Limits on isocurvature perturbations from non-Gaussianity in WMAP temperature anisotropy

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ABSTRACT
We study the effect of primordial isocurvature perturbations on non-Gaussian properties of cosmic microwave background (CMB) temperature anisotropies. We consider generic forms of the non-linearity of isocurvature perturbations which can be applied to a wide range of theoretical models. We derive analytical expressions for the bispectrum and the Minkowski Functionals for CMB temperature fluctuations to describe the non-Gaussianity from isocurvature perturbations. We find that the isocurvature non-Gaussianity in the quadratic isocurvature model, where the isocurvature perturbation $S$ is written as a quadratic function of the Gaussian variable $\sigma$, $S = \sigma^2 - \langle\sigma^2\rangle$, can give the same signal-to-noise ratio as $f_{NL} = 30$ even if we impose the current observational limit on the fraction of isocurvature perturbations contained in the primordial power spectrum $\alpha$. We give constraints on isocurvature non-Gaussianity from Minkowski Functionals using the Wilkinson Microwave Anisotropy Probe (WMAP) 5-year data. We do not find a significant signal of isocurvature non-Gaussianity. For the quadratic isocurvature model, we obtain a stringent upper limit on the isocurvature fraction $\alpha < 0.070$ (95 per cent CL) for a scale-invariant spectrum which is comparable to the limit obtained from the power spectrum.

Key words: methods: analytical – methods: statistical – cosmic microwave background – early Universe.

1 INTRODUCTION
Recent observational progress in cosmology represented by surveys such as Wilkinson Microwave Anisotropy Probe (WMAP) and Sloan Digital Sky Survey has enabled a detailed analysis of cosmic density fields to investigate the physics of the early Universe. In particular, (non-)Gaussianity of primordial density fields has received much attention recently as a key observational probe to differentiate models of the early universe. In the simplest single field inflationary scenario, quantum fluctuations of the inflation during inflation are assumed to be the origin of cosmic density fluctuations, and such a model predicts adiabatic and almost Gaussian primordial fluctuations. However, other generation mechanisms of density fluctuations such as the curvaton scenario (Mollerach 1990; Linde & Mukhanov 1997; Moroi & Takahashi 2001; Enqvist & Sloth 2002; Lyth & Wands 2002), modulated reheating (Kofman 2003; Dvali, Gruzinov & Zaldarriaga 2004) and so on have been proposed. In these mechanisms, the nature of primordial density fluctuations can be very different from that of a simple inflation model, in particular in terms of Gaussianity of the fluctuations. In fact, it has been shown that large non-Gaussianity can be generated in curvaton models (Lyth, Ungarelli & Wands 2003; Bartolo, Matarrese & Riotto 2004; Enqvist & Nurmi 2005; Malik & Lyth 2006; Sasaki, Valiviita & Wands 2006; Suyama & Yamaguchi 2008; Enqvist & Takahashi 2008; Huang 2008; Ichikawa et al. 2008a), modulated reheating scenarios (Zaldarriaga 2004; Ichikawa et al. 2008b; Suyama & Yamaguchi 2008), Dirac–Born–Infeld inflation models (Alishahiha, Silverstein & Tong 2004; Chen et al. 2007b; Arroja, Mizuno & Koyama 2008; Langlois et al. 2008a), Ghost inflation (Arkami-Hamed et al. 2004), ekpyrotic models (Koyama et al. 2007; Buchbinder, Khoury & Ovrut 2008; Lehnerts & Steinhardt 2008), single field inflation with a feature in its potential (Chen, Easther & Lim 2007a), a Gaussian-squared component in curvature

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or entropy perturbations (Linde & Mukhanov 1997; Boubekeur & Lyth 2006; Suyama & Takahashi 2008) and multi-bird inflation (Sasaki 2008; Naruko & Sasaki 2009).

The primordial non-Gaussianity has been quantitatively measured by higher order statistics such as bispectra and Minkowski Functionals from cosmic microwave background (CMB) temperature maps obtained by WMAP (Crennelli et al. 2007; Hikage et al. 2008; Komatsu et al. 2009; Yadav & Wandelt 2008) and also from large-scale structure (Slosar et al. 2008). So far, the observational results are consistent with the Gaussian hypothesis. There is, however, a hint of primordial non-Gaussianity at 2–3σ level (Komatsu et al. 2009; Yadav & Wandelt 2008). Since the non-Gaussianity predicted in the simplest inflation model is too small to be detected by current observations, if a non-Gaussian signal is observed and it originates from primordial fluctuations, other mechanisms beyond the simplest model would be required in the dynamics of the early universe.

As another probe of the early universe, the adiabaticity of primordial density fields has also been the subject of intense study. In fact, current cosmological observations of the temperature-temperature (TT) and the temperature-polarization (TE) spectra of CMB with some other distance measurements such as Type Ia supernovae (SNe) and baryon acoustic oscillation (BAO) have already constrained the fraction of isocurvature perturbations to be less than 10 per cent (e.g. Bean, Dunkley & Pierpaoli 2006; Kawasaki & Sekiguchi 2008; Komatsu et al. 2009). Examples of isocurvature perturbations along with non-Gaussianity have been discussed in the context of non-Gaussian field potentials (Linde & Mukhanov 1997; Peebles 1999; Boubekeur & Lyth 2006; Suyama & Takahashi 2008), the curvaton scenario (Lyth et al. 2003; Bartolo et al. 2004; Beltran 2008; Moroi & Takahashi 2009), modulated reheating (Boubekeur & Crennelli 2006), baryon asymmetry (Kawasaki, Nakayama & Takahashi 2009a) and the axion (Kawasaki et al. 2008, 2009b). In particular, non-Gaussianity generated from isocurvature perturbations has been systematically investigated recently (Kawasaki et al. 2008; Langlois, Vernizzi & Wands 2008b; Kawasaki et al. 2009b).

In this paper, we discuss non-Gaussianity generated from the non-linearity of isocurvature perturbations and study a constraint on non-Gaussianity from isocurvature perturbations using Minkowski Functionals. For this purpose, we derive theoretical expressions for bispectra and Minkowski Functionals to characterize non-Gaussianity in CMB temperature maps generated from primordial mixed perturbations of diabatic and isocurvature components. We consider the non-linearity of isocurvature perturbations in two different forms which are theoretically motivated: one is a Gaussian variable with its quadratic correction (Linear Model). The other form is given as a quadratic of Gaussian variables without a linear term (Quadratic Model). These two generic forms are applicable to a wide range of isocurvature models listed above. Then, we give actual limits on the isocurvature non-Gaussianity using the WMAP 5-year data. As far as we know, this is the first attempt to put a limit on isocurvature perturbations from the non-Gaussianity of CMB anisotropies.

In this paper, we focus on cold dark matter (CDM) isocurvature perturbations. However, it is straightforward to apply our method to other types of isocurvature perturbations including baryon and neutrino ones. We adopt a set of cosmological parameters at the maximum likelihood values for a power-law ΛCDM model obtained from the WMAP 5-year data only fit (Dunkley et al. 2009): Ωm = 0.0432, Ωdm = 0.206, Ωb = 0.7508, H0 = 72.4 km s−1 Mpc−1, τ = 0.089 and nS = 0.961. The total amplitude of primordial power spectra is set to be ΑNL(k = 0.002 Mpc−1) = 2.41 × 10−9.

This paper is organized as follows. In Section 2, we give two different forms of the non-linear isocurvature perturbations called the ‘Linear Model’ and the ‘Quadratic Model’. In Section 3, we derive analytical expressions for bispectra to describe isocurvature non-Gaussianity in CMB temperature anisotropies in the Linear Model. The isocurvature non-Gaussianity in the Quadratic Model is described in Section 4. In Section 5, we present generic perturbative formulae of Minkowski Functionals that can be applied to CMB temperature maps with adiabatic and isocurvature non-Gaussianity. In Section 6, we give limits on isocurvature non-Gaussianity from the WMAP 5-year temperature maps using Minkowski Functionals. In Section 7, we discuss implications of our results for an axion isocurvature model. Section 8 is devoted to a summary and our conclusions.

