A simple analytic model for astrophysical S-factors

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(Dated: October 13, 2010)

We propose a physically transparent analytic model of astrophysical S-factors as a function of a center-of-mass energy \( E \) of colliding nuclei (below and above the Coulomb barrier) for non-resonant fusion reactions. For any given reaction, the \( S(E) \)-model contains four parameters [two of which approximate the barrier potential, \( U(r) \)]. They are easily interpolated among many reactions involving isotopes of the same elements; they give accurate practical expressions for \( S(E) \) with only several input parameters for many reactions. The model reproduces the suppression of \( S(E) \) at low energies (of astrophysical importance) due to the shape of the low-\( r \) wing of \( U(r) \). The model can be used to reconstruct \( U(r) \) from computed or measured \( S(E) \). For illustration, we parameterize our recent calculations of \( S(E) \) (using the São Paolo potential and the barrier penetration formalism) for 946 reactions involving stable and unstable isotopes of C, O, Ne, and Mg (with 9 parameters for all reactions involving many isotopes of the same elements, e.g., \( \text{C+O} \)). In addition, we analyze astrophysically important \( ^{12}\text{C}+^{12}\text{C} \) reaction, compare theoretical models with experimental data, and discuss the problem of interpolating reliably known \( S(E) \) values to low energies (\( E \lesssim 2-3 \) MeV).

PACS numbers: 25.70.Jj;26.50.+x;26.60.Gj,26.30.-k

I. INTRODUCTION

Nuclear reactions are very important for the structure, evolution, nucleosynthesis and various observational manifestations of main-sequence stars, giants and supergiants, presupernovae, white dwarfs and neutron stars. Depending on temperature, density and other parameters, stellar burning may involve many reactions of different nuclei, from light to heavy, and from stable to neutron- and proton-rich ones. Their rates can be calculated using the reaction cross sections \( \sigma(E) \), or related astrophysical S-factors defined as

\[
\sigma(E) = E^{-1} \exp(-2\pi\eta) S(E).  \tag{1}
\]

Here, \( E \) is the center-of-mass energy of the reactants \( [(A_1, Z_1) \text{ and } (A_2, Z_2)] \), \( \eta = \alpha/(\hbar v) = \sqrt{ER/E} \) is the Sommerfeld parameter, \( v = \sqrt{2E/\mu} \) is the relative velocity of the reactants at large separations, \( \alpha = Z_1Z_2e^2 \), \( E_R = \alpha^2\mu/(2\hbar^2) \) is similar to the Rydberg energy in atomic physics, and \( \mu \) is the reduced mass. The factor \( \exp(-2\pi\eta) \) is proportional to the probability of penetration through the Coulomb barrier \( U(r) = \alpha/r \) with zero angular orbital momentum, assuming that this pure Coulomb barrier extends to \( r \to 0 \) (for point-like nuclei); \( E^{-1} \) factorizes out the well-known pre-exponential low-energy dependence of \( \sigma(E) \). The advantage of this approach is that \( S(E) \) is a much more slowly varying function of \( E \) than \( \exp(-2\pi\eta) \) and \( \sigma(E) \).

For astrophysical applications, one needs to know \( S(E) \) for many reactions at low energies, \( E \lesssim 2 \) MeV. Experimental measurements of \( \sigma(E) \) at such energies are mainly not available (because the Coulomb barrier exponentially suppresses low-energy cross sections). Theoretical calculations are model dependent, so that nuclear-physics uncertainties of calculated \( S(E) \) can be substantial. Theoretical calculations show that \( S(E) \) can vary by several orders of magnitude in the energy range of astrophysical importance for a given reaction, and it can vary over many orders of magnitude for different reactions (e.g., \( \text{C+O} \) and references therein). It is the aim of this paper to propose (Sec. II) a physically transparent analytic model of \( S(E) \) for non-resonant reactions between heavy nuclei in order to explain these features, simplify the use of available \( S(E) \)-data, and clarify the problem of interpolating reliably known \( S(E) \) values to low energies of astrophysical importance.

Astrophysical S-factors have been parameterized by different analytic formulae (see [2, 3, 4], for references). We think that our new model is more flexible. It allows one to approximate \( S(E) \) for many reactions with minimum number of fit parameters. For instance, our
recent approximation of \( S(E) \) for 946 reactions involving different isotopes of C, O, Ne, and Mg with 8514 fit parameters is now replaced (Sec. III) with the approximation containing 90 parameters. Moreover, our model directly relates \( S(E) \) with the parameters of the effective potential \( U(r) \) of nucleus-nucleus interaction, and helps to reconstruct (constrain) \( U(r) \) from the \( S(E) \) data (computed or experimental ones; see Secs. III and IV).

II. ANALYTIC MODEL

A. General approach

Let us construct a simplified model of \( S(E) \) at sufficiently low energies \( E \) at which the main contribution to the reaction cross section comes from the s-wave channel. For the reactions involving the nuclei with \( Z_1, Z_2 \sim 6 - 12 \), this is true at the energies of a few tens of MeV. According to the theory of inelastic scattering (e.g., Ref. [8]), a reaction cross-section at such low energies has different energy dependence above and below the Coulomb barrier. Below the barrier (at \( E < E_C \), \( E_C \) being the barrier height) it behaves as

\[
\sigma(E) = S_0 E^{-1} \exp(\Phi(E)),
\]

\[
\Phi(E) = -\frac{2}{\hbar} \int_{r_1}^{r_2} dr \sqrt{2\mu(U - E)},
\]

where \( \Phi(E) \) is the semi-classical exponent argument in the expression for the barrier penetrability. We adopt the semi-classical approximation to calculate this penetrability; \( U = U(r) \) is the effective nucleus-nucleus potential (it is Coulombic at large separations \( r \) but is affected by nuclear forces at small \( r \)); \( r_1 \) and \( r_2 \) are classical turning points. In Eq. (2), \( S_0 \) is a slowly varying function of \( E \) which we treat as a constant. Its order-of-magnitude estimate in terms of physical quantities can be deduced, for instance, from the consideration in Appendix C of Ref. [2].

\[
S_0 \sim \frac{2\pi\hbar^2}{\mu} \sqrt{\frac{E_C}{V_0}},
\]

where \( V_0 \sim 40 \) MeV. The pre-exponent factor \( E^{-1} \) in (2) can be written as \((1/\sqrt{E})(1/\sqrt{E})\), where one factor \( 1/\sqrt{E} \) is a generic feature of low-energy reaction cross sections (neglecting barrier penetration); the extra factor \( 1/\sqrt{E} \) comes from three-dimensional penetrability through the Coulomb barrier [8]. Let us stress that \( S_0 \), in contrast to \( S(E) \), weakly depends on specific reaction.

At \( E > E_C \) the Coulomb barrier is transparent (in the semi-classical approximation), \( \Phi(E) = 0 \), and \( \sigma(E) \propto 1/\sqrt{E} \).

Combining the definition of \( S(E) \), given by Eq. (1), with Eq. (2), and using the above arguments, we present \( S(E) \) in the form

\[
S(E) = S_0 \exp(\Psi(E)),
\]

\[
\Psi(E) = 2\pi\eta + \Phi(E) \quad \text{at} \quad E \leq E_C,
\]

\[
S(E) = S_0 \exp(2\pi\eta) \sqrt{E/E_C} \times [1 + \xi(E - E_C)/E] \quad \text{at} \quad E > E_C.
\]

The last equation is phenomenological and contains a constant parameter \( \xi \). This equation extends Eq. (3) to the energies above the barrier. At energies \( E \gg E_C \) (but still low enough for the model to be valid) we have \( S(E) = S_0 \exp(2\pi\eta)(\xi + 1)/\sqrt{E/E_C} \). Thus, \( \xi \) determines the magnitude of the reaction cross section at \( E \gg E_C \); it is also important for describing the \( S(E) \) behavior at \( E \approx E_C \).

