ON EXTENDING ACTIONS OF GROUPS

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Abstract. We study a problem proposed by E.V. Shchepin, concerning extensions of actions of compact transformation groups, under various assumptions. We present several applications of our methods which we develop in this paper.

1. Introduction

The diagram \( D = \{ X \rightarrow Y \} \) is called admissible if \( p: X \rightarrow X \) is an orbit projection and \( i \) is a topological embedding of the orbit space \( X \) into a hereditarily paracompact space \( Y \). We say that the problem of extending the action is solvable for the admissible diagram \( D \) if there exists an equivariant embedding \( j : X \hookrightarrow Y \) into \( G \)-space \( Y \) (which is called a solution of the problem of extending the action for a given diagram) covering \( i \), that is, the embedding \( \tilde{j} : X \hookrightarrow p(Y) \) of orbit spaces induced by \( j \) coincides with \( i \) (in particular, \( p(Y) = Y \)).

We say that the problem of extending the action (denoted briefly by PEA) is solvable if there exists a solution of the PEA for each admissible diagram \( D \), i.e. the diagram \( D \) yields a commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{j} & Y \\
p \downarrow & & \downarrow p \\
X & \hookrightarrow & Y.
\end{array}
\]

The problem of extending group actions naturally splits into the closed and dense parts – depending on the type of the embedding \( i \). On the other hand, it is clear that the simultaneous solvability of the closed and dense PEA implies the solvability of the PEA in general.

In what follows we shall consider only compact groups \( G \). Motivated by his own results, E.V. Shchepin was the first to state the closed problem of extending group actions.

Proposition 1.1. If the closed problem of extending the group action is solvable for all metric admissible diagrams, i.e. the diagrams in which \( X \) and \( Y \) are metric (it is clear that such a diagram is admissible). Then the following holds:

1. The orbit space of any \( G \)-A[N]E(M)-space \( E \) with metrizable orbits is an A[N]E(M)-space (here the class of all metric \( G \)-spaces is denoted by \( M \)).
Proof. Let $Z \leftarrow A \xrightarrow{\phi} E$ be a partial map with $Z \in \mathcal{M}$. We denote the fiberwise product $A \times_{p} E$ by $A$. Since all orbits of $A$ are metrizable and $A/G = A \in \mathcal{M}$, we have $A \in \mathcal{M}$. In view of $Z \in \mathcal{M}$, the embedding $A \hookrightarrow Z$ is covered by a closed $G$-embedding $A \hookrightarrow Z \in \mathcal{M}$. Consider the partial $G$-map $Z \leftarrow A \xrightarrow{\phi'} E$, where $\phi'$ is parallel to $\phi$. Since $E \in G\text{-}A[N]\text{E}(\mathcal{M})$, the $G$-map $\phi'$ can be $G$-extended into $Z$ [into a $G$-neighborhood]. Passing to the orbit spaces we get the desired extension $\hat{\phi}$ of $\phi$. □

It turns out that the converse fact is also true [5, 6].

**Proposition 1.2.** The validity of the condition (1) implies the solvability of the closed PEA for all metric admissible diagrams.

Proof. Let $T_G$ be the countable product of the metric cone $\text{Con} \mathbb{T}$ over a discrete union of all transitive spaces $G/H \in G\text{-}ANE$. By Theorem 5.1 there exists an isovariant map $f : X \rightarrow T_G$. It is clear that $T_G \in G\text{-AE}(\mathcal{M})$. Hence it follows by (1) that $T_G = T_G/G \in \text{AE}(\mathcal{M})$ and the partial map $Y \leftarrow X \xrightarrow{\mathbb{I}} T_G$ induced by $f$ has an extension $\varphi : Y \rightarrow T_G$. It is easy to verify that the fiberwise product $Y \varphi \times_{p} T_G$ is a desired $G$-space $Y \in \mathcal{M}$. □

The dense PEA arises naturally in theory of equivariant compactifications and it was first stated by L. Zambahidze and Yu. M. Smirnov. By now it has been settled for the class of metric spaces with an action of an arbitrary zero-dimensional compact group [7]. We remark without proof that the dense PEA is intimately connected with the existence of several invariant metrics on a given $G$-space.

**Proposition 1.3.** The dense problem of extending the group action is solvable for all separable metric diagrams, i.e. the diagrams in which $X$ and $Y$ are separable metric spaces, if and only if the following holds:

(2) For each separable metric $G$-space $X$ and each compatible metric $d$ on $X$ there exists a compatible invariant metric $\rho$ on $X$ such that $\mathcal{F}_d = \mathcal{F}_\rho$ (here $\mathcal{F}_d$ is the set of all sequences fundamental with respect to $d$, and $\mathcal{F}_\rho$ is an induced metric on $X$).

The main purpose of the present paper is to present a direct proof of the following theorem which gives the positive answer to the closed PEA under very general assumptions.

**Theorem 1.4.** The closed problem of extending the group action is solvable for all admissible diagrams.

We remark that the closed PEA fails to be solvable in certain simple situations. Let $X = (\text{Con} \mathbb{Z}_2)^{\omega_1}$ be the semi-free $\mathbb{Z}_2$-space, and $i : X \hookrightarrow I^{\omega_1}$ an arbitrary embedding into Tihonoff cube of uncountable weight $\omega_1$ (note that $I^{\omega_1}$ is not hereditarily paracompact). It can be checked that the PEA is unsolvable for the diagram $X \xrightarrow{p} X \xrightarrow{i} I^{\omega_1}$.

If the acting group $G$ is non-metrizable, then some orbits of the solution of the closed PEA for metric admissible diagram $D$ given by Theorem 1.4 can be nonmetrizable. The following result eliminates this defect.
Theorem 1.5. If the $G$-space $X$ in the closed admissible diagram $X \xrightarrow{f_2} X \xleftarrow{f_1} Y$ is stratifiable, then there exists a solution $s: X \to Y$ of the PEA for this diagram such that $Y \subset Y \times Z$ for some metric $G$-space $Z$.

In particular, the direct proof of solvability of the closed PEA in the class of metrizable $G$-spaces, as well as in the class of stratifiable $G$-spaces, follows.

Theorem 1.6 and the method developed for its proof admit various modifications. We endow the set $S$ of all solutions of the closed PEA for an admissible diagram $D = \{X \xrightarrow{h_2} X \xleftarrow{h_1} Y\}$ by the following partial order: $s_2 \geq s_1$ where $\{s_i: X \to Y\}_i \subset S$ if and only if there exists a $G$-map $h: Y_1 \to Y_2$ such that $h \circ s_1 = s_2$ and $h = \text{Id}_Y$.

Theorem 1.7. (On majority) Each finite set of solutions $\{s_i\}_{i=1}^n \subset S$ has a majority, i.e. there exists $s \in S$ such that $s \geq s_i$ for each $i \leq n$.

Theorem 1.8. If the stratifiable $G$-space $X$ is an equivariant absolute extensor for the class of stratifiable spaces, then its orbit space $X$ is an absolute extensor for the class of stratifiable spaces.

In [8] the proof of this theorem was reduced to the Borsuk-Whitehead-Hanner theorem for stratifiable spaces. Unfortunately, the status of the latter theorem is still open in view of the gap in [12]. In connection with establishing Theorem 1.8 it would be interesting to return to [12] and to settle the Borsuk-Whitehead-Hanner theorem for stratifiable spaces overcoming the gap.

Theorem 1.9. If the metric $G$-space $X$ is an equivariant absolute extensor, then for all finitely many $G$-extensions $h_i: Z \to X$ of the partial $G$-map $M \ni Z \to X$ there exists a $G$-extension $h: Z \to X$ of $f$ such that $(G_h(z)) \geq (G_{h_i}(z))$ for any $z \in Z$ and $i$.

Theorem 1.10. Let $G$ be a compact group, $X \in G\text{-ANE}(M)$ and $C \subset \text{Orb} \ G$. Then $X_C = \{x \mid (G_x) \geq (H) \text{ for some } (H) \in C\} \subset X$ is a $G\text{-ANE}(M)$.

Earlier this theorem was proved by M. Murayama [17] for all compact Lie group and one-element collection $C$.

In conclusion we present a result revealing the role of the set of extensor points in the equivariant extensor theory. We say that the $G$-embedding $Y \to X$ is equivariantly homotopically dense, if there exists a $G$-homotopy $F: X \times [0, 1] \to X$ such that $F_0 = \text{Id}$ and

$(\alpha) \quad F_t(X) \subset Y \text{ for each } t > 0$.

Theorem 1.11. (On equivariant homotopy density) Let $X$ be a metric $G\text{-ANE}$-space. Then the subspace $X_C$ of all extensor points is equivariantly homotopically dense in $X$. 
As a immediate consequence we get the following result:

**Theorem 1.12.** Let $\mathcal{X}$ be a metric $G$-ANE-space. Then each $G$-subspace $\mathcal{Y}, \mathcal{X}_E \subset \mathcal{Y} \subset \mathcal{X}$, is a $G$-ANE.

