| = \ell \geq \frac{p^{m_0} + q}{p - 1}$ then $\text{ord}_Q(u_1') < -\ell = \text{ord}_Q(u_1)$.

In this case, we have $u_{Q,\ell} \sim f u_{Q,\ell} := u_1' + u_2$ and $|\text{ord}_Q(u_{Q,\ell}')| < |\text{ord}_Q(u_{Q,\ell})|$. After iterating this procedure for all terms $u_{Q,\ell}$ of $u$, and adding the equivalency relations, we will end up with

$$u = \sum_{Q,\ell} u_{Q,\ell} + h \sim \sum_{Q,\ell} u_{Q,\ell}' + h$$

(11)

Applying the partial fractions decomposition to $u'$, we obtain

$$u' = \sum_{Q,\ell} \hat{u}_{Q,\ell} + \hat{h} + h,$$

(12)

where $\hat{u}_{Q,\ell} \in F(z)$ and $\hat{h} \in F[z]$. Using Lemma 3.5(d) of [PZ04] for the polynomial $\omega := \hat{h} + h$, there is a $\hat{\omega} \in F[z]$ such that $\omega \sim f \hat{\omega}$ and $|\text{ord}_\infty(\hat{\omega})| \leq \Omega$, where $\Omega$ is a bound depending only on $f$. Define $E_{\text{ord}}$ to be

$$E_{\text{ord}}(f) := \max\{\frac{p^{m_0} + q}{p - 1}, \Omega\},$$

(13)

and Item 1 of the Lemma is proved. Item 2 follows clearly.

Note: As seen by its proof, Lemma 2.4 is true for any subring of $\tilde{F}_p(z)$ in the place of $R$. Moreover, observe that the condition $F \subset \tilde{F}_p$ seems to be essential.

3 The order of poles of strongly normalized polynomials is bounded

In this Section, we prove Theorem 1.3. For the rest of this Section, $F$ is an algebraically closed field of characteristic $p > 0$.

3.1 Hasse derivatives and properties.

Hasse derivative or hyperderivative [Has36, Sch76] is a generalization of the derivative for fields of rational functions. It is especially useful in positive characteristic. The $\epsilon - th$ Hasse derivative of $z^j$ is defined as

$$D_\epsilon(z^j) := \left(\frac{j}{\epsilon}\right)z^{j-\epsilon}, \text{ for } j \geq 0.$$
For $\epsilon = 0$, $D_0$ is the identity function and we write $D$ instead of $D_1$. In zero characteristic, it holds that

$$D_\epsilon(z^j) = \frac{1}{\epsilon!} \left( \frac{d}{dz} z^j \right). \quad (15)$$

A set of useful properties is provided in [Jeo11]. Some of these properties are - $f, g$ are rational functions of the variable $z$ over a field $F$:

**P1** The hyperderivative is linear, i.e.,

$$D_\epsilon(f + g) = D_\epsilon(f) + D_\epsilon(g), \quad (16)$$

**P2** The hyperderivative satisfies the Leibniz product formula:

$$D_\epsilon(fg) = \sum_{i+j=\epsilon} D_i(f)D_j(g), \quad (17)$$

where $f, g \in F(z)$ and $i, j, \epsilon \geq 0$.

**P3** If $p > 0$ is the characteristic of $F$, $m, \epsilon \in \mathbb{N}$ and $f \in F(z)$, then

$$D_\epsilon(f^{p^m}) = \begin{cases} (D_j(f))^{p^m} & \text{if } \epsilon = j p^m, \\ 0 & \text{if } \epsilon \not\equiv 0 \pmod{p^m}. \end{cases} \quad (18)$$

**P4** For $\epsilon \in \mathbb{N}$ and $0 \neq f \in F(z)$,

$$D_\epsilon \left( \frac{1}{f} \right) = \frac{\epsilon}{f^j+1} \sum_{\substack{i_1, \ldots, i_j \geq 1 \\ i_1 + \cdots + i_j = \epsilon}} D_{i_1}(f) \cdots D_{i_j}(f). \quad (19)$$

We will apply the above for a field $F$ of characteristic $p$. Let $q = p^s$ for some $s \in \mathbb{N}$. Let $\epsilon \in \{1, 2, \ldots, p^s - 1\}$. Since $\epsilon \not\equiv 0 \pmod{p^s}$, we conclude that, for every $f \in F(z)$, we have

$$D_\epsilon(f^{p^s}) = 0, \quad \forall \epsilon \in \{1, \ldots, q - 1\}. \quad (20)$$

Let $B = \{\beta_1, \ldots, \beta_q\}$ be a basis of $F(z)$ over $F(z^q)$ then, for every $g \in F(z)$, there exist $g_1, \ldots, g_q \in F(z)$ such that

$$g = g_1^q \beta_1 + \cdots + g_q^q \beta_q. \quad (21)$$

Then due to (16, 17) and (20) it holds that

$$D_\epsilon(g) = g_1^q D_\epsilon(\beta_1) + \cdots + g_q^q D_\epsilon(\beta_q), \quad \forall \epsilon \in \{1, \ldots, q - 1\}. \quad (22)$$
3.2 Linear independence criterion in positive characteristic

Now, we present a theorem between Wronskians and linearly independent sets of functions similar to the well known theorem of Calculus. The corresponding theorem regarding the linear independence and the respective Wronskian (with hyperderivatives instead of classic ones), for fields of positive characteristic, was initially described in [GV87] and strengthened in [Wan99]. Recall that we work over $F$ instead of classic ones), for fields of positive characteristic, was in itially described in $F$.

**Theorem 3.1** [Th.1, [GV87]] Let $x_1, \ldots, x_n \in F(z)$. Then $x_1, \ldots, x_n$ are linearly independent over $V_s(F)$ if and only if there exist integers $0 = \epsilon_1 < \epsilon_2 < \cdots < \epsilon_n < p^*$ with $\det (D_{\epsilon_i}(x_j)) \neq 0$.

