Abstract. We show how to construct a stick figure of lines in $\mathbb{P}^3$ using the Hadamard product of projective varieties. Then, applying the results of Migliore and Nagel, we use such a stick figure to build a Gorenstein set of points with given $h$-vector $h$. Since the Hadamard product is a coordinate-wise product, we show, at the end, how the coordinates of the points, in the Gorenstein set, can be directly determined.

1. Introduction

In the last few years, the Hadamard products of projective varieties have been widely studied from the point of view of Projective Geometry and Tropical Geometry. In fact, the Hadamard products of projective varieties and the Hadamard powers of a projective variety are well-connected to other operations of varieties: they are the multiplicative analogs of joins and secant varieties, and in tropical geometry, tropicalized Hadamard products equal Minkowski sums. It is natural to study properties of this new operation, and see its effects on various varieties.

From the point of view of Projective Geometry, several directions of research have been considered. The paper [5], where Hadamard product of general linear spaces is studied, can be considered the first step in this direction. Successively, the first author, with Calussi, Fatàbbi and Lorenzini, in [2], address the Hadamard product of linear varieties not necessarily in general position, obtaining, in $\mathbb{P}^2$ a complete description of the possible outcomes. Then, in [3], they address the Hadamard product of not necessarily generic linear varieties and show that the Hilbert function of the Hadamard product $X \ast Y$ of two varieties, with $\dim(X), \dim(Y) \leq 1$, is the product of the Hilbert functions of the original varieties $X$ and $Y$ and that the Hadamard product of two generic linear varieties $X$ and $Y$ is projectively equivalent to a Segre embedding. An important result contained in [5] concerns the construction of star configurations of points, via Hadamard product. This result found a generalization in [8] where the authors introduce a new construction, using the Hadamard product, to obtain star configurations of codimension $c$ of $\mathbb{P}^n$ and which they called Hadamard star configurations. Successively, Bahmani Jafarloo and Calussi, in [1], introduce a more general type of Hadamard star configuration; any star configuration constructed by their approach is called a weak Hadamard star configuration.

The use of Hadamard products in this context permits a complete control both in the coordinates of the points forming the star configuration and the equations of the hyperplanes involved on it. Thus, the question if other interesting geometrical objects can be obtained by Hadamard products naturally arises. In this paper, we give a first positive answer showing how to construct a Gorenstein set of points in $\mathbb{P}^3$ with given $h$-vector, via Hadamard products.

Our approach is related to the well-known construction of Migliore and Nagel [12], based on Liaison Theory, where the Gorenstein set of points is obtained as the intersection of two aCM curves, linked by a complete intersection which is a stick
The Hadamard product $\star$ is defined as $X \star Y = \{p \star q : p \in X, q \in Y, p \star q \text{ is defined}\}$.

**Remark 2.3.** The Hadamard product $X \star Y$ can be given in terms of composition of the Segre product and projection. Consider the usual Segre product 

$$X \times Y \subset \mathbb{P}^N$$

$$([\alpha_0 : \cdots : \alpha_n], [\beta_0 : \cdots : \beta_n]) \mapsto [\alpha_0\beta_0 : \alpha_0\beta_1 : \cdots : \alpha_n\beta_n]$$
and denote with \(z_{ij}\) the coordinates in \(\mathbb{P}^N\). Let \(\pi : \mathbb{P}^N \to \mathbb{P}^n\) be the projection map from the linear space \(\Lambda\) defined by equations \(z_{ii} = 0, i = 0, \ldots, n\). The Hadamard product of \(X\) and \(Y\) is
\[
X \ast Y = \pi(X \times Y),
\]
where the closure is taken in the Zariski topology.

**Remark 2.4.** Let \(\mathbb{K}[x] = \mathbb{K}[x_0, \ldots, x_n]\) be a polynomial ring over an algebraically closed field.

Let \(I_1, I_2, \ldots, I_r\) be ideals in \(\mathbb{K}[x]\). We introduce \((n + 1)r\) variables, grouped in \(r\) vectors \(y_j = (y_{j0}, \ldots, y_{jn})\), \(j = 1, 2, \ldots, r\) and we consider the polynomial ring \(\mathbb{K}[x, y]\) in all \((n + 1)(r + 1)\) variables.

Let \(I_j(y_j)\) be the image of the ideal \(I_j\) in \(\mathbb{K}[x, y]\) under the map \(x \mapsto y\). Then the Hadamard product \(I_1 \ast I_2 \ast \cdots \ast I_r\) is the elimination ideal
\[
(I_1(y_1) + \cdots + I_r(y_r) + \langle x_i - y_{i1}y_{i2}\cdots y_{ir} \mid i = 0, \ldots, n \rangle) \cap \mathbb{K}[x].
\]
The defining ideal of the Hadamard product \(X \ast Y\) of two varieties \(X\) and \(Y\), that is, the ideal \(I(X \ast Y)\), equals the Hadamard product of the ideals \(I(X) \ast I(Y)\) [5].

As in [5] we give the following definition.

**Definition 2.5.** Let \(H_i \subset \mathbb{P}^n, i = 0, \ldots, n\), be the hyperplane \(x_i = 0\) and set
\[
\Delta_i = \bigcup_{0 \leq j_1 < \cdots < j_n \leq n} H_{j_1} \cap \cdots \cap H_{j_n}.
\]
In other words, \(\Delta_i\) is the \(i\)-dimensional variety of points having at most \(i + 1\) non-zero coordinates. Thus \(\Delta_0\) is the set of coordinates points and \(\Delta_n - 1\) is the union of the coordinate hyperplanes. Note that elements of \(\Delta_i\) have at least \(n - i\) zero coordinates. We have the following chain of inclusions:
\[
(1) \quad \Delta_0 = \{[1 : 0 : \cdots : 0], \ldots, [0 : \cdots : 0 : 1]\} \subset \Delta_1 \subset \cdots \subset \Delta_{n-1} \subset \Delta_n = \mathbb{P}^n.
\]

We end this section recalling some useful results contained in [4] and [5].

**Lemma 2.6.** Let \(L \subset \mathbb{P}^n\) be a linear space of dimension \(m\). Then, for a point \(P \in \mathbb{P}^n\), \(P \ast L\) is either empty or it is a linear space of dimension at most \(m\). If \(P \notin \Delta_{n-1}\), then \(\dim(P \ast L) = m\).

**Lemma 2.7.** Let \(L \subset \mathbb{P}^n\) be a linear space of dimension \(m < n\) and consider points \(P, Q \in \mathbb{P}^n \setminus \Delta_{n-1}\). If \(P \neq Q\), \(L \cap \Delta_{n-m-1} = \emptyset\), and \(\langle P, Q\rangle \cap \Delta_{n-m-2} = \emptyset\), then \(P \ast L \neq Q \ast L\).

**Lemma 2.8.** Let \(P, Q_1, Q_2\) be three points in \(\mathbb{P}^n\) with \(P \notin \Delta_{n-1}\). Then \(P \ast Q_1 = P \ast Q_2\) if and only if \(Q_1 = Q_2\).

If \(I = (i_0, \ldots, i_n)\) is a vector of nonnegative integers, we denote by \(X^I\) the monomial \(x_0^{i_0}x_1^{i_1}\cdots x_n^{i_n}\) and by \([I] = i_0 + \cdots + i_n\). Similarly, if \(P\) is a point of \(\mathbb{P}^n\) of coordinates \([p_0 : p_1 : \cdots : p_n]\), we denote by \(P^I\) the monomial \(X^I\) evaluated at \(P\), that is \(p_0^{i_0}p_1^{i_1}\cdots p_n^{i_n}\).

**Definition 2.9.** Let \(f \in k[x_0, \ldots, x_n]\) be a homogeneous polynomial, of degree \(d\), of the form \(f = \sum_{|I| = d} a_I X^I\) and consider a point \(P \in \mathbb{P}^n \setminus \Delta_n\). The Hadamard transformation of \(f\) by \(P\) is the polynomial
\[
f^P = \sum_{|I| = d} \frac{a_I}{P^I} X^I.
\]

**Theorem 2.10.** Let \(V \subset \mathbb{P}^n\) be a variety and consider a point \(P \in \mathbb{P}^n \setminus \Delta_n\). If \(f_1, \ldots, f_s \subset k[x_0, \ldots, x_n]\) is a generating set for \(I(V)\), that is \(I(V) = \langle f_1, \ldots, f_s\rangle\), then \(f_1^P, \ldots, f_s^P\) is a generating set for \(I(P \ast V)\).
Corollary 2.11. Let $V \subset \mathbb{P}^n$ be a variety. Then for any point $P \in \mathbb{P}^n \setminus \Delta_0$ one has $Q \in V$ if and only if $P \star Q \in P \star V$.

3. Gorenstein points in $\mathbb{P}^3$ from the h-vector

If $X$ is a subscheme of $\mathbb{P}^n$ with saturated ideal $I(X)$, and if $t \in \mathbb{Z}$ then the Hilbert function of $X$ is denoted by

$$h_X(t) = \text{dim}(k[\mathbb{P}^n]_t) - \text{dim}(I(X)_t).$$

If $X$ is arithmetically Cohen-Macaulay (aCM) of dimension $d$ then $A = k[\mathbb{P}^n]/I(X)$ has Krull dimension $d + 1$ and a general set of $d + 1$ linear forms forms a regular sequence for $A$. Taking the quotient of $A$ by such a regular sequence gives a zero-dimensional Cohen-Macaulay ring called the Artinian reduction of $A$. The Hilbert function of the Artinian reduction of $k[\mathbb{P}^n]/I(X)$ is called the $h$-vector of $X$. This is a finite sequence of integers. The $h$-vector can be also defined as the $(d + 1)$-th difference of the Hilbert function of $X$. Thus, when $X$ is a set of points, its $h$-vector is the first difference of its Hilbert function.

Let $n$ and $i$ be positive integers. The $i$-binomial expansion of $n$ is

$$n^{(i)} = \binom{n}{i} + \binom{n-1}{i-1} + \cdots + \binom{n}{j},$$

where $n > n_{i-1} > \cdots > n_j \geq j \geq 1$. The $i$-binomial expansion of $n$ is unique (see, e.g. [7] Lemma 4.2.6). Hence we may define

$$n^{<i>}_{(i)} = \binom{n}{i+1} + \binom{n-1}{i} + \cdots + \binom{n+1}{j+1}.$$

Definition 3.1. Let $\mathbf{h} = (h_0, h_1, \ldots, h_s)$ be a finite sequence of nonnegative integers. Then $\mathbf{h}$ is called an O-sequence if $h_0 = 1$ and $h_{i+1} \leq h^{<i>}_{(i)}$ for all $i$.

