A note on the vacuum structure to lattice Euclidean quantum gravity

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It is shown that the ground state or vacuum to the lattice Euclidean quantum gravity is significantly different from the ground states to the well-known vacua in QED, QCD, et cetera. In the case of the lattice Euclidean quantum gravity, the long-wavelength scale vacuum structure is similar to that in QED, moreover the quantum fluctuations to gravity are very reduced in comparison with the situation in QED. But the small scale (of the order of the lattice scale) vacuum structure to gravity is significantly different from that to the long-wavelength scales: the fluctuation values of geometrical degrees of freedom (tetrads) are commensurable with theirs most probable values.

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I. INTRODUCTION

It has been shown previously that the propagators of doubled (in Wilson sense) fermion quanta in the lattice gravity model decreases exponentially [1]. But in fact the result has been obtained on account of a hypothesis on the vacuum structure. This hypothesis is justified in this paper.

Furthermore, it is shown that the system of gravity coupled with Dirac fermions breaks PT-invariance.

Let’s outline the model of lattice gravity which is studied here. A detailed description of the model and some of its properties are given in [1]-[6].

Denote by $\gamma^a_i(E)$ 4 × 4 Hermitian Dirac matrices, so that

$$
\gamma^a_i(E)\gamma^b_j(E) + \gamma^b_j(E)\gamma^a_i(E) = 2\delta^{ab}, \quad a = 1, 2, 3, 4,
$$

$$
\gamma^5 \equiv \gamma^i(E)\gamma^j(E)\gamma^3 \gamma^4 = (\gamma^5)_i(E). \tag{1.1}
$$

The lower index $(E)$ means that Euclidean signature is used. The physical values are supplied by the index $(E)$ also for Euclidean signature. We use the representation

$$
\gamma^a_i(E) = \begin{pmatrix}
0 & i\sigma^a \\
-i\sigma^a & 0
\end{pmatrix}, \quad \gamma^a_i(E) = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}, \quad a = 1, 2, 3.
$$

The orientable 4-dimensional simplicial complex and its vertices are designated as $\mathfrak{r}$ and $a_{V}$, the indices $V = 1, 2, \ldots, \mathfrak{r} \to \infty$ and $W$ enumerate the vertices and 4-simplices, correspondingly; the pairs of indices $(V_1, V_2)$ enumerate 1-simplices $a_{V_1}, a_{V_2} \in \mathfrak{r}$. It is necessary to use the local enumeration of the vertices $a_V$ attached to a given 4-simplex: the all five vertices of a 4-simplex with index $W$ are enumerated as $a_{W_i}$, $i = 1, 2, 3, 4, 5$.

The later notations with extra index $W$ indicate that the corresponding quantities belong to the 4-simplex with index $W$. The Levi-Civita symbol with in pairs different indexes $\epsilon_{ijklm} = \pm 1$ depending on whether the order of vertices $s_i^W = a_{W_i}a_{W_j}a_{W_k}a_{W_l}a_{W_m}$ defines the positive or negative orientation of 4-simplex $s_i^W$. An element of the compact group $\text{Spin}(4)$ and an element of the Clifford algebra

$$
\Omega(E)_{W} = \Omega^1(E)_{W}ij = \exp \left(\omega(E)_{W}ij\right) \in \text{Spin}(4),
$$

$$
\omega(E)_{W}ij \equiv \frac{1}{2}\sigma^{ab}(E)\omega^{ab}(E)_{W}ij, \quad \sigma^{ab}(E) \equiv \frac{1}{4}[\gamma^a(E), \gamma^b(E)],
$$

$$
\hat{c}(E)_{W}ij = \hat{c}^1(E)_{W}ij = \epsilon^{a}(E)_{W}ij\gamma^a(E) =
\equiv -\Omega(E)_{W}ij\hat{c}(E)_{W}ji\Omega^{-1}(E)_{W}ij, \tag{1.2}
$$

are assigned for each oriented 1-simplex $a_{W_1}a_{W_2}$. The lattice pure gravity action has the form

$$
\mathfrak{R}(E)_g = \frac{1}{5 \cdot 24 \cdot 2} \sum_{W, i, j, k, l, m} \epsilon_{ijklm} \times \tr \gamma^5 \Omega(E)_{W}ij\Omega(E)_{W}kj\Omega(E)_{W}lm\hat{c}(E)_{W}mk\hat{c}(E)_{W}ln, \tag{1.3}
$$

This action is invariant relative to the gauge transformations

$$
\hat{\Omega}(E)_{W}ij = S_{Wi}\Omega(E)_{W}ijS^{-1}_{Wj}, \quad S_{W}i \in \text{Spin}(4),
$$

$$
\hat{c}(E)_{W}ij = S_{Wi}\epsilon(E)_{W}ijS^{-1}_{Wj}. \tag{1.4}
$$

The positively defined Euclidean metric on $\mathfrak{r}$ is defined according to

$$
ds^2(E) = \frac{1}{4} \tr(\epsilon(E)_{W}ij)^2 = (\epsilon^a(E)_{W}ij)^2. \tag{1.5}
$$

