Letter

Generation of entangled states of light using discrete solitons in waveguide arrays

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Abstract

We study the quantum properties of light propagating through an array of coupled nonlinear waveguides and forming a discrete soliton. We demonstrate that it is possible to use certain types of quasi-solitons to form continuous variables entanglement between the certain pair of waveguides. Moreover, there is a possibility to entangle several pairs of waveguides independently. We show that the entanglement is generated for arbitrary high intensity of the input laser field, so it does not require a special material with an extremely high nonlinearity coefficient. Also, absorption in the waveguide media does not influence the discussed process too much.

Keywords: quantum entanglement, discrete solitons, optical arrays

(Some figures may appear in colour only in the online journal)

1. Introduction

On-chip integrated optical waveguide circuits open a wide variety of possibilities to manipulate quantum states of light \cite{1–4}. One of the most promising usages of such devices is the implementation of various quantum informatics algorithms \cite{5}. Entangled states of light are the key elements of quantum information processing. The development of integrated devices that generate such states using ordinary laser sources as their input appears to be an important problem. Recently, there have been proposed a number of schemes \cite{6–10} to form entanglement between optical modes in different waveguides. These schemes are based on down-conversion processes in nonlinear media. They require the use of specific materials and waveguide geometry, which complicates the design of discussed devices. An alternative approach might be to use a nonlinear self-action process which is more common and occurs in all materials with cubic nonlinearity. Currently, the possibility of using this process to generate entangled states is poorly studied. Here we propose to use an array of coupled waveguides with cubic nonlinearity to generate continuous variables entanglement \cite{11}. Such objects are well studied from a classical nonlinear dynamics point of view. It is known that there is a large variety of discrete solitons in such arrays \cite{12, 13}. Taking quantum effects into account, the propagation of such field distributions can lead to nonclassical photon statistics. Such effect is known for continuous solitons \cite{14, 15}. We will show that similar behavior present in the discussed system. The basic idea is that due to nonlinearity, the photon statistics will change as light travels through the waveguide. And the interaction between the optical modes of
the waveguides will transform this non-classical state of light into an entangled one.

2. Methods

We consider an array consisting of \( N \) single-mode waveguides (figure 1). A long laser pulse is sent to the input of each waveguide. Propagation of the pulse may be described in terms of slowly varying field operators \( \hat{\psi}_k(x,t) \), where \( k \) is the index of a waveguide, \( x \) is the coordinate along array. The Hamiltonian of system consists of two parts:

\[
\hat{H} = \hat{H}_i + \hat{H}_f.
\]

(1)

\( \hat{H}_i \) refers to the independent propagation of light along individual waveguides. In the slowly varying envelope and rotating-wave approximations it equals [16]:

\[
\hat{H}_i = \sum_{k=1}^{N} \int dx \left[ i \frac{\partial}{\partial t} \hat{\psi}_k (x,t) + \frac{\partial^2}{\partial x^2} \hat{\psi}_k (x,t) + U \hat{\psi}_k^\dagger \hat{\psi}_k \right],
\]

(2)

\( v \) is group velocity, \( U \) is a nonlinear coefficient describing self-action of the light due to intensity dependent part of the refractive index of the waveguides’ material. Hereinafter we assume that \( h = 1 \). \( \hat{H}_i \) is the Hamiltonian which describes interaction of optical field in different waveguides. In case of weak coupling it equals [17]:

\[
\hat{H}_f = \sum_{k=1}^{N-1} \int dx \kappa \hat{\psi}_k^\dagger \hat{\psi}_{k+1} dx + \text{H.c.}
\]

(3)

