Canonical analysis of the BCEA topological matter model coupled to gravitation in (2+1) dimensions

Laurent Freidel\(^1\), R B Mann\(^2,3\) and Eugeniu M Popescu\(^3\)

\(^1\) Laboratoire de Physique, École Normale Supérieure de Lyon, 46 Allée d’Italie, 69364 Lyon Cedex 07, France
\(^2\) Perimeter Institute for Theoretical Physics, Waterloo, Ontario, N2J 2O9, Canada
\(^3\) Department of Physics, University of Waterloo, Waterloo, Ontario, N2L 3G1, Canada

E-mail: lfreidel@perimeterinstitute.ca, mann@avatar.uwaterloo.ca and empopesc@uwaterloo.ca

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Abstract

We consider a topological field theory derived from the Chern–Simons action in (2+1) dimensions with the \(I(ISO(2, 1))\) group, and we investigate in detail the canonical structure of this theory. Originally developed as a topological theory of Einstein gravity minimally coupled to topological matter fields in (2+1) dimensions, it admits a BTZ black-hole solution, and can be generalized to arbitrary dimensions. In this paper, we further study the canonical structure of the theory in (2+1) dimensions, by identifying all the distinct gauge equivalence classes of solutions as they result from holonomy considerations. The equivalence classes are discussed in detail, and examples of solutions representative of each class are constructed or identified.

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1. Introduction

Because of the difficulties of quantizing gravity in (3+1) dimensions, general relativity in (2+1)-dimensional spacetimes emerged as a lower-dimensional alternative, whose purpose was to help in understanding at least part of the issues involved in the development of a quantum theory of gravitation.

Much simpler than its (3+1)-dimensional counterpart since it has no propagating modes, general relativity in (2+1) dimensions in the absence of matter was shown by Witten [1] to be exactly solvable. If matter is added via the coupling of gravity to pointlike particles, solvability...
is preserved (see [2] for a recent review on the quantization of this system). Unfortunately, when matter is added in the traditional way, by coupling gravity to a field theory, solvability is, even for this simple theory, generally destroyed.

It would appear from the above considerations that the addition of matter automatically destroys the solvability of pure general relativity, and while this statement is indeed true in many instances and for various dimensions of spacetimes, in (2+1) dimensions there is a notable exception. In order to understand this aspect, which is characteristic to (2+1)-dimensional general relativity, it is necessary to make a short digression into how matter is coupled to pure general relativity.

In the Lagrangian formalism, the Hilbert–Einstein action for pure gravity is a functional of a single variable [3], the spacetime metric, and it is through the spacetime metric that matter is usually coupled to the theory. In doing so, the space of states of the theory becomes infinite dimensional, and the solvability of the theory is destroyed. In the Palatini formalism, however, the action functional for pure gravity is considered to be a functional not of a single variable, as in the Hilbert–Einstein case, but of two variables, the spacetime metric and the components of the connection (alternatively, the spacetime metric and the spacetime derivatives of the metric), which are now considered to be independent variables. Correspondingly, in the Palatini formalism there are two possibilities for coupling matter to pure gravity: through the metric and through the connection. Of course, coupling matter through the metric is no different from coupling matter in the Hilbert–Einstein formalism, leading to the same solvability issues, just in a different framework.

However, coupling of matter through the connection yields significantly different results. In this latter context, there is a distinct class of topological field theories where the matter fields introduce only a finite number of degrees of freedom, such that the total space of states remains finite dimensional, rendering these theories solvable both classically and quantum mechanically. It is this particular class of topological field theories that constitutes the notable exception to the (in)solvability versus matter issue mentioned earlier.

In the present paper, we consider one particular such topological matter model which was originally developed by Carlip and Gegenberg [4]. The model (subsequently referred to as the BCEA model) consists of a pair of 1-form matter fields $B, C$ that are minimally coupled to the first-order action of pure gravity through the connection 1-form fields via the covariant derivative (see the following section) and, as mentioned earlier, it is exactly solvable both classically and quantum mechanically. What makes this model interesting is the fact that it is non-trivial for non-trivial topologies of the spacetime foliation. In particular, if the spacelike leaves of the foliation have the topology of a plane with one puncture, it was shown [8] that the model admits a solution that is analogous to the BTZ black hole, and this fact suggests the possibility that at least for this particular case, the model could have a much richer structure. Based on this observation, we have decided, as part of a larger ongoing project, to investigate in detail the general classical structure of the BCEA model in the case where the topology of the spacelike foliation is that of a plane with one puncture, in order to determine all the distinct gauge equivalence classes of solution.

The paper is organized as follows. In section 2, we present a short summary of the important features of the model relevant for our discussion. In section 3, based on holonomy considerations, we perform the canonical analysis of the structure of the model for the case where the leaves of the foliation have the topology of the punctured plane (generalizing the results previously obtained for pure gravity in [9]), and we identify all the distinct sectors of the theory. In section 4, we illustrate the different sectors of the theory by constructing or identifying solutions that are of physical relevance and in section 5 we conclude with some remarks and considerations regarding future work.
2. The BCEA theory

In this section, we present a short review of the BCEA model, insisting on the aspects that are pertinent to the present study. For more details on the theory, the reader is referred to the original work of Carlip and Gegenberg in [4].

The action of the BCEA model in the first-order formalism has the expression:

\[
S[B, C, E, A] = \int_M (E_i \wedge R^i[A] + B_i \wedge D C^i)
\]

where \( M \) is a three-dimensional non-compact spacetime with the topology \( M = R \times \Sigma \) and \( \Sigma \) is a two-dimensional spacelike surface with the topology of a plane with one puncture. The fields \( E_i \) in (1) are \( SO(2, 1) \) 1-forms which, if invertible, correspond to the triads of the spacetime metric and \( R^i[A] \) are the curvature 2-forms associated with the \( SO(2, 1) \) connection 1-forms \( A^i \), with the expression:

\[
R^i[A] = d A^i + \frac{1}{2} \epsilon^{ijk} A_j \wedge A^k.
\]

The \( SO(2, 1) \) 1-forms \( B^i, C^i \) are the topological matter fields that are coupled to the fields \( E^i, A^i \) of pure gravity and \( D C^i \) is the covariant derivative of the field \( C^i \) having the expression:

\[
D C^i = d C^i + \epsilon^{ijk} A_j \wedge C_k.
\]

Throughout the entire paper we adopt the following index convention. Greek indices, taking the values 0, 1, 2, designate the spacetime components of tensors, and are raised and lowered by the spacetime metric \( g_{\alpha \beta} \). Latin lower case indices, also taking the values 0, 1, 2, are \( SO(2, 1) \) indices, \( \eta_{ij} = \text{diag}(-1, 1, 1) \), and \( \epsilon^{ijk} \) is the totally antisymmetric \( SO(2, 1) \) symbol with \( \epsilon^{012} = 1 \).

