Super-Jordanian Quantum Superalgebra $U_h(osp(2/1))$

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Abstract

A triangular quantum deformation of $osp(2/1)$ from the classical $r$-matrix including an odd generator is presented with its full Hopf algebra structure. The deformation maps, twisting element and tensor operators are considered for the deformed $osp(2/1)$. It is also shown that its subalgebra generated by the Borel subalgebra is self-dual.
1 Introduction

Both Lie and quantum superalgebras play important roles in various contexts of theoretical physics. These objects are also of great interest in modern mathematics. The simplest and the most fundamental Lie superalgebra is $osp(2/1)$, and it is in a similar position to $sl(2)$ in the theory of Lie superalgebra. Namely, the understanding of its structure and representations is the basics to the higher rank superalgebras. This means that the quantization of $osp(2/1)$ is also fundamental to study quantum superalgebras. The recent work shows that there exist three distinct bialgebra structures on $osp(2/1)$ and all of them are coboundary [1]. We therefore have three distinct quantization for $osp(2/1)$. Two of them are wellknown [2, 3] but one is still incomplete. Despite a lot of investigations of quantum superalgebras, it is a bit surprising that we do not have a complete list of quantization of $osp(2/1)$. In this paper, we wish to complete the list by giving a full Hopf superalgebra structure of the not yet completed quantization of $osp(2/1)$. We also discuss some algebraic properties of this quantization of $osp(2/1)$ such as twisting element, tensor operators, self-duality and so on.

The plan of this paper is as follows. The next section is a short review of bialgebra structures on $osp(2/1)$ and its quantization. We shall make clear known and unknown results on the quantization. In §3, a quantization of $osp(2/1)$ is presented with deformed commutation relations and a full Hopf structure. This completes the list of quantization of $osp(2/1)$. The quantized $osp(2/1)$ is triangular so that it has a basis satisfying undeformed commutation relations. The maps connecting the bases of deformed and undeformed commutation relations are given in §4. The twisting element for this quantization is investigated and given as power series in the deformation parameter up to order three in §5. In §6, the tensor operators for adjoint representation of the quantized $osp(2/1)$ are given. In §7, we discuss self-duality of the Borel subalgebra of the quantized $osp(2/1)$.

2 Bialgebras on $osp(2/1)$ and quantization

The Lie superalgebra $osp(2/1)$ has three even and two odd elements. Let $J_0$, $J_±$ be even elements and $v_±$ be odd ones, then $osp(2/1)$ is defined by the relations

\begin{align*}
[J_0, v_±] &= \pm \frac{1}{2}v_±, & \{v_+, v_-\} &= -\frac{1}{2}J_0, \\
[J_0, J_±] &= \pm J_±, & [J_+, J_-] &= 2J_0, \\
J_± &= \pm 4v_±^2, & [J_±, v_±] &= 0, & [J_±, v_∓] &= v_±. \tag{2.1}
\end{align*}

The even elements span the $sl(2)$ subalgebra. Note that $osp(2/1)$ and its universal enveloping algebra are generated by $v_±$ and $J_0$. 
The bialgebra structures on $osp(2/1)$ are classified in [1]. The authors, by using computer, show that all the possible bialgebra structures are coboundary and there are three inequivalent classical $r$-matrices.

\begin{align*}
  r_1 &= J_0 \wedge J_+ ,
  \
  r_2 &= J_0 \wedge J_+ - v_+ \wedge v_+ ,
  \
  r_3 &= t(J_0 \wedge J_+ + J_0 \wedge J_- - v_+ \wedge v_- - v_+ \wedge v_- ).
\end{align*}

The parameter $t$ in $r_3$ becomes irrelevant in quantization, since it can be absorbed into a deformation parameter. However, it cannot be removed by a change of the basis in $osp(2/1)$. The $r$-matrices $r_1, r_2$ satisfy the classical Yang-Baxter equation, while $r_3$ satisfies the modified classical Yang-Baxter equation. The quantization of these bialgebras has been considered. The $q$-deformation of $osp(2/1)$ considered in [2] corresponds to $r_3$ and it is a quasi-triangular Hopf superalgebra. The irreducible representations and some applications of the $q$-deformed $osp(2/1)$ are studied in [4]. The $r$-matrix $r_1$ consists of the elements in $sl(2)$ subalgebra and $r_1$ is identical to the classical $r$-matrix of the Jordanian deformation of $sl(2)$. Thus one can quantize $osp(2/1)$ using the inclusion $sl(2) \subset osp(2/1)$. This is done in [3] where the Drinfel’d twist [5] for $sl(2)$ is applied to $osp(2/1)$ and the obtained Hopf superalgebra is triangular. For these two quantum superalgebras, their full Hopf structures, universal $R$-matrices and dual quantum supergroups have been obtained.

On the other hand, the quantization of $r_2$ remains unfinished, though there are two publications [6, 7]. Kulish tried to find a twisting element that gives rise to the universal $R$-matrix whose classical counterpart is $r_2$ [6]. He observed that the quantum $R$-matrix in fundamental representation corresponding to $r_2$ can be obtained from the one for $q$-deformed $osp(2/1)$ by contraction. This $R$-matrix is triangular so that consistent with that $r_2$ solves the classical Yang-Baxter equation. With this $R$-matrix and the FRT-method [8], the subalgebra generated by $J_0$ and $v_+$ is quantized. It is also shown that this $R$-matrix can be decomposed into Jordanian $sl(2)$ part and additional factor. This suggest that the twisting element would be a product of $sl(2)$ Jordanian twist and a factor containing $v_+$. The conjectured form of the twisting element is given in [6], but it contains unknown factors. Juszczak and Sobczyk take the same $R$-matrix to construct the dual quantum supergroup [7]. By the FRT-method, the quantum supergroup was explicitly constructed. Then from the duality, they obtain the quantization of the subalgebra generated by $J_0, J_+$ and $v_+$. In the next section, we shall give full quantization of $osp(2/1)$. 

