On a criterion for the determinate–indeterminate dichotomy of the moment problem

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Received: 17 January 2023 / Accepted: 9 September 2023 / Published online: 4 October 2023
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Abstract
When the classical Hamburger moment problem has solutions, it has either exactly one solution or infinitely many solutions. Correspondingly, the moment problem is said to be either determinate or indeterminate. In terms of Jacobi operators, this dichotomy translates into the operator being either selfadjoint or symmetric nonselfadjoint. In this work, we present a new criterion for the determinate–indeterminate classification which hinges on bases of representation (in Akhiezer–Glazman terminology) for Jacobi operators so that the corresponding matrices have a certain structure.

Keywords  Hamburger moment problem · Bases of matrix representation · Jacobi operators

Mathematics Subject Classification  47B32 · 47B36 · 30E05

1 Introduction

The classical Hamburger moment problem has played a central role in the development of modern mathematical analysis. It consists in finding a Borel measure $\mu$ such that

$$s_k = \int_{\mathbb{R}} t^k d\mu(t). \quad (1)$$

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for a given real sequence of numbers \( \{ s_k \}_{k=0}^{\infty} \). To exclude from our consideration the trivial, degenerate solutions to the moment problem, let us always assume that the solutions have infinite support (which could be bounded or unbounded). The moment problem, although deceptively simple, leads to fundamental questions in various fields of analysis and reveals unexpected connections between seemingly unrelated mathematical constructions and notions. When the classical Hamburger moment problem has solutions, it has either exactly one solution or infinitely many solutions. In the first case, the moment problem is said to be determinate, while in the second case it is said to be indeterminate. This dichotomy is crucial within the moment problem theory; on the one hand, the criteria for distinguishing these cases shed light on the intricacies of the theory and the interconnections between various fields of analysis, and on the other hand, the determinate and indeterminate cases lead to two different facets of the theory.

Due to the inherent richness of the moment problem, one can approach the determinate–indeterminate dichotomy from different viewpoints using different mathematical notions and, consequently, there are numerous criteria for finding out whether the moment problem is determinate or indeterminate. There is a nonexhaustive list of these criteria at the end of Sect. 2, although this is not the main point in this section, but rather the consequences of the one-to-one correspondence between Jacobi matrices and sequences of moments for which the corresponding moment problems admit solutions. On the basis of this correspondence, the determinate–indeterminate dichotomy is transformed into the selfadjoint-nonselfadjoint dichotomy for Jacobi operators. For the passing from matrices to operators, the concept of matrix representation for unbounded closed symmetric operators [2, Sec. 47] is essential.

The criterion presented in this paper is actually an if-and-only-if criterion for resolving the selfadjoint-nonselfadjoint dichotomy for Jacobi operators, however it does not rely on the operator theory techniques nor on the function theoretic methods for establishing selfadjointness or nonselfadjointness (cf. [1, Chs. 3 and 4], [3, Ch. 7 Sec. 1], [25] and [27, Ch. 2]). Instead, this paper relies on the so-called bases of representation for Jacobi operators (Definition 1). The main result can be stated exclusively in terms of these bases, namely:

- If for a Jacobi operator there is more than one basis of representation so that the corresponding matrix is a Jacobi matrix, then the operator is selfadjoint.
- If for a Jacobi operator there is only one basis of representation so that the corresponding matrix is a Jacobi matrix, then the operator is nonselfadjoint.

Due to the relationship between sequences of moments, Jacobi operators, matrices of representation, and measures (see Sect. 2 below), one can paraphrase this criterion as follows:

A moment problem is indeterminate if and only if there is only one basis of representation for the corresponding Jacobi operator so that the matrix representation of the operator with respect to this basis is a Jacobi matrix.

In proving this statement, this work shed light on some new connections between solutions to a moment problem and bases of representation for the corresponding operator, in particular, necessary and sufficient conditions for a basis to be a basis of representation for a Jacobi operator are provided. Furthermore, in the determinate
case, it is shown how to construct a basis of representation so that the corresponding measure has arbitrary index of determinacy.

Let us outline how the material of this work is presented. Section 2 introduces the main objects and the corresponding notation. This section is expository and presents classical results on the Hamburger moment problem and its relation to the theory of Jacobi matrices. Section 3 is a review of the theory of selfadjoint simple operators and tackles the problem of constructing bases of matrix representation for these operators. The index of determinacy and the connection to an algorithm to construct bases of representation for selfadjoint Jacobi operators are given in Sect. 4. Finally, Sect. 5 deals with the case of nonselfadjoint Jacobi operators. This section uses Krein’s representation theory of symmetric operators \[12–15\] and de Branges theory on Hilbert spaces of entire functions \[7\].

2 Jacobi matrices and the Hamburger moment problem

Let us introduce the notions relevant to this paper and lay out the notation. Consider a closed symmetric operator \( A \) in a Hilbert space \( \mathcal{H} \) and an orthonormal basis \( \{ \delta_k \}_{k=1}^{\infty} \) of \( \mathcal{H} \). If the domain of \( A \), denoted by \( \text{dom} \ A \), coincides with the whole space \( \mathcal{H} \) (which implies that \( A \) is bounded since we have assumed it to be closed), then the operator can be uniquely recovered from the numbers

\[
a_{kj} := \langle \delta_k, A \delta_j \rangle; \tag{2}
\]

here and henceforth, the inner product is considered to be antilinear in its first argument. If \( \text{dom} \ A \subsetneq \mathcal{H} \), then the operator \( A \) is not reconstructed uniquely from (2) even when \( \delta_k \in \text{dom} \ A \) for any \( k \in \mathbb{N} \) (\( \mathbb{N} \) denotes the set of positive integers). For this reason, one needs the following:

**Definition 1** An orthonormal basis \( \{ \delta_k \}_{k=1}^{\infty} \) is said to be a basis of representation for the closed operator \( A \) when

(a) \( \delta_k \in \text{dom} \ A \) for all \( k \in \mathbb{N} \);
(b) if there is a closed operator \( B \) such that \( B \delta_k = A \delta_k \), then \( B \supset A \).

When \( \{ \delta_k \}_{k=1}^{\infty} \) is a basis of representation for \( A \), the matrix

\[
[A] = \begin{pmatrix}
    a_{11} & a_{12} & a_{13} & a_{14} & \cdots \\
    a_{21} & a_{22} & a_{23} & a_{24} & \cdots \\
    a_{31} & a_{32} & a_{33} & a_{34} & \cdots \\
    a_{41} & a_{42} & a_{43} & a_{44} & \cdots \\
    \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \tag{3}
\]

with entries given by (2), is the matrix representation of \( A \) with respect to \( \{ \delta_k \}_{k=1}^{\infty} \).
In [2, Sec. 47, Thm. 3], it is established that any closed symmetric operator has a basis of representation. Conversely, if the matrix (3) is Hermitian and satisfies
\[ \sum_{j=1}^{\infty} |a_{jk}|^2 < +\infty, \text{ for all } k \in \mathbb{N}, \]  
then there is a unique closed symmetric operator \( A \) such that \([A]\) is its matrix representation with respect to a given orthonormal basis \( \{\delta_k\}_{k=1}^{\infty} \) of a Hilbert space \( \mathcal{H} \).

Let \( \{q_k\}_{k=1}^{\infty} \) be a sequence of real numbers and \( \{b_k\}_{k=1}^{\infty} \) be a sequence of positive numbers. An infinite matrix of the form
\[
[J] = \begin{pmatrix}
q_1 & b_1 & 0 & 0 & \cdots \\
b_1 & q_2 & b_2 & 0 & \cdots \\
0 & b_2 & q_3 & b_3 & \\
0 & 0 & b_3 & q_4 & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots
\end{pmatrix}
\]
is said to be an infinite Jacobi matrix, or more specifically a semi-infinite Jacobi matrix to emphasize that the diagonals are enumerated by \( \mathbb{N} \) rather than \( \mathbb{Z} \). Since this matrix satisfies (4), upon fixing an orthonormal basis \( \{\delta_k\}_{k=1}^{\infty} \) of a Hilbert space \( \mathcal{H} \), there is a unique closed symmetric operator \( J \), called Jacobi operator, having \([J]\) as its matrix representation with respect to \( \{\delta_k\}_{k=1}^{\infty} \). Usually, one takes \( \mathcal{H} = l_2(\mathbb{N}) \) and \( \{\delta_k\}_{k=1}^{\infty} \) being the so-called canonical basis of \( l_2(\mathbb{N}) \), i.e., \( \delta_k \) is in turn the sequence \( \{\delta_{jk}\}_{j=1}^{\infty} \), where \( \delta_{jk} \) is the Kronecker delta. Henceforth, we assume that these choices for the space and the orthonormal basis are always made.

