Running Coupling Constant of a Gauge Theory in the Framework of the Schwinger-Dyson Equation: Infrared Behavior of Three-Dimensional Quantum Electrodynamics

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Abstract

We discuss how to define and obtain the running coupling of a gauge theory in the approach of the Schwinger-Dyson equation, in order to perform a non-perturbative study of the theory. For this purpose, we introduce the nonlocally generalized gauge fixing into the SD equation, which is used to define the running coupling constant (this method is applicable only to a gauge theory). Some advantages and validity of this approach are exemplified in QED3. This confirms the slowing down of the rate of decrease of the running coupling and the existence of non-trivial infra-red fixed point (in the normal phase) of QED3, claimed recently by Aitchison and Mavromatos, without so many of their approximations. We also argue that the conventional approach is recovered by applying the (inverse) Landau-Khalatnikov transformation to the nonlocal gauge result.

Key words: Schwinger-Dyson equation, nonlocal gauge, quantum electrodynamics, running coupling, renormalization group, fixed point

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1 Introduction

In quantum field theory, a change in the renormalization scale $\mu$ accompanied by suitable change in coupling and mass does not change the theory. The invariance of the theory under such transformation is called the renormalization group (RG) invariance. For a given physical theory, we have, for each value of $\mu$, a definite value of the coupling $g(\mu)$ and $m(\mu)$. These are called the effective (or running) coupling and mass. Physical quantities like the S-matrix are invariant under the change of variable: $(\mu, g(\mu), m(\mu)) \rightarrow (\mu', g(\mu'), m(\mu'))$. This invariance leads to a differential equation for the running coupling and mass, called the RG equation \[1\]. This is the easiest way in practice to compute the effective coupling and mass. The coefficients $\beta, \gamma_m, ...$ in the RG equation are called the RG coefficients which express important properties of a theory.

The most important application of the RG is to compute large-momentum, i.e. ultraviolet (UV) behavior. However, RG methods can be used to compute infrared (IR) behavior too. Certainly this is true in a purely massless theory (or if masses can be neglected), and the IR behavior is computable perturbatively if and only if the theory is not asymptotic free. But in a massive theory it is not useful to take $\mu$ much less than a typical mass, for one obtains logarithms of $m/\mu$ and these prevent simple use of the perturbative method when $\mu \ll m$, see e.g. \[2\].

In four spacetime dimensions, it is well known that the only theories that are asymptotically free are non-Abelian gauge theories with a small enough number of matter fields, see \[3\]. In an asymptotically free theory, the effective coupling $g(\mu)$ goes to zero when $\mu$ goes to infinity, so that short-distance (or high-energy) behavior is computable perturbatively. But, when $\mu$ is small, $g(\mu)$ is large, so that IR behavior cannot be computed reliably by perturbation theory. This is the case for QCD_4.

If we consider a non-asymptotically free theory in four dimensions, like $(\phi^4)_4$ and QED_4, then the effective coupling increases with energy. Thus, in such theories it is impossible to compute the true high-energy behavior by perturbation theory. But when $\mu$ goes to zero, so does the effective coupling. Hence, we can compute IR behavior in such a theory, just as we compute the UV behavior in an asymptotically free theory. However, note that the coupling in QED is $\alpha := e^2/4\pi \sim 1/137$. This is so small that the non-perturbative region in QED_4 does not occur until very many orders of magnitude beyond experimentally accessible energies. Nevertheless, the existence of a non-trivial UV fixed point in a non-asymptotically free theory in four dimensions has been suggested by Miransky \[4\] for strongly coupled QED_4 where the fixed point is expected to be located at $\alpha = \alpha_* \sim O(1)$ of order unity. In the strong coupling region, we need to deal with the theory non-perturbatively.

As well as simulations based on lattice gauge theory \[3\], the Schwinger-Dyson (SD) equation has played an important role in the non-perturbative analytical study of strongly coupled gauge theories. In the actual analysis of the SD equation, an approximation of constant coupling (standing coupling) has been taken together with a bare vertex approximation as the simplest approximation, which is usually called the quenched planar (or ladder) approximation. However, the introduction of running coupling is indispensable to study the unquenched QED, QCD and extended techni-
color theory, etc, see e.g. \[6\]. So far, this has been done in most cases in a somewhat unsatisfactory way, in my opinion, in which the expression for the running coupling constant obtained by (RG-improved) perturbation theory (at most leading logarithm) was substituted into the SD equation. However, the running coupling itself should be calculated within the SD equation approach. Such a kind of calculation was tried, for example, in the massive gauge boson theory in four-dimensions \[7\] and \(N\)-flavor QED, by solving the SD equation for the wavefunction renormalization function \(A(p^2)\) of the fermion, using either a bare vertex or a reasonably simple ansatz for the vertex function. In the original analysis \[13, 14\] of multi-flavor QED, no wavefunction renormalization was assumed from the beginning, based on a naive \(1/N\) argument. Quite recently, self-consistent solutions of QED have been studied more extensively by solving the coupled SD equation under various ansatizes for the vertex \[13\].

The choice of vertex ansatz is the most difficult problem in truncating the infinite hierarchy in the SD equation approach. Even if the vertex ansatz might be reasonable, we need to make a number of approximations in solving the SD (integral) equation, at least analytically. The nature of those approximations is totally different from what one encounters in solving a differential equation. Usually, obtaining an analytical solution for an integral equation is much more difficult than for a differential equation. Therefore, the SD integral equation is often converted into a differential equation by simplifying the kernel, although the integral equation cannot in general be transformed into a differential equation in a mathematically rigorous sense. \[1\]

In this paper we discuss another approach to calculating the running or effective coupling in a gauge theory. In this approach we introduce the nonlocal gauge-fixing \[16, 17, 18\]. Here the existence of gauge degrees of freedom is an essential ingredient, so this approach is applicable to a gauge theory only.

The conventional approach relies on a specific choice of vertex function. Therefore, one can not be free from the criticism whether the result is due to an artifact of the specific ansatz adopted for the vertex function or not. Of course, it is impossible to get rid of this criticism completely. However, we can allow a certain class of ansatz for the vertex function which gives a weaker restriction than the conventional approach. In this paper, we choose the ansatz for the vertex:

\[
\Gamma_\mu(p, q) = \gamma_\mu G(p^2, q^2, k^2) \tag{1.1}
\]

where the function \(G\) is an arbitrary function of the fermion momenta \(p^2, q^2 = (p-k)^2\) and the gauge-boson momentum \(k^2\). We shall see that the resulting running coupling is essentially independent of the explicit form of \(G\), provided the vertex function satisfies the Ward-Takahashi identity. This is a major advantage of this approach.

So far, the nonlocal gauge has been applied to gauge theories in four dimensions \[17, 18\], the gauged Thirring model \[19\] and QED, \[20, 21, 22\]. In those works, a

\[1\] The numerical approach enables us to solve the integral equation as well as the differential equation, although the accuracy and the convergence of the algorithm for solving the integral equation has not been established so well. Justification of the approximations made in the analytical study has been done through the numerical calculations.
The bare vertex approximation has been taken from the beginning. This assumption is not necessary or is weakened as shown in this paper.

The second advantage of this approach is as follows. This approach is free from various approximations which are required to solve the SD equation analytically for the wavefunction renormalization, because we never solve it to obtain the running coupling constant. Instead, we derive a differential equation which should be satisfied by the nonlocal gauge function. The nonlocal gauge is obtained by a simple quadrature without solving any self-consistent equation. This is an advantage of this approach.

We apply this method to study the non-perturbative behavior of QED$_3$, in particular, the IR behavior and the rate of running in the intermediate momentum region which have been extensively studied recently by Aitchison and Mavromatos [23] and their collaborators [24]. We also present the RG-like argument for the behavior of the running coupling within this approach.

This paper is organized as follows. In section 2, we introduce the nonlocal gauge fixing and set up the SD equation under a class of ansatzes for the vertex function in $D(\geq 2)$-dimensional gauge theory.

In section 3, we derive the differential equation which is obeyed by the nonlocal gauge such that there is no wavefunction renormalization $A(p^2) \equiv 1$. Here we introduce the nonlocal gauge with and without IR (not UV) cutoff. The IR cutoff is necessary to discuss the IR behavior of QED$_3$ in the following sections.

In section 4, we discuss how to obtain the non-trivial wavefunction renormalization in the usual (Landau) gauge from the result in the nonlocal gauge. For this, we apply the inverse LK transformation.

In section 5, we define the running coupling through the nonlocal gauge in this approach and compare it with that in the conventional approach. Here we see several advantages of this approach for obtaining the running coupling.

In section 6, we study in detail the behavior of the running coupling of QED$_3$ obtained in the previous section, paying particular attention to the IR behavior. The RG-like interpretation of this result is also given here.

The final section is devoted to conclusion and discussion.

