Four Puzzles of Neutrino Mixing

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Abstract

There are four puzzling questions about by the magnitudes of neutrino mixings and mass splittings. A brief sketch is given of the various kinds of models of neutrino masses and how they answer these questions. Special attention is given to so-called “lopsided” models.
1 Comparison of Quarks and Leptons

Over the last three decades theorists have been trying to understand the spectrum of quark and lepton masses. Although no simple model of the many that have been proposed is uniquely compelling, there are certain basic ideas that seem rather probable and are incorporated in most published models. One of these ideas is that there is a direct relation between the mass ratios and the mixing angles of the quarks. Since the charged leptons exhibit a mass hierarchy very similar to those of the quarks, it was widely expected that the lepton mixing angles would also be like those of the quarks. The discovery that the atmospheric neutrino mixing angle $\theta_{\text{atm}}$ is nearly maximal thus came as a surprise.

In this talk I first review the basic facts about quark masses and mixings and then discuss several features of neutrino mixing that seem at first sight puzzling in light of these facts. I will then show how various types of models explain these puzzling features.

There are two quark mass matrices, $M_U$ and $M_D$, for the up-type quarks ($u, c, t$) and down-type quarks ($d, s, b$) respectively. These are diagonalized by unitary transformations: $V_U^\dagger M_U U_U = \text{diag}(m_u, m_c, m_t)$ and $V_D^\dagger M_D U_D = \text{diag}(m_d, m_s, m_b)$. The mismatch between the unitary transformations of the left-handed quarks gives rise to the CKM matrix, $V_{CKM} = U_U^\dagger U_D$. The CKM angles are $|V_{us}| = \sin \theta_{12} \simeq 0.2$, $|V_{cb}| = \sin \theta_{23} \simeq 0.04$, and $|V_{ub}| = \sin \theta_{13} \simeq 0.003$. The smallness of these angles is presumably due to small ratios of elements in $M_U$ and $M_D$, and is therefore presumably directly related to the smallness of the mass ratios $m_c/m_t$, $m_u/m_c$, $m_s/m_b$, and $m_d/m_s$.

How mass ratios and mixing angles might be directly related can be seen easily from a $2 \times 2$ example [1]. Consider the matrix

$$M = \begin{pmatrix} 0 & \epsilon \\ \epsilon & 1 \end{pmatrix} m.$$  \hspace{1cm} (1)

This is diagonalized by $R(\theta)^T M R(\theta)$, where $R(\theta)$ is the $2 \times 2$ rotation matrix with
\[ \tan 2\theta = 2\epsilon, \text{ or, for small } \epsilon, \theta \approx \epsilon. \] 
The large eigenvalue of \( M \) is obviously \( m_2 \approx m \), while the fact that \( \det M = -\epsilon^2 m \), tells us that other eigenvalue is \( m_1 \approx -\epsilon^2 m \). Consequently, one has that \( \theta \approx \sqrt{|m_1/m_2|} \). This can be compared to the old and famously successful relation for the Cabibbo angle \( \tan \theta_c \approx \sqrt{m_s/m_d} \).

One should note that the matrix in this example is “hierarchical”, by which we mean that the entries get smaller upward and to the left of any diagonal entry. Most realistic models of quark masses and mixings assume such hierarchical mass matrices. For example, a recent model of Babu and Nandi \( \mathbb{[2]} \), which fits the data extremely well, has quark matrices of the form

\[
M_U \sim \begin{pmatrix} \epsilon^6 & \epsilon^4 & \epsilon^4 \\ \epsilon^4 & \epsilon^2 & \epsilon^2 \\ \epsilon^4 & \epsilon^2 & 1 \end{pmatrix} m, \quad M_D \sim \begin{pmatrix} \epsilon^6 & \epsilon^6 & \epsilon^6 \\ \epsilon^6 & \epsilon^4 & \epsilon^4 \\ \epsilon^6 & \epsilon^4 & \epsilon^2 \end{pmatrix} m, \quad (2)
\]

