VANISHING OF STABLE HOMOLOGY WITH RESPECT TO A SEMIDUALIZING MODULE

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Abstract. We investigate stable homology of modules over a commutative noetherian ring $R$ with respect to a semidualizing module $C$, and give some vanishing results that improve/extend the known results. As a consequence, we show that the balance of the theory forces $C$ to be trivial and $R$ to be Gorenstein.

1. Introduction

Stable homology, as a broad generalization of Tate homology to the realm of associative rings, was introduced by Vogel and Goichot [9], and further studied by Celikbas, Christensen, Liang and Piepmeyer [2, 3], and Emmanouil and Manousaki [6]. In their paper [2], it is shown that the vanishing of stable homology over commutative noetherian local rings can detect modules of finite projective (injective) dimension, even of finite Gorenstein dimension, which lead to some characterizations of classical rings such as Gorenstein rings, the original domain of Tate homology. In [6], Emmanouil and Manousaki further investigate stable homology of modules, and give some vanishing results that improve results in [2] by relaxing the conditions on rings and modules.

The study of semidualizing modules was initiated independently by Foxby [8], Golod [10] and Vasconcelos [19]. Over a commutative noetherian ring $R$, a finitely generated $R$-module $C$ is semidualizing if $\text{Hom}_R(C, C) \cong R$ and $\text{Ext}^i_R(C, C) = 0$ for all $i \geq 1$. Examples include finitely generated projective $R$-modules of rank 1.

Modules of finite homological dimension with respect to a semidualizing module have been studied in numerous papers. For example, Takahashi and White [18], and Salimi, Sather-Wagstaff, Tavasoli and Yassemi [15] give some characterizations for such modules in terms of the vanishing of relative (co)homology. In this paper, we show that the vanishing of stable homology can also detect modules of finite homological dimension with respect to a semidualizing module. Our main results are two theorems as shown below; see Theorems 4.6 and 4.7.

Theorem A. Let $R$ be a commutative noetherian ring and let $C$ be a semidualizing $R$-module. For an $R$-module $M$, the following conditions are equivalent.

(i) $\mathcal{F}_C$-pd$_R M < \infty$.

(ii) $\overline{\text{Tor}}^R_{n+1}(C, M, -) = 0$ for each $n \in \mathbb{Z}$.

Date: July 28, 2018.

2010 Mathematics Subject Classification. 13D05, 13D07, 16E05.

Key words and phrases. Stable homology, semidualizing module, proper resolution.

This research was partly supported by NSF of China (Grant Nos. 11761045, 11301240 and 11561039), NSF of Gansu Province (Grant No. 1506RJZA075) and SRF for ROCS, SEM.
(iii) \( \widetilde{\text{Tor}}_n^{PC} X (-, -) = 0 \) for some \( n \geq 0 \).

**Theorem B.** Let \( R \) be a commutative noetherian ring and let \( C \) be a semidualizing \( R \)-module. For an \( R \)-module \( N \), the following conditions are equivalent.

(i) \( \mathcal{I}_C \cdot \text{id}_R N < \infty \).

(ii) \( \widetilde{\text{Tor}}_n^{PC}(-, N) = 0 \) for each \( n \in \mathbb{Z} \).

(iii) \( \widetilde{\text{Tor}}_n^{PC}(-, N) = 0 \) for some \( n < 0 \).

The above two results improve the right and left vanishing results in the introduction of \([2]\). Here the notation \( \mathcal{F}_{\text{C-}\text{pd}_R M}, \mathcal{I}_C \cdot \text{id}_R N \) and \( \widetilde{\text{Tor}}_n^{PC}(-, -) \) can be found in \([2, 2]\) and \([1, 1]\). As a consequence, we show that the isomorphisms \( \widetilde{\text{Tor}}_n^{PC} (M, N) \cong \widetilde{\text{Tor}}_n^{PC} (N, M) \) for all \( R \)-modules \( M \) and \( N \) force \( C \) to be trivial and \( R \) to be a Gorenstein ring; see Corollary \([1, 3]\).

We prove these results using the next characterization of stable (unbounded) tensor product inspired by the work of Emmanouil and Manousaki \([6]\); see Theorem \([3, 5]\).

**Theorem C.** Let \( X \) be a complex of \( R^\circ \)-modules and \( Y \) a bounded above complex of \( R \)-modules with \( \sup \{ i \in \mathbb{Z} | Y_i \neq 0 \} = k \). Then there are isomorphisms of complexes of \( \mathbb{Z} \)-modules
\[
X \circ R Y \cong \lim_{i \in \mathbb{N}} ((X \circ R Y) / (X \circ R Y_{\leq k-i})),
\]
and
\[
X \circ R Y \cong \lim_{i \in \mathbb{N}} (X \circ R Y_{\leq k-i}).
\]

One refers to \([3, 1]\) for the definitions of \( X \circ R Y \) and \( X \circ R Y \), and \( \lim^1 \) is the right derived functor of the limit \( \lim \); see \([3, 2]\).

2. Preliminaries

We begin with some notation and terminology for use throughout this paper.

2.1. Throughout this work, all rings are assumed to be associative rings. Let \( R \) be a ring; by an \( R \)-module we mean a left \( R \)-module, and we refer to right \( R \)-modules as modules over the opposite ring \( R^\circ \). We denote by \( \mathcal{P} \) (resp., \( \mathcal{F} \), \( \mathcal{T} \)) the class of projective \( R \)-modules (resp., flat \( R \)-modules, injective \( R \)-modules).

By an \( R \)-complex we mean a complex of \( R \)-modules. We frequently (and without warning) identify \( R \)-modules with \( R \)-complexes concentrated in degree 0. For an \( R \)-complex \( X \), we set \( \sup X = \sup \{ i \in \mathbb{Z} | X_i \neq 0 \} \) and \( \inf X = \inf \{ i \in \mathbb{Z} | X_i \neq 0 \} \). An \( R \)-complex \( X \) is bounded above if \( \sup X < \infty \), and it is bounded below if \( \inf X > -\infty \). An \( R \)-complex \( X \) is bounded if it is both bounded above and bounded below. The \( n \)-th homology of \( X \) is denoted by \( H_n(X) \). For each \( k \in \mathbb{Z} \), \( \Sigma^k X \) denotes the complex with the degree-\( n \) term \( (\Sigma^k X)_n = X_{n-k} \) and whose boundary operators are \( (-1)^k \partial^X_{n-k} \). We set \( \Sigma M = \Sigma^1 M \).

If \( X \) and \( Y \) are both \( R \)-complexes, then by a morphism \( \alpha : X \rightarrow Y \) we mean a sequence \( \alpha_n : X_n \rightarrow Y_n \) such that \( \alpha_{n-1} \partial^X_n = \partial^Y_n \alpha_n \) for each \( n \in \mathbb{Z} \). A quasi-isomorphism, indicated by the symbol “\( \sim \)”, is a morphism of complexes that induces an isomorphism in homology.
2.2. Let $\mathcal{X}$ be a class of $R$-modules. Following Enochs and Jenda [7], an $\mathcal{X}$-precover of an $R$-module $M$ is a homomorphism $X \to M$ with $X \in \mathcal{X}$ such that the homomorphism $\text{Hom}_R(X', X) \to \text{Hom}_R(X', M)$ is surjective for each $X' \in \mathcal{X}$. $\mathcal{X}$ is called a precovering class if each $R$-module has a $\mathcal{X}$-precover.

For a precovering class $\mathcal{X}$, there is a complex

$$X^+ \equiv \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

with each $X_i$ in $\mathcal{X}$, such that $\text{Hom}_R(X', X^+)$ is exact for each $X' \in \mathcal{X}$. The truncated complex $X \equiv \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow 0$ is called a proper $\mathcal{X}$-resolution of $M$, which is always denoted by $X \rightarrowtail M$. If $\mathcal{X}$ contains all projective $R$-modules, then the complex $X^+$ is exact. In this case, we always denote by $X \rightarrowtail M$ the proper $\mathcal{X}$-resolution of $M$.

