Lattice Quantum Gravity: EDT and CDT

J. Ambjorn\textsuperscript{a,b}

\textsuperscript{a} The Niels Bohr Institute, Copenhagen University
Blegdamsvej 17, DK-2100 Copenhagen \O, Denmark.
email: ambjorn@nbi.dk

\textsuperscript{b} Institute for Mathematics, Astrophysics and Particle Physics (IMAPP)
Radboud University Nijmegen, Heyendaalseweg 135,
6525 AJ Nijmegen, The Netherlands

Abstract

This article is an overview of the use of so-called Euclidean Dynamical Triangulations (EDT) and Causal Dynamical Triangulations (CDT) as lattice regularizations of quantum gravity. The lattice regularizations have been very successful in the case of two-dimensional quantum gravity, where the lattice theories indeed provide regularizations of continuum well defined quantum gravity theories. In four-dimensional spacetime the Einstein-Hilbert action leads to a theory of gravity which is not renormalizable as a perturbative quantum theory around flat spacetime. It is discussed how lattice gravity in the form of EDT or CDT can be used to search for a non-perturbative UV fixed point of the lattice renormalization group in the spirit of asymptotic safety. In this way it might be possible to define a quantum theory of gravity also at length scales smaller than the Planck length.

\textsuperscript{1}This is a contribution to the Handbook of Quantum Gravity which will be published in 2023. It will appear as a chapter in the section of the handbook denoted Causal Dynamical triangulations.
1 Introduction

So far there is no universally accepted quantum theory of four-dimensional gravity. The classical theory of general relativity is not perturbative renormalizable. Therefore, if we think about four-dimensional quantum gravity as small quantum fluctuations around some classical geometry that solves Einstein’s equations, it only makes sense as an effective quantum field theory up to some energy or down to some length-scale, determined by the coupling constants entering in the classical theory (this is discussed in detail in the Section “Effective Quantum Gravity” in the Handbook). Unfortunately, we have presently no experiments which can guide us if we want to approach or go beyond this scale, which is the Planck energy or the Planck length\(^2\). There are examples of other quantum field theories which, when viewed at low energies, appeared to be non-renormalizable, like the theory of weak interactions and the theory of strong interactions. In the case of the weak interactions, the four-fermion interaction, originally suggested to explain the weak interactions, is non-renormalizable. However, we now know that at high energies it is resolved into renormalizable interactions mediated by \(W\) and \(Z\) particles. Similarly, the non-renormalizable non-linear sigma model, which was used to described the low energy \(\pi-\pi\) interaction related to the strong interactions, is a low-energy effective action of an underlying renormalizable quantum field theory of quarks and gluons. From these examples it is tempting to conjecture that the same could happen to quantum gravity, and that the non-renormalizability of gravity would be resolved at larger energy by new degrees of freedom that we have not yet observed. It could be the case, but gravity still looks different from these two examples. In the case of the weak interactions one was led to the four-fermion interaction and not to the renormalizable version of the weak interactions simply because the \(W\) and \(Z\) particles were so heavy that they had not yet been observed. In the case of the strong interactions one was led to a \(\pi-\pi\) interaction because the quark and gluons were not observed, not because they were heavy, but because of quark and gluon confinement. In both cases the starting points were really (effective) quantum theories, the classical aspects of the theories playing minor roles. In gravity the situation is different. We have a classical theory, which seemingly works very well, and this theory even has long-range massless classical excitations propagating with the velocity of light in a classical background geometry, the now famous gravitational waves. It does not seem too promising to try to explain this as a limit of a renormalizable quantum theory constructed from heavy, yet to be observed, fundamental particles, or from “confined” light particles.

String theory is one attempt to provide a quantum theory of gravity. More precisely, closed string theory contains massless spin-two excitations, which can be interpreted as quantum gravity particles and the underlying stringy nature of the theory solves the UV problems associated with quantizing the Einstein-Hilbert action of classical gravity. The original hope was that the (super)string theory would provide us with an explanation of all the particles we actually

\(^2\)If \(G\) denotes Newton’s gravitational constant, \(c\) the velocity of light and \(\hbar\) the Planck constant, the Planck energy is \(E_p = \sqrt{\hbar c^5/G}\) and the Planck length is \(\ell_p = \sqrt{\hbar G/c^3}\).
observe in nature, and at the same time it predicted the existence of particles we have not yet observed. Unfortunately, no clear picture related to the world we observe has yet emerged from string theory, which, since the ambitious start in the 1980s as a Theory of Everything, has developed in many directions. The directions still related to gravity will be described in the Section of the Handbook dedicated to string theory.

Loop quantum gravity is another attempt to circumvent the problem associated with a “naive” quantization of gravity based on the Einstein-Hilbert action. It deals with the UV problems of the naive approach by postulating a new quantization procedure which leads to a Hilbert space quite different from the standard Fock space of particle physics. This procedure defines in principle the physics at the Planck scale, but it becomes difficult to relate the theory to classical gravity as we observe it today. Like string theory it has branched out in a number of different directions, again described in the Handbook in the Section about loop quantum gravity.

Lattice quantum gravity, as described in this Section, is closely related to a field theoretical approach to quantum gravity using what is called asymptotic safety. A special Section in the Handbook is dedicated to the use of asymptotic safety when the quantizing gravity. The hypothesis is that one can use ordinary quantum field theory to quantize gravity and that the UV limit of the theory corresponds to a non-perturbative fixed point. Around this fixed point one cannot apply ordinary perturbation theory, by expanding in a power series of coupling constants appearing in the classical low-energy Lagrangian. Nevertheless it is postulated that there exists a continuum renormalization group flow of the effective action that will reach the UV fixed point by adjusting only a finite number of suitable coupling constants. In this sense the non-perturbative fixed point is similar to a (Gaussian) UV fixed point of a renormalizable quantum field theory. It is in this context that lattice quantum gravity becomes interesting for a number of reasons.

From a Wilsonian point of view lattice field theory is well suited to studying fixed points and renormalization group flows, as well as non-perturbative aspects of the corresponding quantum field theories. The lattice will provide a UV regularization of the quantum field theory in question. Such a UV regularization is usually needed as starting point for defining an interacting quantum field theory. In order to define the corresponding continuum quantum field theory one will in general need to take the UV cut-off, i.e. the lattice spacing, to zero relative to some continuum length scale characterizing the continuum theory. If the theory contains massive particles one can use the inverse mass of such a particle as the length scale (in units where $c = \hbar = 1$). Keeping such a physical length scale fixed while taking the lattice spacing to zero implies that this length scale measured in lattice units will diverge. This is most simply realized in lattice field theories if a correlator of one of the lattice fields for a generic choice of the lattice coupling constants $g_i$ of the theory is decaying exponentially with the distance between the lattice points, in this way defining a correlation length $\xi(g_i)$. A second- (or higher-) order phase transition of the lattice field theory is often characterized by a divergent correlation length. Thus, when trying to find
continuum limits of the lattice field theory, it is natural to look for regions in
the lattice coupling space associated with second- or higher-order phase transi-
tions. One of the assumptions in the Wilsonian approach, if one consider lattice
field Hamiltonians with arbitrary local interactions, i.e. in principle an infinite-
dimensional coupling constant space, is that the critical surface of higher-order
phase transitions has a finite co-dimension. In this case one only has to fine-
tune a finite number of coupling constants to reach the critical surface where the
lattice correlation length is infinite. The way in which one approaches the crit-
ical surface will define the continuum physical parameters of the corresponding
continuum quantum field theory (like the masses of the particles corresponding
to lattice field correlators), and the nature and the number of lattice coupling
constants which need to be fine-tuned to reach the critical surface will depend
on the so-called fixed points of the lattice renormalization group. These fixed
points are located on the critical surface, and in the Wilsonian picture each can
be used to define a continuum quantum field theory.

We want to use this lattice Wilsonian framework to investigate whether we
can define a continuum limit of theories we can denote “lattice gravity” theo-
ries, more precisely, the lattice gravity theories based on Euclidean Dynamical
Triangulations (EDT) or Causal Dynamical Triangulations (CDT). We have a
space of (dimensionless) lattice coupling constants associated with the theory
and want to locate regions in this space where there are second- (or higher-)
order phase transitions. When such regions are localized, we want to under-
stand whether one can approach these phase transition regions such that one
obtains a theory that can be viewed as the quantum theory of gravity. It is
of particular interest if the phase transition surface can be associated with a
UV fixed point, since in this case one might have defined the quantum gravity
theory at arbitrarily short distances.

A number of interesting conceptual problems are associated with quantum
gravity and the Wilsonian lattice renormalization group. The central Wilsonian
idea is that a divergent lattice correlation length of some observable makes it
possible to forget the underlying lattice and that using a limiting procedure
makes it possible to define a continuum quantum field theory. It is also the
reason for the universality associated with the Wilsonian approach: the details
of the local lattice structure as well as the details of the interactions at lattice
distances are often of no consequence for the continuum limit. Global sym-
metries of the interactions might be (and are) important as they can survive
when a continuum limit is taken. However, when trying to apply this line of
reasoning to a lattice gravity theory we are faced with the very simple question:
how does one define a correlation length in a theory of quantum gravity? When
implementing the quantum theory via a path integral, we are instructed to inte-
grate over all geometries, but it is the geometries which define the distances. In
non-gravitational relativistic quantum field theory, correlators are functions of
spacetime points, and the main reason we study these correlators is that their
behavior as a function of the distances between these spacetime points tells us
a lot about the underlying quantum theory. They are also the natural objects
on which the renormalization group acts. Thus it is somewhat disturbing that
it is unclear how to define such correlators in a theory of quantum gravity in a way that relates to distances. A common, and in general healthy, attitude in theoretical physics is to calculate whatever can be calculated and postpone annoying questions like how to define distances in a theory of quantum gravity. However, one nice thing about a lattice theory of gravity is that one is forced to address such questions. Four-dimensional lattice gravity cannot in any way be solved analytically, but one can perform computer simulations of the lattice theory. If one wants to measure anything but the simplest global observables in such computer simulations, one should better have a precise idea of how to define the observable to measure in a sensible way. In the rest of this Chapter we will discuss lattice quantum gravity from this Wilsonian point of view and how one can in principle use the lattice approach to test the asymptotic safety conjecture.

2 A Wilsonian view on two-dimensional EDT and CDT

The EDT partition functions

Two-dimensional gravity is classically a trivial theory since there are no propagating gravitons in two-dimensional spacetime. One reflection of this is that the curvature term in the Einstein-Hilbert action is a topological invariant in two dimensions. Thus, as long as one does not consider topology changes, and we will not do that, the action just contains the cosmological term, a term with no derivatives of the metric. If we consider spacetimes with Euclidean signature, we have the two-dimensional partition function

\[ Z(G, \Lambda, Z_i) = \int \mathcal{D}[g] e^{-S[g]}, \]

\[ S[g] = -\frac{1}{2\pi G} \int d^2 \xi \sqrt{g} (R(\xi) - 2\Lambda) = -\frac{\chi}{G} + \Lambda V[g] + \sum_{i=1}^{n} Z_i L_i[g]. \]

The path integral (1) is over all geometries \([g_{ab}]\) (i.e. metrics \(g_{ab}\) up to diffeomorphism equivalence) on a two-dimensional manifold with \(h\) handles and \(n\) boundaries and with Euler characteristic \(\chi = 2 - 2h - n\). \(\Lambda\) is the cosmological constant (divided by \(\pi G\)) and \(Z_i\) are suitable boundary cosmological constants, which are only introduced for later convenience. \(V[g]\) denotes the two-dimensional volume of the manifold, while \(L_i[g]\) denotes the length of the \(i\)th boundary, all measured in the geometry \([g_{ab}]\). In the following we will ignore the topological term and only consider manifolds with the topology of a sphere.
with boundaries. The partition function (1) can be written as

\[ Z(\Lambda, Z_i) = \int_0^\infty dV \int_0^\infty dL_i e^{-\Lambda V - L_i Z_i} \int D\mathcal{V},L_i[g] \]

\[ = \int_0^\infty dV \int_0^\infty dL_i e^{-\Lambda V - L_i Z_i} N(V,L_i). \]

\[ [g] \] denotes a geometry and in (3) the functional integration is over all geometries which have spacetime volume \( V \) and \( n \) boundaries of lengths \( L_i \). This integration is formally equal to the number of such geometries. In other words, we can compute the partition function of two-dimensional quantum gravity if we can count the number of geometries with a given spacetime volume and given lengths of the boundaries. Moreover, from this perspective the partition function \( Z(\Lambda, Z_i) \) can be viewed as the generating function for the numbers \( N(V,L_i) \), with \( e^{-\Lambda} \) and \( e^{-Z_i} \) playing the role of indeterminates in this generating function. Of course \( N(V,L_i) \) is formally infinite, reflecting the fact that the path integral \( \int D\mathcal{V},L_i[g] \) needs a UV cut off to be defined in the first place. The so-called (Euclidean) dynamical triangulations (EDT) provide a useful regularization. In its simplest version one approximates the integration over geometries by a summation over triangulations constructed from equilateral triangles with link length \( a \), where \( a \) serves as a UV cut-off. To each such triangulation one can associate a piecewise linear geometry by assuming the triangles are flat in the interior. The curvature of the piecewise linear geometry is then naturally located at the vertices. Summing over such triangulations one obtains an approximation to the continuum partition function (1) when one makes the identification

\[ V(T) = \frac{\sqrt{3}}{2} N(T)a^2, \quad L_i(T) = l_i(T)a, \]

where \( N(T) \) is the number of triangles and \( l_i(T) \) the number of boundary links of the \( i \)th boundary in the triangulation \( T \). The lattice gravity partition function can be written as

\[ Z(\mu, \lambda_i) = \sum_T e^{-\mu N(T) - \sum_i \lambda_i l_i(T)} = \sum_{N,l_i} e^{-\mu N - \sum_i \lambda_i l_i} N(N,l_i), \]

\[ \text{Historically, the main interest in the EDT regularization was linked to the use as a regularization of the Polyakov path integral for the bosonic string in } D\text{-dimensional spacetime}[12, 13, 14, 44, 50, 57, 45]. \text{ This path integral can be viewed as two-dimensional quantum gravity coupled to } D\text{ bosonic fields } X_i, \text{ constituting the } D\text{ coordinates of the bosonic string. Unfortunately, the approach did not work when implemented in the simplest way for } D > 1 \text{ as shown in [2]. However, for } D < 1 \text{ it has been very successful, and known as “non-critical” string theory, as will be mentioned below. Pure two-dimensional quantum gravity, which we discuss here, corresponds in this context to } D = 0, \text{ and was first introduced in [48] and discussed in [57]. There are interesting indications that the formalism can be revived as a regularization of bosonic strings for } D > 1 \text{ by taking a new kind of scaling limit [8, 9, 10].} \]

\[ \text{It is assumed that a link on each boundary is marked, in order to avoid symmetry factors appearing in the sum over triangulations.} \]
which is the lattice version of (3) and (4), the integration over geometries being replaced by the summation over equilateral triangles with link length $a$, and with

$$\mu = \Lambda_0 a^2, \quad \lambda_i = Z_i^{(0)} a.$$  

(7)

We call $\Lambda_0$ and $Z_i^{(0)}$ the bare, unrenormalized coupling constants for reasons that will become clear below. By counting the number of triangulations, $N(N, l_i)$, with the topology of a sphere with $n$ boundaries, and performing the sum and eventually taking the limit $a \to 0$, one can then explicitly find the partition function of two-dimensional quantum gravity.

