EXACT ASYMPTOTIC FORMULAS FOR THE HEAT KERNELS OF SPACE AND TIME-FRACTIONAL EQUATIONS

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Abstract. This paper aims to study the asymptotic behaviour of the fundamental solutions (heat kernels) of non-local (partial and pseudo differential) equations with fractional operators in time and space. In particular, we obtain exact asymptotic formulas for the heat kernels of time-changed Brownian motions and Cauchy processes. As an application, we obtain exact asymptotic formulas for the fundamental solutions to the \( n \)-dimensional fractional heat equations in both time and space

\[
\frac{\partial^\beta}{\partial t^\beta} u(t, x) = -(-\Delta_x)^\gamma u(t, x), \quad \beta, \gamma \in (0, 1).
\]

1. Introduction

We are interested in the asymptotic behaviour at zero and at infinity of time- and space-fractional evolution equations. The simplest examples of such equations are

\[
\begin{align*}
(a) \quad \frac{\partial u}{\partial t} = -(-\Delta_x)^\beta u & \quad \text{and} \quad 
(b) \quad \frac{\partial^\beta}{\partial t^\beta} u = \Delta_x u,
\end{align*}
\]

where \( \Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} \) is the Laplace operator on \( \mathbb{R}^n \), \((-\Delta)^\beta \) is the fractional power of the Laplacian of order \( \beta \in (0, 1) \),

\[
-(-\Delta_x)^\beta u(x) = \frac{\beta \Gamma(\beta + \frac{n}{2})}{\pi^{n/2} \Gamma(1 - \beta)} \lim_{\epsilon \to 0} \int_{|y| > \epsilon} \frac{u(x + y) - u(x)}{|y|^{n+2\beta}} \, dy,
\]

and \( \frac{\partial^\beta}{\partial t^\beta} \) is the Caputo derivative of order \( \beta \in (0, 1) \), i.e.

\[
\frac{\partial^\beta f(t)}{\partial t^\beta} = \frac{1}{\Gamma(1 - \beta)} \frac{d}{dt} \int_0^t \frac{f(s) - f(0)}{(t - s)^\beta} \, ds.
\]

Our standard references for fractional derivatives in time is Samko et al. \cite{30}, for the fractional Laplacian in space we use Jacob \cite{15} and Kwasi\'niki \cite{18}. If \( \beta = 1 \), (1.1) becomes the classical heat equation whose fundamental solution is the Gauss kernel

\[
p(t, x, y) = \frac{1}{(4\pi t)^{n/2}} \exp \left( -\frac{|x - y|^2}{4t} \right), \quad t > 0, \quad x, y \in \mathbb{R}^n,
\]

which is also the transition probability density of a Brownian motion \( X = (X_t)_{t \geq 0} \) in \( \mathbb{R}^n \). Over the past years there has been considerable interest in space, time and
space-and-time fractional equations. The paper [13] by Hahn and Umarov explains how such equations arise as Fokker–Planck and Kolmogorov equations related to the solutions of SDEs, in a series of papers Luchko and co-authors [11, 19, 20] study related Cauchy problems, see also Hu et al. [14] for fractional-in-time initial value problems with a pseudo-differential operator in space; Butko [6] investigates Chernoff-type approximations of the semigroups of such equations. Time-and-space fractional Schrödinger equations are discussed by Dubbeldam et al. [9]. There are various generalizations of such problems, e.g. in the direction of fractional stochastic differential equations where a space-time noise term is added on the right-hand side, see e.g. Yan and Yin [34], or in the direction of semi-fractional derivatives (in time) and semi-stable semigroup generators (in space), see Kern et al. [17].

Both equations in (1.1) have interesting probabilistic interpretations. Denote by 
\[ S_t = (S_t)_{t \geq 0} \] a \( \beta \)-stable subordinator \( (0 < \beta < 1) \), i.e. a non-decreasing Lévy process on \( [0, \infty) \) with Laplace transform 
\[ E e^{-rS_t} = e^{-tr^\alpha}, \quad r > 0, \quad t \geq 0. \]
If \( S \) and \( X \) are stochastically independent, the time-changed process 
\[ X_S = (X_{S_t})_{t \geq 0} \]
is a rotationally symmetric \( 2\beta \)-stable Lévy process. By independence, the transition probability density of \( X_{S_t} \) is given by
\[ p^S(t, x, y) = \int_0^\infty p(s, x, y) d_s P(S_t \leq s), \tag{1.3} \]
and Bochner [5] observed that this is the fundamental solution to the space-fractional equation (1.1.a). By \( d_s \) we denote the (generalized) derivative w.r.t. \( s \). This type of time-change is usually called subordination (in the sense of Bochner) and the process \( X_S \) is said to be subordinate to \( X \), cf. [32].

If we perform a time-change with the generalized right-continuous inverse of \( S \),
\[ S_t^{-1} = \inf \{ s \geq 0 : S_s > t \} = \sup \{ s \geq 0 : S_s \leq t \}, \quad t \geq 0, \]
we get a stochastic process \( X_{S^{-1}} = (X_{S_t^{-1}})_{t \geq 0} \) which is trapped whenever \( t \mapsto S_t^{-1} \) is constant. Note that the jumps of \( t \mapsto S_t \) correspond to flat pieces of \( t \mapsto S_t^{-1} \). These traps slow down the original diffusion process \( X \), and in the physics literature \( X_{S^{-1}} \) is commonly referred to as subdiffusion, see e.g. [23, 36, 28] for some applications, [24, 21, 22] for sample path properties and [25] for a representation as scaling limit of a continuous time random walk with heavy-tailed waiting times between the steps.

Since the length of the trapping periods are, in general, not exponentially distributed, we cannot expect that \( X_{S^{-1}} \) is a Markov process. Nevertheless, the transition probability density of each random variable \( X_{S_t^{-1}}, \ t > 0 \), can be expressed as
\[ p^{S^{-1}}(t, x, y) = \int_0^\infty p(s, x, y) d_s P(S_t^{-1} \leq s), \tag{1.4} \]
and it is not hard to see, using the Fourier–Laplace transform, that \( p^{S^{-1}}(t, x, y) \) is the fundamental solution to the time-fractional heat equation (1.1.b), see e.g. [25, Theorem 5.1] or [1] [26].

Already in the simple setting (1.1), the densities \( p^S \) and \( p^{S^{-1}} \) are often not explicitly known – a notable exception is \( p^S \) for the \( \beta = \frac{1}{2} \)-stable subordinator: In this
case $X_S$ is the symmetric Cauchy process and its transition probability density is the Poisson kernel on $\mathbb{R}^n$,

\begin{equation} \label{eq:1.5}
p(t, x, y) = \frac{c(n)}{t^n} \left(1 + \frac{|x - y|^2}{t^2}\right)^{-\frac{n+1}{2}} = \frac{c(n)t}{(t^2 + |x - y|^2)^{(n+1)/2}},
\end{equation}

where $c(n) := \pi^{-(n+1)/2} \Gamma \left(\frac{n+1}{2}\right)$. Therefore, it is important to know the asymptotic behaviour of $p^S$ and $p^{S-1}$ at zero and infinity. For the fundamental solution to (1.1a) the asymptotics of $p^S$ at infinity is known to be

\begin{equation} \label{eq:1.6}
p^S(t, x, y) \sim \frac{c(n, \beta)t}{(|x - y|^2 + t^{1/\beta})^{(n+2\beta)/2}} \quad \text{as} \quad |x - y|^2 t^{-1/\beta} \to \infty,
\end{equation}

where $c(n, \beta) := \beta \pi^{1-n/2} \sin(\pi \beta) \Gamma \left(\frac{n+2\beta}{2}\right) \Gamma(\beta)$. For $d = 1$ this formula is due to Pólya [29] who used Fourier methods, and the case $d \geq 1$ can be found in Blumenthal and Getoor [4]. A beautiful short proof is due to Bendikov [2]. Our approach is similar to Bendikov’s and we show that this method also yields the asymptotics at zero. If we combine these methods, we can obtain the asymptotics of the heat kernels of the following heat equations with fractional operators in both time and space:

\begin{equation} \label{eq:1.7}
\frac{\partial^\beta u}{\partial t^\beta} = -(\Delta_x)^\gamma u, \quad \beta, \gamma \in (0, 1).
\end{equation}