2. Preliminaries

Let $G$ be a compact group. An *action* of $G$ on a space $X$ is a homomorphism $T : G \to \text{Aut}(X)$ of $G$ into the group $\text{Aut}(X)$ of all autohomeomorphisms of $X$ such that the map $G \times X \to X$ given by $(g, x) \mapsto T(g)(x) = g \cdot x$ is continuous. A space $X$ with a fixed action of $G$ is called a $G$-space.

For any point $x \in X$ the *isotropy subgroup* of $x$, or the *stabilizer* of $x$, is defined as $G_x = \{ g \in G \mid g \cdot x = x \}$ and the orbit of $x$ as $G(x) = \{ g \cdot x \mid g \in G \}$. The space of all orbits is denoted by $X/G$ and the natural map $p = p_G : X \to X/G$, given by $p(x) = G(x)$, is called the *orbit projection*. The orbit space $X/G$ is equipped with the quotient topology induced by $p$. In what follows we shall denote $G$-spaces and their orbit spaces variously: $X$, $\mathcal{Y}, \mathcal{Z}, \ldots$ for $G$-spaces, and $X, Y, Z, \ldots$ for their orbit spaces.

The map $f : X \to \mathcal{Y}$ of $G$-spaces is called *equivariant* or *$G$-map*, if $f(g \cdot x) = g \cdot f(x)$ for all $g \in G$ and $x \in X$. Each $G$-map $f : X \to \mathcal{Y}$ induces a map $f : X \to Y$ of orbit spaces by formula $f(G(x)) = G(f(x))$. Equivariant map $f : X \to \mathcal{Y}$ is said to be *isovariant*, if $G_x = G_{f(x)}$ for all $x \in X$.

**Theorem 2.1.** (Equimorphism criterion [11]) An isovariant map $f : \mathcal{X} \to \mathcal{Y}$ inducing the homeomorphism $\tilde{f} : X \to Y$ of orbit spaces is an equimorphism.

Observe that all $G$-spaces and $G$-maps generate a category denoted by $G\text{-TOP}$ or $\text{EQUIV}$, provided that no confusion occurs. If "***" is a well-known notion from nonequivariant topology, then "$G$-***" or "Equiv-***" means the corresponding equivariant analogue. See [11] for more details on compact transformation groups.

The subset $A \subset \mathcal{X}$ is called *invariant* or *$G$-subset*, if $G \cdot A = A$. For each closed subgroup $H$ of $G$ (briefly $H < G$) we define the following sets: $\mathcal{X}^H = \{ x \in \mathcal{X} \mid H \cdot x = x \}$ (which is called $H$-fixed points set); $\mathcal{X}^{(H)} = \bigcup \{ \mathcal{X}_K \mid K < G \text{ and } H' < K \text{ for some subgroup } H' \text{ conjugated with } H \} = G \cdot \mathcal{X}^H$; $\mathcal{X}^{(H)} = \bigcup \{ \mathcal{X}_K \mid K < G \text{ and } K \text{ is conjugated with } H \} = G \cdot \mathcal{X}^H$.

We endow the set $\text{Orb}(G)$ of all conjugate classes of closed subgroups of $G$ with the following partial order: $(K) \leq (H) \iff K \subseteq g^{-1} \cdot H \cdot g$ for some $g \in G$. It is clear that type $\mathcal{X} = \{ (G_x) \mid x \in \mathcal{X} \}$ is a subset of $\text{Orb}(G)$ (hereafter the sign $\equiv$ is used for the introduction of the new objects placed to the left of it). If $\mathcal{C} \subset \text{Orb}(G)$, then $\mathcal{X}^C = \{ x \mid (G_x) \geq (H) \}$ for some $(H) \in \mathcal{C} \subset \mathcal{X}$.

We now introduce several concepts related to extensions of $G$-maps partially defined in the class $\mathcal{K}$ – one of the increasing chain: the class of all metric $G$-spaces $\subset$ the class of all stratifiable $G$-spaces the space $X$ is *stratifiable* if there exists a family $\{ f_U : X \to [0, 1] \mid U \subset X \text{ is an open subset } \}$ of continuous functions such that $f_U^{-1}(0, 1] = U$ and $f_U \leq f_V$ if and only if $U \subset V \subset X$ the class $\mathcal{P}$ of all hereditarily paracompact $G$-spaces (the space is called *hereditarily paracompact* if each its subspace is paracompact, i.e. is equivalent to the paracompactness of each its open subspace).

A space $X$ is called an *absolute neighbourhood extensor for class $\mathcal{K}$*, $X \in G\text{-ANE}(\mathcal{K})$, if each $G$-map $\varphi : \mathcal{A} \to \mathcal{X}$ defined on a closed $G$-subset $\mathcal{A} \subset \mathcal{Z}$ of $G$-space $\mathcal{Z} \in \mathcal{K}$ and called the *partial $G$-map* can be $G$-extended in a $G$-neighborhood $U \subset \mathcal{Z}$.
of \( A, \varphi : U \to X, \varphi \mid_A = \varphi \) (we use the notation \( f \mid_A \) for the restriction of the map \( f \) to \( A \subset X \) or we simply write \( f \mid \) if the set \( A \) in question is clear).

If it is possible to \( G \)-extend \( \varphi \) in \( U = Z \), then \( X \) is called an \textit{equivariant absolute extensor for class} \( K \), \( X \in \text{G-AE}(K) \). If the acting group \( G \) is trivial, then these notions are transformed into the notions of absolute \textit{[neighborhood]} extensors for class \( K = \text{A}[\text{N}]\text{E}(K) \). Since we are mainly interested in equivariant absolute \textit{[neighborhood]} extensors for the class of metrizable \( G \)-spaces, we will briefly denote them as \( \text{G-A}[\text{N}]\text{E} \). The following results are well-known: each Banach \( G \)-space \([13]\), each compactly convex \( G \)-subset of locally convex complete vector \( G \)-space \([11]\), p.155, each linear normed \( G \)-spaces (for acting compact Lie group \( G \)) \([17]\), p.488, are \( G \)-AE-spaces.

We will heavily depend on the the slice theorem \([11]\) which is equivalent to the following assertion: "If \( G \) is a compact Lie group, then the transitive space \( G/H \in \text{G-AE} \) for the class of regular \( G \)-spaces".

We recall the construction of the equivariant absolute extensor for arbitrary compact group \( G \) go back to \([15]\). Recall that \( G \) acts on the space \( X = C(G,Y) \) of all continuous maps endowed with compact-open topology by formula \( (g \cdot f)(h) = f(g^{-1} \cdot h) \) where \( f \in C(G,Y) \) and \( g, h \in G \). If \( Y \) is metric, then \( X \) is also metric.

**Theorem 2.2.** If metric space \( Y \) is AE-space for class \( P \) of paracompact spaces, then \( C(G,Y) \) is a \( G \)-AE-space for \( P \).

By \([14]\) each Banach space \( B \) is a \( G \)-AE for class of paracompact spaces. Hence it follows by Theorem 2.2 that \( C(G,B) \) is \( G \)-AE for \( P \).

**Proposition 2.3.** (Palais Metatheorem \([15]\)) Let \( \mathcal{P}(H) \) be a property which depends on compact Lie group \( H \). Suppose that \( \mathcal{P}(H) \) is true, provided \( \mathcal{P}(K) \) is true for each compact Lie group \( K \) isomorphic to a proper subgroup of \( H \). If \( \mathcal{P}(H) \) is true for trivial group \( H = \{ e \} \), then \( \mathcal{P}(H) \) is true for all compact Lie groups \( H \).

By a fiberwise product of the spaces \( C \) and \( B \) with respect to maps \( g \) and \( f \) we call the subset \( \{(c,b) \mid g(c) = f(b)\} \subset C \times B \) which is denoted by \( C_g \times_f B \). The projections \( D = C_g \times_f B \) onto the factors \( C \) and \( B \) are denoted by \( f : D \to C \) and \( g : D \to A \). These maps \( f \) and \( g \) are called the \textit{maps parallel} to \( f \) and \( g \) respectively, and we write for brevity \( f \| f \) and \( g \| g \). It should remarked that the map \( f \circ g = g \circ f : D \to A \) is the product of \( g \) and \( f \) in the category \( \text{TOP}_A \) of all spaces over \( A \). The most important example of the fiberwise product in the theory of compact transformation groups is supplied by isovariant maps.

**Proposition 2.4.** Let \( h : Y \to X \) be an isovariant map, \( \tilde{h} : Y \to X \) the map of orbit spaces induced by \( h \). Then \( Y \) is the fiberwise product \( Y_h \times_{p_Y} X \), moreover \( h \| h \) and \( p_Y \| p_X \) where \( p_Y, p_X \) are the orbit projections.
there exists a $G$-map $\hat{\phi} : Z \to X$ [\hat{\phi} : U \to X$ defined on a neighborhood $U \subset Z$ of $X$] such that $\hat{\phi} \cdot x = \varphi$ and $f \circ \hat{\phi} = \psi \cdot [\hat{\psi} \cdot x]$. We will say also that $\hat{\phi} : Z \to X$ $[\hat{\phi} : U \to X]$ is a global [local] lifting of $\psi$.

**Definition 2.5.** A morphism $f : X \to Y$ in the category $\text{EQUIV}$ is called equivariantly soft [locally equivariantly soft ] if for each commutative square diagram $D$ in the category $\text{EQUIV}$ the problem of extending of partial lifting is solvable globally [locally ].

