Kaluza-Klein Magnetic Monopole in
Five-Dimensional Global Monopole Spacetime

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Abstract

In this paper we present a solution for Kaluza-Klein magnetic monopole in a five-dimensional global monopole spacetime. This new solution is a generalization of the previous ones obtained by D. Gross and M. Perry (Nucl. Phys. B 226, 29 (1983)) containing a magnetic monopole in a Ricci-flat formalism, and by A. Banerjee, S. Charttejee and A. See (Class. Quantum Grav. 13, 3141 (1996)) for a global monopole in a five-dimensional spacetime, setting specific integration constant equal to zero. Also we analyse the classical motion of a massive charged test particle on this manifold and present the equation for classical trajectory obeyed by this particle.

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1. Introduction

One of the oldest and elegant formalism to unify the gravitation with the gauge theory was proposed by Th. Kaluza [1] many years ago. Kaluza’s conjecture was that the degrees of freedom associated with the gauge field could be accommodated as the new components of the metric tensor in a higher than four dimensional manifold. Specifically, considering the Abelian gauge theory, just one extra dimension would be enough. This extra dimension is compactified on a circle of so small radius that would not be observable at low-energy scale, i.e., smaller than the Planck one. This theory was analyzed by several authors including O. Klein, who clarified many aspects of the structure of the manifold [2].

Five-dimensional Einstein action exhibits, as the low-energy effective theory, the four-dimensional gravity theory coupled with Maxwell one, where all the physical fields do not depend on the fifth coordinate. The generalization of the theory to include non-Abelian gauge fields requires the addition of more than one extra dimensions.

Also one of most important works about Abelian gauge theories was due to the P. M. Dirac many years ago, who proposed a new solution to the Maxwell equations. His new solution for the vector potential corresponds to a point-like magnetic monopole with a singularity string running from the particle’s position to infinity [3]. The most elegant formalism to describe the Abelian point-like magnetic monopole has been developed by Wu and Yang [4]. In their formalism the vector potential is described by a singularity free expression. In order to provide this formalism, Wu and Yang defined the vector potential $A_\mu$ in two overlapping regions, $R_a$ and $R_b$, which cover the whole space.

In their beautiful papers, Gross and Perry [5], and Sorkin [6], independently, presented a soliton-like solution of the five-dimensional Kaluza-Klein theory corresponding to a magnetic monopole. As the Dirac solution, their solutions describes a gauge-dependent string singularity line, if the fifth coordinate is conveniently compactified. Moreover, its magnetic charge has one unit of Dirac charge: $g = 1/2e$ in units $\hbar = c = 1$. These solutions are generalizations of the self-dual Euclidean Taub-NUT solution [7]. Also, Gegenberg and Kunstatter in [8] found another magnetic monopole solution for five-dimensional Kaluza-Klein theory. Their solutions were obtained by applying the static and Ricci-flat requirement on the field
Global monopole is a solution predicted in Grand Unified Theories. It is formed due to a phase transition of a system composed by self-coupling iso-scalar field only. The matter field plays the role of an order parameter which outside the monopole’s core, acquires a non-vanishing value. The simplest theoretical model which gives rise to global monopole has been proposed by Barriola and Vilenkin [9]. This model is composed by triplet Goldstone field $\phi^a$. The original global $O(3)$ symmetry of the physical system is spontaneously broken down to $U(1)$. In four-dimensional spacetime this Lagrangian density reads:

$$L = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^a - \frac{1}{4}\lambda(\phi^a\phi^a - \eta^2)^2$$

with $a = 1, 2, 3$ and $\eta$ being the scale energy where the symmetry is broken. The field configuration which describes a monopole is

$$\phi^a(x) = \eta f(r)\hat{x}^a,$$

where $\hat{x}^a\hat{x}^a = 1$. Coupling this matter field with the Einstein equation, a spherically symmetric regular metric tensor solution is obtained. Barriola and Vilenkin also shown that for points outside the global monopole’s core the geometry of the manifold can be approximately given by the line element

$$ds^2 = -dt^2 + \frac{dr^2}{\alpha^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

with $\alpha^2 = 1 - 8\pi G\eta^2$. This line element represents a three-geometry with a solid angle deficit.