Let us discuss the Wilsonian aspect of the above procedure. From the Wilsonian point of view, the continuum limit should not depend in a crucial way on precisely which class of triangulations one chooses. Similarly, one should be able not only to use triangles, but also squares, pentagons etc. as building blocks, all with link lengths $a$. One then loses the unique piecewise geometry associated with a given graph $T$, but in the Wilsonian spirit one would still assume that for very large graphs one can make an identification like in eq. (5):

$$V(T) \propto N(T)a^2, \quad L_i(T) \propto l_i(T)a,$$

(8)

where $N(T)$ denotes the number of polygons in the graph $T$ and $l_i$ the number of links of the $i$th boundary. This turns out to be true. For a particular set of graphs, so-called bipartite graphs (again with the topology of a sphere with $n$ boundaries), one can even find the corresponding generating function explicitly [15, 17],

$$Z(g, z_i) = \left( \frac{1}{M_1(c^2, g)} \frac{d}{dc^2} \right)^{n-3} \frac{1}{2c^2 M_1(g, c^2)} \prod_{i=1}^{n} \frac{c^2}{(z_i^2 - c^2)^{3/2}}, \quad n \geq 3.$$  

(9)

In this expression we have assigned the indeterminate $g_k = gw_k$ to each 2$k$-edged polygon which enters in the graph, and an indeterminate $1/z_i$ to each link in the $i$th boundary. The relative weights of the polygons are $w_k \geq 0$ and

$$M_1(c^2, g) = \oint_C \frac{dz}{2\pi i} \frac{z V'(z)}{(z^2 - c^2)^{3/2}}, \quad V'(z) = z - \sum_k g_k z^{2k-1},$$

(10)

where the contour $C$ encloses the cut $[-c, c]$ on the real axis and where we assume that only a finite, but in principle arbitrarily large, number of the $w_k$ can be different from zero. Finally, the cut $[-c, c]$ is determined as a function of $g$ by the following equation for $c^2(g)$

$$\oint_C \frac{dz}{2\pi i} \frac{z V'(z)}{(z^2 - c^2(g))^{1/2}} = 2.$$  

(11)

5In this context we define the bipartite graphs as surfaces constructed by gluing together polygons with an even number of links, and where also the boundary loops consist of an even number of links.
We present these explicit formulas because they tell us how to take the continuum limit of the lattice theory. We are interested in a limit where the number $N$ of polygons goes to infinity. To each polygon we associate an indeterminate $g$, and $Z(g, z_i)$ has a convergent power expansion in $g$ for small $g$. Large $N$ will dominate when one reaches the radius of convergence of $Z(g, z_i)$. This occurs either when $M_1(c^2(g), g) = 0$ or when $c^2(g)$ ceases to be an analytic function of $g$. This happens to be at the same point $g_0$. This $g_0(w_j)$ will be a function of the relative weights $w_k$, which in this discussion of convergence we consider fixed. Similarly, we might be interested in the situation where the lattice lengths $l_i$ go to infinity. This happens by the same reasoning when $z_i = c(g)$, where (9) is non-analytic in $z_i$. Denoting $z_0 = c(g_0)$, $g_0(w_j)$ and $z_0(w_j)$ are critical points of our statistical system of graphs. By approaching these critical points according to

$$g = g_0(w_j) e^{-\Lambda a^2} = e^{-\mu}, \quad \frac{1}{z} = \frac{1}{z_0(w_j)} e^{-Z_i a} = e^{-\lambda_i},$$

we can take the continuum limit of (9) by scaling $a \to 0$, and make contact with (4):

$$Z(g, z_i) \propto a^{5 - \frac{7n}{2}} \left( -\frac{d}{d\mu} \right) \left[ 1 + \sum_{k=1}^{n} \frac{1}{Z_k + \sqrt{\Lambda}} \right] \propto a^{5 - \frac{7n}{2}} Z(\Lambda, Z_i),$$

valid for $n \geq 3$. In particular we find from (13) and (4) by inverse Laplace transformation

$$Z(V, L_i) \equiv \mathcal{N}(V, L_i) \propto V^{n - \frac{7}{2}} \sqrt{L_1 \cdots L_n} e^{-L_1 + \cdots + L_n}/4V.$$

This formula is also valid for $n = 0, 1, 2$. From (12), (6) and (7) the relation between $\mathcal{N}(V, L_i)$ and $\mathcal{N}(V, L_i)$ is

$$\mathcal{N}(V, L_i) \propto e^{\mu_0 N + \lambda_i^{(0)}}, \mathcal{N}(V, L_i), \quad e^{\mu_0} = g_0, \quad e^{\lambda_i^{(0)}} = \frac{1}{z_0},$$

which shows that the number of generalized triangulations (bipartite graphs) with spherical topology and $n$ boundaries grows exponentially with $N$, the number of polygons in the graphs. The number of graphs also grows exponentially with $l_i$, the number of boundary links. Eq. (12) can be seen as additive renormalizations of the cosmological and boundary cosmological constants $\Lambda_0$ and $Z_i^{(0)}$.

$$\mu = \Lambda_0 a^2 = \mu_0 + \Lambda a^2, \quad \lambda_i = Z_i^{(0)} a = \Lambda_i^{(0)} + Z_i a.$$

The Wilsonian aspect of the above formulas is the following: we have an infinite-dimensional coupling constant space corresponding to $g_k \geq 0$. The critical surface is defined by $g_k = g_0(w_j) \approx k$, where for given $w_k$, the $g_0(w_j)$ is the critical point discussed above. Thus the critical surface has co-dimension 1 and approaching it for fixed $w_k$ like in (12) leads to the same continuum theory. In this sense it is a beautiful example of Wilsonian universality, but one can ask: where is the divergent correlation length in the lattice theory, leading to
this universality? This is especially interesting since this is a theory of quantum gravity, and as discussed above, we are integrating over to geometries which define length. We will discuss this in the next subsection.

Let us end this subsection with a remark about the critical surface. We have restricted $w_k$ to be larger than or equal to zero and with only a finite number of the $w_k$ different from zero. If one relaxes these constraints, in particular the constraint that the $w_k$ have to be positive, one can obtain different critical behaviors [56, 37] (which can be given the interpretation of matter systems coupled to quantum gravity). One obtains then a picture where a fine-tuning of the bare coupling constants to reach the critical surface might lead to different regions corresponding to different continuum theories. We will not pursue this possibility any further in this Chapter, but only mention that the corresponding continuum quantum field theories are the so-called quantum Liouville theories with different central charge. These Liouville theories arise when quantizing two-dimensional Euclidean gravity coupled to conformal matter. Integrating out the matter fields, while using the conformal gauge for the metric leads to an effective quantum field theory for the conformal factor of the metric, the Liouville quantum field theory, which depends on the central charge of the conformal matter field integrated out [58, 49, 52, 53]. The relation between the central charge $c_L$ of the Liouville theory (which is also a conformal theory, although a somewhat special one) and the central charge $c$ of the matter field is $c_L = 26 - c$. The pure two-dimensional quantum gravity theory we have mainly discussed above corresponds in this notation to a conformal theory with central charge $c = 0$ and thus a Liouville theory with $c_L = 26$. The last 20 years have seen major progress in understanding and formulating the mathematics behind Liouville quantum gravity, and Chapter 7 in this Section of the Handbook, “Lessons from Mathematics of Two-dimensional Quantum Gravity”, will describe this in detail.

A divergent correlation length in 2d EDT

In an ordinary quantum field theory in flat spacetime a correlator is defined by

$$\langle \phi(x) \phi(y) \rangle = \frac{\int D\phi \, e^{-S[\phi]} \, \phi(x) \phi(y)}{\int D\phi \, e^{-S[\phi]}}$$  \hspace{1cm} (17)

By translational and rotational invariance (which we will assume) $\langle \phi(x) \phi(y) \rangle$ is only a function of $|x - y|$, where $x$ and $y$ are spacetime points. We can take advantage of this by averaging over all points $x$ and $y$ separated by a distance $|x - y| = R$ and define

$$\langle \phi\phi \rangle_R = \frac{\int D\phi \, e^{-S[\phi]} \int dx \int dy \, \delta(|x - y| - R) \, \phi(x) \phi(y)}{\int D\phi \, e^{-S[\phi]}},$$  \hspace{1cm} (18)

where formally this average contains a factor $V$, the volume of spacetime, due to translational invariance. We can embed this definition of a correlation function
in a quantum gravity theory

\[ \langle \phi \phi \rangle_R = \frac{\int D[g] D\phi \ e^{-S[g,\phi]} \int dx \int dy \sqrt{g(x)g(y)} \ \delta(D_g(x,y) - R) \ \phi(x)\phi(y)}{\int D[g] D\phi \ e^{-S[g,\phi]}} \]  

(19)

where \( S[g,\phi] \) denotes the combined action of gravity and the field theory, and where \( D_g(x,y) \) is the geodesic distance between spacetime points labelled \( x \) and \( y \). The correlation function \( \langle \phi \phi \rangle_R \) is diffeomorphism-invariant, but non-local. Since the gravity action contains a cosmological constant \( \Lambda \), the average volume of spacetime will be finite and of order \( 1/\Lambda \). Contrary to (18) there is no infinite formal factor in the definition (19). We call \( R \) the quantum geodesic distance.

Note that it will influence \( \langle \phi \phi \rangle_R \) in a potential much more radical way than the \( R \) in (18); when \( R \) is large compared to some appropriate power of \( 1/\Lambda \), it will define the shape of the whole universe in which we measure the correlation.

Thus, in some ways \( R \) is more like a new coupling constant in the theory, in the sense that the average shape of the universe depending it.

In the case of pure gravity, one has no external field \( \phi(x) \), but one could consider curvature–curvature correlators, or simply replace \( \phi(x) \) by \( 1(x) \), which takes the value 1 for all \( x \). This last choice is of interest since the correlator has a clear geometric interpretation and it can be explicitly calculated in the two-dimensional lattice gravity theory. We define the (unnormalized) two-point function corresponding to (19) with \( \phi(x) = 1(x) \) as

\[ G_\Lambda(R) = \int D[g] \ e^{-\Lambda V_g} \int dx \int dy \sqrt{g(x)g(y)} \ \delta(D_g(x,y) - R) \]  

(20)

This is a formal continuum definition and requires a UV cut-off to define the path integral in (20). Again we use EDT and in addition now have to define the geodesic distance between the spacetime points \( x \) and \( y \) in formula (20) in the context of our triangulations. Let us for simplicity consider a triangulation constructed from equilateral triangles. As already mentioned, this triangulation can be viewed as a piecewise linear surface where the geometry is uniquely defined by assuming the triangles are flat in the interior. From such a piecewise linear triangulation, where one knows the length of each link, one can calculate \( D_g(x,y) \). However, an approximate definition, convenient from a calculational point of view, is to define the graph distance between two links as the shortest distance, passing through centers of neighboring triangles, see Fig. 1.

For generic, very large triangulations, and for links with correspondingly large separation, we expect such a distance to be proportional to the “real” geodetic distance. Denote this graph distance between link \( \ell \) and link \( \ell' \) in a

\[ ^6 \text{For such a large } R, \text{ the universe will be quite “elongated”, because by definition at least two points have to be separated a geodesic distance } R. \]

\[ ^7 \text{Chapter 2 in this Section of the Handbook, “Observables and Curvature in CDT”, describes how to introduce curvature in lattice gravity theories.} \]

\[ ^8 \text{One could also have chosen to define the graph distance between two vertices as the shortest link distance between the two vertices. As usual, from a Wilsonian point of view one should be led to the same continuum limit if it exists.} \]
A triangulation $T$ with two links $\ell$ and $\ell'$ separated by a graph distance $D_T(\ell, \ell') = 14$.