2. Results

2.1. General Results. Let $\mathcal{L}$ be the infinitesimal generator of a Feller process $X = (X_t)_{t \geq 0}$ which takes values in a locally compact separable metric space $(M, \rho)$. We assume that $X$ has a transition probability density $p(t, x, y)$; note that $p$ is the fundamental solution to the Kolmogorov backward equation \( \frac{\partial}{\partial t} u \). We may replace in (1.1) the Laplace operator $\Delta$ by the generator $\mathcal{L}$. The resulting equations

\begin{equation} \label{eq:2.1}
(a) \quad \frac{\partial u}{\partial t} = -(-\Delta_x)^\beta u \quad \text{and} \quad (b) \quad \frac{\partial^\beta u}{\partial t^\beta} = \mathcal{L}_x u
\end{equation}

are the Kolmogorov backward equation (2.1a), resp., the master equation (2.1b) of the time-changed processes $X_S$ and $X_{S-1}$, respectively. As before, $S = (S_t)_{t \geq 0}$ is a $\beta$-stable subordinator, and the fundamental solutions to the problems (2.1) are still given by the formulas (1.3) and (1.4), with $p(t, x, y)$ being the probability density of $X_t$.

A deep result by Grigor’yan and Kumagai on two-sided heat kernel estimates [12, Theorem 4.1] shows that under some reasonable conditions, $p(t, x, y)$ satisfies

\begin{equation} \label{eq:2.2}
c_1 \frac{\rho(x, y)}{t^{1/\alpha}} \leq p(t, x, y) \leq C_1 \frac{\rho(x, y)}{t^{1/\alpha}}
\end{equation}

for suitable constants $n, \alpha > 0$, $c_1, c_2, C_1, C_2 > 0$, the metric $\rho(x, y)$ on $M$, and a ‘profile function’ $F$ which is either of exponential type

\begin{equation} \label{eq:2.3}
F(r) = \exp \left[-r^{\alpha/(\alpha-1)}\right] \quad \text{for some} \quad \alpha \geq 2,
\end{equation}

or of polynomial type

\begin{equation} \label{eq:2.4}
F(r) = (1 + r^2)^{-(n+\alpha)/2} \quad \text{for some} \quad \alpha > 0.
\end{equation}
In order to keep the presentation simple, we assume that \( p(t, x, y) \) is of the form
\[
(2.5) \quad p(t, x, y) = \frac{C_1}{t^{n/\alpha}} \left( C_2 \frac{\rho(x, y)}{t^{1/\alpha}} \right)^n, \quad t > 0, \ x, y \in M,
\] where \( n, \alpha, C_1, C_2 > 0 \) are constants, and \( F : [0, \infty) \to (0, \infty) \) is a non-increasing profile function of exponential or polynomial type. This assumption is, if one has in mind [16, Conjecture 1.1] and the concrete examples given there, not artificial. Note that our results hold – with estimates and explicit constants (given in terms of \( c_i \) and \( C_i \)) rather than asymptotics – if we use \( (2.2) \) instead of \( (2.5) \). We leave these obvious adaptations to the reader. Our results might also be interesting for ultrametric spaces \((X, d)\) where heat kernels are often explicitly known and of the form \( p(t, x, y) = t \int_0^{1/d(x,y)} N(x, s) e^{-t s} \, ds \), see Bendikov [3]; \( d_*(x, y) \) is an intrinsic ultrametric which defined on the basis of the underlying ultrametric \( d \).

For example, \( (1.2) \) is exponential with \( M = \mathbb{R}^n, \rho(x, y) = |x - y|, \alpha = 2, C_1 = (4\pi)^{-n/2}, C_2 = 1/2, \) and \( F(r) = e^{-r^2} \), while \( (1.3) \) is of polynomial type with \( M = \mathbb{R}^n, \rho(x, y) = |x - y|, \alpha = 1, C_1 = c(n), C_2 = 1, \) and \( F(r) = (1 + r^2)^{-(n+1)/2} \).

Recently, Chen et al. [7] have established two-sided heat kernel estimates for \( p^{S^{-1}}(t, x, y) \) where \( S \) is a (not necessarily stable) subordinator and under the assumption that the original heat kernel \( p(t, x, y) \) satisfies two-sided estimates of the form \( (2.2) \).

Our aim is to investigate the exact asymptotic behaviour of the heat kernels \( p^S(t, x, y) \) and \( p^{S^{-1}}(t, x, y) \) at zero and at infinity. The setting is as described above, and throughout this subsection we assume that \( S \) is a \( \beta \)-stable subordinator.

**Theorem 2.1** (Asymptotics for subordination). Assume that \( p(t, x, y) \) is given by \((2.5)\) and \((S_t)_{t \geq 0}\) is a \( \beta \)-stable subordinator for some \( \beta \in (0, 1) \).

a) If \( \int_1^\infty s^{n+\alpha\beta-1} F(s) \, ds < \infty \), then as \( \rho(x, y)t^{-1/(\alpha\beta)} \to \infty \),
\[
p^S(t, x, y) \sim C_1 \frac{\alpha \beta}{\Gamma(1 - \beta)} C_2^{-n-\alpha\beta} \int_0^\infty s^{n+\alpha\beta-1} F(s) \, ds \cdot \rho(x, y)^{-n-\alpha\beta} t.
\]

b) As \( \rho(x, y)t^{-1/(\alpha\beta)} \to 0 \),
\[
p^S(t, x, y) \sim C_1 F(0+) \frac{\Gamma \left( \frac{\alpha\beta}{\alpha} \right)}{\beta \Gamma \left( \frac{n}{\alpha} \right)} t^{-n/(\alpha\beta)}.
\]

**Theorem 2.2** (Asymptotics for inverse subordination). Assume that \( p(t, x, y) \) is of the form \((2.5)\) and \((S_t^{-1})_{t \geq 0}\) is an inverse \( \beta \)-stable subordinator for some \( \beta \in (0, 1) \).

