The predual and John-Nirenberg inequalities on generalized BMO martingale spaces

Yong Jiao, Anming Yang, Lian Wu and Rui Yi
Central South University

20 August, 2014

Abstract

In this paper we introduce the generalized BMO martingale spaces by stopping time sequences, which enable us to characterize the dual spaces of martingale Hardy-Lorentz spaces $H^s_{p,q}$ for $0 < p \leq 1, 1 < q < \infty$. Moreover, by duality we obtain a John-Nirenberg theorem for the generalized BMO martingale spaces when the stochastic basis is regular. We also extend the boundedness of fractional integrals to martingale Hardy-Lorentz spaces.

1 Introduction

Basing mainly on the duality, John-Nirenberg inequality and something else, the space BMO (Bounded Mean Oscillation; see [6], [7] and [12]) played a remarkable role in classical analysis and probability. We refer to the monographs [3] and [20] for the function space version, respectively to the monographs [2], [8] and [18] for the martingale version of those theorems.

This paper deals with the John-Nirenberg inequalities and dualities in the martingale theory. Before describing our main results, we recall the classical John-Nirenberg inequalities in the martingale theory. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $\{\mathcal{F}_n\}_{n \geq 0}$ be a nondecreasing sequence of sub-$\sigma$-algebras of $\mathcal{F}$ such that $\mathcal{F} = \sigma(\bigcup_n \mathcal{F}_n)$. We also call $\{\mathcal{F}_n\}_{n \geq 0}$ a stochastic basis (with convention $\mathcal{F}_{-1} = \mathcal{F}_0$). The expectation

Yong Jiao is supported by NSFC (11471337), Hunan Provincial Natural Science Foundation(14JJ1004) and The International Postdoctoral Exchange Fellowship Program.

2000 Mathematics subject classification: Primary 60G46; Secondary 60G42.

Key words and phrases: The predual, John-Nirenberg inequalities; Generalized BMO spaces; Martingale Hardy-Lorentz space; Fractional integral.

Corresponding email: jiaoyong@csu.edu.cn
operator and the conditional expectation operators relative to $$F_n$$ are denoted by $$E$$ and $$E_n$$, respectively. The stochastic basis $$\{F_n\}_{n \geq 0}$$ is said to be regular, if there exist an absolute constant $$R > 0$$ such that

$$f_n \leq R f_{n-1}, \quad (1.1)$$

holds for all nonnegative martingales $$f = (f_n)_{n \geq 0}$$.

A sequence $$f = (f_n)_{n \geq 0}$$ of random variables such that $$f_n$$ is $$F_n$$-measurable is said to be a martingale if $$E(|f_n|) < \infty$$ and $$E_n(f_{n+1}) = f_n$$ for every $$n \geq 0$$. For the sake of simplicity, we assume $$f_0 = 0$$. For $$1 \leq r < \infty$$, the Banach spaces $$BMO_r$$ are defined as follows:

$$BMO_r = \{ f = (f_n)_{n \geq 0} \in L_r : \| f \|_{BMO_r} = \sup_n \| (E_n|f - f_{n-1}|^r)^{1/r} \|_\infty < \infty \}.$$  

Here $$f$$ is $$|f - f_{n-1}|^r$$ means $$f_\infty$$. The usual $$BMO$$ norm corresponds to $$r = 2$$ above, i.e., $$\| f \|_{BMO} = \| f \|_{BMO_2}$$. The John-Nirenberg theorem says that in the sense of equivalent norms,

$$BMO_r = BMO, \quad (1 \leq r < \infty). \quad (1.2)$$

A duality argument yields that (1.1) can be rewritten as follows

$$\| f \|_{BMO} \approx \sup_n \sup_{A \in F_n} E(A)^{-\frac{1}{r}} \left( \int_A |f - f_{n-1}|^r dP \right)^{\frac{1}{r}}. \quad (1.3)$$

Here and in the sequel, $$A \approx B$$ means that there exist two absolute constants $$C_1$$ and $$C_2$$ such that $$C_1 B \leq A \leq C_2 B$$.

The special contribution of this paper is to define the following generalized BMO martingale space $$BMO_{r,q}(\alpha)$$ by stopping time sequences.

**Definition 1.1.** For $$1 \leq r, q < \infty, \alpha \geq 0$$, the generalized BMO martingale space is defined by

$$BMO_{r,q}(\alpha) = \left\{ f \in L_r : \| f \|_{BMO_{r,q}(\alpha)} = \sup_n \left( \frac{\sum_{k \in \mathbb{Z}} 2^k P(\nu_k < \infty)^{1/r} \| f - f_{\nu_k} \|_r}{\left( \sum_{k \in \mathbb{Z}} (2^k P(\nu_k < \infty)^{1+\alpha})^q \right)^{1/q}} \right)^{1/r} < \infty \right\},$$

where the supremum is taken over all stopping time sequences $$\{\nu_k\}_{k \in \mathbb{Z}}$$ such that $$\{2^k P(\nu_k < \infty)^{1+\alpha}\}_{k \in \mathbb{Z}} \in \ell_q$$.

Then the generalized John-Nirenberg theorem, one of our main results, reads as follows.

**Theorem 1.2.** Suppose that the stochastic basis $$\{F_n\}_{n \geq 0}$$ is regular and $$1 \leq q < \infty$$. Then

$$BMO_{r,q}(\alpha) = BMO_{2,q}(\alpha), \quad (1.4)$$

in the sense of equivalent norms for all $$1 \leq r < \infty$$. 

We now explain the relation between (1.1) and (1.4). Let \( \mathcal{T} \) be the set of all stopping times relative to \( \{ \mathcal{F}_n \}_{n \geq 0} \). On one hand, if the stopping time sequence \( \{ \nu_k \}_{k \in \mathbb{Z}} \) reduces to a sequence whose one element is a stopping time \( \nu \) and the others are \( \infty \), then the generalized BMO space \( BMO_{r,q}(\alpha) \) reduces to the following Lipschitz space

\[
BMO_r(\alpha) = \{ f \in L_r : \| f \|_{BMO_r(\alpha)} = \sup_{\nu \in \mathcal{T}} \mathbb{P}(\nu < \infty)^{-\frac{1}{r} - \alpha} \| f - f^\nu \|_r < \infty \}.
\]

On the other hand, if the stochastic basis \( \{ \mathcal{F}_n \}_{n \geq 0} \) is regular, it is not very difficult to check that (1.2) can further be reformulated as

\[
\| f \|_{BMO} \approx \sup_{\nu \in \mathcal{T}} \mathbb{P}(\nu < \infty)^{-\frac{1}{r}} \| f - f^\nu \|_r.
\]  

(1.5)

See also [22] for the facts above. Hence if \( \alpha = 0 \), (1.4) exactly implies (1.5). Consequently, (1.1) can be deduced from (1.4) when the stochastic basis \( \{ \mathcal{F}_n \}_{n \geq 0} \) is regular.

We now turn to the second aim of this paper. The generalized BMO martingale space \( BMO_{r,q}(\alpha) \) defined in this paper enable us to characterize the dualities of martingale Hardy-Lorentz spaces for \( 0 < p \leq 1, 1 < q < \infty \). It is well known that the dual spaces of Lebesgue spaces \( L_p \) or Lorentz spaces \( L_{p,q} \) are trivial when \( 0 < p < 1 \) (see for instance [5] or [9]), namely,

\[
(L_p)^* = (L_{p,q})^* = \{ 0 \}, \quad (0 < p < 1, 0 < q \leq \infty).
\]

However, the dual spaces of martingale Hardy spaces are very different from those of Lebesgue spaces \( L_p \) and Lorentz spaces \( L_{p,q} \). This can be illustrated by the fact that the dual spaces of \( L_p \) and \( L_{p,q} \) (\( 0 < p < 1 \)) are trivial while

\[
(H^s_p)^* = BMO_2(\alpha), \quad (0 < p < 1, \alpha = \frac{1}{p} - 1),
\]

where \( H^s_p \) denotes the martingale Hardy space associated with the conditional quadratic variation, that is,

\[
H^s_p = \{ f = (f_n)_{n \geq 0} : \| f \|_{H^s_p} = \left( \sum_{i=1}^{\infty} \mathbb{E}_{i-1} |d_i f|^2 \right)^{\frac{1}{2}} \|_p < \infty \}.
\]

We refer to [8], [14] and [22] for the fact above. At the same time, Weisz [22] also proved the following duality result for martingale Hardy-Lorentz spaces,

\[
(H^s_{p,q})^* = H^s_{p',q'}, \quad (1 < p < \infty, 1 \leq q < \infty),
\]

where \( p' \) and \( q' \) denote the conjugate numbers of \( p \) and \( q \) respectively; see Section 2 for definition of \( H^s_{p,q} \). But the question how to characterize the dual spaces of martingale Hardy-Lorentz spaces for \( 0 < p \leq 1, 0 < q < \infty \) is still open. We prove that the dual space of martingale Hardy-Lorentz space is the same as the one of martingale Hardy spaces when \( 0 < p, q \leq 1 \), while it needs the new notion \( BMO_{r,q}(\alpha) \) when \( 0 < p \leq 1, 1 < q < \infty \). In Section 4 we shall show
Theorem 1.3. Let $0 < p \leq 1$, $\alpha = \frac{1}{p} - 1$. Then

$$(H_{p,q}^s)^* = \text{BMO}_2(\alpha), \quad 0 < q \leq 1;$$

and

$$(H_{p,q}^s)^* = \text{BMO}_{2,q}(\alpha), \quad 1 < q < \infty.$$ 

This paper will be divided into five further sections. In the next section, some notations and basic knowledge will be introduced. In Section 3, the atomic decompositions of martingale Hardy-Lorentz spaces are formulated. In Section 4, using atomic decompositions in Section 3, we prove some dual theorems of martingale Hardy-Lorentz spaces. By duality, the new John-Nirenberg theorem for the generalized BMO martingale space is proved in Section 5. In the final Section, the boundedness of fractional integrals on martingale Hardy-Lorentz spaces are investigated.

In this paper, the set of integers and the set of nonnegative integers are always denoted by $\mathbb{Z}$ and $\mathbb{N}$, respectively. We use $C$ to denote the absolute constant which may vary from line to line.

2 Notations and preliminaries

We first introduce the distribution function and the decreasing rearrangement. Let $f$ be a measurable function defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We define the distribution function of $f$ by

$$\lambda_s(f) = \mathbb{P}\{\omega \in \Omega : |f(\omega)| > s\}, \quad (s \geq 0).$$

And denote by $\mu_t(f)$ the decreasing rearrangement of $f$, defined by

$$\mu_t(f) = \inf\{s \geq 0 : \lambda_s(f) \leq t\}, \quad (t \geq 0),$$

with the convention that $\inf\emptyset = \infty$.

