Geometry of local quantum dissipation and fundamental limits to local cooling

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We geometrically characterize one-qubit dissipators of a Lindblad type. An efficient parametrization in terms of 6 linear parameters opens way to various optimizations with local dissipation. As an example, we study maximal steady-state singlet fraction that can be achieved with arbitrary local dissipation and two qubit Hamiltonians. We show that this singlet fraction has a discontinuity as one moves from unital to non-unital dissipators and demonstrate that the largest possible singlet fraction is \( \approx 0.654 \). This means that for systems with a sufficiently entangled ground state (e.g., Heisenberg interaction) there is a fundamental limit to the lowest achievable energy. With local dissipation one is unable to cool the system below some non-zero temperature.

Introduction. -- A common goal of quantum information theory \cite{1} is to discover problems for which quantum protocols outperform best classical ones. One can approach this question on a case-by-case basis, studying individual tasks, or, one can ask a broader question, namely, what can be achieved, possibly in an optimal way, by a particular quantum resource. The latter question, which we tackle, boils down to an optimization problem that is often too difficult to solve. An efficient characterization of a given resource is of great help in this respect.

The most general transformations in quantum mechanics are linear completely positive trace-preserving maps (CPM) also called quantum channels \cite{2}. Characterizing CPMs in higher dimensional space is a very hard open problem, though for single qubit it has been solved, leading to geometrical picture of single-qubit channels \cite{2,3}. A simpler subset of possible transformations are those that are solutions of local Markovian master equations of a Lindblad type \cite{4,5}. While generic evolution properties of Lindblad equations have been explored, e.g. \cite{6,7}, lately increasing attention is devoted to specific tasks. On one hand Lindblad equations are a useful tool to study nonequilibrium physics, are a framework that can be experimentally realized, and can be a generic quantum resource, for instance, for universal computation \cite{8} or steady-state manipulation \cite{9}. An interesting question from a practical point of view is what can be achieved with particular limited resources. Pure steady states are relatively well explored, with simple conditions determining when can they be steady states \cite{10,12}. The role of locality on pure steady states has also been studied \cite{13} as well as the existence of translational conservation laws \cite{14}. Very little is on the other hand known about mixed steady states, see though Ref. \cite{15}, where a set of steady states reachable for fixed dissipation and varying \( H \) is studied.

We are first going to characterize one-qubit Lindblad dissipators using linear parameters and study their geometry \cite{2}. An efficient parametrization with a single nontrivial inequality constraint allows then to study various optimization questions. We demonstrate its usefulness by studying the set of, in general mixed, states reachable by arbitrary two qubit Hamiltonian and one-qubit dissipation. We in particular study the overlap of the steady state with a maximally entangled state. We show that this singlet fraction is upper bounded by \((3 + \sqrt{5})/8 \approx 0.654 \). This sheds light on how much can local dissipation influence non-local properties and has a number of implications: local dissipation can produce entangled mixed states even-though it can not produce entangled pure states; if the system’s ground state is a maximally entangled singlet state, like e.g. in the Heisenberg model, then with local dissipation one can not cool the system down to its zero temperature – there will be a “temperature gap” below which one can not go. For the particular setting studied this improves on the 3rd law of thermodynamics and various generic zero-temperature unattainability results \cite{16,17} or, on ground-state cooling limitations in specific situations, e.g. \cite{18}. For quantum tomography our results mean that the singlet fraction can place useful constraints on possible dissipations and that it exhibits a discontinuity at unitality.

Lindblad equation in diagonal form is \cite{4,5}:

\[
\frac{d\rho}{dt} = \mathcal{L}(\rho) = i[\rho, H] + \mathcal{L}_{\text{dis}}(\rho),
\]

where \( \mathcal{L}_{\text{dis}}(\rho) = \sum_j 2L_j \rho L_j^\dagger - \rho L_j^\dagger L_j - L_j^\dagger L_j \rho \) is a superoperator called dissipator that depends on a set of traceless Lindblad operators \( L_j \). Lindblad equation, that generates a CPM map via \( \Lambda_t = e^{Lt} \), can also be written in a non-diagonal form with a dissipator \( \mathcal{L}_{\text{dis}}^{(\text{ind})}(\rho) = \sum_{j,k} g_{j,k} (2L_j \rho L_k^\dagger - \rho L_k^\dagger L_j - L_k^\dagger L_j \rho) \), where \( L_j \) is a complete set of orthogonal traceless operators. Matrix \( g_{j,k} \) is called a GKS matrix \cite{4} and has to satisfy \( g \geq 0 \).