2 Schwinger-Dyson equation

2.1 Introduction of the nonlocal gauge

First, we discuss the SD equation for the fermion propagator. In accord with the bare fermion propagator

$$S_0(p) = (\gamma^\mu p_\mu - m_0)^{-1},$$

we write the full fermion propagator as

$$S(p) = [A(p^2)\gamma^\mu p_\mu - B(p^2)]^{-1}.$$
For a class of gauge theories in $D = d + 1$ dimensional space-time, the SD equation for the full fermion propagator in momentum space is given by

$$S^{-1}(p) = S^{-1}_0(p) + \int \frac{d^Dq}{(2\pi)^D} \gamma_\mu S(q) \Gamma_\nu(p, q) D_{\mu\nu}(p - q),$$

(2.3)

where $\Gamma_\nu(p, q)$ is the full vertex function and $D_{\mu\nu}(p - q)$ is the full gauge boson propagator. We always use $p, q$ for the fermion momentum and $k = p - q$ for the gauge-boson momentum. This class includes QED and the gauged Thirring model [19, 25]. Note that this SD equation can be decomposed into a pair of integral equations for the wavefunction renormalization function $A(p^2)$ and the mass function $B(p^2)$, as shown in the next section. The SD equation for the full fermion propagator $S(p)$ should constitute a closed set of equations together with the SD equations for the full vertex function $\Gamma_\nu(p, q)$ and the full gauge boson propagator $D_{\mu\nu}(p - q)$ which will be specified below.

Next, in order to specify the gauge boson propagator, we discuss the gauge-fixing. In this paper we consider a more general gauge fixing than the usual one, the so-called the nonlocal gauge-fixing. In configuration space, the gauge fixing term in the nonlocal gauge [22] is given by

$$L_{GF} = -\frac{1}{2} F[A(x)] \int d^Dy \frac{1}{\xi(x - y)} F[A(y)],$$

(2.4)

with a gauge-fixing function $F[A]$. In this paper we take the Lorentz-covariant linear gauge:

$$F[A] = \partial^\mu A_\mu.$$  

(2.5)

In momentum representation, the gauge-fixing parameter $\xi$ becomes momentum-dependent, namely, $\xi$ becomes a function of the momentum: $\xi = \xi(k)$. Here it should be noted that $\xi^{-1}(k)$ is the Fourier transform of $\xi^{-1}(x)$:

$$\xi^{-1}(x) = \int \frac{d^Dk}{(2\pi)^D} e^{ikx} \xi^{-1}(k), \quad \xi^{-1}(k) = \int d^Dxe^{-ikx} \xi^{-1}(x),$$

(2.6)

while $\xi(k)$ is not the Fourier transform of $\xi(x)$. If $\xi(k)$ does not have momentum-dependence, i.e., $\xi(k) \rightarrow \xi$, then $\xi^{-1}(x - y) \rightarrow \delta(x - y)\xi^{-1}$ and hence the nonlocal gauge-fixing term (2.4) reduces to the usual gauge-fixing term:

$$L_{GF} = -\frac{1}{2\xi} (F[A(x)])^2.$$  

(2.7)

It is easy to show that the SD equation for the full gauge-boson propagator is given by

$$D_{\mu\nu}^{-1}(k) = D_{\mu\nu}^{(0)}^{-1}(k) - \Pi_{\mu\nu}(k),$$

$$\Pi_{\mu\nu}(k) := e^2 \int \frac{d^Dp}{(2\pi)^D} \text{tr}[\gamma_\mu S(p) \Gamma_\nu(p, p - k) S(p - k)],$$

(2.8)
where the bare gauge-boson propagator \( D^{(0)}_{\mu\nu}(k) \) in the nonlocal gauge (2.3) is given by

\[
D^{(0)-1}_{\mu\nu}(k) = k^2 g_{\mu\nu} - k_\mu k_\nu + \xi(k)^{-1} k_\mu k_\nu. \tag{2.9}
\]

In gauge theory, the vacuum polarization tensor should have the transverse form:

\[
\Pi_{\mu\nu}(k) = \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \Pi(k), \tag{2.10}
\]

provided the gauge invariance is preserved. Hence the full gauge-boson propagator is of the form

\[
D_{\mu\nu}(k) = D_T(k) \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) + \frac{\xi(k)}{k^2} \frac{k_\mu k_\nu}{k^2},
\]

\[
D_T(k) := \frac{1}{k^2 - \Pi(k)}. \tag{2.11}
\]

In this paper we take another form for the full gauge boson propagator:

\[
D_{\mu\nu}(k) = D_T(k^2) \left[ g_{\mu\nu} - \eta(k^2) \frac{k_\mu k_\nu}{k^2} \right], \quad k_\mu := p_\mu - q_\mu. \tag{2.12}
\]

By comparing (2.11) with (2.12), the correspondence between \( \xi \) and \( \eta \) is given in momentum space as follows.

\[
\xi(k) = [1 - \eta(k^2)] [1 - \Pi(k)/k^2]^{-1}, \quad \eta(k^2) = 1 - \xi(k) [1 - \Pi(k)/k^2]. \tag{2.13}
\]

2.2 Vertex ansatz and gauge choice

Finally, we must specify the full vertex function \( \Gamma_\nu(p, q) \). Of course, the vertex function obeys its own SD equation and all the SD equations should be solved simultaneously. In principle, it might be possible to do that. However, it is rather difficult or impossible to actually carry out this scenario, except for some exactly solvable models (see e.g. [25]), due to the infinite hierarchy of the SD equations. Therefore we need to truncate the SD equations so that they become tractable analytically or numerically. One usually adopt an ansatz for the full vertex function \( \Gamma_\mu \), instead of solving the SD equation for the vertex function. Such an ansatz is suggested from various consistency requirements, see e.g. [26]. In this paper we adopt the following simple ansatz:

\[
\Gamma_\mu(p, q) = G(p^2, q^2, k^2) \gamma_\mu, \tag{2.14}
\]

where \( p_\mu, q_\mu \) denotes the momenta of the fermion and \( k_\mu \) the momentum of the gauge boson. Here \( G = G(p^2, q^2, k^2) \) is an arbitrary function of \( p^2, q^2 \) and \( k^2 \) except for a restriction specified below.

The truncation of the set of SD equations in the gauge theory should be done self-consistently in such a way that the resulting truncated SD equations respect gauge
invariance (as much as possible). For example, the vertex function should satisfy the Ward-Takahashi (WT) identity:

\[(p - q)\mu \Gamma_\mu(p, q) = S(p)^{-1} - S(q)^{-1}, \tag{2.15}\]

which is a consequence of the gauge invariance of the theory. When the function \(B(p^2)\) vanishes (in the symmetric phase or normal phase) or is extremely small (in the neighborhood of the critical point in the broken phase or superconducting phase), the WT identity reduces to

\[(p - q)\mu \Gamma_\mu(p, q) \sim [A(p^2)p^\mu - A(q^2)q^\mu]\gamma_\mu. \tag{2.16}\]

This tells us that \(G(p^2, q^2, k^2)\) should be expressed in terms of \(A(p^2), A(q^2)\) and \(A(k^2)\):

\[G(p^2, q^2, k^2) = F[p^2, q^2, k^2, A(p^2), A(q^2)], \tag{2.17}\]

Some examples will be shown below. A restriction for our approach to work is

\[\frac{\partial G(p^2, q^2, k^2)}{\partial k^2}\bigg|_{p^2, q^2} = 0, \tag{2.18}\]

where the derivative is taken with \(p^2\) and \(q^2\) being fixed. This implies that the argument of \(G\) does not depend explicitly on the angle \(\vartheta\) of the inner product \(p \cdot q = pq \cos \vartheta\) which comes from \(k^2 := (p - q)^2 = p^2 + q^2 - 2p \cdot q\). However, \(G\) can depend on \(k^2\) implicitly, through \(A(k^2)\). The reason why we need this restriction will be made clear in the next section. In what follows we restrict our discussion to this case.

The simplest choice is the bare vertex approximation: \(\Gamma_\mu(p, q) \equiv \gamma_\mu\). This approximation is usually called the ladder approximation. And it is said that the ladder approximation breaks gauge invariance (this is obvious, since (2.16) can not be satisfied unless \(A(p^2) \equiv 1\) which is impossible as a solution of the SD equation except Landau gauge.). This statement is sometimes used as indicating that the ladder approximation in the SD equation may lead to wrong result in the analysis of the gauge theory. Here we define the ladder approximation in the SD equation (for the fermion propagator) as an approximation in which the full vertex function is replaced with the bare one. In our definition of ladder approximation, the gauge boson propagator is not necessarily restricted to the bare one.

