Effective multi-quark interactions with explicit breaking of chiral symmetry

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In a long distance Lagrangian approach to the low lying meson phenomenology we present and discuss the most general spin zero multi-quark interaction vertices of non-derivative type which include a set of effective interactions proportional to the current quark masses, breaking explicitly the chiral $SU(3)_L \times SU(3)_R$ and $U_A(1)$ symmetries. These vertices are of the same order in $N_c$ counting as the ‘t Hooft flavor determinant interaction and the eight quark interactions which extend the original leading in $N_c$ four quark interaction Lagrangian of Nambu and Jona-Lasinio. The $N_c$ assignments match the counting rules based on arguments set by the scale of spontaneous chiral symmetry breaking. With path integral bosonization techniques which take appropriately into account the quark mass differences we derive the mesonic Lagrangian up to three-point mesonic vertices. We demonstrate that explicit symmetry breaking effects in interactions are essential to obtain the correct empirical ordering and magnitude of the splitting of certain states such as $m_K < m_{\eta}$ for the pseudoscalars and $m_{\kappa_0} < m_{\eta_0} \sim m_{f_0}$ in the scalar sector, and achieve total agreement with the empirical low lying meson mass spectra. With all parameters of the model fixed by the spectra we analyze further a bulk of two body decays at tree level of the bosonic Lagrangian: the strong decays of the scalars $\sigma \rightarrow \pi \pi$, $f_0(980) \rightarrow \pi \pi$, $\kappa(800) \rightarrow \pi K$, $a_0(980) \rightarrow \pi \eta$, as well as the two photon decays of $a_0(980)$, $f_0(980)$ and $\sigma$ mesons and the anomalous decays of the pseudoscalars $\pi \rightarrow \gamma \gamma$, $\eta \rightarrow \gamma \gamma$ and $\eta' \rightarrow \gamma \gamma$. Our results for the strong decays are within the current expectations and the pseudoscalar radiative decays are in very good agreement with data. The radiative decays of the scalars are smaller than the observed values for the $f_0(980)$ and the $\sigma$, but reasonable for the $a_0$. A detailed discussion accompanies all the results.

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I. INTRODUCTION

A long history of applying the Nambu – Jona-Lasinio (NJL) model in hadron physics shows the importance of the concept of effective multi-quark interactions for modelling QCD at low energies. Originally formulated in terms of the $\gamma_5$ gauge invariant nonlinear four-fermion coupling [1, 2], the model has been extended to the realistic three flavor and color case with $U(1)_A$ breaking six-quark ‘t Hooft interactions [3, 17] and an appropriate set of eight-quark interactions [18]. The last ones complete the number of vertices which are important in four dimensions for dynamical $SU(3)_L \times SU(3)_R$ chiral symmetry breaking [19, 20].

The explicit breaking of chiral symmetry in the NJL model is described by introducing the standard light quark mass term of the QCD Lagrangian (light means consisting of $u, d$ and $s$ quarks), e.g. [21, 22]. The current quark mass dependence is of importance for several reasons, in particular for the phenomenological description of meson spectra and meson-meson interactions, and for the critical point search in hot and dense hadronic matter, where it has a strong impact on the phase diagram [23]. The values of the current quark masses are determined in the Higgs sector of the Standard Model. In this respect they are foreign to QCD and, at an effective description, can be included through the external sources, interacting with the originally massless quark fields. This is why the explicit chiral symmetry breaking (ChSB) by the standard mass term of the free Lagrangian is only a part of the more complicated picture arising in effective models beyond leading order [24]. Chiral perturbation theory [25, 26] gives a well-known example of a self consistent accounting of the mass terms, order by order, in an expansion in the masses themselves. In fact, extended NJL-type models should not be an exception from this rule. If one considers multi-quark effective vertices, to the extent that $1/N_c$ suppressed ‘t Hooft and eight-quark terms are included in the Lagrangian, certain mass dependent multi-quark interactions must be also taken into account.

The aim of the present work is precisely to analyze these higher order terms in the quark mass expansion. Our consideration proceeds along the following steps. We start from the three-flavor NJL-type model with self-interacting massless quarks. The $SU(3)_L \times SU(3)_R$ chiral symmetry of the Lagrangian is known to be dynamically broken to its $SU(3)_V$ subgroup at some scale $\Lambda$, with $\Lambda$ being one of the model parameters. There is also explicit symmetry breaking due to the bare quark masses $\chi_i$, which are taken to transform as $\chi = (3, 3^*)$ under $SU(3)_L \times SU(3)_R$. Since the Lagrangian contains, in general, an unlimited number of non-renormalizable multi-quark and $\chi$-quark interactions (scaled by some powers of $\Lambda$), we formulate the power counting rules to classify these vertices in accordance with their importance for dynamical symmetry breaking. Then we bosonize the theory by using the path-integral method. The functional

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integrals are calculated in the stationary phase approximation and by using the heat kernel technique. As a result one obtains the low-energy meson Lagrangian. At last we fix the parameters of the model by confronting it to the experimental data. In particular, we show the ability of the model to describe the spectrum of the pseudo Goldstone bosons, including the fine tuning of the \(\eta - \eta'\) splitting, and the spectrum of the light scalar mesons: \(\sigma\) or \(f_0(500)\), \(\kappa(800)\), \(f_0(980)\), and \(a_0(980)\).

The coupling constants of multi-quark vertices, fixed from mass-spectra, enter the expressions for meson decay amplitudes and lead to a bulk of model predictions. It is interesting to note that certain multi-quark vertices of the model encode implicitly in the couplings of the tree level bosonized Lagrangian the signature of \(q\bar{q}\) and more complex quark structures which are elsewhere obtained by considering explicitly meson loop corrections, tetraquark configurations and so on [42–46]. It seems appropriate, therefore, to examine the possible physics opportunities connected with the discovery and study of such multi-quark structures in hadrons. For instance, by calculating the mass spectra and the strong decays of the scalars, one can realize which multi-quark interactions are most relevant at the scale of spontaneous ChSB. On the other hand, by analyzing the two photon radiative decays, where a different scale, associated with the electromagnetic interaction, comes into play, one can study the possible recombinations of quarks inside the hadron. We will show, for example, that the \(a_0(980)\) meson couples with a large strength of the multi-quark components to the two kaon channel in its strong decay to two pions, but evidences a dominant \(q\bar{q}\) component in its radiative decay. As opposed to this, the \(\sigma\) and \(f_0(980)\) mesons do not display an enhanced \(q\bar{q}\) component neither in their two photon decays nor in the strong decays.

There are several direct motivations for this work. In the first place, the quark masses are the only parameters of the QCD Lagrangian which are responsible for the explicit ChSB, and it is important for the effective theory to trace this dependence in full detail. In this paper it explicit ChSB, and it is important for the effective theory of the QCD Lagrangian which are responsible for the ex-

The paper is organized as follows. In section II the effective Lagrangian in terms of quark degrees of freedom and bosonic sources with specific quantum numbers is derived using a classification scheme which selects all possible non-derivative vertices according to the symmetries of the strong interaction and which are relevant at the scale \(\Lambda\) of spontaneous chiral symmetry breaking. It is then shown that this scheme can be equally organized in terms of the large \(N_c\) counting rules, which in turn allow to attribute to the couplings of the interactions encoded signatures of \(q\bar{q}\) and more complex structures involving four fermions. We obtain in this section also that a set of interactions lead to the Lagrangian specific Kaplan-Manohar ambiguity associated with the current quark masses.

In section III we proceed to bosonize the multi-quark Lagrangian in two steps. First, we introduce in section III-A a set of auxiliary scalar fields. By these new variables the multi-quark interactions can be brought to the Yukawa form that is quadratic in Fermi fields. Consequently one obtains a Gauss-type integral over quarks, and a set of integrals over auxiliary fields. The latter are evaluated by the stationary phase method. We obtain here the vertices up to the cubic power in the meson fields, needed for the study of the meson spectra and of the two-body decays. Then, in section III-B, we integrate over quark fields. The arising quark determinant of the Dirac operator is a complicated non-local functional of the collective meson fields. We calculate it in the low-energy regime by using the Schwinger-DeWitt technique, based on the heat kernel expansion. In this approximation one can adequately incorporate the effect of different quark masses contained in the modulus of the one-loop quark determinant. We derive the kinetic terms of the collective meson fields, as well as the heat kernel part of contributions to meson masses and interactions. In the end of this section we present the complete bosonized Lagrangian, give the mixing angle conventions used, and the expressions for the strong decay widths. In section III-C we obtain the expressions for the radiative widths of the pseudoscalars and scalars.

In section IV we present the numerical results and discussion, in IV-A for the meson mass spectra and weak decay constants, in IV-B for the strong decays and in IV-C for the radiative decays.

We conclude in section V with a summary of the main results.
II. EFFECTIVE MULTI-QUARK INTERACTIONS

The chiral quark Lagrangian has predictive power for the energy range which is of order $\Lambda \simeq 4\pi f_N \sim 1$ GeV \cite{17}. $\Lambda$ characterizes the spontaneous chiral symmetry breaking scale. Consequently, the effective multi- quark interactions, responsible for this dynamical effect, are suppressed by $\Lambda$, which provides a natural expansion parameter in the chiral effective Lagrangian. The scale above which these interactions disappear and QCD becomes perturbative enters the NJL model as an ultraviolet cut-off for the quark loops. Thus, to build the NJL type Lagrangian we have only three elements: the quark fields $q$, the scale $\Lambda$, and the external sources $\chi$, which generate explicit symmetry breaking effects – resulting in mass terms and mass-dependent interactions.

The color quark fields possess definite transformation properties with respect to the chiral flavor $U(3)_L \times U(3)_R$ global symmetry of the QCD Lagrangian with three massless quarks (in the large $N_c$ limit). It is convenient to introduce the $U(3)$ Lie-algebra valued field $\Sigma = \frac{1}{2}(s_a - ip_a)\lambda_a$, where $s_a = \bar{q}\lambda_a q$, $p_a = \bar{q}\lambda_a i\gamma_5 q$, and $a = 0,1,\ldots,8$, $\lambda_0 = \sqrt{2/3} \times 1$, $\lambda_a$ being the standard $SU(3)$ Gell-Mann matrices for $1 \leq a \leq 8$. Under chiral transformations: $q^a = V_R q_R + V_L q_L$, where $q_R = R e^{ip_a} q_L = P_L q$, and $P_{R,L} = \frac{1}{2} (1 \pm \gamma_5)$. Hence, $\Sigma' = V_R \Sigma V_L^\dagger$, and $\Sigma'' = V_L \Sigma V_R^\dagger$. The transformation property of the source is supposed to be $\chi' = V_R \chi V_L^\dagger$

Any term of the effective multi-quark Lagrangian without derivatives can be written as a certain combination of fields which is invariant under chiral $SU(3)_R \times SU(3)_L$ transformations and conserves $C, P$ and $T$ discrete symmetries. These terms have the general form

$$L_i \sim \bar{g}_i \Sigma^\dagger \chi^\dagger \Sigma \chi,$$

(1)

where $\bar{g}_i$ are dimensionless coupling constants (starting from eq. (21) the dimensional couplings $g_i = \bar{g}_i / \Lambda^\gamma$ will be also considered). Using dimensional arguments we find (in four dimensions) $\alpha + 3\beta - \gamma = 4$, with integer values for $\alpha, \beta$ and $\gamma$.

We obtain a second restriction by considering only the vertices which make essential contributions to the gap equations in the regime of dynamical chiral symmetry breaking, i.e. we collect only the terms whose contributions to the effective potential survive at $\Lambda \rightarrow \infty$. We get this information by contracting quark lines in $L_i$, finding that this term contributes to the power counting of $\Lambda$ in the effective potential as $\sim \Lambda^{2\beta - \gamma}$, i.e. we obtain that $2\beta - \gamma \geq 0$ (we used the fact that in four dimensions each quark loop contributes as $\Lambda^2$).

Combining both restrictions we come to the conclusion that only vertices with

$$\alpha + \beta \leq 4$$

(2)

must be taken into account in the approximation considered. On the basis of this inequality one can conclude that (i) there are only four classes of vertices which contribute at $\alpha = 0$; those are four, six and eight-quark interactions, corresponding to $\beta = 2, 3$ and 4 respectively; the $\beta = 1$ class is forbidden by chiral symmetry requirements; (ii) there are only six classes of vertices depending on external sources $\chi$, they are: $\alpha = 1, \beta = 1, 2, 3$; $\alpha = 2, \beta = 1, 2$; and $\alpha = 3, \beta = 1$.

Let us consider now the structure of multi-quark vertices in detail \cite{33}. The Lagrangian corresponding to the case (i) is well known

$$L_{\text{int}} = \frac{G}{\Lambda^2} \text{tr} \left( \Sigma^\dagger \Sigma \right) + \frac{\bar{g}}{\Lambda^3} \left( \text{det} \Sigma + \text{det} \Sigma^\dagger \right)$$

$$+ \frac{g_1}{\Lambda^4} \left( \text{tr} \Sigma^\dagger \Sigma \right)^2 + \frac{g_2}{\Lambda^5} \left( \text{tr} \Sigma^\dagger \Sigma^\dagger \Sigma \right) . \quad (3)$$

It contains four dimensionful couplings $G, \bar{g}, g_1, g_2$.