The action (1) yields, upon first-order variation (and up to surface terms), the equations of motion:

\[
\begin{align*}
R^i[A] &= 0 \\
D E^i + \epsilon^{ijk} B_j \wedge C_k &= 0 \\
D B^i &= D C^i = 0
\end{align*}
\]

and is invariant under the following 12-parameter infinitesimal gauge transformations:

\[
\begin{align*}
\delta A^i &= D \tau^i \\
\delta B^i &= D \rho^i + \epsilon^{ijk} B_j \tau_k \\
\delta C^i &= D \lambda^i + \epsilon^{ijk} C_j \tau_k \\
\delta E^i &= D \beta^i + \epsilon^{ijk} (E_j \tau_k + B_j \lambda_k + C_j \rho_k)
\end{align*}
\]

where \( \beta^i, \lambda^i, \rho^i, \tau^i \) are 0-form gauge parameters.

The \((2+1)\) canonical splitting induced by the topology of the manifold \( M \) yields four sets of constraints \( J^i, P^i, Q^i, R^i \), which are enforced by the zeroth spacetime components of the form fields \( A^i, E^i, B^i \) and \( C^i \) respectively, acting as Lagrange multipliers. The Lie algebra generated by these constraints is:

\[
\begin{align*}
\{ J^i, J^j \} &= \epsilon^{ijk} J_k \\
\{ J^i, P^j \} &= \epsilon^{ijk} P_k \\
\{ J^i, Q^j \} &= \epsilon^{ijk} Q_k \\
\{ J^i, R^j \} &= \epsilon^{ijk} R_k \\
\{ Q^i, R^j \} &= \epsilon^{ijk} P_k
\end{align*}
\]
with the rest of the Poisson brackets being zero. The algebra (6) is the inhomogenization of the Poincaré algebra determined by \{ J^i, Q^j \} with the Abelian generators \{ P^i, R^i \}, and as such it can be recognized as the Lie algebra of the inhomogenized Poincaré group \( I(ISO(2, 1)) \) [5]. The Hamiltonian of the system is zero on shell, since it depends only on the constraints, and consequently the constraints are preserved in time.

On the Lie algebra of \( I(ISO(2, 1)) \), we can introduce an invariant scalar product\(^4\) for the generators defined as:

\[
\text{tr}(J^i P^j) = \text{tr}(Q^i R^j) = \eta_{ij}
\]

with all the other pairings being zero, and a generalized connection \( \mathcal{A} \) having the expression:

\[
\mathcal{A} = A_i J^i + E_i P^i + B_i Q^i + C_i R^i.
\]

With these definitions, it is straightforward to show that the action (1) of the BCEA model can be written, up to surface terms, as a Chern–Simons theory with the connection \( \mathcal{A} \) and the invariant scalar product (7):

\[
S[\mathcal{B}, \mathcal{C}, \mathcal{E}, \mathcal{A}] = \frac{1}{2} \int_M \text{tr}\left( \mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right).
\]

Moreover, and in order to lay the background for the quantization of this model, by introducing the form fields:

\[
\tilde{\mathcal{A}} = A_i J^i + C_i R^i, \quad \tilde{\mathcal{E}} = E_i P^i + B_i Q^i,
\]

the BCEA model can be written as a BF theory associated with the Poincaré group:

\[
S[\mathcal{B}, \mathcal{C}, \mathcal{E}, \mathcal{A}] = \int_M \text{tr}(\tilde{\mathcal{E}} \wedge F[\tilde{\mathcal{A}}])
\]

where \( F[\tilde{\mathcal{A}}] \) is the curvature of the Poincaré connection \( \tilde{\mathcal{A}} \) and has the expression:

\[
F[\tilde{\mathcal{A}}] = d\tilde{\mathcal{A}} + \frac{1}{2}[\tilde{\mathcal{A}}, \tilde{\mathcal{A}}] = R_i[A] J^i + (DC_i) R^i
\]

with \( R^i[A] \) and \( DC_i \) given by (2) and (3), respectively.

### 3. The canonical structure of the BCEA theory

It was shown in the previous section that the BCEA model can be written as a Chern–Simons theory with the \( I(ISO(2, 1)) \) generalized connection \( \mathcal{A} \) and, therefore, its fundamental degrees of freedom are given by the holonomies of this connection along non-contractible loops in the spacetime \( M \), modulo gauge transformations. The spacetime manifold under consideration has the topology \( M = \mathbb{R} \times \Sigma \), and we restrict ourselves to the case where the leaves \( \Sigma \) of the foliation are two-dimensional spacelike surfaces with the topology of a plane with one puncture.

In this case, there is only one class of non-contractible loops. We can consider these to be loops at fixed time surrounding the puncture without any loss of generality. Consequently, the holonomy of the connection \( \mathcal{A} \) along a loop \( \gamma \) representative of this class is given by the expression:

\[
W_{\mathcal{A}}(\gamma) = \mathcal{P} \exp\left( \oint_\gamma \mathcal{A} \right)
\]

where \( \mathcal{P} \) in the rhs of (13) stands for the usual path ordering.

\(^4\) The most general scalar product that one can introduce can also contain terms of the form \( \tilde{\text{tr}}(J_i Q_j) = \alpha \eta_{ij}, \tilde{\text{tr}}(J_i R_j) = \gamma \eta_{ij} \). One can however redefine the algebra generators such that the algebra is left invariant by this redefinition, and recover the scalar product (7).
Under these circumstances, it is straightforward to show that the explicit group structure of $I(\text{ISO}(2, 1))$ is determined exclusively by the holonomy (13). The generalized connection can be written as $A = A + B + C + E$ with

$$A = A_j^i, \quad B = B_i Q^i, \quad C = C_i R^i, \quad E = E_i P^i,$$

and a direct computation (see appendix A for details) yields for the holonomy of $A$, or equivalently for the general group element of $I(\text{ISO}(2, 1))$, the expression:

$$W_A(\gamma) \equiv G(g, \vec{a}, \vec{b}, \vec{c}) = [\exp(a_i P^i + b_i Q^i + c_i R^i)]g.$$  \tag{15}

The group parameters in (15) are related to the fields of the theory through the relations:

$$a_i = \int_0^1 \tilde{E}_i(s) + \frac{1}{2} \epsilon_{ijk} \int_0^1 \left[ \tilde{B}^j(s), \tilde{C}^k(u) \right] \epsilon(u - s)$$

$$b_i = \int_0^1 \tilde{B}_i(s)$$

$$c_i = \int_0^1 \tilde{C}_i(s)$$

$$g = \mathcal{P} \exp \left[ \int_0^1 A(s) \right]$$ \tag{16}

where in (16) the integrals are taken along a closed loop $\gamma$ with the usual parametrization and $\gamma(0) = \gamma(1)$, and we have used the shorthand $f(s) = \int f_s(\gamma(s))(\gamma'(s))^s(s)$ for the argument of the integrals. The barred fields are defined as $\tilde{E}_i(s) = \tilde{g}^{-1}(s)E_i(s)\tilde{g}(s)$ and similar for the $\tilde{B}$ and $\tilde{C}$, with $\tilde{g}(s) = g^{-1}(s)g$ where $g(s)$ is the Wilson line of the spin connection $A$ along a portion $[0, s]$ of the curve $\gamma$ and has the expression $g(s) = \mathcal{P} \exp \left[ \int_0^s A(u) \right]$.

