Fast Long-Distance Control of Spin Qubits by Photon Assisted Cotunneling

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We investigate theoretically the long-distance coupling and spin exchange in an array of quantum dot spin qubits in the presence of microwaves. We find that photon assisted cotunneling is boosted at resonances between photon and energies of virtually occupied excited states and show how to make it spin selective. We identify configurations that enable fast switching and spin echo sequences for efficient and non-local manipulation of spin qubits. We devise configurations in which the near-resonantly boosted cotunneling provides non-local coupling which, up to certain limit, does not diminish with distance between the manipulated dots before it decays weakly with inverse distance.

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Introduction. Photon assisted tunneling (PAT) is an inelastic process where an electron overcomes a barrier by emission or absorption of photons [1]. It shows up as additional peaks in the tunneling current with sidebands occurring at multiples of the photon frequency [2]. Although observed in superconductors already fifty years ago [3], it took three decades to establish it in other systems as well [4], including semiconductor quantum dots based on two dimensional electron gases [5–11], the structures of primary interest here [12]. With the help of a microwave field, various forms of charge [13–14], spin [15], and photon [16] pumps were demonstrated. The pumping is an adiabatic (parametric) [17–21] or photon-assisted [22,23] process and has potential use in metrology, spintronics, and quantum information processing [24,25].

Rich opportunities offered by a time-dependent control field [29,30] motivate us to investigate microwave-assisted manipulation of spin qubits. In particular, we consider here a linear array of electrically controlled quantum dots [12,31,32] and analyze electron transfer and spin exchange between distant (non-neighboring) quantum dots [33,34]. Such non-local manipulations are higher order processes in the interdot tunneling amplitude (referred to as cotunneling), which proceed through virtually excited dot states (henceforth we refer to these as virtual states for brevity) [35].

We demonstrate that photon assisted cotunneling (PACT) opens up new possibilities due to the straightforward tunability of microwaves. Indeed, in contrast to standard tuning of dot levels by gates [36], PACT allows one to tune to any virtual state [10–42], not just the lowest one. Using Floquet theory, we find that cotunneling amplitudes get boosted near such resonances. Moreover, the virtual states possess spin structure (due to on-site exchange interaction) that results in spin-dependent cotunneling. This dependence can be exploited for, e.g., spin-charge conversion and further be tailored by spin echo protocols. Finally, tuning to a Bloch band of delocalized virtual states produced by an array of coupled dots generates exceptionally long-ranged interactions that enables long-range coupling between distant spin qubits. Overall, we demonstrate that PACT offers striking advantages over standard gate control of spin qubits in terms of speed, long-range coupling, and non-locality.

Model. We investigate PACT first on a linear array of three driven tunnel-coupled dots and subsequently ex-
tend it. The model Hamiltonian,
\[ H = \sum_{\alpha} \left( H_0^\alpha + H_D^\alpha(t) \right) + H_T, \]

is a sum of, respectively, single-dot confinement, driving, and inter-dot tunneling terms. Each dot \( \alpha = A, B, C \) is created by electrostatic confinement, which defines a set of single-particle dot states (with corresponding fermion creation operator \( c_{\alpha i \sigma}^\dagger \)) and energies,
\[ H_0^\alpha = \sum_{i,\sigma} \epsilon_{\alpha i \sigma} c_{\alpha i \sigma}^\dagger c_{\alpha i \sigma} + H_{\text{int}}^\alpha \equiv \sum_k \epsilon_k^\alpha |k\rangle_{\alpha \sigma} \langle k|. \tag{2} \]

Here, the orbital index \( i \) and spin index \( \sigma \) label the states. A uniform magnetic field sets the spin quantization axis. In this work, we neglect spin-orbit and hyperfine interactions and consider only spin-conserving nearest-neighbor tunneling. The total spin is then conserved and the corresponding Zeeman term (not written) commutes with the total Hamiltonian. Instead of specifying the intradot Coulomb interaction \( H_{\text{int}}^\alpha \) for dot \( \alpha \), we introduce a Fock-like basis formed by many-body states \( |k\rangle_{\alpha \sigma} \) with different number of electrons. Specifically, we consider states with zero \(|\{0\}\rangle\), one \(|\{\sigma = \uparrow, \downarrow\}\rangle\), and two electrons. The latter comprise a spin singlet \(|S\rangle\) split by the exchange energy from the unpolarized triplet \(|T_0\rangle\) and the two polarized triplets \(|T_{\pm}\rangle\). Thus, \( k \in \{0; \sigma; S, T_{0, \pm}\}\). The structure is gated such that due to charging energy costs, the doubly occupied states are relevant (and taken into account) only for the middle dot. The total (antisymmetrized) many-body state is written as \(|k_A k_B k_C\rangle \equiv |klm\rangle\) with associated energy \( \epsilon_{klm}\).

The interdot tunneling is described by
\[ H_T = \sum_{\alpha, \beta, \sigma} \tau_{ij}^{\alpha \beta} c_{\alpha i \sigma}^\dagger c_{\beta j \sigma}, \tag{3} \]

where the amplitudes \( \tau_{ij}^{\alpha \beta} \) are non-zero only between neighboring dots and, in general, depend on the single-particle levels they connect.

The oscillating electrostatic potential of the dot \( \alpha \) driven at frequency \( \omega \) with the amplitude \( V_{\alpha} \) shifts simultaneously all energy levels
\[ H_0^\alpha(t) = -\sum_{i,\sigma} e V_{\alpha} \cos(\omega t) c_{\alpha i \sigma}^\dagger c_{\alpha i \sigma}. \tag{4} \]

Here, \( e > 0 \) is the electron charge. This semiclassical description of the electromagnetic field allows us to exploit the Floquet theory to derive the cotunneling amplitudes within a time-independent formalism [44, 45]. We arrive at
\[ \tau_{co} = \sum_{Q, n} \frac{\langle P | H_T | Q \rangle \langle Q | H_T | R \rangle}{\epsilon_P - \epsilon_Q - n \hbar \omega} J_n \left( \frac{e V_{PQ} \hbar \omega}{\hbar \omega} \right) J_{N-n} \left( \frac{e V_{QR} \hbar \omega}{\hbar \omega} \right) \tag{5} \]

for the cotunneling amplitude between the initial state \( P \) and the final state \( R \) proceeding by virtually exciting and de-exciting a state \( Q \) (all being the states \(|klm\rangle\)). During the transition \( N \) photons in total are absorbed, split to \( n \) and \( N-n \) at the two steps. The resonance condition \( \epsilon_P + N \hbar \omega \approx \epsilon_Q \) defines \( N \). The amplitude of an \( n \)-photon process is proportional to the \( n \)-th Bessel function \( J_n \) and depends on the drop of the driving voltage amplitude between the corresponding states, \( V_{PQ} = V_P - V_Q \), with \( V_P \) the sum of the dot driving amplitudes \( V_{\alpha} \) weighted by the dot occupation number of the states forming the total state \( P \) (see SM, Ref. [44] for further details).