However, in our approach we look for a set of consistent solutions for \(A(p^2)\) and \(B(p^2)\) together with the vertex function \(\Gamma_\mu(p, q)\) and the gauge boson propagator \(D_{\mu\nu}(k)\) under the adopted approximation, so that the solutions are consistent with the WT identities. For the bare vertex approximation to be consistent with the WT identity (2.16), therefore, there should be no wavefunction renormalization for the fermion. In other words, the bare vertex approximation should self-consistently yield the result:

\[A(p^2) \equiv 1, \tag{2.19}\]

\footnote{This naming is somewhat misleading, since the SD equation for the fermion propagator does not have an exact graphical representation corresponding to ladders.}
as a solution of SD equation Eq. (2.3). In QED with covariant gauge (2.7), (2.19) is realized only if we take both the Landau gauge and the quenched ladder approximation where the full photon propagator is also replaced with the bare one, i.e. \( \Pi(k) \equiv 0 \). Therefore, within the framework of SD equations in QED, the bare photon propagator in the Landau gauge together with the bare vertex is a set of consistent solutions of the SD equation (the fermion propagator can be non-trivial). This fact is well-known in four dimensions [27] and holds in any dimension \( D > 2 \), see [28].

If gauges other than the Landau gauge are adopted, \( A(p^2) \equiv 1 \) does not hold even in the quenched ladder (bare vertex and bare gauge boson propagator) approximation and we have to solve the coupled equations for \( A \) and \( B \). However, it is known that the solutions in QED obtained in such a scheme are severely gauge-dependent, see e.g. [29]. Indeed, the simplest quenched ladder approximation does not satisfy the WT identity except for the Landau gauge in the sense described above. In order to obtain gauge-parameter independent results in this case, we need to modify the full vertex so as to satisfy the WT identity. In the framework of the SD equation, however, it is rather difficult to obtain gauge-parameter independent results by modifying the vertex function.

As explained above, in the quenched ladder approximation, only the Landau gauge \( \eta(k^2) \equiv 1 \) can give a set of consistent solutions for the SD equation in this sense. However, if the vacuum polarization is included in the photon propagator (i.e. unquenched case \( \Pi(k) \neq 0 \)), the Landau gauge has no longer this property in the SD framework. Adopting the nonlocal gauge enables us to extend this scheme beyond the quenched ladder approximation. In a gauge theory, there is the freedom of choosing such a gauge. The existence of a gauge where \( A(p^2) \equiv 1 \) in the SD framework was shown independently by Georgi, Simmons and Cohen [17] and by Kugo and Mackard [18] in four-dimensional gauge theory. The extension to arbitrary dimension \( D > 2 \) is straightforward, as done in the Appendix of [21]. There the bare vertex was assumed from the beginning. In the next section we re-derive the nonlocal gauge in a more general setting.

As long as the full vertex has the form (2.14), the nonlocal gauge plays the same role as the Landau gauge of the quenched ladder QED. In the next section, we show that the function \( \eta(k^2) \) can be chosen so that the SD equation Eq. (2.3) for the fermion propagator with the vertex function (2.14) leads to the solution Eq. (2.19) for \( A(p^2) \), i.e. no wavefunction renormalization. Therefore, the nonlocal gauge gives the most economical choice, since we have only to solve the single equation for \( B \).

3 In the usual perturbation theory of QED, the Landau gauge is a special gauge in which the vertex correction vanishes. The above example shows that the similar situation occurs also in the framework of the SD equation.

4 It is argued [30] that the quenched limit of gauge-invariant study of QED \( D \gtrsim 4 \) coincides with the Landau-gauge results obtained from the Schwinger-Dyson equation in the quenched ladder (bare vertex) approximation. This suggest that the Landau gauge is the best one in this approximation.

5 In order for the full vertex function to be consistent with the Ward-Takahashi identity, the ansatz for the vertex function should include \( S \), e.g. \( \Gamma_\mu(p, q) = \frac{i}{2}\phi[S^{-1}(p) - S^{-1}(q)] + \Gamma_T^\mu(p, q) \). However, this requirement is not sufficient to determine the vertex uniquely, see ref. [32, 33, 34] and references therein. For an other proposal, see [35].
So far, we have not discussed the explicit dependence of $G$ on $A$. Examples used so far for the ansatz belonging to the class (2.14) are as follows.

$$G(p^2, q^2, k^2) = A(p^2), \quad A(q^2),$$

$$= \frac{1}{2}[A(p^2) + A(q^2)],$$

$$= A(p^2)\theta(p^2 - q^2) + A(q^2)\theta(q^2 - p^2),$$

$$= \left[A(p^2) + c_1 p^2 A'(p^2) + c_2 p^4 A''(p^2) + \ldots \right] \theta(p^2 - q^2) + (p \leftrightarrow q),$$

with some constants $c_i (i = 1, 2, \ldots)$. If we take the nonlocal gauge so that $A(p^2) \equiv 1$, the function $G(p^2, q^2, k^2)$ reduces to 1 uniformly in $p, q, k$ (when $B(p^2)$ is neglected), in order to be consistent with the WT identity. The following ansatzes are incompatible with the WT identity:

$$G(p^2, q^2, k^2) = A^n(p^2), \quad A^n(q^2),$$

$$= \frac{1}{2^n}[A(p^2) + A(q^2)]^n,$$

$$= A(p^2)A(q^2),$$

$$= A(p^2)A(q^2)/A(k^2),$$

(2.21)

for any integer $n \geq 2$.

In the massive fermion phase $B(p) \neq 0$, the small deviation of $G(p^2, q^2, k^2)$ from 1 is $O(B)$ which gives rise to at least $O(B^2)$ terms in the integrand of the SD equation for $B$. In order to study the critical behavior in the neighborhood of the critical point through the solution of the SD equation for $B(p^2)$, we can put $G(p^2, q^2, k^2) \equiv 1$ in the integrand of the integral equation after taking the nonlocal gauge, even if $G(p^2, q^2, k^2)$ might deviate from 1 off the critical point. This is because the order $O(B^2)$ terms are irrelevant to the bifurcation solution (from the trivial one $B(p^2) \equiv 0$) which is sufficient to study the critical behavior.

Therefore, in this approach we do not need the explicit form of the vertex in order to study the critical behavior of the model. This is a further advantage of this approach. For other types of the full vertex with different tensor structure [32, 33, 34], such a convenient gauge is not known and we must solve the coupled equation for $A$ and $B$ as well as $\Gamma_\mu$. In our approach all effects coming from the vertex correction are incorporated into the SD equation by modifying the gauge boson propagator through the nonlocal gauge.

In QED$_3$ it has been shown [22] that the nonlocal gauge reproduces systematically the previous result obtained for the fermion self-energy with corrections up to the next-to-leading order in $1/N$ expansion [3], but in the approach of [4] it was necessary to consider the vertex correction in writing down the SD equation.
3 Derivation of nonlocal gauge

We decompose the SD equation into a pair of integral equations according to the following procedure:

\[
A(p^2) = 1 + \frac{\text{tr}[\Sigma(p^2)\gamma^\mu p^\mu]}{p^2\text{tr}(1)}, \quad (3.1)
\]

\[
B(p^2) = m_0 + \frac{\text{tr}[\Sigma(p^2)]}{p^2\text{tr}(1)}, \quad (3.2)
\]

where \(\Sigma\) denotes the self-energy part:

\[
\Sigma(p^2) := \int \frac{d^Dq}{(2\pi)^D} \gamma_\mu S(q)\Gamma_\nu(q,p)D_{\mu\nu}(p-q). \quad (3.3)
\]

We use the ansatz (2.14) for the vertex function.

Then the SD equation (3.1) for the fermion wave function renormalization \(A\) reads

\[
p^2A(p^2) - p^2 = e^2 \int \frac{d^Dq}{(2\pi)^D} A(q^2)G(p^2,q^2,k^2)
\]

\[
\times k^2D_T(k^2) \left[ (D-2)\frac{p\cdot q}{k^2} + \left( \frac{p\cdot q}{k^2} - 2\frac{p^2q^2 - (p\cdot q)^2}{k^4} \right) \eta(k^2) \right]. \quad (3.4)
\]

On the other hand, the SD equation (3.2) for the fermion mass function \(B\) reads

\[
B(p^2) = m_0 + e^2 \int \frac{d^Dq}{(2\pi)^D} B(q^2)G(p^2,q^2,k^2)
\]

\[
\times D_T(k^2) \left[ (D-2)\frac{p\cdot q}{k^2} + \left( \frac{p\cdot q}{k^2} - 2\frac{p^2q^2 - (p\cdot q)^2}{k^4} \right) \eta(k^2) \right]. \quad (3.5)
\]

