Yukawa Couplings and Effective Interactions
in Gauge-Higgs Unification

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Abstract

The wave functions and Yukawa couplings of the top and bottom quarks in the $SO(5) \times U(1)$ gauge-Higgs unification model are determined. The result is summarized in the effective interactions for $\hat{\theta}_H(x) = \theta_H + H(x)/f_H$ where $\theta_H$ is the Wilson line phase and $H(x)$ is the 4D Higgs field. The Yukawa, $WWH$ and $ZZH$ couplings vanish at $\theta_H = \frac{1}{2}\pi$. There emerges the possibility that the Higgs particle becomes stable.
In the standard model of electroweak interactions the electroweak (EW) symmetry is spontaneously broken by the Higgs field, the mechanism of which is yet to be scrutinized and confirmed by experiments. The Higgs particle is expected to be found at LHC in the coming years. It is not clear at all, however, if the Higgs particle appears as described in the standard model. It is often argued from a theoretical point of view that the naturalness and stability against radiative corrections to the Higgs field indicate the existence of supersymmetry underlying the nature. Other scenarios with the naturalness have also been proposed, among which are the little Higgs theory, the Higgsless model, and the gauge-Higgs unification scenario.[1, 2, 3]

Recently there has been significant progress in the gauge-Higgs unification scenario in which the 4D Higgs field is identified with a part of the extra-dimensional component of gauge fields in higher dimensions.[4]-[37] The Higgs field appears as an Aharonov-Bohm (AB) phase, or a Wilson line phase, in the extra dimension, thereby the EW symmetry being dynamically broken by the Hosotani mechanism.[6, 7, 8] The $SO(5) \times U(1)_X$ gauge-Higgs unification model in the Randall-Sundrum (RS) warped space-time has been extensively studied to give definitive predictions.[9]-[15]

The nature of the Higgs field as an AB phase plays a decisive role here. Let us denote the Wilson line phase along the extra dimension by $\theta_H$. The effective potential $V_{\text{eff}}(\theta_H)$ becomes finite at the one loop level thanks to the AB phase nature of $\theta_H$. The neutral Higgs field $H(x)$ corresponds to four-dimensional fluctuations of $\theta_H$. It immediately follows that the Higgs mass, related to the curvature of $V_{\text{eff}}$ at the minimum, is predicted at a finite value, once the matter content of the theory is specified. Another distinctive prediction is obtained for the Higgs couplings to $W$ and $Z$. In the RS warped spacetime the $WWH$ and $ZZH$ couplings are suppressed by a factor $\cos \theta_H$ compared with those in the standard model.\[1\]

Inclusion of quarks and leptons, particularly of top and bottom quarks, is crucial to have EW symmetry breaking. Medina, Shar, and Wagner (MSW) proposed an $SO(5) \times U(1)_X$ gauge-Higgs unification model with top and bottom quarks in which the EW symmetry breaking is induced.[14] More recently Hosotani, Oda, Ohnuma and Sakamura (HOOS) have proposed a model with simpler matter content and many predictions.[15] It has been shown there that $V_{\text{eff}}(\theta_H)$ is minimized at $\theta_H = \frac{1}{2} \pi$ and the Higgs mass is predicted around

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\[1\]It has been discussed that the suppression occurs in a wider class of models.[38]
50 GeV. The LEP2 bound for the Higgs mass is evaded thanks to the vanishing $ZZH$ coupling at $\theta_H = \frac{1}{2}\pi$.

The purpose of the present paper is two-fold. The Yukawa couplings of quarks to the 4D Higgs field stem from gauge interactions in the extra-dimension. We first evaluate the 4D Yukawa couplings in the HOOS model in the Kaluza-Klein approach by determining the wave functions of the Higgs field and quarks, inserting them into the five-dimensional action, and integrating over the extra-dimensional coordinate. Secondly we develop an effective interaction approach for the Higgs couplings to quarks. As the Higgs field is a fluctuation mode of $\theta_H$, the Yukawa couplings are related to the $\theta_H$-dependence of the masses of quarks in this approach. We shall see that the Yukawa couplings in the HOOS model determined in these two approaches coincide with each other with high accuracy. This establishes the validity of the effective interactions at low energies, which enables us to deduce higher-order Higgs couplings such as $H^n t \bar{t}$ by bypassing laborious procedure of summing over contributions of intermediate Kaluza-Klein (KK) excited states.

We analyze the $SO(5) \times U(1)_X$ model with top and bottom quarks specified in ref. [15], following the notation there. The model is defined in the Randall-Sundrum (RS) warped spacetime whose metric is given by

$$ds^2 = \frac{1}{z^2} \left\{ \eta_{\mu\nu} dx^\mu dx^\nu + \frac{dz^2}{k^2} \right\}$$

for $1 \leq z \leq z_L$. The bulk region $1 < z < z_L$ is an AdS spacetime with the cosmological constant $\Lambda = -6k^2$, being sandwiched by the Planck brane at $z = 1$ and by the TeV brane at $z = z_L$. The warp factor $z_L$ is large, typically around $10^{13}$ to $10^{17}$. The $SO(5) \times U(1)_X$ gauge symmetry is broken to $SO(4) \times U(1)_X$ by the orbifold boundary conditions at the Planck and TeV branes with the parity matrices given by $P_0 = P_1 = \text{diag}(-1, -1, -1, -1, 1)$. The symmetry is further broken to $SU(2)_L \times U(1)_Y$ by additional interactions at the Planck brane.