**Initial-final and virtual state resonance.** Equation (5) applies to a broad range of situations, which we now illustrate on several examples. We first discuss microwave assisted transfer of electrons, which, if spin preserving, can transport spin qubits between spatially separated storage and manipulation domains. Consider the triple dot structure containing two electrons in the "tunneling" configuration, sketched in Fig. 1a. The middle dot is gated below the outer dots and driven at frequency \( \omega \) and amplitude \( V \). We are interested in the photon assisted transfer between the outer dots, with the initial state \( P = |\sigma s 0\rangle \) and the final state \( R = |0 s' s'\rangle \), where \( \sigma, s = \uparrow, \downarrow \) denotes spin. The virtual states \( Q \) then comprise the middle dot either doubly occupied or empty.

We first consider an initial-final state resonance, which refers to a configuration with these states offset in energy. In analogy with the usual PAT, the \( N \)th sideband appears at frequency compensating the energy difference, \( \epsilon_R - \epsilon_P \approx N \hbar \omega \). Assuming this difference is small compared to the charging energy, a typical experimental condition, one can neglect \( n \hbar \omega \) in the first denominator of Eq. (5). This special resonance configuration enables non-local cotunneling [37, 45] but no enhancement is possible since the microwaves suppress the cotunneling amplitude by a factor of order \(|J_N(\sigma)| \leq 1 \) [see Eq. (S25) in SM].

The situation changes dramatically if the photon is matched to virtual states. We call it a "virtual" resonance defined by the initial and final states tuned to degeneracy electrostatically by gates, so that \( N = 0 \) applies in Eq. (5), and there is a value of \( n \) and a set of states \( Q \) for which the offsets
\[ \delta_Q = \epsilon_P - \epsilon_Q + n \hbar \omega \tag{6} \]

are much smaller than all other offsets. Such terms dominate the sum in Eq. (5) and other contributions can be neglected.

The condition on the states \( Q \) being virtual excitations limits the maximal ratio of the photon-assisted amplitude to the non-assisted amplitude to \( eV/2\tau_0 \). To get this estimate, we used Eq. (5) for weak fields \( V \), restricted the sum to a single near-resonant \( Q \) state and denoted the larger of the two matrix elements of \( H_T \) as \( \tau_0 \). We thus obtain our first important result: for \( \tau_0 \ll eV \) (weakly coupled dots), the cotunneling is boosted by microwaves without charging the middle dot.
We now analyze the spin dependence of cotunneling. We first note that the virtual state \( |k\rangle \) (an empty middle dot) in Eq. (41) leads to non-zero cotunneling only between states with simultaneously \( \sigma = s' \) and \( \sigma = s' \). The lack of spin dynamics follows from the lack of any spin structure of \( Q \). We thus consider the more interesting case of the photon frequency matched to doubly occupied states. To grasp qualitative features, we consider a symmetric structure with two single-electron orbitals per dot. This gives us four relevant virtual states \( |0\rangle \) with \( k \) being one of the triplets \( T_{0,\pm} \) or the singlet state \( S \). Within this simple model the three dots with two electrons are described by

\[
\mathcal{H} = \frac{1 + \eta_c}{2} \left( J \sigma_+ \cdot \sigma_+ + J_z \sigma_0^+ \sigma_0^- + (\sigma_0^+ + \sigma_0^-) b + c \right),
\]

where \( J = \frac{\tau^2}{\delta} (\delta_{T_0} - \delta_{S}) \), \( J_z = \frac{\tau^2}{\delta} (\delta_{T_0} - 2\delta_{T_0} - 2\delta_{T_0}) \), \( b = \frac{\tau^2}{\delta} (\delta_{S} - \delta_{T_0}) \), and \( c = \frac{\tau^2}{\delta} (\delta_{S} - \delta_{T_0} - 2\delta_{T_0} - 2\delta_{T_0}) \) with \( \nu_c \equiv \tau \delta_{J} (eV/h\omega) \). The nearest neighbor tunneling amplitude scale is \( \tau = |\tau_{AB}^2| \). The Pauli matrices \( \eta \) act in the pseudo-spin space with up/down being an electron in dot A/C. The superscripts o and i denote the spin of the outer (A/C) and inner (B) dot electron, respectively.

The effective Hamiltonian [see Eq. (7)] generates dynamics with spatial and spin rotations, in general, intertwined. We illustrate the degree of control that microwaves offer on special cases depicted in Fig. 4.

First, at far detuning (with all offsets approximately the same, \( \delta_k \approx \delta \)), the charge oscillates between the outer dots with spins frozen, see Fig. 2a. (Higher photon contributions, which might accidentally break the far-detuning condition for some \( n \), are suppressed exponentially \( J_{n} (eV/h\omega) \approx (eV/2h\omega)^n / n! \) at weak driving \( eV \ll h\omega \).) Second, we take the microwave frequency halfway between the singlet and the triplet states, \( \delta_T = \delta = -\delta_S \), see Fig. 4b. Although spins now rotate, the charge is transferred between the outer dots with spins returned to their initial state at time \( t_0 = \pi \tau_{ij}^2 / 2h\delta \). Compared to the previous case, the transfer is faster since the offset \( \delta \) is smaller. Third, we consider the special case \( |\delta_{T_0,\pm} \gg \delta_{T_0} = \delta = -\delta_{S} \), corresponding to Fig. 4c. This case arises if the \( g \)-factor is state dependent (different \( g \)-factor or Overhauser field for different orbital levels) or if the magnetic field is spatially dependent (the exploitation of which for PACT we leave for subsequent studies), and the external magnetic field tunes the exchange to almost zero by orbital effects \( [48] \). Alternatively, the exchange can also be tuned by gates, if we replace the dot by a singly occupied double-dot and detune it by a bias \( [44] \). At \( t_0 \), the charge is transferred only for unpolarized-spin configurations, effectively allowing for a single-shot measurement of the total spin by a conversion to charge.

