Commutative limit of a renormalizable noncommutative model

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Abstract – Renormalizable $\phi^4$ models on Moyal space have been obtained by modifying the commutative propagator. But these models have a divergent “naïve” commutative limit. We explain here how to obtain a coherent such commutative limit for a recently proposed translation-invariant model. The mechanism relies on the analysis of the ultraviolet/infrared mixing in Feynman graphs at any order in perturbation theory.

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Introduction. – Noncommutative field theory [1] is a subject that lies at the intersection of many different perspectives. It is a natural generalization of Alain Connes noncommutative geometry program [2], and it is an effective regime of string theory [3,4] and of loop quantum gravity [5]. It has also the potential to throw light on difficult nonperturbative physical problems (quantum Hall effect [6–8], quark confinement, and so on). For a recent review, see [9].

The issue of renormalization of noncommutative (Euclidean) field theory on Moyal space, however, was made difficult by the discovery of ultraviolet/infrared (UV/IR) mixing [10]. In the simple case of the $\phi^4$ model, a first solution was obtained by Grosse and Wulkenhaar, who modified the ordinary propagator, adding a harmonic potential term [11]. The resulting theory has a new symmetry called Langmann-Szabo LS duality [12]. This result has sparked a flurry of activity. The corresponding “GW” model (and other related models) has (have) been shown fully renormalizable by all the main methods; furthermore, different field theoretical properties have been exhibited [13–20]. Even more important, it is the first example of a nonsupersymmetric field theory in four dimensions which has a nontrivial UV fixed point [21–23]. It is currently under construction in a nonperturbative sense [24,25], something which has never been fully done for any four-dimensional commutative theory. Noncommutative gauge theories with some version of LS symmetry are actively searched for [26–28]. However, in spite of this great conceptual and mathematical interest the GW model has, for the moment, no direct phenomenological applications. It breaks translation invariance and it seems very difficult to connect it continuously to ordinary field theory at lower energy. In technical terms, the $\theta \to 0$ limit of the GW model seems too singular.

These drawbacks motivated the introduction of another $\phi^4$ model in [29], which is translation invariant and also renormalizable to all orders in perturbation theory. It is not based on the LS symmetry, hence it may not be fully constructible in a nonperturbative sense. However, it may be easier to connect to ordinary high energy physics. To support this idea comes the recent proof [30] that the $\beta$-function of this model is just a rational multiple of the commutative model $\beta$-function. Let us emphasize here that this result, in spite of the title of [30], was proven there at any order in perturbation theory.

Let us further state here that the parametric representation for this model was obtained in [31]. Also some one-loop and higher order Feynman amplitudes were explicitly computed in [32]. Furthermore, the static potential associated to this noncommutative model was calculated in [33]. Moreover, the idea of the translation-invariant scalar model proposed in [29] was also extended to the level of noncommutative gauge fields [34]. For a review of these developments, the interested reader may refer to [35].

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In this paper, we study how in this model the $1/p^2$ counterterms created by the UV/IR mixing graphs can morph into the ordinary mass and wave function counterterms that these graphs generate in the ordinary commutative $\phi_4^4$ as the $\theta$ parameter is turned off. In this way, we show how a renormalizable noncommutative model can become an effective commutative model. This is a small step in the direction of smoothing the road between commutative and noncommutative field theory. Note that such a mechanism could also give insights on the noncommutative limit of the gauge models [34]. Commutative field theory certainly works well at least up to the LHC energies, but a noncommutative field theory regime might be relevant somewhere between the LHC scale and the Planck scale, where gravity has to be quantized.

Scalar field theory on the Moyal space. – In this section, we recall the definition of the noncommutative field theory regime might be relevant somewhere between the LHC scale and the Planck scale, where gravity has to be quantized.

The “naive” $\phi^4$ model. This model is obtained by replacing in the ordinary $\phi^4$ action the pointlike product by the Moyal-Weyl $\star$-product

$$S[\phi] = \int d^4x \left( \frac{1}{2} \partial_{\mu}\phi \star \partial_{\nu}\phi + \frac{1}{2} m^2 \phi \star \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \right)(x),$$

with Euclidean metric. The commutator of the coordinates is

$$[x_{\mu}, x_{\nu}] = i \Theta_{\mu\nu},$$

where

$$\Theta = \begin{pmatrix} 0 & \theta & 0 & 0 \\ -\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta \\ 0 & 0 & -\theta & 0 \end{pmatrix}$$

in which $\theta$ is the noncommutative parameter. In momentum space, action (1) becomes

$$S(\phi) = \frac{1}{(2\pi)^4} \int d^4p \left( \frac{1}{2} p^2 \phi(-p)\phi(p) + \frac{1}{2} m^2 \phi(-p)\phi(p) + \frac{\lambda}{4!} V_{\theta}(\phi, p) \right),$$

where the interaction potential is:

$$V_{\theta}(\phi, p) = \frac{1}{(2\pi)^4} \int d^4qd^4k \ \hat{\phi}(p)\hat{\phi}(q)\hat{\phi}(k)\hat{\phi}(-p-q-k)e^{-\frac{1}{\theta}(p^2 + q^2 + k^2 + (-p-q-k)^2)}.$$

