Lower bounds for the number of closed billiard trajectories of period 2 and 3 in manifolds embedded in Euclidean space

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1 Introduction

Let $M$ be a smooth closed $m$-dimensional manifold embedded in Euclidean space $\mathbb{R}^n$. An ordered set $(x_1, \ldots, x_p) \in M \times \ldots \times M$ is called a periodic (closed) billiard trajectory if the condition

$$\frac{x_i - x_{i+1}}{\|x_i - x_{i+1}\|} + \frac{x_i - x_{i-1}}{\|x_i - x_{i-1}\|} \perp T_{x_i}M$$

holds for each $i \in \mathbb{Z}_p$.

In this paper we find lower bounds for the number of closed billiard trajectories of periods 2 and 3 in an arbitrary manifold.

Let us introduce some notations:

$$M^{\times p} = M \times \ldots \times M$$

is $p$th Cartesian power of the space $M$,

$$\tilde{\Delta}^{(k)} = \bigcup_{i \in \mathbb{Z}_p} \{(x_1, \ldots, x_p) : x_i = x_{i+1} = \ldots = x_{i+p-k-1}\} \subset M^{\times p}$$

is the $k$th diagonal of $M^{\times p}$. Clearly, we have

$$\{(x, x, \ldots, x)\} = \tilde{\Delta}^{(0)} \subset \tilde{\Delta}^{(1)} \subset \ldots \subset \tilde{\Delta}^{(p-2)} \subset \tilde{\Delta}^{(p-1)} = M^{\times p}.$$
By $\tilde{\Delta}$ we denote $\tilde{\Delta}^{(p-2)} = \bigcup_{i \in \mathbb{Z}_p} \{x_i = x_{i+1}\} \subset M^{\times p}$. Let
\[
\tilde{l} = \sum_{i \in \mathbb{Z}_p} \|x_i - x_{i+1}\| : M^{\times p} \to \mathbb{R}
\]
be the length function of a closed polygon. The function $\tilde{l}$ is smooth outside of the diagonal $\tilde{\Delta}$. The symmetric group $S_p$ acts on $M^{\times p}$ by permutations of factors, and so does the dihedral group $D_p \subset S_p$. Let
\[
D^p(M) = M^{\times p}/D_p, \quad \Delta^{(k)} = \Delta^{(k)} / D_p \subset D^p(M), \quad l = \tilde{l} / D_p : D^p(M) \to \mathbb{R}.
\]
By $RD^p(M)$ we denote the reduced dihedral power of $M$, that is $D^p(M)$ with the diagonal $\Delta = \Delta^{(p-2)}$ contracted to a point.

It is well known that $p$-periodical billiard trajectories are the critical points of the function $f$ outside of $\Delta$. Thus our problem is to estimate the minimal number $BT_p(M)$ of critical points of the function $l$ for all generic embeddings of the manifold $M$ in Euclidean space. Here we say that an embedding $M \to \mathbb{R}^n$ is generic if the function $l$ has only non-degenerate critical points outside of $\Delta$.

Recall the following facts:

**Theorem** (see [5]) Let $p$ be a prime integer. Then
\[
BT_p(M) \geq \sum \dim H_q(M^{\times p}/D_p, \Delta; \mathbb{Z}_2).
\]

**Theorem** (M. Farber, S. Tabachnikov, [3], [4]) Let $p$ be an odd prime. Then
\[
\sum \dim \tilde{H}_q(RD^p(S^m); \mathbb{Z}_2) = m(p - 1).
\]

In this paper we estimate the number of closed billiard trajectories of period 2 and 3 for an arbitrary manifold. The estimate for period 2 was proved by P. Pushkar in [2], we give a new proof.

**Theorem** Let $M$ be a smooth closed $m$-dimensional manifold. By $B$ denote the sum of Betti numbers $\sum_{q=0}^m \dim H_q(M; \mathbb{Z}_2)$. Then we have the following estimates for the number of periodical billiard trajectories:
\[
BT_2(M) \geq \frac{B^2 + (m - 1)B}{2},
\]
\[
BT_3(M) \geq \frac{B^3 + 3(m - 1)B^2 + 2B}{6}.
\]
2 A. Dold’s theory of homology of symmetric products

In this section we consider \( \mathbb{Z}_2 \)-modules.

**Definition** An FD-module \( K \) is a sequence of modules \( K_q, q = 0, 1, \ldots \) and module morphisms \( \partial^q_i : K_q \to K_{q-1}, s^q_i : K_q \to K_{q+1}, i = 0, 1, \ldots \). The morphisms \( \partial^q_i \) are called face operators, \( s^q_i \) are called degeneracy operators. The face- and degeneracy- operators satisfy the following axioms:

- \( \partial^q_i = 0, s^q_i = 0 \) if \( i > q \)
- \( \partial^q_{i-1}\partial^q_q = \partial^{q-1}_q\partial^q_i \) if \( i < j \)
- \( s^q_{i+1}s^q_q = s^{q+1}_q s^q_i \) if \( i \leq j \)
- \( \partial^q_{i+1}s^q_q = s^{q-1}_q \partial^q_i \) if \( i < j \)
- \( \partial^{q+1}_q s^q_i = \partial^q_{q+1}s^q_i = id \) if \( i \leq q \)
- \( \partial^{q+1}_q s^q_{i+1} = s^{q-1}_q \partial^q_{i-1} \) if \( i > j + 1 \)

**Definition** Given an FD-module \( K \), put

\[
R(K)_q = \bigcap_{i < q} \ker(\partial^q_i : K_q \to K_{q-1}), \quad q = 0, 1, \ldots
\]

Clearly, \( \partial^q_i(R(K)_q) \subseteq R(K)_{q-1} \) and \( \partial^q_{q-1}\partial^q_q|_{R(K)_q} = 0 \). So \( R \) is a functor from FD-modules to chain modules. Homology groups are

\[
H_q(K) = H_q(R(K)) = \ker \partial^q_q / \text{im} \partial^{q+1}_q.
\]

Another way to define homology groups is as follows. The boundary operator is \( \partial_q = \partial^0_q - \partial^1_q + \partial^2_q - \ldots \), the \( q \)th homology group is \( H_q = \ker \partial_q / \text{im} \partial_{q+1} \). Denote this functor by \( R' \).

**Proposition** (A. Dold, [6])

1. There exists a functor \( R^{-1} \) from chain modules to FD-modules such that \( RR^{-1} \) and \( R^{-1}R \) are naturally equivalent to the respective identity functors.
2. Let $K$ be an FD-module. Chain modules $R(K)$ and $R'(K)$ are homotopically equivalent and therefore they have isomorphic homology groups.