The lattice equivalent of (20) can be written as

$$G_\mu(r) = \sum_T e^{-\mu N(T)} \sum_{\ell, \ell'} \delta_{D_T(\ell, \ell'), r}. \quad (21)$$

Quite remarkably, one can combinatorially calculate the sum of these triangulations [11]. Close to the critical point $\mu_0$ defined in (15), one obtains

$$G_\mu(r) \propto (\mu - \mu_0)^{3/4} \cos \sqrt{\mu - \mu_0} r \sin \sqrt{\mu - \mu_0} r, \quad (22)$$

i.e., an exponential fall-off with a correlation length $\xi(\mu) = 1/\sqrt{\mu - \mu_0}$. Using (16), we can directly read off the continuum limit of (22), provided the geodesic distance $R$ scales anomalously:

$$a^{-3/2} G_\mu(r) \propto G_A(R) = \Lambda^{3/4} \frac{\cos \sqrt{\Lambda} R}{\sin^3 \sqrt{\Lambda} R}, \quad R = a^{1/2} r. \quad (24)$$

Note that the anomalous dimension of $R$ shows that the two-dimensional EDT quantum spacetime is fractal, with Hausdorff dimension $d_h = 4$ at all scales, as first realized in the seminal work [55]. Furthermore,

$$\chi(\mu) = \sum_{r=1}^{\infty} G_\mu(r) = \text{const.} - \frac{1}{6} \sqrt{\mu - \mu_0} + O(\mu - \mu_0) + \cdots \quad (25)$$

$$\equiv \text{analytic} + \frac{1}{(\mu - \mu_0)^\gamma} + \cdots. \quad (26)$$
where $\chi(\mu)$ denotes the susceptibility. The term $(\mu - \mu_0)^{-\gamma}$ is the leading non-analytic term in the expansion of $\chi(\mu)$ around $\mu_0$, and $\gamma$ is called the susceptibility exponent. These notations are inspired by the analogous notations used for spin-spin correlation functions in the theory of critical phenomena. From the definition (21) it follows that $\chi(\mu) \propto d^2Z(\mu)/d\mu^2$, where $Z(\mu)$ is given by (6) with $n = 0$ (no boundaries). We thus obtain

$$Z(\mu) \equiv \text{analytic} + (\mu - \mu_0)^{2-\gamma} + \cdot, \quad \gamma = -\frac{1}{2},$$

(27)

a result which is consistent with (14) and (15). We have identified a divergent correlation length of two-dimensional EDT, and it is directly related to the fractal structure of the corresponding spacetime. The existence of this divergent correlation length explains why the Wilsonian picture works so well in this model. A final remark concerns the quantum geodesic distance $R$ which appears in the definition (20). Eq. (24) shows how the choice of $R$ will affect the general shape of the universe (as already mentioned in footnote 6): for $R \gg 1/\Lambda^{1/4}$ it is a long tube of length $R$ and cross-section proportional to $\Lambda^{-3/4}$.

The generalized two-dimensional CDT theory

As discussed above, the scaling limit for 2d EDT is essentially independent of the choice of weight $w_n$ of the polygons, as long as the weights are non-negative. In a Wilsonian context, a change of universality class is most likely related to a change of some global symmetry. The EDT formalism respects in a formal way the symmetry between space and (Euclidean) time, to a degree that it is unclear how one would actually rotate expressions like the two-point function $G_{\Lambda}(R)$ to spacetimes with a Lorentzian signature. Two-dimensional Causal Dynamical Triangulations (CDT) is a regularization which takes the difference between space and time serious from the outset, and insists on summing over spacetimes which have a well-defined time foliation. It is simplest to implement this in a discretized path integral if one assumes that space has the topology of a circle. Two neighboring spatial slices at discretized integer times $k$ and $k+1$ then consist of $l_k$ and $l_{k+1}$ spatial links, and the two slices are connected by triangles with one spatial link and two time-like links, in such a way that the corresponding two-dimensional triangulation with the spatial slices at $k$ and $k+1$ has the topology of a cylinder, as illustrated in Fig. 2. Clearly, one can in this way iteratively construct a two-dimensional triangulation with spatial slices at $k$, $k = 1, \ldots, s$, consisting of $l_k$ links. This yields a cylinder with an “entrance” spatial loop consisting of $l_1$ and an “exit” spatial loop consisting of $l_s$ links, as also shown in Fig. 2. Like in the EDT case, only the cosmological term will be important if we sum over piecewise linear manifolds with a fixed topology in the path integral. We can write, for a Lorentzian triangulation $T_{\text{lor}}$ of the kind discussed

$$S_{T_{\text{lor}}} (\Lambda, \alpha) = -\Lambda N(T_{\text{lor}}) \frac{\sqrt{4\alpha + 1}}{4} a^2, \quad a_{t}^2 = -\alpha a^2, \quad \alpha > 0.$$

(28)
In (28) we use the explicit area of a triangle with one spatial link of length $a_2^2 = a^2$ and two time-like links with $a_1^2 = -\alpha a^2$ (see [31, 20] for details). If $\alpha > 1/4$ we can perform an analytic continuation in the lower complex $\alpha$-plane to negative $\alpha$ such that

$$S_{T_{\text{lor}}}(\Lambda, \alpha) \rightarrow S_{T_{\text{lor}}}(\Lambda, -\alpha - i\epsilon) = iS_{T_{\text{eucl}}}(\Lambda, \tilde{\alpha}),$$

where the Euclidean triangulation is denoted $T_{\text{eucl}}$, and where

$$S_{T_{\text{eucl}}}(\Lambda, \tilde{\alpha}) = \Lambda N(T_{\text{eucl}}) \frac{\sqrt{4\tilde{\alpha} - 1}}{4} a^2 \quad \tilde{\alpha} = \alpha > \frac{1}{4}. \quad (30)$$

The inequality $\tilde{\alpha} > 1/4$ has the simple geometric interpretation that the sum of lengths of the two “time-like” triangle sides (i.e. $2\sqrt{\alpha} a$) has to be larger than the length $a$ of the space-like side of a triangle in the “flat” Euclidean triangles used in the rotated triangulation.

Eq. (29) is the “usual” formal relation between the Lorentzian and Euclidean actions, such that

$$e^{iS_{T_{\text{lor}}}(\Lambda, \alpha)} = e^{-S_{T_{\text{eucl}}}(\Lambda, \tilde{\alpha})}, \quad \tilde{\alpha} = \alpha > \frac{1}{4}. \quad (31)$$

For each $T_{\text{lor}}$ we perform the rotation to a corresponding $T_{\text{eucl}}$ with the actions related by (29). The important point is that the class of triangulations $\{T_{\text{eucl}}\}$

Figure 2: Left figure: A CDT triangulation (represented as an annulus). Constant time slices corresponding to $k = 1, 2, 3$ are circles. A vertex (or the spatial link to the right of it) on the entrance loop $k = 1$ is marked. Right figure: the corresponding branched polymer (thick black links). An artificial vertex at $k = 0$ connected to each vertex at the $k = 1$ loop ensures a bijection between the CDT triangulations with boundaries at times $k = 1$ and $k = 3$ and so-called rooted branched polymers of height 3 (the root connects the vertex at $k = 0$ to the marked vertex at $k = 1$).
obtained in this way is quite different from the class used in EDT. From now on we will set $\alpha = 1$ since it only contributes a constant of proportionality to the action, where we have anyway already absorbed a factor of proportionality in $\Lambda$. We thus write, as in EDT

$$S_T(\Lambda) = \Lambda N(T)a^2 = \mu N(T), \quad \mu = \Lambda_0 a^2, \tag{32}$$

where the summation is over the triangulations described above, which have the topology of a cylinder, with $s$ spatial slices, where slice 1 consists of $l_1$ spatial links and slice $s$ of $l_s$ spatial links.

The two-dimensional CDT model (and related models) can be solved analytically [7, 51, 3], and rather surprisingly the critical exponents of the model agree with corresponding critical exponents of tree graphs or so-called branched polymers. Later it was understood that this is not a coincidence [46], but that there exists a bijective map of the CDT surface graphs onto so-called rooted branched graphs of height $s + 1$, as illustrated in Fig. 2. This insight highlights the importance of tree-like structures in graphs relevant to quantum gravity. Chapter 5 in this Section of the Handbook, “From Trees to Gravity”, is dedicated to the study of such tree-like graphs.

If we define a slightly modified CDT graph by connecting all vertices at time-slice $k = 1$ to a single vertex at a new time-slice at $k = 0$ and all vertices at time-slice $k = s$ to a single vertex at a new time-slice at $s + 1$ then the graph distance (which we here define to be the shortest link distance) between any vertices will be less or equal to $s + 1$. If we start at the vertex at time 0, then the only vertex where the graph distance to the starting vertex is a local maximum is the vertex at $s + 1$ (and the local maximum is in this case also a global maximum). From a graph point of view this is a rather special situation and one can generalize it to include graphs where a finite number of vertices have a local maximum distance to a starting vertex, even in the limit where the number of vertices goes to infinity. This is the setup of generalized CDT: starting from a vertex or a spatial entrance loop, one moves forward in “proper time”, which is defined as the graph distance from the vertex or the entrance loop (and in the continuum by the geodesic distance from the entrance loop). On the way to the exit spatial loop (or loops), space can branch into several disconnected spatial universes. The ones that do not end in exit loops vanish into the “vacuum”. The distances of these vacuum points to the entrance loop are then local maxima, and the spatial loops that in this way disappear into the vacuum are called baby universes. For graphs consisting of a finite number of

9Again we assume, as in the EDT case, that a boundary link is marked on one of the boundary loops, to avoid symmetry factors occurring in the sum over triangulations.
vertices, there is no real difference between the graphs used in EDT and the ones used in generalized CDT, but the crucial difference comes from requiring that when the number of vertices goes to infinity the number of baby universes stays finite. This will then also be true in the continuum limit and is in contrast to the EDT situation, where the fractal structure with Hausdorff dimension $d_h = 4$ implies that infinitely many baby universes (but most of them with infinitesimal volume) are created in the continuum limit. In the case of generalized CDT one finds $d_h = 2$. Again the discretized model can be solved analytically and one can take the continuum limit\textsuperscript{10} in much the same way as was done in the EDT case \cite{1}. In particular, one can find the continuum version of the generalized CDT two-point function

$$G_\Lambda(\tau) = \frac{\Sigma^3}{\Theta} \left( \frac{\Sigma \sin \Sigma \tau + \Theta \cos \Sigma \tau}{\left( \Sigma \cos \Sigma \tau + \Theta \sin \Sigma \tau \right)} \right)^3$$

where

$$\Sigma = \sqrt{\Lambda} \ H\left(\frac{G_b}{\Lambda^{3/2}}\right), \quad H(0) = 1; \quad \Theta = \sqrt{\Lambda} \ F\left(\frac{G_b}{\Lambda^{3/2}}\right), \quad F(0) = 1.$$ (36)

The functions $H(x)$ and $F(x)$ have a power expansion in $x$, with a radius of convergence $2/3^{3/2}$. A new coupling constant denoted $G_b$ has appeared in (36). It is the coupling constant for a spatial universe to split into two spatial universes. When $G_b/\Lambda^{3/2} \to 0$ we get back to ordinary CDT. When $G_b/\Lambda^{3/2} \to 2/3^{3/2}$, (35) and (36) cease to be valid and one can show that the number of baby universes goes to infinity, indicating that one has a phase transition to ordinary EDT gravity. We have in (35) denoted the geodesic distance entering in the two-point function by $\tau$ (proper time) rather than the $R$ used in (24), to emphasize the origin as a proper time in Lorentzian CDT. Eq. (35) looks superficially like a generalization of eq. (24). However, the important point is that we have $\sqrt{\Lambda} \tau$ as an argument, while in (24) $\sqrt{\Lambda} R$ appears as an argument, capturing the difference in Hausdorff dimension for the two ensembles of geometries. We still have a perfect Wilsonian picture for generalized CDT as embedded in EDT \cite{28, 34}. In EDT one can introduce an additional dimensionless coupling $g_b$, which controls the creation of baby universes, such that for small values of $g_b$ the creation of baby universes is suppressed. One can show that for any finite value of $g_b$ the critical behavior of the system is still that of two-dimensional Euclidean gravity. However, if $g_b$ is scaled to zero at the same time as one approaches the critical surface in the following way (which is a generalization of (16))

$$g_b = G_b a^3, \quad \mu = \mu_0 + \Lambda a^2, \quad \lambda_i = \lambda_i^{(0)} + Z_i a,$$ (37)

one obtains generalized CDT with continuum coupling constants $\Lambda$, $Z_i$ and $G_b$. From a Wilsonian point of view we have an infinite-dimensional critical surface\textsuperscript{10} Rather amazingly, it is possible to solve the model directly in the continuum simply be using that the number of baby universes is finite \cite{23, 26, 27}
and on this critical surface a subspace where \( g_b = 0 \). On this critical subspace there is an asymmetry between “space” and “time”, the same asymmetry that was put in by hand in the original simple CDT model and that was not present in the EDT model. Approaching the subspace as in (37) one will obtain the continuum limit corresponding to generalized CDT, while approaching the critical surface at a point where \( g_b > 0 \), in the way described by (16) leads to the continuum limit of EDT.

