Bending, Buckling and Vibration analysis of third order shear deformation nanoplate based on modified couple stress theory

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1. Introduction

The atomic and molecular scale test is known as the safest method for the study of materials in small-scales. In this method, the nanostructures are studied in real dimensions. The atomic force microscopy (AFM) is used to apply different mechanical loads on nanoplates and measure their responses against those loads in order to determine the mechanical properties of the nanoplate. The difficulty of controlling the test conditions at this scale, high economic costs and time-consuming processes are some setbacks of this method. Therefore, it is used only to validate other simple and low-cost methods.

Atomic simulation is another solution for studying small-scale structures. In this method, the behavior of atoms and molecules is examined by considering the intermolecular and interatomic effects on their motions, which eventually involves the total deformation of the body. In the case of large deformations and multi atomic scale the computational costs is too high, so this method is only used for small deformation problems.
Given the limitations of the aforementioned methods for studying nanostructures, researchers have been looking for simpler solutions for nanostructures. Modeling small-scale structures using continuum mechanics is another solution to this problem. There are a variety of size-dependent continuum theories that consider size effects, some of these theories are; micromorphic theory, microstructural theory, micropolar theory, Kurt's theory, non-local theory, modified couple stress theory and strain gradient elasticity. All of which are the developed notion of classical field theories, which include size effects. Daghigh et al. studied the nonlocal bending and buckling of agglomerated CNT-Reinforced composite nanoplates. They investigated the effect of the parameters, such are degree of agglomeration, nonlocal material scale parameter, temperature, foundation properties, volume fraction of CNTs, and length-to-thickness aspect ratio for the plate [19].

Daikh et al. studied a novel nonlocal strain gradient Quasi-3D bending analysis of sigmoid functionally graded sandwich nanoplates. They investigated the effect of the elastic foundation models, sigmoidal distribution index constant, configuration of sandwich plate, material and length nanoscales, boundary conditions on the static deflection [20].

In this paper, size-dependent nanoplate model is developed to account for the size effect. Hamilton principle is used to derive the equations of motion based on the mentioned theories (i.e. modified couple stress and third order shear deformation theories). In order to investigate the effects of material length scale parameter on deflection, buckling and frequency, analytical solution for a static problem is obtained for a simply supported plate and results are discussed.

2. Modified coupled stress theory

In 2002 Yang et al. [1] proposed a modified couple stress model by modifying the theory proposed by Toppin [2], Mindlin and Thursten [3], Quitter [4] and Mindlin [5] in 1964. The modified couple stress theory consists of one material length scale parameter for projection of the size effect, whereas the classical couple stress theory has two material length scale parameters. In the modified couple stress theory the strain energy density in the three-dimensional vertical coordinates for a body bounded by the volume \( V \) and the area \( \Omega \) [6], is expressed as the follows:

\[
U = \frac{1}{2} \int (\sigma_{ij}E_{ij} + m \gamma_{ij})dV \quad i,j = 1,2,3 \tag{2.1}
\]

Were,

\[
E_{ij} = \frac{1}{2}(u_{ij} + u_{ji}), \gamma_{ij} = \frac{1}{2}(\theta_{ij} + \theta_{ji}) \tag{2.2}
\]

\( \gamma_{ij} \) and \( E_{ij} \) are the symmetric parts of the curvature and strain tensors and \( \theta_{ij} \) and \( u_{ij} \) are the displacement and the rotational vectors, respectively.

\[
\theta = \frac{1}{r} \text{Curl} \; u \tag{2.3}
\]

\[
\sigma_{ii}, \text{ the stress tensor, and } m \text{, the deviatory part of the couple stress tensor, are defined as:}
\]

\[
\sigma_{ij} = \lambda \partial_{k} \delta_{ij} + 2\mu \partial_{ij}, \quad m_{ij} = 2\mu l^2 \chi_{ij} \tag{2.4}
\]

Where \( \lambda \) and \( \mu \) are the lame constants, \( \delta_{ij} \) is the Kronecker delta and \( l \) is the material length scale parameter. From Equations (2.2) and (2.4) it can be seen that \( \gamma_{ij} \) and \( m_{ij} \) are symmetric.

3. Third order shear deformation nanoplate model

In Fig.1 an isotropic rectangular nanoplate with length \( a \), width \( b \) and thickness \( h \) is shown.