The second group (ii) contains eleven terms

$$L_\chi = \sum_{i=0}^{10} L_i,$$

(4)

where

$$L_0 = - \text{tr} \left( \Sigma^\dagger \chi + \chi^\dagger \Sigma \right)$$

$$L_1 = - \frac{\bar{g}_1}{\Lambda} \epsilon_{ijk} e_{mnlt} \Sigma_{im} \chi_{jn} \chi_{kl} + h.c.$$  

$$L_2 = \frac{g_2}{\Lambda^3} e_{ijk} e_{mnlt} \chi_{im} \Sigma_{jn} \Sigma_{kl} + h.c.$$  

$$L_3 = \frac{g_3}{\Lambda^2} \text{tr} \left( \Sigma^\dagger \Sigma \chi \right) + h.c.$$  

$$L_4 = \frac{g_4}{\Lambda^2} \text{tr} \left( \Sigma^\dagger \Sigma \right) \text{tr} \left( \Sigma \chi \right) + h.c.$$  

$$L_5 = \frac{g_5}{\Lambda^2} \text{tr} \left( \Sigma^\dagger \Sigma \Sigma \chi \right) + h.c.$$  

$$L_6 = \frac{\bar{g}_6}{\Lambda^2} \text{tr} \left( \Sigma^\dagger \Sigma \chi + \Sigma \Sigma^\dagger \chi \right)$$  

$$L_7 = \frac{g_7}{\Lambda^2} \left( \text{tr} \Sigma^\dagger \chi + h.c. \right)^2$$  

$$L_8 = \frac{g_8}{\Lambda^2} \left( \text{tr} \Sigma \chi - h.c. \right)^2$$  

$$L_9 = - \frac{g_9}{\Lambda^2} \left( \text{tr} \Sigma \Sigma^\dagger \chi \right) + h.c.$$  

$$L_{10} = - \frac{\bar{g}_{10}}{\Lambda^2} \text{tr} \left( \chi^\dagger \chi \right) \text{tr} \left( \chi \Sigma \right) + h.c.$$  

(5)

Each term in the Lagrangian $L_6$ is hermitian by itself, but because of the parity symmetry of strong interactions, which transforms one of them into the other, they have a common coupling $\bar{g}_6$.

Some useful insight into the Lagrangian above can be obtained by considering it from the point of view of the $1/N_c$ expansion. Indeed, the number of color components of the quark field $q^a$ is $N_c$, hence summing over color indices in $\Sigma$ gives a factor of $N_c$, i.e. one counts $\Sigma \sim N_c$.

The cut-off $\Lambda$ that gives the right dimensionality to the multi-quark vertices scales as $\Lambda \sim N_c^0 = 1$, as a direct consequence of the gap equations (see eq. (37) below), which imply $1 \sim N_c G \Lambda^2$; on the other hand, since
the leading quark contribution to the vacuum energy is known to be of order $N_c$, the first term in (3) is estimated as $N_c$, and we conclude that $G \sim 1/N_c$.

Furthermore, the $U(1)_A$ anomaly contribution (the second term in (3)) is suppressed by one power of $1/N_c$, it yields $\kappa \sim 1/N_c$.

The last two terms in (3) have the same $N_c$ counting as the 't Hooft term. They are of order 1. Indeed, Zweig’s rule violating effects are always of order $1/N_c$ with respect to the leading order contribution $\sim N_c$. This reasoning helps us to find $g_1 \sim 1/N_c^2$. The term with $g_2 \sim 1/N_c^2$ is also $1/N_c$ suppressed. It represents the next to the leading order contribution with one internal quark loop in $N_c$ counting. Such vertex contains the admixture of the four-quark component $\bar{q}q\bar{q}q$ to the leading quark-antiquark structure at $N_c \to \infty$.

Next, all terms in eq. (3) except $L_0$, are of order 1. The argument is just the same as before: this part of the Lagrangian is obtained by successive insertions of the $\chi$-field ($\chi$ counts as $\sim 1$) in place of $\Sigma$ fields in the already known $1/N_c$ suppressed vertices. It means that $\kappa_1, g_2, g_1, g_2, g_7, g_8, g_9 \sim 1/N_c^2$, and $g_3, g_4 \sim 1/N_c^3$.

There are two important conclusions here. The first is that at leading order in $1/N_c$ only two terms contribute: the first term of eq. (3), and the first term of eq. (5). This corresponds exactly to the standard NJL model picture, where mesons are pure $\bar{q}q$ states with constituents which have a non-zero bare mass. At the next to leading order we have thirteen terms additionally. They trace the Zweig’s rule violating effects ($\kappa, \kappa_1, \kappa_2, g_1, g_4, g_7, g_8, g_5, g_9, g_{10}$), and an admixture of the four-quark component to the $\bar{q}q$ one ($g_2, g_3, g_5, g_6, g_8, g_9$). Only the phenomenology of the last three terms from eq. (6) has been studied until now. We must still understand the role of the other ten terms to be consistent with the generic $1/N_c$ expansion of QCD.

The second conclusion is that the $N_c$ counting justifies the classification of the vertices made above on the basis of the inequality (2). This is seen as follows: the equivalent inequality $\left[ (a + \beta)/2 \right] \leq 2$ is obtained by restricting the multi-quark Lagrangian to terms that do not vanish at $N_c \to \infty$ (it follows from (1) that $\beta - \left[ \gamma/2 \right] \geq 0$ by noting that $\bar{g}_i \sim 1/N_c^{[\gamma/2]}$, where $[\gamma/2]$ is the nearest integer greater than or equal to $\gamma/2$).

The total Lagrangian is the sum

$$L = \bar{q}i\gamma^\mu \partial_\mu q + L_{int} + L_\chi. \quad (6)$$

In this $SU(3)_L \times SU(3)_R$ symmetric chiral Lagrangian we neglect terms with derivatives in the multi-quark interactions, as usually assumed in the NJL model. We follow this approximation, because the specific questions for which these terms might be important, e.g. the radial meson excitations, or the existence of some inhomogeneous phases, characterized by a spatially varying order parameter, are not the goal of this work.

Finally, having all the building blocks conform with the symmetry pattern of the model, one is now free to choose the external source $\chi$. Putting $\chi = M/2$, where

$$M = \text{diag}(\mu_u, \mu_d, \mu_s),$$

we obtain a consistent set of explicitly breaking chiral symmetry terms. This leads to the following mass dependent part of the NJL Lagrangian

$$L_\chi \to L_\mu = -\bar{q}m q + \sum_{i=2} g_i \bar{q}q + L_i'$$

(7)

where the current quark mass matrix $m$ is equal to

$$m = M + \frac{\kappa_1}{\Lambda} (\text{det} M) M^{-1} + \frac{g_9}{4\Lambda^2} M^4 + \frac{g_{10}}{4\Lambda^2} (\text{tr} M^2) M,$$

(8)

and

$$L_i' = \frac{g_i}{4\Lambda^2} \epsilon_{ijk} e_{mnl} M_{im} \Sigma_{jm} \Sigma_{kn} + h.c.$$  

(9)

Let us note that there is a definite freedom in the definition of the external source $\chi$. In fact, the sources

$$\chi^{(c_i)} = \chi + \frac{c_1}{\Lambda} (\text{det} \chi) \chi (\chi\chi)^{-1} + \frac{c_2}{\Lambda^2} \chi \chi + \frac{c_3}{\Lambda^2} \text{tr} (\chi\chi) \chi$$

(10)

with three independent constants $c_i$ have the same symmetry transformation property as $\chi$. Therefore, we could have used $\chi^{(c_i)}$ everywhere that we used $\chi$. As a result, we would come to the same Lagrangian with the following redefinitions of couplings

$$\bar{q}_1 \to \bar{q}_1' = \bar{q}_1 + \frac{c_1}{2}, \quad \bar{g}_5 \to \bar{g}_5' = \bar{g}_5 - \bar{q}_2 c_1,$$

$$\bar{g}_7 \to \bar{g}_7' = \bar{g}_7 + \bar{q}_2 c_1, \quad \bar{g}_8 \to \bar{g}_8' = \bar{g}_8 + \bar{q}_2 c_1,$$

$$\bar{g}_9 \to \bar{g}_9' = \bar{g}_9 + c_2 - 2\bar{q}_1 c_1,$$

$$\bar{g}_{10} \to \bar{g}_{10}' = \bar{g}_{10} + c_3 + 2\bar{q}_1 c_1.$$  

(11)

Since $c_i$ are arbitrary parameters, this corresponds to a continuous family of equivalent Lagrangians. This family reflects the known Kaplan – Manohar ambiguity in the definition of the quark mass, and means that several different parameter sets may be used to represent the data. In particular, without loss of generality...
we can use the reparametrization freedom to obtain the set with \( \tilde{s}_3 = \tilde{g}_0 = \tilde{g}_{10} = 0 \).

The effective multi-quark Lagrangian can be written now as

\[
L = \bar{q}(i\gamma^\mu \partial_\mu - m)q + L_{int} + \sum_{i=2}^{8} L_i^{(s,p,\mu)}. \tag{12}
\]

It contains eighteen parameters: the scale \( \Lambda \), three parameters which are responsible for explicit chiral symmetry breaking \( \mu_a, \mu_b, \mu_c \), and fourteen interaction couplings \( \tilde{G}, \tilde{g}_1, \tilde{g}_2, \tilde{g}_3, \ldots, \tilde{g}_{10} \). Three of them, \( \tilde{g}_1, \tilde{g}_3, \tilde{g}_{10} \), contribute to the current quark masses \( m \). Seven more describe the strength of multi-quark interactions with explicit symmetry breaking effects. These vertices contain new details of the quark dynamics which have not been studied yet in any NJL-type models. We shall now see how important they are.

III. BOSONIZATION: MESON MASSES AND DECAYS

A. Stationary phase contribution

The model can be solved by path integral bosonization of the quark Lagrangian [12]. Indeed, following [7] we may equivalently introduce auxiliary fields \( s_a = \bar{q}_a \lambda_a q, p_a = \bar{q}_a \gamma_5 \lambda_a q \), and physical scalar and pseudoscalar fields \( \sigma = \sigma_a \lambda_a, \phi = \phi_a \lambda_a \). In these variables the Lagrangian is a bilinear form in quark fields (once the replacement has been done the quarks can be integrated out giving us the kinetic terms for the physical fields \( \phi \) and \( \sigma \))

\[
L = \bar{q}(i\gamma^\mu \partial_\mu - \sigma - i\gamma_5 \phi)q + L_{aux},
\]

\[
L_{aux} = s_a \sigma_a + p_a \phi_a - s_a m_a + L_{int}(s,p)
+ \sum_{i=2}^{8} L_i^{(s,p,\mu)}. \tag{13}
\]

It is clear, that after the elimination of the fields \( \sigma, \phi \) by means of their classical equations of motion, one can rewrite this Lagrangian in its original form [12]. The term bilinear in the quark fields in (13) will be integrated out using the heat kernel technique in the next subsection. The remaining higher order quark interactions collected in \( L_{aux} \) will be integrated in the stationary phase approximation (SPA). In terms of auxiliary bosonic variables one has

\[
L_{int}(s,p) = L_{4q} + L_{6q} + L_{8q}^{(1)} + L_{8q}^{(2)},
\]

\[
L_{4q}(s,p) = \frac{\tilde{G}}{2\Lambda^2} (s_a^2 + p_a^2),
\]

\[
L_{6q}(s,p) = \frac{\tilde{G}}{4\Lambda^3} A_{abc} s_b s_c - 3 p_b p_c),
\]

\[
L_{8q}^{(1)}(s,p) = \frac{\tilde{g}_1}{4\Lambda^4} (s_a^2 + p_a^2)^2,
\]

\[
L_{8q}^{(2)}(s,p) = \frac{\tilde{g}_2}{8\Lambda^5} [d_{abc} d_{cde} (s_a s_b + p_a p_b) (s_c s_d + p_c p_d)
+ 4 f_{abc} f_{cde} s_a s_c p_b p_d],
\]

and the quark mass dependent part is as follows

\[
\begin{align*}
L'_2 &= \frac{3\tilde{g}_2}{2\Lambda^3} A_{abc} s_b s_c - p_b p_c),
L'_3 &= \frac{\tilde{g}_3}{4\Lambda^4} [d_{abc} d_{cde} (s_a s_b + p_a p_b) - 2 f_{abc} f_{cde} p_b p_c]s_d],
L'_4 &= \frac{\tilde{g}_4}{4\Lambda^5} \mu_b s_b (s_a^2 + p_a^2),
L'_5 &= \frac{\tilde{g}_5}{4\Lambda^6} \mu_b \mu_c \mu_d (d_{abc} d_{cde} - f_{abc} f_{cde}) (s_a s_b - p_a p_c),
L'_6 &= \frac{\tilde{g}_6}{4\Lambda^7} \mu_b \mu_c \mu_d d_{abc} d_{cde} (s_a s_b + p_a p_c),
L'_7 &= \frac{\tilde{g}_7}{4\Lambda^8} (\mu_a s_a)^2,
L'_8 &= -\frac{\tilde{g}_8}{4\Lambda^9} (\mu_a p_a)^2,
\end{align*}
\]

where

\[
A_{abc} = \frac{1}{3!} \epsilon_{ijk} e_{mnl}(\lambda_a) im(\lambda_b) jn(\lambda_c) k\], \tag{16}

and the \( U(3) \) antisymmetric \( f_{abc} \) and symmetric \( d_{abc} \) constants are standard.

