A BORDISM THEORY RELATED TO MATRIX GRASSMANNIANS

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Abstract. In the present paper we study a bordism theory related to pairs \((M, \xi)\), where \(M\) is a closed smooth oriented manifold with a stably trivial normal bundle and \(\xi\) is a virtual SU-bundle of virtual dimension 1 over \(M\). The main result is the calculation of the corresponding ring modulo torsion and the explicit description of its generators.

Introduction

In the present paper we study the bordism theory related to pairs \((M, \xi)\), where \(M\) is a closed smooth oriented manifold with a stably trivial normal bundle and \(\xi\) is a virtual SU-bundle of virtual dimension 1 over \(M\). The bordism is defined with the help of analogous pairs \((W, \sigma)\), where \(W\) is a compact smooth oriented manifold with boundary \(\partial W\) and with a stably trivial normal bundle and \(\sigma\) is a virtual SU-bundle of virtual dimension 1 over \(W\), where the boundary operator \(\partial\) is defined as \(\partial(W, \sigma) = (\partial W, \sigma|_{\partial W})\). A ring structure is induced by the product \((M, \xi) \times (M', \xi') := (M \times M', \xi \boxtimes \xi')\).

Our main result is the calculation of the corresponding graded ring up to torsion elements, which turns out to be the polynomial ring \(\mathbb{Q}[t_2, t_3, \ldots]\), \(\deg t_n = 2n\), and the explicit description of the ring generators which have the form \(t_n = [S^{2n}, \xi^{(n)}]\), where \(\xi^{(n)}\) is the virtual SU-bundle of virtual dimension 1 that is the generator in the multiplicative group of such bundles over \(S^{2n}\), and the brackets \([,]\) denote the corresponding bordism class. Of course, \(S^{2n} = \partial D^{2n+1}\) but it is clear that the bundle \(\xi^{(n)}\) can not be extended to the whole ball.

Note that in contrast to “usual” bordisms, the stabilisation in our case does not correspond to the taking of Whitney sum with trivial bundles but with the tensor product by trivial bundles. Therefore in our case the Thom spaces are not stabilized by usual suspension (see Section 5) and the corresponding limit object is not a suspension spectrum.

It seems that the obtained results are closely related to [1].

1. Main definitions

Consider a pair \((M, \xi)\), where \(M\) is a closed smooth oriented manifold of dimension \(d\) with a stably trivial normal bundle and \(\xi \in \text{KSU}(M)\) is a virtual SU-bundle of virtual dimension 1 (here \(\text{KSU}\) denotes the \(K\)-functor related to SU-bundles).

Pairs \((M, \xi)\) and \((M', \xi')\), \(\dim M' = \dim M = d\) are called bordant if there exists a pair \((W, \sigma)\), where \(W\) is a compact \(d + 1\)-dimensional oriented manifold with boundary \(\partial W\) and with a stably trivial normal bundle and \(\sigma \in \text{KSU}(W)\), \(\dim \sigma = 1\) such that \(\partial W = M \bigcup (-M')\) and \(\sigma|_M = \xi\), \(\sigma|_{M'} = \xi'\) \((-M'\) denotes \(M'\) with reversed orientation).
Clearly that to be bordant is an equivalence relation\(^{1}\) and the corresponding equivalence classes \([M, \xi]\) of pairs \((M, \xi)\), \(\dim M = d\) form an abelian group with respect to the disjoint union which we denote by \(\Omega^d\). The product \([M, \xi] \times [M', \xi'] := [M \times M', \xi \boxtimes \xi']\) equips the direct sum \(\bigoplus_d \Omega^d\) with the structure of the graded ring \(\Omega^*\), (here \(\boxtimes\) denotes the “exterior” tensor product of virtual bundles).

We want to reduce the classification problem of pairs \((M, \xi)\) modulo bordism to the problem of the calculation of the homotopy groups of some Thom space. Let us briefly describe the corresponding argument.

Consider a pair \((M, \xi)\) as above. Let \(\eta \in \text{KSU}(M)\) be the inverse element for \(\xi\) with respect to the tensor product, i.e. \(\xi \otimes \eta = [1]\)\(^2\) where \([n]\) denotes a trivial \(\mathbb{C}^n\)-bundle over \(M\). Let \(k, l\) be a pair of relatively prime positive integers, i.e. their greatest common divisor \((k, l) = 1\). Assume that \(d = \dim M < 2\min\{k, l\}\). Then for virtual bundles \(k\xi, l\eta\) of dimensions \(k\) and \(l\) respectively there are geometric representatives \(\xi_k \to M\), \(\eta_l \to M\), i.e. “genuine” vector bundles, which are unique up to isomorphism. Moreover, \(\xi_k \otimes \eta_l \cong [kl]\) is a trivial bundle of dimension \(kl\). We will show that there is a natural bijection between virtual bundles \(\xi \in \text{KSU}(M)\), \(\dim \xi = 1\) and isomorphism classes of pairs \((\xi_k, \eta_l)\). Furthermore, such pairs are classified by so-called matrix Grassmannian \(\text{Gr}_{k,l}\) (defined below), i.e. there is a natural one-to-one correspondence between isomorphism classes of pairs \((\xi_k, \eta_l)\) over \(M\), \(\dim M < 2\min\{k, l\}\) and the set of homotopy classes \([M, \text{Gr}_{k,l}]\) of maps \(M \to \text{Gr}_{k,l}\). Using this result we will show that there is a natural one-to-one correspondence between bordism classes of pairs \((M, \xi)\), \(\dim M = d\) and homotopy groups \(\pi_{d+2kl}(\Theta_k(l))\) of the Thom space of the trivial \(\mathbb{C}^{kl}\)-bundle \(\vartheta_k(l) \to \text{Gr}_{k,l}\).