B. Model of barrier potential

Let us adopt a simple model of \( U(r) \),

\[
U(r) = \frac{\alpha}{r} \quad \text{at} \quad r \geq R_{C1},
\]

\[
U(r) = E_C \left[ 1 - \beta \left( \frac{r - R_C}{R_C^2} \right) \right] \quad \text{at} \quad r < R_{C1}.
\]

It is a pure Coulomb potential at \( r \geq R_{C1} \) and an inverse parabolic potential at smaller \( r \). The parabolic
The potential $U(r)$ passes through zero at $r = R_{C0} = R_C (1 - \beta^{-1/2})$. Its behavior at smaller $r$ does not affect directly our results. Realistic models should correspond to $\beta \gg 1$ (the low-$r$ slope of $U(r)$ should be sharp). $R_{C0}$ should be positive) which translates into $\delta \ll 1/3$ (because $\beta = 1$ corresponds to $\delta = 1/3$).

For example, in Fig. 1 we plot a model potential $U(r)$ for the $^{40}$Mg+$^{40}$Mg reaction. It will be discussed in Sec. III. In this case, $E_C = 16.27$ MeV and $\delta = 0.0332$, so that Eq. (8) yields $R_{C0} = 12.53$ fm and $R_{C1} = 12.95$ fm. We show the pure Coulomb and parabolic segments (the solid and dashed lines, respectively) by the open dot. The filled dot is the potential maximum. The dotted line is the pure Coulomb potential extended to $r \rightarrow 0$. The thin solid and dashed vertical lines position, respectively, the maximum and separation points. Although $\delta$ is formally small, it produces a noticeable $U(r)$ wing at low $r$.

With the potential (7) the integral (2) is taken analytically. At $E < E_{C1}$ we have

$$\Psi(E) = \Psi_r(E) + \Psi_l(E),$$

$$\Psi_r(E) = 4 \sqrt{\frac{E_R}{E}} \left( \arcsin \sqrt{x_r} + \sqrt{x_r(1-x_r)} \right),$$

$$\Psi_l(E) = -\gamma \sqrt{\frac{E_R}{E}} \left( \frac{E_C-E}{E_C} \right) \times \left( \frac{\pi}{2} + \arcsin x_l + x_l \sqrt{1-x_l^2} \right),$$

where $\gamma = (2 + 3\delta)^{3/2} \sqrt{\delta}(1 + \delta)^2$; $\Psi_r(E)$ and $\Psi_l(E)$ contain the contributions from the integration regions of $R_{C1} \leq r \leq r_2$ and $r_1 \leq r \leq R_{C1}$, respectively; $x_r = R_{C1}/r_2 = R_{C1} E/\alpha = E/E_{C1}$, and $x_l = \delta \sqrt{E_C/(E_C-E)}$. The term $2\pi\eta$ exactly canceled the opposite term which appeared after the integration in $\Phi(E)$. At $E_{C1} \leq E \leq E_C$ we have

$$\Psi(E) = 2\pi\eta + \Psi_l(E) = \pi \left( 2 \sqrt{\frac{E_R}{E}} - \frac{E_C-E}{E_C} \sqrt{\frac{E_R}{E_C}} \right).$$

Equations (9) and (10) fully determine $\Psi(E)$ in (5) in an analytic form. Then Eqs. (4) and (5) give an analytic, easily computable model for $S(E)$. It contains four parameters, $S_0$, $E_C$, $\delta$ and $\xi$; each parameter has simple physical meaning.

For astrophysical applications, one needs $S(E)$ at sub-barrier energies. In this case, it is natural to present $S(E)$ in the form

$$S(E) = S(0) \exp(g_1 E + g_2 E^2 + \ldots),$$

where $g_1$, $g_2$, . . . are some expansion coefficients (determined by low-$r$ behavior of $U(r)$; see below). The main energy dependence of $S(E)$ at $E < E_C$ is thought to be given by $\exp(g_1 E)$. To reduce the energy dependence one often introduces (e.g., [4, 7]) the modified $S$-factor

$$\bar{S}(E) = S(E) \exp(-g_1 E),$$

that is a much less variable function than $S(E)$; the modified $S$-factor is usually treated as constant. Our model differs from the traditional approach: instead of $\bar{S}$ we prefer to introduce $S_0$. Both quantities are nearly constants (at $E \lesssim E_C$) but $S_0$ changes within much narrower limits than $\bar{S}$ for different reactions (Sec. III). In our model, we can use Eq. (3) and expand the exponent argument in powers of $E$. Keeping three lowest expansion terms, at $E$ below $E_C$ we come to Eq. (11) with $S(0) = \bar{S}(0) = S_0 \exp(g_0)$, $g_0$ being the zero-order expansion coefficient (followed by $g_1$, $g_2$, . . .). The coefficients can be presented as $g_i = g_{ir} + g_{il}$ ($i = 0, 1, 2, \ldots$); $g_{ir}$ and $g_{il}$ collect, respectively, the contributions from the right (Coulombic, $r > R_{C1}$) and left (parabolic, $r \leq R_{C1}$) segments of the barrier potential. We obtain

$$g_{0r} = \frac{8 \sqrt{E_R}}{E_{C1}}, \quad g_{1r} = -\frac{4}{3E_{C1}} \sqrt{E_R} \left( \frac{E_R}{E_{C1}} \right)^{1.5},$$

$$g_{0l} = -\gamma \sqrt{\frac{E_R}{E_{C1}}} \left( \frac{E_R}{E_{C1}} \right)^{1.5} \left( \frac{\pi}{2} + \arcsin x_{l0} + x_{l0} \sqrt{1-x_{l0}^2} \right),$$

$$g_{1l} = \gamma \sqrt{\frac{E_R}{E_{C1}}} \left( \frac{\pi}{2} + \arcsin x_{l0} \right),$$

$$g_{2l} = \gamma \sqrt{\frac{E_R}{E_{C1}}} x_{l0} \sqrt{1-x_{l0}^2}.$$
The expansions like (11) have been written long ago (e.g., 2 and references therein) but only for the sharply truncated (rectangular - rt) Coulomb potential,

$$U_{rt}(r) = \frac{\alpha}{r} \quad \text{at } r \geq R_C, \quad U_{rt}(r) = -V_0 \quad \text{at } r < R_C. \quad (16)$$

This potential is obtained from our potential (7) in the limit of $\delta \to 0$, $E_{C1} \to E_C$, and $R_C \to R_C$. In this case one usually used Eq. (11) neglecting $g_2$ and higher-order terms:

$$S_{rt}(E) = S_{rt}(0) \exp(g_{rt}E),$$

$$g_{rt} = g_{rt1} = -\frac{4}{3h} \sqrt{\frac{\mu R_{C1}^2}{2\alpha}}, \quad (17)$$

with $S_{rt}(0) = \widetilde{S}_{rt} = S_{rt0} \exp(g_{r0})$. Now $E_{C1} = E_C = \alpha/R_{C1}$; $R_{C1}$ should be treated as an effective radius of nucleus-nucleus interaction; the parameters $g_{rt}$ are not well defined because the radius $R_C = R_{C1}$ of sharp barrier truncation is unphysical. One has $g_{rt1} < 0$, that is $S_{rt}(E)$ always decreases with increasing $E$ in the model (10). We will show that it is more reasonable to use more advanced models of $U(r)$ with the broadened peak and $g_1 = g_{r1} + g_{rt1}$. 

