STABILITY OF EIGENVALUES FOR VARIABLE EXPONENT PROBLEMS

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ABSTRACT. In the framework of variable exponent Sobolev spaces, we prove that the variational eigenvalues defined by inf sup procedures of Rayleigh ratios for the Luxemburg norms are all stable under uniform convergence of the exponents.

1. INTRODUCTION AND MAIN RESULT

The differential equations and variational problems involving $p(x)$-growth conditions arise from nonlinear elasticity theory and electrorheological fluids, and have been the target of various investigations, especially in regularity theory and in nonlocal problems (see e.g. [1–3, 9, 15, 31] and the references therein). Let $\Omega \subset \mathbb{R}^N$, with $N \geq 2$, be a bounded domain with Lipschitz boundary and let $p : \overline{\Omega} \to \mathbb{R}^+\setminus\{0\}$ be a continuous function such that

\begin{equation}
1 < p_- := \inf_{\Omega} p \leq p(x) \leq \sup_{\Omega} p =: p_+ < N \quad \text{for all } x \in \Omega.
\end{equation}

We also assume that $p$ is log-Hölder continuous, namely

\begin{equation}
|p(x) - p(y)| \leq -\frac{L}{\log |x - y|}
\end{equation}

for some $L > 0$ and for all $x, y \in \Omega$, with $0 < |x - y| \leq 1/2$. From now, we denote by

$$
\mathcal{C} := \{ p \in C(\overline{\Omega}) : p \text{ satisfies (1.1) and (1.2)} \}
$$

the set of admissible variable exponents. The goal of this paper is to study the stability of the (variational) eigenvalues with respect to (uniform) variations of $p$ for the problem

\begin{equation}
- \text{div} \left( p(x) \left| \frac{\nabla u}{K(u)} \right|^{p(x)-2} \frac{\nabla u}{K(u)} \right) = \lambda S(u)p(x) \left| u \right|^{p(x)-2} \frac{u}{k(u)}, \quad u \in W^{1,p(x)}_0(\Omega),
\end{equation}

where we have set

$$
K(u) := \|\nabla u\|_{p(x)}, \quad k(u) := \|u\|_{p(x)}, \quad S(u) := \frac{\int_{\Omega} p(x) \left| \frac{\nabla u}{K(u)} \right|^{p(x)} dx}{\int_{\Omega} p(x) \left| \frac{u}{k(u)} \right|^{p(x)} dx}.
$$

Following the argument contained in [20, Section 3], it is possible to derive equation (1.3) as the Euler-Lagrange equation corresponding to the minimization of the Rayleigh ratio

\begin{equation}
\frac{K(u)}{k(u)} = \frac{\|\nabla u\|_{p(x)}}{\|u\|_{p(x)}}, \quad \text{among all } u \in W^{1,p(x)}_0(\Omega) \setminus \{0\},
\end{equation}

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where \( \| \cdot \|_{p(x)} \) denotes the Luxemburg norm of the variable exponent Lebesgue space \( L^{p(x)}(\Omega) \) (see Section 2). This minimization problem has been firstly introduced in [20] as an appropriate replacement for the inhomogeneous minimization problem

\[
\frac{\int_{\Omega} |\nabla u|^{p(x)} dx}{\int_{\Omega} |u|^{p(x)} dx}, \quad \text{among all } u \in W^{1,p(x)}_0(\Omega) \setminus \{0\},
\]

which was previously considered in [19] to define the first eigenvalue \( \lambda_1 \) of the \( p(x) \)-Laplacian. In [19], sufficient conditions for \( \lambda_1 \) defined in this way to be zero or positive are provided. In particular, if \( p(\cdot) \) has a strict local minimum (or maximum) in \( \Omega \), then \( \lambda_1 = 0 \). Arguing as in [20, Lemma A.1], it can be shown that the functionals \( k \) and \( K \) are differentiable with

\[
\langle K'(u), v \rangle = \frac{\int_{\Omega} p(x) \left| \frac{\nabla u}{K(u)} \right|^{p(x)-2} \frac{\nabla u}{K(u)} \cdot \nabla v \, dx}{\int_{\Omega} p(x) \left| \frac{\nabla u}{K(u)} \right|^{p(x)} \, dx}
\]

for all \( u, v \in W^{1,p(x)}_0(\Omega) \),

\[
\langle k'(u), v \rangle = \frac{\int_{\Omega} p(x) \left| \frac{u}{k(u)} \right|^{p(x)-2} \frac{u}{k(u)} v \, dx}{\int_{\Omega} p(x) \left| \frac{u}{k(u)} \right|^{p(x)} \, dx}
\]

for all \( u, v \in W^{1,p(x)}_0(\Omega) \).

