Realization of the annihilation operator
for generalized oscillator-like system
by a differential operator ¹

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Abstract

This work continues the research of generalized Heisenberg algebras connected with several orthogonal polynomial systems. The realization of the annihilation operator of the algebra corresponding to a polynomial system by a differential operator $A$ is obtained. The important special case of orthogonal polynomial systems, for which the matrix of the operator $A$ in $l^2(Z_+)$ has only off-diagonal elements on the first upper diagonal different from zero, is considered. The known generalized Hermite polynomials give us an example of such orthonormal system. The replacement of the usual derivative by $q$-derivative allows us to use the suggested approach for similar investigation of various "deformed" polynomials.

Contents

1 Introduction 1
2 Background information 3
3 Statement of a problem 5
4 Conditions under which the matrix of an operator $A$ has non zero elements only upper above diagonal 7
5 Generalized Hermite polynomials 9
6 Conditions under which the annihilation operator $A$ can be realized by a differential operator 10
7 Conclusion 11

1 Introduction

The connection of the orthogonal polynomials theory with the group theory as well as the connection the theory of quantum (deformed) groups and algebras with the theory of basic hypergeometric functions (deformed polynomials) [1, 2, 3, 4] is well known [5, 6, 7, 8]. In

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particular, the Hermite polynomials system (after multiplication on \( \exp(-x^2) \)) is a system of eigenfunctions for quantum-mechanical harmonic oscillator energy operator. Besides, these polynomials are connected with representations of the algebra generated by Heisenberg commutation relations (and with representations of the appropriate group as well). Similarly, many known \( q \)-deformed Hermite polynomials \([2, 10, 11]\) make up eigenfunctions systems of the energy operators for the related deformed oscillators \([12, 13, 14]\). Moreover, these polynomials naturally arise from the analysis of irreducible representations of appropriate deformed algebras.

At the standard approach it is suggested that a representation of considered algebra (or group) is prescribed. Then we look for a system of orthogonal polynomials which make up the eigenfunctions system of a significant operator of this algebra (for example, Hamiltonian). In other words, we look for the basis in the representation space, in which this operator take the diagonal form.

In \([15]\) we considered in some sense an inverse task. Let \( \mu \) be a positive Borel measure on the real line \( \mathbb{R}^1 \). We consider a set of real orthonormal polynomials in the space \( L^2(\mathbb{R}^1; \mu(dx)) \). For the orthogonal polynomial system one wants to construct oscillator like algebra, such that these polynomials make up the eigenfunctions system of the energy operator corresponding to this oscillator. By the full solution of this problem it is suggested that a differential or a difference equation for the given system of polynomials is known. In work \([15]\) (strictly) classical polynomials, (namely, the Hermite, Jacobi and Laguerre polynomials) were considered as examples. From these examples it follows that the central problem with obtaining the differential (or difference) equation for the given system of polynomials was in finding a realization of the annihilation (lowering) operator by a differential operator. Note that the operator of a finite order in work \([15]\) occurs only for the standard quantum mechanical oscillator connecting with the orthonormal Hermite polynomial system. In all other cases considered in \([15]\) the annihilation operator connected with the orthonormal classical polynomial system was realized by a differential operator of the infinite order. More exactly these operators were described as rather simple functions of the first order differential operator. Other examples of realization of the annihilation operator by the known functions of the first order differential operator are considered in works \([17, 18, 20]\).

In the present work we shall obtain an expression for an annihilation operator of the generalized deformed oscillator algebra related to a complete system of the real orthonormal polynomials. The above-mentioned examples from \([15]\) are concrete realizations of the general expressions obtained here.

Let us remark that the operators of the finite order arise for the Hermite-Chihara polynomial systems (see, \([21]\) \([3]\) and \([16]\)) and the Hermite-Hahn polynomial systems \([22]\).

It is more interesting to apply the obtained results to investigating a generalized oscillator connected with symmetric basic number \( [a] = \frac{q^a - q^{-a}}{q - q^{-1}} \) of quantum groups theory. This case is still not clearly understood. The research of this case is in a stage of completion now.

