Comparison of Lattice Coulomb
Gauge Wave Functions in Quenched Approximation and with Dynamical Fermions

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ABSTRACT:

We present a comparison of Coulomb gauge wave functions from $6/g^2 = 6.0$ quenched simulations with two simulations which include the effects of dynamical fermions: simulations with two flavors of dynamical staggered quarks and valence Wilson quarks at $6/g^2 = 5.6$ and simulations with two flavors of dynamical Wilson quarks and Wilson valence quarks, at $6/g^2 = 5.3$. The spectroscopy of these systems is essentially identical. Parameterizations of the wave functions are presented which can be used as interpolating fields for spectroscopy calculations. The sizes of particles are calculated using these parameterized wave functions. The resulting sizes are small, approximately half the sizes of the physical states. The charge radius of the neutron, which provides an indication of the asymmetries between the wave functions of up and down quarks, is calculated. Although the size of the nucleon in these simulations is small, the ratio of the charge radius of the neutron to that of the proton is consistent with the physical value. We find no significant differences between the quenched and dynamical simulations.
I. INTRODUCTION

Numerical studies of QCD have become sufficiently fine-grained that it has become possible to investigate the global properties of QCD wave functions directly from Monte Carlo simulations. The goal of these studies is two-fold: First, visualizing wave functions is a powerful diagnostic for lattice studies. A picture of the wave function provides a hint for a good trial wave function for spectroscopy. One can see whether the wave function of a hadron is squeezed by the simulation volume; if it is, then a calculation of spectroscopy may be compromised.

Second, it may be possible to use wave functions for phenomenology, either by directly performing calculations with the wave functions, or by abstracting a continuum model from the wave function, determining its parameters from a small number of lattice measurements, and using the model, rather than expensive lattice simulations, for QCD calculations. Phenomenologically interesting calculations include charge radii and radial moments. Results to date indicate that the wave functions for hadrons in quenched QCD are too small in spatial extent to reproduce quark phenomenology, although ratios of sizes, including the ratio of the charge radius of the neutron to that of the proton, are reasonable.\(^1\)

The subject goes back for many years. In 1985 Velikson and Weingarten\(^2\) studied meson wave functions in SU(2) and Gottlieb\(^3\) carried out the first study of wave functions with SU(3) Wilson fermions. Recently, Chu, Lissia and Negele have investigated gauge invariant wave functions (with a product of links connecting the quarks).\(^4\) Wave functions for heavy quark systems\(^5\) and for heavy-light systems\(^6\) have also recently been reconstructed, and two of us\(^7,1\) have performed an extensive study of wave functions of light quark systems. Wave function methods have also been applied to finite temperature systems, in Ref. 8.

In this paper we extend the wave function calculations of Ref. 1 to systems with dynamical fermions, using lattices generated as part of the High Energy Monte Carlo
Grand Challenge. We parameterize the wave functions for possible use as interpolating fields for spectroscopy. In addition, we compare the charge radii and radial moments determined from the wave functions to the experimentally determined numbers and to the values obtained in the quenched approximation. We note that these wave functions are minimal Fock space wave functions and that the use of the wave function for calculating phenomenological numbers represents an uncontrolled approximation.

The wave function \( \psi_G(r) \) of a meson \( H \) in a gauge \( G \) is defined as

\[
\psi_G(r) = \sum \langle H|q(\vec{x})\bar{q}(\vec{x} + \vec{r})|0 \rangle
\]

where \( q(\vec{x}) \) and \( \bar{q}(\vec{y}) \) are quantum mechanical operators which create a quark and an antiquark at locations \( \vec{x} \) and \( \vec{y} \). (We have suppressed Dirac and color indices.) The wave function can be extracted from a correlation function which is a convolution of quark and antiquark propagators \( G(x,y) \)

\[
C(\vec{r},t) = \sum \Psi(\vec{y}_1, \vec{y}_2)G_q(\vec{y}_1, 0; \vec{x}, t)G_{\bar{q}}(\vec{y}_2, 0; \vec{x} + \vec{r}, t)
\]

where \( \Psi(\vec{y}_1, \vec{y}_2) \) is the \( t = 0 \) operator. At large \( t \) if the mass of the hadron is \( m_H \), then

\[
C(\vec{r},t) \simeq \exp(-m_H t)\psi_G(\vec{r})
\]

and so by plotting \( C(\vec{r},t) \) as a function of \( \vec{r} \) we can reconstruct the wave function up to an overall constant. One can derive a similar expression for baryons, as a function of the two relative coordinates of the three valence quarks.
II. THE SIMULATIONS

Our simulations were performed on Connection Machine CM-2s located at the Supercomputer Computations Research Institute at Florida State University and at the Pittsburgh Supercomputing Center.