Therefore, all critical values of (1.4) are eigenvalues of (1.3) and vice versa. The \( m \)th eigenvalue \( \lambda^{(m)}_{p(x)} \) of (1.3) can be obtained as

\[
\lambda^{(m)}_{p(x)} := \inf_{K \in \mathcal{W}^{(m)}_{p(x)}} \sup_{u \in K} \| \nabla u \|_{p(x)} ,
\]

where \( \mathcal{W}^{(m)}_{p(x)} \) is the set of symmetric, compact subsets of \( \{ u \in W^{1,p(x)}_0(\Omega) : \| u \|_{p(x)} = 1 \} \) such that \( i(K) \geq m \), and \( i \) denotes the Krasnosel’skiǐ genus. In [20] existence and properties of the first eigenfunction were studied, while in [7] a numerical method to compute the first eigenpair of (1.3) was obtained and the symmetry breaking phenomena with respect to the constant case were observed. The growth rate of this sequence of eigenvalues was investigated in [30], getting a natural replacement for the growth estimate for the case \( p \) constant (cf. [21, 22]),

\[
\lambda^{(m)}_p \sim m^{N/p}, \quad \lambda^{(m)}_p := \inf_{K \in \mathcal{W}^{(m)}_p} \sup_{u \in K} \| \nabla u \|_p ^p,
\]

where \( \mathcal{W}^{(m)}_p \) is the set of symmetric, compact subsets of \( \{ u \in W^{1,p}_0(\Omega) : \| u \|_p = 1 \} \).

In this paper we focus on the right continuity of the maps

\[
\mathcal{E}_m : (C(\Omega), \| \cdot \|_{\infty}) \to \mathbb{R}, \quad \mathcal{E}_m(p(\cdot)) := \lambda^{(m)}_{p(x)}, \quad m \geq 1.
\]

We set

\[
\mathcal{S} := \left\{ (p_h) \subset \mathcal{C} : \exists \bar{h} \geq 1 \text{ s.t. } \sup_{h \geq \bar{h}} \sup_{\Omega} p_h < p^*_I, \quad p^*_I := \frac{Np_I}{N-p_I}, \quad p_I := \inf_{h \geq \bar{h}} \inf_{\Omega} p_h \right\}.
\]

We say that \( \mathcal{E}_m \) is right-continuous if

\[
\mathcal{E}_m(p_h(\cdot)) \to \mathcal{E}_m(p(\cdot)), \quad \text{as } h \to \infty,
\]

whenever \( p \in \mathcal{C}, (p_h) \subset \mathcal{S} \), \( p_h \to p \) uniformly in \( \Omega \) and \( p(x) \leq p_h(x) \) for all \( h \in \mathbb{N} \) and \( x \in \Omega \).
We have the following main result.

**Theorem 1.1.** $E_m$ is right-continuous for all $m \geq 1$.

**Remark 1.2.** For a constant $p \in (1, N)$, problem (1.3) reduces to the well-known eigenvalue problem for the $p$-Laplacian operator (see e.g. [24, 25]), namely

$$-\text{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u, \quad u \in W^{1,p}_0(\Omega).$$

In this particular case the continuity of variational eigenvalue has been investigated in [8, 23, 26–29] and, more recently, in [13] in presence of a weight function and including the case where the domain $\Omega$ is unbounded. With exception of [23, 26, 27], all these contributions tackle the problem by studying the $\Gamma$-convergence of the norm functionals.

**Remark 1.3.** As pointed out by Lindqvist [26, see Section 7], already in the constant case, the convergence from below of the $(p_n)$ to $p$ does not guarantee the convergence of the eigenvalues, unless the domain $\Omega$ is sufficiently smooth.

**Remark 1.4.** The same result holds replacing the Krasnosel’skiĭ genus with a general index $i$ with the following properties:

(i) $i(K)$ is an integer greater or equal than 1 and is defined whenever $K$ is a nonempty, compact and symmetric subset of a topological vector space such that $0 \notin K$;

(ii) if $X$ is a topological vector space and $K \subseteq X \setminus \{0\}$ is compact, symmetric and nonempty, then there exists an open subset $U$ of $X \setminus \{0\}$ such that $K \subseteq U$ and $i(K) \leq i(K)$ for any compact, symmetric and nonempty $\hat{K} \subseteq U$;

(iii) if $X, Y$ are two topological vector spaces, $K \subseteq X \setminus \{0\}$ is compact, symmetric and nonempty and $\pi : K \to Y \setminus \{0\}$ is continuous and odd, we have $i(\pi(K)) \geq i(K)$. Examples are the Krasnosel’skiĭ genus and the $\mathbb{Z}_2$-cohomological index [16, 17].

