THE CALKIN ALGEBRA IS $\aleph_1$-UNIVERSAL

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Abstract. We discuss the existence of (injectively) universal C$^*$-algebras and prove that all C$^*$-algebras of density character $\aleph_1$ embed into the Calkin algebra, $\mathcal{Q}(H)$. Together with standard results, this shows that each of the following three assertions is relatively consistent with ZFC: (i) $\mathcal{Q}(H)$ is a $2^{\aleph_0}$-universal C$^*$-algebra. (ii) There exists a $2^{\aleph_0}$-universal C$^*$-algebra, but $\mathcal{Q}(H)$ is not $2^{\aleph_0}$-universal. (iii) A $2^{\aleph_0}$-universal C$^*$-algebra does not exist. We also prove that it is relatively consistent with ZFC that there is no $2^{\aleph_0}$-universal nuclear C$^*$-algebra.

1. Introduction

Let $H$ denote the separable infinite-dimensional complex Hilbert space. The Calkin algebra $\mathcal{Q}(H)$ is the quotient $\mathcal{B}(H)/\mathcal{K}(H)$ of the algebra $\mathcal{B}(H)$ of all bounded linear operators on $H$ over the ideal of all compact operators.

Given a category $C$ of metric structures and a cardinal $\kappa$, an object $A \in C$ is (injectively) $\kappa$-universal if it has density character $\kappa$ and every object $B \in C$ of density character at most $\kappa$ is isometric to a substructure of $A$. The dual notion, surjective universality, is trivialized in the category of unital C$^*$-algebras. Since every unital C$^*$-algebra is generated by its unitary group, the full group C$^*$-algebra associated with the free group $F_\kappa$, $C^*(F_\kappa)$, is surjectively $\kappa$-universal for every infinite cardinal $\kappa$. Similarly in the abelian setting $C([0,1]^\kappa)$ is surjectively $\kappa$-universal.

The question whether the Calkin algebra can be $\aleph_1$-universal for the category of C$^*$-algebras was raised by Piotr Koszmider (personal correspondence) and [28, Question E].

Theorem A. All C$^*$-algebras of density character at most $\aleph_1$ embed into the Calkin algebra. Therefore the Continuum Hypothesis (CH) implies that the Calkin algebra is an $\aleph_1$-universal C$^*$-algebra.

One of the ingredients of our proof is the analysis of the Ext$^w$-group of simple, separable, and unital C$^*$-algebras that tensorially absorb the Cuntz algebra $O_2$. 
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2. The proof of Theorem A

This section is entirely devoted to the proof of Theorem A. Familiarity with model theory, in particular axiomatizability and the different layers of saturation, is required (see [10], or [9] for an overview of the concept of saturation). For information on C*-algebras see [2] and for analytic K-homology see [16].

The main technical difficulty in the proof of Theorem A is posed by the absence of reasonable saturation properties in the Calkin algebra. The simplest instance of this is the fact that the image of the unilateral shift in \( \mathcal{B}(H) \) is a unitary with full spectrum and no square root. As pointed out in the introduction to [25], this implies that \( \mathcal{B}(H) \) is not injective (in a categorical sense) for separable C*-algebras and complicates construction of outer automorphisms of \( \mathcal{B}(H) \). More sophisticated obstructions to saturation in \( \mathcal{B}(H) \) were exhibited in [9, §4] and [13].

A unital C*-algebra \( A \) is purely infinite and simple if it is infinite dimensional and for every nonzero positive \( a \in A \) there is \( x \in A \) such that \( xax^* = 1 \). The Cuntz algebra \( \mathcal{O}_2 \) is the universal C*-algebra generated by two isometries \( s \) and \( t \) satisfying

\[
s^*s = t^*t = 1 = ss^* + tt^*.
\]

Let

\[
\mathcal{O} = \{ A : A \text{ is purely infinite, simple, and } A \otimes \mathcal{O}_2 \cong A \}.
\]

(Since \( \mathcal{O}_2 \) is nuclear, there is no ambiguity in what tensor product is used. In this case there is a unique C*-norm on the algebraic tensor product.)