Now let us consider the leptons. Here again there are two mass matrices, \( M_L \) for the charged leptons (\( e^- \), \( \mu^- \), \( \tau^- \)) and \( M_\nu \) for the neutrinos (\( \nu_e \), \( \nu_\mu \), \( \nu_\tau \)). The matrix \( M_\nu \) is different in two respects from \( M_U \), \( M_D \), and \( M_L \): it has much smaller entries, and it is symmetric, since it is a Majorana matrix connecting left-handed neutrinos to left-handed neutrinos. Nevertheless, as with the quark mass matrices, the lepton mass matrices are diagonalized by unitary transformations that can have a mismatch. That mismatch gives rise to the neutrino mixing matrix sometimes referred to as the MNS matrix \( \mathbb{[3]} \), \( U_{MNS} = U_L^\dagger U_\nu \). In \( (U_{MNS})_{fm} \), \( f = e, \mu, \tau \) and \( m = 1, 2, 3 \). Experimentally one has that \( |U_{\mu 3}| (\equiv \sin \theta_{atm} = \sin \theta_{\ell 23}^f) \approx 0.7, \) \( |U_{e 2}| (\equiv \sin \theta_{sol} = \sin \theta_{\ell 12}^f) = O(1) \) (probably), and \( |U_{e 3}| (\equiv \sin \theta_{\ell 13}^f) \leq 0.15 \). There is still a great deal of uncertainty about the solar mixing angle, but the solution with small \( \theta_{sol} \) (the “SMA” or Small Mixing Angle MSW solution) is disfavored by recent global fits to the data \( \mathbb{[4]} \). The best fits are to the “LMA” or Large Mixing Angle MSW solution and the “LOW” solution. The best-fit value for the LMA solution is \( \tan^2 \theta_{sol} \approx 0.4 \).

The mass splittings needed to fit the atmospheric and solar data are \( \delta m_{atm}^2 = m_3^2 - m_2^2 \approx 3 \times 10^{-3} \text{ eV}^2 \), and \( \delta m_{sol}^2 = m_2^2 - m_1^2 \approx 5 \times 10^{-5} \text{ eV}^2 \) (for LMA, smaller for other solar solutions). The fact that \( \delta m_{sol}^2 \ll \delta m_{atm}^2 \) suggests there is probably
a family hierarchy of neutrino masses, although it also possible that the neutrino masses are nearly degenerate and that only their splittings have a hierarchy.

2 Three Puzzles

In the basic facts about neutrino masses and mixings there are three features that appear puzzling in light of the conventional wisdom about quark masses and mixings.

Puzzle 1: Why are some $\theta^\ell \sim 1$ whereas all $\theta^q \ll 1$? In grand unified theories the quarks and leptons are related, and one expects similar mass ratios and mixing angles for them. In models with flavor symmetry the same flavor symmetries generally control the quark and lepton mass matrices and give them similar structure. Empirically, one indeed sees that the charged leptons have a mass hierarchy qualitatively similar to those of the up-type and down-type quarks. Another similarity is that the 13 mixing angle is by far the smallest in both cases ($|V_{ub}| \ll |V_{us}|, |V_{cb}|$ and $|U_{e3}| \ll |U_{e2}|, |U_{\mu 3}|$).

In light of the expected and actual similarities of quarks and leptons it appears strange that at least one and probably two of the leptonic angles are large, while all the quark angles are very small.

Puzzle 2: How can there be small lepton mass ratios but large leptonic mixing angles? As we have seen, for the quarks the smallness of the mixing angles and mass ratios are generally thought to be related. For the charged leptons the mass ratios are certainly small, and for the neutrinos at least the ratios of mass splittings are small, and yet the leptons are very strongly mixed.

Puzzle 3: Why are two leptonic angles large but the third ($\theta^\ell_{13}$) small? If all the leptonic angles were of order unity it might suggest that all the entries of the neutrino mass matrix $M_\nu$ were of the same order, as would typically be the case if it were a “random” matrix, as has indeed been suggested [5]. However, such a matrix would not generally give a hierarchy of neutrino mass splittings, nor would it generally yield a 13 mixing angle much smaller than the others. The smallness of $\theta^\ell_{13}$ and largeness of
the other leptonic angles suggests that the leptonic mass matrices have quite special forms. To see what those forms might be let us consider the following product of rotation matrices:

\[
R_{23}(\theta_{\text{atm}})R_{12}(\theta_{\text{sol}}) = \begin{pmatrix}
1 & 0 & 0 \\
0 & c_{\text{atm}} & s_{\text{atm}} \\
0 & -s_{\text{atm}} & c_{\text{atm}}
\end{pmatrix}
\begin{pmatrix}
c_{\text{sol}} & s_{\text{sol}} & 0 \\
-s_{\text{sol}} & c_{\text{sol}} & 0 \\
0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
c_{\text{sol}}s_{\text{atm}}c_{\text{sol}} & c_{\text{atm}}s_{\text{sol}} & s_{\text{atm}}s_{\text{sol}} \\
-c_{\text{atm}}s_{\text{sol}} & c_{\text{atm}}c_{\text{sol}} & -s_{\text{atm}}c_{\text{sol}} \\
s_{\text{atm}}s_{\text{sol}} & s_{\text{atm}}c_{\text{sol}} & c_{\text{atm}}
\end{pmatrix}
\]

One sees that even if \( \theta_{\text{atm}} \) and \( \theta_{\text{sol}} \) are both large this matrix has the property that the 13 element vanishes. Thus Puzzle 3 is resolved if one has that \( U_{MNS} \equiv U_L^\dagger U_\nu \cong R_{23}(\theta_{\text{atm}})R_{12}(\theta_{\text{sol}}) \). There are three simple possibilities:

**Solution A:** \( U_L \cong I, U_\nu \cong R_{23}(\theta_{\text{atm}})R_{12}(\theta_{\text{sol}}) \).

Both large mixing angles come from \( M_\nu \), whose diagonalization involves first a large 23 rotation and then a large 12 rotation.

**Solution B:** \( U_L \cong R_{12}(\theta_{\text{sol}})R_{23}(\theta_{\text{atm}}), U_\nu \cong I \).

Both large mixing angles come from \( M_L \), whose diagonalization involves first a large 12 rotation and then a large 23 rotation.

**Solution C:** \( U_L \cong R_{23}(\theta_{\text{atm}}), U_\nu \cong R_{12}(\theta_{\text{sol}}) \).

The large atmospheric angle comes from \( M_L \), and the large solar angle comes from \( M_\nu \).

### 3 How Non-see-saw Models Resolve the Puzzles

Let us first recall how the see-saw mechanism works. The up quarks, down quarks, and charged leptons all have Dirac masses through the Higgs doublet field (or fields) coupling the left-handed fermions to their right-handed partners. If there are right-handed neutrinos, then an analogous Dirac mass matrix \( M_\nu^{\text{Dirac}} \) can exist for the neutrinos as well. However, there can also exist a Majorana mass matrix \( M_R \) connecting the right-handed neutrinos to themselves. These right-handed Majorana masses can
be superlarge as they do not break the gauge symmetries of the Standard Model. Integrating out the superheavy right-handed neutrinos leaves behind light left-handed neutrinos with an effective Majorana mass matrix given by the “see-saw” formula $M_\nu = -M_\nu^{\text{Dirac}} T M_R^{-1} M_\nu^{\text{Dirac}}$. In see-saw models, then, the neutrino masses have fundamentally the same origin as the charged lepton and quark masses, namely they come from the existence of both left- and right-handed components coupled together by the doublet Higgs field (or fields).

In non-see-saw models there are no right-handed neutrinos. The masses of the neutrinos therefore have to arise in some other, completely new way not directly related to mass generation for the charged leptons and quarks. Many such mechanisms have been proposed. Three popular ones are the Zee mechanism, R-parity violation in SUSY models, and triplet Higgs.

In the Zee mechanism, there exists a singly charged, singlet scalar field $h^+$, which can couple to a pair of lepton doublets $(h^+ L_i L_j)$ and to a pair of Higgs doublets $(h^+ H_\alpha H_\beta)$. Obviously, with both types of couplings, there is no way consistently to assign lepton number to $h^+$. Whether one assigns it $L = -2$ or $L = 0$, one of its couplings will violate lepton number by two units, which is what is needed to generate Majorana masses for the left-handed neutrinos. Such masses arise at one-loop.

In theories with low-energy supersymmetry, the neutrinos can acquire mass by coupling to a neutralino that plays the role of right-handed neutrino. The scalar that couples the neutrino to the neutralino is the sneutrino, which is able to obtain a non-zero vacuum expectation value if R-parity is violated. R-parity violation also allows superpotential terms of the type $LQD^c$ and $LLE^c$, which give one-loop neutrino masses when the sleptons and squarks are integrated out.

