DISCREPANCY RESULTS FOR THE VAN DER CORPUT SEQUENCE

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Abstract. Let \( d_N = N D_N(\omega) \) be the discrepancy of the Van der Corput sequence in base 2. We improve on the known bounds for the number of indices \( N \) such that \( d_N \leq \log N/100 \). Moreover, we show that the summatory function of \( d_N \) satisfies an exact formula involving a 1-periodic, continuous function. Finally, we show that \( d_N \) is invariant under digit reversal in base 2.

1. Introduction

Every nonnegative integer \( n \) admits a unique expansion \( n = \sum_{\nu=0}^{\nu} \varepsilon_{\nu} 2^i \) such that \( \nu = 0 \) or \( \varepsilon_{\nu} \neq 0 \). We let \( \varepsilon_{\nu}(n) \) denote the \( \nu \)-th digit in base 2. The Van der Corput sequence is defined via the radical inverse of \( n \) in base 2: define \( \omega_n = \sum_{i=0}^{\nu} \varepsilon_{i}(n)2^{-i-1} \).

Let \( x = (x_n)_{n \geq 0} \) be a sequence in \([0, 1)\). The discrepancy \( D_N(x) \) of \( x \) is defined by
\[
D_N(x) = \sup_{0 \leq a \leq b \leq 1} \left| \frac{A_N(x, a, b)}{N} - \frac{b}{N} \right|
\]
for \( N \geq 1 \), where \( A_N(x, a, b) = |\{ n < N : a \leq x_i < b \}| \). Moreover, we set \( D_0(x) = 0 \). The star-discrepancy (or discrepancy at the origin) of a sequence \( x \) in \([0, 1)\) is defined by \( D_N^*(x) = \sup_{0 \leq b \leq 1} \left| \frac{A_N(x, 0, b)}{N} - \frac{1}{N} \right| \), for \( N \geq 1 \), and we set \( D_0^*(x) = 0 \).

In this paper, we are concerned with the discrepancy of the Van der Corput sequence. We define
\[
d_N = N D_N(\omega),
\]
and we will use this notation throughout this paper. It is well known [2 Théorème 1] that the star discrepancy of the Van der Corput sequence equals its discrepancy: we have \( D_N^*(\omega) = D_N(\omega) \). The Van der Corput sequence is a low discrepancy sequence, that is, we have \( d_N \ll \log N \). More precise results are known: Béjian and Faure [2] proved the following theorem.

Theorem A.
\[
d_N \leq \frac{1}{3} \log_2 N + 1
\]
for all \( N \geq 1 \); moreover
\[
\limsup_{N \to \infty} \left( d_N - \frac{1}{3} \log_2 N \right) = \frac{4}{9} + \frac{1}{3} \log_2 3,
\]
where \( \log_2 \) denotes the logarithm in base 2.

In the proof of these statements, they implicitly show that \( d_N \) is bounded above by the polygonal path connecting the first maxima on the intervals \( I_k = [2^{k-1}, 2^k] \), given by the points \((\frac{1}{2} (2^{k+1} + (-1)^k), \frac{7}{5} + (-1)^k/(9 \cdot 2^{k-1}))\). This should be compared to the argument given by

2010 Mathematics Subject Classification. Primary: 11K38, 11A63; Secondary: 11K31.

Key words and phrases. Van der Corput sequence, irregularities of distribution, digit reversal.

The author acknowledges support by projects F5502-N26 and F5505-N26 (FWF), which are part of the Special Research Program “Quasi Monte Carlo Methods: Theory and Applications”. Moreover, the author thanks the Erwin Schrödinger Institute for Mathematics and Physics where part of this paper was written during his visit for the programme “Tractability of High Dimensional Problems and Discrepancy”.

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Coons and Tyler [5] concerning Stern’s diatomic sequence (also called Stern–Brocot sequence), see also the paper by Coons and Spiegelhofer [3] and the recent paper by Coons [3].

Concerning the “usual” order of magnitude of the discrepancy of the Van der Corput sequence, Drmota, Larcher and Pillichshammer [7, Theorem 2] proved a central limit theorem for $d_N$.

**Theorem B.** For every real $y$, we have

$$\frac{1}{M} \left\lfloor N < M : d_N \leq \frac{1}{4} \log_2 N + y \frac{1}{4\sqrt{3}} \sqrt{\log_2 N} \right\rfloor = \Phi(y) + o(1),$$

where $\Phi(y) = \frac{1}{2\pi} \int_{-\infty}^{y} e^{-t^2/2} dt$.

