SOME INVERSE SPECTRAL RESULTS FOR SEMI-CLASSICAL SCHRÖDINGER OPERATORS

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Abstract. We consider a semi-classical Schrödinger operator, $-\hbar^2 \Delta + V(x)$. Assuming that the potential admits a unique global minimum and that the eigenvalues of the Hessian are linearly independent over $\mathbb{Q}$, we show that the low-lying eigenvalues of the operator determine the Taylor series of the potential at the minimum.

1. Introduction

In this note we will report on some inverse spectral results for the semi-classical Schrödinger operator,

\begin{equation}
P = P(\hbar) = -\hbar^2 \Delta + V(x).
\end{equation}

The potential, $V$, in (1.1) will be assumed to be in $C^\infty(\mathbb{R}^n)$ and have a unique non-degenerate global minimum, $V(0) = 0$, at $x = 0$. We will also assume that, for $\epsilon > 0$ sufficiently small, $V^{-1}(0, \epsilon]$ is compact. Then, for $\hbar$ sufficiently small, the spectrum of $P$ in a small interval, $[0, \delta]$, consists of a finite number of eigenvalues, $E(\hbar)$. In fact, by Weyl’s law

\begin{equation}
\sharp \{ E(\hbar) ; 0 \leq E(\hbar) \leq \delta \} = (2\pi \hbar)^{-n} \left( \text{Vol} \{ 0 \leq \|\xi\|^2 + V(x) \leq \delta \} + o(1) \right)
\end{equation}

(see for instance [2], chapter 9).

We will be concerned below with the question: To what extent do the eigenvalues, (1.2), determine $V$? In particular we will prove:

**Theorem 1.1.** Assume $V$ is symmetric with respect to reflections about the coordinate axes, i.e. for any choice of signs

\[ V(x_1, \ldots, x_n) = V(\pm x_1, \ldots, \pm x_n). \]

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In addition, assume that
\[ V(x) = \sum_{i=1}^{n} u_i x_i^2 + O(|x|^4), \]
where \( u_1, \ldots, u_n \) are linearly independent over the rational numbers. Then
the family of eigenvalues (1.2) (with \( h \) in some neighborhood \((0, h_0)\)) determines the Taylor series of \( V \) at \( x = 0 \). In particular, if \( V \) is real analytic,
these eigenvalues determine \( V \).

Colin de Verdière and Zelditch have proved somewhat similar results for
the Dirichlet problem on convex regions in \( \mathbb{R}^2 \). Namely, suppose \( \Omega \) is a
strictly convex region in the plane which is real-analytic and invariant with
respect to reflections in the \( x \) and \( y \) axes. Zelditch proves that for such a
region the Dirichlet spectrum,
\[ \Delta u = E_i u, \quad 0 \neq u \in L^2(\Omega), \quad u|_{\partial \Omega} = 0 \]
determines \( \Omega \). The idea of his proof is to consider the billiard map, \( B \),
on the co-ball bundle of \( \partial \Omega \). By a theorem of Anderson and Melrose the
singularities of the wave trace (1.3)
\[ \sum_i \cos \sqrt{E_i} t \]
occurs at points \( t = T_\gamma \) where \( T_\gamma \) is the length of a periodic billiard trajectory,
\( \gamma \). In particular, suppose that \( \partial \Omega \) intersects the \( y \) axis at points, \((0, \pm a)\).
Let \( p \) be the fixed point of \( B^2 \) associated with the billiard trajectory, \( \gamma \),
which goes from \((0, a)\) to \((0, -a)\) then reflects and goes back again. Zelditch
proves that the singularities of (1.3) at \( T = 2a, 4a, \) etc. determine the
Birkhoff canonical form of \( B^2 \) at \( p \). Suppose now that near \( x = 0 \) the
boundary component of \( \Omega \) sitting above the \( x \) axis is the graph of a function,
\( f \). Because of the axial symmetries, \( f(x) = f(-x) \), and the boundary
component of \( \Omega \) sitting below the \( x \) axis is the graph of \(-f \). From these two
properties of \( f \), Colin de Verdière deduces that the Birkhoff canonical form
of \( B^2 \) at \( p \) determines the Taylor series of \( f \) at \( x = 0 \) and hence, if \( \Omega \) is real
analytic, determines \( \Omega \). (Colin de Verdière originally used this fact to prove
a somewhat weaker version of Zelditch’s result, namely, that regions of the
type above are spectrally rigid. See [1] and [9] for details.)