3. The functor $R$ moves homotopically equivalent FD-modules (to be defined later) to homotopically equivalent chain modules.

A morphism of FD-modules (or FD-morphism) $i : K \to K'$ is a set of linear mappings $i_q : K_q \to K'_q$ such that $i_{q-1}\partial^i = \partial i_q$ and $i_{q+1}s^i = s^i i_q$. A morphism of FD-modules induces a homomorphism of homology groups $i_*$. 

**Example** Let $\Delta_q$ be the standard $q$-dimensional simplex:

$$\Delta_q = \left\{ (x_0, \ldots, x_q) \in \mathbb{R}^{q+1} : x_i \geq 0, \sum x_i = 1 \right\}.$$ 

Introduce the mappings

$$\varepsilon^i_q : \Delta_q \to \Delta_{q+1}, \quad \varepsilon^i_q(x_0, \ldots, x_q) = (x_0, \ldots, x_{i-1}, 0, x_i, \ldots, x_q),$$

$$\eta^i_q : \Delta_q \to \Delta_{q-1}, \quad \eta^i_q(x_0, \ldots, x_q) = (x_0, \ldots, x_{i-1} + x_i, x_{i+1}, \ldots, x_q).$$

Suppose $X$ is a topological space, $K_q$ is the group of its singular $q$-chains. The elements of $K_q$ are finite sums $\sum a_k f_k$, where $f_k : \Delta_q \to X$ are continuous maps. Now introduce face- and degeneracy-operators:

$$\partial^i_q(f) = f \circ \varepsilon^i_{q-1},$$

$$s^i_q(f) = f \circ \eta^i_{q+1}.$$ 

We have constructed an FD-module $K(X)$ of a topological space $X$. 

Let $x_0$ be a distinguished point of the space $X$. All simplices $\{f : \Delta_q \to x_0\}$ form an FD-submodule $pt(X) \subset K(X)$. The quotient $K(X)/pt(X)$ is called the reduced FD-module of a topological space $X$, its homology $\tilde{H}_*(X)$ is the reduced homology of $X$. It is well known that $\tilde{H}_q(X) \cong H_q(X)$ if $q > 0$, $\dim H_0(X) = \dim \tilde{H}_0(X) + 1$.

The continuous map $f : X \to Y$ induces an FD-morphism $f_* : K(X) \to K(Y)$. The map $f : (X, x_0) \to (Y, y_0)$ of pointed spaces induces a morphism of reduced FD-modules.

Now we define a direct product of FD-modules.
Definition Let $K'$ and $K''$ be FD-modules. Put

$$(K' \times K'')_q = K'_q \otimes K''_q,$$

$$\partial^i_q(a'_q \otimes a''_q) = \partial^i_q(a'_q) \otimes \partial^i_q(a''_q), \ s^i_q(a'_q \otimes a''_q) = s^i_q(a'_q) \otimes s^i_q(a''_q).$$

Let $X'$ and $X''$ be topological spaces. Clearly, $K(X' \times X'') \cong K(X') \times K(X'').$ Indeed, let $\alpha : X' \times X'' \to X'$ and $\beta : X' \times X'' \to X''$ be the projections of the Cartesian product onto each factor. Then $\alpha\# \times \beta\# : K(X' \times X'') \to K(X') \times K(X'')$ is an isomorphism of FD-modules.

Now define an FD-module homotopy. First we construct an FD-module $K(n)$ that is usually called an FD-module of a standard $n$-simplex. The group $K(n)_q$ is free generated by all sets of integers $(w_0, w_1, \ldots, w_q)$ such that $0 \leq w_0 \leq \ldots \leq w_q \leq n$. Face- and degeneracy- operators are given by the formulas

$$\partial^i_q(w_0, \ldots, w_q) = (w_0, \ldots, w_{i-1}, w_{i+1}, \ldots, w_q),$$

$$s^i_q(w_0, \ldots, w_q) = (w_0, \ldots, w_i, w_i, \ldots, w_q).$$

Let $e_i = (i) \in K(n)_1$ be an ith vertex.

Definition Suppose $K'$ and $K''$ are FD-modules, $F^0, F^1 : K' \to K''$ are FD-morphisms. An FD-homotopy is an FD-morphism $\Theta : K(1) \times K' \to K''$ such that

$$\Theta ((s^0_q)qe_i \otimes a_q) = F^i(a_q), \ a_q \in K'_q, \ i = 0, 1.$$ 

If there exists such homotopy $\Theta$, then FD-morphisms $F^0$ and $F^1$ are called homotopic, $F^0 \sim F^1$. If there exist FD-morphisms $F^+ : K' \to K''$ and $F^- : K'' \to K'$ such that $F^+F^- \sim \text{Id}$ and $F^-F^+ \sim \text{Id}$, then FD-modules $K'$ and $K''$ are called homotopically equivalent.

A. Dold has proved the following fact in [6]:

Theorem Two FD-modules are homotopically equivalent if and only if they have isomorphic homology groups.

Let $T$ be a functor on the category of FD-modules.

Definition The functor $T$ preserves homotopy if $T(F^0) \sim T(F^1)$ for each pair $F^0, F^1$ of homotopic FD-morphisms.
Theorem (A. Dold, [6]) Suppose the functor $T$ preserves homotopy. Then
\[ H_*(K') \cong H_*(K'') \Rightarrow H_*(T(K')) \cong H_*(T(K'')), \]
for each pair of FD-modules $K'$, $K''$.

Let $t$ be a functor on the category of $\mathbb{Z}_2$-modules. Suppose $K$ is an FD-module. Put
\[ T(K) = t(K), \quad F_q^i = t(\partial_q^i), \quad D_q^i = t(s_q^i). \]
Thus $T(K)$ is an FD-module with face-operators $F_q^i$ and degeneracy-operators $D_q^i$. We say that the functor $T$ is a prolongation of the functor $t$.

The $p$th direct product $K^{\times p}$ of an FD-module $K$ is a simplest example of a prolongation.

Theorem (A. Dold, [6]) Suppose the functor $T$ is a prolongation of the functor $t$. Then $T$ preserves homotopy.

