Internal Pattern Matching Queries in a Text and Applications*

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Abstract

We consider several types of internal queries: questions about subwords of a text. As the main tool we develop an optimal data structure for the problem called here internal pattern matching. This data structure provides constant-time answers to queries about occurrences of one subword $x$ in another subword $y$ of a given text, assuming that $|y| = O(|x|)$, which allows for a constant-space representation of all occurrences. This problem can be viewed as a natural extension of the well-studied pattern matching problem. The data structure has linear size and admits a linear-time construction algorithm.

Using the solution to the internal pattern matching problem, we obtain very efficient data structures answering queries about: primitivity of subwords, periods of subwords, general sub-string compression, and cyclic equivalence of two subwords. All these results improve upon the best previously known counterparts. The linear construction time of our data structure also allows to improve the algorithm for finding $\delta$-subrepetitions in a text (a more general version of maximal repetitions, also called runs). For any fixed $\delta$ we obtain the first linear-time algorithm, which matches the linear time complexity of the algorithm computing runs. Our data structure has already been used as a part of the efficient solutions for subword suffix rank & selection, as well as substring compression using Burrows-Wheeler transform composed with run-length encoding.

The model of internal queries in texts is connected to the well-studied problem of text indexing. Both models have their origins in the introduction of suffix trees. However, there is an important difference: in our model the size of the representation of a query is constant and therefore enables faster query time. Our results can be viewed as efficient solutions to "internal" equivalents of several basic problems of regular pattern matching and make an improvement in a majority of already published results related to internal queries.

1 Introduction

There are many algorithmic problems concerning subwords (factors, substrings) of a word. In these problems we need to construct a data structure which answers efficiently internal queries, that is, queries specified by subwords of a given word. This constitutes a growing field in the area of

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text processing. Its origins start with the invention of suffix trees that can be used to answer the most basic types of internal queries: equality of subwords and longest common prefix queries, with constant query time and linear space. One of the first studies in this area, on a family of problems of compressibility of subwords, was given by Cormode and Muthukrishnan (SODA'05) [5]; some of these results were later improved in [17]. Other typical problems include: range longest common prefix queries (range LCP) [1, 26], periodicity [18, 9], minimal/maximal suffixes [4, 2] and suffix rank & selection [3].

A related model is text indexing, in which one desires to preprocess a given word for future queries specified by (usually shorter) patterns. In this setting the query time is \( \Omega(m) \), where \( m \) is the size of the query pattern. Our model better fits the scenario when a number of data texts are stored and we only query for subwords of these texts. Indeed, each query can now be specified in constant space and therefore \( o(m) \)-time algorithms answering queries are possible. There is also a study on a related area of cross-document pattern matching, in which we are to index a collection of documents and answer queries for occurrences of a subword of one document in another document [21].

A routine approach to subword-related queries is based on orthogonal range searching [22]. For the current state of knowledge this implies \( \Omega(\log \log n) \) query time and, in most cases, super-linear space. The construction time is \( \Omega(n\sqrt{\log n}) \), most often \( \Omega(n \log n) \). We design tools based on text processing that are better-tailored for subword-related queries and allow to obtain constant query time with linear space and (expected) linear construction time. We benefit from the efficient construction algorithm applying our techniques to problems in an offline setting.

We identify one of the basic problems in this new area, which we call internal pattern matching, and show its usefulness. This problem can be viewed as a direct analogue of the well-studied pattern matching problem. We apply the solution for internal pattern matching problem to find more efficient solutions for generalized substring compression queries and periodicity queries, as well as also for cyclic equivalence (also called conjugacy; see [23]) queries which we introduce in the context of subwords. We use our data structure in a static setting to obtain an efficient algorithm for computing \( \delta \)-subrepetitions which are a generalization of maximal repetitions in words. Moreover, in [3] our solution is combined with some more ideas to obtain efficient solutions for further internal queries.

We consider linearly sortable alphabets, that is, we assume that \( \Sigma \), the set of letters of the input word \( v \), can be sorted in linear time (e.g., \( \Sigma \subseteq \{0, 1, \ldots, |v|^{O(1)}\} \)). A subword of \( v \) is a word of the form \( v[i] \ldots v[j] \). The subwords in each query are represented by a start- and end-position of their occurrence. The results are for word-RAM model with word size \( w = \Omega(\log n) \), with \( n \) being the length of \( v \), and the algorithms are deterministic, unless explicitly stated otherwise.

### 1.1 Previous Work

We say that a positive integer \( q \) is a period of \( v \) if there exists a word \( u \) of length \( q \) such that \( v \) is a prefix of \( u^k \) for a sufficiently large integer \( k \). A word is called periodic if it has a period being at most half of its length. The following three types of internal queries were already studied.

| Period Queries |
|----------------|
| Given a subword \( x \) of \( v \), report all periods of \( x \) (represented by disjoint arithmetic progressions). |

| 2-Period Queries |
|------------------|
| Given a subword \( x \) of \( v \), decide whether \( x \) is periodic and, if so, compute its shortest period. |
**Bounded Longest Common Prefix Queries**

Given two subwords $x$ and $y$ of $v$, find the longest prefix $p$ of $x$ which is a subword of $y$.

Known efficient algorithms for these types of queries apply orthogonal range searching queries in 2-dimensional rank space (i.e., coordinates of points are in the range $[1, n]$, where $n$ is the number of points, and query rectangles are orthogonal).

The **Period Queries** problem was introduced and first studied in [18]. The solutions achieve $O(\log n)$ query time with $O(n \log n)$ space, and $O(Q_{rsucc} \log n)$ query time with $O(n + S_{rsucc})$ space. Here $Q_{rsucc}$ and $S_{rsucc}$ are the query time and space of a data structure for range successor queries; currently the best trade-offs are: $Q_{rsucc} = O(\log^c n)$ for $S_{rsucc} = O(n)$ [25], $Q_{rsucc} = O(\log \log n)$ for $S_{rsucc} = O(n \log \log n)$ [29], and $Q_{rsucc} = O(1)$ for $S_{rsucc} = O(n^{1+\epsilon})$ [10].

**Period Queries**, in spite of their very recent introduction, have already found applications for further internal queries [4] and in computing all subrepetitions in a word, a concept extending the notion of a run in a word [19].

The 2-Period Queries are a special case of the Period Queries briefly introduced in [9]. The best previously known solution employs the techniques from general Period Queries presented in [18] and achieves: $O(Q_{rsucc})$ query time with $O(n + S_{rsucc})$ space, and $O(1)$ query time with $O(n \log n)$ space.

**Bounded Longest Common Prefix Queries** were introduced in [17] as a tool for the following **Generalized Substring Compression Queries**: given two subwords $x$ and $y$ of $v$, compute the part of the LZ77 [30] compression $LZ(y \$ x)$ that corresponds to $x$, where $\$ \notin \Sigma$. This problem was introduced in [5] and was also referred to as substring compression with an additional context substring. In [17] a solution to Bounded Longest Common Prefix Queries with $O(Q_{rsucc} \log |p|)$ query time and $O(n + S_{rsucc})$ space implied an $O(CQ_{rsucc} \log \frac{|x|}{\epsilon})$-time algorithm for Generalized Substring Compression Queries, where $C$ is the number of phrases in the output.

As a by-product of [17], a solution to the following decision version of internal pattern matching problem is obtained: a data structure of size $O(n + S_{rsucc})$ that given subwords $x, y$ checks whether $x$ occurs in $y$ in $O(Q_{rsucc})$ time, provided that $x$ is given by its locus in the suffix tree. All occurrences of $x$ in $y$ can be reported in additional time proportional to the number of these occurrences.

### 1.2 Our Results

We introduce queries that find all occurrences of one subword of $v$ in another subword.

**Internal Pattern Matching (IPM) Queries**

Given subwords $x$ and $y$ of $v$ with $|y| \leq 2|x|$, report all occurrences of $x$ in $y$ (represented as an arithmetic progression).

Our main result is the following theorem together with its corollaries.

**Theorem 1.1.** IPM Queries can be answered in $O(1)$ time by a data structure of size $O(n)$, constructed in $O(n)$ expected time.

**Remark 1.2.** The requirement $|y| \leq 2|x|$ can be dropped at the cost of increasing the query time to $O(|y|/|x|)$ and allowing several arithmetic progressions on the output.

A number of applications of IPM Queries are presented. We introduce two additional types of queries.
which are assigned to all logarithmically many layers of our data structure. In this, we explicitly assign samples only to $k$ is used where they extend to an occurrence of $v$. This leads to a linear-time algorithm finding $\delta$-subrepetitions for a constant $\delta$, which matches an $O(n)$-time algorithm for computing runs [20]. Another application of IPM QUERIES is the following.

**Corollary 1.3.** Using a data structure of $O(n)$ size, which can be constructed in $O(n)$ expected time, one can answer: Prefix-Suffix Queries in $O(1)$ time, 2-Period Queries in $O(1)$ time, Period Queries in $O(\log |x|)$ time, Cyclic Equivalence Queries in $O(1)$ time.

A $\delta$-subrepetition is a generalization of the notion of a run for exponent at least $1 + \delta$, with $\delta \leq 1$. Thus, runs can be seen as 1-subrepetitions. The best previously known algorithm for finding $\delta$-subrepetitions [19] applied Period Queries to achieve $O(n \log n + \frac{n}{\delta} \log \frac{1}{\delta})$ time. With our data structure it improves to $O(n + \frac{n}{\delta} \log \frac{1}{\delta})$. In particular, we obtain the first linear-time algorithm finding $\delta$-subrepetitions for a constant $\delta$, which matches an $O(n)$-time algorithm for computing runs [20].

**Corollary 1.4.** Using a data structure of $O(n + S_{\text{succ}})$ size one can answer Bounded Longest Common Prefix Queries in $O(CQ_{\text{succ}} \log \log |p|)$ time.

The data structure of Corollary 1.4 yields a solution to Generalized Substring Compression Queries with $O(CQ_{\text{succ}} \log \log \frac{|x|}{C})$ query time, as compared to $O(CQ_{\text{succ}} \log \frac{|x|}{C})$-time queries of [17]. Here $C$ is the number of phrases reported.

Wavelet suffix trees, recently introduced by Babenko et al. [3], apply our results for IPM QUERIES to answer subword suffix rank and selection queries. They achieve $O(n)$ space, $O(\log n)$ query time, and $O(n \sqrt{\log n})$ expected construction time. Wavelet suffix trees also support substring compression queries for Burrows-Wheeler transform combined with run-length encoding. These queries take $O(C \log |x|)$ time, where $C$ is the output size and $x$ is the queried subword.

Details on the applications of our data structure can be found in Section 8.

### 1.3 Overview of Data Structure for IPM QUERIES

We use in a novel way an approach to pattern matching by sampling. To search for occurrences of $x$ in $y$, both of which are subwords of $v$, we assign to $x$ a sample $z$, which is a subword of $x$. Then we find those occurrences of $z$ in $y$ which are samples of some subwords of $v$, and finally check which of them extend to an occurrence of $x$ in $y$.

Our goal is to choose the samples assignment so that it can be stored in a small space. For this, we explicitly assign samples only to basic factors, that is, subwords of length $2^k$. This leads to logarithmically many layers of our data structure. In $k$-th layer we use, as samples, $k$-basic factors, which are assigned to all $(k+1)$-basic factors. When we search for a pattern $x$, then the $k$-th layer is used where $k = \lfloor \log |x| - 1 \rfloor$.

This high-level idea is similar to the Locally Consistent Parsing as used by Sahinalp & Vishkin for approximate pattern matching [28]. That technique, however, involves samples whose length might be away by a polynomial factor from the length of the represented fragment, which would
not lead to efficient queries in our setting. Thus, we develop a different approach relying on the basic factors, which gives much more structured samples. As in [28], we have different algorithms for periodic and non-periodic samples.

Non-periodic Queries. In the non-periodic case $y$ may contain only a constant number of occurrences of the sample. Each occurrence of $x$ in $y$ induces an occurrence of the sample $z$. We consider potential locations of $x$ corresponding to all occurrences of $z$ as the sample of some basic factor of $y$. Each of them is verified in constant time using standard techniques ($lcp$-queries).

We use a probabilistic argument to make sure that two adjacent $(k+1)$-basic factors are often assigned the same sample. The expected number of different samples of basic factors of $v$ is $O(n)$. This way not only the samples assignment admits a compact representation, but also we can locate relevant occurrences of samples using a hash table.

This part of the data structure is described in Section 3 under a simplifying additional assumption that $v$ is square-free. In Section 5 this assumption is waived and it is shown how this part of the data structure combines with its periodic counterpart. This is simply a matter of identifying which approach should be used to answer the query and adjusting the complexity analysis.