Once the explicit form of the group elements is known, one can immediately calculate the product of two group elements, the inverse of a group element and the conjugate of a group element by another group element using the Baker–Campbell–Haussdorff formula. If $G_1 = G(g_1, \vec{a}_1, \vec{b}_1, \vec{c}_1)$ and $G_2 = G(g_2, \vec{a}_2, \vec{b}_2, \vec{c}_2)$ are two arbitrary group elements, we find that their product has the expression:

$$G_1 G_2 = G(g', \vec{a}', \vec{b}', \vec{c}')$$

$$g' = g_1 g_2$$

$$\vec{a}' = \vec{a}_1 + (g_1 \cdot \vec{a}_2) + \frac{1}{2} \tilde{b}_1 \times (g_1 \cdot \vec{c}_2) + \frac{1}{2} \tilde{c}_1 \times (g_1 \cdot \vec{b}_2)$$

$$\vec{b}' = \vec{b}_1 + (g_1 \cdot \vec{b}_2)$$

$$\vec{c}' = \vec{c}_1 + (g_1 \cdot \vec{c}_2)$$ \tag{17}

where $(g \cdot \vec{a}) = g^i a^i$ with $g^i$ elements of the matrix of $g$ in the vector representation and $\vec{a} \times \vec{b} = e^{ij} a_j b_i$. By setting $G_1 G_2$ equal to the identity element of the group in (17), it follows immediately that the inverse of an arbitrary group element $G(g, \vec{a}, \vec{b}, \vec{c})$ is given by the expression

$$G^{-1}(g, \vec{a}, \vec{b}, \vec{c}) = G(g^{-1}, -g^{-1} \cdot \vec{a}, -g^{-1} \cdot \vec{b}, -g^{-1} \cdot \vec{c}).$$ \tag{18}

Finally, the conjugate of a group element $G(g, \vec{a}, \vec{b}, \vec{c})$ by another arbitrary group element $K(k, \vec{a}', \vec{b}', \vec{c}')$ is given by

$$K G K^{-1} = (kgk^{-1}, \vec{A}, \vec{B}, \vec{C})$$

$$\vec{A} = (k \cdot \vec{a}) + (1 - kgk^{-1}) \cdot \vec{a} + \frac{1}{2} [(1 + kg^{-1}) \cdot \vec{b}] \times (k \cdot \vec{c})$$

$$+ [(1 + kg^{-1}) \cdot \vec{y}] \times (k \cdot \vec{b}) + (kgk^{-1} \cdot \vec{b}) \times \vec{y} - \vec{b} \times (kgk^{-1} \cdot \vec{y})$$
\[ B = (k \cdot \vec{b}) + (1 - kgk^{-1}) \cdot \vec{b} \]
\[ C = (k \cdot \vec{c}) + (1 - kgk^{-1}) \cdot \vec{c}. \]

Before proceeding with the explicit analysis of the canonical structure of the BCEA
theory, it is necessary to make a few remarks on the methods that will be used for this purpose.
As mentioned earlier in section 2, the fundamental degrees of freedom of the theory are given
by the holonomies of the generalized connection \( A \) along non-contractible loops around the
puncture. Since there is only one class of such loops, the physical configuration space will be
labelled by the conjugacy classes of the \( I(\text{ISO}(2, 1)) \) group which are now to be determined.

Although it is possible in principle to analyse the conjugacy classes of a group directly
on the group, for the general case the analysis is long and tedious. For Lie groups, however,
the analysis can be significantly simplified by restricting to the connected component of the
group and reducing the determination of the conjugacy classes to the determination of the
orbits of the action of the group in some vector representation (usually the adjoint or coadjoint
representation) on its Lie algebra. If this latter orbit approach is used for the analysis of
the canonical structure of the BCEA theory, the physical configuration space will be labelled
correspondingly by the orbits of the group action on its algebra. In our particular case, as
can be seen from (19), it is convenient to use a slightly different version of the orbit approach
described above. Instead of working with a vector representation of the full group acting
on its Lie algebra, we work on the group with the connected component of \( \text{SO}(2, 1) \), whose
conjugacy classes are well known, and use only its vector representation to act on the remaining
ideal of the algebra of \( I(\text{ISO}(2, 1)) \).

Deciding to use an orbit approach for labelling the physical configuration space raises
a very important issue that must be considered, namely the classification of orbits. For any
Lie group acting on some manifold, there is a well-known result of the theory of invariants
that states that there exists a certain number of algebraically independent invariant functions
defined globally on the group that take constant values on the corresponding group orbits
(and hence are independent of the gauge parameters). The number of such invariant functions
depends on both the dimensions of the group and the manifold, and is finite if these dimensions
are finite. Once again, although in principle it is possible to determine such globally invariant
functions on the group, in practice it is easier to work with the well-known Casimir invariants,
which are related to the invariant functions on the group as follows. The Casimir invariants
are elements of the enveloping algebra commuting with all the Lie algebra elements. Now,
since any element of the Lie algebra can be viewed as a linear function on the Lie algebra
if one uses an invariant scalar product, a Casimir invariant can be identified with a function
on the Lie algebra that is invariant under the adjoint action, called a Casimir function. For
instance if \( X = s^i J^i + a^i P^i + b^i Q^i + c^i R^i \) is a Lie algebra element, by using the invariant scalar
product (7), the Lie algebra elements \( P^i \) can be defined as the functions \( P_i(X) = s^i \). The
Casimir functions are therefore the germs of the invariant functions on the group and, hence,
orbit invariants.

