Convergence properties of $\eta \rightarrow 3\pi$ decays in chiral perturbation theory

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Abstract

Theoretical efforts to describe and explain the $\eta \rightarrow 3\pi$ decays reach far back in time. Even today, the convergence of the decay widths and some of the Dalitz plot parameters seems problematic in low energy QCD. In the framework of resummed $\chi$PT, we explore the question of compatibility of experimental data with a reasonable convergence of a carefully defined chiral series, where NNLO remainders are assumed to be small. By treating the uncertainties in the higher orders statistically, we numerically generate a large set of theoretical predictions, which are then confronted with experimental information. In the case of the decay widths, the experimental values can be reconstructed for a reasonable range of the free parameters and thus no tension is observed, in spite of what some of the traditional calculations suggest. The Dalitz plot parameters $a$ and $d$ can be described very well too. When the parameters $b$ and $\alpha$ are concerned, we find a mild tension for the whole range of the free parameters, at less than 2$\sigma$ C.L. This can be interpreted in two ways - either some of the higher order corrections are indeed unexpectedly large or there is a specific configuration of the remainders, which is, however, not completely improbable. Also, the distribution of the theoretical uncertainties is found to be significantly non-gaussian, so the consistency cannot be simply judged by the 1$\sigma$ error bars.
1 Introduction

Theoretical efforts to describe and explain the $\eta \rightarrow 3\pi$ decays reach far back in time. From the very beginning it was known that this is an isospin breaking process, as three isovectors can constitute an isoscalar state only through the fully antisymmetric combination $\epsilon_{abc}\pi^a\pi^b\pi^c$, which together with Bose symmetry and charge conjugation invariance leads to zero contribution to the amplitude.

Initially, the process was considered to be of electromagnetic origin [1, 2], generated by the isospin breaking virtual photon exchange

$$H_{QED}(x) = -\frac{e^2}{2} \int dy D^{\mu\nu}(x - y) T(j_\mu(x)j_\nu(y)).$$

Although calculations applying current algebra and PCAC obtained correct order of magnitude values for the decay rates [1, 2], it was soon pointed out that the decays are almost forbidden in the framework of QED (the Sutherland theorem [3, 4]). The early works [1, 2] related the $\eta - \pi$ matrix elements to the difference of squared kaon masses or kaon and pion masses, respectively, in fact resembling the later Dashen’s theorem, which cannot be justified by electrodynamics [4]. Subsequently it became clear that there has to be a source of isospin breaking beyond the term (1) [5]. As we know now, strong interactions break isospin via the difference between the masses of the $u$ and $d$ quarks

$$H_{IB}^{QCD}(x) = \frac{m_d - m_u}{2}(\bar{d}(x)d(x) - \bar{u}(x)u(x)).$$

The work [5] collected all the relevant current algebra terms contributing to the decays and thus can be considered to be the first to provide the correct leading order calculation. However, the obtained decay rates turned out to be significantly lower than the experimental values, which were just becoming available.

When a systematic approach to low energy hadron physics was born in the form of chiral perturbation theory ($\chi$PT) [6, 7, 8], it was quickly applied to the $\eta \rightarrow 3\pi$ decays [9]. The one loop corrections were very sizable, the result for the decay width of the charged channel was $160\pm50$ eV, compared to the current algebra prediction of 66 eV. However, already at that time there were hints that the experimental value is still much larger. The current PDG value [10] is

$$\Gamma^+_{\exp} = 300 \pm 12 \text{ eV}. \quad (3)$$

For the neutral channel, the current average is [10]

$$\Gamma^0_{\exp} = 428 \pm 17 \text{ eV}. \quad (4)$$

After the effective theory was extended to include virtual photon exchange generated by [1] [11], it was shown that the next-to-leading electromagnetic corrections to the Sutherland's theorem are still very small [12, 13]. Recently it was argued that there is an indication this need not be true for the neutral channel [14], but that is a partial result which has not been finalized yet.

The theory thus seems to converge really slowly for the decays. At last, the two loop $\chi$PT calculation [15] has succeeded to provide a reasonable prediction for the decay widths.

Meanwhile, experimental data are being gathered with increasing precision in order to make more detailed analysis of the Dalitz plot distribution possible. Comparison of the
recent experimental information with the NNLO $\chi$PT results can be found in tables 1 and 2, the conventionally defined Dalitz plot parameters will be introduced in section 2. For the sake of brevity, we added the systematic and statistical uncertainties in squares. As can be seen, a tension between $\chi$PT and experiments appears to be in the charged decay parameter $b$ and the neutral decay parameter $\alpha$.

Alternative approaches were developed in order to model the amplitudes more precisely, namely dispersive approaches [32, 33, 31, 30] and non-relativistic effective field theory [34, 35, 21]. These more or less abandon strict equivalence to $\chi$PT and their success in reproducing a negative sign for $\alpha$ (see table 2) can serve as a motivation to ask what is the culprit of the failure of chiral perturbation theory to do so.

There is a long standing suspicion that chiral perturbation theory might possess slow or irregular convergence in the case of the three light quark flavours [36, 37], the $\eta \rightarrow 3\pi$ decay rates might serve as a prime example. An alternative method, now dubbed resummed $\chi$PT [38, 39], was developed in order to express these assumptions in terms of parameters and uncertainty bands. The starting point is the realization that the standard approach

| $\eta \rightarrow \pi^+ \pi^- \pi^0$ | $a$ | $b$ | $d$ | $f$ | $g$ |
|-------------------------------------|-----|-----|-----|-----|-----|
| Crystal Barrel ’98 [16]             | $-1.22 \pm 0.07$ | $0.22 \pm 0.11$ | $0.06$ (input) | | |
| KLOE ’08 [17]                      | $-1.090 \pm 0.020$ | $0.124 \pm 0.012$ | $0.057 \pm 0.017$ | $0.14 \pm 0.02$ | |
| KLOE ’16 [18]                      | $-1.095 \pm 0.004$ | $0.145 \pm 0.006$ | $0.081 \pm 0.007$ | $0.141 \pm 0.010$ | $-0.044 \pm 0.016$ |
| BESIII ’15 [19]                    | $-1.128 \pm 0.017$ | $0.153 \pm 0.017$ | $0.085 \pm 0.018$ | $0.173 \pm 0.035$ | |
| WASA at COSY ’14 [20]              | $-1.144 \pm 0.018$ | $0.219 \pm 0.042$ | $0.086 \pm 0.025$ | $0.115 \pm 0.037$ | |
| NREFT ’11 [21]                     | $-1.213 \pm 0.014$ | $0.308 \pm 0.023$ | $0.050 \pm 0.003$ | $0.083 \pm 0.019$ | $-0.039 \pm 0.002$ |
| NNLO $\chi$PT ’07 [15]             | $-1.271 \pm 0.075$ | $0.394 \pm 0.102$ | $0.055 \pm 0.057$ | $0.025 \pm 0.160$ | |

Table 1: Recent experimental and theoretical results for the charged channel.

| $\eta \rightarrow \pi^0 \pi^0 \pi^0$ | $\alpha$ |
|-------------------------------------|----------|
| Crystal Barrel ’98 [22]             | $-0.052 \pm 0.020$ |
| SND ’01 [23]                        | $-0.010 \pm 0.023$ |
| Crystal Ball ’01 [24]               | $-0.031 \pm 0.004$ |
| CELSIUS/WASA ’07 [25]               | $-0.026 \pm 0.014$ |
| WASA at COSY ’09 [26]               | $-0.027 \pm 0.009$ |
| Crystal Ball at MAMI-B ’09 [27]     | $-0.032 \pm 0.003$ |
| Crystal Ball at MAMI-C ’09 [28]     | $-0.0322 \pm 0.0025$ |
| KLOE ’10 [29]                       | $-0.0301 \pm 0.0050$ |
| PDG ’14 [10]                        | $-0.0315 \pm 0.0015$ |
| NREFT ’11 [21]                      | $-0.0246 \pm 0.0049$ |
| Prague disp.fit ’11 [30]            | $-0.044 \pm 0.004$ |
| Bern disp.fit ’11 [31]              | $-0.045 \pm 0.010$ |
| NNLO $\chi$PT ’07 [15]             | $+0.013 \pm 0.032$ |

Table 2: Recent experimental and theoretical results for the neutral channel.
to $\chi$PT, as a usual treatment of perturbation series, implicitly assumes good convergence properties and hides the uncertainties associated with a possible violation of this assumption. The resummed procedure uses the same standard $\chi$PT Lagrangian and power counting, but only expansions derived directly from the generating functional are trusted. All subsequent manipulations are carried out in a non-perturbative algebraic way. The expansion is done explicitly to next-to-leading order and higher orders are collected in remainders. These are not neglected, but retained as sources of error, which have to be estimated.

In this paper, we concentrate on the technical details of the resummed $\chi PT$ approach to $\eta \rightarrow 3\pi$ decays and provide a first look at numerical outputs of this formalism. Our goal is to use the resummed framework to analyze the problem from a theoretical point of view. We do not aim to produce an alternative set of predictions, but rather to understand whether the theory, by which we mean $\chi$PT as a low energy representation of QCD, really does have difficulties explaining the data. This is the aim for which we claim the formalism of resummed $\chi PT$ is well suited.

The results of this paper form a basis for further applications, which will follow in separate publications [40]. Namely, the resummed $\chi PT$ approach can be used as a tool for testing various scenarios of the QCD chiral symmetry breaking; preliminary results are already available in [41]. Also, by using complementary information both from inside and outside the $\chi PT$, we can try to address the source of the problem of irregularities of the chiral expansion (see [42] for first results).

The paper is organized as follows. In section 2, we fix our notation and provide a brief review of the kinematics of the decay. A concise summary of the methods of resummed $\chi PT$ is presented in section 3, while a more detailed discussion of the choice of safe observables, their properties and safe manipulation with them is postponed to section 4. The relation between the amplitude and the corresponding safe observables in the presence of $\eta - \pi^0$ mixing is given in section 5. Calculation of the mixing angles, an example of dangerous observables, is presented in section 6. Sections 7, 8 and 9 are devoted to successive steps of the calculations within resummed $\chi PT$, namely the strict expansion, the matching with dispersive representation and the reparameterization in terms of the masses and decay constants, respectively. In section 10, we comment on the treatment of free parameters and the role of the higher order remainders. Numerical results are provided in section 11 and we conclude with a summary in section 12.

An explicit form of the obtained formulae, as well as some other technical details, are postponed to appendices. In appendix A, we present the strict chiral expansion of all the relevant safe observables. Appendices B and C are devoted to the application of the reconstruction theorem to the $\eta \rightarrow 3\pi$ decays and to the matching of the strict expansion with the dispersive approach. Bare expansion of the safe observables under consideration and its reparameterization are summarized in appendices D and E. More detailed discussion of the mixing in resummed $\chi PT$ is presented in appendix F.

2 Notation and kinematics

The S-matrix element of the charged decay $\eta \rightarrow \pi^+\pi^-\pi^0$ can be expressed in terms of the invariant amplitude $A(s, t; u)$ as

$$\langle \pi^+(p_+)\pi^-(p_-)\pi^0(p_0); \text{out}|\eta(p); \text{in} \rangle = i(2\pi)^4\delta(p - p_+ - p_- - p_0)A(s, t; u).$$

(5)
The amplitude is a function of the Mandelstam variables

\[ s = (p - p_0)^2 = (p_+ + p_-)^2 \]
\[ t = (p - p_+)^2 = (p_0 + p_-)^2 \]
\[ u = (p - p_-)^2 = (p_0 + p_+)^2, \]

which satisfy the constraint

\[ s + t + u = 3s_0 = M_{\eta}^2 + 2M_{\pi^+}^2 + M_{\pi^0}^2. \]  

(7)

In what follows, we will work in the first order in the isospin breaking. We will thus not make a difference between the charge and neutral pion masses from now on, because their difference is of the second order in the isospin breaking. In this case, the isospin symmetry and charge conjugation invariance imply (we use the Condon-Shortley convention here)

\[ A(s, t; u) = A(s, u; t) \]
\[ \bar{A}(s, t; u) = -A(s, t; u) - A(t, s; u) - A(u, t; s), \]  

(9)

where \( \bar{A}(s, t; u) \) is the neutral channel amplitude. We can therefore restrict ourselves to the investigation of the charged decay mode only\(^1\).

The Mandelstam variables are bounded as follows

\[ 4M_{\pi}^2 \leq s, t, u \leq (M_{\eta} - M_{\pi})^2. \]  

(11)

For fixed \( s \), the bounds for \( t, u \) are

\[ t_{\text{max}, \text{min}}(s) = \frac{1}{2} \left[ 3s_0 - s \pm \sigma(s) \lambda^{1/2} (s, M_{\eta}^2, M_{\pi}^2) \right] \]  

(12)

\[ u_{\text{max}, \text{min}}(s) = \frac{1}{2} \left[ 3s_0 - s \mp \sigma(s) \lambda^{1/2} (s, M_{\eta}^2, M_{\pi}^2) \right], \]  

(13)

where \( \sigma(s) \) means the velocity of the charged pions in the \( \pi^+\pi^- \) rest frame, i.e.

\[ \sigma(s) = \sqrt{1 - \frac{4M_{\pi}^2}{s}} \]  

(14)

and \( \lambda \) is the Kallen triangle function

\[ \lambda(s, M_{\eta}^2, M_{\pi}^2) = \left( s - (M_{\eta} + M_{\pi})^2 \right) \left( s - (M_{\eta} - M_{\pi})^2 \right). \]  

(15)

For further convenience, we also denote

\[ \Delta_{\eta\pi} = M_{\eta}^2 - M_{\pi}^2. \]  

(16)

\(^1\)However, for the numerical calculation of the decay widths we will hold a distinction in the numerical values of the pion masses for the neutral and the charged decay, and in the position of the Dalitz plot center as well. E.g. for the neutral decay observables we put \( M_{\pi^0} \rightarrow M_{\pi^0} \) and

\[ 3s_0 = M_{\eta}^2 + 3M_{\pi^0}^2. \]  

(10)

For more details, see \(^{10}\).
The differential decay rate is then
\[ \frac{d\Gamma}{d\sigma dt} = \frac{1}{(2\pi)^3 \cdot 32M_\eta^3} |A(s, t; u)|^2. \] (17)

The usual phenomenological parametrization of $|A(s, t; u)|^2$ (known as the Dalitz plot) is given in terms of the variables
\[
x = \sqrt{3} \frac{T_+ - T_-}{Q_\eta} = -\frac{\sqrt{3}}{2M_\eta Q_\eta} (t - u) \quad (18)
\]
\[
y = 3 \frac{T_0}{Q_\eta} - 1 = -\frac{3}{2M_\eta Q_\eta} (s - s_0), \quad (19)
\]
where $T_{0, \pm}$ are the kinetic energies of the final states $\pi^0, \pm$ and
\[
Q_\eta = T_0 + T_+ + T_- = M_\eta - 3M_\pi. \quad (20)
\]

The parametrization then reads
\[
|A(s, t; u)|^2 = |A(s_0, s_0; s_0)|^2 \left( 1 + ay + by^2 + dx^2 + fy^3 + gx^2y + \ldots \right) \quad (21)
\]
and corresponds to the Taylor expansion at the center of the Dalitz plot\footnote{Let us note that beyond the first order in the isospin breaking, which requires to take the neutral and charged pion masses as different, the point $x = y = 0$ does not coincide with the point $s = t = u$ and we have the following formula for $y$
\[
y = -\frac{3}{2M_\eta Q_\eta} ((M_\eta - M_\pi)^2 - s) - 1. \quad (22)\]}

In the case of the neutral decay, the amplitude is symmetric with respect to an exchange of $s$, $t$ and $u$ and it is therefore more convenient to introduce the variable
\[
z = x^2 + y^2 = \frac{3}{2M_\eta^2(M_\eta - M_\pi)^2} (s^2 + t^2 + u^2 - 3s_0^2) \quad (23)
\]
and write the Dalitz plot parametrization in the form
\[
|\overline{A}(s, t; u)|^2 = |A(s_0, s_0; s_0)|^2 \left( 1 + 2\alpha z + 2\beta y(3z - 4y^2) + \gamma z^2 + \ldots \right). \quad (24)
\]

For reasons described below, the basic object of our investigation will be the quantity $G(s, t; u)$
\[
G(s, t; u) \equiv F_\pi^3 F_\eta A(s, t; u), \quad (25)
\]
where $F_\pi$, $F_\eta$ are the pion and eta decay constants. The coefficients $A$, $B$, $C$, $D$ are defined by its expansion at the center of the Dalitz plot
\[
G(s, t; u) = A + B(s - s_0) + C(s - s_0)^2 + D [(t - s_0)^2 + (u - s_0)^2] + O((s - s_0)^3). \quad (26)
\]
These coefficients are related to the Dalitz plot parameters $a$, $b$, $d$, $\alpha$ by means of nonlinear relations

\[
\begin{align*}
    a &= -2R_\eta \text{Re} \left( \frac{B}{A} \right) \\
    b &= R_\eta^2 \left( \frac{B}{A} \right)^2 + \text{Re} \left( \frac{2C + D}{A} \right) \\
    d &= 3R_\eta^2 \text{Re} \left( \frac{D}{A} \right) \\
    \alpha &= \frac{1}{4} \left( d + b - R_\eta^2 \left| \frac{B}{A} \right|^2 \right),
\end{align*}
\]

where

\[
R_\eta = \frac{2}{3} M_\eta Q_\eta.
\]

Note that the last relation for $\alpha$ holds only in the lowest order in the isospin breaking.

### 3 Resummed chiral perturbation theory - the formalism

In this section, we briefly review the formalism of resummed chiral perturbation theory \cite{38, 39}. The general prescription can be summarized in the following points:

- The calculations are based on the standard $\chi$PT Lagrangian and standard chiral power counting given by the Weinberg formula \cite{6}. In particular, the quark masses $m_q$ are counted as $m_q = O(p^2)$.

- The crucial point is an identification of globally convergent observables (named safe observables, i.e. the chiral expansion of which can be trusted) related to the amplitude and other physically relevant observables for the process under consideration. As will be explained in more detail in the next section, these safe observables are related to the Green functions of the quark bilinears by linear operations.

- The next step consists of performing the strict chiral expansion of the safe observables, i.e. an expansion constructed in terms of the parameters of the chiral Lagrangian and strictly respecting the chiral orders. That means, e.g., that the propagators inside the loops carry the $O(p^2)$ masses. The expansion is done up to the $O(p^4)$ order explicitly; the higher orders are collected implicitly in remainders, which arise as additional parameters.

- Then we construct a modified expansion (dubbed bare expansion), which differs from the strict expansion by the location of the branching points of the non-analytical unitarity part of the amplitudes - within the bare expansion they are placed in their physical positions. This can be done either by means of a matching with a dispersive representation or by hand.

- After that we perform an algebraically exact nonperturbative reparametrization of the bare expansion by expressing the $O(p^4)$ LECs $L_4, \ldots, L_8$ in terms of physical values of
experimentally well-confirmed safe observables - the pseudoscalar decay constants and masses. The procedure generates additional higher order remainders. In what follows, we refer to these as indirect remainders.

- The physical amplitude and other relevant observables are then obtained as algebraically exact nonperturbative expressions in terms of the related safe observables and higher order remainders.

- The higher order remainders are explicitly kept and carefully treated by using various information stemming from both inside and outside \( \chi PT \) (order of magnitude estimates, explicit higher order calculations, resonance saturation, etc.).

