Ehresmann theory of connection in a principal bundle - compendium for physicists.

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1 Global theory of connection in principal fibre bundles (compendium).

Differential geometry and tensor analysis give main mathematical tools for relativists. But most of them use, up to now, the old index formalism in local coordinate maps on the spacetime manifold [1,2,3].

This formalism was developed in the past (about 100 years ago) by Italian mathematicians Georgio Ricci Curbastro, Tullio Levi-Civita and Luigi Bianchi and it is exhaustively presented e.g. in [4,5,6,7].

The modern coordinate-free formulation of the formalism created by Charles Ehresmann is still not sufficiently known for majority of relativists.

We would like to give a compendium about these two formulations and connection between them. We will start with foundations of the global Ehresmann theory on fibre bundles and end up with old local index formulation on basic manifold.

Our lecture is founded on standard books [8,9,10,11].

We have assumed that a potential reader knows elements of differentiable manifolds and Lie groups.
1.1 Fibre bundles.

Definition 1. Let $\mathcal{G}$ be a Lie group. A $\mathcal{G}$-space is a differentiable manifold $(M, A_M)$ and the group $\mathcal{G}$ acts as a group of point mappings, i.e. $\mathcal{G}$-space is a pair $[M, \mathcal{G}]$, and there is a smooth mapping 
$$\psi : M \times \mathcal{G} \to M$$

such that $\psi_a \circ \psi_b = \psi_{ba}$ and $\psi_e = \text{id}_M$,
where $\psi_a(p) = \psi(p, a) := pa$, $p \in M$, $a, b \in \mathcal{G}$, and $e$ is a unit element of the group $\mathcal{G}$.

Here $A_M$ means the maximal atlas on $M$.

Remark 1. By $\psi_a(p) = pa$ we mean, that $\mathcal{G}$ acts on the right and we denote it by $R_a$.

Definition 2. An orbit of the point $p \in M$ is the set of points of the manifold defined as follows
$$\{R_a(p) : a \in \mathcal{G}\}$$

and it is denoted by $p\mathcal{G}$.

If $p\mathcal{G} = M$, then $M$ is called a homogenous space and we say, that $\mathcal{G}$ acts on $M$ transitively. It follows, that for any pair of points $p_1, p_2 \in M$ there is $g \in \mathcal{G}$ such that $R_gp_1 = p_2$.

Definition 3. A group of isotropy (stabilizer) of the point $p \in M$ denoted by $\mathcal{G}_p$ is a set
$$\mathcal{G}_p := \{a \in \mathcal{G} : R_a(p) = p\}.$$  
We say, that $p$ is a fixed point of the mapping $R_a$ and $\mathcal{G}_p \subset \mathcal{G}$.

Notice, that a group $\mathcal{G}_p$ is a closed subgroup of the group $\mathcal{G}$.

Definition 4. If $\mathcal{G}_p = \{e\}$ for any point $p \in M$, then we say, that $\mathcal{G}$ acts freely on $M$ (i.e. without fixed points). Then $M$ is called a main space.

If $e \in \mathcal{G}$ is the only element of the group $\mathcal{G}$ for which $R_p$ is an identity, i.e. $R_gx = x$ for every $x \in M$, then we say that the Lie group $\mathcal{G}$ acts on $M$ effectively.

Definition 5. A Lie group $\mathcal{G}$ acts simply transitive or single transitive on $M$, if for any pair of points $p_1, p_2 \in M$ there is only one point $g \in \mathcal{G}$ such that $R_g(p_1) = p_2$. 

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If the Lie group $\mathcal{G}$ acts on $M$ on the right, it assigns for every vector field $A \in \mathfrak{g}$ a vector field $A^*$ on $M$ in the following way:

$$A \leftrightarrow a_t = \exp(tA),$$

where $a_t$ is a 1-parameter group of global transformations of a manifold $M$. It is a subgroup of the group $\mathcal{G}$. The action of the group $a_t$ on $M$ induces on $M$ a vector field tangent to orbits $a_t(p)$, $p \in M$, $t \in \mathbb{R}$. It is a vector field $A^* \in \mathcal{X}(M)$. $\mathcal{X}(M)$ denotes here a Lie algebra of vector fields on the manifold. $\mathfrak{g}$ stands for the group algebra $\mathcal{G}$.

A mapping $\sigma : \mathfrak{g} \to \mathcal{X}(M)$, which for a field $A \in \mathfrak{g}$ assigns a vector field $A^* \in \mathcal{X}(M)$, is a homomorphism of Lie algebras. If the group $\mathcal{G}$ acts on $M$ effectively, then $\sigma : \mathfrak{g} \to \mathcal{X}(M)$ is a monomorphism.

**Definition 6.** A mapping $\phi$ from the algebra $\mathfrak{g}$ onto the algebra $\mathfrak{h}$ is an isomorphism, if:

1. $\phi$ is an isomorphism of vector spaces $\mathfrak{g}$ and $\mathfrak{h}$;
2. $\phi([u,v]) = [\phi(u), \phi(v)]$ for any $u$, $v \in \mathfrak{g}$.

In the above definition $[u,v]$ is a Lie bracket in $\mathfrak{g}$, and $[\phi(u), \phi(v)]$ is a bracket of images of $u$ and $v$ in $\mathfrak{h}$. An isomorphism of the algebra $\mathfrak{g}$ onto itself is called an automorphism.

One can introduce some notions connected with fibre bundle by considering graf $\tilde{f}$ of the mapping $f : M \to N$, where $M$, $N$ are two differential manifolds.

![Figure 1: A trivial fibre bundle.](image)

The above picture shows a trivial fibre bundle connected with graph $\tilde{f}$ of the mapping $f : M \to N$. 

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$E = M \times N$ is called the space of the fibre or just simply the fibre over $M$; $M$ is a base space;

$\pi : E \to M$ is defined as $\pi(p, q) := p$ and it is called a projection (a projection of $E$ onto the base $M$), $p \in M$, $q \in N$, $(p, q) \in E$;

$N$ is a typical fibre;

$E_p := \pi^{-1}(p) \in E$ is a fibre over $p \in M$;

$\tilde{f} : M \to E$ is a section of the bundle $E$ called graf of the mapping $f : M \to N$; $\tilde{f}(p) = [p, f(p)] \in M \times N = E$.

In general $E(M, \pi, N)$ is a fibre bundle over $M$ with a typical fibre $N$ if the projection $\pi : E \to M$ is a smooth surjection of the differentiable manifold $E$ onto the manifold $M$ and if each point $p \in M$ has a neighborhood $U$, such, that there is a diffeomorphism $h : \pi^{-1}(U) \to U \times N$ such that $\pi \circ [h^{-1}(p, y)] = p$, $p \in U$, $y \in N$. This diffeomorphism $h : \pi^{-1}(U) \to U \times N$ is called a local trivialisation of the bundle, and it means, that a piece of $E$ over a sufficiently small $U \subset M$ looks like a product manifold $U \times N$.

**Definition 7.** The bundle $E$ over $M$ is called a trivial bundle if there is a diffeomorphism $h : E \to M \times N$ such that $\pi\left[h^{-1}(p, q)\right] = p$ for $\forall p \in M$, $\forall q \in N$.

The bundle connected with graf $\tilde{f}$ is a trivial bundle.

Another example of the fibre bundle is a tangent bundle $T(M)$ over a differentiable manifold $M$.

Assume that the differentiable manifold $M$ permits a global coordinate system, i.e. an atlas consisting of one chart $\varphi : M \to \mathbb{R}^n$ and let us consider $T(M) := \bigcup_{p \in M} T_p(M)$.

In a coordinate system $(\varphi, M)$ (global coordinate system) a vector field looks as follows:

\[ X = X^i(p) \left( \frac{\partial}{\partial x^i} \right)_p, \quad p \in M. \]

Such field gives a point mapping

\[ (x^i) : M \to \mathbb{R}^n (= N). \]

Let $\varphi : M \to \mathbb{R}^n$ be the chart mentioned before and denote the chart $\hat{\varphi} : E = T(M) \to \mathbb{R}^{2n}$ as follows:

Let $X \in T(M)$, then $X \in T_p(M)$, for some $p \in M$ and $X = X^i(\frac{\partial}{\partial x^i})_p$. Set $\hat{\varphi}(X) := [x^1(p), ..., x^n(p), X^1(p), ..., X^n(p)] \in \mathbb{R}^{2n}$.

$\hat{\varphi}$ is bijective. Moreover, if both $\varphi$ and $\psi$ are $C^k$-compatible (global) charts on $M$, then $\hat{\varphi}$ and $\hat{\psi}$ are $C^k$-compatible on $T(M)$.

We say that the operation " $\hat{}$ " (hat) lifts an atlas $M$ onto the atlas on $T(M)$. Lifted from $M$ onto $T(M)$ the atlas with "hat" changes $T(M)$ into
2n-dimensional differentiable manifold, which is called a tangent bundle of the differentiable manifold \( M \). (In our example with one global chart this bundle is trivial).

Figure 2: Pictures of the fibre bundles \( E(M, \pi, N) \) and \( T(M) \).

### 1.2 Principal fibre bundles.

In the present section we present a definition of principal fibre bundle and some examples of this structure. These bundles play very important role in the differential geometry and they are applied in physics for example in the theory of gauge fields.

**Definition 8.** Let \( M \) be a manifold and \( G \) a Lie group. A (differentiable) principal fibre bundle over \( M \) with group \( G \) consists of a manifold \( P \) and an action of \( G \) on \( P \) satisfying the following conditions:

1. \( G \) acts freely on \( P \) on the right; \((u, a) \in P \times G \to ua := R_a u \in P\);

2. \( M \) is the quotient space of \( P \) by the equivalence relation induced by \( G \), \( M = P/G \), and the canonical projection \( \pi : P \to M \) is differentiable;
1. For each structure group of bundle \( M \), the differential manifold \( M \), manifold linear mapping of \( \sigma \) and \( \chi \) induces a homomorphism \( P \), \( u \in (P) \), \( G \) here identified with the tangent space \( g \) is the Lie algebra of vector fields on \( M \). 

