In various places in the literature, including [2], it is stated that every separable linear order embeds into the real line. This is, however, not the case, at least not with respect to the usual definition of separability.

**Definition 1.** Let $(L, \leq)$ be a linear order. $D \subseteq L$ is dense in $L$ if for all $a, b \in L$ with $a < b$ and $(a, b) \neq \emptyset$ there is $d \in D$ with $a < d < b$. $L$ is separable if it has a countable dense subset.

Two points $x, y \in L$ form a jump if $x < y$ and the open interval $(x, y)$ is empty.

**Lemma 2.** Let $L$ be a suborder of $\mathbb{R}$. Then $L$ is separable and has only countably many jumps.

**Proof.** Let $x_0 < y_0$ and $x_1 < y_1$ be two different jumps in $L$. Then there are $q_0, q_1 \in \mathbb{Q}$ such that for all $i \in 2$, $x_i < q_i < y_i$. Since the two jumps are different, $y_0 \leq x_1$ or $y_1 \leq x_0$. In either case, $q_0 \neq q_1$. It follows that there are not more jumps than rationals, i.e., there are only countably many jumps.

To see that $L$ is separable, choose a countable set $D \subseteq L$ such that for all $q \in \mathbb{Q}$ and all $n > 0$ the following holds: if $L \cap (q - 1/n, q + 1/n) \neq \emptyset$, then $D \cap (q - 1/n, q + 1/n) \neq \emptyset$. It is easily checked that $D$ is dense in $L$.

**Example 3.** Consider the set $\mathbb{R} \times 2$ ordered lexicographically. Then $\mathbb{Q} \times 2$ is dense in $\mathbb{R} \times 2$ and hence $\mathbb{R} \times 2$ is separable. But $\mathbb{R} \times 2$ has uncountably many jumps and therefore does not embed into $\mathbb{R}$.

**Theorem 4.** If $L$ is any separable linear order, then there is an order embedding $e : L \rightarrow \mathbb{R} \times 2$.

**Proof.** Let $D$ be a countable dense subset of $L$. We may assume that $D$ contains the first and last element of $L$ provided they exist. By the saturation of $\mathbb{Q}$, there is an order embedding $i : D \rightarrow \mathbb{Q}$. For each $x \in L$ let

$$e_1(x) = \sup \{i(d) : d \in D \land d \leq x\}.$$
Now $e_1 : L \to \mathbb{R}$ preserves $\le$, but we have $e_1(x) = e_1(y)$ if

$$x > y = \sup\{d \in D : d \le x\}.$$ 

Note that this can only happen if $x$ is the successor of $y$ in $L$ and $x \notin D$.

To correct this failure of injectivity, we embed into $\mathbb{R} \times 2$ rather than $\mathbb{R}$. For $x \in L$ let

$$e(x) = \begin{cases} 
(e_1(x), 0), & \text{if } x = \sup\{d \in D : d \le x\} \text{ and } \\
(e_1(x), 1), & \text{if } x > \sup\{d \in D : d \le x\}.
\end{cases}$$

The proof of Theorem 4 suggests the following notion:

**Definition 5.** Let $L$ be a linear order. A set $D \subseteq L$ is left dense if for all $x \in L$, $x = \sup\{d \in D : d \le x\}$. We define right dense analogously, using inf instead of sup.

$L$ is left (right) separable if $L$ has a countable left (right) dense subset.

**Theorem 6.** For every linear order $L$ the following are equivalent:

1. $L$ is left separable.
2. $L$ is right separable.
3. $L$ is separable and has only countably many jumps.
4. $L$ order embeds into $\mathbb{R}$.

**Proof.** If $L$ is left separable, then the map $e_1$ in the proof of Theorem 4 embeds $L$ into $\mathbb{R}$. This shows the implication from (1) to (4). The implication from (2) to (4) now follows symmetrically. If $L$ embeds into $\mathbb{R}$, then $L$ is separable and has only countably many jumps by Lemma 2. Hence (4) implies (3).