By Macaulay’s theorem we know that O-sequences are the Hilbert functions of standard graded $k$-algebras.

Definition 3.2. Let $\mathbf{h} = (1, h_1, \ldots, h_s, 1)$ be a sequence of nonnegative integers. Then $\mathbf{h}$ is an SI-sequence if:

- $h_i = h_{s-i}$ for all $i = 0, \ldots, s$,
- $(h_0, h_1, \ldots, h_t - h_{t-1}, 0, \ldots)$ is an O-sequence, where $t$ is the greatest integer $\leq \frac{s}{2}$.

Stanley, in [14], characterized the $h$-vectors of all graded Artinian Gorenstein quotients of $k[x_0, x_1, x_2]$, showing that these are SI-sequence and, moreover, any SI-sequence, with $h_1 = 3$, is the $h$-vector of some Artinian Gorenstein quotient of $k[x_0, x_1, x_2]$.

Geramita and Migliore [10], show that every minimal free resolution which occurs for a Gorenstein artinian ideal of codimension 3, also occurs for some reduced set of points in $\mathbb{P}^3$, a stick figure curve in $\mathbb{P}^4$ and more generally a “generalized” stick figure in $\mathbb{P}^n$. In this case the points in $\mathbb{P}^3$, with such minimal free resolution, can be found as the intersection of two stick figures (defined below) which are arithmetically Cohen-Macaulay. It is, however, very hard to see where these points live, that is describe them in term of their coordinates.

We start recall some basic definitions and results that we find in [12], [13], and [14].

Definition 3.3. A generalized stick figure is a union of linear subvarieties of $\mathbb{P}^n$, of the same dimension $d$, such that the intersection of any three components has dimension at most $d - 2$ (the empty set has dimension -1).
In particular, sets of reduced points are stick figure, and a stick figure of dimension $d = 1$ is nothing more than a reduced union of lines having only nodes as singularities.

**Definition 3.4.** Let $C_1$, $C_2$ and $X$ be subschemes of $\mathbb{P}^n$ of the same dimension, where $X$ is a Complete Intersection (arithmetically Gorenstein) such that $I_X \subset I_{C_1} \cap I_{C_2}$. Then $C_1$ is **directly CI-linked** (directly G-linked) to $C_2$ by $X$, if $I_X : I_{C_1} = I_{C_2}$ and $I_X : I_{C_2} = I_{C_1}$.

If $C_1$ is directly linked to $C_2$ by $X$, we will write $C_1 \sim C_2$ and two schemes $C_1$ and $C_2$ are said to be **residual** to each other. If, in addition, $C_1$ and $C_2$ have no common components then we say that they are **geometrically linked** by $X$.

There is a important fact that we will use about Liaison: the possibility to built arithmetically Gorenstein zeroscheme starting from two schemes linked by a Complete Intersection. In fact we have the following theorem.

**Theorem 3.5 (Theorem 4.2.1 in [11]).** Let $C_1$, $C_2$ be two aCM subschemes of $\mathbb{P}^n$ of codimension $c$, with no common components and saturated ideals $I_{C_1}$ and $I_{C_2}$. If we suppose that $X = C_1 \cup C_2$ is a codimension $c$ arithmetically Gorenstein scheme, then $I_{C_1} + I_{C_2}$ is the saturated ideal of a codimension $c + 1$ arithmetically Gorenstein scheme $Y$.

Now we recall how Migliore and Nagel, in Section 6 of [12], find a reduced arithmetically Gorenstein zeroscheme, for the case of $\mathbb{P}^3$, with given $h$–vector. This set of points will result from the intersection of two arithmetically Cohen-Macaulay curves in $\mathbb{P}^3$, linked by a complete intersection curve which is a stick figure.

Let $h = (h_0, h_1, \ldots, h_s) = (1, 3, h_2, \ldots, h_{t-1}, h_t, h_{t+1}, \ldots, h_{t+s}, 3, 1)$ be a $SI$–sequence, and consider the first difference

$$\Delta h = (1, 2, h_2 - h_1, \ldots, h_t - h_{t-1}, 0, 0, \ldots, 0, h_{t-1} - h_t, \ldots, -2, -1).$$

Define two sequences $a = (a_0, \ldots, a_t)$ and $g = (g_0, \ldots, g_{s+1})$ in the following way:

$$a_i = h_i - h_{i-1} \text{ for } 0 \leq i \leq t,$$

and

$$g_i = \begin{cases} 
 i + 1 & \text{for } 0 \leq i \leq t \\
 t + 1 & \text{for } t \leq i \leq s - t + 1 \\
 s - i + 2 & \text{for } s - t + 1 \leq i \leq s + 1
\end{cases}.$$

We observe that $a_1 = g_1 = 2$, $a$ is a $O$–sequence since $h$ is a $SI$–sequence and $g$ is the $h$–vector of a codimension two complete intersection. So, we would like to find two curves $C_1$ and $X$ in $\mathbb{P}^3$ with $h$–vector respectively $a$ and $g$. In particular it is easy to see that, for that $h$–vector $g$, $X$ is a complete intersection of two surfaces in $\mathbb{P}^3$ of degree $t + 1$ and $s - t + 2$.

We can get $X$ as a stick figure by taking, as equations of those surfaces, two polynomials which are the product, respectively, of $A_0, \ldots, A_t$ and $B_0, \ldots, B_{s+t+1}$, all generic linear forms. Considering the entries of $a = (a_0, \ldots, a_t)$, Migliore and Nagel build the stick figure $C_1$ (embedded in $X$), as the union of $a_i$ consecutive lines in $A_i = 0$ (always the first in $B_0 = 0$), that is they take $a_0$ lines given by the intersections of $A_0 = 0$ with $B_0 = 0, \ldots, B_{a_0-1} = 0$, then $a_1$ lines given by the intersections of $A_1 = 0$ with $B_0 = 0, \ldots, B_{a_1-1} = 0$. Here **consecutive** is referred to the indices of the forms $B_0, \ldots, B_{s+t+1}$: two lines are consecutive if they are given by the intersections of a certain $A_i = 0$ with $B_j = 0$ and $B_{j+1} = 0$ for a given $j$. 

with $0 \leq j \leq s - t$. Migliore and Nagel proved that $C_1$, build in this way, is an aCM scheme with h-vector $a$ (Corollary 3.7 in [12]). In this way, if we consider $C_2$, the residual of $C_1$ in $X$, the intersection of $C_1$ and $C_2$ is an arithmetically Gorenstein scheme $Y$ of codimension 3, by Theorem [3,3]. This is also a reduced set of points because $X$, $C_1$ and $C_2$ are stick figures and it has the desired h-vector by the following theorem:

**Theorem 3.6** (Lemma 2.5 in [12]). Let $C_1$, $C_2$, $X$ and $Y$ be defined as above. Let $g = (1, c, g_2, \ldots, g_s, g_{s+1})$ be the h-vector of $X$, and let $a = (1, a_1, \ldots, a_t)$ and $b = (1, b_1, \ldots, b_t)$ be the h-vectors of $C_1$ and $C_2$, then

$$b_i = g_{s+1-i} - a_{s+1-i}$$

for $i \geq 0$. Moreover the sequence $d_i = a_i + b_i - g_i$ is the first difference of the h-vector $h = (h_0, h_1, \ldots, h_s)$ of $Y$.

As a matter of fact we have $d_i = h_i - h_{i-1}$ since:

- for $0 \leq i \leq t$ we have $d_i = a_i = h_i - h_{i-1}$;
- for $t + 1 \leq i \leq s - t$ we have $d_i = b_i - g_i = 0$;
- for $s - t + 1 \leq i \leq s + 1$ we have $d_i = b_i - g_i = -a_{s+1-i} = -(h_{s+1-i} - h_{s-i})$.

**Example 3.7.** Let $h = (1, 3, 4, 3, 1)$ be a SI-sequence. Consider the first difference of $h$, i.e. $\Delta h = (1, 2, 1, -1, -2, -1)$.

So, $t = 2$ and $g = (1, 2, 3, 2, 1)$ is the h-vector of $X$, a stick figure which is the complete intersection of $F_1 = \prod_{i=0}^{2} A_i$ and $F_2 = \prod_{i=0}^{3} B_i$, where $A_i$ and $B_i$ are general linear forms.

Now, we call $L_{i,j}$ the intersection between $A_i = 0$ and $B_j = 0$. Since $a = (1, 2, 1)$, then $C_1 = L_{0,0} \cup L_{1,0} \cup L_{1,1} \cup L_{2,0}$ is the scheme, in $X$, with h-vector $a$.

So, it is clear that the residual $C_2$ of $C_1$ in $X$ is the union of the lines of $X$ which aren’t components in $C_1$. Then the reduced set of points $Y$ with h-vector $(1, 3, 4, 3, 1)$ consists of 12 points which exactly are:

- 3 points on $L_{0,0}$, intersections between $L_{0,0}$ and $L_{0,1}$, $L_{0,2}$ and $L_{0,3}$;
- 2 points on $L_{1,0}$, intersections between $L_{1,0}$ and $L_{1,1}$, $L_{1,3}$;
- 4 points on $L_{1,1}$, intersections between $L_{1,1}$ and $L_{1,2}$, $L_{1,3}$, $L_{0,1}$ and $L_{2,1}$;
- 3 points on $L_{2,0}$, intersections between $L_{2,0}$ and $L_{2,1}$, $L_{2,2}$ and $L_{2,3}$.

4. Planar Complete Intersections via Hadamard Product

In this section we show how to get a zero-dimensional planar complete intersection $Z_{ab}^4$, as the product of two sets of collinear points. Observe that, by Corollary 4.5 in [2], if the two sets of collinear points lie in two general lines, in $\mathbb{P}^3$, then their Hadamard product gives points on a quadric. However, this could also happen when the lines are coplanar, as explained in the following Remark [2]. Hence, for our construction of $Z_{ab}^4$, it is mandatory to carefully choose the coordinates of the points. We start by considering four points in $\mathbb{P}^1$ without zero coordinates.