Spin echo. Spin echo techniques are standard in spin control \([51, 51]\). We demonstrate their usefulness for PACT on the charge transfer discussed above \([58]\). We aim at transport that is robustly spin-preserving (independent on virtual state offsets). It is generated by a propagator \( U_{\text{eff}}(t) \) corresponding to the Hamiltonian in Eq. (7) with terms containing \( \sigma \) matrices being removed. We find that such a propagator is realized by the following echo sequence

\[
U_{\text{eff}}(t) = U(t/4) \Pi^o_x \Pi^o_y U(t/4) \Pi^o_y U(t/4) \Pi^o_x \Pi^o_y U(t/4) \Pi^o_y,
\]

where \( U(t) = \exp(-i\mathcal{H}t/\hbar) \) is the propagator generated by the Hamiltonian at hand, and \( \Pi^o_x \) (\( \Pi^o_y \)) is the inner (outer) spin \( \pi \)-rotation around axis \( n \). For special configurations, the sequence can be further simplified.

Exchange. Next, we consider the standard setup for a spin qubit based quantum processor with single-electron dots (A, B, C) \([12]\). Can microwaves speed-up the interdot spin-spin exchange? The standard derivation for tunnel coupled dots gives the exchange as \( J_{AB} \sim 4\tau_{AB}^2 / U \) with \( U \) being the single dot charging energy. Using cotunneling amplitudes derived in the previous section, the analog of this formula gives a non-local exchange \( J_{AC} \sim 4\tau_{AC}^2 / U \). Such terms indeed arise and are enhanced by microwaves, however, are subdominant. Namely, a resonant microwave field, required to enhance \( \tau_{AC} \), inevitably boosts the inner-outter dot exchange \( J_{AB} \) and \( J_{BC} \) \([52]\). Since these are of order \( \tau^2 \), the non-local exchange given above (also known as superexchange \([12, 53]\)) of order \( \tau_0^2 \sim \tau^4 \) is negligible. It would then seem that the most efficient way to induce non-local operations is to concatenate nearest neighbor ones. Remarkably, we find that one can do better: simultaneously boosted pairwise spin rotations between dots A-B and B-C can conspire to induce a fast and useful interaction between the outer dots A and C without influencing the mediator dot B \([54]\) for specially tuned microwave frequency and/or if assisted by spin echoes. (For yet another alternative, see the long-
distance part below).

To demonstrate this, we adopt the model described previously and, in analogy with the derivation of Eq. (7), derive the Hamiltonian $H_{SWAP} + H_d + H_z$ describing three electron spins in three dots with the middle dot detuned from the aligned outer dots.

$$H_{SWAP} = \tau_0^2 \delta s_1^{-1} - \delta p_1^{-1} \sum_{\alpha=A,C} (\sigma_\alpha^s \sigma^B_\alpha + \sigma_\alpha^s \sigma^B_\alpha),$$

(9)

implements the spin SWAP between outer dots,

$$H_d = -\tau_0^2 d_1^{-1} | \uparrow \uparrow \rangle \langle \uparrow \uparrow | - \tau_0^2 d_1^{-1} | \downarrow \downarrow \rangle \langle \downarrow \downarrow |,$$

(10)

by which the propagator departs from SWAP, and the Zeeman-like term $H_z = -\tau_0^2 \sum_s P_s d_s^{-1}$ is a sum of projectors $P_s$ on subspaces of total spin $z$-projection $s \in \{3/2, 1/2, -1/2, -3/2\}$ with the corresponding inverse energies $E_s^{-1} \in \{2 \delta_{T_+}^{-1}, 2 \delta_{T_+}^{-1} + d_1^{-1}, 2 \delta_{T_+}^{-1} + d_1^{-1}, 2 \delta_{T_+}^{-1}\}$. Here, $d_1^{-1} = (\delta_{T_0}^{-1} + \delta_{S_0}^{-1})/2 - \delta_{T_1}^{-1}$ and $\sigma_+ = \sigma_x \pm i \sigma_y$. Rather than giving analytical results for the propagator in general, we note that in the configuration of Fig. 2, the error-causing term $H_d$ affecting the SWAP is small. Furthermore, $\delta_{T_+} \approx -\delta_{T_0}$ and $d_+ \approx -d_-$ to a very good accuracy. Flipping all spins at time $t/2$ around an axis perpendicular to the $z$-axis then removes $H_z$ (completely) and $H_d$ (in leading order) from the time evolution. We once again find that PACT qualitatively outperforms other schemes at specially designed configurations (here the one of Fig. 2).

**Long distance scaling.** To investigate the PACT amplitudes scaling with distance, we expand the array to $M = 1$ singly occupied dots with uniform interdot tunneling as depicted in Fig. 1. The structure is tuned such that there is a band of virtual states (indexed by $q$) delocalized over the whole linear array (the tuning is detailed in Fig. 1C) with wavefunctions and energies

$$\Psi_q(j) = \sqrt{\frac{2}{M}} \sin \left( \frac{\pi q}{M} j \right), \quad \epsilon_q = \epsilon_B + 4 \tau \sin^2 \left( \frac{\pi q}{2M} \right),$$

(11)

with the position in the array $j; q, j \in [1, M - 1]$. To regularize spurious divergencies of the continuum model adopted below, it is useful to take into account the finite lifetime of the states. We do so phenomenologically by adding an imaginary part to the energy with typical value $\gamma < 1 \mu eV$ for gated quantum dots. The imaginary part of Eq. (12) for $P = R$ then corresponds to the rate $\Gamma$ with which the state leaves the computational space (leakage). For useful manipulations, one has to make sure that $\Gamma \ll \tau_{co}$. With the above, Eq. (12) gives

$$\tau_{co} = \tau_0^2 \sum_q \frac{\Psi_q^\dagger(j_A) \Psi_q(j_C)}{\delta - \epsilon_q + i \gamma},$$