Based on the above considerations, it is very natural to attempt to classify the orbits of the
group action by means of the Casimir invariants or Casimir functions, respectively. And in the
case of compact Lie groups, such a classification is indeed possible since the Casimirs are in a
one-to-one correspondence with the orbits of the action of the group on its Lie algebra. In the
case of non-compact groups, however, like \( I(\text{ISO}(2, 1)) \), things become more complicated.
In this case, the relation between Casimirs and orbits is not one-to-one anymore, and while
the Casimirs (if any) can still label some of the orbits, there are orbits, especially those of
lower dimension, that have no corresponding Casimir (see, for example, [6]). Nevertheless, in
particular cases, and \( I(\text{ISO}(2, 1)) \) is one such case, the orbits of the group action can still be
classified by means of invariants as follows. For orbits corresponding to Casimir invariants of the Lie algebra of the group, they are parametrized as usual by these invariants. For the orbits that do not correspond to Casimir invariants, one can still find either Casimir-like invariants for these orbits—invariants that do not commute with the Lie algebra in the general case, but commute with the Lie algebra on the orbit—and/or scalar invariants, as the case may be (see below). In the following, we adopt this latter method and classify the orbits of the action of \( I(\text{ISO}(2, 1)) \) on its Lie algebra using for this purpose both Casimir invariants and scalar invariants as necessity dictates.

The independent Casimir invariants of the Lie algebra of \( I(\text{ISO}(2, 1)) \) are relatively easy to determine. Following [7], it is straightforward to show that for the adjoint action of \( I(\text{ISO}(2, 1)) \) on its Lie algebra there are only four algebraically independent Casimir invariants given by the expressions:

\[
C_1 = J_\ell P^\ell + Q^\ell R^\ell, \quad C_2 = P_i P^i, \quad C_3 = P_i Q^i, \quad C_4 = P_i R^i
\]  

(20)

where \( X = s^\ell J_\ell + a^i P_i + b^i Q_i + c^i R_i \) is a general Lie algebra element. As a direct consequence of the above number of invariants, the maximal dimension of the corresponding orbits is 8.

We can now resume the analysis of the BCEA theory and proceed to the explicit determination of the adjoint orbits of the \( I(\text{ISO}(2, 1)) \) group. By inspection of (19), the orbits can at once be separated into two major classes, depending on whether the group element \( g \) is the identity element of \( SO(2, 1) \) or not. We investigate each of these cases separately.

(a) The case \( g \neq 1 \)

From the very beginning, this case can be separated into three distinct subcases, depending on whether \( g \) is a rotation, a boost or a null transformation. The corresponding orbits are the most general orbits of the group action, having maximal dimensionality. It should be noted at this time that the sectors of the theory corresponding to these orbits are physically rather trivial in the sense that for such orbits one can always find a gauge in which the dynamics of the \( B, C \) fields of the BCEA theory decouple from the dynamics of Einsteinian gravity.

(a1) If \( g \) is a rotation, we can choose \( k \) such that \( kgk^{-1} = \exp(s J_0) \), and the vector gauge parameters \( \vec{a}, \vec{b}, \vec{c} \) can be chosen such that the vectors \( \vec{A} = a e_0 \), \( \vec{B} = b e_0 \), \( \vec{C} = c e_0 \) are all timelike and parallel to the axis of rotation of \( kgk^{-1} \). The gauge orbit in this case is labelled by four real numbers \((s, a, b, c)\), corresponding to the four Casimir invariants:

\[
P_i P^i = -s^2, \quad P_i J^i + Q_i R^i = -(sa + bc), \quad P_i Q^i = -sc, \quad P_i R^i = -sb.
\]  

(21)

(a2) If \( g \) is a boost (we only consider the connected component of the boost subgroup), i.e. if we can choose \( k \) such that \( kgk^{-1} = \exp(s J_1) \), the vector gauge parameters \( \vec{a}, \vec{b}, \vec{c} \) can be chosen such that the vectors \( \vec{A}, \vec{B}, \vec{C} \) are pure spacelike vectors. The gauge orbit in this case is labelled by four real numbers \((s, a, b, c)\), corresponding to the four Casimir invariants:

\[
P_i P^i = s^2, \quad P_i J^i + Q_i R^i = as + bc, \quad P_i Q^i = sc, \quad P_i R^i = sb.
\]  

(22)

(a3) \( g \) is a null transformation. This case is more complicated, since in this case, the gauge orbit is degenerate relative to the values of the Casimir invariants. When \( g \) is a null transformation (again, we only consider the connected component of the null subgroup), then we can choose \( k \) such that \( kgk^{-1} = \exp\left\{\frac{1}{2} (J_0 + J_1)\right\} \), the vector gauge parameters \( \vec{a}, \vec{b}, \vec{c} \) can be chosen such that each of the vectors \( \vec{A}, \vec{B}, \vec{C} \) can be null and proportional to \((1, 0, -1)\). The gauge orbit in this case is labelled by three real numbers \((a, b, c)\),
and discrete parameters labelling the time orientability of the vectors. But now the values of the independent Casimir invariants are given by all the possible combinations of the numbers:

\[ P_i P^i = 0, \quad J_i P^i + Q_i R^i = \pm a^2, \quad Q_i P^i = \pm b^2 R_i, \quad P^i = \pm c^2. \quad (23) \]

(b) The case \( g = 1 \)

In this case, the condition \( g = 1 \) implies that \( P_i = 0 \) and consequently the highest dimension of the corresponding orbits is only 6. The relevant equations in (19) reduce to:

\[
\begin{align*}
\vec{A} &= (k \cdot \vec{a}) + \vec{\beta} \times (k \cdot \vec{c}) + \vec{\gamma} \times (k \cdot \vec{b}) \\
\vec{B} &= (k \cdot \vec{b}) \\
\vec{C} &= (k \cdot \vec{c})
\end{align*}
\]

(24)

and from the particular form of the system (24) it follows that there are two distinct cases.

(b1) If \( \vec{B} \times \vec{C} \neq 0 \), the vector gauge parameters \( \vec{\beta}, \vec{\gamma} \) in (24) can always be chosen such that \( \vec{A} = 0 \). Using the remaining \( SO(2, 1) \) gauge freedom, one finds that the orbits can be parametrized by three real numbers \( m, n, p \) that correspond to the scalar invariants:

\[ b_i b^i, \quad b_i c^i, \quad c_i c^i. \quad (25) \]

The value of these invariants depends upon whether the vectors \( \vec{b}, \vec{c} \) are spacelike, timelike or null. Furthermore, the orbit also depends on a discrete parameter specifying whether \( \vec{b}, \vec{c} \) are future or past directed when they are timelike or null. The invariants (25) do not correspond to any of the Casimir invariants (20), but as mentioned earlier, for these orbits one can construct the Casimir-like quantities:

\[ Q^i Q_i, \quad Q^i R_i, \quad R^i R_i, \quad (26) \]

which commute with Lie algebra elements if the constraints \( P_i = 0 \) are implemented.

It is important to emphasize that this is the physically most interesting case, since for this configuration we cannot find a gauge where the \( B, C \) fields of the BCEA theory can be decoupled from gravity, i.e. a gauge where \( B \wedge C = 0 \).