In the presence of particle mixing, which is the case of the \( \pi^0 - \eta \) sector treated at the first order in the isospin breaking, the implementation of the procedure is a little bit more complicated. We will therefore give a more detailed explanation of the above points in the following sections.

### 4 Safe observables

The starting point of the formalism of resummed \( \chi PT \) is the generating functional \( Z[v, a, p, s] \) of the correlators of the quark bilinears defined as

\[
Z[v, a, p, s] = \langle 0 | T \exp \left( \frac{i}{\hbar} \int d^4x \left( \bar{q} \gamma \cdot (v + a \gamma^5)q + \bar{q}(s + ip \gamma^5)q \right) \right) | 0 \rangle ,
\]

where \( v, a, p, s \) are the external classical sources and \( q \) stands for the \( SU(N_f) \) multiplet of the quark fields. Pseudo-Goldstone boson (PGB) fields are the only relevant degrees of freedom at energies up to the hadronic scale \( E \ll \Lambda_H \sim 1 \text{GeV} \). The low energy representation of \( Z[v, a, p, s] \) can thus be expressed in terms of the functional integral over the PGB fields

\[
Z[v, a, p, s] = \int DU \exp \left( \frac{i}{\hbar} \left( S^{(2)}[v, a, p, s; U] + \hbar S^{(4)}[v, a, p, s; U] + \ldots \right) \right).
\]

In this expression, the field \( U \) corresponds to the \( N_f \times N_f \) unitary matrix, which can be written, for \( N_f = 3 \), in terms of the pseudoscalar octet fields \( \phi^a \equiv \pi, K, \eta \) as

\[
U = \exp \frac{i}{F_0} \phi^a \lambda^a ,
\]

where \( \lambda^a \) being the Gell-Mann matrices and

\[
\phi^a \lambda^a = \left( \begin{array}{ccc}
\pi^3 + \frac{1}{\sqrt{3}} \eta^8 & \sqrt{2} \pi^+ & \sqrt{2} K^+ \\
-\sqrt{2} \pi^- & -\pi^3 + \frac{1}{\sqrt{3}} \eta^8 & \sqrt{2} K^0 \\
-\sqrt{2} K^- & \sqrt{2} K^0 & -\frac{2}{\sqrt{3}} \eta^8 
\end{array} \right).
\]

\( S^{(n)}[v, a, p, s; U] \) is the action functional of the chiral order \( O(p^n) \). The systematic chiral expansion of \( Z[v, a, p, s] \) is then obtained by means of a loop expansion of (30), which is correlated with the chiral expansion by means of the Weinberg formula \([6]\). In practice, this means
integrating out the quantum fluctuations around the classical solution $\phi^i[v,a,p,s]$ of the lowest order equation of motion (i.e. those derived from the lowest order action $S^{(2)}[v,a,p,s;U]$) order by order in $\hbar$. The result can be then written as (we put $\hbar = 1$ in what follows)

$$Z[v,a,p,s] = \sum_n \frac{F_n}{n!} \int d^4 x_1 \ldots d^4 x_n Z_{i_1i_2\ldots i_n}[v,a,p,s](x_1, \ldots, x_n) \times \phi^{i_1}[v,a,p,s](x_1) \ldots \phi^{i_n}[v,a,p,s](x_n),$$

where the chiral expansion of the coefficient functionals $Z_{i_1i_2\ldots i_n}[v,a,p,s](x_1, \ldots, x_n)$ symbolically reads

$$Z_{i_1i_2\ldots i_n} = Z^{(2)}_{i_1i_2\ldots i_n} + Z^{(4)}_{i_1i_2\ldots i_n} + Z^{(6)}_{i_1i_2\ldots i_n} + \ldots$$

(33)

(34)

(35)

(36)

As can be seen, nothing is assumed about the relative value of $G$ and the leading order (LO) and next-to-leading order (NLO) terms of the chiral expansion, $G^{(2)}$ and $G^{(4)}$, respectively. In particular, the cases when $G^{(2)} \ll G$ or $G^{(2)} \sim G^{(4)}$ are not a priori excluded. This notion can be quantified by assuming that for such a safe observable, denoted generically as $G$ in what follows, the $O(p^6)$ remainder $\Delta_G$, defined by

$$G = G^{(2)} + G^{(4)} + \Delta_G,$$

is reasonably small, typically

$$\delta_G = \frac{\Delta_G}{G} \sim 0.1.$$  

(37)

(38)

(39)

Let us stress that this assumption is a hypothesis which should be questioned and tested.

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The remainder

\[ \Delta_{1/G} = 1/G \left( -\frac{\delta G}{X_G^2} + \left( \frac{1 - X_G}{X_G} \right)^2 \right) \]  

(40)
might be large when \( X_G \ll 1 \), even for a small original remainder \( \Delta_G = G\delta G \). Therefore, in order to get numerically reliable values for such dangerous observables, the formal chiral expansion can not be performed and the original algebraic form has to be preserved by holding the remainders explicit. These parameters then estimate the theoretical uncertainty of the result. For our toy example this means to use, instead of the chiral expansion, an algebraic identity

\[ \frac{1}{G} = \frac{1}{G^{(2)}} + \frac{1}{G^{(4)}} + \Delta_G. \]

(41)

Note that if we dropped the remainder \( \Delta_G \), we could interpret the result of such an approach as a partial resummation of (some of the) higher orders of the chiral expansion

\[ \frac{1}{G} \to \frac{1}{G^{(2)}} \sum_{n=0}^{\infty} (-1)^n \left( \frac{G^{(4)}}{G^{(2)}} \right)^n. \]

(42)

On the other hand, by not neglecting the remainder, we get an exact algebraic identity valid to all orders\(^4\).

5 Amplitudes in terms of safe observables in the presence of mixing

As we have mentioned above, a necessary ingredient of the resummed approach is the identification of the globally convergent safe observables. In this section, we will briefly recapitulate the connection between the safe observables and the amplitudes, and also introduce a generalization for the case of the \( \pi^0-\eta \) mixing.

The key assumption of resummed \( \chi PT \) is that the safe observables are derived by means of linear operations from the generating functional \( Z[v, a, p, s] \). Therefore, the functional derivatives of \( Z[v, a, p, s] \) with respect to the axial vector sources, their Fourier transforms and

\(^4\)Let us note that this approach can be straightforwardly extended to the next-to-next-to-leading order (NNLO). We can write

\[ G = G^{(2)} + G^{(4)} + G^{(6)} + \Delta_G, \]

(43)

where \( \Delta_G \) is now assumed to be a reasonably small \( O(p^8) \) remainder. For the toy example of the chiral expansion of the dangerous observable \( 1/G \), we get

\[ \frac{1}{G} = \frac{1}{G^{(2)}} - \frac{G^{(4)}}{G^{(2)}} \left( \frac{G^{(4)}}{G^{(2)}} \right)^2 - \frac{G^{(6)}}{G^{(2)}} + \Delta_{1/G}. \]

(44)

The \( O(p^8) \) remainder is

\[ \Delta_{1/G} = \frac{1}{G} \left( -\frac{\delta G}{X_G^2} + \left( \frac{1 - X_G}{X_G} \right)^2 \right), \]

(45)

which can be large when \( X_G \ll 1 \) and/or \( X_G \sim 1 \). Therefore, the irregularities in the first two terms of the chiral expansion can produce large higher order \( O(p^8) \) remainders when dangerous operations are carried out, even when performed on globally convergent safe observables. Throwing away the remainders, as usually done within the standard NNLO, is therefore not safe in such a case.
residues at the one particle poles belong to the set of safe observables. Such safe observables are directly connected with the physical amplitudes of the processes with PGB. Indeed, for \( p^2 \rightarrow M_P^2 \), we symbolically have

\[
\int \frac{d^4xe^{ipx} \delta Z[v,a,p,s]}{i\delta a'(x)} = \langle 0| T_{j_{\mu5}^i}(p) \cdots |0 \rangle = \frac{1}{p^2 - M_P^2} \langle 0| j_{\mu5}^i(0)|p, P\rangle \langle p, P| \cdots |0 \rangle + \text{reg},
\]

where \( j_{\mu5}^i \) is the axial vector current, \(|p, P\rangle\) is the one-particle PGB state with mass \( M_P \) and \( \text{reg} = O(1) \) denotes a regular contribution. The PGB states couple to the operators \( j_{\mu5}^i \) and we have the following general relation

\[
\langle 0| j_{\mu5}^i(0)|p, P\rangle = ip_{\mu}F^{Pi}.
\]

Therefore, the residue of \( \langle 0| \tilde{T}_{j_{\mu5}^i}(p_1) \cdots \tilde{T}_{j_{\mu5}^i}(p_n)|0 \rangle \) at the simultaneous poles at \( p_1^2 \rightarrow M_P^2 \) correspond to a safe observable \( G_{P_1P_2 \cdots P_n}(p_1, \ldots, p_n) \), for which we can write (no summation over \( P_k \))

\[
G_{P_1P_2 \cdots P_n}(p_1, \ldots, p_n) = F_{Pi1} \cdots F_{Pni}A_{P_1P_2 \cdots P_n}(p_1, \ldots, p_n).
\]

\( A_{P_1P_2 \cdots P_n}(p_1, \ldots, p_n) \) are the \( S \)-matrix elements with PGB \( P_1P_2 \cdots P_n \) in the in and out states.

In the absence of mixing, when the mass states have definite isospin, we have

\[
\langle 0| j_{\mu5}^i(0)|p, P\rangle = ip_{\mu}F_0 \delta p_P,
\]

where \( F_i \) are the corresponding pseudoscalar decay constants. We obtain \( F_P^2 \) and \( F_P^2M_P^2 \) as the simplest examples of safe observables, related to the residue of \( \langle 0| T_{j_{\mu5}^i}(p)j_{\mu5}^i(0)|0 \rangle \) at \( p^2 \rightarrow M_P^2 \)

\[
\langle 0| T_{j_{\mu5}^i}(p)j_{\mu5}^i(0)|0 \rangle = \frac{p_{\mu}p_{\nu}F_{P\nu}^2}{p^2 - M_P^2} + \text{reg},
\]

\[
\langle 0| T_{j_{\mu5}^i}(p)j_{\mu5}^i(0)|0 \rangle = \frac{M_P^2F_{P\nu}^2}{p^2 - M_P^2} + \text{reg}.
\]

On the other hand, the first powers of the decay constants \( F_P \) and of the masses \( M_P \) cannot be considered as safe observables, as they are linked to \( F_p^2 \) and \( F_p^2M_p^2 \) by nonlinear relations. In the same spirit, the amplitudes \( A_{P_1P_2 \cdots P_n}(p_1, \ldots, p_n) \) do not represent safe observables either, being non-linearly related to \( G_{P_1P_2 \cdots P_n}(p_1, \ldots, p_n) \) and \( F_P^2 \)

\[
A_{P_1P_2 \cdots P_n}(p_1, \ldots, p_n) = F_{P_1}^{-1} \cdots F_{P_n}^{-1}G_{P_1P_2 \cdots P_n}(p_1, \ldots, p_n).
\]

In the case of the \( \pi^0 \cdot \eta \) mixing, when the isospin symmetry is explicitly broken, the matrix \( F Pik \) in the relations (47) is not diagonal. In the first order of the isospin breaking, the non-diagonal terms of the matrix \( F Pik \) directly correspond to the the \( \pi^0 \cdot \eta \) mixing sector. Hence we can define

\[
F = \begin{pmatrix}
F_{\pi^03} & F_{\pi^08} \\
F_{\eta3} & F_{\eta8}
\end{pmatrix} = \begin{pmatrix}
F_\pi & \varepsilon_\pi F_\eta \\
-\varepsilon_\eta F_\pi & F_\eta
\end{pmatrix}.
\]

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$F_{\pi,\eta}$ are the pion and eta decay constants and $\varepsilon_{\pi,\eta}$ are the mixing angles at the leading order in the isospin breaking

$$\varepsilon_{\pi,\eta} = O\left( \frac{1}{R} \right), \quad (54)$$

where

$$R = \frac{m_s - \hat{m}}{m_d - m_u}. \quad (55)$$

As was shown in [8] (see also appendix F for details), the chiral expansion (up to and including the order $O(p^4)$) of the safe observables $G_{t_1t_2...t_n}^{p_1p_2...p_n}$ is related to the chiral expansion of the Fourier transforms of the coefficient functionals $Z_{t_1t_2...t_n}^{0,0,0,0}$, introduced in (33). In the general case, we have the following relation

$$Z_{t_1t_2...t_n}(p_1, \ldots, p_n) = \sum_{P_1...P_n} G_{t_1t_2...t_n}^{p_1p_2...p_n}(p_1, \ldots, p_n), \quad (56)$$

which can also be understood as a definition of an extension of the right hand side off the mass shell.

The $Z_{t_1t_2...t_n}$’s, being linear combinations of the safe observables $G_{t_1t_2...t_n}^{p_1p_2...p_n}$, are therefore safe observables too. In the absence of the mixing, we simply have $G_{t_1t_2...t_n}^{p_1p_2...p_n} = Z_{t_1t_2...t_n}$ and the amplitude $A_{P_1P_2...P_n}$ is given by (52), where the inverse powers of decay constants $F_i$ are assumed not to be expanded but substituted by their physical values. However, for the non-diagonal matrix $F$, we need a nonperturbative inverse of $F$ in order to obtain the amplitude $A_{P_1P_2...P_n}$ from the safe observables $Z_{t_1t_2...t_n}$

$$A_{P_1P_2...P_n}(p_1, \ldots, p_n) = \sum_{j_1...j_n} (F^{-1})^{P_1j_1} \ldots (F^{-1})^{P_nj_n} Z_{j_1j_2...j_n}(p_1, \ldots, p_n). \quad (57)$$

In the $\pi^0$-$\eta$ sector, as we work in the first order of the isospin breaking, we get

$$F^{-1} = \left( \begin{array}{cc} F_\pi^{-1} & -\varepsilon_\pi F_\eta^{-1} \\ \varepsilon_\eta F_\pi^{-1} & F_\eta^{-1} \end{array} \right) + O\left( \frac{1}{R^2} \right). \quad (58)$$

Now, let us apply the above general recipe to the case of the amplitude $A(s, t; u)$ of the decay $\eta(p) \rightarrow \pi^+(p_+)\pi^-(p_-)\pi^0(p_0)$, defined by (5). According to (57) and (58), we then have the following resummed representation of the amplitude $A(s, t; u)$ in terms of safe observables $Z_{83+}(s, t; u)$, $Z_{33+}(s, t; u)$, $Z_{88+}(s, t; u)$ and the matrix $F$ of physical decay constants:

$$F_\pi^3 F_\eta A(s, t; u) \equiv G(s, t; u) = Z_{83+}(s, t; u) - \varepsilon_\pi Z_{33+}(s, t; u) + \varepsilon_\eta Z_{88+}(s, t; u), \quad (59)$$

Note that $F_\pi^{-4} Z_{33+}(s, t; u)$ and $F_\pi^{-2} F_\eta^{-2} Z_{88+}(s, t; u)$ can be identified as the off-shell extensions of the $\pi\pi$ and $\pi\eta$ scattering amplitudes, respectively, calculated in the limit of conserved isospin.

---

$^5$Since we calculate the amplitude at the first order in the isospin breaking, we have put $F_\pi = F_\eta = F_\pi$ on the left hand side of the formula (59).
6 Mixing angles

Provided we knew all the entries of the matrix $F$ \cite{53} from experimental measurements with good enough precision, we could proceed further. However, with the exception of $F_\pi$, this is not the case. Let us therefore calculate the remaining matrix elements $F_{ij}$, which can also be viewed as an illustration of the machinery of resummed $\chi PT$ applied to dangerous observables.

As we have discussed above, the matrix $F$ is directly related to the part of the generating functional $Z[a]$ which is at most quadratic in the fields $\phi^i[a]$. Let us write the generating functional in the form ($Z[a] \equiv Z[v,a,p,s]|_{v=p=s=0}$)

$$Z[a] = \frac{1}{2} \int d^4x [Z_{33}(\partial_3^3[a] - a_3 F_0)^2 + Z_{88}(\partial_8^8[a] - a_8 F_0)^2$$

$$+ 2Z_{38}(\partial_3^3[a] - a_3 F_0) \cdot (\partial_8^8[a] - a_8 F_0)$$

$$- M_{33}^3[a]^2 - M_{88}^8[a]^2 - 2M_{38}^3[a]M_8^8[a] + \ldots + O(p^6),$$

where $F_0$ is the PGB decay constant in the chiral limit and where $Z_{ij}$ and $M_{ij}$ accumulate the $O(p^2)$ and $O(p^4)$ contributions according to \cite{34}

$$Z_{ij} = \delta_{ij} + Z_{ij}^{(4)}$$

$$M_{ij} = \delta_{ij} o_{m_j} + M_{ij}^{(4)}.$$  

Here $o_{m_j}$ denotes the $O(p^2)$ masses. The off-diagonal terms

$$Z_{38} = Z_{38}^{(4)}$$

$$M_{38} = M_{38}^{(2)} + M_{38}^{(4)}$$

are taken at the first order in the isospin breaking,

$$Z_{38}, M_{38} = O \left( \frac{1}{R} \right).$$  

According to our discussion in the previous section, $Z_{ij}$ and $M_{ij}$ represent safe observables.

It can be shown (see appendix \cite{F}) that up to higher order corrections

$$F^T \cdot F = F_0^2 Z + O(p^6).$$

Here

$$Z = \left( \begin{array}{cc} Z_{33}, & Z_{38} \\ Z_{38}, & Z_{88} \end{array} \right)$$

and

$$F^T \cdot \text{diag}(\mathcal{M}_\pi^2, \mathcal{M}_\eta^2) \cdot F = F_0^2 \mathcal{M} + O(p^6),$$

where

$$\mathcal{M} = \left( \begin{array}{cc} \mathcal{M}_{33}, & \mathcal{M}_{38} \\ \mathcal{M}_{38}, & \mathcal{M}_{88} \end{array} \right).$$
The above matrix relations can be written in components

\[ Z_{33} F_0^2 = F_\pi^2 (1 - \delta F_\pi^2) \]  
\[ Z_{88} F_0^2 = F_\eta^2 (1 - \delta F_\eta^2) \]  
\[ Z_{38} F_0^2 = (\varepsilon_\pi F_\pi^2 - \varepsilon_\eta F_\eta^2)(1 - \delta \varepsilon) \]  

and

\[ F_0^2 M_{33} = F_\pi^2 M_\pi^2 (1 - \delta F_\pi^2 M_\pi^2) \]  
\[ F_0^2 M_{88} = F_\eta^2 M_\eta^2 (1 - \delta F_\eta^2 M_\eta^2) \]  
\[ F_0^2 M_{38} = (\varepsilon_\pi F_\pi^2 M_\pi^2 - \varepsilon_\eta F_\eta^2 M_\eta^2)(1 - \delta \varepsilon F), \]  

where we have defined a unique remainders for each observable, e.g.

\[ F_\pi^2 = Z_{33} F_0^2 + F_\pi^2 \delta F_\pi^2 \]  

and similarly in the rest of the cases. As a last step, we algebraically invert these relations in order to find the resummed expressions for \( F_\eta^2 \) and the mixing angles \( \varepsilon_{\pi,\eta} \)

\[ F_\eta^2 = Z_{88} F_0^2 + F_\eta^2 \delta F_\eta^2 \]  
\[ \varepsilon_{\pi,\eta} = - \frac{F_0^2 M_{38} (1 - \delta \varepsilon F M_\pi^2)^{-1} - Z_{38} M_{\eta,\pi}^2 (1 - \delta \varepsilon F)^{-1}}{F_\pi^2 - M_\pi^2 - M_\eta^2}. \]  

The explicit form of the strict chiral expansion of the mixing parameters \( M_{38} \) and \( Z_{38} \) can be found in appendix A, while a detailed analysis of \( F_\eta \) within resummed \( \chi PT \) has been done in [43].

