Hunting $B_c^*$ via Conservation Laws

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Abstract

To distinguish $B_c^{(s)*+}$ and $B_c^{+}$ in the experiments, we propose two methods based on the conservation laws. I. From the angular momentum conservation, a nonzero helicity of $J/\psi$ of $B_c^{(s)*+} \to J/\psi\pi^+$ would be an evidence of $B_c^{s*+}$. II. Since $B_c^{+} \to B^+\phi$ is kinematically forbidden, $B_c^{s*+} \to B^+\phi$ provides a clean channel to probe $B_c^{s*+}$. Particularly, our results show that $B_c^{s*+}$ is promising to be observed at LHC via $B_c^{(s)*+} \to J/\psi\pi^+$. On the other hand, we find that $\mathcal{B}(B_c^{s*+} \to B^+\phi) = (7.0 \pm 3.0) \times 10^{-9}$, which is also feasible to be measured at the forthcoming experiments at HL-LHC and FCC-hh.

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The $B_c$ meson family is unique in the Standard Model (SM) as its members are composed of heavy quarks with two different flavors. The ground state of $B_c$ mesons, unlike charmonium and bottomonium, decay only via the weak interactions. Study on the $B_c$ mesons can deepen our understanding of both the strong and the weak interaction, providing a unique hunting ground for searching new physics beyond the SM.

The ground state of $B_c$ meson was firstly observed by the CDF Collaboration at Fermilab [1] in 1998, and there have been continuous measurements on both the mass [2–4] and the lifetime [5, 6] via the exclusive decay $B_c^+ \to J/\psi \pi^+$ and the semileptonic decay $B_c^+ \to J/\psi l^+ \nu_l$. In 2014, the ATLAS Collaboration reported a structure with the mass of $(6842 \pm 9)$ MeV [7], which is consistent with the value predicted for $B_c(2S)$. In 2019, the excited $B_c(21S_0)$ was confirmed and $B_c^*(2S_1)$ states have been observed in the $B_c^+ \pi^+ \pi^−$ invariant mass spectrum by the CMS and LHCb Collaboration, with their masses determined to be $(6872.1 \pm 2.2)$ MeV and $(6841.2 \pm 1.5)$ MeV [8], respectively. The $B_c(21S_0)^+$ decays to $B_c^*(1S_0)^+ \pi^+ \pi^−$ directly, and the $B_c^*(2S_1)^+$ state decay to $B_c^{*(1S_1)}^+ \pi^+ \pi^−$ followed by $B_c^{*(1S_1)}^+ \to B_c^{+(1S_0)} \gamma$. Since the soft photon in the intermediate decay $B_c^{*(1S_1)}^+ \to B_c^{+(1S_0)} \gamma$ was not reconstructed, the mass of $B_c^*(2S_1)$ meson appears lower than that of $B_c(21S_0)$. This peculiar behaviors of the mass hierarchy makes $B_c^*(1S_1)$ uniquely important in studying $B_c$ meson family.

In the following of this letter, we would abbreviate $B_c^*(1S_1)$ as $B_c^*$ so long as it does not cause confusion. On the mass of $B_c^*$, the theoretical predictions range discrepancy from 6326 to 6346 MeV [10–16], and an experimental measurement is still lacking. The dominant decay mode $B_c^* \to B_c \gamma$ has not yet been observed, partly due to the noisy soft photon background of the hadron collider. To identify $B_c^*$ in the experiments, one of the important task is to distinguish them from $B_c$. In this letter, we propose two methods based on the conservation laws:

• From the angular momentum conservation, the $J/\psi$ can only possess a zero helicity from $B_c^+ \to J/\psi \pi^+$ as $B_c^+$ is spin-0. In contrast, the $J/\psi$ of $B_c^{*+} \to J/\psi \pi^+$ can have either positive, zero, or negative helicities (see Fig. [1]).
FIG. 1: The adjoint decay distributions of $B_c^{(*)+} \to \pi^+ J/\psi (\to l^+ l^-)$, where the blue and the orange represent the possible spin configuration(s) of $B_c^{(*)+}$ and $J/\psi$, with $\otimes$ indicating spin-0 at $\vec{p}_{J/\psi}$ direction.