C. General properties of model S-factors

The proposed model for S-factors is simple and can be analyzed in general form.

a. Super-barrier energies. The model S-factor at $E > E_C$ is given by Eq. (6). It is determined by three constants, $E_C$, $S_0$, and $\xi$, being independent of $\delta$. This S-factor rapidly decreases with the growth of $E$.

Nevertheless, at $E > E_C$ it is better to use the reaction cross-section instead of $S(E)$. This cross section reads $\sigma(E) = S_0 (E_C E)^{-1/2} [1 + \xi (E - E_C)/E]$; it is a slowly varying function of energy. Let us stress again that our Eq. (6) cannot be extended to very large $E$, because the approximation of energy-independent $S_0$ would become questionable. We expect that the term containing $\xi$ phenomenologically accounts for the contribution of higher partial waves $\ell > 0$ at $E \gtrsim E_C$.

b. Subbarrier energies. In this case, $S(E)$ is determined by $S_0$, $E_C$ and $\delta$, being independent of $\xi$. Its main features are illustrated in Fig. 2. The left panel demonstrates the effective potential $U(r)$. The vertical scale gives $U(r)/E_C$, which is independent of $E_C$. The horizontal scale shows $r/R_{C1}^{(0)}$, where $R_{C1}^{(0)} = \alpha/E_C$ for a given $E_C$. The solid line gives the sharply truncated Coulomb potential. The long-dashed, short-dashed, and dotted lines show $U(r)$ for $\delta = 0.01, 0.1,$ and $\frac{1}{4}$, respectively. The higher-$r$ wings of $U(r)$ are pure Coulombic, $U(r) = \alpha/r$, independent of $\delta$. If $E_C$ is fixed but $\delta$ increases, the potential is broadened at small $r$ and becomes, naturally, less transparent. The value $\delta = \frac{1}{4}$ is critical, because nucleus-nucleus binding at small $r$ disappears [we would have $U(r) \geq 0$ at all $r \geq 0$]. This case is unrealistic for exoergic nuclear reactions (with positive $Q$-values) of our study. We show it as a limiting case.

The middle panel of Fig. 2 presents $\sqrt{E_R/E_C} \ln[S(E)/S_0]$ as a function of $E/E_C$ for subbarrier energies $E \leq E_C$. [to visualize the shape of $S(E)$]. As seen from Eqs. (9) and (10), the presented function depends on the only one parameter, $\delta$. The thick lines are calculated from Eqs. (11) and (10) at the same values of $\delta = 0, 0.1, 0.01$, and $\frac{1}{4}$ as in the left panel. Recall that for the sharply truncated Coulomb potential $S(E)$ increases as $E \to 0$. For larger $\delta$, the barrier is less transparent [which can lower $S(E)$ to a great extent]. Moreover, the $S(E)$ shape becomes different because the suppression of $S(E)$ at lower $E$ is naturally stronger. In the range of $0.05 \lesssim \delta \lesssim 0.15$ the $S$-factor turns into a slowly varying function of $E$, and at higher $\delta$ it turns into a function which decreases noticeably with decreasing $E$, showing a pronounced hindrance of $S(E)$ at low $E$.

The thin lines in the middle panel of Fig. 2 present the same function, as the thick lines, but calculated using the low-$E$ asymptote (11). This asymptote is seen to be remarkable accurate for all values of $\delta \leq \frac{1}{4}$, and for all energies $E$ below $E_C$. Actually, the quadratic term $g_2 E^2$ in (11) is small; it is often sufficient to use the reduced expression $S(E) = S_0 \exp(g_0 + g_1 E)$. The quadratic term modifies the exponent argument maximum by 3% at $E = E_C$ and $\delta = 0$. Using Eqs. (14) and (15) one can show that $g_1$ changes sign at $\delta = 0.07745$ (from minus at smaller $\delta$ to plus at higher). This explains a slow energy dependence of $S$ at $0.05 \lesssim \delta \lesssim 0.15$. At $\delta = \frac{1}{4}$ the function $\ln S(E)$ is almost linear in $E$, so that the thick and thin dotted lines nearly coincide.

The right panel of Fig. 2 demonstrates $\log_{10}[S(0)/S_0]$ as a function of $\sqrt{E_R/E_C}$ for the same four values of $\delta$. For a fixed value of $E_R/E_C$, the $S(0)/S_0$ ratio depends only on $\delta$, in our model. The displayed range of $\sqrt{E_R/E_C}$ corresponds to the reactions which we consider in Sec. III. The plot shows the values $S(0)/S_0 = \exp(g_0)$ which determine the zero-energy S-factor, $S(0)$, important for astrophysical applications. For the pure Coulomb potential extended to $r \to 0$, we would have $S(0) = S_0$. The potential cutoff at low $r$ drastically enhances the barrier penetrability, and, hence, $S(0)$ [as well as, generally, low-energy $S(E)$] over $S_0$, up to $\sim$100 orders of magnitude! The highest enhancement is provided by the sharply truncated barrier (10). If we fix $E_C$ (and thus $E_R/E_C$) and increase $\delta$, the enhancement can be strongly reduced. Some examples are given below.
III. EXAMPLE: 946 REACTIONS INVOLVING C, O, NE AND MG ISOTOPES

A. Calculations

For example, let us consider a set of S-factors, which we have calculated recently \[5\] for fusion reactions involving various isotopes of C, O, Ne, and Mg. The calculations include stable, proton-rich, neutron-rich, and very neutron-rich isotopes. Such isotopes can appear during nuclear burning in stellar matter, particularly, in dense matter of white dwarf cores and neutron star envelopes. The calculations were performed using the São Paulo (SP) potential in the context of the barrier penetration model; we employed the NL3-parametrization of nuclear density distributions of reactants within the Relativistic Hartree-Bogoliubov (RHB) approach. The model is based on the standard partial wave decomposition \((\ell = 0, 1, \ldots)\) and considers motion of the nuclei in the effective potential

\[
U_{\text{eff}}(r, E) = U_C(r) + U_{\text{SP}}(r, E) + \frac{\hbar^2 \ell (\ell + 1)}{2\mu r^2},
\]

where \(U_C(r)\), \(U_{\text{SP}}(r, E)\) and the last term are the Coulomb, nuclear and centrifugal potentials, respectively. At low energies \((E \lesssim E_C)\), the main contribution to \(S(E)\) comes from the \(\ell = 0\) (s-wave) channel. The computational scheme is parameter-free and relatively simple for generating a set of data for many non-resonant reactions involving different isotopes.

The reactions in question are summarized in Table I. All isotopes studied were even-even nuclei. We considered 10 reaction types, such as C+C and O+Ne, with the range of mass numbers for both species given in Table I.

For each reaction, we computed \(S(E)\) on a dense grid of \(E\) (with the energy step of 0.1 MeV) from 2 MeV to a maximum value \(E_{\text{max}}\) (also given in Table I) covering wide energy ranges below and above the Coulomb barrier. The last column in Table I presents the number of considered reactions.

