On the Scalar Curvature of Complex Surfaces

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December, 1994

Abstract

Let \((M, J)\) be a minimal compact complex surface of Kähler type. It is shown that the smooth 4-manifold \(M\) admits a Riemannian metric of positive scalar curvature iff \((M, J)\) admits a Kähler metric of positive scalar curvature. This extends previous results of Witten and Kronheimer.

A complex surface is a pair \((M, J)\) consisting of a smooth compact 4-manifold \(M\) and a complex structure \(J\) on \(M\); the latter means an almost-complex structure tensor \(J : TM \to TM, J^2 = -1\), which is locally isomorphic to the usual constant-coefficient almost-complex structure on \(\mathbb{R}^4 = \mathbb{C}^2\). Such a complex surface is called minimal if it contains no embedded copy \(C\) of \(S^2\) such that \(J(C) = C\) and such that \(C \cdot C = -1\) in homology; this is equivalent to saying that \((M, J)\) cannot be obtained from another complex surface \((\tilde{M}, \tilde{J})\) by the procedure of “blowing up a point.”

A Riemannian metric \(g\) on \(M\) is said to be is said to be Kähler with respect to \(J\) if \(g\) is \(J\)-invariant and \(J\) is parallel with respect to the metric connection of \(g\). If such metrics actually exist, \((M, J)\) is then said to be of Kähler type; by a deep result \([1]\) of Kodaira, Todorov, and Siu, this holds iff \(b_1(M)\) is even.

The purpose of the present note is to prove the following:

**Theorem 1** Let \((M, J)\) be a minimal complex surface of Kähler type. Then the following are equivalent:

(a) \(M\) admits a Riemannian metric of positive scalar curvature;

(b) \((M, J)\) admits a Kähler metric of positive scalar curvature;

(c) \((M, J)\) is either a ruled surface or \(\mathbb{C}P^2\).

*Supported in part by NSF grant DMS-9003263.
Here a minimal complex surface \((M, J)\) is said to be \textit{ruled} iff it is the total space of a holomorphic \(\mathbb{CP}_1\)-bundle over a compact complex curve.

The equivalence between (b) and (c) was proved in one of Yau’s first papers \cite{14}. By contrast, the link between (a) and (c) came to light only recently, when Witten \cite{13} discovered that a Kähler surface with \(b^+ > 1\) cannot admit a metric of positive scalar curvature. Kronheimer \cite{6} then used a refinement of Witten’s method to prove that a minimal surface of general type cannot admit positive-scalar-curvature metrics. In essence, what will be shown here is simply that Kronheimer’s method can, with added care, also be applied to the case of minimal elliptic surfaces.

1 Seiberg-Witten Invariants

The ideas presented in this section are fundamentally due to Witten \cite{13}, but much of the formal framework and many technical results are due to Kronheimer-Mrowka \cite{7}. The work of Taubes \cite{12} contains less elementary but more robust proofs of other key results presented here. See also \cite{8}.

Let \((M, g)\) be a smooth compact Riemannian 4-manifold, and suppose that \(M\) admits an almost-complex structure. Then the given component of the almost-complex structures on \(M\) contains almost-complex structures \(J : TM \rightarrow TM\), \(J^2 = -1\) which are compatible with \(g\) in the sense that \(J^*g = g\). Fixing such a \(J\), the tangent bundle \(TM\) of \(M\) may be given the structure of a rank-2 complex vector bundle \(T^{1,0}\) by defining scalar multiplication by \(i\) to be \(J\). Setting \(\bigwedge^0 = \bigwedge^p T^{1,0} \cong \bigwedge^p T^{1,0}\), we may then define rank-2 complex vector bundles \(V_{\pm}\) on \(M\) by

\[
V_+ = \bigwedge^0 \oplus \bigwedge^2, \quad V_- = \bigwedge^1,
\]

and \(g\) induces canonical Hermitian inner products on these bundles.

