Symmetries and scaling in generalised coupled conserved Kardar–Parisi–Zhang equations

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Abstract. We study the noisy nonequilibrium dynamics of a conserved density that is driven by a fluctuating surface governed by the conserved Kardar–Parisi–Zhang equation. We uncover the universal scaling properties of the conserved density. We consider two separate minimal models where the surface fluctuations couple (i) with the spatial variation of the conserved density, and (ii) directly with the magnitude of the conserved density. Both these two models conserve the density, but differ from a symmetry stand point. We use our result to highlight the dependence of nonequilibrium universality classes on the interplay between symmetries and conservation laws.

Keywords: correlation functions, driven diffusive systems, renormalisation group

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1. Introduction

The concept of universality classes, which are parametrised by the space dimensions and the order parameter components, allows one to have a systematic physical understanding of universal scaling properties in equilibrium systems [1, 2]. These universality classes are found to be robust against dynamical perturbations so long as the general conditions for equilibrium are maintained. In contrast, statistical properties of truly nonequilibrium dynamic phenomena in systems with generic non-Gibbsian distribution are found to be strongly sensitive to all kinds of perturbations. Prominent examples are driven diffusive systems [3] and diffusion-limited reactions [4]. For instance, one finds that for the Kardar–Parisi–Zhang (KPZ) equation of surface growth [5, 6], that shows paradigmatic nonequilibrium phase transitions [7], anisotropic perturbations are relevant in \( d > 2 \) spatial dimensions, leading to rich phenomena that include novel universality classes and the possibility of first-order phase transitions and multicritical behavior [8]. Furthermore, novel nonequilibrium scaling behaviour including continuously varying universality classes are often found in multicomponent driven systems [9]. Related physical realisations include driven symmetric mixture of a miscible binary
fluid [10] and magnetohydrodynamic turbulence [11], dynamic roughening of strings moving in random media [12], sedimenting colloidal suspensions [13] and crystals [14].

Conservation laws are known to play significant roles in physical systems. For equilibrium systems, they affect only the dynamical properties [15], whereas for out of equilibrium systems even time independent quantities are affected by conservation laws. This was succinctly brought out by the studies on a conserved version of the KPZ equation (C-KPZ) that shows scaling behaviour distinctly different from the usual KPZ equation [16]. For instance, the KPZ universality class is characterised by the exact relation between the scaling exponents: $\chi_{\text{kpz}} + z_{\text{kpz}} = 2$ [6, 7], where $\chi_{\text{kpz}}$ and $z_{\text{kpz}}$, respectively, are the roughness and dynamic scaling exponents describing the spatial and temporal scaling of the KPZ universality class. In contrast, the C-KPZ equation does not admit any such exact exponent relations [16, 17]. The KPZ equation has subsequently been generalised to multicomponent versions to address different questions of principles. For example, how the surface fluctuations in the KPZ equation control the fluctuations of a conserved scalar density that is dynamically coupled to the KPZ equation has been studied [18] by using a well-known two-component variant of the KPZ equation [12]. It is also known that a breakdown of an external symmetry like parity can lead to novel scaling behaviour [9]. Notable previous works that form a major motivation for our studies here are the studies reported by Drossel and Kardar (hereafter DK) in [19, 20] using a set of coupled generalised KPZ equations for the height field and a density. In particular, DK studied fluctuations in the concentrations of structureless particles advected by a one-dimensional (1d) Burger’s fluid, or equivalently particles sliding on a fluctuating KPZ surface [19]. By retaining feedback from the density fluctuations on the fluctuating KPZ surface, they elucidated various regimes depending upon the choice of parameters for advection or anti-advection. The scaling exponents are obtained. Remarkably, continuously varying scaling exponents are illustrated for the anti-advection case in [19]. In a subsequent study, DK considered the interplay between a fluctuating surface and phase ordering [20], again using a set of coupled generalised KPZ equations for the height field and a nonconserved density. They obtained the relevant scaling exponents and in some cases illustrated continuously varying dynamic exponent in the model. These studies by DK open up the questions: (i) How do the internal symmetries of the equations of motion that control the structure of the nonlinear dynamical cross-coupling terms between the different fields conspire with the conservation laws to determine the universal scaling behaviour? (ii) How does the conservation law for the surface fluctuations affect the dynamics and fluctuations of an attached density?

In order to systematically address these generic issues, we study how a conserved fluctuating surface described by the C-KPZ equation affects the spatio-temporal properties of a conserved scalar density that is dynamically coupled to the fluctuating surface. When there are multiple dynamically coupled fields, with all of them exhibiting dynamical scaling, it is not a priori clear whether or not they should all have the same dynamic exponent; in case of equal dynamic exponents the model is said to display strong dynamic scaling, else weak dynamic scaling ensues [21]. In a study on coupled one-dimensional model, [22] showed the sensitive dependence of the nature of

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3 See, De Dominicis and Peliti L [21]. This is following the terminologies introduced by De Dominicis, Peliti in a similar context in dynamic critical phenomena.
dynamic scaling on the precise forms of the dynamic couplings in the model equations. In a model with several dynamical fields, one must thus distinguish between strong and weak dynamic scaling. These theoretical issues form the major motivation of the present work. Independent of any specific applications, the general importance of our studies here lie in their ability to identify ingredients that may control long-time, large-distance universal scaling behaviour in driven systems.

