SOME AUTOMATIC CONTINUITY
THEOREMS FOR OPERATOR ALGEBRAS
AND CENTRALIZERS OF PEDERSEN’S IDEAL

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Abstract. We prove automatic continuity theorems for “decomposable” or “local” linear transformations between certain natural subspaces of operator algebras. The transformations involved are not algebra homomorphisms but often are module homomorphisms. We show that all left (respectively quasi-) centralizers of the Pedersen ideal of a C*-algebra \(A\) are locally bounded if and only if \(A\) has no infinite dimensional elementary direct summand. It has previously been shown by Lazar and Taylor and Phillips that double centralizers of Pedersen’s ideal are always locally bounded.

§1. Introduction and preliminaries.

In his doctoral dissertation, [27], N.C. Wong showed that both \(Mp\), where \(M\) is a von Neumann algebra and \(p\) a projection in \(M\), and \(A/L\), where \(A\) is a C*-algebra and \(L\) a closed left ideal, can be represented as classes of “admissible” sections of appropriate fields of Hilbert spaces, \(\{H_\varphi : \varphi \in F\}\). In both cases \(F\) is a set of positive functionals and \(H_\varphi\) arises from the GNS construction. Some of Wong’s results concern decomposable linear transformations on \(Mp, A/L\), or related spaces of sections, where a linear transformation \(T\) is called decomposable if it arises from a collection \(\{T_\varphi : \varphi \in F\}\) of linear transformations on \(H_\varphi\). More precisely, \((Tf)(\varphi) = T_\varphi f(\varphi)\) when \(f\) is an admissible section on \(\{H_\varphi\}\) (in particular \(f(\varphi) \in H_\varphi\)).

Wong’s results require that \(T\) be bounded, and it is natural to ask whether decomposable linear transformations are automatically bounded. This question motivated the present paper, and we show that the answer is sometimes, but not always, “yes”. Since our results are in a more general context than indicated above, we now provide definitions of decomposability that do not use, or require any knowledge of, fields of Hilbert spaces or sections. More explicit descriptions of the results will be given after the definitions.

If \(x\) is an element of a von Neumann algebra \(M\), \(in\ x\) denotes the right support projection of \(x\); i.e., the support projection of \(x^*x\).

Definition. 1.1. If \(X\) is a subspace of \(M\) and \(T\) is a linear transformation from \(X\) to \(M\), then \(T\) is called decomposable if \(in\ Tx \leq in\ x, \forall x \in X\).

The proof of the following trivial result is left to the reader.
Lemma 1.2. $T$ is decomposable if and only if $\varphi(x^*x) = 0$ implies $\varphi[(Tx)^*(Tx)] = 0$ for all normal states $\varphi$ on $M$ and all $x$ in $X$.

Remark 1.3 If $T$ is decomposable and $X \subset Mp$ for some projection $p$, then $T(X) \subset Mp$. Also, if $X \subset Mp$ and $T : X \rightarrow Mp$, it is sufficient to verify the condition of 1.2 for $\varphi$ supported by $p$.

Theorem 2.1 below states that if $p$ and $q$ are projections in $M$, then all decomposable linear transformations from $qMp$ to $Mp$ are bounded if and only if the following is true: If $z$ is any central projection of $M$ such that $zpMp$ is finite dimensional, then $zqMp$ is finite dimensional. Results of Wong imply that any bounded decomposable linear transformation from $qMp$ to $Mp$ is of the form $x \mapsto ax$, for some $a$ in $Mq$.

If $A$ is a $C^*$-algebra, then $A^{**}$, the bidual of $A$, can be regarded as a von Neumann algebra, usually called the enveloping $W^*$-algebra of $A$. If $\pi : A \rightarrow B(H)$ is the universal representation of $A$, then $A^{**}$ can be identified with $\pi(A)^\prime\prime$, the double commutant. Also, the normal state space of $A^{**}$ can be identified with the state space of $A$. The reader is referred to standard references, for example sections 3.7 and 3.8 of [22].

Now the concept of decomposability can be applied with $A^{**}$ playing the role of $M$, but for applications it is better to use a weaker concept. Let $P(A)$ denote the set of pure state of $A$.

Definition 1.4. If $X$ is a subspace of $A^{**}$ and $T$ is a linear transformation from $X$ to $A^{**}$, then $T$ is called purely decomposable if $\varphi(x^*x) = 0$ implies $\varphi[(Tx)^*(Tx)] = 0$ for all $\varphi$ in $P(A)$ and all $x$ in $X$.

Remark 1.5. Again it is true that if $X, T(X) \subset A^{**}p$ for some projection $p$ in $A^{**}$, then it is enough to check the condition for pure states supported by $p$. In the main applications this will be so for a special kind of projection $p$. In order to deduce that $T(X) \subset A^{**}p$ from $X \subset A^{**}p$ and pure decomposability, one needs extra hypotheses, which will be satisfied in the main applications.

Certain projections in $A^{**}$ are designated as closed or open, where $p$ is closed if and only if $1 - p$ is open; and there is a one-to-one correspondence between closed left ideals $L$ of $A$ and closed projections $p$ in $A^{**}$, in which $L = \{x \in A : xp = 0\}$ ([1],[14]). Also, $A/L$ can be isometrically identified with $Ap$, where $x + L$ corresponds to $xp$ (cf. [5, Prop. 4.4], [8, Thm. 3.3]). A special case of Theorem 3.6 below is that if $p$ is a closed projection in $A^{**}$ such that there is no infinite dimensional irreducible representation $\pi$ of $A$ with $\pi(p)$ of non-zero finite rank, and if $T$ is a purely decomposable linear transformation from $Ap$ to itself, then $T$ is bounded. The full theorem is stated in a more abstract way. In the cases of most interest, if $T$ is bounded and purely decomposable, then $T$ is a left multiplication, so that in particular $T$ is decomposable (cf. Thm. 5.11 of [27]).

Every $C^*$-algebra $A$ has a smallest dense (two-sided) ideal called the Pedersen ideal and denoted by $K(A)$ (see [22, §5.6]). If $A$ is the commutative $C^*$-algebra $C_0(X)$ for a locally compact Hausdorff space $X$, then $K(A)$ is the set of continuous functions of compact support. Lazar and Taylor [18] and Phillips [25] showed that any double centralizer $(T_1, T_2)$ of $K(A)$ is locally bounded in the following sense: There is a family $\{I_j\}$ of closed two-sided ideals of $A$ such that $(\Sigma I_j)^- = A$ and $T_1$ and $T_2$ are bounded on $K(A) \cap I_j$ for each $j$. The analogous statements for left or quasi-centralizers are false in general, as was pointed out to us by H. Kim, but in Section 4, Theorem 3.6 is used to prove the following: All left (respectively
quasi-) centralizers of $K(A)$ are locally bounded if and only if $A$ is not isomorphic to $A_0 \oplus A_1$, for any infinite dimensional elementary C*-algebra $A_1$. Also, for any $A$ and any left or quasi-centralizer $T$ of $K(A)$, we have:

\[(1)\] $A = A_0 \oplus A_1$, where $A_1$ is a dual C*-algebra and $T$ is locally bounded on $K(A_0)$.

(The meaning of “dual” is that $A_1$ is a $c_0$ direct sum of elementary C*-algebras.)

**Definition 1.6.** If $X$ is a subspace of $A^{**}p$ for an open projection $p$ and $T$ is a linear transformation from $X$ into $A^{**}p$, then $T$ is called local if $xq = 0$ implies $(Tx)q = 0$ for all open projections $q$ in $A^{**}$ such that $q \leq p$ and all $x$ in $X$.

Section 5 deals with $T : X \to A^{**}p \cap A$, where $p$ and $q$ are open projections and $X = A \cap (qA^{**}p)$; i.e., $X$ is the intersection of a closed left ideal and a closed right ideal of $A$. Theorem 5.2 states that local implies (purely) decomposable in this case. We then then deduce from our earlier results:

\[(2)\] The ideal generated by $X$ can be written as $A_0 \oplus A_1 \oplus \cdots \oplus A_n$

such that $A_1, \ldots, A_n$ are elementary and $T|_{A_0 \cap X}$ is bounded.

These theorems were inspired by Peetre’s theorem [23, 24], a special case of which is:

Any local linear operator on $C^\infty(X)$, where $X$ is a differentiable manifold,

\[(3)\] is locally given by differential operators with smooth coefficients.

Actually our result is more directly analogous to the following somewhat easier analogue of Peetre’s theorem (in which $X$ is a topological space):

\[(4)\] Any local linear operator on $C(X)$ is multiplication by a continuous function.

In (3) and (4) $T$ is local if $Tf|_U = 0$ whenever $U$ is an open subset of $X$ such that $f|_U = 0$, or, equivalently, if $\text{supp } Tf \subset \text{supp } f$, where “$\text{supp}$” denotes closed support. There is also a concept of “decomposable” in the context of Peetre’s theorem: In the case of (4) decomposable means that $f(x) = 0$ implies $(Tf)(x) = 0$ (i.e., delete “closed” from “closed support”), and in the case of (3) it means that $(Tf)(x) = 0$ whenever $f$ and its derivatives of order at most $n_x$ vanish at $x$. Since it is considerably easier to prove the conclusions of (3) or (4) for decomposable operators, these results can arguably be interpreted as statements that local implies decomposable. Since it is easy to prove that any local operator which is continuous in a suitable sense is decomposable, (3) and (4) can also be regarded as automatic continuity theorems.

The special case of (4) where $X = \mathbb{R}$ was a Putnam problem in 1966 [26], and the general case of (4) is Theorem 9.8 of Luxemburg [19]. Theorem 6.3 of Neumann and Ptak [20] generalizes the locally compact case of (4). Our technique for proving Theorem 5.2 below is modelled on a proof of (4) which is different from any of the proofs cited above (and harder than the proofs in [19] and [26]), but is partially similar to the proof in [19] of (3) and to the argument on pages 168-169 of [19] (cf. also Lemma 2 of [23]). Our earlier results also use established techniques but are not consciously modelled on any specific theorem in the automatic continuity literature.
We do not know whether there is a good non-commutative ($C^*$-algebraic) analogue of (3). Note, though, that the full version of Peetre’s theorem, which deals with transformations from $C^\infty$ functions of compact support to distributions and differential operators with distribution coefficients, allows a discrete set of exceptional points. It seems intriguing to view the dual $C^*$-algebras $A_1$ of (1) and $A_1 \oplus \cdots \oplus A_n$ of (2) in the same light as this discrete set, despite the fact that the analogy is not very close. Of course, any non-commutative analogue of (3) would deal with Frechet spaces and algebras rather than Banach spaces and algebras as in the present paper. The author at one time hoped that such a generalization of this paper would establish a concept of non-commutative differential operator directly applicable to unbounded derivations of $C^*$-algebras. Although this hope now seems wrong, it still appears likely that Section 5 will be useful. The reader should compare the concepts of locality and decomposability used in this paper with the concept of locality defined by Bratteli, Elliott, and Evans in [6, p.251].