The analysis of the above system in the context of a five-dimensional Einstein equation has been developed by Banerjee at al in [10]. There a family of solutions has been found in the region outside the global monopole, where the Goldstone boson can be approximated by $\phi^a(x) = \eta\hat{x}^a$. They pointed out that, differently from the Barriola and Vilenkin solution, the criteria of uniqueness is lost in five dimensions, and for specific choice of parameters their solution is Schwarzschild-type one.

Here we shall continue the analysis developed by Banerjee at al admitting the presence of a magnetic monopole. As we shall see, the presence of the global monopole system gives rise to a non Ricci-flat solution. So, in order to
make this analysis possible, we shall consider the case where both defects are 
at the same position chosen as the origin of the reference system. In this way 
our solution contains the solutions found by Gross and Perry and Banerjee 
et al as special cases.

The analysis of a system that presents a regular composite topological 
object, which takes into account the presence of a self-gravitating 't Hooft-
Polyakov magnetic monopole in a global monopole spacetime, has been 
developed recently by one of us in [11]. There it was found that at large 
distance the structure of the manifold corresponds to a Reissner-Nordstöm 
spacetime with a solid angle deficit factor. Here we shall present also a 
composite monopole solution in the context of a five-dimensional Einstein 
equation. Our solution contains an Abelian magnetic monopole in the 
presence of a five-dimensional global monopole spacetime.

This paper is organized as follows: In Sec. 2 we briefly review the results 
found by Gross and Perry and Banerjee et al, we introduce the mathematical 
formalism needed to develop our analysis and present the complete system 
that we want to study. Also we present our solutions obtained from the 
Einstein equations in five dimensions taking into consideration the presence 
of the Abelian magnetic monopole. In Sec. 3 we study the classical relativistic 
motion of a test charged particle on this manifold and present the equation 
for the trajectory obeyed by it. Finally we present in Sec. 4 our conclusions 
and most relevant remarks.

2. Composite Monopole

As we have already said, in this section we analyse the physical system given 
by (1) in the context of a five-dimensional Einstein equation. In order to take 
into account the presence of an Abelian magnetic monopole we have to admit 
non-diagonal components to the metric tensor. Following the prescription 
adopted by [5], we shall assume static spherically symmetric structure for 
the four-dimensional spacetime components of the metric tensor. Using the 
coordinates $\dot{x}^A = (t, r, \theta, \phi, \Psi)$, with the index $A$ running from 0 to 4 the 
line element reads:

$$ds^2 = \dot{g}_{AB}d\dot{x}^Ad\dot{x}^B = -B(r)dt^2 + A(r)dr^2 + C(r)r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right)$$
$$+ D(r)(d\Psi + f(\theta)d\phi)^2 . \quad (4)$$
As we can see the structure of the above expression can represent two special cases of solutions of the five-dimensional Einstein equation as shown in the following:

### 2.1 Gross and Perry Solution

Admitting the absence of the global monopole, i.e., taking $\eta = 0$ in (1), the Ricci-flat, $\hat{R}_{AB} = 0$, solution found by Gross and Perry is:

$$A(r) = C(r) = \frac{1}{D(r)} = 1 + \frac{4m}{r}, \quad B(r) = 1, \quad f(\theta) = 4m(1 - \cos \theta). \quad (5)$$

In order to have singularity-free solution, the parameter $m$ is chosen to be $R/8$, being $R$ the radius of the circle of the fifth dimension. Under this condition the magnetic charge of the monopole, $g$, is related with $m$ parameter by $m = g\sqrt{\pi G}$ [5]. Here $G$ denotes Newton’s gravitational constant.

Moreover, Gross and Perry called attention to the fact that although the soliton-solution described by the above expressions presents no gravitational mass, it does possess an inertial mass of order of the Planck one.