We list some properties of distribution functions and decreasing rearrangements in the following proposition. The properties will be used in the proof of theorems in the later sections.

Proposition 2.1. Let $f$ and $g$ be two measurable functions on $(\Omega, \mathcal{F}, \mathbb{P})$, then we have

1. if $|f| \leq |g|$ $\mathbb{P}$-a.e. then $\lambda_s(f) \leq \lambda_s(g)$ for all $s \geq 0$;
2. $\lambda_{s_1+s_2}(f+g) \leq \lambda_{s_1}(f) + \lambda_{s_2}(g)$ for all $s_1, s_2 \geq 0$;
3. $\mu_t(af) = |a|\mu_t(f)$ for all $a \in \mathbb{C}$ and $t \geq 0$;
4. if $|f| \leq |g|$ $\mathbb{P}$-a.e. then $\mu_t(f) \leq \mu_t(g)$ for all $t \geq 0$;
5. $\mu_{t_1+t_2}(f+g) \leq \mu_{t_1}(f) + \mu_{t_2}(g)$ for all $t_1, t_2 \geq 0$. 


The Lorentz space $L_{p,q}(\Omega, \mathcal{F}, \mathbb{P})$, $0 < p < \infty, 0 < q \leq \infty$, consists of those measurable functions $f$ with finite norm or quasinorm $\|f\|_{p,q}$ given by

$$\|f\|_{p,q} = \left( \frac{1}{p} \int_0^\infty \left( \frac{1}{t} \mu_t(f) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}, \quad (0 < q < \infty),$$

$$\|f\|_{p,\infty} = \sup_{t>0} t^\frac{1}{q} \mu_t(f), \quad (q = \infty).$$

It will be convenient for us to use an equivalent definition of $\|f\|_{p,q}$, namely

$$\|f\|_{p,q} = \left( q \int_0^\infty \left( t \mathbb{P}(\{|f(x)| > t\}) \right)^{\frac{q}{2}} \frac{dt}{t} \right)^{\frac{1}{q}}, \quad (0 < q < \infty),$$

$$\|f\|_{p,\infty} = \sup_{t>0} t^\frac{1}{q} \mathbb{P}(\{|f(x)| > t\}), \quad (q = \infty).$$

We recall that Lorentz spaces $L_{p,q}$ increase as the second exponent $q$ increases, and decrease as the first exponent $p$ increases (the second exponent $q$ is not involved). Namely, $L_{p,q_1} \subset L_{p,q_2}$ for $0 < p < \infty$ and $0 < q_1 \leq q_2 \leq \infty$, $L_{p_1,q_1} \subset L_{p_2,q_2}$ for $0 < p_2 < p_1 < \infty$ and $0 < q_1, q_2 \leq \infty$. It is also well known that if $1 < p < \infty$ and $1 \leq q \leq \infty$, or $p = q = 1$, then $\| \cdot \|_{p,q}$ is equivalent to a norm. However, for the other values of $p$ and $q$, $\| \cdot \|_{p,q}$ is only a quasi-norm. In particular, if $0 < q \leq 1$ and $q \leq p < \infty$, then $\| \cdot \|_{p,q}$ is equivalent to a $q$-norm. Hölder’s inequality for Lorentz spaces is the following

$$\|fg\|_{p,q} \leq C \|f\|_{p_1,q_1} \|g\|_{p_2,q_2},$$

where $0 < p, p_1, p_2 < \infty$ and $0 < q, q_1, q_2 \leq \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$.

We now introduce martingale Hardy-Lorentz spaces. Denote by $\mathcal{M}$ the set of all martingales $f = (f_n)_{n \geq 0}$ relative to $\{\mathcal{F}_n\}_{n \geq 0}$ such that $f_0 = 0$. For $f \in \mathcal{M}$, denote its martingale difference by $d_n f = f_n - f_{n-1}$ ($n \geq 0$, with convention $f_{-1} = 0$). Then the maximal function, the quadratic variation and the conditional quadratic variation of a martingale $f$ are respectively defined by

$$f_n^* = \sup_{0 \leq i \leq n} |f_i|, \quad f^* = \sup_{n \geq 0} |f_n|,$$

$$S_n(f) = \left( \sum_{i=1}^{n} |d_i f|^2 \right)^{\frac{1}{2}}, \quad S(f) = \left( \sum_{i=1}^{\infty} |d_i f|^2 \right)^{\frac{1}{2}},$$

$$s_n(f) = \left( \sum_{i=1}^{n} \mathbb{E}_{i-1} |d_i f|^2 \right)^{\frac{1}{2}}, \quad s(f) = \left( \sum_{i=1}^{\infty} \mathbb{E}_{i-1} |d_i f|^2 \right)^{\frac{1}{2}}.$$

Let $\Lambda$ be the collection of all sequences $(\lambda_n)_{n \geq 0}$ of nondecreasing, nonnegative and adapted functions, set $\lambda_\infty = \lim_{n \to \infty} \lambda_n$. For $f \in \mathcal{M}$, $0 < p < \infty, 0 < q \leq \infty$, let

$$\Lambda[Q_{p,q}](f) = \{ (\lambda_n)_{n \geq 0} \in \Lambda : S_n(f) \leq \lambda_{n-1}(n \geq 1), \lambda_\infty \in L_{p,q} \}.$$
A[D_{p,q}](f) = \{ (\lambda_n)_{n \geq 0} \in A : |f_n| \leq \lambda_{n-1}(n \geq 1), \lambda_\infty \in L_{p,q} \}.

We define martingale Hardy-Lorentz spaces as follows. For $0 < p < \infty, 0 < q \leq \infty$,

$H_{p,q}^* = \{ f \in \mathcal{M} : \| f \|_{H_{p,q}^*} = \| f^* \|_{p,q} < \infty \},$

$H_{p,q}^S = \{ f \in \mathcal{M} : \| f \|_{H_{p,q}^S} = \| S(f) \|_{p,q} < \infty \},$

$H_{p,q}^S = \{ f \in \mathcal{M} : \| f \|_{H_{p,q}^S} = \| s(f) \|_{p,q} < \infty \},$

$Q_{p,q} = \{ f \in \mathcal{M} : \| f \|_{Q_{p,q}} = \inf_{(\lambda_n)_{n \geq 0} \in \Lambda(D_{p,q})} \| \lambda_\infty \|_{p,q} < \infty \},$

$D_{p,q} = \{ f \in \mathcal{M} : \| f \|_{D_{p,q}} = \inf_{(\lambda_n)_{n \geq 0} \in \Lambda(D_{p,q})} \| \lambda_\infty \|_{p,q} < \infty \}.$

If taking $p = q$ in the definitions above, we get the usual martingale Hardy spaces. In order to describe the duality theorems, we need to introduce the Lipschitz space $BMO_r(\alpha)$. For $1 \leq r < \infty, \alpha \geq 0$, the Lipschitz space are defined as follows

$BMO_r(\alpha) = \{ f \in L_r : \| f \|_{BMO_r(\alpha)} < \infty \},$

where

$\| f \|_{BMO_r(\alpha)} = \sup_{n \in \mathbb{N}} \sup_{A \in \mathcal{F}_n} \mathbb{P}(A)^{-\frac{\alpha}{r}} \left( \int_A | f - \mathbb{E}_n f |^r d\mathbb{P} \right)^{\frac{1}{r}}.$

Let $\mathcal{T}$ be the set of all stopping times relative to $\{ \mathcal{F}_n \}_{n \geq 0}$. For a martingale $f = (f_n)_{n \geq 0} \in \mathcal{M}$ and a stopping time $\nu \in \mathcal{T}$, we denote the stopped martingale by $f^\nu = (f_n^\nu)_{n \geq 0} = (f_n\wedge \nu)_{n \geq 0}$. Then it is easy to show that

$\| f \|_{BMO_r(\alpha)} = \sup_{\nu \in \mathcal{T}} \mathbb{P}(\nu < \infty)^{-\frac{\alpha}{r}} \| f - f^\nu \|_r, \quad (1 \leq r < \infty, \alpha \geq 0).$

The main new notion of the present paper is the generalized BMO martingale space $BMO_{r,q}(\alpha) (1 \leq r, q < \infty, \alpha \geq 0)$, see Section 1 for the definition. In Definition 1.1, if the stopping time sequence $\{ \nu_k \}_{k \in \mathbb{Z}}$ reduces to a sequence whose one element is a stopping time $\nu$ and the others are $\infty$, then the generalized BMO martingale space $BMO_{r,q}(\alpha)$ reduces to the Lipschitz martingale space $BMO_r(\alpha)$. Obviously, $BMO_{r,q}(\alpha)$ is a subspace of $BMO_r(\alpha)$ and $\| f \|_{BMO_r(\alpha)} \leq \| f \|_{BMO_{r,q}(\alpha)}$.

We will present the atomic decomposition theorems for martingale Hardy-Lorentz spaces in the next section. Now let us introduce the notion of atoms; see for example [22].

**Definition 2.2.** A measurable function $a$ is called a $(p, \infty)$-atom of the first category (or of the second category, or of the third category) if there exists a stopping time $\nu \in \mathcal{T}$ ($\nu$ is called the stopping time associated with $a$) such that

(i) $a_n = \mathbb{E}_n(a) = 0, \ (if \ \nu \geq n), \ (ii) \ \| s(a) \|_\infty \leq \mathbb{P}(\nu < \infty)^{-\frac{1}{p}} \ (or \ (i') \ \| S(a) \|_\infty \leq \mathbb{P}(\nu < \infty)^{-\frac{1}{p}}, \ or \ (i'') \ \| a^* \|_\infty \leq \mathbb{P}(\nu < \infty)^{-\frac{1}{p}}, \ respectively).$
These three category atoms are briefly called \((1, p, \infty)\)-atom, \((2, p, \infty)\)-atom and \((3, p, \infty)\)-atom, respectively.

We conclude this section by two lemmas which are very useful to verify that a function is in Lorentz spaces \(L_{p,q}\), which are respectively from Lemma 1.1 and Lemma 1.2 in [1].