Let $A$ be a collection of four distinct points $A_i = [\alpha_i : \beta_i]$ in $\mathbb{P}^1 \setminus \Delta_0$, for $i = 0, \ldots, 3$, and let

$$
\begin{align*}
\alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 &= 0 \\
\beta_0 x_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 &= 0
\end{align*}
$$

be the equations of a line $L^A$ in $\mathbb{P}^3$.

We define two families of points in $\mathbb{P}^3$ associated to the set $A$ (and hence to the line $L^A$):

$$p_k^A = \left[ \frac{\alpha_0 + k \beta_0}{\alpha_0} : \frac{\alpha_1 + k \beta_1}{\alpha_1} : \frac{\alpha_2 + k \beta_2}{\alpha_2} : \frac{\alpha_3 + k \beta_3}{\alpha_3} \right] \quad k \in \mathbb{N}$$
and

\[ Q_k^A = \left[ \frac{k\alpha_0 + \beta_0}{\beta_0} : \frac{k\alpha_1 + \beta_1}{\beta_1} : \frac{k\alpha_2 + \beta_2}{\beta_2} : \frac{k\alpha_3 + \beta_3}{\beta_3} \right] \quad k \in \mathbb{N}. \]

Note that \( P_0^A = Q_0^A = [1 : 1 : 1 : 1] \).

**Example 4.1.** Consider \( A_0 = [1 : 1] \), \( A_1 = [1 : 2] \), \( A_2 = [1 : 3] \), and \( A_3 = [1 : 4] \) giving, by (4), the line

\[ L^A : \begin{cases} x_0 + x_1 + x_2 + x_3 = 0 \\ x_0 + 2x_1 + 3x_2 + 4x_3 = 0 \end{cases}. \]

One has

\[ P_1^A = [2 : 3 : 4 : 5], \quad P_2^A = [3 : 5 : 7 : 9], \quad P_3^A = [4 : 7 : 10 : 13], \quad P_4^A = [5 : 9 : 13 : 17], \ldots \]

and

\[ Q_1^A = \left[ \frac{3}{2} : \frac{4}{3} : \frac{5}{4} \right], \quad Q_2^A = \left[ \frac{3}{2} : \frac{5}{3} : \frac{7}{4} \right], \quad Q_3^A = \left[ \frac{4}{2} : \frac{5}{3} : \frac{7}{4} \right], \quad Q_4^A = \left[ \frac{5}{3} : \frac{7}{4} : \frac{9}{5} \right], \ldots \]

**Remark 4.2.** The condition that the four points \( A_i \) are distinct implies \( \frac{\alpha_i}{\alpha_j} \neq \frac{\alpha_j}{\alpha_i} \) for any \( 0 \leq i < j \leq 3 \). In particular this fact assure us that \( L^A \cap \Delta_1 = \emptyset \). As a matter of fact, suppose that, for example, \( L^A \) intersects \( \Delta_1 \) in the point \( [0 : 0 : \gamma_2 : \gamma_3] \), with \( \gamma_i \neq 0 \), for \( i = 2, 3 \). Then, from (4), we get

\[ \begin{cases} \alpha_2\gamma_2 + \alpha_3\gamma_3 = 0 \\ \beta_2\gamma_2 + \beta_3\gamma_3 = 0 \end{cases} \]

which gives

\[ \frac{\alpha_2}{\alpha_3} = -\frac{\gamma_3}{\gamma_2} = \frac{\beta_3}{\beta_2} \]

or equivalently \( \frac{\alpha_2}{\alpha_3} = \frac{\beta_3}{\beta_2} \) which implies \( A_2 = A_3 \).

Notice that

\[ P_k^A = [1 + k\frac{\alpha_0}{\alpha_1} : 1 + k\frac{\alpha_2}{\alpha_1} : 1 + k\frac{\alpha_3}{\alpha_1}] = (1-k)[1 : 1 : 1] + k[1 + \frac{\alpha_0}{\alpha_1} : 1 + \frac{\alpha_2}{\alpha_1} : 1 + \frac{\alpha_3}{\alpha_1}] = (1-k)P_0^A + kP_1^A \]

and similarly \( Q_k^A = (1-k)Q_0^A + kQ_1^A \), for all \( k \geq 2 \). Hence the points \( P_k^A \) lie in the line \( \ell^P \) spanned by \( P_0^A \) and \( P_1^A \) and the points \( Q_k^A \) lie in the line \( \ell^Q \) spanned by \( Q_0^A \) and \( Q_1^A \). In particular, for any fixed \( k \), the points \( P_0^A, \ldots, P_k^A \) are collinear and, similarly, the points \( Q_0^A, \ldots, Q_k^A \) are collinear.

Consider now the matrices

\[ M = \begin{pmatrix} \alpha_0/\alpha_1 & \alpha_1/\alpha_1 & \alpha_2/\alpha_1 & \alpha_3/\alpha_1 \\ \alpha_0^2/\alpha_2 & \alpha_1^2/\alpha_2 & \alpha_2^2/\alpha_2 & \alpha_3^2/\alpha_2 \\ \beta_0/\beta_1 & \beta_1/\beta_1 & \beta_2/\beta_1 & \beta_3/\beta_1 \end{pmatrix} \quad N = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_0 & \beta_1 & \beta_2 & \beta_3 \end{pmatrix} \]

and denote by \( |M(i)| \) the determinant of the submatrix of \( M \) with the \( i \)-th column removed and by \( |N(i,j)| \) the determinant of the submatrix of \( N \) with the \( i \)-th and \( j \)-th columns removed.

**Proposition 4.3.** The defining equations in \( k[\mathbb{P}^3] \) of the lines of \( \ell^P \) and \( \ell^Q \) are:

\[ \ell^P : \begin{cases} \sum_{t=0}^3 (-1)^{t+1} \alpha_t \beta_t |M(t+1)| x_t = 0 \\ \sum_{t=1}^3 (-1)^t \alpha_t |N(1,t+1)| x_t = 0 \end{cases}. \]
\[ \ell Q : \begin{cases} 
\sum_{i=0}^{3} (-1)^{i+1} \alpha_i \beta_i |M(t + 1)|x_t = 0 \\
\sum_{i=1}^{3} (-1)^{i} \beta_i |N(1, t + 1)|x_t = 0 
\end{cases} \]

Moreover \( \ell^P \) and \( \ell^Q \) are two distinct coplanar lines.

Proof. We prove the first part of the statement only for \( \ell^P \) since the proof is identical for \( \ell^Q \). The equations of \( \ell^P \) are given by the equation of the plane through \( P_1^A \), \( P_2^A \) and \( Q_1^A \),

\[
\begin{vmatrix}
1 & 1 & 1 & 1 \\
1 + \frac{\alpha_1}{\alpha_0} & 1 + \frac{\alpha_2}{\alpha_1} & 1 + \frac{\alpha_3}{\alpha_2} & 1 + \frac{\alpha_3}{\alpha_3}
\end{vmatrix} = 0
\]

which is, up to rescaling,

\[
\sum_{i=0}^{3} (-1)^{i+1} \alpha_i \beta_i |M(t + 1)|x_t = 0
\]

and by the equation of the plane through \( P_1^A \), \( P_2^A \) and \( [1 : 0 : 0 : 0] \)

\[
\begin{vmatrix}
1 & 1 & 1 & 1 \\
1 + \frac{\beta_1}{\alpha_0} & 1 + \frac{\beta_2}{\alpha_1} & 1 + \frac{\beta_3}{\alpha_2} & 1 + \frac{\beta_3}{\alpha_3}
\end{vmatrix} = 0,
\]

which is, up to rescaling

\[
\sum_{i=1}^{3} (-1)^i \alpha_i |N(1, t + 1)|x_t = 0.
\]

To prove the second part of the statement notice that \( \ell^P \) and \( \ell^Q \) intersect at \([1 : 1 : 1 : 1]\). Thus it is enough to prove that \( \ell^P \) and \( \ell^Q \) are distinct. To this aim, observe that the point \( S_P = \left( \frac{\alpha_1}{\alpha_0} : \frac{\alpha_2}{\alpha_1} : \frac{\alpha_3}{\alpha_2} : \frac{\alpha_3}{\alpha_3} \right) = P_1^A - P_0^A \) lies in \( \ell^P \) and the point \( S_Q = \left( \frac{\alpha_1}{\alpha_0} : \frac{\alpha_2}{\alpha_1} : \frac{\alpha_3}{\alpha_2} : \frac{\alpha_3}{\alpha_3} \right) = Q_1^A - Q_0^A \) lies in \( \ell^Q \). Suppose that \( \ell^P = \ell^Q \). Then the points \([1 : 1 : 1 : 1]\), \( S_P \) and \( S_Q \) would be collinear, that is the matrix

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
\frac{\alpha_1}{\alpha_0} & \frac{\alpha_2}{\alpha_1} & \frac{\alpha_3}{\alpha_2} & \frac{\alpha_3}{\alpha_3}
\end{pmatrix}
\]

would have rank 2. Applying the operations \( R_2 - \frac{\alpha_2}{\alpha_0} R_1 \rightarrow R_2 \) and \( R_3 - \frac{\alpha_3}{\alpha_0} R_1 \rightarrow R_3 \)

(and then \( C_2 - C_1 \)) we get the matrix

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{\alpha_0 \alpha_2 - \alpha_1 \beta_2}{\beta_0 - \beta_2} & \frac{\alpha_0 \alpha_3 - \alpha_1 \beta_3}{\beta_0 - \beta_3} & \frac{\alpha_0 \alpha_3 - \alpha_1 \beta_3}{\beta_0 - \beta_3}
\end{pmatrix}
\]

Then we apply the operation \( R_3 + \frac{\alpha_3}{\beta_0} R_2 \rightarrow R_3 \) obtaining

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{\alpha_0 \alpha_2 - \alpha_1 \beta_2}{\alpha_0 \alpha_1} & \frac{\alpha_0 \alpha_3 - \alpha_1 \beta_3}{\alpha_0 \alpha_1} & \frac{\alpha_0 \alpha_3 - \alpha_1 \beta_3}{\alpha_0 \alpha_1}
\end{pmatrix}
\]

Since \( \frac{\alpha_0}{\beta_0} \neq \frac{\alpha_1}{\beta_0} \), by hypothesis, we can simplify the second row obtaining

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{\alpha_0 \beta_2 - \alpha_1 \beta_2}{\alpha_2 \beta_0 \beta_1} & \frac{\alpha_0 \beta_3 - \alpha_1 \beta_3}{\alpha_2 \beta_0 \beta_1} & \frac{\alpha_0 \beta_3 - \alpha_1 \beta_3}{\alpha_2 \beta_0 \beta_1}
\end{pmatrix}
\]
Thus the matrix would have rank 2 if and only if
\[(α_0β_2 - α_2β_0)(α_1β_2 - α_2β_1) = 0 \text{ and } (α_0β_3 - α_3β_0)(α_1β_3 - α_3β_1) = 0.\]

Such equalities are verified in the following cases
\[\bullet \frac{α_0}{β_0} = \frac{α_2}{β_2} = \frac{α_3}{β_3},\]
\[\bullet \frac{α_0}{β_0} = \frac{α_1}{β_1} \text{ and } \frac{α_1}{β_1} = \frac{α_2}{β_2},\]
\[\bullet \frac{α_0}{β_0} = \frac{α_2}{β_2} \text{ and } \frac{α_1}{β_1} = \frac{α_3}{β_3},\]
which give contradictions since the points \(A_i\) are distinct.

