SOME OPEN PROBLEMS ON LOCALLY FINITE OR LOCALLY NILPOTENT DERIVATIONS AND E-DERIVATIONS

WENHUA ZHAO

Abstract. Let $R$ be a commutative ring and $A$ an $R$-algebra. An $R$-$E$-derivation of $A$ is an $R$-linear map of the form $I - \phi$ for some $R$-algebra endomorphism $\phi$ of $A$, where $I$ denotes the identity map of $A$. In this paper we discuss some open problems on whether or not the image of a locally finite $R$-derivation or $R$-$E$-derivation of $A$ is a Mathieu subspace $[\mathbb{Z}_2, \mathbb{Z}_3]$ of $A$, and whether or not a locally nilpotent $R$-derivation or $R$-$E$-derivation of $A$ maps every ideal of $A$ to a Mathieu subspace of $A$. We propose and discuss two conjectures which state that both questions above have positive answers if the base ring $R$ is a field of characteristic zero. We give some examples to show the necessity of the conditions of the two conjectures, and discuss some positive cases known in the literature. We also show some cases of the two conjectures. In particular, both the conjectures are proved for locally finite or locally nilpotent algebraic derivations and $R$-$E$-derivations of integral domains of characteristic zero.

1. Introduction

Let $R$ be a unital ring (not necessarily commutative) and $A$ an $R$-algebra. We denote by $1_A$ or simply $1$ the identity element of $A$, if $A$ is unital, and $I_A$ or simply $I$ the identity map of $A$, if $A$ is clear in the context.

An $R$-linear endomorphism $\eta$ of $A$ is said to be locally nilpotent (LN) if for each $a \in A$ there exists $m \geq 1$ such that $\eta^m(a) = 0$, and locally finite (LF) if for each $a \in A$ the $R$-submodule spanned by $\eta^i(a)$ ($i \geq 0$) over $R$ is finitely generated.
By an $R$-derivation $D$ of $\mathcal{A}$ we mean an $R$-linear map $D : \mathcal{A} \to \mathcal{A}$ that satisfies $D(ab) = D(a)b + aD(b)$ for all $a,b \in \mathcal{A}$. By an $R$-$\mathcal{E}$-derivation $\delta$ of $\mathcal{A}$ we mean an $R$-linear map $\delta : \mathcal{A} \to \mathcal{A}$ such that for all $a,b \in \mathcal{A}$ the following equation holds:

\[
\delta(ab) = \delta(a)b + a\delta(b) - \delta(a)\delta(b) \tag{1.1}
\]

It is easy to verify that $\delta$ is an $R$-$\mathcal{E}$-derivation of $\mathcal{A}$, if and only if $\delta = I - \phi$ for some $R$-algebra endomorphism $\phi$ of $\mathcal{A}$. Therefore an $R$-$\mathcal{E}$-derivation is a special so-called $(s_1,s_2)$-derivation introduced by N. Jacobson [J] and also a special semi-derivation introduced by J. Bergen in [BE]. $R$-$\mathcal{E}$-derivations have also been studied by many others under some different names such as $f$-derivations in [E1, E2] and $\phi$-derivations in [BFF, BV], etc.

We denote by $\text{End}_R(\mathcal{A})$ the set of all $R$-algebra endomorphisms of $\mathcal{A}$, $\text{Der}_R(\mathcal{A})$ the set of all $R$-derivations of $\mathcal{A}$, and $\mathcal{E}\text{Der}_R(\mathcal{A})$ the set of all $R$-$\mathcal{E}$-derivations of $\mathcal{A}$. Furthermore, for each $R$-linear endomorphism $\eta$ of $\mathcal{A}$ we denote by $\text{Im} \, \eta$ the image of $\eta$, i.e., $\text{Im} \, \eta := \eta(\mathcal{A})$, and $\text{Ker} \, \eta$ the kernel of $\eta$. When $\eta$ is an $R$-derivation or $R$-$\mathcal{E}$-derivation, we also denote by $\mathcal{A}^\eta$ the kernel of $\eta$.

For each $R$-derivation or $R$-$\mathcal{E}$-derivation of an $R$-algebra $\mathcal{A}$, it is easy to see that the kernel $\mathcal{A}^\delta$ is an $R$-subalgebra. Actually, if $\delta = I - \phi$ for some $\phi \in \text{End}_R(\mathcal{A})$, the kernel $\mathcal{A}^\delta$ of $\delta$ coincides with the $R$-subalgebra of the elements of $\mathcal{A}$ that are fixed by $\phi$. The kernels of derivations as well as the kernels of $\mathcal{E}$-derivations (i.e., the subalgebra fixed by algebra endomorphisms) are among the most studied subjects and play important roles in various different areas (e.g., see [N], [E], [E2] and the references therein).

On the other hand, the images, especially, their possible algebraic structures, of derivations or $\mathcal{E}$-derivations have barely been studied. It is presumably because that in general they are not even closed under the multiplication of the algebra. However, recent studies (e.g., see [EWZ], [Z4]–[Z7]) show that the images of certain derivations and $\mathcal{E}$-derivations do possess some algebraic structure. To be more precise, we first need to recall the following notion introduced in [Z2, Z3].

**Definition 1.1.** Let $\vartheta$ represent the words: left, right, or two-sided. An $R$-subspace $V$ of an $R$-algebra $\mathcal{A}$ is said to be a $\vartheta$-Mathieu subspace ($\vartheta$-$\text{MS}$) of $\mathcal{A}$ if for all $a,b,c \in \mathcal{A}$ with $a^m \in V$ for all $m \geq 1$, the following conditions hold:

1) $ba^m \in V$ for all $m \gg 0$, if $\vartheta =$ left;
2) $a^mc \in V$ for all $m \gg 0$, if $\vartheta =$ right;
3) $ba^mc \in V$ for all $m \gg 0$, if $\vartheta =$ two-sided.
A two-sided MS will also be simply called a MS. For an arbitrary ring $B$, the $\vartheta$-MSs of $B$ are defined by viewing $B$ as an algebra over $\mathbb{Z}$. Some more remarks on the notion of MS are as follows.

First, the introduction of the notion in [Z2] and [Z3] is mainly motivated by the Mathieu conjecture in [MO] and the Image conjecture in [Z1], both of which are motivated by and also imply the well-known Jacobian conjecture that was first proposed by O. H. Keller in 1939 [Ke]. See also [BCW] and [E2]. But, a more interesting aspect of the new notion is that it provides a natural but highly non-trivial generalization of the corner-stone notion of ideals of associative algebras.

Second, a Mathieu subspace is also called a Mathieu-Zhao space in the literature (e.g., see [DEZ, EN, EH], etc.) as first suggested by A. van den Essen [E3].

Third, the following notion, first introduced in [Z3], is closely related with MSs, although it is defined for all $R$-subspaces, or even arbitrary subsets, of $R$-algebras.

**Definition 1.2.** [Z3, p. 247] Let $V$ be an $R$-subspace of an $R$-algebra $A$. We define the radical of $V$, denoted by $r(V)$, to be the set of $a \in A$ such that $a^m \in V$ for all $m \gg 0$.

When $A$ is commutative and $V$ is an ideal of $A$, $r(V)$ coincides with the radical of $V$. So this new notion is also interesting on its own right. It is also crucial for the study of MSs. For example, the following lemma can be easily verified, and will be frequently used (implicitly) in this paper.

**Lemma 1.3.** Let $V$ be an $R$-subspace of an $R$-algebra $A$, and $I$ an ideal of $A$. If $I \subseteq V$ and $r(I) = r(V)$, then $V$ is a MS of $A$.

Now we propose the following problems on the image of derivations and $\varepsilon$-derivations.

**Problem 1.4 (LFNED Problem).** Let $R$ be a commutative base ring, $A$ an $R$-algebra and $\delta$ an $R$-derivation or $R$-$\varepsilon$-derivation of $A$.

A) Find the radical of $\delta(I)$ for all one-sided or two sided ideals $I$ of $A$.

B) Decide which $R$-derivations and $R$-$\varepsilon$-derivations of $A$ have the image being a $\vartheta$-MS of $A$.

C) Decide which $R$-derivations and $R$-$\varepsilon$-derivations of $A$ map every $\vartheta$-ideal of $A$ to a $\vartheta$-MS of $A$.

Although the sufficient and necessary conditions for Problem B) and C) are currently far from being clear, based on the studies in [EWZ],
as well as some results that will be shown later in this paper, the following two conjectures seem to be more plausible.

**Conjecture 1.5 (The LFED Conjecture).** Let $K$ be a field of characteristic zero and $A$ a $K$-algebra. Then for every locally finite $K$-derivation or locally finite $K$-$\mathcal{E}$-derivation $\delta$ of $A$, the image $\text{Im} \, \delta := \delta(A)$ of $\delta$ is a (two-sided) MS of $A$.

**Conjecture 1.6 (The LNED Conjecture).** Let $K$ be a field of characteristic zero, $A$ a $K$-algebra and $\delta$ a locally nilpotent $K$-derivation or a locally nilpotent $K$-$\mathcal{E}$-derivation of $A$. Then for every $\vartheta$-ideal $I$ of $A$, the image $\delta(I)$ of $I$ under $\delta$ is a $\vartheta$-MS of $A$.

Throughout this paper we refer to the two conjectures above as the LFED conjecture and the LNED conjecture, respectively.

One motivation of the two conjectures above is that they may provide some new understandings on the LF or LN derivations and $\mathcal{E}$-derivations. Another motivation is that they may produce many non-trivial examples of MSs, which will be beneficial and essential toward the further development of the desired theory of MSs.

Two more remarks on the two conjectures above are as follows. Below we let $K$ be a field of characteristic zero and $A$ a $K$-algebra, unless stated otherwise.

First, by van den Essen’s one-to-one correspondence (see [E1] or [E2, Proposition 2.1.3]) between the set of LN $K$-derivations of $A$ and the set of LN $K$-$\mathcal{E}$-derivations of $A$ and also [Z4, Corollary 2.4], the LN $K$-derivation case and the LN $K$-$\mathcal{E}$-derivation case of Conjecture 1.5 are equivalent to each other. In other words, Conjecture 1.5 holds for all LN $K$-derivations of $A$, if and only if it holds for all LN $K$-$\mathcal{E}$-derivations of $A$.

Second, for every $\vartheta$-MS $V$ of $A$ and idempotent $e \in V$ (i.e., $e^2 = e$), by Definition 1.1 it is easy to see that the principal $\vartheta$-ideal $(e)$ of $A$ generated by $e$ is contained in $V$. Therefore, we have the following weaker versions of Conjectures 1.5 and 1.6.

**Conjecture 1.7 (The Idempotent Conjecture).** Let $K$ be a field of characteristic zero, $A$ a $K$-algebra and $\delta$ a $K$-derivation or $K$-$\mathcal{E}$-derivation of $A$. Then the following two statements hold:

**A)** If $\delta$ is LF, then for all idempotents $e \in \text{Im} \, \delta$, the principle ideal $(e)$ is contained in $\text{Im} \, \delta$;

**B)** If $\delta$ is LN, then for all $\vartheta$-ideal $I$ of $A$ and all idempotents $e \in \delta(I)$, the $\vartheta$-ideal $(e)$ is contained in $\delta(I)$.

Actually, if $A$ is algebraic over $K$, the statements A) and B) in Conjecture 1.7 are respectively equivalent to the LFED conjecture and
the LNED conjecture, due to the following characterization of MSs of \( A \), which is a special case of [Z3, Theorem 4.2].

