Relativistic Kinetics of Phonon Gas in Superfluids

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Abstract

The relativistic kinetic theory of the phonon gas in superfluids is developed. The technique of the derivation of macroscopic balance equations from microscopic equations of motion for individual particles is applied to an ensemble of quasi-particles. The necessary expressions are constructed in terms of a Hamilton function of a (quasi-)particle. A phonon contribution into superfluid dynamic parameters is obtained from energy-momentum balance equations for the phonon gas together with the conservation law for superfluids as a whole. Relations between dynamic flows being in agreement with results of relativistic hydrodynamic consideration are found. Based on the kinetic approach a problem of relativistic variation of the speed of sound under phonon influence at low temperature is solved.

Key words: superfluids, kinetic equation, phonon gas

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I. INTRODUCTION

Over the last years the problems of relativistic superfluid dynamics were discussed repeatedly. Such an interest is well justified. Superfluids turned out to be the object that has revealed some problems of relativistic hydrodynamics and thermodynamics in a new fashion. This has led to earnest discussion in a relativistic generalization of Landau two-fluid model [1, 2, 3, 4, 5].

Usually, two approaches are used in relativistic hydrodynamics, Eckart and Landau-Lifschits ones. In the former the macroscopic velocity of a gas (or liquid) is defined as the unit vector parallel to the particle number flow, and in the latter it does as eigenvector of the energy-momentum tensor. In equilibrium all flows such as a particle number flow and an entropy flow are parallel to this unit vector. When dissipation effects are taken into account all the macroscopic values get additions that are orthogonal to the vector of the macroscopic velocity.

Superfluids fundamentally differs from the perfect fluid. The particle number flow $n^i$ and the entropy flow $S^i$ as well as conjugated dynamic $\mu^i$ and thermal $\Theta^i$ momenta have different directions even in equilibrium. It allows to develop an approach that does not prefer any direction. In context of this approach it becomes possible not only to describe adequately relativistic superfluid dynamics but also to discuss some problems of the traditional approaches [5, 6].

According to [5, 6] the energy-momentum tensor of superfluids at non-zero temperature can be represented in the form

$$T^i_j = n^i \mu_j + S^i \Theta_j - P \delta^i_j,$$

where $P$ is the liquid pressure. Though the notation (1) does not demonstrate the symmetry of the tensor in an explicit form, it really takes place [6, 7].

Among the four flows, $n^i, S^i, \mu^i,$ and $\Theta^i$, only two, $n^i$ and $S^i$ say, are independent. The other two are expressed as

$$\mu^i = B n^i + A S^i, \quad \Theta^i = A n^i + C S^i,$$

with three scalar function $A, B,$ and $C$ named the anomaly, bulk, and caloric coefficients respectively. Consideration of superfluidity supposes the coefficient $A$ should be negative and the coefficients $B$ and $C$ should be positive [5].
Since there is no a preferred flow in the theory developed in [5, 6] the vector of the “hydrodynamic” velocity may be associated with any of them. The choice is made by convenience reason. For example, the velocity may be parallel to the particle number flow when a gas of identical particles is considered, whereas it may be direct along entropy flow if thermodynamic properties are studied. In this work there also exists a preferred vector $V^i$. It has a sense of a superfluid velocity. In context of this paper the superfluid velocity plays a role of a velocity of the carrying medium where quasi-particles, phonons and rotons, are located.

Relativistic superfluid hydrodynamics can be used also for solving more practical problems in different areas apart from studying fundamental matter. This formalism is found to be efficient to describe processes in neutron stars [8, 9, 10, 11] and to construct the cosmological models [12, 13, 14].

Hydrodynamic approach is essentially macroscopic and therefore it does not take into account microscopic processes that have influence on the dynamic and thermal parameters (such as entropy, pressure etc.). It is also clear that hydrodynamics can not describe all the phenomena in superfluids. For example, usage of hydrodynamic equations is restricted by the condition $\omega \tau \ll 1$, where $\omega$ is the sound frequency, $\tau$ is the free path time. This condition is not fulfilled at temperatures close to zero because free path time of the quasi-particles increases. In this temperature range system behavior is described by a kinetic equation.

Meanwhile kinetic phenomena in relativistic superfluids remain beyond the scope of study. The superfluid kinetic theory concerns the problems related to the excitation gas, which significantly determines values of fluid parameters at non-zero temperature. When dissipation is discounted, the quasi-particle gas determines thermal effects in the first place. In hydrodynamics influence of the quasi-particles is masked by macroscopic flows such as an entropy flow. In essence the unit vector in the direction of the entropy flow is considered as a macroscopic velocity of the quasi-particle gas.

The kinetic theory of the quasi-particle gas for non-relativistic superfluids was developed in [15]. To generalize this theory for the relativistic case is the primary purpose of this study. Covariant kinetic theory of the quasi-particles naturally develops and supplements relativistic kinetic theory for the real mass and massless particles.

One of the main goals of kinetic theory is to connect macroscopic variables with intrinsic microscopic parameters. With respect to superfluids this means that a quasi-particle contri-
bution into the superfluid dynamic parameters should be obtained. At the low temperatures we can take into account only phonons (i.e. excitations with the linear dependence between the energy and the momentum) as influence of other quasi-particles may be neglected.

When we deal with quasi-particles in special or general relativity, a main difficulty is to define the macroscopic variables adequately. The usual definitions are inapplicable here, since the quasi-particles being the massless move slower than light. In Sec. III this problem is solved using the Hamilton function for the quasi-particles. Structure of the function is based on the acoustic metrics introduced in Sec. II. This permits to modify the usual constructions of the energy-momentum tensor, the particle number flow and the entropy flow. In Sec. IV balance equations for the quasi-particle gas are derived from the kinetic equations. Simultaneous consideration of the phonon balance equation and conservation laws for all liquid allows to find a phonon contribution to the chemical potential and the pressure of the liquid. In Sec. V the problem of a sound speed variation under influence of the phonon gas at low temperature is investigated for the Minkowski and Robertson-Walker metrics.