We study the coupled nonequilibrium dynamics of a conserved height field \( h \) and a conserved signed density \( \phi \) (that can be positive or negative, e.g. Ising spin-like degrees of freedom) within simple reduced models. In the absence of any general framework for nonequilibrium systems, such simple models are particularly useful to study and answer questions of principle as we illustrate below. More specifically, we consider the nonlinearly coupled dynamics when \( h \) is autonomous, i.e. the time-evolution of \( h \) is independent of the second field \( \phi \) and follows the C-KPZ equation. This models the dynamical evolution of a structureless signed species living on a fluctuating surface with conserved fluctuations. In the absence of the couplings with \( h \), \( \phi \) follows spatio-temporally scale invariant dynamics described by linear equations of motion with exactly known scaling exponents. We consider two different models for conserved \( \phi \)-dynamics: (i) Model I, where the fluctuations of \( h \) couples only with the spatial variation of \( \phi \), given by \( \nabla \phi \), i.e. the dynamics of \( \phi \) is invariant under the shift \( \phi \to \phi + \text{const} \), an internal symmetry that leaves the dynamics unchanged; and (ii) Model II, where the dynamics is not invariant under such a shift of \( \phi \) (i.e. no such invariance, unlike in Model I). Generally, we find that the scaling properties of Model I and Model II are starkly different—the spatio-temporal scaling of \( \phi \) depends crucially on the detailed nature of its symmetry-determined coupling with \( h \). The remainder of the article is organised as follows: in section 2, we introduce Model I, write down the general symmetry permitted equations of motion for \( h \) and \( \phi \) and evaluate the scalings of the model parameters. Then in section 3, we discuss Model II and note the differences between the two models. We finally summarise and conclude in section 4.

2. Model I

The dynamics of \( h \) is simply given by the C-KPZ equation [16]

\[
\frac{\partial h}{\partial t} = -\nabla^2 \left[ \nu \nabla^2 h + \frac{\lambda_1}{2} (\nabla h)^2 \right] + \eta_h,
\]

where \( \eta_h \) is a Gaussian-distributed, zero-mean conserved noise with a variance \( \langle \eta_h(x, t)\eta_h(0, 0) \rangle = -2D_h \nabla^2 \delta(x) \delta(t) \); \( \nu > 0 \) is a damping coefficient and \( \lambda_1 \) is a nonlinear coupling constant [16].

We now write down the dynamical equations of \( \phi \) in the hydrodynamic limit by using symmetry considerations. We demand (i) translational and rotational invariance, (ii) conservation of \( \phi \), and (iii) invariance under \( \phi \to \phi + \text{const} \) for the dynamics of \( \phi \). The last condition can be fulfilled only if derivatives of \( \phi \) appear in the dynamical equations. Furthermore, for simplicity we restrict ourselves to systems that are linear in \( \phi \)-fluctuations, so that the dynamics of \( \phi \) is invariant under the inversion of \( \phi \). The
general form of the relaxational equation of motion for a conserved density $\phi$ is (we ignore any advective processes)

$$\frac{\partial \phi}{\partial t} = \mu \nabla^2 \delta F + NL + \eta_{\phi}. \quad (2)$$

Here, $F$ is a free energy functional that controls the dynamics and thermodynamics of $\phi$ in equilibrium. We choose

$$F = \int d^d \mathbf{x} \left[ r_0 \phi^2 + (\nabla \phi)^2 \right]/2, \quad (3)$$

where we have neglected any nonlinear terms for simplicity; $r_0 = T - T_c$ with $T$ as the temperature and $T_c$ the critical temperature. Furthermore, NL represents conserved nonlinear terms of nonequilibrium origin that are invariant under inversion of $\phi$ as well as a constant shift of $\phi$. We first consider the case with $r_0 = 0$, i.e. $\phi$-fluctuations are critical.

$$\frac{\partial \phi}{\partial t} = -\nabla^2 \left[ \mu \nabla^2 \phi + \lambda_2 (\nabla h) \cdot (\nabla \phi) \right] + \eta_{\phi}. \quad (4)$$

Here, $\eta_{\phi}$ is a Gaussian-distributed, zero-mean conserved noise with a variance $\langle \eta_{\phi}(t) \eta_{\phi}(0) \rangle = -2D_{\phi} \nabla^2 \delta(t)$, $\mu > 0$ is a damping coefficient and $\lambda_2$ is a nonlinear cross-coupling coefficient through which $h$ affects the dynamics of $\phi$. The sign of $\lambda_2$ is arbitrary. Equation (4) corresponds to a current of $\phi$ given by

$$J_{\phi_1} = \nabla [\mu \nabla^2 \phi + \lambda_2 (\nabla h) \cdot (\nabla \phi)]. \quad (5)$$

Thus, the nonequilibrium contribution to $J_{\phi_1}$ can act only when both $\phi$ and $h$ have nonzero gradients and one of these gradients is spatially varying. Note that $J_{\phi_1}$ remains invariant under $\phi \rightarrow \phi + const.$ This is in contrast to the models studied in [19, 20]. Furthermore, in contrast to our model I, the dynamics of the density field in [20] is non-conserved. Clearly, equation (4) is invariant under $\phi \rightarrow \phi + const.$; it is clear from the linearised versions of equations (1) and (4) that the na"ive scaling dimensions of $h$ and $\phi$ are identical. Notice that both equations (1) and (4) do not admit any generalised Galilean invariance; see discussions below and [17] for technical comments.

2.1. Scaling in model I

It is instructive to first consider the linearised version of equation (4) by setting $\lambda_2 = 0$. In that limit, the dynamics of $\phi$ can be solved exactly. In particular, in the Fourier space the correlation function $C_{\phi}(q, \omega) = \langle |\phi(q, \omega)|^2 \rangle$ takes the form

$$C_{\phi}(q, \omega) = \frac{2D_{\phi} q^2}{\omega^2 + \mu^2 q^4}, \quad (6)$$

where $q$ and $\omega$ are the Fourier wavevector and frequency, respectively. Now, correlator (6) corresponds to the dynamic exponent $z_{\phi} = 4$ and roughness exponent $\chi_{\phi} = \frac{2-d}{2}$ for the field $\phi$ [6]. Compare these results with the correlations of $h$ from the linearised version of equation (1). This yields the corresponding dynamic and roughness exponents
for $h$ as $z_h = 4$ and $\chi_h = \frac{2-d}{2}$, respectively. Clearly, $z_\phi = z_h$ at the linear level, implying strong dynamic scaling at the linear level. It is of course well-known that the scaling are affected by relevant (in a scaling sense) nonlinearities [15], and as a result their values at the linear level get modified by the nonlinear effects. For instance, in the lowest order renormalised perturbation theory [16], $z_h = (12 - \epsilon)/3$ with $\epsilon = 2 - d > 0$, where as $z = 4$ for $d \geq 2$ [16]. Whether or not strong dynamic scaling is still observed at the nonlinear level, is a question that we study here.