2 NON-LINEAR ADIABATIC AND ISOCURVATURE PERTURBATIONS

We consider the admixture of an adiabatic perturbation ζ with a CDM isocurvature perturbation S. The curvature perturbation is written up to second order in a local form as

\[ ζ = φ + \frac{3}{5} f_{NL}(φ^2 - ⟨φ^2⟩), \]

where φ is the linear term of ζ that obeys Gaussian statistics. The non-linear parameter fNL represents the quadratic amplitude of the curvature perturbation Φ during the matter era (Komatsu & Spergel 2001), which is related to ζ by Φ = (3/5)ζ.

An isocurvature perturbation S subject to matter and radiation is defined as

\[ S = \frac{δρ_\text{m}}{ρ_\text{m}} - \frac{3δρ_\gamma}{4ρ_\gamma}, \]

where ρm is the matter energy density and ργ is radiation energy density. In this paper, we focus on a CDM isocurvature perturbation.

We consider the non-linearity of the isocurvature perturbation in two different forms. One is a local form similar to equation (1) which has a linear (Gaussian) term with a quadratic correction

**I. Linear Model:** \[ S = η + f_{NL}^{(SO)}(η^2 - ⟨η^2⟩), \]

where η is a Gaussian variable and its non-linearity is characterized by fNL^{(SO)}.

The isocurvature non-Gaussianity in this form was studied in the context of the axion (Kawasaki et al. 2008) and the curvaton scenario (Langlois et al. 2008b).

The other is the case where the linear term is negligible compared with the quadratic term (e.g. Linde & Mukhanov 1997; Peebles 1999; Boubekeur & Lyth 2006; Kawasaki et al. 2008)

**II. Quadratic Model:** \[ S = σ^2 - ⟨σ^2⟩, \]

where σ obeys Gaussian statistics. Linde & Mukhanov (1997) proposed a scenario to generate the quadratic form of isocurvature perturbations by introducing a massive free scalar field oscillating around the vacuum state which has a zero value. In this scenario, the isocurvature fluctuation has a blue spectrum, and thus we consider here a wide range for the spectral index ranging from 1 to 3 in the Quadratic Model. Boubekeur & Lyth (2006) showed that a Gaussian-squared component of the primordial curvature perturbation would be bounded at 10 per cent level by the WMAP bound on the bispectrum.

The auto- and cross-correlation power spectra for fluctuations of I and J are defined as

\[ ⟨I_k J_{k′}⟩ = (2π)^3 δ(k)δ(k + k′)P_{IJ}(k). \]

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and then its dimensionless power is given by
\[ \Delta_{ij}(k) = \frac{P_{ij}(k)k^3}{2\pi^2}, \]

where \( I \) and \( J \) denote \( \zeta \) or \( S \).

We assume the following power-law form of auto- and cross-correlation power spectra for the Gaussian variables \( \phi \) (equation 1), \( \eta \) (equation 3) and \( \sigma \) (equation 4):
\[ \Delta_{XX}(k) = A_X \left( \frac{k}{k_0} \right)^{(\alpha_x - 1)}, \]
\[ \Delta_{YT}(k) = (A_X A_Y)^{1/2} \cos \theta_Y \left( \frac{k}{k_0} \right)^{(\alpha_x + \alpha_Y)/2 - 1}, \]

where \( X \) and \( Y \) denote \( \phi, \eta \) or \( \sigma \). As \( f_{NL}A_\phi^{1/2} \) is observationally limited to be much smaller than unity, the power spectrum of the primordial adiabatic perturbation is given by
\[ \Delta_{\zeta\zeta}(k) \simeq \Delta_{\phi\phi}(k). \]

We define the fraction of isocurvature perturbation as
\[ \alpha = \frac{P_{SS}(k_0)}{P_{\zeta\zeta}(k_0) + P_{SS}(k_0)}, \]
which is the same definition as in, for example, Bean et al. (2006). We here set \( k_0 = 0.002 \text{ Mpc}^{-1} \). The parameter \( \alpha \) is related to another common parameter of adiabaticity \( \delta_\alpha^{(c)} \) (e.g. equation 41 of Komatsu et al. 2009) as \( \delta_\alpha^{(c)} = \left[ \alpha/(1 - \alpha) \right]^{1/2}/3 \). The upper limits of \( \alpha \) from WMAP, BAO and SN combined are given by 0.067 (95 per cent CL) for axion-type \( (\cos \theta_{S} = 0) \) and 0.0367 (95 per cent CL) for curvaton-type \( (\cos \theta_{S} = -1) \) isocurvature perturbations (Komatsu et al. 2009).

3 NON-GAUSSIANITY OF ISOCURVATURE PERTURBATIONS: LINEAR MODEL

3.1 Initial perturbation

In the Linear Model (equation 3), power spectra for the isocurvature perturbation and its cross-term with the adiabatic perturbation become
\[ \Delta_{SS}(k) \simeq \Delta_{\eta\eta}(k), \]
\[ \Delta_{\zeta S}(k) \simeq \Delta_{\phi\eta}(k). \]

Here, we have used the fact that \( f_{NL}^{(ISO)} \lesssim 1/\sqrt{\Delta_{SS}} \simeq 1/\sqrt{\alpha \Delta_{\phi\phi}} \sim 10^6 \). In the case \( f_{NL}^{(ISO)} \) is larger than this upper limit, the linear term of \( \eta \) in equation (3) is negligible so that the model should be described by the quadratic model.

The ratio of the amplitude of the power spectra between \( \phi \) and \( \eta \) is written in terms of \( \alpha \) (equation 10) as
\[ \frac{A_\eta}{A_\phi} = \frac{\alpha}{1 - \alpha}. \]

We define the bispectra of \( \zeta \), \( S \) and their mixed contribution as
\[ \langle I_k, J_k, K_k \rangle \equiv (2\pi)^3 \delta_D^3(k_1 + k_2 + k_3) \times B_{\Delta\Delta\Delta}(k_1, k_2, k_3), \]
where \( I, J \) and \( K \) denote \( \zeta \) or \( S \).

When \( \phi \) and \( \eta \) are initially uncorrelated, the adiabatic and isocurvature modes evolve independently under the linear approximation. There exist bispectra only from each mode given as
\[ B_{\zeta\zeta\zeta}(k_1, k_2, k_3) = 6 f_{NL}^{(ISO)} P_{\phi\phi}(k_1) P_{\phi\phi}(k_2) + P_{\phi\phi}(k_1) P_{\phi\phi}(k_2) + P_{\phi\phi}(k_2) P_{\phi\phi}(k_1). \]
\[ B_{\zeta\zeta\zeta}(k_1, k_2, k_3) = 2 f_{NL}^{(ISO)} P_{\eta\eta}(k_1) P_{\eta\eta}(k_2) + P_{\eta\eta}(k_1) P_{\eta\eta}(k_2) + P_{\eta\eta}(k_2) P_{\eta\eta}(k_1). \]

When \( \phi \) and \( \eta \) are initially correlated, the following cross-correlation terms may become important:
\[ B_{\zeta\zeta\eta}(k_1, k_2, k_3) \equiv B_{\zeta\zeta\eta}(k_1, k_2, k_3) + B_{\zeta\zeta\eta}(k_1, k_2, k_3) + B_{\zeta\zeta\eta}(k_1, k_2, k_3) \]
\[ = 6 f_{NL}^{(ISO)} [P_{\phi\phi}(k_1)] [P_{\phi\phi}(k_2) + P_{\phi\phi}(k_3)] + P_{\phi\phi}(k_2) + P_{\phi\phi}(k_3)] + P_{\phi\phi}(k_1) + P_{\phi\phi}(k_2)]. \]