The 4D Higgs field appears as a zero mode in the $SO(5)/SO(4)$ part of the fifth dimensional component of the vector potential $A^a_z(x, z)$ ($a = 1, \cdots, 4$), which is expanded as

$$A^a_z(x, z) = \phi^a(x) \varphi_H(z) + \cdots , \varphi_H(z) = \sqrt{\frac{2}{k(z_L^2 - 1)} z} .$$

An $SO(4)$ vector $\phi^a$ forms an $SU(2)_L$ doublet $\Phi_H(x)^t = (1/\sqrt{2})(\phi^2 + i\phi^1, \phi^4 - i\phi^3)$ corresponding to the Higgs doublet in the standard model. Without loss of generality one
can assume $\langle \phi^a \rangle = \nu \delta^{a4}$ when the EW symmetry is spontaneously broken by the Hosotani mechanism. Let us denote the generators of $SO(5)/SO(4)$ by $T^a$ ($a = 1, \cdots, 4$). In the vectorial representation $(T^i)_{ab} = (i/\sqrt{2})(\delta_{a5}\delta_{b4} - \delta_{a4}\delta_{b5})$, whereas in the spinorial representation $T^i = (1/2\sqrt{2})I_2 \otimes \tau_i$. The Wilson line phase $\theta_H$ is given by $\exp\{\frac{i}{2}g_H(2\sqrt{2}T^i)\} = \exp\{ig_A \int_1^{z_L} dz \langle A_z \rangle\}$ so that

$$\theta_H = \frac{1}{2} g_A v \sqrt{\frac{z^2_L - 1}{k}} \sim \frac{gv \pi kL}{2 m_{\text{KK}}}. \quad (3)$$

Here the $SO(5)$ gauge coupling constant $g_A$ in five dimensions is related to the four-dimensional $SU(2)_L$ gauge coupling constant $g$ by $g = g_A/\sqrt{L}$ where $L = k^{-1} \ln z_L$ is the size of the fifth dimension in the $y$ ($\equiv k^{-1} \ln z$) coordinate. The Kaluza-Klein mass scale is given by $m_{\text{KK}} = \pi k(z_L - 1)^{-1} \sim \pi k z_L^{-1}$. The $W$ boson mass is approximately given by $m_W \sim \sqrt{k/L z_L^{-1}} |\sin \theta_H|$. The value for $\theta_H$ is dynamically determined such that the effective potential $V_{\text{eff}}(\theta_H)$ is minimized. In the HOOS model $\theta_H = \frac{1}{2} \pi$. With $m_W$ and $z_L$ given, $k$ and $m_{\text{KK}}$ are fixed. For $z_L = 10^{13}$ to $10^{17}$, $k$ ranges from $4.4 \times 10^{15}$ GeV to $5.0 \times 10^{19}$ GeV, but $m_{\text{KK}}$ varies only from 1.38 TeV to 1.58 TeV. Physics predictions do not sensitively depend on the parameter $z_L$ in this range.

The main focus in the present paper is given on fermions and their interactions. Let us consider fermion multiplets containing top and bottom quarks. In the bulk region $1 < z < z_L$ two $SO(5)$ vector multiplets, $\Psi_a$ ($a = 1, 2$), are introduced with the action $L^\text{fermion} = \sum_{a=1}^2 \frac{1}{2} \{\bar{\Psi}_a D(c_a) \Psi_a + \text{h.c.}\}$ where $c_a$ denotes the dimensionless bulk mass parameter. Each of $\Psi_a$’s consists of $SO(4)$ vector and singlet components. The former is decomposed into two $SU(2)_L$ doublets with $SU(2)_R$ charges $T^{3_R} = \pm \frac{1}{2}$:

$$\Psi_1 = \begin{bmatrix} T \\ B \end{bmatrix} \equiv Q_1, \begin{bmatrix} t \\ b \end{bmatrix} \equiv q, t'_{\frac{1}{3}} \quad \frac{1}{3},$$

$$\Psi_2 = \begin{bmatrix} U \\ D \end{bmatrix} \equiv Q_2, \begin{bmatrix} X \\ Y \end{bmatrix} \equiv Q_3, b'_{-\frac{1}{3}}. \quad (4)$$