(b2) If \( \vec{B} \times \vec{C} = 0 \), the vectors \( \vec{B}, \vec{C} \) are parallel, and in this case it is not possible anymore to choose the vector gauge parameter \( \vec{\beta}, \vec{\gamma} \) to cancel out the vector \( \vec{A} \). However, they can be chosen such that they cancel out the component of \( \vec{A} \) that is perpendicular to \( \vec{B}, \vec{C} \), and consequently, by fixing \( \vec{\beta}, \vec{\gamma} \), we can choose without any loss of generality the vector \( \vec{A} \) to be parallel to the vectors \( \vec{B}, \vec{C} \). Taking advantage once again of the remaining \( SO(2, 1) \) gauge freedom, the resulting orbits are parametrized by three real numbers \( m, n, p \) that correspond to the invariant scalars:

\[ a_i b^i, \quad b_i b^i, \quad c_i c^i. \quad (27) \]

Similar to the previous case, the expression of the independent orbit invariants will depend upon whether the vectors \( \vec{a}, \vec{b}, \vec{c} \) are timelike, spacelike or null. Once again, in the timelike and null cases, the orbit will also depend on a discrete parameter labelling the time orientability of such vectors.

4. Examples of solutions

Once the distinct gauge orbits have been determined, the next logical step is to determine or identify solutions of the BCEA theory that will allow us to construct spacetime metrics corresponding to such orbits.
4.1. The point particle solution

As it can be seen from the previous analysis, the most general gauge orbit is eight dimensional and is characterized by four non-zero Casimir invariants. We restrict ourselves to the Casimir invariants (21), i.e. to the case (a.1), and we proceed to construct a solution for the BCEA theory corresponding to this case.

A set of \((B, C, E, A)\) fields compatible with the invariants (21) is given by:

\[
\begin{align*}
E^0 &= a \, d\phi; & E^1 &= 0; & E^2 &= 0 \\
A^0 &= s \, d\phi; & A^1 &= 0; & A^2 &= 0 \\
B^0 &= b \, d\phi; & B^1 &= 0; & B^2 &= 0 \\
C^0 &= c \, d\phi; & C^1 &= 0; & C^2 &= 0
\end{align*}
\]

and these fields are an obvious solution of the BCEA model since they identically satisfy the equations of motion (5). In this form, however, the matrix of the triad form fields is singular, and therefore one cannot directly construct a spacetime metric using the co-triads in (28).

In order to overcome this difficulty, we will use a slight variation of the method described in [10]. By setting \(s = M, a = J\) in (28), the resulting triad and connection form fields:

\[
\begin{align*}
E^0 &= J \, d\phi; & E^1 &= 0; & E^2 &= 0 \\
A^0 &= M \, d\phi; & A^1 &= 0; & A^2 &= 0
\end{align*}
\]

are identical to the triad and connection form fields (3.14) in [10] for a point particle of mass \(M\) and spin \(J\) (which, by an abuse of language, will subsequently be called a flat point particle) in pure Einsteinian gravity [10, 11]. Consequently, the BCEA model admits a flat point-particle solution given by (28) with (29), and this solution corresponds to the most general gauge orbit under consideration.

Since invertibility of the triad form fields is a gauge-dependent property, constructing a solution with invertible triad form fields from (28), (29) is a matter of straightforward calculation. As mentioned earlier, we will use for this purpose an approach similar to that used in [10], the only difference being that instead of using a representation of the \(I(ISO(2, 1))\) Lie algebra generators, we use the gauge transformations (6).

A simple gauge transformation that yields invertible triad form fields is given by:

\[
\beta^0 = t; \quad \beta^1 = \frac{r \cos \phi}{1 - M}; \quad \beta^2 = \frac{r \sin \phi}{1 - M}
\]

with all remaining gauge parameters zero. The resulting gauge-transformed triad fields have the expression:

\[
\begin{align*}
E^0 &= dt + J \, d\phi \\
E^1 &= \frac{1}{1 - M} \cos \phi \, dr - r \sin \phi \, d\phi \\
E^2 &= \frac{1}{1 - M} \sin \phi \, dr + r \cos \phi \, d\phi
\end{align*}
\]

while the rest of the form fields in (28), (29) remain unaffected by this gauge transformation. It is straightforward to show that these gauge-transformed fields are a solution of the BCEA theory. Finally, the triad form fields (32) yield the familiar flat point-particle metric [10]:

\[
dx^2 = -(dt + J \, d\phi)^2 + \frac{dr^2}{(1 - M)^2} + r^2 \, d\phi^2.
\]

The fact that we can recover a flat space solution should not come as a surprise since our analysis shows that in the case where \(A\) is not trivial, i.e. for \(g \neq 1\), we can always chose a gauge where \(\vec{B} \times \vec{C} = 0\). In this gauge, the dynamics of the \(B, C\) fields decouples from the dynamics of 2+1 geometry as it can be seen from (5).
4.2. The BTZ black hole

The geometry which realizes the orbit \((b,l)\) is the BTZ black hole \([12,13]\). The BTZ solution for the BCEA theory with \(A^0 = A^1 = A^2 = 0\) is given by the fields \([8,14]\):

\[
E^0 = 2\sqrt{v^2(r) - 1} \left(\frac{r_+}{l} \, dt - r_- \, d\phi\right)
\]

\[
E^1 = \frac{2\nu}{v(r)} \, d\left[\sqrt{v^2(r) - 1}\right]
\]

\[
E^2 = 2\nu(r) \left(-\frac{r_-}{l} \, dt + r_+ \, d\phi\right)
\]

\[
B^0 = \frac{r_-}{l} \, dt - r_+ \, d\phi
\]

\[
B^1 = -l \, d\left[\nu(r) + \sqrt{v^2(r) - 1}\right]
\]

\[
B^2 = \frac{r_+}{l} \, dt - r_- \, d\phi
\]

\[
C^0 = -\frac{r_-}{l^2} \, dt + \frac{r_+}{l} \, d\phi
\]

\[
C^1 = d\left[\nu(r) - \sqrt{v^2(r) - 1}\right]
\]

\[
C^2 = \frac{r_+}{l^2} \, dt - \frac{r_-}{l} \, d\phi
\]

where

\[
r_+^2 = \frac{Ml^2}{2} \left[1 + \sqrt{1 - (J/Ml)^2}\right]
\]

\[
r_-^2 = \frac{Ml^2}{2} \left[1 - \sqrt{1 - (J/Ml)^2}\right]
\]

are the outer and, respectively, inner horizon radii, satisfying \(r_+ r_- = Jl/2\) and the function \(v(r)\) is given by the expression:

\[
v^2(r) = \frac{r_+^2 - r_-^2}{r_+^2 - r_-^2}.
\]

