Electromagnetic production of pions and quark dynamical mass in Minkowski space

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Mechanism for generation of vector meson resonances is studied in the framework of QCD Dyson-Schwinger equations, which are defined and actually solved in Minkowski space. It is suggested that the timelike pion form factor is generated by the interference of the background quark loops together with the resonant structure predominantly created in the photon-quark-antiquark vertex. It is suggested that QCD Green’s functions involving quark fields are oscillating for timelike arguments, which makes the interference effect among various QCD Green’s functions quite strong and important for the correct description of production processes. A further peculiarities of dynamical chiral symmetry breaking and confinement phenomena as viewed in Minkowski space are discussed.

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I. INTRODUCTION

Probing the charged quarks inside of the hadrons by photons can provide nontrivial information about quark four momentum distribution in hadrons. Resulting charged pion form factor $F_\pi$, as well as of the transition pion form factor represent rather precise data on the electromagnetic structure of the light meson. In the spacelike region the available data [1–5] are represented by a smooth decreasing line, however a certain discrepancy with asymptotic predictions [6] based on the perturbative QCD is exhibited. These day available experimental range $Q^2 \sim 100 GeV^2$ moves the perturbative QCD towards deep spacelike scale.

Contrary to the spacelike behavior, the pion form factor $F_\pi$ in the timelike region represents the amplitudes for the production of $\pi^+\pi^-$ pair with appearance of the striking resonant structure, it has a peak at 770 MeV - the rho meson with a small ($\sim 5\%$) admixture of isoscalar $\omega$ resonance. Other resonant structure, e.g. the deep dip at 1.5 GeV has been found by using Initial Photon State method at BABAR 2012 experiment [5], where also higher energy peaks are observed. Recall here, that the pion form factor can be considered as a boundary value of analytical function, which has a cut on the timelike axis of square of momenta $Q^2$ starting in the branch point $s_{th} = Q^2_{th} = 4m_{\pi}^2$ - the production threshold. On the other side, it is well known that the marriage of PT QCD and dispersion relation method is not possible without a conflict with requirement of the reflection positivity. In other words $\Im F_\pi$ cannot be positive function as sometimes assumed and the known serious attempts to evaluate dispersion relation should explicitly exhibit aforementioned violation of reflection positivity [7]. It is notable, that also precise form factor fit made in [5] does not respect a naive perturbative unitarity. Actually, the complex couplings are used for excited radial states of pion, which spoils the well known relation between the real and the imaginary parts of $F_\pi$.

A reliable low energy description of QCD production processes, which is based on the QCD degrees of freedom was basically lacking and unknown. In Spectral Quark Model (SQM) [8–10] a clever mechanism employed, if correct, is just the opposite: the cuts in the quark propagators conspire to build a pole and threshold in the form factor. Matching together hadronic and quark degrees of freedom is a typical feature of the models [11], which also try to explain behavior of $F_\pi$ at large Minkowski region of $Q^2$. However as in the previous SQM, the dynamics is not obviously governed by QCD gap equations and supporting solution is waiting to be explored.

The Dyson-Schwinger and Bethe-Salpeter equations (DSBSEs) are quantum equations of motions at the level of Green’s functions. This prominent framework of DSBSEs, which provides a traditional tool in the study hadronic physics, is up to a few exceptions implemented with the Euclidean metric due to the known numerical reasons. In QCD, using an Euclidean metric as a definite one, a various approximation of DSBSEs system provided a limited ground for hadronic physics, especially for determination of spectra of ground state mesons, their strong and electroweak decays and the spacelike (Euclidean) form factors [12–14]. In certain belief, similar has been attempted to extract information about excited mesons, however little is known about the implementation of the BSE for production process, where excited states are identified with broad resonances seen experimentally. One is obviously faced with obstacles of technical character: e.g. how to perform the analytical continuation of QCD Green’s function to the large timelike arguments, however also with more principal ambiguities: how to identify excited states with a snapshot of

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resonances, especially when they are tricky hidden in the physical form factors. Roughly say, the pion electromagnetic form factor and associated PDG list of possible $\rho$ and $\omega$ vector resonance participants is the matter. For a practical deal with analytical continuation problem see [17, 20], for recent results for radial excitations based on solution of homogeneous BSE see for instance [21, 23].

In presented paper we continue the study [24] of QCD DSBSEs in Minkowski space, i.e. we are going to ignore the conventional wisdom and does not solve the DSBSEs in the Euclidean space, but instead of, the calculation is performed directly in Minkowski. To show that such, at least one, solution exists, is the main motivations of the author. For a good QCD DSBSEs model one should get light pseudogoldstone boson of broken chiral symmetry-140 MeV heavy pion. Furthermore, as very related consequence, one also should not get stable states with masses exceeding of the mass of several pions. In words, there should not be solution of homogeneous BSE for a heavy real mass for all $\rho$’s, scalars, radially excited pseudoscalars etc.. Instead of, all these broad resonances could be visible in a full color neutral vectors, scalar, pseudoscalar, axialvector, tensor... inhomogeneous BSE. Incorporating the "meson widths" in QCD DSBSE formalism means to consider nonperturbative solution of inhomogeneous BSE. We consider simplified solution of inhomogeneous vector BSE and use it for the calculation of pion form factor in Minkowski space.