We can now write the dynamic generating functional [23], $Z_I$, averaged over the noises $\eta_h$ and $\eta_\phi$, for the coupled system; see also [20] for similar functional approaches

$$Z_I = \int D\hat{h} D\hat{\phi} D\delta D\hat{\phi} \exp[S_I],$$

where $\delta$ and $\hat{\phi}$ are dynamic conjugate fields to $h$ and $\phi$, respectively [23]; $S_I$ is the action functional given by

$$S_I = \int d^d x dt [D_h \hat{h} \nabla^2 \hat{h} + D_\phi \hat{\phi} \nabla^2 \hat{\phi}$$

$$+ \hat{h} \left( \frac{\partial h}{\partial t} + \nabla^2 [\nu \nabla^2 h + \frac{\lambda_1}{2} (\nabla h)^2] \right)$$

$$+ \hat{\phi} \left( \frac{\partial \phi}{\partial t} + \nabla^2 [\mu \nabla^2 \phi + \lambda_2 (\nabla h)(\nabla \phi)] \right)].$$

Nonlinear couplings $\lambda_1, \lambda_2$ preclude any exact enumeration of the relevant correlation functions from the action functional $S_I$ in equation (8). Naturally, perturbative calculations are used. Naive perturbative expansions yield diverging corrections to the measurable quantities. In order to deal with these long wavelength divergences in a systematic manner, we employ Wilson momentum shell dynamic renormalisation group (DRG) [2, 15]. To this end, we first integrate out fields $h(q, \omega), \phi(q, \omega)$ with wavevector $\Lambda/b < q < \Lambda$, $b > 1$, perturbatively up to the one-loop order in (8). Here, $\Lambda$ is an upper cut off for wavevector. This allows us to obtain the ‘new’ model parameters corresponding to a modified action $S_I^\xi$ with an upper cutoff $\Lambda/b < \Lambda$; see appendix for the corresponding one-loop Feynman diagrams.

In order to extract the renormalised parameters, we then rescale wavevectors and frequencies according to $q' = bq$ and $\omega' = b^2 \omega$. Here $b = \exp[l]$ is a dimensionless length scale. In a simple model with a single variable, $z$ becomes the dynamic exponent. For a multivariable problem as ours with the attendant possibility of unequal dynamic exponents for $h$ and $\phi$, the interpretation of $z$ in frequency rescaling as above will be clear as we go along. Under these rescalings, fields $h$ and $\phi$ also scale. We write, in Fourier space, $h(q, \Omega) = \xi_h h(q', \Omega'), \hat{h}(q, \Omega) = \xi_{\hat{h}} \hat{h}(q', \Omega'), \phi(q, \Omega) = \xi_\phi \phi(q', \Omega'), \hat{\phi}(q, \Omega) = \xi_{\hat{\phi}} \hat{\phi}(q', \Omega').$ Using the redundancy [24] of the rescaling factors, $\xi_h, \xi_\phi$ and $\xi_{\hat{\phi}}$, we impose the coefficients of $\int d^d q d\Omega h(-i\Omega) h$ and $\int d^d q d\Omega \phi(-i\Omega) \phi$ to remain unity. This leads to the following condition on the rescaling factors:

$$\xi_h \xi_h = 1 = \xi_{\hat{\phi}} \xi_\phi.$$
In the real space, let \( h(x', t') = \xi^R_h h(x, t) \) and \( \phi(x', t') = \xi^R_\phi \phi(x, t) \). Thus \( \xi^R_h = b^{-(d+\phi)} \xi_h \) and \( \xi^R_\phi = b^{-(d+\phi)} \xi_\phi \), where \( \chi_h \) and \( \chi_\phi \) are roughness exponents \([24]\) associated with \( h \) and \( \phi \), respectively.

2.1.1. Recursion relations and scaling exponents

We set up a perturbative DRG up to the one-loop order, where one-loop fluctuation corrections to the different model parameters are obtained. Notice that there are no fluctuation corrections to \( \lambda_1 \) at this order. In [16], this was ascribed to a modified Galilean invariance. Later on it was argued in [17] that there are indeed corrections to \( \lambda_1 \) at the two-loop order. Such considerations hold for Model I as well. Since we stick to a one-loop order DRG, we ignore such issues here. Following the standard DRG procedure \([15, 25]\), we arrive at the following recursion relations [with \( b = \exp[t] \)]:

\[
\frac{d\nu}{dl} = \nu[z - 4 + g(4 - d)],
\]

\[
\frac{d\mu}{dl} = \mu[z - 4 + \frac{2B^2 g}{P(1 + P)}(4 - d + \frac{2(1 - P)}{1 + P})],
\]

\[
\frac{d\lambda_1}{dl} = \lambda_1[z + \chi_h - 4],
\]

\[
\frac{dD_h}{dl} = D_h[z - 2 - d - 2\chi_h],
\]

\[
\frac{dD_\phi}{dl} = D_\phi[z - 2 - d - 2\chi_\phi],
\]

\[
\frac{d\lambda_2}{dl} = \lambda_2[\chi_h + z - 4 + \frac{2gB(3 + P)}{(1 + P)^2} - \frac{4gB^2}{(1 + P)^2} - \frac{2gB}{1 + P}],
\]

where \( P = \frac{g}{4} \), \( B = \frac{\lambda_2}{2} \) and \( g = \frac{2^n D_h k u^2}{4^n d} \). are the effective dimensionless coupling constants; under rescaling of space and time \( g \) scales as \( b^{2-d} \) implying \( d = 2 \) to be the critical dimension \([16]\). The flow equations for \( g \), \( P \) and \( B \) may be immediately obtained:

\[
\frac{dg}{dl} = g[2 - d + g(d - 4)],
\]

\[
\frac{dP}{dl} = -P g[4 - d - \frac{2B^2}{P(1 + P)}(4 - d + \frac{2(1 - P)}{1 + P})],
\]

\[
\frac{dB}{dl} = B[2gB(3 + P) - \frac{4gB^2}{(1 + P)^2} - \frac{2Bg}{1 + P}] + B^2.
\]