**Lemma 2.3.** Let \(0 < p < \infty, 0 < q \leq \infty\), assume that the nonnegative sequence \(\{\mu_k\}\) satisfies \(\{2^k \mu_k\} \in l^q\). Further suppose that the nonnegative function \(\varphi\) verifies the following property: there exists \(0 < \varepsilon < 1\) such that, given an arbitrary integer \(k_0\), we have \(\varphi \leq \psi_{k_0} + \eta_{k_0}\), where \(\psi_{k_0}\) is essentially bounded and satisfies \(\|\psi_{k_0}\|_\infty \leq C2^{k_0}\), and

\[
2^{k_0 \varphi \mathbb{P}}(\eta_{k_0} > 2^{k_0}) \leq C \sum_{k=k_0}^{\infty} (2^{k\varepsilon} \mu_k)^p.
\]

Then \(\varphi \in L_{p,q}\) and \(\|\varphi\|_{p,q} \leq C\|\{2^k \mu_k\}\|_{l_q}\).

**Lemma 2.4.** Let \(0 < p < \infty\), and let the nonnegative sequence \(\{\mu_k\}\) be such that \(\{2^k \mu_k\} \in l^q, 0 < q \leq \infty\). Further, suppose that the nonnegative function \(\varphi\) satisfies the following property: there exists \(0 < \varepsilon < 1\) such that, given an arbitrary integer \(k_0\), we have \(\varphi \leq \psi_{k_0} + \eta_{k_0}\), where \(\psi_{k_0}\) and \(\eta_{k_0}\) satisfy

\[
2^{k_0 \varphi \mathbb{P}}(\psi_{k_0} > 2^{k_0}) \varepsilon \leq C \sum_{k=-\infty}^{k_0} (2^{k\varepsilon} \mu_k)^p, \quad 0 < \varepsilon < \min(1, \frac{q}{p}),
\]

\[
2^{k_0 \varepsilon \mathbb{P}}(\eta_{k_0} > 2^{k_0}) \leq C \sum_{k=k_0}^{\infty} (2^{k\varepsilon} \mu_k)^p.
\]

Then \(\varphi \in L_{p,q}\) and \(\|\varphi\|_{p,q} \leq C\|\{2^k \mu_k\}\|_{l_q}\).

### 3 Atomic decompositions

The method of atomic decompositions plays an important role in martingale theory; see for instance [10], [13], [15], [16], [21] and [22]. In particular, Jiao, Peng and Liu [11] proved the atomic decompositions of martingale Hardy-Lorentz spaces in 2009. Since \(\| \cdot \|_{p,q}\) is equivalent to a \(q\)-norm just when \(0 < q \leq 1\) and \(q \leq p < \infty\), there is a restrictive condition for the converse part of Theorem 2.1 in [11]. We improve Theorem 2.1 in [11] by using the technical Lemma 2.3 and shows that the converse part of Theorem 2.1 in [11] is true for all \(0 < p < \infty, 0 < q \leq \infty\).

**Theorem 3.1.** If \(f = (f_n)_{n \geq 0} \in H^s_{p,q}(0 < p < \infty, 0 < q \leq \infty)\), then there exists a sequence \((a^k)_{k \in \mathbb{Z}}\) of \((1, p, \infty)\)-atoms and a sequence \((\mu_k)_{k \in \mathbb{Z}}\) \(\in l_q\) of real numbers satisfying \(\mu_k = A \cdot 2^k \mathbb{P} (\nu_k < \infty)\frac{1}{2}\) (where \(A\) is a positive constant and \(\nu_k\) is the stopping time associated with \(a^k\)) such that

\[
f_n = \sum_{k \in \mathbb{Z}} \mu_k a^k_n, \quad a.e., \quad n \in \mathbb{N}, \quad (3.1)
\]
Conversely, if the martingale $f$ has the above decomposition, then $f \in H^p_{p,q}$ and

$$\|f\|_{H^p_{p,q}} \approx \inf \|\mu_k\|_{l_q},$$

where the infimum is taken over all the above decompositions.

Proof. Assume that $f \in H^p_{p,q}$ ($0 < p < \infty, 0 < q \leq \infty$). For each $k \in \mathbb{Z}$, the stopping time is defined as follows

$$\nu_k = \inf\{n \in \mathbb{N} : s_{n+1}(f) > 2^k\}, \quad (\inf \emptyset = \infty).$$

Obviously, the sequence of these stopping times is non-decreasing. Similarly to the proof of Theorem 2.2 in [22] (or see the proof of Theorem 2.1 in [11]), we have

$$\sum_{k \in \mathbb{Z}} (f^{\nu_k+1}_n - f^{\nu_k}_n) = f_n.$$ 

Let

$$\mu_k = 3 \cdot 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}}, \quad a^k_n = \frac{f^{\nu_k+1}_n - f^{\nu_k}_n}{\mu_k}.$$ 

If $\mu_k = 0$, we assume that $a^k_n = 0$. Then for any fixed $k \in \mathbb{Z}$, $a_k = (a^k_n)_{n \geq 0}$ is a martingale. Considering the stopped martingale $f^{\nu_k} = (f^{\nu_k}_n)_{n \geq 0} = (f_{n \wedge \nu_k})_{n \geq 0}$, we have $s(f^{\nu_k}) = s_{\nu_k}(f) \leq 2^k$, $s(f^{\nu_k+1}) \leq 2^{k+1}$. Then

$$s(a^k) \leq \frac{s(f^{\nu_k+1}) + s(f^{\nu_k})}{\mu_k} \leq \mathbb{P}(\nu_k < \infty)^{-\frac{1}{p}},$$

which implies that $(a^k_n)_{n \geq 0}$ is a $L_2$-bounded martingale. So $(a^k_n)_{n \geq 0}$ converges in $L_2$. Denote the limit still by $a_k$, then $\mathbb{E}_n a_k = a^k_n$. If $\nu_k \geq n$, then $a^k_n = 0$, and $\|s(a^k)\|_{\infty} \leq \mathbb{P}(\nu_k < \infty)^{-\frac{1}{p}}$. Thus we conclude that $a_k$ is really a $(1, p, \infty)$-atom. Since $\{\nu_k < \infty\} = \{s(f) > 2^k\}$, we get for $0 < q < \infty$

$$\left(\sum_{k \in \mathbb{Z}} |\mu_k|^q\right)^{\frac{1}{q}} = 3 \left(\sum_{k \in \mathbb{Z}} 2^{kq} \mathbb{P}(\nu_k < \infty)^{\frac{q}{p}}\right)^{\frac{1}{q}} = 3 \left(\sum_{k \in \mathbb{Z}} 2^{kq} \mathbb{P}(s(f) > 2^k)^{\frac{q}{p}}\right)^{\frac{1}{q}} \leq C \left(\sum_{k \in \mathbb{Z}} \int_{2^{k-1}}^{2^k} y^{q-1}dy \mathbb{P}(s(f) > 2^k)^{\frac{q}{p}}\right)^{\frac{1}{q}} \leq C \left(\sum_{k \in \mathbb{Z}} \int_{2^{k-1}}^{2^k} y^{q-1} \mathbb{P}(s(f) > y)^{\frac{q}{p}}dy\right)^{\frac{1}{q}} \leq C \left(\int_0^{\infty} y^{q-1} \mathbb{P}(s(f) > y)^{\frac{q}{p}}dy\right)^{\frac{1}{q}} \leq C\|s(f)\|_{p,q} = C\|f\|_{H^p_{p,q}}.$$
For $q = \infty$,
\[
\|(\mu_k)_{k \in \mathbb{Z}}\|_\infty = \sup_{k \in \mathbb{Z}} |\mu_k| = 3 \cdot \sup_{k \in \mathbb{Z}} 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}}
\]
\[
= 3 \cdot \sup_{k \in \mathbb{Z}} 2^k \mathbb{P}(s(f) > 2^k)^{\frac{1}{p}}
\]
\[
\leq C\|s(f)\|_{p, \infty} = C\|f\|_{H^p_{p, \infty}}.
\]
Consequently, $\|(\mu_k)_{k \in \mathbb{Z}}\|_{l_q} \leq C\|f\|_{H^p_{p, \infty}}$.

Conversely, if the martingale $f$ has the above decomposition, then for an arbitrary integer $k_0$, let
\[
f_n = \sum_{k \in \mathbb{Z}} \mu_k a_k^k = g_n + h_n, \quad (n \in \mathbb{N}),
\]
where $g_n = \sum_{k = -\infty}^{k_0} \mu_k a_k^k$ and $h_n = \sum_{k = k_0}^{\infty} \mu_k a_k^k$. By the sublinearity of the operator $s$, we have $s(f) \leq s(g) + s(h)$. Then
\[
\|s(g)\|_\infty \leq \| \sum_{k = -\infty}^{k_0-1} |\mu_k| s(a^k)\|_\infty \leq \sum_{k = -\infty}^{k_0-1} |\mu_k| \|s(a^k)\|_\infty
\]
\[
\leq \sum_{k = -\infty}^{k_0-1} |\mu_k| \mathbb{P}(|\nu_k| < \infty)^{\frac{1}{p}}
\]
\[
\leq \sum_{k = -\infty}^{k_0-1} A \cdot 2^k = A \cdot 2^{k_0}.
\]
Since $s(a^k) = 0$ on the set $\{\nu_k = \infty\}$, we have $\{s(a^k) > 0\} \subset \{\nu_k < \infty\}$. Then it follows from $s(h) \leq \sum_{k = k_0}^{\infty} |\mu_k| s(a^k)$ that
\[
\{s(h) > 0\} \subset \bigcup_{k = k_0}^{\infty} \{s(a^k) > 0\} \subset \bigcup_{k = k_0}^{\infty} \{\nu_k < \infty\}.
\]
The for each $0 < \varepsilon < 1$, we obtain
\[
2^{k_0+\varepsilon} \mathbb{P}(s(h) > 2^{k_0}) \leq 2^{k_0+\varepsilon} \mathbb{P}(s(h) > 0) \leq 2^{k_0+\varepsilon} \sum_{k = k_0}^{\infty} \mathbb{P}(\nu_k < \infty)
\]
\[
= 2^{k_0+\varepsilon} \sum_{k = k_0}^{\infty} 2^{k_0+\varepsilon} \mathbb{P}(\nu_k < \infty)2^{-k_0\varepsilon}
\]
\[
\leq \sum_{k = k_0}^{\infty} 2^{k_0+\varepsilon} \mathbb{P}(\nu_k < \infty) = \sum_{k = k_0}^{\infty} \left(2^{k_0+\varepsilon} \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}}\right)^p.
\]
By Lemma 2.3, we have $s(f) \in L_{p,q}$ and $\|s(f)\|_{p,q} \leq C \|\{2^k \mathbb{P} (\nu_k < \infty)\}_{k \in \mathbb{Z}}\|_{l_q} \leq C \|[\mu_k]_{k \in \mathbb{Z}}\|_{l_q}$. Then $f \in H^s_{p,q}$ and $\|f\|_{H^s_{p,q}} \leq C \|[\mu_k]_{k \in \mathbb{Z}}\|_{l_q}$. Thus

$$\|f\|_{H^s_{p,q}} \approx \inf \|[\mu_k]_{k \in \mathbb{Z}}\|_{l_q},$$

where the infimum is taken over all the above decompositions. The proof of the theorem is complete.

