A semisimple mod $p$ Langlands correspondence in families for $GL_2(\mathbb{Q}_p)$

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Abstract

This is the sequel to [PS]. Let $F$ be any local field with residue characteristic $p > 0$, and $\mathcal{H}_{p}^{(1)}$ be the mod $p$ pro-$p$-Iwahori Hecke algebra of $GL_2(F)$. In [PS] we have constructed a parametrization of the $\mathcal{H}_{p}^{(1)}$-modules by certain $\hat{GL}_2(F_p)$-Satake parameters, together with an antispherical family of $\mathcal{H}_{p}^{(1)}$-modules. Here we let $F = \mathbb{Q}_p$ (and $p \geq 5$) and construct a morphism from $\hat{GL}_2(F_p)$-Satake parameters to $\hat{GL}_2(F_p)$-Langlands parameters. As a result, we get a version in families of Breuil’s semisimple mod $p$ Langlands correspondence for $GL_2(\mathbb{Q}_p)$ and of Paskunas’ parametrization of blocks of the category of mod $p$ locally admissible smooth representations of $GL_2(\mathbb{Q}_p)$ having a central character. The formulation of these results is possible thanks to the Emerton-Gee moduli space of semisimple $\hat{GL}_2(F_p)$-representations of the Galois group $Gal(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$.

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Introduction

Let $F$ be a local field with ring of integers $\mathcal{O}_F$ and residue field $\mathbb{F}_q$. We let $\mathbb{G}$ be the algebraic group $GL_2$ over $F$ with diagonal torus $T \subset G$. Set $G := GL_2(F)$. Let $\mathcal{H}_{p}^{(1)}$ be the pro-$p$-Iwahori Hecke algebra of $G$, with coefficients in an algebraic closure $\overline{\mathbb{F}_p}$ of $\mathbb{F}_p$. Let $\hat{\mathbb{G}}$ be the Langlands dual group of $\mathbb{G}$ over $\overline{\mathbb{F}_q}$, with maximal torus $\hat{T}$. In this sequel to [PS], we continue to work at $q = q = 0$. That is, we consider the special fibre at $q = 0$ of the Vinberg fibration $V_{\hat{T}} \rightarrow 0$ associated to $\hat{T}$ followed by base change to $\overline{\mathbb{F}_q}$. This yields the $\overline{\mathbb{F}_q}$-semigroup scheme $V_{\hat{T},0} := SingDiag_{2 \times 2} \times \overline{\mathbb{F}_q} G_m$,
where $SingDiag_{2 \times 2}$ represents the semigroup of singular diagonal $2 \times 2$-matrices over $\overline{\mathbb{F}_q}$, cf. [PS] 7.1. Let $T'$ be the finite abelian dual group of $T = T(\mathbb{F}_q)$ and consider the extended semigroup $V_{T,0} := T' \times V_{\hat{T},0}$.
It has a natural $W_0$-action. In [PS 7.2.2] we established the mod $p$ pro-$p$-Iwahori Satake isomorphism

$$\mathcal{H}^{(1)}_{\mathbb{F}_p} : Z(H^{(1)}_{\mathbb{F}_p}) \sim \mathcal{O}(V^{(1)}_{\mathbb{T},0}/W_0)$$

identifying the center $Z(H^{(1)}_{\mathbb{F}_p}) \subset H^{(1)}_{\mathbb{F}_p}$ with the ring of regular functions on the quotient $V^{(1)}_{\mathbb{T},0}/W_0$.

The resulting Satake equivalence $S$ identifies the category of $Z(H^{(1)}_{\mathbb{F}_p})$-modules with the category of $\hat{G}$-Satake parameters, i.e. the category of quasi-coherent sheaves on $V^{(1)}_{\mathbb{T},0}/W_0$, cf. [PS 7.3.2].

We also constructed the mod $p$ antischéral module $M^{(1)}_{\mathbb{F}_p}$, cf. [PS 7.4.1]. This is a distinguished $H^{(1)}_{\mathbb{F}_p}$-action on the maximal commutative subring $A^{(1)}_{\mathbb{F}_p}$ of $H^{(1)}_{\mathbb{F}_p}$. The sheaf $S(M^{(1)}_{\mathbb{F}_p})$, when specialized at closed points of $V^{(1)}_{\mathbb{T},0}/W_0$, gives rise to a dual parametrization of all irreducible $H^{(1)}_{\mathbb{F}_p}$-modules in terms of $\hat{G}$-Satake parameters [PS 7.4.9/7.4.15].

In this sequel to [PS] we construct, in the case $F = \mathbb{Q}_p$ and $p \geq 5$, a morphism $L$ from the space of $\hat{G}$-Satake parameters to the space of $\hat{G}$-Langlands parameters, and prove that the push-forward $L_* S(M^{(1)}_{\mathbb{F}_p})$ interpolates the semisimple mod $p$ local Langlands correspondence $\rho \mapsto \pi(\rho)$ for the group $G$.

To be more precise, let $\zeta : Z(G) \rightarrow \mathbb{F}_p^\times$ be a central character of $G$. There is a natural fibration $\theta : V^{(1)}_{\mathbb{T},0}/W_0 \rightarrow Z(G)^\vee$ where $Z(G)^\vee$ is the group scheme of characters of $Z(G)$, and we put

$$(V^{(1)}_{\mathbb{T},0}/W_0)_\zeta := \theta^{-1}(\zeta).$$

We let from now on $F = \mathbb{Q}_p$ with $p \geq 5$. As a space of $\hat{G}$-Langlands parameters, we may then consider the Emerton-Gee moduli curve $X_{\zeta}$, cf. [Em19], parametrizing (isomorphism classes of) two-dimensional semisimple continuous Galois representations over $\overline{\mathbb{F}}_p$ with determinant $\omega \zeta$:

$$X_{\zeta}(\overline{\mathbb{F}}_p) \cong \{ \hbox{semisimple continuous } \rho : \text{Gal}(\mathbb{Q}_p/\mathbb{Q}_p) \rightarrow \hat{G}(\overline{\mathbb{F}}_p) \hbox{ with } \det \rho = \omega \zeta \} / \sim.$$  

Here $\omega$ is the mod $p$ cyclotomic character. The curve $X_{\zeta}$ is expected to be the underlying scheme of a ringed moduli space for the stack of étale $(\varphi, \Gamma)$-modules $X^\text{det=\omega\zeta}_2$ appearing in [EG19] (see also [EGGS19]). At the moment, it is unclear how to define a replacement for $X_{\zeta}$ when $F/\mathbb{Q}_p$ is a non trivial finite extension, and this is the reason why we have to restrict to the case $F = \mathbb{Q}_p$ (and $p \geq 5$) in our construction of the morphism $L$. Our main result is the following (cf. Theorem 8.3.9).

**Theorem.** Suppose $F = \mathbb{Q}_p$ with $p \geq 5$. There exists a morphism of $\overline{\mathbb{F}}_p$-schemes

$$L_{\zeta} : (V^{(1)}_{\mathbb{T},0}/W_0)_\zeta \rightarrow X_{\zeta}$$

such that the quasi-coherent $\mathcal{O}_{X_{\zeta}}$-module

$$L_{\zeta}^\ast S(M^{(1)}_{\mathbb{F}_p})(V^{(1)}_{\mathbb{T},0}/W_0)_\zeta,$$

equal to the push-forward along $L_{\zeta}$ of the restriction to $(V^{(1)}_{\mathbb{T},0}/W_0)_\zeta \subset V^{(1)}_{\mathbb{T},0}/W_0$ of the Satake parameter $S(M^{(1)}_{\mathbb{F}_p})$, interpolates the $I^{(1)}$-invariants of the semisimple mod $p$ Langlands correspondence

$$X_{\zeta}(\overline{\mathbb{F}}_p) \longrightarrow \text{Mod}_{\zeta}^\text{ladm}(\overline{\mathbb{F}}_p[G]) \longrightarrow \text{Mod}(H_{\mathbb{F}_p}^{(1)}),$$

$$x \quad \mapsto \quad \pi(\rho_x) \quad \mapsto \quad \pi(\rho_x)^{I^{(1)}},$$

in the sense: for all $x \in X_{\zeta}(\overline{\mathbb{F}}_p)$, one has an isomorphism of $H_{\mathbb{F}_p}^{(1)}$-modules

$$\left( (L_{\zeta}^\ast S(M^{(1)}_{\mathbb{F}_p})(V^{(1)}_{\mathbb{T},0}/W_0)_\zeta) \otimes_{\mathcal{O}_{X_{\zeta}}} k(x) \right)^{ss} \cong \left( M^{(1)}_{\mathbb{F}_p} \otimes Z(H_{\mathbb{F}_p}^{(1)}) (A^{(1)}_{\mathbb{F}_p})^{-1}(\mathcal{O}_{L_{\zeta}^\ast x}) \right)^{ss} \cong \pi(\rho_x)^{I^{(1)}}.$$
Here, $\text{Mod}^{\text{ad}}_\zeta(\mathbb{F}_p[G])$ denotes the category of locally admissible smooth $G$-representations over $\mathbb{F}_p$ with central character $\zeta$. The group $I^{(1)} \subset G$ is the standard pro-$p$ Iwahori subgroup and $(\cdot)^{I^{(1)}}$ denotes the functor of $I^{(1)}$-invariants.

As a byproduct of our constructions, we also obtain a version in families of Paškūnas’ parametrization of the blocks of the category $\text{Mod}^{\text{ad}}_\zeta(\mathbb{F}_p[G])$, cf. [Pas13]. See [Sted] for the precise statement.

8 The theory at $q = q_0$: Semisimple Langlands correspondence

We keep the notation from the introduction. In particular, $F$ denotes a local field with ring of integers $o_F$ and residue field $\mathbb{F}_q$ (we switch to $F = \mathbb{Q}_p$ starting from $\mathbb{S}_2$). We also let $k := \mathbb{F}_q$.

8.1 Mod $\rho$ Satake parameters with fixed central character

8.1.1. Let $\omega : \mathbb{F}_q^\times \to k^\times$ be induced by the inclusion $\mathbb{F}_q \subset k$. Then $(\mathbb{F}_q^\times)^r = (\omega)$ is a cyclic group of order $q-1$. An element $\omega^r$ defines a non-regular character of $\mathbb{T}$:

$$\omega^r(t_1, t_2) := \omega^r(t_1)\omega^r(t_2)$$

for all $(t_1, t_2) \in \mathbb{T} = \mathbb{F}_q^\times \times \mathbb{F}_q^\times$. Composing with multiplication in $\mathbb{F}_q^\times$, we get an action of $(\mathbb{F}_q^\times)^r$ on $\mathbb{T}^r$, which factors on the quotient set $\mathbb{T}^r/W_0$:

$$\mathbb{T}^r/W_0 \times (\mathbb{F}_q^\times)^r \to \mathbb{T}^r/W_0, \ (\gamma, \omega^r) \mapsto \gamma \omega^r.$$  

If $\gamma \in \mathbb{T}^r/W_0$ is regular (non-regular), then $\gamma \omega^r$ is regular (non-regular).

8.1.2. Restricting characters of $\mathbb{T}$ to the subgroup $\mathbb{F}_q^\times \simeq \{\text{diag}(a, a) : a \in \mathbb{F}_q^\times\}$ induces a homomorphism $\mathbb{T}^r \to (\mathbb{F}_q^\times)^r$ which factors into a restriction map

$$\mathbb{T}^r/W_0 \to ((\mathbb{F}_q^\times)^r, \gamma \mapsto \gamma|_{\mathbb{F}_q^\times}.$$  

The relation to the $(\mathbb{F}_q^\times)^r$-action on the source $\mathbb{T}^r/W_0$ is given by the formula

$$(\gamma \omega^r)|_{\mathbb{F}_q^\times} = \gamma|_{\mathbb{F}_q^\times} \omega^{2r}.$$  

We describe the fibers of the restriction map $\gamma \mapsto \gamma|_{\mathbb{F}_q^\times}$.

Let $\langle \cdot \rangle|_{\mathbb{F}_q^\times}^{-1}(\omega^{2r})$ be the fibre at a square element $\omega^{2r}$. By the above formula, the action of $\omega^{-r}$ on $\mathbb{T}^r/W_0$ induces a bijection with the fibre $\langle \cdot \rangle|_{\mathbb{F}_q^\times}^{-1}(1)$. The fibre

$$\langle \cdot \rangle|_{\mathbb{F}_q^\times}^{-1}(1) = \{1 \otimes 1\} \coprod \{\omega \otimes \omega^{-1}, \omega^2 \otimes \omega^{-2}, ..., \omega^{\frac{q-1}{2}} \otimes \omega^{-\frac{q-1}{2}}\} \coprod \{\omega^{\frac{q+1}{2}} \otimes \omega^{-\frac{q-1}{2}}\}$$

has cardinality $\frac{q+1}{2}$ and, in the above list, we have chosen a representative in $\mathbb{T}^r$ for each element in the fibre. The $\frac{q-1}{2}$ elements in the middle of this list, i.e. the $W_0$-orbits represented by the characters $\omega^r \otimes \omega^{-r}$ for $r = 1, ..., \frac{q-1}{2}$, are all regular $W_0$-orbits. The two orbits at the two ends of the list are non-regular orbits (note that $\frac{q-1}{2} \equiv -\frac{q-1}{2} \pmod{(q-1)}$). Since the action of $\omega^{-r}$ preserves regular (non-regular) orbits, any fibre at a square element (there are $\frac{q-1}{2}$ such fibres) has the same structure.

On the other hand, let $\langle \cdot \rangle|_{\mathbb{F}_q^\times}^{-1}(\omega^{2r-1})$ be the fibre at a non-square element $\omega^{2r-1}$. The action of $\omega^{-r}$ induces a bijection with the fibre $\langle \cdot \rangle|_{\mathbb{F}_q^\times}^{-1}(\omega^{-1})$. The fibre

$$\langle \cdot \rangle|_{\mathbb{F}_q^\times}^{-1}(\omega^{-1}) = \{1 \otimes \omega^{-1}, \omega \otimes \omega^{-2}, ..., \omega^{\frac{q-1}{2}} \otimes \omega^{-\frac{q+1}{2}}\}$$
has cardinality $\frac{q^d-1}{q^d-q}$ and we have chosen a representative in $\mathbb{T}^\vee$ for each element in the fibre. All elements of the fibre are regular $W_0$-orbits. Since the action of $\omega^{-r}$ preserves regular (non-regular) orbits, any fibre at a non-regular element (there are $\frac{q^d-1}{q^d-q}$ such fibres) has the same structure.

Note that $\frac{q^d-1}{q^d-q} + \frac{q^d-1}{q^d-q} = \frac{q^d-1}{q^d-q}$ is the cardinality of the set $\mathbb{T}^\vee/W_0$.