2.1. $P$-orbit projection. Let $P$ be a normal closed subgroups of $G$ (briefly $P \triangleleft G$) and $\pi : G \to H = G/P$, $\pi(g) = g \cdot P$, be a natural epimorphism. It is clear that $x \sim x' \Leftrightarrow x' \in \pi^{-1}(x)$ is an equivalent relation on the $G$-space $X$. Then the quotient space $X/P$ defined by this relation coincides with $\{ P \cdot x \mid x \in X \}$. It is clear that $Y = X/P$ is an $H$-space: $(g \cdot P) \cdot (P \cdot x) = P \cdot (g \cdot x)$. If $y = P \cdot x$, then the stabilizer $H_y$ coincides with $G_{x \cdot P}$.

The quotient map $f : X \to X/P$ is called the $P$-orbit projection. If $P = G$, then $f$ coincides with the orbit projection $p : X \to X/G$. Since the composition of the $P$-orbit projection $f$ and the $H$-orbit projection $Y$ is perfect, $f$ is a perfect surjection and satisfies the following properties:

1. $f(gx) = \pi(g) \cdot f(x)$ for all $x \in X$ and $g \in G$;
2. $\pi(G_x) = H_{f(x)}$ for all $x \in X$; and
3. If $f(x) = f(x')$, then $x$ and $x'$ belong to the same orbit.

The following fact shows that these properties characterize $P$-orbit projections completely.

**Proposition 2.6.** Let $\pi : G \to H$ be a epimorphism of compact groups, $P = \ker \pi$. The perfect surjection $f : X \to Y$ from the $G$-space $X$ onto the $H$-space $Y$ is the $P$-orbit projection if and only if the properties (1)-(3) for $f$ hold.

**Proof.** We consider the $P$-orbit projection $\varphi : X \to X/P = Z$ and define the map $\theta : Z \to Y$ by the formula $\theta(P \cdot x) = f(x)$. It follows by (1) that $\theta$ is correctly defined and it is equivariant. It follows by perfectness of $f$ that $\theta$ is a perfect surjection, and therefore the induced map $\bar{\theta}$ of orbit space is also perfect and surjective.

Let $z = P \cdot x$ and $z' = P \cdot x'$. If $\bar{\theta}(z) = \bar{\theta}(z')$, then $f(x) = f(x')$. In view of (3), $x$ and $x'$ lie on the same $G$-orbit. Therefore $z$ and $z'$ lie on the same $G/P$-orbit. Hence $\theta$ is a homeomorphism.

The map $\theta$ preserves the orbit type of points as on the one hand, $H_{\theta(z)} = H_{f(x)}$; and on the other hand, $H_z = G_z \cdot P = \pi(G_x)$. By (2) we have $H_{\theta(z)} = H_z$. Therefore $\theta$ is an isovariant map inducing on the orbit spaces a homeomorphism.

By Theorem 2.1 the map $\theta$ is an equimorphism. □

For compact group $G$ we consider the Lie series $\{ P_\alpha \triangleleft G \}$ of normal closed subgroups indexed by ordinals $\alpha < \omega$ [19]. This means that

\begin{align*}
(5) \quad P_1 &= G; P_\beta < P_\alpha \text{ for all } \alpha < \beta; P_\alpha/P_{\alpha + 1} \text{ is a compact Lie group for all } \alpha < \omega; P_\alpha = \cap\{ P_{\alpha} \mid \alpha' < \alpha \} \text{ for each limit ordinal } \alpha, \text{ and also } \cap\{ P_\alpha \mid \alpha < \omega \} = \{ e \}.
\end{align*}

In this case $G$ is the limit $\lim\{ G/P_\alpha, \varphi_\alpha^\beta \}$ of the inverse system of quotient groups $\{ G/P_\alpha \}$ and natural epimorphisms $\varphi_\alpha^\beta : G/P_\beta \to G/P_\alpha, \alpha < \beta$. A more general fact holds.
Lemma 2.7. Let $f^β_α : X_β \to X_α$ be natural projections from $X_β = X/P_β$ into $X_α = X/P_α$. Then $f^β_α$ is a $P_α/P_β$-orbit projection and the map $f : X \to \lim_{←}\{X_α, f^β_α\}$, given by the formula $f(x) = \{P_α : x\}$, is an equimorphism.

The proof of Lemma 2.7 consists of a straightforward application of the equimorphism criterion 2.1. The converse to Lemma 2.7 is also valid.

Lemma 2.8. Let $\{P_α < G\}$ be the Lie series, $P^β_α, α < β$, a kernel of the homomorphism $φ^β_α$, and $g^β_α : Z_β \to Z_α$ a $P^β_α$-orbit projection with $g^β_α \circ g^γ_β = g^γ_α$ for all $α < β < γ$. Then $Z = \lim_{←}\{Z_α, g^β_α\}$ is a $G$-space and $Z_α = Z/P_α$.

3. Extensor subgroups

Definition 3.1. A closed subgroup $H < G$ of a compact group $G$ is called a $P$-subgroup if the transitive space $G/H$ is finite-dimensional and locally connected.

L.S. Pontryagin [19] proved that $H < G$ is a $P$-subgroup if and only if one of the following properties holds:

1. There exists a normal subgroup $P < G$ such that $P < H$ and $G/P$ is a compact Lie group;
2. $G/H$ is a topological manifold.

It is known [19] that each compact group contains arbitrarily small normal $P$-subgroups. Hence the following fact is valid:

Proposition 3.2. Let $G$ be a compact group, $X$ a $G$-space. Then

3. for each neighborhood $O(H) \subset G$ of the subgroup $H < G$ there exists a $P$-subgroup $H' < G$ such that $H \subset H' \subset O(H)$;
4. for each neighborhood $O(x)$ of the point $x \in X$ there exists a normal $P$-subgroup $P < G$ such that the $P$-orbit projection $p : X \to X/P := Y$ has a small inverse image of $y = P : x$, $p^{-1}(y) \subset O(x)$. Moreover, there exists a neighborhood $W \subset X/P$ of $y$ such that $p^{-1}(W) \subset O(x)$.

The property (3) easily implies that

5. $G/H$ is metric if and only if $H < G$ can be presented as intersection of countably many $P$-subgroups.

If $H < G$ is a $P$-subgroup then it follows by (1) and the slice theorem [11] that $G/H$ is $G$-ANE. The proof of the converse fact is based on the existence of a regular $G$-space $Z$ such that the stabilizers of all its points are $P$-subgroups except a nowhere dense orbit $G(z) \cong G/H$. Hence it follows that:

6. $H < G$ is a $P$-subgroup if and only if $G/H$.

The equivalence (6) expresses the main property of $P$-subgroups and it simultaneously justifies the alternative term – extensor subgroups. We also draw reader’s attention to the following result [20]: "If the natural action of the compact group $G$ on $G/H \in$ ANE is effective, then $G$ is a Lie group", from which it follows that

7. $H < G$ is a $P$-subgroup if and only if $G/H$ is an ANE.

Definition 3.3. We call the conjugate class $(H) \subset G$ an extensor subgroup. By $X_E$ we denote the collection of all extensor points of $X$, that is, all points $x \in X$ for which $G_x < G$ is an extensor subgroup.
We say that the $G$-subspace $Y \subset X$ is $G$-dense, if $Y^H \subset X^H$ is dense for each subgroup $H < G$. In [2] it was shown that

(8) $X_\mathcal{E} \subset X$ is $G$-dense if and only if $X$ is a G-ANE for the class of all metric $G$-spaces with zero-dimensional orbit space;

(9) (equivariant Dugunji’s theorem) a linear normed $G$-space $L$ is a $G$-AE if and only if $L_\mathcal{E} \subset L$ is $G$-dense.

There exists an example of a linear normed $G$-space $L \notin G$-AE for which $L_\mathcal{E} \subset L$ is dense (but not $G$-dense).

We list the basic properties of extensor subgroups. It can be shown that

(10) the subgroup $H < G$ is extensor if and only if $G$ admits an orthogonal action on $\mathbb{R}^n$ such that $G/H$ is equimorphic to the orbit of a point.

It is clear that each subgroup $H < G$ in compact Lie group $G$, and also each clopen subgroup $H < G$ are extensor subgroups.

It easily follows by (1) that the property to be an extensor subgroup is inherited by any passage to the larger subgroup. It is well-known that a compact group is a Lie group if and only if it contains no small subgroups [19]. Hence it follows that the quotient group $G/(P_1 \cap P_2), P_i \triangleleft G$, are Lie group if and only if each $G/P_i$ is a Lie group.

**Proposition 3.4.** The intersection of finitely many extensor subgroups is an extensor subgroup.

This proposition cannot be improved: if the extensor subgroup $H < G$ is an intersection of a family $\{H_\alpha < G\}$ of extensor subgroups, then $H$ is an intersection of finitely many subgroups $H_\alpha$.

Since each closed subgroup of compact Lie group is again compact Lie group [19], it easily follows that if $H < G$ is an extensor subgroup, then

(10) $H \cap K < K$ is an extensor subgroup for each $K < G$; and

(11) $H < K$ and $K < G$ are extensor subgroups for each $K < G, H < K < G$.