Thus, the operator \( J \) is the closure of the operator \( J_0 \) whose domain is \( l_\text{fin}(\mathbb{N}) \) (the space of sequences with a finite number of nonzero elements) and satisfies
\[
(J_0 \phi)_1 := q_1 \phi_1 + b_1 \phi_2, \\
(J_0 \phi)_k := b_{k-1} \phi_{k-1} + q_k \phi_k + b_k \phi_{k+1}, \quad k \in \mathbb{N} \setminus \{1\},
\]
for any \( \phi \in l_\text{fin}(\mathbb{N}) \). Also, one verifies that \( J^* = J_0^* \) is the operator defined on the maximal domain, i.e.,
\[
\text{dom } J^* = \left\{ \phi \in l_2(\mathbb{N}) : \sum_{k=2}^{\infty} |b_{k-1} \phi_{k-1} + q_k \phi_k + b_k \phi_{k+1}|^2 < +\infty \right\}.
\]

By setting \( \pi_1 := 1 \), a solution to the equations
\[
z \pi_1 := q_1 \pi_1 + b_1 \pi_2, \\
z \pi_k := b_{k-1} \pi_{k-1} + q_k \pi_k + b_k \pi_{k+1}, \quad k \in \mathbb{N} \setminus \{1\}, \quad z \in \mathbb{C},
\]

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can be found uniquely by recurrence. This solution, \( \pi(z) = (\pi_k(z))_{k=1}^{\infty} \), is a sequence of polynomials of \( z \) called the polynomials of the first kind generated by \( [J] \).

**Remark 1** Since the polynomials’ coefficients are real, if \( \pi(z) \in l_2(\mathbb{N}) \), then \( \pi(\overline{z}) \in l_2(\mathbb{N}) \). Also, it follows from (7) and (8) that \( \pi(z) \in l_2(\mathbb{N}) \) if and only if \( \pi(z) \in \ker(J^* - zI) \). This means, on the one hand, that the deficiency indices of the symmetric operator \( J \) are always equal to each other, i.e., \( n_+(J) = n_-(J) \) and, on the other hand, if \( \pi(z) \in l_2(\mathbb{N}) \) for one nonreal \( z \), then this is true for any nonreal \( z \). When \( \pi(z) \in l_2(\mathbb{N}) \), the deficiency indices are equal to one because any other solution of (8) coincides with \( \pi(z) \) modulo a multiplicative constant (see [1, Ch. 4 Sec. 1.2]). Thus, either \( n_+(J) = n_-(J) = 0 \) or \( n_+(J) = n_-(J) = 1 \). Since \( J \) is closed by definition, the case when \( n_+(J) = n_-(J) = 0 \) corresponds to \( J \) being selfadjoint.

The second-order difference expression (6) (i.e., the matrix (5)) may be either in the limit point case or in the limit circle case. The asymptotic behavior of the sequence of Weyl circles determines the occurrence of one of these two possibilities since either the circles degenerate into a single point or a limit circle [1, Ch. 1 Sec. 3]. For the class of second-order differential expressions pertaining to the Sturm–Liouville operator, the same dichotomy between the limit point and limit circle cases takes place [6, Ch. 9]. Actually, the theory behind the Weyl circles originated in the context of differential equations.

It turns out that the limit point case corresponds to the selfadjoint case, i.e., \( n_+(J) = n_-(J) = 0 \), while the limit circle case occurs when \( n_+(J) = n_-(J) = 1 \). This correspondence is evident from the following expression [1, Eq. 1.21]:

\[
\left( |z - \overline{z}| \sum_{k=1}^{n} \left| \pi_k(z) \right|^2 \right)^{-1},
\]

which gives the \( n \)-th Weyl circle’s radius for \( z \in \mathbb{C} \setminus \mathbb{R} \). Indeed, by Remark 1, selfadjointness of \( J \) is equivalent to the radius vanishing as \( n \to \infty \) in (9) since \( \pi(z) \notin l_2(\mathbb{N}) \) for \( z \in \mathbb{C} \setminus \mathbb{R} \), while nonselfadjointness of \( J \) means that the limit of the sequence of radii (9) is not zero since, in this case, \( \pi(z) \in l_2(\mathbb{N}) \) for \( z \in \mathbb{C} \setminus \mathbb{R} \).

Let us now turn to the moment problem posed at the beginning of Sect. 1. A necessary and sufficient condition for a solution to the Hamburger moment problem to exist [1, Thm. 2.1.1] is that

\[
\det \begin{pmatrix} s_0 & s_1 & \cdots & s_k \\ s_1 & s_2 & \cdots & s_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_k & s_{k+1} & \cdots & s_{2k} \end{pmatrix} > 0
\]

for all \( k \in \mathbb{N} \cup \{0\} \).

For a sequence \( \{s_n\}_{n=0}^{\infty} \) satisfying (10), there is either one solution or more than one solution to the Hamburger moment problem. In the first case, the moment problem is said to be determinate, while in the second case, it is called indeterminate.
As is customary, it is assumed in this paper that the sequence of moments \( \{ s_n \}_{n=0}^{\infty} \) is normalized, i.e., \( s_0 = 1 \). This involves no loss of generality since the general case reduces to the normalized one by dividing the sequence of moments and its solution by \( s_0 \).

There is a one-to-one correspondence between Jacobi matrices (5) and normalized sequences \( \{ s_n \}_{n=0}^{\infty} \) satisfying (10) (see [1, Ch. 1]). Moreover, this bijection pairs every limit point Jacobi matrix with a sequence for which the Hamburger moment problem is determinate and every limit circle Jacobi matrix with a sequence for which the Hamburger moment problem is indeterminate [1, Thm. 2.1.2 and Cor. 2.2.4].

Let us briefly describe how the above mentioned one-to-one correspondence is realized. First, consider the starting point to be an operator \( J \) having the matrix representation (5) with respect to the orthonormal basis \( \{ \delta_k \}_{k=1}^{\infty} \). Since

\[
J \delta_1 = q_1 \delta_1 + b_1 \delta_2, \\
J \delta_k = b_{k-1} \delta_{k-1} + q_k \delta_k + b_k \delta_{k+1}, \quad k \in \mathbb{N} \setminus \{1\},
\]

it is verified that

\[
\delta_k = \pi_k(J) \delta_1. \tag{11}
\]

This means that \( \delta_1 \) is in the domain of any power of the Jacobi operator \( J \). This allows us to define the numbers

\[
s_{k-1} := \langle \delta_1, J^{k-1} \delta_1 \rangle, \quad k \in \mathbb{N}. \tag{12}
\]

If furthermore one sets

\[
\mu(\cdot) := \langle \delta_1, E(\cdot) \delta_1 \rangle, \tag{13}
\]

where \( E \) is either the spectral measure of \( J \) if it is selfadjoint or the spectral measure of a canonical selfadjoint extension\(^1\) of \( J \) otherwise, then (1) holds due to the spectral theorem. This means that the numbers given in (12) form a sequence of moments and \( \mu \), given in (13), is a solution to the moment problem. Note that, by definition, the sequence of moments is normalized.

**Remark 2** The above conclusion is complemented in the classical moment problem theory by showing, on the one hand that if \( J \) is selfadjoint, then \( \mu \) is the unique solution of the moment problem [1, Cor. 2.2.4] and, on the other hand, if \( J \subsetneq J^* \), then different canonical selfadjoint extensions give different solutions of the moment problem. Moreover, there are other solutions apart from the ones given by the canonical selfadjoint extensions (see [1, Ch. 2 Secs. 2 and 3] and [25, Thm. 4]).

Now, let the starting point be any normalized sequence of moments. In this case, it is known that one can construct from this sequence a unique Jacobi matrix using the

\(^1\) A canonical selfadjoint extensions of a symmetric operator is a selfadjoint restriction of its adjoint.
Theorem 1

1. Let $\mu$ be the moment measure of the moment problem $\{s_k\}_{k=0}^\infty$. Then $\mu$ is selfadjoint and completely determined by the measure $\mu$. It is important to bear in mind that in the definition of $M_\mu$, the measure $\mu$ is not necessarily a solution to the moment problem.