Expanding Hermitian density operator \( \rho \) on \( n \) qubits in an orthogonal Hermitian traceless operator basis \( F_j \), \( \rho = \frac{1}{2^n} \mathbb{1} + \sum_j c_j F_j \), so that \( \rho \) is parametrized by real coherence vector \( c \), dissipator \( \mathcal{L}_{\text{dis}}(\rho) = \sum_j c_j^2 F_j \) induces an affine map \( \rho' = \mathcal{M} \rho + \mathcal{I} \), with real matrix \( \mathcal{M} \) and real vector \( \mathcal{I} \). The unitary part \( i[\rho, H] \) of the Lindblad equation induces map \( \rho' = \mathcal{N} \rho \), with \( \mathcal{N} \) being real antisymmetric. We shall discuss generators \( \mathcal{L}_{\text{dis}} \) instead of induced channels \( \Lambda_t \) because of simpler relations and because they form a convex set while Lindblad channels \( \Lambda_t \)
do not, e.g., [19].

We are interested in one-qubit dissipators, for which $\mathbf{M}$ is always symmetric [20], and therefore diagonalizable. Therefore, in an appropriate basis $\mathcal{L}_{\text{dis}}$ can be written in a canonical diagonal form (basis $\{\sigma^x, \sigma^y, \sigma^z, 1\}$),

$$\mathcal{L}_{\text{dis}}^{(d)} = \begin{pmatrix}
  \frac{q_2 + q_3}{2} & 0 & 0 & t_1 \\
  0 & -\frac{q_2 + q_3}{2} & 0 & t_2 \\
  0 & 0 & -\frac{q_2 + q_3}{2} & t_3 \\
  0 & 0 & 0 & 0
\end{pmatrix}. \quad (2)$$

In its canonical form $\mathcal{L}_{\text{dis}}^{(d)}$ a Lindblad dissipator can be parametrized by 6 real parameters, $q_j$ and $t = (t_1, t_2, t_3)$. The central question is, for which values of these parameters (2) is the resulting $\mathcal{L}_{\text{dis}}$ a Lindblad form $\mathcal{L}_{\text{dis}}^{(d)}$, i.e., generates a dynamical semigroup? It is important to note - and this is the main advantage of the parametrization in Eq. (2) - that $\mathcal{L}_{\text{dis}}$ is linear in these parameters, while on the other hand it is quadratic in Lindblad operators $L_j$. Having linear parametrization of $\mathcal{L}_{\text{dis}}$ will greatly simplify all optimization problems.

We write the canonical dissipator $\mathcal{L}_{\text{dis}}^{(d)}$ in terms of a GKS matrix (written in the basis $\{\sigma^x, \sigma^y, \sigma^z\}$)

$$g = \frac{1}{8} \begin{pmatrix}
  q_1 & -it_3 & it_2 \\
  it_3 & q_2 & -it_1 \\
  -it_2 & it_1 & q_3
\end{pmatrix}. \quad (3)$$

Sufficient and necessary condition for $\mathcal{L}_{\text{dis}}^{(d)}$ to be of a Lindblad form is that $g = 0$. If the dissipator is unital, $t = 0$, the condition $g \geq 0$ translates to three simple conditions $q_{1,2,3} \geq 0$. We are now going to show that for non-unital case a single additional condition is necessary.

**Theorem 1.** One-qubit dissipator in the canonical form given by Eq. (3) represents a Lindblad dissipator iff

$$q_i \geq 0,$$  \quad (4)

$$1 \geq \frac{t_1^2}{q_2 q_3} + \frac{t_2^2}{q_1 q_3} + \frac{t_3^2}{q_1 q_2}. \quad (5)$$

(If any of the denominators in the last equation is zero, it must be understood that the corresponding parameter $t_j$ must also be zero, and the term is left out on the RHS)

