Cluster Estimates for Modular Structures

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Abstract. The basic ingredients of Tomita-Takesaki modular theory are used to establish cluster estimates. Applications to thermal quantum field theory are discussed and the convergence of the thermal universal localizing map is proven.

1. Introduction

The cluster theorem of relativistic quantum field theory expresses the decay of correlations between clusters of local observables as the space-like separation between clusters increases. In 3 + 1 dimensions the correlations of the vacuum expectation values of local observables decrease at least like $\delta^{-2}$, where $\delta$ denotes the space-like distance of the clusters. In the presence of a mass-gap one finds an exponential decay like $e^{-M\delta}$, where $M$ is the minimal mass in the theory. The standard argument (see [AHR][F]) is based on the spectral properties of the Hamiltonian and Einstein causality. Thomas and Wichmann [TW] (see also [D]) provided an alternative derivation in case $M > 0$, based on the connection between the representation theory of the Poincaré group and the modular objects for a wedge-shaped region bounded by two characteristic planes. More recently, the author studied the cluster properties of local observables in thermal equilibrium. In this case the spectrum of the effective Hamiltonian (the generator of time translations in the GNS-representation associated with the thermal equilibrium state) does not have a mass gap; not even for free massive particles. But due to the KMS-condition, which characterizes thermal equilibrium states, there still exists a tight connection between the spectral properties of the effective Hamiltonian and the decay of spatial correlations [Jä a].

As far as observable quantities are concerned the results cited so far are sufficiently general to cover the situations of physical interest. In principle, one might insist on a representation independent description of a physical system in terms of a net of (abstract) $C^*$-algebras $\mathcal{O} \to \mathcal{A}(\mathcal{O})$ and a distinguished set of positive linear functionals, representing the local observables and the experimental preparation possibilities, respectively. However, refined mathematical methods for predicting the expectation values of quantities, which can be compared with experimental data, are currently only available, if the observable quantities are represented on some Hilbert space $\mathcal{H}$. Therefore cluster estimates between elements of $\mathcal{B}(\mathcal{H})$, which do not directly refer to local observables, turn out to be useful. For example, the cluster estimates, which will be presented in this letter, provide the clue for a method which explores the connection between KMS-states for different temperatures [Jä b].

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2. Main Theorem

As the title indicates, the cluster estimates, which we will present in this letter, are essentially based on modular theory (see e.g. [Ta][St][BR][KR]). However, our starting point is slightly more general. Given a von Neumann algebra $\mathcal{N}$ acting on some Hilbert space $\mathcal{H}$, we will look out for a one-parameter group $U: t \mapsto U(t)$, a conjugate-linear operator $I$ and a vector $\Psi \in \mathcal{H}$ such that for each $N \in \mathcal{N}$ the function

$$t \mapsto U(t)N\Psi$$

has an analytic continuation into the strip $S(0, 1/2) = \{ z \in \mathbb{C} : 0 \leq \Im z \leq 1/2 \}$ which satisfies

$$IU(i/2)N\Psi = N^*\Psi.$$  \hspace{1cm} (1)

The obvious connection to modular theory is, that if $\mathcal{N}$ is contained in a von Neumann algebra $\mathcal{R}$ with cyclic and separating vector $\Omega$, then a possible choice for $U$, $I$ and $\Psi$ are the modular group $t \mapsto \Delta^it$, the modular conjugation $J$ associated with the pair $(\mathcal{R}, \Omega)$ and $\Psi := \Omega$. Given only $\mathcal{N}$, there is considerable freedom in the choice of the pair $(\mathcal{R}, \Omega)$. In fact, we will not assume that $\mathcal{R}$ is explicitly given. Instead we prefer to specify directly the properties of $I$ and $U$.

Once the objects $\mathcal{N}, U, I, \Psi$ are specified, we will consider the adjoint action of the group $U$ on the von Neumann algebra $\mathcal{M}$ generated by $\mathcal{N}$ and $i(\mathcal{N}) := Ad IN\mathcal{N}$, denoted by $\mathcal{N} \lor i(\mathcal{N})$. We will show that, if there exists another von Neumann algebra $\mathcal{M}$ and some $\delta > 0$ such that

$$\left[ U(t) (\mathcal{N} \lor i(\mathcal{N}))U^{-1}(t), \mathcal{M} \right] = 0 \quad \text{for } |t| < \delta,$$

or equivalently

$$\left[ U(t)NU^{-1}(t), \mathcal{M} \lor i(\mathcal{M}) \right] = 0 \quad \text{for } |t| < \delta,$$

then decent spectral properties of the generator of $U$ can be used to bound the expression

$$\left| (\Psi, NM\Psi) - (\Psi, N\Psi)(\Psi, M\Psi) \right|$$

in terms of inverse powers of the ‘distance’ $\delta$.

We now state our main result.

\hfill \textsuperscript{†} Recall that Tomita’s theorem — the principal result of modular theory — is that the mapping $j: A \mapsto JA^*J$ defines a *-anti-isomorphism from $\mathcal{R}$ onto $\mathcal{R}'$. Given a subalgebra $\mathcal{N} \subset \mathcal{R}$ the algebra $j(\mathcal{N})$ is called the opposite algebra associated to $\mathcal{N}$.
Theorem 2.1. Consider two von Neumann algebras $\mathcal{N}$ and $\mathcal{M}$, both acting on some Hilbert space $\mathcal{H}$. Let $I: \mathcal{H} \to \mathcal{H}$ be a conjugate-linear isometric operator, satisfying the condition $I^2 = 1$ and let $t \mapsto U(t) =: \exp(iHt)$, $t \in \mathbb{R}$, be a strongly continuous one-parameter group of unitaries. Assume these objects satisfy the following three conditions:

(i) (Spectral properties). The generator $H$ of the group $U$ has a unique — up to a phase — normalized eigenvector $\Psi \in \mathcal{H}$ for the simple eigenvalue $\{0\}$ and otherwise continuous spectrum, not necessarily semi-bounded. Furthermore, there exist positive constants $m > 0$ and $C > 0$ such that

$$\|e^{\mp \lambda H} P^\pm N\Psi\| \leq C \cdot \lambda^{-m} \|N\Psi\| \quad \forall N \in \mathcal{N},$$

where $P^\pm$ denote the projections onto the strictly positive resp. negative spectrum of $H$.

(ii) (Analyticity properties). For each $N \in \mathcal{N}$ the vector valued function

$$t \mapsto U(t)N\Psi$$

admits an analytic continuation into the strip $S(0, 1/2) = \{z \in \mathbb{C} : 0 \leq \Im z \leq 1/2\}$. The boundary values are continuous for $\Im z \nearrow 1/2$ and they satisfy

$$IU(i/2)N\Psi = N^*\Psi.$$  

Furthermore

$$IU(z)N\Psi = U(z)IN\Psi \quad \forall z \in S(0, 1/2), \quad N \in \mathcal{N}.$$  

(Note that $1 \in \mathcal{N}$, together with $U(z)\Psi = \Psi$ for all $z \in S_{1/2}$, implies $I\Psi = \Psi$.)

(iii) (Distance). There exists some $\delta > 1$ such that

$$\left[ U(t)NU^{-1}(t), \mathcal{M} \vee i(\mathcal{M}) \right] = 0 \quad \text{for} \quad |t| < \delta,$$

where $i(T) := IT^*I$ for $T \in \mathcal{B}(\mathcal{H})$.

It follows that

$$\left| (\Psi, NM\Psi) - (\Psi, N\Psi)(\Psi, M\Psi) \right| \leq (5e^m + 1)C \cdot n_0^{-m} \left( \|FN\Psi\| \|FM^*\Psi\| + \|FN^*\Psi\| \|FM\Psi\| \right),$$

where $F := P^+ + P^-$ denotes the projection onto the orthogonal complement of $\Psi$ and $n_0 := \inf\{n \in \mathbb{N} : \delta_m \leq n \leq \delta\}$. 

Remark. For δ large (compared to 1/2) the correlations decrease like ∼ δ−\frac{n^2}{m+1}. For small δ the finite width of S(0, 1/2) is clearly reflected in the discrete nature of the bounds, which involve a natural number, namely \( n_0 \).