We note that this implies in particular that $d_N$ is usually of order $\log N$. More precisely, letting $A_{M,y}$ denote the expression on the left hand side of (11), we trivially have $A_{M,y'} \leq A_{M,y}$ if $y' \leq y$. This implies, for any sequence $(y_M)_{M \geq 1}$ of reals such that $y_M \to -\infty$ for $m \to \infty$, that

$$\lim_{M \to \infty} A_{M,y(M)} \leq \lim_{M \to \infty} A_{M,y} = \Phi(y)$$

for all real $y$, therefore this limit equals 0. In particular, if $\delta < 1/4$, the number of integers $N < M$ such that $d_N \leq \delta \log N$ is $o(M)$.

Bounds of this type, with an explicit error term, had been proved earlier: Sós [19] proved such a statement for $\{na\}$-sequences, more generally Tijdeman and Wagner [22] showed that any sequence in $[0,1)$ has almost nowhere small discrepancy. More specifically, they proved the following theorem.

**Theorem C.** Let $\xi$ be a sequence in $[0,1)$. Let $M$ and $N$ be integers with $M \geq 0$ and $N > 1$. Then $D_n(\xi) < \log N/100$ for at most $2N^{5/6}$ integers $n$ with $M < n \leq M + N$.

In fact, it follows from Lemma 2 in their paper [22] that the exponent $5/6$ can be replaced by an arbitrarily small positive value if we demand an arbitrarily small constant in place of $1/100$.

**Corollary.** Let $\xi$ be a sequence in $[0,1)$. For each $\varepsilon > 0$ there exists a constant $\delta > 0$ such that for all integers $M \geq 0$ and $N > 1$ we have $D_n(\xi) < \delta \log N$ for at most $2N^{\varepsilon}$ integers $n$ with $M < n \leq M + N$.

We proceed to the statement of our results.

2. Results

We wish to show that the constant $5/6$ in Theorem C can be improved at least for the Van der Corput sequence.

**Theorem 2.1.** For all large $N$, the number of $n < N$ satisfying $d_n \leq \log N/100$ is bounded above by $N^{0.183}$.

Moreover, Tijdeman and Wagner [22, Theorem 3] showed that for infinitely many $N$ we have $d_n \leq \log N/100$ for more than $N^{1/21}$ integers $n \in [1, N]$. We wish to improve on the exponent $1/21$.

**Theorem 2.2.** For all large $N$, the number of $n < N$ satisfying $d_n \leq \log N/100$ is bounded below by $N^{0.056}$.

It would be interesting to determine, for each given $\varepsilon > 0$, the exact “exponent of strong irregularity” of the Van der Corput sequence. That is, determine the infimum of $\eta$ such that the number of $n < N$ satisfying $d_n \leq \varepsilon \log N$ is bounded by $N^{\eta}$, for all large $N$. By the above results this infimum, for $\varepsilon = 1/100$, lies in $[0.056, 0.183]$. We leave this as an open question.

Next, we consider partial sums

$$S(N) = d_1 + \cdots + d_N.$$
It was shown by Béjian and Faure [2] that
\[ \frac{1}{N} \sum_{k=1}^{N} d_k = \frac{\log_2 N}{4} + O(1), \]
where \( \log_2 N \) denotes the base-2 logarithm of \( N \). We are interested in the error term appearing in this expression. It turns out that there exist an exact formula involving a 1-periodic, continuous function (see, for example, the papers by Delange [6] and Flajolet et al. [11]).

**Theorem 2.3.** There exists a continuous, 1-periodic function \( \psi : \mathbb{R} \to \mathbb{R} \) such that
\[ \frac{1}{N} S(N) = \frac{\log_2 N}{4} + \frac{d_N}{2N} + \psi(\log_2 N). \]
The function \( \psi \) is uniquely determined.

In particular, we obtain the boundedness result of the error term given by Béjian and Faure.

Our third result is concerned with digit reversal: If \( \varepsilon_{\nu} \cdots \varepsilon_0 \) is the proper binary expansion of \( n \), we define
\[ n^R = \sum_{0 \leq i \leq \nu} \varepsilon_{\nu-i} 2^i. \]

**Theorem 2.4.** Assume that \( \alpha, \beta, \gamma \) are complex numbers and that the sequence \( x \) satisfies
\[ x_{2n} = x_n \text{ and } x_{2n+1} = \alpha x_n + \beta x_{n+1} + \gamma \text{ for } n \geq 1. \]
Then for \( n \geq 1 \) we have
\[ x_n = x_n^R. \]
This theorem generalizes Theorem 2.1 in the paper [21] by the author, see also Morgenbesser and the author [16] and the recent paper by the author [20]. We obtain the following, somewhat curious, corollary.