Note that we have used the notation

$$p \wedge q = p_\mu \Theta^{\mu\nu} q_\nu.$$
The “nonplanar” tadpole. The Feynman amplitude $A_G$ of the “nonplanar” tadpole $G_0$ of fig. 2 writes, up to some constant factor,

$$A_G(k) = \int d^4p \frac{e^{ip\wedge k}}{p^2 + m^2}.$$  \hfill (8)

The oscillating factor above is responsible for the convergence of the integral. One can thus interpret $\theta^{-\frac{1}{2}}$ as some kind of UV cutoff. Indeed, when $\theta = 0$ the integral is no longer convergent and (8) will simply correspond to the divergent planar tadpole, leading to the usual mass renormalization. This observation comes from the fact that the amplitude (8) computes to

$$\left\{ \begin{array}{ll}
\frac{\infty}{\sqrt{p^2 + m^2}} K_1(\sqrt{m^2\theta^2 k^2}) & \text{if } \theta = 0, \\
\frac{1}{\sqrt{p^2 + m^2}} & \text{if } \theta \neq 0,
\end{array} \right.$$  \hfill (9)

where again, in the second line above, we have omitted some inessential constants.

General two-point planar irregular graphs. Let us now prove that in the case of a general two-point planar irregular Feynman graph, the considerations of the previous subsection still hold.

As before, let us denote the external moment of a two-point graph by $k$. One has $B = 2$. When shrinking the graph to a rosette (as described in the section “Feynman graphs: planarity and nonplanarity, rosettes”), one can always choose one of the external legs to break the “external face”. The other external leg then breaks an internal face (see for example fig. 3, where this internal face corresponds to line $p_1$).

The general amplitude contains an oscillating factor (see again [36] for further details)

$$e^{-ik\wedge (a^i p_i)}.$$  \hfill (10)

The linear combination $a^i p_j$ corresponds to the sum of momenta of the lines arching over this second external leg, where the coefficients $a_j$ are $\pm 1$. In the example of fig. 3, because of the orientation of the lines, one has to take the internal momenta $p_1$ and $p_3$ with opposite signs: $a^2 p_3 = p_1 - p_3$. A general Feynman amplitude writes

$$A_G = K \int \prod_{i=1}^L d^4p_i \frac{e^{-i k\wedge (a^i p_i)}}{\prod_{i=1}^L(p_i^2 + m^2)}.$$  \hfill (11)

As in the case of the nonplanar tadpole, we now express the integral above in function of the Schwinger parameters $\alpha_\ell$ ($\ell = 1, \ldots, L$). One can then check (see [31] for a more detailed analysis) that after performing the Gaussian integrals over the internal momenta $p_i$, one obtains an integral of the following type:

$$\int_0^\infty U^{-2}(\alpha) e^{-m^2 \sum_{j=1}^L \alpha_j^2 e^{-\frac{p_j(\alpha)}{4 \sum_{\text{overarching } \alpha_\ell}}}}$$  \hfill (12)

where

$$k \circ p = -k_\mu (\Theta^\mu)^{\nu\rho} p_\rho, \quad (\Theta^\mu)^{\nu\rho} = \Theta^{\mu \rho} \Theta^\nu.$$  \hfill (13)

$U(\alpha)$ and $P(\alpha)k^2 = V(\alpha, k)$ are the usual Symanzik (topological) polynomials of the ordinary commutative $4^\ell$ theory (since we deal with a two-point function, there is a single external invariant $k^2$ in factor in $V$). The sum $\sum_{\text{overarching } \alpha_\ell}$ above denotes the sum of the parameters $\alpha$ associated to the tree lines (which were reduced to obtain the rosette) as well as of the parameters $\alpha$ associated to the lines overarching the internal broken face ($\alpha_1 + \alpha_3$ in the example of fig. 3). Note that

$$\left(\text{deg } U\right)^2 = 2(L - (n - 1)).$$  \hfill (14)

Furthermore, since $L = 2n - N/2$ we deal with the two-point function ($N = 2$), one has

$$\left(\text{deg } U\right)^2 = 2(L - (n - 1)) = L + 1.$$  \hfill (15)

Let us recall that the polynomials $U$ and $V$ are explicitly positive.