(12)

with the band detuning $\delta = \epsilon_A - \epsilon_B - n \hbar \omega$ and the manipulated dots positions $j_A, j_C$, and the spatial distance $d = j_C - j_A$. To proceed, we assume the manipulated dots are not too close $\delta d_0$ to the array edges. We can then replace the band wave functions by plane waves and reduce the numerator in Eq. (12) to a phase factor $e^{i \delta d}$. In the continuum limit, appropriate for $M \gg 1$, we estimate

$$\tau_{co} \sim 2 \tau J_n^2 \left( \frac{eV}{\hbar \omega} \right) \sqrt{\frac{\tau}{\delta}} \times \min \{1, \frac{4 d_0}{\pi^2 d} \},$$

(13)

with a crossover distance $d_0 = \sqrt{\tau/4 \delta}$. Both the cotunneling amplitude and spatial range are boosted by tuning the photon energy to the band edge. The minimal allowed value for $\delta$ is set by the leakage. With similar approximations as before we get

$$\Gamma \sim 2 \tau J_n^2 \left( \frac{eV}{\hbar \omega} \right) \sqrt{\frac{\tau}{\delta}} \times \frac{\gamma}{\delta}.$$  

(14)

We singled out the last term being the leakage suppression factor with respect to the cotunneling. The cotunneling is therefore ultimately limited by the virtual state coherence $\gamma$.

The inverse distance decay $1/d$ for $d > d_0$ originates from destructive interferences of the phases $\phi_q$, a general feature $[55-57]$. Such interferences do not influence the incoherent leakage, which therefore dominates at large distances. However, for intermediate distances, Eq. (13) gives the rate for a useful spin-preserving non-local electron transfer or effective spin-spin exchange (depending on how the manipulated dots are gated, as described in Fig.1C). Its constant value for distances up to $d_0$, which can easily be large, is a remarkable result demonstrating that microwaves enable long-distance coherent manipulations in spin qubit arrays.

**Conclusions.** We investigated the photon assisted cotunneling in an array of weakly coupled quantum dots. We found that microwaves may bring substantial advantages for various aspects of spin qubit control, and demonstrated it on specific configurations. Overall, we showed how to use microwaves to increase operation speed, enhance control over the spin, and generate long-range interactions useful for spin qubit manipulations.

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Supplemental Material to ”Fast Long-Distance Control of Spin Qubits by Photon Assisted Cotunneling"

Here we derive the key result, Eq. (5), of the main text, and give some more details on it.

\textbf{SHIRLEY TECHNIQUE FOR FLOQUET THEORY: DERIVATION OF EQ. (5)}

The microscopic Hamiltonian of our system, Eq. (1), is time dependent. In general, a calculation of a propagator for it is much more complicated than for a time independent one. However, since in our case the Hamiltonian contains only discrete frequencies, we can recast the time dependent problem into a time-independent one. The procedure is based on the Floquet theorem \cite{29} and was worked out in the excellent work of J. Shirley \cite{43}. We now restate the results of this work that are of direct relevance for us and refer the reader therein for more.

We restrict ourselves to the case of a single frequency $\omega = 2\pi/T$ present in the Hamiltonian $H(t)$ acting in some Hilbert space spanned by a basis \{\ket{\kappa}\}. The basis is extended into tensor product states $\ket{km} \equiv \ket{\kappa} \otimes \ket{n}$, with $n$ taking integer values from minus to plus infinity. The state $\ket{n}$ is associated with the function $\exp(in\omega t)$.$[^{[a]}]$ A time dependent function $f(t)$ is associated with the matrix elements in the added part of the Hilbert space according to the following rule

$$\langle n|f(t)|m\rangle = \frac{1}{T} \int_0^T dt \ e^{-im\omega t}f(t)e^{im\omega t} \equiv f^{n-m}, \quad (S1)$$

with the last equality sign being a definition of the Fourier transform.$[^{[b]}]$

The propagator evolving the system from time $t_0$ to time $t$ in the original Hilbert space is given by \[Eq. (13)\] in Ref. $^{[43]}$ as

$$U(t,t_0) = \sum_{\kappa\lambda} |\kappa\rangle \langle \lambda| \sum_n e^{in\omega t} \sum_{\kappa\lambda} e^{im\omega t} \times \langle \kappa m|\exp \left( -\frac{i}{\hbar} H_F(t-t_0) \right)|\lambda\rangle, \quad (S2)$$

with the expanded Hilbert space Hamiltonian defined as

$$H_F = -i\hbar \partial_t + H(t). \quad (S3)$$

Its matrix elements follow from \[Eq. (S1)\] as $[Eq. (10)\text{ in Ref. }^{[43]}]$ as

$$\langle kn|H_F|\lambda m\rangle = \langle \kappa|H|\lambda\rangle^{n-m} + nh\omega \delta_{nm} \delta_{\kappa\lambda}, \quad (S4)$$

with $\delta$ the Kronecker delta symbols. The advantage of the described mapping can be appreciated from \[Eq. (S2)\], where the second line takes the form of a propagator of a time-independent problem. Therefore, its calculation is amenable to corresponding perturbative techniques. In another words, the propagator calculation is reduced to a matrix $H_F$ eigenvalue problem.

Let us now consider an illustrative case. Suppose the Hilbert space \{\ket{\kappa}\} consists of three states only and let them be denoted as $\kappa = P, Q, R$. This covers essentially all configurations considered in the main text with the three states being, respectively, an initial, virtual, and final, upon various different identifications of these states with the states $\ket{km}$. As a specific realization, one might consider two electrons in three dots, with the initial state being the right most dot empty, $P = |\sigma s0\rangle$, the final state being the left most dot empty, $R = |0\sigma s\rangle$, and the virtual state being the middle dot empty, $Q = |\sigma 0s\rangle$. The middle dot, driven by microwaves, is gated such that its other states (such as doubly occupied) are far away in energy making their contribution negligible. (The condition will be specified more precisely later). The Hamiltonian restricted to this three state subspace is

$$H(t) = \sum_{\kappa=P,Q,R} \left( \epsilon_\kappa + eV_\kappa \cos(\omega t) \right) |\kappa\rangle \langle \kappa| + \left( \tau_{PQ} |P\rangle \langle Q| + \tau_{QR} |R\rangle \langle Q| + \text{h.c.} \right), \quad (S5)$$

comprising the energies and inter-dot tunneling terms.