II. ACOUSTIC METRICS

In the classical Landau’s theory a superfluid state is governed by quasi-particles. Each quasi-particle has a certain momentum and energy that are functionally related. Such a functional dependence is said to be an energy spectrum of superfluids. In the spectrum of liquid helium one recognizes two kinds of quasi-particles, named phonons and rotons that contribute maximally to thermodynamic properties. Phonons correspond to the linear part of the spectrum whereas rotons belong to the nonlinear part of the spectrum. Processes being due to rotons are inessential at a temperature close to zero and their contribution is neglected. Thus at the low temperature a quasi-particle gas may be considered as a purely phonon gas.

To develop our theory it is convenient to introduce so-called acoustic or phonon metrics

\[ \bar{g}^{ij} = g^{ij} + \left( \frac{c_p^2}{c_s^2} - 1 \right) V_i V_j, \]  

where \( V_i \) is the superfluid velocity normalized by \( V_i V^i = 1 \), and the sound speed \( c_p \) at zero
temperature is defined by the relation
\[ \frac{c_p^2}{c^2} = \frac{dP_0}{d\rho} = \frac{n_{||}}{\rho} \frac{d\mu_0}{dn_{||}}. \] (4)
in which \( n_{||} \equiv n^i V_i, n^i \) is the liquid particle number flow, \( \mu_0 \) is the chemical potential, \( P_0 \) is the pressure, and \( \rho \) is the energy density of the liquid at zero temperature [21]. The acoustic metrics is constructed so that phonon momenta satisfy the relation
\[ \overline{g}^i j p_i p_j = 0, \] (5)
i.e. they are tangent vectors to the characteristic hypersurface of sound propagation in the medium. Along with the tensor \( \overline{g}^j \) the metric tensor with subscripts is involved
\[ \overline{g}_{ij} = g_{ij} + \left( \frac{c_p^2}{c^2} - 1 \right) V_i V_j, \] (6)
which satisfy the condition \( \overline{g}^i j \overline{g}_{jk} = \delta^i_k \).

Notice that the acoustic metrics (3) can be formally derived as the special case of a broader class of a tensor corresponding to an arbitrary energy spectrum. Actually, we can define the tensor
\[ G^{ij} = g^{ij} + \left( \frac{\pi^2}{\varepsilon^2} - 1 \right) V^i V^j, \]
where \( \varepsilon = p_i V^i \) is the energy, and \( \pi^2 = -\Delta^{ij} p_i p_j \equiv -(g^{ij} - V^i V^j) p_i p_j \) is the square of 3-momentum of a quasi-particle in the local rest frame of the medium. In this case \( G^{ij} p_i p_j \equiv 0 \).

If we will establish a functional dependence between \( \varepsilon \) and \( \pi \) we obtain the covariant form of the energy spectrum. However in general, \( G^{ij} \) can not be used as a metrics, since it depends not on only coordinates but also on momenta. Only for phonons with the condition
\[ \varepsilon = \frac{c_p}{c} \pi, \] (7)
\( G^{ij} \) can be considered as a metrics.

The superfluid velocity appearing in the acoustic metrics sets the direction for the “superfluid momentum” \( \mu_j = \mu V_j \), where \( \mu \) is the chemical potential of superfluids [5]. The momentum \( \mu_j \) is gradient of a scalar function (which is interpreted as a phase of a wave function in a quantum description of superfluidity): \( \mu_j = \nabla_j \phi \). This permits to derive the equation
\[ \nabla_{[i} \mu_{j]} = 0, \] (8)
that demonstrates irrotational nature of superfluid moving. Contraction of (8) with $V^i$ gives the relation

$$
\mu D V_j = \Delta^i_j \nabla_i \mu,
$$

(9)

($D \equiv V^i \nabla_i$ is the convective derivative) that can be interpreted as generalization of the non-relativistic equation for the superfluid velocity. The irrotational condition can be obtained in a more traditional form by convolution of (8) with the projector $\Delta^i_k$ and using (9):

$$
\Delta^i_k \nabla^i V_j = 0.
$$

(10)

III. MACROSCOPIC VARIABLES

To describe the phonon gas it is necessary to introduce the usual set of macroscopic parameters, such as particle and entropy flow four-vectors and an energy-momentum tensor, which are constructed using a phonon distribution function $f(x, p)$. If to follow the usual way [16, 19] the macroscopic variables should be defined as

$$
N^i = 2c \int p^j f(x, p) \frac{\Theta(p^0) \delta(g^{ij}p_ip_j)}{(2\pi \hbar)^3} \frac{d^4p}{\sqrt{-g}},
$$

(11a)

$$
T^{ij} = 2c \int p^j p^i f(x, p) \frac{\Theta(p^0) \delta(g^{ij}p_ip_j)}{(2\pi \hbar)^3} \frac{d^4p}{\sqrt{-g}},
$$

(11b)

but these expressions are not applicable in this case since they do not carry required physical content. To illustrate this, let us consider the vector $N^i$ in the flat space-time.

The component $N^0$ must have meaning of a number particle density, i.e.

$$
N^0 = c \int f(x, p) d^3p,
$$

(12)

where the integration is to be taken over usual (three-dimensional) momentum space and $p_0$ is determined by (5). $N^\alpha (\alpha = 1, 2, 3)$ are the components of a particle flow three-vector. Therefore in the medium rest frame the phonon particle flow should be defined as

$$
N^\alpha = c_p \int \frac{p^\alpha}{p} f(x, p) d^3p.
$$

(13)

$\delta$-function provides transformation of (11a) to the usual momentum space

$$
N^i = c \int \frac{g^{ij}p_j}{g^{kk}p_k} f(x, p) d^3p.
$$

(14)
Comparing this expression with (12), one can conclude that $N^i$ defined according (11a) can not be used as a particle number flow.