### 8.1.3. Recall the commutative $k$-semigroup scheme

$$V^{(1)}_{T,0} = \mathbb{T}^\vee \times V_{T,0} = \mathbb{T}^\vee \times \text{SingDiag}_{2 \times 2} \times \mathbb{G}_m$$

together with its $W_0$-action, cf. [PS, 6.2.15]: the natural action of $W_0$ on the factors $\mathbb{T}^\vee$ and $\text{SingDiag}_{2 \times 2}$ and the trivial one on $\mathbb{G}_m$. There is a commuting action of the $k$-group scheme

$$Z^\vee := (F_q^* \vee \times \mathbb{G}_m$$
on $V^{(1)}_{T,0}$; the (constant finite diagonalizable) group $(F_q^*)^\vee$ acts only on the factor $\mathbb{T}^\vee$ and in the way described in [PS, 6.2.15] an element $z_0 \in \mathbb{G}_m$ acts trivially on $\mathbb{T}^\vee$, by multiplication with the diagonal matrix $\text{diag}(z_0, z_0)$ on $\text{SingDiag}_{2 \times 2}$ and by multiplication with the square $z_0^2$ on $\mathbb{G}_m$. Therefore the quotient $V^{(1)}_{T,0}/W_0$ inherits a $Z^\vee$-action. Now, according to [PS, 7.4.7], one has the decomposition

$$V^{(1)}_{T,0}/W_0 = \prod_{\gamma \in (\mathbb{T}^\vee/W_0)_{\text{reg}}} V^{(1)}_{T,0} \prod_{\gamma \in (\mathbb{T}^\vee/W_0)_{\text{non-reg}}} V^{(1)}_{T,0}/W_0.$$

Then the $(F_q^*)^\vee$-action is by permutations on the index set $\mathbb{T}^\vee/W_0$, i.e. on the set of connected components of $V^{(1)}_{T,0}/W_0$: as observed above, it preserves the subsets of regular and non-regular components. The $\mathbb{G}_m$-action on $V^{(1)}_{T,0}/W_0$ preserves each connected component.

### 8.1.4. Recall from [PS, 7.4.7] the antispherical map

$$\text{ASph} : (V^{(1)}_{T,0}/W_0)(k) \longrightarrow \{ \text{left } H^{(1)}_{\mathbb{T}^\vee}(\text{-modules})/\sim \}.$$

The modules in the image of this map are standard modules of length 1 or 2, cf. [PS, 7.4.9] and [PS, 7.4.15].

Let $(\omega^r, z_0) \in Z^\vee(k)$. Then recall that the standard $H^{(1)}_{\mathbb{T}^\vee}(\text{-modules}$ and their simple constituents may be ‘twisted by the character $(\omega^r, z_0)$’ : in the regular case, the actions of $X, Y, U^2$ get multiplied by $z_0, z_0, z_0^2$ respectively and the component $\gamma$ gets multiplied by $\omega^r$, cf. [V04, 2.4]; in the non-regular case, the action of $U$ gets multiplied by $z_0$, the action of $S$ remains unchanged and the component $\gamma$ gets multiplied by $\omega^r$, cf. [V04, 1.6]. This gives an action of the group of $k$-points of $Z^\vee$ on the standard $H^{(1)}_{\mathbb{T}^\vee}(\text{-modules}$ and their simple constituents.

### 8.1.5. Lemma. The map $\text{ASph}$ is $Z^\vee(k)$-equivariant.

**Proof.** Let $(\omega^r, z_0) \in Z^\vee(k)$. Let $v \in (V^{(1)}_{T,0}/W_0)(k)$ and let its connected component be indexed by $\gamma \in \mathbb{T}^\vee/W_0$. Suppose that $\gamma$ is regular, choose an ordering $\gamma = (\chi, \chi^s)$ on the set $\gamma$ and standard coordinates. Then $\text{ASph}(v) = \text{ASph}^\gamma(v)$ is a simple two-dimensional standard $H^{(1)}_{\mathbb{T}^\vee}(\text{-module}$, cf. [PS, 7.4.9], i.e. of the form $M(x, y, z_2, \chi)_{\text{[V04, 3.2]}}$. Then

$$\text{ASph}(\gamma, z_0) = \text{ASph}(\gamma, z_0) \simeq M(z_0, z_0, z_0^2, \chi, \omega^r) \simeq \text{ASph}(\gamma, z_0).$$

Suppose that $\gamma = (\chi)$ is non-regular and choose Steinberg coordinates. (a) If $v \in D(2,\chi)(k)$, then $\text{ASph}(v) = \text{ASph}^\gamma(2)(v)$ is a simple two-dimensional $H^{(1)}_{\mathbb{T}^\vee}(\text{-module$, cf. [PS, 7.4.15$, i.e. of the form $M(z_1, z_2, \chi)_{\text{[V04, 3.2]}}$. Then

$$\text{ASph}(v, (\omega^r, z_0)) \simeq M(z_0z_1, z_0z_2, \chi, \omega^r) \simeq \text{ASph}(v, (\omega^r, z_0)).$$

(b) If $v \in D(1,\chi)(k)$, then the semisimplified module $\text{ASph}(v)^{ss}$ is the direct sum of the two characters in the antispherical pair $\text{ASph}^\gamma(1)(v) = \{(0, z_1), (1, -z_1)\}$ where $z_2 = z_1^2$. Similarly $\text{ASph}(v, (\omega^r, z_0))^{ss}$ is the direct sum of the characters $\{(0, z_0z_1), (1, -z_0z_1)\}$ in the component $\gamma\omega^r$, and hence is isomorphic to $\text{ASph}(v)^{ss}, (\omega^r, z_0)$. \(\square\)
8.1.6. The two canonical projections from $V^{(1)}_{T,0}$ to $T^\vee$ and $G_m$ respectively induce two projection morphisms

$$
\begin{array}{ccc}
V^{(1)}_{T,0}/W_0 & \xrightarrow{pr_{T^\vee/W_0}} & T^\vee/W_0 \\
& \xrightarrow{pr_{G_m}} & G_m.
\end{array}
$$

Then we may compose the map $pr_{T^\vee/W_0}$ with the restriction map $(\cdot)|_{F_q^\times}: T^\vee/W_0 \to (F_q^\times)^\vee$, set

$$
\theta := (\cdot)|_{F_q^\times} \circ pr_{T^\vee/W_0} \circ pr_{G_m}
$$

and view $V^{(1)}_{T,0}/W_0$ as fibered over the space $Z^\vee$:

$$
\begin{array}{ccc}
V^{(1)}_{T,0}/W_0 & \xrightarrow{\rho} & Z^\vee.
\end{array}
$$

The relation to the $Z^\vee$-action on the source $V^{(1)}_{T,0}/W_0$ is given by the formula

$$
\theta(x.(\omega^r, z_0)) = \theta(x)(\omega^{2r}, z_0^2) = \theta(x)(\omega^r, z_0)^2
$$

for $x \in V^{(1)}_{T,0}/W_0$ and $(\omega^r, z_0) \in Z^\vee$. This formula follows from the formula in 8.1.2 and the definition of the $G_m$-action in 8.1.3.

8.1.7. Definition. Let $\zeta \in Z^\vee$. The space of mod $p$ Satake parameters with central character $\zeta$ is the $k$-scheme

$$(V^{(1)}_{T,0}/W_0)_\zeta := \theta^{-1}(\zeta).$$

8.1.8. Let $\zeta = (\zeta|_{F_q^\times}, z_2) \in Z^\vee(k) = (F_q^\times)^\vee \times k^\times$. Denote by $(V^{(1)}_{T,0}/W_0)_{z_2}$ the fibre of $pr_{G_m}$ at $z_2 \in k^\times$. Then by [PS, 7.4.7] we have

$$(V^{(1)}_{T,0}/W_0)_\zeta = \bigoplus_{\gamma \in (T^\vee/W_0)_{\text{reg}}, \gamma|_{F_q^\times} = \zeta|_{F_q^\times}} V^{T,0}_{T,0,z_2} \bigoplus_{\gamma \in (T^\vee/W_0)_{\text{non-reg}}, \gamma|_{F_q^\times} = \zeta|_{F_q^\times}} V^{T,0}_{T,0,z_2}/W_0.
$$

Recall that the choice of standard coordinates $x, y$ identifies

$$V^{T,0}_{T,0,z_2} \simeq \mathbb{A}^1 \cup_0 \mathbb{A}^1$$

with two affine lines over $k$, intersecting at the origin, cf. [PS, 7.4.8]. On the other hand, the choice of the Steinberg coordinate $z_1$ identifies

$$V^{T,0}_{T,0,z_2}/W_0 \simeq \mathbb{A}^1$$

with a single affine line over $k$, cf. [PS, 7.4.10].

8.1.9. Lemma. Let $\zeta, \eta \in Z^\vee$. The action of $\eta$ on $V^{(1)}_{T,0}/W_0$ induces an isomorphism of $k$-schemes

$$(V^{(1)}_{T,0}/W_0)_\zeta \simeq (V^{(1)}_{T,0}/W_0)_{\zeta \eta^2}.$$

Proof. Follows from the last formula in 8.1.6
8.2 Mod \( p \) Langlands parameters with fixed determinant for \( F = \mathbb{Q}_p \)

8.2.1. Notation. In this section, we let \( F = \mathbb{Q}_p \) with \( p \geq 5 \). We fix an algebraic closure \( \overline{\mathbb{Q}_p} \) and let \( \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \) be the absolute Galois group. We normalize local class field theory \( \mathbb{Q}_p^\times \to \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)^{ab} \) by sending \( p \) to a geometric Frobenius. In this way, we identify the \( k \)-valued smooth characters of \( \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \) and of \( \mathbb{Q}_p^\times \). Finally, \( \omega: \mathbb{Q}_p^\times \to k^\times \) denotes the extension of the character \( \omega: \mathbb{F}_p^\times \to k^\times \) satisfying \( \omega(p) = 1 \), and \( \text{ur}(x): \mathbb{Q}_p^\times \to k^\times \) denotes the character trivial on \( \mathbb{F}_p^\times \) and sending \( p \) to \( x \).

8.2.2. Let \( \zeta: \mathbb{Q}_p^\times \to k^\times \) be a character. Recall from [Em10] that the Emerton-Gee moduli curve with character \( \zeta \) is a certain projective curve \( X_{\zeta} \) over \( k \) whose points parametrize (isomorphism classes of) two-dimensional semisimple continuous Galois representations over \( k \) with determinant \( \omega \zeta \):

\[
X_{\zeta}(k) \cong \{ \text{semisimple continuous } \rho: \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to \mathbb{G}(k) \; \text{with } \det \rho = \omega \zeta \}/\sim.
\]

The curve \( X_{\zeta} \) is a chain of projective lines over \( k \) of length \( \frac{\xi-1}{2} \), whose irreducible components intersect at ordinary double points. The sign \( \pm 1 \) is equal to \( -\zeta(-1) \). We refer to \( \zeta \) in the case \( \zeta(-1) = -1 \) resp. \( \zeta(-1) = +1 \) as an even character resp. odd character. There is a finite set of closed points \( X_{\zeta}^{\text{irred}} \subset X_{\zeta} \) which correspond to the classes of irreducible representations. Its open complement \( X_{\zeta}^{\text{red}} = X_{\zeta} \setminus X_{\zeta}^{\text{irred}} \) parametrizes the reducible representations (i.e. direct sums of characters). Let \( \eta: \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to k^\times \) be a character. Since \( \det(\rho \otimes \eta) = (\det \rho)\eta^2 \), twisting representations with \( \eta \) induces an isomorphism

\[
(\cdot) \otimes \eta: X_{\zeta} \xrightarrow{\sim} X_{\zeta}\eta^2.
\]

Hence one is reduced to consider only two ‘basic’ cases: the even case where \( \zeta(p) = 1 \) and \( \zeta|_{\mathbb{F}_p^\times} = 1 \) and the odd case where \( \zeta(p) = 1 \) and \( \zeta|_{\mathbb{F}_p^\times} = \omega^{-1} \). Indeed, if \( \zeta|_{\mathbb{F}_p^\times} = \omega^r \) for some even \( r \), then choosing \( \eta \) with \( \eta(p)^2 = \zeta(p)^{-1} \) and \( \eta|_{\mathbb{F}_p^\times} = \omega^{r/2} \), one finds that \( (\zeta\eta^2)(p) = 1 \) and \( (\zeta\eta^2)|_{\mathbb{F}_p^\times} = 1 \); if \( \zeta|_{\mathbb{F}_p^\times} = \omega^r \) for some odd \( r \), then choosing \( \eta \) with \( \eta(p)^2 = \zeta(p)^{-1} \) and \( \eta|_{\mathbb{F}_p^\times} = \omega^{-r} \), one finds that \( (\zeta\eta^2)(p) = 1 \) and \( (\zeta\eta^2)|_{\mathbb{F}_p^\times} = \omega^{-1} \).

8.2.3. We make explicit some structure elements of \( X_{\zeta} \) in the even case \( \zeta(p) = 1 \) and \( \zeta|_{\mathbb{F}_p^\times} = 1 \). Every irreducible component of \( X_{\zeta} \) is isomorphic to \( \mathbb{P}^1 \) and there are \( \frac{\xi-1}{2} \) components. They are labelled by pairs of Serre weights of the following form:

\[
\begin{align*}
\text{Sym}^0 & \quad | \quad \text{Sym}^{p-3} \otimes \det \\
\text{Sym}^2 \otimes \det^{-1} & \quad | \quad \text{Sym}^{p-5} \otimes \det^2 \\
\text{Sym}^4 \otimes \det^{-2} & \quad | \quad \text{Sym}^{p-7} \otimes \det^3 \\
\vdots & \quad | \quad \vdots \\
\text{Sym}^{p-3} \otimes \det^{\frac{\xi-1}{2}} & \quad | \quad \text{Sym}^0 \otimes \det^{\frac{\xi-1}{2}}.
\end{align*}
\]

