Leptonic CP violation in flipped SU(5) GUT from $Z_{12-I}$ Orbifold Compactification

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We obtain a phenomenologically acceptable PMNS matrix in a flipped SU(5) model inspired by the compactification of heterotic string $E_8 \times E'_8$. To analyze the Jarlskog determinant efficiently, we include the simple Kim-Seo form for the Pontecorbo-Maki-Nakagawa-Sakata matrix. We also noted that $|\delta_{PMNS}| \lesssim 64^\circ$ for the normal hierarchy of neutrino masses with the PDG book parametrization.

PACS numbers: 11.25.Mj, 11.30.Er, 11.25.Wx, 12.60.Jv
Keywords: String compactification, Flipped SU(5) GUT, PMNS matrix, Kim-Seo form, Jarlskog determinant.

I. INTRODUCTION

The most urgent theoretical issue in the standard model (SM) is probing the symmetry structure from which the observed flavor phenomena can be understood. It is desirable if such symmetry results from an ultra-violet completed theory such as from string compactification [1–6]. Most studies along this direction were centered on obtaining three families [7–10].

Now time is ripe enough to study the details of the flavor structure from string compactification. In the quark sector, the Cabibbo-Kobayashi-Maskawa matrix [12, 13] has been studied in our previous paper [14]. In this paper, we present a numerical study on the Pontecorbo-Maki-Nakagawa-Sakata (PMNS) matrix [15, 16] from string compactification via many U(1)’s arising in string compactification. String compactification in our example allows all the needed Yukawa couplings in the SM as non-renormalizable ones [11]. Therefore, the grand design [17] of relating the neutrino masses with the magnitude of the $\mu$ term with renormalizable terms only is not applicable here. The relation might be intertwined in an elaborate way in our model since the $Z_{4R}$ discrete symmetry automatically gives the $\mu$ term at the electroweak scale [18]. Also, we are far from obtaining non-Abelian discrete symmetries such as $A_4$ [19] in this study.

The first step in modeling a flavor structure from string compactification is to allocate the CP phase at some convenient slots in the mass matrices of neutrinos and charged leptons. Toward this, we point out that it is useful to use the Kim-Seo(KS) parametrization [20] of the CKM and PMNS matrices. With the KS parametrization, the CP phase in the Jarlskog triangle can be put in the (31) elements of the CKM and PMNS matrices $V_{KS}$, and the physical magnitude of CP violation is looked upon just from this simple matrix because the Jarlskog determinant is $J = -\Im V_{31}^{KS} V_{22}^{KS} V_{13}^{KS}$ [21]. Toward a model building, the next step is to obtain phenomenologically acceptable mass matrices. Since the Yukawa couplings are too many in standard-like models, here we work in the flipped SU(5) GUT [22, 23] where the number of Yukawa couplings are much less than in the standard-like models. In the flipped SU(5), the neutrino mass matrix turns out to be symmetric and hence we propose in this paper to put the CP phase in the charged lepton mass matrix.

In Sec. II, we recapitulate the simple form of the Jarlskog determinant. In Sec. III, we obtain a useful fit of the PMNS matrix in the KS form from the data presented in the Particle Data Book [24]. As a by-product, we will observe that $|\delta_{PMNS}| < 62.8^\circ$ for the normal hierarchy of neutrino masses with the PDG book parametrization [24]. In Sec. IV, we present possible terms of neutrino and charged lepton mass terms allowed by the quantum numbers of Ref. [11]. Then, we locate possible phases in the complex vacuum expectation values(VEVs) of the SM singlet fields $\sigma_i$. Section V is a brief conclusion.

1 For more references, see Ref. [11].
For the fixed triangle given by (2), the area relation results in

\[ J \equiv \text{Im } V_{i1}^* V_{i2} V_{i3} = 0 \]

where \( R \) are three real numbers in the first row \((2)\), parametrization also. One useful parametrization, the Kim-Seo (KS) parametrization with Det. \( V \) used in the PDG book are \( W \) is defined by the \( V \). In this form, \( J \) relates the entire range of the \( 3 \times 3 \) matrix and hence it can be a theory dependent number. So, it is better to use this form of \( J \). Figure 1 is drawn by considering \( V_{13} \). In the particle data book, the CKM matrix is defined by the \( W_\mu \) coupling \( \mu \) while the PMNS matrix is defined by the \( W_\mu \) coupling \( \mu \). Three real angles used in the PDG book are \( \theta_{12}, \theta_{13}, \) and \( \theta_{23} \) and the phase is denoted as \( \delta \).