2. SU-bundles and matrix Grassmannians

In this section we recall in a suitable form some results from \([3]\).

Recall that the matrix Grassmannian \(\text{Gr}_{k,l}\) is a space which parametrizes unital \(*\)-subalgebras isomorphic to \(M_k(\mathbb{C})\) (“\(k\)-subalgebras”) in a fixed algebra \(M_{kl}(\mathbb{C})\). As a homogeneous space it can be represented in the form \(\text{PU}(kl)/(\text{PU}(k) \otimes \text{PU}(l))\) (here the symbol “\(\otimes\)” denotes the Kronecker product of matrices). In case \((k, l) = 1\) it can also be represented as

\[
\text{SU}(kl)/(\text{SU}(k) \otimes \text{SU}(l)).
\]

The tautological \(M_k(\mathbb{C})\)-bundle \(\mathcal{A}_{k,l} \to \text{Gr}_{k,l}\) is the subbundle of the direct product \(\text{Gr}_{k,l} \times M_{kl}(\mathbb{C})\) consisting of pairs \(\{(x, T) \mid x \in \text{Gr}_{k,l}, T \in M_{k,x} \subset M_{kl}(\mathbb{C})\}\), where \(M_{k,x}\) denotes the \(k\)-subalgebra corresponding to a point \(x \in \text{Gr}_{k,l}\). Let \(\mathcal{B}_{k,l} \to \text{Gr}_{k,l}\) be the \(M_l(\mathbb{C})\)-bundle formed by fiberwise centralizers to the subbundle \(\mathcal{A}_{k,l} \subset \text{Gr}_{k,l} \times M_{kl}(\mathbb{C})\). Clearly, there is the canonical trivialization

\[
\mathcal{A}_{k,l} \otimes \mathcal{B}_{k,l} \cong \text{Gr}_{k,l} \times M_{kl}(\mathbb{C}).
\]

\(^1\)the transitivity follows from the exactness of \(\text{KSU}(W_1 \cup W_2) \to \text{KSU}(W_1) + \text{KSU}(W_2) \to \text{KSU}(W_1 \cap W_2)\)

\(^2\)if \(\xi = 1 + \xi\), where \(\xi \in \text{KSU}(X)\), then \(\eta = 1 - \xi + \xi^2 - \ldots\), but \(\xi^r = 0\) because \(M\) is compact
We are going to prove that for manifolds $M$ Proposition 3. If $\phi$ is unique up to homotopy (see Proposition 4). It requires some preparation.

The map $\lambda_{k, l}$ (6) $Gr^k \times Gr^l \rightarrow Gr_{k, l}$

Let $\xi_{k, l} \rightarrow Gr^k, \eta_{k, l} \rightarrow Gr^l$ be vector $C^k$ and $C^l$-bundles associated with principal bundles (3) and (4). There are isomorphisms $A_{k, l} \cong \text{End}(\xi_{k, l})$, $B_{k, l} \cong \text{End}(\eta_{k, l})$ and the canonical trivialization $\phi: \xi_k \otimes \eta_l \cong M \times C^k$.

Proof. For a given map $f: M \rightarrow Gr_{k, l}$ the triple $(\xi_k, \eta_l, \varphi)$ is defined as follows: $\xi_k := f^*(\xi_{k, l}), \eta_l := f^*(\eta_{k, l})$ and the trivialization $\varphi$ is induced by $f^*$. Conversely, for a given triple $(\xi_k, \eta_l, \varphi)$ over $M$ as in the statement of the proposition the trivialization $\varphi$ determines the trivialization $\text{End}(\varphi): \text{End}(\xi_k \otimes \eta_l) \cong M \times M_{kl}(C)$ of the bundle $

\text{End}(\xi_k \otimes \eta_l) = \text{End}(\xi_k) \otimes \text{End}(\eta_l).$ 

The map $\xi_k \otimes \eta_l$ can be considered as a family of unital $k$-subalgebras in a fixed algebra $M_{kl}(C)$, hence we obtain the required map $f: M \rightarrow Gr_{k, l}$. □

Two triples $(\xi_k, \eta_l, \varphi), (\xi'_k, \eta'_l, \varphi')$ over $M$ are called equivalent if $\xi_k \cong \xi'_k, \eta_l \cong \eta'_l$ and $\varphi$ is homotopic to $\varphi'$ in the class of trivializations.

Corollary 2. There is a natural one-to-one correspondence between equivalence classes of triples $(\xi_k, \eta_l, \varphi)$ over $M$ and the set $[M, Gr_{k, l}]$ of homotopy classes of maps $M \rightarrow Gr_{k, l}$.

Proof easily follows from the previous proposition. □

Let $\lambda_{k, l}: Gr_{k, l} \rightarrow BSU(k)$ be a classifying map for the principal $SU(k)$-bundle (3) (i.e. for the vector bundle $\xi_{k, l}$), $\mu_{k, l}: Gr_{k, l} \rightarrow BSU(l)$ a classifying map for the principal $SU(l)$-bundle (4) (i.e. for the vector bundle $\eta_{k, l}$).

Consider the fibration (cf. (1))

\[
Gr_{k, l} \xrightarrow{\lambda_{k, l} \times \mu_{k, l}} BSU(k) \times BSU(l) \rightarrow BSU(kl).
\]

The map $\lambda_{k, l} \times \mu_{k, l}$ corresponds to the functor $(\xi_k, \eta_l, \varphi) \mapsto (\xi_k, \eta_l)$ which forgets trivialization $\varphi$.