FIG. 2: The quark diagrams for $B_c^{(*)+} \to J/\psi \pi^+$ and $B_c^+ \to B^+ \phi$ at the tree level.

- As $B_c^+ \to B^+ \phi$ is kinematically forbidden, $B_c^{(*)+} \to B^+ \phi$ provides a clean channel.

Their responsible quark diagrams at the tree level are given in Fig. 2, where the hadronizations take place in the blue regions. As the W boson is color blind, the decays are color-allowed and color-suppressed, respectively.

To probe the helicity of $J/\psi$, we utilize the adjoint angular distributions of $B_c^{(*)+} \to J/\psi (\to l^- l^+) \pi^+$. Taking the initial $B_c^{(*)+}$ as unpolarized, the angular distributions of $B_c^{(*)+} \to J/\psi \pi^+$ are given as

$$\frac{\partial \Gamma^{(*)}}{\partial \cos \theta} \propto \sum_{\lambda = \pm, \lambda l = \pm} \left| H^{(*)}_\lambda d^1(\theta)_{\lambda l} \right|^2 \propto 1 - P_2 + \frac{3}{2} \alpha^{(*)} P_2,$$

where $H^{(*)}_\lambda$ are the helicity amplitudes with the subscripts denoting the helicity of $J/\psi$, $d^1(\theta)$ the Wigner $d$-matrix for $J = 1$, $\theta$ defined in the helicity frame of $J/\psi$ (see Fig. 1).
and
\[ P_2 = \frac{1}{2} \left( 3 \cos^2 \theta - 1 \right), \]
\[ \alpha^{(*)} = \frac{|H^+_2|^2 + |H^-_2|^2}{|H^+_2|^2 + |H^-_2|^2 + |H_0^2|^2}. \]  
(2)

Here, \( \alpha \) has the physical meaning of the nonzero-polarized fraction of \( J/\psi \). Notice that \( H_\pm \) are forbidden by the angular momentum conservation, resulting in
\[ \alpha = 0. \]  
(3)

To further extract the helicity information, we define
\[ A^{(*)} = \frac{1}{\Gamma} \left( \int_{|\cos \theta| < x_0} \frac{\partial \Gamma^{(*)}}{\partial \cos \theta} d\cos \theta - \int_{|\cos \theta| > x_0} \frac{\partial \Gamma^{(*)}}{\partial \cos \theta} d\cos \theta \right) = \left( 3x_0 - \frac{3}{2} \right) \alpha^{(*)}, \]  
(4)

where \( x_0 \) is chosen to satisfy
\[ x_0^3 - 3x_0 + 1 = 0, \]  
(5)

which is found to be \( x_0 \approx 0.3473 \).

The experiments of \( B_{c}^{(*)+} \) are polluted by the off-shell contributions from \( B_{c}^+ \) at LHC. Thus, we define the event-average \( \overline{A} \) as
\[ \overline{A} = rA + r^* A* = r^* A^*, \]  
(6)

as well as the event-average nonzero-polarized fraction as
\[ r \frac{\partial \Gamma}{\partial \cos \theta} + r^* \frac{\partial \Gamma^*}{\partial \cos \theta} \propto 1 - P_2 + \frac{3}{2} \alpha P_2, \]  
(7)

with
\[ r^{(*)} = \frac{N_{B_c^{(*)}}}{N_{B_c} + N_{B_c^*}}, \]  
(8)

where \( N_{B_c^{(*)}} \) is the number of the observed events in \( B_c^{(*)+} \rightarrow J/\psi(\rightarrow l^+l^-)\pi^+ \). The second equality in Eq. (6) is attributed to Eq. (3).

