Dressing a new integrable boundary of the nonlinear Schrödinger equation

K.T. Gruner
kgruner@math.uni-koeln.de

Universität zu Köln, Mathematisches Institut,
Weyertal 86-90, 50931 Köln, Germany

August 07, 2020

Abstract

We further develop the method of dressing the boundary for the focusing nonlinear Schrödinger equation (NLS) on the half-line to include the new boundary condition presented by Zambon. Additionally, the foundation to compare the solutions to the ones produced by the mirror-image technique is laid by explicitly computing the change of scattering data under the Darboux transformation. In particular, the developed method is applied to insert pure soliton solutions.

Keywords: NLS equation, integrable boundary conditions, half-line, initial-boundary value problems, soliton solutions, dressing transformation, inverse scattering method.

1 Introduction

The nonlinear Schrödinger equation is one of the well-known examples, where the model of a physical phenomenon incorporates both nonlinearity and dispersion in such a way that a soliton—(i) a wave of permanent form (ii) which is localized (iii) and can interact strongly with other solitons and retain its identity—emerges. For integrable models as the NLS equation is, these special solutions have been worked out extensively and primarily by the inverse scattering method. In this context it should be noted that many physical phenomena naturally arise as initial-boundary value problems, due to the localized character of the problem. Nonetheless, characterizing soliton solutions for these problems is substantially less developed than for the corresponding initial value problems, which is therefore an objective of this work.

Similar to the model of the initial value problem, the integrability is a crucial property to be able to derive soliton solutions. Besides the usually addressed boundary conditions, *The author is partially supported by the SFB/TRR 191 ‘Symplectic Structures in Geometry, Algebra and Dynamics’, funded by the DFG.
the Dirichlet, Neumann and Robin boundary condition [2], in that category for the NLS equation, a new integrable boundary has been derived by dressing the Dirichlet boundary with a defect [6]. To then find soliton solutions in these models different method have been successfully applied.

One of these methods, which is called “dressing the boundary”, utilizes the Darboux transformation in a way which preserves the integrable structures of the system at the boundary. Again, the usual boundary conditions in the case of the NLS equation provided a suitable framework to apply this method [7], after it had already been used to produce results in the case of the sine-Gordon equation on the half-line together with integrable boundary conditions [8]. Most importantly, for these boundary conditions, it has been established that solitons which travel with a specific velocity need to have a counterpart, we call mirror soliton, with equal amplitude and opposite velocity. This pair of solitons stand for the reflection happening at the boundary, where the soliton interchanges its role with the corresponding mirror soliton.

Every soliton has a specific set of so called scattering data in the context of the inverse scattering method, from which it can be described uniquely. In [2], apart from using a different method, the mirror-image technique, to construct soliton solutions in the aforementioned usually addressed models with integrable boundary conditions, relations regarding the scattering data of the soliton and of the corresponding mirror soliton have been established. Having these relations facilitates the analysis and understanding of the soliton behavior.

The paper is organized as follows: In Section 2, we review the inverse scattering method for the NLS equation in order to derive the expression of a one-soliton solution and its dependency on the scattering data. Then, we adapt the method of dressing the boundary to the new boundary conditions in Section 2.2. Using these results, we take pure soliton solutions in Section 3.1, which can be constructed in this theory, and point out common relations between the scattering data of this solution. Finally, we visualize said solutions in Section 3.2.

2 Initial value problem for the NLS

In the following, we give a brief summary of the inverse scattering transform of the focusing NLS equation. As in [2] and [3], it will serve as a guideline in order to implement additional results. Therefore, following the analysis given in [1], we introduce the NLS equation

\[ iu_t + u_{xx} + 2|u|^2u = 0, \]
\[ u(0, x) = u_0(x) \]  

(2.1)

for \( u(t, x): \mathbb{R} \times \mathbb{R} \mapsto \mathbb{C} \) and the initial condition \( u_0(x) \). The equation can be expressed in an equivalent compatibility condition of the following linear spectral problems

\[ \psi_x = U\psi, \]
\[ \psi_t = V\psi, \]  

(2.2)
where \( \psi(t, x, \lambda) \) and the matrix operators

\[
U = -i\lambda \sigma_3 + Q, \quad V = -2i\lambda^2\sigma_3 + \tilde{Q}
\]

are \( 2 \times 2 \) matrices. The potentials \( Q \) and \( \tilde{Q} \) of \( U \) and \( V \) are defined by

\[
Q(t, x) = \begin{pmatrix} 0 & u \\ -u^* & 0 \end{pmatrix}, \quad \tilde{Q}(t, x, \lambda) = \begin{pmatrix} i|u|^2 & 2\lambda u + iu_x \\ -2\lambda u^* + iu_x^* & -i|u|^2 \end{pmatrix}
\]

and \( \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \).

In this context, the matrices \( U \) and \( V \) form a so-called Lax pair, depending not only on \( t \) and \( x \), but also on a spectral parameter \( \lambda \). Hereafter, the asterisk denotes the complex conjugate, \( \mathbb{C}_+ = \{ \lambda \in \mathbb{C} : \Im(\lambda) > 0 \} \) as well as \( \mathbb{C}_- = \{ \lambda \in \mathbb{C} : \Im(\lambda) < 0 \} \) and \( \psi^\dagger \) is the transpose of \( \psi \). For a solution \( \psi(t, x, \lambda) \) of the Lax system (2.2) the compatibility condition \( \psi_{tx} = \psi_{xt} \) for all \( \lambda \in \mathbb{C} \) is equivalent to \( \psi(t, x) \) satisfying the NLS equation (2.1). Moreover, we will refer to \( U \) and \( V \) as the \( x \) and \( t \) part of the Lax pair, respectively.

### 2.1 Inverse scattering method for the NLS equation

In that regard, given a sufficiently fast decaying function \( u(t, x) \rightarrow 0 \) and derivative \( u_x(t, x) \rightarrow 0 \) as \( |x| \rightarrow \infty \), it is reasonable to assume that there exist \( 2 \times 2 \)-matrix-valued solutions, we call modified Jost solutions under time evolution, \( \hat{\psi}(t, x, \lambda) = \psi(t, x, \lambda)e^{i\theta(t, x, \lambda)\sigma_3} \), where \( \theta(t, x, \lambda) = \lambda x + 2\lambda^2t \) of the modified Lax system

\[
\hat{\psi}_x + i\lambda\sigma_3, \hat{\psi} = Q\hat{\psi}, \quad \hat{\psi}_t + 2i\lambda^2\sigma_3, \hat{\psi} = \tilde{Q}\hat{\psi}
\]

with constant limits as \( x \rightarrow \pm\infty \) and for all \( \lambda \in \mathbb{R} \),

\[
\hat{\psi}_{\pm}(t, x, \lambda) \rightarrow 1, \quad \text{as} \quad x \rightarrow \pm\infty.
\]

They are solutions to the following Volterra integral equations:

\[
\hat{\psi}_{-}(t, x, \lambda) = 1 + \int_{-\infty}^{x} e^{-i\theta(0, x-y, \lambda)\sigma_3}Q(t, y)\hat{\psi}_{-}(t, y, \lambda)e^{i\theta(0, x-y, \lambda)\sigma_3} \, dy,
\]

\[
\hat{\psi}_{+}(t, x, \lambda) = 1 - \int_{x}^{\infty} e^{-i\theta(0, x-y, \lambda)\sigma_3}Q(t, y)\hat{\psi}_{+}(t, y, \lambda)e^{i\theta(0, x-y, \lambda)\sigma_3} \, dy.
\]

**Lemma 1.** Let \( u(t, \cdot) \in H^{1,1}(\mathbb{R}) = \{ f \in L^2(\mathbb{R}) : xf, f_x \in L^2(\mathbb{R}) \} \). Then, for every \( \lambda \in \mathbb{R} \), there exist unique solutions \( \psi_{\pm}(t, \cdot, \lambda) \in L^\infty(\mathbb{R}) \) satisfying the integral equations (2.4). Thereby, the second column vector of \( \psi_{-}(t, x, \lambda) \) and the first column vector of \( \psi_{+}(t, x, \lambda) \) can be continued analytically in \( \lambda \in \mathbb{C}_- \) and continuously in \( \lambda \in \mathbb{C}_- \cup \mathbb{R} \), while the first column vector of \( \psi_{-}(t, x, \lambda) \) and the second column vector of \( \psi_{+}(t, x, \lambda) \) can be continued analytically in \( \lambda \in \mathbb{C}_- \) and continuously in \( \lambda \in \mathbb{C}_- \cup \mathbb{R} \).

Analogously, the columns of \( \psi_{\pm}(t, x, \lambda) \) can be continued analytically and continuously into the complex \( \lambda \)-plane, \( \psi_2(t) \) and \( \psi_1(t) \) can be continued analytically in \( \lambda \in \mathbb{C}_- \) and continuously in \( \lambda \in \mathbb{C}_- \cup \mathbb{R} \), while \( \psi_1(t) \) and \( \psi_2(t) \) can be continued analytically in \( \lambda \in \mathbb{C}_+ \) and continuously in \( \lambda \in \mathbb{C}_+ \cup \mathbb{R} \).
The limits of the Jost solutions and the zero trace of the matrix $U$ gives $\det \psi_\pm = 1$ for all $x \in \mathbb{R}$. Further, $\psi_\pm$ are both fundamental matrix solutions to the Lax system (2.2), so there exists an $x$- and $t$-independent matrix $A(\lambda)$ such that

$$
\psi_-(t, x, \lambda) = \psi_+(t, x, \lambda) A(\lambda), \quad \lambda \in \mathbb{R}.
$$

The scattering matrix $A$ is determined by this system and therefore we can also write $A(\lambda) = (\psi_+(t, x, \lambda))^{-1} \psi_-(t, x, \lambda)$, whereas its entries can be written in terms of Wronskians. In particular, $a_{11}(\lambda) = \det[\psi_+^{(1)}(\lambda)]$ and $a_{22}(\lambda) = -\det[\psi_+^{(2)}(\lambda)]$, implying that they can respectively be continued in $\lambda \in \mathbb{C}_+$ and $\lambda \in \mathbb{C}_-$. The eigenfunction inherit the symmetry relation of the Lax pair

$$
\psi_\pm(t, x, \lambda) = -\sigma(\psi_\pm(t, x, \lambda^*))^* \sigma, \quad (2.5)
$$

which directly gives $a_{22}(\lambda) = a_{11}^*(\lambda^*)$ and $a_{21}(\lambda) = -a_{12}^*(\lambda)$. The asymptotic behavior of the modified Jost functions and scattering matrix as $\lambda \to \infty$ is

$$
\hat{\psi}_- = 1 + \frac{1}{2i\lambda} \sigma_3 Q + \frac{1}{2i\lambda} \int_{-\infty}^x |u(t, y)|^2 \, dy + O(1/\lambda^2),
$$

$$
\hat{\psi}_+ = 1 + \frac{1}{2i\lambda} \sigma_3 Q - \frac{1}{2i\lambda} \int_x^{\infty} |u(t, y)|^2 \, dy + O(1/\lambda^2),
$$

and $A(\lambda) = 1 + O(1/k)$.