Now we can prove the following statement:

Lemma 2.1 Let $M_1$ and $M_2$ be polytopes. Then $H_*(M_1; \mathbb{Z}_2) \cong H_*(M_2; \mathbb{Z}_2)$ implies
\[ H_*(M_1 \times M_1 \times M_1/D_3, \Delta_{M_1}; \mathbb{Z}_2) \cong H_*(M_2 \times M_2 \times M_2/D_3, \Delta_{M_2}; \mathbb{Z}_2). \]
As above, the dihedral group $D_3$ acts on $M^{\times 3}$ by permutations of factors, $\Delta_M = \{(x, x, y)\} \subset M^{\times 3}/D_3$.

Proof. Suppose $M$ is a triangulated subset of Euclidean space, $x_0 \in M$ is a distinguished point. Thus $(x_0, x_0, x_0)$ is a distinguished point in the Cartesian cube $M^{\times 3}$. Let $X = RD_3(M)$ be the reduced dihedral cube of $M$.

Let us introduce the following notations:
\[ K = K(M \times M \times M)/pt(M \times M \times M) \cong (K(M)/pt(M))^{\times 3}, \]
\[ K' = K(X)/pt(X). \]
The projection $p : M^{\times 3} \to X$ induces an FD-morphism $p_\# : K \to K'$. 

6
By \( K_{sym} \) denote the FD-submodule of \( K \) generated by all chains of the form \( c_1 \otimes c_2 \otimes c_3 + c_2 \otimes c_3 \otimes c_1 + c_1 \otimes c_2 \otimes c_3 + c_2 \otimes c_1 \otimes c_3 \), and \( c_1 \otimes c_1 \otimes c_3 \). Let \( u : K_{sym} \to K \) be the inclusion, \( \varphi : K \to K/K_{sym} \) be the quotient FD-morphism.

It follows from A. Dold’s theorem that the homology \( H_*(K/K_{sym}) \) can be computed if we know only the homology \( H_*(M) \). Let us prove that

\[
H_*(K/K_{sym}) \cong H_*(K'),
\]

this will complete the proof of the lemma.

First we show that \( \alpha = p_\# \circ \varphi^{-1} \) is an FD-morphism. We need to check that \( u(K_{sym}) \subset \ker p_\# \).

Let \( c = f \otimes g \otimes h, \ c' = g \otimes h \otimes f \) be two elements of \( K_q \). Here \( f, g, h : \Delta_q \to M \) are continuous maps of the standard \( q \)-simplex to the space \( M \). Evidently, \( p_\#(c) = p_\#(c') \), hence \( c + c' \in \ker p_\# \). Now suppose \( c = (\sum f_i) \otimes (\sum g_j) \otimes (\sum h_k), \ c' = (\sum g_j) \otimes (\sum h_k) \otimes (\sum f_i) \), where \( f_i, g_j, h_k : \Delta_q \to M \). Then \( c = \sum_{i,j,k} f_i \otimes g_j \otimes h_k \) and \( c' = \sum_{i,j,k} g_j \otimes h_k \otimes f_i \). We see again that \( p_\#(c + c') = 0 \). Similarly, \( c_1 \otimes c_2 \otimes c_3 + c_2 \otimes c_1 \otimes c_3 \in \ker p_\# \).

Suppose \( c = f \otimes f \otimes g \). Then \( p_\#(c) \) maps the simplex \( \Delta_q \) to the distinguished point, hence \( p_\#(c) = 0 \in K' \). Now suppose \( c = (\sum f_i) \otimes (\sum f_i) \otimes (\sum g_j) = \sum_{i,j} f_i \otimes f_i \otimes g_j \). We see that \( p_\#(c) = 0 \).

Let us show that the FD-morphism \( \alpha \) induces an isomorphism of homology groups. We will construct a triangulation \( \tau \) of the space \( M^{\times 3} \). The idea is as follows. This triangulation will be coherent with the filtration

\[
\{(x, x, x)\} = D^0 \subset \{(x, x, y), (x, y, x), (y, x, x)\} = D \subset M \times M \times M
\]

and invariant under the action of the dihedral group \( D_3 \). Since \( \tau \) is invariant up to the action of \( D_3 \), \( \tau \) induces a triangulation \( \tau/D_3 \) of the space \( D^3(M) = M^{\times 3} \). Since the diagonal \( D \) is triangulated, \( \tau/D_3 \) induces a cell decomposition \( p(\tau) \) of the space \( D^3(M)/D_3 \). All affine maps \( f : \Delta_q \to M^{\times 3} \) such that \( f \) moves vertices of \( \Delta_q \) to vertices of a simplex \( \delta \in \tau \) generate an FD-submodule \( S \subset K \). The image \( p_\#(S) = S' \subset K' \) is an FD-submodule of \( K' \). By \( S_{sym} \) we denote the intersection \( K_{sym} \cap S \). There are inclusions \( i : S_{sym} \to K_{sym} \) and \( j : S \to K \), hence there is an inclusion \( k : S/S_{sym} \to K/K_{sym} \). We show all this FD-modules and morphisms on the following commutative diagram (its columns are exact):

\[
\text{Diagram}
\]
In is well know that the inclusions \( j \) and \( l \) induce isomorphisms of homology groups. It is easy to see that \( i \ast \) is an isomorphism too. Indeed, suppose we have a singular cycle \( \zeta \in \ker \partial_q \subset (K_{sym})_q \). Its 0-skeleton consists of the points on the diagonal and pairs of symmetric points. Thus we can construct a homotopy \( F_t \) of its 0-skeleton such that the image of \( F_t \) belongs to \( K_{sym} \) for each \( t \), and the image of \( F_1 \) belongs to \( S_{sym} \). Then we can extend \( F_t \) to 1-skeleton, and so on. Now two short exact sequences of FD-modules give us two long sequences of homology groups. By the well known 5-lemma, it follows that \( k \ast \) is an isomorphism.

Now we compute \( \ker p \# \). Suppose \( c \in K \), \( p \#(c) = 0 \). Clearly,

\[
c = \sum_m (f_{1m} \otimes f_{2m} \otimes f_{3m} + f'_{1m} \otimes f'_{2m} \otimes f'_{3m}) + \sum_n F_n,
\]

where \( F_n : \Delta_q \to D \) are simplices in the diagonal, \( p \#(f_{1m} \otimes f_{2m} \otimes f_{3m}) = p \#(f'_{1m} \otimes f'_{2m} \otimes f'_{3m}) \). It means that

\[
f_{1m}, f_{2m}, f_{3m}, f'_{1m}, f'_{2m}, f'_{3m} : \Delta_q \to M
\]

and for each point \( P \in \Delta_q \) there exists a permutation \( \sigma \in S_3 \) such that \((f'_{1m}(P), f'_{2m}(P), f'_{3m}(P)) = (f_{\sigma(1)m}(p), f_{\sigma(2)m}(p), f_{\sigma(3)m}(p)) \). Note that this
permutation is the same for all \( p \in \Delta_q \) if the image \((f'_{1m} \otimes f'_{2m} \otimes f'_{3m})(\Delta_q)\) does not intersect the diagonal. In this case \( f_{1m} \otimes f_{2m} \otimes f_{3m} + f'_{1m} \otimes f'_{2m} \otimes f'_{3m} \in \ker \varphi \).