To achieve the convergence of the DSBSEs system, one has not only to ensure that the system is without uncontrolled singularities in all Minkowski space but also one needs to use a right choice of variables. Recall, a naive discretization of Minkowski space momentum integrals leads notoriously to not well defined numerics. Hence to solve the DSBSEs system, for which the method of the iterations is chosen, we take a maximum number of suited Lorentz invariant variable, leaving thus no space for badly convergent numerics. Similar method has been already successful in search of excited states in toy DSBSE with Minkowski metric [24], as well as in the Euclidean theory [25].

II. ELECTROMAGNETIC PION FORM FACTOR FOR TIMELIKE ARGUMENT

The calculations of exclusive hadronic form factors, which are based on the quark and gluonic degrees of freedom were not available due to the complicated nonperturbative nature of low energy QCD. Let us mention the continuation of the perturbative QCD asymptotic behavior [26, 28]

$$F_{\pi}(t) \to \frac{64\pi^2f_{\pi}^2}{(11 - 2/3n_f)t \ln t/A_{QCD}^2}$$

derived for $t = -Q^2 \to \infty$ to the experimentally available region does not reproduce the data for any $Q^2$.

Obviously, one of the main difficulties is the inclusion of resonances at the level of QCD GF’s. Observing the BABAR data [5], resonances are spread over the all experimentally known range of $s$ ($s = Q^2 > 0$). Another, quite important source of aforementioned fail has mathematical origin. In most popular nonperturbative approaches like lattice theory or DSBSEs here, the Euclidean metric is employed from very beginning, however the analytical continuation known from perturbation theory does not apply here.

Actually all sophisticated non-perturbative methods in QCD are applied only in the Euclidean space. Thus for instance DSBSEs provides reasonable meson masses, meson form factors [29, 30] as well as baryon electromagnetic form factor has been studied in spacelike domain [31–33]. Easy and simple models suffice to describe smooth form factor for $Q^2 < 0$ since there are little traces of complicated resonant structure known from the behavior at $Q^2 > 0$.

The shape line of the $\rho$ meson peak- is the most striking property of the pion form factor around $s = (0.8GeV)^2$. Hence a various effective Vector Meson Dominance (VMD) models takes the mass $m_\rho = 775MeV$ as a granted input parameter of a phenomenological effective Lagrangean. Further inclusion of heavier resonances is less certain game. Let us recall here the PDG 2015 list for $\rho(1450)$ where the Breit-Wigner (BW) masses are spread from the value $1265 \pm 75 MeV$ for $e^+e^- \to \pi^+\pi^-$ [34], passing one of the BABAR 2012 data fit, where this first excited $\rho$ has a BW mass $1493 \pm 15 MeV$ for $e^+e^- \to \pi^+\pi^-\gamma$ in [31] and finishes with the highest PDG accepted value $1582 \pm 17 \pm 25 MeV$ for $e^+e^- \to \pi_0\pi_0\gamma$ reaction. Very similar, process dependent list, can be found in PDG 2015 for $\rho(1700)$ (the second $\rho$ excitation this year), recalling highest BABAR fit, which is $1861 \pm 17$ in [5], being fairly apart from $\tau$ lepton decay $\rho$ excitation $1728 \pm 17 \pm 89$ [35] and being thus $100MeV$ higher then the older fits based on $e^+e^- \to \pi^+\pi^-\gamma$ data [34, 36, 37]. In fact, apart of the ground state, the BW mass values for $1^{--}$, isospin 1, light flavoured states listed in PDG are almost continuously spread over the all experimentally available energy region, some of them belong to the one state, some of them to another one.

QCD dual VMD models are effective quantum field theory which deal with a strongly coupled rho meson fields. The classical mass of given vector is identified with some central value of experimentally determined BW parameter. The excitation of spin one fields become a wide resonances due to the quantum effect- in practise, after a single loop is calculated. This large widths of all $\rho$’s makes their labeling by radial quantum number quite puzzling. We will show that this is a large interference effect between various QCD GF’s which makes the situation even more cumbersome.
In the expressions for form factor, the resonances, which originates as a poles in the quark-photon vertex are further weighted by the function which are oscillating in the timelike region. While it does not offer an easier and intuitive picture, the resulting form factor can be calculated in practice. It is a matter of the fact that the first numerical results suggest that there is no one to one correspondence between PDG (VMD modeled) list of resonances and the one we need to use in Green’s functions description in our DSBSEs model. Furthermore, not each DSBSEs resonance need to have its counterpartner in phenomenological VMD (and even in PDG). It will be exemplified explicitly for the case of $F_\pi$.