The proof proceeds in several steps:

(i) we define a function

\[ F_{M,N}: I_{\delta} \to \mathbb{C}, \]  

bounded and analytic in the infinitely often cut plane

\[ I_{\delta} = \mathbb{C}\backslash\{ z \in \mathbb{C} : \Im z = k/2, k \in \mathbb{Z}, |\Re z| \geq \delta \}, \]  

such that

\[ F_{M,N}(0) = (\Psi, MN\Psi). \]  

(ii) we define a function \( f_{M,N} \), analytic on the twofold cut plane

\[ P_{\delta} = \mathbb{C}\backslash\{ z \in \mathbb{C} : \Im z = 0, |\Re z| \geq \delta \} \]  

such that for arbitrary \( n \in \mathbb{N} \):

\[ F_{M,N}(0) - (\Psi, M\Psi)(\Psi, N\Psi) = \sum_{l=-n}^{n} f_{M,N}(il) + \sum_{l=-n}^{n-1} f_{IM^*I,N}(i(l + 1/2)) \]

\[ + (U(-in)P^-N^*\Psi, M\Psi) \]

\[ + (\Psi, MU(-in)P^-N\Psi). \]  

(iii) we derive bounds for the terms on the r.h.s. of (16).

(iv) we optimize the natural number \( n \in \mathbb{N} \) in (16).

**Proof.**

(i) Let \( M \in \mathcal{M} \lor i(\mathcal{M}) \) and \( N \in \mathcal{N} \). Then

\[ \lim_{\Im z \nearrow 1/2} (\Psi, MU(z)N\Psi) = (M^*I\Psi, U(\Re z)U(i/2)N\Psi) \]

\[ = (U(\Re z)IU(i/2)N\Psi, IM^*I\Psi) \]

\[ = (U(\Re z)N^*\Psi, IM^*I\Psi) \]

\[ = (\Psi, U(\Re z)NU(-\Re z)IM^*I\Psi). \]  

From (10) and the fact that \( \mathcal{M} \lor i(\mathcal{M}) \) is invariant under the adjoint action of \( I \), it follows that

\[ [IM^*I, U(t)NU(-t)] = 0 \quad \text{for} \quad |t| < \delta. \]
Thus, for $|\Re z| < \delta$,

$$
\lim_{\Im z \searrow 1/2} (\Psi, MU(z)NU\Psi) = (\Psi, IM^*IU(\Re z)\Psi) = \lim_{\Im z \searrow 1/2} (\Psi, IM^*IU(z - i/2)\Psi).
$$

(19)

Using the Edge-of-the-Wedge Theorem [SW] we conclude that there exists a function

$$
F_{M,N} : \mathcal{I}_{\delta} \rightarrow \mathbb{C},
$$

(20)

bounded and analytic in the infinitely often cut plane

$$
\mathcal{I}_{\delta} = \mathbb{C}\setminus\{z \in \mathbb{C} : \Im z = k/2, k \in \mathbb{Z}, |\Re z| \geq \delta\}
$$

(21)

with the following properties: $F_{M,N}$ is periodic in $\Im z$ with period 1 and, for $-1 \leq \Im z < 1$, $F_{M,N}$ is given by

$$
F_{M,N}(z) = \begin{cases}
(\Psi, IM^*IU(z - i/2)\Psi), & 1/2 \leq \Im z < 1, \\
(\Psi, MU(z)\Psi), & 0 \leq \Im z < 1/2, \\
(\Psi, IM^*IU(z + i/2)\Psi), & -1/2 \leq \Im z < 0, \\
(\Psi, MU(z + i)\Psi), & -1 \leq \Im z < -1/2.
\end{cases}
$$

(22)

(ii) Now let $P^+, P^-$ and $P^{(0)} = |\Psi\rangle\langle\Psi|$ denote the spectral projections onto the strictly positive, the strictly negative and the discrete spectrum $\{0\}$ of $H$. The function $f^{+}_{M,N} : \{z \in \mathbb{C} : \Im z \geq 0\} \rightarrow \mathbb{C}$,

$$
z \mapsto (\Psi, MU(z)P^+NU\Psi) - (\Psi, NU(-z)P^-M\Psi),
$$

(23)

is analytic in the upper half plane $\Im z > 0$, while the function $f^{-}_{M,N} : \{z \in \mathbb{C} : \Im z \leq 0\} \rightarrow \mathbb{C}$,

$$
z \mapsto -(\Psi, MU(z)P^-NU\Psi) + (\Psi, NU(-z)P^+M\Psi),
$$

(24)

is analytic in the lower half plane $\Im z < 0$. Both functions have continuous boundary values for $\Im z \searrow 0$ and $\Im z \nearrow 0$, respectively. Thus

$$
(\Psi, [M, U(t)NU(-t)]\Psi) = f^{+}_{M,N}(t) - f^{-}_{M,N}(t) \quad \forall t \in \mathbb{R}.
$$

(25)

Because of the commutator on the l.h.s., the discrete spectral value $\{0\}$ does not contribute to the r.h.s. By assumption, the l.h.s. vanishes for $|t| < \delta$. Consequently, the boundary values for $\Im z \searrow 0$ and $\Im z \nearrow 0$, respectively, of the functions defined in (23) and (24) coincide for $|\Re z| < \delta$. We conclude that there exists a function $f_{M,N}$, analytic on the twofold cut plane $\mathcal{P}_\delta = \mathbb{C}\setminus\{z \in \mathbb{C} : \Im z = 0, |\Re z| \geq \delta\}$ such that

$$
f_{M,N}(z) = \begin{cases}
f^{+}_{M,N}(z), & \Im z > 0, \\
f^{-}_{M,N}(z), & \Im z < 0.
\end{cases}
$$

(26)
By definition,
\[ (\Psi, MU(z)P^+ N\psi) - f^+_{M,N}(z) = (N^*\psi, U(-z)P^- M\psi) \quad \forall |\Im z| \geq 0. \]  
(27)

The r.h.s. equals
\[ (IU(-z)P^- M\psi, IN^*\psi) = (U(-\bar{z})P^+ IM\psi, U(i/2)N\psi) \]
\[ = (\Psi, IM^* IU(z + i/2)P^+ N\psi) \quad \forall |\Im z| \geq 0. \]  
(28)

Thus
\[ (\Psi, MU(z)P^+ N\psi) = f^+_{M,N}(z) + (\Psi, IM^* IU(z + i/2)P^+ N\psi) \quad \forall |\Im z| \geq 0. \]  
(29)

Since \( IM^* I \in M \vee i(M) \) for \( M \in M \vee i(M) \), we can now iterate this identity:
\[ (\Psi, MU(t)P^+ N\psi) = f_{M,N}(t) + f_{IM^* I,N}(t + i/2) + (\Psi, MU(t + i)P^+ N\psi) \]
\[ = \sum_{l=0}^{n} f_{M,N}(t + il) + \sum_{l=0}^{n-1} f_{IM^* I,N}(t + i(l + 1/2)) \]
\[ + (U(-in)P^- N^*\psi, U(-t)M\psi). \]  
(30)

On the other hand, for \( l + 1/2 < 0 \),
\[ (\Psi, MU(t + il)P^- N\psi) + f_{M,N}(t + il) = \]
\[ = (U(i/2)N^*\psi, U(-t - i(l + 1/2))P^+ M\psi) \]
\[ = (IU(-t - i(l + 1/2))P^+ M\psi, N\psi) \]
\[ = (U(-t + i(l + 1/2))P^- IM^* I\psi, N\psi) \]
\[ = (\Psi, IM^* IU(t + i(l + 1/2))P^- N\psi). \]  
(31)

By iteration
\[ (\Psi, MU(in)P^- U(t)N\psi) + \sum_{l=-n}^{l=-1} f_{M,N}(t + il) + \sum_{l=-n}^{l=-1} f_{IM^* I,N}(t + i(l + 1/2)) = \]
\[ = (\Psi, MU(t)P^- N\psi) \quad \forall n \in \mathbb{N}. \]  
(32)