To give a physical sense to the macroscopic currents, their definitions should be modified. For this purpose we introduce Hamilton function for a quasi-particle

$$\mathcal{H} = \frac{1}{2}g^{ij}p_ip_j,$$

that is equal to zero at the hypersurface defined by the expression (5). Using this function we construct macroscopic variables

$$N^i = c \int \frac{\partial \mathcal{H}}{\partial p_i} f(x,p) \frac{\Theta(p_jV^j)\delta(\mathcal{H})}{(2\pi\hbar)^3} \frac{d^4p}{\sqrt{-g}},$$  \hspace{2cm} (16a)

$$T^i_j = c \int p_j \frac{\partial \mathcal{H}}{\partial p_i} f(x,p) \frac{\Theta(p_jV^j)\delta(\mathcal{H})}{(2\pi\hbar)^3} \frac{d^4p}{\sqrt{-g}},$$  \hspace{2cm} (16b)

$$S^i = -ck_B \int \frac{\partial \mathcal{H}}{\partial p_i} [f \ln f - (1 + f)(1 + f)] \frac{\Theta(p_jV^j)\delta(\mathcal{H})}{(2\pi\hbar)^3} \frac{d^4p}{\sqrt{-g}}.$$  \hspace{2cm} (16c)

Further we will use a compact notation

$$d\mathcal{P} = \Theta(\varepsilon)\delta(\mathcal{H}) \frac{d^4p}{(2\pi\hbar)^3} \sqrt{-g}.$$

The macroscopic flows defined in (16) satisfy all the wished requirements. To verify this fact let us consider the modified vector $N^i$. Integration of (16a) with $\delta$-function leads to the expression (14), but now the acoustic metrics $g^{ij}$ (not $g^{ij}$) is in the numerator, providing fulfilment both (12) and (13).

The determinations (11) usually include $\Theta(p^0)$ that gives the positivity of the particle energy. When we deal with quasi-particles, a role of the energy is assigned to the convolution $p_iV^i \equiv \varepsilon$. That is why it fill the place of the argument to the $\Theta$-function in the expressions (10).

The expressions (10) can be considered as generalization of the traditional definitions of the macroscopic flows. To revert to the usual notations, Hamilton function should be taken in the form

$$\mathcal{H} = \frac{1}{2}g^{ij}p_ip_j - m^2c^2$$

for massive particles, and

$$\mathcal{H} = \frac{1}{2}g^{ij}p_ip_j$$
for massless ones.

The entropy flow definition may be different according to particle statistics \[22\]. Phonons in superfluids are bosons, that leads to the construction in the square brackets of \[16c\].

The energy-momentum tensor defined by \[16b\] is not symmetric in the general case. This implies that the angular momentum of the phonon gas is not conserved \[23\]. This is a quite expected circumstance because the gas is not an isolated system. The liquid is not only a medium for phonons, its parameters are influenced by the quasi-particles. In view of this interrelation, the energy-momentum of the phonon gas is also not conserved as we shall see later. Conservation laws hold for the system as a whole. It will allow us to make conclusion about a phonon contribution in the thermodynamic parameters of superfluids. The angular momentum for all the system is implied to be conserved, therefore the total energy-momentum tensor can be represented in the symmetric form \[6, 23\].

The macroscopic flows \[16\] can be naturally split into contributions parallel and orthogonal to \(V^i\). It is convenient to decompose the phonon energy-momentum tensor as following:

\[
T^i_j = wV^iV_j + Q^iV_j + Q_jV^i + \Pi^i_j + \left(\frac{c^2}{c_p^2} - 1\right) Q_jV^i, \tag{17}
\]

where

\[
w = T^i_jV_jV^i = \frac{c^3}{c_p^2} \int \varepsilon^2 f(x, p) \, d\mathcal{P},
\]

\[
Q_i = T^k_jV^i\Delta_{ki} = c \int \varepsilon\pi_i f(x, p) \, d\mathcal{P},
\]

\[
\Pi^i_j = T^k_i\Delta^l_k\Delta^i_j = c \int \pi^i\pi_j f(x, p) \, d\mathcal{P},
\]

\[
\varepsilon = p_i V^i, \quad \pi_i = \Delta^k_ip_k.
\]

A similar decomposition one can perform for the entropy flow:

\[
S^i = s_{||}V^i + \sigma^i, \quad s_{||} = S^iV_i, \quad \sigma^i = \Delta^i_jS^j. \tag{18}
\]

IV. PHONON CONTRIBUTION INTO SUPERFLUID DYNAMICS

A. Kinetic equation for phonons

The phonon distribution function satisfy Liouville equation \[24, 25\]

\[
\frac{df}{ds} = \frac{\partial f}{\partial x^i} \frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial \mathcal{H}}{\partial x^i} = J(x, f), \tag{19}
\]
where canonical Hamilton equations are used

\[
\frac{dx^i}{ds} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{ds} = -\frac{\partial H}{\partial x^i},
\]

and \(J(x, f)\) is a collision integral for the phonons.

The derivative of the Hamilton function (15) with respect to coordinates can be converted:

\[
\partial H \partial x^i = \frac{1}{2} p_k p_l \partial g^{kl} \partial x^i = -p_k p_s \Gamma^{im} \Gamma_{im} = -\Gamma^{i}_{im} p_k \partial H \partial p_m.
\]

Then the equation (19) takes the form:

\[
\frac{df}{ds} = \partial H \partial p_i \hat{\nabla}^i f = J(x, p),
\]

where

\[
\hat{\nabla}^i = \frac{\partial}{\partial x^i} + \Gamma^{k}_{im} p_k \partial H \partial p_m
\]

is the Cartan derivative [25] and \(\Gamma^{im}_{ij}\) are the Christoffel symbols defined with respect to the metrics \(\mathbf{g}_{ij}\).

One can directly verify that the following equalities are valid:

\[
\hat{\nabla}^i p_k = 0,
\]

\[
\hat{\nabla}^i \partial H \partial p_k = 0,
\]

as well as

\[
\frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g}}{\partial x^i} = \frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g}}{\partial x^i} + \frac{1}{c_p} \frac{\partial c_p}{\partial x^i}.
\]

Taking into account these properties it is easy to show that

\[
\int \frac{\partial H}{\partial p_k} p_{i_1..i_l} \hat{\nabla}_k f(x, p) d\mathcal{P} = \nabla^k \int \frac{\partial H}{\partial p_k} p_{i_1..i_l} f(x, p) d\mathcal{P} - \frac{1}{c_p} \frac{\partial c_p}{\partial x^k} \int \frac{\partial H}{\partial p_k} p_{i_1..i_l} f(x, p) d\mathcal{P},
\]

where \(\nabla^k\) is covariant derivative defined with respect to the metrics \(\mathbf{g}_{ij}\).