2. Preliminary results

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ consists of all measurable functions $u : \Omega \to \mathbb{R}$ having $\varrho_{p(x)}(u) < \infty$, where

$$\varrho_{p(x)}(u) := \int_\Omega |u(x)|^{p(x)} \, dx$$

is the $p(x)$-modular. $L^{p(x)}(\Omega)$ is endowed with the Luxemburg norm $\| \cdot \|_{p(x)}$ defined by

$$\|u\|_{p(x)} := \inf \left\{ \gamma > 0 : \varrho_{p(x)}(u/\gamma) \leq 1 \right\}.$$

The norm $\|u\|_{p(x)}$ is in close relation with the $p(x)$-modular $\varrho_{p(x)}(u)$, as shown for instance by unit ball property [18, Theorem 1.3] which we report here for completeness.

**Proposition 2.1.** Let $p \in L^\infty(\Omega)$ with $1 < p_- \leq p_+ < \infty$. Then, for all $u \in L^{p(x)}(\Omega)$ the following equivalence holds

$$\|u\|_{p(x)} < 1 \quad (= 1; > 1) \iff \varrho_{p(x)}(u) < 1 \quad (= 1; > 1).$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ consists of all $L^{p(x)}(\Omega)$-functions having distributional gradient $\nabla u \in L^{p(x)}(\Omega)$, and is endowed with the norm

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}.$$
Under the smoothness assumption (1.2), we denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{1,p(x)}$ and we endow $W_0^{1,p(x)}(\Omega)$ with the equivalent norm $\|\nabla u\|_{p(x)}$. For further details on the variable exponent Lebesgue and Sobolev spaces, we refer the reader to [15]. We now recall from [12] the notion of $\Gamma$-convergence that will be useful in the sequel.

**Definition 2.2.** Let $X$ be a metrizable topological space and let $(f_h)$ be a sequence of functions from $X$ to $\mathbb{R}$. The $\Gamma$-*lower limit* and the $\Gamma$-*upper limit* of the sequence $(f_h)$ are the functions from $X$ to $\mathbb{R}$ defined by

$$
(\Gamma - \liminf_{h \to \infty} f_h)(u) = \sup_{U \in \mathcal{N}(u)} \left[ \liminf_{h \to \infty} (\inf\{ f_h(v) : v \in U \}) \right],
$$

$$
(\Gamma - \limsup_{h \to \infty} f_h)(u) = \sup_{U \in \mathcal{N}(u)} \left[ \limsup_{h \to \infty} (\inf\{ f_h(v) : v \in U \}) \right],
$$

where $\mathcal{N}(u)$ denotes the family of all open neighborhoods of $u$ in $X$. If there exists a function $f : X \to \mathbb{R}$ such that

$$
\Gamma - \liminf_{h \to \infty} f_h = \Gamma - \limsup_{h \to \infty} f_h = f,
$$

then we write $\Gamma - \lim f_h = f$ and we say that $(f_h)$ $\Gamma$-*converges* to its $\Gamma$-*limit* $f$.

For any $p \in \mathcal{C}$, we define $\mathcal{E}_{p(x)} : L^1(\Omega) \to [0, \infty]$ as

$$
\mathcal{E}_{p(x)}(u) := \begin{cases} 
\|\nabla u\|_{p(x)} & \text{if } u \in W_0^{1,p(x)}(\Omega), \\
+\infty & \text{otherwise}
\end{cases}
$$

and $g_{p(x)} : L^1(\Omega) \to [0, \infty)$ as

$$
g_{p(x)}(u) := \begin{cases} 
\|u\|_{p(x)} & \text{if } u \in L^{p(x)}(\Omega), \\
0 & \text{otherwise}.
\end{cases}
$$

**Proposition 2.3.** The following properties hold:

(a) $g_{p(x)}$ is even and positively homogeneous of degree 1;

(b) for every $b \in \mathbb{R}$ the restriction of $g_{p(x)}$ to $\{u \in L^1(\Omega) : \mathcal{E}_{p(x)}(u) \leq b\}$ is continuous.

