Complex and quaternionic hyperbolic Kleinian groups with real trace fields

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ABSTRACT

Let $\Gamma$ be a nonelementary discrete subgroup of $\text{SU}(n,1)$ or $\text{Sp}(n,1)$. We show that if the trace field of $\Gamma$ is contained in $\mathbb{R}$, then $\Gamma$ preserves a totally geodesic submanifold of constant negative sectional curvature. Furthermore, if $\Gamma$ is irreducible, then $\Gamma$ is a Zariski-dense irreducible discrete subgroup of $\text{SO}(n,1)$ up to conjugation. This is an analog of a theorem of Maskit for general semisimple Lie groups of rank 1.

1. Introduction

The main algebraic objects associated to a Kleinian group $\Gamma$, that is, a discrete subgroup of $\text{PSL}(2,\mathbb{C})$, are its invariant trace field and invariant quaternion algebra. They have played an important role in studying the arithmetic aspects of Kleinian groups, especially of finite-covolume Kleinian groups. For example, the invariant trace field of a finite-covolume Kleinian group is a number field, that is, a finite extension of $\mathbb{Q}$, and the matrix entries of the elements of a finite-covolume Kleinian group are in its trace field. The trace field of $\Gamma$ is not a commensurability invariant, but its invariant trace field and invariant quaternion algebra are commensurability invariants. Note that an arithmetic Kleinian group is determined up to commensurability by its invariant trace field and invariant quaternion algebra (see [13]).

McReynolds (see the forthcoming paper ‘Arithmetic Lattices in $\text{SU}(n,1)$’) introduced the invariant trace field and the invariant algebra for subgroups of $\text{PSU}(n,1)$ in a similar way as for the case of $\text{PSL}(2,\mathbb{C})$. Moreover, he proved that they are commensurability invariants as for Kleinian groups. A central theme in this theory is to study the (invariant) trace field and invariant algebra associated to a subgroup of $\text{PSU}(n,1)$. However, very little is yet known about these algebraic invariants associated to complex hyperbolic Kleinian groups. In particular, Cunha and Gusevskii [2] and Genzmer [5] studied whether a discrete subgroup of $\text{SU}(2,1)$ can be realized over its trace field. The main aim of the paper is to understand the algebraic and geometric features of the discrete subgroups of $\text{SU}(n,1)$ or $\text{Sp}(n,1)$ with real trace fields.

Maskit [10, Theorem V.G.18] characterized the nonelementary discrete subgroups of $\text{SL}(2,\mathbb{C})$ with real trace fields. More precisely, if the trace field of a nonelementary discrete subgroup $\Gamma$ of $\text{SL}(2,\mathbb{C})$ is real, then $\Gamma$ is conjugate to a subgroup of $\text{SL}(2,\mathbb{R})$. In other words, $\Gamma$ is realized over the real field $\mathbb{R}$ up to conjugation. The same question naturally arises as to which discrete subgroups of $\text{SU}(n,1)$ have real trace fields. In fact, an answer for the question has been given in the low-dimensional cases. Cunha and Gusevskii [2] and Fu, Li and Wang [3] independently showed that a nonelementary discrete subgroup of $\text{SU}(2,1)$ with real trace field is conjugate to...
a subgroup of SO(2, 1) or S(U(1) × U(1, 1)). Kim and Kim [9] also proved that a nonelementary discrete subgroup of SU(3, 1) with real trace field is conjugate to a subgroup of SO(3, 1) or SU(2) × SU(1, 1). Note that it does not seem to be easy to extend these approaches to the general case since all the proofs in [2, 3, 9] were based on explicit matrix computations.

In the SU(n, 1) case, the trace field of a discrete subgroup’s being real does not imply that the discrete subgroup is realized over \( \mathbb{R} \) up to conjugation, as in the SL(2, \( \mathbb{C} \)) case. Here is a counterexample. Let \( F_2 \) be a free group with two generators. Let us take a discrete faithful representation \( \rho_1 : F_2 \to SU(1, 1) \) corresponding to a complete hyperbolic structure on a punctured torus and any representation \( \rho_2 : F_2 \to SU(2) \). Define a representation \( \rho : F_2 \to SU(3, 1) \) by \( \rho = \rho_1 \oplus \rho_2 \). Then it is easy to check that \( \rho(F_2) \) is a nonelementary discrete subgroup and, moreover, the trace field of \( \rho(F_2) \) is real, since every element of SU(1, 1) and SU(2) has real trace. However, one can easily make \( \rho(F_2) \) not to be realized over \( \mathbb{R} \) up to conjugation by choosing proper representations \( \rho_1 \) and \( \rho_2 \). In fact, since the choice of \( \rho_2 \) is completely free, one can in this way construct many discrete subgroups of SU(n, 1) which have real trace fields, but are not realized over \( \mathbb{R} \) up to conjugation. For this reason, in the SU(n, 1) case, the trace field being real does not seem to encode the algebraic properties of discrete subgroups. On the other hand, from a geometric point of view, all the previous results so far give a consistent geometric feature, indicating that a discrete subgroup with real trace preserves a totally geodesic submanifold of constant negative sectional curvature, as do Fuchsian groups. In the general setting of SU(n, 1), we obtain the geometric feature of discrete subgroups with real trace fields as follows.

**Theorem 1.1.** Let \( \Gamma \) be a nonelementary discrete subgroup of SU(n, 1). If the trace field of \( \Gamma \) is real, then \( \Gamma \) preserves a totally geodesic submanifold of constant negative sectional curvature in \( H^n_{\mathbb{C}} \).

Assuming that the symmetric metric on the complex hyperbolic \( n \)-space \( H^n_{\mathbb{C}} \) is normalized so that its sectional curvature lies between \(-4\) and \(-1\), it is well known that a totally geodesic submanifold of constant negative sectional curvature is isometric to either a real hyperbolic space of constant sectional curvature \(-1\) or a real hyperbolic 2-plane of constant sectional curvature \(-4\). Note that the first one is isometric to \( H^k_{\mathbb{R}} \) for some \( 2 \leq k \leq n \) and the second one is isometric to \( H^2_{\mathbb{H}} \). Theorem 1.1 is a generalized version of the theorem of Maskit [10, Theorem V.G.18] for SU(n, 1) in the geometric aspect.

We remark here that Fu and Xie [4] gave a sufficient condition for a discrete subgroup of SU(n, 1) to preserve a two-dimensional totally geodesic submanifold in \( H^n_{\mathbb{C}} \). More precisely, they proved that, if \( \Gamma \) is a nonelementary discrete subgroup of SU(n, 1) and all eigenvalues are real for every loxodromic element of \( \Gamma \), then \( \Gamma \) preserves a two-dimensional totally geodesic submanifold in \( H^n_{\mathbb{C}} \). However, this sufficient condition is not a necessary condition, in the sense that there are many nonelementary discrete subgroups \( \Gamma \) of SU(n, 1), so that \( \Gamma \) preserves a two-dimensional totally geodesic submanifold in \( H^n_{\mathbb{C}} \), but does not have all eigenvalues real for every loxodromic element of \( \Gamma \).

In this paper, we also study discrete subgroups of Sp(n, 1) with real trace fields. Since the division ring \( \mathbb{H} \) of the quaternions is not commutative, the situation is quite different from the SU(n, 1) case. For instance, the usual definition of trace is not invariant under conjugation in Sp(n, 1). Nonetheless, it turns out that the trace field of subgroups of Sp(n, 1) is a useful tool in characterizing the discrete subgroups of Sp(n, 1) preserving a totally geodesic submanifold of constant negative sectional curvature in \( H^n_{\mathbb{C}} \), which is not isometric to \( H^2_{\mathbb{H}} \).

Let \( \Gamma \) be a discrete subgroup of Sp(n, 1). Following the definition of the trace field as usual, one can obtain the skew field generated by the traces of all the elements of \( \Gamma \) over \( \mathbb{Q} \). We call this skew field the trace field of \( \Gamma \). Note that the trace field of \( \Gamma \) may be not commutative, and is not invariant under conjugation in Sp(n, 1). Kim [8] showed that a nonelementary discrete
subgroup of $\text{Sp}(2,1)$ with real trace field preserves a copy of $\mathbb{H}^2_\mathbb{R}$ or $\mathbb{H}^1_\mathbb{C}$ in $\mathbb{H}^2_\mathbb{H}$. In accordance with his result, we expect that the trace field of $\Gamma$ being real indicates a specific geometric property of $\Gamma$, as in the $\text{SU}(n,1)$ case, and we will obtain a theorem analogous to Theorem 1.1 for $\text{Sp}(n,1)$.

**Theorem 1.2.** Let $\Gamma$ be a nonelementary discrete subgroup of $\text{Sp}(n,1)$. If the trace field of $\Gamma$ is real, then $\Gamma$ preserves a totally geodesic submanifold of constant negative sectional curvature in $\mathbb{H}^n_\mathbb{H}$, which is not isometric to $\mathbb{H}^1_\mathbb{H}$.

Both Theorems 1.1 and 1.2 imply that if a nonelementary discrete subgroup $\Gamma$ of $\text{SU}(n,1)$ or $\text{Sp}(n,1)$ has a real trace field, $\Gamma$ acts on a real hyperbolic space of dimension at least 2, and thus it may be regarded as a nonelementary discrete subgroup of $\text{SO}(n,1)$. Hence we have the following theorem.

**Theorem 1.3.** Let $\Gamma$ be a nonelementary torsion-free discrete subgroup of $\text{SU}(n,1)$ (respectively, $\text{Sp}(n,1)$) for $n \geq 2$. Then the following are equivalent.

(i) There exists a discrete faithful representation $\rho : \Gamma \to \text{Sp}(n,1)$ such that the trace field of its image group is real.

(ii) There exists a discrete faithful representation $\rho : \Gamma \to \text{SO}(n,1)$ (respectively, $\rho : \Gamma \to \text{O}(n,1)$).

Let $\mathcal{D}(\Gamma)$ be the space of all discrete faithful representations of $\Gamma$ in $\text{Sp}(n,1)$. Then Theorem 1.3 gives a necessary and sufficient condition for the existence of a representation in $\mathcal{D}(\Gamma)$ whose trace field is real. From this point of view, one may get an answer to the question of whether, given a discrete subgroup $\Gamma$ of $\text{SU}(n,1)$ or $\text{Sp}(n,1)$, which is isomorphic to $\Gamma$, cannot be real.

**Corollary 1.4.** Let $\Gamma$ be a nonelementary discrete subgroup of $\text{SU}(n,1)$ or $\text{Sp}(n,1)$ for $n \geq 2$. If the virtual cohomological dimension of $\Gamma$ is greater than $n$, then the trace field of any discrete subgroup of $\text{SU}(n,1)$ or $\text{Sp}(n,1)$, which is isomorphic to $\Gamma$, cannot be real.

**Corollary 1.5.** Let $\Gamma$ be a nonelementary discrete subgroup of $\text{SU}(3,1)$ or $\text{Sp}(3,1)$. Suppose that $\Gamma$ is neither a hyperbolic group nor a relatively hyperbolic group. Then no trace field of a discrete subgroup of $\text{SU}(3,1)$ or $\text{Sp}(3,1)$ isomorphic to $\Gamma$ can be real.

In the $\text{SL}(2, \mathbb{C})$ case, it is not difficult to see that the condition for a discrete group’s being nonelementary is equivalent to the condition for a discrete group being irreducible. Hence one can restate Maskit’s theorem as follows: if the trace field of an irreducible discrete subgroup $\Gamma$ of $\text{SL}(2, \mathbb{C})$ is real, then $\Gamma$ is conjugate to a subgroup of $\text{SL}(2, \mathbb{R})$. From this viewpoint, we can establish another generalized version of Maskit’s theorem for $\text{SU}(n,1)$ and $\text{Sp}(n,1)$ in algebraic terms.