The angles \( \alpha, \beta, \) and \( \gamma \) of Fig. 1 are related to \( \delta \) of \( V \). The same area of the triangle can be given in a different parametrization also. One useful parametrization, the Kim-Seo (KS) parametrization with Det. \( V_{KS} \) = 1, locates three real numbers in the first row \((2)\).

\[
V_{KS} = \begin{pmatrix}
R_1, & R_2, & R_3 \\
T_1, & R_4 + R_5 e^{-i\delta}, & T_2 \\
R_6 e^{i\delta}, & T_3, & T_4
\end{pmatrix},
\]

where \( R_i \) and \( T_i \) are real and complex numbers, respectively. Then, we have \( J = -R_3 R_4 R_6 \sin \delta \). It is remarkable to note that real numbers for one row (shown with the red color) makes it possible to visualize the importance of \( e^{-i\delta} \) in the position \( V_{31} \) \((2)\). A complex (22) element can be always written in the form separating out the term with the factor \( e^{-i\delta} \). Therefore, to use the analyses in the PDG book with the simple form given in Eq. (1), we solve, in view of Fig. 1 the equations for \( \theta_i \) in terms of \( \theta_{ij} \),

\[
\begin{align*}
&c_1 s_1 s_3 = c_{12} c_{13} s_{13}, \\
&c_2^2 c_1^2 s_1^2 s_2^2 + c_1^2 s_1^2 s_3^2 - 2 c_1 c_2 c_3 s_1 s_2 s_3 \cos \alpha = c_{13}^2 c_{23}^2 s_{12} s_{23}, \\
&c_2^2 c_3^2 s_1^2 s_2^2 = c_{13}^2 c_{23}^2 s_{12} s_{23} + c_{12}^2 c_{13}^2 s_{13}^2 - 2 c_{12} c_{13} c_{23} s_{12} s_{23} s_{13} \cos \gamma.
\end{align*}
\]

For the fixed triangle given by \((2)\), the area relation results in

\[ c_2 c_3 s_1 s_2 \sin \alpha = c_{13} c_{23} s_{12} s_{23} \sin \gamma. \]

Since there are four parameters to be determined \( i.e. \theta_{1,2,3} \) and \( \alpha \) from Eq. \((2)\), there is a degree of freedom to define the KS form from the observed angles in the PDG book. Even if we can determine the KS parameters from \((2)\) with one degeneracy parameter, the additional relation \((3)\) has a profound meaning. It must be satisfied for all real values of parameters \( \theta_i, \alpha \) and \( \theta_{ij}, \gamma \). For some angles, therefore, there must be a bound for the relation \((3)\) to be satisfied. Let us fix the parametrization such that the (11) element in the KS form agrees with the (11) element of the PDG.
book, \( c_1 = c_{12}c_{13} \). Then, the four conditions to determine the KS parameters are

\[
\begin{align*}
    c_1 &= c_{12}c_{13}, \\
    s_1s_3 &= s_{13}, \\
    \sqrt{c_1^2c_2^2s_1^2s_2^2 + c_1^2s_1^2s_2^2 - 2c_1c_2c_3s_1^2s_2s_3 \cos \alpha} &= \pm c_{13}c_{23}s_{12}s_{23}, \\
    c_2c_3s_1s_2 &= \pm \sqrt{c_1^2c_2^2c_3^2s_1^2s_2^2 + c_1^2c_2^2s_1^2s_2^2 - 2c_1c_2c_3c_{13}s_{12}s_{23}s_{13}\cos \gamma}.
\end{align*}
\]  

\((4)\)

The second relation of \((4)\) is the important parameter in the neutrino oscillation and hence the condition for the \((11)\) element to reproduce the PDG’s \((11)\) element is intuitive and persuasive. From the known values of \(\theta_{ij}\) and the solutions of \((2)\), \(\gamma\) should be bounded. Especially, it cannot be \(-\frac{\pi}{2}\). The numerical solutions for the angles in the KS form in the PMNS matrix will be presented in Sec. III.

III. DIAGONALIZATION OF MASS MATRICES AND MIXING ANGLES IN THE KS FORM

The charged current (CC) coupling in the lepton sector is

\[
\mathcal{L}_{CC} = -\frac{g}{\sqrt{2}} \sum_{i=\mu,\tau} \bar{l}_i \gamma^\alpha \nu_i W_\alpha^- + \text{h.c.}
\]  

\((5)\)

where the weak eigenstate leptons \(l\) are the defining ones in the CC interaction, and the weak eigenstate leptons \(\bar{\nu}_i\) are related to the mass eigenstate leptons \(l_i^{(\text{mass})}\), \(\nu_i^{(\text{mass})}\), as

\[
l_L = \sum_{j=1}^{3} V_{lj}^{(\text{mass})} \bar{\nu}_j, \quad \nu_L = \sum_{j=1}^{3} V_{lj}^{(\nu)} \nu_j^{(\text{mass})}.
\]  

\((6)\)

Between the mass eigenstates, the CC interaction is given by

\[
\mathcal{L}_{CC} = -\frac{g}{\sqrt{2}} \bar{l}_L^{(\text{mass})} \gamma^\alpha V_\nu^{(\nu)} \nu_L W_\alpha^- + \text{h.c.}
\]  

\((7)\)

The PMNS matrix is given by

\[
V_{PMNS}^\dagger = V^{(\nu)} V^{(\nu)}
\]  

\((8)\)

where \(V^{(\nu)}\) and \(V^{(\nu)}\) are diagonalizing unitary matrices of L-handed charged leptons and neutrino fields. A standard way to parametrize the CC lepton interactions is

\[
\text{CC lepton matrix} = V_{PMNS}^\dagger \begin{pmatrix} 1, & 0 & 0 \\ 0, & e^{i\alpha_{21}/2} & 0 \\ 0, & 0 & e^{i\alpha_{31}/2} \end{pmatrix}
\]  

