A Note on Graviton Amplitudes for New Twistor
String Theories

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Abstract

A new class of twistor string theories were recently constructed by Abou-Zeid, Hull and Mason. The most interesting one of these theories has the particle content of $\mathcal{N} = 8$ supergravity. Arguments are given for the vanishing of the $(+ - -)$-helicity amplitude for gravitons which suggest that this theory describes a chiral $\mathcal{N} = 8$ supergravity rather than Einstein supergravity.
Recently, a new set of twistor string theories has been proposed which has the potential of describing a number of interesting field theories as twistor string theories [1]. This was done by gauging the Berkovits form of the Witten-Berkovits twistor string theory with respect to a holomorphic one-form on twistor space [2, 3]. The Witten-Berkovits string theory describes $\mathcal{N} = 4$ supersymmetric Yang-Mills theory coupled to $\mathcal{N} = 4$ conformal supergravity [4]. There are two reasons why one might seek to modify this string theory. If the twistor string theory is to be used for calculations in Yang-Mills theory, then it is necessary to decouple (or eliminate) the conformal supergravitons which can otherwise contribute in loops. Being conformally coupled, there is no dimensional parameter which can be tuned to eliminate them. Secondly, since conformal gravity is a theory with fourth-order derivatives, the linearized equations of motion have additional solutions which do not have a positive Hilbert space norm and hence leads to loss of unitarity. If a new gauge symmetry can be implemented which eliminates these unphysical modes, then both these problems can be eliminated. Abou-Zeid, Hull and Mason introduced a new gauge symmetry which has the potential for doing just this [1]. A number of different theories, which correspond to different choices of the one-form used for gauging, were obtained; selfdual gravity and selfdual supergravity are among these. Perhaps the most intriguing one was the case of gauging with a weightless one-form which seemed to lead to $\mathcal{N} = 8$ supergravity. A twistor string description of $\mathcal{N} = 8$ supergravity will have implications for the finiteness of certain loop calculations [5] and would also explain the known and suggestive formulae for graviton scattering amplitudes [6]. In this note we will focus on this special case and calculate some graviton scattering amplitudes; we will present evidence that this twistor string theory is actually $\mathcal{N} = 8$ chiral supergravity and not $\mathcal{N} = 8$ Einstein supergravity.

We begin with the basics of Berkovits’ version of the twistor string theory [3]. The target space variables are the supertwistors $Z^I = (\omega^A, \pi_{A'}, \psi^a), A, A' = 1, 2, a = 1, ..., 4,$ and $Y_I$ are the conjugate variables. We also have $\tilde{Z}^I, \tilde{Y}_I$ with $Z^I = \tilde{Z}^I, Y_I = \tilde{Y}_I$ on the world-sheet boundary. For genus zero, the world-sheet may be taken as the upper hemisphere of a $\mathbb{C}P^1$. The action is given by

$$S = \int Y(\bar{\partial} + \bar{A})Z + S_C + \text{tilde part} \quad (1)$$

Here $\bar{A}$ is a $GL(1, \mathbb{C})$ gauge field and the action [1] has the gauge symmetry,

$$Z \rightarrow \lambda Z, \quad Y \rightarrow \lambda^{-1}Y, \quad \bar{A} \rightarrow \bar{A} - \bar{\partial} \log \lambda \quad (2)$$

This symmetry will help to reduce the physical set of variables to the projective supertwistor space.
Consider first the integration over the fields $\bar{A}$ in a functional integral with the action (1). On the two-sphere or $\mathbb{CP}^1$, there are topologically nontrivial $GL(1, \mathbb{C})$ configurations, corresponding to Dirac monopoles.

The space of gauge potentials then splits up into a set of disconnected pieces, one for each value of the monopole number, say, $d$. In each sector, we can write $\bar{A} = \bar{A}_d + \delta \bar{A}$, where $\bar{A}_d$ is a fixed configuration of monopole number $d$ and $\delta \bar{A}$ is a fluctuation (of zero monopole number). In two dimensions, we can always write $\delta \bar{A} = \bar{\partial} \Theta$ for some complex function $\Theta$ on $\mathbb{CP}^1$. This means that configurations within the same connected component can be gauge transformed to $\bar{A}_d$ by the $GL(1, \mathbb{C})$ gauge transformation with parameter $\Theta$.