Let $u(t, \cdot) \in H^{1, 1}(\mathbb{R})$ be generic. That is, $a_{11}(\lambda)$ is nonzero in $\mathbb{C}_+$ except at a finite number of points $\lambda_1, \ldots, \lambda_N \in \mathbb{C}_+$, where it has simple zeros $a_{11}(\lambda_j) = 0$, $a_{11}'(\lambda_j) \neq 0$, $j = 1, \ldots, N$. This set of generic functions $u(t, \cdot)$ is an open dense subset of $H^{1, 1}(\mathbb{R})$ usually denoted by $\mathcal{G}$. By the symmetry mentioned above, $a_{11}(\lambda_j) = 0$ if and only if $a_{22}(\lambda_j^*) = 0$ for all $j = 1, \ldots, N$. At these zeros of $a_{11}$ and $a_{22}$, we obtain for the Wronskians the following relation for $j = 1, \ldots, N$,

$$
\psi_+^{(1)}(t, x, \lambda_j) = b_j \psi_+^{(2)}(t, x, \lambda_j), \quad \psi_+^{(2)}(t, x, \bar{\lambda}_j) = \bar{b}_j \psi_+^{(1)}(t, x, \bar{\lambda}_j),
$$

where we defined $\bar{\lambda}_j = \lambda_j^*$. Whereas for $j = 1, \ldots, N$, the relations then provide residue relations used in the inverse scattering method

$$
\text{Res}_{\lambda = \lambda_j} \left( \frac{\hat{\psi}_+^{(1)}}{a_{11}} \right) = C_j e^{2i\theta(t, x, \lambda_j)} \hat{\psi}_+^{(2)}(t, x, \lambda_j),
$$

$$
\text{Res}_{\lambda = \lambda_j} \left( \frac{\hat{\psi}_+^{(2)}}{a_{22}} \right) = \bar{C}_j e^{-2i\theta(t, x, \bar{\lambda}_j)} \hat{\psi}_+^{(1)}(t, x, \bar{\lambda}_j),
$$

where the weights are $C_j = b_j \left( \frac{da_{11}(\lambda)}{d\lambda} \right)^{-1}$ and $\bar{C}_j = \bar{b}_j \left( \frac{da_{22}(\lambda)}{d\lambda} \right)^{-1}$, and they satisfy the symmetry relations $\bar{b}_j = -b_j^*$ and $\bar{C}_j = -C_j^*$.

The inverse problem can be formulated using the jump matrix

$$
J(t, x, \lambda) = \begin{pmatrix} |\rho(\lambda)|^2 & e^{2i\theta(t, x, \lambda)} \rho^*(\lambda) \\ e^{-2i\theta(t, x, \lambda)} \rho(\lambda) & 0 \end{pmatrix},
$$

for all $x \in \mathbb{R}$.
where the reflection coefficient is \( \rho(\lambda) = \frac{a_{12}(\lambda)}{a_{11}(\lambda)} \) for \( \lambda \in \mathbb{R} \). Defining sectionally meromorphic functions

\[
M_- = (\psi^1_+, \psi^2_- / a_{22}), \quad M_+ = (\psi^1_1 / a_{11}, \psi^2_+),
\]

we can give the method of recovering the solution \( u(t, x) \) from the scattering data.

**Riemann–Hilbert problem 1.** For given scattering data \((\rho, \{\lambda_j, C_j\}_{j=1}^N)\) as well as \( t, x \in \mathbb{R} \), find a \( 2 \times 2 \)-matrix-valued function \( \mathbb{C} \setminus \mathbb{R} \ni \lambda \mapsto \mathcal{M}(t, x, \lambda) \) satisfying

1. \( \mathcal{M}(t, x, \cdot) \) is meromorphic in \( \mathbb{C} \setminus \mathbb{R} \).
2. \( \mathcal{M}(t, x, \lambda) = 1 + \mathcal{O}(\lambda^{-1}) \) as \( |\lambda| \to \infty \).
3. Non-tangential boundary values \( \mathcal{M}_\pm(t, x, \lambda) \) exist, satisfying the jump condition \( \mathcal{M}_+(t, x, \lambda) = \mathcal{M}_-(t, x, \lambda)(1 + J(t, x, \lambda)) \) for \( \lambda \in \mathbb{R} \).
4. \( \mathcal{M}(t, x, \lambda) \) has simple poles at \( \lambda_1, \ldots, \lambda_N, \bar{\lambda}_1, \ldots, \bar{\lambda}_N \) with

\[
\text{Res}_{\lambda = \lambda_j} \mathcal{M}(t, x, \lambda) = \lim_{\lambda \to \lambda_j} \mathcal{M}(t, x, \lambda) = \left( C_j e^{i \theta(t, x, \lambda)_j} \right),
\]

\[
\text{Res}_{\lambda = \bar{\lambda}_j} \mathcal{M}(t, x, \lambda) = \lim_{\lambda \to \bar{\lambda}_j} \mathcal{M}(t, x, \lambda) = \left( 0 \bar{C}_j e^{-2i \theta(t, x, \bar{\lambda}_j)} \right).
\]

After regularization, the Riemann–Hilbert problem can be solved via Cauchy projectors, and the asymptotic behavior of \( \mathcal{M}_\pm(t, x, \lambda) \) as \( \lambda \to \infty \) yields the reconstruction formula

\[
u(t, x) = -2i \sum_{j=1}^N C_j^* e^{-2i \theta(t, x, \lambda_j^*)} [\psi^*_+]_{22}(t, x, \lambda_j)
- \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-2i \theta(t, x, \lambda)} \rho^*(\lambda) [\psi^*_+]_{22}(t, x, \lambda) \, d\lambda.
\]

In the reflectionless case, we have \( \rho(\lambda) = 0 \) for \( \lambda \in \mathbb{R} \) and the one-soliton solution with \( \lambda_1 = \xi + i \eta \) can be calculated as

\[
u(t, x) = -2i \eta \frac{C_1^*}{|C_1|} e^{-i(2\xi x + 4(\xi^2 - \eta^2)t)} \text{sech}(2\eta(x + 4\xi t) - \log \frac{|C_1|}{2\eta})
- \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-2i \theta(t, x, \lambda)} \rho^*(\lambda) [\psi^*_+]_{22}(t, x, \lambda) \, d\lambda.
\]

We change the notation so that \( u(t, x) = u_{1s}(t, x; \xi, \eta, x_1, \phi_1) \) has the following expression

\[
u_{1s}(t, x; \xi, \eta, x_1, \phi_1) = 2\eta e^{-i(2\xi x + 4(\xi^2 - \eta^2)t + (\phi_1 + \pi/2))} \text{sech}(2\eta(x + 4\xi t - x_1)), \quad (2.7)
\]

where \( \phi_1 = \arg(C_1) \) and \( x_1 = \frac{1}{2\eta} \log \frac{|C_1|}{2\eta} \).
2.2 Dressing the boundary

As mentioned in the Introduction, a new integrable boundary condition for the NLS equation on the half-line has been obtained in [6] by dressing a Dirichlet boundary with a “jump-defect”. In this section we want to introduce this model on the half-line and compute soliton solutions via the dressing method. Therefore, consider the NLS equation (2.1) for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}_+$ and complement it with boundary condition at $x = 0$, which, in our notation, are of the following form

$$u_x = \frac{i u_t}{2\Omega} - \frac{u|u|^2}{2\Omega} - \frac{u\alpha^2}{2\Omega},$$

(2.8)

where $\Omega = \sqrt{\beta^2 - |u|^2}$, $\alpha$ and $\beta$ real parameters. Then, the NLS equation has again a corresponding Lax system and the boundary condition can be written in the form of a boundary constraint

$$(K_0)_t(t, x, \lambda)|_{x=0} = (V(t, x, -\lambda)K_0(t, x, \lambda) - K_0(t, x, \lambda)V(t, x, \lambda))|_{x=0},$$

(2.9)

where the boundary matrix $K_0(t, x, \lambda)$ is given by

$$K_0 = \frac{1}{(2\lambda - i|\beta|^2) - \alpha^2} \begin{pmatrix} 4\lambda^2 + 4i\lambda\Omega[0] - (\alpha^2 + \beta^2) & 4i\lambda u[0] \\ 4i\lambda u[0]^* & 4\lambda^2 - 4i\lambda\Omega[0] - (\alpha^2 + \beta^2) \end{pmatrix}.$$ 

(2.10)

The boundary matrix is scaled by $((2\lambda - i|\beta|^2) - \alpha^2)^{-1}$, so that

$$(K_0(t, x, \lambda))^{-1} = K_0(t, x, -\lambda).$$

(2.11)

Note that the scaling could also be chosen as $((2\lambda + i|\beta|^2) - \alpha^2)^{-1}$ leaving the constructed solution unchanged. The property (2.11) is needed in the case of the half-line to properly calculate the zeros and associated kernel vectors of the special solutions. Moreover, contrary to the boundary constraint on two half-lines, the boundary constraint on one half-line has a limitation to the $t$ part of the Lax pair, as already mentioned in [4]. Nevertheless, it is possible to compute soliton solutions in this model. Therefore, we introduce the relation of the boundary matrix $K_0(t, x, \lambda)$ to defect matrices, which are linear in $\lambda$, see [4].

**Proposition 1.** The boundary matrix $K_0(t, x, \lambda)$ can be viewed, up to a function of $\lambda$, as product of two defect matrices

$$2\lambda G_{0, \alpha}(t, x, \lambda) = 2\lambda I + \begin{pmatrix} \alpha \pm i\sqrt{\beta^2 - |u|^2} \\ iu^* \end{pmatrix} \left( \begin{pmatrix} \alpha \mp i\sqrt{\beta^2 - |\tilde{u}|^2} \\ i\tilde{u}^* \end{pmatrix} \right),$$

where $\alpha, \beta \in \mathbb{R} \setminus \{0\}$ and $\tilde{u}$ is subject to the Dirichlet boundary condition. In fact,

$$((2\lambda - i\beta)^2 - \alpha^2)K_0(t, x, \lambda) = 4\lambda^2 G_{0, \alpha}(t, x, \lambda)G_{0, -\alpha}(t, x, \lambda).$$
In particular, it is of importance that the product $K_0(t, x, \lambda)_{\alpha}$ of the two defect matrices $G_{0,\alpha}(t, x, \lambda)$ and $G_{0,-\alpha}(t, x, \lambda)$ is commutative. Thereby, it is comprehensible that a kernel vector for each of the matrices $G_{0,\alpha}(t, x, \lambda)$ and $G_{0,-\alpha}(t, x, \lambda)$ at particular, different $\lambda_1, \lambda_2$ introduce the same kernel vectors for $K_0(t, x, \lambda)$ at these values of $\lambda$. In this approach we will leave out boundary-bound soliton solutions. This is due to the fact that the number of zeros and associated kernel vectors given through the special solutions is halved when working with boundary-bound soliton solutions on one half-line. Referring to the analysis in [4], we also introduce the function space due to the fact that the number of zeros and associated kernel vectors given through the special solutions is halved when working with boundary-bound soliton solutions on one half-line. Referring to the analysis in [4], we also introduce the function space $X = \{ f \in H^1_0(\mathbb{R}^+), \partial_x f \in H^1_0(\mathbb{R}^+) \}$, where $H^1_0(\mathbb{R}^+) = \{ f \in L^2(\mathbb{R}^+) : tf \in L^2(\mathbb{R}^+) \}$ and $H^0_0(\mathbb{R}^+) = \{ f \in L^2(\mathbb{R}^+) : \partial_t f, \partial_x f \in L^2(\mathbb{R}^+) \}$. As in the reference, this function space is essential when it comes to identifying the exact signs of the entries in the diagonal of the constructed boundary matrix corresponding to the dressed solution.