As described, these bundles depend on the choice of a particular almost-complex structure \(J\), but they have a deeper meaning \cite{3} that depends only on the homotopy class \(c\) of \(J\); namely, if we restrict to a contractible open set \(U \subset M\), the bundles \(V_{\pm}\) may be canonically identified with \(S_{\pm} \otimes L^{1/2}\), where \(S_{\pm}\) are the left- and right-handed spinor bundles of \(g\), and \(L^{1/2}\) is a complex line bundle whose square is the ‘anti-canonical’ line-bundle \(L = (\bigwedge^0)^* \cong \bigwedge^{1/2}\).

For each connection \(A\) on \(L\) compatible with the \(g\)-induced inner product, we can thus define a corresponding Dirac operator

\[
D_A : C^\infty(V_+) \rightarrow C^\infty(V_-).
\]

If \(J\) is parallel with respect to \(g\), so that \((M, g, J)\) is a Kähler manifold, and if \(A\) is the Chern connection on the anti-canonical bundle \(L\), then \(D_A = \sqrt{2}(\partial \oplus \bar{\partial})\), where \(\partial : C^\infty(\bigwedge^0) \rightarrow C^\infty(\bigwedge^1)\) is the \(J\)-antilinear part of the exterior
differential \( d \), acting on complex-valued functions, and where \( \overline{\partial} : C^\infty(\wedge^{0,2}) \to C^\infty(\wedge^{0,1}) \) is the formal adjoint of the map induced by the exterior differential \( d \) acting on 1-forms; more generally, \( D_A \) will differ from \( \sqrt{2}(\partial \oplus \partial^*) \) by only \( 0^{th} \) order terms.

In addition to the metric \( g \) and class \( c \) of almost-complex structures \( J \), suppose we also choose some \( \varepsilon \in C^\infty(\wedge^+) \). The perturbed Seiberg-Witten equations

\[
D_A \Phi = 0 \quad (2)
\]

\[
iF_A^+ + \sigma(\Phi) = \varepsilon \quad (3)
\]

are then equations for an unknown smooth connection \( A \) on \( L \) and an unknown smooth section \( \Phi \) of \( V_+ \). Here the purely imaginary 2-form \( F_A^+ \) is the self-dual part of the curvature of \( A \), and, in terms of (1), the real-quadratic map \( \sigma : V_+ \to \wedge^2 \) is given by

\[
\sigma(f,\phi) = (|f|^2 - |\phi|^2)\frac{\omega}{4} + \Im(m(\bar{f}\phi)),
\]

where \( \omega(\cdot,\cdot) = g(J\cdot,\cdot) \) is the ‘Kähler’ form.

For a fixed metric \( g \), let \( \mathcal{M}(g) \) denote the set of pairs \((A, \Phi)\) which satisfy (2), modulo the action \((A, \Phi) \mapsto (A + 2d \log u, u\Phi)\) of the ‘gauge group’ of smooth maps \( u : M \to S^1 \subset \mathbb{C} \). We may then view (3) as defining a map \( \varphi : \mathcal{M}(g) \to C^\infty(\wedge^+) \). One can show [7] that this is a proper map, and so, in particular, has compact fibers.

A solution \((A, \Phi)\) is called \emph{reducible} if \( \Phi \equiv 0 \); otherwise, it is called \emph{irreducible}. Let \( \mathcal{M}^*(g) \) denote the image in \( \mathcal{M}(g) \) of the set of irreducible solutions. Then \( \mathcal{M}(g) \) is a smooth Fréchet manifold, and an index calculation shows that the smooth map \( \varphi : \mathcal{M}^*(g) \to C^\infty(\wedge^+) \) is generically finite-to-one. Let us define a solution \((\Phi, A)\) to be \emph{transverse} if it corresponds to a regular point of \( \varphi \). This holds iff the linearization \( C^\infty(V_+ \oplus \wedge^1) \to C^\infty(\wedge^2) \) of the left-hand-side of (3), constrained by the linearization of (2), is surjective.