**Theorem 1.8.** Let \( K \) be a field (of arbitrary characteristic) and \( A \) a \( K \)-algebra that is algebraic over \( K \). Then a \( K \)-subspace \( V \) of \( A \) is a \( \vartheta \)-MS of \( A \), if and only if for every idempotent \( e \in V \), the principal \( \vartheta \)-ideal \( (e)_{\vartheta} \) of \( A \) generated by \( e \) is contained in \( V \).

**Arrangement:** In Section 2, we mainly give some examples to show the necessity of the conditions in the LFED conjectures 1.5 and the LNED conjectures 1.6. We also give some positive examples with certain weaker conditions. In Section 3 we discuss some positive cases of Conjectures 1.5, 1.6 and 1.7, which are either already known in the literature or can be derived from some other results in the literature.

In Section 4 we discuss the LFED conjectures 1.5 in terms of the decompositions of the \( K \)-algebra \( A \) associated with the Jordan-Chevalley decomposition of the LF \( K \)-derivations and \( K \)-\( E \)-derivations of \( A \). Two other conjectures (see Conjectures 4.1 and 4.4) that are closely related with the semi-simple case of the LFED conjecture are also proposed and discussed.

In Section 5 we show that the LFED conjecture 1.5 holds for \( E \)-derivations associated with some special algebra endomorphisms such as projections and involutions, etc.. In Section 6 we study the LFED conjectures 1.5 and the LNED conjectures 1.6 for algebraic derivations and \( E \)-derivations of domains of characteristic zero. In particular, for integral domains \( A \) of characteristic zero we show that both conjectures hold for LF or LN algebraic derivations and \( E \)-derivations of \( A \) (see Proposition 6.8 and Theorem 6.9).

**Acknowledgment:** The author is very grateful to Professor Arno van den Essen for reading carefully an earlier version of the paper and pointing out some mistakes and typos, and in particular, for sending the author some counter-examples for an earlier (and stronger) version of Conjecture 4.1.

2. Some Examples and Necessity of the Conditions of the LFED and LNED Conjectures

In this section we give some examples to show that the conditions in the LFED conjecture 1.5 and the LNED conjecture 1.6 are necessary. We also give some (positive) examples with some weaker conditions.

Throughout this section \( K \) stands for a field of characteristic zero. All the notations introduced in the previous section will also be in force.
First, the following two examples show that the LF (locally finite) condition is necessary for both the LFED and LNED conjectures.

**Example 2.1.** [EWZ, Example 2.4] Let \( x \) and \( y \) be two commutative free variables and \( D = \partial / \partial x - y^2 \partial / \partial y \). Then \( D \) is not LF and \( \text{Im} \ D \) is not a MS of the polynomial algebra \( K[x, y] \).

**Example 2.2.** Let \( x, y \) be two commutative variables and \( \phi \) the \( K \)-algebra endomorphism of \( K[x, y] \) such that \( \phi(x) = x + 1 \) and \( \phi(y) = y^2 \). Set \( \delta := I - \phi \). Then it is easy to see that \( \delta \) is not LF, and \( \text{Im} \ \delta \neq C[x, y] \), since each \( f \in \text{Im} \ \delta \) with \( \deg_y f \geq 1 \) must have even degree in \( y \). On the other hand, \( 1 = \delta(-x) \in \text{Im} \ \delta \). Then it is easy to check (or by [Z2, Lemma 4.5]) that \( \text{Im} \ \delta \) is not a MS of \( K[x, y] \).

Next, the following two examples show that the LN (locally nilpotent) condition in the LNED conjecture 1.6 is necessary and cannot be replaced by the LF (locally finite) condition.

**Example 2.3.** [Z7, Example 2.4] Let \( x \) be a free variable, \( D = x \frac{d}{dx} \) and \( I = (x^2 - 1)K[x] \). Then \( D \) is LF but the image \( DI \) of \( I \) under \( D \) is not a MS of \( K[x] \).

**Example 2.4.** [Z7, Example 3.6] Let \( K, x, I \) be as in Example 2.3, and \( 0 \neq q \in K \) that is not a root unity. Let \( \phi \in \text{End}_K(K[x]) \) that maps \( x \) to \( qx \) and \( \delta := I - \phi \). Then \( \delta \) is LF but the image \( \delta I \) of \( I \) under \( \delta \) is not a MS of \( K[x] \).

The following two examples show that the base field \( K \) in the LFED and LNED Conjectures cannot be replaced by a field of characteristic \( p > 0 \).

**Example 2.5.** [Z1, Example 2.7] Let \( \mathbb{F} \) be a field of characteristic \( p > 0 \), \( x \) a free variable and \( D := d/dx \). Then \( D \) is LN but \( \text{Im} \ D \) is not a MS of \( \mathbb{F}[x] \).

**Example 2.6.** Let \( \mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z} \), \( x \) a free variable, \( \mathcal{A} = \mathbb{F}_2[x^{-1}, x] \), and \( \phi \in \text{End}_{\mathbb{F}_2}(\mathcal{A}) \) that maps \( x \) to \( x^{-1} \). Then \( \delta := I - \phi \) is LF but \( \text{Im} \ \delta \) is not MS of \( \mathcal{A} \).

**Proof:** It is easy to see that \( \delta \) is LF. To show the second statement, let \( u_m := \delta(x^m) = x^m + x^{-m} \) for all \( m \geq 1 \), and \( V \) be the \( \mathbb{F}_2 \)-subspace of \( \mathcal{A} \) spanned by \( u_m \) (\( m \geq 1 \)). Then \( V \subseteq \text{Im} \ \delta \). By the binomial formula (over \( \mathbb{F}_2 \)) and the fact \( \binom{m}{i} = \binom{m}{m-i} \) for all \( 0 \leq i \leq m \), it is easy to see that for all \( k \geq 1 \), we have \( (x + x^{-1})^{2k-1} \in V \) and \( (x + x^{-1})^{2k} \equiv \binom{2k}{k} \mod V. \)
Note that \(^{(2k)}\) is even, which can actually be seen by letting \(x = 1\) in the equation above. Therefore, \((x + x^{-1})^m \in V \subseteq \text{Im} \delta\) for all \(m \geq 1\).

On the other hand, since \(\phi^2 = 1\), we have \(\phi \delta = \delta\). Therefore, every \(f \in \text{Im} \delta\) is fixed by \(\phi\), i.e., \(\phi(f) = f\). By this fact we see that \(x(x + x^{-1})^m \notin \text{Im} \delta\) for all \(m \geq 1\). Hence \(\text{Im} \delta\) is not a MS of \(\mathcal{A}\). \(\square\)

Although the LFED and LNED Conjectures can not be extended to all the algebras over a field of characteristic \(p > 0\) (as shown by the two examples above), the following example shows that the LFNED problem \([4,4]\) is still interesting for some of these algebras.

**Example 2.7.** Let \(p \geq 2\) be a prime, \(\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}\), \(x\) a free variable and \(\phi\) the Frobenius endomorphism of \(\mathbb{F}_p[x]\), i.e., \(\phi(f) = f^p\) for all \(f \in \mathbb{F}_p[x]\). Set \(\delta := 1 - \phi\). Then \(\mathfrak{r}(\text{Im} \delta) = \{0\}\). Consequently, \(\delta\) maps every \(\mathbb{F}_p\)-subspace of \(\mathbb{F}_p[x]\) to a MS of \(\mathbb{F}_p[x]\).

Since \(a^p = a\) for all \(a \in \mathbb{F}_p\), \(\phi\) is actually an \(\mathbb{F}_p\)-algebra endomorphism of \(\mathbb{F}_p[x]\). Then the conclusion of the example follows from \([Z7]\) Proposition 3.7. But, for the sake of completeness we include here a more straightforward proof.

**Proof of Example 2.7** We first show \(1 \notin \text{Im} \delta\). Otherwise, let \(f \in \mathbb{F}_p[x]\) such that \(\delta f = f - \phi(f) = f(x) - f^p(x) = 1\). Then \(\deg f = 0\), i.e., \(f \in \mathbb{F}_p\). But in this case \(f^p = f\), whence \(f - \phi(f) = 0\). Contradiction.

Now assume \(\mathfrak{r}(\text{Im} \delta) \neq \{0\}\) and let \(0 \neq f(x) \in \mathfrak{r}(\text{Im} \delta)\). Then \(\deg f \geq 1\). Replacing \(f\) by a power of \(f\) we assume \(f^m \in \text{Im} \delta\) for all \(m \geq 1\). Let \(h_m \in \mathbb{F}_p[x]\) \((m \geq 1)\) such that

\[
(2.1) \quad f^m(x) = \delta h_m = h_m(x) - h_m^p(x).
\]

If \(f'(x) = \frac{df}{dx}(x) = 0\), then \(f(x) = \tilde{f}(x^p)\) for some \(\tilde{f}(x) \in \mathbb{F}_p[x]\). By the equation above with \(m = 1\) we have \(h'_1(x) = 0\), whence \(h_1(x) = \tilde{h}_1(x^p)\) for some \(\tilde{h}_1(x) \in \mathbb{F}_p[x]\). The equation above with \(m = 1\) becomes \(\tilde{f}(x^p) = \tilde{h}_1(x^p) - h_1^p(x^p)\). Replacing \(x^p\) by \(x\) we have \(\tilde{f}(x) = \tilde{h}_1(x) - h_1^p(x) \in \text{Im} \delta\).

Applying the same arguments to \(f^m (m \geq 2)\) we see that there exists \(\tilde{f}_m(x) \in \text{Im} \delta\) such that \(\tilde{f}_m(x^p) = f^m(x) = \tilde{f}_m(x^p)\). Hence \(\tilde{f}_m(x) = \tilde{f}^m(x)\) and \(\tilde{f}^m(x) \in \text{Im} \delta\) for all \(m \geq 1\). Note that \(\deg f > \deg \tilde{f} \geq 1\) (since \(\deg f \geq 1\)). Therefore, replacing \(f\) by \(f\) and repeating the same procedure, if necessary, we may assume \(f'(x) \neq 0\). Consequently, by Eq. \((2.1)\) we also have \(\deg h_m \geq 1\) for all \(m \geq 1\).
Now assume $p > 2$. Then $f', (f^2)' \neq 0$, and by Eq. (2.1) with $m = 1, 2$ we have $h_1', h_2' \neq 0$ and

\[(2.2) \quad h_2(x) - h_2^p(x) = (h_1(x) - h_1^p(x))^2 = h_1^2(x) - 2h_1^{p+1}(x) + h_1^{2p}(x).\]

Applying $d/dx$ to the equation above we get

\[(2.3) \quad h_2'(x) = 2h_1(x)h_1'(x) - 2h_1^p(x)h_1'(x).\]

On the one hand, by the two equations above we have

\[(2.4) \quad \deg h_2 = 2 \deg h_1,\]
\[(2.5) \quad \deg h_2' = p \deg h_1 + \deg h_1',\]

which imply

\[p \deg h_1 \leq \deg h_2 \leq \deg h_2' \leq \deg h_2 - 1 = 2 \deg h_1 - 1.\]

Since $\deg h_1 \geq 1$, we get $p < 2$, which is a contradiction.

Therefore, we have $p = 2$. But in this case $f', (f^3)' \neq 0$, and by Eq. (2.1) with $m = 1, 3$ we have $h_1', h_3' \neq 0$ and

\[(2.6) \quad 2 \deg h_3 = 3 \deg f,\]

Applying $d/dx$ to Eq. (2.1) with $m = 3$ and $p = 2$ we get

\[(2.7) \quad \deg h_3'(x) = 2 \deg f + \deg f',\]

By the two equations above we have $\deg f = 2/3 \deg h_3$ and $\deg h_3' = 4/3 \deg h_3 + \deg f' > \deg h_3$, which is a contradiction again. Therefore

\[r(\text{Im } \delta) = \{0\} \quad \square\]

Next, the following example shows that the base field in the LFED Conjecture can not be replaced by an integral domain of characteristic zero.