Similarly, for each \( F_n \) we have \( F_n = g_{1n} \otimes g_{2n} \otimes g_{3n} \), where

\[
g_{1n}, g_{2n}, g_{3n} : \Delta_q \to M
\]

and for each \( P \in \Delta_q \) there exists a pair of indices \( i, j \) such that \( g_{in}(P) = g_{jn}(P) \). If the image \( F_n(\Delta_q) \) does not intersect the small diagonal \( D^0 \), then this pair is the same for all \( p \in \Delta_q \). As before, \( F_n \in \ker \varphi \).

Now we see that all elements of \( \ker \alpha \) have either simplices whose images intersect the diagonal or simplices whose images lie in the diagonal and intersect the small diagonal. Clearly, it follows that \( \ker \alpha \cap \im k = 0 \). Evidently, \( \im \alpha \circ k = \im l \). We obtain that \( \alpha \circ k : S/S_{sym} \to \im(S') \) is an FD-isomorphism. We see that \( l_* \), \( k_* \), \( l_*^{-1} \circ (\alpha \circ k)_* \) are isomorphisms. Consequently, \( \alpha_* = l_* \circ l_*^{-1} \circ (\alpha \circ k)_* \circ k_*^{-1} \) is an isomorphism.

Now let us construct a triangulation of the space \( M \times M \). Suppose the simplices \( L^i_q \) form a triangulation of the space \( M \) (here \( q \) is the dimension, \( i \) is an index). Then the Cartesian cube \( M \times M \times M \) is divided into sets \( L_q \times L_q \times L_q \), \( L_q \times L_q \times L_r \), and \( L_p \times L_q \times L_r \) (here \( p \), \( q \), \( r \) are all distinct).

First consider \( L_p \times L_q \times L_r \). By \( \tau_{pqr} \) denote any triangulation for \( L_p \times L_q \times L_r \), \( p < q < r \). Any triple \( p, q, r \) is a permutation of \( p' < q' < r' \), so a triangulation for \( L_p \times L_q \times L_r \) is obtained by the action of \( D_3 \) from \( \tau_{p'q'r'} \).

Now consider \( L_q \times L_q \times L_q \). There is a filtration

\[
\{(x, y, z) \} = D^0 \subset \{(x, x, y), (x, y, x), (y, x, x)\} = D \subset L_q \times L_q \times L_q.
\]

We triangulate \( L_q \times L_q \times L_q \) according to this filtration and action of the group \( D_3 \). Suppose \( (x, y, z) \in \Delta_q \times \Delta_q \times \Delta_q \). Let the faces of the simplex \( \Delta_q \) have indices \( 0, 1, \ldots, q \). By \( A^a \) denote the distance from \( x \) to \( a \)th face, by \( B^b \) denote the distance from \( y \) to \( b \)th face, by \( C^c \) denote the distance from \( z \) to \( c \)th face, \( 1 \leq a, b, c \leq q \).

Suppose \( 0 \leq k \leq 2q \). Let \( k_1, k_2, k_3 \) be non-negative integers such that \( k_1 + k_2 + k_3 = k \). Suppose \( \vec{a} = (a_1, \ldots, a_{k_1+k_2}) \), \( \vec{b} = (b_1, \ldots, b_{k_1+k_3}) \), \( \vec{c} = (c_1, \ldots, c_{k_2+k_3}) \), where \( 1 \leq a_1, \ldots, a_{k_1+k_2}, b_1, \ldots, b_{k_1+k_3}, c_1, \ldots, c_{k_2+k_3} \leq q \). By \( E_{a,b,c} \) denote the polyhedron given by equations

\[
A^{a_1} = B^{b_1}, \ldots, A^{a_{k_1}} = B^{b_{k_1}},
\]

\[
A^{a_{k_1}+1} = C^{c_1}, \ldots, A^{a_{k_1+k_2}} = C^{c_{k_2}},
\]

\[
A^{a_{k_1+k_2}+1} = D^{d_1}, \ldots, A^{a_{k_1+k_2+k_3}} = D^{d_{k_3}}.
\]

We obtain that \( \ker \alpha \cap \im k = 0 \). Evidently, \( \im \alpha \circ k = \im l \). We obtain that \( \alpha \circ k : S/S_{sym} \to \im(S') \) is an FD-isomorphism. We see that \( l_* \), \( k_* \), \( l_*^{-1} \circ (\alpha \circ k)_* \) are isomorphisms. Consequently, \( \alpha_* = l_* \circ l_*^{-1} \circ (\alpha \circ k)_* \circ k_*^{-1} \) is an isomorphism.
\[ B^{b_{k+1}} = C^{k_2+1}, \ldots, B^{b_{k+k_3}} = C^{c_{k_2+k_3}}. \]

Let \( Q_k \) be the union of all such polyhedra of codimension \( k \):

\[ Q_k = \bigcup_{\vec{a}, \vec{b}, \vec{c} : \text{codim} E_{\vec{a}, \vec{b}, \vec{c}} = k} E_{\vec{a}, \vec{b}, \vec{c}}. \]

Then the filtration \( Q_{2q} \subset Q_{2q-1} \subset \ldots \subset Q_0 \) satisfies the following conditions:

- \( D^0 = Q_{2q} \).
- \( D \subset Q_q \)
- \( \Delta_q \times \Delta_q \times \Delta_q = Q_0 \)
- The filtration is invariant under the action of \( D_3 \).
- The set \( Q_{k+1} \) divides the set \( Q_k \) into pieces. The group \( D_3 \) either transposes a whole piece or does not move each point of a piece.

Now \( Q_{2q} \) is triangulated. According to the last condition the triangulation can be extended from the union of polyhedrons \( Q_k \) to the union of polyhedrons \( Q_{k+1} \).

