Interactive Relay Assisted Source Coding

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Abstract—This paper investigates a source coding problem in which two terminals communicating through a relay wish to estimate one another’s source within some distortion constraint. The relay has access to side information that is correlated with the sources. Two different schemes based on the order of communication, distributed source coding/delivery and two cascaded rounds, are proposed and inner and outer bounds for the resulting rate-distortion regions are provided. Examples are provided to show that neither rate-distortion region includes the other one.

I. INTRODUCTION

Consider a distributed source coding problem in which two terminals A and B, each having access to a correlated source sequence $X^n$ and $Y^n$ respectively, wish to obtain lossless or lossy estimates of each other’s sources. The terminals are only able to interact over rate limited directed links through a relay (R), who observes a correlated sequence $Z^n$. This scenario, which we call interactive relay assisted source coding is illustrated in Fig 1. While in general multiple communication rounds through the relay are possible as in interactive source coding with multiple rounds [1], here we only assume that each link is used once.

One can envision various communication schemes depending on the order of communication. For example, first A and B can communicate with R, which then processes both signals and sends information back to the terminals. This scheme breaks the problem of interactive relay assisted source coding into a distributed source coding (DSC) phase [2] with no reconstruction constraints at the relay, and a delivery (broadcast) phase [3]. We will call this scheme DSC/delivery. Note that while this is similar to the scenario considered in [4], a main difference is that in the delivery phase, the relay has two separate rate-constrained links to the terminals. Another strategy is for A to first forward information to B through R, which then processes and sends information back to A through R. This scheme puts together two cascaded communication rounds [5] - one in each direction - and will be called two cascaded rounds. Our goal in this paper is to study inner and outer bounds for the rate-distortion regions of above two strategies and examine some conditions under which these bounds are tight. We also argue that neither strategy strictly dominates the other in general.

Interactive relay assisted source coding is related to several other problems investigated in the literature. The first step in DSC/delivery is closely related to lossy DSC, which is only solved for certain cases [6], [7]. The complementary delivery problem, which constitutes the broadcast phase of DSC/delivery scheme, is investigated in [3].

The cascade source coding problem, which considers only one round of the two cascaded round scheme (from A to R and then to B) is introduced by Yamamoto [5]. In [5] there is no side information, and R and B are interested in different lossy reconstructions of the source. Vasudevan et al. [8] further investigated this problem by letting R and B to have access to side information Z and Y respectively, where $X - Y = Z$. They considered the cases where A may or may not have access to Z and Y. In [9] B has side information and there are no distortion constraints at R.

This paper is organized as follows. In Section II we present the system model, notation and the two schemes, DSC/delivery and two cascaded rounds. For each scheme we characterize inner and outer bounds in Sections III and IV. In Section V we introduce some examples to show that although these bounds are not tight for general sources, they can be used to prove that neither scheme is in general dominant.

Fig. 1. Interactive relay assisted source coding.

II. PROBLEM SET-UP

A. System Model

We consider interactive relay assisted source coding shown in Fig 1. Terminal A observes source $X^n$, terminal B observes $Y^n$, and the relay, R, observes $Z^n$. The sequences are drawn i.i.d. $\sim p(x,y,z)$. The objective of terminal $j$, $j = A, B$ is to recover the source at terminal $k$, $k = B, A$, $j \neq k$ in a lossy fashion. To achieve this, messages $M_{AR}, M_{RB}, M_{BR},$ and $M_{RA}$ at rates $R_{AR}, R_{RB}, R_{BR},$ and $R_{RA}$ respectively are communicated among the terminals, where $M_{jk}$ is the message sent from $j$ to $k$. We consider the following two schemes, each with a specified communication order:

(a) DSC/delivery

In this scheme, the messages are sent in the following order: First A and B generate and send $M_{AR}$ and $M_{BR}$ respectively, to R. This is the DSC stage. The order of these two messages does not impact the resulting distortions at the terminals. After R gets both
messages it generates $M_{RA}$ and $M_{RB}$ and sends them to $A$ and $B$ respectively. Similar to the DSC stage, these two massages can be sent in any order. More formally the encoding functions are $M_{AR} = \phi^{(n)}_{AR}(X^n)$, $M_{BR} = \phi^{(n)}_{BR}(Y^n)$, $M_{RA} = \phi^{(n)}_{RA}(Z^n, M_{AR}, M_{RB})$ and $M_{RB} = \phi^{(n)}_{RB}(Z^n, M_{AR}, M_{RB})$.

