The students’ understanding of mathematical concepts in resolving the proof of induction

T Wibowo, Fatmawati, and D Yuzianah
Universitas Muhammadiyah Purworejo, Purworejo Jawa Tengah Indonesia
Email: twibowo@umpwr.ac.id

Abstract. The objective of this study is to determine the high school students’ understanding of the mathematical concepts in resolving the proof of induction related to understanding $P(1)$, $P(k)$, and $P(k + 1)$. This research use qualitative methode. The subjects of this study were 2 high school students of grade XI in Purworejo District in the 2018/2019 school year. The technique of taking subjects in this study was purposive. Data collection was done by comprehension tests, observation sheets, and interviews. Data analysis techniques were data reduction, data presentation, and conclusion. The finding shows that understanding the concept is a part of mathematical understanding. Students in this study are able to know the requirements of mathematical induction such as $P(1)$, $P(k)$, and $P(k + 1)$, able to define mathematical induction requirements in writing, be able to apply mathematical induction requirements according to the proving step, and recognize the meaning and mathematical induction interpretation. Mathematical understanding consists of three types. They are translation, interpretation, and extrapolation. Understanding of mathematical concepts in this study on the resolving the proof of induction belongs to the category of extrapolation.

1. Introduction
The human life with use of mathematics. Mathematics can be found in the fields of economic, education, health, politic, and other fields. Understanding of mathematics are important. Understanding of mathematics or not is determined by the students themselves. Students who difficult in understanding of mathematics make they hate mathematics.

Conceptual understanding is comprehension of mathematical concepts, operations and relations [1,2]. Conceptual understanding mean understanding old information to combine with new information so becomes a knowledge and can be used to solve problems. Relations is relates material between one material and another. Conceptual understanding is an essential component of knowledge needed to deal with problems and settings, thats mean conceptual understanding is important component of knowledge to deal with problems [3,4]. Problems can be solved easily if practice frequently. Through practice helps make conceptual understanding by itself. Concept that havedbeen student with a higher level than lower level of student is not communicated by a definition, but only with the example that approriate. Students who have high order thinking sometimes do not need information at length. Only with one example students who have high order thinking quickly understand. Conceptual understanding of mathematics occurs when the teacher is explaining the material or solving mathematical problems and make learning mathematics meaningfully [5].

In the Education Unit Level Curriculum mathematics that the purpose of mathematics at the level of basic education is students have the ability to mathematical understanding which includes: understanding mathematical concepts, explaining the relationship between concepts and applying concepts or algorithms, flexibly, accurate, efficient, and appropriate in solving problems [6,7]. So it can
be concluded that understanding mathematical concepts is part of mathematical understanding. According to Kusmano and Marliyana [8] that understanding ability can be translated into three, namely: a) Translating understanding, translating not only the translation of the meaning of one language into another language. Can also from abstract conception into a model, namely a symbolic model to make it easier for people to learn it, b) Interpretation is broader than translating, this is the ability to recognize and understand the main idea of a communication, c) Extrapolation is different from translating and interpreting, but higher in nature. In addition Miyazaki, et al [9] identifies the ability of mathematical understanding in three types of cognitive behavior, changing one mathematical form to another, interpreting a concept, principle, and mathematical expression, and interpolating and extrapolating a tendency of data.

We can conclude that conceptual understanding is part of mathematical understanding. Understanding of mathematical concepts in mathematical induction settlement is the ability to understand mathematical induction so that a knowledge is formed to complete mathematical induction. According to the author's assumption that mathematical understanding is: 1) translation means knowing mathematically procedurally, 2) interpretation means applying mathematical ideas procedurally, and 3) extrapolation means understanding accompanied by explanation or reason. Therefore, in the completion of mathematical induction can be adjusted with conceptual understanding to be directed to mathematical understanding. The intended mathematical understanding is translation, interpretation, and extrapolation.