**Theorem 1.6.** Let $\Gamma$ be an irreducible discrete subgroup of $\text{SU}(n,1)$ (respectively, $\text{Sp}(n,1)$). Then the trace field of $\Gamma$ is real if and only if $\Gamma$ is conjugate to a Zariski-dense discrete subgroup of $\text{SO}(n,1)$ (respectively, $\text{O}(n,1)$).
2. Preliminaries

2.1. Complex hyperbolic spaces

Let $\mathbb{C}^{n,1}$ be a complex vector space of dimension $n+1$ with a Hermitian form of signature $(n,1)$. An element of $\mathbb{C}^{n,1}$ is a column vector $z = (z_1, \ldots, z_{n+1})^t$. In what follows, we choose the Hermitian form on $\mathbb{C}^{n,1}$ given by the matrix $I_{n,1}$

$$I_{n,1} = \begin{bmatrix} I_n & 0 \\ 0 & -1 \end{bmatrix}.$$ 

Thus $\langle z, w \rangle = w^* I_{n,1} z = z_1 \overline{w}_1 + z_2 \overline{w}_2 + \cdots + z_n \overline{w}_n - z_{n+1} \overline{w}_{n+1}$, where $w^*$ is the Hermitian transpose of $w$.

Let $\mathbb{P} : \mathbb{C}^{n,1} \setminus \{0\} \to \mathbb{C}P^n$ be the canonical projection onto complex projective space. Consider the following subspaces in $\mathbb{C}^{n,1}$:

$$V_0 = \{ z \in \mathbb{C}^{n,1} - \{0\} | \langle z, z \rangle = 0 \}, \quad V_- = \{ z \in \mathbb{C}^{n,1} | \langle z, z \rangle < 0 \}.$$ 

The $n$-dimensional complex hyperbolic space $H^n_{\mathbb{C}}$ is defined as $\mathbb{P}(V_-)$. The boundary $\partial H^n_{\mathbb{C}}$ is defined as $\mathbb{P}(V_0)$.

For a vector $v$ in $\mathbb{C}^{n,1} \setminus \{0\}$, we shall use the notation $\hat{v}$ to denote the point $\mathbb{P}(v)$ in $\mathbb{C}P^n$. If a point $p$ in $\mathbb{C}P^n$ is given, then the inverse space $\mathbb{P}^{-1}(p)$ is one-dimensional. We shall denote a vector in $\mathbb{P}^{-1}(p)$ by $\tilde{p}$ in the situation that the choice of a vector in $\mathbb{P}^{-1}(p)$ makes no confusion, and likewise when the definition of the Bergmann metric is given below.

The Bergmann metric on $H^n_{\mathbb{C}}$ is given by the distance formula

$$\cosh^2 \left( \frac{\rho(p, q)}{2} \right) = \frac{\langle \hat{p}, \hat{q} \rangle \langle \hat{q}, \hat{p} \rangle}{\langle \hat{p}, \tilde{p} \rangle \langle \tilde{q}, \tilde{q} \rangle},$$ 

for $p, q \in H^n_{\mathbb{C}}$. Note that any complex multiplication applied to $\tilde{p}$, or to $\tilde{q}$ in the right-hand side of the above relation will make no difference to its value:

$$\frac{\langle \lambda \hat{p}, \hat{q} \rangle \langle \hat{q}, \lambda \hat{p} \rangle}{\langle \lambda \hat{p}, \lambda \tilde{p} \rangle \langle \lambda \tilde{q}, \lambda \tilde{q} \rangle} = \frac{\lambda \langle \hat{p}, \hat{q} \rangle \lambda \langle \hat{q}, \hat{p} \rangle}{\lambda \langle \hat{p}, \tilde{p} \rangle \langle \tilde{q}, \tilde{q} \rangle} = \frac{\langle \hat{p}, \hat{q} \rangle \langle \hat{q}, \hat{p} \rangle}{\langle \hat{p}, \tilde{p} \rangle \langle \tilde{q}, \tilde{q} \rangle}.$$

Let $U(n,1)$ be the unitary group corresponding to this Hermitian form. Then the holomorphic isometry group of $H^n_{\mathbb{C}}$ is the projective unitary group $PU(n,1)$, and the full isometry group of $H^n_{\mathbb{C}}$ is generated by $PU(n,1)$ and complex conjugation. We denote by $SU(n,1)$ the subgroup of linear transformations in $U(n,1)$ with determinant $1$. We note that this group acts transitively by isometries on $H^n_{\mathbb{C}}$. Then the usual trichotomy which classifies isometries of real hyperbolic spaces also holds here; that is:

1. an isometry is loxodromic if it fixes exactly two points of $\partial H^n_{\mathbb{C}}$;
2. an isometry is parabolic if it fixes exactly one point of $\partial H^n_{\mathbb{C}}$;
3. an isometry is elliptic if it fixes at least one point of $\partial H^n_{\mathbb{C}}$.

In $H^n_{\mathbb{C}}$, it is well known that there are two types of totally geodesic submanifolds $H^k_{\mathbb{C}}$ and $\overline{H}^k_{\mathbb{C}}$. Note that a totally geodesic submanifold of constant negative sectional curvature is either of the form $H^k_{\mathbb{C}}$ or $\overline{H}^k_{\mathbb{C}}$. We say that a discrete group is elementary if its limit set consists of at most two points, and the others are called nonelementary.

Definition 1. Let $\Gamma$ be a subgroup of $SU(n,1)$. Then the trace field of $\Gamma$ is defined as the field generated by the traces of all the elements of $\Gamma$ over the base field $\mathbb{Q}$ of rational numbers.

See [6] and McReynolds (see the forthcoming paper ‘Arithmetic Lattices in $SU(n,1)$’) for more details about the trace field.
2.2. Quaternionic hyperbolic spaces

Let \( \mathbb{H}^{n,1} \) be a quaternionic vector space of dimension \( n + 1 \) with a Hermitian form of signature \((n,1)\). An element of \( \mathbb{H}^{n,1} \) is a column vector \( p = (p_1, \ldots, p_{n+1})^{\mathsf{T}} \). As in the complex hyperbolic case, we choose the Hermitian form on \( \mathbb{H}^{n,1} \) given by the matrix \( I_{n,1} \):

\[
I_{n,1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
\]

Thus \( \langle p, q \rangle = q^{\ast}I_{n,1}p = \overline{q_1}p_1 + \overline{q_2}p_2 + \cdots + \overline{q_{n+1}}p_{n+1} \), where \( q^{\ast} \) is the Hermitian transpose of \( q \). The group \( \text{Sp}(n,1) \) is the subgroup of \( \text{GL}(n + 1, \mathbb{H}) \) which, when acting on the left, preserves the Hermitian form given above.

Let \( \mathbb{P} : \mathbb{H}^{n,1} \setminus \{0\} \rightarrow \mathbb{H}P^n \) be the canonical projection onto quaternionic projective space. Consider the following subspaces in \( \mathbb{H}^{n,1} \):

\[
V_0 = \{ z \in \mathbb{H}^{n,1} - \{0\} \mid \langle z, z \rangle = 0 \}, \quad V_- = \{ z \in \mathbb{H}^{n,1} \mid \langle z, z \rangle < 0 \}.
\]

The \( n \)-dimensional quaternionic hyperbolic space \( \mathbb{H}^n \) is defined as \( \mathbb{P}(V_-) \). The boundary \( \partial \mathbb{H}^n \) is defined as \( \mathbb{P}(V_0) \). There is a metric on \( \mathbb{H}^n \) called the Bergman metric and the isometry group of \( \mathbb{H}^n \) with respect to this metric is \( \text{PSp}(n,1) = \{ [A] : A \in \text{GL}(n + 1, \mathbb{H}), \langle p, q \rangle = \langle Ap, Aq \rangle, p, q \in \mathbb{H}^{n,1} \} = \{ [A] : A \in \text{GL}(n + 1, \mathbb{H}), I_{n,1} = A^{\ast}I_{n,1}A \} \),

where \( [A] : \mathbb{H}P^n \rightarrow \mathbb{H}P^n; x + \mathbb{H} \mapsto (Ax) + \mathbb{H} \) for \( A \in \text{Sp}(n,1) \). Here we adopt the convention that the action of \( \text{Sp}(n,1) \) on \( \mathbb{H}^n \) is on the left and the action of projectivization of \( \text{Sp}(n,1) \) is a right action. In fact, \( \text{PSp}(n,1) \) is the quotient group by the real scalar matrices in \( \text{Sp}(n,1) \). Thus it is not difficult to see that

\[
\text{PSp}(n,1) = \text{Sp}(n,1)/\{ \pm I \}.
\]

Similarly to case of complex hyperbolic space, the totally geodesic submanifolds of quaternionic hyperbolic space are isometric to either \( \mathbb{H}^k_S, \mathbb{H}^k_C \), or \( \mathbb{H}^k_R \) for some \( 1 \leq k \leq n \). Note that a totally geodesic submanifold of constant negative sectional curvature is isometric to either \( \mathbb{H}^k_S \) for some \( 2 \leq k \leq n \), \( \mathbb{H}^1_C \) or \( \mathbb{H}^1_R \). The classification of isometries by their fixed points is exactly the same as in the complex hyperbolic case.

**Definition 2.** Let \( \Gamma \) be a subgroup of \( \text{Sp}(n,1) \). Then the **trace field** of \( \Gamma \) is defined as the skew field generated by the traces of all the elements of \( \Gamma \) over the base field \( \mathbb{Q} \) of rational numbers.

We say that the trace field of \( \Gamma \) is **real** if the trace field of \( \Gamma \) is contained in \( \mathbb{R} \).

2.3. The Zariski topology

Let \( \mathbb{R}[x_{1,1}, \ldots, x_{n,n}] \) denote the set of real polynomials in the \( n^2 \) variables \( \{x_{j,k} \mid 1 \leq j, k \leq n\} \). A subset \( H \) of \( \text{SL}(n, \mathbb{R}) \) is called **Zariski-closed** if there is a subset \( S \) of \( \mathbb{R}[x_{1,1}, \ldots, x_{n,n}] \) such that \( H \) is the zero locus of \( S \). In particular, when \( H \) is a subgroup of \( \text{SL}(n, \mathbb{R}) \), \( H \) is called a real **algebraic group**. It is a standard fact that any Zariski-closed subset of \( \text{SL}(n, \mathbb{R}) \) has only finitely many components. Furthermore, a Zariski-closed subgroup of \( \text{SL}(n, \mathbb{R}) \) is a \( C^\infty \) submanifold of \( \text{SL}(n, \mathbb{R}) \) and, so, a Lie group.

**Definition 3.** The **Zariski closure** of a subset \( H \) of \( \text{SL}(n, \mathbb{R}) \) is the (unique) smallest Zariski-closed subset of \( \text{SL}(n, \mathbb{R}) \) that contains \( H \). We use \( \overline{H} \) to denote the Zariski closure of \( H \).
It is well known that if $H$ is a subgroup of $\text{SL}(n, \mathbb{R})$, then $\overline{H}$ is also a subgroup of $\text{SL}(n, \mathbb{R})$.

**Definition 4.** A subgroup $H$ of $\text{SL}(n, \mathbb{R})$ is *almost Zariski closed* if $H$ is a finite-index subgroup of $\overline{H}$.

We remark that a connected subgroup $H$ of $\text{SL}(n, \mathbb{R})$ is almost Zariski closed if and only if it is the identity component of a Zariski-closed subgroup.

### 2.4. Lie groups

Here we collect some definitions and basic facts in algebraic group theory.

**Definition 5.** Let $U_n$ be the subgroup of $\text{SL}(n, \mathbb{R})$ consisting of lower-triangular matrices with 1 on the diagonal.

(i) A subgroup $U$ of $\text{SL}(n, \mathbb{R})$ is **unipotent** if it is conjugate to a subgroup of $U_n$.

(ii) A subgroup $T$ of $\text{SL}(n, \mathbb{R})$ is a **torus** if $T$ is conjugate (over $\mathbb{C}$) to a group of diagonal matrices, $T$ is connected, and $T$ is almost Zariski closed.

(iii) A closed subgroup $L$ of $\text{SL}(n, \mathbb{R})$ is **semisimple** if its identity component has no nontrivial, connected, abelian, normal subgroups.

(iv) A group $G$ is **solvable** if there is a chain $e = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_k = G$ of subgroups of $G$ such that $G_{i-1}$ is a normal subgroup of $G_i$ and the quotient group $G_i/G_{i-1}$ is abelian for $1 \leq i \leq k$.

(v) A Lie group $G$ is **amenable** if $G$ has a fixed point in every nonempty, compact, convex $G$-space.

