ON OPERADS, BIMODULES AND ANALYTIC FUNCTORS

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Abstract. We develop further the theory of operads and analytic functors. In particular, we introduce the bicategory $\text{Opd}_V$ of operads, that has operads as 0-cells, operad bimodules as 1-cells and operad bimodule maps as 2-cells, and prove that it is cartesian closed. In order to obtain this result, we extend the theory of distributors and the formal theory of monads.

Contents

1. Introduction ................................................................. 1
2. Review of bicategory theory .............................................. 6
3. $V$-categories and presentable $V$-categories .......................... 11
4. Distributors ..................................................................... 14
5. Monoidal $V$-categories and $V$-rigs ................................... 19
6. Monoidal distributors ........................................................ 22
7. Symmetric monoidal $V$-categories and symmetric $V$-rigs ....... 24
8. Symmetric monoidal distributors ......................................... 28
9. Free symmetric monoidal $V$-categories ............................... 29
10. $S$-distributors ............................................................... 32
11. Symmetric sequences and analytic functors ......................... 37
12. Monads, modules and bimodules ........................................ 43
13. Bicategories of bimodules ................................................ 50
14. The bicategory of operads ................................................ 54
15. Monad morphisms and bimodules ...................................... 57
16. Extension and restriction as analytic functors ....................... 61
17. Formal theory of monads in regular bicategories ................... 63
18. Eilenberg-Moore objects in bicategories of bimodules ............. 67
19. Cartesian closed bicategories of bimodules ......................... 75
Acknowledgements ............................................................... 78
Appendix A. A compendium of bicategorical definitions .............. 78
Appendix B. Proof of Theorem 18.11 ....................................... 81
References ............................................................................. 90

1. Introduction

The theory of operads has its roots in algebraic topology [15, 55]. While continuing to be very important in that area (see, e.g., [11, 24, 25, 57]), it has also found applications in several other branches of mathematics, including geometry [30, 37], algebra [50, 56], combinatorics [3, 49] and category theory [4, 7, 48]. See [27, 29, 52, 59] for recent accounts.

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The main goal of this paper is to introduce the bicategory of operads $\text{Opd}_V$ and to show that it is cartesian closed. The bicategory $\text{Opd}_V$ has operads as 0-cells, operad bimodules as 1-cells and operad bimodule maps as 2-cells. Here, by an operad we mean a many-sorted (sometimes called coloured) symmetric operad, enriched over a fixed symmetric monoidal closed presentable category $V$, i.e. a $V$-enriched symmetric multicategory. Such an operad $A$ has a set of objects $|A|$ and, for every tuple $(x_1, \ldots, x_n, x) \in |A|^{n+1}$, an object $A[x_1, \ldots, x_n; x] \in V$ of operations with inputs of sort $x_1, \ldots, x_n$ and output of sort $x$. An operation $f \in A[x_1, x_2; x]$ can be represented as a tree

For operads $A$ and $B$, we write $\text{Opd}_V[A, B]$ for the category of $(B, A)$-bimodules and bimodule maps. Recall that a $(B, A)$-bimodule is a family of objects $M[x_1, \ldots, x_n; y] \in V$, indexed by sequences $(x_1, \ldots, x_n) \in |A|^n$ and $y \in |B|$, subject to suitable functoriality conditions, equipped with a left $B$-action and a right $A$-action commuting with each other. An element $m \in M[x_1, x_2, x_3; y]$ may be represented as a tree:

The right $A$-action is a composition operation for trees of the form

where $f_1 \in A[x_{1,1}, x_{1,2}, x_{1,3}; x_1]$, $f_2 \in A[x_{2,1}, x_{2,2}; x_2]$ and $f_3 \in A[x_{3,1}; x_3]$. The left $B$-action, instead, is a composition operation for trees of the form

where $m_1 \in M[m_{1,1}; y_1]$, $m_2 \in M[x_{2,1}, x_{2,2}; y_2]$ and $g \in B[y_1, y_2; y]$. For example, an operad morphism $u: A \to B$ (i.e. a multifunctor) determines a pair of adjoint bimodules $u^\circ \dashv u_\circ$,
Thus, \(A\) and \(B\) are operads: for two operads \(A\) and \(B\), there is an equivalence \(\text{Opd}_V[A, B] \simeq \text{Alg}_V(A)\), where \(\text{Alg}_V(A)\) denotes the category of \(A\)-algebras. The cartesian closed structure of \(\text{Opd}_V\) can be understood in terms of the categories of algebras for operads: for two operads \(A\) and \(B\), their product \(A \sqcap B\) and the exponential \(B^A\) are characterised by the existence of natural equivalences

\[
\text{Alg}_V(A \sqcap B) \simeq \text{Alg}_V(A) \times \text{Alg}_V(B), \quad \text{Alg}_V(B^A) \simeq \text{Opd}_V[A, B].
\]