For the convenience of the reader in the following section we recall some necessary information from the theory of Jacobi matrices and the classical moment problem.
2 Background information

Let a operator $X$ is given by action on the standard orthonormal basis $\{e_n | n \in \mathbb{Z}_+\}$ in the Hilbert space $H = l^2(\mathbb{Z}_+)$:

$$X e_n = b_n e_{n+1} + a_n e_n + b_{n-1} e_{n-1}, \quad a_n \geq 0, \ b_n \in \mathbb{R}.$$ \hspace{1cm} (1)

Then $X$ can be represent in $H$ by an infinite 3-diagonal matrix

$$X = \begin{pmatrix}
a_0 & b_0 & 0 & 0 & \cdots & \cdots \\
b_0 & a_1 & b_1 & 0 & \cdots & \cdots \\
0 & b_1 & a_2 & b_2 & \cdots & \cdots \\
0 & 0 & b_2 & a_3 & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix},$$ \hspace{1cm} (2)

known as the Jacobi matrix.

With the Jacobi matrix it is related the set of polynomials $P_n(x)$ of degree $N$, satisfying the recurrent relation,

$$b_n P_{n+1}(x) + a_n P_n(x) + b_{n-1} P_{n-1}(x) = x P_n(x),$$ \hspace{1cm} (3)

and subjecting to the following ”initial conditions”

$$P_0(x) = 1, \quad P_{-1}(x) = 0.$$ \hspace{1cm} (4)

Further, we shall assume that $a_n, \ b_n \in \mathbb{R}$ and $a_n = 0, b_n > 0$. Under such conditions polynomials $P_n(x)$ have real coefficients and fulfill the following parity conditions $P_n(-x) = (-1)^n P_n(x)$. Notice that the recurrent relations (3) have two linearly independent solutions. The polynomials $P_n(x)$, which solve to (3) and subject to the initial conditions (4), are called polynomials of the first kind. Besides, there are independent set of solutions consisting from polynomials $Q_n(x)$, which satisfy with the following initial conditions

$$Q_0(x) = 0, \quad Q_1(x) = \frac{1}{b_0}.$$ \hspace{1cm} (5)

For these polynomials the standard agreement $\text{deg} Q_n(x) = n - 1$ is carried out. These polynomials are known as polynomials of the second kind for the Jacobi matrix $X$ (2).

The polynomials of the first and second kind are related by the following equation

$$P_{n-1}(x)Q_n(x) - P_n(x)Q_{n-1}(x) = \frac{1}{b_{n-1}}, \quad (n = 1, 2, 3, \ldots).$$ \hspace{1cm} (6)

It is known that polynomials $\{P_n(x)\}_{n=0}^{\infty}$ are orthonormal

$$\int_{\mathbb{R}} P_n(x) P_m(x) d\sigma(x) = \delta_{n,m}$$ \hspace{1cm} (7)

with respect to a positive Borel measure $\sigma$ on $\mathbb{R}$. In order to evaluate the measure $\sigma$ one must solve the following task.
By $s_n$ we denote the factor $\alpha_0$ at $P_0(t)$ in the expansion

$$t^n = \sum_{k=0}^{n} \alpha_k P_k(t).$$

For the given numerical sequence $\{s_n\}_{n=0}^{\infty}$ it is required to find a positive measure $\sigma$ on the real line such that the following equalities:

$$s_n = \int_{-\infty}^{\infty} t^n d\sigma(t), \quad n = 0, 1, 2, \ldots$$

are hold. This task is known as the degree Hamburger moment problem [24].

If the system (8) defines the measure uniquely, then the moment problem is called determined one. Otherwise there exists an infinite set of such measures, and the moment problem is called undetermined one.

The matrix (2) determines a symmetrical operator $X$ in $H$. It is known that the deficiency indices for the operator $X$ are equal $(0, 0)$ or $(1, 1)$.

**Proposition 2.1.** Let $\overline{X}$ be a closure of an operator $X$ in $H$. The following conditions are equivalent:

1. the deficiency indices of an operator $X$ are equal $(0, 0)$;
2. an operator $\overline{X}$ is selfadjoint in the Hilbert space $H$;
3. the moment problem (3) for the given number set $\{s_n\}_{n=0}^{\infty}$ such that

$$s_n = (e_0, X^n e_0)$$

is determined;
4. a series $\sum_{n=0}^{\infty} |P_n(z)|^2$ is divergent for all $z \in \mathbb{C}$ such that $\text{Im}z \neq 0$.