The evaluation of the pion form factor is a typical quantum field theory problem and how to calculate such transition in BSE approach is generally known \cite{38}. As an additional constraint we should respect (axial) Ward identities- the consequence of (chiral) gauge invariance expressed in the terms of Green’s functions. At least one such an expansion is known \cite{38} and here we will consider only the first term, which defines the so called (dressed) Relativistic Impulse Approximation (RIA)

For the case of electro-production of pions the RIA reads:

$$G^\mu(P_+, P_-) = eF_\pi(q^2)(P_+ - P_-)^\mu$$

$$= \frac{2}{3}ie \int \frac{d^4k}{(2\pi)^4} \text{tr} \left[ \tilde{\Gamma}_{BS}(k_+, P_+)^\mu S(k_1)\Gamma^\nu_{EM}(k_1, k_2)S(k_2)\Gamma_{BS}(k_-, P_-)^\nu S(k_3) \right]$$

$$+ \frac{-1}{3}ie \int \frac{d^4k}{(2\pi)^4} \text{tr} \left[ \Gamma_{BS}(k_-, P_-)^\mu S(k_1)\Gamma^\nu_{EM}(k_1, k_2)S(k_2)\tilde{\Gamma}_{BS}(k_+, P_+)^\nu S(k_3) \right],$$

where in the first line the photon with momentum $Q = P_+ + P_-$ couples to the up-quark with electric charge $2/3e$ while the second line describes triangle diagram where the photon couples to the down-quark. The function $\Gamma_{BS}$ stands for negatively charged outgoing pion, i.e. for BS vertex function with the total momentum $P_- = -P + Q/2$ and relative momentum given by the difference of momenta $k_1$ and $k_3$. The matrix $\Gamma^\mu_{EM}$ is the quark-photon vertex determined by its own inhomogeneous BSE (shown below (2.10)). Analogously we have $P_+ = P + Q/2$, the propagators connecting electromagnetic vertex are $k_{1,2} = k \pm Q/2$. In the isospin limit the propagators of $u$ and $d$ as well as the quark-photon vertices of $u$ and $d$ quarks are identical. By applying charge conjugation one can show that the second line, up to the different prefactor, which turns to be $+1/3e$, is equal to the first one.

All the functions entering expression (2.2) should be self-consistently obtained from the solution of QCD DSBSEs system. Neglecting electroweak interaction, the quark propagator $S$ can be readily written

$$S(p) = [A(p) \not{\!p} - B(p)]^{-1} = \frac{1}{A p^2 - M^2(p)},$$

where the function $M$ is usually called dynamical quark mass and the function $A$ is the inverse of quark renormalization function $Z$. In the next section we present the details of QCD DSBSEs model, which not only provides the correct pion spectrum but which, after substituting the results into Eq. (2.2) provides correct electromagnetic pion form factor.

### A. Quark gap equation and pion BSE

In the following the Minkowski metric is employed. The scalar product of two four-vectors is defined as $p.q = g_{\mu\nu}p^\mu q^\nu$ with metric $g_{\mu\nu} = diag(1, -1, -1, -1)$, thus $p^2 < (>) 0$ for spacelike (timelike) $p$.

The propagator satisfies the DSE (the gap equation), which reads in momentum space

$$S^{-1}(p) = Z_2 \not{\!p} - Z_4 m_4(p) - Z_1 \Sigma(p)$$

$$\Sigma(p) = i \int \frac{d^4k}{(2\pi)^4} \frac{1}{G_{\mu\nu}(p-k)\frac{\lambda^a}{2} \gamma_\mu S(k)\Gamma^a(p, q)},$$

where $m_4(p)$ is the renormalized quark mass at the scale $\mu$, $\Gamma^a(p, q)$ is the quark-gluon vertex satisfying its own SDE and $G_{\mu\nu}$ is gluon propagator.

The BSE vertex function $\Gamma^a_{\pi\alpha}(p, P)$ as well as the BSE wave function $\chi(k, P) = S(k_+)|\Gamma(k, P)\rangle S(k_-)$ is solution of bound state BSE:

$$\Gamma^a_{\pi\alpha}(p, P) = i \int \frac{d^4k}{(2\pi)^4} K(k, p, P) \chi(k, P) \chi(k, P) \Gamma^a_{\pi\alpha},$$
where $K(k, p, P)$ is the renormalized quark-antiquark interaction kernel and where the total momentum satisfies $P^2 = m_{2\pi}^2$. Strictly speaking, the homogeneous BSE (2.5) represents narrow mass approximation at the resonance mass for $P^2 = m_{2\pi}^2$, where $m_{2\pi}$ is the mass of the pion (for $n = 1$) or of the arbitrary excitation $\pi(1300), \pi(1800),...$.

The pion is well established pseudo-Goldstone boson of broken chiral symmetry in QCD. Independently on the metric, in order to get a correct properties of the pion a symmetry preserving truncation of infinite DSEs system is required. The simplest Ward-identity preserving truncation is the so called rainbow-ladder approximation (RLA), which we are going to use in our Minkowski space model as well.