It is well known that abelian groups, compact groups and solvable groups are amenable groups. For more details, see [12, Section 3.4 and 4.4].

### 2.5. Limit set

Let $G$ be a semisimple Lie group of rank 1 and $X$ be the associated symmetric space. For a subgroup $\Gamma$ of $G$, the **limit set** $\Lambda(\Gamma)$ of $\Gamma$ is defined as the accumulation set in $\partial X$ of the $\Gamma$-orbit of any point $x \in X$. It is known that $\Lambda(\Gamma)$ is the smallest closed $\Gamma$-invariant set in $\partial X$.

**Definition 6.** A subgroup $\Gamma$ of $G$ is called **elementary** if its limit set is finite. Otherwise, $\Gamma$ is called nonelementary.

In fact, if $\Gamma$ is elementary, its limit set has 0, 1 or 2 points. If $\Gamma$ is nonelementary, then its limit set is perfect, that is, each point of $\Lambda(\Gamma)$ is an accumulation point of this set. For further details, we refer the reader to [7, Section 3.6 and 4.7].

### 3. Complex hyperbolic Kleinian groups

We are concerned with subgroups of $\text{SU}(n, 1)$ whose trace fields are real. Let us define a subset $\mathcal{R}_{su}$ of $\text{SU}(n, 1)$ by

$$\mathcal{R}_{su} = \{ g \in \text{SU}(n, 1) \mid \text{tr}(g) \in \mathbb{R} \}.$$

Then our starting observation is that $\mathcal{R}_{su}$ is Zariski closed in the following sense: It is well known that complex numbers $a + ib$ can be represented by $2 \times 2$ real matrices that have the
following form: 
\[
\begin{bmatrix}
a & -b \\
b & a \\
\end{bmatrix}.
\]
Via this representation, one can embed SU\((n,1)\) into SL\((2n+2,\mathbb{R})\). Let us denote the embedding by \(\phi : SU(n,1) \rightarrow SL(2n+2,\mathbb{R})\).

**Lemma 3.1.** Let \((\phi \text{ and } R_{su})\) be defined as above. Then SL. \(\phi(R_{su})\) is a Zariski closed subset of SL.

**Proof.** First note that SU\((n,1)\) is a Zariski-closed subgroup of SL\((2n+2,\mathbb{R})\). Let \(g = (g_{m,l})\) be a matrix in SU\((n,1)\), where \(g_{m,l} = a_{m,l} + ib_{m,l}\) for \(a_{m,l}, b_{m,l} \in \mathbb{R}\).

Clearly, \(\text{tr}(g) = g_{1,1} + \cdots + g_{n+1,n+1}\), and hence it is easy to see that \(\text{tr}(g) \in \mathbb{R}\) if and only if \(b_{1,1} + \cdots + b_{n+1,n+1} = 0\). Since \(b_{1,1} + \cdots + b_{n+1,n+1}\) corresponds to a real polynomial with variables in the matrix entries of SL\((2n+2,\mathbb{R})\), \(R_{su}\) is a Zariski-closed subset of SL\((2n+2,\mathbb{R})\).

We consider the Zariski topology on SL\((2n+2,\mathbb{R})\), and then the pullback topology on SU\((n,1)\) under the embedding \(\phi : SU(n,1) \rightarrow SL(2n+2,\mathbb{R})\). Let \(\Gamma\) be a subgroup of SU\((n,1)\) whose trace field is contained in \(\mathbb{R}\). Then \(\Gamma\) is a subset of \(R_{su}\). Since \(R_{su}\) is Zariski closed according to Lemma 3.1, the Zariski closure of \(\Gamma\), denoted by \(\overline{\Gamma}\), is contained in \(R_{su}\). From this observation, we immediately obtain the following corollary.

**Corollary 3.2.** Let \(\Gamma\) be a subgroup of SU\((n,1)\). Then every element of \(\Gamma\) has real trace if and only if every element of \(\overline{\Gamma}\) has real trace.

It is well known that the Zariski closure of a subgroup of SL\((2n+2,\mathbb{R})\) is a Zariski-closed subgroup of SL\((2n+2,\mathbb{R})\), and any Zariski-closed subgroup is a Lie group with finitely many connected components. In particular, the identity component is a normal subgroup and the connected components are the cosets of the identity component.

Corollary 3.2 means that it is sufficient to work with Zariski-closed subgroups of SU\((n,1)\) to characterize the subgroups of SU\((n,1)\) whose trace fields are real. If \(\Gamma\) is nonelementary, its Zariski closure \(\overline{\Gamma}\) cannot have a small dimension.

**Lemma 3.3.** Let \(G\) be a rank 1 semisimple Lie group and \(\Gamma\) be a nonelementary discrete subgroup of \(G\). Then \(\overline{\Gamma}\) has dimension at least 3.

**Proof.** Suppose that \(\overline{\Gamma}\) has dimension at most 2. Denote by \(\overline{\Gamma}^0\) the identity component of \(\overline{\Gamma}\). Then \(\overline{\Gamma}^0\) is a connected real algebraic group with dimension at most 2. According to [1, Corollary 11.6], \(\overline{\Gamma}^0\) is solvable. This implies that \(\overline{\Gamma}\) is virtually solvable since \(\overline{\Gamma}^0\) is a finite-index subgroup of \(\overline{\Gamma}\). Hence \(\overline{\Gamma}\) is amenable. Then, by [11, Theorem 3.3], \(\overline{\Gamma}\) fixes a finite subset of \(X \cup \partial X\). This contradicts the assumption that \(\Gamma\) is nonelementary. Thus \(\overline{\Gamma}\) has dimension at least 3.

To characterize the nonelementary subgroups with real trace fields, we reduce to the problem of characterizing the Zariski-closed subgroups of SU\((n,1)\) with dimension at least 3, for which the trace of every element is a real number. It is much easier to deal with Zariski-closed subgroups with dimension at least 3 than arbitrary nonelementary subgroups. This is the key idea of the present paper. We now recall the structure theorem for almost Zariski-closed groups. We refer the reader to [12, Theorem 4.4.7] for more details.
Theorem 3.4 ([12]). Let $H$ be a connected subgroup of $\text{SL}(m, \mathbb{R})$ that is almost Zariski closed. Then there exist

(i) a semisimple subgroup $L$ of $H$,
(ii) a torus $T$ in $H$, and
(iii) a unipotent subgroup $U$ of $H$,

such that

(i) $H = (LT) \ltimes U$,
(ii) $L, T, and U$ are almost Zariski closed, and
(iii) $L$ and $T$ centralize each other and have finite intersection.

This structure theorem allows us to look at the finer structure of almost Zariski-closed subgroups in the case of $\text{SU}(n, 1)$ as follows.

Lemma 3.5. Let $G$ be a rank 1 semisimple Lie group. Let $H$ be a nonamenable, connected, almost Zariski-closed subgroup of $G$. Let $H = (LT) \ltimes U$ be the decomposition in Theorem 3.4. Then $L$ is a connected, noncompact, semisimple Lie group with real rank 1 and, moreover, $U$ is trivial.

Proof. The connectedness of $L$ follows from the connectedness of $H$. The possible values for the real rank of $L$ are either 0 or 1 since $G$ has real rank 1. If the real rank of $L$ is 0, then $L$ is compact. In this case, since all of $L, T$, and $U$ are amenable, $H$ is amenable. This contradicts the assumption that $H$ is not amenable. Thus the real rank of $L$ must be 1. Note that this is equivalent that $L$ is noncompact.

Now we will prove that $U$ is trivial. Assume that $U$ is not trivial. Then, since every nontrivial unipotent element of $G$ is a parabolic isometry acting on the symmetric space $X$ associated to $G$, each element of $U$ has a unique fixed point on $\partial X$. Moreover, since $U$ is a unipotent subgroup of $G$, $U$ also has a unique fixed point on $\partial X$. Let $\xi$ be the unique fixed point of $U$. Since $U$ is a normal subgroup of $H$, we have $U = lUl^{-1}$ for all $l \in L$. This implies that $U$ fixes $l(\xi)$ for all $l \in L$. Noting that $\xi$ is the unique fixed point of $U$, we have $l(\xi) = \xi$ for all $l \in L$. In other words, $L$ is contained in the stabilizer subgroup of $\xi$ in $G$. However, the subgroup of $G$ stabilizing $\xi$ is amenable, and thus this contradicts the fact that $L$ is not amenable. Therefore, $U$ is trivial.

Let $A$ be an $m \times n$ matrix and let $B$ be an $r \times s$ matrix. Then recall that the direct sum of $A$ and $B$, denoted by $A \oplus B$, is the $(m + r) \times (n + s)$ matrix defined by

$$A \oplus B = \begin{bmatrix}
  a_{1,1} & \cdots & a_{1,n} & 0 & \cdots & 0 \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  a_{m,1} & \cdots & a_{m,n} & 0 & \cdots & 0 \\
  0 & \cdots & 0 & b_{1,1} & \cdots & b_{1,s} \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  0 & \cdots & 0 & b_{r,1} & \cdots & b_{r,s}
\end{bmatrix}.$$}

Proposition 3.6. Let $\Gamma$ be a nonelementary discrete subgroup of $\text{SU}(n, 1)$ with real trace field. Then $\Gamma^0$ is a connected, almost Zariski-closed subgroup, and as such admits a decomposition $\Gamma^0 = (LT) \ltimes U$ as in Theorem 3.4. Let $L_{nc}$ be the noncompact factor of $L$. Then $U$ is trivial and, either $L_{nc}$ is conjugate to $I_{n-m} \oplus \text{SO}(m, 1)$ with $m \geq 2$ and $T$ is conjugate to a torus in $\text{SO}(n - m) \oplus I_{m+1}$, or $L_{nc}$ is conjugate to $I_{n-1} \oplus \text{SU}(1, 1)$ and $T$ is conjugate to a torus in $\text{SO}(n - 1) \oplus I_2$. 


Proof. Since $\Gamma$ is nonelementary, $\Gamma^0\cap\mathfrak{g}$ cannot be amenable. Applying Lemma 3.5 to $\Gamma^0\cap\mathfrak{g}$, it follows that $U$ is trivial and $L$ is a connected, noncompact, semisimple Lie group of real rank 1.

To prove the first statement, note that every element of $\Gamma^0\cap\mathfrak{g}$ has real trace. Applying Corollary 3.2. Let $Y$ be the rank 1 symmetric space associated with $L$. Then $Y$ is a totally geodesic submanifold of $H^n_C$. Due to the classification of totally geodesic submanifolds in $H^n_C$, $Y$ is isometric to either a totally complex geodesic $m$-submanifold $H^m_C$ or a totally real geodesic $m$-submanifold $H^m_R$ for some $1 \leq m \leq n$.

First suppose that $Y$ is isometric to $H^m_C$. Then $L$ is contained in the stabilizer subgroup of $Y$ in $SU(n,1)$. Hence, after conjugation if necessary, we may assume that $L$ is contained in $SU(n-m) \oplus SO(m,1)$. Let $L_c$ be the compact factor of $L$ and let $L_{nc}$ be its noncompact factor. Then $L_c \subset SU(n-m)$ and $L_{nc} \subset SO(m,1)$. Note that the symmetric space associated with $L_{nc}$ is also $Y$. Hence $L_{nc}$ is a connected semisimple Lie group isogenous to $SO(m,1)$. Noting that any connected semisimple Lie group is almost Zariski closed, it is easy to see that $L_{nc}$ is an almost Zariski-closed subgroup of $SO(m,1)$ of finite index. Since $SO(m,1)$ is connected, it has no proper almost Zariski-closed subgroups of finite index. Therefore, $L_{nc}$ has to be equal to $SO(m,1)$. Recall that it is required that $L_{nc}$ is not amenable and every element of $L_{nc}$ has real trace. If $m = 1$, then $SU(n-1) \oplus SO(1,1)$ is amenable, and thus $m \geq 2$. Since every element of $SO(m,1)$ has real trace, $SO(m,1)$ for $m \geq 2$ is a possible semisimple Lie group for $L_{nc}$.