Let the operator $X$ has the deficiency indices $(1, 1)$. Then the operator $\overline{X}$ is not selfadjoint and there exists infinite number of its selfadjoint extensions. In this case series $\sum_{n=0}^{\infty} |P_n(z)|^2 < \infty$ is divergent at all $z \in \mathbb{C}$ such that $\text{Im}z \neq 0$, and moment problem (8) for a set of numbers $s_n$ from (3) is indetermined.

The following proposition gives a sufficient condition under which the operator $X$ has the deficiency indices $(1, 1)$.

**Proposition 2.2.** Under the following conditions

1. there exists $N$ such that for all $n > N$ the inequality

$$b_{n-1} b_{n+1} \leq b_n^2;$$

2. $\sum_{n=0}^{\infty} \frac{1}{b_n} < \infty;$. (11)
the operator $X$ has the deficiency indices equal to $(1, 1)$.

Moreover, we have the following statement.

**Proposition 2.3.** (see [25]). If the conditions (14) and (15) are fulfilled, there is an infinite number of selfadjoint extensions of the operator $X$, and related moment problem (8) for a set of numbers $\{s_n\}_{n=0}^\infty$ from (9) is undetermined.

The polynomials $P_n$ and $Q_n$ can be represented in the form [14]

$$P_n(x; q) = \sum_{m=0}^{\epsilon(\frac{n}{2})} \frac{(-1)^m b_0^{2m-n}}{\sqrt{\{n\}}!} \alpha_{2m-1,n-1} x^{n-2m}$$  \hspace{1cm} (12)

$$\alpha_{-1,n-1} = 1; \quad \alpha_{2m-1,n-1} = \sum_{k_1=2m-1}^{n-1} \sum_{k_2=2m-3}^{k_1-2} \sum_{k_m=1}^{k_{m-1}-2} \{k_1\} \ldots \{k_m\}, \quad m \geq 1$$  \hspace{1cm} (13)

$$Q_{n+1}(x; q) = \sum_{m=0}^{\epsilon(\frac{n}{2})} \frac{(-1)^m b_0^{2m-n}}{\sqrt{\{n+1\}}!} \beta_{2m,n} x^{n-2m}$$  \hspace{1cm} (14)

$$\beta_{0,n} = 1; \quad \beta_{2m,n} = \sum_{k_1=2m-1}^{n} \sum_{k_2=2m-3}^{k_1-2} \sum_{k_m=2}^{k_{m-1}-2} \{k_1\} \ldots \{k_m\}, \quad m \geq 1,$$  \hspace{1cm} (15)

where $\{s\} = \frac{b_0^2}{b_0^2 - 1}$, and the integral part of $x$ is denoted by $\epsilon(x) = \text{Ent}(x)$. Because the representation (12) - (15) of the polynomials $P_n$ and $Q_n$ have not been adequately explored, we shall recall some properties of coefficients $\alpha_{m,n}$ and $\beta_{m,n}$.

From the recurrent relation (3) it follows that the coefficients $\alpha_{m,n}$ and $\beta_{m,n}$ satisfy the conditions:

$$\alpha_{2m-1,n} = \{n\} \alpha_{2m-3,n-2} + \alpha_{2m-1,n-1};$$  \hspace{1cm} (16)

$$\beta_{2m,n} = \{n\} \beta_{2m-2,n-2} + \beta_{2m,n-1}.$$  \hspace{1cm} (17)

From the definitions (13) and (15) of coefficients $\alpha_{k,n}$ and $\beta_{k,n}$ it follows that

$$\alpha_{2k-1,2k-1} = \{2k-1\}!!; \quad \beta_{2k-2,2k-2} = \{2k-2\}!!.$$  \hspace{1cm} (18)

Here

$$\{2n\}!! = \{2n\} \{2n-2\} \ldots \{2\}; \quad \{2n-1\}!! = \{2n-1\} \{2n-3\} \ldots \{1\}.$$

### 3 Statement of a problem

Let $\mathcal{H} = L^2(\mathbb{R}; \mu(dx))$ be a Hilbert space, and $\{\Psi_n(x)\}_{n=0}^\infty$ be a system of polynomials, which are orthonormal with respect to the measure $\mu$. The recurrent relations of these polynomials take the following form:

$$b_n \Psi_{n+1}(x) + b_{n-1} \Psi_{n-1}(x) = x \Psi_n(x), \quad b_n > 0,$$  \hspace{1cm} (19)
\[ \Psi_0(x) = 1, \quad \Psi_{-1}(x) = 0. \] (20)