Lemma 2.1. Every C*-algebra \( A \) embeds into a C*-algebra \( B \in \mathcal{O} \) of the same density character as \( A \). If \( A \) is unital then the embedding can be chosen to be unital.

Proof. The class \( \mathcal{O} \) is separably axiomatizable by ([10, Theorem 2.5.1 and Theorem 2.5.2]). We first consider the case when \( A \) is separable. Then \( A \) is isomorphic to a subalgebra of \( \mathcal{B}(H) \), and the embedding can be chosen to be unital if \( A \) is unital. By the downward Löwenheim–Skolem theorem ([10, Theorem 2.6.2]) we can find a separable elementary submodel \( C \) of \( \mathcal{B}(H) \) into which \( A \) embeds. Then \( C \) is simple and purely infinite, and \( C \otimes \mathcal{O}_2 \) is as required.
Now suppose $A$ is not separable and let $\kappa$ be its density character. Again by the downward Löwenheim–Skolem theorem we can find a separable elementary submodel $A_0$ of $A$. By the first paragraph, we can find a separable $B_0 \in \mathfrak{O}$ into which $A_0$ embeds. By the standard elementary chain argument ([1] Proposition 7.10) we construct a $\kappa$-saturated elementary extension $B_1$ of $B_0$. Writing $A$ as a union of an elementary chain of submodels of density character $< \kappa$ and using the saturation of $B_1$, we can embed $A$ into $B_1$. Being elementarily equivalent to $B_0$, $B_1$ is purely infinite and simple. $B_1$ is not $O_2$-stable, essentially by [14], however $B = B_1 \otimes O_2$ satisfies all requirements.

An alternative (and more natural from the model-theoretic point of view) approach to the proof of Lemma [21] uses the axiomatizable class $\mathfrak{O}' := \{A: A$ is purely infinite, simple, and potentially $O_2$-absorbing} (see [11]) in place of $\mathfrak{O}$.

We shall need semigroups $\text{Ext}^w(A)$ and $\text{Ext}^w_{cpc}(A)$ associated to a separable and unital $C^*$-algebra $A$. An injective unital $^*$-homomorphism $\pi: A \to \mathcal{B}(H)$ is the Busby invariant of an extension of $A$ by $\mathcal{K}(H)$. By a slight abuse of terminology, we say that such $^*$-homomorphism is an extension (see [16] Proposition 2.6.3)). Two extensions $\theta_j: A \to \mathcal{B}(H)$, for $j = 1, 2$ are weakly equivalent if there is a unitary $u \in \mathcal{B}(H)$ such that $\theta_1 = \text{Ad } u \circ \theta_2$.

Since $\mathcal{M}_2(\mathcal{B}(H)) \cong \mathcal{B}(H)$, the set of extensions of $A$ is equipped with the direct sum operation. The set of weak equivalence classes of extensions of $A$ forms a semigroup, denoted $\text{Ext}^w(A)$. An extension $\theta: A \to \mathcal{B}(H)$ is semisplit if there exists a completely positive contraction (c.p.c.) $\varphi: A \to \mathcal{B}(H)$ such that (denoting the quotient map from $\mathcal{B}(H)$ onto $\mathcal{B}(H)$ by $\pi$) $\pi \circ \varphi = \theta$. If $\varphi$ is a unital $^*$-homomorphism then we say that $\theta$ is split. A split extension exists when $A$ is separable, and Voiculescu’s theorem ([16 Theorem 3.4.7]) implies that it acts as the unit in $\text{Ext}^w(A)$. Let

$$\text{Ext}^w_{cpc}(A) := \{\theta \in \text{Ext}^w(A) : \theta \text{ is semisplit}\}.$$  

Stinespring’s theorem ([2 Theorem II.6.9.7]) easily implies that $\text{Ext}^w_{cpc}(A) = \text{Ext}^w(A)^{-1}$, the group of all invertible elements of $\text{Ext}^w(A)$.

The following is a standard application of quasicentral approximate units.

**Lemma 2.2.** Suppose that a separable $C^*$-algebra $A$ is an inductive limit of subalgebras $A_n$, for $n \in \mathbb{N}$. If $\theta: A \to \mathcal{B}(H)$ is an extension such that its restriction to $A_n$ is semisplit for every $n$, then $\theta$ is semisplit.