Next we suppose that $Y$ is isometric to $H^m_R$. As in the previous case, it can be easily seen that $L$ is contained in $SU(n-m) \oplus SU(m,1)$ and $L_{nc} = SU(m,1)$ after conjugation, if necessary. Since every element of $L_{nc}$ must have real trace, the trace of every element of $SU(m,1)$ has to be a real number. This is possible only when $m = 1$. Thus $L_{nc}$ is conjugate to $SU(1,1)$.

Now only the second statement remains. Since $L$ and $T$ centralize each other, $T$ also stabilizes the symmetric space $Y$. Hence $T$ is contained in $SU(n-m)$, after conjugation, if necessary. We set $r = n - m$. Since any torus is contained in a maximal torus, after conjugation, if necessary, we may assume that $T$ is a torus contained in the maximal torus $T_{max}$ defined by

$$T_{max} = \{e^{i\theta_1} \oplus \cdots \oplus e^{i\theta_r} | e^{i\theta_1} \cdots e^{i\theta_r} = 1, \forall \theta_i, \theta_j \in \mathbb{R}\}.$$ 

Let $S$ be a one-dimensional torus in $T$. Then $S$ can be written as

$$S = \{e^{ia_1 t} \oplus \cdots \oplus e^{ia_r t} | e^{ia_1 t} \cdots e^{ia_r t} = 1, t \in \mathbb{R}\}$$

for some $(a_1, \ldots, a_r) \in \mathbb{R}^r$. In order that every element of $S$ have real trace, for all $t \in \mathbb{R}$ we must have

$$\sin a_1 t + \cdots + \sin a_r t = 0.$$

Differentiating both sides repeatedly with respect to $t$ and then putting $t = 0$, it is easy to see that, for all integers $s \geq 0$,

$$a_1^{2s+1} + \cdots + a_r^{2s+1} = 0. \tag{3.1}$$

It is not difficult to see that, up to ordering, any solution of (3.1) is of the form

$$(a_1, -a_1, \ldots, a_k, -a_k, 0, \ldots, 0).$$

Thus $S = \{R(a_1 t) \oplus \cdots \oplus R(a_k t) \oplus I_{r-2k} | t \in \mathbb{R}\}$, where $R(\theta) = e^{i\theta} \oplus e^{-i\theta}$.

In a similar way to one-dimensional torus case, it can be shown that any torus of $T_{max}$ in which every element has real trace is of the form

$$S_1 \oplus \cdots \oplus S_l,$$

where $S_i$ is a one-dimensional torus of the form $R(a_1 t) \oplus \cdots \oplus R(a_k t)$. 

REAL TRACE FIELDS
Noting that
\[
\begin{pmatrix}
 i & i \\
 1 & -1
\end{pmatrix}
\begin{pmatrix}
 e^{i\theta} & 0 \\
 0 & e^{-i\theta}
\end{pmatrix}
\begin{pmatrix}
 i & i \\
 1 & -1
\end{pmatrix}^{-1} =
\begin{pmatrix}
 \cos \theta & -\sin \theta \\
 \sin \theta & \cos \theta
\end{pmatrix},
\]
it easily follows that $T$ is conjugate to a torus in $SO(r)$.

**Remark 1.** In the proof of Proposition 3.6, we used the fact that the trace is invariant under conjugation in $GL(n, \mathbb{C})$. However, the usual definition of trace is not invariant under conjugation in $GL(n, \mathbb{H})$. Hence the proof as in Proposition 3.6 is not available in the case of $Sp(n,1)$.

**Theorem 3.7.** Let $\Gamma$ be a nonelementary discrete subgroup of $SU(n,1)$ with real trace field. Then $\Gamma$ is conjugate to a subgroup of either $S(U(n-m) \oplus O(m,1))$ for $m \geq 2$ or $SU(n-1) \oplus SU(1,1)$.

**Proof.** The Zariski closure $\overline{\Gamma}$ of $\Gamma$ is a Lie group with finitely many components. Hence its identity component $\overline{\Gamma}^0$ is a Lie group with finitely many components. Hence the proof as in Proposition 3.6 is not available in the case of $Sp(n,1)$.

Let $\Gamma$ be a nonelementary discrete subgroup of $SU(n,1)$ with real trace field. Then $\Gamma$ is conjugate to a subgroup of either $S(U(n-m) \oplus O(m,1))$ for $m \geq 2$ or $SU(n-1) \oplus SU(1,1)$.

**Proof.** The Zariski closure $\overline{\Gamma}$ of $\Gamma$ is a Lie group with finitely many components. Hence its identity component $\overline{\Gamma}^0$ is a finite index normal subgroup of $\overline{\Gamma}$ and, moreover, $\overline{\Gamma}$ can be written as
\[
\overline{\Gamma} = \bigcup_{k=1}^r \gamma_k \overline{\Gamma}^0.
\]

According to Proposition 3.6, $\overline{\Gamma}^0$ stabilizes a totally geodesic submanifold $Y$ of $H^n_\mathbb{R}$, which is isometric to either $H^n_{\mathbb{R}}$ for $m \geq 2$ or $H^n_{\mathbb{C}}$. Since $\overline{\Gamma}^0$ is a normal subgroup of $\overline{\Gamma}$, we have
\[
\overline{\Gamma}^0 = \gamma_k \overline{\Gamma}^0 \gamma_k^{-1}
\]
for all $\gamma_k$. This means that $\overline{\Gamma}^0$ stabilizes $\gamma_k(Y)$. Since $\overline{\Gamma}^0$ cannot stabilize two distinct copies of $Y$, we have $\gamma_k(Y) = Y$, that is, $\gamma_k$ also stabilizes $Y$ for all $k = 1, \ldots, r$ and so does $\overline{\Gamma}$.

In the case where $Y$ is isometric to $H^n_{\mathbb{R}}$, the stabilizer group of $Y$ in $SU(n,1)$ is conjugate to $S(U(n-m) \oplus O(m,1))$. Hence $\Gamma \subset S(U(n-m) \oplus O(m,1))$ after conjugation, if necessary. If $Y$ is isometric to $H^n_{\mathbb{C}}$, then the stabilizer group of $Y$ in $SU(n,1)$ is conjugate to $S(U(n-1) \oplus U(1,1))$ and so $\gamma_k \in S(U(n-1) \oplus U(1,1))$ up to conjugation. Write $\gamma_k = \gamma_k^c \gamma_k^{nc}$ for $\gamma_k^c \in U(n-1)$ and $\gamma_k^{nc} \in U(1,1)$. As shown in the proof of Proposition 3.6, $L_{nc} = I_{n-1} \oplus SU(1,1) \subset \Gamma$. Hence the trace of every element of $\gamma_k(I_{n-1} \oplus SU(1,1))$ must be a real number. Let $\gamma_k^{nc} \in e^{i\theta} + SU(1,1)$. Then we have
\[
\text{tr}(\gamma_k^{c}) + te^{i\theta} \in \mathbb{R} \quad \text{for all } t \in [-2,2].
\]
This implies that $\text{tr}(\gamma_k^{c}) \in \mathbb{R}$ and $e^{i\theta} \in \mathbb{R}$, that is, $e^{i\theta} = \pm 1$. Hence $\gamma_k^{nc}$ is an element of $SU(1,1)$. Furthermore, since the determinant of $\gamma_k$ is 1, it follows that $\gamma_k \in SU(n-1)$. This completes the proof.

Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** According to Theorem 3.7, $\Gamma$ preserves a totally geodesic submanifold, which is isometric to $H^n_{\mathbb{R}}$ for some $m \geq 2$ or $H^n_{\mathbb{C}}$. These totally geodesic submanifolds have constant negative sectional curvature. Therefore Theorem 1.1 follows immediately.

In particular, when $\Gamma$ is irreducible, the possible Lie group for $\overline{\Gamma}^0$ is only $SO(n,1)$. Hence Theorem 1.6 follows in the $SU(n,1)$ case.

**Theorem 3.8.** Let $\Gamma$ be an irreducible discrete subgroup of $SU(n,1)$. Then the trace field of $\Gamma$ is real if and only if $\Gamma$ is conjugate to a Zariski-dense discrete subgroup of $SO(n,1)$.
In the case of $n = 2$, we obtain a stronger version of the theorem of Fu, Li and Wang in [3] as a corollary.

**Corollary 3.9.** Let $\Gamma$ be a nonelementary discrete subgroup of $SU(2,1)$. Then the trace field of $\Gamma$ is real if and only if $\Gamma$ is conjugate to a subgroup of either $SO(2,1)$ or $1 \oplus SU(1,1)$.

**Proof.** It follows from Theorem 3.7 that $\Gamma$ is conjugate to a subgroup of $SO(2,1)$ or $SU(1) \oplus SU(1,1)$. Since $SU(1) = \{1\}$, the corollary immediately follows. The converse is trivial.

Note that $SU(2)$ is the following group:

$$SU(2) = \left\{ \begin{bmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{bmatrix} \middle| \alpha, \beta \in \mathbb{C}, \ |\alpha|^2 + |\beta|^2 = 1 \right\}.$$ 

Hence every element of $SU(2)$ has real trace. From this fact, when $n = 3$, we also get the result in [9] as a corollary.

**Corollary 3.10.** Let $\Gamma$ be a nonelementary discrete subgroup of $SU(3,1)$. Then the trace field of $\Gamma$ is real if and only if $\Gamma$ is conjugate to a subgroup of either $SO(3,1)$ or $SU(2) \oplus SU(1,1)$.

**Proof.** Applying Theorem 3.7 to the case of $n = 3$, it follows that $\Gamma$ is conjugate to either $SO(3,1)$ or $S(U(1) \oplus O(2,1))$ or $SU(2) \oplus SU(1,1)$. Since the determinant of every element of $O(2,1)$ is $\pm 1$, it can be easily seen that

$$S(U(1) \oplus O(2,1)) = S(O(1) \oplus O(2,1)) \subset SO(3,1).$$

Thus $\Gamma$ is conjugate to a subgroup of either $SO(3,1)$ or $SU(2) \oplus SU(1,1)$. As observed above, since the trace of every element of $SU(2)$ is a real number, the converse clearly holds.

4. Quaternionic hyperbolic Kleinian groups

As seen in the previous section, the trace field is a useful tool in recognizing subgroups of $SU(n,1)$, that stabilize a totally geodesic submanifold of constant negative sectional curvature. The question naturally arises as to whether this works in the setting of $Sp(n,1)$ or not. The main difficulty in extending the argument in the $SU(n,1)$ case to $Sp(n,1)$ is that the trace is not invariant under conjugation in $Sp(n,1)$. This is due to the noncommutativity of the division ring $\mathbb{H}$ of quaternions. In the $SU(n,1)$ case, if the trace field of a subgroup $\Gamma$ of $SU(n,1)$ is not real, then neither is the trace field of any subgroup conjugate to $\Gamma$, since the trace field is invariant under conjugation. However, this does not work in $Sp(n,1)$. Even if the set of traces of a subgroup $\Gamma$ of $Sp(n,1)$ is not real, it is possible that the trace field of some subgroup conjugate to $\Gamma$ can be real. This is the main difference between $SU(n,1)$ and $Sp(n,1)$. We will give such an example in Section 4.1. Nonetheless, Kim [8] gives a positive answer for $Sp(2,1)$ as follows.

**Theorem 4.1 (Kim).** Let $\Gamma < Sp(2,1)$ be a nonelementary quaternionic hyperbolic Kleinian group containing a loxodromic element fixing 0 and $\infty$. Assume that the sum of the diagonal entries of each element of $\Gamma$ is real. Then $\Gamma$ stabilizes a copy of either $H^3_\mathbb{R}$ or $H^3_\mathbb{C}$.

In accordance with his result, one can expect that the usual definition of trace in $Sp(n,1)$ will also be useful in recognizing subgroups of $Sp(n,1)$, which preserve some specific totally geodesic submanifold.
Throughout this section, \( \Gamma \) denotes a nonelementary discrete subgroup of \( \text{Sp}(n,1) \) with real trace field, and we will stick to the notation used in Lemma 4.2 and denote by \( L_{nc} \) the noncompact factor of \( L \).