The parameters \(M\) and \(J\) in (35) are the quasilocal mass and angular momentum of the black hole, \(l\) is related to the cosmological constant through the relation:

\[
\Lambda = -\frac{1}{l^2} < 0.
\]

In order to calculate the holonomy of the generalized connection \(A\), we choose as a loop a circle of radius \(r_+\) at constant time. In this case, the calculation of the holonomy for the BTZ fields (33) and (34) is straightforward, and we obtain\(^5\):

\[
W_{BTZ}[A] = \exp \left[ 2r_+ P^2 + (r_+ Q^0 - r_- Q^2) - \frac{1}{l} (r_+ R^0 + r_- R^2) \right].
\]

The vectors \(\vec{a}, \vec{b}, \vec{c}\) corresponding to the holonomy (39) have the components:

\[
\vec{a} = (0, 0, 2r_+)
\]

\[
\vec{b} = (r_+, 0, -r_-)
\]

\[
\vec{c} = -\frac{1}{l} (r_+, 0, r_-)
\]

\(^5\) In the expression of the holonomy we have dropped, for simplicity reasons, an overall factor of \(2\pi\) arising from the integral over \(\phi\).
and we can clearly see that this corresponds to the case (b.1) where $g = 1$ and $\vec{b}, \vec{c}$ are not parallel. In this case, the orbit invariants are $b_i b^i, b_i c^i, c_i c^i$ with $\vec{b}, \vec{c}$ being future and past timelike vectors. Using (35), the explicit forms of the invariants in terms of the BTZ black-hole parameters $M, J, l$ are given by:

\begin{align}
    m & \equiv b_i b^i = -[M^2 - (J/l)^2]^{1/2} \\
    n & \equiv b_i c^i = lM \\
    p & \equiv c_i c^i = -l^2[M^2 - (J/l)^2]^{1/2}
\end{align}

and consequently, the gauge orbit corresponding to the BTZ black-hole solution of the BCEA theory is parametrized by three real parameters $(m, n, p)$. Note that the curvature $l$ of the spacetime is a constant of integration and so appears naturally as a dynamical parameter, implying that the parameters $(m, n, p)$ are all independent. This is in contrast to the pure Einstein case, where the curvature of the spacetime occurs as a fixed parameter of the action.

5. Discussion

In this paper, we have studied the phase space structure of 2+1 gravity coupled to a pair of topological matter fields $B, C$. Using the formulation of this theory as a $I(ISO(2, 1))$ Chern–Simons model, we have identified the different sectors of this theory and have constructed or determined corresponding geometries of physical interest. Among the different sectors, two different types of solution emerge as relevant. In the first type, the dynamics $B, C$ fields can be decoupled from the dynamics of the geometry, and the model is equivalent to (2+1)-dimensional gravity in flat space on which form fields are superimposed. In the second type of solution the $B, C$ fields cannot be decoupled from gravity anymore. The dynamics of the fields and of the geometry are strongly interrelated, as one would expect from a theory where gravity is coupled to matter. Illustrative of this case is the BTZ black-hole solution, with the surprising result that the dynamical parameters of the solution include besides the mass and angular momentum of the black hole—the parameters of the traditional (2+1)-dimensional theory of gravity with cosmological constant—the cosmological constant itself.

Now that we have a clear picture of the classical dynamics of the BCEA model, we will need to address its quantization. Since the BTZ black hole is a solution of this model, understanding its quantization should allow us to give a full description of a quantum black hole in the presence of matter fields, a question that so far has never been addressed. Several strategies can be deployed in this direction. Since this theory can be formulated as an $I(ISO(2, 1))$ Chern–Simons model we can perform a Chern–Simons quantization of the model using the description of the classical phase space given here. Alternatively, the BCEA theory can also be formulated as a Poincaré BF theory, which opens the way to spin foam quantization [15].

The most challenging issue that has to be faced in either of these approaches is the presence of asymptotic boundaries. Such boundaries cause the quantization of the theory to become much more involved compared to the closed universe counterpart [17] and, as shown by Carlip and Teitelboim [18] for the case of a black hole, the quantum description of a system on a manifold with asymptotic regions may require one to go beyond the definition of the phase space as the space of classical solutions. All these issues will be dealt with in a future paper.
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Appendix A. Holonomy and gauge group elements

In this appendix, we present the formal derivation of the general expression (15) of the holonomy for the case of the BCEA model. Before proceeding with the explicit calculations, we briefly describe the fundamental idea behind them.

The holonomy of the generalized connection is by definition a path-ordered exponential and, as such, it is a rather cumbersome mathematical object to work with, whereas the group element of the identity component of the gauge group (described by the exponential of its Lie algebra elements) is computationally more convenient. For analysing the conjugacy classes of the holonomy under the action of the gauge group, from the preceding considerations it is only natural to try write the holonomy of the connection in a form involving as many standard exponentials as possible.

To attain this goal, we decompose the holonomy of the generalized connection into a product whose factors are holonomies involving the individual components of this connection. For example, in the case of the BCEA theory, the holonomy (13) will be factorized into a product of four separate holonomies, each involving one of the terms $A, E, B, C$ of the generalized connection as defined in (14). Having performed this decomposition, we then use the structure of the gauge algebra to write as many of these holonomies as possible as standard exponentials, with no path ordering involved. The last step of the method makes use of the Baker–Campbell–Haussdorf formula to replace the products of exponentials of Lie algebra elements by a single exponential, rendering the original holonomy in a form that on one hand is much more amenable to calculations, and on the other hand makes clear(er) its relation with the structure of the gauge group.

Returning now to issue of the derivation of the expression (15), we begin by summarizing a few of the general properties of the Wilson line [16] relevant to our purpose. The Wilson line of a 1-form $B$ along a portion $[0, t]$ of the loop $\gamma$ can be formally defined as the path integral:

$$W[B](t) = \mathcal{P} \exp \left( \int_0^t B(u) \right)$$

where $B(u)$ is the notationally more convenient shorthand for $B_\mu(\gamma(u)) \gamma'^\mu(u) du$. Assuming the usual left to right path ordering along the curve, (A.1) can be rewritten as the formal sum:

$$W[B](t) = 1 + \int_0^t B(s_1) + \cdots + \int_0^t B(s_1) \cdots \int_0^{t-1} B(s_n) + \cdots$$

where $s_1, \ldots, s_n$ are continuous and ordered parameters along the portion of the loop with starting point $\gamma(0)$ and endpoint $\gamma(t)$, $t \in [0, 1]$ and $\gamma(0) = \gamma(1)$. In this context, the Wilson line can be viewed as an incomplete Wilson loop, and consequently the formal definition of the holonomy (13) can be obtained from (A.2) as the particular case with $t = 1$. In all of the following, we will adopt this latter view and do all the calculations for a Wilson line with $t$ arbitrary in the range $(0, 1)$, particularizing in the end the relevant results for a Wilson loop by taking $t = 1$.