The RLA enables us to write down the effective charge $\alpha$:

$$\frac{g^2}{4\pi} \Gamma^\mu(k, p) G_{\mu\nu}(k - p) \rightarrow \gamma^\mu G^{\text{free}}_{\mu\nu}(k - p) \alpha(k - p)$$

(2.6)
defined as a simplified product of the quark-gluon vertex and the gluon propagator. We use the same method of the solution as in the previous paper [24], however here we have include the renormgroup improved UV part as well. The kernel of BSE and DSE is defined as following

$$K(x) = \frac{\alpha(x)}{\pi x} = C_{if} K_{if}(x) + C_{uv} K_{uv}(x)$$

$$K_{if}(x) = -\Theta(-x) \frac{d}{dx} \left[ \exp\left( \frac{x}{a} \xi(x) \right) \right]$$

$$E(x) = \frac{1}{\ln (e + (x/a)^{1/2})}$$

$$G^{-1}(x) = \left( \frac{x}{a} \right)^2 + \frac{a^4}{2a^4}$$

(2.7)

$$K_{uv}(x) = -\Theta(-x) \frac{d}{dx} \ln \left[ e^2 + \left( \frac{x}{A_{QCD}^2} \right)^2 \right]$$

where $x = (k - p)^2$ and the capital letters C stand for the effective infrared and ultraviolet coupling strengths. The later we fix through one loop perturbation theory from which one gets $C_{uv} = \frac{13}{(33-2N_f)2\pi} = 0.0714$. The scale $a$ of dimension $[\text{mass}]^2$ is a parameter of the model and is set up uniquely after the identification of the pion mass. It defines log modulated energy period $\sqrt{\alpha/(2\pi)}$ of the kernel oscillations at the timelike regime and exponential weakening for a large spacelike momenta.

Well known intrinsic property of the dynamical mass function $M$ in Eq. (2.3) is its fast increase in the infrared, where the it reaches some typical constituent quark mass value $M(0) \approx 2m_{\pi}$. Here however, the mass function $M$ is a complex object which phase appears to be sensitive to the ratio of infrared and ultraviolet coupling strength. Let us note, that the physical pole mass defined as $\sum^{-1}(m) = 0$ arising in non-confining theory like QED, does not exists in QCD due to the confinement. Here it is assured by the solution, $M$ becomes complex and oscillating function with imaginary part never small at vicinity of such p where $p = Re M(p)$. To achieve numerical convergence, the correct value of $\beta_{QCD}$ is reached step by step by replacing

$$C_{UV} \rightarrow \kappa C_{UV},$$

where $\kappa$ is a small auxiliary number introduced at the beginning of the iterations and finishes with the value $\kappa = 1$ without lost of numerical stability during the iteration process.

More interesting is the behavior of the function $M$ itself, for small $\kappa$ it has oscillating behavior due to the oscillating kernel. When $C_{UV}$ strength is pronounced the frequency of oscillations is lowered in a special way. For small $\kappa$ the function $M$ is fairly oscillating at the both - spacelike and timelike domain of the Minkowski momenta (see Fig.1) increasing $\kappa$ the oscillations turns to fluctuations at the spacelike domain of momenta where it finnaly leads to occurrence of several extrema in QCD limit $\kappa \approx 1$. If the ratio $K_{UV}/K_{inf}$ was free the details of such behavior would be dependent as well. However, this ratio is fixed by the property of pion (by $m_{\pi}$ and $f_{\pi}$ values) and it seems to be rigid property of our model, that when we finally add the complete UV part in Eq. (2.7) (i.e. when $\kappa = 1$), the solution remains relatively slowly oscillating at the timelike region, while this oscillation completely disappear in the spacelike region of momenta. Assuming small spread freedom given by the choices of $K_{inf}$ and $a$, the open possibility is the behavior of $\Re(3)M$ with $n = 1 \pm 1$ crosses of zero at negative axis of momenta $p^2$, for actual solution see Fig. 2.

Quenched, RLA offers the solution $Z(p) = 1$ in Landau gauge, while the other solutions [39] in the Euclidean space typically give $Z(0) \approx 1/2$ when $Z(\infty) = 1$ in Landau gauge. We have found the renormalization function $Z(p)$ should
not be neglected in our model, since it significantly controls the infrared value $M(p \simeq 0)$. Its introduction costs the numerical stability at some parameters space, and we make a real valued approximative analytical fit of the following form

$$Z(p^2) = \left[ 1 + C_Z \ln^{1/2} \left( e + |p^2/\Lambda^2| \right) \right]^{-1},$$

which is valid for the complete kernel case ($\kappa = 1$) and where reasonably numerical fit gives the constant $C_Z = 2$.

The gap equation has been solved after the renormalization as well as it has been solved in its un-renormalized form. Using a Pauli-Vilars regulator and large spacelike renormalization scale we got basically the same results as for the un-renormalized case. Hence we freely take for renormalization constants $Z_1 = Z_2 = Z_3 = 1$ in thorough the paper. To complete the model, we take the current quark mass $m_c = m(\infty) = 0.01a$ and $C_{if} = 3/100$ and UV par is fully taken into account. For completeness we should mention here, that there exists more then one solution of the
DSBSEs system and in many cases we found evidence for at least another nontrivial solution. To get rid of unphysical solutions, we have calculated BSE as well as the observable $F_2 F_3$ from both of two numerically identified solutions, skipping here completely the discussion about the one, which does not lead to satisfactory description of the pion form factor.