Figure 1. A schematic of the nanoplate and axes

The displacement equations for the third order shear deformation nanoplate are defined as (According to the Reddy shear theory):

\[
\begin{align*}
\begin{pmatrix}
\phi_1(x,y,z) \\
\phi_2(x,y,z) \\
\phi_3(x,y,z) \\
\end{pmatrix} &= \begin{pmatrix}
\frac{1}{3} (x^2 + y^2) \frac{\partial \phi_1}{\partial z} + \frac{1}{3} (x^2 + z^2) \frac{\partial \phi_2}{\partial y} + \frac{1}{3} (y^2 + z^2) \frac{\partial \phi_3}{\partial x} \\
\frac{1}{3} (y^2 + z^2) \frac{\partial \phi_1}{\partial x} + \frac{1}{3} (x^2 + z^2) \frac{\partial \phi_2}{\partial y} + \frac{1}{3} (x^2 + y^2) \frac{\partial \phi_3}{\partial z} \\
\frac{1}{3} (x^2 + z^2) \frac{\partial \phi_1}{\partial y} + \frac{1}{3} (y^2 + z^2) \frac{\partial \phi_2}{\partial x} + \frac{1}{3} (x^2 + y^2) \frac{\partial \phi_3}{\partial z} \\
\end{pmatrix} \\
\end{align*}
\]

(3.1)

Where \( \phi_1 \) and \( \phi_2 \) are rotation of the normal vector around the \( x \), \( y \) and \( w \) are the displacement of the middle surface at the \( z \) axes. The symmetric part of curvature tensor, strain and stress tensor and rotation vector for third order shear deformation nanoplate model are as follows:

\[
\begin{align*}
\begin{pmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\varepsilon_{zz} \\
\end{pmatrix} &= \begin{pmatrix}
\frac{1}{2} \frac{\partial \phi_1}{\partial x} + \frac{1}{2} \frac{\partial \phi_2}{\partial y} + \frac{1}{2} \frac{\partial \phi_3}{\partial z} \\
\frac{1}{2} \frac{\partial \phi_1}{\partial y} + \frac{1}{2} \frac{\partial \phi_2}{\partial x} + \frac{1}{2} \frac{\partial \phi_3}{\partial z} \\
\frac{1}{2} \frac{\partial \phi_1}{\partial z} + \frac{1}{2} \frac{\partial \phi_2}{\partial y} + \frac{1}{2} \frac{\partial \phi_3}{\partial x} \\
\end{pmatrix} \\
\end{align*}
\]

(3.2)
The variation of strain energy is expressed as:

\[
\delta U = \int_\Omega \left[ (\sigma_{xx} \delta E_{xx} + \sigma_{yy} \delta E_{yy} + 2\sigma_{xy} \delta E_{xy}) + 2\sigma_{xx} \delta E_{xx} + 2\sigma_{yy} \delta E_{yy} + m_{xx} \delta x_{xx} + m_{yy} \delta x_{yy} + m_{xy} \delta x_{xy} \right] \, dx \, dy
\]

After substituting and simplify result in Eq. (3.7):

\[
\delta U = \int_\Omega \left[ (E_1 \delta w_{xx} + E_2 \delta w_{yy} + E_3 \delta w_{xy} + E_4 \delta w_{yx}) + E_5 \delta w_{yx} + E_6 \delta \varphi_{xx} + E_7 \delta \varphi_{xy} + E_8 \delta \varphi_{yx} + E_9 \delta \varphi_{yy} + E_{10} \delta \varphi_{xy} + E_{11} \delta \varphi_{yx} + E_{12} \delta \varphi_{yy} + E_{13} \delta \varphi_{xy} + E_{14} \delta \varphi_{yx} + E_{15} \delta \varphi_{yy} \right] \, dx \, dy
\]

The coefficients of variables \(E_i\) are obtain In the appendix A.

\[
l_i = \int \frac{\partial^2 w}{\partial x^2} \, dx \quad (i = 0, 1, 2, n - 1, n, n + 1, 2n - 4, 2n - 2, 2n)
\]

4. Buckling load

For a rectangular plate with length \(a\), width \(b\) and thickness \(h\) with the forces \(P_x, P_y, P_{xy}\) and external force \(q(x, y)\) the buckling force equation can be written as [7, 8]:

\[
P_x \frac{\partial^2 w}{\partial x^2} + 2P_{xy} \frac{\partial^2 w}{\partial x \partial y} + P_y \frac{\partial^2 w}{\partial y^2} = q(x, y)
\]

5. Virtual work of the external forces

In this kind of problems, the virtual work of three kinds of external forces are included in the solutions, if the middle-plane and the middle-perimeter of the plate are shown as \(\Omega\) and \(\Gamma\) respectively, these virtual works are [9]:

1. The virtual work done by the body forces, which is applied on the volume \(V=\Omega \times (-h/2, h/2)\).

2. The virtual work done by the surface tractions at the upper and lower surfaces \(\Omega\).

3. The virtual work done by the shear tractions on the lateral surfaces, \(S=\Gamma \times (-h/2, h/2)\).

If \((f_x, f_y, f_z)\) are the body forces, \((c_x, c_y, c_z)\) are the body couples, \((q_x, q_y, q_z)\) are the forces acting on the \(\Omega\) plane, \((t_x, t_y, t_z)\) are the Cauchy’s tractions and \((S_x, S_y, S_z)\) are surface couples the Variation of the virtual work is expressed as:

\[
\delta w = \int \Omega \left[ (f_x \delta u + f_y \delta V + f_z \delta w + q_x \delta u + q_y \delta V + q_z \delta w + c_x \delta \theta_x + c_y \delta \theta_y + c_z \delta \theta_z) \, dx \, dy \right. \\
+ \left. \int_\Gamma (t_x \delta u + t_y \delta V + t_z \delta w + s_x \delta \theta_x + s_y \delta \theta_y + s_z \delta \theta_z) \, d\Gamma \right]
\]
Given that only external force $q$, is applied in this research, virtual work is as follows:

$$\delta W = \int_0^a \int_0^b q(x,y) \delta w(x,y) \, dx \, dy$$  \hspace{1cm} (5.2)

The kinetic energy variation is expressed as follows:

$$\delta T = \int_A \int_0^b \left( \rho (\dot{u}_1 \dot{u}_1 + \dot{u}_2 \dot{u}_2 + \dot{u}_3 \dot{u}_3) dA \right) \ight) \, dz$$  \hspace{1cm} (5.3)

Where $\rho$ is density. In this study, the equation of motion is derived by Hamilton’s principle. This principle can be expressed as [10]:

$$\int_0^T \left( \delta T - (\delta U - \delta W) \right) \, dt = 0$$  \hspace{1cm} (5.4)

In which $T$ is the kinetic energy, $U$ is the strain energy and $W$ is the work of external forces.

6. The final equation of the nanoplate by applying buckling and external force

By applying the Hamilton’s principle, the main equations are obtained as follows:

$$\left[ \int_{A/2}^{b/2} \left( \frac{\partial^2 E_1}{\partial x^2} + \frac{\partial E_2}{\partial y} + \frac{\partial^2 E_3}{\partial x \partial y} + \frac{\partial E_4}{\partial y} \right) \, dx \right] + P \left[ \frac{\partial^2 w}{\partial x^2} \right]$$

$$+ 2P \rho \left[ \frac{\partial^2 w}{\partial y^2} \right] + P \frac{\partial^2 w}{\partial y^2} = q(x,y) + \rho \alpha w_{xx} - C_w \left[ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right]_{tt}$$

$$+ C_w \rho \left[ \frac{\partial^2 \phi_x}{\partial x^2} + \frac{\partial^2 \phi_y}{\partial y^2} \right]_{tt}$$

$$\left[ \int_{A/2}^{b/2} \left( \frac{\partial^2 E_6}{\partial y^2} + \frac{\partial E_7}{\partial x} - \frac{\partial E_8}{\partial y} - \frac{\partial E_9}{\partial x} + F_{14} \right) \, dx \right] = \rho K_C \phi_{x,tt}$$

$$- C_w \rho \left[ \frac{\partial w}{\partial y} \right]_{tt}$$

$$\left[ \int_{A/2}^{b/2} \left( \frac{\partial^2 E_1}{\partial x^2} + \frac{\partial E_2}{\partial y} + \frac{\partial^2 E_3}{\partial x \partial y} + \frac{\partial E_4}{\partial y} \right) \, dx \right] = \rho K_C \phi_{y,tt} - C_w \rho \left[ \frac{\partial \phi_y}{\partial y} \right]_{tt}$$

$$J_4 = I_4 = C_w I_6$$

$$K_2 = I_2 - 2C_w I_4 - C_w^2 I_6$$  \hspace{1cm} (6.2)

7. Obtaining third order shear deformation nanoplate equations in the general state (including bending, buckling and vibrations)

The general equations of the third order shear deformation nanoplate will be obtained as follows:

$$D_1 \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w}{\partial x^4} + D_3 \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_4 \frac{\partial^4 w}{\partial y^2}$$

$$+ D_5 \frac{\partial^3 \phi_x}{\partial x^3} + D_6 \frac{\partial^3 \phi_x}{\partial x^2 \partial y} + D_7 \frac{\partial^3 \phi_y}{\partial y^3}$$

$$+ D_8 \frac{\partial^3 \phi_x}{\partial x \partial y^2} + D_9 \frac{\partial^3 \phi_y}{\partial y^2} \frac{\partial \phi_y}{\partial y} + D_{10} \frac{\partial^2 \phi_x}{\partial x^2 \partial y} + D_{11} \frac{\partial^2 \phi_y}{\partial y^2} \frac{\partial \phi_y}{\partial y}$$

