Quark-Antiquark Forces From SU(2) And SU(3) Gauge Theories On Large Lattices

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ABSTRACT

We present results on the spin-independent quark-antiquark potential in SU(3) gauge theory from a simulation on a $48^3 \times 64$ lattice at $\beta = 6.8$, corresponding to a volume of $(1.7 \text{ fm})^3$. Moreover, a comprehensive analysis of spin- and velocity-dependent potentials is carried out for SU(2) gauge theory, with emphasis on the short range structure, on lattices with resolutions ranging from .02 fm to .04 fm.

*Invited talk given at “International Workshop on Color Confinement and Hadrons, CONFIMENT 95”, March 22–24, 1995, RCNP Osaka, Japan.
1. Introduction

Quarkonia spectroscopy provides a wealth of information and thus constitutes an important observational window to the phenomenology of strong interactions. Within their non- (or semi-) relativistic setting, potential models have been proven to describe the empirical charmonium and bottomonium spectra remarkably well. The resulting phenomenological potentials can be compared to lattice predictions from the QCD Lagrangian.

Lattice QCD techniques offer on the other hand a direct access to the hadron spectrum, without recourse to the potential picture. In the heavy quark sector (with quark mass $M$), however, the full fledged lattice calculation would require (at present) prohibitively fine lattice resolutions $a < M^{-1}$. It is therefore more practical to expand the QCD Lagrangian in powers of $M^{-1}$ into an effective non-relativistic QCD action (NRQCD) which can be evaluated subsequently by use of lattice methods.

The traditional semi-relativistic potential model approach to spectrum calculations assumes the validity of the instantaneous approximation. This approximation can be established from first principles lattice computations by comparing spectra and wave functions as obtained from NRQCD to results from lattice potentials.

Pioneering attempts to compute corrections to the static potential on the lattice have been launched in the mid eighties. In the meantime, tremendous progress has been achieved, both in computational power and methods. In view of the general interest in the potential formulation of quark binding problems, it appears to be timely for a new lattice determination of spin-dependent (sd) and velocity-dependent (vd) potentials.

As a first step into this direction, we present recent results on the static quark antiquark potential from SU(3) gauge theory, as well as results on sd and vd potentials from SU(2) gauge theory. The limitation to two colours will not yet allow to proceed to spectrum calculations but will hopefully disclose the key features of confinement at work.

2. The Central Quark-Antiquark Potential

In the limit of infinitely heavy colour sources ($M \to \infty$), the Born-Oppenheimer approximation can be applied and — after integrating out the gauge degrees of freedom — the underlying QCD binding problem becomes nonrelativistic.

Theoretically, one expects the leading order (in $M^{-1}$) potential to be dominated by one-gluon exchange at short distances, $V_0(r) = -C_F \frac{\alpha_s(r)}{r}$, with a running coupling $\alpha(r)$ which depends logarithmically on $r$. On the other hand, for large separations $r$, one would expect a bosonic string model to hold which predicts the asymptotic behaviour $V_0(r) = \kappa r - \frac{\pi \cdot 4}{12} + \cdots$ (with string tension $\kappa$). A priori it is not clear
when large or small $r$ behaviour would set in and which form the potential takes in the region of intermediate $r$.

Phenomenologically, the Cornell potential $V_{\text{phen}}(r) = \kappa r - \frac{e}{r}$ has been proven to describe the spin averaged charmonium and bottomonium spectra successfully with one set of parameters, $\kappa$ and $e$, in accord with the flavour independence of strong interactions. It is well known, though, that the binding energies of low lying states are not overly sensitive to the shape of the potential at distances smaller than 0.2 fm or larger than 1 fm.

