WELL-POSEDNESS FOR KDV-TYPE EQUATIONS
WITH QUADRATIC NONLINEARITY

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Abstract. We consider the Cauchy problem of the KdV-type equation
\[ \partial_t u + \frac{1}{3} \partial_x^3 u = c_1 u \partial_x^2 u + c_2 (\partial_x u)^2, \quad u(0) = u_0. \]
Pilod (2008) showed that the flow map of this Cauchy problem fails to be twice differentiable in the Sobolev space \( H^s(\mathbb{R}) \) for any \( s \in \mathbb{R} \) if \( c_1 \neq 0 \). By using a gauge transformation, we point out that the contraction mapping theorem is applicable to the Cauchy problem if the initial data are in \( H^2(\mathbb{R}) \) with bounded primitives. Moreover, we prove that the Cauchy problem is locally well-posed in \( H^1(\mathbb{R}) \) with bounded primitives.

1. Introduction

We consider the Cauchy problem for the Korteweg-de Vries (KdV) type equation
\[ \partial_t u + \frac{1}{3} \partial_x^3 u = c_1 u \partial_x^2 u + c_2 (\partial_x u)^2, \]
where \( u \) is a real valued function and \( c_1 \) and \( c_2 \) are real constants.

If \( c_1 = 0 \), because \( \partial_x u \) satisfies the KdV equation, the results by Kenig et al. [13] and Kishimoto [8] imply that (1.1) is well-posed in the Sobolev space \( H^s(\mathbb{R}) \) for \( s \geq \frac{1}{3} \). On the other hand, Tarama [24] proved that even a linear equation requires a Mizohata-type condition for the well-posedness in \( L^2(\mathbb{R}) \) (see also [18]). Indeed, the linear equation
\[ (\partial_x + \partial_x^3 + a(x) \partial_x^2) u = 0 \]
where \( a \) is smooth with bounded derivatives is well-posed in \( L^2(\mathbb{R}) \) if and only if
\[ \sup_{x_1 \leq x_2} \int_{x_1}^{x_2} a(x) dx < \infty \]
holds. Hence, at least, well-posedness in \( H^s(\mathbb{R}) \) for (1.1) requires some additional conditions. In fact, Pilod [21] showed that the flow map of this Cauchy problem fails to be twice differentiable in \( H^s(\mathbb{R}) \) for any \( s \in \mathbb{R} \) if \( c_1 \neq 0 \).

Local well-posedness was established using the weighted Sobolev spaces \( H^s(\mathbb{R}) \cap L^2(x^{2k} dx) \) for sufficiently large \( s \) and \( k \) by Kenig et al. [12] and Kenig and Staffilani [14]. For the proof, they used a change of dependent variables as in [6, 7]. In these works, the change of dependent variable was called a gauge transformation. By replacing weighted spaces with a spatial summability condition, Harrop-Griffiths [4] proved local well-posedness for (1.1) in a translation invariant space \( l^1 H^s(\mathbb{R}) \) for

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respectively. Thanks to the presence of the gauge transformation for (1.2). On the other hand, the Fourier restriction norm to show an a priori estimate in $H^s(\mathbb{R})$ for (1.2) requires that the primitives of $f$ are bounded. This space is a Banach space equipped with the norm $\|f\|_{X^s} := \|f\|_{H^s} + \left\| \int_{-\infty}^{x} f(y) dy \right\|_{L^2_{\mathbb{R}}}^{1/2}$ for $s > \frac{1}{2}$ (see Proposition 1 in [20]). The following is our main result.

**Theorem 1.1.** The Cauchy problem for (1.1) with $u(0) = u_0$ is local-in-time well-posed in $X^s$ for $s \geq 1$. Moreover, the flow map is (locally) Lipschitz continuous. In addition, the existence time depends only on $\|u_0\|_{X^1}$.

**Remark 1.2.** We note that $l^1 H^s(\mathbb{R})$ is embedded in $H^s(\mathbb{R})$ and that $s > 1$ yields $l^1 H^s(\mathbb{R}) \subset L^1(\mathbb{R})$. Hence, our functions space $X^s$ is bigger than $l^1 H^s(\mathbb{R})$, indeed

$$\sup_{x \in \mathbb{R}} \left\| \int_{-\infty}^{x} f(y) dy \right\| \leq \|f\|_{L^1} \lesssim \|f\|_{l^1 H^s}$$

holds provided that $s > 1$. Moreover, the function $f(x) = \frac{\sin x}{x}$ is an example that $f \in X^s$ for any $s \in \mathbb{R}$, but $f \notin l^1 H^s(\mathbb{R})$. In the quadratic setting, our result is an improvement of that in [4] from the view point both of the integrability and the regularity.

For the proof, we use a gauge transformation as in [20], which makes (1.1) a coupled system of KdV-type equations (see (3.3) and (3.4) below). Roughly speaking, the gauge transformation for (1.1) and (1.2) is defined as

$$u \mapsto e^{\int_{-\infty}^{x} u(t,y) dy} u, \quad u \mapsto e^{i \int_{-\infty}^{x} u(t,y) dy} u,$$

respectively. Thanks to the presence of $i$, the $L^2$-norm is invariant under the gauge transformation for (1.2). On the other hand, the $L^2$-boundedness of the gauge transformation for (1.1) requires that the primitives of $u$ are bounded.

Here, we give an outline of the proof of Theorem 1.1. Our proof depends on the gauge transformation but not on the energy estimate and the Fourier restriction.
norm. To calculate the nonlinear terms, we use the Strichartz estimate, the local smoothing estimate, and the maximal function estimate.

We apply the gauge transformation to rewrite (1.1) to a coupled system of KdV-type equations as mentioned above. First, by using the contraction mapping theorem, we show that the system is well-posed in $X^1 \times H^1$ in [4] which yields that (1.1) is well-posed in $X^2$. Second, we prove the a priori estimate (4.16) in [4] which says that the existence time depends only on $\|u_0\|_{X^1}$ as long as $u$ is a solution to (1.1). Therefore, Theorem 1.1 with $s = 1$ follows from an approximation argument and the fact that the solution to (1.1) exists at least in $X^2$. Because the well-posedness in $X^2$ is required only in this approximation argument, we may use the result in [4] instead of the well-posedness in $X^2$. However, for a self-contained proof of Theorem 1.1, we employ the well-posedness in $X^2$. Third, by applying the fractional Leibniz rule as in [11], we show the well-posedness in $X^s$ for $s \geq 1$ and the persistence property in $X^2$.

We observe that $\|u\|_{L_t^2 L_x^\infty}$ is bounded by the norms of $u$ and the gauge transformed $u$ (Lemma 4.1). Because the quadratic term with derivative in (3.4) vanishes when $c_2 = 0$, the a priori bound (4.16) follows from these facts and a similar argument as in [3]. For $c_2 \neq 0$, by using a gauge transformation, we rewrite (1.1) to an equation which contains no terms of the form $(\partial_x u)^2$. Namely, we apply the gauge transformation twice to obtain Theorem 1.1 in general. This is the reason why we can avoid using the Fourier restriction norm.

Our argument can estimate the difference of two solutions to (1.1), and hence the flow map is (locally) Lipschitz continuous. On the other hand, the flow map is not smooth for low-regularity data even with bounded primitives.

**Proposition 1.3.** If $s < 1$, then the flow map of (1.1) fails to be twice differentiable in $X^s$.

We also consider a semi-linear KdV-type equation with quadratic nonlinearity

\[
\partial_t u + \frac{1}{3} \partial_x^3 u = c_1 u \partial_x^2 u + c_2 (\partial_x u)^2 + c_3 \partial_x u \partial_x^2 u + c_4 (\partial_x^2 u)^2. 
\]

Because $\partial_x^2 u (\partial_x u$ if $c_4 = 0$) satisfies an equation like as (1.1), the same argument as in the proof of Theorem 1.1 yields the following:

**Theorem 1.4.** The Cauchy problem for (1.3) with $u(0) = u_0$ is local-in-time well-posed in $X^3$. Moreover, we can replace $X^3$ by $X^2$ if $c_4 = 0$. In addition, the persistence of regularity holds.

**Remark 1.5.** We can remove the boundedness of primitives if $c_1 = 0$. More precisely, the Cauchy problem for (1.3) is well-posed in $H^2(\mathbb{R})$ and $H^3(\mathbb{R})$ provided that $c_1 = c_4 = 0$ and $c_1 = 0$, respectively.

1.1. **Notation.** We denote the set of nonnegative integers by $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let $P_N$ denote the (inhomogeneous) Littlewood-Paley decomposition:

\[
u = \sum_{N \in \mathbb{N}_0} P_N u.
\]
Let $1 \leq p, q \leq \infty$ and $T > 0$. Define
\[
\|f\|_{L^p_t L^q_x} := \left( \int_{-\infty}^{\infty} \left( \int_{-T}^{T} |f(t,x)|^p dt \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}},
\]
\[
\|f\|_{L^q_t L^p_x} := \left( \int_{-T}^{T} \left( \int_{-\infty}^{\infty} |f(t,x)|^q dx \right)^{\frac{p}{q}} dt \right)^{\frac{1}{p}},
\]
with $T = t$ to indicate the case when $T = \infty$.

We set $L := \partial_t + \frac{1}{3} \partial_x^3$. Let $U(t)$ be the linear propagator of (1.1), that is
\[
U(t) := e^{-\frac{t}{3} \partial_x^3}.
\]
In estimates, we use $C$ to denote a positive constant that can change from line to line. We write $A \lesssim B$ to mean $A \leq CB$ if $C$ is absolute or depends only on parameters that are considered fixed. We define $A \ll B$ to mean $A \leq C^{-1}B$.

2. Lemmas

In this section, we collect some lemmas which are used in the proof.

The first lemma is the Strichartz estimate for the Airy equation.