The following identity, which holds for arbitrary \( n \in \mathbb{N} \) and for \( |t| < \delta \), now follows by addition:
\[ F_{M,N}(t) - (\Psi, M\psi)(\Psi, N\psi) = \sum_{l=-n}^{n} f_{M,N}(t + il) + \sum_{l=-n}^{n-1} f_{IM^* I,N}(t + i(l + 1/2)) \]
\[ + (U(-in)P^- N^*\psi, U(-t)M\psi) \]
\[ + (\Psi, MU(-in)P^- U(t)N\psi). \]  
(33)
(iii) The assumptions on the spectral properties of $H$ allow us to derive bounds on the terms on the r.h.s. of (33). For $\lambda > 0$

$$\sup_{t \in \mathbb{R}} |f_{M,N}(t + i\lambda)| \leq \sup_{t \in \mathbb{R}} \left| \left( \Psi, MU(-i(t + i\lambda))P^+N\Psi \right) \right|$$

$$+ \sup_{t \in \mathbb{R}} \left| \left( U(-i(t - i\lambda))P^-N^*\Psi, M\Psi \right) \right|$$

$$\leq C \cdot \lambda^{-m} \left( \|M^*\Psi\| \|N\Psi\| + \|M\Psi\| \|N^*\Psi\| \right). \quad (34)$$

Similar bounds hold for $\lambda < 0$. Jensen’s inequality can be used to derive the following uniform bounds (see [Jäa], Lemma 2.2):

$$|f_{M,N}(ir)| \leq C \cdot \left( \frac{e}{\delta} \right)^m \left( \|M^*\Psi\| \|N\Psi\| + \|M\Psi\| \|N^*\Psi\| \right) \quad \text{for } 0 \leq r \leq \delta. \quad (35)$$

Recall that $\|IM^*I\| = \|M\|$ and, hence,

$$\left| \left( \Psi, (M - (\Psi, M\Psi))N\Psi \right) \right| = \left| \sum_{l=-n}^{n} f_{M,N}(il) \right| + \left| \sum_{l=-n}^{n-1} f_{IM^*I,N}(i(l + 1/2)) \right|$$

$$+ \left( \|M\Psi\| \|U(-in)P^-N^*\Psi\| + \|M^*\Psi\| \|U(-in)P^-N\Psi\| \right)$$

$$\leq \inf_{n \leq \delta} \left[ (4n + 1) \left( \frac{\delta}{e} \right)^{-m} + n^{-m} \right] \times$$

$$\times C \left( \|M^*\Psi\| \|N\Psi\| + \|M\Psi\| \|N^*\Psi\| \right). \quad (36)$$

(iv) Set $n_\circ := \inf\{n \in \mathbb{N} : \delta^{m-1} \leq n \leq \delta \}$. It follows that

$$\inf_{n \leq \delta} \left[ (4n + 1) \left( \frac{\delta}{e} \right)^{-m} + n^{-m} \right] \leq (5e^m + 1) \cdot n_\circ^{-m}. \quad (37)$$

(v) As noted by Thomas and Wichmann [TW], one can obtain sharper bounds by inserting the self-adjoint projection $F$ onto the orthogonal complement of $\Psi$. This can be achieved here by replacing $M$ by $M - (\Psi, M\Psi)$ and $N$ by $N - (\Psi, N\Psi)$ in (36).

The next theorem concerns certain uniform bounds on the correlations.

**Theorem 2.2.** Consider two von Neumann algebras $\mathcal{N}$ and $\mathcal{M}$, both acting on some Hilbert space $\mathcal{H}$. Let $I: \mathcal{H} \to \mathcal{H}$ be a conjugate-linear isometric operator, satisfying the condition $I^2 = 1$ and let $t \mapsto U(t) := \exp(iHt)$, $t \in \mathbb{R}$, be a strongly continuous one-parameter group of unitaries. Assume these objects satisfy the following three conditions:
(i) (Spectral properties). The generator $H$ of the group $U$ has a unique — up to a phase — normalized eigenvector $\Psi \in \mathcal{H}$ for the simple eigenvalue $\{0\}$ and otherwise continuous spectrum, not necessarily semi-bounded. The maps $\Theta^\pm : \mathcal{N} \to \mathcal{H}$

$$N \mapsto e^{\mp \lambda H} P^\pm N \Psi, \quad \lambda > 0,$$

where $P^\pm$ denote the projections onto the strictly positive resp. negative spectrum of $H$, are nuclear and there exist positive constants $C$ and $m$ such that the nuclear norm $\| \cdot \|_1$ of $\Theta^\pm_\lambda$ satisfies

$$\| \Theta^\pm_\lambda \|_1 \leq C \cdot \lambda^{-m}.$$  \hfill (39)

(ii) (Analyticity properties). For each $N \in \mathcal{N}$ the vector valued function

$$t \mapsto U(t) N \Psi$$

admits an analytic continuation into the strip $S(0, 1/2) = \{ z \in \mathbb{C} : 0 \leq \Im z \leq 1/2 \}$. The boundary values are continuous for $\Im z \nearrow 1/2$ and satisfy

$$IU(i/2) N \Psi = N^* \Psi.$$  \hfill (41)

Furthermore

$$IU(z) N \Psi = U(\bar{z}) IN \Psi \quad \forall z \in S(0, 1/2), \quad N \in \mathcal{N}.$$  \hfill (42)

(iii) (Distance). There exists some $\delta > 1$ such that

$$\left[ U(t) N U^{-1}(t), \mathcal{M} \lor i(\mathcal{M}) \right] = 0 \quad \text{for} \quad |t| < \delta,$$

where $i(T) := IT^* I$ for $T \in \mathcal{B}(\mathcal{H})$.

It follows that for two arbitrary families of operators $M_j \in \mathcal{M}$ and $N_j \in \mathcal{N}$, $j \in \{1, \ldots, j_0\}$

$$\left| \left( \Psi, \sum_{j=1}^{j_0} M_j N_j \right) - \sum_{j=1}^{j_0} (\Psi, M_j \Psi) (\Psi, N_j \Psi) \right| \leq \left( 8e^m + 2 \right) C \cdot n_0^{-m} \left\| \sum_{j=1}^{j_0} M_j N_j \right\|.$$  \hfill (44)

where $n_0 := \inf \{ n \in \mathbb{N} : \delta \frac{m}{m+1} \leq n \leq \delta \}$. 

Proof. The identity (33) generalizes to

\[
\left| \sum_{j=1}^{j_0} (M_j - (\Psi, M_j \Psi) \mathbbm{1}) N_j \Psi \right| \leq \sum_{l=-n}^{n-1} \sum_{j=1}^{j_0} \left| f_{M_j, N_j} (il) \right| + \sum_{j=1}^{j_0} f_{IM_j^*, N} (i(l + 1/2)) + \sum_{j=1}^{j_0} (\Psi, M_j (U(it) P^+ + U(-it) P^-) N_j \Psi) \right|, \tag{45} \]

Introducing sequences of vectors \( \Phi_k^{(\lambda)} \in \mathcal{H} \) and of linear functionals \( \phi_k^{(\lambda)} \in \mathcal{N}^* \) such that

\[
\Theta^+_\lambda (N_j) = \sum_k \phi_k^{(\lambda)} (N_j) \cdot \Phi_k^{(\lambda)}, \tag{46} \]

one finds

\[
\left| \sum_{j=1}^{j_0} (M_j^* \Psi, U(it) P^+ N_j \Psi) \right| = \sum_{k=1}^{\infty} \sum_{j=1}^{j_0} \phi_k^{(n)} (N_j) \cdot (M_j^* \Psi, \Phi_k^{(n)}) \leq \sum_k \|\phi_k^{(n)}\| \|\Phi_k^{(n)}\| \cdot \left| \sum_{j=1}^{j_0} M_j N_j \right| \leq C \cdot n^{-m} \left| \sum_{j=1}^{j_0} M_j N_j \right|. \tag{47} \]

A similar bound holds for the term containing \( P^- \) in (45). The same method can be used to show

\[
\left\{ \sup_{t \in \mathbb{R}} \left| \sum_{j=1}^{j_0} f_{M_j, N_j} (t \pm i\lambda) \right| \right\} \leq 2C \cdot |\lambda|^{-m} \left| \sum_{j=1}^{j_0} M_j N_j \right|. \tag{48} \]