Since $V_s(F)$ is $F(z)$, seen as a vector space over $F(z^{p^*})$, it follows that:

**Corollary 3.2** The set $\{b_1, b_2, \ldots, b_n\} \subset F(z)$ is linearly independent over $V_s(F)$ if and only if there exist integers $0 = \epsilon_1 < \epsilon_2 < \cdots < \epsilon_n < p^*$ with $\det (D_{\epsilon_i}(b_j)) \neq 0$.

3.3 The orders of hyperderivatives

Let $u \in F(z) \setminus \{0\}$. Fix $\lambda \in F$ and drop the subscript in $\text{ord}_\lambda(u)$ for convenience. We will give an upper bound of $\text{ord}(D_{\epsilon}(u))$. Initially, we consider the base $\{1, z - \lambda, (z - \lambda)^2, \ldots, (z - \lambda)^{q-1}\}$ of $F(z)$ over $F(z^{q^*})$ and rewrite $u$ as

$$u = u_0^q + (z - \lambda)u_1^q + (z - \lambda)^2u_2^q + \cdots + (z - \lambda)^{q-1}u_{q-1}^q,$$

for some $u_0, u_1, \ldots, u_{q-1} \in F(z)$. Next, we apply the $\epsilon$-th hyperderivative, for $\epsilon \in \{1, \ldots, q-1\}$, and deduce that

$$D_{\epsilon}(u) = u_0^q D_{\epsilon}(1) + u_1^q D_{\epsilon}((z - \lambda)^1) + \cdots + u_{q-1}^q D_{\epsilon}((z - \lambda)^{q-1}).$$

It is now easy to see, using the composition properties of [Jeo11], that

$$D_{\epsilon}((z - \lambda)^k) = \binom{k}{\epsilon}(z - \lambda)^{k-\epsilon}.$$  \hspace{1cm} (25)

Observe that $D_{\epsilon}((z - \lambda)^k)$ vanishes if $k < \epsilon$.

Since $\text{ord}((c(z - \lambda)^k) = k$, when $k \in \mathbb{N}$ and $c$ is a non-zero constant, for $k \geq \epsilon$, Equation (25) is equivalent to

$$\text{ord}(D_{\epsilon}((z - \lambda)^k)) = k - \epsilon.$$  \hspace{1cm} (26)

We are now ready to prove the following.

**Theorem 3.3** Let $u \in F(z) \setminus \{0\}$, where $F$ is a algebraically closed field of positive characteristic. If $\lambda \in F$, $\epsilon \in \mathbb{N}$ and $u, D_{\epsilon}(u) \neq 0$, then $\text{ord}_\lambda(D_{\epsilon}(u)) \geq \text{ord}_\lambda(u) - \epsilon$.  

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Proof: We write \( u \) as in 23 and observe that the terms on the right hand have pairwise distinct orders at \( \lambda \),

\[
\text{ord} \left( (z - \lambda)^k (u_k)^{q} \right) = k + q \cdot \text{ord}(u_k),
\]

(27)
since \( 0 \leq k < q \). Denote by \( \mu \) the remainder of dividing \( \text{ord}(u) \) by \( q \). It follows that the term in \( \{(z - \lambda)^k u_k^{q} : 0 \leq k < q\} \) with the lowest order is \( (z - \lambda)^\mu u_\mu^{q} \). Therefore, if we rewrite the terms \( \{D_e ((z - \lambda)^k) u_k^{q} : 0 \leq k < q\} \) on the right hand of 24 as \( \{(u^k(z - \lambda)^{k - \mu} u_\mu^{q} : \epsilon \leq k < q \} \) (due to 25) we again notice that all non-vanishing terms have different orders again, equal to the order of the corresponding term of \( u \) minus \( \epsilon \).

If the term \( D_e ((z - \lambda)^\mu) u_\mu^{q} \) does not vanish, it will have the smallest order among all terms, and in this case

\[
\text{ord} \left( D_e(u) \right) = \text{ord} \left( D_e ((z - \lambda)^\mu) u_\mu^{q} \right) = \text{ord} \left( (z - \lambda)^\mu u_\mu^{q} \right) - \epsilon = \text{ord}(u) - \epsilon
\]

(28)

If, however, that term vanishes, some other term with greater order (as a consequence of the analysis above) will give \( \text{ord} \left( D_e(u) \right) \). In this case, \( \text{ord} \left( D_e(u) \right) > \text{ord}(u) - \epsilon \) and our proof is complete.

3.4 The orders of inverse images and Proof of Theorem 1.3

Let \( y \in F(z) \) and let \( x = (x_1, x_2, \ldots, x_n) \) be a solution of

\[
y = f(x) = \sum_{i=1}^{n} f_i(x_i) = \sum_{i=1}^{n} b_i x_i^{p^s} + \sum_{i=1}^{n} \sum_{j=0}^{s-1} c_{i,j} x_i^{p^s-j},
\]

(29)

where \( x_i \in F(z) \), \( n \leq q = p^s \), \( s \in \mathbb{N} \), and \( b_i, c_{i,j} \) belong to \( F[z] \). We assume that the polynomials \( \{b_i : i = 1, \ldots, q\} \) are linearly independent over \( \mathcal{V}_s(F) \). We will prove the following.

Proposition 3.4 Let \( F \) and \( f(\bar{X}) \) as above and let \( \bar{X} = (x_1, x_2, \ldots, x_n) \) a solution of \( f(\bar{X}) = y \), with \( x_i \in F(z) \) for all \( 1 \leq i \leq q \). Let \( y = \frac{y_1}{y_2} \), where \( y_1, y_2 \in F[z] \). Then, there is a constant \( C \), depending only on \( \deg(y_2) \) and the coefficients of \( f \), such that, for any pole \( \lambda \in F \) of some \( x_k \), for all \( 1 \leq i \leq n \), we have \( \text{ord}_{\lambda}(x_i) \geq C \).