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2. Decomposable linear transformations on certain subspaces of von Neumann algebras.

Theorem 2.1. Let $M$ be a von Neumann algebra and $p, q$ projections in $M$. Then the following are equivalent:

(i) All decomposable linear transformations from $qMp$ to $M$ are bounded.

(ii) If $z$ is a central projection of $M$ such that $zpMp$ is finite dimensional, then $zqMp$ is finite dimensional.

Proof. First assume (ii) and let $T : qMp \to M$ be decomposable.

1) If $p_1$ is a subprojection of $p$, then $T$ sends $qMp_1$ into $Mp_1$ and $qM(p - p_1)$ into $M(p - p_1)$. Thus $T(xp_1) = (Tx)p_1$ for $x$ in $qMp$; i.e., $T$ commutes with $R_{p_1}$, the right multiplication by $p_1$.

If $T$ were known to be bounded, it would now be easy to prove that $T$ commutes with all right multiplications by elements of $pMp$. We proceed to prove another partial result.

2) If $p_1$ and $p_2$ are equivalent mutually orthogonal subprojections of $p$ and $u$ is a partial isometry such that $u^*u = p_1$ and $uu^* = p_2$, then $T$ commutes with $R_u$.

To see this, note that $p_1, p_2, u$, and $u^*$ span a copy of $M_2$, the algebra of $2 \times 2$ complex matrices, inside $pMp$. Since $M_2$ is the linear span of its projections, it follows from 1) that $T$ commutes with $R_x$ for all $x$ in $M_2$.

3) If $\{p_i\}$ is a set of mutually orthogonal subprojections of $p$, then there is a constant $C$ such that for all but finitely many $i, ||T_{qM_{p_i}}|| \leq C$.

If this were false, we could change notation and assume a sequence $\{p_n\}$ such that for each $n, ||T_{qM_{p_n}}|| > n2^n$ (this allows the possibility that $T_{qM_{p_n}}$ is unbounded). Then choose $x_n$ in $qM_{p_n}$ such that $||x_n|| = 2^{-n}$ and $||Tx_n|| > n$, and let $x = \Sigma_1^\infty x_n$.

Since $Tx_n = T(xp_n) = (Tx)p_n$, $||Tx_n|| \leq ||Tx||$, $\forall n$, a contradiction.

4) We now use 1) and 3) to show that there is a largest central projection $z$ such that $T_{zqMp}$ is bounded. By Zorn’s lemma there is a maximal collection $\{z_i\}$ of non-zero, mutually orthogonal, central projections of $M$ such that $T_{z_iqMp}$ is bounded, $\forall i$.

Let $z = \Sigma z_i$. Then $zM$ can be identified with $\oplus_i z_i M$, the $\ell^\infty$ direct sum, and $zqMp$ can be identified with $\oplus_i z_i qMp$. Since $T$ commutes with $R_{z_i}, \forall i$, $T(\oplus_i x_i) = \oplus_i T_i x_i$, for $x_i$ in $z_i qMp$ and $T_i = T_{z_i qMp}$. Since $||T_i||$ is bounded by 3), it follows that $T_{zqMp}$ is bounded. Clearly $z$ is as desired.

5) We now change notation (replace $M$ by $M(1 - z)$) and assume $z = 0$. Then by 3), $M$ does not possess an infinite set of non-zero, mutually orthogonal, central projections; i.e., $M$ is the direct sum of finitely many factors. Now we easily reduce to the case where $M$ is a factor and $p, q$ are non-zero. In view of (ii), only three cases are possible:

(a) $pMp$ is properly infinite.

(b) $pMp$ is of type II$_1$

(c) $qMp$ is finite dimensional.

Of course, case (c) is trivial.

6) If $pMp$ is a properly infinite factor, there is a sequence $\{p_n\}$ of mutually orthogonal subprojections of $p$, such that each $p_n$ is equivalent to $p$ and $\Sigma_1^\infty p_n = p$.

By (3), $T_{qM_{p_n}}$ is bounded for at least one $n$. Since 2) applies to the pair $p_n, p - p_n$, also $T_{qM(p - p_n)}$ is bounded. Hence $T$ is bounded.
7) Now assume $pMp$ is a factor of type $\Pi_1$ and $T$ is unbounded. Choose a projection $p_1$ in $pMp$ such that $p_1$ is equivalent to $p - p_1$. Applying 2) to the pair $(p_1, p - p_1)$, we see that both $T|_{qMp_1}$ and $T|_{qM(p - p_1)}$ are unbounded. Next choose a projection $p_2$ in $(p - p_1)M(p - p_1)$ such that $p_2$ is equivalent to $p - p_1 - p_2$ and apply the same argument. Continuing in this manner, we obtain a sequence $\{p_n\}$ of mutually orthogonal subprojections of $p$ such that $T|_{qMp_n}$ is unbounded, $\forall n$, contradicting 3).

Now assume (ii) is false. Then there is central projection $z$ such that $zM$ is a type I factor, $zpMp$ is finite dimensional and $zqMp$ is infinite dimensional. Replacing $M$ with $zM$ and changing notation, we assume $M = B(H)$ for a Hilbert space $H$, $p$ is a non-zero finite rank projection on $H$ and $q$ is an infinite rank projection on $H$. Let $t : qH \to H$ be a discontinuous linear transformation. For each $x$ in $qB(H)p$, $tx$ is a linear transformation whose kernel contains $(1 - p)H$. Therefore $tx \in B(H)$. Thus we can define an unbounded linear transformation $T$ by $Tx = tx$ for $x$ in $qMp$, and clearly $T$ is decomposable.

**Corollary 2.2.** If $p = q$, then all decomposable linear transformations from $qMp$ to $M$ are bounded. In particular, all decomposable linear transformations from $M$ to itself are bounded.

**Remark 2.3.** It is possible to describe the most general decomposable linear transformation $T : qMp \to M$. We have already mentioned that it follows from results of Wong [27] that if $T$ is bounded it is given by a left multiplication. Now by the above proof, the general case is reduced to the case considered in the last paragraph of the proof; and using 2), we see that the unbounded decomposable linear transformations constructed there are the only ones possible. The conclusion is that $M, p$, and $q$ can be identified with $M_0 \oplus B(H_1) \oplus \cdots \oplus B(H_n)$, $p_0 \oplus \cdots \oplus p_n$, $q_0 \oplus \cdots \oplus q_n$, so that $T$ is the left multiplication induced by $t_0 \oplus \cdots \oplus t_n$, where $t_0 \in M_0q_0$, $t_i : q_iH_i \to H_i$ for $i > 0$, and $t_1, \ldots, t_n$ may be discontinuous.

The next corollary is due to Wong [27] in the bounded case.

**Corollary 2.4.** If $M, p$, and $q$ are as above, then a linear transformation $T : qMp \to Mq$ is decomposable if and only if $T$ commutes with $R_x$, $\forall x \in pMp$; i.e., $T$ is a homomorphism of right $pMp$-modules.
§3. Purely decomposable linear transformations in the context of $C^*$-algebras.

The decomposable linear transformations which occur in Wong [27] arise in connection with a closed projection $p$, but the analysis of centralizers of Pedersen’s ideal instead uses open projections. In order to cover both cases, our basic theorem is stated in an abstract way. If $A$ is a $C^*$-algebra and $\pi : A \to B(H)$ a ($\ast$-) representation, $\pi$ can be uniquely extended to a normal representation of $A^{**}$, the enveloping von Neumann algebra. This extension will be denoted by $\pi^{**}$.

**Definition 3.1.** If $A$ is a $C^*$-algebra, $p_1$ and $p_2$ are projections in $A^{**}$, and $X$ is a subspace of $p_1 A^{**} p_2$, then $X$ is said to satisfy $K(p_1, p_2)$ if the following is true: If $\pi : A \to B(H)$ is any irreducible representation such that $\pi^{**}(p_1), \pi^{**}(p_2) \neq 0$, and if $V$ is any non-trivial finite dimensional subspace of $\pi^{**}(p_2) H$, then there is a positive number $k(\pi, V)$ such that for any $t$ in $\pi^{**}(p_1) B(H)$, there is an $x$ in $X$ with $||x|| \leq k(\pi, V) ||t||$ and $\pi^{**}(x)V = tV$. If $X$ satisfies $K(p_1, p_2)$ and $\pi$ is as above, then $k(\pi)$ denotes $\inf \{k(\pi, V) : V \text{ is a 1-dimensional subspace of } \pi^{**}(p_2)H \}$.

**Definition 3.2.** If $p_1$ and $p_2$ are each open or closed, a subspace $X(p_1, p_2)$ is defined as follows:

1. If $p_1$ and $p_2$ are closed, $X(p_1, p_2) = p_1 A p_2$.
2. If $p_1$ and $p_2$ are open, $X(p_1, p_2) = A \cap (p_1 A^{**} p_2)$.
3. If $p_1$ is open and $p_2$ is closed, $X(p_1, p_2) = A p_2 \cap (p_1 A^{**})$.
4. If $p_1$ is closed and $p_2$ is open, $X(p_1, p_2) = p_1 A \cap (A^{**} p_2)$.

Then $X(p_1, p_2)$ is always norm closed. See 4.4 of [5] (cf. also pages 916-918 of [8]) for how to deal with closed projections. In case (a), $p_1 A p_2$ is isometric to $A/L + R$, where $L$ is the left ideal corresponding to $p_2$, $R$ the right ideal corresponding to $p_1$, and $L + R$ is closed by a result of Combes [11]. If $p_1$ and/or $p_2$ is both open and closed, it is not hard to show that both definitions of $X(p_1, p_2)$ agree. (A projection is both open and closed if and only if it is a multiplier of $A$.)

**Proposition 3.3.** If $X(p_1, p_2)$ is as defined above, then $X(p_1, p_2)$ satisfies $K(p_1, p_2)$ with $k(\pi, V) = 1$, $\forall \pi, V$.

**Proof.** Let $\pi, V$, and $t$ be as in 3.1. By the Kadison transitivity theorem [16], there is $a$ in $A$ such that $||a|| \leq ||t||$ and $\pi(a)V = tV$. Let $q_2$ be the projection with range $V$ and $q_1$ the projection with range $t(V)$. If $p_i$, for $i = 1$ or 2, is open then Akemann’s Urysohn lemma [3, Lemma III.1] yields $b_i$ in $A$ such that $b_i = p_i b_i p_i, q_i = \pi(b_i)q_i$, and $||b_i|| = 1$. If $p_i$ is closed, let $b_i = p_i$. Then in all cases let $x = b_1 ab_2$, and it is easy to see that $x \in X(p_1, p_2), ||x|| \leq ||a|| \leq ||t||$, and $\pi^{**}(x)V = \pi(a)V = tV$.

