Operations on $t$-structures and perverse coherent sheaves

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Abstract. We introduce the notions of consistent pairs and consistent chains of $t$-structures and prove that two consistent chains of $t$-structures generate a distributive lattice. The technique developed is then applied to the pairs of chains obtained from the standard $t$-structure on the derived category of coherent sheaves and the dual $t$-structure by means of the shift functor. This yields a family of $t$-structures whose hearts are known as perverse coherent sheaves.

Keywords: derived categories of coherent sheaves, perverse sheaves, $t$-structures.

Dedicated to I. R. Shafarevich on his 90th birthday

Introduction

Suppose that we have an equivalence $\mathcal{D}(\mathcal{A}) \simeq \mathcal{D}(\mathcal{B})$ between the derived categories of Abelian categories $\mathcal{A}$ and $\mathcal{B}$. How does this help us to study $\mathcal{D}(\mathcal{A})$? A real advantage is that we can transfer the standard $t$-structure from $\mathcal{D}(\mathcal{B})$ to $\mathcal{D}(\mathcal{A})$ by means of the equivalence and see how it interacts with the standard $t$-structure in $\mathcal{D}(\mathcal{A})$. In particular, one can try to construct new $t$-structures from these two.

In this paper we consider two binary operations on $t$-structures in a triangulated category $\mathcal{D}$. More precisely, there is a natural partial order on the set of $t$-structures, and the operations are just those of intersection and union as defined in lattice theory. The main problem is to prove the existence of an intersection and a union for two given $t$-structures.

We define lower- and upper-consistent pairs of $t$-structures. These pairs satisfy some elementary conditions guaranteeing the existence of the intersection or union. Then we take a triple of $t$-structures and examine various consistency conditions for pairs in this triple which imply the consistency of the pair formed by a $t$-structure in the triple and the intersection (or union) of the other two. This gives some rules that enable us to iterate the operations of union and intersection.

The lattice of vector subspaces in a vector space does not obey the distributive law but does obey the modular law. The partially ordered set of $t$-structures is somewhat similar. In particular, we prove some versions of the modular law under suitable conditions of consistency in pairs. We consider abstract partially ordered sets with two binary relations subject to certain axioms that hold for the upper and lower consistency. Such sets are called sets with consistencies. We prove that every such set $P$ can be mapped to a universal lattice with consistencies $U(P)$.

AMS 2010 Mathematics Subject Classification. 14F05, 18E30.
A chain in a partially ordered set is a totally ordered set of elements. A theorem of Birkhoff [1] states that two chains in a modular lattice generate a distributive lattice. A pair of chains \( \{a_i\} \) and \( \{b_j\} \) is said to be consistent if all pairs \((a_i, b_j)\) are both upper consistent and lower consistent. We prove that every consistent pair of chains in a set with consistencies generates a distributive lattice \( L \).

We apply the technique developed to the theory of perverse sheaves on schemes of finite type over a field. Consider the derived category \( D(X) = D^b_{\text{coh}}(X) \) of coherent sheaves on such a scheme \( X \). It has a standard \( t \)-structure and possesses the Grothendieck–Serre duality functor \( D : D(X) \to D(X)^{\text{opp}} \), which is defined for \( F \in D(X) \) by the rule

\[
D F = \text{Hom}_X(F, \omega_X),
\]

where \( \omega_X \) stands for the dualizing complex on \( X \). Since \( D \) is an anti-equivalence, it enables us to define another, ‘dual’, \( t \)-structure on \( D(X) \) by transferring the standard \( t \)-structure by means of \( D \). Consider two chains of \( t \)-structures: one of them is obtained by shifts of the standard \( t \)-structure, and the other by shifts of the dual \( t \)-structure. We prove that this pair of chains is consistent. Hence they generate a distributive lattice of \( t \)-structures according to the above-mentioned version of Birkhoff’s theorem. This is a set of perverse \( t \)-structures on coherent sheaves (those independent of the stratification of \( X \)). They were studied independently by Deligne, Bezrukavnikov [2], Kashiwara [3] and Gabber [4].

The hard part in constructing \( t \)-structures with the desired properties is to prove the existence of a functor adjoint to the embedding of an appropriate subcategory \( D^{\leq 0} \) in \( D \). The standard way of producing such a functor is by using limits in the category.

According to the proposal in [5], the derived categories of coherent sheaves can play a crucial role in the Minimal Model Program of birational algebraic geometry. We expect that important ingredients of the Minimal Model Program (such as flips and flops) can be constructed and understood in terms of transformations of \( t \)-structures (see [6] for 3-dimensional flops). This problem requires a consideration of triangulated categories of ‘small’ size, like those arising in ‘realistic’ algebraic geometry: think of the bounded derived categories of complexes of coherent sheaves on algebraic varieties in contrast to the unbounded derived categories of quasi-coherent sheaves. There are no suitable general conditions for the existence of limits in such categories and, therefore, standard constructions often lead us \( a \text{ priori} \) beyond the ‘small’ category. For example, the derived functor of cohomology with support takes coherent complexes to quasi-coherent ones.

One of the main technical problems in studying algebraic geometry by means of triangulated categories is to find appropriate finiteness properties on suitable subcategories of the derived categories of coherent sheaves and to control these properties under the transformations of these subcategories induced by the geometric operations.

We hope that this paper will shed light on the finiteness conditions related to \( t \)-structures. It would be interesting to compare our methods with Bridgeland’s approach, which produces the stability conditions (and hence a wealth of \( t \)-structures) by a deformation argument [7].
I am indebted to Michel van den Bergh for useful discussions and to the referee for useful comments.

This work was done more than 15 years ago and reported for the first time in the Shafarevich seminar at the Steklov Mathematical Institute in the late 1990s. It is my pleasure to dedicate this paper to I. R. Shafarevich on the occasion of his 90th birthday. His influence on Russian algebraic geometry can hardly be overestimated.

§ 1. Consistent systems of $t$-structures

1.1. The partially ordered set of $t$-structures. Let $k$ be a field and let $D$ be a $k$-linear triangulated category with a self-equivalence $T$, called the translation (or shift) functor (see [8]).

Let $C$ be a subcategory of $D$. Its right orthogonal $C^\perp$ (resp. left orthogonal $C^\perp$) is the strictly full subcategory of $D$ consisting of all objects $X$ such that $\text{Hom}(C, X) = 0$ (resp. $\text{Hom}(X, C) = 0$).

We use the standard notation $X[k] := T^k X$, $k \in \mathbb{Z}$.

Recall (see [9]) that a $t$-structure in a triangulated category $D$ is a pair $(D_{\leq 0}, D_{\geq 1})$ of strictly full subcategories satisfying the following conditions.

(i) $TD_{\leq 0} \subset D_{\leq 0}$, $T^{-1}D_{\geq 1} \subset D_{\geq 1}$.

(ii) $\text{Hom}(D_{\leq 0}, D_{\geq 1}) = 0$.

(iii) For every $X \in D$ one can find $X_- \in D_{\leq 0}$, $X_+ \in D_{\geq 1}$ and an exact triangle $X_- \to X \to X_+ \to X_-[1]$.

The main example of a $t$-structure appears when $D$ is equivalent to the bounded derived category $D^b(A)$ of an Abelian category $A$. In this case, $D_{\geq 1}$ (resp. $D_{\leq 0}$) consists of the complexes with cohomology in positive (resp. non-positive) degrees.

Categorically, the main advantage of the notion of a $t$-structure is the uniqueness (up to a unique isomorphism) of the triangle (1) for a given $X$ (this follows from the conditions (i) and (ii)); see [9]. In other words, the formulae

$$\tau_{\leq 0}(X) = X_-, \quad \tau_{\geq 1}(X) = X_+$$

determine the truncation functors $\tau_{\leq 0} : D \to D_{\leq 0}$, $\tau_{\geq 1} : D \to D_{\geq 1}$, which are easily seen to be the right and left adjoint functors to the inclusion functors $i_{\leq 0} : D_{\leq 0} \to D$, $i_{\geq 1} : D_{\geq 1} \to D$ respectively. This suggests the following restatement of the data determining a $t$-structure.

Lemma 1. Determining a $t$-structure in a triangulated category $D$ is equivalent to determining one of the following data:

1) a strictly full subcategory $D_{\leq 0}$ in $D$ which is closed under the action of $T$ and is such that the inclusion functor $i_{\leq 0} : D_{\leq 0} \to D$ has a right adjoint $\tau_{\leq 0}$;

2) a strictly full subcategory $D_{\geq 1}$ in $D$ which is closed under the action of $T^{-1}$ and is such that the inclusion functor $i_{\geq 1} : D_{\geq 1} \to D$ has a left adjoint $\tau_{\geq 1}$.

Proof. Since we have already constructed the adjoint functors for a given $t$-structure, it remains to explain how to construct a $t$-structure from 1), for example.

Put $D_{\geq 1} = (D_{\leq 0})^\perp$. Since $TD_{\leq 0} \subset D_{\leq 0}$, we obviously have $T^{-1}D_{\geq 1} \subset D_{\geq 1}$.
For every $X \in \mc D$ the adjunction morphism $i \leq_0 \tau \leq_0 X \to X$ can be included in a triangle

$$i \leq_0 \tau \leq_0 X \to X \to X_+ \to i \leq_0 \tau \leq_0 X[1].$$

We easily see that the object $X_+$ lies in $\mc D^{\geq 1}$. Hence the triangle is of the form (1). □

**Remark 2.** Note that in general neither the inclusion nor the truncation functor is exact. But one can easily see that if $A \to B \to C$ is an exact triangle with $A, C \in \mc D^{\leq 0}$ (resp. $A, C \in \mc D^{\geq 1}$), then $B \in \mc D^{\leq 0}$ (resp. $B \in \mc D^{\geq 1}$). It follows by ‘rotating the triangle’ that if $A, B \in \mc D^{\leq 0}$ (resp. $B, C \in \mc D^{\geq 1}$), then also $C \in \mc D^{\leq 0}$ (resp. $A \in \mc D^{\geq 1}$).