**Proof.** Let $\delta_n > 0$ be small enough so that for all operators $e$ and $a$ satisfying $0 \leq e \leq 1$, $\|a\| \leq 1$, and $\|[e, a]\| < \delta_n$ we have $\| e^{1/2} a \| < 2^{-n}$. Let $\psi_n$ be a u.c.p. lift for $\theta \upharpoonright A_n$. By the Arveson Extension Theorem ([2 Theorem II.6.9.12]) we can extend $\psi_n$ to a c.p.c. map $\tilde{\psi}_n: A \to \mathcal{B}(H)$. Let $E = \pi^{-1}(\pi(A))$ and let $a_n$, for $n \in \mathbb{N}$, be an enumeration of a dense subset of the unit ball of $A$ whose intersection with the unit ball of $A_n$ is dense for all $n$. By [16 Proposition 3.2.8] we can find a sequence $f_n$, for $n \in \mathbb{N}$,
which is an approximate identity for \( \mathcal{K}(H) \) that is quasicentral for \( E \). By refining this sequence, we may assume that the following conditions hold for all \( i, j, k, \) and \( n \) with \( i, j, k \leq n \).

1. \( \| f_n, \tilde{\psi}_t(a_j) \| < \delta_n \).
2. \( \| (1 - f_n)(\psi_t(a_j) - \tilde{\psi}_t(a_j)) \| < 2^{-n} \), if \( a_j \in A_i \cap A_k \).

The second condition can be assured because the assumptions imply \( \psi_t(a_j) - \tilde{\psi}_t(a_j) \) is compact, and the first condition can be assured by the quasicentrality of the sequence.

Given these conditions we have \( \| ([f_{n+1} - f_n]^{1/2}, \tilde{\psi}_{n}(a_j)] \| < 2^{-n} \) for all \( j \leq n \) and

\[
\psi(a) = \sum_n (f_{n+1} - f_n)^{1/2} \tilde{\psi}_n(a_j)(f_{n+1} - f_n)^{1/2}
\]

is well-defined since the finite partial sums converge in the strong operator topology. Since every partial sum is c.p.c., \( \psi \) is c.p.c. For \( a_j \in A_t \) we also have that \( \psi(a_j) - \tilde{\psi}(a) \) is compact and therefore \( \psi \) is a c.p.c. lift of \( \theta \) as required. \( \square \)

**Proposition 2.3.** Suppose \( A \in \mathbb{D} \) is separable. Then \( \text{Ext}_c^{\text{cpc}}(A) = 0 \)

Proposition 2.3 is proved by putting together several known (and deep) results. An endomorphism \( \phi \) of a \( C^* \)-algebra \( A \) is \emph{asymptotically inner} if there exists a continuous path of unitaries \( u_t \), for \( 0 \leq t < \infty \), such that \( u_0 = 1 \) and \( \phi(a) = \lim_{t \to \infty} (\text{Ad} u_t)a \) for all \( a \in A \).

**Lemma 2.4.** Suppose \( A \cong A \otimes O_2 \) and let \( s \) and \( t \) be the generators of \( O_2 \). Then the endomorphism \( \zeta(a) = (1 \otimes s)a(1 \otimes s^*) + (1 \otimes t)a(1 \otimes t^*) \) of \( A \) is asymptotically inner.

**Proof.** It is well-known that every endomorphism of \( O_2 \) is asymptotically inner. This is a consequence of [24] Lemma 2.2.1: Take \( D = O_2 \), \( m = 2 \), \( \phi = \text{id} \), and note that the assumption on the \( K_1 \)-class of \( u_0 \) is automatic since \( O_2 \) has trivial \( K \)-theory. Since the endomorphism \( \zeta_0(a) = sa^* + ta^* \) of \( O_2 \) is asymptotically inner, so is \( \zeta = \text{id} \otimes \zeta_0 \). \( \square \)

**Proof of Proposition 2.3.** Since \( \text{Ext}^{\text{cpc}}(A) \) is a group, it suffices to prove that each of its elements is idempotent. This follows from a deep theorem of Kasparov, modulo a reformulation of the problem.