B. Balance equations of phonon gas

The relation (27) is very convenient to derive macroscopic equations. Let us multiply the Liouville equation (22) on \(p_{i_1..i_l}\) and integrate it over momenta. If number of the momenta
\[ l = 0 \text{ then we obtain the balance equation} \]
\[ \nabla_i N^i = \int J(x, p) \, dP. \] (28)

Notice that the covariant derivative is associated with the metrics \( g^{ij} \) and a rate of change of the quasi-particles is governed by the collision term of the kinetic equation. Therefore the equation (28) becomes a conservation law only in equilibrium when \( J(x, p) = 0 \).

There is a different situation for phonon energy-momentum balance, that can be obtained setting \( l = 1 \). In this case the integral of the right hand side of the Liouville equation turns to zero by a momentum conservation law and hence we find that

\[ \nabla_i T^i + (\Gamma^m_{ij} - \Gamma^m_{ij}) T_m = 0, \] (29)

where \( \Gamma^m_{ij} \) and \( \Gamma^m_{ij} \) are the Christoffel symbols with respect to the metrics \( g_{ij} \) and \( \bar{g}_{ij} \) correspondingly. The second term in (29) is found using the explicit expression for \( \bar{g}_{ij} \) and decomposition (17). Then the balance equation (29) can be written as

\[ \nabla_i T^i + \left( \frac{c^2}{c_p^2} - 1 \right) Q^i \nabla_j V_i - \frac{w}{c_p} \nabla_j c_p = 0. \] (30)

This means that there is no a phonon energy-momentum conservation even in equilibrium.

Contracting the equation (30) with \( V^j \) and applying the decomposition (17), we find

\[ Dw + w \theta + \nabla_i Q^i - Q^i D V_i - \Pi^{ij} \nabla_i V_j - \frac{w}{c_p} D c_p = 0, \] (31)

where the following notations are introduced:

\[ \theta = \nabla_i V^i = g^{ij} \theta_{ij} = \Delta^{ij} \theta_{ij}, \quad \theta_{ij} = \Delta_{k(i} \nabla^{k} V_{j)}. \] (32)

Next we follow Carter [6], who noticed that there is no evidence of parallelism of entropy and particle flows in the hydrodynamic equations even in equilibrium. Formally we may construct hydrodynamics with unparallel equilibrium flows. Really it is realized in superfluid hydrodynamics. In usual liquid the particle and entropy flows become unparallel in non-equilibrium only, but traditionally a basic thermodynamic relations (that are related to a rest frame of the velocity selected) remain the same. Such an approach seems to be not well consistent since the flows have different directions, and hence, one more independent parameter appears in addition to a usual set of variables. It is an absolute value of difference of the flows. Hence the Gibbs relation have to be supplemented by a term of a kind \( X \delta Y \),
where $Y$ is a new variable and $X$ is a thermodynamically conjugated current. Using a Legendre type transformation the new parameters may be added in the Gibbs relation as $-Y \delta X$. If $Y = \sqrt{y_i y_i}$ then the variation of the scalar can be transformed to the variation of the vector: $X \delta Y = (X y_i / Y) \delta y^i = x_i \delta y^i$.

In our consideration the superfluid velocity and entropy flow have different directions and the Gibbs relation will be used in the following way

$$\delta w = T \delta s_{||} + \sigma^i \delta \tau_i + \mu_1 \delta n_{||}, \quad (33)$$

where $T$ is the temperature, $\tau_i$ is the current conjugated to $\sigma^i$. The third term appears because the phonon energy depends on the liquid particle density. The parameter $\mu_1$ is not the chemical potential of the phonon gas (it equals to zero since the number of quasi-particles is not conserved) and it should be determined as well as $\tau_i$.

Taking into account (18) the entropy production density can be realized in the form

$$\nabla_i S^i = D s_{||} + s_{||} \theta + \nabla_i \sigma^i. \quad (34)$$

The liquid particle number flow can be decomposed at components parallel and orthogonal to the superfluid velocity

$$n^i = n_{||} V^i + H^i, \quad n_{||} = n^i V_i, \quad H^i = \Delta^i j n^j. \quad (35)$$

The corresponding balance equation may be written as

$$\nabla_i n^i = D n_{||} + n_{||} \theta + \nabla_i H^i = 0. \quad (36)$$

$D \omega$ is substituted in the equation (31) using (33). Expressing the convective derivatives $D s_{||}$ and $D n_{||}$ from (34) and (36) we find

$$T \nabla_i S^i = T s_{||} \theta - T \nabla_i \sigma^i + \sigma^i D \tau_i - \mu_1 n_{||} \theta - \mu_1 \nabla_i H^i + w \theta + \nabla_i Q^i - Q^i D V_i - \Pi^{ij} \nabla_i V_j - \frac{w}{c_p} \frac{D c_p}{D n_{||}} = 0. \quad (37)$$

The speed of sound $c_p$ is assumed to depend only on $n_{||}$, and $Q^i = \Theta \sigma^i$, where the proportionality factor $\Theta$ is to be defined. On simple rearrangement, the expression (37) can be written as

$$T \nabla_i S^i = \left\{ \Pi^{ij} - \Delta^j \left( w - T s_{||} - \mu_1 n_{||} + \frac{w n_{||}}{c_p} \frac{\partial c_p}{\partial n_{||}} \right) + \sigma^j \tau^j \right\} \theta_{ij} +$$

$$\{ T - \Theta \} \nabla_i \sigma^i + \left\{ \mu_1 - \frac{w}{c_p} \frac{\partial c_p}{\partial n_{||}} \right\} \nabla_i H^i + \{ \mathcal{L}(\Theta V_i - \tau_i) - \nabla_i \Theta \} \sigma^i. \quad (38)$$
In the expression \((38)\) we have turned from the convective derivatives to the Lie derivatives with respect to the vector \(V^i\). This is due to the fact that the convective derivation is concerned with changes of vector projections onto direction \(V^i\), whereas the Lie derivative does changes of the vectors themselves. The derivatives of two types are related by the law

\[
\mathcal{L}V_i = DV_i, \quad \mathcal{L}\tau_i = D\tau_i + \tau^i\theta_{ij}.
\]

Each term on the right-hand side of \((38)\) includes two factors. One of them consists of space-time derivatives, whereas another algebraically depends on the macroscopic variables.