The proof of the next fact follows from the definition of extensor subgroup and Hurewicz theorem on dimension [9].

**Proposition 3.5.** Let $K < L < G$ and $K < L$ be an extensor subgroup. Then $K < G$ is an extensor subgroup if and only if $L < G$ is an extensor subgroup.

If $G$ is a compact non-Lie group, then by (7) $G \notin$ ANE. It is known that there exists a free $G$-space $X \in$ ANE [4]. Hence each its invariant open subset is not homeomorphic to a product $G \times U$ and therefore in this case the slice theorem fails [22]. But if we weaken the requirement about the slice, then the following is valid. We say that a $G$-map $\alpha : X \to G/H, H < G$, is sliced if $G/H \in$ G-ANE ($\equiv H < G$ is an extensor subgroup).

**Theorem 3.6.** (On approximate slice of $G$-space [1],p.151,[2, 3]) Let a compact group $G$ act on a $G$-space $X$. Then for each neighborhood $O(x)$ of $x \in X$ there exists a neighborhood $V = V(e)$ of the unit $e \in G$, an extensor subgroup $K < G, G_x < K$, and a slice map $\alpha : U \to G/K$ where $U$ is an invariant neighborhood of $x$ such that $x \in \alpha^{-1}(V \cdot [K]) \subset O(x)$.

**Proof of Theorem 3.6.** Let $P \triangleleft G$ be a normal $P$-subgroup, $W$ a neighborhood of $y = P \cdot x \in Y = X/P$ taken from Proposition 3.2(4). Since $Y$ is naturally endowed
with action of compact Lie group $G' = G/P$, and $G'_y = G_x \cdot [P] < G'$, there exists by slice theorem a neighborhood $V(e)$ and a slice map $\alpha : V \rightarrow G'/G'_y \cong G/(G_x \cdot P)$ defined on a $G'$-neighborhood $V$ of the orbit $G'(y)$ such that $\alpha^{-1}(V(e) \cdot [G'_y]) \subset W$. We easily check that $K = G_x \cdot P < G$ is an extensor subgroup and the composition $\alpha \circ p : p^{-1}(V) \rightarrow G/K \in G$-ANE is the desired slice map.

The following result is known for compact Lie groups $[13], 7.6.4.$

**Theorem 3.7.** Let $H$ and $K$ be subgroups of a compact group $G$ such that $H < K$ is an extensor subgroup. Then the natural projection $p : G/K \rightarrow G/H$ is equivariantly locally soft.

### 4. Tube structure of orbit projections

We consider the epimorphism $\pi : G \rightarrow H$ of compact groups with the kernel $P$ being Lie group. Let $K < G$ be an extensor subgroup and $\pi(K) = L < H$. Since $K < \pi^{-1}(L) < G$, it follows by $[3], 10$ that

1. $K < \pi^{-1}(L)$ and $\pi^{-1}(L) < G$ are extensor subgroups, and hence
2. $G/K \in G$-ANE and $G/\pi^{-1}(L) \cong H/L \in H$-ANE.

Let $\kappa : G/K \rightarrow H/L$ be a composition of the natural epimorphism $\alpha : G/K \rightarrow G/\pi^{-1}(L)$ and the isomorphism $\beta : G/\pi^{-1}(L) \rightarrow H/L$. By (1) and Theorem 3.7 we have the following

3. The maps $\alpha : G/K \rightarrow G/\pi^{-1}(L)$ and $\kappa : G/K \rightarrow H/L$ are equivariantly locally soft.

Since $\kappa(g \cdot [K]) = \pi(g) \cdot [L]$ and $\pi(K) = L$, it follows that

4. $\kappa(g \cdot [K]) = [L]$ if and only if $g \in P \cdot K$.

We say that the $P$-orbit projection $f : X \rightarrow Y$ have a $\kappa$-tube structure generated by slice maps $\varphi : X \rightarrow G/K \in G$-ANE and $\psi : Y \rightarrow H/L \in H$-ANE if they close the following diagram $A$ up to commutative one: $\kappa \circ \varphi = \psi \circ f$.

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi} & G/K \\
\downarrow f & & \downarrow \kappa \\
Y & \xrightarrow{\psi} & H/L
\end{array}
\]

The $\kappa$-tube structure on $f$ is said to be nontrivial if $\kappa$ is not bijection. It is equivalent to $K \not\subseteq \pi^{-1}L$ or $P \setminus K \neq \emptyset$.

**Proposition 4.1.** Let $f : X \rightarrow Y$ be a $P$-orbit projection. If $x \notin X^P$, then the restriction of $f$ on the orbit $G(x)$ has a nontrivial tube structure.

**Proof of Proposition 4.1.** Since $P \setminus G_x \neq \emptyset$ then by Proposition 3.2(3) there exists an extensor subgroup $K < G$ such that $P \setminus K \neq \emptyset$ and $G_x < K$. It is clear that $G_{f(x)}$ coincides with $G_x \cdot P$. The following subgroups $G_x < K < K \cdot P$ and $G_x < G_{f(x)} < K \cdot P$ naturally generate $G$-maps $\varphi : G(x) \rightarrow G/K$, $\psi : G(f(x)) \rightarrow G/(K \cdot P)$ and $\kappa : G/K \rightarrow G/(K \cdot P)$, which in its turn generate the nontrivial tube structure on the orbit $G(x)$. □

We consider the diagram $A$ and denote by $S$ and $T$ the $K$-space $\varphi^{-1}([K])$ and the $L$-space $\psi^{-1}([L])$, correspondingly. It is clear that $X$ is $G$-homeomorphic to the twisted product $G \times_K S$ and $Y \cong_H H \times_L T$. The following facts show that if $f$ has
a nontrivial tube structure, then \( f \) is generated by a \( Q \)-orbit projection where \( Q \) is a proper subgroup of \( G \).

**Lemma 4.2.** Let \( f \) has a tube structure given by epimorphism \( \kappa \). Then \( f(S) = T \) and the map \( f : S \to T \) is a \( Q \)-orbit projection where \( Q \) is the kernel of the epimorphism \( \pi' = \pi : K \to L \). If this tube structure is nontrivial, then \( Q \) is a proper subgroup of \( P \).

**Proof.** Let \( y \in T \), \( x \in f^{-1}(y) \) and \( \varphi(x) = g \cdot K \). Then \( |L| = \psi(y) = \psi(f(x)) = \kappa(g \cdot K) \). By (4), \( g \in P \cdot K \). Now it is easily to check that \( x' = g^{-1} \cdot x \in S \) and \( f(x') = y \). Hence \( f(S) = T \). It is clear that the map \( f \mid \) is perfect.

Let us verify the properties (1) - (3) of Proposition 2.6 for restriction \( f : \) the property (1) and the rest of the properties (3) is performed obviously. The property (2) holds, as \( K_{s} = G_{s} \) and \( L_{f(s)} = H_{f(s)} \) for \( s \in S \).

If \( \kappa \) is not a bijection, then \( K_{s} \not\subset \pi^{-1}L \) and hence \( P \setminus K \neq \emptyset \). But \( Q = \ker \pi' \) coincides with \( K \cap P \), and hence it is proper subgroup of \( P \). □

We now consider the converse situation: there exist an epimorphism \( \pi' : K \to L \) of compact groups and a \( Q \)-orbit projection \( f' : S \to T \) where \( S \) is a \( K \)-space, \( T \) is an \( L \)-space and \( Q = \ker \pi' \). Let \( K < G \) and \( L < H \) be extensor subgroups and \( \pi : G \to H \) an epimorphism extending \( \pi' \). By \( \kappa : G/K \to H/L \) we denote the composition of natural epimorphisms \( \alpha : G/K \to G/\pi^{-1}L \) and \( \beta : G/\pi^{-1}L \to H/L \). Then the formula \( f([g,s]_{K}) = [\pi(g), f'(s)]_{L} \) correctly defines the map \( f = \pi \times f' : G \times_{K} S \to H \times_{L} T \). It is straightforwardly checked that the perfectness of \( f' \) implies the same property for \( f \).

**Lemma 4.3.** The map \( f : G \times_{K} S \to H \times_{L} T \) is a \( P \)-orbit projection for \( P = \ker \pi \). In so doing the natural slice maps \( \varphi : G \times_{K} S \to G/K, \psi : H \times_{L} T \to H/L \) and also epimorphism \( \kappa : G/K \to H/L \) set a tube structure on the \( P \)-orbit projection \( f \), that is, \( \kappa \) closes the following diagram up to commutative: \( \kappa \circ \varphi = \psi \circ f \).

**Proof.** Let \( x = [g,s]_{K} \) and \( x' = [g',s']_{K} \in G \times_{K} S \). If \( f(x) = f(x') \), then \( \pi(g') = \pi(g) \cdot l^{-1} \) and \( f'(s') = l \cdot f'(s) \) for \( l \in L \).

Since \( \pi' \) is the epimorphism, \( l = \pi(k), k \in K \), and therefore \( f'(s') = \pi(k) \cdot f'(s) = f'(k \cdot s) \). Since \( f' \) is the \( Q \)-orbit projection, \( s' \) and \( k \cdot s \) lie on the same \( K \)-orbit. Hence it is easily deduced that \( x' \) and \( x \) lie on the same \( G \)-orbit.