**Proof.** $g \geq 0$ iff all eigenvalues $\lambda_j$ of $g$ are non-negative. The characteristic polynomial of (3) is $p(\lambda) := \det (g - \lambda \mathbf{I}) := 512\lambda^3 - 64C\lambda^2 + 8(\lambda - t^2)\lambda - (Q - B)$, where $A := q_1 q_2 + q_1 q_3 + q_2 q_3$, $B := q_1 t_1^2 + q_2 t_2^2 + q_3 t_3^2$, $C := q_1 + q_2 + q_3$, $Q := q_1 q_2 q_3$ and $t^2 := t \cdot t$. Using Descartes’ rule of signs we can infer the maximal number of positive/negative roots by the number of sign changes of coefficients in $p(\lambda)$ and $p(-\lambda)$, respectively. Because $g$ is Hermitian we known that all roots of $p(\lambda)$ are real and we can in fact determine the exact number of negative and positive roots. First, for $g \geq 0$ all diagonal matrix elements $q_j$ must be non-negative, $q_j \geq 0$, which also implies that $C > 0$ (as $\text{tr}(g) = C/8$, $C = 0$ would imply that $\mathcal{L}_{\text{dis}}^{(d)} \equiv 0$). The coefficient in front of $\lambda^3$ is positive and in order to have three non-negative roots one must also have $A \geq t^2$ and $Q \geq B$. $g$ has one zero eigenvalue if $Q = B$, if in addition $A = t^2$, it has two. It is not possible to have $A = t^2$ and $Q \neq B$ because this would imply less than 3 positive roots.

$$Q - B \geq 0,$$  \quad (6)

If all $q_j > 0$ then one can use inequalities like $q_1 q_2 \geq t_1^2$ to show that one can not have $A = t^2$ (while $Q = B$, i.e., $g$ of rank 2, is still possible). The case (ii) is the only possibility when $\mathcal{L}_{\text{dis}}^{(d)}$ can have two-times degenerate steady state.

**Geometry of one-qubit dissipators.** General one-qubit CPM $A$ can be, similarly as Lindblad dissipator $\mathcal{L}_{\text{dis}}$,
brought to a “diagonal” form [2],

\[ \Lambda = \begin{pmatrix} D & \mathbf{v} \\ 0 & 1 \end{pmatrix}, \quad D = \text{diag}(\lambda_1, \lambda_2, \lambda_3). \]  

(6)

Conditions on \( \lambda_j \) and \( v_j \) for which \( \Lambda \) is a CPM are well known [3]. In the unital case \( v_j = 0 \) diagonal \( \lambda_j \) must be contained within a tetrahedron defined by 4 corners at which \( \lambda_j = \pm 1 \) and \( \lambda_1 \lambda_2 \lambda_3 = 1 \), whereas for non-unital channels an additional inequality has to hold [21]. Let us now compare the set of channels obtained from Lindblad dissipators to the set of all CPMs. From the diagonal form \( \mathcal{L}^{(d)} \) in Eq. (2) it is simple to obtain the corresponding Lindblad channel \( \Lambda = e^{\mathcal{L}^{(d)} \tau} \), where without loss of generality we can set \( \tau = 1 \). We get relations \( q_1 = \ln \frac{\lambda_1}{\lambda_3 \lambda_2}, \quad t_1 = -v_1 \frac{\ln \lambda_1}{\ln \lambda_2}, \) and analogously for other components. For unital channels comparison is in Fig. 1. Because for Lindblad dissipators \( q_j \geq 0 \) one always gets \( \lambda_j \geq 0 \), i.e., in the space of \( \lambda_j \) the channel \( \Lambda_1 \) is always in the first octant. Trivially concatenating such Lindblad channel by a rotation by \( \pi \) around one of the three axes one can change the sign of two \( \lambda_j \), obtaining the symmetrical object shown in Figs. 1 and 2. The set of Lindblad channels is understandably smaller than the set of CPMs and is contained inside the tetrahedron. It is bounded by hyperbolic surfaces determined by conditions like \( \lambda_1 \geq \lambda_2 \lambda_3 \) that come due to \( q_j \geq 0 \). In the 1st octant the Lindblad channels fill \( \frac{1}{2} \) of the volume of all CPMs, in the whole tetrahedron this fraction is \( \frac{1}{8} \). For non-unital channels there are three additional “shift” parameters \( v_j \). Plotting the set of CPMs for fixed \( \mathbf{v} \) one gets a “rounded” tetrahedron, see Ref. [21] for equations.