2. Every fibre is diffeomorphic to the structure group \( G \).

3. From local triviality of \( P(M,G,\pi) \) we see, that if \( W \) is a submanifold of \( M \) then \( \pi^{-1}(W)(W,G) \subset P \) is a principal fibre bundle. The bundle \( \pi^{-1}(W)(W,G) \) is called the portion of \( P \) over \( W \) or the restriction of \( P \) to \( W \) or a piece of the bundle \( P \) over \( W \).

Let \( P(M,G) \) be a principal fibre bundle. The action of the group \( G \) on \( P \) induces a homomorphism \( \sigma^* : g \rightarrow \chi(P) \). Here \( g \) is the Lie algebra of \( G \) and \( \chi(P) \) is the Lie algebra of vector fields on \( P \). For each \( u \in P \) let \( \sigma_u \) be the mapping \( a \in G \rightarrow u \cdot a \in P \) (\( \sigma_u : G \rightarrow P \), \( \sigma_u(a) = u a \in P \), \( u \in P \)), then \( (\sigma_u)^* A_e = (\sigma^* A)_u \). Tangent mapping \( \sigma^* \) is a linear mapping of \( g \) into \( \chi(P) \): 

\[
(\sigma_u)^* : T_e(G) \rightarrow T_u(P) \subset \chi(P).
\]

The algebra \( g \) is here identified with the tangent space \( T_e(G) \) in unit element \( e \in G \). \( A_e = A(e) \in T_e(G) \), where \( A \in g \).
Definition 10. For each $A \in \mathfrak{g}$ the field $A^* = \sigma^*(A)$ is called the fundamental vector field corresponding to $A \in \mathfrak{g}$.

Since $G$ acts (vertically) on fibres: $G$ maps each fibre into itself, so the vector $A^*_u$ is tangent to the fibre at each $u \in P$.

Because the dimension of each fibre is equal to that of $\dim \mathfrak{g}$, the mapping $\mathfrak{g} \ni A \to (A^*)_u$ of the algebra $\mathfrak{g}$ into $T^*_u(P)$ is a linear isomorphism of $\mathfrak{g}$ onto the vector space $T^*_u(P)$ tangent to the fibre at $u \in P$ and called the tangent space of vertical vectors.

Fact 1. Let $A^*$ be the fundamental vector field corresponding to $A \in \mathfrak{g}$. Then, for each $a \in G$, the vector field $R_a^*(A^*)$ is the fundamental vector field over $P$, corresponding to the field $\text{Ad}_{a^{-1}}A \in \mathfrak{g}$. $[(\text{Ad}_{a^{-1}})A = (R_a)^*A]$

Here $\text{Ad}$ denotes an adjoint representation of the group $G$ in its own algebra $\mathfrak{g}$.

Remark 3. Fundamental vector fields are important in the theory of connection.

Here we present some examples of principal fibre bundles:

1. The bundle of linear frames $L(M)$ with the structure group $GL(n, \mathbb{R})$.
   Let $(M, A_M)$ be a manifold set on $\mathbb{R}^n$, where $A_M$ is a maximal atlas. A linear frame $u$ at a point $x \in M$ is an ordered basis $(X_1, ..., X_n)$ of the tangent space $T_x(M)$.
   Let $L(M) = \bigcup_{x \in M} \{\text{the set of all linear basis } u \text{ at } x \in M\}$ and let define $\pi : L(M) \to M$ in the following way: $\pi[u(x)] = x \in M$, where $u(x)$ is a basis $T_x(M)$. The general linear group $GL(n, \mathbb{R})$ acts on $L(M)$ on the right as follows: if $a = (a^i_j) \in GL(n, \mathbb{R})$ and $u = (X_1, ..., X_n)$ is a linear frame at $x \in M$, then $ua := (Y_1, ..., Y_n)$, where $Y_i = X_k a^k_i$ is the new linear frame $u'$ at the point $x \in M$. The group $GL(n, \mathbb{R})$ acts freely on $L(M)$. Moreover $\pi(u) = \pi(v)$ if and only if $v = ua$ for some $a \in GL(n, \mathbb{R})$.

Differential structure on $L(M)$
Let $(x^1, ..., x^n)$ be a local coordinate system in a coordinate neighborhood $U \subset M$. Every frame $u$ at $x \in U$ can be expressed uniquely in the form $u = (X_1, ..., X_n)$ with $X_i = X^k_i \frac{\partial}{\partial x^k}$ and $\det[X^k_i] \neq 0$. $\pi^{-1}(U)$ is diffeomorphic with $U \times GL(n, \mathbb{R})$. We can make $L(M)$ into a differentiable manifold by taking $(x^i)$ and $(X^k_i)$ as a local coordinate system in $\pi^{-1}(U)$. $[x^i] = \text{local coordinates on } U \subset M$, a $X^k_i = \text{coordinates (components) of the frame } u \in T_x(M)$ in a natural frame $\{(\frac{\partial}{\partial x^i})_x \text{ given}$
by a local map $(\varphi, U)$. This map gives us local coordinates $(x^i)$ on $U \subset M$. $L(M) [M, GL(n, \mathbb{R})]$ is the principal fibre bundle over $M$ with the structure group $GL(n, \mathbb{R})$. This bundle is called the bundle of linear frames over $M$.

2. The bundle of orthonormal frames $\mathcal{O}[M, O(n)]$.

Let $(M_n, A_M)$ be a Riemann manifold.

**Definition 11.** The basis $\{\vec{e}_i\}(x)_{i=1,2,...,n}$ on $M_n$, $x \in M_n$ is called an orthonormal basis if $g_x(\vec{e}_i, \vec{e}_j) = \langle \vec{e}_i | \vec{e}_j \rangle = \delta_{ij}$ ($g$ - a metric on $M_n$, $g_x := g(x)$).

When we have an orthonormal frame $\{\vec{e}_i\}(x)_{i=1,...,n}$ on $M_n$ we can obtain from it any other orthonormal frame $\{\vec{e}_i'\}_{i'=1,...,n}$ with the help of transformation $\vec{e}_i' = \vec{e}_j A_{ij}^l$, where $[A_{ij}^l] \in O(n) \subset GL(n, \mathbb{R})$. $O(n)$ denotes here an orthonormal group. There is a natural bijection between the collection of orthonormal frames attached at a point $x \in M$ and the group $O(n)$. The sum $\bigcup_{x \in M}(x, o_x)$, where $x \in M_n$ and $o_x$ is a collection of orthonormal frames at $x$ is denoted by $\mathcal{O}(M)$ and it is called the principal bundle of orthonormal frames.

**Definition 12.** The principal bundle of orthonormal frames over a Riemannian manifold $(M_n, g)$ is the set $\mathcal{O}(M)$. This bundle we usually denote by $\mathcal{O}[M_n, O(n), \pi]$ or simply $\mathcal{O}(M)$.

The structure of the manifold is given by local coordinates at $\pi^{-1}(U) : (x^i), (A^l_{ij})$ ($i$, $j$, $i' = 1,...,n$), where $(x^i)$- are local coordinates on $U \subset M_n$, a $(A^l_{ij}) \in O(n)$.

$\pi^{-1}(U) = U \times O(n)$

$\mathcal{O}(M)$ is a restriction of the principal bundle $L(M)$ to the orthogonal group $O(n)$. One can write, that $\mathcal{O}(M) = \{u \in L(M) : g_{ij}(u) = \delta_{ij}\}$, $g_{ij}(u) := g(\vec{e}_i, \vec{e}_j)$. 

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2 A connection in the fibre bundle.

Let $P(M, G)$ be a principal fibre bundle over a manifold $M$ with a structure group $G$. Let $T_u P$ be a tangent space at the point $u \in P$ and let $V_u$ be the subspace of the space $T_u P$ consisting of vectors tangent to the fibre through $u$.

**Definition 13.** A connection $\Gamma$ in the principal fibre bundle $P$ is an assignment of a subspace $H_u$ a space $T_u P$ at each point $u \in P$ such that:

1. $T_u P$ is a direct sum of the subspaces $V_u$ i $H_u$

\[ T_u P = V_u \oplus H_u ; \]

2. $H_u a = (R_a)^* H_u$ for every $u \in P$ and $a \in G$, where $(R_a)^*$ is the tangent mapping to the point transformation $R_a u = R_a (u) = ua$ of the manifold $P$ induced by $a \in G$;

3. an assignment of the subspace $H_u$ is a smooth mapping i.e. a distribution $u \rightarrow H_u$ is differentiable.

The second condition means that the distribution $u \rightarrow H_u$ is invariant by $G$.

In the above definition $V_u$ denotes a vertical subspace of the space $T_u P$, and $H_u$ is a horizontal subspace.

A vector $X \in T_u P$ is called horizontal if it is an element of the subspace $H_u P$ or vertical if it is an element of the subspace $V_u P$. Every vector from the space $T_u P$ can be uniquely written in the following way

\[ X = Y + Z , \]

where $Y$ is a vertical component and $Z$ a horizontal component of the vector $X$ and they are denoted respectively $vX$ i $hX$. The third condition in the above definition means simply that if a vector field $X$ is differentiable on $P$ then $vX$ i $hX$ are also differentiable.
Given a connection \( \Gamma \) in \( P \) we define a 1-form \( \omega \) on \( P \) with values in the Lie algebra \( g \) of the group \( G \) as follows. In the section 1.2 we showed that every vector field \( A \in g \) induces a vector field \( A^* \) on \( P \) which is called the fundamental vector field corresponding to \( A \), and the mapping \( A \rightarrow (A^*)_u \) is a linear isomorphism from the algebra \( g \) onto \( V_u \) for every \( u \in P \). For each \( X \in T_uP \) we define \( \omega(X) \) to be the unique \( A \in g \) such that \( (A^*)_u \) is equal to the vertical component of the vector \( X \). Notice, that \( \omega(X) = 0 \) if and only if the vector \( X \) is horizontal. The form \( \omega \) is called a connection form of the given connection \( \Gamma \).