a) If \( \int_1^\infty s^{n-\alpha-1} F(s) \, ds < \infty \), then as \( \rho(x, y)\alpha/\beta t \to \infty \),
\[
p^{S^{-1}}(t, x, y) \sim \begin{cases} C_1 F(0+) \frac{\Gamma \left( 1 - \frac{\alpha}{\beta} \right)}{\Gamma \left( 1 - \frac{\alpha}{\alpha} \right)} t^{-\beta n/\alpha}, & \text{if } n < \alpha, \\ C_1 \frac{\beta}{\Gamma(1 - \beta)} F(0+) t^{-\beta} \log \left[ \rho(x, y)^{\alpha/\beta} t \right], & \text{if } n = \alpha, \\ C_1 C_2^{-n-\alpha} \frac{\alpha}{\Gamma(1 - \beta)} \int_0^\infty s^{n-\alpha-1} F(s) \, ds \cdot \rho(x, y)^{\alpha-n} t^{-\beta}, & \text{if } n > \alpha. \end{cases}
\]
b) If $F$ is of polynomial type (2.4), then as $\rho(x,y)^{-\alpha/\beta}t \to 0,$

$$p^{S^{-1}}(t,x,y) \sim C_1 \frac{C^{-n-\alpha}}{\beta \Gamma(\beta)} \rho(x,y)^{-n-\alpha} t^{\beta/\alpha}.$$  

c) If $F$ is of exponential type (2.3), then as $\rho(x,y)^{-\alpha/\beta}t \to 0,$

$$p^{S^{-1}}(t,x,y) \sim K_1 \rho(x,y)^{\frac{n(1-\beta)}{\alpha-\beta}} t^{-\frac{n(1-\beta)}{\alpha(\alpha-\beta)}} \exp \left[-K_2 \rho(x,y)^{\frac{\alpha}{\alpha-\beta}} t^{-\frac{\beta}{\alpha-\beta}}\right]$$  

with the constants

$$K_1 := C_1 C_2 \frac{\alpha - 1}{(\alpha - \beta) \beta} \left(\alpha - 1\right)^{\frac{n(\alpha-1)(1-\beta)}{\alpha(\alpha-\beta)}} \beta^{\frac{n(\alpha-1)\beta}{\alpha(\alpha-\beta)}},$$

$$K_2 := C_2^{\frac{\alpha}{\alpha-\beta}} (\alpha - \beta) (\alpha - 1)^{\frac{\alpha - 1}{\alpha-\beta}} \beta^{\frac{\beta}{\alpha-\beta}}.$$  

**Remark 2.3.** We can state Theorem 2.2.c) in the following way:

$$p^{S^{-1}}(t,x,y) \sim K_1 t^{-\frac{\alpha}{\alpha-\beta}} A^{-\frac{n(1-\beta)}{\alpha-\beta}} \exp \left[-K_2 A^{\frac{n}{\alpha-\beta}}\right]$$  

as $A := \rho(x,y)t^{-\beta/\alpha} \to \infty.$

This shows that there exist constants $c_i = c_i(n, \alpha, \beta), i = 1, 2, 3, 4,$ such that

$$c_1 t^{-\frac{\alpha}{\alpha-\beta}} \exp \left[-c_2 A^{\frac{n}{\alpha-\beta}}\right] \leq p^{S^{-1}}(t,x,y) \leq c_3 t^{-\frac{\alpha}{\alpha-\beta}} \exp \left[-c_4 A^{\frac{n}{\alpha-\beta}}\right]$$  

for all $A \geq 1.$

This is in line with the two-sided estimates for $p^{S^{-1}}(t,x,y)$ derived by Chen et al. in [7, Corollary 1.5 (i)].

### 2.2. Time-Changed Brownian Motion.

**Corollary 2.4.** Assume that $p(t,x,y)$ is the Gauss kernel (1.2) and $(S_t)_{t \geq 0}$ is a $\beta$-stable subordinator for some $\beta \in (0,1).$

a) As $|x - y| t^{-1/(2\beta)} \to \infty,$

$$p^S(t,x,y) \sim \frac{\beta 4^\beta \Gamma \left(\frac{n}{2} + \beta\right)}{\pi^{n/2} \Gamma \left(1 - \beta\right)} |x - y|^{-n-2\beta} t.$$  

b) As $|x - y| t^{-1/(2\beta)} \to 0,$

$$p^S(t,x,y) \sim \frac{\Gamma \left(\frac{n+1}{2}\right) \Gamma \left(\frac{n}{2\beta}\right)}{2 \beta \pi^{n+1} / 2 \Gamma(n)} t^{-n/(2\beta)}.$$  

c) As $|x - y|^{-2\beta/3} t \to \infty,$

$$p^{S^{-1}}(t,x,y) \sim \begin{cases} \frac{1}{2 \Gamma \left(1 - \frac{n}{2}\right)} t^{-\beta/2}, & \text{if } n = 1, \\ \frac{\beta}{4 \pi \Gamma(1 - \beta)} t^{-\beta} \log \left[|x - y|^{-2/\beta} t\right], & \text{if } n = 2, \\ \frac{\Gamma \left(\frac{n}{2} - 1\right)}{4 \pi^{n/2} \Gamma(1 - \beta)} |x - y|^{-2-n} t^{-\beta}, & \text{if } n \geq 3. \end{cases}$$  

d) As $|x - y|^{-2\beta/3} t \to 0,$

$$p^{S^{-1}}(t,x,y) \sim K_1 |x - y|^{-\frac{n(1-\beta)}{2-\beta}} t^{-\frac{n\beta}{2\pi(2-\beta)}} \exp \left[-K_2 |x - y|^{\frac{\alpha}{2-\beta}} t^{-\frac{\beta}{2-\beta}}\right],$$
with the constants

\[ K_1 := \frac{1}{\sqrt{\beta(2 - \beta)}} \pi^{-\frac{n}{2}} 2^{-\frac{n}{2} \beta \frac{n}{2(n-1)}} \text{ and } K_2 := (2 - \beta) 2^{-\frac{n}{2} \beta} \pi^{\frac{n}{2}}. \]

**Proof.** Corollary 2.4 follows directly from Theorem 2.1 and 2.2, respectively. For Part b) we use Legendre’s doubling formula for the Gamma function

\[ \Gamma(z) \Gamma \left( z + \frac{1}{2} \right) = 2^{1 - 2z} \sqrt{\pi} \Gamma(2z), \quad z > 0, \tag{2.6} \]

with \( z = \frac{n}{2} \).

Note that Euler’s reflection formula for the Gamma function, \( \Gamma(\beta) \Gamma(1 - \beta) = \frac{\pi}{\sin \pi \beta} \), shows that Corollary 2.4(a) coincides with (1.6), giving an alternative proof for [4, Theorem 2.1].

### 2.3. The Time-Changed Cauchy Process

**Corollary 2.5.** Assume that \( p(t,x,y) \) is the Cauchy kernel \( (1.5) \) and \( (S_t)_{t \geq 0} \) is a \( \beta \)-stable subordinator for some \( \beta \in (0,1) \).

1. As \( |x - y| t^{-1/\beta} \to \infty \),

\[ p^S(t,x,y) \sim \frac{\beta 2^\beta - 1}{\pi^{n/2} \Gamma(1 - \frac{\beta}{2})} |x - y|^{-n-\beta} t. \]

2. As \( |x - y| t^{-1/\beta} \to 0 \),

\[ p^S(t,x,y) \sim \frac{\Gamma \left( \frac{n+1}{2} \right) \Gamma \left( \frac{n}{2} \right)}{\beta \pi^{(n+1)/2} \Gamma(n)} t^{-n/\beta}. \]

3. As \( |x - y|^{-1/\beta} t \to \infty \),

\[ p^{S^{-1}}(t,x,y) \sim \begin{cases} \frac{\beta}{\pi \Gamma(1 - \beta)} t^{-\beta} \log \left( |x - y|^{-1/\beta} t \right), & \text{if } n = 1, \\ \frac{\Gamma \left( \frac{n}{2} \right)}{2\pi^{n/2} \Gamma(1 - \beta)} |x - y|^{-n} t^{-\beta}, & \text{if } n \geq 2. \end{cases} \]

4. As \( |x - y|^{-1/\beta} t \to 0 \),

\[ p^{S^{-1}}(t,x,y) \sim \frac{\Gamma \left( \frac{n+1}{2} \right)}{\pi^{(n+1)/2} \beta \Gamma(\beta)} |x - y|^{-n-1} t^{-\beta}. \]

**Proof.** All assertions follow from the respective cases in Theorems 2.1 and 2.2. For the proof of [a] and [c] we use the well-known integral formula for Euler’s Beta function

\[ B(r,s) = \int_0^\infty z^{r-1} (1 + z)^{-r-s} \, dz = \frac{\Gamma(r) \Gamma(s)}{\Gamma(r+s)}, \quad r, s > 0. \]

Part [a] also needs (2.6) with \( z = \frac{1}{2}(1 - \beta) \).

