A Proposal of Positive-Definite Local Gravitational Energy Density in General Relativity

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Abstract

We propose a 4-dimensional Kaluza-Klein approach to general relativity in the (2,2)-splitting of space-time using the double null gauge. The associated Lagrangian density, implemented with the auxiliary equations associated with the double null gauge, is equivalent to the Einstein-Hilbert Lagrangian density, since it yields the same field equations as the E-H Lagrangian density does. It is describable as a (1+1)-dimensional Yang-Mills type gauge theory coupled to (1+1)-dimensional matter fields, where the minimal coupling associated with the infinite dimensional diffeomorphism group of the 2-dimensional spacelike fibre space automatically appears. The physical degrees of freedom of gravitational field show up as a (1+1)-dimensional non-linear sigma model in our Lagrangian density. Written in the first-order formalism, our Lagrangian density directly yields a non-zero local Hamiltonian density, where the associated time function is the retarded time. From this Hamiltonian density, we obtain a positive-definite local gravitational energy density. In the asymptotically

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flat space-times, the volume integrals of the proposed local gravitational energy density over suitable 3-dimensional hypersurfaces correctly reproduce the Bondi mass and the ADM mass expressed as surface integrals at null and spatial infinity, respectively, supporting our proposal. We also obtain the Bondi mass-loss formula as a negative-definite flux integral of a bilinear in the gravitational currents at null infinity.

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I. INTRODUCTION

The exact correspondence of the Euclidean self-dual Einstein’s equations to the equations of motion of 2-dimensional non-linear sigma models with the target space as the area-preserving diffeomorphism of 2-surface [1,2] has inspired us to look further into the intriguing question whether the full-fledged general relativity of the 4-dimensional space-time can be also formulated as a certain (1+1)-dimensional field theory. Recently we have shown that such a description is indeed possible, and constructed the action principle [3–6] in the framework of the 4-dimensional Kaluza-Klein theory in the (2,2)-splitting. In this approach, the 4-dimensional space-time, at least for a finite range of space-time, is viewed as a fibred manifold that consists of the (1+1)-dimensional “space-time” and the 2-dimensional “auxiliary” fibre space.

There are certain advantages of this 4-dimensional KK approach to general relativity in the above splitting, which led us to develop this formalism. We list a few of them. First of all, in (1+1)-dimensions, there exist a number of field theoretic methods recently developed thanks to the string-related theories. Hopefully the rich mathematical methods in (1+1)-dimensions might also prove useful in studying general relativity in the (2,2)-splitting, classical and quantum. Moreover, in this KK formulation, general relativity can be viewed as a (1+1)-dimensional gauge theory with the prescribed interactions and auxiliary equations. Since the major advantage of gauge theory formulation is that gauge invariant quantities automatically solve Gauss-law equations associated with the gauge invariance, the problem of solving constraints of general relativity could be made even trivial, at least for some of them. Furthermore, this formulation allows us to forget about the space-time picture of general relativity; instead, it enables us to study general relativity much the way as we do for Yang-Mills theories coupled to matter fields in (1+1)-dimensions, putting the space-time physics into a new perspective.

This (1+1)-dimensional method, however, is not entirely new since it was virtually used in analyzing gravitational waves [7,8], and was further developed in the spin-coefficient
formalism \[9–12\] and the null hypersurface formalism \[13–19\]. In these formalisms, a special
gauge which we may call the double null gauge is chosen such that two real dual null
vector fields whose congruences span the (1+1)-dimensional submanifold are singled out,
and the Einstein’s equations are spelled out in that gauge. A characteristic feature of these
formalisms, among others, lies in that the true physical degrees of freedom of gravitational
field show up in the conformal 2-geometry of the transverse 2-surface \[8,13–17,20\]. This
feature, that has been particularly useful for studying the propagation of gravitational waves
in the asymptotically flat space-times, further motivated the canonical analysis \[18,21–26\]
of the null hypersurface formalism, in the hope of getting quantum theory of gravity by
quantizing the true physical degrees of freedom of gravitational field.

In view of these advantages of the KK formalism and the null hypersurface formalism, it
therefore seems worth combining both formalisms to see what could be learnt more about
general relativity. In this article, we shall present such a formalism. In this approach,
it is the (1+1)-dimensional submanifold spanned by two real dual null vector fields that
we imagine as “space-time” and the remaining transverse 2-surface as the “auxiliary” fibre
space. As a by-product of our KK approach in the double null gauge, we obtain a new result
which we report in this article. Namely, we propose a positive-definite local gravitational
energy density in general relativity without referring to the boundary conditions, and show
that the volume integrals of the proposed energy density over suitably chosen 3-dimensional
hypersurfaces correctly yield the Bondi and the ADM surface integral at null and spatial
infinity in the asymptotically flat space-times, respectively. We also obtain the Bondi mass-
loss formula as a negative-definite flux integral of the gravitational degrees of freedom at
null infinity \[7,8,11\]. The proposed local gravitational energy density comes directly from
the local Hamiltonian density of general relativity described as the 4-dimensional KK theory
in the double null gauge. The associated time function is the retarded time, and has the
physical interpretation \[27\] as the phase of the local gravitational radiation in situations
where gravitational waves are present.

This article is organized as follows. In section II, we present the 4-dimensional KK theory
in the (2,2)-splitting, using the double null gauge. It will be seen that, even when the gauge
symmetry is an infinite dimensional symmetry such as the group of diffeomorphisms, the
KK idea is still useful by showing that the KK variables transform properly as gauge fields
and tensor fields under the corresponding gauge transformations \[3–6, 28\]. Next, we present
the Lagrangian density for general relativity in the double null gauge, which is implemented
by the auxiliary equations associated with the double null gauge using the Lagrange multi-
pliers. This Lagrangian density is equivalent to the Einstein-Hilbert Lagrangian density,
since it yields the same field equations as the E-H Lagrangian density does. We shall present
the 10 Einstein’s equations in the double null gauge. Moreover, we shall find that 2 of these
10 equations are in a form of the Schrödinger equation, reminiscent of the Brill wave equa-
tion \[29–31\], i.e. the initial value equation of the axi-symmetric gravitational waves at the
moment of time symmetry.

In section III, we shall present this Lagrangian density in the first-order formalism,
with the retarded time identified as our clock variable. This immediately leads to the local
Hamiltonian density and thus to the local gravitational energy density that we are interested
in. The proposed local gravitational energy density is positive-definite. We shall further show
that the volume integral of the proposed local gravitational energy density over a suitably
chosen 3-dimensional hypersurface become a surface integral, using the vacuum Einstein’s
equations and the Bianchi identities. In the asymptotically flat space-times, this surface
integral becomes the Bondi and the ADM surface integral defined at null and spatial infinity,
respectively. We also derive the Bondi mass-loss formula from the proposed gravitational
energy density.

In Appendix A, we shall describe the general (2,2)-splitting of space-time, and present
the resulting E-H Lagrangian density without picking up a special gauge, as we need it when
we wish to obtain the field equations from the variational principle. In Appendix B, we shall
introduce the covariant null tetrads, as we shall use them in Appendix C where we show
that the proposed volume integrals in section III can be expressed as surface integrals. In
Appendix D, we shall show that \(κ_±^2\), which appears when we discuss the Bondi mass-loss, is
positive-definite.

**II. THE LAGRANGIAN DENSITY IN THE DOUBLE NULL GAUGE**

In this section we combine the null hypersurface formalism with the 4-dimensional Kaluza-Klein approach where space-time is viewed as a fibred manifold, i.e. a local product of the (1+1)-dimensional base manifold and the 2-dimensional fibre space. Let the vector fields \( \partial/\partial X^A = (\partial/\partial u, \partial/\partial v, \partial/\partial y^a) \) \((a = 2, 3)\) span the 4-dimensional space-time. In a Lorentzian space-time we consider here, there always exist two real null vector fields, which we may choose orthogonal to the 2-dimensional spacelike surface \( N_2 \) spanned by \( \partial/\partial y^a \). Following the KK idea \([32]\), the two null vector fields can be represented as the linear combinations of these basis vector fields

\[
\frac{\partial}{\partial u} - A^a_+ \frac{\partial}{\partial y^a}, \quad \text{and} \quad \frac{\partial}{\partial v} - A^a_- \frac{\partial}{\partial y^a}, \tag{2.1}
\]

for some functions \( A^a_\pm(u, v, y) \). Since these null vector fields are assumed to be normal to \( N_2 \), the line element may be written in a manifestly symmetric way as follows;

\[
ds^2 = -2dudv + \phi_{ab}(A^a_+du + A^-adv + dy^a)(A^b_+du + A^-bdv + dy^b), \tag{2.2}
\]

where \( \phi_{ab}(u, v, y) \) is the 2-dimensional metric on \( N_2 \). Notice that, as a consequence of picking up two null vector fields normal to \( N_2 \), 2 out of the 10 metric coefficient functions were gauged away in (2.2). In addition, one more function was removed from (2.2) by choosing the coordinate \( v \) such that \( Cdv' = dv \) for some function \( C \), i.e. by choosing \( C = 1 \). The elimination of these 3 functions may be viewed as a partial gauge-fixing of the space-time diffeomorphism, and may be better understood in terms of the dual metric, which we may write

\[
g^{++} = g^{--} = 0, \quad g^{+-} = g^{-+} = -1, \quad g^{+a} = A^a_-, \quad g^{-a} = A^a_+, \quad g^{ab} = \phi^{ab} - 2A^a_+A^-_. \tag{2.3}
\]
That $g^{++} = g^{--} = 0$ means that $du$ and $dv$ are dual null vector fields, and that $g^{+-} = -1$ is a normalization condition for $v$, given an arbitrary function $u$. We shall call this gauge as the double null gauge[1], and general relativity formulated in this gauge is referred to as the double null formalism [13–17]. The 3-dimensional hypersurface defined by $u = \text{constant}$ is a null hypersurface since it is metrically degenerate; within each null hypersurface the 2-dimensional spacelike space $N_2$ defined by $v = \text{constant}$ is transverse to both $du$ and $dv$.