In order to get the meson spectrum, one first solves the gap equation (2.4), whose solution is required to complete BSE kernel, and then solve the BSE for pion and its radial excitation. Here we consider the dominant component of the BSE vertex function, i.e.

$$\Gamma(k, P)_{\pi_n} = \gamma_5 \Gamma_{E_{\pi_n}}(k, P),$$

(2.9)

which is known as good approximation already from early studies of DSBSEs [40].

The quark propagators need to be evaluated for various arguments $p_{\pm}$ as it appears in the BSE kernel. In order to eliminate one of the sources of potential numerical instability, we avoid the interpolation of numerical data and we make an analytical fit of the DSE solution. The fits, which have been actually used are presented in the Appendix A. A further important facts about Minkowski numerics are discussed in the Section III.

As mentioned the UV part has a large influence on the solution and in fact its presence also change the scale parameter $a$, which appears in $K_{m,n}$. Actually, for the pion mass we get $M_\pi \simeq 0.7a$ for $\kappa = 0$, while we obtain $M_\pi \simeq 0.2a$ for $\kappa = 1$. Contrary to the similar Minkowski model [24] we do not get the solutions for excited states here. By looking the eigenvalue of BSE, we see only the evidence for the first excited state, which is located at $M_\pi \simeq 1300 \pm 100 MeV$. We argue, instead of homogeneous one, the inhomogeneous BSE must be solved to see a broad resonances like $\pi(1300)$ and $\pi(1800)$.

B. Electromagnetic vertex

The information about bound states is encoded in the solution of the following inhomogeneous BSE for the quark-photon vertex:

$$\Gamma_{EM}^\mu(p, P) = \gamma^\mu + i \int \frac{d^4k}{(2\pi)^4} S(k) \Gamma_{EM}^\mu(k, P) S(k) K(k, p, P).$$

(2.10)

The resonances correspond with the complex poles in $\Gamma_{EM}$ in total momenta $P^2$. As there is no stable narrow bound state in this vector channel the resonances do not decouple for a physical, i.e. real valued momenta. The function $\Gamma_{EM}^\mu(p, P)$ appears linearly in (2.22) and thus vectors resonances represent most important contribution to the pion form factor. The spin 1 axial vectors and higher spin tensor resonances contribute to $F_\pi$ as well, however corresponding diagrams are suppressed by $\alpha_{QED}$ since at least two photons should be attached to the loop made out of the quark lines. They do not contribute in RIA and we will neglect their contributions here. We also ignore $\rho\omega$ mixing.

Summation over the color is implicit in Eq. (2.10) and the function $K(k, p, P)$ is the four-quarks scattering kernel, which should be taken in the form defined above (2.6) in order to maintain charge conservation. The solution of (2.10) consists from the twelve tensors accompanied with twelve scalar form factors [41]. The four of them are driven by Ward-Takahashi identity and can be called longitudinal part of the vertex $\Gamma$, the eight remainders are transverse and involves pre-formation a rho meson peak. After the suitable projection, the BSE turns to coupled set of 12 equations (in practice it leaves us with 8 equations as the four form factors are fully determined by QED WT1).

Minkowski space solution of (2.10) remains basically unknown. The author was able to solve the system of equations, where in addition to known longitudinal terms, just the single following transverse component

$$\gamma_T F_3(Q, p) = (\gamma^\mu - \frac{Q Q^\mu}{Q^2}) F_3(Q, p)$$

was considered (we keep notation [41]). It has provided a rather complicated behavior of the function $F_3(Q, p)$ which is spanned above three Lorentz scalars (convenient choice can be associated with fermion legs momenta $q^2_\pm = (q \pm Q/2)^2$ and the photon square of 4-momentum $Q^2$). We found that in the model defined by Eqs. (2.10), (2.4), (2.6), (2.7) the function $F_3$ is oscillating whenever one of the arguments is timelike. Unhappily, the odd attempts to include other Lorentz structure shows the equation system suffer by numerical instabilities and is not yet in the stage which can provide benchmark for precise evaluation of the function $F_3$. Due to this unpleasant fact, we will use a mean value Anstaze for single component $\gamma^\mu$ approximation of Eq. (2.10). For this purpose we take

$$\Gamma_{EM}^\mu(q^2_\pm, Q^2) = \gamma^\mu(1 + \Gamma_T(Q^2))$$

(2.11)
i.e. the averaging is formally taken over the arguments of the quark-antiquark square of momenta $q^2$ and where newly introduced function satisfies $\Gamma_r(0) = 0$ in order to maintain correct pion charge normalization $F_\pi(0) = 1$. Obviously, the electromagnetic vertex does not satisfy WTI, as we have neglected relevant longitudinal terms. While important from theoretical reasoning, these terms gives a second order effect when compared with $\rho$ contribution and we ignore them completely thorough this paper.