Because the remainders are not neglected, the formulae (77) and (78) are exact algebraic identities valid to all orders in the chiral expansion. Let us note that the standard chiral expansion of the denominators in (78) should not be performed because of the possible generation of large \( O(p^6) \) remainders. In this sense, the mixing angles \( \varepsilon_{\pi,\eta} \) are typical examples of dangerous observables.

### 7 Strict chiral expansion for \( G(s, t; u) \)

In the context of resummed \( \chi PT \), we understand strict expansion as the chiral expansion of a safe observable expressed in terms of original parameters of the chiral Lagrangian without any reparametrization in terms of physical observables and without any potentially dangerous operation, like the expansion of the denominators. Also, loop graphs are constructed strictly from propagators and vertices derived from the \( O(p^2) \) part of the Lagrangian. In particular, the propagator masses are held at their LO values, which we denote as \( m_P \) (\( P = \pi, K, \eta \)).

According to (59), the result for the re-scaled amplitude \( G(s, t; u) \) can be expressed in terms of the safe observables \( Z_{ab+,+} \) (where \( a, b = 3, 8 \)) and physical mixing angles \( \varepsilon_{\pi,\eta} \)

\[ G(s, t; u) = Z_{33,3}(s, t; u) - \varepsilon_{\pi} Z_{33,8}(s, t; u) + \varepsilon_{\eta} Z_{88,3}(s, t; u). \]  

The expansion up to \( O(p^4) \) can be written in the form

\[ Z_{ab+,+} = Z_{ab+,+}^{(2)} + Z_{ab+,+}^{(4)} + Z_{ab+,+}^{(4),ct} + Z_{ab+,+}^{(4),tad} + Z_{ab+,+}^{(4),unit} + \Delta Z_{ab+,+}. \]  


where the individual terms on the right hand side represent the $O(p^2)$ contribution, the $O(p^4)$ counterterm, tadpole and unitarity corrections and the $O(p^6)$ remainder, respectively. Let us note that the splitting of the loop correction into the tadpole and the unitarity part is not unique. Here we follow the splitting of the generating functional introduced in [8].

The explicit form of the strict expansion (80) is rather long and is therefore postponed to appendix A. The schematic form of the final result for the amplitude $G_{\text{strict}}(s,t;u)$ is

$$
G_{\text{strict}}(s,t;u) = R(s,t;u) + \sum_{PQ} Q_{PQ}^{(s)}(s,t;u)J_{PQ}(s) + \sum_{PQ} Q_{PQ}^{(t)}(s,t;u)J_{PQ}(t) + \sum_{PQ} Q_{PQ}^{(u)}(s,t;u)J_{PQ}(u) + \Delta G,
$$

(81)

where $R$, $Q_{PQ}^{(s)}$, $Q_{PQ}^{(t)}$ and $Q_{PQ}^{(u)}$ (where $P,Q = \pi,K,\eta$) are second order polynomials in the Mandelstam variables. $J_{PQ}(s)$ is the $\overline{MS}_{\chi PT}$ subtracted scalar bubble

$$
J_{PQ}(s) = -i \int \frac{d^{4-2\varepsilon} k}{(2\pi)^{4-2\varepsilon}} \frac{1}{\left(k^2 - \frac{\alpha^2}{m_P + i0}\right)\left((k-p)^2 - \frac{\alpha^2}{m_Q + i0}\right)} - \lambda,
$$

(82)

where $\alpha m_P$ are $O(p^2)$ masses and

$$
\lambda = \mu^{-2\varepsilon} \left( \frac{1}{\varepsilon} + \ln 4\pi - \gamma + 1 \right).
$$

(83)

In this formula, $d = 4-2\varepsilon$ is the dimension, $\mu$ is the renormalization scale of the dimensional regularization scheme and $s = p^2$.

8 Bare expansion: matching with a dispersive representation

The calculation of the strict expansion is only the first step in the construction of the amplitude within resummed $\chi PT$. Because it strictly respects the chiral orders, it suffers from some pathologies which have to be cured. The most serious one is that the position of the unitarity cuts in the complex $s$, $t$ and $u$ planes is determined by the $O(p^2)$ masses and not by the physical masses of the particles inside the loops. Also, the presence of the $O(p^2)$ masses in the arguments of chiral logarithms can generate undesirable singularities in the $B_0$ dependence of the amplitude. These pathologies can be removed either by hand (by means of some well defined ad hoc prescription, see [44]) or, more systematically, by means of a matching with a dispersive representation of the amplitude, as we have introduced in [45]. The latter procedure, which we term as the construction of the \textit{bare expansion}, is recalled in this section.

Let us briefly introduce the most general model independent form of the amplitude $G(s,t;u)$ to the order $O(p^6)$ respecting unitarity, analyticity and crossing symmetry. Such $G(s,t;u)$ can be constructed using the reconstruction theorem, which has been developed originally for the $\pi\pi$ scattering amplitude [46] and which can be easily generalized to other processes with PGB [47] (see also [45] for a more in-depth discussion of the subtleties connected with applications in resummed $\chi PT$). According to this theorem, we can write the $O(p^4)$ amplitude in the form of a dispersive representation

$$
G(s,t;u) = P(s,t;u) + F^3_\pi F_\eta U(s,t;u) + O(p^6),
$$

(84)

\footnote{Let us remind that $B_0$ is typically taken as a free parameter within resummed $\chi PT$.}
where \( P(s; t; u) \) is a second order polynomial in Mandelstam variables

\[
P(s; t; u) = A_P + B_P(s - s_0) + C_P(s - s_0)^2 + D_P [(t - s_0)^2 + (u - s_0)^2]
\]  

(85)

and \( U(s; t; u) \) represent the unitarity corrections, which can be written in the form

\[
U(s; t; u) = \frac{1}{3} (W_0(s) - W_2(s)) + \frac{1}{2} (3(s - u)W_1(t) + W_2(t)) + \frac{1}{2} (3(s - t)W_1(u) + W_2(u)).
\]  

(86)

As discussed in detail in [46] and [47], the reconstruction theorem, together with the two-particle unitarity relation for the partial waves, can be used for the iterative construction of the amplitude up to and including the order \( O(p^6) \) (for the most general result of such a construction without invoking the Lagrangian formalism, we refer to [30]). Here we use it as a tool for an appropriate modification of the results described in the previous section, i.e. for the construction of the bare expansion.

For this purpose we reconstruct and fix the unitarity part \( U(s; t; u) \) of the amplitude from the \( O(p^2) \) amplitudes calculated within resummed \( \chi PT \). Let us note that there is some ambiguity in the choice of the form of the \( O(p^2) \) amplitudes, which we take as an input for the reconstruction theorem. The reason is that there are (at least) two possibilities how to connect the generic physical \( O(p^2) \) amplitude \( A^{(2)} \) of the process \( AB \to CD \) (which is a dangerous observable) and the corresponding safe observable \( G^{(2)} \), namely either

\[
A^{(2)} = \frac{G^{(2)}}{F_AF_BF_CF_D},
\]

(87)

where \( F_A, \ldots, F_D \) are the physical decay constants or

\[
A^{(2)} = \frac{G^{(2)}}{F_0^4}.
\]

(88)

The choice between this alternatives is in fact a part of the definition of the direct remainders (cf. (95) below). See also [45] and appendix B for more detail.

After calculating the unitarity part, the remaining polynomial part of the amplitude is then fixed by means of matching of the strict chiral expansion obtained in the previous section with the general form (84). The list of all relevant \( O(p^2) \) amplitudes and a reconstruction of \( U(s; t; u) \) from them is given in detail in appendix B. The corresponding \( W_i(s) \) (see (86)) have the following schematic form (cf. (81))

\[
W_i(s) = \sum_{PQ} w_{iPQ}(s) \mathcal{J}_{PQ}(s),
\]

(89)

where \( w_{iPQ}(s) \) is a second order polynomial with coefficients depending on the \( O(p^2) \) parameters of the chiral Lagrangian and

\[
\mathcal{J}_{PQ}(s) = \frac{s}{16\pi^2} \int_{(M_P + M_Q)^2}^\infty \frac{dx}{x} \frac{\lambda^{1/2}(x, M_P^2, M_Q^2)}{x} \frac{1}{x - s - i0}
\]

(90)
is a once subtracted scalar bubble with physical masses $M_P$ and $M_Q$ inside the loop. In \cite{89}, the sum is taken over all two-particle intermediate states $PQ$ in the given channel. Let us note that the general form of the reconstructed $U(s; t; u)$ is similar to the last three terms of the strict expansion of $G(s; t; u)$ \cite{81}, with the exception that $J_{PQ}^r(s)$ with unphysically situated cuts are replaced with $J_{PQ}(s)$ for which the cuts are placed at the physical two particle thresholds $s = (M_P + M_Q)^2$. This enables us to match both representations as follows.

In \cite{81}, let us write $J_{PQ}^r(s) = J_{PQ}^r(s) + J_{PQ}^r(0)$, where the renormalization scale independent part $J_{PQ}^r(s) \equiv J_{PQ}^r(s) - J_{PQ}^r(0)$ of $J_{PQ}^r(s)$ is nothing else but the scalar bubble \cite{82} subtracted at $s = 0$. As a result, we can write

$$G_{\text{strict}}(s; t; u) = G_{\text{pol}}(s; t; u) + F_\pi^3 F_\eta U^r(s; t; u),$$

where

$$G_{\text{pol}}(s; t; u) = G(s; t; u)|_{\overline{J}_{PQ} \rightarrow 0}$$

is a second order polynomial. Let us remark that a polynomial part of the amplitude constructed in this way does not depend on the renormalization scale $\mu$.

The matching can be performed by means of an identification of the polynomial $P(s; t; u)$ from the dispersive representation \cite{84} with $G_{\text{pol}}(s; t; u)$. This means we write the amplitude in the form

$$G(s; t; u) = G_{\text{pol}}(s; t; u) + F_\pi^3 F_\eta U(s; t; u),$$

with the $U(s; t; u)$ from \cite{84} and \cite{86} constructed according to the reconstruction theorem. Note that such a $G(s; t; u)$ satisfies the requirements of the perturbative unitarity exactly. We then get for the polynomial part of the amplitude

$$G_{\text{pol}}(s; t; u) = G^{(2)}(s; t; u) + G^{(4)}_{\text{ct}}(s; t; u) + G^{(4)}_{\text{tad}}(s; t; u) + G^{(4)}_{\text{pol}, u}(s; t; u),$$

where $G^{(4)}_{\text{ct}}$ and $G^{(4)}_{\text{tad}}$ (the counterterm and tadpole contributions) can be found in appendix A, while $G^{(4)}_{\text{pol}, u}$ (the unitarity contribution) in appendix C.

As a last step, we replace, in the expressions for $G_{\text{pol}}(s; t; u)$, the $O(p^2)$ masses inside the chiral logarithms and inside $J_{PQ}^r(0)$ with the physical masses (but we keep them in all other places they appeared). This ad hoc prescription avoids the unwanted logarithmic singularities in the limit $X \rightarrow 0$ in our final formula for the bare expansion of the amplitude $G(s; t; u)$

$$G_{\text{bare}}(s; t; u) = G_{\text{pol}}(s; t; u) + F_\pi^3 F_\eta U(s; t; u) + \Delta_G(s; t; u).$$

The latter formula can be also understood as a definition of the $O(p^6)$ remainder $\Delta_G$. However, because we do not know the detailed analytic structure of $\Delta_G$, we parametrize it in the form of a polynomial in the variables $s$, $t$ and $u$

$$\Delta_G = A\delta_A + B\delta_B(s - s_0) + C\delta_C(s - s_0)^2 + D\delta_D \left[ (t - s_0)^2 + (u - s_0)^2 \right],$$

where the observables $A, \ldots, D$ are the coefficients of the expansion of the amplitude $G(s; t; u)$ at the center of the Dalitz plot

$$G(s; t; u) = A + B(s - s_0) + C(s - s_0)^2 + D \left[ (t - s_0)^2 + (u - s_0)^2 \right] + \Delta_G + O\left( (s - s_0)^3, \ldots \right).$$
The bare expansion of $A, \ldots, D$, derived form (95), is explicitly given in appendix D.

While it is natural to assume the coefficients $A$, $B$, $C$ and $D$ to be safe observables, strictly speaking, this assumption goes beyond the general definition given in section 4. Note that the global convergence of the Green function $G(s,t;u)$ does not automatically imply a convergence of its derivatives at the center of the Dalitz plot. Moreover, the parameters $C$ and $D$ start at $O(p^4)$ and therefore the criterion of the global convergence merely implies that their $O(p^8)$ reminders are reasonably small. The assumption about the natural size of their $O(p^6)$ remainders is thus in fact an additional conjecture about the regularity of the chiral series. Therefore, considering $A$, $B$, $C$ and $D$ as safe observables should rather be taken as a working hypothesis. Actually, one of the issues of this work is probing this assumption.

The conventional Dalitz plot parameters are related to these coefficients by means of the nonlinear relations (27) and thus should be regarded as dangerous observables and expressed nonperturbatively with all the remainders kept explicitly

\begin{align}
a &= -2R_\eta \text{Re} \left( \frac{B}{A} \right) \frac{1 - \delta_A}{1 - \delta_B} \quad (98) \\
b &= R_\eta^2 \left( \frac{B}{A} \right)^2 \left( \frac{1 - \delta_A}{1 - \delta_B} \right)^2 + \text{Re} \left( \frac{2C(1 - \delta_C)^{-1} + D(1 - \delta_D)^{-1}}{A(1 - \delta_A)^{-1}} \right) \quad (99) \\
d &= 3R_\eta^2 \text{Re} \left( \frac{D}{A} \right) \frac{1 - \delta_A}{1 - \delta_D} \quad (100) \\
\alpha &= \frac{1}{4} \left( d + b - R_\eta^2 \left( \frac{B}{A} \right)^2 \left( \frac{1 - \delta_A}{1 - \delta_B} \right)^2 \right). \quad (101)
\end{align}

9 Reparametrization of LECs in terms of physical observables

The approach of resummed $\chi PT$ to the problem of reparametrization of the chiral expansion, i.e. to the exclusion of some parameters of the Lagrangian in terms of physical observables, differs substantially from the one used in standard $\chi PT$. The reason is that the usual reparametrization procedure, consisting of the expansion of the $O(p^2)$ parameters $F_0$ and $B_0m_q$, where $m_q$ are the light quark masses, in terms of $F_\pi$, $F_K$ and pseudoscalar masses $M_\pi$, $M_K$ and $M_\eta$ is in general an operation, which can generate a large higher order remainder. Indeed, on one hand the above mentioned quantities are not safe observables, as we have discussed in the section 5 and thus their expansion in terms of the original parameters of the Lagrangian might include large remainders. On the other hand, even for the safe observables, like $F_\pi^2$ and $F_K^2M_\eta^2$, an inversion of the expansion might generate large remainder as well.

As an illustration, let us assume the following toy example. We can simplify things and assume there is only one leading order parameter $G_0$. Suppose there exists a safe observable $G$, for which the globally convergent expansion has the form

\[ G = G_0 + G^{(4)}(G_0) + G^{(6)}(G_0) + G\delta_G. \]  

The $O(p^4)$ and $O(p^6)$ terms of this expansion generally depend on $G_0$. The usual reparametrization procedure up to the $O(p^6)$ order needs an inversion of this expansion and expressing $G_0$ in terms of $G$

\[ G_0 = G - G^{(4)}(G) - \left[ G^{(6)}(G) - G^{(4)}(G)G^{(4)}(G) \right] + G_0\delta_{G_0}, \]  

18
where we have explicitly grouped the various chiral orders together. The remainder $G_0 \delta G_0$ generated by such a procedure is then dropped. However, we get an identity

$$\delta G_0 = - \frac{1 - X_G}{X_G} - \frac{1}{X_G} X_{G_0} - \frac{1}{X_G} X_{G_0} \overline{X}_{G_0},$$

(104)

where the ratios

$$X_{G_0} = \frac{G(4)(G)}{G},$$

(105)

$$\overline{X}_{G_0} = \frac{G(6)(G) - G(4)'(G)G(4)(G)}{G(4)(G)}$$

(106)

are analogues of (37) and (38) for the expansion (103) and probes the apparent convergence of the inverted expansion. Even if the inverted expansion (103) seems to converge well in the sense that $X_{G_0}$ and $\overline{X}_{G_0}$ are reasonably small, the neglected remainder might be large for $X_G \ll 1$.

Therefore, we keep the dependence on $F_0$ and $B_0 m_q$ in terms of the following parameters

$$X = \frac{2 B_0 F_0^2 \hat{m}}{F_2^2 M_\pi^2},$$

(107)

$$Z = \frac{F_0^2}{F_2^2},$$

(108)

which probe the regularity of the bare chiral expansion of the safe observables $F_\pi^2 M_\pi^2$ and $F_\pi^2$ in the sense of definition (37).

The quark masses are left as free parameters as well, in terms of the ratios

$$r = \frac{m_s}{\hat{m}},$$

(109)

and

$$R = \frac{m_s - \hat{m}}{m_d - m_u}.$$  

(110)

For further convenience, we also introduce the following frequently appearing combination

$$Y = \frac{X}{Z},$$

(111)

which is a measure of the regularity of the expansion of the dangerous observable $M_\pi^2$. Let us note that within the standard approach the parameters $X$, $Z$ and $r$ are fixed by means of the potentially dangerous inverted expansions of the type (103), namely

$$X_{std} = 1 + O(p^2)$$

$$Z_{std} = 1 + O(p^2)$$

$$r_{std} = 2 \frac{M_K^2}{M_\pi^2} - 1 + O(p^2).$$

(112)

---

7 In numerical calculations, we take $r$ from the lattice, which fixes the Kaplan-Manohar ambiguity.

8 At the first order in the isospin breaking, the factor $1/R$ appears only as an overall normalization factor.
Nevertheless, the bare expansion of the safe observables $F_P^2$ and $F_P^2 M_P^2$ can be used for an alternative reparametrization, which is, in contrast to the standard one, safe and do not suffer from dangerous manipulations, which might generate uncontrollably large remainders. Because of the linear dependence of the $O(p^4)$ order on the $L_4$-$L_8$ LECs, it is possible to express these constants by exact algebraic identities in terms of these safe observables and their $O(p^6)$ remainders $\delta_{F_P}$ and $\delta_{F_P M_P}$, schematically

$$L_i = \sum_P a_P F_P^2 (1 - \delta_{F_P}) + \sum_P b_P F_P^2 M_P^2 (1 - \delta_{F_P M_P}).$$

(113)

Due to the linearity of these relations, the corresponding remainders $\Delta_{L_i}$

$$\Delta_{L_i} = - \sum_P (a_P F_P^2 \delta_{F_P} + b_P F_P^2 M_P^2 \delta_{F_P M_P}),$$

(114)

are well under control. The explicit form of these relations have been published in [48], [38] (and also in [45], which is closest to the notation used here).