2
3 Quantum superalgebra $U_h(osp(2/1))$

We denote the generators of the quantized $osp(2/1)$ by $(H, V_\pm)$. Their classical limits are given by $(J_0, v_\pm)$, respectively. We introduce two additional elements $(X, Y)$ whose classical limit are $(J_+, J_-)$, respectively. With the same notations, the classical bialgebra structure is summarized as follows: The classical $r$-matrix reads

$$r = H \otimes X - X \otimes H - 2V_+ \otimes V_+.$$  \hspace{1cm} (3.1)

The classical co-commutators, defined as

$$\delta(g) = [r, (g \otimes 1 + 1 \otimes g)], \quad g \in osp(2/1),$$  \hspace{1cm} (3.2)

are given by

$$\delta(H) = -r,$$
$$\delta(X) = 0,$$
$$\delta(Y) = X \otimes Y - Y \otimes X + 2V_+ \otimes V_- + 2V_- \otimes V_+,$$
$$\delta(V_+) = \frac{1}{2} X \otimes V_+ - \frac{1}{2} V_+ \otimes X,$$
$$\delta(V_-) = \frac{1}{2} X \otimes V_- - \frac{1}{2} V_- \otimes X.$$  \hspace{1cm} (3.3)

We take a quite naive approach for quantization. We have all classical data of $osp(2/1)$ bialgebra. The quantized algebra must have the following properties: (i) The classical co-commutator must be maintained in the classical limit. (ii) The quantum coproduct is a homomorphism of the algebra, and coassociative. (iii) We know from the previous works [6, 7] that $X$ must be primitive. With these informations, one can write down the algebraic relations and the quantum coproducts. A similar approach for quantization is discussed in [9] where quantum coproducts obtained first then commutation relations are derived. In the present case, the commutation relations relations read

$$[H, V_+] = \frac{1}{2} V_+ \cosh(hX), \quad [H, V_-] = -\frac{1}{4} V_- \cosh(hX) - \frac{1}{4} \cosh(hX)V_-,,$$
$$\{V_+, V_-\} = -\frac{1}{2} H, \quad [H, X] = \frac{1}{h} \sinh(hX),$$
$$[H, Y] = -\frac{1}{2} Y \cosh(hX) - \frac{1}{2} \cosh(hX)Y + hV_- \sinh(hX)V_+ - hV_+ \sinh(hX)V_-,$$
$$[X, Y] = 2H, \quad V_+^2 = \frac{1}{4h} \sinh(hX), \quad V_-^2 = -\frac{1}{4} Y,$$
$$[X, V_+] = [Y, V_-] = 0, \quad [X, V_-] = V_+,$$
$$[Y, V_+] = \frac{1}{2} V_- \cosh(hX) + \frac{1}{2} \cosh(hX)V_-.$$  \hspace{1cm} (3.4)
where \( h \) is the deformation parameter and \( h \to 0 \) gives the classical limit. The quantum coproducts \( \Delta \) read

\[
\begin{align*}
\Delta(H) &= H \otimes T^{-1} + T \otimes H + 2hV_+T^{1/2} \otimes V_-T^{-1/2}, \\
\Delta(X) &= X \otimes 1 + 1 \otimes X, \\
\Delta(Y) &= Y \otimes T^{-1} + T \otimes Y + 2hV_+T^{1/2} \otimes T^{-1/2}V_- + 2hT^{1/2}V_- \otimes V_+T^{-1/2}, \\
\Delta(V_\pm) &= V_\pm \otimes T^{-1/2} + T^{1/2} \otimes V_\pm,
\end{align*}
\]

where \( T = \exp(hX) \). We see that there is only one primitive element. It is straightforward to verify that the commutation relations and the coproduct are consistent with the axioms. The counit \( \epsilon \) and the antipode \( S \) follow from the coproduct and they are given by

\[
\begin{align*}
\epsilon(H) &= \epsilon(V_\pm) = \epsilon(X) = \epsilon(Y) = 0, \\
S(H) &= -H - 2hV^2_+, \\
S(V_+) &= -V_+, \\
S(V_-) &= -V_- + \frac{h}{2}V_+ \\
S(X) &= -X, \\
S(Y) &= -Y + hH + h^2V^2_+.
\end{align*}
\]

The relations (3.5)-(3.7) define a quantum superalgebra. We denote this algebra by \( U_h(osp(2/1)) \). By construction, the algebra \( U_h(osp(2/1)) \) is a triangular Hopf superalgebra and a one parameter deformation of the algebra \( U(ops(2/1)) \) so that the algebra \( U_h(osp(2/1)) \) has three generators, \( H \) and \( V_\pm \).

4 Deformation maps

It is known that a triangular Hopf algebra can be obtained from a Lie algebra by Drinfel’d twist \[.\] The algebra obtained by Drinfel’d twist has deformed coproduct and deformed antipode, while the commutation relations and the counit remain undeformed. This can be extended to superalgebras. The quantum superalgebra \( U_h(osp(2/1)) \) defined above is a triangular Hopf superalgebra and has deformed commutation relations. This means that there exist maps that connect generators satisfying deformed commutation relations and undeformed ones. In this section, we seek such deformation maps. We first give a general class of such maps, then give two explicit examples.