Alternatively, we can obtain [a] and [b] from Corollary 2.4(a), [b] if we replace in these formulas \( \beta \) by \( \beta/2 \). This follows from the observation that the Cauchy kernel can be obtained from the Gaussian kernel by subordination with a \( \frac{1}{2} \)-stable subordinator. Since the composition of a \( \frac{1}{2} \)-stable and a \( \beta \)-stable subordinator has the same probability distribution as a \( \frac{\beta}{2} \)-stable subordinator, [a] and [b] are special cases of Corollary 2.4(a), [b].
2.4. Fractional Equations in Both Space and Time. We can combine the previous results which deal with space and time fractionality separately to obtain the following simultaneous space-time fractional asymptotics. As far as we are aware of, this has not yet been considered in the literature.

**Corollary 2.6.** Let \( \beta, \gamma \in (0, 1) \) and denote by \( p(t, x, y) \) the fundamental solution to (1.7).

a) As \(|x - y|^{-2\gamma} t \to \infty\),

\[
p(t, x, y) \sim \begin{cases} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(1 - \frac{1}{2\gamma})}{2\pi\Gamma(1 - \frac{\beta}{2\gamma})} t^{-\beta/(2\gamma)}, & \text{if } n = 1 \text{ and } \gamma \in \left(\frac{1}{2}, 1\right), \\ \frac{\beta}{\pi(1 - \beta)} t^{-\beta} \log \left[|x - y|^{-1/\beta} t\right], & \text{if } n = 1 \text{ and } \gamma = \frac{1}{2}, \\ 2\gamma \Gamma\left(\frac{n - 2}{2}\right) \frac{\pi^{\gamma/\beta}}{2^{1 + 2\gamma} \pi^{n/2} \Gamma(1 - \beta) \Gamma(1 + \gamma)} |x - y|^{2\gamma - n} t^{-\beta}, & \text{if } n > 2\gamma \text{ and } \gamma \in (0, 1). \end{cases}
\]

b) As \(|x - y|^{-2\gamma} t \to 0\),

\[
p(t, x, y) \sim \frac{\gamma^4 \Gamma\left(\frac{n}{2} + \gamma\right)}{\pi^{n/2} \Gamma(1 - \gamma) \beta \Gamma(\beta)} |x - y|^{-n - 2\gamma} t^\beta.
\]

The proof of Corollary 2.6 will be presented in the next section.

If we use \( \gamma = 1/2 \) in Corollary 2.6, we recover Corollary 2.5a) and b).

### 3. Proof of Theorem 2.1, Theorem 2.2 and Corollary 2.6

For the proof of our main results we need some preparations. Let \((S_t)_{t \geq 0}\) be a \( \beta \)-stable subordinator, \( \beta \in (0, 1) \). It is well known that \((S_t)_{t \geq 0}\) has a density \( p_\beta(s), s > 0 \), with respect to Lebesgue measure; moreover, \( p_\beta \) is of class \( C^\infty(0, \infty) \), bounded, unimodal (i.e., it has a unique maximum point) and it has the following asymptotics at zero and infinity, cf. [33] Theorem 4.7.1 (4.7.13) and Theorem 5.4.1,

\[
p_\beta(s) \sim \begin{cases} \frac{1}{\sqrt{2\pi(1 - \beta)}} \beta^{\frac{1}{2(1 - \beta)}} s^{-\frac{\beta}{2(1 - \beta)}} \exp\left[-(1 - \beta) \left(\beta s^{-1}\right)^{\frac{\beta}{1 - \beta}}\right] \quad \text{as } s \to 0, \\ \frac{\beta}{\Gamma(1 - \beta)} s^{\beta - 1}, \quad \text{as } s \to \infty. \end{cases}
\]

This allows us to rewrite \( p_\beta(s) \) for \( s > 0 \) in the following way

\[
(3.1) \quad p_\beta(s) = \frac{1}{\sqrt{2\pi(1 - \beta)}} \beta^{\frac{1}{2(1 - \beta)}} s^{-\frac{\beta}{2(1 - \beta)}} \exp\left[-(1 - \beta) \left(\beta s^{-1}\right)^{\frac{\beta}{1 - \beta}}\right] (1 + \phi_\beta(s)),
\]

\[
(3.2) \quad p_\beta(s) = \frac{\beta}{\Gamma(1 - \beta)} s^{\beta - 1} (1 + \psi_\beta(s)),
\]

where \( \phi_\beta, \psi_\beta : (0, \infty) \to (-1, \infty) \) are continuous functions satisfying

\[
\lim_{s \to 0} \phi_\beta(s) = 0 \quad \text{and} \quad \lim_{s \to \infty} \psi_\beta(s) = 0
\]

and

\[
\lim_{s \to \infty} s^{\frac{\beta(2\gamma - 1)}{2(1 - \beta)}} (1 + \phi_\beta(s)) = \frac{\sqrt{2\pi(1 - \beta)}}{\beta^{\frac{1 - 2\gamma}{2(1 - \beta)}}}.\]
Let us denote by $G_\beta$ the distribution function of $S_1$, i.e.
\[ G_\beta(x) = \mathbb{P}(S_1 \leq x) = \int_0^x p_\beta(s) \, ds, \quad x \geq 0. \]
Because of the scaling property of a $\beta$-stable subordinator, we have for $s, t > 0$,
\[ \mathbb{P}(S_t^{-1} \leq s) = \mathbb{P}(S_s \geq t) = \mathbb{P}(s^{1/\beta} S_1 \geq t) = 1 - \mathbb{P}(S_1 < s^{1/\beta} t) = 1 - G_\beta(s^{1/\beta} t). \]
Combining this with (1.4), we see
\[ p^{s^{-1}}(t, x, y) = -\int_0^\infty p(s, x, y) \, d_s G_\beta(s^{1/\beta} t) = \int_0^\infty p(t^{1/\beta} s, x, y) \, dG_\beta(s), \]
and, similarly,
\[ p^s(t, x, y) = \int_0^\infty p(s, x, y) \, d_s G_\beta(t^{1/\beta} s) = \int_0^\infty p(t^{1/\beta} s, x, y) \, dG_\beta(s). \]

**Proof of Theorem 2.1**. Set $A := \rho(x, y) t^{1/(\alpha\beta)}$; from (3.4) and the assumption on $p(s, x, y)$ we get
\[
 p^s(t, x, y) = C_1 t^{-n/(\alpha\beta)} \int_0^\infty s^{-n/\alpha} F\left(C_2 \frac{\rho(x, y)}{(t^{1/\beta} s)^{1/\alpha}}\right) dG_\beta(s) \\
= C_1 t^{-n/(\alpha\beta)} \int_0^\infty s^{-n/\alpha} F\left(C_2 A s^{-1/\alpha}\right) dG_\beta(s).
\]