**Proof.** (a) follows easily from the definition of $g_{p(x)}$. (b) Let $(u_n) \subset \{u \in L^1(\Omega) : \mathcal{E}_{p(x)}(u) \leq b\}$ converge to $u$ in $L^1(\Omega)$ and consider a subsequence $(u_{n_j})$. By the definition of $\mathcal{E}_{p(x)}$, we know that $(u_{n_j})$ is bounded in $W_0^{1,p(x)}(\Omega)$ which is reflexive, hence there exists a subsequence $(u_{n_{j_k}})$ that converges weakly to $\bar{u}$ in $W_0^{1,p(x)}(\Omega)$. Since $W_0^{1,p(x)}(\Omega)$ is compactly embedded in $L^{p(x)}(\Omega)$, cf. [14, Proposition 2.2 and Lemma 5.5], $u_{n_{j_k}}$ converges strongly to $\bar{u}$ in $L^{p(x)}(\Omega)$. By the arbitrariness of the subsequence $(u_{n_j})$, we get that the whole sequence $u_n \to \bar{u}$ in $L^{p(x)}(\Omega)$ and also in $L^1(\Omega)$. Therefore, $u = \bar{u}$ and the proof is concluded.

**Lemma 2.4.** Let $p, (p_h) \subset \mathcal{C}$ be such that $p_h \to p$ pointwise. Then, for all $w \in C_c^1(\Omega)$

$$
\lim_{h \to \infty} \|\nabla w\|_{p_h(x)} = \|\nabla w\|_{p(x)}.
$$

**Proof.** By means of [15, Corollary 3.5.4], we know that

$$
\|\nabla w\|_{p(x)} \leq \liminf_{h \to \infty} \|\nabla w\|_{p_h(x)}.
$$

It remains to prove that

$$
\|\nabla w\|_{p(x)} \geq \limsup_{h \to \infty} \|\nabla w\|_{p_h(x)}.
$$
If $\nabla w = 0$ in $\Omega$, the conclusion is obvious, so we can assume that $\|\nabla w\|_{p(x)} > 0$. By hypothesis, $p_h \to p$ pointwise and $p_h(x) < N$ for all $h \in \mathbb{N}$ and $x \in \Omega$, hence

$$\left| \frac{\alpha \nabla w}{\|\nabla w\|_{p(x)}} \right|^{p_h(x)} \to \left| \frac{\alpha \nabla w}{\|\nabla w\|_{p(x)}} \right|^{p(x)} \quad \text{for all } x \in \Omega,$$

$$\left| \frac{\alpha \nabla w}{\|\nabla w\|_{p(x)}} \right|^{p_h(x)} \leq \left( 1 + \frac{\alpha \nabla w}{\|\nabla w\|_{p(x)}} \right)^N \in L^1(\Omega) \quad \text{for all } h \in \mathbb{N}.$$

Therefore, by the dominated convergence theorem, we obtain

$$\lim_{h \to \infty} \varrho_{p_h(x)} \left( \frac{\alpha \nabla w}{\|\nabla w\|_{p(x)}} \right) = \varrho_p(x) \left( \frac{\alpha \nabla w}{\|\nabla w\|_{p(x)}} \right) \leq \alpha \varrho_p(x) \left( \frac{\nabla w}{\|\nabla w\|_{p(x)}} \right) = \alpha < 1$$

for all $\alpha \in (0, 1)$. Thus, for $h$ sufficiently large $\varrho_{p_h(x)} \left( \frac{\alpha \nabla w}{\|\nabla w\|_{p(x)}} \right) < 1$, which in turn gives

$$\left\| \frac{\alpha \nabla w}{\|\nabla w\|_{p(x)}} \right\|_{p_h(x)} < 1$$

by Proposition 2.1. Whence

$$\limsup_{h \to \infty} \|\nabla w\|_{p_h(x)} \leq \frac{\|\nabla w\|_{p(x)}}{\alpha} \quad \text{for all } \alpha \in (0, 1)$$

and by the arbitrariness of $\alpha$ the assertion is proved. \hfill \square

**Theorem 2.5.** Let $p, (p_h) \subset \mathcal{C}$ be such that $p_h \to p$ pointwise. Then

$$\varrho_p(x)(u) \geq \left( \Gamma - \limsup_{h \to \infty} \varrho_{p_h(x)}(u) \right) \quad \text{for all } u \in L^1(\Omega).$$