Periodic Queries. In the periodic case we rely on the well-studied theory of repetitions in words. We extend the sample to a run (a maximal repetition) and apply the structure of all runs in $v$ to efficiently find all runs in $y$ that are compatible with the run of the sample.

In a query we use bit vectors storing all periodic basic factors to find a sample within $x$. We develop components to compute the extension to a run of any periodic factor and to find all runs of the same period in $y$. In order to obtain only a constant number of such runs, we need to restrict to long enough runs. This is achieved by introducing a notion of $k$-runs. The space bound of these components relies on an involved combinatorial property of runs, that the sum of their exponents in a word is linear.

The next step, verifying compatibility of runs, uses their Lyndon representations. Such a representation indicates a Lyndon word which corresponds to the period of the run, that is, the lexicographically minimal word which is cyclically equivalent to the period. These representations are also used to find occurrences of $x$ in the runs of $y$, represented by an arithmetic progression.

The periodic part of the data structure is shown in Section 4, with details in Section 6.

Construction. The construction algorithm for the periodic part follows from Section 6. The main challenge related to the non-periodic part construction, the computation of samples assignments, is tackled in Section 7.

An $O(n \log n)$-time and space construction algorithm using the Dictionary of Basic Factors (DBF) would not be difficult to obtain. For a linear-time construction we design a space-efficient version of DBF. We use a simple approach to obtain a set of basic factors likely to be samples. Then we grow this set to make sure it contains all samples and show that in expectation its total size is still linear. Finally we filter out the excessive candidates.

2 Preliminaries

Consider a word $v = v[1]v[2] \ldots v[n]$ of length $|v| = n$, where $v[i] \in \Sigma$. For $1 \leq i \leq j \leq n$, a word $u = v[i] \ldots v[j]$ is called a subword of $v$. By $v[i, j]$ we denote the occurrence of $u$ at position $i$, called
a fragment of $v$. Throughout the paper by $[i, j]$ we denote an integer interval \{i, \ldots, j\}.

The following fact specifies a known efficient data structure for comparison of subwords of a word. It consists of the suffix array with its inverse, LCP table and a data structure for range minimum queries on the LCP table, see [6].

**Fact 2.1 (Equality Testing).** Let $v$ be a word of length $n$. After $O(n)$-time preprocessing, one can find the longest common prefix of two fragments in $O(1)$ time and, in particular, test if these fragments are occurrences of the same subword.

A fragment of $v$ of the form $BF_k(i) = v[i, i+2^k−1]$ is called a $k$-basic fragment. By $n_k = n−2^k+1$ we denote the number of $k$-basic fragments of $v$. A word that occurs as a $k$-basic fragment of $v$ is called a $k$-basic factor of $v$. By $m_k$ we denote the number of different $k$-basic factors of $v$ (note that $m_k \leq n_k$).

The dictionary of basic factors (DBF in short; see [6]), consists of $\lceil \log n \rceil$ layers. The $k$-th layer is a table $DBF_k$ such that $DBF_k[i]$ is an identifier of $BF_k(i)$. The identifiers are consecutive positive integers in $[1, m_k]$ such that $DBF_k[i] \leq DBF_k[i']$ if and only if $BF_k(i) \leq BF_k(i')$.

We say that a positive integer $p$ is a period of $v$ if $v[i] = v[i + p]$ holds for all $i \in [1, n − p]$. The shortest period of $v$ is denoted as per($v$). We call $v$ periodic if $2 \times \text{per}(v) \leq |v|$, and primitive if $\text{per}(v)$ is not a proper divisor of $|v|$.

The following folklore lemma shows that the output of IPM QUERIES indeed fits $O(1)$ space.

**Lemma 2.2 ([27]).** Let $x$, $y$ be words satisfying $|y| \leq 2|x|$. Then the set of positions where $x$ occurs in $y$ forms a single arithmetic progression. Moreover, if there are at least 3 occurrences, the difference of this progression is $\text{per}(x)$.

## 3 Non-Periodic Case

Recall that a word $u$ is called a square if $u = w w$ for a word $w$, and a word is square-free if it does not contain any squares as subwords; see also [23]. In this section, in order to avoid technical details, we assume that $v$ is square-free. This assumption is dropped in Section 5, where we combine the results of this section with the solution to the periodic case given in Section 4. A justification for considering the square-free case is given in the following Fact 3.1; in the solution for general case (Section 5) a similar observation is provided in the non-periodic setting.

A set $X \subseteq \mathbb{N}$ is called $\Delta$-sparse if for any distinct elements $a, b \in X$ it holds that $|a − b| > \Delta$.

**Fact 3.1 (Sparsity of Occurrences).** Let $u$ and $v$ be words and assume $v$ is square-free. Then the set of positions where $u$ occurs in $v$ is $|u|$-sparse.

**Proof.** Occurrences of $u$ at positions $i, j$ such that $i < j \leq |u|$ imply $v[i, j−1] = v[j, 2j−i−1]$, that is $v[i, 2j−i−1] = v[i, j−1]^2$ being a square. \hfill $\square$

By Fact 3.1, IPM QUERIES for a square-free word $v$ return at most one occurrence.

Definition 3.2 introduces the crucial notion of $k$-samples assignment, which is our realization of the idea of pattern matching by sampling.

**Definition 3.2.** Let $v$ be a square-free word. We call sample$_k$ a $k$-samples assignment for $v$, if the following conditions are satisfied:
1. \( \text{sample}_k : [1, n_{k+1}] \rightarrow [1, n_k] \);
2. \( \text{sample}_k(i) \in [i, i+2^k] \) for each \( i \);
3. \( \text{sample}_k(i) = i = \text{sample}_k(i') - i' \) if \( BF_{k+1}(i) = BF_{k+1}(i') \).

The values of \( k \)-samples assignment are called \( k \)-sample positions or sample occurrences of the corresponding \( k \)-basic factors. The set of \( k \)-sample positions is denoted as \( \text{SAMPLES}_k \). We say that a \( k \)-basic fragment is a sample if it starts at a \( k \)-sample position.

Intuitively speaking, \( \text{sample}_k(i) \) assigns to the \((k+1)\)-basic fragment \( BF_{k+1}(i) \) a starting position of a sample which is its \( k \)-basic subfragment. If two \((k+1)\)-basic fragments are occurrences of the same basic factor, their samples start at the corresponding positions. For example, the collection of trivial functions \( \text{sample}_k(i) = i \) is a valid samples assignment. In this case, \( \sum_k |\text{SAMPLES}_k| = \Theta(n \log n) \). However, we need to achieve \( \sum_k |\text{SAMPLES}_k| = \mathcal{O}(n) \) in order to obtain an \( \mathcal{O}(n) \)-size representation of the samples assignment.

### 3.1 Choice of Samples Assignment

Let \( S_m \) be the set of permutations of \([1, m]\). For \( \pi_k \in S_{mk} \) we set

\[
\text{sample}_k(i) = \arg \min \{ \text{ID}_k[j] : j \in [i, i+2^k] \}
\]

where \( \text{ID}_k[j] = \pi_k(DBF_k[j]) \). Below, we prove that this definition satisfies the conditions for a samples assignment; see Figure 1. Moreover, we show that for an appropriate choice of permutations \( \pi_k \), the total size of sets \( \text{SAMPLES}_k \) is \( \mathcal{O}(n) \).

For a function \( g \) defined on an interval \([\ell, r]\), we define the step representation of \( g \) as a collection of pairs \((\ell_i, v_i)\) for \( i = 1, \ldots, q \) such that \( g(x) = v_i \) for \( x \in [\ell_i, \ell_{i+1} - 1] \), with \( \ell_{q+1} = r + 1 \); see Figure 1 for an example. The size of the representation is \( q \), the number of pairs.

\[
\begin{align*}
\text{sample}_3 &: (1, 2), (3, 6) \\
\text{sample}_2 &: (1, 2), (3, 6), (7, 10), (10, 14) \\
\text{sample}_1 &: (1, 2), (3, 4), (4, 6), (7, 9), (8, 10), (11, 12), (12, 14), (15, 17), (16, 18) \\
\text{sample}_0 &: (1, 2), (3, 4), (5, 6), (7, 7), (8, 9), (9, 10), (11, 12), (13, 14), (15, 15), (16, 17), (17, 18), (19, 20)
\end{align*}
\]

\[
\begin{array}{ccccccccccccccccccccccc}
| \text{sample}_3 | & 2 & 2 & 6 & 6 & 6 & 6 & 10 & 10 & 10 & 14 & 14 & 14 & 14 \\
| \text{sample}_2 | & 2 & 2 & 6 & 6 & 6 & 6 & 10 & 10 & 10 & 14 & 14 & 14 & 14 \\
| \text{sample}_1 | & 2 & 2 & 4 & 6 & 6 & 6 & 9 & 10 & 10 & 10 & 14 & 14 & 14 & 17 & 18 & 18 \\
| \text{sample}_0 | & 2 & 2 & 4 & 4 & 6 & 6 & 7 & 9 & 10 & 10 & 12 & 12 & 14 & 14 & 15 & 17 & 18 & 18 & 20 \\
\end{array}
\]

\[
\begin{array}{ccccccccccccccccccccccc}
| \text{word} | & c & a & b & a & c & a & b & c & b & a & c & b & c & a & c & a \\
| \text{pos} | & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
\end{array}
\]

Figure 1: The table presents the full samples assignment for a word \( v = \text{cabacbcabcbacbcaba} \) and \( \pi_k = \text{id} \). The latter means that \( \text{sample}_k \), for each \((k+1)\)-basic fragment, points to its lexicographically smallest \( k \)-basic subfragment. Above the table, there is a compact representation of its contents, i.e., functions \( \text{sample}_k \) are given in a step representation: collection of the sequences of pairs of the type: (argument, value).
Lemma 3.3. Let \( v \) be a square-free word. There exist permutations \( \pi_k \in S_{m_k} \) such that \(\text{sample}_k\) given by (1) form a family of samples assignments for \( v \) and admit step representations of total size \( \mathcal{O}(n) \).

Proof. First, let us prove that \(\text{sample}_k\) given by (1) is a valid samples assignment with respect to Definition 3.2. Clearly properties 1 and 2 are satisfied. For a proof of 3 it suffices to note that \( BF_{k+1}(i) = BF_{k+1}(i') \) implies that for any \( \delta \in [0, 2^k] \) we have \( BF_k(i+\delta) = BF_k(i'+\delta) \), therefore the minimum in (1) for \(\text{sample}_k(i)\) and \(\text{sample}_k(i')\) will represent the corresponding \( k \)-basic fragments.

We apply the probabilistic method to show that one can choose \( \pi_k \) so that \(\text{sample}_k\) admits a step representation of size \( \mathcal{O}(\frac{n}{\log n}) \). Let us choose \( \pi_k \) uniformly at random. First, we claim that \(\text{sample}_k\) is distributed uniformly on \([i, i+2^k]\). Indeed, by Fact 3.1, the identifiers \( DBF_k[i], \ldots, DBF_k[i+2^k] \) are distinct, and after transforming with \(\pi_k\), any relative order of \( ID_k[i], \ldots, ID_k[i+2^k] \) is equally likely. Now, let us bound \( \mathbb{P}[\text{sample}_k(i) \neq \text{sample}_k(i+1)] \). Note that if \(\text{sample}_k(i) \geq i + 1\) and \(\text{sample}_k(i+1) \leq i + 2^k\), then \(\text{sample}_k(i) = \text{sample}_k(i+1) = \arg\min\{ID_k[j] : j \in [i+1, i+2^k]\}\). Consequently,

\[
\mathbb{P}[\text{sample}_k(i) \neq \text{sample}_k(i+1)] \leq \mathbb{P}[\text{sample}_k(i) = i] + \mathbb{P}[\text{sample}_k(i+1) = i + 2^k + 1] = \frac{2}{2^k+1}.
\]

Therefore, the expected size of the step representation of \(\text{sample}_k\) does not exceed \( 1 + \frac{2(n_k-1)}{2^{k+1}} = \mathcal{O}(\frac{n}{\log n}) \). Summing up over all values \( k \), we obtain the desired \( \mathcal{O}(n) \) bound.

3.2 Data Structure and Queries

The data structure for the square-free case uses two auxiliary abstract components defined below. The first of them is based on the well-known \( \text{rank} \) queries on a bit vector. For a bit vector \( B[1..n] \), we define \( \text{rank}_B(i) \) as the number of positions \( j \leq i \) such that \( B[j] = 1 \), and \( \text{select}_B(i) \) as the position of the \( i \)-th 1-bit in \( B \), i.e., the smallest index \( j \) such that \( \text{rank}_B(j) = i \), or \( \perp \) if no such index exists.

