LIM ULRICH SEQUENCE: PROOF OF LECH’S CONJECTURE FOR GRADED BASE RINGS

LINQUAN MA

ABSTRACT. The long standing Lech’s conjecture in commutative algebra states that for a flat local extension \((R, \mathfrak{m}) \to (S, \mathfrak{n})\) of Noetherian local rings, we have an inequality on the Hilbert–Samuel multiplicities: \(e(R) \leq e(S)\). In general the conjecture is wide open as long as \(\dim R > 3\), even in equal characteristic. In this paper, we prove Lech’s conjecture in all dimensions, provided \((R, \mathfrak{m})\) is a standard graded ring over a perfect field (localized at the homogeneous maximal ideal).

We introduce the notions of lim Ulrich and weakly lim Ulrich sequence. Roughly speaking these are sequences of finitely generated modules that are not necessarily Cohen–Macaulay, but asymptotically behave like Ulrich modules. We prove that the existence of these sequences imply Lech’s conjecture. Though the existence of Ulrich modules is known in very limited cases, we construct weakly lim Ulrich sequences for all standard graded domains over perfect field of positive characteristic.

1. INTRODUCTION AND PRELIMINARIES

Around 1960, Lech made the following remarkable conjecture on the Hilbert–Samuel multiplicities [Lec60]:

**Conjecture (Lech’s conjecture).** Let \((R, \mathfrak{m}) \to (S, \mathfrak{n})\) be a flat local extension of Noetherian local rings. Then \(e(R) \leq e(S)\).

This conjecture has now stood for sixty years and remains open in most cases (despite it is very simple to state). In [Lec60, Lec64] the conjecture is proven when \(\dim R \leq 2\) or when the special fiber \(S/\mathfrak{m}S\) is a complete intersection. In [Ma17] the conjecture is proven when \(\dim R = 3\) and \(R\) has equal characteristic. For some other partial progress and related results on Lech’s conjecture, see [Her94, Han99, Han05, Ma14, Ma17]. The main result of this paper settles Lech’s conjecture for a large class of rings, in arbitrary dimension.

**Theorem A** (=Theorem 3.8). Let \((R, \mathfrak{m}) \to (S, \mathfrak{n})\) be a flat local extension of Noetherian local rings. Suppose \((R, \mathfrak{m})\) is a standard graded ring over a perfect field (localized at the homogeneous maximal ideal). Then \(e(R) \leq e(S)\).

Our main new ingredient in the proof of Theorem A (and which we hope to attack Lech’s conjecture in general) is a notion called (weakly) lim Ulrich sequence, which is a special type of (weakly) lim Cohen–Macaulay sequence developed by Bhatt, Hochster and the author in [BHM] (see also [Hoc17]).

Roughly speaking, a sequence of finitely generated modules of maximal dimension is lim Cohen–Macaulay (resp. weakly lim Cohen–Macaulay) if the lengths of their higher Koszul homology modules (resp. their first higher Euler characteristics) with respect to a system of

\[1\] That is, \(\mathbb{N}\)-graded and generated by degree one forms.
parameters grow relative slowly compared to their minimal number of generators. Furthermore, a (weakly) \( \lim \) Cohen–Macaulay sequence is called (weakly) \( \lim \) Ulrich if their minimal number of generators are approaching to their Hilbert–Samuel multiplicities. Clearly, a small (i.e., finitely generated) maximal Cohen–Macaulay module induces a constant \( \lim \) Cohen–Macaulay sequence, and a Ulrich module\(^2\) induces a constant \( \lim \) Ulrich sequence.

One of the main results in \([BHM]\) is that the existence of \( \lim \) Cohen–Macaulay sequences implies Serre’s conjecture on positivity of intersection multiplicities, which greatly extends the earlier observation that the existence of small Cohen–Macaulay modules implies Serre’s conjecture. Similarly, it was an earlier observation of Hochster–Huneke and Hanes that the existence of Ulrich modules implies Lech’s conjecture, see \([Han99]\). We largely generalize this idea and prove the following

**Theorem B** (=**Theorem 2.8**). Let \( (R, \mathfrak{m}) \to (S, \mathfrak{n}) \) be a flat local extension of Noetherian local rings such that \( R \) is a domain and \( \dim R = \dim S \). Suppose \( R \) admits a weakly \( \lim \) Ulrich sequence. Then \( e(R) \leq e(S) \).

Since Ulrich modules were introduced in \([Ulr84]\), a fundamental open question is their existence. One difficulty in the general case is that we do not know the existence of small Cohen–Macaulay modules. However, even if we restrict ourselves to Cohen–Macaulay rings, the existence of Ulrich modules is not known (even in dimension two or graded dimension three!). In fact, only in some very limited cases (e.g., rings with strong combinatorial properties) could we establish the existence of Ulrich modules, and the method is usually difficult: for example see \([HUB91]\) or \([ESW03]\).

The main contribution of this paper shows that, on the other hand, weakly \( \lim \) Ulrich sequences *always* exist for standard graded rings of positive characteristic. This vastly generalizes, in certain sense, our understanding of Ulrich-like modules, and it leads to the aforementioned result on Lech’s conjecture in characteristic \( p > 0 \). The characteristic 0 case of Theorem A then follows from a reduction to characteristic \( p > 0 \) argument.

**Theorem C** (=**Theorem 3.5**). Let \( (R, \mathfrak{m}) \) be a Noetherian standard graded domain over an infinite \( F \)-finite field of characteristic \( p > 0 \) (localized at the homogeneous maximal ideal). Then \( R \) admits a weakly \( \lim \) Ulrich sequence.

It should be pointed out that, even when \( (R, \mathfrak{m}) \) is Cohen–Macaulay, the constructed modules in the weakly \( \lim \) Ulrich sequence in Theorem C are *not* maximal Cohen–Macaulay. Thus it is very important that we allow the weakly \( \lim \) Cohen–Macaulay property, that is, the variations on the asymptotic behavior of the higher Koszul homology modules (rather than requiring them to be zero).

Throughout the rest of this paper, all rings are commutative, Noetherian, with multiplicative identity. We use \( \nu(M) \) or \( \nu_R(M) \) to denote the minimal number of generators of an \( R \)-module \( M \), \( e(I, M) \) to denote the Hilbert–Samuel multiplicity of \( M \) with respect to an \( \mathfrak{m} \)-primary ideal \( I \subseteq R \), \( H_i(x, M) \) to denote the Koszul homology module of \( M \) with respect to a system of parameters \( x \) and \( \chi_1(x, M) := \sum_{i=1}^d (-1)^{i-1} \ell(H_i(x, M)) \) to denote the first higher Euler characteristic. For basic properties on Hilbert–Samuel multiplicities and higher Euler characteristics, we refer to \([Ser65, Eis95, HS06]\).

\(^2\)That is, a small maximal Cohen–Macaulay module whose minimal number of generators equals to its Hilbert–Samuel multiplicity \([Ulr84]\).
2. Weakly lim Cohen–Macaulay and weakly lim Ulrich sequence

In this section we introduce lim Ulrich and weakly lim Ulrich sequence. These definitions depend on the notion of lim Cohen–Macaulay sequence developed in [BHM] as well as its variations. The notion of weakly lim Cohen–Macaulay sequence we define below appeared in [Hoc17, Section 9] (in relation with the monomial property of system of parameters). Here we formally introduce this concept and investigate its properties.