In the work [15] it was shown, that one can construct the oscillator-like algebra \( A_\psi \) corresponding to this polynomial system. In addition, the space \( \mathcal{H} \) is a space of the Fock representation, and the polynomials \( \{ \Psi_n(x) \}_{n=0}^\infty \) make up the Fock basis. The generators \( a_\mu^+, a_\mu^-, N \) of this algebra acts in the Fock representation as follows

\[ \begin{align*}
  a_\mu^+ \Psi_n(x) &= \sqrt{2b_n} \Psi_{n+1}(x), \quad n \geq 0; \\
  a_\mu^- \Psi_n(x) &= \sqrt{2b_{n-1}} \Psi_{n-1}(x), \quad n \geq 1 \\
  N \Psi_n(x) &= n \Psi_n(x),
\end{align*} \] (21)

From these relations it follows that

\[ \begin{align*}
  a_\mu^- a_\mu^+ \Psi_n(x) &= 2b_n^2 \Psi_n(x), \quad a_\mu^+ a_\mu^- \Psi_n(x) = 2b_{n-1}^2 \Psi_n(x), \quad n \geq 0,
\end{align*} \] (23)

and \( b_{-1} = 0 \). Then we have the following commutation relations:

\[ [a_\mu^-, a_\mu^+] = 2(B(N + 1) - B(N)), \quad [N, a_\mu^\pm] = \pm a_\mu^\pm, \] (24)

between the generators of the algebra \( A_\psi \). The function \( B(N) \) is defined by the formula

\[ B(N) \Psi_n(x) = b_{n-1}^2 \Psi_{n-1}(x). \] (25)

If the sequence \( \{ b_n \}_{n=0}^\infty \) satisfies the recurrent relation

\[ b_n^2 - Qb_{n-1}^2 = C(n), \] (26)

then, as follows from (25), the relation

\[ a_\mu^- a_\mu^+ - Qa_\mu^+ a_\mu^- = 2C(N). \] (27)

is fulfilled also.

According to [12] and [13], we have

\[ \Psi_n(x) = \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(-1)^m}{\sqrt{n!}} \alpha_{2m-1, n-1} b_0^{2m-n} x^{n-2m}, \quad n \geq 0. \] (28)

For simplification of further formulas we define the operator \( A \) by a relation \( a^- = \sqrt{2A} \). Then

\[ \begin{align*}
  A \Psi_n &= b_{n-1} \Psi_{n-1}, \quad n \geq 1, \quad A \Psi_0 = 0; \\
  A \Psi_0 &= 0 \Rightarrow a_{00} = 0, \quad n \geq 0
\end{align*} \] (29)

(30)

The main purpose of the present work is to find, using the formulas (29) and (30), the conditions on the coefficients \( a_{ks} \), under which the operator \( A \) take the form

\[ A = \sum_{k,s=0}^\infty a_{ks} x^k \frac{d^s}{dx^s}. \] (31)
Substituting (31) and (3) into (29), one obtains

\[
\sum_{k,s=0}^{\infty} \sum_{m=0}^{\epsilon(n/2)} (-1)^m \alpha_{2m-1,n-1} b_0^{2m-n} x^{n+k-2m-s} \frac{(n-2m)!}{(n-2m-s)!} = \sum_{m=0}^{\epsilon(n-1)} (-1)^m \alpha_{2m-1,n-2} b_0^{2m+1-n} x^{n-1-2m}
\]  

(32)

Equating the coefficients at \(x^t\) in both sides of a relation (32), we receive

\[
a_{n,0} = 0, \quad n \geq 0
\]

\[
\sum_{s=0}^{\infty} \sum_{m=\delta\left(\frac{n-s-1}{2}\right)}^{\epsilon(n-s)} (-1)^m a_{t+s+2m-n,s} \alpha_{2m-1,n-1} b_0^{2m-n} \frac{(n-2m)!}{(n-2m-s)!} = \begin{cases} 
0 & \text{if } n-1-t = 2p+1, \quad p \geq 0 \\
\sqrt{\{n\}} b_{n-1} (-1)^p \alpha_{2p-1,n-2} b_0^{-t} & \text{if } n-1-t = 2p, \quad p \geq 0
\end{cases}
\]  

(33)

where the least integer greater than \(a\) or equal to \(a\) is denoted by \(\delta(a)\).