Hořava-Lifshitz gravity is a continuum theory where one demands that space-time has a time foliation and that the theory is invariant under spatial diffeomorphisms and time redefinitions [54]. Chapter 4 in this Section of the Handbook discusses (among other topics) the relation between two-dimensional CDT and two-dimensional Hořava-Lifshitz gravity. (see also [32]). In the case of higher dimensional CDT no such relation is known to exist.

3 Four-dimensional EDT

While two-dimensional EDT and CDT have a clear Wilsonian interpretation where continuum limits can be defined and continuum correlators can be calculated analytically, the situation is more complicated when one wants to generalize the EDT and CDT formalism to higher-dimensional gravity. First, higher-dimensional gravity is non-renormalizable, and the curvature term that dropped out in two-dimensional quantum gravity is now expected to play a key role. In the Wilsonian context of asymptotic safety\(^{11}\), one needs a non-trivial UV fixed point of the lattice theory if conventional renormalization group logic applies and if one wants the lattice theory to define a quantum continuum theory at all scales. Secondly, contrary to the situation in two dimensions, there is presently no way we can solve the lattice theory analytically. We have to rely on Monte Carlo simulations of the path integral. This implies that we have to use an action with Euclidean signature, since the Monte Carlo simulations need the exponential of the action to have a probability interpretation. Unfortunately, the Euclidean four-dimensional continuum Einstein-Hilbert action is unbounded from below and this will be true also for the lattice action when the lattice volume becomes infinite. Only the measure term in the path integral may save us, if we want to restrict ourselves to the Einstein-Hilbert term as the classical action appearing in the path integral. Alternatively, one could include higher curvature terms in the classical action. Finally, we have to be able to find second or higher order phase transitions for the lattice theory, as discussed above.

Let us define the lattice theory. We follow the two-dimensional theory and consider four-dimensional piecewise linear geometries, constructed by gluing together building blocks consisting of four-simplices, where all link lengths are equal to \( a \), which then serves as our UV cut-off. The only restriction on the gluing is that the gluing locally is such that one has a (piecewise linear) manifold

\(^{11}\)The concept of asymptotic safety and the way it is implemented in the case of gravity is the topic of the Chapter called “Asymptotically Safe Quantum Gravity” in the Handbook.
and that the topology of this piecewise linear manifold is fixed. Most studied is the simplest topology, the four-sphere \( S^4 \), and we will limit ourselves to discussing this case. We consider the four-simplices as flat in the interior. View as a piecewise linear continuum manifold all geodesic distances are well-defined, and thus the geometry of such a triangulation is also fixed without specifying a coordinate system. Summing over combinatorially inequivalent triangulations then results in a summation over a certain class of piecewise linear geometries, and the hope is that in the limit where the lattice spacing \( a \to 0 \), this summation will in some sense be a good representation of the integration over continuous geometries (of which the piecewise linear geometries constitute a subset, that is hopefully dense with respect to a (still) unknown measure).

An obvious question is how to represent the curvature term present in the Einstein-Hilbert action. Regge showed how to define curvature locally on a \( d \)-dimensional piecewise linear manifold [65] constructed from \( d \)-simplices \( \sigma_d \), by locating it on \( (d-2) \)-dimensional subsimplices \( \sigma_d^{d-2} \). The \( d \)-simplices \( \sigma_d \) sharing a \( (d-2) \)-dimensional subsimplex \( \sigma_d^{d-2} \), have dihedral angles
\[
\theta(\sigma_d^{d-2}, \sigma_d^{d-2}),
\]
related to this subsimplex. If the spacetime was flat these dihedral angles would add to \( 2\pi \). The so-called deficit angle \( \epsilon_{\sigma_d^{d-2}} \), associated with the subsimplex \( \sigma_d^{d-2} \) is defined by
\[
\epsilon_{\sigma_d^{d-2}} = 2\pi - \sum_{\{\sigma_d^{d-2} \in \sigma_d\}} \theta(\sigma_d^{d-2}, \sigma_d^{d-2}).
\] (38)

The deficit angle is the angle by which a vector will be rotated when parallel-transported locally around the \( (d-2) \)-simplex in the piecewise linear geometry in the subspace perpendicular to the \( d-2 \)-dimensional simplex. Regge showed that the curvature action and the volume term associated to the piecewise linear manifold \( M \) can be written as
\[
\int_M d^d \xi \sqrt{|g(\xi)|} R(\xi) = 2 \sum_{\sigma_d^{d-2}} \epsilon_{\sigma_d^{d-2}} V_{\sigma_d^{d-2}} \quad \text{and} \quad \int_M d^d \xi \sqrt{|g(\xi)|} = \sum_{\sigma_d} V_{\sigma_d},
\] (39)

where \( V_{\sigma_d^{d-2}} \) denotes the volume of the subsimplex \( \sigma_d^{d-2} \) and \( V_{\sigma_d} \) the volume of the simplex \( \sigma_d \) (in the case there \( d = 2 \) we define \( V_{\sigma_d^{d-2}} = 1 \)).

For the piecewise linear geometries used in EDT this expression simplifies enormously since all dihedral angles are identical, all \( (d-2) \)-volumes are the

---

\( \text{One can study more general models where one relaxes the constraint that the gluing should result in a piecewise linear geometry or that the topology of the manifolds should be fixed. It is possible to formulate such a generalized gluing procedure in different ways and starting with the articles [16, 66], these models are denoted tensor models. If we discuss the gluing of \( d \)-dimensional simplices, the number of tensor indices are equal to the number of \( d-1 \) dimensional subsimplices in the \( d \)-dimensional simplices which constitute the building blocks. For } \( d = 2 \) \text{ we have tensors of rank } 2, \text{ i.e. matrices, and two-dimensional gravity has indeed been studied using matrix models, starting with the work of David [48].}

\( \text{For a given } d \text{-simplex, any of its } (d-2) \text{-subsimplices will be the intersection of precisely two of its } (d-1) \text{-subsimples, and the angle between these two } (d-1) \text{-subsimplices is called the dihedral angle (which is an angle for any } d \geq 2) \).
The same and all $d$-volumes are also equal. In the four-dimensional case, which has our main interest, we have
\[
\theta(\sigma^2, \sigma^4) = \arccos \frac{1}{4}, \quad V_{\sigma^2} = \frac{\sqrt{3}}{2} a^2, \quad V_{\sigma^4} = \frac{\sqrt{5}}{96} a^4.
\] (40)

The Einstein-Hilbert action for a given triangulation $T$ of a closed four-dimensional manifold in EDT can then be written as
\[
S_M(G, \Lambda) = \frac{1}{16\pi G} \int d^4\xi \sqrt{g(\xi)} \left( -R(\xi) + 2\Lambda \right) \to 
S_T(\kappa_2, \kappa_4) = -\kappa_2 N_2(T) + \kappa_4 N_4(T),
\] (41)
\[
\kappa_2 = \frac{1}{8G} \frac{\sqrt{3} a^2}{2}, \quad \kappa_4 = \frac{2\Lambda}{16\pi G} \frac{\sqrt{5} a^4}{96} + 20 \arccos \left( \frac{1}{4} \right) \frac{1}{16\pi G} \frac{\sqrt{3} a^2}{2},
\] (42)

where $N_2(T)$ denotes the number of two-simplices and $N_4(T)$ the number of four-simplices in the triangulation $T$. From the so-called Dehn-Sommerville relations for a closed four-dimensional triangulation $T$ one has that $N_2(T) = 2N_0(T) + 2N_4(T) - 2\chi$, where $N_0(T)$ denotes the number of vertices in the triangulation and $\chi$ is the Euler characteristic of the triangulation. The Euler characteristic only depends on the topology of the triangulation. Thus (41) can be written as
\[
S_T(k_0, k_4) = -k_0 N_0(T) + k_4 N_4(T) + k_0 \chi, \quad k_0 = 2\kappa_2, \quad k_4 = \kappa_4 - 2\kappa_2
\] (43)

where the $\chi$-term is usually ignored since we consider triangulations with a fixed topology.

The EDT partition function of four-dimensional quantum gravity is now obtained by summing over triangulations with the action given by (41) (or (43)):
\[
Z(\kappa_2, \kappa_4) = \sum_T \frac{1}{C_T} e^{\kappa_2 N_2(T) - \kappa_4 N_4(T)} = \sum_{N_4, N_2} e^{\kappa_2 N_2 - \kappa_4 N_4} \mathcal{N}(N_2, N_4),
\] (44)

where the summation is over all abstract triangulations\textsuperscript{14} $T$ of a given four-dimensional manifold, where $C_T$ is a symmetry factor (the order of the automorphism group of the triangulation $T$), and where $\mathcal{N}(N_2, N_4)$ denotes the number of such triangulations with a fixed number of two-simplices and four-simplices, $N_2$ and $N_4$, respectively. As was the case in two dimensions, the partition function is entirely combinatorial: $Z(\kappa_2, \kappa_4)$ is the generating function (with indeterminates $e^{\kappa_2}$ and $e^{-\kappa_4}$) for the number of four-dimensional triangulations with a given topology (here $S^4$) and a given number of four-simplices and two-simplices. It is truly remarkable that four-dimensional quantum gravity in this way is purely “entropic”. Unfortunately, it is not yet possible to

\textsuperscript{14}We use here the notation “abstract triangulation” to emphasize that although we have viewed the triangulations as piecewise linear manifolds and have introduced a link length $a$ as a UV cut-off, in the summation (44) only the labelling as (abstract) triangulations is important. The other aspects will be important when we discuss a continuum limit.
perform this counting analytically. This leaves us presently with Monte Carlo simulations if we want to study the partition function (44).

In view of the unboundedness of the Euclidean Einstein-Hilbert action, the first obvious question one can ask is whether \(Z(<\kappa_2, \kappa_4>)\) is at all well defined for any values of \(\kappa_2\) and \(\kappa_4\). Let us perform the summation over \(N_2\) in (44),

\[
Z(<\kappa_2, \kappa_4>) = \sum_{N_4} e^{-\kappa_4 N_4} \mathcal{N}_{\kappa_2} (N_4), \quad \mathcal{N}_{\kappa_2} (N_4) = \sum_{N_2} e^{\kappa_2 N_2} \mathcal{N} (N_2, N_4).
\] (45)

If \(\mathcal{N}_{\kappa_2} (N_4)\) is exponentially bounded as a function of \(N_4\), i.e. if there exists a constant \(\kappa_4^c (<\kappa_2)\) such that

\[
\mathcal{N}_{\kappa_2} (N_4) \leq e^{(\kappa_4^c(k_2)+\epsilon) N_4}, \quad \text{for all } \epsilon > 0, \quad N_4 > N_4 (\epsilon),
\] (46)

then there is a line \((<\kappa_2, \kappa_4^c(k_2))\) in the \(\kappa_2, \kappa_4\) coupling-constant plane, such that \(Z(<\kappa_2, \kappa_4>)\) is well defined and convergent for \(\kappa_4 > \kappa_4^c (<\kappa_2)\). It is easy to prove that if (46) is valid for \(\kappa_2 = 0\), then it is valid for all \(\kappa_2\) (with a \(\kappa_4^c (<\kappa_2)\) depending on \(\kappa_2\)). However, there is no proof that \(\mathcal{N}_0 (N_4)\) is exponentially bounded. Computer simulations indicate that it is the case [5, 43] and in the following we will assume so. The physics of (45) is all hidden in \(\mathcal{N}_{\kappa_2} (N_4)\), which according to our assumptions can be written as

\[
\mathcal{N}_{\kappa_2} (N_4) = e^{\kappa_4^c(k_2) N_4} H_{\kappa_2} (N_4), \quad H_{\kappa_2} (N_4) \text{ subleading in } N_4,
\] (47)

which implies that the non-trivial continuum physics is to be found in the subleading function \(H_{\kappa_2} (N_4)\). One then obtains

\[
Z(<\kappa_2, \kappa_4>) = \sum_{N_4} e^{-(\kappa_4 - \kappa_4^c(k_2)) N_4} H_{\kappa_2} (N_4).
\] (48)

Given two triangulations \(T (N_4)\) and \(T (N_4')\) there exist local changes in the triangulation \(T_4\) (the so-called Pachner moves) that, when applied a finite number of times, will bring us from \(T (N_4)\) to \(T (N_4')\). These Pachner moves [62] are used in the Monte Carlo simulations, and allow us to have a Monte Carlo algorithm that is ergodic and in principle creates the correct distribution Boltzmann distribution for (44), corresponding to the action (41). Details will be provided in other Chapters of this Section of the Handbook, see “Spectral Observables and Gauge Field Couplings in Causal Dynamical Triangulations”, Chapter 3, and “Semiclassical and Continuum Limits of Four-Dimensional CDT”, Chapter 9. However, an interesting aspect in four dimensions is that \(N_4\) cannot be kept fixed in these Pachner moves, and it is even impossible in principle to calculate the highest \(N_4 (T (N_4), T (N_4'))\) of an intermediate triangulation \(T (N_4)\) that one meets when moving from the triangulation \(T_4 (N_4)\) to \(T_4 (N_4')\) by successive application of the Pachner moves [6]. It is unclear what this implies for the practical ergodicity of the used Monte Carlo simulations. It does not necessarily imply that \(N_4\) is very large, but we cannot in principle provide a general expression for \(N_4 (T (N_4), T (N_4'))\).
In the following we will assume that the Monte Carlo simulations work fine, despite the potential problems mentioned above. In the region of coupling-constant space \((\kappa_2, \kappa_4), \kappa_4 > \kappa_4^*(\kappa_2)\) where the partition function \(Z(\kappa_2, \kappa_4)\) is well defined there is thus no problem with the Euclidean action being unbounded from below. One easily shows, using the so-called Dehn-Sommerville relations, that for a given triangulation \(T\) \[2N_4(T) \leq N_2(T). \tag{49}\]

It is therefore easy to find regions in the coupling-constant space where \(S_T(\kappa_2, \kappa_4)\) given by (41) is less than zero and unbounded from below when \(N_2(T), N_4(T)\) go to infinity. However, in this region of coupling-constant space \(Z(\kappa_2, \kappa_4)\) is not well defined and it is not the region considered in EDT. In fact, in the region of large \(\kappa_2\), which seems most prone to a negative action, the limit \(\kappa_4 \to \kappa_4^*(\kappa_2)\) from above is well understood [4] and corresponds to a continuum theory of fractal geometries known as random continuum trees or branched polymers. This continuum limit does not resemble our present universe. The fractal dimension, the Hausdorff dimension, is two on all scales. The same continuum limit continues with decreasing \(\kappa_2\) until one reaches a critical point \(\kappa_2^c\), where there is a phase transition, such that for \(\kappa_2 < \kappa_2^c\) we encounter a different kind of geometry when \(\kappa_4 \to \kappa_4^*(\kappa_2)\). It is a “crumpled” geometry with infinite Hausdorff dimension [4], where a significant fraction of the four-simplices share a single link and the order of the corresponding two vertices is very high [47]. In particular, this seems to be the entropically preferred type of triangulation if no curvature term is present in the action, i.e. \(\kappa_2 = 0\). Such highly inhomogeneous triangulations also seem unsuited to describe any theory of quantum gravity.