Every $t$-structure generates two sequences of subcategories $\mc D^{< n} := \mc D^{\leq 0}[-n]$, $\mc D^{\geq n} := \mc D^{\geq 1}[1 - n]$. The subcategory $\mc C = \mc D^{\leq 0} \cap \mc D^{\geq 0}$ has the natural structure of an Abelian category (see [9]). It is called the heart of the $t$-structure. The objects lying in the heart (and their shifts) are said to be pure.

When $\mc D = \mc D^b(\mc A)$, the heart of the standard $t$-structure consists of the complexes over $\mc A$ with cohomology in degree 0. Thus the heart is equivalent to the original Abelian category $\mc A$. We identify $\mc A$ with this full subcategory in $\mc D^b(\mc A)$.

If $\mc D$ is triangulated, then the opposite category $\mc D^{op}$ is also naturally triangulated. Namely, $T_{\mc D^{op}} = T_{\mc D}^{-1}$, and the triangles in $\mc D^{op}$ are given by reversing the arrows in the triangles in $\mc D$. A $t$-structure $\mc T$ in $\mc D$ determines a $t$-structure $\mc T^{op}$ in $\mc D^{op}$ by exchanging the roles of $\mc D^{\leq 0}$ and $\mc D^{\geq 1}$. The heart of $\mc T^{op}$ is the opposite Abelian category to the heart of $\mc T$.

An important special case appears when the $t$-structure satisfies $T^{-1} \mc D^{\leq 0} \subseteq \mc D^{\leq 0}$. Then $\mc D^{\leq 0}$ and $\mc D^{\geq 1}$ are triangulated subcategories by Remark 2. We call such $t$-structures translation invariant, or simply $T$-invariant.

A strictly full triangulated subcategory $\mc A$ in $\mc D$ is said to be right admissible (resp. left admissible) [10] if it has a right (resp. left) adjoint to the inclusion $i: \mc A \to \mc D$. For every $T$-invariant $t$-structure, the category $\mc D^{\leq 0}$ (resp. $\mc D^{\geq 1}$) is right admissible (resp. left admissible). This establishes a one-to-one correspondence between $T$-invariant $t$-structures and right admissible (or left admissible) subcategories.

For a $T$-invariant $t$-structure, all the subcategories $\mc D^{< k}, k \in \mathbb Z$, are equal to each other (and the same holds for $\mc D^{> k}, k \in \mathbb Z$). Hence the heart of a $T$-invariant $t$-structure is equal to zero.

Let $\mc D$ be a triangulated category. We consider the following partial order on the set of its $t$-structures (and ignore all possible set-theoretical questions arising since the objects of a category form a class and not a set: this is irrelevant for our discussion). Given any pair $(\mc T_1, \mc T_2)$ of $t$-structures, we say that $\mc T_1 \subseteq \mc T_2$ if $\mc D_1^{< 0} \subseteq \mc D_2^{\leq 0}$. Then also $\mc D_2^{\geq 1} \subseteq \mc D_1^{\geq 1}$. The minimal element $\mathbf 0$ of this ordered set is the $t$-structure with $\mc D^{< 0} = 0$. The maximal element $\mathbf 1$ has $\mc D^{\leq 0} = \mc D$.

The following definition is modelled on the standard notions of union and intersection in partially ordered sets.

**Definition 3.** The (abstract) intersection $\bigcap_{i \in I} \mc T_i$ of a set $\{\mc T_i\}_{i \in I}$ of $t$-structures in $\mc D$ is the supremum of the $t$-structures $\mc T$ in $\mc D$ such that $\mc T \subseteq \mc T_i$ for all $i \in I$. If the supremum exists, it is unique. The (abstract) union $\bigcup_{i \in I} \mc T_i$ is defined in the dual way.
We shall also denote the operations of intersection and union by \( \cdot \) and \(+\) and call them the \textit{product} and \textit{sum}. Indeed, they are just the product and coproduct in the category related to the partially ordered set.

The intersection and union of \( T \)-invariant \( t \)-structures are also \( T \)-invariant whenever they exist.

There is a natural \textit{naive} candidate for \( D \leq 0 \cap (\text{resp. } D \geq 1 \cup) \) for the intersection (resp. union) of \( t \)-structures in a set \( \{T_i\} \): \( D \leq 0 \cap = \bigcap_{i \in I} D \leq 0_{i} \) (resp. \( D \geq 1 \cup = \bigcup_{i \in I} D \geq 1_{i} \)).

There is a duality functor on the category of partially ordered sets. It inverts the partial order and takes unions to intersections and vice versa. We shall often use this duality in what follows.

\textbf{1.2. An example.} The following example of geometric origin yields a pair \((T_1, T_2)\) of \( t \)-structures such that \( D \leq 0_1 \cap D \leq 0_2 \) has no right adjoint to the inclusion. This shows that the naive intersection may not coincide with the abstract one.

Here we assume \( k \) to be algebraically closed and of characteristic zero. Let \( D = D_{\text{coh}}(\mathbb{P}^2) \) be the bounded derived category of coherent sheaves on the projective plane \( \mathbb{P}^2 = \mathbb{P}(V) \), \( \dim V = 3 \). We define two translation invariant \( t \)-structures in \( D \) by declaring the corresponding right-admissible triangulated subcategories \( D \leq 0_1 \) and \( D \leq 0_2 \) to be

\[
D \leq 0_1 = \langle \mathcal{O}(-2), \mathcal{O}(-1) \rangle, \quad D \leq 0_2 = \langle \mathcal{O}(1), \mathcal{O}(2) \rangle,
\]

that is, the minimal strictly full triangulated subcategories generated by these sheaves.

\textbf{Lemma 4.} \textit{The category } \( D \cap = D \leq 0_1 \cap D \leq 0_2 \textit{ is not right admissible.} \)

\textit{Proof.} The category \( D \leq 0_2 \) is equivalent as a triangulated category to \( D^b(\text{mod-}A) \), the bounded derived category of representations of the path algebra of the quiver

\[
\cdot \overset{V}{\longrightarrow} \cdot
\]

(see [10]). The equivalence is given by the functor that sends a complex of representations \( U^* \longrightarrow W^* \) with the structure map \( \varphi: U^* \otimes V \to W^* \) to the complex of sheaves \( U^* \otimes \mathcal{O}(1) \to W^* \otimes \mathcal{O}(2) \) in \( D \leq 0_2 \) with the differential induced by \( \varphi \).

Since the algebra \( A \) is hereditary, every indecomposable object in \( D^b(\text{mod-}A) \) is pure, that is, it can be determined (up to a common shift) by a map \( \varphi: U \otimes V \to W \), where \( U, W \) are vector spaces. The corresponding complex of sheaves in \( D \leq 0_2 \) is of the form

\[ U \otimes \mathcal{O}(1) \to W \otimes \mathcal{O}(2) \]  

(2)

with the components in degrees 0 and 1.

To describe the indecomposable objects in \( D \cap \), we note that an object \( X \in D \) lies in \( D \leq 0_1 \) if and only if \( \text{Ext}^*(\mathcal{O}, X) = 0 \). Applying this to (2), we deduce that the map

\[ \hat{\varphi}: U \otimes V^* \to W \otimes S^2 V^* , \]

which is obtained from \( \varphi \) by partial dualization \( \tilde{\varphi}: U \to W \otimes V^* \) with subsequent tensoring with the identity map in \( V^* \) and partial symmetrization, must be an
isomorphism. Since \( \dim V = 3 \), we see that \( \dim U = 2 \dim W \) for \( X \in \mathcal{D}_\cap = D_1^{<0} \cap D_2^{<0} \).

If we take \( W \) to be one-dimensional, then \( \dim U = 2 \), whence the image of \( \tilde{\varphi} \) annihilates a line in \( V \). It follows that the image of \( \tilde{\varphi} \) consists of the quadrics that vanish on this line, that is, \( \tilde{\varphi} \) is not an isomorphism. Hence there is no indecomposable object in \( \mathcal{D}_\cap \) when \( \dim W = 1 \).

Yet the category \( \mathcal{D}_\cap \) is non-empty. Indeed, take any exceptional vector bundle on \( \mathbb{P}^2 \), that is, a bundle with

\[
\Hom(E, E) = k, \quad \Ext^i(E, E) = 0, \quad i \neq 0.
\]

Consider the sheaf \( F = \End_0 E \) of traceless local endomorphisms of \( E \). It is non-zero if \( \text{rk} E > 1 \). Then it follows from (3) that \( \Ext^i(O, F) = H^i(F) = 0 \). Since \( D_1^{<0} = \mathcal{O}^\perp \) in \( \mathcal{D} \), we have \( F \in D_1^{<0} \). Since \( F^* = F \), we have \( \Ext^i(F, O) = H^i(F^*) = 0 \). Since \( D_2^{<0} = \mathcal{O}^\perp \), we conclude that \( F \in D_2^{<0} \), that is, \( F \in \mathcal{D}_\cap \).

An alternative example of a non-zero object in the category \( \mathcal{D}_\cap \) was suggested by the referee. Let \( F \) be any stable vector bundle of rank 2 on \( \mathbb{P}^2 \) with Chern classes \( c_1(F) = 0 \) and \( c_2(F) = 2 \). Such bundles can be constructed by generic extensions of the form

\[
0 \to O(-1) \to F \to I_Z(1) \to 0,
\]

where \( I_Z \) is the ideal sheaf of the subscheme of a triple of non-collinear points in \( \mathbb{P}^2 \) (this is basically Serre’s construction). One can see from this short exact sequence that \( H^i(F) = 0 \). Since \( F \) is a vector bundle of rank 2 with \( c_1(F) = 0 \), it is self-dual, whence \( H^i(F^*) = 0 \) and, therefore, \( F \in \mathcal{D}_\cap \).