The results of calculations using the SP model have been compared previously \[4, 9, 10\] with experimental data (if available) as well as with theoretical calculations performed using other models such as coupled-channels and fermionic molecular dynamics ones. As detailed in \[10\], the calculated values of \(S(E)\) are uncertain due to nuclear physics effects – due to using the SP model with the NL3 nucleon density distribution. For the reactions involving stable nuclides, typical uncertainties are expected to be within a factor of 2, with maximum up to a factor of 4. For the reactions involving unstable nu-

### Table I: Fusion reactions \((A_1, Z_1) + (A_2, Z_2)\) under consideration (after Ref. \[5\])

| Reaction | \(A_1\) | \(A_2\) | \(E_{\text{max}}\) MeV | Nr. of cases |
|----------|---------|---------|----------------|-------------|
| C+C      | even    | even    | 17.9            | 36          |
| C+O      | 10–24   | 12–28   | 17.9            | 72          |
| C+Ne     | 10–24   | 18–40   | 19.9            | 96          |
| C+Mg     | 10–24   | 20–46   | 19.9            | 126         |
| O+O      | 12–28   | 18–40   | 21.9            | 126         |
| O+Ne     | 12–28   | 18–40   | 21.9            | 128         |
| O+Mg     | 12–28   | 18–40   | 21.9            | 126         |
| Ne+Ne    | 18–40   | 18–40   | 21.9            | 78          |
| Ne+Mg    | 18–40   | 20–46   | 24.9            | 168         |
| Mg+Mg    | 20–46   | 20–46   | 29.9            | 105         |
TABLE II: Fit parameters of $S(E)$ for reactions $(A_1, Z_1) + (A_2, Z_2)$ under consideration

| Reaction type | $R$ | $\Delta R_{1a}$ | $\Delta R_{2a}$ | $\Delta R_{1b}$ | $\Delta R_{2b}$ | $\delta$ | $S_0$ | $\xi_0$ | $\xi_1$ | Max. Rms dev. | dev. |
|---------------|-----|----------------|----------------|----------------|----------------|--------|-------|--------|--------|--------------|------|
| C+C           | 7.4836 | 0.1759 | 0.1759 | 0.0040 | 0.0040 | 0.0040 | 1.3736 | 3.5499 | 0.2658 | 0.35  | 0.11 |
| C+O           | 7.8671 | 0.1740 | 0.1280 | −0.0045 | −0.0310 | 0.0412 | 1.5438 | 5.2576 | 0.2306 | 0.47  | 0.12 |
| C+Ne          | 7.9387 | 0.1720 | 0.1206 | −0.0171 | −0.0035 | 0.0400 | 1.9478 | 3.7661 | 0.2328 | 0.62  | 0.16 |
| C+Mg          | 8.0513 | 0.1705 | 0.1014 | −0.0210 | −0.0186 | 0.0386 | 2.4327 | 4.0059 | 0.1844 | 0.56  | 0.17 |
| O+O           | 8.0641 | 0.1266 | 0.1266 | 0.0037 | −0.0377 | 0.0388 | 2.1998 | 6.0147 | 0.1547 | 0.50  | 0.14 |
| O+Ne          | 8.1191 | 0.1257 | 0.1183 | −0.0461 | −0.0068 | 0.0371 | 2.9486 | 3.5127 | 0.1702 | 0.65  | 0.19 |
| O+Mg          | 8.2404 | 0.1246 | 0.0994 | −0.0500 | −0.0216 | 0.0357 | 3.7433 | 2.8303 | 0.1417 | 0.65  | 0.20 |
| Ne+Ne         | 8.4149 | 0.1175 | 0.1175 | −0.0107 | −0.0107 | 0.0348 | 4.2215 | 0.1225 | 0.1717 | 1.00  | 0.27 |
| Ne+Mg         | 8.2880 | 0.1160 | 0.0987 | −0.0157 | −0.0273 | 0.0339 | 5.2525 | 0.2141 | 0.1342 | 0.86  | 0.28 |
| Mg+Mg         | 8.4509 | 0.0976 | 0.0976 | −0.0288 | −0.0288 | 0.0332 | 5.9785 | 0.5263 | 0.1393 | 0.82  | 0.28 |

C. Discussion

Let us outline the main features of our fits for all 10 reaction types.

We start with the Mg+Mg reactions (Fig. 3). They are characterized by the largest strength of Coulomb interaction (largest product $Z_1Z_2$ and largest $\alpha$). On the left panel of Fig. 3 we compare calculated and fitted $S(E)$ for six selected Mg-Mg reactions. Similar comparison for reactions of other types is given below. The solid curves are our fits; filled dots are calculated $S(E)$. Here and below we plot the calculated data on a rarefied grid (with the energy step of 1 MeV) to simplify the figures. Open dots refer to $E = E_C$ as determined from Eq. (19). For each reaction type we present six $S(E)$ curves chosen in the following way. The lower curve corresponds to the reaction with the lightest isotopes ($^{20}$Mg+$^{20}$Mg in Fig. 3). The upper curve is for the most massive isotopes ($^{40}$Mg+$^{40}$Mg in Fig. 3). The second curve from the bottom is for the most stable isotopes ($^{24}$Mg+$^{24}$Mg); the third curve is for the lightest isotope 1 and the most

neutrons, typical uncertainties were roughly estimated to be as large as one order of magnitude, reaching two orders of magnitude at low energies for the reactions with very neutron-rich isotopes. These uncertainties reflect current state of art in our knowledge of $S(E)$.

B. Fits

In Ref. 8 we fitted the calculated $S(E)$ by a 9-parameter phenomenological analytic expression. These fits are accurate (with maximum relative errors less than 10%) but their use requires extensive tables (of 9×946 = 8514 parameters). Here, we employ our new fit expressions (Sec. II) and show that the same data can be approximated using only 90 fit parameters.

Let us consider reactions of each type (each line in Table I) separately and apply our analytic model (5) and (6) to every reaction. In this way we determine 4 fit parameters ($S_0$, $E_C$, $\delta$, and $\xi$), for each reaction. For instance, we have $4 \times 105 = 420$ parameters for Mg+Mg reactions. However, we notice that we can put $S_0$ and $\delta$ constant for all reactions of a given type (for instance, $S_0 = 5.9785$ MeV b and $\delta = 0.0332$ for all Mg+Mg reactions); this does not increase essentially the fit errors. Such constant $S_0$ and $\delta$ are given in Table I.

Still, we need to specify two parameters, $E_C$ and $\xi$, for every reaction. Collecting the values of $E_C$ and $\xi$ for all reactions of each type, we were able to fit them by analytic expressions

$$E_C = \alpha/R_C^{(0)},$$
$$R_C^{(0)} = R + \Delta R_1 |A_1 - A_{10}| + \Delta R_2 |A_2 - A_{20}|;$$
$$\xi = \xi_0 + \xi_1 (A_1 + A_2),$$

where $A_{10} = 2Z_1$ and $A_{20} = 2Z_2$ are mass numbers of most stable isotopes; $\Delta R_1 = \Delta R_{1a}$ at $A_1 \geq A_{10}$; $\Delta R_1 = \Delta R_{1b}$ at $A_1 < A_{10}$; $\Delta R_2 = \Delta R_{2a}$ at $A_2 \geq A_{20}$; $\Delta R_2 = \Delta R_{2b}$ at $A_2 < A_{20}$. Thus, we have seven new fit parameters $R$, $\Delta R_{1a}$, $\Delta R_{2a}$, $\Delta R_{1b}$, $\Delta R_{2b}$; $\xi_0$ and $\xi_1$ (also given in Table III) for each reaction type, and, hence, 9 parameters in total. Naturally, we have $\Delta R_2 = \Delta R_1$ for the reactions involving isotopes of the same element (e.g., Mg+Mg).

The eleventh column of Table II gives maximum relative deviation of fitted $S(E)$ from calculated ones for all reactions of a given type over all energy grid points (e.g., over $105 \times 280 = 29400$ points for the Mg+Mg reactions). We see that the fitted values of $S(E)$ do not deviate from the calculated ones by more than 100%. Root-mean square (rms) relative deviations, given in the last column, are a factor of 3–4 lower than the maximum ones. This fit accuracy is acceptable because it is well within nuclear physics uncertainties of calculated $S$-factors (Sec. IIIA).
massive isotope 2 ($^{20}\text{Mg}+^{46}\text{Mg}$); and two next curves are for somewhat heavier isotope 1 and lighter isotope 2 ($^{30}\text{Mg}+^{40}\text{Mg} ; ^{40}\text{Mg}+^{40}\text{Mg}$). The general trend is: the higher the reduced mass $\mu$ of the reacting nuclei, the larger $S(E)$.