The $\mathcal{X}$-projective dimension of $M$ is the quantity:

$$\mathcal{X}-\text{pd}_R M = \inf \{ \sup X \mid X \rightarrowtail M \text{ is a proper } \mathcal{X}\text{-resolution of } M \}.$$ 

We define preenveloping classes $\mathcal{Y}$, proper $\mathcal{Y}$-coresolutions and $\mathcal{Y}$-injective dimension, $\mathcal{Y}$-$\text{id}_R M$, of $M$ dually.

When $\mathcal{X}$ is the class of projective (resp. flat) $R$-modules, $\mathcal{X}$-$\text{pd}_R M$ is the classical projective (resp. flat) dimension; we refer the reader to [15, Remark 2.6] for the flat case. Also when $\mathcal{Y}$ is the class of injective $R$-modules, $\mathcal{Y}$-$\text{id}_R M$ is the classical injective dimension.

3. A CHARACTERIZATION OF STABLE (UNBOUNDED) TENSOR PRODUCT

We start by recalling the definition of stable (unbounded) tensor product.

3.1. Let $X$ be an $R^\mathbb{Z}$-complex and $Y$ an $R$-complex. The tensor product $X \otimes_R Y$ is the $\mathbb{Z}$-complex with degree-$n$ term

$$(X \otimes_R Y)_n = \prod_{i \in \mathbb{Z}} (X_i \otimes_R Y_{n-i})$$

and differential given by $\partial X \otimes_R Y(x \otimes y) = \partial X(x) \otimes y + (-1)^{|x|} x \otimes \partial Y(y)$. Following [2, 9], the unbounded tensor product $X \otimes_R Y$ is the $\mathbb{Z}$-complex with degree-$n$ term

$$(X \otimes_R Y)_n = \prod_{i \in \mathbb{Z}} (X_i \otimes_R Y_{n-i})$$

and differential defined as above. $X \otimes_R Y$ is a subcomplex of $X \otimes_R Y$, so we let $X \otimes_R Y$ denote the quotient complex $(X \otimes_R Y)/(X \otimes_R Y)$, which is called the stable tensor product.

We notice that if $X$ or $Y$ is bounded, or if both of them are bounded on the same side (above or below), then the unbounded tensor product coincides with the tensor product, and so the stable tensor product $X \otimes_R Y$ is zero.

3.2. Let $\{\nu^n : X^n \rightarrow X^m\}_{n \geq 0}$ be an $\mathbb{N}$-inverse system of $R$-complexes. For the morphism $1 - \nu : \prod_{i \in \mathbb{N}} X^i \rightarrow \prod_{i \in \mathbb{N}} X^i$ given by $(1 - \nu)(x_i)_i = (x_i - \nu_i (x_{i+1}))_i$ for each $k \in \mathbb{Z}$, where $(x_i)_i \in \prod_{i \in \mathbb{N}} X^k_i$, it is well known that $\text{Ker}(1 - \nu) = \lim_{i \in \mathbb{N}} X^i$ and $\text{Coker}(1 - \nu) = \lim_{i \in \mathbb{N}} X^i$. Here $\lim^1$ is the right derived functor of
the limit lim; see e.g. Emmanouil [5], Roos [14] and Yeh [20] for more details. That is, there is an exact sequence of $R$-complexes

$$0 \to \lim_{i \in \mathbb{N}} X^i \to \prod_{i \in \mathbb{N}} X^i \to \prod_{i \in \mathbb{N}} X^i \to \lim_{i \in \mathbb{N}} X^i \to 0.$$ 

Let $X$ be an $R$-complex, and $X = X^0 \supseteq X^1 \supseteq \cdots$ a filtration. Then the embeddings $\varepsilon^i : X^i \to X^{i-1}$ and the morphisms $\pi^i : X^i/X^i \to X/X^{i-1}$ determine the $\mathbb{N}$-inverse systems

$$\{\varepsilon^{uv} : X^v \to X^u\}_{u \leq v}, \quad \{\pi^{uv} : X^v/X^u \to X/X^u\}_{u \leq v}$$

respectively. For these systems, we have the following result.

3.3 Lemma. Let $X$ be an $R$-complex, and $X = X^0 \supseteq X^1 \supseteq \cdots$ a filtration. Then $\lim_{i \in \mathbb{N}} X^i/X^i = 0$, and there exists an exact sequence

$$0 \to \lim_{i \in \mathbb{N}} X^i \to X \to \lim_{i \in \mathbb{N}} X^i \to \lim_{i \in \mathbb{N}} X^i \to 0.$$ 

Proof. Consider the following commutative diagram with exact rows

$$\begin{array}{cccccc}
0 & \longrightarrow & \prod_{i \in \mathbb{N}} X^i & \longrightarrow & \prod_{i \in \mathbb{N}} X & \longrightarrow & \prod_{i \in \mathbb{N}} X/X^i & \longrightarrow & 0 \\
& \Bigg\downarrow 1-\varepsilon & \Bigg\downarrow 1-id & \Bigg\downarrow 1-\pi & \end{array}$$

$$\begin{array}{cccccc}
0 & \longrightarrow & \prod_{i \in \mathbb{N}} X^i & \longrightarrow & \prod_{i \in \mathbb{N}} X & \longrightarrow & \prod_{i \in \mathbb{N}} X/X^i & \longrightarrow & 0. \\
& \end{array}$$

We notice that the constant $\mathbb{N}$-inverse system $\{X\}$ has $\lim_{i \in \mathbb{N}} X = X$ and $\lim_{i \in \mathbb{N}} X = 0$ since $1-id$ is surjective. Then by 3.2 and the snake lemma, one gets the desired results. \( \square \)

3.4. Let $X$ be an $R^e$-complex and $Y$ an $R$-complex. Fix $k \in \mathbb{Z}$, the filtration

$$Y_{\leq k} \supseteq Y_{\leq k-1} \supseteq Y_{\leq k-2} \supseteq \cdots$$

induces a filtration

$$X \otimes_R Y_{\leq k} \supseteq X \otimes_R Y_{\leq k-1} \supseteq X \otimes_R Y_{\leq k-2} \supseteq \cdots.$$ 

Thus we have two $\mathbb{N}$-inverse systems $\{\varepsilon^{uv} : (X \otimes_R Y_{\leq k-\nu})/(X \otimes_R Y_{\leq k-\nu})\}_{u \leq v}$ and $\{\pi^{uv} : (X \otimes_R Y_{\leq k})/(X \otimes_R Y_{\leq k})\}_{u \leq v}$.

3.5 Theorem. Let $X$ be an $R^e$-complex and $Y$ a bounded above $R$-complex with sup $Y = k$. Then there are isomorphisms of $\mathbb{Z}$-complex

$$X \otimes_R Y \cong \lim_{i \in \mathbb{N}}(X \otimes_R Y_{\leq k-i}),$$

and

$$X \otimes_R Y \cong \lim_{i \in \mathbb{N}}(X \otimes_R Y_{\leq k-i}).$$

Proof. We first prove the case where $k = 0$. In this case, $Y = Y_{\leq 0}$. For each $n \in \mathbb{Z}$,

$$(X \otimes_R Y_{\leq 0})_n = \prod_{p \in \mathbb{Z}}((X_{n+p} \otimes_R (Y_{\leq 0})_{-p}) = \prod_{p \geq 0}(X_{n+p} \otimes_R (Y_{\leq 0})_{-p}),$$