**Theorem 1.** The connection form \( \omega \) of the connection \( \Gamma \) satisfies the following conditions:
1. $\omega(A^*) = A$ for every $A \in \mathfrak{g}$;

2. $(R_a)_* \omega = \text{Ad}(a^{-1})\omega$ i.e. $\omega((R_a)^*X) = \text{Ad}(a^{-1})\omega(X)$ for each $a \in G$ and for each vector field $X$ on $P$, where $\text{Ad}$ denotes the adjoint representation of $G$ in $\mathfrak{g}$.

Conversely, given a 1-form $\omega$ on $P$ with values in $\mathfrak{g}$ and satisfying conditions 1-2, then there exists a unique connection $\Gamma$ in $P$ whose connection form is $\omega$.

$(R_a)_* \omega$ denotes here a pull-back of the form $\omega$ from the point $u \cdot a \in P$ to the point $u \in P$.

Proof:

1. Let $\omega$ be a form of some connection $\Gamma$ on a bundle $P(M, G)$. Then (1.) follows directly from the definition of a connection form.

2. Since every vector field $X$ on $P(M, G)$ is the sum $X = vX + hX$,

it is sufficient to verify (2.) in two special cases:

- $X$ is a horizontal vector field;
- $X$ is a vertical vector field.

Let $X$ be a horizontal vector field. Then the field $(R_a)^*X$ is also horizontal for every $a \in G$ (it follows from (2.) of the definition of a connection $\Gamma$). Hence $\omega((R_a)^*X) \equiv 0$ and $\text{Ad}(a^{-1})[\omega(X)] \equiv 0$.

Now, let $X$ be a vertical field. We can assume that $X$ is a fundamental vector field of $A^*$. Then $R^*_a(A^*)$ is a fundamental vector field corresponding to $\text{Ad}(a^{-1})A$. Therefore we have

$$[(R_a)_\ast \omega]_u(X) = \omega_{ua}[R^*_a(X)] = \omega_{ua}(\text{Ad}(a^{-1})A) = \text{Ad}(a^{-1})A = \text{Ad}(a^{-1})[\omega_u(X)].$$

Conversely, let $\omega$ be the 1-form on $P(M, G)$ with properties (1.) and (2.). We define $H_u := \{X \in T_u(P) : \omega(X) = 0\}$. The distribution $u \to H_u$ defines the connection $\Gamma$ on $P(M, G)$ and $\omega$ is its form.

The projection $\pi : P \to M$ induces a linear mapping (tangent, a differential $\pi$), which we denoted by $\pi^* : T_u(P) \to T_xM$ for each $u \in P$, $x = \pi(u)$.
When on the principal bundle $P(M, G)$ is given a connection $\Gamma$, the differential $\pi^*$ maps isomorphically a horizontal subspace $H_u$ onto $T_x(M)$, $x = \pi(u)$.

The horizontal lift (or just: lift) of a vector field $X$ from $M$ onto $P(M, G)$ is a unique vector field $X^*$ on $P(M, G)$: $X^*$ is horizontal and $\pi^*(X^*_u) = X_{\pi(u)}$ for every $u \in P$.

**Fact 2.** Let $\Gamma$ be the connection in the principal fibre bundle $P$ and let $X$ be a vector field which is defined on the manifold $M$. Then there is uniquely defined a horizontal lift $X^*$ of the vector field $X$. The lift $X^*$ is invariant with respect to $R^*_a$ for each $a \in G$ and $\pi^*(X^*_a) = X_{\pi(u)}$. Conversely, every horizontal vector field $X^*$ on $P$ which is invariant with respect to $G$ is a lift of a vector field $X$ on the manifold $M$.

Figure 4: Picture illustrating Fact 2
**Fact 3.** Let $X^*$ and $Y^*$ be horizontal lifts of the vector fields $X$ and $Y$ respectively. Then:

1. $X^* + Y^*$ is a horizontal lift of the vector field $X + Y$;

2. For every function $f$ on the manifold $M$, $f_*X^*$ is a horizontal lift of the vector field $fX$, where $f_*$ is the function on $P$ defined by the formula $f_*(u) := f \cdot \pi(u)$;

3. the horizontal component of the vector field $[X^*, Y^*]$ is a horizontal lift for $[X, Y]$.

Let $(x^1, ..., x^n)$ be a local coordinate system in a coordinate neighborhood $U \subset M$ and let $X^*_i$ ($i = 1, ..., n$) be the horizontal lifts to $\pi^{-1}(U)$ for the vector fields $X_i = \frac{\partial}{\partial x^i}$ on $U$ ($i = 1, ..., n$). Then the vector fields $X^*_1, ..., X^*_n$ form a basis for the distribution $u \mapsto H_u$ in $\pi^{-1}(U)$.

Let $c : [0, 1] \to M$ be a curve on $M$ and let $e \in P^{-1}(c(0))$.

**Definition 14.** A horizontal lift (a lift) $\bar{c} : [0, 1] \to P$ of the curve $c : [0, 1] \to M$ on $P(M, G)$ is a curve $\bar{c}$ on $P(M, G)$ with following properties:

- $\bar{c}(0) = e$;
- $\pi \cdot \bar{c} = c$;
- $\bar{c}$ is a horizontal curve, i.e. a vector tangent to $\bar{c}$ at the point $\bar{c}(t)$ belongs to $H_{\bar{c}(t)}$ for every $t \in [0, 1]$.

For any $e \in P(M, G)$: $\pi(e) = c(0)$ there is a unique horizontal lift $\bar{c} = c(t)$ of the curve $c : [0, 1] \to M$, which "begins" at the point $e$.

**Remark 4.** A horizontal lift of the curve is strictly associated with a lift of the vector field. Namely, if $X^*$ is a lift of the vector field $X$ on $M$, then the integral curve of the field $X^*$ through $u_0 \in P(M, G)$ is a lift of the integral curve of the field $X$ through the point $x_0 = \pi(u_0) \in M$.

**Definition 15.** We say that $\bar{c}(1)$ is given by the parallel transport of $\bar{c}(0) = e$ along the curve $c : [0, 1] \to M$.

**Remark 5.** In general, even if $c : [0, 1] \to M$ is closed (a loop), $c(0) = c(1)$, then $\bar{c}(t)$ is not closed: $\bar{c}(1) = \bar{c}(0) \cdot a$, where $a \in G$. 
A holonomy group \( \phi(x_0) \) of the connection \( \Gamma \) at the point \( c(0) = x(0) =: x_0 \) consists of elements of the structure group \( G \), which are given by \( \bar{c}(1) = \bar{c}(0) \cdot a \) for all loops \( c : [0, 1] \to M : c(0) = c(1) \), which start and end at the point \( x(0) = c(0) =: x_0 \).

Let us express the form of the connection \( \omega \) on \( P(M, G) \) by the family of forms defined on open sets of the base differentiable manifold \( M \).

Let \( \{U_\alpha\} \) be an open covering \( M \) with the family of diffeomorphisms \( \psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G \) and corresponding to this family the family of transition functions \( \psi_{\alpha\beta}[\pi(u)] := \varphi_\beta(u) \cdot [\varphi_\alpha(u)]^{-1}, \varphi_\alpha : \pi^{-1}(U_\alpha) \to G, \psi_{\alpha\beta} : U_\alpha \cap U_\beta \to G \). For every \( \alpha \) let \( \sigma_\alpha : U_\alpha \to P \) be a section of the bundle \( P \) over \( U_\alpha \), defined as: \( \sigma_\alpha(x) := \psi_\alpha^{-1}(x, e); x \in U_\alpha \), and let \( e \) be the unit element of the group \( G \).

For every nonempty \( U_\alpha \cap U_\beta \) we define \( \theta \)-valued 1-form \( \theta_{\alpha\beta} : T(U_\alpha \cap U_\beta) \to \mathfrak{g} \), \( \theta_{\alpha\beta} := (\psi_{\alpha\beta})_* \theta \) (\( \theta \)-pull back of the form \( \theta \) on \( M \)), and on every \( U_\alpha \) we define \( \mathfrak{g} \)-valued 1-form \( \omega_\alpha = (\sigma_\alpha)_* \omega \).

**Fact 4.** Local forms on \( M \), \( \theta_{\alpha\beta} \) and \( \omega_\alpha \), satisfy on \( U_\alpha \cap U_\beta \) conditions:

\[
\omega_\beta(X) = \text{Ad}(\psi_{\alpha\beta}(X))^{-1}[\omega_\alpha(X)] + \theta_{\alpha\beta}(X)
\]

for each \( X \in T_x(U_\alpha \cap U_\beta) \), \( x \in U_\alpha \cap U_\beta \).

Conversely, every family of 1-forms \( \{\omega_\alpha\} \) with values in \( \mathfrak{g} \) (\( \omega_\alpha \) defined on \( U_\alpha \)), which satisfies the above conditions determines a unique 1-form of the connection \( \omega \) on \( P(M, G) \).

This form generates a family \( \omega_\alpha \) in the way given above.

**Curvature form and structure equations.**

Let \( P(M, G) \) be a principal fibre bundle and let \( \rho \) be a representation of the structure group \( G \) of this bundle in a finite dimensional vector space \( V \):

\( \rho : G \to \text{GL}(V) \); \( \rho(a) : V \to V \) is a linear mapping of \( V \) for each \( a \in G \) such that \( \rho(a \cdot b) = \rho(a) \cdot \rho(b), \rho(a) = A \in \text{GL}(V), \rho(b) = B \in \text{GL}(V), \rho(a \cdot b) = A \cdot B \in \text{GL}(V) \). Here \( \text{GL}(V) \) means the group of linear transformations acting on \( V \).