To get an estimation on the experiments, we calculate the amplitudes within the factorization framework. The helicity amplitudes of \( B_c^{(*)+} \rightarrow J/\psi \pi^+ \) are given as
\[ (2\pi)^4 \delta^4(p_{B_c} - p_{J/\psi} - p_\pi) H_\lambda = i \frac{G_F}{\sqrt{2}} V_{cb} V_{ud} f_s p_\pi^a a_1 \langle J/\psi; p_z; J_z = \lambda | \overline{b} \gamma_{\mu} (1 - \gamma_5) c | B_c^{(*)+}; J_z = \lambda \rangle, \]  
(9)
where $p$ is for the 4-momentum of the hadron in the subscript, $G_F$ and $f_\pi$ the Fermi and the pion decay constants, $a_1$ the effective Wilson coefficient for the color-allowed decays, $J_z$ the angular momentum at the $z$ direction, and $p\hat{z}$ indicates $\vec{p}_{J/\psi}/\hat{z}$.

On the other hand, the helicity amplitudes of $B_c^{s+} \to B^+\phi$ are given as
\[
(2\pi)^4 \delta^4(p_{B_c} - p_{B^+} - p_\phi)H_\lambda = -\frac{G_F}{\sqrt{2}} V_{cs}^* V_{us} f_\phi \epsilon_{\lambda}^{\mu*} a_2 B^+(p\hat{z}) | \bar{u}\gamma_{\mu}(1 - \gamma_5)c | B_c^{(*)+}; J_z = -\lambda, \tag{10}
\]
where $a_2$ is the effective Wilson coefficient for the color-suppressed decays, $f_\phi$ the $\phi$ decay constant, and $\epsilon_{\lambda}^{\mu*}$ the polarization 4-vector of $\phi$ with $\lambda$ its helicity.

Finally, the decay width for $B_c^+ \to J/\psi\pi^+$ is given as
\[
\Gamma = \frac{|\vec{p}_{cm}|}{8\pi M_{B_c}} |H_0|^2, \tag{11}
\]
whereas the decay widths of $B_c^{s+}$ with the daughter vector meson having $\lambda$ helicity are given as
\[
\Gamma_\lambda = \frac{|\vec{p}_{cm}|}{24\pi M_{B_c^*}} |H_{\lambda}^s|^2. \tag{12}
\]
The total decays widths of $B_c^+ \to J/\psi\pi^+$ and $B_c^{s+} \to B^+\phi$ can be easily obtained by adding up the contributions from $\lambda = 0, \pm$.

The meson transition matrix elements require the knowledge of the hadron wave functions. In this work, we employ the ones from the homogeneous bag model, in which the center motions of the hadrons in the original bag model are removed [17]. The bag radius ($R$) and the quark masses can be extracted from the mass spectra, which are found to be [18]
\[
R = (2.81 \pm 0.30) \text{ GeV}^{-1}, \quad M_{u,d} = 0, \quad M_c = 1.641 \text{ GeV}, \quad M_b = 5.093 \text{ GeV}. \tag{13}
\]
The details of the calculation can be found in the supplementary material attached to this letter. In this study, $f_\pi$ and $f_\phi$ are taken from the experiments and the Lattice QCD [19, 20]
\[
f_\pi = 131 \text{ MeV}, \quad f_\phi = (241 \pm 9) \text{ MeV}, \tag{14}
\]
and the effective Wilson coefficients are taken to be

\[ |a_1| = 1.0 \pm 0.1, \quad |a_2| = 0.27 \pm 0.07. \tag{15} \]

The results are given in Table I, where we also include \( \Gamma(B_{c}^{*+} \to B_{c}^{+}\gamma) \), which can be safely approximated as \( 1/\tau \) with \( \tau \) the lifetime of \( B_{c}^{*+} \). Note that a large part of the uncertainties arose from the hadron wave functions is canceled in the branching ratios of \( B_{c}^{*+} \), because the lifetime is calculated by the same hadron wave function.