and for each $i \geq 1$,

$$(X \otimes_R Y_{\leq -i})_n = \prod_{p \in \mathbb{Z}}((X_{n+p} \otimes_R (Y_{\leq -i})_{-p}) = \prod_{p \geq 1}(X_{n+p} \otimes_R (Y_{\leq 0})_{-p}).$$
Thus one gets
\[
((X \otimes_R Y_{\leq 0})/(X \otimes_R Y_{\leq -1}))_n \cong \prod_{p=0}^{n-1} (X_{n+p} \otimes_R (Y_{\leq 0})_{-p}) = \prod_{p=0}^{n-1} (X_{n+p} \otimes_R (Y_{\leq 0})_{-p}).
\]
This implies that
\[
\lim_{n \in \mathbb{N}} ((X \otimes_R Y_{\leq 0})/(X \otimes_R Y_{\leq -1})) = (X \otimes_R Y_{\leq 0}).
\]
Now it is straightforward to verify
\[
X \otimes_R Y_{\leq 0} \cong \lim_{n \in \mathbb{N}} ((X \otimes_R Y_{\leq 0})/(X \otimes_R Y_{\leq -1})).
\]
Since \(\lim_{n \in \mathbb{N}} (X \otimes_R Y_{\leq -1}) = 0\), there is an exact sequence
\[
0 \to X \otimes_R Y_{\leq 0} \to X \otimes_R Y_{\leq 0} \to \lim_{n \in \mathbb{N}} (X \otimes_R Y_{\leq -1}) \to 0
\]
by Lemma 3.3 and the isomorphism proved above. Thus one gets
\[
X \otimes_R Y_{\leq 0} \cong \lim_{n \in \mathbb{N}} (X \otimes_R Y_{\leq -1}).
\]
In the general case, where \(\sup Y = k \in \mathbb{Z}\), we notice that \(Y = \Sigma^k (\Sigma^{-k} Y)_{\leq 0}\) and \((\Sigma^{-k} Y)_{\leq -1} = \Sigma^{-k} Y_{\leq k-1}\). Thus one has,
\[
X \otimes_R Y = \Sigma^k (X \otimes_R (\Sigma^{-k} Y)_{\leq 0})
\]
\[
\cong \Sigma^k \lim_{n \in \mathbb{N}} ((X \otimes_R (\Sigma^{-k} Y)_{\leq 0})/(X \otimes_R (\Sigma^{-k} Y)_{\leq -1}))
\]
\[
\cong \Sigma^k \lim_{n \in \mathbb{N}} ((X \otimes_R \Sigma^{-k} Y_{\leq k})/(X \otimes_R \Sigma^{-k} Y_{\leq k-1}))
\]
\[
\cong \lim_{n \in \mathbb{N}} ((X \otimes_R Y)/(X \otimes_R Y_{\leq k-1})).
\]
and
\[
X \otimes_R Y = \Sigma^k (X \otimes_R (\Sigma^{-k} Y)_{\leq 0})
\]
\[
\cong \Sigma^k \lim_{n \in \mathbb{N}} (X \otimes_R (\Sigma^{-k} Y)_{\leq -1})
\]
\[
\cong \Sigma^k \lim_{n \in \mathbb{N}} (X \otimes_R \Sigma^{-k} Y_{\leq k-1})
\]
\[
\cong \lim_{n \in \mathbb{N}} (X \otimes_R Y_{\leq k-1}),
\]
as desired. \(\square\)

3.6 Corollary. Let \(X\) be an \(R^e\)-complex and \(Y\) a bounded above \(R\)-complex with \(\sup Y = k\). Then there exists an exact sequence
\[
0 \to \prod_{i \in \mathbb{N}} (X \otimes_R Y_{\leq k-1}) \to \prod_{i \in \mathbb{N}} (X \otimes_R Y_{\leq k-1}) \to X \otimes_R Y \to 0.
\]

Proof. Since \(\lim_{n \in \mathbb{N}} (X \otimes_R Y_{\leq k-1}) = 0\) and \(\lim_{n \in \mathbb{N}} (X \otimes_R Y_{\leq k-1}) \cong X \otimes_R Y\) by Theorem 3.5, the desired exact sequence now follows from 3.2. We notice that the map from \(\prod_{i \in \mathbb{N}} (X \otimes_R Y_{\leq k-1})\) to \(\prod_{i \in \mathbb{N}} (X \otimes_R Y_{\leq k-1})\) in the statement is \(1 - \varepsilon\), where \(\varepsilon^n : X \otimes_R Y_{\leq k-u} \to X \otimes_R Y_{\leq k-u}\) for \(u \leq v\) is induced by the filtration \(Y_{\leq k} \supseteq Y_{\leq k-1} \supseteq Y_{\leq k-2} \supseteq \cdots\); see 3.2 and 3.3. \(\square\)

3.7 Corollary. Let \(X\) be an \(R^e\)-complex and \(Y\) a bounded above \(R\)-complex with \(\sup Y = k\). Then for each \(n \in \mathbb{Z}\), there exists an exact sequence
\[
0 \to \lim_{i \in \mathbb{N}} H_{n+1}(X \otimes_R Y_{\leq k-1}) \to H_{n+1}(X \otimes_R Y) \to \lim_{i \in \mathbb{N}} H_n(X \otimes_R Y_{\leq k-1}) \to 0.
\]
In particular, \( H_{n+1}(X \otimes_R Y) = 0 \) if and only if \( \lim_{i \in \mathbb{N}} H_{n+1}(X \otimes_R Y_{\leq k-i}) = 0 \) = \( \lim_{i \in \mathbb{N}} H_n(X \otimes_R Y_{\leq k-i}) \).

**Proof.** By Corollary 3.6 there is an exact sequence

\[
0 \to \prod_{i \in \mathbb{N}} (X \otimes_R Y_{\leq k-i}) \to \prod_{i \in \mathbb{N}} (X \otimes_R Y_{\leq k-i}) \to X \otimes_R Y \to 0.
\]

Thus one gets the following exact sequence

\[
\cdots \to \prod_{i \in \mathbb{N}} H_{n+1}(X \otimes_R Y_{\leq k-i}) \to \prod_{i \in \mathbb{N}} H_{n+1}(X \otimes_R Y_{\leq k-i}) \to H_n(X \otimes_R Y) \to \prod_{i \in \mathbb{N}} H_n(X \otimes_R Y_{\leq k-i}) \to \cdots ,
\]

which yields the desired exact sequence from the definitions of the \( \lim \) and \( \lim^1 \) groups.

**3.8.** Recall that an \( \mathbb{N} \)-inverse system \( \{ \delta_{uv} : M_v \to M_u \}_{u \leq v} \) of \( R \)-modules satisfies the Mittag-Leffler condition if for each \( i \in \mathbb{N} \) there exists an index \( j \in \mathbb{N} \) with \( j \geq i \), such that \( \text{Im} \delta_{ij} = \text{Im} \delta_{ik} \) for each \( k \in \mathbb{N} \) with \( k \geq j \). It is clear that if \( \delta_{i,i+1} \) is surjective for each \( i \gg 0 \) then the \( \mathbb{N} \)-inverse system \( \{ \delta_{uv} : M_v \to M_u \}_{u \leq v} \) satisfies the Mittag-Leffler condition. Grothendieck proved in [11] that if the \( \mathbb{N} \)-inverse system \( \{ \delta_{uv} : M_v \to M_u \}_{u \leq v} \) satisfies the Mittag-Leffler condition then one has \( \lim_{i \in \mathbb{N}} M_i = 0 \). Moreover, following [5, Corollary 6], \( \lim_{i \in \mathbb{N}} M_i^{(n)} = 0 \) if and only if the \( \mathbb{N} \)-inverse system \( \{ \delta_{uv} : M_v \to M_u \}_{u \leq v} \) satisfies the Mittag-Leffler condition.

**3.9 Corollary.** Let \( X \) be an \( R^\circ \)-complex and \( Y \) a bounded above \( R \)-complex with sup \( Y = k \), and let \( n \in \mathbb{Z} \). If \( H_n(X^{(n)} \otimes_R Y) = 0 \), then the \( \mathbb{N} \)-inverse system \( \{ \delta_{uv} : H_n(X^{(n)} \otimes_R Y_{\leq k-v}) \to H_n(X^{(n)} \otimes_R Y_{\leq k-u}) \}_{u \leq v} \) satisfies the Mittag-Leffler condition.

**Proof.** If \( H_n(X^{(n)} \otimes_R Y) = 0 \), then by Corollary 3.7 \( \lim_{i \in \mathbb{N}} H_n(X^{(n)} \otimes_R Y_{\leq k-i}) = 0 \), and so one gets \( \lim_{i \in \mathbb{N}} (H_n(X^{(n)} \otimes_R Y_{\leq k-i}))^{(n)} = 0 \), which implies that the \( \mathbb{N} \)-inverse system \( \{ \delta_{uv} : H_n(X^{(n)} \otimes_R Y_{\leq k-v}) \to H_n(X^{(n)} \otimes_R Y_{\leq k-u}) \}_{u \leq v} \) satisfies the Mittag-Leffler condition; see 3.8.