Using the tree level diagram for $\Gamma^\mu_{EM, r}$, represents the first simplest step. In this, let us call it -Background Relativistic Impulse Approximation-(BRIA) one ignores the resonances completely by taking $\Gamma_r(Q^2) = 0$, i.e. $\Gamma^\mu_{EM, r} = \gamma^\mu$. Obviously BRIA is not satisfactory and for purpose of the next discussion, we review some details of the $F_{\rho_i}$ fit made by BABAR collaboration itself \[5\]. The fit is based on Gounaris-Sakurai VMD model \[42\], where the pion form factor $F_\pi$ is made out of the four $\rho$ BW resonances with masses $m_{\rho_i}(785, 1493, 1861, 2254)$ where each resonance is characterized by its own complex coupling strength. We refer their absolute values

$$|c_i| = (1, 0.158, 0.068, 0.0051) \quad (2.12)$$

for completeness here.

As we will see the DSBSes model we use, will naturally explain these quite distinct couplings as a consequence of the interference effect among quark propagators, the electromagnetic vertex \[2.10\] and pion vertices as well. The effect of the interference among GF’s is abrupt, and as mentioned, the Ansatz for BSE solution, which comply the data, differs from the BABAR fit. The ”spectrum” for the vertex BSE, which works phenomenologically is quite close to the (relativistic) harmonic oscillator

$$M_i \simeq i$$

instead of string inspired Regge trajectory $M_i^2 \simeq i$. To mimic resonances, we use the following form factor

$$\Gamma_r(Q^2) = e^{i \phi_{EM}(Q^2)} T(Q) \Sigma_{\rho_i} \frac{m_{\rho_i}^2}{(Q^2 - m_{\rho_i}^2)^2 + \gamma_{\rho_i}^2 m_{\rho_i}^2} \quad (2.13)$$

where the functional form of the single phase $\phi_{EM}$ is taken to be similar as in the quark-gluon vertex \[2.7\]. Explicitly written the phase we use reads

$$\phi_{EM}(Q^2) = f_c \sqrt{\frac{q^2/a^2}{\ln (e + q^2/a^2)}} \quad (2.14)$$

where $f_c$ is the constant prefactor. In other words, we assume that both gauge vertices are oscillating in the timelike region however their functional forms slightly differ there. Contrary to pure QCD quark-gluon vertex, the color singlet quark-photon vertex includes sign changing enhancements which turns to form factor resonances at the end. The two real constants $m_{\rho_i}, \gamma_{\rho_i}$ are enough for each individual BSE resonance, no special prefactor (the analogue of \[2.12\]) will be needed to explain decreasing crossection at higher $Q^2$. The list of parameters fit is in Tab. I. It is not completely surprising fact, that the list of resonances based on the vector BSE fit can differ from the one based on VMD (and hence PDG as well). For instance, the fit used here does not prefer the $1.5GeV$ heavy resonance used tho generate the pion form factor dip in VMD and there are two lighter vertex resonances instead. This is a certain and notable phenomenological difference between a various VMD and DSBSes model presented here.

### III. NUMERICAL SOLUTION

It is well known empirical fact that a naive integration over the space-time components of the loop momenta does not lead to a convergent solution. This numerical badness is a property of many tested model and is common to the systems which have regular and ultraviolate finite kernels as well. Published DSBSes studies of the pion form factor \[29,50\] and the spectator approximation \[43\] are challenging, but employed for spacelike (Euclidean) $Q^2$ only. A novelty, presented here, is the integration of DSBSes system, which is performed directly in the Minkowski space.

Remind for clarity the procedure, as a first one needs to solve gap equation, then one have to look what are the outcomes for the pion by solving the pseudoscalar homogeneous BSE, after that, when one knows the ratio $m_{\rho_i}/a$ one can calculate the pion form factor. This procedure is usually repeated several times as one needs to improve the
parameters of phenomenological fit $^{2,13}$, which in our case finishes with the values presented in the table. Here we describe all details, without which the numerical solution would be actually impossible.

We use LRA BSDSEs kernel with known primitive function by our construction, which not only improves accuracy but significantly shrink the time of iteration procedure used in our numerical search $^{24}$. After an easy substitution, the angular integration over 3d subspace was done analytically, which leave us with two numerical integrations in DSE as well as in the BSE case. In this way we have avoided the use of angular expansion, which while working in the smooth numerics of Euclidean space, would turn to inconvenient method in the case of Minkowski space.

Further, in order to integrate vector BSE (not shown here), and especially to integrate pion form factor $^{22}$ we always use Lorentz invariants for the numerical integration. Their concrete choice is quite arbitrary as follows from gap/pion equations. However to find convergent integrator for Eq. (2.2) requires more. In this case we have found advantageous to identify integration variables with the arguments of the quark propagators for this purpose.