Our \( B(B_{c}^{+} \to J/\psi\pi^{+}) \) is consistent with the relativistic quark model [21], but two times smaller compared to most of the literature [22], which can be partly attributed to that we use a smaller \( |a_1| \). As our estimation is a more conservative one, the angular analysis is promising to be carried out in the experiments for there are more data points to reconstruct the distribution than we expect.

The decay of \( B_{c}^{*+} \to B^{+}\phi \) is color-suppressed and suffers large uncertainties from \( a_2 \) as well as \( M_{B_{c}^{*+}} \). In particular, as \( M_{B_{c}^{*+}} \) is close to the mass threshold of \( B^{+}\phi \), the decay width can range from 0 to \( 10^{-6} \) eV, depending on \( M_{B_{c}^{*+}} \). The dependency on \( M_{B_{c}^{*+}} \) as well as the uncertainties caused by \( a_2 \) are plotted in Fig. 3. Taking \( M_{B_{c}^{*+}} = 6331 \) MeV, the calculated decay width is given in Tab. I which is consistent with Ref. [23], within the range of the error.
FIG. 3: $\Gamma(B_c^{*+} \rightarrow B^+ \phi)$ versus $M_{B_c^{*+}}$, where the yellow region covers the uncertainty of $a_2$.

From the Table I for $B_c^{*+} \rightarrow J/\psi \pi^+$ we obtain

$$\alpha^* = 0.82 \pm 0.01, \quad A^* = 0.38 \pm 0.01,$$

(16)
in which the theoretical uncertainty is canceled for the correlations between $H^*_\lambda$. The cross section of $B^*_c$ meson at the LHC is expected to be $\sigma(B_c^*) = 29$ nb [24]. At an integrated luminosity of 150 fb$^{-1}$ during LHC Run-2, 300 fb$^{-1}$ during LHC Run-3, and 3000 fb$^{-1}$ after High Luminosity upgrade (HL-LHC) [25], the numbers of $B_c^*$ events are $8.7 \times 10^9$, $1.74 \times 10^{10}$ and $1.74 \times 10^{11}$, resulting in 270, 540 and 5400 events of $B_c^{*+} \rightarrow J/\psi \pi^+$, respectively. Taking the branching ratios $\mathcal{B}(J/\psi \rightarrow l^+l^-) \approx 12\%$ [19], there are expected to be 33, 65 and 650 events of $B_c^{*+} \rightarrow \pi^+J/\psi(\rightarrow l^+l^-)$ being able to be reconstructed at LHC Run-2, LHC Run-3 and HL-LHC, respectively.

By choosing the invariant mass of $J/\psi \pi^+$ between 6325 and 6400 MeV, most of the off-shell contribution from $B_c^+$ would be filtered, in which $N_{B_c}$ is expected to be less than 20 at running LHC from the Fig. 1 of Ref. [27]. Thus, $N_{B_c}$ and $N_{B_c^*}$ can be safely taken as equal in the simulation.

We generate the pseudo-data based on the experimental conditions at LHC Run-2, LHC Run-3 and HL-LHC. The off-shell contributions from $B_c^+$ are also included with $N_{B_c} = N_{B_c^*}$ as discussed in the previous paragraph. The numbers of the events are plotted against $\cos \theta$ in Fig. 4, and the numerical results of $\alpha$ and $A$ are given in

\footnote{In fact, at the bottom right figure, there appears to have a little bump around 6340 MeV.}
FIG. 4: The numbers of the observed events of $B_c^{(*)} \to J/\psi(\to l^+l^-)\pi^+$ plotted against $\cos \theta$. The red points with statistical uncertainties are the pseudo-data generated by the Monte Carlo method for $\alpha^* = 0.82$, and the blue and the red lines are drawn with $\alpha^* = 0$ and $\alpha^* = 0.82$ in Eq. (7), respectively.