**a)** Since $F$ is bounded, \( \int_1^\infty s^{n+\alpha\beta-1} F(s) \, ds < \infty \) implies that \( \int_0^\infty s^{n+\alpha\beta-1} F(s) \, ds < \infty \). From (3.2) and the definition of $G_\beta$ we see
\[
 p^s(t, x, y) = \frac{C_1 \beta}{\Gamma(1 - \beta)} t^{-n/(\alpha\beta)} \int_0^\infty s^{-n/\alpha - \beta - 1} F\left(C_2 A s^{-1/\alpha}\right) \left[1 + \psi_\beta(s)\right] \, ds \\
= \frac{C_1 \alpha \beta}{\Gamma(1 - \beta)} C_2^{-n - \alpha} \frac{t}{\rho(x, y)^n} \int_0^\infty s^{n+\alpha\beta-1} F(s) \left[1 + \psi_\beta \left(C_2^n A^n s^{-n}\right)\right] \, ds.
\]

Since $\psi_\beta$ is bounded on $(0, \infty)$, we may use the dominated convergence theorem to get
\[
 \lim_{A \to \infty} t^{-1} \rho(x, y)^{n+\alpha\beta} p^s(t, x, y) = \frac{C_1 \alpha \beta}{\Gamma(1 - \beta)} C_2^{-n - \alpha} \int_0^\infty s^{n+\alpha\beta-1} F(s) \, ds.
\]

**b)** Using the dominated convergence theorem once again, we see
\[
 \lim_{A \to 0} t^{n/(\alpha\beta)} p^s(t, x, y) = C_1 \lim_{A \to 0} \int_0^\infty s^{-n/\alpha} F\left(C_2 A s^{-1/\alpha}\right) dG_\beta(s) \\
= C_1 F(0+) \int_0^\infty s^{-n/\alpha} dG_\beta(s) \\
= C_1 F(0+) \mathbb{E} S_1^{-n/\alpha} \\
= C_1 F(0+) \frac{\Gamma\left(1 + \frac{n}{\alpha}\right)}{\Gamma\left(1 + \frac{\alpha}{\beta}\right)} = C_1 F(0+) \frac{\Gamma\left(\frac{n}{\alpha}\right)}{\beta \Gamma\left(\frac{\alpha}{\beta}\right)}.
\]
in the equality marked by an asterisk (*) we use Lemma 4.1 from the appendix with \( \kappa = -n/\alpha \) and \( t = 1 \). The last equality follows from the functional equation \( z\Gamma(z) = \Gamma(1+z) \) for the Gamma function.

\[
\begin{align*}
\text{□}
\end{align*}
\]

**Proof of Theorem 2.2.** Define \( A := \rho(x, y)^{-\alpha/\beta} t \). Using (3.3) we get

\[
\begin{align*}
p^{S^{-1}}(t, x, y) &= C_1 t^{-\beta n/\alpha} \int_0^\infty s^{\beta n/\alpha} F\left( C_2 \left( \frac{s}{A} \right)^{\beta/\alpha} \right) dG_\beta(s) \\
&= C_1 t^{-\beta n/\alpha} \int_0^\infty s^{\beta n/\alpha} F\left( C_2 \left( \frac{A}{s} \right)^{\beta/\alpha} \right) dG_\beta(s).
\end{align*}
\]

(a) We begin with the asymptotics for \( A \to \infty \).

Case 1: \( n < \alpha \): If we use in (3.5) the monotone convergence theorem and Lemma 4.1 from the appendix, we obtain

\[
\begin{align*}
\lim_{A \to \infty} t^{\beta n/\alpha} p^{S^{-1}}(t, x, y) &= C_1 \lim_{A \to \infty} \int_0^\infty s^{\beta n/\alpha} F\left( C_2 \left( \frac{s}{A} \right)^{\beta/\alpha} \right) dG_\beta(s) \\
&= C_1 F(0+) \int_0^\infty s^{\beta n/\alpha} dG_\beta(s) \\
&= C_1 F(0+) \Gamma \left( 1 - \frac{n}{\alpha} \right) \Gamma \left( 1 - \frac{\beta n}{\alpha} \right).
\end{align*}
\]

Case 2: \( n = \alpha \): We know that \( \int_1^\infty s^{-1} F(s) \, ds < \infty \). Inserting (3.2) into (3.5) yields

\[
\begin{align*}
\frac{t^\beta}{\log A} p^{S^{-1}}(t, x, y) &= \frac{C_1}{\log A} \int_0^\infty s^\beta F\left( C_2 \left( \frac{s}{A} \right)^{\beta/\alpha} \right) dG_\beta(s) \\
&= \frac{C_1}{\log A} \left( I_1(A, \alpha, \beta) + I_2(A, \alpha, \beta) + I_3(A, \alpha, \beta) \right)
\end{align*}
\]

where

\[
\begin{align*}
I_1(A, \alpha, \beta) &:= \int_0^1 s^\beta F\left( C_2 \left( \frac{s}{A} \right)^{\beta/\alpha} \right) p_\beta(s) \, ds \\
&\leq F(0+) \int_0^1 s^\beta p_\beta(s) \, ds \leq F(0+), \\
I_2(A, \alpha, \beta) &:= \frac{\beta}{\Gamma(1-\beta)} \int_1^\infty s^{-1} F\left( C_2 \left( \frac{s}{A} \right)^{\beta/\alpha} \right) \, ds \\
&= \frac{\alpha}{\Gamma(1-\beta)} \int_{C_2 A^{-\beta/\alpha}}^\infty s^{-1} F(s) \, ds,
\end{align*}
\]

and

\[
\begin{align*}
I_3(A, \alpha, \beta) &:= \frac{\beta}{\Gamma(1-\beta)} \int_1^\infty s^{-1} F\left( C_2 \left( \frac{s}{A} \right)^{\beta/\alpha} \right) \psi_\beta(s) \, ds \\
&= \frac{\alpha}{\Gamma(1-\beta)} \int_{C_2 A^{-\beta/\alpha}}^\infty s^{-1} F(s) \psi_\beta \left( C_2^{-\alpha/\beta} A s^{\alpha/\beta} \right) \, ds.
\end{align*}
\]
Now we can use Lemma 4.4 from the appendix to get
\[
\lim_{A \to \infty} t^\beta \log A p^{S^{-1}}(t, x, y) = C_1 \left( 0 + \frac{\alpha}{\Gamma(1 - \beta)} \cdot \frac{\beta}{\alpha} F(0+) + \frac{\alpha}{\Gamma(1 - \beta)} \cdot 0 \right) = \frac{C_1 \beta}{\Gamma(1 - \beta)} F(0+).
\]