From (33) it follows that the problem of a realization of the annihilation operator by a differential operator (31) was rather cumbersome in general case. For this reason we slightly change the statement of a problem. Namely, we will seek for not an annihilation operator but a “lowering operator” \(A\) (denoted by the same symbol). In other words from (33) we should look for an operator \(A\) such that:

\[
A \Psi_n = \gamma_n \Psi_{n-1}, \quad n \geq 1, \quad A \Psi_0 = 0,
\]

(34)

under the supposition, that the operator (31) works in space \(H\) as a shift operator. Thus we supposed that among coefficients \(\{a_{k,s}\}_{k,s=0}^{\infty}\) in the matrix of this operator only elements on the first upper diagonal are different from zero

\[
a_{n,k} = \begin{cases} 
0 & \text{if } k \neq n+1 \\
0 & \text{if } k = n+1
\end{cases}
\]  

(35)

Let us remark that condition

\[
a_{n,0} = 0, \quad n \geq 0
\]  

(36)

and relations (32) and (33) (with a replacement \(b_{n-1} \rightarrow \gamma_n\) in a right hand side) are hold.

In the following section we shall obtain some consequence of the relations (34) and (35).

4 Conditions under which the matrix of an operator \(A\) has non zero elements only upper above diagonal

Taking into account that all elements \(a_{k,s}\) except for \(a_{k,k+1}\) are equal to zero, and using the relation (33) for \(t = 0, 1, 2, \ldots\) and \(P = 0, 1, 2, \ldots\), we obtain the following conditions on coefficients
where \( k \geq 0 \). Besides, we have the following consistency conditions (for \( p \geq 0, k \geq 0 \))

\[
\frac{\alpha_{2p+1,2p+k+1}}{\alpha_{2p+1,2p+k}} = \frac{\alpha_{2p-1,2p+k} + 1}{\alpha_{2p-1,2p+k+1}} \cdot \frac{\alpha_{1,k+1}}{\alpha_{1,k}}
\]

(40)

As a result we obtain the following theorem.

**Theorem 4.1.** For the operator \( A \) defined by (33) fulfill the conditions (32) and (33) it is necessary and sufficient to have the conditions (32) and (33). Here \( \alpha_{2m-1,n-1} \) are the coefficients of the function \( \Psi_n(x) \) defined by (34). Under these conditions the elements \( a_{k,k+1} \) of the matrix of the operator \( A \) are determined by the relations (39).

**Corollary 4.2.** Under the additional conditions

\[
a_{0,1} \neq 0, \quad 0 = a_{1,2} = a_{2,3} = \ldots = a_{k,k+1} = \ldots,
\]

we have

\[
\sqrt{\{k\}} \gamma_k = k \gamma_1, \quad k \geq 1.
\]

(41)

**Theorem 4.3.** The coefficients \( \alpha_{2m-1,n-1} \) defined by (13) satisfy the conditions (40) iff there exist a positive sequence \( \{v_k\} \), such that

\[
\alpha_{2m-1,n-1} = \frac{(2m-1)!! (v_1 v_2 \ldots v_{n-1})}{(v_1 v_2 \ldots v_{2m-1}) (v_1 v_2 \ldots v_{n-2m-1})}, \quad m \geq 1, n \geq 3
\]

\[
\alpha_{1,n-1} = 1, \quad \alpha_{1,1} = \{1\} = 1.
\]

(42)

Under these conditions the sequence \( \{v_k\} \) has the following properties

1. \( 1 = v_0 \leq v_1 \leq v_2 \leq \ldots \)

2. \( v_{n-2} v_{2p-1} + v_{2p-3} v_{n-2p} = v_n v_{2p-3} + v_{2p-1} v_{n-2p} \) for all \( n \geq 2, p \geq 1, 2p \leq n \) (\( v_1 = 0 \)).

Proof.

The sufficiency of the condition (12) is checked by the direct substitution (12) into (40).