The volume element for the space of gauge potentials is then given as
\[
[d\bar{A}] = \sum_d [d\Theta] \det \bar{\partial}
\] (3)

The invariant integration then reduces to a summation over $d$ with a factor $\det \bar{\partial}$. Using $\bar{\bar{A}} = \bar{A}_d + \bar{\partial} \Theta$ in the action (1) and eliminating $\Theta$ by a $GL(1, \mathbb{C})$ transformation, we see that the correlation function of a set of vertex operators, $V_1, \ldots, V_n$ is given by
\[
\mathcal{M} = \sum_d C_d M_d
\]
with
\[
M_d = \int (\det \bar{\partial}) \ e^{-S_C} \ e^{-\int Y(\bar{\partial} + \bar{A}_d)Z} V_1 V_2 \ldots V_n
\] (4)

where $C_d$ are some normalization constants. Notice that, since we have equal numbers of charged fermionic and bosonic fields in the action (1), there is no anomaly for the $GL(1, \mathbb{C})$ transformation. The factor $\det \bar{\partial}$ is a constant but it does contribute to the conformal anomaly, raising the required central charge of the $S_C$-part of the action to 28.

We now turn to the functional integration over the fields $Z^I, Y_I$. This is most conveniently done by expanding $Z^I$ in modes of $(\partial + A)(\bar{\partial} + \bar{\bar{A}})$ and integrating over the mode coefficients. The key point is that there are zero modes of $(\bar{\partial} + \bar{\bar{A}})$ in the expansion for $Z^I$,
\[
Z^\alpha = \sum_{\{a\}} a_{a_1 a_2 \ldots a_d}^\alpha u^{a_1} u^{a_2} \ldots u^{a_d} + \sum b_{a_1 \ldots a_d i_1 \ldots i_k}^{\alpha, i_1 \ldots i_k} \bar{u}_{i_1} \ldots \bar{u}_{i_k} u^{a_1} \ldots u^{a_d} u^{i_1} \ldots u^{i_k}
\]
\[
\psi^\alpha = \sum_{\{a\}} \gamma^\alpha_{a_1 a_2 \ldots a_d} u^{a_1} u^{a_2} \ldots u^{a_d} + \sum \zeta_{a_1 \ldots a_d i_1 \ldots i_k}^{\alpha, i_1 \ldots i_k} \bar{u}_{i_1} \ldots \bar{u}_{i_k} u^{a_1} \ldots u^{a_d} u^{i_1} \ldots u^{i_k}
\] (5)

The first set of terms on the right hand side correspond to zero modes of $(\bar{\partial} + \bar{\bar{A}})$; these are holomorphic in the $u$'s. These terms give a holomorphic map of $\mathbb{CP}^1$ in twistor space, hence a holomorphic curve of degree $d$. $\{a\}, \{\gamma\}$ are the moduli of this holomorphic curve in supertwistor space. The higher terms on the right hand side of (5) correspond to nonzero
modes of \((\partial + A)(\bar{\partial} + \bar{A})\) and have \(\bar{u}\)'s with \(N_u = N_{\bar{u}} = d\) where \(N_u, N_{\bar{u}}\) are the numbers of \(u\)'s and \(\bar{u}\)'s, respectively. The \(Y_I\)’s have a mode expansion with the mode functions being conjugates of the nonzero modes used for \(Z_I\),

\[
Y_\alpha = \sum \bar{b}_{i_1,\ldots,i_k} \bar{u}_{i_1} \cdots \bar{u}_{i_k} u^{a_1} \cdots u^{a_d} \quad \text{and} \quad Y_a = \sum \bar{\gamma}_{i_1,\ldots,i_k} \bar{u}_{i_1} \cdots \bar{u}_{i_k} u^{a_1} \cdots u^{a_d} \]  

(6)