Lemma 3.4 ([3]). Given a bit vector \( B[1..n] \) packed in \( \frac{n}{\log n} \) machine words, we can extend it in \( \mathcal{O}(\frac{n}{\log n}) \) time with a data structure for \( \text{rank}_B \) and \( \text{select}_B \) queries, occupying \( o\left( \frac{n}{\log n} \right) \) additional space.

| EVALUATOR |
|------------------|
| **Input:** A function \( g : [1, n] \rightarrow U \) that admits a step representation of size \( m \) (the elements of \( U \) fit in \( \mathcal{O}(1) \) words). |
| **Queries:** Given \( i \), compute \( g(i) \). |

Lemma 3.5. For any function \( g \) given in a step representation of size \( m \), there exists an evaluator \( \mathcal{E}(g) \) of size \( \mathcal{O}(m + \frac{n}{\log n}) \) that answers queries in \( \mathcal{O}(1) \) time and can be constructed in \( \mathcal{O}(m + \frac{n}{\log n}) \) time.

Proof. Let \( R = \{(\ell_i, r_i, v_i)\} \) be the step representation of \( g \). We store the values \( v_i \) in an array indexed by \( i \). Let us define a bit vector \( B_R \) setting \( B_R[j] = 1 \) if and only if \( j = \ell_i \) for some \( i \). Bit vector \( B_R \) will be represented in \( \mathcal{O}(\frac{n}{\log n}) \) machine words. Starting from a null vector and setting \( B_R[\ell_i] = 1 \) for all \( i \), we can construct \( B_R \) in \( \mathcal{O}(\frac{n}{\log n} + m) \) time. Observe that \( g(j) = v_i \) where \( i = \text{rank}_B(j) \).

We build the data structure for \( \text{rank} \) queries in \( B_R \), applying Lemma 3.4. In total we obtain the desired \( \mathcal{O}(\frac{n}{\log n} + m) \) bounds on the space and construction time, and constant-time queries.
Lemma 3.6. For a family \( \mathcal{A} = (A_i) \) there exists a locator \( \mathcal{L}(A) \) of size \( O(\sum_i |A_i|) \) that can answer queries in \( O(1) \) time. It can be constructed in \( O(\sum_i |A_i|) \) expected time given \( \{(i,j) : j \in A_i\} \).

Proof. We divide the universe \([1,n]\) into blocks \( B_\ell = [\ell d + 1, (\ell + 1)d] \) for \( \ell \in \mathbb{Z} \). The data structure is based on perfect hashing [14]. For each \( j \in A_i \), we store an entry with key \( (\lceil \frac{j}{d} \rceil, i) \) and value \( j \). For a query we extend \( P \) to \( P' \) such that \( P' \) is composed of full blocks. Note that \( |P'| \leq |P| + 2d = O(d) \). Let \( P' = [\ell d + 1, \ell' d] \). For each \( m \in [\ell + 1, \ell'] \) (note that there are \( O(1) \) such values) we retrieve all entries with the key \( (m,i) \). Clearly, this gives \( P' \cap A_i \). The size of this set is constant by the sparsity condition. Now, it suffices to filter out elements of \( P' \setminus P \) and return the remaining ones.

Theorem 3.7. For a square-free word \( v \) of length \( n \), there exists a data structure of size \( O(n) \) that can answer Internal Pattern Matching Queries in \( O(1) \) time.

Proof. The data structure consists of \( \lfloor \log n \rfloor \) layers. For each layer \( k \) we store an evaluator of the samples assignment and a locator of samples positions, that is \( \mathcal{E}(\text{sample}_k) \) and \( \mathcal{L}(\mathcal{A}^k) \), where \( \mathcal{A}^k_{id} \) is the set of sample occurrences of the \( k \)-basic factor whose identifier \( ID_k \) is \( id \). In the evaluator we additionally store the identifiers \( ID_k \) of the sample positions. We also maintain the global data structure specified in Fact 2.1. If \( \text{sample}_k \) is chosen so that it satisfies Lemma 3.3, the total size of step representations is \( O(n) \), hence \( |\text{SAMPLES}_k| = O(n) \) and the total size of all sets in the families \( \mathcal{A}^k \) is \( O(n) \). This implies that the total size of evaluators and locators is \( O(n) \).

The query algorithm for \( x = v[\ell, r] \) and \( y = v[\ell', r'] \) works as follows. If \( |x| = 1 \), we use a naive algorithm (compare \( x \) with each letter of \( y \)). Otherwise, we use the data structures for the \( k \)-th layer, for \( k = \lfloor \log |x| - 1 \rfloor \); we apply \( \mathcal{E}(\text{sample}_k) \) to obtain \( j = \text{sample}_k(\ell) \) and \( id = ID_k[j] \). We set \( \delta = j - \ell \) and use \( \mathcal{L}(\mathcal{A}^k) \) to compute \( \mathcal{A}^k_{id} \cap [\ell' + \delta, r' + 1 - |x| + \delta] \), i.e., the sample occurrences of the \( k \)-basic factor \( BF_k(j) \) which might be induced by an occurrence of \( x \) in \( y \) (see Figure 2). We obtain a constant number of them and use Fact 2.1 to detect which extend to an actual occurrence of \( x \).

![Figure 2: The query algorithm for \( x = v[\ell, r] \) and \( y = v[\ell', r'] \) finds \( j = \text{sample}_k(\ell) \) and the identifier \( id \) of \( BF_k(j) \), here depicted with a gray rectangle. Then, it finds all sample occurrences of \( BF_k(j) \) which might be induced by occurrences of \( x \) in \( y \). Here, \( BF_k(j) \) occurs at positions \( occ_1 \) and \( occ_2 \). Potential occurrences of \( x \) are marked with dashed rectangles.](image)

4 Overview of Periodic Case

In the periodic case of IPM QUERIES, we assume that \( x \) contains a periodic \( k \)-basic factor for \( k = \lfloor \log |x| - 1 \rfloor \). In the following subsection we recall several known notions and introduce \( k \)-runs,
a central concept for the solution of the periodic case. Afterwards, we describe the main steps of the query algorithm for this case.

4.1 Repetitive Structure of Words

A run (maximal repetition) in \( v \) is a periodic fragment \( \alpha = v[i, j] \) which cannot be extended neither to the left nor to the right without increasing the shortest period \( p = \text{per}(\alpha) \), that is, \( v[i - 1] \neq v[i + p - 1] \) and \( v[j - p + 1] \neq v[j + 1] \), provided that the respective positions exist. We define the exponent of a run as \( \exp(\alpha) = \frac{|\alpha|}{\text{per}(\alpha)} \). The algorithms considered in this paper represent runs together with their periods.

A word that is both primitive and lexicographically minimal in the class of its cyclic rotations is called a Lyndon word. Let \( u \) be a periodic word with the shortest period \( \text{per}(u) = p \). The Lyndon root \( \lambda \) of \( u \) is the Lyndon word that is a cyclic rotation of \( u[1, p] \), i.e., the minimal cyclic rotation of \( u[1, p] \). Then \( u \) can be represented as \( \lambda'\lambda'' \) where \( \lambda' \) is a proper suffix of \( \lambda \), and \( \lambda'' \) is a proper prefix of \( \lambda \). The Lyndon representation of \( u \) is defined as \( (|\lambda'|, k, |\lambda''|) \). We say that two runs are compatible if they have the same Lyndon root.

Fact 4.1 ([7, 9, 11, 20]). In a word of length \( n \) both the number of runs and the sum of their exponents are \( \mathcal{O}(n) \). Moreover, all the runs in a word together with their Lyndon representations can be computed in \( \mathcal{O}(n) \) time.

We say that a run \( \alpha \) extends a fragment \( u \) if \( u \) is a subfragment of \( \alpha \) and \( \text{per}(u) = \text{per}(\alpha) \). For any periodic fragment \( u \), there is a unique run \( \alpha \) extending \( u \), we denote it as \( \text{run}(u) \). If \( u \) is not periodic, we set \( \text{run}(u) = \perp \). A run is called a \( k \)-run if it extends a periodic fragment of length between \( 2^k \) and \( 2^{k+1} - 1 \); see Figure 3. Note that if \( \text{run}(u) = \alpha \neq \perp \), then \( \alpha \) is by definition a \( k \)-run for \( k = \lfloor \log |u| \rfloor \). Also, observe that \( \text{run}(u) = \alpha \) holds if and only if \( u \) is a subfragment of \( \alpha \) of length at least \( 2 \times \text{per}(\alpha) \).

\[
\ldots \ c \ a \ a \ b \ a \ a \ b \ a \ a \ b \ a \ a \ b \ a \ a \ b \ a \ a \ b \ c \ldots \\
\underbrace{\ldots}_{2^2} \quad \underbrace{\ldots}_{2^3} \quad \underbrace{\ldots}_{2^4}
\]

Figure 3: A run \( \alpha \) with \( |\alpha| = 19 \), \( \text{per}(\alpha) = 3 \) and \( \exp(\alpha) = \frac{6}{3} \). It is a \( k \)-run for \( k \in \{2, 3, 4\} \). We have \( \alpha = \text{run}(u) \) for every subword \( u \) of \( \alpha \) of length at least 6.

4.2 Data Structure and Queries

Let \( x \cap y \) denote the common subfragment of overlapping fragments \( x, y \). The following observation shows why \( k \)-runs are useful in the solution of the periodic case; see Figure 4.

Observation 4.2. Assume that \( x = v[\ell, r] \) and \( x' = v[\ell', r'] \) are occurrences of the same subword and that \( x \) contains a periodic \( k \)-basic subfragment \( z \). Let \( \alpha = \text{run}(z) \). Then there exists a \( k \)-run \( \alpha' \) that is compatible with \( \alpha \), such that \( x \cap \alpha \) and \( x' \cap \alpha' \) are the corresponding subfragments of \( x \) and \( x' \).
The runs may have different lengths but their intersections with the occurrences of the subword start and end at the same positions relative to these occurrences.

The query algorithm (pattern matching of a subword $x$ in a subword $y$) works in the following steps:

A. Find a periodic $k$-basic factor $z$ in $x$ for $k = \lfloor \log |x| - 1 \rfloor$.
B. Compute the unique $k$-run $\alpha = \text{run}(z)$.
C. Find all $k$-runs $\alpha'$ overlapping $y$ and satisfying $\text{per}(\alpha) = \text{per}(\alpha')$.
D. Check which $k$-runs $\alpha'$ are compatible with $\alpha$.
E. If $\alpha \cap x \neq x$ then for each $\alpha'$ find the unique candidate for the induced occurrence of $x$.
F. Otherwise, for each $\alpha'$ find an arithmetic progression of occurrences of $x$ in $\alpha' \cap y$.
G. Combine the results for all $k$-runs $\alpha'$ into a single arithmetic progression.

In Section 6 we show that each of the steps can be performed in constant time using data structures of size $O(n)$ which can be constructed in $O(n)$ expected time. This way, we prove the following theorem.

**Theorem 4.3.** There exists a data structure of size $O(n)$ which can be constructed in $O(n)$ expected time, and answers IPM Queries in $O(1)$ time provided that the pattern $x$ contains a periodic $k$-basic factor for $k = \lfloor \log |x| - 1 \rfloor$.

**5 General Case**

In this section we complete the description of the data structure for the general IPM Queries.

The central notion of Section 3 was that of a samples assignment. If we directly generalized Definition 3.2 to arbitrary words with periodicities, the samples assignment would require $\Omega(n \log n)$ space, e.g., for the word $v = a^n$. However, if the query pattern $x = v[\ell, r]$ contains a periodic $k$-basic factor with $k = \lfloor \log |x| - 1 \rfloor$, we can apply the data structure of Theorem 4.3 to answer this query. Thus, if $BF_{k+1}(\ell)$ contains a periodic $k$-basic fragment, we leave $\text{sample}_k(\ell)$ undefined (denoted $\text{sample}_k(\ell) = \bot$) and modify the query algorithm, so that it launches the data structure for periodic case whenever it gets an undefined sample.

Fact 3.1 does not hold for arbitrary word $v$. Nevertheless, if we assume that $u$ is non-periodic, a slightly weaker result still holds.

**Fact 5.1 (Sparsity of Occurrences).** Let $u$ and $v$ be words and assume $u$ is not periodic. Then the set of positions where $u$ occurs in $v$ is $\frac{|u|}{2}$-sparse.