An ansatz for a general class of maps may be assumed as

\[
\begin{align*}
V_+ &= f_1(J_+) v_+ & H &= f_2(J_+) J_0, \\
V_- &= f_3(J_+) v_- + u(J_+) v_+ + w(J_+) v_+ J_0,
\end{align*}
\]

where we introduce \((f_1, f_2, f_3; u, w)\) as functions of \( J_+ \) only. The elements of \( U_h(osp(2/1)) \) with undeformed commutation relations are denoted by the same notations as the classical
\(osp(2/1)\). An additive function \(f(J_+)\) in the expression of \(H\) may be absorbed by a similarity transformation. To ensure correct classical limits, the above introduced functions are required to satisfy the limiting properties:

\[
(f_1, f_2, f_3; u, w) \rightarrow (1, 1, 1; 0, 0)
\]  

as \(h \rightarrow 0\). The operators \(T^\pm \equiv \exp(\pm hX)\) may now be expressed as

\[
T = hJ_+ (f_1(J_+))^2 + \sqrt{1 + h^2 J_+^2 (f_1(J_+))^4},
\]
\[
T^{-1} = -hJ_+ (f_1(J_+))^2 + \sqrt{1 + h^2 J_+^2 (f_1(J_+))^4}.
\]

Substituting the ansatz (4.1) in the defining algebraic relations systematically, we, for a given function \(f_1\), obtain a set of six nonlinear equations for four unknown functions \((f_2, f_3; u, w)\):

\[
f_2(J_+) (2J_+ f'_1(J_+) + f_1(J_+)) - \sqrt{1 + h^2 J_+^2 (f_1(J_+))^4} f_1(J_+) = 0,
\]
\[
2J_+ f_2(J_+) f'_3(J_+) - f_2(J_+) f_3(J_+) + f_1(J_+)^4 f_3(J_+) = 0,
\]
\[
2J_+ f_2(J_+) u'(J_+) + f_2(J_+) u(J_+) + \sqrt{1 + h^2 J_+^2 (f_1(J_+))^4} u(J_+) - \frac{h^2}{2} J_+ (f_1(J_+))^3 = 0,
\]
\[
J_+ f_1(J_+)^3 w(J_+) - f_1(J_+)^4 J_+ f_3(J_+) + f_2(J_+) = 0,
\]
\[
2J_+ f_1(J_+) u(J_+) + J_+^2 f'_1(J_+) w(J_+) - J_+ f_1(J_+) f_3(J_+) + \frac{1}{2} J_+ f_1(J_+) w(J_+) = 0,
\]
\[
f_3(J_+) f'_2(J_+) + J_+ f_2(J_+) w'(J_+) - J_+ w(J_+) f'_3(J_+) + \frac{1}{2} f_2(J_+) w(J_+) + \frac{1}{2} \sqrt{1 + h^2 J_+^2 (f_1(J_+))^4} w(J_+) = 0.
\]

We made an identification \([J_0, f(J_+)] = J_+ \frac{d}{dJ_+} f(J_+)\) on a function \(f(J_+)\). These equations may then be solved consistently and the classical limit (4.2) may be implemented through proper choice of integration constants. This leads to the following solution:

\[
f_2(J_+) = \frac{\sqrt{1 + h^2 J_+^2 (f_1(J_+))^4}}{f_1(J_+) + 2J_+ f'_1(J_+)} f_1(J_+), \quad f_3(J_+) = \frac{1}{f_1(J_+)},
\]
\[
u(J_+) = -\frac{1}{4} w(J_+) + \frac{1}{2} \frac{f'_1(J_+)}{(f_1(J_+))^2} f_2(J_+), \quad w(J_+) = \frac{1}{J_+} - \frac{f_2(J_+)}{f_1(J_+)} f_1(J_+).
\]

We see that for a given function \(f_1\) other functions are uniquely determined. This shows that deformation maps are not unique.
Even for the invertible maps it is useful to consider the general solution starting from the other end. So we consider the following class of inverse maps given by

\[
v_+ = g_1(T) V_+, \quad J_0 = g_2(T) H, \\
v_- = g_3(T) V_+ + a(T) V_+ + b(T) V_+ H,
\]

where \((g_1, g_2, g_3; a, b)\) are functions of \(T\) only. In the classical limit \(h \to 0\), these functions obey the following limiting properties

\[
(g_1, g_2, g_3; a, b) \to (1, 1, 1; 0, 0).
\]

Substituting the above ansatz \(1.6\) in the defining algebra, we, as before, obtain a set of six coupled nonlinear equations:

\[
(T^2 - 1) g_1'(T) g_2(T) + \frac{1}{2} (T + T^{-1}) g_1(T) g_2(T) - g_1(T) = 0,
\]

\[
(T^2 - 1) g_2(T) g_3'(T) - \frac{1}{2} (T + T^{-1}) g_2(T) g_3(T) + g_3(T) = 0,
\]

\[
(T^2 - 1) g_2(T) a'(T) + \left(1 + \frac{1}{2} (T + T^{-1}) g_2(T)\right) a(T) + \frac{h}{4} (T - T^{-1}) g_2(T) g_3(T) = 0,
\]

\[
(T^2 - 1) g_2(T) b'(T) + \left(1 + \frac{1}{2} (T + T^{-1}) g_2(T) - (T^2 - 1) g_2'(T)\right) b(T) + 2 h T g_2'(T) g_3(T) = 0,
\]

\[
(T - T^{-1}) g_1(T) b(T) + 2 h (g_2(T) - 1) = 0,
\]

\[
((T + T^{-1}) g_1(T) + 2 (T^2 - 1) g_1'(T)) b(T) + 8 g_1(T) a(T) - 4 h T g_2'(T) g_3(T) = 0,
\]

where we have used the identity \([H, g(T)] = \frac{1}{2} (T^2 - 1) \frac{d}{dT} g(T)\). Treating the function \(g_1(T)\) as known and taking the boundary conditions \(1.7\) into account the remaining functions may be solved unambiguously as follows:

\[
g_2(T) = 2 \frac{g_1(T)}{(T + T^{-1}) g_1(T) + 2 (T^2 - 1) g_1'(T)}, \quad g_3(T) = \frac{1}{g_1(T)},
\]

\[
a(T) = -\frac{h}{4} \tanh \left(\frac{h X}{2}\right) (g_1(T))^{-1}, \quad b(T) = \frac{2 h}{T - T^{-1}} \frac{1 - g_2(T)}{g_1(T)}.
\]