(b) Two cascaded rounds

The order in which the messages are sent is given by $A$ to $R$, $R$ to $B$, $B$ to $R$ and then finally $R$ to $A$. Similarly, the communication can start from $B$, but in the rest of the paper we will assume $A$ starts the communication. This can be mathematically described by the following encoding functions: $M_{AR} = \phi^{(n)}_{AR}(X^n)$, $M_{BR} = \phi^{(n)}_{BR}(Y^n, M_{AR})$, $M_{RA} = \phi^{(n)}_{RA}(Z^n, M_{AR}, M_{RB})$ and $M_{RB} = \phi^{(n)}_{RB}(Z^n, M_{AR}, M_{BR})$.

The decoding functions in both schemes are given by: $\hat{Y}^n = \Psi_A^{(n)}(X^n, M_{RA})$ and $\hat{X}^n = \Psi_B^{(n)}(Y^n, M_{RB})$. Also note that in both schemes, $M_{jk} \in \{1, 2, \ldots, 2^{nR_{jk}}\}$ where $R_{jk}$ is the directed communication rate between terminals $j$ and $k$.

For any of the schemes the tuple $(\phi^{(n)}_{AR}, \phi^{(n)}_{BR}, \phi^{(n)}_{RA}, \phi^{(n)}_{RB}, \Psi_A^{(n)}, \Psi_B^{(n)})$ is called an $(n, R_{AR}, R_{BR}, R_{RA}, R_{RB}, D_A, D_B)$-code if the produced sequences $\hat{X}^n$ and $\hat{Y}^n$ satisfy $\frac{1}{n} \sum_{i=1}^{n} E_d(Y_i, \hat{Y}_i) \leq D_A$ and $\frac{1}{n} \sum_{i=1}^{n} E_d(B_i, \hat{X}_i) \leq D_B$, where $d_j(\cdot, \cdot)$, $j = A, B$ are single letter distortion functions at the corresponding terminals. The rate and distortion tuple $(R_{AR}, R_{BR}, R_{RA}, R_{RB}, D_A, D_B)$ is achievable if and only if there exists an $(n, R_{AR}+\epsilon, R_{BR}+\epsilon, R_{RA}+\epsilon, R_{RB}+\epsilon, D_A+\epsilon, D_B+\epsilon)$-code.

The set of all achievable $(R_{AR}, R_{BR}, R_{RA}, R_{RB}, D_A, D_B)$ is denoted by $\mathcal{R}_{sch}$, where $sch$ refers to one of the two schemes above. The interactive relay assisted rate-distortion region for that scheme is given by:

$$\mathcal{R}_{sch}(D_A, D_B) = \left\{(R_{AR}, R_{BR}, R_{RA}, R_{RB}) \mid (R_{AR}, R_{BR}, R_{RA}, R_{RB}, D_A, D_B) \in \mathcal{R}_{sch}\right\}$$

Fig. 2. Two terminal source coding with encoder and side information.

B. Notation

We will use the following notation based on the source coding set-up in Fig 2. When all switches are open, this set-up becomes identical to Wyner-Ziv problem [10], with rate-distortion function $R_{WZ}(Y,D)$. We let $R_{X|Z,Y}(D)$ be the rate-distortion function when switch 1 is closed, while switches 2 and 3 are still open. We also define $R_{X|Z,Y}(D)$ to be the rate-distortion function when switches 1 and 2 are closed while 3 is still open. Finally when all switches are closed we get $R_{X|Z}(D)$, the rate-distortion function with side information $YZ$ available both at the encoder and the decoder. With these definitions we have:

$$R_{X|Y,Z}(D) = R_{X|Z,Y}(D) = R_{X|Z,Y}(D) \geq R_{X|Y,Z}(D)$$

III. DSC/Delivery

A. Outer Bound

Theorem 1: If $(R_{AR}, R_{BR}, R_{RA}, R_{RB})$ is in the rate-distortion region of the DSC/delivery scheme, $\mathcal{R}_{DSCD}(D_A, D_B)$, then