Concerning the LECs $L_1$-$L_3$, we don’t have a similar procedure ready at this point. We therefore treat $L_1$-$L_3$ as independent parameters in our approach. Fortunately, as will be demonstrated in section [11], the results depend on these parameters in a very weak way only. This is not surprising in the cases of $L_1$ and $L_2$ which occur only in the combination $(\varepsilon_\pi - \varepsilon_\eta)L_i$. Such terms effectively correspond to a NNLO effect.

To summarize, when concerning the approach to reparametrization of a safe observable $G$, we proceed along the following points:

- dangerous observables are not used
- the potentially dangerous inverted expansions are not performed
- an explicit dependence on the parameters of the $O(p^2)$ Lagrangian is kept ($F_0$ and $B_0 m_q$)
- some of the $O(p^4)$ LECs, namely $L_4, \ldots, L_8$, are expressed in terms of $F_P^2$ and $F_P^2 M_P^2$, using safe nonperturbative algebraical identities
- the result of the reparametrization is then expressed in terms of $O(p^2)$ parameters $X$, $Y$, $r$ and $R$, the remaining $O(p^4)$ LECs $L_1$, $L_2$ and $L_3$ and the direct ($\Delta_G$) and indirect ($\delta_{F_P}$, $\delta_{F_P M_P}$, $\delta_{\varepsilon_F}$, $\delta_{\varepsilon_{FM}}$) remainders

In the case of the observables $A, \ldots, D$, connected with the $\eta \to \pi^+\pi^-\pi^0$, only the polynomial part of $A$ and $B$ depend nontrivially on $L_4$-$L_8$, therefore only these have to be reparametrized. Explicit results are given in appendix [E].

This step completes the construction of the $\eta \to \pi^+\pi^-\pi^0$ amplitude in terms of safe observables within resummed $\chi PT$.

10 Treatment of free parameters and remainders

As we have discussed above, the chiral symmetry breaking parameters $X$ and $Z$ are treated as free. This approach is the opposite of the usual treatment of these parameters within

20
standard $\chi PT$, where they are predicted by means of the chiral expansion order by order, in terms of the physical masses, decay constants and LECs. Because the dependence of the observables on $X$ and $Z$ is held explicitly, it can also serve as a source of information on the mechanism of the chiral symmetry breaking in QCD.

Similarly, the ratio of the quark masses $r = m_s/\bar{m}$ is left as a free parameter too. In what follows, we fix this parameter to a recent averaged lattice QCD value, obtained by FLAG [49]

$$r = 27.5 \pm 0.4.$$  \hfill (115)

Analogously, in the case of the quark mass ratio $R = (m_s - \bar{m})/(m_d - m_u)$, we take an averaged lattice value [49] as well

$$R = 35.8 \pm 2.6.$$  \hfill (116)

A second kind of free parameters present in our calculation are the remainders. We have the direct remainders (96), namely

$$\delta_A, \delta_B, \delta_C, \delta_D,$$  \hfill (117)

which parametrize the unknown higher order contributions to the amplitude $G(s,t;u)$. Then there are the indirect remainders

$$\delta_{F_\pi^2 M_\pi^2}, \delta_{F_\eta^2 M_\eta^2}, \delta_{F_K^2 M_K^2}, \delta_{F_\pi^2 M_\eta^2}, \delta_{F_\pi^2 M_\pi}, \delta_{\varepsilon_{FM}}, \delta_{\varepsilon_F},$$  \hfill (118)

which stem from the higher order contributions to the safe observables $F_\pi^2 M_\pi^2, F_\eta^2, (\varepsilon_{FM} - \varepsilon_{F_\pi^2} M_\eta^2, (\varepsilon_{F_\pi^2} M_\pi^2 - \varepsilon_{F_\eta^2} M_\eta^2)$, i.e. emerge in the procedure of the reparametrization of the strict chiral expansion in terms of the masses and decay constants.

As we have discussed in section 8, the definition of the direct remainders depends on the fixing of the unitarity part of the amplitude, i.e. on the choice between (87) and (88). In the numerical analysis below, we use the alternative (88). The rationale for such a choice is that it is closer to the strict expansion, for which we assume the remainders should respect the global convergence criteria. The possibility (87), on the other hand, leads to a significant suppression of the unitarity corrections due to the factor $F_4^0/\prod_i F_i^2$ appearing in the loop functions, which appears quite unnatural.

In the presented paper, we treat the higher order remainders as a source of uncertainty of the theoretical prediction. Let us remind that contributions of the NNLO LECs $C_i$ are implicitly included in the remainders. Because the $C_i$’s are not well known, they represent an important source of theoretical uncertainty of the standard NNLO $\chi PT$ calculations, which are hard to quantify. In the resummed $\chi PT$ approach this uncertainty is supposed to be under better control as a consequence of the assumption of the global convergence.

Of course, any additional information on the actual values of the remainders both from inside the theory (higher order calculations) or from outside $\chi PT$ (estimates based on resonance contributions or a unitarization of the amplitude) can be used to reduce such an uncertainty.

As a first step, we do not assume any supplemental information. In what follows, we take the remainders as independent, uncorrelated, normally distributed random variables with zero mean value and a standard deviation attributed to them according to a rule based on the general expectation about the convergence of the chiral expansion of the safe observables. In accord with [38], we take

$$\delta^{NLO} = 0.0 \pm 0.3$$  \hfill (119)

$$\delta^{NNLO} = 0.0 \pm 0.1$$  \hfill (120)

\hfill 21
Figure 1: Distribution of the predictions for the Dalitz parameters $b$ and $\alpha$ at $X = Z = 0.5$ (histograms). For comparison, the solid line depicts a normal distribution with the same median and standard deviation around the median.

for NLO ($\delta_C$, $\delta_D$) and NNLO (the rest) remainders, respectively.

Besides the higher order remainders and the parameters $X$, $Z$, $r$ and $R$, the resulting formulae for the amplitude and Dalitz plot parameters depend on the constants $L_1$, $L_2$ and $L_3$. For these a similar reparameterization procedure as that described above for $L_4$–$L_8$ is not available. We therefore collect standard $\chi PT$ fits [50, 51, 52, 53] and by taking the mean and spread of such a set, we obtain an estimate of the influence of these LECs:

\begin{align*}
L_1^r(M_\rho) &= (0.57 \pm 0.18) \times 10^{-3} \\
L_2^r(M_\rho) &= (0.82 \pm 0.27) \times 10^{-3} \\
L_3^r(M_\rho) &= (-2.95 \pm 0.37) \times 10^{-3}.
\end{align*}

We ignore the reported error bars of the fits, as they are relatively small and in some cases there is quite a substantial disagreement among them. The variance we obtain is a source of theoretical uncertainty and we treat it on the same footing as the uncertainty stemming from the higher order remainders.

11 Numerical analysis

As explained above, we treat the remainders, the LECs $L_1$–$L_3$ and the quark mass parameters $r$ and $R$ as normally distributed random variables. This implies that at this stage our predictions are of stochastic nature. In what follows, we therefore numerically generate an ensemble of $4 \times 10^5$ normally distributed random sets of these parameters according to (115), (116), (119), (120), (121) and compute distributions for the observables under interest. Because the observables depend on these random variables in a complex nonlinear manner, the obtained range of theoretical predictions is in general distributed according to non-gaussian distributions (see fig. 1 for an illustration). In particular, the mean value of such a distribution can often be different than the median. The median of the distributions, however, in most of our cases coincide with the value obtained by setting the free parameters to their means very well. We therefore quote the median rather than the mean value in the following and the reported

\footnote{We use the current PDG values [10] for quantities not explicitly discussed above.}
error bars correspond to (a generally non-symmetric) one-sigma contour around it. To get a flavour of the values of the Dalitz plot parameters and of the uncertainty generated by the unknown remainders, as the first step, we provide resummed $\chi PT$ predictions of $a$, $b$, $d$ and $\alpha$ for a set of fixed values of $X$ and $Z$. In Table 3 we set $X$ and $Z$ to values obtained by the most recent standard $\chi PT$ fits [53]. In these fits, the parameters $X$ and $Z$ are obtained from the results for the NLO LECs, while $r$ is fixed to the lattice value (115).

The results collected in Table 3 show values which are consistent with the NNLO $\chi PT$ predictions [15] and with each other as well. However, while the parameters $a$ and $d$ are also compatible with the experimental data, the predictions for the parameters $b$ and $\alpha$ do not encompass the experimental values within the one-sigma uncertainty band. For the parameter $\alpha$, we reproduce the positive sign of the standard $\chi PT$ prediction [15] and the apparent disagreement with the experimental value. We might thus ask a question, whether the suggested tension really implies an incompatibility of the prediction with the experimental data at the indicated confidence level, and if yes, what is the reason for it. One possible explanation could then be that the assumed values of $X$ and $Z$ are not compatible with experiment, another that the assumption about the distributions of the remainders is not adequate and the bare chiral expansion of the apparently safe observables does not satisfy the criteria of global convergence.

Let us therefore take a closer look at the predicted distributions, while assuming a global convergence of the bare expansions and treating the remainders as above, and allow a variation of the parameters $X$ and $Z$ in a wider range. Namely, we will set $Z$ according to two scenarios ($Z=0.5$ and $Z=0.9$) and vary $X$ in the full range $0 < X < 1$. The results are depicted in figures 2 and 3 where we have shown the median (solid line) one-(dashed) and two-(dotted) sigma contours, as well as the experimental value (solid horizontal line with dashed error band). From the figures it is visible that the experimental value of the observable $a$ is compatible within the one-sigma contour for almost all the range of values of $X$ and $Z$, the same is true for the parameter $d$ as well.

As for the Dalitz parameter $b$, its experimental value is located close but inside the two-sigma contour (fig.2). We could thus conclude that we have a marginal compatibility. Note, however, that the theoretical distribution is non-gaussian and strongly constrained from below, see fig.1. Hence the one- and two-sigma contours are very close to each other and it is therefore difficult to make a definite statements on the compatibility of the theory and experiment.

Table 3: Illustration of the predictions of resummed $\chi PT$ for the Dalitz plot parameters in comparison with the experimental data. The input values of the parameters $X$, $Z$ and $r$ are taken from the most recent fit [53]. The upper and lower bounds correspond to a one-sigma interval around the median.

|                | $X_{\text{std}}$ | $Z_{\text{std}}$ | $r_{\text{std}}$ | $a$         | $b$         | $d$         | $\alpha$     |
|----------------|------------------|------------------|------------------|-------------|-------------|-------------|--------------|
| $R\chi PT + \text{BE14}$ | 0.628 | 0.593 | 27.5 | $-1.38^{+0.47}_{-0.56}$ | $0.48^{+0.48}_{-0.27}$ | 0.081$^{+0.060}_{-0.042}$ | 0.012$^{+0.018}_{-0.015}$ |
| $R\chi PT + \text{free fit}$ | 0.452 | 0.482 | 27.5 | $-1.43^{+0.53}_{-0.65}$ | $0.51^{+0.55}_{-0.30}$ | 0.084$^{+0.065}_{-0.047}$ | 0.013$^{+0.020}_{-0.016}$ |
| KLOE'16, '10     | 0.452 | 0.482 | 27.5 | $-1.095(4)$ | 0.145(6)    | 0.081(7)    | $-0.0301(50)$ |
| NNLO $\chi PT$   | $-1.271(75)$    | 0.394(102)       | 0.055(57)        | 0.013(32)   |

\[10^\text{This approach is different from the one we used in the preliminary report [42], where a gaussian distribution was implicitly assumed.}\]
Figure 2: The parameters $a$, $b$ and $d$ as a function of $X$ for $Z = 0.5$ and 0.9. The median (solid line), the one-sigma band (dashed, shadowed) and two-sigma band (dotted) are depicted together with the experimental value (solid horizontal line with dashed error band).
Concerning the neutral decay parameter $\alpha$, the dependence of the median on the parameters of $X$ and $Z$ is relatively mild (fig. 3). The theoretical distribution is non-gaussian again, with a long tail, as can be seen in fig. 1 as well. The experimental value lies inside the two-sigma contour in most of the range of $X$ and $Z$ (with an exception of very low values of $Z$, not shown here), but always very far from the one-sigma one. Note that by assuming a gaussian distribution with the same one sigma contour, one would be tempted to conclude that the two-sigma contour was much more narrow and that the experimental value were clearly incompatible, as was our preliminary result in [42].

Let us now investigate the qualitative predictions of resummed $\chi PT$ for the charged and neutral decay widths, $\Gamma^+$ and $\Gamma^0$, respectively. These are $R$ dependent observables, in contrast to the Dalitz plot parameters discussed above. We use the lattice average value for $R$ [16], with the corresponding error bar as a further source of the uncertainty of the prediction. The results are depicted in fig. 4. As can be seen, the obtained distributions of both widths are strongly $X$ and $Z$ dependent. For a relatively large range of the parameters, we observe good compatibility with the experimental values, while other regions can be excluded at $2\sigma$ C.L. Of course, changing the value of $R$, which is in principle a free parameter in the framework of resummed $\chi PT$, might modify the details of this picture. Qualitatively, however, we expect a similar behavior, as $R$ is present only through an overall normalization factor $1/R$ in the amplitude. The sensitivity of the observables $\Gamma^+$ and $\Gamma^0$ on the chiral symmetry breaking parameters $X$ and $Z$ and the existence of both compatibility and incompatibility regions seems to be promising for a more in-depth analysis of the parametric space of resummed $\chi PT$ with the aim of extracting the values of $X$ and $Z$. This issue we be discussed in a separate paper [40], preliminary results are already available [41].

Overall, we can conclude that there is no indication that the apparently terrible convergence of the decay widths, as discussed in the Introduction, imply a violation of the assumption of the global convergence of the chiral series and a large value of some of the higher order remainders.

For completeness, we have also depicted the $R$ independent ratio $r_T = \Gamma^0/\Gamma^+$, see fig 4. The prediction is compatible with the experimental value $r_T = 1.48 \pm 0.05$ (PDG average

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**Figure 3:** The parameter $\alpha$ as a function of $X$ for $Z = 0.5$ and 0.9. The median (solid line), the one-sigma band (dashed, shadowed) and two-sigma band (dotted) are depicted together with the experimental value (solid horizontal line with dashed error band).
The charged width $\Gamma$ as a function of $X$ for $Z = 0.5$.

The charged width $\Gamma$ as a function of $X$ for $Z = 0.9$.

The neutral width $\Gamma_0$ as a function of $X$ for $Z = 0.5$.

The neutral width $\Gamma_0$ as a function of $X$ for $Z = 0.9$.

The ratio $\frac{\Gamma_0}{\Gamma}$ as a function of $X$ for $Z = 0.5$.

The ratio $\frac{\Gamma_0}{\Gamma}$ as a function of $X$ for $Z = 0.9$.

Figure 4: The charged and neutral decay widths and their ratio as a function of $X$ for $Z = 0.2, 0.5$ and 0.9. The median (solid line), the one-sigma band (dashed, shadowed) and two-sigma band (dotted) are depicted together with the experimental value (solid horizontal line with dashed error band).
The charged width $\Gamma$ as a function of $X$ for $Z=0.5$.

Effect of LECs on Dalitz plot parameter $b$ as a function of $X$ for $Z=0.5$.

Figure 5: The effect of the uncertainty of the LECs on the charged width $\Gamma^+$ and on the parameter $b$ and and as a function of $X$ for $Z = 0.5$. The median (solid line), the one-sigma band (dashed, shadowed) and two-sigma band (dotted) are depicted.

Finally, let’s have a look on the issue of the dependence of the results on the uncertainty stemming from the weak knowledge of the LECs $L_1-L_3$. We can put all the other free parameters to their mean values and leave only the estimated uncertainty (121) at play. We have found the resulting errors to be negligible in all cases, as illustrated on two examples in fig.5.

12 Summary and outlook

The main purpose of this paper was an application of the formalism of resummed $\chi$PT to the $\eta \to 3\pi$ decays and addressing questions concerning convergence properties of various observables related to these decays.

As we have explained in detail, the standard assumption on the convergence of the chiral expansion has to be taken with some care and not all of the observables can be trusted to be automatically well convergent. The working hypothesis of the resummed approach is that only a limited set of safe observables has the property of global convergence, i.e. that the NNLO remainders are of a natural order of magnitude. Observables derived from the safe ones by means of nonlinear relations do not in general satisfy the criteria for global convergence due to the possible irregularities of the chiral series. Therefore, it is necessary to express such dangerous observables in terms of the safe ones in a non-perturbative way. This can be understood as a general procedure in cases when one encounters an expansion with significant irregularities. Also, one has to keep the higher order remainders explicit and not neglect them. In this paper, we have treated them as a source of theoretical uncertainty of the predictions.

As for the observables, we have concentrated on the Dalitz plot parameters $a$, $b$, $d$ of the charged channel, the parameter $\alpha$ of the neutral mode and on the decay widths of both channels. All these observables are dangerous in the above sense. Our results depend, besides the higher order remainders, on several free parameters - the chiral condensate, the chiral decay constant, the strange quark mass and the difference of light quark masses. These are expressed in terms of convenient parameters: $X$, $Z$, $r$ and $R$, respectively. The quark mass parameters have been fixed from lattice QCD averages [49]. There is also a residual
dependence on NLO LECs $L_1-L_3$, which we have shown to be very mild.

We have treated the uncertainties in the higher order remainders and other parameters statistically and numerically generated a large range of theoretical predictions, which have been then confronted with experimental information. Let us stress that at this point our goal is not to provide sharp predictions, as the theoretical uncertainties are large. Nevertheless, in this form, the approach is suitable for addressing questions which might be difficult to ask within the standard framework.

In the case of the decay widths, the experimental values can be reconstructed for a reasonable range of the free parameters and thus no tension is observed, in spite of what some of the traditional calculations suggest \cite{5, 9}. We have found a strong dependence of the widths on $X$ and $Z$ and an appearance of both compatibility ($<1\sigma \text{ C.L.}$) and incompatibility ($>2\sigma \text{ C.L.}$) regions. Such a behavior is not necessarily in contradiction with the global convergence assumption and, moreover, it might be promising for constraining the parameter space and an investigation of possible scenarios of the chiral symmetry breaking \cite{40, 41}.

As for the Dalitz plot parameters, $a$ and $d$ can be described very well too, within $1\sigma \text{ C.L.}$. However, when $b$ and $\alpha$ are concerned, we find a mild tension for the whole range of the free parameters, at less than $2\sigma \text{ C.L.}$ This marginal compatibility is not entirely unexpected. In the case of derivative parameters, obtained by expanding the amplitude in a specific kinematic point, in our case the center of the Dalitz plot, and depending on NLO quantities, the global convergence assumption is questionable, as discussed in section\cite{7}. Also, the distribution of the theoretical uncertainties is found to be significantly non-gaussian, so the consistency cannot be simply judged by the $1\sigma$ error bars.

This paper constitutes the first stage of our effort to gain information from the $\eta \rightarrow 3\pi$ decays. One application is the extraction of the parameters $X$ and $Z$ - the chiral condensate and the chiral decay constant. The theory seems to work well for the decay widths and the Dalitz plot parameter $a$ and thus it seems to be safe to use them for further analysis, which is under preparation \cite{40, 41}. Due to theoretical considerations mentioned above, one should be a bit more careful with regard to the parameter $d$, although it has been reconstructed just fine in this work.