We now give two examples of the maps obtained above. The first choice of \(f_1(J_+)\) and \(g_1(T)\) is

\[
f_1(J_+) = \frac{1}{(1 - 2 h J_+)^{1/4}}, \quad g_1(T) = T^{-1/2},
\]

and the second choice is

\[
f_1(J_+) = \frac{1}{\sqrt{1 - \frac{h^2 J_+^2}{4}}}, \quad g_1(T) = \text{sech} \left(\frac{h X}{2}\right).
\]
The first choice gives the map

\[ V_+ = e^{\sigma/4} v_+, \quad H = e^{-\sigma/2} J_0, \]
\[ V_- = e^{-\sigma/4} v_- + \frac{\hbar}{4} e^{\sigma/4} \tanh \left( \frac{\sigma}{4} \right) v_+ + \frac{\hbar}{\cosh(\sigma/4)} v_+ J_0, \]
\[ X = \frac{1}{2\hbar} \sigma, \quad Y = -4V_-^2, \quad \sigma = -\ln(1 - 2\hbar J_+), \quad (4.12) \]

and its inverse

\[ v_+ = e^{-hX/2} V_+, \quad J_0 = e^{hX} H, \]
\[ v_- = e^{hX/2} V_- - \frac{h}{4} e^{hX/2} \tanh \left( \frac{hX}{2} \right) V_+ - \frac{2he^{hX/2}}{1 + e^{-hX}} V_+ H, \]
\[ J_\pm = \pm 4v_\pm^2. \]

We call the map (4.12) first deformation map. The induced quantum coproducts for \( J_0, J_\pm \) and \( v_\pm \) are obtained from (4.13) and (3.5).

\[ \Delta(J_0) = J_0 \otimes 1 + e^\sigma \otimes J_0 + 2\hbar v_+ e^\sigma \otimes v_+ e^{\sigma/2}, \]
\[ \Delta(J_+) = J_+ \otimes e^{-\sigma} + 1 \otimes J_+, \]
\[ \Delta(v_+) = v_+ \otimes e^{-\sigma/2} + 1 \otimes v_+, \]
\[ \Delta(\sigma) = \sigma \otimes 1 + 1 \otimes \sigma. \quad (4.14) \]

\( \Delta(v_-), \Delta(J_-) \) are quite messy, so we do not give them here. Now \( \sigma \) is the only primitive element. One can see that, with a slight change of conventions, the coproducts (4.14) are identical to the ones in [6] and [7]. Thus the first deformation map connects the basis of \( U_h(osp(2/1)) \) in [6] [7] to ours. The map is also a natural extension of the similar map [10] for Jordanian \( sl(2) \) to \( U_h(osp(2/1)) \).

The second choice (4.11) gives the map

\[ V_+ = \frac{1}{\sqrt{1 - \frac{\hbar^2}{4} J_+^2}} v_+, \quad H = J_0, \quad V_- = \sqrt{1 - \left( \frac{\hbar}{4} J_+ \right)^2} v_- + \frac{\hbar}{4} \frac{J_+}{\sqrt{1 - \left( \frac{\hbar}{4} J_+ \right)^2}} v_+, \]
\[ X = \frac{2}{\hbar} \arctanh \left( \frac{\hbar}{2} J_+ \right) = \frac{1}{\hbar} \ln \left( \frac{1 + \frac{\hbar}{2} J_+}{1 - \frac{\hbar}{2} J_+} \right) \quad Y = -4V_-^2, \quad (4.15) \]

and its inverse

\[ v_+ = \text{sech} \left( \frac{hX}{2} \right) V_+, \quad J_0 = H, \quad v_- = \left( \cosh \frac{hX}{2} \right) V_- - \frac{h}{4} \left( \sinh \frac{hX}{2} \right) V_+, \]
\[ J_+ = \frac{2}{\hbar} \tanh \frac{hX}{2}, \quad J_- = -4v_-^2. \quad (4.16) \]
We call the map (4.15) second deformation map. The map is a natural extension of the similar map [11] for Jordanian $sl(2)$ to $U_h(osp(2/1))$.

The irreducible representations of $U_h(osp(2/1))$ are identical to the ones of the classical $osp(2/1)$, since $U_h(osp(2/1))$ has undeformed commutation relations. The representation matrices for $H, X, Y$ and $V_\pm$ are obtained by the deformation maps. Because deformation maps are not unique, we have many different matrices for a element of $U_h(osp(2/1))$. As examples, the fundamental and the adjoint representation matrices by the first and the second deformation maps are given in Appendix.