Finally, if there is a triplet higgs field $T$ with Standard Model quantum numbers $(1, 3, +1)$, then it can have a renormalizable coupling to a pair of lepton doublets $(TL_i L_j)$ that gives a tree-level neutrino mass.

The great advantage of such non-see-saw mechanisms is that they automatically
provide a very plausible answer to Puzzle 1: the lepton mixing angles differ so dramatically from the quark mixing angles simply because \( M_\nu \) has a very different origin than \( M_U \) and \( M_D \). We will now look at specific non-see-saw ideas to see how they resolve the other Puzzles.

**Inverted Hierarchy Models.** In inverted hierarchy models the neutrino mass matrix has approximately the following form:

\[
M_\nu \approx \begin{pmatrix}
0 & A & B \\
A & 0 & 0 \\
B & 0 & 0
\end{pmatrix},
\]

with \( A \sim B \). This can arise in various ways. In the Zee model the one-loop mass matrix is symmetric with vanishing diagonal elements. If for some reason the 23 (32) elements are smaller than the others, the form in Eq. (4) results. It can also result from an approximate \( L_e - L_\mu - L_\tau \) symmetry.

One can diagonalize the large elements \( A \) and \( B \) in Eq. (4) by two successive large rotations. First, one can rotate in the “23 plane” by angle \( \theta_{23} \approx \tan^{-1}(B/A) = O(1) \) to eliminate the 13 and 31 elements. Then one can rotate in the “12 plane” by \( \theta_{12} \approx \pi/4 \) to eliminate the 12 and 21 elements:

\[
\begin{pmatrix}
0 & A & B \\
A & 0 & 0 \\
B & 0 & 0
\end{pmatrix} \theta_{23} \begin{pmatrix}
0 & \sqrt{A^2 + B^2} & 0 \\
\sqrt{A^2 + B^2} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \theta_{12} \begin{pmatrix}
\sqrt{A^2 + B^2} & 0 & 0 \\
0 & -\sqrt{A^2 + B^2} & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

Note that this sequence of large rotations is precisely Solution A of Puzzle 3. Even though the hierarchy of neutrino masses is inverted here, in the sense that \( m_3 \) is the smallest, the near degeneracy of \( |m_1| \) and \( |m_2| \) gives the correct hierarchy of splittings, \( \delta m^2_{sol} \ll \delta m^2_{atm} \), thus resolving Puzzle 2.

**Factorized Mass Matrix Models.** In some models \( M_\nu \) has approximately the form

\[
M_\nu \approx \begin{pmatrix}
0 & c^2 & d^2 \\
c^2 & B^2 & AB \\
d^2 & AB & A^2
\end{pmatrix},
\]

where \( c, d \ll A \sim B \). One can see that in a sense this form is the opposite of the inverted hierarchy form. (In fact, if \( M_\nu \) has this form, then \( M^{-1}_\nu \) has the inverted
hierarchy form.) We call this form factorized, because if $m_{ij}$ is the 23 block of $M_\nu$, then $m_{ij} \cong a_i a_j$. This form can arise if the dominant contribution to $M_\nu$ comes from integrating out a single heavy fermion $N$ that has Dirac mass $m_i(\nu_i N)$ with the left-handed neutrinos $\nu_2$ and $\nu_3$ (its coupling to $\nu_1$ should be smaller). A notable instance of this occurs in the SUSY models with R-parity violation, where $N$ is the neutralino.

A rotation in the 23 plane by $\theta_{23} = \tan^{-1}(B/A) = O(1)$ eliminates the 23, 32, and 22 elements in Eq. (6). That leaves a matrix whose 12 block has a “pseudo-Dirac” form, with the 12 and 21 elements being larger than the 11 and 22 elements. This block can be diagonalized with a rotation in the 12 plane by $\theta_{12} \cong \pi/4$. The resulting matrix can be diagonalized with only small further rotations. Thus, as in the inverted hierarchy models, one has just the sequence of large rotations corresponding to Solution A of Puzzle 3. The hierarchy of neutrino masses is the “normal” one with $m_1, m_2 \ll m_3$, giving the correct hierarchy of splittings and resolving Puzzle 2.