The set of Lindblad channels for one choice of \( \mathbf{v} \) is shown in Fig. 2. Plotting the set of allowed Lindblad shifts for fixed \( q_j \) one gets an ellipsoid, Eq. (6), while for CPMs this set is in general larger and not an ellipsoid, for details see Appendix.

Application. – The singlet fraction of a 2-qubit steady state \( \rho_\infty \), \( \mathcal{L}(\rho_\infty) = 0 \), is defined as \( F := (\rho^\prime|\rho_\infty|\rho^\prime)/\sqrt{2} \), where \( |\psi\rangle = (|01\rangle - |10\rangle)/\sqrt{2} \). States with high singlet fraction can be used for quantum processing, for instance teleportation [22]. We want to find the maximal singlet fraction \( F \) for a 2-qubit system, maximized over all possible 2-qubit Hamiltonians and 1-qubit dissipators \( \mathcal{L}_\text{dis} \). The following simple theorem will be of help. If \( \rho_{\infty} := \sum j c_j F_j \) is a steady state of Lindblad equation \( \mathcal{L} \) with dissipator \( \mathcal{L}_\text{dis} \), acting as \( \mathbf{M} \mathbf{c} + \tilde{\mathbf{t}} \), and unitary part with \( H \), then \( \rho_{\infty}^\prime := \sum j c_j F_j \) is a steady state of Lindblad equation \( \mathcal{L} \) having the same \( \mathbf{M} \) and \( \mathcal{L} \) as in \( \mathcal{L}_\text{dis} \), acting as \( \mathbf{M} \mathbf{c} + \tilde{\mathbf{t}} \). An important consequence is that for fixed \( \mathbf{M} \) the largest steady-state overlap with any given state is reached at the maximal possible shift \( \tilde{\mathbf{t}} \).

![Fig. 1](image1.png) ![Fig. 2](image2.png)

FIG. 1. (Color online) The set of one-qubit unital Lindblad channels \( \mathcal{L}_\text{dis}^{(d)} \) (colored object). Stripes at the surface are isolines of planes perpendicular to vector \((1,1,1)\). Edges correspond to dissipators \( \mathcal{L}_\text{dis}^{(d)} \) composed of one Lindblad operator, the surface of two, and the interior from three Lindblad operators. Right top: cross-section at \( \lambda_3 = 0.3 \); full rectangle is the tetrahedron’s boundary (CPMs). Right bottom: cross-section perpendicular to \((1,1,1)\) containing point \( 0.4(1,1,1) \); full triangle is tetrahedron’s edge.

FIG. 2. (Color online) The set of one-qubit non-unital Lindblad channels (solid object) and CPMs (transparent outer surface), both for fixed \( \mathbf{v} = 1/2(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \) with \( \theta = \pi/4 \) and \( \phi = \pi/3 \). The surface corresponds to dissipators made from 2, the interior from 3 Lindblad operators. Right: same cross-sections as in Fig. 1. Dotted lines are the tetrahedron edges, full curves crossings with the CPM’s surface.
must hold for all \( k \). If the rank of \( |\psi\rangle \) is maximal (for a given bipartition such that \( L_j \) acts on \( A \)) \( L_j \) must be an identity operator and therefore \( \mathcal{L}_{\text{dis}} \) is zero. In particular, for 1-qubit dissipator pure steady states for any \( n \) are always separable. Steady-state subspace for unital 1-qubit dissipators is therefore spanned by separable states and \( F \) can be at most 1/2. This maximum can be reached only if there is a single Lindblad operator (two different Pauli matrices do not have a common eigenvector), e.g. \( L \sim \sigma^x \), and therefore two \( q_i \) are zero. On the other hand, for nonzero shift length \( t := |t| \) at most one \( q_i \) can be zero. As a consequence, in the limit of small nonzero shift \( t \to \epsilon \) the optimal singlet fraction can be shown to be 1/4. There is a discontinuous transition in the maximal \( F \) from 1/2 to 1/4 as one smoothly moves from unital (\( t = 0 \)) to non-unital dissipators (\( t \neq 0 \)).