**Proposition 2.** Consider a solution $u[0](t, x)$ to the NLS equation on the half-line subject to the new boundary conditions \((2.9)\) with parameters $\alpha, \beta \in \mathbb{R} \setminus \{0\}$ and at $x = 0$ in the function space $u[0](t, 0) \in X$. Take two solutions $\{\psi_1, \psi_2\}$ of the undressed Lax system corresponding to $u[0]$ for $\lambda = \lambda_0 = -\frac{\alpha + i\beta}{2}$ and $\lambda = \lambda_0 = -\lambda_0$. Further, take $N$ solutions $\psi_j$ of the undressed Lax system corresponding to $u[0]$ for distinct $\lambda = \lambda_j \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R} \cup \{0, \lambda_0^2, -\lambda_0, -\lambda_0^2\})$, $j = 1, \ldots, N$. Constructing $K_0(t, x, \lambda)$ as in \((2.10)\) with $u[0]$, $\alpha$ and $\beta$, we assume that there exist paired solutions $\hat{\psi}_j$ of the undressed Lax system corresponding to $u[0]$ for $\lambda = \hat{\lambda}_j = -\lambda_j$, $j = 1, \ldots, N$, satisfying

\[
\hat{\psi}_j|_{x=0} = (K_0(t, x, \lambda_j)\psi_j)|_{x=0}, \quad \hat{\lambda}_j \neq \lambda_j. \tag{2.12}
\]

Then, a $2N$-fold Darboux transformation $D[2N]$ using $\{\psi_1, \hat{\psi}_1, \ldots, \psi_N, \hat{\psi}_N\}$ and their respective spectral parameter lead to the solution $u[2N]$ to the NLS equation on the half-line. In particular, the boundary condition is preserved and we denote such a solution as $u[2N]$ by $\hat{u}[N]$.

**Proof.** With $\{\psi_1, \hat{\psi}_1, \ldots, \psi_N, \hat{\psi}_N\}$, we have $2N$ linear independent solutions to the undressed Lax system \((2.2)\), since $\lambda_1, \ldots, \lambda_N, \hat{\lambda}_1, \ldots, \hat{\lambda}_N$ are distinct due to $\hat{\lambda}_j \neq \lambda_j$. Therefore, the Darboux transformation is uniquely determined and the dressed solution $\hat{u}[N]$ satisfies the NLS equation.

In order to prove that there is a matrix $K_N(t, x, \lambda)$ satisfying

\[
(K_N(t, x, \lambda)|_{x=0} = \left(V[2N](t, x, -\lambda)K_0(t, x, \lambda) - K_N(t, x, \lambda)V[2N](t, x, \lambda)\right)|_{x=0};
\]

it is of advantage to consider the equivalent equality

\[
(D[2N](t, x, -\lambda)K_0(t, x, \lambda))|_{x=0} = (K_N(t, x, \lambda)D[2N](t, x, \lambda))|_{x=0}, \tag{2.13}
\]

where we need to remark that $K_0(t, x, \lambda)$ is multiplied by $((2\lambda - i|\beta|^2 - \alpha^2)/4$ to simplify further notation. In view of this equality, it becomes plausible to assume that the matrix, we wish to find, is of second order in $\lambda$, i.e. $K_N(t, x, \lambda) = \lambda^2 I + \lambda K^{(1)}(t, x) + K^{(0)}(t, x)$. Our goal will be to construct this matrix $K_N(t, x, \lambda)$ as a Darboux transformation with
spectral parameters $\lambda_0$ and $-\lambda_0$ and corresponding kernel vectors which we need to determine in the following paragraph. We will restrict the argumentation to one of the spectral parameters $\lambda_0$ and note that it can be reproduced analogously for the other one $-\lambda_0$.

Since, up to a function of $\lambda$, $K_0(t,x,\lambda) = G_{0,\alpha}(t,x,\lambda)G_{0,-\alpha}(t,x,\lambda)$, we can deduce as in [4] that there exist two vectors $v_0$ and $\hat{v}_0$ at two spectral parameters respectively $\lambda_0$ and $\hat{\lambda}_0$ for which

$$G_{0,\alpha}(t,x,\lambda_0)v_0 = 0, \quad G_{0,-\alpha}(t,x,\hat{\lambda}_0)\hat{v}_0 = 0.$$  

Therefore, $K_0(t,x,\lambda)$ can be seen as two-fold dressing matrix with the inherited kernel vectors of $G_{0,\alpha}$ and $G_{0,-\alpha}$ at respectively $\lambda_0$ and $\hat{\lambda}_0$, so that

$$K_0(t,x,\lambda_0)v_0 = 0, \quad K_0(t,x,\hat{\lambda}_0)\hat{v}_0 = 0.$$  

These kernel vectors $v_0$ and $\hat{v}_0$ are either linear dependent or linear independent of $\{\psi_0, \bar{\psi}_0, \varphi_0, \bar{\varphi}_0\}$. Further, these vectors will serve as a means to construct the kernel vectors for the dressing matrix $K_N(t,x,\lambda)$. Thereby, we distinguish the two cases:

1. The kernel vector $v_0$ of $K_0(t,x,\lambda_0)$ can be expressed as a linear combination of $\{\psi_0, \varphi_0\}$ at $\lambda = \lambda_0$ and $x = 0$. Then, w.l.o.g. $v_0 = \psi_0$, again to simplify notation. Since $\psi_0$ is linearly independent of $\psi_1, \ldots, \psi_N$, it is possible to define a new vector $v'_0 = D[2N](t,x,\lambda_0)v_0$,

which will serve as one of the kernel vectors for the dressing matrix $K_N(t,x,\lambda)$. It is important to note that constructing $K_N(t,x,\lambda)$ in this manner will result in the following relations for the vector $\psi_0$ and the orthogonal vector $\varphi_0$ at $x = 0$:

$$D[2N](t,x,-\lambda_0)K_0(t,x,\lambda_0)\psi_0 = K_N(t,x,\lambda_0)D[2N](t,x,\lambda_0)\psi_0 = 0,$$

$$D[2N](t,x,-\lambda_0^*)K_0(t,x,\lambda_0^*)\varphi_0 = K_N(t,x,\lambda_0^*)D[2N](t,x,\lambda_0^*)\varphi_0 = 0. \quad (2.14)$$

2. The kernel vector $v_0$ of $K_0(t,x,\lambda_0)$ can not be expressed as a linear combination of $\{\psi_0, \varphi_0\}$ at $\lambda = \lambda_0$ and $x = 0$. In this case, making out the kernel vector directly turns out to be not as easy. Since we assumed that the kernel vector $v_0$ and $\{\psi_0, \varphi_0\}$ are linearly independent, we know that at $x = 0$ we can define a vector $\tilde{\psi}_0 = K_0(t,x,\lambda_0)\psi_0 \neq 0$, which, in particular, solves the $t$-part of the undressed Lax system at $x = 0$ and $\lambda = -\lambda_0$, due to $K_0(t,x,\lambda_0)$ satisfying the boundary constraint [2.9]. Similarly, the relations [A.2] for the dressing matrix $D[2N](t,x,\lambda)$ imply that $\tilde{\psi}'_0 = D[2N](t,x,\lambda_0)\psi_0$ and $\tilde{\psi}'_0 = D[2N](t,x,-\lambda_0)\tilde{\psi}_0$ are also solutions the $t$-part of the dressed Lax system at $x = 0$, $\lambda = \lambda_0$ and $\lambda = -\lambda_0$, respectively. Now, connecting these three transformation, we require that there exists a matrix $D[2N](t,x,-\lambda)K_0(t,x,\lambda)(D[2N](t,x,\lambda))^{-1}$, we also call $K_N(t,x,\lambda)$, which then, at $x = 0$, satisfies

$$D[2N](t,x,-\lambda_0)K_0(t,x,\lambda_0)\psi_0 = (K_N(t,x,\lambda_0)D[2N](t,x,\lambda_0)\psi_0) \neq 0,$$

$$D[2N](t,x,-\lambda_0^*)K_0(t,x,\lambda_0^*)\varphi_0 = (K_N(t,x,\lambda_0^*)D[2N](t,x,\lambda_0^*)\varphi_0) \neq 0. \quad (2.15)$$
Further, evaluating the determinant of $K_N(t, x, \lambda)$ at the spectral parameter $\lambda_0$ or $\lambda_0^*$ and $x = 0$, we obtain $\det(K_N(t, x, \lambda_0)) = \det(K_N(t, x, \lambda_0^*)) = 0$. This implies that there exist two kernel vectors of $K_N(t, x, \lambda)$ corresponding to one of the spectral parameters each. Hence, in this case we have found the kernel vector with which we want to construct the dressing matrix $K_N(t, x, \lambda)$, whereas it then satisfies (2.15).

Now, given we have constructed a 2-fold Darboux transformation using $\lambda_0$ and $-\lambda_0$ and the appropriately chosen kernel vectors through the already mentioned procedure, we want to proof that this dressing matrix $K_N(t, x, \lambda)$ indeed satisfies the equality (2.13). First, we will write the equality as matrix polynomials of degree $2N + 2$ in $\lambda$ and denote them as $L(\lambda)$ and $R(\lambda)$. Hence,

$$
L(\lambda) = D[2N](t, 0, -\lambda) K_0(t, 0, \lambda) = \lambda^{2N+2} L_{2N+2} + \lambda^{2N+1} L_{2N+1} + \cdots + \lambda L_1 + L_0,
$$

$$
R(\lambda) = K_N(t, 0, \lambda) D[2N](t, 0, \lambda) = \lambda^{2N+2} R_{2N+2} + \lambda^{2N+1} R_{2N+1} + \cdots + \lambda R_1 + R_0.
$$

The structure of the matrices yields $L_{2N+2} = 1 = R_{2N+2}$, which also stems from the fact that we multiplied $K_0(t, x, \lambda)$ by $((2\lambda - i|\beta|^2 - \alpha^2)/4$. Regarding the property (2.11), the special solutions provide $4N$ zeros and associated kernel vectors

$$
R(\lambda)|_{\lambda=\lambda_j} \psi_j = 0, \quad L(\lambda)|_{\lambda=\lambda_j} \psi_j = 0,
$$

$$
R(\lambda)|_{\lambda=\hat{\lambda}_j} \hat{\psi}_j = 0, \quad L(\lambda)|_{\lambda=\hat{\lambda}_j} \hat{\psi}_j = 0,
$$

at $x = 0$ for $j = 1, \ldots, N$. For $R(\lambda)$ the equalities are clear from the definition of the Darboux transformation and with the assumption (2.12), the equalities for $L(\lambda)$ follow analogously. With the orthogonal vectors $\varphi_j = \sigma_2 \psi_j^*$ and $\hat{\varphi}_j = \sigma_2 \hat{\psi}_j^*$, we obtain

$$
R(\lambda)|_{\lambda=\lambda_j} \varphi_j = 0, \quad L(\lambda)|_{\lambda=\lambda_j} \varphi_j = 0,
$$

$$
R(\lambda)|_{\lambda=\hat{\lambda}_j} \hat{\varphi}_j = 0, \quad L(\lambda)|_{\lambda=\hat{\lambda}_j} \hat{\varphi}_j = 0,
$$

whereby these equalities hold at $x = 0$ for $j = 1, \ldots, N$. This is however not enough to ensure equality in (2.13), since the determinant is of power $4N + 2$ in $\lambda$ and we only have $4N$ zeros. Further, the choice of $K_N$ implies

$$
R(\lambda)|_{\lambda=\lambda_0} \psi_0 = L(\lambda)|_{\lambda=\lambda_0} \psi_0,
$$

$$
R(\lambda)|_{\lambda=\hat{\lambda}_0} \hat{\psi}_0 = L(\lambda)|_{\lambda=\hat{\lambda}_0} \hat{\psi}_0,
$$

and with the orthogonal vectors $\varphi_0 = \sigma_2 \psi_0^*$ and $\hat{\varphi}_0 = \sigma_2 \hat{\psi}_0^*$, we obtain

$$
R(\lambda)|_{\lambda=\lambda_0} \varphi_0 = L(\lambda)|_{\lambda=\lambda_0} \varphi_0,
$$

$$
R(\lambda)|_{\lambda=\hat{\lambda}_0} \hat{\varphi}_0 = L(\lambda)|_{\lambda=\hat{\lambda}_0} \hat{\varphi}_0,
$$