2. Let now $\mu$ be selfadjoint and let $\{s_k\}_{k=0}^\infty$ be its moment sequence. Then $\mu$ is a solution to the moment problem $\{s_k\}_{k=0}^\infty$. The Gram–Schmidt procedure applied to this sequence yields an orthonormal sequence $\{P_k(t)\}_{k=0}^\infty$. The Gram–Schmidt procedure is assumed in this paper to be defined as in [5, Ch. 2 Sec. 2 Thm. 5] which implies that $P_k$ is a polynomial of degree $k$ with positive leading coefficient. The orthonormal sequence of polynomials is uniquely determined by this property [20, Prop. 5.1]. One has the well-known three-term relation theorem (see [11, Sec. 3.1.3], [20, Prop. 5.6], [25, Pag. 92]):

Remark 3

For the Stieltjes moment problem [25, Pag. 83], the determinate/indeterminate dichotomy reduces to the existence of one/multiple nonnegative selfadjoint extensions of the corresponding Jacobi operator [25, Thms. 2 and 3.2]. This paper is not concerned with the Stieltjes moment problem.
Proposition 1 If \( \{ P_{k-1}(t) \}_{k=1}^{\infty} \) is the orthonormal sequence of polynomials defined above, then

\[
\begin{align*}
    t P_0(t) &:= q_1 P_0(t) + b_1 P_1(t), \\
    t P_k(t) &:= b_k P_{k-1}(t) + q_{k+1} P_k(t) + b_{k+1} P_{k+1}(t), \quad k \in \mathbb{N}, \ t \in \mathbb{R},
\end{align*}
\]

(14)

where the sequences \( \{ q_k \}_{k=1}^{\infty} \) and \( \{ b_k \}_{k=1}^{\infty} \) are obtained from the moments \( \{ s_{k-1} \}_{k=1}^{\infty} \) by means of the determinantal formulae mentioned above.

Remark 4 By comparing (8) with (14), one concludes that the sequence \( \{ P_{k-1}(t) \}_{k=1}^{\infty} \) given above is the sequence of polynomials of the first kind generated by the Jacobi matrix with entries given by the coefficients of (14). Moreover, \( \{ P_{k-1}(t) \}_{k=1}^{\infty} \) is an orthonormal basis of \( L_2(\mathbb{R}, \mu) \) if and only if \( \mu \) is such that the polynomials are dense in \( L_2(\mathbb{R}, \mu) \).

Definition 3 Suppose that the sequence \( \{ P_{k-1} \}_{k=1}^{\infty} \) given above is an orthonormal basis of \( L_2(\mathbb{R}, \mu) \). Define the map \( U : L_2(\mathbb{R}, \mu) \rightarrow l_2(\mathbb{N}) \) such that

\[
UP_k = \delta_k, \quad k \in \mathbb{N},
\]

where \( \{ \delta_k \}_{k=1}^{\infty} \) is the canonical basis in \( l_2(\mathbb{N}) \).

By definition, \( U \) realizes an isometric isomorphism between \( L_2(\mathbb{R}, \mu) \) and \( l_2(\mathbb{N}) \). Let \( \tilde{J} \) be the Jacobi operator whose matrix representation is the Jacobi matrix with entries given by the coefficients of (14). From the minimality properties of \( \tilde{J} \) and (14), it follows that \( U^{-1} \tilde{J} U \subset M_\mu \). Now, if \( J = J^* \), then \( U^{-1} \tilde{J} U = M_\mu \) since any selfadjoint operator is a maximal symmetric operator. In this case \( \{ P_{k-1}(t) \}_{k=1}^{\infty} \) is a basis of representation for \( M_\mu \) and its matrix representation is (5). Also, if \( J \subset J^* \), then \( M_\mu \) is a selfadjoint extension of \( U^{-1} \tilde{J} U \) and, therefore, there is a selfadjoint extension \( \tilde{J} \) of \( \tilde{J} \) such that \( U^{-1} \tilde{J} U = M_\mu \). In this case, \( \{ P_{k-1}(t) \}_{k=1}^{\infty} \) is no longer a basis of representation for \( M_\mu \) since the minimality condition (b) of Definition 1 is not satisfied.

Turning to a more particular setting, let \( \mu \) be given by (13). By what has already been said, \( \mu \) is a solution to a Hamburger moment problem. Moreover, because of (11), the polynomials are dense in \( L_2(\mathbb{R}, \mu) \).

The last two paragraph lead to the following classic result in the problem moment theory.

Proposition 2 Let \( \mu(\mathbb{R}) = 1 \). The polynomials are dense in \( L_2(\mathbb{R}, \mu) \) if and only if \( \mu \) is the measure given by (13).

Finally, still assuming that \( \mu \) is given by (13), one has that if the underlying operator \( J \) (see the text below (13)) is selfadjoint, then it is unitarily equivalent to \( M_\mu \) by the spectral theorem and, therefore, \( J = \tilde{J} \). One also arrives at this equality when \( J \subset J^* \).
3 Selfadjoint simple operators

Let $A$ be a selfadjoint operator in a separable Hilbert space $\mathcal{H}$ and $E$ be its spectral measure given by the spectral theorem. For any real Borel set $\partial$ and $h \in \mathcal{H}$, denote by
\[
\mu_h(\partial) := \langle h, E(\partial)h \rangle
\]
the corresponding nonnegative measure. Thus, the spectral theorem allows one to define the operator [5, Ch. 5 Sec. 4]
\[
\phi(A) := \int_{\mathbb{R}} \phi dE, \quad \text{dom } \phi(A) := \{ h \in \mathcal{H} : \phi \in L_2(\mathbb{R}, \mu_h) \}.
\]

The concepts given in Definitions 4 and 5 below play an important role henceforth in this paper and can be found in [2, Sec. 69].

**Definition 4** An element $g \in \mathcal{H}$ is called a generating element of the selfadjoint operator $A$ if the span over all Borel sets $\partial \subset \mathbb{R}$ of $E(\partial)g$ is dense in $\mathcal{H}$. The operator $A$ is said to be simple when it has a generating element.

For any simple operator $A$ and any of its generating elements $g$, there is a unitary map $\Psi_g$ from $L_2(\mathbb{R}, \mu_g)$ onto $\mathcal{H}$ given by
\[
\phi \mapsto \phi(A)g
\]
such that the operator of multiplication $M_{\mu_g}$ (see Definition 2) is transformed into the operator $A$. The unitary map $\Psi_g$ realizes the canonical representation of the simple operator $A$ with respect to $g$ [2, Sec. 69 Thm. 2].

For any Borel measure $\mu$, the operator of multiplication $M_\mu$ is a selfadjoint simple operator. Any function $\eta \in L_2(\mathbb{R}, \mu)$ such that $\eta(t) \neq 0$ for $\mu$–a.e. $t$ is a generating element of $M_\mu$.

**Definition 5** A vector $f$ is a cyclic vector of $A$ when $f \in \text{dom } A^k$ for all $k \in \mathbb{N}$ and
\[
\text{clos span}_{k \in \mathbb{N} \cup \{0\}} A^k f = \mathcal{H}.
\]

A cyclic vector is a generating element [2, Sec. 69 Thm. 3], but the converse is not necessarily true. However, one can always construct a cyclic vector from a generating element. This is done below.

By Definition 5, a function $\eta$ is a cyclic vector of the operator of $M_\mu$ if only if
\[
\text{clos span}_{k \in \mathbb{N} \cup \{0\}} t^k \eta(t) = L_2(\mathbb{R}, \mu).
\]

Therefore, a straightforward consequence of the canonical representation of simple operators is the following lemma.
Lemma 3 Assume $\mu = \mu_g$, with $g$ being a generating element of a simple operator $A$. For the vector $\eta(A)g$ to be a cyclic vector of $A$ it is necessary and sufficient that $\eta$ satisfies (17).

The next statements are used to establish results pertaining to the existence of cyclic vectors. They are based on a reasoning used to prove [1, Thm. 4.2.3]. Although these results are known, their proofs are given below for the sake of completeness.

Lemma 4 Fix a $\sigma$-finite Borel measure $\mu$ on the real line and a function $f$ in the space $L_2(\mathbb{R}, \mu)$. Define

$$G_C(t) := \int_{-\infty}^{t} f(s) d\mu(s) + C, \quad C \in \mathbb{C}. \quad (18)$$

There exist a constant $C_0 \in \mathbb{C}$ such that, if

$$\int_{\mathbb{R}} t^k e^{-\frac{1}{2}t^2} dG_C(t) = 0 \quad (19)$$

for all $k \in \mathbb{N} \cup \{0\}$, then $G_C(t) = 0$ for a.e. $t$ in $\mathbb{R}$.