Our final goal is to clarify the phenomenological role of the mass-dependent terms described by the Lagrangian densities of eq. [15]. We can gain some understanding of this by considering the low-energy meson dynamics which follows from our Lagrangian. For that we must exclude quark degrees of freedom in [13], e.g., by integrating them out from the corresponding generating functional. The standard Gaussian path integral leads us to the fermion determinant, which we expand by using a heat-kernel technique [53, 54]. The remaining part of the Lagrangian, \( L_{aux} \), depends on auxiliary fields which do not have kinetic terms. The equations of motion of such a static system are the extremum conditions

\[
\frac{\partial L}{\partial s_a} = 0, \quad \frac{\partial L}{\partial p_a} = 0, \tag{17}
\]

which must be fulfilled in the neighbourhood of the uniform vacuum state of the theory. To take this into account one should shift the scalar field \( \sigma \rightarrow \sigma + M \). The new \( \sigma \)-field has a vanishing vacuum expectation value \( \langle \sigma \rangle = 0 \), describing small amplitude fluctuations about the vacuum, with \( M \) being the mass of constituent quarks. We seek solutions of eq. (17) in the form:

\[
\begin{align*}
\bar{s}_a &= h_a + h_{ab}^{(1)} \sigma_b + h_{abc}^{(1)} \sigma_c + h_{abc}^{(2)} \phi_b \phi_c + \ldots
\]
\[
\bar{p}_a &= h_a^{(2)} \phi_b + h_{abc}^{(3)} \sigma_c + \ldots \tag{18}
\end{align*}
\]

Eqs. (17) determine all coefficients of this expansion giving rise to a system of cubic equations to obtain \( h_a \), and the full set of recurrence relations to find higher order coefficients in [15]. We can gain some insight into the
physical meaning of these parameters if we calculate the Lagrangian density \( L_{aux} \) on the stationary trajectory. In fact, using the recurrence relations, we are led to the result

\[
L_{aux} = h_a \sigma_a + \frac{1}{2} h^{(1)}_{ab} \sigma_a \sigma_b + \frac{1}{2} h^{(2)}_{ab} \phi_a \phi_b \\
+ \frac{1}{3} \sigma_a \left[ h^{(1)}_{abc} \sigma_b \sigma_c + \left( h^{(2)}_{abc} + h^{(3)}_{pca} \right) \phi_b \phi_c \right] + \ldots
\]

(19)

Indicated are all the terms which are necessary to analyze the mass spectra and two particle decays. Here \( h_a \) define the quark condensates, \( h^{(1)}_{ab} \), \( h^{(2)}_{ab} \) contribute to the masses of scalar and pseudoscalar states, and higher order \( h \)'s are the couplings that measure the strength of the meson-meson interactions. The transition from the Lagrangian \( L_{aux}(s,p) \) in \([13]\) to its form \( L_{aux}(\sigma, \phi) \) in \([19]\) can be viewed as a Legendre transformation.

We proceed now to explain the details of obtaining \( h \). We address first the coefficients \( h_a \), \( h^{(1)}_{ab} \), and \( h^{(2)}_{ab} \). In particular, eq. \([17]\) states that \( h_a = 0 \), if \( a \neq 0,3,8 \), while \( h_\alpha \) (\( \alpha = 0,3,8 \)) after the convenient redefinition to the flavor indices \( i = u, d, s \)

\[
h_i = e_{ai} h_a, \quad e_{ai} = \frac{1}{2\sqrt{3}} \begin{pmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ \sqrt{3} & -\sqrt{3} & 0 \\ 1 & 1 & -2 \end{pmatrix},
\]

(20)

satisfy the following system of cubic equations

\[
\Delta_i + \frac{\kappa_i}{4} t_{ijk} h_j h_k + \frac{h_i}{2} \left( 2G + gh_i h^2 + g_4 h^4 \right) + \frac{g_3}{2} h_i^3 \\
+ \frac{\mu_i}{4} \left[ 3g_3 h_i^2 + g_4 h_i^3 + 2(g_5 + g_6) \mu_i h_i + 4 \gamma_i h_i \right] \\
+ \nu_i t_{ijk} h_j h_k = 0.
\]

(21)

Here \( \Delta_i = M_i - m_i; \) \( t_{ijk} \) is a totally symmetric quantity, whose nonzero components are \( t_{iab} = 1 \); there is no summation over the open index \( x \) but we sum over the dummy indices, e.g. \( h^2 = h_\mu^2 + h_u^2 + h_d^2, \mu h_i = \mu_a h_a + \mu_d h_d + \mu_s h_s \).

In particular, eq. \([8]\) reads in this basis

\[
m_i = \mu_i \left( 1 + \frac{g_9}{4} \mu_i^2 + \frac{g_{10}}{4} \mu_i^2 \right) + \frac{\kappa_i}{2} t_{ijk} h_j h_k.
\]

(22)

For the set \( g_9 = g_{10} = \kappa_1 = 0 \) the current quark mass \( m_i \) coincides precisely with the explicit symmetry breaking parameter \( \mu_i \).

Note that the factor multiplying \( h_i \) in the third term of eq. \([22]\) is the same for each flavor. This quantity also appears in all meson mass expressions, and there is no further dependence on the couplings \( g, g_1, g_4 \) involved for meson states with \( a = 1,2, \ldots, 7 \). Thus there is a freedom of choice which allows to vary these couplings, condensates and quark masses \( \mu_i \), without altering this part of the meson mass spectrum.

To obtain the coefficients \( h^{(i)}_{ab} \), \( i = 1,2 \) in the Lagrangian \( L_{aux} \) \([19]\), it is sufficient to collect in the stationary phase equations \([17]\) only the terms linear in the fields, as can be seen from the structure of the solutions \( \phi_\alpha \). Moreover, for any coefficient multiplying a certain number \( n \) of fields in \( L_{aux} \) it is required to consider terms only up to order \( n - 1 \) in the fields in the expansion \([18]\). For instance, the inverse matrices to \( h^{(1)}_{ab} \) and \( h^{(2)}_{ab} \) are

\[
-2 \left( h^{(1)}_{ab} \right)^{-1} = (2G + gh_i h^2 + g_4 h^4) \delta_{ab} + 4g_1 h_a h_b \\
+ 3A_{abc} (\kappa h_c + 2\kappa_2 \mu_c) + g_2 h_a h_c (d_{abc} d_{cre} + 2d_{ace} d_{brc}) \\
+ g_3 h_c (d_{abe} d_{cde} + d_{ace} d_{bde} + d_{ade} d_{bce}) \\
+ 2g_4 (\mu_a h_b + \mu_b h_a) + g_5 h_c (d_{arc} d_{bce} - d_{are} d_{bce}) \\
+ g_6 h_c (d_{abc} d_{cde} + 4g_7 \mu_a \mu_b).
\]

(23)

\[
2 \left( h^{(2)}_{ab} \right)^{-1} = (2G + gh_i h^2 + g_4 h^4) \delta_{ab} \\
- 3A_{abc} (\kappa h_c + 2\kappa_2 \mu_c) + g_2 h_a h_c (d_{abc} d_{cre} + 2f_{are} f_{bce}) \\
+ g_3 h_c (d_{abe} d_{cde} + f_{arc} f_{bce} + f_{ace} f_{bde}) \\
- g_5 h_c (d_{arc} d_{bce} - f_{arc} f_{bce}) \\
+ g_6 h_c (d_{abc} d_{cde} + 4g_7 \mu_a \mu_b).
\]

(24)

These coefficients are totally defined in terms of \( h_a \) and the parameters of the model. Eqs. \([23]\)-\([24]\) can be easily converted into explicit formulæ for \( h^{(1)}_{ab} \) \( (i = 1,2) \).

Finally, to obtain the \( h^{(i)}_{abc} \), \( i = 1,2,3 \), of the interactions involving three fields in \( L_{aux} \), one equates the factors of \( \sigma_a \sigma_\alpha, \phi_\alpha \phi_\alpha, \phi_\alpha \sigma_\alpha \) in \([17]\) independently to zero. After some algebra, this results into the following expressions

\[
h^{(1)}_{abc} = \left[ \frac{3\kappa}{4} A_{abc} + g_1 (h_a \delta_{bc} + 2h_c \delta_{ab}) \\
+ g_2 h_f (d_{\bar{a}b} d_{\bar{c}f} + \frac{1}{2} d_{\bar{a}f} d_{\bar{c}b}) \\
+ g_4 (m_f (2d_{\bar{a}b} d_{\bar{c}f} d_{\bar{d}p} + d_{\bar{a}b} d_{\bar{c}f} d_{\bar{d}p} + f_{\bar{a}b} f_{\bar{c}d} f_{\bar{d}p}) \\
+ \frac{g_4}{2} (m_f \delta_{bc} + 2m_f \delta_{ab}) \right] h^{(1)}_{aa} h^{(1)}_{bb} h^{(1)}_{cc}
\]

(25)

\[
h^{(2)}_{abc} = \left[ -\frac{3\kappa}{4} A_{abc} + g_1 h_a \delta_{bc} \\
+ g_2 h_f (f_{\bar{a}b} f_{\bar{c}f} + \frac{1}{2} d_{\bar{a}f} d_{\bar{c}b}) \\
- g_4 (m_f (2f_{\bar{a}b} f_{\bar{c}f} d_{\bar{d}p} + f_{\bar{a}b} f_{\bar{c}f} d_{\bar{d}p} + f_{\bar{a}b} f_{\bar{c}d} f_{\bar{d}p}) \\
+ \frac{g_4}{2} m_f \delta_{bc} \right] h^{(1)}_{aa} h^{(2)}_{bb} h^{(2)}_{cc}
\]

(26)

\[
h^{(3)}_{abc} = \left[ -\frac{3\kappa}{4} A_{abc} + 2g_1 h_a \delta_{ba} \\
+ g_2 h_f (d_{\bar{a}b} d_{\bar{c}f} + f_{\bar{a}b} f_{\bar{c}d} f_{\bar{d}p}) \\
+ g_3 (m_f (d_{\bar{a}b} d_{\bar{c}f} d_{\bar{d}p} + f_{\bar{a}b} f_{\bar{c}d} f_{\bar{d}p} + f_{\bar{a}b} f_{\bar{c}d} f_{\bar{d}p}) \\
+ g_4 m_f \delta_{ba} \right] h^{(2)}_{aa} h^{(1)}_{bb} h^{(2)}_{cc}
\]

(27)

Contracting with \( \phi_\alpha \phi_\alpha \) in eq. \([19]\), one sees that the term going with \( h^{(2)}_{ab} \) is simply half the one going with \( h^{(3)}_{bc} \).
and $L_{aux}$ simplifies to

$$L_{aux} = h_a \sigma_a + \frac{1}{2} h^{(1)}_{ab} \sigma_a \sigma_b + \frac{1}{2} h^{(2)}_{ab} \phi_a \phi_b + \sigma_a \left( \frac{1}{3} h^{(1)}_{abc} \sigma_b \sigma_c + h^{(2)}_{abc} \phi_b \phi_c \right) + \ldots$$  \hspace{1cm} (28)

Although there are five parameters $\kappa_1, g_1, g_2, g_3, g_4$ which appear explicitly in $h^{(i)}_{abc}$, they do not represent new freedom to fit the meson interaction dynamics, since they occur also in the $h^{(i)}_{ab}$; through the latter the $h^{(i)}_{abc}$ depend implicitly also on further six parameters $G, \kappa_2, g_5, g_6, g_7, g_8$. All will be fixed by fitting the mass spectra and weak decay constants, see [38] and section IV below.

**B. The heat kernel contribution**

We now turn our attention to the total Lagrangian of the bosonized theory. To write down this Lagrangian we should add the terms coming from integrating out the quark degrees of freedom in (13) to our result (28). Fortunately, the technicalities are known. We use the modified heat kernel technique [43, 44] developed for the case of explicit chiral symmetry breaking. In the isospin limit one can find all necessary details of such calculations for instance in [53]. For future reference we apply it here to obtain the result for the more general case in which the strong isospin symmetry is broken.

From the vacuum to vacuum persistence amplitude in the spontaneous broken phase

$$Z[\sigma, \phi] = \int D\bar{q} Dq \exp \left( i \int d^4x L_q(\sigma, \phi) \right),$$

$$L_q(\sigma, \phi) = \bar{q} (i \gamma^\mu \partial_\mu - M - \sigma - i \gamma_5 \phi) q$$  \hspace{1cm} (29)

the heat kernel result for the integration over the quark degrees of freedom is

$$W[Y] = \ln |\det D| = -\frac{1}{2} \int_0^\infty \frac{dt}{t} \rho(t) \exp(-t D^4_E D_{\bar{E}}),$$

$$D^4_E D_{\bar{E}} = M^2 - \partial^2 + Y, \quad Y = i \gamma_\mu \partial_\mu + i \gamma_5 \partial_5 \phi + \sigma_2 + \{M, \sigma\} + \phi_2 + i \gamma_5 \{\sigma + M, \phi\},$$  \hspace{1cm} (30)

or

$$W[Y] = -\int \frac{d^4x}{32\pi^2} \sum_{i=0}^{\infty} I_{i-1} \text{tr}[b_i]$$  \hspace{1cm} (31)

where $D_E$ stands for the Dirac operator in Euclidean space. We consider the expansion up to the third Seeley-DeWitt coefficient $b_i$

$$b_0 = 1, \quad b_1 = -Y, \quad b_2 = \frac{Y^2}{2} + \frac{\lambda_3}{2} \Delta_{ud} Y + \frac{\lambda_8}{2\sqrt{3}} (\Delta_{us} + \Delta_{ds}) Y,$$  \hspace{1cm} (32)

with $\Delta_{ij} = M_i^2 - M_j^2$. This order of the expansion takes into account the dominant contributions of the quark one-loop integrals $I_i$ ($i = 0, 1, \ldots$); these are the arithmetic average values $I_i = \frac{1}{3} [I_i(M^2_u) + I_i(M^2_d) + I_i(M^2_s)]$

$$J_i(m^2) = \int_0^\infty \frac{dt}{t^2} \rho(t \Delta^2) e^{-t m^2},$$  \hspace{1cm} (33)

with the Pauli-Villars regularization kernel [37, 58]

$$\rho(t \Delta^2) = 1 - (1 + t \Delta^2) \exp(-t \Delta^2).$$  \hspace{1cm} (34)

In the following we need therefore only to know two of them (the lowest order $\sim b_0$ contributes to the effective potential and is not needed in the present study)

$$J_0(m^2) = \Delta^2 - m^2 \ln \left( 1 + \frac{\Lambda^2}{m^2} \right),$$  \hspace{1cm} (35)

and

$$J_1(m^2) = \frac{\Lambda^2}{\Lambda^2 + m^2}.$$  \hspace{1cm} (36)

While both terms proportional to $b_1$ and $b_2$ have contributions to the gap equations and meson masses, only $b_2$ contributes to the kinetic and interaction terms. The $\sigma$ tadpole term must be excluded from the total Lagrangian. This gives us a system of gap equations

$$h_i + \frac{N_c}{6\pi^2} M_i [3I_0 - (3M_i^2 - M^2) I_1] = 0.$$  \hspace{1cm} (37)