**Remark 3.2.** If $q \neq \infty$, then (3.1) holds in $H^s_{p,q}$. Namely, the sum $\sum_{k=m}^{n} \mu_k a^k$ converges to $f$ in $H^s_{p,q}$ as $m \to -\infty$, $n \to \infty$. Indeed,

$$\sum_{k=m}^{n} \mu_k a^k = \sum_{k=m}^{n} (f^{\nu_{k+1}} - f^{\nu_k}) = f^{\nu_{n+1}} - f^{\nu_m}.$$ 

By the sublinearity of $s(f)$ we have

$$\|f - \sum_{k=m}^{n} \mu_k a^k\|_{H^s_{p,q}} = \|s(f - f^{\nu_{n+1}} + f^{\nu_m})\|_{p,q} \leq \|s(f - f^{\nu_{n+1}}) + s(f^{\nu_m})\|_{p,q} \leq C \left( \|s(f - f^{\nu_{n+1}})\|_{p,q} + \|s(f^{\nu_m})\|_{p,q} \right).$$

Since $s(f - f^{\nu_{n+1}})^2 = s(f)^2 - s(f^{\nu_{n+1}})^2$, then $s(f - f^{\nu_{n+1}}) \leq s(f)$, $s(f^{\nu_m}) \leq s(f)$ and

$s(f - f^{\nu_{n+1}})$, $s(f^{\nu_m}) \to 0$ a.e. as $m \to -\infty$, $n \to \infty$. Thus by the Lebesgue convergence theorem, we have

$$\|s(f - f^{\nu_{n+1}})\|_{p,q}, \|s(f^{\nu_m})\|_{p,q} \to 0 \quad \text{as} \quad m \to -\infty, n \to \infty,$$

which means $\|f - \sum_{k=m}^{n} \mu_k a^k\|_{H^s_{p,q}} \to 0$ as $m \to -\infty$, $n \to \infty$. Further, for each $k \in \mathbb{Z}$, $a^k = (a_n^k)_{n \geq 0}$ is $L_2$ bounded, hence $H^s_{p,q} = L_2$ is dense in $H^s_{p,q}$.

**Theorem 3.3.** In Theorem 3.1, if we replace $H^s_{p,q}$, $(1, p, \infty)$-atoms by $Q_{p,q}$, $(2, p, \infty)$-atoms (or $D_{p,q}$, $(3, p, \infty)$-atoms) respectively, then the conclusions still hold.

Proof. The proof is similar to the one of Theorem 3.1, so we give it in sketch, only. If $f = (f_n)_{n \geq 0} \in Q_{p,q}$ (or $D_{p,q}$). The stopping times $\nu_k$ are defined in these cases by

$$\nu_k = \inf \{n \in \mathbb{N} : \lambda_n > 2^k\}, \quad (\inf \emptyset = \infty),$$

where $(\lambda_n)_{n \geq 0}$ is the sequence in the definition of $Q_{p,q}$ (or $D_{p,q}$). Let $a^k$ and $\mu_k$ ($k \in \mathbb{Z}$) be defined as in the proof of Theorem 3.1. Then the conclusions $f_n = \sum_{k \in \mathbb{Z}} \mu_k a^k_n$ ($n \in \mathbb{N}$)

and $\|\{\mu_k\}_{k \in \mathbb{Z}}\|_{l_q} \leq C \|f\|_{Q_{p,q}}$ (or $\|\{\mu_k\}_{k \in \mathbb{Z}}\|_{l_q} \leq C \|f\|_{D_{p,q}}$) still hold.

To prove the converse part, let

$$\lambda_n = \sum_{k \in \mathbb{Z}} \mu_k \chi_{\{\nu_k \leq n\}} \|S(a^k)\|_{\infty} \quad (or \quad \lambda_n = \sum_{k \in \mathbb{Z}} \mu_k \chi_{\{\nu_k \leq n\}} \|(a^k)^n\|_{\infty}).$$

10
Then \((\lambda_n)_{n \geq 0}\) is a nondecreasing, nonnegative and adapted sequence with \(S_{n+1}(f) \leq \lambda_n\) (or \(|f_{n+1}| \leq \lambda_n\)).

For any given integer \(k_0\), let
\[
\lambda_\infty = \lambda_\infty^{(1)} + \lambda_\infty^{(2)},
\]
where
\[
\lambda_\infty^{(1)} = \sum_{k=0}^{k_0-1} \chi_{\{\nu_k < \infty\}} \|S(a^k)\|_\infty \quad \text{(or) \quad} \lambda_\infty^{(1)} = \sum_{k=\infty}^{k_0-1} \chi_{\{\nu_k < \infty\}} \|(a^k)^*\|_\infty,
\]
and
\[
\lambda_\infty^{(2)} = \sum_{k=k_0}^{\infty} \chi_{\{\nu_k < \infty\}} \|S(a^k)\|_\infty \quad \text{(or) \quad} \lambda_\infty^{(2)} = \sum_{k=0}^{\infty} \chi_{\{\nu_k < \infty\}} \|(a^k)^*\|_\infty.
\]
Replacing \(s(g)\) and \(s(h)\) in the proof of Theorem 3.1 by \(\lambda_\infty^{(1)}\) and \(\lambda_\infty^{(2)}\). Using Lemma 2.3, we can obtain \(f \in Q_{p,q}\) (or \(f \in D_{p,q}\)) and \(\|f\|_{Q_{p,q}} \approx \inf \|(\mu_k)_{k \in \mathbb{Z}}\|_{l_{q}}\) (or \(\|f\|_{D_{p,q}} \approx \inf \|(\mu_k)_{k \in \mathbb{Z}}\|_{l_{q}}\)), where the infimum is taken over all the above decompositions. The proof is complete.

4 Duality results

In this section, we prove the predual of the generalized BMO martingale spaces.

**Theorem 4.1.** The dual space of \(H_{p,q}^s\) is \(BMO_2(\alpha)\), \(0 < p, q \leq 1, \alpha = \frac{1}{p} - 1\).

**Proof.** Since \(0 < p, q \leq 1\), we note that by
\[
\|f\|_{H_{p,q}^s} = \|s(f)\|_{p,q} \leq \|s(f)\|_{2,2} = \|f\|_2,
\]
the space \(L_2\) is a subspace of \(H_{p,q}^s\). By the Remark 3.2, we know that \(L_2\) is dense in \(H_{p,q}^s\). For any \(g \in BMO_2(\alpha) \subset L_2\), we show that
\[
\varphi_g(f) = \mathbb{E}(fg), \quad \forall f \in L_2,
\]
is a continuous linear functional on \(L_2\). It follows from Theorem 3.1 that \(f = \sum_{k \in \mathbb{Z}} \mu_k a^k\).

Hence
\[
\varphi_g(f) = \mathbb{E}(fg) = \sum_{k \in \mathbb{Z}} \mu_k \mathbb{E}(a^k g).
\]
By the definition of the atom $a^k$, we have $\mathbb{E}(a^k g) = \mathbb{E}(a^k (g - g^v))$. Using Hölder’s inequality, we obtain

$$ |\varphi_g(f)| = \left| \sum_{k \in \mathbb{Z}} \mu_k \mathbb{E}(a^k (g - g^v)) \right| \leq \sum_{k \in \mathbb{Z}} |\mu_k| |\mathbb{E}(a^k (g - g^v))| $$

$$ \leq \sum_{k \in \mathbb{Z}} |\mu_k| ||a^k||_2 ||g - g^v||_2 = \sum_{k \in \mathbb{Z}} |\mu_k| ||s(a^k)||_2 ||g - g^v||_2 $$

$$ = \sum_{k \in \mathbb{Z}} |\mu_k| ||s(a^k)\chi_{\nu_k < \infty}||_2 ||g - g^v||_2 \leq \sum_{k \in \mathbb{Z}} |\mu_k| ||s(a^k)||_\infty ||\chi_{\nu_k < \infty}||_2 ||g - g^v||_2 $$

$$ \leq \sum_{k \in \mathbb{Z}} |\mu_k| ||p(\nu_k < \infty)\frac{1}{p} ||g - g^v||_2 \leq \sum_{k \in \mathbb{Z}} |\mu_k| ||g||_{BMO_2(\alpha)}. $$

Since $0 < q \leq 1$, we have $|\varphi_g(f)| \leq \left( \sum_{k \in \mathbb{Z}} |\mu_k|^q \right)^{\frac{1}{q}} ||g||_{BMO_2(\alpha)}$, and by Theorem 3.1, we obtain

$$ |\varphi_g(f)| \leq C \|f\|_{H_{p,q}^s} ||g||_{BMO_2(\alpha)}. $$

By density of $L_2$ in $H_{p,q}^s$, $\varphi_g$ can be uniquely extended to a continuous functional on $H_{p,q}^s$.

Conversely, for any $\varphi \in (H_{p,q}^s)^*$, we show that there exists $g \in BMO_2(\alpha)$ such that $\varphi = \varphi_g$ and $||g||_{BMO_2(\alpha)} \leq ||\varphi||$.

Since $L_2$ can be continuously embedded in $H_{p,q}^s$, then there exists $g \in L_2$ such that $\varphi(f) = \mathbb{E}(fg)$, $(f \in L_2)$.

Let $\nu$ be an arbitrary stopping time and

$$ h = \frac{g - g^v}{||g - g^v||_2 \mathbb{P}(\nu < \infty)^{\frac{1}{p} - \frac{1}{q}}}. $$

Then $s(h) = 0$ on $\{\nu = \infty\}$, namely, $s(h) = s(h)\chi_{\nu < \infty}$.

Since $0 < p, q \leq 1$, then there exists $p_1, q_1 > 0$ such that $\frac{1}{p} = \frac{1}{2} + \frac{1}{p_1}, \frac{1}{q} = \frac{1}{2} + \frac{1}{q_1}$.