The \(Y\)'s do not have a zero mode part. The integration over \((b, \bar{b}), (\zeta, \bar{\zeta})\) leads to Wick contractions of \(Y, Z\) in any correlation function. This means that the nonzero mode part of the \(Z\)’s have to be paired (with the \(Y\)’s) to get a nonzero contribution. Thus, once all the \(Y\)'s in the integrand have been Wick-contracted with the \(Z\)'s, the nonzero mode parts of the remaining \(Z\)'s give zero. We may therefore replace the remaining \(Z\)'s by their zero mode part. This leads to a simple rule for calculating correlators:

For any correlator, carry out \(Y-Z\) Wick contractions (integrate over mode coefficients of nonzero modes), then replace all \(Z\)'s by zero modes. Integrate over the coefficients \(\{a\}, \{\gamma\}\) (moduli of holomorphic curves) to obtain the correlator.

It is interesting to see how this works out for the gauge field scattering amplitudes. In this case, the vertex operator is given by

\[
V = \int d\sigma \Phi_{p,\eta}(Z) J(\sigma), \quad \text{where} \quad J(\sigma) = \text{GL}(1, \mathbb{C}) \text{ current from the } SC\text{-part of the action and}
\]

\[
\Phi_{p,\eta}(Z) = \delta(\pi \cdot \bar{p}) \left( \frac{\pi \cdot \alpha}{\bar{p} \cdot \alpha} \right) \exp \left[ (\omega \cdot p + \psi \cdot \eta) \frac{\bar{p} \cdot \alpha}{\pi \cdot \alpha} \right] \]  

(7)

\(\Phi_{p,\eta}(Z)\) is holomorphic in \(Z\) of degree zero and it is of degree \(-2\) in the spinor momentum \(\bar{p}_A\). The Grassmann variable \(\eta^a\) characterizes the helicity state, the term with \(k\) \(\eta\)'s corresponding to the state of helicity \(1 - \frac{k}{2}\). Since there are no \(Y\)'s in \(\Phi\), there are no Wick contractions to be done. In the correlator of \(V\)'s, we may, therefore, replace \(Z\)'s by their zero mode part

\[
Z_{c_{cl}}^\alpha = \sum_{\{a\}} \{a\}^\alpha_{a_1,a_2,\ldots,a_d} u^{a_1} u^{a_2} \cdots u^{a_d} 
\]

\[
\psi_{c_{cl}}^{\alpha} = \sum_{\{a\}} \gamma^\alpha_{a_1,a_2,\ldots,a_d} u^{a_1} u^{a_2} \cdots u^{a_d} \]  

(8)

Notice that these are solutions of the classical equation of motion \(\bar{D}Z = (\bar{\partial} + \bar{A})Z = 0\). We have indicated this by the subscript on the fields \(Z^\alpha, \psi^\alpha\). From the world-sheet point of view the scattering amplitude for gauge particles is entirely classical. Thus the amplitude can be calculated by solving the classical equation of motion \(\bar{D}Z = 0\) and using these in
the product $V_1 V_2 \cdots V_n$. Since there are many classical solutions parametrized by \{a, \gamma\}, we must integrate over the moduli space of classical solutions. The overall $GL(1, \mathbb{C})$ scale invariance implies that one complex scale factor can be removed from the set \{a, \gamma\}. Further, since the solutions are holomorphic in the $u$’s, physical quantities are invariant under the $SL(2, \mathbb{C})$ transformation $u^a \to u^a' = g^b_a u^b$, $g \in SL(2, \mathbb{C})$. This is equivalently expressed in terms of \{a, \gamma\} since

$$Z^I(a, \gamma, gu) = Z^I(a', \gamma', u)$$

Thus we can remove $SL(2, \mathbb{C})$-worth of parameters from \{a, \gamma\}. With the scale factor, this is equivalent to a $GL(2, \mathbb{C})$ symmetry, so that the measure on the moduli space of classical solutions (for the degree $d$ holomorphic curve) is

$$d\mu_d = \frac{d^{4d+4}a \ d^{4d+4} \gamma}{\text{vol}[GL(2, \mathbb{C})]}$$

The scattering amplitude for the degree $d$ curve is given by

$$\langle V_1 V_2 \cdots V_n \rangle_d = \int d\mu_d \int d\sigma_1 d\sigma_2 \cdots d\sigma_n \ [\Phi_{p_1, \eta_1}(Z_{cl}) \Phi_{p_2, \eta_2}(Z_{cl}) \cdots \Phi_{p_n, \eta_n}(Z_{cl})]$$

$$\times \langle J(\sigma_1) J(\sigma_2) \cdots J(\sigma_n) \rangle$$

In the Witten-Berkovits string theory, there are also vertex operators corresponding to conformal supergravitons. We will not need them at this stage.

