Beam splitter as quantum coherence-maker

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Abstract
The aim of this work is to answer the question of how much quantum coherence a beam splitter is able to produce. To this end, we consider as the variables under study both the amount of coherence of the input states as well as the beam splitter characteristics. We conclude that there is an optimal combination of these factors making the gain of coherence maximum. In addition, the two-mode squeezed vacuum arises as the studied state most capable of gaining coherence when passing through a beam splitter. These results are qualitatively equivalent for the l1-norm of coherence and the relative entropy of coherence.

1. Introduction
Quantum coherence is the archetype of quantum correlations. It potentially concentrates all the superposition principle consequences, meaning all the fascinating fundamental implications [1–3] along with the prospering practical applications [4–6]. This general role allows different correlations, such as entanglement or steering, to be generated from coherence [7–9].

Quantum resources theories [10–12] have proved to be a useful support to take advantage of all these quantum features, for example in metrology [13] or quantum information processing [14]. The general scheme of these theories is universal, broadly, to quantify the resource by defining the states lacking it, incoherent states, the operations that do not generate it, free operations, and finally the quantifier that accounts for the amount of resource present [15, 16].

Most of these works focus on abstract situations rather far from practical situations where coherence can manifest. In this regard, quantum optics is a rather privileged arena to test such novel formulations of quantum coherence. Actually, optics is where the most relevant theories of coherence were introduced so far. This is the theory of classical-light coherence, vital for all the scientific and technological applications of interferometry, as well as the quantum theory of optical coherence, introduced by Glauber and Sudarshan [17]. Thus, it may be of practical importance to apply the novel formulations of coherence to the field of quantum optics.

In this work, we focus on characterizing one operation that is not free with regard to quantum coherence, the beam splitter (BS), which is able to enhance the coherence previously present in the system [18, 19]. Beam splitters already display this behaviour of coherence-maker in classical optics, which makes them very intuitive to use in this quantum context and suitable for experimental implementations of coherence-based technologies [20–22].

As an example, in [18], it is shown the increase of coherence of a two-mode squeezed vacuum state (TMSV) when passing through a balanced beam splitter. The choice of a balanced beam splitter is the natural one regarding classical experience. This is also the case of the renowned Hong–Ou–Mandel effect, a corner stone of quantum interference [23]. However, we wonder whether it is the best configuration regarding the gain of these new forms of coherence and whether this optimum coherence-tuning of the beam splitter depends on the input state. Moreover, beam splitters are the key elements involved in generating other quantum correlations from quantum coherence [8, 24] In this context the role of BS is being investigated both from a theoretical and experimental point of view [25].
Accordingly, we thoroughly investigate which choice of the reflectance-transmittance ratio produces maximum coherence for a given input state. As the input states we consider incoherent and coherent states. In this manner we can evaluate the final amount of coherence regarding the initial one. We also divide the problem into subspaces of fixed incoming energy and compare the possible distributions of photons between the two input modes. Since coherence is a basis dependent quantity, throughout this analysis we are working in the photon number basis, so we are always talking about photon-number-coherence. The quantifiers of coherence utilized are the l1-norm and the relative entropy of coherence [16].

2. Beam splitter

The action of the beam splitter can be expressed in matrix form as a relation between the input \(a_{1,2}\) and output \(b_{1,2}\) complex-amplitude operators

\[
\begin{pmatrix}
  b_1 \\
  b_2
\end{pmatrix} =
\begin{pmatrix}
  \tau_1 & \rho_1 \\
  \rho_2 & \tau_2
\end{pmatrix}
\begin{pmatrix}
  a_1 \\
  a_2
\end{pmatrix},
\]

where \(\tau_{1,2}\) and \(\rho_{1,2}\) are the corresponding transmission and reflection coefficients as defined in Figure 1. For a lossless beam splitter, the transformation matrix is unitary, and, without loss of generality regarding our purposes, the coefficients can be chosen so that [26]

\[
\tau_1 = \tau_2 = \cos \theta, \quad \rho_1 = -\rho_2 = -\sin \theta,
\]

where we have introduced the parameter \(\theta\) to express the balance between transmission and reflection, with \(\theta \in [0, \pi/2]\).