If \( \theta : A \to \mathcal{B}(H) \) is a unital representation of \( A \) which is ample (i.e. \( \theta(A) \) has zero intersection with the ideal of compact operators), then the dual algebra of \( A \) is \( \mathcal{D}(A) = \{ b \in \mathcal{B}(H) : [\theta(a), b] \in \mathcal{K}(H) \text{ for all } a \in A \} \) ([16] Definition 5.1.1). Then \( K_0(\mathcal{D}(A)) \cong \text{Ext}_c^{\text{cpc}}(A) \) (this is essentially [16] Proposition 5.1.4 or—modulo passing to \( cpc \mathcal{B}(H) \)—[13] Lemma 3). The group \( K_0(\mathcal{D}(A)) \) (also known as \( K^1(A) \)) is isomorphic to the Kasparov group \( KK^1(A) \) by [16] Theorem 8.4.3].
The semigroup of endomorphisms of $A$ acts on $K^1(A)$ by composition: if $\zeta: A \to A$ and $\theta: A \to \mathcal{F}(A)$ is an extension, then $\zeta \circ \theta$ is an extension of $A$. A theorem of Kasparov ([16, Theorem 9.3.3]) implies that homotopic endomorphisms of $A$ induce the same map on $K^1(A)$. Lemma 2.3 thus implies that $\theta$ and $\theta \oplus \theta$ are equivalent for every $\theta \in \text{Ext}^w_{\text{cpc}}(A)$; this completes the proof.

We are now ready to prove Theorem A. By Lemma 2.1 it suffices to prove that every limit ordinal $\delta < \aleph_1$ for separable $C^*$-algebras ([10, Theorem 2.5.1 and Theorem 2.5.2]).

We want to find extensions $\varphi_\alpha \in \text{Ext}^w_{\text{cpc}}(A_\alpha)$ such for all $\alpha < \beta < \aleph_1$ we have

$$\varphi_\alpha \in \text{Ext}^w_{\text{cpc}}(A_\alpha) \text{ and } \varphi_\beta | A_\alpha = \varphi_\alpha.$$ Choose $\varphi_0 \in \text{Ext}^w_{\text{cpc}}(A_0)$. Suppose $\varphi_\alpha$ has been defined for all $\alpha < \beta$. If $\beta$ is a successor ordinal, let $\alpha$ be such that $\alpha + 1 = \beta$. Fix $\psi \in \text{Ext}^w_{\text{cpc}}(A_\beta)$. Then Proposition 2.3 implies that both $\psi' := \psi | A_\alpha$ and $\varphi_\alpha$ are split. By Voiculescu’s theorem ([16, Theorem 3.4.7]) there exists a unitary $u \in \mathcal{F}(H)$ such that $\varphi_\alpha = \text{Ad} u \circ \psi'$ and $\varphi_\beta = \text{Ad} u \circ \psi$ is as required.

Now suppose $\beta$ is a limit ordinal. Then $\varphi_\beta$ is already defined on a dense subalgebra $\bigcup_{\alpha < \beta} A_\alpha$ of $A_\beta$, and Lemma 2.2 implies that its continuous extension to $A$ is semisplit.

This describes the recursive construction. Since $A = \bigcup_{\alpha < \aleph_1} A_\alpha$, for all $a \in A$ the ordinal $\alpha(a) = \min\{\alpha: a \in A_\alpha\}$ is well-defined. Then $\Phi(a) = \varphi_{\alpha(a)}(a)$ extends each $\varphi_\alpha$, provides the desired embedding of $A$ into $\mathcal{F}(H)$, and completes the proof of Theorem A.

3. Corollaries and related results

Corollary 3.1. Each of the following assertions is relatively consistent with ZFC:

1. $\mathcal{F}(H)$ is a $2^{\aleph_0}$-universal $C^*$-algebra.
2. There exists a $2^{\aleph_0}$-universal $C^*$-algebra, but $\mathcal{F}(H)$ is not $2^{\aleph_0}$-universal.
3. A $2^{\aleph_0}$-universal $C^*$-algebra does not exist.