By Hölder’s inequality we have

$$ ||h||_{H_{p,q}^s} = \frac{||g - g^v||_{H_{p,q}^s}}{||g - g^v||_2 \mathbb{P}(\nu < \infty)^{\frac{1}{p} - \frac{1}{q}}} = \frac{||s(g - g^v)\chi_{\nu < \infty}||_{p,q}}{||g - g^v||_2 \mathbb{P}(\nu < \infty)^{\frac{1}{p} - \frac{1}{q}}} $$$$ \leq \frac{C}{||g - g^v||_2 \mathbb{P}(\nu < \infty)^{\frac{1}{p} - \frac{1}{q}}} ||s(g - g^v)||_{2,2} \|\chi_{\nu < \infty}\|_{p_1,q_1} $$

$$ = \frac{C}{||g - g^v||_2 \mathbb{P}(\nu < \infty)^{\frac{1}{p} - \frac{1}{q}}} \left( \frac{q_1}{p_1} \int_{0}^{\infty} \frac{1}{t^{p_1} - 1} \left( \mathbb{E}(\chi_{\nu < \infty}) \right) \frac{dt}{q_1} \right)^{\frac{1}{q_1}} $$

$$ = \frac{C}{\mathbb{P}(\nu < \infty)^{\frac{1}{p} - \frac{1}{q}}} \mathbb{P}(\nu < \infty)^{\frac{1}{p_1}} = C. $$
Set \( h_0 = h/C \), then \( \|h_0\|_{H_{p,q}} \leq 1 \). Consequently, \( \|\varphi\| \geq |\varphi(h_0)| = E(h_0 g) = E(h_0(g - g'')) = C^{-1}P(v < \infty)^{\frac{1}{2} - \frac{1}{p}}\|g - g''\|_2 \). Taking the supremum over all stopping times, we have \( \|g\|_{BMO_2(\alpha)} \leq C\|\varphi\| \). The proof of the theorem is complete.

Now we investigate the case \( 0 < p \leq 1, 1 < q < \infty \).

**Theorem 4.2.** The dual space of \( H_{p,q}^s \) is \( BMO_{2,q}(\alpha) \), \( 0 < p \leq 1, 1 < q < \infty, \alpha = \frac{1}{p} - 1 \).

Proof. Let \( g \in BMO_{2,q}(\alpha) \subset L_2 \), define \( \varphi_g(f) = E(f g) \), \( f \in L_2 \). Similarly to the proof of Theorem 4.1, by Hölder’s inequality we have

\[
|\varphi_g(f)| = |\sum_{k \in Z} \mu_k E(a_k g)| = |\sum_{k \in Z} \mu_k E(a_k (g - g'^k))| \\
\leq \sum_{k \in Z} |\mu_k| E(|a_k (g - g'^k)|) \leq \sum_{k \in Z} |\mu_k| \|a_k\|_2 \|g - g'^k\|_2 \\
\leq \sum_{k \in Z} |\mu_k| P(\nu_k < \infty)^{\frac{1}{2} - \frac{1}{p}} \|g - g'^k\|_2 = A \sum_{k \in Z} 2^{k} P(\nu_k < \infty)^{\frac{1}{2}} \|g - g'^k\|_2.
\]

By the definition of \( \|\cdot\|_{BMO_{2,q}(\alpha)} \) and Theorem 3.1, then

\[
|\varphi_g(f)| \leq A \left( \sum_{k \in Z} \left( 2^{k} P(\nu_k < \infty)^{\frac{1}{2}} \right)^q \right)^{\frac{1}{q}} \|g\|_{BMO_{2,q}(\alpha)} \leq C \|f\|_{H_{p,q}^s} \|g\|_{BMO_{2,q}(\alpha)}.
\]

Thus \( \varphi_g \) can be uniquely extended to a continuous functional on \( H_{p,q}^s \).

Conversely, if \( \varphi \in (H_{p,q}^s)^* \), we know that there exists \( g \in L_2 \) such that \( \varphi(f) = E(f g) \), \( f \in L_2 \). Let \( \{\nu_k\}_{k \in Z} \) be an arbitrary stopping time sequence such that \( \{2^{k} P(\nu_k < \infty)^{\frac{1}{2}}\}_{k \in Z} \in l_q \) and \( N \) be an arbitrary nonnegative integer. Let

\[
h_k = \frac{|g - g'^k| \text{sign}(g - g'^k)}{\|g - g'^k\|_2}, \quad f = \sum_{k=-N}^{N} 2^{k} P(\nu_k < \infty)^{\frac{1}{2}} (h_k - h'^k).
\]

For an arbitrary integer \( k_0 \) which satisfies \(-N \leq k_0 \leq N \) (for \( k_0 \leq -N, \) let \( G = 0 \) and \( H = f \); for \( k_0 > N, \) let \( H = 0 \) and \( G = f \)), let

\[
f = G + H,
\]

where \( G = \sum_{k=-N}^{k_0-1} 2^{k} P(\nu_k < \infty)^{\frac{1}{2}} (h_k - h'^k) \) and \( H = \sum_{k=k_0}^{N} 2^{k} P(\nu_k < \infty)^{\frac{1}{2}} (h_k - h'^k) \).

Obviously \( \|h_k\|_2 = 1 \), and \( \|G\|_2 \leq 2 \sum_{k=-N}^{k_0-1} 2^{k} P(\nu_k < \infty)^{\frac{1}{2}} \). By the sublinearity of the
operator $s$, we have $s(f) \leq s(G) + s(H)$. Let $\varepsilon = \frac{p}{2}$, then $0 < \varepsilon < \min(1, \frac{q}{p})$. We obtain

$$
2^{k_0}P(s(G) > 2^{k_0}) \leq 2^{k_0} \left( \frac{1}{2^{2k_0}} \|s(G)\|_2^2 \right)^{\varepsilon} \leq C \cdot 2^{k_0(p-2\varepsilon)} \|G\|_2^{2\varepsilon}
$$

$$
\leq C \left( \sum_{k=-N}^{k_0-1} 2^k P(\nu_k < \infty)^{\frac{1}{2}} \right)^{2\varepsilon} = C \left( \sum_{k=-N}^{k_0-1} 2^k P(\nu_k < \infty)^{\frac{p}{2}} \right)^p
$$

$$
\leq C \sum_{k=-N}^{k_0-1} \left( 2^k P(\nu_k < \infty)^{\frac{p}{2}} \right)^p \leq C \sum_{k=-\infty}^{k_0-1} \left( 2^k P(\nu_k < \infty)^{\frac{p}{2}} \right)^p.
$$

On the other hand,

$$
\{s(H) > 0\} \subset \bigcup_{k=k_0}^{N} \{\nu_k < \infty\}.
$$

Then for each $0 < \varepsilon < 1$, we have

$$
2^{k_0}P(s(H) > 2^{k_0}) \leq 2^{k_0} \sum_{k=k_0}^{N} P(\nu_k < \infty)
$$

$$
= 2^{k_0} \sum_{k=k_0}^{N} 2^{k_0} P(\nu_k < \infty) 2^{-k_0} \leq \sum_{k=k_0}^{N} 2^{k_0} P(\nu_k < \infty)
$$

$$
= \sum_{k=k_0}^{N} \left( 2^{k_0} P(\nu_k < \infty)^{\frac{p}{2}} \right)^p \leq \sum_{k=k_0}^{\infty} \left( 2^{k_0} P(\nu_k < \infty)^{\frac{p}{2}} \right)^p.
$$

By Lemma 2.4, we have $s(f) \in L_{p,q}$ and $\|s(f)\|_{p,q} \leq C \|\{2^{k_0} P(\nu_k < \infty)^{\frac{p}{2}}\}_{k \in \mathbb{Z}}\|_{l_q}$. Thus $f \in H_{p,q}^*$ and

$$
\|f\|_{H_{p,q}^*} \leq C \left( \sum_{k \in \mathbb{Z}} \left( 2^{k_0} P(\nu_k < \infty)^{\frac{p}{2}} \right)^q \right)^{\frac{1}{q}}.
$$

Therefore,

$$
\sum_{k=-N}^{N} 2^k P(\nu_k < \infty)^{\frac{1}{2}} \|g - g^{\nu_k}\|_2 = \sum_{k=-N}^{N} 2^k P(\nu_k < \infty)^{\frac{1}{2}} \mathbb{E}(h_k(g - g^{\nu_k}))
$$

$$
= \sum_{k=-N}^{N} 2^k P(\nu_k < \infty)^{\frac{1}{2}} \mathbb{E}((h_k - h_k^{\nu_k})g)
$$

$$
= \mathbb{E}(f g) = \varphi(f) \leq \|f\|_{H_{p,q}^*} \|\varphi\|
$$

$$
\leq C \left( \sum_{k \in \mathbb{Z}} \left( 2^{k_0} P(\nu_k < \infty)^{\frac{p}{2}} \right)^q \right)^{\frac{1}{q}} \|\varphi\|.
$$

Thus we obtain

$$
\frac{\sum_{k=-N}^{N} 2^k P(\nu_k < \infty)^{\frac{1}{2}} \|g - g^{\nu_k}\|_2}{\left( \sum_{k \in \mathbb{Z}} \left( 2^{k_0} P(\nu_k < \infty)^{\frac{p}{2}} \right)^q \right)^{\frac{1}{q}}} \leq C \|\varphi\|.
$$
5 The generalized John-Nirenberg theorem

In this section, we prove the generalized John-Nirenberg theorem by duality when the stochastic basis \( \{F_n\}_{n \geq 0} \) is regular. Some of the dual results are of independent interest. In order to do this, we need the following lemma and we refer to [22] for these facts.

Lemma 5.1. If the stochastic basis \( \{F_n\}_{n \geq 0} \) is regular, then the martingale Hardy-Lorentz spaces \( H^s_{p,q}, H^S_{p,q}, H^s_{p,q}, Q_{p,q} \) and \( D_{p,q} \) are all equivalent for \( 0 < p < \infty, 0 < q \leq \infty \), and \( H^s_{p,q}, H^S_{p,q}, H^s_{p,q}, Q_{p,q}, D_{p,q} \) and \( L_{p,q} \) are all equivalent for \( 1 < p < \infty, 0 < q \leq \infty \).