**Remarks 3.4.**

(a) In case (b) of 3.2, $X(p_1, p_2)$ is a Hilbert $B_1 - B_2$ bimodule, where $B_i$ is the hereditary $C^*$-subalgebra of $A$ supported by $p_i$ (notation: $B_i = her(p_i)$). Also $X(p_1, p_2) = B_1 A B_2 = (B_1 A B_2)^\ast$, and $X(p_1, p_2) = R \cap L$, where $R = A \cap (p_1 A^{**})$, the right ideal corresponding to the open projection $p_1$ (which is the same as the right ideal corresponding to the closed projection $1- p_1$), and $L$ is the left ideal corresponding to the open projection $p_2$. Also $her(p_i) = X(p_i, p_i), L = X(1, p_2)$, and $R = X(p_1, 1)$.

(b) In case (c) of 3.2 it can be shown that $X(p_1, p_2) = Rp_2$ where $R$ is as above; and a similar alternate description can be given in case (d) of 3.2. We have no application in mind for these mixed cases but have included them for completeness, with the feeling that they could prove useful.
(c) The main hypothesis on the domain of the decomposable transformation in the next theorem is \(K(p_1, p_2)\), and the exact form of Definition 3.1 represents a compromise on the part of the author. On the one hand, \(K(p_1, p_2)\) is not a minimal hypothesis, but it is more simply stated than a minimal hypothesis and the additional generality gained from using a weaker hypothesis is probably not important. Also, if \(X\) is norm closed, the existence of \(k(\pi, V)\) could be deduced from the open mapping theorem. On the other hand, Definition 3.1 could be simplified by requiring \(k(\pi, V) = 1\), and the generality lost by thus strengthening the hypothesis is probably not important.

(d) In contrast to the above, the focus on irreducible representations in this section is not an arbitrary decision of the author. It seems necessary to use the Kadison transitivity theorem to obtain an applicable automatic continuity theorem in this context, and this forces us to use irreducible representations.

There is one more technical point needed. Let \(\pi_a\), the reduced atomic representation of \(A\), be the direct sum of one irreducible representation from each unitary equivalence class. Thus for \(x\) in \(A^{**}\), \(||\pi_a^*(x)|| = \sup\{||\pi^*(x)|| : \pi \text{ irreducible}\}.

Pedersen [21] defined a large, norm-closed space \(U\), consisting of self-adjoint elements of \(A^{**}\) and called the set of universally measurable operators, and showed that \(||\pi_a^*(x)|| = ||x||, \forall x \in U\) (cf. [22, §4.3]). Because variants of this concept were needed in Wong [27], it seems desirable to abstract this property also.

Definition 3.5. If \(Y\) is a subspace of \(A^{**}\), \(Y\) is of type \(U\) if \(\pi^*_a(y) = 0\) implies \(y = 0\) for all \(y\) in \(Y\), the norm closure of \(Y\). If there is a constant \(c\) such that \(||y|| \leq c||\pi^*_a(y)||\), for all \(y\) in \(Y\) (or, equivalently, in \(Y^{**}\)), \(Y\) is of strong type \(U\).

Thus any subspace of \(U_C\), the complexification of \(U\), is of strong type \(U\). Unfortunately it is not known whether \(U_C\) is an algebra. A future paper of the author [9] will show that \(U\) is the \(C^*\)-algebra generated by \(A\) and all open or closed projections. Wong [27] uses some variants of the concept of universal measurability and shows the appropriate spaces are of type \(U\). There are a number of results in the literature to the effect that \(U\) or \(U_C\) is “sufficiently large” (these include [12, 2.2.15, 2.4.3], [21, Prop. 3.6], [7, Thm. 4.15], [8, Thm. 4.10], and [4, Lemma 2.1]). Also, if \(A\) is separable, \(\{x \in A^{**} : x\) satisfies the barycenter formula\} is a weakly sequentially closed \(C^*\)-algebra of strong type \(U\) which contains \(U_C\). This algebra is large enough for all imaginable applications. The upshot is that the “type \(U\)” hypothesis in the next theorem does not impede applicability.

Theorem 3.6. Assume \(A\) is a \(C^*\)-algebra, \(X\) is a subspace of \(A^{**}\) which satisfies \(K(p_1, p_2)\), where \(p_1\) and \(p_2\) are projections in \(A^{**}\), and \(T : X \to A^{**}\) is a purely decomposable linear transformation.

(a) For each irreducible representation \(\pi : A \to B(H_{\pi})\) such that \(\pi^*(p_1), \pi^*(p_2) \neq 0\), there is a linear transformation \(t_\pi : \pi^*(p_1)H_{\pi} \to H_{\pi}\) such that \(\pi^*(Tx) = t_\pi \pi^*(x), \forall x \in X\). Also if \(\pi^*(p_1)\) or \(\pi^*(p_2)\) is 0, then \(\pi^*[T(X)] = \{0\}\).

(b) If \(T\) is bounded, then \(t_\pi\) is bounded and \(||t_\pi|| \leq k(\pi)||T||\).

(c) If \(X\) is norm closed and either \(\pi^*(p_2)\) has infinite rank or \(\pi^*(p_1)\) has finite rank, then \(t_\pi\) is bounded.

(d) Assume that \(X\) is norm closed, \(T(X)\) is of type \(U\), and for each irreducible representation \(\pi\) of \(A\) at least one of the following holds:

(i) \(\pi^*(p_2) = 0\)

(ii) \(\pi^*(p_2)\) has infinite rank

(iii) \(\pi^*(p_1)\) has finite rank.
Then $T$ is bounded.

**Remark.** The main part of the sufficient condition for automatic continuity given by part (d) is analogous to that in Theorem 2.1, but it is not necessary. In the following two sections, in the context of open projections $p_1, p_2$, we will obtain necessary and sufficient conditions for automatic continuity, but we have not attempted this in the context of closed projections.

**Proof.** (a) If $v \in H_\pi$, $x \in X$, and $\pi^*(x)v = 0$, then $\pi^*(Tx)v = 0$, since $T$ is purely decomposable. ($|\pi^*(a)v|^2 = \varphi_v(a^*a)$, $\forall a \in A^*$, where $\varphi_v$ is a multiple of a pure state, $\varphi_v = (\pi(\cdot)v, v)$). Thus we can define a linear transformation $t_v$ on $\{\pi^*(x)v : x \in X\}$ by $t_v\pi^*(x)v = \pi^*(Tx)v$. If $v \notin \pi^*((1 - p_2)H_\pi$, it follows from $K(p_1, p_2)$ that the domain of $t_v$ is $\pi^*(p_1)H_\pi$ and clearly $t_vv = t_v$ for non-zero $c$ in $\mathbb{C}$ and $w$ in $\pi^*(1 - p_2)H_\pi$. If $v_1$ and $v_2$ are linearly independent modulo $\pi^*((1 - p_2)H_\pi$ and $w$ is in $\pi^*(p_1)H_\pi$, then by $K(p_1, p_2)$ there is $x$ in $X$ such that $\pi^*(x)v_1 = \pi^*(x)v_2 = w$. Thus $\pi^*(x)(v_1 - v_2) = 0$ and since $T$ is purely decomposable, $\pi^*(Tx)(v_1 - v_2) = 0$. Since $t_vw = \pi^*(Tx)v_1$, this implies $t_vw = t_vv_2$. Thus we can take $t_\pi$ to be the common value of $t_v$. The last sentence is clear.

(b) Let $v$ be a unit vector in $\pi^*(p_2)H_\pi$ and $V = \mathbb{C}v$. For any $w$ in $\pi^*(p_1)H_\pi$, there is $x$ in $X$ such that $\pi^*(x)v = w$ and $|w| \leq k(\pi, V)||w||$. Then $||t_\pi w|| = ||\pi^*(Tx)v|| \leq ||T||||v|| \leq k(\pi, V)||T||||w||$. Thus $||t_\pi v|| \leq k(\pi, V)||T||$ for all such $V$, and hence $||t_\pi v|| \leq k(\pi)||T||$.

(c) It is trivial that $t_\pi$ is bounded if $\pi^*(p_1)$ has finite rank. Thus assume $\pi^*(p_2)$ has infinite rank and $t_\pi$ is unbounded. Choose an orthonormal sequence $\{e_n\}$ in $\pi^*(p_2)H_\pi$, and let $V_n = \text{span}\{e_1, \ldots, e_n\}$. We choose recursively $x_n$ in $X$ such that $|x_n| < 2^{-n}, \pi^*(x_n)e_k = 0$ for $k < n$, and $||t_\pi \pi^*(x_1 + \cdots + x_n)e_n|| > n$. If $x_k$ has been chosen for $k < n$, we choose $w$ in $\pi^*(p_2)H_\pi$ such that $||w|| < 2^{-n}k(\pi, V_n)^{-1}$ and $||t_\pi w|| > n + ||t_\pi \pi^*(x_1 + \cdots + x_{n-1})e_n||$. Then by $K(p_1, p_2)$ we can find $x_n$ in $X$ such that $\pi^*(x_n)e_n = w, \pi^*(x_n)e_k = 0$ for $k < n$, and $||x_n|| \leq k(\pi, V_n)||w|| < 2^{-n}$. Now let $x = \sum_{k=1}^\infty x_k = y_n + z_n$, where $y_n = \sum_{k=1}^n x_k$ and $z_n = \sum_{k=n+1}^\infty x_k$. Then $\pi^*(z_n)e_n = 0$. Hence $||t_\pi \pi^*(x)e_n|| = ||t_\pi \pi^*(y)e_n|| > n$. Then $t_\pi \pi^*(x)$ is unbounded, which contradicts the fact that $t_\pi \pi^*(x) = \pi^*(Tx)$.

(d) This now follows from the closed graph theorem. If $T$ is not bounded, there is a sequence $\{x_n\}$ in $X$ such that $x_n \to 0$ and $Tx_n \to y$, for some non-zero $y$ in $T(X)^-$. Then there is an irreducible $\pi$ such that $\pi^*(y) \neq 0$. By the last sentence of (a) this implies $\pi^*(p_1), \pi^*(p_2) \neq 0$. Hence $t_\pi$ is defined and by (c) it is bounded. Hence $\pi^*(y) = \lim t_\pi \pi^*(x_n) = \lim t_\pi \pi^*(x_n) = 0$, a contradiction.

Wong [27] considers a closed projection $p$ in $A^*$ and two spaces of “sections”, denoted $S$ and $W$. $S$ can be identified with $Ap$ and $W$ with $\{x \in A^*p : y^*x \in pAp, \forall y \in Ap\}$.

**Corollary 3.7.** If there is no infinite dimensional irreducible representation $\pi$ of $A$ such that $\pi^*(p)$ has non-zero finite rank, then any purely decomposable linear transformation from $S$ to $W$ or from $W$ to $S$ is bounded. (A fortiori, the same is true for transformations from $S$ or $W$ to $S$.)