Here $N_c = 3$ is the number of colors, and $M^2 = M^2_u + M^2_d + M^2_s$. Combining all terms of the total Lagrangian $L = L_{\text{kin}} + L_{\text{mass}} + L_{\text{int}}$ that contribute to the kinetic terms $L_{\text{kin}}$ and meson masses $L_{\text{mass}}$ one gets

$$L_{\text{kin}} + L_{\text{mass}} = \frac{N_c I_3}{16\pi^2} \text{tr} \left[ (\partial_\mu \sigma)^2 + (\partial_\mu \phi)^2 \right] + \frac{N_c I_0}{4\pi^2} (\sigma_2^2 + \phi_2^2)$$

$$- \frac{N_c I_1}{12\pi^2} \left\{ \begin{array}{l} 2 (M_u + M_d)^2 - M_u M_d - M_u^2 \phi_2^2 + \phi_2^2 \\
2 (M_u + M_s)^2 - M_u M_s - M_s^2 \phi_2^2 + \phi_2^2 \\
2 (M_d + M_s)^2 - M_d M_s - M_s^2 \phi_2^2 + \phi_2^2 \\
\frac{1}{2} [\sigma_2^2 (8 M_u^2 - M_d^2 - M_s^2) + \phi_2^2 (8 M_d^2 - M_u^2 - M_s^2)] \\
\frac{1}{2} [\phi_2^2 (2 M_u^2 - M_d^2 - M_s^2) + \phi_2^2 (2 M_d^2 - M_u^2 - M_s^2)] \\
\frac{1}{2} [\phi_2^2 (M_u - M_d)^2 + M_u M_d - M_u^2] \phi_2^2 + \phi_2^2 \\
\frac{1}{2} [2 (M_u - M_d)^2 + M_u M_d - M_u^2] \phi_2^2 + \phi_2^2 \\
\frac{1}{2} [2 (M_d - M_s)^2 + M_d M_s - M_d^2] \phi_2^2 + \phi_2^2 \\
\frac{1}{2} [2 (M_u - M_s)^2 + M_u M_s - M_u^2] \phi_2^2 + \phi_2^2 \end{array} \right\}$$  \hspace{1cm} (38)
The kinetic term requires a redefinition of meson fields,
\[ \sigma_a = g \sigma_a^R, \quad \phi_a = g \phi_a^R \quad \gamma^2 = \frac{4 \pi^2}{N_c}, \]
(39)
to obtain the standard factor 1/4. The flavor and charged fields are related through
\[
\frac{\lambda_u}{\sqrt{2}} \phi_a = \left( \begin{array}{c} \frac{\phi_u}{\sqrt{2}} \\ \pi^+ \\ K^+ \\ \frac{\phi_u}{\sqrt{2}} \\ \frac{\phi_d}{\sqrt{2}} \\ K^0 \\ \frac{\phi_u}{\sqrt{2}} \\ \frac{\phi_u}{\sqrt{2}} \\ K^- \end{array} \right) \frac{\lambda_d}{\sqrt{2}} \phi_a = \left( \begin{array}{c} \frac{\phi_s}{\sqrt{2}} \\ a^+ \\ \kappa^+ \\ \frac{\phi_s}{\sqrt{2}} \\ \frac{\phi_s}{\sqrt{2}} \\ \kappa^0 \\ \frac{\phi_s}{\sqrt{2}} \\ \frac{\phi_s}{\sqrt{2}} \\ \kappa^- \end{array} \right)
\]
(40)
and in particular for the diagonal components
\[
\phi_u = \phi_3 + \sqrt{2} \phi_0 + \phi_8 = \phi_3 + \eta_{ns},
\]
\[
\phi_d = -\phi_3 + \sqrt{2} \phi_0 + \phi_8 = -\phi_3 + \eta_{ns},
\]
\[
\phi_s = \sqrt{2} \phi_0 - 2 \phi_8 = \sqrt{2} \eta_s
\]
(41)
and similar for the scalar fields. Here we also introduce the \( \eta_{ns} \) and \( \eta_s \) which stand for the flavor components of the physical \( \eta, \eta' \) states in the nonstrange and strange basis. In addition to the flavor mixing in the \( \eta, \eta' \) channels the isospin breaking induces a coupling between the \( \pi^0 \) and these states
\[ \pi^0 = \phi_3 + \epsilon \eta + \epsilon' \eta'. \]
(42)
To get the physical \( \pi^0, \eta \) and \( \eta' \) mesons and correspondingly the scalar \( a_0^0(980) \), \( \sigma \) and \( f_0(980) \) mesons one may proceed as in [59]. Since \( \phi_3 \) couples weakly to the \( \eta_{ns} \) and \( \eta_s \) states (decoupling in the isospin limit) while the \( \eta - \eta' \) mixing is strong, it is appropriate to use isoscalar \( \eta_{ns}, \eta_s \) and isovector \( \phi_3 \) combinations as a starting point for an unitary transformation to the physical meson states \( \pi^0, \eta, \eta' \). In this case the corresponding unitary matrix \( U \) can be linearized in the \( \pi^0 - \eta \) and \( \pi^0 - \eta' \) mixing angles \( \epsilon_1, \epsilon_2 \sim \mathcal{O}(\epsilon), \epsilon \ll 1 \). Precisely [59]
\[
\begin{pmatrix} \pi^0 \\ \eta' \end{pmatrix} = U(\epsilon_1, \epsilon_2, \psi) \begin{pmatrix} \phi_3 \\ \eta_{ns} \\ \eta_s \end{pmatrix},
\]
(43)
where
\[
U = \begin{pmatrix} 1 & \epsilon_1 + \epsilon_2 \cos \psi & -\epsilon_2 \sin \psi \\ -\epsilon_2 - \epsilon_1 \cos \psi & \cos \psi & -\sin \psi \\ \epsilon_1 \sin \psi & \sin \psi & \cos \psi \end{pmatrix}
\]
(44)
In particular, in eq. (42) \( \epsilon = \epsilon_2 + \epsilon_1 \cos \psi, \epsilon' = \epsilon_1 \sin \psi \).

In the isospin limit we use the mixing angle conventions summarized in the Appendix B of [58]. We have the following different possibilities of relating the physical states \( (X, X) \) with the states of the strange-nonstrange basis
\[ \begin{pmatrix} \bar{X} \\ X \end{pmatrix} = R_{\psi} \begin{pmatrix} X_{ns} \\ X_s \end{pmatrix} = R_{\psi} \begin{pmatrix} -X_s \\ X_{ns} \end{pmatrix}, \]
(45)
where the orthogonal \( 2 \times 2 \) matrix \( R_{\psi} \) is
\[
R_{\psi} = \begin{pmatrix} \cos \psi & -\sin \psi \\ \sin \psi & \cos \psi \end{pmatrix},
\]
(46)
or of the singlet-octet basis
\[
\begin{pmatrix} \bar{X} \\ X \end{pmatrix} = R_{\theta} \begin{pmatrix} X_s \\ X_0 \end{pmatrix}.
\]
(47)
Here \( \theta \), being a solution of the equation \( \tan 2\theta = x \), is the principal value of arctan \( x \), i.e. belongs to the interval \( -(\pi/4) \leq \theta \leq (\pi/4) \). The angle \( \psi \) is related with \( \theta \) by the equation \( \psi = \theta + \theta_{id} \), where \( \theta_{id} (\theta_{id} + \theta_{id} = \pi/2) \) is determined by the equations \( \sin \theta_{id} = \sqrt{2/3}, \cos \theta_{id} = 1/\sqrt{3} \), therefore \( \psi = \theta + \arctan \sqrt{2} = \theta + 54.74^\circ \). It means that \( \psi \) is restricted to the range \( 9.74^\circ \leq \psi \leq 99.74^\circ \). If the value of \( \psi \) leaves the range, we must resort to the angle \( \psi = \pi/2 - \theta - \theta_{id} \), taking values in the interval \( -80.26^\circ \leq \psi \leq 9.74^\circ \). These two angles correspond to two alternative phase conventions for a strange \( \bar{s}s \)-component. As a result of the following numerical calculations, in the case of the pseudoscalars the identification of the physical states is \( X = \eta, X = \eta' \) and for the scalars \( X = f_0(980), X = \sigma \).

We turn to the interaction terms of the heat kernel action in [30]. The only contribution comes from \( Y/2 \) in the term proportional to \( b_2 \) and reads
\[
L_{int}^{(hk)} = -\frac{N_c}{2 \pi^2} I_1 M_4 \left[ d_{abc}d_{cep} \sigma_b (\sigma_c \sigma_e + \phi_c \phi_e) + 2f_{cep} \sigma_b \sigma_c \phi_e \right],
\]
(48)
which must be added to the interaction piece stemming from [28], yielding the total interaction Lagrangian
\[
L_{int} = L_{int}^{(hk)} + \sigma_a \left( \frac{1}{3} h^{(1)}_{abc} \sigma_b \sigma_c + h^{(2)}_{abc} \phi_b \phi_c \right),
\]
(49)
Note that all dependence on the parameters of the explicit symmetry breaking quark interactions is explicitly absorbed in the bosonized Lagrangian through the matrices \( h^{(1,2)}_{ab} \) for the meson mass spectra [38] and through the \( h^{(1,2,3)}_{abc} \) for the meson interaction Lagrangian [49]. In other words, the formal structure of the Lagrangian [28] in comparison to the case without these interactions remains unchanged. This differs from the heat kernel Lagrangian where the information about the difference in constituent quark masses leads to a resummation of the heat kernel series for the modified Seeley-DeWitt coefficients \( b_1, b_2, b_3 \) [54,56]. The parameters of these two seemingly separated sectors of the Lagrangian, i.e. the constituent
quark masses and scale parameter $\Lambda$ for the heat kernel Lagrangian on one hand, and the multiquark interaction couplings for the SPA piece on the other hand, are con-
nected through the gap equations \[(37)\] which must be solved self-consistently with the SPA equations \[(21)\].

In the remaining of this subsection we discuss the scheme in which the strong decay widths of the scalar mesons are calculated. Given the complexity of the La-
grangian, we will restrict our study of the decays to the tree level bosonic couplings \[(48), (49)\]. To deal in an ap-
proximate way with the proximity of particle thresholds to the resonance mass we shall resort to the widely ac-
cepted Flatté type distribution \[(50)\]. Other closed bosonic channel contributions will not be taken into consideration for simplicity, since the ratios of couplings in the concur-
ring closed channels to the nominal one turn out to be numerically less relevant in our fits.

The strong decay width of the scalar meson $S$ in two pseudoscalars $P_1, P_2$ are thus obtained as

$$ \Gamma_{\beta} = \frac{|\vec p_{\beta}|}{8\pi m_S^2} |g_\beta|^2 \equiv \bar g_\beta |\vec p_{\beta}| \quad \text{(50)} $$

with

$$ |\vec p_{\beta}| = \sqrt{\frac{(m_S^2 - (m_1 + m_2)^2)(m_S^2 - (m_1 - m_2)^2)}{4m_S^2}} $$

where index $\beta$ specifies all necessary kinematic characteristics of the channel $S \to P_1 P_2$, and the masses $m_s, m_1, m_2$ of the states. We introduce also a shorthand notation for the dimensionless quantity $g_\beta$ in eq.\[(50)\]. In this definition we include all flavor and symmetry factors associated with the final state.

The so obtained widths are valid in the Breit-Wigner resonance scheme, which is known to be an incomplete description for decays with the resonance mass close to the threshold of particle emission. We use Flatté distributions in the cases of the $a_0(980)$ and $f_0(980)$ decays to accomodate the threshold effects associated with the two kaon production, on grounds of analyticity and unitarity at the threshold. Close to this threshold the elastic scattering cross section for $\pi\eta$ in the case of $a_0$ or $\pi\pi$ for $f_0$ is parametrized by a two-channel resonance

$$ \sigma_{el} = 4\pi|f_{el}|^2, \quad f_{el} = \frac{1}{|\vec p_{\beta}| m_R - s - i m_R (\Gamma_{\beta} + \Gamma_{sKK}^S)} \quad \text{(51)} $$

with the index $\beta$ designating here either the $a_0\pi\eta$ or the $f_0\pi\pi$ channels and

$$ \Gamma_{sKK}^S = \begin{cases} \bar g_K \sqrt{\frac{s}{4} - m_K^2} & \text{above threshold} \\ \bar g_K \sqrt{m_K^2 - \frac{s}{4}} & \text{below threshold} \end{cases} \quad \text{(52)} $$

where $\bar g_K^S$ stands for the coupling of $S$ to the two kaons, in this case $S = a_0$ or $f_0$. Here $m_R$ is the nominal resonance mass and $s = (p_1 + p_2)^2$, where $p_1, p_2$ are the 4-momenta of $P_1$ and $P_2$. Near the $K\bar K$ threshold only the width $\Gamma_{sKK}^S$ is expected to vary strongly; the width $\Gamma_\beta$ are approximated by a constant value in this region, taken to be \[(50)\] evaluated at $s = m_R^2$, since the $\pi\eta$ and $\pi\pi$ thresholds lie further away from the resonance. The numerical results are presented and discussed in the section IV.