Checking the proof of [5, Lemma 4.1], one gets the following result.

**3.10 Lemma.** Let \( \{ \delta_{uv} : X_v \to X_u \}_{u \leq v} \) be an \( \mathbb{N} \)-inverse system of \( R \)-modules satisfying the Mittag-Leffler condition. If \( \lim_{i \in \mathbb{N}} X_i = 0 \), then one has
\[
\text{colim}_{i \in \mathbb{N}} \text{Hom}_{\mathbb{Z}}(X_i, \mathbb{Q}/\mathbb{Z}) = 0.
\]

The next proposition will be used to prove our main results advertised in the introduction.

**3.11 Proposition.** Let \( X \) be an \( R^\circ \)-complex and \( Y \) a bounded above \( R \)-complex with sup \( Y = k \), and let \( n \in \mathbb{Z} \). If \( H_n(X^{(n)} \otimes_R Y) = 0 = H_{n+1}(X \otimes_R Y) \), then one has
\[
\text{colim}_{i \in \mathbb{N}} H_{-n}(\text{Hom}_{\mathbb{R}}(X, \text{Hom}_{\mathbb{Z}}(Y, \mathbb{Q}/\mathbb{Z})_{\geq i-k})) = 0
\]
and
\[
\text{colim}_{i \in \mathbb{N}} H_{-n}(\text{Hom}_{\mathbb{R}}(Y_{\leq k-i}, \text{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z}))) = 0.
\]
Hom as in [15, Lemma 3.1(c)], and (d) is from [18, Lemma 2.1(b)].

In the next lemma, (a) and (b) can be found in [15, Lemma 3.1], (c) can be proved one has
\[ \text{colim}_{i \in \mathbb{N}} H_{-n}(\text{Hom}_Z(X \otimes_R Y_{\leq -i}, Q/\mathbb{Z})) \cong \text{colim}_{i \in \mathbb{N}} \text{Hom}_Z(H_n(X \otimes_R Y_{\leq -i}), Q/\mathbb{Z}) = 0. \]

Now the desired equations hold by the adjoint isomorphism. \qed

We end this section with the following result that will be used in the next section.

3.12 Proposition. Let X be an \( R^c \)-complex, let Y be a bounded \( (R, S^c) \)-complex, and let Z be an S-complex. Then there is an isomorphism of \( Z \)-complexes,
\[ (X \otimes_R Y) \otimes_S Z \to X \otimes_R (Y \otimes_S Z), \]
which is functorial in \( X, Y \) and \( Z \).

Proof. Consider the commutative diagram of \( Z \)-complexes:
\[
\begin{array}{cccccc}
0 & \to & (X \otimes_R Y) \otimes_S Z & \to & (X \otimes_R Y) \otimes_S Z & \to & (X \otimes_R Y) \otimes_S Z & \to & 0 \\
\downarrow & & \downarrow & & \downarrow \alpha & & \downarrow & & \downarrow \\
0 & \to & X \otimes_R (Y \otimes_S Z) & \to & X \otimes_R (Y \otimes_S Z) & \to & X \otimes_R (Y \otimes_S Z) & \to & 0.
\end{array}
\]

We notice that \( X \otimes_R Y = X \otimes_R Y \) and \( Y \otimes_S Z = Y \otimes_S Z \) since \( Y \) is bounded. Then the second vertical map \( \alpha \) is an isomorphism by [2, Proposition A4]. The first one is clearly an isomorphism. So one gets an isomorphism
\[ (X \otimes_R Y) \otimes_S Z \to X \otimes_R (Y \otimes_S Z), \]
which is clearly functorial in \( X, Y \) and \( Z \). \qed

4. STABLE HOMOLOGY WITH RESPECT TO A SEMIDUALIZING MODULE

Convention. In this section, \( R \) is a commutative noetherian ring, and \( C \) is a semidualizing \( R \)-module.

4.1 Definition. Let \( \mathcal{X} \) (resp., \( \mathcal{Y} \)) be a precovering (resp., preenveloping) class of \( R \)-modules. For \( R \)-modules \( M \) and \( N \), let \( X \to M \) be a proper \( \mathcal{X} \)-resolution of \( M \), and \( N \to Y \) be a proper \( \mathcal{Y} \)-coresolution of \( N \). For each \( n \in \mathbb{Z} \), the \( n \)th stable homology of \( M \) and \( N \) with respect to \( \mathcal{X} \) and \( \mathcal{Y} \) is
\[ \text{Tor}^n_{\mathcal{X}\mathcal{Y}}(M, N) = H_{n+1}(X \otimes_R Y). \]

4.2. Following [7, Section 8.2], any two proper \( \mathcal{X} \)-resolutions of \( M \), and similarly any two proper \( \mathcal{Y} \)-coresolutions of \( N \), are homotopy equivalent. Thus by [2, 1.5(d)], the above definition is independent of the choices of (co)resolutions. We notice that \( \text{Tor}^n_{\mathcal{X}\mathcal{Y}}(M, N) \) is the classical stable homology, \( \text{Tor}^n_{R}(M, N) \), of \( M \) and \( N \) defined by Goichot [9]; see also [2].

We denote by \( \mathcal{P}_C \) (resp., \( \mathcal{F}_C, \mathcal{I}_C \)) the class of \( R \)-modules \( C \otimes_R P \) (resp., \( C \otimes_R F \), \( \text{Hom}_R(C, I) \)) with \( P \) projective (resp., \( F \) flat, \( I \) injective). Then \( \mathcal{P}_C \) and \( \mathcal{F}_C \) are precovering and \( \mathcal{I}_C \) is preenveloping; see e.g. Holm and White [12, Proposition 5.3]. In the next lemma, (a) and (b) can be found in [15, Lemma 3.1], (c) can be proved as in [15, Lemma 3.1(c)], and (d) is from [18, Lemma 2.1(b)].
4.3 Lemma. Let $M$ be an $R$-module.
(a) If $F \rightarrowrightarrow \Hom_R(C, M)$ is a proper flat (resp., projective) resolution, then $C \otimes_R F \rightarrowrightarrow \Hom_R(C, M)$ is a proper $\mathcal{F}_C$ (resp., $\mathcal{P}_C$)-resolution of $M$.
(b) If $G \rightarrowrightarrow \Hom_R(C, M)$ is a proper $\mathcal{F}_C$ (resp., $\mathcal{P}_C$)-resolution of $M$, then $\Hom_R(C, G) \rightarrowrightarrow \Hom_R(C, M)$ is a proper flat (resp., projective)-resolution of $\Hom_R(C, M)$.
(c) If $C \otimes_R M \rightarrowrightarrow I$ is an injective resolution of $C \otimes_R M$, then $M \rightarrowrightarrow \Hom_R(C, I)$ is a proper $\mathcal{I}_C$-coresolution.
(d) If $M \rightarrowrightarrow J$ is a proper $\mathcal{I}_C$-coresolution of $M$, then $C \otimes_R M \rightarrowrightarrow C \otimes_R J$ is an injective resolution of $C \otimes_R M$.

4.4 Proposition. Let $M$ and $N$ be $R$-modules. Then there are isomorphisms
$$\Tor^R_n\mathcal{P}_C\mathcal{I}_C(M, N) \cong \Tor^R_n(\Hom_R(C, M), C \otimes_R N) \cong \Tor^R_n\mathcal{P}_C\mathcal{I}_C(M, N),$$
which are functorial in $M$ and $N$.

**Proof.** Let $P \rightarrowrightarrow \Hom_R(C, M)$ be a projective resolution of $\Hom_R(C, M)$, and let $C \otimes_R N \rightarrowrightarrow I$ be an injective resolution of $C \otimes_R N$. Then by Lemma 4.3(a)(c), $C \otimes_R P \rightarrowrightarrow M$ is a proper $\mathcal{P}_C$-resolution of $M$, and $N \rightarrowrightarrow \Hom_R(C, I)$ is a proper $\mathcal{I}_C$-coresolution, and so one gets
$$\Tor^R_n\mathcal{P}_C\mathcal{I}_C(M, N) = H_{n+1}((C \otimes_R P) \otimes_R \Hom_R(C, I))$$
$$\cong H_{n+1}(P \otimes_R (C \otimes_R \Hom_R(C, I)))$$
$$\cong H_{n+1}(P \otimes_R I)$$
$$\cong \Tor^R_n(\Hom_R(C, M), C \otimes_R N),$$
where the first isomorphism follows from Proposition 4.3 and the second one holds since $I$ is a complex of injective $R$-modules.