Necessity. Let the coefficients \( \alpha_{2m-1,n-1} \) determined by relations (13) are satisfied the conditions (10). Let us put \( v_0 = a_{1,1} = 1, \quad v_1 \geq 1 \), and define the remaining terms of a sequence \( \{v_k\} \) from the equalities

\[
\alpha_{1,n} = \frac{v_{n-1} v_n}{v_1}, \quad n \geq 1.
\]

(43)

Using relations (40), we immediately check the validity of the formulas (12).

An example of a polynomial system of the above-mentioned type give us the generalized Hermite polynomials (entered in one of notes in the monography [21] and explicitly investigated in [3]). We consider these polynomials in the next section.
5 Generalized Hermite polynomials

The more complete exposition of the material considered in this section can be found in [16].

Let us consider the Hilbert space

\[ H_\gamma = L^2(R; |x|^{\gamma} \Gamma(\frac{1}{2}(\gamma + 1)))^{-1} \exp(-x^2) dx, \quad \gamma \geq -1. \]  

(44)

According to the method suggested in [15], we shall construct a canonical orthonormal polynomial system \( \{\psi_n(x)\}_{n=0}^\infty \), which is complete in the space \( H_\gamma \). The polynomials \( \psi_n(x) \) will satisfy the recurrent relations (19) and (20) with coefficients \( \{b_n\}_{n=0}^\infty \), which are given by the following formulas:

\[ b_{n-1} = \frac{1}{2} \left\{ \begin{array}{ll}
\sqrt{n} & n = 2m, \\
\sqrt{n + \gamma} & n = 2m + 1.
\end{array} \right. \]  

(45)

The polynomials \( \psi_n(x) \) are defined by the relations (42), (18) and the formulas (12), where \( \{s\} = \frac{b_{2s-1}^2}{b_0^2} \). Besides, the sequence \( \{v_n\}_{n=0}^\infty \) is given by the formulas

\[ v_n = \left\{ \begin{array}{ll}
\frac{\gamma + n + 1}{\gamma + 1} & n = 2m, \\
\frac{n + 1}{\gamma + 1} & n = 2m + 1.
\end{array} \right. \]  

(46)

It is clear that \( v_0 = 1, \quad v_1 = \frac{2}{\gamma + 1} = b_0^{-2} \).

The appropriate ”lowering” operator \( A \) is defined by relations (31) and (34), where

\[ a_{0,1} = 1, \quad a_{m-1,m} = \frac{(-2)^{m-1}}{m!} \frac{\gamma}{\gamma + 1}, \quad m \geq 2. \]  

(47)

The sequence \( \{\gamma_n\}_{n=0}^\infty \) included in (34) is given by the expressions:

\[ \gamma_n = \sqrt{2} \left\{ \begin{array}{ll}
\sqrt{n} & n = 2m, \\
\sqrt{n + \gamma} & n = 2m + 1.
\end{array} \right. \]  

(48)

The annihilation operator \( a^-_\mu \) of the generalized oscillator algebra is equal to

\[ a^-_\mu = \frac{\gamma + 1}{\sqrt{2}} A. \]  

(49)

It is valid the following commutation relations for the position operator \( X_\mu \) and the number operator \( N \):

\[ X_\mu \frac{d}{dx} - N = (a^-_\mu)^2. \]  

(50)
Using the eigenvalue equation for a Hamiltonian \( H_\mu = a^-_\mu a^+_\mu + a^-_\mu a^-_\mu \), and taking into account the equality \( a^+_\mu a^-_\mu = 2B(N) \), one can obtain the following relation:

\[
a^-_\mu a^+_\mu = 2B(N + I).
\] (51)

From (50) – (51), it follows the differential equation

\[
x\psi''_n + (\gamma - 2x^2)\psi'_n + (2nx - \frac{\theta_n}{x})\psi_n = 0, \quad n \geq 0,
\] (52)

where \( \theta_n = \frac{\gamma(1 - (-1)^n)}{2} \). This equation coincides with the known differential equation for generalized Hermite polynomials ([3] see also [21]).