The quenched data set consists of 41 lattices of size $16^4$ sites at a coupling $6/g^2 = 6.0$ separated by 500 evolutionary sweeps (100 passes through the lattice of a pattern of four overrelaxed sweeps and one Kennedy-Pendleton heat bath sweep). We recorded propagators with hopping parameters equal to $\kappa = 0.145, 0.152, 0.153, 0.154, 0.155$ with the corresponding pion masses in lattice units range from $m_\pi a = 0.82$ to $m_\pi a = 0.28$, where $a$ is the lattice spacing.

The simulations with two flavors of dynamical staggered quarks use the Hybrid Molecular Dynamics algorithm. The lattice size is $16^3 \times 32$ sites and the lattice coupling $6/g^2 = 5.6$. The dynamical quark mass is $am_q = 0.01$. A subset of the data (whose spectroscopic analysis is described in Ref. 12) was taken for this analysis. It consists of 20 lattices spaced 80 simulation time units apart (with the normalization of Ref. 13). We computed spectroscopy with staggered sea quarks at three values of the Wilson quark hopping parameter: $\kappa = 0.1565, 0.1585, \text{ and } 0.1600$. The pseudoscalar mass in lattice units ranges from about 0.22 to 0.45.

The simulations with two flavors of Wilson sea quarks used the Hybrid Monte Carlo algorithm. The lattice size is again $16^3 \times 32$ and the lattice coupling is $6/g^2 = 5.3$. Again, a subset of the whole data set (whose spectroscopic analysis will be described in Ref. 15) was taken consisting of 19 lattices spaced 65 Hybrid Monte Carlo time units apart. Only one hopping parameter was studied: $\kappa = 0.1670$, corresponding to a pion mass in lattice units of about 0.47.
All spectroscopy in the three data sets was extracted using identical methods and computer programs. We gauge fixed lattices to Coulomb gauge using an overrelaxation algorithm. Our criterion for gauge fixing was that the average change in the trace of a spacelike link was less than $\text{Tr}\delta U = 10^{-5}$. The sources for the quarks are Gaussians centered about some origin on a single time slice. Our inversion technique is conjugate gradient with preconditioning via ILU decomposition by checkerboards. We used a fast matrix inverter written in CMIS (Connection Machine Instruction Set).

We employ relativistic wave functions. The baryon wave functions are:

**Proton:**

$$|P, s\rangle = (uC\gamma_5 d)u_s = (u_1 d_2 - u_2 d_1 + u_3 d_4 - u_4 d_3)u_s$$

**Delta:**

$$|\Delta, s\rangle = (u_1 d_2 + u_2 d_1 + u_3 d_4 + u_4 d_3)u_s$$

We measured meson correlation functions using spin structures for the source of $\bar{\psi}\gamma_5 \psi$ for the pion and $\bar{\psi}\gamma_0\gamma_3 \psi$ for the rho. At the wave function we used the same spin structure for the pion and $\bar{\psi}\gamma_3 \psi$ for the rho.

We include the full covariance matrix in order to get a meaningful estimate of the goodness of fit. Reference 20 discusses this fitting procedure in detail.

The rho mass was used to fix the spacing on the dynamic staggered lattices at $a^{-1} = 2140 \text{ MeV}$, which differs only slightly from the lattice spacing in quenched QCD at $6/g^2 = 6.0$ of $a^{-1} = 2312 \text{ MeV}$. Comparison of the two numbers suggests that the spacing on the dynamic staggered lattices is 8% larger than on the quenched lattices. Fixing the lattice spacing to the proton mass yields a value of $a^{-1} = 1800 \text{ MeV}$ on the dynamic staggered fermion lattices and $a^{-1} = 1991 \text{ MeV}$ on the quenched lattices. We use $a^{-1} = 2000 \text{ MeV}$ to estimate dimensionful quantities for both the dynamical staggered fermion data and the quenched simulation.
We have recently extended the dynamic Wilson spectroscopy to a second value of the Wilson hopping parameter at $\kappa = 0.1675$. The rho mass fixes the lattice spacing to $a^{-1} = 1640$ MeV, indicating that the spacing of the dynamic Wilson lattice is around 30% larger than the spacing of the dynamic staggered lattice. As a lattice problem we can analyze the wave functions on the dynamical Wilson fermion lattices and compare their properties (such as their sizes) to those in the quenched and staggered dynamical simulations when the lattice masses are similar, providing another comparison of the lattice spacings. Indeed, we have a similar problem comparing the quenched and staggered dynamical fermion simulations: all the bare parameters are different. However, when we compare mass ratios (via Edinburgh plots, for example), we see that the data sets are not dissimilar.