Our proof of Theorem 1.1 will involve roughly the same kind of argument.
Namely, we will prove:
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Theorem 1.2. Modulo the assumption on \(V\) above, the Birkhoff canonical form of the classical Hamiltonian

\[
H(x, \xi) = \sum_{i=1}^{n} \xi_i^2 + V(x)
\]

at \(x = 0, \xi = 0\) determines the Taylor series of \(V\) at \(x = 0\).

We will then make use of some recent results of Iantchenko-Sjöstrand-Zworski to show that if \(\psi \in C^\infty(\mathbb{R})\) is a bump function with \(\psi(s) = 1\) for \(s < 1/2\) and \(\psi(s) = 0\) for \(s > 1\), then for \(\epsilon > 0\) sufficiently small, the trace of the operator

\[
\psi\left(\epsilon^{-1}P(h)\right) \exp ith^{-1}P(h)
\]

(which only involves the eigenvalues of \(P(h)\) in a neighborhood of zero) determines the Birkhoff canonical form of \(H\) at \((x, \xi) = (0, 0)\).

We will review these results in the next section and prove Theorem 1.2 in \(\S 3\). In section 4 we indicate how to extend Theorem 1.1 to the case when the configuration space is a Riemannian manifold.

2. Microlocal Birkhoff canonical forms

If \(V\) is a potential with the properties described in Theorem 1.1, then microlocally in a neighborhood of \((x, \xi) = (0, 0)\) the Schrödinger operator (1.1) can be conjugated by a unitary FIO to a rather simple “quantum Birkhoff normal form”. More explicitly, there exist neighborhoods, \(O_1\) and \(O_2\), of \((x, \xi) = (0, 0)\) in \(\mathbb{R}^{2n}\), a canonical transformation

\[
\kappa : O_1 \to O_2, \quad \kappa(0, 0) = (0, 0),
\]

and a quantization of \(\kappa\) by a unitary Fourier integral operator, \(U\), such that microlocally on \(O_2\)

\[
UPU^{-1} = p(h^2D_1^2 + x_1^2, \ldots, h^2D_n^2 + x_n^2, h) + Q_\infty(h) + R
\]

where:

1. The smooth function \(p\) is an \(h\)-admissible symbol admitting an asymptotic expansion

\[
p(s_1, \ldots, s_n, h) \sim \sum_{j=0}^{\infty} h^j p_j(s_1, \ldots s_n)
\]
where the $p_j$ are smooth functions of $n$ variables, and the operator
\[ p(h^2 D_1^2 + x_1^2, \ldots, h^2 D_n^2 + x_n^2, h) \]
is the Weyl quantization of $p(\xi_1^2 + x_1^2, \ldots, \xi_n^2 + x_n^2)$.

(2) $p_0$ is of the form:
\[ p_0(w) = \sum_k u_k w_k^2 + \cdots \]
the dots indicating quartic and higher order terms.

(3) $Q_\infty$ is of order $\infty$ in $h$.

(4) The symbol of $R$ vanishes to infinite order at $(x, \xi) = (0, 0)$.

We will give a brief sketch of how to prove this in the next section.