First let us choose the c.m.s. frame defined by the photon momentum $Q^\mu = (Q,0,0,0)$, then we can take $x = (Q/2-k)^2$; $y = (Q/2+k)^2$, where the inverse substitution gives $-k = \sqrt{(k_0 - Q/2)^2 - x}$ and $k_0 = (y - x)/2Q - Q/2$. In this way one directly gets for the Lorentz invariant measure the following expression

$$\int d^4k = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \frac{\sqrt{D}}{2Q} \Theta(D) \int d\Omega_3,$$

where $D = \left(\frac{y-x}{2Q} - Q/2\right)^2$. Outgoing pions are physical, i.e. their four momenta satisfy $(p \pm Q/2)^2 = m_\pi^2$, which implies that relative momentum is purely spacelike. In our notation $p^\mu = (0,0,0,p_z)$ and for instance the product $p.k$ is given as the following

$$k.p = -\sqrt{D}\sqrt{Q^2/4 - m_\pi^2}\cos \theta.$$

Actually, the formal mathematical operation expressed by the Eq. (3.1) defines new measure, where only the right hand side works numerically after the integrals are discretized to the sums (for which we use the Simpson method with Gaussian distribution of mesh points). While not needed here, let us note for completeness, that the Eq. (2.10) has to be solved off-shell, thus an analogue pair of $x, y$ internal variable, say: $x' = (p-Q/2)^2$, $y' = (p+Q/2)^2$ external variables are required for the introduction of the time component of external relative four-momenta $p_\sigma = (y' - x')/2Q$. Without aforementioned substitution one does not get convergent integrals in the equation for form factor and numerical noise with unwanted oscillations would prevent the stability of the iteration process employed (the all badness of using l.h.s. (3.1) can be seen by evaluating the form factor (2.2), where obviously no iteration is needed). The reason why a naive integration over the Cartesian coordinates does not work, while the integration with Lorentz invariants works numerically remains unknown mystery, at least to the author. All relevant codes are available at the author’s web page (??).

For the same reasons as in the case of the BSE solution we use the analytical fit for the all functions entering the integrand of $F_\pi$. For BS vertex we use chiral approximation $\Gamma(p, P \to 0) = M(p)/f_\pi$, improved due to the small nonzero current mass in the following way: $\Gamma_\pi(p, P^2 = m_\pi^2) = \frac{M(p)}{f_\pi} \left(\frac{L_\pi}{L_\pi + P^2}\right)^{1/2}$, where we take $L = 1\,\text{GeV}$ for our model.

After evaluating the traces in (2.2), the numerical integration is performed as explained above. Resulting form factor is shown in Fig. 3 for several values of $f_\pi$ and $f_\pi^\text{eff}$ (see Eq. (2.14)). The approximative equality $f_\pi \approx f_\pi^\text{eff}$ is affirming similarity of oscillating behavior in the both GF’s: in the quark-photon and in the quark-gluon vertex.

We have found $F$ does not possesses spectral representation with $4m_\pi^2$ threshold value, instead of, it is complex function everywhere (up to $Q^2 = 0$ where the normalization condition $F(0) = 1$ ensures reality). Actually, after the correct normalization, the imaginary and real parts of $F$ are not related via dispersion relation. Obviously, $F$ is not direct observable in quantum field theory, but the scalar product $FF^*$ should be. At this pragmatic level, ignoring the question of analytical unitarity here, the same physics will be certainly achieved when

$$F_{\text{DSE}/\text{BSE}}(q^2) = F_{an}(q^2) e^{i\delta_c(q^2)},$$

where $\delta_c(q^2)$ is some running phase showing an allowed deviation from an ideal function $F_{an}$, assuming $F_{an}$ is the analytical function with the usual cut, and it exists.

IV. SUMMARY, CONCLUSION AND PROSPECTS

In this paper we have overcome usual numerical obstacles and solve the DSBSEs system directly in Minkowski space. To reach numerical stability, we use maximal number of Lorentz invariants as integration variables which turns
usually unstable Minkowski numerics to the stable one. The numerics is at least enlightening and quite promising for future applications. The pion form factor has been calculated in RIA with phenomenological vector Bethe-Salpeter vertex contribution. It shows the importance of the interference effect between Bethe-Salpeter $\rho$ mesons peaks and the background quark loop. The physics of resonances is quantum field phenomena where the interference effects do not play negligible role and can explains the complex couplings used in VMD models. However, the setup and shapes of resonances in DSBSEs approach is still based on phenomenology and a solution of more complete vector BSE is required in the future study. Presented numerical calculations shows oscillation as well as it suggest the complexity.