\textbf{Key Example} Let \((M, g, J)\) be a Kähler surface, and let \( s \) denote the scalar curvature of \( g \). Set \( \varepsilon = s + 1|\omega|/4 \), set \( \Phi = (1, 0) \in \wedge^{0,0} \oplus \wedge^{0,2} \), and let \( A \) be the Chern connection on the anti-canonical bundle. Since \( iF_A^+ \) is just the Ricci form of \((M, g, J)\), it follows that \( iF_A^+ + \sigma(\Phi) = s\omega/4 + \omega/4 = \varepsilon \), and \((\Phi, A)\) is thus an irreducible solution of equations (2) and (3).

The linearization of (2) at this solution is just

\[
(\overline{\partial} \oplus \overline{\partial}^*)(u + \psi) = -\frac{1}{2} \alpha, \quad (4)
\]

where \((u, \psi) \in C^\infty(V_+)\) is the linearization of \( \Phi = (f, \phi) \) and \( \alpha \in \wedge^{0,1} \) is the \((0, 1)\)-part of the purely imaginary 1-form which is the linearization of \( A \).
Linearizing (3) at our solution yields the operator

$$(v, \psi, \alpha) \mapsto id + (\alpha - \bar{\alpha}) + \frac{1}{2}(\Re ev)\omega + \Im m\psi.$$ 

Since the right-hand-side is a real self-dual form, it is completely characterized by its component in the $\omega$ direction and its $(0,2)$-part. The $\omega$-component of this operator is just

$$(v, \psi, \alpha) \mapsto \Re e \left[ -\bar{\partial}^* \alpha + \frac{v}{2} \right],$$

while the $(0,2)$-component is

$$(v, \psi, \alpha) \mapsto i\bar{\partial} \alpha - \frac{i}{2}\psi.$$ 

Substituting (4) into these expressions, we obtain the operator

$$C^\infty(C \oplus \Lambda^{0,2}) \to C^\infty(R \oplus \Lambda^{0,2})$$

$$(v, \psi) \mapsto (\Re e \left[ \Delta + \frac{1}{2} \right] v, -i \left[ \Delta + \frac{1}{2} \right] \psi),$$

which is surjective because $\Delta + \frac{1}{2}$ is a positive self-adjoint elliptic operator. The constructed solution is therefore transverse. 

For any metric $g$, let $c_1^+(L)$ denote the image of $c_1(L) \in H^2(M, R)$ under orthogonal projection to the linear subspace $H^+(g)$ of deRham classes which are represented by self-dual 2-forms with respect to $g$. Given any $\varepsilon \in C^\infty(\Lambda^+)$, let $\varepsilon_H$ denote its harmonic part; this is a closed self-dual 2-form, since the Laplacian commutes with the Hodge star operator.

**Lemma 1** Let $g$ be any Riemannian metric on $M$, and let $\varepsilon \in C^\infty(\Lambda^+)$. Suppose that $[\varepsilon_H] \neq 2\pi c_1^+$ in deRham cohomology. Then every solution of (3) and (4) is irreducible.

**Proof.** If $\Phi \equiv 0$, (3) says that $c_1(L)$ is represented by $\varepsilon/2\pi$ plus an anti-self-dual form. Taking the harmonic part of this representative and projecting to the self-dual harmonic forms then yields $c_1^+ = [\varepsilon_H]/2\pi$. 

**Definition 1** If $g$ is a smooth Riemannian metric on $M$ and $\varepsilon \in C^\infty(\Lambda^+)$ is such that $[\varepsilon_H] \neq 2\pi c_1^+$, then, with respect to $c = [J]$, we will say that $(g, \varepsilon)$ is a good pair. The path components of the manifold of all good pairs $(g, \varepsilon)$ will be called chambers. 

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\footnote{with respect to the intersection form}
Lemma 2 If $b^+(M) > 1$, there is exactly one chamber. If $b^+(M) = 1$, there are exactly two chambers.

Proof. The projection $(g, \varepsilon) \mapsto g$ factors through the rank-$b^+$ vector bundle $H^+$ over the space of Riemannian metrics via a map $(g, \varepsilon) \mapsto (g, \varepsilon_H)$ with connected fibers. Now $2\pi c_1^+(g)$ is a smooth section of $H^+$, and since the space of Riemannian metrics is path-connected, the complement of this section is connected if $b^+ > 1$, and has exactly two components if $b^+ = 1$. The result follows.