**Proof.** Since $S = X(1, p)$ and $W \supset S, S$ and $W$ satisfy $K(1, p)$ by 3.3, and both are norm closed. $W$ is of type $U$ by [27].

**Corollary 3.8.** If $p$ is a closed projection in $A^*$ and $T : pAp \to UC$ is a purely decomposable linear transformation, then $T$ is bounded. In particular, any purely decomposably linear transformation from $A$ to $UC$ is bounded.
Remark 3.9. If $T$ is decomposable, $T(X) \subset A^{**}p_2$. We briefly discuss the question of proving a priori that purely decomposable implies decomposable. If $x$ is in $X$ and $q = \text{in } x$, pure decomposability implies that $\pi_a(\{Tx(1-q)\}) = 0$. If $(Tx)(1-q)$ is contained in some space of type $\mathcal{U}$, then this implies that $(Tx)(1-q) = 0$; i.e. in $Tx \leq \text{in } x$. Now if the $C^*$-algebra generated by $x$ is contained in $\mathcal{U}_\mathcal{C}$ then $q \in \mathcal{U}_\mathcal{C}$ by [22, 4.5.15]. Under suitable universal measurability hypotheses on the elements of $X$ and $T(X)$, this sort of argument can be used to prove decomposability (cf. the comments after Definition 3.5). If we want only to show that $T(X) \subset A^{**}p_2$, it is enough to assume (or prove) $T(X)(1-p_2)$ is of type $\mathcal{U}$.

It has already been mentioned that in many cases a bounded purely decomposable linear transformation can be proved to be a left multiplication. This in particular would be an a posteriori proof of decomposability. It is proved in [27] that transformations of the four types mentioned in 3.7 are left multiplications (if bounded). We content ourselves here with proving the easiest abstract result of this sort. Of course this is not an automatic continuity result, but when applicable it allows the conclusion of Theorem 3.6(d) to be strengthened.

Proposition 3.10. Assume $A$ is a $C^*$-algebra, $X$ is a subspace of $A^{**}$ which satisfies $K(p_1,p_2)$ with $k(\pi)$ bounded independently of $\pi$, where $p_1$ and $p_2$ are projections in $A^{**}, T : X \to A^{**}$ is a bounded, purely decomposable linear transformation, and span$\{\{Tx)x_2 \ldots x_n : x_i \in X \text{ for } i \text{ odd}, x_i \in X^* \text{ for } i \text{ even, } n \geq 1\}$ is of strong type $\mathcal{U}$. Then there is $t$ in $A^{**}$ such that $Tx = tx, \forall x \in X$.

Proof. Let $Y = \text{span}\{x_1x_2 \ldots x_n : x_i \in X \text{ for } i \text{ odd}, x_i \in X^* \text{ for } i \text{ even, } n \geq 1\}$ and let $Z$ be the linear span above. Define a linear map $S : Y \to Z$ by $S(x_1x_2 \ldots x_n) = (Tx_1)x_2 \ldots x_n$. We show that $S$ is bounded, and hence well-defined. For $\pi$ irreducible with $\pi^*(p_1), \pi^*(p_2) \neq 0$ and $y \in Y, ||\pi^*(Sy)|| = ||t_\pi \pi^*(y)|| \leq ||t_\pi|| \cdot ||y|| \leq k(\pi)||T|| \cdot ||y|| \leq c_1||T|| \cdot ||y||$, by 3.6 (a), (b) and hypothesis. If $\pi^*(p_1)$ or $\pi^*(p_2)$ is 0, then $\pi^*(X) = \pi^*(T(X)) = \{0\}$, and hence $\pi^*(Sy) = 0$. Thus $||Sy|| \leq c_2||\pi_a^*(Sy)|| \leq c_1c_2||T|| \cdot ||y||$, since $Z$ is of strong type $\mathcal{U}$. Then $S$ extends to a bounded linear map from $Y$ to $Z$, also denoted by $S$. Let $\{e_i\}$ be an approximate identity of the $C^*$-algebra generated by $XX^*$. Then for $x \in X, e_i x \in Y, S(e_i x) = S(e_i)x$, and $e_i x \to x$ in norm. Hence $Tx = Sx = \lim S(e_i x) = \lim S(e_i)x$. Now since $||Se_i|| \leq c_1c_2||T||$, $\{Se_i\}$ has a weak cluster point $t$ in the von Neumann algebra $A^{**}$. Then $Tx = tx, \forall x \in X$, and also $||t|| \leq c_1c_2||T||$. (The earlier results and remarks of this section show that in many cases $c_1c_2 = 1$.)
§4. Centralizers of Pedersen’s ideal.

If $B$ is an algebra, a left centralizer of $B$ is a linear transformation $T : B \to B$ such that $TR_b = R_bT$ for every right multiplication $R_b$, $b$ in $B$. Right centralizers are defined similarly. A quasi-centralizer is a bilinear map $T : B \times B \to B$ such that $T(b_1b_2,b_3) = b_1T(b_2,b_3)$ and $T(b_1, b_2b_3) = T(b_1, b_2)b_3$ for all $b_1, b_2, b_3$ in $B$. A double centralizer is a pair $(T_1, T_2)$ of linear maps such that $b_1(T_1b_2) = (T_2b_1)b_2$ for all $b_1, b_2$ in $B$. If the left and right annihilators of $B$ are trivial, it then follows that $T_1$ is a left centralizer and $T_2$ a right centralizer. Note that $B$ should be assumed non-unital, since the theory of centralizers is trivial if $B$ is unital.

If $C$ is a super-algebra of $B$, then a left multiplier of $B$ (in $C$) is an element $c$ of $C$ such that $cB \subset B$. Similarly, $c$ is a right multiplier if $Bc \subset B$, a quasi-multiplier if $Bc, Bc \subset B$, and a multiplier if $Bc, Bc \subset B$. Every left multiplier induces a left centralizer (namely, $Lc(B)$), every quasi-multiplier induces a quasi-centralizer, etc.

If $A$ is a $C^*$-algebra, when $B$ above is taken to be $A$, it is standard to take $C$ to be $A^{**}$. In this case the sets of multipliers of the various types are denoted $LM(A), RM(A), QM(A)$, and $M(A)$. Of course, $M(A) = LM(A) \cup RM(A)$ and $LM(A), RM(A) \subset QM(A)$. Also, every left centralizer of $A$ is induced from a left multiplier, every quasi-centralizer from a quasi-multiplier, etc., and moreover the norms agree. If $\pi : A \to B(H)$ is any faithful, non-degenerate representation, then $\pi^{**}$ maps $LM(A)$ isometrically onto the set of left multipliers of $\pi(A)$ in $B(H)$, and similarly for $RM(A), QM(A), M(A)$. References for the above, which has been sketched only very briefly, are Johnson [15], Busby [10], and section 3.12 of [22].

In this section the role of $B$ above is played by $K(A)$, Pedersen’s ideal, and centralizers of the various types may not be bounded. Of course, any bounded centralizer extends by continuity to a centralizer of $A$. If $A$ is the commutative $C^*$-algebra $C_0(X)$, then the centralizers (of any type) of $A$ can be identified with $C_b(X)$, the set of bounded continuous functions on $X$, and the centralizers (of any type) of $K(A)$ can be identified with $C(X)$, the set of arbitrary continuous functions.

Lazar and Taylor [18] and Phillips [25] showed that any double centralizer $T = (T_1, T_2)$ of $K(A)$ is locally bounded in the following sense: There is a family $\{I_j\}$ of (closed two-sided) ideals of $A$ such that $A = (\Sigma I_j)^-$ and $T|_{K(A) \cap I_j}$ is bounded for each $j$. For each $j$, $T$ induces an element of $M(I_j)$; and $M(I_j)$ will be identified with a subset of $A^{**}$, since $A^{**}$ is canonically isomorphic to $I_j^{**} \oplus (A/I_j)^{**}$. Phillips [25] identified $\Gamma(K(A))$, the set of double centralizers of $K(A)$, with the inverse limit of $\{M(I_c)\}$ for a suitable directed family $\{I_c\}$ of ideals of $A$. It is also possible to identify $\Gamma(K(A))$ with an appropriate set of unbounded operators (on the Hilbert space of the universal representation of $A$) affiliated with the von Neumann algebra $A^{**}$, and this approach to $\Gamma(K(A))$ and related concepts is taken by H. Kim [17].

Kim asked whether quasi-centralizers of $K(A)$ would also be locally bounded and then answered this question negatively by showing that it is false for infinite dimensional elementary $C^*$-algebras. (See 4.6 below and note that since elementary $C^*$-algebras are simple, local boundedness would imply boundedness.) Infinite dimensional elementary $C^*$-algebras are the easiest examples of non-unital, non-commutative $C^*$-algebras, but they also turn out to be essentially the only counter-examples for this question.

In this section we will frequently use some widely known facts about $C^*$-algebras concerning the spectrum of a $C^*$-algebra $A$ (the spectrum, denoted by $\hat{A}$, is the
Lemma 4.3. Assume algebras (also called liminal C*-algebras). References for this are sections 3.1 to 3.5, 4.2, and 4.4 of [13] or sections 4.1, 6.1, and 6.2 of [22]. This material, as well as material on the Pedersen ideal contained in section 5.6 of [22] may be used without explicit citation.

Proposition 4.1. \( K(A) \), for a C*-algebra \( A \), satisfies \( K(1,1) \) with \( k(\pi, V) = 1, \forall \pi, V \).

Proof. Let \( \pi: A \to B(H) \) be irreducible, let \( V \) be a finite-dimensional subspace of \( H \), and let \( t \) be in \( B(H) \). By the Kadison transitivity theorem there is \( a \) in \( A \) such that \( ||a|| \leq ||t|| \) and \( \pi(a)_{|V} = t_{|V} \). By the same theorem there is \( b \) in \( A \) such that \( b^* = b, \pi(b)v = v, \forall v \in V, \) and \( ||b|| = 1 \). Let \( f: \mathbb{R} \to [0,1] \) be a continuous function such that \( f(1) = 1 \) and \( f \) vanishes in a neighborhood of 0. If \( c = f(b) \), then \( c \in K(A) \) and \( c \) satisfies the properties given above for \( b \). Then for \( x = ac, x \in K(A), ||x|| \leq ||a|| \leq ||t||, \) and \( \pi(x)_{|V} = t_{|V} \).

Proposition 4.2. If \( T \) is a left centralizer of \( K(A) \), then \( T \) is decomposable and a fortiori purely decomposable.

Remark. It is also true that any purely decomposable \( T : K(A) \to K(A) \) is a left centralizer. The proof is similar to that of (iv) \( \Rightarrow \) (i) in 5.5 below.