### C. A note on radiative decays

Additional information on the structure of the mesons is obtained through the study of their radiative decays. We consider in this work the two photon decays at the quark one-loop order of the scalar and pseudoscalar mesons. The corresponding integrals are finite. A direct extension of the heat kernel Lagrangian to incor-
porate the coupling to the electromagnetic interaction shows that there is no contribution up to the order $b_2$ of the Seeley-DeWitt coefficients for the scalar decays. The anomalous pseudoscalar - two photon decays belong to the imaginary part of the action and are not contemplated by the heat kernel techniques considered, which apply only to the real part. By the Adler-Bardeen theorem \[(61–63)\] they are fully determined by the three-point function Feynman amplitudes involving one quark loop; higher orders only redefine the couplings. There is however a source of uncertainty which resides in the model dependent determination of the coupling of the $\eta$ and $\eta'$ mesons to the quarks. In our approach they are calculated within the heat kernel technique outlined in sec-
ction III.B. Regarding the scalar meson two photon decays, they are also most simply evaluated through the three-point Feynman amplitudes, keeping only the con-
tribution corresponding to the first non-vanishing order in the heat kernel action, that is the term involving the Seeley-DeWitt coefficient $b_3$. From now on we will consider the case with exact $SU(2)$ isospin symmetry, i.e. $\mu_u = \mu_d = \mu \neq \mu_s$, and $M_u = M_d = M \neq M_s$. With the standard electromagnetic coupling to quarks $\mathcal{L}_q = -e\bar q\gamma^\mu q A_\mu$, $Q = \frac{1}{2} (\lambda_3 + \frac{1}{\sqrt{3}} \lambda_8)$ and using the Pauli-Villars regularization, the scalar meson photon photont amplitude $A$: $S(s) \to \gamma(p_1, \epsilon^\mu_1) + \gamma(p_2, \epsilon^\mu_2)$ is obtained in terms of the gauge invariant tensor $\mathcal{L}_{\mu\nu} = (p_1^\mu p_2^\nu - \frac{1}{2} s g^{\mu\nu})$, with $s = (p_1 + p_2)^2$

$$ A^{\mu\nu}_{s\gamma\gamma} = \mathcal{L}^{\mu\nu} A_{s\gamma\gamma}; \quad S = \sigma, f_0(980), a_0(980) $$

$$ A_{s\gamma\gamma} = \frac{5}{9} T_u \cos \psi - \frac{\sqrt{3}}{9} T_s \sin \psi $$

$$ A_{f_0\gamma\gamma} = \frac{5}{9} T_u \sin \psi - \frac{\sqrt{3}}{9} T_s \cos \psi $$

$$ A_{a_0\gamma\gamma} = \frac{1}{3} T_u $$

where

$$ T_i = 32\pi g M_i Q_3(s, M_i), \quad i = (u,s) $$
\[ Q_3(s, M_i) = \frac{iN_c}{16\pi^2} \int_0^1 dx \int_0^{1-x} dy (1 - 4xy) \times \int_0^\infty dt \rho(t\Lambda^2) e^{-t(M_i^2 - xys)} \] (54)

\[ \alpha = \frac{2}{\pi} \] is the fine structure constant and \( g \) the field normalization defined in [39]. The factors of \( T_i \) result from the flavor traces and projection to the physical states with the angle \( \psi \) defined in (45). The result for the integral \( Q_3(s, M_i) \) with the Pauli-Villars kernel \( \rho(t\Lambda^2) \), eq. (34), has been evaluated in [64]. To obtain the dominant contribution, i.e. the first non-vanishing order in the heat kernel series, one needs to express the integrals \( Q_3(s, M_i) \) as the following averaged sum evaluated at \( s = 0 \) [55, 56]:

\[ Q_3(0, M_i) \rightarrow Q_3(0, M_u, M_s) = \frac{1}{3} (2Q_3(0, M_u) + Q_3(0, M_s)) + O(b_3) \] (55)

where the term \( O(b_3) \) is discarded as it belongs to the next order in the heat kernel series [30] and

\[ Q_3(0, M_i) = -\frac{N_c}{48\pi^2 M_i^2} \left( \frac{\Lambda^2}{\Lambda^2 + M_i^2} \right)^2, \] (56)

or, in the notation of (33), we have that

\[ Q_3(0, M_i) = -\frac{N_c}{48\pi^2} J_2(M_i^2). \] (57)

Finally the decay widths for the scalar mesons in the narrow width approximation are given as (see also [54])

\[ \Gamma_{S\gamma\gamma} = \frac{m_S^3}{64\pi} |A_{S\gamma\gamma}|^2 \] (58)

The anomalous decay of the pseudoscalars \( P = (\pi^0, \eta, \eta') \) in two photons \( P(p) \rightarrow \gamma(p_1, c_{\gamma 1}) + \gamma(p_2, c_{\gamma 2}) \) has the same Lorentz structure in all channels and reads

\[ A_{\mu\nu\gamma\gamma} = \epsilon^{\mu\nu\alpha\beta} p_1\alpha p_2\beta A_{\mu\nu\gamma\gamma} \]

\[ A_{\eta\gamma\gamma} = -\frac{5}{9} T^P_u \sin \psi_P - \frac{\sqrt{2}}{9} T^P_s \cos \psi_P \]

\[ A_{\eta'\gamma\gamma} = \frac{5}{9} T^P_u \cos \psi_P - \frac{\sqrt{2}}{9} T^P_s \sin \psi_P \]

\[ A_{\pi^0\gamma\gamma} = \frac{1}{3} T^P \] (59)

where \( \psi_P \) stands for the mixing angle in the pseudoscalar channels, eq. (15) and

\[ T^P_i(s, M_i) = 32\pi^2 g M_i I_P(s, M_i) \]

\[ I_P(s, M) = -\frac{N_c}{16 \pi^2} \int_0^1 dx \int_0^{1-x} dy \int_0^\infty dt e^{-t(M^2 - xys)} \] (60)

and the contribution to the imaginary part of the heat kernel action is

\[ I_P(0, M) = -\frac{N_c}{32\pi^2 M^2}. \] (61)

At this stage one sees that the only parameter dependence in the radiative decays of the scalars and pseudoscalars enters through the wave function normalization \( g \), common to all decays considered, and through the constituent quark masses; there is also an explicit dependence on the scale \( \Lambda \) in the case of the scalar decays through the factor \( \left( \frac{\Lambda^2}{\Lambda^2 + M^2} \right)^2 \) in (56). The PCAC hypothesis establishes a relation between \( g \), the weak pion and kaon decay couplings and the constituent quark masses (see also [66] below)

\[ f_\pi = \frac{M}{g}, \quad f_K = \frac{M + M_s}{2g}. \] (62)

These identities allow to eliminate all dependence on the constituent quark masses from the pseudoscalar radiative decays, leading to

\[ T^P(0, M) = \frac{N_c}{\pi f_\pi}, \quad T^P(0, M_s) = \frac{N_c}{\pi (2f_K - f_\pi)}. \] (63)

One obtains then the celebrated relation \( A_{\pi\gamma\gamma} = \frac{g}{\pi f_\pi} \) for the \( \pi^0 \) decay amplitude [61]. The Adler-Bardeen theorem allows to infer that the study and measurement of the anomalous decays are a reliable means of determination of the mixing angle of the \( \eta \) and \( \eta' \) mesons, which must comply with the mixing angle determination extracted from the mass spectrum. One should also stress that with the present model Lagrangian one is able to account properly for the \( SU(3) \) breaking effects in the description of the weak decay constants \( f_\pi \) and \( f_K \), in addition to having the correct empirical \( \eta \) and \( \eta' \) meson masses (see section IV), which has been an open problem until now. This is important for the numerical consistency in the amplitudes [63].

The respective widths are calculated as

\[ \Gamma_{P\gamma\gamma} = \frac{|\vec{p}|^3}{8\pi} |A_{P\gamma\gamma}|^2 \] (64)

with \( |\vec{p}| = \sqrt{m_P^2/4} \) and \( m_P \) the pseudoscalar mass. The numerical results are presented in section IV.

IV. Fixing Parameters, Numerical Results and Discussion

A. Meson Spectra and weak decays

In the chiral limit, \( m_u = m_d = m_s = 0 \), the Lagrangian [38] leads to the conserved vector, \( V^\mu_\mu \), and axial-vector, \( A^b_\mu \), currents. The matrix elements of axial-vector currents

\[ \langle 0 | A^a_\mu(0) | 0 | b \rangle = i g_A f^{ab} \] (65)

define the weak and electromagnetic decay constants of physical pseudoscalar states (see details in [52]). Now let us fix the values of the various quantities introduced. After choosing the set \( n_1 = g_9 = g_{10} = 0 \) we
still have to fix fourteen parameters: $\Lambda, \hat{m}, m_s, G, \kappa, \kappa_2$ and $g_1, \ldots, g_7$. There are two intrinsic restrictions of the model, namely, the stationary phase \[21\] and the gap \[37\] equations, which as mentioned above must be solved self-consistently. This is how the explicit symmetry breaking is intertwined with the dynamical symmetry breaking and vice versa. We use \[37\] to determine $\hat{\kappa}, \hat{h}_s$ through $\Lambda, M_s$ and $M$. The ratio $M_s/M$ is related to the ratio of the weak decay constants of the pion, $f_\pi = 92$ MeV, and the kaon, $f_K = 113$ MeV. Here we obtain

$$\frac{M_s}{M} = 2\frac{f_K}{f_\pi} - 1 = 1.46. \quad (66)$$

Furthermore, the two eqs. \[21\] can be used to find the values of $\Lambda$ and $M$ if the parameters $\hat{m}, m_s, G, \kappa, \kappa_2, g_1, \ldots, g_7$ are known. Thus, together with $g_8$ we have at this stage thirteen couplings to be fixed. Let us consider the current quark masses $\hat{m}$ and $m_s$ to be an input. Their values are known, from various analyses of the chiral treatment of the light pseudoscalars, to be around $\hat{m} = 4$ MeV and $m_s = 100$ MeV \[65\]. Then the remaining eleven couplings can be found by comparing with empirical data. One should stress the possibility (which did not exist before the inclusion of mass-dependent interactions) to fit the low lying pseudoscalar spectrum, $m_\pi = 138$ MeV, $m_K = 494$ MeV, $m_\eta = 547$ MeV, $m_{\eta'} = 958$ MeV, the weak pion and kaon decay constants, $f_\pi = 92$ MeV, $f_K = 113$ MeV, and the singlet-octet mixing angle $\theta_p = -15^\circ$ to perfect accuracy, see Table I.

One can deduce that the couplings $\kappa_2$ and $g_8$ are essential to improve the description in the pseudoscalar sector; in particular, $g_8$ is responsible for fine tuning the $\eta - \eta'$ mass splitting, see also Table II, where the input $g_8$ between set (b) and sets (a,c,d) is due to the input $\theta_p = -15^\circ$ versus $\theta_p = -12^\circ$ respectively.

The remaining five conditions are taken from the scalar sector of the model. Unfortunately, the scalar channel in the region about 1 GeV became a long-standing problem of QCD. The abundance of meson resonances with $0^{++}$ quantum numbers shows that one can expect the presence of non-$q\bar{q}$ scalar objects, like glueballs, hybrids, multiquark states and so forth \[11\]. This creates known difficulties in the interpretation and classification of scalars. For instance, the numerical attempts to organize the $U(3)$ quark-antiquark nonet based on the light scalar mesons, $\sigma$ or $f_0(600)$, $a_0(980)$, $\kappa(800)$, $f_0(980)$, in the framework of NJL-type models have failed (see, e.g. \[8\] [10] [58] [66] - [68]). The reason is the ordering of the calculated spectrum which typically is $m_\sigma < m_{a_0} < m_\kappa < m_{f_0}$, as opposed to the empirical evidence: $m_\kappa < m_{a_0} \simeq m_{f_0}$.

On the other hand, it is known that a unitarized non-relativistic meson model can successfully describe the light scalar meson nonet as $q\bar{q}$ states with a meson-meson admixture \[33\]. Another model which assumes the mixing of $q\bar{q}$-states with others, consisting of two quarks and two antiquarks, $q^2\bar{q}^2$ \[29\], yields a possible description of the $0^{++}$ meson spectra as well \[38\] [39]. The well known model of Close and Törnqvist \[40\] is also designed to describe two scalar nonets (above and below 1 GeV). The light scalar nonet below 1 GeV has a core made of $q^2\bar{q}^2$ states with a small admixture of a $q\bar{q}$ component, rearranged asymptotically as meson-meson states. These successful solutions seemingly indicate on the importance of certain admixtures for the correct description of the light scalars. Our model contains such admixtures in the form of the appropriate effective multi-quark vertices with the asymptotic meson states described by the bosonized $q\bar{q}$ fields. We have found, that the quark mass dependent interactions can solve the problem of the light scalar spectrum and these masses can be understood in terms of spontaneous and explicit chiral symmetry breaking only. Indeed, one can easily fit the data: $m_\pi = 600$ MeV, $m_{a_0} = 980$ MeV, $m_\kappa = 850$ MeV, $m_{f_0} = 980$ MeV. In this case we obtain for the singlet-octet mixing angle $\theta_p$ roughly $\theta_s = 19^\circ$ \[48\]. Without changing the mass spectra better fits for the strong radiative decays of the scalars are obtained with $\theta_s = 25^\circ \pm 28^\circ$, in the next subsection.