The subscript $\frac{1}{3}$ or $-\frac{1}{3}$ indicates the $U(1)_X$ charge $Q_X$. The electric charge is given by $Q_E = T^{3_L} + T^{3_R} + Q_X$. The orbifold boundary condition is given by $\Psi_a(x, y_j - y) = P_j \Gamma^5 \Psi_a(x, y_j + y)$ in the $y$ coordinate with $(y_0, y_1) = (0, L)$. This leads to zero modes in $Q_{aR}$, $q_{aL}$, $t'_R$, and $b'_R$, where the subscripts $L$ and $R$ denote the left- and right-handed components in four dimensions, respectively.
In addition to the bulk fermions, three right-handed multiplets localized on the Planck brane, belonging to \( (\frac{1}{2}, 0) \) representation of \( SU(2)_L \times SU(2)_R \), are introduced;
\[
\hat{\chi}^{1R} = \left( \hat{T}_R \right)_{7/6} , \quad \hat{\chi}^{2R} = \left( \hat{U}_R \right)_{1/6} , \quad \hat{\chi}^{3R} = \left( \hat{X}_R \right)_{-5/6} . \tag{5}
\]
Here the subscripts \( 7/6 \) etc. represent the \( U(1)_X \) charges. The brane fermions \( \hat{\chi}^a_R \) have, besides gauge invariant kinetic terms on the Planck brane, mass terms with \( q_L \) and \( Q_aL \) given by
\[
L_{\text{mass}}^{\text{brane}} = - i \delta(y) \left\{ \sum_{\alpha=1}^{3} \mu_\alpha \hat{\chi}^\dagger_{\alpha R} Q_\alpha L + \bar{\mu} \hat{\chi}^\dagger_{2R} q_L \right\} + (\text{h.c.}) . \tag{6}
\]
The four brane mass parameters, \( \mu_\alpha \) and \( \bar{\mu} \) have dimensions of (mass)\(^{1/2}\). We suppose that \( \mu_\alpha^2, \bar{\mu}^2 \gg m_{KK} \). In this case the only relevant parameter for the spectrum at low energies turns out the ratio \( \bar{\mu}/\mu_2 \sim m_b/m_t \).

In ref. \cite{15} the spectrum of various fields were determined in the twisted gauge achieved by a gauge transformation
\[
\Omega(z) = \exp \left\{ i \theta(z) \sqrt{2} T^4 \right\} , \quad \theta(z) = \frac{z_2 L}{z \tau - 1} \theta_H . \tag{7}
\]
In the twisted gauge \( \bar{A}_M = \Omega A_M \Omega^\dagger - (i/g) \Omega \partial_M \Omega^\dagger \) and the background field vanishes, \( \langle \bar{A}_M \rangle = 0 \), but the boundary conditions at \( z = 0 \) get twisted from the original ones.

The fields in the bulk satisfy the free equations in the linear approximation. The equations in the bulk for the fermion fields \( \bar{\Psi} = z^{-2} \Omega \Psi \) with the bulk mass parameter \( c \) simplify to
\[
\left\{ \begin{pmatrix} \sigma \partial \\ \bar{\sigma} \partial \end{pmatrix} - k \begin{pmatrix} D_-(c) \\ D_+(c) \end{pmatrix} \right\} \begin{pmatrix} \bar{\Psi}_R \\ \bar{\Psi}_L \end{pmatrix} = 0 \tag{8}
\]
where \( D_\pm(c) = \pm (d/dz) + (c/z) \). Various fields mix among themselves through the brane mass terms in (6) and the twisted boundary conditions caused by \( \Omega(z) \) in (7). The \( z \)-dependence of the solutions to (8) is expressed in terms of the Bessel functions. The basis functions are given by
\[
\begin{pmatrix} C_L \\ S_L \end{pmatrix} (z; \lambda, c) = \pm \frac{\pi}{2} \lambda \sqrt{z_2 z_L} F_{c+1/2, c+1/2} (\lambda z, \lambda z_L) ,
\]
\[
\begin{pmatrix} C_R \\ S_R \end{pmatrix} (z; \lambda, c) = \mp \frac{\pi}{2} \lambda \sqrt{z_2 z_L} F_{c-1/2, c+1/2} (\lambda z, \lambda z_L) , \tag{9}
\]
where \( F_{\alpha,\beta}(u,v) = J_\alpha(u)Y_\beta(v) - Y_\alpha(u)J_\beta(v) \). They satisfy the relations \( S_L(z;\lambda,-c) = -S_R(z;\lambda,c) \) and \( C_L C_R - S_L S_R = 1 \). They also obey the boundary conditions that \( C_R = C_L = 1 \), \( D_- C_R = D_+ C_L = 0 \), \( S_R = S_L = 0 \) and \( D_- S_R = D_+ S_L = \lambda \) at \( z = z_L \). Further \( D_\pm \) links them by \( D_+(C_L,S_L) = \lambda(S_R,C_R) \) and \( D_-(C_R,S_R) = \lambda(S_L,C_L) \).