Proof. If \( x \) is in \( K(A) \), then \( L \subset K(A) \), for \( L \) the closed left ideal generated by \( x \), by [18, Prop. 3.3]. There is \( y \) in \( L \) such that \( x = y(x^*x)^\frac{1}{2} \) by [22, 1.4.5]. Since also \( (x^*x)^\frac{1}{2} \in K(A), Tx = (Ty)(x^*x)^\frac{1}{2}. \) Therefore \( in T x \leq in (x^*x)^\frac{1}{2} = in x. \)

Lemma 4.3. Assume \( L \) is a closed left ideal of the C*-algebra \( A, L \) generates \( A \) as a (closed, two-sided) ideal, and \( T : L \to L \) is a purely decomposable linear map. If \( T \) is not bounded, then there is a direct sum decomposition, \( A = A_0 \oplus A_1, \) such that \( A_1 \) is an elementary C*-algebra for which the corresponding \( t_{\pi} \) (notation as in 3.6) is not bounded.

Remark. In the next section lemmas 4.3 and 4.4 will be generalized by replacing \( L \) with \( X(p_1, p_2) \) for open projections \( p_1 \) and \( p_2 \). The same arguments apply except that some details must be added to the “preliminary” paragraph in the proof below dealing with cutting down to an ideal. These additional details will be provided in the next section.

Proof. Some explanation of the statement of the lemma is in order. First, \( L = X(1, p) \) for some open projection \( p \) and hence \( t_{\pi} \) is defined for each irreducible \( \pi \). (Note: \( \pi**(p) = 0 \) implies \( \pi(L) = 0, \) which is impossible by hypothesis.) Second, if \( A = A_0 \oplus K(H) \), then an irreducible representation \( \pi: A \to B(H) \) is defined as projection onto the second summand. Obviously, \( \pi(A) = K(H) \).

Now because of the one-to-one correspondence between ideals of \( A \) and open subsets of \( \hat{A} \), direct sum decompositions of \( A \) correspond one-to-one to separations of \( \hat{A}, \hat{A} = S_0 \cup S_1 \). If \( A_1, \) where \( A_1 = S_1, \) is to be elementary, then \( S_1 \) must consist of a single clopen point. Conversely, if \( \pi \) is an irreducible representation such that \( \{[\pi]\}, \) where \( [\pi] \) denotes the equivalence class of \( \pi, \) is a clopen subset of \( \hat{A} \) and \( \pi(A) = K(H_\pi) \), then \( \pi \) corresponds to a direct sum decomposition of the desired type. (The last condition is necessary because it is not known whether \( \hat{A}_1 = \{\text{one point}\} \) implies \( A_1 \) elementary.)

In the course of the proof we will replace \( A \) by an ideal, and therefore we include one more preliminary paragraph. If \( I \) is an ideal of \( A \), then \( L \cap I \) is a closed
left ideal of $I$ and $L \cap I$ generates $I$ as an ideal (for example, because $I = IAI = I(ALA)^{-} I \subset (IALAI)^{-} \subset (ILI)^{-}$ and $IL \subset L \cap I$). Also, $T$ maps $L \cap I$ into itself, since $L \cap I = L \cap [\{ \text{kernel } \pi : [\pi] \in (A/I)^{*} \}]$ and $T$ maps $L \cap \text{kernel } \pi$ into itself by pure decomposability. Now for $\pi$ in $\hat{I}$, $t_{\pi}$ is the same whether computed relative to $L \cap I$ or $L$ (the domain of $t_{\pi}$ is $H_{\pi}$). Finally, any direct sum decomposition, $I = I_{0} \oplus \mathcal{K}(H_{\pi})$, leads also to $A = A_{0} \oplus \mathcal{K}(H_{\pi})$ provided $\pi(A) = \mathcal{K}(H_{\pi})$. The reason is that $\pi(A) = \mathcal{K}(H_{\pi})$ implies $[\pi]$ is closed in $\hat{A}$, and $[\pi]$ open in $\hat{I}$ implies $[\pi]$ open in $\hat{A}$.

Now since $T$ is not bounded, by the closed graph theorem there is a sequence $\{x_{n}\}$ in $L$ such that $x_{n} \to 0$ and $Tx_{n} \to y \neq 0$. Of course, $y$ is in $L$, and since $\pi(y) = \lim \pi(Tx_{n}) = \lim t_{\pi}\pi(x_{n})$, $t_{\pi}$ must be unbounded for all $\pi$ such that $\pi(y) \neq 0$. If $I$ is the ideal generated by $y$, this means that $t_{\pi}$ is unbounded whenever $[\pi]$ is in $\hat{I}$. By 3.6 (c), if $t_{\pi}$ is not bounded, then $\pi^{**}(p)$ has finite rank and hence $\pi(x)$ has finite rank for all $x$ in $L$. Since $L$ generates $A$ as an ideal, this implies $\pi(A) \subset \mathcal{K}(H_{\pi})$, which implies $\pi(A) = \mathcal{K}(H_{\pi})$.

We replace $A$ by $I$ and change notation. Thus we now have that $t_{\pi}$ is unbounded for all $\pi$ and that $A$ is a CCR algebra. Then $A$ has a non-zero ideal which has Hausdorff spectrum, and we may cut down and change notation once more. Thus, now $\hat{A}$ is Hausdorff and $t_{\pi}$ is unbounded for all $\pi$.

To complete the proof, we need only show that $\hat{A}$ has an isolated point, and we do this by showing that $\hat{A}$ is finite. In fact, if $\hat{A}$ has infinitely many points, then there is a sequence $\{U_{n}\}$ of non-empty, mutually disjoint open sets. If $I_{n}$ is the ideal corresponding to $U_{n}$, then $t_{\pi}$ is unbounded for all irreducible representations $\pi$ of $I_{n}$ (i.e., all irreducibles $\pi$ of $A$ such that $[\pi] \in U_{n}$, or equivalently $\pi(I_{n}) \neq \{0\}$). By 3.6 (b) this means $T|_{L \cap I_{n}}$ is not bounded. Thus we may choose $x_{n}$ in $L \cap I_{n}$ such that $||x_{n}|| < 2^{n}$ and $||Tx_{n}|| > n$. Let $x = \sum_{1}^{\infty}x_{n}$, an element of $L$. For $[\pi]$ in $U_{n}$, $\pi(Tx) = t_{\pi}(x) = t_{\pi}\pi(x_{n})$, since $\pi|_{I_{n}} = 0$ for $m \neq n$. Thus $||Tx|| \geq \sup_{U_{n}}||t_{\pi}\pi(x_{n})|| = \sup_{U_{n}}||\pi(Tx_{n})|| = ||Tx_{n}|| > n$, where one of the equalities uses the fact that $Tx_{n} \in I_{n}$. This contradiction completes the proof.

**Corollary 4.4.** Assume $L$ is a closed left ideal of the $C^{*}$-algebra $A$, $L$ generates $A$ as an ideal, and $T : L \to L$ is a purely decomposable linear map. Then there is a finite set $F = \{[\pi_{1}], \ldots, [\pi_{n}]\}$ of clopen points of $\hat{A}$ such that $\pi_{i}(A) = \mathcal{K}(H_{\pi_{i}})$ and $t_{\pi}$ is bounded whenever $[\pi] \notin F$.

**Proof.** Let $U = \{[\pi] : [\pi]$ is a clopen point of $\hat{A}, \pi(A) = \mathcal{K}(H_{\pi})$, and $t_{\pi}$ is not bounded$\}$ If $U$ is infinite, there is a sequence $\{[\pi_{n}]\}$ of distinct points in $U$. Then the argument in the last paragraph of the proof of 4.3 produces a contradiction, with $U_{n} = \{[\pi_{n}]\}$ and $I_{n}$ the elementary direct summand corresponding to $\pi_{n}$. Thus $U$ is finite, and we have a direct sum decomposition, $A = A_{0} \oplus \mathcal{K}(H_{\pi_{1}}) \oplus \cdots \oplus \mathcal{K}(H_{\pi_{n}})$, where $U = \{[\pi_{1}], \ldots, [\pi_{n}]\}$. The proof of 4.3 shows that the hypotheses are still satisfied if we replace $(A, L)$ by $(A_{0}, A_{0} \cap L)$; and since $U \cap \hat{A}_{0} = \emptyset$, 4.3 then implies that $T|_{A_{0} \cap L}$ is bounded. Then 3.6 (b) implies that $t_{\pi}$ is bounded for $\pi \notin \pi_{1}, \ldots, \pi_{n}$.

**Theorem 4.5.** If $A$ is a $C^{*}$-algebra and $T$ is a left centralizer of $K(A)$, then there is a direct sum decomposition, $A = A_{0} \oplus A_{1}$, such that $A_{1}$ is a dual $C^{*}$-algebra and $T|_{K(A_{0})}$ is locally bounded.

**Remark.** $K(A) = K(A_{0}) \oplus K(A_{1})$. 
Proof. By 4.1, 4.2, and 3.6 (a), a linear transformation \( t_\pi : H_\pi \to H_\pi \) is defined for each irreducible \( \pi \). Suppose \( t_\pi \) is not bounded. Let \( x \) be an element of \( K(A) \) such that \( \pi(x) \neq 0 \), let \( L_x \) be the closed left ideal generated by \( x \), and let \( I_x \) be the closed two-sided ideal generated by \( x \) (or by \( L_x \)). Then \( L_x \subseteq K(A) \) and the proof of 4.2 shows that \( T(L_x) \subseteq L_x \) (in the notation of 4.2, \( (x^*x)\hat{T} \) is in \( L \)). Thus we can apply 4.4 with \( (L_x, I_x) \) in the role of \( (L, A) \) to obtain that \( [\pi] \) is a clopen point of \( \hat{I}_x \), and hence an open point of \( \hat{A} \), and \( \pi(I_x) = K(H_\pi) \). In particular, \( \pi(x) \in K(H_\pi) \), and since this holds for any \( x \) in \( K(A) \) with \( \pi(x) \neq 0 \), we see that \( \pi(K(A)) \subseteq K(H_\pi) \), and thus \( \pi(A) = K(H_\pi) \). This implies that \( [\pi] \) is closed in \( \hat{A} \) and \( \pi \) corresponds to an elementary direct summand of \( A \).

Now let \( U = \{ [\pi] \in \hat{A} : t_\pi \) is not bounded \}. By the above, \( U \) is open and we now prove \( U \) closed. Assume \( [\pi] \) is not in \( U \) for some irreducible \( \pi \), and choose \( x \) in \( K(A) \) such that \( \pi(x) \neq 0 \). As above, we can cut down to \( L_x \) and \( I_x \), and then 4.4 implies that \( U \cap \hat{I}_x \) is finite. Since each point of \( U \) is closed, this implies that \( [\pi] \) is not in \( \overline{U} \).