In order to see whether this 4-dimensional KK program is justifiable in the absence of any Killing symmetry, as is the case here, we have to first examine the transformation properties of $\phi_{ab}$ and $A_\pm^a$ in (2.2) under the action of some group of transformations associated with $N_2$. The most natural group of transformations associated with $N_2$ is the diffeomorphisms of $N_2$, i.e. $\text{diff}N_2$. Under the $\text{diff}N_2$ transformation

$$y^a' = y^a(u,v,y), \quad u' = u, \quad v' = v,$$

these fields must transform as

$$\phi'_{ab}(u,v,y') = \frac{\partial y^c}{\partial y'^a} \frac{\partial y^d}{\partial y'^b} \phi_{cd}(u,v,y),$$

$$A'_\pm^a(u,v,y') = \frac{\partial y'^a}{\partial y^c} A^c_\pm(u,v,y) - \partial_\pm y'^a,$$

so that the line element $ds^2$ remains invariant [13,14]. Under the corresponding infinitesimal transformation

$$\delta y^a = \xi^a(u,v,y), \quad \delta u = \delta v = 0,$$

where $\xi^a$ is an arbitrary function, we find

$$\delta \phi_{ab} = -[\xi, \phi]_{ab},$$

$$\delta A^a_\pm = -D_\pm \xi^a = -\partial_\pm \xi^a + [A_\pm, \xi]^a,$$

1We notice that this double null gauge is valid only for a finite range of space-time. See for instance [13,14,30].
where $A_\pm := A_\pm^a \partial_a$ and $\xi := \xi^a \partial_a$. Here the brackets are the Lie derivatives associated with $\text{diff} N_2$,

$$\left[ \xi, \phi \right]_{ab} = \xi^c \partial_c \phi_{ab} + (\partial_a \xi^c) \phi_{cb} + (\partial_b \xi^c) \phi_{ac},$$ (2.8a)

$$[A_\pm, \xi]^a = A_\pm^c \partial_c \xi^a - \xi^c \partial_c A_\pm^a.$$ (2.8b)

This observation tells us two things. First, $\text{diff} N_2$ is the residual symmetry \cite{11} of the line element (2.2) which survives even after the double null gauge was chosen. Second, $\text{diff} N_2$ should be viewed as a local\footnote{Local means local in the (1+1)-dimensional “space-time”.} gauge symmetry of the Yang-Mills type, since $A_\pm^a$ and $\phi_{ab}$ transform as a gauge field and a tensor field under the $\text{diff} N_2$ transformations, respectively.

This feature is rather surprising, since the KK variables were often thought to be useful for higher dimensional gravity theories where the degrees of freedom associated with the extra dimensions are suppressed in one way or another\footnote{See however \cite{28,33}.}. In our 4-dimensional KK approach to general relativity, we leave all the “internal” degrees of freedom intact so that all the fields in (2.2) depend on all of the coordinates $(u, v, y^a)$. In spite of these generalities, the Lagrangian density associated with the metric (2.2) can be still identified, à la Kaluza-Klein, as a gauge theory Lagrangian density defined on the (1+1)-dimensional “space-time”, with $\text{diff} N_2$ as the associated local gauge symmetry, as we shall see shortly.

The metric (2.2) in the double null gauge can be obtained from the general KK line element

$$ds^2 = \gamma_{\mu\nu} dx^\mu dx^\nu + \phi_{ab} (A_\mu^a dx^\mu + dy^a) (A_\nu^b dx^\nu + dy^b),$$ (2.9)

where $\mu, \nu = 0, 1$ and $a, b = 2, 3$. From this we obtain (2.2) by introducing the retarded and advanced coordinate $(u, v)$ and the fields $A_\pm^a$,

$$u = \frac{1}{\sqrt{2}} (x^0 - x^1), \quad v = \frac{1}{\sqrt{2}} (x^0 + x^1),$$ (2.10a)

$$A_\pm^a = \frac{1}{\sqrt{2}} (A_0^a \mp A_1^a),$$ (2.10b)
assuming the (1+1)-dimensional “space-time” metric $\gamma_{\mu\nu}$ to be

$$
\gamma_{+-} = -1, \quad \gamma_{++} = \gamma_{--} = 0,
$$

(2.11)
in the $(u,v)$-coordinates, where $+(-)$ represents $u(-v)$. In Appendix A, the general E-H Lagrangian density [3–5] for the metric (2.9) and the prescription how to obtain it are presented. Using the “ansatz” (2.11), we can easily show that the general E-H Lagrangian density (A32) reduces to the following expression corresponding to the metric (2.2). If we neglect the auxiliary equations associated with the double null gauge (2.11) for the moment, it is given by [3]

$$
\mathcal{L}_0 = \sqrt{\phi} \left\{ \frac{1}{2} \phi_{ab} F_+^a F_+^b + \frac{1}{2} \phi^{ab} \phi^{cd} \left\{ (D_+ \phi_{ac})(D_- \phi_{bd}) - (D_+ \phi_{ab})(D_- \phi_{cd}) \right\} \right\},
$$

(2.12)

where we ignored the surface terms. Here $\phi = \det \phi_{ab}$, and $F_+^a$ is the diff $N_2$-valued field strength, and $D_\pm \phi_{ab}$ is the diff$N_2$-covariant derivative defined as

$$
F_+^a = \partial_+ A^a - \partial_- A_+^a - [A_+, A_-]^a, \quad (2.13a)
$$

$$
D_\pm \phi_{ab} = \partial_\pm \phi_{ab} - [A_\pm, \phi]_{ab}, \quad (2.13b)
$$

where $[A_+, A_-]^a$ and $[A_\pm, \phi]_{ab}$ are the Lie derivatives defined as

$$
[A_+, A_-]^a = A_+^c \partial_c A_-^a - A_-^c \partial_c A_+^a, \quad (2.14a)
$$

$$
[A_\pm, \phi]_{ab} = A_\pm^c \partial_c \phi_{ab} + (\partial_a A_\pm^c) \phi_{cb} + (\partial_b A_\pm^c) \phi_{ac}, \quad (2.14b)
$$

respectively (see Appendix A). Recall that, in the double null formalism of general relativity, the transverse 2-metric with a unit determinant, i.e. the conformal 2-geometry, is the two physical degrees of freedom of gravitational field [8,13–17]. Since we are essentially reformulating the double null formalism from the KK point of view in this article, we would like to see first of all what the Lagrangian density looks like when written in terms of the conformal 2-geometry. Let us therefore decompose the 2-metric $\phi_{ab}$ into the conformal classes

$$
\phi_{ab} = \Omega \rho_{ab}, \quad (\Omega > 0 \quad \text{and} \quad \det \rho_{ab} = 1), \quad (2.15)
$$
where $\rho_{ab}$ is the conformal 2-geometry\(^4\) of the transverse 2-surface $N_2$. If we define $\sigma$ by $\sigma := \ln \Omega$, the second term in (2.12) becomes

$$
K := \frac{1}{2} \sqrt{\phi} \phi^{cd} \left\{ (D_+ \phi_{ac}) (D_- \phi_{bd}) - (D_+ \phi_{ab}) (D_- \phi_{cd}) \right\} \\
= -\Omega^{-1} (D_+ \Omega) (D_- \Omega) + \frac{1}{2} \Omega \rho^{ab} \rho^{cd} (D_+ \rho_{ac}) (D_- \rho_{bd}) \\
= -e^\sigma (D_+ \sigma) (D_- \sigma) + \frac{1}{2} e^\sigma \rho^{ab} \rho^{cd} (D_+ \rho_{ac}) (D_- \rho_{bd}),
$$

(2.16)

where $D_\pm \Omega$, $D_\pm \sigma$, and $D_\pm \rho_{ab}$ are the diff$N_2$-covariant derivatives

$$
D_\pm \Omega = \partial_\pm \Omega - [A_\pm, \Omega],
$$

(2.17a)

$$
D_\pm \sigma = \partial_\pm \sigma - [A_\pm, \sigma],
$$

(2.17b)

$$
D_\pm \rho_{ab} = \partial_\pm \rho_{ab} - [A_\pm, \rho]_{ab},
$$

(2.17c)

and $[A_\pm, \Omega]$, $[A_\pm, \sigma]$, and $[A_\pm, \rho]_{ab}$ are given by

$$
[A_\pm, \Omega] = A_\pm^a \partial_a \Omega + (\partial_a A_\pm^a) \Omega,
$$

(2.18a)

$$
[A_\pm, \sigma] = A_\pm^a \partial_a \sigma + \partial_a A_\pm^a,
$$

(2.18b)

$$
[A_\pm, \rho]_{ab} = A_\pm^c \partial_c \rho_{ab} + (\partial_a A_\pm^c) \rho_{cb} + (\partial_b A_\pm^c) \rho_{ac} - (\partial_c A_\pm^c) \rho_{ab},
$$

(2.18c)

respectively. Here $\partial_a A_\mu^a$-terms are included in the Lie derivatives, since $\Omega$ and $\rho_{ab}$ are tensor densities of weight $-1$ and +1 under diff$N_2$, respectively. Thus (2.12) becomes

$$
\mathcal{L}_0 = \frac{1}{2} e^{2\sigma} \rho_{ab} F_+^{a} F_-^{b} \ - e^\sigma (D_+ \sigma) (D_- \sigma) + \frac{1}{2} e^\sigma \rho^{ab} \rho^{cd} (D_+ \rho_{ac}) (D_- \rho_{bd}).
$$

(2.19)

In order to find the correct variational principle that yields all the 10 Einstein’s equations, however, we must implement (2.19) with the auxiliary equations associated with the double

\(^4\)This may be viewed as a finite analogue of the physical transverse traceless degrees of freedom of the spin-2 fields propagating in the flat space-time \([14]\). The double null gauge may be also viewed as an analogue of the Coulomb gauge in Maxwell’s theory, where the physical degrees of freedom are the transverse traceless vector potentials \([34]\).
null gauge. These equations can be found by first varying the general E-H Lagrangian density (A32) with respect to \(\gamma^{++}\), \(\gamma^{--}\), and \(\gamma^{+-}\), and then plugging the double null gauge (2.11) into the resulting equations. They are found to be

\[
C_{\pm} := D_{\pm}^2 \sigma + \frac{1}{2} (D_{\pm} \sigma)^2 + \frac{1}{4} \rho^{ab} \rho^{cd} (D_{\pm} \rho_{ac}) (D_{\pm} \rho_{bd}) = 0, \tag{2.20a}
\]

\[
C_0 := -\frac{1}{2} e^\sigma \rho_{ab} F_+^a F_+^b + D_+ D_- \sigma + D_- D_+ \sigma + 2 (D_+ \sigma) (D_- \sigma) + R_2
\]

\[
= 0, \tag{2.20b}
\]

respectively, where \(R_2 := \phi^{ac} R_{ac}\) is the scalar curvature of \(N_2\). Remarkably, the equations \(C_{\pm} = 0\) in (2.20a) may be written in a form of the Schrödinger equation. Using the identity

\[
e^\sigma \left( D_{\pm}^2 \sigma + \frac{1}{2} (D_{\pm} \sigma)^2 \right) = 2 \ D_{\pm}^2 e^{\sigma/2}, \tag{2.21}\]

the equations \(C_{\pm} = 0\) become

\[
D_{\pm}^2 e^{\sigma/2} + \kappa_{\pm}^2 e^{\sigma/2} = 0, \quad \text{where} \quad \kappa_{\pm}^2 := \frac{1}{8} \rho^{ab} \rho^{cd} (D_{\pm} \rho_{ac}) (D_{\pm} \rho_{bd}) \geq 0. \tag{2.22}\]

That \(\kappa_{\pm}^2\), a bilinear in the currents of the gravitational degrees of freedom, is positive-definite can be shown easily (see Appendix D). The equations (2.22) are the analogues of the Brill wave equation\(^5\) [29–31], as they are of the Schrödinger equation type for a wave function corresponding to a state of zero energy in the potential \(-\kappa_{\pm}^2\), coupled to the external gauge fields \(A_\pm^a\). Thus, viewed as a scattering problem, the scattering data \(e^{\sigma/2}\) in (2.22) is an auxiliary field that can be determined by the potential \(-\kappa_{\pm}^2\) up to

\(^5\)Our Schrödinger equations look like one-dimensional wave equations coupled to gauge fields, but actually they are 3-dimensional partial differential equations like the Brill wave equation, due to the Lie derivatives along \(A_\pm = A_\pm^a \partial_a\) in (2.22). The wave function in (2.22) is related to the conformal factor of the 2-dimensional wavefront, the spatial projection of the null hypersurface \(u = \text{constant}\), rather than that of spacelike hypersurface. But it should be also mentioned that, for the metric (2.2), the area measure of the 2-dimensional wavefront and the volume measure of 3-dimensional null hypersurface \(u = \text{constant}\) are the same [33].
some integral “constant” functions. The generic behaviors of solutions of the 2 Einstein’s equations \( C_{\pm} = 0 \) are therefore expected either of the scattering type, or of the bound-state or resonance type, corresponding to the asymptotically flat space-times or spatially closed universes, respectively, on a par with the Brill wave equation.