By the previous proposition, we immediately get the following

**Corollary 4.4.** One has \(P_i^A \neq Q_j^A\), for every \(i, j \geq 1\).

**Proof.** Suppose that, for some \(i, j \geq 1\) one has \(P_i^A = Q_j^A\). Then \(ℓ^P\) and \(ℓ^Q\) would intersect in the points \([1, 1, 1, 1]\) and \(P_i^A(= Q_j^A)\) giving \(ℓ^P = ℓ^Q\), which is a contradiction, by Proposition 4.3.

**Example 4.5.** Consider Example 4.1. In this case one has
\[M = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 1 & 4 & 9 & 16 \end{pmatrix} \quad N = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}\]
from which we get
\[|M(1)| := -2 \quad |M(2)| := -6 \quad |M(3)| := -6 \quad |M(4)| := -2\]
and
\[|N(1, 2)| := 1 \quad |N(1, 3)| := 2 \quad |N(1, 4)| := 1 .\]

Hence the line \(ℓ^P\) through the points \(P_k^A\) is defined, up to rescaling, by the equations
\[ℓ^P : \begin{cases} 2x_0 - 12x_1 + 18x_2 - 8x_3 = 0 \\ -x_1 + 2x_2 - x_3 = 0 \end{cases}\]
and the line \(ℓ^Q\) through the points \(Q_k^A\) is defined, up to rescaling, by the equations
\[ℓ^Q : \begin{cases} 2x_0 - 12x_1 + 18x_2 - 8x_3 = 0 \\ -2x_1 + 6x_2 - 4x_3 = 0 \end{cases} .\]

We add, now, another condition on the points in \(A\). Let \(W_i\) be the point \([1 : -i]\), then we define the set of points \(W\) as
\[W = \bigcup_{i \in N^*} (W_i \cup W_+),\]
where \(N^* = N \setminus \{0\}\).

**Remark 4.6.** It is easy to verify that if \(A_i \notin W\), for \(i = 0, \ldots, 3\), then \(P_i^A \notin Δ_2\), for any \(i\), and \(Q_j^A \notin Δ_2\), for any \(j\), that is such points do not have any zero coordinate. This fact will be fundamental in the successive parts of the paper in order to apply Lemmas 2.6, 2.7, and 2.8. Clearly, \(A_i \notin W\) if its coordinates are both strictly positive.
Denote by $I(u) = \{i_0, i_1, \ldots, i_{n-1}\}$ a set of nonnegative integers with $0 = i_0 < i_1 < \cdots < i_{n-1}$. Given positive integers $a$ and $b$, we define the set of points $Z_{a,b}^4$ by the pair-wise Hadamard product of points $P_i^4$ and $Q_j^4$ as

$$Z_{a,b}^4 = \{P_i^4 \ast Q_j^4 : i \in I(a), j \in I(b)\}.$$  

We can represent these sets in matrix form as:

\[
\begin{pmatrix}
P_{i_0}^4 \ast Q_{j_0}^4 & P_{i_0}^4 \ast Q_{j_1}^4 & \cdots & P_{i_0}^4 \ast Q_{j_{n-1}}^4 \\
P_{i_1}^4 \ast Q_{j_0}^4 & P_{i_1}^4 \ast Q_{j_1}^4 & \cdots & P_{i_1}^4 \ast Q_{j_{n-1}}^4 \\
\vdots & \vdots & \ddots & \vdots \\
P_{i_{n-1}}^4 \ast Q_{j_0}^4 & P_{i_{n-1}}^4 \ast Q_{j_1}^4 & \cdots & P_{i_{n-1}}^4 \ast Q_{j_{n-1}}^4
\end{pmatrix}
\]

Observe that, by the conditions on $I(a)$ and $I(b)$ one has

$$P_i^4 = P_0^4 = [1 : 1 : 1] = Q_0^4 = Q_{0,0}^4.$$

**Theorem 4.7.** If $A_i \notin W$, for $i = 0, \ldots, 3$, then, for any positive integers $a$ and $b$, $Z_{a,b}^4$ is a planar complete intersection of $ab$ points.

**Proof.** Consider $i, k \in I(a)$ and $j, l \in I(b)$. We prove first that $P_i^4 \ast Q_j^4 = P_k^4 \ast Q_l^4$ if and only if $i = k$ and $j = l$, implying that $Z_{a,b}^4$ is a set of cardinality $ab$.

Suppose that $P_i^4 \ast Q_j^4 = P_k^4 \ast Q_l^4$ and distinguish two cases. First, we consider the case in which two indices are equal. Suppose, for example, that $i = k$ and $j \neq l$, i.e. $P_i^4 \ast Q_j^4 = P_i^4 \ast Q_l^4$. Since, by Remark 1.6, $P_i^4 \in \Delta_2$, one has, by Lemma 2.8, that $Q_j^4 = Q_l^4$, which is a contradiction since $j \neq l$. The same approach works if $i \neq k$ and $j = l$. Let us consider the case $i \neq k$ and $j \neq l$. Looking at the coordinates, the condition $P_i^4 \ast Q_j^4 = P_k^4 \ast Q_l^4$, is

\[
\begin{pmatrix}
(a_0 + i\beta_0)(a_0 + j\beta_0) \\
a_1 + i\beta_1 \\
a_2 + i\beta_2 \\
a_3 + i\beta_3
\end{pmatrix} =
\begin{pmatrix}
(a_0 + k\beta_0)(a_0 + l\beta_0) \\
a_1 + k\beta_1 \\
a_2 + k\beta_2 \\
a_3 + k\beta_3
\end{pmatrix}
\]

or equivalently

\[
\begin{pmatrix}
(a_s + i\beta_s)(ja_s + \beta_s) \\
(a_s + k\beta_s)(la_s + \beta_s)
\end{pmatrix} = \lambda
\]

for some $\lambda \neq 0$. This implies

\[
\frac{(a_0 + i\beta_0)(ja_0 + \beta_0)}{(a_0 + k\beta_0)(la_0 + \beta_0)} = \frac{(a_s + i\beta_s)(ja_s + \beta_s)}{(a_s + k\beta_s)(la_s + \beta_s)}
\]

for $s = 1, \ldots, 3$.

Hence $[a_0 : \beta_0]$, $[a_1 : \beta_1]$, $[a_2 : \beta_2]$ and $[a_3 : \beta_3]$ must satisfy

\[(5) \quad (a_0 + i\beta_0)(ja_0 + \beta_0)(a_s + k\beta_s)(la_s + \beta_s) - (a_s + i\beta_s)(ja_s + \beta_s)(a_0 + k\beta_0)(la_0 + \beta_0) = 0
\]

for $s = 1, \ldots, 3$. If we rewrite (5) as an equation in $a_s$, we get $\tau_2 a_s^2 + \tau_1 a_s + \tau_0 = 0$ where

\[
\tau_2 = (a_0 + i\beta_0)(ja_0 + \beta_0)l - (a_0 + k\beta_0)(la_0 + \beta_0)j,
\]

\[
\tau_1 = [(a_0 + i\beta_0)(ja_0 + \beta_0)(kl + 1) - (a_0 + k\beta_0)(la_0 + \beta_0)(ij + 1)]\beta_s,
\]

\[
\tau_0 = [(a_0 + i\beta_0)(ja_0 + \beta_0)k - (a_0 + k\beta_0)(la_0 + \beta_0)i]\beta_s^2.
\]

The discriminant of $\tau_2 a_s^2 + \tau_1 a_s + \tau_0$ turns to be equal to

\[
\beta_s^2(ja_0^2 - la_0^2 - ijl + jkl^2 + 2jk a_0 \beta_0 - 2ik a_0 \beta_0 - iij k a_0 \beta_0)^2
\]
which gives, after some tedious computation, the solutions of $\alpha_s$ as

$$\alpha_s = \frac{\alpha_0 \beta_s}{\beta_0} \text{ or } \alpha_s = \rho \beta_s, \text{ for } s = 1, \ldots, 3,$$

where

$$\rho = \frac{(jk - il)\alpha_0 + (ijk - i - ikl + k)\beta_0}{(ijl - j + l - jkl)\alpha_0 + (il - jk)\beta_0}.$$