The partition function (transition amplitude) is defined
as follows:

\[ \Omega(E)_{ij} = \text{const} \cdot \left( \prod_{(v_1 v_2)} \int d\Omega(E)(v_1 v_2) \times \right) \exp \left( \mathcal{A}(E)_{ij} \right). \] (1.6)

Here \( d\Omega(\{v_1 v_2\}) \) is invariant measure on the group \( \text{Spin}(4) \),

\[ d\epsilon_{(v_1 v_2)} = d\epsilon_{(v_1 v_2)}^1 \wedge d\epsilon_{(v_1 v_2)}^2 \wedge d\epsilon_{(v_1 v_2)}^3 \wedge d\epsilon_{(v_1 v_2)}^4. \]

Now let us pass on to the limit of slowly varying fields, when the action (1.3) reduces to the well known continuous gravity action. This transition have meaning together with the transition to Minkowski signature. As a result the compact gauge group \( \text{Spin}(4) \) transforms into the non-compact group \( \text{Spin}(3,1) \).

Firstly let us perform the following deformations of contours integration in integral (1.6) and matrix substitutions:

\[ \omega^{ab}_{(E)Wij} = i\omega^{ab}_{ij}, \quad \omega^{ab}_{(E)Wij} = -\omega^{ab}_{ij}, \]

\[ e^{a}_{(E)Wij} = e^{a}_{ij}, \quad e^{a}_{(E)Wij} = i e^{a}_{ij}, \]

\[ \gamma^{4}_{(E)} = \gamma^{0}, \quad \gamma^{\alpha}_{(E)} = i \gamma^{\alpha} \rightarrow \gamma^{5} = i \gamma^{0} \gamma^{1} \gamma^{2} \gamma^{3}, \]

\[ \sigma^{4a}_{(E)} = i \sigma^{0a}, \quad \sigma^{ab}_{(E)} = -\sigma^{ab}, \] (1.7)

where

\[ \sigma^{ab} \equiv \frac{1}{4} [\gamma^{a}, \gamma^{b}], \quad a, b \ldots = 0, 1, 2, 3, \quad \alpha, \beta, \ldots = 1, 2, 3, \]

\[ \frac{1}{4} \text{tr} \, \gamma^{a} \gamma^{b} = \eta^{ab} = \text{diag}(1, -1, -1, -1), \]

\[ \text{tr} \, \gamma^{5} \gamma^{a} \gamma^{b} \gamma^{c} \gamma^{d} = 4i \varepsilon_{abcd}, \quad \varepsilon_{0123} = 1, \] (1.8)

and the variables \( \omega^{ab}_{Wij}, \gamma^{ab}_{Wij} \) are real quantities (while the variables \( \omega^{ab}_{(E)Wij} \) and \( e^{a}_{(E)Wij} \) become complex quantities). As a result of (1.7) we have

\[ \omega^{ab}_{(E)Wij} \equiv \frac{1}{2} \omega^{ab}_{ij} \sigma^{ab}_{ij} \equiv \omega_{Wij}, \]

\[ \dot{e}^{a}_{(E)Wij} \equiv \dot{e}^{a}_{ij} \sigma^{a}_{ij} \equiv \dot{e}_{Wij}, \]

\[ d\Omega_{(E)ij} \equiv \exp \left( \frac{1}{2} \omega^{ab}_{ij} \sigma^{ab}_{ij} \right) = \exp \left( \frac{1}{2} \omega^{ab}_{ij} \sigma^{ab}_{ij} \right) \equiv \Omega_{Wij} \in \text{Spin}(3,1). \] (1.10)

Thus, the metric acquires Minkowski signature (1.10) and according to (1.10) the holonomy elements \( \Omega_{Wij} \) become the elements of the non-compact group \( \text{Spin}(3,1) \).

In order to pass to the long-wavelength limit let’s consider a certain 4D sub-complex of complex \( \mathcal{K} \) with the trivial topology of four-dimensional disk. Realize geometrically this sub-complex in \( \mathbb{R}^{4} \). Suppose that the geometric realization is the triangulation of a compact part of \( \mathbb{R}^{4} \), so that in \( \mathbb{R}^{4} \) we have \( (\text{int} \, s_{1}^{W}) \cap (\text{int} \, s_{2}^{W}) = \emptyset \) for \( W \neq W' \) and the sizes of these simplices are commensurable. Thus each vertex of the sub-complex acquires the coordinates \( x^{\mu} \) which are the coordinates of the vertex image in \( \mathbb{R}^{4} \):

\[ x^{\mu}_{W_{i}} = x^{\mu}_{\mathcal{K}}(a_{W_{i}}) \equiv x^{\mu}(a_{V}), \quad \mu = 0, 1, 2, 3. \] (1.11)

We stress that the correspondence between the vertices from the considered subset and the coordinates (1.11) is one-to-one. It is evident that the four vectors

\[ dx^{\mu}_{W_{mi}} \equiv x^{\mu}_{W_{m} - x^{\mu}_{W_{mi}}}, \quad i = 1, 2, 3, 4 \] (1.12)

are linearly independent. Here, the differentials of coordinates (1.12) correspond to one-dimensional simplices \( a_{W_{m}W_{mi}} \), so that, if the vertex \( a_{W_{m}} \) has coordinates \( x^{\mu}_{W_{m}} \), then the vertex \( a_{W_{mi}} \) has the coordinates \( x^{\mu}_{W_{m}} + dx^{\mu}_{W_{mi}} \).