The correct variational principle is now given by

\[
\mathcal{L} = \frac{1}{2}e^{2\sigma}\rho_{ab}F_{+}^{a}F_{-}^{b} - e^{\sigma}(D_{+}\sigma)(D_{-}\sigma) + \frac{1}{2}e^{\sigma}\rho^{ab}\rho^{cd}(D_{+}\rho_{ac})(D_{-}\rho_{bd}) + \sum_{\alpha=\pm,0} \lambda^{\alpha}C_{\alpha}, \tag{2.23}
\]

where \( \lambda^{\alpha}\)’s are the Lagrange multipliers which should be put to zero after variation. The equations of motions for \( A_{\pm}^{a}, \sigma, \) and \( \rho_{ab} \) (subject to \( \det \rho_{ab} = 1 \)) can be obtained by varying (2.23), with \( \lambda^{\alpha} = 0 \). Here we present the results only;

\[
(a) \quad D_{-}(e^{2\sigma}\rho_{ab}F_{+}^{b}) + e^{\sigma}(D_{-}\sigma)(\partial_{a}\sigma) - \partial_{a}(e^{\sigma}D_{-}\sigma) - \frac{1}{2}e^{\sigma}\rho^{bc}\rho^{de}(D_{-}\rho_{bd})(\partial_{a}\rho_{ce}) + \partial_{b}(e^{\sigma}\rho^{bc}D_{-}\rho_{ac}) = 0; \tag{2.24a}
\]

\[
(b) \quad D_{+}(e^{2\sigma}\rho_{ab}F_{+}^{b}) - e^{\sigma}(D_{+}\sigma)(\partial_{a}\sigma) + \partial_{a}(e^{\sigma}D_{+}\sigma) + \frac{1}{2}e^{\sigma}\rho^{bc}\rho^{de}(D_{+}\rho_{bd})(\partial_{a}\rho_{ce}) - \partial_{b}(e^{\sigma}\rho^{bc}D_{+}\rho_{ac}) = 0; \tag{2.24b}
\]

\[
(c) \quad (D_{+}\sigma)(D_{-}\sigma) + 2D_{+}\rho_{ac}(D_{-}\rho_{bd}) + e^{\sigma}\rho_{ab}F_{+}^{a}F_{-}^{b} = 0; \tag{2.24c}
\]

\[
(d) \quad D_{+}(e^{\sigma}\rho^{ac}D_{-}\rho_{bc}) - \frac{1}{2}e^{2\sigma}(\rho_{bc}F_{-}^{a}F_{-}^{c} - \frac{1}{2}\delta_{bc}^{a}\rho_{de}F_{-}^{+}F_{-}^{d}) = 0. \tag{2.24d}
\]

where the symmetric symbol is normalized such that \((\alpha\beta) = (\alpha\beta + \beta\alpha)/2\). Together with the 3 equations \( C_{\pm} = 0, C_{0} = 0 \) that we obtain by varying (2.23) with respect to \( \lambda^{\pm}, \lambda^{0} \), these field equations are identical to the 10 Einstein’s equations spelled out in the double null gauge, which we obtain by first varying the general E-H Lagrangian density (A32) in Appendix A, and then imposing the double null gauge (2.11). Therefore the Lagrangian density (2.23) is equivalent to the general E-H Lagrangian density (A32), with the understanding that \( \lambda^{\alpha}\)’s are to be set to zero after variation.

The Lagrangian density (2.23) may be naturally interpreted as the Yang-Mills type Lagrangian density on the (1+1)-dimensional “space-time”, interacting with the (1+1)-dimensional “matter” fields \( \sigma \) and \( \rho_{ab} \). The corresponding local gauge symmetry is the
built-in diff$N_2$, and the “matter” fields couple to the diff$N_2$-valued gauge fields through the minimal couplings. In addition, each term in (2.23), including the auxiliary equations, is manifestly invariant under the diff$N_2$ transformations. Therefore (2.23) should be duly regarded as a gauge theory formulation of the vacuum general relativity [4–6].

That $\rho_{ab}$ is the physical degrees of freedom can be also seen as follows. In this (1+1)-dimensional interpretation, the diff$N_2$-valued gauge fields $A_\pm^a$ are auxiliary fields since they have no propagating (i.e. no transverse traceless) degrees of freedom. Moreover, as we have seen already, $\sigma$ is also an auxiliary field that is determined by $\rho_{ab}$ through the equations $C_\pm = 0$. This confirms that the two physical degrees of freedom of gravitational field are indeed contained in $\rho_{ab}$. It seems appropriate to notice here that, in the propagating equations of motion (2.24d) for $\rho_{ab}$, the source term is given by

$$\frac{1}{2} e^{2\sigma} \rho_{bc} F_{+-}^a F_{+-}^c,$$

(2.25)

whose trace is precisely the local gravitational energy density, as we shall see in the next section. This indicates that the local energy density in general relativity indeed plays the analogous role as the local charge density does in Maxwell’s theory.

III. THE LOCAL GRAVITATIONAL ENERGY DENSITY

In this section, we shall find the local Hamiltonian density of general relativity. This can be obtained simply by writing the local Lagrangian density (2.23) in the first-order form using a suitable time coordinate. The most natural time in this formulation seems the retarded time $u$ [11,27]. With the retarded time as our clock, the first term in (2.23) may be written as

$$\mathcal{L}_{YM} := \frac{1}{2} e^{2\sigma} \rho_{ab} F_{+-}^a F_{+-}^b$$

$$= e^{2\sigma} \rho_{ab} F_{+-}^b \left( \partial_+ A_-^a - \partial_- A_+^a - [A_+, A_-]^a - \frac{1}{2} F_{+-}^a \right).$$

(3.1)

In terms of the phase space variables $(\Pi_a, A_-^a)$, where $\Pi_a$ is defined as
\[ \Pi_a = e^{2\sigma} \rho_{ab} F_{+-}^b \]  

(3.2)

this can be written as

\[ \mathcal{L}_{YM} = \Pi_a \partial_+ A_-^a - \frac{1}{2} e^{-2\sigma} \rho^{ab} \Pi_a \Pi_b + A_+^a D_- \Pi_a, \]  

(3.3)

ignoring the surface terms. Here \( D_- \Pi_a \) is the \( \text{diffN}_2 \)-covariant derivative of the density \( \Pi_a \) defined as

\[ D_- \Pi_a = \partial_- \Pi_a - [A_-, \Pi]_a, \]  

(3.4)

where \([A_-, \Pi]_a \) is the Lie derivative of \( \Pi_a \),

\[ [A_-, \Pi]_a = A_-^c \partial_c \Pi_a + (\partial_a A_-^c) \Pi_c + (\partial_c A_-^a) \Pi_a. \]  

(3.5)

The second and third term in (2.23) are already in the first-order form, apart from the terms proportional to \( A_+^a \) whose variation yields the Gauss-law equations associated with the residual \( \text{diffN}_2 \) invariance. Putting these all together, the Lagrangian density (2.23) can be written in the following Hamiltonian form\(^6\)

\[ \mathcal{L} = \Pi_a \partial_+ A_-^a - e^\sigma (D_- \sigma)(\partial_+ \sigma) + \frac{1}{2} e^\sigma \rho^{ab} \rho^{cd} (D_- \rho_{bd})(\partial_+ \rho_{ac}) \]

\[ - \frac{1}{2} e^{-2\sigma} \rho^{ab} \Pi_a \Pi_b + A_+^a C_a + \sum_{\alpha = \pm, 0} \lambda^\alpha C_\alpha, \]  

(3.6)

where \( C_a \) is given by

\[ C_a = D_- \Pi_a + e^\sigma (D_- \sigma) (\partial_a \sigma) - (\partial_a (e^\sigma D_- \sigma)) - \frac{1}{2} e^\sigma \rho^{bc} \rho^{de} (D_- \rho_{bd})(\partial_a \rho_{ce}) \]

\[ + \partial_b (e^\sigma \rho^{bc} D_- \rho_{ac}), \]  

(3.7)

which is the same as (2.24a) if we use (3.3). From (3.6) the local gravitational Hamiltonian density \( \mathcal{H} \) is given by

\[^6\text{Notice that the Hessian of this Lagrangian density is zero, so that the usual method of Hamiltonization does not work. This is the reason that we did not introduce the momenta conjugate to} \sigma \text{ and} \rho_{ab}. \text{See for instance [36–39].} \]
\[ \mathcal{H} = \frac{1}{2} e^{-2\sigma} \rho^{ab} \Pi_a \Pi_b - A_+^a C_a - \sum_{\alpha = \pm, 0}^{} \lambda^\alpha C_\alpha. \]  

(3.8)

Using the 5 equations \( C_a = 0, \ C_\pm = 0, \) and \( C_0 = 0, \) the local Hamiltonian density (3.8) becomes

\[ \mathcal{E} = \frac{1}{2} e^{-2\sigma} \rho^{ab} \Pi_a \Pi_b = \frac{1}{2} e^{2\sigma} \rho^{ab} F_+^a F_+^b \geq 0, \]

(3.9)

which is positive-definite for any \( \sigma \) and \( \Pi_a, \) since the conformal 2-metric \( \rho^{ab} \) has a positive-definite signature. Thus, at least formally, we have obtained, in the double null gauge, a positive-definite local gravitational energy density for the vacuum general relativity! The time function associated with this non-zero local energy density is the retarded time \( u \) that we may choose at will\(^7\). This is our proposal of the positive-definite local gravitational energy density in this article. This seems to be against the usual argument that, in general relativity, local gravitational energies can not be defined because they can be always “transformed” away due to the equivalence principle, let alone the positive-definiteness. In our definition of the local gravitational energy density, however, the “field strength” \( F_+^a \) in (2.13a) is the coefficient of the commutator of the two null vector fields \( \partial_\pm - A_\pm^a \partial_a, \) which measures the twist of the parallelogram made of two successive parallel transports of these null vector fields along each other. Certainly the twist of this null parallelogram can not be “transformed” away even in a local Lorentz frame, and thus can serve as a measure of gravitational energy associated with the parallelogram surrounding the space-time point under consideration. The proposed local gravitational energy density is just the square of this local “field strength” multiplied by the canonical integration measure.