**Proof.** If $u$ occurs in $v$ at positions $i$ and $i + d \leq i + |u|$, then $u[j + d] = u[j]$ for any $j \in [1, |u| - d]$, i.e., $d$ is a period of $u$. Thus $d > \frac{|u|}{2}$ since $u$ is not periodic. \qed
Lemma 5.3. Let $v$ be an arbitrary word. We call $\text{sample}_k$ a $k$-samples partial assignment for $v$, if the following conditions are satisfied:

1. $\text{sample}_k : [1, n_{k+1}] \to [1, n_k] \cup \{\bot\};$
2. $\text{sample}_k(i) \in [i, i + 2^k] \cup \{\bot\}$ for each $i$;
3. $\text{sample}_k(i) = \bot$ if and only if $BF_{k+1}(i)$ contains a periodic $k$-basic factor;
4. $\text{sample}_k(i) - i = \text{sample}_k(i') - i'$ or $\text{sample}_k(i) = \text{sample}_k(i') = \bot$ if $BF_{k+1}(i) = BF_{k+1}(i')$.

The notions of sample positions $\text{SAMPLES}_k$ and sample occurrences carry on. Generalizing the approach of Section 3, we choose permutations $\pi_k \in S_{n_k}$, define $ID_k[j] = \pi_k(DBF_k[j])$, and set

$$\text{sample}_k(i) = \begin{cases} \bot & \text{if } BF_k(j) \text{ is periodic for some } j \in [i, i + 2^k], \\ \text{argmin}\{ID_k[j] : j \in [i, i + 2^k]\} & \text{otherwise.} \end{cases}$$

Like in the square-free case, we can choose $\pi_k$ so that $\text{sample}_k$ admit step representations of total size $O(n)$. The following lemma is a general version of Lemma 3.3

Lemma 5.3. Let $v$ be an arbitrary word. There exist permutations $\pi_k \in S_{n_k}$ such that $\text{sample}_k$ given by (5) form a family of samples partial assignments for $v$ and admit step representations of total size $O(n)$.

Proof. It is easy to see that $\text{sample}_k$ given as (5) satisfies the conditions required for a samples partial assignment (see Definition 5.2). As in the proof of Lemma 3.3, we use the probabilistic method to show that if $\pi_k$ is drawn uniformly at random, then the expected size of the step representation of $\text{sample}_k$ is $O\left(\frac{n}{2^k}\right)$. This will imply that for some choices of $\pi_k$ the actual size is $O\left(\frac{n}{2^k}\right)$, which sums up to $O(n)$ for all values of $k$.

First, let us prove that the number of blocks of consecutive $\bot$'s in $\text{sample}_k$ is $O\left(\frac{n}{2^k}\right)$. Observe that if $BF_k(j)$ is periodic, then $\text{sample}_k(i) = \bot$ for all $i \in [j - 2^k, j] \cap [1, n_k]$. Thus each block of $\bot$’s, possibly except for the first and the last one, contains at least $2^k + 1$ positions. Consequently, the number of such blocks is at most $2 + \frac{n_k}{2^k} = O\left(\frac{n}{2^k}\right)$.

Now, as in the proof of Lemma 3.3 we analyze the distribution of $\text{sample}_k(i)$ provided that $\text{sample}_k(i) \neq \bot$ (the latter does not depend on the choice of $\pi_k$). Consider identifiers $DBF_k[i], \ldots, DBF_k[i + 2^k]$. While they are not necessarily distinct, Fact 5.1 implies that any identifier may occur at most twice in the sequence, i.e., there are at least $2^k - 1 + 1$ distinct values. Any of them is equally likely to become the minimum after mapping with $\pi_k$, and thus $P[\text{sample}_k(i) = j] \leq \frac{1}{2^{k-1}+1}$ for any $j \in [i, i + 2^k]$. We still have that if $\text{sample}_k(i) \neq \bot$ and $\text{sample}_k(i + 1) \neq \bot$, then

$$P[\text{sample}_k(i) \neq \text{sample}_k(i + 1)] \leq P[\text{sample}_k(i) = i] + P[\text{sample}_k(i + 1) = i + 2^k + 1] \leq 2 \cdot \frac{1}{2^{k-1}+1}.$$

Consequently, in the blocks of non-$\bot$’s, $\text{sample}_k$ in expectation changes at most $\frac{n_k}{2^{k-1}+1} = O\left(\frac{n}{2^k}\right)$ times. Combined with the bound on the number of such blocks, we conclude that the expected size of the step representation of the whole assignment $\text{sample}_k$ is bounded by $O\left(\frac{n}{2^k}\right)$. □

Theorem 5.4. For any word $v$ of length $n$ there exists a data structure of size $O(n)$ that can answer Internal Pattern Matching Queries in $O(1)$ time.

Proof. The proof is analogous to that of Theorem 3.7. Apart from the data structure of Theorem 4.3 for the periodic case, we use the same components: the data structure from Fact 2.1, evaluators
\( \mathcal{E}(\text{sample}_k) \) and locators \( \mathcal{L}(A^k) \) (constructed for samples partial assignments \( \text{sample}_k \)). The locators \( \mathcal{L}(A^k) \) are now defined for \( d = 2^{k-1} \), with \( 2^{k-1} \)-sparsity of \( A^k \) being a consequence of Fact 5.1.

Relative to the square-free case, there are just two differences in the query algorithm for \( x = v[\ell, r] \) and \( y = v[\ell', r'] \). First, if \( \text{sample}_k(\ell) \) (for \( k = [\log |x|-1] \)) turns out to be undefined, we find out that we are in the periodic case, so we launch the component responsible for that case. The second modification concerns the very last step, returning the output. In Section 3 we were guaranteed that there is at most one occurrence, so returning it as an arithmetic progression was trivial. Here, we either pass the arithmetic progression obtained from the periodic case, or obtain a constant number of occurrences, which, by Lemma 2.2, also form an arithmetic progression.

\[ \square \]

\section{Full Description of Periodic Case}

In this section we describe the implementation of steps A-G of the query algorithm for periodic case presented in Section 4. We also develop the underlying auxiliary data structures. The particular steps are encapsulated in Lemmas 6.8-6.15 of Section 6.2. This part is preceded with several combinatorial results. The main of these results, specified by Fact 6.5, is a combinatorial bound related to \( k \)-runs, which we use extensively to show that our data structures use \( O(n) \) space.

\subsection{Combinatorial Toolbox}

We start by recalling three classic lemmas.

\textbf{Lemma 6.1} (Periodicity Lemma \cite{13, 24}). Let \( v \) be a word with periods \( p \) and \( q \). If \( p + q \leq |v| \), then \( \gcd(p, q) \) is also a period of \( v \).

\textbf{Lemma 6.2} (Synchronization Property \cite{6}). Let \( \lambda \) be a primitive non-empty word. Then \( \lambda \) has exactly two occurrences in \( \lambda \lambda \).

\textbf{Lemma 6.3} (Three Squares Lemma \cite{12, 8}). Let \( v_1, v_2, v_3 \) be words such that \( v_2^2 \) is a prefix of \( v_2^3 \), \( v_2^3 \) is a prefix of \( v_3^2 \), and \( v_1 \) is primitive. Then \( |v_1| + |v_2| \leq |v_3| \).

The remainder of this section is devoted to runs. By \( \mathcal{R}(v) \) we denote the set of all runs in \( v \), and by \( \mathcal{R}_k(v) \) the set of \( k \)-runs in \( v \). Note that \( \bigcup \mathcal{R}_k(v) = \mathcal{R}(v) \), but the sum is not necessarily disjoint. The following observation gives an alternative definition of \( k \)-runs, which is more convenient for the proof of the subsequent fact.

\textbf{Observation 6.4}. A run \( \alpha \) is a \( k \)-run if and only if \( |\alpha| \geq 2^k \) and \( \per(\alpha) < 2^k \).

\textbf{Fact 6.5}.

\[ (a) \sum_k \sum_{\alpha \in \mathcal{R}_k(v)} \frac{|\alpha|}{2^k} = O(n) \quad (b) \sum_k |\mathcal{R}_k(v)| = O(n). \]

\textbf{Proof}. Let us fix a run \( \alpha \). We have

\[ \sum_{k : \alpha \in \mathcal{R}_k(v)} \frac{|\alpha|}{2^k} \leq \sum_{k : \per(\alpha) < 2^k} \frac{|\alpha|}{2^k} = \sum_{k = [\log \per(\alpha)] + 1}^{\infty} \frac{|\alpha|}{2^k} \leq \frac{|\alpha|}{\per(\alpha)} = \exp(\alpha). \]
Summing up over all runs and applying the bound for the sum of exponents of runs (Fact 4.1), we get (a). As for part (b), if $\alpha$ is a $k$-run, then $|\alpha| \geq 2^k$, so $\frac{|\alpha|}{2^k} \geq 1$. Thus (b) is a consequence of (a):

$$\sum_k |\mathcal{R}_k(v)| \leq \sum_k \sum_{\alpha \in \mathcal{R}_k(v)} \frac{|\alpha|}{2^k} = \mathcal{O}(n).$$

Finally, we note that runs sharing a common period cannot overlap too much.

Observation 6.6. Let $\alpha \neq \alpha'$ be runs with period $p$. Then $|\alpha \cap \alpha'| < p$.

6.2 Implementation of Periodic Case

Recall that in Section 4 we have listed the steps of the query algorithm for the periodic case. In this section we introduce algorithms and data structures that perform the respective steps.

In this section we introduce algorithms and data structures that perform the respective steps.

The data structure for Step A of the query algorithm is given in Lemma 6.8. Its proof uses the following auxiliary Lemma 6.7.

Let $R$ be a finite set of integers. We call $B$ a block of $R$ if $B$ is an inclusion-wise maximal interval contained in $R$. A block representation of $R$ is a sorted list of all its blocks.

Lemma 6.7. For $k \in [0, \lfloor \log n \rfloor]$, let $P_k = \{ i \in [1, n_k] : BF_k(i) \text{ is periodic} \}$. In $\mathcal{O}(n)$ time we can compute all sets $P_k$, each of them represented both in the block representation and as a bit vector.

Proof. Each periodic $k$-basic fragment is induced by a unique $k$-run. Moreover, for a fixed $k$-run $\alpha$ the set of positions where periodic $k$-basic fragments induced by $\alpha$ start, forms an interval. For $\alpha = [i, j]$, the interval is $[i, j - 2^k + 1]$ if $\text{per}(\alpha) \leq 2^{k-1}$, and $\emptyset$ otherwise.

In order to compute the block representation of $P_k$, it suffices to sort these intervals and join some of them so that we get blocks of $P_k$. This takes $\mathcal{O}(n + \sum_k |\mathcal{R}_k(v)|) = \mathcal{O}(n)$ time in total by Fact 6.5(b). The bit vector representing $P_k$ can be obtained from a block representation in time proportional to the total size of both representations. We start with a null vector, and then for each block $B$ of $P_k$ we set all bits corresponding to that block. Using bit-operations we can do this in $\mathcal{O}(1 + \frac{|B|}{\log n})$ time. For a fixed $k$, the former terms sum up to the number of blocks (the size of the block representation) and the latter to $\mathcal{O}(\frac{|P_k|}{\log n})$ (the size of the bit vector representation). In total we obtain $\mathcal{O}(n)$ construction time over all values $k$.

Lemma 6.8 (Step A). There exists a data structure of $\mathcal{O}(n)$ size that, given an integer $k$ and a fragment $v[\ell, r]$, in constant time finds a periodic $k$-basic fragment which is a subfragment of $v[\ell, r]$ or states that no such basic fragment exists. The data structure can be constructed in $\mathcal{O}(n)$ time.

Proof. The data structure contains $\lfloor \log n \rfloor$ layers. The $k$-th layer consists of the data structure for successor queries for $P_k$, i.e., queries asking to determine $\min(P_k \cap [i, n])$. Observe that this value is actually equal to $\text{select}_{B_k}(\text{rank}_{B_k}(i - 1) + 1)$, where $B_k$ is a bit-vector representation of $P_k$. Thus, for such queries it suffices to use the data structure for $\text{rank}$ and $\text{select}$ queries, which by Lemma 3.4 can be constructed in $\mathcal{O}(n)$ time from the bit-vector representation of $P_k$ built using Lemma 6.7.

Lemma 6.9 (Step B). There exists a data structure of $\mathcal{O}(n)$ size, which given a fragment $u$ returns $\text{run}(u)$ in constant time. Moreover, the data structure can be constructed in $\mathcal{O}(n)$ time.
Figure 5: $\text{run}(u) = \alpha$. If $u$ is $k$-periodic then $\alpha$ is a $k$-run.

Proof. We say that a word $u$ is $k$-periodic, if $u$ is periodic, $|u| \geq 2^k$ and $\text{per}(u) < 2^k$. We start with two auxiliary claims.