Table II. Our analysis show that there would be a 1.5\textsigma signal of nonzero $\overline{\alpha}$ at LHC Run-2, and a 5\textsigma signal at HL-LHC, which would be a solid evidence of $B_c^*$.

|                | $N_{B_c} = N_{B_c}^*$ | $\overline{\alpha}$ | $\overline{\alpha}$ |
|----------------|------------------------|-----------------------|-----------------------|
| LHC Run-2      | 33                     | 0.48 ± 0.39           | 0.15 ± 0.10           |
| LHC Run-3      | 65                     | 0.43 ± 0.26           | 0.17 ± 0.07           |
| HL-LHC         | 650                    | 0.44 ± 0.10           | 0.19 ± 0.03           |

On the other hand, at the forthcoming experiments at FCC-hh [26], the number of $B_c^*$ events are expected to be $10^{12}$. Hence, there would be about 2000 and $10^5$
$B^{*+}_c \rightarrow B^+ \phi$ events at HL-LHC and FCC-hh, respectively, which would be sufficient for the experiments to determine the mass.

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Supplementary material for “Hunting $B^*_c$ via Conservation Laws”

In this supplementary material, we give the meson wave functions of the homogeneous bag model, which are used in the calculation of the transition matrix elements in the main text. In the original version of the bag model, both the asymptotic freedom and the confinement of the QCD are described by the bag radius, $R$. The quarks are confined in the bag but moving freely within it, satisfying the free Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \quad \text{for } r < R. \quad (17)$$

For low-lying hadrons, we can take the wave functions to be spherical, and we arrive that

$$\psi(x)_q = \phi_q(\vec{x})e^{-iE_q t} = N \begin{pmatrix} \omega_{q+j_0(p_q r)} \chi \\ i\omega_{q-j_1(p_q r)} \hat{r} \cdot \vec{\sigma} \chi \end{pmatrix} e^{-iE_q t} \quad \text{for } r < R, \quad (18)$$

where $q$ is the quark flavor, $N$ the normalizing constant, $\chi$ the two component spinor, $p_q$ the magnitude of the 3-momentum, and $\omega_{q\pm} = \sqrt{1 + m_q/E_q}$ with $E_q$ the quark energy. The anti-quark wave functions are obtained by taking the charge conjugate.

At the boundary of the bag the current shall vanish, which give us the boundary condition, read as

$$\hat{r} \cdot (\bar{\psi} \vec{\gamma} \psi) = 0, \quad \text{at } |\vec{x}| = R. \quad (19)$$

In analogy to the familiar infinite square well, $p_q$ is quantized, satisfying

$$\tan(p_q R) = \frac{p_q R}{1 - m_q R - E_q R}. \quad (20)$$

We concern the low-lying hadrons only and therefore take the minimum of $p_q$. At the massless and the heavy quark limits we have

$$\lim_{m_q R \to 0} p_q R = 2.0428, \quad \lim_{m_q R \to \infty} p_q R = \pi, \quad (21)$$

respectively. A meson can be constructed by confining a quark and a anti-quark to a same bag. By considering the bag energy, zero point energy, and the interaction between quarks, the bag model can successfully explain most of the low-lying hadron masses as well as the ratios of the magnetic dipole moments [18].
However, despite the success on the hadron masses, the wave functions of the bag model are problematic when it comes to decays. As the description of a static bag is essentially localized, the hadron wave function can not be the momentum eigenstates, and thus the transition matrix elements can not be consistently calculated. This problem has been resolved with the linear superposition of infinite bags by one of the authors (Liu), and with it the experimental branching ratios of $Λ_b \to Λ^{+}_c π^+$ and $Λ_b \to p π^+$ can be well explained [17].