**Lemma 2.1** (Lemma 2.4 in [9]). Let $2 \leq q, r \leq \infty$ and $0 \leq s \leq \frac{1}{q} + \frac{3}{r} = \frac{1}{2}$. Then,
\[
\| |\partial_x|^s U(t) u_0 \|_{L^q_t L^r_x} \lesssim \| u_0 \|_{L^2}.
\]

The second lemma is the local smoothing effect of Kato-type (see, for example, Theorem 3.5 in [11]).

**Lemma 2.2.** For any $u_0 \in L^2(\mathbb{R})$, we have
\[
\| \partial_x U(t) u_0 \|_{L^\infty_t L^2_x} \lesssim \| u_0 \|_{L^2}.
\]

The third lemma is the maximal function estimates.

**Lemma 2.3** (Corollary 2.9 in [10]). Let $s > \frac{3}{4}$. Then for any $u_0 \in H^s(\mathbb{R})$ and any $\rho > \frac{3}{4}$,
\[
\| U(t) u_0 \|_{L^2_t L^\infty_x} \lesssim (T)^\rho \| u_0 \|_{H^s}.
\]

3. Well-posedness via the contraction mapping theorem

In this section, by using the iteration argument, we show that (1.1) is locally well-posed in $X^2$.

First, we observe some formal calculations. Let $\Lambda$ and $v$ be real valued functions.

A direct calculation shows
\[
e^\Lambda L (e^{-\Lambda} v) = L v + \left( \partial_x \Lambda \partial_x^2 \Lambda - \frac{1}{3} (\partial_x \Lambda)^3 - \Lambda \Lambda \right) v + \left( - \partial_x^2 \Lambda + (\partial_x \Lambda)^2 \right) \partial_x v - \partial_x \Lambda \partial_x^2 v.
\]

Let $u$ be a solution to (1.1) and set $v = \partial_x u$. Then, (1.1) yields
\[
L v = \partial_x L u = (c_1 + 2c_2) \partial_x u \partial_x^2 u + c_1 u \partial_x^3 u.
\]
To cancel out the worst part, we set $\Lambda(t, x) = c_1 \int_{-\infty}^{T} u(t, y)dy$. Since

$$
\mathcal{L}\Lambda = c_1 \int_{-\infty}^{x} (Lu)(t, y)dy = c_1^2 \int_{-\infty}^{x} (u\partial_y^2 u)dy + c_1 c_2 \int_{-\infty}^{x} (\partial_y u)^2 dy
$$

(3.2)

$$
= c_1^2 u\partial_x u + c_1(-c_1 + c_2) \int_{-\infty}^{x} (\partial_y u)^2 dy,
$$

with $v = \partial_x u$ leads to the following:

$$
e^{\Lambda t} \mathcal{L} (e^{-\Lambda} \partial_x u) = 2c_2 \partial_x u\partial_y^2 u + c_1^2 u^2 \partial_y^2 u + c_1(c_1 - c_2) \partial_x u \int_{-\infty}^{x} (\partial_y u)^2 dy - \frac{c_1^3}{3} u^3 \partial_x u.
$$

Hence, by setting $v := e^{-\Lambda} \partial_x u$, we have

$$
(\mathcal{L}u = c_1 e^{\Lambda t} (\partial_x v + c_1 u v) + c_2 e^{2\Lambda t} v^2,
$$

(3.3)

$$
(\mathcal{L}v = 2c_2 e^{\Lambda t} (\partial_x v + c_1 u v) + c_1^2 u^2 \partial_x v + c_1(c_1 - c_2) v \int_{-\infty}^{x} e^{2\Lambda t} v^2 dy + \frac{2}{3} c_1^3 u^3 v.
$$

3.1 Proof of Theorem 1.1 with $s = 2$. Let $\varepsilon > 0$ be sufficiently small. We define the function space $X_T$ for $T > 0$ by

$$
X_T := \{ f \in L^\infty([-T, T]; L^2(\mathbb{R})) : \| f \|_{X_T} < \infty \},
$$

(3.5)

$$
\| f \|_{X_T} := \| f \|_{L^\infty_x L^2_t} + \| f \|_{L^q_x L^r_t} + \| \partial_x f \|_{L^q_x L^r_t} + \| (\partial_x)^\frac{3}{2} - \varepsilon f \|_{L^q_x L^r_t}.
$$

Lemmas 2.1, 2.3 yield that

$$
\| U(t)u_0 \|_{X_T} \leq C_1 \| u_0 \|_{L^2}
$$

for $0 < T < 1$. In addition, an interpolation shows that

$$
\| u \|_{L^q_x L^r_t} \lesssim \| u \|_{X_T}
$$

for any $2 \leq q, r \leq \infty$ with $\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$. In particular, $(q, r) = (12, 4), (9, 6), (8, 8)$ are allowed. Furthermore, for such $(q, r)$, $1 \leq q' < q$, and $0 < T < 1$, we have

$$
\| u \|_{L^{q'}_x L^r_t} \leq T^\frac{1}{q'} \| u \|_{L^q_x L^r_t} \lesssim \| u \|_{X_T}.
$$

We will apply the contraction mapping theorem in the space

$$
Y_T := \left\{ (u, v) \in X_T \times X_T : (\partial_x)u \in X_T, (\partial_x)v \in X_T, \sup_{|t| \leq T, x \in \mathbb{R}} \left| \int_{-\infty}^{x} u(t, y)dy \right| < \infty \right\}
$$

equipped with the norm

$$
\|(u, v)\|_{Y_T} := \| (\partial_x)u \|_{X_T} + \| (\partial_x)v \|_{X_T} + \sup_{|t| \leq T, x \in \mathbb{R}} \left| \int_{-\infty}^{x} u(t, y)dy \right|.
$$
We define \( \Psi_{u_0}(u, v) := (\Psi_{u_0}^{(1)}(u, v), \Psi_{u_0}^{(2)}(u, v)) \) by

\[
\Psi_{u_0}^{(1)}(u, v) := \mathcal{U}(t)u_0 + \int_0^t \mathcal{U}(t-t') \left\{ c_1 e^{\Delta} u (\partial_x v + c_1 u v) + c_2 e^{2\Delta} v^2 \right\} (t', x) dt',
\]

\[
\Psi_{u_0}^{(2)}(u, v) := \mathcal{U}(t)v_0 + \int_0^t \mathcal{U}(t-t') \left\{ 2c_2 e^{\Delta} v (\partial_x v + c_1 u v) + c_1^2 u^2 \partial_x v + c_1 (1 - c_2) v \int_0^t e^{2\Delta} v^2 dy + \frac{2}{3} c_1^3 u^3 \right\} (t', x) dt',
\]

where \( \Lambda(t, x) := c_1 \int_{-\infty}^t u(t, y) dy \) and \( v_0 := e^{-c_1 \int_{-\infty}^t u_0(y) dy} \partial_x u_0 \).

Let \( 0 < T < 1 \) be determined later. Then, Hölder’s inequality yields that

\[
\|u \partial_x v\|_{L^2_t H^2_x} \lesssim \|u \partial_t^2 v\|_{L^2_t H^2_x} + \|\partial_x u \partial_x v\|_{L^2_t H^2_x} + \|u \partial_x v\|_{L^2_t H^2_x} \lesssim \|u\|_{L^2_t L^2_x} \|\partial_x^2 v\|_{L^2_t H^2_x} + \|\partial_x u\|_{L^2_t L^2_x} \|\partial_x v\|_{L^2_t H^2_x} + \|u\|_{L^2_t L^2_x} \|\partial_x v\|_{L^2_t H^2_x} \lesssim \|u\|_{X_T} \|\partial_x v\|_{X_T}.
\]

Since (3.6)
\[
\|e^{\Delta} f\|_{H^1} \lesssim \|e^{\Delta} f\|_{L^2_{T,x}} \left( \|u\|_{L^2_{T,x}} + 1 \right) \|f\|_{H^1} \lesssim \|e^{\Delta} f\|_{L^2_{T,x}} \left( \|\partial_x u\|_{X_T} + 1 \right) \|f\|_{H^1},
\]

we use (3.5) to obtain the following:

\[
\|\partial_x \Psi_{u_0}^{(1)}(u, v)\|_{X_T} - C_1 \|u_0\|_{H^1} \lesssim T^{\frac{1}{2}} \left( \|e^{\Delta} u \partial_x v\|_{L^2_t H^2_x} + \|e^{\Delta} u^2 v\|_{L^2_t H^2_x} + \|e^{2\Delta} v^2\|_{L^2_t H^2_x} \right) \lesssim T^{\frac{1}{2}} \left( \|e^{\Delta} u\|_{L^2_{T,x}} + \|e^{2\Delta} u^2\|_{L^2_{T,x}} \right) (\|u, v\|^2_{Y_T} + \|(u, v)\|^2_{Y_T}).
\]

