A simple proof of a theorem of Jean Bourgain*

by G. Pisier

Abstract. We give a simple proof of Bourgain’s disc algebra version of Grothendieck’s theorem, i.e. that every operator on the disc algebra with values in $L_1$ or $L_2$ is 2-absolutely summing and hence extends to an operator defined on the whole of $C$. This implies Bourgain’s result that $L_1/H^1$ is of cotype 2. We also prove more generally that $L_r/H^r$ is of cotype 2 for $0 < r < 1$.

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In this note, we give a very simple proof (compared to the preceding known proofs) of Bourgain’s version of Grothendieck’s theorem for the disc algebra. As far as we know, the currently known proofs are essentially the original one in [B1], the simpler one in [BD], and several new proofs given recently by Kisliakov, in [K1,K2].

We first recall the definition of a \( q \)-absolutely summing (in short \( q \)-summing) operator for \( 1 \leq q < \infty \). Let \( u : X \to Y \) be an operator between two Banach spaces. We say that \( u \) is \( q \)-summing if there is a constant \( C \) such that for all finite sequences \( x_1, x_2, \ldots, x_n \) in \( X \), we have

\[
(\sum \|u(x_i)\|^q)^{1/q} \leq C \sup\{(\sum |x^*(x_i)|^q)^{1/q} \mid x^* \in X^*, \|x^*\| \leq 1\}.
\]

We denote by \( \pi_q(u) \) the smallest possible constant \( C \). Let us denote by \( A \) the disc algebra. Then if \( u : A \to Y \) is \( q \)-summing, by Pietsch’s factorisation theorem, there is a probability measure \( \lambda \) on the unit circle such that

\[
\forall f \in A \quad \|u(f)\| \leq \pi_q(u)(\int |f|^q d\lambda)^{1/q}.
\]

We refer e.g. to [P1] for more information on this notion.

We will prove

**Bourgain’s Theorem:** There is a constant \( K \) such that any bounded operator \( u : A \to \ell_2 \) is 2-summing and satisfies:

\[
\pi_2(u) \leq K\|u\|.
\]

Also, \( u \) extends to a bounded operator \( \hat{u} : C(T) \to \ell_2 \) such that

\[
\|\hat{u}\| \leq K\|u\|.
\]

Moreover, the same result holds for all operators \( u : A \to Y \) if \( Y = \ell_1 \), or more generally, whenever \( Y \) is a Banach space of cotype 2.

Let us recall here the definitions of the \( K_t \) and \( J_t \) functionals which are fundamental in the real interpolation method. Let \( A_0, A_1 \) be a compatible couple of Banach (or quasi-Banach) spaces. For all \( x \in A_0 + A_1 \) and for all \( t > 0 \), we let

\[
K_t(x, A_0, A_1) = \inf \left( \|x_0\|_{A_0} + t\|x_1\|_{A_1} \mid x = x_0 + x_1, x_0 \in A_0, x_1 \in A_1 \right).
\]
For all $x \in A_0 \cap A_1$ and for all $t > 0$, we let

$$J_t(x, A_0, A_1) = \max(\|x_0\|_{A_0}, t\|x_1\|_{A_1}).$$

Recall that the (real interpolation) space $({A_0, A_1})_{\theta,p}$ is defined as the space of all $x$ in $A_0 + A_1$ such that $\|x\|_{\theta,p} < \infty$ where

$$\|x\|_{\theta,p} = \left(\int (t^{-\theta}K_t(x, A_0, A_1))^p dt/t\right)^{1/p}.$$

We refer to [BeL] for more details.