Lemma 3 Suppose that $b^+(M) = 1$, $c_1(L) \neq 0$, and $c_1^2(L) \geq 0$. Then $(g,0)$ is a good pair for any metric $g$. In particular, the chamber containing $(g,0)$ is independent of $g$, and will be called the preferred chamber.

Proof. Our hypotheses say the intersection form on $H^2$ is a Lorentzian inner product and that $c_1(L)$ is a non-zero null or time-like vector. Since $2\pi c_1^+(g)$ is the projection of $2\pi c_1(L)$ onto a time-like line, it thus never equals $0 = [0_H]$.

Definition 2 Let $(M,c)$ be a compact 4-manifold equipped with a homotopy class $c = [J]$ of almost-complex structures. A good pair $(g,\varepsilon)$ will be called excellent if $\varepsilon$ is a regular value of the map $\wp: M^*(g) \to C^\infty(\Lambda^+)$.

Notice that $(g,\varepsilon)$ is excellent iff every solution of (2) and (3) with respect to $(g,\varepsilon)$ is irreducible and transverse.

Definition 3 Let $(M,c)$ be a compact 4-manifold equipped with a class $c = [J]$ of almost-complex structures. If $(g,\varepsilon)$ is an excellent pair on $M$, we define its (mod 2) Seiberg-Witten invariant $n_c(M,g,\varepsilon) \in \mathbb{Z}_2$ to be

$$n_c(M,g,\varepsilon) = \#\{\text{gauge classes of solutions of (2) and (3)}\} \mod 2$$

calculated with respect to $(g,\varepsilon)$.

Lemma 4 If two excellent pairs are in the same chamber, they have the same Seiberg-Witten invariant $n_c$.

Proof. Any two such pairs can be joined by a path of good pairs which is transverse to $\wp$. This gives a cobordism between the relevant solution spaces.
Definition 4 Let \((M, c)\) be a smooth 4-manifold equipped with a class of almost-complex structures. If \(b^+(M) > 1\), the Seiberg-Witten invariant \(n_c(M)\) is defined to be \(n_c(M, g, \varepsilon)\), where \((g, \varepsilon)\) is any excellent pair. If \(b^+(M) = 1\), \(c_1(L) \neq 0\), and \(c_2^1(L) \geq 0\), then the Seiberg-Witten invariant \(n_c(M)\) is defined to be \(n_c(M, g, \varepsilon)\), where \((g, \varepsilon)\) is any excellent pair in the preferred chamber.

Theorem 2 Let \((M, J)\) be a compact complex surface which admits a Kähler metric \(g\); let \(c = [J]\). Then there is a chamber for which the Seiberg-Witten invariant \(n_c\) is non-zero. Moreover, if \(c_1 \cdot [\omega] < 0\), where \([\omega]\) is the Kähler class of \(g\), then the chamber in question contains the good pair \((g, 0)\).

Proof. Set \(\varepsilon = (s + 1)\omega/4\), where \(s\) is the scalar curvature of the Kähler metric \(g\). Then \(\varepsilon_H = (s_0 + 1)\omega/4\), where the average value \(s_0\) of the scalar curvature of \(g\) is given by

\[
s_0 = \frac{\int_M s \, d\mu}{\int_M d\mu} = \frac{2\int_M \rho \wedge \omega}{\int_M \omega \wedge \omega/2} = \frac{8\pi c_1 \cdot [\omega]}{[\omega]^2}
\]

because the Ricci form \(\rho\) represents \(2\pi c_1\). On the other hand, \(2\pi c_1^+\) is represented by the harmonic form \(s_0\omega/4\), so we always have \(2\pi c_1^+ \neq [\varepsilon_H]\). This shows that \((g, \varepsilon)\) is a good pair. Moreover, if \(c_1 \cdot [\omega] < 0\), then \(s_0 < 0\), and \((g, t\varepsilon)\) is a good pair for all \(t \in [0, 1]\). Thus it suffices to show that \(n_c(M, g, \varepsilon) \equiv 1 \mod 2\).