The singlet fraction \( F \) is invariant to any local rotation \( U \otimes U \) (\( U \) is a 1-qubit unitary) and we can always rotate dissipator to a basis in which the shift vector has only one non-zero component, say \( t_1 \neq 0, t_{2,3} = 0 \). Due to symmetry reasons, in the optimal case, we expect \( \mathcal{L}_{\text{dis}}^{(d)} \) in this basis to have \( q_2 = q_3 \) and possibly different \( q_1 \). Because \( t_1 \) also has to be maximal we have \( \rho_\infty \) in addition \( q_2 q_1 = t_1^2 = 1 \) (because \( H \) is arbitrary we can set \( t_1 \) without loss of generality). One can argue (see Appendix) that for such \( \mathcal{L}_{\text{dis}}^{(d)} \) (acting on the 1st qubit) the optimal steady state will be of form \( \rho_\infty = \frac{1}{2} 1\_1 1\_2 + c_1 1\_1 \sigma^2_2 + c_4 1\_1 \sigma^2_3 + c_6 1\_2 \sigma^2_3 + c_{11} 1\_2 \sigma^2_3 + c_{13} 1\_1 \sigma^2_3 + c_{18} 1\_1 \sigma^2_2 + c_{12} 1\_2 \sigma^2_2 + c_{19} 1\_1 \sigma^2_2 + c_{21} 1\_2 \sigma^2_2 + c_{18} 1\_2 \sigma^2_2 + c_{21} 1\_1 \sigma^2_2 + c_{21} 1\_2 \sigma^2_2 \). One could try to maximize \( F \) subject to necessary conditions \( \text{tr}(\rho_\infty^{(d)} \mathcal{L}_{\text{dis}}^{(d)}(\rho_\infty)) = 0, r = 1, 2, 3 \), we found it though more convenient (see Appendix) to use an alternative approach. Operator equation \( \mathcal{L}(\rho_\infty) = 0 \) represents a set of 15 equations that are linear in parameters \( d_j \) of \( H = \sum_j d_j F_j \) as well as in \( c_j \). They can be written as a matrix equation \( G d = f \), where the matrix \( G \) as well as the inhomogeneous part \( f \) depend on \( c \) (but not on \( d \)). The equation has a solution for \( d \) only if \( f \) is orthogonal to the kernel of \( G \). The kernel of \( G \) can be explicitly calculated, resulting in three constraints (see Appendix). The fidelity \( F = 1/4 + c_1 + c_6 + c_{11}, \) subject to these constraints, can be analytically maximized. For fixed \( q_{2,3} \) the optimum is always at \( q_1 = 0 \), i.e., for single Lindblad operator. The absolute maximum is reached at \( q_2 = q_3 = 1 \) when \( F_{\text{max}} = (1 + \varphi)/4 \approx 0.6545 \), where \( \varphi = (1 + \sqrt{5})/2 \) is the golden mean. Steady state \( \rho_\infty \) of this optimal case is of rank 4 and is entangled. Such optimal \( \rho_\infty \) can not be obtained exactly at \( q_2 = 1 \), but only in the limit of \( q_2 \to 1 \), which however poses no serious obstacles, see Appendix.

As we see, local non-unital dissipation can create mixed entangled steady-states, even though it can not create pure entangled steady-states. It is known that dissipation can mitigate entanglement destruction, for instance, when common environment is coupled to non-interacting systems \([23]\), or high-temperature entanglement in driven \([24]\) or even steady states \([25]\). The fact that \( F_{\text{max}} \) is bounded away from 1 has some important consequences. For Hamiltonians whose non-degenerate ground-state has overlap with a maximally entangled state larger than \( F_{\text{max}} \) (for instance, the Heisenberg \( H \)) using local dissipation one can not “cool” the systems down to its ground state. There is a fundamental limit to local quantum cooling – a minimal temperature below which one can not cool. We expect similar result to hold also in more than 2-qubit systems. In the setting studied this improves on various unattainability results for certain specific situations \([18]\). Note that in classical setting, using local cooling by e.g. Langevin bath, there are no obstacles to the lowest achievable energy. Another interesting consequence is that it can put a constraint on possible dissipation: for instance, measuring \( F \) in the steady state to be larger than \( F_{\text{max}} \) one can infer that, even if nothing is known about \( H \) or \( \mathcal{L}_{\text{dis}} \), dissipation can not be local. Also, the maximal \( F \) is very sensitive to unitality. For strictly unital local dissipation it is 1/2, whereas for infinitesimally weak violation it drops to 1/4.