**Example 2.8.** Let $t, x, y$ be commutative free variables; $R = \mathbb{C}[t^{-1}, t]; A = R[x, y]$; and $\phi \in \text{End}_R(A)$ that maps $x \to 2x$ and $y \to ty$. Then it is easy to verify that $1 - \phi$ is LF and Im $(1 - \phi)$ is the $R$-subspace spanned by $(1 - 2^a t^b)x^a y^b$ for all $a, b \in \mathbb{Z}$. In particular, $x^m \in \text{Im } (1 - \phi)$ for all $m \geq 1$, since $(1 - 2^m t)$ is invertible in $R$. But for all $m \geq 1$, $x^m y \notin \text{Im } (1 - \phi)$, since $(1 - 2^m t)$ is not invertible in $R$. Therefore, \text{Im } (1 - \phi) is not a MS of $A$.

On the other hand, the following example shows that Problem 1.4 is also interesting for some algebras over an integral domain.
Example 2.9. Let $a \in \mathbb{Z}$, $x$ be a free variable, and $\phi_a$ the $\mathbb{Z}$-algebra endomorphism of $\mathbb{Z}[x]$ that maps $x$ to $ax$. Then $\text{Im}(I - \phi_a)$ is the $\mathbb{Z}$-subspace spanned by $(1 - a^n)x^n$ for all $n \geq 1$. More precisely,

\begin{equation}
\text{Im}(I - \phi_a) = \begin{cases} 
x\mathbb{Z}[x] & \text{if } a = 0; \\
\{0\} & \text{if } a = 1; \\
2x\mathbb{Z}[x^2] & \text{if } a = -1; \\
\text{Span}_\mathbb{Z}\{(1 - a^n)x^n \mid n \geq 1\} & \text{otherwise.}
\end{cases}
\end{equation}

The radical of $\text{Im}(I - \phi)$ is given by

\begin{equation}
\mathfrak{r}(\text{Im}(I - \phi_a)) = \begin{cases} 
x\mathbb{Z}[x] & \text{if } a = 0; \\
\{0\} & \text{otherwise.}
\end{cases}
\end{equation}

Consequently, for all $a \in \mathbb{Z}$ the image $\text{Im}(I - \phi_a)$ is MS of $\mathbb{Z}[x]$.

**Proof:** Eq. (2.8) is obvious and the last statement can be easily verified by Eq. (2.9) and Definition 1.1. To show Eq. (2.9), the cases $a = 0, \pm 1$ are straightforward. So we assume $|a| \geq 2$. Note that $1 - a^n$ in this case is not invertible in $\mathbb{Z}$ for any $n \geq 1$. Note also that $\text{Im}(I - \phi)$ is a homogeneous $\mathbb{Z}$-subspace or $\mathbb{Z}$-submodule of $\mathbb{Z}[x]$.

Let $u \in \mathfrak{r}(\text{Im}(I - \phi))$. Then $\deg u \geq 1$, for $\text{Im}(I - \phi)$ obviously does not contain any nonzero constant. Replacing $u$ by a power of $u$ we assume $u^m \in \text{Im}(I - \phi)$ for all $m \geq 1$. Let $bx^d$ be the leading term of $u$. Then $(bx^d)^m \in \text{Im}(I - \phi)$ for all $m \geq 1$, whence $(1 - a^{md}) | b^m$. Set $\tilde{a} := a^d$. Then $(1 - \tilde{a}^m) | b^m$ for all $m \geq 1$. Consequently, there are only finitely many distinct primes $p$ such that $p$ divides $1 - \tilde{a}^m$ for some $m \geq 1$.

On the other hand, for all co-prime $m, k \geq 1$ there exist $u(t), v(t) \in \mathbb{Q}[t]$ such that

\[(1 - t^m)u(t) + (1 - t^k)v(t) = 1 - t.\]

Furthermore, by going through the Euclidean algorithm for $1 - t^m$ and $1 - t^k$ it is easy to see that we can actually choose $u(t), v(t) \in \mathbb{Z}[t]$.

Replacing $t$ by $\tilde{a}$ in the equation above we see that the integers $| (1 - \tilde{a}^m)/(1 - \tilde{a}) |$ and $| (1 - \tilde{a}^k)/(1 - \tilde{a}) |$ are co-prime for all co-prime $m, k \geq 1$. Since for all distinct $m \geq 1$ the integers $1 - \tilde{a}^m$ are all distinct, it is easy to see that there are infinitely many distinct primes $p$ such that $p$ divides $1 - \tilde{a}^m$ for some $m \geq 1$. Contradiction. \qed

3. Some Known Cases of the LFED and LNED Conjectures

In this section we discuss some cases of conjectures 1.5, 1.6 and 1.7 that either are known in the literature or can be proved from some
results in the literature. Throughout this section $K$ denotes a field of characteristic zero and $A$ a $K$-algebra.

We start with the following example. Although it is trivial, it can be read as a first positive sign for the LFED conjecture.

**Example 3.1.** Let $R$ be a unital commutative ring containing $\mathbb{Q}$, $x$ a free variable and $D$ an arbitrary $R$-derivation of the polynomial algebra $R[x]$. Write $D = a(x)\frac{d}{dx}$ for some $a(x) \in R[x]$. Then $\text{Im} D$ is the principal ideal of $R[x]$ generated by $a(x)$, and hence a MS of $R[x]$.

Furthermore, for the univariate polynomial algebra $K[x]$ the following theorem is proved in [Z7].

**Theorem 3.2.** 1) The LFED conjecture holds for all $K$-derivations and $K$-$\mathcal{E}$-derivations (not necessarily LF) of $K[x]$.

2) The LNED conjecture holds for all $LN$ $K$-derivations of $K[x]$.

3) The LNED conjecture holds for all $LN$ $K$-$\mathcal{E}$-derivations $\delta$ of $K[x]$ and the ideals $I$ of $K[x]$ that are generated by a polynomial $u \in K[x]$ with either $u = 0$, or $\deg u \leq 1$, or $u$ has at least one repeated root in the algebraic closure of $K$.

For multivariate polynomial algebras the following theorem is proved in [EWZ], which can be re-stated as follows.

**Theorem 3.3.** [EWZ Theorem 3.1] The LFED conjecture holds for all LF $K$-derivations of the polynomial algebra over $K$ in two commutative free variables.

For (multivariate) Laurent polynomial algebras the following theorem is proved in [Z6].

**Theorem 3.4.** Let $x = (x_1, x_2, \ldots, x_n)$ be $n$ commutative free variable and $K[x^{-1}, x]$ the Laurent polynomial algebra in $x$ over $K$. Then the following statements hold:

1) $K[x^{-1}, x]$ has no nonzero locally nilpotent $K$-derivations or $K$-$\mathcal{E}$-derivations. Hence, the LNED conjecture holds (trivially) for $K[x^{-1}, x]$.

2) if $n \leq 2$, then the LFED conjecture holds for $K[x^{-1}, x]$.

Next, we discuss some cases of the LFED and LNED conjectures for algebraic $K$-algebras. First, we have the following

**Theorem 3.5.** Both the LFED conjecture and the LNED conjecture hold for all local $K$-algebras $A$ that are algebraic over $K$.

**Proof:** Note first that by [Z3, Theorem 7.6] the proper MSs of $A$ are characterized as follows:
Let $\delta$ be a $K$-derivation or $K$-$E$-derivation of $A$. Assume first that $\delta$ is LF. If $1 \notin \text{Im} \delta$, then by the fact $(\ast)$ above $\text{Im} \delta$ is a MS of $A$. If $1 \in \text{Im} \delta$, then $\text{Im} \delta = A$ by [Z4, Proposition 1.4]. Hence the LFED conjecture holds for $A$.

Now assume that $\delta$ is LN. Then $1 \in A$ if $\delta \in \text{Der}_K(A)$, and by [E2, Proposition 2.1.3] and [Z4, Corollary 2.4], it is also the case if $\delta \in \text{Eder}_K(A)$. If $1 \in \text{Im} \delta$, then $\delta s = 1$ for some $s \in A$. Then by [Z4, Proposition 3.2] $s$ is transcendental over $K$. Contradiction.

Therefore $1 \notin \text{Im} \delta$. Then by the fact $(\ast)$ above $\delta$ actually maps every $K$-subspace of $A$ to a MS of $A$. In particular, the LNED conjecture holds for $A$ as well. $\blacksquare$

Next, the following two theorems are proved in [Z5].

**Theorem 3.6.** Both the LFED conjecture and the LNED conjecture hold for all finite dimensional $K$-algebras.

**Theorem 3.7.** Let $A$ be a $K$-algebra such that every $K$-subalgebra generated by finitely many elements of $A$ is finite dimensional, and $\delta$ a $K$-derivation or $K$-$E$-derivation of $A$. Then the following statements hold:

1) if $\delta$ is LN, then $\delta$ maps every $K$-subspace of $A$ to a MS of $A$. In particular, the LNED conjecture holds for $A$;

2) if $\delta$ is a LF $K$-derivation, or a LF $K$-$E$-derivation of the form $\delta = 1 - \phi$ for some surjective $\phi \in \text{End}_K(A)$, then $\text{Im} \delta$ is a MS of $A$, i.e., the LFED conjecture holds for $\delta$.

For commutative algebraic $K$-algebras we here give a different proof for the proposition below, which is stronger for the $K$-derivation case than that of the theorem above.

**Proposition 3.8.** Let $A$ be a commutative $K$-algebra that is algebraic over $K$, and $\delta$ an arbitrary $K$-derivation, or a LN $K$-$E$-derivation of $A$. Then $\text{Im} \delta$ does not contain any nonzero idempotent of $A$. Consequently, $\delta$ maps every $K$-subspace of $A$ to a MS of $A$.

**Proof:** Let $e \in \text{Im} \delta$ be an idempotent, $u \in A$ such that $\delta u = e$, and $p(t) = t^d + \sum_{i=0}^{d-1} c_i t^i \in K[t]$ the minimal polynomial of $u$.

We first consider the $K$-derivation case. Let $\delta = D \in \text{Der}_K(A)$. Since $A$ is commutative, $De = De^2 = 2eDe$, whence $(1 - 2e)De = 0$. Note that $1 - 2e$ is a unit of $A$, for $(1 - 2e)^2 = 1$. Hence $De = 0$. Consequently, $D^k(u^k) = k! e$ and $D^m(u^k) = 0$ for all $m > k \geq 1$. Therefore $0 = D^d p(u) = de$, whence $e = 0$. Therefore $\text{Im} D$ does not
contain any nonzero idempotent of \( \mathcal{A} \), and by Theorem 1.8, \( D \) maps every \( K \)-subspace of \( \mathcal{A} \) to a MS of \( \mathcal{A} \).

To show the \( K \)-derivation case, by a similar argument as above it suffices to show that \( \delta e = 0 \) and \( \delta^k(u^k) = k!e \) for all \( k \geq 1 \).

First, by van den Essen’s one-to-one correspondence between the set of LN \( K \)-derivations of \( \mathcal{A} \) and the set of LN \( K \)-\( E \)-derivations of \( \mathcal{A} \) (see [E1] or [E2, Proposition 2.1.3]), there exists a LN \( K \)-derivation \( D \) of \( \mathcal{A} \) such that \( \delta = I - eD = \sum_{i=1}^{\infty} \frac{D^i}{i!} \). By the argument in the first part of the second paragraph of this proof we have \( De = 0 \), regardless \( e \in \text{Im} \, D \) or not. Hence we also have \( \delta e = 0 \).

Next, we use the induction to show that \( \delta^k(u^k) = k!e \) for all \( k \geq 1 \).