3. Incoherent input state

We start this analysis by considering a beam splitter illuminated by a number state on both input ports.

The corresponding incoming number state \(|n_1\rangle_1 \otimes |n_2\rangle_2 = |n_1, n_2\rangle\) can be expressed in terms of the vacuum as

\[
|n_1, n_2\rangle = \frac{1}{\sqrt{n_1!n_2!}} a_1^\dagger n_1 a_1 a_2^\dagger n_2 |0, 0\rangle.
\]

The output state can be readily obtained in the number basis by inverting the input-output relation (1) to express the input modes \(a_{1,2}\) in terms of the output modes \(b_{1,2}\), and translating the result to equation (3) to get

\[
|\text{out}\rangle_{n_1n_2} = \frac{1}{\sqrt{n_1!n_2!}} (\cos \theta b_1^\dagger + \sin \theta b_2^\dagger)^{n_1} (\sin \theta b_1^\dagger - \cos \theta b_2^\dagger)^{n_2} |0, 0\rangle.
\]

Equivalently, the decomposition formulas for the su(2) Lie algebras may be used [27]. After some simple algebra we get that the output state (4) becomes

\[
|\text{out}\rangle_{n_1n_2} = \sum_{j=0}^{n_1+n_2} c_j |j, n_1 + n_2 - j\rangle,
\]
where
\[ c_j = \sqrt{n_1!n_2!j!(n_1 + n_2 - j)!} \sum_{k=\text{max}(0,j-n_2)}^{n_2} \frac{(-1)^k \cos^{n_1+2k-j} \sin^{n_2-2k+j} \theta}{(n_1 - k)!k!(n_2 + k - j)!(j - k)!}. \]  
(6)

In the particular case of \( n_1 = 0 \) or \( n_2 = 0 \), we get that \(|\text{out}\rangle\) are SU(2) coherent states \([28]\).

As suitable coherence measures we have the \( l_1 \)-norm of coherence, \( C_{lh} \), and the relative entropy of coherence, \( C_S \), which we will use in their forms adapted to pure states \([10,16]\)

\[ C_H = \left( \sum_j |c_j|^2 \right)^2 = 1, \]  
(7)

\[ C_S = -\sum_j |c_j|^2 \ln |c_j|^2, \]  
(8)

respectively. If we restrict ourselves to a subspace of fixed energy, \( n_1 + n_2 = \) constant, the maximum coherence holds for \(|c_j|^2 = \) constant \([29,30]\), this is for the phase-like states

\[ |\phi\rangle = \frac{1}{\sqrt{n_1 + n_2 + 1}} \sum_{j=0}^{n_1+n_2} e^{i\phi_j} |j, n_1 + n_2 - j\rangle, \]  
(9)

where \( \phi_j \) are arbitrary phases \([31,32]\). The corresponding maximum of coherence is

\[ C_{H}^{\text{max}} = n_1 + n_2, \]  
(10)

leading to a curious but accidental identification of coherence with energy. In the case of the relative entropy of coherence, the maximum value is

\[ C_{S}^{\text{max}} = -\ln \left[ \frac{1}{n_1 + n_2 + 1} \right]. \]  
(11)

As we are about to see, not all incoming states can reach this maximum coherence, irrespective of the beam splitter parameters. This idea leads us to an alternative expression for the coherence as the maximum overlap between the system state \(|\psi\rangle\), assumed pure, and the phase-like states \(|\phi\rangle\) when \( \phi \) is varied

\[ C_H = |\sqrt{n_1 + n_2 + 1}| \max_{\phi} \langle \phi | \psi \rangle |^2 - 1. \]  
(12)

### 3.1. One photon

This case, \( n_1 + n_2 = 1 \), is a rather simple situation since essentially there is just a single configuration, the input number state \(|1, 0\rangle\). The corresponding output state, omitting irrelevant relative phases, is the split photon

\[ |\text{out}\rangle_{1,0} = \cos \theta |1, 0\rangle + \sin \theta |0, 1\rangle, \]  
(13)

that shows how the maximum coherence holds for a 50% beam splitter \( \theta = \pi / 4 \), for which \(|\text{out}\rangle_{1,0}\) becomes a phase-like state \(|\phi\rangle\) in equation (9).

