On quantum Hall effect: Covariant derivatives, Wilson lines, gauge potentials, lattice Weyl transforms, and Chern numbers

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Abstract
We show that the gauge symmetry of the nonequilibrium quantum transport of Chern insulator in a uniform electric field is governed by the Wilson line of parallel transport operator coupled with the dynamical translation operator. This is dictated by the minimal coupling of derivatives with gauge fields in $U(1)$ gauge theory. This parallel transport symmetry consideration leads to the integer quantum Hall effect in electrical conductivity obtained to first-order gradient expansion of the nonequilibrium quantum transport equations.

1 Introduction
In previous papers [1, 2, 3], we make use of the gapped energy-band structure of solids under external electric field to derive the integer quantum Hall effect (IQHE) of Chern insulator. We employ the real-time superfield and lattice Weyl transform nonequilibrium Green’s function (SFLWT-NEGF) [4] quantum transport formalism [5, 6] to the first-order gradient expansion to derive the topological Chern number of the IQHE for two-dimensional systems, as an integral multiple of quantum conductance, also known as the minimal contact conductance in mesoscopic physics [4].

We find that the quantization of Hall effect occurs strictly not to first order in the electric field per se but rather to first-order gradient expansion in the nonequilibrium quantum transport equation. The Berry connection and Berry curvature is the fundamental physics [7] behind the exact quantization of Hall
conductance in units of $e^2/h$, which also happens to coincide with the source and drain contact conductance per spin in a closed circuit of mesoscopic quantum transport [4].

In Ref. [1], we have shown that the $(p,q,E,t)$ phase-space is renormalized to that of $(\vec{K}, \varepsilon)$ phase-space, where
\begin{align}
\vec{K} &= \vec{p} + e\vec{F}t, \\
\varepsilon &= E_0 + e\vec{F} \cdot \vec{q}
\end{align}
(1)
and
\begin{align}
\vec{K} &= \vec{p} + e\vec{F}t, \\
\varepsilon &= E_0 + e\vec{F} \cdot \vec{q}
\end{align}
(2)

Here, the uniform electric field, $\vec{F}$, is in the $x$-direction, and the Hall current is in the $y$-direction.

We have identified the topological invariant in $(\vec{K}, \varepsilon)$ phase-space nonequilibrium quantum transport equation leading to IQHE. Moreover, the formula is also applicable to gapped Landau-level structure of a free electron gas in intense magnetic field [8] since the variable $\vec{K}$ can incorporates the external vector potential, and its corresponding parallel transport, if present. Note that the change of variables from $\vec{K}$ to $\vec{p}$ in the integration over the whole Brillouin zone has a Jacobian unity. It was shown in previous paper [1] that the correct expression for the IQHE conductivity given by
\[
\sigma_{yx} = \frac{e^2}{h} \sum_\alpha \Delta \phi_{\text{total}} \frac{2\pi}{2\pi} = \sum_\alpha \frac{e^2}{h} n_\alpha
\]

In the present paper, we show that Eqs. (1) and (2) directly arise from the local gauge symmetry under a uniform electric field of localized Wannier functions centered on lattice sites. These follow from covariant derivatives leading to the dynamical spatio-temporal finite translation operators that is coupled with the Wilson line parallel-transport operators.

## 2 Translation operators in uniform electric fields

In gauge theory, the minimally-coupled gauge covariant derivative is defined as
\[
D_\mu = \partial_\mu + i\alpha^\mu A_\mu
\]
where $A_\mu$ is the electromagnetic four "vector" potential, and $\alpha^\mu$ is the serve as a coupling "charge". In electrodynamics, $\alpha^{x,y,z} = \frac{e}{hc}$, whereas $\alpha^0 = \frac{e}{h}$. Therefore a finite translation operator $T (q_\mu)$ by the four coordinates $q_\mu$ can be written as
\[
T (q_\mu) = \exp [q_\mu (D_\mu)] = \exp [q_\mu (\partial_\mu + i\alpha^\mu A_\mu)]
\]
where we have used the Einstein summation convention.
2.1 Wilson lines

The part of $T(q_{\mu})$ given by $\exp [q_{\mu} (i\alpha^{\mu} A_{\mu})]$ represent the Wilson lines. In general, for non-Abelian gauge theories, the Wilson line is often written as,

$$W_C = \mathcal{P} \exp \left[ i \int_{q=0}^{q} A_{\mu} dx^{\mu} \right]$$

where $\alpha$’s are absorbed in $A_{\mu}$, $\mathcal{P}$ is the path ordering operator since the $A_{\mu}$’s are generally non-Abelian and do not commute, e.g., like the Pauli matrices.