9
at \( x = 0 \). At this point, it is important that all vectors are linearly independent. In view of the additional vectors from the construction of \( K_N(t,x,\lambda) \), at \( \lambda = \lambda_0 \), we see that in both cases either a linear combination of \( \psi_0 \) and \( \varphi_0 \) or \( \psi_0 \) itself is given. Therefore, this provides a linear independent vector and the same is true for the second spectral parameter. As mentioned in the second case, these vectors are not necessarily kernel vectors for \( L(\lambda) \) and \( R(\lambda) \). However, they are indeed kernel vectors of the difference

\[
C(\lambda) = L(\lambda) - R(\lambda) = \lambda^{2N+1}C_{2N+1} + \cdots + C_0, 
\]

which can therefore be calculated explicitly with the given amount of zeros and kernel vectors

\[
(C_{2N+1}, \ldots, C_0) \begin{pmatrix} \lambda_0^{2N+1} \psi_0 & \cdots & (\hat{\lambda}_N)^{2N+1} \tilde{\varphi}_N \\ \vdots & \ddots & \vdots \\ \psi_0 & \cdots & \tilde{\varphi}_N \end{pmatrix} = 0.
\]

Consequently, the matrix coefficients of \( C(\lambda) \) are zero and so is \( C(\lambda) \), implying \( L(\lambda) = R(\lambda) \). Furthermore, this equality gives us that in both cases, the kernel vectors are indeed as described in the first case equal to \( D[2N](t,x,\lambda_0)\psi_0 \) and \( D[2N](t,x,\hat{\lambda}_0)\tilde{\varphi}_0 \) with \( \psi_0 \) and \( \tilde{\varphi}_0 \) kernel vectors of \( K_0(t,x,\lambda) \) respectively at \( \lambda = \lambda_0 \) and \( \lambda = -\lambda_0 \). Even though, the linear dependence of the kernel vector to the pair \( \{\psi_0, \tilde{\varphi}_0\} \) is not implied in both cases.

Given \( K_N(t,x,\lambda) \) of the form \( \lambda^2 I + \lambda K^{(1)}(t,x) + K^{(0)}(t,x) \), we want to determine the matrix coefficients in terms of the solution at \( x = 0 \) to confirm that the boundary conditions are preserved. Thereby, the symmetry of \( V^*(t,x,\lambda^*) = \sigma V(t,x,\lambda)\sigma^{-1} \), where \( \sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \), is inherited by \( K_N(t,x,\lambda) \), so that we can identify

\[
K_N(t,0,\lambda) = \lambda^2 I + \lambda \begin{pmatrix} K^{(1)}_{11}(t) & K^{(1)}_{12}(t) \\ -(K^{(1)}_{12}(t))^* & (K^{(1)}_{11}(t))^* \end{pmatrix} + \begin{pmatrix} K^{(0)}_{11}(t) & K^{(0)}_{12}(t) \\ -(K^{(0)}_{12}(t))^* & (K^{(0)}_{11}(t))^* \end{pmatrix}.
\]

The equality \( L_{2N+1} = R_{2N+1} \) gives at \( x = 0 \) on the off-diagonal of \( K^{(1)}(t,0) \) that \( K^{(1)}_{12}(t) = i\Omega[2N](t,0) \) and \( K^{(1)}_{21}(t) = -(K^{(1)}_{12}(t))^* = i\Omega^*[2N](t,0) \). For the entries on the diagonal of \( K^{(1)}(t,x) \), we obtain from the same equality

\[
K^{(1)}_{11}(t) = i\Omega[0] - 2(\Sigma_1)_{11},
\]

\[
(K^{(1)}_{11}(t))^* = -i\Omega[0] - 2(\Sigma_1^*)_{11},
\]

where \( \Sigma_1 \) is defined as the matrix coefficient of \( \lambda^{2N-1} \) of the matrix \( D[2N](t,x,\lambda) \), see (A.24). To determine the remaining entries of the matrix coefficients, we need to extract information from the determinant of \( K_N(t,x,\lambda) \), which can be calculated as

\[
\det(K_N(t,0,\lambda)) = \det(D[2N](t,0,-\lambda)) \det(K_0(t,0,\lambda)) \det((D[2N](t,0,\lambda))^{-1}),
\]

and using that \( \det(D[2N](t,0,-\lambda)) = \det(D[2N](t,0,\lambda)) \), due to the fact that for each spectral parameter \( \lambda_j \) we also use \( -\lambda_j \) for \( j = 1, \ldots, N \), we obtain

\[
= \det(K_0(t,0,\lambda)) = \lambda^4 - \frac{\alpha^2 - \beta^2}{2} \lambda^2 + \frac{(\alpha^2 + \beta^2)^2}{16}.
\]

10
Formally, calculating the determinant of the matrix $K_N(t,0,\lambda)$ in polynomial form as above, we can match the coefficients such that

$$
\text{tr}(K^{(1)}(t,0)) = 0,
$$

$$
\text{tr}(K^{(0)}(t,0)) + \text{det}(K^{(1)}(t,0)) = -\frac{\alpha^2 - \beta^2}{2},
$$

$$
2\Re(K^{(1)}_{11}(t)(K^{(1)}_{11}(t))^*) - 2(K^{(0)}_{12}(t))^*\Im(u[2N](t,0)) = 0,
$$

$$
\text{det}(K^{(0)}(t,0)) = \frac{(\alpha^2 + \beta^2)^2}{16}.
$$

Combining the first line in (2.17) with the expressions we have for $K^{(1)}_{11}(t)$ and its complex conjugate, see (2.16), we can deduce that $\Re(K^{(1)}_{11}(t)) = 0$. Further, evaluating the next equality of (2.13) at $x = 0$, which is $L_{2N} = R_{2N}$, implies

$$
L_{2N} = \Sigma_2 - i\Sigma_1 \left( \Omega[0] u[0]^* - \Omega[0] \right) - \frac{\alpha^2 + \beta^2}{4} = \Sigma_2 + K^{(1)}(t,0)\Sigma_1 + K^{(0)}(t,0) = R_{2N},
$$

where again $\Sigma_2$ is the matrix coefficient of $\lambda^{2N-2}$ of the matrix $D[2N](t,x,\lambda)$. Matching the $(12)$-entry of this equality, we derive

$$
(u[2N] - u[0]) \frac{\Omega[0]}{2} - iu[0](\Sigma_1)_{11} = -\frac{i}{2} K^{(1)}_{11}(t)(u[2N] - u[0]) + iu[2N](\Sigma^*_1)_{11} + K^{(0)}_{12}(t),
$$

and using the expressions in (2.16) we have for $(\Sigma_1)_{11}$ and $(\Sigma^*_1)_{11}$, we obtain after cancellation

$$
0 = K^{(0)}_{12}(t) - iu[2N]\Re(K^{(1)}_{11}(t)).
$$

However, we already calculated that $\Re(K^{(1)}_{11}(t))$ needs to be zero in order for the determinants to be equal. Hence, also $K^{(0)}_{12}(t)$ and thereby the off-diagonal of $K^{(0)}(t,0)$ vanishes. It follows by the third equation of (2.17) that $\Im(K^{(1)}_{11}(t)) = 0$ and then, by the fourth equation we have $K^{(0)}(t,0) = \pm\frac{\alpha^2 + \beta^2}{4}1$. To verify that it is indeed minus as for $K_0(t,0,\lambda)$, we confirm with the equality of $L_0 = R_0$ at $x = 0$, which is

$$
-\frac{\alpha^2 + \beta^2}{4} \Sigma_{2N} = K^{(0)}(t,0)\Sigma_{2N},
$$

where $\Sigma_{2N}$ is the zero-th order matrix coefficient of the dressing matrix $D[2N](t,x,\lambda)$. For this to be satisfied for all $t \in \mathbb{R}_+$, we need to have $K^{(0)} = -\frac{\alpha^2 + \beta^2}{4}1$. Thereby, we obtain $\text{tr}(K^{(0)}(t,0)) = -\frac{\alpha^2 + \beta^2}{2}$. Thus, the second equation of (2.17) implies that

$$
K^{(1)}_{11}(t) = \pm i\sqrt{\beta^2 - |u[2N]|^2},
$$

$$
(K^{(1)}_{11}(t))^* = \mp i\sqrt{\beta^2 - |u[2N]|^2}.\]
Now, we need to determine the sign of the diagonal entries of $K^{(1)}(t, 0)$ to be able to constitute that $K_N(t, 0, \lambda)$ preserves the boundary constraint, i.e. we need to show that the signs coincide with the signs in the same entry of $K_0(t, 0, \lambda)$ in front of $\Omega[0]$.