The component with label " \( \text{Sym}^0 \mid \text{Sym}^{p-3} \otimes \det \)" intersects the next component at the point of \( X_{\zeta}^{\text{irred}} \) parametrizing the irreducible Galois representation whose associated Serre weights are \( \{\text{Sym}^0 \otimes \det^{-1}, \text{Sym}^{p-3} \otimes \det\} \). The component with label " \( \text{Sym}^2 \otimes \det^{-1} \mid \text{Sym}^{p-5} \otimes \det^2 \)" intersects the next component at the point of \( X_{\zeta}^{\text{irred}} \) parametrizing the irreducible Galois representation whose associated Serre weights are \( \{\text{Sym}^2 \otimes \det^{-2}, \text{Sym}^{p-5} \otimes \det^3\} \). Continuing in this way, one finds \( \frac{\xi-1}{2} \) points of \( X_{\zeta}^{\text{irred}} \) which correspond to the \( \frac{\xi-1}{2} \) double points of the chain \( X_{\zeta} \). There are two more points in \( X_{\zeta}^{\text{irred}} \): they are smooth points, each one lies on one of the two ‘exterior’ components and corresponds there to the irreducible Galois representation whose associated Serre weights are \( \{\text{Sym}^0, \text{Sym}^{p-1}\} \) and \( \{\text{Sym}^0 \otimes \det^{\frac{\xi-1}{2}}, \text{Sym}^{p-1} \otimes \det^{\frac{\xi-1}{2}}\} \) respectively. So \( X_{\zeta}^{\text{irred}} \) has cardinality \( \frac{\xi-1}{2} \). Suppose we are on one of the two exterior components \( \mathbb{P}^1 \). There is a canonical affine coordinate \( z_1 \) on the open complement of the double point, identifying this open complement with \( \mathbb{A}^1 \). We call the four points where \( z_1 = \pm 1 \) the four exceptional points of \( X_{\zeta} \).
8.2.4. We make explicit some structure elements of $X_\xi$ in the odd case $\zeta(p) = 1$ and $\zeta|_{\mathbb{F}_p} = \omega^{-1}$.
Every irreducible component of $X_\xi$ is isomorphic to $\mathbb{P}^1$ and there are $\frac{\dim X}{2}$ components. They are labelled by pairs of Serre weights of the following form:

| $\text{Sym}^{p-2}$ | $\text{Sym}^{-1}$ |
| $\text{Sym}^{p-4} \otimes \text{det}$ | $\text{Sym}^{1} \otimes \text{det}^{-1}$ |
| $\text{Sym}^{p-6} \otimes \text{det}^{2}$ | $\text{Sym}^{3} \otimes \text{det}^{-2}$ |
| $\vdots$ | $\vdots$ |
| $\text{Sym}^{1} \otimes \text{det}^{\frac{p-3}{2}}$ | $\text{Sym}^{p-4} \otimes \text{det}^{\frac{p+1}{2}}$ |
| $\text{Sym}^{-1} \otimes \text{det}^{\frac{1}{2}}$ | $\text{Sym}^{p-2} \otimes \text{det}^{\frac{1}{2}}$ |

The component with label $\text{" Sym}^{p-2} \mid \text{" Sym}^{-1} \text{"}$ intersects the next component at the point of $X_\xi^{\text{irred}}$ parametrizing the irreducible Galois representation whose associated Serre weights are $\{$Sym$^1 \otimes \text{det}^{-1}, \text{Sym}^{p-2}\}$. The component with label $\text{" Sym}^{p-4} \otimes \text{det} \mid \text{Sym}^{1} \otimes \text{det}^{-1} \text{"}$ intersects the next component at the point of $X_\xi^{\text{irred}}$ parametrizing the irreducible Galois representation whose associated Serre weights are $\{$Sym$^3 \otimes \text{det}^{-2}, \text{Sym}^{p-4} \otimes \text{det}\}$. Continuing in this way, one finds $\frac{\dim X}{2}$ points of $X_\xi^{\text{irred}}$, which correspond to the $\frac{\dim X}{2}$ double points of the chain $X_\xi$. There are no more points in $X_\xi^{\text{irred}}$ and $X_\xi^{\text{irred}}$ has cardinality $\frac{\dim X}{2}$. Suppose we are on one of the two exterior components $\mathbb{P}^1$. There is a canonical affine coordinate $t$ on the open complement of the double point, identifying this open complement with $\mathbb{A}^1$. We call the four points where $t = \pm 2$ the four exceptional points of $X_\xi$.\footnote{The Galois representations living on the two exterior components in the odd case are unramified (up to twist), i.e. of type $\rho = \begin{pmatrix} \text{unr}(x) & 0 \\ 0 & \text{unr}(x^{-1}) \end{pmatrix} \otimes \eta$ and $t$ equals the ‘trace of Frobenius’ $x + x^{-1}$. Hence $t = \pm 2$ if and only if $x = \pm 1$.}

8.2.5. Definition. The category of quasi-coherent modules on the Emerton-Gee moduli curve $X_\xi$ will be called the category of mod $p$ Langlands parameters with determinant $\omega_\xi$, and denoted by $\text{LP}_{G,0,\omega_\xi}$:

$$\text{LP}_{G,0,\omega_\xi} := \text{QCoh}(X_\xi).$$

8.3 A semisimple mod $p$ Langlands correspondence in families for $F = \mathbb{Q}_p$

8.3.1. Let us consider $W$ to be a subgroup of $G$, by sending $s$ to the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and by identifying the group $\Lambda$ with a subgroup of $T$ via $(1,0) \mapsto \text{diag}(\omega^{-1}, 1)$ and $(0,1) \mapsto \text{diag}(1,\omega^{-1})$. We obtain for example (recall that $u = (1,0)s \in W$)

$$u = \begin{pmatrix} 0 & \omega^{-1} \\ 1 & 0 \end{pmatrix}, \quad u^{-1} = \begin{pmatrix} 0 & 1 \\ \omega & 0 \end{pmatrix}, \quad us = \begin{pmatrix} \omega^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \quad su = \begin{pmatrix} 1 & 0 \\ 0 & \omega^{-1} \end{pmatrix}.$$

Moreover, $u^2 = \text{diag}(\omega^{-1},\omega^{-1})$. Since

$$\begin{pmatrix} 0 & \omega^{-1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \omega & 0 \end{pmatrix} = \begin{pmatrix} d & \omega^{-1}c \\ \omega b & a \end{pmatrix},$$

the element $u \in G$ normalizes the group $I^{(1)}$.

8.3.2. Let $\text{Mod}^{\text{sm}}(k[G])$ be the category of smooth $G$-representations over $k$. Taking $I^{(1)}$-invariants yields a functor $\pi \mapsto \pi^{I^{(1)}}$ from $\text{Mod}^{\text{sm}}(k[G])$ to the category $\text{Mod}(\mathcal{H}^{(1)}_f)$. If $F = \mathbb{Q}_p$, it induces a bijection between the irreducible $G$-representations and the irreducible $\mathcal{H}^{(1)}_f$-modules, under which supersingular representations correspond to supersingular Hecke modules $\mathcal{V}_0$.\footnote{Note that our element $u$ equals the element $u^{-1}$ in $[\text{Be11}, \text{Br07}]$ and $[\text{V04}].$
For future reference, let us recall the $I^{(1)}$-invariants for some classes of representations. If $\pi = \text{Ind}^G_B(\chi)$ is a principal series representation with $\chi = \chi_1 \otimes \chi_2$, then $\pi^{I^{(1)}}$ is a standard module in the component $\gamma := \{\chi|_{\overline{T}}, \chi^*|_{\overline{T}}\}$.

In the regular case, one chooses the ordering $(\chi|_{\overline{T}}, \chi^*|_{\overline{T}})$ on the set $\gamma$ and standard coordinates $x, y$. Then
\[
\text{Ind}^G_B(\chi)^{I^{(1)}} = M(0, \chi(su), \chi(u^2), \chi|_{\overline{T}}) = M(0, \chi_2(\overline{w}^{-1}), \chi_1(\overline{w}^{-1})\chi_2(\overline{w}^{-1}), \chi|_{\overline{T}})
\]

In the non-regular case, one has
\[
\text{Ind}^G_B(\chi)^{I^{(1)}} = M(\chi(su), \chi(u^2), \chi|_{\overline{T}}) = M(\chi_2(\overline{w}^{-1}), \chi_1(\overline{w}^{-1})\chi_2(\overline{w}^{-1}), \chi|_{\overline{T}}).
\]

These standard modules are irreducible if and only if $\chi \neq \chi^*$.\footnote{Our formulas differ from [V04] 4.2/4.3 by $\chi(\cdot) \leftrightarrow \chi(\cdot)^{-1}$, since we are working with left modules; also compare with the explicit calculation with right convolution given in [V04] Appendix A.5.}

Let $F = \mathbb{Q}_p$. If $\pi = \pi(r, 0, \eta)$ is a standard supersingular representation with parameter $r = 0, \ldots, p - 1$ and central character $\eta : \mathbb{Q}^\times_p \to k^\times$, then $\pi^{I^{(1)}}$ is a supersingular module in the component $\gamma = \{\chi, \chi^*\}$ represented by the character $\chi := (\omega^r \otimes 1) \cdot (\eta|_{\overline{F}_q})$, cf. [Br07] 5.1/5.3. If $\pi$ is the trivial representation $\mathbb{1}$ or the Steinberg representation $\text{St}$, then $\gamma = 1$ and $\pi^{I^{(1)}}$ is the character of $0, 1$ or $(-1, -1)$ respectively.

8.3.3. Let $\pi \in \text{Mod}^{sm}(k[G])$. Since $u \in G$ normalizes the group $I^{(1)}$, one has $I^{(1)}uI^{(1)} = uI^{(1)}$. It follows that the convolution action of the Hecke operator $U$ (resp. $U^2$) on $\pi^{I^{(1)}}$ is therefore induced by the action of $u$ (resp. $u^2$ on $\pi$). Similarly, the group $I^{(1)}$ is normalized by the Iwahori subgroup $I$ and $I/I^{(1)} \simeq \mathbb{T}$. It follows that the convolution action of the operators $T_t, t \in \mathbb{T}$ on $\pi^{I^{(1)}}$ is the factorization of the $\mathbb{T}(o_F)$-action on $\pi$.

8.3.4. We identify $F^\times$ with the center $Z(G)$ via $a \mapsto \text{diag}(a, a)$. A (smooth) character
\[
\zeta : Z(G) = F^\times \longrightarrow k^\times
\]
is determined by its value $\zeta(\overline{w}^{-1}) \in k^\times$ and its restriction $\zeta^F|_k$. Since the latter is trivial on the subgroup $1 + \overline{w}o_F$, we may view it as a character of $F^\times_q$; we will write $\zeta^F_q$ for this restriction in the following. Thus the group of characters of $Z(G)$ gets identified with the group of $k$-points of the group scheme $Z^\vee = (F^\times_q)^\vee \times G_m$:
\[
Z(G)^\vee \xrightarrow{\sim} Z^\vee(k), \quad \zeta \mapsto (\zeta|_{\overline{F}_q}, \zeta(\overline{w}^{-1})).
\]

8.3.5. Lemma. \textit{Suppose that $\pi \in \text{Mod}^{sm}(k[G])$ has a central character $\zeta : Z(G) \to k^\times$. Then the Satake parameter $S(\pi^{I^{(1)})}$ of $\pi^{I^{(1)}} \in \text{Mod}(\mathcal{H}^{I^{(1)}})$ has central character $\zeta$, i.e. it is supported on the closed subscheme
\[
(V^{I^{(1)}}_{\mathbb{T}, 0}/W_0)\zeta|_{\overline{F}_q}, \zeta(\overline{w}^{-1}) \subset V^{I^{(1)}}_{\mathbb{T}, 0}/W_0.
\]

Proof. If $M$ is any $\mathcal{H}^{I^{(1)}}$-module, then
\[
M = \bigoplus_{\gamma \in \mathbb{T}^\vee/W_0} \varepsilon_\gamma M = \bigoplus_{\lambda \in \mathbb{T}^\vee/W_0} \oplus_{\lambda \in \mathbb{T}^\vee/W_0} \varepsilon_\lambda M,
\]
and $\mathbb{T} \subset \overline{F}_q[\mathbb{T}] \subset \mathcal{H}^{I^{(1)}}$ acts on $\varepsilon_\lambda M$ through the character $\lambda : \mathbb{T} \to \overline{F}_q^\times$. Now if $M = \pi^{I^{(1)}}$, then the $\mathbb{T}$-action on $M$ is the factorization of the $\mathbb{T}(o_F)$-action on $\pi$, cf. 8.3.3. In particular, the restriction of the $\mathbb{T}$-action along the diagonal inclusion $\overline{F}_q^\times \subset \mathbb{T}$ is the factorization of the action of the central subgroup $o_F^\times \subset Z(G)$ on $\pi$, which is given by $\zeta^F_q$ by assumption. Hence
\[
\varepsilon_\lambda M \neq 0 \quad \Longrightarrow \quad \forall \lambda \in \gamma, \quad \lambda|_{\overline{F}_q} = \zeta|_{\overline{F}_q} \quad \text{i.e.} \quad \gamma|_{\overline{F}_q} = \zeta|_{\overline{F}_q}.
\]
Moreover, the element $u^2 = \text{diag}(\varpi^{-1}, \varpi^{-1}) \in Z(G)$ acts on $\pi$ by multiplication by $\zeta(\varpi^{-1})$ by assumption. Therefore, by [S,3,3] the Hecke operator $z_2 := U^2 \in \mathcal{H}_{\varpi}^{(1)}$ acts on $\pi^{(1)}$ by multiplication by $\zeta(\varpi^{-1})$. Thus we have obtained that $S(\pi^{(1)})$ is supported on

$$
\prod_{\gamma \in (T^\vee/W_0)_{\text{reg}}, |\gamma|_{\varpi} = \zeta(\varpi^{-1})} V_{T.0, \zeta(\varpi^{-1})} \prod_{\gamma \in (T^\vee/W_0)_{\text{non-reg}}, |\gamma|_{\varpi} = \zeta(\varpi^{-1})} V_{T.0, \zeta(\varpi^{-1})/W_0} = (V_{T.0}^{(1)}/W_0)(\zeta|_{\varpi}^{-1}) \cdot \zeta(\varpi^{-1})
$$

Next, recall the twisting action of the group $Z^\vee(k)$ on the standard $\mathcal{H}_{\varpi}^{(1)}$-modules and their simple constituents [S,1,4].

**8.3.6. Proposition.** Let $\pi \in \text{Mod}^{\text{adm}}(k[G])$ be irreducible or a reducible principal series representation. Let $\eta : F^\times \to k^\times$ be a character. Then

$$
(\pi \otimes \eta)^{(1)} = \pi^{(1)} \cdot (\eta|_{\varpi}^{-1}, \eta(\varpi^{-1}))
$$

as $\mathcal{H}_{\varpi}^{(1)}$-modules.

**Proof.** An irreducible locally admissible representation, being a finitely generated $k[G]$-module, is admissible [Em10, 2.2.19]. A principal series representation (irreducible or not) is always admissible [Em10, 4.1.7]. The list of irreducible admissible smooth $G$-representations is given in [H11b, Thm. 1.1]. There are four families: principal series representations, supersingular representations, characters and twists of the Steinberg representation.

We first suppose that $\pi$ is a principal series representation (irreducible or not), i.e. of the form $\text{Ind}_B^G(\chi)$ with a character $\chi = \chi_1 \otimes \chi_2$. Then $\pi \otimes \eta \simeq \text{Ind}_B^G(\chi_1 \eta \otimes \chi_2 \eta)$. We use the results from [S,3,2]. The modules $\pi^{(1)}$ and $(\pi \otimes \eta)^{(1)}$ are standard modules in the components $\gamma := \{\chi|_T, \chi^\times|_T\}$ and $\gamma(\eta|_{\varpi}^{-1})$ respectively. Suppose that $\gamma$ is regular. We choose the ordering $(\chi|_T, \chi^\times|_T)$ and standard coordinates $x, y$. Then

$$
\text{Ind}_B^G(\chi)^{(1)} = M(0, \chi_2(\varpi^{-1}), \chi_1(\varpi^{-1}) \chi_2(\varpi^{-1}), \chi|_T)
$$

and

$$
\text{Ind}_B^G(\chi_1 \eta \otimes \chi_2 \eta)^{(1)} = M(0, \chi_2(\varpi^{-1}) \eta(\varpi^{-1}), \chi_1(\varpi^{-1}) \chi_2(\varpi^{-1}) \eta(\varpi^{-2}), (\chi|_T).(\eta|_{\varpi}^{-1})).
$$

This shows $(\pi \otimes \eta)^{(1)} = \pi^{(1)} \cdot (\eta|_{\varpi}^{-1}, \eta(\varpi^{-1}))$ in the regular case. Suppose that $\gamma$ is non-regular. Then

$$
\text{Ind}_B^G(\chi)^{(1)} = M(\chi_2(\varpi^{-1}), \chi_1(\varpi^{-1}) \chi_2(\varpi^{-1}), \chi|_T)
$$

and

$$
\text{Ind}_B^G(\chi_1 \eta \otimes \chi_2 \eta)^{(1)} = M(\chi_2(\varpi^{-1}) \eta(\varpi^{-1}), \chi_1(\varpi^{-1}) \chi_2(\varpi^{-1}) \eta(\varpi^{-2}), (\chi|_T).(\eta|_{\varpi}^{-1})).
$$

This shows $(\pi \otimes \eta)^{(1)} = \pi^{(1)} \cdot (\eta|_{\varpi}^{-1}, \eta(\varpi^{-1}))$ in the non-regular case.