### 2.2 Banerjee *et al* Solution

In the absence of the magnetic monopole, $f(\theta)$ is taken to be zero. In this case (4) becomes diagonal. The solutions found by Banerjee *et al* to the Einstein equations in the presence of the system described by (1) were obtained in the region outside the global monopole, where, approximately, $f(r) \simeq 1$. Doing this, they were able to find a family of solutions:

$$B(r) = \left(\alpha^2 - \frac{2GM}{r}\right)^a, \quad A(r) = \left(\alpha^2 - \frac{2GM}{r}\right)^{-(a+b)},$$

$$C(r) = \left(\alpha^2 - \frac{2GM}{r}\right)^{(1-a-b)}, \quad D(r) = \left(\alpha^2 - \frac{2GM}{r}\right)^b, \quad (6)$$

where $\mathcal{M}$ is a constant of integration, and $a$ and $b$ are two dimensionless parameters which obey the consistency condition $a^2 + ab + b^2 = 1$. So, differently from the solution found by Barriola and Vilenkin in a four-dimensional spacetime, in this formalism the uniqueness criteria to the
solution is lost. For the particular choice \( a = 1 \) and \( b = 0 \), the above solution can be understood as a five-dimensional extension of the Barriola and Vilenkin one.

### 2.3 The Model

After the above review about these two models we shall present our model. We shall consider the physical system which presents an Abelian magnetic monopole in a five-dimensional global monopole manifold. We shall see that due to the presence of latter system, the Ricci-flat condition is not longer fulfilled. In order to proceed with our investigation, we first present the non-vanishing components of the energy-momentum tensor associated with the global monopole Lagrangian density (1), considered as an external source:

\[
T_{00} = -\eta^2 B(r) \left[ \frac{f'^2(r)}{2A(r)} + \frac{f'^2(r)}{r^2C(r)} + \frac{\lambda \eta^2}{4} (f'^2(r) - 1)^2 \right] ,
\]

\[
T_{11} = -\eta^2 \left[ \frac{f'^2(r)}{2} - A(r) \frac{f'^2(r)}{r^2C(r)} - \frac{\lambda \eta^2}{4} A(r)(f'^2(r) - 1)^2 \right] ,
\]

\[
T_{22} = \eta^2 r^2 C(r) \left[ \frac{f'^2(r)}{2A(r)} + \frac{\lambda \eta^2}{4} (f'^2(r) - 1)^2 \right] ,
\]

\[
T_{33} = \eta^2 r^2 C(r) \sin^2 \theta \left[ \frac{f'^2(r)}{2A(r)} + \frac{\lambda \eta^2}{4} (f'^2(r) - 1)^2 \right]
\]
\[+ \eta^2 D(r) f'^2(\theta) \left[ \frac{f'^2(r)}{2A(r)} + \frac{f'^2(r)}{r^2C(r)} + \frac{\lambda \eta^2}{4} (f'^2(r) - 1)^2 \right] ,
\]

\[
T_{34} = T_{43} = \eta^2 D(r) f(\theta) \left[ \frac{f'^2(r)}{2A(r)} + \frac{f'^2(r)}{r^2C(r)} + \frac{\lambda \eta^2}{4} (f'^2(r) - 1)^2 \right] ,
\]

\[
T_{44} = \eta^2 D(r) \left[ \frac{f'^2(r)}{2A(r)} + \frac{f'^2(r)}{r^2C(r)} + \frac{\lambda \eta^2}{4} (f'^2(r) - 1)^2 \right] . \tag{7}
\]

The equation for the Higgs field in the metric (4) gives rise to the following radial differential equation:

\[
\frac{1}{A(r)} f''(r) + \left[ \frac{2}{A(r)r} + \frac{1}{2B(r)C^2(r)D(r)} \left( \frac{B(r)C^2(r)D(r)}{A(r)} \right)' \right] f'(r)
\]
\[ - \frac{2}{C(r)r^2} f(r) - \lambda \eta^2 f(r)(f'^2(r) - 1) = 0 . \tag{8}
\]
In four-dimensional flat spacetime the similar differential equation has no analytical solution; however it is shown [9] that for radial distance \( r \) larger than the monopole’s core, \( \delta \simeq \lambda^{-1/2}\eta^{-1} \), \( f(r) \simeq 1 \), and for \( r \to 0 \), \( f(0) = 0 \). We observe that in this present context the same boundary conditions can be applied to the function \( f(r) \). So in the follows we shall analyse the complete system in the region outside the global monopole’s core. In this way the components for the energy-momentum tensor become much simpler. They read:

\[
\begin{align*}
T_{00} &= -\eta^2 \frac{B(r)}{r^2C(r)}, \\
T_{11} &= \eta^2 \frac{A(r)}{r^2C(r)}, \\
T_{22} &= 0, \\
T_{33} &= \eta^2 \frac{f^2(\theta)D(r)}{r^2C(r)}, \\
T_{34} &= \eta^2 \frac{D(r)f(\theta)}{r^2C(r)}, \\
T_{44} &= \eta^2 \frac{D(r)}{r^2C(r)}. 
\end{align*}
\]

