INARIANT HOCHSCHILD COHOMOLOGY OF SMOOTH
FUNCTIONS

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Abstract. Given an action of a Lie group on a smooth manifold, we discuss
the induced action on the Hochschild cohomology of smooth functions, and
notions of invariance on this space. Depending on whether one considers in-
variance of cochains or invariance of cohomology classes, two different spaces
of invariants arise. We perform a general comparison of these notions, and
give an interpretation of the lower orders of the invariant cohomology spaces
and conclude as our main result that for proper group actions both spaces
are isomorphic. As a corollary and a geometric interpretation, an invariant
version of the Hochschild-Kostant-Rosenberg theorem is given, identifying the
cohomology of invariant cochains with invariant multivector fields. Using this
theorem, we shortly discuss the invariant Hochschild cohomology in the case
of homogeneous spaces.

1. Introduction

Hochschild cohomology, initially investigated by Hochschild in [10] and sig-
nificantly developed by Gerstenhaber in [4, 5, 6, 7, 8], is found to be a valuable tool in
defformation theory [13, 14]. Precisely, if one is interested in formal deformations
of a certain algebra \(A\), it is well-known that the lower orders of the corresponding
Hochschild cohomology \(HH^\bullet(A, A)\) characterize both the equivalence classes of
infinitesimal deformations and the obstructions to order-by-order continuation of
an arbitrary formal deformation, see [5].

As one motivating example, in formal deformation quantization, originally consi-
dered in [1], one is interested in constructing deformations of the algebra of smooth
functions \(C^\infty(M)\) on some manifold \(M\) in order to equip this commutative algebra
with a new, non-commutative multiplication. Indeed, here one is able to explicitly
calculate continuous Hochschild cohomology by means of the Hochschild-Kostant-
Rosenberg theorem, whereby the cohomology groups can be identified with multi-
vector fields on the manifold (see [11] for the original formulation of this theorem
in a purely algebraic setting and [14] for the differential case). However, to gain a
more refined, maybe even finite-dimensional classification, one needs to add further
information to the setting.

In the following, we want to consider the case where desirable cochains are
invariant under a certain symmetry of the manifold, here modeled as the action
of a Lie group. This idea is again motivated by deformation quantization, where

\begin{footnotesize}
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\item[Date:] 13-08-2018.
\item[Key words and phrases.] Differential geometry, homological algebra, deformation quantization.
\item[1]One such piece of information is the choice of a Poisson structure on the manifold, which one
requires deformations to respect in some sense. Notable mentions here are Fedosov’s results [3] in
the symplectic, and Kontsevich’s results [13, 14] in the general Poisson case.
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the manifold is considered a physical phase space, which in practice often admits certain symmetries (e.g., translational, rotational, scaling). The requirement of a given deformation to also reflect this symmetry then naturally leads one to consider invariant Hochschild cochains.

For this document, which began as the author’s Master thesis, we begin by describing the Hochschild cohomology in the most general setting in Section 2, while noting the most important specialties that arise in the case of the smooth functions. In Section 3, we describe how to equip this space with reasonable notions of invariance, and end up with two different notions: invariance of Hochschild cochains on one hand and invariance of Hochschild cohomology classes on the other. The invariant cochains naturally inherit the structure of a cochain complex, and we denote the cohomology of invariant cochains by \( HH^*_G(\mathcal{A}, N) \), whereas the space of invariant classes of the original cohomology is denoted \( HH^*(\mathcal{A}, N)^G \). There is a natural morphism

\[
\iota : HH^*_G(\mathcal{A}, N) \to HH^*(\mathcal{A}, N)^G
\]

relating both notions. Injectivity and surjectivity of this map yield nontrivial statements about how the differential interacts with the group action, and will be our focus of investigation in this document. In Section 4, we give explicit interpretations of the lower orders of the cohomology of invariant cochains, analogous to the ones given for the usual Hochschild cohomology in the context of deformation theory.

In Section 5, we prove our main result, which is that \( \iota \) is an isomorphism in the case where \( \mathcal{A} = C^\infty(M) \) is the space of smooth functions on a manifold and the group is acting properly on \( M \). We construct an averaging operator on the space of Hochschild cochains to prove injectivity and use the Hochschild-Kostant-Rosenberg theorem to prove surjectivity. In particular, this isomorphism allows us to formulate an invariant Hochschild-Kostant-Rosenberg theorem, relating cohomology of invariant cochains to invariant multivector fields on the given manifold. A few technical calculations are moved to the appendix.

2. Hochschild cohomology

We begin by recalling some basic definitions about the Hochschild complex and its cohomology.

Throughout this paper, we fix some field \( \mathbb{K} \), an associative \( \mathbb{K} \)-algebra \( \mathcal{A} \) and an \( \mathcal{A} \)-bimodule \( N \). While we want to keep the definitions general for now, our main results will concern the smooth functions \( C^\infty(M) = \mathcal{A} = N \) on a smooth manifold \( M \) and \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \).

**Definition 2.1 (Hochschild complex).** Define the space of Hochschild cochains via

\[
HC^*(\mathcal{A}, N) := \bigoplus_{n=0}^\infty HC^n(\mathcal{A}, N) := N \oplus \text{Hom}_\mathbb{K}(\mathcal{A}, N) \oplus \text{Hom}_\mathbb{K}(\mathcal{A} \otimes^2 \mathcal{A}, N) \oplus \text{Hom}_\mathbb{K}(\mathcal{A} \otimes^3 \mathcal{A}, N) \oplus \ldots.
\]
Define the Hochschild (co-)differential $\delta : \text{HC}^n(\mathcal{A}, N) \to \text{HC}^{n+1}(\mathcal{A}, N)$ for elements $\phi \in \text{HC}^n(\mathcal{A}, N)$ via
\[
(\delta \phi)(a_0, \ldots, a_n) := a_0 \cdot \phi(a_1, \ldots, a_n) + (-1)^n \phi(a_0, \ldots, a_{n-1}) \cdot a_n
\]
\[+ \sum_{i=0}^{n-1} (-1)^{i+1} \phi(a_0, \ldots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \ldots, a_n).
\]

We call the pair $(\text{HC}^*(\mathcal{A}, N), \delta)$ the Hochschild complex of $\mathcal{A}$ with values in $N$, which is pictured via
\[
0 \to N = \text{HC}^0(\mathcal{A}, N) \xrightarrow{\delta} \text{HC}^1(\mathcal{A}, N) \xrightarrow{\delta} \text{HC}^2(\mathcal{A}, N) \to \ldots
\]
Remember that $\delta^2 = 0$. The Hochschild cohomology $\text{HH}^*(\mathcal{A}, N)$ is defined to be the usual graded vector space
\[
\text{HH}^*(\mathcal{A}, N) := \ker \delta / \text{Im} \delta.
\]

**Remark 2.2.** In the case $N = \mathcal{A}$, where $\mathcal{A}$ is regarded an $\mathcal{A}$-bimodule by algebra multiplication, this space can be given the structure of a super Lie algebra, and the cohomology can be equipped with the structure of a Gerstenhaber algebra. While we do not use this fact here, we want to mention that all further results about group actions and invariants will be compatible with the Lie algebra/Gerstenhaber structures, so any time we talk about morphisms, they can be regarded as morphisms in the category of either super Lie algebras or Gerstenhaber algebras.