\((9)\)

where the first factor called the PMNS matrix is usually written as \(24,28\),

\[
V_{PMNS}^\dagger \simeq \begin{pmatrix} C_{12}C_{13}, & S_{12}C_{13}, & S_{13}e^{-i\delta} \\ -S_{12}C_{23} - C_{12}S_{23}S_{13}e^{i\delta}, & C_{12}C_{23} - S_{12}S_{23}S_{13}e^{i\delta}, & S_{23}C_{13} \\ S_{12}S_{23} - C_{12}C_{23}S_{13}e^{i\delta}, & -C_{12}S_{23} - S_{12}C_{23}S_{13}e^{i\delta}, & C_{23}C_{13} \end{pmatrix},
\]  

\((10)\)

where \(C_{ij} = \cos \theta_{ij}\), \(S_{ij} = \sin \theta_{ij}\), \(\theta_{ij} = [0, \frac{\pi}{2}]\), and the angle \(\delta = [0, 2\pi]\) is the Dirac CP violation phase, and \(\alpha_{21}, \alpha_{31}\) are two Majorana CP violation phases. The second factor of \((9)\) contains the Majorana phases which may be determined by heavy neutrinos in the seesaw mechanism. The best fit(BF) real angles of the PMNS matrix are \(24,\)

\[
\begin{align*}
    \Theta_{12} &= 0.5764 \quad [C_{12} = 0.8385, \quad S_{12} = 0.5450], \\
    \Theta_{23} &= 0.7101 \quad [C_{23} = 0.7583, \quad S_{23} = 0.6519], \\
    \Theta_{13} &= 0.1472 \quad [C_{13} = 0.9892, \quad S_{13} = 0.1466],
\end{align*}
\]  

\((11)\)

\(2\) Compare with the CKM matrix \(V_{CKM} = V^{(u)} V^{(d)}\) defined from the \(W_\mu^+\) coupling, 

\[-\frac{g}{\sqrt{2}} \bar{u}_L^{(\text{mass})}\gamma^\alpha V^{(u)} \nu_L d^{(\text{mass})} W_\alpha^- + \text{h.c.}\]
and we have the following bound from Fig. 2

$$-78.29^\circ +2.35^\circ < \gamma < +78.29^\circ -2.35^\circ$$  \hspace{1cm} (12)

from which we obtain

$$V^{\dagger}_{PMNS} \approx \begin{pmatrix} 0.8294, & 0.5391, & 0.1466 \cos \delta_{KS} \\ -0.4132 - 0.08015 \sin \delta_{KS}, & 0.6358 - 0.0521 \cos \delta_{KS}, & 0.6449 \\ 0.3553 - 0.0932 \sin \delta_{KS}, & -0.5466 - 0.0606 \cos \delta_{KS}, & 0.7501 \end{pmatrix}$$  \hspace{1cm} (13)

where we used the central values for the allowed angles, $\theta_{12} = 0.5758(= 32.99^\circ), \theta_{13} = 0.1471(= 8.428^\circ)$ and $\theta_{23} = 0.7101(= 40.69^\circ)$, for the normal hierarchy\(^3\) of neutrino masses $m_1 < m_2 < m_3$. However, it is useful to have real numbers in one row of the PMNS matrix as in [20].

$$V^{\dagger}_{KS} = \begin{pmatrix} C_1, & S_1 C_3, & S_1 S_3 \\ -C_2 S_1, & C_1 C_2 C_3 + S_2 S_3 e^{-i \delta_{KS}}, & C_1 C_2 S_3 - S_2 C_3 e^{-i \delta_{KS}} \\ -e^{i \delta_{KS}} S_1 S_2, & -C_2 S_3 + C_1 S_2 C_3 e^{i \delta_{KS}}, & C_2 C_3 + C_1 S_2 S_3 e^{i \delta_{KS}} \end{pmatrix}$$  \hspace{1cm} (14)

where $\text{Det} V_{KS} = 1$. Then, the phase appearing in the (31) element is the key, viz. $J = -\text{Im} V^{KS}_{31} V^{KS}_{22} V^{KS}_{13} = -C_1 C_2 C_3 S_1^2 S_2 S_3 \sin \delta_{KS}$ [21].

\(^3\) Here we cite, for simplicity of presentation, mainly the numbers for normal hierarchy of neutrino masses except in Fig. 3.
FIG. 3: Same as in Fig. 2 except for the inverted hierarchy [24], $m_3 < m_1 < m_2$, where Eq. (12) becomes $-78.80^\circ + 0.94^\circ < \gamma < +78.80^\circ + 3.03^\circ$ and the lower limit of $\gamma$ becomes $-62.64^\circ + 1.48^\circ < \gamma$.