The marginal compatibility in the case of the parameters $b$ and $\alpha$ can be interpreted in two ways - either some of the higher order corrections are indeed unexpectedly large or there is a specific configuration of the remainders, which is, however, not completely improbable. This warrants a further investigation of the higher order remainders by including additional information. Work is under way in analyzing $\pi\pi$ rescattering effects and resonance contributions, some preliminary results can be found in \cite{42}.

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A  Explicit form of the strict expansion of $G(s, t; u)$

In this appendix, we give a summary of formulae for various contributions to the strict chiral expansion of the amplitude $G(s, t; u)$ and to the mixing parameters $Z_{38}$ and $\mathcal{M}_{38}$. We write the amplitude $G(s, t; u)$ in the form

$$G(s, t; u) = Z_{38+} (s, t; u) - \varepsilon \pi Z_{33+} (s, t; u) + \varepsilon \eta Z_{88+} (s, t; u)$$  \hspace{1cm} (122)

and split the expansion of $Z_{ab+-}$ up to $O(p^4)$ according to

$$Z_{ab+-} = Z_{ab+-}^{(2)} + Z_{ab+-,ct}^{(4)} + Z_{ab+-,tad}^{(4)} + Z_{ab+-,unit}^{(4)} + \Delta Z_{ab+-},$$  \hspace{1cm} (123)

where the individual terms denote the $O(p^2)$, $O(p^4)$ counterterms, the tadpoles, the unitary contributions and the $O(p^6)$ remainder, respectively.

A.1  $O(p^2)$ contribution $Z_{ab+-}^{(2)}$

$$Z_{38+}^{(2)} (s, t; u) = -\frac{F_0^2 m_r}{6\sqrt{3} \pi} (r-1)$$  \hspace{1cm} (124)

$$Z_{33+}^{(2)} (s, t; u) = \frac{F_0^2}{3} \left( 3(s-s_0) + \frac{m_r^2}{\pi} \right)$$  \hspace{1cm} (125)

$$Z_{88+}^{(2)} (s, t; u) = \frac{1}{3} F_0^2 m_r^2$$  \hspace{1cm} (126)

A.2  $O(p^4)$ counterterm contributions $Z_{ab+-,ct}^{(4)}$

$$Z_{38+,-ct}^{(4)} (s, t; u) = -\frac{2m_r^2}{3\sqrt{3} \pi} \left[ 6L_4(s-2M_r^2) + 3L_5(s-2s_0) + 4m_r^2 \left( L_6(r+8) - 2L_7(r-4) + 8L_8 \right) \right]$$  \hspace{1cm} (127)

$$Z_{33+,-ct}^{(4)} (s, t; u) = 4(2L_1 + L_3)(s-M_r^2-M_r^2)(s-2M_r^2) + 4L_2 \left[ (t-M_r^2-M_r^2)(t-2M_r^2) + (u-M_r^2-M_r^2)(u-2M_r^2) \right] + 8L_4 m_r^2 \left[ s(r+4) - s_0(r+5) \right] + 4L_5 m_r^2 (4s-5s_0) + \frac{16 m_r^4}{3 \pi} \left[ L_6(r+8) + 4L_8 \right]$$  \hspace{1cm} (128)

$$Z_{88+,-ct}^{(4)} (s, t; u) = \frac{4}{3} (6L_1 + L_3)(s-M_r^2-M_r^2)(s-2M_r^2) + \frac{4}{3} (3L_2 + L_3) \left[ (t-M_r^2-M_r^2)(t-2M_r^2) + (u-M_r^2-M_r^2)(u-2M_r^2) \right]$$
\[ + \frac{8}{3} L_4 \frac{m_\pi^2}{\pi} \left[ (s - 2M_\pi^2)(2r + 4) - 3\Delta_{\pi\eta} \right] \\
- 4L_5 \frac{m_\pi^2}{\pi} s_0 \\
+ \frac{16}{3} m_\pi \left[ L_6(5r + 4) - 4L_7(r - 1) + 4L_8 \right] \] (129)

A.3 \( O(p^4) \) tadpole contributions \( Z_{ab,+,tad}^{(4)} \)

\[
Z_{83+,+,tad}^{(4)} = \frac{F_0^2 \frac{m_\pi^2}{\pi}}{6\sqrt{3} \pi} \left[ \frac{m_\pi}{\mu} (3r - 4) + 2\mu_K(r - 1) \right] \\
+ \frac{1}{3} \mu_\eta (r + 2) + 2R\Delta\mu_K \\
+ \frac{\sqrt{3}}{4} F_0^2 (s - s_0) \Delta\mu_K 
\]
(130)

\[
Z_{33+,+,tad}^{(4)} = -\frac{F_0^2 \frac{m_\pi^2}{\pi}}{3} \left[ 3\mu_\pi + 2\mu_K + \frac{1}{3} \mu_\eta \right] \\
- 7F_0^2 (s - s_0) \left[ \mu_\pi + \frac{1}{2} \mu_K \right] 
\]
(131)

\[
Z_{88+,+,tad}^{(4)} = -\frac{F_0^2 \frac{m_\pi^2}{\pi}}{3} \left[ 3\mu_\pi + 2\mu_K + \frac{1}{3} \mu_\eta \right] 
\]
(132)

We denote

\[
\mu_P = \frac{m_P^2}{\pi} \ln \left( \frac{m_P^2}{\mu^2} \right) = \frac{m_P^2}{2F_0^2} \left( J_{PP}(0) + \frac{1}{16\pi^2} \right) 
\]
(133)

\[
\Delta\mu_K = \mu_{K^0} - \mu_{K^+} = \frac{1}{32\pi^2 F_0^2} \Delta m_K^2 \left( \ln \frac{m_K^2}{\mu^2} + 1 \right) = -\frac{1}{2F_0^2} \Delta m_K^2 J_{K\bar{K}}(0) 
\]
(134)

\[
\Delta m_K^2 = \Delta m_{B0} = \frac{B_0 \tilde{m}(r - 1)}{R} 
\]
(135)

\[
\Delta M_K^2 = M_{K^0}^2 - M_{K^+}^2 
\]
(136)

\[
\Delta_{\pi\eta} = M_\eta^2 - M_\pi^2 
\]
(137)

\[
\Sigma_{\pi\eta} = M_\eta^2 + M_\pi^2 
\]
(138)

A.4 \( O(p^4) \) unitarity contributions \( Z_{ab,+,unit}^{(4)} \)

\[
Z_{83+,+,unit}(s, t, u) = -\frac{1}{4\sqrt{3} R} m_\pi^2 \left( s - 2M_\pi^2(2r - 3) + m_\pi^2(3r - 4) \right) J_{\pi\pi}(s) \\
- \frac{1}{8\sqrt{3} R} m_\pi^2 (r - 1) \left[ s - 2M_\pi^2 + 2m_\pi^2 \right] J_{K\bar{K}}(s) 
\]
where

$$\Delta J_{KK}(s) = J_{KK}^r(s) - J_{KK}^r(s)$$

$$= -2\Delta m^2_K J_{KK}^r(s) - J_{KK}^r(4m_K^2) + O\left(\frac{1}{R^2}\right)$$ (140)

$$Z^{(4)}_{33+-, \text{unit}}(s, t; u) = \frac{1}{2} \left[ s - 2M^2_\pi + 2m^2_\pi \right] \left[ s - 2M^2_\pi + 3m^2_\pi \right] J_{\pi\pi}^r(s)$$

$$- \frac{1}{2} \Delta_{\pi\eta} \left[ s - 2M^2_\pi + 2m^2_\pi \right] J_{\pi\pi}^r(s)$$

$$+ \frac{1}{8} \left[ s - 2M^2_\pi + 2m^2_\pi \right]^2 J_{KK}^r(s)$$

$$- \frac{1}{8} \Delta_{\pi\eta} \left[ s - 2M^2_\pi + 2m^2_\pi \right] J_{KK}^r(s)$$

$$+ \frac{1}{18} \Delta_{\pi\eta} \left[ s - 2M^2_\pi + m^2_\pi \right] J_{\pi\eta}^r(s)$$

$$+ \frac{1}{12} \left[ (s - u)(t - 4m^2_\pi) + 3(t - 2M^2_\pi)^2 \right] J_{\pi\pi}^r(t)$$

$$+ \frac{1}{4} \Delta_{\pi\eta} \left[ t - 2M^2_\pi \right] J_{\pi\pi}^r(t)$$

$$+ \frac{1}{288\pi^2} (s - u)(t - 6m^2_\pi) + 3(s - u)F^2_0 \mu_\pi$$

$$+ \frac{1}{24} (s - u) \left[ t - 4m_K^2 \right] J_{KK}^r(t)$$

$$+ \frac{1}{576\pi^2} (s - u) \left[ t - 6m_K^2 \right] + \frac{1}{6} (s - u) F^2_0 \mu_K$$
\[ Z^{(4)}_{\text{unit}}(s, t, u) = \frac{1}{3} m_\pi^2 \left[ -s - 2M_\pi^2 + \frac{3}{2} \right] J_{\pi\pi}(s) + \frac{1}{8} \left[ s - 2M_\pi^2 - 2m_\pi \right] \times \left[ 3s - 6M_\eta^2 + 4m_\eta - \frac{2}{3} \right] J_{KK}(s) + \frac{3}{8} \Delta_{\pi\eta} \left[ s - 2M_\pi^2 + \frac{2}{3} \right] J_{KK}(s) + \frac{4}{9} \left[ m_\eta - \frac{1}{4} \right] J_{\eta\eta}(s) + \frac{1}{9} m_\pi J_{\pi\eta}(t) + \frac{3}{8} \left[ t - M_\pi^2 - M_\eta^2 + \frac{2}{3} \right] ^2 J_{KK}(t) + \frac{3}{8} \Delta_{\pi\eta} \left[ t - M_\pi^2 - M_\eta^2 + \frac{2}{3} \right] J_{KK}(t) + \frac{1}{9} m_\pi J_{\eta\eta}(u) + \frac{3}{8} \left[ u - M_\pi^2 - M_\eta^2 + \frac{2}{3} \right] ^2 J_{KK}(u) + \frac{3}{8} \Delta_{\pi\eta} \left[ u - M_\pi^2 - M_\eta^2 + \frac{2}{3} \right] J_{KK}(u) \] (142)

Up to now, we have kept the masses at their \( O(p^2) \) values in all the loop functions \( J_{pQ}(s) \).

### A.5 Mixing parameters \( Z_{38} \) and \( M_{38} \)

The strict chiral expansion of the parameters \( Z_{38} \) and \( M_{38} \) to \( O(p^4) \) reads

\[ Z_{38} = \frac{8(1-r)B_0 \tilde{m}}{\sqrt{3} F_0^2 R} L_5 + \frac{\sqrt{3} \Delta \mu_K}{R} \] (143)

\[ M_{38} = \frac{(1-r)B_0 \tilde{m}}{\sqrt{3} R} + \frac{32B_0^2 \tilde{m}^2(1-r)}{\sqrt{3} F_0^2 R} \left[ (r+2)L_6 - 2(r-1) \right] + 2L_7 + 2L_8 \]

\[ + \frac{B_0 \tilde{m}}{\sqrt{3} R} \left[ 2 \Delta \mu_K + \frac{1}{3} \mu_\eta (r+2) + 2 \mu_K (r-1) + \mu_\pi (3r-4) \right] \] (144)
B Reconstruction of the unitarity part $U(s, t; u)$

According to the reconstruction theorem (for more details on the general method see [46, 47] and for the application to resummed $\chi$PT, see [45]), we get the following general formula for the unitarity part $U(s, t; u)$ of the amplitude

$$U(s, t; u) = \frac{1}{3} (W_0(s) - W_2(s)) + \frac{1}{2} (3(s - u)W_1(t) + W_2(t)) + \frac{1}{2} (3(s - t)W_1(u) + W_2(u)).$$  \hfill (145)

Here, $W_i(s)$ are uniquely defined up to a subtraction polynomial by appropriately subtracted dispersion integrals with discontinuities

$$\text{disc} W_0(s) = -32\pi\sqrt{3}\theta(s - 4M_\pi^2)\text{disc} A_{0,0}^0(s)$$

$$\text{disc} W_1(s) = \mp 32\sqrt{3}\theta(s - 4M_\pi^2)\frac{A_{1,1}^{1/1}(s)}{A_{1/1}^{1/1}(s, M_\pi^2, M_\pi^2)\sigma(s)}$$

$$\text{disc} W_2(s) = 32\pi\sqrt{\frac{3}{2}}\theta(s - 4M_\pi^2)\text{disc} A_{0,2}^0(s).$$  \hfill (146)

In the above expressions, $A_{l,l_3}^I(s)$ corresponds to an $l$–th partial wave amplitude in the channel $\eta\pi_{l_3} \to (\pi\pi)^{l,l_3}$, with fixed isospin and its third component in the final state. For the isospin decomposition, we use the Condon-Shortley phase convention

$$A_{0,0}^0(s, t; u) = -\frac{1}{\sqrt{3}} [3A(s, t; u) + A(t, s; u) + A(u, t; s)]$$  \hfill (147)

$$A_{2,0}^2(s, t; u) = \sqrt{\frac{2}{3}} [A(t, s; u) + A(u, t; s)]$$  \hfill (148)

$$A_{1,1}^{1/1}(s, t; u) = \mp \frac{1}{\sqrt{2}} [A(t, s; u) - A(u, t; s)].$$  \hfill (149)

The discontinuities of $W_i(s)$ (where $I = 0, 1, 2$) are fixed by unitarity. Up to kinematic factors, they correspond to two-particle intermediate state contributions to the right hand cut discontinuities of $A_{l,l_3}^I$

$$\text{disc} A_{l,l_3}^I(s) = \sum_{PQ} \theta(s - (M_P + M_Q)^2)$$

$$\times 2 \left( \frac{N \cdot N''}{N}\right) \frac{\lambda_{1/2}^{1/2}(s, M_P^2, M_Q^2)}{s} A_{l,l_3}^{\eta\pi_{l_3}\to PQ}(s) A_{l,l_3}^{\pi\pi\to PQ}(s) A_{l,l_3}^{Q\to PQ}(s)^*.$$  \hfill (150)

Here $N', N''$ and $N$ are normalization factors of the expansion of the amplitudes $A_{l,l_3}^{\eta\pi_{l_3}\to PQ}$, $A_{l,l_3}^{Q\to PQ}$ and $A_{l,l_3}^{PQ\to PQ}$ to the partial waves $A_{l,l_3}^{\eta\pi_{l_3}\to PQ}(s)$, $A_{l,l_3}^{PQ\to PQ}(s)$ and $A_{l,l_3}^{PQ\to PQ}(s)$, respectively. Schematically

$$A(s, t; u) = 32\pi N \sum_{l=0}^{\infty} (2l + 1) A_l(s) P_l(\cos \theta).$$  \hfill (151)
$S$ is a symmetry factor of the intermediate state $PQ$. As a result of the reconstruction, we get $W_i(s)$ as a sum of the contributions of the two-particle intermediate states in each channel

$$W_0(s) = W_{0}^{(4)\pi\pi}(s) + W_{0}^{(4)\pi\eta}(s) + W_{0}^{(4)\eta\eta}(s) + W_{0}^{(4)K\overline{K}}(s) \quad (152)$$

$$W_1(s) = W_{1}^{(4)\pi\pi}(s) + W_{1}^{(4)K\overline{K}}(s) \quad (153)$$

$$W_2(s) = W_{2}^{(4)\pi\pi}(s) + W_{2}^{(4)\eta\eta}(s) + W_{0}^{(4)K\overline{K}}(s). \quad (154)$$

For the reconstruction of the $O(p^4)$ functions $W_{I}^{(4)PQ}(s)$, with a help of $[146]$ and $[150]$, we need a complete set of coupled $O(p^2)$ amplitudes. The relevant $O(p^2)$ amplitudes in the $\pi\pi$, $\pi\eta$, $\eta\eta$ and $K\overline{K}$ channels, as well as the explicit form for $W_{I}^{(4)PQ}(s)$, are given in the following subsections.

As explained in detail in $[15]$, we use two possible ways how to treat the $O(p^2)$ amplitudes entering the reconstruction theorem. The reason is that there are two possibilities how to connect the generic physical $O(p^2)$ amplitude $A^{(2)}$ of the process $AB \rightarrow CD$ (which is a dangerous observable) and the corresponding safe observable $G^{(2)}$. In what follows, we give the formulae in accord with the choice

$$A^{(2)} = \frac{G^{(2)}}{F_AF_BF_CF_D}, \quad (155)$$

where $F_P$ is the physical decay constant of the PGB $P$. The second possibility corresponds to a replacement of $F_P \rightarrow F_0$ in the above formula. For this second possibility, the $W_{I}^{(4)PQ}(s)$ are easily obtained form the results presented below by means of a substitution of $F_P \rightarrow F_0$ on the right hand side of the expressions for $F_{\pi}^{\alpha}F_{\eta}W_{I}^{(4)PQ}(s)$.