$$+ D_{12} \frac{\partial^2 \phi_x}{\partial x \partial y} + D_{13} \frac{\partial^2 \phi_y}{\partial y}$$  \hspace{1cm} (7.1)

$$- D_4 \frac{\partial^2 w}{\partial x^2 \partial y^2} + D_5 \frac{\partial^2 \phi_x}{\partial x^2 \partial y} + D_6 \frac{\partial^2 \phi_x}{\partial x^2 \partial y} + D_7 \frac{\partial^2 \phi_y}{\partial y^2} + D_8 \frac{\partial^2 \phi_x}{\partial x \partial y^2} + D_9 \frac{\partial^2 \phi_y}{\partial y^2}$$

$$+ D_{10} \frac{\partial w}{\partial x^2} + D_{11} \frac{\partial w}{\partial y^2} - D_{12} \frac{\partial \phi_x}{\partial x} - D_{13} \frac{\partial \phi_y}{\partial y}$$

$$= \rho \frac{\partial^2 w}{\partial x \partial y^2}$$  \hspace{1cm} (7.2)

$$- D_4 \frac{\partial^2 w}{\partial x^2 \partial y^2} + D_5 \frac{\partial^2 \phi_x}{\partial x^2 \partial y} + D_6 \frac{\partial^2 \phi_x}{\partial x^2 \partial y} + D_7 \frac{\partial^2 \phi_y}{\partial y^2}$$

$$+ D_{10} \frac{\partial w}{\partial x^2} + D_{11} \frac{\partial w}{\partial y^2} - D_{12} \frac{\partial \phi_x}{\partial x} - D_{13} \frac{\partial \phi_y}{\partial y}$$

$$= \rho \frac{\partial^2 w}{\partial y^2}$$  \hspace{1cm} (7.3)

8. Navier Solution Method

The Navier solution method is applicable to rectangular plates with simply supported boundary conditions on all edges. The displacement functions of the middle surface can be expanded in the forms of double trigonometric series as follows [9, 11]:

$$W(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn} \sin \alpha x \sin \beta y e^{i \omega t}$$

$$\phi_x(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \phi_{mn} \cos \alpha x \sin \beta y e^{i \omega t}$$

$$\phi_y(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \phi_{mn} \sin \alpha x \cos \beta y e^{i \omega t}$$  \hspace{1cm} (8.1)

load can also be calculated from the following equation:

$$q = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Q_{mn} \sin \alpha x \sin \beta y e^{i \omega t}$$

$$Q_{mn} = \frac{4}{ab} \int_0^a \int_0^b q(x,y) \sin \alpha x \sin \beta y \, dx \, dy$$

Where $\alpha = \frac{mn}{a}$, $\beta = \frac{mn}{b}$, $i = \sqrt{-1}$
9. Obtaining the matrix of third order shear deformation nanoplate equations

After solving the equation using the Navier method the general matrix of third order shear deformation nanoplate equations will be obtained as follows:

\[
\begin{pmatrix}
R_1 & R_2 & R_3 \\
R_4 & R_5 & R_6 \\
R_7 & R_8 & R_9
\end{pmatrix} - \alpha^2 \begin{pmatrix}
G_1 & G_2 & G_3 \\
G_4 & G_5 & G_6 \\
G_7 & G_8 & G_9
\end{pmatrix} \begin{pmatrix}
W_{mn} \\
X_{mn} \\
Y_{mn}
\end{pmatrix} = \begin{pmatrix}
Q_{mn,0} \\
0 \\
0
\end{pmatrix}
\]  

(9.1)

The coefficients of variables \( R_i \) and \( G_i \) are obtained in the appendix C.

In this paper, graphene is considered for the material of the nanoplate. A single layer graphene sheet has the following properties [10]:

\[ E = 1.067 \times 10^6 \text{Pa}, \nu = 0.25, h = 0.34 \text{nm}, \rho = 2250 \text{kg/m}^3 \]

Also, the relationship between \( E, \mu \) and \( \nu \) can be written as:

\[ \lambda = \frac{\nu E}{(1 + \nu)(1 - 2
\nu)} \quad \mu = \frac{E}{2(1 + \nu)} \]

Where \( E \) is the Young modulus and \( \mu \) and \( \lambda \) are Lame coefficients [12]. Also, \( q = 1 \text{N/m}^2 \).

10. Results and discussion

The results are obtained using MATLAB software. All boundary conditions are also considered as simply supported. Table 1 compares the dimensionless static deflections of nanoplates subjected to a sinusoidal load. According to the table 1, the dimensionless static deflections of Kirchhoff nanoplate has the highest value and the Mindlin nanoplate has the lowest. Table 2 compares the dimensionless static deflections of the third order shear deformation nanoplate subjected to the uniform load for different value of length/width ratio. It is observed that except classical mode (\( l = 0 \)), by increasing the length scales parameter/thickness ratio, the dimensionless static deflections is decreased. Also by increasing the length/width ratio, it is increased.