At the DRG fixed point (FP), \( dg/dl = 0 = dB/dl = dP/dl \). Then, we have from equation (16), \( B^* = 1, P^* = 1 \) and \( g^* = \frac{2-d}{3(4-d)} \) or \( g^* = 0 \) at the FP. Linear stability analysis reveals that \( g^* = \frac{2-d}{3(4-d)} \) is the stable FP for \( d < 2 \); for \( d \geq 2 \), \( g^* = 0 \) [16]. For
$d < 2$ and with these values of $B^*, P^*$ and $g^*$ at the stable DRG FP, we note that both (10) and (11) yield the choice $z = \frac{10 + d}{3}$ at the DRG FP make both $d\nu(l)/dl$ and $d\mu(l)/dl$ zero. This in turn implies that both $h$ and $\phi$ have the same dynamic exponent $z_h = z_\phi = \frac{10 + d}{3}$. Thus, Model I displays \textit{strong dynamic scaling}. Furthermore, by using (13) and (14) at the stable DRG FP for $d < 2$, we obtain $\chi_h = \chi_\phi = \frac{2 - d}{3}$, $d < 2$. Also, expectedly in contrast to the results in [20], the flat phase of the CKPZ equation becomes unstable below $d = 2$ and not below $d = 4$, due to the roughness exponent becoming positive below $d = 2$ [16]; equivalently, $d = 2$ is the critical dimension for the CKPZ equation. For $d \geq 2$, the nonlinearities are irrelevant (in a RG sense) and hence the results from the linear theory holds. Note that the nonlinearities become irrelevant in [20] only above $d = 6$.

2.1.2. Model I with $\lambda_1 = 0$ Consider now the limiting case with $\lambda_1 = 0$. Thus, $h$ evolves linearly with $z_h = 4$ and $\chi_h = \frac{2 - d}{2}$ known exactly. The flow equations simplify to

\[
\begin{align*}
\frac{d\mu}{dl} &= \mu[z\phi - 4 + \frac{\lambda_2^2 D_h K_d}{2\nu\mu(\nu + \mu)d}(4 - d + \frac{2(\nu - \mu)}{\nu + \mu})], \\
\frac{dD_\phi}{dl} &= D_\phi[z\phi - 2 + d - 2\chi\phi], \\
\frac{d\lambda_2}{dl} &= \lambda_2[\chi_h + z\phi - 4 - \frac{\lambda_2^2 D_h K_d \Lambda^2}{\nu(\mu + \nu)^2d}].
\end{align*}
\] (17)

In obtaining the flow equations (17), we have rescaled time $t$ that corresponds to a dynamic exponent $z_\phi$.

Clearly, there are positive corrections to $\mu$. Thus, scale-dependent $\mu(l) \gg \nu(l) = \nu$, as the DRG FP is approached. Thus, we already conclude that $z_\phi < z_h = 4$. Hence, \textit{weak dynamic scaling} is expected, implying $\nu(l)/\mu(l) \to 0$ as $l \to \infty$. In that limit we find from the above flow equations

\[
\begin{align*}
\frac{d\mu}{dl} &= \mu[z - 4 + \frac{\lambda_2^2 D_h K_d}{2\nu\mu^2d}(2 - d)], \\
\frac{d\lambda_2}{dl} &= \lambda_2[\chi_h + z - 4 - \frac{\lambda_2^2 D_h K_d \Lambda^2}{\nu\mu^2d}].
\end{align*}
\] (18) (19)

We identify an effective coupling constant $\tilde{g} = \frac{\lambda_2^2 D_h K_d \Lambda^2}{\nu\mu^2d}$ that scales as $t^{2 - d}$ under the rescaling of space and time. This shows that $d = 2$ is the critical dimension, such that for $d < 2$ fluctuation corrections should be relevant in the long wavelength limit. The DRG flow equation for $\tilde{g}$ is

\[
\frac{d\tilde{g}}{dl} = \tilde{g}[2 - d + 2\tilde{g}(d - 3)].
\] (20)

At the DRG FP, $d\tilde{g}/dl = 0$, yielding $\tilde{g} = \frac{2 - d}{2(3 - d)}$ as the stable FP for $d < 2$, where as $\tilde{g} = 0$ is the stable FP for $3 > d > 2$. The apparent singularity in $\tilde{g}$ at $d = 3$ is likely
to be an artifact of a low order perturbation theory used here [26]. This then implies
\[ z_\phi = 4 + \frac{(2-d)^2}{2d-6} = 4 + O(\epsilon)^2, \]
where \( z_\phi \) differs from \( z_h \) by \( O(\epsilon)^2 \). Since our one-loop analysis is valid only up to \( O(\epsilon) \), we set \( z_\phi = z_h \) at this order, restoring strong dynamic scaling. Whether or not this remains true at higher order remains to be checked. On the whole, thus, within a one-loop approximation Model I displays strong dynamic scaling independent of whether nonlinear effects are considered in the dynamics of \( h \), i.e. \( \lambda_1 = 0 \) or not. Whether or not this remains true at higher order remains to be seen. In contrast, [19] finds both equal (\( z_h = z_\phi \)) and unequal (\( z_h \neq z_\phi \)) dynamic exponents at \( d = 1 \), depending upon the details of the nonlinear couplings. Furthermore, Model I has \( d = 2 \) as the critical dimension, similar to the KPZ equation [7], or the conserved KPZ equation [16]. In contrast, the interplay between the KPZ surface fluctuations and phase separation dynamics tend to make the critical dimension higher, as reported in [20]. Unsurprisingly, the scaling behaviour of Model I is completely different from those in [20].