Separating the angle \(\vartheta\) defined by

\[
k^2 := (q-p)^2 = x + y - 2\sqrt{xy}\cos \vartheta, \quad x := p^2, \quad y := q^2, \quad (3.6)
\]

we find (for \(D > 2\))

\[
xA(x) = x
\]

\[
= C_D e^2 \int_0^\infty dy \frac{y^{(D-2)/2}A(y)}{yA^2(y) + B^2(y)} \int_0^\pi d\vartheta \sin^{D-2} \vartheta G(p^2,q^2,k^2)
\]

\[
\times k^2D_T(k^2) \left\{ \cos \vartheta \sqrt{\frac{xy}{k^2}}[D - 2 + \eta(k^2)] - 2\frac{xy - (\sqrt{xy}\cos \vartheta)^2}{k^4} \eta(k^2) \right\}, \quad (3.7)
\]

where

\[
C_D := \frac{1}{2^D\pi(D+1)/2\Gamma(D-1/2)}. \quad (3.8)
\]

We follow the same procedure as that given in [18] and Appendix of ref. [21]. We perform the angular integration by parts according to

\[
\int_0^\pi d\vartheta \sin^{D-2} \vartheta \cos \vartheta \sqrt{xy} f(z)
\]

\[
= \frac{\sqrt{xy}}{D-1} \left[ \sin^{D-1} \vartheta f(z) \right]_0^\pi - \frac{2xy}{D-1} \int_0^\pi d\vartheta \sin^{D} \vartheta \frac{\partial}{\partial z} f(z), \quad (3.9)
\]
where $z := k^2 = (q - p)^2$ and the differential with respect to $z$ is done with $x$ and $y$ being fixed. Thus we find

$$x A(x) - x = -\frac{C_D e^2}{D - 1} \int_0^{\Lambda^2} \frac{d^D q}{(2\pi)^D q^2 + B^2(q^2)} \left( \frac{B(q^2)}{q^2} \right) \int_0^{\pi} d\vartheta \sin^D \vartheta \times \int_0^{\pi} d\vartheta \sin^D \vartheta \times \left[ \frac{\partial}{\partial z} \{ G(x, y, z)[D - 2 + \eta(z)]D_T(z) \} + (D - 1) \frac{G(x, y, z)D_T(z)\eta(z)}{z} \right]$$

This is further rewritten as

$$x A(x) - x = -\frac{C_D e^2}{D - 1} \int_0^{\Lambda^2} \frac{d^D q}{(2\pi)^D q^2 + B^2(q^2)} \left( \frac{B(q^2)}{q^2} \right) \int_0^{\pi} d\vartheta \sin^D \vartheta \times \int_0^{\pi} d\vartheta \sin^D \vartheta \times \left[ \frac{\partial}{\partial z} \{ G(x, y, z)[D - 2 + \eta(z)]D_T(z) \} + (D - 1) \frac{G(x, y, z)D_T(z)\eta(z)}{z} \right].$$

From this equation, it turns out that the requirement $A(p^2) \equiv 1$ is achieved irrespective of $B$, if

$$\frac{\partial G(x, y, z)}{\partial z} = 0$$

and $\eta(k^2)$ satisfies the following differential equation:

$$\frac{\partial}{\partial z} [z^{D-1}D_T(z)\eta(z)] = -(D - 2)z^{D-1} \frac{\partial}{\partial z} D_T(z).$$

Thus, once the function $D_T(k^2)$ is given, we can find the nonlocal gauge $\eta(k^2)$ by solving Eq. (3.13), so that $A(k^2) \equiv 1$ follows under the ansatz for the vertex function (2.14), provided that $G$ does not depend explicitly on $k^2$. Then we have only to solve Eq. (3.3) for the fermion mass function $B(p^2)$.

$$B(p^2) = m_0 + e^2 \int \frac{d^D q}{(2\pi)^D q^2 + B^2(q^2)} \frac{B(q^2)}{q^2} G(p^2, q^2, k^2)D_T(k^2)[D - \eta(k^2)],$$

or

$$B(x) = m_0 + C_D e^2 \int_0^{\Lambda^2} \frac{d^D y}{(2\pi)^D y^2 + B^2(y)} \frac{B(y)}{y} G(x, y, z)K(x, y),$$

where the kernel is given by

$$K(x, y) := \int_0^{\pi} d\vartheta \sin^{D-2} \vartheta D_T(k^2)[D - \eta(k^2)].$$

The quenched ladder QED in the covariant gauge is recovered by taking $\Pi(k) \equiv 0$, i.e. $D_T(z) = 1/z$. In this case the nonlocal gauge reduces to the local gauge: $\eta(k^2) \equiv \eta = 1 - \xi$. This reproduces the well-known result: $A(p^2) \equiv 1$ in the gauge $\eta = 1$, i.e. Landau gauge $\xi = 0$ for $D > 2$. Especially, for $D = 2$, $A(x) \equiv 1$ is satisfied in the gauge $\eta \equiv 0$, i.e. Feynman gauge $\xi = 1$, as can be seen from Eq. (3.4). However, in $D = 2$, the SD equation can be exactly solved in an arbitrary gauge, see [29].
3.1 Nonlocal gauge without IR cutoff

The differential equation (3.13) is a first order differential equation and is simply solved by choosing a boundary condition. Integrating both sides of Eq. (3.13) from 0 to $k^2$, we obtain

$$
\eta(k^2) = -\frac{D-2}{(k^2)^{D-1}D_T(k^2)} \int_0^{k^2} dz D'_T(z) z^{D-1},
$$

(3.17)

where the prime denotes the differential with respect to $z$. Here we have assumed the boundary condition:

$$
[z^{D-1}D_T(z)\eta(z)]_{z=0} = 0,
$$

(3.18)

so as to eliminate the $1/z^{D-1}$ singularity in $\eta(z)$. Alternatively, we can write

$$
\eta(z) = (D-2) \left[ \frac{D-1}{z^{D-1}D_T(z)} \int_0^z dt D_T(t) t^{D-2} - 1 \right],
$$

(3.19)

where we have assumed that

$$
[z^{D-1}D_T(z)]_{z=0} = 0.
$$

(3.20)

This should be checked after having obtained the function $\eta(z)$.

3.2 Nonlocal gauge with IR cutoff

For later convenience, we introduce an IR cutoff $\epsilon$ in the nonlocal gauge by integrating (3.13) from $\epsilon^2$ to $k^2$:

$$
\eta(k^2) = \frac{\epsilon^{2(D-1)}D_T(\epsilon^2)}{(k^2)^{D-1}D_T(k^2)} \eta(\epsilon^2) - \frac{D-2}{(k^2)^{D-1}D_T(k^2)} \int_{\epsilon^2}^{k^2} dz D'_T(z) z^{D-1},
$$

(3.21)

where $\eta(\epsilon^2)$ is undetermined. Note that Eq. (3.13) is rewritten as

$$
\eta(z) = -z^{D-1}D_T(z)\eta'(z) + (D-2)z^{D-1}D'_T(z).
$$

(3.22)

Therefore, if we impose the flatness condition $\eta'(\epsilon^2) = 0$ on $\eta(k^2)$ at $k^2 = \epsilon^2$ as a boundary condition, the value $\eta(\epsilon^2)$ is determined as

$$
\eta(\epsilon^2) = -\frac{(D-2)(\epsilon^2)^{D-1}D'_T(\epsilon^2)}{[z^{D-1}D_T(z)]'|_{z=\epsilon^2}}.
$$

(3.23)

Hence we arrive at the expression of the nonlocal gauge:

$$
\eta(k^2) = -(D-2)\epsilon^{2(D-1)}D_T(\epsilon^2) \frac{(\epsilon^2)^{D-1}D'_T(\epsilon^2)}{[z^{D-1}D_T(z)]'|_{z=\epsilon^2}} + \frac{D-2}{(k^2)^{D-1}D_T(k^2)} \int_{\epsilon^2}^{k^2} dz D'_T(z) z^{D-1},
$$

(3.24)

This means the flatness of the effective coupling, see section 6.
or
\[
\eta(k^2) \equiv 1 - \bar{\xi}(k^2) = -(D - 2) \epsilon^{2(D-1)} D_T(\epsilon^2) \left[ \frac{(\epsilon^2)^{D-1} D_T(\epsilon^2)}{(k^2)^{D-1} D_T(k^2)} \right]_{z=\epsilon^2} - 1 \right.
- (D - 2) + \frac{(D - 2)(D - 1)}{(k^2)^{D-1} D_T(k^2)} \int_{\epsilon^2}^{k^2} dz D_T(z) z^{D-2}. \tag{3.25}
\]
Note that \( \bar{\xi}(k^2) := 1 - \eta(k^2) \) is in general different from \( \xi(k^2) \).