The isomorphism $\Tor^R_n\mathcal{P}_C\mathcal{I}_C(M, N) \cong \Tor^R_n(\Hom_R(C, M), C \otimes_R N)$ can be proved similarly by taking a proper flat resolution $F \rightarrowrightarrow \Hom_R(C, M)$ and using Lemma 4.3(a) and Proposition 2.6.

Now it is straightforward to verify that the desired isomorphisms are functorial in $M$ and $N$. \qed

4.5 Lemma. Let $M$ be an $R$-module and let $n \in \mathbb{Z}$.
(a) If $\Tor^R_{n-1}\mathcal{P}_C\mathcal{I}_C(-, M) = 0$, then $\Tor^R_n\mathcal{P}_C\mathcal{I}_C(-, M) = 0$.
(b) If $\Tor^R_{n+1}\mathcal{P}_C\mathcal{I}_C(M, -) = 0$, then $\Tor^R_n\mathcal{P}_C\mathcal{I}_C(M, -) = 0$.

**Proof.** (a) For an $R$-module $M'$, by Proposition 5.3(b) there is a complex $0 \rightarrowrightarrow K \rightarrowrightarrow P \rightarrowrightarrow M' \rightarrowrightarrow 0$ with $P \in \mathcal{P}_C$ such that the sequence
$$0 \rightarrowrightarrow \Hom_R(P', K) \rightarrowrightarrow \Hom_R(P', P) \rightarrowrightarrow \Hom_R(P', M') \rightarrowrightarrow 0$$
is exact for each $P' \in \mathcal{P}_C$. In particular, the sequence
$$0 \rightarrowrightarrow \Hom_R(C, K) \rightarrowrightarrow \Hom_R(C, P) \rightarrowrightarrow \Hom_R(C, M') \rightarrowrightarrow 0$$
is exact. Since $\Hom_R(C, P)$ is projective, one gets
$$\Tor^R_n(\Hom_R(C, M'), C \otimes_R M) \cong \Tor^R_{n-1}(\Hom_R(C, K), C \otimes_R M),$$
and so by Proposition 4.4, \( \widetilde{\text{Tor}}^P_{n+1}(M, M) = 0 \), which yields
\[ \widetilde{\text{Tor}}^P_{n+1}(-, M) = 0. \]

(b) Let \( N \) be an \( R \)-module. Then by [12, Proposition 5.3(c)] there is a complex
\[ 0 \to N \to I \to K \to 0 \]
with \( I \in \mathcal{I}_C \) such that the sequence
\[ 0 \to \text{Hom}_R(K, I') \to \text{Hom}_R(I, I') \to \text{Hom}_R(N, I') \to 0 \]
is exact for each \( I' \in \mathcal{I}_C \). Since \( C' = \text{Hom}_R(C, \mathbb{Q}/\mathbb{Z}) \) is in \( \mathcal{I}_C \), the sequence
\[ 0 \to \text{Hom}_R(K, C') \to \text{Hom}_R(I, C') \to \text{Hom}_R(N, C') \to 0 \]
is exact, which implies that the sequence
\[ 0 \to C \otimes_R N \to C \otimes_R I \to C \otimes_R K \to 0 \]
is exact. We notice that \( C \otimes_R I \) is injective. Thus one gets
\[ \widetilde{\text{Tor}}^R_n(\text{Hom}_R(C, M), C \otimes_R N) \cong \widetilde{\text{Tor}}^R_{n+1}(\text{Hom}_R(C, M), C \otimes_R K), \]
and so by Proposition 4.4, \( \widetilde{\text{Tor}}^P_{n+1}(M, N) = 0 \), which yields
\[ \widetilde{\text{Tor}}^P_{n+1}(M, -) = 0. \]

Now we are in a position to give the main results of this section described in the introduction.

4.6 Theorem. For an \( R \)-module \( N \), the following conditions are equivalent:

(i) \( \mathcal{I}_C \cdot \text{id}_R N < \infty \).
(ii) \( \widetilde{\text{Tor}}^P_n(-, N) = 0 \) for each \( n \in \mathbb{Z} \).
(iii) \( \widetilde{\text{Tor}}^P_n(-, N) = 0 \) for some \( n < 0 \).

Proof. (i) \( \Rightarrow \) (ii): Since \( \mathcal{I}_C \cdot \text{id}_R N < \infty \), there is a proper \( \mathcal{I}_C \)-coresolution \( N \to I \) with \( I \) bounded. Thus for each \( R \)-module \( M \) with \( P \to M \) a proper \( \mathcal{P}_C \)-resolution, one has \( \widetilde{\text{Tor}}^P_n(M, N) = H_{n+1}(P \otimes_R I) = 0 \).

(ii) \( \Rightarrow \) (iii) is clear.

(iii) \( \Rightarrow \) (i): We first notice that \( \widetilde{\text{Tor}}^P_0(-, N) = 0 = \widetilde{\text{Tor}}^P_{-1}(-, N) = 0 \) by Lemma 4.5.

Let \( M \) be an \( R \)-module and \( F \xrightarrow{\sim} \text{Hom}_R(C, M) \) a proper flat resolution. Then by Lemma 4.3(a), \( C \otimes_R F \to M \) is a proper \( \mathcal{F}_C \)-resolution of \( M \). Let \( N \to I \) be a proper \( \mathcal{I}_C \)-resolution of \( N \). Since \( \widetilde{\text{Tor}}^P_{n+1}(M, N) = \widetilde{\text{Tor}}^P_{n+1}(M, K) = 0 \) by Proposition 4.4, one gets \( H_1((C \otimes_R F) \otimes_R I) = 0 \).

On the other hand, one has \( \widetilde{\text{Tor}}^P_{n+1}(M^{(n)}, N) = 0 \), so by Proposition 4.4
\[ \widetilde{\text{Tor}}^P_{n+1}((\text{Hom}_R(C, M))^{(n)}, C \otimes_R N) = 0. \]

Note that \( F \xrightarrow{\sim} \text{Hom}_R(C, M) \) is a flat resolution, and so \( F^{(n)} \xrightarrow{\sim} (\text{Hom}_R(C, M))^{(n)} \) is a flat resolution of \( (\text{Hom}_R(C, M))^{(n)} \). Since \( C \otimes_R N \xrightarrow{\sim} C \otimes_R I \) is an injective resolution by Lemma 4.3(d), one gets \( H_0(F^{(n)} \otimes_R (C \otimes_R I)) = 0 \); see [2, Proposition 2.6]. Thus we have
\[ H_0((C \otimes_R F^{(n)}) \otimes_R I) \cong H_0((C \otimes_R F^{(n)}) \otimes_R I) \cong H_0(F^{(n)} \otimes_R (C \otimes_R I)) = 0, \]
where the second isomorphism follows from Proposition 5.12.

Now by Proposition 5.11, one gets
\[ \operatorname{colim}_{i \in \mathbb{N}} H_0(\text{Hom}_R(C \otimes_R F_{i}, \text{Hom}_Z(I, \mathbb{Q}/\mathbb{Z})_{\leq i})) = 0. \]
We notice that \( C \otimes_R F \to M \) is a proper \( \mathcal{F}_C \)-resolution of \( M \), and \( \text{Hom}_R(I, \mathbb{Q}/\mathbb{Z}) \to \text{Hom}_R(N, \mathbb{Q}/\mathbb{Z}) \) is a proper \( \mathcal{F}_C \)-resolution of \( \text{Hom}_R(N, \mathbb{Q}/\mathbb{Z}) \). Then by Proposition A.13 one gets
\[
\widetilde{\text{Ext}}^0_{\mathcal{F}_C}(M, \text{Hom}_R(N, \mathbb{Q}/\mathbb{Z})) = 0
\]
for each \( R \)-module \( M \). Thus \( \mathcal{F}_C\text{-pd}_R \text{Hom}_R(N, \mathbb{Q}/\mathbb{Z}) < \infty \) by Proposition A.11 and so \( \mathcal{I}_C\text{-id}_R N < \infty \); see Sather-Wagstaff, Sharif and White [16, Lemma 4.2]. \( \square \)

4.7 Theorem. For an \( R \)-module \( M \), the following conditions are equivalent:

(i) \( \mathcal{F}_C\text{-pd}_R M < \infty \).
(ii) \( \widetilde{\text{Tor}}^n_{\mathcal{F}_C}(\mathcal{I}_C, -) = 0 \) for each \( n \in \mathbb{Z} \).
(iii) \( \widetilde{\text{Tor}}^n_{\mathcal{F}_C}(\mathcal{I}_C, -) = 0 \) for some \( n \geq 0 \).