The Wilson loop is defined as the average of the trace of Wilson lines around a closed loop, where the average is taken using the Chern-Simons Lagrangian which we will not go into. Moreover, we will not be dealing with non-Abelian $A_{\mu}$’s and our Wilson lines are explicitly gauge invariant, being expressed in terms of the gauge-field strength.

The importance of the Wilson line as a parallel transport operator in the gauge independent formulation of gauge theories has been emphasized by Mandelstam [9] and further developed by Wu and Yang [10]. The term given by $\int A_{\mu} dx^{\mu}$ is the electromagnetic Berry phase. Precisely, the definition of the Berry phase $\phi$ reads

$$\exp i\phi \left| \psi (x + q_x, t + T) \right\rangle = T(q_{\mu}) \left| \psi (x, t) \right\rangle = \exp [q_{\mu} (\partial_{\mu} + i\alpha^{\mu} A_{\mu})] \left| \psi (x, t) \right\rangle$$

$$= \exp \left\{ [q_{x} \cdot \partial_{x} + T\partial_{t}] + i\phi \right\} \left| \psi (x, t) \right\rangle = \exp \left\{ \frac{i}{\hbar} \int_{0}^{T} \hat{P} \cdot dx - i\hat{H} (t') dt' \right\} \left| \psi (x, t) \right\rangle$$

where $\hat{P}$ is the momentum operator, $\hat{H}$ is the energy operator, $A_{\mu}$ is the electromagnetic Berry connection. The last line serves as a generalization in terms of quantum mechanical operators, $\hat{P}$ and $\hat{H}$. The extra phase acquired in translation given by

$$\phi = \int \alpha^{\mu} A_{\mu} dx^{\mu}$$

is the geometrical phase. In this paper, it is identified with the generalized Peierls phase factor in energy band dynamics of solid state physics. In quantum physics, the adiabatic contour integral of Eq. (4) is now generally known as the Berry phase and $A_{\mu}$ is now generally known mathematically as the connection.

2.1.1 Peierls phase factor

The factor $\exp i\phi$ on the left side of Eq. (3) is the Peierls-phase factor well-known in energy band dynamics of solids [11].
2.2 Gauge potential in uniform electric fields

In this section, we determined the gauge potential \( A_\mu = (A_0, A_x, A_y, A_z) = (A_0, \vec{A}) \), where \( A_0 \) is the time component. We have

\[
\vec{A} = \vec{F}ct
\]

where \( \vec{F} \) is the uniform electric field. Clearly, \( \alpha^\mu A_\mu = \frac{\epsilon}{\hbar c} \vec{A} \) has the same units as \( \vec{\partial} \) and \( \frac{\epsilon}{c} \vec{A} \) has the same dimensional units as the momentum, i.e.,

\[
\hat{P} + \frac{\epsilon}{c} \vec{A}
\]

represents the minimal coupling in quantum electrodynamics.

For \( A_0 \), we have

\[
\alpha^0 A_0 = -\frac{1}{\hbar} e \vec{F} \cdot \vec{q}
\]

has the same units as \( \partial_t \) and hence from Eq. (3), we have

\[
\frac{1}{\hbar} \left( E_0 + e \vec{F} \cdot \vec{q} \right)
\]

represents the coupling of the zero-field energy, \( E_0 \), to the electric field at arbitrary lattice site \( \vec{q} \). Note that the negative sign of Eq. (5) is dictated by the equation for the force as the negative gradient of the potential, namely,

\[
e \vec{F} = -\nabla \left( -e \vec{F} \cdot \vec{q} \right)
\]

Therefore under uniform electric fields, we have the Wilson lines for our Abelian gauge potential given by

\[
W_C = \exp \left[ i \int \alpha^\mu A_\mu dx^\mu \right] = \exp \left[ i \left( \alpha^0 A_0 t + \alpha \vec{A} \cdot \vec{q} \right) \right] = \exp \left[ i \left\{ -\frac{1}{\hbar} e \vec{F} \cdot \vec{q} \right\} t + \left\{ \frac{e}{\hbar c} \vec{F}ct \cdot \vec{q} \right\} \right]
\]

Thus, the Wilson lines serves as the generalization of Peierls phase factor in solid state physics.

3 Alternative derivation of gauge potentials and Peierls phase factor

In contrast to the derivation given above, which directly use the covariant derivative in \( U(1) \) gauge theory, here we will use a self-consistent Heisenberg equation for the spatio-temporal displacement operators to determine the gauge
potentials and hence the generalized Peierls phase factor. Remarkably, the self-consistent Heisenberg equation of motion automatically fixes the negative sign of the scalar gauge of Eq. (5).