Apart the BABAR fit, we did not list here a huge list of references devoted to pion form factors, however for a recent treatment of old-fashioned VMD model within the use the dispersion relation technique see [7, 44, 45], for the use of ChPT see for instance [46, 47] and Regge approach used in the pion form factor calculations can be found in [48–50], recall here also Resonance ChPT [51, 52] used in hadronic $\tau$ decay Monte Carlo simulations [53]. Most of the approaches do not contradict with analyticity, while the disagreement of QCD with spectrality is known for years. However here, if one is willing to accept the scenario proposed, the analytical structure of QCD GF’s requires further attention. As QCD Green’s functions here, are not an analytical continuations of their Euclidean counter-partners it leads us to the result for hadronic form factor, which does not comply with the dispersion relation as well as with Perturbative Theory Integral Representation [54]. As a consequence, the pion form factor does not have spectral representation, and it suffers by the presence of unobservable ”confining” phase.

Very interesting would be a study of heavier mesons. Especially, for bottomonia we expect the aforementioned interference phenomena should be less pronounced due to the more ”particle” nature of heavy $b$ quarks. However even there, the so called hindered transitions [55, 56] are very good experimental candidates, where overlaps of GF’s with a large interference effect can be expected. Various $XYZ$ candidates [58], reminding the famous state $X(3872)$ [59] observed by many experimental collaborations [60], shows up as enhancement appearing in more then one spin channel [61, 62]. Obviously, without necessity of fit to a simple quark-antiquark bound state picture, such enhancements or experimental bumps can simply originate in the interference effects of QCD Green’s functions.

V. APPENDIX A

The following fits of quark DSE solutions have been found useful for evaluation of electromagnetic form factors. They fit the solutions for different $\kappa s$, hence different ratios $m_\pi/a$, where $a$ is the interaction kernel parameter and $m_\pi$ is the pion mass.

For the case with no no mixing in DSBSEs kernels ($(\kappa = 0)$), one gets $m_\pi/a \simeq 0.7$ and the following fit has been found for the mass function:
\[ \text{Re} M(x) = -\frac{S(x)}{0.1 + x} \cos \omega(x) + m_c Z(x) \]
\[ \text{Im} M(x) = -\frac{S(x)}{0.1 + x} \sin \omega(x) \]
\[ L(x) = \ln \left[ e + \sqrt{T/1.8 + x/1.4 + (x/30)^2} \right] \]
\[ S(x) = a L(x) Z(x) \quad \omega(x) = f_q\sqrt{x} \]

(5.1)

for spacelike momenta \( q^2 < 0 \), where we use shorthand notation \( x = -q^2/a^2 \) and the fit:

\[ \text{Re} M(x) = \frac{2T(x)}{1 + x} \cos \omega(x) - T(x) \sin \omega(x) \]
\[ \text{Im} M(x) = -\frac{2T(x)}{1 + x} \sin \omega(x) - T(x) \cos \omega(x) \]
\[ T(x) = 1.7a \ln^{1.3} (e + x^2), \quad (5.2) \]

which is valid for timelike momenta, where for now \( x = q^2/a^2 \). For an achieved accuracy of the fit see Fig. 1. The numerical fit gives \( f_q = 8 \) for the kernel with \( \kappa = 0 \).

For combined kernel with \( \kappa = 1 \) one gets \( m_\pi/a \simeq 0.2 \) and the spacelike part must be reconsidered. The following fit has been used for \( \kappa \simeq 1 \):

\[ \text{Re} M(x) = \frac{c_1 a}{c_2 a + x} \]
\[ \text{Im} M(x) = -\frac{c_1 a}{c_2 a + x} \]

(5.3)

for spacelike momenta \( q^2 < 0 \), where typically we took \( c_2 \simeq 0.01 - 1 \) and \( c_1 \simeq 0.02 - 2 \).

The fit for the timelike \( M \) remains the same as in (5.1) even for combined kernel \( \kappa \simeq 1/2 - 1 \), since we have found that variation of \( f_q \) was sufficient for our purpose. For \( \kappa \approx 1 \) we found the quark mass frequency lies in the interval \( f_q = (2, 3) \) very independently from the value \( C_{\text{inf}} \) used in the kernel. Combinations of these two fits (5.1) and (5.3), assuming simultaneous readjustment of the parameter \( f_q \) provides high quality fit for any value of \( \kappa \), providing thus the solution in the range \( m_\pi/a \approx 0.1 - 0.7 \). Let us stress, the pion form factor has been evaluated for the real QCD strength, i.e. \( \kappa = 1 \).

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TABLE I: First two lines provide two parametric fit characteristic for six resonances of the model. $m_{Bfit}$ are the values of complex BW fit made by BABAR. The last line are masses of $\rho$ resonances presented in PDG 2016, related with $\pi^+\pi^-$ mode in PDG (there are also $\rho(1570)$ and $\rho(1900)$ states in PDG which are not measured in two pions production). Instead of one, our model have two resonances

| i | m  | $\gamma$ | $m_{Bfit}$ | $m_{PDG}$ |
|---|----|---------|-----------|-----------|
| 1 | 775 | 220     | 775       | 775       |
| 2 | 1000 | 250     | 1493      | 1450      |
| 3 | 1240 | 340     | 1861      | 1700      |
| 4 | 1850 | 250     | 2254      | 2150      |
| 5 | 2350 | 250     | -         | -         |
| 6 | 2950 | 350     | -         | -         |