Similarly to the \( \text{SU}(n,1) \) case, we define a subset \( \mathcal{R}_{sp} \) of \( \text{Sp}(n,1) \) by

\[
\mathcal{R}_{sp} = \{ g \in \text{Sp}(n,1) \mid \text{tr}(g) \in \mathbb{R} \}.
\]

It is a standard fact that \( \text{Sp}(n,1) \) can be embedded in \( \text{SL}(4n, \mathbb{R}) \) by identifying \( \mathbb{H}^n \) with \( \mathbb{R}^{4n} \). More precisely, a quaternion \( a + bi + cj + dk \) can be written as a \( 4 \times 4 \) real matrix,

\[
\begin{bmatrix}
a & b & c & d \\
-b & a & -d & c \\
-c & d & a & -b \\
-d & -c & b & a
\end{bmatrix}.
\]

By a similar proof to that of Lemma 3.1, it follows that \( \mathcal{R}_{sp} \) is Zariski closed in \( \text{SL}(4n, \mathbb{R}) \). Furthermore, it is not difficult to see that Corollary 3.2 and Lemma 3.5 also work in the setting of \( \text{Sp}(n,1) \) by following their proofs in the \( \text{SU}(n,1) \) case. Hence we have the following lemma.

**Lemma 4.2.** Let \( \Gamma \) be a nonelementary discrete subgroup of \( \text{Sp}(n,1) \) with real trace field. Then every element of the Zariski closure \( \overline{\Gamma} \) of \( \Gamma \) also has real trace. Furthermore, there exist a connected real rank 1 semisimple subgroup \( L \) of \( \overline{\Gamma}^0 \) and a torus \( T \) in \( \overline{\Gamma}^0 \), such that \( L \) and \( T \) centralize each other and have finite intersection, and \( \overline{\Gamma}^0 = LT \).

If the trace was invariant under conjugation in \( \text{Sp}(n,1) \), then one could exclude \( \text{SU}(m,1) \) for \( 2 \leq m \leq n \) and \( \text{Sp}(k,1) \) for \( 1 \leq k \leq n \) from the list of possible Lie groups for \( L_{nc} \), as was done in the \( \text{SU}(n,1) \) case. Unfortunately, the trace is not invariant. This makes it difficult to find all possible Lie groups for \( L_{nc} \). First, we will start with the cases \( \text{Sp}(1,1) \) and \( \text{Sp}(2,1) \), and then we will deal with the general case.

### 4.1. The case of \( \text{Sp}(1,1) \)

Recall that

\[
\text{Sp}(1,1) = \{ g \in M_2(\mathbb{H}) \mid g^* I_{1,1} g = I_{1,1} \},
\]

where \( g^* \) is the conjugate transposed matrix of \( g \). A straightforward computation shows that

\[
\text{Sp}(1,1) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{H}) \mid |a|^2 - |c|^2 = |d|^2 - |b|^2 = 1, \ ab = \bar{c}d \right\}.
\]

For a matrix \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{Sp}(1,1) \), its inverse \( A^{-1} \) is written as

\[
A^{-1} = \begin{bmatrix} \bar{a} & -\bar{c} \\ -b & d \end{bmatrix}.
\]

In addition, it can be easily seen that \( |a| = |d| \) and \( |b| = |c| \).

Let \( \Gamma \) be a nonelementary discrete subgroup of \( \text{Sp}(1,1) \) such that \( \text{tr}(\gamma) \in \mathbb{R} \) for all \( \gamma \in \Gamma \). According to Lemma 4.2, \( \overline{\Gamma}^0 = LT \). Let \( Y \) be the symmetric space associated with \( L \). Then \( Y \) is a totally geodesic submanifold of \( \mathbb{H}^3 \), and hence \( Y \) is isometric to either \( \mathbb{H}^3_1 \) or \( \mathbb{H}^3_3 \). Noting that \( T \) centralizes \( L \) and \( \overline{\Gamma}^0 \) is a normal subgroup of \( \overline{\Gamma} \), it can be easily shown that \( \overline{\Gamma} \) stabilizes \( Y \). If \( Y = \mathbb{H}^3_1 \), then \( \overline{\Gamma} = \text{Sp}(1,1) \). However, the set of traces of \( \text{Sp}(1,1) \) is not contained in \( \mathbb{R} \). For example, take

\[
g = \begin{bmatrix} i & 0 \\ 0 & j \end{bmatrix} \in \text{Sp}(1,1),
\]

Then \( \text{tr}(g) = i + j \) is not real. For this reason, \( \overline{\Gamma} \) can never be \( \text{Sp}(1,1) \), and thus \( Y \) cannot be \( \mathbb{H}^3_1 \).
Now we suppose that $Y$ is isometric to $\mathbb{H}_C^1$. By similar reasoning as in the proof of Proposition 3.6, the noncompact simple factor $L_{nc}$ of $L$ is conjugate to $SU(1,1)$. Let $L_{nc} = gSU(1,1)g^{-1}$ for some $g \in Sp(1,1)$. Since every element of $\mathbb{T}$ has real trace, every element of $gSU(1,1)g^{-1}$ also has real trace. Note that, although the set of traces of $SU(1,1)$ is contained in $\mathbb{R}$, the set of traces of $gSU(1,1)g^{-1}$ may be not contained in $\mathbb{R}$ for some $g \in Sp(1,1)$. Here is an example. Let
\[
g = \begin{bmatrix}
\frac{1 + i}{\sqrt{2}} & 0 \\
0 & \frac{j + k}{\sqrt{2}}
\end{bmatrix} \in Sp(1,1).
\]
Any element of $SU(1,1)$ can be written as
\[
\begin{bmatrix}
z & w \\
\bar{w} & \bar{z}
\end{bmatrix},
\]
where $z$ and $w$ are complex numbers with $|z|^2 - |w|^2 = 1$. Then a straightforward computation shows that
\[
tr\left(\begin{bmatrix}
\frac{1 + i}{\sqrt{2}} & 0 \\
0 & \frac{j + k}{\sqrt{2}}
\end{bmatrix} \begin{bmatrix}
z & w \\
\bar{w} & \bar{z}
\end{bmatrix} \begin{bmatrix}
\frac{1 + i}{\sqrt{2}} & 0 \\
0 & \frac{j + k}{\sqrt{2}}
\end{bmatrix}^{-1}\right) = \frac{1}{2} \{(1 + i)z(1 - i) - (j + k)\bar{z}(j + k)\} = 2z.
\]
Since $z$ can be any complex number with $|z| \geq 1$, the set of traces of $gSU(1,1)g^{-1}$ is equal to $\{z \in \mathbb{C} | |z| \geq 2\}$, which is not contained in $\mathbb{R}$.

Consider the subset $\mathcal{R}$ of $Sp(1,1)$ defined by
\[
\mathcal{R} = \{g \in Sp(1,1) | tr(gAg^{-1}) \in \mathbb{R} \text{ for all } A \in SU(1,1)\}. \tag{4.1}
\]
One can see that $SU(1,1) \subset \mathcal{R}$. We show that the converse is also true up to the left action of $Sp(1)$ on $SU(1,1)$.

**Proposition 4.3.** Every element of $gSU(1,1)g^{-1}$ has real trace if and only if $g \in Sp(1) \cdot SU(1,1)$.

**Proof.** It is sufficient to show that $\mathcal{R} = Sp(1) \cdot SU(1,1)$. First of all, we observe that if $g \in \mathcal{R}$, then $hg \in \mathcal{R}$ for any unit quaternion number $h$ as follows: If $g \in \mathcal{R}$, then $tr(gug^{-1}) \in \mathbb{R}$ for all $u \in SU(1,1)$. It is easy to check that, for any unit quaternion number $h$,
\[
tr((hg)u(hg)^{-1}) = tr(hg)u(g^{-1}h) = h \cdot tr(gug^{-1}) \cdot \bar{h} = |h|^2 tr(gug^{-1}) = tr(gug^{-1}) \in \mathbb{R}
\]
for all $u \in SU(1,1)$. Hence $hg \in \mathcal{R}$. This implies that if $g \in \mathcal{R}$, then every element of the $Sp(1)$-orbit of $g$ under the left action of $Sp(1)$ on $Sp(1,1)$ is an element of $\mathcal{R}$. Since the trace is invariant in $SU(1,1)$ under conjugation by any element of $SU(1,1)$, it is clear that $SU(1,1) \subset \mathcal{R}$. It follows from the above observation that $Sp(1) \cdot SU(1,1) \subset \mathcal{R}$. From now on, we will show its converse $\mathcal{R} \subset Sp(1) \cdot SU(1,1)$.

Put
\[
g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Sp(1,1), \quad |a|^2 - |c|^2 = |d|^2 - |b|^2 = 1, \quad \bar{a}b = \bar{c}d.
\]
Recall that $|a||b| = |c||d|$. We may assume that $a$ is a nonnegative real number by multiplying $g$ by some unit quaternion scalar.
Case 1: The entries $a,b,c,d$ of $g$ satisfy $|a||b| = |c||d| = 0$.

Since $|a|^2 - |c|^2 = |d|^2 - |b|^2 = 1$, neither $|a|$ nor $|d|$ can ever be 0. Hence we have $b = c = 0$ and $|a| = |d| = 1$. Moreover, since $a$ is a nonnegative real number, $a = 1$. By the assumption that every element of $gSU(1,1)g^{-1}$ has real trace,

$$\text{tr} \left( \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} \bar{a} & 0 \\ 0 & \bar{d} \end{bmatrix} \right) = ai\bar{a} - did\bar{d} = i - did\bar{d} \in \mathbb{R}.$$ 

Noting that $i - did\bar{d} = -(i - did\bar{d})$, it is easy to see that $i - did\bar{d} = 0$. Writing $d = d_1 + d_2i + d_3j + d_4k$,

$$did\bar{d} = (d_1 + d_2i + d_3j + d_4k)i(d_1 - d_2i - d_3j - d_4k) = (d_1^2 + d_3^2) = 2(d_1d_4 + d_2d_3)x + 2(-d_1d_3 + d_2d_4)x = i.$$ 

Now we have the following equations:

$$d_1^2 + d_2^2 - d_3^2 - d_4^2 = 1,$$

$$d_1d_4 + d_2d_3 = 0,$$

$$d_1d_3 - d_2d_4 = 0. \quad (4.2)$$

Then we have $d_1d_2(d_3^2 + d_4^2) = 0$ from the last two equations of (4.2). If $d_1 = 0$, then $d_2$ can never be 0 due to the fact that $d_2^2 - d_3^2 - d_4^2 = 1$. Hence $d_3 = d_4 = 0$. In a similar way, if $d_2 = 0$, then $d_3 = d_4 = 0$. If $d_3^2 + d_4^2 = 0$, then clearly $d_3 = d_4 = 0$. Thus in either case, $d_3 = d_4 = 0$. This implies $d \in \mathbb{C}$ with $|d| = 1$. Let $d = e^{2i\theta}$. Then

$$g \in \text{Sp}(1) : \begin{bmatrix} 1 & 0 \\ 0 & e^{2i\theta} \end{bmatrix} = \text{Sp}(1) \cdot \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \subset \text{Sp}(1) \cdot \text{SU}(1,1).$$

Case 2: The entries $a,b,c,d$ of $g$ satisfy $|a||b| = |c||d| \neq 0$.

Recall again that any element of $\text{SU}(1,1)$ can be written as

$$\begin{bmatrix} z & w \\ \bar{w} & \bar{z} \end{bmatrix}, \quad z,w \in \mathbb{C}, \quad |z|^2 - |w|^2 = 1.$$ 

Let $z = z_1 + iz_2$ and $w = w_1 + iw_2$. Then

$$\text{tr} \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} z & w \\ \bar{w} & \bar{z} \end{bmatrix} \begin{bmatrix} \bar{a} & -\bar{c} \\ -b & \bar{d} \end{bmatrix} \right) = z_1(|a|^2 - |b|^2 - |c|^2 + |d|^2) + z_2(ai\bar{a} + bi\bar{b} - ci\bar{c} - di\bar{d}) + w_1(-a\bar{b} + b\bar{a} + c\bar{d} - d\bar{c}) + w_2(-ai\bar{b} - b\bar{a} + ci\bar{d} + d\bar{c}) \in \mathbb{R}$$

for all real numbers $z_1, z_2, w_1,$ and $w_2$ with $z_1^2 + z_2^2 - w_1^2 - w_2^2 = 1$. Hence

$$ai\bar{a} + bi\bar{b} - ci\bar{c} - di\bar{d} \in \mathbb{R},$$

$$a\bar{b} - b\bar{a} - c\bar{d} + d\bar{c} \in \mathbb{R},$$

$$ai\bar{b} + b\bar{a} - ci\bar{d} + d\bar{c} \in \mathbb{R}. \quad (4.3)$$

All three numbers in (4.3) satisfy $\bar{q} = -q$, and so they are actually zero. Since $a$ is real and $\bar{a}b = \bar{c}d$, if follows that $b = \bar{c}d/a$. Hence

$$0 = \bar{a}b - b\bar{a} - c\bar{d} + d\bar{c} = \bar{d}c - \bar{c}d - c\bar{d} + d\bar{c} = (\bar{d}c - \bar{c}d) - (d\bar{c} - \bar{c}d).$$

This implies that $\bar{d}c - \bar{c}d$ is real. Furthermore, it can be shown by a direct computation that the real part of $\bar{d}c - \bar{c}d$ is zero for all quaternions $c$ and $d$. Hence $d\bar{c} - \bar{c}d = 0$. 