**Corollary 2.5.**
\[ ND_N(\omega) = N^R D_{N^R}(\omega). \]

We note, however, that this digit reversal property seems to be restricted to base 2. That is, the Van der Corput sequence in base \( q \), where \( q \geq 3 \), does not seem to satisfy an analogous property with respect to digit reversal in base \( q \). We refer the reader to [9, 10, 13, 17] concerning results on the discrepancy and diaphony of digital sequences. Among these one can find explicit formulas for the star discrepancy analogous to (3). For illustration, we list the first values of \( d_N = ND_N(\omega) \).

| \( N \) | 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 |
| --- | --- |
| \( d_N \) | 0 1 \( \frac{3}{2} \) 1 \( \frac{7}{4} \) \( \frac{3}{2} \) \( \frac{7}{4} \) 1 \( \frac{15}{8} \) \( \frac{7}{8} \) \( \frac{17}{8} \) \( \frac{3}{2} \) \( \frac{17}{8} \) \( \frac{7}{8} \) \( \frac{15}{8} \) |

Apart from the identity \( d_N = d_{2^k-N} \), which is valid for \( 2^{k-1} \leq N \leq 2^k \) and which can be shown easily by induction, we see the notable identity \( d_{2^k} = 25 \). Note that \( 19^R = 25 \).

The remainder of this paper is dedicated to the proofs of our results.

3. PROOFS

We will use the following explicit formula due to Béjian and Faure [2].
\[ d_N = \sum_{j=1}^{\infty} \left\| \frac{N}{2^j} \right\|. \]

Based on this result Béjian and Faure proved that \( d_N \) satisfies the following recurrence:
\[ d_0 = 0, \quad d_1 = 1, \quad d_{2N} = d_N, \quad d_{2N+1} = \frac{d_N + d_{N+1} + 1}{2}, \]
which is valid for all $N \geq 0$.

We note that $(d_n)_{n \geq 0}$ is a 2-regular sequence in the sense of Allouche and Shallit [1]. Moreover, the recurrence is of the discrete divide-and-conquer type [8, 12].

3.1. Proof of Theorems 2.1 and 2.2. In order to prove these theorems, we state a couple of lemmas. We let $|N|_01$ denote the number of occurrences of 01 in the binary expansion of $N$.

**Lemma 3.1.** We have

\[ \frac{1}{2} |N|_{01} \leq d_N \leq 2|N|_{01}. \] (5)

**Proof.** We use the formula $d_N = \sum_{j=1}^{\infty} \| \frac{N}{2^j} \|$. Assume that $m = |N|_01$. For $0 \leq k < m$ let $a_k$ be the index corresponding to the beginning of the $k$-th block of 1s, and $b_k$ be the index corresponding to the end. We prove the first inequality first. We have

\[ \sum_{j=0}^{a_k+2} \| \frac{N}{2^j} \| = \sum_{j=0}^{a_k+2} \| \frac{N}{2^{a_k+2+j}} \| \geq \sum_{j=0}^{a_k+2} \| \frac{1}{2^j} \| = 1/2, \]

moreover for $0 \leq k < m - 1$

\[ \| \frac{N}{2^{b_k+1}} \| \geq \| 1/2 + 1/8 + 1/16 + \cdots \| = 1/4 \]

and for $1 \leq k < m$

\[ \| \frac{N}{2^{a_k+2}} \| \geq \| 1/4 \| = 1/4. \]

To conclude the proof of the first inequality, we note that the indices $b_k + 1$ and $a_k + 2$ are pairwise different.

As for the second inequality, we bound the contribution of each block of 1s by 2 as follows. For simplicity of the argument, we set $b_{-1} = \infty$. We have

\[ d_N = \sum_{j=1}^{\infty} \| \frac{N}{2^j} \| = \sum_{-1 \leq k < m-1} \left( \sum_{j=a_k+2}^{b_k} \| \frac{N}{2^j} \| + \sum_{j=b_k+1}^{a_{k+1}+1} \| \frac{N}{2^j} \| \right). \]

The summands are bounded above by geometric series with quotient $q = 1/2$, which yields the second inequality. \(\square\)

We note that the constant 2 is optimal, which can be seen by considering integers having the binary expansion $(0^*1)^k$ and letting $s \to \infty$. The constant 1/2 probably can be improved, but not beyond 2/3, which follows by considering integers of the form $(01)^k$ and letting $k \to \infty$. The next lemma is concerned with counting occurrences of 01 in the binary expansion.