**Definition 16.** Pseudotensorial form of the type \( (\rho, V) \) and of the degree \( r \) on \( P(M, G) \) is a \( V \)-valued \( r \)-form \( \varphi \) on \( P(M, G) \) with property \( (R_a)_* \varphi = \rho(a^{-1}) \cdot \varphi \), for \( a \in G \).
In the extended form there is
\[[R_u] \varphi](x_1, ..., x_r) = \rho(a^{-1}) \cdot \varphi_u(x_1, ..., x_r), \text{ where } x_1, ..., x_r \in T_u(P) \times ... \times T_u(P), \rho(a^{-1}) \in GL(V).

**Definition 17.** A form \( \varphi \) of degree \( r \) and of the type \( (\rho, V) \) on \( P(M, G) \) is tensorial, if it is a horizontal form, i.e. if \( \varphi(X_1, ..., X_r) = 0 \) if at least one of the tangent vectors \( X_i \) \( (i = 1, ..., r) \) on \( P(M, G) \) is vertical, i.e. tangent to a fibre.

Let \( H \) be a connection on \( P(M, G) \). Let \( V_u \) and \( H_u \) be a vertical (\( V_u \)) and a horizontal space (\( H_u \)) of the tangent space \( T_u(P) \) respectively and let \( h: T_u(P) \to H_u \) be a projection onto \( H_u \) (\( h \) assigns for every \( X \in T_u(P) \) a horizontal component \( hX \)).

**Fact 5.** If \( \varphi \) is a pseudotensorial \( r \)-form of the type \( (\rho, V) \) on \( P(M, G) \), then:

1. The form \( \varphi \cdot h \) defined by \( \varphi \cdot h(X_1, ..., X_r) := \varphi(X_1, ..., hX_r), X_i \in T_u(P) \) is a tensorial form of the type \( (\rho, V) \) on \( P(M, G) \);
2. \( d\varphi \) is a pseudotensorial \( (r+1) \)-form of the type \( (\rho, V) \);
3. \( D\varphi := (d\varphi) \cdot h \) is a tensorial \( (r+1) \)-form of the type \( (\rho, V) \).

**Definition 18.** A form \( D\varphi := (d\varphi) \cdot h = d\varphi(hX_1, ..., hX_r, hX_{r+1}) \) is called an exterior covariant differential of an \( r \)-form \( \varphi \) and the operation \( D \) is called an exterior covariant differentiation.

If \( \rho \) is an adjoint representation of the group \( G \) in its algebra \( \mathfrak{g} \), then pseudotensorial form of the type \( (\rho, \mathfrak{g}) \) is called a form of the type \( AdG \).

An example: a connection form \( \omega \) is a pseudotensorial \( 1 \)-form of the type \( AdG \).

**Fact 6.** \( D\omega = (d\omega)h = d\omega(hX) \) is a tensorial \( 2 \)-form of the type \( AdG \) which is called a curvature form of the connection \( \omega \). We denote it by \( 2 \)-form \( \Omega \).

**Definition 19.** Let \( \omega \) be a connection form and \( \Omega \) its curvature form. Then an equation \( \Omega(X, Y) = d\omega(X, Y) + \frac{1}{2}[\omega(X), \omega(Y)] \) is a structure equation of the connection \( \omega \), \( X, Y \in T_u(P) \) and \( u \in P(M, G) \).

**Theorem 2.** Let \( e_1, ..., e_r \) be a basis of a Lie algebra \( \mathfrak{g} \) of a Lie group \( G \), and let \( C^i_{jk} = -C^k_{ij} \) \( (i, j, k = 1, ..., r) \) be structure constants of the algebra \( \mathfrak{g} \) with respect to this basis, i.e. \( [e_j, e_k] = C^i_{jk}e_i, (i, j, k = 1, ..., r) \). Let \( \omega = \omega^i e_i \), \( \Omega = \Omega^i e_i \) for \( i, j = 1, ..., r \). Then the structure equation of the connection \( \omega \) can be expressed as follows:

\[ d\omega = -\frac{1}{2} C^i_{jk} \omega^j \wedge \omega^k + \Omega^i \] \( (i = 1, ..., r) \).
Proof:

\[
(d\omega^i)\bar{e}_i = -\frac{1}{2}[\omega^i \bar{e}_i, \omega^k \bar{e}_k] + \Omega^i \bar{e}_i
\]

\[
[\omega^i \bar{e}_i, \omega^k \bar{e}_k] = [\bar{e}_i, \bar{e}_k] \omega^i \wedge \omega^k
\]

\[
(d\omega^i)\bar{e}_i = -\frac{1}{2}[\bar{e}_i, \bar{e}_k] \omega^i \wedge \omega^k + \Omega^i \bar{e}_i
\]

\[
[\bar{e}_i, \bar{e}_k] = C_{ik}^l \bar{e}_l
\]

\[
(d\omega^i)\bar{e}_i = -\frac{1}{2}C_{ik}^l \omega^l \wedge \omega^k \bar{e}_i + \Omega^i \bar{e}_i
\]

\[
d\omega^i \bar{e}_i = -\frac{1}{2}C_{ik}^l \omega^l \wedge \omega^k \bar{e}_i + \Omega^i \bar{e}_i
\]

\[
d\omega^i \bar{e}_i = (\frac{1}{2}C_{ik}^l \omega^l \wedge \omega^k + \Omega^i) \bar{e}_i,
\]

or after missing basis \(\{e_i\}\)

\[
d\omega^i = -\frac{1}{2}C_{ik}^l \omega^l \wedge \omega^k + \Omega^i
\]

We have an important Bianchi identity: \(\mathcal{D}\Omega \equiv 0\).

Proof:

\[
\mathcal{D}\Omega(X, Y, Z) := d\Omega \cdot h(X, Y, Z) = d\Omega \cdot h = d\Omega(hX, hY, hZ).
\]

To prove Bianchi identity it is sufficient to show that \(d\Omega(X, Y, Z) \equiv 0\), where \(X, Y, Z\) are horizontal vector fields. We apply the external differentiation \(d\) to structure equations \(d\omega^i = -\frac{1}{2}C_{ik}^l \omega^l \wedge \omega^k + \Omega^i\), and then we obtain \(d \cdot d\omega^i \equiv 0 = -\frac{1}{2}C_{ik}^l (d\omega^l \wedge \omega^k - \omega^l \wedge d\omega^k) + d\Omega^i\). Since \(\omega^i(X) = 0\) for each horizontal field \(X\), then also \((d\omega^i \wedge \omega^k - \omega^l \wedge d\omega^k)(X, Y, Z) \equiv 0\) because we in here have expressions \(\omega^k(Z), \omega^i(X)\). Thus we obtain

\[
d\Omega(hX, hY, hZ) = 0 \implies d\Omega(hX, hY, hZ) = \mathcal{D}\Omega(X, Y, Z) \equiv 0
\]

### 2.1 Linear connection

Now we are going to consider a connection in the bundle of linear frames \(L(M)\). Take \(P\) which will denote \(L(M)\) in our further considerations, let denote by \(G\) the general linear group \(GL(n, \mathbb{R})\), where \(n = dimM\).
Definition 20. A canonical form $\theta$ on $P$ is a 1-form on $P$ with values in $\mathbb{R}^n$ and it is defined as follows

$$\theta(X) = u^{-1}(\pi^*(X)) \quad \text{for } X \in T_u(P),$$

where $u$ is a linear mapping $u : \mathbb{R}^n \to T_{\pi(u)}(M)$, $x = \pi(u)$, $\pi^*(X) \in T_x(M)$.

$\theta$ is also called a soldering (or a solder form). It joins the bundle of linear frames $P[= L(M)]$ with the base manifold $M$ and causes that the geometrical structure of the base is determined by the geometrical structure of the bundle (and conversely). It follows from the fact that $M$ is modelled on $\mathbb{R}^n$, and $\theta$ has values in $\mathbb{R}^n$. Other general principal fibre bundles over $M$ do not have such a form. In a natural basis $\{\xi_i\}_{i=1,...,n}$ of the space $\mathbb{R}^n$, $\theta = \theta^i\xi_i$, where $\theta^i$ is a set of $n$ 1-forms with values in $\mathbb{R}$. If $X \in T_u[L(M)]$, then $\theta^i(X)$ is the i-th component of the projection of $X$ onto $M$ in the basis $u = (X_1,...,X_n)$ of the space $T_x(M)$: $X_i = u(e_i); x = \pi(u)$.

Fact 7. A canonical form $\theta$ in $P[= L(M)]$ is a tensorial 1-form (horizontal) of the type $[id, \mathbb{R}^n]$.

Proof:
Let $X$ be a vertical vector at $u \in P$. Then $\pi^*(X) = 0; \theta(X) = u^{-1}[\pi^*(X)] = u^{-1}(0) = 0$. Hence $\theta$ is a tensorial 1-form (horizontal). If $X \in T_u(P)$ is any vector at $u \in P$ and $a \in G$ is any element of $G \equiv GL(n, \mathbb{R})$, then $R_a^*(X)$ is a vector at $u \cdot a \in P$. Therefore $[(R_a)_*\theta](X) = \theta[R_a^*(X)] = (ua)^{-1}[\pi^*(R_a^*(X))] = a^{-1} \cdot u^{-1}[\pi^*(X)] = a^{-1} \cdot \theta(X) = \rho(a^{-1})\theta(X)$. Since $\rho(a^{-1}) = a^{-1}$, $\rho = id$. Thus $\theta$ is a tensorial 1-form of the type $(id, \mathbb{R}^n)$ on $P$.

Definition 21. A connection in the bundle $L(M)$ is called a linear connection of the manifold $M$.

A linear connection $\Gamma$ of the manifold $M$ allows to assign for every $\xi \in \mathbb{R}^n$ a horizontal vector field $B(\xi)$ on the bundle $P[= L(M)]$ as follows: for each $u \in L(M)$ $[B(\xi)]_u$ there is a unique horizontal vector at $u$ such that $\pi^*([B(\xi)]_u) = u(\xi)$, $u(\xi) \in T_x(M), x = \pi(u)$.