Let $T$ be the circle group equipped with its normalized Haar measure $m$. Let $1 \leq p \leq \infty$. When $B$ is a Banach space we denote by $L_p(B)$ the usual space of Bochner-$p$-integrable $B$-valued functions on $(T, m)$, so that when $p < \infty$, $L_p \otimes B$ is dense in $L_p(B)$. We denote by $H^p(B)$ the subspace of $L_p(B)$ formed by all the functions $f$ such that their Fourier transform vanishes on the negative integers. When $B$ is one dimensional, we write $H^p$ instead of $H^p(B)$. When $0 < p < 1$, we define $H^p$ as the closure in $L_p$ of the linear span of the functions $\{e^{int} | n \geq 0\}$. We refer to [G,GR] for basic information on $H^p$-spaces.

The next proposition although very simple is the key new ingredient in our proof. We refer to [P2] for more applications of the same idea to the interpolation spaces between $H^p$ spaces.

**Proposition 1:** Let $1 \leq p \leq q \leq \infty$. Consider a compatible couple of Banach spaces $(A_0, A_1)$, the following are equivalent:

(i) There is a constant $C$ such that

$$\forall f \in H^p(A_0) + H^q(A_1), \quad \forall t > 0, \quad K_t(f, H^p(A_0), H^q(A_1)) \leq CK_t(f, L^p(A_0), L^q(A_1)).$$

(ii) There is a constant $C$ such that

$$\forall f \in [L^p(A_0)/H^p(A_0)] \cap [L^q(A_1)/H^q(A_1)], \quad \forall t > 0, \quad \exists \hat{f} \in L^p(A_0) \cap L^q(A_1)$$

representing the equivalence class of $f$ and satisfying

$$J_t(\hat{f}, L^p(A_0), L^q(A_1)) \leq CJ_t(f, L^p(A_0)/H^p(A_0), L^q(A_1)/H^q(A_1)).$$
representing the equivalence class of \( f \) and satisfying
\[
\|\hat{f}\|_{L^p(A_0)} \leq C\|f\|_{L^p(A_0)/H^p(A_0)} \quad \text{and} \quad \|\hat{f}\|_{L^q(A_1)} \leq C\|f\|_{L^q(A_1)/H^q(A_1)}.
\]

In the above statement we regard the spaces \( L^p(A_0)/H^p(A_0) \) and \( L^q(A_1)/H^q(A_1) \) as included via the Fourier transform \( f \rightarrow (\hat{f}(-1), \hat{f}(-2), \hat{f}(-3), ...) \) in the space of all sequences in \( A_0 + A_1 \). In this way, we may view these quotient spaces as forming a compatible couple for interpolation. (For the subspaces \( H^p(A_0), H^q(A_1) \), there is no problem, we may clearly consider them as a compatible couple in the obvious way.)

**Proof:** For brevity, we will denote simply \( L^p/H^p(A_0) \) instead of \( L^p(A_0)/H^p(A_0) \), we will also write \( L^p/H^p \) instead of \( L^p(A_0) \) \( H^p(A_0) \) no confusion should arise. The proof is routine. We only indicate the argument for \( (i) \Rightarrow (ii) \Rightarrow (iii) \) which is the one we use below.