Thus, \(A \sqcap B\) is the operad whose algebras are pairs consisting on an \(A\)-algebra and a \(B\)-algebra, while \(B^A\) is the operad whose algebras are \((B, A)\)-bimodules. We have \(|A \sqcap B| = |A| \sqcup |B|\), an instance of a duality phenomenon which pervades the paper.

\[
\begin{align*}
\text{Opd}_V & \{ \text{Objects: operads } (X, A), (Y, B), (Z, C), \ldots \} \\
& \text{Hom-categories: } \text{Opd}_V[(X, A), (Y, B)] = \text{Sym}_V[X, Y]^B_A
\end{align*}
\]

\[
\begin{align*}
\text{CatSym}_V & \{ \text{Objects: small } V\text{-categories } X, Y, Z, \ldots \} \\
& \text{Hom-categories: } \text{CatSym}_V[X, Y] = \text{CAT}_V[S(X)^{\text{op}} \otimes Y, V]
\end{align*}
\]

\[
\begin{align*}
\text{Sym}_V & \{ \text{Objects: sets } X, Y, Z, \ldots \} \\
& \text{Hom-categories: } \text{Sym}_V[X, Y] = \text{CAT}[S(X)^{\text{op}} \times Y, V]
\end{align*}
\]

**Table 1.** Overview of the main bicategories in the paper.

In order to construct the bicategory \(\text{Opd}_V\) and prove that it is cartesian closed, we introduce an auxiliary bicategory, called the bicategory of symmetric sequences between small \(V\)-categories and denoted \(\text{CatSym}_V\), and show that it is cartesian closed, extending the main result of [26] to the enriched setting. More explicitly, the 0-cells of \(\text{CatSym}_V\) are small \(V\)-categories, the 1-cells \(F: X \to Y\) are symmetric sequences between \(X\) and \(Y\), i.e. \(V\)-functors \(F: S(X)^{\text{op}} \otimes Y \to V\), where \(S(X)\) is the free symmetric monoidal \(V\)-category on \(X\) (which we will define explicitly in Section 9), and the 2-cells are \(V\)-natural transformations. For small \(V\)-categories \(X\) and \(Y\), their product \(X \sqcap Y\) and the exponential \(Y^X\) in \(\text{CatSym}_V\) are given by the formulas:

\[
X \sqcap Y = \text{def } X \sqcup Y, \quad Y^X = \text{def } S(X)^{\text{op}} \otimes Y,
\]

where \(X \sqcup Y\) denotes the coproduct of \(X\) and \(Y\) in the 2-category of small \(V\)-categories. We then consider the sub-bicategory \(\text{Sym}_V\) of \(\text{CatSym}_V\) spanned by sets (viewed as small discrete \(V\)-categories). The 1-cells of this bicategory, called symmetric sequences between sets, can be seen as the many-sorted generalisation of single-sorted symmetric sequences, while the composition operation in \(\text{Sym}_V\) is a generalisation of the tensor product of the substitution monoidal structure on the category of single-sorted symmetric sequences [27, 40, 57, 58]. Explicitly, for sets \(X, Y\) and \(Z\), the composite in \(\text{Sym}_V\) of symmetric sequences \(F: X \to Y\) and \(G: Y \to Z\) is given by...
the formula

\[(G \circ F)(\varpi; z) := \bigg( \prod_{m \in \mathbb{N}} \int_{(y_1, \ldots, y_m) \in S^m(Y)} G[y_1, \ldots, y_m; z] \otimes \prod_{i=1}^m \int_{\varpi_i \in S(X)} \prod_{j=1}^m \varpi_m \otimes F(\varpi_1; y_1) \otimes \cdots \otimes F(\varpi_m; y_m), \]

where \((\varpi; z) \in S(X)^{op} \times Y\). By virtue of these definitions, the general notion of a monad in a bicategory (which we will recall in Section 12) reduces in \(\text{Sym}_V\) exactly to the notion of an operad (cf. [4]). This generalises the fact that a single-sorted operad is a monoid with respect to the substitution monoidal structure in the category of single-sorted symmetric sequences.