Definition 22. A vector field $B(\xi)$ is called a standard horizontal vector field on $L(M)$, corresponding to $\xi \in \mathbb{R}^n$.

Remark 6. A standard horizontal vector field depends on the choice of a connection in the bundle $L(M)$. (Fundamental fields on $L(M)$ did not depend on the connection.)

Fact 8. A standard horizontal vector field has the following properties:
1. if \( \theta \) is a canonical 1-form on the bundle \( L(M) \), then \( \theta[B(\xi)] = \xi \) for \( \xi \in \mathbb{R}^n \);

2. \( R^*_{a}[B(\xi)] = B(a^{-1}\xi) \) for \( a \in G \) and \( \xi \in \mathbb{R}^n \), \( a^{-1}(\xi) \in \mathbb{R}^n \), \( G = GL(n, \mathbb{R}) \);

3. if \( \xi \neq 0 \), then \( B(\xi) \) is everywhere \( \neq 0 \).

**Remark 7.** Conditions \( \theta(B(\xi)) = \xi \) and \( \omega(B(\xi)) = 0 \) (where \( \omega \) is a connection form) determine \( B(\xi) \) for each \( \xi \in \mathbb{R}^n \).

Let \((B_1, ..., B_n)\) be standard horizontal vector fields corresponding to the natural basis \( \vec{e}_1, ..., \vec{e}_n \) in \( \mathbb{R}^n \) and let \( \{E^i_j\} \) be fundamental vector fields, which correspond to the basis \( \{E_i\} \) in \( gl(n, \mathbb{R}) \). Then \( \{B_i, E^i_j\} \) and \( \{\theta^i, \omega^j\} \) are dual to each other in the following sense

\[
\theta^k(B_i) = \delta^k_i, \quad \theta^k(E^j_i) = 0, \\
\omega^k_i(B_i) = 0, \quad \omega^k_i(E^j_i) = \delta^k_i \delta^j_i.
\]

**Remark 8.** \( n^2 + n \) many vector fields \( \{B_k, E^i_j, i, j, k = 1, ..., n\} \) introduce in \( L(M) \) teleparallelism, i.e. that \( n^2 + n \) many vectors \([B_k, (E^i_j)]\) form a basis of \( T_u(P) \) for each \( u \in P \equiv L(M) \). It follows from the above remark, that the bundle \( T[L(M)] \), which is tangent to \( L(M) \), is trivial.

**Definition 23.** A 2-form \( \Theta := D\theta \) is called a 2-form of torsion of the linear connection \( \omega \) on \( L(M) \). \( \Theta \) is a tensorial 2-form on \( L(M) \) of the type \((id, \mathbb{R}^n)\).

**Fact 9.** If \( A^* \) is a fundamental vector field corresponding to \( A \in \mathfrak{g} \) and if \( B(\xi) \) is a standard vector field corresponding to \( \xi \in \mathbb{R}^n \), then

\[
[A^*, B(\xi)] = B(A\xi),
\]

where \( A\xi \) indicates the image of \( \xi \) with respect to \( A \in \mathfrak{g} = gl(n, \mathbb{R}) \) (a Lie algebra of all matrices of the dimension \( n \times n \)), which acts in \( \mathbb{R}^n \).

**Fact 10.** (Structure equations of the linear connection) Let \( \omega \), \( \Theta \) and \( \Omega \) be a connection form, a torsion form and a curvature form of the linear connection \( \Gamma \) on the manifold \( M \) respectively. We have the first structure equation:

\[
\Theta(X, Y) = d\theta(X, Y) + \frac{1}{2}\left(\omega(X) \cdot \theta(Y) - \omega(Y) \cdot \theta(X)\right),
\]

and the second structure equation:

\[
\Omega(X, Y) = d\omega(X, Y) + \frac{1}{2}\left[\omega(X), \omega(Y)\right],
\]

where \( X, Y \in T_u(P) \) and \( u \in P[\equiv L(M)] \).
In the basis of $\mathbb{R}^n$ and $\mathfrak{gl}(n, \mathbb{R})$ we have $\Theta^i = d\theta^i + \omega^i_j \wedge \theta^j$, $\Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j$.

Proof:

$\Theta(X,Y) = d\theta(X,Y) + \frac{1}{2} \left( \omega(X) \cdot \theta(Y) - \omega(Y) \cdot \theta(X) \right)$,

$X, Y \in T_u(P), P \equiv L(M)$ - a principal bundle of linear frames over $M$, $\theta(X) \in \mathbb{R}^n$, $\omega(X) \in \mathfrak{gl}(n, \mathbb{R})$.

In the basis of $\mathbb{R}^n$ and $\mathfrak{gl}(n, \mathbb{R})$ we have:

$\theta = \theta^i \vec{e}_i$, $\Theta = \theta^i \vec{e}_i$, $\omega = \omega^i_j E_i^j$, $\Omega = \Omega^i_j E_i^j$ for $(i, j = 1, \ldots, n)$.

$\Theta^i(X,Y)\vec{e}_i = d\theta^i(X,Y)\vec{e}_i + \frac{1}{2} \left( \omega^i_j(X) \cdot \theta^j(Y) - \omega^j_i(Y) \wedge \theta^i(X) \right)\vec{e}_i$,

$\Theta^j(X,Y)\vec{e}_j = d\theta^j(X,Y)\vec{e}_j + \omega^j_i(X) \wedge \theta^i(Y)\vec{e}_i$.

Hence $\Theta^i = d\theta^i + \omega^i_j \wedge \theta^j$.

$\theta$ - a canonical 1-form in the principal bundle $L(M)$, $\Theta = \mathcal{D}\theta$ - 2-form of torsion of the linear connection.

$\Omega(X,Y) = d\omega(X,Y) + \frac{1}{2} \left[ \omega(X), \omega(Y) \right]$, $\omega = \omega^i_j E_{ij} = \omega^i_j E_i^j$,

$\Omega^i_j(X,Y)E_i^j = d\omega^i_j(X,Y)E_i^j + \frac{1}{2} \left[ \omega^i_j(X)E_i^j, \omega^k_l(Y)E_k^l \right]$,

$= d\omega^i_j(X,Y)E_i^j + \frac{1}{2} \left[ E_i^j, E_k^l \right] \omega^i_j \wedge \omega^k_l(X,Y)$.

We made use of the fact:

$\left[ \omega(X), \omega(Y) \right] = \left[ \omega^i_j(X)E_i^j, \omega^k_l(X)E_k^l \right]$,

$= \left[ E_i^j, E_k^l \right] \omega^i_j \wedge \omega^k_l(X,Y)$,

$X, Y \in T_u(L(M))$. Then we use the fact, that the commutator

$\left[ E_i^j, E_k^l \right] = \delta^i_k E_i^j - \delta^i_k E_k^j$,

and obtain:

$\Omega^i_j(X,Y)E_i^j = d\omega^i_j(X,Y)E_i^j + \frac{1}{2} \left( \delta^i_k E_i^j - \delta^i_k E_k^j \right) \omega^i_j \wedge \omega^k_l(X,Y)$,

$= d\omega^i_j(X,Y)E_i^j + \frac{1}{2} \left( \omega^k_l(X,Y)E_i^j - \omega^i_j \wedge \omega^j_l(X,Y)E_i^j \right)$,

$= d\omega^i_j(X,Y)E_i^j + \frac{1}{2} \left( \omega^k_j \wedge \omega^k_j - \omega^k_j \wedge \omega^k_j \right) (X,Y)E_i^j$,

$= d\omega^i_j(X,Y)E_i^j + \omega^i_j \wedge \omega^k_l(X,Y)E_i^j$.
From there after missing the basis \( \{ E_i^j \} \) we get
\[
\Omega^i_{\ j} = d\omega^i_{\ j} + \omega^i_{\ k} \wedge \omega^k_{\ j}.
\]

The expressions \( \Theta^i = d\theta^i + \omega^i_{\ l} \wedge \theta^l \) and \( \Omega^i_{\ j} = d\omega^i_{\ j} + \omega^i_{\ k} \wedge \omega^k_{\ j} \) are used to practical computations of the curvature form or the torsion form of the linear connection.

**Theorem 3. (Bianchi identity)** For the linear connection we have:

the first identity of Bianchi

\[
3 D\Theta(X, Y, Z) \equiv \Omega(X, Y) \theta(Z) + \Omega(Y, Z) \theta(X) + \Omega(Z, X) \theta(Y)
\]

or equivalently in the terms of basis of the algebra’s \( gl(n, \mathbb{R}) \) and \( \mathbb{R}^n \)

\[
D\Theta^i \equiv \Omega^i_k \wedge \theta^k,
\]

where \( X, Y, Z \in T_u(P), P \equiv L(M) \).

\( \theta = 1 \) – a canonical form on the bundle \( L(M) \).

the second identity of Bianchi

\[
D\Omega \equiv 0
\]

or in the algebra basis \( gl(n, \mathbb{R}) \), \( D\Omega^i_k = 0 \).

**Remark 9.** Since \( \Omega^i_{\ j}, \Theta^i, \theta^l \) are horizontal, we can decompose \( \Omega^i_{\ j}, \Theta^i \) into components in the basis \( \theta^i \wedge \theta^l \), \( i, l = 1, \ldots, n \).

\[
\Omega^i_{\ j} = \frac{1}{2} R^i_{\ jkl} \theta^k \wedge \theta^l,
\]

\[
\Theta^i = \frac{1}{2} Q^i_{\ kl} \theta^k \wedge \theta^l.
\]

The above decompositions define the following tensors: curvature tensor \( R^i_{\ jkl} = -R^i_{\ jlk} \) and torsion tensor \( Q^i_{\ kl} = -Q^i_{\ lk} \) of the linear connection [on the bundle \( L(M) \)].

### 2.2 Metric connection.

In the previous section we defined a linear connection. A linear connection of a manifold \( M \) defines for every curve \( \tau = x_t, 0 \leq t \leq 1 \) a parallel displacement of a tangent space \( T_{x_0}(M) \) onto a tangent space \( T_{x_1}(M) \). The tangent spaces are considered here as a vector spaces and the parallel displacement is a linear isomorphism between them.