Assume \( (i) \). Let \( f \) be as above such that \( J_t(f, L^p/H^p(A_0), L^q/H^q(A_1)) < 1 \). Then let \( g_p \in L^p(A_0) \) and \( g_q \in L^q(A_1) \) be such that
\[
\|g_p\|_{L^p} < 1, \quad \|g_q\|_{L^q} < t^{-1}, \quad f = g_p + H^p(A_0), \quad f = g_q + H^q(A_1).
\]
Therefore, \( g_p - g_q \) must be in \( H^p + H^q \) and
\[
K_t(g_p - g_q, L^p(A_0), L^q(A_1)) \leq \|g_p\|_{L^p} + t\|g_q\|_{L^q} < 2.
\]
By \( (i) \), we have \( K_t(g_p - g_q, H^p, H^q) < 2C' \), hence there are \( h_p \in H^p(A_0) \) and \( h_q \in H^q(A_1) \) such that \( g_p - g_q = h_p - h_q \) and \( \|h_p\|_{H^p} + t\|h_q\|_{H^q} < 2C' \). Now if we let \( \hat{f} = g_p - h_p = g_q - h_q \), then we find that \( \hat{f} \in L^p(A_0) \cap L^q(A_1), f = \hat{f} + H^p(A_0) \) in the space \( L^p/H^p(A_0) \) and \( f = \hat{f} + H^q(A_1) \) in the space \( L^q/H^q(A_1) \) and moreover
\[
J_t(\hat{f}, L^p, L^q) \leq \max(\|\hat{f}\|_{L^p}, t\|\hat{f}\|_{L^q}) \leq 1 + 2C'.
\]
By homogeneity this completes the proof of \( (i) \Rightarrow (ii) \) with \( C \leq 1 + 2C' \). To check \( (ii) \Rightarrow (iii) \), simply write \( (ii) \) with \( t = (\|f\|_{L^p/H^p(A_0)})/(\|f\|_{L^q/H^q(A_1)})^{-1} \).
Remark: It is well known that the Hilbert transform is a bounded operator on all the (so-called mixed norm) spaces of the form $L^p(\ell^q)$ for all $1 < p, q < \infty$. (Apparently this goes back to [BB]. We refer to [GR] for more information and references). Therefore, the orthogonal projection from $L^2(\ell^2)$ onto $H^2(\ell^2)$ is bounded simultaneously on all the spaces $L^p(\ell^q)$ for $1 < p, q < \infty$. It follows immediately that if $1 < p_0, p_1, q_0, q_1 < \infty$ there is a constant $C$ such that $\forall f \in H^{p_0}(\ell^{q_0}) + H^{p_1}(\ell^{q_1}), \forall t > 0$,

$$K_t(f, H^{p_0}(\ell^{q_0}), H^{p_1}(\ell^{q_1})) \leq C' K_t(f, L^{p_0}(\ell^{q_0}), L^{p_1}(\ell^{q_1})).$$

Proposition 2: There is a constant $C'$ such that for all $t > 0$ and all $f \in H^1(\ell_1) + H^1(\ell_2)$ we have

$$K_t(f, H^1(\ell_1), H^1(\ell_2)) \leq CK_t(f, L^1(\ell_1), L^1(\ell_2)).$$

For the proof of Prop. 2, we will use the

Sublemma:

$$H^1(\ell_{4/3}) \subset (H^1(\ell_1), H^1(\ell_2))_{1/2, \infty},$$

and the inclusion is bounded with norm less than a constant $K$.

Proof: Take a function $f = (f_n)$ in the unit ball of $H^1(\ell_{4/3})$ and factor it as $f = (g_n h_n)$ with $g = (g_n)$ in the unit ball of $H^2(\ell_2)$ and $h = (h_n)$ in the unit ball of $H^2(\ell_4)$. This is easy to do by factoring out the Blaschke product of each component $f_n$ and raising the factor without zero to the appropriate power. (More precisely, write $f_n = B_n F_n$ where $B_n$ is a Blaschke product and where $F_n$ does not have zeros in $D$, let $F$ be an outer function such that we have $|F| = (\sum |F_n|^{4/3})^{3/4}$ on the unit circle, then let

$$g_n = B_n (F_n/F)^{2/3} F^{1/2} \text{ and } h_n = (F_n/F)^{1/3} F^{1/2}.$$  

This factorisation has the properties claimed for $g$ and $h$.)

Recall the inclusion (which obviously follows from the above remark)

$$H^2(\ell_2) = (H^2(\ell_{4/3}), H^2(\ell_4))_{1/2} \subset (H^2(\ell_{4/3}), H^2(\ell_4))_{1/2, \infty}.$$

Then, by interpolation, since the operator of coordinatewise multiplication by $h = (h_n)$ maps $H^2(\ell_{4/3})$ into $H^1(\ell_1)$ and $H^2(\ell_4)$ into $H^1(\ell_2)$, we obtain the announced inclusion. q.e.d.
**Proof of Prop. 2:** Consider \( f = (f_n) \in H^1(\ell_1) + H^1(\ell_2) \) such that

\[ K_t(f, L^1(\ell_1), L^1(\ell_2)) < 1. \]