5 Twisting elements for the maps

Twisting elements are the most fundamental objects for triangular quantum (super) algebras, since all such deformations are caused by appropriate twists. In other words, the twisting element contains all information about the deformation. The universal $R$-matrix is immediately constructed from the twisting element. In this section, the twisting element for $U_h(osp(2/1))$ is investigated and it is given as a power series in the deformation parameter $h$. The series up to $O(h^3)$ is explicitly given. The higher order terms can, in principle, be obtained in a similar way. In many literatures, twisting elements are given in terms of the classical generators. Here the twisting element for $U_h(osp(2/1))$ is presented in terms of deformed generators $(H, X, Y, V_\pm)$, since, as seen in the previous section, the coproducts for deformed generators are far simpler in form than the undeformed ones. Thus the twisting element has deformation map dependence in its expression. We shall give the twisting element for the first and second deformation maps, respectively. The relation of deformation maps to twisting elements is studied in [12, 13] for Jordanian deformation of $sl(2)$ and $gl(2)$.

The general structure of the twisting element corresponding to the given maps may be described as follows. Let $m$ be a deformation map and $m^{-1}$ be its inverse

$$m : (V_\pm, H) \rightarrow (v_\pm, J_0), \quad m^{-1} : (v_\pm, J_0) \rightarrow (V_\pm, H)$$

(5.1)

the classical ($\Delta_0$) cocommutative and the quantum ($\Delta$) non-cocommutative coproducts may be related via the twisting element $G$ as

$$G \Delta \circ m^{-1}(\phi) G^{-1} = (m^{-1} \otimes m^{-1}) \circ \Delta_0(\phi) \quad \forall \phi \in U(osp(2/1)), \quad (5.2)$$

where the twisting element $G$ satisfies the cocycle condition

$$(G \otimes 1)((\Delta \otimes \text{id})G) = (1 \otimes G)((\text{id} \otimes \Delta)G). \quad (5.3)$$
In the present case we, corresponding to the maps discussed in the previous section, obtain a series expansion of the twisting element in powers of the deformation parameter $h$:

$$G = 1 \otimes 1 + hG_1 + h^2G_2 + h^3G_3 + \cdots .$$  \hfill (5.4)

For the first map these expansion coefficients of the twisting element reads

$$G_1 = 2X \otimes H + 2V_+ \otimes V_+,$$
$$G_2 = \frac{G^2_1}{2!} + 2X \otimes XH + XV_+ \otimes V_+,$$
$$G_3 = \frac{G^3_1}{3!} + \frac{1}{2}G_1(2X \otimes XH + XV_+ \otimes V_+) + \frac{1}{2}(2X \otimes XH + XV_+ \otimes V_+)G_1$$
$$+ X \otimes X^2H - \frac{1}{4}V_+ \otimes X^2V_+ - \frac{1}{12}X^2V_+ \otimes V_+ + \frac{5}{6}XV_+ \otimes XV_+$$  \hfill (5.5)

and for the second map they are given by

$$G_1 = X \otimes H - H \otimes X + 2V_+ \otimes V_+,$$
$$G_2 = \frac{G^2_1}{2!} + \frac{1}{4}(H \otimes X^2 + X^2 \otimes H + 2XV_+ \otimes V_+ - 2V_+ \otimes XV_+),$$
$$G_3 = \frac{G^3_1}{3!} + \frac{1}{8}G_1(H \otimes X^2 + X^2 \otimes H + 2XV_+ \otimes V_+ - 2V_+ \otimes XV_+)$$
$$+ \frac{1}{8}(H \otimes X^2 + X^2 \otimes H + 2XV_+ \otimes V_+ - 2V_+ \otimes XV_+)G_1$$
$$- \frac{1}{24}(2XH \otimes X^2 - 2X^2 \otimes XH + X^2H \otimes X - X \otimes X^2H - 6XV_+ \otimes XV_+)$$
$$- \frac{1}{12}(V_+ \otimes X^2V_+ + X^2V_+ \otimes V_+ + 2XV_+ \otimes XV_+).$$  \hfill (5.6)

It may be checked that the twist operators corresponding to the two maps described above satisfy the cocycle condition (5.3) upto the desired order in $h$. Moreover, following [12], we may interrelate the above two twist operators pertaining to two different maps.

Let us, for the purpose of avoiding confusion, denote two mapping functions, given in (4.10) and (4.11), by $g_1$ and $\hat{g}_1$; and the corresponding twist operators, given in (5.5) and (5.6), by $G$ and $\hat{G}$, respectively. Following [12] these mapping functions may be related by a similarity relation:

$$U^{-1}g_1(T)U = \hat{g}_1(T) \Rightarrow U^{-1}TU = f_1(\hat{g}_1(T)).$$  \hfill (5.7)

As discussed in [12], for the purpose of demonstrating the equivalence of two maps, it is sufficient to ensure the relation (5.7). The transforming operator $U$ may be expressed in a series by direct computation:

$$U = \exp \left(-hX - \frac{1}{4}(hX)^2 + \frac{5}{24}(hX)^3 \right)H .$$  \hfill (5.8)
Now the two relevant twist operators, described above, may be related à la [12] as
\[ \hat{G} = (U^{-1} \otimes U^{-1}) G (\Delta(U)). \] (5.9)

6 Tensor operators

We obtained the twisting element \( G \) up to \( O(h^3) \) in the previous section. The higher order computation seems to be difficult. As mentioned in the previous section, it is important to obtain a closed form of the twisting element. Let us now recall that tensor operators for twisted algebras are obtained from the undeformed ones by using the twisting element [14]. Let \( t, t_0 \) be tensor operators for deformed and undeformed algebras, respectively. Then
\[ t = \mu (id \otimes S)(F(t_0 \otimes 1)F^{-1}), \] (6.1)
where \( F = G^{-1} \) is the twisting element and \( \mu \) is the product of algebra, \( \mu(a \otimes b) = ab \). Thus obtaining the explicit form of tensor operators could be the first step to obtain a closed form of twisting elements. Note also that tensor operators for various Lie algebras appear in many context of physics. The consideration of tensor operators has physical importance.