**Definition 2.1.** Let \((R, \mathfrak{m})\) be a local ring of dimension \(d\). A sequence of finitely generated \(R\)-modules \(\{M_n\}\) of dimension \(d\) is called \(\text{lim} \) Cohen–Macaulay, if there exists a system of parameters \(\underline{x} = x_1, \ldots, x_d\) of \(R\) such that for all \(i \geq 1\), \(\ell(H_i(\underline{x}, M_n)) = o(\nu(M_n))\). \(\{M_n\}\) is called weakly \(\text{lim} \) Cohen–Macaulay, if there exists a system of parameters \(\underline{x} = x_1, \ldots, x_d\) of \(R\) such that \(\chi_1(\underline{x}, M_n) = o(\nu(M_n))\).

**Remark 2.2.** It is worth to point out that, under the above definitions, there do exist weakly lim Cohen–Macaulay sequences that are not \(\text{lim} \) Cohen–Macaulay, see [Hoc17] paragraph before Conjecture 10.1.

We begin by collecting some simple facts about weakly lim Cohen–Macaulay sequence.

**Lemma 2.3.** Let \((R, \mathfrak{m}) \to (S, \mathfrak{n})\) be a flat local extension of local rings such that \(\dim R = \dim S\). If \(\{M_n\}\) is a (weakly) lim Cohen–Macaulay sequence for \(R\), then \(\{M_n \otimes_R S\}\) is a (weakly) lim Cohen–Macaulay sequence for \(S\).

**Proof.** This is immediate by noting that, since \((R, \mathfrak{m}) \to (S, \mathfrak{n})\) is flat local with \(\dim R = \dim S\), we have
\[
\ell_S(H_i(\underline{x}, M_n \otimes_R S)) = \ell_R(H_i(\underline{x}, M_n)) \cdot \ell_S(S/\mathfrak{m}S)
\]
and \(\nu_S(M_n \otimes_R S) = \nu_R(M_n)\).

**Lemma 2.4.** Let \((R, \mathfrak{m})\) be a local ring of dimension \(d\). Then a sequence of finitely generated modules \(\{M_n\}\) of dimension \(d\) is weakly lim Cohen–Macaulay if and only if there exists a system of parameters \(\underline{x} = x_1, \ldots, x_d\) of \(R\) such that
\[
\lim_{n \to \infty} \frac{e(\underline{x}, M_n)}{\ell(M_n/(x)M_n)} = 1 \quad \text{(or equivalently,} \quad \lim_{n \to \infty} \frac{\chi_1(\underline{x}, M_n)}{\ell(M_n/(x)M_n)} = 0)\).
\]
Moreover, if \(R\) is a domain and \(\{M_n\}\) is weakly lim Cohen–Macaulay, then there exists a constant \(C\) such that for all \(n\),
\[
\text{rank}_R(M_n) \leq \nu(M_n) \leq C \cdot \text{rank}_R(M_n).
\]
In particular, if \(R\) is a domain then we can use \(\text{rank}_R(M_n)\) in place of \(\nu(M_n)\) in the definition of lim Cohen–Macaulay and weakly lim Cohen–Macaulay sequence.

**Proof.** The first conclusion is clear since \(\nu(M_n) \leq \ell(M_n/(x)M_n) \leq \nu(M_n) \cdot \ell(R/(x))\) (thus asymptotically it doesn’t matter whether we use \(\ell(M_n/(x)M_n)\) or \(\nu(M_n)\) in the denominator).

To see the second conclusion, we note that
\[
\text{rank}_R(M_n) \cdot e(\underline{x}, R) = e(\underline{x}, M_n) = \ell(M_n/(x)M_n) - \chi_1(\underline{x}, M_n) \geq \nu(M_n) - \chi_1(\underline{x}, M_n).
\]
Dividing by \(\nu(M_n)\) we obtain that
\[
\frac{\text{rank}_R(M_n) \cdot e(\underline{x}, R)}{\nu(M_n)} \geq 1 - \frac{\chi_1(\underline{x}, M_n)}{\nu(M_n)}.
\]
Since \( \{M_n\} \) is weakly lim Cohen–Macaulay, the right hand side tends to 1 when \( n \to \infty \). Thus there exists \( \epsilon > 0 \) such that for all \( n \) sufficiently large,

\[
\nu(M_n) \leq (1 + \epsilon)e(\underline{x}, R) \operatorname{rank}_R(M_n).
\]

We now simply pick \( C \gg (1 + \epsilon)e(\underline{x}, R) \) that also works for all small values of \( n \).

In [BHM], it is proved that the definition of lim Cohen–Macaulay sequence is independent of the choice of the system of parameters \( \underline{x} \). Here we prove the analogous statement for weakly lim Cohen–Macaulay sequence. The proof is quite non-obvious (however, we point out that this is needed, because eventually we can only show that our construction leads to weakly lim Cohen–Macaulay sequence: see Theorem 3.5).

**Proposition 2.5.** Let \((R, \mathfrak{m})\) be a local ring of dimension \( d \). If \( \{M_n\} \) is a weakly lim Cohen-Macaulay sequence, then

\[
\lim_{n \to \infty} \frac{e(\underline{x}, M_n)}{\ell(M_n/\langle x \rangle M_n)} = 1 \quad \text{(or equivalently,} \quad \lim_{n \to \infty} \frac{\chi_1(\underline{x}, M_n)}{\ell(M_n/\langle x \rangle M_n)} = 0) \tag{1}
\]

for every system of parameters \( \underline{x} = x_1, \ldots, x_d \) of \( R \). As a consequence, if \( \{M_n\} \) is a weakly lim Cohen-Macaulay sequence, then \( \chi_1(\underline{x}, M_n) = o(\nu(M_n)) \) for every system of parameters \( \underline{x} = x_1, \ldots, x_d \) of \( R \).

**Proof.** We first note that if (1) holds for \( \underline{x} = x_1, \ldots, x_d \), then it holds for \( \underline{x}^t = x_1^t, \ldots, x_d^t \). This is because \( e(\underline{x}^t, M) = t_1 \cdots t_d \cdot e(\underline{x}, M) \) while \( \ell(M_n/\langle x \rangle M_n) \leq t_1 \cdots t_d \cdot \ell(M_n/\langle x \rangle M_n) \), and the limit in (1) is always \( \leq 1 \).

We next note that given two system of parameters \( \underline{x} = x_1, \ldots, x_d \) and \( y = y_1, \ldots, y_d \) of \( R \), we can always connect \( x, y \) by a chain of system of parameters such that each two consecutive only differ by one element. Thus it suffices to show that if (1) holds for \( x, x_2, \ldots, x_d \), then it holds for \( y, x_2, \ldots, x_d \). By the discussion in the first paragraph we can replace \( x \) by \( x^t \) for \( t \gg 0 \) to assume that \( (x, x_2, \ldots, x_d) \subseteq (y, x_2, \ldots, x_d) \), and thus by a change of variables we may assume \( x = y \). Thus it is enough to prove that if (1) holds for \( y, x_2, \ldots, x_d \), then it holds for \( y, x_2, \ldots, x_d \).