The most striking way we have found to display spectroscopy from the three data sets is to plot the vector and baryon masses as a function of the pion mass in lattice units. This we do in Fig. 1. We see that the three data sets resemble each other rather closely, though the Wilson dynamical fermion particles appear to be about fifteen per cent heavier than the quenched and staggered dynamical spectroscopy at the same value of the pion mass.
III. GLOBAL VIEWS OF WAVE FUNCTIONS

A hierarchy of particle sizes emerges from a comparison of the wave functions. To facilitate this comparison the meson wave functions plotted in Fig. 2 have been normalized so that the value at zero separation is one. The baryon wave functions show greater fluctuation in the normalization than the meson wave functions. We have normalized the baryon wave functions in Fig. 2 on a lattice-by-lattice basis. Our justification for presenting baryons in this way is that the resulting plot is consistent with that obtained from a correlated fit to the data, and doing so helps the viewer to see qualitative features. The meson wave functions show the amplitude as the quark is pulled apart from the antiquark along a principal axis. The baryon plots are of the wave functions for the unique flavor quark when the two like-flavor quarks are fixed to be at the same site. The pion and proton wave functions are smallest; the rho is largest, and the delta is next largest.

The wave functions for the hadrons made of the lightest valence quarks are very large. Of course, because of the periodic boundary conditions in the spatial directions of the lattice, the wave functions shown at \( r = 8 \) are twice the size they would be on an infinite size lattice. Nevertheless, the \( \kappa = 0.1600 \) rho has only fallen to twenty five per cent of its peak value by \( r = 8 \).

The pion wave functions in the staggered and Wilson simulations are compared in Fig. 3. The pion wave function on the staggered lattices is relatively insensitive to the value of the quark mass. The pion wave function on the Wilson lattices is dramatically smaller, measured in lattice units, although the masses in lattice units on the Wilson and the \( \kappa = 0.1565 \) staggered dynamic fermion lattices are similar. Our wave function analysis supports the 30% difference in the lattice spacings indicated by fixing the lattice spacings to the rho mass, as shown in the next sections.

All of the spectroscopy with dynamical staggered fermions was originally performed
on spatial lattices with $12^3$ sites. Both baryons showed strong finite-size effects: their masses fell by about fifteen per cent when they were recomputed on a $16^3$ lattice. Neither meson showed an appreciable change in mass with lattice size. It is difficult to reconcile this behavior with the observed hierarchy of wave functions: why are finite size effects not largest for the rho meson? Some physics which governs the energy of a particle in a finite simulation volume is not being included in the minimum Fock space wave function.
IV. COMPARISONS OF QUENCHED
AND DYNAMICAL SIMULATIONS

A. Meson Properties

In this and the following section we analyze the wave functions. We parameterize the wave functions for possible use as interpolating fields for spectroscopy. In addition, we compare the charge radii and radial moments determined from the wave functions to the experimentally determined numbers and to the values obtained in the quenched approximation. All of the data in quenched approximation has been presented and more completely discussed in Ref. 1. We remind the reader that these wave functions are minimal Fock space wave functions and that the use the wave function for calculating phenomenological numbers represents an uncontrolled approximation.

The second moment of the pion, \( \langle r_\pi^2 \rangle \), has been determined experimentally to have the value of 0.405 \( \pm 0.024 \) fm\(^2\).\(^{23}\) The second moment is defined in the quark model as

\[
\langle r^2 \rangle = \sum_i q_i \langle (\vec{r}_i - \vec{R})^2 \rangle
\]

(5)

where \( \vec{R} \) is the location of the center of mass and \( q_i \) is the charge of the \( i^{th} \) valence quark. In terms of the wave function \( \Psi \) this is

\[
\langle r^2 \rangle = \frac{\int d^3 \vec{x} (\Psi^2(\vec{x}))}{\int d^3 \vec{x} \Psi^2(\vec{x})}. \tag{6}
\]

In order to evaluate the second moment from our data we suggest a parameterization of the wave function, make a correlated fit of the parameters to the data, and integrate analytically to obtain the second moment at each value of the hopping parameter.

Our parametrization of the meson wave function is

\[
\Psi(r) = x_1 \exp(-x_2 r x_3)
\]

(7)

10
where \( r \) is the separation between quark and antiquark. The periodic boundary conditions are treated by including an additional term with \((L - r)\) substituted in place of \( r \), where \( L \) is the length of the lattice, as in Ref. 1. A full correlated fit of these three parameters to a subset of the data is made. We choose to use the points along principal axes of the lattice in the fit. The resulting fit parameters are listed in Table 1 and are plotted in Fig. 4.