We now treat the case where $\pi$ is a character or a twist of the Steinberg representation. Consider the exact sequence

$$
1 \to \mathbb{1} \to \text{Ind}_B^G(1) \to \text{St} \to 1.
$$

According to [VOL, 4.4] the sequence of invariants

$$
(S) : 1 \to \mathbb{1}^{(1)} \to \text{Ind}_B^G(1)^{(1)} \to \text{St}^{(1)} \to 1
$$

is still exact and $\mathbb{1}^{(1)}$ resp. $\text{St}^{(1)}$ is the trivial character $(0, 1)$ resp. sign character $(-1, -1)$ in the Iwahori component $\gamma = 1$. Tensoring the first exact sequence with $\eta$ produces the exact sequence

$$
1 \to \eta \to \text{Ind}_B^G(1) \otimes \eta \to \text{St} \otimes \eta \to 1.
$$

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Since the restriction $\eta|_{k^*}$ is trivial on $1 + \varpi_0 F$, one has $(\eta \circ \det)|_{I(1)} = 1$ and so, as a sequence of $k$-vector spaces with $k$-linear maps, the sequence of invariants

$$1 \to \eta^{(1)} \to (\text{Ind}_G^H(1) \otimes \eta)^{(1)} \to (\text{St} \otimes \eta)^{(1)} \to 1$$

coincides with the sequence $(S)$. It is therefore an exact sequence of $\mathcal{H}_{\pi_q}^{(1)}$-modules, with outer terms being characters of $\mathcal{H}_{\pi_q}^{(1)}$. From the discussion above, we deduce

$$(\text{Ind}_G^H(1) \otimes \eta)^{(1)} = \text{Ind}_G^H((1)^{(1)}(\eta|_{F_q^*, \eta(\varpi)^{-1}}) = M(\eta(\varpi)^{-1}, \eta(\varpi^{-2}), 1, (\eta|_{F_q^*})).$$

It follows then from [V04, 1.1] that $\eta^{(1)}$ must be the trivial character $(0, \eta(\varpi^{-1}))$ in the component $1, (\eta|_{F_q^*})$ and $(\text{St} \otimes \eta)^{(1)}$ must be the sign character $(-1, -\eta(\varpi^{-1}))$ in the component $1, (\eta|_{F_q^*})$. This implies 

$$\eta^{(1)} = \mathbb{I}^{(1)}(\eta|_{F_q^*}, \eta(\varpi)^{-1}) \text{ and } (\text{St} \otimes \eta)^{(1)} = \mathbb{I}^{(1)}(\eta|_{F_q^*}, \eta(\varpi)^{-1}).$$

This proves the claim in the cases $\pi = \mathbb{I}$ or $\pi = \text{St}$. If, more generally, $\pi = \eta'$ is a general character of $G$, then

$$(\pi \otimes \eta)^{(1)} = (\eta' \eta)^{(1)} = \mathbb{I}^{(1)}((\eta' \eta)|_{F_q^*}, (\eta' \eta)(\varpi)^{-1}) = \pi^{(1)}((\eta' \eta)|_{F_q^*}, \eta(\varpi)^{-1}).$$

On the other hand, if $\pi = \text{St} \otimes \eta'$ is a twist of Steinberg, then

$$(\pi \otimes \eta)^{(1)} = (\text{St} \otimes (\eta' \eta))^{(1)} = (\eta' \eta)(\varpi)^{-1} = \mathbb{I}^{(1)}(\eta|_{F_q^*}, (\eta' \eta)(\varpi)^{-1}) = \pi^{(1)}((\eta|_{F_q^*}, \eta(\varpi)^{-1}).$$

It remains to treat the case where $\pi$ is a supersingular representation. In this case $\pi \otimes \eta$ is also supersingular and the two modules $\pi^{(1)}$ and $(\pi \otimes \eta)^{(1)}$ are supersingular $\mathcal{H}_{\pi_q}^{(1)}$-modules [V04, 4.9]. Let $\gamma$ be the component of the module $\pi^{(1)}$. By 8.3.3, the component of $(\pi \otimes \eta)^{(1)}$ equals $\gamma(\eta|_{F_q^*})$. Moreover, if $U^2$ acts on $\pi^{(1)}$ via the scalar $z_2 \in k^*$, then $U^2$ acts on $(\pi \otimes \eta)^{(1)}$ via $z_2(\eta \circ \det)(u^2) = z_2 \eta(\varpi)^{-2}$. From 8.3.3. Since the supersingular modules are uniquely characterized by their component and their $U^2$-action, we obtain $(\pi \otimes \eta)^{(1)} = \pi^{(1)}((\eta|_{F_q^*}, \eta(\varpi)^{-1})$, as claimed.

8.3.7. Let $F = \mathbb{Q}_p$ with $p \geq 5$. We let $\text{Mod}_{\rho}^{\text{adm}}(k[G])$ be the full subcategory of $\text{Mod}_{\rho}^{\text{adm}}(k[G])$ consisting of locally admissible representations having central character $\zeta$. By work of Paskunas [Pas13], the blocks $b$ of the category $\text{Mod}_{\rho}^{\text{adm}}(k[G])$, defined as certain equivalence classes of simple objects, can be parametrized by the set of isomorphism classes $[\rho]$ of semisimple continuous Galois representations $\rho : \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \text{G}(k)$ having determinant $\det \rho = \omega \zeta$, i.e. by the $k$-points of $X_{\zeta}$. There are three types of blocks. Blocks of type 1 are supersingular blocks. Each such block contains only one irreducible $G$-representation, which is supersingular. Blocks of type 2 contain only two irreducible representations. These two representations are two generic principal series representations of the form $\text{Ind}_G^H(\chi_1 \otimes \chi_2 \omega^{-1})$ and $\text{Ind}_G^H(\chi_2 \otimes \chi_1 \omega^{-1})$ (where $\chi_1 \chi_2 \neq 1, \omega \pm 1$). There are four blocks of type 3 which correspond to the four exceptional points. In the even case, each such block contains only three irreducible representations. These representations are of the form $\eta, \text{St} \otimes \eta$ and $\text{Ind}_G^H(\omega \otimes \omega^{-1}) \otimes \eta$. In the odd case, each block of type 3 contains only one irreducible representation. It is of the form $\text{Ind}_G^H(\chi \otimes \chi \omega^{-1})$.

8.3.8. Let $F = \mathbb{Q}_p$ with $p \geq 5$. Paskunas’ parametrization $[\rho] \mapsto b_{[\rho]}$ is compatible with Breuil’s semisimple mod $p$ local Langlands correspondence

$$\rho \mapsto \pi(\rho)$$

for the group $G$ [Br07] [Bel11], in the sense that if $\rho$ has determinant $\omega \zeta$, then the simple constituents of the $G$-representation $\pi(\rho)$ lie in the block $b_{[\rho]}$ of $\text{Mod}_{\rho}^{\text{adm}}(k[G])$.

The correspondence and the parametrizations (for varying $\zeta$) commute with twists: for a character $\eta : \mathbb{Q}_p^* \to k^*$, $\pi(\rho \otimes \eta) = \pi(\rho) \otimes \eta$ and $b_{[\rho] \otimes \eta} = b_{[\rho \otimes \eta]}$.
8.3.9. Theorem. Suppose $F = \mathbb{Q}_p$ with $p \geq 5$. Fix a character $\zeta : Z(G) = \mathbb{Q}_p^\times \rightarrow k^\times$, corresponding to a point $(\zeta|_{\mathbb{F}_p^\times}, \zeta(p^{-1})) \in Z^\vee(k)$ under the identification $Z(G)^\vee \cong Z^\vee(k)$ from 8.3.2. Let $(V_{T,0}^{(1)}/W_0^\zeta)$ be the space of mod $p$ Satake parameters with central character $\zeta$ and $X_\zeta$ be the moduli space of mod $p$ Langlands parameters with determinant $\omega_\zeta$.

There exists a morphism of $k$-schemes

$$L_\zeta : (V_{T,0}^{(1)}/W_0)^\zeta \longrightarrow X_\zeta$$

such that the quasi-coherent $O_{X_\zeta}$-module

$$L_\zeta \ast S(M_{\mathbb{F}_p}^{(1)})_{(V_{T,0}^{(1)}/W_0)^\zeta}$$

equal to the push-forward along $L_\zeta$ of the restriction to $(V_{T,0}^{(1)}/W_0)^\zeta \subset V_{T,0}^{(1)}/W_0$ of the Satake parameter of the mod $p$ antispherical module $M_{\mathbb{F}_p}^{(1)}$ interpolates the $I^{(1)}$-invariants of the semisimple mod $p$ Langlands correspondence

$$X_\zeta(k) \longrightarrow \text{Mod}_{\zeta}(k[G]) \longrightarrow \text{Mod}(\mathcal{H}_{\mathbb{F}_p}^{(1)})$$

$$x \longmapsto \pi(\rho_x) \longmapsto \pi(\rho_x)^{I^{(1)}}$$
in the sense that for all $x \in X_\zeta(k)$,

$$\left( (L_\zeta \ast S(M_{\mathbb{F}_p}^{(1)})_{(V_{T,0}^{(1)}/W_0)^\zeta} \otimes_{O_{X_\zeta}} k(x) \right)^{ss} = \left( M_{\mathbb{F}_p}^{(1)} \otimes_{Z(\mathcal{H}_{\mathbb{F}_p}^{(1)})} (\mathcal{H}_{\mathbb{F}_p}^{(1)})^{-1}(O_{L_\zeta(-1)}) \right)^{ss} \cong \pi(\rho_x)^{I^{(1)}}$$
in $\text{Mod}(\mathcal{H}_{\mathbb{F}_p}^{(1)})$.

8.3.10. The connected components of $(V_{T,0}^{(1)}/W_0)^\zeta$ are either regular and then of type $\mathbb{A}_1 \cup_0 \mathbb{A}_1$, or non-regular and then of type $\mathbb{A}_1$. The morphism $L_\zeta$ appearing in the theorem depends on the choice of an order of the two affine lines in each regular component. It is surjective and quasi-finite. Moreover, writing $L_\gamma$ for its restriction to the connected component $(V_{T,0}^{(1)}/W_0)^\zeta \subset (V_{T,0}^{(1)}/W_0)^\zeta$, one has:

(e) Even case. All connected components are of type $\mathbb{A}_1 \cup_0 \mathbb{A}_1$, except for the two ‘exterior’ components which are of type $\mathbb{A}_1$. $L_\gamma$ is an open immersion for any $\gamma$.

(o) Odd case. All connected components are of type $\mathbb{A}_1 \cup_0 \mathbb{A}_1$. $L_\zeta$ is an open immersion on all connected components, except for the two ‘exterior’ ones. On an ‘exterior’ component $\gamma$, the restriction of $L_\zeta$ to one irreducible component $\mathbb{A}_1$ is an open immersion, and its restriction to the open complement $\mathbb{G}_m$ is a degree 2 finite flat covering of its image, with branched locus equal to the intersection of this image with the exceptional locus of $X_\zeta$.

8.3.11. Note that the semisimple mod $p$ Langlands correspondence associates with any semisimple $\rho : \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \rightarrow \tilde{G}(k)$ a semisimple smooth $G$-representation $\pi(\rho)$ of length 1, 2 or 3, hence whose semisimple $\mathcal{H}_{\mathbb{F}_p}^{(1)}$-module of $I^{(1)}$-invariants $\pi(\rho)^{I^{(1)}}$ has length 1, 2 or 3. On the other hand, the antispherical map

$$\text{ASph} : (V_{T,0}^{(1)}/W_0)(k) \longrightarrow \{ \text{left } \mathcal{H}_{\mathbb{F}_p}^{(1)} \text{-modules} \}$$

has an image consisting of $\mathcal{H}_{\mathbb{F}_p}^{(1)}$-modules of length 1 or 2, cf. [PS 7.4.9] and [PS 7.4.15]. Theorem 8.3.9 combined with the properties 8.3.10 of the morphism $L_\zeta$ provide the following case-by-case elucidation of the $\mathcal{H}_{\mathbb{F}_p}^{(1)}$-modules $\pi(\rho)^{I^{(1)}}$.

8.3.12. Corollary. Let $x \in X_\zeta(k)$, corresponding to $\rho_x : \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \rightarrow \tilde{G}(k)$. Then the $\mathcal{H}_{\mathbb{F}_p}^{(1)}$-module $\pi(\rho_x)^{I^{(1)}}$ admits the following explicit description.
(i) If \( x \in X^\mathrm{red}_\zeta(k) \), then the fibre \( L^{-1}_\zeta(x) = \{v\} \) has cardinality 1 and 
\[ \pi(\rho_x)^I \simeq \text{ASph}(v). \]

It is irreducible and supersingular.

(ii) If \( x \in X^\mathrm{red}_\zeta(k) \setminus \{\text{the four exceptional points}\} \), then \( L^{-1}_\zeta(x) = \{v_1, v_2\} \) has cardinality 2 and 
\[ \pi(\rho_x)^I \simeq \text{ASph}(v_1) \oplus \text{ASph}(v_2). \]

It has length 2.

(iii) If \( x \in X^\mathrm{red}_\zeta(k) \) is exceptional in the even case, then \( L^{-1}_\zeta(x) = \{v_1, v_2\} \) has cardinality 2 and 
\[ \pi(\rho_x)^I \simeq \text{ASph}(v_1)^\text{ss} \oplus \text{ASph}(v_2). \]

It has length 3.

(iii) If \( x \in X^\mathrm{red}_\zeta(k) \) is exceptional in the odd case, then \( L^{-1}_\zeta(x) = \{v\} \) has cardinality 1 and 
\[ \pi(\rho_x)^I \simeq \text{ASph}(v) \oplus \text{ASph}(v). \]

It has length 2.

8.3.13. Now we proceed to the proof of 8.3.9, 8.3.10 and 8.3.12.

We start by defining the morphism \( L_\zeta \) at the level of \( k \)-points. Let \( v \in (V^\mathrm{ss}_q(W_0)_\zeta(k)) \) and let its connected component be indexed by \( \gamma \in \mathbb{T}_\gamma/W_0 \).