The modification of the action corresponding to the new set of twistor string theories is given by \[1\]

$$S = \int [Y \bar{D}Z + \bar{B}_i K_i] + S_C + \text{tilde part}$$

where $K_i$ is a holomorphic one-form on twistor space; it is of the form $K_i = k_i I \partial Z^I$. The action \[12\] has a new gauge invariance given by the transformation

$$\delta \bar{B}_i = \delta \Lambda_i, \quad \delta Z^I = 0, \quad \delta Y_I = k_i I \partial \Lambda_i + 2 \Lambda_i k_i I, J \partial Z^J$$

The one-forms $K_i$ can, in general, be taken to carry $GL(1, \mathbb{C})$ charge $h_i$. Also, in general, one can take the number of fermionic components $\psi^a$ to be $\mathcal{N}$, rather than 4. The conformal and $GL(1, \mathbb{C})$ anomalies are then given by

$$C = 4 - \mathcal{N} - 28 + C_C - 2(\delta - n)$$

$$\kappa = 4 - \mathcal{N} - \sum \epsilon_i(h_i)^2$$

(14)
Here $\delta$ is the number of bosonic one-forms and $n$ is the number of fermionic one-forms among the chosen set of $K_i$'s. Further, $\epsilon_i = 1$ for bosonic $K_i$ and $\epsilon_i = -1$ for fermionic $K_i$. With these formulae, one can see that there are many anomaly-free solutions, giving many new string theories. For example, the choice $\mathcal{N} = 0$ and $K = \epsilon^{A'B'} \pi_A d\pi_{B'}$ leads to selfdual gravity (with no supersymmetry). Many other cases are given by Abou-Zeid, Hull and Mason in reference [1]. Perhaps the most interesting case corresponds to the $\mathcal{N} = 4$ (so that we are back to the $\mathcal{N} = 4$ supertwistor space) with

$$K = w(\pi) \epsilon^{A'B'} \pi_A d\pi_{B'}$$

where $w(\pi)$ is of degree $-2$. The gauging is thus done with a weightless one-form. In this case $h = 0$ and $\delta = 1, n = 0$. This seems to lead to $\mathcal{N} = 8$ supergravity; the field content of the theory is expressed in terms of $\mathcal{N} = 4$ multiplets.

There are two types of vertex operators for gravitons,

$$V_f = \int f^I Y_I, \quad V_g = \int g_I \partial Z^I$$

For these operators to be physical, the functions $f^I, g_I$ must obey the constraints

$$\partial f^I = 0, \quad g_I Z^I = 0$$

There is also an equivalence relation $f^I \sim f^I + \delta f^I, g_I \sim g_I + \delta g_I$, for functions differing by the transformations,

$$\delta f^I = \lambda Z^I, \quad \delta g_I = \partial_I \chi$$

for arbitrary $\lambda, \chi$. The requirements (17) and (18) are the same as in the original Witten-Berkovits theory and lead to conformal supergravity multiplet. The new gauge symmetry leads to the further condition

$$\epsilon^{A'B'} f_{A'} \pi_{B'} = 0$$

and the equivalence relation $g^A' \sim g^A' + \delta g^A'$,

$$\delta g^A' = \epsilon^{A'B'} \pi_{B'} \xi$$

for some function $\xi$. The vertex operator which obeys these conditions may be obtained as

$$V_f = \int d\sigma \epsilon^{AB} \frac{\partial h}{\partial \omega^B} Y_A + \int d\sigma f^a Y_a$$

The function $h$ is of degree of homogeneity 2,

$$h = h_2 + h_{3/2}^a \psi^a + \cdots + h_0 \psi^1 \psi^2 \psi^3 \psi^4$$
This gives an $\mathcal{N} = 4$ graviton multiplet starting with the graviton of helicity +2, represented by $h^2$. The $f^a Y_a$-term will correspond to an $\mathcal{N} = 4$ gravitino multiplet starting with the $^{3\over 2}$-helicity state.