Proof: Consider 29 and move the terms of lower degree on the right hand side:

\[
b_1 x_1^{q} + \cdots + b_n x_n^{q} = u := - \sum_{i=1}^{n} \sum_{j=1}^{s} c_{i,j} x_i^{p^s-j} + y.
\]

(30)

By Corollary 3.2 since \( \{b_1, \ldots, b_n\} \) are linearly independent over \( \mathcal{V}_s(F) \), there exist integers \( 0 = \epsilon_1 < \epsilon_2 < \cdots < \epsilon_n < p^s \) with \( \det \left( D_{e_i}(b_{j}) \right) \neq 0 \).

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For every \( \epsilon \in \{ \epsilon_i : 1 \leq i \leq n \} \), we apply the hyperderivative \( D_\epsilon \) on both sides of \( \text{(30)} \). We get

\[
\sum_{i=1}^{n} x_i^q D_\epsilon(b_i) = D_\epsilon(u) = \sum_{i=1}^{n} \sum_{j=1}^{s-1} \gamma_{i,j} x_i^{p^s-j} + \sum_{i=1}^{n} \gamma_i D_\epsilon(x_i) + D_\epsilon(y),
\]

(31)

where \( \gamma_{i,j}, \gamma_i \) all polynomials of \( c_{i,j} \) and their hyperderivatives.

Consider the system obtained by writing Equation \( \text{(31)} \) for \( \epsilon = \epsilon_1, \epsilon_2, \ldots, \epsilon_n \),

\[
\begin{align*}
\sum_{i=1}^{n} x_i^q D_{\epsilon_1}(b_i) &= D_{\epsilon_1}(u) \\
\vdots & \\
\sum_{i=1}^{n} x_i^q D_{\epsilon_n}(b_i) &= D_{\epsilon_n}(u)
\end{align*}
\]

(32)

If we consider the variables \( x_i^q \) on the left hand side of the system as the unknowns then the determinant of the system is \( W := \det(D_\epsilon(b_j)) \neq 0 \).

Let \( J \) be an index such that \( \text{ord}(x_J) \leq \text{ord}(x_i) \) for every \( i \in \{1, \ldots, n\} \). Notice that \( \text{ord}(x_J) < 0 \) since we assumed that \( \lambda \) is a pole of some \( x_k \). Applying Crammer’s rule for the \( J \)-th index, we get that \( W x_J^q = \Lambda \), where

\[
\begin{vmatrix}
\begin{array}{cccccc}
b_1 & \cdots & b_{J-1} & u & b_{J+1} & \cdots & b_n \\
D_{\epsilon_2}(b_1) & \cdots & D_{\epsilon_2}(b_{J-1}) & D_{\epsilon_2}(u) & D_{\epsilon_2}(b_{J+1}) & \cdots & D_{\epsilon_2}(b_n) \\
D_{\epsilon_3}(b_1) & \cdots & D_{\epsilon_3}(b_{J-1}) & D_{\epsilon_3}(u) & D_{\epsilon_3}(b_{J+1}) & \cdots & D_{\epsilon_3}(b_n) \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
D_{\epsilon_n}(b_1) & \cdots & D_{\epsilon_n}(b_{J-1}) & D_{\epsilon_n}(u) & D_{\epsilon_n}(b_{J+1}) & \cdots & D_{\epsilon_n}(b_n)
\end{array}
\end{vmatrix} = \sum_{i=1}^{n} \sum_{j=1}^{s-1} \delta_{i,j} x_i^{p^s-j} + \sum_{i=1}^{n} \sum_{k=1}^{n} \delta_i D_{\epsilon_k}(x_i) + \sum_{k=1}^{n} \zeta_k D_{\epsilon_k}(y),
\]

\[
\text{(33)}
\]

where \( \delta_{i,j}, \delta_i \) and \( \zeta_k \) are polynomials that can be determined from \( f \).

We are now interested in determining bounds for \( \text{ord} \) of each term on the right hand side of \( \text{(33)} \).

Regarding the terms of the form \( \delta_{i,j} x_i^{p^s-j} \), where \( 1 \leq i \leq n \) and \( 1 \leq j \leq s-1 \), it holds that

\[
\text{ord}(\delta_{i,j} x_i^{p^s-j}) \geq p^{s-1} \text{ord}(x_J) + \Delta,
\]

\[
\text{(34)}
\]

where \( \Delta := \min_{i,j}\{\text{ord}(\delta_{i,j})\} \), since \( \text{ord}(x_i) \geq \text{ord}(x_J) \) and \( \text{ord}(x_J) < 0 \).

Regarding the terms of the form \( \delta_i D_{\epsilon_k}(x_i) \), where \( 1 \leq i \leq n \) and \( 1 \leq k \leq n \), it holds that

\[
\text{ord}(\delta_i D_{\epsilon_k}(x_i)) \geq p^{s-1} \text{ord}(x_J) + E - (q - 1),
\]

\[
\text{(35)}
\]
where $E := \min_i \{\text{ord}(\delta_i)\}$. To prove this, we have used Theorem 3.3 along with the relations $\epsilon_k \leq q - 1$, $\text{ord}(x_i) \geq \text{ord}(x_J)$ and $\text{ord}(x_J) < 0$.

Since $\Delta, E \geq 0$, all terms $\delta_{i,j}(x_i)^{p^e-j}$ and $\delta_{i}D_{\epsilon_k}(x_i)$ have $\text{ord}_\lambda$ greater than or equal to $p^{s-1}\text{ord}_\lambda(x_J)+1-q$. Let $\Phi_2$ be the order at $\lambda$ of the term $\sum_{k=1}^n \zeta_i D_{\epsilon_k}(y)$ appearing in 33. Then, clearly, it holds that $\Phi_2 \geq -\deg(y_2)-(q-1)$, due to Theorem 3.3. Also observe that, from the relation $Wx_J^q = \Lambda$, it follows that $\text{ord}_\lambda(W) + q\text{ord}_\lambda(x_J) = \text{ord}_\lambda(\Lambda)$.

We now compare $\Phi_2$ and $p^{s-1}\text{ord}_\lambda(x_J)+1-q$ and distinguish two cases.