To make the PMNS matrix with one row real from the numbers given in Eq. (15), we present numerical solutions of Eq. (4) in Fig. 2. 4 The BF real angles from [24] determine $\Theta_1$ and $\Theta_3$ accurately,$$
\Theta_1 = 0.5928 \ [C_1 = 0.8294, \ S_1 = 0.5587],
\Theta_3 = 0.2656 \ [C_3 = 0.9649, \ S_3 = 0.2625],$$ but $\Theta_2$ is can be 0.5377 or 1.0331. For $\alpha = -\pi/2$ (corresponding to $\gamma = -62.8^\circ$) and $\Theta_2 = 0.5377$, we have
$$V_{KS}^\dagger = \begin{pmatrix}
0.82939, & 0.53909, & 0.14663 \\
-0.47985, & 0.68740 + 0.13441 e^{-i\delta_{KS}}, & 0.18697 - 0.49417 e^{-i\delta_{KS}} \\
-0.28611 e^{i\delta_{KS}}, & -0.22543 + 0.40986 e^{i\delta_{KS}}, & 0.82880 + 0.11148 e^{-i\delta_{KS}}
\end{pmatrix},$$
$$J = -\text{Im} V_{KS}^{\dagger} V_{K^\ast S}^{\dagger} V_{K^\ast S} = -2.8838 \times 10^{-2} \sin \delta_{KS}.$$ Namely, to have $J$ given in Eq. (16) for $\delta_{KS} = -\pi/2$ compared to $J$ of Eq. (13), we have the minimum allowed value $\gamma = -62.8^\circ$ which is inside the region given in Eq. (12). In Fig. 2 we mark the $+10\%$ band from this value, $\gamma = [62.8^\circ, 56.52^\circ]$, as the pink band. In the third quadrant, the band becomes anti-symmetric to the curve in the first quadrant, $\gamma = [-62.8^\circ -1.25^\circ, -56.56]$. In Fig. 3 we present an inverted hierarchy solution for $m_3 < m_1 < m_2$.

We used $\dagger$ notation in (14) since the definition of the PMNS matrix is given by $W^-\mu$ coupling and the CKM matrix is given by $W^+\mu$ coupling. To compare both with the $W^-\mu$ coupling, factoring out the Majorana phases, let us consider the PMNS parametrization with $\dagger$ of Eq. (14),
$$V_{KS} = \begin{pmatrix}
C_1, & -C_2 S_1, & -S_1 S_2 e^{-i\delta_{KS}} \\
S_1 C_3, & C_1 C_2 C_3 + S_2 S_3 e^{i\delta_{KS}}, & -C_2 S_3 + C_1 S_2 C_3 e^{-i\delta_{KS}} \\
S_1 S_3, & C_1 C_2 S_3 - S_2 C_3 e^{i\delta_{KS}}, & C_2 C_3 + C_1 S_2 S_3 e^{-i\delta_{KS}}
\end{pmatrix},$$
$$4 \text{ An approximate analytic solution near the dodeca symmetric point was given before [30].}$$
FIG. 4: The CKM and PMNS unitary triangles with one common angle [31]: (a) \( \alpha \) in the KS form, and (b) \( \gamma \) in the SV form.

To build a model, leading to (17), one must find out the mass matrices \( M^{(\nu)} \) and \( M^{(l)} \) with appropriate insertions of \( e^{\pm i\delta_{KS}} \).

As suggested in [31], if we use the KS parametrization for the CKM matrix given in [20] and again the KS parametrization for the PMNS matrix [30] given in Eq. (17) and the same CP phase \( \alpha \) appears in the CKM and PMNS phases, we expect the unitary triangles take the forms given in Fig. 4 (a). If we use the Maiani-Chau-Keung (MCK) parametrization for the CKM matrix and the Schechter-Valle (SV) parametrization for the PMNS matrix given in [24, 28] and the same CP phase \( \gamma \) appears in the CKM and PMNS phases, we expect the unitary triangles take the forms given in Fig. 4 (b). The CKM unitary triangle is known rather accurately but the PMNS unitary triangle is not known accurately, chiefly because the error bars allowed for \( \gamma \) is large: e.g. for the normal hierarchy \( \delta_{CP} = -1.728^{+0.085}_{-0.055} \) [29]. These unitary triangles are defined by CC interactions, and determined chiefly by the decay processes in the quark sector and by neutrino oscillations in the lepton sector.

IV. SUGGESTION FROM THE FLIPPED SU(5) MODEL

If we consider only the SM particles, neutrino masses arise from the diagram shown in Fig. 5. Any further attachments to this diagram are SM singlet scalars. If we consider the quantum numbers under SU(2)_W \times U(1)_Y, two neutrinos have \( 1_{-1} \oplus 1^*_{+1} \) where \( ^* \) means that the 3rd component of the weak isospin is +1. Possible scalar attachments must carry quantum number \( 1^*_{+1} \) or \( 3_{+1} \), and \( 1^*_{+1} \) is ruled out because \( (1_{+1}) \) breaks U(1)_em. \( 3^*_{+1} \) allows the scalar attachments, shown as \( H_u \oplus H_u \) in Fig. 5. Depending on details of high energy fields, implied by the question mark in the gray, two types of neutrino masses are named, Type I seesaw [33] and Type II seesaw [34]. Type III seesaw [35] requires more light particles at the electroweak scale. From the SU(5)_{lep} spectra shown in Ref. [32], we note that there is no SU(2) triplet representation; hence only Type I seesaw is allowed from our string compactification.