In the \( Q_{EM} = \frac{2}{3} \) sector (the top sector) \( U, B, t, t', \tilde{U}_R \) and \( \tilde{B}_R \) mix with each other. The top quark component \( t(x) \) in four dimensions is contained in these fields in the form

\[
\begin{pmatrix}
\tilde{U}_L \\
(\tilde{B}_L \pm i_L)/\sqrt{2} \\
\tilde{p}_L
\end{pmatrix}
(x,z) = \sqrt{k}
\begin{pmatrix}
a_U C_L(z;\lambda,c_2) \\
a_B \pm i L(z;\lambda,c_1) \\
a_v S_L(z;\lambda,c_1)
\end{pmatrix}
t_L(x)
\]

\[
\begin{pmatrix}
\tilde{U}_R \\
(\tilde{B}_R \pm i_R)/\sqrt{2} \\
\tilde{p}_R
\end{pmatrix}
(x,z) = \sqrt{k}
\begin{pmatrix}
a_U S_R(z;\lambda,c_2) \\
a_B \pm i R(z;\lambda,c_1) \\
a_v C_R(z;\lambda,c_1)
\end{pmatrix}
t_R(x).
\]

The brane fermions are related to the bulk fermions by

\[
\hat{U}_R(x) = \frac{2}{\mu^2} U_R(x,1^+) = \frac{2}{\mu^*} t_R(x,1^+) , \quad \hat{B}_R(x) = \frac{2}{\mu_1^*} B_R(x,1^+) \quad (11)
\]

as follows from the equations of motion. We note that \( U_R, t_R \) and \( B_R \) develop discontinuities at the Planck brane. The top quark mass is given by \( m_t = k\lambda \). The coefficients \( a_j \)'s are common to both left- and right-handed components as a consequence of the equations of motion in the bulk \((\bar{\sigma}\partial \tilde{U}_R = k D_+ \tilde{U}_L \text{ etc.})\) with the normalization \( \bar{\sigma} \partial t_R(x) = m_t t_L(x) \).

The eigenvalue \( \lambda \) and coefficients \( a_j \)'s are determined from the boundary conditions. The details of the computations were given in ref. [15]. Let us denote \( s_H = \sin \theta_H, c_H = \cos \theta_H, \) and \( C_L^{(j)} = C_L(1;\lambda,c_j) \) etc. The coefficients satisfy \( s_H a_B \pm i C_L^{(1)} = c_H a_v S_L^{(1)} \) and

\[
K \begin{bmatrix}
a_U \\
(a_B + i + c_H^{-1} a_B - i) / \sqrt{2} \\
(a_B + i + c_H^{-1} a_B - i) / \sqrt{2}
\end{bmatrix} = 0 ,
\]

\[
K = \begin{bmatrix}
\lambda s_R^{(2)} - \frac{|\mu_2|^2}{2k} C_L^{(2)} & -\frac{|\mu_2|^2}{2k} C_L^{(1)} & 0 \\
\frac{\mu^* \mu_2}{2k} C_L^{(2)} & \lambda s_L^{(1)} - \frac{|\mu|^2}{2k} C_L^{(1)} & -\frac{\lambda s_H^2}{2} S_L^{(1)} \\
0 & -\lambda s_H^2 S_L^{(1)} & 2\lambda s_L^{(1)} - \frac{|\mu_1|^2}{k} C_L^{(1)}
\end{bmatrix}
\]

(12)
Table I: With the value of $z_L$ given, $k, \lambda, c_1 = c_2 = c$ are determined. Input parameters are the $W$ boson mass $m_W = 80.40$ GeV and the top quark mass $m_t = 172$ GeV

| $z_L = e^{kL}$ | $k$(GeV) | $\lambda(\theta_H = \pi/2)$ | $c$ | $m_{KK}$(TeV) |
|----------------|----------|-----------------------------|-----|----------------|
| $10^{15}$      | $4.70 \times 10^{17}$ | $3.66 \times 10^{-16}$ | 0.432 | 1.48 |
| $10^{10}$      | $3.83 \times 10^{12}$ | $4.49 \times 10^{-11}$ | 0.396 | 1.20 |

Figure 1: The $\theta_H$-dependence of $\lambda z_L$ of the top quark for $z_L = 10^{10}$ and $z_L = 10^{15}$. The top mass is given by $m_t = \lambda k$. The plots fit well with $\kappa \sin \theta_H$ as in (14).

where $\bar{S}^{(1)} = S^{(1)}_R + (s_H^2/2S^{(1)}_L)$. The top mass, or the eigenvalue $\lambda$, is determined by the condition $\det K = 0$. When $|\mu_j|^2, |\tilde{\mu}|^2 \gg m_{KK}$, the equation is approximated, to high accuracy, by

$$|\mu_2|^2 C^{(2)}_L \left\{ S^{(1)}_R + \frac{s_H^2}{2S^{(1)}_L} \right\} + |\tilde{\mu}|^2 C^{(1)}_L S^{(2)}_R = 0 \quad .$$

(13)

The first term in (13) dominates over the second. With given $z_L, c_1$ is fixed so as to reproduce the observed $m_t \sim 172$ GeV at $\theta_H = \frac{1}{2} \pi$. See Table I. With these parameters fixed, the $\theta_H$-dependence of $m_t$ is determined numerically, which is depicted in Fig. I for $z_L = 10^{10}$ and $10^{15}$. The curves fit well with

$$m_t \sim \frac{m_{KK}}{\sqrt{2\pi}} \sqrt{1 - 4c_1^2} |\sin \theta_H|$$

with an error of $2.0\% \sim 4.0\%$. The top mass $m_t = \lambda k$ vanishes at $\theta_H = 0$ as the chiral symmetry is restored. The effective potential $V_{\text{eff}}(\theta_H)$ is evaluated from the $\theta_H$-dependence of the mass spectrum. It was found that the contribution from the top quark dominates over those from gauge fields and other fermions. $V_{\text{eff}}$ is minimized at $\theta_H = \pm \frac{1}{2} \pi$.