## 4 Gauge choice and LK transformation

It is worth remarking that the SD equation and the WT identity are form-invariant under the Landau-Khalatnikov (LK) transformation [16]:

\[
\begin{align*}
D'_{\mu\nu}(x) &= D_{\mu\nu}(x) + \partial_\mu \partial_\nu f(x), \\
S'(x, y) &= e^{i2[f(o)-f(x-y)]} S(x, y), \\
\mathcal{V}'_\nu(x, y, z) &= e^{i2[f(o)-f(x-y)]} \mathcal{V}_\nu(x, y, z) \\
&+ S(x, y) e^{i2[f(o)-f(x-y)]} \partial_\nu^z[f(x-z) - f(z-y)].
\end{align*}
\tag{4.26}
\]

where

\[
\begin{align*}
D_{\mu\nu}(x, y) &= \langle 0|T[A_\mu(x) A_\nu(y)]|0 \rangle, \\
S(x, y) &= \langle 0|T[\bar{\Psi}(x) \Psi(y)]|0 \rangle, \\
\mathcal{V}_\nu(x, y, z) &= \langle 0|T[\bar{\Psi}(x) \Psi(y) A_\nu(z)]|0 \rangle. \tag{4.27}
\end{align*}
\]

This can be easily shown in coordinate space where the SD equation has the following form:

\[
(i\partial - m) S(x, y) = \delta^D(x - y) + i e^2 \gamma^\mu \langle \Psi(x) \bar{\Psi}(y) A_\mu(x) \rangle, \tag{4.28}
\]

and

\[
\begin{align*}
D^{-1}_{\mu\nu}(x, z) &= D^{-1}_{\mu\nu}(0, x, z) - \Pi_{\mu\nu}(x, z), \\
\Pi_{\mu\nu}(x, z) &= (g_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu) \Pi(x - z) \\
&= i e^2 \int d^D z_1 d^D z_2 \text{tr}[\gamma_\mu \gamma_\nu S(x, z_1) \Gamma_{\nu}(z_1, z_2; z) S(z_2, x)], \tag{4.29}
\end{align*}
\]

where

\[
\langle \Psi(x) \bar{\Psi}(y) A_\mu(z) \rangle = \int d^D x' d^D y' d^D z' S(x, x') \Gamma_\nu(x', y'; z') S(y', y) D_{\mu\nu}(z', z). \tag{4.30}
\]

Therefore, if we know a consistent set of solutions (for the full gauge boson propagator, the full fermion propagator and the full vertex function) of SD equation in a single gauge, the solutions in other gauges are obtained through the LK transformation.

So far, the LK transformation has been used to transform the Landau gauge result \( \xi = 0 \) into other gauges \( \xi \neq 0 \). Note that it is possible to perform the inverse
LK transformation (from non-Landau gauge to Landau gauge). It turns out that the inverse LK transformation is obtained from (4.26) by replacing $f(x)$ with $-f(x)$. Furthermore, the LK transformation and its inverse allows us to deal with the nonlocal gauge fixing, since $f(x)$ is an arbitrary function. Therefore, the Landau gauge result is recovered from the nonlocal gauge by choosing

$$f(x) = \int \frac{d^Dk}{(2\pi)^D} e^{ik\cdot x} \frac{\xi(k)}{k^4},$$

where $\xi(k^2)$ is related to $\eta(k^2)$ by (2.13). The inverse LK transformation enables us to compare the result of this approach with the conventional one.

Using the inverse LK transformation, we can obtain the fermion propagator $S_L(x)$ in the (usual) Landau gauge $\xi = 0$ from the fermion propagator $S_{nlg}(x)$ in the nonlocal gauge

$$S_L(x) = e^{\Delta(x)} S_{nlg}(x),$$

where

$$\Delta(x) = e^2 \int \frac{d^Dk}{(2\pi)^D} (e^{ik\cdot x} - 1)f(k), \quad f(k) := \frac{\tilde{\xi}(k^2)}{k^4[1 - \Pi(k)/k^2]}.$$  \hspace{1cm} (4.33)

In general, the fermion propagator in configuration space is written in the form:

$$\tilde{S}(x) = i\gamma^\mu x_\mu P(x) + Q(x),$$  \hspace{1cm} (4.34)

in accord with $S(p) = [A(p^2)\gamma^\mu p_\mu + B(p^2)]^{-1}$. Taking into account the relation

$$S(p) = \int d^D x e^{ip\cdot x} \tilde{S}(x) = \frac{A(p^2)\gamma^\mu p_\mu + B(p^2)}{A^2(p^2)p^2 + B^2(p^2)},$$

we obtain

$$\frac{A(p^2)p^2}{A^2(p^2)p^2 + B^2(p^2)} = i \int d^D x e^{ip\cdot x}(p\cdot x)P(x),$$

$$\frac{B(p^2)}{A^2(p^2)p^2 + B^2(p^2)} = \int d^D x e^{ip\cdot x}Q(x).$$

(4.36)

Therefore, $A(p^2)$ and $B(p^2)$ in the Landau gauge are obtained by substituting $P(x) = e^{\Delta(x)} P_{nlg}(x)$ and $Q(x) = e^{\Delta(x)} Q_{nlg}(x)$ into (4.36). In order to accomplish this, we must obtain the expression of the fermion propagator $S_{nlg}(x)$ in configuration space.

In the massless fermion phase $B(x) \equiv 0$, the fermion propagator $S_{nlg}(p)$ in the nonlocal gauge is nothing but the free massless propagator $S_{nlg}(p) \equiv S_0(p) = 1/(\gamma^\mu p_\mu)$. In configuration space the free mass propagator is given by

$$\tilde{S}_0(x) = \int \frac{d^D p}{(2\pi)^D} e^{ip\cdot x} \frac{\gamma^\mu p_\mu}{p^2} = i\gamma^\mu x_\mu P_0(x), \quad P_0(x) = \frac{\Gamma(D/2)}{2\pi^{D/2}|x|^{D/2}}.$$  \hspace{1cm} (4.37)
Therefore, the full fermion propagator in Landau gauge is given by

\[ S_L(p) = \frac{\gamma^\mu p_\mu}{A(p^2)p^2}, \quad A^{-1}(p^2) = i \int d^D x e^{ip \cdot x} e^{\Delta(x)} (p \cdot x) P_0(x). \tag{4.38} \]

Note that the vacuum polarization \( \Pi(k) \) determines \( \tilde{\xi}(k) \) and then \( \Delta(x) \). Therefore, the specification of the vacuum polarization is crucial. In the quenched limit \( \Pi(k) \equiv 0 \), \( \tilde{\xi}(k) \equiv 0 \) and hence \( S_L \equiv S_0 \). This is a consistency check.

In the massless fermion phase, the vacuum polarization can be calculated easier than the massive case. The explicit expression depends crucially on the spacetime dimension in question. The nonlocal gauge function \( \tilde{\xi}(k) \) is expected to have a finite range, namely, \( |\tilde{\xi}(k)| < c \) uniformly in \( k \), except for the neighborhood of \( k = 0 \). So \( \Pi(k) \) is expected not to change the UV behavior of \( f(k) \) qualitatively, see (4.33). However, in lower dimensions \( 2 < D < 4 \), \( \Pi(k) \) can influence the IR behavior of \( f(k) \) considerably. This is discussed in section 5.3 in more detail.

## 5 Running coupling constant and nonlocal gauge

### 5.1 Wavefunction renormalization and running coupling

Suppose that the SD equation (3.4) for \( A \) has been solved in the Landau gauge \( \eta(k^2) \equiv 1 \) under the ansatz (2.14) for the vertex function (when \( B^2 \) is neglected). Then the SD equation (3.5) for \( B \) can be written as

\[ B(p^2) = m_0 + e^2 \int \frac{d^D q}{(2\pi)^D} \frac{B(q^2)}{q^2} \left[ \frac{G(p^2, q^2, k^2)}{A^2(q^2)} \right] (D - 1) D_T(k^2), \tag{5.1} \]

where \( B^2(q^2) \) in the denominator is neglected according to the bifurcation method [36, 37].

In the paper [11], the following ansatz for the vertex is adopted in order to look for the solution in QED\(_3\):

\[ G(p^2, q^2, k^2) = A(q^2)^n, \tag{5.2} \]

where \( n \) is an integer \((n = 1, 2, ...\)). Under this ansatz, the approximate solution of the SD equation for \( A \) was obtained in the usual Landau gauge:

\[ A(p^2) = \left( 1 + \frac{2 - n}{3} K t \right)^{1/(2-n)}, \tag{5.3} \]

where

\[ K := \frac{8}{\pi^2 N_f}, \tag{5.4} \]

and

\[ t := \ln \frac{p}{\alpha}, \quad \alpha := \frac{e^2 N_f}{8}. \tag{5.5} \]
Based on this solution, it was pointed out that one can define a running coupling constant $K(t)$ whose actual running is given by

$$K(t) := A_{sol}^{n/2}(p^2)K = \frac{K}{1 + \frac{2-n}{3}Kt}$$

(5.6)

due to wavefunction renormalization. Indeed, in the absence of wavefunction renormalization $A(p^2) \equiv 1$ and the coupling $K(t)$ does not run, i.e. $K(t) \equiv K$.