- If $p^{s-1}\text{ord}_\lambda(x_J)+1-q < \Phi_2$, then $\text{ord}_\lambda(\Lambda) \geq p^{s-1}\text{ord}_\lambda(x_J)+1-q$. In this case, we conclude that
  \[
  \text{ord}_\lambda(x_J) \geq \frac{1-q-\text{ord}_\lambda(W)}{q-p^{s-1}} \geq \frac{1-q-\deg_z(W)}{q-p^{s-1}}. \tag{36}
  \]

- If $p^{s-1}\text{ord}_\lambda(x_J)+1-q \geq \Phi_2$, then $\text{ord}_\lambda(\Lambda) \geq \Phi_2 \geq -\deg(y_2)-(q-1)$. It follows that
  \[
  \text{ord}_\lambda(x_J) \geq \frac{1-q-\deg_z(W)-\deg(y_2)}{q}. \tag{37}
  \]

Take $C$ to be the minimum of the bounds of $\text{ord}_\lambda(x_J)$ in 36 and 37. Note that $\text{ord}_\lambda(x_i) \geq \text{ord}_\lambda(x_J) \geq C$ for every $1 \leq i \leq n$ and the proof is complete. \hfill \Box

Proof:[Proof of Theorem 1.1] It suffices to establish the conclusion of Proposition 3.4 not only for affine poles but also for the pole at infinity. Here is how: Let $\eta \in F$ such that none of the coefficients of $f$ has a zero at $\eta$. Apply the automorphism $z \mapsto 1/(z - \eta)$ and observe that coefficients $b_i$ map to $\tilde{b}_i/(z - \eta)^{\deg(b_i)}$, where $\tilde{b}_i \in F[z]$. Let $M := \max_i \{\deg(b_i)\}$. Clear the denominators by multiplying with $(z - \eta)^M$ and apply Proposition 3.4 for $\lambda = \eta$. \hfill \Box

### 4 The Proof of Theorem 1.1

We will now prove a series of propositions that will be used to prove Theorem 1.1.

**Proposition 4.1** Let $f$ and $h$ be additive polynomials of degree $p^e$, in $m$ and $n$ variables respectively, where $m+n = p^e$. Let $H$ and $G$ be additive polynomials in only $F$-variables. Assume that, for some $N \in \mathbb{N}$ and for some $e \in \mathbb{F}_p[z]$, it holds that

\[
R = \text{Im}(f) + \text{Im}(h) + \text{Im}_F\left(\frac{1}{e^N}G\right). \tag{38}
\]

Then, the formula $u \in \text{Im}(f) + \text{Im}_F(H)$ is equivalent, in $\mathcal{R}$, to the formula $\phi_1(u)$, defined as

\[
\forall x = (x_1, \ldots, x_m) \forall y \forall \gamma \left[ (\gamma \in F \land u = f(x) + h(y) + \frac{1}{e^N}G(\gamma)) \rightarrow \pi_1(y, \gamma) \right], \tag{39}
\]
where $\pi_1(y, \gamma)$ is the formula

$$
\pi_1(y, \gamma) : \exists w \exists \alpha [\alpha \in F \land f(w) + h(y) + \frac{1}{eN} \hat{G}(\gamma) = H(\alpha)].
$$

(40)

In the writing of the formulas above we mean that the variables of the tuples $x, y, w, \alpha, \beta$ and $\gamma$ are pairwise distinct.

Note that essentially $e = s_1 s_2 \cdots s_n$. The proof is similar to the one of Claim 4.2 (iii) of [PZ04]. For completeness, we include a detailed proof in Section A.2.

We will be referring to an existential formula $\phi(u, \{v_j\}_{j \in J})$, as in 5.

**Proposition 4.2** With notation as above, the formula $\phi(u, \{v_j\}_{j \in J})$ is equivalent, in $R$, to the formula $\phi_2$, given by

$$
\phi_2 : u \in Im(f) + Im_F(H) \land
\forall w \forall \beta [(\beta \in F \land f(w) + H(\beta) = u) \rightarrow \pi_2(\{v_j\}_{j \in J, w, \beta})],
$$

where

$$
\pi_2(\{v_j\}_{j \in J, w, \beta}) : \exists t \exists \gamma [\gamma \in F \land f(t) + H(\gamma) = 0 \land
\{\land_{j \in J} e_j(t) + G_j(\gamma) \neq v_j - e_j(w) - G_j(\beta)\} \land P_\sigma(\beta + \gamma)].
$$

Here, $P_\sigma(\beta + \gamma)$ denotes the result of substitution in $P_\sigma(\alpha)$ of the tuple of variables $\alpha$ by the array of variables $\beta + \gamma$ where $+$ implies component-wise addition. The variables of the tuples $\alpha$, $\beta$ and $\gamma$ are pairwise distinct.

The proof is similar to the one of Claim 4.1 of [PZ04]. For completeness, we include a detailed proof in Section A.3. Note that Proposition 4.2 holds even if the ring $R$ is replaced by any subring of $F(z)$ that contains $F[z]$.

**Proposition 4.3** A bounded existential $\mathcal{L}_p(z)^e$-formula is equivalent in $R$ to a universal $\mathcal{L}_p(z)^e$-formula.

**Proof:** We will prove the proposition for any formula of the form

$$
\pi(\{v_j\}_{j \in J, \beta}) : \exists \gamma [\gamma \in F \land \{\land_{j \in J} G_j(\beta, \gamma) \neq v_j\} \land P_\sigma(\beta, \gamma)],
$$

(43)

where $v_j$ are terms of $\mathcal{L}_p(z)$ and the $G_j$ are additive polynomials in the variables of the tuples of $F$-variables $\beta$ and $\gamma$. Let $M$ be the maximum of the degrees of the $G_j$ with respect to the variable $z$. Pick an element $Q$ of $\mathbb{F}_p[z]$, which is not divisible by any polynomial that is irreducible in $F[z]$ and invertible in $R$. For each of the terms $v_j$, we construct the term

$$
t_j := \sum_{i=0}^{M} \mu_{i,j} Q^i,
$$

(44)
where each $\mu_{i,j}$ is a term of the form $\mu_{i,j} = \sum_{k=0}^{d} \mu_{i,j,k} z^k$ with $d := \deg(Q)$, and $\mu_{i,j,k}$ are new $F$-variables. For each $j \in J$, let $y_j$ be a new variable (ranging over $R$).

