Theoretical investigation of the perturbed artificial satellite problem using oblate continued fraction potential

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ABSTRACT

In this research article, a new idea concerning the application of concept of the continued fractions to the expansion of the geopotential is treated. The perturbed motion of the satellite in the oblate gravitational field of the Earth using a potential of a continued fractions procedure is studied. To compare with the usual perturbation theory, the continued fraction series is truncated beyond the third term. The first oblateness term of the potential is retained. The equations of motion are constructed using the Lagrange planetary equations. The integrals and averages required for solving the problem is outlined as well as the different cases are highlighted. The perturbations in all orbital elements due to considered model of continued fractions are evaluated. The new findings of the manuscript contrast with classical oblate potential field, the semi-major axis, eccentricity, and inclination, nonzero averaged terms in perturbations are obtained. Extra nonzero averaged perturbing terms in the argument of perigee and ascending node are revealed in addition to the classical perturbations.

1. Introduction

The realistic calculations of orbital elements are not a deterministic one. No unique methods of integration can be relied on for complete regime of orbital elements without any bugs. In investigation about the most ideal track of calculations, one is always looking for new methods that fulfill this target or even to interpret some uninterpreted phenomena e.g. the secular decay of the semi-major axis of LAGEOS satellite by a rate of 1.1 mm/day. Many force models are advised and examined to interpret these phenomena. But unfortunately, most of these trials indeed fail to interpret such phenomenon. In that way one can suggest different potential forms to account for unpredictable phenomena.

The Earth’s potential due to the sphericity of the central body is considered as the most important perturbations of the two-body problem. There are several ways to express the Earth’s potential. This form includes the two-body part of the potential is the first term. A detailed discussion of the harmonic gravity terms can be found in Roy [1]

\[ U_{B} = -\frac{\mu}{r} \left[ 1 + \sum_{n=2}^{\infty} \frac{J_{n}}{n} \frac{R_{n}(\sin \phi)}{r} \cos n \lambda \sin m \lambda + \sum_{n=2}^{\infty} \frac{C_{n,m}}{n} \cos n \lambda \sin m \lambda + \sum_{n=2}^{\infty} \frac{S_{n,m}}{n} \sin n \lambda \right] \]

where \(J_n\) are the zonal harmonics, \(C_{n,m}\), \(S_{n,m}\) are the sectorial harmonics of the gravitational field, \(R_{n}\) is the mean equatorial radius of the Earth, \(P_{n}(\sin \phi)\) are the Legendre polynomials, \(P_{m}^{n}\) are the associated Legendre functions, \(\phi\) is the satellite’s geocentric latitude, and \(\lambda\) is the satellite’s geocentric longitude. The leading term is due to the potential of the point mass. The dominant terms beyond the leading term are due to \(J_{2}\) zonal harmonic which is the most important effect. Many interested researchers investigated the effect of these perturbations on the motion of satellites, e.g. Iorio [2], Renzetti [3], Abd El-Salam et. al. [4], Abd El-Bar and Abd El-Salam [5], Abd El-Salam and Abd El-Bar [6] and Elshaboury and Mostafa [7].

The problem of computing the perturbed orbit due to oblateness of the Earth has received considerable attention since the beginning of the space era to the date. It is worth noting to sketch here some important works [8, 9]. Some of these studies of this problem by means of general perturbation theories are Struble [10], Arsenault et al. [11] and Sagovac [12].

Mihail Bărbosu et al. [13] considered a generalization of the famous Lennard–Jones potential to study the two-body problem associated to this potential. They investigated all possible situations created by the interplay among the constants of integration and the field parameters. They obtained the global flow on the zero energy manifold of the two-body problem given by the sum of the Newtonian potential and the two
anisotropic perturbations corresponding to the
generalized Lennard–Jones potential, and illustrated it in
both 3D and 2D. This flow exhibits a great variety of
orbits, a homoclinic one is included. All phase portraits
are interpreted in terms of physical trajectories.

Abd El-Salam et al. [14] have used a scheme of con-
tinued fraction to reformulate the two-body problem, they
obtained the integrals of motion in this model.

Finally, the literature is wealth on the modified grav-
itational potentials and their effects, and it is worth
noting to highlight some recent as well as important
works [15–22]. On the other hand, the continued frac-
tion to reformulate the two-body problem,

\[ U = \frac{\mu}{||r - R_\oplus||} + R \]  

where \( \frac{\mu}{||r - R_\oplus||} \) is the potential of the unperturbed
problem, \( R_\oplus \) is the equatorial radius of the Earth, \( r \) is
the mutual distance between the bodies and \( \mu = 0.39860 \times 10^5 \text{km}^3/\text{s}^2 \). The \( c \)'s to be adjusted later, and \( R \) represents as usual the perturbation in celestial
mechanics, here it can be interpreted as the terms of
the continued fraction expansion beyond the first one.