In the continuous limit, the holonomy group elements (1.2) are close to the identity element, so that the quantities \( \omega^{ab}_{ij} \) tend to zero being of the order of \( O(dx^{\mu}) \). Thus one can consider the following system of equation for \( \omega_{W\mu} \)

\[ \omega_{W\mu} \, dx^{\mu}_{W_{mi}} = \omega_{W_{mi}}, \quad i = 1, 2, 3, 4. \] (1.13)

In this system of linear equation, the indices \( W \) and \( m \) are fixed, the summation is carried over the index \( \mu \), and index runs over all its values. Since the vectors (1.12) are linearly independent, the quantities \( \omega_{W\mu} \) are defined uniquely. Suppose that a one-dimensional simplex \( X_{W_{m}} \) belongs to four-dimensional simplices with indices \( W_{1}, W_{2}, \ldots, W_{r} \). Introduce the quantity

\[ \omega_{\mu} \left( \frac{1}{2} \left( x_{W_{m}} + x_{W_{i}} \right) \right) \equiv \frac{1}{r} \left\{ \omega_{W_{1}\mu} + \ldots + \omega_{W_{r}\mu} \right\}, \] (1.14)

which is assumed to be related to the midpoint of the segment \( [x^{\mu}_{W_{m}}, x^{\mu}_{W_{i}}] \). Recall that the coordinates \( x^{\mu}_{W} \), as well as the differentials (1.12) depend only on vertices but not on the higher dimensional simplices to which these vertices belong. According to the definition, we have the following chain of identities:

\[ \omega_{W_{1}m_{i}} \equiv \omega_{W_{2}m_{i}} \equiv \ldots \equiv \omega_{W_{r}m_{i}}. \] (1.15)

It follows from (1.12) and (1.13), (1.15) that

\[ \omega_{\mu} \left( x_{W_{m}} + \frac{1}{2} \, dx_{W_{mi}} \right) \, dx^{\mu}_{W_{mi}} = \omega_{W_{mi}}. \] (1.16)

The value of the field element \( \omega_{\mu} \) in (1.16) is uniquely defined by the corresponding one-dimensional simplex.

Next, we assume that the fields \( \omega_{\mu} \) smoothly depend on the points belonging to the geometric realization of
each four-dimensional simplex. In this case, the following formula is valid up to $O((d x)^2)$ inclusive

$$
\Omega_{Wmi} \Omega_{Wij} \Omega_{Wjm} =
\exp \left[ \frac{1}{2} \mathcal{R}_{\mu\nu}(x_{Wm}) d x_{Wmi}^\mu d x_{Wmj}^\nu \right],
$$

(1.17)

where

$$
\mathcal{R}_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu + [\omega_\mu, \omega_\nu].
$$

(1.18)

When deriving formula (1.20), we used the Hausdorff formula.

In exact analogy with (1.13), let us write out the following relations for a tetrad field without explanations

$$
\hat{e}_{Wm} d x_{Wmi}^\mu = \hat{e}_{Wmi}.
$$

(1.19)

With the help of Eqs. (1.9), (1.10), (1.11), (1.12), (1.13), we rewrite the action (1.21) in the case of Minkowski signature and long-wavelength limit as follows:

$$
\mathcal{A}_{(E)g} = i \mathcal{A}_g, \quad \mathcal{A}_g = -\frac{1}{4 l_P^2} \varepsilon_{abcd} \int \mathcal{A}^{ab} \wedge \mathcal{E}^c \wedge \mathcal{E}^d.
$$

(1.20)

The expression (1.20) is Hilbert-Einstein action in the Palatini form.

We need also the fermion part of the lattice action:

$$
\mathcal{A}_{(E)\Psi} = \frac{1}{2 \cdot 2^4} \sum_{W} \sum_{i,j,k,l,m} \varepsilon_{ijklm} \times
$$

$$
\times \text{tr} \gamma^5 \hat{\Theta}_{(E)Wmi} \hat{e}_{Wm}\hat{e}_{Wmj} \hat{e}_{Wmk} \hat{e}_{Wnl} \hat{e}_{Wnl},
$$

$$
\hat{\Theta}_{(E)Wij} = \frac{i}{2} \gamma^a(E) \Psi_{(E)Wij}^\dagger \gamma^a(E) \Omega_{(E)Wij} \Psi_{(E)Wij} - \Psi_{(E)Wij}^\dagger \Omega_{(E)Wij} \gamma^a(E) \Psi_{(E)Wij}.
$$

(1.21)

Here $\Psi_{(E)}$ and $\Psi_{(E)}$ are independent Dirac Grassmann variables assigned to each vertex, they are in mutual Hermitian involution.