**Conclusion.**– We have geometrically characterized single-qubit Lindblad dissipators opening door for various optimizations. To demonstrate its applicability we have calculated the maximal singlet fraction achievable by one-qubit dissipation, showing that there exists a fundamental limit to local quantum cooling.

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**APPENDIX**

Maximal singlet fraction for GKS matrix with \( t_2 = t_3 = 0 \)

We want to find out the maximal singlet fraction \( F \) for \( \mathcal{L}_{\text{dis}} \) with \( q_2 q_3 = 1, t_1 = 1 \) and \( t_2 = t_3 = 0 \), for which there are two Lindblad operators, \( L_1 = (\sqrt{2} \sigma^x_1 + i \sigma^y_1 \sqrt{2})/\sqrt{5} \) and \( L_2 = \sqrt{\frac{2}{5}} \sigma^x_2 \). Necessary conditions for \( \rho \) to be a steady state of Lindblad equation is that \( C_r := \text{tr}(\rho_\infty^{(d)} \mathcal{L}_{\text{dis}}^{(d)}(\rho_\infty)) = 0 \) for \( r = 1, 2, 3 \) \([15]\). Let us parametrize \( \rho \) as \( \rho_\infty = \frac{1}{2} 1\_1 1\_2 - \sum_{j,k=0}^3 c_m \sigma^j_1 \sigma^k_2 \), where \( m = j + 4 k + 1 \) and \( \sigma^\varphi = 0 \ldots 3 \) denotes \( \{ \sigma^x, \sigma^y, \sigma^z, 1 \} \), respectively. For such parametrization \( F \) is

\[
F = \frac{1}{4} + c_1 + c_6 + c_{11}, \tag{7}
\]

i.e., it is given by a sum of coefficients in front of \( \sigma^x_1 \sigma^\varphi_2, \sigma^y_1 \sigma^\varphi_2, \) and \( \sigma^z_1 \sigma^\varphi_2 \). Conditions \( C_r = 0 \) result in equations that are of \( (r + 1)\)-th order in unknown coefficients \( c_m \). One could use the method of Langrange multipliers to solve this constrained optimization problem. However, besides practical solvability issues, there is a fundamental
difficulty that the domain of allowed c’s is not bounded
with $C_{1,2,3} = 0$ (the reason is that $\mathcal{L}_{\text{dis}}^{(d)}$ acts only on single qubit). As a consequence, for instance, solving the resulting Euler-Lagrange equations for optimization of $F$ subject to only $C_1 = 0$, gives a solution for which $F = \frac{1}{2}$, which, as we shall see, is not the correct maximum. To make the domain bounded one could add an additional constraint, for instance $I := \text{tr}(\rho_\infty^2) \leq 1$. One difficulty with using only $C_r = 0$ though would still remain. $C_r = 0$ are necessary and sufficient if the steady state $\rho$ has non-degenerate spectrum, but only necessary otherwise. As it turns out, the optimal $\rho_\infty$ has in our case a degenerate spectrum (one eigenvalue is twice degenerate). We shall therefore use a slightly different approach, where though conditions $C_r = 0$ will still be used to first infer that a number of coefficients $c_m$ are zero in the optimal case.