**Claim 6.10.** Let $u$ be a periodic fragment of $v$. Then $\text{run}(u)$ is the unique run $\alpha$ such that $\text{per}(\alpha) \leq |u| - \text{per}(u)$ and $u \cap \alpha = u$. Moreover, $\text{per}(\text{run}(u)) = \text{per}(u) \leq \frac{|u|}{2}$ and, if $u$ is $k$-periodic, then $\text{run}(u)$ is a $k$-run.

**Proof.** The latter statement is an immediate consequence of the definitions. As for the former statement, suppose that $\beta$ is a different run satisfying the aforementioned conditions. Note that both $\text{per}(\alpha)$ and $\text{per}(\beta)$ are periods of $u$ and $\text{per}(\alpha) + \text{per}(\beta) \leq |u|$. Periodicity Lemma (Lemma 6.1) implies that $\text{per}(\alpha) = \text{per}(\beta)$. However, $u$ is a subfragment of $\alpha \cap \beta$, which results in a contradiction with Observation 6.6.

**Claim 6.11.** Let $u_1, u_2, u_3$ be $k$-periodic fragments of $v$, all starting at the same position $i$. Then $\text{run}(u_1), \text{run}(u_2), \text{run}(u_3)$ cannot be all distinct.

**Proof.** For a proof by contradiction suppose that runs $\alpha_i = \text{run}(u_i)$ are pairwise distinct. Observation 6.6 implies that these runs must have pairwise distinct periods $p_i$. Without loss of generality assume $p_1 < p_2 < p_3$, i.e., $u_1, u_2, u_3$ are three periodic subwords of length at least $2^k$ with different shortest periods $p_1 < p_2 < p_3 < 2^k$ starting at the position $i$ in $v$. By the Three Squares Lemma (Lemma 6.3) we conclude that $p_1 + p_2 \leq p_3 < 2^k$. Now consider the $k$-basic fragment $u$ starting at position $i$. We have $2p_1 < 2^k$, therefore $u$ is a $k$-periodic fragment. Observe that both $\alpha_1$ and $\alpha_2$ satisfy the statement of Claim 6.10, which implies that $\alpha_1 = \text{run}(u) = \alpha_2$, a contradiction.

We proceed with the proof of the lemma. Consider a function $R_k : [1, n_k] \rightarrow 2^{|R_k(v)|}$ which assigns to a position $i$ the set of $k$-runs inducing a $k$-periodic fragment starting at position $i$. Claim 6.11 implies that $|R_k(i)| \leq 2$ for each $i$. Note that for $\alpha = v[i, j]$ we have $\alpha \in R_k(i')$ if and only if $i' \in [i, \min(j - 2\text{per}(\alpha), j - 2^k) + 1]$.

For a $k$-run $\alpha = v[i, j]$ define $\text{beg}_k(\alpha) = i$ and $\text{end}_k(\alpha) = \min(j - 2\text{per}(\alpha), j - 2^k)$. Observe that $R_k(i) \neq R_k(i - 1)$ implies $i = \text{beg}_k(\alpha)$ or $i = \text{end}_k(\alpha)$ for some $k$-run $\alpha$. Thus $R_k$ admits a step representation of size at most $2|R_k(v)|$.

Summing up over all layers $k$, by Fact 6.5(b) this implies that the total size of representations of functions $R_k$ is $O(n)$, which means that evaluators $E(R_k)$ take $O(n)$ space in total. The step representation of $R_k$ can be easily constructed by an algorithm which traverses the word from left to right maintaining a set $A$ of (at most 2) $k$-runs. At each position $i$ it removes from $A$ all $k$-runs $\alpha$ with $\text{end}_k(\alpha) = i$ and adds those with $\text{beg}_k(\alpha) = i$. Such events can be prepared and sorted in $O(n)$ time simultaneously for all $k$.

We answer queries as follows. If $u = v[i, j]$ is periodic, then it is $k$-periodic for $k = \lceil \log |u| \rceil$, so the run inducing $v[i, j]$ belongs to $R_k(i)$. We use $E(R_k)$ to find all such $k$-runs in constant time.
Finally, we check if \( \alpha = \text{run}(u) \) using the characterization of Claim 6.10, i.e., verifying that \( \alpha \) covers \( u \) and \( \text{per}(\alpha) \leq \frac{|u|}{2} \).

**Step C** is performed using auxiliary data structures that we call \( k \)-run locators.

**Lemma 6.12 (Step C).** There exist \( k \)-run locators \( K_k(v) \) that answer queries in \( O(1) \) time, take \( O(n) \) space in total, and can be constructed in \( O(n) \) expected total time.

**Proof.** The data structure is similar to LOCATOR, see Lemma 3.6. Let us divide \([1, n]\) into blocks \( B_1, \ldots, B_m \) of size \( 2^k \) (the last one possibly shorter). Note that \( m = O(\frac{n}{2^k}) \). We build a perfect hash table \([14]\): for each \( k \)-run \( \alpha \) and each index \( i \) such that \( B_i \) overlaps with \( \alpha \), we store an entry with key \( (i, \text{per}(\alpha)) \) and value \( \alpha \). Note that the number of entries is bounded by \( \sum_{\alpha \in R_k(v)} (\frac{\alpha}{2^k} + 2) \), which, by Fact 6.5(a), is \( O(n) \) when summed over all \( k \). As we shall see, there are at most 4 values for a fixed key. Any \( k \)-run intersecting a fixed block \( B_i \) must contain the first or the last position in that block, and by the following claim each of these positions might be contained in at most two \( k \)-runs of period \( p \).

**Claim 6.13.** Any position lies within at most two runs of period \( p \).

**Proof.** Consider three distinct runs \( \alpha_1 = v[i_1, j_1] \), \( \alpha_2 = v[i_2, j_2] \) and \( \alpha_3 = v[i_3, j_3] \) with period \( p \). Assume that \( i_1 \leq i_2 \leq i_3 \). From Observation 6.6 we get

\[
    i_3 > j_2 - p + 1 = j_2 - i_2 + 1 - p + i_2 = |\alpha_2| - p + i_2 \geq p + i_2 > p + j_1 - p + 1 > j_1.
\]

Thus \( i_3 > j_1 \) which means that \( \alpha_1 \) and \( \alpha_3 \) do not intersect. \( \square \)

We extend the query range \( P \) to \( P' \) so that \( P' = B_i \cup \ldots \cup B_j \) (with \( j - i = O(1) \)) and find all \( k \)-runs of period \( p \) overlapping \( P' \). As we have just shown, there are \( O(1) \) such \( k \)-runs, so we can easily check which of them overlap with \( P \); also duplicates can be removed in constant time. \( \square \)

In **Step D** we are given a constant number of runs \( \alpha' \) and we are to check which of them are compatible with the run \( \alpha \), that is, which of them have the same Lyndon root as \( \alpha \). Recall from Fact 4.1 that we can store each run together with its Lyndon representation.

**Lemma 6.14 (Step D).** Using a data structure of \( O(n) \) size, that can be constructed in \( O(n) \) time, one can check if two runs, given with their Lyndon representations, are compatible in \( O(1) \) time.

**Proof.** A Lyndon representation of a run indicates its Lyndon root, that is, given the Lyndon representation we can find the fragment of \( v \) where the Lyndon root occurs. Therefore, the data structure of Fact 2.1 lets us test in constant time if the Lyndon roots of the two runs are equal. \( \square \)

In **Step E** we assume that \( \alpha \) does not cover \( x \), i.e., \( x \cap \alpha \neq x \). By Observation 4.2, if for an occurrence \( x' \) of \( x \) the corresponding run is \( \alpha' \), then \( \alpha' \) also does not cover \( x' \), and \( x \cap \alpha \) and \( x' \cap \alpha' \) are the corresponding subfragments of \( x \) and \( x' \), respectively. Therefore, for a fixed \( \alpha' \) we have a
single possible location of \( x' \). One can use Fact 2.1 to verify in constant time whether \( x \) actually occurs there.

**Step F** assumes that \( \alpha \) covers \( x \). By Observation 4.2 any occurrence \( x' \) of \( x \) is covered by a compatible run \( \alpha' \). To find all occurrences of \( x \) within \( \alpha' \), we apply the following Lemma 6.15 for \( x \) and \( y \cap \alpha' \). The Lyndon representations of both subwords can be easily computed from the Lyndon representations of \( \alpha \) and \( \alpha' \).

**Lemma 6.15 (Step F).** Let \( x \) and \( y \) be periodic with common Lyndon root. Then the set of positions where \( x \) occurs in \( y \) is an arithmetic progression that can be computed in \( O(1) \) time given the Lyndon representations of \( x \) and \( y \).

**Proof.** Let \( \lambda \) be the common Lyndon root of \( x \) and \( y \) and let their Lyndon representations be \((p, k, s)\) and \((p', k', s')\) respectively. Synchronization Property (Lemma 6.2) implies that \( \lambda \) occurs in \( y \) only at positions \( i \) such that \( i \equiv p' + 1 \pmod{|\lambda|} \). Consequently, \( x \) occurs in \( y \) only at positions \( i \) such that \( i \equiv p' - p + 1 \pmod{|\lambda|} \). Clearly \( x \) occurs in \( y \) at all such positions \( i \) within the interval \([1, |y| - |x| + 1]\). Therefore, it is a matter of simple calculations to compute the arithmetic progression of these positions.

**Step G** is to merge a constant number of arithmetic progressions of occurrences of \( x \) in \( y \). By Lemma 2.2, the result of this operation is a single arithmetic progression. Moreover, in case the number of elements in the resulting progression is greater than 2, we know the difference of this progression. Hence, we can combine the subsequent arithmetic progressions in constant time.

To construct the data structure for periodic case, we compute the set of all runs, together with their Lyndon representations, using Fact 4.1. The construction of the respective components of the data structure comes down to running the algorithms provided by the appropriate lemmas of this section.

### 7 Construction of the Data Structure

In this section we complete the solution for IPM QUERIES by providing an \( O(n) \) expected time construction algorithm of the data structure. The construction of the part of the data structure responsible for the periodic case has already been described in Section 6. Therefore, in this section we focus on the non-periodic counterpart, in particular, the construction of the samples partial assignment.

First in Section 7.1 we sketch the main steps of the construction algorithm. Section 7.2 describes an involved tool for the construction, that is, an \( O(n) \)-time realization of the Dictionary of Basic Factors, based on suffix trees and a previous result by Gawrychowski [15]. Additional combinatorial and probabilistic tools that guarantee correctness and efficiency of our approach are given in Section 7.3. Finally Section 7.4 and Section 7.5 contain the main parts of the construction algorithm.

### 7.1 Overview

We start with a sketch of our approach to constructing \( \text{sample}_k \). First assume that the word \( v \) is square-free (we have been working under this assumption in Section 3). We construct, in two steps, a candidate set \( C_k \), which is a small superset of \( \text{SAMPLES}_k \). First, we find \( A_k = \)
Figure 6: $C = \text{FillGaps}(A, 4, [1, 34])$, where $A = \{7, 9, 13, 20, 21, 26, 32, 34\}$ and $C = \{1, \ldots, 7, 9, 13, \ldots, 21, 26, \ldots, 32, 34\}$. $C$ is obtained from $A$ by inserting all maximal subintervals of the domain that are disjoint with $A$ and contain more than 4 integers (in this example, 17 elements are inserted).

\[
\{j \in [1, n_k] : \text{ID}_k[j] \leq \ell_k\}
\]

for an appropriate parameter $\ell_k$. Then we extend $A_k$, setting $C_k = \text{FillGaps}(A_k, 2^k, [1, n_k])$, where FillGaps is defined as:

\[
\text{FillGaps}(A, \Delta, I) = A \cup \bigcup \{[i, i+\Delta] : [i, i+\Delta] \subseteq I \setminus A\};
\]

see also Figure 6. As we shall see in Section 7.3 in a more abstract setting, the set $C_k$ generated this way is always a superset of $\text{SAMPLES}_k$ and its expected size is $O(\frac{n^k}{2^k})$. Thus the total size of sets $C_k$ is $O(n)$. Finally we show that given any superset of $\text{SAMPLES}_k$ it is easy to construct $\text{sample}_k$.

If $v$ is allowed to be arbitrary, the main idea of the construction algorithm remains unchanged: we are still looking for a candidate set $C_k$ that is a small superset of $\text{SAMPLES}_k$. As previously we start with $A_k = \{j : \text{ID}_k[j] \leq \ell_k\}$ but we need to use FillGaps much more carefully: instead of taking $I = [1, n_k]$, we apply it separately for each maximal block of positions where non-periodic $k$-basic fragments start. In particular, we need to use some techniques from the periodic case to find these blocks and to identify positions $i$ where $\text{sample}_k(i)$ is defined.

