STABILIZATION OF WEAKLY COUPLED VISCOELASTIC KIRCHHOFF PLATE AND WAVE EQUATIONS

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Abstract

In this paper, we consider a weakly coupled system consisting of a viscoelastic Kirchhoff plate equation involving free boundary conditions and the viscoelastic wave equation with Dirichlet boundary conditions in a bounded domain. By assuming a more general type of relaxation functions, we establish explicit and general decay rate results, using the multiplier method and some properties of the convex functions.
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ABSTRACT. In this paper, we consider a weakly coupled system consisting of a viscoelastic Kirchhoff plate equation involving free boundary conditions and the viscoelastic wave equation with Dirichlet boundary conditions in a bounded domain. By assuming a more general type of relaxation functions, we establish explicit and general decay rate results, using the multiplier method and some properties of the convex functions.

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Keywords: Kirchhoff plate equation, wave equation, weakly coupled equations, relaxation function.

1. INTRODUCTION

We consider the following weakly coupled system of Kirchhoff plate and wave equations

\[
\begin{aligned}
&u_{tt} - \gamma \Delta u_{tt} + \Delta^2 u - \int_0^t g_1(t-s) \Delta^2 u(s) \, ds + \alpha v = 0 \quad \text{in} \quad \Omega \times (0, \infty) \\
v_{tt} - \Delta v + \int_0^t g_2(t-s) \Delta v(s) \, ds + \alpha u = 0 \quad \text{in} \quad \Omega \times (0, \infty) \\
u = \partial_n u = 0 \quad \text{on} \quad \Gamma_0 \times (0, \infty) \\
B_1 u - B_1 \left\{ \int_0^t g_1(t-s) u(s) \, ds \right\} = 0 \quad \text{on} \quad \Gamma_1 \times (0, \infty) \\
B_2 u - \gamma \partial_n u_{tt} - B_2 \left\{ \int_0^t g_1(t-s) u(s) \, ds \right\} = 0 \quad \text{on} \quad \Gamma_1 \times (0, \infty) \\
v = 0 \quad \text{on} \quad \Gamma \times (0, \infty) \\
u(0) = u^0, \quad v_t(0) = u^1, \quad v(0) = \nu^0, \quad v_t(0) = \nu^1 \quad \text{in} \quad \Omega,
\end{aligned}
\]

where \( \Omega \) is an open set of \( \mathbb{R}^2 \) with regular boundary \( \Gamma = \partial \Omega = \Gamma_0 \cup \Gamma_1 \) (class \( C^4 \) will be enough) such that \( \Gamma_0 \cap \Gamma_1 = \emptyset \), the initial data \( u^0, u^1, v^0 \) and \( v^1 \) lie in appropriate Hilbert space, the constant \( \gamma > 0 \) is the rotational inertia of the plate and the constant \( 0 < \mu < \frac{1}{2} \) is the Poisson coefficient. The boundary operators \( B_1, B_2 \) are defined by

\[
\begin{aligned}
B_1 &= \Delta u + (1 - \mu) B_1 u, \\
B_2 &= \partial_n \Delta u + (1 - \mu) B_2 u,
\end{aligned}
\]

and

\[
\begin{aligned}
B_1 u &= 2 \nu_1 \nu_2 u_{x_1 x_2} - \nu_1^2 u_{x_2 x_2} - \nu_2^2 u_{x_1 x_1}, \\
B_2 u &= \partial_{\tau} \left( (\nu_2^2 - \nu_1^2) u_{x_1 x_2} + \nu_1 \nu_2 (u_{x_2 x_2} - u_{x_1 x_1}) \right),
\end{aligned}
\]

where \( \nu = (\nu_1, \nu_2) \) is the unit outer normal vector to \( \Gamma \) and \( \tau = (-\nu_2, \nu_1) \) is a unit tangent vector.

The coupling parameter \( \alpha \) is assumed to satisfy:

\[
|\alpha| < \lambda_0 \eta, \tag{1.2}
\]
where $\lambda_0^2$ is the first eigenvalue of the operator $-\Delta$ with Dirichlet boundary conditions, and $\eta^2$ is the coercivity constant of the operator $A = \Delta^2$ defined as follows:

$$
\hat{A} : D(\hat{A}) \subset L^2(\Omega) \to L^2(\Omega),
$$

with domain

$$
D(\hat{A}) = \{ u \in H^4(\Omega) \cap H^2_0(\Omega) : \Delta u + (1 - \mu)B_1 u = \partial_\nu \Delta u + (1 - \mu)B_2 u = 0 \text{ on } \Gamma_1 \}.
$$

It is clear that $\hat{A}$ is then positive definite and self-adjoint. We define

$$
V = \{ u \in H^2(\Omega) : u = \partial_\nu u = 0 \text{ on } \Gamma_0 \},
$$

and

$$
H^1_{\Gamma_0}(\Omega) = \{ u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_0 \}.
$$

We have for all $u, v \in V$:

$$
\langle \hat{A}^{\frac{1}{2}} u, \hat{A}^{\frac{1}{2}} v \rangle_{L^2(\Omega)} = \langle \hat{A} u, v \rangle_{V' \times V} = a(u, v),
$$

where $a : V \times V \to \mathbb{C}$ is a bilinear form defined by

$$
a(u, v) = \int_\Omega \{ u_{x_1 x_1} v_{x_1 x_1} + u_{x_2 x_2} v_{x_2 x_2} + 2(1 - \mu) u_{x_1 x_2} v_{x_1 x_2} + \mu (u_{x_1 x_1} v_{x_2 x_2} + u_{x_2 x_2} v_{x_1 x_1}) \} \, dx.
$$

We first recall the following Green’s formula (see [9]):

$$
a(u, v) = \int_\Omega \Delta^2 uv \, dx + \int_\Gamma (B_1 u \partial_\nu v - B_2 uv) \, d\Gamma, \quad \forall u \in H^4(\Omega), \quad v \in H^2(\Omega).
$$

For further purposes, we need a weaker version of it. Indeed as $\mathcal{D}(\Omega)$ is dense in $E(\Delta^2, L^2(\Omega)) := \{ u \in H^2(\Omega) \mid \Delta^2 u \in L^2(\Omega) \}$ equipped with its natural norm, we deduce that $u \in E(\Delta^2, L^2(\Omega))$ (see Theorem 5.6 in [19]) satisfies $B_1 u \in H^{-\frac{1}{2}}(\Gamma)$ and $B_2 u \in H^{-\frac{3}{2}}(\Gamma)$ with

$$
a(u, v) = \int_\Omega \Delta^2 uv \, dx + \langle B_1 u, \partial_\nu v \rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)} - \langle B_2 u, v \rangle_{H^{-\frac{3}{2}}(\Gamma), H^{\frac{5}{2}}(\Gamma)}, \quad \forall v \in H^2(\Omega).
$$

Now, with the parameter $\gamma > 0$, we define a space $W = H^1_{\Gamma_0, \gamma}(\Omega)$ equivalent to $H^1_{\Gamma_0}(\Omega)$ with its inner product being

$$
\langle u_1, u_2 \rangle_{H^1_{\Gamma_0, \gamma}(\Omega)} = \langle u_1, u_2 \rangle_{L^2(\Omega)} + \gamma \langle \nabla u_1, \nabla u_2 \rangle_{L^2(\Omega)} \quad \forall u_1, u_2 \in H^1_{\Gamma_0}(\Omega),
$$

and with its dual (pivotal with respect to $L^2$ inner product) denoted as $H^{-1}_{\Gamma_0, \gamma}(\Omega)$.

We first recall some results for a single wave equation and Kirchhoff plate equation. For viscoelastic wave equation, we refer to [5][10][11][13][18] and references therein, in which the authors proved that the energy decays exponentially if the relaxation function $g$ decays exponentially and polynomially if $g$ decays polynomially. In [6], Cavalcanti et al. considered the following wave equation

$$
u_{tt} - \kappa_0 \Delta u + \int_0^t \text{div}[a(x)g(t - s)\nabla u(s)] \, ds + f(u) + b(x)h(u_t) = 0,
$$

where frictional damping was also considered. They proved an exponential stability result for $g$ decaying exponentially and $h$ having linear and polynomial stability result for $g$ decaying polynomially and $h$ having a polynomial growth near zero. We mention, in the case where $\kappa_0 = 1$ and $f = h = 0$, that the uniform decay of solutions was obtained in [15]. For viscoelastic Kirchhoff plate equation, in [4], the authors showed
exponential and polynomial decay of the solutions to viscoelastic plate equation. They considered a relaxation function satisfying
\[-d_0 g(t) \leq g'(t) \leq -d_1 g(t), \quad 0 \leq g''(t) \leq d_2 g(t),\]
for some positive constant \(d_i, \quad i = 0, 1, 2\). Park et al. [20] obtained a general decay for weak viscoelastic Kirchhoff plate equations with delay boundary conditions.