Moreover, we observe the following estimates:

\[
\|u \partial_x v\|_{L^2_t H^2_x} \lesssim \|u \partial_t^2 v\|_{L^2_t H^2_x} + \|\partial_t v\|^2_{L^2_t H^2_x} + \|u \partial_x v\|_{L^2_t H^2_x} \lesssim \|u\|_{L^2_t L^2_x} \|\partial_t^2 v\|_{L^2_t H^2_x} + \|\partial_t v\|^2_{L^2_t H^2_x} + \|u\|_{L^2_t L^2_x} \|\partial_x v\|_{L^2_t L^2_x} \lesssim \|\partial_x v\|^2_{X_T},
\]

\[
\|u v^2\|_{L^2_t H^2_x} \lesssim \|u \partial_t v\|_{L^2_t H^2_x} + \|\partial_t u v^2\|_{L^2_t H^2_x} + \|u v^2\|_{L^2_t H^2_x} \lesssim \|u\|_{L^2_t L^2_x} \|\partial_t v\|_{L^2_t H^2_x} + \|\partial_t u\|_{L^2_t L^2_x} \|v\|_{L^2_t L^2_x} \leq \|u\|_{L^2_t L^2_x} \|v\|_{L^2_t L^2_x} \lesssim \|\partial_x v\|^2_{X_T},
\]

\[
\|u v\|_{L^2_t H^2_x} \lesssim \|u \partial_t v\|_{L^2_t H^2_x} + \|\partial_t u v\|_{L^2_t H^2_x} + \|u v\|_{L^2_t H^2_x} \lesssim \|u\|_{L^2_t L^2_x} \|\partial_t v\|_{L^2_t H^2_x} + \|\partial_t u\|_{L^2_t L^2_x} \|v\|_{L^2_t L^2_x} + \|u\|_{L^2_t L^2_x} \|v\|_{L^2_t L^2_x} \lesssim \|\partial_x v\|^2_{X_T},
\]
Accordingly, (3.5) and (3.6) imply that
\[ \|u^2 \partial_x v\|_{L^2_x H^1_x} \leq \|u^2 \partial_x v\|_{L^2_x} + \|u \partial_x u \partial_x v\|_{L^2_x} + \|u^2 \partial_x v\|_{L^2_x} \]
\[ \leq \|u\|_{L^\infty_T} \left( \|u\|_{L^2_T} \|\partial_x^2 v\|_{L^\infty_T} + \|\partial_x u\|_{L^4_T} \|\partial_x v\|_{L^2_T} + \|u\|_{L^\infty_T} \|\partial_x v\|_{L^2_T} \right) \]
\[ \leq \|(\partial_x v) u\|_{L^2_T} \|(\partial_x v)\|_{L^2_T} \]
\[ \|v\|_{-\infty} \int_{-\infty}^x e^{2\Lambda} v^2 dy \|_{L^2_T H^1_x} \]
\[ \leq \|e^{2\Lambda}\|_{L^\infty_T} \left( \|v\|_L^2 T \|\partial_x v\|_{L^2_T} + \|v\|_T^3 \right) \]
\[ \leq \|u^2\|_{L^1_T H^1} \leq \|u^2 \partial_x v\|_{L^2_T} + \|u^2 \partial_x u \partial_x v\|_{L^2_T} + \|u^2 \partial_x v\|_{L^2_T} \]
\[ \leq \|u\|_T^3 \|\partial_x v\|_{L^2_T} + \|u\|_T^2 \|\partial_x v\|_{L^2_T} + \|u\|_T^3 \|\partial_x v\|_{L^2_T} \]
\[ \leq \|(\partial_x v) u\|_{L^2_T} \|(\partial_x v)\|_{L^2_T} . \]

Accordingly, (3.5) and (3.6) imply that
(3.8)
\[ \|(\partial_x \Phi^{(2)}_{u_0})(u, v)\|_{L^2_T} - C_1 \|v_0\|_{H^1} \]
\[ \leq T^\frac{2}{\Lambda} \left( e^{2\Lambda} \|u\|_{L^2_T} + e^{2\Lambda} \|u v\|_{L^2_T} + \|u^2 \partial_x v\|_{L^2_T} + \int_{-\infty}^x e^{2\Lambda} v^2 dy \right) \]
\[ \leq T^\frac{2}{\Lambda} \left( e^{2\Lambda} \|L^\infty_T - 1\| (\|(u, v)\|_{L^2_T} + \|(u, v)\|_{L^2_T}) \right) . \]

Since \( \Phi^{(1)}_{u_0} (u, v) \) satisfies
\[ \left( \partial_t + \frac{1}{3} \partial_x^3 \right) \Phi^{(1)}_{u_0} (u, v) = c_1 e^{2\Lambda} (\partial_x v + c_1 uv) + c_2 e^{2\Lambda} v^2, \]
the fundamental theorem of calculus shows
(3.9)
\[ \int_{-\infty}^x \Phi^{(1)}_{u_0} (u, v)(t, y) dy - \int_{-\infty}^x u_0(y) dy \]
\[ = \int_{0}^{t} \frac{d}{dt} \int_{-\infty}^x \Phi^{(1)}_{u_0} (u, v)(\tau, y) dy d\tau \]
\[ = - \frac{1}{3} \int_{0}^{t} \partial_x^2 \Phi^{(1)}_{u_0} (u, v)(\tau, x) d\tau + \int_{0}^{t} \int_{-\infty}^x \left( c_1 e^{2\Lambda} (\partial_x v + c_1 uv) + c_2 e^{2\Lambda} v^2 \right) dy d\tau \]
\[ = - \frac{1}{3} \int_{0}^{t} \partial_x^2 \Phi^{(1)}_{u_0} (u, v)(\tau, x) d\tau + c_1 \int_{0}^{t} e^{2\Lambda} uv d\tau - c_1 \int_{0}^{t} \int_{-\infty}^x c_1 e^{2\Lambda} \partial_x uv dy d\tau \]
\[ + c_2 \int_{0}^{t} \int_{-\infty}^x e^{2\Lambda} v^2 dy d\tau , \]
which leads to the following:

\( (3.10) \)

\[
\left\| \int_{-\infty}^{x} \Psi_{u_0}^{(1)}(u,v)(t,y)dy \right\|_{L_{T,x}^{\infty}} \leq \left\| \int_{-\infty}^{x} u_0(y)dy \right\|_{L_{T,x}^{\infty}}
\]

\( \lesssim \| \partial_x^2 \Phi_{u_0}^{(1)}(u,v) \|_{L_{T,x}^{\infty}L_T^2} + \| e^A u v \|_{L_{T,x}^{\infty}L_T^2} + \| e^A \partial_x u v \|_{L_{T,x}^{\infty}L_T^2} + \| e^{2A} v^2 \|_{L_{T,x}^{\infty}L_T^2} \)

\( \lesssim T^4 \left( \| \partial_x^2 \Phi_{u_0}^{(1)}(u,v) \|_{L_{T,x}^{\infty}L_T^2} + \| e^A \|_{L_{T,x}^{\infty}L_T^2} \| u \|_{L_{T,x}^{\infty}L_T^\infty} \| v \|_{L_T^{\infty}L_T^\infty}
\]

\[
+ \| e^A \|_{L_{T,x}^{\infty}L_T^2} \| \partial_x u \|_{L_T^{\infty}L_T^2} \| v \|_{L_T^{\infty}L_T^2} + \| e^{2A} \|_{L_{T,x}^{\infty}L_T^2} \| v \|_{L_T^{\infty}L_T^2}^2 \right) \]

\( \leq C_0 T^4 \left( \| \partial_x \Phi_{u_0}^{(1)}(u,v) \|_{X_T} + \| e^A \|_{L_{T,x}^{\infty}L_T^\infty} \| (\partial_x) u \|_{X_T} \| (\partial_x) v \|_{X_T} + \| e^{2A} \|_{L_{T,x}^{\infty}L_T^2} \| v \|_{X_T}^2 \right) .
\]

Therefore, \( (3.7) \), \( (3.8) \), and \( (3.10) \) yield that

\( (3.11) \)

\[
\| \Phi_{u_0}(u,v) \|_{Y_T} \leq 2C_1 \left( \| u_0 \|_{X^1} + \| v_0 \|_{H^1} \right) + C_2 T^4 \| e^{2|c_1|} \|_{Y_T} \left( \| (u,v) \|_{Y_T} + \| (\bar{u},v) \|_{Y_T} \right)
\]

provided that \( C_0 T^4 < \frac{1}{2} \). A similar calculation leads to the estimate for the difference.

Here, we set a closed ball \( Y_T \) of \( Y \) by

\[ B_T := \{ (u,v) \in Y_T : \| (u,v) \|_{Y_T} \leq 3C_1 (\| u_0 \|_{X^1} + \| v_0 \|_{H^1}) \} . \]

Then, \( \Phi_{u_0} \) is a contraction mapping on \( Y_T \) if \( T \) is small depending only on \( \| u_0 \|_{X^1} \) and \( \| v_0 \|_{H^1} \).

If \( u_0 \in X^2 \), we have \( v_0 = e^{-c_1 \int_{-\infty}^{x} u_0(y)dy} \partial_x u_0 \in H^1(\mathbb{R}) \). Because \( (u,v) \) is a solution to \( (3.3) \)–\( (3.4) \), the equation \( w(t,x) = e^{-c_1 \int_{-\infty}^{x} u(t,y)dy} \partial_x u(t,x) \) holds, which implies the well-posedness in \( X^2 \) of the Cauchy problem for \( (1.1) \).

For the reader’s convenience, we give the proof of this fact. Let \( w := \partial_x u - e^A v \). By \( (3.3) \), a direct calculation shows that

\[
L \partial_x u = e^A \left( c_1 u \partial_x^2 v + c_1 \partial_x u \partial_x v + 2c_1^2 u^2 \partial_x v + 2c_1^2 u \partial_x uv + c_1^3 u^3 v \right) + 2c_2 e^{2A} \left( \partial_x u v + c_1 uv^2 \right) ,
\]

\[
\int_{-\infty}^{x} Ludy \, dy = -c_1 \int_{-\infty}^{x} e^A \partial_y u v dy + c_1 e^A uv + c_2 \int_{-\infty}^{x} e^{2A} v^2 dy
\]

\[
= -c_1 \int_{-\infty}^{x} e^A uv dy + c_1 e^A uv - (c_1 - c_2) \int_{-\infty}^{x} e^{2A} v^2 dy .
\]

From \( (3.1) \) and \( (3.4) \), we have

\[
e^{-A} L(e^A v) = Lw + \left( c_1^2 u \partial_x u + c_3^3 u^3 + L \right) v + (c_1 \partial_x u + c_1^2 u^2) \partial_x v + c_1 u \partial_x^2 v
\]

\[
= 2c_2 e^{2A} (\partial_x v + c_1 uv) + 2c_1^2 u^2 \partial_x v + c_1^3 u^3 v
\]

\[
+ \left( c_1^2 u \partial_x u - c_1^2 \int_{-\infty}^{x} e^A uv dy + c_1^2 e^A uv \right) v + c_1 \partial_x u \partial_x v + c_1 u \partial_x^2 v
\]

Accordingly, we obtain

\[
Lw = c_1^2 e^{2A} uv w + c_1^2 e^A v \int_{-\infty}^{x} e^A uv dy .
\]
The same calculation as in (3.8) leads to
\[ \|w\|_{X_T} \lesssim T^{\frac{2}{3}}e^{2\|A\|^2L_{T,x}} (\|u\|_{X_T} + \|v\|_{X_T}) \|w\|_{X_T} \].