7.2 Space-efficient DBF

The standard implementation of the Dictionary of Basic Factors uses $\Theta(n \log n)$ space [6]. We introduce its compact version which provides the same operations as regular DBF but with linear space and construction time. Its main component is the suffix tree augmented with a data structure of Gawrychowski [15], which efficiently locates basic factors in the suffix tree.

| CompactDBF |
|---|
| **Input:** a word $v$ of length $n$ |
| **Queries:** for an integer $k$: |
| 1 given a position $i$ return $DBF_k[i]$, |
| 2 given an identifier $j$ report $\{i : DBF_k[i] = j\}$, |
| 3 return $m_k$, the number of distinct identifiers in $DBF_k$. |

Before we proceed with a solution, we recall a number of tools related to suffix trees. The suffix trie of $v$ is the trie of all suffixes of $v$. Each subword $x$ of $v$ corresponds to a unique node in the suffix trie, called the locus of $x$, such that $x$ is spelled by the letters on the path from the root to that node.

The suffix tree of $v$ [6, 16], denoted $T(v)$, is the compacted suffix trie of $v$, i.e., nodes that are neither branching (with 2 or more children) nor terminal (loci of suffixes of $v$) are dissolved. The dissolved nodes are called implicit, the remaining nodes are called explicit. An implicit node $x$ can
be represented as a pair \((u, d)\), where \(u\) is the lowest explicit descendant of \(x\), and \(d\) is the distance (the number of letters) from \(x\) to \(u\). The pair \((u, d)\) is called the locus of the subword corresponding to \(x\), and \(u\) is called its explicit locus.

The suffix tree of \(v\) takes linear space and can be constructed in linear time provided that the letters of \(v\) can be sorted in linear time [6]. The following result is due to Gawrychowski [15].

**Lemma 7.1.** The suffix tree \(T(v)\) can be preprocessed in \(O(|v|)\) time so that given integers \(i, k\) the locus of the basic factor \(BF_k(i)\) can be determined in \(O(1)\) time.

We say that an explicit node \(u\) is a \(k\)-basic node if \(u\) is an explicit locus of a \(k\)-basic factor. Observe that there is a natural bijection between identifiers in \(DBF_k\) and the \(k\)-basic nodes. While storing it explicitly for all \(k\) takes too much space, we shall devise an alternative way to evaluate it.

There are up to \(2n\) explicit nodes, let us assign them pre-order identifiers \(id\). Note that such identifiers also preserve lexicographic order of the corresponding subwords. We have the following observation.

**Observation 7.2.** If the explicit locus of \(BF_k(i)\) is \(u\), then \(DBF_k[i]\) is the number of \(k\)-basic nodes with identifiers not exceeding \(id(u)\).

Due to the observation, it suffices to store a bit vector \(B_k\) such that \(B_k[i] = 1\) if and only if the explicit node \(u\) with \(id(u) = i\) is \(k\)-basic. Then rank queries on \(B_k\) (see Lemma 3.4) can be used to determine the identifier of \(u\) in \(DBF_k\). Similarly, select queries on \(B_k\) let us find the \(id\) of the node which corresponds to a given identifier in \(DBF_k\). In order to be able to report the occurrences of the \(k\)-basic factor with \(O(1)\)-time delay, for each explicit node we maintain pointers to the leftmost and rightmost terminal node in the corresponding subtree. Additionally, we maintain a linked list of terminal nodes in lexicographic order of their labels.

We use Lemma 3.4 to efficiently construct the data structures for rank and select queries on \(B_k\), but first we need to determine these bit vectors. Note that a single node can be \(k\)-basic for many values of \(k\), but these values form a range, since the set of lengths of subwords, for which \(u\) is an explicit locus, forms a range. We can construct such ranges for each explicit node. For a range \([k_1, k_2]\) of a node \(u\), \(id(u) = i\), we construct events \((k_1, i)\) and \((k_2 + 1, i)\). Then \(B_k\) can be computed from \(B_{k-1}\) by flipping all bits \(i\) for which \((k, i)\) is an event, with \(B_{-1}\) defined as a null vector. The number of events is linear, so in \(O(n)\) time we can construct all vectors \(B_k\) and equip them with the data structure for rank and select queries.

Now, answering queries is simple: for (1) we find the explicit locus of \(BF_k(i)\) using Lemma 7.1, and then determine its identifier using a rank\(_{B_k}\) query. For (2) we use a select\(_{B_k}\) query to get an explicit locus \(u\). Note that the corresponding basic factor occurs at position \(i\) if and only if the suffix \(v[i, n]\) has its locus in the subtree rooted in \(u\). Thus, it suffices to visit all terminal nodes in the subtree rooted in \(u\). We use the pointers to leftmost and rightmost terminal node and the linked list of terminal nodes to visit them with \(O(1)\)-time delay. Finally for (3) it suffices to note that \(m_k\) is the number of \(k\)-basic nodes, which is the total number of 1-bits in \(B_k\). This concludes the implementation of CompactDBF and gives the following result.

**Lemma 7.3.** For a word \(v\) of length \(n\) there exists CompactDBF \(D(v)\) which takes \(O(n)\) space, can be constructed in \(O(n)\) time and can answer (1) and (3) queries in \(O(1)\) time, and (2) queries with \(O(1)\)-time delay per item reported.
Recall that in order to define \texttt{sample}, we use identifiers $ID_k[j] = \pi_k(DBF_k[j])$, where $\pi_k$ is a permutation of $S_{mk}$. \texttt{RANDOMIZEDDBF} is a modification of \texttt{COMPACTDBF}, which instead of $DBF_k[i]$ operates on $ID_k[i] = \pi_k(DBF_k[i])$ as identifiers for queries (1) and (2), where for each level $k$, the permutation $\pi_k$ is drawn uniformly at random from $S_{mk}$.

**Lemma 7.4.** For a word $v$ of length $n$ there exists a \texttt{RANDOMIZEDDBF} $D^*(v)$ which takes $O(n)$ space, can be constructed in $O(n)$ expected time and can answer (1) and (3) queries in $O(1)$ time, and (2) queries in with $O(1)$-time delay per item reported.

**Proof.** To obtain random identifiers, it suffices to randomly shuffle identifiers $id$ in the previous construction (in particular the bit vectors $B_k$ are indexed using these random identifiers). Then for each $k$ the identifiers of $k$-basic nodes also form a random order, since a (uniformly) random order of a set induces a uniformly random order of any subset. \hfill $\square$

\texttt{RANDOMIZEDDBF} is the source of randomization in our construction algorithm. Note that for different values of $k$, permutations $\pi_k$ are not independent. Actually, we could not draw them independently, as this would require $\Omega(\log^2 n)$ bits of randomness as opposed to $O(n \log n)$ we can get during the $O(n)$-time construction.

### 7.3 Probabilistic Tools

Let $a$ be a sequence of length $n$ over $[1,m]$ and let $\pi \in S_m$. We say that the sequence $a$ is $\Delta$-diverse if for each element $\sigma \in [1,m]$ the set $\{i : a_i = \sigma\}$ is $\Delta$-sparse. Fix a positive integer $\Delta$ and for $i \in [1,n-\Delta]$ define

$$f_{\pi}(i) = \text{argmin}\{\pi(a_j) : j \in [i,i+\Delta]\}.$$ 

In case of ties, which are possible if $a$ is not $\Delta$-diverse, we take the leftmost index. We say that the values of $f_{\pi}$ are local $\pi$-minima; see Figure 7.

**Observation 7.5.** Assignment $\texttt{sample}_k$ is a local $\pi_k$-minima function for $\pi_k$ such that $ID_k[j] = \pi_k(DBF_k[j])$, $\Delta = 2^k$ and sequence $a = DBF_k$, which is $\Delta/2$-diverse by Fact 5.1.

**Lemma 7.6.** Assume that $a$ is a $\frac{\Delta}{2}$-diverse sequence of length $n$ over $[1,m]$ and let $\pi$ be a permutation of $[1,m]$ drawn uniformly at random. Let $A = \{i : \pi(a_i) \leq \ell\}$ for a parameter $\ell$, and $C = \text{FillGaps}(A, \Delta, [1,n])$. Then

(a) $C$ contains all local $\pi$-minima,
(b) if $\ell = \lceil \frac{2m \log \Delta}{\Delta} \rceil$, then $\mathbb{E}[|C|] = O(\frac{n \log \Delta}{\Delta})$.

![Figure 7](image.png)

Figure 7: An illustration of $f_{\pi}$ for $m = 4$, $\Delta = 4$, $\pi = (3,2,1,4)$ and an example sequence $a$. Shades of gray represent intervals in a step representation of $f_{\pi}$: $(1,2), (3,4), (5,7), (7,11)$. Note that $a$ is 2-diverse.
Proof of Lemma 7.6(a). Assume \( j = f_\pi(i) \) is a local \( \pi \)-minimum. If \( \pi(a_j) \leq \ell \), then \( j \in A \), so in particular \( j \in C \). Otherwise, not only \( \pi(a_j) > \ell \), but for any \( i' \in [i, i + \Delta] \) we have \( \pi(a_{i'}) > \ell \). Thus \([i, i + \Delta] \subseteq [1, n] \setminus A\), i.e., the FillGaps operation adds this interval, and in particular \( j \), to \( C \). \( \square \)

Before we give the proof of Lemma 7.6(b), let us recall a standard fact and apply it in an auxiliary claim.

Fact 7.7. Let \( U \) be a set, \( T \) be its subset of size \( t \), and let \( S \) be drawn uniformly at random from the family of subsets of \( U \) of size \( s \). Then \( \mathbb{P}[S \cap T = \emptyset] \leq (1 - \frac{t}{|U|})^s \leq \exp\left(-\frac{st}{|U|}\right) \).

Claim 7.8. Let \( P \subseteq [1, n] \) be an interval of size \( \left\lceil \frac{\Delta}{2} \right\rceil \). Then \( \mathbb{P}[P \cap A = \emptyset] \leq \frac{2}{\Delta} \).

Proof. Let \( V_P = \{a_i : i \in P\} \). Observe that \( \frac{\Delta}{2} \)-diversity implies that \( |V_P| = |P| \leq m \). Note that \( P \cap A = \emptyset \) if and only if \( \pi(V_P) \cap [1, \ell] = \emptyset \), or equivalently \( V_P \cap \pi^{-1}([1, \ell]) = \emptyset \). Observe that \( L = \pi^{-1}([1, \ell]) \) is a subset of \([1, m]\) of size \( \ell \) drawn from the uniform distribution. Thus by Fact 7.7

\[
\mathbb{P}[P \cap A = \emptyset] = \mathbb{P}[V_P \cap L = \emptyset] \leq \exp\left(-\frac{|P|}{m}\right) \leq \exp\left(-\frac{2\ell}{{\Delta}} + \frac{|P|}{m}\right) = \exp\left(-\frac{2|P|\log \Delta}{m} - \frac{|P|}{m}\right) \leq \exp(- \log \Delta + 1) \leq \frac{2}{\Delta}. \]

Proof of Lemma 7.6(b). First, let us bound the expected size of \( A \).

\[
\mathbb{E}[|A|] = \sum_{j=1}^{n} \mathbb{P}[^{P}\pi(a_j) \leq \ell] = \sum_{j=1}^{n} \frac{\ell}{m} = \frac{\ell n}{m} = \frac{2m \log \Delta}{\Delta} \leq \frac{2n \log \Delta}{\Delta}.
\]

Let us consider a position \( j \in C \setminus A \). By definition of FillGaps there must be an integer interval \( R \) such that \( j \in R \subseteq [1, n] \setminus A \) and \( |R| > \Delta \). Let us define \( R_{\leq j} = [j - \left\lceil \frac{\Delta}{2} \right\rceil + 1, j] \) and \( R_{\geq j} = [j, j + \left\lceil \frac{\Delta}{2} \right\rceil - 1] \). Note that \( R_{\leq j} \subseteq R \) or \( R_{\geq j} \subseteq R \). Moreover, \( R_{\leq j} \subseteq R \) implies \( R_{\leq j} \subseteq [1, n] \) and \( R_{\leq j} \cap A = \emptyset \), by Claim 7.8 this holds with probability at most \( \frac{2}{\Delta} \). A similar reasoning holds for \( R_{\geq j} \).