Since the Wilson line (A.2) has a continuous dependence on the parameter $t$ (for reasons of brevity, the dependence of the Wilson lines on the parameter $t$ will be assumed as implicit,
and will be dropped notationally unless absolutely necessary), it is straightforward to show that $W[B]$ satisfies the differential equation:

$$\frac{d}{dt}W[B] = BW[B]$$

(A.3)

with the initial condition $W[B](0) = 1$. Alternatively, equation (A.3) can be considered as the definition of the Wilson line, with the solution given by (A.2), and in the following we will use both viewpoints interchangeably, depending on the context.

Having a definition of the Wilson line, we will now derive the ‘composition’ law for Wilson lines, i.e. we derive the formula that will allow us to decompose the holonomy of the generalized connection into a product of holonomies of the individual fields that enter additively in the expression of the generalized connection. For this purpose, we introduce $W^{-1}[B]$ as the inverse of the Wilson line, and we can write:

$$\frac{d}{dt}\{W^{-1}[B]W[B]\} = \frac{d}{dt}\{W^{-1}[B]\}W[B] + W^{-1}[B]\frac{d}{dt}\{W[B]\} = 0.$$  

(A.4)

Using (A.3), (A.4) straightforwardly yields a differential definition of the inverse Wilson line which is analogous to (A.3):

$$\frac{d}{dt}W^{-1}[B] = -W^{-1}[B]B.$$  

(A.5)

Once we have established the differential definition of the inverse holonomy, we can use (A.3) and (A.5) to establish the following relation:

$$\frac{d}{dt}\{W^{-1}[B_1]W[B_1 + B_2]\} = \{W^{-1}[B_1]B_2W[B_1]\}W^{-1}[B_1]W[B_1 + B_2].$$  

(A.6)

According to the differential definition of the holonomy, it follows from (A.6) that:

$$W^{-1}[B_1]W[B_1 + B_2] = W\{W^{-1}[B_1]B_2W[B_1]\}$$  

(A.7)

and by multiplying (A.7) to the left with $W[B_1]$, we finally obtain the ‘composition’ law for holonomies:

$$W[B_1 + B_2] = W[B_1]W[W^{-1}[B_1]B_2W[B_1]]$$  

(A.8)

which allows one to write the holonomy of a sum of two connections as the product of holonomies.

We can now specialize to the case of the BCEA theory and use the composition law (A.8) to write the holonomy (13) as a product of holonomies. With the notation (14), we then have:

$$W[A] = W[A + E + B + C] = gW[\tilde{E} + \tilde{B} + \tilde{C}]$$  

(A.9)

or, in a more explicit form,

$$W[A](t) = g(t)P\exp\left[\int_0^t \tilde{E}(s) + \tilde{B}(s) + \tilde{C}(s)\right]$$  

(A.10)

where in (A.9), (A.10) we have used the notation

$$g(t) = P\exp\left[\int_0^t A(s)\right]$$

$$\tilde{E}(s) = g^{-1}(s)E(s)g(s)$$

$$\tilde{B}(s) = g^{-1}(s)B(s)g(s)$$

$$\tilde{C}(s) = g^{-1}(s)C(s)g(s).$$  

(A.11)

Before proceeding any further with the decomposition of the second factor in the last equality of (A.9), it is necessary to make some comments regarding the effects of the
conjugation by $g$ on the fields $E, B, C$. We first remark that, path ordering or not, $g$ is an element of group $SO(2,1)$. Now, according to the structure of the Lie algebra (6) of $I[ISO(2,1)]$, the adjoint action of $SO(2,1)$ leaves invariant the subalgebras with the corresponding vector subspaces $\text{Span}[P^i], \text{Span}[Q^i], \text{Span}[R^i]$, which means that in fact the tilded fields in (A.9), (A.11) have the forms:
\[
\tilde{E} = E_i P^i; \quad \tilde{B} = B_i Q^i; \quad \tilde{C} = C_i R^i.
\] (A.12)

We can now proceed with the decomposition of the second factor in the last equality in (A.9) and we have:
\[
W[\tilde{E} + \tilde{B} + \tilde{C}] = W[\tilde{E}]W[ W^{-1}[\tilde{E}] (\tilde{B} + \tilde{C}) W[\tilde{E}]].
\] (A.13)

Based on the above remarks, and since $[P, P] = 0$, the holonomy of the field $\tilde{E}$ becomes the holonomy of an Abelian connection, and as such reduces to the standard exponential:
\[
W[\tilde{E}](t) = \exp \left[ \int_0^t \tilde{E}(s) \right].
\] (A.14)

Furthermore, since $[P, Q] = [P, R] = 0$, the adjoint action of $W[\tilde{E}]$ is trivial, and the decomposition of the holonomy in the lhs of (A.13) can be rewritten in the form:
\[
W[\tilde{E} + \tilde{B} + \tilde{C}] = \left\{ \exp \left[ \int_0^t \tilde{E}(s) \right] \right\} W[\tilde{B} + \tilde{C}].
\] (A.15)

Similarly, we can write
\[
W[\tilde{B} + \tilde{C}] = W[\tilde{B}]W[ W^{-1}[\tilde{B}] \tilde{C} W[\tilde{B}]]
\] (A.16)

and since $[Q, Q] = 0$, the holonomy of $\tilde{B}$ also reduces to the standard exponential
\[
W[\tilde{B}](t) = \exp \left[ \int_0^t \tilde{B}(s) \right].
\] (A.17)

The processing of the second factor in the product in the rhs of (A.16) is more involved, since now $[Q, R] \sim P$. Nevertheless, using the Baker–Campbell–Haussdorff formula it is rather straightforward to show that this factor can be written in the form:
\[
W[ W^{-1}[\tilde{B}] \tilde{C} W[\tilde{B}]] = W[\tilde{C} - [\tilde{B}, \tilde{C}]]
\] (A.18)

where in (A.18) we have used the notation
\[
[\tilde{B}, \tilde{C}] \equiv [\tilde{B}, \tilde{C}](s) = \int_0^s du[\tilde{B}(u), \tilde{C}(s)]
\] (A.19)

with $[,]$ the Lie algebra commutator. Noting that $[\tilde{B}, \tilde{C}]$ is an element of $\text{Span}[P^i]$ while $\tilde{C}$ is an element of $\text{Span}[Q^i]$, using arguments similar to those used in the derivation of (A.15), the rhs of (A.18) can be rewritten in the form:
\[
W[\tilde{C} - [\tilde{B}, \tilde{C}]][t] = \exp \left[ \int_0^t \tilde{C}(s) \right] \exp \left[ - \int_0^t [\tilde{B}, \tilde{C}](s) \right].
\] (A.20)