The case \( k = 1 \) is trivial. Assume that \( \delta^i(u^i) = i!\delta^i(u) \) for all \( 1 \leq i \leq k \). It is easy to check inductively that for all \( x, y \in \mathcal{A} \) and \( n \geq 1 \), we have

\[
\delta^n(xy) = \sum_{i=0}^{n} \binom{n}{i} \delta^i(x)(I - \delta)^i(\delta^{n-i}(y)).
\]  

Letting \( n = k + 1 \), \( x = u \) and \( y = u^k \) in the equation above we have

\[
\delta^{k+1}(u^{k+1}) = \sum_{i=0}^{k+1} \binom{k+1}{i} \delta^i(u)(I - \delta)^i(\delta^{k+1-i}(u^k)).
\]

Then by the facts \( \delta u = e \) and \( \delta e = 0 \), the only nonzero term in the sum above is the \( i = 1 \) term. Therefore by the induction assumption we have

\[
\delta^{k+1}(u^{k+1}) = (k + 1)e(I - \delta)(\delta^k(u^k)) = (k + 1)!e(I - \delta)(\delta^k(u^k)) = (k + 1)!e.
\]

Hence, by induction we have \( \delta^k(u^k) = k!e \) for all \( k \geq 1 \), as desired. \( \square \)

Note that the arguments in the proof of Proposition 3.8 above go through equally well for every \( K \)-derivation or \( K \)-\( E \)-derivation \( \delta \) of \( \mathcal{A} \) and all idempotents \( e \in \mathcal{A}^\delta \cap \text{Im} \, \delta \), regardless of the commutativity of \( \mathcal{A} \). Therefore, we also have the following

**Corollary 3.9.** Let \( \mathcal{A} \) be a \( K \)-algebra (not necessarily commutative) that is algebraic over \( K \), and \( \delta \) an arbitrary \( K \)-derivation or \( K \)-\( E \)-derivation of \( \mathcal{A} \). Then \( \mathcal{A}^\delta \cap \text{Im} \, \delta \) does not contain any nonzero idempotent of \( \mathcal{A} \).

For the \( K \)-algebras that are not algebraic over \( K \), we have the following theorem proved in [Z4].

**Theorem 3.10.** Let \( K \) be a field of characteristic zero and \( \mathcal{A} \) a \( K \)-algebra (not necessarily unital or commutative). Then the following statements hold:
1) for every LF $D \in \text{Der}_K(A)$ and an idempotent $e \in A^D \cap \text{Im} \, D$, we have $(e) \subseteq \text{Im} \, D$;
2) for every LF $\delta \in \mathcal{E}\text{der}_K(A)$ and an idempotent $e \in A^\delta \cap \text{Im} \, \delta$, we have $eA, Ae \subseteq \text{Im} \, \delta$. Furthermore, if $\delta$ is LN, we also have $(e) \subseteq \text{Im} \, \delta$.

Note that, if $A$ is commutative, then for an arbitrary $K$-derivation or a LN $K$-$\mathcal{E}$-derivation $\delta$ of $A$, we have that all idempotents of $A$ lie in $A^\delta$, as shown in the proof of Proposition 3.8. Therefore, from this fact and the theorem above we immediately have the following

**Corollary 3.11.** Assume that $A$ is commutative (but not necessarily algebraic over $K$). Then Conjecture 1.7 holds for all LF $D \in \text{Der}_K(A)$ and all LN $\delta \in \mathcal{E}\text{-Der}_K(A)$.

The next case of Conjecture 1.7 follows (somewhat unexpectedly) from the following classical Singer-Wermer Theorem in the theory of Banach algebras.

**Theorem 3.12.** Let $A$ be a commutative Banach algebra and $D$ an arbitrary derivation. Then $\text{Im} \, D$ is contained in the Jacobson radical $\mathfrak{J}(A)$ of $A$.

The theorem above was first proved by I. M. Singer and J. Wermer [SW] in 1955 for all continuous derivations, and in the same paper they also conjectured that the continuous condition is not necessary. More than thirty years later it was shown by M. P. Thomas [T] in 1988 that it is indeed the case.

Note that for all unital rings $R$ and nonzero idempotents $e \in R$, $1_R - e$ is also an idempotent, and can not be invertible. Then by [Pi, Proposition 4.3] the Jacobson radical $\mathfrak{J}(R)$ of $R$ does not contain any nonzero idempotent of $R$. From this general fact and Theorem 3.12 we immediately have the following

**Corollary 3.13.** Let $A$ be a commutative Banach algebra and $D$ an arbitrary derivation. Then $\text{Im} \, D$ does not contain any nonzero idempotent $e$ of $A$. In particular, Conjecture 1.7 holds for all derivations of $A$.

Note that there are also many results in the literature on the generalizations of the Singer-Wermer Theorem to certain derivations of some other algebras (e.g., see the survey paper [MM] and the book [Pa, Section 6.4], and also the references therein). For example, it was shown in [MR] that every centralizing derivation $D$ (i.e., for all $u \in A$, $[u, Du]$ lies in the center of $A$) of an arbitrary Banach algebra $A$ has its image contained in the Jacobson radical of $A$. Hence Corollary 3.13...
and Conjecture 1.7 also hold for centralizing derivations of all Banach algebras.

Conversely, Conjecture 1.7 and more generally, the LFED and LNED conjectures in some sense provide some generalizations of the Singer-Wermer Theorem to (noncommutative) Banach algebras and also some other more general algebras (e.g., the normed algebras, etc.) in the general theory of Banach algebras.

4. The LFED Conjecture from a Different Point of View

Throughout this section $K$ stands for a field of characteristic zero and $A$ for a $K$-algebra. In this section we discuss the LFED conjecture 1.5 in terms of the decompositions of $A$ associated with the Jordan-Chevalley decompositions of LF $K$-derivations and $K$-$\varepsilon$-derivations of $A$.

We first assume that $K$ is algebraically closed. For each LF $K$-linear endomorphism $\psi$ of $A$, let $\Lambda$ be the set of eigenvalues of $\psi$ and $A_\lambda := \sum_{i=1}^{\infty} \ker (\lambda I - \psi)^i$ for all $\lambda \in \Lambda$. Then it is well-known (e.g., see [E2, Proposition 1.3.8], [H, Proposition 4.2]) that $A$ can be decomposed as

$$A = \bigoplus_{\lambda \in \Lambda} A_\lambda. \quad (4.1)$$

Furthermore, $\psi$ is said to be semi-simple if $A_\lambda$ ($\lambda \in \Lambda$) coincides with the eigenspace of $\psi$ corresponding to the eigenvalue $\lambda$.

With the decomposition as in Eq. (4.1) it can be readily verified (e.g., see the proof of [Z4, Lemma 3.5 or 4.1]) that the image $\text{Im} \, \psi$ can be decomposed as

$$\text{Im} \, \psi = \psi(A_0) \bigoplus_{0 \neq \lambda \in \Lambda} A_\lambda. \quad (4.2)$$

If $\psi$ is a (LF) $K$-derivation of $A$, then by setting $A_\gamma = 0$ for all $\gamma \notin \Lambda$ we have $A_\lambda A_\mu \subseteq A_{\lambda + \mu}$ for all $\lambda, \mu \in \Lambda$, i.e., the decomposition in Eq. (4.1) is a so-called additive algebra grading of $A$. In particular, $A_0$ is a $\psi$-invariant $K$-subalgebra of $A$, and the restriction $\psi |_{A_0}$ is a LN $K$-derivation of the $K$-algebra $A_0$. Then by Eq. (4.2) the image of the LF $K$-derivation $\psi$ of the $K$-algebra $A$ is completely determined by the image of the LN $K$-derivation $\psi |_{A_0}$ of the $K$-algebra $A_0$.

Similarly, if $\psi$ is a (LF) $K$-algebra endomorphism of $A$, then by setting $A_\gamma = 0$ for all $\gamma \notin \Lambda$ we have $A_\lambda A_\mu \subseteq A_{\lambda \mu}$ for all $\lambda, \mu \in \Lambda$, i.e., the decomposition in Eq. (4.1) is a so-called multiplicative algebra grading of $A$. In particular, $A_1$ is a $\psi$-invariant $K$-subalgebra of $A$, and the restriction $\psi |_{A_1}$ is a $K$-algebra endomorphism of the $K$-algebra $A_1$ such that $I_{A_1} - \psi |_{A_1}$ is a LN $K$-$\varepsilon$-derivation of $A_1$. 
Now set $\delta := 1 - \psi$. Then $\delta$ is a LF $K$-$E$-derivation of $\mathcal{A}$, and $\delta |_{\mathcal{A}_1}$ is a LN $K$-$E$-derivation of $\mathcal{A}_1$. By Eq. (4.2) with $\psi$ replaced by $\delta$ we have

\[ \text{Im} \delta = \delta(\mathcal{A}_1) \oplus \bigoplus_{1 \neq \lambda \in \Lambda} \mathcal{A}_\lambda. \]  

(4.3)

Therefore, the image of the LF $K$-$E$-derivation $\delta$ of the $K$-algebra $\mathcal{A}$ is completely determined by the image of the LN $K$-$E$-derivation $\delta |_{\mathcal{A}_1}$ of the $K$-algebra $\mathcal{A}_1$.

One special but important case is when the $K$-linear map $\psi$ is a semi-simple $K$-derivation or a semi-simple $K$-endomorphism of $\mathcal{A}$, i.e., $\mathcal{A}_\lambda (\lambda \in \Lambda)$ in Eq. (4.4) is the eigenspace of $\psi$ corresponding to the eigenvalue $\lambda$ of $\psi$. In this case the $K$-subspaces $\psi(A_0)$ in Eq. (4.2) and $\delta(A_1)$ in Eq. (4.3) are respectively equal to zero. Based on this observation and also the LFED conjecture we propose the following what we call the Grading conjecture.

**Conjecture 4.1. (The Grading Conjecture)** Let $K$ be a field of characteristic zero (not necessarily algebraically closed), $\mathcal{A}$ a $K$-algebra and $(\Lambda, \cdot)$ a monoid with the unit $e$. Assume that $\mathcal{A}$ has the following $K$-algebra grading with respect to the monoid $(\Lambda, \cdot)$:

\[ \mathcal{A} = \bigoplus_{\lambda \in \Lambda} \mathcal{A}_\lambda, \]

(4.4)
i.e., $\mathcal{A}_\lambda$ is a $K$-subspace of $\mathcal{A}$ for each $\lambda \in \Lambda$, and $\mathcal{A}_\lambda \mathcal{A}_\mu \subseteq \mathcal{A}_{\lambda \cdot \mu}$ for all $\lambda, \mu \in \Lambda$. Then the $K$-subspace $\bigoplus_{e \neq \lambda \in \Lambda} \mathcal{A}_\lambda$ is a MS of $\mathcal{A}$.

Two remarks on the conjecture above are as follows.

First, it is easy to see that the set of all semi-simple $K$-derivations (resp., $K$-algebra endomorphisms) of $\mathcal{A}$ is in one-to-one correspondence with the set of all additive (resp., multiplicative) algebra gradings of $\mathcal{A}$ with the index monoid $(\Lambda, \cdot)$ being a sub-monoid of $(K, +)$ (resp. $(K, \cdot)$). Therefore, the semi-simple $K$-derivation (resp., $K$-$E$-derivation) case of the LFED conjecture is equivalent to the case of the Grading conjecture under the extra assumption that the index monoid $(\Lambda, \cdot)$ is a sub-monoid of $(K, +)$ (resp. $(K, \cdot)$). For convenience, we refer to these two cases of the Grading conjecture respectively as the additive Grading conjecture and the multiplicative Grading conjecture.