Therefore,

\[
\mathbb{E}[|C \setminus A|] = \sum_{j=1}^{n} \mathbb{P}[j \in C \setminus A] \leq \sum_{j=1}^{n} \frac{4}{\Delta} = \mathcal{O}\left(\frac{n}{\Delta}\right).
\]

Consequently,

\[
\mathbb{E}[|C|] = \mathbb{E}[|A|] + \mathbb{E}[|C \setminus A|] = \mathcal{O}\left(\frac{n \log \Delta}{\Delta} + \frac{n}{\Delta}\right) = \mathcal{O}\left(\frac{n \log \Delta}{\Delta}\right). \quad \square
\]

Part (a) of Lemma 7.6 justifies the correctness of our approach to the construction, part (b) is central to the \( \mathcal{O}(n) \) bound for the expected running time.

7.4 Computing Candidates

As indicated in the beginning of this section, the crucial step of the construction algorithm is building candidate sets \( C_k \), supersets of \( \text{SAMPLES}_k \) whose total size is \( \mathcal{O}(n) \).

Lemma 7.9. Let \( v \) be an arbitrary word of length \( n \). There exists an algorithm which returns sets \( C_k \subseteq [1, n_k] \) together with identifiers \( ID_k[j] \) for \( j \in C_k \) such that:

- there exist permutations \( \pi_k \in S_{m_k} \) such that \( ID_k[j] = \pi_k(DBF_k[j]) \) and, for the samples partial assignment \( \text{sample}_k \) defined using (5) with these identifiers, \( C_k \supseteq \text{SAMPLES}_k \).
The expected running time of the algorithm is $O(n + \sum_k |C_k|) = O(n)$.

Proof. The algorithm is based on an instance of \textsc{RandomizedDBF} $D^*(v)$, which in particular makes a random choice of the underlying permutations $\pi_k$. We start with presenting our construction of sets $C_k$, then prove its correctness using the results of Section 7.3 and conclude with an efficient implementation of our algorithm.

Recall that in Section 6 we have defined $P_k$ as the set of positions $j$ such that $BF_k(j)$ is periodic. By $N_k$ we denote its complement $[1, n_k] \setminus P_k$. The candidate sets $C_k$ are constructed in two steps. First we set $A_k = \{j \in N_k : ID_k[j] \leq \ell_k\}$ for $\ell_k = \left\lfloor \frac{kn_k}{2^k} \right\rfloor$. Then for each block $B$ of $N_k$ we compute $\text{FillGaps}(A_k \cap B, 2^k, B)$ and return the union of these sets as $C_k$.

Consider a block $B'$ of the set of positions where \text{sample}_k is defined. Note that if we extend $B'$ by $2^k$ positions to the right, we obtain a block $B$ of $N_k$. Moreover, \text{sample}_k for positions in $B'$ assigns local $\pi_k$-minima for $DBF_k$ restricted to $B$. By Fact 5.1, $DBF_k$ restricted to $B$ is $2^k - 1$-diverse. Consequently, we can apply Lemma 7.6, which gives $E[|B \cap C_k|] = O\left(\frac{|B|k}{2^k}\right)$. By linearity of expectation we conclude that $E[|C_k|] = O\left(\frac{kn_k}{2^k}\right)$ and $E[\sum_k |C_k|] = O(n)$. This proves the correctness of our algorithm, it remains to provide an efficient implementation.

Lemma 6.7 lets us quickly compute sets $P_k$, both in a block representation and as bit vectors. The former can be easily transformed to a block representation of $N_k$ and the latter can be used to test in $O(1)$ time for $j \in [1, n_k]$ whether $BF_k(j)$ is periodic. We construct $A_k$ separately for every $k$. We use a type (3) query on $D^*(v)$ to determine $m_k$, which is necessary to compute $\ell_k$. Then for each identifier $\leq \ell_k$ we use a type (2) query on $D^*(v)$ to get one occurrence of the corresponding $k$-basic factor. We use the bit-vector representation of $P_k$ to test if this $k$-basic factor is periodic. For non-periodic basic factors we proceed with the execution of the type (2) query adding to $A_k$ all the positions where the basic factor occurs. This way $A_k$ is constructed in $O\left(1 + |A_k| + \frac{kn_k}{2^k}\right) = O\left(|A_k| + \frac{kn_k}{2^k}\right)$ time. Then we simultaneously sort all $A_k$'s, which increases the running time by a single $O(n)$ term.

Once $A_k$ are sorted we apply the FillGaps operations, again independently for each $k$. We simultaneously traverse $A_k$ and the blocks of $N_k$. This lets us determine all blocks of $N_k \setminus A_k$, and add to $C_k$ all elements of those blocks of size at least $2^k + 1$. This is equivalent to running $\text{FillGaps}(A_k \cap B, 2^k, B)$ separately for every block $B$ of $N_k$. Apart from $O(|C_k|)$ time to traverse $A_k$ and actually fill the gaps, this procedure requires additional time proportional to the number of blocks of $N_k$. However, since these representations for all sets $N_k$ were constructed in $O(n)$ total time, this extra cost sums up to $O(n)$. Finally, we equip each $j \in C_k$ with $ID_k[j]$ using type (1) query on $D^*(v)$.

Unfortunately, the bound on $\sum_k |C_k|$ provided by Lemma 7.9 holds only in expectation, and we are to construct a data structure with a guaranteed $O(n)$ size bound. Nevertheless, it easy to modify this algorithm so that $\sum_k |C_k|$ is guaranteed to be $O(n)$.

**Lemma 7.10.** The algorithm of Lemma 7.9 can be modified so that $\sum_k |C_k|$ is guaranteed to be $O(n)$, with the running time still $O(n)$ in expectation.

Proof. We run the algorithm of Lemma 7.9 and repeat until the actual value of $\sum_k |C_k|$ does not exceed twice the expectation. If the random bits used by subsequent iterations are independent, by Markov inequality each iteration succeeds with probability at least $\frac{1}{2}$. Consequently, the probability
that the $i$-th iteration is performed is at most $\frac{1}{k}$. The expected running time of a single iteration is $O(n)$, so the total expected running time is $O(n)$. \hfill \square

### 7.5 From Candidates to Samples

The main part of the construction algorithm is building a small-sized step representation of a samples partial assignment $\text{sample}_k$. An appropriate assignment is already implied by the output of the algorithm of Lemma 7.10, which gives sets $C_k$ equipped with identifiers $ID_k[j]$ for $j \in C_k$. The subsequent step involves the an abstract function, whose implementation is based on the following folklore result:

**Fact 7.11.** A simple queue can be augmented so that it can return its minimal element (settling ties arbitrarily) and all operations enqueue, dequeue and find-min on the queue work in $O(1)$ amortized time.

| Function Slider |
|------------------|
| **Input:** Positive integers $d \leq m$ and a set $A$ of pairs $(q, p)$ with $q \in \mathbb{Z}$ and $p \in [1, m]$. |
| **Output:** A step representation of $G : [1, m - d] \rightarrow A$ defined as follows: $G(i)$ is the lexicographically smallest pair $(q, p) \in A$ among pairs with $p \in [i, i + d]$, ⊥ if no such pair exists. |

**Lemma 7.12.**_slider can be implemented in $O(|A|)$ time, provided that pairs in $A$ are sorted by $p$ in the input.

**Proof.** We traverse all values $i \in [1, m - d]$ in the increasing order maintaining $Q_i = \{(q, p) \in A : p \in [i, i + d]\}$ stored in the augmented queue of Fact 7.11, with minima computed according to the lexicographic order on pairs.

Note that $G(i) = \min Q_i$, moreover $Q_i$ can be obtained from $Q_{i-1}$ by enqueueing all pairs with their second coordinate equal to $i + d$ and dequeueing all pairs with their second coordinate equal to $i - 1$. We can store these operations as events and sort the events (merging the lists of events of both kinds), so that we perform any work only for $i$ with some events associated. For every such $i$ we evaluate $G(i)$ using the find-min query, and if the value is different than previously, we start a new interval in the step representation of $G$. \hfill \square

Finally the construction of the data structure from the candidate sets works as follows. We run Slider for $\{(ID_k[j], j) : j \in C_k\}$ and $d = 2^k$, which gives a step representation of a function

$$G(i) = \begin{cases} \perp & \text{if } C_k \cap [i, i + 2^k] = \emptyset, \\ \min\{(ID_k[j], j) \in C_k : j \in [i, i + 2^k]\} & \text{otherwise}. \end{cases}$$

Since $C_k$ is guaranteed to contain all sample positions, whenever $\text{sample}_k(i) = j \neq \perp$ we have $G(i) = (ID_k[j], j)$. Thus, in order to construct a step representation of $\text{sample}_k$, it suffices to find all the maximum intervals where $\text{sample}_k$ is defined, use a representation of $G$ in these intervals and set $\perp$ elsewhere.

Recall that we have defined $N_k = \{j : BF_k(j) \text{ is non-periodic}\}$ and block representations of all sets $N_k$ can be obtained in $O(n)$ time (see Lemma 7.9). Also, note that $\text{sample}_k(i) \neq \perp$ if and only if $[i, i + 2^k] \subseteq N_k$. Therefore, it suffices to take all intervals in the representation of $N_k$, remove those of length at most $2^k$ and trim the remaining by $2^k$ positions from the right. Consequently, a
step representation of sample_k can be constructed in time proportional to |C_k| and the size of the block representation of N_k. Both terms sum up to O(n) for all values of k.

Once we have sample_k, we can run the construction algorithm of evaluator E(sample_k). We also prepare the set \{(ID_k[j], j)\}, which is passed to the construction algorithm of a locator L(A^k), where A^k is an indexed family of sets A^k_{id}, with A^k_{id} defined as the set of all sample occurrences of the k-basic factor whose identifier ID_k is id.

Finally, we construct the global components, i.e., the data structure of Theorem 4.3 for the periodic case and the component of Fact 2.1. This way we obtain the result announced already in the introduction that concludes the whole description of the data structure for IPM Queries.

**Theorem 1.1.** INTERNAL PATTERN MATCHING QUERIES can be answered in O(1) time by a data structure of size O(n), constructed in O(n) expected time.

Unfortunately, for some input strings our construction algorithm with \frac{1}{n^{o(1)}} probability works in \omega(n) time. We leave as an open problem to improve it to run in O(n) time with high probability, or design a deterministic O(n \log^{O(1)} n)-time construction.

8 Applications

Before we proceed with the applications of our data structure for IPM QUERIES, let us state a stronger version of Fact 2.1. For words x, y denote the length of their longest common prefix by lcp(x, y).

**Fact 8.1.** Let v be a word of length n. After O(n)-time preprocessing the following queries can be answered in O(1) time for any words x, y represented as concatenations of O(1) subwords of v:

(a) compute lcp(x, y),
(b) decide if x = y,
(c) for an integer p \leq |x| find the longest prefix of x which has period p.

We introduce one more combinatorial observation, which we use for each of the applications.

**Observation 8.2.** Let u, u' be words such that lcp(u, u') \geq q. Let d (d') be the length of the longest prefix of u (resp. u') which has period q. If d \neq d', then lcp(u, u') = \min(d, d'). Otherwise, lcp(u, u') \geq d.

8.1 Prefix-Suffix Queries

We show the solution for PREFIX-SUFFIX QUERIES (and furthermore PERIOD QUERIES and 2-PERIOD QUERIES) using IPM QUERIES. Let us recall the definition of the former.

**Definition 8.3.** If a word z is simultaneously a prefix of a word x and a suffix of a word y, we call it a prefix-suffix of a pair \{(x, y)\}. 

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We assume that $|x|, |y| \geq d$, since otherwise there are no prefix-suffixes to report. Let $x'$ be the prefix of $x$ of length $d$ and $y'$ be the suffix of $y$ of length $\min(2d, |y|)$. Observe that any prefix-suffix implies an occurrence of $x'$ in $y'$. We find all such occurrences with a single IPM Query. Let $y_i$ be the suffix of $y$ corresponding to the $i$-th occurrence of $x'$. If there are at most two occurrences, we use Fact 8.1 to verify which $y_i$ is a prefix-suffix.

Otherwise, we conclude that $x'$ is periodic and its period $q$ is, by Lemma 2.2, the difference in the arithmetic progression of occurrences. Let $y_i$ be the suffix of $y'$ starting with the $i$-th leftmost occurrence of $x'$ in $y'$. We use Fact 8.1 to find the longest prefixes $d'$ and $d_1$ of $x$ and $y_1$, respectively, that have period $q$. We will answer the Prefix-Suffix Query in constant time based on their lengths (see Figure 8).