In the homogeneous bag model, the meson wave functions at rest are given as

$$\Psi(x_{q_1}, x_{q_2}) = N \int d^3 \vec{x} q_1 (\vec{x}_{q_1} - \vec{x}) \phi^c q_2 (\vec{x}_{q_2} - \vec{x}) e^{-i(E_{q_1} t_{q_1} + E_{q_2} t_{q_2})}, \quad (22)$$

where $N$ is the normalizing constant, and $c$ in the superscript denotes the charge conjugate. The wave function in Eq. (22) is manifestly invariant under the space translation and therefore describes a meson at rest. The wave functions with nonzero momenta can be easily obtained by Lorentz boost.

By demanding the normalization condition

$$\langle p | p' \rangle = 2p^0 (2\pi)^3 \delta^3 (p - p'), \quad (23)$$

we find

$$\frac{1}{N^2} = 2M \int d^3 \vec{x} \Delta \prod_{i=1,2} d^3 x_{q_i} \phi^c_{q_i} \left( \vec{x}_{q_i} + \frac{1}{2} \vec{x} \Delta \right) \phi_{q_i} \left( \vec{x}_{q_i} - \frac{1}{2} \vec{x} \Delta \right), \quad (24)$$

with $p$ and $M$ the hadron momentum and mass, respectively.

With the wave functions, the meson transition matrix elements can be computed straightforwardly. For simplicity we take $B^{(s)+}_{c} \to J/ψ π^−$ as an example. The results of $B^{(s)+}_{c} \to J/ψ π^+$ can be obtained by taking CP conjugate as CP is conserved in $b \to c$ transition. The meson transition matrix elements read as

$$\int \langle J/ψ | \bar{c} γ^\mu b(x)e^{ip_{x}x} | B^{(s)+}_{c} \rangle d^4 x = Z \int d^3 \vec{x} \Delta V^\mu(\vec{x} \Delta) D_c(\vec{x} \Delta), \quad (25)$$

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with

\[
Z \equiv (2\pi)^4 \delta^4(p_{B_c} - p_{J/\psi} - p_\pi) \frac{N_{B_c} N_{J/\psi}}{\gamma},
\]

\[
D_c(x_\Delta) \equiv \int d^3 \bar{x} \phi^\dagger_c \left( \bar{x} + \frac{1}{2} x_\Delta \right) \phi_c \left( \bar{x} - \frac{1}{2} x_\Delta \right) e^{-2iE_c \bar{v} \cdot \bar{x}},
\]

\[
V^\mu(x_\Delta) = \int d^3 \bar{x} \phi^\dagger_c \left( \bar{x} + \frac{1}{2} x_\Delta \right) \gamma^0 \gamma^\mu \phi_b \left( \bar{x} - \frac{1}{2} x_\Delta \right) e^{i(M_{J/\psi} + M_{B_c} - E_c - E_b) \bar{v} \cdot \bar{x}}.
\] (26)

Here, the calculation is taken at the Briet frame where $B_c^-$ and $J/\psi$ have the velocity $-\bar{v}$ and $\bar{v}$ respectively. Although the derivation is quite tedious (see Ref. [17] for an example), their physical meaning can be easily understood as the following:

- $Z$ is the overall normalizing constant along with the momentum conservation.
- $D_c(x_\Delta)$ is the overlapping coefficient attributed by the spectator quark between the initial and the final states. Note that their centers of the bags are separated at a distance of $x_\Delta$.
- $V^\mu(x_\Delta)$ is the matrix element of the weak current at the quark level, where the centers of the bags are separated at a distance of $x_\Delta$.

Here, the exponential in the integrals of $D_c(x_\Delta)$ and $V^\mu(x_\Delta)$ would oscillate violently at large velocity, leading to a suppression which is a punishment for not being at the same speed.

The matrix elements of $B_c^{*+} \rightarrow B^+ \phi$ can be calculated similarly.