We obtain and understand the empirical mass assignment inside the light scalar nonet as a consequence of the quark-mass dependent interactions, i.e. as the result of some predominance of the explicit chiral symmetry breaking terms over the dynamical chiral symmetry breaking ones for these states. Indeed, let us consider the difference

$$m_{a_0}^2 - m_\kappa^2 = 2g_2^2 \left( \frac{1}{H_{a_0}} - \frac{1}{H_\kappa} \right) - 2(M_s + 2\hat{M})(M_s - \hat{M}). \quad (67)$$

The sign of this expression is a result of the competition of two terms. In the chiral limit both of them are zero, since at $\bar{\mu}, \mu_s = 0$ we obtain $\hat{M} = M_s$ and $H_{a_0} = H_\kappa$, for $M_{a_0}$ and $H_\kappa$ being positive. The splitting $H_\kappa > H_{a_0}$ is a necessary condition to get $m_{a_0} > m_\kappa$. The following terms contribute to the difference

$$H_\kappa - H_{a_0} = \kappa(h_s - \hat{h}) + 2\kappa_2(\mu_s - \hat{\mu}) - g_2(h_s^2 + hh_s - 2\hat{h}^2) + \frac{g_3}{2} \left( 2\mu_s h_s + \mu_s h + \hat{\mu}h_s - 4\hat{\mu}\hat{h} \right) + g_5\hat{\mu}(\mu_s - \hat{\mu}) + \frac{g_6}{2} \left( \mu_s^2 - \hat{\mu}^2 \right). \quad (68)$$

Accordingly, from this formula we deduce the “anatomy” of the numerical fit, e.g. for set (d) (see next subsection):

$$m_{a_0}^2 - m_\kappa^2 = \begin{align*}
& (0.006)_\kappa + (0.046)_{\kappa_2} + (6 \times 10^{-4})_{g_2} \\
& + (0.938)_{g_3} + (0.003)_{g_5} + (-0.316)_{g_6} \\
& - (0.44)_{\hat{M}} = 0.24 \text{ GeV}^2,
\end{align*} \quad (69)$$

where the contributions of terms with corresponding coupling (see eq. \[68\]) are indicated in square brackets. The last number, marked by $\hat{M}$, is the value of the last term from \[67\]. It is a contribution due to the dynamical chiral symmetry breaking (in the presence of an explicit chiral
TABLE I: The same values for the pseudoscalar and scalar masses (except for $m_{a}$) and weak decay constants (all in MeV) are used as input (marked with *) for different sets of the model. Parameter sets (a),(b),(c),(d) of all following tables differ by varying the mixing angles and $m_{a}$: sets (a), (b) and (d) with $m_{a} = 550$ MeV versus set (c) with $m_{a} = 600$ MeV, sets (a),(c) and (d) with $\theta_{p} = -12^\circ$ versus set (b) with $\theta_{p} = -15^\circ$. The scalar mixing angle is kept constant, $\theta_{S} = 25^\circ$, in (a),(b),(c) and increased to $\theta_{S} = 27.5^\circ$ in set (d).

| Sets | $\tilde{m}$ | $m_{a}$ | $M$ | $M_{s}$ | $\Lambda$ | $G$ | $-\kappa$ | $g_{1}$ | $g_{3}$ | $g_{5}$ |
|------|-------------|---------|-----|---------|----------|-----|-----------|--------|--------|--------|
| a    | 4.0*        | 100*    | 372 | 541     | 830      | 9.74| 121.1     | 3136   | 133    |
| b    | 4.0*        | 100*    | 372 | 542     | 829      | 9.83| 118.5     | 3305   | -158   |
| c    | 4.0*        | 100*    | 370 | 539     | 830      | 10.45| 120.3     | 2081   | 102    |
| d    | 4.0*        | 100*    | 373 | 544     | 828      | 10.48| 122.0     | 3284   | 173    |

TABLE II: Parameter sets of the model: $\tilde{m}, m_{a},$ and $\Lambda$ are given in MeV. The couplings have the following units: $[G] = \text{GeV}^{-2}$, $[\kappa] = \text{GeV}^{-5}$, $[g_{1}] = [g_{2}] = \text{GeV}^{-8}$. We also show here the values of constituent quark masses $\tilde{M}$ and $M_{s}$ in MeV. See also caption of Table I

| Sets | $\tilde{m}$ | $m_{a}$ | $\tilde{M}$ | $M_{s}$ | $\Lambda$ | $G$ | $\kappa$ | $g_{1}$ | $g_{3}$ | $g_{5}$ |
|------|-------------|---------|-------------|---------|----------|-----|--------|--------|--------|--------|
| a    | 6.14        | 3638    | 657         | 210     | 1618     | 105 | -65    | 0*     | 0*     |
| b    | 5.61        | 6472    | 702         | 210     | 1668     | 100 | -38    | 0*     | 0*     |
| c    | 6.12        | 6214    | 464         | 207     | 1598     | 133 | -66    | 0*     | 0*     |
| d    | 6.17        | 6497    | 1235        | 213     | 1642     | 13.3| -64    | 0*     | 0*     |

TABLE III: Explicit symmetry breaking interaction couplings. The couplings have the following units: $[\kappa_{1}] = \text{GeV}^{-1}$, $[\kappa_{2}] = \text{GeV}^{-3}$, $[g_{1}] = [g_{2}] = \text{GeV}^{-6}$, $[g_{3}] = [g_{7}] = [g_{9}] = \text{GeV}^{-4}$, $[g_{5}] = [g_{10}] = \text{GeV}^{-8}$. See also caption of Table I

| Sets | $\kappa_{1}$ | $\kappa_{2}$ | $g_{3}$ | $g_{4}$ | $g_{5}$ | $g_{6}$ | $-\kappa$ | $g_{1}$ | $g_{3}$ | $g_{5}$ |
|------|--------------|--------------|--------|--------|--------|--------|-----------|--------|--------|--------|
| a    | 0*           | 6.14         | 6338   | 657    | 210    | 1618   | 105       | -65    | 0*     | 0*     |
| b    | 0*           | 5.61         | 6472   | 702    | 210    | 1668   | 100       | -38    | 0*     | 0*     |
| c    | 0*           | 6.12         | 6214   | 464    | 207    | 1598   | 133       | -66    | 0*     | 0*     |
| d    | 0*           | 6.17         | 6497   | 1235   | 213    | 1642   | 13.3      | -64    | 0*     | 0*     |

TABLE IV: Strong decays of the scalar mesons, $m_{R}$ is the resonance mass in MeV, $\Gamma^{BW}$ and $\Gamma^{FI}$ are the Breit-Wigner width and the Flatté distribution width in GeV, $R^{S} = \frac{\bar{g}_{R}^{S}}{g_{R}}$

| Set  | Decays | $m_{R}$ | $\Gamma^{BW}$ | $\Gamma^{FI}$ | $\bar{g}_{3}$ | $\bar{g}_{K}$ | $R^{S}$ | $\theta_{p}$ | $\theta_{S}$ |
|------|--------|---------|---------------|---------------|--------------|-------------|---------|-------------|-------------|
| a    | $\sigma \to \pi \pi$ | 550     | 465           | 1.95          | 0.97         | 0.497       | -12     | 25          |
|      | $\kappa \to K \pi$ | 850     | 310           | 1.2           | 0            |             |         |             |             |
| a    | $\alpha_{0} \to \eta \pi$ | 980     | 419           | 45            | 1.32         | 2.69        | 2.05    |             |             |
| b    | $\sigma \to \pi \pi$ | 550     | 465           | 1.955         | 0.986        | 0.504       | -15     | 25          |
|      | $\kappa \to K \pi$ | 850     | 310           | 0             | 0            |             |         |             |             |
| c    | $\sigma \to \pi \pi$ | 600     | 635           | 2.39          | 1.52         | 0.61        | -12     | 25          |
|      | $\kappa \to K \pi$ | 850     | 310           | 1.2           | 0            |             |         |             |             |
| d    | $\sigma \to \pi \pi$ | 550     | 461           | 1.94          | 0.63         | 0.33        | -12     | 27.5        |
|      | $\kappa \to K \pi$ | 850     | 310           | 1.2           | 0            |             |         |             |             |
related with and without the explicit symmetry breaking, a very subtle interplay takes place involving parameters. The remaining parameters have much less significant changes. We conclude that the model parameters. We study the effect of having the masses both decrease to \( \hat{\eta} \). As a result we obtain that the constituent quark masses agree with the ones found in previous considerations. Their values are not much affected by the quark mass effects. We have already learned (as seen again in Table II) that higher values of \( g_1 \) lead to the lower \( \eta \) mass. This eight-quark interaction violates Zweig’s rule, since it involves \( q \bar{q} \) annihilation.

Table II contains the standard set of parameters, which are responsible for the explicit chiral symmetry breaking effects in the interactions. Largest variations are observed in the couplings \( g_4 \) and \( g_7 \) in set (d) as compared to sets (a-c) and \( g_4 \) in \( \eta \) between set (b) and the other sets. In the former case it is related with the change of the scalar mixing angle and in the latter with the change in the pseudoscalar mixing angle. The coupling \( g_7 \) is seen to occur only in \( (\eta') \), thus it probes the mass spectrum of the scalars, whereas \( g_4 \) appears only in \( (\eta') \), related to the mass spectrum of the pseudoscalars. With all observables kept fixed, except the mixing angle, changes in these couplings are obviously related to them. Regarding \( g_4 \) it enters in both mass spectra. Comparing sets (a) and (c) where both \( \theta_s \) and \( \theta_P \) are the same, but the \( \eta \) mass different, show that \( g_4 \) responds also to the change in the \( \eta \) mass.

The calculated values of quark condensates are approximately the same for all sets: \( \langle \bar{u}u \rangle \hat{=} 232 \text{ MeV} \), and \( \langle s\bar{s} \rangle \hat{=} 204 \text{ MeV} \). Our calculated values for the constituent quark masses agree with the ones found in \[8\] \[10\] \[47\], showing their insensitivity to the new mass-dependent corrections.

In Table IV are shown the results for the strong decay widths of the scalar mesons for the four different sets. The experimental status is as follows. The mass and width of the \( \eta \) meson quoted until recently had a large uncertainty, \( m_\eta = (400 \div 1200) \text{ MeV} \) and a full width \( \Gamma_\eta = (600 \div 1000) \text{ MeV} \). Presently \[65\] it has been narrowed to \( m_\eta = (400 \div 550) \text{ MeV} \) and \( \Gamma_\eta = (400 \div 700) \text{ MeV} \). The result based on the average over the dispersion angles, as described in the caption of Table IV with all other meson masses and weak decay constants remaining fixed to the values there indicated.

### Table V: Radiative decays of the scalar mesons \( \Gamma_{S\gamma\gamma} \) in KeV, \( m_R \) is the resonance mass in MeV.

| Decays         | Set a | \( m_R \) | \( \Gamma_{S\gamma\gamma} \) | Set b | \( m_R \) | \( \Gamma_{S\gamma\gamma} \) | Set c | \( m_R \) | \( \Gamma_{S\gamma\gamma} \) | Set d | \( m_R \) | \( \Gamma_{S\gamma\gamma} \) |
|----------------|-------|----------|-----------------|-------|----------|-----------------|-------|----------|-----------------|-------|----------|-----------------|
| \( \sigma \rightarrow \gamma \gamma \) | 550   | 0.212    | 550             | 0.212 | 600      | 0.277           | 550   | 0.210    |                 |       |          |                 |
| \( f_0 \rightarrow \gamma \gamma \)    | 980   | 0.055    | 980             | 0.055 | 980      | 0.055           | 980   | 0.080    |                 |       |          |                 |
| \( a_0 \rightarrow \gamma \gamma \)    | 980   | 0.389    | 980             | 0.386 | 980      | 0.392           | 980   | 0.383    |                 |       |          |                 |

### Table VI: Anomalous decays \( \Gamma_{P\gamma\gamma} \) for sets (a) and (c) in KeV, corresponding to \( \theta_P = -12^\circ \), \( m_R \) is the particle mass in MeV. [For set (b), corresponding to \( \theta_P = -15^\circ \), we have \( \Gamma_{P\gamma\gamma} = 0.6 \text{ KeV} \), \( \Gamma_{S\gamma\gamma} = 4.8 \text{ KeV} \).]

| Decays        | Set a | \( m_R \) | \( \Gamma_{P\gamma\gamma} \) | \( \Gamma_{S\gamma\gamma} \) |
|---------------|-------|----------|-----------------|-----------------|
| \( \pi^0 \rightarrow \gamma \gamma \) | 136   | 0.00798  |                  |                 |
| \( \eta \rightarrow \gamma \gamma \)  | 547   | 0.5239   |                  |                 |
| \( \eta' \rightarrow \gamma \gamma \) | 958   | 5.225    |                  |                 |

B. Strong decays

Let us now show the result of our global fitting of the model parameters. We study the effect of having a slightly different \( m_\sigma \) mass, sets (a), (b) and (d) with \( m_\sigma = 550 \text{ MeV} \) versus set (c) with \( m_\sigma = 600 \text{ MeV} \), as well as having different pseudoscalar and scalar mixing parameters. We deviate (arbitrarily) the values of \( f_K \) and \( \eta \) from their empirical values, keeping the remaining observables fixed.

Let’s consider first the weak decays. We take set (d) as reference and change in the input data only \( f_K \) and \( f_\eta \) as well as the ordering \( m_K < m_\eta \). For these cases many parameters are at work simultaneously. To illustrate this trend, we set \( \eta \) and change in the input only the \( \eta \) mass. The largest deviation in absolute value is for \( g_2 \), which increases by 50%, followed by \( g_1 \) which decreases by 40%. The parameters \( \{ g_7, \kappa_2, g_3, g_4, g_6, \kappa \} \) decrease in the given order by \( \{ 27, 25, 25, 22, 18, 15 \} \) parts in hundred, and \( g_8 \) increases by 28%. The remaining parameters have much less significant changes. We conclude that a very subtle interplay takes place involving parameters related with and without the explicit symmetry breaking also in this case.

As for \( m_K < m_\eta \): we take again set (d) as reference and change in the input only the \( \eta \) mass, lowering it to \( \eta = 490 \text{ MeV} \). In this case the largest changes are observed in \( \{ g_7, g_6, g_2 \} \), with an increase of \( \{ 168, 162, 93 \} \) per cent and a decrease in \( \kappa_2 \) by 73%, while a lesser increase in \( \{ g_4, g_6, \kappa \} \) of \( \{ 29, 25, 20 \} \) and decrease of \( g_3 \) by 16 per cent is registered.