Case 3: \( n > \alpha \): We know that \( \int_0^\infty s^{n-\alpha-1} F(s) \, ds < \infty \). Since \( F \) is bounded, this yields \( \int_0^\infty s^{n-\alpha-1} F(s) \, ds < \infty \). Using (3.5) and (3.2) we get
\[
\rho(x, y)^{n-\alpha} t^\beta p^{S^{-1}}(t, x, y) = \frac{C_1 \beta}{\Gamma(1 - \beta)} A^{\beta(1-n/\alpha)} \int_0^\infty s^{\beta n/\alpha-1-\beta} \psi_s \left( C_2 \left( \frac{s}{A} \right)^{\beta/\alpha} \right) \left[ 1 + \psi_s(s) \right] \, ds
\]
\[
= \frac{C_1 C_2^{\alpha-n/\alpha}}{\Gamma(1 - \beta)} \int_0^\infty s^{-n-1} F(s) \left[ 1 + \psi_s \left( C_2^{-\alpha/\beta} A s^{\alpha/\beta} \right) \right] \, ds.
\]
Since \( \psi_s \) is bounded on \((0, \infty)\), the dominated convergence theorem gives
\[
\lim_{A \to \infty} \int_0^\infty s^{-n-1} F(s) \psi_s \left( C_2^{-\alpha/\beta} A s^{\alpha/\beta} \right) \, ds = 0.
\]

Therefore, we have
\[
\lim_{A \to \infty} \rho(x, y)^{n-\alpha} t^\beta p^{S^{-1}}(t, x, y) = C_1 \frac{C_2^{\alpha-n/\alpha}}{\Gamma(1 - \beta)} \int_0^\infty s^{n-1} F(s) \, ds.
\]

[b] Now we consider the limit \( A \to 0 \) for the profile \( F(r) = (1 + r^2)^{-(n+\alpha)/2} \) where \( \alpha > 0 \). Applying in (3.5) the dominated convergence theorem and Lemma 4.1 from the appendix gives
\[
\lim_{A \to 0} \rho(x, y)^{n+\alpha} t^{-\beta} p^{S^{-1}}(t, x, y)
\]
\[
= C_1 \lim_{A \to 0} A^{-\frac{2\alpha}{\beta}} \int_0^\infty s^{\beta n/\alpha} \left( 1 + C_2^2 \left( \frac{s}{A} \right)^{2\beta/\alpha} \right)^{-(n+\alpha)/2} \, dG(s)
\]
\[
= C_1 \lim_{A \to 0} \int_0^\infty s^{\beta n/\alpha} \left( A^{2\beta/\alpha} + C_2^2 s^{2\beta/\alpha} \right)^{-(n+\alpha)/2} \, dG(s)
\]
\[
= C_1 C_2^{-n/\alpha} \int_0^\infty s^{-\beta} \, dG(s)
\]
\[
= C_1 C_2^{-n/\alpha} \mathbb{E} S_1^{-\beta} = C_1 C_2^{-n/\alpha} \frac{\Gamma(2)}{\Gamma(1 + \beta)} = \frac{C_1 C_2^{-n/\alpha}}{\beta \Gamma(\beta)}.
\]

[c] Finally we consider \( A \to 0 \) for the profile \( F(r) = \exp[-r^{\alpha/(\alpha-1)}] \) with \( \alpha \geq 2 \). Combining (3.5) and (3.1) yields
\[
p^{S^{-1}}(t, x, y) = C_1 t^{-\frac{\alpha}{\beta}} \int_0^\infty s^{-\frac{\alpha}{\beta}} \exp \left[ -C_2^{-\alpha/\beta} A^{-\frac{\alpha}{\beta}} s^{\frac{\alpha}{\beta}} \right] p_s(s) \, ds
\]
\[
= \frac{C_1}{\sqrt{2\pi(1 - \beta)}} t^{-\frac{\alpha}{\beta}} \int_0^\infty s^{-\frac{\alpha}{\beta}} \exp \left[ -C_2^{-\alpha/\beta} A^{-\frac{\alpha}{\beta}} s^{\frac{\alpha}{\beta-1}} - (1 - \beta) \beta^{\frac{\alpha}{\beta-1}} s^{-\frac{\alpha}{\beta-1}} \right] (1 + \phi_\beta(s)) \, ds.
\]
The claim follows with Lemma 4.6 from the appendix. \( \square \)
Proof of Corollary 2.6. In abuse of notation we denote by \( p_{2\gamma}(t, x, y) = p_{2\gamma}(t, |x - y|) \) the heat kernel of the \( n \)-dimensional rotationally symmetric \( 2\gamma \)-stable Lévy process \( X \). We know that the fundamental solution to \( (1.7) \) can be written as \( p(t, x, y) = p_{2\gamma}^{S^{-1}}(t, x, y) \), where \( S \) is a \( \beta \)-stable subordinator which is independent of \( X \). On the other hand, it follows from the scaling property that
\[
p_{2\gamma}(t, x, y) = t^{-n/(2\gamma)} p_{2\gamma}(1, t^{-1/(2\gamma)} |x - y|),
\]
which yields that \( p_{2\gamma}(t, x, y) \) is of the form \( (2.5) \) with \( M = \mathbb{R}^n \), \( \rho(x, y) = |x - y| \), \( C_1 = C_2 = 1 \), \( \alpha = 2\gamma \), and \( F(r) = p_{2\gamma}(1, r) \).

\[ a) \text{Using Corollary 2.4[b]} \] with \( t = 1 \) and \( \beta \) replaced by \( \gamma \), we find that
\[
F(0+) = p_{2\gamma}(1, 0+) = \frac{\Gamma \left( \frac{n+1}{2} \right) \Gamma \left( \frac{n}{2\gamma} \right)}{2^{\gamma(1+1/2)} \Gamma(n)}. 
\]

Moreover,
\[
E|X_1|^{-2\gamma} = \int_{\mathbb{R}^n} |x|^{-2\gamma} p_{2\gamma}(1, |x|) \, dx = \frac{2 \pi^{n/2}}{\Gamma \left( \frac{n}{2\gamma} \right)} \int_0^\infty s^{n-2\gamma-1} p_{2\gamma}(1, s) \, ds,
\]
which, together with the moment formula for stable Lévy processes in Lemma 4.2 in the appendix, implies that
\[
\int_0^\infty s^{n-2\gamma-1} F(s) \, ds = \frac{\Gamma \left( \frac{n}{2} \right)}{2\pi^{n/2}} E|X_1|^{-2\gamma} = \frac{\Gamma \left( \frac{n-2\gamma}{2} \right)}{2^{1+2\gamma} \pi^{n/2} \Gamma(1+\gamma)}.
\]

It remains to apply Theorem 2.2[a] to get the first asymptotic formula.

\[ b) \text{Applying Corollary 2.4[a]} \] with \( t = 1 \) and \( \beta \) replaced by \( \gamma \), we have
\[
p_{2\gamma}(1, r) \sim \frac{\gamma 4^{\gamma} \Gamma \left( \frac{n}{2} + \gamma \right)}{\pi^{n/2} \Gamma(1-\gamma)} r^{-n-2\gamma} \quad \text{as } r \to \infty.
\]

Let \( A := |x - y|^{-2\gamma/\beta} \). Then it holds from \( (3.5) \) and the dominated convergence theorem that
\[
\lim_{A \to 0} |x - y|^{n+2\gamma t^{-\beta}} p(t, x, y) = \lim_{A \to 0} |x - y|^{n+2\gamma t^{-\beta}} p_{2\gamma}^{S^{-1}}(t, x, y)
\]
\[
= \lim_{A \to 0} |x - y|^{n+2\gamma t^{-\beta}} \int_0^\infty s^{n/(2\gamma)} p_{2\gamma} \left( 1, \left( \frac{s}{A} \right)^{\beta/(2\gamma)} \right) \, dG_\beta(s)
\]
\[
= \lim_{A \to 0} \int_0^\infty s^{-\beta} \left( \frac{s}{A} \right)^{(n+2\gamma)\beta/(2\gamma)} p_{2\gamma} \left( 1, \left( \frac{s}{A} \right)^{\beta/(2\gamma)} \right) \, dG_\beta(s)
\]
\[
= \frac{\gamma 4^{\gamma} \Gamma \left( \frac{n}{2} + \gamma \right)}{\pi^{n/2} \Gamma(1-\gamma)} \int_0^\infty s^{-\beta} \, dG_\beta(s)
\]
\[
= \frac{\gamma 4^{\gamma} \Gamma \left( \frac{n}{2} + \gamma \right)}{\pi^{n/2} \Gamma(1-\gamma)} E S_1^{-\beta}.
\]

Combining this with Lemma 4.1 in the appendix, we obtain the second formula. \( \square \)
4. Appendix

We will need a moment formula for stable subordinators which can be found in Sato [31, Eq. (25.5), p. 162] (without proof but references to the literature). The following short and straightforward derivation seems to be new.