Table 1. Comparison of dimensionless static deflections of nanoplates subjected to a sinusoidal load for different values of \( a/b \) (\( a/h = 30, q = 1 \times 10^8 \text{N/mm}^2, l/h = 1 \)).

| \( a/b \) | Kirchhoff plate | Mindlin plate | Third order shear deformation plate | N order shear deformation plate (\( n = 5 \)) |
|---|---|---|---|---|
| 1.0 | 0.2 | 0.07226 | 0.19912 | 0.19907 |
| 1.5 | 0.2 | 0.07212 | 0.19927 | 0.19923 |
| 2.0 | 0.2 | 0.07204 | 0.19935 | 0.19931 |

Table 2. Dimensionless static deflections of the third order shear deformation nanoplate subjected to the uniform load for different value of length/width ratio (\( q = 1 \times 10^8 \text{N/mm}^2, a/h = 30 \)).

| \( a/b \) | \( l/h \) |
|---|---|
| 0.0 | 0.5 | 1.0 | 2.0 |
| 1.0 | 1.00000 | 0.49874 | 0.19922 | 0.05856 |
| 1.5 | 1.00000 | 0.49911 | 0.19945 | 0.05864 |
| 2.0 | 1.00000 | 0.49923 | 0.19952 | 0.05866 |

Table 3 compares the static deflections of the third order shear deformation nanoplate subjected to the sinusoidal load for different values of length/width ratio. It is observed that by increasing the length scales parameter/thickness ratio, the static deflections is decreased. Also by increasing the length/width ratio, it is increased.

Table 3. Static deflections of the third order shear deformation nanoplate subjected to the sinusoidal load for different values of length/width ratio (\( q = 1 \times 10^8 \text{N/mm}^2, a/h = 30 \)).

| \( a/b \) | \( l/h \) |
|---|---|
| 1.0 | 0.070630 | 0.142905 | 21.1039 | 21.1039 |
| 1.5 | 0.52515 | 0.71284 | 10.5297 | 10.5297 |
| 2.0 | 1.4064 | 2.8477 | 4.2070 | 4.2070 |

Figure 2 shows the dimensionless critical buckling load of the third order shear deformation nanoplate for biaxial buckling and different value of length/thickness ratio. It is observed that by increasing the length scales parameter/thickness ratio the dimensionless critical buckling load is increased. Also except for the classical mode (\( l = 0 \)), by increasing the length/thickness ratio, it is decreased. Dimensionless critical buckling load for uniaxial buckling and different nanoplate dimensionless critical buckling load for uniaxial buckling and different nanoplate.

Figure 2. Comparison of the dimensionless critical buckling load of the third order shear deformation nanoplate for biaxial buckling and different values of length/thickness ratio (\( a/h = 1 \)).

Table 4 compares the dimensionless critical buckling load for uniaxial buckling and different nanoplate. According to Table 4:

- By increasing the length/thickness ratio of the Mindlin nanoplate, the dimensionless critical buckling load for uniaxial buckling is increased.
- By increasing the length/thickness ratio of the third and fifth order shear deformation nanoplates, the dimensionless critical buckling load for uniaxial buckling is slightly decreased.
- By increasing the length/thickness ratio of the Kirchhoff nanoplate, the dimensionless critical buckling load for uniaxial buckling is fixed.

Table 4. Comparison of the dimensionless critical buckling load for biaxial buckling and different nanoplate. According to Table 4:

- By increasing the length/thickness ratio of the Mindlin nanoplate, the dimensionless critical buckling load for uniaxial buckling is increased.
- By increasing the length/thickness ratio of the third and fifth order shear deformation nanoplates, the dimensionless critical buckling load for uniaxial buckling is slightly decreased.
- By increasing the length/thickness ratio of the Kirchhoff nanoplate, the dimensionless critical buckling load for uniaxial buckling is fixed.
Table 4. Dimensionless critical buckling load for uniaxial buckling and different nanoplates (a / b = 1, l / h = 1)

| a/h | Kirchhoff nanoplate | Mindlin nanoplate | Third order shear deformation nanoplate | N order shear deformation nanoplate |
|-----|-------------------|------------------|--------------------------------------|-----------------------------------|
| 05  | 5.0000            | 10.1594          | 5.6521                               | 5.6937                            |
| 10  | 5.0000            | 12.8100          | 5.1723                               | 5.1826                            |
| 20  | 5.0000            | 13.6820          | 5.0437                               | 5.0463                            |
| 30  | 5.0000            | 13.8568          | 5.0195                               | 5.0206                            |
| 40  | 5.0000            | 13.9191          | 5.0110                               | 5.0116                            |
| 50  | 5.0000            | 13.9481          | 5.0070                               | 5.0074                            |

Figure 3 shows the critical buckling load of the third order shear deformation nanoplate for uniaxial buckling and different value of length/thickness ratio. It is observed that by increasing the length scales parameter/thickness ratio the critical buckling load is increased. Also, by increasing the length/ thickness ratio, it is decreased.