Let $T > 0$ be the smallest period of the classical flow in the region of
phase space $\{ H(x, \xi) = \|\xi\|^2 + V(x) < \epsilon \}$. Consider now the (smoothing)
operator, (1.5), with $t \in (0, T)$. We claim that its trace,
\[
\Tr(t, h) = \sum_j e^{-ith^{-1}E_j(h)} \psi(\epsilon^{-1}E_j(h))
\]
has an asymptotic expansion as $h \to 0$,
\[ \Tr(t, h) \sim a_0(t) + ha_1(t) + \cdots \]
Indeed for each $t$ the operator $\psi(P(h))e^{-ith^{-1}P(h)}$ is an $h$-Fourier integral
operator whose canonical relation is contained in
\[
\{(x, \xi, x', \xi') : (x', \xi') = f_t(x, \xi), H(x, \xi) \leq \epsilon \}
\]
where $f_t$ is the classical flow. The lemma of stationary phase yields the
desired expansion of the trace.

The expansion, (2.4), is unchanged if we replace $P$ by the operator (2.1).
The trace of the resulting operator, as an asymptotic series in $h$, is equal to
the asymptotic expansion in powers of $h$ of the expression
\[ \sum_{k=(k_1, \ldots, k_n)} \psi(\epsilon^{-1}p((2k + 1)h, h)) \exp(\epsilon^{-1}t p((2k + 1)h, h)). \]
where we have set $p((2k + 1)h, h) = p((2k_1 + 1)h, \ldots, (2k_n + 1)h, h)$. Since
$\psi$ is identically equal to one in a neighborhood of zero, as a formal power
series in $h$ the expansion of $\Tr(t, h)$ is equal to
\[ \sum_{k=(k_1, \ldots, k_n)} \exp(ih^{-1}t \sum_{j=0}^\infty h^j p_j((2k + 1)h)). \]
Using “Kronecker type” theorems of Stark [3] and Fried [4], Iantchenko-Sjöstrand-Zworski, [7], show that one can extract from this expression the Taylor series of the functions, $p_j(x_1^2 + \xi_1^2, \ldots, x_n^2 + \xi_n^2)$, at $(x, \xi) = (0, 0)$ and, in particular, of the function, $p_0$, which is the “classical” Birkhoff normal form of the classical Hamiltonian (1.4). To be more specific, they show that the coefficients of the expansion of the trace are of the form

$$a_l(t) = q_l(it^{-1}\partial_{\mu})\prod_{j=1}^{n} \frac{1}{\sinh(t\mu_j/2)}|_{\mu=0}. \tag{2.7}$$

for some polynomials $q_l$. In the setting of [7] $t$ is an integer, and the authors show that from the asymptotics of (2.7) as $t \to \infty$ one can recover the coefficients of the polynomial, $q_l$. (This uses heavily the “Kronecker type” theorems cited above.) This is the key step in the reconstruction process. (For more details see §3 of [7].)

In the present case we only know the trace for $t \in (0, T)$ but, since (2.7) is an analytic function of $t$, we know the coefficients of the expansion for all $t$. Therefore we can conclude:

**Theorem 2.1.** (7) The eigenvalues, (1.2), determine the microlocal Birkhoff normal form of $P$ (and, in particular, the classical Birkhoff normal form of the Hamiltonian (1.4)).

This result is a semi-classical version of earlier results of this type by Guillemin [5] and Zelditch [8]. (In fact, by the standard trick of reformulating high-energy asymptotics as small $\hbar$ asymptotics, this result gives an alternative, somewhat simpler, proof of these earlier results.)

### 3. The proof of Theorem 1.2

#### 3.1. Classical Birkhoff canonical forms

We begin by recalling the construction of the classical Birkhoff canonical form. Conjugating the Hamiltonian (1.4) by the linear symplectomorphism

$$x_i \mapsto u_i^{1/2}x_i, \quad \xi_i \mapsto u_i^{-1/2}\xi_i, \quad i = 1, \ldots, n$$

one can assume without loss of generality that

$$H = \sum_i u_i(x_i^2 + \xi_i^2) + V(x_1^2, \ldots, x_n^2)$$

where $V(s_1, \ldots, s_n) = O(s^2)$. 