We now take \( G = O(-1)[1] \). If \( F \in \mathcal{D}_\cap \) is of the form (2), then

\[
\Hom^1(F, G) = W^* \quad \text{Hom}^i(F, G) = 0, \quad i \neq 1.
\]

Suppose that \( \mathcal{D}_\cap \) is right admissible and \( \tau \) is the right adjoint to the embedding functor \( \mathcal{D}_\cap \to \mathcal{D} \). Put \( G' = \tau G \). Then \( G' \in \mathcal{D}_\cap \). If \( G' \) is of the form \( U' \otimes O(1) \to W' \otimes O(2) \), where \( F \) is of the form (2), then \( \Hom^i(F, G') \) is the cohomology of the complex

\[
U^* \otimes U' \oplus W^* \otimes W' \to U^* \otimes W' \otimes V^* \quad (5)
\]

with two non-trivial components in degrees 0 and 1. Since \( \dim U = 2 \dim W \) and \( \dim U' = 2 \dim W' \), the Euler characteristic of this complex is equal to \( -\dim W \cdot \dim W' \).

We conclude from the adjunction property and (4) that

\[
-\dim W = \chi(F, G) = \chi(F, G') = -\dim W \cdot \dim W'.
\]

Hence \( \dim W' \) must be equal to 1, which has already been seen to be impossible.

Generally, \( G' \), being an object in \( \mathcal{D}_\cap \), can be decomposed as a sum of objects of the form (2) and their translations. Every non-trivial translation would give a non-trivial contribution to \( \Hom^i(F, G') \) for \( i \neq 1 \) because these \( \Hom \)-groups are calculated by translations of the complexes (5). But this contradicts the adjunction property for \( \tau \). Hence the object \( G' \) cannot exist. □

One can similarly show that the abstract intersection \( T_1 \cap T_2 \) is equal to 0.
1.3. Consistent t-structures and operations. Here we define lower- and upper-consistent pairs of t-structures. These notions can be generalized in various ways, as we hope to discuss elsewhere.

We endow the truncation and inclusion functors of t-structures with appropriate superscripts.

Definition 5. A pair \((T_1, T_2)\) of t-structures is said to be lower consistent if
\[
\tau_{\leq 0}^1 D_{\leq 0} \subseteq D_{\leq 0},
\]
and upper consistent if
\[
\tau_{\geq 1}^2 D_{\geq 1} \subseteq D_{\geq 1}.
\]
A pair is said to be consistent if it is simultaneously lower and upper consistent.

The intersections and unions of consistent t-structures exist and are naive.

Proposition 6. Let \((T_1, T_2)\) be a lower-consistent (resp. upper-consistent) pair of t-structures. Then the formulae
\[
D_{\leq 0} = D_{\geq 0}^\leq \cap D_{\leq 0}, \quad D_{\geq 1} = (D_{\leq 0})^\perp_{\geq 1}
\]
with the truncation functor \(\tau_{\leq 0}^\cap := \tau_{\leq 0}^1 \tau_{\leq 0}^2\) (resp. the formulae
\[
D_{\geq 1} = D_{\geq 1}^{\geq 1} \cap D_{\leq 1}, \quad D_{= 0} = \perp (D_{\geq 1}^{\geq 1})
\]
with the truncation functor \(\tau_{\geq 1}^\cup := \tau_{\geq 1}^2 \tau_{\geq 1}^3\) determine a new t-structure in \(D\).

Proof. For a lower-consistent pair \((T_1, T_2)\) of t-structures, the category \(D_{\leq 0} = D_{\leq 0}^\leq \cap D_{\leq 0}\) is of the form 1) in Lemma 1. Indeed, this category is obviously preserved by the shift functor \(T\). Moreover, the image of the functor \(\tau_{\leq 0}^\cap := \tau_{\leq 0}^1 \tau_{\leq 0}^2\) lies in \(D_{\leq 0}\) since the pair is lower consistent. We easily see that this functor is right adjoint to the inclusion functor \(i_{\leq 0}^\cap : D_{\leq 0} \to D\). A similar proof works for upper-consistent pairs. \(\square\)

The example in § 1.2 yields an inconsistent pair of t-structures.

We now specify conditions on a sequence of t-structures under which the operations of intersection and union can be iterated. Our first result says that intersection is an associative operation on ordered sequences with pairwise lower-consistent elements, and the lower-consistency is preserved at the intermediate steps.

Proposition 7. Let \((T_1, T_2, T_3)\) be a triple of t-structures with lower-consistent pairs \((T_i, T_j)\) for \(i < j\). Then the pairs \((T_1 \cap T_2, T_3)\) and \((T_1, T_2 \cap T_3)\) are lower consistent. Moreover, \((T_1 \cap T_2) \cap T_3\) coincides with \(T_1 \cap (T_2 \cap T_3)\).

Proof. By Proposition 6, the truncation functor for the t-structure \(T_1 \cap T_2\) is the composite of \(\tau_{\leq 0}^1\) and \(\tau_{\leq 0}^2\). Since both functors preserve \(D_{\leq 0}^\leq\), it follows that the pair \((T_1 \cap T_2, T_3)\) is lower consistent.

Furthermore, \(D_{\leq 0}^\leq = D_{\leq 0} \cap D_{\leq 0}^\leq\). The categories \(D_{\leq 0}^\leq\) and \(D_{\leq 0}^\leq\) are preserved by \(\tau_{\leq 0}^1\). Hence \(\tau_{\leq 0}^1 D_{\leq 0}^\leq \subseteq D_{\leq 0}^\leq\) and, therefore, the second pair \((T_1, T_2 \cap T_3)\) is lower consistent.

Finally, for both \((T_1 \cap T_2) \cap T_3\) and \(T_1 \cap (T_2 \cap T_3)\), the categories \(D_{\leq 0}\) coincide with \(D_{\leq 0} \cap D_{\leq 0}^\leq \cap D_{\leq 0}^\leq\). It follows that these t-structures coincide. \(\square\)
Let enable one to construct the new that show that for every finite sequence Denoting the truncation functors for the we have \((\tau_\cap, \tau_\cup)\). The pair \((\tau_\cap, \tau_\cup)\) one can construct a new consistent. \(\text{Remark} (\text{for convenience we draw the octahedron as a square}):\) in the following ‘octahedron’ all of whose columns and rows are exact triangles (for convenience we draw the octahedron as a square):

\[
\begin{array}{ccc}
\tau_\leq & \to & \tau_\leq \cap X \\
\tau_\leq \cap & \to & X \\
\tau_\leq \cap & \to & \tau_\leq \cap \cup X \\
\tau_\leq \cap \cup & \to & X \\
\tau_\leq \cap \cup & \to & \tau_\leq \cap \cup \cap X \\
\tau_\leq \cap \cup \cap & \to & \tau_\leq \cap \cup \cap \cap X \\
\tau_\leq \cap \cup \cap \cap & \to & 0 \\
\end{array}
\]

We claim that \(\tau_\cap \cup \cup \cap \leq \cap D_1^{\geq 1} \subset D_1^{\geq 1}\). Indeed, take \(X \in D_1^{\geq 1}\). Since the pair \((T_1, T_3)\) is upper consistent, we have \(\tau_\leq \cap \leq \cap X \in D_1^{\geq 1}\). It follows from the middle row in (6) and Remark 2 that \(\tau_\leq \cap \leq \cap X = D_1^{\geq 1}\). Since the pair \((T_1, T_2)\) is upper consistent, we deduce that \(\tau_\leq \cap \leq \cap \leq \cap X \in D_1^{\geq 1}\). It now follows from the upper row in (6) that \(\tau_\cap \leq \cap X \in D_1^{\geq 1}\). \(\square\)

Thus the hypotheses of Proposition 9 enable one to construct the new \(T_1 \cup (T_2 \cap T_3)\).
The following proposition is proved by duality.

**Proposition 10.** Let \((T_1, T_2, T_3)\) be a triple of \(t\)-structures such that the pairs \((T_1, T_3)\) and \((T_2, T_3)\) are lower consistent and \((T_1, T_2)\) is upper consistent. Then the pair \((T_1 \cup T_2, T_3)\) is lower consistent.

Every pair \((T_1, T_2)\) of \(t\)-structures with \(T_1 \subseteq T_2\) is easily seen to be lower and upper consistent. Moreover, \((T_2, T_1)\) is also lower and upper consistent. In particular, the pairs \((T_1, T_1[k])\) are consistent for all \(k \in \mathbb{Z}\). We note that the truncation functors commute:

\[
\tau^1_{\leq 0} \leq 0 \tau^2_{\leq 0} = \tau^1_{\leq 0} = \tau^2_{\leq 0} \tau^1_{\leq 0}, \quad \tau^1_{\geq 1} \tau^2_{\geq 1} = \tau^2_{\geq 1} = \tau^2_{\geq 1} \tau^1_{\geq 1}.
\]

We conclude this subsection by proving another lemma (to be used below) on the commutativity of the truncation functors for an ordered pair of \(t\)-structures.

**Lemma 11.** Let \((T_1, T_2)\) be an ordered pair of \(t\)-structures with \(T_1 \subseteq T_2\). Then the truncation functors commute:

\[
\tau^2_{\leq 0} \tau^1_{\geq 1} = \tau^1_{\geq 1} \tau^2_{\leq 0}.
\]

**Proof.** For every object \(X \in \mathcal{D}\) there is an isomorphism \(\tau^2_{\geq 1} \tau^1_{\geq 1} X = \tau^2_{\geq 1} X\). Hence, by the octahedron axiom, we have the following diagram with exact rows and columns:

\[
\begin{array}{ccc}
\tau^2_{\leq 0} \tau^1_{\geq 1} X & \rightarrow & \tau^1_{\geq 1} X \\
\downarrow & & \downarrow \\
\tau^2_{\leq 0} X & \rightarrow & \tau^2_{\geq 1} X \\
\downarrow & & \downarrow \\
\tau^1_{\leq 0} X & \sim & \tau^1_{\leq 0} X \\
\downarrow & & \downarrow \\
& & 0 \\
\end{array}
\] (7)

The objects \(\tau^1_{\geq 1} X\) and \(\tau^2_{\geq 1} \tau^1_{\geq 1} X\) in the upper row lie in \(\mathcal{D}_{\geq 1}^{\geq 1}\). Hence the third object \(\tau^2_{\leq 0} \tau^1_{\geq 1} X\) lies in \(\mathcal{D}_{\leq 0}^{\geq 1}\). Then the lower (resp. upper) object in the left column lies in \(\mathcal{D}_{\leq 0}^{\leq 0}\) (resp. \(\mathcal{D}_{\leq 1}^{\geq 1}\)). Therefore this column is the canonical decomposition of the middle object \(\tau^2_{\leq 0} X\) with respect to the first \(t\)-structure. This yields a canonical isomorphism \(\tau^2_{\leq 0} \tau^1_{\geq 1} X \simeq \tau^1_{\geq 1} \tau^2_{\leq 0} X\). \(\square\)

1.4. **The standard postulates for lattices.** Lattice theory enables one to interpret certain partially ordered sets (called lattices) as abstract algebras whose union (sum) and intersection (product) operations satisfy the so-called standard postulates [1]. This interpretation is useful for manipulations with lattices, such as taking quotients.