**Definition 24.** A metric or a metric tensor of the class \( C^k \) on a differential manifold \( (M_n, A_M) \) is a tensorial field \( g \) of the class \( C^k \) and the type \((0, 2)\) satisfying conditions \([1],[2],[4]\):
1. \( g \) is symmetric, i.e. \( \forall \ x \in M_n \) a tensor \( g_x \) (\( g_x = g \) at the point \( x \)) is a symmetric tensor \( g_x(u,v) = g_x(v,u), u,v \in T_x(M_n), x \in M_n \).

2. \( \forall \ x \in M_n \) a bilinear form \( g_x \) is nondegenerate, i.e. \( g_x(u,v) = 0, v, u \in T_x(M_n), \forall v \in T_x(M_n) \iff u = 0. \) The form \( g_x(u,v) \) defines a scalar product in \( T_x(M_n) \). In the terms of the components of the vectors \( u \) and \( v \) it has a form \( g_x(u,v) = g_{ik}u^i v^k \), where \( g_{ik} = g_x(\partial_i, \partial_k) \).

Definition 25. A differential manifold with such defined metric is called a Riemannian manifold. It is also said, that \( g \) equips \( M_n \) with a Riemannian structure [8], [9], [11].

Definition 26. A Riemannian manifold is called a proper Riemannian manifold if and only if \( \forall 0 \neq v \in T_x(M_n) \) and \( x \in M_n \) it holds that \( g_x(v,v) > 0. \) In the other case we say about a pseudoriemannian manifold [8], [9], [11].

Equivalent definitions of a metric field \( g \) which are more suitable in the theory of a linear connection on the bundle \( L(M) \) [11].

- A section of a bundle of symmetrical tensors of the type \((0,2)\) associated with a principal bundle of linear frames \( L[M, GL(n, \mathbb{R})] \);

- A set of \( \frac{n(n+1)}{2} \) functions \( g_{ij} : L(M) \to \mathbb{R} \):
  \[
g_{ij}(u \cdot a) = A^k_i A^l_j g_{kl}(u), \]
  where \( g_{kl}(u) := g(\vec{e}_k, \vec{e}_l) \) (a scalar product \((\vec{e}_k, \vec{e}_l)\)), \( u = \{\vec{e}_i\} \) is a frame at \( x = \pi(u), (i,j,k = 1, ..., n), a = (A^i_j) \in GL(n, \mathbb{R}) \).

- A reduction of a principal bundle \( L(M) \) to the orthogonal group \( O(n) \), which is a subgroup of \( GL(n, \mathbb{R}) \). Such reduction gives a bundle of an orthonormal frames \( O[M, O(n)] \subset L(M) \). (It is a subbundle of a principal bundle \( L[M, GL(n, \mathbb{R})] \)). The bundle \( O[M, O(n)] \) defines \( g \): if \( u = \{\vec{e}_i\} \in O[M, O(n)] \) is given, then at \( x = \pi(u) \) \( g = e^1 \otimes e^1 + ... + e^n \otimes e^n \), where \( e^i(\vec{e}_j) = \delta^i_j \) \{\( \vec{e}_i\) is a frame in \( T_x(M) \); \{\( e^j\) is a dual basis to \( \{\vec{e}_i\} \), i.e. a basis in \( T^*_x(M) \)}. Conversely, if there is given a metric \( g \) on \( M \), then \( O[M, O(n)] \) is defined as a set of all orthonormal frames with respect to \( g \):
  \[
  O[M, O(n)] = \{u \in L(M) : g_{ij}(u) = \delta_{ij}\},
  \]

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\[ u = (e_1, ..., e_n) =: \{e_i\}, \]

\[ g_{ij}(u) = g(\vec{e}_i, \vec{e}_j). \]

\( O[M, O(n)] \) is a set of pairs \((x, O_x)\), where \( x \in M_n \) and \( O_x \) is any orthonormal frame at a point \( x \).

**Remark 10.** If the metric \( g \) on \( M \) is indefinite, i.e. if the quadratic form \( g_{ij}(x) dx^i dx^j \) is not positive definite and has a signature \((k, l)\), then we construct over \( M \) a principal bundle of pseudoorthonormal frames \( O[M, O(k, l)] \) with a pseudoorthogonal structure group \( O(k, l) \). \( O(k, l) \) extends a special pseudoorthogonal group \( SO(k, l) \) by including reflections.

In the physical spacetime \((M_4, g_L) \) \((k = 1, l = 3)\) a pseudoorthogonal group is the Lorentz group \([\mathcal{L} \equiv SO(1, 3)]\). Here we consider a bundle of tetrads \( O[M_4, \mathcal{L}] \) over the spacetime and a dual bundle of Lorentzian corepers \( P(M_4, \mathcal{L}) \).

\( \mathcal{L} \) denotes here a Lorentz group which is isomorphic to the group \( SO(1, 3) \) and \( g_L \) means the Lorentzian metric on \( M_4 \). A tetrad at the point \( x \in (M_4, g_L) \) is a basis \( \{\vec{e}_I\} \in T_x(M_4, g_L) : g_L(e_I, e_K) = \eta_{IK} = \text{diag}(1, -1, -1, -1) \), and a Lorentzian coreper \( \{\vartheta^K\} \) at \( x \in M_4 \) is a set of four 1-forms: \( g = \eta_{IK} \vartheta^I \otimes \vartheta^K \), \( \vartheta^K(e_I) = \delta^K_I \).

Orthonormal tetrads and Lorentzian corepers are very useful in GR. Here and in the future we will denote tetrads and cotetrads indices by using big Latin letters.

**Definition 27.** A linear connection \( \Gamma \) on the principal bundle \( L(M) \) is called compatible with the metric \( g \) or a metric connection \( \iff \) for every \( e \in O[M, O(n)] \) \( H_e \subset T_e O[M, (n)] \). For such connection \( Dg = 0 \). (\( = Dg_{ij} = 0 \); where \( Dg_{ij} = dg_{ij} - \omega_{ij} - \omega_{ji} \)).

Below we give the other formulation of the compatibility of the connection \( \Gamma \) on \( L(M) \) with metric \( g \). This formulation is better adjusted to the bundle \( L(M) \).

**Definition 28.** \( \Gamma \) is compatible with \( g \iff \omega \) is \( \mathfrak{o}(n) \)-valued on \( O[M_n, O(n)] \), \( \mathfrak{o}(n) \) stands for the algebra of the orthogonal group \( O(n) \), i.e. \( \omega \) when restricted to \( O[M, O(n)] \) takes values in \( \mathfrak{o}(n) \).

From all metric connections on the bundle \( L(M) \) the most important is Riemannian connection, which is also called Levi-Civita connection. It is a metric connection (\( Dg = 0 \); \( Dg_{ij} = dg_{ij} - \omega_{ij} - \omega_{ji} \)), whose torsion equals to zero (\( \Theta = D\theta = 0 \)). This connection is unique and it is completely determined by a metric \( g \) and its partial derivatives [J. A. Schouten’s theory; see, e.g. [4,5]]. In a relativistic theory of gravity we usually restrict to metric connections.
Let \( P_1(M_1, G_1) \) and \( P_2(M_2, G_2) \) be principal fibre bundles.

**Definition 29.** A homomorphism of principal fibre bundles \( P_1 \) and \( P_2 \) is a triple of mappings \( (h, k, f) \) such that \( h : P_1 \rightarrow P_2 \), \( k : G_1 \rightarrow G_2 \) and \( f : M_1 \rightarrow M_2 \), where \( k \) is a homomorphism of Lie groups, such that the following diagram commutes.

\[
\begin{array}{ccc}
P_1 \times G_1 & \xrightarrow{h \times k} & P_2 \times G_2 \\
\downarrow R_1 & & \downarrow R_2 \\
P_1 & \xrightarrow{p_1} & P_2 \\
\pi_1 & \xrightarrow{f} & \pi_2 \\
M_1 & \rightarrow & M_2
\end{array}
\]

**Theorem 4.** If there is a homomorphism of bundles between \( P_1(M_1, G_1) \) and \( P_2(M_2, G_2) \), then the connection on the bundle \( P_1(M_1, G_1) \) uniquely determines a connection on the bundle \( P_2(M_2, G_2) \).

Structure equations of Riemannian connection on \( L(M) \).

\[
\Omega(X, Y) = d\omega(X, Y) = \frac{1}{2} [\omega(X), \omega(Y)] \quad \text{2-nd structure equation}
\]

\[
d\theta(X, Y) + \frac{1}{2}(\omega(X) \cdot \theta(Y)) - \omega(Y) \cdot \theta(X) = 0 \quad \text{1-st structure equation}
\]

\( X, Y \in T_u(L(M)) \).

In basis of \( \mathbb{R}^n \) and \( gl(n, \mathbb{R}) \) we obtain:

\[
\Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j,
\]

\[
d\theta^i + \omega^i_k \wedge \theta^k = 0.
\]

\( \theta^i \) is the \( i \)-th component in the natural basis of \( \mathbb{R}^n \) of canonical 1-form \( \theta \) on \( L(M) \).

Bianchi identities:

1-st \( \Omega(X, Y) \cdot \theta(Z) + \Omega(Y, Z) \cdot \theta(X) + \Omega(Z, X) \cdot \theta(Y) \equiv 0 \),  
2-nd \( \mathcal{D}O\Omega(X, Y, Z) \equiv 0 \)

or in the basis of algebra \( gl(n, \mathbb{R}) \) \( \mathcal{D}\Omega^i_k = 0 \).

Expressions in the local charts on \( M \).