Thus \( U \) induces a direct sum decomposition, \( A = A_0 \oplus A_1 \), where \( A_1 = U \), and clearly \( A_1 \) is dual. We replace \( A \) by \( A_0 \) and change notation. Thus from now on we assume \( t_\pi \) bounded for all \( \pi \). Now complete the proof by showing that \( T|_{K(A) \cap I_x} \) is bounded for all \( x \) in \( K(A) \). Since \( T(L_x) \subseteq L_x \), the closed graph theorem implies \( T|_{L_x} \) is bounded (cf. 3.6 (d)). Now 3.6 (b) implies that \( \|t_\pi\| \leq \|T|_{L_x}\| \) whenever \( [\pi] \in \hat{I}_x \). Note that \( T[K(A) \cap I_x] \subseteq I_x \), since \( Ty \in L_y \) for all \( y \) in \( K(A) \). Then for \( y \) in \( K(A) \cap I_x \), \( \|Ty\| = \sup_{y \in I_x} \|\pi(Ty)\| = \sup_{y \in I_x} \|t_\pi \pi(y)\| \leq \|T|_{L_x}\| \|y\| \). In other words, \( \|T|_{K(A) \cap I_x}\| = \|T|_{L_x}\| \).

If \( A \) is the elementary \( X^* \)-algebra \( K(H) \), then \( K(A) = F(H) \), the set of bounded, finite rank operators on \( H \). \( F(H) \) is spanned by (rank one) operators of the form \( v \times w, v, w \in H \), where \( (v \times w)u = (u, w) v \).

Proposition 4.6. (H. Kim). Let \( A = K(H) \).

(a) The left centralizers of \( K(A) \) correspond one-to-one to linear transformations \( t : H \to H \), as follows:

\[
T(v \times w) = (tv) \times w.
\]

(b) The quasi-centralizers of \( K(A) \) correspond one-to-one to sesqui-linear forms \( f : H \times H \to \mathbb{C} \), as follows:

\[
T(v_1 \times w_1, v_2 \times w_2) = f(v_2, w_1)v_1 \times w_2, \quad \text{or} \quad p\hat{T}(x, y)q = f(w, x^*v)v \times w, \quad \text{if} \quad p = v \times v, \quad q = w \times w.
\]

Sketch of proof. (a) It is easy to see that for any \( t \) the formula given extends by linearity to a left centralizer. Conversely, for any left centralizer \( T \) and any \( w \) in \( H \), it is easy to see that a linear transformation \( t \) exists as in the formula. Using \( R_u \), for \( u \) a rank one operator, one easily shows that \( t \) is independent of \( w \). (Of course, 3.6 (a) also applies.)

(b) It is easy to see that for any \( f \), the formula given extends by bilinearity to a quasi-centralizer. Conversely, for any quasi-centralizer \( T \) and any \( v_1, w_2 \) in \( H \), it is easy to see that a sesqui-linear form \( f \) exists as in the formula. Using \( L_{u_1}, R_{u_2} \) for rank one operators \( u_1, u_2 \), one easily shows that \( f \) is independent of \( v_1, w_2 \).
Now if \( A = A_0 \oplus A_1 \), it is easy to see that any left centralizer of \( K(A) \) is of
the form \( T_0 \oplus T_1 \), where \( T_j \) is a left centralizer of \( K(A_j) \). Also if \( A_1 = \oplus_i \mathcal{K}(H_i) \),
a \( c_0 \)-direct sum (i.e., if \( A_1 \) is dual), then \( K(A_1) = \oplus_i \mathcal{F}(H_i) \), an algebraic direct
sum (each element has only finitely many non-zero terms). Thus 4.5 and 4.6 (a)
describe all left centralizers of \( K(A) \), but note that the decomposition, \( A = A_0 \oplus A_1 \),
depends on the particular \( T \).

**Corollary 4.7.** For any \( C^* \)-algebra \( A \) the following are equivalent:

(i) Every left centralizer of \( K(A) \) is locally bounded.

(ii) \( A \) has no infinite dimensional elementary direct summand.

**Theorem 4.8.** If \( A \) is a \( C^* \)-algebra and \( T \) is a quasi-centralizer of \( K(A) \), then there
is a direct sum decomposition, \( A = A_0 \oplus A_1 \), such that \( A_1 \) is a dual \( C^* \)-algebra and
\( T|_{K(A_0) \times K(A_0)} \) is locally bounded.

*Proof.* In the first three steps \( \pi \) denotes a fixed irreducible representation which
does not correspond to an elementary direct summand of \( A \) and the subscript “\( \pi \)”
is supressed.

1) For each \( x \) in \( K(A) \), \( T(x, \cdot) \) is a left centralizer of \( K(A) \). By the proof of 4.5,
there is \( t_x \) in \( B(H) \) such that \( \pi[T(x, y)] = t_x \pi(y) \), \( \forall y \in K(A) \). Since \( (x, y) \mapsto
T(y^*, x^*)^* \) is also a quasi-centralizer, the same argument gives for each \( y \) in \( K(A) \)
a \( u_y \) in \( B(H) \) such that \( \pi[T(x, y)] = \pi(x)u_y \), \( \forall x \in K(A) \).

2) We define a linear transformation \( s : H \rightarrow H \), as follows:
\[
sv = w \text{ if and only if } t_x v = \pi(x)w, \quad \forall x \in K(A).
\]

Since \( \pi(x)w_1 = \pi(x)w_2 \), \( \forall x \in K(A) \), implies \( w_1 = w_2 \), \( s \) is well-defined on its
domain. To see that \( s \) is defined on all of \( H \), choose \( y \) in \( K(A) \) and \( v_0 \) in \( H \)
such that \( \pi(y)v_0 = v \), for given \( v \). Then \( t_x v = \pi[T(x, y)]v_0 = \pi(x)u_yv_0 \), so that
\( sv = u_yv_0 \). It is now clear that \( s \) is linear and also that \( \pi[T(x, y)] = \pi(x)s\pi(y) \).

3) We show that \( s \) is bounded. As in the proof of 4.5, choose a closed left
ideal \( L \) and a closed two-sided ideal \( I \) such that \( L \subset K(A) \), \( L \) generates \( I \), and
\( \pi \) is non-trivial on \( I \). Consider the bilinear function \( \pi[T(x, y)] \) for \( x \) in \( L^* \) and
\( y \) in \( L \). By 1) this function is separately continuous. It is then a well known
consequence of the uniform boundedness principle that there is a constant \( c \) such that
\[
|\pi[T(x, y)]| \leq c|x||y|, \quad \forall x \in L^*, \quad \forall y \in L.
\]

Let \( p \) be the open projection such that \( L = X(1, p) \) and note that \( \pi^{**}(p) \neq 0 \). Choose a unit vector \( v_0 \) in \( \pi^{**}(p) H \).
For given vectors \( v, w \) in \( H \), choose, using 3.3, \( x, y \) in \( L \) such that
\( ||x|| = ||v||, ||y|| = ||w||, \pi(x)v_0 = v, \) and \( \pi(y)v_0 = w \). Then since \( \pi[T(x, y)] = \pi(x)^s\pi(y) \),
we have \( ||sv, v|| = ||p_0 \pi(x)^s\pi(y)p_0 || \leq c ||v|| ||w|| \), where \( p_0 = v_0 \times v_0 \). Thus \( ||s|| \leq c \).

4) If \( \pi \) is an irreducible which does correspond to an elementary direct summand,
then 4.6 (b) gives a sesqui-linear form \( f_\pi : H_\pi \times H_\pi \rightarrow \mathbb{C} \). Let \( U = \{[\pi] \in \hat{A} : f_\pi \)
(as above) is not bounded}. An argument similar to part of the proof of 4.5 will
show that \( U \) is closed provided we prove the following: For each \( z \) in \( K(A) \), \( U \cap \hat{I}_z \)
is finite. If this is false, there is a sequence \( \{[\pi_n] \} \) of distinct points in \( U \cap \hat{I}_z \). Let
\( I_n \) be the elementary direct summand corresponding to \( \pi_n \) and let \( L_n = \hat{I}_z \cap I_n \).
(Each \( I_n \) is an ideal of \( A \) and \( \hat{I}_z = \{[\pi_n] \} \). Although each \( I_n \) is a direct summand
of \( A \), it is not valid to write \( A = A_0 \oplus [\oplus_n I_n] \). For each \( n \) choose \( v_n, w_n \) in \( H_{\pi_n} \)
such that \( ||v_n||, ||w_n|| < 2^{-n} \) and, \( |f_{\pi_n}(w_n, v_n)| > n \). Also choose a unit vector
\( u_n \) in \( \pi_{\pi_n}^{**}(p_n) H_{\pi_n} \), where \( p_n \) is the open projection such that \( L_n = X(1, p) \). Then
choose \( x_n, y_n \) in \( L_n \) such that \( \|x_n\| = \|v_n\|, \|y_n\| = \|w_n\|, \pi_n(x_n)u_n = v_n \), and \( \pi_n(y_n)u_n = w_n \). Then \( \|T(x_n^*, y_n)\| \geq \|q_n T(x_n^*, y_n) q_n\| = \|f_{\pi_n}(w_n, v_n)\| > n \), where \( q_n = u_n \times u_n \). Let \( x = \Sigma_1^\infty x_n \) and \( y = \Sigma_1^\infty y_n \), so that \( x, y \in L_z \subset K(A) \).

The fact that \( T(x', y') \in L_{y'} \quad \forall x', y' \in K(A) \), and similarly \( T(x', y') \in R_{x'} \), the closed right ideal generated by \( x' \), \( \forall x', y' \in K(A) \). If we apply this for \( x' = \Sigma_{k \neq n} x_k^*, \ y' = \Sigma_{k \neq n} y_k \), we see that \( \pi_n[T(x^*, y)] = T(x_n^*, y_n) \), which contradicts the fact that \( \|\pi_n[T(x^*, y)]\| \leq \|T(x^*, y)\|, \quad \forall n \).

5) We can now write \( A = A_0 \oplus A_1 \), where \( \hat{A}_1 = U \) and \( A_1 \) is dual. Replace \( A \) by \( A_0 \) and change notation, so that from now on \( U = \emptyset \). Then for each irreducible \( \pi \), we either have \( s_\pi \) in \( B(H_\pi) \) satisfying the formula in 2) or a bounded bilinear form \( f_\pi \) as in 4). In the latter case there is \( s_\pi \) in \( B(H_\pi) \) such that \( f_\pi(v, w) = (s_\pi v, w) \), so that for all \( \pi, \pi[T(x, y)] = \pi(x)s_\pi \pi(y), \quad \forall x, y \in K(A) \).

6) To complete the proof, we show that \( T \) is bounded on \( [K(A) \cap I_x] \times [K(A) \cap I_x] \) for each \( x \) in \( K(A) \). For any \( y \) in \( K(A) \) and \( \pi \) in \( \hat{I}_x \), the transformation \( t_x \) for the left centralizer \( T(y, \cdot) \) is \( \pi(y)s_\pi \), which is bounded. Therefore \( T(y, \cdot) \) is bounded on \( L_x \) by the proof of 4.5. Similarly, \( T(\cdot, y) \) is bounded on \( L_x^* \), and as above there is a constant \( c \) such that \( \|T(y, z)\| \leq c\|y\| \|z\|, \quad \forall y \in L_x^*, \forall z \in L_x \). As in 3) above, we then see that \( \|s_\pi\| \leq c \) whenever \( [\pi] \in \hat{I}_x \). Now arguments already given show that \( T(y, z) \in I_x \) whenever \( x, y \) are in \( K(A) \cap I_x \), so that \( T \) is bounded (by \( c \)) on \( [K(A) \cap I_x] \times [K(A) \cap I_x] \).