### 3.2. Two photons

In this situation where \( n_1 + n_2 = 2 \), we have just two meaningful cases, say SU(2) coherent states \(|2, 0\rangle\) and SU(2) squeezed states \(|1, 1\rangle\) \([28,33,34]\). The output states are respectively

\[ |\text{out}\rangle_{2,0} = \cos^2 \theta |2, 0\rangle + \sqrt{2} \sin \theta \cos \theta |1, 1\rangle + \sin^2 \theta |0, 2\rangle, \]  
(14)

and

\[ |\text{out}\rangle_{1,1} = \sqrt{2} \sin \theta \cos \theta |2, 0\rangle + (\cos^2 \theta - \sin^2 \theta) |1, 1\rangle - \sqrt{2} \sin \theta \cos \theta |0, 2\rangle. \]  
(15)

In figure 2 we represent the coherence \( C_H \) for these states as a function of \( \theta \), blue solid line for the \(|\text{out}\rangle_{2,0}\) and red dashed line for \(|\text{out}\rangle_{1,1}\), while the green line marks the maximum coherence in equation (10).

We can appreciate that varying the transmittance-reflection ratio of the beam splitter, the \(|\text{out}\rangle_{1,1}\) state can get larger coherence than the SU(2) coherent state \(|\text{out}\rangle_{2,0}\), which betrays its name a bit.

Moreover, it can be clearly seen that the SU(2)-coherent case \(|\text{out}\rangle_{2,0}\) finds its maximum coherence for a balanced beam splitter \( \theta = \pi / 4 \). On the other hand, the coherence for the \(|\text{out}\rangle_{1,1}\) case finds its maximum for an unbalanced disposition, that can be found analytically to be given by the \( \theta \) satisfying the equality \( \tan(2\theta) = \pm \sqrt{2} \). For these optimum beam splitters we have that the output are actually phase-like states \(|\phi\rangle\).
while for a balanced beam splitter $\theta = \pi/4$ we will have the N00N state

$$|\text{out}_{1,1}\rangle = \frac{1}{\sqrt{2}}(-|2, 0\rangle + |1, 1\rangle \mp |0, 2\rangle),$$

(16)

This result is reproduced if we consider the relative entropy of coherence. As expected, the absolute values of $C_H$ and $C_S$ are different, however the tendencies and the optimum configurations of the beam splitter are alike.

We can highlight the clearly smoother behaviour around the balanced BS.

### 3.3. Larger photon numbers

For photon number larger than two, we present numerical computations confirming the main results already commented. For the case of four photons displayed in figure 3(a), the larger coherence is obtained by the twin-photon states $|\text{out}_{2,2}\rangle$ emerging from an unbalanced beam splitter. Such states $|\text{out}_{2,2}\rangle$ are no longer phase-like states, equation (9), and so the absolute maximum coherence in equation (10) is not reached. Nevertheless, it can be shown numerically that for the optimum case, this is $|\text{out}_{2,2}\rangle$ at optimum $\theta$, there is a symmetric split of the photons between the output modes, this is

$$|\langle n_1, n_2|\text{out}_{2,2}\rangle| = |\langle n_5, n_4|\text{out}_{2,2}\rangle|.
$$

(18)

This symmetry is specially useful in relation to the sensitivity of two-path interferometers [35].

Regarding odd number of photons, that is $n_1 + n_2 = 2k + 1$ for integer $k$, the situation is quite similar. The maximum coherence is obtained for input states closest to the equal splitting of the photons between the input modes, say $|k + 1, k\rangle$, and unbalanced beam splitter $\theta = \pi/4$, as shown in figure 3(b) for five photons.

The case of three photons is special, as shown in figure 3(c), since both states $|\text{out}_{3,0}\rangle$ (blue solid line) and $|\text{out}_{2,1}\rangle$ (red dashed line) reach the maximum coherence for the balanced beam splitter $\theta = \pi/4$, although neither do they achieve the maximum coherence.