Note that finite displacement operators are characteristically exponential operators amenable to Fourier series expansion. Remarkably, phase factors are generally acquired due to displacement or motion in parameter space in the presence of electromagnetic fields. This is ubiquitous in solid-state physics (e.g., Peierls phase factor in magnetic fields) before the Berry connection become mainstream and fashionable. To begin, we write our bare Hamiltonian of electrons as

\[ H = H_o - e \vec{F} \cdot \vec{r}, \]

where \( H_o \) is the periodic Hamiltonian in the absence of the electric field, \( \vec{F} \).

In this section, we want to show that the displacement operator,

\[ \hat{T}(q) = \exp \left[ i \frac{\hbar}{\epsilon} (\vec{q} \cdot -i \hbar \nabla \vec{r}) \right], \]

acquires a phase factor in the presence of electric field, and is given by

\[ \tilde{T}(q) = \exp \left[ i \frac{\hbar}{\epsilon} \left( \vec{q} \cdot \left( \frac{\epsilon}{c} \vec{F} c t + (\vec{q} \cdot -i \hbar \nabla \vec{r}) \right) \right], \]

\[ = \exp \left[ i \frac{\hbar}{\epsilon} \left( \vec{q} \cdot \left( \frac{\epsilon}{c} \vec{F} c t + (\vec{q} \cdot \hat{P}) \right) \right] \], \quad (6) \]

where we indicate by \( \tilde{T} \) the translation operator \( \hat{T} \) with a phase factor.

Similarly, for the time displacement operator, we will show that

\[ \hat{T}(t) = \exp \left[ -i \frac{\hbar}{\epsilon} \left( t \cdot i \hbar \frac{\partial}{\partial t} \right) \right], \quad (7) \]

also acquires a phase factor and is given by

\[ \tilde{T}(t) = \exp \left[ -i \frac{\hbar}{\epsilon} \left( t \left( \epsilon \vec{F} \cdot \vec{q} \right) + (t \mathcal{H}) \right) \right], \quad (8) \]

where \( \tilde{T}(t) \) differs from \( \hat{T}(t) \) by a phase factor, and \( i \hbar \frac{\partial}{\partial t} : = \mathcal{H} \) is the energy operator of the system. In other words, displacement in space by lattice vector \( q \) acquires phase factors equal to \( \exp \left( i \frac{\hbar}{\epsilon} \vec{F} \cdot \vec{q} \right) \). Similarly, displacement in time by \( t \) acquires phase factor equal to \( \exp \left( -i \frac{\hbar}{\epsilon} \left( \epsilon \vec{F} \cdot \vec{q} \right) t \right) \). In the absence of the electric field these phase factors give unity, e.g., \( \hat{T}(t) \Rightarrow_{\vec{F} \rightarrow 0} \hat{T}(t) \), reduces to ordinary translation operator.

The physics behind these phase factors is dictated by a selfconsistent translation of local functions in space and time under a uniform electric field, \( \vec{F} \). These follow from the self-consistent Heisenberg equation of motion for quantum operators. For efficient bookkeeping and for ease taking lattice Weyl transform,
it is usually more convenient to attach these phase factors to displaced local functions themselves, Eq. (3). This yields a generalization of Peierls phase factor, well-known for solid-state problems in magnetic fields. The derivation goes as follows.

### 3.0.1 Nonlocality in coordinates

The nature of the derivatives of exponential displacement operator is determined, e.g., by the following operation,

\[
i\hbar \frac{\partial}{\partial t} \hat{T} (q, r - 0) = i\hbar \frac{\partial \phi}{\partial t} \hat{T} (q, r - 0),
\]

\[
\frac{\partial}{\partial t} \hat{T} (q) = \frac{\partial \phi}{\partial t} \hat{T} (q),
\]

\[
\frac{d\hat{T}}{\hat{T} (q)} = d\phi,
\]

where specifically \( \hat{T} (-q) \) is a translation of the center coordinate of a localized Wannier function centered in the origin to another lattice point \( q \), yielding \( W_\lambda (r - q) \). Therefore \( \hat{T} (-q) \) resembles the physical process of transferring a localized function centered at the origin to a localized function centered at another lattice point \( q \). Here we define total \( \phi \) as undetermined for the moment as

\[
\phi = -i \frac{\hbar}{\hbar} \left( f (q, t) + (\vec{q} \cdot \hat{P}) \right),
\]