This leaves us with \(\kappa_2^c\) as the only point where one might be able to obtain an interesting theory of quantum gravity. Potentially, this is a good scenario: at the phase transition point the typical geometries one would encounter for \(N_4 \to \infty\) could be geometries with a Hausdorff dimension between \(d_h = 2\) for \(\kappa_2 > \kappa_2^c\) and the \(d_h = \infty\) for \(\kappa_2 < \kappa_2^c\). If the phase transition was a second-order transition, this scenario could be reasonable, since then one might hope for a smooth transition between the two extreme limits, \(d_h = 2\) and \(d_h = \infty\).

Unfortunately, the Monte Carlo simulations show that the transition is a first-order transition and the geometry at the transition point seems not to be a smooth interpolation between the two types of geometry [41].

This situation does not necessarily imply that an interesting continuum limit cannot be found, but in a Wilsonian context it implies that we have to use a more general action than (41). Adding a suitable term, which could be a measure term or some higher curvature term, in this now three-dimensional coupling constant space, the critical point \(\kappa_2^c\) will turn into a critical line. If the critical line ends, the endpoint would be a candidate for a second-order transition point. Also new phases might appear, and in such a more complicated landscape there might be different second-order (see [42] for the most recent results).
4 Four-dimensional CDT

Four-dimensional Causal Dynamical Triangulations (CDT) is a generalization of the simplest version of the two-dimensional CDT described above. The aim is to perform the path integral over geometries on a manifold \([0, 1] \times \Sigma\), where \([0, 1]\) denotes a time interval and \(\Sigma\) is a three-dimensional spatial manifold. We discretize the time, the discrete times labelled by \(t_k\). At each discretized time \(t_k\) we have a spatial manifold \(\Sigma\), on which we apply the EDT formalism and assign a piecewise linear geometry constructed by gluing together three-dimensional simplices (tetrahedra) with link lengths \(a\), such that the topology of the three-dimensional triangulation \(T^3\) (where the superscript “3” means that the triangulation is three-dimensional, not that it is a three-torus) matches that of the manifold \(\Sigma\). Since the geodesic distances on \(T^3\) are uniquely determined, so is the (piecewise linear) geometry. Given a three-dimensional triangulation \(T^3_k\) at time \(t_k\) and a three-dimensional triangulation \(T^3_{k+1}\) at time \(t_{k+1}\), we connect these by four-simplices, such that we obtain a four-dimensional triangulation with boundaries \(T^3_k\) and \(T^3_{k+1}\), and such that the topology of this four-dimensional triangulation is \([0, 1] \times \Sigma\). The four-dimensional simplices filling out the “slab” between \(T^3_k\) and \(T^3_{k+1}\) can be of four kinds, depending on how many vertices the four-dimensional simplices share with a tetrahedron belonging to \(T^3_k\). If the four-simplex shares four vertices with \(T^3_k\), i.e. is a tetrahedron in \(T^3_k\), and thus one vertex with the triangulation \(T^3_{k+1}\) we denote it a \(T(4,1)\) simplex. If it shares three vertices with a tetrahedron in \(T^3_k\), i.e. forms a triangle in \(T^3_k\), and two vertices with \(T^3_{k+1}\), i.e. forms a link in \(T^3_{k+1}\), we denote it a \(T(3,2)\) simplex. The four-simplices \(T(2,3)\) and \(T(1,4)\) are defined similarly.

The whole construction is clearly a generalization of the construction of two-dimensional CDT, where in an analogous notation we would have two kinds of two-simplices, \(T(2,1)\) and \(T(1,2)\). The links of the four-dimensional simplices which are also links in \(T^3_k\) and \(T^3_{k+1}\) are assigned a positive length \(a\), while the links connecting vertices in \(T^3_k\) to vertices in \(T^3_{k+1}\) are viewed as time-like, i.e. we write, in analogy with (28),

\[
a^2 = -\alpha a^2, \quad \alpha > 0.
\]  

(50)

A complete triangulation of the manifold \([0, 1] \times \Sigma\) is now obtained by repeating the above procedure for \(k = 1, 2, \ldots, s\), yielding a four-dimensional triangulation with spatial boundaries \(T^3_1\) and \(T^3_s\) and spatial slices \(T^3_k\), \(1 < k < s\). The corresponding Regge action for such a geometry is still very simple, although slightly more complicated than (41), since we have introduced a parameter \(\alpha\), which will allow us to perform a rotation of the geometry with Lorentzian signature to one with Euclidean signature. The Lorentzian action for such a Lorentzian triangulation \(\mathcal{T}_{\text{lor}}\), expressed using the notation from (41) can be
written as [20, 31]

\[ S_{\text{tor}} = \kappa_2 \sqrt{4\alpha + 1} \left[ \frac{\pi}{2} N_2^{\text{TL}} + \right. \]

\begin{align*}
N_4^{(4,1)} &\left( -\frac{\sqrt{3}}{4\sqrt{\alpha + 1}} \arcsin \frac{1}{2\sqrt{2\sqrt{3\alpha + 1}}} - \frac{3}{2} \arccos \frac{2\alpha + 1}{2(3\alpha + 1)} \right) + \\
N_4^{(3,2)} &\left( \frac{\sqrt{3}}{4\sqrt{\alpha + 1}} \arcsinh \frac{\sqrt{3} \sqrt{12\alpha + 7}}{2(3\alpha + 1)} + \right.
\end{align*}

\begin{align*}
&\frac{3}{4} \left( 2 \arccos \frac{\sqrt{2\sqrt{3\alpha + 1}}}{2\sqrt{2\alpha + 1}} + \arccos \frac{4\alpha + 3}{4(2\alpha + 1)} \right) \\
&- \kappa_4 \left( N_4^{(4,1)} \frac{\sqrt{8\alpha + 3}}{96} + N_4^{(3,2)} \frac{\sqrt{12\alpha + 7}}{96} \right). \\
\end{align*}

\( N_4^{(4,1)} \) and \( N_4^{(3,2)} \) denote the total number of four-simplices of types \( T^{(4,1)} \) and \( T^{(1,4)} \) and of types \( T^{(3,2)} \) and \( T^{(2,3)} \), respectively, in the triangulation \( T_{\text{tor}} \). \( N_2^{\text{TL}} \) denotes the number of time-like triangles in the triangulation, i.e. triangles with one space-like link and two time-like links. Of course \( \kappa_2 \) also multiplies the number of \( N_2^{\text{SL}} \) of space-like triangles, but this number has been expressed in terms of the numbers \( N_4^{(4,1)} \) and \( N_4^{(3,2)} \) of four-simplices, by virtue of the special time-slicing structure present for a CDT triangulation. Finally, we have ignored a Regge boundary action term, coming from the two boundaries, since in the actual computer simulations we replace the manifold \( [0, 1] \times \Sigma \) with the manifold \( S^1 \times \Sigma \). As we will see, the set-up of the computer simulations will be such that in most cases there will be no difference between choosing \( [0, 1] \) or \( S^1 \). The action is written in a way that makes it real for all positive \( \alpha \) and purely imaginary for \( \alpha < -7/12 \). Of course our starting point is a Lorentzian geometry with \( \alpha > 0 \), but now, like in the two-dimensional case, we can make a rotation to Euclidean geometry by performing a rotation \( \alpha \rightarrow -\alpha \) in the lower complex \( \alpha \)-plane, assuming \( \alpha > 7/12 \). One then obtains the action

\[ S_{\text{eucl}} = -\kappa_2 \sqrt{4\alpha - 1} \left[ \pi \left( N_0 - \chi + \frac{1}{2} N_4^{(4,1)} + N_4^{(3,2)} \right) + \right. \]

\begin{align*}
N_4^{(4,1)} &\left( -\frac{\sqrt{3}}{4\sqrt{\alpha - 1}} \arcsin \frac{1}{2\sqrt{2\sqrt{3\alpha - 1}}} + \frac{3}{2} \arccos \frac{2\alpha - 1}{6\alpha - 2} \right) + \\
N_4^{(3,2)} &\left( \frac{\sqrt{3}}{4\sqrt{\alpha - 1}} \arccos \frac{6\alpha - 5}{6\alpha - 2} + \frac{3}{4} \arccos \frac{4\alpha - 3}{8\alpha - 4} + \\
&\frac{3}{2} \arccos \frac{1}{2\sqrt{2\sqrt{\alpha - 1}\sqrt{3\alpha - 1}}} \right) + \kappa_4 \left( N_4^{(3,2)} \frac{\sqrt{12\alpha - 7}}{96} + N_4^{(4,1)} \frac{\sqrt{8\alpha - 3}}{96} \right). \\
\end{align*}

Analogous to the two-dimensional case, we have

\[ S_{\text{minor}}(\kappa_2, \kappa_4, \alpha) \rightarrow S_{\text{tor}}(\kappa_2, \kappa_4, -\alpha) = i S_{\text{eucl}}(\kappa_2, \kappa_4, \tilde{\alpha}), \quad \alpha = \tilde{\alpha} > \frac{7}{12}. \]
In the same way as the constraint $\tilde{\alpha} > 1/4$ in the two-dimensional case was linked to the triangle inequality (and still is in (52)), the inequality $\tilde{\alpha} > 7/12$ is linked to the geometry of a $T^{(3,2)}$ simplex: for $\tilde{\alpha} = 7/12$ the “time-like” distance between the opposing spatial link and spatial triangle in the simplex becomes zero. In (52) we have replaced $N^2_{TL}$ by $N_0 - \chi + \frac{1}{2} N^{(4,1)}_4 + N^{(3,2)}_4$, where $N_0$ denotes the number of vertices in the triangulation $T$ and $\chi$ the Euler characteristic of the manifold. This relation again follows from the Dehn-Sommerville relations for four-dimensional CDT triangulations. One can check that for $\tilde{\alpha} = 1$ one precisely recovers the EDT expression (43).

In the Monte Carlo simulations, using (generalized\textsuperscript{15}) Pachner moves to change the triangulations, the topology of the triangulations is kept fixed and we can ignore the $\chi$-term. We will do that in the following.

Let us again stress that while an expression like (52) looks somewhat complicated because of the $\tilde{\alpha}$-dependence, it is, like (43), exceedingly simple, since the action of a triangulation $T$ just depends on the three global numbers $N_0(T)$, $N^{(4,1)}_4(T)$ and $N^{(3,2)}_4(T)$. Again the partition function for CDT quantum gravity is simply the generating function for the number of triangulations with given $N_0$, $N^{(4,1)}_4$ and $N^{(3,2)}_4$, with suitable indeterminates, now depending not only on $\kappa_2$ and $\kappa_4$, but also on $\tilde{\alpha}$. By redefining the coupling constants we can make this simplicity explicit by writing

$$S_T(k_0, k_4, \Delta) = -(k_0 + 6\Delta) N_0(T) + k_4 \left( N^{(4,1)}_4(T) + N^{(3,2)}_4(T) \right) + \Delta N^{(4,1)}_4(T)$$ \hspace{1cm} (54)

This action is still formally equal to the Regge version of the Einstein-Hilbert action for a piecewise linear manifold constructed as described above, where the spatial links have length $a$ and the “time-like” links a length $\sqrt{\tilde{\alpha}} a$. It is still true that $k_0 \propto a^2/G$, while $\Delta$ is a rather complicated function of $k_0$, $k_4$ and $\tilde{\alpha}$ such that $\Delta = 0$ corresponds to $\tilde{\alpha} = 1$. However, from the computer simulations to be discussed below, it will be clear we cannot maintain such an interpretation. It is thus a more fruitful, Wilsonian interpretation of (54) to say that our starting point is the Regge action with

$$\alpha = \tilde{\alpha} = 1, \quad \text{and} \quad \Delta \text{ is an independent coupling constant.}$$ \hspace{1cm} (55)

In this way $\Delta = 0$ will correspond to the Euclidean Einstein-Hilbert action (43), but where the geometries have a time foliation coming from the Lorentzian geometries described above, and the new coupling constant $\Delta$ is a Wilsonian enlargement of the coupling-constant space from $(k_0, k_4)$ to $(k_0, k_4, \Delta)$. We will explore this space in the search for potentially interesting phase transitions of the lattice system, which could be associated with a UV fixed point for quantum gravity\textsuperscript{16}. In this context let us mention that we can of course rewrite (54) as

$$S_T(\tilde{k}_0, k_{4,1}, k_{32}) = -\tilde{k}_0 N_0(T) + k_{41} N^{(4,1)}_4(T) + k_{32} N^{(3,2)}_4(T)$$ \hspace{1cm} (56)

\textsuperscript{15}We have to use slightly generalized Pachner moves to preserve the CDT foliation structure\textsuperscript{[31]}.