For coupled wave system, a general model on coupled wave equations with weak dampings is given by:
\[
\begin{align*}
\dot{u}_t - \Delta u + \int_0^t g_1(t - s) \Delta u(s) ds + h_1(u_t) &= f_1(u, v), \\
\dot{v}_t - \Delta v + \int_0^t g_2(t - s) \Delta v(s) ds + h_2(v_t) &= f_2(u, v).
\end{align*}
\]
(1.5)

In [8], Han and Wang established several results related to local existence, global existence and finite time blow-up (the initial energy \(E(0) < 0\), by taking \(h_1(u_t) = |u_t|^{m-1} u_t \) and \(h_2(v_t) = |v_t|^{r-1} v_t\). Latter on, Houari et al. [21] improved the last results by considering a larger class of initial data for which the initial energy can take positive values. Messaoudi and Tatar [14] considered a coupled system only with viscoelastic terms, and proved exponential decay and polynomial decay results. Al-Gharabli and Kafini considered the system in [14] and established a more general decay result, see [1]. Mustafa [16] considered the following problem
\[
\begin{align*}
\dot{u}_t - \Delta u + \int_0^t g_1(t - s) \Delta u(s) ds + f_1(u, v) &= 0, \\
\dot{v}_t - \Delta v + \int_0^t g_2(t - s) \Delta v(s) ds + f_2(u, v) &= 0,
\end{align*}
\]
and proved the well-posedness and energy decay result. The decay result was improved by Messaoudi and Hassan in their recent paper [12], where they established a new general decay result for wider class of relaxation functions.

Finally, let us mention the work of Hajej et al. [7] in which the authors studied the indirect stabilization (only one equation of the coupled system is damped) of a coupled wave equation and Kirchhoff plate equation without viscoelastic terms (\(g_1 = g_2 = 0\)) and with frictional damping. Motivated by this work, we intend to study the stability of this coupled system but only with the presence of viscoelastic terms in the two equations with a wider class of relaxation functions.

The paper is organized as follows. In Section 2, we introduce some assumptions needed in this paper to establish our main results. The well-posedness of the problem is proved in the third section. In Section 4, we state and establish the general decay result of the energy.

2. Preliminaries

In this section, we present some fundamental materials needed for the proof of our main results. We assume that

(A1): \(g_i : \mathbb{R}_+ \to (0, +\infty)\) (for \(i = 1, 2\)) are two non-increasing \(C^1\) functions such that
\[
1 - \int_0^\infty g_i(\tau) d\tau = l_i > 0.
\]
(2.1)

(A2): There exists a positive \(C^1\) function \(H : (0, +\infty) \to (0, +\infty)\), and \(H\) is linear or strictly increasing and strictly convex \(C^2\) function on \((0, r), (r < 1)\), with \(H(0) = H'(0) = 0\), such that
\[
g_i'(t) \leq -\xi_i(t) H(g_i(t)), \quad \forall \ t \geq 0,
\]
(2.2)
where $\xi_1$ and $\xi_2$ are positive non-increasing differentiable functions.

To simplify calculations in our analysis, we introduce the following notations:

$$
(g_1 \Box u)(t) = \int_0^t g_1(t-s)a(u(t)-u(s), u(t)-u(s))\,ds,
$$

$$
(g_2 \circ v)(t) = \int_0^t g_2(t-s)\|v(t)-v(s)\|^2\,ds.
$$

We introduce the energy functional by

$$
E(t) = \frac{1}{2} \left( 1 - \int_0^t g_1(s)\,ds \right) a(u, u) + \frac{1}{2}\|u_t\|^2 + \frac{\gamma}{2}\|\nabla u_t\|^2 + \frac{1}{2}(g_1 \Box u)(t) + \frac{1}{2} \left( 1 - \int_0^t g_2(s)\,ds \right) \|\nabla v\|^2
$$

or

$$
E(t) = \frac{1}{2} (g_1 \Box u)(t) - \frac{1}{2} g_1(t)a(u, u) + \frac{1}{2}(g_2 \circ \nabla v)(t) - \frac{1}{2} g_2(t)\|\nabla v(t)\|^2 \leq 0.
$$

**Proof.** In (1.1), multiplying the first equation by $u_t$ and the second one by $v_t$, add the resulting equations and integrate by parts over $\Omega$ to obtain

$$
\frac{d}{dt} \left\{ \frac{1}{2}\|u_t\|^2 + \frac{\gamma}{2}\|\nabla u_t\|^2 + \frac{1}{2}a(u, u) + \frac{1}{2}\|v_t\|^2 + \frac{1}{2}\|\nabla v\|^2 + \alpha \int_\Omega uv\,dxdy \right\}
$$

$$
- \int_0^t g_1(t-s)a(u(s), u_s)\,ds - \int_0^t g_2(t-s)\int_\Omega \nabla v(s)\nabla v_t(t)\,dxdy = 0
$$

By the virtue of Lemma 2.1 in [4], we have

$$
a \left( \int_0^t g_1(t-s)u(s)\,ds, u_t \right) = -\frac{1}{2}g_1(t)a(u, u) - \frac{d}{dt} \left\{ (g_1 \Box u)(t) - \left( \int_0^t g_1(s)\,ds \right) a(u, u) \right\}
$$

$$
+ \frac{1}{2}(g_1 \Box u)(t),
$$

for any $u \in C^1(0, T; H^2(\Omega))$.

Besides, a direct computation shows that

$$
\int_0^t g_2(t-s)\int_\Omega \nabla v(s)\nabla v_t(t)\,dxdy = \frac{1}{2}(g_2 \circ \nabla v)(t) - \frac{1}{2} g_2(t)\|\nabla v(t)\|^2 - \frac{1}{2} \left\{ (g_2 \circ \nabla v)(t) - \left( \int_0^t g_2(s)\,ds \right) \|\nabla v(t)\|^2 \right\}.
$$

Replacing (2.6) and (2.7) in (2.5), we get the desired result.

We will use $C$ and $c$, throughout this paper, to denote a generic positive constants.
3. Global existence

We start this section by giving the definition of a weak solution of the problem (1.1).

**Definition 3.1.** Let $T > 0$. A pair of functions $(u, v)$ such that

$$u \in C([0, T], V) \cap C^1([0, T], W), \ v \in C([0, T], H_0^1(\Omega)) \cap C^1([0, T], L^2(\Omega)),$$

is called a weak solution of the problem (1.1) if

$$\int_{\Omega} u_{tt} w dx + \gamma \int_{\Omega} \nabla u_{tt} \nabla w dx + a(u, w) - \int_{0}^{t} g_1(t - s)a(u(s), w) ds + \alpha \int_{\Omega} v w dx = 0,$$

$$u(x, 0) = u_0(x), \ u_t(x, 0) = u_1(x)$$

and

$$\int_{\Omega} v_{tt} y dx + \int_{\Omega} \nabla v \nabla y dx - \int_{0}^{t} g_2(t - s) \int_{\Omega} \nabla v(s) \nabla y dx ds + \alpha \int_{\Omega} u y dx = 0,$$

$$v(x, 0) = v_0(x), \ v_t(x, 0) = v_1(x)$$

for all test functions $w \in V, y \in H_0^1$ and almost all $t \in [0, T]$.

Now, we state the local existence theorem.

**Theorem 3.2.** Suppose (A1)-(A2) hold and let $(u_0, u_1) \in V \times W$ and $(v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$. Then, problem (1.1) has a unique local weak solution on $[0, T]$, for any $T > 0$.

**Proof.** The existence is proved using the Faedo-Galerkin method. In order to do so, let $\{w_j\}_{j=1}^{\infty}$ and $\{y_j\}_{j=1}^{\infty}$ be a basis of $V$ and $H_0^1$ respectively. Define $V_m = \text{span}\{w_1, w_2, \ldots, w_m\}$ and $Y_m = \text{span}\{y_1, y_2, \ldots, y_m\}$. The projection of the initial data on the finite dimensional subspaces $V_m$ and $Y_m$ is given by

$$u_0^m(x) = \sum_{j=1}^{m} a_j w_j, \ u_1^m(x) = \sum_{j=1}^{m} b_j w_j, \ v_0^m(x) = \sum_{j=1}^{m} c_j y_j, \ v_1^m(x) = \sum_{j=1}^{m} d_j y_j,$$

such that

$$u_0^m, v_0^m \to (u_0, v_0) \quad \text{in} \quad V \times H_0^1(\Omega), \quad \text{and} \quad (u_1^m, v_1^m) \to (u_1, v_1) \quad \text{in} \quad W \times L^2(\Omega).$$