All other characterization properties for \( P \)-orbit projections from Proposition 2.6 is checked straightforwardly and we leave it to the reader. □

The following theorem on extension of tube structure of maps has a highly important role in inductive argument.

**Theorem 4.4.** Let \( B \) be a commutative diagram,

\[
\begin{array}{ccc}
\mathfrak{X} = \text{Cl} \mathfrak{X} & \overset{J}{\longrightarrow} & \mathfrak{W} \\
\downarrow f & & \downarrow \hat{f} \\
\mathfrak{Y} = \text{Cl} \mathfrak{Y} & \overset{I}{\longrightarrow} & \mathfrak{Z}
\end{array}
\]

in which \( f : \mathfrak{X} \to \mathfrak{Y} \) and \( \hat{f} : \mathfrak{W} \to \mathfrak{Z} \) are \( P \)-orbit projections. If \( f \) has a \( \kappa \)-tube structure generated by slice maps \( \varphi : \mathfrak{X} \to G/K \) and \( \psi : \mathfrak{Y} \to H/L \), then there exist an invariant neighborhoods \( \mathfrak{B}, \mathfrak{Y} \subset \mathfrak{B} \subset \mathfrak{Z} \), and \( A := \hat{f}^{-1}(\mathfrak{B}), \mathfrak{X} \subset A \subset \mathfrak{W} \), such that the \( P \)-orbit projection \( f : A \to \mathfrak{B} \) has a \( \kappa \)-tube structure generated by slice maps \( \hat{\varphi} : A \to G/K \) and \( \hat{\psi} : \mathfrak{B} \to H/L \), which are extensions of \( \varphi \) and \( \psi \) respectively.
Proof of Theorem 5.1. Let $A$ be a diagram which generates the $\kappa$-tube structure on $f$. Since the subgroup $H < G$ is extensor, it follows by (10) from Section 4 that $G$ admits an orthogonal action on $\mathbb{R}^n$ such that $G/K$ is equivomorph to the orbit of a point. Hence we can assume without loss of generality that $G/K$ is an orbit in $H/L \times \mathbb{R}^N$, moreover, $\kappa$ coincides with the restriction of the projection $pr_1 : H/L \times \mathbb{R}^N \to H/L$ on $G/K$. Represent the composition $X \xrightarrow{\sigma} G/K \xrightarrow{\kappa} H/L \times \mathbb{R}^N$ as $(\varphi_1, \varphi_2)$.

Since by (2) the map $\kappa : G/K \to H/L$ is equivariently locally soft, there exists a fiberwise equivariant retraction $r : \mathbb{V} \to G/K$ of an invariant neighborhood $\mathbb{V} \subset H/L \times \mathbb{R}^N$ of $G/K$ such that $pr_1 \circ r = pr_1$. In view of $H/L \in H-ANE$ and $\mathbb{R}^N \in G-AE$ there exist a local $H$-extension $\hat{\psi} : \mathbb{E}' \to H/L, \mathbb{Y} \subset \mathbb{B}' \subset \mathbb{Z}$, of $\psi$ and a $G$-extension $\hat{\chi} : \mathbb{W} \to \mathbb{R}^N$ of $\varphi_2 : X \to \mathbb{R}^N$.

Let $\hat{A} := \hat{f}^{-1}(\mathbb{B}')$. Consider the $G$-map $\sigma : \hat{A} \to H/L \times \mathbb{R}^N$ given by the formula $\sigma = (\hat{\psi} \circ \hat{f}) \times \hat{\chi}$. It is clear that $\hat{A} = \hat{\sigma}^{-1}(\mathbb{V})$ and $\mathbb{B} := \hat{f}(\hat{A})$ are invariant neighborhoods of $X$ and $Y$ respectively. We assert that $\hat{\varphi} := \hat{r} \circ \sigma : \hat{A} \to G/K$ and $\hat{\psi} := \hat{\psi} \circ \hat{r}$ generate a $\kappa$-tube structure on $\hat{f} \restriction_{\hat{A}} : \hat{\kappa} \circ \hat{\varphi} = \hat{f} \restriction_{\hat{A}} \circ \hat{\psi}$. \(\square\)

5. Equivariant homotopy density

We postpone the proof of Theorem 1.11 to the end of the section in view of the necessity of certain auxiliary facts. For a compact group $G$ we denote by $\mathbb{T}$ the discrete union of all transitive spaces $G/H \in G-ANE$. It is clear that $\mathbb{T}$ is metrizable, each point of the metric cone $\text{Con} \mathbb{T}$ over $\mathbb{T}$ is an extensor and its orbit space is the cone over a discrete space.

Theorem 5.1. Let $X$ be a metric $G$-subspace of the $G$-space $\mathbb{Y}$ whose orbit space $Y$ is metrizable. Then for each nested family $\{\mathbb{V}_n \subset \mathbb{Y}\}_{n=1}^{\infty}$ of invariant neighborhoods of $X$ there exists a $G$-map $f : \mathbb{V} \to \mathbb{T}_G := (\text{Con} \mathbb{T})^\omega$ such that

(a) the restriction of $f$ on $X$ is isovariant;

(b) $f(\mathbb{Y} \setminus \{\mathbb{V}_n \mid n \geq 1\}) \subset (\mathbb{T}_G)^\varepsilon$.

Proof of Theorem 5.1. We can assume without loss of generality that $\mathbb{X}$ contains no isolated points.

It otherwise should pass from $X$ and $\mathbb{Y}$ to $X \times [0,1]^\omega$ and $\mathbb{Y} \times [0,1]^\omega$ respectively.

Since $Y = \mathbb{Y}/G$ is metric, there exists a family $\mathcal{B} = \{W_\mu \mid \mu \in \mathcal{M}\}$ of open subsets of $Y$ intersecting $X$ such that

(1) $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$, where $\mathcal{B}_n = \{W_\mu \mid \mu \in \mathcal{M}_n \subset \mathcal{M}\}$ is a discrete family, $\prod_{n=1}^{\infty} M_n = \mathcal{M}$;

(2) the body of $\mathcal{B}_n$ is contained in $\mathbb{V}_n$ for each $n$;

(3) the restriction $\mathcal{B} \restriction_X$ generates the basis of $X$.

By $\mathbb{W}_\mu$ we denote $p^{-1}W_\mu$ where $p : \mathbb{Y} \to Y$ is the orbit projection. Since $\mathbb{W}_\mu$ has a trivial slice, the number

$i(\mu) := \inf \{\text{diam} \left( X \cap \varphi^{-1}(g \cdot \{H]\} \right) \mid \varphi : \mathbb{W}_\mu \to G/H \text{ is a slice map and } g \in G \} \geq 0$

is correctly defined (here we take the diameter with respect to the compatible invariant metric $\varphi$ existing on $X$ by [13]). By (c), $i(\mu) > 0$. It follows by invariance of $\sigma$ that the diameters of $X \cap \varphi^{-1}(g \cdot \{H]\}) = X \cap g \cdot \varphi^{-1}(\{H]\})$ and $X \cap \varphi^{-1}(\{H]\})$ are equal. Hence it is sufficient to take $g = e$ in the definition of $i(\mu)$. Of particular interest
is those slice map \( \varphi_\mu : \mathbb{W}_\mu \to G/H_\mu, \mu \in M_n \), for which \( \text{diam}(X \cap \varphi_\mu^{-1}([H_\mu])) < j(\mu) = 2i(\mu) \). It is easy to see that \( \varphi_\mu \) satisfies the following important property.

**Lemma 5.2.** If \( \mu \in M_n \), then \( \text{diam}(X \cap \varphi_\mu^{-1}([H_\mu])) < 2 \text{diam}(X \cap \varphi^{-1}([H])) \) for each slice map \( \varphi : \mathbb{W}_\mu \to G/H_\mu \).

We consider the \( G \)-map \( \psi_\mu \equiv \text{Con} \varphi_\mu : Y \to \text{Con} G/H_\mu \) coinciding with \( (\varphi_\mu, \xi_\mu) \) on \( \mathbb{W}_\mu \), and with the vertex \( \{\ast\} \) on the complement to \( \mathbb{W}_\mu \) (here \( \xi_\mu : Y \to [0, 1] \)) is a function constant on orbits and such that \( \xi_\mu^{-1}(0) = Y \setminus \mathbb{W}_\mu \). It is clear that \( \psi_\mu^{-1}(G/H_\mu \times (0, 1]) = \mathbb{W}_\mu \) and \( G_{\psi_\mu(y)} = G_{\psi_\mu(y)} \) for all \( y \in \mathbb{W}_\mu \).

Since the family \( \mathcal{B}_n \) is discrete, it is easy to see that the formulae \( \psi_n |\mathbb{W}_\mu = \psi_\mu \) for \( \mu \in M_n \) and \( \psi_n |\mathbb{W}_{\mu_i} \setminus \mathbb{W}_\mu = \{\ast\} \) correctly define the continuous \( G \)-map \( \psi_n : Y \to \text{Con} T \). It is clear that \( \psi_n^{-1}(\ast) = Y \setminus \mathbb{W}_\mu | \mu \in M_n \). It turns out that the desired \( G \)-map \( f \) is the diagonal product \( \Delta \psi_n : Y \to (\text{Con} T)^n = T^G \).