In order to appreciate what this really means, however, we have to first define the volume integral of the local gravitational energy density \( \mathcal{E} \) over a 3-dimensional hypersurface defined

\[^7\]For the asymptotically flat space-times, the number of possible choices of the retarded time \( u \) is equal to the number of an arbitrary, monotonically increasing function of three variables \( (u, y^a) \)

[11]. This may be true for other space-times as well.
by $u = \text{constant}$, and evaluate it for the asymptotically flat space-times, since the total gravitational energy is well-defined only for the asymptotically flat space-times. As we now show, for the asymptotically flat space-times, the volume integrals of the proposed local energy density over suitably chosen 3-dimensional hypersurfaces can be re-expressed as the Bondi and ADM surface integral at null and spatial infinity, respectively. We shall also derive the Bondi mass loss-formula as a negative-definite flux integral of a bilinear in the gravitational currents at null infinity.

### A. The Bondi Mass

In this subsection we wish to show that, for the asymptotically flat space-times, the volume integral of (3.9) over the $u = \text{constant}$ null hypersurface is precisely the Bondi mass as measured at null infinity. Let us first notice that the volume integral

$$
E = \frac{1}{2} \int d^2y \, e^{2\sigma} \rho_{ab} F_+^a F_-^b \geq 0,
$$

is positive-definite for any topology of $N_2$. In order to express (3.10) as a surface integral, it is necessary to write it in a slightly different form using the field equations. For this let us consider the following identity

$$
D_+ D_- \sigma - D_- D_+ \sigma = -F_+^a \partial_a \sigma - \partial_a F_+^a.
$$

Using (3.11), the integral of the equation $C_0 = 0$ in (2.20) over $N_2$ with the integration measure $e^\sigma$ may be written as

$$
\frac{1}{2} \int d^2y \, e^{2\sigma} \rho_{ab} F_+^a F_-^b = \int d^2y \, e^\sigma \left\{ R_2 + 2(D_+ \sigma)(D_- \sigma) + 2D_+ D_- \sigma \right\} + \int d^2y \, \partial_a (e^\sigma F_+^a).
$$

The last term in (3.12) is zero for any 2-surface $N_2$ that we assume compact without boundary. Thus (3.10) becomes

$$
E = \int d^2y \, e^\sigma \left\{ R_2 + 2(D_+ \sigma)(D_- \sigma) + 2D_+ D_- \sigma \right\}.
$$
Let us also integrate the equation (2.24c) over $N_2$, using (3.11), to obtain

$$
\int d^2 y \ e^\sigma \{(D_+\sigma)(D_-\sigma) + 2D_+D_-\sigma\}
= - \int d^2 y \ e^{2\sigma} \rho_{ab} F^a_+ F^b_+ - \frac{1}{2} \int d^2 y \ e^\sigma \rho^{ab} \rho^{cd}(D_+\rho_{ac})(D_-\rho_{bd})
- \int d^2 y \ \partial_a(e^\sigma F^a_+)
,$$

(3.14)

where the last term may be also dropped. Thus the volume integral (3.13) becomes

$$
E = \int dv d^2 y \ e^\sigma \{R^2 + (D_+\sigma)(D_-\sigma) - \frac{1}{2} \rho_{ab} \rho^{cd}(D_+\rho_{ac})(D_-\rho_{bd})\}
- \int dv d^2 y \ e^{2\sigma} \rho_{ab} F^a_+ F^b_+,
$$

(3.15)

or,

$$
E = \frac{1}{2} \int dv d^2 y \ e^{2\sigma} \rho_{ab} F^a_+ F^b_+
= \frac{1}{3} \int dv d^2 y \ e^\sigma \{R^2 + (D_+\sigma)(D_-\sigma) - \frac{1}{2} \rho_{ab} \rho^{cd}(D_+\rho_{ac})(D_-\rho_{bd})\}.
$$

(3.16)

To show that this can be expressed as a surface integral, the covariant null tetrad notation that we described in Appendix B is useful. Let us notice that the Gauss equation in the (2,2)-splitting of space-time is given by

$$
R^2 + (D_+\sigma)(D_-\sigma) - \frac{1}{2} \rho_{ab} \rho^{cd}(D_+\rho_{ac})(D_-\rho_{bd}) = h^{AC} h^{BD} C_{ABCD},
$$

(3.17)

where $C_{ABCD}(A, B, \cdots = 0, 1, 2, 3)$ is the conformal curvature tensor, and $h_{AB}$ is the 2-metric on the transverse surface $N_2$, i.e. the covariant form of $\phi_{ab}$. Then the volume integral $E$ may be written as

---

8Notice that this energy integral is different from the one in Hayward’s paper [40]. The energy density he proposed is the minus of $L_0$ in (2.19), modulo the Euler density, and is not positive-definite. This difference may be traced back to the fact that in his paper both $u$ and $v$ are treated as the time variables.

9Here we used the vacuum Einstein’s equations.
\[ E = \frac{1}{3} \int d\nu d\varphi \, e^{\sigma} h^{AC} h^{BD} C_{ABCD}. \]  
\hfill (3.18)

In Appendix C, we have shown that, using the Bianchi identity
\[ \nabla_{[M} C_{AB]CD} = 0, \]  
\hfill (3.19)
this can be expressed as the surface integral
\[ E = \frac{1}{3} \lim v \int d\nu d\varphi \, e^{\sigma} h^{AC} h^{BD} C_{ABCD}, \]  
\hfill (3.20)
where \( \lim \) means that the integral over \( N_2 \) is to be evaluated at the limiting boundary value(s) of \( v \). This expression picks up the coefficient of \( 1/v \)-term, and becomes precisely the Bondi mass\(^10\) \[ [40–43] \] in the limit as \( v \) approaches to infinity! Notice that the parameter \( v/\sqrt{2} \) becomes the area radius in the limit \( v \to \infty \) (keeping \( u = u_0 = \text{constant} \)), as our metric (2.2) approaches to
\[ ds^2 \to -2d\nu dv + \frac{1}{2}(v - u)^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \]  
\hfill (3.21)
at null infinity.

B. The Bondi Mass-Loss Formula

In this subsection we shall continue to obtain the Bondi mass-loss formula in the presence of the gravitational radiation in the asymptotically flat space-times. For this, we may simply take a \( u \)-derivative of the integral (3.20), using suitable vacuum Einstein’s equations. However, since we wish to account for the mass-loss in terms of the physical degrees of freedom \( \rho_{ab} \), we shall work with the volume integral (3.13).

Recalling that \( \text{diff}\, N_2 \) is the residual gauge symmetry of the metric (2.2), we may fix this symmetry by choosing \( A_+^a = 0 \)\(^11\) by a suitable coordinate transformation on \( N_2 \). Then the

\(^{10}\) We have a factor of \( 1/3 \), which could be taken care of by a suitable normalization.

\(^{11}\) This is equivalent to the assumption that \( \partial/\partial u \) is a twist-free null vector field \[ [4,5,44] \]. However, there could be some topological obstructions against globalizing this choice.
equation $C_+ = 0$ in (2.22) reduces to the following Schrödinger equation

$$\partial^2 e^{\sigma/2} + \kappa_+^2 e^{\sigma/2} = 0,$$

where $\kappa_+^2 := \frac{1}{8} \rho^{ab} \rho^{cd} (\partial_+ \rho_{ac})(\partial_+ \rho_{bd}) \geq 0$. (3.22)

In the following we shall use this equation when we examine the rate of change in $E$ as the retard time $u$ advances. Let us further notice that the metric (2.2) becomes, in the gauge $A_+^a = 0$,

$$ds^2 = -2dudv + e^\sigma \rho_{ab}(A_+^a dv + dy^a)(A_+^b dv + dy^b).$$ (3.23)

With $y^a = (\vartheta, \varphi)$, where $\vartheta, \varphi$ are the angles of $S_2$, we find by comparing (3.23) with (3.21) the following asymptotic behaviors of the metric as $v$ approaches to infinity,

$$e^\sigma = O(v^2), \quad \rho_{ab} = O(1), \quad A_+^a = O(1/v^2).$$ (3.24)

In the gauge $A_+^a = 0$, the volume integral (3.13) becomes

$$E = \int dv^2 y e^\sigma \{ R_2 + 2(\partial_+ \sigma)(D_- \sigma) + 2\partial_+ D_- e^\sigma \}
= \int dv^2 y \{ e^\sigma R_2 + 2\partial_+ D_- e^\sigma \},$$ (3.25)

where we used the identity

$$\partial_+ D_- e^\sigma = e^\sigma (\partial_+ \sigma)(D_- \sigma) + e^\sigma \partial_+ D_- \sigma.$$ (3.26)

The first term in (3.25) is the $v$-integration of the Euler number $\chi$, where

$$\chi = \frac{1}{4\pi} \int d^2 y e^\sigma R_2 = 2, 0, -2(g - 1),$$ (3.27)

for $N_2 = S_2, T_2$, and the 2-surface $\Sigma_g$ of genus $g$, respectively. Since the $u$-derivative of the Euler integral is zero, the rate of change in $E$ as $u$ increases comes from the second term in (3.25). For the asymptotically flat space-times, we may assume $N_2 = S_2$ so that

---

\(^{12}\)This gauge choice is only for convenience, since we already obtained the covariant expression of the Bondi mass in the previous subsection.
\[\frac{dE}{du} = 2 \int_{S^2} d^2 y \, \partial^2_+ (D_- e^\sigma)\]
\[= 2 \int_{v=\infty, S^2} d^2 y \, (\partial^2_+ e^\sigma) - 2 \int_{v=v_0, S^2} d^2 y \, (\partial^2_+ e^\sigma) \left(1 + O(v_0^{-1})\right),\]
(3.28)

where the domain of the \(v\)-integration was chosen from \(v_0\) to \(\infty\), and we used the asymptotic behaviors (3.24). Here \(v_0\) is some point that lies sufficiently far away from the sources of gravitational waves along the out-going null direction such that the gravitational waves are contained entirely in the range \(v_0 < v \leq \infty\) at the instant \(u = \) constant. In this asymptotic region, we may also assume the out-going null condition [7,8,11],
\[e^\sigma = \frac{1}{2} v^2 \sin \vartheta \left\{1 + \frac{f(u, \vartheta, \varphi)}{v} + O\left(\frac{1}{v^2}\right)\right\},\]
(3.29)

for some function \(f(u, \vartheta, \varphi)\). From this, it is found that
\[\partial^2_+ e^\sigma = 2 e^{\sigma/2} \partial^2_+ e^{\sigma/2} \left(1 + O(1/v)\right)\]
\[= -\frac{1}{4} e^\sigma \rho^{ab} \rho^{cd} (\partial_+ \rho_{ac})(\partial_+ \rho_{bd}) \left(1 + O(1/v)\right),\]
(3.30)

where in the second line we used the Schrödinger equation (3.22). Thus \(dE/du\) becomes
\[\frac{dE}{du} = -\frac{1}{2} \lim_{v \to \infty} \int_{S^2} d^2 y \, e^\sigma \rho^{ab} \rho^{cd} (\partial_+ \rho_{ac})(\partial_+ \rho_{bd})\]
\[+ \frac{1}{2} \int_{v=v_0, S^2} d^2 y \, e^\sigma \rho^{ab} \rho^{cd} (\partial_+ \rho_{ac})(\partial_+ \rho_{bd}) \left(1 + O(v_0^{-1})\right).\]
(3.31)