Now consider \( L_q \times L_q \times L_r \). We must construct a triangulation that is invariant up to \( D_2 \)-action by permuting two \( L_q \) in the Cartesian product and coherent with the filtration

\[ \{(x, x, y)\} \subset L_q \times L_q \times L_r. \]

First we construct a triangulation for \( L_q \times L_q \) such that it is coherent with the filtration

\[ \{(x, x)\} \subset L_q \times L_q \]

and invariant under the action of the group \( D_2 \). We can do it in similar way as for \( L_q \times L_q \times L_q \). By \( \text{sk}_j(L_r) \) we denote a union of all \( j \)-faces of the simplex \( L_r \). Now we see that for \( L_q \times L_q \times \text{sk}_0(L_r) \) the triangulation is constructed. Evidently, we can extend this triangulation from \( L_q \times L_q \times \text{sk}_j(L_r) \) to \( L_q \times L_q \times \text{sk}_{j+1}(L_r) \).

This completes the proof. \( \square \)

**Remark** In the same way, if \( H_*(X_1; \mathbb{Z}_2) \cong H_*(X_2; \mathbb{Z}_2) \), then

\[ H_*(X_1 \times X_1/\mathbb{Z}_2, \{(x, x)\}; \mathbb{Z}_2) \cong H_*(X_2 \times X_2/\mathbb{Z}_2, \{(x, x)\}; \mathbb{Z}_2). \]
3 Proof of P. Pushkar’s theorem

Let $M$ be an $m$-dimensional smooth closed manifold embedded in Euclidean space $\mathbb{R}^n$, $k_q = \dim H_q(M; \mathbb{Z}_2)$. We need to calculate the sum

$$\sum \dim H_q(M \times^2 \mathbb{Z}_2, \Delta; \mathbb{Z}_2).$$

From the results of the previous section, it follows that this sum equals

$$\sum \dim H_q(X \times^2 \mathbb{Z}_2, \Delta; \mathbb{Z}_2),$$

where $X$ is a bouquet of spheres

$$X = S^m \vee S^{m-1}_1 \vee \ldots \vee S^{m-1}_{k_m-1} \vee \ldots \vee S^1_1 \vee \ldots \vee S^1_{k_1}.$$

First let us prove the following

**Lemma 3.1**

$$\sum \dim H_q(S^m \times S^m / \mathbb{Z}_2, \Delta; \mathbb{Z}_2) = m + 1,$$

$$H_*(S^m \times S^m / \mathbb{Z}_2, \Delta; \mathbb{Z}_2) = \{0, \ldots, 0, \mathbb{Z}_2, \ldots, \mathbb{Z}_2\}.$$  

**Proof.** We construct a cell decomposition of the reduced Cartesian square $S^m \times S^m / \tilde{\Delta}$. (Let us recall that $\tilde{\Delta} = \{(x, x)\}$). We consider $S^m$ as an $m$-dimensional cube $[0, 1]^m$ with the boundary contracted to a point. So $S^m \times S^m$ is a Cartesian product of two such cubes. Let $\varphi_1, \ldots, \varphi_m$ be the coordinates on the first cube, $\psi_1, \ldots, \psi_m$ be the coordinates on the second cube.

The cells are

$$e'_i = \{\varphi_1 \ast \psi_1, \ldots, \varphi_m \ast \psi_m\},$$

where $\ast$ is one of the signs $>, <, =$. Of course, not all of $\ast$ are $=$, since we have contracted the diagonal $\tilde{\Delta}$. The codimension of a cell is $\#(\ast)$. There are two $m$-dimensional cells:

$$e^m_1 = \{\varphi_1 = 0, \ldots, \varphi_m = 0\},$$

$$e^m_2 = \{\psi_1 = 0, \ldots, \psi_m = 0\}.$$ 

Evidently, they do not belong to the boundaries of $(m + 1)$-dimensional cells.
This decomposition is invariant under the $D_2$-action, hence it induces a cell decomposition of the reduced dihedral square $RD^2(S^m)$. Evidently, the chain complex for $RD^2(S^m)$ is a direct sum of two subcomplexes. One of them is generated by all $e^q_i = \{\varphi_1\psi_1, \ldots, \varphi_m\psi_m\}$, let us denote it by $b(RD^2(S^m))$, and another consists of one element $e^m = \{\varphi_1 = 0, \ldots, \varphi_m = 0\}$, denote it by $s(RD^2(S^m))$.

Obviously, $H_m(S^m \times S^m/\mathbb{Z}_2, \Delta) \cong \mathbb{Z}_2$: there is only one cell of dimension $m$, its boundary is 0.

Consider the Smith exact sequence of the Cartesian square $S^m \times S^m$ and the group $\mathbb{Z}_2$ acting on it (see [3]):

\[
\ldots \rightarrow H_q(S^m \times S^m/\mathbb{Z}_2, \Delta) \oplus H_q(S^m) \rightarrow H_q(S^m \times S^m) \rightarrow H_q(S^m \times S^m/\mathbb{Z}_2, \Delta) \rightarrow H_{q-1}(S^m \times S^m/\mathbb{Z}_2, \Delta) \oplus H_{q-1}(S^m) \rightarrow \ldots
\]

Since $H_q(S^m \times S^m) \cong H_q(S^m) \cong 0$ if $q \neq 0, m, 2m$, we have

\[
0 \rightarrow H_q(S^m \times S^m/\mathbb{Z}_2, \Delta) \rightarrow H_{q-1}(S^m \times S^m/\mathbb{Z}_2, \Delta) \rightarrow 0,
\]

where $q = m + 2, \ldots, 2m - 1$.

Thus we see that $H_{m+1}(S^m \times S^m/\mathbb{Z}_2, \Delta) \cong \ldots \cong H_{2m-1}(S^m \times S^m/\mathbb{Z}_2, \Delta)$.

Consider the end of the Smith sequence:

\[
0 \rightarrow H_{m+1}(S^m \times S^m/\mathbb{Z}_2, \Delta) \rightarrow H_m(S^m \times S^m/\mathbb{Z}_2, \Delta) \oplus \mathbb{Z}_2 \rightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \rightarrow H_m(S^m \times S^m/\mathbb{Z}_2, \Delta) \rightarrow 0.
\]

Since $H_m(S^m \times S^m/\mathbb{Z}_2, \Delta) \cong \mathbb{Z}_2$, we see that $H_{m+1}(S^m \times S^m/\mathbb{Z}_2, \Delta) \cong \mathbb{Z}_2$.