For $V_g$, we may choose a representative $g_A$ of the form $g_A = \omega_A g$, because of the equivalence relation (18). Further, the new equivalence relation (20) tells us that $g^{A'} = \tau^{A'} g$ where $\tau \cdot \pi \neq 0$. The constraint $g_I Z^I = 0$ then implies that $\tilde{g}$ is determined by $g_a \psi^a$. Thus the physical degrees of freedom are contained in $g$, which has degree of homogeneity $-2$, and in $g_a$, which is of degree $-1$. We can write

$$g = g_0 + g_{-1/2} \psi^a + \cdots + g_{-2} \psi^1 \psi^2 \psi^3 \psi^4$$

(23)

This describes an $\mathcal{N} = 4$ negative helicity graviton multiplet. Likewise, $g_a$ will describe a negative helicity gravitino multiplet. There are also vertex operators for vector multiplets. All of these together can make up the $\mathcal{N} = 8$ supergravity multiplet.

The explicit form of the vertex operators, particularly the graviton part, is useful for explicit calculations. They can be taken as

$$h_2 = \delta(\pi \cdot \tilde{p}) \left( \frac{\pi \cdot \alpha}{\tilde{p} \cdot \alpha} \right)^3 \exp \left[ (\omega \cdot p + \psi \cdot \eta) \frac{\tilde{p} \cdot \alpha}{\pi \cdot \alpha} \right]$$

(24)

$$f^A = \epsilon^{AB} p_B \frac{\tilde{p} \cdot \alpha}{\pi \cdot \alpha} h_2 \equiv \epsilon^{AB} K_B h_2$$

$$g_{-2} = \delta(\pi \cdot \tilde{p}) \left( \frac{\tilde{p} \cdot \alpha}{\pi \cdot \alpha} \right)^5 \exp \left[ \omega \cdot p \frac{\tilde{p} \cdot \alpha}{\pi \cdot \alpha} \right]$$

(25)

The three-graviton amplitude for the choice of helicities (+++) was calculated by AHM in their original paper. This is given by the correlator $\langle V_f(1)V_g(2)V_g(3) \rangle$ and is nonzero for curves of degree zero. Explicitly,

$$\langle V_f(1)V_g(2)V_g(3) \rangle_{d=0} = \left( \frac{12}{31} \langle 23 \rangle \right)^2 \delta^{(4)}(p_1 + p_2 + p_3)$$

(26)

This result is in agreement with Einstein supergravity. While this is an encouraging result, it is necessary to calculate amplitudes for other helicities, at least the (+ − −)-amplitude, to see whether the theory is truly Einstein supergravity or some chiral version of it. We shall now calculate the latter amplitude. This is contained in the correlator $\langle V_f(1)V_g(2)V_g(3) \rangle$. This amplitude cannot get contributions from curves of degree zero, but can have nonzero contributions from curves of degree one. In a direct calculation of this amplitude, we will encounter only one Wick contraction $\langle YZ \rangle$ since there is only one factor of $V_f$. Therefore, this amplitude, considered as a correlator in the world-sheet field theory, will be entirely

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classical. We can calculate it by solving the equations of motion for the action (12) with vertex operator sources,

\[ S = \int \left[ Y \bar{D}Z + \bar{B}_i K_i \right] + S_C + \oint \lambda g_I \partial Z^I + \oint \rho f_I Y_I + \text{tilde part} \] (27)

The equation of motion for \( Z^I \) is given by

\[ (\bar{D}Z)^I + \rho f_I \delta(|\sigma| - \sigma_0) = 0 \] (28)

The \( \delta \)-function localizes the source term on the boundary. For \( \bar{A}'s \) of degree one, the solution to the equation (28) is of the form

\[ Z^I = a^I_\alpha u^\alpha - \int \left( \frac{1}{D} \right)_{12} \left[ \rho f^I(a^I_\alpha u^\alpha) \right]_{2} + \cdots \] (29)