From now on we use \( x^- \) to denote \( x_2, \ldots, x_d \). For each \( M \in \{M_n\} \), we have

\[
\ell(M/(yz, x^-)M) - \chi_1((yz, x^-), M) = e((yz, x^-), M) \\
\leq e(yz, M/(x^-)M) = \ell(M/(yz, x^-)M) - \ell(\operatorname{Ann}_{M/x^-} yz).
\]

Thus \( \ell(\operatorname{Ann}_{M/x^-} yz) \leq \chi_1((yz, x^-), M) \). Since we assume (1) holds for \( (yz, x^-) \), we have

\[
\lim_{n \to \infty} \frac{\ell(\operatorname{Ann}_{M_n/x^-M_n} yz)}{\nu(M_n)} \leq \ell(R/(yz, x^-)) \cdot \lim_{n \to \infty} \frac{\chi_1((yz, x^-), M_n)}{\ell(M_n/(yz, x^-)M_n)} = 0.
\]

Since \( \operatorname{Ann}_{M_n/x^-M_n} y \) and \( \operatorname{Ann}_{M_n/x^-M_n} z \) are submodules of \( \operatorname{Ann}_{M_n/x^-M_n} yz \), we have

\[
\lim_{n \to \infty} \frac{\ell(\operatorname{Ann}_{M_n/x^-M_n} y)}{\nu(M_n)} = 0, \quad \text{and} \quad \lim_{n \to \infty} \frac{\ell(\operatorname{Ann}_{M_n/x^-M_n} z)}{\nu(M_n)} = 0.
\]
At this point, we look at the long exact sequence of the Koszul homology:

\[ 0 \to H_d((yz, x^-), M) \to H_{d-1}(x^-, M) \to H_{d-1}((yz, x^-), M) \to \cdots \]

Recall that if \( N \) is any finitely generated \( R \)-module and \( w \in R \) is such that \( \ell(N/wN) < \infty \), then \( e(w, N) = \ell(N/wN) - \ell(\Ann w). \) Thus taking the alternating sum of lengths and multiplicities of the above long exact sequence, we get:

\[
(2.5.2) \quad \sum_{j=1}^{d-1} (-1)^{j-1} e(yz, H_j(x^-, M)) = \chi_1((yz, x^-), M) - \ell(\Ann_{x^-} M yz).
\]

The same argument shows that

\[
(2.5.3) \quad \sum_{j=1}^{d-1} (-1)^{j-1} e(y, H_j(x^-, M)) = \chi_1((y, x^-), M) - \ell(\Ann_{x^-} y), \text{ and}
\]

\[
(2.5.4) \quad \sum_{j=1}^{d-1} (-1)^{j-1} e(z, H_j(x^-, M)) = \chi_1((z, x^-), M) - \ell(\Ann_{x^-} z).
\]

Since we assume \([\dag]\) holds for \((yz, x^-)\), applying (2.5.2) for each \( M \in \{M_n\} \) shows that

\[
(2.5.5) \quad \lim_{n \to \infty} \frac{\sum_{j=1}^{d-1} (-1)^{j-1} e(yz, H_j(x^-, M_n))}{\nu(M_n)} \leq \ell(R/(yz, x^-)) \cdot \lim_{n \to \infty} \frac{\chi_1((yz, x^-), M_n)}{\nu(M_n)} = 0.
\]

Thus by (2.5.1) (2.5.3) (2.5.4) and the non-negativity of \( \chi_1 \), we have

\[
(2.5.6) \quad \lim_{n \to \infty} \frac{\sum_{j=1}^{d-1} (-1)^{j-1} e(y, H_j(x^-, M_n))}{\nu(M_n)} = \lim_{n \to \infty} \frac{\chi_1((y, x^-), M_n)}{\nu(M_n)} \geq 0, \text{ and}
\]

\[
(2.5.7) \quad \lim_{n \to \infty} \frac{\sum_{j=1}^{d-1} (-1)^{j-1} e(z, H_j(x^-, M_n))}{\nu(M_n)} = \lim_{n \to \infty} \frac{\chi_1((z, x^-), M_n)}{\nu(M_n)} \geq 0.
\]

Finally, we recall that \( e(y z, N) = e(y, N) + e(z, N) \) for any finitely generated \( R \)-module \( N \) such that \( \ell(N/(yz)N) < \infty \). Thus by (2.5.5), we know that

\[
0 = \lim_{n \to \infty} \frac{\sum_{j=1}^{d-1} (-1)^{j-1} e(y z, H_j(x^-, M_n))}{\nu(M_n)} = \lim_{n \to \infty} \frac{\sum_{j=1}^{d-1} (-1)^{j-1} e(y, H_j(x^-, M_n))}{\nu(M_n)} + \lim_{n \to \infty} \frac{\sum_{j=1}^{d-1} (-1)^{j-1} e(z, H_j(x^-, M_n))}{\nu(M_n)}.
\]

\[\text{To see this, we can complete } R \text{ and } N. \text{ Let } V \text{ be a coefficient ring of } R \text{ and we can view } N \text{ as a module over the regular ring } A = V[y, z][y, z] \text{ with } \ell(N/(yz)N) < \infty. \text{ Since the multiplicity is the same as the Euler characteristic computed over } A, \text{ the desired formula follows from the additivity of } \chi^A(-, N) \text{ applied to the short exact sequence } 0 \to A/y \to A/yz \to A/z \to 0.\]
But by (2.5.6) and (2.5.7), the last two limits above are both non-negative, hence they are both zero. But then by (2.5.6) and (2.5.7) again, we have

$$\lim_{n \to \infty} \frac{\chi_1((y, x^\cdot), M_n)}{\nu(M_n)} = \lim_{n \to \infty} \frac{\chi_1((z, x^\cdot), M_n)}{\nu(M_n)} = 0.$$  

Therefore by Lemma 2.4 (†) holds for the system of parameters \((y, x^\cdot)\). The last conclusion follows from Lemma 2.4. This finishes the proof. 

We need the following important consequence of the above proposition.

**Corollary 2.6.** Let \((R, m)\) be a local ring of dimension \(d\) with an infinite residue field and let \(\{M_n\}\) be a weakly lim Cohen–Macaulay sequence. Then

$$\lim_{n \to \infty} \frac{e(m, M_n)}{\nu(M_n)} \geq 1.$$  

**Proof.** Let \(\tilde{z} = z_1, \ldots, z_d\) be a minimal reduction of \(m\). Since \(\{M_n\}\) is weakly lim Cohen–Macaulay, by Proposition 2.5 we know that \(\chi_1(\tilde{z}, M_n) = o(\nu(M_n))\). Therefore

$$\lim_{n \to \infty} \frac{e(m, M_n)}{\nu(M_n)} = \lim_{n \to \infty} \frac{e(\tilde{z}, M_n)}{\nu(M_n)} = \lim_{n \to \infty} \frac{\ell(M_n/(\tilde{z}) M_n)}{\nu(M_n)} \geq 1.$$  

Finally, we introduce lim Ulrich and weakly lim Ulrich sequence.