3.2 Bispectra of CMB temperature anisotropy

Harmonic coefficients \( a_{lm} \) of CMB temperature anisotropies \( \Delta T / T \) are defined as
\[ a_{lm} = \int d\hat{n} \frac{\Delta T}{T} (\hat{n}) Y^*_{lm}(\hat{n}). \]

Introducing the radiative transfer function \( g^\zeta_l \) (or \( g^S_l \)), they are related to \( \zeta \) (or \( S \)) as
\[ a_{lm} = a_{lm}^\zeta + a_{lm}^S, \]
\[ a_{lm}^\zeta = 4\pi(-i)^l \int \frac{d^3k}{(2\pi)^3} \hat{k} \times \hat{n} g^\zeta_l(k) Y^*_{lm}(\hat{k}). \]
\[ a_{lm}^S = 4\pi(-i)^l \int \frac{d^3k}{(2\pi)^3} S_k g^S_l(k) Y^*_{lm}(\hat{k}). \]

The angular power spectra of adiabatic and isocurvature components are
\[ C^\zeta_l = \frac{2}{\pi} \int k^2 dk P_{\zeta\zeta}(k) g^\zeta_l^2(k)^2, \]
\[ C^S_l = \frac{2}{\pi} \int k^2 dk P_{SS}(k) g^S_l^2(k)^2. \]

Fig. 1 shows the angular power spectrum from adiabatic and isocurvature perturbations. The radiation transfer functions for adiabatic and isocurvature perturbations are computed using the publicly
available cmbfast code (Seljak & Zaldarriaga 1996). The index of spectral curvature perturbations in the Linear Model is \( n_s = 1 \). We set \( \alpha = 0.067 \) for \( n_s = 1 \) at \( k_0 = 0.002 \) Mpc\(^{-1}\), which is a 2\( \sigma \) upper limit from the WMAP 5-yr paper (Komatsu et al. 2009).

The total angular bispectrum of CMB is written as the sum of bispectra with different combinations of adiabatic and isocurvature components:

\[
\mathcal{B}_{lll} = \sum_{ijk} \mathcal{B}^{ijk},
\]

where each component is defined as

\[
\langle a_{lm1} a_{lm2} a^K_{lm3} \rangle = \delta_{lll} \mathcal{B}^{ijk},
\]

\[
G^{lllm} = \int \mathcal{Y}_l(\hat{\mathbf{n}}) \mathcal{Y}_{lm}(\hat{\mathbf{n}}) \mathcal{Y}_{lm}(\hat{\mathbf{n}}) \tag{27}
\]

\[
= \frac{\sqrt{(2l_1+1)(2l_2+1)(2l_3+1)}}{4\pi} \left( \begin{array}{ccc}
1 & l_2 & l_3 \\
0 & 0 & 0 \\
\end{array} \right)
\times \left( \begin{array}{ccc}
l_1 & l_2 & l_3 \\
m_1 & m_2 & m_3 \\
\end{array} \right),
\]

where \( I \) and \( J \) denote \( \zeta \) or \( S \). Then, \( \mathcal{B}^{ijk} \) can be written as

\[
b^{ijk} = \mathcal{B}^{ijk} = \frac{8}{\pi^2} \int r^2 dr \int k^2 dk_1 \int k^2 dk_2 \int k^2 dk_3 \ g_{l_1}(k_1) g_{l_2}(k_2) g_{l_3}(k_3) B_{ijk}(k_1, k_2, k_3).
\]

The explicit form of adiabatic, isocurvature bispectra and their cross-correlations is given in Appendix A.

Fig. 2 shows each component of the CMB bispectrum for equilateral triangles (\( l_1 = l_2 = l_3 \)). The purely adiabatic component \( b^{SSS}_{lll} \) with \( \eta_{NL} = 50 \) is plotted with a solid line. A long-dashed line represents a mixed component \( b^{CSS}_{lll} \) with \( |\cos \phi_{\phi_0}| = 1 \) and the curvaton-type upper limit \( \alpha = 0.0037 \). The other mixed component \( b^{SCS}_{lll} \) with \( |\cos \phi_{\phi_0}| = 0.1 \) and the axion-type limit \( \alpha = 0.067 \) is plotted with a short-dashed line. The pure isocurvature component \( b^{SSS}_{lll} \) with the axion-type limit \( \alpha = 0.067 \) is plotted with dotted lines. The non-linearity of isocurvature perturbation \( f^{(ISO)}_{NL} \) is set to be \( 10^4 \). The spectral index of isocurvature perturbation is 1.

We numerically estimate how large \( f^{(ISO)}_{NL} \), the non-Gaussianity from isocurvature perturbations, should be to obtain the same values of the signal-to-noise ratio of the CMB bispectrum as the one derived from the non-Gaussianity from purely adiabatic perturbations characterized by \( \eta_{NL} \). The signal-to-noise ratio is defined as

\[
\left( \frac{S}{N} \right)^2 = \sum_{2\leq l_1 \leq l_2 \leq l_3} \frac{I_{l_1l_2l_3}^2}{C_l C_{l_2} C_{l_3} \Delta l_{l_1l_2l_3}},
\]

where \( C_l \) is \( C_{lll} \) of purely adiabatic perturbations (\( \alpha = 0 \)) including noise and \( I_{l_1l_2l_3} \) is defined as

\[
I_{l_1l_2l_3} = \frac{\sqrt{(2l_1+1)(2l_2+1)(2l_3+1)}}{4\pi} \left( \begin{array}{ccc}
l_1 & l_2 & l_3 \\
0 & 0 & 0 \\
\end{array} \right)
\times \left( \begin{array}{ccc}
l_1 & l_2 & l_3 \\
m_1 & m_2 & m_3 \\
\end{array} \right).
\]

The factor \( \Delta l_{l_1l_2l_3} \) is equal to 6 (\( l_1 = l_2 = l_3 \)), 2 (\( l_1 = l_3 \) or \( l_2 = l_3 \)) and 1 (\( l_1 \neq l_2 \neq l_3 \)). WMAP beam window functions are included both in the bispectrum and in the power spectrum. The homogeneous noise distribution for the WMAP 5-year data is used to estimate the noise.

The pure isocurvature term \( b^{SSS}_{lll} \) with \( n_S = 1 \) has non-Gaussianity corresponding to

\[
f_{NL} \approx 15 \left( \frac{\alpha}{0.067} \right)^2 \left( \frac{f^{(ISO)}_{NL}}{10^4} \right).
\]

where the above equations are valid when \( \alpha \ll 1 \). It is found that isocurvature non-Gaussianity reaches \( f_{NL} \sim 10 \) only if the non-linearity in isocurvature perturbations \( f^{(ISO)}_{NL} \) is of the order of \( 10^4 \) because isocurvature non-Gaussianity in the Linear Model is suppressed by \( \alpha^2 \).

When there is a weak correlation between \( \phi \) and \( \eta \) (\( |\cos \phi_{\phi_0}| \ll 1 \)), \( b^{SSS}_{lll} \) has non-Gaussianity corresponding to

\[
f_{NL} \approx 9.7 \left( \frac{\alpha}{0.067} \right)^{3/2} \left( \frac{1}{0.1} \right) \left( \frac{f^{(ISO)}_{NL}}{10^4} \right).
\]
If the correlation is strong \(|\cos \theta_{\phi} \simeq 1|\) (e.g. curvaton-type isocurvature perturbation), the other correlated term \(f_{\xi}^{(S)}\) becomes important with a more severe limit \(\alpha < 0.0037\) for curvaton-type isocurvature perturbations:

\[
    f_{NL} \simeq 7.2 \left( \frac{\alpha}{0.0037} \right) \cos^3 \theta_{\phi} \left( \frac{f_{\xi}^{(S)}}{10^4} \right).
\]

(34)

In both cases, however, isocurvature perturbations need to have a strong non-linearity \(f_{\xi}^{(S)} \sim 10^4\) in order to generate non-Gaussianity corresponding to \(f_{NL} \sim 10\). Our result is consistent with previous theoretical estimations (Kawasaki et al. 2008; Langlois et al. 2008b).