Second, by the equivalences mentioned above, some of the known cases of the LFED conjecture discussed in Section 3 and also some of those that will be proved in the next two sections can be translated or re-formulated as certain cases of the additive and multiplicative Grading conjectures. For example, both the additive and multiplicative Grading conjectures hold for the univariate polynomial algebra $K[x]$ by Theorem 3.2 and all local algebraic $K$-algebra by Theorem 5.5.
and all finite dimensional $K$-algebras by Theorem 3.6 etc. They also hold for the $K$-algebras $A$ in Theorem 3.7, which can be shown in the following.

**Corollary 4.2.** Let $A$ be as in Theorem 3.7. Assume that $A$ has a $K$-linear decomposition as in Eq. (4.1) (with $\Lambda \subseteq K$). Then the following statements hold:

1) if the decomposition in Eq. (4.1) is an additive algebra grading of $A$, then the $K$-subspace $\bigoplus_{0 \neq \lambda \in \Lambda} A_\lambda$ is a MS of $A$;

2) if the decomposition in Eq. (4.1) is a multiplicative algebra grading of $A$, then the $K$-subspace $\bigoplus_{1 \neq \lambda \in \Lambda} A_\lambda$ is a MS of $A$.

In other words, both the additive and multiplicative Grading conjectures hold for $A$.

**Proof:** 1) Define $D : A \rightarrow A$ by setting $Du = \lambda u$ for all $\lambda \in \Lambda$ and $u \in A_\lambda$. Since the decomposition in Eq. (4.1) is an additive algebra grading of $A$, it is easy to see that $D$ is a LF $K$-derivation of $A$ with $\text{Im } D = \bigoplus_{0 \neq \lambda \in \Lambda} A_\lambda$. Then by Theorem 3.7 2) the statement follows.

2) Define $\phi : A \rightarrow A$ by setting $\phi(u) = \lambda u$ for all $\lambda \in \Lambda$ and $u \in A_\lambda$. Since the decomposition in Eq. (4.1) is a multiplicative algebra grading of $A$, it is easy to see that $\phi$ is a LF $K$-algebra endomorphism of $A$. Set $\delta := I - \phi$. Then $\delta$ is a LF $K$-$\mathcal{E}$-derivation of $A$ with $\text{Im } \delta = \bigoplus_{1 \neq \lambda \in \Lambda} A_\lambda$.

Note that $A_0 = \text{Ker } \phi$ and is an ideal of $A$. Set $\bar{A} := A/A_0$ and $\bar{\delta} := I_{\bar{A}} - \bar{\phi}$, where $\bar{\phi}$ is the $K$-algebra endomorphism of $\bar{A}$ induced by $\phi$. We may identify $\bar{A}$ with the $K$-subalgebra $\bigoplus_{0 \neq \lambda \in \Lambda} A_\lambda$ of $A$. Then under this identification $\bar{\phi}(u) = \lambda u$ for all $0 \neq \lambda \in \Lambda$ and $u \in A_\lambda$. In particular, $\bar{\phi}$ is a LF $K$-algebra automorphism of $\bar{A}$ and $\bar{\delta} = I_{\bar{A}} - \bar{\phi}$ is a LF $K$-$\mathcal{E}$-derivation of $\bar{A}$. Then by Theorem 3.7 2) $\text{Im } \bar{\delta}$ is a MS of $\bar{A}$. Note that $\text{Im } \delta = \text{Im } \delta/A_0$ and the ideal $A_0$ is obviously contained in $\text{Im } \delta$. Then, by $[Z3$, Proposition 2.7] $\text{Im } \delta$ is a MS of $A$, whence the statement follows. □

Besides the cases above, we also have the following cases of the additive and multiplicative Grading conjectures in a more general setting.

**Proposition 4.3.** Let $K$ be a field of characteristic zero and $A$ a $K$-algebra with a decomposition as in Eq. (4.1) (with $\Lambda \subseteq K$). Let $H_1$ (resp., $H_2$) be the sub-monoid of the abelian group $(K, +)$ (resp., $(K \setminus \{0\}, \cdot)$) generated by elements $0 \neq \lambda \in \Lambda$ (resp., $0, 1 \neq \lambda \in \Lambda$). Then the following statements hold:

1) if the decomposition in Eq. (4.1) is an additive algebra grading of $A$ and $0 \notin H_1$, then for every $\vartheta$-$\text{MS } V$ of $A_0$, the $K$-subspace $V \oplus \bigoplus_{0 \neq \lambda \in \Lambda} A_\lambda$ is a $\vartheta$-$\text{MS }$ of $A$;
2) if the decomposition in Eq. (4.7) is a multiplicative algebra grading of \( A \) and \( 1 \not\in H_2 \), then for every \( \vartheta \)-MS \( V \) of \( A_1 \), the \( K \)-subspace \( V \oplus \bigoplus_{1 \neq \lambda \in \Lambda} A_\lambda \) is a \( \vartheta \)-MS of \( A \).

Proof: Note that the \( K \)-subspace \( \bigoplus_{0 \neq \lambda \in \Lambda} A_\lambda \) in statement 1) under the condition \( 0 \not\in H_1 \) is an ideal of \( A \), and the same for the \( K \)-subspace \( \bigoplus_{1 \neq \lambda \in \Lambda} A_\lambda \) in statement 2) under the condition \( 1 \not\in H_2 \). Then both statements 1) and 2) follow directly from [Z3, Proposition 2.7].

Next, we discuss an important special case of the multiplicative Grading conjecture. Let \( z = (z_1, z_2, \ldots, z_n) \) be \( n \) commutative or noncommutative free variables and \( A[z^{-1}, z] \) the algebra of Laurent polynomials in \( z \) over a \( K \)-algebra \( A \). Let \( q = (q_1, q_2, \ldots, q_n) \in K^n \) be such that \( 0 \not\in K \) (1 \( \leq i \leq n \)) and \( q^\alpha := \prod_{i=1}^{n} q_i^{\alpha_i} \neq 1 \) for all \( 0 \not\in \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}^n \) (e.g., let \( q_i \) (1 \( \leq i \leq n \)) be \( n \) distinct prime integers).

Let \( \Lambda = \{q^\alpha \mid \alpha \in \mathbb{Z}^n\} \) and \( V_\lambda (\lambda \in \Lambda) \) be the \( K \)-subspace formed by all \( f(z) \in A[z^{-1}, z] \) such that \( f(q_1z_1, q_2z_2, \ldots, q_nz_n) = \lambda f(z) \). Then it is easy to see that \( A[z^{-1}, z] \) can be decomposed as

\[
(4.5) \quad A[z^{-1}, z] = \oplus_{\lambda \in \Lambda} V_\lambda,
\]

which is a multiplicative algebra grading of \( A[z^{-1}, z] \) with \( V_0 = A \).

Then the multiplicative Grading conjecture for the multiplicative \( K \)-algebra grading of \( A[z^{-1}, z] \) in Eq. (4.5) becomes the following

**Conjecture 4.4.** Let \( K \) be a field of characteristic zero, \( A \) a \( K \)-algebra, and \( z = (z_1, z_2, \ldots, z_n) \) \( n \) commutative or noncommutative free variables. Denote by \( M \) the \( K \)-subspace of the Laurent polynomial algebra \( A[z^{-1}, z] \) consisting of the Laurent polynomials with no constant term. Then \( M \) is a MS of \( A[z^{-1}, z] \).

One known case of the conjecture above is as follows. Let \( z_i \) (1 \( \leq i \leq n \)) be commutative free variables, \( A = K \). Then the conjecture above in this case follows directly from the following remarkable Duistermaat-van der Kallen Theorem [DK], which is also the special case of the Mathieu conjecture [MO] for complex tori.

**Theorem 4.5.** Let \( K \) be a field of characteristic zero, \( z = (z_1, \ldots, z_n) \) commutative free variables and \( M \) the \( K \)-subspace of \( K[z^{-1}, z] \) of the Laurent polynomials with no constant term. Then \( \tau(M) \) consists of \( f \in K[z^{-1}, z] \) such that 0 does not lie in the polytope of \( f \). Consequently, \( M \) is a MS of \( K[z^{-1}, z] \).
From the discussion above, we see that the conjecture 4.4 and more generally, the Grading conjecture 4.1 can be viewed as some natural generalizations of the Duistermaat-van der Kallen Theorem.

5. The LFED Conjecture for Some Special $\mathcal{E}$-Derivations

Throughout this section $R$ denotes a unital commutative ring and $\mathcal{A}$ an $R$-algebra. We denote by nil($\mathcal{A}$) the set of all nilpotent elements of $\mathcal{A}$ (although $\mathcal{A}$ may not be commutative).

We shall show the LFED Conjecture 1.5 for the $R$-$\mathcal{E}$-derivations associated with some special $R$-algebra endomorphisms of $\mathcal{A}$. We start with the following lemma, which will also play an important role in the next section of this paper.

**Lemma 5.1.** Let $A, B, C, D$ be four commuting $R$-module endomorphisms of $\mathcal{A}$ such that $AB = 0$ and $AD + BC = I$. Then $Im \ A = Ker \ B$.

**Proof:** Since $BA = AB = 0$, we have $Im \ A \subseteq Ker \ B$. Now let $a \in Ker \ B$. Then $a = (AD + BC)(a) = (AD + CB)(a) = A(D(a))$. Therefore $a \in Im \ A$, whence the lemma follows. $\square$

Next, we consider $R$-$\mathcal{E}$-derivations associated with $R$-projections (i.e., $\phi \in End_R(\mathcal{A})$ with $\phi^2 = \phi$) and $R$-involutions (i.e., $\phi \in End_R(\mathcal{A})$ with $\phi^2 = I$) of $\mathcal{A}$.

**Proposition 5.2.** Let $\phi$ be an $R$-algebra endomorphism of $\mathcal{A}$. Then the following statements hold:

1) If $\phi^2 = \phi$, then $Im \ (I - \phi) = Ker \ \phi$.

2) If $\phi^2 = I$ and $2 \cdot 1_R$ is a unit of $R$, then

$$Im \ (I - \phi) = Ker \ (I + \phi),$$

$$r(Im \ (I - \phi)) = nil \ (\mathcal{A}).$$

In both cases above, $Im \ (I - \phi)$ is a MS of $\mathcal{A}$.

**Proof:** 1) Since $\phi^2 = \phi$, by Lemma 5.1 above with $A = I - \phi$, $B = \phi$ and $C = D = I$, we have $Im \ (I - \phi) = Ker \ \phi$, which is an ideal of $\mathcal{A}$, and hence also a MS of $\mathcal{A}$.

2) Since $2 \cdot 1_R$ is a unit of $R$, we may apply Lemma 5.1 with $A = I - \phi$, $B = I + \phi$ and $C = D = \frac{1}{2}I$, from which we get Eq. (5.1).