Considering the SM singlet attachments to Fig. 5 let us consider the neutrino mass operator allowed in the Z_{12-I} compactification. Firstly, the diagonal masses are

\[
M_{33}^{\nu} \propto \frac{1}{M_3^2} \int d^2\bar{\vartheta} \, 5_{+3}(U_3, 0; +1)5_{+3}(U_3, 0; +1)5_{-2}(T_6, \frac{1}{3}; -2)5_{-2}(T_6, \frac{1}{3}; -2)\mathbf{10}_{-1}(T_3, \frac{1}{3}; +4)\mathbf{10}_{-1}(T_3, 0; +4)
\]

\[
M_{22}^{\nu} \propto \frac{1}{M_2^2} \int d^2\vartheta d^2\bar{\vartheta} \, 5_{+3}(T_4, \frac{1}{4}; -1)5_{+3}(T_4, \frac{1}{4}; -1)5_{-2}(T_6, \frac{1}{3}; -2)5_{-2}(T_6, \frac{1}{3}; -2)\mathbf{10}_{-1}(T_3, \frac{1}{3}; +4)\mathbf{10}_{-1}(T_3, 0; +4)\mathbf{10}_{0}(\sigma_5, T_6, \frac{1}{2}; +4)
\]

where the last number after \( ; \) is the \( Q_R \) charge, and \( \hat{M}_3 \) and \( \hat{M}_2 \) are determined by ? in Fig. 5. We need \( Q_R = 2 \)
FIG. 5: A neutrino interaction with the SM fields only.

modulo 4 for $d^2 \varphi$ integration and $Q_R = 0$ modulo 4 for $d^2 \varphi d^2 \bar{\varphi}$ integration. $M_{11}', M_{12}', M_{23}'$ have the same structure as $M_{52}'$. The quantum numbers are listed in Tables I and II. Note that the selection rule is making the phase an integer multiple, which is satisfied above, viz. $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$ and $\frac{2}{4} + \frac{1}{4} + \frac{1}{3} + \frac{1}{3} = 2$. Then, the above masses are estimated as

$$M_{33}' \sim \frac{v_{EW}^2 M_{10}^2}{M_3^3}, \quad M_{22}' \sim \frac{v_{EW}^2 M_{10}^2 |\sigma_5|}{M_5^2}.$$  \hspace{1cm} (19)

Then, neutrino mixing masses are generally of order $v_{EW}^2 / M$ since the SM singlet VEVs can be at the GUT scale without breaking $Z_{4R}$.

For the off-diagonal masses between $U_3$ and $T_4^0$ neutrinos, we need $d^2 \varphi d^2 \bar{\varphi}$ integration,

$$M_{32}', M_{31}' \propto \frac{1}{M^6} \int d^2 \varphi d^2 \bar{\varphi} 5_{+3}(U_3, 0; +1)5_{+3}(T_4^0, \frac{1}{4}; -1)5_{-2}(T_6, \frac{1}{3}; -2)5_{-2}(T_6, \frac{1}{3}; -2) \cdot \langle \bar{T}_{0-1} T_3, 0; +4 \rangle \langle \bar{T}_{0-1} T_3, 0; +4 \rangle \cdot 1_0(\sigma_1, T_1^0, \frac{1}{4}; +4)^* 1_0(\sigma_1, T_3, \frac{2}{3}; -4)^*.$$ \hspace{1cm} (20)

Then, the above mass mixing is estimated as

$$M_{13,23}' \sim \frac{v_{EW}^2 M_{10}^2 |\sigma_1 \sigma_{14}|}{M_5^6},$$ \hspace{1cm} (21)

where $M$ is some mass scale determined by the above equations. Note that $\Sigma_2, \Sigma_3$ and $\sigma_1$ can have the GUT scale VEVs because all of them carry $Q_R = 0$ modulo 4, and we obtain a similar order of mass for all of $M_{11,12,22,31,32}'$ and $M_{33}'$.

Comparing $M_{11}'$, $M_{22}'$, $M_{33}'$:

$$\frac{M_{11}'}{M_{33}'} \approx \frac{|\sigma_5|}{M} \quad \frac{M_{31}'}{M_{33}'} \approx \frac{|\sigma_1 \sigma_{14}|}{M^2},$$ \hspace{1cm} (22)

we note that the neutrino mass hierarchy favors the normal hierarchy (in the sense that $\nu_3$ is the heaviest) if the VEVs of $\sigma$ singlets are comparably small, $|\sigma_1|, |\sigma_3| < M$.

Since we obtained all entries in the neutrino mass matrix, here we investigate how the CP phase can be inserted in the mass matrix of the $Q_{em} = -1$ leptons and in the neutrino mass matrix.

A. Neutrino mass matrix inspired by flipped SU(5)

In Ref. [11] based on the flipped SU(5) model of [36], a possible identification $Z_{4R}$ has been achieved, forbidding dimension-5 B violating operators but allowing the electroweak scale $\mu$ term and dimension-5 L violating Weinberg operator. The $Z_{4R}$ quantum numbers, $Q_{4R}$, of the SM fields and neutral singlets ($\sigma$’s), are presented in Ref. [11]. In the flipped SU(5), the neutrino masses arise in the form

$$-L_{\nu}^I = f_{I J}(\sigma) 5_{+3}^i 5_{+3}^j 5_{-2}^k (H_u) 5_{-2}^l (H_u) |\bar{T}_{0-1} (H_{GUT}) \bar{T}_{0-1} (H_{GUT})|_{ijkl} + h.c.,$$ \hspace{1cm} (23)

where the couplings $f_{I J}(\nu)$ are complex parameters, $I$ and $J$ are flavor indices, $i, j, k, l, m$ are SU(5) indices, and the subscript is the $U(1)_X$ quantum number of SU(5)$_{lep}$. $5_{-2}$ is usually denoted as $H_{uL}$, and $\bar{T}_{0-1}$, together with $10_{+1},$
is the ten-plet needed for breaking the rank 5 gauge group SU(5)×U(1) at a GUT scale down to the rank 4 SM gauge group. These quantum numbers in SU(5)up are given in Ref. [11].