4 NON-GAUSSIANITY OF ISOCURVATURE PERTURBATIONS II: QUADRATIC MODEL

4.1 Initial perturbation

In the Quadratic Model (equation 4), the auto- and cross-correlation power spectra of \(\zeta\) and \(S\) become

\[
    P_{\zeta S}(k) = 2 \int_{L^{-1}} \frac{d^3 p}{(2\pi)^3} P_{\sigma \sigma}(p) P_{\sigma \sigma}(|k + p|),
\]

(35)

\[
    P_{\zeta S}(k) = \frac{6}{5} f_{NL} \int_{L^{-1}} \frac{d^3 p}{(2\pi)^3} P_{\sigma \sigma}(p) P_{\sigma \sigma}(|k + p|),
\]

where a finite box-size \(L\) gives an infrared cut-off. To avoid assumptions at scales far beyond the present horizon \(H^{-1}\), \(L\) should be set not too much bigger than \(H^{-1}\) (Lyth 2007). Hereafter we set \(L = 30\) Gpc.

Using the power-law form as in equation (7), the isocurvature power spectra are written as

\[
    \Delta_{SS}(k) = A^2 F(n_s) \left( \frac{k}{k_0} \right)^{2(n_s - 1)},
\]

(37)

where the factor \(F\) is approximately given by

\[
    F(n_s) = \begin{cases} 
    4 \log(kL) & (n_s = 1), \\
    (4(n_s - 1) - 1/(1 - kL)^{1-n_s}) & (n_s \neq 1).
    \end{cases}
\]

(38)

As we take \(L\) to be much larger than the range of scales we are interested in, the \(k\) dependence in \(F\) is weak.

As \(S\) is the square of a Gaussian variable, the cross-correlation coefficient becomes nearly zero regardless of \(\cos \theta_{\phi}\):

\[
    \frac{\Delta_{\zeta S}}{(\Delta_{\zeta \zeta} \Delta_{SS})^{1/2}} \simeq f_{NL}(A_\zeta F)^{1/2} \cos^3 \theta_{\phi} < f_{NL}(A_\zeta F)^{1/2} \lesssim \mathcal{O}(10^{-2}).
\]

(39)

Following the definition of \(\alpha\) in equation (10), the ratio between \(A_\zeta\) and \(A_\phi\) becomes

\[
    \frac{A_\zeta}{A_\phi} = \left( \frac{\alpha}{1 - \alpha} \right)^{1/2} (A_\zeta F)^{-1/2}.
\]

(40)

The amplitude of \(\sigma\) has an additional factor \((A_\zeta F)^{-1/2} \sim \mathcal{O}(10^4)\) relative to the amplitude of \(\phi\) because the isocurvature perturbation is given as quadratic in \(\sigma\) (equation 4).

The bispectra from isocurvature perturbations are written as

\[
    B_{SSS}(k_1, k_2, k_3) = \frac{8}{3} \int L^{-1} \frac{d^3 p}{(2\pi)^3} P_{\sigma \sigma}(p)
    \times [P_{\sigma \sigma}(k_1 + p) P_{\sigma \sigma}(k_2 - p)]
    + P_{\sigma \sigma}(k_2 + p) P_{\sigma \sigma}(k_1 - p)
    + P_{\sigma \sigma}(k_1 + p) P_{\sigma \sigma}(k_3 - p)],
\]

(41)

4.2 Bispectra for CMB temperature anisotropy

The angular power spectra for isocurvature perturbations in the Quadratic Model are plotted in Fig. 1 for different spectral indices \(n_s = 1, 1.5\) and 2. We set \(\alpha = 0.067\) for \(n_s = 1\) as in the Linear Model. The power spectrum in the Quadratic Model is slightly different from that in the Linear Model due to the weak scale dependence of \(F\) (equation 38). For other spectral indexes, we set \(\alpha = 0.008\) (\(n_s = 1.5\)) and 0.001 (\(n_s = 2\)) so that the amplitude of \(C_l\) around \(l = 200\) becomes roughly the same.

The angular bispectrum from the isocurvature term \(B_{SSS}\) is approximately given by putting \(k_1\) as one of the wavenumbers of \(P_{\sigma \sigma}\) in equation (42):

\[
    \frac{B_{SSS}}{B_{\zeta \zeta}} = 2 \left( \frac{\alpha}{1 - \alpha} \right)^{1/2} \frac{f_{NL}}{\mathcal{O}(10^{-2})} \left( A_\zeta F \right)^{1/2}.
\]

(45)

\[
    \frac{B_{SSS}}{B_{\zeta \zeta}} = 3 \left( \frac{\alpha}{1 - \alpha} \right)^{1/2} \cos^2 \theta_{\phi} (A_\zeta F)^{-1/2}.
\]

(46)

\[
    \frac{B_{SSS}}{B_{\zeta \zeta}} = 3 \left( \frac{\alpha}{1 - \alpha} \right)^{1/2} \cos^2 \theta_{\phi}.
\]

(47)

The bispectra \(B_{\zeta \zeta S}\) and \(B_{SSS}\) have an additional factor \((A_\zeta F)^{-1/2} \sim \mathcal{O}(10^4)\) relative to \(B_{\zeta \zeta}\). This is because the isocurvature perturbation has no Gaussian part (equation 4) and thus its cubic term does not vanish unlike the adiabatic case. The isocurvature non-Gaussianity can therefore be substantial even if \(\alpha \sim 0.067\). We neglect the last term \(B_{\zeta \zeta S}\) due to the current observational limit on \(\alpha < 0.067\).

When we extract the dominant contributions around the poles, equation (41) is approximately written as (Kawasaki et al. 2008)

\[
    B_{SSS}(k_1, k_2, k_3) \approx 2 \Delta_{\sigma \sigma}(k_1) F(n_s) P_{\sigma \sigma}(k_2) P_{\sigma \sigma}(k_3)
    + P_{\sigma \sigma}(k_2) P_{\sigma \sigma}(k_3) + P_{\sigma \sigma}(k_1) P_{\sigma \sigma}(k_3),
\]

(42)

where \(k_b = \text{Min}(k_1, k_2, k_3)\).

The cross-correlation terms are

\[
    B_{\zeta \zeta S}(k_1, k_2, k_3) = \frac{24}{5} f_{NL} \int \frac{d^3 p}{(2\pi)^3} P_{\sigma \sigma}(p)
    \times [P_{\sigma \sigma}(k_1 + p) P_{\sigma \sigma}(k_2 - p)]
    + P_{\sigma \sigma}(k_2 + p) P_{\sigma \sigma}(k_3 - p)
    + P_{\sigma \sigma}(k_1 + p) P_{\sigma \sigma}(k_3 - p)].
\]

(43)

The amplitude of each bispectrum component relative to the pure adiabatic one at \(k_b\) is estimated as

\[
    \frac{B_{SSS}}{B_{\zeta \zeta}} = \frac{5}{3} f_{NL} \left( \frac{\alpha}{1 - \alpha} \right)^{3/2} (A_\zeta F)^{-1/2}.
\]

(45)

\[
    \frac{B_{\zeta \zeta S}}{B_{\zeta \zeta}} = \frac{5}{3} f_{NL} \left( \frac{\alpha}{1 - \alpha} \right)^{1/2} \cos^2 \theta_{\phi} (A_\zeta F)^{-1/2}.
\]

(46)

\[
    \frac{B_{SSS}}{B_{\zeta \zeta}} = 3 \left( \frac{\alpha}{1 - \alpha} \right)^{1/2} \cos^2 \theta_{\phi}.
\]

(47)
The isocurvature bispectrum in the Quadratic Model depends on $\alpha$. The isocurvature bispectrum depends on $\alpha$.