### B.1 $\pi\pi$ intermediate state

The $\eta\pi^{I_3} \rightarrow (\pi\pi)^{I,I_3}$ partial wave amplitudes $A_{I}^{I,I_3}(s)$ are

$$F_{\pi}^{3}F_{\eta}A_{0}^{0,0}(s) = \frac{F_{\eta}^{2}}{16\pi^{3}} \left[ \varepsilon_{\pi}(s-s_{0}) + \frac{5}{18} B_{0} \tilde{m} \left( \sqrt{3} \frac{(r-1)}{R} + 6(\varepsilon_{\pi} - \varepsilon_{\eta}) \right) \right], \quad (156)$$

$$F_{\pi}^{3}F_{\eta}A_{1}^{1,1}(s) = \pm \frac{F_{\eta}^{2}}{96\pi^{3/2}} \varepsilon_{\pi} \lambda^{1/2}(s,M_{\pi}^{2},M_{\eta}^{2}) \sigma(s), \quad (157)$$

$$F_{\pi}^{3}F_{\eta}A_{0}^{2,0}(s) = \frac{F_{\eta}^{2}}{32\pi^{3}} \left[ \frac{2}{3} \varepsilon_{\pi}(s-s_{0}) - \frac{2}{9} B_{0} \tilde{m} \left( \sqrt{3} \frac{(r-1)}{R} + 6(\varepsilon_{\pi} - \varepsilon_{\eta}) \right) \right], \quad (158)$$

while the $(\pi\pi)^{I} \rightarrow (\pi\pi)^{I}$ partial wave amplitudes $A_{I}(s)$ can be written as

$$F_{\pi}^{4}A_{0}^{0}(s) = \frac{F_{\eta}^{2}}{16\pi} \left[ (s - \frac{4}{3} M_{\pi}^{2}) + \frac{5}{6} m_{\pi}^{2} \right], \quad (159)$$

$$F_{\pi}^{4}A_{1}^{1}(s) = \frac{F_{\eta}^{2}}{96\pi}(s - \frac{4}{3} M_{\pi}^{2}), \quad (160)$$

$$F_{\pi}^{4}A_{0}^{0}(s) = -\frac{F_{\eta}^{2}}{32\pi} \left[ (s - \frac{4}{3} M_{\pi}^{2}) - \frac{2}{3} m_{\pi}^{2} \right]. \quad (161)$$
We then get by using (146) and (150)

\[
F_\pi^3 F_\eta W_0^{(4)\eta\pi}(s) = -\frac{F_0^4}{F_\pi^2} \frac{\varepsilon_\pi(s - s_0) + \frac{5}{18} B_0 \hat{m}}{3 \sqrt{3} R} \left[ \left( \frac{r - 1}{R} + 6\varepsilon_\pi - 6\varepsilon_\eta \right) \right] + \frac{5 \varepsilon_\eta (1 + 8r)}{6 \sqrt{3} R} - 6\varepsilon_\pi \] (162)

\[
F_\pi^3 F_\eta W_1^{(4)\eta\pi}(s) = -\frac{F_0^4}{F_\pi^2} \frac{\varepsilon_\pi(s - s_0)}{18} \left[ \left( \frac{r - 1}{R} + 6\varepsilon_\pi - 6\varepsilon_\eta \right) \right] (s) (163)

\[
F_\pi^3 F_\eta W_2^{(4)\eta\pi}(s) = -\frac{F_0^4}{F_\pi^2} \frac{\varepsilon_\pi(s - s_0)}{18} \left[ \left( \frac{r - 1}{R} + 6\varepsilon_\pi - 6\varepsilon_\eta \right) \right] (s). (164)

### B.2 \( \eta\pi \) intermediate state

For the \( \eta\pi \rightarrow \eta\pi \) amplitude \( A_0^{\eta\pi}(s) \) (S-wave only), we get at \( O(p^2) \)

\[
F_\pi^2 F_\eta^2 A_0^{\eta\pi}(s) = \frac{F_0^2}{32\pi} \frac{\hat{m}_\pi^2}{3} \] (165)

and therefore, according to (146) and (150) and with a help of (156)

\[
F_\pi^3 F_\eta W_0^{(4)\eta\pi}(s) = -\frac{F_0^4}{F_\pi^2} \frac{\hat{m}_\pi^2}{3} \left[ \left( \frac{r - 1}{R} + 6\varepsilon_\pi - 6\varepsilon_\eta \right) \right] \] (166)

\[
F_\pi^3 F_\eta W_2^{(4)\eta\pi}(s) = -\frac{F_0^4}{F_\pi^2} \frac{\hat{m}_\pi^2}{3} \left[ \left( \frac{r - 1}{R} + 6\varepsilon_\pi - 6\varepsilon_\eta \right) \right] \] (167)

### B.3 \( \eta\eta \) intermediate state

For the \( \eta\pi \rightarrow \eta\eta \) amplitude \( A_0^{\eta\pi\rightarrow\eta\eta}(s) \) (S-wave only), we have

\[
F_\pi F_\eta A_0^{\eta\pi\rightarrow\eta\eta}(s) = \frac{F_0^2}{32\pi} \frac{\hat{m}_\pi^2}{6} \left[ \left( \frac{r - 1}{R} + \frac{2}{3} \varepsilon_\eta (1 + 8r) - 6\varepsilon_\pi \right) \right] \] (168)

and the \( \eta\eta \rightarrow (\pi\pi)^I.I_3 \) amplitude \( A_0^{\eta\eta\rightarrow(\pi\pi)^I.I_3}(s) \) (S-wave, \( I, I_3 = (0, 0) \) only) is

\[
F_\pi^2 F_\eta^2 A_0^{\eta\eta\rightarrow(\pi\pi)^I.I_3}(s) = -\frac{F_0^2}{32\pi} \frac{\hat{m}_\pi^2}{3 \sqrt{3}} \] (169)

We then get

\[
F_\pi^3 F_\eta W_0^{(4)\eta\eta}(s) = \frac{F_0^4}{F_\pi^3} \frac{\hat{m}_\pi^2}{12} \left[ \left( \frac{r - 1}{R} + \frac{2}{3} \varepsilon_\eta (1 + 8r) - 6\varepsilon_\pi \right) \right] \] (170)
In particular, because breaking, must be taken into account. Let and also the mass difference \( \Delta M^2_K = M^2_{K^0} - M^2_{K^\pm} \), which is of the first order in the isospin breaking, must be taken into account. Let

\[
\langle KK_{\text{out}} | \eta\pi(I_3)_{\text{in}} \rangle = i(2\pi)^4\delta^{(4)}(P_f - P_a)A_{KK}^{I_4}(s,t;u) \tag{172}
\]

and

\[
\langle \pi\pi(I_3)_{\text{out}} | K_{\text{in}} \rangle = i(2\pi)^4\delta^{(4)}(P_f - P_a)A_{KK}^{I_4}(s,t;u). \tag{173}
\]

Then it follows from the isospin decomposition of the amplitudes

\[
A_{KK}^{0}(s,t;u) = C^0(s,t;u) \pm B^0(s,t;u) = 16\pi \sum_l (2l + 1)(C_l^0(s) \pm B_l^0(s))P_l(\cos \theta_0), \tag{174}
\]

\[
A_{KK}^{\pm 1}(s,t;u) = C^\pm 1(s,t;u) \pm B^\pm 1(s,t;u) = 16\pi \sum_l (2l + 1)(C_l^1(s) \pm B_l^1(s))P_l(\cos \theta_1). \tag{175}
\]

Here \( C^{I_3} \) is the isospin conserving \( \Delta I = 0 \) and \( B^{I_3} \) is the isospin breaking \( \Delta I = 1 \) part of the amplitudes, and, to the first order in the isospin breaking (i.e. up to the corrections \( O((\Delta M^2_K)^2) \)),

\[
\cos \theta_0 = \frac{t - u + \frac{1}{4}\Delta M^2_K \Delta}{\lambda^{1/2}(s,M^2_{K^0},M^2_{K^\pm})} \left( 1 - \frac{4M^2_K}{s} \right)^{-1/2} \left( 1 \pm \frac{\Delta M^2_K}{s - 4M^2_K} \right) \tag{176}
\]

\[
\cos \theta_1 = \frac{t - u + \frac{1}{4}\Delta M^2_K \Delta}{\lambda^{1/2}(s,M^2_{K^0},M^2_{K^\pm})} \left( 1 - \frac{4M^2_K}{s} \right)^{-1/2}. \tag{177}
\]

In particular, because \( C^{\pm 1}(s,t;u) = C^{\pm 1}(s,u;t) \) as a consequence of the \( C \)-symmetry, we have \( C^\pm 1(s) = 0 \). In the same way

\[
A_{KK}^{I_4}(s,t;u) = \pm C^{I_4}(s,t;u) \pm B^{I_4}(s,t;u) = 16\pi \sqrt{2} \sum_l (2l + 1)(\pm C_l^{I_4}(s) \pm B_l^{I_4}(s))P_l(\cos \theta_0'), \tag{178}
\]

\[
A_{KK}^{1,\pm 1}(s,t;u) = C^{1,\pm 1}(s,t;u) \pm B^{1,\pm 1}(s,t;u) = 16\pi \sqrt{2} \sum_l (2l + 1)(C_l^{1,\pm 1}(s) \pm B_l^{1,\pm 1}(s))P_l(\cos \theta_1'), \tag{179}
\]

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where once again, $C^{I,I_3}$ and $B^{I,I_3}$ mean isospin conserving and breaking parts, respectively, and to the first order in the isospin breaking

\[
\cos \theta_0 = \frac{t - u}{\sqrt{s^2 - 4M^2_K}} \left( 1 - \frac{4M^2_K}{s} \right)^{-1/2} \left( 1 \pm \frac{\Delta M_K^2}{s - 4M^2_K} \right) \tag{180}
\]

\[
\cos \theta_1 = \frac{t - u}{\lambda^{1/2}(s, M^2_K, M^2_{K^*})} \left( 1 - \frac{4M^2_K}{s} \right)^{-1/2}. \tag{181}
\]

Once again, due to the $C$–invariance, $B^{1,\pm 1}(s, t; u) = B^{1,\pm 1}(s, u; t)$, so that $B_1^{1,\pm 1}(s) = 0$. Using the following formulae, valid up to the $O((\Delta M_K^2)^2)$ corrections,

\[
\frac{\lambda^{1/2}(s, M^2_K, M^2_{K^*})}{s} = \sqrt{1 - \frac{4M^2_K}{s}}, \tag{183}
\]

we can write for the contribution of the $K\bar{K}$ intermediate states to the discontinuities of the isospin partial waves along the right hand cut up to the first order in the isospin breaking

\[
disc A_0^{1,0}(s) = \frac{1}{\sqrt{2}} \sqrt{1 - \frac{4M^2_K}{s}} \left[ \frac{2\Delta M_K^2}{s - 4M^2_K} C_0^0(s) C_0^{I,0}(s) 
\right.
\nonumber
\]

\[
+ 2C_0^0(s) B_0^{I,0}(s) + 2B_0^{I,0}(s) C_0^{I,0}(s) \right] \tag{184}
\]

\[
disc A_1^{1,\pm 1}(s) = \pm \frac{1}{\sqrt{2}} \sqrt{1 - \frac{4M^2_K}{s}} \left[ \frac{2\Delta M_K^2}{s - 4M^2_K} B_1^{1,\pm 1}(s) C_1^{I,\pm 1}(s) \right]. \tag{185}
\]

We further need the $\eta\pi \to K\bar{K}$ amplitudes, for which we get

\[
F_\pi F_\eta F_{K^*} C^0(s, t; u) = -\frac{\sqrt{3}F_0^2}{4} \left[ (s - \frac{1}{3}M_\eta^2 - \frac{1}{3}M_\pi^2 - \frac{2}{3}M_K^2) - \frac{1}{3}(2m_K^2 - m_\eta^2 - m_\pi^2) \right] \tag{186}
\]

\[
F_\pi F_\eta F_{K^*} B^0(s, t; u) = \frac{3F_0^2}{4} \left[ \varepsilon_\eta \left( \frac{1}{3} \varepsilon_\pi \right) \right] \left[ (s - \frac{1}{3}M_\eta^2 - \frac{1}{3}M_\pi^2 - \frac{2}{3}M_K^2) - \frac{2}{3} \varepsilon_\eta \left( \frac{2m_K^2 - m_\eta^2 - m_\pi^2}{R} \right) \right] \tag{187}
\]

\[
F_\pi F_\eta F_{K^*} C^{\pm 1}(s, t; u) = -\frac{\sqrt{6}F_0^2}{4} \left[ (s - \frac{1}{3}M_\eta^2 - \frac{1}{3}M_\pi^2 - \frac{2}{3}M_K^2) + \frac{2}{3} \varepsilon_\eta \left( \frac{2m_K^2 - m_\eta^2 - m_\pi^2}{R} \right) \right] \tag{188}
\]

\[
F_\pi F_\eta F_{K^*} B^{\pm 1}(s, t; u) = \frac{F_0^2}{2\sqrt{2}} \varepsilon_\pi(t - u) \tag{189}
\]

\[37\]
and also the $KK \to (\pi\pi)^{I,J_3}$ amplitudes, which read

$$F_{\pi K}^2 C^{0,0}(s,t;u) = -\frac{\sqrt{3} F_0^2}{4} \left( s - \frac{2}{3} M_\pi^2 - \frac{2}{3} M_K^2 \right) - \frac{F_0^2}{2\sqrt{3}} (\frac{\alpha^2}{m_K^2} - \frac{\alpha^2}{m_\pi^2})$$  \hspace{1cm} (190)

$$F_{\pi K}^2 B^{0,0}(s,t;u) = \frac{F_0^2}{\sqrt{2}} \varepsilon_\eta \left( s - \frac{2}{3} M_\pi^2 - \frac{2}{3} M_K^2 \right) - \frac{F_0^2}{9} \varepsilon_\eta (m_K^2 - m_\pi^2)$$

$$- \frac{5 F_0^2}{12\sqrt{3} R} (m_K^2 - m_\pi^2) + \frac{F_0^2}{4\sqrt{3}} \Delta M_K^2$$  \hspace{1cm} (191)

$$F_{\pi K}^2 C^{2,0}(s,t;u) = 0$$  \hspace{1cm} (192)

$$F_{\pi K}^2 B^{2,0}(s,t;u) = -\frac{F_0^2}{\sqrt{2}} \varepsilon_\eta \left( s - \frac{2}{3} M_\pi^2 - \frac{2}{3} M_K^2 \right) + \frac{\sqrt{2} F_0^2}{9} \varepsilon_\eta (m_K^2 - m_\pi^2)$$

$$+ \frac{F_0^2}{3\sqrt{6} R} (m_K^2 - m_\pi^2)$$  \hspace{1cm} (193)

$$F_{\pi K}^2 C^{1,1}(s,t;u) = -\frac{1}{2 F_{\pi K}^2} (t - u)$$  \hspace{1cm} (194)

$$F_{\pi K}^2 B^{1,1}(s,t;u) = 0$$  \hspace{1cm} (195)

Putting all these ingredients together, with the help of (146) and (150), we get the final result

$$F_{\pi K}^3 W_0^{(4)K_K}(s) = \frac{3\sqrt{3} F_0^4}{8 F_{\pi K}^2} \Delta M_K^2 \left[ \left( s - \frac{1}{3} M_\eta^2 - \frac{1}{3} M_\pi^2 - \frac{2}{3} M_K^2 \right) \right.$$  \hspace{1cm} (196)

$$- \frac{1}{3} (2m_K^2 - m_\pi^2 - m_\eta^2) \right]$$

$$\times \left[ \left( s - \frac{2}{3} M_\pi^2 - \frac{2}{3} M_K^2 \right) + \frac{2}{3} (m_K^2 - m_\pi^2) \right]$$

$$\times \left( J_{KK}(s) - J_{KK}(4M_K^2) \right) \frac{3\sqrt{3} F_0^2}{4 F_{\pi K}^2} \left( \frac{3}{2} \left( \varepsilon_\eta - \frac{1}{3} \varepsilon_\pi \right) \right) \left( s - \frac{1}{3} M_\eta^2 - \frac{1}{3} M_\pi^2 - \frac{2}{3} M_K^2 \right)$$

$$- \frac{1}{8\sqrt{3} R} (2m_K^2 - m_\pi^2 - m_\eta^2 - 2R \Delta M_K^2)$$

$$- \frac{1}{6} \varepsilon_\pi (m_K^2 + m_\pi^2) + \frac{1}{6} \varepsilon_\eta (3m_K^2 - m_\pi^2)$$

$$\times \left[ \left( s - \frac{2}{3} M_\pi^2 - \frac{2}{3} M_K^2 \right) + \frac{2}{3} (m_K^2 + m_\pi^2) \right] J_{KK}(s)$$

$$+ \frac{1}{2} \left( \varepsilon_\eta (s - \frac{2}{3} M_\pi^2 - \frac{2}{3} M_K^2) - \frac{1}{9} \varepsilon_\eta (m_K^2 - m_\pi^2) \right)$$

$$- \frac{5}{12\sqrt{3} R} (m_K^2 - m_\pi^2) + \frac{1}{4\sqrt{3}} \Delta M_K^2$$

$$\times \left[ \left( s - \frac{1}{3} M_\eta^2 - \frac{1}{3} M_\pi^2 - \frac{2}{3} M_K^2 \right) - \frac{1}{3} (2m_K^2 - m_\pi^2 - m_\eta^2) \right] J_{KK}(s) \right)$$

$$\left. \left. \right. \right]$$

$$F_{\pi K}^2 F_0 W_1^{(4)K_K}(s) = -\frac{F_0^2}{F_{\pi K}^2} \varepsilon_\pi (s - 4M_K^2) J_{KK}(s)$$  \hspace{1cm} (197)
\[ F_2^3 F_1 W^{(4)RR}(s) = \frac{F_1^2}{F_2^3} \left[ \sqrt{\frac{3}{\pi}} \varepsilon_\eta (s - \frac{2}{3} M_\pi^2 - \frac{2}{3} M_K^2) - \frac{2}{9} \varepsilon_\eta (\bar{m}_K - \bar{m}_\pi) \right. \\
\left. - \frac{1}{3\sqrt{3} R} (\bar{m}_K - \bar{m}_\pi) \right] \\
\times \left[ (s - \frac{1}{3} M_\eta^2 - \frac{1}{3} M_\pi^2 - \frac{2}{3} M_K^2) - \frac{1}{3} (2\bar{m}_K - \bar{m}_\pi - \bar{m}_\eta) \right] J_{KK}(s). \]

(198)

C Unitarity contribution to the polynomial part

In this appendix, we summarize the result of the matching of the strict expansion with the dispersive reconstruction of the amplitude, as explained in section 8. Let us remind that the resulting polynomial part of the amplitude can be written in the form

\[ G_{\text{pol}}(s, t; u) = G^{(2)}(s, t; u) + G^{(4)}_{ct}(s, t; u) + G^{(4)}_{\text{tad}}(s, t; u) + G^{(4)}_{\text{pol}, u}(s, t; u), \]

(199)

where the listed contributions correspond to the leading order, counterterms, tadpoles and unitarity part, respectively. The strict expansion of the former three contributions can be found in appendix A. Here we will concentrate on the unitarity contribution

\[ G^{(4)}_{\text{pol}, u}(s, t; u) = G^{(4)}_{\text{pol}, u83}(s, t; u) - \varepsilon_\pi G^{(4)}_{\text{pol}, u33}(s, t; u) + \varepsilon_\eta G^{(4)}_{\text{pol}, u88}(s, t; u), \]

(200)

defined as

\[ G^{(4)}_{\text{pol}, u83}(s, t; u) = Z_{83+,-,\text{unit}}(s, t; u)|_{\mathcal{J}^P Q \to 0}, \]

(201)
\[ G^{(4)}_{\text{pol}, u33}(s, t; u) = Z_{33+,-,\text{unit}}(s, t; u)|_{\mathcal{J}^P Q \to 0}, \]

(202)
\[ G^{(4)}_{\text{pol}, u88}(s, t; u) = Z_{88+,-,\text{unit}}(s, t; u)|_{\mathcal{J}^P Q \to 0}. \]

(203)

As a result, we get

\[ G^{(4)}_{\text{pol}, u83}(s, t; u) = -\frac{1}{8\sqrt{3} R} m_\pi^2 \left[ \frac{2}{3} (s - 2 M_\pi^2) (2r - 3) + 2 \bar{m}_\pi (3r - 4) + \Delta_{\pi\eta} - s \right] J_{\pi\pi}^r(0) \]

\[ -\frac{1}{16 \sqrt{3} R} m_\pi^2 (r - 1) \left[ 8 \bar{m}_\pi^2 - 3 M_\eta^2 - M_\pi^2 - s \right] J_{KK}^r(0) \]

\[ -\frac{(r + 2)}{36 \sqrt{3} R} m_\pi^2 J_{\pi\eta}^r(0) \]

\[ -\frac{1}{8 \sqrt{3} R} m_\pi^2 \left[ \frac{4}{3} m_\pi^2 r + 3 (s - s_0) \right] J_{\pi\eta}^r(0). \]