With the help of Eqs. (1.9), (1.10), (1.11), and the substitution

$$
\Psi_{(E)}^\dagger = \Psi_{(E)} \gamma^0 = \mathcal{V},
$$

(1.22)

the fermion Euclidean lattice action (1.21) transforms into long-wavelength action for Minkowski signature:

$$
\mathcal{A}_{(E)\Psi} = i \mathcal{A}_\Psi, \quad \mathcal{A}_\Psi = \frac{1}{6} \varepsilon_{abcd} \int \Theta^a \wedge \mathcal{E}^b \wedge \mathcal{E}^c \wedge \mathcal{E}^d,
$$

$$
\Theta^a = \frac{i}{2} \left[ \mathcal{V} \gamma^a \partial_\mu \mathcal{V} \Psi - (\partial_\mu \mathcal{V}) \gamma^a \Psi \right] d x^\mu,
$$

$$
\partial_\mu = (\partial_\mu + \omega_\mu).
$$

(1.23)

The organization of the paper is as follows. In Section II the well known picture of long-wavelength scale quantum vacuum structure is outlined shortly. The discourse is based on the work [7]. In Section III the small scale quantum vacuum structure is studied. It is shown that the fluctuation values of tetrads $e_{V_1 V_2}$ are commensurable with theirs most probable values. The interpretation and discussion of the obtained results are made in Section IV.

II. LONG-WAVELENGTH SCALE VACUUM STRUCTURE

Let’s consider integral (1.6) in the long-wavelength limit. In this case we will make the substitutions

$$
\mathcal{A}_{(E)} \rightarrow i \mathcal{A},
$$

$$
\left( \prod_{(V_1 V_2)} d \Omega_{(E)(V_1 V_2)} \right) \rightarrow \prod_{x, \mu, (a < b)} d \omega^{ab}_\mu(x), \quad -\infty < \omega^{ab}_\mu(x) < \infty,
$$

$$
\left( \prod_{(V_1 V_2)} d e_{(E)(V_1 V_2)} \right) \rightarrow \prod_{x, \mu, a} d e^a_\mu(x).
$$

(2.1)

According to (1.18), (2.20) and (2.1) the integral (1.6) over the variables $\omega^{ab}_\mu(x)$ is Gaussian, so that its value (up to a pre-exponential factor $F\{\epsilon\}$ which is the functional of $\{\epsilon\}$) is equal to

$$
\mathcal{U} = \text{const} \cdot \int d \mu \{\epsilon\} \exp(i \mathcal{A}) \bigg|_{(\partial \mathcal{A}/\partial \omega = 0)}.
$$

(2.2)

Here $d \mu \{\epsilon\}$ is the measure which will be not interesting further, especially for the reason the integral is nonrenormalizable.

It is well known that the Hilbert-Einstein-Palatini action is equal to the Hilbert-Einstein action

$$
\mathcal{A} = -\frac{1}{l_P^2} \int d(4) x \sqrt{g} \mathcal{R}.
$$

(2.3)

on the hypersurface

$$
\frac{\partial \mathcal{A}}{\partial \omega^{ab}_\mu(x)} = 0.
$$

(2.4)

Therefore the functional integral (2.2) can be rewritten as

$$
\mathcal{U} = \text{const} \cdot \int d \mu \{g\} \exp(i \mathcal{A}),
$$

(2.5)

where the action $\mathcal{A}$ is given by (2.3) and $d \mu \{g\}$ is the functional measure calculated for the metric variables

$$
\mathcal{g}_{\mu\nu} = \eta_{ab} e^{a}_\mu e^{b}_\nu.
$$

Here the explicit form of $d \mu \{g\}$, including the gauge fixing factors and the corresponding ghosts measure, is of no interest. But to review the vacuum quantum fluctuations, the action (2.3) will be considered to some extent. For this end we extract some formulæ (without derivation) from the work [7].

Let’s redefine the metric tensor $g_{\mu\nu}$ in the action (2.3) as $\gamma_{\mu\nu}$ and then expand it into the sum of classical and quantum parts:

$$
\gamma_{\mu\nu} = g_{\mu\nu} + l_P h_{\mu\nu}.
$$

(2.6)

In what follows $g_{\mu\nu}$ means classical field satisfying classical Einstein equation, and $h_{\mu\nu}$ is quantum field. Further
all quantities (e.g. $\Gamma_{\nu}^{\mu}, \Omega_{\nu}^{\rho}$, and so on) which are not underlined are constructed with the use of $g_{\mu\nu}$, by a general rule, the symbol $\nabla_\mu$ means covariant derivative with the use of $\Gamma^\nu_{\nu\lambda}$, and lowering and raising of indices for the field $h_{\mu\nu}$ are made with the help of metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$.