The exponent \( x_3 \) is close to \( 3/2 \) for the rho meson wave function calculated with dynamic fermions, as it is for the quenched rho. This is the value obtained as the solution to the nonrelativistic wave equation in a linear potential, and thus may be an indication of a potential which is approximately linear in the quark separation.

The second moments of the mesons calculated from correlated fit parameters on both staggered and Wilson lattices are shown in Fig. 5. The mass of the dynamic staggered pion, at \( am_\pi = 0.45 \), is comparable to the mass of the dynamic Wilson pion, at \( am_\pi = 0.47 \). The second moments of the dynamic staggered mesons are approximately twice as large as the second moments of the dynamic Wilson mesons; the ratios for both pion and rho meson are \( 2.1 \pm 0.1 \). This difference is largely explained by the 30% difference in lattice spacings found by fixing the lattice spacings to the rho mass.

The correlated fit parameters scale in a way which is roughly consistent with a 30% difference in the lattice spacings. The value of the exponent \( x_3 \) is unchanged by a rescaling of the lattice spacing, but the exponential falloff \( x_2 \) is rescaled as

\[
x_2' = (a'/a)^3 x_2.
\]

In fact the values of \( x_3 \) are not very different in the dynamic staggered and dynamic Wilson simulations. We would expect the ratio of \( x_2 \) for the Wilson point to the \( \kappa = 0.1565 \) staggered point to be around 1.38 for the pion and around 1.49 for the rho meson based on the scaling relation of Eqn. 8. From Table 1 we find the ratio of exponential falloofs to be \( 1.44 \pm 0.02 \) for the pion, which is a bit higher than the anticipated value and would
suggest a 40% difference in the lattice spacings. For the rho meson the ratio is 1.29 ± 0.06, which is lower than the anticipated value and would suggest a 20% difference in the lattice spacings.

Extrapolating the second moment of the dynamic staggered pion linearly in $\kappa$ to $\kappa_c$, we find $\langle (r/a)^2 \rangle_\pi = 4.79 \pm 0.20$, a figure which is three standard deviations below the quenched value of 6.24 ± 0.25. The corresponding number for the dynamic staggered rho is $\langle (r/a)^2 \rangle_\rho = 8.91 \pm 0.72$, which is around one standard deviation below the quenched value of 10.3 ± 1.1. The ratio of the moments is $\langle (r/a)^2 \rangle_\pi / \langle (r/a)^2 \rangle_\rho = 1.86 \pm 0.17$, which is consistent with the quenched ratio of 1.65 ± 0.19. The quenched pion second moment is 30% larger than that of the dynamic pion. This difference in size could be completely explained by a 15% difference in lattice spacings, and could be at least partially explained by the approximately 8% difference in the quenched and dynamic staggered lattice spacings determined through fixing the lattice spacing to the mass of the rho meson. The second moment of the quenched rho meson is 15% larger than that of the dynamic rho, a difference which is perfectly accounted for by an 8% difference in the lattice spacings.

The pion moment on the staggered fermion lattices converts approximately to the dimensionful number of $\sqrt{\langle (r/a)^2 \rangle_\pi} = 0.21$ fm, which is one third of the physical value, and for the rho meson $\sqrt{\langle (r/a)^2 \rangle_\rho} = 0.29$ fm, using $a^{-1} = 2000$ MeV.

The only qualitative feature we observe for the dynamic mesons which may differ from the quenched mesons is the dependence of the pion size on the value of the hopping parameter. The quenched pion’s size was observed to grow consistently larger with decreasing quark mass. The dynamic pion at $\kappa = 0.1600$ appears to be no larger than the dynamic pions at the two smaller $\kappa$ values, as is seen in Fig. 5.

The second moments of the mesons can be calculated directly from the data using
discrete lattice sums, as

\[ <r^2> = \left( \sum_{lattices} \sum_{\vec{s}} (s/2)^2 \Psi^*(\vec{s}) \Psi(\vec{s}) \right) / \left( \sum_{lattices} \sum_{\vec{s}} \Psi^*(\vec{s}) \Psi(\vec{s}) \right). \] (9)

These second moments calculated from discrete lattice sums are shown in Fig. 6. Using this method to compute the second moments of mesons on the staggered lattices at all three \( \kappa \) values, a linear extrapolation in \( \kappa \) to \( \kappa_c \) results in a pion second moment of \( \langle (r/a)^2 \rangle_{\pi} = 6.32 \pm 0.58 \), one and a half standard deviations below the corresponding quenched number of \( 8.05 \pm 0.65 \). The second moment of the rho meson is found to be \( \langle (r/a)^2 \rangle_{\rho} = 13.7 \pm 0.6 \), which is consistent with the quenched number of \( 14.0 \pm 0.8 \). In contrast to the suggestion that the second moment of the pion as derived from correlated fits is independent of quark mass, the second moment from the discrete lattice sums rises steadily with decreasing quark mass. This method of obtaining radial moments does not compensate for the contributions to the wave function from image particles, nor does it account for the considerable tails of the wave functions which extend beyond the lattice. Radial moments derived from correlated fits to the data, as outlined near the beginning of this section, are free of these two problems.