Computing the solutions of (53) for $s = 1, \ldots, 3$, we obtain that $P_i^A \ast Q_j^A = P_k^A \ast Q_l^A$ if one of the following cases is verified

i) $\alpha_1 = \frac{\alpha_0 \beta_1}{\beta_0}$, $\alpha_2 = \frac{\alpha_0 \beta_2}{\beta_0}$, $\alpha_3 = \frac{\alpha_0 \beta_3}{\beta_0};$

ii) $\alpha_1 = \frac{\alpha_0 \beta_1}{\beta_0}$, $\alpha_2 = \frac{\alpha_0 \beta_2}{\beta_0}$, $\alpha_3 = \rho \beta_3;$

iii) $\alpha_1 = \frac{\alpha_0 \beta_1}{\beta_0}$, $\alpha_2 = \rho \beta_2$, $\alpha_3 = \frac{\alpha_0 \beta_3}{\beta_0};$

vi) $\alpha_1 = \frac{\alpha_0 \beta_1}{\beta_0}$, $\alpha_2 = \rho \beta_2$, $\alpha_3 = \rho \beta_3;$

v) $\alpha_1 = \rho \beta_1$, $\alpha_2 = \frac{\alpha_0 \beta_2}{\beta_0}$, $\alpha_3 = \frac{\alpha_0 \beta_3}{\beta_0};$

vi) $\alpha_1 = \rho \beta_1$, $\alpha_2 = \frac{\alpha_0 \beta_2}{\beta_0}$, $\alpha_3 = \rho \beta_3;$

vii) $\alpha_1 = \rho \beta_1$, $\alpha_2 = \rho \beta_2$, $\alpha_3 = \frac{\alpha_0 \beta_3}{\beta_0};$

viii) $\alpha_1 = \rho \beta_1$, $\alpha_2 = \rho \beta_2$, $\alpha_3 = \rho \beta_3.$

However, all cases implies that there are at least two pairs of indices $(p_1, p_2)$ and $(p_3, p_4)$ with $\frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2}$ and $\frac{\alpha_3}{\beta_3} = \frac{\alpha_4}{\beta_4}$ which is a contradiction since the points $A_i$ must be distinct. Hence $P_i^A \ast Q_j^A = P_k^A \ast Q_l^A$ if and only if $i = k$ and $j = l$ and then $Z_{a,b}^A$ consists of $ab$ points.

To prove that $Z_{a,b}^A$ is a planar complete intersection, notice first that, since $P_i^A \notin \Delta_2$, for $i \in \mathcal{I}(a)$ and $Q_j^A \notin \Delta_2, \text{ for } j \in \mathcal{I}(b)$, we can apply Lemma 2.6 obtaining that $P_i^A \ast \ell^Q$ is a line for $i \in \mathcal{I}(a)$ and $Q_j^A \ast \ell^P$ is a line for $j \in \mathcal{I}(b)$. Moreover, by Corollary 2.11 one has

$$P_i^A \ast Q_j^A \in \mathcal{Q}_j \ast \ell^P \text{ for } i \in \mathcal{I}(a)$$

$$P_i^A \ast Q_j^A \in \mathcal{P}_j \ast \ell^Q \text{ for } j \in \mathcal{I}(b)$$

that is the points $P_{i_1}^A \ast Q_{j_1}^A, \ldots, P_{i_{n-1}}^A \ast Q_{j_n}^A$ lie in the line $Q_j \ast \ell^P$, for $j \in \mathcal{I}(b)$ and similarly the points $P_{i_1}^A \ast Q_{j_1}^A, \ldots, P_{i_{n-1}}^A \ast Q_{j_n}^A$ lie in the line $P_i \ast \ell^Q$, for $i \in \mathcal{I}(a).$

For any $i$ and $j$ the lines $P_i \ast \ell^Q$ and $Q_j \ast \ell^P$ clearly intersect in the point $P_i^A \ast Q_j^A$, hence, as $i$ varies in $\mathcal{I}(a)$ and $j$ varies in $\mathcal{I}(b)$, the $ab$ intersections $P_i \ast \ell^Q \cap Q_j \ast \ell^P$ give the $ab$ points in $Z_{a,b}^A$. \hfill \Box

In Figure 1 we can see four different examples of $Z_{a,b}^A$. The example in (i) is for $a = b = 2$ with $\mathcal{I}(a) = \mathcal{I}(b) = \{0, 1\}$ and the white points are represented to show the behaviour of the families of points $P_i^A$ and $Q_j^A$. The example in (ii) is for $a = 4$ and $b = 5$ with $\mathcal{I}(a) = \{0, 1, 2, 3\}$ and $\mathcal{I}(b) = \{0, 1, 2, 3, 4\}$. The examples in (iii) and (iv) are for $a = 2$ and $b = 3$, but, while in (iii) we use $\mathcal{I}(a) = \{0, 1\}$ and $\mathcal{I}(b) = \{0, 1, 2\}$, in (iv) we use $\mathcal{I}(a) = \{0, 2\}$ and $\mathcal{I}(b) = \{0, 2, 4\}$.

**Remark 4.8.** In [2], the authors prove that the Hadamard product of two generic lines is a quadric surface. This leads to the question if the Hadamard product of two coplanar lines is a plane. The following example shows that even when the lines $\ell$ and $\ell'$ are coplanar, $\ell \ast \ell'$ might still be a quadric. Consider the points

$$S_1 = [1, 1, 1, 1], \ S_2 = [3, \frac{3}{2}, 5, \frac{7}{2}], \ S_3 = [\frac{3}{2}, 3, \frac{4}{3}, \frac{7}{3}]$$

and the coplanar lines

$$\ell = S_1 \ast S_2 \quad \ell' = S_1 \ast S_3.$$
One has

$$\ell : \begin{cases} 3x_1 + 4x_2 - 7x_3 = 0 \\ 7x_0 - 4x_1 - 3x_2 = 0 \end{cases}$$

$$\ell' : \begin{cases} x_1 + 24x_2 - 25x_3 = 0 \\ 10x_0 - x_1 - 9x_2 = 0 \end{cases}$$

Using the Singular procedure \texttt{HPr}, described in Section 5 of [2], we can easily see that $\ell \ast \ell'$ is a quadric:

> ring R=0,(x(0..3)),dp;
> ideal J1=3*x(1)+4*x(2)-7*x(3),7*x(0)-4*x(1)-3*x(2);
> ideal J2=x(1)+24*x(2)-25*x(3), 10*x(0)-x(1)-9*x(2);
> ideal K=HPr(J1,J2,3);
> K;

\begin{align*}
K[1] & = 1120*x(0)^2-68*x(0)*x(1)+x(1)^2+1056*x(0)*x(2)-
& -30*x(1)+x(2)+216*x(2)-2-3500*x(0)+x(3)+110*x(1)+x(3)-
& -1530*x(0)+x(3)+2625*x(3)-2
\end{align*}

On the other hand, our construction shows that there are cases in which $\ell \ast \ell'$ is a plane. As an example consider the points

$$A_0 = [1, 2], A_1 = [2, 1], A_2 = [1, 3], A_3 = [2, 5]$$

giving

$$P_0^A = Q_0^A = [1, 1, 1, 1], P_1^A = [3, \frac{3}{2}, 4, \frac{7}{2}], Q_1^A = [\frac{3}{2}, 3, \frac{4}{3}, \frac{7}{3}]$$.
For the lines \( \ell^P = P^A_0 \star P^A_1 \) and \( \ell^Q = Q^A_0 \star Q^A_1 \), we know, by Theorem 4.7 that \( \ell \star \ell' \) is a plane. If we write down the equations of the two lines

\[
\ell^P : \begin{cases} 
    x_1 + 4x_2 - 5x_3 = 0 \\
    5x_0 - 2x_1 - 3x_2 = 0
\end{cases}
\]

\[
\ell^Q : \begin{cases} 
    x_1 + 24x_2 - 25x_3 = 0 \\
    10x_0 - x_1 - 9x_2 = 0
\end{cases}
\]

we can do a direct check in Singular:

> ring R=0,(x(0..3)),dp;
> ideal I1=x(1)+4*x(2)-5*x(3),5*x(0)-2*x(1)-3*x(2);
> ideal I2=x(1)+24*x(2)-25*x(3), 10*x(0)-x(1)-9*x(2);
> ideal K=HP(11,I2,3);
> K;

Here we have two examples of coplanar lines with different behaviour of their Hadamard product. In these examples the lines are generated by respectively the following points

\[ S_1 = [1,1,1,1], \quad S_2 = [3,\frac{1}{2},5,\frac{7}{2}], \quad S_3 = [\frac{1}{2},3,\frac{4}{5},\frac{7}{5}] \]

\[ P^A_0 = [1,1,1,1], \quad P^A_1 = [3,\frac{1}{2},4,\frac{7}{2}], \quad Q^A_1 = [\frac{1}{2},3,\frac{4}{5},\frac{7}{5}] \]

Notice that \( S_1 = P^A_0 \), and \( S_3 = Q^A_1 \) while \( S_2 \) and \( P^A_1 \) differ only by an entry.

**Corollary 4.9.** Let \( Z^A_{a,b} \) as in Theorem 4.7 and let

\[
h = \sum_{t=0}^{3} (-1)^{t+1} \alpha_t |M(t + 1)|x_t = 0
\]

\[
f = \sum_{t=1}^{3} (-1)^{t} \alpha_t |N(1, t + 1)|x_t = 0
\]

\[
g = \sum_{t=1}^{3} (-1)^{t} \beta_t |N(1, t + 1)|x_t = 0.
\]

Then the ideal of \( Z^A_{a,b} \) is generated by \( h, f \star Q^A_0 \cdots \star Q^A_{n-1}, g \star P^A_0 \cdots \star P^A_{n-1} \).

**Proof.** Recall that \( h \) and \( f \) are the equation of \( \ell^P \) and \( h \) and \( g \) are the equations of \( \ell^Q \). By Theorem 2.10 the equations of \( Q^A_j \star \ell^P \) are given by \( h \star Q^A_j \) and \( f \star Q^A_j \), and the equations of \( P^A_i \star \ell^Q \) are given by \( h \star P^A_i \) and \( g \star P^A_i \). By Theorem 4.7, since all the lines \( Q^A_j \star \ell^P \) and \( P^A_i \star \ell^Q \) are coplanar, then one of the generators for the ideal of each of them can be chosen to be the equation of the plane \( H \) where they lie. Since this plane contains \( P^A_0 \star Q^A_0 = [1 : 1 : 1] \), \( P^A_1 \star Q^A_0 = P^A_1 \) and \( P^A_0 \star Q^A_1 = Q^A_1 \), we get that the equation of \( H \) is exactly \( h \). Thus the ideal of \( Q^A_j \star \ell^P \) is generated by \( h \) and \( f \star Q^A_j \), while the ideal of \( P^A_i \star \ell^Q \) is generated by \( h \) and \( g \star P^A_i \), from which we get, by Theorem 4.7, that \( Z^A_{a,b} \) is generated by \( h, f \star Q^A_0 \cdots \star Q^A_{n-1}, g \star P^A_0 \cdots \star P^A_{n-1} \). \( \square \)

5. Stick figures of lines via Hadamard product

In this section we show how to get, via the Hadamard product, the stick figure of lines, in \( \mathbb{P}^3 \), required for the construction in 12.