Consider $5^I_{i^+3}5^J_{j^+3} \in \text{Eq. (28)}$ which is symmetric under $I$ and $J$. Thus, the neutrino mass matrix is symmetric. The Majorana phase factored in Eq. (9) is from the heavy neutrinos, which does not affect our study of CC interactions shown in Eq. (10). As in the quark case, we assume that the neutrino mass matrix determining the PMNS matrix is real. Thus, $V^{(\nu)}$ can be considered to be an orthogonal matrix $O^{(\nu)}$.

### B. Mass matrix of charged leptons inspired by flipped SU(5)

We can always take $U^{(\nu)}$ as a real matrix $O^{(\nu)}$. Thus, the PMNS matrix given in (14) can be represented as

$$V_{KS}^{(\nu)} = V^{(e)}O^{(\nu)T} = \begin{pmatrix} q_{11}T_{11} + q_{12}T_{12} + q_{13}T_{13}, & q_{11}T_{21} + q_{12}T_{22} + q_{13}T_{23}, & q_{11}T_{31} + q_{12}T_{32} + q_{13}T_{33}, \\
q_{21}T_{11} + q_{22}T_{12} + q_{23}T_{13}, & q_{21}T_{21} + q_{22}T_{22} + q_{23}T_{23}, & q_{21}T_{31} + q_{22}T_{32} + q_{23}T_{33}, \\
q_{31}T_{11} + q_{32}T_{12} + q_{33}T_{13}, & q_{31}T_{21} + q_{32}T_{22} + q_{33}T_{23}, & q_{31}T_{31} + q_{32}T_{32} + q_{33}T_{33} \end{pmatrix}$$

(24)

where the elements $V_{ij}^{(\nu)} = q_{ij}$ and $O_{ij}^{(\nu)} = r_{ij}$ are complex and real numbers numbers, respectively. Comparing with Eq. (14), $q_{11}, q_{12}$ and $q_{13}$ are required to be real.

The unitary matrices relating the weak eigenstates $l$ and mass eigenstates $i$ of the charged leptons are named as $V$ for L-handed fields and $U$ for R-handed fields,

$$l_L = \sum_{j=1}^{3} V_{ij}^{(\nu)} l^\text{mass}_L, \quad l_R = \sum_{j=1}^{3} U_{ij}^{(\nu)} l^\text{mass}_R.$$

(25)

The mass matrix $l_L^\text{mass} (\tilde{m}_e, \tilde{m}_\mu, 1) l_R^\text{mass}$ (where $\tilde{m}_l = m_l/m_\nu$) in the mass eigenstate basis becomes

$$\tilde{l}_L(V^{(\nu)}M^{mass}U^{(\nu)}T)l_R$$

(26)

in the weak eigenstate basis. Since R-handed leptons are not participating in the CC interactions, the lepton R-handed unitary matrix $U^{(\nu)}$ can be taken as the identity matrix. Thus, the mass matrix in the weak basis becomes

$$M^{(l)} = V^{(l)} \begin{pmatrix} \tilde{m}_e, & 0, & 0 \\
0, & \tilde{m}_\mu, & 0 \\
0, & 0, & 1 \end{pmatrix} U^{(l)\dagger} = \begin{pmatrix} q_{11}\tilde{m}_e, & q_{12}\tilde{m}_\mu, & q_{13} \\
q_{21}\tilde{m}_e, & q_{22}\tilde{m}_\mu, & q_{23} \\
q_{31}\tilde{m}_e, & q_{32}\tilde{m}_\mu, & q_{33} \end{pmatrix} = \begin{pmatrix} \text{real}, & \text{real}, & \text{real} \\
\text{complex}, & \text{complex}, & \text{complex} \end{pmatrix}$$

(27)

where $V_{ij}^{(l)} = q_{ij}$ and we obtained $q_{11}, q_{12}$ and $q_{13}$ are real numbers.

We show that the quantum numbers of the model presented in [11] allows an effective mass matrix form Eq. (27) for the charged leptons.

$$-L_I^{lJ} = \sum_{I,J} \left\{ \langle \sigma \rangle \right\} 5^I_{i^+3} l^J_{j^+3} \bar{5}_{-3} \int_{H(H_{\text{GUT}})}^{H_{\text{GUT}}} + \text{h.c.},$$

(28)

which arises from, for example for the (22), (33) and (32) elements, viz. Tables II and III,

$$1/M_4 \int d^2 \phi d^2 \bar{\phi} \bar{\theta}_2 (T^0_4, 1/4) \mu^c (T^0_4, 1/4) H_d (T_6, 1/3) \Sigma_2 (T_3, 0) \Sigma_1^* (T_3, 2/3) \sigma_5^* (T_6, 1/2)$$

(29)

which is allowed with $Q_R = 0$ (needed for D-terms) and $Q_R = 2$ (needed for F-terms) modulo 4, respectively. The BSM fields in Eq. (29) carry $Q_R = 4$ and $Z_{4R}$ is not broken by the mass terms of the charged leptons.