The second moments of the mesons on the staggered lattices are slightly smaller than the second moments which were calculated in the quenched approximation. The ratios of the second moments of rho meson to pion are completely consistent between the two simulations. It is possible that all of the difference in the sizes can be ascribed to a difference in the lattice spacings. As observed for the quenched pion, the pion on a lattice containing dynamic staggered fermions has a size which is approximately one third that of the physical pion.

B. Baryon Properties

Charge radii for the baryons cannot be calculated by discrete lattice sum owing to the
limited subset of the data which has been recorded, but the charge radii can be calculated by parametrizing the wave function, inserting the resulting expression for the wave function in the analytic expression for the charge radius and integrating. We do this to compare the charge radii of the baryons on dynamic lattices with charge radii in the quenched approximation, as well as for comparison with the physical values.

The integral for the charge radius of a baryon is written in terms of two relative coordinates, one of which is the separation between two quarks of flavor $a$ ($\vec{r}_{aa}$), and the second of which is a vector reaching from midway between the $a$ quarks to a quark of flavor $b$ ($\vec{r}_{cb} = \vec{r}_b - \frac{1}{2}(\vec{r}_a + \vec{r}_{a'})$). In terms of these variables the integral for the charge radius is

$$\langle r^2 \rangle = \frac{\int d^3\vec{r}_{aa} \int d^3\vec{r}_{cb} |\Psi(\vec{r}_{aa}, \vec{r}_{cb})|^2 \sum_{q=1,2,3} e_q r_q^2 (\vec{r}_{aa}, \vec{r}_{cb})}{\int d^3\vec{r}_{aa} \int d^3\vec{r}_{cb} |\Psi(\vec{r}_{aa}, \vec{r}_{cb})|^2}$$

where $r_q$ is the distance from the center of mass to the location of a particular quark.

We compare the probability for the two quarks of flavor $a$ to be at the same position, with the quark of flavor $b$ out at some distance $r$. If SU(6) were unbroken then this would be equal to the probability for one of the $a$ quarks to be at the same position as the $b$ quark, with the second $a$ quark out at the same distance $r$. Swaths of such points are compared in Fig. 7. The nucleon wave function amplitudes at separations 2, 4 and 6 for the two orientations differ by about three standard deviations, indicating a negative charge radius for the neutron. The delta wave function amplitudes for the two orientations differ by about $\frac{3}{2}\sigma$, indicating a slight positive charge radius for the delta of quark content $ddu$. This is unanticipated and may represent a statistical fluctuation. We note again that the baryon wave function points are normalized on a lattice-by-lattice basis.

We use a wave function which is a product of three exponentials, with each exponential being a function of the separation between one of the pairs of quarks. Our wave function is, for two quarks of flavor “$a$” and one quark of flavor “$b$”

$$\Psi(r_{aa}, r_1, r_2) = N \exp(-x_{aa} r_{aa}^y) \exp(-x_{ab} r_{1}^y) \exp(-x_{ab} r_{2}^y)$$

(11)
where $r_{aa}$ is the relative separation of the two $a$ quarks (as in Eqn. 10), $r_1$ is the separation of the $b$ quark from an $a$ quark and $r_2$ is the separation between the $b$ quark and the other $a$ quark.

We store data at four separations between the like-flavor quarks (quarks of flavor $a$ in Eqn. 11), and for each of those four values we store the amplitude for the other quark (flavor $b$ in Eqn. 11) to be anywhere on the lattice. In order to calculate charge radii we make use of a very limited but symmetric subset of the data, using the four data points for which the two quarks of flavor $a$ are at zero relative separation and the quark of flavor $b$ is at separations of 0, 2, 4 and 6 from the $aa$ pair, as well as the points for which one $a$ quark is at zero relative separation from the $b$ quark and the second quark of flavor $a$ is at separations of 2, 4 and 6 from the $ab$ pair.

For this subset of the data the parametrized wave function of Eqn. 11 can be simplified. The wave function for a quark of flavor $b$ relative to an $aa$ diquark can be written as

$$\Psi_b(r_b) = N \exp(-x_b r_b^y).$$

and the wave function for a quark of flavor $a$ relative to an $ab$ diquark is written as

$$\Psi_a(r_a) = N \exp(-x_a r_a^y).$$

Full correlated fits are made of the parameters of Eqns. 12 and 13 to this limited subset of the data. The data points and the functional form which has been fit to those points are illustrated in Fig. 8 for the $\kappa = 0.1585$ nucleon.