**Flavor Democracy Models.** A third kind of model, about which many papers have been written [8], assumes that the $M_L$, $M_U$, and $M_D$ all have approximately the “democratic” form in which all the entries are equal. The matrix $M_\nu$, is assumed to be approximately diagonal. The CKM angles thus end up being small due to cancellation, whereas the large leptonic angles that come from diagonalizing $M_L$ do not get cancelled. The democratic form can be diagonalized by the sequence of rotations

$$
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{pmatrix}
\theta_{12}
\begin{pmatrix}
0 & 0 & 0 \\
0 & 2 & \sqrt{2} \\
0 & \sqrt{2} & 1 \\
\end{pmatrix}
\theta_{23}
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 3 \\
\end{pmatrix},
$$

where $\theta_{12} = \tan^{-1} 1 = \pi/4$ and $\theta_{23} = \tan^{-1}\sqrt{2}$. This sequence of large rotations is just that of Solution B of Puzzle 3. The flavor democracy models also give the “normal” neutrino mass hierarchy, resolving Puzzle 2.

In all three kinds of non-see-saw model we have discussed we see that $\theta_{sol} \cong \pi/4$ (maximal mixing), whereas $\theta_{atm}$ is only predicted to be large, but not nearly maximal (though it may be by accident). Curiously, the empirical situation is just the reverse. It is $\theta_{atm}$ that is observed to be close to maximal. (The best-fit value is $\sin^2 2\theta_{atm} =$}
This is our fourth puzzle:

**Puzzle 4:** Why is $\theta_{\text{atm}}$ so close to maximal? It is not an accident that many models predict $\theta_{\text{sol}}$ to be nearly maximal while very few models exist where $\theta_{\text{atm}}$ is. The reason is essentially the following. The simplest way to arrange that a mixing angle is nearly maximal is to assume that the relevant $2 \times 2$ block of the $3 \times 3$ mass matrix is pseudo-Dirac. The diagonalization of such a matrix leads to nearly degenerate masses, which is to say, very small $\delta m^2$. For example, suppose one considers the matrix

$$M_\nu \cong \begin{pmatrix} 0 & m & 0 \\ m & 0 & 0 \\ 0 & 0 & m' \end{pmatrix}. \quad (8)$$

This gives $\theta_{\text{sol}} \cong \pi/4$, $m_1 \cong m_2 \cong m$, and $m_3 \cong m'$. Thus one has $\delta m_{\text{sol}}^2 \ll \delta m_{\text{atm}}^2$ as desired. Now consider the matrix

$$M_\nu \cong \begin{pmatrix} m' & 0 & 0 \\ 0 & 0 & m \\ 0 & m & 0 \end{pmatrix}. \quad (9)$$

This gives $\theta_{\text{atm}} \cong \pi/4$, but now $m_3 \cong m_2 \cong m$ and $m_1 \cong m'$, so that $\delta m_{\text{atm}}^2 \ll \delta m_{\text{sol}}^2$. Thus, models with nearly maximal $\theta_{\text{sol}}$ tend to give $\delta m_{\text{sol}}^2 \ll \delta m_{\text{atm}}^2$ as observed, whereas with nearly maximal $\theta_{\text{atm}}$ will tend to give the wrong result $\delta m_{\text{atm}}^2 \ll \delta m_{\text{sol}}^2$.

### 4 See-saw Models

See-saw models have three great advantages over non-see-saw models. First, they do not have to invent an exotic mechanism for generating neutrino mass. There is nothing exotic about right-handed neutrinos, which indeed have to exist in most kinds of gauge-unified models ($SU(5)$ being an exception). Indeed, grand unification, which is well motivated on other grounds, naturally leads to see-saw neutrino masses. Second, see-saw/GUT models beautifully explain the scale of neutrino mass. Writing the heaviest neutrino mass as $m_3 = m_1^2/M_R$ (GUTs typically relate the neutrino Dirac masses to the up-type quark masses), and taking $m_3 \cong \sqrt{m_3^2 - m_2^2} = \sqrt{\delta m_{\text{atm}}^2} \cong 0.06$
eV, one finds, $M_R \sim 10^{15}$ GeV, which is very close to the GUT scale known from running of the gauge couplings. By contrast, in non-see-saw models the neutrino mass scale depends on many parameters about which virtually nothing is known even as to their order of magnitude. Third, see-saw/GUT models tend to be far more predictive than most non-see-saw schemes.