It is claimed that the running coupling constant of QED$_3$ obtained in such a way corresponds to the asymptotically free case for $n < 2$ (here $n = 1$ is likely to be the physical one due to the WT identity). If we accept Eq. (5.6) at face value, the running coupling $K(t)$ diverges in the IR region for $n < 2$. In this case, $K(t)$ becomes strong enough to be able to cause chiral symmetry breaking and to make the bound state $\bar{\Psi}\Psi$. It should be noted that the asymptotic freedom defined here is valid only if $t_0 := \ln \epsilon/\alpha < t_\infty := -3/[(2-n)K]$. Therefore this asymptotic freedom would disappear for sufficiently large IR cutoff $\epsilon$ or small UV cutoff $\alpha$. This gives a possible (physical) explanation for the controversy among the results [14, 8] on the phase structure of QED$_3$. For more details, see ref. [11].

The above observation is based on the approximate solution (5.3) for $A$ in the Landau gauge $\xi = 0$ under the vertex ansatz (5.2). Quite recently, the rate of running of the coupling constant $K(t)$ has been studied in more detail by Aitchison and Mavromatos [23] and a subsequent paper [24] where the validity of all the approximations made in solving the SD equation for $A$ in the paper [11] were reexamined thoroughly under the same vertex ansatz.

### 5.2 Running coupling through nonlocal gauge

Now we study the running coupling based on the nonlocal gauge. In the nonlocal gauge, the function $B$ obeys the SD equation (3.14). If we choose the nonlocal gauge $\xi(k)$ such that $A(p^2) \equiv 1$, the function $G(p^2, q^2, k^2)$ should be replaced with 1, i.e. $G(p^2, q^2, k^2) \equiv 1$ from the consistency with the WT identity. In this case, the wavefunction renormalization disappears and the SD equation for $B$ reduces to

$$B(p^2) = m_0 + e^2 \int \frac{d^Dq}{(2\pi)^D} \frac{B(q^2)}{q^2 + B^2(q^2)} [D - \eta(k^2)] D_T(k^2).$$

(5.7)

This should be compared with the SD equation (5.1). Then the running coupling constant is given by

$$K(t)/K = \frac{D - \eta(k^2)}{D - 1} = \frac{D - 1 + \tilde{\eta}(k^2)}{D - 1}.$$  

(5.8)

Here note that the argument of the running coupling constant is the gauge-boson momentum, which is required for the nonlocal gauge to be consistent with the axial WT identity as well as the vector WT identity [13]. Therefore, the problem of finding the running coupling constant reduces to finding the nonlocal gauge.

Therefore we can see the following advantage or benefits of this approach:
1) We do not have to assume a particular ansatz for the vertex. We can study a class of vertices of the form: $\Gamma_\mu(p,q) = \gamma_\mu G(p^2,q^2,k^2)$. The consistency with the WT identity requires that $G(p^2,q^2,k^2) \to 1$ as $A(p^2) \to 1$. This class of vertex ansatz includes the previous one (5.2) with $n = 1$.

2) Under this ansatz for the vertex, we need not to solve the SD equation for $A(p^2)$. This releases us from worrying about the validity of a number of approximations which are required to solve the SD equation for $A$. Instead, we obtain the differential equation for the nonlocal gauge. This is solved by simple quadrature. In particular, we do not have to perform any angular integration. So the Higashijima-like approximation [38] is unnecessary to separate the kernel of the integral equation (at least in the normal phase).

3) If necessary, we can recover the solution in the usual gauge (e.g. Landau gauge), by making use of the (inverse) LK transformation as discussed in the previous section. This enables us to compare the result in this approach with the conventional result.

4) We can study the effect of a cutoff in more detail.

The final point needs more explanations. In this approach it is not necessary to introduce the UV cutoff $\alpha$ in QED$_3$. In the analysis of QED$_3$ [11], the existence of the IR cutoff $\epsilon$ was essential as well as the UV cutoff $\alpha$. Whether or not there exists a "finite" critical number of flavors $N_f^c$ above which ($N_f > N_f^c$) spontaneous chiral symmetry breaking disappear depends crucially on the existence of the infrared cutoff and the ratio $\epsilon/\alpha$. It is expected that the asymptotic freedom claimed above would disappear for sufficiently large IR cutoff $\epsilon$ or small UV cutoff $\alpha$ and hence this leads to a finite critical number of flavors $N_f^c < \infty$ for sufficiently large $\epsilon/\alpha$. This observation is based on a naive analogy with QCD in 3+1 dimensions (QCD$_4$). In QCD$_4$, the theory has only one phase, the chiral-symmetry breaking and confining phase, due to asymptotic freedom where the running of the gauge coupling constant is essential [38]. On the other hand, QED$_4$ is not asymptotically free and has a critical gauge coupling constant $\epsilon_c$ above which ($\epsilon > \epsilon_c$) a fermion mass is dynamically generated and chiral symmetry is spontaneously broken. Therefore, the IR behavior of the running coupling constant in QED$_3$ is very important to resolve the phase structure of QED$_3$.

On the the hand, QED$_3$ has already been studied in the nonlocal gauge [21, 22] where we have found a phase transition at a finite critical number of flavors $N_f^c < \infty$ which separates the chiral-symmetric phase from the spontaneous-chiral-symmetry-breaking phase. In this analysis we did not introduce the infrared cutoff from the beginning. Apparently, the results of [11] and [2] contradict with each other, since the very small IR cutoff (compared with UV cutoff $\alpha$) should lead to the asymptotic freedom and no phase transition, i.e. non-existence of a finite critical number of flavors according to [11]. In this paper we resolve this apparent contradiction. For this, we introduce the infrared cutoff in the setting up of nonlocal gauge. The result is given in the next sections.
5.3 Inverse LK transformation and wavefunction renormalization

Thanks to the inverse LK transformation, we can obtain the fermion propagator $S_L(x)$ in the (usual) Landau gauge $\xi = 0$ from the fermion propagator $S_{nl}(x)$ in the nonlocal gauge. In $D = 3$ dimensions, the free massless propagator in the configuration space is given by

$$S_0(x) = \frac{\gamma^\mu x_\mu}{4\pi |x|^3}. \quad (5.9)$$

For $D = 3$, the massless fermion generates the vacuum polarization (at one-loop):

$$\Pi(k) = -\alpha k. \quad (5.10)$$

By substituting the vacuum polarization function (5.10) and $S_{nl}(x) = S_0(x)$ given by (5.3) into (4.32) with the nonlocal gauge function $\tilde{\xi}(k^2)$ obtained explicitly in the next section, we can get the fermion propagator $S_L(x)$ in the Landau gauge in configuration space.

If $\xi$ was a constant, we would have for $D = 3$

$$\Delta(x) = e^2 \int \frac{d^3k}{(2\pi)^3} (e^{ikx} - 1) \frac{\xi}{k^4} = -\frac{e^2}{8\pi} \xi |x|. \quad (5.11)$$

According to (4.38), it is not difficult to show that the wavefunction renormalization function is obtained as

$$A^{-1}(p^2) = 1 - \frac{e^2\xi}{8\pi p} \arctan\left(\frac{8\pi p}{e^2\xi}\right). \quad (5.12)$$

In the IR region, we find that

$$A^{-1}(p^2) = \frac{1}{3} \left(\frac{8\pi p}{e^2\xi}\right)^2 + O(p^4). \quad (5.13)$$

This is totally different from what we expect based on the nonlocal gauge.

In the nonlocal gauge, the integrand $f(k)$ of $\Delta(x)$ in the UV region $k \to \infty$ exhibits the behavior

$$f(k) \sim \frac{\tilde{\xi}(k^2)}{k^4} \to 0 \quad (k \to \infty), \quad (5.14)$$

so the effect of $\Pi$ is neglected in this region. However, in the IR region $k \to 0$, the presence of $\Pi(k)$ totally changes the situation:

$$f(k) \sim \frac{\tilde{\xi}(k^2)}{\alpha k^3} \quad (k \to 0). \quad (5.15)$$

In either case, the situation does not resemble the constant $\xi$ case. If the integration is performed numerically in (4.38), we will be able to obtain the wavefunction renormalization function $A(p^2)$ in the Landau gauge (the corresponding vertex function can be also obtained from the LK transformation). The numerical result will be given elsewhere.
6 Running coupling constant of QED$_3$

In $D = 3$ dimensions, the running coupling (5.8) is obtained by shifting and scaling the nonlocal gauge according to

$$K(t)/K = 1 + \frac{\bar{\xi}(k^2)}{2}. \quad (6.1)$$

The exact non-trivial wavefunction renormalization in the Landau gauge is obtained from (4.38). Under a specific ansatz (5.2), it is obtained from (5.6) and (5.8) as

$$A(p^2) = \left(1 + \frac{\bar{\xi}(p^2)}{2}\right)^{1/(n-2)}, \quad (6.2)$$

and in particular for $n = 1$

$$A(p^2) = \left(1 + \frac{\bar{\xi}(p^2)}{2}\right)^{-1}. \quad (6.3)$$

Therefore, we study the behavior of the nonlocal gauge function $\bar{\xi}(k^2)$ in what follows.