Since all the entries of the mass matrix $M^{(l)}$ are allowed, we show below how the required form (27) results. Because of the degeneracy of the SM fields in the sector $T_4$, the mass matrix can be written as

$$\sim \begin{pmatrix} r_1 e^{i\phi_1}, & r_2 e^{i\phi_2}, & \bar{r}_1 e^{i\phi_1}, & \bar{r}_2 e^{i\phi_2} \\
r_1 e^{i\phi_1}, & r_2 e^{i\phi_2}, & \bar{r}_1 e^{i\phi_1}, & \bar{r}_2 e^{i\phi_2} \\
r_3 e^{i\phi_1}, & r_4 e^{i\phi_2}, & \bar{r}_3 e^{i\phi_1}, & \bar{r}_4 e^{i\phi_2} \end{pmatrix}.$$  

(30)
Redefining the L-handed and R-handed phases, in the $i$-th family.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
State$(P + kV_G)$ & $\Theta_k$ & $R_X$ (Sect.) & $Q_R$ & $Q_1$ & $Q_2$ & $Q_3$ & $Q_4$ & $Q_5$ & $Q_{\text{anoon}}$ & $Q_{18}$ & $Q_{20}$ & $Q_{22}$ \\
\hline
$\xi_3$ & $(+++--;--+) (0^5)'$ & 0 & $\bar{T}_{0}^{-1}(U_3)$ & +1 & -6 & -6 & 0 & 0 & 0 & -13 & +1 & -1 & +1 \\
$\eta_3$ & $(---++;++) (0^3)'$ & 0 & $5_{33}(U_3)$ & +1 & +6 & +6 & 0 & 0 & 0 & -1 & +1 & -1 & +1 \\
$\tau^c$ & $(+++--;--+) (0^3)'$ & 0 & $1_{-5}(U_3)$ & +1 & -6 & +6 & 0 & 0 & 0 & +5 & +1 & -1 & +1 \\
$\xi_2$ & $(+++--;-\frac{1}{6},-\frac{1}{6},-\frac{1}{6}) (0^3)'$ & $\frac{1}{\sqrt{2}}$ & $5_{33}(T_5^0)$ & -1 & -2 & -2 & 0 & 0 & 0 & -3 & -1 & -1 & -1 \\
$\bar{\eta}_2$ & $(---++;--+) (0^3)'$ & $\frac{1}{\sqrt{2}}$ & $5_{33}(T_5^0)$ & -1 & -2 & -2 & 0 & 0 & 0 & -3 & -1 & -1 & -1 \\
$\mu^c$ & $(+++--;-\frac{1}{6},-\frac{1}{6},-\frac{1}{6}) (0^3)'$ & $\frac{1}{\sqrt{2}}$ & $1_{-5}(T_4^0)$ & -1 & -2 & -2 & 0 & 0 & 0 & -3 & -1 & -1 & -1 \\
$\xi_1$ & $(---++;--+) (0^3)'$ & $\frac{1}{\sqrt{2}}$ & $5_{33}(T_5^0)$ & -1 & -2 & -2 & 0 & 0 & 0 & -3 & -1 & -1 & -1 \\
$\xi_1$ & $(---++;--+) (0^3)'$ & $\frac{1}{\sqrt{2}}$ & $5_{33}(T_5^0)$ & -1 & -2 & -2 & 0 & 0 & 0 & -3 & -1 & -1 & -1 \\
$e^c$ & $(+++--;-\frac{1}{6},-\frac{1}{6},-\frac{1}{6}) (0^3)'$ & $\frac{1}{\sqrt{2}}$ & $5_{33}(T_5^0)$ & -1 & -2 & -2 & 0 & 0 & 0 & -3 & -1 & -1 & -1 \\
$H_{aL}$ & $(+10000; 0000) (0^5; -\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2})$ & $\frac{1}{\sqrt{2}}$ & $2 \cdot 5_{-2}(T_6)$ & -2 & 0 & 0 & -12 & 0 & 0 & 0 & -1 & -1 & -1 \\
$H_{dL}$ & $(-10000; 0000) (0^5; +\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$ & $\frac{1}{\sqrt{2}}$ & $2 \cdot 5_{+2}(T_6)$ & +2 & 0 & 0 & +12 & 0 & 0 & 0 & -1 & -1 & -1 \\
\hline
\end{tabular}
\caption{U(1) charges of matter fields in the SM. $\xi_i$ and $\bar{\eta}_i$ contain the left-handed quark and lepton doublets, respectively, in the $i$-th family.}
\end{table}

Redefining the L-handed and R-handed phases,

\begin{equation}
I'_L = \left( \begin{array}{ccc}
0 & 0 & e^{i\phi_5} \\
0 & e^{i\phi_5} & 0 \\
e^{i\phi_5} & 0 & 1 \\
\end{array} \right) I_L, \quad I'_R = \left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & e^{i\phi_6} \\
\end{array} \right) I_R,
\end{equation}

we obtain the mass matrix for the choice of $\phi_5 = -\phi_1$ and $\phi_6 = \phi_2 - \phi_1$,

\begin{equation}
\sim \left( \begin{array}{ccc}
r_1 & r_1 & r_2 \\
r_1 & r_1 & r_2 \\
r_3 e^{i\phi_3} & r_3 e^{i\phi_3} & r_4 e^{-i\phi_2} \\
\end{array} \right).
\end{equation}