Fix arbitrary \( x \in X \) and \( \alpha > 0 \). By Theorem 5.6 on approximate slice there exists a \( G \)-map \( r : U(x) \to G/H \in G \)-ANE of some neighborhood \( U(x) \subset Y \) for which \( r^{-1}([H]) \) has diameter less than \( \alpha/2 \). Since by (3) the restriction \( \mathcal{B} \times X \) is the basis of \( X \), there exists an \( \mu \in M_n \) such that \( x \in \mathbb{W}_\mu \subset U(x) \). We note that \( \text{diam}(r^{-1}([H])) < \alpha/2 \) implies \( i(\mu) < \alpha/2 \), and hence it follows by Lemma 5.2 that

\[
(\text{diam}(X \cap \varphi_\mu^{-1}([H_\mu]))) < i(\mu) < \alpha.
\]

Therefore \( \text{diam}(X \cap \varphi_\mu^{-1}(g \cdot [H_\mu])) < \alpha \) for all \( g \in G \). Hence it follows the existence of a sequence of slice maps \( \{\varphi_\mu : \mathbb{W}_\mu \to G/H_\mu | i \geq 1, \mu_i \in M_n \} \) such that for \( a_i = \varphi_\mu(x), i \geq 1 \), we have \( \varphi_\mu^{-1}(a_i) \subset \mathbb{W}_\mu \), and

\[
(\text{diam}(X \cap \varphi_\mu^{-1}(a_i))) \to 0.
\]

Since \( \varphi_\mu(x) = a_i \) and \( G_{a_i} = G_{\varphi_\mu(x)} = G_{\psi_\mu(x)} \supseteq G_{f(x)} \supseteq G_x \), \( G_x \subset \cap \{G_{a_i} | i \geq 1 \} \). To prove that the stabilizers of \( x \) and \( f(x) \) are equal, it is sufficient to establish the converse inclusion \( \cap \{G_{a_i} | i \geq 1 \} \subset G_x \). If \( g \in \cap \{G_{a_i} | i \geq 1 \} \), then \( g \cdot x \in \varphi_\mu^{-1}(a_i) \) for all \( i \geq 1 \), and by virtue of \( (e) \), we have \( x = g \cdot x \). Hence \( g \in G_x \).

It now remains to check \( (b) \). If \( g \notin G_n \), then \( y \notin G_{n+m} \) and hence it follows from (2) that all coordinates of \( f(y) \) excepting the first \( n \) ones coincide with the vertex \( \{\ast\} \). Hence \( G_{f(y)} \) is an extensor subgroup as the intersection of finitely many of extensor subgroups (Proposition 5.4). \( \square \)

Theorem 5.2 implies an important result on the structure of solutions of the closed PEA.

**Proposition 5.3.** For each solution \( s' : X \hookrightarrow Y' \in \mathcal{M} \) of PEA for closed admissible diagram \( D \) there exists a solution \( s : X \hookrightarrow Y \) of PEA for \( D \) such that \( s \geq s' \) and \( Y \setminus X \subset Y' \).

**Proof of Proposition 5.3** By Theorem 5.1 there exists a \( G \)-map \( f : Y' \to T^G \) such that \( f \mid X \) is isovariant and \( f(Y' \setminus X) \subset (T^G)_e \). We consider the fiberwise product \( Y := Y_f \times p \) \( T_G \in \mathcal{M} \) where \( p : T_G \to T^G/G \) is the orbit projection.

Since \( f \mid X \) is isovariant, \( X \subset Y \). It is easy to check that \( s : X \hookrightarrow Y \) covers \( X \hookrightarrow Y \) and the natural \( G \)-map \( h : Y' \to Y, h(y') = (p(y'), f(y')) \), satisfies \( h \circ s' = s \) and \( h = \text{Id}_{Y} \). It is clear that \( Y \setminus X \subset Y_e \). \( \square \)

We note that each closed \( G \)-embedding \( s : \mathbb{A} \hookrightarrow \mathbb{Z} \) is the solution of the closed PEA for diagram \( D \equiv \{ \mathbb{A} \hookrightarrow \mathbb{A} \hookrightarrow \mathbb{Z} \} \). As an easy application of Proposition 5.3 to \( D \), we get
Proposition 5.4. For each partial G-map $M \ni Z \leftrightarrow A \xrightarrow{f} X \in G$-AE($M$) there exists a G-extension $\hat{f} : Z \to X$ such that $\hat{f}(Z \setminus A) \subset X_P$. The similar result takes place for a G-AE-space.

To prove Theorem 1.11 we apply Proposition 5.4 to the partial G-map $X \times [0,1] \leftrightarrow X \times \{0\} \xrightarrow{f} X \in G$-AE. The case $X \in G$-AN is proved analogically. □

6. Extensions of P-orbit projections

We treat the closed problem of extending the action in the general context. Let $f : X \to Y$ be a P-orbit projection for the kernel $P$ of the epimorphism $\pi : G \to H$ of compact groups, and $i : Y \hookrightarrow Z$ an arbitrary $H$-embedding of $Y$ into the equivariantly hereditarily paracompact, i.e. each open invariant subset is paracompact (it is equivalent to the hereditary paracompactness of the orbit space), $H$-space $Z$.

The resulting diagram $D := \{X \xrightarrow{f} Y \xrightarrow{i} Z\}$ is called H-admissible. It is clear that the induced diagram $X \xrightarrow{\pi \circ f} Y \xrightarrow{i} Z$ is G-admissible or merely admissible. We say that

1. The general problem on extending of action (GPEA) is solvable for H-admissible diagram $D$ if there exist a $G$-embedding $j : X \hookrightarrow W$ into a $G$-space $W$ and a P-orbit projection $\hat{f} : W \to Z$ such that $\hat{f} \circ j = i \circ f$;

2. The GPEA is locally solvable for H-admissible diagram $D$ if it is solvable for some $H$-admissible diagram $X \xrightarrow{f} Y \xrightarrow{i} E$ where $Y \subset \text{Int} E \subset Z$;

3. The general problem on extending of action is solvable for P-orbit projections if GPEA is solvable for each $H$-admissible diagram.

It will be shown below that the main result – Theorem 1.4 is reduced to the following key theorem on solvability of the closed GPEA.

Theorem 6.1. The closed GPEA is solvable for all P-orbit projections, provided that the kernel $P$ is a compact Lie group.

We note that the content of Theorems 1.4 and 6.1 is identical in the case of $G = P$. First we consider the simplest case of Theorem 6.1 and give the complete its proof in the following section.

Lemma 6.2. The GPEA for each $H$-admissible diagram $X \xrightarrow{f} Y \xrightarrow{i} Z$ is solvable, provided $i$ is open.

Hence it easily follows that

4. If the GPEA is locally solvable for H-admissible diagram $D$, then the GPEA is solvable for this diagram $D$.

Proof of Lemma 6.2. We construct the $G$-space $W$ in a such manner that $W \setminus X = Z \setminus Y$. With this goal in view, we set $W := X \cup (Z \setminus Y)$. The basis of $W$ is generated by all open sets in $X$ and by sets $\{O := f^{-1}(O \cap Y) \cup (O \setminus Y) \mid O \subset Z\}$ arbitrary open set}. The action of $G$ on $W$ is defined as follows. It coincides with the action of $G$ on $X$, and it is given by the formula $g \cdot y = \pi(g) \cdot y$, $y \in Z \setminus Y$, on $Z \setminus Y$. We can easily verify the continuity of this action. It is clear that the $G$-map $\hat{f} : W \to Z$ coincides with $f$ on $X$ and with $\text{Id}$ on $Z \setminus Y$ is a $P$-orbit projection extending $f$. □
Let $X \xrightarrow{f} Y \xrightarrow{i} Z$ be an $H$-admissible diagram. We note that $\hat{A} = f^{-1}(f(\hat{A}))$ for all $\hat{A} \subset X$, and also $G_x = P \cdot G_x = \pi^{-1}(\pi(G_x))$ for $P < G_x$. Hence it follows by perfectness of $f$ that

(5) $f \mid_{X'} : X' \to f(X')$ is an equimorphism of closed subsets of $X$ and $Y$.

Let $X' := X \setminus X', Z' := Z \setminus \text{Cl}_Z(f(X'))$ and $Y' := Y \cap Z' = Y \setminus f(X')$. It is clear that

(6) $(X')^P = \emptyset$ and the map $f \mid : X' \to Y'$ is a $P$-orbit projection.

The following assertion reduces the investigation of the closed GPEA to the case of absence of $P$-fixed points in $X$.

**Lemma 6.3.** If the closed GPEA for all $H$-admissible diagram $X \xrightarrow{f} Y \xrightarrow{i} Z$ with empty $P$-fix point set $X^P$ is solvable, then it is solvable for all $H$-admissible diagram.