The bicategory \(\text{Opd}_V\) is the bicategory \(\text{Bim}(\text{Sym}_V)\) of monads and bimodules in \(\text{Sym}_V\). In general, for a regular bicategory \(\mathcal{E}\) (i.e. a bicategory whose hom-categories have reflexive coequalizers and whose composition functors preserve coequalizers in each variable) there is a bicategory \(\text{Bim}(\mathcal{E})\) which has monads in \(\mathcal{E}\), monad bimodules and monad bimodule maps as 0-cells, 1-cells and 2-cells, respectively [13, 17, 28, 53]. Indeed, as we prove in Section 14, \(\text{Sym}_V\) is a regular bicategory. The composition of bimodules in \(\text{Opd}_V\) is a generalisation to many-sorted operads of the circle-over construction for single-sorted operads defined by Charles Rezk in [57].

An overview of the definition of the bicategories \(\text{Sym}_V, \text{CatSym}_V\) and \(\text{Opd}_V\) is given in Table 1.

Our strategy to prove that \(\text{Opd}_V\) is cartesian closed is to establish the following three facts.

**Theorem A.** The bicategory \(\text{CatSym}_V\) is cartesian closed.

**Theorem B.** If a regular bicategory \(\mathcal{E}\) is cartesian closed, then \(\text{Bim}(\mathcal{E})\) is cartesian closed.

**Theorem C.** The inclusion \(\text{Sym}_V \subseteq \text{CatSym}_V\) induces an equivalence of bicategories

\[\text{Bim}(\text{Sym}_V) \simeq \text{Bim}(\text{CatSym}_V)\]

i.e. \(\text{Opd}_V \simeq \text{Bim}(\text{CatSym}_V)\).

The combination of Theorem A and Theorem B implies that \(\text{Bim}(\text{CatSym}_V)\) is cartesian closed, which in turn implies that \(\text{Opd}_V\) is cartesian closed by Theorem C.

In order to prove Theorem B and Theorem C, we develop some aspects of the formal theory of monads (in the sense of [61]) in regular bicategories. In particular, we establish a universal property of \(\text{Bim}(\mathcal{E})\), namely that of being the Eilenberg-Moore completion of \(\mathcal{E}\) as a regular bicategory (a notion that we will define precisely in Section 18), which is a special case of a result obtained independently by Richard Garner and Michael Shulman in the context of the theory of categories enriched in a bicategory [28].

Our approach to the definition of the bicategory \(\text{CatSym}_V\) and to the proof that it is cartesian closed (Theorem A) differs significantly from the one adopted in [26] in the non-enriched case. In particular, its construction and the proof that it has finite products follow immediately from the results in the first part of the paper. There, we develop further the theory of distributors (also known as bimodules or profunctors) [9, 47] and introduce and study the notions of a (lax) monoidal distributor and of a symmetric (lax) monoidal distributor, and show how they can be seen as morphisms of appropriate bicategories. These bicategories and the bicategory \(\text{CatSym}_V\) are defined using the notion of a Gabriel factorisation of a homomorphism (which we define in Section 2), rather than via the theory of pseudo-distributive laws [20, 53, 54] as done in [26]. Furthermore, our proof that the bicategory \(\text{CatSym}_V\) is cartesian closed is organized into several simple observations on symmetric monoidal distributors and does not involve lengthy coend calculations like the proof of the corresponding fact in [26].
This paper introduces also some natural extensions of the notion of an analytic functor defined by the second-named author in [36]. Recall that a single-sorted symmetric sequence $F : S(1) \to V$ (where $S(1)$ is the category of natural numbers and permutations) defines an analytic functor $F : V \to V$, given by

$$F(T) = \underset{n \in \mathbb{N}}{\sum} F[n] \otimes T^n,$$

where $T^n$ denotes the $n$-fold tensor product of $T$ with itself and $\Sigma_n$ is the $n$-th symmetric group. In particular, when $V = \text{Set}$, one obtains the notion of an analytic functor introduced in [36]. The substitution monoidal structure on the functor category $[S(1), V]$ corresponds to the composition monoidal structure on the functor category $[V, V]$, in the sense that the function sending a single-sorted symmetric sequence to its analytic functor becomes a (strong) monoidal functor. Extending this idea, for every symmetric sequence $F : X \to Y$ we define an associated analytic functor $F : R^X \to R^Y$, for every symmetric monoidal closed presentable $V$-category $R = (\mathcal{R}, \circ, e)$, defined by letting

$$F(T)(y) = \int_{x \in S(X)} F[x; y] \otimes T(x_1) \circ \ldots \circ T(x_n).$$