We are going to prove that for manifolds $M$ of dimension $\text{dim } M < 2 \min\{k, l\}$ such a trivialization $\varphi$ is unique up to homotopy (see Proposition 4). It requires some preparation.

Proposition 3. If $(km, ln) = 1$ then the embedding $Gr_{k, l} \rightarrow Gr_{km, ln}$ induced by a unital $*$-homomorphism $M_{kl}(C) \rightarrow M_{klmn}(C)$ induces an isomorphism of homotopy groups up to dimension $\sim 2 \min\{k, l\}$. 

\[
\xi_{k, l} \sim \eta_{k, l} \sim \xi_{k' = 2k \times l} \sim \eta_{k' = 2k \times l} \sim \xi_{k'' = 2k' \times l} 
\]

\[
\xi_{k, l} \sim \eta_{k, l} \sim \xi_{k' = 2k \times l} \sim \eta_{k' = 2k \times l} \sim \xi_{k'' = 2k' \times l} 
\]
Proof follows from the representation (11) and the sequence of homotopy groups of the corresponding fibration, see [3]. □

The proven proposition implies that the homotopy type of the direct limit $\lim_{(k_i, l_i)} \text{Gr}_{k_i, l_i}$ does not depend on the choice of a sequence of pairs $\{k_i, l_i\}$ of positive integers provided $(k_i, l_i) = 1$, $k_i|k_{i+1}$, $l_i|l_{i+1} \forall i$ and $k_i, l_i \to \infty$ when $i \to \infty$. This homotopy type we will denote by $\text{Gr}$.

The space $\text{Gr}$ has the natural $H$-space structure induced by maps $\text{Gr}_{k, l} \times \text{Gr}_{m, n} \to \text{Gr}_{km, ln}$, $(km, ln) = 1$ defined by the tensor product of matrix algebras $M_{kl}(\mathbb{C}) \times M_{mn}(\mathbb{C}) \to M_{kl}(\mathbb{C}) \otimes M_{mn}(\mathbb{C})$. By $\text{Gr}$ we will also denote this $H$-space.

Put $\text{BSU}(k^n) := \lim_{n} \text{BSU}(k^n)$, $\text{BSU}(l^n) := \lim_{n} \text{BSU}(l^n)$, where direct limits are taken over maps induced by tensor products. We consider these spaces as $H$-spaces with the multiplication induced by the tensor product of the corresponding bundles.

A simple calculation with homotopy groups shows that $\text{Gr}$ has the same homotopy groups as $\text{BSU}_\otimes$ and the maps $\lambda_{k^n, l^n} : \text{Gr} \to \text{BSU}(k^n)$, $\mu_{k^n, l^n} : \text{Gr} \to \text{BSU}(l^n)$ are the localizations over $k$ and $l$ respectively (in the sense that $k$ and $l$ become invertible). Moreover, these localizations are $H$-spaces homomorphisms. This implies that $\text{Gr}$ is isomorphic to $\text{BSU}_\otimes$ as an $H$-space (recall that the product in $\text{BSU}_\otimes$ is induced by the tensor product of virtual bundles of virtual dimension 1).

**Proposition 4.** Assume that $\dim M < 2 \min\{k, l\}$. Then for a classifying map $M \to \text{BSU}(k) \times \text{BSU}(l)$ of a pair $(\xi_k, \eta_l)$ such that $\xi_k \otimes \eta_l \cong [kl]$ a lift $M \to \text{Gr}_{k, l}$ in (6) exists and is unique up to homotopy.

**Proof.** Assume that a map $\bar{f} = \bar{f}_1 \times \bar{f}_2 : M \to \text{BSU}(k) \times \text{BSU}(l)$ classifies the pair of bundles $(\xi_k, \eta_l)$ as in the proposition statement, i.e. $\xi_k = \bar{f}_1^*(\xi_k, l)$, $\eta_l = \bar{f}_2^*(\eta_k, l)$, and moreover $\xi_k \otimes \eta_l \cong [kl]$. Then $\otimes \circ \bar{f} \simeq \ast$ (see (9)) and it follows from the exactness of (6) that there exists some lift $f : M \to \text{Gr}_{k, l}$, i.e. $\lambda_{k, l} \circ f \simeq \bar{f}_1$, $\mu_{k, l} \circ f \simeq \bar{f}_2$.

In order to prove the uniqueness up to homotopy of the lift provided $\dim M < 2 \min\{k, l\}$ let us use the above introduced direct limits. Recall that $\lambda_{k^n, l^n} : \text{Gr} \to \text{BSU}(k^n)$, $\mu_{k^n, l^n} : \text{Gr} \to \text{BSU}(l^n)$ are the localizations over $k$ and $l$ respectively. Together with the condition $(k, l) = 1$ this implies that the map

$$\lambda_{k^n, l^n} \times \mu_{k^n, l^n} : \text{Gr} \to \text{BSU}(k^n) \times \text{BSU}(l^n)$$

(see (6)) induces an injective homomorphism of groups

$$[M, \text{Gr}] \to [M, \text{BSU}(k^n) \times \text{BSU}(l^n)]$$

of homotopy classes of maps. Now using Proposition 3 we obtain the required assertion. □

Recall (see the end of the previous section) that given a virtual bundle $\xi \in \text{KSU}(M)$, $\dim \xi = 1$ and a pair $k, l$, $(k, l) = 1$, $\dim M < 2 \min\{k, l\}$ we can find a unique up to isomorphism pair of geometric bundles $\xi_k, \eta_l$ such that $\xi_k \otimes \eta_l \cong [kl]$ which (according to the proven proposition) defines a classifying map $f : M \to \text{Gr}_{k, l}$ unique up to homotopy.
Conversely, for a given map \( f : M \to \text{Gr}_{k,l} \) we want to define a virtual bundle \( \xi \in \text{KSU}(M) \), \( \dim \xi = 1 \).