Observe that, by Fact 1.6, for each $j \in J$ and any value of the term $v_j$ over $R$, there are values of the variables of $\mu_{i,j}$ over $F$ so that $v_j = t_j + Q^{N+1}y_j$ holds true. And, for those values and any value of the variables of the tuples $\beta$ and $\gamma$, the sub-formula $G_j(\beta, \gamma) \neq v_j$ is equivalent to $y_j \neq 0 \lor G_j \neq t_j$. Hence $\pi$ is equivalent to the formula

$$
\chi(\{v_j\}_{j \in J}, \beta) := \forall \mu \forall y \ (\beta \in F \land \mu \in F \land \{ \land_{j \in J} v_j = t_j + Q^{N+1}y_j \} ) \rightarrow 
(\lor_{K \subseteq J} \{ \land_{j \in K} y_j \neq 0 \} \land \{ \land_{j \in J \setminus K} y_j = 0 \} \land \pi_K[v_j/t_j]),
$$

where

- $\mu$ stands for the tuple of all variables $\mu_{i,j,k}$,
- $y$ stands for the tuple of variables $y_j$,
- the index $K$ of $\lor_K$ ranges over all subsets of the set $J$ of indices in $\pi$ and,
- for each subset $K$ of $J$, the formula $\pi_K[v_j/t_j]$ is the formula that results from $\pi$ by deleting the inequalities $G_j(\beta, \gamma) \neq v_j$ for which $j \in K$ and replacing each term $v_j$, for which $j \in J \setminus K$, by $t_j$.

We prove the equivalence of $\pi$ and $\chi$:

Assume that $\pi(\{v_j\}_{j \in J}, \beta)$ is true for some set of values $\tilde{v}_j$ of the terms $v_j$ and $\tilde{\beta}$ of the $F$-variables $\beta$. Write each $v_j$ as $\tilde{v}_j = \tilde{t}_j + Q^{N+1}\tilde{y}_j$, using Fact 1.6. Let $K$ be the set of indices $j$ for which $\tilde{y}_j \neq 0$. Then, for each $j \in J \setminus K$, we have $\tilde{y}_j = 0$. Then, since $\pi(\{\tilde{v}_j\}_{j \in J}, \tilde{\beta})$ is true, it holds that there are $\tilde{\gamma}$ over $F$ so that the inequalities $G_j(\tilde{\beta}, \tilde{\gamma}) \neq \tilde{v}_j$ hold for $j \in J \setminus K$, hence, for those $j$, we have $\tilde{y}_j = 0$ and $G_j(\tilde{\beta}, \tilde{\gamma}) \neq \tilde{t}_j$ holds true and $P_\sigma(\beta, \gamma)$ is true; Hence $\pi_K[v_j/t_j]$ is true. So $\chi(\{v_j\}_{j \in J}, \beta)$ is true. We leave the converse to the reader. Now observe that each sub-formula $\pi_K[v_j/t_j]$ of $\pi$ is equivalent in $\mathcal{R}$ to a formula of the form $P_\tau$, for some formula $\tau$ of $\mathcal{L}_p$. The same holds true for equations of the form $H(\beta) = 0$ that may appear in the quantifier free part of $\pi$. So the result holds in the desired generality.

Proof] of Theorem 1.1

By Lemma 2.1 given an additive polynomial $f$ as in Proposition 4.1 or 4.2 there is a proper transformation $\xi(Y, \delta)$ such that, setting $\hat{f}(y) + G(\delta) = f(\xi(Y, \delta))$, the additive polynomial $\hat{f}$ is strongly normalized. Hence, since a proper transformation is onto, we may assume that the $f$ appearing in Propositions 4.1 or 4.2 are strongly normalized.

Lemma 2.4 implies that, since the additive polynomial $f$ is strongly normalized, there are $h$ and $G$ so that the assumption of Proposition 4.1 is satisfied for $e \in \mathbb{F}_p[z]$ and $N \in \mathbb{N}$. Observe that, due to Theorem 1.3 the variables of $w$ in $\pi_1$ can be substituted by a finite tuple of $F$-variables. Hence, formula $\pi_1(y, \gamma)$ is equivalent to a formula of the form $P_\tau$, for some formula $\tau$ of $\mathcal{L}_p$. It follows that $\phi_1(u)$ is equivalent to a universal $\mathcal{L}_p(z)^+$-formula.

In Proposition 4.2 observe that, from Theorem 1.3 and the equation $f(t) + H(\gamma) = 0$ in $\pi_2$, the variable $t$ may be substituted by a finite tuple of $F$-variables. Hence the formula $\pi_2(\{v_j\}_{j \in J}, w, \beta)$ is equivalent in $\mathcal{R}$ to a bounded existential formula.
So far we have proved that any existential $\mathcal{L}_p(z)^e$-formula is equivalent in $\mathcal{R}$ to a disjunction of bounded existential formulas, in which equations do not occur. From Proposition 4.3 it follows that formulas $\pi_1$ of Proposition 4.1 and $\pi_2$ of Proposition 4.2 are equivalent to universal $\mathcal{L}_p(z)^e$-formulas, and actually to ones with no more free variables than those present in $\pi_1$ and $\pi_2$ respectively.

By induction on the number of alterations of quantifiers of an arbitrary $\mathcal{L}_p(z)^e$-formula in prenex form Theorem 1.1 is proved.