By \( w(0) = 0 \), the standard continuity argument shows that \( w(t) = 0 \) for \( |t| \leq T \).

Therefore, we obtain that \( w(t, x) = e^{-c_1 \int_{-\infty}^{t} u(\lambda, y)dy} \partial_x u(t, x) \) for \( |t| \leq T \).

4. WELL-POSEDNESS FOR (1.1) IN \( \Lambda^1 \)

We first consider the special case \( c_2 = 0 \), because the general case is a bit complicated. In §4.1, we show the well-posedness in \( \Lambda^1 \) under \( c_2 = 0 \). In §4.2, we observe the persistency of regularity for \( c_2 = 0 \). Finally, in §4.3, we prove Theorem 1.1 without \( c_2 = 0 \).

4.1. Proof of Theorem 1.1 under \( c_2 = 0 \)

Let \( c_2 = 0 \) and \( u_0 \in \Lambda^1 \). Then, there exists a sequence \( \{u_{0,n}\} \subset \Lambda^2 \) such that \( u_{0,n} \) converges to \( u_0 \) in \( \Lambda^1 \). Without loss of generality, we may assume that \( \|u_{0,n}\|_{\Lambda^1} \leq \|u_0\|_{\Lambda^1} \) holds for any \( n \in \mathbb{N} \). By the well-posedness in \( \Lambda^2 \), there exist \( T_n > 0 \) and the solution \( u_n \in C([-T_n, T_n]; \Lambda^2) \), where \( T_n \) depends on \( \|u_{0,n}\|_{\Lambda^2} \).

Set \( \Lambda_n(t, x) := c_1 \int_{-\infty}^{t} u_n(t, y)dy \) and \( v_n := e^{-\Lambda_n} \partial_x u_n \). First, we observe the following bound.

**Lemma 4.1.**
\[ \|u_n\|_{L^2 T_{\Lambda^1}^2} \lesssim e^{\frac{3}{2}\|\Lambda_n\|_{L^1 T}} (\|u_n\|_{X_T} + \|u_n\|_{X_T}^2 + \|v_n\|_{X_T}^2) \].

**Proof.** The low frequency part is easily handled:
\[ \|P_1 u_n\|_{L^2 T_{\Lambda^1}^2} \lesssim \|\partial_x\|^{-1}\|P_1 u_n\|_{L^2 T_{\Lambda^1}^2} \lesssim \|u_n\|_{X_T} \].

We use the Littlewood-Paley decomposition to estimate the high frequency part:
\[ \|P_{>1} u_n\|_{L^2 T_{\Lambda^1}^2} \lesssim \|P_{>1} \partial_x\|^{-1} (e^{\Lambda_n} v_n) \|_{L^2 T_{\Lambda^1}^2} \lesssim \sum_{N_1, N_2 \in 2^{\mathbb{N}}} \|\partial_x\|^{-1} (P_{N_1} e^{\Lambda_n} P_{N_2} v_n) \|_{L^2 T_{\Lambda^1}^2}. \quad (4.1) \]

For \( N_1 \gtrsim N_2 \), we have
\[ \|\partial_x\|^{-1} (P_{N_1} e^{\Lambda_n} P_{N_2} v_n) \|_{L^2 T_{\Lambda^1}^2} \lesssim \|P_{N_1} e^{\Lambda_n} P_{N_2} v_n\|_{L^2 T_{\Lambda^1}^2} \lesssim N_1^{-\frac{1}{2} + \varepsilon} \|P_{N_1} e^{\Lambda_n}\|_{L^p T_{\Lambda^1}} \|\partial_x\|^{-1}\|P_{N_2} v_n\|_{L^2 T_{\Lambda^1}^2} \lesssim N_1^{-\frac{1}{2} + \varepsilon} \|P_{N_1} e^{\Lambda_n}\|_{L^p T_{\Lambda^1}} \|\partial_x\|^{-1}\|P_{N_2} v_n\|_{L^2 T_{\Lambda^1}^2} \lesssim N_1^{-\frac{1}{2} + \varepsilon} \|\Lambda_n\|_{L^p T_{\Lambda^1}^2} \|v_n\|_{X_T} \lesssim \|\Lambda_n\|_{L^p T_{\Lambda^1}^2} \|v_n\|_{X_T}. \]

Here, we have used the Gagliardo-Nirenberg type inequality in the last inequality as follows:
\[ \|u_n\|_{L^p T_{\Lambda^1}^2} \lesssim \|u_n\|_{L^p T_{\Lambda^1}} \|\partial_x u_n\|_{L^2 T_{\Lambda^1}} \lesssim \|e^{\Lambda_n}\|_{L^p T_{\Lambda^1}} \|u_n\|_{L^p T_{\Lambda^1}} \|v_n\|_{L^p T_{\Lambda^1}^2}. \quad (4.2) \]

When \( N_1 \ll N_2 \), because the frequency of the product of the two functions is around \( N_2 \), we have
\[ \|\partial_x\|^{-1} (P_{N_1} e^{\Lambda_n} P_{N_2} v_n) \|_{L^2 T_{\Lambda^1}^2} \lesssim N_2^{-\frac{1}{2} + \varepsilon} \|e^{\Lambda_n}\|_{L^p T_{\Lambda^1}} \|\partial_x\|^{-1}\|P_{N_2} v_n\|_{L^2 T_{\Lambda^1}^2} \lesssim N_2^{-\frac{1}{2} + \varepsilon} \|e^{\Lambda_n}\|_{L^p T_{\Lambda^1}} \|v_n\|_{X_T}. \]
Hence, by using \((N_1 + N_2)^{-\frac{1}{2} + \epsilon}\), we can sum up the summation with respect to \(N_1\) and \(N_2\) in (4.1). Therefore, we obtain the desired bound.

Lemma 4.1 and 4.2 yield that

\[
\|u_n\partial_x v_n\|_{L^2_{T,x}} + \|u_n^2 v_n\|_{L^2_{T,x}}
\]

(4.3)

\[
\lesssim \|u_n\|_{L^2_{T,x}} \|\partial_x v_n\|_{L^2_{T,x}} + \|u_n\|_{L^2_{T,x}}^2 \|v_n\|_{L^2_{T,x}}
\]

Since (4.2) yields that

\[
\|u_n\|_{L^2_{T,x}} \lesssim e^{\frac{\Lambda_n}{2}} \left(\|u_n\|_{X_T} + \|\partial_x u_n\|_{X_T} + \|v_n\|_{X_T}^2\right) \|v_n\|_{X_T},
\]

(4.4)

\[
\lesssim \|u_n\|_{L^2_{T,x}} \|v_n\|_{L^2_{T,x}} \|\partial_x v_n\|_{L^2_{T,x}} + \|u_n\|_{L^2_{T,x}}^2 \|v_n\|_{L^2_{T,x}}
\]

\[
+ \|e^{\frac{\Lambda_n}{2}}\|_{L^2_{T,x}} \|v_n\|_{L^2_{T,x}}^2 \left(\|u_n\|_{X_T}^2 + \|v_n\|_{X_T}^2 + \|u_n\|_{X_T}^3\right) \|v_n\|_{X_T},
\]

(4.5)

by (3.3), we have

\[
\left\| \int_{-\infty}^{\infty} u_n(t,y)dy \right\|_{L^2_{T,x}} - \left\| \int_{-\infty}^{\infty} u_0,n(y)dy \right\|_{L^2_{T,x}}
\]

(4.6)

\[
\lesssim \|\partial^2_x u_n\|_{L^2_{T,x}} + \|u_n \partial_x u_n\|_{L^2_{T,x}} + \|\partial_x u_n\|_{L^2_{T,x}}^2
\]

\[
\lesssim \|e^{\Lambda_n}\|_{L^2_{T,x}} \left(\|\partial_x v_n\|_{L^2_{T,x}} + \|u_n v_n\|_{L^2_{T,x}} + \|\partial_x u_n\|_{L^2_{T,x}}^2\right)
\]

\[
\lesssim \left(T \frac{1}{2}\right)^2 \|e^{\Lambda_n}\|_{L^2_{T,x}} \left(1 + \|u_n\|_{X_T} + \|v_n\|_{X_T}\right) \|v_n\|_{X_T}.
\]

We set

\[
\|u\|_{Z_T} := \|u\|_{X_T} + \left\| e^{-c_1} \int_{-\infty}^{\infty} u(t,y)dy \partial_x u \right\|_{X_T} + \left\| \int_{-\infty}^{\infty} u(t,y)dy \right\|_{L^2_{T,x}}.
\]

Because \(u_n\) and \(v_n\) satisfy (3.3), (3.4) with \(c_2 = 0\), the estimates (5.5), (4.3), (4.4), and (4.6) yield that

\[
\|u_n\|_{Z_T} \leq C_1 \|u_0,n\|_{X_T} + C_2 T_n \left(1 + \|u_n\|_{Z_T} \right) \left(1 + \|u_n\|_{Z_T}^3\right).
\]

(4.7)

For simplicity, we set

\[
\|u\|_{\tilde{Z}_T} = \|u\|_{Z_T} + \left\| \partial_x (e^{-c_1} \int_{-\infty}^{\infty} u(t,y)dy \partial_x u) \right\|_{X_T}.
\]