Theorem 5.2. If the stochastic basis \( \{F_n\}_{n \geq 0} \) is regular, then

\[
(H^s_{p,q})^* = BMO_{r,q}(\alpha), \quad (0 < p \leq 1, 1 < q, r < \infty, \alpha = \frac{1}{p} - 1).
\]

Proof. Let \( g \in BMO_{r,q}(\alpha) \subset L_r \) and \( r' \) be the conjugate number of \( r \), then \( 1 < r' < \infty \). Define \( \varphi_g(f) = \mathbb{E}(fg), f \in L_{r'} \). Note that \( L_{r'} = H^s_{r',r'} \subset H^s_{p,q} \). By Theorem 3.1 there exists a sequence \( (a_k)_{k \in \mathbb{Z}} \) of \((1,p,\infty)\)-atoms and a sequence of real numbers \( (\mu_k)_{k \in \mathbb{Z}} \) satisfying \( \mu_k = A \cdot 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}} \) (where \( A \) is a positive constant and \( (\nu_k)_{k \in \mathbb{Z}} \) is the corresponding stopping time sequence) such that \( f = \sum_{k \in \mathbb{Z}} \mu_k a^k \) and \( \|\mu_k\|_{l_q} \leq C\|f\|_{H^s_{p,q}} \). By Hölder’s inequality we can obtain

\[
|\varphi_g(f)| = \left| \sum_{k \in \mathbb{Z}} \mu_k \mathbb{E}(a_k^g) \right| = \left| \sum_{k \in \mathbb{Z}} \mu_k \mathbb{E}(a_k^g g - g^{\nu_k}) \right| \leq \sum_{k \in \mathbb{Z}} |\mu_k| \mathbb{E}(a_k^g(g - g^{\nu_k})) \\
\leq \sum_{k \in \mathbb{Z}} |\mu_k| \|a_k^g\|_r \|g - g^{\nu_k}\|_r \leq C \sum_{k \in \mathbb{Z}} |\mu_k| \|a_k^g\|_r \|g - g^{\nu_k}\|_r \\
\leq C \sum_{k \in \mathbb{Z}} |\mu_k| \mathbb{P}(\nu_k < \infty)^{\frac{1}{p'}} \|g - g^{\nu_k}\|_r = C \cdot A \sum_{k \in \mathbb{Z}} 2^k \mathbb{P}(\nu_k < \infty)^{1-\frac{1}{p'}} \|g - g^{\nu_k}\|_r.
\]

By the definition of \( \| \cdot \|_{BMO_{r,q}(\alpha)} \), we obtain

\[
|\varphi_g(f)| \leq C \cdot A \left( \sum_{k \in \mathbb{Z}} (2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p'}})^q \right)^{\frac{1}{q}} \|g\|_{BMO_{r,q}(\alpha)} \leq C\|f\|_{H^s_{p,q}} \|g\|_{BMO_{r,q}(\alpha)}.
\]

Thus \( \varphi_g \) can be extended to a continuous functional on \( H^s_{p,q} \).

Conversely, if \( \varphi \in (H^s_{p,q})^* \). By the regularity of the stochastic basis \( \{F_n\}_{n \geq 0} \), we have \( L_{r'} = H^s_{r',r'} \subset H^s_{p,q} \), then \( (H^s_{p,q})^* \subset (L_{r'})^* = L_r \). Thus there exists \( g \in L_r \) such that \( \varphi(f) = \varphi_g(f) = \mathbb{E}(fg), (f \in L_{r'}) \).
Let \( \{\nu_k\}_{k \in \mathbb{Z}} \) be an arbitrary stopping time sequence such that \( \{2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}}\}_{k \in \mathbb{Z}} \in l_q \) and \( N \) be an arbitrary nonnegative integer. Let

\[
h_k = \frac{|g - g^{\nu_k}|^{r-1} \text{sign}(g - g^{\nu_k})}{\|g - g^{\nu_k}\|_{r'}^{r-1}}, \quad f = \sum_{k=-N}^{N} 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}} (h_k - h_k^{\nu_k}).
\]

For an arbitrary integer \( k_0 \) which satisfies \(-N \leq k_0 \leq N\) (for \( k_0 \leq -N \), let \( G = 0 \) and \( H = f \); for \( k_0 > N \), let \( H = 0 \) and \( G = f \)), let

\[
f = G + H,
\]

where \( G = \sum_{k=-N}^{k_0-1} 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}} (h_k - h_k^{\nu_k}) \) and \( H = \sum_{k=k_0}^{N} 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}} (h_k - h_k^{\nu_k}) \).

Obviously, \( \|h_k\|_{r'} = 1 \) and \( \|G\|_{r'} \leq 2 \sum_{k=-N}^{k_0-1} 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}} \). By the sublinearity of the operator \( s \), we have \( s(f) \leq s(G) + s(H) \). Let \( \varepsilon = \frac{2}{r} \), then \( 0 < \varepsilon < \min(1, \frac{2}{p}) \). By Lemma 5.1 we have

\[
2^{k_0 \varepsilon} \mathbb{P}(s(G) > 2^{k_0})^\varepsilon \leq 2^{k_0 \varepsilon} \left( \frac{1}{2^{k_0 \varepsilon} \|s(G)\|_{r'}} \right)^\varepsilon \leq C \cdot 2^{k_0 (p - \varepsilon)} \|G\|_{r'}^{\varepsilon} = C \left( \sum_{k=-N}^{k_0-1} 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}} \right)^p \leq C \sum_{k=-N}^{k_0-1} (2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}})^p.
\]

On the other hand, \( \{s(H) > 0\} \subseteq \bigcup_{k=k_0}^{N} \{\nu_k < \infty\} \). Then for each \( 0 < \varepsilon < 1 \), we have

\[
2^{k_0 \varepsilon} \mathbb{P}(s(H) > 2^{k_0}) \leq 2^{k_0 \varepsilon} \mathbb{P}(s(H) > 0) \leq 2^{k_0 \varepsilon} \sum_{k=k_0}^{N} \mathbb{P}(\nu_k < \infty) \leq \sum_{k=k_0}^{N} (2^{k_0 \varepsilon} \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}})^p \leq \sum_{k=k_0}^{\infty} (2^{k_0 \varepsilon} \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}})^p.
\]

By Lemma 2.4, we have \( s(f) \in L_{p,q} \) and \( \|s(f)\|_{p,q} \leq C \|\{2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}}\}_{k \in \mathbb{Z}}\|_{l_q} \). Thus \( f \in H_{p,q}^s \) and

\[
\|f\|_{H_{p,q}^s} \leq C \left( \sum_{k \in \mathbb{Z}} (2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}})^q \right)^{\frac{1}{q}}.
\]
Consequently,
\[
\sum_{k=-N}^{N} 2^k \mathbb{P}(\nu_k < \infty)^{1 - \frac{1}{q}} \|g - g^{\nu_k}\|_r = \sum_{k=-N}^{N} 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{q}} \mathbb{E}(h_k(g - g^{\nu_k})) = \sum_{k=-N}^{N} 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{q}} \mathbb{E}((h_k - h_k^{\nu_k})g) = \mathbb{E}(fg) = \varphi(f) \leq \|f\|_{H_{p,q}^1} \|\varphi\| \leq C \left( \sum_{k \in \mathbb{Z}} (2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{q}})^q \right)^{\frac{1}{q}} \|\varphi\|.
\]
Thus we obtain
\[
\frac{\sum_{k=-N}^{N} 2^k \mathbb{P}(\nu_k < \infty)^{1 - \frac{1}{q}} \|g - g^{\nu_k}\|_r}{\left( \sum_{k \in \mathbb{Z}} (2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{q}})^q \right)^{\frac{1}{q}}} \leq C \|\varphi\|.
\]
Taking \( N \to \infty \) and the supremum over all of such stopping time sequences such that \( \{2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{q}}\}_{k \in \mathbb{Z}} \) in \( l_q \), we get \( \|g\|_{BMO_{r,q}(\alpha)} \leq C \|\varphi\| \). The proof is complete.

It should be mentioned that the proof method of Theorem 5.2 is not available for \( r = 1 \). In this case, we need new insight. Let the dual space of \( D_{p,q} \) be \( D_{p,q}^* \). Let us denote by \( (D_{p,q}^*)_1 \) those elements \( \varphi \) from \( D_{p,q}^* \) for which there exists \( g \in L_1 \) such that \( \varphi(f) = \mathbb{E}(fg) \), \( f \in L_\infty \). Namely
\[
(D_{p,q}^*)_1 = \{ \varphi \in D_{p,q}^* : \exists g \in L_1 \ s.t. \ \varphi(f) = \mathbb{E}(fg), \ \forall f \in L_\infty \}.
\]

**Theorem 5.3.** \( (D_{p,q}^*)_1 = BMO_1(\alpha), \ (0 < p, q \leq 1, \alpha = \frac{1}{p} - 1) \).

Proof. Let \( g \in BMO_1(\alpha) \subset L_1 \). Define \( \varphi_g(f) = \mathbb{E}(fg), \ (f \in L_\infty) \). By Theorem 3.3, there exists a sequence \( (\alpha^k)_{k \in \mathbb{Z}} \) of \( (3, p, \infty) \)-atoms and a sequence of real numbers \( (\mu_k)_{k \in \mathbb{Z}} \) satisfying \( \mu_k = A \cdot 2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}} \) (where \( A \) is a positive constant and \( (\nu_k)_{k \in \mathbb{Z}} \) is the corresponding stopping time sequence) such that \( f = \sum_{k \in \mathbb{Z}} \mu_k \alpha^k \) and \( \|\mu_k\|_{l_q} \leq C \|f\|_{D_{p,q}} \). By Hölder’s inequality we obtain
\[
|\varphi_g(f)| = |\sum_{k \in \mathbb{Z}} \mu_k \mathbb{E}(\alpha^k g)| = |\sum_{k \in \mathbb{Z}} \mu_k \mathbb{E}(\alpha^k (g - g^{\nu_k}))| \leq \sum_{k \in \mathbb{Z}} |\mu_k|\mathbb{E}(|\alpha^k (g - g^{\nu_k})|) \leq \sum_{k \in \mathbb{Z}} |\mu_k|\|\alpha^k\|_\infty \|g - g^{\nu_k}\|_1 \leq \sum_{k \in \mathbb{Z}} |\mu_k|\|(\alpha^k)^*\|_\infty \|g - g^{\nu_k}\|_1 \leq \sum_{k \in \mathbb{Z}} |\mu_k|\mathbb{P}(\nu_k < \infty)^{-\frac{1}{p}} \|g - g^{\nu_k}\|_1 \leq \sum_{k \in \mathbb{Z}} |\mu_k|\|g\|_{BMO_1(\alpha)}.
\]
Since $0 < q \leq 1$, then

$$|\varphi_g(f)| \leq \left( \sum_{k \in \mathbb{Z}} |\mu_k|^q \right)^{\frac{1}{q}} \|g\|_{BMO_1(\alpha)} \leq C \|f\|_{D_{p,q}} \|g\|_{BMO_2(\alpha)}.$$ 

Then $\varphi_g$ can be extended to a continuous functional on $D_{p,q}$, and $\varphi_g \in (D_{p,q})_1$.