$$R_{AR} \geq I(X; U_1 | Z, Y)$$
$$R_{BR} \geq I(Y; U_2 | Z, X)$$
$$R_{RA} \geq I(Y; V_2 | X, U_1)$$
$$R_{RB} \geq I(X; V_1 | Y, U_2)$$

for some $(U_1, U_2, V_1, V_2)$ where $U_1 - X - (Y, Z), U_2 - Y - (X, Z)$, and $(V_1, V_2) - (Z, U_1, U_2) - (X, Y)$ and decoding functions $Y = \Psi_A(X, U_1, Y_2)$ and $Y = \Psi_B(Y, U_2, V_1)$ such that $E_d(A, Y) \leq D_A$ and $E_d(B, X) \leq D_B$.

Proof: We first lower bound $R_{AR}$ as follows:

$$n R_{AR} \geq H(M_{AR}|Z^n, Y^n) - I(X^n; M_{AR}|Z^n, Y^n)$$
$$\geq H(X^n|Z^n, Y^n) - H(X^n|M_{AR}, Z^n, Y^n)$$
$$\geq \sum_{i=1}^{n} I(X_i; U_{2,i} | Z_i, Y_i)$$

where $U_{1,i} = (M_{AR}, X_{i-1}, Y^{n}_{i-1})$. Similarly $n R_{BR} \geq \sum_{i=1}^{n} I(Y_i; U_{2,i} | Z_i, X_i)$ with $U_{2,i} = (M_{BR}, X^{n}_{i+1}, Y^{n}_{i-1})$. For the delivery phase we have:

$$n R_{RA} \geq H(M_{RA}|M_{AR}, X^n) - I(Y^n; M_{RA}|M_{AR}, X^n)$$
$$= \sum_{i=1}^{n} I(Y_i; M_{RA} | M_{AR}, X^n, Y^n_{i+1})$$
$$\geq \sum_{i=1}^{n} I(Y_i; M_{RA} | X^n_{i+1} | M_{AR}, X^n, Y^n_{i+1})$$

where $V_{2,i} = M_{RA}$. Similarly with $V_{1,i} = M_{RB}$, we have $n R_{RB} \geq \sum_{i=1}^{n} I(V_{1,i} | X_i, U_{1,i}, Y_{2,i})$. It is easy to verify that $U_{1,i}, U_{2,i}, V_{1,i} and V_{2,i}$ satisfy the required Markov conditions.

Define $\Psi^{(n)}_{A,i}$ to be the function that maps $(X^n, M_{RA})$ to the $i^{th}$ symbol of $\Psi^{(n)}_{B,i}(X^n, M_{RA})$ for the code at hand. Similarly let $\Psi^{(n)}_{B,i}$ be the function that maps $(Y^n, M_{RB})$ to the $i^{th}$ symbol of $\Psi^{(n)}_{B,i}(Y^n, M_{RB})$. We next argue that there exist deterministic functions $\Psi_{A,i}(X_i, U_{1,i}, V_{2,i})$ and $\Psi_{B,i}(Y_i, U_{2,i}, V_{1,i})$ that achieve distortions no larger than what
\( \Psi_{A,i}^{(n)}(X^n, M_{RA}) \) and \( \Psi_{B,i}^{(n)}(Y^n, M_{RB}) \) can achieve respectively. We define \( D_{A,i} \) as
\[
D_{A,i} \triangleq E\left[ d_A(Y_i, \hat{Y}_i) \right] = E_{X^n, Y^n, M_{RA}} \left[ d_A(Y_i, \Psi_{A,i}^{(n)}(X^n, M_{RA})) \right]
\]

\( \Psi_{A,i}^{(n)}(x^n, M_{RA}) \triangleq \arg\min_{x_{i+1}^{n-i}} \mathbb{E}(d_A(Y_i, X^n_i)) \)

Let
\[
\tilde{x}_{i+1}^n(x^i, y_{i+1}^i, m_{AR}, m_{RA}) \triangleq \arg\min_{x_{i+1}^{n-i}} \mathbb{E}(d_A(Y_i, X^n_i)) \]

and define
\[
\Psi_{A,i}(x_i, u_{1,i}, v_{2,i}) \triangleq \Psi_{A,i}^{(n)}(x^i, \tilde{y}_{i+1}^i, m_{AR}, m_{RA}), m_{RA}) \]

It is easy to show that \( Y_i = (X_i, U_{1,i}, V_{2,i}) - X_i^n \). Using this Markov chain we have:
\[
E\left[ d_A(Y_i, \Psi_{A,i}(X_i, U_{1,i}, V_{2,i})) \right] \leq D_{A,i}
\]

Similarly we can define \( \Psi_{B,i}(Y_i, U_{2,i}, V_{1,i}) \). Using these single letter decoders the rest of the proof follows from convexity of the proposed outer bound.