Now let $a \in r(Im \ (I - \phi))$. Replacing $a$ by a power of $a$ we may assume that $a, a^2 \in Im \ (I - \phi)$. Then by Eq. (5.1) we have $a, a^2 \in Ker \ (I + \phi)$, whence $\phi(a) = -a$ and $\phi(a^2) = -a^2$. But since $\phi$ is an $R$-algebra endomorphism of $\mathcal{A}$, we also have $\phi(a^2) = \phi(a)^2 = (-a)^2 = a^2$. Hence $a^2 = -a^2$ and $a^2 = 0$, for $2 \cdot 1_R$ is a unit of $R$. Therefore
Lemma 5.3. Let \( \phi \) be an \( R \)-algebra endomorphism of \( \mathcal{A} \). Set \( \text{Ker} \geq 1 \phi := \sum_{i \geq 1} \text{Ker} \phi^i \) and \( \bar{A} := A / \text{Ker} \geq 1 \phi \). Denote by \( \pi \) the quotient map from \( A \) to \( \bar{A} \) and \( \bar{\phi} \) the induced map of \( \phi \) from \( A \) to \( \bar{A} \). Then

1) \( \text{Ker} \geq 1 \phi \subseteq \text{Im} (I - \phi) \).
2) \( \bar{\phi} \) is injective.
3) The following equations hold:

\[
\text{Eq. (5.3)} \quad \pi(\text{Im} (I_{\bar{A}} - \phi)) = \text{Im} (I_{\bar{A}} - \bar{\phi}).
\]

\[
\text{Eq. (5.4)} \quad \text{Im} (I_{\bar{A}} - \phi) = \pi^{-1}(\text{Im} (I_{\bar{A}} - \bar{\phi})).
\]

\[
\text{Eq. (5.5)} \quad \text{t}(\text{Im} (I_{\bar{A}} - \phi)) = \pi^{-1}(\text{t}(\text{Im} (I_{\bar{A}} - \bar{\phi}))).
\]

4) \( \text{Im} (I - \phi) \) is a MS of \( A \), if and only if \( \text{Im} (I_{\bar{A}} - \bar{\phi}) \) is a MS of \( \bar{A} \).

Proof: 1) Let \( a \in \text{Ker} \geq 1 \phi \). Then \( \phi^k(a) = 0 \) for some \( k \geq 1 \). Let \( v = \sum_{i=0}^{\infty} \phi^i(a) \), which is a well-defined element of \( \mathcal{A} \). Then \( (I - \phi)v = (I - \phi)(\sum_{i=1}^{\infty} \phi^i)(a) = a \). Therefore \( a \in \text{Im} (I - \phi) \).

2) Let \( a \in A \) such that \( \bar{\phi}(\pi(a)) = 0 \). Since \( \bar{\phi} \pi = \pi \phi \), we have \( \pi(\phi(a)) = 0 \), i.e., \( \phi(a) \in \text{Ker} \geq 1 \phi \). Then \( \phi^{k+1}(a) = \phi^k(\phi(a)) = 0 \) for some \( k \geq 1 \). Therefore, \( a \in \text{Ker} \geq 1 \phi = \text{Ker} \pi \), whence \( \pi(a) = 0 \) and \( \bar{\phi} \) is injective.

3) Since \( \pi \phi = \bar{\phi} \pi \), we have \( \pi(I_{\bar{A}} - \phi) = (I_{\bar{A}} - \bar{\phi}) \pi \), from which and the surjectivity of \( \pi \) we have Eq. (5.3). To show Eq. (5.4), first, by Eq. (5.3) we have \( \text{Im} (I_{\bar{A}} - \phi) \subseteq \pi^{-1}(\text{Im} (I_{\bar{A}} - \bar{\phi})) \). Let \( a \in \pi^{-1}(\text{Im} (I_{\bar{A}} - \bar{\phi})) \). Then \( \pi(a) \in \text{Im} (I_{\bar{A}} - \bar{\phi}) \), i.e., there exists \( b \in A \) such that \( \pi(a) = (I_{\bar{A}} - \bar{\phi})(\pi(b)) = \pi(I_{\bar{A}} - \phi)(b) \).

Set \( c := (I_{\bar{A}} - \phi)(b) \). Then \( c \in \text{Im} (I_{\bar{A}} - \phi) \) and \( a - c \in \text{Ker} \pi \). Since \( \text{Ker} \pi = \text{Ker} \geq 1 \phi \), by statement 1) we have \( a - c \in \text{Im} (I_{\bar{A}} - \phi) \). Hence \( a = (a - c) + c \in \text{Im} (I_{\bar{A}} - \phi) \), and Eq. (5.4) follows.
To show Eq. (5.5), first by Eq. (5.3) we immediately have
\[ \pi(r(\text{Im}(I_A - \phi))) \subseteq r(\text{Im}(I_A - \bar{\phi})). \]
\[ r(\text{Im}(I_A - \phi)) \subseteq \pi^{-1}(r(\text{Im}(I_A - \bar{\phi}))). \]

Now let \( a \in \pi^{-1}(r(\text{Im}(I_A - \bar{\phi}))) \). Then \( \bar{a} := \pi(a) \in r(\text{Im}(I_A - \bar{\phi})) \), i.e., \( \pi(a^m) = \bar{a}^m \in \text{Im}(I_A - \bar{\phi}) \), and hence \( a^m \in \pi^{-1}(\text{Im}(I_A - \bar{\phi})) \), for all \( m \gg 0 \). Then by Eq. (5.4), \( a^m \in \text{Im}(I_A - \phi) \) for all \( m \gg 0 \). Hence \( a \in r(\text{Im}(I_A - \phi)) \) and Eq. (5.5) follows.

4) follows directly from statement 1), Eq. (5.3) and Proposition 2.7 in [Z3]. \( \square \)

Now we consider the following special family of \( \mathcal{E} \)-derivations.

**Proposition 5.4.** Assume that \( A \) is commutative and torsion-free as a \( \mathbb{Z} \)-module, i.e., no \( 0 \neq m \in \mathbb{Z} \) is a zero-divisor of \( A \). Let \( \phi \in \text{End}_R(A) \) such that \( \phi^i = \phi^j \) for some \( 1 \leq i < j \). Set \( \delta := I - \phi \) and \( \text{Ker}_{\geq 1} \phi := \sum_{k \geq 1} \text{Ker} \phi^k \). Then
\[
(5.6) \quad r(\text{Im} \delta) = r(\text{Ker} \phi^i) = r(\text{Ker}_{\geq 1} \phi).
\]
Consequently, \( \text{Im} \delta \) is a MS of \( A \).

**Proof:** First, the case \( \phi = 0 \) or \( I \) is trivial. So we assume \( \phi \neq 0, I \). Second, since \( \phi^i = \phi^j \) with \( i < j \), we have \( \phi^i = \phi^m \) for all \( m \geq 1 \) of the form \( m = i + q(j - i) \) (\( q \geq 0 \)). Then for each \( k \geq i \), choosing \( q \) large enough such that \( k \leq m := i + q(j - i) \) we have
\[ \text{Ker} \phi^i \subseteq \text{Ker} \phi^k \subseteq \text{Ker} \phi^m = \text{Ker} \phi^i. \]

Hence \( \text{Ker} \phi^i = \text{Ker} \phi^k \) for all \( k \geq i \) and \( \text{Ker} \phi^i = \text{Ker}_{\geq 1} \phi \).

Let \( \pi \) be the quotient map from \( A \) to \( \bar{A} := A/\text{Ker}_{\geq 1} \phi \), and \( \bar{\phi} \) the \( R \)-algebra endomorphism of \( \bar{A} \) induced by \( \phi \). Since \( \pi^{-1}(\text{nil}(\bar{A})) = r(\text{Ker} \pi) = r(\text{Ker}_{\geq 1} \phi) \), by Eq. (5.5) it suffices to show \( r(\text{Im}(I_A - \bar{\phi})) = \text{nil}(\bar{A}) \).

Furthermore, by replacing \( A \) by \( \bar{A} \) and \( \phi \) by \( \bar{\phi} \), and by Lemma 5.3, 2) we may assume that \( \bar{\phi} \) is injective, and only need to show the following equation:
\[
(5.7) \quad r(\text{Im} \delta) = \text{nil}(A).
\]

First, by Definition 1.2 \( \text{nil}(A) \) is obviously contained in \( r(\text{Im} \delta) \). Conversely, let \( a \in r(\text{Im} \delta) \). Replacing \( a \) by a power of \( a \) we assume that \( a^m \in \text{Im} \delta \) for all \( m \geq 1 \).

Second, under the injective assumption on \( \phi \), the condition \( \phi^i = \phi^j \) with \( i < j \) implies \( I = \phi^n \), where \( n := j - i \). Then \( n \geq 2 \), for we have assumed \( \phi \neq I \).
Since \( I - \phi^n = 0 \), we have \( g(\phi)\delta = g(\phi)(I - \phi) = 0 \), where \( g(t) := \sum_{k=0}^{n-1} t^k \). Hence \( \text{Im} \, \delta \subseteq \text{Ker} \, g(\phi) \). Therefore \( a^m \in \text{Ker} \, g(\phi) \) for all \( m \geq 1 \). Set \( b_i = \phi^i(a) \) for all \( 0 \leq i \leq n - 1 \). Then for all \( m \geq 1 \), we have

\[
(5.8) \quad b_0^m + b_1^m + \cdots + b_{n-1}^m = 0.
\]

Note that the left-hand side of Eq. (5.8) is the value at \( b_i \) \((0 \leq i \leq n-1)\) of the \( m \)-th power sum symmetric polynomial \( p_m(x) := \sum_{i=0}^{n-1} x_i^m \). It is well-known (e.g., see \([Ma]\), or \([Wiki]\) and references therein.) that each elementary symmetric polynomial \( e_m \) \((m \geq 1)\) can be written as a polynomial in \( p_m \) \((m \geq 1)\) with coefficients in \( \mathbb{Q} \). Therefore, for all \( m \geq 1 \), the values \( e_m(b_i; 0 \leq i \leq n-1) \) \((in \mathcal{A})\) of \( e_m \) at \( b_i \) \((0 \leq i \leq n-1)\), when viewed as elements of \( \mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{A} \), are all equal to zero.

On the other hand, since \( \mathcal{A} \) is a torsion-free \( \mathbb{Z} \)-module (and \( \mathbb{Z} \) is a PID), \( \mathcal{A} \) is also a flat \( \mathbb{Z} \)-module (e.g., see \([Bo, \text{Chapter I, \S 2.4, Prop. 3}]\)). In particular, the homomorphism \( \mathcal{A} = \mathbb{Z} \otimes_{\mathbb{Z}} \mathcal{A} \to \mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{A} \) is injective. Therefore, \( e_m(b_i; 0 \leq i \leq n-1) \) \((in \mathcal{A})\) of \( e_m \) at \( b_i \) \((0 \leq i \leq n-1)\), when viewed as elements of \( \mathcal{A} \) are also equal to zero. Consequently, \( \prod_{i=0}^{n-1} (t - b_i) = t^n \) in \( \mathcal{A}[t] \). Letting \( t = b_0 = a \) we get \( a^n = 0 \), whence \( a \in \text{nil} \, (\mathcal{A}) \), as desired. \( \square \)

From Proposition 5.4 or from its proof above we immediately have the following

**Corollary 5.5.** Assume that \( \mathcal{A} \) is commutative and torsion-free as a \( \mathbb{Z} \)-module. Then for every finite order \( R \)-algebra automorphism \( \phi \) of \( \mathcal{A} \), we have

\[
(5.9) \quad \tau(\text{Im} \, (I - \phi)) = \text{nil} \, (\mathcal{A}).
\]

In particular, \( I - \phi \) maps every \( R \)-subspace of \( \mathcal{A} \) to a MS of \( \mathcal{A} \).

### 6. Some Cases for Algebraic Derivations and \( \mathcal{E} \)-Derivations of Domains

Throughout this section \( R \) stands for a unital commutative ring that contains \( \mathbb{Z} \) as a subring, and \( \mathcal{A} \) a unital \( R \)-algebra that is torsion-free as a \( \mathbb{Z} \)-module. For convenience, we also assume \( \mathbb{Z} \subseteq R \subseteq \mathcal{A} \). If \( \mathcal{A} \) has no left or right zero-divisors, we say \( \mathcal{A} \) is a domain.

Recall that an \( R \)-derivation or \( R-(\mathcal{E}) \)-derivation \( \delta \) of \( \mathcal{A} \) is algebraic over \( R \) if there exists a nonzero polynomial \( f(t) \in R[t] \) such that \( f(\delta) = 0 \). When the base ring \( R \) is clear in the context, we also simply say that \( \delta \) is algebraic.