**Lemma 3.2.** For $k, \ell \geq 0$ set

\[ a_{k,\ell} = \left| \{ n \in [2^k, 2^{k+1}) : |n|_1 = \ell \} \right|. \]

Then

\[ a_{k,\ell} = \left( \frac{k + 1}{2\ell - 1} \right). \]

**Proof.** We are interested in the set $A$ of integers $n \in [2^k, 2^{k+1})$ having exactly $\ell$ blocks of consecutive 1s. We define a bijection $\varphi$ from $A$ onto the set of $2\ell - 1$-element subsets of $\{0, \ldots, k\}$ as follows. The binary expansion $\varepsilon_k \cdots \varepsilon_0$ of $n$ consists of $\ell$ blocks of consecutive 1s and $\ell - 1$ or $\ell$ blocks of consecutive 0s. Let $\varphi(n)$ consist of those indices $i \in \{0, \ldots, k\}$ corresponding to the rightmost element of a block of 1s or to the rightmost element of one of the first $\ell - 1$ blocks of 0s. It is clear how to construct the inverse function. \(\square\)
We are interested in the quantity

\[ A_{k,\varepsilon} = \left| \{ N \in [2^k, 2^{k+1}) : d_N \leq \varepsilon \log N \} \right|. \]

By (5) and Lemma 3.2 we have

\[
A_{k,\varepsilon} \leq \left| \{ N \in [2^k, 2^{k+1}) : |N|_0 \leq 2\varepsilon \log 2^{k+1} \} \right|
\]

(6)

\[
= \sum_{\ell=0}^{2\varepsilon(k+1)\log 2} a_k,\ell = \sum_{\ell=0}^{2\varepsilon(k+1)\log 2} \left( k+1 \right) \left( \frac{1}{2\ell-1} \right)
\]

and

\[
A_{k,\varepsilon} \geq \left| \{ N \in [2^k, 2^{k+1}) : |N|_0 \leq (\varepsilon/2) \log 2^k \} \right|
\]

(7)

\[
= \sum_{\ell=0}^{(\varepsilon/2)k\log 2} a_k,\ell = \sum_{\ell=0}^{(\varepsilon/2)k\log 2} \left( k+1 \right) \left( \frac{1}{2\ell-1} \right).
\]

We are therefore interested in large deviations of the binomial distribution. To this end, we state the following lemma.

**Lemma 3.3.** Assume that \( k, \ell \geq 1 \) are integers, \( \alpha, \beta \in (0, 1/e) \) real numbers, where \( e = 2.71828 \ldots \), and \( \alpha k \leq \ell \leq \beta k \). Then

\[
\frac{1}{3\sqrt{k}} \left( \frac{\alpha^{-\alpha}}{(1-\alpha)^{1-\beta}} \right)^k \leq \left( \frac{k}{\ell} \right) \leq \left( \frac{\beta^{-\beta}}{(1-\beta)^{1-\alpha}} \right)^k.
\]

**Proof.** For all \( n \geq 1 \), we have the estimate (see Robbins [18])

\[
\sqrt{2\pi n^{n+1/2}} e^{-n} \leq n! \leq e(n+1/2) e^{-n}.
\]

Noting that the maximum of \( \ell \mapsto \ell^{-1/2}(k-\ell)^{-1/2} \) for \( 1 \leq \ell \leq k\alpha \) is attained at \( \ell = 1 \), it follows that

\[
\left( \frac{k}{\ell} \right) \leq \frac{\sqrt{k}}{\sqrt{k-1} \cdot 2\pi (k-1)^{1/2-1}} \leq \frac{k^k}{\ell^\ell (k-\ell)^{-\ell-1}} = \left( \frac{k}{\ell} \right) \left( 1 + \frac{\ell}{k-\ell} \right)^{k-\ell}.
\]

Since \( \ell \mapsto (k/\ell)^\ell \) is increasing for \( \ell \in [1, n/e] \), it follows that

\[
\left( \frac{k}{\ell} \right) \leq \left( \frac{k}{\ell} \right) \left( 1 + \frac{k\beta}{k(1-\beta)} \right)^{k(1-\alpha)}.
\]

This implies the second inequality. The proof of the first inequality is similar. \( \square \)