where \( f (q, t) \) is to be determined. Note that the presence of \( f (q, t) \) is needed for selfconsistency in the presence of electric field. Now consider the second term in the exponent, namely,

\[
\phi_2 = -i \frac{\hbar}{\hbar} \vec{q} \cdot \hat{P} = -i \frac{\hbar}{\hbar} \left( -i \hbar \frac{\partial}{\partial \vec{r}} \right),
\]

leading to Fourier series expansion of the translation, \( \hat{T} (-q) \). Then, we obtain

\[
-i \hbar \frac{\partial \phi_2}{\partial t} = [H, \phi_2] = \frac{i}{\hbar} \left[ \frac{\partial}{\partial \vec{r}}, \left( -i \hbar \frac{\partial}{\partial \vec{r}} \right) \right] = e \vec{F} \cdot (-\vec{q}),
\]

(9)

since \( -i \hbar f (q, t) \) commutes with the Hamiltonian. Therefore, we have

\[
\frac{d\hat{T} (-q)}{\hat{T} (q)} = d\phi = \frac{i}{\hbar} e \vec{F} \cdot (-\vec{q}) dt.
\]

(10)

We may thus write

\[
-i \hbar \frac{\partial}{\partial t} \hat{T} (q, t) = [H, \hat{T} (q, t)] = e \vec{F} \cdot (\vec{q}) \hat{T} (\vec{q}, t).
\]

(11)
Therefore
\[
\hat{T}(q) = \exp \left( \frac{i}{\hbar} \left[ \left( e\vec{F}\Delta t \right) \cdot \vec{q} + \hat{P} \cdot \vec{q} \right] \right),
\]
\[= \exp \left( \frac{i}{\hbar} \left[ \hat{P} + e\vec{F}t \right] \cdot \vec{q} \right).\] (12)

This means that a displacement by \(\vec{q}\) of localized function acquires a phase factor given by
\[
\text{Peierls phase factor} = \exp \left( \frac{i\hbar}{\hbar} \left( \hat{P} + e\vec{F}t \right) \cdot \vec{q} \right) = \exp \left( \frac{i\hbar}{\hbar} \right) \left( e\vec{F}t \cdot \vec{q} \right).\] (13)

We can also deduce from Eq. (9) the relation for the momentum operator,
\[
\frac{i}{\hbar} [\hat{H}, \tilde{T}(\vec{q})] = e\vec{F} \cdot \vec{q} \tilde{T}(\vec{q}),
\]
\[= \frac{\partial \hat{P}}{\partial t} \cdot (q), \implies \frac{\partial \hat{P}}{\partial t} = e\vec{F},
\]
\[\Rightarrow \hat{P} = \hat{P}_0 + e\vec{F}t.\] (14)

which indicates a covariant derivative in spatial coordinates,
\[
\vec{D}_\mu = \partial_\mu + ie\hbar\vec{A}
\]
with
\[
\vec{A}_\mu = \vec{F}ct
\]
as obtained before.

3.0.2 Simultaneous eigenvalues for \(\hat{H}\) and \(\tilde{T}(\vec{q}, t)\)

One very important conclusion is implied in Eq. (11), which we rewrite here for convenience
\[
\left[ \hat{H}, \tilde{T}(\vec{q}, t) \right] = e\vec{F} \cdot \vec{q} \tilde{T}(\vec{q}, t).\] (15)

What the above relation means is that if \(\tilde{T}(\vec{q}, t)\) is diagonal then \([\hat{H}, \tilde{T}(\vec{q}, t)]\) is also diagonal. But if \(\tilde{T}(\vec{q}, t)\) is diagonal, then \(\hat{H}\) is also diagonal with the same eigenvalues. The eigenfunction of \(\tilde{T}(\vec{q}, t)\) is labeled by the quantum label \(\vec{K} = \vec{p}_0 + e\vec{F}t\). This implies that \(\hat{H}\) is also diagonal in \(\vec{K}\). The electric Bloch function labeled by \(B(k_0 + \frac{e}{\hbar}k\vec{F}t, \cdot, \cdot)\) is the eigenfunction of \(\tilde{T}(\vec{q}, t)\) as well as that of the renormalized
\[
H_{\text{renormalized}} \left( \vec{K}, \vec{Q} \right) \iff W_n \left( \vec{K}, \mathcal{E} \right),\] (16)

where the double pointed arrow denotes lattice Weyl correspondence, see Eq. (27) below, using the localized electric Wannier function.