Conceptual understanding is needed in mathematics learning. Conceptual understanding is the goal of learning mathematics. Through conceptual understanding students can solve mathematical problems. Understanding of mathematical induction many use symbols and logical statements rather than numbers. Proof more complicated than other work. Proof and proving in mathematics is necessity at school [10-13]. The teaching and learning of proof is recognised internationally as a key component of mathematics education and thus of mathematics curricula [14]. The proof explains that mathematical statements are true with certain axiom assumptions. Zaslavsky, et al [15] argues that teaching proof is to shed light on the origins and connections of mathematical knowledge. Mathematical induction is one of lesson in mathematics for student of senior high school. According to Ron and Dreyfus [16] that proof by mathematical induction is a method to prove statements that are true for every natural number can be interpreted as proof of mathematical induction is a method of proving a true number for each natural number.

Based on the observation that students confused in proof of mathematics induction, when starting where students assume $P(1)$ means substituting $n = 1$. In the step of induction students assume $P(k)$ means substituting $n = k$, and $P(k + 1)$ means that substituting $k = k + 1$. Therefore, the author will do research on high school students about mathematical induction to know how far the understanding of $P(1)$, $P(k)$, and $P(k + 1)$ The aim of research is to know the understanding of the mathematical concepts of high school students in resolving the proof of induction related to the understanding of $P(1)$, $P(k)$, and $P(k + 1)$.

2. Method
The type of research is qualitative. The study was held in Purworejo Regency for 8 (eight) months. The subject of this study was eleventh grade high school students in Purworejo Regency. Subjects as informants to get much information. Researchers used purposive to take subjects in research.

The purposive technique, also called judgment sampling, is the deliberate choice of a participant due to the qualities the participant possesses [17]. This study aim to know the understanding of students' concepts. This research need subjects who have requirements. The requirements are students have received mathematical induction and high school students who have a high ability to resolve proof of induction. Researchers do not choose students who have medium and low abilities because conceptual understanding is high order thinking ability. If the data source has not been able to provide satisfactory data, a larger number of data sources are needed. This will continue until there are no different answers found by the researcher until saturated data was obtained. The primary data source of this study is
students who are able to resolved mathematical induction with high ability and subject to participate in data collection during the study.

Data collection in this study uses triangulation. In this study used technical triangulation. The other method to collection data is test, observation, and interview. Researchers use concept understanding tests. This test to help data collection. The test was given one question on concept understanding of mathematical. This study uses participant observation. The things that are observed are the student's process when working on the problem. In this study researchers used interviews. Moleong [18] said the main instrument in qualitative research was the researcher himself. The supporting instrument in this study is a mathematical induction of test instrument. The test instrument consists of one question for taking subjects and one question for research. The questions used for taking subject different from the questions that used for research. To strengthen the validity of the instrument, the validity of the instrument is needed, including the research test, and the observation guideline. Data analysis techniques were data reduction, data presentation, and conclusion [19].

3. Result and Discussion
Research about conceptual understanding of mathematical concepts on high school students in resolving proof of induction has the aim to know understanding of mathematical concepts on high school students in completing proof of induction related to understanding \( P(1) \), \( P(k) \), and \( P(k + 1) \). Taking of data in this study was collected on several subjects of class XI Senior High School in Purworejo. This research get five prospective subjects who meet based on the conceptual understanding test worksheet, observation sheet and interview. These data have almost the same results and can be said to be identical. For five prospective subjects with consideration of the test worksheet, observation sheet, and interview, the researcher took two prospective subjects to be research subject to represent the data expected by the researcher. The subject uses purposive technique.

The results of the test, observation, and interview. These three forms of data will be used as benchmarks for researcher to conclude how to understand the mathematical concepts of high school students in resolving mathematical induction. The test instrument used in the form of a test that refers to the conceptual understanding indicators and observation sheets used to observe what is written on the subject work sheet. The interview is done after the student has finished working on. Here's one question in Figure 1 about understanding the mathematical concepts that are done by the subject:

\[
\text{Prove that } 2 + 7 + 12 + \ldots + (5n - 3) = \frac{n(5n - 1)}{2} \text{ applies correctly to } n \geq 1 \text{ using mathematical induction!}
\]