We search a solution of the form

$$u^m(x, t) = \sum_{j=1}^{m} h_j(t) w_j(x), \quad v^m(x, t) = \sum_{j=1}^{m} k_j(t) y_j(x),$$

which satisfy the approximate problem in $V_m$ and $Y_m$ respectively

$$\int_{\Omega} u_{tt}^m w dx + \gamma \int_{\Omega} \nabla u_{tt}^m \nabla w dx + a(u^m, w) - \int_{0}^{t} g_1(t - s)a(u^m(s), w) ds + \alpha \int_{\Omega} v^m w dx = 0,$$

$$u^m(0) = u_0^m, \ u_t^m(0) = u_1^m, \ v^m(0) = v_0^m, \ v_t^m(0) = v_1^m.$$

This system leads to a system of ODEs for unknown functions $h_j(t)$ and $k_j(t)$. Based on standard existence theory for ODE, one can conclude the existence of a solution $(u^m, v^m)$ of (3.2) on a maximal interval $[0, t_m), 0 < t_m \leq T$ for each $m \geq 1$. In fact, $t_m = T$ and the local solution is uniformly bounded independent
of \( m \) and \( t \). To show this, we take \( w = u^m \) in the first equation of (3.2) and \( y = v^m \) in the second one. Summing the resulting equations and integrating by parts over \( \Omega \) to get

\[
\frac{d}{dt} E^m(t) = \frac{1}{2} (g_1 \Box u^m) (t) - \frac{1}{2} g_1(t) a(u^m, u^m) + \frac{1}{2} (g_2 \triangledown v^m)(t) - \frac{1}{2} g_2(t) \| \triangledown v^m(t) \|^2,
\]

where

\[
E^m(t) = \frac{1}{2} \left( 1 - \int_0^t g_1(s) ds \right) a(u^m, u^m) + \frac{1}{2} \| u^m \|^2 + \frac{\gamma}{2} \| \triangledown u^m \|^2 + \frac{1}{2} (g_1 \Box u^m)(t) + \frac{1}{2} \| v^m \|^2 + \alpha \int_\Omega u^m v^m dx.
\]

Noting, by (3.1), that \((u^0_0, v^0_0)\) and \((u^1_0, v^1_0)\) are bounded, respectively, in \( V \times H_0^1(\Omega) \) and \( W \times L^2(\Omega) \), integrate (3.3) over \((0, t), 0 < t < t_m\), to obtain

\[
E^m(t) \leq E^m(0) \leq C,
\]

where \( C \) is a positive constant independent of \( t \) and \( m \). Thus, we can extend \( t_m \) to \( T \) and, in addition, we have

\[
\begin{align*}
(u^m) & \text{ is a bounded sequence in } L^\infty(0, T; V) \\
(u^m_t) & \text{ is a bounded sequence in } L^\infty(0, T; W) \\
(v^m) & \text{ is a bounded sequence in } L^\infty(0, T; H_0^1(\Omega)) \\
(v^m_t) & \text{ is a bounded sequence in } L^\infty(0, T; L^2(\Omega)).
\end{align*}
\]

Therefore, there exists a subsequence of \((u^m)\) and \((v^m)\), still denoted by \((u^m)\) and \((v^m)\) respectively, such that

\[
\begin{align*}
u^m & \to u \text{ weakly star in } L^\infty(0, T; V) \text{ and weakly in } L^2(0, T; V) \\
u^m_t & \to u_t \text{ weakly star in } L^\infty(0, T; W) \text{ and weakly in } L^2(0, T; W) \\
v^m & \to v \text{ weakly star in } L^\infty(0, T; H_0^1(\Omega)) \text{ and weakly in } L^2(0, T; H_0^1(\Omega)) \\
v^m_t & \to v_t \text{ weakly star in } L^\infty(0, T; L^2(\Omega)) \text{ and weakly in } L^2(0, T; L^2(\Omega)).
\end{align*}
\]

Now, integrate (3.2) over \((0, t)\) to obtain

\[
\begin{align*}
\int_\Omega u^m_t w dx + \gamma \int_\Omega \triangledown u^m \triangledown w dx + \int_0^t a(u^m, w) ds - \int_0^t \int_0^s g_1(s - \tau) a(u^m(\tau), w) d\tau ds \\
+ \alpha \int_0^t \int_\Omega v^m w dx ds = \int_\Omega u^m_0 w dx + \gamma \int_\Omega \triangledown u^m_0 \triangledown w dx \\
+ \int_\Omega v^m_0 y dx + \int_0^t \int_\Omega \triangledown v^m \triangledown y dx ds - \int_0^t \int_0^s g_2(s - \tau) \int_\Omega \triangledown v^m(\tau) \triangledown y dx d\tau ds \\
+ \alpha \int_0^t \int_\Omega u^m y dx ds = \int_\Omega v^m_0 y dx.
\end{align*}
\]
Using (3.5) and letting $m \to \infty$, we get for all $w \in V$ and $y \in H_0^1$ 

$$
\int_{\Omega} u_t w d\mathbf{x} + \gamma \int_{\Omega} \nabla u_t \nabla w d\mathbf{x} - \int_{\Omega} u_1 w d\mathbf{x} - \gamma \int_{\Omega} \nabla u_1 \nabla w d\mathbf{x}
$$

(3.7)

$$
= - \int_0^t a(u, w) ds + \int_0^t \int_0^s g_1(s - \tau) a(u^m(\tau), w) d\tau ds - \alpha \int_0^t \int_{\Omega} w d\mathbf{x} ds
$$

$$
\int_{\Omega} v_t y d\mathbf{x} - \int_{\Omega} v_1 y d\mathbf{x} = - \int_0^t \int_{\Omega} \nabla v \nabla y d\mathbf{x} ds
$$

$$
+ \int_0^t \int_0^s g_2(s - \tau) \int_{\Omega} \nabla v(\tau) \nabla y d\mathbf{x} d\tau ds - \alpha \int_0^t \int_{\Omega} u y d\mathbf{x} ds
$$

Using the fact that the right hand side of the first equation and the second one in (3.7) is an absolutely continuous function, hence it is differentiable almost everywhere, we obtain

$$
\int_{\Omega} u_t w d\mathbf{x} + \gamma \int_{\Omega} \nabla u_t \nabla w d\mathbf{x} + a(u, w) - \int_0^t g_1(t - s) a(u(s), w) ds + \alpha \int_{\Omega} w d\mathbf{x} = 0, \ \forall \ w \in V
$$

$$
\int_{\Omega} v_t y d\mathbf{x} + \int_{\Omega} \nabla v \nabla y d\mathbf{x} - \int_0^t g_2(t - s) \int_{\Omega} \nabla v(s) \nabla y d\mathbf{x} ds + \alpha \int_{\Omega} u y d\mathbf{x} = 0, \ \forall \ y \in H_0^1(\Omega)
$$

Regarding the initial conditions, we can also use (3.7) to verify that

$$
u(0) = u_0, \ u_t(0) = u_1, \ v(0) = v_0, \ v_t(0) = v_0.$$

For uniqueness, let us assume that $(u_1, v_1), (u_2, v_2)$ are two weak solutions of (1.1). Then, $(p, q) = (u_1 - u_2, v_1 - v_2)$ satisfies,

$$
\begin{align*}
\frac{\partial p}{\partial t} - \gamma \Delta p_t + \Delta^2 p - \int_0^t g_1(t - s) \Delta^2 p(s) ds + \alpha q &= 0, \ \text{in} \ L^2(0, T; V') \\
\frac{\partial q}{\partial t} - \Delta q + \int_0^t g_2(t - s) \Delta q(s) ds + \alpha p &= 0, \ \text{in} \ L^2(0, T; H^{-1}(\Omega))
\end{align*}
$$

(3.8)

$$
p(0) = p_t(0) = q(0) = q_t(0) = 0.
$$

We shall use the Visik-Ladyzenskaya method. We consider, for each $s \in [0, T]$ the following functions

$$
\psi(t) = \begin{cases} 
- \int_t^s p(\xi) d\xi, & 0 \leq t \leq s \\
0, & s \leq t \leq T
\end{cases}
$$

and

$$
\varphi(t) = \begin{cases} 
- \int_t^s q(\xi) d\xi, & 0 \leq t \leq s \\
0, & s \leq t \leq T
\end{cases}
$$

The derivatives (in the distributions sens) of $\psi$ and $\varphi$ are given by

$$
\psi'(t) = \begin{cases} 
p(t), & 0 \leq t \leq s \\
0, & s \leq t \leq T
\end{cases}
$$

and

$$
\varphi'(t) = \begin{cases} 
q(t), & 0 \leq t \leq s \\
0, & s \leq t \leq T
\end{cases}
$$

It is clear that

$$
\psi, \psi' \in L^\infty(0, T; V) \ \text{and} \ \varphi, \varphi' \in L^\infty(0, T; H_0^1(\Omega)),
$$

which implies that

$$
\psi \in C^0([0, T]; V) \ \text{and} \ \varphi \in C^0([0, T]; H_0^1(\Omega)).
$$
By composing the first equation in (3.8) by $\psi$ and the second equation by $\varphi$, we get

$$
\int_0^s ((I - \gamma \Delta) p_{tt}, \psi(t))_{V^\prime \times V} dt + \int_0^s (\Delta^2 p, \psi(t))_{V^\prime \times V} dt - \int_0^s \int_0^t g_1(t - \tau)(\Delta^2 p(\tau), \psi(t))_{V^\prime \times V} d\tau dt + \alpha \int_0^s (q, \psi(t))_{V^\prime \times V} dt = 0
$$