Baryon correlated fit parameters are presented in Table 2 and in Figs. 9, 10 and 11. The value of the exponent $y$ lies in a narrow range for all of the baryons, regardless of the composition of the lattice or of the identity of the baryon. The magnitude of $y$ varies only slightly with quark mass. The exponential falloffs $x_{aa}$ and $x_{ab}$ tend towards slightly larger values on the dynamic staggered lattices than on the quenched lattices, similarly
suggesting a small difference in the lattice spacings in the two formulations. The magnitude of the exponential falloffs between a pair of $aa$ quarks and an $ab$ pair is undifferentiated for the delta within each of the four dynamic fermion simulations. This indicates a neutral charge radius for the delta, despite the hint of a positive charge radius from the wave function points of Fig. 7. The magnitude of the exponential falloffs between one pairing of the quarks is substantially different from that between the other pairing of quarks for the nucleon within each simulation, indicating a statistically significant negative charge radius for the neutron.

The baryon correlated fit parameters are consistent with a 30% difference in the lattice spacings. For a 30% difference in lattice spacings we expect the ratio of exponential falloffs on the dynamic Wilson lattices to that on the dynamic staggered lattices at $\kappa = 0.1565$ to be around 1.42 for the nucleon, based on the scaling relation of Eqn. 8. In fact we find the ratio of $x_{aa}$ parameters to be $1.18 \pm 0.18$, and the ratio for the parameter $x_{ab}$ is $1.41 \pm 0.06$. Both of these figures compare well with the expected value of 1.42. For the delta the anticipated ratio of exponential falloffs is 1.41. We find the ratio of $x_{aa}$’s for the delta is $1.3 \pm 0.3$, and the ratio of $x_{ab}$’s is $1.4 \pm 0.2$, both of which are consistent with the expected ratio.

The charge radii for the proton and neutron are presented in Table 3. A linear extrapolation in $\kappa$ to $\kappa_c$ of the proton charge radius yields a value of $\langle (r/a)_p^2 \rangle = 15.1 \pm 3.4$, slightly below but consistent with the charge radius in the quenched approximation of $16.6 \pm 3.1$. The ratio of the charge radius of the neutron to that of the proton is in Table 3 and is also displayed in Fig. 12. The dynamic data are consistent with the quenched data except at $\kappa = 0.1600$, where the point falls one standard deviation low. A linear extrapolation in $\kappa$ of the ratio of the charge radii on the dynamic staggered lattices gives $\langle r_n^2 \rangle / \langle r_p^2 \rangle = -0.21 \pm 0.04$. This ratio is slightly more than one standard deviation above the experimental figure of $-0.146 \pm 0.005^{+0.24}$ and is evidently pulled in that direction by the
point at $\kappa = 0.1600$. The charge radius of the proton calculated with dynamic staggered fermions is $\sqrt{\langle r_p^2 \rangle} = 0.38$ fm using $a^{-1} = 2000$ MeV, half the physical size of $\sqrt{\langle r_p^2 \rangle} = 0.81$ fm.$^{25}$

The charge radius of the dynamic staggered proton at $\kappa = 0.1565$ is $\langle (r/a)_p^2 \rangle = 10.5 \pm 1.2$ while the dynamic Wilson proton has a charge radius of $\langle (r/a)_p^2 \rangle = 8.1 \pm 1.8$. The 30% difference in lattice spacings derived from the rho mass translates into a ratio of 1.69 for the charge radii. The ratio of the calculated charge radii is $1.30 \pm 0.34$, slightly below the anticipated value.

The baryon wave functions calculated on dynamic staggered fermion lattices are not substantially different from the quenched baryon wave functions. The size of the proton calculated with minimal Fock space wave functions is half the size of the physical proton, using quenched or dynamic staggered lattices. The baryon wave functions calculated on dynamic Wilson fermion lattices are smaller (in lattice units) than their counterparts in the other two formulations. The magnitude of the difference in size is consistent with the difference in lattice spacings which results from fixing the lattice spacings to the rho mass.
V. CONCLUSIONS

No dramatic differences are seen between wave functions in the quenched approximation and wave functions in full QCD. The second moments of the pion and rho meson in lattice units are $3\sigma$ and $\frac{3}{2}\sigma$ smaller than on the quenched lattices, respectively. We believe most of this difference can be accounted for by a rescaling of the lattice spacing of about 8%.