(4.7)
Since Lemma 4.1 and 4.2 lead to (4.8)
\[ \| \partial_x \left( e^{-\Lambda_n} \left( c_n^2 u_n^2 \partial_x^2 u_n + c_n^2 \partial_x^3 u_n \int_{-\infty}^{x} (\partial_y u_n)^2 dy - \frac{c_n^3}{3} u_n^3 \partial_x u_n \right) \right) \|_{L^2_T,L^\infty_x} \]
\[ \lesssim \| u_n^2 \partial_x^2 v_n \|_{L^2_T,L^\infty_x} + \| u_n^3 \partial_x v_n \|_{L^2_T,L^\infty_x} + \| \partial_x v_n \int_{-\infty}^{x} e^{2\Lambda_n} v_n^2 dy \|_{L^2_T,L^\infty_x} \]
\[ + \| e^{\Lambda_n} \|_{L^\infty_T} \left( \| u_n \partial_x v_n \|_{L^2_T,L^\infty_x} + \| u_n^2 v_n^2 \|_{L^2_T,L^\infty_x} + \| e^{\Lambda_n} \|_{L^\infty_T} \| v_n^3 \|_{L^2_T,L^\infty_x} \right) \]
\[ \lesssim \| u_n \|_{L^2_T,L^\infty} \| \partial_x v_n \|_{L^2_T,L^\infty} + \| u_n \|_{L^6_T,L^\infty} \| \partial_x v_n \|_{L^2_T,L^\infty} \]
\[ + \| u_n^2 \|_{L^2_T,L^\infty} \| \partial_x v_n \|_{L^2_T,L^\infty} \| v_n \|_{L^2_T,L^\infty} \]
\[ + e^{2|\Lambda_n|} \| \partial_x v_n \|_{L^2_T,L^\infty} \| u_n \|_{L^2_T,L^\infty} \| \partial_x v_n \|_{L^2_T,L^\infty} \]
\[ \lesssim e^{2|\Lambda_n|} \| \partial_x v_n \|_{L^2_T,L^\infty} \| u_n \|_{L^2_T,L^\infty} \]
we have
\[ \| u_n \|_{\tilde{Y}^{L^\infty}_T} \leq C_1 \| u_{0,n} \|_{X^\infty} + C_2 \| u_0 \|_{X^\infty} \]
Here, we set
\[ T^* := \frac{1}{10} \left( C_2 e^{10|c_1|C_1 \| u_0 \|_{X^\infty} \left( 1 + (4C_1 \| u_0 \|_{X^\infty})^3 \right) \right)^2, \]
which is independent of n. By \| u_{0,n} \|_{X^\infty} \leq 2 \| u_0 \|_{X^\infty} , the continuity argument shows
\[ \| u_n \|_{Z^{(0)}_{T_n}} \leq 3C_1 \| u_0 \|_{X^\infty} , \| u_n \|_{\tilde{Y}^{L^\infty}_{T^{(0)}}} \leq 3C_1 \| u_{0,n} \|_{X^\infty}, \]
where \( T^{(0)}_n := \min(T_n, T^*) \). Then, Theorem 4.1 yields that there exists \( \rho_n \) depending on \( \| u_0 \|_{X^\infty} \) and \( \| u_{0,n} \|_{X^\infty} \) such that \( u_n \) satisfies (4.1) on \( [-T_n + \rho_n, T_n + \rho_n] \). Because we can apply the estimates (4.3), (4.4), (4.6), and (4.8) as long as \( u_n \) is a solution to (4.1), we obtain
\[ \| u_n \|_{Z^{(0)}_{T_n + \rho_n}} \leq C_1 \| u_{0,n} \|_{X^\infty} \]
\[ + C_2 (T_n + \rho_n)^\frac{1}{2} e^{\frac{5}{2} |c_1| \| u_n \|_{Z^{(0)}_{T_n + \rho_n}} \| u_n \|_{Z^{(0)}_{T_n + \rho_n}} \left( 1 + \| u_n \|_{Z^{(0)}_{T_n + \rho_n}}^3 \right) \}, \]
\[ \| u_n \|_{\tilde{Y}^{L^\infty}_{T_n + \rho_n}} \leq C_1 \| u_{0,n} \|_{X^\infty} \]
\[ + C_2 (T_n + \rho_n)^\frac{1}{2} e^{\frac{5}{2} |c_1| \| u_n \|_{Z^{(0)}_{T_n + \rho_n}} \| u_n \|_{\tilde{Y}^{L^\infty}_{T_n + \rho_n}} \left( 1 + \| u_n \|_{Z^{(0)}_{T_n + \rho_n}}^3 \right) \} \].
By setting \( T^{(1)}_n := \min(T_n + \rho_n, T^*) \), these bounds show that
\[ \| u_n \|_{Z^{(1)}_{T^{(1)}_n}} \leq 3C_1 \| u_0 \|_{X^\infty} , \| u_n \|_{\tilde{Y}^{L^\infty}_{T^{(1)}_n}} \leq 3C_1 \| u_{0,n} \|_{X^\infty} \]
By repeating this procedure k-times, we can extend this bound to that for \( T^{(k)}_n := \min(T_n + k\rho_n, T^*) \) and \( k \in \mathbb{N} \). In particular, because there exists an integer \( k_n \) such that \( T^{(k_n)}_n = T^* \), we obtain
\[ \| u_n \|_{Z^{(k_n)}_{T^*}} \leq 3C_1 \| u_0 \|_{X^\infty} , \| u_n \|_{\tilde{Y}^{L^\infty}_{Z^{(k_n)}_{T^*}}} \leq 3C_1 \| u_{0,n} \|_{X^\infty} \]
for any \( n \in \mathbb{N} \).
Next, we consider the estimate for the difference. By (4.9), (4.5), (4.9), and taking $T^*$ small if necessary, we have
\[
\| \Lambda_n - \Lambda_m \|_{L^\infty_T \mathcal{X}^r_x} \leq \left\| \int_{-\infty}^x (u_{0,n}(y) - u_{0,m}(y))dy \right\|_{L^\infty_T} \\
\lesssim \| \partial_t^2 u_n - \partial_t^2 u_m \|_{L^\infty_T L^1_T} + \| u_n \partial_x u_n - u_m \partial_x u_m \|_{L^\infty_T L^1_T} \\
\quad + \left\| (\partial_x u_n)^2 - (\partial_x u_m)^2 \right\|_{L^1_T \mathcal{X}^r_x} \\
\lesssim \| \langle \partial_x \rangle^2 \|_{L^\infty_T \mathcal{X}^r_x} \| \partial_x \mathcal{U}_n \|_{L^\infty_T L^1_T} + \| \langle \partial_x \rangle^2 \|_{L^\infty_T \mathcal{X}^r_x} \| \partial_x \mathcal{U}_m \|_{L^\infty_T L^1_T} \\
\quad + \| \langle \partial_x \rangle^2 \|_{L^\infty_T \mathcal{X}^r_x} \| \partial_x \mathcal{U}_n \|_{L^\infty_T L^1_T} + \| \langle \partial_x \rangle^2 \|_{L^\infty_T \mathcal{X}^r_x} \| \partial_x \mathcal{U}_m \|_{L^\infty_T L^1_T} \\
\quad + \| \langle \partial_x \rangle^2 \|_{L^\infty_T \mathcal{X}^r_x} \| \partial_x \mathcal{U}_n \|_{L^\infty_T L^1_T} + \| \langle \partial_x \rangle^2 \|_{L^\infty_T \mathcal{X}^r_x} \| \partial_x \mathcal{U}_m \|_{L^\infty_T L^1_T} \\
\leq \frac{1}{2} \left( \| \Lambda_n - \Lambda_m \|_{L^\infty_T \mathcal{X}^r_x} + \| u_n - u_m \|_{\mathcal{X}^r_x} + \| \mathcal{U}_n - \mathcal{U}_m \|_{\mathcal{X}^r_x} \right).
\]
Because the remaining cases are similarly handled, we obtain
\[
\| \Lambda_n - \Lambda_m \|_{L^\infty_T \mathcal{X}^r_x} + \| u_n - u_m \|_{\mathcal{X}^r_x} + \| \mathcal{U}_n - \mathcal{U}_m \|_{\mathcal{X}^r_x} \lesssim \| u_{0,n} - u_{0,m} \|_{\mathcal{X}^1}.
\]
Therefore, \( \{ u_n \} \) is a Cauchy sequence and the limit \( u \) is in \( C([-T^*, T^*]; \mathcal{X}^1) \). Hence, we conclude that (1.1) is well-posed in \( \mathcal{X}^1 \) if \( c_2 = 0 \).

### 4.2. Persistence of regularity

Let \( c_2 = 0, s \geq 1 \), and \( u_0 \in \mathcal{X}^s \). The well-posedness in (1.1) says that there exist the time \( T > 0 \) and the solution \( u \in C([-T, T]; \mathcal{X}^s) \). We prove that the solution has regularity, i.e., \( u \in C([-T, T]; \mathcal{X}^r) \), where \( T \) depends only on \( \| u_0 \|_{\mathcal{X}^1} \). For simplicity, we set \( r := s - 1 \geq 0 \). We apply Lemmas 2.1 and 2.2 and Stein’s interpolation theorem 24 as in (11) to obtain
\[
\| \langle \partial_x \rangle^\theta \mathcal{U}(t) u_0 \|_{L^{2 \theta}_x L^{2 \theta}_T} \lesssim \| u_0 \|_{L^2}
\]
for \( 0 < T < 1 \) and \( 0 < \theta < 1 \). Hence, by (3.3), we have
\[
\| \mathcal{U}(t) u_0 \|_{\mathcal{X}^r_T} \leq C_1 \| u_0 \|_{H^r}
\]
for \( 0 < T < 1 \). We also use the following norms:
\[
\| u \|_{\mathcal{X}^s_T} := \| u \|_{\mathcal{X}^s_T} + \left\| e^{-c_1 \int_0^t u(t,y)dy} \partial_x u \right\|_{\mathcal{X}^s_T},
\| u \|_{L^2 T^*} := \| u \|_{L^2 T^*} + \left\| \int_0^T u(t,y)dy \right\|_{L^2 T^*}.
\]

We observe a product estimate in the Sobolev space, while similar estimates are known (see, for example, Theorem 4 of §4.6.2 in [22], Theorem A.1 in [16], and Lemma 2.2 in [17]).
Lemma 4.2. For $r \geq 0$, we have
\[ \|f g\|_{H^r} \lesssim \|f\|_{H^r} \|g\|_{L^\infty} + \|f\|_{H^{r-\lceil r \rceil}} \|g\|_{H^{(r + 1)\lceil r \rceil}}, \]
where $\lceil r \rceil$ means the largest integer less than or equal to $r$.