Therefore, a similar analysis as in [4] is needed, where we use the fact that under the Darboux transformation functions $u[0](\cdot, 0)$ in the function space $X$ are mapped onto functions, here $u[2N](\cdot, 0)$, which lie in the function space $X$. Further, we have that $K_0(t, 0, \lambda)$ has a positive sign in the $(11)$-entry in front of $\Omega[0]$. As before, we have the kernel vectors $v_0$ and $\tilde{v}_0$ of $K_0(t, x, \lambda)$ at $x = 0$ and respectively $\lambda = \lambda_0$ and $\lambda = \tilde{\lambda}_0$. Then, for $K_0(t, 0, \lambda)$ multiplied by $\frac{(2\lambda - i|\beta|)^2 - \alpha^2}{4}$, we have as $t$ goes to infinity that

$$
\lim_{t \to \infty} K_0(t, 0, \lambda) = \text{diag}(\lambda^2 + i|\beta|\lambda - \frac{(\alpha^2 + \beta^2)}{4}, \lambda^2 - i|\beta|\lambda - \frac{(\alpha^2 + \beta^2)}{4})
$$

$$
= \begin{cases} 
\text{diag}((\lambda - \lambda_0)(\lambda - \tilde{\lambda}_0^*), (\lambda - \lambda_0^*)(\lambda - \tilde{\lambda}_0)), & \text{if } \beta > 0, \\
\text{diag}((\lambda - \lambda_0^*)(\lambda - \tilde{\lambda}_0), (\lambda - \lambda_0)(\lambda - \tilde{\lambda}_0^*)), & \text{if } \beta < 0.
\end{cases}
$$

In turn, this implies that the limits of the kernel vectors of $K_0(t, 0, \lambda)$ are

$$
v_0 \sim \begin{cases} 
e_1, & \text{if } \beta > 0, \\
e_2, & \text{if } \beta < 0,
\end{cases} \quad \tilde{v}_0 \sim \begin{cases} 
e_2, & \text{if } \beta > 0, \\
e_1, & \text{if } \beta < 0,
\end{cases}
$$

as $t$ goes to infinity, where $e_1$ and $e_2$ are unit vectors. Since the dressing matrix $D[2N](t, x, \lambda)$ is also diagonal as $t$ goes to infinity, see [3], the kernel vectors $v_0^0$, $\tilde{v}_0^0$ of $K_N(t, 0, \lambda)$ inherit the long time behavior of their corresponding vector. Therefore, the signs can be determined to be positive in the $(11)$-entry and negative in the $(22)$-entry of $K^{(1)}$. In conclusion, if we assume $u[0](\cdot, 0) \in X$, we can find a Darboux transformation $K_N(t, x, \lambda)$ for which $V[2N]$ satisfies (2.9) regarding $x = 0$, where $K_N(t, x, \lambda)$ is similar to $K_0(t, x, \lambda)$ with an updated function $\tilde{u}[N]$. □

**Remark 1.** Similar to the analysis of the long time behavior of the kernel vectors, one could look at the long time behavior of the dressing matrix $D[2N](t, x, \lambda)$ to deduce the same result through the equality of $K_N(t, x, \lambda)$ with the product of matrices $D[2N](t, x, -\lambda)K_0(t, x, \lambda)(D[2N](t, x, \lambda))^{-1}$. Nevertheless, this is closely related to one another, since the limit behavior of the kernel vectors of $D[2N](t, x, \lambda)$ determines the distribution of factors $\lambda - \lambda_j$, $\lambda - \tilde{\lambda}_j$, $\lambda - \lambda_j^*$ and $\lambda - \tilde{\lambda}_j^*$ for $j = 1, \ldots, N$ in the diagonal entries as $t$ goes to infinity.

Therewith, we have shown that the method of dressing the boundary can be applied to the new boundary conditions constituted in [6]. Unlike in [7], where the boundary matrices $G_0(\lambda)$ and $G_N(\lambda) = G_0(\lambda)$ are provided at the beginning and the proof is to check that they satisfy the equality (2.13) together with the dressing matrix $D[2N](t, x, \lambda)$, we only provide $K_0(t, x, \lambda)$ and the proof is to construct a suitable boundary matrix $K_N(t, x, \lambda)$ satisfying the equality (2.13). Afterwards, we need to verify that the constructed matrix $K_N(t, x, \lambda)$ is in terms of the solution space indeed the boundary matrix we need for the boundary constraint with respect to the dressed solution $\tilde{u}[N]$. The
reason why we need a different approach is due to the boundary matrix. In the case of the Robin boundary condition, the structure of the boundary matrix $G_0(\lambda)$ is such that when comparing $L(\lambda)$ with $R(\lambda)$ it is already clear with regard to $\lambda$ that the $(2N+1)$-th and zero-th order matrix coefficients are equal. Hence, the zeros and associated kernel vectors of the dressing matrix $D[2N](t,x,\lambda)$ are sufficient to derive the equality of $L(\lambda)$ and $R(\lambda)$. However defining $K_N(t,x,\lambda)$ similarly to $K_0(t,x,\lambda)$ as in (2.10) with $u[0]$ and $\Omega[0]$ updated to $u[2N]$ and $\Omega[2N]$, we only have the equality of the $(2N+1)$-th order matrix coefficient and consequently the zeros and associated kernel vectors of the dressing matrix $D[2N](t,x,\lambda)$ are insufficient to derive the equality. Even though the new boundary condition does fit in the proof initially suggested in [7], the Robin boundary condition very well fits into this one as we will put forward in Appendix [B].

3 Soliton solutions

3.1 Relations between scattering data

In this section, we want to derive relations for multi-soliton solutions between the weights $C_j$ and $\tilde{C}_j$, which correspond to a pair of zeros $\lambda_j$ and $\hat{\lambda}_j$ of $a_{11}(\lambda)$, for $j = 1, \ldots, N$.

Consider the zero seed solution $u[0] = 0$ and $\mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R}) \ni \lambda_0 = -\frac{\alpha + i\beta}{2}$. Also,

$$K_0(t,x,\lambda) = \frac{1}{(2\lambda - i|\beta|)^2 - \alpha^2} \begin{pmatrix} 4\lambda^2 + 4i\lambda|\beta| - (\alpha^2 + \beta^2) & 0 \\ 0 & 4\lambda^2 - 4i\lambda|\beta| - (\alpha^2 + \beta^2) \end{pmatrix}.$$ 

Then, in order to apply Proposition [2], we take distinct $\lambda = \lambda_j \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R} \cup \{\lambda_0, \lambda_0^*,-\lambda_0,-\lambda_0^*\})$, $j = 1, \ldots, N$, and at the corresponding spectral parameter a solution $\psi_j$, $j = 0, \ldots, N$, to the Lax system regarding $u[0]$. The solutions $\psi_j$ to the zero seed solution are readily produced

$$\psi_j = \begin{pmatrix} u_j \\ v_j \end{pmatrix} = e^{-i(\lambda_j x + 2\lambda_j^2 t)\sigma_3} \begin{pmatrix} u_j \\ v_j \end{pmatrix},$$

where $(u_j, v_j)^T \in \mathbb{C}^2$ for $j = 0, \ldots, N$. Particularly, the choice for $\psi_0$ will be $(u_0, v_0) = (1,0)$ or $(u_0, v_0) = (0,1)$ respectively for $\beta > 0$ or $\beta < 0$ inspired by the first case in the proof of Proposition [2]. Now, given the relation (2.12), we also take solutions to the same Lax system at $\lambda = \hat{\lambda}_j = -\lambda_j$ defined by $\hat{\psi}_j = e^{-i(\hat{\lambda}_j x + 2\hat{\lambda}_j^2 t)\sigma_3} (\hat{u}_j, \hat{v}_j)^T$, $j = 1, \ldots, N$. Whereas, the relation is equivalent to

$$\frac{\hat{u}_j}{\hat{v}_j} = \frac{(2\lambda_j + i|\beta|)^2 - \alpha^2 u_j}{(2\lambda_j - i|\beta|)^2 - \alpha^2 v_j}, \quad j = 1, \ldots, N.$$

Note that if $\lambda_j \in \mathbb{C}_+$, then $\hat{\lambda}_j \in \mathbb{C}_-$ which, in turn, implies that $\hat{\psi}_j$ has opposite limit behavior as $\psi_j$ for $x \to \pm \infty$. Such that in order to apply Theorem [1] to the Darboux transformation corresponding to $\hat{\lambda}_j$ and $\hat{\psi}_j$, we instead use the counterpart $\tilde{\lambda}_j^*$ and $\tilde{\varphi}_j$. Since with $\hat{\lambda}_j^* \in \mathbb{C}_+$, the vector $\tilde{\varphi}_j = e^{-i(\hat{\lambda}_j^* x + 2\hat{\lambda}_j^2 t)\sigma_3} (\hat{\psi}_j^*, \hat{\psi}_j^*)^T$, admits the same limit
behavior as \( \psi_j \) for \( x \to \pm \infty \). Similar to Remark 3 following Theorem 1 we can deduce for a two-fold Darboux transformation consisting of \( \{ \lambda_1, \psi_1, \hat{\lambda}_1, \hat{\psi}_1 \} \) that the weights in the scattering data can be calculated as

\[
C_1^{(2)} = \frac{v_1}{u_1} \frac{(\lambda_1 - \lambda_1^*)(\lambda_1 - \hat{\lambda}_1)}{\lambda_1 - \lambda_1^*}, \quad C_2^{(2)} = -\frac{\hat{u}_1^*}{\hat{v}_1^*} \frac{(\hat{\lambda}_1^* - \lambda_1^*)(\hat{\lambda}_1^* - \hat{\lambda}_1)}{\lambda_1 - \lambda_1^*}.
\]

This results in

\[
C_1^{(2)}(C_2^{(2)})^* = -4\lambda_1^2 \cdot \frac{(2\lambda_1 + i|\beta|) - \alpha^2}{(2\lambda_1 - i|\beta|)^2 - \alpha^2} \cdot \Re(\lambda_1)^2 \cdot \Im(\lambda_1)^2,
\]

where it is obvious that the factor \( \frac{(2\lambda_1 + i|\beta|)^2 - \alpha^2}{(2\lambda_1 - i|\beta|)^2 - \alpha^2} \) is the only difference to the analogous result in the case of the Robin boundary condition, where we have \( \frac{i\alpha - 2\lambda_1}{m + 2\alpha} \). Nevertheless, by defining \( \lambda_1 = \xi_1 + i\eta_1 \) and \( \hat{\lambda}_1 = -\xi_1 + i\eta_1 \) as well as the corresponding weights \( C_1 = 2\eta_1 e^{2\eta_1 x_1 + i\phi_1} = C_1^{(2)} \) and \( \hat{C}_1 = 2\eta_1 e^{2\eta_1 \hat{x}_1 + i\hat{\phi}_1} = C_2^{(2)} \), we obtain a relation between the initial positions and phases of the two inserted solitons

\[
x_1 + \hat{x}_1 = \frac{1}{2\eta_1} \log \left( 1 + \frac{\eta_1^2}{\xi_1^2} \right) + \frac{1}{4\eta_1} \log \left( \frac{4\xi_1^2 - \alpha^2 - (2\eta_1 + |\beta|)^2 + (4\xi_1(2\eta_1 + |\beta|))^2}{4\xi_1^2 - \alpha^2 - (2\eta_1 - |\beta|)^2 + (4\xi_1(2\eta_1 - |\beta|))^2} \right),
\]

\[
\phi_1 - \hat{\phi}_1 = 2 \arg(\lambda_1) + \arg \left( \frac{4\xi_1^2 - \alpha^2 - (2\eta_1 + |\beta|)^2 + i4\xi_1(2\eta_1 + |\beta|)}{4\xi_1^2 - \alpha^2 - (2\eta_1 - |\beta|)^2 + i4\xi_1(2\eta_1 - |\beta|)} \right) + \pi.
\]