Using the fact that $\psi(s) = \varphi(s) = p(t) = q(t) = 0$, $\psi'(t) = p(t)$ and $\varphi'(t) = q(t)$ in $[0, s]$, integrating by parts and adding the resulting equations to obtain

$$
- \int_0^s \int_0^t g_1(t - \tau)(\Delta^2 p(\tau), \psi(t))_{V^\prime \times V} d\tau dt + \alpha \int_0^s (p(t), \psi(t))_{H^1_0(\Omega)} dt = 0
$$

which gives by using (2.6) and (2.7)

$$
\frac{1}{2} \frac{d}{dt} \left\{ - \int_0^s \|p\|^2_W dt + \int_0^s \left( 1 - \int_0^t g_2(t) d\tau \right) a(\psi, \psi) dt + \int_0^s (g_1 \Box \psi(t)) dt - \int_0^s \|q\|^2_{H^1_0(\Omega)} dt \right\}
$$

$$
+ \int_0^s \left( 1 - \int_0^t g_2(t) d\tau \right) \|\varphi(t)\|^2_{H^1_0(\Omega)} dt
$$

$$
= - \frac{1}{2} \alpha \int_0^s ((q(t), \psi(t))_{L^2(\Omega)} + (p(t), \varphi(t))_{L^2(\Omega)}) dt.
$$

Now using the fact that $g_i, -g_i', a(\psi, \psi) \geq 0$ for $i = 1, 2$, and $W, H^1_0 \subset L^2(\Omega)$, we obtain the existence of a positive constant $C$ such that

$$
\frac{1}{2} \|p(s)\|^2 + \frac{1}{2} \|\psi(0)\|^2 + \frac{1}{2} \|q(s)\|^2 + \frac{1}{2} \|\varphi(0)\|^2
$$

(3.10)

$$
\leq C \int_0^s ((\|q(t)\| \|\psi(t)\| + \|p(t)\| \|\varphi(t)\|)) dt
$$

Finally let $p_1(t) = \int_0^t p(\xi) d\xi$ and $q_1(t) = \int_0^t q(\xi) d\xi$. We have, for all $t \in [0, s]$

$$
\psi(t) = p_1(t) - p_1(s), \quad \psi(0) = -p_1(s),
$$

and

$$
\varphi(t) = q_1(t) - q_1(s), \quad \varphi(0) = -q_1(s).
$$
Consequently, (3.10) becomes
\[
\frac{1}{2} \|p(s)\|^2 + \frac{1}{2} \|p_1(s)\|^2 + \frac{1}{2} \|q(s)\|^2 + \frac{1}{2} \|q_1(s)\|^2
\leq C \int_0^s \left( \|q(t)\| \|p_1(t) - p_1(s)\| + \|p(t)\| \|q_1(t) - q_1(s)\| \right) dt
\]
\[
\leq C \left\{ \int_0^s \|q(t)\| \|p_1(t)\| dt + \int_0^s \|q(t)\| \|p_1(s)\| dt
+ \int_0^s \|p(t)\| \|q_1(t)\| dt + \int_0^s \|p(t)\| \|q_1(s)\| dt \right\}
\]
\[
\leq C \left\{ \int_0^s \|q(t)\| \|p_1(t)\| dt + \int_0^s \sqrt{2C_s} \|q(t)\| \frac{1}{\sqrt{2C_s}} \|p_1(s)\| dt
+ \int_0^s \|p(t)\| \|q_1(t)\| dt + \int_0^s \sqrt{2C_s} \|p(t)\| \frac{1}{\sqrt{2C_s}} \|q_1(s)\| dt \right\}
\]
\[
\leq \frac{C}{2} \int_0^s \|p(t)\|^2 dt + \frac{C}{2} \int_0^s \|p_1(t)\|^2 dt + TC^2 \int_0^s \|p(t)\|^2 dt + \frac{1}{4} \|p_1(s)\|^2
+ \frac{C}{2} \int_0^s \|q(t)\|^2 dt + \frac{C}{2} \int_0^s \|q_1(t)\|^2 dt + TC^2 \int_0^s \|q(t)\|^2 dt + \frac{1}{4} \|q_1(s)\|^2,
\]
which implies that
\[
\frac{1}{4} \|p(s)\|^2 + \frac{1}{4} \|p_1(s)\|^2 + \frac{1}{4} \|q(s)\|^2 + \frac{1}{4} \|q_1(s)\|^2
\leq C \int_0^s \left( \|p(t)\|^2 + \|p_1(t)\|^2 + \|q(t)\|^2 + \|q_1(t)\|^2 \right) dt
\]

By using Gronwall’s Lemma, we deduce that
\[
\frac{1}{4} \|p(s)\|^2 + \frac{1}{4} \|p_1(s)\|^2 + \frac{1}{4} \|q(s)\|^2 + \frac{1}{4} \|q_1(s)\|^2 \leq 0.
\]

Then, we get that
\[
p(s) = q(s) = 0, \quad \text{in} \quad L^2(\Omega), \quad \forall \ s \in (0, T),
\]
and since \(p(0) = q(0) = 0\) we obtain that
\[
p(s) = q(s) = 0, \quad \text{in} \quad L^2(\Omega), \quad \forall \ s \in [0, T],
\]
which means that \((u_1, v_1) = (u_2, v_2)\). This completes the proof.

\[\square\]

4. General decay

The main result of this paper reads as follows.

**Theorem 4.1.** Let \((u_0, u_1) \in V \times W\) and \((v_0, v_1) \in H^1_0(\Omega) \times L^2(\Omega)\). Assume that (A1) and (A2) hold. Then there exist positive constants \(\beta_1\) and \(\beta_2\) such that the energy \(E(t)\) satisfies for any \(t > g^{-1}(r)\),

\[
E(t) \leq \beta_2 H^{-1}_1 \left( \beta_1 \int_{g^{-1}(r)}^t \xi(s) ds \right),
\]

where \(\xi(t) = \min\{\xi_1(t), \xi_2(t)\}\), \(g(t) = \max\{g_1(t), g_2(t)\}\) and \(H_1(t) = \int_t^r \frac{1}{sH(s)} ds\).
Remark 4.2. ([17])

1. The following Jensen’s inequality is critical to prove our main result. Let \( G \) be a convex increasing function on \([a, b]\), \( h : \Omega \to [a, b] \) and \( m \) are integrable functions on \( \Omega \) such that \( m(x) \geq 0 \) and 
\[
\int_{\Omega} m(x) \, dx = k > 0,
\]
then Jensen’s inequality states that 
\[
G\left[ \frac{1}{k} \int_{\Omega} h(x)m(x) \, dx \right] \leq \frac{1}{k} \int_{\Omega} G[h(x)]m(x) \, dx.
\]

2. From (A2), we infer that \( \lim_{t \to +\infty} g_i(t) = 0 \). Then there exists some \( t_1 \geq 0 \) large enough such that 
\[
g_i(t_1) = r \Rightarrow g_i(t) \leq r, \quad \forall \ t \geq t_1.
\]

Since \( H \) is positive continuous function and \( g_i, \xi_i \) are positive non-increasing continuous functions, we can get for every \( t \in [0, t_1] \), 
\[
0 < g_i(t_1) \leq g_i(t) \leq g_i(0) \quad \text{and} \quad 0 < \xi_i(t_1) \leq \xi_i(t) \leq \xi_i(0), \quad i = 1, 2,
\]
which implies for some positive constants \( a_i \) and \( b_i \),
\[
a_i \leq \xi_i(t)H(g_i(t)) \leq b_i, \quad i = 1, 2.
\]

This shows that for every \( t \in [0, t_1] \),
\[
g_i'(t) \leq -\xi_i(t)H(g_i(t)) \leq -\frac{a_i}{g_i(0)}g_i(0) \leq -\frac{a_i}{g_i(0)}g_i(t), \quad i = 1, 2.
\]

3. If different functions \( H_1 \) and \( H_2 \) have the properties mentioned in (A2) such that \( g_1'(t) \leq -H_1(g_1(t)) \)
and \( g_2'(t) \leq -H_2(g_2(t)) \), then there exists \( r < \min\{r_1, r_2\} \) small enough so that, say, \( H_1(t) \leq H_2(t) \)
on the interval \((0, r)\). Thus the function \( H(t) = H_1(t) \) satisfies (A2) for both functions \( g_1 \) and \( g_2 \) for all \( t \geq t_1 \).