Here we can design another form of (2) depending on
the concept of the continued fraction series (1) as

\[ U = \frac{\mu}{||r - R_\oplus||} + \frac{c_1 \mu}{||r - R_\oplus||^2} + \frac{c_2 \mu^2}{||r - R_\oplus||^3} + \ldots \]

3. Problem statement

In this paper, we will apply definition for continued
fraction to the potential function of a perturbed two-
body problem to compute the perturbing effects in the
orbital elements. We will truncate the continued fraction
series at some relevant terms depending on rig-
orous computation steps. The potential function of a
perturbed two-body problem could be written as

\[ U = \frac{\mu}{||r - R_\oplus||} + R \]  

and retaining the first three terms only of (3) we get

\[ U = \left( \frac{\mu}{||r - R_\oplus||} + \frac{c_1 \mu}{||r - R_\oplus||^2} \right) \left( 1 + \frac{\mu (c_2 + c_1)}{||r - R_\oplus||^2} \right)^{-1} \]

which can be written as

\[ U = \frac{\mu}{||r - R_\oplus||} \left( 1 - \frac{\alpha \mu}{||r - R_\oplus||^2} \right) \left( 1 - \frac{\alpha \mu}{||r - R_\oplus||^2} \right)^{-1} \]

Setting \( \alpha = c_1 + c_2 \) and retaining the first two terms
in the binomial expansion yields

\[ U = \frac{\mu}{||r - R_\oplus||} - \frac{c_1 \mu^2}{||r - R_\oplus||^3} - \frac{c_2 \alpha \mu^3}{||r - R_\oplus||^5} \]

Using \( ||r - R_\oplus|| = \left( 1 - 2 \frac{R_\oplus}{r} \cos \theta + \frac{R_\oplus^2}{r^2} \right)^{1/2} \), then

\[ U = \frac{\mu}{1 - 2 \frac{R_\oplus}{r} \cos \theta + \frac{R_\oplus^2}{r^2}} \left( 1 - 2 \frac{R_\oplus}{r} \cos \theta + \frac{R_\oplus^2}{r^2} \right)^{1/2} \]

Equation (6) can be written as

\[ U = \frac{\mu}{1 - 2 \frac{R_\oplus}{r} \cos \theta + \frac{R_\oplus^2}{r^2}} \left( 1 - 2 \frac{R_\oplus}{r} \cos \theta + \frac{R_\oplus^2}{r^2} \right)^{-1/2} \]

\[ - \frac{c_1 \mu^2}{r^3} \left( 1 - 2 \frac{R_\oplus}{r} \cos \theta + \frac{R_\oplus^2}{r^2} \right)^{-3/2} \]

\[ \times \frac{c_2 \alpha \mu^3}{r^5} \left( 1 - 2 \frac{R_\oplus}{r} \cos \theta + \frac{R_\oplus^2}{r^2} \right)^{-5/2} \]
The included Legendre polynomials are given by

\[ \left(1 - 2 \frac{R_B}{r} \cos \theta + \frac{R_B^2}{r^2}\right)^{-1/2} \]

could be expressed in terms of the generating function of Legendre polynomials as

\[
\left(1 - 2 \frac{R_B}{r} \cos \theta + \frac{R_B^2}{r^2}\right)^{-1/2} = \sum_{n=0}^{\infty} \left(\frac{R_B}{r}\right)^n J_n P_n(\cos \theta). \tag{8}
\]

Therefore using (8), the continued fraction potential (7) can be written in the form

\[
U = \frac{\mu}{r} \sum_{n=0}^{\infty} \left(\frac{R_B}{r}\right)^n J_n P_n(\cos \theta)
- \frac{c_1 \mu^2}{r^3} \left(\sum_{n=0}^{\infty} \left(\frac{R_B}{r}\right)^n J_n P_n(\cos \theta)\right)^3
- \frac{c_2 \alpha \mu^3}{r^5} \left(\sum_{n=0}^{\infty} \left(\frac{R_B}{r}\right)^n J_n P_n(\cos \theta)\right)^5 \tag{9}
\]

where \(J_n\) are the oblateness (dynamical shape) parameters; their numerical values are found easily from any geophysical data of the Earth. With \(J_2 \cong 10^{-3}\), we can assume that \(J_2\) as small parameter of the problem and is being of order 1.