**Definition 12.** A lattice is a partially ordered set in which the union and intersection of any two elements exist.
We recall the standard postulates for the operations on a partially ordered set. In the following formulae, we use the notation \((\cdot, +)\) with the usual priority of \(\cdot\) over \(+\) and suppose that both sides of the equations are well defined for the elements \(T, T_1, T_2, T_3\) of the partially ordered set (this condition obviously holds for lattices):

\[
T \cdot T = T, \quad T + T = T, \quad \tag{8}
\]
\[
T_1 \cdot T_2 = T_2 \cdot T_1, \quad T_1 + T_2 = T_2 + T_1, \quad \tag{9}
\]
\[
(T_1 \cdot T_2) \cdot T_3 = T_1 \cdot (T_2 \cdot T_3), \quad (T_1 + T_2) + T_3 = T_1 + (T_2 + T_3), \quad \tag{10}
\]
\[
T_1 \cdot (T_1 + T_2) = T_1 + T_1 \cdot T_2 = T_1. \quad \tag{11}
\]

**Proposition 13** [1]. Lattices are identified with abstract algebras with binary operations \(\cdot\) and \(+\) satisfying the standard postulates (8)–(11). The partial order of elements in such an algebra can be recovered by the rule

\[
x \leq y \quad \text{if and only if} \quad x \cdot y = x. \tag{12}
\]

If 0 and/or 1 exist, they can be added as nullary operations.

An equivalence relation \(\sim\) on a lattice \(L\) is called a congruence if \(p \sim q\) implies that \(p + l \sim q + l\) and \(p \cdot l \sim q \cdot l\) for all \(l \in L\). Given pairs of equivalences \(p_i \sim q_i, i \in I\), we define the congruence generated by them as the minimal congruence containing all of them. It is the intersection of all congruences that contain \(\{p_i \sim q_i\}_{i \in I}\). The intersection of any number of congruences (as subsets of \(L \times L\)) is again a congruence.

If \(\sim\) is a congruence on a lattice \(L\) (regarded as an abstract algebra), then the quotient algebra \(L' = L/\sim\) is well defined. Being an abstract algebra of the same type (that is, the standard postulates hold), it is again a lattice.

Note that for \(x', y' \in L\) we have \(x' \leq y'\) in \(L'\) if and only if there are \(x, y \in L\) such that \(x \sim x', y \sim y'\) and \(x \leq y\) in \(L\). Indeed, if \(x' \leq y'\) in \(L'\), then \(x' \sim x'y'\) by (12). Adding \(y'\) to both sides, we get \(x' + y' \sim x'y' + y' = y'\). Hence we can take \(x = x'y'\) and \(y = x' + y'\) for the required \(x, y\).

Unfortunately, I know of no ‘sufficiently large’ class of \(t\)-structures which constitutes a lattice. It would be interesting to find such a class.

### 1.5. Modular laws.

The following distributive inequalities hold in any partially ordered set (assuming that both sides are well defined):

\[
(T_1 + T_2) \cdot T_3 \geq T_1 \cdot T_3 + T_2 \cdot T_3, \tag{13}
\]
\[
T_1 + T_2 \cdot T_3 \leq (T_1 + T_2) \cdot (T_1 + T_3). \tag{14}
\]

The distributive laws (that is, equations in (13) and (14)) generally do not hold for \(t\)-structures.

**Example 14.** Let \(T_1\) and \(T_2\) be the \(T\)-invariant \(t\)-structures in the example in §1.2. We define a \(T\)-invariant \(t\)-structure \(T_3\) by putting

\[
D_{3}^{\leq 0} = \langle \mathcal{O} \rangle.
\]
Then \( T_1 + T_2 = 1 \) because \( D_{1}^{\geq 1} \cap D_{2}^{\geq 1} = \langle O(-3) \rangle \cap \langle O \rangle = 0 \). We also have \( T_1 \cdot T_3 = T_2 \cdot T_3 = 0 \) since \( D_{1}^{\leq 0} \cap D_{3}^{\leq 0} = D_{2}^{\leq 0} \cap D_{3}^{\leq 0} = 0 \). It follows that the distributive law

\[
(T_1 + T_2) \cdot T_3 = T_1 \cdot T_3 + T_2 \cdot T_3
\]
does not hold: the left-hand side is \( T_3 \) while the right-hand side is \( 0 \).

If \( T_1 \leq T_3 \), then (13) and (14) yield the following self-dual inequality:

\[
(T_1 + T_2) \cdot T_3 \geq T_1 + T_2 \cdot T_3.
\]  (15)

Replacing the inequality by equality in (15), we get the modular law.

Under appropriate consistency assumptions we shall prove a number of results that may be regarded as modular laws for \( t \)-strictures. Our first theorem is self-dual.

**Theorem 15** (modular law 1). Let \( (T_1, T_2, T_3) \) be a triple of \( t \)-structures such that \( T_1 \leq T_3 \), the pair \( (T_1, T_2) \) is upper consistent and \( (T_2, T_3) \) is lower consistent. Then

(i) the pair \( (T_1 + T_2, T_3) \) is lower consistent,
(ii) the pair \( (T_1, T_2 \cdot T_3) \) is upper consistent,
(iii) \( (T_1 + T_2) \cdot T_3 = T_1 + T_2 \cdot T_3 \).

**Proof.** Since \( T_1 \leq T_3 \), the pair \( (T_1, T_3) \) is lower consistent. Hence the lower consistency of \( (T_1 + T_2, T_3) \) follows from Proposition 10. In a similar vein, the pair \( (T_1, T_3) \) is upper consistent, whence \( (T_1, T_2 \cdot T_3) \) is upper consistent by Proposition 9.

Thus both sides of the equation in (iii) are well defined.

We already have inequality (15). The inequality opposite to (15) is by definition equivalent to the inclusion of subcategories

\[
D_{(1+2)3}^{\leq 0} \subseteq D_{1+2:3}^{\leq 0},
\]  (16)

which is in turn equivalent to the following assertion.

(*) If \( X \in D_{(1+2)3}^{\leq 0} \) and \( Y \in D_{1+2:3}^{\geq 1} \), then \( \text{Hom}(X, Y) = 0 \).

To prove (*), we decompose \( X \) into an exact triangle

\[
\tau_{\leq 0}^{1} X \to X \to \tau_{\geq 1}^{1} X.
\]  (17)

Since \( T_1 \subseteq (T_1 + T_2)T_3 \), the pair \( (T_1, (T_1 + T_2)T_3) \) is lower consistent. Hence \( \tau_{\leq 0}^{1} X \in D_{(1+2)3}^{\leq 0} \). Then Remark 2 shows that \( \tau_{\geq 1}^{1} X \in D_{(1+2)3}^{\leq 0} \).

Since \( T_1 \subseteq T_1 + T_2 T_3 \), we have \( \text{Hom}(D_{1}^{\leq 0}, D_{1+2:3}^{\geq 1}) = 0 \). It follows that \( \text{Hom}(\tau_{\leq 0}^{1} X, Y) = 0 \). Applying the functor \( \text{Hom}(\cdot, Y) \) to the triangle (17), we see that it suffices to prove that \( \text{Hom}(\tau_{\geq 1}^{1} X, Y) = 0 \). Thus the proof of (*) reduces to the case when \( X \in D_{(1+2)3}^{\leq 0} \cap D_{1+2:3}^{\geq 1}, Y \in D_{1+2:3}^{\geq 1} \).

We now decompose \( Y \) with respect to the third \( t \)-structure:

\[
\tau_{\leq 0}^{3} Y \to Y \to \tau_{\geq 1}^{3} Y.
\]  (18)

Since \( T_{1+2:3} \subseteq T_3 \), the pair \( (T_{1+2:3}, T_3) \) is upper consistent. Hence \( \tau_{\geq 1}^{3} Y \in D_{1+2:3}^{\geq 1} \).

By Remark 2 we again have \( \tau_{\leq 0}^{3} Y \in D_{1+2:3}^{\geq 1} \). Since \( T_{(1+2)3} \subseteq T_3 \), we have
Hom(\(D_{(1+2)3}^{\leq 0}, D_{3}^{\geq 1}\)) = 0. It follows that Hom(\(X, \tau_{\geq 1}^{3}Y\)) = 0. Applying Hom(\(X, \cdot\)) to (18), we see that it suffices to show that Hom(\(X, \tau_{\leq 0}^{2}Y\)) = 0. Hence our proof of (*) reduces to the following assertion.

(**) If \(X \in D_{(1+2)3}^{\leq 0}) \cap D_{1+2}^{\geq 1}\) and \(Y \in D_{3}^{\leq 0} \cap D_{1+2,3}^{\geq 1}\), then Hom(\(X, Y\)) = 0.