\( \kappa \) is the coefficient describing interaction of the light in neighbouring waveguides due to modes overlap. Using Hamiltonian equation (1) one can derive Heisenberg equations:

\[
\begin{align*}
\frac{\partial}{\partial t} \hat{\psi}_k (x,t) &= -i \left[ \hat{H}_i, \hat{\psi}_k (x,t) \right] \\
&= -i k \left[ \hat{\psi}_{k-1} (x,t) + \hat{\psi}_{k+1} (x,t) \right] \\
&\quad - iU \hat{\psi}_k^\dagger (x,t) \hat{\psi}_k (x,t).
\end{align*}
\]

(4)

To simplify further calculations, we perform them in a moving frame using substitution \( t_0 = t - x/v \). Also we normalize distance \( x \to z = \kappa x/v \):

\[
\begin{align*}
i \frac{\partial}{\partial z} \hat{\psi}_k^M (z,t_0) &= \hat{\psi}_{k-1}^M (z,t_0) + \hat{\psi}_{k+1}^M (z,t_0) \\
&\quad + L \hat{\psi}_k^{\dagger M} (z,t_0) \hat{\psi}_k^M (z,t_0) \hat{\psi}_k^{\dagger M} (z,t_0),
\end{align*}
\]

(5)

where \( L = Uv/\kappa \) and

\[
\hat{\psi}_k^M (z,t_0) = \hat{\psi}_k \left( \frac{v}{\kappa}, t - \frac{z}{\kappa} \right).
\]

(6)

In equation (5), \( t_0 \) is just a parameter, so in further calculations we omit it assuming that it is always the same and corresponds to the center of the input pulses. To calculate the evolution of light field along the array based on equation (5) we assume that the quantum state of light is Gaussian for every value of \( v \). This means, that the Sudarshan-Glauber P distribution has a Gaussian profile. For such distributions cumulants \([18]\) \( \ll X_1 \cdot X_2 \cdot X_3 \gg \) and \( \ll X_1 \cdot X_2 \cdot X_3 \cdot X_4 \gg \) equal to zero if \( \hat{X} \) takes a value from the set \( \left\{ \hat{\psi}_k^M, \hat{\psi}_k^{\dagger M} \right\} \) in such a way that the products are normally ordered. This fact allows using the following relations:

\[
\begin{align*}
\langle X_1 \cdot X_2 \cdot X_3 \rangle &= \langle \hat{X}_1 \rangle \langle \hat{X}_2 \rangle \langle \hat{X}_3 \rangle \\
&\quad + \langle \hat{X}_1 \rangle \langle \hat{X}_2 \rangle - 2 \langle \hat{X}_1 \rangle \langle \hat{X}_3 \rangle \\
&\quad + \langle \hat{X}_1 \rangle \langle \hat{X}_2 \rangle - 2 \langle \hat{X}_1 \rangle \langle \hat{X}_3 \rangle \langle \hat{X}_4 \rangle.
\end{align*}
\]

(7)

\[
\begin{align*}
\langle \hat{X}_1 \rangle \langle \hat{X}_2 \rangle \langle \hat{X}_3 \rangle &= \langle \hat{X}_1 \rangle \langle \hat{X}_2 \rangle \langle \hat{X}_3 \rangle \\
&\quad + \langle \hat{X}_1 \rangle \langle \hat{X}_2 \rangle - 2 \langle \hat{X}_1 \rangle \langle \hat{X}_3 \rangle \\
&\quad + \langle \hat{X}_1 \rangle \langle \hat{X}_2 \rangle - 2 \langle \hat{X}_1 \rangle \langle \hat{X}_3 \rangle \langle \hat{X}_4 \rangle.
\end{align*}
\]

(8)