We start with the definition of tensor operators. Let \( a, b \) be elements of \( U_h(osp(2/1)) \) and \( t, s \) be a operator acting on a given space. We assume that \( (a \otimes b)(t \otimes s) = (-)^{p(b)p(t)} at \otimes bs \), where \( p(b) \) and \( p(t) \) take \( \pm 1 \) depending on even or odd. The adjoint action of \( a \) on \( t \) is defined by
\[ \text{ad } a(t) = \mu(id \otimes S)\Delta(a)(t \otimes 1). \] (6.2)

It follows from this definition that
\[ \text{ad } ab(t) = \text{ad } a \circ \text{ad } b(t). \] (6.3)

Thus ad gives a representation of the algebra
\[ \text{ad } [a, b](t) = [\text{ad } a, \text{ad } b](t), \quad \text{ad } \{a, b\}(t) = \{\text{ad } a, \text{ad } b\}(t). \] (6.4)

Let \( \mathcal{I} \) be a set of indices, \( \{ t_i, \ i \in \mathcal{I} \} \) be a set of operators and \( D(a) \) be a representation matrix of \( a \). If the operators \( t_i \) form a representation basis of the algebra, namely
\[ \text{ad } a(t_i) = \sum_{j \in \mathcal{I}} D(a)_{ji} t_j, \] (6.5)
we call $t_i$ tensor operators for the representation $D$.

The explicit expressions of $U_h(osp(2/1))$ adjoint action for an even operator $t$ are given by

$$
\text{ad}X(t) = [X, t],
\text{ad}V_+(t) = [V_+T^{-1/2}, T^{1/2}t]T^{1/2},
\text{ad}H(t) = [HT^{-1}, Tt]T - 2h(TtV_+^2 + V_+T^{1/2}tV_+T^{1/2}),
\text{ad}V_-(t) = [V_-T^{-1/2}, T^{1/2}t]T^{1/2} + \frac{h}{2}T^{1/2}tV_+,
\text{ad}Y(t) = [YT^{-1}, Tt]T + \frac{h}{2}\{HT^{-1}, Tt\}T - 2hT^{1/2}(V_+tV_+ + V_-tV_+)T^{1/2}
$$

$$
-\frac{h}{2}\text{ad}H(t),
$$

and for an odd operator $t$

$$
\text{ad}X(t) = [X, t],
\text{ad}V_+(t) = \{V_+T^{-1/2}, T^{1/2}t\}T^{1/2},
\text{ad}H(t) = [HT^{-1}, Tt]T - 2h(TtV_+^2 - V_+T^{1/2}tV_+T^{1/2}),
\text{ad}V_-(t) = \{V_-T^{-1/2}, T^{1/2}t\}T^{1/2} - \frac{h}{2}T^{1/2}tV_+,
\text{ad}Y(t) = [YT^{-1}, Tt]T + \frac{h}{2}\{HT^{-1}, Tt\}T + 2hT^{1/2}(V_+tV_+ + V_-tV_+)T^{1/2}
$$

$$
-\frac{h}{2}\text{ad}H(t).
$$

Let us consider the tensor operators for the adjoint representation, since for the representation tensor operators are given in terms of the elements of algebra itself. For comparison, we start with tensor operators for the classical $osp(2/1)$. With the index set $I = \{1, 1/2, 0, -1/2, -1\}$, it is obvious that the tensor operators for the adjoint representation (A.3) of $osp(2/1)$ are given by

$$
t_1 = J_+, \quad t_{1/2} = v_+, \quad t_0 = J_0, \quad t_{-1/2} = v_-, \quad t_{-1} = J_-.
$$

Since the first and second deformation maps give the different form of matrices for $H, X, Y$ and $V_\pm$, we have two different expressions of tensor operators for $U_h(osp(2/1))$. Of course, these two expressions are related each other. The adjoint representation matrices by the first deformation map are found in (A.4). The explicit forms of the tensor operators become

$$
t_1 = \frac{1}{\hbar}T^{-1}\sinh(\hbar X),
$$

11
\[ t_{1/2} = V_+ T^{-3/2}, \]
\[ t_0 = HT^{-1} + \frac{3}{4} T^{-1} \sinh(hX), \]
\[ t_{-1/2} = V_- T^{1/2} + 2hV_+ T^{-1/2} + \frac{h}{2} V_+ T^{-1/2} \sinh(hX), \]
\[ t_{-1} = YT + 2hH^2 - 2hV_- V_+ T + hH (\cosh(hX) + T^{-1}) \]
\[ + \frac{h}{8} (5T^{-1} - 3 \sinh(hX)) \sinh(hX). \quad (6.9) \]

The tensor operators for the adjoint representation (A.5) by the second deformation map are given by

\[ \tau_1 = t_1 = \frac{1}{h} T^{-1} \sinh(hX), \]
\[ \tau_{1/2} = t_{1/2} = V_+ T^{-3/2}, \]
\[ \tau_0 = t_0 = HT^{-1} + \frac{3}{4} T^{-1} \sinh(hX), \]
\[ \tau_{-1/2} = t_{-1/2} - \frac{h}{2} t_{1/2} = V_- T^{1/2} + 2hV_+ T^{-1/2} + \frac{h}{2} V_+ T^{-1/2} (\sinh(hX) - T^{-1}), \]
\[ \tau_{-1} = t_{-1} - 2ht_0 + \frac{h^2}{2} t_1 \]
\[ = YT + 2hH^2 - 2hV_+ V_+ T + hH \sinh(hX) - \frac{3h}{8} \sinh(hX) \cosh(hX). \quad (6.10) \]

The relation to (6.9) is also given in (6.10). It can be seen that the first three tensor operators, that reduced to the Borel subalgebra of \( osp(2/1) \) in the classical limit, are identical in both expressions.