Hence

\[
\left| \sum_{j=1}^{j_0} (M_j - (\Psi, M_j \Psi) \mathbbm{1}) N_j \Psi \right| \leq \left| \sum_{l=-n}^{n-1} \sum_{j=1}^{j_0} f_{M_j, N_j} (il) \right| + \sum_{l=-n}^{n-1} \sum_{j=1}^{j_0} f_{IM_j^*, N} (i(l + 1/2)) + \sum_{j=1}^{j_0} (\Psi, M_j (U(it) P^+ + U(-it) P^-) N_j \Psi) \right| \leq \inf_{n \leq \delta} \left( 4n \left( \frac{\delta}{\epsilon} \right)^{-m} + n^{-m} \right) \cdot 2C \cdot \left| \sum_{j=1}^{j_0} M_j N_j \right|. \tag{49} \]

Once again, \( n_0 := \inf \{n \in \mathbb{N} : \delta \frac{m}{n} \leq n \leq \delta \} \) provides a convenient choice for \( n \in \mathbb{N} \) in (49). \( \square \)
3. Applications to Thermal Field Theories

We start from a local quantum field theory \( \mathcal{O} \rightarrow \mathcal{A}(\mathcal{O}) \), specified by a net of (abstract) \( C^* \)-algebras \( \mathcal{A}(\mathcal{O}) \) and a one parameter group of automorphisms \( \tau \), representing the time evolution (as described in the monograph by Haag [H]). The Hermitian elements of \( \mathcal{A}(\mathcal{O}) \) are interpreted as the observables which can be measured at times and locations in \( \mathcal{O} \). The time evolution acts geometrically, i.e.,

\[
\tau_t(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\mathcal{O} + te) \quad \forall t \in \mathbb{R},
\]

where \( e \) is a unit vector denoting the time direction with respect to a given Lorentz-frame. The net \( \mathcal{O} \rightarrow \mathcal{A}(\mathcal{O}) \), for mathematical convenience embedded in the \( C^* \)-algebra

\[
\mathcal{A} = \bigcup_{\mathcal{O} \subset \mathbb{R}^4} \mathcal{A}(\mathcal{O})^{C^*},
\]

is subject to a number of physically significant conditions which play no role for the present result except for the principle of locality: If two local regions \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) are space-like separated by some distance \( \delta \), then the corresponding algebras commute, i.e.,

\[
\mathcal{A}(\mathcal{O}_2) \subset \mathcal{A}^c(\mathcal{O}_1),
\]

where \( \mathcal{A}^c(\mathcal{O}_1) := \{ a \in \mathcal{A} : [a, b] = 0 \ \forall b \in \mathcal{A}(\mathcal{O}_1) \} \).

Thermal equilibrium states are characterized by the KMS-condition [HHW]. Given a KMS-state \( \omega_\beta \), the GNS-representation \( (\pi_\beta, \Omega_\beta, \mathcal{H}_\beta) \) gives rise to a thermal field theory, specified by a net of von Neumann algebras

\[
\mathcal{O} \rightarrow \mathcal{R}_\beta(\mathcal{O}) := \pi_\beta(\mathcal{A}(\mathcal{O}))''.
\]

Due to the time invariance of \( \omega_\beta \), the time evolution can be unitarily implemented in the representation \( \pi_\beta \). It coincides — up to rescaling — with the modular group

\[
t \mapsto \Delta^{it}_\beta
\]

associated with the GNS-vector \( \Omega_\beta \) and the von Neumann algebra \( \mathcal{R}_\beta := \pi_\beta(\mathcal{A})'' \).

3.1 A First Example

Let \( \mathcal{O} \) be some open and bounded space-time region. Since \( \mathcal{R}_\beta(\mathcal{O}) \subset \mathcal{R}_\beta \), condition (ii) of Theorem 2.1 is automatically satisfied, if \( \exp(-\beta H) \) and \( I \) are identified with the modular objects \( (\Delta_\beta, J_\beta) \) associated with the pair \( (\mathcal{R}_\beta, \Omega_\beta) \) and \( \Psi := \Omega_\beta \). The action of the modular group on the algebra

\[
\mathcal{M}_\beta(\mathcal{O}) := \mathcal{R}_\beta(\mathcal{O}) \vee j_\beta(\mathcal{R}_\beta(\mathcal{O})),
\]

is defined as

\[
\Delta^{it}_\beta.
\]
where \( j_\beta(T) := J_\beta T^* J_\beta \) for all \( T \in \mathcal{B}(\mathcal{H}_\beta) \), is geometrical, i.e.,

\[
\Delta^t_\beta M_\beta(\mathcal{O}) \Delta^{-t}_\beta = M_\beta(\mathcal{O} + t\beta e) \quad \forall t \in \mathbb{R}.
\]

(56)

In fact, the map \( \mathcal{O} \to M_\beta(\mathcal{O}) \) preserves inclusions and respects the local structure, i.e.,

\[
M_\beta(\mathcal{O}_1) \subset M_\beta(\mathcal{O}_2)' \quad \text{if} \quad \mathcal{O}_1 \subset \mathcal{O}_2'.
\]

(57)

If the effective Hamiltonian \( H \) — the generator of the modular group — has decent infrared properties (as required in condition (i)), then Theorem 2.1 provides cluster estimates between elements in

\[
\mathcal{N} := \mathcal{R}_\beta(\mathcal{O}) \quad \text{and} \quad \mathcal{M} := \left( \bigvee_{|t|<\delta} M_\beta(\mathcal{O} + t\beta e) \right)'
\]

(58)

in the vector state induced by \( \Omega_\beta \). Note that if \((\mathcal{O} + t\beta e) \subset \hat{\mathcal{O}} \) for all \(|t| < \delta\), then \( M_\beta(\hat{\mathcal{O}}') \subset \mathcal{M} \).

The reader may have noticed that condition (ii) of Theorem 2.1 fails, if we try to set \( \mathcal{N} := M_\beta(\mathcal{O}) \), \( U(t) := \Delta^t_\beta \) and \( \Psi := \Omega_\beta \): for an element \( A \in \mathcal{R}_\beta(\mathcal{O}) \) the function

\[
t \mapsto \Delta^t_\beta A \Omega_\beta, \quad t \in \mathbb{R},
\]

(59)

allows an analytic continuation into the strip \( S(0,1/2) \), whereas for an element \( B \in j_\beta(\mathcal{R}_\beta(\mathcal{O})) \) the function

\[
t \mapsto \Delta^t_\beta B \Omega_\beta, \quad t \in \mathbb{R},
\]

(60)

allows an analytic continuation into the strip \( S(-1/2,0) \). For an arbitrary element \( N \in M_\beta(\mathcal{O}) \) the function

\[
t \mapsto \Delta^t_\beta N \Omega_\beta, \quad t \in \mathbb{R},
\]

(61)

will in general not allow any analytic continuation. This is why we had to restrict ourselves in this example to the case \( \mathcal{N} \subset \mathcal{R}_\beta \).

Remark. Previous cluster estimates [Jä a] only covered the case when both \( \mathcal{N} \) and \( \mathcal{M} \) are subalgebras of \( \mathcal{R}_\beta \). Although the description of a physical system should not depend on non-observable, auxiliary quantities, it is interesting to note that the spatial correlations between elements in \( \mathcal{R}_\beta(\mathcal{O}) \) and \( j_\beta(\mathcal{R}_\beta(\hat{\mathcal{O}}')) \) in the vector state induced by the KMS-vector \( \Omega_\beta \) decrease as the space-like distance between \( \mathcal{O} \) and \( \hat{\mathcal{O}}' \) increases. This suggests a geometric interpretation of the map \( \mathcal{O} \to j_\beta(\mathcal{R}_\beta(\mathcal{O})) \); it is not just an abstract labeling.
3.2 A Refined Example

If the net $\mathcal{O} \rightarrow \mathcal{R}_\beta(\mathcal{O})$ satisfies the split property, then we can improve the previous result: we will show that Theorem 2.2 allows us to estimate the correlations between elements in

$$\mathcal{N} := \mathcal{M}_\beta(\mathcal{O}) \quad \text{and} \quad \mathcal{M} := \mathcal{M}_\beta(\hat{\mathcal{O}})'$$

in the vector state induced by the KMS-vector $\Omega_\beta$, as the space-like distance between $\mathcal{O}$ and $\hat{\mathcal{O}}'$ increases. The proof of this result is involved and we proceed in several steps.