For Item 2 of Theorem 1.1, it follows from Item 1 that any $\mathcal{L}_p(z)^e$-sentence is equivalent to an existential formula like $\phi$ in 5 which is, in addition, a sentence. This means that, in this case, the terms $u$ and $v_j$ are concrete elements of $\mathbb{F}_p[z]$. By Lemma 2.1 we may assume, without loss of generality, that the additive polynomial $f$ is strongly normalized. Re-enumerate the variables of $x$ so that $x = (x_1, \ldots, x_k, x_{k+1}, \ldots, x_m)$ and $x_{k+1}, \ldots, x_m$ are exactly the variables of $x$ which occur in $f$ with non-zero highest degree coefficient. Then, by Theorem 1.3 for any value $\tilde{x}$ of the tuple $x$ which is a solution of the equation $f + H = u$, the heights of $\tilde{x}_{k+1}, \ldots, \tilde{x}_m$ are effectively bounded, hence, the variables $x_{k+1}, \ldots, x_m$ may be substituted by (existentially quantified) $F$-variables. Therefore, we may assume that the sentence $\phi$ has no equations and amounts to the solvability of the system of inequalities $g_j + G_j \neq v_j$, together with $P_\sigma$. Clearly, because $\mathcal{R}$ is an infinite domain, all inequalities in which some of the variables $x_1, \ldots, x_k$ occurs with a non-zero coefficient may be satisfied simultaneously. All the inequalities in which none of the variables $x_1, \ldots, x_k$ occurs is clearly equivalent to a formula of the form $P_\tau$. Hence $\phi$ is equivalent in $\mathcal{R}$ to a formula of the form $P_\tau$, for some sentence $\tau$ of $\mathcal{L}_p$.

\[ \Box \]

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References

[Has36] Helmut Hasse. “Theorie der höheren Differentiale in einem algebraischen Funktionenkörper mit vollkommener Konstantenkörper bei beliebiger Charakteristik.” In: Journal für die reine und angewandte mathematik 1936.175 (Jan. 1936), pp. 50–54.

[Rob51] Raphael M Robinson. “Undecidable rings”. In: Transactions of the American Mathematical Society 70.1 (1951), pp. 137–159.
[AK65] James Ax and Simon Kochen. “Diophantine Problems Over Local Fields I”. In: *American Journal of Mathematics* 87.3 (1965), pp. 605–630. ISSN: 00029327, 10806377. URL: [http://www.jstor.org/stable/2373065](http://www.jstor.org/stable/2373065).

[Maz75] Barry Mazur. “Eigenvalues of Frobenius acting on algebraic varieties over finite fields”. In: *Algebraic geometry (Proc. Sympos. Pure Math., Vol. 29, Humboldt State Univ., Arcata, Calif., 1974)*. 1975, pp. 231–261.

[Sch76] Wolfgang M Schmidt. *Equations over finite fields: an elementary approach*. Vol. 536. Springer, 1976.

[Den79] Jan Denef. “The Diophantine Problem for Polynomial Rings of Positive Characteristic”. In: *Logic Colloquium '78*. Ed. by Maurice Boffa, Dirkvan Dalen, and Kenneth Mcaloon. Vol. 97. Studies in Logic and the Foundations of Mathematics. Elsevier, 1979, pp. 131–145.

[Phe85] Thanases Pheidas. “The Diophantine problem for addition and divisibility in polynomial rings (decidability, undecidability)”. PhD thesis. Purdue University, 1985.

[GV87] Arnaldo Garcia and J F Voloch. “Wronskians and linear independence in fields of prime characteristic”. In: *manuscripta mathematica* 59.4 (Dec. 1987), pp. 457–469.

[Shl93] Alexandra Shlapentokh. “Diophantine relations between rings of S-integers of fields of algebraic functions in one variable over constant fields of positive characteristic”. In: *J. Symbolic Logic* 58.1 (1993), pp. 158–192.

[CH99] Zoé Chatzidakis and Ehud Hrushovski. “Model theory of difference fields”. In: *Trans. Amer. Math. Soc.* 351.8 (1999), pp. 2997–3071. ISSN: 0002-9947.

[Wan99] Julie Tzu-Yueh Wang. “A note on Wronskians and the ABC theorem in function fields of prime characteristic”. In: *manuscripta mathematica* 98.2 (Feb. 1999), pp. 255–264.

[PZ00] Thanases Pheidas and Karim Zahidi. “Undecidability of existential theories of rings and fields: a survey”. In: *Contemporary mathematics - American Mathematical Society* 270 (2000), pp. 49–105. ISSN: 1098-3627.

[Hru04] Ehud Hrushovski. “The elementary theory of the Frobenius automorphisms”. In: *arXiv preprint math/0406514* (2004).

[PZ04] Thanases Pheidas and Karim Zahidi. “Elimination theory for addition and the Frobenius map in polynomial rings”. In: *Journal of Symbolic Logic* 69.4 (2004), pp. 1006–1026.

[PZ08] Thanases Pheidas and Karim Zahidi. “Decision problems in algebra and analogues of Hilbert’s Tenth Problem”. eng. In: *Model theory with Applications to Algebra and Analysis*. Cambridge University Press, 2008, pp. 207–236. ISBN: 9780521709088.
[Poo08] Bjorn Poonen. “Undecidability in number theory”. In: Notices of the American Mathematical Society 55 (Mar. 2008), pp. 344–350.

[Sir10] Alla Sirokofrkich. “On an exponential predicate in polynomials over finite fields”. In: Proceedings of the American Mathematical Society 138.7 (2010), pp. 2569–2583.

[Jeo11] Sangtae Jeong. “Calculus in positive characteristic p”. In: Journal of Number Theory 131.6 (June 2011), pp. 1089–1104.

[Koe18] Jochen Koenigsmann. “Decidability in local and global fields”. In: Proceedings of the International Congress of Mathematicians (ICM 2018). World Scientific, 2018, pp. 45–59.

[Ona18] Gönenç Onay. $\mathbb{F}_p((X))$ is decidable as a module over the ring of additive polynomials. 2018. arXiv: [1806.03123]

[CMS21] Jennifer Chubb, Russell Miller, and Reed Solomon. “Model completeness and relative decidability”. In: Archive for Mathematical Logic (2021), pp. 1–15.
A Appendix - Omitted Proofs

A.1 Proof of Fact 1.6

Proof: To prove Item 1, consider a $u = \frac{a}{b} \in R$ with $a$ and $b$ coprime polynomials of $F[z]$ and let $c \in F[z] \setminus \{0\}$. Then $b$ is invertible in $R$. By the elementary algebra of polynomials we know that there are $v', r' \in F[z]$, with $\deg(v') < \deg(b)$ and $\deg(r') < \deg(c)$, such that $v'c + r'b = 1$. So $\frac{a}{b} = v'c + ar'$. Divide the polynomial $ar'$ by $c$ - using euclidean division of polynomials - to obtain the result.