**Remark 2.** In general, we can construct a 2N-Darboux transformation using the information given by \( \{ \lambda_1, \psi_1, \ldots, \lambda_N, \psi_N, \hat{\lambda}_1, \hat{\psi}_1, \ldots, \hat{\lambda}_N, \hat{\psi}_N \} \), where \( \lambda_j = \xi_j + i\eta_j \) and consequently \( \hat{\lambda}_j = -\xi_j + i\eta_j \) for \( j = 1, \ldots, N \) with corresponding solutions to the undressed Lax system as above. Then for \( j = 1, \ldots, N \), the relation for a pair of initial positions \( x_j \) and \( \hat{x}_j = x_{N+j} \) as well as phases \( \phi_j \) and \( \hat{\phi}_j = \phi_{N+j} \) amounts to

\[
x_j + \hat{x}_j = \frac{1}{2\eta_j} \log \left( 1 + \frac{\eta_j^2}{\xi_j^2} \right) + \frac{1}{4\eta_j} \log \left( \frac{4\xi_j^2 - \alpha^2 - (2\eta_j + |\beta|)^2 + (4\xi_j(2\eta_j + |\beta|))^2}{4\xi_j^2 - \alpha^2 - (2\eta_j - |\beta|)^2 + (4\xi_j(2\eta_j - |\beta|))^2} \right)
\]

\[
- \frac{1}{2\eta_j} \sum_{k=1}^{N'} \log \left[ (\xi_j - \xi_k)^2 + (\eta_j - \eta_k)^2 \right] \cdot \left[ (\xi_j + \xi_k)^2 + (\eta_j + \eta_k)^2 \right]
\]

\[
\phi_j - \hat{\phi}_j = 2 \arg(\lambda_j) + \arg \left( \frac{4\xi_j^2 - \alpha^2 - (2\eta_j + |\beta|)^2 + i4\xi_j(2\eta_j + |\beta|)}{4\xi_j^2 - \alpha^2 - (2\eta_j - |\beta|)^2 + i4\xi_j(2\eta_j - |\beta|)} \right) + \pi
\]

\[
- \sum_{k=1}^{N'} \arg \left( \frac{(\xi_j - \xi_k) + i(\eta_j - \eta_k)}{(\xi_j + \xi_k) + i(\eta_j + \eta_k)} \right) \cdot \left[ (\xi_j + \xi_k) + i(\eta_j + \eta_k) \right] \cdot \left[ (\xi_j - \xi_k) + i(\eta_j - \eta_k) \right],
\]

whereas the product of a pair of weights \( C_j \) and \( \hat{C}_j = C_{N+j} \) is

\[
C_j \hat{C}_j^* = -4\lambda_j^2 \frac{(2\lambda_j + i|\beta|)^2 - \alpha^2}{(2\lambda_j - i|\beta|)^2 - \alpha^2} \cdot \Re(\lambda_j)^2 \cdot \Im(\lambda_j)^2
\]

\[
\cdot \left( \prod_{k=1}^{N'} \frac{(\lambda_j - \lambda_k)(\lambda_j + \lambda_k)}{(\lambda_j + \lambda_k)(\lambda_j - \lambda_k)} \right)^2,
\]

where the prime indicates that the term with \( k = j \) is omitted from the sum and product.
Incidentally by the argumentation of the proof of Proposition 2, it follows that the boundary matrix $K_N(t,x,\lambda)$ corresponding to the dressed solution $\hat{u}[N]$ has kernel vectors $\psi_0' = D[2N](t,x,\lambda_0)\psi_0$ and the orthogonal vector as before $\varphi_0' = D[2N](t,x,\lambda_0^*)\varphi_0$ respectively at the spectral parameter $\lambda = \lambda_0$ and $\lambda_0^*$ as described in the first case. Additionally, note that the second case, in which the solution $\psi_0$ and the kernel vector $\upsilon$ are linearly independent, can also occur. An example is given by the non-zero seed solution $u[0] = \rho e^{2i\rho^2t}$ with constant background $\rho > 0$ in the case of the Robin boundary condition, where the solutions to the Lax system can not be connected to the kernel vectors.

3.2 Soliton reflection

The Darboux transformation presented in Appendix A gives the algebraic means to derive $N$-soliton solutions simply by calculating the $(12)$-entry of the projector matrices $(P[j])_{12}$ for $j = 1,\ldots,N$ recursively and then sum them up or by the direct calculation of the quotient of two $2N \times 2N$ matrices, which represents the $(12)$-entry of the sum of projector matrices, i.e. $(\Sigma_1)_{12}$, as presented in [7]. Motivated by Section 3.1, the pure soliton solutions in the case of the new boundary condition, which we can obtain, are constructed by choosing pairs of spectral parameter $\lambda_j$ and $\hat{\lambda}_j^*$, $j = 1,\ldots,N$, and associated constants $u_j$, $v_j$, $-\hat{v}_j^*$ and $\hat{u}_j^*$ as explained therein.

For $N = 1$, consider the spectral parameter $\lambda_1 = \xi_1 + i\eta_1$, where it is comprehensible that, with regard to (2.7), $\xi_1$ and $\eta_1$ respectively describe the velocity and the amplitude of the physical one-soliton. Further, the quotient of the constants $u_1$ and $v_1$ is highly related to the initial position $x_1$ and phase $\phi_1$ of the soliton. Consequently, the mirror soliton corresponding to $\hat{\lambda}_1^* = -\xi_1 + i\eta_1$ has opposite velocity to and the same amplitude as the physical soliton. Particularly, we have visualized said behavior in Figures 1 and 2. Whereas, the Dirichlet boundary condition $u(t,0) = 0$ occur as a special case of the new boundary condition (2.8), when for example $|\alpha| \to \infty$, $|\beta| \to \infty$ or $\beta \to 0$. Indeed, structurally these cases correspond to the boundary matrix $K_0(t,x,\lambda) = 1$. Thereby, we plotted in Figure 1 on the left the reflection of a one-soliton solution $|\hat{u}[1](t,x)|$ subject to the Dirichlet boundary condition as well as on the right a contour plot, which includes the mirror soliton (dashed). In Figure 2 we chose particular parameter $\alpha = 1$ and $\beta = 2$ to plot an example of a one-soliton solution $|\hat{u}[1](t,x)|$ in the case of soliton reflection with respect to the new boundary condition in three dimensions on the left and as a contour plot together with the mirror soliton on the right. It is observable that in these cases the physical soliton and the mirror soliton change roles after the usual soliton interaction with the physical soliton visible before and the mirror soliton visible after the interaction with the boundary. Additionally, in the case of the Dirichlet boundary condition the interaction of the pair of solitons is such that the whole solution is zero at the boundary $x = 0$.

Subsequently, we used the mentioned algorithm to include higher order soliton solutions in the results. First of all, inspired by the breather in the case of the Dirichlet boundary condition, see Figure 6 in [2], we plotted a similar breather solution as soliton reflection in the case of the new boundary condition with parameter $\alpha = 1$ and $\beta = 2$ on
**Fig. 1.** Dirichlet boundary condition at \( x = 0 \): one-soliton reflection with \( \xi_1 = 1, \eta_1 = 1, x_1 = 5 \) and \( \phi_1 = 0 \). Left: 3D plot of \( |\hat{u}[1](t,x)| \). Right: contour plot showing the mirror soliton (dashed) to the left of \( x = 0 \).

**Fig. 2.** New boundary condition \((\alpha = 1, \beta = 2)\) at \( x = 0 \): one-soliton reflection with \( \xi_1 = 1, \eta_1 = 1, x_1 = 5 \) and \( \phi_1 = 0 \). Left: 3D plot. Right: contour plot.

**Fig. 3.** New boundary condition \((\alpha = 1, \beta = 2)\) at \( x = 0 \): two-soliton reflection with \( \xi_1 = \xi_2 = 1/2, \eta_1 = 1/2, \eta_2 = 3/2, x_1 = x_2 = 5 \) and \( \phi_1 = \phi_2 = 0 \). Left: 3D plot. Right: contour plot.
Fig. 4. New boundary condition ($\alpha = 1$, $\beta = 2$) at $x = 0$: three-soliton reflection with $\xi_1 = 3/2$, $\xi_2 = 1/2$, $\xi_3 = 5/4$, $\eta_1 = 1$, $\eta_2 = 3/4$, $\eta_3 = 1/2$, $x_1 = 5$, $x_2 = 8$, $x_3 = 11$ and $\phi_1 = \phi_2 = \phi_3 = 0$. Left: 3D plot. Right: contour plot.

The left and the contour together with the mirror soliton on the right of Figure 3. As one would suspect, the main difference can be observed at the boundary $x = 0$. Ultimately, we went one step further and even plotted the reflection of a three-soliton solution in the case of the new boundary condition, again with the same parameters, on the left and its contour including the mirror soliton on the right of Figure 4. The choice of parameters, which is needed in order to comply with the conditions of Proposition 2, is described in Section 3.1.

Conclusion

In this work, we further developed the method of dressing the boundary to be applicable to the NLS equation on the half-line with the new boundary condition. The boundary condition corresponds to a time dependent gauge transformation (2.10) and this time dependence together with the polynomial degree with respect to the spectral parameter of the transformation thin out the solution space for the new boundary condition. As we have seen in [4], for the time dependence we need the solution to go to zero as the time goes to infinity. Moreover, the polynomial degree disables the consideration of boundary-bound solitons. Nonetheless, we were able to show that it is possible to construct reflected pure soliton solutions of arbitrary (even) order in this model and to visualize the result.
Appendices

A Darboux transformation

The Darboux transformation can be viewed as gauge transformation acting on forms of the Lax pair $U, V$. For that, the undressed Lax system $U[0], V[0]$ and $\psi[0]$ and the transformed, structural identical system as $U[N], V[N]$ and $\psi[N]$ with $N \in \mathbb{N}$. The transformed vector $\psi[N] = D[N]\psi[0]$ satisfies the transformed system

$$\psi[N]_x = U[N]\psi[N], \quad \psi[N]_t = V[N]\psi[N],$$

(A.1)

whereas they are connected by

$$D[N]_x = U[N]D[N] - D[N]U[0], \quad D[N]_t = V[N]D[N] - D[N]V[0].$$

(A.2)

For given $N$ distinct column vector solutions $\psi_j = (\mu_j, \nu_j)^T$ of the undressed Lax system (2.2) evaluated at $\lambda = \lambda_j, j = 1 \ldots N$, we construct an iteration of the one-fold dressing matrix $D[1]$ in the following sense

$$D[N] = ((\lambda - \lambda_N^*)\mathbb{I} + (\lambda_N^* - \lambda_N)P[N]) \cdots ((\lambda - \lambda_1^*)\mathbb{I} + (\lambda_1^* - \lambda_1)P[1]),$$

where $P[j]$ are projector matrices defined by

$$P[j] = \frac{\psi_j[j - 1]\psi_j[j - 1]^\dagger}{\psi_j[j - 1]\psi_j[j - 1]^\dagger}, \quad \psi_j[j - 1] = D[j - 1]|_{\lambda = \lambda_j}\psi_j.$$

(A.3)

Then to reconstruct the solution $u[N]$, we need to insert $\psi[N] = D[N]\psi[0]$ into the transformed Lax system (A.1) and extract the information of the coefficient of $\lambda^{N-1}$ of the first line. Therefore, we need the coefficient of $\lambda^{N-1}$ of $D[N]$ which we denote by $\Sigma_1$, i.e.

$$\Sigma_1 = \sum_{j=1}^N -\lambda_j^* \mathbb{I} + (\lambda_j^* - \lambda_j)P[j].$$

(A.4)

Consequently, the reconstruction formula can be computed as

$$Q[N] = Q[0] - i \sum_{j=1}^N (\lambda_j - \lambda_j^*)[\sigma_3, P[j]].$$

This calculation can be used to recursively construct $N$-soliton solutions. Especially, we use it to compute the solutions in Section 3.2. Moreover, the change of the scattering data under Darboux transformations has been investigated, among others, for the NLS equation, see [5]. With that book serving as a basis, we give a brief overlook for the relevant theorem in our notation.
A.1 Change of scattering data under Darboux transformations