Let’s substitute the quantity $\Omega_{\mu\nu}$ (see (2.3)) for the variable $g_{\mu\nu}$ in (2.8) and expand the action into a series in $h_{\mu\nu}$. In the first order we have

$$\mathfrak{A}^{(1)} = \int d^{4}x \sqrt{-g} h^{\mu\nu} \left[ \mathfrak{R}_{\mu\nu} - \frac{1}{2} \delta^{\mu\nu} \mathfrak{R} \right].$$

(2.7)

Since the field $g_{\mu\nu}$ is classical, so we have

$$\mathfrak{R}_{\mu\nu} - \frac{1}{2} \delta_{\mu\nu} \mathfrak{R} = 0 \leftrightarrow \mathfrak{R}_{\mu\nu} = 0. \quad (2.8)$$

To write out the effective second order quantum field contribution into the action (2.3) one must use the Faddeev-Popov technique: the gauge fixing contribution must be added to the second order quantum field contribution into the action (2.3). In our case the gauge fixing condition is as follows

$$\left( \nabla_\nu h^\nu_{\mu} - \frac{1}{2} \partial_\mu h^\nu_{\nu} \right) = 0. \quad (2.9)$$

$$\mathfrak{A}^{(2)} = \int d^{4}x \sqrt{-g} h_{\mu\nu} \times$$

$$\times \left\{ - \frac{1}{4} \left[ g^\mu^\nu g_\nu^\sigma - \frac{1}{2} g^\mu^\nu g_\nu^\lambda \right] \nabla_\sigma \nabla^\sigma - \frac{1}{2} \mathfrak{R}^\mu_\nu \right\} h_{\lambda\rho}. \quad (2.10)$$

Note also that the energy (Hamiltonian operator) of the dynamic system (2.10) is positively defined at least for the small curvature tensor. In the stationary case ($\partial g_{\mu\nu}/\partial x^\mu = 0$) the system (2.10) transforms into the system of independent harmonic oscillators, of course.

Further, quantum fluctuations “die out” for long-wavelength scales in the nonrenormalizable theories. In the considered case the action (2.10) leads to the following form for the correlator:

$$h(0)h(x) = O(1/x^2). \quad (2.11)$$

Therefore according to Eqs. (2.9) and (2.11) the quantum corrections to the classical field $g_{\mu\nu}$ at the scale $\sim x$ are of the order of

$$\delta g_{\mu\nu} \sim (x^2/x), \quad (2.12)$$

and

$$\langle \delta g_{\mu\nu} \rangle/g_{\mu\nu} = O(l_P/x) \rightarrow 0 \quad \text{as} \quad x \rightarrow \infty. \quad (2.13)$$

It follows from here that the classical theory of gravity is adequate model for long-wavelength scales since the metric quantum fluctuations can be neglected. It is known that the electromagnetic (Yang-Mills) field long-wavelength quantum fluctuations are more essential (most crucial).

On the contrary, quantum fluctuations reconstruct radically the theory at small scales (the lattice in our case).

### III. SMALL SCALE VACUUM STRUCTURE

The action (1.3) can be rewritten briefly as

$$\mathfrak{A}(E) = \frac{1}{2l_P^2} \sum_{\mathcal{M}(\Omega)} \mathcal{M}(\Omega)_{\mathcal{R}} e_{\mathcal{R}} e_\mathcal{T}. \quad (3.1)$$

Here the indices $\mathcal{R}, \mathcal{T}, \ldots$ enumerate the totality of indices $\{a, \{V_1V_2\}\}$, i.e. they enumerate the totality of variables $\{e_{\{V_1V_2\}}\} = \{e_{\mathcal{R}}\}$, and the matrix $\mathcal{M}(\Omega)_{\mathcal{R}\mathcal{T}}$ is the real functional of holonomy group elements $\Omega_{\mathcal{R}}$ and

$$\mathcal{M}_{\mathcal{R}\mathcal{T}} = \mathcal{M}_{\mathcal{T}\mathcal{R}}, \quad (3.2)$$

$$\mathcal{M}_{\mathcal{R}\mathcal{R}} = 0 \quad \text{(no summation)} \rightarrow \text{tr} \mathcal{M} = 0. \quad (3.3)$$

To understand the logic of the subsequent consideration, let us demonstrate the idea on a simple example. Let the partition function $f(x)$ be Gaussian:

$$f(x) = \frac{1}{\sqrt{2\pi a^2}} \exp \left( -\frac{x^2}{2a^2} \right), \quad \int_{-\infty}^{\infty} f(x) dx = 1. \quad (3.4)$$

We have:

$$\langle x \rangle = 0, \quad \langle x^2 \rangle \equiv \int_{-\infty}^{\infty} x^2 f(x) dx = a^2. \quad (3.5)$$

On the other hand, the result (3.5) can be obtained with the help of statistical mechanics methodology. For that purpose one must minimize the free energy of the system which is equal in our case to

$$F(x) = \frac{x^2}{2a^2} - \ln \Delta \Gamma, \quad \Delta \Gamma = \int_{-x}^{x} dp = 2|x| \rightarrow \ln \Delta \Gamma = \frac{1}{2} \ln(4a^2), \quad \frac{\partial F(x)}{\partial x} = \frac{x}{a^2} - \frac{1}{x}. \quad (3.6)$$

Thus the minimum value of free energy takes place at the points

$$\frac{\partial F(x)}{\partial x} \bigg|_{x=x_0} = 0 \rightarrow x_0^2 = a^2, \quad x_0 = \pm a. \quad (3.7)$$