We now show the the energy variable \(\mathcal{E}\) in Eq. (16) does incorporate the coordinates through the gauge potential \(A_0\).
Because nonlocal arguments in time acquire phase factor also, the energy variable of the theory now varies with $\mathbf{eF} \cdot \mathbf{q}$, as discussed next. From Eq. (7) for a displacement in time,

$$\hat{T}(t) = \exp \left( \frac{i}{\hbar} (\mathcal{H}t) \right)$$

or

$$\hat{T}(t) \equiv \mathcal{F} \exp \left( \frac{-i}{\hbar} \left( \mathbf{f}(q,t) + \mathbf{eF} \cdot \mathbf{n} \right) \right) = \mathcal{F} \exp \left( \frac{i}{\hbar} \left( \mathbf{f}(q,t) \right) \right),$$

where $\mathcal{F}$ incorporates the scalar gauge potential, $A_0$. Let

$$\mathcal{F} = \exp \left[ i\alpha A_0 t \right] = \exp \left[ -\frac{i}{\hbar} \left( f_t (q,t) \right) \right],$$

is a phase factor to be determined. The total phase is

$$\phi_t (q) = -\frac{i}{\hbar} \left( f_t (q,t) + t \left( \mathbf{i} \frac{\partial}{\partial t} \right) \right).$$

From the equation of motion,

$$-i\hbar \frac{\partial \hat{T}(t)}{\partial q} = -i\hbar \frac{\partial \phi_t (q)}{\partial q} \hat{T}(t),$$

$$-i\hbar \frac{\partial \phi_t (q)}{\partial q} = \mathbf{Re} \left[ \mathbf{f} (t'), t \left( \mathbf{i} \frac{\partial}{\partial t'} \right) \right] = \left[ \mathbf{p} + \frac{\mathbf{eF}}{c} \mathbf{n} \right] = -e\mathbf{F} t.$$

Therefore

$$-i\hbar \frac{\partial \phi_t (q)}{\partial q} = -e\mathbf{F} t,$$

$$\frac{\partial \phi_t (q)}{\partial q} = -\frac{i}{\hbar} e\mathbf{F} t.$$

Thus, we obtained,

$$\frac{\partial \hat{T}(t)}{\partial q} = \left( -\frac{i}{\hbar} e\mathbf{F} t \right) \hat{T}(t),$$

and hence,

$$d \ln \hat{T}(t) = \left( -\frac{i}{\hbar} e\mathbf{F} t \right) \cdot dq,$$

$$\ln \hat{T}(t) = \left( -\frac{i}{\hbar} e\mathbf{F} t \right) \cdot \Delta q.$$
Hence a displacement in time carries a phase factor given by \( \exp \left( -\frac{i}{\hbar} (e\vec{F} \cdot \Delta \vec{q}) \right) \) and
\[
\tilde{T}(t) = \exp \left\{ -\frac{i}{\hbar} \left[ t (e\vec{F} \cdot \Delta \vec{q}) + t \left( i\hbar \frac{\partial}{\partial t'} \right) \right] \right\} \\
= \exp \left\{ -\frac{i}{\hbar} \left[ t (e\vec{F} \cdot \vec{q}) + t \left( i\hbar \frac{\partial}{\partial t'} \right) \right] \right\} \tag{20}
\]
Once more this gives support to the covariant derivative,
\[
D_0 = \partial_t + i\alpha^0 A_0 
\]
where
\[
i\alpha^0 A_0 = - \frac{e\vec{F} \cdot \vec{q}}{\hbar} 
\]
Therefore,
\[
-i\hbar \frac{\partial}{\partial \vec{q}} \tilde{T}(\vec{q}, t) = \left[ \vec{K}(t'), \tilde{T}(\vec{q}, t) \right] = \left( -e\vec{F}t \right) \tilde{T}(\vec{q}, t) \tag{21}
\]
Now from second line of Eq. (19), we have
\[
-\frac{i}{\hbar} \left[ \vec{K}(t'), (t) i\hbar \frac{\partial}{\partial t'} \right] = \left[ \vec{p} + e\vec{F}t', (t) \frac{\partial}{\partial t'} \right] = -e\vec{F}t, \\
\left[ \vec{K}(t'), \mathcal{H} \right] = -i\hbar e\vec{F} = -i\hbar \frac{\partial \mathcal{E}}{\partial \vec{q'}},
\]
which leads to the expression for in Eq. (17) for \( f_t \) in the phase factor as
\[
\frac{\partial \mathcal{E}}{\partial \vec{q}} = e\vec{F} \implies f_t = e\vec{F} \cdot \vec{q}.
\]
All these results, Eq. (12) and Eq. (20), lead to the time-dependent wave vector,
\[
\hbar \vec{k} = \hbar \vec{k}_0 + he\vec{F}t
\]
and to the position-dependent energy,
\[
\mathcal{E} = \mathcal{E}_0 + e\vec{F} \cdot \vec{q}, \tag{22}
\]
respectively. Again, for for taking lattice Weyl transform, it is more convenient to attach these phase factor to the displaced local functions themselves. This allows us to generalize the Peierls phase factor to space and time displacements, originally well-known for solid-state problems for magnetic fields.