In equilibrium the entropy production density is equal to zero. Since spacelike components of the flows \(S^i\) and \(n^i\) are independent, each term should be put to be zero separately. This leads to

\[
\Pi^{ij} = \Delta^{ij}(w - Ts^{||}) - \sigma^i\tau^j,
\]

\[
T = \Theta, \tag{40}
\]

\[
\mu_1 = \frac{w}{c_p} \frac{\partial c_p}{\partial n^{||}}, \tag{41}
\]

\[
\nabla_i \Theta - \mathcal{L}(\Theta V^i - \tau_i) = 0. \tag{42}
\]

The first three conditions give the connection between variables involved in the energy-momentum tensor \((17)\). The last one is non-dissipative heat conduction equation.

In non-equilibrium the entropy production density is positive due to additional contributions in the currents. It is usually supposed that for small deviations from equilibrium these contributions depend only linearly on the gradients of the equilibrium quantities \([16, 22]\). According this approach the right hand side of the equation \((43)\) becomes proportional to the non-equilibrium component in \(\sigma^i\), corresponding to an irreversible heat flow. Since this dependence emerges in the next order of magnitude with respect to the equilibrium values, it does not affect on the relations \((40)\)–\((42)\) (we disregard viscous effects).

In the context of the discussion regarding infinite propagation velocity for heat signal in relativistic thermodynamics, Carter suggested \([6]\) another way to obtain the positivity of the entropy production. This approach is based on the assumption that heat inertia is nonzero and the energy-momentum tensor is constructed as a purely algebraic function of independent currents such as a particle number flow and a heat current. This statement looks like Fok’s “physical principle” which implies that energy-momentum tensor components must contain the state functions only and may not include gradients as well as coordinates in the
Taking into account Carter’s ansatz, the relations (40)–(42) remain valid as before and the equation (43) transforms into
\[ \nabla_i \Theta - \mathcal{L}(\Theta V_i - \tau_i) = -Y_{ij} \sigma^j, \] (44)
where \( Y_{ij} \) is a positive semidefinite resistivity matrix. The equation (44) leads to a hyperbolic equation for heat conduction automatically without using an additional phenomenological parameter.

For our purposes the result regarding relations (40)–(42) is more necessary than different forms of the heat conduction equation and it is the same at least for the considered approaches, however the later seems to be more consecutive in our case.

C. Phonon contribution into superfluid dynamics

In previous section phonon gas was considered. If it is in solids, then it is believed that phonon gas does not affect the moving of the medium. In superfluids, phonons not only determine thermal parameters but also change dynamic parameters of the medium. Thus considering superfluids as closed system containing two parts, liquid (medium) and phonons, we should assume that tensor (17) is contained in the total energy-momentum tensor (1). Because of \( \mathcal{T}^{ij} \) is the energy-momentum tensor of all the system it obeys the covariant conservation law
\[ \nabla_i \mathcal{T}^{ij} = 0. \] (45)
At \( T = 0 \) the energy-momentum tensor (11) takes the form
\[ \mathcal{T}^i_j = n^i \mu_j - P_f \delta^i_j, \] (46)
where \( P_f \) is the pressure of the liquid properly, without a phonon influence.

Using (40) the energy-momentum tensor (17) can be represented in the form
\[ T^i_j = S^i \Theta^*_j - P_{ph} \delta^i_j. \] (47)
In this case
\[ \Theta^*_j = T V_j + \tau_j, \] (48)
\[ \tau_j = \frac{T c^2}{s || c_p^2} \sigma^j, \] (49)
\[ P_{ph} = T s || - w. \] (50)
Since the thermodynamic properties of superfluids are completely determined by the phonons, and the entropy flow vector appears in the second term of (11) only, it is reasonable to identify $\Theta_j^*$ with $\Theta_j$.

The residual part has the form (46) with $P_{ph} + P_f = P$, but parameters appearing in (46) change their values under the phonon influence. To obtain the phonon contribution into the dynamic parameters one should eliminate the phonon balance equation (30) from the conservation law (45), that yields

$$\nabla_i (n^i \mu_j - P_f \delta^i_j) - \left( \frac{c_p^2}{c_p^2 - 1} \right) Q^i \nabla_j V_i + \frac{w}{c_p} \nabla_j c_p = 0.$$  \hspace{1cm} (51)

One transforms this expression using the condition (8):

$$n_\parallel \nabla_j \left( \mu - \frac{w}{c_p} \frac{\partial c_p}{\partial n_\parallel} \right) - \nabla_j \left( P_f - \frac{wn_\parallel}{c_p} \frac{\partial c_p}{\partial n_\parallel} \right) + \left( \mu H^i - \frac{c_p^2}{c_p^2 - 1} Q^i \right) \theta_{ij} = 0.$$ \hspace{1cm} (52)

The most obvious way to provide the fulfillment of this relation is to equate the last term to zero separately. This means that vectors $H^i$ are proportional to $\sigma^i$:

$$\mu H^i = \left( \frac{c_p^2}{c_p^2 - 1} \right) T \sigma^i.$$\hspace{1cm} (53)

In the residual part the expressions in parenthesis should be identified as the chemical potential $\mu_0$ and the liquid pressure $P_0$ at zero temperature that are connected by $n_\parallel \delta \mu_0 = \delta P_0$.