Proof If one defines

$$C_0 := \frac{-1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left( \int_{-\infty}^{t} f(s) d\mu(s) \right) e^{-\frac{1}{2}t^2} dt,$$

then

$$\int_{\mathbb{R}} G_C(t)e^{-\frac{1}{2}t^2} dt = 0. \quad (20)$$

Integrating (19) by parts, one arrives at

$$\int_{\mathbb{R}} \left( kt^k - t^{k+1} \right) G_C(t)e^{-\frac{1}{2}t^2} dt = 0. \quad (21)$$

Substituting $k = 0$ in this equation, one obtains

$$\int_{\mathbb{R}} G_C(t)te^{-\frac{1}{2}t^2} dt = 0. \quad (22)$$

Using (20) and (22), it follows from (21) by recurrence that

$$\int_{\mathbb{R}} G_C(t)t^k e^{-\frac{1}{2}t^2} dt = 0, \quad \forall k \in \mathbb{N} \cup \{0\}. \quad (23)$$

By the closure of the Chebyshev–Hermite functions in $L_2(\mathbb{R})$ (see [26, Thm. 5.7.1] and [2, Sec. 11.C]), one concludes from (23) that $G_C(t) = 0$ for a.e. $t \in \mathbb{R}$. \qed

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Lemma 5 Let \( \mu \) be an arbitrary finite Borel measure. If \( \eta(t) = \exp(-\alpha t^2) \) with \( \alpha \geq 1/2 \), then \( \eta \) satisfies (17).

Proof Since \( \mu \) is finite, \( t^k \eta(t) \) is in \( L_2(\mathbb{R}, \mu) \) for any \( k \in \mathbb{N} \cup \{0\} \). Suppose that \( \phi \) in \( L_2(\mathbb{R}, \mu) \) is orthogonal to all functions \( t^k \eta(t) \), i.e., for all \( k \in \mathbb{N} \cup \{0\} \),

\[
0 = \int_{\mathbb{R}} \phi(t) t^k e^{-\alpha t^2} \, d\mu(t) = \int_{\mathbb{R}} t^k e^{-\frac{1}{2}t^2} \, dG_C(t),
\]

where

\[
G_C(t) = \int_{-\infty}^{t} \phi(s) e^{-(\alpha - \frac{1}{2})s^2} \, d\mu(s) + C
\]

with \( C \) being an arbitrary constant. By Lemma 4, there is a constant \( C_0 \) such that \( G_{C_0}(t) = 0 \) for a.e. \( t \in \mathbb{R} \). Thus,

\[
\|\phi\|^2 = \int_{\mathbb{R}} e^{(\alpha - \frac{1}{2})t^2} \phi(t) \, dG_{C_0}(t) = 0.
\]

\( \square \)

The conclusion of Lemma 5 motivates the following definition:

Definition 6 Let \( g \) be a generating element for the selfadjoint operator \( A \). For any \( \alpha \geq 0 \), define

\[
\eta(\alpha, g) := \exp(-\alpha A^2)g.
\]

In this paper, the vector \( \eta(\alpha, g) \) is referred to as the Stone vector of order \( \alpha \) obtained from the generating element \( g \).

The combination of Lemmas 3 and 5 yields the following assertion which is the first step in generalizing one of Stone’s classical results (see [1, Thm. 4.2.3]).

Corollary 6 For any generating element \( g \) of a simple selfadjoint operator \( A \), any Stone vector \( \eta(\alpha, g) \) is a cyclic vector of \( A \) for all \( \alpha \geq 1/2 \).

Remark 5 Let \( J \) be the operator whose matrix representation is (5) with respect to the canonical basis \( \{\delta_k\}_{k=1}^{\infty} \). If \( J \) is selfadjoint, then it follows from (11) that \( J \) is simple and \( \delta_1 \) is a cyclic vector of it. If \( J \nsubseteq J^* \), then \( \delta_1 \) is a cyclic vector for each of the selfadjoint extensions of \( J \) (and, therefore, each selfadjoint extension is simple).

The next proposition amounts, in a certain sense, to the converse of the assertion in the preceding remark.
Proposition 7 Let $A$ be a simple operator and $\delta$ a cyclic vector of it. Construct the sequence $\{\delta_k\}_{k=1}^{\infty}$ by applying the Gram–Schmidt procedure to the sequence $\{A^{k-1}\delta\}_{k=1}^{\infty}$. If $B$ is the minimal closed operator such that $B\delta_k = A\delta_k$ for all $k \in \mathbb{N}$ (cf. Definition 1), then the matrix representation of $B$ with respect to $\{\delta_k\}_{k=1}^{\infty}$ is a semi-infinite Jacobi matrix (see (5)).

Proof First note that, by the Gram–Schmidt algorithm (see [5, Ch. 2 Sec. 2 Thm. 5]), one has $\delta_1 = \delta/\|\delta\|$, so $\delta_1$ is a normalized cyclic vector. Taking this into account, the proof reduces to a well known assertion [11, Sec. 3.1.3] on orthogonal polynomials by means of the canonical representation of $A$ with respect to $\delta_1$. Indeed, using the map introduced in (16), one has

$$\Psi_{\delta_1}^*(A^{k-1}\delta_1) = t^{k-1}$$

for any $k \in \mathbb{N}$. The unitarity of the map given in (16) and Definition 5 imply that the sequence $\{t^{k-1}\}_{k=1}^{\infty}$ is total in $L_2(\mathbb{R}, \mu_{\delta_1})$. Therefore, the Gram–Schmidt procedure applied to $\{t^{k-1}\}_{k=1}^{\infty}$ yields an orthonormal basis $\{P_{t^{k-1}}\}_{k \in \mathbb{N}}$ in $L_2(\mathbb{R}, \mu_{\delta_1})$ (see the end of Sect. 2). Clearly, $\delta_k = \Psi_{\delta_1} P_{t^{k-1}}$. Therefore, on the basis of Proposition 1, one concludes that the numbers

$$a_{jk} := \langle \delta_j, A\delta_k \rangle = \langle P_{t^{k-1}} t P_{t^{k-1}} \rangle_{L_2(\mathbb{R}, \mu_{\delta_1})}, \quad j, k \in \mathbb{N}, \quad (24)$$

generate a semi-infinite Jacobi matrix. To finish the proof notice that, by Definition 1, $B$ is the operator whose matrix representation has the entries (24).

Remark 6 In the assertion of Proposition 7, it could be that $B \subsetneq A$, i.e., the orthonormal basis obtained from the Gram–Schmidt procedure applied to $\{A^{k-1}\delta\}_{k=1}^{\infty}$ is not necessarily a basis of representation for $A$. An example of this has already appeared at the end of Sect. 2. Indeed, let $J$ be the operator whose matrix representation is (5) with respect to the canonical basis $\{\delta_k\}_{k=1}^{\infty}$ and assume that $J \subsetneq J^*$. By Remark 5, $\delta_1$ is a cyclic vector of $J$, a fixed selfadjoint extension of $J$. Moreover, it follows from (11) and Remark 4 that the basis $\{\delta_k\}_{k=1}^{\infty}$ is obtained from the Gram–Schmidt procedure applied to $\{J^{k}\delta_1\}$ since $J \supset J$. Note that $\{\delta_k\}_{k=1}^{\infty}$ is the basis of representation for $J$, but not for $J$.

The next assertion is a generalization of a classical result by Stone on simple operators (see [1, Thm. 4.2.3]).

Proposition 8 For any simple selfadjoint operator, there is an uncountable set of bases of matrix representation such that the corresponding matrix representation of the operator with respect to each of the bases is a Jacobi matrix.

Proof Let $A$ be a simple operator and $g$ a generating element of it. If $\alpha \geq \frac{1}{2}$, then $\eta(\alpha, g)$ given in Definition 6 is a cyclic vector of $A$. Due to Proposition 7, if $\{\delta_k(\alpha, g)\}_{k=1}^{\infty}$ is the orthonormal basis obtained from applying the Gram–Schmidt procedure to the sequence $\{A^{k-1}\eta(\alpha, g)\}_{k=1}^{\infty}$, then

$$\{\delta_j(\alpha, g), A\delta_k(\alpha, g)\}$$
is a Jacobi matrix which will be denoted by \([A](\alpha, g)\).

It remains to prove that \([A](\alpha, g)\) is the matrix representation of \(A\) with respect to the orthonormal basis \(\{\delta_k(\alpha, g)\}_{k=1}^{\infty}\). According to Definition 1, this boils down to showing that \(A\) is the minimal closed operator associated with the matrix \([A](\alpha, g)\).