To be definite, let us take $\mu_j, \tilde{\mu} > 0$ given by

$$\mu_1^2 = \mu_2^2 = 10^{10} \text{GeV} , \quad \tilde{\mu}^2 = 5.96 \times 10^6 \text{GeV} ,$$

(15)
which, a posteriori, leads to the value $m_b/m_t \sim 4.2/172$ for $c_1 = c_2$. With the value $\lambda$ for the top quark, $\lambda S_R/[(\mu_2^2/2k)C_L]$ in the matrix $K$ in (12), for instance, is $O(10^{-15})$ so that the equation (12) is well approximated by

$$
\begin{pmatrix}
|\mu_2|^2 C_L^{(2)} & \mu_2^* \tilde{\mu} C_L^{(1)} \\
\tilde{\mu}^* \mu_2 C_L^{(2)} & |\mu_2|^2 C_L^{(1)} \\
0 & 0
\end{pmatrix} \begin{pmatrix} a_U \\ a_{B+t} - c_H^{-1} a_{B-t} \end{pmatrix} / \sqrt{2} \sim 0.
$$

It follows that

$$
[a_{B-t}, a_U, a_{U'}] \sim \begin{pmatrix} -c_H, -\sqrt{2} \tilde{\mu} C_L^{(1)} / \mu_2 C_L^{(2)}, -s_H C_L^{(1)} / S_L^{(1)} \end{pmatrix} a_{B+t}.
$$

The coefficient $a_{B+t}$ is determined so as to have canonical normalization for the kinetic term of $t_L(x)$. Note that $\lambda$ depends on $\theta_H$.

In the $Q_{EM} = -\frac{1}{3}$ sector (the bottom sector) $b, D, X, b', \hat{D}_R$ and $\hat{X}_R$ mix with each other. As in the top sector, the bottom quark component $b(x)$ in four dimensions appears as

$$
\begin{pmatrix}
\tilde{b}_L \\
(\hat{D}_L \pm \hat{X}_L) / \sqrt{2}
\end{pmatrix} (x, z) = \sqrt{k} \begin{pmatrix} a_b C_L(z; \lambda, c_1) \\ a_{D+X} C_L(z; \lambda, c_2) \\
a_{U'} S_L(z; \lambda, c_2)
\end{pmatrix} b_L(x)
$$

$$
\begin{pmatrix}
\tilde{b}_R \\
(\hat{D}_R \pm \hat{X}_R) / \sqrt{2}
\end{pmatrix} (x, z) = \sqrt{k} \begin{pmatrix} a_b S_R(z; \lambda, c_1) \\ a_{D+X} S_R(z; \lambda, c_2) \\
a_{U'} C_R(z; \lambda, c_2)
\end{pmatrix} b_R(x).
$$

The brane fermions are related to the bulk fermions by

$$
\hat{D}_R(x) = 2 \mu_2^* R_2 D_R(x, 1^+) = 2 \mu_3^* b_R(x, 1^+) , \quad \hat{X}_R(x) = 2 \mu_3^* X_R(x, 1^+)
$$

The equation corresponding to (12) is obtained by replacing $(U, B, t)$ by $(b, D, X)$ and interchanging $(c_1, c_2)$, $(\mu_1, \mu_3)$ and $(\mu_2, \tilde{\mu})$. In the same approximation as in the top case the bottom mass and the coefficients $a_j$'s are found, for $0 < c_1, c_2 < \frac{1}{2}$, to be

$$
m_b \sim \sqrt{\frac{1 + 2c_2}{1 + 2c_1}} \frac{\mu}{\mu_2} |\tilde{\mu}| c_1^{1-c_2} m_t
$$