Therefore, a lattice determination of the QCD potential in the intermediate $r$ region is of particular interest. In Fig. 1 we show high statistics results on the central potential from SU(3) gauge theory on a $48^3 \times 64$ lattice at $\beta = 6.8$, which corresponds to a lattice resolution $a \approx 0.035$ fm. The data for $r \geq 0.3$ fm has been fitted to a Cornell type parametrization (solid curve). At small $r$ we observe deviations from this parametrization. This is in accord with running coupling effects. Note, that in the lattice computation vacuum polarization due to sea quarks has been neglected and thus no string breaking occurs.

Before one can proceed to predict spin-averaged quarkonia spectra from such lattice potentials, one has to investigate possible corrections to the infinite mass limit at realistic charm and bottom quark masses. In solving the Schrödinger equation with our potential, we find the average speed of the quarks (in units of the speed of light) to be $\langle v^2 \rangle \approx 0.25$ and $\langle v^2 \rangle \approx 0.09$ for the charmonium and bottomonium ground states, respectively. This leads us to expect that, at least in the case of charmonium,
the phenomenological potential, $V_{\text{phen}}(r)$, (which has been tuned to reproduce the empirical spectrum) might deviate by substantial $O(v^2)$ corrections from the static potential, $V_0(r)$. Needless to say, that fine and hyperfine splittings have their origin in such $v$ or, equivalently, $1/M$ corrections.

3. $1/M$-Corrections

Starting from a Foldy-Wouthuysen transformation of the Euclidean quark propagator in an external gauge field, the asymptotic ($t \to \infty$) expression $W(r, t) \propto \exp(-V_0(r)t)$ can be derived for the static potential, $V_0(r)$, where $W(r, t)$ denotes the expectation value of the familiar Wilson loop with spatial extent $r$ and temporal extent $t$. Perturbing the propagator in terms of the inverse quark masses $M_1^{-1}$ and $M_2^{-1}$ around its static solution, one arrives at the semi-relativistic Hamiltonian (in the CM system, $p_1 = -p_2 = p$),

$$H = \frac{p^2}{2} \left( \frac{1}{M_1} + \frac{1}{M_2} \right) - \frac{(p^2)^2}{8} \left( \frac{1}{M_1} + \frac{1}{M_2} \right) + V(r, L, S_1, S_2, p)$$

where the potential consists of a central part, sd and vd corrections,

$$V(r, L, S_1, S_2, p) = V_0(r) + V_{sd}(r, L, S_1, S_2) + V_{vd}(r, p)$$

The sd contributions have been derived by Eichten, Feinberg and Gromes while the vd terms have been elaborated by Barchielli et al.:

$$V_{sd}(r, L, S_1, S_2) = \frac{l_1 s_1 + l_2 s_2}{2r} (V'_0(r) + 2V'_1(r)) + \frac{l_1 s_2 + l_2 s_1}{r} V'_2(r)$$

$$+ \left( \frac{(s_1 r)(s_2 r)}{r^2} - \frac{s_1 s_2}{3} \right) V_3(r) + \frac{s_1 s_2}{3} V_4(r)$$

with

$$V_{vd}(r, p) = \frac{1}{8} \left( \frac{1}{M_1^2} + \frac{1}{M_2^2} \right) \nabla^2 (V_0(r) + V_a(r))$$

$$+ \left\{ v^i_1, v^j_2, \delta_{ij} V_b(r) + \left( \frac{\delta_{ij}}{3} - \frac{r_i r_j}{r^2} \right) V_c(r) \right\}_W$$

$$+ \sum_{k=1}^2 \left\{ v^i_k, v^j_k, \delta_{ij} V_d(r) + \left( \frac{\delta_{ij}}{3} - \frac{r_i r_j}{r^2} \right) V_e(r) \right\}_W$$

with $v_i = p_i / M_i$, respectively. $\{\cdot, \cdot, \cdot\}_W$ denotes Weyl ordering of the three arguments.

The sd and vd potentials $V'_0, V'_1, V'_2, V_3, V_4$ and $V_a - V_e$ can be computed from lattice correlation functions in Euclidean time, $C(T)$. This is done by measuring expectation values of Wilson loops with two colour field insertions (ears) within the
temporal transporters, divided by the corresponding loop without ears ($T$ denotes the ear to ear temporal distance). One ear excites the gluon field between the two charges while the second one returns the field into its ground state. To obtain the potentials, an integration over all possible interaction times (temporal positions of the ears) has to be performed.