The angle \(\theta = \frac{\pi}{2} - \delta\) is the co-declinations, i.e. Equation (9) can be written in the form

\[
U = -\frac{\mu}{r} - \frac{c_1 \mu^2}{r^3} - \frac{c_2 \alpha \mu^3}{r^5}
+ J_2 \left(\frac{\mu}{r} + 3 \frac{c_1 \mu^2}{r^3} + 5 \frac{c_2 \alpha \mu^3}{r^5}\right) \frac{R_B}{r} P_2(\sin \delta) \tag{10}
\]

The included Legendre polynomials are given by \(P_2(\sin \delta) = \frac{1}{2}(3\sin^2 \delta - 1)\). Define the Keplerian orbital elements \((a, e, i, \omega, \Omega, f)\), where \(a\) is the semi-major axis, \(e\) is the eccentricity, \(i\) is the angle between the orbital plane and the fundamental plane (here is the equatorial plane), called the orbital inclination, \(\omega\) is the angle measured on the orbit from the ascending node to the satellite, \(\Omega\) is the angle measured on the fundamental plane from the vernal point of equinox to the ascending node (the point of intersection between the fundamental plane and the orbit plane), and \(f\) is the angle measured on the orbit from the major axis to the position of the satellites, it is called the true anomaly, and it is called the argument of perigee. Adopting the notation \(F_f = if + f\omega\) using \(\sin \delta = \sin i \sin(f + \omega)\), the Legendre polynomials \(P_2(\sin \delta)\) can be written in terms of orbital elements as \(P_2(\sin \delta) = \frac{1}{2}(3\sin^2 \delta - 2) - 3\sin^2 \cos F_{22}\), where \(C = \cos i, S = \sin i\).

Using the above-mentioned notations, the potential (10) can be expressed in terms of orbital elements as

\[
U = -\frac{\mu}{a} \left(\frac{a}{r}\right) - \frac{c_1 \mu^2}{a^3} \left(\frac{a}{r}\right)^3 - \frac{c_2 \alpha \mu^3}{a^5} \left(\frac{a}{r}\right)^5
+ \frac{1}{4} J_2 \left(\frac{R_B}{a}\right)^2 \left(\frac{\mu}{a}\right)^3 + \left(3 \frac{c_1 \mu^2}{a^3}\right) \left(\frac{a}{r}\right)^5
+ \left(5 \frac{c_2 \alpha \mu^3}{a^5}\right) \left(\frac{a}{r}\right)^7 \right\} \left(3\sin^2 \delta - 2\right) - 3\sin^2 \cos F_{22}\right) \tag{11}
\]

Setting \(A_0 = -1, A_1 = -c_1, A_2 = -c_2 \alpha, B_3 = 1, B_1 = 3c_1, B_2 = 5c_2 \alpha\), Equation (11) can be rewritten as

\[
U = \sum_{n=0}^{2} \frac{\mu^{n+1}}{a^{2n+1}} \left[ A_n \left(\frac{a}{r}\right)^{2n+1} + B_n J_2 \left(\frac{R_B}{a}\right)^2 \left(\frac{a}{r}\right)^{2n+3}\right]
\times \left((3\sin^2 \delta - 2) - 3\sin^2 \cos F_{22}\right) \tag{12}
\]

### 4. Lagrange planetary equations

Lagrange was the first to derive the rates of change of the osculating orbital elements in a system of six ordinary differential equations, known as the Lagrange planetary equations. He studied the planetary motion around the Sun and disturbed by a small perturbation due to another gravitational attraction of the planets. See Roy [1], Brouwer and Clemence [25] and Taff [9].

\[
\frac{da}{dt} = 2 \frac{\partial U}{\partial aM}, \tag{13}
\]

\[
\frac{de}{dt} = \frac{1 - e^2}{na^2} \frac{\partial U}{\partial aM} - \frac{\sqrt{1 - e^2}}{na^2} \frac{\partial U}{\partial a}, \tag{14}
\]

\[
\frac{di}{dt} = \cot i \frac{\partial U}{\partial aM} - \frac{1}{na^2} \frac{\partial U}{\partial a} \tag{15}
\]

\[
\frac{d\omega}{dt} = \frac{\sqrt{1 - e^2}}{na^2} \frac{\partial U}{\partial a} - \frac{\cot i}{na^2} \frac{\partial U}{\partial a} \tag{16}
\]

\[
\frac{d\Omega}{dt} = \frac{1}{na^2} \frac{\partial U}{\partial a} \frac{\partial U}{\partial a} \tag{17}
\]

\[
\frac{dM}{dt} = n - 2 \frac{\partial U}{\partial aM} \frac{\partial U}{\partial a} \frac{\partial U}{\partial a} \frac{\partial U}{\partial a} \frac{\partial U}{\partial a} \tag{18}
\]

where \(M\) is the mean anomaly of the satellite, it is the angular distance from the perigee of a fictitious body moving in a circular orbit, and \(n\) is the mean motion of the satellite which is the angular speed required by the satellite to complete one orbit.