Note that $\Theta^2 = -\theta^2 \text{Id}$ so that integral (11) writes

$$\int_0^\infty U^{-2}(\alpha) e^{-m^2 \sum_{j=1}^L \alpha_j^2 e^{-\frac{p_j(\alpha)}{4 \sum_{\text{overarching } \alpha_\ell}}}}.$$

Finally, since $L = 2n - N/2$ and we deal with the two-point function ($N = 2$), one has

$$\left(\text{deg } U\right)^2 = 2(L - (n - 1)) = L + 1.$$  \hfill (16)

Let us recall that the polynomials $U$ and $V$ are explicitly positive.

Note that $\Theta^2 = -\theta^2 \text{Id}$ so that integral (11) writes

$$\int_0^\infty U^{-2}(\alpha) e^{-m^2 \sum_{j=1}^L \alpha_j^2 e^{-\frac{p_j(\alpha)}{4 \sum_{\text{overarching } \alpha_\ell}}}} \prod_{i=1}^L \text{d} \alpha_i.$$  \hfill (17)
In the UV regime ($\alpha_\ell \to 0$), one needs to consider only

$$\int_0^\infty U^{-2}(\alpha)e^{-m^2 \sum_{j=1}^L \alpha_j^2} \times \exp \left(-\frac{\theta^2 k^2}{4 \sum_{\text{overarching } \ell} \alpha_{\ell}}\right) \prod_{i=1}^L d\alpha_i. \quad (16)$$

We now analyze this integral by performing the following change of variable:

$$\alpha_i = k^2 \alpha_i^\prime, \quad i = 1, \ldots, L. \quad (17)$$

Using (14), the integral (16) becomes

$$\int \frac{1}{k^2} \int_0^\infty U^{-2}(\alpha')e^{-m^2 k^2 \sum_{j=1}^L \alpha_j^2} \times \exp \left(-\frac{\theta^2}{4 \sum_{\text{overarching } \ell} \alpha_{\ell}^\prime}\right) \prod_{i=1}^L d\alpha_i^\prime. \quad (18)$$

This remaining integral is convergent. We have thus established the asymptotic behavior of $A_G$ to be $\text{const}/k^2$ at small $k$, as for the “nonplanar” tadpole. Moreover, when $\theta \to 0$ we recover easily on (11) the usual mass and wave function divergences associated to $G$.

**The noncommutative model and its renormalization.** The translation-invariant $\phi_4^2$ model defined in [29] has action

$$A_{\phi}[\phi] = \int d^4p \left(\frac{1}{2} p_\mu p^\mu \phi + \frac{1}{2} m^2 \phi \phi \right) + \frac{1}{2} a \left(\frac{1}{2} p_\mu p^\mu \phi + \frac{\lambda}{4!} V_0 \right), \quad (19)$$

where $a$ is some dimensionless parameter. The propagator is

$$C(p) = \frac{1}{p^2 + \mu^2 + \frac{a}{4!} p^2}. \quad (20)$$

We further choose $a > 0$ so that this propagator is positively defined.

Let us recall from [29] Table 1 summarizing the renormalization of the model (19).

| Planar regular | Renormalization (mass and wave function) | Four-point function |
|----------------|-----------------------------------------|---------------------|
|                | Convergent                              | Convergent          |
| Planar irregular | Finite renormalization ($a$)             |                     |
| Nonplanar      | Convergent                              |                     |

**The commutative limit.** In the limit $\theta \to 0$, eqs. (19) or (20) no longer make sense. This phenomenon takes place because the limit is done in a too direct, “naive” way. One should proceed as indicated by the previous analysis. As we have seen, when $\theta \to 0$ the convergent integrals of planar irregular two-point graphs become the usual divergent ones responsible for “some part” of the mass and wave function renormalizations. Furthermore, recall that the integrals of nonplanar graphs also become divergent when $\theta \to 0$. We thus propose the following action with UV cutoff $\Lambda$:

$$S_{\Lambda,\phi}[\phi] = \int d^4p \left[\frac{1}{2} \eta^{-1}(p/\Lambda)p_\mu p^\mu \phi + \frac{1}{2} m^2 \phi \phi + \frac{\lambda}{4!} V_0 \right. \left. + \frac{1}{2} \delta_z p^2 \phi^2 + \frac{1}{2} \delta_z' (1 - T(\Lambda, \theta)) p^2 \phi^2 \right. \left. + \frac{1}{2} \delta_v \frac{1}{2} k^2 \phi^2 T(\Lambda, \theta) + \frac{1}{2} a \frac{1}{2} k^2 \phi^2 T(\Lambda, \theta) \right. \left. \right.$$