**Proof.** Suppose that $\varrho_p(x)(u) < \infty$ (otherwise (2.2) is obvious) and take $b \in \mathbb{R}$ such that $b > \varrho_p(x)(u)$. Let $\delta > 0$ and $w \in C^1_c(\Omega)$ with $\|w - u\|_1 < \delta$ and $\|\nabla w\|_{p(x)} < b$, then $\|\nabla w\|_{p_h(x)} \to \|\nabla w\|_{p(x)}$ by Lemma 2.4. Therefore,

$$b > \lim_{h \to \infty} \varrho_{p_h(x)}(w),$$

and in turn

$$b > \limsup_{h \to \infty} (\inf \{ \varrho_{p_h(x)}(v) : \|v - u\|_1 < \delta \}).$$

By the arbitrariness of $\delta > 0$ we get

$$b \geq \left( \Gamma - \limsup_{h \to \infty} \varrho_{p_h(x)}(u) \right)$$

and since $b > \varrho_p(x)(u)$ is arbitrary, we obtain (2.2). \hfill \square

**Lemma 2.6.** Let $p, q : \Omega \to [1, \infty)$ be measurable functions with $p(x) \leq q(x)$ for a.a. $x \in \Omega$. Then $L^{q(x)}(\Omega) \to L^{p(x)}(\Omega)$ with embedding constant less or equal than

$$C(\|\cdot\|_p, q) := \left[ \left( \frac{p}{q} \right)_+ + \left( 1 - \frac{p}{q} \right)_+ \right] \max \{ \|\cdot\|_{1/p - 1/q_+}, \|\cdot\|_{1/p - 1/q_-} \}.\,$$

In particular, $C(\|\cdot\|_p, q) \to 1$ whenever $q_j$ converges uniformly to $p$.

**Proof.** For all $u \in L^{q(x)}(\Omega)$, by Hölder’s inequality [15, cf. (3.2.23)]

$$\|u\|_{p(x)} \leq \left[ \left( \frac{p}{r} \right)_+ + \left( \frac{p}{q} \right)_+ \right] \|1\|_{r(x)\{u\|_{q(x)}}$$
where \(1/r := 1/p - 1/q\) a.e. in \(\Omega\). Moreover, by [15, Lemma 3.2.5], we get
\[
\|1\|_{r(x)} \leq \max\{\|\Omega\|^{1/r-}, \|\Omega\|^{1/r+}\},
\]
which concludes the proof.

\[\square\]

**Theorem 2.7.** Let \(p, (p_h) \subset \mathcal{C}\) be such that \(p(x) \leq p_h(x)\) for all \(h \in \mathbb{N}\) and \(x \in \Omega\), and \(p_h \to p\) uniformly in \(\Omega\). Then
\[
\varepsilon_{p(x)}(u) \leq \left(\Gamma - \lim inf_{h \to \infty} \varepsilon_{p_h(x)}(u)\right) \quad \text{for all } u \in L^1(\Omega).
\]

**Proof.** If \((\Gamma - \lim inf_{h \to \infty} \varepsilon_{p_h(x)}(u)) = +\infty\) there is nothing to prove. In the other case, take \(b \in \mathbb{R}\) such that \(b > (\Gamma - \lim inf_{h \to \infty} \varepsilon_{p_h(x)}(u))\). By virtue of [12, Proposition 8.1-(b)] there exists a sequence \((u_h) \subset L^1(\Omega)\) such that \(u_h \to u\) in \(L^1(\Omega)\) and
\[
\left(\Gamma - \lim inf_{h \to \infty} \varepsilon_{p_h(x)}(u)\right) = \lim inf_{h \to \infty} \varepsilon_{p_h(x)}(u_h).
\]
Hence, there is a subsequence \((p_{h_n})\) for which
\[
\sup_{n \in \mathbb{N}} \varepsilon_{p_{h_n}(x)}(u_{h_n}) < b.
\]
Let \((v_n) \subset C^1_c(\Omega)\) verify
\[
\|v_n - u_{h_n}\|_1 < \frac{1}{n}, \quad \varepsilon_{p_{h_n}(x)}(v_n) < b \quad \text{for all } n \in \mathbb{N}.
\]
Then \(v_n \to u\) in \(L^1(\Omega)\) and, by the embedding \(W^{1,p_{h_n}}_0(\Omega) \hookrightarrow W^{1,p(x)}_0(\Omega)\),
\[
b > \|\nabla v_n\|_{p_{h_n}(x)} \geq \frac{\|\nabla v_n\|_{p(x)}}{C(|\Omega|, p, p_{h_n})} \quad \text{for all } n \in \mathbb{N},
\]
where \(C(|\Omega|, p, p_{h_n})\) is given in (2.3) with \(q = p_{h_n}\) and \(C(|\Omega|, p, p_{h_n}) \leq 2(1 + |\Omega|) < \infty\) for all \(n\). Therefore, \((v_n)\) is bounded in the reflexive space \(W^{1,p(x)}_0(\Omega)\) and so there exists a subsequence \((v_{n_j})\) such that \(v_{n_j} \rightharpoonup u\) in \(W^{1,p(x)}_0(\Omega)\). By Lemma 2.6 and the uniform convergence of \(p_{h_n}\) to \(p\),
\[
\lim_{j \to \infty} C(|\Omega|, p, p_{h_n}) = 1,
\]
and, together with the weak lower semicontinuity of the norm, we get
\[
b \geq \lim inf_{j \to \infty} \frac{\|\nabla v_{n_j}\|_{p(x)}}{C(|\Omega|, p, p_{h_n})} \geq \|\nabla u\|_{p(x)} = \varepsilon_{p(x)}(u).
\]
In conclusion, by the arbitrariness of \(b\), we obtain (2.4).