In order to prove Theorems 2.1 and 2.2 we combine Lemma 3.3 and the estimates (6), (7). From (6) and the lemma, we obtain for large \( k, \beta = 4 \log 2/100 \) and \( \alpha = \beta - \delta \), where \( \delta > 0 \) is small,

\[
A_{k,1/100} \leq 2/100(k+1) \log 2 + 1 \leq \frac{2\varepsilon}{\varepsilon k} \max_{\ell \leq \frac{k+k}{2\ell-1}} \left( \frac{k+1}{2\ell-1} \right) \leq 2/100(k+1) \log 2 + 1 \left( \beta^{-\beta}/(1-\beta)^{1-\alpha} \right)^{k+1}
\]

for all \( k \geq 1 \). Since

\[
\left| \{ n < N : d_n \leq \log n/100 \} \right| \leq A_{0,1/100} + \cdots + A_{L,1/100},
\]

where \( 2^L \leq N < 2^{L+1} \), we easily obtain the first theorem by noting that \( \log(\beta^{-\beta}/(1-\beta)^{1-\alpha}) \log(2) < 0.183 \).
To prove Theorem 2.2, we note that for large $k$ we obtain from (7) and Lemma 3.3 setting $\beta = \log 2/100$ and $\alpha = \beta - \delta$,

$$A_{k,1/100} \geq \left( \frac{k + 1}{2(1/200)k \log 2} - 1 \right) \geq \frac{1}{3} \frac{\alpha - \alpha}{(1 - \alpha)^{1 - \beta}}^k.$$ 

This implies the statement of Theorem 2.2 noting that $\log(\beta^{-\delta}/(1 - \beta)^{1 - \beta})/\log(2) > 0.056$.

3.2. Proof of Theorem 2.3. We define $S'(N) = S(N) - d_N/2 = d_1 + \cdots + d_{N-1} + d_N/2$. By splitting the sum into even and odd indices and using the recurrence (4), we obtain

$$S'(2N) = \sum_{k=1}^{N-1} d_{2k} + \sum_{k=0}^{N-1} d_{2k+1} + \frac{d_2N}{2} = S'(N) + \frac{1}{3} \sum_{k=0}^{N-1} (d_k + d_{k+1} + 1)$$

(8)

$$= S'(N) + \frac{1}{3} \sum_{k=1}^{N} d_k + \frac{1}{3} \sum_{k=1}^{N} d_k + \frac{N}{2} = 2S'(N) + \frac{N}{2}.$$

Define

$$R(N) = \frac{1}{N} S'(N) - \frac{1}{4} \log_2 N.$$

By a simple calculation using (3) we obtain

$$R(2N) = R(N).$$

We may therefore define a 1-periodic function $\psi$ defined on the set $\{\log_2 N : N \in \mathbb{N}\} + \mathbb{Z}$ as follows: if $x = \log_2 N + \ell$, where $N$ is odd and $\ell \in \mathbb{Z}$, we set $\psi(x) = R(N)$. Using the identity $R(2N) = R(N)$, it is easy to see that this is well-defined, moreover (2) holds.

To prove Theorem 2.2, we note that for large $k$ we obtain

$$\psi_k([\log_2(x)]) = \frac{1}{x} F_k(x) - \frac{1}{4} \log_2 x \quad \text{for} \quad 2^{k-1} \leq x < 2^k.$$

Moreover, set $\psi_k(1) = F_k(2^k)/2^k - k/4$. We have $\psi_k(0) = \psi_k(1) = 1/2$ by (3). Note that each $z \in [0,1)$ is hit exactly once by the function $\{\log_2(x)\}$, therefore $\psi_k$ is uniquely determined. Moreover the height of the jumps of $\psi_k : [0,1] \to \mathbb{R}$ is bounded by $O(k/2^k)$. We first show pointwise convergence of the sequence $(\psi_k)_k$. The statement is clear for $z = \{\log_2 N\}$ and also for $z = 1$. Assume that $z \in [0,1)$ is not of this form. Choose, for each $k \geq 1$,

$$N_k = \max \{ N \in [2^{k-1}, 2^k] : \{\log_2 N_k\} \leq z \}.$$

We consider the sequence of values $\psi_k(\{\log_2 N_k\})$. Note that $N_{k+1} \in \{2N_k, 2N_k + 1\}$. Trivially, we have $|\psi_{k+1}(\{\log_2 (2N_k)\}) - \psi_k(\{\log_2 N_k\})| = 0$. By (3) we have

$$|\psi_{k+1}(\{\log_2 (2N_k + 1)\}) - \psi_k(\{\log_2 N_k\})|$$

$$= \frac{1}{2N_k + 1} S'(2N_k + 1) - \frac{1}{4} \log_2(2N_k + 1) - \left( \frac{1}{N_k} S'(N_k) - \frac{1}{4} \log_2 N_k \right)$$