With scattering data \((\rho, \{\lambda_j, C_j\}_{j=1}^N)\), \(\lambda_j \in \mathbb{C}_+\) for all \(j = 1, \ldots, N\), we want to give the relevant information needed to retrace the change of scattering data under Darboux transformations. It is of renewed importance that the solution and its derivative with respect to \(x\) connected to the scattering data is a sufficiently fast decaying function for \(|x| \to \infty\). Then, given a spectral parameter \(\lambda_0 \in \mathbb{C}_+ \setminus \{\lambda_1, \ldots, \lambda_N\}\) and a column solution of the undressed Lax system

\[
\psi_0 = u_0 \psi_-^{(1)}(t, x, \lambda_0) + v_0 \psi_+^{(2)}(t, x, \lambda_0) = u_0 \psi_-^{(1)}(t, x, \lambda_0) e^{-i\theta(t, x, \lambda_0)} + v_0 \psi_+^{(2)}(t, x, \lambda_0) e^{i\theta(t, x, \lambda_0)}.
\]

Defining the ratio of the second and the first component to be

\[
q = \frac{[\hat{\psi}_-]_{21}(t, x, \lambda_0) + \frac{u_0}{v_0} [\hat{\psi}_+]_{22}(t, x, \lambda_0) e^{2i\theta(t, x, \lambda_0)}}{[\hat{\psi}_-]_{11}(t, x, \lambda_0) + \frac{u_0}{v_0} [\hat{\psi}_+]_{12}(t, x, \lambda_0) e^{2i\theta(t, x, \lambda_0)}},
\]

we obtain, in turn, an expression for the ratio of \(v_0\) and \(u_0\), i.e.

\[
-v_0 \frac{u_0}{v_0} = \frac{[\hat{\psi}_-]_{21}(t, x, \lambda_0) - q[\hat{\psi}_-]_{11}(t, x, \lambda_0)}{[\hat{\psi}_+]_{22}(t, x, \lambda_0) - q[\hat{\psi}_+]_{12}(t, x, \lambda_0)} e^{-2i\theta(t, x, \lambda_0)}.
\]

Also, the one-fold Darboux transformation corresponding to \(\lambda_0\) and \(\psi_0\) takes the form

\[
D[1] = \lambda I + \frac{1}{1 + |q|^2} \left( -\lambda_0 - \lambda_0^* |q|^2 \begin{pmatrix} (\lambda_0^* - \lambda_0) q^* \\ -\lambda_0^* - \lambda_0 |q|^2 \end{pmatrix} \right).
\]

The properties of the Jost functions imply

\[
\lim_{x \to \infty} q = \infty, \quad \lim_{x \to -\infty} q = 0.
\]

Thereby, adding a pole to the scattering data under Darboux transformations can be explained by the following

**Theorem 1.** Let the scattering data \(a_{11}(\lambda), \lambda \in \mathbb{C}_+ \cup \mathbb{R}, a_{12}(\lambda), \lambda \in \mathbb{R}\) and \(b_j\) for \(j = 1, \ldots, N\) be given. Applying the Darboux transformation with \(\lambda_0 \in \mathbb{C}_+ \setminus \{\lambda_1, \ldots, \lambda_N\}\) and \(\psi_0 = u_0 \psi_-^{(1)}(t, x, \lambda_0) + v_0 \psi_+^{(2)}(t, x, \lambda_0)\), where \(u_0, v_0 \in \mathbb{C} \setminus \{0\}\), we add an eigenvalue to the scattering data leaving the original eigenvalues unchanged. Further,

\[
a'_{11}(\lambda) = \frac{\lambda - \lambda_0}{\lambda - \lambda_0^*} a_{11}(\lambda), \quad \lambda \in \mathbb{C}_+ \cup \mathbb{R}, \quad \rho'(\lambda) = \frac{\lambda - \lambda_0^*}{\lambda - \lambda_0} \rho(\lambda), \quad \lambda \in \mathbb{R},
\]

\[
a'_{12}(\lambda) = a_{12}(\lambda), \quad \lambda \in \mathbb{R},
\]

\[
b'_j = b_j, \quad j = 1, \ldots, N,
\]

\[
b'_0 = -\frac{v_0}{u_0},
\]

\[
C'_j = \frac{\lambda_j - \lambda_0^*}{\lambda_j - \lambda_0} C_j, \quad j = 1, \ldots, N,
\]

\[
C'_0 = -\frac{v_0}{u_0} \frac{\lambda_0 - \lambda_0^*}{a_{11}(\lambda_0)}.
\]
Proof. The scattering data rely heavily on the Jost functions. That is why, the first step is to find the behavior of the Jost functions in the transformed system. Therefore, we need to see what the limit values of the Darboux transformation are

\[
\lim_{x \to -\infty} D[1] = \text{diag}(\lambda - \lambda_0^*, \lambda - \lambda_0), \quad \lim_{x \to +\infty} D[1] = \text{diag}(\lambda - \lambda_0, \lambda - \lambda_0^*).
\]

Then, we can deduce that the transformed Jost functions can be expressed through

\[
(\psi_-^{(1)})'(t, x, \lambda) = \frac{D[1]}{\lambda - \lambda_0^*} \psi_-^{(1)}(t, x, \lambda), \quad (\psi_+^{(2)})'(t, x, \lambda) = \frac{D[1]}{\lambda - \lambda_0^*} \psi_+^{(2)}(t, x, \lambda),
\]

which also is passed onto \((\hat{\psi}_-^{(1)})'\) and \((\hat{\psi}_+^{(2)})'\). As already mentioned in Section 2,\(a_{11}(\lambda) = \text{det}[\psi_-^{(1)} | \psi_+^{(2)}]\). It follows that for \(\lambda \in \mathbb{C}_+ \cup \mathbb{R}\), the limit values of \([\hat{\psi}_-^{11}]\) and \([\hat{\psi}_+^{22}]\) are \(a_{11}(\lambda)\) as \(x\) goes to \(+\infty\) and \(-\infty\), respectively. So that we have \(a_{11}'(\lambda) = \lim_{x \to -\infty} ([\hat{\psi}_-^{11}])' = \frac{\lambda - \lambda_0}{\lambda - \lambda_0^*} a_{11}(\lambda).\)

Analogously, we find for \(a_{12}(\lambda)\) that

\[
a_{12}(\lambda) = [\psi_+]_{22} [\psi_-]_{12} - [\psi_+]_{12} [\psi_-]_{22} = ([\hat{\psi}_+]_{22} [\hat{\psi}_-]_{12} - [\hat{\psi}_+]_{12} [\hat{\psi}_-]_{22}] e^{2i\theta(t, x, \lambda)},
\]

and therefore the limit values of \([\hat{\psi}_-^{12}]\) and \(-[\hat{\psi}_+^{12}]\) behave as \(a_{12}(\lambda)e^{-2i\theta(t, x, \lambda)}\) as \(x\) goes to \(+\infty\) and \(-\infty\), respectively. Consequently,

\[
a_{12}'(\lambda) = \lim_{x \to -\infty} ([\hat{\psi}_-^{12}])' = a_{12}(\lambda).
\]

Also resulting in \(\rho'(\lambda) = \frac{\lambda - \lambda_0^*}{\lambda - \lambda_0} \rho(\lambda)\). Since the Jost functions we relate in order to obtain \(b_j\) are changed identically with \(D[1]/(\lambda - \lambda_0^*)\), \(b_j\) remain unchanged, i.e. \(b_j' = b_j\) for \(j = 1, \ldots, N\). Then, by the definition of \(C_j\) we can calculate

\[
C_j' = b_j' \left( \frac{da_{11}'(\lambda_j)}{d\lambda} \right)^{-1} = \frac{\lambda_j - \lambda_0^*}{\lambda_j - \lambda_0} C_j, \quad j = 1, \ldots, N.
\]

At the new eigenvalue \(\lambda = \lambda_0\), we have that the transformed Jost function are also identically changed by

\[
\frac{D[1](t, x, \lambda_0)}{(\lambda_0 - \lambda_0^*)} = \frac{1}{1 + |q|^2} \begin{pmatrix} |q|^2 & -q^* \\ -q & 1 \end{pmatrix}.
\]

Hence, as we calculated already in (A.5), we obtain

\[
b_0' = \left( \frac{[\psi_-]_{21}}{[\psi_+]_{22}} \right)'(t, x, \lambda_0) = \frac{[\psi_-]_{21}(t, x, \lambda_0) - q[\psi_-]_{11}(t, x, \lambda_0)}{[\psi_+]_{22}(t, x, \lambda_0) - q[\psi_+^{12}(t, x, \lambda_0)} = -\frac{v_0}{u_0}.
\]

Subsequently, the weight for the added eigenvalue is readily obtained by

\[
C_0' = b_0' \left( \frac{da_{11}'(\lambda_0)}{d\lambda} \right)^{-1} = -\frac{v_0 \lambda_0 - \lambda_0^*}{u_0 a_{11}(\lambda_0)}.
\]

\[\square\]
Remark 3. A particular example is dressing a pure soliton solution from the zero seed solution for which \( a_{11}(\lambda) = 1, \ a_{12}(\lambda) = 0 \), whereby \( \rho(\lambda) = 0 \). Then, inserting poles \( \lambda_1, \ldots, \lambda_N \in \mathbb{C}_+ \) with corresponding \( u_j, v_j \in \mathbb{C} \setminus \{0\} \), \( j = 1, \ldots, N \), results in the (relevant) scattering data

\[
a_{11}^{(N)}(\lambda) = \prod_{j=1}^{N} \frac{\lambda - \lambda_j}{\lambda - \lambda_j^*}, \quad a_{12}^{(N)}(\lambda) = 0, \quad G_j^{(N)} = \frac{-v_j}{u_j} \prod_{k=1}^{N} (\lambda_j - \lambda_k^*) \left( \prod_{k=1}^{N} (\lambda_j - \lambda_k) \right)^{-1},
\]

where the prime indicates that the term with \( k = j \) is omitted from the product.

B Robin boundary condition

As mentioned in Section 2, the proof, which we tailored to fit the new boundary condition, is also applicable to the Robin boundary condition. In fact, we will proceed and write it down explicitly not only for the convenience of the interested reader but also since it was a guiding step between the proof for the defect conditions connecting two half-lines, see [4], and Proposition 2.

B.1 Dressing the Robin boundary condition

As before, we look at the NLS equation (2.1) for \( (t, x) \in \mathbb{R}_+ \times \mathbb{R}_+ \) and complement it with Robin boundary condition at \( x = 0 \), which, in our notation, is

\[
u_x = \alpha u, \quad \alpha \in \mathbb{R}.
\]

(B.1)

Then, the NLS equation has again a corresponding Lax system and the boundary condition can be written in the form of a boundary constraint

\[
0 = (V(t, x, -\lambda)G_0(\lambda) - leftrightarrow t, x, \lambda))|_{x=0},
\]

where the boundary matrix \( G_0(\lambda) \) is given by

\[
G_0(\lambda) = \frac{1}{i\alpha + 2\lambda} \begin{pmatrix} i\alpha - 2\lambda & 0 \\ 0 & i\alpha + 2\lambda \end{pmatrix}
\]

(B.3)

and is, in particular, independent of \( t \) and \( x \). Similarly to the boundary matrix (2.10), \( G_0(\lambda) \) is scaled by \( (i\alpha + 2\lambda)^{-1} \), so that \( (G_0(\lambda))^{-1} = G_0(-\lambda) \), which is crucial for the proof of Proposition 3.