We see that our $S(E)$-model reproduces the data reasonably (uniformly) well for all Mg+Mg reactions. All these fits are done with the same $\delta = 0.0332$ and $S_0=5.9785$ MeV b. Fitting the same data with the model (11) of rectangular potential would be much less accurate. This point is also illustrated on the left panel of Fig. 3. The dotted lines are our best fits of the same data with the rectangular potential. Such a potential leads to a faster growth of $S(E)$ at $E \rightarrow 0$; it strongly (up to several orders of magnitude) overestimates low-energy $S(E)$; fit errors become much higher.

In the right panel of Fig. 3 we display the effective potential $U(r)$ for the $^{46}\text{Mg}+^{46}\text{Mg}$ reaction (involving the most massive Mg isotopes from our collection). The solid line is our model $U(r)$, reconstructed by fitting calculated values of $S(E)$ with our model. It is given by Eq. (7) and plotted also in Fig. 1, with the fit parameters deduced from our fits (Table II). The dotted line is our model $U(r)$ for the best fit with $\delta = 0$. The long-dashed and short-dashed lines are the effective potentials $U_{\text{eff}}(r,E)$ given by Eq. (13) and used in original SP calculations. They depend on $E$ and are plotted for $E=10$ and 2 MeV, respectively. We see that the reconstructed potential with $\delta = 0.0332$ is remarkably close to the original ones. Therefore, by fitting the available $S(E)$ data (calculated or experimental ones) with our $S(E)$-model, one can reconstruct the effective potential $U(r)$. On the other hand, the model with $\delta = 0$ gives the potential (the dotted line in the right panel of Fig. 3) with unreasonably high $E_C = 20.14$ MeV that is sharply truncated at too large $R_C = 10.29$ fm. Naturally, this potential strongly overestimates $S(E)$ at low $E$.

The value of $S(0)$ for the $^{46}\text{Mg}+^{46}\text{Mg}$ reaction is approximately 83 orders of magnitude higher than our fit value $S_0=5.9785$ MeV b. This huge difference is solely attributed to the definitions of $S(0)$ and $S_0$. While $S_0$ is related to the penetrability of the real barrier (truncated at small $r$), $S(0)$ is defined through the penetrability of the barrier which remains pure Coulomb to $r \rightarrow 0$. These penetrabilities are drastically different.

Figure 1 gives nine plots, similar to that on the left panel of Fig. 3 for the reactions of nine types. These reactions are ordered (from Ne+Mg, Ne+Ne and O+Mg in the top row to C+Ne, C+O and C+C in the bottom row) in such a way to have progressively lower strength of Coulomb nucleus-nucleus interaction (lower $Z_1Z_2$). Generally, the fits seem satisfactory and reasonably uniform.

It is remarkable that the fit parameter $S_0$ takes on the values in a narrow range from $\approx 1.4$ to $\approx 6$ MeV b for all reactions of our study, while the astrophysical $S$-factor varies over many orders of magnitude. Our approach opens a possibility to extrapolate the values of $S_0$ to a wider class of reactions without performing new calculations of $S(E)$. One can also extrapolate the values of $\delta$, $E_C$ and $\zeta$, and obtain thus $S(E)$-factors for new reactions. Note that we could have fitted all the data (Table I) by using one and the same $\delta \approx 0.04$, and the fit accuracy would be nearly the same as in our present fits.

Our model $S(E)$ is flexible to describe different $S(E)$ curves. We believe that the description of $S(E)$ in terms of four well defined parameters ($S_0$, $\delta$, $E_C$, and $\zeta$) is physical and sufficient for the majority of applications. However, our interpolation of $E_C$ and $\zeta$ by Eqs. (19) and (20) can be regarded only as a reasonably successful phenomenological fit. We expect that, while doing more accurate fitting of $S(E)$ for these or other reactions in the future, one can find better (and physically meaningful) interpolation expressions for $E_C$ and $\zeta$ as functions of $A_1$ and $A_2$, and consider $\delta$ and $S_0$ as functions of $A_1$ and $A_2$ as well. One can also improve our fit at $E \gtrsim E_C$ by going beyond the semi-classical approximation and by replacing the phenomenological $S(E)$ dependence with the results of more accurate consideration. We think that the $S(E)$-factor at $E > E_C$ can be calculated in a more rigorous form and expressed through $E_C$, $S_0$ and $\delta$ without introducing an additional parameter like $\zeta$.

Note that for each reaction type we consider some reactions involving proton-rich nuclei (e.g., $^{10}\text{C}+^{10}\text{C}$) and many reactions involving neutron-rich nuclei (e.g., $^{24}\text{C}+^{24}\text{C}$). For each reaction type, we have observed a change in the behavior of fit parameters on $A_1$ and $A_2$ while crossing the stability line ($A \approx 2Z$). Because we include only a few proton-rich isotopes, we do not recommend to extrapolate our fits to the region of proton-rich nuclei (that would require calculations of $S$-factors for more proton-rich isotopes). We stress that our data sets include only even-even nuclei. In the future we can additionally calculate the $S$-factors for reactions involving other nuclei (even-odd or odd-odd) and approximate them with our model in a similar fashion. We do not recommend directly extrapolating our present results (Table II) to these reactions.

IV. $^{12}\text{C}+^{12}\text{C}$ REACTION

Finally, let us discuss the quality of our $S(E)$ model for the $^{12}\text{C}+^{12}\text{C}$ reaction. The reaction is very important for astrophysics of compact stars – for late stellar burning, ignition of type Ia supernovae and triggering explosive events such as superbursts (e.g., 11, 12 and references therein). Our results are summarized in Fig. 5. On the left panel we show $S(E)$-data; on the right panel we give corresponding effective potentials $U(r)$.

The $^{12}\text{C}+^{12}\text{C}$ reaction cross section has been measured by many groups. The cross section contains resonances and the non-resonant part. We can compare our theoretical model with the non-resonant contribution. We take experimental $S(E)$ from Refs. 16, 17, 18, 22. The data cover the energy range from $\approx 2.1$ MeV to 20 MeV. The data are nonuniform and not fully consistent, especially at lower $E$, where experimental $S(E)$ measure-
ements are most difficult and experimental uncertainties are high. Experimental values of $S(E)$ seem reasonably accurate at $E \gtrsim 4$ MeV, moderately uncertain in the $3-4$ MeV range, and rather uncertain at lower $E$. The presence of low-energy resonances (e.g.,\cite{7,21} and references therein) complicates data analysis. To compare with the SP theoretical calculations, which neglect the resonances, and to smooth out the effect of experimental uncertainties we bin the experimental data (with the bin width $\Delta E = 1$ MeV). The binned data are shown by crosses in the left panel of Fig. 5. Theoretical SP $S(E)$ values are presented by filled dots.

The short-dashed line in the left panel of Fig. 5 is our 4-parameter best fit of experimental $S(E)$ factors. In this case we have $E_C = 6.25$ MeV, $\delta = 0.0763$, $S_0 = 0.735$ MeV b, and $\chi = 13.05$. Reduced chi-square (per one degree of freedom) is 0.41, which means that the fit is acceptable. The solid line is our fit to the SP data. It is seen to be in reasonable agreement with the experimental data. It agrees also with the fit to the experimental data at those energies at which the data are reliable. The dotted curve is another fit to the SP data, this time assuming a rectangular barrier $^{15}$ ($\delta = 0$). It is in poor agreement with the experimental and SP data.