Since there are no propagating gravitational degrees of freedom in the region \(v \leq v_0\) the \(currents\) of gravitational waves must vanish in this region, so that
\[\rho^{ab} \partial_+ \rho_{ac} = 0 \quad \text{for} \quad v \leq v_0.\]
(3.32)

Thus the second term in (3.31) vanishes, and we finally have
\[\frac{dE}{du} = -\frac{1}{2} \lim_{v \to \infty} \int_{S^2} d^2 y \, e^\sigma \rho^{ab} \rho^{cd} (\partial_+ \rho_{ac})(\partial_+ \rho_{bd}) \leq 0.\]
(3.33)

Apart from the integration measure \(e^\sigma\), this flux integral over \(S^2\) at null infinity is expressed entirely in terms of the physical degrees of freedom, and is negative-definite. This is precisely the Bondi mass-loss formula! It must be stressed that the gravitational energy carried
away to null infinity by the gravitational radiation is given in a bilinear combination of the
gravitational currents, in excellent accordance with our experience that observables are very
often expressed in bilinears of the physical fields. This strongly supports the view held by
the geometric quantization school [36–38] that, in general relativity, as in other non-linear
field theories, it is the current rather than the conformal 2-geometry that should be regarded
as the fundamental physical field.

The total radiated gravitational energy to null infinity between the null time interval \( u_0 \)
and \( u \) can be obtained by integrating (3.33), and is given by

\[
E(u) - E(u_0) = -\frac{1}{2} \lim_{v \to \infty} \int_{u_0}^{u} du \int_{S^2} d^2y \, e^{\sigma} \rho^{ab} \rho^{cd} (\partial_+ \rho_{ac})(\partial_+ \rho_{bd}).
\]  

(3.34)

C. The ADM Mass

Now we shall show that the volume integral of the proposed local gravitational energy
density (3.10) over a spacelike hypersurface reproduces the ADM surface integral. Let us
make the following coordinate transformation

\[
r = -\frac{1}{2} u + v.
\]  

(3.35)

In the new coordinates \((u, r, y^a)\), the metric (2.2) becomes

\[
ds^2 = -du^2 - 2dudr + e^{\sigma} \rho_{ab} \left\{ (A_+^a + \frac{1}{2} A_-^a)du + A_-^a dr + dy^a \right\}
\[
\{ (A_+^b + \frac{1}{2} A_-^b)du + A_-^b dr + dy^b \}.
\]  

(3.36)

Since we still have the residual gauge symmetry associated with the \( \text{diff} N_2 \) invariance in
(3.36), we may well fix this residual symmetry by choosing

\[
A_+^a + \frac{1}{2} A_-^a = 0.
\]  

(3.37)

Then (3.36) reduces to

\[
ds^2 = -du^2 - 2dudr + e^{\sigma} \rho_{ab} (A_-^a dr + dy^a)(A_-^b dr + dy^b).
\]  

(3.38)
In the limit as $r \to \infty$, both (3.36) and (3.38) approach over to the flat space-time metric
\[
ds^2 \to -du^2 - 2dudr + \frac{1}{2}r^2(d\vartheta^2 + \sin^2\vartheta d\psi^2),
\]
showing that $r/\sqrt{2}$ becomes the area radius in this limit. Moreover the $u$-coordinate is the proper time at each point on the $u = \text{constant}$ hypersurface, suggesting that the volume integral be defined over the spacelike hypersurface $u = \text{constant}$ in the new coordinates, since the ADM surface integral is associated with a unit time translation at spatial infinity.

To find the relevant Hamiltonian for the ADM mass we need to write the local Lagrangian density in the new coordinates. The local Lagrangian density can be found directly from (2.23)
\[
L = \frac{1}{2}e^{2\sigma}\rho_{ab}F_{+}^{a}F_{+}^{b} - e^{\sigma}(D_{+}\sigma)(D_{-}\sigma) + \frac{1}{2}e^{\sigma}\rho^{ab}\rho^{cd}(D_{+}\rho_{ac})(D_{-}\rho_{bd}) + \sum_{\alpha = \pm, 0} \lambda^{\alpha} C_{\alpha}|_{\partial_{-} = \partial_{r}}.
\]
with $\partial_{-}$ replaced by $\partial_{r}$ everywhere. Thus the relevant volume integral is given by
\[
E = \frac{1}{2} \int drd^2y \ e^{2\sigma}\rho_{ab}F_{+}^{a}F_{+}^{b} |_{\partial_{-} = \partial_{r}} = \frac{1}{3} \int drd^2y \ e^{\sigma}\left\{R_{2} + (D_{+}\sigma)(D_{-}\sigma) - \frac{1}{2}\rho^{ab}\rho^{cd}(D_{+}\rho_{ac})(D_{-}\rho_{bd})\right\} |_{\partial_{-} = \partial_{r}} = \frac{1}{3} \int drd^2y \ e^{\sigma}h^{AC}h^{BD}C_{ABCD}.
\]
(3.41)

Repeating the same reasoning as in Appendix C, with $\tilde{\Omega}^{-1} = r$, we find that the volume integral (3.41) becomes the surface integral,
\[
E = \frac{1}{3} \lim_{r \to \infty} r \int d^2y \ e^{\sigma}h^{AC}h^{BD}C_{ABCD},
\]
(3.42)
which becomes precisely the covariant expression of the ADM mass of the asymptotically flat space-times in the limit as $r$ approaches to infinity! [40–43]

IV. DISCUSSIONS

In this article, we combined the double null formalism of general relativity with the KK formalism in the (2,2)-splitting, and proposed a local gravitational energy density of general
relativity. As we have seen so far, there are a number of notable features of this description which deserve further remarks. First of all, this formalism explicitly brings out the gauge theory aspects of general relativity of the 4-dimensional space-times. Although it has been realized for a long time that the local diffeomorphism invariance in general relativity is on a par with the local gauge symmetry in gauge theories, it seems fair to say that the full-fledged gauge theory formulation of general relativity is still lacking. Our 4-dimensional KK approach to general relativity in the (2,2)-splitting, using the double null gauge, seems to provide such a formulation, as we have described in this article. Thus we may well take care of the Gauss-law equations associated with the \( \text{diff}N_2 \) invariance by considering the \( \text{diff}N_2 \) invariant quantities only.

Moreover, this formalism shows that, in the double null gauge, local gravitational energy density of general relativity can be well-defined, and moreover, is positive-definite. The volume integral of this local gravitational energy density over the 3-dimensional null and spacelike hypersurface correctly reproduces the Bondi and ADM surface integral at null and spatial infinity, respectively. The Bondi mass-loss due to the gravitational radiation in the asymptotically flat space-times is given by a negative-definite flux integral of the bilinear in the gravitational currents at null infinity. It should be mentioned that the proposed gravitational energy density can be also used to define quasi-local gravitational energies for a finite region of a given 3-dimensional hypersurface in a straightforward way \[ 40, 45–49 \].

The non-zero local Hamiltonian density proposed in this article also has a direct bearing to the problem of time \[ 50, 54 \]. The time associated with the non-zero Hamiltonian is the retarded time \( u \), which has the physical meaning as the phase of the gravitational radiation when gravitational waves are present. The canonical analysis of our formalism is under progress \[ 55 \], which will shed further light on this important issue.

In addition, this formalism does seem to indicate the intriguing possibility that quantum general relativity of the 4-dimensional space-time may be regarded as a (1+1)-dimensional quantum field theory. For instance, one might even speculate that quantum gravity might be a finite theory, given that the renormalizability depends critically on the dimensions of
“space-time”. However, it must be addressed that this formalism is for the vacuum general relativity only. It certainly is an interesting question to see whether this formalism can be extended to include matter fields. We leave this problem for the future investigation.

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**APPENDIX A: THE E-H LAGRANGIAN DENSITY IN THE GENERAL (2,2)-SPLITTING**

In this appendix, we shall make a general (2,2)-decomposition of space-time, and obtain the corresponding E-H Lagrangian density $\mathcal{L}$ without picking up a particular gauge. By examining the transformation properties of the metric in the (2,2)-splitting under the $\text{diff}N_2$ transformations, we shall find that each field can be identified either as a scalar, a tensor, or a gauge field with respect to the $\text{diff}N_2$ transformations, respectively, suggesting that the KK program works even in the absence of any Killing vector fields. Then we simplify the general E-H Lagrangian density by introducing the double null gauge to obtain $\mathcal{L}_0$ in (2.12) [3], which also has the $\text{diff}N_2$ symmetry as the residual symmetry.

The 4-dimensional space-time may be regarded as a fibred manifold, i.e., a local product of two 2-dimensional submanifolds $M_{1+1} \times N_2$, for which we introduce two pairs of the basis
vector fields $\partial_\mu = \partial/\partial x^\mu (\mu = 0, 1)$ and $\partial_a = \partial/\partial y^a (a = 2, 3)$, respectively. The corresponding metrics on $M_{1+1}$ and $N_2$ will be denoted as $\gamma_{\mu\nu}$ and $\phi_{ab}$, respectively. Then the general line element of the 4-dimensional space-time can be written as

$$ds^2 = \gamma_{\mu\nu} dx^\mu dx^\nu + \phi_{ab}(A_\mu^a dx^\mu + dy^a)(A_\nu^b dx^\nu + dy^b).$$

(A1)