Let \( m, n \) be another pair of positive integers such that \((km, ln) = 1 = (k, m)\), \( \dim M < 2 \min\{m, n\}\). Consider the diagram

\[
\text{Gr}_{k,l} \overset{i}{\rightarrow} \text{Gr}_{km,ln} \overset{j}{\leftarrow} \text{Gr}_{m,n},
\]

where maps \( i \) and \( j \) are induced by matrix algebra homomorphisms. It follows from Proposition \( 3 \) that for \( f := f_{k,l} \) there exists a unique up to homotopy map \( f_{m,n} : M \to \text{Gr}_{m,n} \) such that \( i \circ f_{k,l} \simeq j \circ f_{m,n} : M \to \text{Gr}_{km,ln} \). Moreover, \( i^*(\xi_{km,ln}) \cong \xi_{k,l} \otimes [m], j^*(\xi_{km,ln}) \cong \xi_{m,n} \otimes [k] \).

Hence for the bundle \( \xi_k := f_{k,l}^*(\xi_{k,l}) \) over \( M \) there exists the bundle \( \xi_m := f_{m,n}^*(\xi_{m,n}) \) such that \( \xi_k \otimes [m] \cong \xi_m \otimes [k] \), hence the relation \( m\xi_k = k\xi_m \) in the K-functor.

Suppose \( u, v \) be a pair of integers such that \( uk + vm = 1 \) (recall that we have chosen \( m \) such that \((k, m) = 1\)). Then \( \xi_k = uk\xi_k + vm\xi_k = uk\xi_k + vk\xi_m = k(u\xi_k + v\xi_m) \). Suppose \( \xi := u\xi_k + v\xi_m \), then \( \xi \in \text{KSU}(M) \), \( \dim \xi = 1 \). Moreover, \( m\xi = um\xi_k + vm\xi_m = (uk + vm)\xi_m = \xi_m \). Thereby to a map \( f : M \to \text{Gr}_{k,l} \) we assign a virtual bundle \( \xi \in \text{KSU}(M) \) of virtual dimension 1, and hence we have a bijection \([M, \text{Gr}_{k,l}] \overset{\cong}{\to} 1 + \text{KSU}(M)\). It is easy to see that this bijection can be extended to the group isomorphism \([M, \text{Gr}] \overset{\cong}{\to} (1 + \text{KSU}(M))^\times = [M, \text{BSU}_\otimes] \) (this time without any condition on \( \dim M \)). In particular, we again have established the H-space isomorphism \( \text{Gr} \cong \text{BSU}_\otimes \).

In particular, we have proven the following theorem.

**Theorem 5.** For any pair \((M, \xi)\) such that \( \dim M < 2 \min\{k, l\} \) there exists a unique up to homotopy map \( f_{\xi} : M \to \text{Gr}_{k,l} \) representing \( \xi \) (in the sense that \( \xi \) can be uniquely restored by the pair \( f_{\xi}^*(\xi_{k,l}), f_{\xi}^*(\eta_{k,l}) \)).

Note that two pairs \((\xi_k, \eta_l)\) and \((\xi_m, \eta_n)\) provided \((km, ln) = 1\) correspond to the same bundle \( \xi \) if \( \xi_k \otimes [m] \cong \xi_m \otimes [k], \eta_k \otimes [n] \cong \eta_n \otimes [l] \) (cf. \( 17 \)). In general (without assumption \((km, ln) = 1\)) we have to take the transitive closure of this relation.

### 3. Bordism of Triples

In this section using the obtained results we replace pairs \((M, \xi)\) by some triples \((M, \xi_k, \eta_l)\) of more geometric nature.

Let \( M \), \( \dim M = d \) be a smooth oriented manifold with a stably trivial normal bundle, \( f : M \to \mathbb{R}^{d+N} \) a smooth embedding, in addition we assume that the trivial normal bundle \( \nu \cong M \times \mathbb{R}^N \) is equipped with an almost complex structure \( \Rightarrow 2 \mid N \) and moreover there is a representation \( \nu \cong \xi_k \otimes \eta_l \Rightarrow N = 2kl \) as the tensor product of (complex) vector bundles \( \xi_k, \eta_l \) over \( M \) of dimensions \( k, l, (k, l) = 1 \) and with structural groups \( \text{SU}(k) \) and \( \text{SU}(l) \) respectively. If \( d < 2 \min\{k, l\} \), then, according to the previous section, the pair \((\xi_k, \eta_l)\) determines a classifying map \( M \to \text{Gr}_{k,l} \) which is unique up to homotopy. Therefore we can replace pairs \((M, \xi)\) by equivalent triples \((M, \xi_k, \eta_l)\).