Proof. (1) Assume CH. Theorem A implies that $\mathcal{F}(H)$ is $2^{\aleph_0}$-universal.

(2) We shall prove that the Proper Forcing Axiom, PFA, implies the conclusion. If $2^\kappa = 2^{\aleph_0}$ for all $\kappa < 2^{\aleph_0}$, then [1] Proposition 7.10] implies that the theory of $\mathcal{F}(H)$ has a saturated model $B$ of density character $2^{\aleph_0}$. By Lemma 2.1 (and its proof), such $B$ is a $2^{\aleph_0}$-universal $C^*$-algebra. It is well-known that PFA implies $2^{\aleph_1} = 2^{\aleph_0} = \aleph_2$ (e.g. see [20]).

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2 Readers concerned with the consistency strength issues may rest assured that only a small fragment of PFA with zero large cardinal strength is required in [28].
By [25] Corollary 5.3.14 and Theorem 5.3.15 (also [22]) PFA implies that there exist a closed subset $X$ of $\beta \mathbb{N} \setminus \mathbb{N}$ such that $C(X)$ does not embed into the Calkin algebra. Such an $X$ can be chosen as follows. Let $Z_0 = \{S \subseteq \mathbb{N} : \lim_{n \to \infty} |S \cap n|/n = 0\}$, the ideal of asymptotic density zero sets. (Any other dense analytic P-ideal would do in place of $Z_0$; see [28].) Identifying $\beta \mathbb{N}$ with the set of all ultrafilters on $\mathbb{N}$, we may let $X = \{U \in \beta \mathbb{N} : U \cap Z_0 = \{\emptyset\}\}$. By standard forcing techniques ([20]) the assertion $\aleph_2^\mathbb{N} < 2^\aleph_0 < 2^{\aleph_1}$ is relatively consistent with ZFC. (For example, start from a model of GCH, add $\aleph_4$ Cohen subsets of $\aleph_1$, then add $\aleph_3$ Cohen reals.) Therefore $\kappa = 2^{\aleph_0}$ and $\lambda = \aleph_1$ satisfy the following inequality ($\lambda^+$ denotes the least cardinal greater than $\lambda$):

$$\lambda^+ < \kappa < 2^\lambda.$$  

If such $\lambda$ exists then $\kappa$ is said to be far from GCH ([19]). We shall prove that (3) and $\kappa^{\aleph_0} = \kappa$ together imply that there is no $\kappa$-universal C$^*$-algebra.

By [6, §3.3] every Banach space embeds isometrically into an operator space (and therefore into a C$^*$-algebra) of the same density character. Hence a $\kappa$-universal C$^*$-algebra would also be a $\kappa$-universal Banach space. But, by [27, Corollary 2.4], (3) implies that there is no (isometrically) $\kappa$-universal Banach space, and this concludes the proof.

**Remark 3.2.** A sketch of an alternative, self-contained (and, we believe, more informative) proof that the condition (3) in Corollary 3.1 together with $\kappa^{\aleph_0} = \kappa$ implies there is no $\kappa$-universal C$^*$-algebra is in order. Instead of using [27, Corollary 2.4], we follow the lines of its proof. By [19, Theorem 3.10], (3) implies that there is no $\kappa$-universal linear order, and moreover that any theory $T$ with the order property does not have a $2^{\aleph_0}$-universal model. As in [12, Lemma 5.3], consider the following condition in the language of C$^*$-algebras

$$\varphi(x, y) = \max(|1 - \|x\|, |1 - |y|, \|xy - y\|).$$

For $x$ and $y$ in a C$^*$-algebra $A$, $\varphi(x^*x, y^*y) = 0$ if and only $\|x\| = \|y\| = 1$ and in the second dual $A^{**}$ of $A$ the support projection of $y^*y$ is below the spectral projection of $x^*x$ corresponding to 1. Therefore $\varphi(x, y)$ defines a partial order, $\preceq_\varphi$, on $A$. Every infinite-dimensional C$^*$-algebra contains an infinite $\preceq_\varphi$ chain (consider a masa or see [12, Lemma 5.3]). Thus every infinite-dimensional C$^*$-algebra has the Strict Order Property ([27, Definition 1.1]). Since $\varphi$ is quantifier-free, an embedding of $A$ into $B$ is an embedding of the poset $(A, \preceq_\varphi)$ into $(B, \preceq_\varphi)$. Hence if $C$ is a $\kappa$-universal C$^*$-algebra for some cardinal $\kappa$, then every linear ordering of cardinality $\kappa$ embeds into the linearization of $(C, \preceq_\varphi)$. Hence the latter is a $\kappa$-universal linear ordering.