For the uniqueness statement, assume in the above that $c$ is not divisible by any irreducible factor in $F[z]$ which is invertible in $R$. Assume that there are $v, \hat{v}$ in $R$ and $r$ and $\hat{r}$ in $F[z]$ such that $u = vc + r = \hat{v}c + \hat{r}$. Then $0 = (v - \hat{v})c + (r - \hat{r})$ with $\deg(r - \hat{r}) < \deg(c)$. Then there is a $\hat{b} \in F[z]$, which is invertible in $R$, such that $\hat{b}(v - \hat{v}) \in F[z]$. Hence $0 = \hat{b}(v - \hat{v})c + \hat{b}(r - \hat{r})$. Hence $c$ divides $\hat{b}(r - \hat{r})$ in $F[z]$. But by hypothesis $c$ is co-prime to $\hat{b}$. Hence $c$ divides $r - \hat{r}$ in $F[z]$, so, necessarily $r - \hat{r} = 0$.

For Item 2, iterate the conclusion of Item 1 $N$ times, each time applying it to the ‘quotient’ $v$ of the previous step.

Item 3 follows from Item 1. \hfill ∎

A.2 Proof of Proposition 4.1

Proof: $(\rightarrow)$ Say that for some $\bar{u} \in R$, for some value $\bar{x}$ of the tuple of variables $x$ and for some value $\bar{\alpha}$ (over $F$) of the tuple of variables $\alpha$ we have $\bar{u} = f(\bar{x}) + H(\bar{\alpha})$.

Consider any value $\bar{\tilde{x}}$ of the tuple $\bar{x}$, any value $\tilde{y}$ of the tuple $y$ and any value (over $F$) of the tuple of variables $\tilde{\gamma}$ of the tuple $\gamma$ for which

$$
\bar{u} = f(\bar{x}) + h(\tilde{y}) + \frac{1}{e^N} G(\tilde{\gamma}).
$$

Set $\tilde{\bar{w}} = \bar{x} - \tilde{x}$ where $-$ denotes component-wise subtraction. Then, by the additivity of $f$ we have

$$
f(\tilde{\bar{w}}) + h(\tilde{y}) + \frac{1}{e^N} G(\tilde{\gamma}) = f(\bar{x} - \tilde{x}) + h(\tilde{y}) + \frac{1}{e^N} G(\tilde{\gamma}) =
$$

$$
f(\bar{x}) - f(\tilde{x}) + h(\tilde{y}) + \frac{1}{e^N} G(\tilde{\gamma}) = \bar{u} - f(\tilde{x}) = H(\bar{\alpha}).
$$

$(\leftarrow)$ By hypothesis there are $\bar{x}, \tilde{y}$ and $\tilde{\gamma}$ so that $u = f(\bar{x}) + h(\tilde{y}) + \frac{1}{e^N} G(\tilde{\gamma})$. Let $\bar{u} \in R$ and assume that $\phi_3(\bar{u})$. Then there is a $w$ and a $\bar{\alpha}$ such that $f(w) + h(\tilde{y}) + \frac{1}{e^N} G(\tilde{\gamma}) = H(\bar{\alpha})$.

Set $\bar{x} = \bar{x} - \tilde{w}$. By the additivity of $f$ and we have

$$
f(\bar{x}) + H(\bar{\alpha}) = f(\bar{x}) - f(\tilde{w}) + H(\bar{\alpha}) =
$$

$$
[\bar{u} - h(\tilde{y}) - \frac{1}{e^N} G(\tilde{\gamma})] - [H(\bar{\alpha}) - h(\tilde{y}) - \frac{1}{e^N} G(\tilde{\gamma})] + H(\bar{\alpha}) = \bar{u},
$$

hence $\bar{u} \in Im(f) + Im_F(H)$. \hfill ∎
A.3 Proof of Proposition 4.2

Proof: Assume that \( \tilde{u} \) and \( \tilde{v}_j \) are given values of the terms \( u \) and \( v_j \), respectively. Assume that for some value \( \tilde{x} \) of the array of variables \( x \) over \( R \) and for some value \( \tilde{\alpha} \) of the array of variables \( \alpha \) over \( F \) the statement \( \psi(\tilde{x}, \tilde{\alpha}) \) is true in \( \mathcal{R} \), with \( \psi \) as in \( \Box \) i.e.,

\[
 f(\tilde{x}) + H(\tilde{\alpha}) = \tilde{u} \land \bigwedge_{j \in J} e_j(\tilde{x}) + G_j(\tilde{\alpha}) \neq \tilde{v}_j \land P_\sigma(\tilde{\alpha})
\]

(48) holds.

Let \( \tilde{w} \) be a tuple of elements of \( R \) and \( \tilde{\beta} \) be a tuple of elements of \( F \) such that \( f(\tilde{w}) + H(\tilde{\beta}) = \tilde{u} \) is true in \( \mathcal{R} \).

Define \( \tilde{t} = \tilde{x} - \tilde{w} \) and \( \tilde{\gamma} = \tilde{\alpha} - \tilde{\beta} \), where \( - \) denotes component-wise subtraction. Then, by the additivity of \( f \), \( H \), \( e_j \) and \( G_j \) we have

\[
 f(\tilde{t}) + H(\tilde{\gamma}) = f(\tilde{x}) - f(\tilde{w}) + H(\tilde{\alpha}) - H(\tilde{\beta}) = \tilde{u} - \tilde{u} = 0,
\]

(49) and for each \( j \in J \) the following holds:

\[
 e_j(\tilde{t}) + G_j(\tilde{\gamma}) = e_j(\tilde{x}) + G_j(\tilde{\alpha}) - e_j(\tilde{w}) - G_j(\tilde{\beta}) \neq \tilde{v}_j - e_j(\tilde{w}) - G_j(\tilde{\beta}).
\]

(50)

Moreover \( P_\sigma(\tilde{\alpha}) \) holds true, hence \( P_\sigma(\tilde{\beta} + \tilde{\gamma}) \) holds true. It follows that \( \mathcal{R} \models \phi \rightarrow \phi_2 \).