i.) We will investigate the following geometrical situation: we consider four space-time regions $\mathcal{O}, \mathcal{O}_1, \mathcal{O}_2$ and $\hat{\mathcal{O}}$ such that

$$\mathcal{O} \subset \mathcal{O}_1 + t\beta \cdot e \subset \mathcal{O}_2 \subset \hat{\mathcal{O}} \quad \forall |t| < \delta, \quad \delta > 0. \quad (63)$$

*Notation.* By $\mathcal{O} \subset \mathcal{O}_1$ — note that $t = 0$ was not excluded in (63) — we mean that the closure of the open and bounded space-time region $\mathcal{O}$ lies in the interior of the open space-time region $\mathcal{O}_1$. We will also use the abbreviations $\Gamma := (\mathcal{O}, \mathcal{O}_1)$ and $\Lambda := (\mathcal{O}_2, \hat{\mathcal{O}})$.

ii.) The split property for the net $\mathcal{O} \rightarrow \mathcal{R}_\beta(\mathcal{O})$ provides us with two type I factors $\mathcal{S}_\Gamma$ and $\mathcal{S}_\Lambda$ such that

$$\mathcal{R}_\beta(\mathcal{O}) \subset \mathcal{S}_\Gamma \subset \mathcal{R}_\beta(\mathcal{O}_1 + te) \subset \mathcal{R}_\beta(\mathcal{O}_2) \subset \mathcal{S}_\Lambda \subset \mathcal{R}_\beta(\hat{\mathcal{O}}) \quad \forall |t| < \delta. \quad (64)$$

iii.) The existence of the type I factors $\mathcal{S}_\Gamma$ and $\mathcal{S}_\Lambda$ implies (see [Jäd]) that there exist two vectors $\Omega_\Gamma$ and $\Omega_\Lambda \in \mathcal{H}_\beta$ in the natural positive cones

$$\mathcal{P}_\beta^2(\mathcal{R}_\beta(\mathcal{O}) \vee \mathcal{R}_\beta(\mathcal{O}_1)'), \Omega_\beta) \quad \text{and} \quad \mathcal{P}_\beta^2(\mathcal{R}_\beta(\mathcal{O}_2) \vee \mathcal{R}_\beta(\hat{\mathcal{O}})'), \Omega_\beta), \quad (65)$$

respectively, such that

$$(\Omega_\Gamma, AB\Omega_\Gamma) = (\Omega_\beta, A\Omega_\beta)(\Omega_\beta, B\Omega_\beta) \quad (66)$$

for all $A \in \mathcal{R}_\beta(\mathcal{O})$ and all $B \in \mathcal{R}_\beta(\mathcal{O}_1)'$ and

$$(\Omega_\Lambda, CD\Omega_\Lambda) = (\Omega_\beta, C\Omega_\beta)(\Omega_\beta, D\Omega_\beta) \quad (67)$$

for all $C \in \mathcal{R}_\beta(\mathcal{O}_2)$ and all $D \in \mathcal{R}_\beta(\hat{\mathcal{O}})'$. $\Omega_\Gamma$ is cyclic and separating for $\mathcal{R}_\beta(\mathcal{O}) \vee \mathcal{R}_\beta(\mathcal{O}_1)'$ and $\Omega_\Lambda$ is cyclic and separating for $\mathcal{R}_\beta(\mathcal{O}_2) \vee \mathcal{R}_\beta(\hat{\mathcal{O}})'$. Because of the Reeh-Schlieder property of $\Omega_\beta$ (see [Jä e]), the product vectors $\Omega_\Gamma$ and $\Omega_\Lambda$ are cyclic even for $\mathcal{M}_\beta(\mathcal{O})$.

iv.) The existence of $\Omega_\Lambda$ implies that the von Neumann algebra generated by $\mathcal{R}_\beta(\mathcal{O}_2)$ and $j_\beta(\mathcal{R}_\beta(\mathcal{O}_2)) \subset \mathcal{R}_\beta(\hat{\mathcal{O}})$, namely

$$\mathcal{M}_\beta(\mathcal{O}_2) := \mathcal{R}_\beta(\mathcal{O}_2) \vee j_\beta(\mathcal{R}_\beta(\mathcal{O}_2)), \quad (68)$$
is naturally isomorphic to the $W^*$-tensor product of $\mathcal{R}_\beta(\mathcal{O}_2)$ and $j_\beta(\mathcal{R}_\beta(\mathcal{O}_2))$:

$$\mathcal{M}_\beta(\mathcal{O}_2) \cong \mathcal{R}_\beta(\mathcal{O}_2) \otimes j_\beta(\mathcal{R}_\beta(\mathcal{O}_2)). \quad (69)$$

v.) For $A \in \mathcal{R}_\beta(\mathcal{O}_2)$ and $B \in j_\beta(\mathcal{R}_\beta(\mathcal{O}_2))$ the function

$$t \mapsto \Delta^{it}_\beta A \Omega_\beta \otimes \Delta^{-it}_\beta B \Omega_\beta, \quad t \in \mathbb{R},$$

allows an analytic continuation into the strip $S(0, 1/2)$.

vi.) Since $j_\beta(\mathcal{R}_\beta(\mathcal{O}_2)) \subset \mathcal{R}_\beta(\hat{\mathcal{O}})'$, the product vector $\Omega_\Lambda$ can be utilized to specify the isomorphism (69): The unitary operator $V_\Lambda: \mathcal{H}_\beta \to \mathcal{H}_\beta \otimes \mathcal{H}_\beta$ defined by linear extension of

$$V_\Lambda C D \Omega_\Lambda = C \Omega_\beta \otimes D \Omega_\beta,$$

where $C \in \mathcal{R}_\beta(\mathcal{O}_2)$ and $D \in \mathcal{R}_\beta(\hat{\mathcal{O}})'$, satisfies

$$V_\Lambda \mathcal{R}_\beta(\mathcal{O}_2) V_\Lambda^* = \mathcal{R}_\beta(\mathcal{O}_2) \otimes \mathbb{1} \quad \text{and} \quad V_\Lambda \mathcal{R}_\beta(\hat{\mathcal{O}})' V_\Lambda^* = \mathbb{1} \otimes \mathcal{R}_\beta(\hat{\mathcal{O}})' \quad (72)$$

and maps the type I factor $\mathcal{S}_\Lambda$ onto $\mathcal{B}(\mathcal{H}_\beta) \otimes \mathbb{1}$. We emphasize that our specification of the isomorphism (69) depends on the choice of both $\mathcal{O}_2$ and $\hat{\mathcal{O}}$.

vii.) Given the isometry $V_\Lambda$ specified in (71), a one-parameter group of unitaries $t \mapsto \Delta^{-it}_\Lambda: \mathcal{H}_\beta \to \mathcal{H}_\beta$ and an anti-unitary operator $J_\Lambda: \mathcal{H}_\beta \to \mathcal{H}_\beta$ are specified by linear extension of

$$\Delta^{-it}_\Lambda C D \Omega_\Lambda := V_\Lambda^* \left( \Delta^{-it}_\beta C \Omega_\beta \otimes \Delta^{-it}_\beta D \Omega_\beta \right), \quad t \in \mathbb{R},$$

and, respectively,

$$J_\Lambda C D \Omega_\Lambda := V_\Lambda^* \left( J_\beta C \Omega_\beta \otimes J_\beta D \Omega_\beta \right),$$

where $C \in \mathcal{R}_\beta(\mathcal{O}_2)$ and $D \in \mathcal{R}_\beta(\hat{\mathcal{O}})'$. By definition, $J^2_\Lambda = \mathbb{1}$ and $J_\Lambda \Omega_\Lambda = \Omega_\Lambda$. Moreover, $\Omega_\Lambda = V_\Lambda^*(\Omega_\beta \otimes \Omega_\beta)$ is the unique — up to a phase — normalized eigenvector for the simple eigenvalue $\{0\}$ of the generator of the group $t \mapsto \Delta^{-it}_\Lambda$.