To this aim, we consider, for a suitable choice of \( I(a) \) and \( I(b) \), the set \( Z^A_{a,b} \) defined in the previous section and the line \( L^A \) defined in 13 and we take their Hadamard product \( Z^A_{a,b} \star L^A \).

Before proving that \( Z^A_{a,b} \star L^A \) is a stick figure, we need two preliminary lemmas.
Lemma 5.1. If $A_i \notin W$, for $i = 0, \ldots, 3$, then $\ell^P \cap \Delta_0 = \emptyset$ and $\ell^Q \cap \Delta_0 = \emptyset$.

Proof. We prove the statement only for the $\ell^P$ since the proof is identical for $\ell^Q$. Suppose that, for example, $\ell^P$ intersects $\Delta_0$ in the point $E_0 = [1 : 0 : 0 : 0]$. Notice that $P^A_i \neq E_0$, for all $j$, since $A_i \notin W$ for $i = 0, \ldots, 3$. In particular, $P^A_i$ has all coordinates different from zero. Since we are assuming that $E_0 \in \ell^P$, $E_0$ can be written as a linear combination of $P^A_0$ and $P^A_1$, that is

$$[1 : 0 : 0 : 0] = \lambda [1 : 1 : 1 : 1] + \mu \left[1 + \frac{\beta_0}{\alpha_0} : 1 + \frac{\beta_1}{\alpha_1} : 1 + \frac{\beta_2}{\alpha_2} : 1 + \frac{\beta_3}{\alpha_3}\right]$$

which is possible only if

$$\frac{\lambda}{\mu} = -\frac{\beta_1 + \alpha_1}{\alpha_1} \quad \text{and} \quad \frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2} = \frac{\alpha_3}{\beta_3}$$

which is a contradiction since the points $A_i$ are distinct. \hfill \Box

Lemma 5.2. Let $r_{ijkl}$ be the line through $P^A_i \ast Q^A_j$ and $P^A_k \ast Q^A_l$. If $A_i \notin W$, for $i = 0, \ldots, 3$, then $r_{ijkl} \cap \Delta_0 = \emptyset$.

Proof. We distinguish three cases:

1. $i = k$ and $j \neq l$,
2. $i \neq k$ and $j = l$,
3. $i \neq k$ and $j \neq l$.

If we are in case (1), the line through $P^A_i \ast Q^A_j$ and $P^A_k \ast Q^A_l$ is the line $P^A_i \ast \ell^Q$ which does not intersects $\Delta_0$ by Corollary 2.11 and Lemma 5.1. Similarly, if we are in case (2), the line through $P^A_i \ast Q^A_j$ and $P^A_k \ast Q^A_l$ is the line $Q^A_j \ast \ell^P$ which, again, does not intersects $\Delta_0$ by Corollary 2.11 and Lemma 5.1.

For case (3), suppose that $r_{ijkl}$ intersects $\Delta_0$ in $E_0 = [1 : 0 : 0 : 0]$ (the other cases being similar). Notice that $P^A_i \ast Q^A_j \neq E_0$, and $P^A_k \ast Q^A_l \neq E_0$ since $A_i \notin W$ for $i = 0, \ldots, 3$. Since we are assuming that $E_0 \in r_{ijkl}$, $E_0$ can be written as a linear combination of $P^A_i \ast Q^A_j$ and $P^A_k \ast Q^A_l$

$$[1 : 0 : 0 : 0] = \lambda \left[\frac{(\alpha_0 + i \beta_0)(\alpha_0 + j \beta_0)}{\alpha_0 \beta_0} : \frac{(\alpha_1 + i \beta_1)(\alpha_1 + j \beta_1)}{\alpha_1 \beta_1} : \frac{(\alpha_2 + i \beta_2)(\alpha_2 + j \beta_2)}{\alpha_2 \beta_2} : \frac{(\alpha_3 + i \beta_3)(\alpha_3 + j \beta_3)}{\alpha_3 \beta_3}\right] + \mu \left[\frac{(\alpha_0 + k \beta_0)(\alpha_0 + l \beta_0)}{\alpha_0 \beta_0} : \frac{(\alpha_1 + k \beta_1)(\alpha_1 + l \beta_1)}{\alpha_1 \beta_1} : \frac{(\alpha_2 + k \beta_2)(\alpha_2 + l \beta_2)}{\alpha_2 \beta_2} : \frac{(\alpha_3 + k \beta_3)(\alpha_3 + l \beta_3)}{\alpha_3 \beta_3}\right]$$

and looking at all the coordinates but the first, this means

$$\lambda \frac{(\alpha_s + i \beta_s)(\alpha_s + \beta_s)}{\alpha_s \beta_s} = -\mu \frac{(\alpha_s + k \beta_s)(\alpha_s + \beta_s)}{\alpha_s \beta_s} \quad \text{for} \ s = 1, 2, 3$$

or equivalently

$$-\frac{\mu}{\lambda} = \frac{\alpha_s + i \beta_s}{\alpha_s + k \beta_s} \quad \text{for} \ s = 1, 2, 3$$

which gives rise to the same set of equations [5] of Theorem 4.7. Arguing as in the proof of Theorem 4.7, but considering that now we have one less equation (since we are not considering the first coordinate), we get that any non-zero solution of [5] requires that there is a pair $(i_1, i_2)$ of indices such that $\frac{\alpha_{i_1}}{\beta_{i_1}} = \frac{\alpha_{i_2}}{\beta_{i_2}}$ which is a contradiction since the points $A_i$ are distinct. \hfill \Box

Coming back to $Z^A_{a,b} \ast L^A$, we first prove that no pairs of points $P^A_i \ast Q^A_j, P^A_k \ast Q^A_l \in Z^A_{a,b}$, can give $P^A_i \ast Q^A_j \ast L^A = P^A_k \ast Q^A_l \ast L^A$.

Proposition 5.3. In the same hypothesis of Theorem 4.7, $Z^A_{a,b} \ast L^A$ is a set of $ab$ distinct lines, for any choice of positive integers $a$ and $b$ and sets $I(a)$ and $I(b)$.
Proof. Since, by hypothesis, $P_i^A \cdot Q_j^A \notin \Delta_2$, by Lemma 2.7, one has that $P_i^A \cdot Q_j^A \cdot L^A$ is a line for all $i$ and $j$ with $i \in \mathcal{I}(a)$ and $j \in \mathcal{I}(b)$. Let us show now that if $P_i^A \cdot Q_j^A \neq P_k^A \cdot Q_l^A$ then $P_i^A \cdot Q_j^A \cdot L^A \neq P_k^A \cdot Q_l^A \cdot L^A$. We distinguish three cases.

If $i = k$ and $j \neq l$ then $P_i^A \cdot Q_j^A \neq P_k^A \cdot Q_l^A$. By Lemma 5.1 $\ell_Q \cap \Delta_0 = \emptyset$ which implies $P_i^A \cdot \ell_Q \neq P_i^A \cdot Q_l^A$. Since

i) $P_i^A \cdot Q_j^A \notin \Delta_2$,

ii) $L^A \notin \Delta_1$,

iii) $\langle P_i^A \cdot Q_j^A, P_i^A \cdot Q_l^A \rangle = P_i^A \cdot \ell_Q$,

we can apply Lemma 2.7 obtaining $P_i^A \cdot Q_j^A \cdot L^A \neq P_i^A \cdot Q_l^A \cdot L^A$.

The case $i \neq k$ and $j = l$ is similar to the previous one. The same proof, but using the line $\ell_P$, gives $P_i^A \cdot Q_j^A \cdot L^A \neq P_i^A \cdot Q_j^A \cdot L^A$.

Finally if $i \neq k$ and $j \neq l$ then $P_i^A \cdot Q_j^A \neq P_k^A \cdot Q_l^A$. By Lemma 5.2 $r_{ijkl} \cap \Delta_0 = \emptyset$. Since

i) $P_i^A \cdot Q_j^A, P_k^A \cdot Q_l^A \notin \Delta_2$,

ii) $L^A \notin \Delta_1$,

iii) $\langle P_i^A \cdot Q_j^A, P_k^A \cdot Q_l^A \rangle = r_{ijkl}$.

we can again apply Lemma 2.7 obtaining $P_i^A \cdot Q_j^A \cdot L^A \neq P_k^A \cdot Q_l^A \cdot L^A$.

Thus we conclude that $Z_{a,b}^A \cdot L^A$ consists of $ab$ distinct lines. □

We study now the intersection properties of the set $Z_{a,b}^A \cdot L^A$. More precisely we have the following.