Here, h.c. stands for Hermitian conjugate. Our choice of driving the middle dot at potential amplitude $V$ gives $V_P = V$, $V_Q = 0$, and $V_R = V$. Finally,

$$\tau_{\kappa\lambda} = \langle \kappa|H_T|\lambda\rangle, \quad (S6)$$

denotes the tunneling amplitudes.

Next, we calculate the matrix $H_F$ with $H(t)$ given by

\[\text{[a]}\] Change of the value of this index corresponds to a change in the number of photons, with the correspondence explained in Ref. $^{[43]}$. We will also use the word photons in this sense.

\[\text{[b]}\] In the case of multiple frequencies, the procedure is straightforwardly generalized, with the extended Hilbert space state being $|\kappa \otimes n_1 \otimes n_2 \otimes \ldots \otimes n_m\rangle$, with $m$ the number of discrete frequencies. The right hand side of \[Eq. (S1)\] is then to be understood as taking the limit $T \to \infty$. 


Eq. (S5) according to Eqs. (S1) and (S4). We obtain

$$
\begin{pmatrix}
\cdots & \mathcal{P}_0 & \mathcal{Q}_0 & \mathcal{R}_0 & \mathcal{P}_1 & \mathcal{Q}_1 & \mathcal{R}_1 & \mathcal{P}_2 & \mathcal{Q}_2 & \mathcal{R}_2 \\
\mathcal{P}_0 & \epsilon_{\mathcal{P}_0} & \epsilon_{\mathcal{Q}_0} & \epsilon_{\mathcal{R}_0} & \epsilon_{\mathcal{P}_1} & \epsilon_{\mathcal{Q}_1} & \epsilon_{\mathcal{R}_1} & \epsilon_{\mathcal{P}_2} & \epsilon_{\mathcal{Q}_2} & \epsilon_{\mathcal{R}_2} \\
\mathcal{Q}_0 & \tau_{\mathcal{P}_\mathcal{Q}} & \epsilon_{\mathcal{Q}_0} & \tau_{\mathcal{Q}_\mathcal{R}} & \epsilon_{\mathcal{P}_1} & \tau_{\mathcal{Q}_\mathcal{R}} & \epsilon_{\mathcal{R}_1} & \tau_{\mathcal{P}_\mathcal{Q}} & \epsilon_{\mathcal{Q}_2} & \tau_{\mathcal{Q}_\mathcal{R}} \\
\mathcal{R}_0 & \tau_{\mathcal{Q}_\mathcal{R}} & \epsilon_{\mathcal{R}_0} & \epsilon_{\mathcal{R}_1} & \epsilon_{\mathcal{P}_1} & \tau_{\mathcal{Q}_\mathcal{R}} & \epsilon_{\mathcal{R}_2} & \tau_{\mathcal{Q}_\mathcal{R}} & \epsilon_{\mathcal{R}_1} & \epsilon_{\mathcal{R}_2} \\
\mathcal{P}_1 & \epsilon_{\mathcal{P}_1} & \tau_{\mathcal{P}_\mathcal{Q}} & \epsilon_{\mathcal{P}_0} & \epsilon_{\mathcal{P}_1} & \epsilon_{\mathcal{P}_0} & \epsilon_{\mathcal{P}_1} & \epsilon_{\mathcal{P}_0} & \epsilon_{\mathcal{P}_1} & \epsilon_{\mathcal{P}_0} \\
\mathcal{Q}_1 & \epsilon_{\mathcal{Q}_1} & \tau_{\mathcal{Q}_\mathcal{R}} & \epsilon_{\mathcal{Q}_1} & \epsilon_{\mathcal{Q}_1} & \epsilon_{\mathcal{Q}_1} & \epsilon_{\mathcal{Q}_1} & \epsilon_{\mathcal{Q}_1} & \epsilon_{\mathcal{Q}_1} & \epsilon_{\mathcal{Q}_1} \\
\mathcal{R}_1 & \epsilon_{\mathcal{R}_1} & \epsilon_{\mathcal{R}_1} & \epsilon_{\mathcal{R}_1} & \epsilon_{\mathcal{R}_1} & \epsilon_{\mathcal{R}_1} & \epsilon_{\mathcal{R}_1} & \epsilon_{\mathcal{R}_1} & \epsilon_{\mathcal{R}_1} & \epsilon_{\mathcal{R}_1} \\
\mathcal{P}_2 & \epsilon_{\mathcal{P}_2} & \tau_{\mathcal{P}_\mathcal{Q}} & \epsilon_{\mathcal{P}_2} & \epsilon_{\mathcal{P}_2} & \epsilon_{\mathcal{P}_2} & \epsilon_{\mathcal{P}_2} & \epsilon_{\mathcal{P}_2} & \epsilon_{\mathcal{P}_2} & \epsilon_{\mathcal{P}_2} \\
\mathcal{Q}_2 & \epsilon_{\mathcal{Q}_2} & \tau_{\mathcal{Q}_\mathcal{R}} & \epsilon_{\mathcal{Q}_2} & \epsilon_{\mathcal{Q}_2} & \epsilon_{\mathcal{Q}_2} & \epsilon_{\mathcal{Q}_2} & \epsilon_{\mathcal{Q}_2} & \epsilon_{\mathcal{Q}_2} & \epsilon_{\mathcal{Q}_2} \\
\mathcal{R}_2 & \epsilon_{\mathcal{R}_2} & \epsilon_{\mathcal{R}_2} & \epsilon_{\mathcal{R}_2} & \epsilon_{\mathcal{R}_2} & \epsilon_{\mathcal{R}_2} & \epsilon_{\mathcal{R}_2} & \epsilon_{\mathcal{R}_2} & \epsilon_{\mathcal{R}_2} & \epsilon_{\mathcal{R}_2} \\
\end{pmatrix}
$$

where the matrix indexes are indicated by the row and column labels. For space reasons we introduced a short hand notation for the energies

$$
\epsilon_{\kappa \nu} = \epsilon_{\kappa} + n \hbar \omega. \quad (S7)
$$

What we give in Eq. (S7) is a finite block of an infinite matrix, which is symbolized by the three dots in the left upper corner. The index \(n\) takes negative as well as positive integer values, so that the first state which is not shown to the left in the first row would have index \(\mathcal{R}(-1)\), while the next one continuing to the right would be \(\mathcal{P}3\), and so on. Even though the matrix is infinite, the calculations are tractable because it has a periodic structure already visible in Eq. (S7).

$$
\langle \kappa(n + m)|H_F|\lambda(n' + m)\rangle = \langle \kappa n \mid m \hbar \omega + H_F \mid \lambda n' \rangle. \quad (S9)
$$

Namely, upon a shift of the the integer index the matrix elements are identical up to adding a constant on the diagonal.