2.1.3. Model I with \( r_0 > 0 \)

We now briefly discuss the dynamics of \( \phi \) for \( r_0 > 0 \), i.e. \( \phi \)-fluctuations are noncritical. This generates a linear \( \nabla^2 \phi \) term in equation (2) leading to
\[
\frac{\partial \phi}{\partial t} = -\nabla^2 \left[ -\mu_1 \phi + \mu \nabla^2 \phi + \lambda_2 (\nabla h) \cdot (\nabla \phi) \right] + \eta_\phi,
\]
where \( \mu_1 = \mu r_0 > 0 \). Equation (21) gives \( z_\phi = 2 \) at the linear level, corresponding to weak dynamic scaling, since the \( \mu_1 \nabla^2 \phi \)-term is more relevant than the \( -\mu \nabla^4 \phi \)-term (in a scaling sense). However, with the existing form of the \( \lambda_2 \)-nonlinear term, corrections to the propagator are still all at \( O(q^4) \). This implies that there are no fluctuation-corrections to the \( \mu_1 \)-term, yielding \( z_\phi = 2 \) exactly even at the nonlinear level, and hence weak dynamic scaling prevails. This together with the exact knowledge of \( \chi_\phi \) from the non-renormalisation of \( D_\phi \) yield the scaling exponents of \( \phi \) exactly, which are identical to their values in the corresponding linear theory. Dynamics of \( \phi \), then, is totally unaffected by the nonequilibrium drive when \( \phi \)-fluctuations are noncritical.

3. Model II

In Model I above, height fluctuations couple with \( \nabla \phi \), the local spatial variation in \( \phi \). In contrast, we now consider the case when \( \nabla h \) couples directly with \( \phi \); consequently the dynamics is not invariant under a constant shift of \( \phi \): \( \phi \rightarrow \phi + \text{const.} \). We again consider the case where the dynamics of \( h \) is autonomous, i.e. unaffected by \( \phi \). Thus, the dynamical equation of \( h \) is still given by equation (1). The dynamical equation for \( \phi \) is still given by equation (2), while the nonequilibrium terms \( \text{NL} \) should now include conserved terms that break the symmetry under a constant shift of \( \phi \) as well. We continue to assume that the dynamics in linear in \( \phi \). Furthermore, we now set \( r_0 > 0 \), i.e. \( T > T_c \) (hence \( \phi \) is noncritical); we briefly discuss the \( r_0 \) case at the end. With all

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4 See, Frey and Tauber [26]. We believe this is analogous to the singularity in the effective coupling constant in the KPZ equation for \( 1 < d < 2 \) within a one-loop approximation. A two-loop approximation removes this problem.
these, the most general equation for $\phi$ to the leading order in nonlinearities and spatial gradients in the hydrodynamic limit is now given by

$$\frac{\partial \phi}{\partial t} = \tilde{\mu} \nabla^2 \phi + g_2 \nabla(\phi \nabla h) + \eta_\phi. \quad (22)$$

Here, $\tilde{\mu} > 0$ is a damping coefficient, $g_2$ is a nonlinear coupling constant; noise $\eta_\phi$ is same as that in Model I. Notice that the nonlinear coupling term $g_2$ is identical to the one introduced in [19]. Equation (22) corresponds to a current

$$J_{\phi_2} = -\tilde{\mu} \nabla \phi - g_2 \phi \nabla h. \quad (23)$$

Thus, the nonequilibrium parts in $J_{\phi_2}$ contribute wherever there is a local tilt in the surface given by $\nabla h$ with a local $\phi$ [19]. This distinguishes the nonequilibrium effects of Model II from Model I. Both equations (22) and (23) are clearly not invariant under $\phi \rightarrow \phi + \text{const.}$. At this stage it is convenient to split $\phi$ as a sum of its mean $\phi_0 = \int d^d x \phi(x,t)/V$ and a zero-mean fluctuating part; here $V$ is the system volume. This clearly generates a linear term proportional to $\phi_0 \nabla^2 h$. Such a term manifestly breaks the symmetry under inversion of $\phi$. In effect, $\phi_0$ now parametrises the dynamics of $\phi$. We set $\phi_0 = 0$ and denote the fluctuating part with zero-mean by $\phi$ below. This then restores the symmetry under inversion of $\phi$. Notice that equation (22) is invariant under $x \rightarrow -x$.

3.1. Scaling in model II

Similar to our analysis for Model I, we first consider the linearised version of (22) that can be solved exactly. The correlation function $C_\phi(q,\omega)$ then takes the exact form:

$$C_\phi(q,\omega) = \frac{2D_\phi q^2}{\omega^2 + \tilde{\mu}^2 q^4}. \quad (24)$$

Equation (24) implies that $z_\phi = 2$ and roughness exponent $\chi_\phi = -\frac{d}{2}$ for the density field $\phi$. Thus, at the linear level, this clearly implies weak dynamic scaling since $z_h = 4 \neq z_\phi$ at the linear level. As before, we study whether and if so, how the nonlinear effects modify these scaling behaviors, and in particular if weak dynamic scaling can get further reinforced (larger differences between $z_h$ and $z_\phi$) or otherwise by nonlinear effects.

The action functional $S_{II}$ for Model II is given by

$$S_{II} = \int d^d x \int dt [D\hat{h} \nabla^2 \hat{h} + D_\phi \hat{\phi} \nabla^2 \hat{\phi} + \hat{h} \left( \frac{\partial \hat{h}}{\partial t} + \nabla^2 [\nu \nabla^2 h + \frac{\lambda_1}{2} (\nabla h)^2] \right) + \hat{\phi} \left( \frac{\partial \hat{\phi}}{\partial t} - \tilde{\mu} \nabla^2 \hat{\phi} - g_2 \nabla(\phi \nabla h) \right)]. \quad (25)$$

As in Model I, nonlinearities preclude any exact enumeration of the scaling exponents. We again resort to perturbative DRG up to the one-loop order as for Model I.