We will prove inductively that for $N = 1, 2, \ldots$ there exists a neighborhood, $\mathcal{O}$, of $x = 0 = \xi$, and a canonical transformation, $\kappa : \mathcal{O} \to \mathbb{R}^{2n}$, $\kappa(0,0) = (0,0)$, such that

\begin{equation}
\kappa^* H = \sum_{i=1}^{N} H_i + R_{N+1} + R'_{N+1},
\end{equation}

where:

(a) The $H_i$’s are homogeneous polynomials of degree $2i$ of the form

\begin{equation}
H_i = H_i(x_1^2 + \xi_1^2, \ldots, x_n^2 + \xi_n^2),
\end{equation}

with

\begin{equation}
H_1 = \sum_{i=1}^{n} u_i (x_i^2 + \xi_i^2).
\end{equation}

(b) $R_N$ is homogeneous of degree $2N$ and of the form:

\begin{equation}
R_N = V_N + R_N^*,
\end{equation}

where $V_N$ consists of the terms homogeneous of degree $2N$ in the Taylor series of $V(x_1^2, \ldots x_n^2)$ at $x = 0$, and $R_N^*$ is an artifact of the previous inductive steps.

(c) $R'_N$ vanishes to order $2N + 2$ at the origin and is of the form

\begin{equation}
R'_N = V - \sum_{k=2}^{N} V_k + S_N
\end{equation}

where $S_N$ is another artifact of the inductive process. In addition, $R'_N$ is even.

We will also show that this induction argument is such that one can read off from the $H_i$’s the first $N$ terms in the Taylor expansion of $V(s_1, \ldots s_n)$ at $s = 0$.

For $N = 1$ in (3.2) these assertions are true with $\kappa$ the identity, $R_2 = V_2$ consists of the quartic terms in the Taylor series of $V$, and $R'_2 = V - V_2$ (in particular $R'_2 = 0 = S_2$).

Let us suppose that these assertions are true for $N - 1$ ($N \geq 2$), so that

\begin{equation}
\kappa^* H = \sum_{i=1}^{N-1} H_i + \tilde{R}_N
\end{equation}
with \( \tilde{R}_N = R_N + R'_N \), as above. We look for a homogeneous polynomial of degree \( 2N \), \( G_N \), such that

\[
\{ H_1, G_N \} = R_N - H_N(x_1^2 + \xi_1^2, \ldots, x_n^2 + \xi_n^2)
\]

where \( H_N(s_1, \ldots, s_n) \) is a homogeneous polynomial of degree \( N \) in \( s \). Introducing complex coordinates, \( z_i = x_i + \sqrt{-1}\xi_i \), the Hamiltonian vector field

\[
\mathcal{V} = \sum_i \frac{\partial H_1}{\partial \xi_i} \frac{\partial}{\partial x_i} - \frac{\partial H_1}{\partial x_i} \frac{\partial}{\partial \xi_i}
\]

becomes the vector field

\[
\sqrt{-1}\sum_i u_i \left( z_i \frac{\partial}{\partial z_i} - \bar{z}_i \frac{\partial}{\partial \bar{z}_i} \right)
\]

in these coordinates, and

\[
x_i^2 + \xi_i^2 = z_i \bar{z}_i = |z_i|^2.
\]

Thus for \(|\alpha| + |\beta| = 2N\),

\[
L_V(z^\alpha \bar{z}^\beta) = \sqrt{-1} \left( \sum_i u_i (\alpha_i - \beta_i) \right) z^\alpha \bar{z}^\beta.
\]

Suppose \( R_N = \sum_{|\alpha| + |\beta| = 2N} c_{\alpha, \beta} z^\alpha \bar{z}^\beta \). Letting

\[
G = \frac{1}{\sqrt{-1}} \sum_{\alpha \neq \beta} {c_{\alpha, \beta}} \langle u, \alpha - \beta \rangle z^\alpha \bar{z}^\beta
\]