Evaluating the action on this solution, we find

\[ S = \oint \lambda g_{I\ell} \partial Z^I_{\ell} \]

\[ = \oint \lambda g_I \partial Z^I \] \[ \left. \right|_{Z=au} \] \[ = \lambda \rho \frac{1}{2} (\partial_I g_{I\ell} - \partial_{I\ell} g_I) \left( \frac{1}{D} \right)_{12} f^I(Z(2)) \] \[ \left. \right|_{Z=au} + \cdots \] (30)

The scattering amplitude is given by integral of \( e^{-S} \) over the moduli of solutions, and for \( \langle V_f V_g V_g \rangle \), we are interested in the term with two powers of \( \lambda \) and one power of \( \rho \). Thus

\[ \langle V_f(1) V_g(2) V_g(3) \rangle_{d=1} = \int d\mu \oint f^I(1) \left( \frac{1}{D} \right)_{12} \frac{1}{2} \left[ (\partial_I g_{I\ell} - \partial_{I\ell} g_I) \partial Z^I \right]_{2} (g_K \partial Z^K)_{3} + (2 \leftrightarrow 3) \] (31)

The remaining integration is over the moduli space of the classical solutions, namely, over \( a^I_\alpha \). For this purpose, we need to identify the physical set of classical solutions. In the Witten-Berkovits string theory, the \( GL(2, \mathbb{C}) \) invariance allowed us to choose \( \pi_{A'} = a_{A'\alpha} u^\alpha = u_{A'} \).

In other words, we could set \( a_{A'\alpha} = \delta_{A'\alpha} \), which led to \( \omega^A = x^{AA'} u_{A'} = x^{AA'} \pi_{A'} \). The moduli space of the line, for the bosonic part, is thus a copy of spacetime in twistor space. In the present case, the situation is somewhat different. The new gauge symmetry gives the constraint

\[ w(Z) \epsilon^{AB'} \pi_A \partial \pi_{B'} = 0 \] (32)

This may be viewed as the equation of motion for \( \bar{B} \). Using \( \pi_{A'} = a_{A'\alpha} u^\alpha \), and the fact that \( w(Z) \) must be chosen to have singularities away from the world-sheet on \( \mathbb{CP}^1 \), this equation gives

\[ \det a = \frac{1}{2} \epsilon^{AB'} \epsilon^{\alpha\beta} a_{A'\alpha} a_{B'\beta} = 0 \] (33)
By $SL(2, \mathbb{C})$ invariance we can bring the solution set of this condition to the form

$$a_{A'\alpha} = \begin{pmatrix} 1 & 0 \\ \zeta & 0 \end{pmatrix}$$  \hspace{1cm} (34)

In other words, $\pi_1' = u^1$, $\pi_2' = \zeta u^1$, where $\zeta$ is the same for all vertex operators. Unlike the previous case, we do not get a copy of spacetime in the twistor space. It is clear that the amplitude (31) will, therefore, vanish. This can be seen in detail as follows. The $SL(2, \mathbb{C})$ transformation which brings $a_{A'\alpha}$ to the form (34) can be written as

$$g = \begin{pmatrix} (1 - a_{12} \chi)/a_{11} & -a_{12} \\ \chi & a_{11} \end{pmatrix}$$  \hspace{1cm} (35)

$\chi$ is still a free parameter and we see the form (34) is unchanged by transformations of the form

$$g = \begin{pmatrix} 1 & 0 \\ \chi & 1 \end{pmatrix}$$  \hspace{1cm} (36)

For $u^\alpha$ this leads to the change $u^1 \to u^1$, $u^2 \to \chi u^1 + u^2$, so that $\sigma = u^2/u^1$ changes as $\sigma \to \sigma + \chi$. Thus we can use this freedom to set one of the $\sigma$'s, say $\sigma_3$ to zero. Further, we write $\omega^A = a^A_{\alpha} u^\alpha = u^1 b^a + \sigma \bar{b}^A$ and $\psi^a = \gamma^a_\alpha u^\alpha$. The integration over the fermionic variables $\gamma^a_\alpha$ leads to a factor $(u_2 \cdot u_3)^4 \sim (\sigma_2 - \sigma_3)^4$. Putting all these together, the correlator in (31) becomes