We will accomplish this by showing that, with respect to \((g, \varepsilon)\) and up to gauge equivalence, there is exactly one solution of the perturbed Seiberg-Witten equations, namely the transverse solution described in the Key Example. Indeed, suppose that \(\Phi = (f, \phi)\) is any solution of equations (2) and (3), and let \(\hat{\Phi} = (f, -\phi)\). The Weitzenböck formula for the twisted Dirac operator and equation (3) thus tell us that

\[
0 = D_A^* D_A \Phi = \nabla^* \nabla \Phi + \frac{s}{4} \Phi + \frac{1}{2} F_A : \Phi
= \nabla^* \nabla \Phi + \frac{s}{4} \Phi + \frac{i}{2} \sigma(\Phi) \cdot \Phi - \frac{i}{2} \varepsilon \cdot \Phi
= \nabla^* \nabla \Phi + \frac{1}{4} (s + |\Phi|^2) \Phi - \frac{1}{4} (s + 1) \Phi
\]

because the \(\pm 1\)-eigenspaces of Clifford multiplication on \(V_+\) by \(-2i\omega\) are respectively \(\Lambda^{0,0}\) and \(\Lambda^{0,2}\). Projecting to the \(\nabla\)-invariant sub-bundle \(\Lambda^{0,0} \subset V_+\) now yields

\[
0 = 4\nabla^* \nabla f - f + |\Phi|^2 f.
\]

At the maximum of \(|f|^2\), we thus have

\[
0 \leq 2\Delta \langle f, f \rangle = 4\langle \Delta f, f \rangle - 4\langle \nabla f, \nabla f \rangle \leq |f|^2 - |\Phi|^2 |f|^2 \leq (1 - |f|^2) |f|^2.
\]
because $|\Phi|^2 = |f|^2 + |\phi|^2 \geq |f|^2$. This gives us the inequality
\[ |f|^2 \leq 1 \quad (5) \]
at all points of $M$, with equality only when $\phi = 0$ and $\nabla f = 0$.

On the other hand, the closed 2-form $iF_A$ is in the same cohomology class as the Ricci form $\rho$, which satisfies $\langle \omega, \rho \rangle = s/2$. Writing $iF_A = \rho + d\beta$ for some 1-form $\beta$, we have
\[
\int_M (|f|^2 - |\phi|^2)\,d\mu = 2 \int_M \langle \omega, \sigma(f, \phi) \rangle d\mu \\
= 2 \int_M \langle \omega, -iF_A + \varepsilon \rangle d\mu \\
= 2 \int_M \langle \omega, -\rho - d\beta \rangle d\mu + \int_M \langle \omega, (s + 1)\frac{\omega}{2} \rangle d\mu \\
= -\int_M s \, d\mu - 2 \int_M \langle d^* \omega, \beta \rangle d\mu + \int_M (s + 1) d\mu \\
= \int_M 1 \, d\mu,
\]
which is to say that
\[
\int_M (|f|^2 - 1) d\mu = \int_M |\phi|^2 d\mu \geq 0.
\]
The $C^0$ estimate (5) thus implies that $|f|^2 \equiv 1$, $\phi \equiv 0$, and $\nabla f \equiv 0$. The connection $\nabla$ induced on the $\Lambda^{0,0}$ by $A$ is therefore flat and trivial, and $A$ is thus gauge equivalent to the Chern connection on $L$. Hence our solution coincides, up to gauge transformation, with that of the example; in particular, every solution with respect to $(g, \varepsilon)$ is irreducible and transverse, and $(g, \varepsilon)$ is an excellent pair. But since there is only one gauge class of solutions with respect to $(g, \varepsilon)$, it follows that $n_c = 1 \text{ mod } 2$ for the chamber containing $(g, \varepsilon)$.