When $n_\sigma$ is equal to 1, $h_i(y)$ is analytically given as $2y/(1 - y^2)$.

In the left-hand panel of Fig. 3, we compare the bispectrum of the adiabatic component with those of the isocurvature components with different spectral indexes $n_{\sigma}$ of 1, 1.5, and 2. We set $f_{NL} = 50$ and $\alpha = 0.067$ for $n_\sigma = 1$, $\alpha = 0.008$ for $n_\sigma = 1.5$, and $\alpha = 0.001$ for $n_\sigma = 2$ (same as Fig. 1). The box-size $L$ is set to be 30 Gpc.

Computationally, it is very difficult to evaluate the full expression at high $\ell$. Thus, we check the validity of the pole approximations at low $\ell$, less than 10. The thick lines represent the full calculations within a factor of 2 at least for $\ell \leq 10$. The full calculation has comparable or larger amplitude for all $n_{\sigma}$, and a slightly steeper slope than the pole approximation. This indicates that the isocurvature bispectrum at higher $l$ would become larger than the pole approximation and then the proper signal may be larger than the pole approximation. The amplitude of the isocurvature bispectrum is proportional to $\alpha^{3/2}$ and therefore the effect on the estimation of $\alpha$ is suppressed at the power of two-third.

The isocurvature bispectrum in the Quadratic Model depends on the assumed box-size $L$, which is set to be 30 Gpc in this analysis. Equation (45) indicates that the ratio of the isocurvature component in the primordial perturbation at $k = k_0 = 0.002\,\text{Mpc}^{-1}$ is proportional to a box-size-dependent factor $L^{-1/2}$ (equation 38). When $L$ is set to be 10 times larger (300 Gpc), the amplitude decreases by 20 per cent for $n_{\sigma} = 1.5$ per cent for $n_{\sigma} = 1.5$ and 1 per cent for $n_{\sigma} = 2$. We find that the overall amplitude of the CMB bispectrum also decreases at the same level at $l > 10$, while additional large-scale power at more than 30 Gpc slightly increases the amplitude of the bispectrum at smaller $l$. The isocurvature bispectrum depends on $\alpha^{3/2}$ and therefore the error in $\alpha$ increases $(1/0.8)^{3/2} \approx 16$ per cent for $n_{\sigma} = 1$. 

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Using equation (30), we estimate the isocurvature non-Gaussianity in terms of $f_{NL}$:

$$ f_{NL} = 30 \left( \frac{\alpha}{0.067} \right)^{3/2}. \quad (56) $$

It is found that the isocurvature non-Gaussianity in the Quadratic Model can reach $f_{NL} \sim 30$ given the current 2σ limit on $\alpha$.

The angular bispectrum from the correlation term $B_{ijl(S)}$ is given as

$$ b_{ijl}^{(S)} = 2 \int r^2 dr \left[ b_{ijl}^{(S)}(r) b_{klm}^{(S)}(r) b_{nlj}^{(S)}(r) \right]. \quad (57) $$

The correlated term $b_{ijl}^{(S)}$ is plotted in the right-hand panel of Fig. 3 for different spectral indexes $n_\alpha = 1, 1.5$ and 2. The correlated coefficient $\cos \theta_{\phi\sigma}$ is set to be 0.1. Using equation (30), the isocurvature non-Gaussianity with $n_\alpha = 1$ corresponds to

$$ f_{NL} = 240 \left( \frac{\alpha}{0.067} \right)^{1/2}. \quad (58) $$

If $\phi$ and $\sigma$ are strongly correlated initially, the correlation term can generate substantial non-Gaussianity, while the effect of the correlation on $C_l$ is negligible as shown in equation (39).

## 5 Perturbative Formulae of Minkowski Functionals

We adopt the perturbation formulae for Minkowski Functionals developed by Matsubara (2003) to describe the non-Gaussianity of the CMB temperature anisotropy (Hikage, Komatsu & Matsubara 2006; Hikage et al. 2008). We separate the analytical formulae of Minkowski Functionals into the amplitude and a function of $v$, which is defined as $\Delta T/T$ deviation, as follows:

$$ V_k(v) = A_k(v). \quad (59) $$

The amplitude $A_k$ is given using the angular power spectrum $C_l$ as

$$ A_k = \frac{1}{(2\pi)^{k+1/2}} \frac{\sigma_1}{\sigma_0} \left( \frac{\sigma_1}{\sqrt{2}\sigma_0} \right)^k, \quad (60) $$

$$ \sigma^2 \equiv \frac{1}{4\pi} \sum_l (2l+1)(l+1)! C_l W_l^2, \quad (61) $$

where $\sigma_0 = \pi^{1/2}/\Gamma(k/2 + 1)$ gives $\sigma_0 = 1, \omega_1 = 2, \omega_2 = \pi$, and $W_l$ represents the smoothing kernel determined by the pixel and beam window functions and an additional smoothing. In our analysis, we add a Gaussian kernel $W_l = \exp[-l(l+1)\omega^2_l/2]$ where $\omega$ is a smoothing scale. For weakly non-Gaussian fields, the function $v_k(v)$ can be divided into the Gaussian term $v_k^{(G)}$ and the non-Gaussian term $\Delta v_k^{(G)}$:

$$ v_k(v) = v_k^{(G)}(v) + \Delta v_k(v, f_{NL}). \quad (62) $$

$$ v_k^{(G)} = e^{-v^2/2} H_{k-1}(v), \quad (63) $$

$$ \Delta v_k(v, f_{NL}) = e^{-v^2/2} \left\{ \frac{1}{6} S^{(0)} H_{k-2}(v) + \frac{k}{3} S^{(1)} H_k(v) + \frac{k(k-1)}{6} S^{(2)} H_{k-2}(v) \right\} \sigma_0 + \mathcal{O}(\sigma_0^2), \quad (64) $$

where $H_n(v)$ is the $n$th Hermite polynomials. The non-Gaussian term $\Delta v_{10}$ is characterized by three skewness parameters $S^{(i)}$ at the lowest order in $\sigma_0$. The skewness parameters $S^{(i)}$ are given by the integral of the reduced bispectrum as

$$ S^{(0)} = \frac{3}{2\pi\sigma_0} \sum_{2 \leq l_1 \leq l_2 \leq l_3} I_{l_1l_2l_3}^2 b_{l_1l_2} W_{l_1} W_{l_2} W_{l_3}, \quad (65) $$

$$ S^{(1)} = \frac{3}{8\pi\sigma_0^3} \sum_{2 \leq l_1 \leq l_2 \leq l_3} \left[ I_{l_1}(l_1 + 1) + I_{l_2}(l_2 + 1) + I_{l_3}(l_3 + 1) \right] $$

$$ \times I_{l_2l_3}^2 b_{l_2l_3} W_{l_1} W_{l_2} W_{l_3}, \quad (66) $$

$$ S^{(2)} = \frac{3}{4\pi\sigma_0^4} \sum_{2 \leq l_1 \leq l_2 \leq l_3} \left\{ I_{l_1}(l_1 + 1) + I_{l_2}(l_2 + 1) - I_{l_3}(l_3 + 1) \right\} $$

$$ \times I_{l_3}(l_3 + 1) + (\text{cycl}) I_{l_1l_2l_3}^2 b_{l_1l_2l_3} W_{l_1} W_{l_2} W_{l_3}, \quad (67) $$

where $I_{l_1l_2}$ is as previously defined in equation (31).