**Remark 1:** The following simple outer bound:
\[
R_{AR} \geq R_X^{WZ}(D_B) \]
\[
R_{BR} \geq R_{YX}^{WZ}(D_A) \]
\[
R_{RA} \geq R_{YX}^{WZ}(D_A) \]
\[
R_{RB} \geq R_{XY}^{WZ}(D_B) \]

which can be found by considering the cuts between each terminal \( j \) and super-node \( (k, R_j) \), \( j, k = A, B \) and \( j \neq k \) is looser than the bound in Theorem 1. This is because the cut-set bound is found by optimizing each rate separately with respect to auxiliary random variables, while in Theorem 1 a joint optimization is considered.

**B. Inner Bound**

**Theorem 2:** Any rate vector \( (R_{AR}, R_{BR}, R_{RA}, R_{RB}) \) satisfying
\[
R_{AR} \geq I(X; U_1, Z, U_2, Q) \quad (1)
\]
\[
R_{BR} \geq I(Y; U_2, Z, U_1, Q) \quad (2)
\]
\[
R_{RA} \geq I(\hat{X}^2, Z, U_1, U_2, Q) \quad (3)
\]
\[
R_{RB} \geq I(U_2, Z, V_2, X, U_1, Q) \quad (4)
\]

for some \( p(x, y, z)p(q)p(u_1|x, q)p(u_2|y, q)p(v_1, v_2|u_1, u_2, z, q) \) and some decoding functions \( \hat{Y} = \Psi_B(Y, V_2) \) and \( \hat{X} = \Psi_A(X, V_2) \) such that \( Ed_A(x, \hat{Y}) \leq D_B \) and \( Ed_B(X, \hat{X}) \leq D_B \) and then a proper description of it is forwarded to \( B \). Finally, recompression is done by first recovering the information and then putting it together with the side information available at

**Proof:** Terminals \( A \) and \( B \) use Berger-Tung coding \( \mathcal{E} \) to encode their sources into \( U_1 \) and \( U_2 \) respectively. The relay exploits its side information \( Z \) and recovers \( (U_1, U_2) \) provided that \( (1)-(4) \) are satisfied. The relay then uses Wyner-Ziv coding to communicate with each terminal leading to (5) and (6).

**IV. TWO CASCADED ROUNDS**

**A. Outer Bound**

**Theorem 3:** If \( (R_{AR}, R_{RB}, R_{BR}, R_{RA}) \) is in the rate-distortion region of the two cascaded rounds scheme, \( \mathcal{R}_{TCR}(D_A, D_B) \), then
\[
R_{AR} \geq R_{WZ}^{WZ}(D_B) \quad (6)
\]
\[
R_{RB} \geq R_{WZ}^{WZ}(D_B) \quad (7)
\]
\[
R_{BR} \geq R_{WZ}^{WZ}(D_B) \quad (8)
\]
\[
R_{RA} \geq R_{WZ}^{WZ}(D_B) \quad (9)
\]

**Proof:** Considering the cut between \( A \) and the super-node \( (R, B) \), we get (6). The inequality (7) comes from considering the cut between super-source \( (A, R) \) and the terminal \( B \). For communication between \( B \) and \( R \), consider the cut between \( B \) and the super-node \( (R, A) \), and assume that \( X \) is known to all terminals, which leads to (8). Now consider the cut that separates \( A \) and the super-source \( (R, B) \). With \( X \) known to all terminals, having the extra side information \( Z \) at the encoder does not help, hence we get (9).