viii.) Assume the maps $\Theta^\pm_\Lambda: \mathcal{R}_\beta(\mathcal{O}_1) \to \mathcal{H}_\beta$

$$A \mapsto e^{\mp \lambda H_\beta} P^\pm A \Omega_\beta, \quad \lambda > 0,$$

where $P^\pm$ denote the projections onto the strictly positive resp. negative spectrum of $H_\beta$, are nuclear, and that there exist positive constants $C(\mathcal{O}_1)$ and $m$ such that the nuclear norm $\| \cdot \|_1$ of $\Theta^\pm_\Lambda$ satisfies

$$\| \Theta^\pm_\Lambda \|_1 \leq C(\mathcal{O}_1) \cdot \lambda^{-m}. \quad (76)$$
The split property for the net $\mathcal{O} \rightarrow \mathcal{R}_\beta(\mathcal{O})$ is a consequence of this nuclearity condition [Jäd]. Now let $Q^\pm$ denote the projections onto the strictly positive resp. negative spectrum of $K_\beta$. It follows that the maps $\theta^\pm_\lambda : \mathcal{M}_\beta(\mathcal{O}_1) \rightarrow \mathcal{H}_\beta$,

$$N \mapsto e^{\mp \lambda K_\beta} Q^\pm N \Omega_\Lambda, \quad \lambda > 0,$$

are nuclear. This can be seen as follows: Let $A \in \mathcal{R}_\beta(\mathcal{O}_1)$ and $B \in j(\mathcal{R}_\beta(\mathcal{O}_1))$. By definition,

$$\Delta_\lambda^A B \Omega_\Lambda = V^* \left( \Delta_\lambda^A \Omega_\beta \otimes \Delta_\lambda^- B \Omega_\beta \right).$$

(78)

The maps $A \mapsto \Delta_\lambda^A P^\pm A \Omega_\beta$, $A \in \mathcal{R}_\beta(\mathcal{O}_1)$, and $B \mapsto \Delta_\lambda^B P^\pm B \Omega_\beta$, $B \in j(\mathcal{R}_\beta(\mathcal{O}_1))$, are nuclear for $\lambda > 0$. The tensor product of two nuclear maps is again a nuclear map and the norm is bounded by the product of the nuclear norms [P]. Furthermore,

$$Q^+ = V^* \left( P^+ \otimes |\Omega_\beta\rangle \langle \Omega_\beta| \right) V + V^* \left( |\Omega_\beta\rangle \langle \Omega_\beta| \otimes P^+ \right) V$$

$$+ V^* \left( P^+ \otimes P^+ \right) V + V^* \left( P^- \otimes P^- \right) V$$

(79)

and

$$Q^- = V^* \left( P^- \otimes |\Omega_\beta\rangle \langle \Omega_\beta| \right) V + V^* \left( |\Omega_\beta\rangle \langle \Omega_\beta| \otimes P^- \right) V$$

$$+ V^* \left( P^+ \otimes P^- \right) V + V^* \left( P^- \otimes P^+ \right) V.$$ (80)

For $\lambda$ large, the leading contributions to the nuclear norm of the map given in (77) come from the terms involving the projection $|\Omega_\beta\rangle \langle \Omega_\beta|$ onto the simple eigenvalue $\{0\}$ on one side of the tensor product. Thus

$$\|\theta^\pm_\lambda\| \leq 2C(\mathcal{O}_1) \cdot \lambda^{-m} + O\left(\lambda^{-2m}\right),$$

(81)

where $C(\mathcal{O}_1)$ is the constant appearing in the nuclearity condition (76) for $\mathcal{R}_\beta(\mathcal{O}_1)$.

ix.) The analyticity properties required in condition (ii) of Theorem 2.2 are satisfied: For each $N \in \mathcal{N} := \mathcal{M}_\beta(\mathcal{O})$ the vector valued function

$$t \mapsto \Delta_\lambda^n N \Omega_\Lambda$$

(82)

admits an analytic continuation into the strip $S(0, 1/2) = \{z \in \mathbb{C} : 0 \leq \Im z \leq 1/2\}$. The boundary values are continuous for $\Im z \uparrow 1/2$ and, since

$$J_\lambda \Delta_\lambda^{1/2} C D \Omega_\Lambda = V_\lambda^* \left( C^* \Omega_\beta \otimes D^* \Omega_\beta \right)$$

$$= C^* D^* \Omega_\Lambda = (C D)^* \Omega_\Lambda$$

(83)

for all $C \in \mathcal{R}_\beta(\mathcal{O})$ and $D \in j(\mathcal{R}_\beta(\mathcal{O}))$, they satisfy

$$J_\lambda \Delta_\lambda^{1/2} N \Psi = N^* \Psi.$$ (84)
Furthermore,
\[ J_{\Lambda} \Delta_{\Lambda}^{z} N_{\Omega_{\Lambda}} = \Delta_{\Lambda}^{z} J_{\Lambda} N_{\Omega_{\Lambda}} \quad \forall z \in S(0, 1/2), \quad N \in \mathcal{M}_{\beta}(\mathcal{O}). \quad (85) \]

Note that \( J_{\Lambda} \) and \( \Delta_{\Lambda} \) are not the modular objects associated to the pair \( (\mathcal{M}_{\beta}(\mathcal{O}), \Omega_{\Lambda}) \).

x.) By definition (73)
\[ \Delta_{\Lambda}^{it} \mathcal{M}_{\beta}(O_1) \Delta_{\Lambda}^{-it} \subset \mathcal{M}_{\beta}(O_1 + t\beta \cdot e) \quad \text{for} \quad |t| < \delta. \quad (86) \]

Inspecting the definition (74) of \( J_{\Lambda} \) we find
\[ J_{\Lambda} \mathcal{M}_{\beta}(\hat{O})^{' J_{\Lambda}} = V_{\Lambda} (1 \otimes J_{\beta} \mathcal{M}_{\beta}(\hat{O})^{' J_{\beta}}) V_{\Lambda} = \mathcal{M}_{\beta}(\hat{O}). \quad (87) \]

Therefore
\[ \left[ \Delta_{\Lambda}^{it} \mathcal{M}_{\beta}(O_1) \Delta_{\Lambda}^{-it}, \mathcal{M}_{\beta}(\hat{O})^{' j_{\Lambda} (\mathcal{M}_{\beta}(\hat{O})^{'})} \right] = 0 \quad \text{for} \quad |t| < \delta \quad (88) \]

reduces to
\[ \left[ \Delta_{\Lambda}^{it} \mathcal{M}_{\beta}(O_1) \Delta_{\Lambda}^{-it}, \mathcal{M}_{\beta}(\hat{O})^{'} \right] = 0 \quad \text{for} \quad |t| < \delta, \quad (89) \]

which is a direct consequence of (86).

xi.) The product vector \( \Omega_{\Gamma} \) satisfies
\[ (\Omega_{\Gamma}, SR\Omega_{\Gamma}) = (\Omega_{\beta}, S\Omega_{\beta})(\Omega_{\beta}, R\Omega_{\beta}) \quad \forall S \in S_{\Gamma}, \quad R \in S_{\Gamma}'. \quad (90) \]

Consequently, there exists an isometry \( W_{\Gamma} \) such that
\[ W_{\Gamma} R \Omega_{\beta} = R \Omega_{\Gamma} \quad \forall R \in S_{\Gamma}'. \quad (91) \]

Inspecting (91), we notice that \( W_{\Gamma} \in S_{\Gamma} \). Moreover,
\[ (\Omega_{\Gamma}, T\Omega_{\Gamma}) = (\Omega_{\Lambda}, T\Omega_{\Lambda}) \quad \forall T \in S_{\Gamma} \lor R_{\beta}(\hat{O})^{'} \quad (92) \]

If \( M \in \mathcal{M}_{\beta}(\hat{O})^{'} \) and \( N \in \mathcal{M}_{\beta}(\mathcal{O}) \), then \( MN \in S_{\Gamma} \lor R_{\beta}(\hat{O})^{'} \). Hence
\[ (\Omega_{\beta}, MN\Omega_{\beta}) - (\Omega_{\beta}, M\Omega_{\beta})(\Omega_{\beta}, N\Omega_{\beta}) = \]
\[ = (\Omega_{\Gamma}, MW_{\Gamma} NW_{\Gamma}^{*} \Omega_{\Gamma}) - (\Omega_{\Lambda}, M\Omega_{\Lambda})(\Omega_{\Lambda}, NW_{\Gamma}^{*} \Omega_{\Lambda}) \quad (93) \]

We note that \( W_{\Gamma} NW_{\Gamma}^{*} \subset S_{\Gamma} \lor M_{\beta}(\mathcal{O}) \subset M_{\beta}(\mathcal{O}_1) \).