Such an approach is not only possible but it allows the equation (52) to be carried out for an arbitrary state equation. Thus, at the temperature close to zero the chemical potential and the pressure of the superfluid are determined by the equations

$$\mu = \mu_0 + \frac{w}{c_p} \frac{\partial c_p}{\partial n_\parallel}, \hspace{1cm} P_f = P_0 + \frac{wn_\parallel}{c_p} \frac{\partial c_p}{\partial n_\parallel}.$$\hspace{1cm} (54)

These expressions coincide exactly with the nonrelativistic ones obtained for the phonon gas [17]. The additional term in the chemical potential has the same form as $\mu_1$ introduced in the Gibbs relation. This result could be predicted $a\ priori$ since the physical sense of this factor is a variation of an energy of the one liquid particle being due to the phonon contribution under changing a number of the particles.
Taking into account the relations (18), (35), (48), and (53), two flows, \( \mu^i \) and \( \Theta^i \) say, may be expressed in terms of \( n^i \) and \( S^i \). Comparing them with (2) one can obtain the expressions for the coefficients \( A, B, \) and \( C \):

\[
\begin{align*}
A &= -\mu T \frac{c^2/c_p^2 - 1}{\mu n_\parallel - s_\parallel T \left(c^2/c_p^2 - 1\right)}; \\
B &= \mu^2 \left\{ \mu n_\parallel - s_\parallel T \left(c^2/c_p^2 - 1\right) \right\}^{-1}, \\
C &= \frac{T c^2}{s_\parallel c_p^2} \left\{ \mu n_\parallel + s_\parallel T \left(1 - c_p^2/c^2\right) \right\}.
\end{align*}
\]

Since superfluid dynamics implies that \( A < 0 \) and \( B, C > 0 \), the denominator in (55) have to be positive. These coefficients were estimated in the work \cite{7} in which the phonon gas in superfluids is considered from the statistical standpoint. This result can be derived from (55) under the condition \( \mu n_\parallel \gg s_\parallel T (c^2/c_p^2) \).

V. SOUND IN SUPERFLUIDS AT LOW TEMPERATURE

A. Minkowski metrics

Let us consider sound propagation in superfluids being in a flat space-time at the temperature close to zero. In this temperature range a free path of the quasi-particle increases rapidly and equilibrium in the phonon gas has no time to be established. Under these conditions, the hydrodynamic equations for the phonon gas are inapplicable. Properties of this gas are described by the kinetic equation, in which the collision integral is negligibly small. Thus the phonon distribution function satisfies the Liouville equation (19) with the right-hand side being equal to zero. The Liouville equation must be supplemented by equations of motion of the medium where the phonons are situated. These equations are the conservation law of the particle number flow (36) and the irrotational condition for the superfluid velocity (9).

Solution will be sought in the form

\[
\begin{align*}
f &= f_0 + f_1 e^{ik_i x_i}, \\
n_\parallel &= n_0 + n_1 e^{ik_i x_i}, \\
V^j &= V_0^j + v^j e^{ik_i x_i},
\end{align*}
\]

where \( f_1, n_1, v^j \) are the small addition to the constant equilibrium values \( f_0, n_0, V_0^j \).
The equilibrium distribution function must turn the collision integral into zero. The solution of the equation \( J(x, p) = 0 \) for the phonon gas (that obeys the Bose-Einstein statistics) is well-known (see e.g. Ref. [27])

\[
    f_0 = \frac{1}{e^{\beta k} - 1},
\]

where \( \beta^k = \beta V_0^k, \beta = c(k_B T)^{-1} \).

After linearization, the following system of equations is derived from (19), (36) and (9)

\[
    f_{1k_i} \frac{\partial H}{\partial p_i} + \frac{\partial f_0}{\partial \epsilon} k_i \frac{\partial H}{\partial p_i} v^j + \frac{\partial f_0}{\partial \epsilon} k_i V_0^j \left\{ \left( \frac{c^2}{c_p^2} - 1 \right) \epsilon p_j v^j + \frac{\epsilon^2 \partial(c^2/c_p^2)}{2 \partial n_{\parallel}} n_1 \right\} = 0, \quad (58a)
\]

\[
    k_{\parallel} n_1 + n_0 k_j v^j + \frac{(c^2/c_p^2 - 1)c}{\mu} \int \epsilon k_j \pi^j f_1 d\mathcal{P} = 0, \quad (58b)
\]

\[
    \mu k_{\parallel} v_j = \frac{c^2}{c_p^2} \frac{\mu_0}{n_{\parallel}} \kappa_j n_1 + \frac{\partial}{\partial n_{\parallel}} \left( \frac{w}{c_p} \frac{\partial c_p}{\partial n_{\parallel}} \right) \kappa_j n_1 + \frac{c^2 \partial c_p}{c_p^3 \partial n_{\parallel}} \kappa_j \int \epsilon^2 f_1 d\mathcal{P}, \quad (58c)
\]

where \( k_{\parallel} = k_i V_0^i, \kappa_j = \Delta_j^i k_i \). In the equation (58b) we have taken into account the expression (53) and the fact that \( Q_i = 0 \) in equilibrium.

The equations (58c) allow to conclude that \( v_j \) is parallel to \( \kappa_j \) and only one equation remains by entering a new variable \( v \) so that \( v_j = v \kappa_j \).