In the case $\mathcal{A} = N = C^\infty(M)$ of smooth functions on a smooth manifold $M$, which is the one we are most interested in, we can apply analytical methods to the Hochschild cochains if we restrict to “analytically interesting” cochains. Hence, denote by $\text{HC}^*_{\text{cont}}(C^\infty(M))$ the Hochschild complex of continuous cochains with respect to the Fréchet topology on $C^\infty(M)$. Accordingly, denote by $\text{HH}^*_{\text{cont}}(C^\infty(M))$ the corresponding cohomology.

**Remark 2.3.** Similarly, one can restrict to local, differential, and differential, vanishing on constants cochains. From [20] p.413f.] we cite
\[
\text{HC}^*_{n.c.diff}(C^\infty(M)) \subset \text{HC}^*_{\text{diff}}(C^\infty(M)) \subset \text{HC}^*_{\text{cont}}(C^\infty(M)) \subset \text{HC}^*_{\text{diff}}(C^\infty(M)),
\]
and they all have well-defined cohomologies, which have analogous notation.

3. **GROUP ACTIONS AND INVARIANT COHOMOLOGY**

3.1. **Definitions.** We will now define our notions of group actions on the different spaces and according invariant spaces. Let $G$ be some group acting on both $\mathcal{A}$ and $N$, in the sense that
\[
1) \text{ the group acts on } \mathcal{A} \text{ via algebra isomorphisms},
\]
\[
2) \text{ the group acts on } N \text{ via isomorphisms with respect to the Abelian group structure of } N, \text{ and, denoting the actions by } \triangleright,
\]
\[
(g \triangleright (a \cdot n)) = (g \triangleright a) \cdot (g \triangleright n), \quad (g \triangleright (n \cdot a)) = (g \triangleright n) \cdot (g \triangleright a),
\]
for all $g \in G, a \in \mathcal{A}, n \in N$.

Specifically in the previously mentioned case $\mathcal{A} = N = C^\infty(M)$, if we assume the existence of an action on the manifold, desired actions on $\mathcal{A}$ and $N$ are induced by pullback.

We can lift these actions to the space of Hochschild cochains:
Definition 3.1 (Space of invariant Hochschild cochains). Given actions on $\mathcal{A}$ and $N$ as above, define for all $n \in \mathbb{N}_0$
\[ \triangleright : G \times HC_n(\mathcal{A}, N) \to HC_n(\mathcal{A}, N) \]
(3.2) \[ (g \triangleright \phi)(a_1, \ldots, a_n) := g \triangleright (\phi(g^{-1} \triangleright a_1), \ldots, \phi(g^{-1} \triangleright a_n)). \]

We define the space of invariant Hochschild cochains via

(3.3) \[ HH_G^\ast(\mathcal{A}, N) := (HC^\ast(\mathcal{A}, N))^G = \{ \phi \in HC^\ast | g \triangleright \phi = \phi \quad \forall g \in G \}. \]

The Hochschild differential commutes with the group action and as such, we can restrict the Hochschild differential to a map

(3.4) \[ \delta : HH_G^\ast(\mathcal{A}, N) \to HH_G^\ast(\mathcal{A}, N). \]

Hence, the space of invariant cochains inherits the structure of a complex, and we can define the associated cohomology:

Definition 3.2 (Invariant Hochschild complex). The tuple \((HC_G^\ast(\mathcal{A}, N), \delta)\) is called the invariant Hochschild complex of $\mathcal{A}$ and $N$, equivalently pictured as the sequence

(3.5) \[ 0 \to N^G = HC_0^0(\mathcal{A}, N) \xrightarrow{\delta} HC_1^0(\mathcal{A}, N) \xrightarrow{\delta} HC_2^0(\mathcal{A}, N) \to \ldots \]

One then declares the cohomology of invariant cochains $HH_G^\ast(\mathcal{A}, N)$ to be the graded vector space defined by

(3.6) \[ HH_G^\ast(\mathcal{A}, N) := \frac{\ker \delta |_{HC_G^\ast(\mathcal{A}, N)}}{\text{Im} \delta |_{HC_G^\ast(\mathcal{A}, N)}}. \]

Like with standard Hochschild cohomology, for the case $\mathcal{A} = N = C^\infty(M)$ denote the invariant complexes with analytical properties (continuous, local etc.) and the respective cohomologies with the according subscript, e.g.

(3.7) \[ HC_G^\ast,\text{cont}(C^\infty(M)), HH_G^\ast,\text{cont}(C^\infty(M)) \]

and so on, whenever the action is compatible with the respective property.

One can also consider invariance on the level of cohomology rather than on the cochains: The action on the cochains canonically descends to one on the equivalence classes, so one may define the space of invariant classes

(3.8) \[ HH^\ast(\mathcal{A}, N)^G := \{ [\phi] \in HH^\ast(\mathcal{A}, N) | g \triangleright [\phi] = [\phi] \quad \forall g \in G \}. \]

Analogous spaces can be defined in the case $N = \mathcal{A} = C^\infty(M)$ for the analytical subcomplexes.

Note the difference between $HH_G^\ast(\mathcal{A}, N)$ and $HH^\ast(\mathcal{A}, N)^G$: when one wants to limit one’s framework to invariant cochains, e.g. when considering invariant deformations in deformation theory, the interesting space to consider is the cohomology of invariant cochains $HH_G^\ast(\mathcal{A}, N)$, as we will see in Section 4.

In general, however, we will see that this space cannot easily be related to the original Hochschild cohomology, hence, one is opening a whole new can of cohomological worms. The space of invariant classes $HH^\ast(\mathcal{A}, N)^G$, in contrast, is simply a subspace of the original cohomology.