To prove the converse, let $\varphi \in (D_{p,q})_1$, then there exists $g \in L_1$ such that $\varphi(f) = \mathbb{E}(fg)$, $(f \in L_\infty)$. Let $h = \text{sign}(g - g^\nu)$, $a = \frac{1}{2} \mathbb{P}(\nu < \infty)^{-\frac{1}{p}}(h - h^\nu)$, where $\nu \in \mathcal{T}$ is an arbitrary stopping time. Then $a$ is a $(3, p, \infty)$-atom.

Let $\mu = 2A \cdot \mathbb{P}(\nu < \infty)^\frac{1}{p}$, let $h_0 = \mu a = A(h - h^\nu)$. Considering the atomic decomposition of $h_0$, by Theorem 3.2 we have $h_0 \in D_{p,q}$ and $\|h_0\|_{D_{p,q}} \leq C|\mu| = 2CA \cdot \mathbb{P}(\nu < \infty)^\frac{1}{p}$, then $\|h - h^\nu\|_{D_{p,q}} \leq 2C \cdot \mathbb{P}(\nu < \infty)^\frac{1}{p}$. Thus we have

$$\mathbb{P}(\nu < \infty)^{-\frac{1}{p}}\|g - g^\nu\|_1 = \mathbb{P}(\nu < \infty)^{-\frac{1}{p}}\mathbb{E}(h(g - g^\nu)) = \mathbb{P}(\nu < \infty)^{-\frac{1}{p}}\mathbb{E}((h - h^\nu)g) \leq \mathbb{P}(\nu < \infty)^{-\frac{1}{p}}\|h - h^\nu\|_{D_{p,q}} \|\varphi\| = 2C\|\varphi\|.$$ 

Taking the supremum over all stopping times, then we obtain $\|g\|_{BMO_1(\alpha)} \leq C\|\varphi\|$. The proof of the theorem is complete.

Now we consider $(D_{p,q})_1$, $(0 < p \leq 1, 1 < q < \infty)$. We have the following theorem.

**Theorem 5.4.** $(D_{p,q})_1 = BMO_{1,q}(\alpha)$, $(0 < p \leq 1, 1 < q < \infty, \alpha = \frac{1}{p} - 1)$.

Proof. Let $g \in BMO_{1,q}(\alpha) \subset L_1$, then

$$\|g\|_{BMO_{1,q}(\alpha)} = \sup_{k \in \mathbb{Z}} \frac{\sum_{k \in \mathbb{Z}} 2^k \|g - g^{\nu_k}\|_1}{\left( \sum_{k \in \mathbb{Z}} (2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}})^q \right)^{\frac{1}{q}}} < \infty,$$

where the supremum is taken over all stopping time sequences $\{\nu_k\}_{k \in \mathbb{Z}} \subset \mathcal{T}$ such that $\{2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}}\}_{k \in \mathbb{Z}} \in l_q$. Define $\varphi_g(f) = \mathbb{E}(fg)$, $(f \in L_\infty)$. Similarly to the proof of Theorem 4.3, by Hölder’s inequality we can obtain

$$|\varphi_g(f)| = \left| \sum_{k \in \mathbb{Z}} \mu_k \mathbb{E}(a^k g) \right| = \left| \sum_{k \in \mathbb{Z}} \mu_k \mathbb{E}(a^k (g - g^{\nu_k})) \right| \leq \sum_{k \in \mathbb{Z}} |\mu_k| \|a^k\|_{\infty} \|g - g^{\nu_k}\|_1 \leq \sum_{k \in \mathbb{Z}} |\mu_k| \mathbb{P}(\nu_k < \infty)^{-\frac{1}{p}} \|g - g^{\nu_k}\|_1 = A \sum_{k \in \mathbb{Z}} 2^k \|g - g^{\nu_k}\|_1.$$ 

By the definition of $\| \cdot \|_{BMO_{1,q}(\alpha)}$, we obtain

$$|\varphi_g(f)| \leq A \left( \sum_{k \in \mathbb{Z}} (2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p}})^q \right)^{\frac{1}{q}} \|g\|_{BMO_{1,q}(\alpha)} \leq C \|f\|_{D_{p,q}} \|g\|_{BMO_{1,q}(\alpha)}.$$
Thus $\varphi_g$ can be extended to a continuous functional on $D_{p,q}$. Moreover, $\varphi_g \in (D^*_{p,q})_1$.

Conversely, if $\varphi \in (D^*_{p,q})_1$, then there exists $g \in L_1$ such that $\varphi(f) = \mathbb{E}(fg)$, $(f \in L_\infty)$. Let $\{\nu_k\}_{k \in \mathbb{Z}}$ be an arbitrary stopping time sequence such that $\{2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p'}}\}_{k \in \mathbb{Z}} \in l_q$. Let

$$h_k = \text{sign}(g - g^{\nu_k}), \quad a^k = \frac{1}{2}(h_k - h_k^{\nu_k})\mathbb{P}(\nu_k < \infty)^{-\frac{1}{p'}}.$$ 

then $a^k$ is a $(3, p, \infty)$-atom.

Let $f^N = \sum_{k=-\infty}^{N} 2^{k+1} \mathbb{P}(\nu_k < \infty)^{\frac{1}{p'}} a^k$, where $N$ is an arbitrary nonnegative integer. By Theorem 3.3 we have $f^N \in D_{p,q}$ and

$$\|f^N\|_{D_{p,q}} \leq C \left( \sum_{k=-\infty}^{N} \mathbb{E} \left( (2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p'}})^q \right)^{\frac{1}{q'}} \right)^\frac{1}{q} \leq C \left( \sum_{k \in \mathbb{Z}} \mathbb{E} \left( (2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p'}})^q \right)^{\frac{1}{q'}} \right)^\frac{1}{q} \|\varphi\|.$$

Consequently,

$$\sum_{k=-\infty}^{N} 2^k \|g - g^{\nu_k}\|_1 = \sum_{k=-\infty}^{N} 2^k \mathbb{E}(h_k(g - g^{\nu_k})) = \sum_{k=-\infty}^{N} 2^k \mathbb{E}((h_k - h_k^{\nu_k})g) = \mathbb{E}(f^N g) = \varphi(f^N) \leq \|f^N\|_{D_{p,q}} \|\varphi\| \leq C \left( \sum_{k \in \mathbb{Z}} \mathbb{E} \left( (2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p'}})^q \right)^{\frac{1}{q'}} \right)^\frac{1}{q} \|\varphi\|.$$

Thus we have

$$\frac{\sum_{k=-\infty}^{N} 2^k \|g - g^{\nu_k}\|_1}{\left( \sum_{k \in \mathbb{Z}} \mathbb{E} \left( (2^k \mathbb{P}(\nu_k < \infty)^{\frac{1}{p'}})^q \right)^{\frac{1}{q'}} \right)^\frac{1}{q}} \leq C \|\varphi\|.$$

This shows $\|g\|_{BMO_{\alpha}(\mathbb{R})} \leq C \|\varphi\|$. The proof is complete.

**Proposition 5.5.** If the stochastic basis $\{\mathcal{F}_n\}_{n \geq 0}$ is regular, for $0 < p \leq 1, 0 < q < \infty$, then $(D^*_{p,q})_1 = D^*_{p,q}$.

Proof. Since $0 < p \leq 1$, then by Lemma 5.1, $L_2$ can also be embedded continuously in $D_{p,q}$. Then $D^*_p \subseteq (L^*_2) = L_2$. Let $\varphi$ be an arbitrary element of $D^*_{p,q}$, then there exists $g \in L_2 \subset L_1$ such that $\varphi = \varphi_g$. By the definition of $(D^*_{p,q})_1$, we have $\varphi \in (D^*_{p,q})_1$, then $D^*_{p,q} \subseteq (D^*_{p,q})_1$. And the inclusion relation $(D^*_{p,q})_1 \subseteq D^*_{p,q}$ is evident. Hence we obtain

$$(D^*_{p,q})_1 = D^*_{p,q}, \quad (0 < p \leq 1, 0 < q < \infty).$$

The proof of the proposition is complete.
We now are a position to prove Theorem 1.2.

Proof of Theorem 1.2. It follows from Theorem 4.2 and Theorem 5.2 that
\[ BMO_{r,q}(\alpha) = BMO_{2,q}(\alpha), \quad 1 < r < \infty. \]

For \( r = 1 \), combining Theorem 4.2, Lemma 5.1, Theorem 5.4 with Proposition 5.5, we get
\[ BMO_{1,q}(\alpha) = BMO_{2,q}(\alpha). \]

6 Boundedness of fractional integrals on martingale Hardy-Lorentz spaces

As we know, Chao and Ombe [14] introduced the fractional integrals for dyadic martingales. Recently, Nakai and Sadasue [17] extended the notion of fractional integrals to more general martingales. Sadasue [19] proves the boundedness of fractional integrals on martingale Hardy spaces for \( 0 < p \leq 1 \). We now extend the boundedness of fractional integrals to martingale Hardy-Lorentz spaces. In this section, we suppose that every \( \sigma \)-algebra \( \mathcal{F}_n \) is generated by countable atoms, where \( B \in \mathcal{F}_n \) is called an atom, if any \( A \subset B \) with \( A \in \mathcal{F}_n \) satisfies \( \mathbb{P}(A) < \mathbb{P}(B) \), then \( \mathbb{P}(A) = 0 \). Denote by \( A(\mathcal{F}_n) \) the set of all atoms in \( \mathcal{F}_n \). Without loss of generality, we always suppose that the constant in (1.3) satisfying \( R \geq 2 \).

Now we give the definition of fractional integral as follows.

**Definition 6.1.** For \( f = (f_n)_{n \geq 0} \in \mathcal{M} \), \( \alpha > 0 \), the fractional integral \( I_{\alpha} f = \left((I_{\alpha} f)_n\right)_{n \geq 0} \) of \( f \) is defined by
\[
(I_{\alpha} f)_n = \sum_{k=1}^{n} b_{k-1}^\alpha d_k f.
\]
where \( b_k \) is an \( \mathcal{F}_k \)-measurable function such that \( \forall B \in A(\mathcal{F}_k), \forall \omega \in B, b_k(\omega) = \mathbb{P}(B) \).

In order to prove the boundedness of fractional integrals, we need the following lemmas.