**Figure 1.** The questions given to the subject

Subjects were given time to work on concept comprehension questions for 20 minutes. The following results from subject work. Analysis is done based on the results of the work, interview, and observation as long as the subject is working on the problem. Based on the results of data collection obtained from research. The following is an analysis of the data on understanding the mathematical concepts of subject (S1).
In Figure 2, S1 writes the induction requirement, namely \( n = 1 \). S1 can use \( n = 1 \) to get the same result between the right and left sections. S1 provides information for \( n = 1 \). It shows that S1 knows the induction requirements for \( n = 1 \). From interview conversation between the researcher (P) and the following subject (S1) obtain:

P: "... Why write \( n = 1 \) ?"
S1: "Because the terms of induction \( n \geq 1 \) and I take the number one".
P: "How to use \( n = 1 \) ?"
S1: "From \( 5n - 3 = \frac{n(5n-1)}{2} \) to \( n \) is replaced by 1 obtained answer 2".
P: "Why do you give correct information?"
S1: "Because the results are the same for \( n = 1 \), so I give correct information".

The interview above shows that S1 writes \( n = 1 \) based on the conditions given in the question. How to use \( n = 1 \) is done by substituting /replacing \( n \) with number one. S1 also provides information for \( n = 1 \) seen from the results. Other information is also obtained through observation. Observation is done by the researcher when S1 works. The following are the observations on S1.

| No | Aspek yang Diamati                        | Keterlaksanaan |
|----|------------------------------------------|----------------|
| 1. | Students write mathematical induction requirements \( n = 1 \) | \( \checkmark \) |
| 2. | Students write mathematical induction requirements \( n = k \) | \( \checkmark \) |
| 3. | Students write mathematical induction conditions \( n = k + 1 \) | \( \checkmark \) |
| 4. | Students write down proved to be true for \( n = 1 \) | \( \checkmark \) |
| 5. | Students write assumed to be true for \( n = k \) | \( \checkmark \) |
| 6. | Students write down proven true for \( n = k + 1 \) | \( \checkmark \) |
| 7. | Students describe the usage of \( n = 1 \) | \( \checkmark \) |
| 8. | Students describe the usage of \( n = k \) | \( \checkmark \) |
| 9. | Students describe the usage of \( n = k + 1 \) | \( \checkmark \) |
| 10. | Students write conclusions from the results of the proof | \( \checkmark \) |

Figure 3. S1 observation results for \( n = 1 \)

The results in Figure 3 indicate that S1 writes the induction requirements for \( n = 1 \). S1 writes correctly instead of being proven correct for \( n = 1 \) on the worksheet. However, in this case the researcher still considers that S1 still writes proven to be true because it has almost the same meaning. S1 describes the use of \( n = 1 \) correctly.
From Figure 4, shows that S1 writes the induction condition, namely \( n = k \). S1 can use \( n = k \). It is seen that S1 write \( n = k \). S1 also provides information for \( n = k \). This shows that S1 knows the induction requirements for \( n = k \). From interview conversation with subjek obtain data:

P : "Why write \( n = k \)?
S1 : "If \( n = 1 \) is correct, then it is assumed that \( n = k \) is also correct".
P : "What is mean by \( n = k \)?"
S1 : "\( k \) can be replaced with other natural numbers, for example 2, 3, etc."
P : "How to use \( n = k \)?"
S1 : "\( n \) is replaced by \( k \), just like when we do \( n = 1 \)"
P : "Explain the purpose of the result assumption \( n = 1 \)"
S1 : "If \( n = 1 \) is correct, we assume for \( n = k \) right".

The interview above shows that S1 write \( n = k \) as the correct assumption of \( k \) if it is replaced by an original number. How to use \( n = k \) is done by substituting/ replacing \( n \) with \( k \). S1 also gives an explanation for \( n = k \) seen from experience working \( n = 1 \), so that it has an assumption for \( n = k \).