(where we are using the hypothesis of the linear independence of the \( u_i \) over \( \mathbb{Q} \)), we get from (3.7)\[
\{ H_1, G_N \} = L_V G = R_N - H_N
\]

where

\[
H_N = \sum_{\alpha = \beta, |\alpha| = N} c_{\alpha, \beta} z^\alpha \bar{z}^\beta.
\]

These arguments prove:

**Lemma 3.1.** There is a unique homogeneous polynomial, \( G_N \), linear combination of monomials \( z^\alpha \bar{z}^\beta \) with \(|\alpha| + |\beta| = 2N \) and \( \alpha \neq \beta \), such that (3.4) holds (and \( H_N \) consists of the “diagonal” monomials of \( R_N \).)

For future reference we make a little more explicit the form of \( G_N \). Note that

\[
x_i^{2k_i} = \left( \frac{z_i + \bar{z}_i}{2} \right)^{2k_i} = \left( \frac{1}{2} \right)^{2k_i} \binom{2k_i}{k_i} |z_i|^2 + F_{k_i} + \overline{F_{k_i}},
\]
where
\[ F_{k_i} = \left( \frac{1}{2} \right)^{2k_i} \sum_{0 \leq r < k_i} \binom{2k_i}{k_i} z_1^{2k_i-r} \bar{z}_r. \]

This shows that
\[ x_1^{2k_1} \cdots x_n^{2k_n} = \left( \frac{2k_1}{k_1} \right) \cdots \left( \frac{2k_n}{k_n} \right) |z_1|^{2k_1} \cdots |z_n|^{2k_n} + \cdots \]
where the dots are a linear combination of monomials of the form \( z^\alpha \bar{z}^\beta \) with \( \alpha \neq \beta \). Hence by (3.6) and (3.7) there exists a homogeneous polynomial, \( G_k = G_{k_1, \ldots, k_n} \) of degree 2N such that
\[ (3.8) \quad x_1^{2k_1} \cdots x_n^{2k_n} = \left( \frac{2k_1}{k_1} \right) \cdots \left( \frac{2k_n}{k_n} \right) |z_1|^{2k_1} \cdots |z_n|^{2k_n} + \{H_1, G_k\}. \]

As we’ll see below, this implies that, in solving (3.4), we can keep track of the Taylor coefficients of \( V_N \) as terms in \( H_N \) which are not artifacts of the previous steps in our induction.

Let \( W \) be the Hamiltonian vector field
\[ W = \sum_i \frac{\partial G}{\partial \xi_i} \frac{\partial}{\partial x_i} - \frac{\partial G}{\partial x_i} \frac{\partial}{\partial \xi_i}. \]

We will use the fact that, for any homogeneous function \( F \), the Taylor series of the pull-back \( (\exp W)^*(F) \) at the origin is given by the expansion
\[ (\exp W)^*(F) \sim \sum_{k=0}^{\infty} (\text{ad}_G)^k(F), \]
where \( \text{ad}_G(F) = \{G, F\} \). (Note that if \( G \) is homogeneous of degree \( l \) and \( F \) is homogeneous of degree \( m \), then \( \{G, F\} \) is homogeneous of degree \( m+l-2 \).)

With this in mind, by (3.2)
\[ (\exp W)^* \kappa^* H = \kappa^* H + \{G, \kappa^* H\} + \cdots = \kappa^* H + \{G, H_1\} + \cdots \]
\[ = \kappa^* H + H_N - R_N + \cdots = \sum_{i=1}^{N} H_N + \cdots \]
where the dots represent terms that vanish to order \( 2N+2 \) at \( (x, \xi) = (0,0) \). In fact, a calculation shows that for \( N \geq 3 \) the sum of the terms homogeneous of degree \( 2N+2 \), \( R_N \), equals
\[ R_{N+1} = \{H_2, G\} + (R_N)_{2N+2} \]
where \((R_N)_{2N+2}\) is the sum of the terms homogeneous of degree \( 2N+2 \) in the Taylor expansion of \( R_N' \). (For \( N = 2 \) one has a couple of harmless
additional terms, see below.) Thus if we replace $\kappa$ by $\kappa \exp(W)$, the inductive assumptions hold with $N - 1$ replaced with $N$.