By classical factorisation theory, each \( f_n \) can be factored as \( f_n = B_n F_n \) where \( B_n \) is a Blaschke product and where \( F_n \) does not have zeros in \( D \) so that the analytic function \((F_n)^p\) makes sense for any \( p > 0 \). (Alternatively, we could use the inner-outer factorisation instead.) Let us simply denote by \( F^{1/2} \) the sequence of analytic functions \( F^{1/2} = (F_n^{1/2})_{n \geq 1} \). Note that any assumption of the form (1) depends only on the values of each \( |f_n| \) on the boundary. Now, on the boundary we have \( |f_n|^{1/2} = |F_n|^{1/2} \), so that (1) implies obviously

\[ K_{t^{1/2}}(F^{1/2}, L^2(\ell_2), L^2(\ell_4)) < 2^{1/2}. \]

Therefore, by the above Remark,

\[ K_{t^{1/2}}(F^{1/2}, H^2(\ell_2), H^2(\ell_4)) < 2^{1/2}C, \]

where \( C \) is a numerical (absolute) constant. Hence, there is a decomposition \( F^{1/2} = g_0 + g_1 \) with

\[ \|g_0\|_{H_2(\ell_2)} + t^{1/2}\|g_1\|_{H_2(\ell_4)} < 2^{1/2}C. \]

Let us now return to \( f = (f_n) = (B_n(g_0_n + g_1_n)^2) \). Let us simply denote by \( g_0g_1 \) the sequence \((g_0_n g_1_n)_{n \geq 1}\), similarly, we also denote by \( g_0^2 \) and \( g_1^2 \) the sequences of squares. Observe that by (3) and by Hölder, we have

\[ \|g_0g_1\|_{H^1(\ell_{4/3})} < 2C^2 t^{-1/2}, \]

which implies by the sublemma

\[ t^{-1/2}K_t(g_0g_1, H^1(\ell_1), H^1(\ell_2)) < 2C^2 K t^{-1/2}. \]

After simplification

\[ K_t(g_0g_1, H^1(\ell_1), H^1(\ell_2)) < 2C^2 K. \]
On the other hand we have clearly by (3)

\[ K_t(g_0^2 + g_1^2, H^1(\ell_1), H^1(\ell_2)) \leq 2C^2 + 2C^2 = 4C^2. \]

Therefore, we conclude by the triangle inequality (and the fact that Blaschke products are of unit norm in \( H^\infty \))

\[ K_t(f, H^1(\ell_1), H^1(\ell_2)) \leq K_t(g_0^2 + g_1^2, H^1(\ell_1), H^1(\ell_2)) + K_t(2g_0g_1, H^1(\ell_1), H^1(\ell_2)) \leq 4C^2 + 4C^2K. \]

By homogeneity, this completes the proof. q.e.d.

**Corollary:** There is a constant \( C \) such that for all \( 1 < p < 2 \) and all \( f \in L^1/H^1(\ell_p) \) we have

\[ \|f\|_{L^1/H^1(\ell_p)} \leq C\|f\|_{L^1/H^1(\ell_1)}^\theta \|f\|_{L^1/H^1(\ell_1)}^{1-\theta}, \]

where \( 1/p = \theta/2 + (1-\theta)/1 \).

**Proof:** By Prop. 2 and Prop. 1, there is a constant \( C \) such that every \( f \in L^1/H^1(\ell_1) \) admits a lifting \( \hat{f} \in L^1(\ell_1) \) such that we have *simultaneously*

\[ \|\hat{f}\|_{L^1(\ell_1)} \leq C\|f\|_{L^1/H^1(\ell_1)} \]

\[ \|\hat{f}\|_{L^1(\ell_2)} \leq C\|f\|_{L^1/H^1(\ell_2)}. \]

Then (4) is an immediate consequence of Hölder’s inequality. q.e.d.