The natural question arises whether the two notions of invariance can be related. As coboundaries in $HC_G^\ast(\mathcal{A}, N)$ can also be considered coboundaries in $HC^\ast(\mathcal{A}, N)$, there is a well-defined map

(3.9) \[ \iota : HH_G^\ast(\mathcal{A}, N) \to HH^\ast(\mathcal{A}, N)^G, [\phi] \mapsto [\phi]. \]
The main topic of this document will be proving that this map is an isomorphism in the case where we assume $\mathcal{A} = \mathbb{N} = C^\infty(M)$, proper actions, and a restriction to continuous cochains.

4. INTERPRETATION OF THE LOWER ORDER COHOMOLOGY SPACES

As with non-invariant Hochschild cohomology, it is possible to give lower orders of invariant Hochschild cohomology an interpretation. One quickly finds

\[
\begin{align*}
\text{HC}_0^G(\mathcal{A}, \mathbb{N}) &= \{ n \in \mathbb{N} \mid a \cdot n = n \cdot a \text{ and } g \cdot n = 0 \ \forall \ g \in G, a \in \mathcal{A} \}, \\
\text{HC}_1^G(\mathcal{A}, \mathbb{N}) &= \text{Der}(\mathcal{A}, \mathbb{N})^G / \text{InnDer}(\mathcal{A}, \mathbb{N})^G
\end{align*}
\]

in the lowest orders, where $Z_A(N)$ is the center of $N$ with respect to the $(\mathcal{A}, \mathcal{A})$-bimodule structure, and $^G$ denotes restricting to invariant objects. In the context of deformation theory, there are also direct interpretations for the orders 2 and 3, which we will translate into the invariant context in the following. We shortly recall a few necessary definitions of deformation theory, which can be looked up in more detail in \cite[Chapter 6]{20}. Fix an associative $\mathbb{K}$-algebra $\mathcal{A}$ and denote its multiplication by $\mu_0 : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$. Also, denote by $\mathcal{A}[\lambda]$ the algebra of formal power series in $\mathcal{A}$ in a parameter $\lambda$. In the following we set $\text{HC}^\bullet(\mathcal{A}) := \text{HC}^\bullet(\mathcal{A}, \mathcal{A})$.

Definition 4.1. A formal deformation of the algebra $\mathcal{A}$ is an associative map $\mu : \mathcal{A}[\lambda] \otimes \mathcal{A}[\lambda] \to \mathcal{A}[\lambda]$ which can be written in the form

\[
\mu = \mu_0 + \lambda \mu_1 + \lambda^2 \mu_2 + \ldots
\]

with maps $\mu_i \in \text{HC}^2(\mathcal{A}, \mathcal{A})$. A formal deformation up to order $k \in \mathbb{N}$ is defined analogously replacing $\mathcal{A}[\lambda]$ by $\mathcal{A}[\lambda]/(\lambda^{k+1})$.

Two formal deformations $\mu, \tilde{\mu}$ of the same algebra are equivalent if there exists an algebra isomorphism $\Phi : (\mathcal{A}[\lambda], \mu) \to (\mathcal{A}[\lambda], \tilde{\mu})$ which is the identity in the zeroth order. Such a $\Phi$ is called an equivalence transformation.

Recall also the existence of the Gerstenhaber bracket (see \cite{4} for the original work on this),

\[
[\cdot, \cdot] : \text{HC}^\bullet(\mathcal{A}) \otimes \text{HC}^\bullet(\mathcal{A}) \to \text{HC}^\bullet(\mathcal{A}),
\]

making the Hochschild complex into a Lie superalgebra (if one defines the grading to give $\phi \in \text{HC}^i(\mathcal{A}, \mathcal{A})$ the degree $i + 1$). It has the following properties:

- A map $\mu \in \text{HC}^2(\mathcal{A}, \mathcal{A})$ is associative if and only if $[\mu, \mu] = 0$.
- The Hochschild differential $\delta$ can be written in the form $\delta \phi = -[\phi, \mu_0]$.

We do not want to give an explicit formula for this bracket here, but let us remark that the Gerstenhaber bracket of two cochains is simply a linear combination of partial compositions of the cochains in one another at different arguments of the cochains. From this, one can derive that the bracket is $G$-equivariant in the sense that

\[
g \cdot [\phi, \psi] = [g \cdot \psi, g \cdot \phi]
\]

for all $\phi, \psi \in \text{HC}^\bullet(\mathcal{A})$.

Now, let us describe the obstructions given by the spaces $\text{HH}^2_G(\mathcal{A})$ and $\text{HH}^3_G(\mathcal{A})$. For this, we consider the well-known statements about the non-invariant Hochschild
cohomology, see for example [20] p.402ff., and formulate them for the invariant setting.

**Proposition 4.2** (Obstructions in $\text{HH}^3_G(\mathcal{A})$). Denote by

$$(4.4) \quad \mu^{(k)} = \mu_0 + \cdots + \lambda^k \mu_k$$

an invariant formal deformation of $\mu_0$ up to order $k$, so $\mu_i \in \text{HC}^2_G(\mathcal{A}, \mathcal{A})$ for all $i = 1, \ldots, k$. Then

$$(4.5) \quad R_{k+1} = \frac{1}{2} \sum_{l=1}^{k} [\mu_l, \mu_{k+1-l}]$$

is an invariant Hochschild 3-cocycle, and $\mu^{(k)}$ can be continued to an invariant associative deformation of order $k+1$ if and only if $R_{k+1} = \delta \mu_{k+1}$ for some $\mu_{k+1} \in \text{HC}^2_G(\mathcal{A}, \mathcal{A})$. In this case $\mu^{(k+1)} := \mu^{(k)} + \lambda^{k+1} \mu_{k+1}$ yields such a continuation.

**Proof.** This is a straightforward corollary of Proposition 6.2.19 of [20] p.402. \qed

As such, this proposition makes it clear that for the order-by-order continuation of invariant cochains, the cohomology of invariant cochains $\text{HH}^*_G(\mathcal{A})$ is indeed the correct obstruction space to consider, rather than the space of invariant classes $\text{HH}^*_G(\mathcal{A})^G$.

**Proposition 4.3** (Obstructions in $\text{HH}^2_G(\mathcal{A})$). Given two invariant formal deformations $\mu, \tilde{\mu}$ which are identical up to order $k$, their difference $\mu_{k+1} - \tilde{\mu}_{k+1}$ is an invariant cocycle. Furthermore, there exists an equivalence transformation up to order $k+1$ of the form $S = \exp(\lambda^{k+1}[T_{k+1}, \cdot])$ with an invariant $T_{k+1} \in \text{HC}^1_G(\mathcal{A}, \mathcal{A})$ if and only if $\mu_{k+1} - \tilde{\mu}_{k+1} = \delta T_{k+1}$ for some $\tilde{T}_{k+1} \in \text{HC}^1_G(\mathcal{A}, \mathcal{A})$. In this case both operators $\tilde{T}_{k+1}, T_{k+1}$ can be chosen to be identical.

