Let $G$, $H$ be two groups. Denote by $H^G$ the group of all maps $f: G \to H$ with finite support, i.e., such that $f(x) = 1$ for all but a finite set of elements of $G$. Recall that their (restricted regular) wreath product $W = H \wr G$ is defined as the semidirect product $H^G \times G$ with the natural action of $G$ on $H^G: f^g(a) = f(a^g)$ [1, p. 175].

We are going to find a set of generators and relations for $H \wr G$ knowing those for $G$ and $H$. Then we shall extend this result to the multiple wreath products $\wr_{i=1}^n G_k = ((\ldots ((G_1 \wr G_2) \wr G_3) \ldots) \wr G_n$.

If $x = \{x_1, x_2, \ldots, x_n\}$ are generators for $G$ and $R = \{R_1, R_2, \ldots, R_m\}$ are defining relations for this set of generators, we write $G = \langle x_1, x_2, \ldots, x_n R_1, R_2, \ldots, R_m \rangle$ or $G = \langle x | R \rangle$. A presentation is called minimal if neither of the generators $x_1, x_2, \ldots, x_n$ nor of the relations $R_1, R_2, \ldots, R_m$ can be excluded. We call the set of generators $x$ conormal if neither element $x \in x$ belongs to the normal subgroup $N_c$ generated by all $y \in x \setminus \{x\}$. For instance, any minimal set of generators of a finite group $G$ is conormal since their images are linear independent in the factorgroup $G/G^f(G, G)$ [1] (Theorem 5.48).

**Theorem 1.** Let $G = \langle x | R(x) \rangle$, $H = \langle y | S(y) \rangle$ be presentations of $G$ and $H$. Choose a subset $T \subseteq G$ such that $T \cap T^{-1} = \emptyset$ and $T \cup T^{-1} = G \setminus \{1\}$, where $T^{-1} = \{t^{-1} | t \in T\}$. Then the wreath product $W = H \wr G$ has a presentation of the form

$$W = \langle x, y | R(x), S(y), [y, t^{-1}] = 1 \text{ for all } y, z \in y, t \in T \rangle. \tag{1}$$

If the given presentations of $G$ and $H$ are minimal and the set of generators $y$ is conormal, the presentation (1) is minimal as well.

**Theorem 2.** Let $G_i := \langle x_i | R_i(x_i) \rangle$ be presentations of the groups $G_i$, $1 \leq i \leq m$. For $1 < i \leq m$ choose a subset $T_i \subseteq G_i$ such that $T_i \cap T_i^{-1} = \emptyset$ and $T_i \cup T_i^{-1} = G_i \setminus \{1\}$. Then the wreath product $W = \wr_{i=1}^n G_i$ has a presentation of the form

$$W = \langle x_i, 1 \leq i \leq m | R_i(x_i), 1 \leq i \leq m, [x, t^{-1}] = 1 \text{ for all } x, y \in \bigcup_{i<j} x_i, t \in T_j \rangle. \tag{2}$$

If all given presentations of $G_i$ are minimal and the sets of generators $x_i$, $1 \leq i < n$, respectively.
are conormal, the presentation (2) is minimal as well.

In what follows, we keep the notations of Theorem 1. Note that \( H^G = \bigoplus_{a \in G} H(a) \), where \( H(a) \) is a copy of the group \( H \); the elements of \( H(a) \) will be denoted by \( h(a) \), where \( h \) runs through \( H \). Then \( h(a)^h = h(ah) \) and \( H^G = \langle y(a) | S(y(a)), [y(a), z(b)] = 1 \rangle \), where \( a, b \in G, a \neq b \).

The following lemma is quite evident.

**Lemma 1.** Suppose a group \( G \) acting on a group \( N \). Let \( G = \langle x | R(x) \rangle, N = \langle y | S(y) \rangle \) be presentations of \( G \) and \( N \), and \( y^c = w_{ys}(y) \) for each \( x \in x, y \in y \). Then their semidirect product \( N \rtimes G \) has a presentation

\[
N \rtimes G := \langle x, y | R(x), S(y), x^{-1} y x = w_{ys}(y) \text{ for all } x \in x, y \in y \rangle.
\]

Note that this presentation may not be minimal even if both presentations for \( G \) and \( N \) were so, since some elements of \( y \) may become superfluous.

**Corollary 1.** The wreath product \( W = H \wr G \) has indeed a presentation (1).

**Proof.** Lemma 1 gives a presentation

\[
W := \langle x, y | R(x), S(y), y a x = y a x \text{ for } x \in x, y \in y \rangle.
\]

Using the last relations, we can exclude all generators \( y(a) \) for \( a \neq 1 \); we only have to replace \( y(a) \) and \( z(b) \) by \( a^{-1} y(1)a \) and \( b^{-1} z(1)b \). So we shall write \( h \) instead of \( h(1) \) for \( h \in H \); especially, the relations for \( y(a) \) and \( z(b) \) are rewritten as \( [a^{-1} y a, b^{-1} z b] = 1 \). The latter is equivalent to \( [y, t^{-1} z t] = 1 \), where \( t = b a^{-1} \neq 1 \).

Moreover, the relations \( [y, t^{-1} z t] = 1 \) and \( [z, t y t^{-1}] = 1 \) are also equivalent; therefore we only need such relations for \( t \in T \).