Remark 4. The kernel vectors for \( G_0(\lambda) \) can be easily obtained at \( \lambda = \frac{i\alpha}{2} \) and \( \lambda = -\frac{i\alpha}{2} \). We have respectively \( e_1 \) and \( e_2 \).
B.2 Dressing the boundary

In this approach we will also leave out boundary-bound soliton solutions. Due to the fact that for boundary-bound soliton solutions the proof needs to be slightly changed. The proof will be not as detailed as for Proposition 2. Nonetheless, we will point out the differences rather than the similarities.

**Proposition 3.** Consider a solution $u(0)(t,x)$ to the NLS equation on the half-line subject to the Robin boundary conditions \( \text{(B.2)} \) with parameter $\alpha \in \mathbb{R}$. Take one solution $\psi_0$ of the undressed Lax system corresponding to $u[0]$ for $\lambda = \lambda_0 = -\frac{i\alpha}{2}$. Further, take $N$ solutions $\psi_j$ of the undressed Lax system corresponding to $u[0]$ for distinct $\lambda = \lambda_j \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$, $j = 1, \ldots, N$. Constructing $G_0(\lambda)$ as in \( \text{(B.3)} \) with $\alpha$, we assume that there exist paired solutions $\hat{\psi}_j$ of the undressed Lax system corresponding to $u[0]$ for $\lambda = \lambda_j = -\lambda_j$, $j = 1, \ldots, N$, satisfying

$$
\hat{\psi}_j|_{x=0} = (G_0(\lambda_j)\psi_j)|_{x=0}, \quad \hat{\lambda}_k \neq \lambda_j.
$$

Then, a $2N$-fold Darboux transformation $D[2N]$ using $\{\psi_1, \psi_1, \ldots, \psi_N, \hat{\psi}_N\}$ and their respective spectral parameter lead to the solution $u[2N]$ to the NLS equation on the half-line. In particular, the boundary condition is preserved and we denote such a solution $u[2N]$ by $\tilde{u}[N]$.

**Proof.** Defining the matrix $G_N(t,\lambda) = 2\lambda \sigma_3 + G^{(0)}(t)$ similar to $K_N(t,x,\lambda)$ through the transformed kernel vectors $D[2N](t,x,\lambda_0)e_1$ and $D[2N](t,x,\lambda_0^*)e_2$ at $x = 0$ and respectively $\lambda = \lambda_0$ and $\lambda = \lambda_0^*$, we can derive that the equality

$$
(D[2N](t,x,-\lambda)G_0(\lambda))|_{x=0} = (G_N(t,\lambda)D[2N](t,x,\lambda))|_{x=0}
$$

holds, whereas $G_0(\lambda)$ is multiplied by $2\lambda + i\alpha$.

To reconstruct the expression of $G_N(t,\lambda)$, we analyze the equality \( \text{(B.5)} \). In particular for the equality of the matrix coefficients regarding $\lambda$ of order 2$N$, we obtain for the off-diagonal entries of $G^{(0)}(t)$ that $G^{(0)}_{12}(t) = 0$ and $G^{(0)}_{21}(t) = 0$. Then, for the diagonal entries, we need to evaluate the determinant of $G_N(t,\lambda)$ in two ways. Firstly, as a product of matrices

$$
det(G_N(t,\lambda)) = det(D[2N](t,0,-\lambda)) det(G_0(\lambda)) det((D[2N](t,0,\lambda))^{-1}) = -4\lambda^2 - \alpha^2.
$$

Secondly, through the definition $G_N(t,\lambda) = 2\lambda \sigma_3 + G^{(0)}(t)$ and the partial result for the off-diagonal entries, we obtain consequently

$$
G^{(0)}_{11}(t) - G^{(0)}_{22}(t) = 0, \quad det(G^{(0)}(t)) = -\alpha^2.
$$

Hence, we need to have $G^{(0)}(t) = \pm i\alpha \mathbb{1}$. However, by the equality \( \text{(B.5)} \) of the matrix coefficients regarding $\lambda$ of zero-th order, we can verify, that the structure of $G_0(\lambda)$ is preserved, since we need to have $G^{(0)}(t) = -i\alpha \mathbb{1}$ in order for

$$
- \alpha \Sigma_{2N} = G^{(0)}(t)\Sigma_{2N}
$$

\( \Box \)

22
This proposition serves as a means to insert solitons in the case of the zero seed solution and also the non-zero seed solution with constant background as presented in [7]. Similarly, to the argumentation therein, we would also need to proof distinctly the dressing of boundary-bound solitons with pure imaginary spectral parameters $\lambda_j \in i\mathbb{R}$, $j = 1, \ldots, N$. However, since the reasoning is similar and it is presumably not applicable in the case of the new boundary condition, we omit that consideration here.

Nonetheless, we hereby gave the proof to dress solitons corresponding to $\lambda_j \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$, $j = 1, \ldots, N$, in the dressing the boundary method we adapted to the new boundary conditions. It turns out that the equality (B.7), from which we then follow the definite form of $G_N(\lambda)$, functions as an intermediate step between the ideas in the proof for the defect conditions connecting two half-lines [4] and for the new boundary conditions, see Proposition 2.

### B.3 Relations between scattering data

Consider the zero seed solution $u[0] = 0$ and $\mathbb{C} \setminus \mathbb{R} \ni \lambda_0 = -\frac{i\alpha}{2}$. Following the steps in Section 3.1, we take $\psi_j = e^{-i(\lambda_j x + 2\lambda_j^2 t)}(u_j, v_j)\sigma_3$ for $\lambda = \lambda_j$ and with respect to the relation $\psi_j|_{x=0} = (G_0(\lambda_j)\psi_j)|_{x=0}$, $-\lambda_k \neq \lambda_j$ for all $j, k \in \{1, \ldots, N\}$, we also take $\hat{\psi}_j = e^{-i(\hat{\lambda}_j x + 2\hat{\lambda}_j^2 t)}(\hat{u}_j, \hat{v}_j)\sigma_3$ for $\lambda = \hat{\lambda}_j = -\lambda_j$, whereas

$$\frac{\hat{u}_j}{\hat{v}_j} = \frac{i\alpha - 2\lambda_j u_j}{i\alpha + 2\lambda_j v_j}, \quad j = 1, \ldots, N.$$  

Analogously, if we apply a two-fold Darboux transformation consisting of $\{\lambda_1, \psi_1, \hat{\lambda}_1, \hat{\psi}_1\}$, we obtain the scattering data, which particularly results in the relation

$$C_1^{(2)}(C_2^{(2)})^* = -4\lambda_1^2 \cdot \frac{i\alpha - 2\lambda_1}{i\alpha + 2\lambda_1} \cdot \frac{\Im(\lambda_1)}{\Re(\lambda_1)^2},$$  

(B.8)

which is up to notation the same as in [2]. To align the notation, one would need to complex conjugate (B.8) and then it would be equal to the equation (2.36) in their paper with $k_1 = -\lambda_1^*$. This is due to the differently defined potential $\tilde{Q}$ of the matrix $V$, which as a consequence gives the existence of Jost solutions with different asymptotic behavior and continuations into different parts of the complex plane. Analogously to Section 3.1, we obtain the following relations between the initial positions and phases of $2N$ inserted solitons.

**Remark 5.** In general, we can construct a $2N$-Darboux transformation using the information given by $\{\lambda_1, \psi_1, \ldots, \lambda_N, \psi_N, \hat{\lambda}_1, \hat{\psi}_1, \ldots, \hat{\lambda}_N, \hat{\psi}_N\}$, where $\lambda_j = \xi_j + i\eta_j$ and consequently $\hat{\lambda}_j^* = -\xi_j + i\eta_j$ for $j = 1, \ldots, N$ with corresponding solutions to the undressed Lax system as above. Then for $j = 1, \ldots, N$, the relation for a pair of initial
positions \( x_j \) and \( \hat{x}_j = x_{N+j} \) as well as phases \( \phi_j \) and \( \hat{\phi}_j = \phi_{N+j} \) amounts to

\[
x_j + \hat{x}_j = \frac{1}{2\eta_j} \log \left( 1 + \frac{\eta_j^2}{\xi_j^2} \right) + \frac{1}{4\eta_j} \log \left( \frac{(2\xi_j)^2 + (\alpha - 2\eta_j)^2}{(2\xi_j)^2 + (\alpha + 2\eta_j)^2} \right)
- \frac{1}{2\eta_j} \sum_{k=1}^{N} \log \left[ \frac{[(\xi_j - \xi_k)^2 + (\eta_j - \eta_k)^2][(\xi_j + \xi_k)^2 + (\eta_j + \eta_k)^2]}{[(\xi_j + \xi_k)^2 + (\eta_j + \eta_k)^2][(\xi_j - \xi_k)^2 + (\eta_j + \eta_k)^2]} \right],
\]

\[
\varphi_j - \hat{\varphi}_j = 2 \arg(\lambda_j) + \arg\left( \frac{2\xi_j + i(2\eta_j - \alpha)}{2\xi_j + i(2\eta_j + \alpha)} \right)
- \sum_{k=1}^{N} \arg\left( \frac{[(\xi_j - \xi_k) + i(\eta_j - \eta_k)][(\xi_j + \xi_k) + i(\eta_j - \eta_k)]}{[(\xi_j + \xi_k) + i(\eta_j + \eta_k)][(\xi_j - \xi_k) + i(\eta_j + \eta_k)]} \right),
\]

whereas the product of a pair of weights \( C_j, \hat{C}_j = C_{N+j} \) is

\[
C_j \hat{C}_j = -4\lambda_j^2 \left( \frac{i\alpha - 2\lambda_j (2\eta_j)^2}{i\alpha + 2\lambda_j (2\xi_j)^2} \right) \left[ \prod_{k=1}^{N} \frac{(\lambda_j - \lambda_k^*) (\lambda_j + \lambda_k)}{(\lambda_j - \lambda_k^*) (\lambda_j + \lambda_k^*)} \right]^2.
\]

References

[1] M. J. Ablowitz, B. Prinari, and A. D. Trubatch. Discrete and continuous nonlinear Schrödinger systems, Volume 302. Cambridge University Press, 2004.

[2] G. Biondini and G. Hwang. Solitons, boundary value problems and a nonlinear method of images. Journal of Physics A: Mathematical and Theoretical, 42(20), 2009.

[3] A. S. Fokas and B. Pelloni. Unified transform for boundary value problems: Applications and advances. SIAM, 2014.

[4] K. T. Gruner. Soliton solutions of the nonlinear Schrödinger equation with defect conditions. arXiv preprint arXiv:1908.05101, 2019.

[5] C. Gu, A. Hu, and Z. Zhou. Darboux transformations in integrable systems: theory and their applications to geometry, volume 26. Springer Science & Business Media, 2006.

[6] C. Zambon. The classical nonlinear Schrödinger model with a new integrable boundary. Journal of High Energy Physics, 2014(8):36, 2014.

[7] C. Zhang. Dressing the boundary: On soliton solutions of the nonlinear Schrödinger equation on the half-line. Studies in Applied Mathematics, 2018.

[8] C. Zhang, Q. Cheng, and D.-J. Zhang. Soliton solutions of the sine-Gordon equation on the half line. Applied Mathematics Letters, 86:64–69, 2018.