If we let $N$ tend to infinity in (3.2) we obtain the Birkhoff canonical form

$$
\sum_{i=1}^{\infty} H_i(x_1^2 + \xi_1^2, \ldots, x_n^2 + \xi_n^2)
$$

for the classical Hamiltonian (1.4).

3.2. Recovering the Taylor series of $V$. We now prove that if we are given the sequence of functions $\{H_i\}$ we can recover the Taylor series of $V$ at the origin.

We begin by noticing that $H_1$ consists precisely of the quadratic terms in the Taylor series. Moreover, the function $G_2$ in the first step of the inductive procedure above satisfies

$$\{H_1, G_2\} = V_2 - H_2$$

(recall that $R_2 = V_2$, i.e. $R_2^g = 0$). It is clear from (3.8) that the information in $V_2$ is encoded in $H_2$; explicitly, if

$$V_2 = \sum_{k,|k|=2} c_k x^{2k},$$

then by (3.8)

$$H_2 = \sum_{k,|k|=2} c_k \binom{2k}{k} |z|^{2k}$$

where we have left $x^{2k} = x_1^{2k_1} \cdots x_n^{2k_n}$, etc. Therefore the quartic term, $V_2$, is determined by $H_2$. This implies that $G_2$ is also determined (see Lemma 3.1).

If we now conjugate $H = H_1 + V$ by the exponential of the Hamilton vector field of $G_2$, we obtain:

$$\kappa^* H = H_1 + R_2 + R_2' + \{G_2, H_1\} + \{G_2, R_2\} + \{G_2, \{G_2, H_1\}\} + O(8) = H_1 + H_2 + V_3 + \{G_2, R_2\} + \{G_2, \{G_2, H_1\}\} + O(8),$$

where $O(8)$ stands for a function that vanishes to order eight at the origin. (We have: $R_2' = V - V_2 = V_3 + O(8)$.) Therefore

$$R_3 = V_3 + \{G_2, R_2\} + \{G_2, \{G_2, H_1\}\},$$
the last two terms constituting $R_3^2$, the first artifact of the inductive process. Notice, however, that $R_3^2$ is known to us since $G_2$ was determined in the previous step.

The next inductive step involves the equation
\[
\{H_1, G_3\} = V_3 + R_3^2 - H_3,
\]
where one should notice that $R_3^2$, and therefore its “diagonal” monomials, are known. Since $H_3$ is also known, arguing exactly as before (appealing to (3.8)), we see that $V_3$ and $G_3$ are determined by $H_1$, $H_2$ and $H_3$. It is clear now that one can continue indefinitely in this fashion.

Q. E. D.

3.3. Appendix: The existence of a quantum Birkhoff canonical form. The existence of a quantum Birkhoff canonical form for the Schrödinger operator, (1.1), can be proved by essentially the same methods. Namely, let $p_0(s_1, \ldots, s_n)$ be a $C^\infty$ function with the same Taylor series as the series, $\sum_i H_i(s)$. Then, as we have just seen, there exists a canonical transformation, $\kappa$, conjugating (1.4) to the Birkhoff canonical form, $p_0(x_1^2 + \xi_1^2, \ldots, x_n^2 + \xi_n^2)$, modulo an error term which vanishes to infinite order at $x = \xi = 0$. Let $U$ be a Fourier integral operator quantizing $\kappa$. Then
\[
U \rho U^{-1} = p_0(h^2 D_1^2 + x_1^2, \ldots, h^2 D_n^2 + x_n^2) + hQ + R
\]
where the symbol of $R$ vanishes to infinite order at $x = \xi = 0$. We will prove by induction that there exists an F. I. O., $U_N$, such that microlocally near $x = \xi = 0$
\[
U_N \rho U_N^{-1} = p_0(h^2 D_1^2 + x_1^2, \ldots, h^2 D_n^2 + x_n^2) + h^N Q + R
\]
and the symbol of $R$ vanishes to infinite order at $x = \xi = 0$. Assuming this assertion is true for $N - 1$ let us prove it for $N$. Let the symbol of $Q$ be of the form $q = q_0(x, \xi) + hq'(x, \xi, h)$. We claim