\textsuperscript{16}In this sense the situation becomes similar to the EDT situation, where one has to enlarge the $(k_0, k_4)$ coupling-constant space defined in (43) by some new coupling constant in order to obtain an interesting result, as already mentioned.
to emphasize that the action is the most general action that only depends linearly on the global number of simplices or subsimplices in a CDT triangulation\(^{17}\).

To summarize, our four-dimensional CDT partition function is

\[
Z(k_0, k_4, \Delta) = \sum_T \frac{1}{C_T} e^{-S_T(k_0, k_4, \Delta)}
\]

\[
= \sum_{N_0, N_4^{(4,1)}, N_4^{(3,2)}} e^{\left[(k_0+6\Delta)N_0-k_4\left(N_4^{(4,1)}+N_4^{(3,2)}\right)-\Delta N_4^{(4,1)}\right]} \mathcal{N}(N_0, N_4^{(4,1)}, N_4^{(3,2)}),
\]

where the summation is over CDT triangulations and \(\mathcal{N}(N_0, N_4^{(4,1)}, N_4^{(3,2)})\) denotes the number of such triangulations with \(N_0\) vertices, \(N_4^{(4,1)}\) simplices of type \(T^{(4,1)}\) plus type \(T^{(1,4)}\), and \(N_4^{(3,2)}\) simplices of type \(T^{(3,2)}\) plus type \(T^{(2,3)}\).

We now turn to the discussion of the phase diagram of this statistical system.

**Search for a UV fixed point in CDT**

The enlargement of the CDT coupling-constant space with the coupling constant \(\Delta\) leads to an amazingly complex phase diagram\(^{18}\) shown in Fig. 3. It shows the \((k_0, \Delta)\) coupling constant plane. As discussed above, it is impossible to keep \(N_4 = N_4^{(4,1)} + N_4^{(3,2)}\) fixed in the Monte Carlo simulations. However, they can be conducted in such a way that measurements of the observables used to identify the phase transitions are performed for a given chosen value of \(N_4\). In this way \(k_4\) does not enter actively as a coupling constant influencing the observables\(^{19}\). This is why the figure only shows the \((k_0, \Delta)\) coupling-constant plane. The physics related to the coupling-constant \(k_4\) can be recovered by performing measurements for many different values of \(N_4\). Explicitly we have

\[
Z(k_0, \Delta, k_4) = \sum_{N_4} e^{-k_4N_4} Z_{N_4}(k_0, \Delta),
\]

\[
Z_{N_4}(k_0, \Delta) = \sum_T \frac{1}{C_T} e^{(k_0+\Delta)N_0}\left(T(N_4)\right)-\Delta N_4^{(4,1)}\left(T(N_4)\right),
\]

where the summation is over all triangulations \(T(N_4)\) with a fixed number \(N_4\) of four-simplices.

Fig. 3 shows the phase transition lines between the various phases, denoted \(A, B, C_b\) and \(C_{dS}\). Here the subscript \(dS\) stands for “de Sitter”, and \(b\) for

\(^{17}\)See [31] for a classification of time- and spacelike (sub)simplices of a CDT configuration, and the constraints these numbers satisfy. There are 10 different types of (sub)simplices and 7 constraints.

\(^{18}\)The phase diagram presented in the first articles [21, 29, 30, 31] was simpler since it missed the \(C_b\) phase, discovered in [35, 36, 39].

\(^{19}\)In many of the simulations it has been more convenient to instead keep \(N_4^{(4,1)}\) fixed at the measurements.
Figure 3: The CDT phase diagram. Phase transition between phase $C_{dS}$ and $C_b$ is second order when the topology of a spatial slice is $S^3$, as is the transition between $C_b$ and $B$. The transition between $C_{dS}$ and $A$ and the transition between $A$ and $B$ are first-order transitions. The transition between $C_{dS}$ and $B$ is still under investigation.

The order of the transition between phase $C_{dS}$ and phase $B$ is most likely also a higher-order transition, but it is not entirely settled yet.
matter was added to the model, and maybe in this way be important for the first galaxy formations. In particular this would present an intriguing scenario if the phase transition line could be associated with a UV fixed point. First of all, as already mentioned, a UV fixed point in a theory of gravity is central to the asymptotic safety scenario, and finding it in our lattice approach is central to the idea that one can use the lattice theory as a non-perturbative definition of quantum gravity. Secondly, it is also often assumed that the “origin” of the Universe is associated with the theory of gravity at short distances (some kind of Big Bang scenario), and this should then naturally relate to physics close to the UV fixed point. Inhomogeneity as part of this UV physics could then be important for the formation of structure in a universe with matter.

Therefore, confronted with a second-order phase transition line, the phase transition line between the \( C_b \) and \( C_{dS} \) phases, the obvious question of interest is whether or not there is a non-perturbative UV fixed point associated with this line.

Search for a UV fixed point in a \( \phi^4 \) theory

In order to address this question, let us step back and briefly recall how it has been addressed in ordinary \( \phi^4 \) scalar field theory in four-dimensional flat (Euclidean) spacetime. Consider the scalar \( \phi^4 \) theory defined on a four-dimensional hyper-cubic lattice with lattice spacing \( a \). We denote the integer lattice coordinates of the vertices by \( n = (n^1, \ldots, n^4) \) and the spacetime coordinates of these lattice points by \( x_n = an \). A lattice scalar field \( \phi \) takes values on the lattice vertices and we use the notation \( \phi(n) \) or \( \phi(x_n) \). The action is

\[
S[\phi, \mu, \lambda; a] = \sum_n a^4 \left( \frac{1}{2} \sum_{i=1}^{4} \frac{(\phi(n+i\hat{i}) - \phi(n))^2}{a^2} + \frac{1}{2} \mu \frac{1}{a^2} \phi^2(n) + \frac{1}{4!} \lambda \phi^4(n) \right), \quad (60)
\]

where \( \hat{i} \) denotes the unit vector in direction \( i \).

The theory has two dimensionless lattice coupling constants \( \mu \) and \( \lambda \). In this coupling-constant space there is a phase transition line between a symmetric phase where \( \langle \phi(n) \rangle = 0 \) and a symmetry-broken phase where \( \langle \phi(n) \rangle \neq 0 \). The symmetry broken is \( \phi(n) \rightarrow -\phi(n) \). This transition line is a second-order phase transition line, and the correlation length between the fields at different lattice points diverges when one approaches the transition line. The question is whether this phase transition line be used to define a non-perturbative UV fixed point for the \( \phi^4 \) quantum field theory. The tentative continuum quantum field theory is defined by its two renormalized continuum coupling constants \( m_R \) and \( \lambda_R \), the continuum mass and the continuum \( \phi^4 \) coupling constant. They can be extracted from the two-point correlator and the four-point correlator. The lattice two-point function is characterized by a lattice correlation length \( \xi \). We

\[\text{Since the Higgs field } \phi \text{ in the Standard Model is governed by a } \phi^4 \text{ field theory (embedded in a larger theory), the existence or non-existence of such a UV fixed point is actually important for Standard Model.}\]
can write

\[ \xi(\mu, \lambda) = \lim_{|n-n'| \to \infty} -\log \frac{\langle (\phi(n') - \langle \phi \rangle)(\phi(n) - \langle \phi \rangle) \rangle}{|n-n'|}, \quad m_R = \frac{1}{a_\xi}. \]  

(61)

We assume for simplicity that the coupling constants are chosen such that we are in the symmetric phase, where \( \langle \phi \rangle = 0 \). Let us not discuss in detail how to define \( \lambda_R \) (for details see for instance [61]), but only state that insisting that \( \lambda_R \) is constant defines a path \((\mu(\xi), \lambda(\xi))\) in the lattice coupling-constant space \((\mu, \lambda)\). For each point on this path we can calculate a correlation length \( \xi \) using (61), and we use these \( \xi \) as a parametrization of the path. If the path meets the second-order transition line at a point \((\mu_c, \lambda_c)\), it implies that \( \xi \to \infty \) at this point. This point can serve as a UV fixed point, since we now demand that \( m_R \) is constant along the path, i.e. the lattice spacing \( a \) becomes a function of \( \xi \) via

\[ a(\xi) = \frac{1}{m_R \xi}, \quad \text{i.e.} \quad a(\xi) \to 0 \quad \text{for} \quad \xi \to \infty. \]  

(62)

We conclude that by following a path in the bare, dimensionless coupling-constant space, where continuum observables are kept fixed, one is led to a UV fixed point, provided it exists. If the UV point does not exist, the path will be such that \( \xi \) never reaches infinity, no matter where we start in the bare coupling constant space. According to (62) this implies that we cannot remove the UV cut-off \( a \).

The approach to the UV fixed point is governed by the \( \beta \)-function\footnote{The \( \beta \)-function is a function of \( \lambda \) and \( \mu \), but close to the fixed point one can ignore the \( \mu \)-dependence.}, which relates the change in \( \lambda \) to the change in \( a(\xi) = 1/(m_R \xi) \) as we move along the trajectory of constant \( m_R, \lambda_R \),

\[ -a \frac{d\lambda}{da} \bigg|_{m_R, \lambda_R} = \xi \frac{d\lambda}{d\xi} \bigg|_{m_R, \lambda_R} = \beta(\lambda). \]  

(63)

Since \( \lambda(\xi) \) stops changing when \( \xi \to \infty \), we have \( \beta(\lambda_c) = 0 \), and expanding the \( \beta \)-function to first order one finds

\[ \lambda(\xi) = \lambda_c + \text{const.} \xi^{\delta(\lambda_c)}, \quad \beta'(\lambda) = \frac{d\beta}{d\lambda}. \]  

(64)

It follows from (64) that \( \beta'(\lambda_c) < 0 \) at a UV fixed point.

The correlation length \( \xi \) clearly plays a major role in the above scenario. It will be convenient to replace it with a finite lattice volume by using so-called finite-size scaling. Assume we have a finite hypercubic lattice. The volume is then \( V = N a^4 \), where \( N \) is the number of hypercubes. We keep the ratio between the linear size of the lattice and the correlation length fixed,

\[ \frac{\xi}{N^{1/4}} = \frac{1}{(a(\xi)m_R)N^{1/4}} = \frac{1}{m_R V^{1/4}}. \]  

(65)

Thus, if we are moving along a trajectory with constant \( m_R \) and \( \lambda_R \) in the bare \((\mu, \lambda)\)-coupling constant plane and change \( N \) according to (65), the finite
continuum volume stays fixed. Assuming that there is a UV fixed point, such
that \( a(\xi) \to 0 \), we see that \( N \) can go to infinity even if \( V \) stays finite, and that
the correlation length \( \xi \) in (64) can be substituted by a dependence on the linear
size \( N^{1/4} \) in lattice units of the spacetime, leading to
\[
\lambda(N) = \lambda^c + \text{const.} \frac{N^\beta'(\lambda^c)}{4}.
\] (66)

As we saw above, the absence of a UV fixed point could be deduced by the
absence of a divergent correlation length along a trajectory of constant physics
in the \((\mu, \lambda)\)-plane. In the finite-size scaling scenario this is restated as \( N \) not
going to infinity along any such curve of constant physics. In the case of a
\( \phi^4 \) theory in four-dimensional spacetime this is exactly what happens, and the
conclusion is that there is no UV fixed point in the theory [59, 60]. The second-
order transition line of the theory is related to the IR limit of the theory where
\( \lambda_R = 0 \).