From Eqs. (3.5) and (3.7) we have

$$\langle x^2 \rangle = x_0^2. \quad (3.8)$$

The interpretation of this result is evident: the most probable values (but not the mean value) for random variable $x$ with the partition function (3.4) are equal to the values $x_0 = \pm a$ (see (3.7)) due to the entropy $\ln \Delta \Gamma$, but the mean value $\langle x \rangle = 0$. So one can make the statement that the random variable $x$ fluctuates approximately within limits of

$$-a \lesssim x \lesssim a \quad (3.9)$$

with the mean values $\langle x \rangle$ and $\langle x^2 \rangle$ given by (3.5).
Further we use the statistical mechanics methodology in studies of the system (1.6), especially for the reason that some eigenvalues of the matrix (3.2) are negative. In the considered case the free energy has the form

\[ F(e) = \frac{1}{2l_P} \sum_{\Omega} \mathcal{M}(\Omega)_{R \tau} e_R e_T - \ln \Delta \Gamma, \]  
(3.10)

According to Eqs. (1.6) and (3.6) we define the phase space volume as

\[ \Delta \Gamma = \int [e_R] \prod_{R} (\Lambda d e_R) = \prod_{R} (2 | e_R |) = \prod_{R} 4e_R. \]  
(3.11)

Therefore the most probable values (but not the mean value) for random variables \( e_R \) satisfy the system of equations

\[ \frac{\partial F}{\partial e_R} = 0 \rightarrow \frac{1}{l_P} \sum_{\tau} \mathcal{M}(\Omega)_{R \tau} e_T = \frac{1}{e_R}. \]  
(3.12)

Only real solutions for Eq. (3.12) are interesting here.

Let’s modify, in some sense, the used approach.

Since the real matrix \( \mathcal{M}(\Omega) \) is symmetric matrix, so it can be diagonalized. Let \( \{ V^{(A)} \} \) be a complete orthonormal set of the real eigenvectors for matrix \( \mathcal{M}(\Omega) \):

\[ \sum_{\tau} \mathcal{M}(\Omega)_{R \tau} V^{(A)} = \lambda_A V^{(A)}_R, \quad \sum_{R} V^{(A)}_R V^{(B)}_R = \delta^{A B}. \]  
(3.13)

In consequence of Eqs. (3.2) and (3.3) all eigenvalues \( \lambda_A \) are real and

\[ \sum_{A} \lambda_A = 0. \]  
(3.14)

The last equation means that some of the eigenvalues are positive, while the rest of them are negative (or zeroth).

Any configuration \( \{ e_R \} \) can be presented as

\[ e_R = \sum_{A} p_A V^{(A)}_R, \]  
(3.15)

and the set of real numbers \( \{ p_A \} \) is in one-to-one correspondence with the set of real numbers \( \{ e_A \} \). Therefore the set of numbers \( \{ p_A \} \) can be considered as a set of independent variables. As a consequence of Eqs. (3.15) we have

\[ d \mu_e = \prod_{R} (\Lambda d e_R) = \prod_{A} (\Lambda d p_A) \rightarrow \Delta \Gamma = \prod_{A} 4p_A^2. \]  
(3.16)

So in new variables

\[ F(p) = \frac{1}{2l_P} \sum_{A} \lambda_A p_A^2 - \ln \left( \prod_{A} \sqrt{4p_A^2} \right), \]  
(3.17)

\[ \frac{\partial F}{\partial p_A} = 0 \rightarrow \frac{\lambda_A}{l_P} p_A = \frac{1}{p_A}, \]  
(3.18)

Introduce the designations \( \lambda_A(>0) \) for

\[ \lambda_A(>0) > 0, \quad \lambda_A(<0) < 0. \]  
(3.19)

From Eqs. (3.18) we obtain:

\[ p_A = \pm \frac{l_P}{\sqrt{\lambda_A(>0)}}, \quad p_A = \pm \frac{i l_P}{\sqrt{-\lambda_A(<0)}}. \]  
(3.20)

The solution (3.20) means that the set of variables \( \{ e_R \} \) is not real. To remove this inconsistency one must redefine the phase space volume \( \Delta \Gamma \). For that let us choose some two-dimensional subspace \( S^{(2)} \) of the linear space (3.15) with the basis \( \{ V^{(A)}_R \} \). The basis vectors in \( S^{(2)} \) are \( V^{(A)(>0)}_R \) and \( V^{(A)(<0)}_R \) with the corresponding eigenvalues \( \lambda_A(>0) \) and \( \lambda_A(<0) \). For brevity sake denote these quantities as \( V^{(1)}_R, V^{(2)}_R \) and \( \lambda_1 > 0, \lambda_2 < 0 \), correspondingly. Introduce new basis in \( S^{(2)} \):

\[ V^{(\pm)}_R = \frac{1}{\sqrt{2}} \left( V^{(1)}_R \pm V^{(2)}_R \right), \]  
\[ \sum_R \left( \frac{V^{(+)}_R}{V^{(-)}_R} \right)^2 = \sum_R \left( \frac{V^{(-)}_R}{V^{(+)}_R} \right)^2 = 1, \quad \sum_R V^{(+)}_R V^{(-)}_R = 0. \]  
(3.21)