Let \( W \), \( \dim W = d + 1 \) be a compact oriented manifold with boundary \( \partial W \) and with trivial normal bundle \( \nu_W \) for an embedding \( F : W \to \mathbb{R}^{d+1+N}, F(\partial W) \subset \mathbb{R}^{d+N} \), moreover, \( \nu_W = \sigma_k \otimes \rho_l \) for some vector bundles \( \sigma_k, \rho_l \) with structural groups \( \text{SU}(k), \text{SU}(l) \) respectively. Then we can
define a boundary operator as follows:

\[ \partial(W, \sigma_k, \rho_l) = (\partial W, \sigma_k|_{\partial W}, \rho_l|_{\partial W}). \]

In particular, for \( W = M \times I \), \( \sigma_k = \hat{\xi}_k := \pi^*(\xi_k) \), \( \rho_l = \hat{\eta}_l := \pi^*(\eta_l) \), where \( \pi \) is the projection onto the first factor \( M \times I \to M \) we have:

\[ \partial(M \times I, \hat{\xi}_k, \hat{\eta}_l) = (M, \xi_k, \eta_l) \sqcup (-M, \xi_k, \eta_l). \]

Furthermore, we can define an equivalence relation: two triples are bordant if they become isomorphic (in the natural sense) after taking the disjoint union with boundaries. It follows from the previous section that there is a natural one-to-one correspondence between bordism classes of triples \((M, \xi_k, \eta_l)\) and bordism classes of pairs \((M, \xi)\) as we have defined in Section 1.

In order to take into account the possibility of choices of pairs of bundles \((\xi_k, \eta_l)\) of different dimensions \(k, l\), related to a virtual bundle \(\xi\), we have to extend the equivalence relation. It is generated by the equivalence between \((M, \xi_k, \eta_l)\) and \((M, \xi_k \otimes [m], \eta_l \otimes [n])\) provided \((km, ln) = 1\) (cf. Proposition 3).

### 4. Thom spaces

Suppose we are given a triple \((M, \xi_k, \eta_l)\) and a bundle \(\nu \cong \xi_k \otimes \eta_l \cong M \times \mathbb{R}^N\) as in the previous section. Then, according to Proposition 4, we have a unique up to homotopy map \(f: M \to \text{Gr}_{k,l}\) which classifies the pair \((\xi_k, \eta_l)\). That is we have the map of trivial bundles

\[
\begin{array}{ccc}
\nu & \xrightarrow{\partial_{k,l}} & \text{Gr}_{k,l} \\
\downarrow & & \downarrow \\
M & \xrightarrow{f} & \text{Gr}_{k,l}
\end{array}
\]

which is compatible with the representations \(\nu = \xi_k \otimes \eta_l\), \(\partial_{k,l} = \xi_k \otimes \eta_{k,l}\) in the form of tensor products (see [5]), i.e. \(f^*(\partial_{k,l}) = f^*(\xi_k) \otimes f^*(\eta_l) \cong \xi_k \otimes \eta_l = \nu\), and it can be extended to the map \(\varphi\) of their one-point compactifications, i.e. the Thom spaces \(\varphi: T(\nu) \to T(\partial_{k,l})\). Then the composition of the map \(S^{d+N} \to T(\nu)\) \((N = 2kl)\) contracting the complement to a tubular neighborhood for the embedded manifold \(M \subset S^{d+N}\) to the base point with the map \(\varphi\) defines some map \(S^{d+N} \to T(\partial_{k,l})\). It is easy to see that a bordism between \((M, \xi_k, \eta_l)\) and some other triple determines a homotopy \(S^{d+N} \times I \to T(\partial_{k,l})\). So we can assign some element of \(\pi_{d+N}(T(\partial_{k,l}))\) to the bordism class of a triple \((M, \xi_k, \eta_l)\).

The standard argument using t-regularity to the smooth submanifold \(\text{Gr}_{k,l} \subset T(\partial_{k,l}) - \{\ast\}\) (where \(\{\ast\}\) is the base point of the Thom space) shows that, conversely, we can assign the bordism class of some triple \((M, \xi_k, \eta_l)\) to an element of the group \(\pi_{d+N}(T(\partial_{k,l}))\).

Thus we have proven the following theorem.

**Theorem 6.** The above described correspondence defines an isomorphism between the group of bordism classes of triples \((M, \xi_k, \eta_l), \dim M = d\) and the homotopy group \(\pi_{d+N}(T(\partial_{k,l}))\).
According to the previous results (concerning the relation between virtual bundles $\xi$ of virtual dimension 1 with pairs $(\xi_k, \eta_l)$) we also have an isomorphism between the group of bordisms of pairs $(M, \xi)$ as in Section 1 and the homotopy group $\pi_{d+N}(T(\vartheta_{k,l}))$.

5. Stabilization

In contrast with “usual” bordism theories, in our case the stabilization is related to the tensor product of bundles, therefore we have to use another functor instead of the suspension.

According to the above theorem, for any element of $\pi_{d+N}(T(\vartheta_{k,l}))$, $N = 2kl$, $d < 2\min\{k, l\}$ there exists a well-defined bordism class $[M, \xi_k, \eta_l]$, $\dim M = d$. Consider the triple $(M, \xi_k \otimes [m], \eta_l \otimes [n])$, $(km, ln) = 1$ and the corresponding map $S^{d+2klmn} \to T(\vartheta_{km, ln})$. It is easy to see that the corresponding element $\pi_{d+2klmn}(T(\vartheta_{km, ln}))$ is well defined by the bordism class of the triple $(M, \xi_k, \eta_l)$, and therefore we obtain a homomorphism $\pi_{d+N}(T(\vartheta_{k,l})) \to \pi_{d+2klmn}(T(\vartheta_{km, ln}))$.

**Proposition 7.** If $d < 2\min\{k, l\}$ then the above defined homomorphism $\pi_{d+N}(T(\vartheta_{k,l})) \to \pi_{d+2klmn}(T(\vartheta_{km, ln}))$ is an isomorphism.