$$\langle V_f(1)V_g(2)V_g(3) \rangle_{d=1} = \int \frac{[d\zeta/db]}{GL(1, \mathbb{C})} \delta(\pi \cdot \bar{p}_1)\delta(\pi \cdot \bar{p}_2)\delta(\pi \cdot \bar{p}_3) \frac{(\bar{p}_2 \cdot \alpha)^5(\bar{p}_3 \cdot \alpha)^5}{(\bar{p}_1 \cdot \alpha)^3(\pi \cdot \alpha)^7}$$
$$\times \exp \left( b^A[K_{1A} + K_{2A} + K_{3A}] \right) \times \mathcal{I}$$
$$+ (2 \leftrightarrow 3)$$  \hspace{1cm} (37)

where $\mathcal{I} = \mathcal{I}_1 + \mathcal{I}_2$, with

$$\mathcal{I}_1 = (2 + b \cdot K_2) \int [d\sigma_1 d\sigma_2 d\bar{b}] \frac{\sigma_2^2}{(\sigma_1 - \sigma_2)} K_1 \cdot \bar{b} \cdot \bar{b} e^{(\sigma_1 b \cdot K_1 + \sigma_2 \bar{b} \cdot K_2)}$$
$$\mathcal{I}_2 = \int [d\sigma_1 d\sigma_2 d\bar{b}] \frac{\sigma_2^4}{(\sigma_1 - \sigma_2)} K_1 \cdot \bar{b} \cdot \bar{b} \left[ \sigma_2 \bar{b} \cdot K_2 \right] e^{(\sigma_1 \bar{b} \cdot K_1 + \sigma_2 \bar{b} \cdot K_2)}$$  \hspace{1cm} (38)

Also, $K_A$ stands for $p_A \bar{p} \cdot \alpha / \pi \cdot \alpha$.

Consider now the evaluation of $\mathcal{I}_1$. The factors of $\bar{b}$ inside the integral can be replaced by derivatives with respect to $K_1$ of the exponential by defining

$$Q = (2 + b \cdot K_2) K_1 \cdot \frac{\partial}{\partial K_1} b \cdot \frac{\partial}{\partial K_1}$$  \hspace{1cm} (39)
We can then write, denoting $\sigma_1 - \sigma_2 = \sigma_{12}$,
\[
\mathcal{I}_1 = Q \int [d\sigma_1 d\sigma_2 d\tilde{b}] \frac{\sigma_2^4}{\sigma_{12}^2 \sigma_1^4} e^{(\sigma_1 b, K_1 + \sigma_2 b, K_2)}
\]
\[
= Q \int [d\sigma_1 d\sigma_2] \frac{\sigma_2^4}{\sigma_{12}^2 \sigma_1^4} \delta(\sigma_1 K_{11} + \sigma_2 K_{21}) \delta(\sigma_1 K_{12} + \sigma_2 K_{22})
\]
\[
= -Q \int d\sigma_2 \sigma_2 \frac{K_{11}^2}{(K_{11} + K_{21})} \delta(\sigma_2 [K_{22} - K_{21} K_{12}/K_{11}])
\]
\[
= -Q \int \frac{K_{11}^2}{(K_{11} + K_{21}) K_1 \cdot K_2} d\sigma_2 \sigma_2 \delta(\sigma_2) = 0 \quad (40)
\]
We have used a continuation to real values for the $\sigma$-integrals, as is done for the twistor amplitudes for the gauge particles. In $\mathcal{I}_2$ we have another factor with $\tilde{b}$ which gives an additional $1/\sigma_1$; but there is also a factor of $\sigma_2$ multiplying it, so that the powers of $\sigma_2$ in the last integral do not change and we get $\mathcal{I}_2 = 0$ as well. The scattering amplitude thus vanishes.

The theory we are discussing has the particle content of $\mathcal{N} = 8$ supergravity. The vanishing of the $(+--)$-amplitude is inconsistent with Einstein supergravity, rather it is suggestive of the theory being a chiral version of $\mathcal{N} = 8$ supergravity, rather like the theory introduced by Siegel [7].

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