The corollary is proved.

**Lemma 2.** Suppose that \( y \) is a conormal set of generators of the group \( H, u, v \in y \), and consider the group \( H'_{u,v} = (H \ast H')/N_{u,v} \), where \( \ast \) denotes the free product of groups, \( H' \) is a copy of the group \( H \) whose elements are denoted by \( h' \) (\( h \in H \)), and \( N_{u,v} \) is the normal subgroup of \( H \ast H' \) generated by the commutators \( [y, z] \) with \( y, z \in y, (y, z) \neq (u, v) \). Then \( [u, v'] \neq 1 \) in \( H_{u,v} \).

**Proof.** Let \( C = H/N_{u,v}, C' = H'/N_{u,v}', P = C \ast C', \overline{u} = u N_{u,v}, \overline{v} = v N_{u,v}' \). Consider the homomorphism \( \varphi \) of \( H \ast H' \) to \( P \) such that

\[
\varphi(y) = \begin{cases} 1 & \text{if } y \in y_{u,v}, \\ \overline{u} & \text{if } y = u, \end{cases}
\]

\[
\varphi(z') = \begin{cases} 1 & \text{if } z \in y_{u,v}' , \\ \overline{v}' & \text{if } z = v. \end{cases}
\]

Obviously, \( \varphi \) is well defined and \( \varphi([y, z]) = 1 \) if \( (y, z) \neq (u, v) \), so it induces a homomorphism \( H_{u,v} \to P \). Since \( \varphi([u, v']) = [\overline{u}, \overline{v}'] \neq 1 \), it accomplishes the proof.

Now fix elements \( c \in T, u, v \in y \), and let \( K_{c,u,v} \) be the group with a presentation

\[
K_{c,u,v} := \langle y(a), a \in G | S(y(a)), [y(a), z(ta)] = 1 \text{ for all } y, z \in y, a \in G, t \in T, (t, y, z) \neq (c, u, v) \rangle.
\]
Corollary 2. Let the set of generators $y$ be conormal. Then $[u(1), v(c)] \neq 1$ in the group $K_{c,u,v}$.

Proof. There is a homomorphism $\psi : K_{c,u,v} \to H_{u,v}$, where $H_{u,v}$ is the group from Lemma 2, mapping $u(1) \mapsto u$, $v(c) \mapsto v'$, $y(a) \to 1$ in all other cases. Then $\psi([u(1), v(c)]) = [u, v'] \neq 1$, so $[u(1), v(c)] \neq 1$ as well.

Corollary 3. If the given presentations of $G$ and $H$ are minimal and the set of generators $y$ is conormal, the presentation (1) is minimal.

Proof. Obviously, we can omit from (1) neither of generators $x$ nor of the relations $R(x)$, $S(y)$. So we have to prove that neither relation $[u, c^{-1}v c] = 1$ ($u, v \in y, c \in T$) can be omitted as well. Consider the group $K = K_{c,u,v}$ of Corollary 2. The group $G$ acts on $K$ by the rule: $h(a)^{y} = h(a) g$. Let $Q = K \rtimes G$. Then, just as in the proof of Corollary 1, this group has a presentation

$$Q := \langle x, y | R(x), S(y), [y, t^{-1} z t] = 1 \text{ for all } y, z \in y, t \in T, (t, y, z) \neq (c, u, v) \rangle,$$

where $y = y(1)$ for all $y \in y$, but $[u, c^{-1} v c] = [u(1), v(c)] \neq 1$.

The corollary is proved.

Now for an inductive proof of Theorem 2 we only need the following simple result.

Lemma 3. If the sets of generators $x$ of $G$ and $y$ of $H$ are conormal, so is the set of generators $x \cup y$ of $H \wr G$.

Proof. Since $G = (H \wr G)/\hat{H}$, where $\hat{H}$ is the normal subgroup generated by all $y \in y$, it is clear that neither $x \in x$ belongs to the normal subgroup generated by $(x \setminus \{x\}) \cup y$. On the other hand, there is an epimorphism $H \wr G \to C \wr G$, where $C = H/N_y$ for some $y \in y$; in particular, $C \neq \{1\}$ and is generated by the image $\bar{y}$ of $y$. Since $C$ is commutative, the map $C \wr G \to C$, $(f(x), g) \mapsto \prod_{x \in G} f(x)$ is also an epimorphism mapping $\bar{y}$ to itself. The resulting homomorphism $H \wr G \to C$ maps all $x \in x$ as well as all $z \in y \setminus \{y\}$ to 1 and $y$ to $\bar{y} \neq 1$, which accomplishes the proof.

Example 1. The wreath product $C_n \wr C_m$, where $C_n$ denotes the cyclic group of order $n$, has a minimal presentation

$$C_n \wr C_m := \langle x, y | x^m = 1, y^n = 1, [y, x^{-k} y x^k] = 1 \text{ for } 1 \leq k \leq m/2 \rangle.$$

(Possibly, $m = \infty$ or $n = \infty$, then the relation $x^m = 1$ or, respectively, $y^n = 1$ should be omitted.)

Received 18.04.08

ISSN 1027-3190. Укр. мат. журн., 2008, т. 60, № 7