To derive a closed set of equations for quantities \( \alpha_k = \sqrt{L} (\hat{\psi}_k^M) \), \( \Delta^a_{k,l} = L (\hat{\psi}_k^M \hat{\psi}_l^M) - \alpha_k \alpha_l \) and \( \Delta^b_{k,l} = L (\hat{\psi}_k^M \hat{\psi}_l^M) - \alpha_k^2 \alpha_l^2 \) [19]:

\[
\begin{align*}
i \frac{\partial}{\partial z} \alpha_k &= (\Delta^a_{k,k+1} + \Delta^a_{k,k-1} - \Delta^a_{k,k+1} - \Delta^a_{k,k-1}) \\
&\quad + 2 \left( \Delta^b_{k,k+1} + \alpha_k^2 - |\alpha_k|^2 \right) \Delta^b_{k,l} \\
&\quad + \alpha_k^2 \Delta^b_{k,k+1} \Delta^b_{k,k-1} + \alpha_k^2 \Delta^b_{k,k+1} \Delta^b_{k,k-1}.
\end{align*}
\]

(9)

(8)

Before we start solving the equation (8) we need to define boundary conditions at plane \( z = 0 \). We assume that the quantum state of light is a multi-mode coherent state at the input of the array. For such sort of states \( \Delta^a_{k,k} \) and \( \Delta^b_{k,k} \) equal zero for every \( k, l = 1 \ldots N \). So it is enough to specify the distribution of values \( \alpha_k (z = 0) \). Before we do so, it should be noted that if in equation (8) set \( L = 0 \), the system of equations will reduce to:

\[
i \frac{\partial}{\partial z} \alpha_k = (\alpha_{k-1} + \alpha_{k+1}) + |\alpha_k|^2 \alpha_k.
\]

(9)

We want to emphasize here that \( L = 0 \) does not mean that we are neglecting nonlinearity. Instead, it means that the number of photons in optical modes tends to infinity, which can be interpreted as the transition to the classical physics limit. Equation (9) have solutions that are called discrete...
solitons [12]. To find them let us look for solution having the form \( \alpha_k = \beta_k e^{-\omega k} \). Here \( \beta_k \) are real numbers that do not depend on \( z \). For \( \alpha_k \) to satisfy equation (9) these numbers should obey next equations:

\[
-\omega \beta_k + \beta_{k+1} + \beta_{k+1}^2 = 0. \tag{10}
\]

For equation (10) there is a number of solutions sets parameterized by parameter \( \omega \) and which exist only when \( \omega > 2 \). Some of these sets are linearly stable if \( \omega \) is large enough and we will use them as boundary conditions for equation (8). On the top row of figure 2 some examples of such solutions are depicted. It should be noted that the equation (9) can be considered a discrete analog of the nonlinear Schrödinger equation (NLSE). There is a well-known soliton solution for this equation, quantum properties of which have been studied in several papers [14, 15]. Discrete analog of such solution is shown in figure 2(a). In the case of the infinite number of waveguides and \( \omega \to 2 \) mentioned discrete soliton should become similar to NLSE-soliton and have the same quantum properties. In further discussion, we will focus on situations where discrete effects are significant. This means that we will use discrete solitons corresponding to large values of \( \omega \) and having no continuous analogs, as shown in figures 2(b) and (c).

By solving equation (8) we want to study the quantum properties of the light field. First of all, we are interested in the entanglement that forms between the modes of different optical waveguides. We will limit our consideration only by studying bipartite entanglement [20, 21]. In the case of Gaussian quantum states, all properties of the state can be characterized in terms of covariance matrix [22, 23]. Such sort of matrix can be obtained in the experiment using the standard technique of homodyne detection [23]. For two waveguides with indexes \( k \) and \( l \) the covariance matrix \( \sigma \) is defined as:

\[
\sigma_{m,n} = \langle \hat{\xi}_m \hat{\xi}_n + \hat{\xi}_n \hat{\xi}_m \rangle / 2 - \langle \hat{\xi}_m \rangle \langle \hat{\xi}_n \rangle, \tag{11}
\]

where indexes \( m \) and \( n \) take value from 1 to 4, and \( \hat{\xi}_m \) and \( \hat{\xi}_n \) are the corresponding components of a four-dimensional vector \( \hat{\xi} = (\hat{q}_k, \hat{p}_k, \hat{q}_l, \hat{p}_l) \).