Lemma 4.1. The moments of order $\kappa \in (-\infty, \beta)$ of a $\beta$-stable subordinator $(S_t)_{t \geq 0}$ exist and are given by

$$E S_t^\kappa = \frac{\Gamma \left(1 - \frac{\kappa}{\beta}\right)}{\Gamma(1 - \kappa)} t^{\kappa/\beta}, \quad t > 0.$$  

Proof. Since $S_t$ has the same probability distribution as $t^{1/\beta} S_1$, it is enough to consider $t = 1$. Recall that the Laplace transform of $S_1$ is $E e^{-t S_1} = e^{-t^\beta}, \quad t > 0$. Substituting $\lambda = S_1$ in the well-known formula [32, p. vii]

$$\lambda - r = 1 \int_0^{\infty} e^{-\lambda x} x^{r-1} dx, \quad \lambda > 0, r > 0,$$

and taking expectations yields, because of Tonelli’s theorem,

$$E S_1^{-r} = \frac{1}{\Gamma(r)} \int_0^{\infty} E e^{-x S_1} x^{r-1} dx = \frac{1}{\Gamma(r)} \int_0^{\infty} e^{-x^\beta} x^r dx.$$

Now we change variables according to $y = x^\beta$, and get

$$E S_1^{-r} = \frac{1}{\Gamma(r)} \int_0^{\infty} e^{-y} y^{r-1} dy = \frac{1}{\Gamma(r)} \cdot \frac{r}{\beta} \cdot \Gamma \left(\frac{r}{\beta}\right) = \frac{\Gamma \left(1 - \frac{\kappa}{\beta}\right)}{\Gamma(1 - \kappa)}.$$

Setting $\kappa = -r$ proves the assertion for $\kappa \in (-\infty, 0)$. Note that this formula extends (analytically) to $-r = \kappa < \beta$. Alternatively, use the very same calculation and the formula [32, p. vii]

$$(4.1) \quad \lambda^r = \frac{r}{\Gamma(1 - r)} \int_0^{\infty} (1 - e^{-\lambda x}) x^{-r-1} dx, \quad \lambda > 0, r \in (0, 1),$$

to get the assertion for $\kappa \in (0, \beta)$. \hfill $\square$

The following theorem is known in the literature in dimension $n = 1$, see [31, p. 163]. The multivariate setting and the short proof via subordination are new.

Lemma 4.2. Let $(X_t)_{t \geq 0}$ be a rotationally symmetric $\alpha$-stable Lévy process on $\mathbb{R}^n$ with $0 < \alpha < 2$. For any $\kappa \in (-n, \alpha)$,

$$E |X_t|^\kappa = \frac{\Gamma \left(\frac{n+\kappa}{2}\right) \Gamma \left(1 - \frac{\kappa}{\alpha}\right)}{\Gamma \left(\frac{n}{2}\right) \Gamma \left(1 - \frac{n}{2}\right)} t^{\kappa/\alpha}, \quad t > 0.$$

If $\kappa \leq -n$ or $\kappa \geq \alpha$ the moments are infinite.

Proof. Let $(B_t)_{t \geq 0}$ be a Brownian motion on $\mathbb{R}^n$ (starting from zero) with transition probability density given by (1.2), and $(S_t)_{t \geq 0}$ be an independent $\alpha/2$-stable subordinator, that is an increasing Lévy process. From Bochner’s subordination is well known that the time-changed process $B_{S_t}, \quad t \geq 0,$ is a rotationally symmetric $\alpha$-stable Lévy process on $\mathbb{R}^n$. 

For any $\kappa > -n$ and $t > 0$, we have

\[
E|B_t|^\kappa = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} |x|^\kappa \exp \left( -\frac{|x|^2}{4t} \right) \, dx
= \frac{2^{1-n}t^{-n/2}}{\Gamma\left(\frac{n}{2}\right)} \int_0^\infty r^{n+\kappa-1} \exp \left( -\frac{r^2}{4t} \right) \, dr
= \frac{2^\kappa \Gamma\left(\frac{n+\kappa}{2}\right)}{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{1}{2}\right)} t^{\kappa/2}.
\]

Let $E^B$ and $E^S$ denote the expectations w.r.t. $(B_t)_{t \geq 0}$ and $(S_t)_{t \geq 0}$, respectively. Using Lemma 4.1, we obtain that for any $\kappa \in (-n, \alpha)$ and $t > 0$,

\[
E|B_t|^\kappa = E^S \left[ E^B|B_t|^\kappa \right] = \frac{2^\kappa \Gamma\left(\frac{n+\kappa}{2}\right)}{\Gamma\left(\frac{n}{2}\right) \Gamma(1-\frac{\alpha}{2})} t^{\kappa/\alpha}.
\]

If $\kappa \leq -n$ or $\kappa \geq \alpha$, we have $E|X_t|^\kappa = \infty$, see [8, Theorem 3.1.e) and Remark 3.2.d)].

**Remark 4.3.** We want to sketch another, slightly more general proof of Lemma 4.2 which avoids the subordination argument. Combining the well-known formulas

\[
|x|^\kappa = \frac{2^\kappa - 1}{\pi^{n/2} \Gamma\left(\frac{n+\kappa}{2}\right)} \int_{\mathbb{R}^n \setminus \{0\}} (1 - \cos(x \cdot \xi))|\xi|^{-\kappa-n} \, d\xi, \quad \kappa \in (0, 2),
\]

\[
|x|^\kappa = \frac{2^\kappa \Gamma\left(\frac{n+\kappa}{2}\right)}{\pi^{n/2} \Gamma\left(-\frac{\kappa}{2}\right)} \int_{\mathbb{R}^n \setminus \{0\}} |\xi|^{-n-\kappa} e^{-ix \cdot \xi} \, d\xi, \quad \kappa \in (-n, 0)
\]

(the second formula is to be understood in the sense of L. Schwartz distributions) with an Abel-type convergence factor argument and Fubini’s theorem, also yields the moment formula of Lemma 4.2.