(204)
\[ G_{\text{pol,}u33}^{(4)}(s, t; u) = \left\{ \frac{1}{2} \left[ s - 2M^2 + m^2 \right] \left[ s - 2M^2 + 3m^2 \right] \right. \]
\[ - \frac{1}{12} \left[ 12m^2(s - s_0) - (s - u)t - (s - t)u \right] \]
\[ + \frac{1}{4} \left[ (t - 2M^2)^2 + (u - 2M^2)^2 \right] \]
\[ - \frac{1}{4} \Delta_{\pi\eta} \left[ s - 8M^2 + 4m^2 + 3s_0 \right] J_{\pi\pi}(0) \]
\[ + \left\{ s - 2M^2 + 2m^2 \right\} \left[ s - M^2 - M^2 + 2m^2 \right] \]
\[ - \frac{1}{24} \left[ 12m^2(s - s_0) - (s - u)t - (s - t)u \right] \}
\[ J_{KK}^r(0) \]
\[ + \frac{1}{18} \frac{\hat{m}^2}{\pi} J^r_{\pi\eta}(0) + F_{\pi}^2(s - s_0) \left( \mu_\pi + \frac{1}{2} \mu_K \right) \]
\[ + \frac{1}{192\pi^2} (s - u) \left[ t - 2m_K^2 - 4m^2 \right] \]
\[ + \frac{1}{192\pi^2} (s - t) \left[ u - 2m_K^2 - 4m^2 \right] \]
\[ (205) \]

and

\[ G_{\text{pol,}u88}^{(4)}(s, t; u) = \frac{1}{3} \frac{\hat{m}^2}{\pi} \left[ s - 2M^2 + 3m^2 \right] J^r_{\pi\pi}(0) \]
\[ + \frac{2}{9} \frac{\hat{m}^2}{\pi} J^r_{\pi\eta}(0) + \frac{2}{9} \frac{\hat{m}^2}{\pi} \left[ \frac{3}{2} \left( \frac{\hat{m}^2}{\pi} - \frac{1}{4} \frac{\hat{m}^2}{\pi} \right) \right] J^r_{\pi\eta}(0) \]
\[ + \left\{ \frac{1}{8} \left[ s - 2M^2 + 2m^2 \right] \right. \]
\[ \times \left[ 3s - 6M^2 + 4m^2 - \frac{2}{3} m^2 \right] \]
\[ + \frac{3}{8} \left[ t - M^2 - M^2 + \frac{2}{3} m^2 \right] \]
\[ + \frac{3}{8} \left[ u - M^2 - M^2 + \frac{2}{3} m^2 \right] \}
\[ J^r_{KK}(0) \]
\[ + \frac{3}{8} \Delta_{\pi\eta} \left[ -M^2 - M^2 + \frac{10}{3} m^2 \right] J^r_{KK}(0). \]
\[ (206) \]

D \quad \text{Bare expansion of the observables} \ A, \ldots, \ D

The observables \( A, \ldots, D \) correspond to an expansion of \( G(s, t; u) \) in the center of the Dalitz plot

\[ G(s, t; u) = A + B(s - s_0) + C(s - s_0)^2 + D \left[ (t - s_0)^2 + (u - s_0)^2 \right] + \ldots. \]
\[ (207) \]

The splitting of the amplitude

\[ G(s, t; u) = Z_{33}^{(-)}(s, t; u) - \epsilon_\pi Z_{33}^{(0)}(s, t; u) + \epsilon_\eta Z_{88}^{(-)}(s, t; u), \]
\[ (208) \]
and the further splitting into the polynomial and the unitarity parts (which corresponds to \( \text{(95)} \))

\[
Z_{ab+}(s, t; u) = Z_{ab+, \text{pol}}(s, t; u) + Z_{ab+, \text{unitary}}(s, t; u),
\]

(209)

induce analogous splitting for \( A, \ldots, D \). We, therefore, write

\[
\mathcal{C} = (\mathcal{C}_{83}^p + \mathcal{C}_{83}^u) - \varepsilon_\pi (\mathcal{C}_{33}^p + \mathcal{C}_{33}^u) + \varepsilon_\eta (\mathcal{C}_{88}^p + \mathcal{C}_{88}^u),
\]

(210)

where \( \mathcal{C} = A, \ldots, D \). \( \mathcal{C}_{ab}^p \) stems from the polynomial part and \( \mathcal{C}_{ab}^u \) from the unitarity corrections. In the following formulae, we have abbreviated \( L_i^r(\mu) \equiv L_i \). The results are given in the following subsections (we write \( \Delta_0 = m_\eta^2 - m_\pi^2, \Delta_{PQ} = M_P^2 - M_Q^2, \Sigma_{PQ} = M_P^2 + M_Q^2, \Delta M_K^2 = (M_{K0}^2 - M_{K1}^2)_{QCD} \)).

### D.1 Polynomial contributions

\[
A_{83}^p = \frac{-F_0^2(m_\eta^2 - m_\pi^2)}{4\sqrt{3}R}
+ \frac{1}{16\sqrt{3}R} m_\pi M_\eta^2 \left( 4 (J_{\pi\pi}^r(0) + 8L_4 - 4L_5) - 5J_{KK}^r(0) \right)
- \frac{\sqrt{3}}{16R} m_\pi M_\eta^2 \left( J_{KK}^r(0) + 4 (J_{\pi\pi}^r(0) + 8L_4 + 4L_5) \right)
- \frac{1}{12\sqrt{3}R} \left[ 9J_{KK}^r(0) + 3 (J_{\pi\pi}^r(0) + 6J_{\pi\pi}^r(0) + 96L_6 + 192L_7 + 128L_8) + J_{\eta\eta}^r(0) \right]
+ \frac{1}{16\sqrt{3}R} \left[ 5J_{KK}^r(0) - 4 (J_{\pi\pi}^r(0) + 8L_4 - 4L_5) \right]
+ \frac{\sqrt{3}}{16R} m_\eta M_\pi^2 \left( J_{KK}^r(0) + 4 (J_{\pi\pi}^r(0) + 8L_4 + 4L_5) \right)
+ \frac{1}{48\sqrt{3}R} \left[ 45J_{KK}^r(0) + 2 \left( -J_{\eta\eta}^r(0) + 2J_{\pi\pi}^r(0) + 44J_{\pi\pi}^r(0) + 720L_6 + 864L_7 + 768L_8 \right) \right]
- \frac{1}{48\sqrt{3}R} \left[ 9J_{KK}^r(0) + 2 \left( J_{\eta\eta}^r(0) + 144 (L_6 - 2L_7) \right) \right]
+ \frac{25m_\pi^4}{768\sqrt{3}^2R} - \frac{7m_\pi^2 m_\eta^2}{384\sqrt{3}^2R} - \frac{11m_\eta^4}{768\sqrt{3}^2R}
\]

(211)

\[ -A_{33}^p = \frac{1}{3} F_6^2 m_\pi + \frac{1}{12} m_\pi^2 M_\eta^2 \left( J_{KK}^r(0) + 4 (J_{\pi\pi}^r(0) + 8L_4 + 4L_5) \right)
+ \frac{1}{4} m_\pi M_\eta^2 \left( J_{KK}^r(0) + 4 (J_{\pi\pi}^r(0) + 8L_4 + 4L_5) \right)
- \frac{1}{36} m_\pi m_\eta \left( 9J_{KK}^r(0) + 2 J_{\eta\eta}^r(0) + 288L_6 \right)
+ \frac{1}{36} M_\eta^4 \left( J_{KK}^r(0) + 8 (J_{\pi\pi}^r(0) + 8L_1 + 8L_2 + 4L_3) \right)
- \frac{1}{12} M_\pi^2 M_\eta^2 \left( J_{KK}^r(0) + 8 (J_{\pi\pi}^r(0) + 8L_1 + 8L_2 + 4L_3) \right)
- \frac{1}{36} m_\pi^4 \left( 21J_{KK}^r(0) + 2J_{\eta\eta}^r(0) + 72J_{\pi\pi}^r(0) + 1440L_6 + 768L_8 \right)
\]
\[
A_{88}^p = \frac{1}{3} F_0^2 \frac{m_\pi^2}{m_\eta}
- \frac{1}{36} m_\pi^2 M_\eta^2 (25 J_{KK}^r(0) - 4 J_{\pi\pi}^r(0) + 192 L_4 + 48 L_5)
- \frac{1}{12} m_\pi^2 M_\pi^2 (5 J_{KK}^r(0) + 4 J_{\pi\pi}^r(0) + 48 L_5)
+ \frac{1}{36} m_\pi^2 m_\eta^2 (45 J_{KK}^r(0) + 10 J_{\eta\pi}^r(0) + 1440 L_6 - 1152 L_7)
+ \frac{1}{6} m_\pi^2 M_\eta^2 (J_{KK}^r(0) + 16 L_4) - \frac{1}{2R} m_\pi^2 M_\pi^2 (J_{KK}^r(0) + 16 L_4)
- \frac{1}{36} M_\eta^4 (9 J_{KK}^r(0) + 32 (L_1 + L_2) + L_3)
+ \frac{1}{12} M_\pi^2 M_\eta^2 (9 J_{KK}^r(0) + 32 (L_1 + L_2) + L_3)
+ \frac{1}{36} m_\pi^4 (9 J_{KK}^r(0) + 4 (9 J_{\pi\pi}^r(0) + 8 L_6 + 32 L_7) + 2 J_{\pi\eta}^r(0) + 192 L_8) - 2 J_{\eta\eta}^r(0))
+ \frac{7m_\pi^4}{192\pi^2} + \frac{11m_\pi^2 m_\eta^2}{576\pi^2}
\]

\[
B_{88}^p = \frac{\sqrt{3}}{16R} m_\pi^2 (J_{KK}^r(0) - 2 J_{\pi\eta}^r(0) + 6 J_{\pi\pi}^r(0) + 32 L_4 + 16 L_5)
- \frac{\sqrt{3}}{16R} m_\eta^2 (J_{KK}^r(0) + 4 (J_{\pi\pi}^r(0) + 8 L_4 + 4 L_5))
\]

\[
B_{33}^p = F_0^2 + \frac{1}{4} m_\pi^2 (3 J_{KK}^r(0) + 16 (J_{\pi\pi}^r(0) + 7 L_4 + 4 L_5))
\frac{3}{4} m_\pi^2 J_{KK}^r(0) + 16 L_4) - \frac{4}{3} (2L_1 - L_2 + L_3) (M_\eta^2 + 3M_\pi^2)
+ \frac{M_\eta^2}{192\pi^2} + \frac{M_\pi^2}{64\pi^2} + \frac{9m_\pi^2}{64\pi^2} + \frac{3m_\eta^2}{64\pi^2}
\]

\[
B_{88}^p = \frac{1}{6} m_\pi^2 (J_{KK}^r(0) + 2 J_{\pi\pi}^r(0) + 48 L_4) + \frac{1}{2} m_\eta^2 (J_{KK}^r(0) + 16 L_4)
+ \frac{1}{3} (4L_2 - 8 L_1) (M_\eta^2 + 3M_\pi^2)
\]

\[
C_{88}^p = 0
\]

\[
C_{33}^p = -\frac{1}{96\pi^2} + \frac{1}{24} (J_{KK}^r(0) + 8 J_{\pi\pi}^r(0) + 192 L_1 + 96 L_3)
\]

\[
C_{88}^p = \frac{1}{24} (9 J_{KK}^r(0) + 32 (6 L_1 + L_3))
\]

\[
D_{88}^p = 0
\]

\[
D_{33}^p = \frac{1}{24} (J_{KK}^r(0) + 8 J_{\pi\pi}^r(0) + 96 L_2) + \frac{1}{192\pi^2}
\]

42
\[ D_{88}^p = \frac{1}{24} (9J_{KK}(0) + 96L_2 + 32L_3) \]
\[ B_{s3}^u = \frac{F^4}{F_K^2} \frac{1}{288} \bar{J}_K \left( s_0 \right) \left( -4 \Delta_{K\pi} + 8 \hat{m}_{\pi}^2 + 3 \Delta_0 \right) \]
\[ \times \left( -4M_K^2 + 2 \left( 4 \hat{m}_{\pi}^2 + \Sigma_{\eta\pi} \right) + 3 \Delta_0 \right) + \frac{F^4}{F_K^2} \frac{5}{18} \bar{J}_{\eta\pi} \left( s_0 \right) \hat{m}_{\pi}^2 \left( \hat{m}_{\pi}^2 + \Delta_{\eta\pi} \right) \]
\[ + \frac{F^4}{F_K^2} \frac{1}{24} \bar{J}_K \left( s_0 \right) \left( -16M_K^2 + 6s_0 + 8 \hat{m}_{\pi}^2 + 3 \Delta_0 \right) + \frac{F^4}{F_K^2} \frac{1}{6} \bar{J}_{\pi\pi} \left( s_0 \right) \left( M_{\eta}^2 - 5M_K^2 + 10 \hat{m}_{\pi}^2 \right) \]
\[ + \frac{F^4}{F_K^2} \frac{1}{2} \bar{J}_{\pi\pi} \left( s_0 \right) \hat{m}_{\pi}^2 \] (227)

\[ B_{s3}^u = \frac{F^4}{F_K^2} \frac{1}{12} \bar{J}_K \left( s_0 \right) \Delta_0 \left( 4 \hat{m}_{\pi}^2 + 3s_0 - 4M_K^2 \right) \]
\[ + \frac{F^4}{F_K^2} \frac{1}{18} \bar{J}_{\pi\pi} \left( s_0 \right) \hat{m}_{\pi}^2 \left( \Delta_{\eta\pi} - 8 \Delta_{K\pi} + 4 \hat{m}_{\pi}^2 \right) \] (227)
\[
\frac{4R}{\sqrt{3}} C_{83}^{\eta} = \frac{F^4}{F_K^4} \frac{1}{12} J_{KK}^\eta(s_0) \Delta_0^2 \\
+ \frac{F^4}{F_K^4} \frac{1}{18} J_{\eta\eta}^\eta(s_0) m_\pi^2 \left( 3m_\pi^2 + 4\Delta_0 \right) \\
+ \frac{F^4}{F_K^4} \frac{1}{18} J_{\eta\pi}^\eta(s_0) m_\pi^2 \left( \alpha^2 + \Delta_{\eta\pi} \right) \\
+ \frac{F^4}{F_K^4} \frac{1}{6} J_{KK}(s_0) \left( 4m_\pi^2 + 3\Delta_0 \right) + \frac{F^4}{F_K^4} \frac{5}{6} J_{\pi\pi}(s_0) m_\pi^2
\]
\[- \frac{F_4^4}{F_\pi^3} \hat{J}'(M_\pi^2, M_\pi^2, s_0) \Delta_0 \]

\[
\frac{4R}{\sqrt{3}} C_{33}^{\eta_0} = \frac{F_4^4}{F_\pi^5} \frac{1}{144\sqrt{3}} \hat{J}''(M_K^2, M_K^2, s_0) \left( -4 \Delta_{K\pi} + 8 m_\pi^2 + 3 \Delta_0 \right) \\
\times \left( -4 M_K^2 + 2 \left( 4 m_\pi^2 + \Sigma_{\eta\pi} \right) + 3 \Delta_0 \right) \\
+ \frac{F_4^4}{F_\pi^5} \frac{1}{18\sqrt{3}} \hat{J}'(M_K^2, M_K^2, s_0) \left( 12 M_K^2 + 3 s_0 + 24 m_\pi^2 + 9 \Delta_0 \right) \\
+ \frac{F_4^4}{F_\pi^5} \frac{2}{9\sqrt{3}} \hat{J}''(M_\pi^2, M_\pi^2, s_0) \frac{M_\pi^4}{m_\pi} + \frac{F_4^4}{F_\pi^5} \frac{1}{3\sqrt{3}} \hat{J}''(M_\eta^2, M_\eta^2, s_0) \frac{m_\pi^4}{m_\eta}
\]

\[
\frac{4R}{\sqrt{3}} C_{88}^{\eta_0} = \frac{F_4^4}{F_\pi^5} \frac{1}{6\sqrt{3}} \hat{J}'(M_K^2, M_K^2, s_0) \\
\times \left( -12 M_K^2 + 9 s_0 + 16 m_\pi^2 + 9 \Delta_0 \right) \\
+ \frac{F_4^4}{F_\pi^5} \frac{1}{36\sqrt{3}} \hat{J}''(M_K^2, M_K^2, s_0) M_K^4 \\
\times \left( 18 s_0 + 32 m_\pi^2 + 18 \Delta_0 \right) \\
+ \frac{F_4^4}{F_\pi^5} \frac{1}{3\sqrt{3}} \hat{J}''(M_K^2, M_K^2, s_0) M_K^4 \\
+ \frac{F_4^4}{F_\pi^5} \frac{1}{144\sqrt{3}} \hat{J}''(M_\pi^2, M_\pi^2, s_0) M_\pi^4 \\
\times \left( 36 M_\eta^2 + 112 m_\pi^2 + 51 \Delta_0 - 12 \Sigma_{\eta\pi} \right) \\
+ \frac{F_4^4}{F_\pi^5} \frac{1}{144\sqrt{3}} \hat{J}''(M_K^2, M_K^2, s_0) \\
\times \left( 16 m_\pi^2 M_\eta^2 + 15 \Delta_0 M_\eta^2 + 36 M_\pi^4 + 64 m_\pi^4 + 96 m_\pi^2 \Delta_0 + 3 \Delta_0 \Sigma_{\eta\pi} + 27 \Delta_0^2 \right) \\
+ \frac{F_4^4}{F_\pi^5} \frac{2}{9\sqrt{3}} \hat{J}''(M_\pi^2, M_\pi^2, s_0) \frac{m_\pi^4}{m_\eta}
\]

\[
\frac{4R}{\sqrt{3}} C_{88}^{\eta_0} = \frac{F_4^4}{F_\pi^5} \frac{1}{9\sqrt{3}} \hat{J}'(M_\eta^2, M_\eta^2, s_0) \frac{3 m_\pi^2}{3 m_\pi^2 + 4 \Delta_0}
\]
In this appendix, we list our final formulae for the Dalitz plot expansion parameters $A_{ab}$ and $B_{ab}$.

In this appendix, we list our final formulae for the Dalitz plot expansion parameters $A_{ab}$ and $B_{ab}$ (for the definition see appendix D), expressed in terms of the physical masses and

\begin{align}
D_{53}^a &= -\frac{F^4}{F^4_{\pi} 48\sqrt{3}R^2} J''_{\pi K} (s_0) \Delta_0 + \frac{F^4}{F^4_{\pi} 48\sqrt{3}R^2} J''_{\pi K} (s_0) \Delta_0 (4\Delta_{K\pi} + \Delta_0) \\
D_{33}^a &= -\frac{F^4}{F^4_{\pi} 72} J''_{\pi K} (s_0) (12M_K^2 - 3s_0) + \frac{F^4}{F^4_{\pi} 18\sqrt{3}R} J''_{\pi K} (s_0) \frac{m_{\pi}}{2} \left( 2m_{\pi}^2 - \Delta_{\eta\pi} \right) \\
D_{58}^a &= \frac{F^4}{F^4_{\pi} 8} J''_{\pi K} (s_0) \left( 3s_0 - 4M_K^2 - \Delta_0 \right) + \frac{F^4}{F^4_{\pi} 18} J''_{\pi K} (s_0) \frac{m_{\pi}}{2} \left( 2m_{\pi}^2 - \Delta_{\eta\pi} \right) \\
&- \frac{F^4}{F^4_{\pi} 36} J''_{\pi K} (s_0) \frac{m_{\pi}}{2} + \frac{F^4}{F^4_{\pi} 24} J''_{\pi K} (s_0) + \frac{F^4}{F^4_{\pi} 8} J''_{\pi K} (s_0)
\end{align}

Note that all the above observables are renormalization scale independent, which can be verified by using the explicit $\mu$ dependence of the chiral logarithms and $L_i$.