We claim that
\[
\begin{align*}
a) & \quad D_{(1+2)3}^{\leq 0} \cap D_{1+2}^{\geq 1} \subset D_{2}^{\leq 0}; \\
b) & \quad D_{3}^{\leq 0} \cap D_{1+2,3}^{\geq 1} \subset D_{2}^{\geq 1}.
\end{align*}
\]
Indeed, since the pair \((T_{1}, T_{2})\) is upper consistent, we have \(\tau_{\geq 1}^{1+2} = \tau_{\geq 1}^{2} \tau_{\geq 1}^{1}\) by Proposition 6. It follows that if \(X \in D_{1+2}^{\geq 1}\), then \(\tau_{\geq 1}^{1+2}X = \tau_{\geq 1}^{2}X\). If, in addition, \(X \in D_{(1+2)3}^{\leq 0}\), then we have \(\tau_{\geq 1}^{2}X = \tau_{\geq 1}^{1+2}X = 0\) because \(T_{(1+2)3} \subseteq T_{1+2}\). This proves a).

Dually, since the pair \((T_{2}, T_{3})\) is lower consistent, we have \(\tau_{\leq 0}^{2:3} = \tau_{\leq 0}^{2} \tau_{\leq 0}^{3}\) by Proposition 6. Hence \(\tau_{\leq 0}^{2:3}Y = \tau_{\leq 0}^{2}Y\) for all \(Y \in D_{3}^{\leq 0}\). If, in addition, \(Y \in D_{1+2,3}^{\geq 1}\), then \(\tau_{\leq 0}^{2}Y = \tau_{\leq 0}^{2:3}Y = 0\) because \(T_{2:3} \subseteq T_{1+2:3}\). This proves b).

Since (**) obviously follows from a) and b), the theorem is proved. □

We shall also use another incarnation of the modular law.

**Theorem 16** (modular law 2). Let \((T_{1}, T_{2}, T_{3})\) be a triple of \(t\)-structures such that \(T_{1} \leq T_{3}\), the pair \((T_{1}, T_{2})\) is upper consistent and \((T_{3}, T_{2})\) is lower consistent. Then
\[
\begin{align*}
\text{(i)} & \quad \text{the pair } (T_{1}, T_{2} \cdot T_{3}) \text{ is upper consistent}, \\
\text{(ii)} & \quad \text{the pair } (T_{3}, T_{1} + T_{2}) \text{ is lower consistent}, \\
\text{(iii)} & \quad (T_{1} + T_{2}) \cdot T_{3} = T_{1} + T_{2} \cdot T_{3}.
\end{align*}
\]

We note that in contrast to Theorem 15 the lower consistency of \((T_{3}, T_{1} + T_{2})\) is not a formal consequence of the propositions in §1.3.

**Proof of Theorem 16.** Since \(T_{1} \leq T_{3}\), the pair \((T_{1}, T_{3})\) is upper consistent. Hence Proposition 9 is applicable to the triple \((T_{1}, T_{3}, T_{2})\) and yields that the pair \((T_{1}, T_{2} \cdot T_{3})\) is upper consistent. This proves part (i) and the existence of \(T_{1+2:3}\).

We know from Proposition 6 that \(\tau_{\leq 0}^{2:3} = \tau_{\leq 0}^{2} \tau_{\leq 0}^{3}\). Then, for every \(X \in D\), the octahedron axiom yields the following diagram with exact rows and columns:

\[
\begin{array}{cccccc}
\tau_{\geq 1}^{3} \tau_{\leq 0}^{2}X & \rightarrow & \tau_{\geq 1}^{2:3}X & \rightarrow & \tau_{\geq 1}^{2}X \\
\tau_{\leq 0}^{2}X & \rightarrow & X & \rightarrow & \tau_{\geq 1}^{2}X \\
\tau_{\leq 0}^{3} \tau_{\leq 0}^{2}X & \sim & \tau_{\leq 0}^{3} \tau_{\leq 0}^{2}X & \rightarrow & 0
\end{array}
\]

By Proposition 6, \(\tau_{\geq 1}^{1+2} = \tau_{\geq 1}^{2} \tau_{\geq 1}^{1}\) and \(\tau_{\geq 1}^{1+2:3} = \tau_{\geq 1}^{2:3} \tau_{\geq 1}^{1}\). Hence, substituting \(X = \tau_{\geq 1}^{1}Y\) in the upper row of the octahedron (19), we get an exact triangle,

\[
\tau_{\geq 1}^{3} \tau_{\leq 0}^{2} \tau_{\geq 1}^{1}Y \rightarrow \tau_{\geq 1}^{2:3+1}Y \rightarrow \tau_{\geq 1}^{1+2}Y.
\]
Now, substituting $Y = \tau_{\leq 0}^{1+2} Z$ in (20), we get a natural isomorphism,

$$\tau_{\leq 0}^3 \tau_{\leq 0}^{1+2} \tau_{\leq 1}^1 Z \simeq \tau_{\geq 1}^{1+2} \tau_{\leq 0}^{1+2} Z.$$ 

It follows that for every $Z \in D$ we have

$$\tau_{\leq 0}^3 \tau_{\leq 0}^{1+2} \tau_{\leq 0}^{1+2} Z = 0.$$

Clearly, $T_{1+2,3} \leq T_3$. Hence Lemma 11 yields that $\tau_{\leq 0}^3$ and $\tau_{\geq 1}^{1+2}$ commute. It follows that

$$\tau_{\geq 1}^{1+2} \tau_{\leq 1}^1 \tau_{\leq 0}^{1+2} = 0.$$

Equivalently, the image of $\tau_{\leq 0}^{1+2} \tau_{\leq 0}^{1+2}$ lies in $D_{1+2,3}^{\leq 0}$. On the other hand, $D_{1+2,3}^{\leq 0} \subset D_{1+2}^{\leq 0} \cap D_3^{\leq 0}$. Hence the image of $\tau_{\leq 0}^{1+2} \tau_{\leq 0}^{1+2}$ lies in $D_{1+2}^{\leq 0} \cap D_3^{\leq 0}$. Then $\tau_{\leq 0}^3 D_{1+2}^{\leq 0} \subset D_{1+2}^{\leq 0}$. This proves the lower consistency of $(T_3, T_1 + T_2)$.

By Proposition 6, $\tau_{\leq 0}^{1+2} \tau_{\leq 0}^{1+2}$ is the truncation functor for $T_{(1+2),3}$. Its image is $D_{1+2,3}^{\leq 0} = D_{1+2}^{\leq 0} \cap D_3^{\leq 0}$. We have already proved that this image lies in $D_{1+2,3}^{\leq 0}$. Combining this with the opposite inclusion (15), we get part iii). □

The following assertion is dual to Theorem 16.

**Theorem 17** (modular law 2'). Let $(T_1, T_2, T_3)$ be a triple of $t$-structures such that $T_1 \leq T_3$, the pair $(T_2, T_1)$ is upper consistent and $(T_2, T_3)$ is lower consistent. Then

(i) the pair $(T_2 \cdot T_3, T_1)$ is upper consistent,
(ii) the pair $(T_1 + T_2, T_3)$ is lower consistent,
(iii) $(T_1 + T_2) \cdot T_3 = T_1 + T_2 \cdot T_3$.

§ 2. Sets with consistencies and pairs of chains

For the sake of clear and logical exposition, we formalize the properties of consistency proved in § 1 and give a convenient pictorial description. Then we prove that consistent pairs of chains generate distributive lattices.

**2.1. Sets with consistencies.** By a set with consistencies we mean a partially ordered set containing 0 and 1 and endowed with two binary (non-reflexive) relations (called the upper and lower consistencies) such that for every lower-consistent (resp. upper-consistent) pair, the abstract intersection (resp. union) exists and satisfies the following axioms (where the prime means the dual axiom).

SC1. If $a \leq b$, then the pairs $(a, b)$ and $(b, a)$ are upper and lower consistent.
SC2. Persistence of lower consistency under intersection as in Proposition 7.
SC2'. Persistence of upper consistency under union as in Proposition 8.
SC3. Persistence of upper consistency under intersection as in Proposition 9.
SC3'. Persistence of lower consistency under union as in Proposition 10.
SC4. Modular law as in Theorem 15.
SC5. Modular law as in Theorem 16.
SC5'. Modular law as in Theorem 17.
In this formalism we omit any mention of $t$-structures from all quoted assertions. We shall sometimes refer directly to these assertions instead of the corresponding axioms for general sets with consistencies.

Note that the standard postulates (8)–(11) automatically hold for sets with consistencies. It follows from axiom SC1 that 0 and 1 form a lower- and upper-consistent pair with any element.

In contrast to lattices, a set with consistencies is not an abstract algebra.

2.2. Diagrams. It is instructive to use the following pictorial description of consistencies and operations with them.

Let $V$ be a set with consistencies. We consider a graph $\Gamma(V)$ whose vertices are labelled by elements of $V$. Draw a solid arrow

$$a \rightarrow b$$

if the pair $(a, b)$ is lower consistent in $V$, and a dashed arrow

$$a \dashrightarrow b$$

if $(a, b)$ is upper consistent.

Then the operations of union and intersection are interpreted as the contraction of solid or dashed arrows. For the intersection this takes the form

$$a \rightarrow b \Rightarrow a \cdot b,$$

and for the union it takes the form

$$a \dashrightarrow b \Rightarrow a + b.$$

The duality of $t$-structures described in §1.1 replaces solid arrows by dashed ones (and vice versa) and reverses their direction.

We shall actually consider subgraphs of $\Gamma(V)$ which ensure the possibility of iterating contractions.

Axioms SC2–SC3$'$ in the definition of a set with consistencies provide us with patterns of connecting a new vertex (resulting from the contraction of an arrow) to other (old) vertices of the graph by certain arrows. Axiom SC2 (Proposition 7) may be regarded as the possibility of contracting two edges of a triangle that are solid arrows:

$$a \rightarrow b \cdot c \iff b \rightarrow a \cdot b \rightarrow c.$$

Axiom SC2$'$ describes the same picture with dashed arrows. Axiom SC3 (Proposition 9) reads

$$a \dashrightarrow b \cdot c \Rightarrow a \dashrightarrow b \dashrightarrow c.$$
Axiom SC3’ reads

\[
\begin{array}{c}
\text{b} \\
\text{a} \\
\end{array} \quad \Rightarrow \quad \begin{array}{c}
\text{a + b} \\
\text{c} \\
\end{array}
\]

Note that there are no implications obtained from the last two pictures by replacing solid arrows by dashed ones and vice versa (and preserving the directions of arrows).