Finite-size scaling analysis in CDT

We now want to apply the above formalism to CDT [33, 40] and in addition take
advantage of the time-slice structure present in CDT. In fact, one does precisely
that in Monte Carlo simulations on an ordinary hypercubic lattice. Instead of
the point-point correlator \( \langle \phi(n)\phi(n') \rangle \) used in (61), one averages the positions \( n \)
and \( n' \) over positions \( n(t_k) \) belonging to a hyperplane located at “time” \( t_k = ka \)
and positions \( n'(t'_{k'}) \) located at “time” \( t' = k'a \), where the two hyperplanes are
separated by a lattice distance \( d = |k - k'| \). This reduces the fluctuations of the
measured correlator and it also has the advantage that power corrections to the
exponential fall-off of the correlator are absent, making it easier to determine
the correlation length (see again [61] for details). Thus one replaces (61) by
\[
\left\langle \sum_{n(t_k)} \phi(n(t_k)) \sum_{n'(t'_{k'})} \phi(n'(t'_{k'})) \right\rangle = \text{const.} \, e^{-|k-k'|/\xi}. \tag{67}
\]

In our CDT theory of pure geometry we do not have a field \( \phi(n) \) at our dis-
posal, but as in the two-dimensional case we can use the “unit” field \( 1(n) \) which
assigns the value 1 to each four-simplex. Each three-simplex \( T^3 \) in the three-
dimensional triangulation \( T^d_k \) corresponding to time \( t_k \) belongs to two four-
simplices \( T^{(4,1)}(T^3) \) and \( T^{(1,4)}(T^3) \). We can then write
\[
\sum_{\mathcal{N}_{t_k}} \phi(n(t_k)) \to \frac{1}{2} \sum_{T^3 \in T^d_k} \left( 1(T^{(4,1)}(T^3)) + 1(T^{(1,4)}(T^3)) \right) = \mathcal{N}_3(t_k), \tag{68}
\]
where \( \mathcal{N}_3(t_k) \) is the number of three-simplices in the spatial slice at time \( t_k \).
Since \( \langle \mathcal{N}_3(t_k) \rangle > 0 \) we have a situation like in the broken phase of a \( \phi^4 \) theory:
connected correlators have to be expanded around \( \langle \phi \rangle \neq 0 \). Here one has
to expand around \( \langle \mathcal{N}_3(t_k) \rangle > 0 \) (see eq. (71) below). Moreover, it turns out
that \( \langle \mathcal{N}_3(t_k) \rangle \) will be a highly non-trivial function of \( t_k \), see Fig. 4, once the
translational invariance of the action in t is dealt with in the proper way. In the symmetric phase of the $\phi^4$ theory all correlators with an odd number of fields will be zero, and the two- and the four-point correlators are needed in order to study the renormalization flow of $\mu$ and $\lambda$. Here, in CDT, we can extract information of the flow of $k_0$ and $\Delta$ by considering only one- and two-point correlators because the one-point function $\langle N_3(t) \rangle$ has a non-trivial dependence on $t$.

\[ \langle N_3(t) \rangle_{N_4} \propto N_4^{3/4} H \left( \frac{k}{N_4^{1/4}} \right), \quad H(0) = 1 \]  

(69)

\[ \langle N_3(t) N_3(t') \rangle_{N_4}^c \propto N_4 F \left( \frac{k}{N_4^{1/4}}, \frac{k'}{N_4^{1/4}} \right), \]  

(70)

where the connected correlator, as in (61), is defined by

\[ \langle N_3(t) N_3(t') \rangle_{N_4}^c = \langle (N_3(t) - \langle N_3(t) \rangle) (N_3(t') - \langle N_3(t') \rangle) \rangle_{N_4}. \]

(71)

In particular we have

\[ \langle \delta N_3(t) \rangle_{N_4} = \sqrt{\langle N_3^2(t) \rangle_{N_4}} \propto \sqrt{N_4} F \left( \frac{k}{N_4^{1/4}}, \frac{k}{N_4^{1/4}} \right), \quad F(0, 0) = 1. \]  

(72)

Monte Carlo simulations confirm the functional form alluded to in (69) and (70) when we are in phase $C_{dS}$. In particular we have measured with high precision [25, 24]

\[ H \left( \frac{k}{N_4^{1/4}} \right) \propto \cos^3 \left( \frac{k}{\omega N_4^{1/4}} \right) \]  

(73)

where $\omega$ depends on $k_0$ and $\Delta$. This functional form is the reason why we call phase $C_{dS}$ the “de Sitter” phase. It is the functional form that $N_3(t)$ would have for a four-sphere where $t$ denotes the geodesic distance from the three-equator. It is valid for

\[ -\frac{\pi}{2} \omega N_4^{1/4} < k < \frac{\pi}{2} \omega N_4^{1/4}, \]  

(74)

where we for convenience have chosen to locate the maximum of $\langle N_3(t) \rangle_{N_4}$ at $k = 0$. In the computer simulations which resulted in this distribution, the lattice time extension was chosen larger than $\pi \omega N_4^{1/4}$. Outside the region (74) one finds $N_3(t) \approx 0$ which is of the order of the cut-off, i.e. the smallest $S^3$ one can create by gluing together 5 tetrahedra). This is why it does not matter whether we choose the time direction to correspond to $S^1$ or to $[0, 1]$. Fig. 4 shows the measured three-volume profile (69), as well as the theoretical curve (73), and finally the fluctuations (72) around the measured three-volume profile.

\footnote{The non-trivial dependence on $t$ is shown in Fig. 4, and it only appears after the zero mode corresponding to translational invariance in t has been eliminated. The condition $H(0) = 1$ then refers to the time $t_0 = 0$ that is chosen as the maximum of the “blob” $\langle N_3(t) \rangle$.}
Eqs. (69) and (72) allow us in principle to follow a path of constant continuum physics in the \((k_0, \Delta)\) coupling-constant space, which might lead to a UV fixed point, in the spirit of the finite-size scaling discussion for the \(\phi^4\) theory. We define the continuum three-volume of a time slice as

\[
V_3(t_k) = \frac{\sqrt{2}}{12} a^3 N_3(t_k), \quad t_k = k a. \tag{75}
\]

First, for fixed \((k_0, \Delta)\), taking \(N_4 \to \infty\), we see that

\[
\left. \frac{\delta V_3(t_k)}{V_3(t_k)} \right|_{N_4} = \frac{\left. \langle \delta N_3(t_k) \rangle \right|_{N_4}}{\langle N_3(t_k) \rangle} \propto \frac{1}{N_4^{1/4}} \to 0 \quad \text{for} \quad N_4 \to \infty. \tag{76}
\]

The simplest interpretation of this result is that for fixed \((k_0, \Delta)\) we should view the lattice spacing \(a\) in (75) as constant. Then \(N_4 \to \infty\) implies that \(V_3 \to \infty\) and for large continuum \(V_3(t)\) one expects that the fluctuations will be small relative to \(V_3(t)\). However, in the spirit of finite-size scaling, we are interested in a limit where \(V_3(t)\) stays finite when \(N_4 \to \infty\). This is clearly a limit where also the fluctuations around \(V_3(t)\) will stay finite, since they then represent the "real" continuum fluctuations around \(V_3(t)\), and they should be independent of...
\( N_4 \) for sufficient large \( N_4 \). We therefore require

\[
\frac{\delta V_3(t)}{V_3(t)} = \frac{\langle \delta N_3(t) \rangle}{\langle N_3(t) \rangle} = \text{const.} \quad \text{for fixed} \quad t = k a \propto k N_4^{-1/4}. \quad (77)
\]

According to (76), we can only obtain this by changing \((k_0, \Delta)\) when we change \( N_4 \). We thus have precisely the picture advocated in the \( \phi^4 \) case: the requirement of constant continuum physics leads to a path in the bare, dimensionless lattice coupling-constant space when we increase \( N_4 \). If this path continues to \( N_4 \rightarrow \infty \), then according to (75) the lattice spacing \( a \rightarrow 0 \) and we might reach a UV fixed point \((k_c^0, \Delta_c^r)\).

In [33] it was attempted to follow this program, but it was before the \( C_b \)-\( C_{dS} \) phase transition line was discovered and the \( N_4 \) used might have been too small, so the question about the existence of a UV fixed point associated with the \( C_b \)-\( C_{dS} \) phase transition line is still open.

A different approach, in some sense closer to the renormalization group approach, is to use the Monte Carlo simulation data to construct an effective action for the three-volume \( V_3(t_k) \). The corresponding effective action can be viewed as a kind of minisuperspace action. However, the action is (numerically) derived from the full quantum theory, so the geometric degrees of freedom different from \( N_3(t) \) have not been ignored, but rather (numerically) integrated out. Chapter 9, “Semiclassical and Continuum Limits of CDT” in this Section of the Handbook will discuss this in detail. So far, it has not been possible to identify in an unambiguous way a UV fixed point, but the formalism for doing so now exists, as explained above, and future computer simulations can hopefully clarify if the fixed point exists or not.

**Future perspectives**

In discussing four-dimensional CDT we have focused on how one can in principle locate a UV fixed point. If it exists and if it is non-trivial, it would be a strong indication that there exists a non-perturbative, unitary quantum field theory of Lorentzian geometries at all length scales. It could be the quantum theory of GR. Of course one would have to provide convincing arguments in favor of such an interpretation. The CDT theory is most likely unitary when rotated back from Euclidean spacetime to Lorentzian spacetime, since one can show that the Euclidean rotated version of CDT is reflection positive, a property that usually ensures that when one rotates back to Lorentzian signature, one obtains a unitary theory. We expect a quantum theory of GR to be unitary, so a putative continuum 4d CDT theory passes that test. It also makes it unlikely that the continuum limit of 4d CDT should be some generic \( R^2 \) version of GR, since these theories typically will be non-unitary theories. Another test is that classical GR should emerge from the quantum effective action in the limit where \( \hbar \rightarrow 0 \). Such a test could be performed if one could construct the effective action of the quantum theory. As mentioned the effective action has been constructed for the three-volume \( V_3(t) \) and it is closely related to a GR
minisuperspace action. However, the real test would be to construct the full quantum effective action from the MC data and take the $\hbar \to 0$ limit, and we do not yet know how to do that. Maybe the simplest way to relate the lattice theory associated with a UV fixed point to continuum gravity theories is to determine the critical exponents related to the lattice fixed point and compare with similar analytic renormalization group calculations. In principle the functional renormalization group approach should provide us with unique critical exponents if a UV quantum gravity fixed point exists. In practice the calculations in both approaches will have error-bars and the comparison might not be easy.

There is a number of conceptional issues in a theory of quantum gravity. We tried to illustrate these in the case of solvable two-dimensional gravity models. How do we talk about distances in a theory of quantum gravity where in the path integral we integrate over the geometries that determine distances? Does it make sense to talk about arbitrarily small distances, much smaller than the quantum fluctuations of geometries? The solvable models of two-dimensional gravity encourage us to believe that it can make sense, and that we are not forced to endorse the common statement that "the concept of geometry has to break down at short distances". Of course the situation could be different in four dimensions and there is little hope that we can solve the four-dimensional theories analytically. However, the Monte Carlo simulations of the four-dimensional quantum gravity models might teach us how we should think about geometry at the shortest distances. In the two-dimensional models there have been very fruitful interplays between pure theory and Monte Carlo simulations of the models. What can relatively easily be measured in the Monte Carlo simulations are the fractal dimensions of the spacetime geometries, both the so-called Hausdorff dimension and spectral dimension 24. Let us just mention that the measurement of the spectral dimension in 4d CDT resulted in the surprising result that the dimension seems to be scale-dependent [22]. Inspired by this, similar results have been obtained analytically in a number of quantum gravity models. This is an example of a fruitful interplay between numerical studies and analytic calculations also in higher than two dimensions.

Presently we do not know for certain if there exists a UV fixed point that will allow us to define an “ordinary” quantum field theory of quantum gravity at all scales. However, the lattice efforts will not be wasted even if it should turn out that such a fixed point does not exist. First of all there exists most likely in ordinary continuum gravity an effective quantum field theory up to energies of the order of the Planck energy. There is no conceptual problems also using such an effective theory in a cosmological context. It will be a quantum field theory with a cut-off. The lattice theories, both four-dimensional EDT and CDT will be such theories, where the lattice spacing acts as the cut-off. Such cut-off theories can still provide us with a lot of non-perturbative information about (the effective theory of) quantum gravity, since non-perturbative informa-

\footnote{This is discussed Chapter 3, “Spectral Observables and Gauge Field Couplings in Causal Dynamical Triangulations” in this Section of the Handbook.}
tion is not necessarily linked to short-distance phenomena. In fact, an example of this is provided by one of the first cases where non-perturbative topological aspects entered into quantum field theory, namely in the three-dimensional Georgi-Glashow model. This model is invariant under local $SO(3)$ gauge transformations. It has a non-Abelian gauge field and a three-component scalar field that transforms as a vector under the action of the $SO(3)$ gauge group. When rotated to Euclidean space-time the model has classical monopole solutions, that act as instantons. The Higgs potential is such that the $SO(3)$ symmetry is broken down to a $U(1)$ symmetry with an associated Abelian gauge field, while the two other gauge field components combine to a massive charged, so-called $W$ particle. The monopoles will result in confinement of particles with $U(1)$ charge [64]. Amazingly, a pure $U(1)$ lattice gauge theory will produce exactly the same non-perturbative physics because it has lattice monopoles [63]. These monopoles have a mass which diverges as $1/a$, the lattice spacing, while the mass of the monopoles in the Georgi-Glashow model is proportional to $m_w/e^2$, where $m_w$ is the mass of the $W$ particle and $e$ the charge associated to the $U(1)$ symmetry. The lattice physics is therefore identical to the physics of the Georgi-Glashow model for distances larger than the lattice spacing $a$, as long as $a \sim e^2/m_w$. However, we cannot take $a < e^2/m_w$ and still capture the physics of the Georgi-Glashow models. When $a \to 0$ the lattice monopoles will be infinitely heavy and decouple, and we will just obtain a theory with a free photon. If we have a situation in quantum gravity, where there is no UV fixed point, there might be new degrees of freedom for distances shorter than the Planck length, and we will not be able to represent them correctly when our lattice spacing $a$ is less than the Planck length. But physics at scales larger than the Planck length, including some non-perturbative physics caused by these unknown degrees of freedom, could still be correctly described by our lattice models. In the case of CDT we have, as mentioned above, constructed an effective minisuperspace action, which may describe physics well all the way down to the Planck length and allow us to study universes which are not much larger than the Planck length, and discover corrections to the simplest minisuperspace models. In addition, we can also study the non-perturbative interaction between matter fields and gravity in a full quantum context\footnote{The non-trivial interaction between geometry and matter is described in Chapter 8, “Scalar Fields in Four-Dimensional CDT”, of this Section of the Handbook}. But hopefully there is a UV fixed point. Then such studies will not be confined to distances larger than the Planck length.

References

[1] J. Ambjorn and T. G. Budd, J. Phys. A: Math. Theor. 46 (2013), 315201, doi:10.1088/1751-8113/46/31/315201,[arXiv:1302.1763 [hep-th]].