At first glance, Puzzle 1 seems especially puzzling in the context of see-saw/GUT models, since grand unification closely relates quarks and leptons. And it is certainly true that historically the great majority of GUT models predicted leptonic mixing angles of the same order as the CKM angles. Looking more closely, however, we see that this need not be the case. Indeed, there is a beautiful way to resolve Puzzle 1 in the see-saw/GUT framework.

All grand unified gauge groups contain $SU(5)$ as a subgroup, and $SU(5)$ relates down-type quarks to charged leptons having the opposite chirality. The $\bar{5}$ contains $\ell_L^-$ and the charge conjugate of $d_R$, while the $10$ contains $d_L$ and the charge conjugate of $\ell_R^-$. As a consequence, what is related by $SU(5)$ is $\theta_{d_L} \longleftrightarrow \theta_{\ell_R}$ and $\theta_{d_R} \longleftrightarrow \theta_{\ell_L}$. Since the CKM angles are the mixings of left-handed quarks, and the MNS angles are the mixings of left-handed leptons, $SU(5)$ does not relate them to each other. Rather, it relates the CKM angles to some unobserved mixing of right-handed leptons, and the MNS angles to some unobserved mixing of right-handed quarks. Consequently, it is perfectly possible for the CKM angles to be small and the “corresponding” MNS angles large if the mass matrices $M_L$ and $M_D$ are highly left-right asymmetric or “lopsided” \[9\]. We can see this in a simple toy model.

Consider an $SU(5)$ model with mass terms for the second and third family given by $\lambda(10_3 \bar{5}_3 + \sigma 10_3 \bar{5}_2 + \epsilon 10_2 \bar{5}_3)(\bar{5}_H)$, with $\epsilon \ll \sigma \sim 1$. The mass matrices $M_L$ and $M_D$ appear in the terms $\bar{\ell}_R_i(\bar{M}_L)_{ij}\ell_L^j + \bar{d}_R_i(\bar{M}_D)_{ij}d_L^j$, so they have the form

$$M_L = \begin{pmatrix} - & - & - \\ - & 0 & \epsilon \\ - & \sigma & 1 \end{pmatrix} m, \quad M_D = \begin{pmatrix} - & - & - \\ - & 0 & \sigma \\ - & \epsilon & 1 \end{pmatrix} m, \quad (10)$$

where the dashes are small entries for the first family coming from other terms.
Note the left-right transposition between $M_L$ and $M_D$, whose origin we have already explained. The 32 entry in $M_L$ is $\sigma$, which gives an $O(1)$ contribution to the MNS angle $U_{\mu 3}$. The 32 entry in $M_D$, on the other hand, is small and gives only a small contribution to the CKM angle $V_{ub}$. (The 23 entries control mixings of right-handed fermions.)

There is an interesting feature of the quark and lepton mixings that is explained very elegantly by such lopsided “textures” as in Eq. (10). Many models are based on symmetric “textures” that are extensions of the $2 \times 2$ matrix shown in Eq. (1). As we saw, such textures tend to relate the mixing angles to the square-roots of mass ratios. A typical prediction for the quarks is

$$V_{cb} = \sqrt{|m_s/m_b|} - e^{i\phi} \sqrt{|m_c/m_t|}. \quad (11)$$

The first term on the right-hand side is about 0.14, and the second about 0.05, whereas $V_{cb} \approx 0.04$, so that the prediction for $V_{cb}$ of such symmetric-texture models tends to be about a factor of 2 or 3 too large. The analogous relation for leptons is