The preceding corollary implies immediately

**Proposition 3:** There is a constant \( C \) such that, for all Banach spaces \( Y \), for all \( 2 < q < \infty \) and all 2-summing operator \( u : A \to Y \), we have

\[ \pi_q(u) \leq C\pi_2(u)^\theta \|u\|^{1-\theta}, \]

where \( 1/q = \theta/2 + (1-\theta)/\infty \).
Proof: We first claim that for any \( n > 1 \) and for any \( x_1, x_2, \ldots, x_n \) in \( A \), we have

\[
\sum_{1}^{n} \| u(x_i) \| \leq \lambda \|( \sum |x_i|^q )^{1/q} \|_\infty,
\]

where \( \lambda \leq Cn^{1/q'} \). Indeed, let us denote by \( \lambda(q, n) \) the best constant in this inequality. Assume w.l.o.g. that \( u \) is the adjoint of an operator \( v : Y^* \to L^1/H^1 \). Let \( p = q' \). By duality, we find

\[
\lambda(q, n) = \sup \{ \|( v(y_i) \|_{L^1/H^1(\ell^q_n)} \}
\]

where the sup runs over all \( n \)-tuples \( (y_i) \) in \( Y \) such that \( \sup \| y_i \| \leq 1 \). Therefore, (4) immediately yields \( \lambda(q, n) \leq C\lambda(2, n)\theta \lambda(\infty, n)^{1-\theta} \leq C(n^{1/2} \pi_2(u))^{\theta} (n\|u\|)^{1-\theta} \), hence

(6)
\[
\lambda(q, n) \leq Cn^{1/q'} \pi_2(u)^{\theta} \| u \|^{1-\theta}.
\]

For simplicity, let \( B = C\pi_2(u)^{\theta} \| u \|^{1-\theta} \). By (6), we have for any \( x_1, x_2, \ldots, x_n \) in \( A \),

(7)
\[
\sum_{1}^{n} \| u(x_i) \| \leq B \|( \sum |x_i|^q )^{1/q} \|_\infty.
\]

Now let us rewrite (7) for a sequence composed of \( x_1/(k_1)^{1/q} \) repeated \( k_1 \)-times, \( x_2/(k_2)^{1/q} \) repeated \( k_2 \)-times, etc.. We obtain

\[
(\sum k_i)^{-1/q'} \sum k_i^{1/q'} \| u(x_i) \| \leq B \|( \sum |x_i|^q )^{1/q} \|_\infty.
\]

Clearly, since the sequences of the form \((\sum k_i)^{-1}k_i\) are obviously dense in the set of all sequences \( (\alpha_i) \) such that \( \sum \alpha_i = 1 \), we obtain

\[
\sum (\alpha_i)^{1/q'} \| u(x_i) \| \leq B \|( \sum |x_i|^q )^{1/q} \|_\infty.
\]

Taking the supremum over all such \( (\alpha_i) \), we finally obtain the announced result (5). q.e.d.

We now recall a classical inequality due to Khintchine, concerning the Rademacher functions \( r_1, r_2, \ldots, r_n, \ldots \) defined on the Lebesgue interval. For every \( q > 2 \), there is a constant \( B_q \) such that for all finite sequences of scalars \( (\alpha_i) \), we have

\[
(\int |\sum \alpha_i r_i|^q dt)^{1/q} \leq B_q (\sum |\alpha_i|^2)^{1/2}.
\]
The following is a known result of Maurey [M].