As an example, for nonequilibrium translational symmetric and steady-state condition.
\[
\langle q_1, t_1 | \mathcal{O} | q_2, t_2 \rangle = e^{i\pi \vec{F} t_1 (\vec{q}_2 - \vec{q}_1)} e^{-i\pi \vec{F} t_2 (\vec{q}_2 - \vec{q}_1)} \mathcal{O}(\vec{q}_2 - \vec{q}_1, t_2 - t_1) \tag{23}
\]
where
\[
\vec{q} = \frac{1}{2} (\vec{q}_1 + \vec{q}_2), \\
t = \frac{1}{2} (t_1 + t_2)
\]
4 Lattice Weyl transforms

Using the four dimensional notation: \( p = (\vec{p}, E) \) and \( q = (q, t) \), the lattice Weyl transform (LWT), \( A(p, q) \) of any operator \( \hat{A} \) is defined by

\[
A_{\lambda\lambda'}(p, q) = \sum_v e^{i\pi p \cdot v} \langle q - v, \lambda | \hat{A} | q + v, \lambda' \rangle = \sum_u e^{i\pi q \cdot u} \langle p + u, \lambda | \hat{A} | p - u, \lambda' \rangle
\]  

(24)

Writing Eq. (24) explicitly, we have

\[
A_{\lambda\lambda'}(\vec{p}, \vec{q}; E, t) = \sum_{\vec{v}; \tau} e^{i\pi \vec{p} \cdot \vec{v}} e^{-i\pi \vec{E} \cdot \vec{q} (t_1 - t_2)} A(\vec{q}_1 - \vec{q}_2, t_1 - t_2) = e^{i\pi \vec{F} \cdot \vec{q}_2} e^{-i\pi \vec{E} \cdot \vec{q} \tau} A_{\lambda\lambda'}(\vec{q}_1 - \vec{q}_2, t_1 - t_2).
\]

(25)

Using the form of matrix elements in Eq. (23), we have

\[
\langle \vec{q} - \vec{v}; t - \frac{\tau}{2}, \lambda | \hat{A} | \vec{q} + \vec{v}; t + \frac{\tau}{2}, \lambda' \rangle = e^{i\pi \vec{F} \cdot (\vec{q}_1 - \vec{q}_2)} e^{-i\pi \vec{E} \cdot (\vec{q}_1 - \vec{q}_2) \tau} A_{\lambda\lambda'}(\vec{q}_1 - \vec{q}_2, t_1 - t_2).
\]

(26)

Thus

\[
A_{\lambda\lambda'}(\vec{p}, \vec{q}; E, t) = \sum_{\vec{v}; \tau} e^{i\pi \vec{p} \cdot \vec{v}} e^{i\pi \vec{F} \cdot (2\vec{v})} e^{-i\pi \vec{E} \cdot (\vec{q}_1 - \vec{q}_2) \tau} A_{\lambda\lambda'}(\vec{q}_1 - \vec{q}_2, t_1 - t_2),
\]

\[
= \sum_{\vec{v}; \tau} e^{i\pi \vec{v} \cdot \vec{p}} e^{i\pi \vec{F} \cdot (2\vec{v})} e^{-i\pi \vec{E} \cdot (\vec{q} + \vec{v}) \tau} A_{\lambda\lambda'}(\vec{q}_1 - \vec{q}_2, t_1 - t_2),
\]

\[
= A_{\lambda\lambda'} \left[ \left( \vec{p} + e \vec{F} t \right), \left( E + e \vec{F} \cdot \vec{q} \right) \right],
\]

\[
= A_{\lambda\lambda'} \left( \vec{K}; \mathcal{E} \right).
\]

(27)

Hence the expected dynamical variables in the phase space including the time variable occurs in particular combinations of \( \vec{K} \) and \( \mathcal{E} \). Therefore, besides the crystal momentum varying in time as

\[
\vec{K} = \vec{p}_0 + e \vec{F} t,
\]

the energy variable vary with \( \vec{q} \) as

\[
\mathcal{E} = E_0 + e \vec{F} \cdot \vec{q}.
\]