$$U_{\mu 3} = \sqrt{|m_\mu/m_\tau|} - e^{i\phi'} \sqrt{|m_2/m_3|}. \quad (12)$$

Here the first term on the right-hand side is 0.24, and the second less than about 0.1 (assuming hierarchical neutrino masses, so that $m_3 \approx \sqrt{\delta m^2_{atm}}$ and $m_2 \approx \sqrt{\delta m^2_{sol}}$), whereas $U_{\mu 3} \approx 0.7$. Thus the prediction for $U_{\mu 3}$ in such symmetric-texture models tends to be about a factor of 2 or 3 too small. That symmetric textures give $V_{cb}$ too large and $U_{\mu 3}$ too small by about the same factor is readily explained by the assumption that the textures are in reality not symmetric but lopsided. We see from Eq. (10) that for the lopsided textures the mass ratios of second family to third family fermions are of order $\sigma \epsilon$. (The third eigenvalue is $\approx m$ while the product of the second and third eigenvalues is just the determinant of the 23 block or $-\sigma \epsilon m^2$.) Since $V_{cb} \sim \epsilon$ and $U_{\mu 3} \sim \sigma$, one expects $V_{cb} \sim \sqrt{\epsilon/\sigma} \sqrt{m_s/m_b}$ and $U_{\mu 3} \sim \sqrt{\sigma/\epsilon} \sqrt{m_\mu/m_\tau}$. This is just what is observed if $\sqrt{\sigma/\epsilon} \sim 2$ or 3. In other words, in lopsided models the smallness
of $V_{cb}$ and the largeness of $U_{\mu 3}$ are seen to be two sides of the same coin. (In realistic lopsided models the textures are similar in form but not exactly the same as Eq. (10); however, the qualitative argument just given still applies.)

I said that see-saw/GUT models are in general more predictive than non-see-saw models. And, indeed, simple and highly predictive $SO(10)$ models that are very similar (for the second and third families) to the toy model just described have been constructed [10]. In fact, many models based on lopsided mass matrices now exist in the literature [6].

Note the very important point that in such lopsided models the large atmospheric neutrino angle comes from the charged lepton mass matrix $M_L$ rather than from $M_\nu$. This shows how such models resolve Puzzle 2. In lopsided models the reason why some of the neutrino mixing angles can be large even though all the neutrino mass ratios are small is that large neutrino mixing angles can be caused by large off-diagonal elements in $M_L$ (here $\sigma$) whereas the neutrino mass ratios obviously are determined entirely by $M_\nu$.

How can lopsided models resolve Puzzle 3? There are two interesting and simple possibilities. One possibility is that the large $\theta_{atm}$ arises from $M_L$ as just described, but that the large $\theta_{sol}$ arises from $M_\nu$. This corresponds to Solution C. Such models are very easy to construct [11]. The other possibility is that both of the large angles $\theta_{atm}$ and $\theta_{sol}$ come from lopsidedness in $M_L$ [12]. Consider the following matrix

$$M_L = \begin{pmatrix} - & - & \epsilon \\ - & - & \rho' \\ \rho' & \rho & 1 \end{pmatrix} m, \quad (13)$$

where $\epsilon \ll \rho' \sim \rho \sim 1$ and the dashes represent elements yet smaller than $\epsilon$. A rotation in the 12 plane by $\theta_{atm} \equiv \tan^{-1}(\rho'/\rho)$ brings the matrix to the form shown in Eq. (8) with $\sigma = \sqrt{\rho^2 + \rho'^2}$. Then a rotation in the 23 plane by angle $\theta_{23} \equiv \tan^{-1} \sigma$, as in the toy model, eliminates the large 32 element. The further rotations required to diagonalize $M_L$ are small. This sequence of large rotations in the charged lepton sector gives just Solution B of Puzzle 3. (It should be noted that, as in Eq. (10), the
matrix $M_D$ will have the large elements appearing transposed compared to $M_L$, so that they only affect mixings of right-handed quarks.)

Very few models in the literature attempt to explain why $\theta_{\text{atm}}$ is nearly maximal (Puzzle 4). It turns out that within the framework of lopsided models it is not difficult to obtain $\theta_{\text{atm}} \cong \pi/4$ \cite{12}. Consider a model with $M_L$ having the form in Eq. (9), where some nonabelian symmetry relates $\mu_L^-\tau_L^-$ so that $\rho = 1$. That would give the relations $\tan \theta_{\text{sol}} \cong \rho'$ and $\tan \theta_{\text{atm}} \cong \sqrt{1 + \rho'^2}$, which imply the interesting relation $\tan^2 \theta_{\text{atm}} \cong 1 + \tan^2 \theta_{\text{sol}}$, or equivalently $\sin^2 2\theta_{\text{atm}} \cong (1 + \tan^2 \theta_{\text{sol}})/(1 + \frac{1}{2}\tan^2 \theta_{\text{sol}})^2$. Even for the best-fit LMA value $\tan^2 \theta_{\text{sol}} \approx 0.4$ this gives $\sin^2 2\theta_{\text{atm}} \cong 0.97$.

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