Note that for each $i$, the length of the longest prefix of $y_i$ which admits period $q$ is $d_i = d_1 - (i - 1)q$. Also, observe that $y_i$ is a prefix-suffix if and only if $lcp(x, y_i) = |y_i|$. We apply Observation 8.2 to conclude the following. If $d_1 < |y_1|$, then the only prefix-suffix might be $y_i$ with $d_i = d'$, which we can verify using Fact 2.1. Otherwise, $d_1 = |y_1|$ and thus $y_i$ is a prefix-suffix whenever $d' \geq d_i = |y_i|$, i.e., for $i \leq 1 + \frac{d_1 - d'}{q}$.

Consequently, the data structure of Theorem 1.1 can answer Prefix-Suffix Queries in $O(1)$ time, which gives the following result.

**Theorem 8.4.** Using a data structure of $O(n)$ size, which can be constructed in $O(n)$ expected time, one can answer Prefix-Suffix Queries in $O(1)$ time.

**Period Queries**

Given a subword $x$ of $v$, report all periods of $x$ (represented by disjoint arithmetic progressions).

**Theorem 8.5.** Using a data structure of size $O(n)$, which can be constructed in $O(n)$ expected time, one can answer Period Queries in $O(\log |x|)$ time.

**Proof.** Period Queries can be answered using the data structure for Prefix-Suffix Queries. To compute all periods of $x$ we use Prefix-Suffix Queries to find all borders of $x$ (words which are simultaneously prefixes and suffixes of $x$) of length between $2^k - 1$ and $2(2^k - 1)$ for each $k \in [0, \lfloor \log(|x| + 1) \rfloor]$. Lengths of borders can be easily transformed to periods, since any word $x$ has period $p$ if and only if it has a border of length $|x| - p$. \hfill \square

**2-Period Queries**

Given a subword $x$ of $v$, decide whether $x$ is periodic and, if so, compute its shortest period.

While 2-Period Queries can be easily reduced to Prefix-Suffix Queries (asking for borders of $x$ of length at least $\frac{|x|}{2}$), our techniques give a much simpler solution, with an additional merit in the form of deterministic construction algorithm.
Theorem 8.6. Using a data structure of size $O(n)$, which can be constructed in $O(n)$ time, one can answer 2-Period Queries in $O(1)$ time.

Proof. Recall that for any periodic subword $x$ we have defined $\text{run}(x)$ as the run extending $x$, and as $\bot$ if $x$ is not periodic. Lemma 6.9 gives a data structure computing $\text{run}(x)$ in $O(1)$ time, that can be constructed in $O(n)$ deterministic time. Moreover, if $x$ is periodic then $\text{per}(x) = \text{per}(\text{run}(x))$.

8.2 Cyclic Equivalence Queries
We define $\text{Rot}(u) = u[n]u[1]\ldots u[n-1]$. Additionally, for an integer $r$ we write $\text{Rot}(u, r)$ to denote $\text{Rot}$ applied $r$ times on $u$. Note that $\text{Rot}(u, r) = \text{Rot}(u, r')$ if $r \equiv r' \pmod{n}$. We say that $w$ is a cyclic rotation of $u$ if there exists an integer $r$ such that $w = \text{Rot}(u, r)$. Let us recall the statement of Cyclic Equivalence Queries.

Cyclic Equivalence Queries
Given subwords $x$ and $y$ of $v$, decide whether $x$ is a cyclic rotation of $y$ and, if so, report all corresponding cyclic shift values (represented as an arithmetic progression).

We proceed with an algorithm answering this type of queries. We can clearly assume that $|x| = |y|$. Let us denote the common length of $x$ and $y$ by $d$, and the desired set of cyclic shifts \{ $r \in [0, d - 1] : y = \text{Rot}(x, r)$ \} by $R(x, y)$. The following observation not only is useful for computing $R(x, y)$, but combined with Lemma 2.2 also proves that this set indeed forms an arithmetic progression.

Observation 8.7. Let $x, y$ be words of common length. Then $R(x, y)$ is equal to the set of positions among $\{ 0, \ldots, |y| - 1 \}$ where $x$ occurs in $yy$.

Below, we give an algorithm which computes $R(x, y) \cap [0, \left\lfloor \frac{d}{2} \right\rfloor]$, i.e., cyclic shifts not exceeding $\frac{d}{2}$. Note that $y = \text{Rot}(x, r)$ if and only if $x = \text{Rot}(y, d - r)$, so running this algorithm both for $(x, y)$ and $(y, x)$ lets us easily retrieve $R(x, y)$.

Let $x'$ be the prefix of $x$ of length $\left\lceil \frac{d}{2} \right\rceil$. Note that any occurrence of $x$ in $yy$ at position $\leq \frac{d}{2}$ induces an occurrence of $x'$ in $y$. We use IPM Queries to find all positions where $x'$ occurs in $y$, each of them is a candidate shift value. If the number of occurrences is constant (at most 2), we can verify each candidate, using Fact 8.1(b) to test whether $x$ actually occurs in $yy$ at the appropriate position.

Otherwise, Lemma 2.2 guarantees that the occurrences lie at positions $j_1, \ldots, j_m$ which form an arithmetic progression with difference $q = \text{per}(x')$. We need to find out at which of these positions $x$ actually occurs in $yy$. We apply Fact 8.1(c) to find two values: $d'$, the length of the longest prefix of $x$ which admits period $q$, and $d_1$, the length of the longest prefix of $y_1 = y[j_1, |y|]y$ which admits

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Figure 9: Cyclic Equivalence Queries
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period \( q \) (see Figure 9). Observe that for any \( i \in [1, m] \) the longest prefix of \( y_i = y[j_i, |y|]y \) which admits period \( q \) has length \( d_i = d_1 - (i - 1)q \).

Due to Observation 8.2 we have two cases:

- If \( d' < |x| \) then, among the candidates considered, \( x \) may occur in \( yy \) only at position \( j_i \) which satisfies \( d_i = d' \). We check this candidate using Fact 8.1(b),
- Otherwise, \( q \) is a period of \( x \), and \( x \) occurs in \( yy \) at all positions \( j_i \) which satisfy \( |x| \geq d_i \).

Thus, the data structure for IPM Queries accompanied with the one of Fact 8.1 can answer Cyclic Equivalence Queries.

**Theorem 8.8.** Using a data structure of \( \mathcal{O}(n) \) size, which can be constructed in \( \mathcal{O}(n) \) expected time, one can answer Cyclic Equivalence Queries in \( \mathcal{O}(1) \) time.

### 8.3 Generalized Substring Compression

In this section we improve the results of [17] for Generalized Substring Compression Queries.

| Generalized Substring Compression Queries |
|------------------------------------------|
| Given two subwords \( x \) and \( y \) of \( v \), compute \( LZ(x|y) \), that is, the part of the LZ77 [30] compression \( LZ(y$sx) \) corresponding to \( x \), where $ \notin \Sigma \). |

We actually provide a more efficient algorithm for the following auxiliary problem, introduced in [17].

| Bounded Longest Common Prefix Queries |
|---------------------------------------|
| Given two subwords \( x \) and \( y \) of \( v \), find the longest prefix \( p \) of \( x \) which is a subword of \( y \). |

We use as a black-box several results developed in [17] for the original solution for Bounded Longest Common Prefix Queries. The authors of [17] define the following auxiliary problem.

| Interval Longest Common Prefix Queries |
|----------------------------------------|
| Given an interval \([\ell, r]\) and a subword \( x \) of \( v \), find the longest prefix \( p \) of \( x \) which occurs at some position \( t \in [\ell, r] \) in \( v \). |

**Lemma 8.9** ([17]). Using a data structure of \( \mathcal{O}(n + S_{rsucc}) \) size, one can answer Interval Longest Common Prefix Queries in \( \mathcal{O}(Q_{rsucc}) \) time, provided that \( x \) is given by its locus in the suffix tree \( T(v) \).

**Lemma 8.10** ([17]). For a word \( v \) of length \( n \) there exists a data structure of size \( \mathcal{O}(n + S_{rsucc}) \), such that given subwords \( x, y \) one can decide whether \( x \) occurs in \( y \) in \( \mathcal{O}(Q_{rsucc}) \) time, provided that \( x \) is given by its locus in the suffix tree \( T(v) \).

We have already referred to the result of Lemma 8.10 as a decision version of IPM Queries. Recall that \( S_{rsucc} \) and \( Q_{rsucc} \) were defined in the Introduction as the matching space and query-time bounds for orthogonal range successor queries.

Additionally, recall the following result due to Gawrychowski [15] already introduced in Section 7.2.

**Lemma 7.1.** The suffix tree \( T(v) \) can be preprocessed in \( \mathcal{O}(|v|) \) time so that given integers \( i, k \) the locus of the basic factor \( BF_k(i) \) can be determined in \( \mathcal{O}(1) \) time.
We proceed with the solution for BOUNDED LONGEST COMMON PREFIX QUERIES. Assume \( x = v[\ell', r'] \) and \( y = v[\ell, r] \). First, we search for the largest \( k \) such that the prefix of \( x \) of length \( 2^k \) (i.e., \( BF_k(\ell') \)) occurs in \( y \). We use a variant of binary search involving exponential search (also called galloping search), which requires \( O(\log K) \) steps where \( K \) is the optimal value of \( k \). At each step for a fixed \( k \) we need to decide if \( BF_k(\ell') \) occurs in \( y \). This can be done in \( O(Q_{rsucc}) \) time: we find the locus of \( BF_k(\ell') \) using Lemma 7.1 and then apply Lemma 8.10.

At this point we have an integer \( K \) such that the optimal prefix \( p \) has length \( 2^K \leq |p| < 2^{K+1} \). So far the time is \( O(Q_{rsucc} \log K) = O(Q_{rsucc} \log \log |p|) \).

Let \( p' \) be the prefix obtained from INTERVAL LONGEST COMMON PREFIX QUERY for \( x \) and \([\ell, r - 2^{K+1}] \). Note that \( BF_{K+1}(\ell') \) does not occur in \( x \), so \(|p'| < 2^{K+1} \) and thus the occurrence of \( p' \) starting within \([\ell, r - 2^{K+1}] \) lies within \( y \). Thus, \(|p'| \leq |p| \); moreover, if \( p \) occurs at a position within \([\ell, r - 2^{K+1}] \), then \( p = p' \).

The other possibility is that \( p \) occurs in \( y \) only near its end, i.e., within the suffix of \( y \) of length \( 2^{K+1} \), which we denote as \( y' \). We use a similar approach as for PREFIX-SUFFIX QUERIES with \( d = 2^K \) to detect \( p \) in this case. We define \( x' \) as the prefix of \( x \) of length \( d \), and \( y' \) as the suffix of \( y \) of length \( \min(2d, |y|) \). Note that an occurrence of \( p \) must start with an occurrence of \( x' \), so we find all occurrences of \( x' \) in \( y' \). If there are at most two of them, we check them manually using Fact 8.1(a).

Otherwise, we know \( q = \operatorname{per}(x') \) and compute \( d' \), the length of the longest prefix of \( x \) which admits period \( q \), and \( d_1 \), the length of the longest prefix of \( y_1 \) which admits period \( q \) (see Figure 8).

Observation 8.2 lets us restrict our attention to \( y_1 \) (which maximizes \( \min(d', d_i) \)) and \( y_i \) such that \( d' = d_i \) (if any). Thus, even if there are more occurrences of \( x' \) in \( y' \), we need to consider only 2 of them.

Consequently, we can always choose the final solution as the best among three candidates: one obtained from the INTERVAL LONGEST COMMON PREFIX QUERY, and two corresponding to the occurrences of \( x' \) in \( y' \), with the actual lengths obtained using lcp queries.

Thus, the data structure for IPM QUERIES, accompanied with the suffix tree \( T(v) \) and the data structure of Lemma 7.1, as well as the data structures of Lemmas 8.9, 8.10 and Fact 8.1, gives the following result.

**Lemma 8.11.** Using a data structure of \( O(n + S_{rsucc}) \) size, the BOUNDED LONGEST COMMON PREFIX QUERIES can be answered in \( O(Q_{rsucc} \log \log |p|) \) time.

Lemma 8.11 yields the following corollary.

**Theorem 8.12.** Using a data structure of \( O(n + S_{rsucc}) \) size, one can answer GENERALIZED SUBSTRING COMPRESSION QUERIES in \( O(CQ_{rsucc} \log \log |p|) \) time, where \( C \) is the number of phrases reported.

**Proof.** The algorithm for GENERALIZED SUBSTRING COMPRESSION QUERIES is the same as the one presented in [17], it just uses our solution for BOUNDED LONGEST COMMON PREFIX QUERIES instead of the original one. Its running time is \( O(\sum_{i=1}^{C} Q_{rsucc} \log \log |p_i|) \), where \( p_1, \ldots, p_C \) are lengths of the output phrases. Using Jensen’s inequality for the \( \log \log \) function we derive the desired bound. \( \square \)
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