Our 4-parameter fit to SP data gives (Table $^{11}$) $E_C = 6.93$ MeV, $\delta = 0.04$, $S_0 = 1.37$ MeV b, $\chi = 9.93$. At subbarrier energies this fit is well described by Eqs. $^{11}$ and $^{13}$, with the expansion terms $g_i$ given by $^{13}$ and $^{15}$. This yields

$$S_{SP}(E) = 1.7 \times 10^{16} \exp(-0.302E - 0.021E^2) \text{ MeV b,}$$

(21)

where $E$ is in MeV. Note that the contribution $g_{i}\delta$ of the parabolic segment of the potential barrier $U(r)$ to $g_i = g_{ir} + g_{i\delta}$ is generally substantial. For instance, $g_0 = 42.615 - 5.555 = 37.060$. This expansion term is absorbed in $S(0) = 1.7 \times 10^{16}$ MeV b; the value $g_{i\delta} = -5.555$ reduces considerably $S(0)$. In the next term $g_{r} = -1.045 + 0.743 = -0.302$ the parabolic $U(r)$ part ($g_{ir} = 0.743$) is very essential. In the quadratic term $g_{2} = -0.023 + 0.002 = -0.021$ it is less important, and the term itself is rather unimportant in $^{21}$.

A careful analysis of experimental data of different groups has recently been performed by Aguilera et al. $^{21}$. The authors rescaled some data trying to obtain a unified description of $S(E)$ including resonant structures. Then they took several theoretical models of non-resonant $S_{nr}(E)$ and compared them with their full unified experimental $S(E)$ data. The difference $S(E) - S_{nr}(E)$ was treated as the resonance contribution. The best theoretical $S_{ns}(E)$ model was claimed to be that obtained with the KNS (Krappe-Nix-Sierk $^{22}$) barrier potential $U_{KNS}(r)$. It gives most reasonable reproduction of resonant structures after subtracting non-resonant term.

It is customary (e.g., $^{2,6,7,12,21}$) to approximate the non-resonant $S$-factor for the $^{12}$C+$^{12}$C reaction at

![Graph](image-url)

FIG. 3: (Color online) Left: $S$-factors for six Mg+Mg reactions. Filled dots are the São Paulo (SP) calculations; solid lines are our fits (Table $^{11}$); open dots show the fit values of $E_C$; dotted lines are best fits of $S(E)$ assuming $\delta = 0$. Right: The effective potential $U(r)$ for the $^{46}$Mg+$^{46}$Mg reaction. The solid line, marked as Fit, is reconstructed from the calculated $S(E)$ using our model $^{17}$ for $U(r)$; the dotted line is the same but assuming rectangular potential ($\delta = 0$). The long-dashed and short-dashed lines show the effective potential used in the original SP calculations of $S(E)$ at $E = 10$ and 2 MeV, respectively.
FIG. 4: (Color online) Each of nine panels shows $S$-factors for six reactions of one type (as in the left panel of Fig. 3). The reaction types are Ne+Mg, Ne+Ne, O+Mg, O+Ne, C+Mg, O+O, C+Ne, C+O, and C+C.

Subbarrier energies by

$$S_{nr}(E) = \tilde{S}_a \exp(-0.46E), \quad (22)$$

where $\tilde{S}_a$ is a parameter [analogous to $\tilde{S}$, defined by (12), with the specific value of $g_1 = -0.46$ discussed below]. The results of Aguilera et al. [21] can be approximated in this way using their Fig. 11 [in our notations, that figure gives $S(E) = S_{nr}(E) \exp(0.46E)$]. Their best $S_{nr}(E)$ (given by the KNS model) in the energy range from $\approx 3.5$ to $\approx 5.5$ MeV (below the barrier but at those $E$ where experimental data are reliable) can be approximated by (22) with $\tilde{S}_a \approx (1.4 - 1.7) \times 10^{16}$ MeV b (earlier studies gave $\tilde{S}_a \approx 3 \times 10^{16}$ MeV b; e.g., [6, 24]). However, this approximation becomes inaccurate at lower $E$. We remark, that at any $E \lesssim 5.5$ MeV, the best (KNS) model of Ref. [23] is accurately approximated by $S_{KNS}(E) \approx 0.83 \times 10^{16} \exp(-0.32E)$ MeV b, in reasonable agreement with our approximation (21) of SP data. Nevertheless, another acceptable $S_{nr}(E)$ model of Ref. [21], based on the proximity-adiabatic (PA) approach [25], is approximated at the same energies $E \lesssim 5.5$ MeV as $S_{PA}(E) \approx 1.6 \times 10^{16} \exp(-0.46E)$ MeV b, in agreement with (22).

These different approximations reflect uncertainties in our knowledge of $S_{nr}(E)$. One should be careful in using (22) for the $^{12}$C+$^{12}$C reaction. The factor $g_1$ in the
The exponent argument can be different from \( g_1 = -0.46 \) (can be closer to \(-0.3\)). The value \( g_1 = -0.46 \) was first introduced by Patterson et al. [6]. The authors claimed that it came from the model of rectangular potential barrier. According to Eq. (17), in this model we would have \( g_{\text{rect}} = -0.05 R_C^{3/2} \), where \( R_C = R_{C1} \) is in fm. Taking the standard value \( R_C \approx 2 \times 1.3 A^{1/3} = 6 \) fm we would get \( g_{\text{rect}} = -0.73 \), noticeably different from \(-0.46\). Taking \( g_{\text{rect}} = -0.46 \) one would have \( R_C = 4.4 \) fm [6], an unrealistically small radius of the potential well \( U(r) \). In fact, Patterson et al. [6] obtained \( g_1 = -0.46 \) by fitting a restricted set of experimental \( S(E) \) values available by 1969. They discussed possible variations of \( g_1 \) but the discussion has been forgotten, whereas their best value \( g_1 = -0.46 \) is widely cited in the literature.

The right panel of Fig. 5 presents the effective potentials \( U(r) \) corresponding to \( S(E) \) models on the left panel. The short-dashed curve gives \( U(r) \) that is calculated from Eq. (17) with the parameters \( E_C = 6.25 \) MeV and \( \delta = 0.0763 \) inferred from the fit to experimental \( S(E) \). It is our reconstruction of the real potential \( U(r) \) (whose details are still unknown) using our analytic \( S(E) \) model. The long-dashed line is the theoretical SP \( U_{\text{eff}}(r) \) (that is almost independent of \( E \) for the \(^{12}\text{C} + ^{12}\text{C} \) reaction); it was used to calculate the SP \( S(E) \). The solid curve is \( U(r) \) reconstructed from the full \( (E_C = 6.93 \) MeV, \( \delta = 0.04 \)) fit to our calculated SP \( S(E) \). The dotted curve is a similar reconstruction but based on the rectangular \( U(r) \) model [Eq. (18), \( \delta = 0 \)]. The dotted curve looks unrealistic – it gives too large \( V_C = 9.18 \) MeV, indicating once more that the model of rectangular potential is too crude. Three other potentials are remarkably close even in this, most difficult \(^{12}\text{C} + ^{12}\text{C} \) case, complicated by pronounced resonance structures of experimental \( S(E) \). This fact confirms that the SP model is generally a valid tool for studying non-resonant fusion reactions. Moreover, we see that our analytic \( S(E) \) model can help to reconstruct \( U(r) \) from experimental data. Let us add that the solid, short-dashed and long-dashed \( U(r) \) curves are also close to effective potentials, particularly, to \( U_{\text{KNS}}(r) \) and \( U_{\text{PA}}(r) \), used by Aguilera et al. [21] to approximate non-resonant contribution to \( S(E) \) in experimental data. As seen from Fig. 8 and Table 3 in [21], the basic parameters \( (E_C and R_C) \) of \( U_{\text{KNS}}(r) \) and \( U_{\text{PA}}(r) \) are fairly close, but \( U_{\text{KNS}}(r) \) has a slightly more extended low-\( r \) wing which, however, changes \( S_{\text{nr}}(E) \) behavior at low \( E \) (from \( g_{1\text{KNS}} = -0.32 \) to \( g_{1\text{PA}} = -0.46 \)).