1. Suppose that \( \gamma \) is regular. Then \( \text{ASph}(v) = \text{ASph}^\gamma(v) \) is a simple two-dimensional \( \mathcal{H}^\gamma_{rig} \)-module, cf. [PS, 7.4.9]. Let \( \pi \in \text{Mod}^{\text{red}}(k[G]) \) be the simple module, unique up to isomorphism, such that \( \pi^I \simeq \text{ASph}^\gamma(v) \), cf. 8.3.2. Then \( \pi \in \text{Mod}^{\text{ad}}(k[G]) \) with 
\[ \zeta = (\zeta|_{\mathcal{F}}^\gamma, \zeta(p^{-1})) = (\gamma|_{\mathcal{F}}^\gamma, z_2) \]
by 8.3.3. Let \( b \) be the block of \( \text{Mod}^{\text{ad}}(k[G]) \) which contains \( \pi \). We define \( L_\zeta(v) \) to be the point of \( X_\zeta(k) \) which corresponds to \( b \).

2. Suppose that \( \gamma \) is non-regular.

(a) If \( v \in D(1)_{\gamma}(k) \), then \( \text{ASph}(v) = \text{ASph}^\gamma(2)(v) \) is a simple two-dimensional \( \mathcal{H}^\gamma_{rig} \)-module, cf. [PS, 7.4.15]. As in the regular case, there is a simple module \( \pi \), unique up to isomorphism, such that \( \pi^I \simeq \text{ASph}^\gamma(2)(v) \). It has central character \( \zeta = (\gamma|_{\mathcal{F}}^\gamma, z_2) \) and there is a block \( b \) of \( \text{Mod}^{\text{ad}}(k[G]) \) which contains \( \pi \). We define \( L_\zeta(v) \) to be the point of \( X_\zeta(k) \) which corresponds to \( b \).

(b) If \( v \in D(2)_{\gamma}(k) \), then \( \text{ASph}(v)^\text{ss} \) is the direct sum of the two characters forming the antispherical pair \( \text{ASph}^\gamma(1)(v) = \{(0, z_1), (\omega, -z_1)\} \) where \( z_2 = z_1^2 \), cf. [PS, 7.4.15]. As in the regular case, there are two simple modules \( \pi_1 \) and \( \pi_2 \), unique up to isomorphism, such that \( \pi_1^I \simeq (0, z_1) \) and \( \pi_2^I \simeq (\omega, -z_1) \) and \( \pi_1, \pi_2 \) have central character \( \zeta = (\gamma|_{\mathcal{F}}^\gamma, z_2) \). Moreover, we claim that there is a unique block \( b \) of \( \text{Mod}^{\text{ad}}(k[G]) \) which contains both \( \pi_1 \) and \( \pi_2 \). Indeed, if \( \gamma = \{1 \otimes 1\} \) and \( z_1 = 1 \), then \( \pi_1 = 1 \) and \( \pi_2 = \text{St} \), cf. 8.3.2. Then by 8.3.3 it follows more generally that if \( \gamma = \{\omega' \otimes \omega''\} \), then \( \pi_1 = \eta \) and \( \pi_2 = \text{St} \otimes \eta \) with \( \eta = (\eta|_{\mathcal{F}}^\gamma, \eta(p^{-1})) = (\omega', z_1) \). Consequently, \( \pi_1, \pi_2 \) are contained in a unique block \( b \) of type 3, cf. 8.3.7. We define \( L_\zeta(v) \) to be the point of \( X_\zeta(k) \) which corresponds to \( b \).

Thus we have a well-defined map of sets \( L_\zeta: (V^\mathrm{ss}_q(W_0)_\zeta(k)) \to X_\zeta(k) \).

We show property (i) of 8.3.12 Let \( x \in X^\mathrm{red}_\zeta(k) \) and suppose \( L_\zeta(v) = x \). Then \( b_x \) is a supersingular block, contains a unique irreducible representation \( \pi \), which is supersingular, and \( \pi = \pi(\rho_x) \), cf. 8.3.7. By definition of \( L_\zeta \), one has \( \text{ASph}(v) \simeq \pi^I \). Since the antispherical
map $\text{ASph}$ is $1 : 1$ over supersingular modules, cf. [PS 7.4.9] and [PS 7.4.15], such a preimage $v$ of $x$ exists and is uniquely determined by $x$. Summarizing, we have $L_\zeta^{-1}(x) = \{v\}$ and $\text{ASph}(v) \simeq \pi(\rho_v)^{(1)}$. This is property (i).

As a next step, we take a second character $\eta : \mathbb{Q}_p^\times \rightarrow k^\times$ and show that the diagram

$$
\begin{array}{ccc}
(V^{(1)}_{T,0}/W_0)_\zeta(k) & \xrightarrow{L_\zeta} & X_\zeta(k) \\
\eta \simeq & & \simeq (\circ \eta) \\
(V^{(1)}_{T,0}/W_0)_{\zeta \eta^2}(k) & \xrightarrow{L_{\zeta \eta^2}} & X_{\zeta \eta^2}(k)
\end{array}
$$

commutes. Here, the vertical arrows are the bijections coming from $S.1.9$ and $S.2.2$. To verify the commutativity, let $v \in (V^{(1)}_{T,0}/W_0)_\zeta(k)$ and let its connected component be indexed by $\gamma \in T^\vee/W_0$. Suppose that $\gamma$ is regular or that $\gamma$ is non-regular with $v \in D(2)_\gamma(k)$. Let $\pi$ be the simple $G$-module with $\pi^{(1)} \simeq \text{ASph}(v)$ and let $b_{[\rho]}$ be the block corresponding to the point $L_\zeta(v)$. By the equivariance property $S.1.9$, one has $\text{ASph}(v \circ \eta) \simeq \text{ASph}(v) \circ \eta$. Taking $\pi$-invariants is compatible with twist, cf. $S.3.6$, and so $L_{\pi \eta^2(v \circ \eta)}$ corresponds to the block which contains the representation $\pi \circ \eta$, i.e. to $b_{[\rho \circ \eta]} = b_{[\rho \circ \eta \circ \eta]}$, cf. $S.3.5$ and so $L_{\pi \eta^2(v \circ \eta)} = [\rho \circ \eta] = L_\zeta(v \circ \eta)$.

If $v \in D(1)^\vee_\gamma(k)$, let $\pi_1$ and $\pi_2$ be the simple modules such that $(\pi_1 \circ \pi_2)^{v \circ \eta}_\gamma \simeq \text{ASph}(v \circ \eta)^{v \circ \eta}$. As before, we conclude from $\text{ASph}(v \circ \eta)^{v \circ \eta} \simeq \text{ASph}(v)^{v \circ \eta} \circ \eta$ that $L_{\pi \eta^2(v \circ \eta)}$ corresponds to the block which contains $\pi_1 \circ \eta$ and $\pi_2 \circ \eta$ and that $L_{\pi \eta^2(v \circ \eta)} = L_\zeta(v \circ \eta)$. The commutativity of the diagram is proved.

Thus, we are reduced to prove that the map $L_\zeta$ comes from a morphism of $k$-schemes satisfying $S.3.9$ and the remaining parts of $S.3.12$ in the two basic cases of a character $\zeta$ such that $\zeta(p^{-1}) = 1$ and $\zeta|_{\mathbb{Q}_p^\times} \in \{1, \omega^{-1}\}$. This is established in the next two subsections.

### 8.4 The morphism $L_\zeta$ in the basic even case

Let $\zeta : \mathbb{Q}_p^\times \rightarrow k^\times$ be the trivial character. Here we show that the map of sets $L_\zeta : (V^{(1)}_{T,0}/W_0)_\zeta(k) \rightarrow X_\zeta(k)$ that we have defined in $S.3.13$ satisfies properties (ii) and (iii) of $S.3.12$ and we define a morphism of $k$-schemes $L_\zeta : (V^{(1)}_{T,0}/W_0)_\zeta \rightarrow X_\zeta$ which coincides with the previous map of sets at the level of $k$-points. By construction, it will have the properties $S.3.10$. This will complete the proof of $S.3.12$, $S.3.10$ and $S.3.9$ in the case of an even character.

#### 8.4.1. We verify the properties (ii) and (iii). We work over an irreducible component $\mathbb{P}^1$ with label " $\text{Sym}^r \otimes \text{det}^a \mid \text{Sym}^{p-r-3} \otimes \text{det}^{r+1+a}$" where $0 \leq r \leq p-3$ and $0 \leq a \leq p-2$, cf. $S.2.2$. On this component, we choose an affine coordinate $x$ around the double point having $\text{Sym}^r \otimes \text{det}^a$ as one of its Serre weights. Away from this point, we have $x \neq 0$ and the corresponding Galois representation has the form

$$
\rho_x = \begin{pmatrix}
\text{unr}(x)\omega^{r+1} & 0 \\
0 & \text{unr}(x^{-1})
\end{pmatrix} \otimes \eta
$$

with $\eta = \omega^a$. By [Br97 1.3] or [Br07 4.11], we have

$$
\pi(\rho_x) = \pi(x, r, \eta)^{\text{ss}} \oplus \pi([p - 3 - r], x^{-1}, \omega^{r+1}\eta)^{\text{ss}} =: \pi_1 \oplus \pi_2
$$

where $[p - 3 - r]$ denotes the unique integer in $\{0, \ldots, p - 2\}$ which is congruent to $p - 3 - r$ modulo $p - 1$. Now suppose that $L_\zeta(v) = x$. We distinguish two cases.

1. The generic case $0 < r < p - 3$. In this case, the point $x$ lies on one of the "interior" components of the chain $X_\zeta$, which has no exceptional points. The length of $\pi(\rho_x)$ is 2. Indeed, $\pi_1 = \pi(x, r, \eta)$ and $\pi_2 = \pi(p - 3 - r, x^{-1}, \omega^{r+1}\eta)$ are two irreducible principal series representations [Br97 Thm. 4.4]. The block $b_{\rho_x}$ is of type 2 and contains only these two irreducible representations, cf. $S.3.7$, $S.5.8$. We may write

$$
\pi_1 = \text{Ind}^G_B(\chi) \otimes \eta
$$

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with \( \chi = \text{unr}(x) \otimes \omega^r \text{unr}(x^{-1}) \), according to [Br07, Rem. 4.4(ii)]. By our assumptions on \( r \), the character \( \chi|_{\Gamma} = 1 \otimes \omega^r \) is regular (i.e. different from its \( s \)-conjugate). We conclude from \( \text{S.3.6} \) and \( \text{S.3.3} \) that \( \pi_{1}^{(i)} \) is a simple 2-dimensional standard module in the regular component represented by the character \((1 \otimes \omega^r).((\eta_{|\Gamma_p^2}) \otimes (\eta_{|\Gamma_p^2}) \omega^r) \in \mathbb{T}^\vee \). Similarly, we may write

\[
\pi_2 = \text{Ind}(\gamma/G)(1) \otimes \omega^{r+1} \eta
\]

where now \( \chi = \text{unr}(x^{-1}) \otimes \omega^{p-3-r} \text{unr}(x). \) By our assumptions on \( r \), the character \( \chi|_{\Gamma} = 1 \otimes \omega^{p-3-r} \) is regular and we conclude, as above, that the \( I^{(1)} \)-invariants \( \pi_{1}^{(i)} \) form a simple 2-dimensional standard module in the regular component represented by the character \((\eta_{|\Gamma_p^2}) \omega^{r+1} \otimes (\eta_{|\Gamma_p^2}) \omega^{r+1} \omega^{p-3-r} \in \mathbb{T}^\vee \). Note that the component of \( \pi_{1}^{(i)} \) is different from the component of \( \pi_{2}^{(i)} \), by our assumptions on \( r \).

We conclude from \( L_\zeta(v) = x \) that either \( \text{ASph}(v) = \pi_{1}^{(i)} \) or \( \text{ASph}(v) = \pi_{2}^{(i)} \). Since for \( \gamma \) regular, the map \( \text{ASph} \gamma \) is a bijection onto all simple \( \mathcal{H}_{\Gamma_p^2}' \)-modules, cf. [PS 7.4.9], one finds that \( L_\zeta^{-1}(x) = \{v_1, v_2\} \) has cardinality 2 and

\[
\text{ASph}(v_1) \oplus \text{ASph}(v_2) \simeq \pi(\rho_x)_{I^{(i)}}.
\]

This settles property (ii) of \( \text{S.3.12} \) in the generic case.

2. The boundary cases \( r \in \{0, p-3\} \). In this case, the point \( x \) lies on one of the two ‘exterior’ components of \( X_\zeta \). On such a component, we will denote the variable \( x \) rather by \( z_1 \), which is the notation \( \lambda \) we used already in \( \text{S.2.3} \).

(a) Suppose that \( z_1 \neq \pm 1 \). The length of \( \pi(\rho_{z_1}) \) is 2. Indeed, as in the generic case, \( \pi_1 = \pi(r, z_1, \eta) \) and \( \pi_2 = \pi(p-3-r, z_1^{-1}, \omega^{r+1} \eta) \) are two irreducible principal series representations. The block \( b_{z_1} \) is of type 2 and contains only these two irreducible representations. It follows, as above, that their invariants \( \pi_{1}^{(i)} \) and \( \pi_{2}^{(i)} \) are simple 2-dimensional standard modules, in the components represented by \((\eta_{|\Gamma_p^2}) \otimes (\eta_{|\Gamma_p^2}) \omega^r \in \mathbb{T}^\vee \) and \((\eta_{|\Gamma_p^2}) \omega^{r+1} \otimes (\eta_{|\Gamma_p^2}) \omega^{r+1} \omega^{p-3-r} \in \mathbb{T}^\vee \) respectively. Since \( r \in \{0, p-3\} \), one of these components is regular, the other non-regular. In particular, the two components are different. We conclude from \( L_\zeta(v) = z_1 \) that either \( \text{ASph}(v) = \pi_{1}^{(i)} \) or \( \text{ASph}(v) = \pi_{2}^{(i)} \). Since for non-regular \( \gamma \), the map \( \text{ASph} \gamma(2) \) is a bijection from \( D(2)(k) \) onto all simple standard \( \mathcal{H}_{\Gamma_p^2}' \)-modules, cf. [PS 7.4.15], we may conclude as in the generic case: \( L_\zeta^{-1}(z_1) = \{v_1, v_2\} \) has cardinality 2 and

\[
\text{ASph}(v_1) \oplus \text{ASph}(v_2) \simeq \pi(\rho_{z_1})_{I^{(i)}}.
\]

This settles property \( \text{S.3.12} \) (ii) in the remaining case \( z_1 \neq \pm 1 \).