and

$$
[a_{D+X}, a_{D-X}, a_{U'}] \sim \begin{pmatrix} -1, c_H, s_H C_L^{(2)} / S_L^{(2)} \end{pmatrix} a_b.
$$
With the wave functions of the top and bottom quarks at hand, one can evaluate their Yukawa couplings in two manners. In the Kaluza-Klein approach we insert the wave functions into the five-dimensional Lagrangian density $\mathcal{L}_\text{fermion} + \mathcal{L}_\text{brane}$ and integrate over the fifth dimensional coordinate to obtain four-dimensional Lagrangian. The part \(k^{-1} \sum_{j=1}^{2} \overline{\Psi}_j (\gamma \partial)_{d=4} \Psi_j\) gives the four-dimensional kinetic terms for the top and bottom quarks. The part with the covariant derivative in the fifth coordinate
\[
\sum_{j=1}^{2} \left\{-i \overline{\Psi}_j^L \left(D_-(c_j) + ig_A \tilde{A}_z \right) \Psi_j^R + i \overline{\Psi}_j^R \left(D_+(c_j) - ig_A \tilde{A}_z \right) \Psi_j^L \right\}
\]
generates both the masses and Yukawa couplings of the top and bottom quarks. The 4D Higgs field is contained in the gauge potential $A_z$. The vev $v$ of $\phi^i(x)$ in (2) is related to $\theta_H$ by (3) and its fluctuation around $v$ corresponds to the neutral Higgs field $H(x)$. Hence the relevant part of the gauge potential is expressed as
\[
A_z(x,z) = \hat{\theta}_H(x) \cdot \frac{2\sqrt{2} z}{z_L^2 - 1} \cdot T^i + \ldots
\]
in the original gauge where
\[
\hat{\theta}_H(x) = \theta_H + \frac{H(x)}{f_H}, \quad f_H = \frac{2}{g_A} \sqrt{k} \sqrt{\frac{1}{z_L^2 - 1}} \sim \frac{2}{\sqrt{k L \pi g}} m_{KK} .
\]
In the twisted gauge defined in (7), $A_z^c = \langle \tilde{A}_z \rangle$ vanishes, $A_z(x,z)$ being expanded as in (23) with $\hat{\theta}_H$ replaced by $H(x)/f_H$.

The Yukawa coupling originates from $g_A(\Psi_L^+ A_z \Psi_R + \Psi_R^+ A_z \Psi_L)$ or $g_A(\tilde{\Psi}_L^+ \tilde{A}_z \tilde{\Psi}_R + \tilde{\Psi}_R^+ \tilde{A}_z \tilde{\Psi}_L)$, whereas the mass term comes from $-i \Psi_L^+(D_- + ig_A A_z^c) \Psi_R + i \Psi_R^+(D_+ - ig_A A_z^c) \Psi_L$ in the original gauge or $-i \tilde{\Psi}_j^L D_- \tilde{\Psi}_j^R + i \tilde{\Psi}_j^R D_+ \tilde{\Psi}_j^L$ in the twisted gauge. The terms involving $D_\pm$ are important. With the wave function in (2), (10) and (18) inserted, $\varphi_H(z) \tilde{\Psi}_j^L T^i \tilde{\Psi}_j^R$ (or $\varphi_H(z) \tilde{\Psi}_j^R T^i \tilde{\Psi}_j^L$) has different $z$-dependence from $\tilde{\Psi}_j^L D_- \tilde{\Psi}_j^R$ ($\tilde{\Psi}_j^R D_+ \tilde{\Psi}_j^L$). After the integration over $z$, the Yukawa coupling is not proportional to the fermion mass in the RS spacetime. We also recall that a large gauge transformation generates $\theta_H \rightarrow \theta_H + 2\pi$ so that the mass spectrum remains invariant under the shift $\theta_H \rightarrow \theta_H + 2\pi$, or equivalently under $H(x) \rightarrow H(x) + 2\pi f_H$. The mass is a periodic, nonlinear function of $\theta_H$. (There is no level-crossing in the RS spacetime.) The nonlinearity in the relation between the Yukawa coupling and mass is confirmed by direct evaluation described below.
Let us define the normalized coefficients \( a^L_j, a^R_j \) by

\[
(a^L_U, a^L_B, a^L_t) = \left( \sqrt{N_{CL}^{(2)}}, \sqrt{N_{CL}^{(1)}}, \sqrt{N_{SR}^{(1)}} \right) a_u',
\]

\[
(a^R_U, a^R_B, a^R_t) = \left( \sqrt{N_{SR}^{(2)}}, \sqrt{N_{SR}^{(1)}}, \sqrt{N_{CL}^{(1)}} \right) a_v',
\]

where \( N_{CL}^{(j)} = \int_{z=1}^{z_L} dz C_L(z; \lambda, c_j)^2 \) etc.. Then the free part of the Lagrangian for the top quark is found to be

\[
L_{\text{free}}^4 \sim -P_L \bar{t}_L^\dagger \sigma \partial t_L + P_R \bar{t}_R^\dagger \sigma \partial t_R + \lambda k \frac{P_L + P_R}{2} (i t_L^\dagger t_R - i t_R^\dagger t_L),
\]

\[
P_{L,R} = |a^L_{U,R}|^2 + |a^L_{B+1}|^2 + |a^L_{B-1}|^2 + |a^R_{U,R}|^2.
\]

The contributions coming from the brane mass term \( L_{\text{brane}}^\text{mass} \) turn out \( O(10^{-15}) \) smaller than \( P_L \) and \( P_R \), and can be ignored.