**Proof.** 1) Surjectivity. Using the $t$-regularity argument, we see that every element of $\pi_{d+2klmn}(T(\vartheta_{km, ln}))$ comes from some triple $(M, \xi_{km}, \eta_{ln})$, $\dim M = d$. Since, according to Proposition 3 the inclusion $Gr_{k,l} \to Gr_{km, ln}$ for $(km, ln) = 1$ is a homotopy equivalence up to dimension $2\min\{k, l\}$, we see that for $d < 2\min\{k, l\}$ a classifying map $M \to Gr_{km, ln}$ for the pair $(\xi_{km}, \eta_{ln})$ comes from some map $M \to Gr_{k,l}$, i.e. the triple $(M, \xi_{km}, \eta_{ln})$ comes from some triple $(M, \xi_k, \eta_l)$ as described above.

2) Injectivity. Given a homotopy between two maps $S^{d+2klmn} \to T(\vartheta_{km, ln})$ we have the corresponding bordism given by a $d+1$-dimensional manifold with boundary. Using the same argument as in item 1), we see that already the corresponding maps $S^{d+N} \to T(\vartheta_{k,l})$ are homotopic. □

**Remark 8.** Note that for $d < 2\min\{m, n\}$ we also have an isomorphism $\pi_{d+2mn}(T(\vartheta_{m,n})) \to \pi_{d+2klmn}(T(\vartheta_{km, ln}))$. Hence the group $\pi_{d+2mn}(T(\vartheta_{m,n}))$ does not depend on the choice of $m, n$, $(m, n) = 1$.

So, the bordism group $\Omega^d$ can be defined as the direct limit $\lim_{(k, l)=1} \pi_{d+2kl}(T(\vartheta_{k,l}))$ which is stabilized from some dimension.

6. The ring structure

Let $(M, \xi_k, \eta_l)$, $(M', \xi'_m, \eta'_n)$ be two triples as above. Then

$$(M, \xi_k, \eta_l) \times (M', \xi'_m, \eta'_n) := (M \times M', \xi \otimes \xi', \eta \otimes \eta')$$

is a new triple of the same kind (here “$\otimes$” denotes the “exterior” tensor product of bundles).

If $f : S^{d+2k} \to T(\vartheta_{k,l})$, $f' : S^{d'+2mn} \to T(\vartheta_{m,n})$ classify triples $(M, \xi_k, \eta_l)$, $(M', \xi'_m, \eta'_n)$ respectively, then the triple $(M \times M', \xi_k \otimes \xi'_m, \eta_l \otimes \eta'_n)$ is classified by some map $S^{d+d'+2klmn} \to T(\vartheta_{km, ln})$.

Note that the stabilization introduced in the previous section corresponds to the product by $(M', \xi'_m, \eta'_n) = (pt, \mathbb{C}^n, \mathbb{C}^n)$. 

It is easy to see that the introduced product of triples defines the structure of a graded ring on their bordism classes, moreover (because of the $H$-space isomorphism $\text{Gr} \cong \text{BSU}_\otimes$) it coincides with the one introduced in Section 1 on the bordism group of pairs $(M, \xi)$, $\xi \in \text{KSU}(M)$, $\dim \xi = 1$.

7. The calculation of the ring $\Omega^* \otimes \mathbb{Q}$

In this section we compute the ring $\Omega^* \otimes \mathbb{Q}$. First let us formulate two obvious corollaries from the classical Theorems [2].

**Theorem 9.** Since the trivial bundle $\vartheta_{k,l} \to \text{Gr}_{k,l}$, clearly, is orientable, we have the Thom isomorphism $H_d(\text{Gr}_{k,l}, \mathbb{Z}) \cong H_{d+2kl}(T(\vartheta_{k,l}), \mathbb{Z})$.

**Theorem 10.** Since the Thom space $T(\vartheta_{k,l})$ is $(2kl - 1)$-connected, we see that the Hurewicz homomorphism $\pi_{d+2kl}(T(\vartheta_{k,l})) \to H_{d+2kl}(T(\vartheta_{k,l}), \mathbb{Z})$ is a $C$-isomorphism for $d < 2kl - 1$. Here $C$ is the Serre class of finite abelian groups.

Since the space $\text{Gr}_{k,l}$ is homotopy equivalent to $\text{BSU}$ up to dimension $\sim 2 \min\{k, l\}$, we see that in this dimensions $\text{rk} H_d(\text{Gr}_{k,l}, \mathbb{Z})$ is equal to 0 for $d$ odd and the number of partitions $\frac{d}{2}$ into the sum of 2, 3, 4, . . . for $d$ even. Actually, we will show that $\Omega^* \otimes \mathbb{Q} \cong \mathbb{Q}[t_2, t_3, t_4, \ldots]$, where $\deg t_n = 2n$. Moreover, one can take the bordism class of the triple $(S^{2n}, \xi_k, \eta_n)$, $n < \min\{k, l\}$ as $t_n$, where $(\xi_k, \eta_n)$ is the generator (i.e. its classifying map $S^{2n} \to \text{Gr}_{k,l}$ represents the generator in $\pi_{2n}(\text{Gr}_{k,l}) \cong \mathbb{Z}$). In other words, the pair $(\xi_k, \eta_n)$ corresponds to the generator $\xi \in \text{KSU}(S^{2n})$, $\dim \xi = 1$ according to the correspondence described in Section 2.

In order to calculate $\Omega^* \otimes \mathbb{Q}$ it is sufficient to consider only rational characteristic classes. By analogy with Pontryagin’s theorem, we can prove the following result:

**Theorem 11.** For a pair $(M, \xi)$ as in Section 1 and an arbitrary characteristic class $\alpha(\xi) \in H^d(M, \mathbb{Q})$ of the bundle the characteristic number $\langle \alpha(\xi), [M] \rangle \in \mathbb{Q}$ (where $[M] \in H_d(M, \mathbb{Q})$ is the fundamental homology class of the manifold $M$) depends only on the bordism class $[M, \xi]$.