Remark 5.1. For the generators \( a^+_\mu, a^-_\mu \) of the Heisenberg algebras \( A_\mu \), corresponding to the system of the generalized Hermite polynomials, we have (see [15]):

\[
[a^-_\mu, a^+_\mu] = (\gamma + 1)I - 2(2B(N) - N).
\] (53)

"The energy levels" for Hamiltonian of the related oscillator are equal to:

\[
\lambda_0 = \gamma + 1, \quad \lambda_n = 2n + \gamma + 1, \quad n \geq 1.
\] (54)

Finally, we have for the momentum operator the following formula:

\[
P_\mu = i\left(\frac{d}{dx} + X^{-1}_\mu \Theta_N - X_\mu\right),
\] (55)

where

\[
\Theta_N = 2B(N) - N.
\] (56)

6 Conditions under which the annihilation operator \( A \) can be realized by a differential operator

Now we consider the relations (32) and (33) without the supposition (35) relative to the coefficients \( a_{k,s} \).

Let us introduce

\[
C_{k+2m-2w,k+2p+1-2w} = \frac{(k + 2p + 1 - 2m)!}{b_0^{k+2p+1-2m}}\alpha_{k+2m-2w,k+2p+1-2w}
\] (57)

Then the following assertion is hold.

Theorem 6.1. The operator \( A \) defined by the formula (31) satisfy the relation (29) iff the elements \( a_{k,s} \) of this operator matrix satisfy the following conditions:

- For all \( t = 2l, n = 2p + 2l + 1, \quad p \geq 0, \quad l \geq 0 \) the following relation

\[
\sum_{w=0}^{l-1} \sum_{m=0}^{w} (-1)^m \frac{\alpha_{2m-1,2p+2l}}{(2w-2m)!} B_1 + \sum_{w=1}^{p+l} \sum_{m=w-l}^{w} (-1)^m \frac{\alpha_{2m-1,2p+2l}}{(2w-2m)!} B_2 = (-1)^p \frac{(2l+2p+1)!!}{b_0^p},
\] (58)
where
\[ B_1 = \left( C_{2l+2m-2w,2p+2l+1-2w} + \frac{(2p + 2l + 1 - 2m)}{b_0} C_{2l+2m-2w-1,2p+2l-2w} \right), \]
\[ B_2 = \left( C_{2l+2m-2w,2p+2l+1-2w} + \frac{(2p + 2l + 1 - 2m)}{b_0} C_{2l+2m-2w-1,2p+2l-2w} \right). \]

must be hold.

- For all \( t = 2l + 1, n = 2p + 2l + 2, p \geq 0, l \geq 0 \) the following relation
\[ \sum_{w=0}^{l-1} \sum_{m=0}^{w} (-1)^m \frac{\alpha_{2m-1,2p+2l+1}}{(2w-2m)!} B_3 + \sum_{w=l}^{p+l} \sum_{m=w-l}^{w} (-1)^m \frac{\alpha_{2m-1,2p+2l+1}}{(2w-2m)!} = \]
\[ (-1)^p \{2l + 2p + 2\} \frac{\alpha_{2p-1,2p+2l}}{b_0^{2l+1}} \]
should be fulfilled, where
\[ B_3 = \left( C_{2l+2m-2w+1,2p+2l+2-2w} + \frac{(2p + 2l + 2 - 2m)}{(2w + 1 - 2m) b_0} C_{2l+2m-2w,2p+2l+1-2w} \right), \]
\[ B_4 = \left( C_{2l+2m-2w+1,2p+2l+2-2w} + \frac{(2p + 2l + 2 - 2m)}{(2w + 1 - 2m) b_0} C_{2l+2m-2w,2p+2l+1-2w} \right), \]

and the coefficients \( \alpha_{2mp-1,n-1} \) are given by (13).

7 Conclusion

From the results given above it follows that in the general case a orthonormal polynomial system satisfies differential (or difference) equation of a finite order. Therefore it is interesting to describe such orthonormal polynomial system, for which one can obtain a differential (or difference) equation of a finite order. It is desirable that we have to deal only with differential equation of the second order. The elementary example of such class of polynomials selected by the condition (35), is given by Hermite - Chihara polynomials [3, 21] considered in detail in [16].

Unfortunately, the relations (57 – 59) are rather complicated in the general case. Therefore generally local formulas representing an annihilation operator (and also for a momentum operator and Hamiltonian) by a differential operator are too cumbersome.

Finally, we note the that the formalism put forward in the present work holds true also after replacement usual derivation by some of its \( q \)-analog. The last circumstance provides a useful guide to investigating of deformed polynomials.

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