Putting together all the results of the above derivations, the holonomy of the BCEA connection $\mathcal{A}$ can be written in the form
\[
W[\mathcal{A}](t) = g(t) \exp \left[ \int_0^t \tilde{E}(s) \right] \exp \left[ \int_0^t \tilde{B}(s) \right] \exp \left[ \int_0^t \tilde{C}(s) \right] \exp \left[ - \int_0^t [\tilde{B}, \tilde{C}](s) \right].
\] (A.21)

The last step in the processing of the holonomy that remains to be completed, according to the program described at the beginning of this appendix, is to rewrite the products of exponentials
in the rhs of (A.21) as a single exponential. This can of course be achieved by using once
again the Baker–Campbell–Haussdorf formula, and we finally obtain for the holonomy of
the connection \( A \) the expression:

\[
W[A](t) = g(t) \exp \left\{ \int_0^t \left[ \tilde{E}'(s) + \tilde{B}(s) + \tilde{C}(s) \right] \right\} \tag{A.22}
\]

where we have:

\[
\int_0^t \tilde{E}'(s) = \int_0^t \tilde{E}(s) - \int_0^t \int_0^t \left[ \tilde{B}(u), \tilde{C}(s) \right] + \frac{1}{2} \int_0^t \int_0^t \left[ \tilde{B}(s), \tilde{C}(u) \right]. \tag{A.23}
\]

The rhs of (A.23) can be further processed as follows. The second term can be rewritten as:

\[
\int_0^t \int_0^t \left[ \tilde{B}(u), \tilde{C}(s) \right] \Theta(s - u) \tag{A.24}
\]

with \( \Theta(s - u) \) the usual step function, and by using the identities \( \Theta(s - u) + \Theta(u - s) = 1 \), \( \Theta(u - s) - \Theta(s - u) = \epsilon(u - s) \), (A.23) can be put in the more compact form:

\[
\int_0^t \tilde{E}'(s) = \int_0^t \tilde{E}(s) + \frac{1}{2} \int_0^t \int_0^t \left[ \tilde{B}(s), \tilde{C}(u) \right] \epsilon(u - s) \tag{A.25}
\]

where \( \epsilon(u - s) \) is the unit antisymmetric function of its arguments (see the second identity
above).

It is rather clear that the expression (A.22) with the fields given by (A.11) and (A.25)
is the simplest form in which we can cast the Wilson line of the BCEA connection \( A \). It
contains a path-ordered exponential through \( g(t) \) and a standard exponential, and any attempt
to further process the former will only complicate the expression of the Wilson line due to the
commutation relations for the generators of the Lorentz subalgebra.

Once we have obtained the simplest form of the Wilson line for the BCEA model, as
mentioned earlier, we can obtain the holonomy of the connection (Wilson loop) along the loop \( \gamma \) by setting \( t = 1 \) in the above results. By doing so, we obtain:

\[
W[A] = g \exp \left\{ \int_0^1 \left[ \tilde{E}'(s) + \tilde{B}(s) + \tilde{C}(s) \right] \right\} \tag{A.26}
\]

with

\[
\int_0^1 \tilde{E}'(s) = \int_0^1 \tilde{E}(s) + \frac{1}{2} \int_0^1 \int_0^1 \left[ \tilde{B}(s), \tilde{C}(u) \right] \epsilon(u - s)
\]

\[
g = P \exp \left[ \int_0^1 A(s) \right]
\]

\[
g(s) = P \exp \left[ \int_0^s A(u) \right] \tag{A.27}
\]

\[
\tilde{E}(s) = g^{-1}(s)E(s)g(s)
\]

\[
\tilde{B}(s) = g^{-1}(s)B(s)g(s)
\]

\[
\tilde{C}(s) = g^{-1}(s)C(s)g(s)
\]

and for the remainder of the appendix, we will use the notation \( g \equiv g(1) \).

With the holonomy in the form (A.26), it is only a matter of conjugation by \( g \) to recast it
in the form (15). Indeed, (A.26) can immediately be rewritten as:

\[
W[A] = g \exp \left\{ \int_0^1 \left[ \tilde{E}'(s) + \tilde{B}(s) + \tilde{C}(s) \right] \right\} g \tag{A.28}
\]

\[
= \left[ \exp \left\{ \int_0^1 \left[ E'(s) + B(s) + C(s) \right] \right\} \right] g
\]
and introducing now the notation \( \tilde{g}(s) = g(s)g^{-1} \), the quantities appearing in the rhs of (A.28) are given by expressions similar to those in (A.27):

\[
\begin{align*}
\int_0^1 \tilde{E}'(s) &= \int_0^1 \tilde{E}(s) + \frac{1}{2} \int_0^1 [\tilde{B}(s), \tilde{C}(u)] \epsilon(u - s) \\
g &= \mathcal{P} \exp \left[ \int_0^1 A(s) \right] \\
g(s) &= \mathcal{P} \exp \left[ \int_0^s A(u) \right] \\
\tilde{E}(s) &= \tilde{g}^{-1}(s)E(s)\tilde{g}(s) \\
\tilde{B}(s) &= \tilde{g}^{-1}(s)B(s)\tilde{g}(s) \\
\tilde{C}(s) &= \tilde{g}^{-1}(s)C(s)\tilde{g}(s).
\end{align*}
\]

(A.29)

It should be noted at this time that by using the properties of the holonomy with respect to the multiplication of paths, once could write of paths to write \( g \) in a more compact form as a single path integral. However, for the purpose of the present work (A.28) and (A.29) all that we need, and such a refinement is unnecessary.

Separating now in the exponential in (A.28), the components of the fields along the generators of the Lie algebra of \( I[ISO(2, 1)] \) and using the commutation relations between these generators, it is straightforward to rewrite (A.28) in the form:

\[
W[A] = [\exp[a_i P^i + b_j Q^j + c_i R^i]]g = [\exp[a \cdot P + b \cdot Q + c \cdot R]]g
\]

(A.30)

with \( a_i, b_j, c_i \) given by the relations:

\[
\begin{align*}
a_i &= \int_0^1 \tilde{E}_i(s) + \frac{1}{2} \epsilon_{ijk} \int_0^1 [\tilde{B}^j(s), \tilde{C}^k(u)] \epsilon(u - s) \\
b_j &= \int_0^1 \tilde{B}_j(s) \\
c_i &= \int_0^1 \tilde{C}_i(s)
\end{align*}
\]

(A.31)

and the barred quantities defined in (A.29). The expressions in (A.30) and (A.31) are exactly the expressions (15) and (16) of the group element and group parameters that we have used in section 3.

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