Again, 4.8 and 4.6 (b) describe all quasi-centralizers of \( K(A) \).

**Corollary 4.9.** For any \( C^* \)-algebra \( A \), the following are equivalent:

(i) Every quasi-centralizer of \( K(A) \) is locally bounded.

(ii) \( A \) has no infinite dimensional elementary direct summand.
§5. Local linear transformations in the context of $C^*$-algebras and open projections.

If $\pi : A \to B(H)$ is an irreducible representation, then there is an isomorphism between $A^{**}$ and $B(H) \oplus M$, where $M = \text{kernel } \pi^{**}$. To any rank one projection $p$ in $B(H)$ corresponds a minimal (non-zero) projection in $A^{**}$, and all minimal projections in $A^{**}$ arise in this way. There is a one-to-one correspondence between pure states of $A$ and minimal projections in $A^{**}$ as follows: Any $\varphi$ in $P(A)$ is given by $\varphi(a) = (\pi(a)v, v)$ for some irreducible $\pi$ and some unit vector $v$ in $H_\pi$, and $\varphi$ corresponds to the projection $p$ with range $Cv$. Then $p$ is called the support projection of $\varphi$ because it is the smallest projection such that $\varphi(px) = \varphi(x)$, $\forall x \in A^{**}$. As usual we make no notational distinction between states of $A$ and the corresponding normal states of $A^{**}$.

**Lemma 5.1.** Assume $A$ is a $C^*$-algebra, $p$ is a closed projection in $A^{**}$, $\varphi$ is an element of $P(A)$ such that $\varphi(p) = 0$, and the support projection, $p_0$, of $\varphi$ is not in $A$. Then there is a net $\{\varphi_i\} \subset P(A)$ such that $\varphi_i \perp \varphi$ (i.e., $p_ip_0 = 0$, where $p_i$ is the support projection of $\varphi_i$), $\varphi_i(p) = 0$, and $\varphi_i \to \varphi$ weak$^*$.

**Remark.** Intuitively, this lemma deals with the concept of an isolated point, or rather a non-isolated point.

**Proof.** We reduce to the unital case as follows: Let $\tilde{A}$ be the result of adjoining an identity to $A$ and note that $\tilde{A}^{**} \cong A^{**} \oplus \mathbb{C}$. Replace $p$ by $p \oplus 1$, a closed projection in $\tilde{A}^{**}$. Since $P(A)$ can be identified with $\{\varphi \in P(\tilde{A}) : \varphi(0 \oplus 1) = 0\}$, the desired conclusion for $A$ follows from the conclusion for $\tilde{A}$. Thus for the remainder of the proof we assume $A$ unital.

Now consider the hereditary $C^*$-algebra $B$ corresponding to the open projection $1 - p_0$. By Akemann’s Urysohn lemma [2] or [3] there is $e$ in $B$ such that $p \leq e \leq 1 - p_0 \leq 1$. Let $\{f_j\}_{j \in D_1}$ be an (increasing) approximate identity for $B$. Then $\{e_j\}$ is also an approximate identity, where $e_j = e + (1 - e)^{\frac{1}{2}}f_j(1 - e)^{\frac{1}{2}}$. Let $D = D_1 \times \mathbb{N}$ and for $i = (j, n)$ in $D$ choose a pure state $\varphi_i$ of $B$ supported by $E_{[0, \frac{1}{2}]}(e_j)$, where the last symbol denotes a spectral projection (in $B^{**}$) of $e_j$. If this were impossible, i.e., if the closed projection $E_{[0, \frac{1}{2}]}(e_j)$ were zero, then $e_j$ would be invertible in $B$ and $B$ would be a unital $C^*$-algebra. This would imply $p_0 \in A$, a contradiction. Since $p \leq E_{[1]}(e_j), \varphi_i(p) = 0$ for $n > 1$, and clearly $\varphi_i \perp \varphi$. We see as follows that $\varphi_i \to 0$ in the weak$^*$ topology of $B^*$: Given $b \in B$, $b \geq 0$, $b \approx e_j \frac{1}{n} be_j$ for $j$ sufficiently large. Hence $\varphi_{j,n}(b) \approx \varphi_{j,n}(e_j \frac{1}{n} be_j) \leq ||b|| \varphi_{j,n}(e_j) \leq \frac{1}{n}||b||$. This means that any weak$^*$ cluster point of $\{\varphi_i\}$ in $A^*$ is supported by $p_0$ and hence $\varphi_i \to \varphi$.

**Theorem 5.2.** Assume $A$ is a $C^*$-algebra, $p_1$ and $p_2$ are open projections in $A^{**}$, and $T : X(p_1, p_2) \to X(1, p_2)$ is a local linear transformation. Then $T$ is purely decomposable.

**Proof.** It is sufficient to show $\varphi(x^*x) = 0$ implies $\varphi[(Tx)^*(Tx)] = 0$ for $\varphi$ a pure state supported by $p_2$ and $x \in X(p_1, p_2)$. Let $p_0$ be the support projection of $\varphi$, so that $p_0 \leq p_2$. Since for all $x, \varphi(x^*x) = 0 \iff xp_0 = 0$, the result to be proved is immediate if $p_0$ is open. Therefore we may assume $p_0 \notin A$.

Suppose then that $\varphi(x^*x) = 0$ and $\varphi[(Tx)^*(Tx)] > 1$. We will construct recursively pure states $\varphi_n$, open projections $q_n$, and elements $y_n$ of $X(p_1, p_2)$ such that $\overline{\varphi}_n \perp \overline{\varphi}_m$ for $n \neq m$, $\overline{\varphi}_n \perp p_0$, $\overline{\varphi}_n \leq p_2$, $\varphi_n$ is supported by $q_n$, $y_n q_k = 0$ for $k < n$, $||y_n|| < 2^{-n}$, and $||T(y_1 + \cdots + y_n)w_{\varphi_n}|| > n$. Here, for any projection
If $q$ in $A^{**}$. $\overline{q}$ denotes the smallest closed projection majorizing $q$ (cf. [1]), and for any state $\psi$ of $A$, $w_\psi$ denotes the cyclic vector in the GNS construction, so that $\|yw_\psi\| = [\psi(y^* y)]^{1/2}$ for any $y$ in $A^{**}$. Then if $y = \Sigma_{i=1}^\infty y_n$, $y q_n = (y_1 + \cdots + y_n) q_n$. Hence by locality $\|(Ty_1 + \cdots + y_n) w_{\varphi_n}\| > n$. This is a contradiction, since $\|(Ty) w_{\varphi_n}\| \leq \|(Ty)\|, \forall n$.

Assume we already have $\varphi_1, \varphi_2, \cdots, \varphi_n, q_1, \cdots, q_{n-1}$, and $y_1, \cdots, y_{n-1}$ $(n \geq 1)$. Choose $\lambda > \|(Ty_1 + \cdots + y_{n-1})\| + n (\lambda > 1$ if $n = 1)$. Let $\delta = \lambda^{-1}2^{-n}$ and choose a continuous function $f_\delta : [0, \infty) \to [0, 1]$ such that $f_\delta = 1$ in a neighborhood of 0 and $f_\delta(t) = 0$ for $t \geq \delta$. Let $x' = x f_\delta(|x|)$, where $|x| = (x^* x)^{1/2}$. Then $\|x'\| < \delta$ and $(x' - x) q = 0$ where $g$ is an open projection of the form $E_{[0,\epsilon)}(|x|)$. Since $p_0 \leq q$, it follows from locality that $[T(x' - x)] p_0 = 0$. Thus, $\|(Ty') w_{\varphi}\| > \|(Ty) w_{\varphi}\| > 1$. Now by Akemann's Urysohn lemma choose $h$ in $A$ such that $p_0 \leq h \leq p_2 - (\overline{q}_1 + \cdots + \overline{q}_{n-1})$. The fact that the projection on the right is open, i.e., $1 - p_2 + \overline{q}_1 + \cdots + \overline{q}_{n-1}$ is closed, follows from [1]. Let $g : [0, 1] \to [0, 1]$ be a continuous function such that $g(0) = 0$ and $g = 1$ in a neighborhood of 1. Then $y_n = \lambda x' g(h)$. Then $\|y_n\| < 2^{-n}$, and by locality $\|(Ty_n) w_{\varphi}\| = \lambda \|(Ty') w_{\varphi}\| > \lambda$. Now use 5.1 to find $\varphi_n$ in $P(A)$ such that $\varphi_n \perp \varphi$, $\varphi_n (1 - p_2 + \overline{q}_1 + \cdots + \overline{q}_{n-1}) = 0$ and $\varphi_n [(Ty_n)^* (Ty_n)] > \lambda^2$. Therefore $\|(Ty_n) w_{\varphi_n}\| > \lambda$ and hence $\|(Ty_1 + \cdots + y_n) w_{\varphi_n}\| > n$. Finally, if $r_n$ is the support projection of $\varphi_n$, then we can find, by Akemann's Urysohn lemma, $h'$ in $A$ such that $r_n \leq h' \leq t_2 - (\overline{q}_1 + \cdots + \overline{q}_{n-1}) - p_0$. Then let $q_n = E_{[\frac{1}{2}, 1]}(h')$, so that $\overline{q}_n \leq E_{[\frac{1}{2}, 1]}(h')$. This completes the recursive construction and the proof of the theorem.

**Corollary 5.3.** Under the same hypotheses the ideal of $A$ generated by $X(p_1, p_2)$ can be written as a direct sum, $A_0 \oplus A_1 \oplus \cdots \oplus A_n$, such that $X(p_1, p_2) = \oplus_i^n [X(p_1, p_2) \cap \ A_i]$, $A_i = K(H_i)$ for $i > 0$, and $T(x_0 x_1 \cdots x_n) = t_0 x_0 \oplus t_1 x_1 \cdots \oplus t_n x_n$ for some $t_0$ in $A_0^{**}$ and (possibly discontinuous) linear transformations $t_i : \pi_i^{**}(p_1) H_i \to H_i$, where $\pi_i$ is the irreducible representation corresponding to $A_i$.

**Proof.** There is no loss of generality in assuming $X(p_1, p_2)$ generates $A$ as an ideal. Note that if $p_1 = 1$, then $X(p_1, p_2)$ is a left ideal, and we are in the situation of 4.3 and 4.4. The main point is to generalize 4.3 and 4.4, which will give us the desired direct sum decomposition so that $T|_{X(p_1, p_2) \cap A_0}$ is bounded, and $t_i = t_{\pi_i}$ for $i > 0$.