By [18], $\mathcal{O}_2$ is an $\aleph_0$-universal nuclear (and even exact) C$^*$-algebra.

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\footnote{See Remark 3.2 for a sketch of a self-contained proof}
Corollary 3.3. If $2^{\aleph_0}$ is far from GCH then there is no $2^{\aleph_0}$-universal nuclear C*-algebra.

Proof. By Remark 3.2 the theory of $C([0,1])$ has the Strict Order Property witnessed by a quantifier-free formula. Since every abelian C*-algebra is nuclear (and being abelian is axiomatizable, [10] Theorem 2.5.1), the corollary follows by the argument of Remark 3.2. □

Although CH implies that there exists $\aleph_1$-universal linear ordering, and the existence of an $\aleph_1$-universal linear ordering is even relatively consistent with the negation of CH ([26]), it is not clear whether the existence of an $\aleph_1$-universal nuclear C*-algebra is relatively consistent with ZFC.

We could not find a proof of the following lemma in the literature; it is included for the reader’s convenience.

Lemma 3.4. Suppose $A$ is a C*-algebra and $B$ is a C*-subalgebra of $A$ that contains an approximate unit for $A$. Then the inclusion from $B$ into $A$ extends to an injection from $\mathcal{M}(B)$ into $\mathcal{M}(A)$, and $\mathcal{M}(B)/B$ is isomorphic to a subalgebra of $\mathcal{M}(A)/A$.

Proof. We can identify $A$ with a nondegenerate subalgebra of $\mathcal{B}(H)$ and $\mathcal{M}(A)$ with the idealizer of $A$ in $\mathcal{B}(H)$, $\{c \in \mathcal{B}(H) : cA \subseteq A\}$ ([2], II.7.3.5)]. Since $B$ has an approximate unit for $A$ it is also nondegenerate in $\mathcal{B}(H)$ and $\mathcal{M}(B)$ can be identified with the idealizer of $B$ in $\mathcal{B}(H)$. Fix an approximate unit $(e_\lambda)$ for $A$ included in $B$. If $c \in \mathcal{M}(B)$ and $a \in A$, then $ca = \lim_\lambda ce_\lambda a$. Since $ce_\lambda \in B$ for all $\lambda$, $ca$ is a limit of a Cauchy net in $A$ and therefore in $A$. Similarly $ac \in A$, and since $c \in \mathcal{M}(B)$ was arbitrary we have $\mathcal{M}(B) \subseteq \mathcal{M}(A)$. Since $\mathcal{M}(B) \cap A = B$, $\mathcal{M}(B)/B$ is isomorphic to a subalgebra of $\mathcal{M}(A)/A$. □

From Theorem [A] and Lemma 3.3 we immediately have the following.

Corollary 3.5. Let $A$ be a unital separable C*-algebra. If $\mathcal{D}(H)$ is $2^{\aleph_0}$-universal, then so is the corona of $A \otimes \mathcal{K}(H)$. □

We record an easy consequence of a trick first used in [24, Theorem 4.3.11].

Proposition 3.6. If $\kappa < 2^{\aleph_0}$, there is no $\kappa$-universal C*-algebra.