[2] J. Ambjorn and B. Durhuus, Phys. Lett. B 188 (1987), 253-257, doi:10.1016/0370-2693(87)90016-5
[3] J. Ambjorn and A. Ipsen, Phys. Lett. B 724 (2013), 150-154, doi:10.1016/j.physletb.2013.06.005, [arXiv:1302.2440 [hep-th]].

[4] J. Ambjorn and J. Jurkiewicz, Phys. Lett. B 278 (1992), 42-50, doi:10.1016/0370-2693(92)90709-D

[5] J. Ambjorn and J. Jurkiewicz, Phys. Lett. B 335 (1994), 355-358, doi:10.1016/0370-2693(94)90363-8, [arXiv:hep-lat/9405010 [hep-lat]].

[6] J. Ambjorn and J. Jurkiewicz, Phys. Lett. B 345 (1995), 435-440, doi:10.1016/0370-2693(94)01666-Z, [arXiv:hep-lat/9411008 [hep-lat]].

[7] J. Ambjorn and R. Loll, Nucl. Phys. B 536 (1998), 407-434, doi:10.1016/S0550-3213(98)00692-0, [arXiv:hep-th/9805108 [hep-th]].

[8] J. Ambjorn and Y. Makeenko, Phys. Rev. D 93 (2016) no.6, 066007, doi:10.1103/PhysRevD.93.066007, [arXiv:1510.03390 [hep-th]].

[9] J. Ambjorn and Y. Makeenko, Phys. Lett. B 756 (2016), 142-146, doi:10.1016/j.physletb.2016.02.075, [arXiv:1601.00540 [hep-th]].

[10] J. Ambjorn and Y. Makeenko, Phys. Rev. D 96 (2017) no.8, 086024, doi:10.1103/PhysRevD.96.086024, [arXiv:1704.03059 [hep-th]].

[11] J. Ambjorn and Y. Watabiki, Nucl. Phys. B 445 (1995), 129-144, doi:10.1016/0550-3213(95)00154-K, [arXiv:hep-th/9501049 [hep-th]].

[12] J. Ambjorn, B. Durhuus and J. Frohlich, Nucl. Phys. B 257 (1985), 433-449, doi:10.1016/0550-3213(85)90356-6.

[13] J. Ambjorn, B. Durhuus, J. Frohlich and P. Orland, Nucl. Phys. B 270 (1986), 457-482, doi:10.1016/0550-3213(86)90563-8.

[14] J. Ambjorn, B. Durhuus and J. Frohlich, Nucl. Phys. B 275 (1986), 161-184, doi:10.1016/0550-3213(86)90594-8.

[15] J. Ambjorn, J. Jurkiewicz and Y. M. Makeenko, Phys. Lett. B 251 (1990), 517-524, doi:10.1016/0370-2693(90)90790-D

[16] J. Ambjorn, B. Durhuus and T. Jonsson, Mod. Phys. Lett. A 6 (1991), 1133-1146, doi:10.1142/S0217732391001184.

[17] J. Ambjorn, B. Durhuus and T. Jonsson, Quantum Geometry: A Statistical Field Theory Approach, Cambridge Univ. Press, 1997, ISBN 978-0-521-01736-7, 978-0-521-46167-2, 978-0-511-88535-8, doi:10.1017/CBO9780511524417

[18] J. Ambjorn, M. Carfora and A. Marzuoli, Springer Lect. Notes Phys. Monogr. 50 (1997), doi:10.1007/978-3-540-69427-4, [arXiv:hep-th/9612069 [hep-th]].
[19] J. Ambjorn, M. Carfora, D. Gabrielli and A. Marzuoli, Nucl. Phys. B 542 (1999), 349-394, doi:10.1016/S0550-3213(98)00830-X, [arXiv:hep-lat/9806035 [hep-lat]].

[20] J. Ambjorn, J. Jurkiewicz and R. Loll, Nucl. Phys. B 610 (2001), 347-382, doi:10.1016/S0550-3213(01)00297-8, [arXiv:hep-th/0105267 [hep-th]].

[21] J. Ambjorn, J. Jurkiewicz and R. Loll, Phys. Rev. D 72 (2005), 064014, doi:10.1103/PhysRevD.72.064014, [arXiv:hep-th/0505154 [hep-th]].

[22] J. Ambjorn, J. Jurkiewicz and R. Loll, Phys. Rev. Lett. 95 (2005), 171301, doi:10.1103/PhysRevLett.95.171301, [arXiv:hep-th/0505113 [hep-th]].

[23] J. Ambjorn, R. Loll, W. Westra and S. Zohren, JHEP 12 (2007), 017, doi:10.1088/1126-6708/2007/12/017, [arXiv:0709.2784 [gr-qc]].

[24] J. Ambjorn, A. Gorlich, J. Jurkiewicz and R. Loll, Phys. Rev. Lett. 100 (2008), 091304, doi:10.1103/PhysRevLett.100.091304, [arXiv:0712.2485 [hep-th]].

[25] J. Ambjorn, A. Gorlich, J. Jurkiewicz and R. Loll, Phys. Rev. D 78 (2008), 063544, doi:10.1103/PhysRevD.78.063544, [arXiv:0807.4481 [hep-th]].

[26] J. Ambjorn, R. Loll, Y. Watabiki, W. Westra and S. Zohren, JHEP 05 (2008), 032, doi:10.1088/1126-6708/2008/05/032, [arXiv:0802.0719 [hep-th]].

[27] J. Ambjorn, R. Loll, Y. Watabiki, W. Westra and S. Zohren, Phys. Lett. B 665 (2008), 252-256, doi:10.1016/j.physletb.2008.06.026, [arXiv:0804.0252 [hep-th]].

[28] J. Ambjorn, R. Loll, Y. Watabiki, W. Westra and S. Zohren, Phys. Lett. B 670 (2008), 224-230, doi:10.1016/j.physletb.2008.11.003, [arXiv:0810.2408 [hep-th]].

[29] J. Ambjorn, A. Gorlich, S. Jordan, J. Jurkiewicz and R. Loll, Phys. Lett. B 690 (2010), 413-419, doi:10.1016/j.physletb.2010.05.054, [arXiv:1002.3298 [hep-th]].

[30] J. Ambjorn, S. Jordan, J. Jurkiewicz and R. Loll, Phys. Rev. Lett. 107 (2011), 211303, doi:10.1103/PhysRevLett.107.211303, [arXiv:1108.3932 [hep-th]].

[31] J. Ambjorn, A. Goerlich, J. Jurkiewicz and R. Loll, Phys. Rept. 519 (2012), 127-210, doi:10.1016/j.physrep.2012.03.007, [arXiv:1203.3591 [hep-th]].

[32] J. Ambjorn, L. Glaser, Y. Sato and Y. Watabiki, Phys. Lett. B 722 (2013), 172-175, doi:10.1016/j.physletb.2013.04.006, [arXiv:1302.6359 [hep-th]].
[33] J. Ambjorn, A. Görlich, J. Jurkiewicz, A. Kreienbuehl and R. Loll, Class. Quant. Grav. **31** (2014), 165003 doi:10.1088/0264-9381/31/16/165003 [arXiv:1405.4585 [hep-th]].

[34] J. Ambjorn, T. Budd and Y. Watabiki, Phys. Lett. B **736** (2014), 339-343, doi:10.1016/j.physletb.2014.07.047, [arXiv:1406.6251 [hep-th]].

[35] J. Ambjorn, D. N. Coumbe, J. Gizbert-Studnicki and J. Jurkiewicz, JHEP **08** (2015), 033, doi:10.1007/JHEP08(2015)033, [arXiv:1503.08580 [hep-th]].

[36] J. Ambjorn, D. Coumbe, J. Gizbert-Studnicki and J. Jurkiewicz, Phys. Rev. D **93** (2016) no.10, 104032 doi:10.1103/PhysRevD.93.104032 [arXiv:1603.02076 [hep-th]].

[37] J. Ambjorn, T. Budd and Y. Makeenko, Nucl. Phys. B **913** (2016), 357-380, doi:10.1016/j.nuclphysb.2016.09.013, [arXiv:1604.04522 [hep-th]].

[38] J. Ambjorn, J. Gizbert-Studnicki, A. Görlich, J. Jurkiewicz, N. Klitgaard and R. Loll, Eur. Phys. J. C **77** (2017) no.3, 152, doi:10.1140/epjc/s10052-017-4710-3, [arXiv:1610.05245 [hep-th]].

[39] J. Ambjorn, D. Coumbe, J. Gizbert-Studnicki, A. Görlich and J. Jurkiewicz, Phys. Rev. D **95** (2017) no.12, 124029, doi:10.1103/PhysRevD.95.124029, [arXiv:1704.04373 [hep-lat]].

[40] J. Ambjorn, J. Gizbert-Studnicki, A. Görlich, J. Jurkiewicz and R. Loll, Front. in Phys. **8** (2020), 247, doi:10.3389/fphy.2020.00247, [arXiv:2002.01693 [hep-th]].

[41] P. Bialas, Z. Burda, A. Krzywicki and B. Petersson, Nucl. Phys. B **472** (1996), 293-308, doi:10.1016/0550-3213(96)00214-3, [arXiv:hep-lat/9601024 [hep-lat]].

[42] S. Bassler, J. Laiho, M. Schiffer and J. Unmuth-Yockey, Phys. Rev. D **103** (2021), 114504, doi:10.1103/PhysRevD.103.114504 [arXiv:2103.06973 [hep-lat]].

[43] S. Bilke and G. Thorleifsson, Phys. Rev. D **59** (1999), 124008, doi:10.1103/PhysRevD.59.124008, [arXiv:hep-lat/9810049 [hep-lat]].

[44] A. Billoire and F. David, Nucl. Phys. B **275** (1986), 617-640, doi:10.1016/0550-3213(86)90577-8

[45] D. V. Boulatov, V. A. Kazakov, I. K. Kostov and A. A. Migdal, Nucl. Phys. B **275** (1986), 641, doi:10.1016/0550-3213(86)90578-X

[46] B. Durhuus, T. Jonsson and J. F. Wheater, J. Statist. Phys. **139** (2010), 859, doi:10.1007/s10955-010-9968-x, [arXiv:0908.3643 [math-ph]].
[47] S. Catterall, G. Thorleifsson, J. B. Kogut and R. Renken, Nucl. Phys. B 468 (1996), 263-276, doi:10.1016/0550-3213(96)00130-7, [arXiv:hep-lat/9512012 [hep-lat]].

[48] F. David, Nucl. Phys. B 257 (1985), 543-576, doi:10.1016/0550-3213(85)90363-3

[49] F. David, Mod. Phys. Lett. A 3 (1988), 1651, doi:10.1142/S0217732388001975

[50] F. David, J. Jurkiewicz, A. Krzywicki and B. Petersson, Nucl. Phys. B 290 (1987), 218-230, doi:10.1016/0550-3213(87)90186-6

[51] P. Di Francesco, E. Guitter and C. Kristjansen, Nucl. Phys. B 549 (1999), 657-667, doi:10.1016/S0550-3213(99)00187-X, [arXiv:cond-mat/9902082 [cond-mat]].

[52] J. Distler and H. Kawai, Nucl. Phys. B 321 (1989), 509-527, doi:10.1016/0550-3213(89)90354-4

[53] J. Distler, Z. Hlousek and H. Kawai, Int. J. Mod. Phys. A 5 (1990), 1093, doi:10.1142/S0217732389000507

[54] P. Horava, Phys. Rev. D 79 (2009), 084008, doi:10.1103/PhysRevD.79.084008, [arXiv:0901.3775 [hep-th]].

[55] H. Kawai, N. Kawamoto, T. Mogami and Y. Watabiki, Phys. Lett. B 306 (1993), 19-26, doi:10.1016/0370-2693(93)91131-6, [arXiv:hep-th/9302133 [hep-th]].

[56] V. A. Kazakov, Mod. Phys. Lett. A 4 (1989), 2125, doi:10.1142/S0217732389002392

[57] V. A. Kazakov, A. A. Migdal and I. K. Kostov, Phys. Lett. B 157 (1985), 295-300, doi:10.1016/0370-2693(85)90669-0

[58] V. G. Knizhnik, A. M. Polyakov and A. B. Zamolodchikov, Mod. Phys. Lett. A 3 (1988), 819, doi:10.1142/S0217732388000982

[59] M. Luscher and P. Weisz, Nucl. Phys. B 290 (1987), 25-60, doi:10.1016/0550-3213(87)90177-5

[60] M. Luscher and P. Weisz, Nucl. Phys. B 295 (1988), 65-92, doi:10.1016/0550-3213(88)90228-3

[61] I. Montvay and G. Munster, Quantum fields on a lattice, Cambridge University Press, 1997, ISBN 978-0-521-59917-7, 978-0-511-87919-7, doi:10.1017/CBO9780511470783

[62] U. Pachner, Udo (1991), European Journal of Combinatorics, 12 (2): 129–145, doi:10.1016/s0195-6698(13)80080-7
[63] A. M. Polyakov, Phys. Lett. B 59 (1975), 82-84, doi:10.1016/0370-2693(75)90162-8

[64] A. M. Polyakov, Nucl. Phys. B 120 (1977), 429-458, doi:10.1016/0550-3213(77)90086-4

[65] T. Regge, Nuovo Cim. 19 (1961), 558-571, doi:10.1007/BF02733251

[66] N. Sasakura, Mod. Phys. Lett. A 6 (1991), 2613-2624, doi:10.1142/S0217732391003055