**Proof.** The fact that $\mu_{k+1} - \tilde{\mu}_{k+1}$ is a cocycle is a corollary of Satz 6.2.22 ii) of [20] p.404. The statement about the coboundary follows directly from writing out the definition of an equivalence transformation, namely $\tilde{\mu} = \exp(\lambda^{k+1}[T_{k+1}, \cdot]) \mu$ in order $\lambda^{k+1}$. \qed

**Remark 4.4.** Note that the above proposition does not classify arbitrary equivalences up to order $k+1$, but only those with this specific form. However, in the case $k = 0$, where we consider infinitesimal deformations, every equivalence up to order 1 is of the form $\exp(\lambda[T_1, \cdot])$, and the only additional requirement is that $T_1$ be invariant.

This proposition also lays out a deeper understanding for the difference of the two notions of invariance of Hochschild cohomology: In the case $\text{HH}^2_G(\mathcal{A}) = 0$ but $\text{HH}^2_G(\mathcal{A})^G \neq 0$, not all infinitesimal deformations are equivalent, but all invariant infinitesimal deformations are, and their equivalences can be chosen to have invariant exponent as well. In the opposite case $\text{HH}^2_G(\mathcal{A}) = 0$ and $\text{HH}^2_G(\mathcal{A})^G = 0$, again all invariant deformations are equivalent, but not necessarily using invariant equivalences. Hence, in the deformation-theoretic setting, the choice of what “invariant Hochschild cohomology” means relates to how one defines equivalences of invariant deformations: One may require the equivalences themselves to be invariant under the group action, making $\text{HH}^2_G(\mathcal{A})^G$ the relevant space, or keep the full set of equivalences of the non-invariant scenario, so that $\text{HH}^2_G(\mathcal{A})^G$ contains the appropriate information.
5. Results for proper actions

5.1. Injectivity of \( \iota \). Let us now restrict to the case \( N = \mathcal{A} = C^\infty(M) \) of smooth functions on a smooth manifold \( M \), where the bimodule structure is given by ordinary multiplication of functions. Let \( G \) be a Lie group acting smoothly on \( M \). By pullback, this induces an action of \( G \) on \( C^\infty(M) \), which in turn induces an action on \( \text{HC}^\bullet(C^\infty(M)) \) as discussed in Section 3. Recall that an action \( \triangleright : G \times M \to M \) on a manifold \( M \) is called proper if the map

\[
\triangleright : G \times M \to M \times M, \quad (g, m) \mapsto (g \triangleright m, m),
\]

is proper, meaning preimages of compact sets are compact under \( \triangleright \). This includes a large class of actions, e.g. actions of compact groups or also the natural action of a Lie group \( G \) on a homogeneous space \( G/H \) whenever \( H \) is a compact Lie subgroup.

On manifolds, properness of an action is equivalent to the following statement: for any two convergent sequences \( \{x_i\}_{i \in \mathbb{N}} \) and \( \{g_i \triangleright x_i\}_{i \in \mathbb{N}} \), the sequence \( \{g_n\}_{n \in \mathbb{N}} \) has a convergent subsequence (see for example [17, p.59]).

We will explicitly construct averaging operators for actions of this kind on the space of Hochschild cochains. To construct this operator we will use partitions of unity (see for example [17, p.13]). Recall that for every open cover of a smooth manifold, there exists a smooth partition of unity with compact support subordinate to it.

We further require a \( G \)-invariant analogue:

**Proposition 5.1** (Existence of \( G \)-invariant partitions of unity). [17, p.61] Let \( \triangleright : G \times M \to M \) be a proper, smooth action of a Lie group \( G \) on a manifold \( M \), and let \( \{O_a\}_{a \in I} \) be an open cover of \( M \) by \( G \)-invariant subsets. Then there exists a subordinate partition of unity \( \{\chi_n\}_{n \in \mathbb{N}} \) consisting of \( G \)-invariant functions \( \chi_n \in C^\infty(M)^G \).

Note that from this proposition one does not necessarily gain a partition of unity with functions in \( C^\infty_0(M) \), as compact support is in general not compatible with \( G \)-invariance. Thus the compactly supported partitions and the invariant partitions should be viewed as separate results, and indeed, we will need both in the following. First, we first prove there is a way to average these kinds of cochains:

**Lemma 5.2.** Let \( G \) be a Lie group acting smoothly and properly on the smooth manifold \( M \). Given a left invariant volume form \( \Omega \in \Gamma^\infty(\Lambda^\dim G^* G) \), a compactly supported function \( \xi \in C^\infty_0(M) \) and a continuous cochain \( \beta \in \text{HC}^k_{\text{cont}}(C^\infty(M)) \) for \( k \in \mathbb{N}_0 \), the formula

\[
(\xi \cdot \beta)^{av} : C^\infty(M)^{\otimes k} \times M \to \mathbb{R},
\]

\[
(\xi \cdot \beta)^{av}(f_1, \ldots, f_k)(p) := \int_G (g \triangleright (\xi \cdot \beta))(f_1, \ldots, f_k)(p)\Omega(g)
\]

defines an element \( (\xi \cdot \beta)^{av} \in \text{HC}^k_{\text{cont}}(C^\infty(M)) \). This averaging is linear and commutes with the differential in the following sense:

\[
(\xi \cdot \delta \beta)^{av} = \delta((\xi \cdot \beta)^{av}) \quad \forall \beta \in \text{HC}^\bullet_{\text{cont}}(C^\infty(M)).
\]

**Proof.** To show that this map is well-defined it suffices to show that the integrand is zero outside of a compact domain. Restricting to an arbitrary open subset \( U \subset M \) with compact closure \( \overline{U} \), we want to show that the set \( G_{U, \xi} \subset G \) of group elements
which can have non-vanishing contribution to the integration is compact. Analyze this set:
\[
G_{U,\xi} = \{ g \in G \mid \exists p \in U : g^{-1} \triangleright p \in \text{supp } \xi \}
\]
\[
= \{ g \in G \mid \exists p \in U : \pi(g^{-1}, p) \in \text{supp } \xi \times U \}
\]
\[
= \text{pr}_G \left( (\pi \circ (\text{inv } \times \text{id}))^{-1}(\text{supp } \xi \times U) \right).
\]

Note that $\xi$ has compact support, $\text{inv} : G \to G, g \mapsto g^{-1}$ is a homeomorphism, and $\pi$ is a proper map, so the argument of the projection $\text{pr}_G : G \times M \to G$ in the above equation is a compact set and the image $G_{U,\xi}$ of the projection is also a compact set. This implies for every $p \in U$

\[
(5.4) \quad (\xi \cdot \beta)_{av}^{G}(f_1, \ldots, f_{k-1})(p) = \int_{G_{U,\xi} \in \pi} \left( g \triangleright (\xi \cdot \beta) \right)(f_1, \ldots, f_{k-1})(p) \Omega(g),
\]

wherefore the integral is well-defined as an integral of a smooth function over a compact set.