Let us express \( f_1 \) from the equation (58a) and substitute it in the integrals in (58b) and (58c). It is convenient to perform the integration in the superfluids local rest frame, marked by the condition \( V^i = \delta_0^i \). In this frame

\[
    f_1 = \frac{\partial f_0}{\partial \epsilon} \frac{\epsilon^2}{p} \left\{ \left( k_{\parallel} - \frac{c}{c_p} k \cos \vartheta \right) \frac{c}{c_p} k v \cos \vartheta - \frac{k_{\parallel}}{c} \frac{\partial(c^2/c_p^2)}{\partial n_{\parallel}} \right\} \left\{ \frac{c}{c_p} k_{\parallel} - k \cos \vartheta \right\}^{-1}, \quad (59)
\]

where \( k = (\kappa_j \kappa^j)^{1/2}, p = (\pi_i \pi^i)^{1/2} \), and \( \vartheta \) is the angle between the vectors \( \kappa_j \) and \( \pi_j \). Having integrated over the angles we will deduce the system of equations

\[
    \left\{ \frac{c_p^2}{c^2 n_{\parallel}} + \frac{\partial}{\partial n_{\parallel}} \left( \frac{w}{c_p} \frac{\partial c_p}{\partial n_{\parallel}} \right) + \frac{k_{\parallel}}{c} \left( \frac{1}{c_p} \frac{\partial c_p}{\partial n_{\parallel}} \right)^2 I \right\} n_1 = \left\{ \mu k_{\parallel} - \frac{k_{\parallel}^2}{c k} \left( \frac{c^2}{c_p^2} - 1 \right) \frac{\partial c_p}{\partial n_{\parallel}} I \right\} v = 0, \quad (60a)
\]

\[
    \left\{ \mu k_{\parallel} - \frac{k_{\parallel}^2}{c k} \left( \frac{c^2}{c_p^2} - 1 \right) \frac{\partial c_p}{\partial n_{\parallel}} I \right\} n_1 = \left\{ \mu n_0 k^2 + \frac{k_{\parallel}^3}{c} \frac{c_p}{c^2} \left( \frac{c^2}{c_p^2} - 1 \right)^2 I \right\} v = 0, \quad (60b)
\]

where

\[
    I = -2 \pi c \ln \frac{c_k k_{\parallel} + c_p k}{c_k - c_p k} + \frac{p^4}{(2\pi \hbar)^3} \frac{\partial f_0}{\partial \epsilon} dp = \frac{\pi^2 (k_B T)^4 c^2}{15 (\hbar c_p)^3 c_p} \ln \frac{c_k k_{\parallel} + c_p k}{c_k - c_p k}. \quad (61)
\]
Only logarithmic terms are retained after integrating over the angle $s$. They are large in comparison with another terms because $c k \| \approx c_p k$ as we will see below.

Having equated the determinant of the system (60) to zero we will obtain the dispersion relation

$$\frac{k^2}{k^2 c_p^2} = 1 + \frac{I}{\mu n \|} \left\{ \frac{n \|}{c_p} \frac{\partial c_p}{\partial n \|} + 1 - \frac{c_p^2}{c^2} \right\}^2. \quad (62)$$

The relativistic addition to the sound speed

$$\frac{\delta c_p}{c_p} = \frac{I}{2 \mu n \|} \left\{ \frac{n \|}{c_p} \frac{\partial c_p}{\partial n \|} + 1 - \frac{c_p^2}{c^2} \right\}^2 \quad (63)$$

differs from the nonrelativistic one

$$\delta c_p = \frac{\mu n \|}{30 \hbar^2 \rho} \left( \frac{k_B T}{c_p} \right)^4 \left\{ \frac{\rho}{c_p} \frac{\partial c_p}{\partial \rho} + 1 \right\}^2 \ln \frac{\omega + k c_p}{\omega - k c_p}. \quad (64)$$

by the ratio of the light speed square to the sound speed one in the braces and the production $\mu n \|$ instead of nonrelativistic mass density $\rho$.

The relativistic and nonrelativistic chemical potentials are related by $\mu = mc^2 + \mu_{NR}$ (see e.g. Ref. 28). Thus the classical formula (64) takes into account only rest energy of the superfluids particles whereas (63) includes also thermodynamic contribution in energy.

To reduce (63) to (64) it is necessary that the conditions $c_p \ll c$ and $\mu_{NR} \ll mc^2$ are performed.

To compare (63) and (64) in ultrarelativistic limit is incorrect in general, since the latter expression is pure nonrelativistic. However it may be useful to show how the sound speed is changed when nonrelativistic matter transforms to ultrarelativistic one. For the latter the sound speed $c_p = c/\sqrt{3}$ and

$$\frac{(\delta c_p)_{UR}}{(\delta c_p)_{NR}} = \frac{4 \rho c^2}{9 \mu n \|.} \quad (65)$$

This ratio demonstrates decrease of sound speed addition when relativistic effects become essential.

B. Robertson-Walker metrics

In this section we consider the expanding universe with Robertson-Walker metrics that comprises, at least in part, superfluid matter. Similar models are regarded earlier for a Bose-Einstein condensate whose properties intimately related to the superfluidity and
superconductivity. In particular it is found that the coherence length of the cosmic Bose-Einstein condensate is equal to the Jeans wavelength. Thereby the condensate can form gravitationally stable structures. If a superfluid state is realized in the cosmic matter, then the question naturally arises as to whether the excitations have an influence on this scale. Since the Jeans scale varies directly as the speed of sound, evaluation of $\delta c_p$ may be useful to estimate the extent of this influence.

For the present moment we leave aside the questions of the universe evolution under the superfluids influence and here we study behavior of the sound speed near equilibrium.

Before to study the weakly non-equilibrium state of the phonon gas it is necessary to examine a solution of the equations (19), (36) and (9) at equilibrium.

The phonon gas is supposed to be at rest with respect to the superfluids frame and the distribution function depends on only one variable

$$f_0(x, p) = f_0(p_k \beta^k), \quad (66)$$

and has the form (57) turning the collision integral into zero. For the Robertson-Walker metrics with the scale factor $a$ the dispersion relation (7) takes the form

$$\varepsilon = \frac{c_p p}{c \ a}. \quad (67)$$

The kinetic equation (22) leads to a law connecting the variations of the temperature and the sound speed in time

$$\frac{\dot{\beta}}{\beta} - \frac{\dot{a}}{a} + \frac{\dot{c}_p}{c_p} = 0 \quad \Rightarrow \quad \frac{\beta c_p}{a} = \text{const.} \quad (67)$$

The equations of motion (36) and (9) of the liquid take the form

$$\Delta^k \nabla_k \mu = \mu D V_j = 0, \quad (68a)$$

$$D n_0 + n_0 \theta = D n_0 + \frac{\dot{a}}{a^3} n_0 = 0, \quad (68b)$$

where the dot denotes the time derivative. It follows from the equations (68) that the chemical potential is an arbitrary function of time and the equilibrium particle number density of the liquid varies according the law $n_0 a^3 = \text{const.}$

Let us look for the weakly non-equilibrium solution in the form (56). However, unlike the flat metrics case, both the equilibrium values $f_0$, $n_0$, $V_0^j$ and the small additions $f_1$, $n_1$, $v^i$
depend on time. These quantities vary slower than the exponent power, i.e.
\[ \frac{\partial f_1}{\partial x^i} \ll k_i f_1, \]
and so on.