Table 5. Comparison of the dimensionless critical buckling load of the third order shear deformation nanoplate for biaxial buckling and different buckling modes (a / b = 1, h = 0.34)

| Mode | l/h   | 0.0  | 0.5  | 1.0  | 2.0  |
|------|-------|------|------|------|------|
| P_{11}| 1.0000| 2.0050| 5.0195| 17.0765|
| P_{12}| 1.0000| 2.0125| 5.0486| 17.1908|
| P_{21}| 1.0000| 2.0125| 5.0486| 17.1908|
| P_{22}| 1.0000| 2.0199| 5.0774| 17.3044|

Figures 4 to 7 show the dimensionless frequencies \( \omega_{11} / \omega_{ct}, \omega_{12} / \omega_{ct}, \omega_{21} / \omega_{ct}, \omega_{22} / \omega_{ct} \) of the third order shear deformation nanoplate for different values of length/thickness ratio. It is observed that by increasing the length scales parameter/thickness ratio, the dimensionless frequencies are increased. Also except for classical mode (l = 0) by increasing the length/ thickness ratio, it is decreased. As well as for first mode, it is minimum.
Table 6. Comparison of the frequencies of the third order shear deformation nanoplate for different value of length scales parameter/thickness ratio (THz) (a / b = 1, a / h = 30)

| Mode | l/h | 0    | 0.5  | 1    | 2    |
|------|-----|------|------|------|------|
| \(\omega_{11}\) | 13.9441 | 19.7447 | 31.2407 | 57.6223 |
| \(\omega_{12}\) | 34.6497 | 49.1546 | 77.8533 | 143.6613 |
| \(\omega_{21}\) | 34.6497 | 49.1546 | 77.8533 | 143.6613 |
| \(\omega_{22}\) | 55.1098 | 78.3225 | 124.1752 | 229.2384 |
| \(\omega_{33}\) | 121.6342 | 173.8911 | 276.5826 | 511.3107 |

Table 7. Comparison of frequencies for the different nanoplates (a / b = 0.5, l / h = 1)

| Mode | a/h | 20     | 30     | 40     |
|------|-----|--------|--------|--------|
| Mindlin plate | \(\omega_{11}\) | 280.4153 | 128.0217 | 72.7219 |
| Kirchhoff plate | \(\omega_{11}\) | 279.4825 | 124.7977 | 70.2985 |
| | \(\omega_{12}\) | 588.5686 | 264.0744 | 149.0415 |
| | \(\omega_{22}\) | 690.3772 | 310.2573 | 175.2090 |
| Third order shear deformation plate | \(\omega_{11}\) | 174.0385 | 77.8533 | 43.8941 |
| | \(\omega_{12}\) | 276.5826 | 124.1752 | 70.1049 |
| | \(\omega_{21}\) | 576.6542 | 261.4753 | 148.1887 |
| | \(\omega_{22}\) | 674.3836 | 306.7113 | 174.0385 |

Figure 7. Comparison of the dimensionless frequencies (\(\omega_{22}\)) of the third order shear deformation nanoplate for different values of length/thickness ratio (a / b = 1, h = 0.34)

Table 6 shows that by increasing the length scales parameter/thickness ratio, the frequencies of different modes (\(\omega_{11}, \omega_{12}, \omega_{21}, \omega_{22}\)) are increased. Table 7 shows the frequency of different modes (\(\omega_{11}, \omega_{12}, \omega_{21}, \omega_{22}\)) for different nanoplates. According to the table, the frequency of Mindlin nanoplate is maximum and for third order shear deformation nanoplate is minimum. The results of this paper have been verified by being compared to references [13-18] and good agreement is attained between the results.

11. Conclusion

In this paper, bending, buckling and vibrations of the third order shear deformation nanoplate were studied. As shown in tables and figures, by increasing the length scales parameter/thickness ratio, the dimensionless static deflection of nanoplate subjected to a sinusoidal load is decreased. It is also increased by increasing the length/width ratio. By increasing the length scales parameter/thickness ratio the dimensionless critical buckling load for biaxial buckling is increased. It’s also observed that this value decreases by increasing the length/thickness ratio, except for the classical mode. As discussed before, by increasing the length scales parameter/thickness ratio, the dimensionless frequencies are increased, except for the classical mode (l = 0), which is decreased by increasing the length/ thickness ratio. And the minimum value of this parameter is obtained for the first mode.