Note also that for continuous cochains $\beta$, this averaging still yields continuous cochains, which is treated in Lemma [A.1]. Furthermore, let us show that the map $\{ (\xi \cdot \beta)_{av}^{G}(f_1, \ldots, f_k) \}$ is a smooth function for all $f_i \in C^\infty(M)$: Since the unaveraged $(\xi \cdot \beta)(f_1, \ldots, f_k)$ is a smooth function, this is implied if the order of integration in the parameter $g \in G$ and differentiation in the parameter $p \in M$ can be reversed. Since around every point we can restrict the integration to a compact domain, the function is continuous in $g$ and smooth in $p$. Then the smoothness follows from the Leibniz integral rule for general measure theoretic spaces.

The facts that this averaging is linear and commutes with the Hochschild differential are straightforward calculations. This concludes the proposition.

We now proceed to proving the injectivity:

**Proposition 5.3.** For a Lie group $G$ acting properly and smoothly on a manifold $M$, taking the induced actions on $HC^\bullet(C^\infty(M))$ and $HH^\bullet(C^\infty(M))$, one finds that the map

\[
(5.5) \quad \iota : HH^\bullet_{G,\text{cont}}(C^\infty(M)) \to HH^\bullet_{\text{cont}}(C^\infty(M))^G
\]

is injective.

**Proof.** Injectivity is equivalent to the following: Given an invariant coboundary $\phi \in HC^{k+1}_{G,\text{cont}}(C^\infty(M))$ with

\[
(5.6) \quad \phi = \delta \psi, \quad \psi \in HC^k_{G,\text{cont}}(C^\infty(M)),
\]

there also exists an invariant $\hat{\psi} \in HC^k_{G,\text{cont}}(C^\infty(M))$ with $\phi = \delta \hat{\psi}$. First, choose a countable open cover $\{O_n\}_{n \in \mathbb{N}}$ of $M$ with compact closure $\overline{O_n}$ for all $n \in \mathbb{N}$ and a subordinate partition of unity $\{\xi_n\}_{n \in \mathbb{N}}$ with

\[
(5.7) \quad \text{supp } \xi_n \subset O_n.
\]

Choose any left invariant volume form so that we can use Lemma [5.2] to define the averages of $\xi_n \cdot \psi \in HC^k(C^\infty(M))$ and $\xi_n \cdot 1 = \xi_n \in C^\infty(M) = HC^0(C^\infty(M))$, so

\[
(5.8) \quad (\xi_n \cdot \psi)_{av}^{\xi_n}, \quad \xi_n^{av} := (\xi_n \cdot 1)_{av}^{\xi_n}.
\]

Furthermore, define the sets $U_n := (\xi_n^{av})^{-1}((0, \infty))$. They are open and $G$-invariant. Also, since for every $p \in M$ there exists a $\xi_n$ with $\xi_n(p) \neq 0$ and thus
\( \xi_n^{av}(p) \neq 0 \), so the \( U_n \) cover \( M \). Thus by Proposition 5.1 there exists a \( G \)-invariant partition of unity \( \{ \chi_n \}_{n \in \mathbb{N}} \) subordinate to the \( U_n \), satisfying \( \text{supp} \chi_n \subset U_n \). In particular this means that for every \( p \in M \) there exists a \( \chi_n \) with \( \chi_n(p) \neq 0 \) and by construction also \( \xi_n^{av}(p) \neq 0 \).

Using these functions, \( \phi = \delta \psi \) implies

\[
(5.9) \quad \xi_n \cdot \phi = \xi_n \cdot \delta \psi = \delta(\xi_n \cdot \psi).
\]

This equation can now be averaged; as \( \phi \) is already invariant, and \( \delta \) commutes with the averaging integral, it follows that

\[
(5.10) \quad \xi_n^{av} \cdot \phi = (\xi_n \cdot \phi)^{av} = (\delta(\xi_n \cdot \psi))^{av} = \delta((\xi_n \cdot \psi)^{av}).
\]

This cannot yet be summed over \( n \), as neither side has to be locally finite. However, multiplication with elements \( \chi_n \) of the \( G \)-invariant partition of unity yields a well-defined, locally finite sum:

\[
(5.11) \quad \sum_{n \in \mathbb{N}} \chi_n \cdot \xi_n^{av} \cdot \phi = \sum_{n \in \mathbb{N}} \delta(\chi_n \cdot (\xi_n \cdot \psi)^{av}).
\]

As such, the sum can be interchanged with the linear operator \( \delta \), giving the equation

\[
(5.12) \quad \phi \cdot \sum_{n \in \mathbb{N}} \chi_n \cdot \xi_n^{av} = \delta \left( \sum_{n \in \mathbb{N}} \chi_n \cdot (\xi_n \cdot \psi)^{av} \right).
\]

Now \( \sum_{n \in \mathbb{N}} \chi_n \cdot \xi_n^{av} > 0 \), as for every \( p \in M \) there exists an \( n \in \mathbb{N} \) with \( \chi_n(p) > 0 \) and \( \xi_n(p) > 0 \), so \( \xi_n^{av}(p) > 0 \). It follows that

\[
(5.13) \quad \phi = \delta \left( \frac{\sum_{n \in \mathbb{N}} \chi_n \cdot (\xi_n \cdot \psi)^{av}}{\sum_{n \in \mathbb{N}} \chi_n \cdot \xi_n^{av}} \right).
\]

By Lemma A.2 the cochain \( \tilde{\psi} := \frac{\sum_{n \in \mathbb{N}} \chi_n \cdot (\xi_n \cdot \psi)^{av}}{\sum_{n \in \mathbb{N}} \chi_n \cdot \xi_n^{av}} \) is continuous. As \( \tilde{\psi} \) now only consists of \( G \)-invariant functions and cochains, the statement is shown.

\[ \Box \]

Remark 5.4. The above statement also holds when \( \text{cont} \) is replaced by \( \text{loc, diff or n. c. diff} \); one only needs to check that the averaging procedure \( \psi \mapsto \psi^{av} \) leaves these properties untouched, they then carry through the rest of the construction without problem.