### 5. The required partial derivatives

In what follows, we will compute the partial derivatives required to solve the system (13)–(18). Now consider
the potential function given by (12) we obtain the following non-vanishing results:

\[
\frac{\partial U}{\partial a} = \sum_{n=0}^{\infty} \frac{\mu^{n+1}}{a^{2n+1}} \left( 2n+1 \right) A_n \left( \frac{a}{r} \right)^{2n+1} \sin \theta \\
- \left( 2n+3 \right) \frac{B_n}{4} J_2 \left( \frac{R_\oplus}{a} \right)^2 \left( \frac{a}{r} \right)^{2n+3} \\
\times \left( 3S^2 - 2 \right) \sin F_{1.0} - 3S^2 \sin F_{2.2} \right] \tag{19}
\]

\[
\frac{\partial U}{\partial e} = \sum_{n=0}^{\infty} \frac{\mu^{n+1}}{a^{2n+1}} \left( 2n+1 \right) A_n \left( \frac{a}{r} \right)^{2n+1} \cos f \\
+ \left( 2n+3 \right) \frac{B_n}{4} J_2 \left( \frac{R_\oplus}{a} \right)^2 \left( \frac{a}{r} \right)^{2n+3} \\
\times \left( 3S^2 - 2 \right) \sin F_{1.0} - 3S^2 \sin F_{2.2} \right] \tag{20}
\]

\[
\frac{\partial U}{\partial \iota} = \sum_{n=0}^{\infty} \frac{\mu^{n+1}}{a^{2n+1}} \left( 2n+1 \right) A_n \left( \frac{a}{r} \right)^{2n+1} \sin 2\iota \left( 1 - \cos F_{2.2} \right) \tag{21}
\]

\[
\frac{\partial U}{\partial \omega} = \sum_{n=0}^{\infty} \frac{\mu^{n+1}}{a^{2n+1}} \left( 2n+1 \right) A_n \left( \frac{a}{r} \right)^{2n+1} \sin F_{2.2} \tag{22}
\]

\[
\frac{\partial U}{\partial M} = \sum_{n=0}^{\infty} \frac{\mu^{n+1}}{a^{2n+1}} \left( 2n+1 \right) A_n \left( \frac{a}{r} \right)^{2n+1} \sin F_{1.0} \\
- \left( 2n+3 \right) \frac{B_n}{4} J_2 \left( \frac{R_\oplus}{a} \right)^2 \left( \frac{a}{r} \right)^{2n+3} \\
\times \left( 3S^2 - 2 \right) \sin F_{1.0} - 3S^2 \sin F_{2.2} \right] \tag{23}
\]

6. Construction of the equation of motion

Using the relation \( \frac{df}{dt} = \frac{\sqrt{\mu a(1-e^2)}}{r} \) to change the dependence from the explicit time to the true anomaly \( f \), then \( \frac{df}{dM} = \frac{df}{d\sigma} \frac{d\sigma}{dM} \), \( \sigma : a, e, i, \omega, M \). Also substituting the partial derivatives given by (19)–(23) into the Lagrange planetary equations (13)–(18) yields

\[
\frac{\partial a}{\partial t} = \frac{\mu}{2 \sqrt{\mu a(1-e^2)}} \left( \frac{R_\oplus}{a} \right)^2 \frac{2 (2n+1) A_n \left( \frac{a}{r} \right)^{2n+1}}{\sum_{n=0}^{\infty} \frac{\mu^{n+1}}{a^{2n+1}}} \\
\times \left( \left( 2n+3 \right) \frac{B_n}{4} J_2 \left( \frac{R_\oplus}{a} \right)^2 \left( \frac{a}{r} \right)^{2n+3} \\
\times \left( 3S^2 - 2 \right) \sin F_{1.0} - 3S^2 \sin F_{2.2} \right] \tag{24}
\]

\[
\frac{\partial e}{\partial t} = \frac{\mu}{2 \sqrt{\mu a(1-e^2)}} \left( \frac{R_\oplus}{a} \right)^2 \frac{2 (2n+1) A_n \left( \frac{a}{r} \right)^{2n+1}}{\sum_{n=0}^{\infty} \frac{\mu^{n+1}}{a^{2n+1}}} \\
\times \left( \left( 2n+3 \right) \frac{B_n}{4} J_2 \left( \frac{R_\oplus}{a} \right)^2 \left( \frac{a}{r} \right)^{2n+3} \\
\times \left( 3S^2 - 2 \right) \sin F_{1.0} - 3S^2 \sin F_{2.2} \right] \tag{25}
\]

\[
\frac{\partial \iota}{\partial t} = \frac{\mu}{2 \sqrt{\mu a(1-e^2)}} \left( \frac{R_\oplus}{a} \right)^2 \frac{2 (2n+1) A_n \left( \frac{a}{r} \right)^{2n+1}}{\sum_{n=0}^{\infty} \frac{\mu^{n+1}}{a^{2n+1}}} \\
\times \left( \left( 2n+3 \right) \frac{B_n}{4} J_2 \left( \frac{R_\oplus}{a} \right)^2 \left( \frac{a}{r} \right)^{2n+3} \\
\times \left( 3S^2 - 2 \right) \sin F_{1.0} - 3S^2 \sin F_{2.2} \right] \tag{26}
\]