## 7 Self-duality of Borel subalgebra

It is known that the Jordanian quantization of \( sl(2) \) Borel subalgebra generated by \( J_0, J_+ \) is self-dual [1]. We show the similar is true for \( U_h(osp(2/1)) \). Let \( \mathcal{B} \) be a quantization of the Borel subalgebra of \( osp(2/1) \) generated by \( H, X \) and \( V_+ \) and \{ \( E_{k\ell m} = H^k X^\ell V_+^m, \ k, \ell \in \mathbb{Z}_{\geq 0}, \ m = 0, 1 \} \) be its basis. Let \{ \( e^{k\ell m} = H^k X^\ell V_+^m, \ k, \ell \in \mathbb{Z}_{\geq 0}, \ m = 0, 1 \} \) be the basis of the algebra \( \mathcal{B}^* \) dual to \( \mathcal{B} \) such that \( < E_{k\ell m}, e^{pq r} > = \delta_{k,p} \delta_{\ell,q} \delta_{m,r} \). Then by definition of the duality

\[ E_{k\ell m} e^{k'\ell' m'} = f_{k\ell m, k'\ell' m'}^{pq r} e^{pq r}, \quad \Delta(E_{k\ell m}) = g^{pq r, k'\ell' m'}_{k\ell m} E^{pq r} \otimes E^{k'\ell' m'}, \]
\[ e^{pq r} e^{p'q' r'} = g^{pq r, p'q' r'}_{k\ell m} e^{k\ell m}, \quad \Delta(e^{pq r}) = f^{pq r, k'\ell' m'}_{k\ell m} e^{k\ell m} \otimes e^{k'\ell' m'}, \quad (7.1) \]

where the sum over the repeated indices is understood. We follow [16] in order to determine the commutation relations and the coproducts for \( x = e^{100}, y = e^{010} \) and \( z = e^{001} \).
It is not difficult to see that
\[
\begin{align*}
f_{k-100,100} &= k\delta_{\ell,0}\delta_{m,0}, \quad k = 1, 2, \cdots \\
f_{k00,001} &= \delta_{\ell,0}\delta_{m,1}, \quad k = 0, 1, 2, \cdots \\
f_{k00,001} &= (-1)^{k+1}2h\delta_{\ell,0}\delta_{m,0}, \quad k = 1, 2, \cdots 
\end{align*}
\]
It follows that
\[
\begin{align*}
e^{k00} &= \frac{z^k}{k!}, \quad e^{k01} = \frac{x^kz}{k!}, \quad z^2 = 2h(1 - e^{-x}). \quad (7.2)
\end{align*}
\]
We also see that
\[
\begin{align*}
f_{010,010} &= f_{010,010} = 1, \quad f_{100,010} = -f_{010,100} = (-1)^k h, \quad k = 1, 2, \cdots (-1)^k, \\
f_{k00,100} &= f_{k00,100} = 0, \quad \text{otherwise} \\
f_{100,001} &= f_{001,100} = 1, \quad f_{100,001} = f_{100,001} = 0, \quad \text{otherwise} 
\end{align*}
\]
and
\[
\begin{align*}
f_{001,010} &= f_{001,010} = 1, \quad f_{010,010} = -f_{001,010} = (-1)^k \frac{h}{2}, \quad k = 0, 1, 2, \cdots \\
f_{010,001} &= f_{010,001} = 0, \quad \text{otherwise} 
\end{align*}
\]
It follows the commutation relations
\[
[x, y] = 2h(e^{-x} - 1), \quad [x, z] = 0, \quad [y, z] = hxe^{-x}. \quad (7.3)
\]
The coproducts for \(x, y\) and \(z\) are obtained from the observation
\[
\begin{align*}
g_{100,000} &= g_{000,100} = 1, \quad g_{100}^{pqr} = 0, \quad \text{otherwise} \\
g_{000,010} &= 1, \quad g_{010}^{k00} = (-1)^k, \quad k = 0, 1, 2, \cdots \\
g_{001,010} &= \frac{1}{4} \left(-\frac{1}{2}\right)^k, \quad k = 0, 1, 2, \cdots \\
g_{010}^{pqr} &= 0, \quad \text{otherwise} \\
g_{001}^{001} &= 1, \quad g_{001}^{k00} = \left(-\frac{1}{2}\right)^k, \quad k = 0, 1, 2, \cdots \\
g_{001}^{pqr} &= 0, \quad \text{otherwise} 
\end{align*}
\]
The coproducts read
\[
\begin{align*}
\Delta(x) &= x \otimes 1 + 1 \otimes x, \quad \Delta(y) = y \otimes e^{-x} + 1 \otimes y + \frac{1}{4} z \otimes z e^{-x/2}, \quad \Delta(z) = z \otimes e^{-x/2} + 1 \otimes z. \quad (7.4)
\end{align*}
\]
We have a unique primitive element. Note that \( x, y \) are even and \( z \) are odd. The counit and antipode are calculated from the coproducts

\[
\epsilon(x) = \epsilon(y) = \epsilon(z) = 0, \\
S(x) = -x, \quad S(y) = -ye^x + \frac{h}{2} (e^x - 1), \quad S(z) = -ze^{x/2}.
\]