In Fig. 4, we plot the three skewness parameters $S^{(i)}$ from each component of bispectrum in the Quadratic Model as a function of $\theta_i$. Upper panels show the skewness of isocurvature components with different spectral indices $n_\alpha = 1, 1.5$ and 2 in comparison with the adiabatic case. The skewness from a mixed component $b_{ijl}^{(S)}$ is plotted in the lower panels. The smaller scale (higher $l$) information in the bispectrum is reflected in skewness parameters with a higher number (i.e. $S^{(2)}$ rather than $S^{(0)}$) because it is weighted towards higher $l$. We can extract further detailed scale-dependent information by changing the smoothing scale $\theta_i$.

Fig. 5 illustrates an example of non-Gaussian effects on each Minkowski Functional $\Delta v_i$ (equation 64) at different smoothing scales $\theta_i = 10, 20, 40, 70$ and 100 arcmin. Here, we consider isocurvature non-Gaussianity in the Quadratic Model with $n_\alpha = 1$ and $\alpha = 0.067$. It is found that the non-Gaussian effect on Minkowski Functionals is 1 per cent or less.

## 6 Limits on Isocurvature Non-Gaussianity from WMAP Data

We use the WMAP 5-year data to constrain the non-Gaussianity associated with primordial isocurvature perturbations. We use the linearly co-added maps for $Q$ and $V$ frequency bands with $N_{side} = 512$. The co-added maps are masked with the $Kq75$ galaxy mask including the point-source mask provided by Gold et al. (2009), which leaves 71.8 per cent of the sky available for the data analysis. The field is smoothed with a Gaussian filter at a scale of $\theta_i$. We obtain the normalized Minkowski Functional (equation 64) for the WMAP data using the same procedure described in Hikage et al. (2008).

We use Bayes’s theorem to find a probability of $\alpha$ or other combined parameters, together denoted by $x$, from the observed set of $\Delta v^{(obs)}$ as follows:

$$ P(x|\Delta v^{(obs)}) \propto L(\Delta v^{(obs)}|x) P(x), \quad (68) $$

where $L$ is the likelihood function of $\Delta v^{(obs)}$ when a non-Gaussian parameter has a value $x$ and the probability $P(x)$ represents the prior for $x$. In general, we need to analyse all non-Gaussian components together; however we consider the cases in which a single non-Gaussian term dominates over the other terms for simplicity; in the Linear Model, the pure isocurvature term (equation A2) dominates when there is no correlation between $\phi$ and $\eta$. If there exists a weak correlation, the correlation term of $(\xi SS)$ (equation A8) becomes...
The skewness parameters $S^{(i)}$ (left-hand panel: $k = 0$, centre: $k = 1$, right-hand panel: $k = 2$) of each component in the Quadratic Model plotted as a function of $\theta_1$. The skewness values for a pure isocurvature component are plotted in upper panels, while those for a mixed component are plotted in lower panels. The parameters of isocurvature perturbation are $n_x = 1$ with $\alpha = 0.067$ (long-dashed), $n_x = 1.5$ with $\alpha = 0.008$ (short-dashed) and $n_x = 2$ with $\alpha = 0.001$ (dotted). For comparison, the adiabatic skewness with $n_\phi = 0.96$ and $f_{NL} = 50$ is plotted with solid lines in all panels.

Non-Gaussian term of Minkowski Functionals $\Delta v_k$ (equation 64) from isocurvature bispectrum in Quadratic Model (left-hand panel: $k = 0$, centre: $k = 1$, right-hand panel: $k = 2$). The isocurvature perturbation has a spectral index $n_x = 1$ and the fraction $\alpha$ is 0.067. The smoothing scales are shown by different lines: $\theta_1 = 10$ (solid), $\theta_1 = 20$ (long-dashed), $\theta_1 = 40$ (short-dashed), $\theta_1 = 70$ (dotted) and $\theta_1 = 100$ (dot-dashed) arcmin.

The likelihood function is computed by

$$-2 \ln L(\Delta v_{(obs)}|x) \propto \sum_{ij} \left[ \Delta v_{ij}^{(obs)} - \Delta v_{ij}^{(theory)}(x) \right] C_{ij}^{-1} \times \left[ \Delta v_{ij}^{(obs)} - \Delta v_{ij}^{(theory)}(x) \right],$$

(69)

where $i$ and $j$ denote the binning number of threshold values $\nu$, different kinds of Minkowski Functional $k$ and smoothing scales parametrized with $\theta$. We choose 18 threshold values at an equal spacing in the range of $\nu$ from -3.6 to 3.6. The full covariance matrix $C_{ij}$ is estimated from 1000 Gaussian simulation maps from purely adiabatic perturbations. They include the pixel and beam window function, $Kq75$ survey mask and inhomogeneous noise for WMAP 5-year maps. Applying our procedure to limit the non-Gaussianity from curvature perturbations by putting $x$ as $f_{NL}$, we obtain $-63 < f_{NL} < 76$ at 95 per cent CL (Hikage et al. in preparation).

Table 1 lists the mean and 1σ error of the isocurvature non-Gaussianity in the Linear Model characterized by $\alpha \cos^2 \theta_\phi f_{NL}^{(ISO)}$, $\alpha^{3/2} \cos \theta_\phi f_{NL}^{(ISO)}$ and $\alpha^2 f_{NL}^{(ISO)}$. Significant isocurvature non-Gaussian signals are not found. If isocurvature perturbations exist and there is no correlation between $\phi$ and $\eta$, the non-linear parameter $f_{NL}^{(ISO)}$ is constrained from isocurvature non-Gaussianity for a fixed $\alpha$ to be

$$f_{NL}^{(ISO)} = (-3300 \pm 13000)(\alpha/0.067)^{-2}.$$  

(70)

If there is a strong correlation between $\phi$ and $\eta$ represented by curvaton-type isocurvature perturbations ($\cos \theta_{\phi\eta} = -1$), the correlated term $(\zeta \zeta S)$ becomes important and then its non-Gaussianity is limited as

$$f_{NL}^{(ISO)} = (4900 \pm 43000)(\alpha/0.0037)^{-1}.$$  

(71)

Table 2 lists the maximum likelihood value $\alpha_{ML}$, at which $P(\alpha)$ has a maximum value, and the 95 per cent confidence limit of $\alpha$ when the pure isocurvature term in the Quadratic Model dominates.
Table 1. Mean values and 1σ uncertainties of each isocurvature non-Gaussianity parameter in the Linear Model; the pure isocurvature term \( \alpha^2 f_{\text{NL}}^{(\text{ISO})} \) (equation A2) and the correlated terms with adiabatic perturbations \( \alpha^{3/2} \cos \theta_{\phi \sigma} f_{\text{NL}}^{(\text{ISO})} \) (equation A8) and \( \cos^2 \theta_{\phi \sigma} f_{\text{NL}}^{(\text{ISO})} \) (equation A7). The limits are obtained from Minkowski Functionals for WMAP 5-year data at different smoothing scales \( \theta_s \) and their combination.

| \( \theta_s \) (arcmin) | \( \alpha^2 f_{\text{NL}}^{(\text{ISO})} \) | \( \alpha^{3/2} \cos \theta_{\phi \sigma} f_{\text{NL}}^{(\text{ISO})} \) | \( \alpha \cos^2 \theta_{\phi \sigma} f_{\text{NL}}^{(\text{ISO})} \) |
|------------------------|-------------------------------|---------------------------------|-------------------------------|
| 100                    | 33 ± 84                        | 26 ± 62                         | 91 ± 220                      |
| 70                     | 36 ± 74                        | 28 ± 56                         | 104 ± 210                     |
| 40                     | 71 ± 78                        | 56 ± 61                         | 256 ± 230                     |
| 20                     | 0 ± 89                         | -3 ± 66                         | 73 ± 240                      |
| 10                     | 1 ± 91                         | -30 ± 60                        | 41 ± 230                      |
| 10, 20, 40, 70, 100    | -15 ± 60                       | -18 ± 43                        | 18 ± 160                      |