Let \(B\) be the operator whose matrix representation is \([A](\alpha, g)\) (on account of what is said in the paragraph below (5) such operator is univocally determined by the matrix and this operator is symmetric). Assume that \(h := \phi(A)g\) is orthogonal to \((B - iI)\delta_k(\alpha, g)\) for all \(k \in \mathbb{N}\), then

\[
0 = \langle h, (B - iI)\delta_k(\alpha, g) \rangle = \langle h, (A - iI)\delta_k(\alpha, g) \rangle
\]

\[
= \left\{ \phi(A)g, (A - iI)P_{k-1}(A)e^{-\alpha A^2}g \right\}
\]

\[
= \int_{\mathbb{R}} \overline{\phi(t)}(t - i)P_{k-1}(t)e^{-\alpha t^2}d\mu_g(t),
\]

where the second equality holds since \(B \subset A\). In the third equality, one uses Definition 6 and the fact that \(\delta_k(\alpha, g) = \Psi_{\eta(\alpha, g)}(P_{k-1})\) (see the proof of Proposition 7). In the last equality, one recurs to the isometric property of \(\Psi_g\). Thus, for any \(k \in \mathbb{N}\), one has

\[
0 = \int_{\mathbb{R}} \frac{\overline{\phi(t)}}{t + i}P_{k-1}(t)e^{-\alpha t^2}d\mu_g(t)
\]

\[
= \int_{\mathbb{R}} \left| t + i \right|^2 e^{-(\alpha - \frac{1}{2})t^2} |\phi(t)|^2 d\mu_g(t),
\]

This implies that \(\|\phi\|_{L^2(\mathbb{R}, \mu_g)} = 0\). Thus, one concludes that the deficiency space of \(B\) on the upper half-plane is trivial, and therefore, \(B\) is maximal which, in turn, means that it does not have proper symmetric extensions. \(\square\)

**Remark 7** For any generating element \(g\) of \(A\) and \(\alpha \geq \frac{1}{2}\), the definition of Stone vectors \(\eta(\alpha, g)\) by means of a Gaussian function guarantees not only cyclicity, but also the fact that \(\{\delta_k(\alpha, g)\}_{k=1}^{\infty}\) is a basis of representation for \(A\) (cf. Remark 6).

The following assertion gives necessary and sufficient conditions for a cyclic vector \(\delta\) of \(A\) to generate, through the Gram–Schmidt procedure applied to the sequence \(\{A^{k-1}\delta\}_{k=1}^{\infty}\), a basis of representation for \(A\).
Proposition 9 Let $A$ be a simple operator in $\mathcal{H}$ and $\delta$ a cyclic vector of it. The Gram–Schmidt procedure applied to the sequence $\{A^{k-1}\delta\}_{k=1}^{\infty}$ yields a basis of representation for $A$ if and only if

$$\text{clos span}_{k \in \mathbb{N}} \{(A - iI)A^{k-1}\delta\} = \mathcal{H}. \quad (25)$$

Proof Let $\{\delta_k\}_{k=1}^{\infty}$ be the orthonormal basis obtained by the Gram–Schmidt procedure applied to the sequence $\{A^{k-1}\delta\}_{k=1}^{\infty}$. Thus, $\delta_k = P_{k-1}(A)\delta$, where $P_k$ is a polynomial of degree $k$ (see the proof of Proposition 7). Denote by $B$ the minimal closed operator so that $B\delta_k = A\delta_k$. Assume first that (25) holds. If the vector $h$ is such that $\langle h, (B - iI)\delta_k \rangle$ vanishes for all $k \in \mathbb{N}$, then

$$\langle h, (A - iI)A^{k-1}\delta \rangle = 0, \quad \forall k \in \mathbb{N}. \quad (26)$$

Therefore, it follows from (25) and (26) that $h = 0$. Since $\text{ran}(B - iI)$ contains $\text{span}_{k \in \mathbb{N}}(B - iI)\delta_k$, one concludes that the closed symmetric operator $B$ is maximal, and therefore, $B = A$.

Now suppose that $B = A$ and (25) does not hold, i.e., there is a nonzero vector $h \in \mathcal{H}$ so that $h \perp (A - iI)A^{k-1}\delta$ for all $k \in \mathbb{N}$. This implies that

$$\langle h, (B - iI)\delta_k \rangle = 0, \quad \forall k \in \mathbb{N}, \quad (27)$$

since $\delta_k$ is a polynomial of $A$ applied to $\delta$. By Proposition 7, $B$ is a Jacobi operator so one can denote the entries of the corresponding matrix as in (5). Therefore, by writing $h = \sum_{k=1}^{\infty} h_k \delta_k$, one obtains from (27) that

$$ih_1 := q_1h_1 + b_1h_2,$$
$$ih_k := b_{k-1}h_{k-1} + q_kh_k + b_kh_{k+1}, \quad k \in \mathbb{N} \setminus \{1\}.$$ 

By the assumption that $B = A$, $B$ is selfadjoint, and therefore,

$$\sum_{k=1}^{\infty} |h_k|^2 = +\infty.$$ 

This contradicts the fact that $h$ is a nonzero element of $\mathcal{H}$.

To close up this section, we put its results in the context of Jacobi operators in the selfadjoint and nonselfadjoint cases.

A straightforward consequence of Proposition 8 and Remark 5 is the following:

Proposition 10 Let $J$ be the operator whose matrix representation is (5) with respect to the canonical basis $\{\delta_k\}_{k=1}^{\infty}$. If (5) is in the limit point case, then, for any $\alpha \geq \frac{1}{2}$, the basis $\{\delta_k(\alpha, g)\}_{k=1}^{\infty}$, constructed from any generating element $g$, is a basis of matrix representation for $J$ and the corresponding matrix $[J](\alpha, g)$ is a Jacobi matrix in the limit point case.
Remark 8 If the hypotheses of the preceding proposition hold, then, for each generating element $g$ of $J$ and $\alpha \geq \frac{1}{2}$, the Jacobi matrix $[J](\alpha, g)$ generates a sequence of moments $\{s_k(\alpha, g)\}_{k=0}^{\infty}$ so that the solution to the corresponding moment problem is unique. By the properties of the spectral measure (see [5, Ch.5 Sec. 3 Lem.3]), this unique solution is given by

$$
\mu_{\eta(\alpha, g)}(\partial) = \int_{\partial} e^{-2\alpha t^2} d\mu_g(t)
$$

for any Borel set $\partial$.

According to Remark 5, if $J$ is a nonselfadjoint Jacobi operator, then, for each selfadjoint extension $\tilde{J}$ and $\alpha \geq \frac{1}{2}$, the Jacobi matrix $[\tilde{J}](\alpha, \delta_1)$ is associated with a determinate moment problem whose unique solution is

$$
\mu_{\eta(\alpha, \delta_1)}(\partial) = \int_{\partial} e^{-2\alpha t^2} d\tilde{\mu}_{\delta_1}(t)
$$

for any Borel set $\partial$, where $\tilde{\mu}_{\delta_1}$ is given by (15) with $h = \delta_1$ and $E$ being the spectral measure of $\tilde{J}$.

4 Index of determinacy and bases of representation

In the previous section, bases of representation for a selfadjoint Jacobi operator $J$ were constructed from an arbitrary generating element of it by means of the Stone vectors (Definition 6). The matrices representing $J$ with respect to these bases were Jacobi matrices. The function involved in Definition 6 guarantee not only that the Stone vector is a cyclic vector, but also that the basis obtained from it is a basis of representation.

In this section, an alternative method is used for the construction of bases of representation for a selfadjoint Jacobi operator so that the corresponding matrix is a Jacobi matrix. This method is related to the so-called index of determinacy [4] of a solution to the moment problem.

Let $\mu$ be a Borel measure. For any Borel set $\partial$, denote

$$
\mu_n(\partial) := \int_{\partial} (1 + x^2)^n d\mu(x), \quad n \in \mathbb{N}.
$$

Note that $\mu_n$ ($m \in \mathbb{N}$) is obtained by applying the transformation of the right-hand side of (30) with $n = 1$ to the measure $\mu_{m-1}$.

Two classical results pertaining to the density of polynomials in $L_2$ spaces correspond to Proposition 2 and the following statement known as M. Riesz theorem.

Proposition 11 The measure $\mu$ is the solution to a determinate Hamburger moment problem if and only if the polynomials are dense in $L_2(\mathbb{R}, \mu_1)$.

The proof of this assertion is found in [18] (see also [4, Lem. A] and [20, Cor. 6.11]). Note that, by this proposition, if $\mu$ is the solution to a determinate Hamburger moment...
problem, then the polynomials are dense in $L_2(\mathbb{R}, \mu)$. This can also be established from the results and argumentation of Sect. 2.