**Definition 2.7.** Let \((R, m)\) be a local ring of dimension \(d\). A sequence of finitely generated \(R\)-modules \(\{U_n\}\) of dimension \(d\) is called \(\text{lim Ulrich}\) (resp. \(\text{weakly lim Ulrich}\)) if it is lim Cohen–Macaulay (resp. weakly lim Cohen–Macaulay) and

$$\lim_{n \to \infty} \frac{e(m, U_n)}{\nu(U_n)} = 1.$$  

The following is the main result of this section.

**Theorem 2.8.** Let \((R, m) \to (S, n)\) be a flat local extension of local rings such that \(R\) is a domain and \(\dim R = \dim S\). Suppose \(R\) admits a weakly lim Ulrich sequence \(\{U_n\}\). Then \(e(R) \leq e(S)\).

**Proof.** We can replace \(S\) by \(S[t]_{\|t\|}\) to assume \(S\) has an infinite residue field. Since \(R\) is a domain and \(\{U_n\}\) is a weakly lim Ulrich sequence, we have:

$$e(R) = \lim_{n \to \infty} \frac{e(m, U_n)}{\text{rank}_R U_n} = \lim_{n \to \infty} \frac{\nu_R(U_n)}{\text{rank}_R U_n} = \lim_{n \to \infty} \frac{\nu_S(U_n \otimes_R S)}{\text{rank}_R U_n} \leq \lim_{n \to \infty} \frac{e(n, U_n \otimes_R S)}{\text{rank}_R U_n} = e(S)$$

where the only \(\leq\) follows from Corollary 2.6 because \(\{U_n \otimes_R S\}\) is a weakly lim Cohen–Macaulay sequence over \(S\) by Lemma 2.3. □

We end this section with a proposition which follows from more general results in [BHM]. As this work is still in the stage of preparation, we give the proof of the proposition for the sake of completeness.
**Proposition 2.9.** Let \((R, m)\) be a local domain of dimension \(d\) and let \(\{M_n\}\) be a sequence of finitely generated modules of dimension \(d\). Suppose \(H^i_m(M_n)\) has finite length for all \(n\) and all \(j < d\). Then \(\{M_n\}\) is a lim Cohen–Macaulay sequence (and hence a weakly lim Cohen–Macaulay sequence) if
\[
\ell(H^i_m(M_n)) = o(\text{rank}_RM_n)
\]
for all \(j < d\).

**Proof.** Let \(x = x_1, \ldots, x_d\) be a system of parameters of \(R\). We have
\[
H_i(x, M_n) = h^{-i}(K_\bullet(x, R) \otimes_R M_n) = h^{-i}(K_\bullet(x, R) \otimes_R R\Gamma_m(M_n)).
\]
Therefore we have a spectral sequence:
\[
H_{j+i}(x, H^i_m(M_n)) \Rightarrow H_i(x, M_n).
\]
If \(j = d\), then \(j + i > d\) when \(i \geq 1\). So for all \(i \geq 1\) we have
\[
\ell(H_i(x, M_n)) \leq \sum_{j=0}^{d-1} \ell(H_{j+i}(x, H^j_m(M_n))) \leq \sum_{j=0}^{d-1} 2^d \cdot \ell(H^j_m(M_n)) = o(\text{rank}_RM_n).
\]
This implies that \(\{M_n\}\) is lim Cohen–Macaulay by Lemma 2.4. \(\square\)

### 3. Main result for graded rings

In this section we prove our main results. We start with a Segre product construction which will play a crucial role in our construction of weakly lim Ulrich sequence.

**Setting 3.1.** We fix an infinite field \(k\) of characteristic \(p > 0\) and let \(q = p^e\) (eventually we will let \(e \to \infty\) so one should think \(q\) being very large). We consider
\[
W^n_q := k[x_1, y_1]#k[x_2, y_2](q)\cdots#k[x_n, y_n]((n-1)q),
\]
which is a rank one module over the ring
\[
T_n = k[x_1, y_1]#k[x_2, y_2]\cdots#k[x_n, y_n].
\]
We note that \(T_n\) is a standard graded ring of dimension \(n + 1\): the degree \(j\) part is spanned by monomials whose total degree in \(x_i\) and \(y_i\) is \(j\) for each \(1 \leq i \leq n\). Hence \(T_n\) is module-finite over \(A_n = k[z_1, z_2, \ldots, z_{n+1}]\) where \(z_1, \ldots, z_{n+1}\) are general homogeneous degree one elements in \(T_n\). We will view \(W^n_q\) as a graded module over \(A_n\) that sits in non-negative degrees (because \(k[x_1, y_1]\) only lives in non-negative degrees). We abuse notations a bit and let \(m\) denote the homogeneous maximal ideal of \(A_n\). Since \(W^n_q\) is torsion-free and reflexive, we have \(H^0_m(W^n_q) = H^1_m(W^n_q) = 0\).

The next lemma on the degrees and dimensions of local cohomology modules of \(W^n_q\) is elementary. In fact, since \(W^n_q\) is explicitly described, precise dimensions of each degree of its local cohomology modules can be computed (geometrically, this simply corresponds to the sheaf cohomology of \(O_{P^1}(t) \boxtimes O_{P^1}(t + q) \boxtimes \cdots \boxtimes O_{P^1}(t + (n-1)q)\) on a product of projective lines when \(t, q\) vary). We are not interested in the precise formulas so we state what we need.

**Lemma 3.2.** With notations as in Setting 3.1, we have
(a) For each \(2 \leq j \leq n\), \(H^j_m(W^n_q)\) sits in degrees \(-(j-2)q - 2, \ldots, -(j-1)q\).
(b) \(H^{n+1}_m(W^n_q)\) sits in degrees \(- (n-1)q - 2\).
(c) Fix a negative integer $-r$, then as $q \to \infty$,
\[
\dim_k H^2_m(W^n_q)_{-r}, \dim_k H^3_m(W^n_q)_{-r}, \ldots, \dim_k H^{n+1}_m(W^n_q)_{-(n-1)q-r}
\]
grow like $o(q^n)$ while $\dim_k H^{n+1}_m(W^n_q)_{-(n+t)q-r}$ grows like $o(q^{n+1})$ for each fixed $t \geq 0$.

Proof. We use induction on $n$, the case $n = 1$ is obvious. Now suppose the lemma is proven for $n-1$. Since $W^n_q = W^{n-1}_q \# k[x_n, y_n]((n-1)q)$, it follows from the Kunneth formula for local cohomology (see [GW78]) that
\[
\begin{align*}
H^j_m(W^n_q) &= H^j_m(W^{n-1}_q) \# (k[x_n, y_n]((n-1)q)), \text{ for all } j \leq n \\
H^{n+1}_m(W^n_q) &= H^{n+1}_m(W^{n-1}_q) \# H^1_m(k[x_n, y_n]((n-1)q)).
\end{align*}
\]

Note that we are abusing notations a bit here and simply use $\mathfrak{m}$ to denote the homogeneous maximal ideal over the corresponding ring. Also note that we are ignoring terms that are 0 coming from the inductive hypothesis when applying the Kunneth formula.