Proof. This follows from [17] Theorem 2.3 and Remark 2.10. The space OS$_3$ of three-dimensional operator spaces can be equipped by a metric $\delta_{cb}$ such that the space of all operator spaces that embed into a C*-algebra $A$ has density at most equal to the density character of $A$ ([17] Proposition 2.6(a))], and the space $(\text{OS}_3, \delta_{cb})$ has density character $2^{\aleph_0}$. □

This is analogous to the fact that the space $D(T)$ of quantifier-free types in models of theory $T$ has density $2^{\aleph_0}$ whenever it is nonseparable; see [19].
4. Remarks on universality in related categories

**Isomorphic embeddings of Banach spaces.** Theorem [A] was inspired by [3, Theorem 1.4], where the analogous statement for $2^{\aleph_0}$-universal Banach spaces was proved. Brech and Koszmider constructed a forcing extension in which an isometrically $2^{\aleph_0}$-Banach space exists, but $\ell_\infty/c_0$ is not isometrically, or even isomorphically, $2^{\aleph_0}$-universal Banach space. The result of [27, Corollary 2.4] used in the proof of Corollary 3.1 was improved in [3, Theorem 1.3], where it was proved that consistently there is no isomorphically $2^{\aleph_0}$-universal Banach space.

**Linear orders.** The existence of universal linear orders is a well-studied subject ([19]). Much attention has been devoted to the question of $2^{\aleph_0}$-universality of $\mathcal{P}(\mathbb{N})/\text{Fin}$. Since the Calkin algebra is its noncommutative analogue (see e.g. [29]), we shall concentrate on the role of $\mathcal{P}(\mathbb{N})/\text{Fin}$. While it is consistent that CH fails and $\mathcal{P}(\mathbb{N})/\text{Fin}$ is $2^{\aleph_0}$-universal ([21]), it is not clear whether the assertion `$\mathcal{L}(H)$ is a $2^{\aleph_0}$-universal $C^*$-algebra' is relatively consistent with the failure of CH. Even the (probably much easier) problem, whether for a given $C^*$-algebra $A$ there exists a ccc forcing notion (see [20, §III.2]) that forces an embedding of $A$ into $\mathcal{L}(H)$ appears to be nontrivial. Notably, the structure of the small category of linear orders that embed into $\mathcal{P}(\mathbb{N})/\text{Fin}$ is remarkably malleable in ZFC (see [7, §1]) and very rigid if a fragment of PFA holds ([8]).

**Surjective universality for compact Hausdorff spaces.** A compact Hausdorff space $X$ is said to be $\kappa$-universal if it is surjectively universal among compact Hausdorff spaces of weight $\kappa$. By Gelfand–Naimark duality, this is equivalent to $C(X)$ being an injectively universal unital abelian $C^*$-algebra. CH implies that $\beta\mathbb{N}\setminus\mathbb{N}$ is an $\aleph_1$-universal compact Hausdorff space (Parovičenok’s theorem) and that $\beta\mathbb{R}_+\setminus\mathbb{R}_+$ is an $\aleph_1$-universal connected compact Hausdorff space ([5]). As in Corollary 3.1, PFA implies that $\beta\mathbb{N}\setminus\mathbb{N}$ is not $2^{\aleph_0}$-universal because it does not map onto the Stone space of the Lebesgue measure algebra ([4]).

**$\Pi_1$-factors.** In [23] it was proved that there is no $\kappa$-universal $\Pi_1$-factor for any $\kappa < 2^{\aleph_0}$. As in Corollary [3.1], $\kappa < \kappa = \kappa$ implies there is a $\kappa$-universal $\Pi_1$-factor. The theory of $\Pi_1$-factors has the Order Property ([12, Lemma 3.2]) but it is not known whether it has the Strict Order Property. If it does, the argument from Remark 3.2 would imply that the existence of a cardinal $\lambda$ such that $\lambda^+ < 2^{\aleph_0} < 2^\lambda$ implies there is no $2^{\aleph_0}$-universal $\Pi_1$-factor. As a curiosity, we note that Connes’ Embedding Problem has the positive solution if and only if the Continuum Hypothesis implies that an ultrapower of the hyperfinite $\Pi_1$-factor is a $2^{\aleph_0}$-universal $\Pi_1$-factor. Similarly, Kirchberg’s Embedding Problem ([15]) has the positive solution if and only if the Continuum Hypothesis implies that an ultrapower of $\mathcal{O}_2$ is $2^{\aleph_0}$-universal $C^*$-algebra.
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