The contribution of the subspace \( S^{(2)} \) into the space (3.15) is described as follows:

\[ \Delta \epsilon_R = x^{(+)}(+) + x^{(-)}(-), \]  
(3.22)

and so the phase space volume \( \Delta \Gamma \) becomes proportional to the factor

\[ \Delta \gamma = \sqrt{\left( 4x^{(+)}x^{(-)} \right)^2} \]  
(3.23)

instead of the factor \( \sqrt{(4p_1 p_2)}^2 \). Two real variables \( x^{(\pm)} \) are new degrees of freedom in the subspace \( S^{(2)} \). Now one must look for the stationary points of the contribution into free energy

\[ \Delta F(e) = \frac{1}{2l_P} \sum_{R \tau} \mathcal{M}(\Omega)_{R \tau} \Delta \epsilon_R \Delta \epsilon_T - \ln \Delta \gamma = \]  
\[ \frac{1}{4l_P} \left( (\lambda_1 + \lambda_2) \left( x^{(+)} + x^{(-)} \right)^2 + 2(\lambda_1 - \lambda_2) x^{(+)} x^{(-)} \right) - \frac{1}{2} \ln \left( 4x^{(+)}x^{(-)} \right)^2. \]  
(3.24)

The stationary points are the solutions of the system of equations

\[ \frac{\lambda_1 + \lambda_2}{2l_P} x^{(+)} + \frac{\lambda_1 - \lambda_2}{2l_P} x^{(-)} = \frac{1}{x^{(+)}}, \]  
\[ \frac{\lambda_1 + \lambda_2}{2l_P} x^{(-)} + \frac{\lambda_1 - \lambda_2}{2l_P} x^{(+)} = \frac{1}{x^{(-)}}. \]  
(3.25)
1) Consider the ordinary case

\[ \lambda_1 + \lambda_2 \neq 0. \]  
(3.26)

In this case there are only two real solutions

\[ x_{(+)} = x_{(-)} = \pm \frac{lp}{\sqrt{\lambda_1}}. \]  
(3.27)

So, according to (3.21), (3.22) and (3.27)

\[ \Delta e_R = \pm \frac{2}{\lambda_1} lp V_R^{(1)}. \]  
(3.28)

2) The degenerate case:

\[ \lambda_1 + \lambda_2 = 0. \]  
(3.29)

This case is realised on the subspace of measure zero in the full connection space with measure \( \left( \prod_{1 \leq i < j \leq 2} \Omega (v_i, v_j) \right) \) (see (1.6)). In the case we have the real solution family (hyperbola in the plane \( (x_{(+)}, x_{(-)}) \))

\[ x_{(+)} x_{(-)} = \frac{2 l_p^2}{\lambda_1 - \lambda_2} = l_p^2. \]  
(3.30)

In this case according to (3.21), (3.22) and (3.30)

\[ \Delta e_R = \frac{1}{\sqrt{2}} \left( x_{(+)} + \frac{l_p^2}{\lambda_1} x_{(+)} \right) V_R^{(1)} + \frac{1}{\sqrt{2}} \left( x_{(+)} - \frac{l_p^2}{\lambda_1} x_{(+)} \right) V_R^{(2)}, \quad x_{(+)} \neq 0. \]  
(3.31)

According to (3.24), (3.28) and (3.30) in both cases 1) and 2) the "energy" takes the same value on these solutions:

\[ \frac{1}{2l_p^2} \sum_{R \in T} \mathcal{M}(\Omega)_{R T} \Delta e_R \Delta e_T = 1. \]  
(3.32)

Note that the considered theory (1.6) is local. This means that the correlations between the different space-time regions drop quickly with their space separation increase. It follows from here that each space-time region contains some number of configurations \( V^2_R \) with positive eigenvalues \( \lambda_A(>0) \). Thus, the probability of the event when there is only a finite number of configurations with positive eigenvalues \( \lambda_A(>0) \) is equal to zero in the infinite space-time. In other words, there are infinite number of \( \lambda_A(>0) \) and infinite number of \( \lambda_A(<0) \). Therefore, each configuration with negative eigenvalue \( \lambda_A(<0) \) can be "coupled" with a configuration with positive eigenvalue \( \lambda_A(>0) \) in a manner described above. According to Eq. (3.28) (the possibility (3.31) we ignore) the most probable values for random variables \( e_R \) are

\[ e_R^{\text{probable}} = lp \sum_{A, \lambda_A(>0)} \pm \sqrt{\frac{2}{\lambda_A(>0)}} V_R^{(A)(>0)}, \]  
(3.33)

where the signs in all summands are mutually independent, so that there are \( 2^{N_{(>0)}} \) the most probable values for random variables \( e_R \). \( N_{(>0)} \) is the number of vectors \( V^{(A)(>0)}_R \). Moreover, according to (3.32) the probabilities of all these probable values are equal.

It is important that the obtained result (3.33) does not depend on the method of "coupling" of the vectors \( V^{(A)(>0)}_R \) and \( V^{(A)(<0)}_R \); the vector pair \( \left( V^{(A)(>0)}_R, V^{(A)(<0)}_R \right) \) can be chosen arbitrarily.