From [3.2.1] part (a) and (b) are clear by the inductive hypothesis. For example, when $j = n$, $H^n_m(W^{n-1}_q)$ lives in degree $\leq -(n-2)q - 2$ while $k[x_n, y_n]((n-1)q)$ lives in degree $\geq -(n-1)q$, which shows that $H^n_m(W^n_q)$ lives in degree $-(n-2)q - 2, \ldots, -(n-1)q$. To establish part (c), we note that by [3.2.1] and the induction hypothesis, for $j \leq n$,
\[
\dim_k H^j_m(W^n_q)_{-(j-2)q-r} = \dim_k H^j_m(W^{n-1}_q)_{-(j-2)q-r} \cdot \dim_k (k[x_n, y_n]((n+1-j)q-r)
\]
\[
= o(q^{n-1}) \cdot ((n+1-j)q-r+1) = o(q^n).
\]

For the top local cohomology, again by [3.2.1] and the induction hypothesis,
\[
\dim_k H^{n+1}_m(W^n_q)_{-(n+t)q-r} = \dim_k H^{n+1}_m(W^{n-1}_q)_{-(n+t)q-r} \cdot \dim_k H^2_m(k[x_n, y_n])_{(t+1)q-r}
\]
\[
= o(q^n) \cdot ((t+1)q+r-1).
\]

This gives $o(q^n)$ for $t = -1$ and $o(q^{n+1})$ for $t \geq 0$. $\square$

The following immediate consequence is what we will need in the sequel. We adopt the following notation: if $M$ is a $\mathbb{Z}$-graded module, then $M_{a (\mod q)} := \oplus_{i \in \mathbb{Z}} M_{a+iq}$

**Corollary 3.3.** With notations as in Setting 3.1, for any fixed negative integer $-r$ and any $0 \leq j \leq n$,
\[
\dim_k H^j_m(W^n_q)_{-r (\mod q)} = o(q^n) \text{ as } q \to \infty.
\]

Proof. This follows directly from part (a) and (c) of Lemma 3.2. $\square$

**Remark 3.4.** We caution the reader that the degree range in Lemma 3.2 and Corollary 3.3 are important: it is not true that $\dim_k H^j_m(W^n_q)_{t} = o(q^n)$ for every $t$ and every $0 \leq j \leq n$.

Now we state and prove our main result on weakly lim Ulrich sequence.

**Theorem 3.5.** Let $(R, \mathfrak{m})$ be a standard graded domain over an infinite $F$-finite field $k$ of characteristic $p > 0$ (localized at the homogeneous maximal ideal). Then $R$ admits a weakly lim Ulrich sequence.

Proof. Let $\dim R = d$. We will assume $d \geq 2$ to avoid some tautology in the construction (if $d = 1$, then it is easy to see that $\mathfrak{m}^N$ is a Ulrich module for $N \gg 0$ so $R$ trivially admits a weakly lim Ulrich sequence). Since $R$ is standard graded and $k$ is infinite, there exists homogeneous degree one elements $z_1, \ldots, z_d$ of $R$ that forms a minimal reduction of $\mathfrak{m}$. We
identify the subring $A := k[z_1, \ldots, z_d]$ with the ring $A_{d-1}$ as in Setting 3.1. Thus we have a sequence of finitely generated modules $\{W^d_q\}$ over $A$ where $q = p^e$. We will show that the following sequence:

$$U_e := F^e_e((R \otimes_A W^d_q)_{-1}(\text{mod} \; q))$$

is a weakly lim Ulrich sequence over $R$.

Note that the $R$-module structure on $U_e$ is well-defined: under the $e$-th Frobenius push-
forward, $x \in R$ acts as $x^q$ so elements in $R \otimes_A W^d_q$ of degree $\equiv -1 \; (\text{mod} \; q)$ are preserved under the $R$-action. Also note that we take the degree $\equiv -1 \; (\text{mod} \; q)$ in the definition of $U_e$ just for simplicity: in fact the proof will show that any fixed negative integer $-r$ will work (on the other hand, non-negative integers will not work!).

We first consider the special case that $R$ is Cohen–Macaulay: the proof in this case is substantially less technical while revealing the idea behind the construction. It is worth to point out that in this case, we can actually show that $\{U_e\}$ is lim Ulrich. However, we also point out that the individual $U_e$ is not Cohen–Macaulay.

The case $R$ is Cohen–Macaulay. Since $R$ is Cohen–Macaulay and is a graded module-
finite extension of the polynomial ring $A$, we know $R \cong \oplus_{i=1}^s A(-a_i)$ as a graded $A$-module where $s = \text{rank}_A R$ and $a_i \geq 0$ for each $i$. Thus we have

$$U_e \cong \oplus_{i=1}^s F^e_e(W^d_q(-a_i)_{-1}(\text{mod} \; q)) \cong \oplus_{i=1}^s F^e_e((W^d_q)_{-1-a_i}(\text{mod} \; q))$$

as graded $A$-modules. Recall that $W^d_q$ is a rank one module over $T_{d-1}$, and

$$\dim_k(T_{d-1})_t = (t + 1)^{d-1} = t^{d-1} + o(t^{d-1}),$$

thus the multiplicity of $T_{d-1}$ as an $A$-module is $(d-1)!$. Therefore

$$e(\underline{z}, W^d_q) = e(m_A, W^d_q) = (d-1)!.$$

It follows that $\text{rank}_A W^d_q = (d-1)!$. Thus for every fixed negative integer $-r$, the rank of $(W^d_q)_{-r}(\text{mod} \; q)$ as a module over the $q$-th Veronese subring of $A$ is equal to $(d-1)!$ (to see this, let $A^{(q)}$ denote the $q$-th Veronese subring of $A$, we claim $(W^d_q)_{-r}(\text{mod} \; q) \otimes_{A^{(q)}} \text{Frac}(A) = W^d_q \otimes_A \text{Frac}(A)$ is free over $A$, every homogeneous element of the latter tensor product can be written as $w/x$, where $w \in W^d_q$ and $x \in A$, we can pick $y \in A$ such that $\deg w + \deg y \equiv -r \; (\text{mod} \; q)$ since $A$ is generated in degree one, thus $w/x = w y/x y$ so it is in the former tensor product).

Thus the rank of $F^e_e((W^d_q)_{-r}(\text{mod} \; q))$ over $A^{(q)}$ is equal to $(d-1)!q^{d+\alpha}$ where $\alpha = \log_p [k : k^p]$. Therefore, since $\text{rank}_{A^{(q)}} A = q$, for every fixed negative integer $-r$, we have

$$\text{rank}_A F^e_e((W^d_q)_{-r}(\text{mod} \; q)) = (d-1)!q^{d+\alpha-1}.$$

To show $\{U_e\}$ is (weakly) lim Cohen–Macaulay, by Proposition 2.9 it is enough to prove that for every fixed negative integer $-r$ and each $j \leq d-1$,

$$\ell(H^j_m(F^e_e((W^d_q)_{-r}(\text{mod} \; q)))) = o(\text{rank}_A F^e_e((W^d_q)_{-r}(\text{mod} \; q))) = o(q^{d+\alpha-1}).$$