Let us now suppose that the dots are tuned close to a single photon initial-final state resonance, corresponding to \(N = 1\) in the notation of Eq. (5) or denoting the corresponding energy explicitly as \(\epsilon\).

$$
\epsilon = \epsilon_\mathcal{R} \approx \epsilon_\mathcal{P} + \hbar \omega. \quad (S10)
$$

We are interested in the dynamics of the system starting in the initial state \(\mathcal{P}\). Associating it with \(\mathcal{P}1\) in the matrix \(H_F\) [the choice of the value of \(n\) is arbitrary, because of the periodic structure given in Eq. (S9)], we note that this state is, by Eq. (S10), degenerate with state \(\mathcal{R}0\). Together they span a degenerate subspace which we denote by projector

$$
P = |\mathcal{P}1\rangle \langle \mathcal{P}1| + |\mathcal{R}0\rangle \langle \mathcal{R}0|. \quad (S11)
$$

If the matrix elements of \(H_F\) between a state from subspace \(\mathcal{P}\) and another one from its complement \(Q = 1 - \mathcal{P}\) are much smaller than the difference of the corresponding diagonal entries, the dynamics produced by \(H_F\) will be well approximated by restricting the basis to the degenerate subspace and taking the effects of other states perturbatively. We calculate the matrix elements of the effective Hamiltonian \(H_F\) for the subspace \(\mathcal{P}\) using the following formula derived by the Brillouin-Wigner perturbation method

$$
H_F = \mathcal{P} H_F \mathcal{P} + \mathcal{P} H_F^Q \frac{1}{\epsilon - H_F^Q} \mathcal{Q} H_F^Q \mathcal{P}. \quad (S12)
$$

The matrix divisions should be understood as \(\mathcal{X} = X \cdot Y^{-1}\), and \(H_F^Q\) is the diagonal/off-diagonal part of the matrix \(H_F\). The energy \(E\) is the eigenvalue of the eigenstate of \(H_F\), by which the equation is a self-consistent one (non-linear) in principle. However, this drawback is in practice not substantial, as one can solve order by order in the off-diagonal elements of \(H_F\). The consecutively higher orders are indexed by the summation index \(p\). To derive the results of the main text, we need only the lowest order of this formula, \(p = 0\) (the second order in off-diagonal matrix elements) for which one can replace \(E\) by \(\epsilon\) and get an ordinary (linear) Schröedinger equation with the Hamiltonian

$$
H_F \approx \mathcal{P} H_F \mathcal{P} + \mathcal{P} H_F^Q \frac{1}{\epsilon - H_F^Q} \mathcal{Q} H_F^Q \mathcal{P}. \quad (S13)
$$

The formula is valid if the off-diagonal elements and the energy differences of states within the subspace \(\mathcal{P}\) are much smaller than the energy denominator, which is, respectively, required for the convergence of the sum over \(p\) in Eq. (S12), and for the replacement \(E \rightarrow \epsilon\). Importantly, Eq. (S13) is valid for any dimension of \(\mathcal{P}\) and \(\mathcal{Q}\) and not only for our specific example of three states. In practical calculations, it is usually straightforward to identify the most relevant virtual states that are to be retained in the subspace \(\mathcal{Q}\). The majority of states can be neglected. Some states strictly do not contribute as there is no non-zero matrix element connecting them to the subspace \(\mathcal{P}\) (such as states with different number of electrons, or different spin). The contribution of other states is subdominant in requiring more dot-dot hoppings (this is the case of the super-exchange discussed in the main text page 3) or is negligible due to a large energy cost [the denominator in Eq. (S13)], which would be the case, e.g., for states involving excited single particle orbitals.

We now return to our example, with \(H_F\) given in Eq. (S7). Trying to apply Eq. (S13), however, we find that there is no term contributing to the the matrix element \(\langle \mathcal{P}1|H_F|\mathcal{R}0\rangle\) in the second order of \(H_F\). [c] Using

[c] To go from initial to the final state, it is needed to nearest-neighbor tunnel twice and absorb photon(s) once. Each of these corresponds to an off diagonal element in \(H_F\), at minimum three off-diagonal terms together.
Eq. (S12) in the next order p = 1, we get

$$\langle P|H_P|R0\rangle \approx \frac{eV}{2\hbar\omega} \tau_{pq} \tau_{QR} - \frac{eV}{2\hbar\omega} \tau_{pq} \tau_{QR}.$$  (S14)

This result shows that, apart from generating contributions in higher orders only, the matrix in Eq. (S2) potentially breaks the assumption of the off-diagonal elements being small compared to the energy differences. Namely, even though we always assume neighboring dots are gated such that the tunneling amplitude is small compared to the detuning of the nearest empty state where an electron can hop in, the appearance of the ratio $eV/\hbar\omega$ would restrict the validity of our results to weak driving only, $eV \ll \hbar\omega$. To account for both of these issues, we introduce a unitary transformation as the last step necessary to obtain the cotunneling amplitudes given in the main text.