**Definition 7** The index of determinacy of a solution $\mu$ to a Hamburger moment problem is

$$\text{ind } \mu := \sup\{n \in \mathbb{N} : \text{the polynomials are dense in } L_2(\mathbb{R}, \mu_n)\}.$$ 

It is not excluded that $\text{ind } \mu$ could be $\infty$, which takes place when the polynomials are dense in $L_2(\mathbb{R}, \mu_n)$ for any $n \in \mathbb{N}$.

This definition differs from the one in [4]. If $\mu$ has index $n$ according to [4, Eq. 1.1], then $\text{ind } \mu = n + 1$ by Definition 7. The index of determinacy in this paper is so that any determinate measure has positive index of determinacy. Indeed, by Proposition 11, the index of determinacy makes sense only for solutions to determinate moment problems and for any such solution $\mu$, $\text{ind } \mu \geq 1$. Note that the index of determinacy decreases one unit each time the transformation (30) with $n = 1$ is applied. Also, it follows from Propositions 2 and 11 that if $\text{ind } \mu = 1$, then $\mu_1$ is given by (13) with $E$ being the spectral measure of a canonical selfadjoint extension of a nonselfadjoint Jacobi operator and $\mu_2$ is such that the polynomials are no longer dense in $L_2(\mathbb{R}, \mu_2)$.

By reverting the transformation (30), one can increase the index of determinacy of a given measure. Indeed, let $J$ be a nonselfadjoint Jacobi operator in $l_2(\mathbb{N})$ and $[J]$ its matrix representation with respect to the canonical basis $\{\delta_k\}_{k=1}^{\infty}$. Fix a canonical selfadjoint extension $\tilde{J}$ of $J$ and denote by $\mu$ the measure given by (13) with $E$ being the spectral measure of $\tilde{J}$. Now, for any Borel set $\partial$ and $n \in \mathbb{N}$, define

$$\nu_n(\partial) := \frac{1}{C_n} \int_\partial (1 + x^2)^{-n} d\mu(x), \quad \text{where} \quad C_n := \int_\mathbb{R} (1 + x^2)^{-n} d\mu(x).$$

Due to Propositions 2 and 11, one verifies that according to Definition 7 $\text{ind } \nu_n = n$ for any $n \in \mathbb{N}$.

Let $\{R_{k-1}\}_{k=1}^{\infty}$ be the orthonormal sequence of polynomials in $L_2(\mathbb{R}, \nu_1)$ obtained from monomials by the Gram–Schmidt procedure. Define

$$f_{k-1}(t) := \frac{1}{\sqrt{C_1(t-i)}} R_{k-1}(t), \quad \text{for all} \quad k \in \mathbb{N}.$$  \hspace{1cm} (31)

Thus,

$$\langle f_k, f_j \rangle_{L_2(\mathbb{R}, \mu)} = \frac{1}{C_1} \int_\mathbb{R} (1 + t^2)^{-1} R_k(t)R_j(t) d\mu(t) = \int_\mathbb{R} \overline{R_k(t)} R_j(t) d\nu_1(t) = \delta_{jk}$$  \hspace{1cm} (32)

so that $\{f_{k-1}\}_{k=1}^{\infty}$ is orthonormal in $L_2(\mathbb{R}, \mu)$. This orthonormal system is also complete due to the fact that if, for any $k \in \mathbb{N},$

$$0 = \langle f_{k-1}, h \rangle_{L_2(\mathbb{R}, \mu)} = \frac{1}{\sqrt{C_1}} \int_\mathbb{R} (t + i)^{-1} \overline{R_{k-1}(t)} h(t) d\mu(t),$$
then \( h(t) = 0 \) for \( \mu \)-a.e. \( t \in \mathbb{R} \). This follows from the density of polynomials in \( L_2(\mathbb{R}, \mu) \) and the properties of the resolvent of \( M_\mu \).

It is equally straightforward to establish that \( f_{k-1} \) is in the domain of the multiplication operator \( M_\mu \) (see Definition 2) for any \( k \in \mathbb{N} \) since \( R_{k-1} \) is in the domain of \( M_{\nu_1} \) for any \( k \in \mathbb{N} \).

Thus, the orthonormal basis \( \{f_{k-1}\}_{k=1}^\infty \) in \( L_2(\mathbb{R}, \mu) \) satisfies (a) of Definition 1 with respect to the operator of multiplication \( M_\mu \).

**Proposition 12** If \( \mu \) is given by (13) with \( J \subsetneq J^* \), then the orthonormal basis \( \{f_{k-1}\}_{k=1}^\infty \) defined above is a basis of representation for the operator \( M_\mu \).

**Proof** Item (a) of Definition 1 has already been established. Let us show that the sequence \( \{f_{k-1}\}_{k=1}^\infty \) is obtained by the Gram–Schmidt procedure applied to the sequence \( \{t^k - 1(\sqrt{C_1(t - i)})^{-1}\}_{k=1}^\infty \) in \( L_2(\mathbb{R}, \mu) \). Indeed, proceeding as in (32), one has for the first step of the Gram–Schmidt algorithm [5, Ch. 2 Sec. 2 Thm. 5]

\[
(\sqrt{C_1(t-i)})^{-1} \left[ t - \left( t(\sqrt{C_1(t-i)})^{-1} \right) \right]_{L_2(\mathbb{R}, \mu)} = (\sqrt{C_1(t-i)})^{-1} [t - \langle t, 1 \rangle_{L_2(\mathbb{R}, \nu_1)}].
\]

The expression in the square brackets of (33) is the first step of Gram–Schmidt procedure applied to the sequence \( \{t^k - 1(\sqrt{C_1(t - i)})^{-1}\}_{k=1}^\infty \) in \( L_2(\mathbb{R}, \nu_1) \). By induction, taking into account (31), one verifies that \( \{f_{k-1}\}_{k=1}^\infty \) is the result of orthonormalizing the sequence \( \{t^k - 1(\sqrt{C_1(t - i)})^{-1}\}_{k=1}^\infty \). In particular, this shows that \( f_0 = (\sqrt{C_1(t-i)})^{-1} \) is a cyclic vector of \( M_\mu \) since \( \{f_{k-1}\}_{k=1}^\infty \) is total in \( L_2(\mathbb{R}, \mu) \).

Now, for any \( k \in \mathbb{N} \), one has

\[
[(M_\mu - i I)M_\mu^{-1}f_0](t) = t^{k-1}/\sqrt{C_1},
\]

which, on the basis of Proposition 2, implies that (25) holds for \( M_\mu \). Thus, Proposition 9 leads to the desired conclusion. \( \Box \)

As a consequence of Propositions 7 and 12, the matrix representation of \( M_\mu \) with respect to \( \{f_{k-1}\}_{k=1}^\infty \) is a Jacobi matrix. This matrix can be found by observing that the sequence \( \{f_{k-1}\}_{k=1}^\infty \) satisfies the same three-term recurrence relation that the sequence of polynomials \( \{R_{k-1}\}_{k=1}^\infty \) does. Since \( \text{ind} \nu_1 = 1 \), the coefficients of the recurrence relation form a Jacobi matrix in the limit point case, which is denoted by \( \hat{J} \). Hence, the matrix representation of the operator of multiplication \( M_\mu \) with respect to \( \{f_{k-1}\}_{k=1}^\infty \) is \( \hat{J} \). Taking into account that \( \hat{J} \) is the selfadjoint extension corresponding to the measure \( \mu \), one has that \( U^{-1}\hat{J}U = M_\mu \), where \( U \) is the map given in Definition 3 (see the last paragraph of Sect. 2). Thus, if one defines

\[
\omega_k := Uf_{k-1}, \quad \text{for all} \quad k \in \mathbb{N},
\]

then \( \{\omega_k\}_{k=1}^\infty \) is a basis of representation for \( \hat{J} \) and the corresponding matrix is \( [\hat{J}] \).

As has been said before (see Remark 6), the canonical basis \( \{\delta_k\}_{k=1}^\infty \) in \( l_2(\mathbb{N}) \) is not
a basis of representation for \( \widehat{J} \). Note that the sequence \( \{\omega_k\}_{k=1}^{\infty} \) is not in \( \text{dom} \ J \) since otherwise the minimality condition (b) of Definition 1 for \( \widehat{J} \) is violated.

It is worth mentioning that the measure \( \nu_1 \), which gives rise to the matrix \( [\widehat{J}] \), has the smallest index that a determinate measure could have. Furthermore, for any \( n \in \mathbb{N} \), by using \( \nu_n \) and modifying accordingly (31), one can construct a basis of matrix representation for \( \widehat{J} \) so that the corresponding matrix is a Jacobi matrix.