6.1 Running coupling constant (I)

From Eq. (3.25), the nonlocal gauge of QED$_3$ with an IR cutoff $\epsilon$ is given by

$$\bar{\xi}_\epsilon(k^2) = \xi^a_\epsilon(k^2) + \xi^b_\epsilon(k^2),$$

$$\xi^a_\epsilon(k^2) := 2 - \frac{2}{(k^2)^2 D_T(k^2)} \int_{\epsilon^2}^{k^2} dz D_T(z) z,$$

$$\xi^b_\epsilon(k^2) := \frac{\epsilon^4 D_T(\epsilon^2)}{(k^2)^2 D_T(k^2)} \left[ \frac{(\epsilon^2)^2 D_T(\epsilon^2)}{[z^2 D_T(z)]'|_{z=\epsilon^2} - 1} \right]. \quad (6.4)$$

Here we decomposed the nonlocal gauge into two pieces: the first piece $\xi^a_\epsilon$ reduces in the limit $\epsilon \to 0$ to the nonlocal gauge without an IR cutoff, while the second piece $\xi^b_\epsilon$ comes from the flatness condition $\xi'(\epsilon^2) = 0$ and vanishes in the limit $\epsilon \to 0$. First of all, we consider the following integral

$$J_1(k^2; \epsilon) := \int_{\epsilon^2}^{k^2} dz \frac{z}{z + \alpha \sqrt{z}}$$

$$= k^2 - 2\alpha \sqrt{k^2} - \epsilon^2 + 2\alpha \sqrt{\epsilon^2} + 2\alpha^2 \ln \frac{k + \alpha}{\sqrt{\epsilon^2} + \alpha}. \quad (6.5)$$

This has the expansion:

$$J_1(k^2; \epsilon) = (-\epsilon^2 + 2\alpha \sqrt{\epsilon^2} + 2\alpha^2 \log(\alpha) - 2\alpha^2 \log(\alpha + \sqrt{\epsilon^2})) + \frac{2k^3}{3\alpha} + O(k^4). \quad (6.6)$$

Hence the first piece $\xi^a_\epsilon$ of the nonlocal gauge (6.4) given by

$$\xi^a_\epsilon(k^2) = 2 - 2\frac{k^2 + \alpha \sqrt{k^2}}{(k^2)^2} J_1(k^2; \epsilon^2) \quad (6.7)$$
has the following expansion around $k = 0$:

$$
\xi^a(k^2) = \frac{2 \alpha}{k^3} \left( \epsilon^2 - 2 \alpha \sqrt{\epsilon^2} + 2 \alpha^2 \log(1 + \sqrt{\epsilon^2}/\alpha) \right) + \frac{2}{k^2} \left( \epsilon^2 - 2 \alpha \sqrt{\epsilon^2} + 2 \alpha^2 \log(1 + \sqrt{\epsilon^2}/\alpha) \right) + \frac{2}{3} - \frac{k}{3\alpha} + \frac{k^2}{5\alpha^2} - \frac{2k^3}{15\alpha^3} + O(k^4). \tag{6.8}
$$

Note that the introduction of the IR cutoff $\epsilon$ generates IR singular terms like $2\alpha k^{-3}$, $2\alpha k^{-2}$ where the coefficient $c$ is always positive. Such a singular behavior in the IR region disappears if we put $\epsilon = 0$ from the beginning, and $\xi^a$ reduces to

$$
\tilde{\xi}_0(k^2) = \frac{2}{3} - \frac{k}{3\alpha} + \frac{k^2}{5\alpha^2} - \frac{2k^3}{15\alpha^3} + O(k^4). \tag{6.9}
$$

This is nothing but the result obtained in [21] (by setting the Chern-Simons coefficient $\theta$ equal to zero: $\theta = 0$ in eq. (29) of [21]). On the other hand, the second piece is always negative and singular at $k = 0$:

$$
\xi^b(k^2) = -\frac{4}{k^4} \left( \frac{k^2 + \alpha \sqrt{k^2}}{3 \alpha + 2 \sqrt{\epsilon^2}} \right)^3 = -\frac{4\epsilon^3}{(3\alpha + 2\sqrt{\epsilon^2})} \left( \frac{\alpha}{k^3} + \frac{1}{k^2} \right). \tag{6.10}
$$

By adding (6.8) and (6.10), we find that the singular part with negative power of $k$ in the nonlocal gauge is negative.

In the region $k/\alpha \gg 1$, $\xi^a$ is dominant, because $\xi^b$ decreases more rapidly than $\xi^a$ which behaves as

$$
\xi^a(k^2) = \frac{2 \alpha}{k} + \frac{4 \alpha^2 + 2 \epsilon^2 - 4 \alpha \sqrt{\epsilon^2} + 4 \alpha^2 \log((\alpha + \sqrt{\epsilon^2})/k)}{k^2} + O(\frac{1}{k})^3. \tag{6.11}
$$

It turns out that the $\xi^a$ alone is monotonically decreasing in $k$ with a maximum value 2 at $k = \epsilon$ (and diverges monotonically as $k \to 0$) which rather enhances the effective coupling compared with $\xi_0(0) = 2/3$. This tendency agrees with the observation made in the previous paper [11] where the running of the coupling is terminated at $k = \epsilon$ and the flatness of the coupling $K(k^2) = K(\epsilon^2)$ for $k \leq \epsilon$ is assumed a priori, which is borrowed from the QCD$_4$ analysis. If we consider only the $\xi^a$ piece, there is a discontinuity in the derivative $\xi^a'(k)$ at $k = \epsilon$, since $\xi^a'((\epsilon + 0)) < 0$ and $\xi^a'((\epsilon - 0)) = 0$. However, the inclusion of $\xi^b(< 0)$ which is necessary to satisfy $\xi^b(\epsilon) = 0$ (continuity of $\xi^b(k)$ at $k = \epsilon$) considerably changes the situation. The inclusion of such a term causes a slowing down of the rate of decrease of the effective coupling constant $K(t)$. This tendency agrees with the recent analysis of [24]. The the nonlocal gauge function and the running coupling constant are monotonically decreasing in $k$ for $k > \epsilon$ and have upper bonds:

$$
\dot{\xi}_e(\epsilon^2) = \xi^a(\epsilon^2) + \xi^b(\epsilon^2) = 2 - \frac{4(\alpha + \epsilon)}{3\alpha + 2\epsilon} = \frac{2}{3 + 2\epsilon/\alpha} < \frac{2}{3}. \tag{6.12}
$$

For $k < \epsilon$, the nonlocal gauge $\tilde{\xi}_e$ decreases as $k$ decreases. In Figure 1, the nonlocal gauge is plotted for $\epsilon/\alpha = 0.1$. The running coupling constant is obtained from (6.11), and the wavefunction renormalization in the Landau gauge is obtained from (5.3).
6.2 Running coupling constant (II)

In this section, we consider another way of introducing the IR cutoff. We introduce the IR cutoff \( \delta \) in the gauge-boson propagator:

\[
D_T(k^2) = \frac{1}{k^2 + \alpha k + \delta^2}.
\]

The cutoff \( \delta \) plays the same role as the gauge-boson mass and seems to be more natural than the previous one. This choice of \( D_T \) is equivalent to the gauged Thirring model in the nonlocal \( R_\xi \) gauge \cite{19} where \( \delta^2 = e^2 G_T^{-1} \) for the Thirring coupling \( G_T \).