Having repeated the procedures of the previous section, we will obtain that
\[ f_1 = \beta a f_0 \frac{\varepsilon^2}{p} \left\{ \left( k_{||} - \frac{c}{a c_p} k \cos \vartheta \right) \frac{c}{a c_p} k v \cos \vartheta - \frac{k_{||}}{2} \frac{\partial (c^2/c_p^2)}{\partial n_{||}} \right\} \left\{ \frac{c}{c_p} k_{||} - \frac{k}{a} \cos \vartheta \right\}^{-1}, \quad (69) \]
and integration over the angles gives the following result
\[ I = -\frac{2\pi \beta}{a^5} \ln \frac{ack_{||} + c_p k}{ack_{||} - c_p k} \int \frac{p^4}{(2\pi T)^3} f'_0 \, dp \]
\[ = \frac{\pi^2}{15} \frac{(k_B T)^4 c^2}{(h c_p)^3} \ln \frac{ack_{||} + c_p k}{ack_{||} - c_p k}, \]
where the prime denotes the derivative with respect to the argument. To write down the dispersion relation for the Robertson-Walker metrics, one should add \( a^2 \) to the numerator of the right hand side of the expression (62).

In view of (67) and (68b) \( \delta c_p \propto 1/\mu a \) (logarithmic dependence is neglected, since it related with dispersion of phonons). Further consideration is caused by behavior of the chemical potential, that can not be determined without additional assumptions. As possible models one can consider the simple equations of state that allows to examine the evolution of \( \delta c_p \).

(i) Linear barotropic equation of state takes the form \( P_0 = (c_p^2/c^2) \mu_0 n_{||} \) for superfluid matter at zero temperature where \( c_p = \text{const.} \) Integrating the relation (4) one obtains \( \mu_0 \propto n_{||} c_p^2/c^2 \) and therefore \( \delta c_p \propto a^{3c_p^2/c^2 - 1} \).

In the ultrarelativistic limiting case the addition to the sound speed is independent from the scale factor \( a \). Thus the sound speed is constant but does not coincide with one at zero temperature. For the nonrelativistic limiting case \( \delta c_p \propto a^{-1} \) and universe inflationary implies decay of the phonon contribution into the sound speed.

(ii) Polytropic equation of state \( P_0 = K n_{||}^\gamma \) (\( \gamma > 1 \) is supposed in judgements) in combination with the relations (4) permits to obtain the expressions:
\[ \frac{c_p^2}{c^2} = \frac{K \gamma n_{||}^{-1}}{mc^2 + K \gamma n_{||}^{-1}/(\gamma - 1)}, \]
\[ \mu_0 = mc^2 + \frac{\gamma}{\gamma - 1} K n_{||}^{-1}, \]
which is similar to the expressions for nonrelativistic polytropes \[30\] because of their temperature independence.

In the nonrelativistic area \((mc^2 \gg Kn^{\gamma-1})\) the sound speed \(c_p \propto a^{-3(\gamma-1)/2}\) and \(\delta c_p \propto a^{-1}\). One can see that for \(\gamma > 5/3\) the sound speed decreases slower then the phonon addition. In general, the same behavior also takes place in the relativistic area \((mc^2 \sim Kn^{\gamma-1})\) especially for the large values of \(a\). When \(mc^2 \ll Kn^{\gamma-1}\) and the scale factor is not so large, then \(c_p = \text{const}\) and \(\delta c_p \propto a^{3\gamma-4}\). The value \(\gamma = 4/3\) leads us again to the ultrarelativistic limits described in (i).

VI. CONCLUSION

In this paper we introduced the principle of constructing macroscopic flows of the phonon gas in superfluids using the phonon distribution function. These definitions can be applied to arbitrary quasi-particles without any changes. Nevertheless we restrict ourselves by consideration of the phonon gas. In this case there is no dependence on the quasi-particle energy in \(g^{ij}\) so this tensor can be used as an effective second metrics.

This allowed us to apply the well developed technique to derive the macroscopic balance equations for the phonon gas, which is the part of the hydrodynamic conservation laws. Based on this statement the phonon contribution in superfluid dynamics was obtained.

It is worth noting that using the expression (16b) instead of (11b), one can draw an analogy with the works \[31, 32, 33\] (see also references therein) where a distinction between momentum and pseudomomentum is discussed. This problem is considered for dielectric media and electromagnetic waves in \[31, 32\] and for acoustic waves in \[33\]. In these papers the mentioned problem is studied for an individual optical or acoustic vibrations. In contrast to such approach, the tensors (11b) and (16b) do not contain parameters of individual phonons but describe the quasi-particle ensemble. Therefore, it is reasonable to consider concepts of momentum and pseudomomentum in order to compare their applicability to description of macroscopic objects.

Obtained in this paper kinetic equation was solved directly to determine the addition to the speed of sound at temperature close to zero.

The result for the flat space-time is the relativistic generalization of the classical Khalatnikov’s one \[15\], whereas results from Sec. V.B should be reviewed from the other standpoint.
They are certainly preliminary and estimating ones, however they demonstrate that including phonons into cosmological models can be used to fit necessary parameters properly.

The present research was carried out for the low temperature quasi-particle gas, when purely phonon processes dominate, whereas contributions of quasi-particles corresponding to another parts of the spectrum are neglected. To develop the similar theory for another sorts of quasi-particles, such as rotons, special techniques are needed because of the nonlinear dependence between the energy and momentum of a quasi-particle.

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