In contrast to the construction given above, the measures appearing in Sect. 3 have infinite index of determinacy. This is asserted in the following proposition.

**Proposition 13** For any \( \alpha > 0 \), the measure \( \mu_{\eta}(\alpha, g) \) given in (29) has infinite index of determinacy.

**Proof** Denote by \( \chi_t \) the measure

\[
\chi_t(\partial) := \begin{cases} 
1 & t \in \partial \\
0 & t \notin \partial,
\end{cases}
\]

where \( \partial \subset \mathbb{R} \) is a Borel set.

If one assumes that \( \mu_{\eta}(\alpha, g) \) has finite index, then the measure is discrete [4, Cor. 3.4] and according to [4, Thm. 3.9] there is a finite collection of real numbers \( t_1, \ldots, t_n \) out of the support of \( \mu_{\eta}(\alpha, g) \) so that, for any positive numbers \( a_1, \ldots, a_n \),

\[
\tilde{\mu} := \mu_{\eta}(\alpha, g) + \sum_{k=1}^{n} a_k \chi_{t_k}
\]

is indeterminate. But, one verifies

\[
\int_{\mathbb{R}} e^{|t|} d\tilde{\mu}(t) = \int_{\mathbb{R}} e^{|t|} - 2\alpha t^2 d\mu_g(t) + \sum_{k=1}^{n} a_k e^{|t_k|} < +\infty.
\]

Whence, by [8, Thm. 5.2], one concludes that \( \tilde{\mu} \) is determinate which is a contradiction. \( \square \)

## 5 Non-selfadjoint Jacobi operators

This section begins with an account on some of the remarkable properties of the class of entire operators [12] to which the class of nonselfadjoint Jacobi operators belongs. One of these properties leads to the fact that a nonselfadjoint Jacobi operator has a unique matrix representation being a Jacobi matrix.

**Definition 8** A closed operator \( A \) in a Hilbert space \( \mathcal{H} \) is said to be regular when for any \( z \in \mathbb{C} \), there is a constant \( C > 0 \) (which depends on \( z \)) such that

\[
\|(A - zI)\phi\| \geq C \|\phi\|
\]

for all \( \phi \in \text{dom} \ A \).

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The fact that an operator is regular means that its spectral kernel is empty, therefore, every regular symmetric operator is completely nonselfadjoint (i.e., there is no invariant subspace of the operator in which it induces a selfadjoint operator). Indeed, since any part of an operator with empty spectral kernel has empty spectral kernel, this part cannot be selfadjoint. It is noteworthy that there are completely nonselfadjoint operators which are not regular.

Completely nonselfadjointness of a closed symmetric operator $A$ means that \[ \bigcap_{z \in \mathbb{C} \setminus \mathbb{R}} \ker(A - zI) = \{0\}, \tag{34} \]

since the l.h.s of (34) is the maximal invariant subspace in which $A$ is selfadjoint [10, Thm. 1.2.1].

**Definition 9** An antilinear map $\mathcal{I}$ of $\mathcal{H}$ onto itself being an involution (i.e., $\mathcal{I}^2 = I$) and such that

\[ \langle \mathcal{I}\phi, \mathcal{I}\psi \rangle = \langle \psi, \phi \rangle \quad \text{for any } \phi, \psi \in \mathcal{H} \tag{35} \]

is called a *conjugation* (see [19, Sec. 13.1] and [28, Eq. 8.1]).

A procedure for constructing a conjugation commuting (see [28, Eq. 8.1]) with all canonical selfadjoint extensions of a symmetric completely nonselfadjoint operator $A$ with one-dimensional deficiency spaces is presented in [22, Prop. 2.3]. This conjugation commutes with $A$ (see the proof of [19, Prop. 13.25(ii)]). Conversely, if a conjugation commutes with a symmetric operator $A$ with deficiency indices $n_+(A) = n_-(A) = 1$, then the conjugation commutes with all canonical selfadjoint extensions of $A$ (see [19, Prop. 13.25 (iv)] and [25, Cor. 2.5]).

The following statement is motivated by Krein’s representation theory of symmetric operators [12–15]. The assertion’s constructive proof can be found in [22, Prop. 2.12].

**Proposition 14** If $A$ is a regular, symmetric operator such that $n_+(A) = n_-(A) = 1$ and $\mathcal{I}$ is a conjugation that commutes with $A$, then there is a vector function $\xi_A : \mathbb{C} \to \mathcal{H}$ with the following properties:

(a) $\xi_A$ is entire and zero-free.
(b) $\xi_A(z) \in \ker(A^* - zI)$ for each $z \in \mathbb{C}$.
(c) For all $z \in \mathbb{C}$, $\mathcal{I}\xi_A(z) = \xi_A(\overline{z})$.

Having fixed the involution $\mathcal{I}$, the function $\xi_A$ is uniquely determined modulo a multiplicative scalar factor being an entire, zero-free function which turns out to be real (see [22, Rem. 2.13] and [24, Lem. 3]). Recall that a complex valued function $f$ of complex variable satisfying

\[ \overline{f(z)} = f(\overline{z}) \tag{36} \]

is called *real*; thus, a real function is real on the real line.
Remark 9 It is worth mentioning that if, for a closed symmetric operator $A$ with $n_+(A) = n_-(A) = 1$, the equality (34) holds and there is a function $\xi_A$ satisfying (a)–(c) of Proposition 14, then the operator is regular. This is proven by means of the functional model given in [22, Sec. 2.3] and [23, Sec. 4]. This functional model gives a de Branges space [7] for any such operator $A$ having such a function $\xi_A$.

It follows from [9, Sec. 2.2] and [21, Sec. 2] that for any regular, symmetric operator $A$ with $n_+(A) = n_-(A) = 1$, there is $\gamma$ in $\mathcal{H}$ such that

$$\mathcal{H} = \text{ran}(A - zI) \oplus \text{span}\{\gamma\}$$

for all $z \in \mathbb{C} \setminus S_\gamma$, where $\text{card } S_\gamma \leq \text{card } \mathbb{N}$ (see [9, Sec. 2.2], [21, Sec. 2]). The set $S_\gamma$ turns out to be at most countable since it is the zero set of the analytic function $\langle \xi_A(\cdot), \gamma \rangle$, which does not vanish identically due to the fact that $\gamma \neq 0$ and $\{\xi_A(z)\}_{z \in \mathbb{C} \setminus \mathbb{R}}$ is a total set in $\mathcal{H}$ (cf. (34)). The vector $\gamma$ is said to be a gauge of $A$.

Remark 10 The set $S_\gamma$ is the zero set of the function $\langle \xi_A(\cdot), \gamma \rangle$ due to the property (b) of the function $\xi_A$.

The gauge $\gamma$ can be chosen in such a way so that the exceptional set $S_\gamma$ lies entirely on the real line [21, Lem. 2.1] or completely outside the real line [21, Thm. 2.2]. This last assertion was first stated without proof in [14, Thm. 8].

Definition 10 A regular, symmetric operator $A$ such that $n_+(A) = n_-(A) = 1$ is said to be entire if there exists a gauge $\gamma$ so that $S_\gamma = \emptyset$. In this case, $\gamma$ is an entire gauge of $A$.

A straightforward consequence of this definition is that $A$ is an entire operator and $\gamma$ its entire gauge if and only if the entire function

$$t(\cdot) := \langle \xi_A(\cdot), \gamma \rangle$$  \hspace{2cm} (37)

is a zero-free function. Another property of this function is given by the following assertion.

Lemma 15 The function $t$ given in (37) is real if and only if $\mathcal{I}\gamma = \gamma$.

Proof Assume that $t$ is real. Thus,

$$\langle \gamma, \xi_A(\overline{z}) \rangle = \langle \xi_A(z), \gamma \rangle$$

$$= \langle \mathcal{I}\gamma, \mathcal{I}\xi_A(z) \rangle$$

$$= \langle \mathcal{I}\gamma, \xi_A(\overline{z}) \rangle,$$ \hspace{2cm} (38)

where the first equality is (36) and in the second and third equalities, one uses (35) and Proposition 14(c), respectively. Since $\{\xi_A(z)\}_{z \in \mathbb{C} \setminus \mathbb{R}}$ is a total set in $\mathcal{H}$ (cf. (34)), the last equality in (38) means that $\mathcal{I}\gamma = \gamma$. 