Proposition 4: Let $X$ be any Banach space. Let $Y$ be a Banach space of cotype 2, i.e. such that there is a constant $C_2$ satisfying, for all $n$ and for all $n$-tuples $y_1, y_2, \ldots, y_n$ in $Y$,

$$
(\sum \|y_i\|^2)^{1/2} \leq C_2 (\int \| \sum r_i y_i \|^2 dt)^{1/2}.
$$

Then, for every $q > 2$, every $q$-summing operator $u : X \to Y$ is actually 2-summing, and moreover

$$
\pi_2(u) \leq B_q C_2 \pi_q(u).
$$

Proof: Let $x_1, x_2, \ldots, x_n$ be a finite subset of $X$ such that $\sum |x^*(x_i)|^2 \leq 1$ for all $x^*$ in the unit ball of $X^*$. Then, by the above Khintchine Inequality, we have for all $x^*$ in the unit ball of $X^*$,

$$
(\int |\sum r_i x^*(x_i)|^q dt)^{1/q} \leq B_q.
$$

Hence, by the definition of $\pi_q(u)$ (note that the integral below is actually an average over $2^n$ choices of signs),

$$
(\int \| \sum r_i u(x_i) \|^q dt)^{1/q} \leq \pi_q(u) B_q.
$$

Hence, by the definition of the cotype 2,

$$
(\sum \|u(x_i)\|^2)^{1/2} \leq C_2 B_q \pi(q) (u).
$$

By homogeneity, this proves Proposition 4. q.e.d.

We can now complete the

Proof of Bourgain’s Theorem: We use the same general line of attack as Bourgain. This approach is based on an extrapolation trick which originates in the work of Maurey [M] and has been used several times before Bourgain’s work (especially by the author) to prove various extensions of Grothendieck’s Theorem. (The latter theorem corresponds to the case $A = C$, $Y = \ell_1$ in the above statement, see [P1].) In this approach, the crucial point reduces to showing (5). Indeed, assuming (5), it is easy to conclude: By Prop. 4, we have $\pi_2(u) \leq C_2 B_q \pi_q(u)$, hence by (4), $\pi_2(u) \leq CC_2 B_q \pi_2(u) \theta \|u\|^{1-\theta}$ hence if we assume a priori that $\pi_2(u)$ is finite, we obtain

$$
\pi_2(u) \leq (CC_2 B_q)^{1/1-\theta} \|u\|,
$$

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which establishes the announced result in the case of a 2-summing operator. Hence in particular (8) holds if $u$ is of finite rank, so that we can easily conclude, since $A$ has the Metric Approximation Property, that it actually holds for arbitrary operators. Finally, the last assertion follows from a well known factorisation property of 2-summing operators, due to Pietsch, cf. e.g. [P1], chapter 1. q.e.d.

Remark 5: There is also a slightly different way to prove (8). One can use a simple interpolation argument to prove that for any $n > 1$ and for any $x_1, x_2, \ldots, x_n \in A$, we have

$$
\|(u(x_i))\|_{\ell_{q,\infty}(Y)} \leq B\|(x_i)\|_{L_{\infty}(\ell_{q,\infty})},
$$

where $B$ is as above. Then, we may apply this replacing $x_1, x_2, \ldots, x_n$ by the $2^n$-tuple formed by the $2^n$ choices of signs $\sum r_i(t)x_i$. After normalisation by a factor $2^{-n/q}$, we obtain

$$
\|(\sum r_i u(x_i))\|_{L_{q,\infty}(dt;Y)} \leq B\sum r_i x_i \|_{L_{\infty}(L_{q,\infty}(dt))}.
$$

But then, we observe that Khintchine’s inequality implies a fortiori the equivalence of $\| \sum r_i x_i \|_{L_{\infty}(L_{q,\infty}(dt))}$ with $\|(x_i)\|_{L_{\infty}(\ell_2)}$. This immediately leads to (8) by the same argument as above.

Remark 6: As is well known, it follows by standard arguments from Bourgain’s theorem as stated above that $L_1/H^1$ is a cotype 2 space. This can be derived as in [B1] from a result of Wojtaszczyk which ensures that $L_1/H^1$ is isomorphic to $L_1/H^1(\ell_1)$. Alternately, if one wishes to avoid the use of the latter result, one can observe that our proof of Bourgain’s Theorem is valid with essentially the same proof with $L_1/H^1(\ell_1)$ instead of $L_1/H^1$.