It is sometimes convenient to put the vertex marked with the intersection (resp. union) of \(t\)-structures at the midpoint of the corresponding solid (resp. dashed) arrow.

2.3. Universal lattices with consistencies. A morphism of sets with consistencies is a morphism of the corresponding partially ordered sets that takes every upper-consistent (resp. lower-consistent) pair to an upper-consistent (resp. lower-consistent) pair. A set with consistencies which is a lattice is called a lattice with consistencies. Morphisms of lattices with consistencies are defined in the obvious way.

A morphism \(\phi: P \to L\), where \(P\) is a partially ordered set and \(L\) is a lattice, is an order-preserving map such that \(\phi(x + y) = \phi(x) + \phi(y)\) and \(\phi(xy) = \phi(x)\phi(y)\) whenever \(x + y\) or \(xy\) exists.

Consider a partially ordered set \(P\) endowed with two arbitrary binary operations. We are interested in morphisms \(P \to L\) of \(P\) to a lattice \(L\) with consistencies such that the image of any pair of elements of \(P\) belonging to the first (resp. second) relation is lower consistent (resp. upper consistent) in \(L\). Such a morphism \(P \to U\) is said to be universal if for every other morphism \(P \to L\) of this kind there is a unique morphism \(U \to L\) of lattices with consistencies that makes the following diagram of morphisms commutative:

\[
\begin{array}{ccc}
U & \to & L \\
\downarrow & & \downarrow \\
P & \to & U \\
\end{array}
\]

Then we call \(U = U(P)\) the universal lattice with consistencies generated by \(P\).

Omitting the consistency relations in the definitions above, we get the notion of the universal lattice \(L(P)\) generated by a partially ordered set \(P\). The universal lattice \(L(P)\) exists by the classical results in [1]. Indeed, we recall that lattices are nothing but abstract algebras satisfying the standard postulates. Consider the free abstract algebra \(F(P)\) generated by \(P\) as a disjoint set, and take its quotient with respect to the congruence generated by the relations \(x + y = “x + y”\) and \(x \cdot y = “x \cdot y”\) for all \(x\) and \(y\) whose union “\(x + y”\) (resp. intersection “\(x \cdot y”\) exists in \(P\) (see §1.4). This quotient is \(L(P)\).

A lattice is said to be distributive if

\[
\begin{align*}
x(y + z) &= xy + xz, \\
x + yz &= (x + y)(x + z)
\end{align*}
\]

for all \(x, y, z\). These postulates are mutually dual and each of them implies the other.
One can similarly define the universal distributive lattice $D(P)$ generated by a partially ordered set $P$. Namely, $D(P)$ is the quotient of $L(P)$ with respect to the congruence generated by the equivalences $(x + y)z \sim xz + yz$ for all $x, y, z \in L(P)$.

**Proposition 18.** Let $P$ be a partially ordered set with two binary relations. Then there is a universal lattice with consistencies generated by $P$.

*Proof.* We consider the universal lattice $L(P)$ generated by $P$ and endow it with the following two binary relations, which will be called upper and lower consistency respectively. First, we require that the images in $L(P)$ of all pairs in $P$ belonging to the first (resp. second) given binary relation are upper consistent (resp. lower consistent). Second, we extend the two binary relations of consistency by adding the minimal set of pairs that satisfy axioms SC1–SC3' and those parts of axioms SC5, SC5' that imply consistencies (we recall that not all of these implications follow formally from the other axioms).

Note that this procedure does not make $L(P)$ into a set with consistencies because the modular equations in axioms SC4–SC5' are not yet satisfied. To achieve this, we consider the quotient lattice $L_1(P)$ of $L(P)$ with respect to the congruence generated by the equivalences $(t_1 + t_2)t_3 \sim t_1 + t_2t_3$ for all triples $(t_1, t_2, t_3) \in L(P)$ that satisfy the hypotheses of one of the modular laws SC4–SC5'. Endow $L_1(P)$ with the consistency relations inherited from $L(P)$. Clearly, axiom SC1 follows from the fact (proved in §1.4) that $x' \leq y'$ in $L_1(P)$ if and only if there are $x, y \in L(P)$ such that $x \sim x'$, $y \sim y'$ and $x \leq y$ in $L(P)$.

But the other axioms need not hold in $L_1(P)$. Therefore we subject $L_1(P)$ (with its consistency relations) to the same procedure as we did for $L(P)$. Namely, we extend the binary relations appropriately and take a similar quotient. This produces $L_2(P)$ from $L_1(P)$. Iterating this process, we get a sequence of lattices $L_i(P)$ along with maps $P \to L_i(P)$ which are compatible with the quotient maps. Then $U(P) = \lim L_i(P)$ with the two colimit binary relations is the universal lattice with consistencies.

Indeed, consider any morphism $P \to L$, where $L$ is a lattice with consistencies and the morphism sends pairs from the first (resp. second) binary relation on $P$ to lower-consistent (resp. upper-consistent) pairs in $L$. Since the morphism $P \to L(P)$ is universal, there is a lattice homomorphism $L(P) \to L$ that makes the corresponding diagram commutative. This homomorphism clearly descends to homomorphisms $L_i(P) \to L$ whose limit is a homomorphism $U(P) \to L$. It is easy to see that this homomorphism is unique. □

A distributive lattice may be regarded as a lattice with consistencies where all pairs of elements are upper and lower consistent. Since modularity follows from distributivity, this is indeed a well-defined lattice with consistencies. By the universality of $U(P)$, we have a canonical homomorphism

$$\psi: U(P) \to D(P).$$

This is clearly an epimorphism since the congruences used to construct $U(P)$ as a quotient of $L(P)$ in Proposition 18 are among those used to construct $D(P)$ as a quotient of $L(P)$.
2.4. Consistent pairs of chains. A chain in a partially ordered set is a totally ordered sequence of elements. A pair of chains  in a set with consistencies is called a consistent pair of chains if the pairs , are upper and lower consistent for all pairs of indices . Our aim in this subsection is to prove that a consistent pair of chains generates (by means of union and intersection) a distributive lattice.

There is a theorem of Birkhoff [1] that two chains in a modular lattice generate a distributive lattice. Passing to sets with consistencies, we encounter two new difficulties (compared to the hypotheses of Birkhoff’s theorem). First, the intersection and union do not exist a priori. Second, the modular laws hold only under certain consistency conditions.

The notions of sum and intersection of any number of elements in a partially ordered set are defined intrinsically. For example, the sum of the elements of a given subset is the element which is greater than any element in but smaller than any other element with this property. The sum may not exist. The following trivial observation strengthens slightly the associativity postulates (10).

**Lemma 19.** Let be a triple of elements in a partially ordered set. If exists, then the existence of either side of the following equation implies that of the other and the equation holds:

\[(a + b) + c = a + b + c.\] (21)

In the dual assertion we replace + by ·.

**Proof.** This is obvious. □

**Remark 20.** In the constructions of this subsection, Lemma 19 is the only tool for deducing the existence of the union (resp. intersection) for pairs of elements (here ) which are not a priori upper consistent (resp. lower consistent). A typical situation is as follows. We are given a triple of elements (say, -structures) with for . Then exists by Proposition 10 and Lemma 19. Hence neither nor is a priori lower consistent.

Throughout this subsection we assume that there is a consistent pair of chains and . We put

\[u_{ij} = a_ib_j, \quad \nu_{ij} = a_i + b_j.\]

**Lemma 21.** If , then the pair is upper consistent. Dually, the pair is lower consistent for .

**Proof.** We can assume that . Indeed, if , then , whence is upper consistent.

Applying axiom SC3 to the triple , we see that the pair is upper consistent.

Hence axiom SC5‘ (modular law 2‘) is applicable to the triple . We conclude that the pair is upper consistent. The rest follows by duality. □
We recall that a decomposition \( x = x_1 + \cdots + x_n \) is said to be \textit{irreducible} if \( x \neq x_1 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_n \) for all \( i \).

**Proposition 22.** Any finite sum of the elements \( u_{ij} \) exists and has a decomposition of the form
\[
  u = u_{i_1 j_1} + \cdots + u_{i_k j_k}, \quad i_1 < \cdots < i_k, \quad j_1 < \cdots < j_k. \tag{22}
\]

Dually, any finite product of the \( v_{ij} \) exists and is of the form
\[
  v = v_{i_1 j_1} \cdots v_{i_k j_k}, \quad i_1 < \cdots < i_k, \quad j_1 < \cdots < j_k. \tag{23}
\]

**Proof.** Every sum of the form (22) exists. Indeed, the summation can be done step by step from left to right using Lemma 21 and axiom SC2'. The rest of the proof is by induction on the number of summands \( u_{ij} \). Suppose that we add an element \( u_{ij} \) to a sum \( u \) of the form (22). It suffices to prove the existence of
\[
  u' = u + u_{ij}. \tag{24}
\]

Indeed, if this sum exists, then we can (if necessary) reduce the decomposition of \( u' \) to get an irreducible decomposition. Every irreducible decomposition is automatically of the form (22) because otherwise there would exist summands \( u_{pq} \) and \( u_{rs} \) with \( u_{pq} \leq u_{rs} \).

If \( i \leq i_t \) and \( j \geq j_t \) for some \( t \), then \( u_{ij} \geq u_{i_t j_t} \). Similarly, if \( i \geq i_t \) and \( j \leq j_t \) for some \( t \), then \( u_{ij} \leq u_{i_t j_t} \). In both cases, one of the two elements in the sum for \( u' \) (either \( u_{ij} \) or \( u_{i_t j_t} \)) can be absorbed by the rule (11). Then the sum \( u' \) exists by Lemma 19 and the induction hypothesis.

If none of the inequalities \( i \leq i_t \) and \( j \geq j_t \) hold for some \( t \), then we can order all the summands in the decomposition (24) for \( u' \) to get a decomposition of the form (22). We have already seen that every such sum exists. \( \square \)

A chain is said to be \textit{extended} if it contains \( 0 \) and \( 1 \) as extreme elements.