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**Figure 2.** $C_H$ for the states $|\text{out}_{2,0}\rangle$ (blue solid line) and $|\text{out}_{1,1}\rangle$ (red, dashed line) as functions of the BS’s parameter $\theta$. The green, dotted line marks the maximum coherence (10). $|\text{out}_{1,1}\rangle$ achieves the maximum coherence but not for the balanced BS. The coherence of $|\text{out}_{2,0}\rangle$ is optimum after a balanced BS but it does not reach the maximum.

**Figure 3.** Plots of the $l_1$-norm of coherence for different incoherent input states as functions of the parameter $\theta$. The green, dotted lines mark the maximum coherence in each subspace (10), and the thicker gray line shows the coherence achieved by the optimal configuration of incoherent inputs states and BS parameters. (a) $C_H$ for the states $|\text{out}_{4,0}\rangle$ (blue solid line), $|\text{out}_{3,1}\rangle$ (red, dashed line), and $|\text{out}_{2,2}\rangle$ (black, dotted-dashed line). (b) $C_H$ for the states $|\text{out}_{5,0}\rangle$ (blue solid line), $|\text{out}_{4,1}\rangle$ (red, dashed line), and $|\text{out}_{3,2}\rangle$ (black, dotted-dashed line). (c) $C_H$ for the states $|\text{out}_{3,0}\rangle$ (blue solid line) and $|\text{out}_{2,1}\rangle$ (red, dashed line).
The coherence obtained when the output state is a phase-like state, this is the maximum coherence on each subspace, scales linearly with the total number of photons equation (10). However, the overlap between the possible outcomes and the phase-states decreases with the total energy so equation (12) becomes more and more distant to the ideal maximum coherence. We show this evolution in figure 4.

The results of this section are also reproduced by the relative entropy of coherence. For the sake of comparison we replicate in figure 5 the cases of 3, 4 and 5 photon-subspaces.

Likewise, we can see how the maximum $C_{\alpha}$ achieved by this incoming states distances from the maximum available coherence in each subspace, equation (11). This comparison is presented in figure 6.

4. Two mode squeezed vacuum

We take advantage of the previously computed coherence for the number states to calculate the coherence obtained when each port of the beam splitter is illuminated with one of the modes of the two mode squeezed vacuum state (TMSV),

$$|\xi\rangle = \sqrt{1 - \xi^2} \sum_{n=0}^{\infty} \xi^n |n, n\rangle,$$  \hspace{1cm} \hspace{1cm} (19)

where $\xi$ is the squeezing parameter, considered real without loss of generality. Therefore, we can calculate its coherence as
In figure 7 it is represented the final coherence as a function of the parameter $\theta$ for different squeezing parameters $\xi$. The minimum coherence, at $\theta = 0$ coincides with the coherence of the squeezed vacuum without any beam splitter transformation [19].

\[
C_H = \left[ \frac{1}{\xi^2 + 1} \right]^{1/2} - 1.
\]  

(20)

In figure 7 it is represented the final coherence as a function of the parameter $\theta$ for different squeezing parameters $\xi$. The minimum coherence, at $\theta = 0$ coincides with the coherence of the squeezed vacuum without any beam splitter transformation [19].

\[
C_H = \frac{2\xi}{1 - \xi}.
\]  

(21)

Once more, the results are replicated by the relative entropy of coherence (see figure 8).

Since the coherence of the initial state is not zero, we can define the coherence gained when it goes through the beam splitter as a function of the incoming amount of coherence. To this end we define the gain in coherence as the next percentage

\[
\mathcal{G} = \frac{C_H_{\text{max}}}{C_H_{\text{in}}} \times 100.
\]  

(22)

Different definitions of this concept have been developed [36]. We can compute $\mathcal{G}$ depending on the squeezing of the incoming state, and thus on the mean number of photons. In figure 9 it is shown how this gain grows rapidly with the squeezing parameter.