E  Reparameterization of $A_{ab}^p$ and $B_{ab}^p$

In this appendix, we list our final formulae for the Dalitz plot expansion parameters $A_{ab}^p$ and $B_{ab}^p$ (for the definition see appendix D), expressed in terms of the physical masses and

\begin{align}
D_{83}^a &= -\frac{F^4}{F^4_{\pi} 32R} J''_{KK} (s_0) \Delta_0 + \frac{F^4}{F^4_{\pi} 128\sqrt{3}R} J''_{KK} (s_0) \Delta_0 (4\Delta_{K\pi} + \Delta_0) \\
D_{33}^a &= -\frac{F^4}{F^4_{\pi} 72} J''_{KK} (s_0) (12M_K^2 - 3s_0) + \frac{F^4}{F^4_{\pi} 18\sqrt{3}R} J''_{KK} (s_0) \frac{m_{\pi}}{2} \left( 2m_{\pi}^2 - \Delta_{\eta\pi} \right) \\
D_{58}^a &= \frac{F^4}{F^4_{\pi} 8} J''_{KK} (s_0) \left( 3s_0 - 4M_K^2 - \Delta_0 \right) + \frac{F^4}{F^4_{\pi} 18} J''_{KK} (s_0) \frac{m_{\pi}}{2} \left( 2m_{\pi}^2 - \Delta_{\eta\pi} \right) \\
&- \frac{F^4}{F^4_{\pi} 36} J''_{KK} (s_0) \frac{m_{\pi}}{2} + \frac{F^4}{F^4_{\pi} 24} J''_{KK} (s_0) + \frac{F^4}{F^4_{\pi} 8} J''_{KK} (s_0)
\end{align}
decay constants, the LECs $L_1$, $L_2$ and $L_3$, the parameters $X (Y)$, $Z$ and $r$ and the indirect remainders (namely $\delta_{F_\pi M_\eta^2}$, $\delta_{F_\pi^2}$, $\delta_{F_\pi M_\eta K}$, $\delta_{F_K M_\eta^2}$, $\delta_{F_\pi^2 M_\eta^2}$, $\delta_{F_\pi M_\eta}$, $\delta_{\varepsilon FM}$ and $\delta_{\varepsilon F}$). We use the notation

\[ r_s^* = \frac{2F_K^2 M_K^2}{F_\pi^2 M_\pi^2} - 1 \]

(235)

\[ \epsilon(r) = \frac{2r_s^* - r}{r^2 - 1} \]

(236)

\[ \eta(r) = \frac{2}{r - 1} \left( \frac{F_K^2}{F_\pi^2} - 1 \right) \]

(237)

\[ \Delta_{GMO} = \frac{3F_\eta^2 M_\eta^2 + F_\pi^2 M_\eta^2 - 4F_K^2 M_K^2}{F_\pi^2 M_\pi^2} \]

(238)

\[ \frac{4R}{3\sqrt{3}} A_{s3}^p = -\frac{2}{9} X (r - 1) F_\pi^2 M_\pi^2 \]

\[ -\frac{2}{9} (r) \frac{(r - 1)(r + 1)}{(r + 2)} F_\pi^2 M_\pi^2 + \frac{1}{9} \eta(r) \frac{(r - 1)}{(r + 2)} F_\pi^2 ((r + 4) M_\eta^2 + 3 M_\pi^2 r) \]

\[ + \frac{2}{9} \Delta_{GMO} \frac{(r - 4)}{(r - 1)} F_\pi^2 M_\pi^2 \]

\[ + \frac{2}{9} (Z - 1) \frac{(r - 1)}{(r + 2)} F_\pi^2 (M_\eta^2 - 3 M_\pi^2) + \frac{2}{9} (X - 1) (r + 8) \frac{(r - 1)}{(r + 2)} F_\pi^2 M_\pi^2 \]

\[ + \frac{1}{18} Y \frac{M_\eta^2 M_\pi^2}{(r + 2)} \]

\[ \times (2 (3r^2 - 8) J_{KK}(0) - (r + 4)(2r + 1) J_{K\eta}(0) + (9 - 4r)r + 20) J_{\pi\pi}(0)) \]

\[ - \frac{1}{6} Y \frac{M_\eta^2 M_\pi^2}{(r + 2)} (2r (J_{KK}(0) + J_{\eta\eta}(0) - 2 J_{\pi\pi}(0)) + J_{K\eta}(0) - J_{\pi\pi}(0)) \]

\[ + \frac{1}{9} Y^2 \frac{M_\eta^2}{(r + 2)} \]

\[ \times [(2(r(3r + 4) + 8) J_{KK}(0) - (r(9r + 8) + 16) J_{\pi\pi}(0)) \]

\[ + r(5r + 4) J_{\eta\eta}(0) - 2r (r + 2) J_{\pi\pi}(0)] \]

\[ + \frac{M_\pi^2}{864\pi^2} Y (r - 1) (5 M_\pi^2 (22 Y - 9) - 3 M_\eta^2) \]

\[ - \frac{2}{9} \frac{((r + 4) M_\eta^2 + 3 M_\pi^2 r)}{(r + 2)} F_\pi^2 \delta_{F_\pi} + \frac{2}{9} \frac{((2r + 3) M_\eta^2 + 3 M_\pi^2)}{(r + 2)} F_\pi^2 \delta_{F_\pi} \]

\[ - \frac{2}{3} \frac{(r - 4)}{(r - 1)} F_\pi^2 M_\eta^2 M_\eta M_\pi - \frac{2}{9} \frac{(r(r + 3) + 3) - 16}{(r^2 + r - 2)} F_\pi^2 M_\eta^2 \delta_{F_\pi} M_\pi \]

\[ + \frac{8}{9} \frac{(2r + 1)(r^2 + r - 8)}{(r^3 + 2r^2 - r - 2)} F_K^2 M_K^2 \delta_{F_K M_K} \]

(239)
\[
\begin{align*}
&\frac{4}{\sqrt{3}} A_{ss}^\rho = \frac{4}{3\sqrt{3}} X F_\pi^2 M_\pi^2 + \frac{4}{3\sqrt{3}} \frac{e(r) r(2r+1)}{(r+2)} F_\pi^2 M_\pi^2 \\
&+ \frac{2}{9\sqrt{3}} \frac{\pi(r)}{(3-3r)} F_\pi^2 M_\pi^2 \\
&+ \frac{8}{\sqrt{3}} \Delta_{GMO} \frac{1}{(3-3r)} M_\pi^2 \\
&+ \frac{4}{9\sqrt{3}} \frac{M_\eta^2 (2r-5) - 3 M_\pi^2 (2r+1)}{(r+2)} F_\pi^2 \\
&+ \frac{4}{9\sqrt{3}} \frac{(X-1)(5r+4)}{(r+2)} F_\pi^2 M_\pi^2 \\
&- \frac{1}{9\sqrt{3}} (9 J_{KK}(0) + 32 (2 L_1 + L_2 + L_3)) M_\eta^2 (M_\eta^2 - 3 M_\pi^2) \\
&+ \frac{1}{9\sqrt{3}} Y \frac{M_\pi^2}{(r^2 + r - 2)} \\
&\times [3 M_\pi^2 (2 (8 - 5r^2) J_{KK}(0) + (7 - 2r) r + 4) J_{\eta\eta}(0) + (r(12r - 7) - 20) J_{\pi\pi}(0)] \\
&- M_\eta^2 (2 (r(r+12) - 16) J_{KK}(0) + (-14r^2 + r + 4) J_{\eta\eta}(0) + (r(12r - 25) + 28) J_{\pi\pi}(0))] \\
&+ \frac{2}{9\sqrt{3}} \frac{Y^2 M_\pi^4}{(r^2 + r - 2)} [6(r(3r - 2) - 4) J_{KK}(0) + 4 (r^2 + r - 2) J_{\eta\eta}(0) \\
&+ (5r - 2) r + 4) J_{\pi\pi}(0) + 9((2 - 3r)r + 4) J_{\pi\pi}(0)] \\
&+ \frac{M_\pi^2}{432\sqrt{3}\pi^2} \frac{Y [3(19 - 4r) M_\eta^2 + M_\pi^2 (9(4r + 11) - 2(44r + 67) Y)]}{(r^2 + r - 2)} F_K^2 \delta_{FK} \\
&- \frac{4}{9\sqrt{3}} \frac{((4 - 7r) M_\eta^2 + 3 M_\pi^2 (r - 4))}{F_K^2 \delta_{FK}} F_\pi^2 M_\pi^2
\end{align*}
\]
\[
\begin{align*}
\frac{4}{\sqrt{3}} B_{s3}^p &= -\frac{1}{3} \eta(r) \frac{r(r-1)}{(r+2)} F_\pi^2 \\
&+ \frac{2}{3} (Z-1) \frac{(r-1)}{(r+2)} F_\pi^2 \\
&+ \frac{1}{6} Y \frac{M_\pi^2}{(r+2)} \left[ 2r^2 J^r_{KK}(0) + 2 (2r^2 + r + 3) J^r_{\pi\pi}(0) \\
&+ (2r + 1) r^r_{\eta\eta}(0) - 3(r + 2) J^r_{\eta\eta}(0) \right] \\
&+ \frac{5}{96 \pi^2} Y (r-1) M_\pi^2 \\
&+ \frac{2}{3} \frac{1}{(r+2)} \left( r F_\pi^2 \delta_{FK} - F_\pi^2 \delta_{Fp} \right) \\
\end{align*}
\]

\[
\begin{align*}
\frac{4}{\sqrt{3}} B_{s8}^p &= \frac{4}{\sqrt{3}} Z F_\pi^2 + \frac{4}{\sqrt{3}} \eta(r) \frac{r}{(r+2)} F_\pi^2 \\
&- \frac{4}{\sqrt{3}} (Z-1) \frac{(r+4)}{(r+2)} F_\pi^2 \\
&- \frac{16}{\sqrt{3}} ((2L_1 - L_2) + L_3) s_0 \\
&- \frac{2}{\sqrt{3} \pi^2} Y \frac{M_\pi^2 r^r}{(r^2 + r - 2)} \left[ 2r (J^r_{KK}(0) + J^r_{\eta\eta}(0) - 2 J^r_{\pi\pi}(0)) \\
+ J^r_{\eta\eta}(0) - J^r_{\pi\pi}(0) \right] \\
&+ \frac{s_0}{16 \sqrt{3} \pi^2} - \frac{5 M_\pi^2}{8 \sqrt{3} \pi^2} Y \\
&- \frac{8}{\sqrt{3} (r^2 + r - 2)} F_\pi^2 \delta_{FK} - \frac{4}{\sqrt{3} (r^2 + r - 2)} F_\pi^2 \delta_{Fp} \\
\end{align*}
\]
\[ -\frac{8}{3\sqrt{3}} \frac{(r + 1)}{(r - 1)} F^2 \pi \delta F, \]  

(244)

F \hspace{1cm} \pi^0 - \eta \text{ mixing at } O(p^4)

In this appendix, we discuss the interrelation between the safe observable \( G(s, t; u) \) and the scattering amplitude in the presence of the \( \pi^0 - \eta \) mixing in more detail. We can write the generating functional in the form

\[
Z[a] = \frac{1}{2} \int d^4x \left[ Z_{33} \left( \partial \pi_3[a] - a_3 F_0 \right)^2 + Z_{88} \left( \partial \eta_8[a] - a_8 F_0 \right)^2 + 2Z_{38} \left( \partial \pi_3[a] - a_3 F_0 \right) \cdot \left( \partial \eta_8[a] - a_8 F_0 \right) \right. \\
\left. - M_{33} \pi_3[a]^2 - M_{88} \eta_8[a]^2 - 2M_{38} \pi_3[a] \eta_8[a] \right] + \ldots + O(p^6),
\]

(245)

where we have restricted ourselves to the \( \pi_3 - \eta_8 \) sector. Let us remind that we treat the mixing parameters \( Z_{38}, M_{38} \) at the leading order in the \( 1/R \) expansion. The corresponding equations of motion are equivalent, up to higher order corrections, to a stationarity condition for the generating functional itself, namely

\[
\frac{\delta Z[a]}{\delta \phi} = 0,
\]

(246)

or explicitly

\[
Z_{33} \partial^2 \pi_3[a] + Z_{38} \partial^2 \eta_8[a] + M_{33} \pi_3[a] + M_{38} \eta_8[a] = F_0 Z_{33} \partial \cdot a_3 + F_0 Z_{38} \partial \cdot a_8 + \ldots \\
Z_{38} \partial^2 \pi_3[a] + Z_{88} \partial^2 \eta_8[a] + M_{38} \pi_3[a] + M_{88} \eta_8[a] = F_0 Z_{38} \partial \cdot a_3 + F_0 Z_{88} \partial \cdot a_8 + \ldots
\]

(247)

(248)

We can diagonalize the kinetic terms by means of an orthogonal transformation \( O^T O = 1 \)

\[
Z = O^T \cdot \text{diag}(Z_3, Z_8) \cdot O,
\]

(249)

where the eigenvalues satisfy \( Z_3, Z_8 > 0 \). After rescaling, we get

\[
Z = O'^T \cdot 1 \cdot O',
\]

(250)

with

\[
O' = \text{diag}(Z_3^{1/2}, Z_8^{1/2}) \cdot O.
\]

(251)

Subsequently, we diagonalize the transformed mass terms \( (O'^{-1})^T M O'^{-1} \) with another orthogonal transformation \( O''^T O'' = 1 \), which does not effect the kinetic term

\[
(O'^{-1})^T \cdot M \cdot O'^{-1} = O''^T \cdot \text{diag}(M_{\pi^0}^2, M_{\eta}^2) \cdot O''.
\]

(252)

As a result, we can write

\[
M = G^T \cdot \text{diag}(M_{\pi^0}^2, M_{\eta}^2) \cdot G,
\]

(253)

where the matrix \( G \) reads

\[
G = O'' \cdot O' = O'' \cdot \text{diag}(Z_3^{1/2}, Z_8^{1/2}) \cdot O.
\]

(254)

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Then we can rewrite the generating functional in terms of the physical fields $\pi^0$ and $\eta$, according to

$$Z[a] = \frac{1}{2} \int d^4x [(\partial \pi^0[a] - a^0 F_0)^2 + (\partial \eta[a] - a^\eta F_0)^2 - M_{\pi^0}^2 \pi^0[a]^2 - M_{\eta}^2 \eta[a]^2] + \ldots + O(p^6),$$  

with

$$\begin{pmatrix} \pi^0[a] \\ \eta[a] \end{pmatrix} = G \begin{pmatrix} \pi_3[a] \\ \eta_8[a] \end{pmatrix},$$  

$$\begin{pmatrix} a^\pi \\ a^\eta \end{pmatrix} = G \begin{pmatrix} a_3 \\ a_8 \end{pmatrix}.$$  

In terms of this fields, the equations of motion has become diagonal

$$\begin{align*}
(\partial^2 + M_{\pi^0}^2) \pi^0[a] &= F_0 \partial \cdot a^\pi + \ldots, \\
(\partial^2 + M_\eta^2) \eta[a] &= F_0 \partial \cdot a^\eta + \ldots,
\end{align*}$$  

and therefore the functional derivatives of their solutions with respect to $a_3$ are

$$\begin{align*}
\frac{\delta \pi^0[a]}{\delta a_3} &= \frac{F_0}{\partial^2 + M_{\pi^0}^2} \delta \partial \cdot a^\pi + \ldots = F_0 G^\pi_{\pi^0} \frac{\partial}{\partial^2 + M_{\pi^0}^2} + \ldots, \\
\frac{\delta \eta[a]}{\delta a_3} &= \frac{F_0}{\partial^2 + M_\eta^2} \delta \partial \cdot a^\eta + \ldots = F_0 G^\eta_{\eta} \frac{\partial}{\partial^2 + M_\eta^2} + \ldots,
\end{align*}$$  

or more generally

$$\frac{\delta P[a]}{\delta a_i} = F_0 G^{P_i} \frac{\partial}{\partial^2 + M_P^2} + \ldots.$$  

For the second functional derivatives, we therefore get

$$\frac{1}{i} \frac{\delta^2 Z[a]}{\delta a_i \delta a_j} \bigg|_{a=0} = i F_0^2 \sum_P \frac{\partial}{\partial^2 + M_P^2} G^{P_i} G^{P_j} + \ldots$$  

and thus

$$\langle 0 | j^{\mu}_5(0) | p, P \rangle = i p_\mu F_0 G^{P_i}.$$  

We can identify the elements of the matrix $G$ to be

$$G^{P_i} = \frac{F^{P_i}}{F_0}.$$  

The entries of the matrix $F$ in terms of the physical decay constants are

$$F = \begin{pmatrix} F_{\pi^0} & F_{\pi^0} \\ F_{\eta^3} & F_{\eta^8} \end{pmatrix} = \begin{pmatrix} F_{\pi} & \varepsilon_{\pi} F_{\pi} \\ -\varepsilon_{\eta} F_{\eta} & F_{\eta} \end{pmatrix},$$  

where $\varepsilon_i = O(1/R)$ are the mixing angles. The inverse matrix to the first order in the isospin breaking then takes the form

$$F^{-1} = \begin{pmatrix} F_{\pi}^{-1} & -\varepsilon_{\pi} F_{\pi}^{-1} \\ \varepsilon_{\eta} F_{\pi} & F_{\eta}^{-1} \end{pmatrix} + O\left( \frac{1}{R^2} \right).$$  

52
The LSZ formulae give, for $p^2 \rightarrow M_P^2$

$$\langle 0 | \hat{j}^{i}_{\mu 5}(p) \cdots | 0 \rangle = \frac{i}{p^2 - M_P^2} \langle 0 | j^{i}_{\mu 5}(0) | p, P \rangle \langle p, P | \cdots | 0 \rangle + \text{reg.}$$

$$= \frac{i}{p^2 - M_P^2} i F_0 G^{P_i} p_\mu \langle p, P | \cdots | 0 \rangle + \text{reg.} \quad (268)$$

Writing the generating functional in the physical basis

$$Z[a] = \sum_n \int d^4 x_1 \cdots d^4 x_n P_1[a](x_1) \cdots P_n(x_n) V_n(x_1, \ldots, x_n)[a], \quad (269)$$

we symbolically get (tilde denotes a Fourier transform here)

$$\langle 0 | \hat{j}^{i}_{\mu 5}(p) \cdots | 0 \rangle = \frac{i}{p^2 - M_P^2} i F_0 G^{P_i} \tilde{V}_n + \text{reg.} \quad (270)$$

$$= \frac{i}{p^2 - M_P^2} i F_0 G^{P_i} \tilde{V}_n + \text{reg.} \quad (271)$$

This means that in order to extract the physical amplitudes from the generating functional, we can diagonalize the $O(p^4)$ kinetic and mass terms and then use the generating functional as a non-local Lagrangian. The diagonalization is achieved by the substitution

$$\pi_3 = \frac{F_0}{F_\pi} \pi^0 - \varepsilon_\pi \frac{F_0}{F_\eta} \eta \quad (272)$$

$$\eta_8 = \frac{F_0}{F_\eta} \eta + \varepsilon_\eta \frac{F_0}{F_\pi} \pi^0 \quad (273)$$

in the generating functional $Z[a]$.

Alternatively, one can work in the $\pi_3, \eta_8$ basis. The following relation between the safe observable $G$, in terms of the original fields, and the physical amplitude is then obtained

$$G^{i_1 i_2 i_3 i_4}(s, t; u) = \sum_{P_1 P_2 P_3 P_4} F^{i_1 P_1} \cdots F^{i_4 P_4} A_{P_1 P_2 P_3 P_4}(s, t; u). \quad (274)$$

Solving this relation algebraically with respect to the amplitude $A_{P_1 P_2 P_3 P_4}(s, t; u)$ is equivalent to using the diagonalization procedure in the first approach.
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