(9)

From the five-dimensional Einstein equation,

\[
\hat{R}_{AB} = 8\pi G_K \left( \hat{T}_{AB} - \frac{\hat{g}_{AB}}{3} \hat{T} \right),
\]

(10)

where \( G_K \) is the five-dimensional gravitational constant, we find that the only non-vanishing components of the Ricci tensor are:

\[
R_{22} = \alpha^2 - 1, \\
R_{33} = (\alpha^2 - 1) \sin^2 \theta. 
\]

(11)

Here \( \alpha^2 = 1 - 8\pi G_K \eta^2 \). However it is possible to relate the five-dimensional gravitational coupling constant with the Newton’s one \( G \) by

\[
G_K = 2\pi RG. 
\]

(12)

Defining the energy scale \( \eta \) in the five-dimensional spacetime as the ordinary one in four-dimensions divided by the \( \sqrt{2\pi R} \), we re-obtain for the parameter \( \alpha \) the same expression as given before to the Barriola and Vilenkin model.

Now, to complete this analysis, we have to find solutions for the functions \( B(r), A(r), C(r), D(r) \) and \( f(\theta) \) compatible with the above results. Because we want that our expression would reproduce a four dimensional generalization of the self-dual Euclidean Taub-NUT solution in presence of a global monopole, we must have \( B(r) = 1 \). Moreover, because it must approach asymptotically to the five-dimensional extension of the Barriola and Vilenkin solution found in Ref. [10], \( \alpha^2 \) should be a multiplicative factor.
Although the existence of the magnetic monopole depends on the topology of the spacetime, in this case, supported by previous analysis about composite topological defect [11], we can infer that the presence of the global monopole does not modify the configuration of the magnetic monopole. Taking all these informations in consideration we find:

$$A(r) = (\alpha^2)^{-\frac{1}{2}} \left(1 + \frac{4m}{\alpha r}\right),$$

$$C(r) = (\alpha^2)^{-\frac{1}{2}} \left(1 + \frac{4m}{\alpha r}\right),$$

$$D(r) = (\alpha^2)^{-\frac{1}{2}} \left(1 + \frac{4m}{\alpha r}\right)^{-1},$$

$$f(\theta) = 4m(1 - \cos \theta)$$

for any value of the parameter $a$. At this point we could think that our solutions, as the Banerjee et al ones, represent a family of independent solutions. However, this is not true: by a global scale transformation on the metric tensor, $\hat{g}_{AB} \rightarrow \alpha^{-1}\hat{g}_{AB}$ and redefining the time coordinate appropriately, the following line element is obtained:

$$ds^2 = -dt^2 + V(r) \left(\frac{dr^2}{\alpha^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2)\right) + V(r)^{-1}(d\Psi + 4m(1 - \cos \theta)d\phi)^2$$

with

$$V(r) = 1 + \frac{4m}{\alpha r}. \tag{15}$$

This solution presents the most relevant properties associated with the global and magnetic monopoles: i) it can be considered as a five-dimensional extension of the Barriola and Vilenkin solution, in the sense that the space section asymptotically presents a solid angle $\Omega = 4\pi \alpha^2$, consequently smaller than the ordinary one, and ii) also presents an Abelian magnetic monopole.