**Lemma 6.2.** Let \( \{\mathcal{F}_n\}_{n \geq 0} \) be regular, \( f \in \mathcal{M} \) and \( \alpha > 0 \). Let \( R \) be the constant in (1.3). If there exists \( B \in \mathcal{F} \) such that \( f^* \leq \chi_B \). Then there exists a positive constant \( C_{\alpha} \) independent of \( f \) and \( B \) such that
\[
(I_{\alpha} f)^* \leq C_{\alpha}\mathbb{P}(B)^\alpha\chi_B.
\]

For the proof of Lemma 6.2, see [19], Lemma 3.5.

In the next lemma, we regard \( (3, p, \infty) \)-atom \( a \) as a martingale by \( a = (a_n)_{n \geq 0} = (E_n(a))_{n \geq 0} \), so we can consider the fractional integral \( I_{\alpha} a = \left((I_{\alpha} a)_n\right)_{n \geq 0} \).
Lemma 6.3. Let \( \{F_n\}_{n \geq 0} \) be regular and \( R \) be the constant in (1.3). If \( 0 < p_1 < p_2 < \infty, \alpha = \frac{1}{p_1} - \frac{1}{p_2}, 0 < q_2 \leq \infty, \) and \( a \) is a \((3, p_1, \infty)\)-atom as in Definition 2.2. Then we have

\[
||I_a a||_{H^*_{p_2, q_2}} \leq C_\alpha,
\]

where \( C_\alpha \) is the same constant as in Lemma 6.2.

Proof. Let \( \nu \) be the stopping time associated with \( a \). Then we have \( a^* \leq \mathbb{P}(\nu < \infty)^{-\frac{1}{p_1}} \chi_{\{\nu < \infty\}} \). Therefore \((\mathbb{P}(\nu < \infty)^{-\frac{1}{p_1}} a)^* = \mathbb{P}(\nu < \infty)^{-\frac{1}{p_1}} a \leq \chi_{\{\nu < \infty\}} \). By Lemma 6.2 we can obtain \((I_a(\mathbb{P}(\nu < \infty)^{-\frac{1}{p_1}} a))^* \leq C_\alpha \mathbb{P}(\nu < \infty)^\alpha \chi_{\{\nu < \infty\}} \). Then

\[
(I_a a)^* \leq C_\alpha \mathbb{P}(\nu < \infty)^\alpha \mathbb{P}(\nu < \infty)^{-\frac{1}{p_1}} \chi_{\{\nu < \infty\}} = C_\alpha \mathbb{P}(\nu < \infty)^{-\frac{1}{p_2}} \chi_{\{\nu < \infty\}}.
\]

By Proposition 2.1, we have

\[
\mu_t((I_a a)^*) \leq \mu_t(C_\alpha \mathbb{P}(\nu < \infty)^{-\frac{1}{p_2}} \chi_{\{\nu < \infty\}}) = C_\alpha \mathbb{P}(\nu < \infty)^{-\frac{1}{p_2}} \chi_{\{0,\mathbb{P}(\nu < \infty)\}}(t).
\]

For \( 0 < q_2 < \infty \), then

\[
||I_a a||_{H^*_{p_2, q_2}}^q = ||(I_a a)^*||_{p_2, q_2}^q = \frac{q_2}{p_2} \int_0^\infty \left( \mu_t((I_a a)^*) \right)^{\frac{q_2}{p_2}} dt 
\leq \frac{q_2}{p_2} \int_0^\infty t^{\frac{q_2}{p_2} - 1} \left( C_\alpha \mathbb{P}(\nu < \infty)^{-\frac{1}{p_2}} \chi_{\{0,\mathbb{P}(\nu < \infty)\}}(t) \right)^{\frac{q_2}{p_2}} dt 
= \frac{q_2}{p_2} \int_0^{\mathbb{P}(\nu < \infty)} t^{\frac{q_2}{p_2} - 1} C_\alpha^{q_2} \mathbb{P}(\nu < \infty)^{-\frac{q_2}{p_2}} dt 
= C_\alpha^{q_2}.
\]

For \( q_2 = \infty \), then

\[
||I_a a||_{H^*_{p_2, \infty}} = ||(I_a a)^*||_{p_2, \infty} = \sup_{t > 0} t^{\frac{1}{p_2}} \mu_t((I_a a)^*) 
\leq \sup_{t > 0} C_\alpha t^{\frac{1}{p_2}} \mathbb{P}(\nu < \infty)^{-\frac{1}{p_2}} \chi_{\{0,\mathbb{P}(\nu < \infty)\}}(t) 
= C_\alpha.
\]

Therefore \( ||I_a a||_{H^*_{p_2, q_2}} \leq C_\alpha \), where \( C_\alpha \) is the same constant as in Lemma 6.2. The proof of is complete.

Theorem 6.4. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete and nonatomic probability space, and \( \{F_n\}_{n \geq 0} \) be a regular stochastic basis, let \( 0 < q_1 \leq 1, q_2 \leq q_1 \leq p_2, 0 < p_1 < p_2 < \infty, \alpha = \frac{1}{p_1} - \frac{1}{p_2} \), then there exists a constant \( C \) such that

\[
||I_a f||_{H^*_{p_2, q_2}} \leq C ||f||_{H^*_{p_1, q_1}},
\]

for all \( f \in H^*_{p_1, q_1} \).
Proof. For $f \in H^*_{p_1,q_1}$. Since $\{F_n\}_{n \geq 0}$ is regular, by Theorem 3.3 and Lemma 5.1, there exists a sequence $(a_k^k)_{k \in \mathbb{Z}}$ of $(3,p_1,\infty)$-atoms and and a real number sequence $(\mu_k)_{k \in \mathbb{Z}} \in l_{q_1}$ such that

$$f_n = \sum_{k \in \mathbb{Z}} \mu_k a_n^k, \quad (n \in \mathbb{N}),$$

and

$$\|(\mu_k)_{k \in \mathbb{Z}}\|_{l_{q_1}} \leq C \|f\|_{H^*_{p_1,q_1}}.$$ 

Then by Lemma 6.3, we have

$$\|I_\alpha f\|_{q_1}^{q_1} = \|(I_\alpha f)^*\|_{p_2,q_2} = \|(I_\alpha \sum_{k \in \mathbb{Z}} \mu_k a_k^k)^*\|_{p_2,q_2}$$

$$\leq \sum_{k \in \mathbb{Z}} |\mu_k| \|(I_\alpha a_k^k)^*\|_{p_2,q_2} \leq C \sum_{k \in \mathbb{Z}} |\mu_k| \|(I_\alpha a_k^k)^*\|_{p_2,q_1}$$

$$\leq C \sum_{k \in \mathbb{Z}} q_1 \|(I_\alpha a_k^k)^*\|_{p_2,q_1} \leq C \cdot C_{q_1}^{q_1} \|f\|_{H^*_{p_1,q_1}}.$$ 

Thus we have

$$\|I_\alpha f\|_{H^*_{p_2,q_2}} \leq C \|f\|_{H^*_{p_1,q_1}}.$$ 

The proof of the theorem is complete.

Remark 6.5. In Theorem 6.4, if we consider the special case $p_1 = q_1 = p, p_2 = q_2 = q$, then we obtain the boundedness of fractional integrals on martingale Hardy spaces for $0 < p \leq 1$, Theorem 3.1 in [19] due to Sadasue.

References

[1] W.Abu-Shammala, A.Torchinsky. The Hardy-Lorentz Spaces $H^{p,q}(\mathbb{R}^n)$. Studia Math, 182: 283-294, 2007.

[2] R.F.Bass. Probabilistic techniques in analysis. Springer-Verlag, New York, 1995.

[3] C.Bennett, R.Sharpley. Interpolation of operators. Academic Press, New York, 1988.

[4] J-A.Chao, H.Ombe. Commutators on Dyadic Martingales. Proc. Japan Acad., 61, Ser. A:35-38, 1985.

[5] J.B.Conway. A Course in Functional Analysis. Springer-Verlag, New York, 1985.

[6] R.Fefferman. Characterization of bounded mean oscillation. Bull. Amer. Math. Soc. 77:587-588, 1971.
[7] R.Fefferman. Bounded mean oscillation on the polydisk. Ann. of Math. 110:395-406, 1979.

[8] A.M.Garsia. Martingale Inequalities, Seminar Notes on Recent Progress. Math. Lecture Notes Series, 1973.

[9] L.Grafakos. Classical Fourier Analysis, Second Edition. Springer, New York, 2008.

[10] Y.L.Hou, Y.B.Ren. Weak martingale Hardy spaces and weak atomic decompositions. Science in China: Series A Mathematics, Vol.49 (7):912-921, 2006.

[11] Y.Jiao, L.H.Peng, P.D.Liu. Atomic decompositions of Lorentz martingale spaces and applications. J.Funct.Spaces Appl, 7(2):153-166, 2009.

[12] F.John, L.Nirenberg. On functions of bounded mean oscillation. Comm. Pure Appl. Math. 14:415-426, 1961.

[13] P.D. Liu, Y.L Hou. Atomic decomposition of Banach-space-valued martingales. Science in China: Series A Mathematics 42:38-47, 1999.

[14] R.L.Long. Martingale Spaces and Inequalities. Beijing, Peking University Press, 1993.

[15] T.Ma, P.D.Liu. Atomic decompositons and duals of weak Hardy spaces of B-valued martingales. Acta Mathematica Scientia,29B(5):1439-1452, 2009.

[16] T.Miyamoto, E.Nakai, G.Sadasue. Martingale Orlicz-Hardy spaces. Math.Nachr. 285: 670-686, 2012.

[17] E.Nakai, G.Sadasue. Martingale Morrey-Campanato spaces and fractional integrals. Journal of Functional Spaces and Application. Article ID 673929, 29 pages (doi:10.1155/2012/673929), 2012.

[18] K.E.Petersen. Brownian motion, Hardy spaces and bounded mean oscillation. London Math. Soc. Lecture Notes Series 28, Cambridge Univ. Press, Cambridge, 1977.

[19] G.Sadasue. Fractional integrals on martingale Hardy spaces for $0 < p \leq 1$. Memoirs of Osaka Kyoiku University. Ser.III. Vol.60(1):1-7, 2011.

[20] E.M.Stein. Singular Integrals and Differentiability Properties of Functions. Princeton University Press, Princeton, 1970.

[21] F.Weisz. Martingale Hardy spaces for $0 < p \leq 1$. Probab. Theory Related Fields. 84:361-376, 1990.

[22] F.Weisz. Martingale Hardy Spaces and their Applications in Fourier Analysis. Lecture Notes in Mathematics, Vol 1568. Springer-Verlag, Berlin, Heidelberg, 1994.