Thus \( B^* \) is a Hopf algebra defined by the relations (7.2), (7.3), (7.4) and (7.5). It is easy to see that the algebra \( B \) is isomorphic to \( B^* \). The isomorphism \( \rho \) is given by

\[
\rho(H) = \frac{1}{2h} ye^{x/2}, \quad \rho(X) = \frac{1}{2h} x, \quad \rho(V_\pm) = \frac{1}{4h} ze^{x/4}.
\]

8 Concluding remarks

We gave an explicit Hopf algebra structure of quantization of \( osp(2/1) \) with the classical \( r \)-matrix \( r_2 \). It is a distinct triangular deformation of \( osp(2/1) \) from the one by embedding of \( sl(2) \) into \( osp(2/1) \). We now have all Hopf algebra structures obtained from the bialgebras on \( osp(2/1) \). However, one should not say that quantization of \( osp(2/1) \) is completed, since the universal \( R \)-matrix of our \( U_h(osp(2/1)) \) and the twisting element have not been evaluated as closed form expressions. There may be some approaches to find the objects, for instance, power series expansion in \( \S 5 \) or tensor operators as mentioned in \( \S 6 \). Recall also that because of the self-duality of the quantized Borel subalgebra of \( sl(2) \) the quantum double construction from the Borel subalgebra gives rise to the universal \( R \)-matrix of whole Jordanian \( sl(2) \). It is observed that the quantized Borel subalgebra \( B \) of \( osp(2/1) \) is self-dual in \( \S 7 \). We expect that the quantum double construction gives the universal \( R \)-matrix of the whole \( U_h(osp(2/1)) \). We hope that one can write down a closed form of the twisting element and the universal \( R \)-matrix by an approach mentioned above. This will be a future work.

Appendix

In this appendix, the fundamental and the adjoint representation matrices for \( osp(2/1) \) and \( U_h(osp(2/1)) \) are summarized.

(1) fundamental representation of \( osp(2/1) \)

\[
J_0 = \frac{1}{2} \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1
\end{pmatrix}, \quad v_+ = \frac{1}{2} \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}, \quad v_- = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 \\
-1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}, \quad J_+ = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad J_- = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}.
\]

(A.1)
(2) fundamental representation of $U_h(osp(2/1))$ by first deformation map

\[
H = \frac{1}{2} \begin{pmatrix} 1 & 0 & h \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad V_+ = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad V_- = \frac{1}{2} \begin{pmatrix} 0 & -\frac{h}{2} & 0 \\ 0 & 0 & -\frac{h}{2} \\ -1 & 0 & -\frac{h}{2} \end{pmatrix}, \quad (A.2)
\]

\[
J_+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad J_- = \begin{pmatrix} -\frac{h}{2} & 0 & -\frac{h^2}{4} \\ 0 & 0 & 0 \\ 1 & 0 & \frac{h}{2} \end{pmatrix}.
\]

(3) fundamental representation of $U_h(osp(2/1))$ by second deformation map

The second deformation map does not deform the fundamental representation. The matrices for $H, V_\pm$ and $J_\pm$ are the same as (A.1).

(4) adjoint representation of $ops(2/1)$

The representation basis is the algebra itself. The basis is taken as $(J_+, v_+, J_0, v_-, J_-)$.

\[
J_+ = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad v_+ = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (A.3)
\]

\[
J_0 = \text{diag} (1, \frac{1}{2}, 0, -\frac{1}{2}, -1),
\]

\[
v_- = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 \end{pmatrix}, \quad J_- = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.
\]

(5) adjoint representation of $U_h(osp(2/1))$ by first deformation map

\[
X = \begin{pmatrix} 0 & 0 & -1 & 0 & -2h \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad V_+ = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{h}{4} & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & -\frac{h}{2} \\ 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (A.4)
\]

\[
H = \begin{pmatrix} 1 & 0 & 0 & 0 & -h^2 \\ 0 & 0 & 0 & \frac{h}{2} & 0 \\ 0 & 0 & 0 & 0 & 2h \\ 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad V_- = \begin{pmatrix} 0 & 0 & 0 & \frac{5h^2}{16} & 0 \\ 0 & 0 & 0 & \frac{3h^4}{4} & 0 \\ -1 & 0 & -\frac{h}{4} & 0 & -\frac{h^2}{8} \\ -\frac{1}{2} & 0 & \frac{3h^4}{4} & 0 & h \\ 0 & 0 & 0 & -\frac{1}{2} & 0 \end{pmatrix}, \quad (A.4)
\]

\[
Y = \begin{pmatrix} 0 & 0 & \frac{5h^2}{8} & 0 & \frac{5h^3}{4} \\ 0 & -\frac{h}{2} & 0 & -\frac{3h^2}{4} & 0 \\ 0 & 0 & 0 & \frac{5h^3}{4} & 0 \\ 0 & 0 & 1 & \frac{b}{2} & 0 \\ -2 & 0 & -2h & 0 & \frac{13h^2}{4} \end{pmatrix}.
\]
adjoint representation of $U_h(osp(2/1))$ by second deformation map

\[ X = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad V_+ = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \]

\[ H = \text{diag}(1, \frac{1}{2}, 0, -\frac{1}{2}, -1), \]

\[ V_- = \begin{pmatrix} 0 & 0 & 0 & -\frac{h^2}{16} & 0 \\ -1 & 0 & 0 & 0 & -\frac{h^2}{8} \\ 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{2} & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & \frac{h^2}{8} & 0 & 0 \\ 0 & 0 & 0 & \frac{-h^2}{2} & 0 \\ -2 & 0 & 0 & 0 & -\frac{h^2}{4} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \]

$X, V_+$ and $H$ remain undeformed by this map.

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