As it was pointed out by Gross and Perry in Ref. [5], the gauge field associated with the magnetic monopole

$$A_{\phi} = 4m(1 - \cos \theta) \tag{16}$$

\footnote{The complete set of field equations derived from (10) presents very long expressions, even considering specific ansatz to the unknown functions. This is the reason why we decided do not include this set of differential equation in our paper.}
presents a singularity at $\theta = \pi$. However, this singularity is gauge dependent if the period of the compactified coordinate $\Psi$ is equal to $16\pi m$. This is the geometric description of the Dirac quantization. Adopting this period for the extra coordinate, it is possible to provide the Wu and Yang formalism to describe the four-vector potential, $A_\mu$, associated with the Abelian magnetic monopole without line of singularity. In order to do that it is necessary to construct two overlapping regions, $R_a$ and $R_b$, which cover the whole space section of the manifold. Using spherical coordinate system, with the monopole at origin the only non-vanishing components for the vector potential are

\begin{align*}
(A_\phi)_a &= 4m(1 - \cos \theta) , \quad R_a : 0 \leq \theta < \frac{1}{2} \pi + \delta , \\
(A_\phi)_b &= -4m(1 + \cos \theta) , \quad R_b : \frac{1}{2} \pi - \delta < \theta \leq \pi , \quad (17)
\end{align*}

with $0 < \delta < \pi/2$. In the overlapping region, $R_{ab}$, the non-vanishing components are related by a gauge transformation. Using the appropriate normalization factor $[5]$, one can rewrite the above vector potential in terms of the physical one, $A^{ph}_\phi$:

$$
\sqrt{16\pi G} (A^{ph}_\phi)_a = \sqrt{16\pi G} \left[ (A^{ph}_\phi)_b + \frac{i}{e} S \partial_\phi S^{-1} \right] ,
$$

where $S = e^{2i\omega \phi}$, $\omega = -eg = -n/2$ in units $\hbar = c = 1$ and $g$ being the monopole strength. In terms of non-physical vector potential this gauge transformation corresponds to subtract the quantity $8m$, which compensates the changing in the fifth coordinate $\Psi' = \Psi + 8m \phi$. Also we must say that the same Ricci tensors (11) are obtained for both expressions of the four-vector potential.

Before to finish this section three important remarks about the solution should be made: i) The radial function $V(r)$ in the line element (14) differs from the similar one found by Gross and Perry by $\alpha$ factor multiplying the radial coordinate in the denominator. The obtained magnetic field is

$$
\vec{B} = \vec{\nabla} \times \vec{A} = \frac{1}{V(r)} \frac{4m}{r^2} \hat{r} = -\frac{1}{\sqrt{V}} \vec{\nabla} V(r) ,
$$

which asymptotically gives rise to the usual Dirac magnetic monopole. Moreover, we can see by calculating the total magnetic flux on a spherical
surface concentric with the monopole, that $\Phi_B = 4m(4\pi)$. \(\text{ii})\) Changing the sign of $m$ in $V(r)$, we obtain another solution of the field equations. \(\text{iii})\) Finally, we want to emphasize that the solutions found for the components of the metric tensor are valid only in the region outside the global monopole.

3. Analysis of the Motion of a Charged Particle in the Manifold

As we have already said, the line element (14) is the five-dimensional extension of the Barriola and Vilenkin solution in the presence of an Abelian magnetic monopole. The classical motion of a test massive particle in this manifold can be analysed by a Lagrangian obtained by differentiating this quantity with respect to some affine parameter $\xi$:

$$L = -\dot{t}^2 + \left(1 + \frac{4m}{\alpha r}\right) \left(\frac{\dot{r}^2}{\alpha^2} + r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2)\right) + \left(1 + \frac{4m}{\alpha r}\right)^{-1} (\dot{\Psi} + 4m(1 - \cos \theta)\dot{\phi})^2.$$  \hspace{1cm} (20)

Because the above Lagrangian does not depend explicitly on the coordinates $t$, $\Psi$ and $\phi$, three constants of motion can be promptly identified:

$$\dot{t} = a , \hspace{1cm} (21)$$

$$V^{-1}(r) \left[\dot{\Psi} + 4m(1 - \cos \theta)\dot{\phi}\right] = \kappa , \hspace{1cm} (22)$$

and

$$V(r) r^2 \sin^2 \theta \dot{\phi} + V^{-1}(r) \left[\dot{\Psi} + 4m(1 - \cos \theta)\dot{\phi}\right] 4m(1 - \cos \theta) = h . \hspace{1cm} (23)$$