In this section we mainly consider some cases of Problem 1.4 for algebraic derivations and \( \mathcal{E} \)-derivations of domains. In particular, we
show that both the LFED conjecture and the LNED conjecture hold for all LF or LN algebraic derivations and $\mathcal{E}$-derivations of integral domains of characteristic zero (see Theorem \ref{6.9}). The proof will be divided into several lemmas and propositions, some of which will be proved in more general settings.

**Lemma 6.1.** Assume further that $A$ is reduced, i.e., $A$ has no nonzero nilpotent element. Then $A$ has no nonzero nilpotent $R$-derivations or $R$-$\mathcal{E}$-derivations.

**Proof:** Here, we only show the $R$-$\mathcal{E}$-derivation case. The $R$-derivation case can be proved similarly.

Assume otherwise and let $\phi \in \text{End}_R(A)$ such that the $R$-$\mathcal{E}$-derivation $\delta := 1 - \phi$ is nonzero and nilpotent. Let $k \geq 2$ be the least positive integer such that $\delta^k = 0$. Then there exists $u \in A$ such that $\delta^{k-1}u \neq 0$.

By Eq. \eqref{3.1} it is easy to see that for all $m \geq 1$ and $v \in A$ with $\delta^2v = 0$, we have

\begin{equation}
\delta^m(\nu v) = (\delta^m u)v + m(\delta^{m-1}u - \delta^m u)\delta v.
\end{equation}

Then by letting $m = k$ and $v = \delta^{k-2} u$, and applying the assumption $\delta^k = 0$ we get

\[0 = \delta^k(u(\delta^{k-2} u)) = k(\delta^{k-1} u)^2.\]

Since $A$ is reduced and torsion-free as a $\mathbb{Z}$-module, we have $\delta^{k-1} u = 0$. Contradiction. \hfill $\square$

Next, let us recall the following proposition proved in \cite[Theorem 4.6]{Z8}.

**Proposition 6.2.** Let $R$ be a unital integral domain of characteristic zero and $A$ a unital reduced $R$-algebra (not necessarily commutative) that is torsion-free as an $R$-module. Then $A$ has no nonzero $R$-derivation that is locally algebraic over $R$. In particular, $A$ has no nonzero $R$-derivation that is algebraic over $R$.

**Remarks 6.3.** 1) Proposition \ref{6.2} does not always hold for $\mathcal{E}$-derivations, e.g., taking $\phi$ to be a non-identity finite order automorphism of $A$, if there is any.

2) Assume further that $A$ is a domain of characteristic zero. Then by Proposition \ref{6.2} both the LFED and LNED Conjectures hold (trivially) for $R$-derivations of $A$ that are algebraic over $R$.

Next we consider algebraic $\mathcal{E}$-derivations of domains of characteristic zero.
Lemma 6.4. Assume further that $R$ is an integral domain of characteristic zero, and $A$ is a domain (containing $R$). Let $0, 1 \neq \phi \in \mathcal{E}nd_R(A)$ be algebraic over $R$, and $f(t)$ a minimal polynomial of $\phi$, i.e., $f(t)$ has the least degree among all $0 \neq g(t) \in R[t]$ with $g(\phi) = 0$. Then $f(t) = (1 - t)h(t)$ for some $h(t) \in R[t]$ with $\deg h \geq 1$ and $h(1) \neq 0$.

Proof: Let $K_R$ be the field of fractions of $R$ and $\bar{K}_R$ be the algebraic closure of $K_R$. Decompose $f(t)$ in $\bar{K}_R[t]$ as
\[
f(t) = (1 - t)^k h(t)
\]
for some $k \geq 0$ and $h(t) \in \bar{K}_R[t]$ such that $h(1) \neq 0$. Since the leading coefficient of $(1 - t)^k$ is a unit in $R$, by going through the division of $f(t)$ by $(t - 1)^k$, it is easy to see that $h(t)$ actually lies in $R[t]$.

Since $\phi(1)$ is an idempotent of $A$ and $A$ is a domain, we have $\phi(1) = 0$ or $1$. Since $\phi \neq 0$ by assumption, we have $\phi(1) = 1$. Applying $0 = f(\phi)$ to $1$ we get $f(1) = 0$, whence $k \geq 1$. Furthermore, since $\phi \neq 1$, we also have $\deg h \geq 1$.

Let $\bar{A} = \bar{K}_R \otimes_R A$. Since $A$ is a domain containing $R$, and hence torsion-free as an $R$-module, the standard map $A \cong R \otimes_R A \to \bar{K}_R \otimes_R A$ is injective, for by [AM] Prop. 3.3] $K_R \otimes_R A$ is isomorphic to the localization $S^{-1}A$ with $S = R \setminus \{0\}$. Since every field is absolutely flat, the standard map $\bar{K}_R \otimes_R A \to \bar{K}_R \otimes_{\bar{K}_R}(K_R \otimes_R A) = \bar{K}_R \otimes_R A$ is also injective. Therefore, we may view $A$ as an $R$-subalgebra of $\bar{A}$ in the standard way and extend $\phi \bar{K}_R$-linearly to a $\bar{K}_R$-algebra endomorphism of $\bar{A}$, which we will denote by $\bar{\phi}$.

Since $\phi$ is algebraic over $R$, $\bar{\phi}$ is algebraic over $\bar{K}_R$. Then $\bar{A}$ can be decomposed as a direct sum of the generalized eigen-subspaces of $\bar{\phi}$ (e.g., see [H Proposition 4.2]). More precisely, let $r_i$ ($1 \leq i \leq \ell$) be all the distinct roots of $f(t)$ in $\bar{K}_R$ with multiplicity $m_i$. Set $\bar{A}_i = \text{Ker}(r_i I_{\bar{A}} - \bar{\phi})^{m_i}$ for all $1 \leq i \leq \ell$. Then we have
\[
\bar{A} = \bigoplus_{i=1}^{\ell} \bar{A}_i.
\]

Furthermore, the decomposition above is actually an algebra grading of $\bar{A}$, i.e., $\bar{A}_i \bar{A}_j \subseteq \bar{A}_{ij}$ for all $1 \leq i, j \leq \ell$. In particular, $\bar{A}_1$ is a nonzero $R$-subalgebra of $\bar{A}$, and hence also a unital domain over $R$, for $1 \in \bar{A}_1$.

Note also that $\bar{A}_1$ is $\phi$-invariant and hence also $h(\bar{\phi})$-invariant. Furthermore, since $h(1) \neq 0$, the restriction of $h(\bar{\phi})$ on $\bar{A}_1$ is injective. Otherwise, there would exist $0 \neq a \in \bar{A}_1$ such that $h(\bar{\phi})(a) = 0$. Since $(I_{\bar{A}} - \bar{\phi})^{m_1}(a) = 0$, and $h(t)$ and $(1 - t)^{m_1}$ are co-prime, we have $a = 0$. Contradiction.

Now, since $f(\bar{\phi})|_{\bar{A}_1} = h(\phi)|_{\bar{A}_1}(I_{\bar{A}_1} - \bar{\phi})^k|_{\bar{A}_1} = 0$, we have $(I_{\bar{A}_1} - \bar{\phi})^k|_{\bar{A}_1} = 0$, i.e., $(I_{\bar{A}_1} - \bar{\phi})$ is a nilpotent $R$-$\mathcal{E}$-derivation of $\bar{A}_1$. Then
I_{\bar{A}} - \bar{\phi} = 0 \text{ by Lemma } 6.4. \text{ Consequently, } \bar{f}(\bar{\phi}) = 0, \text{ where } \bar{f}(t) = (1-t)h(t). \text{ Hence we also have } \bar{f}(\phi) = 0. \text{ Since } h(t) \in R[t] \text{ as pointed above, we have } \bar{f}(t) \in R[t]. \text{ Then by the choice of } f(t), \text{ we have } f(t) = \bar{f}(t), \text{ whence } k = 1, \text{ as desired. } \square

Corollary 6.5. Let R and A be as in Lemma 6.4. Then A has no nonzero locally nilpotent R-E-derivation that is algebraic over R.

Proof: Let $\delta \in Eder_R(A)$ be LN and algebraic over R. Write $\delta = I - \phi$ for some $\phi \in End_R(A)$. Then $\phi = I - \delta$ is also algebraic over R. Let $f(t) \in R[t]$ be a minimal polynomial of $\phi$. Then for each $a \in A$, we have $f(\phi)(a) = 0$ and $\delta^k(a) = (1 - \phi)^k(a) = 0$ for some $k \geq 1$.

Let $K_R$ be the field of fractions of $R$, $B := K_R \otimes_R A$, and $\bar{\phi}$ and $\bar{\delta}$ the $K_R$-linear extension maps of $\phi$ and $\delta$, respectively, from $B$ to $B$. As pointed out in the proof of Lemma 6.4, we may identify $A$ as an $R$-subalgebra of $B$.

With the setting above, we have $\bar{\delta} = I_B - \bar{\phi}$, $f(\bar{\phi})(a) = 0$ and $\bar{\delta}^k(a) = (I_{\bar{A}} - \bar{\phi})^k(a) = 0$. By Lemma 6.4 $\gcd(f(t), (1 - t)^k) = 1 - t$ in $K_R[t]$. Hence there exist $u(t), v(t) \in K_R[t]$ such that $u(t)f(t) + v(t)(1 - t)^k = 1 - t$. Consequently, $(I_B - \bar{\phi})(a) = 0$. Since $a \in A$, we further have $\delta(a) = (I - \phi)(a) = (I_B - \bar{\phi})(a) = 0$. Therefore, $\delta = 0$ and the corollary follows. \square

From now on we focus on the E-derivations of integral domains of characteristic zero.

Lemma 6.6. Assume further that R is an integral domain of characteristic zero, and A is an integral domain containing R. Let $\phi \in End_R(A)$ and $g(t) = \sum_{i=0}^d c_it^i \in R[t]$ with $c_r, c_d \neq 0$. Then for each $a \in A$ such that $a^m \in \ker g(\phi)$ for all $m \geq 1$, the following statements hold:

1) $\phi^i(a) = \phi^j(a)$ for some $r < i < j \leq d$;
2) if $g(1) \neq 0$, then $\phi^k(a) = 0$ for some $r < k \leq d$.

Proof: If $\phi = 0$, the lemma is trivial. So we assume $\phi \neq 0$. If $d = r$, then $g(\phi) = c_r\phi^r$ and $\ker g(\phi) = \ker \phi^r$. So we have $\phi^r(a) = 0$, whence both statements 1) and 2) hold. So we assume $r < d$.

Set $b_i := \phi^i(a)$ for all $r \leq i \leq d$. Since $g(\phi)(a^m) = 0$ and $b_i^m = \phi^i(a)^m = \phi^i(a^m)$ for all $m \geq 1$, we have

\[
(6.4) \quad c_r b_i^m + c_{r+1} b_{i+1}^m + \cdots + c_d b_d^m = 0.
\]

Since $A$ is an integral domain and not all coefficients $c_i$’s are zero, the vandemonde determinant $\prod_{r \leq i < j \leq d}(b_j - b_i) = 0$, whence $b_i - b_j = 0$, i.e., $\phi^i(a) = \phi^j(a)$, for some $r \leq i < j \leq d$. So statement 1) holds.
To show statement 2), assume otherwise, i.e., $\phi^i(a) \neq 0$ for all $r \leq i \leq d$. Let $u_k (1 \leq k \leq \ell)$ be all distinct (nonzero) elements of $b_i = \phi^i(a)$ ($r \leq i \leq d$). For each $1 \leq k \leq \ell$, let $B_k$ be the subset of $r \leq i \leq d$ such that $\phi^i(a) = u_k$, and set $\tilde{c}_k := \sum_{i \in B_k} c_i$. Then $\sum_{k=1}^{\ell} \tilde{c}_k = g(1) \neq 0$, whence $\tilde{c}_k (1 \leq k \leq \ell)$ are not all zero.