Clearly, different segments of \( S(E) \) are determined by different parts of \( U(r) \). The range of \( E \) below \( E_C \) down to 3 – 4 MeV is controlled by \( U(r) \) at \( r \gtrsim 6 \) fm [not far from the \( U(r) \)-peak]. In this case, different theoretical \( U(r) \)-models give reasonably similar non-resonant \( S(E) \)-factors, which generally agree with (accurate) experimental data. At lower energies, \( E \sim 1 – 3 \) MeV, which are important for \(^{12}\text{C} \) burning in stellar matter in the thermonuclear regime, experimental values of \( S(E) \) are either uncertain or not available. These values are sensitive to the sharp low-\( r \) wing of the \( U(r) \) potential \( (r \sim 5 – 6 \) fm) which is not very well constrained by theory and experiment. However, because the 1 – 3 MeV energy range is close to the range of higher \( E \), where the \( S(E) \)-factor is well studied, one expects that an extrapolation from
higher $E$ to the 1 - 3 MeV range is more or less reliable. Finally, lowest energies $E \lesssim 1$ MeV are important for pycnonuclear burning of $^{12}\text{C}$ in dense stellar matter (e.g., [4, 9]). In this case, $S(E)$ is controlled by the very steep low-$r$ slope of $U(r)$ and seems rather uncertain. It can be affected by slight variations of the $U(r)$ slope (as discussed above taking KNS and AP models in [21] as an example). Extrapolations to these energies can be inaccurate. The problem is further complicated by the oblateness of $^{12}\text{C}$ nuclei in the ground state. Our analysis is based on the approximation of spherically symmetric nuclei and spherically symmetric potential $U(r)$. In the presence of oblateness, $S(E)$-factors depend on orientations of colliding nuclei. This effect is beyond the scope of the present paper. Note, however, that according to calculations [20] the oblateness increases low-energy $S(E)$ for the $^{12}\text{C}+^{12}\text{C}$ reaction by a factor of $\sim 1.7$ [which is within theoretical uncertainties of non-resonant $S(E)$].

We have also compared calculated and fitted $S(E)$-factors with experimental data for the $^{16}\text{O}+^{16}\text{O}$ and $^{16}\text{O}+^{16}\text{O}$ reactions. We have carried out SP calculations based on the RHB approach outlined in Sec. [III] as well as SP calculations which employ two-parameter Fermi (2pF) parametrization of nuclear density distributions of reactants. 2pF calculations agree with experimental data better than RHB ones (although the accuracy of both approaches is sufficient for many applications). If we restricted ourselves to reactions involving stable nuclei, the 2pF parametrization would be more accurate. However, our main goal was to obtain a uniform set of theoretical $S(E)$ data for a large collection of nuclei involving unstable ones (to simulate nuclear burning in neutron stars and white dwarfs). In this case, SP calculations based on the RHB approach are favorable.

V. CONCLUSIONS

We have suggested (Sec. [III]) a simple model with physically meaningful parameters to describe the astrophysical $S$-factor as a function of center-of-mass energy $E$ of reacting heavy nuclei for non-resonant fusion reactions. Our main conclusions are as follows:

- For any reaction, the model gives $S(E)$ in an analytic form in terms of four parameters. They are $E_C$, the height of the Coulomb barrier; $S_0$ that characterizes the strength of the nucleus-nucleus interaction neglecting Coulomb interaction; $\delta$ that describes the peak broadening of the effective barrier potential $U(r)$; and $\xi$ to describe the transition from subbarrier energies to $E \gg E_C$. The model is expected to be sufficiently accurate for energies below and above $E_C$ (up to a few $E_C$).

- As an example, we have applied our model to describe the $S$-factors for 946 fusion reactions involving various isotopes of C, O, Ne, and Mg, from the stability valley to very neutron-rich nuclei. In Ref. [3] these $S$-factors were calculated using the SP method and the barrier penetration model. They were fitted by a phenomenological formula containing 9 fit parameters for every reaction (8514 parameters in total). With the present analytic model, we can fit the same data set using 90 fit parameters (Sec. [III] Table III). The fit accuracy is worse than in Ref. [3] but is well within estimated nuclear-physics uncertainties of calculated $S(E)$ (Sec. [III]).

- We have also compared our model (Sec. [IV]) for the $^{12}\text{C}+^{12}\text{C}$ reaction (that is most important for neutron stars and white dwarfs) with experimental $S(E)$ data and discussed the problem of extrapolation of experimental $S(E)$ to low energies of astrophysical importance.

- Our analytic $S(E)$-model is easy for implementing into computer codes, which calculate nuclear reaction rates and simulate various nuclear burning phenomena in astrophysical environment; it is not as costly for CPU time as reading large tables can be. The $S(E)$ dependence for any reaction is determined by the values of $E_C$, $S_0$, $\delta$ and $\xi$ through Eqs. (5), (6) and (9). As a rule, one needs only subbarrier $S(E)$ to calculate nuclear reaction rates in stellar matter. In this case, it is sufficient to use a simplified expression (11), where the coefficients $g_1$, $g_2$, and $g_3$ are given by Eqs. (13) and (14).

- The analytic model is practical for describing large uniform sets of $S(E)$ data (for instance, many reactions involving isotopes of the same elements). The parameters $E_C$, $S_0$, $\delta$ and $\xi$ vary slowly from one reaction to another, and are easily interpolated (by analytic expressions) over large sets; $S_0$ and $\delta$ can be set constant for many reactions. Analytic interpolations can be used to extrapolate $S(E)$ to other reactions of the same type.

- The functional form of our analytical $S(E)$ is flexible to describe qualitatively different behaviors of $S(E)$. Particularly, by varying $\delta$ we can obtain either growth or decrease of $S(E)$ as $E \to 0$. The decrease is realized at not too small $\delta$ and may explain the low-energy hindrance of $S(E)$ whose signature was observed in some reactions (e.g., [11, 27] and references therein). The low-energy behavior of $S(E)$ is indeed very sensitive to the parameter $\delta$. The model of sharply truncated Coulomb potential (10), that is widely used in the astrophysical literature (e.g., Refs. [1, 2]), can be inaccurate in extrapolating calculated or measured $S(E)$ to low $E$ (Secs. [III] and [IV]).

- Fitting a given $S(E)$ (computed or measured in laboratory) with our analytic model can be used to reconstruct the effective potential $U(r)$ (Figs. 3 and 5). Of course, the real potential $U(r)$ can be too
complicated to be exactly described by our model potential [7]. However, we expect that this potential allows one to reproduce correct $S(E)$ behavior in many cases.

There is no doubt that the $U(r)$-peak is not sharp, but broadened. Roughly speaking, this broadening is twofold. First, the potential peak becomes smoother. Second, the low-$r$ wing of $U(r)$ becomes less steep. Clearly, the second effect has stronger impact on the low-energy $S(E)$ than the first one. In our model, both effects are described by one and the same parameter $\delta$. We can complicate the model by introducing new parameters but think that the present version is good as the first step. Let us add that our model is useful for reactions between heavy nuclei. Astrophysical $S$-factors for reactions involving light nuclei contain strong resonances (e.g., [24]) which are not described by our model.