Proposition 5.4. Assume that $1 \notin \mathcal{I}(a) \cup \mathcal{I}(b)$. In the same hypothesis of Theorem 4.7 let $P_i^A \cdot Q_j^A$ and $P_k^A \cdot Q_l^A$ in $Z_{a,b}^A$. Then $P_i^A \cdot Q_j^A \cdot L^A \cap P_k^A \cdot Q_l^A \cdot L^A = \emptyset$ if and only if $i = k$ or $j = l$. Moreover,

i) if $j \neq l$, the intersection $P_i^A \cdot Q_j^A \cdot L^A \cap P_i^A \cdot Q_l^A \cdot L^A$ is given by

$$
\begin{align*}
&\frac{(a_0+i_3)\frac{j_0+\beta_0}{j_0+\beta_0}}{\alpha_0\beta_0(\alpha_0\beta_0-\alpha_1\beta_0)(\alpha_0\beta_0-\alpha_2\beta_0)(\alpha_0\beta_0-\alpha_3\beta_0)} \\
&\frac{(a_0+i_3)\frac{j_0+\beta_0}{j_0+\beta_0}}{\alpha_1\beta_1(\alpha_1\beta_1-\alpha_0\beta_1)(\alpha_1\beta_1-\alpha_2\beta_1)(\alpha_1\beta_1-\alpha_3\beta_1)} \\
&\frac{(a_0+i_3)\frac{j_0+\beta_0}{j_0+\beta_0}}{\alpha_2\beta_2(\alpha_2\beta_2-\alpha_0\beta_2)(\alpha_2\beta_2-\alpha_1\beta_2)(\alpha_2\beta_2-\alpha_3\beta_2)} \\
&\frac{(a_0+i_3)\frac{j_0+\beta_0}{j_0+\beta_0}}{\alpha_3\beta_3(\alpha_3\beta_3-\alpha_0\beta_3)(\alpha_3\beta_3-\alpha_1\beta_3)(\alpha_3\beta_3-\alpha_2\beta_3)(\alpha_3\beta_3-\alpha_3\beta_3)} \\
&\vdots
\end{align*}
$$

ii) if $i \neq k$, the intersection $P_i^A \cdot Q_j^A \cdot L^A \cap P_k^A \cdot Q_l^A \cdot L^A$ is given by

$$
\begin{align*}
&\frac{(a_0+i_3)\frac{j_0+\beta_0}{j_0+\beta_0}}{\alpha_0\beta_0(\alpha_0\beta_0-\alpha_1\beta_0)(\alpha_0\beta_0-\alpha_2\beta_0)(\alpha_0\beta_0-\alpha_3\beta_0)} \\
&\frac{(a_0+i_3)\frac{j_0+\beta_0}{j_0+\beta_0}}{\alpha_1\beta_1(\alpha_1\beta_1-\alpha_0\beta_1)(\alpha_1\beta_1-\alpha_2\beta_1)(\alpha_1\beta_1-\alpha_3\beta_1)} \\
&\frac{(a_0+i_3)\frac{j_0+\beta_0}{j_0+\beta_0}}{\alpha_2\beta_2(\alpha_2\beta_2-\alpha_0\beta_2)(\alpha_2\beta_2-\alpha_1\beta_2)(\alpha_2\beta_2-\alpha_2\beta_2)(\alpha_2\beta_2-\alpha_3\beta_2)} \\
&\frac{(a_0+i_3)\frac{j_0+\beta_0}{j_0+\beta_0}}{\alpha_3\beta_3(\alpha_3\beta_3-\alpha_0\beta_3)(\alpha_3\beta_3-\alpha_1\beta_3)(\alpha_3\beta_3-\alpha_2\beta_3)(\alpha_3\beta_3-\alpha_3\beta_3)} \\
&\vdots
\end{align*}
$$
Proof. By Theorem 2.10 one has that the equations of $P_k^A \star Q_j^d \star L^A$ are
\[
\begin{align*}
(a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3) \in (P_k^A \star Q_j^d) & = 0 \\
(\beta_0x_0 + \beta_1x_1 + \beta_2x_2 + \beta_3x_3) \in (P_k^A \star Q_j^d) & = 0
\end{align*}
\]
which can be written explicitly as
\[
\begin{pmatrix}
\alpha_0^2 \beta_0 + \alpha_1^2 \beta_1 + \alpha_2^2 \beta_2 + \alpha_3^2 \beta_3 \\
\alpha_0 \beta_1 + \alpha_1 \beta_0 + \alpha_2 \beta_2 + \alpha_3 \beta_3 \\
\alpha_0 \beta_2 + \alpha_1 \beta_1 + \alpha_2 \beta_0 + \alpha_3 \beta_3 \\
\alpha_0 \beta_3 + \alpha_1 \beta_3 + \alpha_2 \beta_2 + \alpha_3 \beta_1
\end{pmatrix} = 0
\]
Similarly the equations of $P_k^A \star Q_j^d \star L^A$ are
\[
\begin{pmatrix}
(a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3) \in (P_k^A \star Q_j^d) & = 0 \\
(\beta_0x_0 + \beta_1x_1 + \beta_2x_2 + \beta_3x_3) \in (P_k^A \star Q_j^d) & = 0
\end{pmatrix}
\]
which can be written explicitly as
\[
\begin{pmatrix}
\alpha_0^2 \beta_0 + \alpha_1^2 \beta_1 + \alpha_2^2 \beta_2 + \alpha_3^2 \beta_3 \\
\alpha_0 \beta_1 + \alpha_1 \beta_0 + \alpha_2 \beta_2 + \alpha_3 \beta_3 \\
\alpha_0 \beta_2 + \alpha_1 \beta_1 + \alpha_2 \beta_0 + \alpha_3 \beta_3 \\
\alpha_0 \beta_3 + \alpha_1 \beta_3 + \alpha_2 \beta_2 + \alpha_3 \beta_1
\end{pmatrix} = 0
\]
Passing to the system of the two lines $P_k^A \star Q_j^d \star L^A$ and $P_k^A \star Q_j^d \star L^A$ one has the following matrix of coefficients
\[
M = \begin{pmatrix}
\alpha_0^2 \beta_0 & \alpha_1^2 \beta_1 & \alpha_2^2 \beta_2 & \alpha_3^2 \beta_3 \\
\alpha_0 \beta_1 & \alpha_1 \beta_0 & \alpha_2 \beta_2 & \alpha_3 \beta_3 \\
\alpha_0 \beta_2 & \alpha_1 \beta_1 & \alpha_2 \beta_0 & \alpha_3 \beta_3 \\
\alpha_0 \beta_3 & \alpha_1 \beta_3 & \alpha_2 \beta_2 & \alpha_3 \beta_1
\end{pmatrix}
\]
Clearly this matrix has rank greater than or equal to 3, otherwise the two lines $P_k^A \star Q_j^d \star L^A$ and $P_k^A \star Q_j^d \star L^A$ will be coincident, in contradiction with Proposition 5.3.

Computing the determinant of $M$ one has
\[
\det(M) = \left(\prod_{i=0}^3 a_ib_i\right) \left(\prod_{0 \leq s \leq r \leq 3} (a_s b_r - a_r b_s)\right) (i - k)(j - l)(j - k)(i - l)
\]
By definition of the points $A_i$ and by Remark 4.2 we know that the two terms $\left(\prod_{i=0}^3 a_ib_i\right)$ and $\left(\prod_{0 \leq s \leq r \leq 3} (a_s b_r - a_r b_s)\right)$ are different from 0. By the condition 1 $\notin \mathcal{I}(a) \cup \mathcal{I}(b)$ one as that $(j - k)(i - l) \neq 0$ for all $i, k \in \mathcal{I}(a)$ and all $j, l \in \mathcal{I}(b)$. Hence $M$ has rank 4 when $i \neq k$ and $j \neq l$ and has rank 3 when $i = k$ or $j = l$, which concludes the first part of the proof. The second part of the proof follows directly substituting the values in [10] in the system [11] taking $i = k$, and the values in [10] in the same system [11] but taking $j = l$. □

Remark 5.5. Although, by Proposition 5.3 any choice of the sets $\mathcal{I}(a)$ and $\mathcal{I}(b)$ always gives a set of $ab$ distinct lines, the condition $1 \notin \mathcal{I}(a) \cup \mathcal{I}(b)$ of the previous proposition is mandatory to avoid extra intersections among the lines in $Z_{a,b}^A \star L^A$. In fact, without this condition $Z_{a,b}^A \star L^A$ could be still a stick figure, but it is not complete intersection.

As a corollary we get the following fact.
Corollary 5.6. Assume that $1 \notin \mathcal{I}(a) \cup \mathcal{I}(b)$. With the same hypothesis of Theorem 4.7 one has:

- $P_{\alpha_i}^A \ast Q_{\beta_j}^A \ast L^A, \ldots, P_{\alpha_{i-1}}^A \ast Q_{\beta_j}^A \ast L^A$ are coplanar for all $j \in \mathcal{I}(b)$;
- $P_{\beta_1}^A \ast Q_{\alpha_i}^A \ast L^A, \ldots, P_{\beta_k}^A \ast Q_{\alpha_i}^A \ast L^A$ are coplanar for all $i \in \mathcal{I}(a)$.

We have now all ingredients to state the main result of this section.

Theorem 5.7. Assume that $1 \notin \mathcal{I}(a) \cup \mathcal{I}(b)$. With the same hypothesis of Theorem 4.7 $Z_{a,b}^A \ast L^A$ is a stick figure of ab lines in $\mathbb{P}^3$. Moreover $Z_{a,b}^A \ast L^A$ is a complete intersection.

Proof. By Proposition 5.3 we know that $Z_{a,b}^A \ast L^A$ consists of ab distinct lines. By Corollary 5.6 it follows that $Z_{a,b}^A \ast L^A$ is a complete intersection. By the first part of Proposition 5.3 we know that two lines in $Z_{a,b}^A$ intersect in a space of dimension at most 0. By the second part of Proposition 5.3 we know that the coordinates of the point of intersection of two lines $P_{\alpha_i}^A \ast Q_{\beta_j}^A \ast L^A$ and $P_{\alpha_i}^A \ast Q_{\beta_j}^A \ast L^A$ (resp. $P_{\beta_1}^A \ast Q_{\alpha_i}^A \ast L^A$ and $P_{\beta_1}^A \ast Q_{\alpha_i}^A \ast L^A$) are dependent of the indices $i,j$ and $l$ (resp. $i,j$ and $k$) assuring us that three lines in $Z_{a,b}^A \ast L^A$ intersect in a space of dimension at most -1. Hence $Z_{a,b}^A \ast L^A$ satisfies the conditions to be a stick figure of lines and the statement is proved.

\[\square\]

6. Gorenstein sets of points

As a final step of our construction, we apply the procedure described in Section 3 to our stick figure to get a Gorenstein set of points in $\mathbb{P}^3$ with a given $h$-vector.

Again, let

$$h = (h_0, h_1, \ldots, h_s) = (1, 3, h_2, \ldots, h_{t-1}, h_t, h_{t+1}, \ldots, h_{t+2})$$

be a SI-sequence, and consider the first difference

$$\Delta h = (1, 2, h_2 - h_1, \ldots, h_{t+1} - h_{t}, 0, \ldots, 0, h_{t+2} - h_{t+1}, -2, -1).$$

Define the two sequences $\mathbf{a} = (a_0, \ldots, a_t)$ and $\mathbf{g} = (g_0, \ldots, g_{s+1})$ as expressed in (2) and (3). As already said in Section 3 $\mathbf{g}$ is the $h$-vector of a complete intersection, $X$, of two surfaces in $\mathbb{P}^3$ of degree $t+1$ and $s-t+2$.