Other information is also obtained through observation. Observation is done by the researcher when S1 works. The following are the observations on S1.

| No | Aspek yang Diamati                                      | Keterlaksanaan |
|----|----------------------------------------------------------|----------------|
| 1. | Students write mathematical induction requirements \( n = 1 \) | Ya             |
| 2. | Students write mathematical induction requirements \( n = k \) | Ya             |
| 3. | Students write mathematical induction conditions \( n = k + 1 \) | Ya             |
| 4. | Students write down proved to be true for \( n = 1 \)      | Ya             |
| 5. | Students write assumed to be true for \( n = k \)          | Ya             |
| 6. | Students write down proven true for \( n = k + 1 \)        | Ya             |
| 7. | Students describe the usage of \( n = 1 \)                 | Ya             |
| 8. | Students describe the use of \( n = k \)                   | Ya             |
| 9. | Students describe the use of \( n = k + 1 \)               | Ya             |
| 10. | Students write conclusions from the results of the proof    | Ya             |

**Figure 5.** S1 observation results for \( n = k \)

From Figure 5, indicate that S1 write the induction requirements for \( n = k \). S1 write correctly if it is not proven true for \( n = k \) on the worksheet. However, in this case the researcher still considers that S1 still write proven to be true because it has almost the same meaning. S1 describes the use of \( n = k \) correctly.
Figure 6. S1 work results related to $n = k + 1$

From Figure 6, shows that S1 write the induction condition that is $n = k + 1$. S1 can use $n = k + 1$ to get the same result between the left and right segments. It is seen that S1 rewrite the question before using $n = k + 1$. S1 also give an explanation for $n = k + 1$. It shows that S1 knows the induction requirement for $n = k + 1$. In addition, data obtained from the interview:

P : "Why write $n = k + 1$?"
S1 : "Because after $k"  
P : "How to use $n = k + 1$?"
S1 : "If there is $n = k$, then there is $n = k + 1$ which mean that after $5k - 3$ there are $5(k + 1)$ -3. If $\frac{k(5k-1)}{2}$ then $\frac{k+1(5(k+1)-1)}{2}$. So that from $2 + 7 + 12 + \ldots + (5k - 3) + (5k + 1) -3$ $\frac{k+1(5(k+1)-1)}{2}$ becomes $\frac{k+1(5(k+1)-1)}{2} + 5k - 2 = \frac{k+1(5(k+1)-1)}{2}$".

P : "What is the result?"
S1 : "The result are the same for both segments"
P : "Explain the information you write"
S1 : "Because $n = k$ is still an assumption, to prove whether $k$ is right or wrong by substituting $n = k + 1$, and the result are correct for both segments".

The interview above shows that S1 write $n = k + 1$. How to use $n = k + 1$ is done by substituting / replacing $n$ with $k + 1$. S1 also gives information proven to be true for $n = k + 1$ seen from the final result of the two sections. In addition, data is also obtained through observation. Observation is done by the researcher when S1 works. The following are the observations on S1.
The result of Figure 7, indicate that S1 write the induction requirement for \( n = k + 1 \). S1 write that it is proven correct on the worksheet. S1 describes the use of \( n = k + 1 \) correctly so that the same result is obtained between the right and left segments.

From Figure 8, shows that S1 conclude by looking at the final result of the application of \( n = k + 1 \) for the same right and left segments. This shows that S1 can make conclusions from previous information. From interview conversation also obtained:

\[ P : \text{"What can you conclude?"} \]

\[ S1 : \text{"} 2 + 7 + 12 + \ldots + (5n - 3) = \frac{n(5n-1)}{2} \text{ is correct because it can be proved that } n = 1, n = k, \text{ and } n = k + 1 \text{ is the same result. So } 2 + 7 + 12 + \ldots + (5n - 3) = \frac{n(5n-1)}{2} \text{ applies to all natural numbers".} \]

The interview above shows that S1 write conclusion, S1 conclusion based on the information contained in the problem. In addition, observation is done by the researcher when S1 work. Following are the observation on S1.

| No | Aspek yang Diamati                              | Keterlaksanaan |
|----|-----------------------------------------------|----------------|
| 3  | Students write mathematical induction conditions \( n = k + 1 \) | √              |
| 4  | Students write down proved to be true for \( n = 1 \) | √              |
| 5  | Students write assumed to be true for \( n = k \) | √              |
| 6  | Students write down proven true for \( n = k + 1 \) | √              |
| 7  | Students describe the usage of \( n = 1 \)            |                |
| 8  | Students describe the use of \( n = k \)              |                |
| 9  | Students describe the use of \( n = k + 1 \)          |                |
| 10 | Students write conclusions from the results of the proof | √              |

Figure 7. S1 observation results for \( n = k + 1 \)