\[
\frac{\partial \omega}{\partial t} = \frac{\mu}{2 \sqrt{\mu a(1-e^2)}} \left( \frac{R_\oplus}{a} \right)^2 \frac{2 (2n+1) A_n \left( \frac{a}{r} \right)^{2n+1}}{\sum_{n=0}^{\infty} \frac{\mu^{n+1}}{a^{2n+1}}} \\
\times \left( \left( 2n+3 \right) \frac{B_n}{4} J_2 \left( \frac{R_\oplus}{a} \right)^2 \left( \frac{a}{r} \right)^{2n+3} \\
\times \left( 3S^2 - 2 \right) \sin F_{1.0} - 3S^2 \sin F_{2.2} \right] \tag{27}
\]
\[
\frac{\partial \Omega}{\partial f} = \frac{1}{n a^2 \sqrt{\mu a(1 - e^2)}} \sin i \sum_{n=0}^{2} \mu_n^{n+1} 3 B_n 4 \sin 2i(1 - \cos F_{2,2})
\]

(28)

\[
\frac{\partial M}{\partial f} = \frac{1}{n a^2 \sqrt{\mu a(1 - e^2)}} \sin i \sum_{n=0}^{2} \mu_n^{n+1} \frac{\partial}{\partial f} \left[ \left( \frac{a}{r} \right)^{2n+3} \sin 2i(1 - \cos F_{2,2}) \right]
\]

(29)

7. Integrals and averages

To evaluate the integrals of these quantities

\[
\left( \frac{a}{r} \right)^i \cos F_{n,m} = \left( \frac{a}{r} \right)^i \left[ \cos n f \cos m \omega + \sin n f \sin m \omega \right],
\]

(30)

\[
\left( \frac{a}{r} \right)^i \sin F_{n,m} = \left( \frac{a}{r} \right)^i \left[ \sin n f \cos m \omega + \cos n f \sin m \omega \right],
\]

(31)

with respect to \( i \), therefore integrals of the form \( (a/r)^i \cos nf \) and \( (a/r)^i \sin nf \) follows directly.

7.1. For \( i \geq 2 \)

Let

\[
P_{i,n}^f = \int \left( \frac{a}{r} \right)^i \cos nf \; dl, \quad P_{i,n}^\psi = \int \left( \frac{a}{r} \right)^i \sin nf \; dl,
\]

(32)

where \( i, n \) are integers and \( i \geq 2 \). Using the relations, see Ahmed [26]

\[
\eta_{-i} = \left( \frac{1}{\sqrt{1 - e^2}} \right)^i, \quad \eta_{-1,1} = \frac{1}{\sqrt{1 - e^2}},
\]

\[
\Psi = \eta_{-2} f(1 + e \cos f), \quad \Psi = \left( \frac{a}{r} \right)^i,
\]

and

\[
dl = \eta_{-1,1} \Psi \; df.
\]

we can write (32) in the form

\[
P_{i,n}^f = \int \eta_{-1,1} \Psi^{-i} \cos nf \; df,
\]

(33)

The integrals in (33) can be written as

\[
P_{i,n}^f = \eta_{-1,1} \Psi^{-i} \cos nf \; df
\]

or

\[
P_{i,n}^\psi = \eta_{-1,1} \Psi^{-i} \sin nf \; df
\]

(34)

It is clear that for \( i < 2 \), the above expression will not lead to closed expressions, we have

\[
\cos k f = 2^{1-k} \sum_{r=0}^{N} \left( 1 - \frac{1}{2} \delta_{k}^r \right) C_k^r \cos(k - 2r)f,
\]

\[
N = \begin{cases} k/2, & k \text{ even} \\ (k - 1)/2, & k \text{ odd} \end{cases}
\]

(35)

Then substituting equation (35) into (34) gives

\[
P_{i,n}^f = \eta_{-1,1} \Psi^{-i} \cos nf \; df
\]

(36)

\[
P_{i,n}^\psi = \eta_{-1,1} \Psi^{-i} \sin nf \; df
\]

(37)

7.2. Averages of \( \Psi^i \cos nf \) and \( \Psi^i \sin nf \)

Let

\[
f_{i,n}^f = \frac{1}{2\pi} \int_0^{2\pi} \Psi^i \cos nf \; dl
\]

(38)

\[
f_{i,n}^\psi = \frac{1}{2\pi} \int_0^{2\pi} \Psi^i \sin nf \; dl.
\]

(39)