**Theorem 3** Let $M$ be a compact 4-manifold which admits a class $c = [J]$ of almost-complex structures and has $b^+ > 0$. If $g$ is a metric of positive scalar curvature on $M$, then $(g, 0)$ is in the closure of a chamber for which $n_c = 0$.

**Proof.** Suppose not. For every $\varepsilon > 0$, there is a $\varepsilon$ such that $\text{sup} |\varepsilon| < 2\varepsilon$ and such that $(g, \varepsilon)$ is an excellent pair. If $n_c(M, g, \varepsilon) \neq 0$, there is a solution $\Phi \neq 0$ of equations (2) and (3) with respect to $g$ and $\varepsilon = 0$. The Weitzenböck formula
\[
0 = D^*_\Lambda D_A \Phi = \nabla^* \nabla \Phi + \frac{s + |\Phi|^2}{4} \Phi - \frac{i}{2} \varepsilon \cdot \Phi
\]
then implies that
\[ 0 > \int_M \left( \frac{s - \epsilon}{4} \right) |\Phi|^2 d\mu. \]
Taking \( \epsilon < \min s \) then yields a contradiction.

2 Surface Classification and Scalar Curvature

Recall \([1]\) that the Kodaira dimension \( \text{Kod}(M, J) \in \{-\infty, 0, 1, 2\} \) of a compact complex surface \((M, J)\) is defined to be \( \limsup(\log h^0(M, O(L^{\otimes m}))/\log m) \).

The following well-known degree argument may be found e.g. in \([14]\).

**Lemma 5** Let \([\omega]\) be a Kähler class on a compact complex surface \((M, J)\) of \( \text{Kod} \geq 0 \). Then \( c_1 \cdot [\omega] \leq 0 \), with equality iff \((M, J)\) is a minimal surface of \( \text{Kod} = 0 \).

**Proof.** If \( \text{Kod}(M, J) \geq 0 \), some positive power \( \kappa m \) of the canonical line bundle has a holomorphic section. Let \( D \) be the holomorphic curve, counted with appropriate multiplicity, where this section vanishes. The homology class \([D] \in H_2(M)\) is then the Poincaré dual of \( c_1(\kappa m) = -mc_1(L) \). The area of \( D \) is thus
\[ \int_D \omega = -mc_1 \cdot [\omega] \]
which shows that \( c_1 \cdot [\omega] \leq 0 \), with equality iff \( D = \emptyset \). Since the latter happens iff \( \kappa m \) is holomorphically trivial, the result follows.

This leads us directly to a result first discovered by Kronheimer \([6]\).

**Proposition 1** (Kronheimer) Let \((M, J)\) be a minimal complex surface of \( \text{Kod} = 2 \). Then \( M \) does not admit a Riemannian metric of positive scalar curvature.

**Proof.** Such a surface is automatically \([1]\) of Kähler type, and has \( c_1^2 > 0 \). The Seiberg-Witten invariant \( n_c(M) \) is of \((M, J)\) is thus well-defined by Lemma \([3]\) and is non-zero by virtue of Theorem \([2]\). The result therefore follows by Theorem \([3]\).

Similar reasoning yields

**Proposition 2** Let \((M, J)\) be a minimal complex surface of Kähler type with \( \text{Kod} = 1 \). Then \( M \) does not admit a Riemannian metric of positive scalar curvature.
Proof. Such a surface must have $c_1^2 = 0$ and $c_1 \neq 0$. The Seiberg-Witten invariant $n_c(M)$ of $(M, J)$ is thus well-defined by Lemma 3, and the conclusion now follows by the same argument used above.

The next case is actually covered by existing results [10, 2, 11], but a Seiberg-Witten proof is given for the sake of completeness.

**Proposition 3** Let $(M, J)$ be a minimal complex surface of Kähler type such that Kod$(M) = 0$. Then $M$ does not admit a Riemannian metric of positive scalar curvature.