Now assume that \( \phi_2 \) is true in \( \mathcal{R} \) for the given values \( \tilde{u} \) and \( \tilde{v}_j \) of \( u \) and \( v_j \), respectively. Since \( \tilde{u} \in \text{Im}(f) + \text{Im}_F(H) \) there are values \( \tilde{w} \) over \( R \) and \( \tilde{\beta} \) over \( F \) of the variables \( w \) and \( \beta \) such that \( \tilde{\beta} \in F \land f(w) + H(\tilde{\beta}) = \tilde{u} \). Since \( \phi_2 \) is true, there are values \( \tilde{t} \) over \( R \) and \( \tilde{\gamma} \) over \( F \) of the variables \( t \) and \( \gamma \), respectively, so that \( f(\tilde{t}) + H(\tilde{\gamma}) = 0 \land \bigwedge_{j \in J} e_j(\tilde{t}) + G_j(\tilde{\gamma}) \neq \tilde{v}_j - e_j(\tilde{w}) - G_j(\tilde{\beta}) \land P_\sigma(\tilde{\beta} + \tilde{\gamma}) \). Define \( \tilde{x} = \tilde{w} + \tilde{t} \) and \( \tilde{\alpha} = \beta + \tilde{\gamma} \). Obviously \( \psi(x, \alpha) \) of \( \Box \) is true for the values \( \tilde{x} \) over \( R \) of \( x \) and \( \tilde{\alpha} \) over \( F \) of \( \alpha \). It follows that \( \mathcal{R} \models \phi_2 \rightarrow \phi \).

\[ \Diamond \]

A.4 Outline of the proof of Lemma 2.1

Proof: We present the proof of the statement of the Lemma up to the point that the resulting \( \tilde{f} \) is \( p \)-free. Consider a given additive polynomial \( f(x) = \sum_{i=1}^{m_0} f_i(x_i) \) of degree \( p^s \) where each \( f_i(x_i) \) is an additive polynomial of the variable \( x_i \) only and of degree \( p^{s_i} \) and coefficients \( b_i \in \mathbb{F}_p[z] \). Assume that the set \( B = \{ b_i z^{p^s} \mid i = 1, \ldots, m_0, \ 0 \leq j < p^{s_i} \} \) is linearly dependent in \( \mathcal{V}_s(\mathbb{F}_p) \). Then there are \( c_{i,j} \in \mathbb{F}_p[z] \), not all equal to 0, so that \( \sum_{i,j} c_{i,j} b_i z^{p^s} = 0 \), where the index \( j \) ranges as in the set \( B \). Re-enumerating the indices \( i \) we may assume that some \( c_{i,j} \) is not equal to 0 and for each \( i \) for which there is a \( j \) so that \( c_{i,j} \neq 0 \) we have \( s_i \neq s_i \). Let

\[
 c := \sum_{\mu=0}^{p^{s_1}-1} c_{1,\mu} z^{\mu}.
\]

(51)
Notice that, since the set \( \{1, \ldots, z_{p^s-1} \} \) is linearly independent in \( V_{s-s_1}(\mathbb{F}_p) \) (see Fact 1.8) we have \( c \neq 0 \). We apply the proper transformation: \( x_1 = cy_1 + H \) and, for \( i \neq 1 \),

\[
x_i = y_i + \sum_{j=0}^{p^s-s_i-1} c_{i,j} z^j y_1^{p^s_1-s_i},
\]

(52) where \( H = \alpha_0 + \alpha_1 z + \cdots + \alpha_{\ell-1} z^{\ell-1}, \) \( \ell \) is the degree of \( c \) and the \( \alpha_k \) are new and pairwise distinct \( F \)-variables. The polynomial that results from the transformation has the form \( \hat{f} + G \), where \( \hat{f} \) has variables the unrestricted (i.e. not \( F \)-variables) \( y_i \) and \( G \) is an additive polynomial in the \( F \)-variables \( \alpha_k \). We observe that the coefficient of \( y_1^{p^s_1} \) in \( \hat{f} \) is \( b_1 c^{p^s_1} + \sum_{i \neq 1} \sum_{j=0}^{p^s-s_i-1} c_{i,j} z^j y_1^{p^s_1-s_i} = 0 \). Then \( \hat{f} \) has degree in \( y_1 \) less than the degree of \( f \) in \( x_1 \) and for all \( i > 1 \) it has degree in \( y_i \) equal to the degree of \( f \) in \( x_i \). Work by induction on the sum of the degrees of \( f \) in each of its (unrestricted) variables. The proper transformation \( \xi \) is onto, by Facts 1.6 and 1.8, so \( \text{Im}(f) = \text{Im}(\hat{f}) + \text{Im}_F(G) \).

Now consider a given additive polynomial \( f \), as above, which is \( p \)-free. It is easy to see that substituting each variable \( x_i \) by

\[
x_{i,0}^{p^s-s_i} + \cdots + z^k x_{i,k}^{p^s-s_i} + z^{p^s-s_i-1} x_{i,p^s_1-s_i-1},
\]

(53) where the \( x_{i,k} \) are new variables, results in a normalized additive polynomial \( \tilde{f} \): The degree of \( \tilde{f} \) with respect to each of its variables is \( s \) and the set of its leading coefficients is linearly independent in \( V_s(\mathbb{F}_p) \). By Fact 1.8 \( f \) and \( \tilde{f} \) have the same image.

To convert a normalized additive polynomial to a strongly normalized one takes the application of a sequence of proper transformations - it is left to the reader or see [PZ04], Lemma 3.3.