Lemma 3.2. There exist $C^\infty$ functions, $a(x, \xi)$ and $p_N(s_1, \ldots, s_n)$, such that
\[
\{p_0, a\} = q_0 - p_N(x_1^2 + \xi_1^2, \ldots, x_n^2 + \xi_n^2) + r
\]
where $r$ vanishes to infinite order at $x = \xi = 0$. 


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Proof. By (2.3), $p_0 = H_1 + p_0'$ where $p_0'(s) = O(s^2)$. Writing (3.12) in the form

$$\{H_1, a\} = q_0 - \{p_N, a\} + r$$

it is clear that the homogeneous terms of degree $k$ in the Taylor series of $a$ and $p_N$ can be determined from the previous terms by (3.4). □

Now let $A = a^W(x, \hbar D)$ be the Weyl quantization of $a$, and let $V = \exp(-\sqrt{-\hbar N} A)$. Then

$$V(U_{N-1}P U_{N-1}^{-1})V^{-1} = U_{N-1}P U_{N-1}^{-1} + \sqrt{-\hbar} h^N [U_{N-1}P U_{N-1}^{-1}, A] + h^{N+1}Q' + R'$$

where the symbol of $R'$ vanishes to infinite order at $x = \xi = 0$. In view of (3.11) and (3.12) the right hand side of this identity can be rewritten in the form

$$\sum_{j=1}^N h^2 p_j(h^2 D_1^2 + x_1^2, \ldots, h^2 D_n^2 + x_n^2) + h^{N+1}Q'' + R''$$

where the symbol of $R''$ vanishes to infinite order at $x = \xi = 0$. Thus with $U_{N-1}$ replaced by $VU_{N-1}$ the identity (3.11) is valid with $N - 1$ replaced by $N$, and letting $N$ tend to infinity one obtains the Birkhoff canonical form for $P$ described in the previous section.

4. An extension to the Riemannian case

Let $M$ be an $n$-dimensional Riemannian manifold, $\Delta$ its Laplace-Beltrami operator, and $V : M \to \mathbb{R}$ a smooth function. We consider the Schrödinger operator $P = \hbar^2 \Delta + V$ on $L^2(M)$. We assume for simplicity that the spectrum of $P$ is discrete. In this section we describe an extension of Theorem 1.1 to $P$.

We will assume that $V$ has a unique global minimum, $m \in M$, and that it is non-degenerate. Then the Hessian of $V$ at $m$ is a well-defined positive-definite quadratic form on $T_m M$. Using the metric, we can speak of the eigenvalues, $u_1, \ldots, u_n$, of the Hessian. We assume that these eigenvalues are linearly independent over $\mathbb{Q}$. In particular all eigenvalues are distinct, and therefore $T_m M$ splits naturally as an orthogonal direct sum of lines (spanned by eigenvectors of the Hessian). Denote by $G \cong \mathbb{Z}^2^n$ the group of linear transformations of $T_m M$, generated by the reflections $v_i \mapsto -v_i$, where $v_1, \ldots, v_n$ is a basis of eigenvectors of the Hessian.