But since $H^j_m(F^e_e((W^d_q)_{-r}(\text{mod} \; q))) = F^e_e(H^j_m(W^d_q)_{-r}(\text{mod} \; q))$ and under the Frobenius push-forward $F^e_e$, the lengths get multiplied by $p^\alpha$, (3.5.2) follows from Corollary 3.3.
Finally, to show \( \{ U_e \} \) is (weakly) \( \lim \) Ulrich, we note that

\[
e(m, U_e) = e(\mathbb{Z}, U_e) = \sum_{i=1}^{s} e(m, F^e_s((W_q^{d-1})_{-1-a_i(mod q)})
\]

\[
= \sum_{i=1}^{s} \text{rank}_A F^e_s((W_q^{d-1})_{-1-a_i(mod q)}) = (d - 1)!sq^{d+\alpha - 1}
\]

by \( (3.5.1) \). On the other hand, since \( R \otimes A W_q^{d-1} \) lives in non-negative degrees, \( m^{[q]} : (R \otimes A W_q^{d-1}) \) lives in degree \( \ge q \). Therefore by the definition of \( U_e \), we know that

\[
\nu_R(U_e) \ge \text{dim}_k F^e_s ((R \otimes A W_q^{d-1})_{q-1}).
\]

However, by the definition of \( W_q^{d-1} \) as in \( \text{Setting 3.1} \) for every fixed negative integer \( -r \), we know that

\[
\dim_k (W_q^{d-1})_{q-r} = (q-r+1)(2q-r+1) \cdots ((d-1)q-r+1) = (d-1)!q^{d-1} + o(q^{d-1}).
\]

Therefore, since \( a_i \ge 0 \), we have

\[
\dim_k F^e_s ((R \otimes A W_q^{d-1})_{q-1}) = \sum_{i=1}^{s} \dim_k F^e_s ((W_q^{d-1})_{q-1-a_i}) = (d - 1)!sq^{d+\alpha - 1} + o(q^{d+\alpha - 1}).
\]

Putting the above together, we have

\[
\lim_{e \to \infty} \frac{e(m, U_e)}{\nu_R(U_e)} \le \frac{e(m, U_e)}{\dim_k F^e_s ((R \otimes A W_q^{d-1})_{q-1})} = 1.
\]

Since we know the above limit is always \( \ge 1 \) by \( \text{Corollary 2.6} \) (because we have shown that \( \{ U_e \} \) is a (weakly) \( \lim \) Cohen–Macaulay), this shows the above limit is equal to 1 and hence \( \{ U_e \} \) is a (weakly) \( \lim \) Ulrich sequence.

**The general case.** To handle the general case we first observe that our argument in the Cohen–Macaulay case proves that for every fixed negative integer \( -r \), \( F^e_s ((W_q^{d-1})_{-r(mod q)}) \) is a \( \lim \) Cohen–Macaulay sequence over \( A \) (see \( (3.5.1) \) \( (3.5.2) \) and \( \text{Proposition 2.9} \)). In particular, we have (dropping \( F^e_s \) results in dividing lengths by \( q^\alpha \))

\[
(3.5.4) \quad \ell \left( \frac{(W_q^{d-1})_{-r(mod q)}}{(\mathbb{Z}^q)(W_q^{d-1})_{-r(mod q)}} \right) = (d - 1)!q^{d-1} + o(q^{d-1}).
\]

On the other hand, we know that \( \dim_k (W_q^{d-1})_{q-r} = (d - 1)!q^{d-1} + o(q^{d-1}) \) by \( (3.5.3) \) and that \( (W_q^{d-1})_{q-r} \cap (\mathbb{Z}^q)(W_q^{d-1})_{-r(mod q)} = 0 \) for degree reason (recall that \( W_q^{d-1} \) only lives in non-negative degrees). This together with \( (3.5.4) \) imply that

\[
(3.5.5) \quad \dim_k \left( (W_q^{d-1}/(\mathbb{Z}^q)W_q^{d-1})_{-r(mod q), \ne q-r} \right) = o(q^{d-1}).
\]

We now prove that \( \{ U_e \} \) is a weakly \( \lim \) Cohen–Macaulay sequence. Let \( s = \text{rank}_A R \). We have a degree-preserving short exact sequence

\[
(3.5.6) \quad 0 \to \oplus_{i=1}^{s} A(-b_i) \to R \to C \to 0
\]
where $C$ has dimension less than $d$ (note that $b_i \geq 0$ for all $i$). The rank of $U_e$ over $A$ is the same as the rank of

$$F^e_\ast \left( \bigoplus_{i=1}^{s} (\mathbb{A}/(b_i)) \otimes_A W^{d-1}_{q} \right)_{1 (\text{mod } q)} \cong \bigoplus_{i=1}^{s} F^e_\ast \left( (W^{d-1}_{q})_{1 - b_i (\text{mod } q)} \right)$$

over $A$. Therefore by (3.5.1) we still have

$$\text{rank}_A U_e = e(\bar{z}, U_e) = (d - 1)! sq^{d+\alpha - 1}.$$

Thus to show $\{U_e\}$ is weakly lim Cohen–Macaulay, it is enough to show $\ell(U_e/(\bar{z})U_e) \leq (d - 1)! sq^{d+\alpha - 1} + o(q^{d+\alpha - 1})$ by Lemma 2.4 (applied to $\bar{z} = \bar{z}$). Dropping $F^e_\ast$, this comes down to prove that

$$\ell \left( \frac{(R \otimes_A W^{d-1}_{q})_{1 (\text{mod } q)}}{\left( \bar{z} \right) A \otimes W^{d-1}_{1 (\text{mod } q)} \right) \right) \leq (d - 1)! sq^{d-1} + o(q^{d-1}).$$

From (3.5.6) we obtain an exact sequence:

$$\frac{\bigoplus_{i=1}^{s} (W^{d-1}_{q})_{1 - b_i (\text{mod } q)}}{ \left( \bar{z} \right) \bigoplus_{i=1}^{s} (W^{d-1}_{q})_{1 - b_i (\text{mod } q)} } \rightarrow \frac{(R \otimes_A W^{d-1}_{q})_{1 (\text{mod } q)}}{\left( \bar{z} \right) (R \otimes_A W^{d-1}_{q})_{1 (\text{mod } q)} } \rightarrow \frac{(C \otimes_A W^{d-1}_{q})_{1 (\text{mod } q)}}{\left( \bar{z} \right) (C \otimes_A W^{d-1}_{q})_{1 (\text{mod } q)} } \rightarrow 0.$$

By (3.5.4) in order to establish (3.5.7) it is enough to show that

$$\ell \left( \frac{(C \otimes_A W^{d-1}_{q})_{1 (\text{mod } q)}}{\left( \bar{z} \right) (C \otimes_A W^{d-1}_{q})_{1 (\text{mod } q)} } \right) = o(q^{d-1}).$$

Since $C$ is a finitely generated graded $A$-module of dimension less than $d$ and lives in non-negative degrees, it has a graded filtration by $(A/P_i)(-c_i)$, where $P_i$ are nonzero homogeneous prime ideals of $A$ and $c_i \geq 0$. So it is enough to prove that for any fixed homogeneous prime ideal $P \subseteq A$ and any $c \geq 0$, we have

$$\ell \left( \frac{(W^{d-1}_{q})_{1 - c (\text{mod } q)}}{(P W^{d-1}_{q})_{1 - c (\text{mod } q)} + \left( \bar{z} \right) (W^{d-1}_{q})_{1 - c (\text{mod } q)} } \right) = o(q^{d-1}).$$

At this point, we invoke (3.5.5). Thus in order to establish the above, it is enough to show that

$$\dim_k(W^{d-1}_q/PW^{d-1}_q)_{q-1-c} = o(q^{d-1}).$$

Fix $0 \neq z \in P$ of degree $a > 0$. Since $W^{d-1}_q/zW^{d-1}_q \rightarrow W^{d-1}_q/PW^{d-1}_q$ and $W^{d-1}_q$ is torsion-free, we know that

$$\dim_k(W^{d-1}_q/PW^{d-1}_q)_{q-1-c} \leq \dim_k(W^{d-1}_q/zW^{d-1}_q)_{q-1-c} = \dim_k(W^{d-1}_q)_{q-1-c} - \dim_k(W^{d-1}_q)_{q-1-c-a} = o(q^{d-1})$$

where the last equality follows from (3.5.3). This completes the proof of (3.5.7) and hence we have established that $\{U_e\}$ is weakly lim Cohen–Macaulay.