In order to prove the main Theorem (4.1), we need to introduce a several Lemmas. To this end, let us introduce the functionals
\[
I(t) = \int_{\Omega} (uu_t + \gamma \nabla u \nabla u) \, dx + \int_{\Omega} vv_t \, dx,
\]
and
\[
K(t) = -\int_{\Omega} u_t \int_{0}^{t} g_1(t - \tau)(u(t) - u(\tau)) \, d\tau \, dx - \gamma \int_{\Omega} \nabla u_t \int_{0}^{t} g_1(t - \tau)\nabla (u(t) - u(\tau)) \, d\tau \, dx
\]
\[
- \int_{\Omega} v_t \int_{0}^{t} g_2(t - \tau)(v(t) - v(\tau)) \, d\tau \, dx.
\]

Lemma 4.3. Assume that that (A1) and (A2) hold. Then the functional \( I(t) \) introduced in (4.4) satisfies, along the solution, the estimate
\[
I'(t) \leq \int_{\Omega} |u_t|^2 \, dx + \gamma \int_{\Omega} |\nabla u_t|^2 \, dx - \frac{l_1}{2} a(u, u) + \frac{1 - l_1}{2l_1} (g_1 \Box u)(t)
\]
\[
+ \int_{\Omega} |v_t|^2 \, dx - \frac{l_2}{2} \int_{\Omega} |\nabla v|^2 \, dx + \frac{1 - l_2}{2l_2} (g_2 \circ \nabla v)(t) - 2\alpha \int_{\Omega} uv \, dx
\]
Proof. Direct differentiation of $I$, using (1.1), yields

$$ I'(t) = \int_{\Omega} |u_t|^2 \, dx + \int_{\Omega} u u_{tt} \, dx + \gamma \int_{\Omega} |\nabla u_t|^2 \, dx + \int_{\Omega} \nabla u \nabla u_{tt} \, dx + \int_{\Omega} |v_t|^2 \, dx + \int_{\Omega} v v_{tt} \, dx $$

$$ = \int_{\Omega} |u_t|^2 \, dx + \int_{\Omega} |\nabla u_t|^2 \, dx + a(u, u) + \int_{0}^{t} g_1(t-\tau)a(u(\tau), u(t)) \, d\tau $$

$$ + \int_{\Omega} |v_t|^2 \, dx - \int_{\Omega} |\nabla v|^2 \, dx + \int_{0}^{t} g_2(t-\tau) \int_{\Omega} \nabla v(\tau) \nabla v(t) \, d\tau - 2\alpha \int_{\Omega} uv \, dx \tag{4.7} $$

By using the Cauchy-Schwarz inequality, Young’s inequality and the fact that $\int_{0}^{t} g_1(\tau) \, d\tau \leq \int_{0}^{+\infty} g_1(\tau) \, d\tau = 1 - l_1$, we obtain

$$ \int_{0}^{t} g_1(t-\tau)a(u(t), u(\tau)) \, d\tau $$

$$ = \int_{0}^{t} g_1(t-\tau)a(u(\tau) - u(t), u(t)) \, d\tau + \int_{0}^{t} g_1(t-\tau)a(u(t), u(t)) \, d\tau $$

$$ \leq \int_{0}^{t} g_1(t-\tau) \{a(u(\tau) - u(t), u(\tau) - u(t))\}^{1/2} \{a(u(t), u(t))\}^{1/2} \, d\tau + \left( \int_{0}^{t} g_1(\tau) \, d\tau \right) a(u(t), u(t)) $$

$$ \leq \frac{l_1}{2} a(u(t), u(t)) + \frac{1}{2l_1} \left( \int_{0}^{t} \sqrt{g_1(t-\tau)} \left\{ g_1(t-\tau)a(u(\tau) - u(t), u(\tau) - u(t)) \right\}^{1/2} \, d\tau \right)^2 $$

$$ + (1 - l_1)a(u(t), u(t)) $$

$$ \leq \left( 1 - \frac{l_1}{2} \right) a(u(t), u(t)) + \frac{1 - l_1}{2l_1} (g_1 \Box u)(t) \tag{4.8} $$

Besides, we have (see for example [10]) that

$$ \int_{0}^{t} g_2(t-\tau) \int_{\Omega} \nabla v(\tau) \nabla v(t) \, dx \, d\tau $$

$$ \leq (1 - \frac{l_2}{2}) \int_{\Omega} |\nabla v|^2 \, dx + \frac{1 - l_2}{2l_2} (g_2 \circ \nabla v)(t). \tag{4.9} $$

Inserting (4.8) and (4.9) in (4.7), the assertion of the lemma is established. \qed
Lemma 4.4. Assume that (A1) and (A2) hold. Then, the functional introduced in (4.3) satisfies, along the solution, the estimate

$$K'(t) \leq \left(\delta - \int_0^t g_1(\tau)d\tau\right) \left(\int_\Omega |u_t|^2 dx\right)$$

$$+ \left\{\delta(C + 1) + \delta(1 - l_1)(\delta + 2(1 - l_1)) - \int_0^t g_1(\tau)d\tau\right\} a(u, u)$$

$$+ \gamma \left(\delta - \int_0^t g_1(\tau)d\tau\right) \left(\int_\Omega |\nabla u|^2 dx\right)$$

$$+ \left\{(1 - l_1)(C + 2) + \delta(1 - l_1) + (1 - l_1)^2\right\} (g_1 \square u)(t)$$

$$- \frac{C g_1(0)}{2\delta} (g_1 \square u)(t) + \left(\delta - \int_0^t g_2(\tau)d\tau\right) \left(\int_\Omega |v_t|^2 dx\right)$$

$$+ \delta \left(C + 1 + 2(1 - l_2)^2\right) \left(\int_\Omega |\nabla v|^2 dx\right)$$

$$+ \left\{(1 - l_2)(C + 1) + (2\delta + \frac{1}{4\delta})(1 - l_2)\right\} (g_2 \circ \nabla v)(t)$$

$$- \frac{C g_2(0)}{4\delta} (g_2 \circ \nabla v)(t), \quad \forall \delta > 0.$$
Proof. By exploiting equation (1.1) and integrating by parts, we have

\[
K'(t) = \left( -\int_0^t g_1(\tau)d\tau \right) \left( \int_{\Omega} |u_t|^2 dx \right) - \int_{\Omega} u_t \int_0^t g_1'(t-\tau)(u(t) - u(\tau))d\tau dx \\
- \int_{\Omega} g_1(t-\tau)(u(t) - u(\tau))d\tau dx \\
- \gamma \int_{\Omega} \nabla u_t \int_0^t g_1(t-\tau)\nabla(u(t) - u(\tau))d\tau dx \\
- \gamma \int_{\Omega} \nabla u_t \int_0^t g_1'(t-\tau)\nabla(u(t) - u(\tau))d\tau dx - \gamma \left( \int_0^t g_1(\tau)d\tau \right) \left( \int_{\Omega} |\nabla u_t|^2 dx \right) \\
- \int_{\Omega} v_t \int_0^t g_2(t-\tau)(v(t) - v(\tau))d\tau dx \\
- \int_{\Omega} v_t \int_0^t g_2'(t-\tau)(u(t) - u(\tau))d\tau dx - \left( \int_0^t g_2(\tau)d\tau \right) \left( \int_{\Omega} |v_t|^2 dx \right) \\
= \left( -\int_0^t g_1(\tau)d\tau \right) \left( \int_{\Omega} |u_t|^2 dx \right) - \int_{\Omega} u_t \int_0^t g_1'(t-\tau)(u(t) - u(\tau))d\tau dx \\
+ a \left( u, \int_0^t g_1(t-\tau)(u(t) - u(\tau))d\tau \right) \\
- \int_{\Omega} g_1(t-\tau)a \left( u(\tau), \int_0^t g_1(t-\tau)(u(t) - u(\tau))d\tau \right) d\tau \\
+ \alpha \int_{\Omega} v \int_0^t g_1(t-\tau)(u(t) - u(\tau))d\tau dx \\
- \gamma \int_{\Omega} \nabla u_t \int_0^t g_1'(t-\tau)\nabla(u(t) - u(\tau))d\tau dx - \gamma \left( \int_0^t g_1(\tau)d\tau \right) \left( \int_{\Omega} |\nabla u_t|^2 dx \right) \\
+ \int_{\Omega} \nabla v \int_0^t g_2(t-\tau)\nabla(v(t) - v(\tau))d\tau dx \\
- \int_{\Omega} \left( \int_0^t g_2(t-\tau)\nabla v(\tau)d\tau \right) \left( \int_0^t g_2(t-\tau)\nabla(u(t) - u(\tau))d\tau \right) dx \\
+ \alpha \int_{\Omega} u \int_0^t g_2(t-\tau)(v(t) - v(\tau))d\tau dx - \left( \int_0^t g_2(\tau)d\tau \right) \left( \int_{\Omega} |v_t|^2 dx \right) \\
- \int_{\Omega} v_t \int_0^t g_2'(t-\tau)(u(t) - u(\tau))d\tau dx \\
(4.11)
\]