We then show that the composite of the analytic functors associated to two symmetric sequences $F : X \to Y$ and $G : Y \to Z$ in $\text{CatSym}_V$, thus extending the single-sorted case (which arises by taking $X = Y = 1$) and the case $V = R = \text{Set}$, discussed in [26]. Extending this idea even further, for operads $A$ and $B$ and a $(B, A)$-bimodule $F$, we define an associated analytic functor $F : \text{Alg}_R(A) \to \text{Alg}_R(B)$ for every symmetric monoidal closed presentable $V$-category $R$, where we write $\text{Alg}_R(A)$ and $\text{Alg}_R(B)$ for the categories of $A$-algebras and $B$-algebras, respectively, in $\mathcal{R}$. Again, composition in $\text{Opd}_V$ corresponds to composition of these analytic functors. As we show in Section 16 examples of such analytic functors include the well-known extension and restriction functors associated to an operad morphism [27]. An overview of the various types of analytic functors discussed in the paper is presented in Table 2.

| Opd$V$ | Operad bimodule $F : (X, A) \to (Y, B)$ |
| --- | --- |
| | Symmetric sequence $F : X \to Y$ with $(B, A)$-bimodule structure |
| | Analytic functor $\text{Alg}_R(F) : \text{Alg}_R(A) \to \text{Alg}_R(B)$ |
| CatSym$V$ | Categorical symmetric sequence $F : X \to Y$ |
| | $V$-functor $F : S(X)^{\text{op}} \otimes Y \to V$ |
| | Analytic functor $F : \mathcal{R}^X \to \mathcal{R}^Y$ |
| Sym$V$ | Symmetric sequence $F : X \to Y$ |
| | Functor $F : S(X)^{\text{op}} \times Y \to V$ |
| | Analytic functor $F : \mathcal{R}^X \to \mathcal{R}^Y$ |

Table 2. Analytic functors associated to the morphisms of the main bicategories.
Organisation of the paper. The paper is organized as follows. Section 2 reviews some material from the theory of bicategories that will be used in the paper. In particular, we present the notion of a Gabriel factorisation of a homomorphism. For the convenience of the reader, Appendix A recalls the definitions of the main bicategorical notions with which we work. The rest of the paper is organized in two parts.

The first part includes sections 3 to 11. Sections 3 and 4 recall the definition of the bicategory of distributors, sections 5 and 6 introduce the bicategory of (lax) monoidal distributors, while sections 7 and 8 define the bicategory of symmetric (lax) monoidal distributors. These bicategories are defined in a uniform way using the notion of a Gabriel factorisation. Sections 9, 10, 11 are devoted to constructing the bicategory of symmetric sequences between small $\mathcal{V}$-categories and to proving that it is cartesian closed. First, in section 9 we lift the adjunction between the 2-category of small $\mathcal{V}$-categories and the 2-category of symmetric monoidal $\mathcal{V}$-categories to an adjunction between the bicategory of distributors and the bicategory of symmetric monoidal distributors. The left adjoint of the lifted adjunction is then used in Section 10 to define the bicategory of $S$-distributors via a Gabriel factorisation. Finally, section 11 defines the bicategory of symmetric sequences between small $\mathcal{V}$-categories as the opposite of the bicategory of $S$-distributors (see Table 3) and establishes that it is cartesian closed.

| $\mathcal{E}$   | $\mathcal{E}^{\text{op}}$ |
|-----------------|-------------------------|
| $S$-Dist$_\mathcal{V}$ | CatSym$_\mathcal{V}$    |
| $S$-Mat$_\mathcal{V}$ | Sym$_\mathcal{V}$      |

Table 3. Some bicategories and their opposites.

The second part of the paper includes sections 12 to 19. Sections 12 and 13 recall the notions of a monad, bimodule and bimodule map in a bicategory $\mathcal{E}$ and we describe, under the assumption that $\mathcal{E}$ is regular, the bicategory $\text{Bim}(\mathcal{E})$. In Section 14 we prove that Sym$_\mathcal{V}$ is regular and define the bicategory of operads Opd$_\mathcal{V}$. Section 15 investigates the relationship between bicategories of bimodules and bicategories of monads, defined as in [61]. Section 16 applies these results to show that the extension and restriction functors between categories of operad algebras are examples of analytic functors. Sections 17 and 18 develop some aspects of the formal theory of monads in regular bicategories. Section 19 establishes that if $\mathcal{E}$ is a cartesian closed regular bicategory and applies this result to show that the bicategory Opd$_\mathcal{V}$ is cartesian closed.