Finally, we prove that $\{U_e\}$ is weakly lim Ulrich. Again since $R \otimes_A W^{d-1}_q$ only lives in non-negative degrees, $m^{[q]} \cdot (R \otimes_A W^{d-1}_q)$ lives in degree $\geq q$. Thus by the definition of $U_e$, we know that

$$\nu_R(U_e) \geq \dim_k F^e_\ast \left( (R \otimes_A W^{d-1}_q)_{q-1} \right).$$

Thus it remains to show that

$$\dim_k(R \otimes_A W^{d-1}_q)_{q-1} \geq (d - 1)! sq^{d-1} + o(q^{d-1}),$$

where $C$ has dimension less than $d$ (note that $b_i \geq 0$ for all $i$). The rank of $U_e$ over $A$ is the same as the rank of

$$F^e_\ast \left( \bigoplus_{i=1}^{s} (\mathbb{A}/(b_i)) \otimes_A W^{d-1}_{q} \right) \cong \bigoplus_{i=1}^{s} F^e_\ast \left( (W^{d-1}_{q})_{1 - b_i (\text{mod } q)} \right)$$

over $A$. Therefore by (3.5.1) we still have

$$\text{rank}_A U_e = e(\bar{z}, U_e) = (d - 1)! sq^{d+\alpha - 1}.$$
because this then implies that \( \dim_k F^s_r \left( (R \otimes_A W_q^{d-1})_{q-1} \right) \geq (d-1)!sq^{d+\alpha-1} + o(q^{d+\alpha-1}) \) while \( e(m, U_e) = e(z, U_e) = (d - 1)!sq^{d+\alpha-1} \). To establish \( (3.5.8) \) is trickier as \( R \otimes_A W_q^{d-1} \) is no longer a direct sum of graded shifts of \( W_q^{d-1} \). We need the following claim.

**Claim 3.6.** Let \( M \) be a finitely generated graded \( A \)-module which sits in non-negative degrees. Then for any fixed negative integer \(-r\) and any \( i \geq 1 \), we have

\[
\ell(\Tor^A_i(M, W_q^{d-1})_{-r} \mod q) = \ell(h^{-i}(M \otimes_A W_q^{d-1})_{-r} \mod q) = o(q^{d-1}).
\]

**Proof of Claim.** Since all the lower local cohomology modules of \( W_q^{d-1} \) have finite length, \( W_q^{d-1} \) is Cohen–Macaulay on the punctured spectrum of \( A \). Since \( A \) is regular, this means \( W_q^{d-1} \) is finite free on the punctured spectrum of \( A \) and hence \( \Tor^A_i(M, W_q^{d-1}) \) has finite lengths for all \( i \geq 1 \). A simple spectral sequence argument shows that

\[
h^{-i}(M \otimes_A W_q^{d-1}) = h^{-i}(R\Gamma_m(M \otimes_A W_q^{d-1})) = h^{-i}(M \otimes_A R\Gamma_m(W_q^{d-1})) \text{ for all } i \geq 1.
\]

As a consequence, we have a degree-preserving spectral sequence:

\[
\Tor^A_{j+i}(M, H^j_m(W_q^{d-1})) \Rightarrow h^{-i}(M \otimes_A W_q^{d-1}).
\]

Next we consider a minimal graded finite free resolution of \( M \) over \( A \):

\[
(3.6.1) \quad 0 \rightarrow \oplus_l A(-a_{nl}) \rightarrow \cdots \rightarrow \oplus_l A(-a_{1l}) \rightarrow \oplus_l A(-a_{q1}) \rightarrow 0
\]

where \( n = \pd A M \) and all the \( a_{ij} \) are non-negative integers (since \( M \) lives in non-negative degrees). If \( j \leq d - 1 \), then using the above free resolution to compute \( \Tor^A_{j+i}(M, H^j_m(W_q^{d-1})) \), we see that

\[
\ell(\Tor^A_{j+i}(M, H^j_m(W_q^{d-1})_{-r} \mod q)) \leq \sum_l \dim_k H^j_m(W_q^{d-1})_{-r-a_{1j+l} \mod q} = o(q^{d-1})
\]

by Corollary 3.3. But if \( j = d \), then \( j + i \geq d + 1 \) so \( \Tor^A_{j+i}(M, H^j_m(W_q^{d-1})) = 0 \) since \( A \) is regular of dimension \( d \). Therefore all the \( E_2 \)-contributions of \( h^{-i}(M \otimes_A W_q^{d-1})_{-r \mod q} \) have lengths \( o(q^{d-1}) \). This completes the proof of the claim.

Now we return to the proof of the theorem, the short exact sequence \( (3.5.6) \) induces:

\[
\Tor^A_1(C, W_q^{d-1})_{q-1} \rightarrow \oplus_{i=1}^s W_q^{d-1}(-b_i)_{q-1} \rightarrow (R \otimes_A W_q^{d-1})_{q-1} \rightarrow (C \otimes_A W_q^{d-1})_{q-1} \rightarrow 0.
\]

It follows that

\[
\dim_k (R \otimes_A W_q^{d-1})_{q-1} \geq \sum_{i=1}^s \dim_k (W_q^{d-1})_{q-1-b_i} - \dim_k \Tor^A_1(C, W_q^{d-1})_{q-1} = (d - 1)!sq^{d-1} + o(q^{d-1})
\]

where the last equality follows from \( (3.5.3) \) and Claim 3.6. This completes the proof of \( (3.5.8) \) and hence \( \{U_e\} \) is a weakly lim Ulrich sequence, as desired.

**Remark 3.7.** We suspect the sequence \( \{U_e\} \) constructed in Theorem 3.5 is in fact lim Ulrich beyond the Cohen–Macaulay case. We hope to investigate this in future work. On the other hand, we also believe that the weakly lim Ulrich condition may be more flexible to work with (and easier to construct in practice).
Theorem 3.8. Let \((R, \mathfrak{m}) \to (S, \mathfrak{n})\) be a flat local extension of local rings. Suppose \((R, \mathfrak{m})\) is a standard graded ring over a perfect field \(k\) (localized at the homogeneous maximal ideal). Then \(e(R) \leq e(S)\).