5.2. Surjectivity of \( \iota \). For the surjectivity of the natural map \( \iota \), one would have to show that every invariant class contains an invariant cochain. In analogy to the proof of injectivity, the intuitive route would be to take an arbitrary cochain from the invariant class and average it to gain an invariant one. However, it is not clear why the averaged cochain is still in the same class. With the notation of the proof for injectivity, one would have to show that

\[
(5.14) \quad \int_G (g \triangleright \xi_n)(\phi - g \triangleright \phi) \Omega(g)
\]

is a coboundary if the expression \( \phi - g \triangleright \phi \) is a coboundary for all \( g \in G \). While it is true that for every \( g \in G \) there exists a cochain \( \psi_g \) so that \( \phi - g \triangleright \phi = \delta \psi_g \), there is no control over the map \( g \mapsto \psi_g \).

We will take another route using the celebrated Hochschild-Kostant-Rosenberg theorem, see [11] for its original formulation in terms of algebraic varieties. The following version is phrased in terms of smooth manifolds.
We will use some standard notation from differential geometry: If \( \pi : E \to M \) is a smooth vector bundle, denote by \( \Gamma^\infty(E) \) its smooth sections, and by \( \Lambda^k E \) its \( k \)-th exterior power. For a smooth section \( X \in \Gamma^\infty(E) \) and any \( p \in M \), we denote by \( X_p \) its value at the point \( p \). The map \( d \) is the usual Cartan differential and \( i_\alpha \) for some smooth form \( \alpha \in \Gamma^\infty(T^*M) \) denotes the interior product map, i.e. contraction with the form \( \alpha \). For a smooth map \( f \in C^\infty(M) \) and a vector field \( X \in \Gamma^\infty(TM) \), the expression \( X(f) \in C^\infty(M) \) denotes \( X \) acting on \( f \) as a derivation in the canonical way. We also denote the space of multivector fields by \( \mathfrak{X}^\bullet(M) = \bigoplus_{k=0}^{\infty} \Gamma^\infty(\Lambda^k TM) \).

Let us now recall the Hochschild-Kostant-Rosenberg theorem, which we cite from [20]. Proofs are found in [14] for the differential case, [2] for the local case, and [16] and [18] for the continuous case.

**Theorem 5.5** (Hochschild-Kostant-Rosenberg (HKR)). [20, p.417] Let \( M \) be a smooth manifold. Define

\[
U : \mathfrak{X}^\bullet(M) \to \text{HC}^\bullet_{\text{cont}}(C^\infty(M)), \quad X \mapsto U(X),
\]

\[
(5.15)
\]

This induces an isomorphism

\[
\mathcal{U} : \mathfrak{X}^\bullet(M) \to \text{HH}^\bullet_{\text{cont}}(C^\infty(M)),
\]

where \( \text{cont} \) can be replaced with \( \text{loc}, \text{diff}, \text{n.c. diff} \).

**Remark 5.6.** For factorizing multivector fields \( X_1 \wedge \cdots \wedge X_n \), the above definition yields

\[
U(X_1 \wedge \cdots \wedge X_n)(f_1, \ldots, f_n) = \frac{1}{k!} \sum_{\sigma \in S_k} X_{\sigma(1)}(f_1) \cdots X_{\sigma(k)}(f_k),
\]

where \( X_i(f_j) \) is just the derivation associated to \( X_i \) acting on \( f_j \) and \( S_k \) denotes the permutation group on \( k \) letters. Note that the Serre-Swan theorem of differential geometry implies that this formula fully defines the map on all multivector fields.

Note that while the proof of injectivity is fairly straightforward, the surjectivity of this map heavily relies on the analytical/topological structure of the Hochschild cochains and is highly technical. To the knowledge of the author, neither a proof nor a counterexample to the HKR theorem for non-continuous Hochschild cohomology has been found.

We can define a group action on the space of multivector fields which corresponds to the action on the Hochschild complex. Define for every \( g \in G \) and \( X \in \Gamma^\infty(TM) \) the action

\[
(5.18)
\]

and accordingly actions on higher order multivector fields via

\[
(5.19)
\]

Using the explicit formula of \( U \) above, we find

\[
(5.20)
\]

for all \( g \in G \).
Proposition 5.7. For a smooth action of $G$ on $M$, using the induced actions on $\HC^\bullet(C^\infty(M))$ and $\HH^\bullet(C^\infty(M))^G$, the natural map

$$\iota : \HH^\bullet_{G, \text{cont}}(C^\infty(M)) \to \HH^\bullet_{\text{cont}}(C^\infty(M))^G$$

(5.21)

is surjective. Here, cont can be replaced with loc, diff, n.c. diff.

Proof. Recall that surjectivity of $\iota$ means that every invariant class contains an invariant cocycle. Given any $[\phi] \in \HH^\bullet(C^\infty(M))^G$, for all $g \in G$ there exists some $\psi_g \in \HC^\bullet(C^\infty(M))$ so that

$$g \triangleright \phi = \phi + \delta(\psi_g).$$

(5.22)

By the HKR theorem, there exists a unique multivector field $X \in \mathfrak{x}^\bullet(M)$ with

$$U(X) = \phi + \delta(\xi).$$

(5.23)

Now, for any $g \in G$ we find

$$U(g \triangleright X) = g \triangleright U(X) = g \triangleright \phi + g \triangleright \delta(\xi)$$

$$= \phi + \delta(g \triangleright \phi + g \triangleright \xi) = U(X) + \delta(\psi_g + g \triangleright \xi - \xi).$$

(5.24)

This means that $U(X)$ and $U(g \triangleright X)$ lie in the same equivalence class, which, by injectivity of the HKR isomorphism $\mathcal{U}$, is only possible if $X = g \triangleright X$. It follows that $U(X)$ is an invariant cocycle lying in the same equivalence class as $\phi$, which proves surjectivity of $\iota$. □

To summarize: If we assume a proper group action and restrict to any one of the analytical subcomplexes of $\HH^\bullet(C^\infty(M))$ where the HKR theorem is applicable, we gain isomorphy of the spaces:

Theorem 5.8. For a proper, smooth action of a Lie group $G$ on a manifold $M$, taking the induced actions on $\HC^\bullet(C^\infty(M))$ and $\HH^\bullet(C^\infty(M))^G$, one finds that the natural map

$$\iota : \HH^\bullet_{G, \text{cont}}(C^\infty(M)) \to \HH^\bullet_{\text{cont}}(C^\infty(M))^G, \ [\phi] \mapsto [\phi]$$

(5.25)

is a well-defined isomorphism. Here, cont can be replaced by loc, diff, n.c. diff.