The wave functions calculated on dynamic Wilson fermion lattices are substantially smaller in lattice units than the corresponding wave functions in the other two formulations. When we convert to physical units the size of any of the particles is roughly independent of the formulation of the lattice.

The ratio of the charge radius of the neutron to that of the proton with staggered dynamic fermions is consistent with the experimental ratio, as was found in the quenched simulation. In contrast, the sizes of the particles derived from the wave functions in all three simulations are smaller than the physical states, smaller by around a factor of two. A more desirable method of calculating radial moments and charge radii may be to use structure functions rather than wave functions, as discussed in Ref. 1.

The rho meson wave function in Coulomb gauge is larger than that for the pion, proton or delta. This would lead one to expect that finite size effects in spectroscopy studies would be greatest for the rho. That this is not true is another indication that some physics which impacts spectroscopy is not included in the minimum Fock space wave function.
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FIGURE CAPTIONS

1. Comparison of quenched Wilson spectroscopy at $6/g^2 = 6.0$ (squares) with Wilson valence spectroscopy from a dynamical staggered fermion simulation at $6/g^2 = 5.6$ (octagons) and dynamical Wilson fermion simulation at $6/g^2 = 5.3$ (diamond).

2. Coulomb gauge wave functions at time $t = 6$, for separations of $(x, 0, 0)$. For the baryons the two like-flavor quarks are pinned to the same site while the non-like quark is at separation $(x, 0, 0)$ relative to the other two. The meson data has been normalized after averaging. The baryon data has been normalized on a lattice-by-lattice basis. Particles are labeled by boxes for pion, diamonds for rho, octagons for proton and crosses for delta. (a)—(c) are simulations with dynamical staggered quarks and Wilson valence quarks, at Wilson hopping parameters of $\kappa = 0.1565, 0.1585$ and 0.1600, respectively. (d) is with Wilson dynamical and valence quarks at $\kappa = 0.1670$.

3. Pion wave function on lattices with staggered fermions at all three $\kappa$ values ($\kappa = 0.1565, 0.1585, 0.1600$) and with Wilson fermions (at $\kappa = 0.1670$).

4. Correlated fit parameters for meson wave functions (as in Eqn. 7) from correlated fits to the data. (a) exponential falloff $x_2$, (b) exponent $x_3$.

5. Second moment of mesons (as in Eqn. 6), using parameterized wave functions of Eqn. 7. Crosses indicate staggered dynamic fermions, squares indicate Wilson dynamic fermions, diamonds are quenched. (a) pion, (b) rho.

6. Meson second moments as calculated through discrete lattice sums, via Eqn. 9. Crosses indicate staggered dynamic fermions, squares indicate Wilson dynamic fermions, diamonds are quenched. (a) pion, (b) rho.

7. Falloff of wave function of $b$ quark (crosses) with separation from an $aa$ diquark, and of an $a$ quark from an $ab$ diquark (octagons). (a) nucleon, (b) delta. Data are at
\( \kappa = 0.1585 \), with dynamic staggered fermions.

8. Nucleon wave function data points at \( \kappa = 0.1585 \) (with dynamic staggered fermions) with functional forms of Eqn. 12 and Eqn. 13 overplotted. Crosses represent wave function for quark of flavor \( b \) relative to \( aa \) diquark, octagons represent wave function for quark of flavor \( a \) relative to \( ab \) diquark.

9. Exponential falloffs \( x_{aa} \) and \( x_{ab} \) which parameterize the nucleon wave function, as in Eqn. 11.

10. Exponential falloffs \( x_{aa} \) and \( x_{ab} \) which parameterize the wave function of the delta, as in Eqn. 11.

11. Value of the exponent \( y \) which parametrizes the baryon wave functions, as in Eqn. 11. (a) nucleon, (b) delta.

12. Ratio of charge radii of neutron to proton as a function of pion mass. Horizontal dashed line is the experimental ratio. Crosses indicate staggered dynamic fermions, squares indicate Wilson dynamic fermions, diamonds are quenched.

**TABLE CAPTIONS**

1. Correlated Fit Parameters for Mesons
2. Correlated Fit Function Parameters for Baryons
3. Charge Radii of Baryons
TABLES