**Proof.** Since $Z'$ is hereditarily paracompact, the diagram $X' \xrightarrow{f} Y' \xrightarrow{i'} Z'$ is $H$-admissible. Since $(X')^P = \emptyset$, the lemma can be applied and hence the closed GPEA is solvable: there exist a $G$-embedding $j' : X' \hookrightarrow W'$ into $G$-space $W'$ and a $P$-orbit projection $\hat{f} : W' \to Z'$ such that $\hat{f} \circ j' = i' \circ f \mid$. We apply Lemma 6.2 to the natural open $H$-embedding $i'' : Z' \hookrightarrow Z$: the GPEA for diagram $W' \xrightarrow{i''} Z'' \xrightarrow{i''} Z$ is solvable. Therefore the GPEA is solvable for arbitrary $H$-admissible diagram.

Let $D = \{X \xrightarrow{f} Y \xrightarrow{i} Z\}$ be a closed $H$-admissible diagram. Each representation of $Z$ as a union of closed $G$-subspaces $Z_1$ and $Z_2$ generates three closed $H$-admissible diagrams $X_i \xrightarrow{f} Y_i \xrightarrow{i} Z_i$, $i = 0, 1, 2$, where $Z_0 = Z_1 \cap Z_2$, $Y_i = Z_i \cap Y$ and $X_i = f^{-1}(Y_i)$. Suppose that $s_i : X_i \hookrightarrow W_i$, $i = 0, 1, 2$, are solutions of the closed GPEA for these $H$-admissible diagrams, so that $s_0$ is the restriction of $s_j$ on $X_0$ for each $j = 1, 2$ (assuming that $W_0$ is naturally contained in $W_j$). The following fact is evident:

**Lemma 6.4.** Let $W$ be a natural gluing of $G$-spaces $W_1$ and $W_2$ along $W_0$. Then $W_i \subset W$, $i = 1, 2$, are closed invariant subsets of $G$-space $W$, and there exists a solution $s : X \hookrightarrow W$ of the closed GPEA for $D$ such that $s_i$ is the restriction of $s$ on $X_i$ for each $i = 0, 1, 2$.

As an easy corollary of Lemma 6.4 we get

**Proposition 6.5.** Let $\mathbb{F}$ be a closed $G$-subset of $Z$ such that $Z = Y \cup \mathbb{F}$. If the closed GPEA is solvable for $H$-admissible diagram $f^{-1}(X \cap \mathbb{F}) \xrightarrow{f} Y \cap \mathbb{F} \xrightarrow{i} \mathbb{F}$, then the closed GPEA is solvable for $D$.

In conclusion, we explain the reduction of Theorem [1.4] to Theorem 6.1.

**Proposition 6.6.** The validity of Theorem [6.1] implies the validity of Theorem [1.4].

**Proof.** Let $\{P_\alpha \triangleleft G\}$ be a Lie series of $G$ with $P_1 = G$. We represent $X$ as $\lim\{X_\alpha, f_\alpha^\beta\}$ (see the notation from Lemma [2.7]). Since $X_1 = X$ and $Y_1 = Y$, the embedding $i : X \hookrightarrow Y$ can be identified with a $G/P_1$-embedding $i_1 : X_1 \hookrightarrow Y_1$. Since $P_\alpha/P_{\alpha+1}$ is a compact Lie group, Theorem 6.1 can be applied many times. Hence it follows by transfinite induction on $\alpha$ that there exist a $G/P_\alpha$-embedding $i_\alpha : X_\alpha \hookrightarrow Y_\alpha$ into $G/P_\alpha$-spaces $Y_\alpha$ and $P_\alpha^{\alpha+1}$-orbit projection $f_\alpha^{\alpha+1} : Y_{\alpha+1} \to Y_\alpha$. 


such that \( \tilde{f}_{\alpha+1} \circ i_{\alpha+1} = i_{\alpha} \circ f_{\alpha+1} \) for each \( \alpha < \omega \). Then \( X = \lim_{\alpha} \{ X_\alpha, f_\alpha \} \) naturally lies in the G-space \( Y = \lim_{\alpha} \{ Y_\alpha, \tilde{f}_\alpha \} \) and \( X \hookrightarrow Y \) is the desired G-embedding. \( \square \)

7. PROOF OF THEOREM 6.3

We use the argument based on Palais metatheorem \[28\] If \( |P| = 1 \) then \( \pi \) is the isomorphism that trivializes the situation under consideration. We suppose now that for each proper subgroup \( Q < P \) the closed GPEA for each \( Q \)-orbit projection is solvable and show that it is solvable for each \( P \)-orbit projection \( f : X \rightarrow Y \). By Lemma 6.3 it is sufficient to study a \( G \)-space \( X \) without \( P \)-fixed points, \( X^P = \emptyset \).

First we consider the case of the \( P \)-orbit projection \( f \) having a nontrivial tube structure.

**Lemma 7.1.** If \( f \) has a nontrivial tube structure, then the closed GPEA is solvable for each \( H \)-admissible diagram \( X \hookrightarrow Y \hookrightarrow Z \).

**Proof of Lemma 7.1.** Consider the slice maps \( \varphi : X \rightarrow G/K \) and \( \psi : Y \rightarrow H/L \) from commutative diagram \( A \) which generate the nontrivial tube structure on \( f \). Since \( H/L \in G \)-ANE, \( \psi \) has an \( H \)-extension \( \hat{\psi} : U \rightarrow H/L \) in some invariant neighborhood \( U, Y \subset U \subset \mathbb{Z} \).

We note that by Lemma 4.2 the map \( f \restriction : S = \varphi^{-1}[K] \rightarrow T = \psi^{-1}[L] \) is a \( Q \)-orbit projection for proper compact group \( Q < P \). Since \( T' = \hat{\psi}^{-1}[L] \) is equivariantly hereditarily paracompact, the diagram \( S \hookrightarrow T \hookrightarrow T' \) is \( L \)-admissible. In view of the inductive hypothesis, the closed GPEA is solvable for all \( Q \)-orbit projections, and hence it is solvable for \( S \hookrightarrow T \hookrightarrow T' \). Hence there exist a \( K \)-embedding \( j' : S \hookrightarrow S' \) into \( K \)-space \( S' \) and a \( Q \)-orbit projection \( f' : S' \rightarrow T' \) such that \( f' \circ j' = i \circ f \restriction \).

Next, by Lemma 4.3 the map \( \hat{f} = \pi \times j' : G \times_K S' \rightarrow H \times_L T' = U \) is a \( P \)-orbit projection, and, as is easy to see, \( \hat{f} \) solves the closed GPEA for \( H \)-admissible diagram \( X \hookrightarrow Y \hookrightarrow Z \). We apply Lemma 6.2 for diagram \( G \times_K S' \hookrightarrow H \times_L T' = U \hookrightarrow Z \) with open embedding \( U \hookrightarrow Z \) and complete the proof. \( \square \)

The last case of the proof of Theorem 6.3 consists in consideration of a \( P \)-orbit projection \( f : X \rightarrow Y \) with \( X^P = \emptyset \) and a closed \( H \)-embedding \( \gamma \hookrightarrow Z \).

**Lemma 7.2.** There exist a closed neighborhood \( E, Y \subset E \subset Z \) and its locally finite closed \( H \)-cover \( \sigma = \{ F_\gamma \subset E \} \subset \Gamma \in \text{cov } E \) such that for each \( \gamma \in \Gamma \)

1. the map \( g_\gamma = f \restriction : \gamma \hookrightarrow \gamma \cap Y \) has a nontrivial tube structure where \( \gamma \hookrightarrow \gamma \cap Y \).

**Proof.** Since \( X^P = \emptyset \), Proposition 4.1 implies that for each \( x \in X \), \( f \restriction_{G(x)} \) has a nontrivial tube structure. Hence it follows by Theorem 4.3 on extension of tube structure of maps that there exists a locally finite open \( H \)-cover \( \omega = \{ U_\alpha \} \subset \text{cov } Y \) such that

2. Each \( P \)-orbit projection \( f \restriction : f^{-1}(U_\alpha) \rightarrow U_\alpha \) has a nontrivial tube structure.

Let \( \nu \) be a family of open \( H \)-sets of \( Z \) the restriction of which on \( Y \) coincides with \( \omega \). Since \( Z \) is hereditarily paracompact, the body \( \cup \nu \) is paracompact. Hence there exists a closed locally finite \( H \)-cover \( \sigma' \subset \cup \nu \) refining \( \nu \). It is clear that \( \sigma = \{ F \subset \sigma' \mid F \cap Y \neq \emptyset \} \) and \( E = \cup \sigma \) are desired. \( \square \)
To argue by the new transfinite induction, we well-order the set \( \Gamma \) indexing the elements of \( \{ F_\gamma \} \). Without loss of generality we can assume that \( \Gamma \) has the maximal element \( \omega \). If \( Q_\gamma := \cup \{ F_\gamma | \gamma' \leq \gamma \} \), then it is obvious that \( Q_\omega = \mathbb{E} \) is the body of locally finite increasing system of closed subsets \( \{ Q_\gamma \} \), moreover \( Q_\gamma \cap F_\gamma = Q_\gamma \) for each \( \gamma = \gamma' + 1 \).