Using Young’s inequality and Cauchy Schwarz’s inequality, we obtain for any \( \delta > 0 \)

\[
- \int_{\Omega} u_t \int_0^t g_1'(t-\tau)(u(t) - u(\tau))d\tau dx \\
\leq \delta \int_{\Omega} |u_t|^2 + \frac{1}{4\delta} \int_{\Omega} \left( \int_0^t g_1'(t-\tau)(u(t) - u(\tau))d\tau \right)^2 dx \\
\leq \delta \int_{\Omega} |u_t|^2 - \frac{g_1(0)}{4\delta} \int_{\Omega} g_1'(t-\tau) \int_{\Omega} |u(t) - u(\tau)|^2 dx d\tau \\
(4.12)
\]

\[
\leq \delta \int_{\Omega} |u_t|^2 - \frac{Cg_1(0)}{4\delta} (g_1 \Box u)(t),
\]
and

\[
a(u, \int_0^t g_1(t - \tau)(u(t) - u(\tau))d\tau) = \int_0^t g_1(t - \tau)a(u(t), u(t) - u(\tau))d\tau
\]

\[
\leq \int_0^t g_1(t - \tau)[a(u(t), u(\tau))]^{\frac{1}{2}}[a(u(t) - u(\tau), u(t) - u(\tau))]^{\frac{1}{2}}d\tau
\]

\[
\leq \delta a(u, u) + \frac{1}{4\delta} \left\{ \int_0^t g_1(t - \tau)[a(u(t) - u(\tau), u(t) - u(\tau))]^{\frac{1}{2}}d\tau \right\}^2
\]

(4.13)

Besides, we have

\[
- \int_0^t g_1(t - \tau)a(u(\tau), \int_0^t g_1(t - \tau)(u(t) - u(\tau))d\tau)d\tau
\]

\[
\leq \left( \int_0^t g_1(t - \tau)[a(u(\tau), u(\tau))]^{\frac{1}{2}}d\tau \right) \left( \int_0^t g_1(t - \tau)[a(u(t) - u(\tau), u(t) - u(\tau))]^{\frac{1}{2}}d\tau \right)
\]

\[
\leq \delta \left( \int_0^t g_1(t - \tau)[a(u(\tau), u(\tau))]^{\frac{1}{2}}d\tau \right)^2 + \frac{1}{4\delta} \left( \int_0^t g_1(t - \tau)[a(u(t) - u(\tau), u(t) - u(\tau))]^{\frac{1}{2}}d\tau \right)^2
\]

(4.14) \leq \delta \left( \int_0^t g_1(t - \tau)[a(u(\tau), u(\tau))]^{\frac{1}{2}}d\tau \right)^2 + \frac{1-l_1}{4\delta} (g_1 \Box u)(t).

Now, we will estimate the term \( \left( \int_0^t g_1(t - \tau)[a(u(\tau), u(\tau))]^{\frac{1}{2}}d\tau \right)^2 \). We have

\[
\left( \int_0^t g_1(t - \tau)[a(u(\tau), u(\tau))]^{\frac{1}{2}}d\tau \right)^2
\]

\[
\leq (1-l_1) \int_0^t g_1(t - \tau)a(u(\tau), u(\tau))d\tau
\]

\[
=(1-l_1) \int_0^t g_1(t - \tau)(a(u(t) - u(\tau), u(t) - u(\tau)) + 2a(u(t), u(\tau)) - a(u(t), u(\tau)))d\tau
\]

\[
=(1-l_1)(g_1 \Box u)(t) - (1-l_1) \left( \int_0^t g_1(\tau)d\tau \right)a(u, u) + 2(1-l_1) \int_0^t g_1(t - \tau)a(u(t), u(\tau))d\tau
\]

\[
\leq (1-l_1)(g_1 \Box u)(t) - (1-l_1) \left( \int_0^t g_1(\tau)d\tau \right)a(u, u) + \delta (1-l_1)a(u, u)
\]

\[
+ \frac{(1-l_1)^2}{\delta} (g_1 \Box u)(t) + 2(1-l_1)^2 a(u, u)
\]

(4.15) \leq (1-l_1) \left( \delta + 2(1-l_1) - \int_0^t g_1(\tau)d\tau \right)a(u, u) + \left( 1 - l_1 + \frac{(1-l_1)^2}{\delta} \right) (g_1 \Box u)(t)
Inserting (4.15) in (4.14), we obtain

\[- \int_0^t g_1(t - \tau) a \left( u(\tau), \int_0^\tau g_1(t - \tau)(u(t) - u(\tau))d\tau \right) d\tau \]

\[\leq \delta (1 - l_1) \left( \delta + 2(1 - l_1) - \int_0^t g_1(\tau)d\tau \right) a(u, u)\]

\[+ \left( \delta (1 - l_1) + (1 - l_1)^2 + \frac{(1 - l_1)}{4\delta} \right) (g_1 \square u)(t)\]

(4.16)

Next, we have

\[\alpha \int_\Omega v \int_0^\tau g_1(t - \tau)(u(t) - u(\tau))d\tau dx \]

\[\leq \delta \int_\Omega |v|^2 dx + \frac{C(1 - l)}{4\delta} (g_1 \square u)(t)\]

(4.17)

The term \(-\gamma \int_\Omega \nabla u_t \int_0^t g_1'(t - \tau) \nabla (u(t) - u(\tau))d\tau dx\) can be estimated as follows:

\[-\gamma \int_\Omega \nabla u_t \int_0^t g_1'(t - \tau) \nabla (u(t) - u(\tau))d\tau dx \]

\[\leq \gamma \delta \int_\Omega |\nabla u_t|^2 - \frac{Cg(0)}{4\delta} (g_1 \square u)(t)\]

(4.18)

Besides, we obtain that

\[\alpha \int_\Omega u \int_0^\tau g_2(t - \tau)(v(t) - v(\tau))d\tau dx \]

\[\leq C\delta a(u, u) + \frac{C(1 - l_2)}{4\delta} (g_2 \circ \nabla v)(t)\]

(4.19)

The remaining terms can be estimated as, for example, in [10] (see estimates (3.14), (3.15) and (3.16) in the mentioned paper).

\[\int_\Omega \nabla v \int_0^\tau g_2(t - \tau) \nabla (v(t) - v(\tau))d\tau dx \]

\[\leq \delta \int_\Omega |\nabla v|^2 dx + \frac{1 - l_2}{4\delta} (g_2 \circ \nabla v)(t),\]

(4.20)

\[- \int_\Omega \left( \int_0^t g_2(t - \tau) \nabla v(t)d\tau \right) \left( \int_0^t g_2(t - \tau) \nabla (v(t) - v(\tau))d\tau \right) \]

\[\leq (2\delta + \frac{1}{4\delta})(1 - l_2)(g_2 \circ \nabla v)(t) + 2\delta (1 - l_2)^2 \int_\Omega |\nabla v|^2 dx,\]

(4.21)

and

\[- \int_\Omega v_t \int_0^t g_2'(t - \tau)(u(t) - u(\tau))d\tau dx \]

\[\leq \delta \int_\Omega |v_t|^2 dx - \frac{Cg_2(0)}{4\delta} (g_2' \circ \nabla v)(t).\]

(4.22)
By combining (4.11)–(4.22), we get the desired estimate.

Now we define the functional $F(t)$ by

$$F(t) = NE(t) + N_1 I(t) + N_2 K(t),$$

where $N, N_1$ and $N_2$ are positive constants that will be chosen later. It is easy to verify that for $N$ large enough, we have $F \sim E$, i.e.

$$c_1 E(t) \leq F(t) \leq c_2 E(t),$$

for some $c_1, c_2 > 0$.

**Lemma 4.5.** The functional $F$ satisfies

$$F'(t) \leq - \left( \int_\Omega |u_t|^2 \, dx + \int_\Omega |v_t|^2 \, dx + \gamma \int_\Omega |\nabla u_t|^2 \, dx \, dy \right)$$

$$+ 4(1 - l) \left( a(u, u) + \int_\Omega |\nabla v|^2 \, dx \right) + c \left( (g_1 \Box u)(t) + (g_2 \circ \nabla v)(t) \right) - 2\alpha N_1 \int_\Omega uv \, dx, \quad \forall t \geq t_1,$$

where $t_1$ was introduced in (4.2) and $c > 0$.

**Proof.** Let

$$g_0 = \min \left\{ \int_0^{t_1} g_1(s) \, ds, \int_0^{t_1} g_2(s) \, ds \right\} > 0,$$

and $l = \min \{l_1, l_2\}$.