Consider a pair $(M, \xi)$ as above and the Chern character $ch(\xi) = 1 + ch_2(\xi) + ch_3(\xi) + \ldots$, $ch_n(\xi) \in H^{2n}(M, \mathbb{Q})$ ($ch_1(\xi) = 0$ because $\xi$ is a virtual SU-bundle).

**Remark 12.** If a pair $(M, \xi)$ corresponds to a triple $(M, \xi_k, \eta_n)$, then $ch(\xi) = \frac{ch(\xi_k)}{k}$. Note that $\frac{ch(\xi_k)}{k}$ is well-defined if pairs $(\xi_k, \eta_n)$, $(\xi_m, \eta_n)$ are equivalent in the sense that their classifying maps $M \to \text{Gr}_{k,l}$, $M \to \text{Gr}_{m,n}$ are homotopic as maps to $\text{Gr}_{km,ln}$, see (7).

Let $\{(S^{2n}, \xi^{(n)} | n \geq 2\}$ be the collection of pairs such that $ch_n(\xi^{(n)}) = \eta_n$, where $\eta_n \in H^{2n}(S^{2n}, \mathbb{Z}) \subset H^{2n}(S^{2n}, \mathbb{Q})$ is the generator (recall that the Chern character takes integer values on spheres). Then elements $\xi^{(n)} \in 1 + \text{K\bar{S}}U(S^{2n})$ themselves are generators (note that $(1 + \text{K\bar{S}}U(S^{2n}))^\times \cong \mathbb{Z}$). Let $\xi^{(n)}$ be the $k$-th power of the bundle $\xi^{(n)}$, then $ch_n(\xi^{(n)k}) = k\eta_n$. (Indeed, $\xi^{(n)} = 1 + \bar{\xi}^{(n)}$, $(1 + \bar{\xi}^{(n)})^k = 1 + k\bar{\xi}^{(n)}$, because $\bar{\xi}^{(n)2} = 0$ in the ring $\text{K\bar{S}}U(S^{2n})$).

**Proposition 13.** $[S^{2n}, \xi^{(n)k}] = k[S^{2n}, \xi^{(n)}]$ in the group $\Omega^{2n} \otimes \mathbb{Q}$.

**Proof.** We have: $\langle ch_n(\xi^{(n)k}), [S^{2n}] \rangle = k = k\langle ch_n(\xi^{(n)}), [S^{2n}] \rangle$. From the other hand, $H^*(\text{BSU}, \mathbb{Q}) = \mathbb{Q}[ch_2, ch_3, \ldots]$, hence the products of the form $ch_{n_1}\ldots ch_{n_r}$, $2 \leq n_1 \leq \ldots \leq
$n_r$, $n_1 + \ldots + n_r = n$, $r \geq 1$ form an additive basis of $H^{2n}(\text{BSU}, \mathbb{Q})$. The required assertion now follows from Theorems 9 and 10, additivity of characteristic numbers (with respect to the addition of bordism classes) and from the fact that $\text{ch}_m(\xi^{(n)k}) = 0$ for $m \neq n$. □

Note that the existence of a bordism $(S^{2n}, \xi^{(n)k}) \sqcup (S^{2n}, \xi^{(n)^l}) \sim (S^{2n}, \xi^{(n)+l})$ in $\Omega^{2n}$ can be perceived from a geometric argument.

Note also that for the inverse element $-[S^{2n}, \xi^{(n)}] = [-S^{2n}, \xi^{(n)}]$, where $\xi^{(n)}(n)$ is the inverse element for $\xi^{(n)}$ in the group $(1 + \text{KSU}(S^{2n}))^{\times}$. This is connected with the existence of the orientation-reversing diffeomorphism (for instance, the antipodal map) $f : S^{2n} \to S^{2n}$ such that $\xi^{(n)}(n) \cong f^*(\xi^{(n)})$. Thus, $-[S^{2n}, \xi^{(n)}] = [S^{2n}, \xi^{(n)}]^{-1}$.

Now we want to prove that the classes $[S^{2n_1} \times \ldots \times S^{2n_r}, \xi^{(n_1)} \otimes \ldots \otimes \xi^{(n_r)}] = [S^{2n_1}, \xi^{(n)}] \ldots [S^{2n_r}, \xi^{(n)}], 2 \leq n_1 \leq \ldots \leq n_r$, $n_1 + \ldots + n_r = n$, $r \geq 1$ form an additive basis of $\Omega^{2n} \otimes \mathbb{Q}$.

**Proposition 14.** $\langle \text{ch}_{n_1} \ldots \text{ch}_{n_r}(\xi^{(n_1)} \otimes \ldots \otimes \xi^{(n_r)}) \rangle \neq 0$ only if the partition $n_1 \ldots n_r$ of $n$ is a refinement of the partition $m_1 \ldots m_s$.

**Proof.** We have:

$$\text{ch}(\xi^{(n_1)} \otimes \ldots \otimes \xi^{(n_r)}) = (1 + \iota_{n_1}) \otimes \ldots \otimes (1 + \iota_{n_r}),$$

whence $\text{ch}_m(\xi^{(n_1)} \otimes \ldots \otimes \xi^{(n_r)})$ is the degree $2m$ homogeneous component of the right-hand side of (3). Multiplying the obtained expressions, we get the required assertion. □

Let $p'(n)$ be the partition number of writing $n$ as a sum of numbers $2, 3, \ldots , n$ (with 1 omitted).