Formally this is quite similar to the “dimensional reduction” in KK theory, where $M_{1+1}$ is regarded as the $(1+1)$-dimensional “space-time” and $N_2$ as the “internal” fibre space. In the standard KK reduction one assumes a restriction on the metric, namely, an isometry condition, to make $A_\mu^a$ a gauge field associated with the isometry group. Here, however, we do not assume such isometry conditions, and allow all the fields to depend on both $x^\mu$ and $y^a$. Nevertheless $A_\mu^a(x, y)$ can still be identified as a gauge field, but now associated with an infinite dimensional diffeomorphism group $\text{diff}N_2$. To show this, let us consider the following diffeomorphism of $N_2$,

$$y^a = y^a(x, y), \quad x^\mu = x^\mu. \quad (A2)$$

Under this transformation, we find

$$\gamma'_{\mu\nu}(x, y') = \gamma_{\mu\nu}(x, y), \quad (A3)$$

$$\phi'_{ab}(x, y') = \frac{\partial y^c}{\partial y'^a} \frac{\partial y^d}{\partial y'^b} \phi_{cd}(x, y), \quad (A4)$$

$$A'^a_\mu(x, y') = \frac{\partial y'^a}{\partial y'^c} A^c_\mu(x, y) - \partial_\mu y'^a, \quad (A5)$$

such that the line element (A1) is invariant. Under the corresponding infinitesimal transformation

$$\delta y^a = \xi^a(x, y), \quad \delta x^\mu = 0, \quad (A6)$$

these become

$$\delta \gamma_{\mu\nu} = -[\xi, \gamma_{\mu\nu}] = -\xi^c \partial_c \gamma_{\mu\nu}, \quad (A7)$$

$$\delta \phi_{ab} = -[\xi, \phi]_{ab} = -\xi^c \partial_c \phi_{ab} - (\partial_\alpha \xi^c) \phi_{cb} - (\partial_b \xi^c) \phi_{ac}, \quad (A8)$$

$$\delta A^a_\mu = -\partial_\mu \xi^a + [A_\mu, \xi]^a = -\partial_\mu \xi^a + (A_\mu^c \partial_c \xi^a - \xi^c \partial_\mu A^a_\mu), \quad (A9)$$
where the bracket represents the Lie derivative that acts on the “internal” indices $a, b, \text{etc,}$ only. Notice that the Lie derivative, an infinite dimensional generalization of the finite dimensional matrix commutators, appears naturally. Associated with this $\text{diff}N_2$ transformation, the $\text{diff}N_2$-covariant derivative $D_\mu$ is defined by

$$D_\mu = \partial_\mu - [A_\mu,],$$  \hspace{1cm} (A10)

where the bracket is again the Lie derivative along $A_\mu = A_\mu^a \partial_a$. With this definition, we have

$$\delta A_\mu^a = -D_\mu \xi^a,$$ \hspace{1cm} (A11)

which clearly indicates that $A_\mu^a$ is the gauge field valued in the infinite dimensional Lie algebra associated with $\text{diff}N_2$. Moreover the transformation properties (A7) and (A8) show that $\gamma_{\mu\nu}$ and $\phi_{ab}$ are a scalar and a tensor field, respectively, under $\text{diff}N_2$. The field strength $F_{\mu\nu}^a$ corresponding to $A_\mu^a$ can now be defined as

$$[D_\mu, D_\nu] = -F_{\mu\nu}^a \partial_a = -\{\partial_\mu A_\nu^a - \partial_\nu A_\mu^a - [A_\mu, A_\nu]^a\} \partial_a,$$ \hspace{1cm} (A12)

which transforms covariantly under the infinitesimal transformation (A3),

$$\delta F_{\mu\nu}^a = -[\xi, F_{\mu\nu}]^a.$$ \hspace{1cm} (A13)

To obtain the E-H Lagrangian density, we have to first compute connections and curvature tensors. For this purpose it is convenient to introduce the following horizontal lift basis $\hat{\partial}_A = (\hat{\partial}_\mu, \hat{\partial}_a)$ where \[32\]

$$\hat{\partial}_\mu := \partial_\mu - A_\mu^a \partial_a, \quad \hat{\partial}_a := \partial_a.$$ \hspace{1cm} (A14)

From the following commutation relations

$$[\hat{\partial}_A, \hat{\partial}_B] = f_{AB}^C \hat{\partial}_C,$$ \hspace{1cm} (A15)

we find the structure functions $f_{AB}^C(x, y)$

26
\[ f_{\mu \nu}^a = - F_{\mu \nu}^a, \]
\[ f_{\mu a}^b = - f_{a \mu}^b = \partial_a A_\mu^b, \]
\[ f_{AB}^C = 0, \quad \text{otherwise.} \quad (A16) \]

The virtue of this basis is that it brings the metric \((A1)\) into a block diagonal form

\[ \hat{g}_{AB} = \begin{pmatrix} \gamma_{\mu \nu} & 0 \\ 0 & \phi_{ab} \end{pmatrix}, \quad (A17) \]

which drastically simplifies the computation of the scalar curvature.

The connection coefficients and the curvature tensors in this basis are given by \([32,56]\)

\[
\hat{\Gamma}_ABC^D = \frac{1}{2} \hat{g}^{CD} \left( \partial_A \hat{g}_{BD} + \partial_B \hat{g}_{AD} - \partial_D \hat{g}_{AB} \right) + \frac{1}{2} \hat{g}^{CD} \left( f_{ABD} - f_{BDA} - f_{ADB} \right),
\]
\[
\hat{R}_{ABC}^D = \partial_B \hat{\Gamma}_{AC}^D - \partial_A \hat{\Gamma}_{BC}^D + \hat{\Gamma}_{BE}^D \hat{\Gamma}_{AC}^E - \hat{\Gamma}_{AE}^D \hat{\Gamma}_{BC}^E + f_{AB}^E \hat{\Gamma}_{EC}^D,
\]
\[
\hat{R}_{AC} = \hat{g}^{BD} \hat{R}_{ABCD},
\]
\[
R = \hat{g}^{AC} \hat{R}_{AC}, \quad (A18)
\]

where \(f_{ABC} := \hat{g}^{CD} f_{AB}^D\). In components the connection coefficients are given by

\[
\hat{\Gamma}_\mu^\alpha = \frac{1}{2} \gamma_{\alpha \beta} \left( \partial_\mu \gamma_{\nu \beta} + \partial_\nu \gamma_{\mu \beta} - \partial_\beta \gamma_{\mu \nu} \right),
\]
\[
\hat{\Gamma}_\mu^\alpha = - \frac{1}{2} \delta^ab \partial_b \gamma_{\mu \nu} + \frac{1}{2} F_{\mu \nu}^a,
\]
\[
\hat{\Gamma}_\mu^\nu = \hat{\Gamma}_a^\mu = \frac{1}{2} \gamma^{\mu \nu} \partial_\alpha \gamma_{\mu \alpha} + \frac{1}{2} \gamma^{\mu \alpha} \phi_{ab} F_{\mu a}^b,
\]
\[
\hat{\Gamma}_\mu^a = \frac{1}{2} \delta^bc \partial_\mu \phi_{bc} + \frac{1}{2} \partial_\mu A_\nu^c - \frac{1}{2} \phi_{ae} \partial_\nu \phi_{bc} A_\mu^e,
\]
\[
\hat{\Gamma}_{ab}^\mu = - \frac{1}{2} \gamma^{\mu \nu} \partial_\nu \phi_{ab} + \frac{1}{2} \gamma^{\mu \nu} \partial_\nu A_{ab}^c + \frac{1}{2} \gamma^{\mu \nu} \phi_{bc} \partial_\nu A_{ab}^c,
\]
\[
\hat{\Gamma}_{ab}^c = \frac{1}{2} \phi^cd \left( \partial_a \phi_{bd} + \partial_b \phi_{ad} - \partial_d \phi_{ab} \right), \quad (A19)
\]

The following identities are also useful:

\[
\hat{\Gamma}_\mu^\nu = \frac{1}{2} \gamma^{\alpha \beta} \partial_\nu \gamma_{\alpha \beta}, \quad \hat{\Gamma}_\nu^\beta = \frac{1}{2} \gamma^{\alpha \beta} \partial_\alpha \gamma_{\nu \beta}, \quad \hat{\Gamma}_\nu^a = \frac{1}{2} \phi^{ab} \partial_\nu \phi_{ab} - \partial_\beta A_\nu^a,
\]
\[
\hat{\Gamma}_{\nu a} = \frac{1}{2} \phi^{ab} \partial_\nu \phi_{ab}, \quad \hat{\Gamma}_{ab}^a = \frac{1}{2} \phi^{ac} \partial_b \phi_{ac}. \quad (A20)
\]
The Ricci tensors and the scalar curvature are given by

\[ \hat{R}_{\mu\nu} = \hat{R}_{\mu\nu}^a + \hat{R}_{\mu\nu}^b, \quad \hat{R}_{ac} = \hat{R}_{abc}^b + \hat{R}_{aac}^c, \quad R = \gamma^{\mu\nu} \hat{R}_{\mu\nu} + \phi^{ac} \hat{R}_{ac}. \quad \text{(A21)} \]

Thus, in order to obtain the E-H Lagrangian density, we need to calculate \( \gamma^{\mu\nu} \hat{R}_{\mu\nu} \) and \( \phi^{ac} \hat{R}_{ac} \) only. Let us first define \( R'_{\mu\nu} \) and \( R_{ac} \) as follows;

\[
\begin{align*}
R'_{\mu\nu} &= \hat{\partial}_a \hat{\Gamma}_{\mu\nu}^a - \hat{\partial}_\mu \hat{\Gamma}_{\alpha\nu}^a + \hat{\Gamma}_{\beta\alpha}^\beta \hat{\Gamma}_{\mu\nu}^\alpha - \hat{\Gamma}_{\mu\beta}^\alpha \hat{\Gamma}_{\alpha\nu}^\beta, \\
R_{ac} &= \hat{\partial}_b \hat{\Gamma}_{ac}^b - \hat{\partial}_a \hat{\Gamma}_{bc}^b + \hat{\Gamma}_{db}^d \hat{\Gamma}_{ac}^b - \hat{\Gamma}_{ad}^d \hat{\Gamma}_{bc}^d. \quad \text{(A22)}
\end{align*}
\]

Notice that formally \( R'_{\mu\nu} \) is identical to the Ricci tensor of \( M_{1+1} \), except that \( \hat{\partial}_\mu \) was used instead of \( \partial_\mu \). For this reason it might be called the “gauged” Ricci tensor of \( M_{1+1} \), whereas \( R_{ac} \) is the usual Ricci tensor of \( N_2 \). After a long computation we obtain \( \gamma^{\mu\nu} \hat{R}_{\mu\nu} \) and \( \phi^{ac} \hat{R}_{ac} \) as follows;

\[
\begin{align*}
\gamma^{\mu\nu} \hat{R}_{\mu\nu} &= \gamma^{\mu\nu} R'_{\mu\nu} - \frac{1}{2} \gamma^{\mu\nu} \gamma^{\alpha\beta} \phi_{ab} F_{\mu\alpha} F_{\nu\beta} - \frac{1}{4} \gamma^{\mu\nu} \phi^{ab} \phi^{cd} (\partial_\mu \phi_{ac})(\partial_\nu \phi_{bd}) \\
&\quad + \frac{1}{4} \gamma^{\mu\nu} \phi^{ab} \phi^{cd} (\partial_\mu \phi_{ab})(\partial_\nu \phi_{cd}) + \gamma^{\mu\nu} \phi^{bc} (\partial_\mu \phi_{ac})(\partial_\beta A^a_\nu) \\
&\quad - \gamma^{\mu\nu} \phi^{ab} (\partial_\mu \phi_{ab})(\partial_\alpha A^c_\nu) - \gamma^{\mu\nu} (\partial_\alpha A^a_\mu)(\partial_\nu A^b_\nu) - \frac{1}{2} \gamma^{\mu\nu} (\partial_\alpha A^b_\mu)(\partial_\beta A^a_\nu) \\
&\quad - \frac{1}{2} \gamma^{\mu\nu} \phi^{ab} \phi^{cd} (\partial_\alpha A^c_\mu)(\partial_\beta A^d_\nu) \\
&\quad - (\hat{\nabla}_\mu + \hat{\Gamma}_{\mu\nu}^c)(\frac{1}{2} \gamma^{\mu\nu} \phi^{ab} \partial_\nu \phi_{ab} - \gamma^{\mu\nu} \partial_\alpha A^a_\nu) \\
&\quad - (\hat{\nabla}_a + \hat{\Gamma}_{\alpha\nu}^\alpha)(\frac{1}{2} \phi^{ab} \phi^{cd} \partial_\nu \phi_{cd}) \quad \text{(A23)}
\end{align*}
\]