3.1.1. Rescaling of fields and parameters: recursion relations and scaling exponents

We enumerate the one-loop corrections to the various model parameters in Model II. See

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appendix for the one-loop diagrams for Model II. We rescale time by a factor that corresponds to a dynamic exponent $z$. The interpretation of $z$ as the dynamic exponent of $h$ or $\phi$ will become clear below. The recursion relations for the model parameters are given by:

$$
\frac{d\nu}{dl} = \nu[z - 4 + g(4 - d)],
$$
$$
\frac{d\mu}{dl} = \mu[z - 2 + g_3],
$$
$$
\frac{d\lambda_1}{dl} = \lambda_1[z + \chi_h - 4],
$$
$$
\frac{dD_h}{dl} = D_h[-d + z - 2 - 2\chi_h],
$$
$$
\frac{dD_\phi}{dl} = D_\phi[-d + z - 2 - 2\chi_\phi + g_3],
$$
$$
\frac{dg_2}{dl} = g_2[z - 2 + \chi_h - g_3],
$$

(26)

where $g = \frac{\lambda^2 D_h K_d \Lambda^2}{4\nu^2 d}$ (same as in Model I) and $g_3 = \frac{2g_2 D_h K_d \Lambda^2}{\nu^2 d} > 0$ are the effective dimensionless coupling constants for Model II that scales as $b^{2-d}$ under rescaling of space and time, suggesting $d = 2$ to be the critical dimension (same as in Model I). Now, if we set $\frac{d\mu}{dl} = 0$ at the DRG FP, we find

$$z = 2 - g_3.
$$

(27)

Since this choice of $z$ leaves $\mu(l)$ scale independent, we identify it as the dynamic exponent of $\phi$: $z_\phi = z$. Notice that this choice for $z$ does not leave $\nu(l)$ scale independent, and hence cannot be the dynamic exponent $z_h$ of $h$ (which is anyway known independently), $z_h = (10 + d)/3$ for $d \leq 2$ and $z_h = 4$ for $d \geq 2$). With this choice for $z$, instead $\nu(l)$ scales as $l^{-4 + g(4-d)}$ [22]. This scale-dependent $\nu(l)$ together with the identification $l = -\ln q$ may be used to extract $z_h$ from the formal definition

$$
C_h(q, \omega) = \frac{2D_h q^2}{\omega^2 + \nu(q^2) q^2 \omega^2},
$$

(28)

see [22] for more details. This line of argument yields the same value for $z_h$, as already obtained above.

Equation (27) combined with $\frac{dD_\phi}{dl} = 0$, gives $\chi_\phi = -\frac{d}{2}$ at the DRG FP. Note that $\chi_h$ and $z_h$ retain respectively the same values as in Model I.

Further, the DRG flow equation for $g_3$ is

$$
\frac{dg_3}{dl} = 2g_3(\chi_h - 2g_3).
$$

(29)

Thus, we have at the FP, $g_3 = 0, \frac{d}{2}$. Since $\chi_h = (2 - d)/3$, $g_3 = (2 - d)/6$ gives the stable FP for $d < 2$; for $d \geq 2$, $g_3 = 0$ at the stable FP. Thus for $d \geq 2$, the coupling becomes irrelevant in the dynamics of $\phi$ and the results from the linear equation holds. But for $d < 2$, nonlinear effects are relevant and $z_\phi = 2 - g_3 = 2 - \frac{d}{2} = \frac{10 + 4d}{6}$. In general, for any $d$ weak dynamic scaling prevails in the system.
3.1.2. Dynamics of φ when $\lambda_1 = 0$ We now consider the case when $\lambda_1 = 0$ for model II. In this limit, $\frac{dw}{dt} = 0 \Rightarrow z_h = 4$. From the flow equation of $g_2$ and $g_3$, we have

$$\frac{dg_2}{dt} = g_2(z - 2 + \chi_h - g_3) \quad (30)$$

and

$$\frac{dg_3}{dt} = 2g_3(\chi_h - 2g_3). \quad (31)$$

Now in the limit of $\lambda_1 = 0$, we have

$$\chi_h = \frac{2 - d}{2}. \quad (32)$$

Using the fact that at FP, either $g_3 = 0$ (stable for $d \geq 2$), or $g_3 = \frac{\chi_h}{2}$ (stable for $d < 2$) and the value of $\chi_h$ given by equation 32, we arrive at two values for $z_\phi$.

For $g_3 = 0$, $z_\phi = 2$ and $\chi_\phi = -d/2$, valid for $d \geq 2$. When $g_3 = \frac{\chi_\phi}{2}$, $z_\phi = \frac{2 + d}{2}$ and $\chi_\phi = -d/2$, valid for $d < 2$. Thus $\chi_\phi = -d/2$ for all $d$.

The results from Model II are complementary to those in [19]. While [19] considered non-conserved dynamics for $h$ and conserved dynamics for $\phi$ at $d = 1$, we have considered conserved dynamics for both the fields $h$ and $\phi$ in Model II in general $d$ dimensions, with the dynamics of $h$ being treated as autonomous for simplicity. Nonetheless, Model II and the studies in [19] display universal scaling very different from each other—a hallmark of the models being nonequilibrium and unlike equilibrium models where a conservation law can affect only the dynamic scaling behaviour; see, e.g. model A and model B in the language of [15]. In contrast, none of the scaling exponents in the two studies have any simple relations. While the roughness exponent takes the value $\frac{d-2}{2}$ for all $d$ in our Model II, it can take several values depending upon the model parameters in [19]. More importantly, [19] shows the possibility of both strong and weak dynamic scalings. In sharp contrast, our Model II only gives weak dynamic scaling for any $d$ and for both the stable and unstable FP values of the coupling constant, $g_3$. Thus comparison between the results of [19] and Model II significantly establishes how conservation laws can lead to entirely different physical outcomes, even though the coupling between the degrees of freedom can have the same structure.