If $L$ is separable and has only countably many jumps, let $D \subseteq L$ be countable, dense, and such that for each jump $x < y$, $x, y \in D$. It is easily checked that $D$ is both left and right dense. Hence (3) implies both (1) and (2). □

Note that if $x < y$ is a jump of a linear order $L$ and there is an automorphism $\varphi$ of $L$ that maps $x$ to $y$, then $y$ and hence $x$ is isolated. It follows that a separable homogeneous linear order either has no jumps and therefore embeds into $\mathbb{R}$ or all its elements are isolated and hence the linear order is isomorphic to the integers $\mathbb{Z}$. It follows that $\mathbb{R}$ is universal for homogeneous separable linear orders, but no universal separable linear order is homogeneous.

Another way to analyze the situation is this: given a linear order $L$, we consider two subsets definable without parameters, namely the set $J_l(L)$ of left partners of a jump and the set $J_r(L)$ of right partners of a jump. Also, there is a binary relation that can be defined without parameters, namely the relation $J(L)$ where $(x, y) \in J(L)$ if $x < y$ or $y < x$ is a jump.
Every automorphism of $L$ preserves $J_\ell(L)$, $J_r(L)$, and the relation $J(L)$. Therefore it makes sense to define homogeneity as follows:

**Definition 7.** A linear order $L$ is **jump homogeneous** if for all finite sets $A, B \subseteq L$ every bijection $b : A \to B$ that preserves the relations $<$, $J_\ell(L)$, $J_r(L)$, and $J(L)$ extends to an automorphism of $L$.

**Lemma 8.** The linear order $\mathbb{R} \times 2$ is jump homogeneous.

*Proof.* Let $A, B \subseteq \mathbb{R}$ be finite sets and let $f : A \to B$ a bijection that preserves $<$, $J_\ell(L)$, $J_r(L)$, and $J(L)$. Note that $J(L)$ is an equivalence relation. Since $f$ preserves $J(L)$, it induces a bijection $\overline{f} : A/J(L) \to B/J(L)$. Since $\mathbb{R}$ is homogeneous and $(\mathbb{R} \times 2)/J(L) \cong \mathbb{R}$, $\overline{f}$ extends to an automorphism $\overline{f}$ of $(\mathbb{R} \times 2)/J(L)$. Now $\overline{f}$ lifts to an automorphism $g$ of $\mathbb{R} \times 2$. Since $f$ preserves $J_\ell(L)$ and $J_r(L)$, $g$ extends $f$. □

We now discuss the existence of a universal separable linear order of size $\aleph_1 < 2^{\aleph_0}$.

**Definition 9.** A set $S \subseteq \mathbb{R}$ is $\aleph_1$-dense if for all $x, y \in \mathbb{R}$ with $x < y$, $S \cap (x, y)$ is of size $\aleph_1$.

Baumgartner showed the following [1]:

**Theorem 10.** It is consistent with $2^{\aleph_0} = \aleph_2$ that any two $\aleph_1$-dense set of reals are order isomorphic.

**Corollary 11.** It is consistent that there is a universal separable linear order of size $\aleph_1 < 2^{\aleph_0}$.

*Proof.* By Baumgartner’s result, we can assume that any two $\aleph_1$-dense sets of reals are isomorphic and $\aleph_1 < 2^{\aleph_0}$. Let $L$ be a separable linear order of size $\aleph_1$. By Theorem 4, $L$ is isomorphic to a subset of $\mathbb{R} \times 2$. Let $S \subseteq \mathbb{R}$ be of size $\aleph_1$ such that $L$ embeds into $S \times 2$. By enlarging $S$ if necessary, we may assume that $S$ is $\aleph_1$-dense.

It follows that every separable linear order of size $\aleph_1$ embeds into an order of the form $S \times 2$ where $S$ is an $\aleph_1$-dense subset of $\mathbb{R}$. But by our assumption, any two $\aleph_1$-dense subsets of $\mathbb{R}$ and therefore also any two linear orders of the form $S \times 2$ with $S \subseteq \mathbb{R}$ $\aleph_1$-dense are isomorphic. It follows that every linear order of the form $S \times 2$ with $S$ $\aleph_1$-dense is universal for separable linear orders of size $\aleph_1$. □

**References**

[1] J. Baumgartner, *All $\aleph_1$-dense sets of reals can be isomorphic*, Fund. Math. 79 (1973), no. 2, 101–106

[2] S. Geschke, M. Kojman, *Metric Baumgartner theorems and universality*, Math. Res. Lett. 14 (2007), no. 2, 215–226
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