Let us add that nuclear reaction rates in dense stellar matter (especially, in the cores of white dwarfs and envelopes of neutron stars [28]) can be greatly affected by plasma screening of the Coulomb interaction and by the transition to pycnonuclear burning regime (where zero-point vibrations of nuclei in a strongly coupled plasma of ions become important). These plasma physics effects were described by Salpeter and Van Horn [29] (also see [1] and references therein). They modify the interaction potential $U(r)$ but mainly at sufficiently large $r$, typically higher than nucleus sizes, while we focus on the nuclear physics effects which influence $U(r)$ at lower $r$. It is widely thought that the plasma physics and nuclear physics effects are distinctly different and can be considered separately. However, we notice that in very dense and not too hot stellar matter both effects can become interrelated (and should be considered together).

We expect that the broadening of the $U(r)$ peak is especially important for pycnonuclear reactions in the inner crust of accreting neutron stars in X-ray transients [32–34]. They are compact binary systems containing a neutron star and a low-mass companion. Pycnonuclear reactions are thought to be responsible for deep crustal heating of accreted matter. The heating can power thermal surface emission of these neutron stars that is observed in quiescent states of transients (see, e.g., Refs. [32–34]). Pycnonuclear reactions occur at high densities and involve very neutron-rich nuclei (e.g., $^{34}\text{Ne}+^{34}\text{Ne}$ at $\rho \approx 2 \times 10^{12} \text{ g cm}^{-3}$, according to Ref. [33]) immersed in a sea of free neutrons (e.g., $^{33}\text{Ne}$). The $U(r)$-peak should be broadened not only by a diffusive structure of neutron-rich nuclei (that is taken into account in the $\Sigma P$ calculations) but also by the presence of free neutrons. The latter effect is unexplored, but it can affect $S(E)$; nuclear reaction rates, the deep crustal heating and its observational manifestations (e.g., Ref. [33]).

Acknowledgments

The authors are grateful to Andrey Chugunov for critical remarks. This work was partly supported by the Joint Institute for Nuclear Astrophysics (NSF-PHY-0822648), the U.S. Department of Energy under the grant DE-FG02-07ER41459, the Russian Foundation for Basic Research (grants 08-02-00837 and 09-02-12080), by the State Program “Leading Scientific Schools of Russian Federation” (Grant NSh 2600.2008.2), and by the CompStar Program.

[1] E. M. Burbidge, G. R. Burbidge, W. A. Fowler, and F. Hoyle, Rev. Mod. Phys. 29, 547 (1957).
[2] W. A. Fowler and F. Hoyle, Astrophys. J. Suppl. 9, 201 (1964); Appendix C.
[3] D. D. Clayton, Principles of Stellar Evolution and Nucleosynthesis (University of Chicago Press, Chicago, 1983).
[4] D. G. Yakovlev, L. R. Gasques, M. Beard, M. Wiescher, and A. V. Afanasjev, Phys. Rev. C 74, 035803 (2006).
[5] M. Beard, A. V. Afanasjev, L. C. Chamon, L. R. Gasques, M. Wiescher, and D. G. Yakovlev, ADNDT 96, 541 (2010).
[6] J. R. Patterson, H. Winkler, and C. S. Zaidins, Astrophys. J. 157, 367 (1969).
[7] T. Spillane et al., Phys. Rev. Lett. 98, 122501 (2007).
[8] L. D. Landau and E. M. Lifshitz, Quantum Mechanics (Pergamon, Oxford, 1976).
[9] L. R. Gasques, A. V. Afanasjev, E. F. Aguilera, M. Beard, L. C. Chamon, P. Ring, M. Wiescher, and D. G. Yakovlev, Phys. Rev. C 72, 025806 (2005).
[10] L. R. Gasques, A. V. Afanasjev, M. Beard, J. Lubian, T. Neff, M. Wiescher, and D. G. Yakovlev, Phys. Rev. C 76, 045802 (2007).
[11] L. R. Gasques, E. F. Brown, A. Chieffi, C. L. Jiang, M. Limongi, C. Rolfs, M. Wiescher, and D. G. Yakovlev, Phys. Rev. C 76, 035802 (2007).
[12] R. L. Cooper, A. W. Steiner and E. F. Brown, Astrophys. J. 702, 660 (2009).
[13] M. G. Mazarakis and W. E. Stephens, Phys. Rev. C 7, 1280 (1973).
[14] M. D. High and B. Cucuc, Nucl. Phys. A282, 181 (1977).
[15] K. U. Kettner, H. Lorenz-Wirzba, and C. Rolfs, Z. Phys. A298, 65 (1980).
[16] H. W. Becker, K. U. Kettner, C. Rolfs, and H. P. Trautvetter, Z. Phys. A303, 305 (1981).
[17] K. A. Erb and D. A. Bromley, Phys. Rev. C 23, 2781 (1981).
[18] B. Dasmahapatra, B. Cucuc, and F. Lahlou, Nucl. Phys. A384, 257 (1982).
[19] L. J. Satkowiak, P. A. DeYoung, J. J. Kolata, and M. A. Xapsos, Phys. Rev. C 26, 2027 (1982).
[20] P. Rosales et al., Rev. Mex. Fis. 49, 17 (2003).
[21] E. F. Aguilera et al., Phys. Rev. C 73, 064601 (2006).
[22] L. Barrón-Palos et al., Nucl. Phys. A779, 318 (2006).
[23] H. J. Krappe, J. R. Nix and A. J. Sierk, Phys. Rev. C 20, 992 (1979).
[24] G. R. Caughlan and W. A. Fowler, ADNDT 40, 283 (1988).
[25] J. Blocki and W. J. Światecki, Ann. Phys. (NY) 132, 53 (1981).
[26] V. Yu. Denisov and N. A. Pilipenko, Phys. Rev. C 81, 025805 (2010).
[27] C. L. Jiang, K. E. Rehm, B. B. Back, and R. V. F. Janssens, Phys. Rev. C 75, 015803 (2007).
[28] S. L. Shapiro and S. A. Teukolsky, *Black Holes, White Dwarfs, and Neutron Stars* (Wiley-Interscience, New York, 1983).
[29] E. E. Salpeter and H. M. Van Horn, Astrophys. J. 155, 183 (1969).
[30] A. I. Chugunov, H. E. DeWitt, and D. G. Yakovlev, Phys. Rev. D 76, 025028 (2007).
[31] A. I. Chugunov and H. E. DeWitt, Phys. Rev. C 80, 014611 (2009).
[32] S. Gupta, E. F. Brown, H. Schatz, P. Möller, and K.-L. Kratz, Astrophys. J. 662, 1188 (2007).
[33] P. Haensel and J. L. Zdunik, Astron. Astrophys. 229, 117 (1990).
[34] P. Haensel and J. L. Zdunik, Astron. Astrophys. 404, L33 (2003).
[35] E. F. Brown and L. Bildsten, Astrophys. J. 496, 915 (1998).
[36] K. P. Levenfish and P. Haensel, Astrophys. Space Sci. 308, 457 (2007).
[37] P. Haensel, A. Y. Potekhin, and D. G. Yakovlev, *Neutron Stars. 1. Equation of State and Structure* (Springer, New York, 2007).
[38] D. G. Yakovlev, L. Gasques, and M. Wiescher, Mon. Not. Roy. Astron. Soc. 371, 1322 (2006).