3.1.3. Model II with $r_0 = 0$ We briefly discuss what happens when $r_0 = 0$, i.e. the $\phi$-fluctuations are critical. With $r_0 = 0$, the $\mu \nabla^2 \phi$-term in equation (22) is to be replaced by a $\nabla^4 \phi$-term. Nonetheless, with the existing nonlinear term in (22), the lowest order corrections to the propagator are at $O(q^2)$, thus generating a $\nabla^2 \phi$-term in the fluctuation corrected equation. All our results for Model II with $r_0 > 0$ derived above then immediately follow. It is however possible to start with a specific bare $r_0$, so that the fluctuation-corrected $r_0$ vanishes. The relaxation of $\phi$-fluctuations will now be controlled by a (subleading to a bare $\mu \nabla^2 \phi$-term) $\nabla^4 \phi$-term, with a $z_\phi = 4$ at the linear level (as in Model I with $r_0 = 0$). However, the fluctuation corrections to this $\nabla^4 \phi$-term are expected to be different from those in Model I with $r_0 = 0$, owing to the different form of the nonlinear term in equation (22). We do not discuss the details here.
4. Conclusions and outlook

We have thus investigated how the presence or absence of an internal symmetry affects the universal scaling properties in the noisy dynamics of a conserved scalar density driven by a fluctuating conserved KPZ surface \( h \). We make a particularly simple choice for internal symmetry, viz invariance under a constant shift of \( \phi \). To this end, we consider two specific reduced models, Model I and Model II, to address how the interplay between the symmetries that control the structure of the nonlinear terms and conservation laws control the universal scaling properties. In Model I, \( h \)-fluctuations couple with \( \nabla \phi \), rendering the ensuing dynamics of \( \phi \) independent of \( \phi_0 \), the mean of \( \phi \). Model I is constructed in a way to respect the invariance under inversion of \( \phi \). At the linearised level, both \( h \) and \( \phi \) dynamics display strong dynamic scaling with a single dynamic exponent \( z = 4 \). Beyond the linearised theory, the scaling exponents depend crucially on the details of the nonlinear couplings, and also whether \( \phi \) is a critical field or noncritical. The relevant scaling exponents are evaluated in a one-loop DRG calculation for critical \( \phi \)-fluctuations; for noncritical \( \phi \)-fluctuations, the scaling exponents are unaffected by the nonlinearity and known exactly, that corresponds to weak dynamic scaling. For critical \( \phi \)-fluctuations, strong dynamic scaling ensues.

We have studied another model, Model II, where \( h \)-fluctuations directly couple with \( \phi \). Thus in contrast to Model I, Model II does not remain invariant under a constant shift of \( \phi \). As a result, \( \phi_0 \), the mean of \( \phi \), parametrises the dynamics of \( \phi \). We focus on the particular case where \( \phi_0 = 0 \). This restores the symmetry of the model under inversion of \( \phi \)-fluctuations. In Model II (with \( r_0 > 0 \)), even at the linear level weak dynamic scaling follows \(( z_\phi = 2 )\), a feature that holds good even when the nonlinear effects are taken into account. Furthermore, if we assume \( \phi_0 \neq 0 \), we obtain additional linear term proportional to \( \nabla^2 h \) in (22). Given that the dynamics of \( h \) is independently known (being autonomous), this term effectively acts like an additional additive noise in the problem, whose correlation is not \( \delta \)-correlated in space and time. This is likely to affect the scaling properties of \( \phi \) in nontrivial ways. Comparison of the results from Model I and Model II thus establish the significance of the internal symmetry under constant shifts of \( \phi \) in determining the scaling properties. It would be of interest to construct equivalent discrete lattice-gas models and study these issues there. The specific microscopic rules for the lattice-gas models for Model I and Model II may be formed from the nonequilibrium contributions to the currents (5) and (23) respectively. We welcome further work along this direction.

We now make a brief general comparison of our studies here with those on the generalised coupled KPZ equations by DK. First and foremost, DK allowed for the feedback of the density fluctuations on the height fluctuations, whereas in our case, the dynamics of the surface is assumed to be autonomous, independent of the density fluctuations. Furthermore, the surface fluctuations in both Model I and Model II in our studies are conserved, and hence slower than the nonconserved height fluctuations in the models of DK. This is reflected in the generic higher values of the dynamic exponents \( z_h \) in our studies.

For reasons of simplicity, we have ignored a \( \phi^4 \) term in the free energy (3)\(^5\) while setting up Model I or Model II. Such a term, if included, will generate a \( \sim \phi^4 \) term in the dynamics of \( \phi \) in both Model I and Model II. Clearly, this term manifestly breaks

\(^5\) Without the \( \phi^4 \)-term in (3), it becomes admittedly artificial, but still suffices for our purposes here.
the $\phi \rightarrow \phi + \text{const}$, symmetry of Model I. Thus all the couplings present in Model II that also break the invariance under a constant shift of $\phi$ should now be included, and Model I will effectively reduce to Model II (albeit at $r_0 = 0$). In case of Model II, a $\phi^2$-term in the dynamics of $\phi$ will lead to competition with the already existing nonlinearities in Model II; the resulting scaling behaviour can further be investigated within the framework of one-loop RG (not done here).

Our consideration of the the dynamics of $h$ as autonomous is clearly a limiting case. More generally, generic nonlinear feedback of $\phi$ on the dynamics of $h$ may be present. Again due to symmetry reasons the nonlinear structure of the feedback should differ from Model I to Model II. It would be interesting to see whether and how the feedback may alter the conclusions drawn above. Furthermore, our equations of motion are all invariant under spatial inversion. An interesting generalisation would be to allow terms that violate this invariance under spatial inversion. Such terms may potentially lead to generation of underdamped kinematic waves, absent in Model I or Model II. Kinematic waves can be important, e.g. these waves lead to weak dynamic scaling in [22]. Whether similar breakdown of strong dynamic scaling occurs in Model I in the presence of kinematic waves remains to be investigated. The dynamical field $\phi$ being a conserved density follows a conservation law form of equation of motion. Similar to [20], phase ordering dynamics on a conserved KPZ surface may be studied by making $\phi$ a nonconserved density. A nontrivial variant of this would be to consider $\phi$ to be a broken symmetry mode that follows a nonconserved equation of motion, but executes a scale-invariant dynamics. When such a broken symmetry variable is driven by a conserved KPZ field, the emerging scaling properties are likely to be quite different from what is reported here. Lastly, coupled systems with linear instabilities may be considered, such that the nonequilibrium steady states may even involve patterns. We look forward to future work in these directions.