Thus we have

\[
H_m(S^m \times S^m/\mathbb{Z}_2, \Delta) \cong \ldots \cong H_{2m-1}(S^m \times S^m/\mathbb{Z}_2, \Delta) \cong \mathbb{Z}_2.
\]

Now consider the beginning of the Smith sequence:

\[
0 \rightarrow H_{2m}(S^m \times S^m/\mathbb{Z}_2, \Delta) \rightarrow \mathbb{Z}_2 \rightarrow H_{2m}(S^m \times S^m/\mathbb{Z}_2, \Delta) \rightarrow H_{2m-1}(S^m \times S^m/\mathbb{Z}_2, \Delta) \rightarrow 0.
\]

We already know that $H_{2m-1}(S^m \times S^m/\mathbb{Z}_2, \Delta) \cong \mathbb{Z}_2$. Computing the Euler characteristic we obtain that $H_{2m}(S^m \times S^m/\mathbb{Z}_2, \Delta) \cong \mathbb{Z}_2$. This completes the proof. □
Now let us prove P. Pushkar’s estimate.

**Lemma 3.2** Let $M$ be an $m$-dimensional manifold, $k_i = \dim H_i(M; \mathbb{Z}_2)$, $i = 0, 1, \ldots, m$, $B = \sum k_i$. Then

$$\sum_{i=1}^{m} ik_i = \frac{mB}{2}.$$ 

**Proof.** Using Poincaré duality, we have

$$\sum_{i=1}^{m} ik_i = \sum_{i=0}^{m} ik_i = \frac{1}{2} \sum_{i=0}^{m} (ik_i + (m - i)k_{m-i}) = \frac{1}{2} \sum_{i=0}^{m} (ik_i + (m - i)k_i) = \frac{mB}{2}.$$ 

$\square$

**Theorem** Let $B = \sum \dim H_q(M; \mathbb{Z}_2)$. Then

$$BT_2(M) \geq \frac{B^2 + (m - 1)B}{2}.$$ 

**Proof.** Let us recall that $X$ is a bouquet of spheres such that $H_*(X; \mathbb{Z}_2) \cong H_*(M; \mathbb{Z}_2)$. Taking into account all arguments above, we see that it is enough to prove that

$$(*) \quad \sum \dim H_q(X^{x2}/\mathbb{Z}_2, \Delta; \mathbb{Z}_2) = \frac{B^2 + (m - 1)B}{2}.$$ 

We construct a cell decomposition of the space $RD^2(X)$. This space is the union of its subsets:

$$RD^2(X) = \bigcup_{1 \leq p \leq m, 1 \leq i \leq k_p} A_i^p \cup \bigcup_{1 \leq p \leq m, 1 \leq i < j \leq k_p} B_{ij}^p \cup \bigcup_{1 \leq p < q \leq m, 1 \leq i \leq k_p, 1 \leq j \leq k_q} C_{ij}^{pq},$$

where

- $A_i^p$ is the set of points $(x, x') \in RD^2(X)$ such that $x, x' \in S_i^p$. Thus $A_i^p$ is homeomorphic to $RD^2(S^p)$.

- $B_{ij}^p$ is the set of points $(x_1, x_2) \in RD^2(X)$ such that $x_1 \in S_i^p, x_2 \in S_j^p$. Thus $B_{ij}^p$ is homeomorphic to $S^p \times S^p$. 

13
• $C_{ij}^{pq}$ is the set of points $(x, y) \in RD^2(X)$ such that $x \in S_p^i$, $y \in S_q^j$. Thus $C_{ij}^{pq}$ is homeomorphic to $S_p^i \times S_q^j$.

For each $A_p^i$ we have constructed a cell decomposition earlier. For each $B_p^i$, $C_{ij}^{pq}$ we have a standard decomposition:

$$S_p^i = e_0^i \cup e_p^i, \quad S_q^j = e_0^j \cup e_q^j, \quad S_p^i \times S_q^j = e_0^i \times e_0^j \cup e_0^i \times e_q^j \cup e_p^i \times e_0^j \cup e_p^i \times e_q^j$$

with all boundaries equal to 0. By $s(B_p^i)$ we denote a chain complex generated by one cell $e_p^i \times e_p^i$, by $s(C_{ij}^{pq})$ we denote a chain complex generated by one cell $e_p^i \times e_q^j$.

Notice that these decompositions coincide on intersections of the sets $A_p^i$, $B_p^i$, $C_{ij}^{pq}$.

Now we see that a chain complex for $RD^2(X)$ is a direct sum of its subcomplexes:

$$\bigoplus_{1 \leq p \leq m} \left( b(A_p^i) \oplus s(A_p^i) \right) \bigoplus_{1 \leq i \leq k_p} s(B_p^i) \bigoplus_{1 \leq p < q \leq m} s(C_{ij}^{pq})$$

Now we compute contributions to the sum of Betti numbers $(\ast)$ for each of these direct summands.

• $b(A_p^i) \oplus s(A_p^i)$: The sum of Betti numbers of each $A_p^i$ is $p + 1$. So we have a contribution $\sum_{i=1}^{m} (i + 1)k_i$.

Let us transform this expression using the previous lemma.

$$\sum_{p=1}^{m} (p + 1)k_p = B - 1 + \frac{mB}{2}.$$ 

• $s(B_{ij}^p)$: Each of $s(B_{ij}^p)$ gives us 1 to the sum of Betti numbers $(\ast)$, so totally we have

$$\frac{1}{2} \sum_{p=1}^{m} k_p(k_p - 1).$$

• $s(C_{ij}^{pq})$:

$$\sum_{p<q}^{m} k_p k_q.$$
Thus we have
\[ \sum \dim H_q(X \times^2 \mathbb{Z}_2, \Delta; \mathbb{Z}_2) = B - 1 + \frac{mB}{2} + \frac{1}{2} \sum_{p=1}^{m} k_p(k_p - 1) + \sum_{p<q} k_p k_q = \]
\[ B - 1 + \frac{mB}{2} + \frac{1}{2} \left( \sum_{p=1}^{m} k_p \right)^2 \frac{1}{2} \sum_{p=1}^{m} k_p = \frac{(B - 1)^2 + (m + 1)B - 1}{2} = \frac{B^2 + (m - 1)B}{2}. \]

This completes the proof. \( \Box \)

4 Estimate for the number of 3-periodical trajectories

As above, a smooth closed manifold \( M \) is embedded in Euclidean space \( \mathbb{R}^n \). Let \( k_q = \dim H_q(M; \mathbb{Z}_2) \). We need to compute the sum
\[ \sum \dim H_q(M \times^3 S^3, \Delta; \mathbb{Z}_2). \]

It equals to
\[ \sum \dim \tilde{H}_q \left( RD^3(X); \mathbb{Z}_2 \right), \]
where \( X \) is a bouquet of spheres:
\[ X = S^m \vee S^{m-1}_1 \vee \ldots \vee S^{m-1}_{k_m} \vee \ldots \vee S^1_1 \vee \ldots \vee S^1_{k_1}, \]