Putting $b = \bar{c}d/a$ in $a\bar{b} + \bar{b}a - c\bar{d} - d\bar{c} = 0$,

$$0 = a\bar{b} + \bar{b}a - c\bar{d} - d\bar{c} = ic\bar{c} + \bar{c}d - c\bar{d} - d\bar{c} = (ic\bar{c} - c\bar{d}) - (ic\bar{c} - c\bar{d}).$$

This implies $(ic\bar{c} - c\bar{d}) \in \mathbb{R}$. It is easy to check that the real part of $(ic\bar{c} - c\bar{d})$ is zero, and thus $ic\bar{c} - c\bar{d} = 0$. Since $\bar{d}c = \bar{c}d$,

$$ic\bar{c} - c\bar{d} = (ic - ci)d = 0.$$

Finally, we have $ci = ic$ since $d$ cannot be 0. It is not difficult to see that if $ci = ic$, then $c$ is a complex number. If $c$ is a real number, then $b = (c/a)d$, and so we have

$$0 = a\bar{b} + \bar{b}a - c\bar{d} - d\bar{c} = (a^2 - c^2)i + \left(\frac{c}{a}\right)^2 d\bar{d} - d\bar{c} = i - \frac{d\bar{c}}{a^2}.$$

This means that

$$\frac{d}{a}i\left(\frac{\bar{d}}{a}\right) = i.$$

As before, it follows that $d/a$ is a complex number and so is $d$. From $b = (c/a)d$, it follows that $b$ is a complex number. Therefore, all numbers $a$, $b$, $c$, and $d$ are complex numbers.

If $c$ is a complex number which is not real, then $\bar{d}c = \bar{c}d$ implies that $d$ is a complex number. Since $d$ and $c$ are complex numbers, $b = \bar{c}d/a$ is also a complex number. Hence all $a$, $b$, $c$, and $d$ are complex numbers.

In either Case 1 or 2, we prove that $g$ is a complex matrix in $Sp(1,1)$, and thus $g \in U(1,1)$. It is easy to see that

$$g \in Sp(1) \cdot U(1,1) = Sp(1) \cdot SU(1,1).$$

Therefore, $\mathcal{R} \subset Sp(1) \cdot SU(1,1)$. Finally, we have $\mathcal{R} = Sp(1) \cdot SU(1,1)$. \hfill \Box

**Theorem 4.4.** Let $\Gamma$ be a nonelementary discrete subgroup of $Sp(1,1)$. Then the trace field of $\Gamma$ is real if and only if $\Gamma$ is conjugate to a subgroup of $SU(1,1)$ by an element of $Sp(1)$.

**Remark 2.** Regarding $\Gamma$ as a nonelementary discrete subgroup of $PSp(1,1)$ acting on the quaternionic hyperbolic space $H^1_{\mathbb{H}}$, it stabilizes $H^1_{\mathbb{R}}$ up to the left action of $Sp(1)$ on $H^1_{\mathbb{C}}$. Note that the right action of $Sp(1)$ on $H^1_{\mathbb{C}}$ is trivial, but the left action of $Sp(1)$ on $H^1_{\mathbb{R}}$ is not.

**4.2. The case of $Sp(2,1)$**

Similarly to the $SU(n,1)$ case, it can be seen that $L_{nc}$ is conjugate to either $SO(2,1)$, $1 \oplus SU(1,1)$, $SU(2,1)$, $1 \oplus Sp(1,1)$, or $Sp(2,1)$, and every element of $L_{nc}$ has real trace. Then one can notice easily that $L_{nc}$ cannot be conjugate to $Sp(2,1)$. This follows from the fact that $L_{nc} = Sp(2,1)$ if $L_{nc}$ is conjugate to $Sp(2,1)$ and the set of traces of $Sp(2,1)$ is not contained in $\mathbb{R}$.

In the case of $SU(2,1)$ and $1 \oplus Sp(1,1)$, one cannot easily deduce that $L_{nc}$ cannot be $SU(2,1)$ or $1 \oplus Sp(1,1)$, even if the set of traces for them is not contained in $\mathbb{R}$. This is because the trace is not invariant under conjugation in $Sp(2,1)$. One needs to check whether the set of traces is contained in $\mathbb{R}$ for all groups conjugate to $SU(2,1)$, or $1 \oplus Sp(1,1)$ in $Sp(2,1)$ in order to exclude them.

**Lemma 4.5.** The set of traces of $g(1 \oplus Sp(1,1))g^{-1}$ is not contained in $\mathbb{R}$ for any $g \in Sp(2,1)$. 

Proof. Assume that the set of traces of \( g(1 \oplus \text{Sp}(1,1))g^{-1} \) is contained in \( \mathbb{R} \) for some \( g \in \text{Sp}(2,1) \). Put

\[
g = \begin{bmatrix}
a_{1,1} & a_{1,2} & a_{1,3} \\
a_{2,1} & a_{2,2} & a_{2,3} \\
a_{3,1} & a_{3,2} & a_{3,3}
\end{bmatrix}.
\]

For any \( q \in \text{Sp}(1) \), \( \left[ \begin{array}{c} q \\ t_{xy} \\ b \\ 0 \\ 1 \end{array} \right] \) is an element of \( \text{Sp}(1,1) \). Hence,

\[
\text{tr} \left( \begin{bmatrix}
a_{1,1} & a_{1,2} & a_{1,3} \\
a_{2,1} & a_{2,2} & a_{2,3} \\
a_{3,1} & a_{3,2} & a_{3,3}
\end{bmatrix} \right) \left[ \begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0
\end{array} \right] = \begin{bmatrix}
\tilde{a}_{1,1} & \tilde{a}_{1,2} & \tilde{a}_{1,3} \\
\tilde{a}_{2,1} & \tilde{a}_{2,2} & \tilde{a}_{2,3} \\
\tilde{a}_{3,1} & \tilde{a}_{3,2} & \tilde{a}_{3,3}
\end{bmatrix} \in \mathbb{R}
\]

for all unit quaternion numbers \( q \). By a direct computation, it can be shown that the above condition is equivalent to

\[
a_{1,2}q\tilde{a}_{1,2} + a_{2,2}q\tilde{a}_{2,2} - a_{3,2}q\tilde{a}_{3,2} \in \mathbb{R}
\]

for all unit quaternion numbers \( q \). From the fact that \( g \in \text{Sp}(2,1) \), it follows that

\[
|a_{1,2}|^2 + |a_{2,2}|^2 - |a_{3,2}|^2 = 1.
\]

Claim. There does not exist a triple of quaternion numbers \( (x, y, z) \in \mathbb{H}^3 \) with \( |x|^2 + |y|^2 - |z|^2 = 1 \) such that \( xq\tilde{x} + yq\tilde{y} - zq\tilde{z} \in \mathbb{R} \) for all unit quaternions \( q \).

Proof of the Claim. Assume that there exists \( (x, y, z) \in \mathbb{H}^3 \) such that \( xq\tilde{x} + yq\tilde{y} - zq\tilde{z} \in \mathbb{R} \) for all unit quaternions \( q \). This is equivalent to

\[
\begin{align*}
ix\tilde{x} + iy\tilde{y} - iz\tilde{z} & \in \mathbb{R}, \\
xj\tilde{x} + yj\tilde{y} - zj\tilde{z} & \in \mathbb{R}, \\
xk\tilde{x} + yk\tilde{y} - zk\tilde{z} & \in \mathbb{R}.
\end{align*}
\]

Put \( x = x_1 + x_2i + x_3j + x_4k \), \( y = y_1 + y_2i + y_3j + y_4k \), and \( z = z_1 + z_2i + z_3j + z_4k \). It follows from \( ix\tilde{x} + iy\tilde{y} - iz\tilde{z} \in \mathbb{R} \) that

\[
\begin{align*}
(x_1^2 + x_2^2 - x_3^2 - x_4^2) + (y_1^2 + y_2^2 - y_3^2 - y_4^2) - (z_1^2 + z_2^2 - z_3^2 - z_4^2) & = 0, \\
(x_1x_4 + x_2x_3) + (y_1y_4 + y_2y_3) - (z_1z_4 + z_2z_3) & = 0, \\
(x_2x_4 - x_1x_3) + (y_2y_4 - y_1y_3) - (z_2z_4 - z_1z_3) & = 0.
\end{align*}
\]

Similarly, \( xj\tilde{x} + yj\tilde{y} - zj\tilde{z} \in \mathbb{R} \) and \( xk\tilde{x} + yk\tilde{y} - zk\tilde{z} \in \mathbb{R} \) imply that

\[
\begin{align*}
(x_1^2 - x_2^2 + x_3^2 - x_4^2) + (y_1^2 - y_2^2 + y_3^2 - y_4^2) - (z_1^2 - z_2^2 + z_3^2 - z_4^2) & = 0, \\
(x_1x_4 - x_2x_3) + (y_1y_4 - y_2y_3) - (z_1z_4 - z_2z_3) & = 0, \\
(x_1x_2 + x_3x_4) + (y_1y_2 + y_3y_4) - (z_1z_2 + z_3z_4) & = 0,
\end{align*}
\]

and

\[
\begin{align*}
(x_1^2 - x_2^2 - x_3^2 + x_4^2) + (y_1^2 - y_2^2 - y_3^2 + y_4^2) - (z_1^2 - z_2^2 - z_3^2 + z_4^2) & = 0, \\
(x_1x_3 + x_2x_4) + (y_1y_3 + y_2y_4) - (z_1z_3 + z_2z_4) & = 0, \\
(x_1x_2 - x_3x_4) + (y_1y_2 - y_3y_4) - (z_1z_2 - z_3z_4) & = 0.
\end{align*}
\]
In addition, \(|x|^2 + |y|^2 - |z|^2 = 1\) gives
\[
(x_1^2 + x_2^2 + x_3^2 + x_4^2) + (y_1^2 + y_2^2 + y_3^2 + y_4^2) - (z_1^2 + z_2^2 + z_3^2 + z_4^2) = 1.
\]
Summarizing these equations, one gets the following equations:
\[
\begin{align*}
    x_1x_2 + y_1y_2 - z_1z_2 &= 0, \\
    x_1x_3 + y_1y_3 - z_1z_3 &= 0, \\
    x_1x_4 + y_1y_4 - z_1z_4 &= 0, \\
    x_2x_3 + y_2y_3 - z_2z_3 &= 0, \\
    x_2x_4 + y_2y_4 - z_2z_4 &= 0, \\
    x_3x_4 + y_3y_4 - z_3z_4 &= 0.
\end{align*}
\]  \(\text{(4.4)}\)

Let \(v_i = (x_i, y_i, z_i)\) for each \(i = 1, \ldots, 4\). Let \(\mathbb{R}^{2,1}\) be a three-dimensional Minkowski space with a bilinear form \(\langle \cdot, \cdot \rangle_{2,1}\) defined by
\[
\langle (x, y, z), (x', y', z') \rangle_{2,1} = xx' + yy' - zz'.
\]
Then all equations in (4.4) mean that \(\{v_1, v_2, v_3, v_4\}\) is the set of nontrivial positive vectors that are pairwise orthogonal with respect to the bilinear form \(\langle \cdot, \cdot \rangle_{2,1}\). However, this is impossible due to the signature of \(\langle \cdot, \cdot \rangle_{2,1}\) and the dimension of \(\mathbb{R}^{2,1}\). Therefore, there does not exist any solution for (4.4). \(\square\)

The claim implies that, for any \(g \in \text{Sp}(2,1)\), the set of traces of \(g(1 \oplus \text{Sp}(1,1))g^{-1}\) can be never contained in \(\mathbb{R}\). This completes the proof. \(\square\)

Next we move to the case that \(L_{nc}\) is conjugate to \(\text{SU}(2,1)\).