\[
\begin{align*}
\phi^{ac} \hat{R}_{ac} &= \phi^{ac} R_{ac} + \frac{1}{4} \gamma^{\mu\nu} \gamma^{\alpha\beta} \phi_{ab} F_{\mu\alpha} F_{\nu\beta} - \frac{1}{4} \phi^{ab} \gamma^{\mu\nu} \gamma^{\alpha\beta} (\partial_\alpha \gamma_{\mu\nu})(\partial_\beta \gamma_{\mu\nu}) \\
&\quad + \frac{1}{4} \phi^{ab} \gamma^{\mu\nu} \gamma^{\alpha\beta} (\partial_\alpha \gamma_{\mu\nu})(\partial_\beta \gamma_{\mu\nu}) \\
&\quad - (\hat{\nabla}_\mu + \hat{\Gamma}_{\mu\nu}^c)(\frac{1}{2} \gamma^{\mu\nu} \phi^{ab} \partial_\nu \phi_{ab} - \gamma^{\mu\nu} \partial_\alpha A^a_\nu) \\
&\quad - (\hat{\nabla}_a + \hat{\Gamma}_{\alpha\nu}^\alpha)(\frac{1}{2} \phi^{ab} \phi^{cd} \partial_\nu \phi_{cd}) \quad \text{(A24)}
\end{align*}
\]

Here the derivatives \( \hat{\nabla}_\mu \) and \( \hat{\nabla}_a \) are compatible with the metrics \( \gamma_{\alpha\beta} \) and \( \phi_{bc} \) in the horizontal lift basis, respectively, such that

\[
\begin{align*}
\hat{\nabla}_\mu \gamma_{\alpha\beta} &= \hat{\partial}_\mu \gamma_{\alpha\beta} - \hat{\Gamma}_{\mu\alpha}^\delta \gamma_{\delta\beta} - \hat{\Gamma}_{\mu\beta}^\delta \gamma_{\alpha\delta} = 0, \\
\hat{\nabla}_a \phi_{bc} &= \hat{\partial}_a \phi_{bc} - \hat{\Gamma}_{ab}^d \phi_{dc} - \hat{\Gamma}_{ac}^d \phi_{bd} = 0. \quad \text{(A25)}
\end{align*}
\]
For an object of the mixed-type such as $X_{b\cdots a\cdots}$ in this basis, these derivatives act as follows;

$$
\hat{\nabla}_\mu X_{b\cdots a\cdots} = \hat{\partial}_\mu X_{b\cdots a\cdots} - \hat{\Gamma}_{a\mu}^b X_{b\cdots a\cdots} + \hat{\Gamma}_{b\mu}^\alpha X_{b\cdots a\cdots} + \cdots, \\
\hat{\nabla}_a X_{b\cdots a\cdots} = \partial_a X_{b\cdots a\cdots} - \hat{\Gamma}_{ab}^d X_{d\cdots a\cdots} + \hat{\Gamma}_{ad}^c X_{b\cdots d\cdots} + \cdots.
$$

(A26)

For instance, the followings are true,

$$
\hat{\nabla}_\mu (\phi^{ab} \partial_\nu \phi_{ab}) = \hat{\partial}_\mu (\phi^{ab} \partial_\nu \phi_{ab}) - \hat{\Gamma}_{a\mu}^\alpha (\phi^{ab} \partial_\alpha \phi_{ab}), \\
\hat{\nabla}_a (\gamma^{\mu\nu} \partial_b \gamma_{\mu\nu}) = \partial_a (\gamma^{\mu\nu} \partial_b \gamma_{\mu\nu}) - \hat{\Gamma}_{ab}^c (\gamma^{\mu\nu} \partial_c \gamma_{\mu\nu}), \\
\hat{\nabla}_\mu (\partial_a A^{\alpha}_\nu) = \hat{\partial}_\mu (\partial_a A^{\alpha}_\nu) - \hat{\Gamma}_{\mu\alpha}^\alpha \partial_a A^{\alpha}_\nu,
$$

(A27)

which we used in (A23) and (A24). The scalar curvature $R$ of the metric (A1) becomes, using (A23) and (A24),

$$
R = \gamma^{\mu\nu} R^{\mu\nu} + \phi^{ac} R_{ac} - \frac{1}{4} \gamma^{\mu\nu} \gamma^{\alpha\beta} \phi_{ab} F_{\mu\alpha}^a F_{\nu\beta}^b \\
- \frac{1}{4} \gamma^{\mu\nu} \phi^{ab} \phi^{cd} \left\{ (D_\mu \phi_{ac})(D_\nu \phi_{bd}) - (D_\mu \phi_{ab})(D_\nu \phi_{cd}) \right\} \\
- \frac{1}{4} \phi^{ab} \gamma^{\mu\nu} \gamma^{\alpha\beta} \left\{ (\partial_a \gamma_{\mu\alpha})(\partial_b \gamma_{\nu\beta}) - (\partial_a \gamma_{\mu\nu})(\partial_b \gamma_{\alpha\beta}) \right\} \\
- (\hat{\nabla}_\mu + \hat{\Gamma}_{\mu\alpha}^\alpha) j^{\mu} - (\hat{\nabla}_a + \hat{\Gamma}_{\alpha\alpha}^a) j^a.
$$

(A28)

Here $j^{\mu}$ and $j^a$ are defined as

$$
j^{\mu} = \gamma^{\mu\nu} \phi^{ab} \partial_\nu \phi_{ab} - 2 \gamma^{\mu\nu} \partial_a A^{\alpha}_\nu, \\
j^a = \phi^{ab} \gamma^{\mu\nu} \partial_b \gamma_{\mu\nu},
$$

(A29)

and $D_\mu \phi_{ab}$ is the diff$N_2$-covariant derivative

$$
D_\mu \phi_{ac} = \hat{\partial}_\mu \phi_{ac} - (\partial_a A^{\epsilon}_\mu) \phi_{\epsilon c} - (\partial_c A^{\epsilon}_\mu) \phi_{a\epsilon} \\
= \partial_\mu \phi_{ac} - [A_\mu, \phi]_{ac},
$$

(A30)

where $[A_\mu, \phi]_{ac}$ is the Lie derivative of $\phi_{ac}$ along $A_\mu = A^{\epsilon}_\mu \partial_\epsilon$,

$$
[A_\mu, \phi]_{ac} = A^{\epsilon}_\mu \partial_\epsilon \phi_{ac} + (\partial_a A^{\epsilon}_\mu) \phi_{\epsilon c} + (\partial_c A^{\epsilon}_\mu) \phi_{a\epsilon}.
$$

(A31)
Thus the E-H Lagrangian density in this (2,2)-splitting finally becomes

\[
\mathcal{L}' = \sqrt{-\gamma} \sqrt{\phi} R
\]

\[
= \sqrt{-\gamma} \sqrt{\phi} \left[ \gamma^{\mu\nu} R_{\mu\nu} + \phi^{ac} R_{ac} - \frac{1}{4} \gamma^{\mu\nu} \gamma^{\alpha\beta} \phi_{ab} F_{\mu\alpha} F_{\nu\beta} \right]
\]

\[
- \frac{1}{4} \gamma^{\mu\nu} \phi^{ab} \phi^{cd} \left\{ (D_{\alpha} \phi_{\mu\nu})(D_{\psi} \phi_{\theta}) - (D_{\mu} \phi_{\alpha\nu})(D_{\psi} \phi_{\theta}) \right\}
\]

\[
- \frac{1}{4} \phi^{ab} \gamma^{\mu\nu} \phi_{\alpha\beta} \left\{ (\partial_{\alpha} \gamma_{\mu\nu})(\partial_{\beta} \gamma_{\psi}) - (\partial_{\mu} \gamma_{\alpha\nu})(\partial_{\beta} \gamma_{\psi}) \right\}
\]

\[- \sqrt{-\gamma} \sqrt{\phi} \left\{ (\hat{\nabla}_{\mu} + \hat{\Gamma}_{\mu} \phi)^{c} j^{\mu} + (\hat{\nabla}_{a} + \hat{\Gamma}_{a} \phi)^{a} j^{a} \right\},
\]

(A32)

where \( \gamma = \det \gamma_{\mu\nu} \) and \( \phi = \det \phi_{ab} \). It can be shown that the last two terms in (A32) are total divergences \( [3 \text{-} 5] \),

\[
\sqrt{-\gamma} \sqrt{\phi} \left( \hat{\nabla}_{\mu} + \hat{\Gamma}_{\mu} \phi \right)^{c} j^{\mu} = \partial_{\mu} \left( \sqrt{-\gamma} \sqrt{\phi} j^{\mu} \right) - \partial_{a} \left( \sqrt{-\gamma} \sqrt{\phi} A_{\mu}^{a} j^{\mu} \right);
\]

(A33)

\[
\sqrt{-\gamma} \sqrt{\phi} \left( \hat{\nabla}_{a} + \hat{\Gamma}_{a} \phi \right)^{a} j^{a} = \partial_{a} \left( \sqrt{-\gamma} \sqrt{\phi} j^{a} \right);
\]

(A34)

using (A20) and the following identities

\[
\hat{\nabla}_{\mu} j^{\mu} = \hat{\partial}_{\mu} j^{\mu} + \hat{\Gamma}_{\alpha}^{a} j^{a};
\]

\[
\hat{\nabla}_{a} j^{a} = \partial_{a} j^{a} + \hat{\Gamma}_{b}^{a} j^{b}.
\]

(A35)

In the \((u,v)\) coordinates

\[
u = \frac{1}{\sqrt{2}} (x^{0} - x^{1}), \quad v = \frac{1}{\sqrt{2}} (x^{0} + x^{1}),
\]

(A36)

the following substitution

\[
\gamma_{+-} = -1, \quad \gamma_{++} = \gamma_{--} = 0,
\]

(A37)

together with

\[
A_{\pm}^{a} = \frac{1}{\sqrt{2}} (A_{0}^{a} \mp A_{1}^{a}),
\]

(A38)

leads to the double null gauge (2.2) and enormously simplifies the E-H Lagrangian density (A32). The resulting expression is \( \mathcal{L}_{0} \) in (2.12).
APPENDIX B: THE COVARIANT NULL TETRADS