Proof. Since every minimal prime of \(R\) is homogeneous, by the same argument as in [Ma17 Lemma 2.2], we may assume \((\hat{R}, \mathfrak{m})\) is a standard graded domain and \(\dim R = \dim S\). We can further assume that \(k\) is infinite and \(F\)-finite by replacing \(R\) and \(S\) by \(R[t]/mR[t]\) and \(S[t]/nS[t]\). The conclusion in characteristic \(p > 0\) now follows from Theorem 2.8 and Theorem 3.5 (note that we only need to assume \(k\) is \(F\)-finite).

Next we suppose \(k\) has characteristic 0 and \(R \to S\) is a counter-example to the theorem. Then \(\hat{R} \to \hat{S}\) is a flat local extension with \(e(\hat{R}) > e(\hat{S})\). Applying the argument in [Ma17 Lemma 5.1], we may assume \(\mathfrak{n} \cong R/\mathfrak{m} \cong S/\mathfrak{n}\) is algebraically closed and \(\hat{R} \to \hat{S}\) is module-finite (note that \(R\) is still standard graded over \(k\)). Now applying the reduction procedure in [Ma17 Subsection 5.1][4], there exists a pointed \(\acute{\text{e}}\text{tale}\) extension \(R'\) of \(R_\mathfrak{m}\) and a finite flat extension \(S'\) of \(R'\) such that \(e(\hat{R}) = e(\hat{R'}) > e(\hat{S})\). But then by inverting elements if necessary, we may assume that we have

\[
R \to R'' = \left(\frac{R[x]}{f}\right)_\mathfrak{n} \to S''
\]

such that \(R''\) is standard \(\acute{\text{e}}\text{tale}\) over \(R\) near a maximal ideal \(\mathfrak{n}''\) lying over \(\mathfrak{m}\), \(R'' \to S''\) is finite flat with a maximal ideal \(\mathfrak{n}'' \in S''\) lying over \(\mathfrak{n}''\), and that \(e(\mathfrak{n}, R) = e(\mathfrak{n}'', R'') > e(\mathfrak{n}'', S'').\)

We can reduce this set up to characteristic \(p \gg 0\) as in [Ma17 Subsection 5.2] to obtain

\[
R_\kappa \to R''_\kappa \to S''_\kappa
\]

with \(\mathfrak{n}''_\kappa\) a maximal ideal of \(S''_\kappa\) lying over the homogeneous maximal ideal \(\mathfrak{m}_\kappa\) of \(R_\kappa\), such that \((R_\kappa)_{\mathfrak{m}_\kappa} \to (S''_\kappa)_{\mathfrak{n}''_\kappa}\) is flat and \(e((R_\kappa)_{\mathfrak{m}_\kappa}) > e((S''_\kappa)_{\mathfrak{n}''_\kappa})\) (note that \(R_\kappa \to R''_\kappa\) is always flat since \(f\) is a monic polynomial in \(x\)). Thus we arrive at a counter-example (with \((R_\kappa, \mathfrak{m}_\kappa)\) standard graded over an \(F\)-finite field \(\kappa\)) in characteristic \(p > 0\), which is a contradiction.

Lastly, we mention that in [BHM], it is proven that every complete local domain of characteristic \(p > 0\) with an \(F\)-finite residue field admits a \(\text{lim}\) Cohen–Macaulay sequence \(\{F^e R\}\), which follows from standard methods in tight closure theory [HH90]. The results of this paper suggest the following \textit{challenging} question, in which a positive answer would settle Lech’s conjecture in characteristic \(p > 0\) when the residue field is \(F\)-finite (then the equal characteristic 0 case follows by [Ma17 Section 5]).

Question 3.9. Does every complete local domain of characteristic \(p > 0\) with an \(F\)-finite residue field admit a \(\text{lim}\) Ulrich sequence, or at least a weakly \(\text{lim}\) Ulrich sequence?

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\[4\]In [Ma17], we are not assuming \(R\) is the completion of a finite type algebra therefore we choose a complete regular local ring \(A\) inside \(R\) and descend data to the Henselization of the localization of a polynomial ring, while here \(R\) is finite type (in fact standard graded) over \(k\) so we can run the same argument over \(R\), the counter-example then descends to the Henselization of \(R_\mathfrak{m}\) and thus to a pointed \(\acute{\text{e}}\text{tale}\) extension of \(R_\mathfrak{m}\).
References

[BHM] B. Bhatt, M. Hochster, and L. Ma: Lim Cohen-Macaulay sequence, in preparation.

[Eis95] D. Eisenbud: Commutative algebra with a view toward algebraic geometry, Springer-Verlag, New York, 1995.

[ESW03] D. Eisenbud, F.-O. Schreyer, and J. Weyman: Resultants and Chow forms via exterior syzygies, J. Amer. Math. Soc. 16 (2003), no. 3, 537–579. 1969204

[GW78] S. Goto and K. Watanabe: On graded rings I, J. Math. Soc. Japan 30 (1978), no. 2, 179–213.

[Han99] D. Hanes: Special conditions on maximal Cohen-Macaulay modules, and applications to the theory of multiplicities, Thesis, University of Michigan (1999).

[Han05] D. Hanes: On the Cohen-Macaulay modules of graded subrings, Trans. Amer. Math. Soc. 357 (2005), no. 2, 735–756. 2095629

[Her94] B. Herzog: Kodaira-Spencer maps in local algebra, Springer-Verlag, Berlin, 1994.

[HUB91] J. Herzog, B. Ulrich, and J. Backelin: Linear maximal Cohen-Macaulay modules over strict complete intersections, J. Pure Appl. Algebra 71 (1991), no. 2-3, 187–202.

[Hoc17] M. Hochster: Homological conjectures and lim Cohen-Macaulay sequences, Homological and computational methods in commutative algebra, Springer INdAM Ser., vol. 20, Springer, Cham, 2017, pp. 173–197. 3751886

[HH90] M. Hochster and C. Huneke: Tight closure, invariant theory, and the Briançon-Skoda theorem, J. Amer. Math. Soc. 3 (1990), no. 1, 31–116.

[HS06] C. Huneke and I. Swanson: Integral closure of ideals, rings, and modules, London Mathematical Society Lecture Note Series, vol. 336, Cambridge University Press, Cambridge, 2006. 2266432

[Lec60] C. Lech: Note on multiplicities of ideals, Ark. Mat 4 (1960), 63–86.

[Lec64] C. Lech: Inequalities related to certain couples of local rings, Acta. Math 112 (1964), 69–89.

[Ma14] L. Ma: The Frobenius endomorphism and multiplicities, Thesis, University of Michigan (2014).

[Ma17] L. Ma: Lech’s conjecture in dimension three, Adv. Math. 322 (2017), 940–970. 3720812

[Ser65] J.-P. Serre: Algèbre locale. Multiplicités, Cours au Collège de France, 1957–1958, rédigé par Pierre Gabriel. Seconde édition, 1965. Lecture Notes in Mathematics, 11, Springer-Verlag, Berlin-New York, 1965, pp. vii+188 pp.

[Ulr84] B. Ulrich: Gorenstein rings and modules with high numbers of generators, Math. Z. 188 (1984), no. 1, 23–32. 767359

Department of Mathematics, Purdue University, West Lafayette, IN 47907

E-mail address: ma326@purdue.edu