### B. Strong decays

Let us now show the result of our global fitting of the model parameters. We study the effect of having a slightly different \( m_\sigma \) mass, sets (a), (b) and (d) with \( m_\sigma = 550 \text{ MeV} \) versus set (c) with \( m_\sigma = 600 \text{ MeV} \), as well as having different pseudoscalar and scalar mixing parameters. We deviate (arbitrarily) the values of \( f_K \) and \( f_\eta \) as well as the ordering \( m_K < m_\eta \). For these cases many parameters are at work simultaneously. To illustrate this trend, we set \( \eta \) and change in the input only the \( \eta \) mass. The largest deviation in absolute value is for \( g_2 \), which increases by 50%, followed by \( g_1 \) which decreases by 40%. The parameters \( \{ g_7, \kappa_2, g_3, g_4, g_6, \kappa \} \) decrease in the given order by \( \{ 27, 25, 25, 22, 18, 15 \} \) parts in hundred, and \( g_8 \) increases by 28%. The remaining parameters have much less significant changes. We conclude that a very subtle interplay takes place involving parameters related with and without the explicit symmetry breaking also in this case.
TABLE VII: The coefficients $\text{coef}^{HK}$ and $\text{coef}^{SPA}$ of the heat kernel and of the SPA contributions to the total value of the coupling $g_{S\pi f p_z}$ resulting from the interaction Lagrangian for the open decay channels. Values are for the neutral channels. Units are in GeV.

| $g_{S\pi f p_z}$ | $\text{coef}^{HK}/g^3$ | $\text{coef}^{SPA}/g^3$ | total/$g^3$ |
|-----------------|------------------------|------------------------|------------|
| $\sigma\pi^0\pi^0$ | -0.0450 | 0.0215 | -0.0235 |
| $f_0\pi^0\pi^0$ | -0.0061 | -0.0047 | -0.0109 |
| $\kappa^0K\pi^0$ | 0.0660 | -0.0257 | 0.0403 |
| $a_0^0\rho\pi^0$ | -0.0666 | -0.0178 | -0.0844 |

TABLE VIII: The coefficients $\text{coef}^{HK}$ and $\text{coef}^{SPA}$ of the heat kernel and of the SPA contributions to the total value of the coupling $g_{S\pi f p_z}$ resulting from the interaction Lagrangian for the open decay channels. Values are for the neutral channels. Units are in GeV.

| $g_{S\pi f p_z}$ | $\text{coef}^{HK}/g^3$ | $\text{coef}^{SPA}/g^3$ | total/$g^3$ |
|-----------------|------------------------|------------------------|------------|
| $\sigma K\pi$ | -0.041 | 0.0178 | -0.0232 |
| $f_0 K\pi$ | 0.118 | -0.081 | 0.0372 |
| $a_0^0 K\pi$ | 0.0246 | 0.0968 | 0.121 |

analysis of [69, 72] leads even to a very sharp value for the pole position $M - i\Gamma/2 = (446 \pm 6) - (276 \pm 5)$ MeV. The mass and full width of the $f_0(980)$ meson are quoted as $m_{f_0(980)} = 990 \pm 20$ MeV and $\Gamma_{f_0(980)} = 40 \pm 100$ MeV and for the $a_0(980)$ meson as $m_{a_0(980)} = 980 \pm 20$ MeV and $\Gamma_{a_0(980)} = 50 \pm 100$ MeV. The results for the $\kappa(800)$ quoted in the PDG table from a Breit-Wigner fit have the pole at $(764 \pm 63^{+71}_{-54}) - i(306 \pm 149^{+143}_{-82})$ MeV.

We obtain that the $\sigma$ mass and $\sigma \to \pi \pi$ decay are within the recent limits for sets (a-b) and (d) while set (c) has a mass larger than the upper limit by $\sim 50$ MeV. While in set (a-b) and (d) the calculated width is smaller than the nominal mass of the resonance, the opposite behavior is seen in set (c). The coupling strength $g_{\sigma \pi \pi}$ increases comparing e.g. set(a) to (c) explaining the larger width, however the ratio $R^\sigma = \frac{g_{\sigma \pi \pi}}{g_{\sigma \pi \eta}}$ of the $\sigma$ to kaon and to the pion couplings also increases by 20%. The obtained ratios for $R^\sigma$ are in agreement with the experimental value $R^\sigma_{exp} = 0.5 \pm 0.1$ in [73] for sets (a-c) and slightly below for set (d). We expect some effect on the width if these channels were taken into account, but only a moderate one since the coupling to pions dominates, $R^\sigma \sim 0.3 \pm 0.5$.

The decay width for $\kappa(800) \to K\pi \sim 310$ MeV is smaller roughly by a factor two than the quoted central value but lies still within the limits. The ratio of the couplings $\frac{g_{\kappa \pi \eta}}{g_{\kappa \pi \pi}} \frac{m_{\kappa}^2}{m_\eta^2} = 1.5$ (the ratio of meson masses corrects for the different definitions of the couplings in [73]) is within the experimental values in [73], as opposed to the $qq$ and $q^2\bar{q}^2$ model approaches considered in the same paper.

The widths of the $a_0(980) \to \pi \eta$ and $f_0(980) \to \pi \pi$ decays are well accomodated within a Flatté description. We read the width at half maximum of the elastic cross section in Figs. 1 and 2, respectively. Note the huge reduction in width in the case of the $a_0(980)$ meson when the kaon channels are taken into account. This possibility was already noticed by Flatté in his analysis [60]. This is explained in our description by the ratio $R^{a_0} \sim 2$ showing the dominant component to be in the coupling to the kaons.

As demonstrated in [74] the ratio $R^S = \frac{g_{S\pi \eta}}{g_{S\pi \pi}}$ of the couplings is a relatively stable quantity in despite of the large fluctuations in the experimental values extracted for the individual couplings. Our calculated $R^S$ are compatible with the indicated values in [74]. It should be emphasized that the ratio $R^{f_0} = \frac{g_{f_0 \pi \eta}}{g_{f_0 \pi \pi}}$ is strongly dependent on the mixing angle $\theta_S$ of the scalar sector. As can be seen comparing sets (a-c) with set (d) the increase in $\theta_S$ is responsible for the larger ratio $R^{f_0} = 3.9$ in set (d), which agrees well with the experimental value $R^{f_0}_{exp} = 4.21 \pm 0.46$ of BES [75]. An often considered quantity is the crossed ratio $r = \frac{R^{f_0}}{R^{\pi \pi}}$, usually assumed to be larger than unity. The $a_0(980)$ does not depend on the $\theta_S$ mixing angle (an eventual correlation with the $f_0(980)$ meson through isospin mixing is discarded here), but does depend on the pseudoscalar $\theta_P$ angle through its decay into the $\pi\eta$. The $\theta_P$ is fixed in the pseudoscalar sector to yield the correct $\pi\eta$ and $\eta'$ masses, as well as their radiative two photon decay widths. Therefore the ratio $R^{a_0}$ of the $a_0$ couplings to kaons and to the $\pi\pi$ channels remains approximately constant for all parameter sets ($R^{a_0}_{exp}^{-1} \sim 0.5$. This value is not too bad in comparison with the experimental quoted ratio ($R^{a_0}_{exp}^{-1} = 0.75 \pm 0.11$). Requiring the ratio $r > 1$ constrains further the angle to be larger than $\theta_S \sim 26^\circ$.

On the other hand the ratio $R^{f_0}$ increases until $\theta_S$ reaches ideal mixing. In the interval $\theta_{id} < \theta_S < \frac{\pi}{2}$ it decreases but stays much larger than the experimental accepted ratio, e.g. at $\theta_S = 44^\circ$ one has $R^{f_0} \sim 11$. The combined requirement $r > 1$ and $R^{f_0}_{exp}$ confines the mixing angle to the narrow window $27^\circ < \theta_S < 28^\circ$. From the point of view of the calculated strong decay widths however the somewhat smaller angle $\theta_S = 25^\circ$ is also acceptable. Our interval of values for the mixing angle
25° < θ_S < 28°, corresponding to −10.3° < ψ < −7.3° are within the values −14° < ψ < −3° estimated in [77], more specifically ψ ∼ −9° if a Flatté distribution is used in a complementarity approach of Chiral Perturbation Theory and the Linear Sigma Model.

C. Radiative decays

The two photon decays of the pseudoscalars are in very good agreement with data, (Table [VI], the π⁰ and η in two photons are within the experimental error bars, the η' decay lies 10% above the upper limit for sets (a), (c) and (d), i.e. θ_P = −12°. In the case of set (b), θ_P = −15°, the result for the η' decay is at the upper margin, and for the η about 10% above the upper boundary.

For the radiative widths of the σ, see Table [V] there is a large spread in the experimental data from different facilities. Our results for σ → γγ only account for about 20% of the value (1.2 ± 0.4) keV [79] obtained from the nucleon electromagnetic polarizabilities, which is one of the lower estimates for this width. For the f₀(980) → γγ the PDG average is quoted as (0.29±0.07) keV. Sets (a-c) yield approximately 20% and set (e) 30% of this value. These results meet the current expectations that a direct coupling to the photons via a quark loop are not sufficient to account for the observed radiative widths of these mesons.

A natural question arises then why in our approach the strong widths can be described reasonably well in all channels and the radiative ones fall short of the empirical values for the σ, f₀ decays. This can be understood: only the strong decays probe directly the multi-quark couplings gₘ contained in the stationary phase (SPA) piece [28] of the total interaction Lagrangian [49]. Since this part of the Lagrangian has no derivative terms only the heat kernel (HK) Lagrangian involves the electromagnetic interaction, after minimal coupling. The information of the SPA conditions which leaks through the gap equations to the electromagnetic sector is rather weak; it is contained only in the wave function normalization which is the same for all mesons, and the quark constituent masses and scale Λ which remain approximately constant in all parameter sets. Thus, effectively, the two photon decays of the scalars yield a clean signature whether the electromagnetic decay of the mesons proceeds dominantly through a qq channel or not.

This in turn ties up with the strength distribution in the HK and SPA contributions to the coupling gₘP shown in Tables VII and VIII for set (d). The HK piece relates directly to the meson-qq channel, the SPA part to the higher order multiquark interactions.

Consider first the a₀ meson: the calculated a₀(980) → γγ ∼ 0.39 keV overestimates the present average PDG value 0.21±0.08 and points within our approach to the dominance of the direct one quark loop coupling to photons of this meson.

This is corroborated by the fact that the large bare width that we obtain for the a₀ → πη decay is shown to stem mainly from the HK coefficient represented with 80% of the total strength, see Table VIII. The a₀ meson in the q̅q picture is composed only of u and d quarks, thus its coupling to the K̅K mesons requires a flavor change at the kaon vertices, as opposed to the ηπ case. As can be seen from a similar decomposition in HK and SPA contributions of the a₀K̅K coupling in Table VIII it is much more favorable to couple to the kaons through the multiquark vertices, which now represent 80% of the total strength instead. Therefore for the overall strong decay width it is important to take this mode into account through the two-channel Flatté distribution. From the point of view of the two photon decay of a₀, we note that a πη loop does not couple directly to two photons [80] and the decay proceeds through the quark loop of u or d quarks with the large strength of the corresponding HK component. To access the dominant SPA component the two photon decay would have to proceed through coupling to the K̅K loop, a sub-leading process in N_c counting as compared to the direct q̅q loop. Furthermore, due to the relatively large mass of the kaons, this loop is not expected to contribute significantly.

Now let us analyze the σ, f₀ channels: there are substantial contributions or cancellations from the SPA part. For the f₀ππ and f₀K̅K cases, one sees that the strength in the SPA coefficient is in magnitude about ² of the HK coefficient for both cases, but changes relative sign in the latter. In the σππ and σK̅K cases, the cancellations occur in both cases, with the SPA piece contributing about half of the HK part. There is a subtle interplay about the HK and SPA coefficients which finally add up to the correct description of the mass spectra and strong decays of these mesons. The lack of a pronounced dominance of the HK has as consequence that the q̅q coupling of these mesons to the photons represents only a fraction of the total width. The remaining strength must derive from the multiquark channels which should be included in an extra step, taking into account explicitly meson loop contributions.

Regarding the strong decay of the f₀, one can further infer that because of the stronger participation of the multi-quark interactions and because of cancellations in the kaon channel as opposed to the pion channel, a coupling to the kaon channel through the Flatté approach is not imperative to obtain a reasonable magnitude of the width, as seen from the Table IV.

Rescattering effects have been shown in several approaches to yield the main contribution, e.g. for the σ → γγ extracted from the dispersion analysis of γγ → π⁰π⁰ [80]. Claims for a tetraquark structure [29] of the σ meson were forwarded e.g. in [81], and in [82] interpreted as pion and kaon loop contributions. Our approach sheds light on these phenomena from a different angle.

Finally we mention that the radiative decays of the scalar mesons have been calculated a long time ago in a variant of the NJL model, with and without meson loop contributions, [83]. The amplitudes differ from ours...
in two key aspects: we use the unified description for all non-anomalous decays based on the generalized heat kernel approach which leads (i) to a common wave function normalization for all mesons that implies the reduction factor of ~ \( \frac{2}{3} \) in the amplitude and in the case of the radiative decays to (ii) the regularized one loop integrals carrying the factors \( \left( \frac{\Lambda^2}{\Lambda^2 + M^2} \right)^2 \), in despite of the integrals being finite. The latter reduces the amplitude by approximately half. The combined effect is a dramatic reduction by a factor ~ 10 in the decay widths, as compared to \( \text{SM} \) for the quark loop contribution. Thus caution must be used when it comes to interpret and comparing our numerical results with seemingly related model calculations, e.g. [84], [85].

Summarizing the results of sections IV B. and C., the strong decays calculated from our tree level meson couplings encode leading and higher order \( N_c \) and multi-quark effects in combinations that account for the main bulk of the empirical widths. The two photon decays of the scalars at leading order of the bosonized Lagrangian yield complementary information, testing whether the direct one quark loop coupling to photons is the dominant decay process. We obtained that the \( a_0 \) meson decay into two photons proceeds mainly through the \( q\bar{q} \) loop, whereas for the \( \sigma, f_0 \) mesons we conclude that higher order multi-quark interactions are necessary to account for the observed widths. This does not mean that the \( a_0 \) meson is mainly a \( q\bar{q} \) state, but that the multi-quark component with the large strength in the two kaon channel, important for the reduction of the \( a_0\pi\eta \) strong decay width, is not the leading process in the two photon decay of this meson.