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APPENDIX A

\[ E_1 = \frac{\partial^2 w}{\partial x^2} \left[ \lambda (C_2 - C_1) + \frac{\partial^2 l^2}{\partial x^2} (1 + C_2) - \frac{\partial^2 l^2}{\partial y^2} (1 + C_1) \right] + \frac{\partial^2 w}{\partial y^2} \left[ \lambda (C_1 - C_2) - \frac{\partial^2 l^2}{\partial x^2} (1 + C_1) \right] + \frac{\partial^2 l^2}{\partial x \partial y} \left[ (\lambda + 2\mu)(C_1 - C_2) - \frac{\partial^2 l^2}{\partial x^2} (1 + C_1) \right] + \frac{\partial^2 l^2}{\partial y^2} \left[ -2\mu C_2 C_1 - \frac{\partial^2 l^2}{\partial y^2} (1 + C_2) \right] \right] \] (A-1)

\[ E_2 = \frac{\partial^2 w}{\partial y^2} \left[ (\lambda + 2\mu)(C_1 - C_2) + \frac{\partial^2 l^2}{\partial y^2} (1 + C_2) \right] + \frac{\partial^2 w}{\partial x^2} \left[ -2\mu C_2 C_1 - \frac{\partial^2 l^2}{\partial x^2} (1 + C_1) \right] \] (A-2)

\[ E_3 = \frac{\partial^2 w}{\partial x^2} \left[ 4\mu C_2^2 + \mu l^2 (1 + C_2) \right] + \frac{\partial^2 w}{\partial y^2} \left[ -2\mu C_2 C_1 - \frac{\partial^2 l^2}{\partial y^2} (1 + C_2) \right] \] (A-3)

\[ E_4 = \left( \frac{\partial w}{\partial x} + \phi_x \right) \left[ \mu C_2^2 + \frac{\partial^2 l^2}{\partial y^2} \right] \] (A-4)

\[ E_5 = \left( \frac{\partial w}{\partial y} + \phi_y \right) \left[ \mu C_2^2 + \frac{\partial^2 l^2}{\partial x^2} \right] \] (A-5)

\[ E_6 = \left( \frac{\partial w}{\partial x} + \phi_x \right) \left[ \frac{\partial^2 l^2}{\partial y^2} \right] \] (A-6)

\[ E_7 = \left( \frac{\partial w}{\partial y} + \phi_y \right) \left[ \frac{\partial^2 l^2}{\partial x^2} \right] \] (A-7)

\[ E_8 = \left( \frac{\partial w}{\partial x} + \phi_x \right) \left[ \frac{\partial^2 l^2}{\partial x^2} \right] \] (A-8)

\[ E_9 = \left( \frac{\partial w}{\partial y} + \phi_y \right) \left[ \frac{\partial^2 l^2}{\partial y^2} \right] \] (A-9)

\[ E_{10} = \frac{\partial^2 w}{\partial x^2} \left[ (\lambda + 2\mu)(C_1^2 - C_2^2) - \frac{\partial^2 l^2}{\partial y^2} (1 + C_2) \right] \] (A-10)

\[ E_{11} = \frac{\partial^2 w}{\partial y^2} \left[ (\lambda + 2\mu)(C_1^2 - C_2^2) - \frac{\partial^2 l^2}{\partial x^2} (1 + C_2) \right] \] (A-11)

\[ E_{12} = \frac{\partial^2 w}{\partial x^2} \left[ -2\mu C_2 C_1 - \frac{\partial^2 l^2}{\partial y^2} (1 + C_2) \right] \] (A-12)

\[ E_{13} = \frac{\partial^2 w}{\partial y^2} \left[ 2\mu C_2 C_1 - \frac{\partial^2 l^2}{\partial x^2} (1 + C_2) \right] \] (A-13)

\[ E_{14} = \frac{\partial w}{\partial x} \left[ \mu (1 - C_2^2) + \frac{\partial^2 l^2}{\partial y^2} \right] \] (A-14)

\[ E_{15} = \frac{\partial w}{\partial y} \left[ \mu (1 - C_2^2) + \frac{\partial^2 l^2}{\partial x^2} \right] \] (A-15)
Where:

\[ C_1 = x - \frac{4}{3} \left( \frac{1}{h} \right)^2 x^3 \]  
(A-14)

\[ C_2 = \frac{4}{3} \left( \frac{1}{h} \right)^2 x^3 \]  
(A-15)

\[ C_3 = \frac{4}{3} \left( \frac{1}{h} \right)^2 x^4 \]  
(A-16)

\[ C_4 = 4 \left( \frac{1}{h} \right)^2 \]  
(A-17)