5. Coherent state

We compute the coherence provided by the beam splitter when it is illuminated with coherent states on each port.
We consider the same beam splitter introduced in equations (1)–(2). The input state in terms of the vacuum state is

\[ |0, 0, 23\rangle \]

and the output becomes

\[ |\text{out}| = |\alpha_1 \cos \theta - \alpha_2 \sin \theta, \alpha_2 \cos \theta + \alpha_1 \sin \theta\rangle. \]

We compute the coherence of the outcome state when the input state is a coherent state of \( \bar{N} = 4 \) on one input port and the vacuum state on the other. In figure 10 it can be seen how the optimum beam splitter configuration is the one that allows a symmetrical output, with the same mean number of photons on each mode.

It can be seen that the minimum of \( C_H \) is the coherence of the single-mode coherent state \( C_{H_{\text{co}}}=C_h(\alpha = \sqrt{\bar{N}}) \) and it appears when the output state is of the form \( |\alpha = \sqrt{\bar{N}}, 0\rangle \). As in the incoherent case, this result can be reproduced by the relative entropy of coherence.

We may analyze the advantage of the beam splitter in terms of coherence by computing the gain in coherence (22). In figure 11 it can be seen how the gain of coherence increases with the initial coherence of the single-mode state but the growth ratio is smaller for high \( \bar{N} \).

5.1. Comparison with TMSV

In the following we compare the performance of the beam splitter when illuminated by the coherent state and by the TMSV when these present identical: (a) minimum amount of coherence, \( C_{H_{\text{co}}} \), and (b) mean number of photons, \( \bar{N} \).

As introduced, we are considering two states with the same minimum amount of coherence, e.g., \( C_{H_{\text{co}}} = 3.0 \). The corresponding coherent state before the beam splitter is \( |\alpha_1 = \sqrt{0.83}, 0\rangle \), and the gain in coherence caused by the transformation is
However, if we consider a TMSV with the same minimum coherence available, this is $|\xi = 0.6\rangle$, the gain produced by the beam splitter is remarkably higher,

$$G_\xi \approx 364\%.$$  

Regarding the energy, the mean number of photons of the TMSV is

$$\bar{N} = \frac{\xi^2}{1 - \xi^2},$$

thus, for $\bar{N} = 1$ the gain in coherence caused by the beam splitter is

$$G_\xi \approx 470\%$$

whereas for a coherent state with $|\alpha|^2 = 1$ it is

$$G_\alpha \approx 260\%.$$  

Therefore, the TMSV is able to gain more coherence since the squeezing allows it to better resemble the constant statistics of phase-like states.

6. Conclusions

We have performed a detailed study of the role of beam splitters as quantum coherence makers, obtaining the optimum configuration of the reflectance-transmitance parameters for several incoming states. The optimum configuration is such that the outcoming state is as similar as possible to the phase-like states. We have investigated how the amount of energy of the input state concerns the maximum coherence achievable by the system. By studying, at a fixed energy, the two mode squeezed vacuum along with the coherent state cases we conclude that the beam splitter generates considerably more coherence when illuminated by the TMSV. In the same way, the gain in coherence of the TMSV is higher if we consider fixed the minimum coherence available.
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Data availability statement

No new data were created or analysed in this study.