**B. Inner Bound**

**Theorem 4:** Any rate vector \( (R_{AR}, R_{RB}, R_{BR}, R_{RA}) \) satisfying:
\[
R_{AR} \geq I(X; U_0, U_1, W_1|Z) + I(X; V_1|Y, U_1) \quad (10)
\]
\[
R_{BR} \geq I(Y; U_2, W_2|X, U_0, U_1, W_1) \quad (11)
\]
\[
R_{RA} \geq I(Y; U_2, W_2|X, U_0, U_1, W_1) \quad (12)
\]
\[
R_{RB} \geq I(Z; U_2, W_2|X, U_0, U_1, W_1, V_1) \quad (13)
\]

for some \( p(x, z)p(u_0, u_1, v_1) \) and some decoding functions \( \hat{Y} = \Psi_B(Y, V_2) \) and \( \hat{X} = \Psi_A(X, V_2) \) such that \( Ed_A(x, \hat{Y}) \leq D_B \) and \( Ed_B(X, \hat{X}) \leq D_B \) is in the rate-distortion region of two cascaded rounds scheme, \( \mathcal{R}_{TCR}(D_A, D_B) \).

**Proof:** From \( A \) to \( R \) to \( B \) we consider a communication scheme consisting of four separate flows: private message for \( R \), simple forward, recover and forward and recompression at \( R \). As the name suggests private message is the part of information that is only intended to be received by \( R \). Simple forward refers to \( R \) forwarding the received description without recovering the underlying information stream. In recover and forward, first the information is recovered at \( R \) and then a proper description of it is forwarded to \( B \). Finally, recompression is done by first recovering the information and then putting it together with the side information available at
\( R \) to compress it again. Note that availability of private and common (forwarded) messages at \( R \) facilitate communication by providing side information for the \( B \) to \( R \) to \( A \) communication.

In order to accomplish the above, terminal \( A \) uses a coding scheme similar to Wyner-Ziv by first covering \( X \) by \( U_0, V_1, U_2 \) and \( W_1 \) and then creating proper descriptions via binning. The four codebooks correspond to private message, simple forward, recover and forward and recompression schemes respectively. Therefore the binning for \((U_0, U_1, W_1)\)-triplet is done with respect to the side information at \( R \), which results in the first term in (10) and for \( V_1 \), the binning is done with respect to \((Y, U_1)\). Upon receiving these bin indices, \( R \) forwards the bin index corresponding to \( V_1 \) to \( B \), and then uses its side information to recover \( U_0, U_1 \) and \( W_1 \). The relay keeps \( U_0 \) as its own private message. To forward \( U_1 \), \( R \) creates a new description of it with respect to the side information at \( B \) which can be done with any rate higher than \( I(X; U_1|Y) \). Putting this together with the rate needed to forward the description of \( V_1 \) to \( B \), gives the first term in (11). The recompression is done by treating \((Z, W_1)\) as a new super-source at the relay and using Wyner-Ziv coding by first covering it by \( S_1 \) and then creating a proper description of it by binning it with respect to \((Y, U_1, V_1)\) resulting in the second term in (11). Upon receiving these, \( B \) first uses its side information, \( Y \) to recover \( U_1 \) and \( V_1 \). It then uses \((Y, U_1, V_1)\) to recover \( S_1 \).

Similarly the communication from \( B \) to \( R \) to \( A \) is carried out by considering \( Y \) as the source, \((U_1, V_1, S_1)\) as the transmitter’s side information, \((Z, U_0, U_1, W_1, S_1)\) as the side information at \( R \) and \((X, U_0, U_1, V_1, W_1)\) as the side information at \( A \). Note that private message is only needed to improve the side information at \( R \) for further communication from either terminal to \( R \). Hence, there is no need to generate and send a private message. Using similar arguments, it is easy to show the inequalities (12) and (13).

\[ R_{AR} \geq I(X; U_1|Y) \]

\[ R_{BR} \geq I(Y; U_2|X, Z, U_1) \]

\[ \text{Lemma 1} \]

\[ R_{AR} \geq I(X; U_1|Y) \]

\[ R_{BR} \geq I(Y; U_2|X, Z, U_1) \]

\[ \text{Theorem 3} \]

\[ R_{AR} \geq I(X; U_1|Y) \]

\[ R_{BR} \geq I(Y; U_2|X, Z, U_1) \]

\[ \text{Theorem 3} \]

\[ R_{AR} \geq I(X; U_1|Y) \]

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