(b) Suppose now that \( z_1 = \pm 1 \), i.e. we are at one of the four exceptional points. We will verify property (iie). The length of \( \pi(\rho_{z_1}) \) is 3. Indeed, the representation \( \pi(0, \pm 1, \eta) \) is a twist of the representation \( \pi(0, 1, 1) \) (note that \( \pi(r, z_1, \eta) \simeq \pi(r, -z_1, \text{unr}(-1) \eta) \) according to [Br07, Rem. 4.4(v)])], which itself is an extension of \( \mathbb{1} \) by \( \text{St} \), cf. [Br07, Thm. 4.4(iii)]. As in the case (a), the representation \( \pi_2 = \pi(p-3, \pm 1, \omega^r \eta) \) is an irreducible principal series representation. The block \( b_{z_1} \) is of type 3 and contains only these three irreducible representations. The invariants \( \pi_{1}^{(i)} \) form a direct sum of two antispherical characters in a non-regular component \( \gamma \), whereas the invariants \( \pi_{2}^{(i)} \) form a simple standard module in a regular component, as before. Since for non-regular \( \gamma \), the map \( \text{ASph} \gamma(1) \) is a bijection from \( D(1)(k) \) onto all antispherical pairs of characters of \( \mathcal{H}_{\Gamma_p^2}' \), cf. [PS 7.4.15], we may conclude that \( L_\zeta^{-1}(z_1) = \{v_1, v_2\} \) has cardinality 2 with \( v_1 \in D(1)(k) \) and \( \text{ASph} \gamma(1)(v_1)_{ss} \simeq \pi_{1}^{(i)} \). In particular,

\[
\text{ASph}(v_1)_{ss} \oplus \text{ASph}(v_2) \simeq \pi(\rho_x)_{I^{(i)}}.
\]

This settles property \( \text{S.3.12} \) (iiiie).
8.4.2. We define a morphism of $k$-schemes $L_\zeta : (V_{\tilde T,0}/W_0)_\zeta \to X_\zeta$ which coincides on $k$-points with the map of sets $L_\zeta : (V_{\tilde T,0}/W_0)_\zeta(k) \to X_\zeta(k)$. We work over a connected component of $(V_{\tilde T,0}/W_0)_\zeta$, indexed by some $\gamma \in \mathbb{T}^v/W_0$. Let $v$ be a $k$-point of this component.

Since $\zeta|_p = 1$, the connected components of $(V_{\tilde T,0}/W_0)_\zeta$ are indexed by the fibre $(\cdot)|_{\mathbb{F}_p}(1)$. This fibre consists of the $\frac{p^3-3}{2}$ regular components, represented by the characters of $T$

$$\chi_k = \omega^k \otimes \omega^{-k}$$

for $k = 1, \ldots, \frac{p^3-3}{2}$, and of the two non-regular components, given by $\chi_0$ and $\chi_{\mathbb{Z}/2}$, cf. [S.12] We distinguish two cases. Note that $z_2 = \zeta(p^{-1}) = 1$.

1. The regular case $0 < k < \frac{p^3-1}{2}$. We fix the order $\gamma = (\chi_k, \chi_k^2)$ on the set $\gamma$ and choose the standard coordinates $x, y$. According to [PS] 7.4.8, our regular connected component identifies with two affine lines intersecting at the origin:

$$V_{\tilde T,0,1} \simeq \mathbb{A}^1 \cup_0 \mathbb{A}^1.$$ 

Suppose that $v = (0, 0)$ is the origin, so that $A\text{Sph}(v)$ is a supersingular module. Let $\pi(r, 0, \eta)$ be the corresponding supersingular representation. It corresponds to the irreducible Galois representation $\rho(r, \eta) = \text{Ind}(\omega^r \otimes \eta)$ in the notation of [He11] 1.3, whence $L_\zeta(v) = [\rho(r, \eta)]$. According to [S.32] the component of the Hecke module $\pi(r, 0, \eta)^{(1)}$ is given by $(\omega^r \otimes 1) \cdot (\eta|_{\mathbb{F}_p})$. Setting $\eta|_{\mathbb{F}_p} = \omega^a$, this implies $(\omega^r \otimes 1) \cdot (\eta|_{\mathbb{F}_p}) = \omega^{a+r} \otimes \omega^a = \chi_k$ and hence $a = -k$ and $r = 2k$. Therefore the Serre weights of the irreducible representation $\rho(r, \eta)$ are $\{\text{Sym}^{2k} \otimes \det^{-k}, \text{Sym}^{p-3-2k} \otimes \det^{k+1}\}$, cf. [Br07] 1.9.

Comparing these pairs of Serre weights with the list [S.23] shows that the $\frac{p^3-3}{2}$ points

$$\{\text{origin} (0, 0) \text{ on the component } (\chi_k, \chi_k^2)\}$$

for $0 < k < \frac{p^3-1}{2}$ are mapped successively to the $\frac{p^3-3}{2}$ double points of the chain $X_\zeta$.

Fix $0 < k < \frac{p^3-1}{2}$ and consider the double point

$$Q = L^c(\text{origin} (0, 0) \text{ on the component } \gamma = (\chi_k, \chi_k^2)).$$

As we have just seen, $Q$ lies on the irreducible component $\mathbb{P}^1$ whose label includes the weight $\text{Sym}^{2k} \otimes \det^{-k}$ (i.e. on the component " $\text{Sym}^{2k} \otimes \det^{-k}$ | $\text{Sym}^{p-3-2k} \otimes \det^{k+1}$"). We fix an affine coordinate on this $\mathbb{P}^1$ around $Q$, which we will also call $x$ (there will be no risk of confusion with the standard coordinate above!). Away from $Q$, the affine coordinate $x \neq 0$ parametrizes Galois representations of the form

$$\rho_x = \left(\begin{array}{cc} \text{unr}(x) \omega^{2k+1} & 0 \\ 0 & \text{unr}(x^{-1}) \end{array}\right) \otimes \eta$$

with $\eta := \omega^{-k}$. As we have seen above, $\pi(\rho_x) = \pi(2k, x, \eta) \oplus \pi(p-3-2k, x^{-1}, \omega^{r+1}\eta) =: \pi_1 \oplus \pi_2$. Moreover, $\pi_1 = \text{Ind}_{\mathbb{F}_p}(\chi) \otimes \eta$ with $\chi = \text{unr}(x) \otimes \omega^{2k} \text{unr}(x^{-1})$. Since

$$(1 \otimes \omega^{2k}) \cdot (\eta|_{\mathbb{F}_p}) = \omega^{-k} \otimes \omega^k = \chi_k^2 \in \mathbb{T}^v,$$

we deduce from the regular case of [S.32] that

$$\pi_1^{(1)} = M(0, x, 1, \chi_k^2)$$

is a simple 2-dimensional standard module. Note that $M(0, x, 1, \chi_k^2) = M(x, 0, 1, \chi_k)$ according to [V04] Prop. 3.2.

Now suppose that $v = (x, 0), x \neq 0$, denotes a point on the $x$-line of $\mathbb{A}^1_0 \cup_0 \mathbb{A}^1_1$. In particular, $A\text{Sph}^o(v) = M(x, 0, 1, \chi_k)$. By our discussion, the point $L_\zeta((x, 0))$ corresponds to the block which contains $\pi_1$. Since $\pi_1$ lies in the block parametrized by $[\rho_x]$, cf. [S.33] it follows that

$$L_\zeta((x, 0)) = [\rho_x] = x \in \mathcal{G}_m \subset \mathbb{P}^1 \subset X_\zeta.$$
Since $(0,0)$ maps to $Q$, i.e. to the point at $x = 0$, the map $L\zeta$ identifies the whole affine $x$-line $\mathbb{A}^1 = \{(x,0) : x \in k\} \subset V_{T,0,1}$ with the affine $x$-line $\mathbb{A}^1 \subset \mathbb{P}^1 \times X\zeta$.

On the other hand, the double point $Q$ lies also on the irreducible component $\mathbb{P}^1$ whose labelling includes the other weight of $Q$, i.e. the weight $\text{Sym}^{p-1-2k} \otimes \text{det}^k$. We fix an affine coordinate $y$ on this $\mathbb{P}^1$ around $Q$. Away from $Q$, the coordinate $y \neq 0$ parametrizes Galois representations of the form

$$\rho_x = \begin{pmatrix} \text{unr}(y)\omega^{p-2k} & 0 \\ 0 & \text{unr}(y^{-1}) \end{pmatrix} \otimes \eta$$

with $\eta := \omega^k$. As in the first case, $\pi(\rho_y)$ contains $\pi_1 := \pi(p-1-2k,y,\eta) = \text{Ind}_B^G(\chi) \otimes \eta$ as a direct summand, where now $\chi = \text{unr}(y) \otimes \omega^{p-1-2k} \text{unr}(y^{-1})$. Since $$(1 \otimes \omega^{p-1-2k})(\eta|_{\mathbb{P}^1}) = \omega^k \otimes \omega^{-k} = \chi_k \in \mathbb{T}^\vee,$$ we deduce, as above, that $\pi_1^{(1)} = M(0,y,1,\chi_k)$ is a simple 2-dimensional standard module.

Now suppose that $v = (0,y), y \neq 0$, denotes a point on the $y$-line of $\mathbb{A}^1 \cup_0 \mathbb{A}^1$. In particular, $\text{ASph}^\vee(v) = M(0,y,1,\chi_k)$. By our discussion, the point $L\zeta((0,y))$ corresponds to the block which contains $\pi_1$. Since $\pi_1$ lies in the block parametrized by $[\rho_y]$, cf. [8.3.8], it follows that

$$L\zeta((0,y)) = [\rho_y] = y \in G_m \subset \mathbb{P}^1 \subset X\zeta.$$ Since $(0,0)$ maps to $Q$, i.e. to the point at $y = 0$, the map $L\zeta$ identifies the whole affine $y$-line $\mathbb{A}^1 = \{(0,y) : y \in k\} \subset V_{T,0,1}$ with the affine $y$-line $\mathbb{A}^1 \subset \mathbb{P}^1 \subset X\zeta$.

In this way, we get an open immersion of each regular connected component of $(V_{T,0,1}^1)/W_0)\zeta$ in the scheme $X\zeta$, which coincides on $k$-points with the restriction of the map of sets $L\zeta$.

2. The non-regular case $k \in \{0, \frac{p-1}{2}\}$. We choose the Steinberg coordinate $z_1$. According to [PS] 7.4.10, our non-regular connected component identifies with an affine line:

$$V_{T,0,z_1}/W_0 \simeq \mathbb{A}^1.$$ Suppose that $v = (0)$ is the origin, so that $\text{ASph}(v)$ is a supersingular module. Let $\pi(r,0,\eta)$ be the corresponding supersingular representation so that $L\zeta(v) = [\rho(r,\eta)]$. Exactly as in the regular case, we may conclude that the Serre weights of the irreducible representation $\rho(r,\eta)$ are $\{\text{Sym}^k \otimes \text{det}^{-k}, \text{Sym}^{p-1-2k} \otimes \text{det}^k\}$. For the two values of $k = 0$ and $k = \frac{p-1}{2}$ we find $\{\text{Sym}^0, \text{Sym}^{p-1}\}$ and $\{\text{Sym}^0 \otimes \text{det}^{1/2}, \text{Sym}^{p-1} \otimes \text{det}^{1/2}\}$ respectively. Comparing with the list [8.2.3] shows that the 2 points

$$\{\text{origin } (0) \text{ on the component } (\chi_k = \chi_k^0)\}$$

for $k \in \{0, \frac{p-1}{2}\}$ are mapped to the 2 smooth points in $X\zeta^{\text{irred}}$, which lie on the two ‘exterior’ components of $X\zeta$, cf. [8.2.3].

Fix $k \in \{0, \frac{p-1}{2}\}$ and consider the point

$$Q = L\zeta(\text{origin } (0) \text{ on the component } \gamma = (\chi_k = \chi_k^0)).$$

As we have just seen, $Q$ lies on an ‘exterior’ irreducible component $\mathbb{P}^1$ whose label includes the weight $\text{Sym}^0 \otimes \text{det}^k$. We fix an affine coordinate on this $\mathbb{P}^1$ around $Q$, which we call $z_1$ (there will be no risk of confusion with the Steinberg coordinate above!). Away from $Q$, the affine coordinate $z_1 \neq 0$ parametrizes Galois representations of the form

$$\rho_{z_1} = \begin{pmatrix} \text{unr}(z_1)\omega & 0 \\ 0 & \text{unr}(z_1^{-1}) \end{pmatrix} \otimes \eta$$

with $\eta := \omega^k$. As in the regular case, $\pi(\rho_{z_1}) = \pi(0,z_1,\eta)^{ss} \oplus \pi(p-3,z_1^{-1},\omega\eta)^{ss}$. Moreover, $\pi(0,z_1,\eta) = \text{Ind}_B^G(\chi) \otimes \eta$ with $\chi = \text{unr}(z_1) \otimes \text{unr}(z_1^{-1})$. Since

$$(1 \otimes 1)(\eta|_{\mathbb{P}^1}) = \omega^k \otimes \omega^{-k} = \chi_k = \chi_k^0 \in \mathbb{T}^\vee,$$

$^5$The representations $\pi(0,z_1,\eta)$ constitute the unramified principal series of $G$. 

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we deduce from the non-regular case of [S3.12] that \( \pi(0, z_1, \eta)^{(1)} = M(z_1, 1, \chi_k) \) is a 2-dimensional standard module. Moreover, the standard module is simple if and only if \( \chi \neq \chi^* \), i.e. if and only if \( z_1 \neq \pm 1 \).

Now let \( v = z_1 \neq 0 \) denote a nonzero point on our connected component \( \mathbb{A}^1 = V_{T, 0, 1}/W_0 \).

Suppose that \( z_1 \neq \pm 1 \), i.e. \( v \in D(2) \gamma \). In particular, \( \text{ASph}(v) = M(z_1, 1, \gamma) \) is irreducible. By our discussion, the point \( L_\zeta(z_1) \) corresponds to the block (a block of type 2) which contains \( \pi(0, z_1, \eta) \).

Suppose that \( z_1 = \pm 1 \), i.e. \( v \in D(1) \gamma \). In particular, \( \text{ASph}(v) = M(z_1, 1, \chi_k) \) and again, \( L_\zeta(z_1) \) corresponds to the block (now a block of type 3) which contains the simple constituents of \( \pi(0, z_1, \eta)^{ss} \). In both cases, we conclude

\[ L_\zeta(z_1) = [\rho_{z_1}] = z_1 \in \mathbb{G}_m \subset \mathbb{P}^1 \subset X_\zeta. \]

Since \( (0) \) maps to \( Q \), i.e. to the point at \( z_1 = 0 \), the map \( L_\zeta \) identifies the whole \( z_1 \)-line \( \mathbb{A}^1 = V_{T, 0, 1}/W_0 \) with the \( z_1 \)-line \( \mathbb{A}^1 \subset \mathbb{P}^1 \subset X_\zeta \).

In this way, we get an open immersion of each non-regular connected component of \( (V_{T, 0}/W_0)_\zeta \) in the scheme \( X_\zeta \), which coincides on \( k \)-points with the restriction of the map of sets \( L_\zeta \).