5.3. Invariant multivector fields and an invariant HKR map. We briefly want to look into an invariant analogue of the HKR map. Let us first define the space of invariant multivector field, motivated by the action on multivector fields which we derived earlier:

Definition 5.9 ($G$-invariant multivector fields). For a smooth group action $\Phi^M$ of a Lie group $G$, define the $G$-invariant multivector fields on $M$ by

$$\mathfrak{x}^\bullet(M)^G := \{ X \in \mathfrak{x}^\bullet(M) \mid g \triangleright X = X \ \forall g \in G \}.$$  

(5.26)

As the HKR map $\mathcal{U}$ intertwines the actions of $G$ on $\HH^\bullet(\mathfrak{x}^\bullet)$ and $\mathfrak{x}^\bullet(M)$, it descends to an isomorphism on the invariant spaces. We can summarize all the results for the $C^\infty(M)$ case in the following theorem:

Theorem 5.10 (Invariant HKR). For a smooth action of a group $G$ on a manifold $M$, using the induced actions on $\mathfrak{x}^\bullet(M)$ and $\HH^\bullet(C^\infty(M))$, the HKR map $\mathcal{U}$ from Equation (5.15) induces an isomorphism to the space of invariant classes

$$\mathcal{U} : \mathfrak{x}^\bullet(M)^G \to \HH^\bullet_{\text{cont}}(C^\infty(M))^G.$$  

(5.27)
If \( G \) acts properly on \( M \), by Theorem 5.8 this induces an isomorphism
\[
\mathcal{U} : \mathfrak{X}^\bullet(M)^G \rightarrow \text{HH}^\bullet_{G, \text{cont}}(C^\infty(M)).
\]
Here, \( \text{cont} \) can be replaced with \( \text{loc}, \text{diff}, n_c, \text{diff} \).

**Example 5.11** (Homogeneous spaces). By the invariant HKR Theorem, we see that whenever the given action is smooth, proper and transitive, the invariant cohomology \( \text{HH}^\bullet_{G} (C^\infty(M)) \) can be identified with a subspace of \( \Lambda^\bullet T_p^* M \) for an arbitrary point \( p \in M \), as an invariant vector field is in this case already fully determined by its value on at a single point. As such, the cohomology spaces become finite-dimensional.

In particular, when \( M \) is represented as a homogeneous space, i.e. \( M = G/H \) with \( H \) a closed Lie subgroup of the Lie group \( G \), one has a proper action of \( G \) on \( M \) if and only if the stabilizer \( H \) is compact (one implication is a general property of proper maps, the other implication can be shown using the definition of properness via sequences in \( G \) and \( G/H \)). In this case, if \( g \) and \( h \) are the corresponding Lie algebras of \( G \) and \( H \), we have
\[
\text{HH}^\bullet_{G} (C^\infty(G/H)) \cong (\Lambda^\bullet g/h)^H,
\]
where the \( H \) invariance is to be understood with respect to the adjoint action of \( G \) restricted to \( H \).

**Appendix A. Continuity of properly averaged cochains**

In this appendix, we want to show that the averaging procedures we defined in Section 5 map continuous cochains to continuous cochains. The notion of continuity is here induced by the topology of uniform convergence on compact subsets on \( C^\infty(M) \). The necessary theory about this locally convex topology can, for example, be found in [12, 15, 19]. We shortly recall the construction of the seminorms on this space.

For any compact set \( K \) lying within a chart \((U, x)\) and any \( l \in \mathbb{N} \) define the corresponding seminorms as
\[
p_{k, K, U, x}(f) := \max_{p \in K, x_1, \ldots, x_k} \left| \frac{\partial^k (f \circ x^{-1})}{\partial x^{i_1} \cdots \partial x^{i_k}} (x(p)) \right|.
\]
(A.1)

For details see for example [9]. Note that if our manifold is equipped with a Lie group action, we can also define an action on the space of seminorms: for all \( f \in C^\infty(M) \) set
\[
(g \triangleright p_{k, K, U, x})(g \triangleright f) := p_{k, K, U, x}(f).
\]
(A.2)

This implies:
\[
g \triangleright p_{k, K, U, x} = p_{k, g \triangleright K, g \triangleright U, g \triangleright x}
\]
(A.3)

where the diffeomorphism \( M \to M, p \to g \triangleright p \) carries the chart \((U, x)\) to another chart \((g \triangleright U, g \triangleright x)\). This notational trick will greatly simplify some of the expressions in the following proof:

**Lemma A.1** (Continuity of averaged cochains). Let \( R \) be a compact subset of a Lie group \( G \) with an action \( \Phi^M : G \times M \to M \) on a smooth manifold \( M \). For a continuous \( \psi \in \text{HC}_{\text{cont}}^{\infty}(C^\infty(M)) \), the map \( \psi^{av} := \int_R (g \triangleright \psi) \Omega(g) \) is also continuous, where the integration is performed with respect to some invariant measure as in Section 5.
Proof. Continuity of $\psi$ is equivalent to the following: for every seminorm $p_{k,K,U,x}$ there exist finitely many seminorms $p_{i,L_i,U_i,x_i}$ with $i = 1, \ldots, n$ and

$$p_{k,K,U,x}(\psi(f_1, \ldots, f_n)) \leq p_{l_1,L_{l_1},U_{l_1},x_1}(f_1) \cdots p_{l_n,L_{l_n},U_{l_n},x_n}(f_n) \quad \text{(A.4)}$$

Then, for every $g \in R$, there also exist such $l_i, L_i, U_i, x_i$ so that

$$p_{k,K,U,x}((g \triangleright \psi)(f_1, \ldots, f_n)) = (g^{-1} \triangleright p_{k,K,U,x})(\psi(g^{-1} \triangleright f_1, \ldots, g^{-1} \triangleright f_n)) \leq p_{l_1,L_{l_1},U_{l_1},x_1}(g^{-1} \triangleright f_1) \cdots p_{l_n,L_{l_n},U_{l_n},x_n}(g^{-1} \triangleright f_n) = (g \triangleright p_{l_1,L_{l_1},U_{l_1},x_1})(f_1) \cdots (g \triangleright p_{l_n,L_{l_n},U_{l_n},x_n})(f_n).$$