\[\square\]

**Lemma 2.8.** Let \(p, (p_h) \subset \mathcal{C}\) be such that \(p_h \to p\) pointwise, \(u \in L^{p(x)}(\Omega)\), \(u_h \in L^{p_h(x)}(\Omega)\) for all \(h\), and \(u_h \to u\) a.e. in \(\Omega\). Then
\[
\|u\|_{p(x)} \leq \lim inf_{h \to \infty} \|u_h\|_{p_h(x)}.
\]

**Proof.** Suppose that \(\lim inf_{h \to \infty} \|u_h\|_{p_h(x)} < \infty\) (otherwise there is nothing to prove) and take any \(\alpha \in \mathbb{R}\) such that \(\alpha > \lim inf_{h \to \infty} \|u_h\|_{p_h(x)}\). Then there exists a subsequence \((p_{h_j})\) for which \(\|u_{h_j}\|_{p_{h_j}(x)} < \alpha\) for all \(j\). Hence, \(\varepsilon_{p_{h_j}(x)}(u_{h_j}/\alpha) < 1\) and by Fatou’s Lemma
\[
\int_\Omega \left|\frac{u}{\alpha}\right|^{p(x)} dx \leq \lim inf_{j \to \infty} \int_\Omega \left|\frac{u_{h_j}}{\alpha}\right|^{p_{h_j}(x)} dx \leq 1.
\]
Thus, by Proposition 2.1, \(\|u/\alpha\|_{p(x)} \leq 1\), that is \(\|u\|_{p(x)} \leq \alpha\). The conclusion follows by the arbitrariness of \(\alpha\).

\[\square\]
Lemma 2.9. Let $p, (p_h) \subset \mathcal{C}$ and $p_h \to p$ pointwise and suppose that for some $\tilde{h} \in \mathbb{N}$
\begin{equation}
(2.5) \quad p_I := \inf_{h \geq \tilde{h}} (p_h)_- > 1
\end{equation}
and
\begin{equation}
(2.6) \quad ps := \sup_{h \geq \tilde{h}} (p_h)_+ < p^*_I, \quad \text{where } p^*_I = \frac{Np_I}{N - p_I}.
\end{equation}

Then, for every sequence $(u_h)$ such that $u_h \in W^{1, p_h}_0(\Omega)$ for all $h$ and $u_h \rightharpoonup u$ in $W^{1, p_I}_0(\Omega)$, there exists a subsequence $(u_{h_n})$ for which
\begin{equation}
\lim_{n \to \infty} \mathcal{E}_{p_{h_n}}(u_{h_n}) = \mathcal{E}(u).
\end{equation}

Proof. By (2.5) and Lemma 2.6, $u_h \in W^{1, p_I}_0(\Omega)$ for all $h \geq \tilde{h}$. Since $p_I < p^*_I$, $W^{1, p_I}_0(\Omega)$ is compactly embedded in $L^{p_S}(\Omega)$ and so $u_h \to u$ in $L^{p_S}(\Omega)$. Then there exists a subsequence $(u_{h_n})$ and a function $v \in L^{p_S}(\Omega)$ for which $u_{h_n} \to u$ and $|u_{h_n}| \leq |v|$ a.e. in $\Omega$. Whence, a.e.,
\begin{equation}
\lim_{n \to \infty} |u_{h_n}|_{p_{h_n}(x)} = |u|_{p(x)},
\end{equation}
\begin{equation}
|u_{h_n}|_{p_{h_n}(x)} \leq 1 + |v|_{p_S} \in L^1(\Omega) \quad \text{for all } n \in \mathbb{N}.
\end{equation}