\[ C_5 = -8x \left( \frac{1}{h} \right)^2 \]  
(A-18)

\[ C_6 = \frac{4}{3} \left( \frac{1}{h} \right)^2 \]  
(A-19)

\[ C_7 = \mu h \]  
(A-20)

\[ C_8 = \frac{h^2}{2} \]  
(A-21)

\[ C_9 = \frac{h}{2\sqrt{2}} (\lambda + 2\mu) \]  
(A-22)

\[ C_{10} = (\lambda + 2\mu) \frac{h^3}{6} \]  
(A-23)

\[ C_{11} = \mu t^2 \frac{4}{3h} \]  
(A-24)

\[ C_{12} = \frac{1}{\lambda} t^2 \frac{h}{h} \]  
(A-25)

**APPENDIX B**

\[ D_1 = 2C_{12} + t^2C_7 + \frac{3}{2} C_8 + 2C_9 \]  
(B-1)

\[ D_2 = \frac{1}{2}D_1 = C_{12} + C_9 + \frac{3}{2}C_7 + \frac{3}{2}C_8 \]  
(B-2)

\[ D_3 = -\mu t^2 + 2C_7 - C_9 - C_{11} \]  
(B-3)

\[ D_4 = C_9 - C_{10} + \frac{3}{2}C_8 - C_{12} \]  
(B-4)

\[ D_5 = 3C_{12} - \frac{3}{2}t^2C_7 + \frac{3}{2}C_8 - (\lambda + \mu)I_2 + 2(\lambda + \mu)C_4 I_4 \]  
(B-5)

\[ - (\lambda + \mu)C_6 I_6 \]  
(B-5)

\[ D_6 = -\mu t^2 + 2\mu tC_4 I_4 - \mu C_6 I_6 - 4C_{12} + 2t^2C_7 - t^2C_8 \]  
(B-6)

\[ D_7 = \frac{1}{2}t^2 J_2 - \frac{1}{2}tC_6 I_6 + \frac{1}{2} t^2 C_6 I_6 \]  
(B-7)

\[ D_8 = -\lambda + 2\mu I_2 + 2C_{10} - C_9 - C_{12} + \frac{1}{2}t^2C_7 - \frac{1}{2}t^2C_8 \]  
(B-8)

\[ D_9 = \frac{1}{4} \frac{t^2 C_8}{2} - \frac{3}{2} t^2 C_7 - \frac{3}{2} C_{12} - (\lambda + \mu)I_2 \]  
(B-9)

\[ - (\lambda + \mu)C_6 I_6 + 2(\lambda + \mu)C_4 I_4 \]  
(B-10)

\[ D_{14} = 3t^2 C_7 - \frac{3}{2} t^2 C_8 + \frac{3}{2} t^2 C_7 I_4 - \mu I_2 - \mu C_6 I_6 + 2\mu C_4 I_4 \]  
(B-10)

\[ D_{14} = 4C_{12} \]  
(B-11)

\[ D_{12} = \mu C_6 I_6 - \rho C_6 I_6 \]  
(B-12)

\[ D_{13} = \rho t^2 - 2\rho C_4 I_4 - \rho C_6 I_6 \]  
(B-13)

**APPENDIX C**

\[ R_1 = D_1 t^2 \beta^2 + D_2 t^4 + D_3 t^2 - D_3 t^2 - D_3 \beta^2 - P_3 \alpha^2 \]  
(C-1)

\[ R_2 = R_3 = D_3 t^2 \alpha^2 + D_4 t^2 \beta - D_3 \beta \]  
(C-2)

\[ R_3 = R_4 = D_2 t^2 \alpha^2 + D_4 t^2 \beta - D_3 \beta \]  
(C-3)

\[ R_5 = -D_3 t^2 - D_4 t^4 - D_4 \beta^2 - D_5 \alpha^2 - D_5 \beta^2 - D_3 \]  
(C-4)

\[ R_6 = D_4 t^2 \beta^2 + D_4 t^2 \beta - D_6 \alpha \]  
(C-5)

\[ R_7 = -D_3 t^2 \beta - D_4 t^2 \beta - D_6 \alpha \]  
(C-6)

\[ R_8 = D_6 t^2 \alpha^2 + D_4 t^2 \beta - D_6 t^2 \beta - D_6 \beta^2 - D_6 \]  
(C-7)

\[ G_1 = -D_1 \alpha^2 - D_1 \beta^2 - \rho \]  
(C-8)

\[ G_2 = G_3 = D_4 t^2 \beta \]  
(C-9)

\[ G_4 = G_5 = D_4 t^2 \beta \]  
(C-10)

\[ G_6 = G_8 = -D_1 \beta^2 \]  
(C-11)

\[ G_6 = G_8 = 0 \]  
(C-12)