Actually, we can generalize Bourgain’s theorem as follows:

**Theorem 7:** Let $0 < r < 1$. Then any operator $u : c_0 \to L_r/H^r$ is 2-summing. Moreover, there is a constant $C_r$ such that every operator $u : c_0 \to L_r/H^r$ is 2-summing and satisfies $\pi_2(u) \leq C_r\|u\|$. Finally, $L_r/H^r$ is of cotype 2.

**Proof:** We only sketch the argument. (It might very well be that this result follows from the other proofs, however it seems to have passed unnoticed so far.) Consider an operator $u : \ell_\infty^n \to L_r/H^r$. We will show that there is a constant $C_r$ independent of $n$ such that,

$$
(9) \forall m \ \forall x_1, x_2, \ldots, x_m \in \ell_\infty^n, \ \|(u(x_i))\|_{L_r/H^r(\ell_2^n)} \leq C_r\|u\||(x_i)\|_{\ell_\infty^n(\ell_2^n)}.
$$
We argue similarly as above, but in a dual setting. Let \( r \leq p \leq \infty \). We denote by \( C_p(u) \) the smallest constant \( C \) such that

\[
\forall m \quad \forall x_1, x_2, \ldots, x_m \in \ell_\infty^n, \quad \|(u(x_i))\|_{L_r/H^r(\ell_p^n)} \leq C\|x_i\|_{\ell_p^n(\ell_\infty^n)}.
\]

Obviously, we have \( C_r(u) = \|u\| \). Choose \( p \) such that \( 1 < p < 2 \). Let \( \theta \) be such that \( 1/p = (1 - \theta)/r + \theta/2 \). A simple adaptation of Proposition 1 and 2 yields a simultaneous "good" lifting for the couple \( L_r/H^r(\ell_r^m), L_r/H^r(\ell_2^m) \), and the corresponding extension of (4). It follows that we have for some constant \( C' \) (independent of \( m \))

\[
\|(u(x_i))\|_{L_r/H^r(\ell_p^m)} \leq C'C_2(u)\theta \|u\|^{1 - \theta} m^{1/p} \sup \|x_i\|_{\ell_\infty^n}.
\]

As a consequence, if \( B' = C'C_2(u)\theta \|u\|^{1 - \theta} \), we have

\[
(10) \quad \|(u(x_i))\|_{L_r/H^r(\ell_p^m)} \leq B'\|(x_i)\|_{\ell_p^m, (\ell_\infty^n)}.
\]

It is easy to check that for some constant \( C'' \) (independent of \( m \) or \( n \)) we have,

\[
\|(x_i)\|_{\ell_p^m, (\ell_\infty^n)} \leq C'' m^{1/p - 1/2} \|(x_i)\|_{\ell_2^m(\ell_\infty^n)}
\]

so that (10) gives after normalisation (here \( L_p^m \) denotes the \( L_p \)-space relative to \( \{1, 2, \ldots, m\} \) equipped with the uniform probability measure)

\[
(11) \quad \|(u(x_i))\|_{L_r/H^r(L_p^m)} \leq B'C''\|(x_i)\|_{L_p^m(\ell_\infty^n)}.
\]

Let \( K = B'C'' \). Now we take \( m = 2^k \), we replace \( (x_i) \) by the \( 2^k \) "choices of signs" \( x_t = \sum_1^k r_i(t)x_i \) and we use the dualisation of Khintchine’s inequality in \( L_p \) which says that, if \( p > 1 \), the quotient of \( L_p \) by the orthogonal of the Rademacher functions can be identified with \( \ell_2 \). If we simply denote by \( Q(n) \) the quotient space of \( L_2(\ell_\infty^n) \) by the subspace of all functions "orthogonal" to the Rademacher functions (i.e. which have a zero integral against any Rademacher function), we can deduce from (11)

\[
(12) \quad \|(u(x_i))\|_{L_r/H^r(\ell_2^k)} \leq K\|(x_i)\|_{Q(n)}.
\]
But on the other hand by a known reformulation of Grothendieck's theorem (see [P1], corollary 6.7, p. 77), we have

\[ \|(x_i)\|_{Q(n)} \leq K' \|(x_i)\|_{\ell_\infty^m(\ell_2^n)} \]  \hspace{1cm} (13)

where \(K'\) is a numerical constant. Therefore, (12) implies

\[ C_2(u) \leq KK'. \]