The equation (23) can be written in a simpler form if we use the definition of the constant $\kappa$ given in (22). Adopting the notation given in the paper by Gross and Perry, this constant is the ratio of the charge of the test particle to its mass: $\kappa = q/M$. Including in (23) the definition for the physical magnetic charge and recognizing $q\sqrt{16\pi G}$ as the physical charge of the particle, we can identify the $z$–component of the conserved total angular momentum associated with a charged particle in this manifold as:

$$J_z = \frac{M}{2} \hbar = V(r) Mr^2 \sin^2 \theta \dot{\phi} + \omega \cos \theta , \hspace{1cm} (24)$$
where \( \omega = -ge \). (In the deduction of the above expression we discarded the constant \( 4m\kappa \) in (23).) Moreover the classical equation of motions to the polar and radial variables can be obtained, respectively, by the Euler-Lagrange formalism and by imposing that the Lagrangian above is a constant \( \epsilon \). This constant can be 0, 1 and \(-1\), respectively, if the geodesic associated with the motion of the particle is null, for massless particle, spacelike and timelike. Finally these equations are:

\[
\ddot{\theta} + \left[ \frac{2}{r} + \frac{V'(r)}{V(r)} \right] \dot{r} \dot{\theta} - \sin \theta \cos \theta \dot{\phi}^2 - \frac{4m\kappa \sin \theta}{V(r)r^2} \dot{\phi} = 0 ,
\]

and

\[
\frac{V(r)}{a^2} r^2 + V(r)r^2\dot{\theta}^2 + V(r)r^2 \sin^2 \theta \dot{\phi}^2 + \kappa^2 V(r) = \epsilon + a^2 .
\]

Schwinger et al [15] shown many years ago that the conserved total angular momentum, associated with an electric charged particle in the presence of a magnetic monopole, is given by

\[
\vec{J} = \vec{l} + \omega \hat{r} ,
\]

with \( \vec{l} \) being the ordinary orbital angular momentum.\(^2\) Because \( \vec{J} \cdot \vec{r} = \omega \), the motion of the particle is confined to a cone of half-polar angle \( \theta_0 \) given by:

\[
\cot \theta_0 = \frac{|\omega|}{l}.
\]

This means that a particular choice of coordinate system has been adopted, and in this system the direction of the \( z \)-axis is parallel to the vector \( \vec{J} \). The ordinary angular momentum vector \( \vec{l} \) is given in terms of the unity vector \( \hat{\theta} \) only: \( \vec{l} = -l\hat{\theta} \). In the present case that we are analysing we can observe that \( \theta = \text{const} \) is solution of (25) providing

\[
\dot{\phi} = -\frac{4mq}{MV(r)r^2 \cos \theta} = \frac{\omega}{MV(r)r^2 \cos \theta} .
\]

This result is compatible with what we expected in the sense that the motion of the particle here is also constrained to a cone. (In particular

\(^2\)In fact it was H. Poincaré [16] who first investigate the classical motion of an electron in the presence of a magnetic pole. J. Schwinger and collaborators generalized this analysis to two dyons.
for $r \to \infty$, $V(r) \to 1$, so the above expression reproduces the angular velocity associated with a charged particle in a flat spacetime in the presence of a magnetic monopole.) Moreover, by making a specific choice for the coordinate system we can infer, from (29), that the first term on the right hand side of (24) corresponds to the conserved $z-$component of the ordinary angular momentum in this manifold and consequently

$$
\dot{\phi} = \frac{l}{MV(r)r^2 \sin \theta}.
$$

(30)

Finally the equation of motion relating the radial coordinate with the azimuthal angle can be obtained combining (26) and (30) as

$$
\frac{r^2}{\dot{\phi}^2} = \left( \frac{dr}{d\phi} \right)^2 = \frac{\alpha^2 \sin^2 \theta V(r) r^4}{l^2} \left[ (\epsilon + a^2) M^2 - q^2 V(r) \right] - \alpha^2 \sin^2 \theta r^2.
$$

(31)

Defining a new variable $u = 1/r$ we can express the above equation in a simpler form:

$$
\left( \frac{du}{d\phi} \right)^2 = A - Bu - Cu^2,
$$

(32)

with

$$
A = \frac{\alpha^2 M^2}{J^2} \left[ (\epsilon + a^2) - \frac{q^2 M^2}{M^2} \right],
$$