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Now, assume that $I\gamma = \gamma$. Using a similar reasoning, one has
\[
\langle \xi_A(z), \gamma \rangle = \langle I\gamma, I\xi_A(z) \rangle = \langle \gamma, \xi_A(z) \rangle,
\]
which is (36). \qed

It is established in [9, Ch. 2 Sec. 4.1] that, for any entire operator $A$, the vector-valued function $\xi_A$ and the gauge $\gamma$ can be chosen so that the scalar function $t$ given in (37) is a real constant (see also [12, Sec. 2]). Below, it will be shown that the “natural” choice of the gauge $\gamma$ and the function $\xi_A$ for an entire operator is the one for which $t$ is a real constant. However, having done this choice for $\gamma$ and $\xi_A$, one is interested in the behavior of the zero-free entire function $\langle \xi_A(z), \tilde{\gamma} \rangle$, where the entire gauge $\gamma$ has been substituted by another entire gauge $\tilde{\gamma}$. To study this function, let us recall two notions related to the theory of growth of entire functions.

A function of at most exponential type is a function of at most order one and normal type [16, Ch. 1 Sec. 20]. The dependence of the growth of a function $f$ of exponential type on the direction in which the independent variable tends to infinity is given by the function
\[
\mathcal{h}_f(\theta) := \limsup_{r \to \infty} \frac{\log |f(re^{i\theta})|}{r}, \quad \theta \in [0, 2\pi),
\]
which is the so-called indicator function of $f$ (see [16, Ch. 1 Sec. 15] and [17, Ch. II.9 Sec. 45]).

In [15, Sec. 8] (see also [9, Ch. 2 Sec. 5]), the following assertion is established.

**Proposition 16** Let $A$ be an entire operator and pick the corresponding function $\xi_A$ and gauge $\gamma$ so that the function $t$ given in (37) is a real constant. Then, for any $\phi \in \mathcal{H}$, the function $f(\cdot) := \langle \xi_A(\cdot), \phi \rangle$ is at most of exponential type and its indicator function obeys
\[
\mathcal{h}_f(\theta) = \begin{cases} 
\alpha(\frac{\pi}{2}) \sin \theta & \text{if } 0 \leq \theta \leq \pi \\
-\alpha(\frac{\pi}{2}) \sin \theta & \text{if } \pi < \theta \leq 2\pi.
\end{cases}
\]

The proof of the first part of Proposition 16 is found in the paragraph preceding [15, Lem. 8.1] (see also [9, Eq. 5.1]), where implicitly it is used that (37) is a constant. As regards the second part, see the proof of [15, Lem. 8.1] or the proof of [9, Ch. 2 Lem. 5.1].

The following assertion exhibits a property of entire operators which is crucial for this section. It is related to [12, Thm. 1] whose proof can be found in [21, Prop. 4.6].

**Proposition 17** Let $A$ be an entire operator. If there are two entire gauges $\gamma_1, \gamma_2$ of $A$ and two functions $\xi_A^{(1)}, \xi_A^{(2)}$ satisfying (a)–(c) of Proposition 14 so that $\langle \xi_A^{(1)}(\cdot), \gamma_1 \rangle$ and $\langle \xi_A^{(2)}(\cdot), \gamma_2 \rangle$ are real constants, then there is a real constant $C$ such that $\gamma_1 = C\gamma_2$. 

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Lemma 15 yields that $I\gamma_1 = \gamma_1$ and $I\gamma_2 = \gamma_2$. Thus, again by Lemma 15, the zero-free function $f(\cdot) := \langle \xi_A^{(1)}(\tau), \gamma_2 \rangle$ is real and has the form $\exp(g(\cdot))$, where $g$ is a real entire function. Furthermore, since $\langle \xi_A^{(1)}(\tau), \gamma_1 \rangle$ is a real constant, the first part of Proposition 16 implies that the function $f$ is a function of at most exponential type, and therefore, the function $g$ is a polynomial of the first degree, whence $\langle \xi_A^{(1)}(\tau), \gamma_2 \rangle = C \exp((a + ib)z)$ for all $z \in \mathbb{C}$ ($a, b \in \mathbb{R}$).

On the one hand, it follows from the second part of Proposition 16 that the indicator function of $f$ is (39); on the other hand, the indicator function of $\exp((a + ib)\cdot)$ has the form

$$h(\theta) = a \cos \theta - b \sin \theta.$$  

Comparing (39) with (40), one arrives at the conclusion that $a = 0$. Finally, it follows from the reality of the function $\langle \xi_A^{(1)}(\tau), \gamma_2 \rangle$ that $b = 0$ and $C$ is real. \hfill \Box

Any nonselfadjoint Jacobi operator is regular. This is a classical result of the moment problem (or Jacobi operator) theory. It is shown by establishing that the spectra of its selfadjoint extensions do not intersect (see the proof of [1, Thm. 4.2.4] and [25, Thm. 5]). Recall that the spectral kernel of an operator is contained in the spectral kernel of its extension, thus if a point is in the spectral kernel of a symmetric operator, then this point is in the spectrum on any of its selfadjoint extensions.

Proposition 18 Any nonselfadjoint Jacobi operator is an entire operator and $\delta_1$ (the first element of the canonical basis in $l_2(\mathbb{N})$) is an entire gauge of the operator.

Proof It has been established that any nonselfadjoint operator $J$ having the matrix representation (5) with respect to the orthonormal basis $\{\delta_k\}_{k=1}^\infty$ has deficiency indices $n_+(J) = n_-(J) = 1$ and it is regular. As mention in Remark 1, the vector-valued function $\pi$ satisfies (b) of Proposition 14. In view of Remark 10, for finishing the proof, it only remains to note that $\langle \pi(\tau), \delta_1 \rangle \equiv 1$. \hfill \Box

Remark 11 When $J \subsetneq J^*$, the function $\pi$ is actually the function $\xi_J$ given in Proposition 14. Indeed, apart from satisfying (b) of Proposition 14 (see the proof above), $\pi$ complies with (a) since the zeros of polynomials of the first kind interlace [1, Thm. 1.2.2]. The property (c) is a consequence of the reality of the polynomials’ coefficients. Note that the function $t$ given in (37), with $\gamma = \delta_1$ and $\xi_J = \pi$, is a real constant ($\equiv 1$). Thus, due to Remark 9, the functional model [23] yields a de Branges space for the operator $J$. On the basis of de Branges space theory [7], one knows that the absolutely continuous measure $\mu = \lambda/E$, where $\lambda$ is the Lebesgue measure and $E$ is the Hermite–Biehler function of the de Branges space, is a solution of the corresponding indeterminate moment problem. Note that the measures given by selfadjoint extensions of $J$ are all discrete.

Proposition 19 There is only one basis of representation (modulo reflection\footnote{Reflection means that every element of the orthonormal basis is multiplied by $-1$.}) with respect to which any nonselfadjoint Jacobi operator has a Jacobi matrix as its matrix.
representation. The Jacobi matrix representing a nonselfadjoint Jacobi operator is unique.

Proof Let $J$ be nonselfadjoint such that it has the matrix representation (5) with respect to the orthonormal basis $\{\delta_k\}_{k=1}^{\infty}$. As has been shown, $\delta_1$ determines all the elements of the orthonormal basis and, consequently, the entries of the Jacobi matrix. Indeed, the vectors $\delta_2, \delta_3, \ldots$ are obtained by applying the Gram–Schmidt procedure to the sequence $\{J^{k-1}\delta_1\}_{k=1}^{\infty}$ (see the proof of Proposition 8). Likewise, as asserted in Remark 8, the entries of the matrix can be obtained from the moments $\langle \delta_1, J^{k-1}\delta_1 \rangle$, $k \in \mathbb{N}$. Now, suppose that for $J$ there is another orthonormal basis $\{\tilde{\delta}_k\}_{k=1}^{\infty}$ with respect to which $J$ has a Jacobi matrix representation. As a consequence of Proposition 18, the vectors $\delta_1$ and $\tilde{\delta}_1$ are entire gauges of $J$ satisfying the hypothesis of Proposition 17. Therefore, $\tilde{\delta}_1 = C\delta_1$, where $C \in \mathbb{R}$. Since $\|\delta_1\| = \|\tilde{\delta}_1\| = 1$, one concludes that $C$ is either 1 or $-1$.

Acknowledgements D.H.B. is supported with a postdoctoral fellowship by DGAPA-UNAM at IIMAS-UNAM. L.O.S. has been partially supported by CONACyT Ciencia de Frontera 2019 No 304005. The authors thank Professor R. Szwarz for his comments and interest in this work. The authors express their gratitude to the anonymous referees for the comments and suggestions that have led to an improved presentation of this work.

Data availability Not applicable.

Declarations

Conflict of interest The authors have no conflicts of interest to declare. All the co-authors have seen and agree with the contents of the manuscript and there is no financial interest to report. We certify that the submission is original work and is not under review at any other publication.

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