Moreover, if \( M \) is finitely generated, then (i)–(iii) are equivalent to

(i') \( \mathcal{P}_C\text{-pd}_R M < \infty \).

Proof. (i) \( \Rightarrow \) (ii): Since \( \mathcal{F}_C\text{-pd}_R M < \infty \), there is a proper \( \mathcal{F}_C \)-resolution \( F \to M \) with \( F \) bounded. Thus for each \( R \)-module \( N \) with \( N \to I \) a proper \( \mathcal{I}_C \)-coresolution, one has \( \widetilde{\text{Tor}}^n_{\mathcal{F}_C}(\mathcal{I}_C, N) \cong \text{Tor}^n_{\mathcal{F}_C}(\mathcal{I}_C, N) = H_{n+1}(F \otimes_R I) = 0 \) by Proposition 4.4.

(ii) \( \Rightarrow \) (iii) is clear.

(iii) \( \Rightarrow \) (i): We first notice that \( \widetilde{\text{Tor}}^n_{\mathcal{F}_C}(\mathcal{I}_C, -) = 0 = \widetilde{\text{Tor}}^n_{\mathcal{F}_C}(\mathcal{I}_C, -) = 0 \) by Lemma 4.5.

Let \( F \xrightarrow{\zeta} \text{Hom}_R(C, M) \) be a proper flat resolution of \( \text{Hom}_R(C, M) \). Then \( C \otimes_R F \to M \) is a proper \( \mathcal{F}_C \)-resolution by Lemma 4.3(a), and

\[
C \otimes_R \text{Hom}_R(M, \mathbb{Q}/\mathbb{Z}) \cong \text{Hom}_R(C, M, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\zeta} \text{Hom}_R(F, \mathbb{Q}/\mathbb{Z})
\]
is an injective resolution of \( C \otimes_R \text{Hom}_R(M, \mathbb{Q}/\mathbb{Z}) \), and so

\[
\text{Hom}_R(M, \mathbb{Q}/\mathbb{Z}) \to \text{Hom}_R(C, \text{Hom}_R(F, \mathbb{Q}/\mathbb{Z})) \cong \text{Hom}_R(C \otimes_R F, \mathbb{Q}/\mathbb{Z})
\]
is a proper \( \mathcal{I}_C \)-coresolution of \( \text{Hom}_R(M, \mathbb{Q}/\mathbb{Z}) \) by Lemma 4.3(c).

Let \( N \) be an \( R \)-module, and let \( C \otimes_R N \xrightarrow{\zeta} I \) be an injective resolution of \( C \otimes_R N \). Then \( N \to \text{Hom}_R(C, I) \) is a proper \( \mathcal{I}_C \)-coresolution by Lemma 4.3(c), and

\[
C \otimes_R N^{(N)} \cong (C \otimes_R N)^{(N)} \xrightarrow{\zeta} I^{(N)}
\]
is an injective resolution of \( C \otimes_R N^{(N)} \), and so

\[
N^{(N)} \to \text{Hom}_R(C, I^{(N)}) \cong (\text{Hom}_R(C, I))^{(N)}
\]
is a proper \( \mathcal{I}_C \)-coresolution by Lemma 4.3(c).

Since \( \widetilde{\text{Tor}}^0_{\mathcal{F}_C}(\mathcal{I}_C, N) = 0 = \widetilde{\text{Tor}}^0_{\mathcal{F}_C}(\mathcal{I}_C, N^{(N)}) \) by Proposition 4.4, one gets \( H_1((C \otimes_R F) \otimes_R \text{Hom}_R(C, I)) = 0 \), and

\[
H_0((C \otimes_R F)^{(N)} \otimes_R \text{Hom}_R(C, I)) \cong H_0((C \otimes_R F) \otimes_R (\text{Hom}_R(C, I))^{(N)}) = 0
\]
by Proposition 3.12. Now using Proposition 3.11 one gets
\[
\text{colim}_{i \in \mathbb{N}} H_0(\text{Hom}_R(\text{Hom}_R(C, I)_{\leq -i}, \text{Hom}_Z(C \otimes_R F, \mathbb{Q}/\mathbb{Z}))) = 0,
\]
and so \( \text{Ext}^0_{\mathcal{I}_C}(N, \text{Hom}_Z(M, \mathbb{Q}/\mathbb{Z})) = 0 \) for each \( R \)-module \( N \) by Proposition A.14. Thus \( \mathcal{I}_C\text{-id}_R \text{Hom}_Z(M, \mathbb{Q}/\mathbb{Z}) < \infty \) by Proposition A.12 and so \( \mathcal{F}_C\text{-pd}_R M < \infty \); see [16, Lemma 4.2].
Finally, if $M$ is finitely generated, then by [13, Theorem 5.5] the conditions (i) and (i') are equivalent.

As a corollary of the above theorems, we give a balance result for stable homology with respect to a semidualizing module.

4.8 Corollary. The following conditions are equivalent for a local ring $R$:

(i) $\widetilde{\text{Tor}}_n^{PC \mathcal{I}C}(M, N) \cong \widetilde{\text{Tor}}_n^{PC \mathcal{I}C}(N, M)$ for all $R$-modules $M$ and $N$, and for each $n \in \mathbb{Z}$.

(ii) $\mathcal{I}_C \cdot \text{id}_R C < \infty$.

(iii) $C \cong R$ and $R$ is Gorenstein.

Proof. (i) $\Rightarrow$ (ii): Since $C$ is $C$-projective, one gets

$\widetilde{\text{Tor}}_n^{PC \mathcal{I}C}(M, C) \cong \widetilde{\text{Tor}}_n^{PC \mathcal{I}C}(C, M) = 0$

for all $R$-modules $M$ and for each $n \in \mathbb{Z}$, and so $\mathcal{I}_C \cdot \text{id}(C) < \infty$ by Theorem 4.6.

The implication (ii) $\Rightarrow$ (iii) follows from Sather-Wagstaff and Yassemi [17, Lemma 2.11], and (iii) $\Rightarrow$ (i) holds by [2, Corollary 4.7].

Appendix. Stable cohomology

The next definitions of bounded and stable Hom-complexes can be found in Avramov and Veliche [1], and [9].

A.1. For $R$-complexes $X$ and $Y$, the bounded Hom-complex $\overline{\text{Hom}}_R(X, Y)$ is the subcomplex of $\text{Hom}_R(X, Y)$ with degree-$n$ term

$\overline{\text{Hom}}_R(X, Y)_n = \bigsqcup_{i \in \mathbb{Z}} \text{Hom}_R(X_i, Y_{n+i}).$

We denote by $\overline{\text{Hom}}_R(X, Y)$ the quotient complex $\text{Hom}_R(X, Y)/\overline{\text{Hom}}_R(X, Y)$, which is called the stable Hom-complex.

A.2 Proposition. Let $X$ and $Z$ be an $R$-complex and an $S$-complex, respectively, and let $Y$ be a bounded $(S, R^o)$-complex. Then there are isomorphisms of $\mathbb{Z}$-complexes,

$\overline{\text{Hom}}_S(Y \otimes_R X, Z) \cong \overline{\text{Hom}}_R(X, \text{Hom}_S(Y, Z))$ and

$\overline{\text{Hom}}_S(Y \otimes_R X, Z) \cong \overline{\text{Hom}}_R(X, \text{Hom}_S(Y, Z))$, which are functorial in $X, Y$ and $Z$.