(33)

$$
B = \frac{4m \alpha M^2}{J^2} \left[ (\epsilon + a^2) - 2 \frac{q^2 M^2}{M^2} \right],
$$

(34)

and

$$
C = \frac{\omega^2 + \alpha^2 l^2}{J^2}.
$$

(35)

Admitting that the solution of (32) has the form:

$$
u(\phi) = D + E \cos(\lambda \phi),
$$

(36)

we found that the constants are given by:

$$
D = -\frac{2mM^2 \alpha}{\omega^2 + \alpha^2 l^2} \left[ \epsilon + a^2 - 2 \frac{q^2 M^2}{M^2} \right],
$$

(37)
\[ E^2 = \frac{4m^2M^4\alpha^2}{(\omega^2 + \alpha^2l^2)^2} \left[ \epsilon + a^2 - 2\frac{q^2}{M^2} \right]^2 + \frac{\alpha^2M^2}{\omega^2 + \alpha^2l^2} \left[ \epsilon + a^2 - \frac{q^2}{M^2} \right] \]  
\hspace*{1cm} (38)

and

\[ \lambda^2 = \frac{\omega^2 + \alpha^2l^2}{J^2}, \]  
\hspace*{1cm} (39)

which is smaller than unity.

Finally in order to have trajectories equation unbounded from below, i.e., that admit that \( r \) goes to infinity, we must have \(|E| \geq |D|\), which imply \( \epsilon + a^2 - q^2/M^2 \geq 0 \). So this equation of motion corresponds to the movement of a test particle constrained to a cone, where its radial coordinate increases without limit.

4. Concluding Remarks

In this work we have presented an exact solutions of five-dimensional Einstein equation which admits a magnetic monopole in a point-like global monopole spacetime. Our solution is a generalization of the previous ones found by Gross and Perry and Barnerjee et all. The latter in the \( M = 0 \) limit.

Although the solution presented by Gross and Perry corresponds to a point-like configuration of magnetic monopole, it is a regular solution in the sense that it has a finite inertial mass. Our solution, on the other hand, is valid only in the region outside the global monopole. Admitting a point-like configuration to the latter, we can observe that because \( g_{00} = -1 \), the gravitational mass associated with our solution is zero, although it possesses a finite inertial mass.

As we have said our solutions to the components of the metric tensor and the radial function \( V(r) \), were obtained in the region outside the global monopole’s core. In a pure global monopole system in four-dimensional spacetime, the exact solution for the equations of motion, considering the region near the monopole’s core, can only be obtained numerically [12, 13]. So we do not expect to find for this more general system analytical solutions either. Moreover, as to the global monopole system, numerical calculation indicates the existence of a small negative gravitational mass [12] to this object. On the other hand, considering also the presence of a Non-Abelian magnetic monopole in the system, positive effective gravitational mass to this composite topological object has been found [11, 14]. So, these aspects
suggest similar properties to this present composite monopole. These are points to be investigated in the future.

In spite of our solution to the metric tensor (13) presents an explicit dependence on the parameter $a$, we cannot say that we have found a family of independent solutions. The numerical factors which appear in those components can be gauged away by a redefinition of a coordinate system. Consequently all the invariants of the respective manifold do not depend on it.

By a direct calculation we found that the solid angle associated with the space section of (14) presents a radial dependence. This fact is a consequence of the long range effect of the radial function $V(r) = 1 + 4m/\alpha r$. The solid angle is:

$$\Omega = 4\pi \alpha^2 \frac{1}{\left[1 + \frac{2m}{\alpha r} V^{-1/2}(r) \ln(1 + \frac{\alpha r}{2m} (1 + V^{1/2}(r)))\right]^2},$$

which asymptotically reproduces the well known result found in the pure global monopole spacetime. So at spatial infinity (14) approaches to the global monopole metric solution (3) in the presence of the Dirac magnetic monopole.

The analysis of the classical trajectories of a massive charged particles in this manifold has also been performed. We observe that, as in the flat three-dimensional case, this particle has its motion confined to a cone with radial coordinate increasing without limit, indicating that there is no bound states.

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