$$= \frac{1}{2N_k} \left( 2S'(N_k) + \frac{N_k}{2} + \frac{d_{2N_k} + d_{2N_k + 1}}{2} \right) + \left( \frac{1}{2N_k + 1} - \frac{1}{N_k} \right) S'(2N_k + 1)$$

$$+ \frac{1}{4} \left( \log_2(2N_k) - \log_2(2N_k + 1) \right) - \frac{1}{4} \left( \log_2(2N_k) - \log_2(2N_k) - S'(N_k) \right)$$

$$= \frac{d_{2N_k} + d_{2N_k + 1}}{4N_k} - \frac{S'(2N_k + 1)}{2N_k} + \frac{1}{4} (\log_2(2N_k) - \log_2(2N_k + 1)).$$
Using the estimate $S'(N_k) = O(N_k \log(N_k))$, which follows from Theorem A, we obtain

$$
|\psi_{k+1}(\{\log_2 N_{k+1}\}) - \psi_k(\{\log_2 N_k\})| \leq C \frac{\log N_k}{N_k} \leq C \frac{k}{2^k}, \tag{10}
$$

where the constant $C$ is independent of $z$.

Moreover, let $x \in [2^{k-1}, 2^k)$ be such that $z = \{\log_2 x\}$. Note that $N_k < x < N_k + 1$. We have

$$
|\psi_k(\{\log_2 x\}) - \psi_k(\{\log_2 N_k\})| \leq \frac{1}{x} S'(\lfloor x \rfloor) - \frac{1}{N_k} S'(N_k) + \frac{1}{4} |\log_2 x - \log_2 N_k|
$$

$$
\leq S'(N_k) \left( \frac{1}{N_k} - \frac{1}{x} \right) + \frac{1}{4N_k} \leq C' \frac{\log N_k}{N_k} \leq C \frac{k}{2^k}, \tag{11}
$$

where the constant $C$, without loss of generality, is the same as in (10).

We define $K_k = C \sum_{i \geq 1} \frac{1}{4^i}$. We note that $K_k \to 0$ as $\ell \to \infty$. Let $I_k$ be the symmetric interval of length $2K_k$ around $\psi_k(\{\log_2 N_k\})$. By (11) and the triangle inequality we have $I_k+1 \subseteq I_k$, moreover (11) implies $\psi_k(z) \in I_k$. By the nested intervals theorem the sequence $(\psi_k)_{k \geq 1}$ converges pointwise to a function that we call $\psi$. Since both of $\psi_k(z)$ and $\psi(z)$ lie in the interval $\psi_k(\{\log_2 N_k\}) \pm K_k = I_k$, the number $\psi_k(z)$ lies in the interval $\psi(z) \pm 2K_k$ for all $z \in [0, 1)$ and $k \geq 1$, therefore the sequence $(\psi_k)_{k \geq 1}$ of functions converges uniformly to $\psi$. We need to show continuity of $\psi$. Let $z \in [0, 1]$ and assume that $\varepsilon > 0$. Choose $k$ so large that the height of the jumps of $\psi_k$ is bounded by $\varepsilon/3$ and also such that $\sup_{0 \leq y \leq 1} |\psi(y) - \psi_k(y)| < \varepsilon/3$. Let $\delta$ be so small that $\psi_k$ has at most one jump in the interval $[z - \delta, z + \delta] \cap [0, 1]$. Application of the triangle inequality and noting that $\psi_k$ is nonincreasing between the jumps finishes the proof of continuity. Moreover, we have $\psi_k(0) = \psi_k(1) = 1/2$, therefore the continuation to $\mathbb{R}$ is continuous.

We note that similar reasoning can be applied to Stern’s diatomic sequence defined by $s_1 = 1$, $s_{2n} = s_n$ and $s_{2n+1} = s_n + s_n+1$ for $n \geq 1$. The partial sums $S'(N) = s_1 + \cdots + s_{N-1} + s_N/2$ satisfy $S'(2N) = 3S'(N)$, moreover the maximum of $s_n$ on dyadic intervals $[2^k, 2^{k+1})$ is $F_k$, where $F_k$ is the $k$-th Fibonacci number (see Lehmer [14] and Lind [15]). We obtain a representation of the partial sums $S(N) = s_1 + \cdots + s_N$:

$$S_N = N^{\log_2 3} \psi(\log_2 N) + \frac{s_N}{2},$$

where $\psi$ is continuous and 1-periodic.