(28)

(29)

This is the result of covariant derivatives for energy band dynamics in the presence of uniform electric fields. Thus, differentiation with respect to coordinate and time variables are now relegated to differentiation with respect to energy \( \mathcal{E} \)
and $\vec{K}$, respectively,

$$\frac{\partial}{\partial t} = \frac{\partial \vec{K}}{\partial t} \cdot \frac{\partial}{\partial \vec{K}}, \quad (30)$$

$$\frac{\partial}{\partial \vec{q}} = \frac{\partial \vec{E}}{\partial \vec{q}} \cdot \frac{\partial}{\partial \vec{E}}, \quad (31)$$

$$\frac{\partial}{\partial \vec{K}} = \frac{\partial \vec{E}}{\partial \vec{K}} \cdot \frac{\partial}{\partial \vec{E}}, \quad (32)$$

where $v_g$ is the group velocity. The LWT of the effective or renormalized lattice Hamiltonian $H_{\text{eff}} \Rightarrow H(\vec{p}, \vec{q}; E_0, t)$ can therefore be analyzed on $(\vec{K}, E)$-space as

$$H(\vec{p}, \vec{q}; E_0, t) \Rightarrow H(\vec{K}, E). \quad (33)$$

The last line is by virtue of Eqs. (28)–(29). Of course in the absence of the electric field, the dependence in phase space becomes the familiar $H(\vec{p}, \omega)$ for translationally symmetric and steady-state system. But with $\vec{F} \neq 0$ all gauge invariant quantities are functions of $(\vec{K}, E)$ such as the electric Bloch function [11, 6] or Houston wavefunction [12] and electric Wannier function, i.e., the electric-field dependent generalization of Wannier function. In particular, the Weyl transform of a commutator,

$$W[H, G^\prec] = \sin \Lambda \left\{ H(\vec{K}, E) G^\prec (\vec{K}, E) \right\}, \quad (34)$$

where $\Lambda$ is the Poisson bracket operator. We can therefore write the Poisson bracket operator $\Lambda$, as

$$\Lambda = \frac{\hbar}{2} \left[ \frac{\partial^{(a)} \partial^{(b)}}{\partial t \partial E} - \frac{\partial^{(a)} \partial^{(b)}}{\partial E \partial t} \right],$$

$$= \frac{\hbar}{2} \frac{\partial \vec{K}}{\partial t} \cdot \left[ \frac{\partial^{(a)} \partial^{(b)}}{\partial \vec{K} \partial E} - \frac{\partial^{(a)} \partial^{(b)}}{\partial E \partial \vec{K}} \right],$$

$$= \frac{\hbar}{2} e^F \cdot \left[ \frac{\partial^{(a)} \partial^{(b)}}{\partial \vec{K} \partial E} - \frac{\partial^{(a)} \partial^{(b)}}{\partial E \partial \vec{K}} \right]. \quad (35)$$

on $(\vec{K}, E)$-phase space.

5 Application to nonequilibrium quantum transport equation

The ballistic phase-space quantum transport equation [11] generally reads,

$$\frac{\partial}{\partial t} G^\prec (\vec{p}, \vec{q}; E, t) = \frac{2}{\hbar} \sin \Lambda \left\{ H(p, q) G^\prec (p, q) \right\} \quad (36)$$
where in the right side of Eq. \((6)\) the 4-dimensional notation of phase space is employed. Under a uniform electric field, this simplifies to

\[
\frac{\partial}{\partial t} G^< \left( \vec{\kappa}, \mathcal{E} \right) = \sin \Lambda \ \left\{ H \left( \vec{\kappa}, \mathcal{E} \right) G^< \left( \vec{\kappa}, \mathcal{E} \right) \right\}
\]

(37)

If we expand Eq. \((36)\) to first-order in the gradient, i.e., \(\sin \Lambda \approx \Lambda\), the phase-space transport equation \((4)\) can be written in a compact form as

\[
\frac{\partial}{\partial t} G^< \left( \vec{\kappa}, \mathcal{E} \right) = \frac{2 \hbar}{\alpha^2} e F_y \cdot \left[ \frac{\partial^{(a)} \partial^{(b)}}{\partial \kappa_x^a \partial \mathcal{E}^b} - \frac{\partial^{(a)} \partial^{(b)}}{\partial \mathcal{E}^a \partial \kappa_x^b} \right] H^{(a)} \left( \vec{\kappa}, \mathcal{E} \right) G^{<(b)} \left( \vec{\kappa}, \mathcal{E} \right).
\]

(38)