IV. CONCLUSION AND DISCUSSION

Eq. (3.33) shows the following:

(i) The fluctuation values of tetrads \( e_{v_i v_j} \) are commensurable with theirs most probable values; in pure gravity theory the fluctuations are symmetric about the zero \( e_{v_i v_j} \rightarrow -e_{v_i v_j} \) and the corresponding mean values

\[ \langle e_{v_i v_j} \rangle = 0. \]  
(4.1)

The obtained result justify the choice of model for propagation of irregular (doubled in Wilson sense) fermion quanta [18]: the propagation of irregular quanta on irregular "breathing" lattice is similar to the Markov process of a random walks. So it turns out that the propagator of irregular modes on irregular lattice decreases very quickly (exponentially); the doubled irregular modes are "bad" quasiparticles [1].

(ii) In the gravity theory coupled with fermions with the action \( \mathcal{A}_c = \mathcal{A}_c(E) + \mathcal{A}_c(E) \) the fluctuations symmetric about the zero \( e_{v_i v_j} \rightarrow -e_{v_i v_j} \) is broken. Indeed, the fermion part of action (1.2) is proportional to the third (but not second) power of the variables \( \{e\} \). The considered system is modelled by the partition function \( f(x) = A \exp \left( -\frac{2}{3} x^2 + \frac{6}{3} x^3 \right) \). Then one can easily find with the help of perturbation theory that \( \langle x \rangle = b/a^2 \).

So we have

\[ \langle e_{v_i v_j} \rangle \neq 0. \]  
(4.2)

(iii) It appears that the inequalities (4.2) cause the violation of PT-invariance. Indeed, the PT-transformation for the Dirac fermions and the holonomy elements in Euclidean signature is of the form of

\[ \Psi_{(E)\bar{V}}^{PT} = U_{PT} \left( \Psi_{(E)\bar{V}} \right)^t, \quad \Psi_{(E)\bar{V}}^{PT} = -\left( \Psi_{(E)\bar{V}} \right)^t U_{PT}^{-1}, \]

\[ U_{PT}^{-1} a_{(E)}^a U_{PT} = \left( a_{(E)}^a \right)^t, \quad U_{PT}^{-1} \sigma_{(E)}^{ab} U_{PT} = \left( \sigma_{(E)}^{ab} \right)^t, \]

\[ U_{PT} = a_{(E)}^\gamma. \]  
(4.3)

Here the upper index \( ^t \) means matrix transformation. Under the PT-transformation we have also:

\[ e^{PT} = -e, \quad U_{PT}^{-1} \Omega_{(E)\bar{V}ij} U_{PT} = \left( \Omega_{(E)\bar{V}ji} \right)^t. \]  
(4.4)
With the help of (1.3) and (1.4) the quantity (1.21) transforms as follows:

$$\hat{\Theta}_{(E)}^{PT} w_{ij} = - \hat{\Theta}_{(E)} w_{ij}. \tag{4.5}$$

The transformations (1.4) and (1.5) leaves the total action invariant:

$$\mathcal{A}_{(E)}^{PT} = \mathcal{A}_{(E)}. \tag{4.6}$$

This means that there is a global PT-invariance of the considered lattice theory.

Now let’s prepare a local state which is not PT-invariant. Denote the vacuum state as $|0\rangle$ and the combined creation operator of the state as $C^\dagger$. By definition $C \neq C^{PT}$. The experimenter can prepare the states $|C\rangle = C^\dagger |0\rangle$ and $|C^{PT}\rangle = (C^{PT})^\dagger |0\rangle$, but not the state $|C^{PT}\rangle = (C^{PT})^\dagger |0\rangle^{PT}$ since the vacuum state can not be changed by the efforts of experimenter. Therefore the reflection $\epsilon \rightarrow - \epsilon$ is absent in the experiment, and the evolutions of the states $|C\rangle$ and $|C^{PT}\rangle$ are not connected by the PT-transformation. But the evolutions of the states $|C\rangle$ and $|C^{PT}\rangle$ would be identical after replacing all degrees of freedom by the corresponding PT-transformed degrees of freedom.

The problem of discrete PT-symmetry in gravity is studied intensively. See e.g. the works [8]-[17].

(iv) The important question arises: how does the integration over holonomy elements $\{\Omega_{(E)}(\nu_1, \nu_2)\}$ in (1.6) affect the conclusions of the paper?

It seems that the conclusion in item (i) does not change. Indeed, the conclusion is true for almost all configurations of the holonomy elements $\{\Omega_{(E)}(\nu_1, \nu_2)\}$ (except for the configuration subset of measure zero). Therefore the summation over holonomy elements does not change the result but can only heighten an effect.

It seems also that the integration over holonomy elements does not violate the conclusions in items (ii) and (iii).

We give the following justification of the statement. Let’s perform Wilson recursion procedure. This means that the integration over short-wavelength degrees of freedom is made in the integral (1.7) including also the holonomy elements. Thus the effective action depending on more long-wavelength degrees of freedom arises. Then we repeat this action many times. As a result we come to the long-wavelength action (1.20), (1.21). But all statements made in Points (ii) and (iii) remain valid also for the action (1.20), (1.21). Note that Eq. (2.3) is equivalent to the following one:

$$d e^a + \omega^a_b \wedge e^b = 0.$$