### 8.5 The morphism \( L_\zeta \) in the basic odd case

Let \( \zeta := \omega^{-1} : Q^*_p \to k^* \). Here we show that the map of sets \( L_\zeta : (V_{T, 0}/W_0)_\zeta(k) \to X_\zeta(k) \) that we have defined in [S3.13] satisfies properties (ii) and (iii) of [S3.12] and we define a morphism of \( k \)-schemes \( L_{\hat{\zeta}} : (V_{T, 0}/W_0)_\zeta \to X_\zeta \) which coincides with the previous map of sets at the level of \( k \)-points. By construction, it will have the properties [S3.10] This will complete the proof of [S3.12] [S3.13] and [S3.9] in the case of an odd character.

#### 8.5.1. We verify properties (ii) and (iii). We work over an irreducible component \( \mathbb{P}^1 \) with label " \( \text{Sym}^r \otimes \det^a \mid \text{Sym}^{r-3} \otimes \det^{r+1+a} \) " where \( 1 \leq r \leq p - 2 \) and \( 0 \leq a \leq p - 2 \), cf. [S2.4] We distinguish two cases.

1. **The generic case** \( r \neq p - 2 \). In this case, the irreducible component of \( X_{\zeta} \) we consider is an 'interior' component and has no exceptional points. On this component, we choose an affine coordinate \( x \) around the double point having \( \text{Sym}^r \otimes \det^a \) as one of its Serre weights. Away from this point, we have \( x \neq 0 \) and the corresponding Galois representation has the form

\[ \rho_{x} = \begin{pmatrix} \text{unr}(x) \omega^{+1} & 0 \\ 0 & \text{unr}(x^{-1}) \end{pmatrix} \otimes \eta \]

with \( \eta = \omega^a \). As before, we have

\[ \pi(\rho_x) = \pi(r, x, \eta)^{ss} \oplus \pi([p - 3 - r], x^{-1}, \omega^{+1 + 1})^{ss}. \]

The length of \( \pi(\rho_x) \) is 2. Indeed, by our assumptions on \( r \), the principal series representations \( \pi(r, x, \eta) \) and \( \pi(p - 3 - r, x^{-1}, \omega^{+1 + 1}) \) are irreducible and the block \( b_x \) contains only these two irreducible representations. We may follow the argument of the generic case of [S4.1] word for word and deduce property [S3.12] (ii).

2. **The two boundary cases** \( r = p - 2 \). In this case, the irreducible component is one of the two 'exterior' components with labels " \( \text{Sym}^{p-2} \mid \text{Sym}^{-1} \) " or " \( \text{Sym}^{-1} \det^{-1} \mid \text{Sym}^{p-2} \det^{-1} \) ".

Points of the open locus \( X_{\zeta}^{\text{red}} \) lying on such a component correspond to twists of unramified Galois representations of the form

\[ \rho_{x + x^{-1}} = \begin{pmatrix} \text{unr}(x) & 0 \\ 0 & \text{unr}(x^{-1}) \end{pmatrix} \otimes \eta \]

with \( \eta = 1 \) or \( \eta = \omega^{\pm 1} \). Let us concentrate on one of the two components, i.e. let us fix \( \eta \).

Mapping an unramified Galois representation \( \rho_{x + x^{-1}} \) to \( t := x + x^{-1} \in k \) identifies this open locus with the \( t \)-line \( \mathbb{A}^1 \subset \mathbb{P}^1 \). We have

\[ \pi(\rho_t) = \pi(p - 2, x, \eta)^{ss} \oplus \pi(p - 2, x^{-1}, \eta)^{ss} =: \pi_1 \oplus \pi_2 \]

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since \([p - 3 - (p - 2)] = p - 2\) (indeed, \(p - 3 - (p - 2) = -1 \equiv p - 2 \mod (p - 1)\)). The length of \(\pi(\rho_t)\) is 2. Indeed, \(\pi_1 = \pi(p - 2, x, \eta)\) and \(\pi_2 = \pi(p - 2, x^{-1}, \eta)\) are two irreducible principal series representations and the block \(B_t\) contains only these two irreducible representations. They are isomorphic if and only if \(x = \pm 1\), i.e. if and only if \(t = \pm 2\) is an exceptional point. In this case, \(B_t\) contains only one irreducible representation and is of type 3, otherwise it is of type 2.

We may write

\[
\pi_1 = \text{Ind}_{B_t}^G(\chi) \otimes \eta
\]

with \(\chi = \text{unr}(x) \otimes \omega^{p-2} \text{unr}(x^{-1})\). Similarly for \(\pi_2\). The character \(\chi|_{\rho(t)} = 1 \otimes \omega^{p-2}\) is regular (i.e. different from its \(s\)-conjugate) and we are in the regular case of 8.3.2. We conclude that \(\pi_1^{(t)} = M(0, x, 1, (1 \otimes \omega^{p-2}), \eta)\) and \(\pi_2^{(t)} = M(0, x^{-1}, 1, (1 \otimes \omega^{p-2}), \eta)\) are both simple 2-dimensional standard modules in the regular component \(\gamma\) represented by the character \((1 \otimes \omega^{p-2}).(\eta|_{\rho(t)} = (\eta|_{\rho(t)} \otimes (\eta|_{\rho(t)}) \omega^{p-2} \in T^\vee\). They are isomorphic if and only if \(t = \pm 2\). We choose an order \(\gamma = ((\eta|_{\rho(t)} \otimes (\eta|_{\rho(t)}) \omega^{p-2} \otimes (\eta|_{\rho(t)})\) on the set \(\gamma\). Then from \(L_\zeta(v) = t\) we get that either \(\text{ASph}^\gamma(v) = \pi_1^{(t)}\) or \(\text{ASph}^\gamma(v) = \pi_2^{(t)}\). Since for regular \(\gamma\), the map \(\text{ASph}^\gamma\) is a bijection onto all simple \(H^\vee\)-modules, cf. [PS 7.4.9], one finds that \(L_\zeta^{-1}(t) = \{v_1, v_2\}\) has cardinality 2 if \(t \neq \pm 2\) and then

\[
\text{ASph}(v_1) \oplus \text{ASph}(v_2) \cong \pi(\rho_t)^{f(t)}
\]

This settles property 8.3.12 (ii). In turn, if \(t = \pm 2\) is an exceptional point, then \(L_\zeta^{-1}(t) = \{v\}\) has cardinality 1 and

\[
\text{ASph}(v) \oplus \text{ASph}(v) \cong \pi(\rho_t)^{f(t)}
\]

This settles property 8.3.12 (iii)ii.

8.5.2. We define a morphism of \(k\)-schemes \(L_\zeta : (V^{(t)}_{T,0}/W_0)_\zeta \to X_\zeta\) which coincides on \(k\)-points with the map of sets \(L_\zeta : (V^{(t)}_{T,0}/W_0)_\zeta(k) \to X_\zeta(k)\). We work over a connected component of \((V^{(t)}_{T,0}/W_0)_\zeta\), indexed by some \(\gamma \in T^\vee/W_0\). Let \(v\) be a \(k\)-point of this component.

Since \(\zeta|_{\rho(t)} = \omega^{-1}\), the connected components of \((V^{(t)}_{T,0}/W_0)_\zeta\) are indexed by the fibre \((\gamma)|_{\rho(t)}^{-1}(\omega^{-1})\). This fibre consists of the \(\frac{p-1}{2}\) regular components, represented by the characters

\[
\chi_k = \omega^{-k} \otimes \omega^{-k}
\]

for \(k = 1, \ldots, \frac{p-1}{2}\), cf. 8.1.2. Recall that \(z_2 = \zeta(p) = 1\).

Fix an order \(\gamma = (\chi_k, \chi_k^*)\) on the set \(\gamma\) and choose standard coordinates \(x, y\). According to [PS 7.4.8], our regular connected component identifies with two affine lines intersecting at the origin:

\[
V_{T,0,1} \simeq \mathbb{A}^1 \cup_0 \mathbb{A}^1.
\]

Suppose that \(v = (0, 0)\) is the origin, so that \(\text{ASph}(v)\) is a supersingular module. Let \(\pi(r, 0, \eta)\) be the corresponding supersingular representation. It corresponds to the irreducible Galois representation \(\rho(r, \eta)\) in the notation of [He 1.1.3], whence \(L_\zeta(v) = [\rho(r, \eta)]\). According to the component of \(\pi(r, 0, \eta)^{f(t)}\) is given by \((\omega^r \otimes 1 \cdot (\eta|_{\rho(t)})\). Setting \(\eta|_{\rho(t)} = \omega^a\), this implies \((\omega^r \otimes 1 \cdot (\eta|_{\rho(t)}) = \omega^{r+a} \otimes \omega^a = \chi_k\) and hence \(a = -k\) and \(r = 2k - 1\). The Serre weights of the irreducible representation \(\rho(r, \eta)\) are therefore \([\text{Sym}^{2k-1} \otimes \det^{-k}, \text{Sym}^{2k-2k} \otimes \det^{k-1}]\), cf. [Br 1.9].

Comparing these pairs of Serre weights with the list [S.24] shows that the \(\frac{p-1}{2}\) points

\[
\{\text{origin} (0, 0) \text{ on the component } (\chi_k, \chi_k^*)\}
\]

for \(k = 1, \ldots, \frac{p-1}{2}\) are mapped successively to the \(\frac{p-1}{2}\) double points of the chain \(X_\zeta\). We distinguish two cases.

1. The generic case \(1 < k < \frac{p-1}{2}\). In this case, the argument proceeds as in the regular case of 8.3.2. Consider the double point

\[
Q = L_\zeta(\text{origin} (0, 0) \text{ on the component } \gamma = (\chi_k, \chi_k^*)).
\]
As we have just seen, \( Q \) lies on an ‘interior’ irreducible component \( \mathbb{P}^1 \) whose label includes the weight \( \text{Sym}^{2k-1} \otimes \det^{-k} \). We fix an affine coordinate on this \( \mathbb{P}^1 \) around \( Q \), which we will also call \( x \). Away from \( Q \), the affine coordinate \( x \neq 0 \) parametrizes Galois representations of the form

\[
\rho_x = \left( \begin{array}{c} \text{unr}(x)\omega^{2k} & 0 \\ 0 & \text{unr}(x^{-1}) \end{array} \right) \otimes \eta
\]

with \( \eta := \omega^{-k} \). As we have seen above, \( \pi(\rho_x) = \pi(2k-1, x, \eta) \otimes \pi(p-3-2k+1, x^{-1}, \omega^{2k}\eta) =: \pi_1 \otimes \pi_2 \). Moreover, \( \pi_1 = \text{Ind}_{B}^{G}(\chi) \otimes \eta \) with \( \chi = \text{unr}(x) \otimes \omega^{2k-1} \text{unr}(x^{-1}) \). Since

\[
(1 \otimes \omega^{2k-1}).(\eta|_{\mathbb{F}_p}) = \omega^{-k} \otimes \omega^{k-1} = \chi^k \in \mathbb{T}^v,
\]

we deduce from the regular case of \([5.3.2]\) that \( \pi_{1}^{((1))} = M(0, x, 1, \chi^k) \) is a simple 2-dimensional standard module. Note that \( M(0, x, 1, \chi_k) \) according to [V04, Prop. 3.2].

Now suppose that \( v = (x, 0), \) \( x \neq 0, \) denotes a nonzero point on the \( x \)-line of \( \mathbb{A}^1 \cup_0 \mathbb{A}^1 \). In particular, \( \text{ASph}^v(v) = M(x, 0, 1, \chi_k) \). Our discussion shows that the point \( L_{\zeta}((x, 0)) \) corresponds to the block which contains \( \pi_1 \). Since \( \pi_1 \) lies in the block parametrized by \( [\rho_x] \), cf. \([5.3.3]\) it follows that

\[
L_{\zeta}((x, 0)) = [\rho_x] = x \in \mathbb{G}_m \subset \mathbb{P}^1.
\]

Since \((0, 0)\) maps to \( Q \), i.e. to the point at \( x = 0 \), the map \( L_{\zeta} \) identifies the whole affine \( x \)-line \( \mathbb{A}^1 = \{(x, 0) : x \in k\} \subset V_{\mathbb{T}, 0, 1} \) with the affine \( x \)-line \( \mathbb{A}^1 \subset \mathbb{P}^1 \subset X_{\zeta} \).

On the other hand, the double point \( Q \) also lies on the irreducible component whose labelling includes the other weight of \( Q \), i.e. the weight \( \text{Sym}^{p-2k} \otimes \det^{-k} \). We fix an affine coordinate \( y \) on this \( \mathbb{P}^1 \) around \( Q \). Away from \( Q \), the coordinate \( y \neq 0 \) parametrizes Galois representations of the form

\[
\rho_y = \left( \begin{array}{c} \text{unr}(y)\omega^{p-2k+1} & 0 \\ 0 & \text{unr}(y^{-1}) \end{array} \right) \otimes \eta
\]

with \( \eta := \omega^{-k} \). As in the first case, \( \pi(\rho_y) \) contains \( \pi_1 := \pi(p - 2k, y, \eta) = \text{Ind}_{B}^{G}(\chi) \otimes \eta \) as a direct summand, where now \( \chi = \text{unr}(y) \otimes \omega^{p-2k} \text{unr}(y^{-1}) \). Since

\[
(1 \otimes \omega^{p-2k}).(\eta|_{\mathbb{F}_p}) = \omega^{k-1} \otimes \omega^{-k} = \chi_k \in \mathbb{T}^v,
\]

we deduce from the regular case of \([5.3.2]\) that \( \pi_{1}^{((1))} = M(0, y, 1, \chi_k) \) is a simple 2-dimensional standard module.

Now suppose that \( v = (0, y), y \neq 0, \) denotes a nonzero point on the \( y \)-line of \( \mathbb{A}^1 \cup_0 \mathbb{A}^1 \). In particular, \( \text{ASph}^v(v) = M(0, y, 1, \chi_k) \). Our discussion shows that the point \( L_{\zeta}((0, y)) \) corresponds to the block which contains \( \pi_1 \), parametrized by \( [\rho_y] \). Hence

\[
L_{\zeta}((0, y)) = [\rho_y] = y \in \mathbb{G}_m \subset \mathbb{P}^1.
\]

Since \((0, 0)\) maps to \( Q \), i.e. to the point at \( y = 0 \), the map \( L_{\zeta} \) identifies the whole \( y \)-line \( \mathbb{A}^1 = \{(0, y) : y \in k\} \subset V_{\mathbb{T}, 0, 1} \) with the affine \( y \)-line \( \mathbb{A}^1 \subset \mathbb{P}^1 \subset X_{\zeta} \).

In this way, we get an open immersion of each connected component \( (V_{\mathbb{T}, 0, 1}^2/W_0)_\zeta \) of \((V_{\mathbb{T}, 0, 1}^1/W_0)_\zeta \) such that \( \gamma = (\chi_k, \chi_k^s) \) with \( 1 < k < \frac{p-1}{2} \), in the scheme \( X_{\zeta} \), which coincides on \( k \)-points with the restriction of the map of sets \( L_{\zeta} \).