The averages (38) and (39) will be evaluated for different \( i \) and \( n \).
7.3. For $i = 0$

The average values may be evaluated using the formula \[27,28]:
\[
\rho_{i,0} = \left( \frac{e^{1/1} - 1}{1 - \eta_{1,-1}} \right)^{-n}(1 + n\eta_{1,-1}), \quad \rho_{i,0} = 0. \quad (40)
\]

The following special cases are of particular interest
\[
\rho_{i,0} = 1
\]
\[
\rho_{i,1} = -e
\]
\[
\rho_{i,2} = \left[ 1 - \eta_{1,-2} + 2\eta_{1,-3} \right] e^{-2}
\]
\[
\rho_{i,3} = \left[ 1 - 2\eta_{1,-2} - 3\eta_{1,-3} + 3\eta_{1,-4} \right] e^{-3}
\]
\[
\rho_{i,4} = \left[ 1 - 3\eta_{1,-2} + 2\eta_{1,-3} - 6\eta_{1,-4} \right] e^{-4}
\]

7.4. For $i = 1$

The averages may be evaluated using the relation
\[
\rho_{i,1} = \eta_{1,-1} \left[ \rho_{i,0} + \frac{e}{2} \rho_{i,0} + \frac{e}{2} \rho_{i,1} \right] \quad (41)
\]
where the $\rho_{i,0}$ are defined by Equation (40).
\[
\rho_{i,0} = \eta_{1,-2}
\]
\[
\rho_{i,1} = \eta_{1,-2} \left[ \rho_{i,0} + \frac{e}{2} \rho_{i,0} + \frac{e}{2} \rho_{i,1} \right]
\]
\[
\rho_{i,2} = \eta_{1,-2} \left[ \frac{e}{2} + \frac{e}{2} (1 - 3\eta_{1,-2} + 2\eta_{1,-3}) \right]
\]
\[
\rho_{i,3} = \eta_{1,-2} \left[ \frac{e}{2} (1 - 12\eta_{1,-2} + 12\eta_{1,-3} - 3\eta_{1,-4}) \right]
\]
\[
\rho_{i,4} = \eta_{1,-2} \left[ \frac{e}{2} (1 - 2\eta_{1,-2} + 6\eta_{1,-3} - 3\eta_{1,-4}) \right]
\]
\[
\rho_{i,5} = \eta_{1,-2} \left[ \frac{e}{2} (1 - 8\eta_{1,-2} + 2\eta_{1,-3}) \right]
\]

7.5. For $i \geq 2$

A direct consequence of Equations (38) and (39) is that
\[
\langle \Psi_i \sin \eta \rangle = 0, \quad \forall i, n \quad (42)
\]
\[
\langle \Psi_i \cos \eta \rangle = 0, \quad \forall n > i - 2 \quad (43)
\]

7.6. For $n \leq i - 2$

The required averages are obtained from Equations (38) and (39).

The following special cases are of particular interest
\[
\rho_{i,0} = \eta_{i-1,-1}, \quad \rho_{i,0} = \eta_{i-1,-3}, \quad \rho_{i,1} = \frac{1}{2} e^{i-3,3},
\]
\[
\rho_{i,0} = \eta_{i-1,-5}, \quad \rho_{i,0} = \eta_{i-1,-7}, \quad \rho_{i,0} = \eta_{i-1,-9}, \quad \rho_{i,0} = \eta_{i-1,-11}, \quad (r)_f = a \sqrt{1 - e^2}, \quad (r^2)_f = a^2 \sqrt{1 - e^2}.
\]

8. Perturbations in the orbital elements

After some lengthy computations and using the obtained averages in the previous section and its sections, we can obtain the following final expressions for the perturbations in the orbital elements

8.1. Perturbation in the semi-major axis

The semi-major axis suffers no perturbations in classical unperturbed potential, while when considering the
continued fraction potential, we obtained a nonzero averaged perturbation.

\[ \Delta \alpha_f = \frac{J_2 R_0^2 e}{2n(1 - e^2)a\sqrt{\mu a}} \left\{ \frac{e}{\sqrt{1 - e^2}} \left[ -3S^2 \sin 2\omega + \left( 5B_1 \frac{\mu}{a^2} e\eta_{-5,5} + 7B_2 \frac{\mu^3}{a^3} e^3 \eta_{-7,9} \left( 2e + \frac{3}{2} e^3 \right) \right) \right] \right\} \]

\[ + 6\sqrt{1 - e^2} S^2 \left( B_1 \frac{\mu}{a^2} e^2 \eta_{-11,11} (2 + e^2) \right) \sin 2\omega \] (44)

8.2. Perturbation in the eccentricity

The eccentricity suffers no perturbations in classical unperturbed potential, while when considering the continued fraction potential, we obtained a nonzero averaged perturbation.