As already remarked, the only parts of the proofs of 4.3 and 4.4 that require change are the justifications of cutting down to an ideal $I$ of $A$. The fact that $X(p_1, p_2) \cap I$ generates $I$ as an ideal can be proved as follows (it requires no proof for readers familiar with the theory of imprimitivity bimodules): Let $B_i = \operatorname{her}(p_i)$, so that $X(p_1, p_2) = [B_1 AB_2]^{**}$, and let $Y = X(p_1, p_2) \cap I$. Then $[Y I]^{**} \supset [I (B_1 IB_2)^* I]^{**} \supset [(IA)B_1 (AIA)B_2 (AI)]^{**} \supset I (AB_1 A)^* I (AB_2 A)^* I = IAIAI = I$. Here we used the fact that $B_i$ generates $A$ as an ideal, since $X(p_1, p_2)$ is clearly contained in the ideal generated by $B_i$. The fact that $T(Y) \subset X(1, p_2) \cap I$ follows from pure decomposability. We must show that for an irreducible $\pi$ which is non-trivial on $I$, $t_{\pi}$ is the same whether computed for $A$ and $X(p_1, p_2)$ or $I$ and $Y$. Since the first version of $t_{\pi}$ is an extension of the second, we need only show the domains are the same. Now $Y = X(z p_1, z p_2)$, where $z$ is the open central projection corresponding to $I$; i.e., $I = \{ a \in A : za = a \}$. If $\pi$ is non-trivial on $I$, then $\pi^{**}(z) = 1$ and hence $\pi^{**}(z p_1) = \pi^{**}(p_1)$.

Now we have $A = A_0 \oplus \cdots \oplus A_n$ and the fact that $X(p_1, p_2) = \oplus_i^n [X(p_1, p_2) \cap A_i]$
is immediate from the fact that $p_i$ can be written, $p_i = \oplus_{j=0}^n p_{ij}$ with $p_{ij}$ in $A_j^{**}$. Since $T|_{X(p_1, p_2) \cap A_0}$ is bounded, 3.10 yields $t_0$ in $A_0^{**}$ with the desired property.

**Remark.** Of course, $t_0$ is not an arbitrary element of $A_0^{**}$, since necessarily $t_0[X(p_1, p_2) \cap A_0] \subset A_0$. One should regard $t_0$ as a kind of left multiplier (this interpretation possibly should be used only in the special case where $T(X(p_1, p_2)) \subset X(p_1, p_2)$). It has already been mentioned that $X(p_1, p_2)$ is a Hilbert $B_1 - B_2$ bimodule (an imprimitivity bimodule, even, if $X(p_1, p_2)$ generates $A$ as an ideal), and Hilbert bimodules have many properties in common with $C^*$-algebras. However, every left centralizer of a $C^*$-algebra is bounded by [22, 3.12.2], and in certain cases left centralizers of $X(p_1, p_2)$, of the sort considered in 5.3, are not bounded. The exceptions occur only when the ideal generated by $X(p_1, p_2)$ has an elementary direct summand $K(H_*)$ such that $\pi^{**}(p_2)$ has finite rank and $\pi^{**}(p_1)$ has infinite rank.

**Corollary 5.4.** If $p_1 = p_2$, then $T$ is bounded. In particular, any local linear transformation from $A$ to itself is bounded.

**Remark.** Since $T$ is given only as a local linear transformation, not as a left centralizer, the last sentence of 5.4 is a new result. Of course, a posthiori, $T$ is a left centralizer.

**Proposition 5.5.** Assume $A$ is a $C^*$-algebra, $p_1$ and $p_2$ are open projections in $A^{**}$, and $T$ is a linear transformation from $X(p_1, p_2)$ to $X(1, p_2)$. Then the following are equivalent:

(i) $T(xb) = (Tx)b$, for all $b$ in $\text{her}(p_2)$, the hereditary $C^*$-subalgebra of $A$ supported by $p_2$, and all $x$ in $X(p_1, p_2)$.

(ii) $T$ is decomposable.

(iii) $T$ is local.

(iv) $T$ is purely decomposable.

**Proof.** (i) $\Rightarrow$ (ii). We can write $x = y|x|^{\frac{1}{2}}$ for some $y$ in $X(p_1, p_2)$ (cf. [22, 1.4.5]). Then $|x|^{\frac{1}{2}} \in \text{her}(p_2)$, and hence $Tx = (Ty)|x|^{\frac{1}{2}}$. Since in $x = in \ |x|^{\frac{1}{2}}$, this shows that in $Tx \leq in \ x$.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (iv) is 5.2.

(iv) $\Rightarrow$ (i). If $\pi$ is irreducible and $x, b$ are as in (i), then by 3.6 (a), $\pi[T(xb)] = t_\pi \pi(xb)$ and $\pi[(Tx)b] = \pi(Tx)\pi(b) = t_\pi \pi(x)\pi(b) = t_\pi \pi(xb)$ (in case $\pi^{**}(p_1)$ or $\pi^{**}(p_2)$ is 0, both above are 0). Since there are enough irreducible representations to distinguish elements of $A$ (i.e., $A$ is of type $U$), this implies $T(xb) = (Tx)b$.

**Remark 5.6.** It is possible to combine the subjects of Sections 4 and 5 by considering local operators $T$ defined on $X(p_1, p_2) \cap K(A)$. The analogues of 5.2 and 5.5 are true in this context and also the analogue of 5.3 and 4.5: The ideal of $A$ generated by $X(p_1, p_2)$ can be written as $A_0 \oplus A_1$ such that $A_1$ is dual and $T|_{X(p_1, p_2) \cap K(A) \cap A_0}$ is locally bounded. The proofs do not require much in the way of new arguments:

1. In the proof of 5.2, $y$ is in the closed right ideal generated by $x$ and hence $x$ in $K(A)$ implies $y$ in $K(A)$.

2. For $x$ in $K(A) \cap X(p_1, p_2)$, let $q$ be the support projection of $x^{**}x$. Then $q$ is open, $q \leq p_2$, and $x \in X(p_1, q) \subset K(A)$. This provides a substitute for $L_x$, as used in the proof of 4.5. ($L_x = X(1, q)$.)

3. If $I$ is an ideal of $A$, then $K(A) \cap I$ may be strictly larger then $K(I)$. Thus if we have a direct sum decomposition, $A_0 \oplus A_1$, as above, it is not obvious how to describe the domain of $T$ in terms of $A_0$ and $A_1$. Nevertheless, the fact that
$K(A)$ is a union of closed left ideals can be used to show that the domain of $T$ is compatible with the direct sum decomposition. Also, writing $A_1 = \bigoplus \mathcal{K}(H_i)$ and using the same idea, we can show that every element of $K(A) \cap X(p_1, p_2) \cap A_1$ has only finitely many non-zero components (provided each $t_i$ is discontinuous). If this were false a construction like the one in the last paragraph of the proof of 4.3 could be carried out (within the set $X(p_1, q)$ of point 2 above) to give a contradiction. (Thus, \textit{a posteriori}, $K(A) \cap X(p_1, p_2) \cap A_1 \subset K(A_1)$.}
Centralizers of Pedersen’s Ideal

References

1. C.A. Akemann, *The general Stone-Weierstrass problem*, J. Funct. Anal. 4 (1969), 277-294.
2. ———, *Left ideal structure of C*-algebras*, J. Funct. Anal. 6 (1970), 305-317.
3. ———, *A Gelfand representation theory for C*-algebras*, Pac. J. Math. 39 (1971), 1-11.
4. C.A. Akemann, J. Anderson, and G.K. Pedersen, *Approaching infinity in C*-algebras*, J. Operator Theory 21 (1989), 255-271.
5. C.A. Akemann, G.K. Pedersen, and J. Tomiyama, *Multipliers of C*-algebras*, J. Funct. Anal. 13 (1973), 277-301.
6. O. Bratteli, G.A. Elliott, and D.E. Evans, *Locality and differential operators on C*-algebras*, J. Diff. Eqs. 64 (1986), 221-273.
7. L.G. Brown, *Close hereditary C*-subalgebras and the structure of quasi-multipliers*, MSRI preprint #11211-85.
8. ———, *Semicontinuity and multipliers of C*-algebras*, Canad. J. Math. 40 (1988), 865-988.
9. ———, *A large C*-algebra of universally measurable elements*, in preparation.
10. R.C. Busby, *Double centralizers and extensions of C*-algebras*, Trans. Amer. Math. Soc. 171 (1972), 195-234.
11. F. Combes, *Sur les faces d’une C*-algèbre*, Bull. Sci. Math. 93 (1969), 37-62.
12. ———, *Quelques propriétés des C*-algèbres*, Bull. Sci. Math. 94 (1970), 165-192.
13. J. Dixmier, *Les C*-algèbres et leurs représentations*, Gauthier-Villars, Paris, 1964.
14. E.G. Effros, *Order ideals in a C*-algebra and its dual*, Duke Math. J. 30 (1963), 391-412.
15. B.E. Johnson, *An introduction to the theory of centralizers*,Proc. London Math. Soc. 14 (1964), 299-320.
16. R.V. Kadison, *Irreducible operator algebras*, Proc. Nat. Acad. Sci. USA 43 (1957), 273-276.
17. Hyoungsoon Kim, *Semicontinuity for unbounded operators affiliated with operator algebras*, doctoral dissertation, Purdue University (1992).
18. A.J. Lazar and D.C. Taylor, *Multipliers of Pedersen’s ideal*, Memoirs Amer. Math. Soc. 169 (1976).
19. W.A. J. Luxemburg, *Some aspects of the theory of Riesz spaces*, Univ. of Arkansas LNM 4 (1979).
20. M. Neumann and V. Ptak, *Automatic continuity, local type, and causality*, Studia Math. 82 (1985), 61-90.
21. G. K. Pedersen, *Applications of weak* * semicontinuity in C*-algebra theory*, Duke Math. J. 39 (1972), 431-450.
22. ———, *C*-algebras and their automorphism groups*, Academic Press, London, 1979.
23. J. Peetre, *Une caractérisation abstraite des opérateurs différentiels*, Math. Scand. 7 (1959), 211-218.
24. ———, *Récifiation à l’article “Une caractérisation abstraite des opérateurs différentiels”*, Math. Scand. 8 (1960), 116-120.
25. N.C. Phillips, *A new approach to the multipliers of Pedersen’s ideal*, Proc. Amer. Math. Soc. 104 (1988), 861-867.
26. *The William Lowell Putnam Mathematical Competition, Problems and Solutions: 1965-1984*, Ed. G.L. Alexanderson, L.F. Klosinski, L.C. Larson, Math. Assoc. of Amer. (1985).
27. Ngai-ching Wong, *The left quotient of a C*-algebra and its representation through a continuous field of Hilbert spaces*, doctoral dissertation, Purdue University (1991).