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Appendix A. Model I: results

In the Fourier space, the propagators and correlators of $h$ and $\phi$, respectively, thus take the following form:

\[
\langle \hat{h}(q, \omega)h(-q, -\omega) \rangle = \frac{-1}{-i\omega + \nu q^4}
\]

\[
\langle h(q, \omega)h(-q, -\omega) \rangle = \frac{2Dhq^2}{\omega^2 + \nu^2q^8}
\]

\[
\langle \hat{\phi}(q, \omega)\phi(-q, -\omega) \rangle = \frac{-1}{-i\omega + \mu q^4}
\]

\[
\langle \phi(q, \omega)\phi(-q, -\omega) \rangle = \frac{2D\phi q^2}{\omega^2 + \mu^2q^8}
\]

(A.1)
A.1. One loop corrections to the model parameters

The corrections of the model parameters and the corresponding relevant Feynman diagrams for Model I are given below:

\begin{align*}
\nu^\lessgtr & = \nu - \frac{\lambda_1^2 D_h K_d}{\nu^2} \left[ \int_{\Lambda/b}^{\Lambda} \frac{dq q^{d-1}}{4q^2} - \int_{\Lambda/b}^{\Lambda} \frac{dq q^{d-1}}{q^2 d} \right] \tag{A.2} \\
\mu^\lessgtr & = \mu - \frac{\lambda_2^2 D_h K_d}{\nu(\nu + \mu)} \left[ \frac{1}{2} - \frac{2 + (\nu - \mu)/\nu (\nu + \mu)}{d} \right] \int_{\Lambda/b}^{\Lambda} \frac{dq q^{d-1}}{q^2} \tag{A.3}
\end{align*}

Figure A1 shows the relevant Feynman diagram for one-loop correction to \( \nu \).

\begin{align*}
\lambda_1^\lessgtr & = \lambda_1 \\
D_h^\lessgtr & = D_h \\
D_\phi^\lessgtr & = D_\phi
\end{align*}

(A.4)  \hspace{1cm} (A.5)  \hspace{1cm} (A.6)
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\[ \lambda^< = \lambda_1 = \frac{g_2^2 D_h K_d}{\nu \mu d} \]  

\[ \mu^< = \tilde{\mu} + \frac{g_2^2 D_h K_d}{\nu \mu d} \int_{\Lambda/b}^{\Lambda} \frac{dq^{d-1}}{q^2} \]  

\[ \lambda^<_2 = \lambda_2 + \frac{D_h \lambda_2^3 \lambda_1 K_d (3 \nu + \mu)}{2 \nu^2 (\nu + \mu)^2 d} - \frac{D_h \lambda_2^3 K_d}{\nu (\nu + \mu)^2 d} \]  

\[ - \frac{D_h \lambda_2^3 \lambda_1 K_d}{2 \nu^2 (\nu + \mu)d^d} \int_{\Lambda/b}^{\Lambda} \frac{dq^{d-1}}{q^2} \]  

\[ \nu^< = \nu - \frac{\lambda_1^2 D_h K_d}{\nu^2} \left[ \int_{\Lambda/b}^{\Lambda} \frac{dq^{d-1}}{4 q^2} - \int_{\Lambda/b}^{\Lambda} \frac{dq^{d-1}}{q^2 d} \right] \]  

Relevant one-loop corrections to \( \lambda^<_2 \) are given in figure A3.

### A.2. Model II results

The propagators and correlators for the system are given by

\[ \langle \hat{h}(q, \omega) h(-q, -\omega) \rangle = \frac{-1}{-i \omega + \nu q^4} \]  

\[ \langle h(q, \omega) h(-q, -\omega) \rangle = \frac{2 D_h q^2}{\omega^2 + \nu^2 q^8} \]  

\[ \langle \hat{\phi}(q, \omega) \phi(-q, -\omega) \rangle = \frac{-1}{-i \omega + \tilde{\mu} q^2} \]  

\[ \langle \phi(q, \omega) \phi(-q, -\omega) \rangle = \frac{2 D_\phi q^2}{\omega^2 + \tilde{\mu}^2 q^4} \]  

As before, we now find corrections to the bare model parameters by evaluating integrals up to one-loop order from wavevector \( \Lambda/b \) to \( \Lambda \). This leads to the following results:

\[ \nu^< = \nu - \frac{\lambda_2^2 D_h K_d}{\nu^2} \left[ \int_{\Lambda/b}^{\Lambda} \frac{dq^{d-1}}{4 q^2} - \int_{\Lambda/b}^{\Lambda} \frac{dq^{d-1}}{q^2 d} \right] \]  

\[ \tilde{\mu}^< = \tilde{\mu} + \frac{g_2^2 D_h K_d}{\nu \mu d} \int_{\Lambda/b}^{\Lambda} \frac{dq^{d-1}}{q^2} \]  

\[ \lambda^<_1 = \lambda_1 \]  

Figure A4. One loop correction to \( \tilde{\mu} \).

Figure A5. One loop correction to \( D_\phi \).
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\[ D_h^\omega = D_h \]
\[ D_\phi^\omega = D_\phi + \frac{g_2^2 D_\phi D_h K_d}{\nu \bar{\mu}^2 d} \int_{\Lambda/b}^{\Lambda} dq \frac{q^{d-1}}{q^2} \]
\[ g_2^\omega = g_2 - \frac{g_2^3 D_h K_d}{\nu \bar{\mu}^2 d} \int_{\Lambda/b}^{\Lambda} dq \frac{q^{d-1}}{q^2} \]

The relevant Feynman diagrams for \( \bar{\mu}, D_\phi \) and \( g_2 \) are given by figures A4–A6, respectively.

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