**Theorem 15.** (cf. [2]) $p'(n) \times p'(n)$-matrix consisting of characteristic numbers

$$\langle \text{ch}_{m_1} \ldots \text{ch}_{m_s}(\xi^{(n_1)} \otimes \ldots \otimes \xi^{(n_r)}) \rangle \neq 0 \text{ for } [S^{2n_1} \times \ldots \times S^{2n_r}],$$

where $m_1 \ldots m_s$ and $n_1 \ldots n_r$ runs over all partitions of $n$ into a sum of positive integers $\neq 1$ is nonsingular.

**Proof.** There is a partial order on the set of partitions of $n$ defined by refinement. Extending it to a total order, we obtain the corresponding $p'(n) \times p'(n)$-matrix consisting of numbers as in the statement of the theorem. According to the previous proposition, this matrix is a lower triangular with zeros over the main diagonal, while its diagonal elements

$$\langle \text{ch}_{n_1} \ldots \text{ch}_{n_r}(\xi^{(n_1)} \otimes \ldots \otimes \xi^{(n_r)}) \rangle \neq 0$$

clearly are nonzero. Hence the asserted nonsingularity. □

**Example 16.** Take $n = 6$. We have 4 partitions which we order as follows: (2 2 2), (3 3), (2 4), 6. For $\xi^{(2)} \otimes \xi^{(2)} \otimes \xi^{(2)}$ over $S^4 \times S^4 \times S^4$ we have:

$$\text{ch}(\xi^{(2)} \otimes \xi^{(2)} \otimes \xi^{(2)}) = (1 + \iota_2) \otimes (1 + \iota_2) \otimes (1 + \iota_2) = 1 \otimes 1 \otimes 1 + \iota_2 \otimes 1 \otimes 1 + 1 \otimes \iota_2 \otimes 1 + 1 \otimes 1 \otimes \iota_2 +$$

$$+ \iota_2 \otimes \iota_2 \otimes 1 + \iota_2 \otimes 1 \otimes \iota_2 + 1 \otimes \iota_2 \otimes \iota_2 + \iota_2 \otimes \iota_2 \otimes \iota_2.$$
whence

\[ ch_2(\xi(2) \boxtimes \xi(2) \boxtimes \xi(2)) = \iota_2 \otimes 1 \otimes 1 + 1 \otimes \iota_2 \otimes 1 + 1 \otimes 1 \otimes \iota_2; \]

\[ ch_4(\xi(2) \boxtimes \xi(2) \boxtimes \xi(2)) = \iota_2 \otimes \iota_2 \otimes 1 + \iota_2 \otimes 1 \otimes \iota_2 + 1 \otimes \iota_2 \otimes \iota_2; \]

\[ ch_6(\xi(2) \boxtimes \xi(2) \boxtimes \xi(2)) = \iota_2 \otimes \iota_2 \otimes \iota_2; \]

and

\[ ch_3(\xi(2) \boxtimes \xi(2) \boxtimes \xi(2)) = 0 = ch_5(\xi(2) \boxtimes \xi(2) \boxtimes \xi(2)). \]

We have:

\[ ch_2 ch_2 ch_2 = 3! \iota_2 \otimes \iota_2 \otimes \iota_2 = 6 \iota_2 \otimes \iota_2 \otimes \iota_2; \]

\[ ch_2 ch_4 = 3 \iota_2 \otimes \iota_2 \otimes \iota_2; \]

\[ ch_6 = \iota_2 \otimes \iota_2 \otimes \iota_2; \]

\((ch_n := ch_n(\xi(2) \boxtimes \xi(2) \boxtimes \xi(2))). \) Therefore the corresponding characteristic numbers are 6, 3, 1 respectively.

Reasoning in this way we obtain the following table of characteristic numbers:

|     | \(S^4 \times S^4 \times S^4\) | \(S^6 \times S^6\) | \(S^4 \times S^8\) | \(S^{12}\) |
|-----|-------------------------------|-------------------|------------------|--------|
| 2 2 2 | 6 | 0 | 0 | 0 |
| 3 3 | 0 | 2 | 0 | 0 |
| 2 4 | 3 | 0 | 1 | 0 |
| 6 | 1 | 1 | 1 | 1 |

(9)

Note that, in particular, the class \([S^{2m} \times S^{2n}, \xi^{(m)k} \boxtimes \xi^{(n)}]\) is equal to the class \([S^{2m} \times S^{2n}, \xi^{(m)} \boxtimes \xi^{(n)k}]\) in \(\Omega^{2(m+n)} \otimes \Bbb Q\).

**Corollary 17.** The bordism classes \([S^{2n_1} \times \ldots \times S^{2n_r}, \xi^{(n_1)} \boxtimes \ldots \boxtimes \xi^{(n_r)}] = [S^{2n_1}, \xi^{(n_1)}] \ldots [S^{2n_r}, \xi^{(n_r)}], \) 2 \(\leq n_1 \leq \ldots \leq n_r, \) \(n_1 + \ldots + n_r = n, \) \(r \geq 1\) form an additive basis of \(\Omega^{2n} \otimes \Bbb Q\).

Put \(t_n := [S^{2n}, \xi^{(n)}], \) deg \(t_n = 2n.\) The previous results imply the following theorem:

**Theorem 18.** The graded algebra \(\Omega^* \otimes \Bbb Q\) is isomorphic to the polynomial algebra \(\Bbb Q[t_2, t_3, t_4, \ldots],\) where deg \(t_n = 2n.\)

Note that \([S^4, \xi^{(2)}]\) is a nondivisible element of \(\Omega^4\) because \(ch_2\) on SU-bundles coincides with the second Chern class \(c_2\) and therefore \(ch_2 \in H^4(BSU, \Bbb Z),\) while \(\langle ch_2(\xi^{(2)}), S^4\rangle = 1.\)

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