Hence, $g \triangleright \psi$ is a continuous map. Similar inequalities can be used for the averaging procedure:

$$p_{k,K,U,x}(\psi_{av}(f_1, \ldots, f_n)) = \max_{\begin{array}{c} p \in K \\ i_1, \ldots, i_k \in \mathbb{N} \end{array}} \left| \int_R \frac{\partial^k (g \triangleright \psi)(f_1, \ldots, f_n) \circ x^{-1}}{\partial x^{i_1} \cdots \partial x^{i_k}}(x(p))\Omega(g) \right|$$

$$\leq \int_R \max_{\begin{array}{c} p \in K \\ i_1, \ldots, i_k \in \mathbb{N} \end{array}} \left| \frac{\partial^k (g \triangleright \psi)(f_1, \ldots, f_n) \circ x^{-1}}{\partial x^{i_1} \cdots \partial x^{i_k}}(x(p)) \right| \Omega(g)$$

$$= \int_R p_{k,K,U,x}((g \triangleright \psi)(f_1, \ldots, f_n)) \Omega(g) \leq \text{Vol}(R) \cdot \max_{g \in R} p_{k,K,U,x}((g \triangleright \psi)(f_1, \ldots, f_n)) \leq \text{Vol}(R) \cdot \max_{g \in R} (g \triangleright p_{l_1,L_{l_1},U_{l_1},x_1})(f_1) \cdots \cdots (g \triangleright p_{l_n,L_{l_n},U_{l_n},x_n})(f_n),$$

where the last inequality is due to Inequality \textbf{A.3}. Note that the last maximum is indeed a maximum, because using Equation \textbf{A.2} we can move the $g$-dependence of the seminorm into its argument, so that the expression becomes a continuous function in $g$. Now, for every compact set $g \triangleright L_i$ in a coordinate patch, using an exhaustion of the coordinate patch by compact sets, one can construct a set $\hat{L}_i(g)$, so that $g \triangleright L_i$ lies in the interior of $\hat{L}_i(g)$ and $\hat{L}_i(g)$ is still a subset of the coordinate patch. The interiors of the $\hat{L}_i(g)$ for all $g \in R$ then yield an open cover of $R \triangleright L_i$, so the maximum over all the $g \in R$ in the above chain of inequalities is assumed in one of the $\hat{L}_i(g)$. However, as $R \triangleright L_i$ is a compact set, finitely many $\hat{L}_i(g_{t_i}), t_i = 1, \ldots, r_i$ are sufficient to cover $R \triangleright L_i$, so the maximum is assumed in one of the finitely many $\hat{L}_i(g_{t_i})$. The choice of the $\hat{L}_i(g_{t_i})$ does not depend on the $f_1, \ldots, f_n$, so one finds for every $i = 1, \ldots, n$

$$\text{max}_{g \in R} (g \triangleright p_{l_i,L_{l_i},U_{l_i},x_i})(f_i) \leq \text{max}_{t_i = 1, \ldots, r_i} p_{l_i,\hat{L}_i(g_{t_i}),g_{t_i} \triangleright U_{t_i},g_{t_i} \triangleright x_i}(f_i) \quad \text{(A.6)}$$

and finally

$$p_{k,K,U,x}(\psi_{av}(f_1, \ldots, f_n)) \leq \text{Vol}(R) \cdot \left( \text{max}_{t_1 = 1, \ldots, r_1} p_{l_1,\hat{L}_1(g_{t_1}),g_{t_1} \triangleright U_{t_1},g_{t_1} \triangleright x_1}(f_1) \right) \cdots \cdots \left( \text{max}_{t_n = 1, \ldots, r_n} p_{l_n,\hat{L}_n(g_{t_n}),g_{t_n} \triangleright U_{t_n},g_{t_n} \triangleright x_n}(f_n) \right).$$
Finite maxima of seminorms are again seminorms, and multiplication of a seminorm with non-negative constants again yields a seminorm. As such, continuity of the averaged map is shown. □

**Lemma A.2** (Continuity of locally finite sums). Consider a partition of unity \( \{\chi_i\}_{i \in I} \) of a smooth manifold \( M \). Then the locally finite sum

\[
\sum_{i \in I} \chi_i \cdot \psi_i
\]

of continuous cochains \( \psi_i \in HC^n(C^\infty(M)) \) is again a continuous cochain.

**Proof.** The calculation will in the following only be done for seminorms in zeroth order of differentiation. In higher orders of differentiation one only receives factors with the corresponding maxima of the derivatives of \( \chi_i \), and via product rule one receives multiple summands, for which just the same calculations can be done. We also drop all references to charts \( (U, x) \), as they do not play a role in the following.

Let \( K \) be some compact set within a coordinate patch

\[
p_{0,K} \left( \sum_{i \in I} \chi_i \cdot \psi_i(f_1, \ldots, f_n) \right) = \max_{x \in K} \left\{ \sum_i \chi_i(x) \psi_i(f_1, \ldots, f_n)(x) \right\}.
\]

For every \( x \in K \), because of the locally finiteness, there exists an open neighbourhood \( U_x \subset K \), so that the sum is finite on \( U_x \). The union of these neighbourhoods is an open cover of \( K \), by compactness of \( K \) there then exist finitely many \( U_{x_j}, j = 1, \ldots, m \), so that \( K = \bigcup_{j=1}^{m} U_{x_j} \). It follows that the maximum is assumed in one of the finitely many \( U_{x_j} \), and by consequence in one of the finitely many \( U_{x_j} \), which are compact as a closed subset of the compact \( K \). The largest index \( l \), so that the \( \chi_i \) do not vanish on \( U_{x_j} \), is denoted by \( l_j \), with which one can write:

\[
p_{0,K} \left( \sum_{i \in I} \chi_i \cdot \psi_i(f_1, \ldots, f_n) \right) = \max_{j=1}^{m} \max_{x \in U_{x_j}} \left\{ \sum_{i=1}^{l_j} \chi_i(x) \cdot \psi_i(f_1, \ldots, f_n)(x) \right\}
\]

\[
\leq \max_{j=1}^{m} \max_{i=1}^{l_j} \max_{x \in U_{x_j}} |\psi_i(f_1, \ldots, f_n)(x)|
\]

\[
= \max_{j=1}^{m} \max_{i=1}^{l_j} p_{0,U_{x_j}}(\psi_i(f_1, \ldots, f_n)).
\]

Note that the choice of \( U_{x_j} \) does not depend on the \( f_i \). As finite maxima of seminorms yield seminorms again, the above inequality proves continuity of the locally finite sum. □

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