By using (2.4), (4.6), (4.10), we get for any $t \geq t_1$

$$F'(t) = NE'(t) + N_1 I'(t) + N_2 K'(t)$$

$$\leq - \left( N_2(g_0 - \delta) - N_1 \right) \left( \int_\Omega |u_t|^2 \, dx + \int_\Omega |v_t|^2 \, dx + \gamma \int_\Omega |\nabla u_t|^2 \, dx \right)$$

$$- \left( \frac{N_1}{2} - N_2 \delta \left( C + 1 + 2(1 - l)^2 \right) - N_2 \delta^2 (1 - l) + N_2 g_0 \right) a(u, u)$$

$$- \left( \frac{N_1}{2} - N_2 \delta \left( C + 1 + 2(1 - l)^2 \right) \right) \int_\Omega |\nabla v|^2 \, dx$$

$$+ \left( \frac{N_1 (1 - l)}{2l} + N_2 \left( \frac{C + 2}{4\delta} (1 - l) + \delta (1 - l) + (1 - l)^2 \right) \right) (g_1 \Box u)(t)$$

$$+ \left( \frac{N_1 (1 - l)}{2l} + N_2 \left( \frac{C + 1}{4\delta} (1 - l) + \left( 2\delta + \frac{1}{4\delta} \right) (1 - l) \right) \right) (g_2 \circ \nabla v)(t)$$

$$+ \left( \frac{N}{2} - \frac{CN_2 g_0}{2\delta} \right) \left\{ (g_1' \Box u)(t) + (g_2' \circ \nabla v)(t) \right\} - 2\alpha N_1 \int_\Omega uv \, dx \quad (4.26)
Thus, \((4.25)\) is established.

\[ F'(t) \leq - \left( \frac{N_2g_0 - \frac{l}{4(C + 1 + 2(1-l)^2)}}{4(C + 1 + 2(1-l)^2)} - N_1 \right) \left( \int_{\Omega} |u_1|^2 \, dx + \int_{\Omega} |v_1|^2 \, dx + \gamma \int_{\Omega} |\nabla u_1|^2 \, dx \right) \]

\[ \quad - \left( \frac{N_1l}{2} - \frac{l}{4} - \frac{l^2(1-l)}{16N_2(C + 1 + 2(1-l)^2)^2} + N_2g_0 \right) a(u, u) \]

\[ \quad - \left( \frac{N_1l}{2} - \frac{l}{4} \right) \int_{\Omega} |\nabla v|^2 \, dx \]

\[ + \left( \frac{N_1(1-l)}{2l} + \frac{N_2^2(C + 2)(1-l)(C + 1 + 2(1-l)^2)}{l} \right) \]

\[ (4.27) \]

\[ + \left( \frac{l(1-l)}{4(C + 1 + 2(1-l)^2)} + N_2(1-l)^2 \right) (g_1 \Box u)(t) \]

\[ + \left( \frac{N_1(1-l)}{2l} + \frac{N_2^2(C + 2)(1-l)(C + 1 + 2(1-l)^2)}{l} + \frac{\frac{l(1-l)}{2(C + 1 + 2(1-l)^2)}}{l} \right) (g_2 \circ \nabla v)(t) \]

\[ + \left( \frac{N}{2} - \frac{2Cg_0N_2^2(C + 1 + 2(1-l)^2)}{l} \right) \{ (g_1' \Box u)(t) + (g_2' \circ \nabla v)(t) \} - 2\alpha N_1 \int_{\Omega} uv \, dx \]

At this point, we choose \(N_1\) large enough so that

\[ \frac{N_1l}{2} - \frac{l}{4} > 4(1-l), \]

then \(N_2\) large enough such that

\[ N_2g_0 - \frac{\frac{l}{4(C + 1 + 2(1-l)^2)}}{4(C + 1 + 2(1-l)^2)} - N_1 > 1, \]

and

\[ N_2g_0 - \frac{l^2(1-l)}{16N_2(C + 1 + 2(1-l)^2)^2} > 0, \]

Now, choosing \(N\) large enough such that

\[ \frac{N}{2} - \frac{2Cg_0N_2^2(C + 1 + 2(1-l)^2)}{l} > 0. \]

Thus, \((4.25)\) is established.

\[ \square \]

**Proof of Theorem (4.1).** Taking into account \((2.4)\) and \((4.3)\), we obtain that for any \(t \geq t_1\)

\[ \int_{0}^{t_1} g_1(s)a(u(t) - u(t - s), u(t) - u(t - s)) \, ds \]

\[ \leq - \frac{g_1(0)}{a_1} \int_{0}^{t_1} g_1'(s)a(u(t) - u(t - s), u(t) - u(t - s)) \, ds \leq -cE'(t), \]

and

\[ \int_{0}^{t_1} g_2(s) \int_{\Omega} |\nabla v(t) - \nabla v(t - s)|^2 \, dx \, ds \]

\[ \leq - \frac{g_2(0)}{a_2} \int_{0}^{t_1} g_2'(s) \int_{\Omega} |\nabla v(t) - \nabla v(t - s)|^2 \, dx \, ds \leq -cE'(t). \]
Therefore, (4.25) yields for some \( m > 0 \) and all \( t \geq t_1 \),
\[
F'(t) \leq -mE(t) + c(g_1 \Box u)(t) + c(g_2 \circ \nabla v)(t)
\]
\[
\leq -mE(t) - cE'(t) + c \int_{t_1}^{t} g_1(s)a(u(t) - u(t - s), u(t) - u(t - s)) \, ds
\]
(4.28)
\[
+c \int_{t_1}^{t} g_2(s) \int_{\Omega} |\nabla v(t) - \nabla v(t - s)|^2 \, dx \, ds.
\]

Denote \( \mathcal{L}(t) = F(t) + cE(t) \). It is obvious that \( \mathcal{L}(t) \) is equivalent to \( E(t) \). It follows from (4.28) that
\[
\mathcal{L}'(t) \leq -mE(t) + c \int_{t_1}^{t} g_1(s)a(u(t) - u(t - s), u(t) - u(t - s)) \, ds
\]
(4.29)
\[
+c \int_{t_1}^{t} g_2(s) \int_{\Omega} |\nabla v(t) - \nabla v(t - s)|^2 \, dx \, ds.
\]

Next we distinguish the following two cases.

**Case 1.** The function \( H(t) \) is linear.

We multiply (4.29) by \( \xi(t) \) and use Assumption (A2) and (2.4) to get
\[
\xi(t)\mathcal{L}'(t) \leq -m\xi(t)E(t) + c\xi(t) \int_{t_1}^{t} g_1(s)a(u(t) - u(t - s), u(t) - u(t - s)) \, ds
\]
\[
+c\xi(t) \int_{t_1}^{t} g_2(s) \int_{\Omega} |\nabla v(t) - \nabla v(t - s)|^2 \, dx \, ds
\]
\[
\leq -m\xi(t)E(t) + c \int_{t_1}^{t} \xi_1(s)g_1(s)a(u(t) - u(t - s), u(t) - u(t - s)) \, ds
\]
\[
+c \int_{t_1}^{t} \xi_2(s)g_2(s) \int_{\Omega} |\nabla v(t) - \nabla v(t - s)|^2 \, dx \, ds
\]
\[
\leq -m\xi(t)E(t) - c \int_{t_1}^{t} g_1'(s)a(u(t) - u(t - s), u(t) - u(t - s)) \, ds
\]
\[
-c \int_{t_1}^{t} g_2'(s) \int_{\Omega} |\nabla v(t) - \nabla v(t - s)|^2 \, dx \, ds
\]
(4.30)
\[
\leq -m\xi(t)E(t) - cE'(t).
\]

Denote \( \mathcal{F}(t) = \xi(t)\mathcal{L}(t) + cE(t) \sim E(t) \). Then we have, from (4.30) and the fact that \( \xi \) is non-increasing, that for any \( t \geq t_1 \),
\[
\mathcal{F}'(t) \leq -m\xi(t)E(t).
\]

Using the fact that \( \mathcal{F} \sim E \), we obtain
\[
\mathcal{F}'(t) \leq -c_1\mathcal{F}(t),
\]
for some positive constant \( c_1 \). By applying Gronwall’s Lemma, we obtain the existence of a constant \( c_2 > 0 \) such that
\[
\mathcal{F}(t) \leq c_2e^{-c_1 \int_{t_1}^{t} \xi(s) \, ds},
\]
which yields to
\[ E(t) \leq c_3 e^{-c_1 \int_{t_1}^{t} \xi(s) \, ds}, \]
for some constant \( c_3 > 0 \).