In this appendix, we describe the kinematics of a Lorentzian space-time of 4-dimensions using the covariant null tetrads \[9–12,19\]. This allows us to compare the variables in the traditional double null hypersurface formulation and our KK variables directly. Moreover, in order to express the volume integral (3.18) as a surface integral, it is better to use the covariant null tetrad notation. Let the two real dual null tetrads \(l_A\) and \(n_A\) \((A=0,1,2,3)\) be the gradient fields for some scalar functions \(u\) and \(v\),

\[
l_A = \nabla_A u, \quad n_A = \nabla_A v, \quad (B1)
\]

so that \(\nabla_A [Bl_A] = \nabla_A [Bn_A] = 0\). The dual vector fields \(du\) and \(dv\) are related to the dual null tetrads by

\[
du = l_A dX^A, \quad dv = n_A dX^A. \quad (B2)
\]

We also have the vector fields \(\partial/\partial u\), \(\partial/\partial v\), and \(\partial/\partial y^a\) \((a = 2, 3)\) which we may write

\[
\frac{\partial}{\partial u} = u^A \frac{\partial}{\partial X^A}, \quad \frac{\partial}{\partial v} = v^A \frac{\partial}{\partial X^A}, \quad \frac{\partial}{\partial y^a} = y_a^A \frac{\partial}{\partial X^A}. \quad (B3)
\]

If we choose the basis vector fields of space-time so that \(\nabla_A = (\partial/\partial u, \partial/\partial v, \partial/\partial y^a)\), the components of the dual null tetrads and vector fields are given by

\[
l_A = (1, 0, 0, 0) \quad n_A = (0, 1, 0, 0),
\]

\[
u^A = (1, 0, 0, 0) \quad v^A = (0, 1, 0, 0) \quad y_a^A = \delta_a^A. \quad (B4)
\]

From this it follows that

\[
l_A v^A = n_A u^A = 1, \quad n_A v^A = n_A u^A = 0. \quad (B5)
\]

Since \(l_A l^A = n_A n^A = 0\), the null tetrad \(l^A\) and \(n^A\) may be chosen as

\[
l = l^A \frac{\partial}{\partial X^A} = -\frac{\partial}{\partial v} + A_a \frac{\partial}{\partial y^a}, \quad n = n^A \frac{\partial}{\partial X^A} = -\frac{\partial}{\partial u} + A_a \frac{\partial}{\partial y^a}, \quad (B6)
\]

i.e.
\[ l^A = (0, -1, A_-), \quad n^A = (-1, 0, A_+) \] \hspace{1cm} (B7)

such that

\[ l_A n^A = -1. \] \hspace{1cm} (B8)

\((n \text{ and } l \text{ are the minus of the horizontal lift vector fields } \hat{\partial}_\pm = \partial_\pm - A_\pm^a \partial_a \text{ in the } (u, v)\text{-coordinates, respectively.})\) The condition \((B8)\) is equivalent to the previous normalization condition \(g^{+-} = -1\) in \((2.3)\), and means that, given an arbitrary function \(u\), the function \(v\) must be chosen in such a way that the normalization condition \((B8)\) is satisfied. We still have the freedom to orient \(l^A\) and \(n^A\) in space-time, and we fix this freedom by demanding that \(l^A\) and \(n^A\) are normal to the 2-dimensional spacelike surface \(N_2\) whose tangent vector fields are \(\partial/\partial y^a\),

\[ h_{AB} l^B = h_{AB} n^B = 0, \] \hspace{1cm} (B9)

where \(h_{AB}\) is the metric on \(N_2\) (\(h_{AB}\) is the covariant expression of \(\phi_{ab}\)). Using \(h_{AB}\) and \(l_A\), \(n_A\), the space-time metric \(g_{AB}\) may be written as

\[ g_{AB} = h_{AB} - (l_A n_B + n_A l_B). \] \hspace{1cm} (B10)

From these relations, we easily find that \((B10)\) is identical to the metric \((2.2)\)

\[ ds^2 = -2dudv + \phi_{ab}(A_+^a du + A_-^a dv + dy^a)(A_+^b du + A_-^b dv + dy^b). \] \hspace{1cm} (B11)

**APPENDIX C: THE BONDI SURFACE INTEGRAL**

We now show that the volume integral \((3.18)\)

\[ E = \frac{1}{3} \int dudv dy^a h^{AC} h^{BD} C_{ABCD} \] \hspace{1cm} (C1)

can be expressed as a surface integral, using the Bianchi identity of the conformal curvature tensor
\[ \nabla_{[M} C_{AB]CD} = 0. \quad (C2) \]

Let us notice that, due to the Bianchi identity, the following is true for any scalar function \( \tilde{\Omega} \),
\[ \nabla_{[M} \left( \tilde{\Omega}^{-1} C_{AB]CD} \right) = \left( \nabla_{[M} \tilde{\Omega}^{-1} \right) C_{AB]CD}. \quad (C3) \]

If we contract (C3) by \( h^{AC} h^{BD} \), it becomes
\[ h^{AC} h^{BD} \nabla_{[M} \left( \tilde{\Omega}^{-1} C_{AB]CD} \right) = h^{AC} h^{BD} \left( \nabla_{[M} \tilde{\Omega}^{-1} \right) C_{AB]CD}. \quad (C4) \]

Let us choose \( \tilde{\Omega}^{-1} \) as a function of \((u, v)\) only, and let \( \nabla_M = \nabla_- \). Then the r.h.s. of (C4) becomes
\[ h^{AC} h^{BD} \left( \nabla_- \tilde{\Omega}^{-1} \right) C_{AB]CD} = \frac{1}{3} h^{AC} h^{BD} \left( \nabla_- \tilde{\Omega}^{-1} \right) C_{ABCD}, \quad (C5) \]

since \( h^{AB} \nabla_B \tilde{\Omega}^{-1} = 0 \). The l.h.s. of (C4) becomes
\[ h^{AC} h^{BD} \nabla_- \left( \tilde{\Omega}^{-1} C_{AB]CD} \right) = \nabla_- \left( h^{AC} h^{BD} \tilde{\Omega}^{-1} C_{AB]CD} \right) \]
\[ - \nabla_- \left( h^{AC} h^{BD} \right) \tilde{\Omega}^{-1} C_{AB]CD}. \quad (C6) \]

Using \( h^{AC} = g^{AC} + l^A n^C + n^A l^C \) and the properties of the conformal curvature tensor
\[ g^{AC} C_{ABCD} = g^{BD} C_{ABCD} = 0, \]
\[ C_{ABCD} = C_{[AB][CD]}, \quad (C7) \]

the second term in the r.h.s. of (C6) may be written as
\[ \nabla_- \left( h^{AC} h^{BD} \right) \tilde{\Omega}^{-1} C_{AB]CD} = 2 \nabla_- \left( l_A n_B \right) \tilde{\Omega}^{-1} C^{ABCD}. \quad (C8) \]

Since \( l_A \) and \( n_A \) are non-zero only when \( A, B \) are + or −, we have
\[ \nabla_- \left( l_A n_B \right) \tilde{\Omega}^{-1} C^{ABCD} = 0, \quad (C9) \]

due to the repeated indices in the anti-symmetric symbol. Therefore the l.h.s. of (C4) becomes
\[ h^{AC} h^{BD} \nabla_{\tilde{\Omega}^{-1} (\tilde{\Omega}^{-1} C_{AB})} = \nabla_{\tilde{\Omega}^{-1} (h^{AC} h^{BD} \tilde{\Omega}^{-1} C_{AB})}. \]  

(C10)

Thus (C4) becomes, using (C5) and (C10),

\[ h^{AC} h^{BD} (\nabla_{\tilde{\Omega}^{-1}}) C_{ABCD} = \nabla_{\tilde{\Omega}^{-1} (h^{AC} h^{BD} \tilde{\Omega}^{-1} C_{ABCD})} + \nabla_A (h^{AC} h^{BD} \tilde{\Omega}^{-1} C_{B-CD}) + \nabla_B (h^{AC} h^{BD} \tilde{\Omega}^{-1} C_{-ACD}). \]  

(C11)

Integrating (C11) over the \( u = \text{constant} \) hypersurface with the canonical measure \( \sqrt{h} \), it becomes, using \( \sqrt{g} = \sqrt{h} \) for the metric (B10),

\[
\int d^2 y \sqrt{h} h^{AC} h^{BD} (\nabla_{\tilde{\Omega}^{-1} C_{ABCD}}) = \int d^2 y \sqrt{h} (h^{AC} h^{BD} \tilde{\Omega}^{-1} C_{ABCD}),
\]

(C12)

since the surface integrals coming from the last two terms in the r.h.s. of (C11) vanish for any 2-surface \( N_2 \) compact without boundary. Here \( \text{lim} \) means that the integral over \( N_2 \) must be evaluated at the limiting boundary value(s) of \( v \). Now let us choose \( \tilde{\Omega} \) such that \( \tilde{\Omega}^{-1} = v \). Then the identity (C12) becomes

\[
\int d^2 y \sqrt{h} h^{AC} h^{BD} C_{ABCD} = \lim v \int d^2 y \sqrt{h} h^{AC} h^{BD} C_{ABCD}.
\]

(C13)

Since \( \sqrt{h} = e^\sigma \) in our previous notation, the volume integral (3.18) becomes

\[
E = \frac{1}{3} \int d^2 y \ e^\sigma h^{AC} h^{BD} C_{ABCD} = \frac{1}{3} \lim v \int d^2 y \ e^\sigma h^{AC} h^{BD} C_{ABCD},
\]

(C14)

which is the desired expression for the Bondi surface integral as \( v \) approaches to infinity.

**APPENDIX D: THE POSITIVE-DEFINITENESS OF \( \kappa^2_\pm \)**

Here we show that \( \kappa^2_\pm \) is positive-definite. Let us introduce super indices \( A', B' \) for the symmetric combinations \( (ac) \) and \( (bd) \), respectively, so that

\[
\rho_{A'} := \rho_{ac}, \quad \rho_{B'} := \rho_{bd}.
\]

(D1)

Define the supermetric \( G_{A'B'} \) and its inverse \( G^{A'B'} \) by
\[ G_{A'B'} := \frac{1}{2}(\rho_{ab}\rho_{cd} + \rho_{ad}\rho_{cb}), \quad G^{A'B'} := \frac{1}{2}(\rho^{ab}\rho^{cd} + \rho^{ad}\rho^{cb}), \]  

such that

\[ G_{A'E'} G_{E'B'} = \delta_{A'B'}, \quad \text{where} \quad \delta_{A'} = \frac{1}{2}(\delta_a \delta_c + \delta_a \delta_c). \]  

This supermetric raises and lowers the super indices

\[ G^{A'B'} \rho_{B'} = \rho_{A'}, \quad G_{A'B'} \rho_{B'} = \rho_{A'}, \]  

and has a positive-definite signature since it becomes

\[ G^{A'B'} = \text{diag}(+1, +1/2, +1) \quad \text{for} \quad \rho^{ab} = \delta^{ab}. \]  

Therefore it follows that

\[ \kappa^2_{\pm} = \frac{1}{8} \rho^{ab} \rho^{cd} (D_{\pm \rho_{ac}})(D_{\pm \rho_{bd}}) \]
\[ = \frac{1}{8} G^{A'B'} (D_{\pm \rho_{A'}})(D_{\pm \rho_{B'}}) \geq 0. \]
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