In this case, the nonlocal gauge reads

\[
\tilde{\xi}_\delta(k^2) = 2 - 2 \frac{k^2 + \alpha k + \delta^2}{(k^2)^2} J_2(k^2, \delta),
\]

with

\[
J_2(k^2; \delta) := \int_0^{k^2} dz \frac{z}{z + \alpha \sqrt{z} + \delta^2}
\]

\[
= \int_0^k dr \frac{2r^3}{r^2 + \alpha r + \delta^2}
\]

\[
= (k - 2\alpha)k + (\alpha^2 - \delta^2) \ln |(k^2 + \alpha k + \delta^2)/\delta^2| + \alpha(3\delta^2 - \alpha^2)[I_B(k) - I_B(0)],
\]

where the indefinite integral \( I_B(k) \) is defined by

\[
I_B(k) := \int_0^k dr \frac{1}{r^2 + \alpha r + \delta^2}.
\]

The indefinite integral \( I_B(k) \) can be calculated: For \( \alpha^2 > 4\delta^2 \),

\[
I_B(k) = \frac{-1}{\sqrt{\alpha^2 - 4\delta^2}} \ln \frac{(2k + \alpha + \sqrt{\alpha^2 - 4\delta^2})^2}{|k^2 + \alpha k + \delta^2|},
\]

and for \( \alpha^2 < 4\delta^2 \),

\[
I_B(k) = \frac{2}{\sqrt{4\delta^2 - \alpha^2}} \arctan \frac{2k + \alpha}{\sqrt{4\delta^2 - \alpha^2}}.
\]

When \( \alpha^2 > 4\delta^2 \), a careful analysis shows that all the terms in \( J_2 \) up to \( O(k^3) \) cancel, namely,

\[
J_2(k^2, \delta) = \frac{1}{2\delta^2} k^4 - \frac{2\alpha}{5\delta^4} k^5 + \frac{\alpha^2 - \delta^2}{3\delta^6} k^6 + O(k^7).
\]

This implies the following expansion of the nonlocal gauge \( \tilde{\xi}_\delta(k^2) \) around \( k = 0 \):

\[
\tilde{\xi}_\delta(k^2) = 1 - \frac{\alpha}{5\delta^2} k + \frac{2\alpha^2 - 5\delta^2}{15\delta^4} k^3 - \frac{2\alpha(5\alpha^2 - 17\delta^2)}{105\delta^6} k^3 + O(k^4).
\]
This result shows that there is no singularity in the nonlocal gauge (6.14) at $k = 0$, in sharp contrast with the nonlocal gauge $\tilde{\xi}_0(k^2)$. In the region $k/\alpha \ll 1$, the nonlocal gauge behaves as follows: at $\delta/\alpha = 0.1$

$$
\tilde{\xi}_0(k^2) = 1 - 20(k/\alpha) + 1300(k/\alpha)^2 - 92000(k/\alpha)^3 + O((k/\alpha)^4),
$$

(6.21)

and at $\delta/\alpha = 0.01$

$$
\tilde{\xi}_0(k^2) = 1 - 2000(k/\alpha) + 1.333 \times 10^7(k/\alpha)^2 - 9.52057 \times 10^{10}(k/\alpha)^3 + O((k/\alpha)^4).
$$

(6.22)

It should be remarked that, in the limit $(\delta \to 0)$, the nonlocal gauge (6.20) does not reduce to the expected form (6.9). There is a discontinuity between the nonlocal gauge $\tilde{\xi}_\delta(k^2)$ with an IR cutoff $\delta$ and the nonlocal gauge $\tilde{\xi}_0(k^2)$ obtained without an IR cutoff from the beginning. Such a discontinuity does not exist for the nonlocal gauge $\tilde{\xi}_\epsilon(k^2)$ with IR cutoff $\epsilon$. Such a situation can be seen in the finite temperature case: there may occur some discontinuity between the zero-temperature limit of finite-temperature calculation and the corresponding result evaluated at zero-temperature. In this context $\delta$ can be interpreted as a plasmon mass. This discontinuity is very similar to the result found by Aitchison et al. [24]. The non-local gauge $\tilde{\xi}_\delta(k^2)$ decreases more slowly than claimed in the previous paper [11]. The nonlocal gauge is plotted in Figure 2.

6.3 RG-like point of view

By using the idea of RG, we show that the IR and UV behaviors of the nonlocal gauge $\tilde{\xi}_\delta(k^2)$ and the effective running coupling constant can be easily analyzed without performing any integration. For this, we calculate the derivative of the nonlocal gauge function and express the result in terms of the nonlocal gauge function itself:

$$
\beta(\tilde{\xi}) := -\frac{d}{d \ln(k/\mu)} \tilde{\xi}(k^2) = 4 + \frac{2k^2 + 3\alpha k + 4\delta^2}{k^2 + \alpha k + \delta^2} (\tilde{\xi}(k^2) - 2).
$$

(6.23)

From this equation, we can observe the following:

1) without IR cutoff ($\delta = 0$), in the IR limit $k \to 0$

$$
\beta(\tilde{\xi}) = 4 + 3(\tilde{\xi}(k^2) - 2) = 3\tilde{\xi}(k^2) - 2.
$$

(6.24)

This implies $\tilde{\xi}(k^2) \to 2/3 = \tilde{\xi}_0(0)$ in the IR limit $k \to 0$.

2) with IR cutoff ($\delta \neq 0$), in the IR limit $k \to 0$

$$
\beta(\tilde{\xi}) = 4 + 4(\tilde{\xi}(k^2) - 2) = 4\tilde{\xi}(k^2) - 4.
$$

(6.25)

This implies $\tilde{\xi}(k^2) \to 1$ in the IR limit $k \to 0$.

These results seem to show the existence of non-trivial IR fixed point for the running coupling constant $\tilde{K} := K(t)$ defined by $K(t)/K = 1 + \tilde{\xi}(k^2)/2$ at $\tilde{K} = 4/3K$ and $\tilde{K} = 3/2K \sim O(1/N)$ corresponding to 1) and 2) respectively.
In particular,
3) in the UV limit $k \to \infty$ (irrespective of the value of the IR cutoff),

$$\beta(\tilde{\xi}) = 4 + 2(\tilde{\xi}(k^2) - 2) = 2\tilde{\xi}(k^2).$$ \hspace{1cm} (6.26)

This implies $\tilde{\xi}(k^2) \to 0$ in the UV limit $k \to \infty$.

Especially,
4) in the quenched limit ($N_f \to 0$ and $\delta = 0$),

$$\beta(\tilde{\xi}) = 4 + 2(\tilde{\xi}(k^2) - 2) = 2\tilde{\xi}(k^2).$$ \hspace{1cm} (6.27)

for any $k$. This shows that the constant solution is possible: $\tilde{\xi}(k^2) \equiv 0$.

It is straightforward to generalize the above argument to arbitrary dimension.

7 Conclusion and discussion

In this paper we have discussed an alternative approach for obtaining the effective or running coupling constant and the RG property of gauge theory in the SD framework, which is appropriate for non-perturbative study of gauge theory. This approach can be applied only to a gauge theory, as we use the the gauge invariance as an essential ingredient in the approach. We argued that this approach is superior in several respects to the previous conventional approach in which the SD integral equation has been solved to obtain the wavefunction renormalization function $A(p^2)$ under a specific vertex ansatz to define the running coupling. In this approach we do not solve the SD equation for $A$, instead we obtain the differential equation which the nonlocal gauge function must satisfy in order that $A(p^2) \equiv 1$.

The validity of this approach has been exemplified in the study of the IR behavior of the running coupling constant and non-trivial IR fixed point in QED$_3$. This approach confirms a recent result: the slowing down of the rate of decrease of the running coupling constant and the existence of non-trivial IR fixed point, as claimed in \cite{23,24}, but here without relying on a specific ansatz of vertex and on a number of approximations adopted to solve the SD equation in the conventional approach \cite{11}.

In this paper we have not studied the dynamical mass generation and spontaneous chiral symmetry breaking \cite{14} in QED$_3$ by solving the SD equation for $B$ in the nonlocal gauge. The nonlocal gauge $\xi_0(k^2)$ does not qualitatively change the previous result \cite{22} in the nonlocal gauge $\xi_0(k^2)$ without IR cutoff. For example, there should exist a certain finite critical number of flavors $N_f^c$ above which dynamical mass is not generated and chiral symmetry is restored.

The relationship between our approach and the conventional one can be seen by using the LK transformation. Under an LK transformation, the SD equation and the WTI identity are form-invariant. By making use of an LK transformation, the non-trivial wavefunction renormalization $A(p^2)$ in the Landau gauge can be obtained from the nonlocal gauge, as shown in section 4.3. Thus the conventional picture can be recovered by the (inverse) LK transformation from our approach, if desired. It would be interesting to apply this procedure to the case of QED$_3$ for which Maris
[13] has recently given a rather full discussion of the SD equation in the conventional framework.

The nonlocal gauge has been applied to gauge theories in four dimensions [14, 15], QED$_3$ [20, 21, 22] and the gauged Thirring model [19] under a bare vertex approximation. Obviously, this approach is not so systematic as the perturbative method. The nonlocal gauge must be obtained case by case and there is no guarantee that such a gauge does exist beyond this order of truncation of SD equation. Nevertheless it is sufficiently interesting to warrant the extension of this approach to finite temperature gauge field theory [39]. In such a case, the nonlocal gauge may only be obtained approximately, as in the case for QED$_3$ with a Chern-Simons term [21, 22].

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Figure Captions

Fig.1: Plot of nonlocal gauge given by (6.4) as a function of $k/\alpha$. Three graphs correspond to $\xi^a(k)$ (above), $\xi_0(k)$ (middle) and $\xi_\epsilon(k)$ (below). Here we have chosen $\epsilon/\alpha = 0.1$.

Fig.2: Plot of nonlocal gauge given by (6.14) as a function of $k/\alpha$. Two graphs correspond to $\epsilon/\alpha = 0.1$ (above) and $\epsilon/\alpha = 0.01$ (below).