Maximizing $F$ subject to $C_1 = 0$ as well as $I = (\frac{1}{4} + 4\sum_m c_m^2) \leq 1$ we have a quadratic maximization problem that can be solved exactly. First, one can observe that in $C_1$ some coefficients come only in perfect squares, e.g., $2(q_3 + q_4)(c_2^4 + c_4^4 + c_6^4 + c_8^4) + 2(q_2 + q_1)(c_2^4 + c_4^4 + c_6^4 + c_8^4)$. Therefore, if we have a solution with nonzero $c_{2,10,14,3,15}$ it is always better, meaning we will have higher $F$, to set them to zero and instead increase $c_{6,11}$. In the maximum we will always have $c_{2,3,7,10,14,15} = 0$. Then, using Lagrange multipliers one can also show that $c_{5,8,9,12}$ must as well be zero: one has a homogeneous set of linear equations for these coefficients with the only solution being $c_{5,8,9,12} = 0$, unless a Langrange multiplier takes a special value in which case though equations for $c_{1,4}$ do not have a solution. Therefore, in the optimal situation only five $c_{1,4,6,11,13}$ are nonzero. The fact that only $c_{1,4,6,11,13}$ appear in the steady state with optimal singlet fraction is actually not very surprising. $F$ depends on $c_{6,11}$ so these coefficients will likely appear in the optimal $\rho$. Then, dissipator $\mathcal{L}_{\text{dis}}^{(d)}$ couples $I_1I_2$ to $\sigma_1^x I_2$ as well as $I_1I_2^* \sigma_2^x \sigma_2^y$. Coefficients $c_{1,4,6,11,13}$ therefore represent in a way a “minimal” set that can satisfy all constraints. Incorporating constraints $C_2$ and $C_3$ into analytical argument is harder so we rather show results of numerical optimization. In Fig. 3 we show the dependence of optimal $F$ on the norm $||c_A||^2 = 4 \sum_m (2,3,5,7,8,9,10,12,14,15) c_m^2$. We can see that the maximum is reached for $||c_A|| = 0$, as it was already the case for the single constraint $C_1 = 0$. Adding constraints $C_2 = 0$ and $C_3 = 0$, as well as having $q_2 \neq q_3$ and $q_1 \neq 0$, therefore does not change the conclusion that in the optimum only $c_{1,4,6,11,13}$ are nonzero. Observe also from Fig. 3 that adding condition $C_3 = 0$ to $C_{1,2} = 0$ adds very little, for instance, we can analytically show that for $||c_A|| = 0$ one gets $F_{\text{opt}} \approx 0.65496$ when $C_{1,2} = 0, I \leq 1$, while $F_{\text{opt}} \approx 0.65451$ when $C_{1,2,3} = 0, I \leq 1$.

We are therefore left with 5 nonzero coefficients $c_{1,4,6,11,13}$. Stationary Lindblad equation $\mathcal{L}(\rho_\infty) = 0$ can be written as a matrix equation $G(c)\mathbf{d} = f(c)$, for un

![Figure 3. Optimal singlet fraction $F$ for different fixed norm $||c_A||^2$. Circles show the case with constraints $C_1 = 0, C_2 = 0, I \leq 1$; full curve (almost overlapping with circles) the case for $C_1 = 0, C_2 = 0, C_3 = 0, I \leq 1$; $q_2 = q_3 = 1, q_1 = 0$.](image)

known parameters $d$ of the Hamitonian $H = \sum_j d_j F_j$. The equation has a solution provided $f(c)$ is orthogonal to the kernel of $G(c)$. These give sufficient and necessary conditions on coefficients $c$ in order that the corresponding $\rho_\infty$ is a steady state. Because we have only 5 remaining unknown $c$’s $G$ is sufficiently simple so that its kernel, being in general of size 3, can be analytically calculated. First, for fixed $q_{2,3}$ in optimum one always has $q_1 = 0$, i.e. rank 1 GKS matrix with one Lindblad operator. The three kernel conditions are in this case $q_{23}c_1 - 4c_4 = 0, 1 + 2q_{23}c_{13} + D(c_6^2 - c_4^2) = 0$ and $q_2c_{11} + D(c_4c_6 + c_{11}c_{13}) = 0$, with $D := q_3c_6/(c_4c_{11} + c_6c_{13})$ and $q_{23} := q_2 + q_3$. These are now sufficiently simple so that $F$ can be analytically maximized. We also note that kernel conditions are stronger than $C_{1,2,3} = 0$. The dependence of $F$ on $q_2$ is shown in Fig. 4. Not very surprisingly, the maximum is achieved for $q_2 = q_3 = 1$. Only at that point is $\mathcal{L}_{\text{dis}}^{(d)}$ invariant to rotations about the $x$-axis, similarly as is the singlet state (for $q_2 \neq 1$ the symmetry between $y$ and $z$ is lost). The value of the maximal singlet fraction is $F_{\text{max}} = \frac{3 + \sqrt{3}}{8}$, reached

![Figure 4. The optimal singlet fraction of the steady state for 1-qubit $\mathcal{L}_{\text{dis}}^{(d)}$ with $t_2 = t_3 = q_1 = 0, t_1 = 1, q_3 = 1/q_2$.](image)
at $q_2 = q_3 = 1$, while the values of coefficients are $c_1 = c_4 = -c_{13} = (5 + \sqrt{5})/40$, $c_6 = c_{11} = 1/(4\sqrt{5})$.