**Lemma 4.4.** Let $\psi : [1, \infty) \to \mathbb{R}$ be a bounded function such that $\lim_{s \to \infty} \psi(s) = 0$ and $\omega : [0, \infty) \to (0, \infty)$ a non-increasing function satisfying $\int_1^\infty s^{1-\omega(s)} \, ds < \infty$. For any $c > 0$ and $\delta > 0$ one has

\[
\lim_{A \to \infty} \frac{1}{\log A} \int_{cA^{-\delta}}^{\infty} s^{-1} \omega(s) \, ds = \delta \omega(0+)
\]

and

\[
\lim_{A \to \infty} \frac{1}{\log A} \int_{cA^{-\delta}}^{\infty} s^{-1} \omega(s) \psi\left(c^{-1/\delta} A^{1/\delta}\right) \, ds = 0.
\]
Lemma 4.5. Assume that 

\[ r \text{ method, see e.g. de Bruijn [10, Section 4.2, pp. 63–65].} \]

and this completes the proof. \( \Box \)

Lemma 4.6. Let \( C \) and \( C \) and \( C \) are given by \( I \)

First, we prove that \( I \)

Proof. \( I \)

The following asymptotic formula for integrals can be proved by the Laplace method, see e.g. de Bruijn [10] Section 4.2, pp. 63–65] for \( r_0 = 0 \).

Lemma 4.5. Assume that \( -\infty \leq v < w \leq \infty, h \in C^2(v, w), \) and \( \int_v^w e^{-h(r)} dr < \infty. \)

Let \( r_0 \in (v, w). \) If \( h(r_0) \geq 0, h''(r_0) > 0, \) and \( h \) is strictly decreasing on \( (v, r_0) \) and strictly increasing on \( [r_0, w) \), then

\[
\int_v^w e^{-Ch(r)} dr \sim e^{-Ch(r_0)} \sqrt{\frac{2\pi}{Ch''(r_0)}} \text{ as } C \to \infty.
\]

Lemma 4.6. Let \( \phi : (0, \infty) \to (-1, \infty) \) be a continuous function such that \( \phi(0+) = 0 \) and \( \lim_{s \to \infty} s^{\theta}(1 + \phi(s)) < \infty \) for some \( \theta \in \mathbb{R}. \) For all constants \( a \in \mathbb{R} \) and \( b, c, d > 0 \) the following asymptotics holds

\[
\int_0^\infty s^a e^{-B s^b - c s^d} (1 + \phi(s)) ds \sim I(B) \text{ as } B \to \infty
\]

where the value \( I(B) \) is given by

\[
I(B) := \sqrt{\frac{2\pi}{b + d}} (bB)^{-\frac{(a+1)}{2(b+d)}} (cd)^{-\frac{b}{2(b+d)}} \exp \left[ -(b + d) \left( b^{-1} c \right)^{\frac{d}{d+1}} (d^{-1} B)^{\frac{b}{d+1}} \right].
\]

Proof. First, we prove that

\[ (4.2) \quad \int_0^\infty s^a e^{-B s^b - c s^d} ds \sim I(B) \text{ as } B \to \infty. \]

If \( a \neq -1, \) changing variables according to \( r = (c^{-1} B)^{(a+1)/(b+d)} s^{a+1} \) gives

\[
\int_0^\infty s^a e^{-B s^b - c s^d} ds = \frac{1}{|a + 1|} (cB^{-1})^{\frac{a+1}{a+1}} \int_0^\infty \exp \left[ -e^{-\frac{1}{c} B^{\frac{1}{c}}} B^{\frac{d}{d+1}} (r^{\frac{b}{d+1}} + r^{-\frac{d}{d+1}}) \right] dr.
\]

Lemma 1.5 with \( v = 0, \ w = \infty, \ h(r) = r^{b/(a+1)} + r^{-d/(a+1)}, \ r_0 = (b^{-1} d)^{(a+1)/(b+d)} \) and \( C = c^{b/(b+d)} B^{d/(b+d)} \) yields (4.2).

If \( a = -1, \) we change variables according to \( r = \log s + (b + d)^{-1} \log(c^{-1} B) \), and use Lemma 1.5 with \( v = -\infty, \ w = \infty, \ h(r) = e^{br} + e^{-dr}, \ r_0 = (b + d)^{-1} \log(b^{-1} d) \) and \( C = c^{b/(b+d)} B^{d/(b+d)} \) to obtain (4.2).
We still have to check that
\[
\lim_{B \to \infty} B^{\frac{2(a+1)+d}{2(b+\theta)}} \exp \left[ (b + d) (b^{-1} c) \frac{b}{b + d} (d^{-1} B) \frac{d}{d + b} \right] \int_0^\infty s^a e^{-B s^b - c s^d} \phi(s) \, ds = 0.
\]
To this end, we fix \( n \in \mathbb{N} \) and observe that
\[
I_1(n, B) := \int_0^{1/n} s^a e^{-B s^b - c s^d} |\phi(s)| \, ds \leq \left\| \phi \mathbb{1}_{(0,1/n)} \right\|_\infty \int_0^{1/n} s^a e^{-B s^b - c s^d} \, ds
\leq \left\| \phi \mathbb{1}_{(0,1/n)} \right\|_\infty \int_0^\infty s^a e^{-B s^b - c s^d} \, ds.
\]
Moreover, set
\[
I_2(n, B) := \int_{1/n}^\infty s^a e^{-B s^b - c s^d} |\phi(s)| \, ds.
\]
By our assumption, there exists a constant \( C(n) > 0 \) depending on \( n \) such that \( 1 + \phi(s) \leq C(n) s^\theta \) for all \( s \geq 1/n \). Thus,
\[
|\phi(s)| \leq 1 + (1 + \phi(s)) \leq (ns)^{\theta v_0} + C(n) s^\theta \leq C(n, \theta) s^{\theta v_0}, \quad s \geq 1/n,
\]
where \( C(n, \theta) := n^{\theta v_0} + C(n) n^{(-\theta) v_0} \). Using the dominated convergence theorem we deduce
\[
B^{\frac{2(a+1)+d}{2(b+\theta)}} \exp \left[ (b + d) (b^{-1} c) \frac{b}{b + d} (d^{-1} B) \frac{d}{d + b} \right] I_2(n, B)
\leq C(n, \theta) \int_{1/n}^\infty B^{\frac{2(a+1)+d}{2(b+\theta)}} s^{a+(\theta v_0)} \exp \left[ (b + d) (b^{-1} c) \frac{b}{b + d} (d^{-1} B) \frac{d}{d + b} - B s^b - c s^d \right] ds
\xrightarrow{B \to \infty} C(n, \theta) \cdot 0 = 0.
\]
Combining these calculations gives
\[
B^{\frac{2(a+1)+d}{2(b+\theta)}} \exp \left[ (b + d) (b^{-1} c) \frac{b}{b + d} (d^{-1} B) \frac{d}{d + b} \right] \cdot \int_0^\infty s^a e^{-B s^b - c s^d} \phi(s) \, ds
\leq B^{\frac{2(a+1)+d}{2(b+\theta)}} \exp \left[ (b + d) (b^{-1} c) \frac{b}{b + d} (d^{-1} B) \frac{d}{d + b} \right] \cdot (I_1(n, B) + I_2(n, B))
\leq B^{\frac{2(a+1)+d}{2(b+\theta)}} \exp \left[ (b + d) (b^{-1} c) \frac{b}{b + d} (d^{-1} B) \frac{d}{d + b} \right]
\times \left( \left\| \phi \mathbb{1}_{(0,1/n)} \right\|_\infty \int_0^\infty s^a e^{-B s^b - c s^d} \, ds + I_2(n, B) \right)
\xrightarrow{B \to \infty} \left\| \phi \mathbb{1}_{(0,1/n)} \right\|_\infty \sqrt{\frac{2\pi}{b + d}} B^{\frac{2(a+1)+d}{2(b+\theta) b}} (cd)\frac{2(a+1)+d}{2(b+\theta) b} 0 \xrightarrow{n \to \infty} 0 + 0 = 0.
\]
This completes the proof. \( \Box \)

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