These observations are summarized as follows:
Corollary 3.1. Consider a quadruple of space-time regions \( \Lambda := (\mathcal{O}, \mathcal{O}_1, \mathcal{O}_2, \hat{\mathcal{O}}) \) such that
\[
\mathcal{O} \subset \subset \mathcal{O}_1 + t \in \mathcal{O}_2 \subset \subset \hat{\mathcal{O}} \quad \forall |t| < \delta, \quad \delta > 0.
\] (94)
Assume the maps \( \Theta^\pm_\lambda : \mathcal{R}_\beta(\mathcal{O}) \to \mathcal{H}_\beta \)
\[
A \mapsto e^{\mp \lambda H_\beta} P^\pm \! A \Omega_\beta, \quad \lambda > 0,
\] (95)
where \( P^\pm \) denote the projections onto the strictly positive resp. negative spectrum of \( H_\beta \), are nuclear and that there exist positive constants \( C(\mathcal{O}_1) \) and \( m \) such that the nuclear norm \( \| \Theta^\pm_\lambda \|_1 \) of \( \Theta^\pm_\lambda \) satisfies
\[
\| \Theta^\pm_\lambda \|_1 \leq C(\mathcal{O}_1) \cdot \lambda^{-m}.
\] (96)
It follows that for two arbitrary families of operators \( N_j \in \mathcal{M}_\beta(\mathcal{O}) \) and \( M_j \in \mathcal{M}_\beta(\hat{\mathcal{O}})' \), \( j \in \{1, \ldots, j_0\} \)
\[
\left| \left( \Omega_\beta, \sum_{j=1}^{j_0} M_j \Omega_\beta \right) - \sum_{j=1}^{j_0} \left( \Omega_\beta, M_j \Omega_\beta \right) \right|
\]
\[
\leq (8e^m + 2) C \cdot n_\circ^{-m} \sum_{j=1}^{j_0} M_j N_j + O(n_\circ^{-2m}),
\] (97)
where \( n_\circ := \inf \{ n \in \mathbb{N} : \delta \frac{m+1}{m+1} \leq n \leq \delta \} \).

4. The Convergence of the Universal Localizing Map

In the vacuum sector, the split property allows us to approximate global quantities (e.g., the generators of symmetry transformations) by local ones [BDL]. As the space-time regions tend to \( \mathbb{R}^4 \), the local quantities converge to the global ones [D'ADF].

The situation in the thermal case is slightly more involved. Set \( \Upsilon_i := (\mathcal{O}_i, \hat{\mathcal{O}}_i) \) for \( \mathcal{O}_i \subset \subset \hat{\mathcal{O}}_i \) and \( i \in \mathbb{N} \). Assume \( \mathcal{O}_i, \hat{\mathcal{O}}_i \) tend to \( \mathbb{R}^4 \) as \( i \to \infty \). If the sequence of split inclusion
\[
\mathcal{R}_\beta(\mathcal{O}_i) \subset \mathcal{S}_\Upsilon_i \subset \mathcal{R}_\beta(\hat{\mathcal{O}}_i)
\] (98)
is used to define a sequence of ‘universal localizing maps’, then this sequence can not converge against the identity as \( i \to \infty \) (see [D'ADF]). Therefore we have to look out for different split inclusions. The tensor product of two type I factors is again of type I [S 71, Prop. 2.6.2]. It follows that there exist type I factors, namely \( \mathcal{K}_\Upsilon_i := \mathcal{S}_\Upsilon_i \cup J_\beta \mathcal{S}_\Upsilon_i J_\beta \), such that
\[
\mathcal{M}_\beta(\mathcal{O}_i) \subset \mathcal{K}_\Upsilon_i \subset \mathcal{M}_\beta(\hat{\mathcal{O}}_i).
\] (99)
This split inclusion is standard* [DL]. Consequently, there exists a unique vector $\eta_{\Upsilon_i}$ in the natural positive cone $\mathcal{P}^\bowtie(M_{\beta}(O_i) \vee M_{\beta}(\hat{O}_i)', \Omega_{\beta})$ such that

$$\quad (\eta_{\Upsilon_i}, M_i N_i \eta_{\Upsilon_i}) = (\Omega_{\beta}, M_i \Omega_{\beta})(\Omega_{\beta}, N_i \Omega_{\beta}) \quad (100)$$

for all $N_i \in M_{\beta}(O_i)$ and all $M_i \in M_{\beta}(\hat{O}_i)'$. The product vectors $\eta_{\Upsilon_i}$ can now be utilized in the following

**Definition.** Consider two open and bounded space-time regions $O$ and $\hat{O}$ such that the closure of $O$ is contained in the interior of $\hat{O}$. The universal localizing map $\psi_{\Upsilon}: \mathcal{B}(\mathcal{H}_{\beta}) \to \mathcal{K}_\Upsilon$ associated with the pair $\Upsilon := (O, \hat{O})$ is given by

$$\quad T \mapsto U^{-1}_{\Upsilon} (T \otimes \mathbb{1}) U_{\Upsilon}, \quad T \in \mathcal{B}(\mathcal{H}_\beta), \quad (101)$$

where $U_{\Upsilon}: \mathcal{H}_{\beta} \to \mathcal{H}_{\beta} \otimes \mathcal{H}_{\beta}$ is given by linear extension of

$$\quad U_{\Upsilon} M N \eta_{\Upsilon} := M \Omega_{\beta} \otimes N \Omega_{\beta} \quad (102)$$

for all $N \in M_{\beta}(O)$ and all $M \in M_{\beta}(\hat{O})'$.

The map $\psi_{\Upsilon}$ is a $\ast$-isomorphism of $\mathcal{B}(\mathcal{H}_\beta)$ onto the canonical type I factor $\mathcal{K}_\Upsilon$ between $M_{\beta}(O)$ and $M_{\beta}(\hat{O})$ and it acts trivial on $M_{\beta}(O)$.

**Proposition 4.1.** Consider a sequence of pairs $\Upsilon_i := (O_i, \hat{O}_i)$ of space-time regions with diameters $r_i$ and $\hat{r}_i$, respectively, such that

$$\hat{r}_i = r_i^\gamma \quad \text{and} \quad \gamma > \frac{d(m + 1)}{m^2}. \quad (103)$$

Assume the maps $\vartheta_{\lambda, O_i}: \mathcal{R}_{\beta}(O_i) \to \mathcal{H}_{\beta}$

$$\quad A \mapsto \Delta^{\pm \lambda} P^{\pm} A \Omega_{\beta}, \quad \lambda > 0, \quad (104)$$

are nuclear and satisfies

$$\quad \|\vartheta_{\lambda, O_i}\|_1 \leq \text{const} \cdot r_i^d \lambda^{-m}. \quad (105)$$

It follows that $\Psi_{\Upsilon_i}$ converges pointwise strongly to the identity, i.e.,

$$\quad s - \lim_{i \to \infty} \Psi_{\Upsilon_i}(T) = T \quad \forall T \in \mathcal{B}(\mathcal{H}_\beta). \quad (106)$$

---

* A split inclusion $A \subset B$ is called standard, if there exists a vector $\Omega$ which is cyclic for $A' \wedge B$ as well as for $A$ and $B$. 
Proof. The norm distance between $\eta_{\mathcal{Y}_i} \in \mathcal{P}_i(\mathcal{M}(\mathcal{O}_i) \lor \mathcal{M}(\hat{\mathcal{O}}_i)', \Omega_\beta)$ and $\Omega_\beta$ is bounded by

$$\|\eta_{\mathcal{Y}_i} - \Omega_\beta\|^2 \leq \sup_{\|M_i\|=1} \left| (\eta_{\mathcal{Y}_i}, M_i\eta_{\mathcal{Y}_i}) - (\Omega_\beta, M_i\Omega_\beta) \right|,$$

where the supremum has to be taken w.r.t. $M_i \in \mathcal{M}(\mathcal{O}_i) \lor \mathcal{M}(\hat{\mathcal{O}}_i)'$ [BR, 2.5.31]. Consequently, the convergence of the universal localizing maps follows from Corollary 3.3: For $\delta_i := \hat{r}_i - r_i$ sufficiently large

$$\|\eta_{\mathcal{Y}_i} - \Omega_\beta\|^2 \leq \text{const}' \cdot r_i^d \delta_i^{\frac{m^2}{m+1}}. \quad (108)$$