We introduce a new basis (with states denoted by a tilde) by the following formula (this step goes beyond Ref. 43)

$$|\tilde{\kappa}n\rangle = \sum_m J_m(eV/\hbar\omega)|\kappa(n-m)\rangle,$$  (S15)

with $V_\kappa$ the driving amplitude of the particular state, and $J_m(x)$ the Bessel function of the first kind. Using the sum rule for the Bessel functions [see Eq. (S28) and Eq. (S30)], one can check that the new basis is also orthonormal and the matrix relating the old and new basis is, therefore, unitary with matrix elements

$$\langle \lambda m|\tilde{\kappa}n\rangle = \delta_{\kappa\lambda} J_{n-m}(eV/\hbar\omega).$$  (S16)

Using the time dependent representation of $|n\rangle$,

$$|\tilde{\kappa}n\rangle \to \sum_m J_m\left(\frac{eV}{\hbar\omega}\right)e^{i(n-m)\omega t}|\kappa\rangle = e^{i\omega t}|\tilde{\kappa}\rangle,$$  (S17)

along with the notation

$$|\tilde{\kappa}(t)\rangle \equiv |\kappa\rangle \exp\left\{-\frac{i}{\hbar} \int_0^t dt' eV_\kappa \cos(\omega t')\right\},$$  (S18)

one can understand the choice in Eq. (4) as accommodating the basis to include the accumulated phase from the oscillating part of the energy.

In this basis the matrix $H_F$ takes the form

\[
\begin{array}{cccccccc}
\cdots & \tilde{P}0 & \tilde{Q}0 & \tilde{R}0 & \tilde{P}1 & \tilde{Q}1 & \tilde{R}1 & \tilde{P}2 & \tilde{Q}2 & \tilde{R}2 \\
\tilde{P}0 & \epsilon_0 & \tau_{PQ}^{(0)} & \tau_{PR}^{(0)} & \tau_{PQ}^{(-1)} & \tau_{PR}^{(-1)} & \tau_{PQ}^{(-2)} & \tau_{PR}^{(-2)} \\
\tilde{Q}0 & \tau_{QP}^{(0)} & \epsilon_0 & \tau_{QR}^{(0)} & \tau_{QP}^{(1)} & \tau_{QR}^{(1)} & \tau_{QP}^{(2)} & \tau_{QR}^{(2)} \\
\tilde{R}0 & \tau_{RP}^{(0)} & \tau_{RP}^{(0)} & \epsilon_0 & \tau_{RR}^{(0)} & \tau_{RP}^{(-1)} & \tau_{RR}^{(-1)} & \tau_{RP}^{(-2)} & \tau_{RR}^{(-2)} \\
\tilde{P}1 & \tau_{RP}^{(-1)} & \tau_{RP}^{(-1)} & \tau_{RP}^{(-1)} & \epsilon_1 & \tau_{RP}^{(0)} & \tau_{RP}^{(0)} & \tau_{RP}^{(0)} & \tau_{RP}^{(0)} \\
\tilde{Q}1 & \tau_{QP}^{(-1)} & \tau_{QP}^{(-1)} & \tau_{QP}^{(-1)} & \tau_{QP}^{(0)} & \epsilon_1 & \tau_{QP}^{(-0)} & \tau_{QP}^{(-0)} & \tau_{QP}^{(-0)} \\
\tilde{R}1 & \tau_{QR}^{(-1)} & \tau_{QR}^{(-1)} & \tau_{QR}^{(-1)} & \tau_{QR}^{(-1)} & \tau_{QR}^{(-0)} & \epsilon_1 & \tau_{QR}^{(-0)} & \tau_{QR}^{(-0)} \\
\tilde{P}2 & \tau_{RP}^{(-2)} & \tau_{RP}^{(-2)} & \tau_{RP}^{(-2)} & \tau_{RP}^{(-2)} & \tau_{RP}^{(-2)} & \tau_{RP}^{(-2)} & \epsilon_2 & \tau_{RP}^{(-2)} & \tau_{RP}^{(-2)} \\
\tilde{Q}2 & \tau_{QP}^{(-2)} & \tau_{QP}^{(-2)} & \tau_{QP}^{(-2)} & \tau_{QP}^{(-2)} & \tau_{QP}^{(-2)} & \tau_{QP}^{(-2)} & \tau_{QP}^{(-2)} & \epsilon_2 & \tau_{QP}^{(-2)} & \tau_{QP}^{(-2)} \\
\tilde{R}2 & \tau_{QR}^{(-2)} & \tau_{QR}^{(-2)} & \tau_{QR}^{(-2)} & \tau_{QR}^{(-2)} & \tau_{QR}^{(-2)} & \tau_{QR}^{(-2)} & \tau_{QR}^{(-2)} & \tau_{QR}^{(-2)} & \epsilon_2 & \tau_{QR}^{(-2)} \\
\end{array}
\]

The terms $eV/2$ were removed from the off-diagonal on the expense of generating more tunneling elements. The inter-dot tunneling Hamiltonian now has elements between states with different integer indexes\[d\]

$$\tau_{\kappa\lambda}^{(n-m)} = \langle \tilde{\kappa}n|H_T|\tilde{\lambda}m\rangle = \tau_{\kappa\lambda} J_{m-n}(eV_\kappa/\hbar\omega),$$  (S20)

with $V_{\kappa\lambda} = V_\kappa - V_\lambda$ the voltage amplitude drop between the two states. The amplitude of the total many-body state $\kappa = |k_\alpha k_B k_C\rangle$ is defined as $V_\kappa = V_\lambda n(k_\alpha) + V_B n(k_B) + V_C n(k_C)$ with $n(k)$ the number of electrons in the state $|k\rangle$. Since the Bessel functions are not larger than one for any real parameter, the matrix $H_F$ in the newly adopted basis is suitable for perturbative calculations even for a strong driving, $eV \gg \hbar\omega$.

We note that such complete removal of the driving terms proportional to the voltage is possible because of the form of the driving part of the Hamiltonian that we chose in Eq. (4). By that we neglect the spatial deformation of dot states induced by the electric field. This simplification can make a qualitative difference only if such terms would break some symmetry which blocks tunnelings.\[30\] There is no such symmetry in our case. Also, this simplification is not essential for using the Shirley technique.

The propagator in the transformed basis is

$$U(t, t_0) = \sum_{\kappa\lambda n} |\tilde{\kappa}(t)\rangle \langle \tilde{\lambda}(t_0)|e^{i\omega(t-t_0)}|U_F|\tilde{\lambda}(0)\rangle,$$  (S21)

where the bra vector $\langle \tilde{\lambda}(t)|$ is defined by a complex conjugation of Eq. (S13), and $U_F = \exp\{-i/\hbar(H_F(t-t_0))\}$. This is a complete analogue to Eq. (S2).