**Case 2:** \( H \) is nonlinear. First, we define the following quantities
\[
I_1(t) = \frac{\kappa}{l} \int_0^t \left[ a(u(t), u(t)) + a(u(t-s), u(t-s)) \right] ds, \quad t > 0
\]
and
\[
I_2(t) = \frac{\kappa}{l} \int_0^t \int_\Omega \left| \nabla v(t) - \nabla v(t-s) \right|^2 dx ds, \quad t > 0.
\]
Then, we have
\[
I_1(t) \leq \frac{2\kappa}{l} \int_0^t \left[ a(u(t), u(t)) + a(u(t-s), u(t-s)) \right] ds \\
\leq \frac{4\kappa}{lt} \left( \int_0^t (E(t) + E(t-s)) ds \right) \\
\leq \frac{8\kappa}{lt} \int_0^t E(s) ds \\
\leq \frac{8\kappa}{lt} \int_0^t E(0) ds = \frac{8\kappa}{l} E(0) < +\infty,
\]
and likewise we have
\[
I_2(t) \leq \frac{8\kappa}{l} E(0) < +\infty.
\]
Thus, choosing \( 0 < \kappa < 1 \) small enough so that, for all \( t > 0 \),
\[
(4.31) \quad I_i(t) < 1, \quad \text{for } i = 1, 2.
\]
Also, we define \( \lambda_1(t) \) and \( \lambda_2(t) \) by
\[
\lambda_1(t) = -\int_0^t g_1'(s) a(u(t) - u(t-s), u(t) - u(t-s)) ds,
\]
and
\[
\lambda_2(t) = -\int_0^t g_2'(s) \int_\Omega \left| \nabla v(t) - \nabla v(t-s) \right|^2 dx ds.
\]
It is obvious that \( \lambda_i(t) \leq -cE'(t), \quad i = 1, 2. \)
Since $H$ is strictly convex on $(0, r]$ and $H(0) = 0$, then $H(\theta x) \leq \theta H(x)$, provided that $0 \leq \theta \leq 1$ and $x \in (0, r]$. This, together with (A1), [4.31] and Jensen’s inequality, leads to

$$\lambda_1(t) = \frac{1}{\kappa I_1(t)} \int_0^t I_1(t) (-g_1'(s) - \kappa a(u(t) - u(t-s), u(t) - u(t-s))ds$$

$$\geq \frac{1}{\kappa I_1(t)} \int_0^t I_1(t) \xi_1(s) H(g_1(s)) \kappa a(u(t) - u(t-s), u(t) - u(t-s))ds$$

$$\geq \frac{\xi_1(t)}{\kappa I_1(t)} \int_0^t H(I_1(t)g_1(s)) \kappa a(u(t) - u(t-s), u(t) - u(t-s))ds$$

$$= \frac{\xi_1(t)}{\kappa} \int_0^t g_1(s)a(u(t) - u(t-s), u(t) - u(t-s))ds$$

$$= \frac{\xi_1(t)}{\kappa} \int_0^t g_1(s)a(u(t) - u(t-s), u(t) - u(t-s))ds$$

where $\overline{H}$ is an extension of $H$ such that $\overline{H}$ is strictly increasing and strictly convex $C^2$ fonction on $(0, +\infty)$. This implies that

$$\int_0^t g_1(s)a(u(t) - u(t-s), u(t) - u(t-s))ds \leq \frac{1}{\kappa} \overline{H}^{-1} \left( \frac{\kappa \lambda_1(t)}{\xi_1(t)} \right).$$

Similarly, we have

$$\int_0^t g_2(s) \int_{\Omega} |\nabla v(t) - \nabla v(t-s)|^2 dx ds \leq \frac{1}{\kappa} \overline{H}^{-1} \left( \frac{\kappa \lambda_2(t)}{\xi_2(t)} \right).$$

We infer from (4.29) that for any $t \geq t_1$

$$L'(t) \leq -mE(t) + c\overline{H}^{-1} \left( \frac{\kappa \lambda_1(t)}{\xi_1(t)} \right) + c\overline{H}^{-1} \left( \frac{\kappa \lambda_2(t)}{\xi_2(t)} \right).$$

For $\varepsilon_0 < r$, using (4.32) and the fact that $E' \leq 0$, $\overline{H}' > 0$, $\overline{H}'' > 0$, we find that the functional $K_1$, defined by

$$K_1(t) = \overline{H} \left( \varepsilon_0 \frac{E(t)}{E(0)} \right) L(t) + E(t),$$

is equivalent to $E(t)$ and satisfies

$$K'_1(t) = \varepsilon_0 \frac{E'(t)}{E(0)} \overline{H}'' \left( \varepsilon_0 \frac{E(t)}{E(0)} \right) L(t) + \overline{H} \left( \varepsilon_0 \frac{E(t)}{E(0)} \right) L'(t) + E'(t)$$

$$\leq -mE(t) \overline{H}' \left( \varepsilon_0 \frac{E(t)}{E(0)} \right) + c\overline{H} \left( \varepsilon_0 \frac{E(t)}{E(0)} \right) \overline{H}^{-1} \left( \frac{\kappa \lambda_1(t)}{\xi_1(t)} \right) + c\overline{H} \left( \varepsilon_0 \frac{E(t)}{E(0)} \right) \overline{H}^{-1} \left( \frac{\kappa \lambda_2(t)}{\xi_2(t)} \right).$$

Now let $\overline{H}'$ be the convex conjugate of $\overline{H}$ in the sense of Young (see [2]). Then

$$\overline{H}'(s) = s(\overline{H}')^{-1}(s) - \overline{H}((\overline{H}')^{-1}(s)),$$

which satisfies

$$AB_i \leq \overline{H}'(A) + \overline{H}(B_i), \quad i = 1, 2,$$
with \( A = \overline{H'} \left( \varepsilon_0 \frac{E(t)}{E(0)} \right) \) and \( B_i = \overline{H}^{-1} \left( \frac{\varepsilon \lambda_i(t)}{\xi_i(t)} \right), \ i = 1, 2. \)

Using (4.33), (4.34) and (4.35), we obtain

\[
K'_1(t) \leq -mE(t)\overline{H}' \left( \varepsilon_0 \frac{E(t)}{E(0)} \right) + c\varepsilon_0 \frac{E(t)}{E(0)} \overline{H}' \left( \varepsilon_0 \frac{E(t)}{E(0)} \right) + c\varepsilon_0 \frac{E(t)}{E(0)} \left( \frac{\lambda_1(t)}{\xi_1(t)} + \frac{\lambda_2(t)}{\xi_2(t)} \right)
\]

Multiplying the last inequality by \( \xi(t) \) and using the fact that, as \( \varepsilon_0 \frac{E(t)}{E(0)} < r, \overline{H}' \left( \varepsilon_0 \frac{E(t)}{E(0)} \right) = H' \left( \varepsilon_0 \frac{E(t)}{E(0)} \right) \) and \( \lambda_i(t) \leq -cE'(t) \) (for \( i = 1, 2 \)), we get

\[
(4.36) \quad \xi(t)K'_1(t) \leq -mE(t)\xi(t)H' \left( \varepsilon_0 \frac{E(t)}{E(0)} \right) + c\varepsilon_0 \frac{E(t)}{E(0)} \xi(t)H' \left( \varepsilon_0 \frac{E(t)}{E(0)} \right) - cE'(t).
\]

Consequently, letting \( K_2 = \xi K_1 + cE \), we have: \( \alpha_1 K_2(t) \leq E(t) \leq \alpha_2 K_2(t), \) for some \( \alpha_1, \alpha_2 > 0. \)

Hence, we conclude that, for some constant \( \beta_1 > 0 \) and for all \( t \geq t_1 \)

\[
(4.37) \quad K'_2(t) \leq -\beta_1 \xi(t) \frac{E(t)}{E(0)} H' \left( \varepsilon_0 \frac{E(t)}{E(0)} \right) := -\beta_1 \xi(t) H_2 \left( \frac{E(t)}{E(0)} \right),
\]

where \( H_2(t) = tH'(\varepsilon_0 t). \) Since \( H'_2(t) = H'(\varepsilon_0 t) + \varepsilon_0 tH''(\varepsilon_0 t), \) then, using the strict convexity of \( H(t) \) on \( (0, r], \)

we reach that \( H'_2(t), H_2(t) > 0 \) on \( (0, 1]. \) Thus, with \( R(t) = \frac{\alpha_1 K_2(t)}{E(0)}, \)

and using the fact that \( K_2 \sim E \) and (4.37), we have

\[
(4.38) \quad R(t) \sim E(t),
\]

and for some \( \beta_2 > 0, \)

\[
R'(t) \leq -\beta_2 \xi(t) H_2(R(t)), \forall t \geq t_1.
\]

Integrating the latter over \((t_1, t)\) yields

\[
\int_{t_1}^{t} \frac{-R'(s)}{H_2(R(s))} ds \geq \beta_2 \int_{t_1}^{t} \xi(s) ds \Rightarrow \int_{\xi_0 R(t_1)}^{\xi_0 R(t)} \frac{1}{s H'(s)} \sigma(t) ds \geq \beta_2 \int_{t_1}^{t} \xi(s) ds.
\]

Lastly, since the function \( H_1 \) given by \( H_1(t) = \int_{t_0}^{t} \frac{1}{s H'(s)} ds, \) is strictly decreasing on \( (0, r] \) and \( \lim_{t \rightarrow 0} H_1(t) = +\infty, \)

we deduce that

\[
R(t) \leq \frac{1}{\varepsilon_0} H_1^{-1} \left( \beta_1 \int_{t_1}^{t} \xi(s) ds \right).
\]

Combining the latter with (4.38), one can claim that (4.1) holds. \( \square \)

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