Proof. We use the paraproduct decomposition:
\[ (4.11) \quad f g = \sum_{N_1, N_2 \in 2^{\mathbb{N}}} P_{N_1} f P_{N_2} g + \sum_{N_1, N_2 \in 2^{\mathbb{N}}} P_{N_1} f P_{N_2} g =: I + II. \]

We note that the first term on the right hand side is written as follows:
\[ \langle \partial_x \rangle^r \sum_{N_1, N_2 \in 2^{\mathbb{N}}} P_{N_1} f P_{N_2} g = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^2} e^{i \xi \eta} \sigma(\xi, \eta) (\partial_x)^r f(\xi) \hat{g}(\eta) d\xi d\eta, \]
where $\sigma(\xi, \eta) := \frac{(\xi + \eta)^r}{(\xi^2 + \eta^2)^{\frac{r}{2}}} \phi \left( \frac{\eta}{\xi} \right)$ and $\phi$ is a smooth function with supp $\phi \subset [-\frac{1}{2}, \frac{1}{2}]$. A direct calculation shows that
\[ |\partial_x^\alpha \partial_y^\beta \sigma(\xi, \eta)| \lesssim_{\alpha, \beta} (|\xi| + |\eta|)^{-\alpha - \beta} \]
for $(\xi, \eta) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ and $\alpha, \beta \in \mathbb{N}_0$. Accordingly, we can apply Coifman-Meyer’s Fourier multiplier theorem (see [1]) to obtain
\[ \|I\|_{H^r} \lesssim \|f\|_{H^r} \|g\|_{L^\infty}. \]

The second term on the right hand side of (4.11) is calculated as follows:
\[
\begin{align*}
\|II\|_{H^r} & \lesssim \sum_{N_1, N_2 \in 2^{\mathbb{N}}} N_2^{1 \lceil r \rceil} \|P_{N_1} f P_{N_2} g\|_{L^2} \\
& \lesssim \sum_{N_1 \in 2^{\mathbb{N}}} \|P_{N_1} f\|_{L^2} \|P_{N_1} g\|_{L^\infty} \\
& \quad + \sum_{N_1 \in 2^{\mathbb{N}}} \sum_{N_2 \in 2^{\mathbb{N}}} N_1^{1 + \lceil r \rceil + \frac{1}{2}} N_2^{-\lceil r \rceil - 1} \left\|P_{N_1} (\partial_x)^{-\lceil r \rceil + 1} f\right\|_{L^2} \left\|P_{N_2} (\partial_x)^{-\lceil r \rceil + 1} g\right\|_{L^2} \\
& \lesssim \|f\|_{L^2} \|g\|_{L^\infty} + \|f\|_{H^{\lceil r \rceil - \lceil r \rceil}} \|g\|_{H^{(r + 1)\lceil r \rceil}}.
\end{align*}
\]

Thanks to
\[ \|e^A\|_{H^k} \lesssim \|e^A\|_{L^\infty} \|u\|_{H^{k-1}} (1 + \|u\|^{k-1}_{L^2 \cap H^{k-2}}) \]
for $k \in \mathbb{N}$, Lemma 4.2 leads to
\[ (4.12) \quad \|e^A f\|_{H^r} \lesssim \|e^A\|_{L^\infty} \left( \|f\|_{H^r} + \|u\|_{H^r} \left( 1 + \|u\|^{\lceil r \rceil}_{L^2 \cap H^{\lceil r \rceil - 1}} \right) \|f\|_{H^{\lceil r \rceil - \lceil r \rceil}} \right). \]

We show a generalized version of Lemma 4.3.

Lemma 4.3. For $r \geq 0$, we have
\[ \|\langle \partial_x \rangle^r u\|_{L^2_x L^\infty_t} \lesssim e^{2\frac{\|A\|_{L^\infty_x}}{\sqrt{r}}} \left( \|u\|_{X^\text{max}(r - \frac{1}{2} + 2r, 0)} + \|u\|^{\lceil r \rceil + 3}_{X^\text{max}(r - \frac{1}{2} + 2r, 0)} \right). \]
When $N$ for (4.13)

Proof. As in (4.1), we have

\[ \|P_{>1}(\partial_x)^{r-1}(e^A_P N_P v)\|_{L^2_x L^\infty_T} \lesssim N_{1, N_2} \|P_{>1}(\partial_x)^{r-1}(e^A P_N v)\|_{L^2_x L^\infty_T}. \] (4.13)

For $N_1 \gtrsim N_2$, we have

\[ \|P_{>1}(\partial_x)^{r-1}(P_{N_1} e^A P_N v)\|_{L^2_x L^\infty_T} \lesssim \|\partial_x^{[r]}(P_{N_1} e^A P_N v)\|_{L^2_x L^\infty_T} \lesssim N_1^{r-4+\varepsilon} \|\partial_x^{[r]+1} P_{N_1} e^A\|_{L^\infty_{x,T}} \|\partial_x^{r-4-\varepsilon} P_N v\|_{L^2_x L^\infty_T} \lesssim N_1^{r-4+\varepsilon} \|\partial_x^{[r]}(e^A u)\|_{L^\infty_{x,T}} \|v\|_{X_T}. \]

When $[r] \geq 1$, Sobolev’s embedding and (4.2) yield that

\[ \|\partial_x^{[r]}(e^A u)\|_{L^\infty_{x,T}} \lesssim \|\partial_x^{[r]-1}(e^A u^2)\|_{L^\infty_{x,T}} + \|\partial_x^{[r]-1}(e^A u^2)\|_{L^\infty_{x,T}} \lesssim |e^A u^2|_{H^{[r]-4+\varepsilon}} + |u_{[r]} e^A v|_{H^{[r]-4+\varepsilon}} \lesssim |e^A|_{L^\infty_{x,T}} \|b\|_{L^\infty_{x,T}} \left( 1 + \|u\|_{L^\infty_{x,T}}^{[r]} \right) \lesssim e^{2|A|_{L^\infty_{x,T}}} \left( \|u\|_{X_T^{[r]-4+\varepsilon}} + \|u\|_{L^\infty_{x,T}}^{[r]} \right). \]

Hence, we have

\[ \|P_{>1}(\partial_x)^{r-1}(P_{N_1} e^A P_N v)\|_{L^2_x L^\infty_T} \lesssim N_1^{r-4+\varepsilon} e^{2|A|_{L^\infty_{x,T}}} \left( \|u\|_{X_T^{[r]-4+\varepsilon}} + \|u\|_{L^\infty_{x,T}}^{[r]} \right). \]

For $N_1 \gtrsim N_2$, we have

\[ \|\partial_x^{r-1}(P_{N_1} e^A P_N v)\|_{L^2_x L^\infty_T} \lesssim N_2^{r-4+\varepsilon} \|P_{N_1} e^A\|_{L^\infty_{x,T}} \|\partial_x^{r-4-\varepsilon} P_N v\|_{L^2_x L^\infty_T} \lesssim N_2^{r-4+\varepsilon} \|e^A\|_{L^\infty_{x,T}} \|v\|_{X_T}. \]

For $N_1 \ll N_2$, the frequency of the product of the two functions is around $N_2$. For $0 \leq r < \frac{1}{4} - \varepsilon$, we have

\[ \|\partial_x^{r-1}(P_{N_1} e^A P_N v)\|_{L^2_x L^\infty_T} \lesssim N_2^{r-4+\varepsilon} \|P_{N_1} e^A\|_{L^\infty_{x,T}} \|\partial_x^{r-4-\varepsilon} P_N v\|_{L^2_x L^\infty_T} \lesssim N_2^{r-4+\varepsilon} \|e^A\|_{L^\infty_{x,T}} \|v\|_{X_T}. \]

Hence, we can sum up the summation with respect to $N_1$ and $N_2$ in (4.13). Therefore, we obtain the desired bound. \[\square\]
Let $\tilde{r} := r - [r]$. The fractional Leibniz rule (see Appendix in [11]), Lemma 4.3, and an interpolation argument yield that
\[
\|\partial_x^{[r]}(u \partial_x v)\|_{L^2_{T,x}} \lesssim \sum_{k=0}^{[r]} \left( \|\partial_x^{[r]-k} u \partial_x^{k+1} v\|_{L^2_{T,x}} + \|\partial_x^{[r]-k} u\|_{L^2_{T,x}} + \|\partial_x^{[r]-k} \partial_x^{k+1} v\|_{L^2_{T,x}} \right)
\]

\[
\lesssim \sum_{k=0}^{[r]} \left( \|\partial_x^{[r]-k} u\|_{L^2_{T,x}} \left( \|\partial_x^{[r]-k} u\|_{L^2_{T,x}} + \|\partial_x^{[r]-k} \partial_x^{k+1} v\|_{L^2_{T,x}} \right) \right)
\]

\[
\lesssim \sum_{k=0}^{[r]} \left( \|\partial_x^{[r]-k} u\|_{L^2_{T,x}} \left( \|\partial_x^{[r]-k} u\|_{L^2_{T,x}} + \|\partial_x^{[r]-k} \partial_x^{k+1} v\|_{L^2_{T,x}} \right) \right)
\]

\[
\lesssim \sum_{k=0}^{[r]} \left( \|\partial_x^{[r]-k} u\|_{L^2_{T,x}} \left( \|\partial_x^{[r]-k} u\|_{L^2_{T,x}} + \|\partial_x^{[r]-k} \partial_x^{k+1} v\|_{L^2_{T,x}} \right) \right)
\]