**Proof.** Any such an $M$ is finitely covered by a surface $\tilde{M}$ with $b^+ = 3$; in fact, $\tilde{M}$ is either a K3 surface or a 4-torus. The Seiberg-Witten invariant $n_c(\tilde{M})$ is thus well-defined by Lemma 2 and is non-zero by virtue of Theorem 2. By Theorem 3 $\tilde{M}$ does not admit a metric with $s > 0$. The result therefore follows because any metric on $M$ can be pulled back to $\tilde{M}$.

Our next result immediately implies Theorem 1:

**Theorem 4** Let $(M, J)$ be a minimal surface of Kähler type. If $M$ admits a Riemannian metric of positive scalar curvature, then $(M, J)$ is either $\mathbb{CP}^2$ or a ruled surface. As a consequence, $(M, J)$ therefore carries Kähler metrics of positive scalar curvature.

**Proof.** By the proceeding Propositions, $(M, J)$ must have Kodaira dimension $-\infty$. The Kodaira-Enriques classification [1] thus says that $(M, J)$ is either $\mathbb{CP}^2$ or a ruled surface. Now [14] any minimal ruled surface admits Kähler metrics of positive scalar curvature; indeed, if $M \cong \mathbb{P}(E)$, where $\varpi : E \to C$ is rank-2 holomorphic vector bundle, then, for any Kähler form $\omega_C$ on the Riemann surface $C$ and any Hermitian norm $h : E \to \mathbb{R}$ on the complex vector bundle $E$, the $(1, 1)$-form

$$\omega = \varpi^* \omega_C + \epsilon (i\partial \overline{\partial} \log h)$$

is a Kähler form on $M$ with positive scalar curvature if $\epsilon > 0$ is sufficiently small. Since the Fubini-Study metric on $\mathbb{CP}^2$ is also a Kähler metric of positive scalar curvature, the result follows.

Since the minimality hypothesis only features as a technicality in connection with the $b^+ = 1$ case, the following conjecture now seems extremely credible:

**Conjecture 1** Let $(M, J)$ be a compact complex surface with $b_1$ even. Then the following are equivalent:

(a) $M$ admits a Riemannian metric of positive scalar curvature;
(b) \((M, J)\) admits a Kähler metric of positive scalar curvature;

(c) \((M, J)\) is either \(\mathbb{CP}^2\) or a blow-up of some minimal ruled surface.

However, even (b) ⇔ (c) is only known ‘generically;’ cf. [4, 5, 9].

References

[1] W. Barth, C. Peters, and A. Van de Ven, Compact Complex Surfaces, Springer-Verlag, 1984.

[2] M. Gromov and H.B. Lawson, “Spin and Scalar Curvature in teh Presence of the Fundamental Group,” Ann. Math. 111 (1980) 209–230.

[3] N.J. Hitchin, Harmonic Spinors, Adv. Math. 14 (1974) 1–55.

[4] N.J. Hitchin, On the Curvature of Rational Surfaces, Proc. Symp. Pure Math. 27 (1975) 65–80.

[5] J.-S. Kim, C. LeBrun and M. Pontecorvo, Scalar-Flat Kähler Surfaces of All Genera, preprint, 1994.

[6] P. Kronheimer, private communication.

[7] P. Kronheimer and T. Mrowka, The Genus of Embedded Surfaces in the Complex Projective Plane, Math. Res. Lett. to appear.

[8] C. LeBrun, Einstein Metrics and Mostow Rigidity, Math. Res. Lett. to appear.

[9] C. LeBrun and S. Simanca, On Kähler Surfaces of Constant Positive Scalar Curvature, J. Geom. Analysis, to appear.

[10] A. Lichnerowicz, Spineurs Harmoniques, C.R. Acad. Sci. Paris 257 (1963) 7–9.

[11] R. Schoen and S.-T. Yau, The Structure of Manifolds of Positive Scalar Curvature, Man. Math. 28 (1979) 159–183.

[12] C.H. Taubes, The Seiberg-Witten Invariants and Symplectic Forms, Math. Res. Lett. to appear.

[13] E. Witten, Monopoles and Four-Manifolds, preprint, 1994.

[14] S.-T. Yau, On the Curvature of Compact Hermitian Manifolds, Inv. Math. 25 (1974) 213-239.