**Lemma 4.6.** The set of traces of \(g\text{SU}(2,1)g^{-1}\) is not contained in \(\mathbb{R}\) for any \(g \in \text{Sp}(2,1)\).

**Proof.** Assume that the set of traces of \(g\text{SU}(2,1)g^{-1}\) is contained in \(\mathbb{R}\) for some \(g \in \text{Sp}(2,1)\). Let us retain the notation for the matrix form of \(g\) used in Lemma 4.5. For any unit complex numbers \(z, w \in \mathbb{C}\), note that
\[
\begin{bmatrix}
  z & 0 & 0 \\
  0 & w & 0 \\
  0 & 0 & \bar{z}\bar{w}
\end{bmatrix} \in \text{SU}(2,1).
\]
By the assumption, we have
\[
\text{tr}
\begin{bmatrix}
  a_{1,1} & a_{1,2} & a_{1,3} \\
  a_{2,1} & a_{2,2} & a_{2,3} \\
  a_{3,1} & a_{3,2} & a_{3,3}
\end{bmatrix}
\begin{bmatrix}
  z & 0 & 0 \\
  0 & w & 0 \\
  0 & 0 & \bar{z}\bar{w}
\end{bmatrix}
\begin{bmatrix}
  \bar{a}_{1,1} & \bar{a}_{2,1} & -\bar{a}_{3,1} \\
  \bar{a}_{1,2} & \bar{a}_{2,2} & -\bar{a}_{3,2} \\
  -\bar{a}_{1,3} & -\bar{a}_{2,3} & \bar{a}_{3,3}
\end{bmatrix}
\]
\[
= (a_{1,1}\bar{z}\bar{a}_{1,1} + a_{2,1}\bar{z}\bar{a}_{2,1} - a_{3,1}\bar{z}\bar{a}_{3,1}) + (a_{1,2}w\bar{a}_{1,2} + a_{2,2}w\bar{a}_{2,2} - a_{3,2}w\bar{a}_{3,2})
- (a_{1,3}\bar{z}\bar{w}\bar{a}_{1,3} + a_{2,3}\bar{z}\bar{w}\bar{a}_{2,3} - a_{3,3}\bar{z}\bar{w}\bar{a}_{3,3})
\]
is a real number for all \(z, w \in \mathbb{C}\) with \(|z| = |w| = 1\).

Taking \((z, w) = (1, i), (i, 1),\) and \((i, i)\), we obtain sequentially the following conditions:
\[
\begin{align*}
    (a_{1,2}i\bar{a}_{1,2} + a_{2,2}i\bar{a}_{2,2} - a_{3,2}i\bar{a}_{3,2}) + (a_{1,3}i\bar{a}_{1,3} + a_{2,3}i\bar{a}_{2,3} - a_{3,3}i\bar{a}_{3,3}) & \in \mathbb{R}, \\
    (a_{1,1}i\bar{a}_{1,1} + a_{2,1}i\bar{a}_{2,1} - a_{3,1}i\bar{a}_{3,1}) + (a_{1,3}i\bar{a}_{1,3} + a_{2,3}i\bar{a}_{2,3} - a_{3,3}i\bar{a}_{3,3}) & \in \mathbb{R}, \\
    (a_{1,1}i\bar{a}_{1,1} + a_{2,1}i\bar{a}_{2,1} - a_{3,1}i\bar{a}_{3,1}) + (a_{1,2}i\bar{a}_{1,2} + a_{2,2}i\bar{a}_{2,2} - a_{3,2}i\bar{a}_{3,2}) & \in \mathbb{R}.
\end{align*}
\]
This implies that
\[
\begin{align*}
(a_{1,1}i\bar{a}_{1,1} + a_{2,1}i\bar{a}_{2,1} - a_{3,1}i\bar{a}_{3,1}) & \in \mathbb{R}, \\
(a_{1,2}i\bar{a}_{1,2} + a_{2,2}i\bar{a}_{2,2} - a_{3,2}i\bar{a}_{3,2}) & \in \mathbb{R}, \\
(a_{1,3}i\bar{a}_{1,3} + a_{2,3}i\bar{a}_{2,3} - a_{3,3}i\bar{a}_{3,3}) & \in \mathbb{R}.
\end{align*}
\] (4.5)

It is not difficult to check that the real part of \(a_{1,1}i\bar{a}_{1,1} + a_{2,1}i\bar{a}_{2,1} - a_{3,1}i\bar{a}_{3,1}\) is 0 and, hence, \(a_{1,1}i\bar{a}_{1,1} + a_{2,1}i\bar{a}_{2,1} - a_{3,1}i\bar{a}_{3,1} = 0\). Similarly, the other terms in (4.5) are also 0. Since \(|a_{1,1}|^2 + |a_{2,1}|^2 - |a_{3,1}|^2 = |a_{1,2}|^2 + |a_{2,2}|^2 - |a_{3,2}|^2 = 1\) and \(|a_{1,3}|^2 + |a_{2,3}|^2 - |a_{3,3}|^2 = -1\), one can note that \((a_{1,3}, a_{2,3}, a_{3,3}) \in \mathbb{H}^3\) is a solution of the following system of quaternionic equations:
\[
\begin{aligned}
&\begin{cases}
    xi\bar{x} + yi\bar{y} - zи = 0, \\
    |x|^2 + |y|^2 - |z|^2 = -1.
\end{cases}
\end{aligned}
\] (4.6)

However, we will show that there is no solution for (4.6). To show this, first observe that \(z \neq 0\) and, so,
\[
\bar{z} xi\bar{x} + \bar{z} yi\bar{y} z - |z|^4 i = 0.
\]

Put
\[
t = \frac{\bar{z} x}{|z|^2}, \quad s = \frac{\bar{z} y}{|z|^2}.
\]
Then the system of Equations (4.6) can be rewritten as
\[
\begin{aligned}
&\begin{cases}
    ti\bar{i} + si\bar{s} - i = 0, \\
    |z|^2(|t|^2 + |s|^2 - 1) = -1.
\end{cases}
\end{aligned}
\] (4.7)

Let \(t = t_1 + t_2 i + t_3 j + t_4 k\) and \(s = s_1 + s_2 i + s_3 j + s_4 k\). From the first equation in (4.7), we have
\[
(t_1^2 + t_2^2 - t_3^2 - t_4^2) + (s_1^2 + s_2^2 - s_3^2 - s_4^2) = 1.
\]
The second equation in (4.7) implies \(|t|^2 + |s|^2 - 1 < 0\). Then
\[
0 > |t|^2 + |s|^2 - 1
= (t_1^2 + t_2^2 + t_3^2 + t_4^2) + (s_1^2 + s_2^2 + s_3^2 + s_4^2) - 1
= (t_1^2 + t_2^2 + s_1^2 + s_2^2) + (t_3^2 + t_4^2 + s_3^2 + s_4^2) - 1
= (t_3^2 + t_4^2 + s_3^2 + s_4^2) + 1 + (t_1^2 + t_2^2 + s_1^2 + s_2^2) - 1
= 2(t_3^2 + t_4^2 + s_3^2 + s_4^2).
\]
This is impossible. Thus there does not exist a solution for (4.6). Finally, we can conclude that, for any \(g \in \text{Sp}(2, 1)\), the set of traces of \(g\text{SU}(2, 1)g^{-1}\) is not contained in \(\mathbb{R}\), which completes the proof. □

Now the cases of \(\text{SO}(2, 1)\) and \(1 \oplus \text{SU}(1, 1)\) remain. In either case, every element has real trace. Hence it is possible that \(L_{nc}\) is conjugate to either \(\text{SO}(2, 1)\) or \(1 \oplus \text{SU}(1, 1)\).

**Theorem 4.7.** Let \(\Gamma\) be a nonelementary discrete subgroup of \(\text{Sp}(2, 1)\) with real trace field. Then \(\Gamma\) stabilizes a totally geodesic submanifold in \(\mathbb{H}_3\), which is isometric to either \(\mathbb{H}_R^2\) or \(\mathbb{H}_C^1\).

**Proof.** As seen above, \(L_{nc}\) is conjugate to either \(\text{SO}(2, 1)\) or \(1 \oplus \text{SU}(1, 1)\). Then \(L_{nc}\) stabilizes a totally geodesic submanifold in \(\mathbb{H}_3\) that is isometric to either \(\mathbb{H}_R^2\) or \(\mathbb{H}_C^1\). Hence, by a similar proof to that of Theorem 3.7, the Zariski closure \(\overline{\Gamma}\) of \(\Gamma\) also stabilizes the totally geodesic submanifold. This completes the proof. □
Remark 3. In [8], Kim assumed that \( \Gamma \) contains a loxodromic element fixing 0 and \( \infty \). If the trace was invariant under conjugation, Kim’s result could be applied to arbitrary nonelementary quaternionic hyperbolic Kleinian groups. However, the trace is not invariant under conjugation in \( \text{Sp}(n, 1) \), and thus it is not easy to apply the method in [8] to arbitrary nonelementary quaternionic hyperbolic Kleinian groups even in the case of \( \text{Sp}(2, 1) \). In Theorem 4.7, we do not assume that \( \Gamma \) contains a loxodromic element fixing 0 and \( \infty \). This is one advantage of our approach over the approach in [8] in characterizing nonelementary discrete subgroups of \( \text{Sp}(n, 1) \) with real trace fields. Another advantage of our approach is that it makes it possible to treat this problem in the general case of \( \text{Sp}(n, 1) \).

4.3. General case

Recall that by the same argument as in the \( \text{SU}(n, 1) \) case, the identity component \( \overline{\Gamma} \) of the Zariski closure of \( \Gamma \) is decomposed into \( \overline{\Gamma} = LT \) as in Lemma 4.2. Then \( L_{nc} \) is conjugate to either \( I_{n-m} \oplus \text{Sp}(m, 1) \), \( I_{n-m} \oplus \text{SU}(m, 1) \) or \( I_{n-m} \oplus \text{SO}(m, 1) \) for some \( m \geq 1 \). Furthermore, it is required that every element of \( L_{nc} \) has real trace. Hence one can expect that the possible Lie groups among them for \( L \) be conjugate to either \( I_{n-m} \oplus \text{SO}(m, 1) \) or \( I_{n-1} \oplus \text{SU}(1, 1) \), as in the case of \( \text{SU}(n, 1) \), because the set of traces for the other Lie groups is not contained in \( \mathbb{R} \). However, it does not seem easy to check whether the set of traces is contained in \( \mathbb{R} \) for all groups conjugate to the other Lie groups. Fortunately, we will find two criteria for \( L_{nc} \).

Let \( d_n \) denote a diagonal matrix of size \( n+1 \) with ordered diagonal entries, \( 1, \ldots, 1, i = \sqrt{-1} \).

We obtain the first criterion for \( L_{nc} \) as follows.

**Proposition 4.8 (Criterion I).** Let \( G \) be a subgroup of \( \text{Sp}(n, 1) \) containing \( d_n \). Then, for any \( g \in \text{Sp}(n, 1) \), the set of traces of \( gGg^{-1} \) is not contained in \( \mathbb{R} \).

**Proof.** Let \( a_{i,j} \) denote the \((i, j)\)th entry of \( g \). Then \( g^{-1} \) can be written as

\[
g^{-1} = \begin{bmatrix}
\bar{a}_{1,1} & \cdots & \bar{a}_{n,1} & -\bar{a}_{n+1,1} \\
\vdots & \ddots & \vdots & \vdots \\
\bar{a}_{1,n} & \cdots & \bar{a}_{n,n} & -\bar{a}_{n+1,n} \\
-\bar{a}_{1,n+1} & \cdots & -\bar{a}_{n,n+1} & \bar{a}_{n+1,n+1}
\end{bmatrix}.
\]

To prove the theorem, it is sufficient to show that \( \text{tr}(gd_n g^{-1}) \) is not real for all \( g \in \text{Sp}(n, 1) \).