Figure 8. S1 work result related to conclusions

Figure 9. S1 observation result

Figure 9 result of observation, S1 write the result of verification. This shows that S1 is able to understand the information about the problem. The first thing the subject does when working on mathematical induction is to write \( n=1 \) means that the subject knows the basics/basic requirements that exist in mathematical induction based on the requirements of the given, then the subject describes the use of \( n=1 \) means that the subject can apply induction based on his knowledge, as well as the subject giving information for \( n=1 \) means that the subject can define mathematical induction requirements in writing. Next the subject write \( n=k \) means that the subject know the requirement in mathematical
induction based on the step of induction, then the subject describes the use of $n=k$ means that the subject applies the induction according to the mathematical induction of proving step, and gives an explanation for $n=k$ means that the subject can define mathematical induction requirements in writing. Next the subject write $n=k+1$ means that the subject knows the requirement in mathematical induction based on the induction step, then the subject describes the use of $n=k+1$ means that the subject applies the induction according to the mathematical induction proving step, and gives an explanation for $n=k+1$ means that the subject can define mathematical induction requirements in writing. Finally the subject of writing a conclusion from the results of the proof means that the subject can recognize the meaning and interpretation of mathematical induction.

In resolving of mathematical induction the subject use induction requirement of the problem given in the form of $n = 1, n = k$, and $n = k + 1$. It is proved by the subject who wrote on the answer sheet the results of the question. This means that the subject passes a mathematical understanding of the type of translation. According to Kusmanto and Marliyana [8] that translating (understanding) translates here not only the transfer (meaning) of the meaning of one language into another, it can also be from an abstract conception into a model, namely a symbolic model for make it easier for people to learn it. Subject are able to translate symbols on induction and are able to apply or use on condition that induction is proofed by the description of the subject work related to $n = 1, n = k$, and $n = k + 1$. This indicates that the subject passes a mathematical understanding of the type of interpretation. According to Kusmanto and Marliyana [8] that interpreting this ability is broader than translating, this is the ability to recognize and understand the main idea of communication.

The subject is able to understand the induction problem given to be able to describe the use of $n = 1, n = k$, and $n = k + 1$, and after completing work is able to make a conclusion based on the previous data. This shows that the subject is included in a mathematical understanding of the type of extrapolation. However, this is again reminiscent of the subject included in translation and interpretation, that the subject included in translation and interpretation can be said to be a subject including extrapolation. The subject is said to be extrapolation if it meets translation and interpretation. According to Kusmanto and Marliyana [8] that extrapolating (extrapolation) is rather different from translating and interpreting, but higher in nature. In the Education Unit Level Curriculum mathematics that the purpose of mathematics at the level of basic education is students have the ability to mathematical understanding which includes: understanding mathematical concepts, explaining the relationship between concepts and applying concepts or algorithms, flexibly, accurate, efficient, and appropriate in solving problems [6]. So it can be concluded that understanding mathematical concepts is part of mathematical understanding. Therefore, from the results of the study it was found that subjects included in the category of understanding concepts also met the criteria for mathematical understanding of the type of extrapolation. Thus, understanding mathematical concepts is part of a mathematical understanding of the type of extrapolation.

4. Conclusion

Students in this study are able to know the requirements of mathematical induction such as $P(1)$, $P(k)$, and $P(k+1)$, able to define mathematical induction requirements in writing, be able to apply mathematical induction requirements according to the proving step, and recognize the meaning and interpretation of mathematical induction. Students can resolve mathematical induction based on their understanding. Mathematical understanding consists of three types. They are translation, interpretation, and extrapolation. Mathematical understanding is: 1) translation means knowing procedurally mathematical induction, 2) interpretation means applying procedurally mathematical induction, and 3) extrapolation means understanding mathematical induction with an explanation or reason. Students included in translation and interpretation can be said that students are included in extrapolation. Or students are said to be extrapolation if they have fulfilled translation and interpretation.

Based on the research that students can understanding and knowledge in applying and understanding mathematical induction according the step of induction resolving. The results showed that students who category of conceptual understanding also can be criteria of extrapolation. Understanding of mathematical concepts in this study on the resolving the proof of induction belongs to the category of extrapolation.
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