We know that (see [3], [4])
\[ \sum \dim \tilde{H}_q \left( RD^3(S^m); \mathbb{Z}_2 \right) = 2m. \]

First let us prove the following

Lemma 4.1 Let \( M \) be \( m \)-dimensional manifold, \( k_i = \dim H_i(M; \mathbb{Z}_2) \), \( i = 0, 1, \ldots, m \), \( B = \sum k_i \). Then
\[ \sum_{1 \leq i \leq m} ik_i^2 + \sum_{1 \leq i \neq j \leq m} ik_i k_j = \frac{mB^2 - mB}{2}. \]
Proof. Taking into account Poincaré duality $k_i = k_{m-i}$, we have

$$\sum_{1 \leq i \leq m} ik_i^2 + \sum_{1 \leq i \neq j \leq m} ik_i k_j =$$

$$= \sum_{0 \leq i \leq m} ik_i^2 + \sum_{0 \leq i \neq j \leq m} ik_i k_j - \sum_{1 \leq i \leq m, j=0} ik_i k_j =$$

$$= \sum_{0 \leq i,j \leq m} ik_i k_j - \sum_{0 \leq i \leq m} ik_i =$$

$$= \frac{1}{2} \sum_{0 \leq i,j \leq m} (i + m - i)k_i k_j - \frac{1}{2} \sum_{0 \leq i \leq m} (i + m - i)k_i =$$

$$= \frac{mB^2 - mB}{2}.$$

\[\square\]

Theorem. Suppose $B = \sum_{i=0}^{m} k_i$. Then we have an estimate for the number of 3-periodical trajectories in $M$

$$BT_3(M) \geq \frac{B^3 + 3(m - 1)B^2 + 2B}{6}.$$

Proof. Let us construct a cell decomposition of the reduced dihedral cube of $X$. The space $RD^3(X)$ is a union of its subsets:

$$RD^3(X) = \bigcup_{1 \leq p \leq m} A_p^i \cup \bigcup_{1 \leq p \leq m} B_{ij}^p \cup \bigcup_{1 \leq i \leq m} C_{ijk}^p \cup \bigcup_{1 \leq p \neq q \leq m, 1 \leq i \leq k_p} D_{ij}^{pq} \cup \bigcup_{1 \leq p \neq q \leq m, 1 \leq i \leq k_p} E_{ijk}^{pq} \cup \bigcup_{1 \leq p \neq q \neq r \leq m, 1 \leq i \leq k_p, 1 \leq j \leq k_q, 1 \leq k \leq k_r} F_{ijk}^{pqr}$$

Here

- $A_p^i$ is the set of points $(x, x', x'') \in RD^3(X)$ such that $x, x', x'' \in S_i^p$. Thus it is homeomorphic to $RD^3(S_p)$. 

16
• $B_{ij}^p$ is the set of points $(x_1, x_1', x_2) \in RD^3(X)$ such that $x_1, x_1' \in S_i^p$, $x_2 \in S_j^p$. Thus it is homeomorphic to

$$RD^2(S^p) \times S^p / \{(x, O, O)\},$$

where $O$ is the distinguished point of the bouquet $X$.

• $C_{ijk}^p$ is the set of points $(x_1, x_2, x_3) \in RD^3(X)$ such that $x_1 \in S_i^p$, $x_2 \in S_j^p$, $x_3 \in S_k^p$. Thus it is homeomorphic to

$$S^p \times S^p \times S^p / \tilde{\Delta}_{S^p}.$$  

• $D_{pq}^{ijk}$ is the set of points $(x, x', y) \in RD^3(X)$ such that $x, x' \in S_i^p$, $y \in S_j^q$. Thus it is homeomorphic to

$$RD^2(S^p) \times S^q / \{(x, O, O)\}.$$ 

• $E_{pq}^{ijk}$ is the set of points $(x, y, z) \in RD^3(X)$ such that $x \in S_i^p$, $y \in S_j^q$, $z \in S_k^q$. Thus it is homeomorphic to

$$S^p \times S^q \times S^q / \{(x, O, O)\}, \{(O, y, O)\}, \{(O, O, z)\}.$$ 

Let us construct a cell decomposition for each $A_i^p = RD^3(S^p)$ like we have done it for $RD^2(S^p)$. We consider a $p$-sphere as a $p$-dimensional cube $[0, 1]^{x_p}$ with the boundary contracted to a point. Let $\varphi_1^1, \varphi_2^2, \varphi_3^3$ be the coordinates on three such cubes. There are the following cells. First

$$U^d_{\beta} = \{\varphi_1^{i_1} \ast \varphi_1^{j_1} \ast \varphi_1^{k_1}, \ldots, \varphi_m^{i_m} \ast \varphi_m^{j_m} \ast \varphi_m^{k_m}\},$$

where $\ast$ is one of the signs $< \text{ or } =$, $(i_\alpha, j_\alpha, k_\alpha)$ is a permutation of 1, 2, 3. Secondly

$$V^d_{\beta} = \{\varphi_1^1 \ast \varphi_1^2, \ldots, \varphi_m^1 \ast \varphi_m^2, \varphi_3^3 = 0\},$$

where $\ast$ is one of the signs $< \text{ or } > \text{ or } =$. Clearly, the cells of the kind $U$ do not belong to the boundaries of the cells of the kind $V$. Similarly, the cells of
the kind \( V \) do not belong to the boundaries of the cells of the kind \( U \). It is evident for \( p > 1 \). For \( p = 1 \) all boundary operators here vanish.

We see that a chain complex for \( A^p_i \) is a direct sum of two subcomplexes \( u(A^p_i) \) and \( v(A^p_i) \).

Now we construct a cell decomposition for \( B^p_{ij} \) and \( D^{pq}_{ij} \). Here we have the same cells "of the kind \( V \)"

\[
V^d_\beta = \{ v_1^1 * v_1^2, \ldots, v_m^1 * v_m^2, \psi = 0 \},
\]

where \( v_1^1 \) and \( v_2^1 \) are the coordinates on \( i \)th \( p \)-sphere, \( \psi \) are the coordinates on \( j \)th \( p \)- or \( q \)-sphere. Besides, there are the cells

\[
W^d_\beta = \{ v_1^1 * v_1^2, \ldots, v_m^1 * v_m^2, 0 < \psi_k < 1 \},
\]

\[
\tilde{W}^d_\beta = \{ v_1^1 = 0, 0 < v_1^2 < 1, 0 < \psi_1 < 1 \}.
\]

As above, the cells of each of these three kinds form a chain subcomplex. Thus a chain complex for \( RD^3(X) \) is a direct sum \( u(B^p_{ij}) \oplus w(B^p_{ij}) \oplus \tilde{w}(B^p_{ij}) \) and so is one for \( \tilde{W}^3 \).