**Proposition 23.** Let \( a_1 \geq \cdots \geq a_n \) and \( b_1 \leq \cdots \leq b_m \) be a consistent pair of extended chains. We choose two ordered sets of indices \( I = \{i_1 \leq \cdots \leq i_k\} \), \( i_t \in [1, n] \), and \( J = \{j_1 \leq \cdots \leq j_k\} \), \( j_t \in [1, m] \), and put
\[
  r_{IJ} = a_{i_1} (b_{j_1} + a_{i_2}) \cdots (b_{j_{k-1}} + a_{i_k}) b_{j_k},
\]
\[
  s_{IJ} = a_{i_1} b_{j_1} + a_{i_2} b_{j_2} + \cdots + a_{i_k} b_{j_k}.
\]

Then
1. \( r_{IJ} = s_{IJ} \),
2. \( a_i \longrightarrow s_{IJ} \longrightarrow b_j \) for all \( (i, j) \in [1, n] \times [1, m] \).

**Remark 24.** If we extend the chains \( \{a_i\}_{i \in I} \) and \( \{b_j\}_{j \in J} \) by \( 0 \), then the formula for \( r_{IJ} \) is simply the product of suitable \( v_{ij} \). We allow the indices in \( \{a_i\} \) and \( \{b_j\} \) to coincide in order to simplify the presentation of the inductive proof of Proposition 23.
Proof of Proposition 23. Both of the elements \( r_{IJ} \) and \( s_{IJ} \) exist by Proposition 22.

We prove parts (i), (ii) by simultaneous induction on \( k = |I| = |J| \). For \( k = 1 \) we have \( r_{IJ} = a_{i_1} b_{j_1} = s_{IJ} \) and part (ii) follows from axioms SC2 and SC3'.

Assume that part (ii) holds for \( |I| = |J| = k - 1 \).

Put \( I' = I \setminus i_k \) and \( J' = J \setminus j_k \). Clearly, \( s_{I',J'} \leq a_{i_k} \). By the induction hypothesis we have \( a_{i,j} \dashv \rightarrow s_{I',J'} \). Hence axiom SC5' (Theorem 17) is applicable to the triple \((s_{I',J'}, a_{i_k}, b_{j_k})\), and we get

\[
(r_{I',J'} + a_{i_k}) b_{j_k} = (s_{I',J'} + a_{i_k}) b_{j_k} = s_{I',J'} + a_{i_k} b_{j_k} = s_{IJ}.
\]  

(25)

We claim that the left-hand side of (25) is equal to \( r_{IJ} \). Indeed, decompose \( r_{I',J'} \) into a product \( r_{I',J'} = q b_{j_{k-1}} \), where

\[ q = a_{i_1} (b_{j_1} + a_{i_2}) \cdots (b_{j_{k-2}} + a_{i_{k-1}}). \]

Since the second chain is extended, its last element \( b_m \) is equal to 1. Then \( q = s_{I',J''}', \) where \( J'' = \{j_1, \ldots, j_{k-2}, m\} \). Applying the induction hypothesis to the subsets \( I' \) and \( J'' \), we get \( q \rightarrow b_{j_{k-1}} \). Since we also have \( a_{i_k} \leq q \) and \( a_{i_k} \rightarrow b_{j_{k-1}} \), axiom SC5 is applicable to the triple \((a_{i_k}, b_{j_{k-1}}, q)\):

\[ r_{I',J'} + a_{i_k} = q b_{j_{k-1}} + a_{i_k} = q(b_{j_{k-1}} + a_{i_k}). \]

It follows that the left-hand side of (25) is equal to \( r_{IJ} \). This proves part (i).

Since \( a_{i_k} \rightarrow s_{I',J'} \), the induction hypothesis shows that one can apply axioms SC2' and SC3' to get

\[ a_i \rightarrow (s_{I',J'} + a_{i_k}) \rightarrow b_j \]

for all \((i,j)\). Since \( s_{I',J'} + a_{i_k} \rightarrow b_{j_k} \), one can then apply axioms SC2 and SC3 to get

\[ a_i \rightarrow (s_{I',J'} + a_{i_k}) b_{j_k} \rightarrow b_j. \]

It follows from (25) that the middle term here coincides with \( s_{IJ} \). This proves part (ii). \( \square \)

Theorem 25. A consistent pair of chains in a set with consistencies generates in this set a distributive lattice all of whose elements are of the form (22).

Proof. Adding, if necessary, 0 and 1 as extreme elements, we can assume that the chains are extended. Then all \( a_i \) and \( b_j \) are among the \( u_{ij} \). By Proposition 22, any sum of \( u_{ij} \) exists and is of the form (22).

We claim that any product of elements of the form (22) exists and is of the same form. Indeed, by Proposition 23, any sum of the form (22) may be rewritten as the product of some of the \( v_{ij} \) (the elements \( a_{i_1} \) and \( b_{j_k} \) in the formula for \( r_{IJ} \) may be interpreted as \( v_{i_1 m} \) and \( v_{1 j_k} \) respectively). The product of these elements exists by Proposition 22 and is of the form (23), which may in turn be rewritten as a sum of the form (22) by Proposition 23. This proves that the set of elements of the form (22) is a lattice \( L \). This is the lattice generated by the pair of chains.

Consider the partially ordered set \( P = \sigma \cup \tau \), which is the disjoint union of the chains \( \sigma = \{a_1 \geq \cdots \geq a_n\} \) and \( \tau = \{b_1 \leq \cdots \leq b_m\} \) as an abstract partially ordered set. Let \( D(P) \) be the universal distributive lattice generated by \( P \).
We endow $P$ with two coinciding binary relations which are formed by all the pairs $(a_i, b_j)$. Let $U(P)$ be the universal lattice with consistencies generated by $P$ (this lattice exists by Proposition 18) and let $\psi: U(P) \to D(P)$ be the canonical epimorphism. We claim that it is an isomorphism.

Indeed, since $U(P)$ is a quotient of the universal lattice $L(P)$, it contains only elements of the form (22). To prove that $\psi$ is an isomorphism, it thus suffices to construct a distributive lattice generated by the two chains in such a way that all elements of the form (22) are taken by $\psi$ to distinct elements.

Such a lattice may be realized as a sublattice of subsets in the finite set of integer points in the rectangle $\{(x,y) \in [1,m] \times [-n,1]\}$. The elements $a_i$ (resp. $b_j$) are represented by the subsets $\{ y \leq -i \}$ (resp. $\{ x \leq j \}$). Each element of the lattice is the set of integer points under a staircase descending from the point $(1,0)$ to the point $(m,-n)$. Being a sublattice of the lattice of subsets, it is distributive and the images of distinct elements of the form (22) under $\psi$ are obviously distinct. Hence $\psi$ is injective. It follows that the lattice $U(P)$ is distributive and, by the universality property, $L$ is also distributive. \[ \square \]

Remark 26. An element $x$ of a lattice is said to be indecomposable if $x = a + b$ implies that $x = a$ or $x = b$. Otherwise $x$ is said to be decomposable. By another theorem of Birkhoff [1], every element of a finite distributive lattice can be written uniquely as an irreducible sum of indecomposable elements. Clearly, all indecomposable elements are among the $u_{ij}$. We easily see that an element $u_{ij}$ is indecomposable in $L$ if and only if $u_{ij} = u_{i+1,j} + u_{i,j-1}$.

§ 3. Perverse coherent sheaves

3.1. Grothendieck duality and the dual $t$-structure. Let $k$ be a field of characteristic zero. In this section we assume for simplicity that all schemes are of finite type over $k$. Given such a scheme $X$, we consider the bounded derived category $D(X) := D_{\text{coh}}^b(X)$ of coherent sheaves. We shall construct some perverse $t$-structures in $D(X)$ by using the technique developed in §§1, 2.

Let $T$ be the standard $t$-structure in $D(X)$ with $D_{\leq 0}$ (resp. $D_{\geq 0}$) consisting of the complexes of coherent sheaves on $X$ whose cohomology sheaves are trivial in positive (resp. negative) degrees.

We denote the $i$th cohomology sheaf of a complex $F \in D(X)$ by $H^i(F)$, and let $O_X$ be the structure sheaf of rings on $X$. We write $f_*$, $f^*$, $\mathcal{H}om$ for the derived push-forward and pull-back functors and the derived local Hom functor. Given a functor $\Phi: D(X) \to D(Y)$, we write $\mathbb{R}^k\Phi := \mathcal{H}^k \cdot \Phi$ for the corresponding cohomology functors $D(X) \to \text{Coh}(Y)$. Here $\Phi$ is usually the derived functor of a left-exact functor $\text{Coh}(X) \to \text{Coh}(Y)$. If $\Phi$ is the derived functor of a right-exact functor $\text{Coh}(X) \to \text{Coh}(Y)$, then we follow the tradition of writing the cohomology functors as $\mathbb{L}^k\Phi := \mathcal{H}^{-k} \cdot \Phi$.

For every morphism $f: X \to Y$ there is a twisted inverse image functor $f^!: D_{\text{qcoh}}^+(Y) \to D_{\text{qcoh}}^+(X)$ between the lower-bounded derived categories of quasi-coherent sheaves (see [11] and [12]). It is defined by the formula

$$f^!(\cdot) = f^*(-) \otimes f^!(O_Y),$$
where the object \( f^!(\mathcal{O}_Y) \) is defined by applying the functor right adjoint to \( f_* \) to the structure sheaf \( \mathcal{O}_Y \). When \( f \) is proper and of finite Tor-dimension, \( f^! \) coincides with the right adjoint for \( f_* \), and \( f^! \) takes \( \mathcal{D}(Y) \) to \( \mathcal{D}(X) \). In the particular case when \( f \) is a closed embedding, this property gives an obvious construction for \( f^! \) at least locally over the base. When \( f \) is smooth of relative dimension \( n \), \( f^! \) is defined via twisting by the relative canonical class \( \omega_{X/Y} \):

\[
f^!(\mathcal{O}_Y) = f^*(\mathcal{O}_Y) \otimes \omega_{X/Y}[n].
\]

If \( f \) is projective, it can be written as \( f = pi \), where \( i \) is a closed embedding and \( p \) is smooth. Then \( f^! = p^! i^! \). We also note that \( f \) is automatically of finite Tor-dimension if \( Y \) is a smooth variety.