With the electric field in the \(x\)-direction, then we have

\[
G^< \left( \vec{\kappa}, \mathcal{E} \right) = e \left| F_y \right| \int dt \left[ \frac{\partial^{(a)} \partial^{(b)}}{\partial \kappa_x^a \partial \mathcal{E}^b} - \frac{\partial^{(a)} \partial^{(b)}}{\partial \mathcal{E}^a \partial \kappa_x^b} \right] H^{(a)} \left( \vec{\kappa}, \mathcal{E} \right) G^{<(b)} \left( \vec{\kappa}, \mathcal{E} \right).
\]

(39)

The Hall current in the \(y\)-direction is given by the following equation,

\[
J_y = \frac{a^2}{(2\pi\hbar)} \left| \int d\vec{\kappa}_x d\vec{\kappa}_y \left( \frac{e}{a^2} \frac{\partial \mathcal{E}}{\partial \vec{\kappa}_y} \right) \left( -i G^< \left( \vec{\kappa}, \mathcal{E} \right) \right) \right|
\]

which leads to \(J_y = \sigma_{yx} \left| F_y \right|\),

\[
\sigma_{yx} = e^2 \frac{1}{(2\pi\hbar)^2} \int \int d\vec{\kappa}_x d\vec{\kappa}_y dt \times \left[ \frac{\partial^{(a)} \partial^{(b)}}{\partial \kappa_x^a \partial \kappa_y^b} - \frac{\partial^{(a)} \partial^{(b)}}{\partial \kappa_y^a \partial \kappa_x^b} \right] H^{(a)} \left( \vec{\kappa}, \mathcal{E} \right) \left( -i G^{<(b)} \left( \vec{\kappa}, \mathcal{E} \right) \right). \quad (40)
\]

By reverting to its equivalent matrix element expression and integrating with respect to time, we obtain

\[
\sigma_{yx} = \frac{e^2}{\hbar} \sum_{\alpha} f \left( E_{\alpha} \right) \frac{i}{(2\pi)} \int \int dk_x dk_y \left[ \left\langle \alpha, \frac{\partial}{\partial \vec{k}_x}, \vec{\kappa}, \mathcal{E} \right| \left\langle \alpha, \frac{\partial}{\partial \kappa_y}, \vec{\kappa}, \mathcal{E} \right\rangle - \left\langle \alpha, \frac{\partial}{\partial \kappa_y}, \vec{\kappa}, \mathcal{E} \right| \left\langle \alpha, \frac{\partial}{\partial \vec{k}_x}, \vec{\kappa}, \mathcal{E} \right\rangle \right]. \quad (41)
\]

At low temperature, we can just write Eq. \((41)\) as,

\[
\sigma_{yx} = \frac{ie^2}{2\pi\hbar} \left( \frac{1}{\alpha} \right) \int \int_{\text{occupied BZ}} dk_x dk_y \left[ \nabla_{\vec{\kappa}} \times \left\langle \alpha, \vec{k}, \frac{\partial}{\partial \vec{k}_x}, \vec{\kappa} \right| \left\langle \alpha, \vec{k}, \frac{\partial}{\partial \kappa_y} \right\rangle \right]_{\text{plane}},
\]

\[
= \frac{e^2}{2\pi\hbar} \left( \frac{i}{\alpha} \right) \int dk_z \left[ \left\langle \alpha, \vec{k}, \frac{\partial}{\partial k_z} \right| \left\langle \alpha, \vec{k}, \frac{\partial}{\partial \kappa} \right\rangle \right]_{\text{contour}}. \quad (42)
\]

where \(\left\langle \alpha, \vec{k}, \frac{\partial}{\partial \kappa} \right| \left\langle \alpha, \vec{k} \right\rangle\) is the Berry connection in a band and \(\int dk_z \left[ \left\langle \alpha, \vec{k}, i\frac{\partial}{\partial k_z} \right| \left\langle \alpha, \vec{k} \right\rangle \right]\) is the Berry phase. Thus, Eq. \((42)\) yields

\[
\sigma_{yx} = \sum_{\alpha} \frac{e^2}{\hbar} n_{\alpha}
\]

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6 Concluding remarks

In this paper, we have shown that the direct route to generalized Peierls phase factor or Wilson lines for crystalline solid under uniform electric fields is embodied in the use of covariant derivatives (representing minimal coupling in $U(1)$ gauge theory) in the finite translation operators. The present calculation thus readily leads to the phase-space topological invariant in nonequilibrium quantum transport equation which yields the IQHE of electrical conductivity upon reverting to equivalent matrix elements expression [1].

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