Recall that \( D_- S_R = \lambda C_L \) and \( D_+ C_L = \lambda S_R \), from which it follows that \( N_{CL} = N_{SR} + \lambda^{-1} S_RC_L |_{z=1} \). Hence

\[
P_L = P_R + \frac{1}{\lambda} \left\{ S_{CL}^{(2)} |a_U|^2 + S_{CL}^{(1)} \left( |a_{B+1}|^2 + |a_{B-1}|^2 \right) + S_{CL}^{(1)} |a_{L}|^2 \right\}
\]

\[
= P_R + \frac{2}{\lambda} |a_{B+1}|^2 C_L \left\{ S_{SR}^{(1)} + \frac{s_H^2}{2 s_{LR}^{(1)}} + \frac{\mu_l^2}{\mu_2^2} S_{CL}^{(1)} \right\}
\]

\[
= P_R.
\]

The relations \( (17) \) and \( C_L C_R - S_L S_R = 1 \) have been used in the second equality. The last equality follows from the relation \( (13) \) determining the mass spectrum. Let us adopt the normalization \( P_L = P_R = 1 \) with which the top mass appears as \( \lambda k \) in \( (26) \) as it should.

The coefficients \( a^L_L \) and \( a^R_R \) represent how much portion of each field contains the left- and right-handed top quark, respectively.

Similarly the normalized coefficients \( a^L_b, a^L_{D\pm X}, a^L_{b'}, a^R_{L} \) are determined. The numerical values are tabulated in Table \( \text{I} \). The numerical values for the dominant terms \( (a^L_{B\pm 1}, a^L_{b'}, a^L_{D\pm X}, a^L_{b}, a^L_{D\pm X}, \text{ and } a^R_{b'}) \) do not vary very much with \( z_L \) in the range \( 10^{10} \) to \( 10^{15} \). In the \( \theta_H = 0 \) limit, the four-dimensional \( t_L(x) \) and \( t_R(x) \) are mostly contained in the five-dimensional \( t \) and \( t' \), respectively. At \( \theta_H = \frac{1}{2} \pi \), \( t_L(x) \) resides in the \( (B + t)/\sqrt{2} \) and \( t' \) components, whereas \( t_R(x) \) remains in \( t' \). The four-dimensional \( b_L(x) \) and \( b_R(x) \) are mostly contained, for any value of \( \theta_H \), in the five-dimensional \( b \) and \( b' \), respectively.
and the wave functions (10) into (22) in the twisted gauge, one finds, for the top quark, integrating over $\theta$ Table II: The coefficients (25) of the wave functions of the top and bottom quarks at $\theta_H = 0$ and $\frac{1}{2}\pi$, evaluated for $c_1 = c_2 = 0.43$, $z_L = 10^{15}$, and $\mu_j, \bar{\mu}$ in (15).

The Yukawa couplings are evaluated in the same manner. Inserting $\tilde{A}_z^4 = H(x)\varphi_H(z)$ and the wave functions (10) into (22) in the twisted gauge, one finds, for the top quark,

$$\sqrt{\det g} \, \mathcal{L}_Y = -\frac{i}{2} g_A H \varphi_H(z) \left\{ \tilde{t}_R^\dagger (\tilde{t}_L - \tilde{B}_L) + \tilde{t}_L^\dagger (\tilde{t}_R - \tilde{B}_R) \right\} - (\text{h.c.})$$

$$= -\frac{i}{\sqrt{2}} g_A k a_{\ell} a_{\ell'} \varphi_H(z) H(x) \left\{ t_R^\dagger t_L(x) - t_L^\dagger t_R(x) \right\}. \quad (28)$$

The overall phase of the $a_j$'s has been taken to be real. By making use of (17) and integrating over $z$, the 4D Yukawa coupling constant in $\mathcal{L}_{\text{Yukawa}}^{4D} = i y H(t_L^\dagger t_R - t_R^\dagger t_L)$ is found to be

$$y(\theta_H) = \frac{g \sqrt{k L(z_L^2 - 1)} s_{\text{H}} c_H C_L^{(1)}}{4 S_L^{(1)} \bar{P}},$$

$$\bar{P} = \frac{1 + c_{\text{H}}^2}{2} N_{S_L}^{(1)} + \frac{c_{\text{H}}^2}{2} \left( \frac{C_L^{(1)}}{S_L^{(1)}} \right)^2 N_{S_L}^{(1)} + \frac{\bar{\mu}^2}{|\mu|^2} \left( \frac{C_L^{(1)}}{C_L^{(2)}} \right)^2 N_{S_L}^{(2)}. \quad (29)$$

Note that $s_{\text{H}}/N_{S_L}^{(1)}$ remains finite in the $s_{\text{H}} \to 0$ limit. The $\theta_H$-dependence of $y(\theta_H)$ for the top quark is depicted in fig. 2 which is well approximated by the cosine curve. It is seen that $y$ vanishes at $\theta_H = \frac{1}{2}\pi$. The result for the bottom quark is similar to that for the top quark, with a magnitude scaled down by a factor $m_b/m_t$.

So far we have evaluated the masses and Yukawa couplings of the top and bottom quarks in the Kaluza-Klein approach. One can develop an effective interaction approach
Figure 2: The $\theta_H$-dependence of the Yukawa coupling for the top quark for $z_L = 10^{15}$. The curve is well approximated by a cosine curve. The curve has little dependence on $z_L$.