**Theorem 4.1.** Assume that there exists a neighborhood, $U$, of $m$ such that:
(1) $G$ acts on $\mathcal{U}$ by isometries.

(2) $m$ is a fixed point of the action, and the infinitesimal action on $T_m$ agrees with the original linear action of $G$.

(3) $V|_\mathcal{U}$ is invariant under $G$.

Assume furthermore that $V(m) = 0$. Then, assuming that we know the Riemannian metric on $M$, the eigenvalues (1.2) determine the infinite jet of $V$ at $m$.

The proof is a straightforward generalization of the one given above in the Euclidean case. We will say a few words on how to show that the Birkhoff normal form of the classical Hamiltonian,

$$H(x, \xi) = \|\xi\|^2 + V(x)$$

(where $(x, \xi) \in T^*M$), determines the Taylor series of $V$ at $m$.

Introduce geodesic normal coordinates centered at $m$, $x = (x_1, \ldots, x_n)$, such that the corresponding frame at $m$ consists of eigenvectors of the Hessian. In these coordinates the classical Hamiltonian takes the form

$$H(x, \xi) = \sum_{i,j=1}^{n} g^{ij}(x)\xi_i\xi_j + V(x)$$

where:

1. $g^{ij}(x) = \delta_{ij} + O(|x|^2)$,
2. $g^{ij}(\pm x_1, \ldots, \pm x_n) = g^{ij}(x_1, \ldots, x_n)$,
3. $V(\pm x_1, \ldots, \pm x_n) = V(x_1, \ldots, x_n)$.

(The reason for (2) and (3) is that in the chosen coordinates $G$ acts by: $(x_1, \ldots, x_n) \mapsto (\pm x_1, \ldots, \pm x_n)$.) Proceeding as in §3, we can rewrite the Hamiltonian in the form

$$H(x, \xi) = \sum_{i,j=1}^{n} h^{ij}(x_1^{2}, \ldots, x_n^{2})\xi_i\xi_j + V(x_1^{2}, \ldots, x_n^{2})$$

where $V(s_1, \ldots, s_n) = O(s^2)$ and $h^{ij}(s_1, \ldots, s_n) = O(s)$.

The construction of the Birkhoff normal form $\sum_j H_j$ proceeds as before, except that the polynomials $R_N$ are now of the form

$$R_N = \sum_{i,j=1}^{n} h^{ij}_{N-1}(x_1^{2}, \ldots, x_n^{2})\xi_i\xi_j + V_N(x_1^{2}, \ldots, x_n^{2}) + R_N^x$$

where $h^{ij}_{N}(s_1, \ldots, s_n)$ consists of the terms homogeneous of degree $N$ in the Taylor series of $h^{ij}(s_1, \ldots, x_n)$. $H_N$ consists of the “diagonal” monomials in
$R_N$, as before (that is, monomials of the form $c_{z} z_{\alpha \bar{z}^\alpha}$ in complex coordinates $z_j = x_j + \sqrt{-1} \xi_j$).

We need to show that the sequence $\{H_j\}$ determines the Taylor series of $V$. This is possible because we are assuming that we know the metric, and therefore we know the sum $\sum_{i,j=1}^n h_{ij} N^{-1} (x_1^2, \ldots, x_n^2) \xi_i \xi_j$ appearing in $R_N$.

For example, $H_2$ consists of the diagonal terms in

$$R_2 = -\frac{1}{3} \sum_{i,j,k,l} R_{ikjl}(0) x_k x_l \xi_i \xi_j + V_2(x_1^2, \ldots, x_n^2).$$

If we subtract from $H_2$ the (known) diagonal terms in $-\frac{1}{3} \sum_{i,j,k,l} R_{ikjl}(0) x_k x_l \xi_i \xi_j$, we obtain the diagonal terms in $V_2(x_1^2, \ldots, x_n^2)$, which, as we have seen, determine the quartic terms in the Taylor expansion of $V$. The higher-degree cases are no different.

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