The minimal distance of an ear to an “end” of the Wilson loop, occurring within the integration, $\Delta T$, represents the time the gluon field has at its disposal to decay into the ground state after creation and, therefore, governs excited state contaminations. The spatial transporters within the Wilson loops are smeared to suppress such pollutions from the beginning, allowing us to reduce $\Delta T$. As an additional technical trick we integrate out temporal links analytically which results in reduced statistical fluctuations. Finally, we exploit transfer matrix techniques to obtain asymptotic results from a finite integration range (in $T$). Details about this procedure will be published elsewhere.

4. Matching of the Effective Hamiltonian to QCD

The sd and vd potentials are computed from amplitudes of correlation functions rather than from eigenvalues of the transfer matrix. This gives rise to renormalizations in respect to the corresponding continuum potentials. A different way to illustrate the necessity of renormalization is the fact that the colour electric (magnetic) “ears”,

$$gF_{\mu\nu} = \frac{1}{2ia^2} \left( U_{\mu\nu} - U_{\mu\nu}^{\dagger} \right) + \mathcal{O}(a^2),$$  \hspace{1cm} (7)

which have been inserted into Wilson loops, explicitly depend on the lattice scale, $a$, and discretization.

However, renormalization is not a pure lattice problem in this case. By truncating the expansion of the QCD Lagrangian in powers of $M^{-1}$ at a given order, the ultraviolet behaviour is changed in respect to the full theory. Therefore, the effective Lagrangian has to be matched to full QCD at a renormalization scale $\mu < M$, giving rise to renormalization constants $c_i(\mu, M)$, connecting a QCD potential $V_i(r; M)$ to the corresponding potential, computed in the framework of the effective theory, e.g.

$$V_i(r; M) = c_i(\mu, M)V_i(r; \mu, M).$$

This problem, which becomes visible beyond the tree-level, has been approached systematically for sd potentials in the context of heavy quark effective theory by Chen et al.

Up to an additional $\delta^3(r)$-like factor, that originates from mixing of the dimension 6 spin-spin interaction term with a four fermion contact term, a result is obtained which can be rewritten into Eq. (3) by substituting the naive potentials $V_i$ by renormalized potentials $V_{i,\text{ren}}$ which can be related to each other by a renormalization

\textsuperscript{a}On the lattice, $\mu \simeq \pi/a$. 

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constant, $c_3$,

$$V_{1,\text{ren}}'(r) = c_3(\mu, M)V_1'(r, \mu) + (c_3(\mu, M) - 1) V_0'(r), \quad (8)$$

$$V_{2,\text{ren}}'(r) = c_3(\mu, M)V_2'(r, \mu), \quad (9)$$

$$V_{3/4,\text{ren}}'(r) = c_3^2(\mu, M)V_{3/4}'(r, \mu). \quad (10)$$

The constant $c_3$ is known to one-loop perturbation theory. As $c_3$ is likely to be dominated by higher order perturbative and nonperturbative uncertainties in the low energy regime of interest, we apply the (nonperturbative) HM renormalization procedure introduced by Huntley and Michael. This procedure has originally been invented to remove the continuum-lattice renormalization problem but — as a by-product — also cures the matching problem of the effective heavy quark theory. The success of this approach can be checked numerically in two ways, namely (a) by varying the lattice resolution $a$ and checking scaling of the results and (b) by testing against the Gromes relation between spin-orbit potentials and the central potential (which does not undergo renormalization);

$$V_0'(r) = V_{2,\text{ren}}'(r) - V_{1,\text{ren}}'(r). \quad (11)$$

\begin{itemize}
  \item[c^b] For simplicity we assume $M = M_1 = M_2$.
  \item[c^c] Two additional constraints, relating vd potentials to the central potential, have been found:
\end{itemize}
Fig. 3. Scaling plot of the long range spin-orbit potential $V_1'$ (in units of the string tension, $\kappa$). The curve corresponds to a one-parameter fit of the form $V_1'(r) = -\kappa - a/r^2$.