2. The two boundary cases \( k \in \{1, \frac{p-1}{2}\} \). Consider the double point

\[
Q = L_{\zeta}(\text{origin} (0, 0)) \text{ on the component } \gamma = (\chi_k, \chi_k^s).
\]

As we have just seen, \( Q \) lies on an ‘interior’ irreducible component \( \mathbb{P}^1 \) whose label includes the weight \( \text{Sym}^1 \otimes \det^{-1} \) (for \( k = 1 \)) or the weight \( \text{Sym}^1 \otimes \det^{-1} \) (for \( k = \frac{p-1}{2} \)). We fix an affine coordinate on this \( \mathbb{P}^1 \) around \( Q \), which we will call \( z \). Away from \( Q \), the coordinate \( z \neq 0 \) parametrizes Galois representations of the form

\[
\rho_z = \left( \begin{array}{c} \text{unr}(z)\omega^2 & 0 \\ 0 & \text{unr}(z^{-1}) \end{array} \right) \otimes \eta
\]
with \( \eta = \omega^{-1} \) or \( \eta = \omega^{\frac{p-1}{2}} \).

Let \( k = 1 \), i.e. \( \eta = \omega^{-1} \). Following the argument in the generic case for word, we may conclude that \( L_\zeta \) identifies the \( x \)-line \( A^1 = \{(x,0) : x \in k\} \subset V_{T,0} \) with the \( z \)-line \( A^1 \subset P^1 \subset X_\zeta \).

Let \( k = \frac{p+1}{2} \), i.e. \( \eta = \omega^{\frac{p+1}{2}} \). As in the generic case, we may conclude that \( L_\zeta \) identifies the \( y \)-line \( A^1 = \{(0,y) : y \in k\} \subset V_{T,0} \) with the \( z \)-line \( A^1 \subset P^1 \subset X_\zeta \).

On the other hand, the double point \( Q \) lies also on the irreducible component \( P^1 \) whose labelling includes the other weight of \( Q \), i.e. the weight \( \text{Sym}^{p-2} \) (for \( k = 1 \)) or the weight \( \text{Sym}^{p-2} \otimes \chi \) (for \( k = \frac{p+1}{2} \)). These are the two ‘exterior’ components. Points of the open locus \( X_\zeta^{\text{red}} \) lying on such a component correspond to unramified (up to twist) Galois representations of the form

\[
\rho_t = \begin{pmatrix} \text{unr}(z) & 0 \\ 0 & \text{unr}(z^{-1}) \end{pmatrix} \otimes \eta
\]

where \( \eta = 1 \) (for \( k = 1 \)) or \( \eta = \omega^{\frac{p-1}{2}} \) (for \( k = \frac{p+1}{2} \)) and with \( t = z + z^{-1} \in A^1 \subset P^1 \). As in the boundary case of [8.5.1], we have \( \pi(\rho_t) = \pi(p - 2, z, \eta) \oplus \pi(p - 2, z^{-1}, \eta) =: \pi_1 \oplus \pi_2 \) and these are irreducible principal series representations. We may write \( \pi_1 = \text{Ind}_D^G(\chi) \otimes \eta \) with \( \chi = \text{unr}(z) \otimes \omega^{p-2} \text{unr}(z^{-1}) \). The character \( \chi_{y,x} = 1 \otimes \omega^{p-2} \) is regular (i.e. different from its s-conjugate) and we are in the regular case of [8.5.2]. We conclude that

\[
\pi_t^{(1)} = M(0, z, 1, (1 \otimes \omega^{p-2}) \eta)
\]
is a simple 2-dimensional standard module in the regular component represented by the character

\[
(1 \otimes \omega^{p-2}) \cdot (\eta|_{T_P}) = (\eta|_{T_P}) \otimes (\eta|_{T_P}) \omega^{p-2} = (\eta|_{T_P}) \otimes (\eta|_{T_P}) \omega^{-1} \in T^\vee.
\]

This latter character equals \( \chi_1 \) for \( \eta = 1 \) and \( (\chi_1)^\ast \) for \( \eta = \omega^{\frac{p-1}{2}} \) (indeed, note that \( \omega^{\frac{p-1}{2}} \equiv -\frac{1}{2} \mod p - 1 \)).

Now suppose that \( k = 1 \), i.e. \( \eta = 1 \). Let \( v = (0, y), y \neq 0 \), be a nonzero point on the \( y \)-line of \( A^1 \cup_0 A^1 \). In particular, \( \text{ASph}^y(v) = M(0, y, 1, \chi_1) \). Our discussion shows that the point \( L_\zeta((0,y)) \) corresponds to the block which contains \( \pi_1 \), i.e. which is parametrized by \( [\rho_t] \). It follows that

\[
L_\zeta((0,y)) = [\rho_t] = t = y + y^{-1} \in A^1 \subset P^1.
\]

Since \( (0,0) \) maps to \( Q \), i.e. to the point at \( t = \infty \), the map of sets \( L_\zeta \) maps the \( k \)-points of the whole affine \( y \)-line \( A^1 = \{(0,y) : y \in k\} \subset V_{T,0} \) to the \( k \)-points of the whole ‘left exterior’ component \( P^1 \subset X_\zeta \) via the formula

\[
A^1 \quad \longrightarrow \quad P^1
\]

\[
y \quad \longmapsto \quad \begin{cases} y + y^{-1} & \text{if } y \neq 0 \\ \infty = Q & \text{if } y = 0. \end{cases}
\]

This formula is algebraic: indeed, for \( y \in A^1 \setminus \{\pm i\} \) (where \( \pm i \) are the roots of the polynomial \( f(y) = y^2 + 1 \)), we have \( y + y^{-1} \neq 0 \) and \( (y + y^{-1})^{-1} = y/(y^2 + 1) \), which is equal to \( 0 \) at \( y = 0 \). Moreover, it glues at the origin \( (0,0) \) with the open immersion of the \( x \)-line of \( V_{T,0} \) onto \( A^1 \cup_0 A^1 \) in \( X_\zeta \) defined above, since both map \( (0,0) \) to \( Q \). We take the resulting morphism of \( k \)-schemes \( A^1 \cup_0 A^1 \rightarrow X_\zeta \) as the definition of \( L_\zeta \) on the connected component \( \left(V_{(1)}^{(1)} / W_0\right)_\zeta \) of \( \left(V_{(1)}^{(1)} / T_0\right)_\zeta \).

Note that its restriction to the open subset \( \{y \neq 0\} \) in the \( y \)-line \( A^1 \) is the morphism \( G_m \rightarrow A^1 \) corresponding to the ring extension

\[
k[t] \rightarrow k[y, y^{-1}] = k[t][y]/(y^2 - ty + 1),
\]

and that the discriminant \( t^2 - 4y^2 - ty + 1 \in k[t][y] \) vanishes precisely at the two exceptional points \( t = \pm 2 \).

Suppose \( k = \frac{p+1}{2} \), i.e. \( \eta = \omega^{\frac{p+1}{2}} \). Let \( v = (x,0), x \neq 0 \), denote a nonzero point on the \( x \)-line of \( A^1 \cup_0 A^1 \). In particular,

\[
\text{ASph}^y(v) = M(0, x, 1, (\chi_1)^\ast) = M(x, 0, 1, \chi_1^\ast).
\]
8.6.2. Definition. Let \[ \text{Morphism of } \gamma \text{ with determinant } \omega \zeta \] be the line of \( V_{T,0} \) to the \( k \)-points of the whole ‘right exterior’ component \( \mathbb{P}^1 \subset X_\zeta \) via the formula

\[
\begin{align*}
\mathbb{A}^1 & \longrightarrow \mathbb{P}^1 \\
x & \longmapsto \begin{cases} 
 x + x^{-1} & \text{if } x \neq 0 \\
 \infty = Q & \text{if } x = 0.
\end{cases}
\end{align*}
\]

This formula is algebraic. Moreover, it glues at the origin \( (0,0) \) with the open immersion of the \( y \)-line of \( V_{T,0} \) in \( X_\zeta \) defined above, since both map \( (0,0) \) to \( Q \). We take the resulting morphism of \( k \)-schemes \( \mathbb{A}^1 \rightarrow X_\zeta \) as the definition of \( L_\zeta \) on the connected component \( \{ (x,0) : y \in k \} \subset V_{T,0} \) of the whole ‘right exterior’ component \( \mathbb{P}^1 \subset X_\zeta \).

8.6 A mod \( p \) Langlands parametrization in families for \( F = \mathbb{Q}_p \)

In this subsection we continue to assume that \( F = \mathbb{Q}_p \) with \( p \geq 5 \).

8.6.1. Recall the mod \( p \) parametrization functor \( P : \text{Mod}(\mathcal{H}_p^{(1)}) \rightarrow \text{SP}_{\mathcal{G},0} \) from [PS 7.3.6]. For \( \zeta \in Z^\vee(k) \), let \( \text{Mod}_\zeta(\mathcal{H}_p^{(1)}) \) be the full subcategory of \( \text{Mod}(\mathcal{H}_p^{(1)}) \) whose objects are the \( \mathcal{H}_p^{(1)} \)-modules whose Satake parameter is supported on the closed subscheme \( (V_{T,0}^{(1)}/W_0)_\zeta \subset V_{T,0}^{(1)}/W_0 \).

A \( \mathcal{H}_p^{(1)} \)-module \( M \) lies in the category \( \text{Mod}_\zeta(\mathcal{H}_p^{(1)}) \) if and only if: \( M \) is only supported in \( \gamma \)-components where \( \gamma|_{\mathbb{F}_p} = \zeta|_{\mathbb{F}_p} \) and the operator \( U^2 \) acts on \( M \) via the \( \mathbb{G}_m \)-part of \( \zeta \). Set \( \text{SP}_{\mathcal{G},0,\zeta} := \text{QCoh}((V_{T,0}^{(1)}/W_0)_\zeta) \), the category of quasi-coherent modules on the \( k \)-scheme \( (V_{T,0}^{(1)}/W_0)_\zeta \). Then \( P \) induces a mod \( p \) \( \zeta \)-parametrization functor

\[
P_\zeta : \text{Mod}_\zeta(\mathcal{H}_p^{(1)}) \longrightarrow \text{SP}_{\mathcal{G},0,\zeta}.
\]

For \( \zeta \in Z^\vee(k) \), also recall the category \( \text{LP}_{\mathcal{G},0,\zeta} := \text{QCoh}(X_\zeta) \) of mod \( p \) Langlands parameters with determinant \( \omega \zeta \) from [8.2.5]; it induces the functor

\[
L_\zeta : \text{SP}_{\mathcal{G},0,\zeta} \longrightarrow \text{LP}_{\mathcal{G},0,\zeta}
\]

push-forward along the \( k \)-morphism \( L_\zeta : (V_{T,0}^{(1)}/W_0)_\zeta \rightarrow X_\zeta \) from [8.3.9].

Finally recall that for \( \zeta \in Z^\vee(k) \), the functor of \( I^{(1)} \)-invariants \( (\cdot)^{I^{(1)}} : \text{Mod}^{\text{sm}}(k[G]) \rightarrow \text{Mod}(\mathcal{H}_p^{(1)}) \) induces a functor

\[
(\cdot)^{I^{(1)}} : \text{Mod}^{\text{sm}}(k[G]) \rightarrow \text{Mod}_\zeta(\mathcal{H}_p^{(1)}),
\]

by [8.3.2].

8.6.2. Definition. Let \( \zeta \in Z^\vee(k) \). The mod \( p \) \( \zeta \)-Langlands parametrization functor is the functor

\[
L_\zeta P_\zeta := L_\zeta \circ P_\zeta : \\
\text{Mod}_\zeta(\mathcal{H}_p^{(1)})
\]

\[
\longrightarrow \\
\text{LP}_{\mathcal{G},0,\zeta}.
\]
Identifying $\zeta$ with a central character of $G$, the functor $\mathrm{L}_G \mathcal{P}_\zeta$ extends to the category $\mathrm{Mod}^\text{em}_G(k[G])$ by precomposing with the functor $(-)^{(1)}_\zeta: \mathrm{Mod}^\text{em}_G(k[G]) \to \mathrm{Mod}_\mathbb{G}_0$:

$$\mathrm{L}_G \mathcal{P}_\zeta \circ (-)^{(1)}_\zeta: \mathrm{Mod}^\text{em}_G(k[G]) \to \mathrm{Lp}_{\mathbb{G}_0, \zeta}.$$ 

8.6.3. Theorem. Suppose $F = \mathbb{Q}_p$ with $p \geq 5$. Fix a character $\zeta: Z(G) = \mathbb{Q}_p^\times \to k^\times$, corresponding to a point $(\zeta|_\mathbb{Z}^\times, (p^{-1})) \in Z^\vee(k)$ under the identification $Z(G)^\vee \cong Z^\vee(k)$ from 8.3.4.

The mod $p \zeta$-Langlands parametrization functor $L_p \rho_{\zeta}$ interpolates the Langlands parametrization of the blocks of the category $\mathrm{Mod}^\text{em}_G(k[G])$, cf. 8.3.7: for all $x \in X_{\zeta}(k)$ and for all $\pi \in b_{\rho_{\zeta}}$,

$$L_p \mathcal{P}_\zeta(\pi^{(1)}_{\zeta}) = \begin{cases} i_x \ast (\pi^{(1)}_{\zeta}) & \text{if } x \text{ is not an exceptional point in the odd case} \\ i_x \ast (\pi^{(1)}_{\zeta}) \oplus 2 & \text{otherwise} \end{cases} \in \mathrm{Lp}_{\mathbb{G}_0, \zeta}$$

where $i_x : \text{Spec}(k) \to X_{\zeta}$ is the $k$-point $x$.

Proof. By definition of a block of a category as a certain equivalence class of simple objects [Pas13], if $\pi \in b_{\rho_{\zeta}}$ then in particular $\pi$ is simple. Then $\pi^{(1)}_{\zeta}$ is simple too, and hence has a central character. Therefore $P_{\zeta}(\pi^{(1)}_{\zeta})$ is the underlying $k$-vector space of $\pi^{(1)}_{\zeta}$ supported at the $k$-point $v \in (V^{(1)}_0/W_0)_\zeta$ corresponding to its central character under the isomorphism $\mathcal{H}^{(1)}_p$, which lies on some connected component $\gamma$. Suppose $\dim_k(\pi^{(1)}_{\zeta}) = 2$. Then $\pi^{(1)}_{\zeta}$ is isomorphic to the simple standard module of $\mathcal{H}^{(1)}_p$ with central character $v$, i.e. to $\text{ASph}^\vee(v)$, and hence $L_{\zeta}(v) = x$ by definition of the map of sets $L_{\zeta}(k)$. Suppose $\dim_k(\pi^{(1)}_{\zeta}) = 1$. Then $\pi^{(1)}_{\zeta}$ is one of the two antipathery characters of $\mathcal{H}^{(1)}_p$, whose restriction to the center $Z(\mathcal{H}^{(1)}_p)$ is equal to $v$, i.e. it is one of the simple constituents of $(\text{ASph}^\vee(v))^\text{em}$, and hence again $L_{\zeta}(v) = x$ by definition of the map of sets $L_{\zeta}(k)$. Now if $x$ is not an exceptional point in an odd case, then $L_{\zeta}$ is an open immersion at $v$, and otherwise it has ramification index 2 at $v$. The theorem follows. \hfill \Box

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