3.3. **Proof of Theorem 2.4** The proof is an adaptation of the proof of [21, Theorem 2.1]. The central property that we will need in our proof is given by the following lemma.

**Lemma 3.4.** Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & \beta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$v = (\alpha \beta \gamma) \quad \text{and} \quad w = (1 \ 1 \ 1)^T.$$
Then the following identities for $1 \times 3$-matrices hold.

$$v \begin{pmatrix} A \end{pmatrix} = 0 \cdot v \begin{pmatrix} A \end{pmatrix} + (\beta^2 + \beta + 1) v \begin{pmatrix} A \end{pmatrix} + (-\beta^2 - \beta) v \begin{pmatrix} A \end{pmatrix},$$

$$w^T A^T A^T = 0 \cdot w^T A^T A^T + (\beta^2 + \beta + 1) w^T A^T + (-\beta^2 - \beta) w^T,$$

$$v \begin{pmatrix} A \end{pmatrix} = (\beta + 1) v \begin{pmatrix} A \end{pmatrix} + (-\beta) v \begin{pmatrix} B \end{pmatrix} + 0 \cdot v \begin{pmatrix} B \end{pmatrix},$$

$$w^T A^T A^T B^T = (\beta + 1) w^T A^T B^T + (-\beta) w^T B^T + 0 \cdot w^T,$$

$$v \begin{pmatrix} A \end{pmatrix} = (\beta + 1) v \begin{pmatrix} A \end{pmatrix} + 0 \cdot v \begin{pmatrix} A \end{pmatrix} + (-\beta) v \begin{pmatrix} A \end{pmatrix},$$

$$w^T A^T B^T A^T = (\beta + 1) w^T B^T A^T + 0 \cdot w^T A^T + (-\beta) w^T,$$

$$v \begin{pmatrix} A \end{pmatrix} = (\alpha + 1) v \begin{pmatrix} A \end{pmatrix} + (-\alpha) v \begin{pmatrix} A \end{pmatrix} + 0 \cdot v \begin{pmatrix} A \end{pmatrix},$$

$$w^T A^T B^T B^T = (\alpha + 1) w^T A^T B^T + (-\alpha) w^T A^T + 0 \cdot w^T,$$

$$v \begin{pmatrix} A \end{pmatrix} = (\alpha + 1) v \begin{pmatrix} B \end{pmatrix} + (-\alpha) v \begin{pmatrix} A \end{pmatrix} + 0 \cdot v \begin{pmatrix} A \end{pmatrix},$$

$$w^T B^T A^T B^T = (\alpha + 1) w^T A^T B^T + 0 \cdot w^T B^T + (-\alpha) w^T,$$

$$v \begin{pmatrix} A \end{pmatrix} = (\alpha + 1) v \begin{pmatrix} A \end{pmatrix} + 0 \cdot v \begin{pmatrix} B \end{pmatrix} + (-\alpha) v \begin{pmatrix} B \end{pmatrix},$$

$$w^T B^T B^T A^T = (\alpha + 1) w^T B^T A^T + 0 \cdot w^T B^T + 0 \cdot w^T,$$

$$v \begin{pmatrix} A \end{pmatrix} = 0 \cdot v \begin{pmatrix} B \end{pmatrix} + (\alpha^2 + \alpha + 1) v \begin{pmatrix} B \end{pmatrix} + (-\alpha^2 - \alpha) v \begin{pmatrix} B \end{pmatrix},$$

$$w^T B^T B^T B^T = 0 \cdot w^T B^T B^T + (\alpha^2 + \alpha + 1) w^T B^T + (-\alpha^2 - \alpha) w^T.$$

The proof is too trivial and tiresome to reproduce here. □

The proof of Theorem 2.4 is by induction. Set $A(0) = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & \beta & \gamma \\ 0 & 0 & 1 \end{pmatrix}$, and $A(1) = \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & 1 & \gamma \end{pmatrix}$. As in [21], we have for odd $n \geq 3$ such that $n = (\varepsilon_0 \cdots \varepsilon_0)_2$,

$$x_n = \begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix} A(\varepsilon_1) \cdots A(\varepsilon_{\nu-1}) \begin{pmatrix} 1 & 1 & 1 \end{pmatrix},$$

and the statement of the theorem is equivalent to the assertion that

$$A(\varepsilon_1) \cdots A(\varepsilon_{\nu-1}) \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T = \begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix} A(\varepsilon_1) \cdots A(\varepsilon_{\nu-1}) \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$$