On the other hand, Eq. (6.4) above can be re-written as

$$(6.5) \quad \tilde{c}_1 u_1^m + \tilde{c}_2 u_2^m + \cdots + \tilde{c}_\ell u_\ell^m = 0.$$ 

Since $A$ is an integral domain and $u_k (1 \leq k \leq \ell)$ are distinct nonzero elements of $A$, by using the vandemonde determinant we see that $\tilde{c}_k = 0$ for all $1 \leq k \leq \ell$. Contradiction. \hfill \Box

**Corollary 6.7.** Assume that $R$ is an integral domain of characteristic zero, and $A$ is an integral domain (containing $R$). If $A$ is finitely generated as an $R$-algebra, then for every $\phi \in \text{End}_R(A)$ that is algebraic over $R$, we have $\phi^i = \phi^j$ for some $1 \leq i < j$.

**Proof:** Let $0 \neq f(t) \in R[t]$ such that $f(\phi) = 0$, and $a_k \in A$ ($1 \leq k \leq n$) that generate $A$ as an $R$-algebra. Hence Ker $f(\phi) = A$ and $a_k \in \text{r(Ker } f(\phi))$ for all $1 \leq k \leq n$. By lemma 6.6 for each $1 \leq k \leq n$ there exists $1 \leq i_k < j_k$ such that $\phi^{i_k}(a_k) = \phi^{j_k}(a_k)$. Applying some powers of $\phi$ to the equation above we may assume that $i_k (1 \leq k \leq n)$ are all equal to one another. We denote this integer by $i$.

Set $j = i + \prod_{k=1}^{n} (j_k - i)$. Then it is easy to see that $\phi^i(a_k) = \phi^j(a_k)$ for all $1 \leq k \leq n$. Since $A$ as an $R$-algebra is generated by $a_k (1 \leq k \leq n)$ and $\phi$ is an $R$-algebra endomorphism, we have $\phi^i = \phi^j$, as desired. \hfill \Box

Now, we are ready to show the main results of this section.

**Proposition 6.8.** Assume that $R$ is an integral domain of characteristic zero, $A$ an integral domain containing $R$, and $\phi$ an $R$-endomorphism of $A$ that is algebraic over $R$. Set Ker $\geq 1 \phi := \sum_{i \geq 1} \text{Ker } \phi^i$. Then we have

$$(6.6) \quad \text{r(Im } (I - \phi)) = \text{r(Im } \geq 1 \phi).$$

Consequently, Im $(I - \phi)$ is a MS of $A$.

**Proof:** The case $\phi = 0$ or $I$ is trivial, so we assume $\phi \neq 0, I$. By Lemma 5.3, 1) we have Ker $\geq 1 \phi \subseteq \text{Im } (I - \phi)$, whence $\text{r(Ker } \geq 1 \phi) \subseteq \text{r(Im } (I - \phi))$.

Conversely, let $a \in \text{r(Im } (I - \phi))$ and $f(t)$ be a minimal polynomial of $\phi$. Replacing $a$ by a power of $a$ we assume that $a^m \in \text{Im } (I - \phi)$ for all $m \geq 1$. Let $K_R$ be the field of fractions of $R$, $B := K_R \otimes_R A$ and
the $K_R$-linear extension of $\phi$ for $\mathcal{B}$ to $\mathcal{B}$. As pointed out as in the proof of Lemma 6.4, we may identify $\mathcal{A}$ as an $R$-subalgebra of $\mathcal{B}$.

By Lemma 6.4, $f(t) = (t-1)h(t)$ for some $h(t) \in R[t]$ such that $h(1) \neq 0$ and $\deg h \geq 1$. Then there exist $u(t), v(t) \in K_R[t]$ such that $(1-t)u(t) + h(t)v(t) = 1$. Then by Lemma 5.1 with $A = I_B - \bar{\phi}$, $B = h(\bar{\phi})$, $C = u(\bar{\phi})$ and $D = v(\bar{\phi})$, we have $\text{Im} (I_B - \bar{\phi}) = \text{Ker} h(\bar{\phi})$, whence $h(\phi)(a^m) = h(\bar{\phi})(a^m) = 0$ for all $m \geq 1$. Applying Lemma 6.6 2) with $g(t) = h(t)$ we have $\phi^k(a) = 0$ for some $k \geq 0$. If $k = 0$, then $a = 0$, and if $k \geq 1$, $a \in \text{Ker} \phi^k$. In either case $a \in \text{Ker} \geq 1 \phi$, whence Eq. (6.6) follows.

The statement that $\text{Im} (I - \phi)$ is a MS of $\mathcal{A}$ follows directly from Eq. (6.6), Lemma 5.3 1) and Lemma 1.3.

Theorem 6.9. Assume that $R$ is an integral domain of characteristic zero, and $\mathcal{A}$ is an integral domain (containing $R$). Then the LFED conjecture (resp., the LNED conjecture) holds for all (resp., locally nilpotent) $R$-derivations and $R$-$E$-derivations of $\mathcal{A}$ that are algebraic over $R$.

Proof: By Proposition 6.2, $\mathcal{A}$ has no nonzero $R$-derivation that is algebraic over $R$. Hence the $R$-derivation case of the corollary holds.

By Corollary 6.5, $\mathcal{A}$ has no nonzero locally nilpotent $R$-$E$-derivation that is algebraic over $R$. Hence the $R$-$E$-derivation case of the LNED conjecture in the corollary holds. The $R$-$E$-derivation case of the LFED conjecture in the corollary follows directly from Proposition 6.8.

References

[AM] M. F. Atiyah and I. G. Macdonald, Introduction to Commutative Algebra. Addison-Wesley Publishing Co., 1969. [MR0242802].

[BCW] H. Bass, E. Connell and D. Wright, The Jacobian Conjecture, Reduction of Degree and Formal Expansion of the Inverse. Bull. Amer. Math. Soc. 7, (1982), 287–330.

[Be] J. Bergen, Derivations in Prime Rings. Canad. Math. Bull. 26 (1983), 267–270.

[Bo] N. Bourbaki, Commutative Algebra, Chapters 17. Translated from the French. Reprint of the 1989 English translation. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1998. [MR1727221].

[BFF] M. Brešar, A. Fošner and M. Fošner, A Kleinecke-Shirokov Type Condition with Jordan Automorphisms. Studia Math. 147 (2001), no. 3, 337-242.

[BV] M. Brešar and AR Villena, The Noncommutative SingerWermer Conjecture and $\phi$-Derivations. J. London Math. Soc. 66 (2002), 710-720.
H. Derksen, A. van den Essen and W. Zhao, The Gaussian Moments Conjecture and the Jacobian Conjecture. To appear in Israel J. Math.. See also arXiv:1506.05192 [math.AC].

J. J. Duistermaat and W. van der Kallen, Constant Terms in Powers of a Laurent Polynomial. Indag. Math. (N.S.) 9 (1998), no. 2, 221–231. [MR1691479].

A. van den Essen, The Exponential Conjecture and the Nilpotency Subgroup of the Automorphism Group of a Polynomial Ring. Prepublications. Univ. Autònoma de Barcelona, April 1998.

A. van den Essen, Polynomial Automorphisms and the Jacobian Conjecture. Prog. Math., Vol.190, Birkhäuser Verlag, Basel, 2000.

A. van den Essen, Introduction to Mathieu Subspaces. “International Short-School/Conference on Affine Algebraic Geometry and the Jacobian Conjecture” at Chern Institute of Mathematics, Nankai University, Tianjin, China. July 14-25, 2014.

A. van den Essen and L. C. van Hove, Mathieu-Zhao Spaces. To appear.

A. van den Essen and S. Nieman, Mathieu-Zhao Spaces of Univariate Polynomial Rings with Non-zero Strong Radical. J. Pure Appl. Algebra, 220 (2016), no. 9, 3300–3306.

A. van den Essen and W. Zhao, Mathieu Subspaces of Univariate Polynomial Algebras. J. Pure Appl. Algebra. 217 (2013), no.7, 1316-1324. See also arXiv:1012.2017 [math.AC].

A. van den Essen, D. Wright and W. Zhao, Images of Locally Finite Derivations of Polynomial Algebras in Two Variables. J. Pure Appl. Algebra 215 (2011), no.9, 2130-2134. [MR2786603]. See also arXiv:1004.0521 [math.AC].

G. Freudenburg, Algebraic Theory of Locally Nilpotent Derivations. Encyclopaedia of Mathematical Sciences, 136. Invariant Theory and Algebraic Transformation Groups, VII. Springer-Verlag, Berlin, 2006. [MR2259515].

J. E. Humphreys, (1972), Introduction to Lie Algebras and Representation Theory. Graduate Texts in Mathematics, Springer, 1972.

A. Nowicki, Polynomial derivations and their rings of constants. N. Copernicus University Press, Toruń, 1994.
[Pa] T. W. Palmer, *Banach Algebras and the General Theory of $\ast$-algebras. Vol. I. Algebras and Banach algebras*. Encyclopedia of Mathematics and its Applications, 49. Cambridge University Press, Cambridge, 1994.

[Pi] R. S. Pierce, *Associative Algebras*. Graduate Texts in Mathematics, 88. Springer-Verlag, New York-Berlin, 1982. [MR0674652].

[T] M. P. Thomas, *The Image of a Derivation is Contained in the Radical. Ann. of Math. (2) 128* (1988), no. 3, 435-460. [MR0970607].

[SW] I. M. Singer and J. Wermer, *Derivations on commutative normed algebras. Math. Ann. 129*, (1955). 260-264. [MR0070061].

[Wiki] https://en.wikipedia.org/wiki/Newton’s_identities.

[Z1] W. Zhao, *Images of Commuting Differential Operators of Order One with Constant Leading Coefficients. J. Alg. 324* (2010), no. 2, 231–247. [MR2651354]. See also arXiv:0902.0210 [math.CV].

[Z2] W. Zhao, *Generalizations of the Image Conjecture and the Mathieu Conjecture. J. Pure Appl. Alg. 214* (2010), 1200-1216. See also arXiv:0902.0212 [math.CV].

[Z3] W. Zhao, *Mathieu Subspaces of Associative Algebras. J. Alg. 350* (2012), no. 2, 245-272. See also arXiv:1005.4260 [math.RA].

[Z4] W. Zhao, *Idempotents in Intersection of the Kernel and the Image of Locally Finite Derivations and $E$-derivations. Eur. J. Math. 4* (2018), no. 4, 1491-1504. See also arXiv:1701.05993 [math.RA].

[Z5] W. Zhao, *The LFED and LNED Conjectures for Algebraic Algebras. Linear Algebra Appl. 534* (2017), 181-194. See also arXiv:1701.05990 [math.RA].

[Z6] W. Zhao, *The LFED and LNED Conjectures for Laurent Polynomial Algebras. Under submission*. See also arXiv:1701.05997 [math.AC].

[Z7] W. Zhao, *Images of Ideals under Derivations and $E$-Derivations of Univariate Polynomial Algebras over a Field of Characteristic Zero. Under submission*. See also arXiv:1701.06123 [math.AC].

[Z8] W. Zhao, *The Radical of the Kernel of a Certain Differential Operator and Applications to Locally Algebraic Derivations. Under submission*. See also arXiv:1701.06124 [math.RA].

Department of Mathematics, Illinois State University, Normal, IL 61761. Email: wzhao@ilstu.edu