Recalling the value of \(K\) and \(B'\), we conclude that

\[ C_2(u) \leq K'C'C'_2(u)\|u\|^{1-\theta}, \]

so that we again conclude by "extrapolation" that \(C_2(u) \leq K''\|u\|\) for some constant \(K''\) depending only on \(r\). Combining (12) and (13) with this last estimate, we obtain the announced result (9) with \(C_r = KC'C''\). Since there is obviously a norm one inclusion of \(L_r/H^r(\ell_2^n)\) into \(\ell_2^n(L_r/H^r)\), we have \(\pi_2(u) \leq C_2(u) \leq C_r\|u\|\), and this completes the proof for \(X = \ell_\infty^n\), (with a constant \(C_r\) bounded independently of \(n\)). By density, this is enough to prove the case of an operator defined on \(c_0\). Finally, the cotype 2 property can be proved as indicated in Remark 6, by observing that the first part of Theorem 7 remains valid with \(L_r/H^r(l_r)\) (or equivalently \(l_r(L_r/H^r)\)) in the place of \(L_r/H^r\). We then follow a standard argument, given elements \(x_1, x_2, \ldots, x_n \in L_r/H^r\), we consider the operator \(u : \ell_\infty^n \to L_r(L_r/H^r)\) defined by \(u(a_1, a_2, \ldots, a_n) = \sum a_i r_i x_i\), where \(r_1, r_2, \ldots, r_n\) are the Rademacher functions as before. We have

\[ (\sum \|x_i\|^2)^{1/2} \leq \pi_2(u) \leq C_r\|u\|, \]  \hspace{1cm} (14)

but it is well known that there is a constant \(B_r\), depending only on \(r\) such that

\[ \|u\| \leq B_r \sum r_i(t)x_i\|_{L_r(dt; L_r/H^r)}, \]

therefore, (14) implies that \(L_r/H^r\) is a cotype 2 space. q.e.d.
Final Remarks:

1) As a corollary, one obtains that every rank \( n \) operator on \( A \) extends to the whole of \( C(T) \) with norm at most \( C \log n \) for some constant \( C \). This follows from Bourgain’s theorem and a previous result of Mityagin and Pelczyński, see [B1] for the deduction.

2) The preceding argument shows that

\[
H^\infty(\ell_p, \infty) = (H^\infty(\ell_1), H^\infty(\ell_\infty))_{\theta, \infty}
\]

where \( 1/p = 1 - \theta, 0 < \theta < 1 \). But this kind of result is not really new. It can be derived from the remarks on interpolation spaces included in [B1] using a rather simple factorisation argument, such as for instance the one used for Theorem 2.7 in [HP]. More results along this line have been obtained by Xu [X]. In [P2], we will give a more systematic treatment of results such as (15), in more general cases for the real interpolation method with arbitrary parameters.

3) We should mention that while Kisliakov’s recent proof of Bourgain’s theorem seems more complicated than the above, it also yields more information (on the so-called \((p,q)\)-summing operators) which do not follow from our approach, cf. [K2,K3]. Moreover, although the above argument applies also for an operator defined on \( H^\infty \) and with values in a cotype 2 space \( Y \) with the Bounded Approximation Property, it is a well known drawback of the ”extrapolation method” that it does not apply to the case of a linear operator from \( H^\infty \) into its dual, although that case was settled in [B2].

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Texas A. and M. University
College Station, TX 77843, U.S.A.
and
Université Paris 6
Equipe d’Analyse, Boîte 186,
75230 Paris Cedex 05, France