\[
\hat{q}_{k,l} = \frac{1}{\sqrt{2}} (\hat{\psi}_{k,l}^v + \hat{\psi}_{k,l}^v), \quad \hat{p}_{k,l} = \frac{i}{\sqrt{2}} (\hat{\psi}_{k,l}^v - \hat{\psi}_{k,l}^v). \tag{12}
\]

To retrieve information about entanglement between waveguides from the covariance matrix we use a quantity called logarithmic negativity \( (E_N) \) [22]:

\[
E_N = -\frac{1}{2} \sum l \log_2 \min \{1, 2 |l| \}, \tag{13}
\]

where \( l \) are symplectic eigenvalues of matrix \( \sigma \).

Before we discuss the results of calculations for quantum entanglement, we should do a remark about the Gaussian approximation we are using. The problem is that equation (5) do not preserve the property of a quantum state to be Gaussian. But the initial quantum state is Gaussian. So we may assume that our approximation is valid for a certain distance of pulse propagation. And we need to estimate this distance to make sure the results we get are correct. To solve this problem we consider the next order of cumulants expansion. We get \( \langle X_1 X_2 X_3 \rangle \) be non-zero but higher order cumulants still remain zero. In this assumption, we can derive the set of equations describing the evolution of third-order cumulants to compare them with zero. The result equations are too huge to derive them manually, so we have made a program that derives them using symbolic calculations in a form suitable for further numeric solution [24]. To determine whether the Gaussian approximation is still valid we calculate the next quantity:
Here we divide the maximum value between all cumulants we need to derive equation (8) by the maximum value of averages which we calculate using the fact that the corresponding cumulant is zero. Thus, the Gaussian approximation is valid when the value of ‘Err’ is much less than one. Dependency of ‘Err’ on the distance of light propagation along the array is shown on figure 3(a) by red dotted line. It is calculated for discrete soliton having filed distribution like the one shown on figure 2(b) and parameters: \( \omega = 10.0, L = 0.01 \). From the curve one may notice that the defined quantity grows very slow for \( z < 1 \), but after that point it increases dramatically. This means that our approximation is valid until the propagation distance is less than one. This threshold distance depends weakly on the discrete soliton type and parameters of the system \( \omega \) and \( L \). All further calculations will be limited by this distance.

### 3. Results and discussion

Our calculations show that the distribution of the average photons number between waveguides does not change on propagation distance for which Gaussian approximation is valid. It remains the same as it is on the input of an array. At the same time, the discussed distance is enough for the quantum state to change enough to form entangled states. An example of the evolution of logarithmic negativity \( (E_N) \) is shown in figure 3(a) with a black solid line. It is calculated for two central waveguides of a discrete soliton with an amplitude distribution as in figure 2(b), and the parameters are \( \omega = 10.0 \) and \( L = 0.01 \). The discussed curve shows that \( E_N \) reaches its maximum value rather quickly. Further evolution shows the quasi oscillation behavior of the discussed quantity. To explain such sort of behavior we can approximately separate two physical phenomena occurring in the system. First, the nonlinear self-action of light in every waveguide causes the quantum state of light to become squeezed [25]. Second, interacting optical modes of neighboring waveguides behave like a beam splitter, with a transition/reflection coefficient that depends on the length of the interaction. So, the result of the light evolution in the array can be described as a result of interference of the two squeezed optical modes on a beam splitter. The result of such interaction can be a two-mode squeezed state, which is entangled, or just a separable state of two independent squeezed states. It depends both on the parameters of the input states and transition/reflection coefficient of the beam splitter [26]. Squeezing parameters are more important for understanding the curve in figure 3(a). They are correlated with the value of additional phase that light gain due to nonlinear self-action, and thus oscillate with \( \omega \) frequency, which approximately agree with the distance between two maximums in figure 3(a). The maxima point is characterized by the greatest degree of entanglement that can be achieved in the system. For a given propagation distance, we will discuss further results. Distributions of \( E_N \) calculated for different pairs of waveguides and three types of discrete solitons are shown in figure 2 on the bottom line. These images show that entanglement is formed mainly between central waveguides having maximal intensity. For discrete analog of NLSE-soliton (figures 2(a) and (d)) bipartite entanglement is present only between the central waveguide and its nearest neighbors. Moreover, as \( \omega \) increases, the distribution of the average number of photons degenerates to a situation where almost all photons are in one waveguide. Of course, in such a situation, there is no entanglement in the system. For the case of sign-changing discrete solitons like depicted in figures 2(b) and (c) bipartite entanglement is present only for the pair of optical modes having maximal average photon number (see figures 2(e) and (f)).