Let \( (M_n, A_M) \) be a differential manifold and \( (U, \varphi) \) a local map on \( M_n \) with local coordinates \( (x^1, ..., x^n) \) on \( U \subset M_n \). Here \( A_M \) means the maximal atlas on \( M_n \). Let denote by \( \{X_i = \frac{\partial}{\partial x^i}\} \) (\( i = 1, ..., n \)) vector fields of natural basis (\( \equiv \) coordinates basis) on \( U \). Every linear frame \( u = (X_1, ..., X_n) \) at the point \( x \in U \) can be uniquely given in the form \( X_k = X^i_k \partial_i \), \( \{\partial_k\} \) is a natural basis at \( x \in V \), \( \det(X^i_k) \neq 0 \) (\( i, k = 1, ..., n \)) \( \{x^i, X^i_k\} \) are the local coordinates at \( \pi^{-1}(U) \subset L(M) \) (\( i, j, k = 1, ..., n \)). Let
\[ Y^j_k \] be an inverse matrix to the matrix \([X^j_k]\) : \[ X^j_i Y^i_k = Y^j_i X^k_j = \delta^j_k \] and let \((\vec{e}_1, \ldots, \vec{e}_n)\) be a natural basis at \(\mathbb{R}^n\). Let \(\theta = \theta^j \vec{e}_j\), where \(\theta\) is a canonical 1-form (soldering form) on the principal fibre bundle of linear frames \(L(M)\).

**Fact 11.** Forms \(\theta^i = \theta^j \vec{e}_j\) are expressed in the terms of local coordinates as follows \(\{x^i, X^j_k\}\)

\[ \theta^i = Y^i_j dx^j. \]

In the natural basis \(\{X_i = \partial_i\}\), \(Y^i_k = \delta^i_k\) and \(\theta^i = dx^i\).

Forms \(\theta^i\) are the forms on \(U\) which are dual to the basis \(X_k\) on \(U \subset M_n\), i.e. \(\theta^i(X_k) = \delta^i_k\).

Let \(\omega\) be a 1-form of linear connection of the manifold \((M_n, A_M) : \omega = \omega^i_j E^j_i\), where \(\{E^j_i\}\) is a natural basis of the algebra \(\mathfrak{gl}(n, \mathbb{R})\) of a Lie group \(GL(n, \mathbb{R})\). Let \(\sigma : U \rightarrow L(M)\) be a section of the principal bundle \(L[M, GL(n, \mathbb{R})]\) over \(U \subset M_n\), which attaches for every \(x \in U\) a linear frame \(\{X_i(x)\}\) \((i = 1, \ldots, n)\).

Denote \(\omega_U := \sigma_* \omega\), where \(\omega_U\) is a pull-back of the connection form \(\omega\) from \(L(M)\) bundle onto \(U\). \(\omega_U\) is a 1-form on \(U \subset M_n\) with values in the algebra \(\mathfrak{gl}(n, \mathbb{R}) : \omega_U = \omega^i_U E^j_i\). Let decompose 1-forms \(\omega_U^{-1} k\) in a natural cobasis \(\{dx^i = \sigma_* \theta^j\}\) of 1-forms on \(U\):

\[ \omega_U^{-1} k := \Gamma^i_k dx^i. \]

\(\{\Gamma^i_k\}(i, j, k = 1, \ldots, n)\) are the components of the pull-back \(\omega_U^{-1} k\) of the linear connection \(\Gamma\) on \(L(M)\) in the local coordinates on the base manifold \(M_n\).

It can be shown, that \(\Gamma^i_k\) undergo the following transformation law on the intersection of two local charts on \(M_n\):

\[
\Gamma^i_{j'}(P) = \frac{\partial x^{i'}}{\partial x^i}(P) \frac{\partial x^k}{\partial x^j}(P) \frac{\partial x^j}{\partial x^{j'}}(P) \Gamma^i_k(P) + \frac{\partial^2 x^{i'}}{\partial x^k \partial x^j}(P) \frac{\partial x^j}{\partial x^{j'}}(P) \frac{\partial x^k}{\partial x^i}(P) \tag{1}
\]

\(P \in U \cap V\). On \(U\) we have local coordinates \(\{x^i\}(i = 1, \ldots, n)\) and on \(V\) local coordinates \(\{x^{i'}\}(i' = 1, \ldots, n)\).

The above formula is the very known transformation law of the linear connection components \(\Gamma^i_{kl}\) in the local charts on \((M_n, A_M)\). It is obtained \[8\] from the transformation law given in Fact 4.

Thus from the general global theory of connection (Ehresmann) we have got the local index theory of connection on the differentiable manifold \((M_n, A_M)\). This theory was fully developed in the past mainly by J.A.Schouten. It is also possible to show the inverse fact: the local components \(\Gamma^i_{kl}\) which satisfy (1) determine a unique linear connection \(\Gamma\) in \(L(M)\).

We had on \(L(M)\):

\[ \Theta^i = \frac{1}{2} Q^i_{kl} \theta^k \wedge \theta^l, \]

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\[
\Omega^i_j = \frac{1}{2} R^i_{jkl} \theta^k \wedge \theta^l.
\]

Let \( \sigma : U \to L(M) \) be a local section of \( L(M) \) over \( U \).

We define:

\[
\theta^i_U := \sigma^* \theta^i(= dx^i),
\]

\[
\Theta^i_U := \sigma^* \Theta^i := \frac{1}{2} \tilde{Q}^i_{kl} dx^k \wedge dx^l,
\]

\[
\Omega^i_{U j} := \sigma^* \Omega^i_{j} = \frac{1}{2} \tilde{R}^i_{jkl} dx^k \wedge dx^l.
\]

It appears [8], that:

\[
\tilde{Q}^i_{kl} = -\tilde{Q}^i_{lk} = \Gamma^i_{lk} - \Gamma^i_{kl},
\]

\[
\tilde{R}^i_{jkl} = -\tilde{R}^i_{jlk} = \Gamma^i_{jl,k} - \Gamma^i_{jk,l} + \Gamma^i_{mk} \Gamma^m_{jl} - \Gamma^i_{ml} \Gamma^m_{jk}.
\]

The above formulas are standard expressions for components of the torsion and curvature tensors in the local charts on the differentiable manifold \((M_n, A_M)\).

**Remark 11.** 1. For the local forms \( \Omega^i_{U j}, \Theta^i_U, \omega^i_{U k}, \theta^i_U \), we have the same relations as on the bundle \( L(M) \)

\[
\Omega^i_{U j} = d\omega^i_{U j} + \omega^i_{U k} \wedge \omega^i_{U j},
\]

\[
\Theta^i_U = d\theta^i_U + \omega^i_{U k} \wedge \theta^i_U.
\]

2. For the Riemannian connection (and for the pseudoriemannian connection) we have in the local coordinates on the Riemannian (pseudoriemannian) differentiable manifold \((M_n, \tilde{g})\): \( \sigma^* \theta^i = dx^i, \Theta^i = 0 \implies \Theta^i_U = 0 \implies \tilde{Q}^i_{kl} = 0, \tilde{D} \tilde{g}_{ik} = 0 \equiv dx^i \tilde{g}_{ik}: = 0 \equiv dx^i(\tilde{g}_{ik} - \Gamma^p_{ik} \tilde{g}_{pk} - \Gamma^p_{kj} \tilde{g}_{ip}) \), where \( \tilde{g}_{ik}(x) = \sigma^* g_{ik} = g_{ik} \sigma, g_{ik} : L(M) \to \mathbb{R}^{n(n+1)/2} \) of functions on \( L(M) \), \( \tilde{g}_{ik} \) is a metric on \( U \subset M_n \) and \( \Gamma^i_{kl} = \Gamma^i_{lk} = \{i_{kl}\} = \frac{1}{2} g^{im}(\tilde{g}_{km,l} + \tilde{g}_{lm,k} - \tilde{g}_{kl,m}) \). These formulas are well-known for the local Riemannian geometry on the base \((M_n, \tilde{g})\).

Bianchi identities for Riemannian (Levi-Civita) connection in the local charts on \((M_n, \tilde{g})\) and in terms of the Riemannian curvature tensor:

These are the identities

\[
\bar{R}^k_{[mrl]} \equiv 0 \quad (the \ first)
\]

or after extending alternation

\[
\bar{R}^k_{mrl} + \bar{R}^k_{rml} + \bar{R}^k_{lmr} \equiv 0.
\]

\[
\bar{R}^k_{[rs;lm]} \equiv 0 \quad (the \ second)
\]
Definition 30. A pseudotensorial $q$-form of the type $\rho$ on $L(M)$ is a $q$-form $\alpha$ on $L(M)$ such that

$$(R_a)_{\alpha}(X_1, \ldots, X_q) = \rho_{a^{-1}} \cdot [\alpha(X_1, \ldots, X_q)],$$

$a \in GL(n, \mathbb{R})$, $X_1, \ldots, X_q \in T_a[L(M)]$.

Definition 31. hor $\cdot \alpha := \alpha \cdot h(X_1, \ldots, X_q) = \alpha(hX_1, \ldots, hX_q)$.

Definition 32. A form $\alpha$ is tensorial ($\equiv$ horizontal) $\iff$ hor $\cdot \alpha \equiv \alpha$.

Tensorial forms on $L(M)$ with values in $T^r_s$ are used in theoretical physics to describe material fields. In the case of a manifold $(M_n, A_m)$ the space $V_n$ is $T_x(M)$ and $V^*_n = T^*_x(M)$.

Definition 33. An exterior covariant differential of the tensorial $q$-form of the type $\rho$ on $L(M)$ is a tensorial $(q+1)$-form of the type $\rho$ denoted by $\mathcal{D}\alpha$, where

$$\mathcal{D}\alpha := \text{hor} \cdot d\alpha = d\alpha(hX_1, hX_2, \ldots, hX_q, hX_{q+1}).$$

Hereafter we limit ourselves to the tensorial $q$-forms on $L(M)$ with values in $T^r_s = \otimes T^r_x(M_n) \otimes \otimes T^*_x(M_n)$.