By assumption

$$\hat{r}_i = r_i^\gamma \quad \text{with} \quad \gamma > \frac{d(m+1)}{m^2}. \quad (109)$$

It follows that the r.h.s. of (108) tends to zero as $i \to \infty$. Moreover, for $T \in \mathcal{B}(\mathcal{H}_\beta)$ and $N \in \mathcal{M}_\beta(\mathcal{O})$ we find

$$U_{\mathcal{Y}_i}^{-1}(TN\Omega_\beta \otimes \Omega_\beta) \to TN\Omega_\beta \quad \text{as} \quad i \to \infty. \quad (110)$$

Applying the definition of $\psi_{\mathcal{Y}_i}$ we conclude that

$$\psi_{\mathcal{Y}_i}(T)N\eta_{\mathcal{Y}_i} \to TN\Omega_\beta \quad \text{as} \quad i \to \infty. \quad (111)$$

Now $\eta_{\mathcal{Y}_i}$ tends to $\Omega_\beta$, $\Omega_\beta$ is cyclic for $\mathcal{M}_\beta(\mathcal{O})$ and the maps $\psi_{\mathcal{Y}_i}, i \in \mathbb{N}$, are uniformly bounded, hence $\psi_{\mathcal{Y}_i}(T)$ converges strongly to $T$ as $i \to \infty$ [D'ADFL].

5. Appendix: Wedge-shaped Regions

We consider a local quantum field theory in the vacuum representation $\pi$, specified by a net of von Neumann algebras $\mathcal{O} \to \mathcal{R}(\mathcal{O}) := \pi(\mathcal{A}(\mathcal{O}))''$. Let

$$W := \{x \in \mathbb{R}^4 : |x_0| < x_3\} \quad (112)$$

denote a wedge-shaped region in Minkowski space. We assume that the modular operator $\Delta_W$ associated with the algebra $\mathcal{R}(W)$ and the vacuum vector $\Omega$ implements the Lorentz boosts, i.e.,

$$\Delta_{W}^{it} \mathcal{R}(\mathcal{O}) \Delta_{W}^{-it} = \mathcal{R}(M(\pi t)\mathcal{O}), \quad \mathcal{O} \subset W, \quad (113)$$

where $M(t)$ denotes the Lorentz matrix

$$M(t) := \begin{pmatrix}
\cosh(t) & 0 & 0 & \sinh(t) \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\sinh(t) & 0 & 0 & \cosh(t)
\end{pmatrix}. \quad (114)$$
Bisognano and Wichmann [BW a,b] have shown that (113) is generically the case, if the net of local observables is constructed from a Wightman field theory. Moreover, Borchers [Bo] has shown that in 1+1 space-time dimensions equation (113) is a consequence of the standard assumptions of algebraic quantum field theory. The algebra $\mathcal{R}(W)$ associated with a wedge is of type $\text{III}_1$ [Dr], thus the spectrum of $\log \Delta_W$ has a (up to a phase) unique eigenvector for the simple eigenvalue $\{0\}$, namely the vacuum vector $\Omega$, and otherwise continuous spectrum on the whole real axis.

The conditions (i-iii) of Theorem 2.1 are fulfilled if we consider two algebras $\mathcal{R}(O_1) \subset \mathcal{R}(W)$ and $\mathcal{R}(O_2) \subset \mathcal{R} := \pi(\mathcal{A})''$, such that

$$\Delta_W^{it} \mathcal{R}(O_1) \Delta_W^{-it} \subset \mathcal{R}(O_2)'', \quad |t| < \delta, \quad \delta > 0,$$

and put

$$\mathcal{N} := \mathcal{R}(O_1), \quad \mathcal{M} := \mathcal{R}(O_2), \quad \Psi := \Omega,$$

$$U(t) := \Delta_W^{it}, \quad I := J_W.$$ (116)

Spectral information on the vacuum Hamilton operator $\mathbf{H}$ can be used to estimate

$$\|e^{\mp \lambda H} P^\pm N\Psi\|, \quad N \in \mathcal{R}(O_1).$$ (117)

One can proceed along the lines discussed in [BD’AL], see for example, the last inequality on p. 125. But since we do not expect any improvement of the existing vacuum cluster theorems, we do not dwell on this point.

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References

[AHR] Araki, H., Hepp, K., Ruelle, D., On the asymptotic behavior of Wightman functions in space-like directions, Helv. Phys. Acta 35, 164–174 (1962)

[Bo] Borchers, H.J., The CPT-theorem in two-dimensional theories of local observables, Commun. Math. Phys. 143, 315–332 (1992)

[BDL] Buchholz, D., Doplicher, S. and Longo, R., On Noether’s theorem in quantum field theory, Ann. Phys. 170, 1–17 (1986)

[BR] Bratteli, O. and Robinson, D.W., Operator Algebras and Quantum Statistical Mechanics I,II, Springer-Verlag, New York-Heidelberg-Berlin, 1981.

[BW a] Bisognano, J. and Wichmann, E.H., On the duality condition for a Hermitian scalar field, J. Math. Phys. 16, 985–1007 (1975)
Bisognano, J. and Wichmann, E.H., *On the duality condition for quantum fields*, J. Math. Phys. 17, 303–321 (1976)

Davidson, D.R., *Cluster estimates and analytic wavefunctions*, hep-th/9406104.

Driessler, W., *Comments on lightlike translations and applications in relativistic quantum field theory*, Commun. Math. Phys. 44, 133–141 (1975)

D’Antoni, C., Doplicher, S., Fredenhagen, K., and Longo, R., *Convergence of local charges and continuity properties of $W^*$-inclusions*, Commun. Math. Phys. 110, 321–344 (1987) and Erratum Commun. Math. Phys. 116, 321–344 (1988)

Doplicher, S. and Longo, R., *Standard and split inclusions of von Neumann algebras*, Invent. Math. 73, 493–536 (1984)

Fredenhagen, K., *A remark on the cluster theorem*, Commun. Math. Phys. 97, 461–463 (1985)

Haag, R., *Local Quantum Physics: Fields, Particles, Algebras*, Springer-Verlag, Berlin-Heidelberg-New York, 1992.

Haag, R., Hugenholtz, N.M. and Winnink, M., *On the equilibrium states in quantum statistical mechanics*, Commun. Math. Phys. 5, 215–236 (1967)

Jäkel, C.D., *Decay of spatial correlations in thermal states*, Ann. l’Inst. H. Poincaré 69, 425–440 (1998)

Jäkel, C.D., *On the relation between KMS states for different temperatures*, hep-th/9803245.

Jäkel, C.D., *Two algebraic properties of thermal quantum field theories*, hep-th/9901015.

Jäkel, C.D., *Nuclearity and split for thermal quantum field theories*, hep-th/9811227.

Jäkel, C.D., *The Reeh-Schlieder property for thermal field theories*, hep-th/9904049.

Kadison, R.V. and Ringrose, J.R., *Fundamentals of the theory of operator algebras II*, Academic Press, New York, 1986.

Pietsch, A., *Nuclear locally convex spaces*, Springer-Verlag, Berlin-Heidelberg-New York, 1972.

Sakai, S., *$C^*$-Algebras and $W^*$-Algebras*, Springer-Verlag, Berlin-Heidelberg-New York, 1971.

Strătilă, S., *Modular Theory in Operator Algebras*, Abacus Press, 1981.

Streater, R.F. and Wightman, A.S., *PCT, Spin and Statistics and all that*, Benjamin, New York, 1964.

Takesaki, M., *Tomita’s Theory of Modular Hilbert Algebras and its Application*, Springer, Lecture Notes in Mathematics, Berlin, 1970.

Thomas, L.J. and Wichman, E.H., *About matrix elements of spatial translations in local field theories*, Lett. Math. Phys. 28, 49–58 (1993)