\[ \Delta e_f = \frac{J_2 R_0^2 e}{4nae\sqrt{\mu a}} \left\{ 3S^2 e \left[ \frac{5B_1 \mu}{a^4} e\eta_{-5,5} + \frac{21B_2 \mu^3}{2a^6} e^3 \eta_{-7,9} (4 + e^2) + 7B_2 \mu^3}{3a^6} e^3 \eta_{-9,9} \right] \right\} \]

\[ + 6(1 - e^2) S^2 \left( \frac{3B_1 \mu}{a^4} e^2 \eta_{-11,7,7} e^2 \eta_{-7,7} + \frac{5B_2 \mu^3}{a^6} \right) \]

\[ \times e^2 \eta_{-11,11} (2 + e^2) \right\} \sin 2\omega \] (45)

8.3. Perturbation in the inclination

The inclination suffers no perturbations in classical unperturbed potential, while when considering the continued fraction potential, we obtained a nonzero averaged perturbation.

\[ \Delta I_f = \frac{J_2}{2nae^2} \left\{ \frac{9B_2 R_0^2 \mu}{8na^2 \sqrt{\mu a}} \left( \frac{e^2}{1 - e^2} \eta_{-7,7} \sin 2\omega \right) \right\} \] (46)

8.4. Perturbation in the argument of perigee

The argument of perigee suffers a definite periodic perturbation in the case of the classical unperturbed potential, but when considering the continued fraction potential, we obtained extra nonzero averaged perturbating terms.

\[ \Delta \omega_f = -\frac{1}{ne\sqrt{\mu a}} A_0 \frac{\mu}{a} e - 5 \frac{1}{ne\sqrt{\mu a}} A_2 \frac{\mu^3}{a^3} e\eta_{-5,5} \]

\[ + \frac{1}{ne\sqrt{\mu a}} B_0 \left( \frac{e}{2} - \frac{6S^2}{4e} \left( \eta_{-1,1} - \eta_{-2,2} \right) \right) \]

\[ - \frac{1}{2e^3} (23 - e^2) - \frac{9}{2e^3} \eta_{-2,2} - \frac{3}{e^3} \eta_{-3,3} \]

\[ + \frac{1}{2e^3} (-1 + 15\eta_{-2,2} - 40\eta_{-3,3} + 45\eta_{-4,4} - 24\eta_{-5,5} + 5\eta_{-6,6}) + \frac{1}{4} B_1 \frac{\mu^2}{a^3} e\eta_{-5,5} \]

\[ \times (10 - 3S^2 + 2\eta_{-3,3}) + B_2 \frac{\mu^3}{a^3} \left( \frac{1}{2} e\eta_{-7,7} \right) \]

\[ + \frac{3}{4} e^3 \eta_{-7,7} + 7e\eta_{-9,9} - \frac{21}{4} e^3 \eta_{-9,9} \]

\[ - \frac{1}{42} e^3 \eta_{-18,18} + \frac{9}{4} e^3 \eta_{-7,7} \]

\[ - \frac{9}{16} e^3 S^2 \eta_{-9,9} + \frac{1}{8} e^3 \eta_{-7,7} \right\} \cos 2\omega \]

\[ - \frac{6S^2}{4} \left( B_0 \frac{\mu}{a} \left[ \eta_{-2,2} - \left( 1 + e^2 \right) \eta_{-2,2} \right] \right) \]

\[ - \frac{2}{e^3} \eta_{-1,1} + \frac{3e^2}{2} \cos 2\omega \] (47)

8.5. Perturbation in the longitude of the ascending node

The longitude of the ascending node suffers a definite periodic perturbation in the case of the classical unperturbed potential, but when considering the continued fraction potential, we obtained extra nonzero averaged perturbing terms.

\[ \Delta \Omega_f = \frac{J_2}{2na^3 \sqrt{\mu a}} \left\{ 3B_0 \mu \frac{R_0^2}{a} C \right\} \]

\[ + \frac{3R_0^2}{2na^2 \sqrt{\mu a}} \left( 1 - e^2 \right) J_2 \]

\[ \times \left\{ \frac{\mu^2}{a^3} B_1 \eta_{-3,3} + \frac{\mu^3}{a^5} B_2 \eta_{-7,7} \left( 1 + \frac{3}{2} e^2 \right) \right\} \]

\[ - \frac{3\mu^3}{4a^5} B_2 e^2 \eta_{-7,7} \cos 2\omega \] (48)

8.6. Perturbation in the mean anomaly

The mean anomaly suffers a definite secular perturbation in the case of the classical unperturbed potential, but when considering the continued fraction potential,
we obtained extra nonzero averaged perturbing terms.

\[
\Delta M_f = \frac{na^2}{\sqrt{\mu a}} - \frac{2}{n} \frac{1}{\sqrt{\mu a(1 - e^2)}} \left[-A_0 \frac{\mu}{a} \sqrt{1 - e^2} \right.