Proof. For every $n \in \mathbb{Z}$ one has,

$\overline{\text{Hom}}_S(Y \otimes_R X, Z)_n = \prod_{h \in \mathbb{Z}} \text{Hom}_S((Y \otimes_R X)_h, Z_{n+h})$

$= \prod_{h \in \mathbb{Z}} \text{Hom}_S(\prod_{q \in \mathbb{Z}} (Y_q \otimes_R X_{h-q}), Z_{n+h})$

$\cong \prod_{h \in \mathbb{Z}} \prod_{q \in \mathbb{Z}} \text{Hom}_S(Y_q \otimes_R X_{h-q}, Z_{n+h})$

$= \prod_{p \in \mathbb{Z}} \prod_{q \in \mathbb{Z}} \text{Hom}_S(Y_q \otimes_R X_p, Z_{n+p+q}).$
On the other hand, for every \( n \in \mathbb{Z} \) one has,
\[
\widetilde{\text{Hom}}_R(X, \text{Hom}_S(Y, Z))_n = \prod_{p \in \mathbb{Z}} \text{Hom}_R(X_p, \text{Hom}_S(Y, Z)_{n+p})
\]
\[
= \prod_{p \in \mathbb{Z}} \text{Hom}_R(X_p, \prod_{q \in \mathbb{Z}} \text{Hom}_S(Y_q, Z_{n+p+q}))
\]
\[
\cong \prod_{p \in \mathbb{Z}} \prod_{q \in \mathbb{Z}} \text{Hom}_R(X_p, \text{Hom}_S(Y_q, Z_{n+p+q})).
\]

Here the isomorphisms in the above computations hold since \( Y \) is bounded.

We notice that there is a natural isomorphism of \( \mathbb{Z} \)-modules
\[
\rho_{Y,X} : \text{Hom}_S(Y \otimes_R X, Z_{n+p+q}) \to \text{Hom}_R(X_p, \text{Hom}_S(Y_q, Z_{n+p+q})).
\]

Thus one gets an isomorphism of \( \mathbb{Z} \)-complexes
\[
\rho_{Y,X,Z} : \text{Hom}_S(Y \otimes_R X, Z) \to \text{Hom}_R(X, \text{Hom}_S(Y, Z)).
\]

It is straightforward to verify that \( \rho_{Y,X,Z} \) is functorial in \( X, Y \) and \( Z \).

For the second isomorphism in the statement, consider the following commutative diagram of \( \mathbb{Z} \)-complexes:
\[
\begin{array}{ccc}
0 & \to & \widetilde{\text{Hom}}_S(Y \otimes_R X, Z) \\
\rho \downarrow & & \downarrow \varrho \\
0 & \to & \text{Hom}_R(X, \text{Hom}_S(Y, Z))
\end{array}
\]
\[
\begin{array}{ccc}
\text{Hom}_R(X, \text{Hom}_S(Y, Z)) & \to & \text{Hom}_R(X, \text{Hom}_S(Y, Z)) \\
\varrho & & \varrho \\
0 & & 0
\end{array}
\]

Since \( \rho \) and \( \varrho \) are isomorphisms, one gets an isomorphism
\[
\widetilde{\text{Hom}}_S(Y \otimes_R X, Z) \to \text{Hom}_R(X, \text{Hom}_S(Y, Z)),
\]
which is clearly functorial in \( X, Y \) and \( Z \).

\( \square \)

**A.3.** Let \( \mathcal{X} \) be a precovering class of \( R \)-modules, and let \( X_M \to M \) and \( X_N \to N \) be proper \( \mathcal{X} \)-resolutions of \( R \)-modules \( M \) and \( N \), respectively. For each \( n \in \mathbb{Z} \), the \( n \)-th stable cohomology of \( M \) and \( N \) with respect to \( \mathcal{X} \) is
\[
\widetilde{\text{Ext}}_\mathcal{X}^n(M, N) = H_{-n}(\text{Hom}_R(X_M, X_N)).
\]

Dually, let \( \mathcal{Y} \) be a preenveloping class of \( R \)-modules, and let \( M \to Y_M \) and \( N \to Y_N \) be proper \( \mathcal{Y} \)-coresolutions of \( M \) and \( N \), respectively. For each \( n \in \mathbb{Z} \), the \( n \)-th stable cohomology of \( M \) and \( N \) with respect to \( \mathcal{Y} \) is
\[
\widetilde{\text{Ext}}_\mathcal{Y}^n(M, N) = H_{-n}(\text{Hom}_R(Y_M, Y_N)).
\]

**A.4.** Any two proper \( \mathcal{X} \)-resolutions of \( M \), and similarly any two proper \( \mathcal{Y} \)-coresolutions of \( N \), are homotopy equivalent; see [7, Section 8.2]. Thus the above definitions are independent of the choices of (co)resolutions. We notice that \( \widetilde{\text{Ext}}_\mathcal{X}^n(M, N) \) is the classical stable cohomology, \( \text{Ext}_R^n(M, N) \), of \( M \) and \( N \); see [1] and [9]. Also \( \widetilde{\text{Ext}}_\mathcal{Y}^n(M, N) \) is the cohomology given by Nucinkis [13].
Stable cohomology with respect to proper flat (injective) resolutions. The proof of the next result can be modelled along the argument in the proof of [13, Theorem 3.6], when the argument is applied to the functor $\operatorname{Ext}^n_F(M, -)$ that is computed by $H_{-i}(\operatorname{Hom}_R(F, -))$, where $F \xrightarrow{\sim} M$ is a proper flat resolution.

A.5 Proposition. For an $R$-module $M$, the following conditions are equivalent:

(i) $\operatorname{id}_R M \leq \infty$.

(ii) $\operatorname{Ext}^n_F(M, -) = 0 = \operatorname{Ext}^n_F(-, M)$ for each $n \in \mathbb{Z}$.

(iii) $\operatorname{Ext}^n_F(M, M) = 0$.

Dually, we have the next result that is proved by Nucinkis in [13, Theorem 3.7].

A.6 Proposition. For an $R$-module $N$, the following conditions are equivalent:

(i) $\operatorname{id}_R N < \infty$.

(ii) $\operatorname{Ext}^n_F(N, -) = 0 = \operatorname{Ext}^n_F(-, N)$ for each $n \in \mathbb{Z}$.

(iii) $\operatorname{Ext}^0_F(N, N) = 0$.

A.7 Proposition. Let $M$ and $N$ be $R$-modules with proper flat resolutions $F \xrightarrow{\sim} M$ and $F' \xrightarrow{\sim} N$, respectively. For every $n \in \mathbb{Z}$ there is an isomorphism

$$\operatorname{Ext}^n_F(M, N) \cong \operatorname{colim}_{i \in \mathbb{N}} H_{-n}(\operatorname{Hom}_R(F, F'_i)).$$

Proof. Set $\Omega_{s}M = \operatorname{Coker}(F_{s+1} \to F_s)$ and $\Omega_{s}N = \operatorname{Coker}(F'_{s+1} \to F'_s)$. Using a similar proof as proved in [13, Theorem 3.6], one gets a natural isomorphism

$$\operatorname{colim}_{i \in \mathbb{N}} \operatorname{Ext}^i_F(M, \Omega_{i-n}N) \cong \operatorname{colim}_{i \in \mathbb{N}} \operatorname{Hom}_R(\Omega_i M, \Omega_{i-n}N)/\operatorname{FHom}_R(\Omega_i M, \Omega_{i-n}N).$$

Here $\operatorname{FHom}_R(\Omega_{i}M, \Omega_{i-n}N)$ denotes the set of all homomorphisms of $R$-modules $f \in \operatorname{Hom}_R(\Omega_i M, \Omega_{i-n}N)$ factoring through a flat $R$-module. As proved in [13, Theorem 4.4] (see also [13, B.2]), one gets an isomorphism

$$\widetilde{\operatorname{Ext}}^n_F(M, N) \cong \operatorname{colim}_{i \in \mathbb{N}} \operatorname{Hom}_R(\Omega_i M, \Omega_{i-n}N)/\operatorname{FHom}_R(\Omega_i M, \Omega_{i-n}N).$$

On the other hand, we notice that $\Sigma^{-i}F'_i \xrightarrow{\sim} \Omega_{i}N$ is a proper flat resolution. Thus one has,

$$\operatorname{colim}_{i \in \mathbb{N}} \operatorname{Ext}^i_F(M, \Omega_{i-n}N) \cong \operatorname{colim}_{i \in \mathbb{N}} \operatorname{Ext}^{i+n}_F(M, \Omega_{i}N) \cong \operatorname{colim}_{i \in \mathbb{N}} H_{-i-n}(\operatorname{Hom}_R(F, \Sigma^{-i}F'_{i})) \cong \operatorname{colim}_{i \in \mathbb{N}} H_{-n}(\operatorname{Hom}_R(F, F'_{i})).$$

where the second isomorphism follows from Christensen, Frankild and Holm [4, Proposition 2.6]. Now one gets the isomorphism in the statement. □

Dually, one gets the following result, which is proved in [6, Proposition 1.1(iii)].