Notice that

\[
v(B^p_{ij}) = v(D^{pq}_{ij}) = v(A^p_i),
\]

\[
\tilde{w}(B^p_{ij}) = \tilde{w}(D^{pq}_{ij}), \quad \tilde{w}(D^{pq}_{ij}) = \tilde{w}(D^{pq}_{ij}).
\]

Clearly, each \( \tilde{w}(\ldots) \) consists of only one cell with boundary equal to 0.

Finally, \( C^p_{ijk}, E^{pq}_{ijk}, \) and \( F^{pq}_{ijk} \) are Cartesian products of spheres, so here we have a standard cell decomposition:

\[
S^0 = e^0 \cup e^p, \quad S^q = e^0 \cup e^q, \quad S^r = e^0 \cup e^r,
\]

\[
S^p \times S^q \times S^r = O \cup e^0 \times e^q \times e^r \cup e^0 \times e^p \times e^q \times e^r \cup e^p \times e^q \times e^r.
\]

We see that \( e^0 \times e^q \times e^r, e^p \times e^0 \times e^r, \) and \( e^p \times e^q \times e^0 \) are cells "of the kind \( \tilde{W}^3 \)." The cell \( e^p \times e^q \times e^r \) with vanishing boundary forms a chain subcomplex. We denote this subcomplex by \( s(C^p_{ijk}), s(E^{pq}_{ijk}), \) or \( s(F^{pq}_{ijk}) \).

Now we see that the chain complex for \( RD^3(X) \) is a direct sum of subcomplexes:

\[
\bigoplus_{1 \leq i < j < k \leq p} (u(A^p_i) \oplus v(A^p_i)) \bigoplus_{1 \leq i < j < k \leq p} w(B^p_{ij}) \bigoplus_{1 \leq i < j < k \leq p} \tilde{w}(B^p_{ij}) \bigoplus_{1 \leq i < j < k \leq p} s(C^p_{ijk})
\]
Now let us consider all these subcomplexes and compute their contributions to the sum of Betti numbers $\sum \dim \tilde{H}_q(RD^3(X); \mathbb{Z}_2)$.

- $u(A^p_i) \oplus v(A^p_i)$: $2 \sum_{p=1}^m pk_p = mB$. Here we have

$$mB.$$

- $w(B^p_{ij})$: $\sum_{p=1}^m pk_p(k_p - 1)$. Indeed, for each $p$ we have $k_p(k_p - 1)$ sets $B^p_{ij}$; the cells of the kind $W$ form the reduced dihedric square of the $p$-sphere without the unique $p$-dimensional cell that forms a subcomplex $\tilde{w}(B^p_{ij})$.

- $\tilde{w}(B^p_{ij})$: $\frac{1}{2} \sum_{p=1}^m k_p(k_p - 1)$. Here $\frac{1}{2}$ appears, since $\tilde{w}(B^p_{ij}) = \tilde{w}(B^p_{ji})$.

- $w(D^pq_{ij})$: $\sum_{p \neq q} pk_pk_q$.

- $\tilde{w}(D^pq_{ij})$: $\sum_{1 \leq p < q \leq m} k_p k_q$.

- $s(C^p_{ijk})$: Here we have subcomplexes consisting of one cell. Thus we have a contribution

$$\frac{1}{6} \sum_{p=1}^m k_p(k_p - 1)(k_p - 2).$$

- $s(E^pq_{ijk})$: Here we have

$$\frac{1}{2} \sum_{p \neq q} k_p(k_p - 1)k_q.$$
• \(s(I_{ijk}^{pqr})\): Here we have
\[
\sum_{p<q<r} k_pk_qk_r.
\]

Taking into account two computational lemmas, we have
\[
mB + \sum_{p=1}^{m} pk_p(k_p - 1) + \frac{1}{2} \sum_{p=1}^{m} k_p(k_p - 1) + \sum_{p\neq q} pk_pk_q + \sum_{1\leq p<q\leq m} k_pk_q + \]
\[
+ \frac{1}{6} \sum_{p=1}^{m} k_p(k_p - 1)(k_p - 2) + \frac{1}{2} \sum_{p\neq q} k_p(k_p - 1)k_q + \sum_{p<q<r} k_pk_qk_r =
\]
\[
= mB + \frac{mB^2}{2} - \frac{mB}{2} + \frac{(B - 1)^2}{2} - \frac{B - 1}{2} + \frac{(B - 1)^3}{6} - \frac{(B - 1)^2}{2} + \frac{B - 1}{3} =
\]
\[
= \frac{1}{6}(B^3 - 3B^2 + 3B - 1 + 3mB^2 - B + 1) =
\]
\[
= \frac{B^3 + 3(m - 1)B^2 - 2B}{6}.
\]

This completes the proof. \(\square\)
References

[1] G. Birkhoff. Dynamical systems. New York, 1927.

[2] P. Pushkar. Periodic trajectories of billiard dynamical systems. Ph.D. thesis. Moscow, 1998.

[3] M. Farber, S. Tabachnikov. Topology of cyclic configuration spaces and periodic trajectories of multi-dimensional billiards. arXiv:math.DG/9911226.

[4] M. Farber, S. Tabachnikov. Periodic trajectories in 3-dimensional convex billiards. arXiv:math.DG/0106049.

[5] F. Duzhin. On the lower bounds for the number of periodic billiard trajectories in manifolds embedded in Euclidean space. arXiv:math.AT/0112178.

[6] A. Dold. Homology of symmetric products and other functors of complexes. Annals of Mathematics, vol. 68, No. 1, July, 1958.

[7] S. Eilenberg, S. MacLane. On the groups $H(\Pi, n)$. Annals of Mathematics, vol. 58, No. 1, July, 1953.

[8] O. Viro, D. Fuchs. Topology-2. In: Encyclopedia of Math. Sciences, vol. 24.

[9] P. Griffiths, J. Harris. Principles of Algebraic Geometry. John Wiley & Sons, 1978.

[10] A. Fomenko, D. Fuchs. A Course in Homotopic Topology.

[11] B. Dubrovin, S. Novikov, A. Fomenko. Modern Geometry.