![FIG. 1: The \( \pi\pi \) cross section as function \( E = \sqrt{s} - 2m_K \) for the \( a_0 \) resonance channel from the Flatté distribution (solid line) with parameters of set (b), \( \hat{g}_{0,\pi\pi} = 1.44, \hat{g}_{K}^{a_0} = 2.8, \hat{R}^{a_0} = 1.944 \). The width read at half peak value is \( \Gamma^{F_1} = 50 \) MeV. Dashed line corresponds to the single \( \pi\eta \) channel.](image1)

![FIG. 2: The \( \pi\pi \) cross section as function \( E = \sqrt{s} - 2m_K \) for the \( f_0 \) resonance channel from the Flatté distribution (solid line) with parameters of set (b), \( \hat{g}_{f_0,\pi\pi} = 0.23, \hat{g}_{K}^{f_0} = 0.31, \hat{R}^{f_0} = 1.36 \). The width read at half peak value is \( \Gamma^{F_1} = 60 \) MeV. Dashed line corresponds to just the two pion channel.](image2)

V. CONCLUDING REMARKS

In this paper we have generalized the effective multi-quark Lagrangians of the NJL type by including higher order terms in the current quark-mass expansion. The procedure is based on the very general assumption that the scale of spontaneous chiral symmetry breaking determines the hierarchy of local multi-quark interactions. As a consequence, one can distinguish a finite subset of vertices which are responsible for the explicit chiral symmetry breaking at each order considered. We have classified these vertices at next to leading order and studied the phenomenological consequences of their inclusion in the Lagrangian.

We are led to a subset of ten quark-mass dependent interactions which enter the Lagrangian at the same order as the ’t Hooft determinant and eight quark terms previously analyzed in the literature. From these, three are related with the Manohar-Kaplan ambiguity, and the remaining seven with genuinely new vertices. These new terms carry either signatures of violation of the Zweig-rule or of admixtures of \( q^2\bar{q}^2 \) states to the quark-antiquark ones and are thus potentially interesting candidates in the quest of analyzing the structure and interaction dynamics of the low lying mesons.

We have derived the bosonized Lagrangian up to cubic order in the meson fields, from which we obtain the meson spectra and their two body strong, weak and electromagnetic decays. Here are our main conclusions:

1. We fit the low lying pseudoscalar spectrum (the pseudo Goldstone \( 0^{-+} \) nonet) and weak decay constants of the pion and the kaon to perfect accuracy. The fitting of the \( \eta - \eta' \) mass splitting together with the overall successful description of the whole set of low-energy pseudoscalar characteristics is actually a solution for a long
standing problem of NJL-type models. We have found that the quark mass dependent interaction terms mainly responsible for the fit belong to the class of OZI-violating interactions. They represent additional corrections to the ’t Hooft $U_A(1)$ breaking mechanism. In the interaction terms independent of the quark masses, we observe however that the $g_2$ coupling of the non OZI-violating $8q$ interactions carrying the signature of the $q^2q^2$ states are also relevant in fitting the $f_8$, $f_K$ values as well as for the ordering $m_{K^0} < m_{\eta'}$.

(2) We are also capable to describe the spectrum of the light scalar nonet. In this case we identify the quark-mass independent interaction term studied in earlier papers dictates the mass of the $\sigma(500)$ meson, we conclude that these states are strongly affected by OZI-violating short range forces.

(3) Regarding the mixing angle of the singlet-octet scalar states $\theta_S$ we have found that its value is particularly sensitive to the interaction term proportional to $g_1$, which is OZI-violating. Together with the result that the strength $g_1$ of the eight quark OZI-violating and quark mass independent interaction term studied in earlier papers dictates the mass of the $\sigma(500)$ meson, we conclude of accounting for the decay widths within the actual margins of empirical data. We corroborate other model calculations in which the coupling of the $f_0(980)$ and $a_0(980)$ mesons to the $KK$ channel is needed for the description of the decays $f_0(980) \to \pi\pi$ and $a_0(980) \to \pi\eta$. We find that this coupling is most crucial for the latter process.

(5) The radiative decays of the scalar mesons into two photons show that the main channel for the $a_0(980)$ decay proceeds through coupling to a quark-antiquark state, while the radiative decays of singlet-octet states $\sigma, f_0$ must proceed through more complex structures. We refer to the full discussion given in sections IV B and IV C.

(6) Finally, the radiative decays of the pseudoscalars are in very good agreement with data.

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[1] Y. Nambu, G. Jona-Lasinio, Phys. Rev. 122, 345 (1961); 124, 246 (1961).
[2] V. G. Vaks, A. I. Larkin, Zh. Éksp. Teor. Fiz. 40, 282 (1961) [Sov. Phys. JETP 13, 192 (1961)].
[3] G. ’t Hooft, Phys. Rev. D 14, 3432 (1976).
[4] G. ’t Hooft, Phys. Rev. D 18, 2199 (1978).
[5] V. Bernard, R. L. Jaffe, U.-G. Meißer, Phys. Lett. B 198, 92 (1987).
[6] V. Bernard, R. L. Jaffe, U.-G. Meißer, Nucl. Phys. B 308, 753 (1988).
[7] H. Reinhardt and R. Alkofer, Phys. Lett. B 207, 482 (1988).
[8] S. Klimt, M. Lutz, U. Vogl, W. Weise, Nucl. Phys. A 516, 429 (1990).
[9] U. Vogl, M. Lutz, S. Klimt, W. Weise, Nucl. Phys. A 516, 469 (1990).
[10] U. Vogl, W. Weise, Progr. Part. Nucl. Phys. 27, 195 (1991).
[11] M. Takizawa, K. Tsushima, Y. Kohyama, K. Kubodera, Nucl. Phys. A 507, 611 (1990).
[12] S. P. Klevansky, Rev. Mod. Phys. 64, 649 (1992).
[13] T. Hatsuda, T. Kunihiro, Phys. Rep. 247, 221 (1994).
[14] V. Bernard, A. H. Blin, B. Hiller, U.-G. Meißer, M. C. Ruivo, Phys. Lett. B 305, 163 (1993).
[15] V. Dmitrasinovic, Nucl. Phys. A 686, 379 (2001).
[16] M. C. Birse, T. D. Cohen, J. A. McGovern, Phys. Lett. B 388, 137 (1996).
[17] K. Naito, M. Oka, M. Takizawa, T. Umekawa, Progr. Theor. Phys. 109, 969 (2003).
[18] A. A. Osipov, B. Hiller, J. da Providência, Phys. Lett. B 634, 48 (2006).
[19] A. A. Andrianov, V. A. Andrianov, Theor. Math. Phys. 94, 3 (1993).
[20] A. A. Andrianov, V. A. Andrianov, Int. J. of Mod. Phys. A 8, 1981 (1993).
[21] D. Ebert, H. Reinhardt, Nucl. Phys. B 271, 188 (1986).
[22] B. Jijins, C. Bruno, E. de Rafael, Nucl. Phys. B 390, 501 (1993).
[23] Z. Zhang, T. Kunihiro, arXiv:1005.1882 (2010).
[24] J. Gasser, H. Leutwyler, Phys. Lett. B 87, 77 (1982).
[25] Weinberg, Physica A 96, 327 (1979).
[26] H. Pagels, Phys. Rev. C 16, 219 (1975).
[27] J. Gasser, H. Leutwyler, Ann. of Phys. 158, 142 (1984).
[28] J. Gasser, H. Leutwyler, Nucl. Phys. B 250, 465 (1985).
[29] R. J. Jaffe, Phys. Rev. D 15, 267 (1977); R. J. Jaffe, Phys. Rev. D 15, 281 (1977).
[30] D. Black, A. H. Fariborz, E. Sannino, J. Schechter, Phys. Rev. D 59, 074026 (1999).
[31] D. Wong, K. F. Liu, Phys. Rev. D 21, 2039 (1980).
[32] S. Narrison, Phys. Lett. B 175, 88 (1986).
[33] E. van Beveren, T. A. Rijken, K. Metzger, C. Dullemond, G. Rupp, J. E. Ribeiro, Zeit. Phys. C 30, 615 (1986).
[34] J. I. Latorre, P. Pascoal, J. Phys. G 11, L231 (1985).
[35] M. Alford, R. L. Jaffe, Nucl. Phys. B 509, 312 (1998).
[36] N. N. Achasov, S. A. Devyanyin, D. N. Shestakov, Z. Phys. C 16, 55 (1984).
[37] N. Isgur, J. Weinstein, Phys. Rev. D 40, 2236 (1990).
[38] A. H. Fariborz, R. Jora, J. Schechter, Phys. Rev. D 77,
[39] A. H. Fariborz, R. Jora, J. Schechter, Phys. Rev. D 79, 074014 (2009).

[40] F. E. Close, N. A. Törnqvist, J. Phys. G: Nucl. Part. Phys. 28, R249 (2002); arXiv:hep-ph/0204205.

[41] E. Klempt, A. Zaitsev, Phys. Rep. 454, 1 (2007).

[42] A. Ospiov, B. Hiller, A. H. Blin, J. da Providência, Phys. Lett. B 650, 262 (2007).

[43] R. Gatto, M. Ruggieri, Phys. Rev. D 82, 054027 (2010).

[44] R. Gatto, M. Ruggieri, Phys. Rev. D 83, 034016 (2011).

[45] M. Frasca, M. Ruggieri, Phys. Rev. D 83, 094024 (2011).

[46] R. Gatto, M. Ruggieri, arXiv:1207.3190 to appear in Lect. Notes Phys. "Strongly interacting matter in magnetic fields" (Springer), edited by D. Kharzeev, K. Landsteiner, A. Schmitt, H.-U. Yee.

[47] A. Manohar, H. Georgi, Nucl. Phys. B 234, 189 (1984).

[48] A. A. Osipov, B. Hiller, A. H. Blin, Eur. Phys. J. A 49, 14 (2013).

[49] D. B. Kaplan, A. V. Manohar, Phys. Rev. Lett. 56, 2004 (1986).

[50] H. Leutwyler, Nucl. Phys. B 337, 108 (1990).

[51] J. F. Donoghue, D. Wyler, Phys. Rev. D 45, 892 (1992).

[52] H. Leutwyler, Phys. Lett. B 374, 163 (1996).

[53] A. A. Osipov, B. Hiller, A. H. Blin, J. da Providência, Ann. of Phys. 322, 2021 (2007).

[54] A. A. Osipov, B. Hiller, Phys. Lett. B 515, 458 (2001).

[55] A. A. Osipov, B. Hiller, Phys. Rev. D 63, 094009 (2001).

[56] A. A. Osipov, B. Hiller, Phys. Rev. D 64, 087701 (2001).

[57] M. K. Volkov, A. A. Osipov, Sov. J. Nucl. Phys. 41, 500 (1985).

[58] A. A. Osipov, H. Hansen, B. Hiller, Nucl. Phys. A 745, 81 (2004).

[59] P. Kroll, Int. J. Mod. Phys. A 20, 331 (2005).

[60] S. M. Flatté, Phys. Lett. B 63, 224 (1976); S. M. Flatté, Phys. Lett. B 63, 228 (1976).

[61] S. Adler, Phys. Rev. 177, 2426 (1969).

[62] J. S. Bell, R. Jackiw, Nuovo Cimento A 60, 47 (1969).

[63] S. L. Adler and W. A. Bardeen, Phys. Rev. 182, 1517 (1969).

[64] B. Bajc, A. H. Blin, B. Hiller, M. C. Nemes, A. A. Osipov, M. Rosina, Nucl. Phys. A 604, 406 (1996).

[65] J. Beringer et al., Particle Data Group, Phys. Rev. D 86, 010001 (2012).

[66] M. K. Volkov, Ann. Phys. 157, 282 (1984).

[67] M. K. Volkov, Fiz. Elem. Chasits At. Yadra 17, 433 (1986).

[68] M. X. Su, L. Y. Xiao, H. Q. Zheng, Nucl. Phys. A 792, 288 (2007).

[69] G. Colangelo, J. Gasser and H. Leutwyler, Nucl. Phys. B 603, 125 (2001).

[70] I. Caprini, G. Colangelo and H. Leutwyler, Phys. Rev. Lett. 96, 132001 (2006).

[71] R. Garcia-Martin, R. Kaminski, J. R. Pelaez and J. Ruiz de Elvira, Phys. Rev. Lett. 107, 072001 (2011).

[72] B. Moussalam, Eur. Phys. J. C 71, 1814 (2011).

[73] D.V. Bugg, Eur. Phys. J. C 47, 57 (2006); D. V. Bugg, Eur. Phys. J. C 47, 45 (2006).

[74] V. Baru, J. Haidenbauer, C. Hanhart, A. Kudryavtsev, U.-G. Meißner, Eur. Phys. J. A 23, 523 (2005).

[75] M. Ablikim et al., Phys. Let. B 607, 43 (2005).

[76] B. Hyams et al., Nucl. Phys. B 64, 134 (1973).

[77] A. Bramon, R. Escrivan, J.L. Lucio M, M. Napsuciale, G. Pancheri, Eur. Phys. J. C 26, 253 (2002).

[78] N.N. Achasov and G. N. Shestakov, Phys. Rev. D 81, 094029 (2010).

[79] J. Bernabeu and J. Prades, Phys. Rev. Lett. 100, 241804 (2008).

[80] J. A. Oller, L. Roca, C. Schat, Phys. Lett B 659, 201 (2008).

[81] F. Giacosa, Phys. Rev. D 74, 014028 (2006).

[82] N.N. Achasov and G. N. Shestakov, Phys. Rev. D 77, 074020 (2008).

[83] D. Ebert, T. Feldmann, M. K. Volkov, Int. J. Mod. Phys. A 12, 4399 (1997).

[84] M. K. Volkov, E. A. Kuraev, Yu. M. Bystritsky, arXiv:0904.2484.

[85] M. Schumacher, J. Phys. G 38, 083001 (2011).

[86] In absence of the vector mesons, Their inclusion leads to this possibility through additional vertices $V\gamma$, where $V,P$ stand for vector and pseudoscalar mesons respectively [78].