TABLE 1

| $\kappa$   | sea quarks | pion $x_2$ | pion $x_3$ | rho $x_2$ | rho $x_3$ |
|------------|------------|------------|------------|-----------|-----------|
| 0.145      | quenched   | 0.2069(16) | 1.274(9)   | 0.0972(14)| 1.514(11) |
| 0.152      | quenched   | 0.1970(10) | 1.247(7)   | 0.0727(12)| 1.534(15) |
| 0.153      | quenched   | 0.1964(9)  | 1.241(7)   | 0.0698(14)| 1.534(17) |
| 0.154      | quenched   | 0.1961(9)  | 1.234(7)   | 0.0674(18)| 1.532(21) |
| 0.155      | quenched   | 0.1960(10) | 1.228(8)   | 0.0658(26)| 1.529(28) |
| 0.1565     | staggered  | 0.234(2)   | 1.216(7)   | 0.094(3)  | 1.45(2)   |
| 0.1585     | staggered  | 0.232(2)   | 1.211(8)   | 0.086(4)  | 1.46(3)   |
| 0.1600     | staggered  | 0.234(2)   | 1.213(11)  | 0.080(5)  | 1.48(3)   |
| 0.1670     | Wilson     | 0.337(5)   | 1.253(9)   | 0.121(4)  | 1.58(2)   |
TABLE 2

(a): Nucleon

| $\kappa$ | sea quarks | $x_a$ | $x_b$ | $x_{aa}$ | $x_{ab}$ | $y$     |
|----------|------------|-------|-------|----------|----------|--------|
| 0.145    | quenched   | 0.158(3) | 0.177(3) | 0.069(3) | 0.089(1) | 1.398(12) |
| 0.152    | quenched   | 0.135(3) | 0.156(2) | 0.057(3) | 0.078(1) | 1.378(11) |
| 0.153    | quenched   | 0.132(3) | 0.153(2) | 0.055(3) | 0.077(1) | 1.372(12) |
| 0.154    | quenched   | 0.129(3) | 0.151(3) | 0.054(4) | 0.076(1) | 1.365(14) |
| 0.155    | quenched   | 0.125(4) | 0.149(3) | 0.050(5) | 0.074(2) | 1.361(19) |
| 0.1565   | staggered  | 0.158(4) | 0.187(4) | 0.065(4) | 0.094(2) | 1.363(12) |
| 0.1585   | staggered  | 0.148(4) | 0.180(4) | 0.058(5) | 0.090(2) | 1.355(12) |
| 0.1600   | staggered  | 0.123(10)| 0.170(5) | 0.038(10)| 0.085(3) | 1.35(4)   |
| 0.1670   | Wilson     | 0.208(9)| 0.264(8)| 0.076(10)| 0.132(4) | 1.33(3)   |

(b): Delta

| $\kappa$ | $x_a$ | $x_b$ | $x_{aa}$ | $x_{ab}$ | $y$     |
|----------|------|------|----------|----------|--------|
| 0.145    | 0.150(3) | 0.141(3) | 0.079(3) | 0.071(1) | 1.410(13) |
| 0.152    | 0.124(3) | 0.116(2) | 0.066(3) | 0.058(1) | 1.376(15) |
| 0.153    | 0.122(3) | 0.113(2) | 0.065(4) | 0.057(1) | 1.367(17) |
| 0.154    | 0.120(4) | 0.111(2) | 0.064(4) | 0.056(1) | 1.357(19) |
| 0.155    | 0.119(6) | 0.109(3) | 0.064(6) | 0.055(2) | 1.346(26) |
| 0.1565   | 0.159(6) | 0.155(5) | 0.081(6) | 0.078(3) | 1.323(23) |
| 0.1585   | 0.152(8) | 0.154(6) | 0.075(9) | 0.077(3) | 1.31(3)   |
| 0.1600   | 0.159(20)| 0.142(9) | 0.088(21)| 0.071(4) | 1.26(6)   |
| 0.1670   | 0.217(17)| 0.226(26)| 0.104(22)| 0.113(13)| 1.28(11)  |


| $\kappa$ | sea quarks | $\langle \frac{r}{a} \rangle_n^2$ | $\langle \frac{r}{a} \rangle_p^2$ | $\langle r_n^2 \rangle / \langle r_p^2 \rangle$ |
|---------|------------|-----------------|-----------------|------------------|
| 0.145   | quenched   | -0.73(38)       | 9.1(12)         | -0.080(34)        |
| 0.152   | quenched   | -1.27(59)       | 12.4(18)        | -0.102(37)        |
| 0.153   | quenched   | -1.37(64)       | 13.1(20)        | -0.104(38)        |
| 0.154   | quenched   | -1.55(71)       | 14.0(23)        | -0.111(38)        |
| 0.155   | quenched   | -1.83(82)       | 15.0(29)        | -0.122(38)        |
| 0.1565  | staggered  | -1.23(33)       | 10.5(12)        | -0.117(23)        |
| 0.1585  | staggered  | -1.63(43)       | 12.1(15)        | -0.135(24)        |
| 0.1600  | staggered  | -3.5(18)        | 17.4(61)        | -0.204(44)        |
| 0.1670  | Wilson     | -1.28(48)       | 8.1(18)         | -0.158(33)        |
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