Assume that \( \text{tr}(gd_n g^{-1}) \in \mathbb{R} \) for some \( g \in \text{Sp}(n, 1) \). By a direct computation, \( \text{tr}(gd_n g^{-1}) \in \mathbb{R} \) is equivalent to the following condition:

\[
a_{1,n+1}i\bar{a}_{1,n+1} + \cdots + a_{n,n+1}i\bar{a}_{n,n+1} - a_{n+1,n+1}i\bar{a}_{n+1,n+1} \in \mathbb{R}.
\]

In fact, since \( \bar{q}i\bar{q} = -qi\bar{q} \) for any \( q \in \mathbb{H} \), we have

\[
a_{1,n+1}i\bar{a}_{1,n+1} + \cdots + a_{n,n+1}i\bar{a}_{n,n+1} - a_{n+1,n+1}i\bar{a}_{n+1,n+1} = 0.
\]

Since \( g^*I_{n,1}g = I_{n,1} \),

\[
|a_{1,n+1}|^2 + \cdots + |a_{n,n+1}|^2 - |a_{n+1,n+1}|^2 = -1.
\]

Hence \( (a_{1,n+1}, \ldots, a_{n+1,n+1}) \in \mathbb{H}^{n+1} \) is a solution of the following quaternionic system of equations:

\[
\begin{cases}
x_1i\bar{x}_1 + \cdots + x_ni\bar{x}_n - x_{n+1}i\bar{x}_{n+1} = 0, \\
|x_1|^2 + \cdots + |x_n|^2 - |x_{n+1}|^2 = -1.
\end{cases}
\]

As in the \( \text{Sp}(2, 1) \) case, we will prove that there is no solution for (4.8) in \( \mathbb{H}^{n+1} \). From the second equation in (4.8), it follows that \( |x_{n+1}| \geq 1 \) and so \( x_{n+1} \neq 0 \). Putting \( t_m = \bar{x}_{n+1}x_m/|x_{n+1}|^2 \)
for each $m = 1, \ldots, n$, the system of equations (4.8) can be reformulated as follows:

\[
\begin{cases}
t_1i^1 + \cdots + t_ni^n - i = 0, \\
|t_{n+1}|^2(|t_1|^2 + \cdots + |t_n|^2 - 1) = -1.
\end{cases}
\] (4.9)

Let $t_m = t_{m,1} + t_{m,2}i + t_{m,3}j + t_{m,4}k$ for $m = 1, \ldots, n$. Then the first equation in (4.9) gives rise to

\[
\sum_{m=1}^{n} (t_{m,1}^2 + t_{m,2}^2) - \sum_{m=1}^{n} (t_{m,3}^2 + t_{m,4}^2) = 1.
\]

At the same time, the second equation in (4.9) gives rise to

\[
\sum_{m=1}^{n} (t_{m,1}^2 + t_{m,2}^2) + \sum_{m=1}^{n} (t_{m,3}^2 + t_{m,4}^2) - 1 < 0.
\]

Thus we finally get

\[
2 \sum_{m=1}^{n} (t_{m,3}^2 + t_{m,4}^2) < 0.
\]

This is impossible. Therefore, there is no solution to (4.8). This implies that $\text{tr}(gd_n g^{-1})$ cannot be a real number for any $g \in \text{Sp}(n, 1)$, which completes the proof.

**Corollary 4.9.** Let $m \geq 1$. Then, for any subgroup of $\text{Sp}(n, 1)$ conjugate to $I_{n-m} \oplus \text{Sp}(m, 1)$, the set of traces is not contained in $\mathbb{R}$.

**Proof.** Clearly, $d_n \in I_{n-m} \oplus \text{Sp}(m, 1)$ for any $1 \leq m \leq n$. By Proposition 4.8, the corollary follows immediately.

Criterion I does not give any information for $I_{n-m} \oplus \text{SU}(m, 1)$ because of $d_n \notin I_{n-m} \oplus \text{SU}(m, 1)$ for any $m \geq 1$. We need another criterion. Let

\[
c_1 = \begin{bmatrix} 1 & 0 & 0 \\
0 & i & 0 \\
0 & 0 & -i \end{bmatrix}, \quad c_2 = \begin{bmatrix} i & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -i \end{bmatrix}, \quad c_3 = \begin{bmatrix} i & 0 & 0 \\
0 & -i & 0 \\
0 & 0 & 1 \end{bmatrix}.
\]

Note that $c_1$, $c_2$, and $c_3$ are elements of $\text{SU}(2, 1)$. The second criterion is as follows.

**Proposition 4.10** (Criterion II). Let $G$ be a subgroup of $\text{Sp}(n, 1)$ containing $I_{n-2} \oplus c_1$, $I_{n-2} \oplus c_2$, and $I_{n-2} \oplus c_3$. Then for any $g \in \text{Sp}(n, 1)$, the set of traces of $gGg^{-1}$ is not contained in $\mathbb{R}$.

**Proof.** Suppose that there exists an element $g \in \text{Sp}(n, 1)$ such that the set of traces of $gGg^{-1}$ is contained in $\mathbb{R}$. Since $I_{n-2} \oplus c_1$, $I_{n-2} \oplus c_2$, and $I_{n-2} \oplus c_3$ are elements of $G$, the traces of their conjugates by $g$ are real. By a direct computation, $\text{tr}(g(I_{n-2} \oplus c_1)g^{-1}) \in \mathbb{R}$ is equivalent to

\[
\lambda_n + \lambda_{n+1} \in \mathbb{R},
\]

where $\lambda_m = a_{1,m}i\bar{a}_{1,m} + \cdots + a_{n,m}i\bar{a}_{n,m} - a_{n+1,m}i\bar{a}_{n+1,m}$ for each $m$. Similarly, the conditions $\text{tr}(g(I_{n-2} \oplus c_2)g^{-1}) \in \mathbb{R}$ and $\text{tr}(g(I_{n-2} \oplus c_3)g^{-1}) \in \mathbb{R}$ are equivalent to $\lambda_{n-1} + \lambda_{n+1} \in \mathbb{R}$ and $\lambda_{n-1} + \lambda_{n} \in \mathbb{R}$, respectively. Thus, $\lambda_{n-1}$, $\lambda_n$, and $\lambda_{n+1}$ are real numbers. Moreover, $\lambda_m = -\lambda_m$ for each $m = n-1, n, n+1$. Hence we have

\[
\lambda_{n-1} = \lambda_n = \lambda_{n+1} = 0.
\]
Note that $\lambda_{n+1} = 0$ implies that $(a_{1,n+1}, \ldots, a_{n+1,n+1}) \in \mathbb{H}^{n+1}$ is a solution to the system of Equations (4.8). However, by the proof of Proposition 4.8, there is no solution for (4.8). Therefore, we can conclude that there is no $g \in \text{Sp}(n,1)$ such that the set of traces of $gGg^{-1}$ is contained in $\mathbb{R}$.

\begin{proof}

Corollary 4.11. Let $m \geq 2$. Then, for any subgroup of $\text{Sp}(n,1)$ conjugate to $I_n - m \oplus \text{SU}(m,1)$, the set of traces is not contained in $\mathbb{R}$.

Proof. Note that $I_{n-2} \oplus c_1, I_{n-2} \oplus c_2$, and $I_{n-2} \oplus c_3$ are elements of $I_{n-2} \oplus \text{SU}(2,1)$. For $m \geq 2$, since $I_{n-2} \oplus \text{SU}(2,1) \subset I_{n-m} \oplus \text{SU}(m,1)$, it follows that $I_{n-2} \oplus c_1, I_{n-2} \oplus c_2$, and $I_{n-2} \oplus c_3$ are also elements of $I_{n-m} \oplus \text{SU}(m,1)$ for $m \geq 2$. Hence the corollary follows from Proposition 4.10.

Due to Criteria I and II, we can conclude that any possible simple Lie group of rank 1 for $L_{nc}$ is conjugate to either $I_{n-m} \oplus \text{SO}(m,1)$ for $m \geq 2$ or $I_{n-1} \oplus \text{SU}(1,1)$.

Theorem 4.12. Let $\Gamma$ be a nonelementary discrete subgroup of $\text{Sp}(n,1)$ with real trace field. Then $\Gamma$ is conjugate to a subgroup of $\text{Sp}(n-m) \oplus O(m,1)$ for $m \geq 2$ or $\text{Sp}(n-1) \oplus \text{SU}(1,1)$.

Proof. As mentioned before, $L_{nc}$ is conjugate to either $I_{n-m} \oplus \text{SO}(m,1)$ for $m \geq 2$ or $I_{n-1} \oplus \text{SU}(1,1)$. Then $L$ preserves a totally geodesic submanifold $Y$ of $\mathbb{H}^m_{\mathbb{R}}$, which is isometric to either $\mathbb{H}^m_{\mathbb{R}}$ for $m \geq 2$ or $\mathbb{H}^1_{\mathbb{R}}$. Since $L$ and $T$ centralize each other, $T$ also preserves $Y$ and thus $\Gamma^c$ preserves $Y$. Finally, $\Gamma$ preserves $Y$ since $\Gamma^c$ is a normal subgroup of $\Gamma$. Hence we conclude that $\Gamma$ preserves a totally geodesic submanifold in $\mathbb{H}^m_{\mathbb{R}}$, which is isometric to $\mathbb{H}^m_{\mathbb{R}}$ for $m \geq 2$ or $\mathbb{H}^1_{\mathbb{R}}$. The stabilizer subgroup of $\mathbb{H}^m_{\mathbb{R}}$ in $\text{Sp}(n,1)$ is conjugate to $\text{Sp}(n-m) \oplus \text{O}(m,1)$, and the stabilizer subgroup of $\mathbb{H}^1_{\mathbb{R}}$ in $\text{Sp}(n,1)$ is conjugate to $\text{Sp}(n-1) \oplus \text{SU}(1,1)$. This completes the proof.

If $\Gamma$ is an irreducible discrete subgroup of $\text{Sp}(n,1)$ with real trace field, then $\Gamma$ should be conjugate to a Zariski-dense subgroup of $O(n,1)$ by Theorem 4.12. Hence Theorem 1.6 follows for the $\text{Sp}(n,1)$ case.

Proof of Theorem 1.2. It follows from Theorem 4.12 that $\Gamma$ preserves a totally geodesic submanifold in $\mathbb{H}^m_{\mathbb{R}}$, which is isometric to one of $\mathbb{H}^m_{\mathbb{R}}$ for $m \geq 2$ and $\mathbb{H}^1_{\mathbb{R}}$. They are totally geodesic submanifolds of constant negative sectional curvature in $\mathbb{H}^m_{\mathbb{R}}$ which are not isometric to $\mathbb{H}^1_{\mathbb{R}}$. Thus Theorem 1.2 follows immediately.

Proof of Theorem 1.3. It is sufficient to prove the theorem in the case of $\text{Sp}(n,1)$, because $SU(n,1) \cap O(n,1) = SO(n,1)$. Let $\rho : \Gamma \to \text{Sp}(n,1)$ be a discrete faithful representation such that the trace field of $\rho(\Gamma)$ is real. First assume that $\rho(\Gamma) \leq \text{Sp}(n-m) \oplus O(m,1)$. Then the representation $\rho$ can be decomposed into $\rho_c \oplus \rho_{nc}$, where $\rho_c : \Gamma \to \text{Sp}(n-m)$ and $\rho_{nc} : \Gamma \to O(m,1)$. Since $\text{Sp}(n-m)$ is compact, $\rho_{nc}$ must be a discrete faithful representation in order that $\rho$ is discrete and faithful. Thus we obtain a discrete faithful representation $\rho_{nc} : \Gamma \to O(m,1)$.

In the case that $\rho(\Gamma) \leq \text{Sp}(n-1) \oplus SU(1,1)$, the projection of $\rho$ onto the $SU(1,1)$ factor is a discrete faithful representation for a similar reason as in the previous case. By composing this with the Lie group homomorphism from $SU(1,1)$ to $SO(2,1)$, we also obtain a discrete faithful representation $\Gamma \to SO(2,1)$. Therefore (i) implies (ii). The converse is clear.
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