Fig. 4. The spin-spin potential $V_4$ in lattice units.
The present simulations have been performed on $32^4$ and $48^4$ lattices at $\beta = 2.74$ and $\beta = 2.96$ which corresponds to lattice spacings $a \approx 0.041$ fm and $a \approx 0.020$ fm, respectively. The above physical scales have been adjusted such that the string tension comes out to be $\sqrt{\kappa} = 440$ MeV. In Fig. 3 we check our data on $V'_2 - V'_1$ (in units of the string tension, $\kappa$) against the force, obtained from a fit to the central potential, $V_0(r)$. As can be seen, the two data sets scale nicely onto each other and reproduce the central potential up to lattice artefacts at small $r$.

5. Results

In the continuum, tree-level perturbation theory yields the following expectations for the spin-orbit and spin-spin potentials,

$$V'_1 = 0 \quad V'_2 = \frac{e}{r^2} \quad V'_3 = \frac{3e}{r^3} \quad V'_4(r) = 8\pi\delta^3(r) \quad (12)$$

where $e = C_F\alpha$ and $C_F = 3/4$ for SU(2). Note, that the first spin-orbit potential does not contain vector exchange contributions and, due to the Gromes relation, should be of the form (assuming Eq. (12) to be valid and a Cornell parametrization of the central potential), $V'_1 = -\kappa$.

The data for $V'_1$ is displayed in Fig. 3. Our lattice resolution enables us to establish an attractive short range contribution to $V'_1$ that can be well fitted to a Coulomb ($1/r^2$) form (in addition to the constant long-range term, which is in agreement with the string tension, $\kappa$). This term amounts to about one quarter of the Coulomb-like...
contribution to the static potential.

We confirm the second spin-orbit potential $V_2'$ to be definitely of short range nature. Moreover, up to the first data point, it qualitatively agrees with tree-level lattice perturbation theory. This agreement can be made quantitative by allowing for a running coupling parameter. The same holds true for the spin-spin potential $V_3$.

The remaining spin-spin potential, $V_4$, exhibits oscillatory behaviour as a lattice artefact (Fig. 4) and can largely be understood as a $\delta$-contribution, according to Eq. (12). The tree-level lattice perturbative expectation $8\pi c\delta L^3(r/a)$ is indicated by squares in the figure. The normalization has been obtained from a $c/r^2$ fit to the $V_2'$ data points. The error bands without symbols are obtained by using lattice single gluon exchange with an infra-red protected two-loop running coupling in momentum space. The range corresponds to different reasonable choices of the QCD $\Lambda$-parameter. Apart from the dominant $\delta$-like contribution, another very short ranged contribution seems to exist.

Very recently, we have computed the vd potentials for the first time. We find reasonable signals for long range forces, as illustrated in Fig. 5.

6. Summary and Conclusions

We have studied central and spin-dependent forces in SU(2) gauge theory in a high statistics lattice simulation. We find reliable renormalized potentials with good scaling behaviour. There is clear evidence for a short range scalar exchange contribution in the first spin-orbit potential at the level of 20–25 % of the Coulomb part of the central potential. The other sd potentials are found to be short ranged and are well approximated by perturbation theory.

We are encouraged from the results of a feasibility study of velocity-dependent potentials. An extension of the present investigations to the case of interest, SU(3) gauge theory, is in progress.

7. Acknowledgements

We thank DFG for supporting the Wuppertal CM-2 and CM-5 projects (grants Schi 257/1-4 and Schi 257/3-2). We are grateful for computing time on the CM-5 at GMD in Birlinghoven and appreciate support by EU (grant CHRX-CT92-0051).

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