Because Sobolev’s embedding and (4.12) imply that
\[
\|\partial_x^{[r]} u\|_{L^\infty_T L^\infty_x} \lesssim \|\partial_x^{[r]} (e^A v)\|_{L^\infty_T L^\infty_x} \lesssim \|e^A v\|_{H^{-\frac{1}{2},+}_x} \lesssim \|e^A v\|_{L^\infty_T L^\infty_x} \left( 1 + \|v\|_{H^{(k-1)}_T} \right)
\]

for $k \in \mathbb{N}$, the same calculation as above leads to
\[
\|\partial_x^{[r]} (u^2 \partial_x v)\|_{L^2_{T,x}} \lesssim \sum_{k=0}^{[r]} \left( \|\partial_x^{[r]-k} (u^2) \partial_x^{k+1} v\|_{L^2_{T,x}} + \|\partial_x^{[r]-k} (u^2)\|_{L^2_{T,x}} + \|\partial_x^{[r]-k} \partial_x^{k+1} v\|_{L^2_{T,x}} \right)
\]

Lemma 4.2 and (4.12) show that
\[
\|v \int_{-\infty}^{x} e^{2\Lambda} v^2 \, dy\|_{L^2_{T,x}} \lesssim \|v\|_{L^2_{T,x}} \|e^{2\Lambda}\|_{L^\infty_{T,x}} \|v\|_{L^2_{T,x}} + \|\|v\|_{L^2_{T,x}} \|e^{2\Lambda} v^2\|_{L^2_{T,x}} \|v\|_{L^2_{T,x}}
\]

\[
\lesssim \|e^{2\Lambda}\|_{L^2_{T,x}} \left( \|u\|_{\chi_{r-1,0}^{\max(r-0,0)}} + \|u\|_{Z^{r+2}_{T}} \right) \|u\|_{\tilde{X}_T^{r}}.
\]

Because the remaining terms on the right hand side of (3.3) and (3.4) with $c_0 = 0$ more easily handled, the estimates (4.10) and (4.11) yield that
\[
\|u\|_{Z_T^r} \leq C_1 \|u_0\|_{\chi^{r}} + C_2 T^\frac{3}{2} e^{3c_1 \|u\|_{Z_T^r}} \left( \|u\|_{Z^{r+3}_{T}^{\max(r-\frac{3}{5},1)}} + \|\|u\|_{Z^{r+3}_{T}^{\max(r-\frac{3}{5},1)}} \right) \|u\|_{Z_T^r}.
\]

The persistence property follows from this a priori bound with a standard continuity argument.
4.3. Proof of Theorem 1.1 without $c_2 = 0$. The first term on the right hand side of (3.4) causes some technical difficulty, because it has a quadratic term with one derivative. However, by using a gauge transformation, we cancel out this term. As in the previous subsection, the well-posedness is reduced to show an a priori estimate as in (4.17).

Let $\Xi(t, x) = c_2 \int_{-\infty}^{x} u(t, y) d y$. Then, (1.1), (3.1), and (3.2) yield

$$
e^{-\Xi} L (e^{-\Xi} u) = (c_1 - c_2) u \partial_x^2 u - (c_1 - 2 c_2) c_2 u^2 \partial_x u + (c_1 - c_2) c_2 u \int_{-\infty}^{x} (\partial_y u)^2 d y - \frac{c_3^2}{3} u^4.$$

Set $u := e^{-\Xi}$. Since

$$\partial_x u = e^{-\Xi} \partial_x u = e^{-\Xi} \partial_x u + c_2 e^{2\Xi} u^2,$$

$$\partial_x^2 u = e^{-\Xi} (\partial_x^2 u + c_2 u \partial_x u) + c_2 e^{2\Xi} (2u \partial_x u + 2c_2 u u^2),$$

we have

$$\begin{align*}
L u &= (c_1 - c_2) u (e^{-\Xi} \partial_x^2 u + 3c_2 e^{2\Xi} u \partial_x u + 2c_2^2 e^{3\Xi} u^3) \\
&\quad - (c_1 - 2c_2) c_2 e^{2\Xi} u^2 (e^{-\Xi} \partial_x u + c_2 e^{2\Xi} u^2) \\
&+ (c_1 - c_2) c_2 u \int_{-\infty}^{x} (e^{-\Xi} \partial_y u + c_2 e^{2\Xi} u^2)^2 d y - \frac{c_3^2}{3} e^{3\Xi} u^4 \\
&= (c_1 - c_2) e^{-\Xi} \partial_x^2 u + (2c_1 - c_2) c_2 e^{2\Xi} u^2 \partial_x u \\
&\quad + (c_1 - c_2) c_2 u \int_{-\infty}^{x} (e^{-\Xi} \partial_y u + c_2 e^{2\Xi} u^2)^2 d y + \left(c_1 - \frac{c_2}{3}\right) c_2^2 e^{3\Xi} u^4.
\end{align*}$$

A direct calculation shows that

$$L \partial_x u = (c_1 - c_2) e^{-\Xi} \partial_x^3 u + (c_1 - c_2) e^{-\Xi} \partial_x u \partial_x^2 u + \mathcal{N},$$

where $\mathcal{N}$ is a linear combination of

$$e^{2\Xi} u^2 \partial_x^2 u, \quad e^{2\Xi} (\partial_x u)^2, \quad e^{3\Xi} u^3 \partial_x u, \quad \partial_x u \int_{-\infty}^{x} (e^{-\Xi} \partial_y u + c_2 e^{2\Xi} u^2)^2 d y, \quad e^{4\Xi} u^5.$$

Moreover, let $\Theta := (c_1 - c_2) \int_{-\infty}^{x} (e^{-\Xi} u)(t, y) d y = (c_1 - c_2) \int_{-\infty}^{x} u(t, y) d y$ and $v := e^{-\Theta} \partial_x u$. Because

$$\begin{align*}
L \Theta &= (c_1 - c_2) c_1 u \partial_x u - (c_1 - c_2)^2 \int_{-\infty}^{x} (\partial_y u)^2 d y \\
&= (c_1 - c_2) c_1 e^{-\Xi} u (e^{-\Xi} \partial_x u + c_2 e^{2\Xi} u^2) - (c_1 - c_2)^2 \int_{-\infty}^{x} (e^{-\Xi} \partial_y u + c_2 e^{2\Xi} u^2)^2 d y,
\end{align*}$$

(3.1) and (4.15) imply that $L v$ is equal to a linear combination of

$$e^{2\Xi} u^2 \partial_x v, \quad e^{2\Xi + \Theta} u^2, \quad e^{3\Xi} u^3 v, \quad v \int_{-\infty}^{x} (e^{\Xi + \Theta} v + c_2 e^{2\Xi} u^2)^2 d y, \quad e^{4\Xi - \Theta} u^5.$$

In addition, (4.14) is written as follows:

$$\begin{align*}
L u &= (c_1 - c_2) e^{\Xi + \Theta} \partial_x v + c_2^2 e^{2\Xi + \Theta} u^2 v + (c_1 - c_2) c_2 u \int_{-\infty}^{x} (e^{\Xi + \Theta} v + c_2 e^{2\Xi} u^2)^2 d y \\
&\quad + \left(c_1 - \frac{c_2}{3}\right) c_2^2 e^{3\Xi} u^4.
\end{align*}$$
Here, we define the norm
\[
\|u\|_{\tilde{Z}_T} := \left\| e^{-c_2 \int_{-\infty}^{x} u(t,y)dy} \right\|_{X_T} + \left\| e^{-(c_1-c_2) \int_{-\infty}^{x} u(t,y)dy} \partial_x \left( e^{-c_2 \int_{-\infty}^{x} u(t,y)dy} \right) \right\|_{X_T} + \left\| \int_{-\infty}^{x} u(t,y)dy \right\|_{L_{t,x}^{\infty}}.
\]
Then, (3.5), (4.3), and (4.4) yield that
\[
(4.16) \quad \|u\|_{\tilde{Z}_T} \leq C_1 \|u_0\|_{X^1} + C_2 T^4 e^{5(|c_1|+|c_2|)} \|u\|_{\tilde{Z}_T} \left( 1 + \|u\|_{X_T}^4 \right)
\]
as long as \(u\) is a solution to (1.1). Hence, the same argument as in (4.11) shows that the existence time \(T\) depends only on \(\|u_0\|_{X^1}\). Moreover, (1.1) is well-posed in \(X^1\). Because the persistency follows from the same argument as in (4.12) we omit the details here.

5. Well-posedness for the quadratic KdV-type equation

In this section, we consider the Cauchy problem for the semi-linear KdV-type equation with quadratic nonlinearity. Let \(u\) be a solution to (1.3). Then, \(\partial_x u\) and \(\partial_x^2 u\) satisfy the following equations:

\[
(5.1) \quad \mathcal{L} \partial_x u = (c_1 + 2c_2) \partial_x u \partial_x^2 u + (c_1 + c_3 \partial_x u) \partial_x^2 u + c_3 (\partial_x^3 u)^2 + 2c_4 \partial_x^3 u \partial_x u^3 u,
\]

\[
(5.2) \quad \mathcal{L} \partial_x^2 u = (c_1 + 2c_2) (\partial_x^2 u)^2 + (2c_1 + c_2) \partial_x u + 3c_3 \partial_x^2 u \partial_x u + (c_1 + c_3 \partial_x u + 2c_4 \partial_x^2 u) \partial_x^2 u + 2c_4 (\partial_x^3 u)^2.
\]

Set \(J := 2c_4 \partial_x u\) and \(w := e^{-3} \partial_x^2 u\). Then, (5.1), (5.1), and (5.2) yield that

\[(5.3) \quad \mathcal{L} w = (c_1 + c_3 \partial_x u) \partial_x^2 w + N_1,
\]
where \(N_1\) is a linear combination of forms

\[f_1 \partial_x w, \quad f_1 f_2 \partial_x w, \quad e^{-3} f_1 f_2, \quad e^{-3} f_1 f_2 f_3, \quad e^{-3} f_1 f_2 f_3 f_4\]

for \(f_j \in \{u, \partial_x u, e^3 w\}\).