The optimal point of $q_2 = q_3 = 1$ for which $F = F_{\text{max}}$ is in fact degenerate: kernel of $G$ is in this special case larger and $G(c)d = f(c)$ has no solutions. Optimality can be reached only in the limit $q_2 \to 1$, which, because dependence of $F$ close to $q_2 = 1$ is quadratic (see Fig. 1), poses no serious obstacle. One possibility is to take $c_1 = 1/2 q_2^2 (1 + 2q_2^2/\kappa)/(1 + q_2^2)^2$, $c_4 = 1/2 q_2(1 + 2q_2^2/\kappa)/(1 + q_2^2)$, $c_6 = 1/4 (1 + 1/(4\sqrt{5})/40, c_{11} = 1/(1 + q_2^2)/\kappa$, $c_{13} = -c_4$, and $H = (\kappa - 2q_2^2)/(4(1 - q_2^2)(1 + q_2^2)^2 q_2^2 \sigma_y^2 \sigma_y^2 + \sigma_y^2 \sigma_y^2)$, where $\kappa := \sqrt{1 + 4q_2^2 + 10q_2^4 + 4q_2^6 + q_2^8}$ resulting in a singlet fraction $F = q_2^2/(4q_2^2 + q_2^4 + 1 - \kappa)$ (which is not the optimal one for $q_2 \neq 1$, but approaches the one for $q_2 \to 1$). Taking $q_2 = 1 - \epsilon$, in the limit $\epsilon \to 0$, the expressions simplify to $H = \sqrt{1/\kappa} (\sigma_y^2 \sigma_y^2 + \sigma_y^2 \sigma_y^2)$, and $c_1 = (5 + \sqrt{5})/40(1 - \epsilon^2/2)$, $c_4 = (5 + \sqrt{5})/40$, $c_6 = (1 + \epsilon)/(4\sqrt{5})$, $c_{11} = (1 + \epsilon)/(4\sqrt{5})$, all written to the lowest order in $\epsilon$.

**Comparing Lindblad and quantum channels**

In the main text we have compared the set of single-qubit Lindblad channels to the set of general qubit channels for the unital case in Fig. 1 and for a fixed shift vector $\mathbf{v}$ (in Fig. 2). It is instructive to compare the two sets in a non-unital case also for channel’s fixed generalized singular values $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ (in Fig. 3).

These sets are shown in Fig. 5. What is shown is a polar plot of the maximal allowed length of the shift vector $\mathbf{v}$, such that the resulting map is completely positive or can be generated by Lindblad evolution [5]. Maximal allowed length of $\mathbf{v}$ in a given direction is equal (up to a scale) to the distance between the shown surface and its center, i.e., $\lambda$, around which the “ball” is plotted. For Lindblad channels, in the space of generator shifts $\mathbf{t}$, this set would be an ellipsoid [5]. In the space of channel shifts $\mathbf{v}$ it is a smooth ellipsoid-like shape seen in top Fig. 5. For general channels (CPMs) the set, which is larger or equal to the Lindblad one, can be seen in bottom Fig. 4. The boundary of this CPM set is determined by Eq. (30) from Ref. [21] and can have non-smooth edges.

**FIG. 5.** (Color online) The sets of allowed channel (6) shifts $\mathbf{v}$ (scaled by factor $0.3$) for 7 different diagonal values $\lambda$, namely for $(0, 0, 0)$, $(0, 0, 0.6)$, $(0, 0, 0.9)$, $(0.5, 0.5, 0.5)$, $(0.5, 0.5, 0.8)$, $(0.5, 0, 0)$ and $(-0.6, -0.6, 0.4)$. Top: Lindblad channels. Bottom: CPMs. What is shown is a polar plot: maximal $\mathbf{v}$ in a given direction, centered around fixed $\lambda$. In the Lindblad case the surface of “balls” corresponds to dissipators with 2 Lindblad operators, the interior to 3. On the diagonal $\lambda \propto (1, 1, 1)$ the set of allowed $\mathbf{v}$ is the same (perfect balls) for CPMs and Lindblad channels, everywhere else the Lindblad set is smaller.

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