In our opinion, solitons of this kind are the most interesting from a practical point of view. Our calculations show that when the complex amplitude of a soliton changes sign several times (as in figure 2(c)), entanglement is formed only between distinct pairs of waveguides. In practice, this fact can be used to construct an optical integrated circuit generating multiple entangled states in parallel.

For most optical materials, the cubic nonlinearity coefficient is relatively small. So, to form a discrete soliton, it is necessary to use laser pulses with a large number of photons. It is important to understand what happens to the discussed entangled states when the intensity of the incoming light increases and we move to the limit that is usually considered...
classical. To achieve this goal we should perform calculations for $L$ tending to zero. The result is shown in figure 3(b) by the black line. An interesting fact here is that the quantum state remains entangled as the intensity of the incoming light increases. This result can be easily explained. If we look at equation (8) we can notice that variables $\Delta k_{ij}$ and $\Delta \eta_{ij}$ are small for small values of $L$. We can linearize them and get a system of inhomogeneous equations. It is clear that solution is proportional to source term $\Gamma_0/\kappa$. The same time, the correlation matrix equation (11) which determines entanglement, is proportional to $\sigma \sim \Delta k_{ij}/L$. These two facts in combination lead to that $\sigma$ does not depend on $L$. From the point of view of quasiprobability distributions, this means that the evolution of quadratures does not depend on the position of the distribution center. The last thing we want to mention here is the influence of absorption on the process of entangled states formations. To consider absorption, we use a standard model of interaction with a reservoir of an infinite number of harmonic oscillators [16]. This will result in additional terms in equation (8) which have the form $-\Gamma_0/\kappa$, $-\Gamma \Delta k_{ij}$ and $-\Gamma \Delta \eta_{ij}$ for corresponding equations sets. Here $\Gamma$ is the absorption coefficient. In the presence of absorption, the evolution of entanglement remains qualitatively the same. The only thing that changes is the maximum value of logarithmic negativity that can be achieved in the system. In figure 3(b) dependency of this quantity on the intensity of input beams is shown for several values of $\Gamma$. The entangled states of light are still formed even for relatively high absorption coefficients. In real experiment setups coupling coefficient $\kappa/\sigma$ can be about 1 cm$^{-1}$ [27]. For such value $\Gamma = 0.3$ corresponds to absorption 0.3 cm$^{-1}$ which is very high for typical optical materials.

4. Conclusion

In conclusion, we underline the achieved results. We have considered the quantum properties of discrete solitons propagating through an array of cubic nonlinear waveguides. Certain types of such field distributions form entangled states of light between optical modes of multiple pairs of coupled waveguides. We propose that they can be used to create integrated optical devices for generating continuous-variable entangled states. The advantage of such a scheme is that discussed quasisolitons are stable and do not require the light field amplitude distribution on the input of the array to match the ideal one with high precision. Also we have shown that entangled states are formed even in the case of using input sources with high intensities. This means that the proposed scheme does not require any specific materials with extremely high nonlinear coefficients. Absorption which is present in all real materials also does not influence the entangled states’ generation process too much. There is an open question of the influence of the other sources of noise on the discussed process, such as Raman scattering or fluctuations of the coupling coefficient. However, this is the topic for future studies.

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