In conclusion, by the dominated convergence theorem we obtain
\begin{equation}
\lim_{n \to \infty} \int_\Omega |u_{h_n}|_{p_{h_n}(x)} dx = \int_\Omega |u|_{p(x)} dx,
\end{equation}
namely the assertion. \hfill \Box

Remark 2.10. Condition (2.5) is valid for instance when $p_h \to p$ uniformly or when $p_h \searrow p$ pointwise, while assumption (2.6) is a consequence of (2.5) when $p_I > N/2$, being $p_S \leq N$.

Theorem 2.11. Let $p, (p_h) \subset \mathcal{C}, p_h \to p$ pointwise and let (2.5)-(2.6) hold for some $\tilde{h} \in \mathbb{N}$. Then, for every subsequence $(p_{h_n})$ and for every sequence $(u_n) \subset L^1(\Omega)$ verifying
\begin{equation}
(2.7) \quad \sup_{n \in \mathbb{N}} \mathcal{E}_{p_{h_n}(x)}(u_n) < \infty,
\end{equation}
there exists a subsequence $(u_{n_j})$ such that, as $j \to \infty,$
\begin{align*}
u_{n_j} &\to u \quad \text{in } L^1(\Omega),
g_{p_{h_{n_j}}(x)}(u_{n_j}) \to g_{p(x)}(u).
\end{align*}

Proof. For all $h_n \geq \tilde{h}$, $W^{1, p_{h_n}(x)}_0(\Omega) \hookrightarrow W^{1, p_I}_0(\Omega)$ with embedding constant less than or equal to $2(1 + |\Omega|)$ (cf. [15, Corollary 3.3.4]), then
\begin{equation}
\|\nabla u_n\|_{p_I} \leq 2(1 + |\Omega|)\|\nabla u_n\|_{p_{h_n}(x)} \leq 2b(1 + |\Omega|),
\end{equation}
where
\begin{equation}
b := \sup_{n \in \mathbb{N}} \mathcal{E}_{p_{h_n}(x)}(u_n).
\end{equation}

Since $W^{1, p_I}_0(\Omega)$ is reflexive, $(u_n)$ admits a subsequence $(u_{n_j})$ weakly convergent to $u$ in $W^{1, p_I}_0(\Omega)$. Thus, $u_{n_j} \to u$ in $L^1(\Omega)$ and up to a subsequence $u_{n_j} \to u$ a.e. in $\Omega$. For the second part of the statement, we have to prove that $\|u_{n_j}\|_{p_{h_{n_j}}(x)} \to \|u\|_{p(x)}$. By Lemma 2.8 we know that
\begin{equation}
\|u\|_{p(x)} \leq \liminf_{j \to \infty} \|u_{n_j}\|_{p_{h_{n_j}}(x)}.
\end{equation}
Now, for every real number
\[ \alpha < \limsup_{j \to \infty} \| u_{n_j} \|_{p_{h_{n_j}}}(x), \]
there exists a subsequence, still denoted by \((p_{h_{n_j}})\), for which \( \alpha < \| u_{n_j} \|_{p_{h_{n_j}}}(x) \) for all \( j \), and so
\[ 1 < \frac{\| u_{n_j} \|_{p_{h_{n_j}}}(x)}{\alpha} \]
by virtue of Proposition 2.1. Therefore, Lemma 2.9 yields
\[ 1 \leq \lim_{j \to \infty} \phi_{p_{h_{n_j}}}(x) \left( \frac{u_{n_j}}{\alpha} \right) = \phi_p(x) \left( \frac{u}{\alpha} \right), \]
that is \( \| u \|_p(x) \geq \alpha \) again by unit ball property. The conclusion follows by the arbitrariness of \( \alpha \).

We need to show that the minimax values with respect to the \( W_0^{1,p(x)}(\Omega) \)-topology are equal to those with respect to the weaker topology \( L^1(\Omega) \). To this aim, let \( \mathcal{W}^{(m)}_{p} \) be the family of those subsets \( K \) of
\[ \{ u \in W_0^{1,p(x)}(\Omega) : g_p(u) = 1 \} \]
which are compact and symmetric (i.e. \( K = -K \)), for which \( \inf K \geq m \) with respect to the norm topology of \( W_0^{1,p(x)}(\Omega) \), where \( \inf \) denotes the Krasnosel’skii genus. Furthermore, denote by \( K^{(m)}_{s,p(x)} \) the family of compact and symmetric subsets \( K \) of
\[ \{ u \in L^1(\Omega) : g_p(u) = 1 \} \]
such that \( \inf K \geq m \), with respect to the topology of \( L^1(\Omega) \).