Hence we consider, as $X$, the stick figure $Z_{t+1, s-t+2}^A \ast L^A$ (for a suitable choice of $\mathbf{a}$, $\mathcal{I}(t+1)$ and $\mathcal{I}(s-t+2)$ with the hypotheses of Theorems 4.7 and 5.7).

If we set

$$\mathcal{I}(t+1) = \{u_0, \ldots, u_t\} \text{ and } \mathcal{I}(s-t+2) = \{v_0, \ldots, v_{s-t+1}\},$$

then the aCM scheme $C_1$ with $h$-vector $\mathbf{a}$ is given by the following set of lines in $Z_{t+1, s-t+2}^A \ast L^A$:

$$P_{u_i}^A \ast Q_{v_j}^A \ast L^A \text{ for } j = 0, \ldots, a_i - 1 \text{ and } i = 0, \ldots, t.$$ 

and, obviously, the residual scheme $C_2$ is the set of lines in $Z_{t+1, s-t+2}^A \ast L^A$ and not in $C_1$.

We use the following notation for the points of intersections of lines in the stick figure $Z_{t+1, s-t+2}^A \ast L^A$:

$$G_{i(j,k)} = P_{u_i}^A \ast Q_{v_j}^A \ast L^A \cap P_{u_i}^A \ast Q_{v_k}^A \ast L^A$$

and

$$G_{(i,k)} = P_{u_i}^A \ast Q_{v_j}^A \ast L^A \cap P_{u_k}^A \ast Q_{v_j}^A \ast L^A.$$
Theorem 6.1. Let \( h, a \) and \( g \) be as above. Then the set of points

\[
\begin{cases}
G_{i(j,k)} & \text{with } 0 \leq j \leq a_i - 1 \text{ and } a_i \leq k \leq s - t + 1 \text{ for } i = 0, \ldots, t \\
G_{(i,k)} & \text{with } \min\{a_i, a_k\} \leq j \leq \max\{a_i, a_k\} - 1 \text{ for } 0 \leq i < k \leq t
\end{cases}
\]

is a Gorenstein zeroscheme with \( h \)-vector \( h \).

Proof. This follows directly from Theorem 5.3 and Theorem 3.6.

Using the description, in Proposition 5.4 of intersections in the stick figure, we can state the previous theorem in terms of the coordinates of the points in the desired Gorenstein set.

Consider \( \mathcal{I}(t+1) \) and \( \mathcal{I}(s-t+2) \) as in (12). Denote by \([V_{i(j,k)}]_0\) the point whose coordinates are

\[
[V_{i(j,k)}]_0 = -\frac{(a_0 + u_i \beta_0)(v_j a_0 + \beta_0)(v_k a_0 + \beta_0)}{a_0 \beta_0(a_0 \beta_1 - a_1 \beta_0)(a_0 \beta_2 - a_2 \beta_0)(a_0 \beta_3 - a_3 \beta_0)}
\]

and by \([V_{i(k,j)}]_0\) the point whose coordinates are

\[
[V_{i(k,j)}]_0 = -\frac{(a_0 + u_i \beta_0)(a_0 + u_k \beta_0)(v_j a_0 + \beta_0)}{a_0 \beta_0(a_0 \beta_1 - a_1 \beta_0)(a_0 \beta_2 - a_2 \beta_0)(a_0 \beta_3 - a_3 \beta_0)}
\]

(13)

Corollary 6.2. Let \( h \) be an admissible \( h \)-vector for a Gorenstein zeroscheme in \( \mathbb{P}^3 \) of the form \( h = (h_0, \ldots, h_s) = (1, 3, h_2, \ldots, h_{t-1}, h_t, h_2, \ldots, h_t, h_{s-1}, \ldots, 3, 1) \) and let \( a_i = h_i - h_{i-1} \) for \( 0 \leq i \leq t \). Fix four distinct points \( A_i = [a_i : \beta_i] \) in \( \mathbb{P}^1 \) \( \setminus \{ \Delta_0 \cup W \} \), for \( i = 0, \ldots, 3 \) and fix the sets of nonnegative integers \( \mathcal{I}(t+1) = \{ u_0, \ldots, u_t \} \) and \( \mathcal{I}(s-t+2) = \{ v_0, \ldots, v_{s-t+1} \} \) with \( 0 \in \mathcal{I}(t+1) \cap \mathcal{I}(s-t+2) \) and \( 1 \notin \mathcal{I}(t+1) \cup \mathcal{I}(s-t+2) \). Then the set of points

\[
\begin{cases}
[V_{i(j,k)}] & \text{with } 0 \leq j \leq a_i - 1, a_i \leq k \leq s - t + 1, \text{ for } i = 0, \ldots, t \\
[V_{(i,k)}] & \text{with } \min\{a_i, a_k\} \leq j \leq \max\{a_i, a_k\} - 1, \text{ for } 0 \leq i < k \leq t
\end{cases}
\]

is a Gorenstein zeroscheme with \( h \)-vector \( h \).

Example 6.3. Let \( h \) be \( h \)-vector \((1, 3, 4, 3, 1)\) of Example 5.7. One has \( t = 2, s = 4 \) and \( a = (1, 2, 1) \).

Fix

\[
A_0 = [1 : 1], A_1 = [1 : 2], A_2 = [1 : 3], A_3 = [1 : 4]
\]
and

\[ I(t + 1) = \{u_0, u_1, u_2\} = \{0, 2, 4\} \]

\[ I(s - t + 2) = \{v_0, v_1, v_2, v_3\} = \{0, 2, 4, 6\}. \]

Substituting these values in (13) and (14) we get, by Corollary 6.2, that the Gorenstein set of points with h-vector \((1, 3, 4, 3, 1)\) is given by

\[
\begin{bmatrix}
-\frac{(2k+1)(2j+1)(2i+1)}{(2k+3)(2j+3)(2i+1)}
-\frac{(2k+2)(2j+2)(4i+1)}{(2k+4)(2j+4)(8i+1)}
\end{bmatrix}
\]

with \(0 \leq j \leq a_i - 1, a_i \leq k \leq 3\), for \(i = 0, 1, 2\)

and

\[
\begin{bmatrix}
-\frac{(2k+1)(2j+1)(2i+1)}{(4k+1)(2j+1)(2i+1)}
-\frac{(6k+1)(2j+3)(6i+1)}{(6k+3)(2j+3)(6i+1)}
\end{bmatrix}
\]

with \(\text{min}\{a_i, a_k\} \leq j \leq \text{max}\{a_i, a_k\} - 1\), for \(0 \leq i < k \leq 2\).

We can check in \texttt{Singular} if this set of points is Gorenstein. The procedure \texttt{IP(n,M)} computes the ideal of a set of points given in matrix form \(M\), where each column of \(M\) represents a point. The procedure \texttt{HF(n,I,t)} computes the Hilbert function of an ideal \(I\), in degree \(t\). The integer \(n\) refers to the number of variables in the polynomial ring \(k[x_0, \ldots, x_n]\).

```plaintext
int n=3;
ring R=0,(x(0..n)),dp;

matrix G[4][12]=-1/2,-5/6,-7/6,-5/2,-7/2,-15/2,-21/2,-5/2,-25/6,
-35/6,-3/2,-15/2,2,3,4,15,20,30,40,18,27,36,5,45,-5/2,-7/2,
-9/2,-49/2,-65/2,-245/6,-105/2,-65/2,-91/2,-117/2,-35/6,
-455/6,1,4/3,5/3,12,15,18,45/2,17,68/3,85/3,9/4,153/4;

ideal I=IP(n,G);
HF(n,I,0);
1
HF(n,I,1);
4
HF(n,I,2);
8
HF(n,I,3);
11
HF(n,I,4);
12
```
Hence, the first difference of the Hilbert function for this set of points is exactly $(1, 3, 4, 3, 1)$.

References

[1] I. Bahmani Jafarloo and G. Calussi, Weak Hadamard star configurations and apolarity, Rocky Mountain J. Math. 50(3), 851–862 (2020).
[2] C. Bocci, G. Calussi, G. Fatabbi and A. Lorenzini, On Hadamard product of linear varieties, Journal of Algebra and its applications, 16(8), 155–175 (2017).
[3] C. Bocci, G. Calussi, G. Fatabbi and A. Lorenzini, The Hilbert function of some Hadamard products, Coll. Mathematica, 69(2), 205–220 (2018).
[4] C. Bocci and E. Carlini, Hadamard products of hypersurfaces, arXiv:2109.09548.
[5] C. Bocci, E. Carlini and J. Kileel, Hadamard Products of Linear Spaces, J. of Algebra 448, 595–617 (2016).
[6] D. Buchsbaum and D. Eisenbud, Algebra Structures for Finite Free Resolutions and some Structure Theorems for Ideals of Codimension 3, Amer. J. of Math., 99, 447–485 (1977).
[7] W. Bruns, J. Herzog, Cohen-Macaulay rings, Revised edition, Cambridge Studies in Advanced Mathematics 39, Cambridge Univ. Press, Cambridge, 1998.
[8] E. Carlini, M. V. Catalisano, E. Guardo and A. Van Tuyl, Hadamard star configurations, Rocky Mountain J. Math. 49(2), 419–432 (2019).
[9] S. Diesel, Irreducibility and Dimension Theorems for Families of Height 3 Gorenstein Algebras, Pacific J. of Math., 172, 365–397 (1966).
[10] A. V. Geramita and J. C. Migliore, Reduced Gorenstein Codimension Three Subschemes of Projective Space, Proc. Amer. Math. Soc., 125, 643–950 (1997).
[11] J. C. Migliore, Introduction to Liaison Theory and Deficiency Modules, Birkhäuser, 1998.
[12] J. Migliore and U. Nagel, Reduced arithmetically Gorenstein schemes and simplicial polytopes with maximal Betti numbers Adv. Math. 180(1), 1–63 (2003).
[13] C. Peskine and L. Szpiro, Liaison des Variétés Algébriques I, Inv. Math., 26, 271–302 (1974).
[14] R. Stanley, Hilbert Functions of Graded Algebras, Advances in Math., 28, 57–83 (1978).