$$\left. + \frac{1}{2} \delta_v \frac{1}{2} k^2 \phi^2 (1 - T(\Lambda, \theta)) + \frac{\delta_{\phi}}{4!} (1 - T(\Lambda, \theta)) V_0 \right) + \frac{\delta_{\phi}}{4!} T(\Lambda, \theta) V_0, \quad (21)$$

where we have written the counterterms associated to (19). The cutoff $\Lambda$ is some UV scale with the dimension of a momentum. The function $\eta^{-1}(p/\Lambda)$ is a standard momentum-space UV cutoff that truncates momenta higher than $\Lambda$ in the propagator (20). For this, $\eta(p)$ could be a fixed smooth function with compact support interpolating smoothly between value $1$ for $|p/\mu| < 1/2$ and 0 for $|p/\mu| > 1$. Furthermore, $T(\Lambda, \theta)$ is some smooth function satisfying the following conditions:

$$\lim_{\theta \to 0} T(\Lambda, \theta) \frac{1}{\theta^2} = 0, \quad (22)$$

$$\lim_{\Lambda \to \infty} T(\Lambda, \theta) = 1. \quad (23)$$

There are of course infinitely many functions which satisfy these conditions, for instance a possibility is

$$T(\Lambda, \theta) = 1 - e^{-\Lambda^2 \theta^2}$$

(where the factor in the exponential has been chosen to be dimensionless).

Let us now comment on action (21):

- the term $\delta_z p^2$ corresponds to the planar regular graphs, whereas $\delta_z' p^2$ corresponds to the planar irregular and nonplanar graphs;
- the terms $\delta_v, \delta_v'$ and, $\delta_v''$ are the mass counterterms associated to the planar regular graphs, the planar irregular graphs, and the nonplanar graphs, respectively.
the term $\delta_a$ is the counterterm associated to the parameter $a$, and

the term $\delta_{\lambda'}$ corresponds to the planar regular graphs, whereas $\delta_{\lambda''}$ corresponds to the planar irregular and nonplanar graphs.

Note that the $\delta_a$ term comes into the picture only when $\theta \neq 0$ (thanks to the $T$-function). Thus, the $T$-function introduced here switches between the mass counterterm $\delta_{m''}$ (which is not present when $\theta \neq 0$) and the counterterm $\delta_a$. It is the main ingredient of the mechanism that we propose here.

By taking the limit $\theta \to 0$, one obtains, using (22), the usual commutative action

$$
limit_{\theta \to 0} S_{A, \theta} = S_A = \int d^4 p \left[ \frac{1}{2} p^2 \phi^2 + \frac{1}{2} m^2 \phi^2 + \lambda \frac{V}{4!} \right]
$$

$$
+ \frac{1}{2} \delta_Z p^2 \phi^2 + \frac{1}{2} \delta_m \phi^2 + \frac{\delta_{\lambda'}}{4!} V
$$

(24)

where we have rewritten

$$
\delta_{m''} + \delta_{m'''} + \delta_{m''''} = \delta_m
$$

(25)

as the usual total mass counterterm, and $V$ is the interaction potential obtained in the commutative limit (i.e., when the noncommutative Moyal-Weyl product simply becomes the usual pointlike multiplication of fields). Analogous relations as (25) also hold for $\delta_Z$ and $\delta_{\lambda}$.

Let us now make a few remarks. First, we recall that in [31] it was proved that the convergence of the nonplanar graphs improves proportionally to the value of their genus $g$. Here, we do not take this phenomenon into consideration, since what we are interested in is to show a mechanism which will transform these graphs that diverge when $\theta \to 0$, without looking into the details of the degree of convergence of the "initial" graph (i.e., when $\theta \neq 0$).

Indeed, when $\theta \neq 0$, the counterterms $\delta_{Z''}$, $\delta_{m'''}$, $\delta_{m''''}$, and $\delta_{\lambda''}$ no longer survive (as requested by the renormalization results recalled in the previous section). When the parameter $\theta$ is switched off, all these counterterms come back to life and sum up in relations of type (25).

Let us end this paper by concluding that the limit mechanism proposed here has nothing arbitrary, but is simply dictated in a natural way by the behavior of the two-point planar irregular Feynman amplitudes (as studied in the section “UV/IR mixing as insight for the commutative limit”).

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