- 3A_1 \frac{\mu^3}{a^3} \eta_{-2,2} - 5A_0 \frac{\mu^3}{a^3} \eta_{-3,3} - \frac{J_2 R_{\oplus}^2}{4a} \n
\times \left\{ \left( 3S^2 - 2 \right) \left( B_0 \frac{\mu}{a} \eta_{-2,2} + B_1 \frac{\mu^2}{a^3} \eta_{-3,3} \right.ight.

+ B_2 \frac{\mu^3}{a^5} \left( 1 + \frac{3}{2} e^2 \right) \eta_{-7,7} \n
- 3S^2 \left( B_0 \frac{\mu}{a} \eta_{-2,2} \left( \frac{1}{2a^2} \left( 1 - 12 \eta_{-2,2} + 12 \eta_{-3,3} \right. \right. \n
- 3\eta_{\omega,4} - e^2 \right) + \left. 3 \frac{B_3 \mu^3}{a^3} e^2 \eta_{-7,7} \right) \cos 2 \omega \left. \right\} \n
- \frac{1}{n} e^2 \frac{1}{\sqrt{\mu a(1 - e^2)}} \n
\times \left[ -e \left( A_0 \frac{\mu}{a} + 5A_2 \frac{\mu^3}{a^3} \eta_{-5,5} + \frac{J_2}{4} \left( \frac{R_{\oplus}}{a} \right)^2 \right) \n
\times \left( 3S^2 - 2 \right) e \left( -5B_1 \frac{\mu^2}{a^3} \eta_{-5,5} \right. \n
+ 7B_2 \frac{\mu^3}{a^5} \left( 2 + \frac{3}{2} e^2 \right) \eta_{-9,9} \n
- 3S^2 e \left( -5B_1 \frac{\mu^2}{a^3} \eta_{-5,5} \right. \n
+ 7B_2 \frac{\mu^3}{a^5} \left( 2 + \frac{4}{2} e^2 \right) \n
\times \eta_{-9,9} \cos 2 \omega \right. \n
- \frac{6}{4} J_2 \left( \frac{R_{\oplus}}{a} \right)^2 \left( \frac{S^2}{1 - e^2} \right) \n
\times \left\{ -\frac{1}{2e^3} B_0 \frac{\mu}{a} \eta_{-2,2} \left( e^2 - 1 + 2\eta_{\omega} \right. \n
+ 4\eta_{-3,3} - 9\eta_{\omega,4} + 4\eta_{-5,5} \right) + \frac{1}{2} B_1 \n
\times \frac{\mu^2}{a^3} \eta_{-3,3} + 3 \frac{B_2 \mu^3}{a^5} e \left( 3 + 1 e^2 \right) \eta_{-7,7} \n
+ \frac{e}{2} \left( B_0 \frac{\mu}{a} \eta_{-2,2} \left( 1 - \frac{1}{2e^4} \left( 1 - 5 \eta_{-2,2} \right. \n
+ 15\eta_{\omega,4} - 16\eta_{-3,3} + 5\eta_{6,6} - 6) \n
+ e^2 \left( -1 + 6\eta_{\omega,2} - 8\eta_{-3,3} + 3\eta_{\omega,4} - 4) \right) \right) \n
+ B_1 \frac{\mu^2}{a^3} \eta_{-3,3} + B_2 \frac{\mu^3}{a^5} \left( 1 + \frac{3}{2} e^2 \right) \eta_{-7,7} \left. \right\} \n
\times \cos 2 \omega \right] \quad (49)
\]

9. Conclusion

We can point out the original contribution of this idea as follows. The geopotential is suited to accept the continued fraction procedure. Since the continued fractions continues infinite time of fractions schemes, the series is truncated beyond the third-order term as well as the first oblateness term of the potential Lagrange planetary equations is modified with new terms arising from the adopted continued fractions scheme. The equations of motion of a satellite move in a continued fraction potential field are derived. To solve these equations, the required integrals and averages are outlined, and different cases are highlighted using Ahmed [26]. The perturbations in all orbital elements due to considering the model of continued fractions are computed. The new results to highlight are that the semi-major axis, eccentricity, inclination are suffering perturbations when considering the continued fraction potential in contrast to classical oblate potential. The argument of perigea also suffers a definite perturbation in the case of the classical unperturbed potential, but when considering the continued fraction potential, extra nonzero averaged perturbing terms is obtained. All these findings are new in comparison with the last contribution to the authors [16]. In that article, the authors have treated the problem of two bodies generally and no perturbations have been applied; it checked only some major concepts of the dynamics of unperturbed two-body problem, mainly, it has reviewed the integrals of motion, e.g., the conservation of mechanical energy, angular momentum, the eccentricity vector.

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