Let \(K := c_1 \int_{-\infty}^{x} u(t,y)dy + c_3 u\) and \(w := e^{-3} \partial_x w\). Because

\[
\mathcal{L} \int_{-\infty}^{x} u(t,y)dy = c_1 u \partial_x u - (c_1 - c_2) \int_{-\infty}^{x} (\partial_y u)^2 dy + c_4 \int_{-\infty}^{x} (\partial_y u)^2 dy,
\]

(3.1) and (5.3) imply that \(\mathcal{L} w\) is equal to a linear combination of forms

\[f_1 \partial_x w, \quad f_1 f_2 \partial_x w, \quad \int_{-\infty}^{x} (\partial_y u)^2 dy \quad w, \quad \int_{-\infty}^{x} e^{3+K} w^2 dy \quad w, \quad e^{-3-K} g_1 g_2, \quad e^{-3-K} g_1 g_2 g_3, \quad e^{-3-K} g_1 g_2 g_3 g_4, \quad e^{-3-K} g_1 g_2 g_3 g_4 g_5\]

for \(f_j \in \{u, \partial_x u, e^3 w\}\), \(g_k \in \{u, \partial_x u, e^3 w, e^{3+K} w\}\). Moreover, (1.3), (5.1), and (5.3) are written as follows:

\[
\mathcal{L} u = c_1 e^3 w w + c_2 (\partial_x u)^2 + c_3 e^3 \partial_x u w + c_4 e^{23} w^2,
\]

\[
\mathcal{L} \partial_x u = (c_1 + 2c_2) e^3 \partial_x u w + (c_1 + c_3 \partial_x u) e^3 (e^K w + 2c_4 w^2) + c_3 e^{23} w^2 + 2c_4 e^{23} w (e^K w + 2c_4 w^2),
\]

\[
\mathcal{L} w = (c_1 u + c_3 \partial_x u) e^K \partial_x w + N_1,
\]
where $\tilde{N}_1$ is a linear combination of forms
\[ e^R f_1 w, \quad e^R f_1 f_2 w, \quad e^{-3} f_1 f_2, \quad e^{-3} f_1 f_2 f_3, \quad e^{-3} f_1 f_2 f_3 f_4 \]
for $f_j \in \{ u, \partial_x u, e^w \}$. Hence, we can apply the contraction mapping theorem as in [33] to obtain well-posedness in $X^4$ of (1.3).

We define the norm as follows:
\[
\| u \|_{Z_T} := \| u \|_{X_T} + \| \partial_x u \|_{X_T} + \| e^{-c_4 \partial_x u} \partial_x^2 u \|_{X_T}
\]
\[ + \left| \int_{-\infty}^{x} u(t, y) dy \right|_{X_T} + \left| c_1 \int_{-\infty}^{x} u(t, y) dy \right|_{L^\infty_{T,x}}. \]

Because
\[
\|3\|_{L^\infty_{T,x}} \leq 2|c_4| \| \partial_x u \|_{L^\infty_{T,x}} \lesssim \| u \|_{L^\infty_{T,x} H^2},
\]
\[
|\bar{R}|_{L^\infty_{T,x}} \leq |c_1| \left( \left| \int_{-\infty}^{x} u(t, y) dy \right|_{L^\infty_{T,x}} + \| u \|_{L^\infty_{T,x} H^2} \right) \lesssim \| u \|_{L^\infty_{T,x} X^4},
\]
and a similar calculation as in (4.3) and (4.4) yield that
\[
\| u \|_{Z_T} \leq C_1 \| u_0 \|_{X^1} + C_2 T^\frac{1}{4} e^{C_3 \| u \|_{Z_T}} \| u \|_{Z_T} \left( 1 + \| u \|_{Z_T}^4 \right).
\]

When $c_4 = 0$, we set
\[
\| u \|_{Z_T} := \| u \|_{X_T} + \| e^{-c_3 u} \partial_x u \|_{X_T} + \left| \int_{-\infty}^{x} u(t, y) dy \right|_{X_T}
\]
\[ + \left| c_1 \int_{-\infty}^{x} u(t, y) dy \right|_{L^\infty_{T,x}}.
\]

Then, the same argument as above shows that
\[
\| u \|_{Z_T} \leq C_1 \| u_0 \|_{X^1} + C_2 T^\frac{1}{4} e^{C_3 \| u \|_{Z_T}} \| u \|_{Z_T} \left( 1 + \| u \|_{Z_T}^4 \right).
\]

Remark 5.1. When $c_4 = 0$, the boundedness of primitives is not necessary, because $\int_{-\infty}^{x} u(t, y) dy$ disappears in $\| u \|_{Z_T}$ and $\| u \|_{Z_T'}$.

6. Irregular Flow Maps

6.1. On the condition for initial data. For $c_1 \neq 0$, Pilod [21] proved that the flow map fails to be twice differentiable in $H^s(R)$ for any $s \in \mathbb{R}$. Here, we briefly observe that our result does not contradict to Pilod’s result.

For simplicity, we consider (1.1) with $c_1 \neq 0$ and $c_2 = 0$. Pilod put the following sequence of the initial data:
\[
u_{0,N} := \mathcal{F}^{-1} \left[ N^{1} \left[ -N^{-2}, 1 \right] + N^{-s+1} 1 \left[ -N^{-N^{-2}}, 0 \right] \right],
\]
for any $N \geq 1$. Then, $\| u_{0,N} \|_{H^s} \lesssim 1$.

If $\xi_1 \in \left[ -2N^{-2}, 2N^{-2} \right]$ and $\xi - \xi_1 \in \left[ -N^{-2}, N^{-2} \right]$, then $\xi \in \left[ N - 2N^{-2}, N + 2N^{-2} \right]$ and
\[
|\xi^3 | \lesssim 3 \| \xi_1 \| \lesssim 1.
\]

Accordingly, for $0 < T \ll 1$, we have
\[
\left| \int_{0}^{T} \mathcal{U}(t - t') (\mathcal{U}(t) u_{0,N}(x)) \mathcal{U}(t') \partial_x^2 u_{0,N}(x) dt' \right|_{L^\infty_{T,x} H^s}
\]
\[ \gtrsim T \| N^{-s+2} \mathcal{F}^{-1} \left[ 1 \left[ -N^{-N^{-2}}, 0 \right] \right] \|_{L^\infty_{T,x} H^s} \gtrsim TN, \]

which shows the flow map fails to be twice differentiable in $H^s(\mathbb{R})$.

By a simple calculation, the initial datum is written as follows:

$$u_{0,N}(x) = \sqrt{\frac{2}{\pi}} \left(1 + 2N^{-s} \cos Nx\right) N \frac{\sin N^{-2}x}{x}.$$

Since

$$\int_{-\infty}^{\infty} u_{0,N}(y) dy = \sqrt{\frac{2}{\pi}} N \left(\int_{-\infty}^{\infty} \frac{\sin y}{y} dy + 2N^{-s} \int_{-\infty}^{\infty} \frac{\sin y \cos N^3 y}{y} dy\right)$$

$$= \sqrt{2\pi} N,$$

this sequence is not bounded in $X^s$. In other words, we can avoid the worst interaction because of $\sup_{x \in \mathbb{R}} \left|\int_{-\infty}^{x} u_{0}(y) dy\right| < \infty$.

6.2. Not $C^2$ in $X^s$. For simplicity, we assume that $c_1 = 1$ and $c_2 = 0$. We set

$$u_{0,N} := \mathcal{F}^{-1} \left[N^{-s+\frac{2}{3}} 1_{[-N-N^{-a},-N+N^{-a}] \cup [N-N^{-a},N+N^{-a}]}\right]$$

for any $N \gg 1$ and $a > 0$. Then, $\|u_{0,N}\|_{H^s} \lesssim 1$. Since

$$u_{0,N}(x) = 2 \sqrt{\frac{2}{\pi}} N^{-s+\frac{2}{3}} \frac{\sin N^{-2}ax}{x} \cos Nx,$$

a direct calculation shows

$$\int_{-\infty}^{x} u_{0,N}(y) dy = \sqrt{\frac{2}{\pi}} N^{-s+\frac{2}{3}} \int_{-\infty}^{x} \left\{\frac{\sin (N+N^{-a})y}{y} - \frac{\sin (N-N^{-a})y}{y}\right\} dy$$

$$= \sqrt{\frac{2}{\pi}} N^{-s+\frac{2}{3}} \int_{(N-N^{-a})x}^{(N+N^{-a})x} \frac{\sin y}{y} dy.$$

The mean value theorem for integrals yields

$$\sup_{x \in \mathbb{R}} \left|\int_{-\infty}^{x} u_{0,N}(y) dy\right| \lesssim N^{-s+\frac{2}{3} - 1}.$$

Therefore, $\{u_{0,N}\}$ is a bounded sequence in $X^s$ provided that $s > -\frac{2}{3} - 1$.

On the other hand, for $0 < T \ll 1$, we have

$$\sup_{t \in [-T,T], x \in \mathbb{R}} \left|\int_{-\infty}^{t} \int_{-\infty}^{t} \mathcal{U}(t-t') (\mathcal{U}(t') u_{0,N}(y) \mathcal{U}(t') \partial_x^2 u_{0,N}(y)) dt' dy\right|$$

$$\gtrsim \sup_{t \in [-T,T]} \left|\mathcal{F} \left[\int_{0}^{t} \mathcal{U}(t-t') (\mathcal{U}(t') u_{0,N}(x) \mathcal{U}(t') \partial_x^2 u_{0,N}(x)) dt'\right] (0)\right|$$

$$\gtrsim T \left|\mathcal{F} \left[u_{0,N} \partial_x^2 u_{0,N}\right] (0)\right| \gtrsim TN^{-2s+2},$$

which shows the flow map fails to be twice differentiable in $X^s$ for $s < 1$.

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