We now evaluate off-diagonal elements of the effective Hamiltonian in the transformed basis. Using Eq. (S19) in Eq. (S13) we get

$$\langle \tilde{P}|H_P|R0\rangle = \sum_{n=-\infty}^{\infty} \frac{\langle n|\tau^{(0)}_{PQ}\tau^{(1-n)}_{QR}\rangle}{\epsilon - \epsilon_Q - n\hbar\omega}.$$  (S22)

The validity of the formula follows from conditions on the validity of Eq. (S13), which were stated therein. In terms of the parameters used here, both the photon assisted tunneling amplitudes and the degeneracy detuning should be smaller than the energy denominator

$$|\tau^{(n)}_{PQ}|, |\tau^{(1-n)}_{QR}|, |\epsilon_R - \epsilon_F - \hbar\omega| \ll |\epsilon - \epsilon_Q - n\hbar\omega|$$  (S23)

for each term in the summation over $n$, which is the photon index.

Equation (S22) takes the microwave field into account to all orders, enumerated by index $n$. It would be difficult

---

\[d\] In the original basis a time independent operator is diagonal in the photon index, which is no more the case in the transformed basis.
to include such higher order processes using Eq. (S7), as they correspond to higher order of perturbation expansion in Eq. (S12). The correspondence between the two bases can be established upon expanding the Bessel functions in their argument. For weak driving the tunneling amplitudes \(\tau^{(n)}\) fall off exponentially with \(|n|\). In such a case, retaining only the leading order terms, \(n = 0, 1\), using Eq. (S20), expanding the Bessel functions up to the lowest order, and using Eq. (S31), Eq. (S22) reduces to Eq. (S13).

We can now easily generalize to \(N\) photon resonance, and to more intermediate states \(Q\). The former means that the initial and final state energies differ by \(N\hbar\omega\), and the generalization amounts to replacing the index 1 by \(N\) in Eq. (S22). The latter means a summation over the intermediate states, as contributions from subspace \(Q\) are additive in Eq. (S13). With these generalizations we get

\[
\langle \mathcal{P} N | H_P | \mathcal{R} 0 \rangle = \sum_{Q} \sum_{n=-\infty}^{\infty} \frac{\tau^{(n)}_P \tau^{(n)}_R}{\epsilon - \epsilon_Q - n\hbar\omega} J_N \left( \frac{\nu_{PR}}{\hbar\omega} \right). \tag{S24}
\]

Denoting the left hand side as \(\tau_{ca}\), and relating to the parameters of the original Hamiltonian using Eqs. (S6) and (S20), we obtain Eq. (5) of the main text.

We remind that the “initial-final” state resonance used in the main text refers to a configuration in which the initial and final states differ in energy. The system can make the transition between these two states only if a non-zero number of photons is absorbed in total, with the photon number \(N\) given by the resonance condition \(N\hbar\omega\) equal to the initial and final state energy difference. In the often met case of the initial to final state detuning being much smaller than the virtual states offsets, so that one can neglect the photon energies in the denominator of Eq. (S24), the sum over the photon index can be evaluated using Eq. (S28) to arrive at

\[
\langle \mathcal{P} N | H_P | \mathcal{R} 0 \rangle \approx \left( \sum_{Q} \frac{\tau_{Q} \tau_{R}}{\epsilon - \epsilon_Q} \right) J_N \left( \frac{\nu_{PR}}{\hbar\omega} \right). \tag{S25}
\]

This result corroborates the corresponding claims of the main text, showing explicitly that in this configuration the photon assisted cotunneling is proportional to a cotunneling without any driving (the first bracket) times a suppression factor depending on the relative amplitude of driving of the initial and final states. The role of microwaves is to allow for observation of the cotunneling by coupling the resonant states \(|\mathcal{P} N\rangle\) and \(|\mathcal{R} 0\rangle\). The result also shows that in this regime driving the mediating states is ineffective.

The “virtual resonance”, on the other hand, corresponds to \(N = 0\), so that energy does not have to be provided by microwaves in order for the transition to occur. Rather, the resonance now means that the microwave frequency is such that a virtual state \(Q\) (or a few states) becomes quasi-degenerate with the initial and final state energies upon adding the energy of \(n\) photons. Based on a much smaller energy denominator, the sum in Eq. (S24) can be approximated by this single term \(n\) and state \(Q\).

Once the cotunneling amplitude is calculated, the dynamics of the system can be found using the effective Hamiltonian \(H_P\), which in the subspace \(|\mathcal{P} N\rangle, |\mathcal{R} 0\rangle\) takes the form

\[
H_P = \begin{pmatrix} \epsilon_P + N\hbar\omega \tau_{ca} & \tau_{ca} \epsilon_R \\ \tau_{ca} \epsilon_R & \epsilon_R \end{pmatrix}, \tag{S26}
\]

and the propagator in the original Hilbert space basis \(|\mathcal{P}, |\mathcal{R}\rangle\), with the phase factors defined in Eq. (S18), takes the form

\[
U(t_0, t) = \begin{pmatrix} e^{iN\omega(t-t_0)} & 0 \\ 0 & 1 \end{pmatrix} \exp \left\{ -\frac{\hbar i}{\hbar} H_P(t-t_0) \right\}. \tag{S27}
\]

Close to resonance \(\epsilon_R \approx \epsilon_P + N\hbar\omega\) the system will display Rabi oscillations with frequency \(\tau_{ca}\). This finishes the derivation and interpretation of the photon assisted cotunneling amplitudes.

**Bessel functions properties**

Here we list a few properties of Bessel functions which are needed in the discussed derivations,

\[
J_n(z_1 \pm z_2) = \sum_{k=-\infty}^{\infty} J_{n-k}(z_1) J_k(z_2), \tag{S28}
\]

\[
J_{-n}(x) = (-1)^n J_n(x), \tag{S29}
\]

\[
J_n(0) = \delta_{n,0}, \tag{S30}
\]

\[
J_n(x) \approx (x/2)^n/n!. \tag{S31}
\]

In the last equation, the result is given in the leading order expansion around \(x = 0\) assuming \(n\) to be a positive integer.