Table 2. Limits on \( \alpha \) for the Quadratic Model from Minkowski Functionals for WMAP 5-year data at different smoothing scales \( \theta_s \). The listed values are maximum likelihood values \( \sigma_{\text{ML}} \) and 95 per cent CL on \( \alpha \). We neglect non-Gaussianity from curvature perturbations and their cross-correlation. We impose a non-negative condition on \( \alpha \) following from its definition in equation (10).

| \( \theta_s \) (arcmin) | \( \alpha \) |       | \( \alpha \) |       | \( \alpha \) |
|------------------------|-------------|-------|-------------|-------|-------------|
|                        | \( n_s = 1 \) | 95 per cent CL | \( n_s = 1.5 \) | 95 per cent CL | \( n_s = 2 \) | 95 per cent CL |
| 100                    | 0.037       | <0.115 | 0.022       | <0.100 | 0.005       | <0.047 |
| 70                     | 0.036       | <0.105 | 0.029       | <0.079 | 0.013       | <0.030 |
| 40                     | 0.051       | <0.112 | 0.010       | <0.057 | 0           | <0.014 |
| 20                     | 0           | <0.101 | 0           | <0.048 | 0           | <0.0085 |
| 10                     | 0           | <0.093 | 0           | <0.047 | 0           | <0.0071 |
| 10, 20, 40, 70, 100    | 0           | <0.070 | 0           | <0.042 | 0           | <0.0064 |

We do not find a significant non-zero value of \( \alpha \). The upper limit of \( \alpha \) is given by

\[
\alpha < 0.070 \quad (n_s = 1),
\alpha < 0.042 \quad (n_s = 1.5),
\alpha < 0.0064 \quad (n_s = 2),
\]

at 95 per cent CL. Our constraint on \( \alpha \) for \( n_s = 1 \) is comparable to that from the joint limit from WMAP (TT and TE spectra) + BAO + SN, which is smaller than 0.067 (95 per cent CL), obtained by Komatsu et al. (2009). The limits on the correlated term \( \alpha^{3/2} \cos \theta_{\phi \sigma} \) are also listed in Table 3. Our result roughly agrees with Boubekeur & Lyth (2006) who showed the fraction of a Gaussian-squared component of CMB isocurvature bispectra is limited to be smaller than 0.18, which corresponds to \( \alpha < 0.031 \), when \( |f_{\text{NL}}| < 100 \). The difference from their analysis is that we calculate a Gaussian-squared component of CMB isocurvature bispectra including its full angular transfer function.

Table 3. Same as Table 2 but for the parameter \( \alpha^{3/2} \cos \theta_{\phi \sigma} \) where the correlation term \( b_{l1l2l3}^{(c)} \) in the Quadratic Model dominates. We also impose a non-negative condition on \( \alpha^{3/2} \cos \theta_{\phi \sigma} \).

| \( \theta_s \) (arcmin) | \( \alpha^{3/2} \cos \theta_{\phi \sigma} \) | 95 per cent CL | \( \alpha^{3/2} \cos \theta_{\phi \sigma} \) | 95 per cent CL | \( \alpha^{3/2} \cos \theta_{\phi \sigma} \) | 95 per cent CL |
|------------------------|-------------------------------|----------------|-------------------------------|----------------|-------------------------------|----------------|
| 100                    | 0.018                         | <0.095         | 0.018                         | <0.12          | 0.010                         | <0.089         |
| 70                     | 0.021                         | <0.092         | 0.021                         | <0.11          | 0.014                         | <0.073         |
| 40                     | 0.045                         | <0.12          | 0.040                         | <0.14          | 0.017                         | <0.075         |
| 20                     | 0.012                         | <0.098         | 0.026                         | <0.11          | 0.013                         | <0.041         |
| 10                     | 0                             | <0.071         | 0.003                         | <0.063         | 0.0026                        | <0.019         |
| 10, 20, 40, 70, 100    | 0                             | <0.049         | 0                             | <0.039         | 0                             | <0.012         |

7 IMPLICATIONS FOR AXION ISOCURVATURE

Here, we briefly discuss implications of our results in some explicit models. Although for the linear model, the constraint from non-Gaussianity obtained here is not as strong as that from the power spectrum, the constraint for the quadratic model is severe as we showed in the previous section. Thus, we consider the case of the axion as discussed in Kawasaki et al. (2008). Assuming that Pecci–Quinn symmetry has already been spontaneously broken during inflation, the mean value of the axion field can be written as

\[
\alpha = f_a \theta_s,
\]

where \( f_a \) is the axion decay constant and \( \theta_s \) is the phase of the axion. During inflation, the axion field has quantum fluctuations

\[
\delta a = \frac{H_{\text{inf}}}{2\pi},
\]

where \( H_{\text{inf}} \) is the Hubble parameter during inflation. When the average value of the axion is much less than the mean-square
inhomogeneity of \( a \), i.e. when \( f_a \theta_a \leq H_{\text{inf}}/2\pi \), the density fluctuation of \( a \) is written as
\[
\frac{\delta \rho_a}{\rho_a} = \left( \frac{\delta a}{a} \right)^2.
\] (75)

Thus power spectrum for isocurvature fluctuations can be written as
\[
k^3 P_{\text{NL}}(k) = \left( \frac{\Omega_h^2}{\Omega_{\text{cdm}} h^2} \right)^2.
\] (76)

where \( \Omega_h^2 \) is the energy density of the axion at present. When \( f_a \theta < H_{\text{inf}}/2\pi \), the abundance of the axion is given by (Turner 1986)
\[
\Omega_h^2 = 0.2 \left( \frac{f_a}{10^{13} \text{ GeV}} \right)^{-0.825} \left( \frac{H_{\text{inf}}/2\pi}{10^{12} \text{ GeV}} \right)^2.
\] (77)

Since the total amplitude of primordial perturbations should be
\[
\Delta_{\text{tot}} \simeq \Delta_{\text{ci}} + \Delta_{\text{SS}} = k^3/2\pi^2 (P_{\text{ci}} + P_{\text{SS}}) = 2.4 \times 10^{-9}
\]
which is required from WMAP5, by using the constraint on \( \alpha \) presented in the previous section, \( \alpha < 0.070 \), we obtain a limit for the Hubble parameter during inflation
\[
H_{\text{inf}} < 1.7 \times 10^{10} \left( \frac{f_a}{10^{13} \text{ GeV}} \right)^{0.41} \text{ GeV},
\] (78)

where we adopt \( \Omega_{\text{cdm}} h^2 = 0.108 \) which is the mean value for a \( \Lambda \)CDM model from the WMAP5 analysis. The condition \( f_a \theta_a < H_{\text{inf}}/2\pi \) gives
\[
\theta_a < 2.7 \times 10^{-3} \left( \frac{f_a}{10^{13} \text{ GeV}} \right)^{-0.59}.
\] (79)

Although our results obtained in this paper may also have implications for other models with isocurvature fluctuations, a detailed study of this issue will be given elsewhere.

8 SUMMARY AND CONCLUSIONS

We have explored the effect of non-Gaussianity from primordial isocurvature perturbations on CMB temperature anisotropies. Considering the linear and quadratic forms of isocurvature perturbations, which are applicable to a wide range of theoretical models, we derived theoretical expressions for bispectra and Minkowski Functionals of CMB temperature maps with isocurvature non-Gaussianity. We find that the amplitude of a quadratic correction \( f_{\text{NL}} \), in the Linear Model (a Gaussian variable plus its quadratic correction) needs to be of the order of \( 10^5 \) to generate CMB non-Gaussianity at a level of \( f_{\text{NL}} \sim 10 \). The isocurvature non-Gaussianity in the Quadratic Model (quadratic in a Gaussian variable without linear terms) can reach \( f_{\text{NL}} \sim 30 \) while respecting the current upper limit on the isocurvature contribution to the power spectrum \( \alpha < 0.067 \). Isocurvature perturbations provide a possible mechanism to explain primordial non-Gaussianity recently suggested from the observed bispectrum of the CMB (Komatsu et al. 2009; Yadav & Wandelt 2008).