By transfinite induction on \( \gamma \) we construct closed neighborhoods \( R_\gamma, Y \subset R_\gamma \subset Q_\gamma \cup Y \), such that

\begin{enumerate}
\item If \( \gamma_0 \) is the minimal element of \( \Gamma \), then \( R_{\gamma_0} = Y \cup F_{\gamma_0} \);
\item For all \( \gamma \in \Gamma \), the closed GPEA for \( H \)-admissible diagram \( X \xrightarrow{f} Y \xrightarrow{i} R_\gamma \) is solvable, that is, there exist a closed \( G \)-embedding \( j_\gamma : X \hookrightarrow W_\gamma \) into a \( G \)-space \( W_\gamma \) and a \( P \)-orbit projection \( f_\gamma : W_\gamma \to R_\gamma \) such that \( f_\gamma \circ j_\gamma = i \circ f \);
\item \( R_1 \subset R_{\gamma_2}, W_1 \subset W_{\gamma_2} \) and \( f_\gamma |_{W_1} = f_\gamma |_{W_{\gamma_2}} \) for all \( \gamma_1 < \gamma_2 \);
\item \( R_\gamma \setminus R_{\gamma_1} \subset \cup \{ F_\gamma | \gamma_1 < \gamma \leq \gamma_2 \} \) for all \( \gamma_1 < \gamma_2 \).
\end{enumerate}

By (5) the stabilization condition of constructed neighborhoods \( R_\gamma \) follows: if \( \Gamma_\alpha := \{ \gamma \in \Gamma | \alpha \cap F_\gamma \neq \emptyset \}, \alpha \subset \mathbb{Z} \), is finite, then for each \( \gamma' \in \Gamma \) the intersection \( \alpha \cap R_{\gamma'} \) coincides with \( \alpha \cap R_\gamma \) where \( \gamma' \) is \( \max \{ \gamma \in \Gamma_\alpha | \gamma \leq \gamma' \} \).

It is clear that \( R_\omega \subset Q_\omega = \mathbb{E} \) is a closed neighborhood of \( Y \) in \( \mathbb{Z} \). Hence it follows by (3) that the closed GPEA for diagram \( D = \{ X \xrightarrow{f} Y \xrightarrow{i} \mathbb{Z} \} \) is locally solvable. We take into account the property (4) from the previous section and conclude that the closed GPEA for \( D \) is solvable, which completes the proof of Theorem 6.1.

The base of the inductive argument is easily established with help of Lemma 7.1.

**Lemma 7.3.** The closed GPEA for diagram \( X \xrightarrow{f} Y \xrightarrow{i} R_{\gamma_0} = Y \cup F_{\gamma_0} \) is solvable.

**Proof.** By Lemmas 7.1 and 7.2 the closed GPEA for \( H \)-admissible diagram \( V_{\gamma_0} \xrightarrow{g_{\gamma_0}} F_{\gamma_0} \cap Y \xrightarrow{f_{\gamma_0}} R_{\gamma_0} = \mathbb{E} \cup F_{\gamma_0} \) is solvable. To complete the proof we apply Proposition 6.5 to \( H \)-admissible diagram \( X \xrightarrow{f} Y \xrightarrow{i} R_{\gamma_0} \). □

The **inductive step** consists of the following proposition.

**Lemma 7.4.** Assume that for \( \gamma' \in \Gamma \), \( \gamma' < \omega \), we have defined a closed neighborhood \( R_{\gamma'}, Y \subset R_{\gamma'} \subset Q_{\gamma'} \cup \mathbb{Y} \), a \( G \)-embedding \( j_\gamma : X \hookrightarrow W_\gamma \) into a \( G \)-space \( W_\gamma \) and a \( P \)-orbit projection \( f_\gamma : W_\gamma \to R_\gamma \) satisfying (3). Then for \( \gamma = \gamma' + 1 \in \Gamma \)

\begin{enumerate}
\item There exist a closed neighborhood \( R_\gamma, Y \subset R_\gamma \subset Q_\gamma \cup \mathbb{Y} \), a \( G \)-embedding \( j_\gamma : X \hookrightarrow W_\gamma \) into a \( G \)-space \( W_\gamma \) and a \( P \)-orbit projection \( f_\gamma : W_\gamma \to R_\gamma \) satisfying (3) such that \( R_{\gamma'} \subset R_\gamma, W_{\gamma'} \) naturally lies in \( W_\gamma \), \( f_\gamma |_{W_{\gamma'}} = f_{\gamma'} \) and \( R_\gamma \setminus R_{\gamma'} \subset F_\gamma \).
\end{enumerate}

**Proof of Lemma 7.4.** Since \( g_{\gamma} : \mathbb{Y} \to F_\gamma \cap Y \) has a nontrivial tube structure, it follows by Theorem 4.1 that

\begin{enumerate}
\item There exists a closed invariant neighborhood \( S \subset R_{\gamma'} \) of \( F_{\gamma'} \cap R_{\gamma'} \) such that the \( P \)-orbit projection \( h = f_{\gamma'} |_T : T \to S \) where \( T := (f_{\gamma'})^{-1}(S) \subset W_{\gamma'} \) has a nontrivial tube structure.
\end{enumerate}

Let \( \hat{S} \subset R_{\gamma'} \cup F_{\gamma} \) be a closed neighborhood of \( F_{\gamma} \cap R_{\gamma'} \) such that \( \hat{S} \cap R_{\gamma'} = S \). Hence it follows by Lemma 7.4 that the closed GPEA is solvable for \( H \)-admissible diagram \( T \xrightarrow{h} S \xrightarrow{\hat{h}} \hat{S} \). To complete the proof we apply Proposition 6.6 to \( H \)-admissible
diagram $\mathcal{W}_\gamma \xrightarrow{f_\gamma'} \mathbb{R}_\gamma \rightarrow \mathbb{R}_\gamma = \mathbb{R}_\gamma \cup \hat{S}$. The verification of all details is evident and we leave it to the reader. □

Let $\gamma \in \Gamma$ be a limit ordinal. Since \{F$_\gamma$\}$\gamma' < \gamma$ is a locally finite $H$-cover of $\mathcal{Y} \cup \mathcal{Q}_\gamma$, it follows by (5) that the increasing system of $G$-subspaces \{W$_\gamma \subset \mathcal{W}_\gamma$\}$\gamma' < \gamma < \gamma$ and the system of $P$-orbit projections \{f$_\gamma : \mathcal{W}_\gamma \rightarrow \mathbb{R}_\gamma$\}$\gamma' < \gamma$ are locally stratifiable. In view of this remark, we set $\mathcal{W}_\gamma = \bigcup\{\mathcal{W}_\gamma \mid \gamma' < \gamma\}$, and $f_\gamma : \mathcal{W}_\gamma \rightarrow \mathbb{R}_\gamma = f_\gamma'$ on $\mathcal{W}_\gamma$ for all $\gamma' < \gamma$. It is easily checked that $X \rightarrow \mathcal{W}_\gamma$ and $\hat{f}_\gamma$ solve the closed $G$-PGEA for $H$-admissible diagram $X \rightarrow \mathcal{Y} \rightarrow \mathbb{R}_\gamma$ in such a manner that $\mathcal{W}_\gamma \subset \mathcal{W}_\gamma$ and $\hat{f}_\gamma = f_\gamma |_{\mathcal{W}_\gamma}$ for all $\gamma' < \gamma$.

8. Proof of Theorem 1.5

By Theorem 1.4 there exists a solution $s' : X \rightarrow \mathcal{Y}$ covering $X \rightarrow Y$. Since $Y$ is paracompact and $p_{\mathcal{Y'}} : \mathcal{Y'} \rightarrow Y$ is perfect, $\mathcal{Y'}$ is also paracompact.

Since $X$ is stratifiable, there exists a continuous bijection of $X$ onto a metric space \cite{10}. Hence there exists a continuous map $\varphi : X \rightarrow B$ to the Banach space $B$ such that

1) its restriction on each orbit is an embedding.

Consider the map $\psi : X \rightarrow C(G, B)$ given by the formula $\psi(x)(g) = \varphi(g^{-1}x), g \in G, x \in X,$ and which is by (1) isovariant. Since by Theorem 2.2 $C(G, B)$ is $G$-ANE for the class of paracompact spaces and $\mathcal{Y'}$ is paracompact, there exists a $G$-map $\psi : \mathcal{Y'} \rightarrow C(G, B)$ extending $\psi$.

We consider the fiberwise product $\mathcal{Y} = Y_\theta \times_p C(G, B)$ where $p : C(G, B) \rightarrow C(G, B)/G$ is the orbit projection and $\theta : \mathcal{Y} \rightarrow C(G, B)/G$ is the map induced by $\theta$. The $G$-space $\mathcal{Y}$ is stratifiable as a subset of the product of two stratifiable spaces. Since $\psi$ is isovariant, the $G$-space $X$ is naturally contained in the $G$-space $\mathcal{Y}$, $s : X \rightarrow \mathcal{Y}$, moreover $s$ covers $X \rightarrow Y$. Finally, the natural $G$-map $h : \mathcal{Y'} \rightarrow \mathcal{Y}, h(y') = (p_{\mathcal{Y'}}(y'), \psi(y'))$, satisfies the properties $h \circ s' = s$ and $\hat{h} = \text{Id}_Y$. □

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