A.8 Proposition. Let $M$ and $N$ be $R$-modules with injective resolutions $M \xrightarrow{\sim} I$ and $N \xrightarrow{\sim} I'$, respectively. For every $n \in \mathbb{Z}$ there is an isomorphism

$$\widetilde{\operatorname{Ext}}^n_F(M, N) \cong \operatorname{colim}_{i \in \mathbb{N}} H_{-n}(\operatorname{Hom}_R(I_{\leq -i}, I')).$$
Stable cohomology with respect to a semidualizing module. In this subsection, we assume that $R$ is a commutative noetherian ring, and let $C$ be a semidualizing $R$-module.

A.9 Lemma. Let $M$ and $N$ be $R$-modules. Then there is an isomorphism
\[ \widetilde{\Ext}^n_{\mathcal{F}_C}(M, N) \cong \widetilde{\Ext}^n_F(\Hom_R(C, M), \Hom_R(C, N)), \]
which is functorial in $M$ and $N$.

Proof. Let $F \xrightarrow{\sim} \Hom_R(C, M)$ and $F' \xrightarrow{\sim} \Hom_R(C, N)$ be proper flat resolutions of $\Hom_R(C, M)$ and $\Hom_R(C, N)$, respectively. Then by Lemma 4.3(a), $C \otimes_R F \to M$ and $C \otimes_R F' \to N$ are proper $\mathcal{F}_C$-resolutions of $M$ and $N$, respectively. Thus one has,
\[ \widetilde{\Ext}^n_{\mathcal{F}_C}(M, N) = H_{-n}(\widetilde{\Ext}^{n}(F, C \otimes_R F')) \]
\[ \cong H_{-n}(\widetilde{\Ext}^{n}(F, F')) \]
\[ \cong \Ext^n_R(\Hom_R(C, M), \Hom_R(C, N)), \]
where the first isomorphism follows from Proposition A.2, and the second one holds since $F'$ is a complex of flat $R$-modules. It is straightforward to verify that the desired isomorphism is functorial in $M$ and $N$. □

The next result can be proved dually using Lemma 4.3(c) and Proposition A.2.

A.10 Lemma. Let $M$ and $N$ be $R$-modules. Then there is an isomorphism
\[ \Ext^n_{\mathcal{I}_C}(M, N) \cong \Ext^n_C(\mathcal{I} \otimes_R M, C \otimes_R N), \]
which is functorial in $M$ and $N$.

A.11 Proposition. For an $R$-module $M$, the following conditions are equivalent.
\begin{enumerate}
    \item[(i)] $\mathcal{F}_C$-pd$_R M < \infty$.
    \item[(ii)] $\widetilde{\Ext}^n_{\mathcal{F}_C}(M, -) = 0 = \widetilde{\Ext}^n_{\mathcal{F}_C}(-, M)$ for each $n \in \mathbb{Z}$.
    \item[(iii)] $\widetilde{\Ext}^0_{\mathcal{F}_C}(M, M) = 0$.
\end{enumerate}

Proof. (i) $\implies$ (ii): Since $\mathcal{F}_C$-pd$_R M < \infty$, there is a proper $\mathcal{F}_C$-resolution $F \to M$ with $F$ bounded, and so $\Hom_R(F, -) = 0 = \Hom_R(-, F)$. Thus one gets $\widetilde{\Ext}^n_{\mathcal{F}_C}(M, -) = 0 = \widetilde{\Ext}^n_{\mathcal{F}_C}(-, M)$ for each $n \in \mathbb{Z}$.
\[ (ii) \implies (iii) \]
\[ (iii) \implies (i): By Proposition A.9, one gets $\widetilde{\Ext}^0_F(\Hom_R(C, M), \Hom_R(C, M)) \cong \widetilde{\Ext}^0_{\mathcal{F}_C}(M, M) = 0$, and so $\text{fd}_R \Hom_R(C, M) < \infty$ by Proposition A.5. Thus one gets $\mathcal{F}_C$-pd$_R M < \infty$; see [15 Proposition 5.2(b)]. □

The next result can be proved dually using Propositions A.6 and A.10 and [18 Theorem 2.11(b)].

A.12 Proposition. For an $R$-module $N$, the following conditions are equivalent.
\begin{enumerate}
    \item[(i)] $\mathcal{I}_C$-id$_R N < \infty$.
    \item[(ii)] $\widetilde{\Ext}^n_{\mathcal{I}_C}(N, -) = 0 = \widetilde{\Ext}^n_{\mathcal{I}_C}(-, N)$ for each $n \in \mathbb{Z}$.
\end{enumerate}
\( (iii) \) \( \widetilde{\text{Ext}}_{I_C}^0(N, N) = 0. \)

**A.13 Proposition.** Let \( M \) and \( N \) be \( R \)-modules with proper \( F_C \)-resolutions \( F \rightarrow M \) and \( F' \rightarrow N \), respectively. For every \( n \in \mathbb{Z} \) there is an isomorphism

\[
\widetilde{\text{Ext}}_{F_C}^n(M, N) \cong \text{colim}_{i \in N} \text{H}_{-n}(\text{Hom}_R(F, F'_{\geq i})).
\]

**Proof.** By Lemma [4.3(b)], \( \text{Hom}_R(C, F) \xrightarrow{\sim} \text{Hom}_R(C, M) \) and \( \text{Hom}_R(C, F') \xrightarrow{\sim} \text{Hom}_R(C, N) \) are proper flat resolutions of \( \text{Hom}_R(C, M) \) and \( \text{Hom}_R(C, N) \), respectively. Thus we have,

\[
\widetilde{\text{Ext}}_{F_C}^n(M, N) \cong \widetilde{\text{Ext}}_{F}(\text{Hom}_R(C, M), \text{Hom}_R(C, N))
\]

\[
\cong \text{colim}_{i \in N} \text{H}_{-n}(\text{Hom}_R(\text{Hom}_R(C, F), \text{Hom}_R(C, F'_{\geq i}))
\]

\[
= \text{colim}_{i \in N} \text{H}_{-n}(\text{Hom}_R(\text{Hom}_R(C, F), \text{Hom}_R(C, F'_{\geq i})))
\]

\[
\cong \text{colim}_{i \in N} \text{H}_{-n}(\text{Hom}_R(C \otimes_R \text{Hom}_R(C, F), F'_{\geq i}))
\]

\[
\cong \text{colim}_{i \in N} \text{H}_{-n}(\text{Hom}_R(F, F'_{\geq i}))
\]

where the first isomorphism follows from Proposition [A.9], the second one follows from Proposition [A.7] and the last one holds since \( F \) is a complex of \( C \)-flat \( R \)-modules. \( \square \)

Dually, we have the following result.

**A.14 Proposition.** Let \( M \) and \( N \) be \( R \)-modules with proper \( I_C \)-coresolutions \( M \rightarrow I \) and \( N \rightarrow I' \), respectively. For every \( n \in \mathbb{Z} \) there is an isomorphism

\[
\widetilde{\text{Ext}}_{I_C}^n(M, N) \cong \text{colim}_{i \in N} \text{H}_{-n}(\text{Hom}_R(I_{\leq-i}, I')).
\]

**Acknowledgments**

We thank Ioannis Emmanouil and Panagiota Manousaki for making [6] available to us and for discussions regarding this work. We also thank the anonymous referees for several corrections and valuable comments that improved the presentation at several points.

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