We give limits on isocurvature non-Gaussianity from Minkowski Functionals for the WMAP 5-year data. In the Quadratic Model of isocurvature perturbations, we obtain a stringent limit \( \alpha < 0.070 \) (95 per cent CL) from the non-Gaussianity, which is comparable to the current constraints from WMAP TT and TE spectra, BAO and SN combined \( \alpha < 0.067 \) (95 per cent CL). We apply our results to a QCD axion isocurvature model and then obtain a limit for the Hubble parameter and the phase of the axion.

We estimate isocurvature non-Gaussianity in the Quadratic Model using the pole approximation (equation 42). The validity can be checked by comparing Minkowski Functionals with non-Gaussian simulations with a Gaussian-squared perturbation. We plan to perform this analysis in the near future.

We employ Minkowski Functionals to characterize non-Gaussianity in CMB maps. The application of our work to other higher order statistics is important to utilize non-Gaussian information in a more complete manner. Different statistics are sensitive to different aspects of density fields and they are affected by possible observational systematics (e.g. foregrounds and point sources) in a different way. There is actually some friction between the limits on \( f_{\text{NL}} \) from Minkowski Functionals and the bispectrum (Hikage et al. 2008); Minkowski Functionals analysis indicates a maximum likelihood value around nearly 0, while bispectrum analysis favours a more positive \( f_{\text{NL}} \) around 50 or more. Complementary analyses with different statistical approaches will provide a more robust way to analyze primordial non-Gaussianity.

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where
\[
\begin{align*}
 b^{c,\phi\phi}_{L_1}(r) &= \frac{2}{\pi} \int k^2 dk P_{\phi\phi}(k) g^c_{L_1}(k) j_1(k r), \\
 b^{S,\eta\eta}_{L_1}(r) &= \frac{2}{\pi} \int k^2 dk P_{\eta\eta}(k) g^S_{L_1}(k) j_1(k r), \\
 b^{ \eta \eta}_{NL}(r) &= \frac{2}{\pi} \int k^2 dk g^\eta_{NL}(k) j_1(k r), \\
 b^{S}_{NL}(r) &= \frac{2}{\pi} \int k^2 dk g^S_{NL}(k) j_1(k r).
\end{align*}
\]

The angular bispectra from the correlation terms are given by
\[
\begin{align*}
 b^{(S)ISO}_{l_1 l_2 l_3} &= b^{c,\phi\phi}_{l_1 l_2 l_3} + b^{S,\eta\eta}_{l_1 l_2 l_3} + b^{ \eta \eta}_{NL l_1 l_2 l_3} \\
 &= 2 f_{NL}^{(ISO)} \int r^2 dr \left[ b^{c,\phi\phi}_{l_1}(r) b^{c,\phi\phi}_{l_2}(r) b^{c,\phi\phi}_{l_3}(r) \\
 &\quad + b^{S,\eta\eta}_{l_1}(r) b^{S,\eta\eta}_{l_2}(r) b^{S,\eta\eta}_{l_3}(r) \\
 &\quad + b^{ \eta \eta}_{NL l_1}(r) b^{ \eta \eta}_{NL l_2}(r) b^{ \eta \eta}_{NL l_3}(r) \\
 &\quad + \frac{6}{5} f_{NL} b^{c,\phi\phi}_{l_1}(r) b^{S,\eta\eta}_{l_2}(r) b^{S,\eta\eta}_{l_3}(r) \\
 &\quad + b^{S,\eta\eta}_{l_1}(r) b^{S,\eta\eta}_{l_2}(r) b^{S,\eta\eta}_{l_3}(r) \\
 &\quad + \frac{6}{5} f_{NL} b^{ \eta \eta}_{NL l_1}(r) b^{ \eta \eta}_{NL l_2}(r) b^{ \eta \eta}_{NL l_3}(r) \\
 &\quad + 2 f_{NL}^{(ISO)} \int r^2 dr \left[ b^{c,\phi\phi}_{l_1}(r) b^{S,\eta\eta}_{l_2}(r) b^{S,\eta\eta}_{l_3}(r) \\
 &\quad + b^{S,\eta\eta}_{l_1}(r) b^{S,\eta\eta}_{l_2}(r) b^{S,\eta\eta}_{l_3}(r) \\
 &\quad + b^{ \eta \eta}_{NL l_1}(r) b^{ \eta \eta}_{NL l_2}(r) b^{ \eta \eta}_{NL l_3}(r) \\
 &\quad + \frac{6}{5} f_{NL} b^{c,\phi\phi}_{l_1}(r) b^{S,\eta\eta}_{l_2}(r) b^{S,\eta\eta}_{l_3}(r) \\
 &\quad + b^{S,\eta\eta}_{l_1}(r) b^{S,\eta\eta}_{l_2}(r) b^{S,\eta\eta}_{l_3}(r) \\
 &\quad + \frac{6}{5} f_{NL} b^{ \eta \eta}_{NL l_1}(r) b^{ \eta \eta}_{NL l_2}(r) b^{ \eta \eta}_{NL l_3}(r) \right],
\end{align*}
\]

APPENDIX A: ADIABATIC, ISOCRUVATURE
AND MIXED COMPONENTS OF CMB
ANGULAR BISPECTRUM

Explicit forms of the angular bispectra from pure adiabatic and
isocurvature mode are calculated as (Komatsu & Spergel 2001)
\[
\begin{align*}
 b^{c,\phi\phi}_{l_1 l_2 l_3} &= \frac{6}{5} f_{NL} \int r^2 dr \left[ b^{c,\phi\phi}_{l_1}(r) b^{c,\phi\phi}_{l_2}(r) b^{c,\phi\phi}_{l_3}(r) \\
 &\quad + b^{S,\eta\eta}_{l_1}(r) b^{S,\eta\eta}_{l_2}(r) b^{S,\eta\eta}_{l_3}(r) \\
 &\quad + b^{ \eta \eta}_{NL l_1}(r) b^{ \eta \eta}_{NL l_2}(r) b^{ \eta \eta}_{NL l_3}(r) \right],
\end{align*}
\]

\[
\begin{align*}
 b^{S,\eta\eta}_{l_1 l_2 l_3} &= 2 f_{NL}^{(ISO)} \int r^2 dr \left[ b^{S,\eta\eta}_{l_1}(r) b^{S,\eta\eta}_{l_2}(r) b^{S,\eta\eta}_{l_3}(r) \\
 &\quad + b^{ \eta \eta}_{NL l_1}(r) b^{ \eta \eta}_{NL l_2}(r) b^{ \eta \eta}_{NL l_3}(r) \right],
\end{align*}
\]

where
\[
\begin{align*}
 b^{c,\phi\phi}_{L_1}(r) &= \frac{2}{\pi} \int k^2 dk P_{\phi\phi}(k) g^c_{L_1}(k) j_1(k r), \\
 b^{S,\eta\eta}_{L_1}(r) &= \frac{2}{\pi} \int k^2 dk P_{\eta\eta}(k) g^S_{L_1}(k) j_1(k r), \\
 b^{ \eta \eta}_{NL}(r) &= \frac{2}{\pi} \int k^2 dk g^\eta_{NL}(k) j_1(k r), \\
 b^{S}_{NL}(r) &= \frac{2}{\pi} \int k^2 dk g^S_{NL}(k) j_1(k r).
\end{align*}
\]