For a proper morphism \( f \) of finite Tor-dimension there is a duality isomorphism for local homomorphism functors:

\[
f_* \mathcal{H}om_X(\mathcal{F}, f^! \mathcal{G}) = \mathcal{H}om_Y(f_* \mathcal{F}, \mathcal{G}),
\]

which is natural with respect to \( \mathcal{F} \in \mathcal{D}(X) \), \( \mathcal{G} \in \mathcal{D}(Y) \) [13].

We write \( \omega_X^i \) for the dualizing complex on \( X \). By definition, \( \omega_X^i = \pi_X^* \mathcal{O}_{pt} \), where \( \pi_X : X \to pt \) is the projection to a point \( pt \). For a smooth variety \( X \) of dimension \( n \) we have \( \omega_X^i = \omega_X[n] \), where \( \omega_X = \Omega_X^n \) is the canonical sheaf of differential forms of top degree. For general \( X \) it is known that \( \omega_X^i \) lies in \( \mathcal{D}(X) \).

Given a closed embedding \( i : X \to Y \) of \( X \) in a smooth scheme \( Y \), we have

\[
\omega_X^i = i^! \omega_Y^i = i^! \omega_Y[\dim Y].
\]

The category \( \mathcal{D}(X) \) possesses a contravariant involutive exact functor \( D = D_X \). Its action on \( \mathcal{F} \in \mathcal{D}(X) \) is defined by

\[
D\mathcal{F} = D_X \mathcal{F} = \mathcal{H}om_X(\mathcal{F}, \omega_X^i).
\]

The functor \( D \) is compatible with the triangulated structure. In other words, \( DT \) is naturally isomorphic to \( T^{-1} D \), and \( D \) takes every exact triangle \( A \to B \to C \) to an exact triangle \( DC \to DB \to DA \).

If \( j : U \to X \) is an open embedding, then the dualizing complex over \( U \) is obtained by restriction of the dualizing complex on \( X \):

\[
\omega_U^i = j^* \omega_X^i.
\]

It follows that \( D \) is local, that is, for every open embedding as above, we have

\[
j^* D_X = D_U j^*.
\]

We write \( D^{(i)} \mathcal{F} := \mathcal{H}^i(D\mathcal{F}) \) for the cohomology sheaves of \( D\mathcal{F} \). The functor \( D \) preserves \( \mathcal{D}(X) \), and \( D \cdot D \) is naturally isomorphic to the identity functor [13].

Using \( D \), we can define another \( t \)-structure \( \widetilde{T} = (\widetilde{D}^{\leq 0}, \widetilde{D}^{\geq 1}) \) on \( \mathcal{D}(X) \) by requiring that

\[
\mathcal{F} \in \widetilde{D}^{\leq 0} \iff D\mathcal{F} \in \mathcal{D}^{\geq 0},
\]

\[
\mathcal{F} \in \widetilde{D}^{\geq 0} \iff D\mathcal{F} \in \mathcal{D}^{\leq 0}.
\]

Since \( D \) is an anti-equivalence, \( \widetilde{T} \) is a \( t \)-structure. We call it the dual \( t \)-structure. Note that the dualizing complex is a pure object for the dual \( t \)-structure.
3.2. Consistent pairs of chains and perverse sheaves. We denote the truncation functors for $T$ (resp. $\tilde{T}$) by $\tau_{\leq k}$, $\tau_{\geq k}$ (resp. $\tilde{\tau}_{\leq k}$, $\tilde{\tau}_{\geq k}$).

It follows from the definition of $\tilde{T}$ that

$$
\tilde{\tau}_{\leq r} = D\tau_{\geq -r}, \quad \tilde{\tau}_{\geq r} = D\tau_{\leq -r}.
$$

(33)

**Proposition 27.** Let $X$ be a scheme of finite type over $k$. Suppose that $G \in D^{\geq 0}(X)$. Then $D(\tau_{\leq r}DG) \in D^{\geq 0}(X)$ for every $r \in \mathbb{Z}$.

**Proof.** In view of (30), the statement of the proposition is local. Therefore we may assume that $X$ is affine and (being of finite type) embeddable in a smooth variety over $k$. We fix a closed embedding $i: X \to Y$ of $X$ in a smooth variety $Y$ of dimension $l$. Since $i$ is a proper morphism of finite Tor-dimension, we see from (26) and (27) that, for every $G \in D(X)$,

$$
i_*DG = i_*\mathcal{H}om_X(G, \omega_X) = i_*\mathcal{H}om_X(G, i^!\omega_Y) = \mathcal{H}om_Y(i_*G, \omega_Y[l]).
$$

Since the functor $i_*$ is exact with respect to the standard $t$-structure, we have

$$
i_*D^{(k)}G = \mathbb{R}^k i_*DG = \mathbb{R}^k \mathcal{H}om_Y(i_*G, \omega_Y[l]).
$$

(34)

Suppose that $G \in D^{\geq 0}(X)$. Then $\mathbb{R}^k i_*G$ is equal to zero for $k < 0$.

Consider the spectral sequence with $E^{m,j}_2 = \mathcal{E}xt^m_Y(\mathbb{R}^j i_*G, \omega_Y)$ that converges to $\mathbb{R}^m \mathcal{H}om_Y(i_*G, \omega_Y[l])$. Since $\omega_Y$ is locally free, the sheaves $\mathcal{E}xt^j_Y(\mathcal{F}, \omega_Y)$ have supports of codimension $\geq k$ in $Y$ for every coherent sheaf $\mathcal{F}$ on $Y$ (compare [14], p. 142). Here we adopt the convention that the empty set (the support of the zero sheaf) has infinite codimension.

Since $E^{m,j}_2 = 0$ for $j > 0$, it follows from the spectral sequence that the support of $\mathbb{R}^m \mathcal{H}om_Y(i_*G, \omega_Y[l])$ has codimension $\geq s + l$. By (34), the same restriction on the codimension of the support holds for $i_*D^{(s)}G$. Since the cohomology sheaves of the truncation $\tau_{\leq r}i_*DG$ either coincide with those of $i_*DG$ or vanish, the codimension of the support of $\mathcal{H}^s(\tau_{\leq r}i_*DG)$ in $Y$ is not smaller than $s + l$ for every $r \in \mathbb{Z}$.

Consider the spectral sequence with $E^{j,s}_2 = D^j \mathcal{H}^{-s}(\tau_{\leq r}DG)$ that converges to the cohomology sheaves $\mathcal{H}^{j+s}(D(\tau_{\leq r}DG))$.

Using (34) and the exactness of $i_*$, we get

$$
i_*D^j \mathcal{H}^{-s}(\tau_{\leq r}DG) = \mathbb{R}^j \mathcal{H}om_Y(i_*\mathcal{H}^{-s}(\tau_{\leq r}DG), \omega_Y[l])
$$

$$= \mathcal{E}xt^{j+1}_Y(\mathcal{H}^{-s}(\tau_{\leq r}i_*DG), \omega_Y).
$$

(35)

Since $\mathcal{E}xt^j_Y(\mathcal{F}, \mathcal{E}) = 0$ for every locally free sheaf $\mathcal{E}$ and every coherent sheaf $\mathcal{F}$ whose support has codimension greater than $k$, we see that the right-hand side of (35) is trivial for $j < -s$.

Then it follows from the spectral sequence that

$$
i_* \mathcal{H}^l(D(\tau_{\leq r}DG)) = 0.
$$

□
We now regard the set of $t$-structures in $\mathcal{D}(X)$ as a set with consistencies, where the binary relations of consistency are defined as in §1.3. The sequences $\mathcal{T}[r]$ and $\mathcal{\tilde{T}}[r]$ may be regarded as chains in this partially ordered set. An ordered pair of $t$-structures is said to be $T$-consistent if the pair of chains obtained from these $t$-structures by applying the translation functor is consistent.

**Theorem 28.** The pair $(\mathcal{T}, \mathcal{\tilde{T}})$ of $t$-structures is $T$-consistent.

**Proof.** We must prove the upper and lower consistency of the pair $(\mathcal{T}[r], \mathcal{\tilde{T}}[r])$ for every $r \in \mathbb{Z}$.

If we take $F \in \mathcal{D}^{\leq 0}$ and put $G = DF$, then $G \in \mathcal{D}^{\geq 0}$. By Proposition 27, $D\tau_{\leq r}DG = D\tau_{\leq r}F$ belongs to $\mathcal{D}^{\geq 0}$. Hence $\tau_{\leq r}D^{\leq 0} \subset D^{\leq 0}$. This proves the lower consistency.

By (33) we have $\tau_{\geq r}D^{\geq 0} = D\tau_{\leq -r}D^{\geq 0} \subset D^{\geq 0}$. This proves the upper consistency. □

By Theorem 28, the pair of chains $\mathcal{T}[r]$ and $\mathcal{\tilde{T}}[r]$ is consistent. Hence, by Theorem 25, we obtain a distributive lattice of $t$-structures. The $t$-structures occurring in this lattice are in fact coherent versions of perverse $t$-structures (compare [2]).

We recall that the naive intersection (resp. union) of $t$-structures is defined by intersecting the subcategories $\mathcal{D}^{\leq 0}$ (resp. $\mathcal{D}^{\geq 1}$).

**Proposition 29.** Any (multiple) intersection or union of $t$-structures in the resulting lattice is naive.

**Proof.** The intersection and union of consistent pairs are naive. By Remark 20, it now suffices to verify that the union and intersection are naive in the situation of Lemma 19. Thus the desired result is a consequence of the following restatement of Lemma 19 for $t$-structures. If $a + b$ exists and is naive, then the existence and naivity of the element on one side of equation (21) imply that the element on the other side exists and is naive and the equation holds. The proof of this restatement is obvious. □

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Received 2/JUL/12
Translated by THE AUTHOR

7/OCT/12