for all $\nu \geq 1$ and all finite sequences $(\varepsilon_1, \ldots, \varepsilon_{\nu-1})$ in $\{0,1\}$. This can be checked for $\nu \leq 3$ by simple calculation. Let therefore $\nu \geq 4$. Assume that $\varepsilon_1 \varepsilon_2 \varepsilon_3 = 000$. We consider the first pair of identities in Lemma 9.3. We multiply the first of these equations by $A(\varepsilon_1) \cdots A(\varepsilon_{\nu-1})$ from the right and the second one by $A(\varepsilon_1) \cdots A(\varepsilon_{\nu-1})^T v^T$, also from the right. Then the left hand sides give the two constituents of (13), and the right hand sides are equal by the induction hypothesis. The other 7 cases are analogous, and the proof Theorem 2.4 is complete. □

References

[1] J.-P. Allouche and J. Shallit, The ring of $k$-regular sequences, Theoret. Comput. Sci., 98 (1992), pp. 163–197.

[2] R. Béjian and H. Fauré, Discrépance de la suite de Van der Corput, in Séminaire Delange-Pisot-Poitou, 19e année: 1977/78, Théorie des nombres, Fasce. 1, Secrétariat Math., Paris, 1978. Exp. No. 13.

[3] M. Coons, Proof of Northshield’s conjecture concerning an analogue of Stern’s sequence for $\mathbb{Z}[\sqrt{2}]$, 2017. Preprint, http://arxiv.org/abs/1709.01987.

[4] M. Coons and L. Spiegelhofer, The maximal order of hyper-$(b, ary)$-expansions, Electron. J. Combin., 24 (2017). Paper 1.15.
[5] M. Coons and J. Tyler, The maximal order of Stern’s diatomic sequence, Mosc. J. Comb. Number Theory, 4 (2014), pp. 3–14.
[6] H. Delange, Sur la fonction sommatoire de la fonction “somme des chiffres”, Enseignement Math. (2), 21 (1975), pp. 31–47.
[7] M. Drmota, G. Larcher, and F. Pillichshammer, Precise distribution properties of the van der Corput sequence and related sequences, Manuscripta Math., 118 (2005), pp. 11–41.
[8] M. Drmota and W. Szpankowski, A master theorem for discrete divide and conquer recurrences, J. ACM, 60 (2013). Art. 16.
[9] H. Faure, Décréances de suites associées à un système de numération (en dimension un), Bull. Soc. Math. France, 109 (1981), pp. 143–182.
[10] ———, Discrepancy and diaphony of digital (0, 1)-sequences in prime base, Acta Arith., 117 (2005), pp. 125–148.
[11] P. Flajolet, P. Grabner, P. Kirschenhofer, H. Prodinger, and R. F. Tichy, Mellin transforms and asymptotics: digital sums, Theoret. Comput. Sci., 123 (1994), pp. 291–314.
[12] P. J. Grabner and H.-K. Hwang, Digital sums and divide-and-conquer recurrences: Fourier expansions and absolute convergence, Constr. Approx., 21 (2005), pp. 149–179.
[13] G. Larcher and F. Pillichshammer, Sums of distances to the nearest integer and the discrepancy of digital nets, Acta Arith., 106 (2003), pp. 379–408.
[14] D. H. Lehmer, On Stern’s Diatomic Series, Amer. Math. Monthly, 36 (1929), pp. 59–67.
[15] D. A. Lind, An extension of Stern’s diatomic series, Duke Math. J., 36 (1969), pp. 55–60.
[16] J. F. Morgenbesser and L. Spiegelhofer, A reverse order property of correlation measures of the sum-of-digits function, Integers, 12 (2012). Paper No. A47.
[17] F. Pillichshammer, On the discrepancy of (0, 1)-sequences, J. Number Theory, 104 (2004), pp. 301–314.
[18] H. Robbins, A remark on Stirling’s formula, Amer. Math. Monthly, 62 (1955), pp. 26–29.
[19] V. T. Sós, On strong irregularities of the distribution of \{na\} sequences, in Studies in pure mathematics, Birkhäuser, Basel, 1983, pp. 685–700.
[20] L. Spiegelhofer, A digit reversal property for an analogue of Stern’s sequence, 2017. Preprint, http://arxiv.org/abs/1709.05651.
[21] ———, A digit reversal property for Stern polynomials, 2017. Preprint, http://arxiv.org/abs/1610.00108.
[22] R. Tijdeman and G. Wagner, A sequence has almost nowhere small discrepancy, Monatsh. Math., 90 (1980), pp. 315–329.

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