[12, 13, 38] to concisely summarize the results. It enables us for deducing the Higgs couplings in higher order as well.

In the original gauge $\theta_H$ and $H(x)$ always appear in the combination $\hat{\theta}_H(x)$ in (24). Therefore the effective local interactions at low energies, which manifest significant deviation from the standard model, can be written in the form

$$L_{\text{eff}} = -V_{\text{eff}}(\hat{\theta}_H) - m_W(\hat{\theta}_H)^2 W^\dagger_\mu W^\mu - \frac{1}{2} m_Z(\hat{\theta}_H)^2 Z_\mu Z^\mu - \sum_f m_f(\hat{\theta}_H) \bar{\psi}_f \psi_f.$$  

(30)

The key feature is that $\theta_H$ is a phase variable so that $L_{\text{eff}}$ is periodic in $\hat{\theta}_H$ with a period $2\pi$. The first term is the effective potential for $\hat{\theta}_H$. As shown in ref. [10], $V_{\text{eff}}$ is finite and the value of $\theta_H$ is unambiguously determined by the location of its global minimum. The Higgs mass $m_H$, given by $m_H^2 = V_{\text{eff}}^{(2)}(\theta_H)/f^2_H$, is predicted to be finite. $m_W(\hat{\theta}_H)$ and $m_Z(\hat{\theta}_H)$ in the $SO(5) \times U(1)_X$ model in the RS spacetime has been evaluated in refs. [10, 11];

$$m_W(\hat{\theta}_H) \sim \cos \theta_W m_Z(\hat{\theta}_H) \sim \frac{1}{2} g f_H \sin \hat{\theta}_H$$  

(31)

where $m_W = m_W(\theta_H)$, $m_Z = m_Z(\theta_H)$, and $\theta_W$ is the Weinberg angle. Expanding $m_W(\hat{\theta}_H)^2$ and $m_Z(\hat{\theta}_H)^2$ in (30) in a power series in $H$, one finds that $WWH$ and $ZZH$ couplings are suppressed by a factor $\cos \theta_H$ compared with those in the standard model. For the $WWHH$ and $ZZHH$ couplings the suppression factor becomes $\cos 2\theta_H$. As demonstrated by Sakamura, it includes the contributions of the KK towers of $W$ and $Z$ in the intermediate states. [13] The effective interactions contain contributions coming from heavy KK excited states.
We apply the same argument to the last term in (30). In this approach the Yukawa coupling $y_f H \overline{\psi}_f \psi_f$ is related to the mass by

$$y_f(\theta_H) = \frac{1}{f_H} \frac{d m_f(\theta_H)}{d \theta_H}. \quad (32)$$

The top quark mass $m_t(\theta_H)$ is determined from (13) as a function of $\theta_H$. Its derivative $d m_t(\theta_H)/d \theta_H$ is compared with the Yukawa coupling $y_t(\theta_H)$ in (29) determined in the Kaluza-Klein approach. We have numerically confirmed that the equality (32) between the two holds with an error less than 0.3% in the entire region of $\theta_H$, which establishes the validity and usefulness of the effective interaction approach. As is seen in fig. 1 the mass $m_t(\theta_H)$ reaches the maximum at $\theta_H = \frac{1}{2} \pi$. The relation (32) implies that the Yukawa coupling $y_t(\theta_H)$ vanishes there, which, independently, is shown in the Kaluza-Klein approach as well. In the effective interaction approach the $HH \overline{\psi}_f \psi_f$ coupling, is given by $m_f^{(2)}(\theta_H)/f_H^2$. In the HOOS model $m_f(\theta_H) \sim \kappa_f \sin \hat{\theta}_H$ and $\theta_H = \frac{1}{2} \pi$. Although the Yukawa coupling $y_f$ vanishes, the $HH \overline{\psi}_f \psi_f$ coupling is nonvanishing ($\sim -m_f/f_H^2$). The KK excited states of $\psi_f$ contribute in the intermediate states for the $HH \overline{\psi}_f \psi_f$ coupling.

In this paper we have given detailed analysis of the Yukawa couplings in the $SO(5) \times U(1)$ gauge-Higgs unification model, particularly in the HOOS model[15]. We have determined the wave functions of the top and bottom quarks in the extra-dimensional space, with which the Yukawa couplings are evaluated numerically in the Kaluza-Klein approach. We have also shown that all the results are concisely cast in the form of the effective interactions.

The phenomenological implication is significant. In the gauge-Higgs unification scenario the large deviation from the standard model of electroweak interactions appears in the Higgs couplings. All of the $WWH$, $ZZH$, and Yukawa couplings are suppressed by a factor $\cos \theta_H$, which can be checked in the forthcoming experiments at LHC. In the HOOS model, in particular, $\theta_H = \frac{1}{2} \pi$ is dynamically realized, leading to completely new phenomenology. The Higgs particle becomes stable in the low energy effective theory at the tree level. It is interesting to see whether or not the Higgs particle can decay at all through heavy KK excited states. We will come back on this issue in a separate paper in more detail.

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