The above mass matrix form is simple enough to assign phases in the SM singlet fields, $\sigma_i$. From Eq. (29), we can choose the following phase for the singlets, $\langle \sigma_1 \rangle \sim e^{i\phi_3}$, $\langle \sigma_5 \rangle \sim e^{i\phi_6}$ and $\langle \sigma_2 \rangle \sim e^{i\phi_2}$. Determining these phases is postponed until a sufficiently accurate value of the PMNS phase is known.

\section{CONCLUSION}

After presenting a useful parametrization in the Kim-Seo form of the PMNS matrix, we obtained the mass matrix forms of neutrinos and charged leptons from symmetries allowed in a compactified string \cite{11}. The flipped SU(5) model compactified on $\mathbb{Z}_{12-1}$ is simple enough to draw this analysis up to satisfying all data on the PMNS matrix. In the flipped SU(5), the $d$-type quark mass matrix and the neutrino mass matrix are symmetric. These matrices are set to be real. We have shown that the CP phase $\delta_{\text{PMNS}}$ in the PMNS matrix can be introduced from the charged lepton mass matrix.

\section*{Acknowledgments}

J.E.K. thanks Carlos Muñoz for the helpful discussion during his visit of UAM, Madrid, Spain, where this work was initiated. This work is supported in part by the IBS (IBS-R017-D1-2014-a00) and by the National Research Foundation (NRF) grant NRF-2018R1A2A3074631.
TABLE II: U(1) charges of L-handed neutral scalars (but $\sigma_{7,8}$ for R-handed). We kept up to one oscillators represented as *Number of resulting fields* (number of oscillating mode). For example, $n(11)$ means that there results $n$ multiplicities with one oscillator $1_{11}$. For $Q_{18,20,22}$ charges, here we listed only those of L-handed fields, participating in the Yukawa couplings. $\sigma_{2,3,4,11,15,21,22,23,24}$ have phase $\Theta_i = 0$, which can be used to break $Z_{4R}$ down to $Z_{2R}$.

[1] P. Candelas, G. T. Horowitz, A. Strominger, and E. Witten, Vacuum configurations for superstrings, Nucl. Phys. B 258 (1985) 46 [doi:10.1016/0550-3213(85)90602-9].
[2] L. J. Dixon, J. A. Harvey, C. Vafa, and E. Witten, Strings on orbifolds. 2., Nucl. Phys. B 274 (1986) 285 [doi:10.1016/0550-3213(86)90287-7].
[3] L. E. Ibanez, H. P. Nilles, and F. Quevedo, Orbifolds and Wilson lines, Phys. Lett. B 187 (1987) 25 [doi:10.1016/0370-2930(87)90666-9].
[4] H. Kawai, D. C. Lewellen, and S. H. H. Tye, Construction of fermionic string models in four-dimensions, Nucl. Phys. B 288 (1987) 1 [doi:10.1016/0550-3213(87)90208-2].
[5] I. Antoniadis, C. P. Bachas, and C. Kounnas, Four-dimensional superstrings, Nucl. Phys. B 289 (1987) 87 [doi:10.1016/0550-3213(87)90372-5].
[6] D. Gepner, Supersymmetry in compactified string theory and superconformal models, Nucl. Phys. B 296 (1988) 757 [doi:10.1016/0550-3213(88)90397-5].
[7] L. E. Ibanez, J. E. Kim, H. P. Nilles, and F. Quevedo, Orbifold compactifications with three families of SU(3) × SU(2) × U(1)*, Phys. Lett. B 191 (1987) 292 [doi:10.1016/0370-2693(87)90255-3].
RevD.22.2227;
G. Lazarides, Q. Shafi and C. Wetterich, Proton lifetime and fermion masses in an SO(10) model, Nucl. Phys. B 181 (1981) 287 [doi:10.1016/0550-3213(81)90354-0];
R. N. Mohapatra and G. Senjanovic, Neutrino masses and mixings in gauge models with spontaneous parity violation, Phys. Rev. D 23 (1981) 165 [doi:10.1103/PhysRevD.23.165];
E. Ma and U. Sarkar, Neutrino masses and leptogenesis with heavy Higgs triplets, Phys. Rev. Lett. 80 (1998) 5716 [arXiv:hep-ph/9802445];
R. N. Mohapatra and P. Pal, Massive neutrinos in physics and astrophysics, (World Scientific, Singapore, 1991), p. 127.
[35] R. Foot, H. Lew, X. G. He, and G. C. Joshi, Seesaw neutrino masses induced by a triplet of leptons, Z. Phys. C 44 (1989) 441 [doi:10.1007/BF01415558].
[36] J. H. Huh, J. E. Kim, and B. Kyae, SU(5)_{lep} x SU(5)' from Z_{12−1}, Phys. Rev. D 80 (2009) 115012 [arXiv: 0904.1108 [hep-ph]].