**Theorem 2.12.** Let \( p \in \mathcal{C} \) and \( \mathcal{E}_{p(x)} : L^1(\Omega) \to [0, \infty] \) be the function defined in (2.1). Then, \( \mathcal{E}_{p(x)} \) is convex, even and positively homogeneous of degree 1. Moreover, for every integer \( m \geq 1 \), we have
\[ (2.8) \quad \inf_{K \in \mathcal{C}^{(m)}_{s,p(x)}} \sup_{K \in \mathcal{C}^{(m)}_{s,p(x)}} \mathcal{E}_{p(x)} = \inf_{K \in \mathcal{W}^{(m)}_{p}} \sup_{K \in \mathcal{W}^{(m)}_{p}} \mathcal{E}_{p(x)}. \]

**Proof.** The fact that \( \mathcal{E}_{p(x)} \) is convex, even and positively homogeneous of degree 1 follows easily by the definition. Furthermore, by Proposition 2.3-(b) we know that for all \( b \in \mathbb{R} \) the restriction of \( g_p(u) \) to the set \( \{ v \in L^1(\Omega) : \mathcal{E}_{p(x)}(v) \leq b \} \) is \( L^1(\Omega) \)-continuous. *A fortiori* the restriction of \( g_p(u) \) to the same set is continuous with respect to the stronger topology \( W_0^{1,p(x)}(\Omega) \) and the conclusion follows by [13, Corollary 3.3]. \( \square \)

3. Proof of Theorem 1.1

Due to Proposition 2.3 and the first part of Theorem 2.12, the functionals \( \mathcal{E}_{p(x)} \), \( g_p(x) \), \( \mathcal{E}_{p_h(x)} \) and \( g_{p_h}(x) \) for all \( h \in \mathbb{N} \) satisfy all the structural assumptions required in Section 4 of [13]. Moreover, by Theorems 2.5 and 2.7, we know that
\[ \mathcal{E}_{p(x)}(u) = \left( \Gamma - \lim_{h \to \infty} \mathcal{E}_{p_h(x)} \right)(u) \quad \text{for all } u \in L^1(\Omega). \]

Therefore, together with Theorem 2.11, all the hypotheses of [13, Corollary 4.4] are verified and so we can infer that
\[ \inf_{K \in \mathcal{K}^{(m)}_{s,p(x)}} \sup_{u \in K} \mathcal{E}_{p(x)}(u) = \lim_{h \to \infty} \left( \inf_{K \in \mathcal{K}^{(m)}_{s,p_h(x)}} \sup_{u \in K} \mathcal{E}_{p_h(x)}(u) \right). \]
Finally, by (2.8) the last equality reads as

$$\lambda_{p(x)}^{(m)} = \inf_{K \in W_{a,p,q}^{m}} \sup_{u \in K} E_p(x)(u) = \lim_{h \to \infty} \left( \inf_{K \in W_{a,p,q}^{m}} \sup_{u \in K} E_p(u) \right) = \lim_{h \to \infty} \lambda_{p_h}^{(m)}$$

which proves the assertion.

**Remark 3.1.** Significant progresses were recently achieved in the framework of regularity theory for minimisers of a class of double phase integrands of the Calculus of Variations, see [4–6,10,11] and the references therein. The model case is

$$u \mapsto \int_{\Omega} (|\nabla u|^p + a(x)|\nabla u|^q) dx, \quad q > p, \ a(\cdot) \geq 0,$$

and it can be embedded into the class of Musielak-Orlicz spaces, with Orlicz norm

$$\|u\|_{L^H} = \inf \left\{ \lambda > 0 : \int_{\Omega} H(x, \frac{|u(x)|}{\lambda}) dx \leq 1 \right\}, \quad H(x, s) := s^p + a(x)t^q, \ t \geq 0, \ x \in \Omega.$$

For a given topological index, such as the Krasnosel’skii genus or the $\mathbb{Z}_2$-cohomological index, we plan to investigate in a forthcoming paper the asymptotic growth, the stability of the nonlinear eigenvalues $\lambda_{a,p,q}^{(m)}$, and the basic properties of the first eigenvalue.

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