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References

[1] Karnieli A et al 2021 The coherence of light is fundamentally tied to the quantum coherence of the emitting particle Sci. Adv. 7 18
[2] Yuan X et al 2019 Quantum coherence and intrinsic randomness Adv. Quantum Technol. 2 1900053
[3] Schlosshauer M 2004 Decoherence, the measurement problem, and interpretations of quantum mechanics Rev. Mod. Phys. 76 4
[4] Caravelli F et al 2021 Energy storage and coherence in closed and open quantum batteries Quantum 5 505
[5] Mitchell M T 2019 Quantum thermal absorption machines: refrigerators, engines and clocks Contemp. Phys. 60 164–87
[6] Lüders G et al 2021 Quantifying quantum coherence in polariton condensates PRX Quantum 2 030320
[7] Kim S, Xiong C, Kumar A and Wu J 2021 Converting coherence based on positive-operator-valued measures into entanglement Phys. Rev. A 103 052418
[8] Wang Z-X, Sang S, Ma T et al 2016 Gaussian entanglement generation from coherence using beam-splitters Sci. Rep. 6 38002
[9] Ma J et al 2016 Converting coherence to quantum correlations Phys. Rev. Lett. 116 160407
[10] Streltsov A, Adesso G and Plenio M B 2017 Colloquium: quantum coherence as a resource Rev. Mod. Phys. 89 041003
[11] Chitambar E and Gour G 2019 Quantum resource theories Rev. Mod. Phys. 91 012001
[12] Tan K C and Jeong H 2019 Resource theories of nonclassical light Quantum Reports 1 151–61
[13] GE W et al 2020 Operational resource theory of nonclassicality via quantum metrology Phys. Rev. Lett. 2 023400
[14] Nie Y Q et al 2019 Quantum coherence witness with untrusted measurement devices Phys. Rev. Lett. 123 090502
[15] Horodecki R, Horodecki P, Horodecki M and Horodecki K 2009 Quantum entanglement Rev. Mod. Phys. 81 865
[16] Baumgratz T, Cramer M and Plenio M B 2014 Quantifying Coherence Phys. Rev. Lett. 113 140401
[17] Mandel L and Wolfe F 1995 Optical Coherence and Quantum Optics (Cambridge, U.K: Cambridge University Press) (https://doi.org/10.1017/CBO9781139644105)
[18] Zhang Y R, Shao L H, Li Y and Fan H 2016 Quantifying coherence in infinite-dimensional systems Phys. Rev. A 93 012334
[19] Ares L and Luis A 2021 Distance-based approaches to quantum coherence and nonclassicality Phys. Rev. A 106 012415
[20] Wu K D, Streltsov A, Regula B, Xiang G Y and Guo G C 2021 Experimental progress on quantum coherence: detection, quantification, and manipulation Adv. Quantum Technol. 4 2100040
[21] Takahashi M, Iana S and Streltsov A 2022 Creating and destroying coherence with quantum channels Phys. Rev. A 105 L060401
[22] Wu K D, Theurer T, Xiang G Y, Li Ch F, Guo G C, Plenio M B and Strehlov A 2020 Quantum coherence and state conversion: theory and experiment npj Quantum Inf. 6 32
[23] Hong C K, Ou Z Y and Mandel L 1987 Measurement of subpicosecond time intervals between two photons by interference Phys. Rev. Lett. 59 2044–6
[24] Goldberg A Z and Heshami K 2021 How squeezed states both maximize and minimize the same notion of quantumness Phys. Rev. A 104 032425
[25] Kang H, Liu Y, Han D, Wang N and Su X 2021 Experimental demonstration of the conversion of local and correlated Gaussian quantum coherence Opt. Lett. 46 3817–20
[26] Luis A and Sánchez-Soto L I 1995 A quantum description of the beam splitter Quantum Semiclass. Opt. 7 153–60
[27] Ban M 1993 Decomposition formulas for su(1, 1) and su(2) Lie algebras and their applications in quantum optics J. Opt. Soc. Am. B 10 1347–59
[28] Arecchi F T, Courtens E, Gilmore R and Thomas H 1972 Atomic coherent states in quantum optics Phys. Rev. A 6 2211–37
[29] Cheng S and Hall M J W 2015 Complementarity relations for quantum coherence Phys. Rev. A 92 042101
[30] Streltsov A et al 2018 Maximal coherence and the resource theory of purity New J. Phys. 20 053058
[31] Santhanam T S 1976 Canonical commutation relation for operators with bounded spectrum Phys. Lett. A 56 345–6
[32] Luis A and Sánchez-Soto L I 2000 Quantum phase difference, phase measurements and stokes operators Progress in Optics ed E Wolf vol. 41 (Amsterdam: Elsevier) 421–81
[33] Agarwal G S and Puri R R 1994 Atomic states with spectroscopic squeezing Phys. Rev. A 49 4968–71
[34] Briët C and Mann A 1996 Nonclassical interferometry with intelligent light Phys. Rev. A 54 4505–18
[35] Hofmann H F 2009 All path-symmetric pure states achieve their maximal phase sensitivity in conventional two-path interferometry Phys. Rev. A 79 033822
[36] García-Díaz M, Egloff D and Plenio M B 2016 A note on coherence power of n-dimensional unitary operators Quantum Info Comput. 16 1282–94