Let $\sigma : U \to L(M)$ be a local section of the bundle such that $L(M) : \pi \cdot [\sigma(x)] = x$, where $x \in U \subset M_n$, $\sigma(x) = \{x_1(x), \ldots, x_n(x), X_1(x), \ldots, X_n(x)\} \in L(M)$, $\{X_1(x), \ldots, X_n(x)\}$ is a linear basis in $T_x(M)$.

Then $\Lambda = \sigma_* \alpha := \alpha \cdot \sigma$ is a tensorial $q$-form on $U \in M_n$ with values in $V = T^r_s = \otimes T^r_x(M_n) \otimes \otimes T^*_x(M_n)$, where $x \in U \subset M_n$.

This form can be written in the local chart on $U \subset M_n$ in the form

$$\Lambda = \Lambda_{i_1 \ldots i_q}^{j_1 \ldots j_r} (x) \partial_{j_1} \otimes \ldots \otimes \partial_{j_r} \otimes dx^{i_1} \otimes \ldots \otimes dx^{i_q} \otimes dx^{k_1} \wedge \ldots \wedge dx^{k_q}.$$
Mappings \( \{ \partial_{j_1} \otimes \ldots \otimes \partial_{j_r} \otimes dx^{i_1} \otimes \ldots \otimes dx^{i_s} \} \) form the natural basis in \( T^*_r = \otimes T_x(M_n) \otimes \otimes T^*_x(M_n) \) and mappings \( (dx^{k_1} \wedge \ldots \wedge dx^{k_q}) \) form the natural basis \( q \)-form on \( U \subset M_n \).

In practice we write the above tensorial \( q \)-form of the type \( (r, s) \) as follows

\[
q^r \Lambda_{j_1 \ldots j_r}^{i_1 \ldots i_s} B_{j_1 \ldots j_r}^{i_1 \ldots i_s},
\]

where \( B_{j_1 \ldots j_r}^{i_1 \ldots i_s} = \partial_{j_1} \otimes \ldots \otimes \partial_{j_r} \otimes dx^{i_1} \otimes \ldots \otimes dx^{i_s} \), remembering that every component of \( \Lambda_{j_1 \ldots j_r}^{i_1 \ldots i_s} \) is \( q \)-form

\[
\Lambda_{i_1 \ldots i_s}^{j_1 \ldots j_r} = \sum_{k_1 < k_2 < \ldots < k_q} \Lambda_{i_1 \ldots i_s}^{j_1 \ldots j_r} dx^{k_1} \wedge \ldots \wedge dx^{k_q}
\]

We define the exterior differential \( d \Lambda \)

\[
d \Lambda = d \Lambda_{i_1 \ldots i_s}^{j_1 \ldots j_r} \otimes B_{j_1 \ldots j_r}^{i_1 \ldots i_s},
\]

where \( d \Lambda_{i_1 \ldots i_s}^{j_1 \ldots j_r} \) means Cartan derivative of the \( q \)-form

\[
\Lambda_{i_1 \ldots i_s}^{j_1 \ldots j_r} = \sum_{k_1 < k_2 < \ldots < k_q} \Lambda_{i_1 \ldots i_s}^{j_1 \ldots j_r} dx^{k_1} \wedge \ldots \wedge dx^{k_q}
\]

The action \( d \) is linear and \( d \cdot d \Lambda = 0 \) for \( \Lambda \). But it appears that \( d \Lambda \) is not the tensorial \( (q + 1) \)-form with values in \( T^*_r = \otimes T_x(M_n) \otimes [\otimes T^*_x(M_n)] \), \( x \in M_n \).

An exterior covariant differential \( D \Lambda \) is the tensorial \( (q+1) \)-form on \( U \subset M_n \) with values in \( T^*_r \).

One can show that [9]

\[
D \Lambda = D \Lambda_{i_1 \ldots i_s}^{j_1 \ldots j_r} \otimes B_{j_1 \ldots j_r}^{i_1 \ldots i_s},
\]

where \( D \Lambda_{i_1 \ldots i_s}^{j_1 \ldots j_r} = d \Lambda_{i_1 \ldots i_s}^{j_1 \ldots j_r} + \omega^{j_1}_{i_1} \wedge \Lambda_{i_2 \ldots i_s}^{j_2 \ldots j_r} + \omega^{j_2}_{i_2} \wedge \Lambda_{i_1 \ldots i_s}^{j_1 \ldots j_r} \)

\[
+ \ldots + \omega^{j_r}_{i_r} \wedge \Lambda_{i_1 \ldots i_{r-1}}
\]

By \( \omega^j_{i} \) one should understand the local forms \( \omega^j_{i} = \sigma^j_{i} dx^j \).

In the natural cobasis on \( U \subset M_n \) we have \( \omega^i_k = \Gamma^i_{kl} dx^l \).

In nonholonomic Lorentzian coreper \( (\partial^I) : \quad g = \eta_{IK} \partial^I \otimes \partial^K \) we put \( \omega^I_K = \gamma^I_{KL} dx^L \).
In this case for curvature components we have
\[ \Gamma^i_{kl}(x) = h^i_A(x)\gamma^A_{BC}(x)h^B_k h^C_l(x) + h^i_A(x)\partial_k h^A_l(x), \]
where matrices \((h^i_a)\) and \(h^a_B\) are defined as follows
\[ \partial^i = h^i_a(x)dx^a \]
\[ h^a_B(x)h^a_B(x) = \delta^a_B \equiv h^a_B(x)h^D_a(x) = \delta^D_B. \]
The inverse relation has the form
\[ \gamma^A_{BC} = h^a_C h^b_B h^d_A \Gamma^{ab}_{bg} - h^a_C h^b_B \partial_b (h^A_a). \]

For components of a tensor-valued form one has simpler relations, e.g. for curvature components we have
\[ R_{ABCD} = h^a_A h^b_B h^c_C h^d_D R_{abcd}, \]
\[ R_{abcd} = h^A_a h^B_b h^C_c h^D_d R_{ABCD}. \]

If \(\alpha\) is a tensorial 0-form with values in \(T^r_s\), then \(\Lambda = \sigma_s \alpha\) is a tensorial field of the type \((r, s)\) on \(U \subset M_n\), i.e. \(\Lambda\) is a section of tensor bundle of the type \((r, s)\).

Then in the natural basis and cobasis we have on \(U \subset M_4\)
\[ \Lambda = \Lambda_{i_1...i_s}^{j_1...j_r} \otimes B_{j_1...j_r}^{i_1...i_s}. \]
In this case
\[ \mathcal{D}\Lambda = \mathcal{D}\Lambda_{i_1...i_s}^{j_1...j_r} \otimes B_{j_1...j_r}^{i_1...i_s}, \]
where \(\mathcal{D}\Lambda_{i_1...i_s}^{j_1...j_r} (x) = d\Lambda_{i_1...i_s}^{j_1...j_r} (x) + \omega^j_{p} \Lambda_{i_1...i_s}^{j_1...j_r} \otimes B_{j_1...j_r}^{i_1...i_s} - \omega^j_{i_s} \Lambda_{i_1...i_s}^{j_1...j_r} - ... - \omega^j_{i_1} \Lambda_{i_1...i_s}^{j_1...j_r}. \)

If we decompose the connection 1-form \((\omega^i_k)\) in the natural cobasis \(U \subset M_4\)
\[ \omega^i_k = \Gamma^i_{kl} dx^l \]
then we obtain that
\[ \mathcal{D}\Lambda_{i_1...i_s}^{j_1...j_r} = \Lambda_{i_1...i_s}^{j_1...j_r} \otimes dx^k. \]
We can see that \(\mathcal{D}\Lambda_{i_1...i_s}^{j_1...j_r}\) are components of the absolute differential of the tensorial field \(\Lambda\), and \(\Lambda_{i_1...i_s}^{j_1...j_r} \otimes dx^k\) are components of the covariant derivative of this field. One can read from the last formula that
\[ \Lambda_{i_1...i_s}^{j_1...j_r} = \Lambda_{i_1...i_s}^{j_1...j_r} + \Gamma^j_{pl} \Lambda_{i_1...i_s}^{pj_2...j_r} + ... + \Gamma^j_{pl} \Lambda_{i_1...i_s}^{j_1...p} \]
\[ - \Gamma^j_{pl} \Lambda_{i_{kl}...i_s}^{j_1...j_r} - ... - \Gamma^j_{pl} \Lambda_{i_{kl}...i_s}^{j_1...j_r}. \]
If $\alpha$ is an ordinary Cartan q-form with values in $\mathfrak{g}$, then $\Lambda = \sigma \alpha$ is an ordinary Cartan q-form on $U \subset M_4$:

$$\Lambda = \sum_{i_1 < i_2 < \ldots < i_q} \Lambda_{i_1 \ldots i_q} dx^{i_1} \wedge \ldots \wedge dx^{i_q}.$$ 

Then $D\Lambda = d\Lambda$, where $d$ denotes Cartan exterior differential. We have

$$d\Lambda\overset{\text{q}}{=} \sum_{i_1 < i_2 < \ldots < i_q} d\Lambda_{i_1 \ldots i_q} dx^{i_1} \wedge \ldots \wedge dx^{i_q}. \quad (2)$$

$d\Lambda_{i_1 \ldots i_q}$ denotes here ordinary differential of the components $\Lambda_{i_1 \ldots i_q}$ of the q-form $\Lambda$. From (2) one can obtain the formula for practical calculations [9]

$$d\Lambda\overset{\text{q}}{=} \sum_{i_0 < i_1 < \ldots < i_q} \left( \sum_{\alpha=0}^{q} (-1)^\alpha \partial_{i_0} \Lambda_{i_0 i_1 \ldots i_q} \right) dx^{i_0} \wedge \ldots \wedge dx^{i_q}. \quad (3)$$

"\overset{\text{q}}{=}" over the index means that this index should be left.

From the above formulas we can see that the exterior covariant differential $D$ is the most general differentiation on the manifold. Namely, the absolute differential of the tensorial field $D$ and the external differential $d$ of the external Cartan form are the particular examples of this differentiation. All formulas given above are used in standard computations in General Relativity (generally: in theoretical physics).
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