2. Review of bicategory theory

This section reviews the notions and results of bicategory theory that will be used in the remainder of the paper. We also recall some basic examples of bicategories. Further examples will be given in the next sections. For the convenience of the reader, the definitions of bicategory, homomorphism, pseudo-natural transformation and modification are recalled in Appendix A.

Additional information on the theory of bicategories may be found in [16, Volume I, Chapter 7] and [8, 43, 62].

Basics. We write $\text{CAT}$ for the category of locally small categories and functors. Recall that a 2-category is a category enriched over $\text{CAT}$. We denote the category of small categories by $\text{Cat}$. For a bicategory $\mathcal{E}$, we write $\mathcal{E}[X, Y]$ for the hom-category between two objects $X, Y \in \mathcal{E}$. A bicategory $\mathcal{E}$ is said to be locally small when $\mathcal{E}[X, Y]$ is a small category for every $X, Y \in \mathcal{E}$. A
morphism, or a 1-cell \( F: X \to Y \) is an object of the category \( \mathcal{E}[X,Y] \), and a 2-cell \( \alpha: F \to F' \) is a morphism of the category \( \mathcal{E}[X,Y] \). We write \( 1_X: X \to X \) for the identity morphism on an object \( X \in \mathcal{E} \). The composition operation of 2-cells, i.e. the the composition operation of the hom-categories of \( \mathcal{E} \), is usually referred to as the vertical composition in \( \mathcal{E} \) and its effect on \( \alpha: F \to F' \), \( \beta: F' \to F'' \) is written \( \beta \cdot \alpha: F \to F'' \). The identity arrow of an object \( F \in \mathcal{E}[X,Y] \) is called an identity 2-cell of \( \mathcal{E} \) and written \( 1_F: F \to F \). We refer to the composition operation

\[
(-) \circ (-): \mathcal{E}[Y,Z] \times \mathcal{E}[X,Y] \to \mathcal{E}[X,Z]
\]
as the horizontal composition of \( \mathcal{E} \). The horizontal composite of \( F: X \to Y \) and \( G: Y \to Z \) is denoted \( G \circ F: X \to Z \). The horizontal composition of \( \alpha: F \to F' \) with \( \beta: G \to G' \) is written \( \beta \circ \alpha: G \circ F \to G' \circ F' \). This 2-cell is the common value of the composites in the following naturality square

\[
\begin{array}{ccc}
G \circ F & \xrightarrow{G \circ \alpha} & G \circ F'' \\
\beta \circ F & \downarrow & \beta \circ F' \\
G' \circ F & \xrightarrow{G' \circ \alpha} & G' \circ F'.
\end{array}
\]

We say that a bicategory is strict if its composition operation is strictly associative and if the units \( 1_X \) are strict. A strict bicategory is the same thing as a 2-category. Given two bicategories \( \mathcal{E} \) and \( \mathcal{F} \), their cartesian product \( \mathcal{E} \times \mathcal{F} \) is the bicategory with \( \text{Obj}(\mathcal{E} \times \mathcal{F}) = \text{def} \mathcal{E} \times \mathcal{F} \) and

\[
(\mathcal{E} \times \mathcal{F})[(X,Y),(X',Y')] = \text{def} \mathcal{E}[X,X'] \times \mathcal{F}[Y,Y'],
\]
for \( X, X' \in \mathcal{E} \) and \( Y, Y' \in \mathcal{F} \). Composition is defined in the obvious way.

**Example 2.1.** A monoidal category \( \mathcal{C} = (\mathcal{C}, \otimes, I) \) can be identified with a bicategory, here denoted \( \Sigma(\mathcal{C}) \), which has a single object and \( \mathcal{C} \) as its hom-category. The horizontal composition of \( \Sigma(\mathcal{C}) \) is then given by the tensor product of \( \mathcal{C} \). Every bicategory with one object is of the form \( \Sigma(\mathcal{C}) \) for some monoidal category \( \mathcal{C} \).

For a bicategory \( \mathcal{E} \), we write \( \mathcal{E}^{\text{op}} \) for the opposite bicategory of \( \mathcal{E} \), which is obtained by formally reversing the direction of the morphisms of \( \mathcal{E} \), but not that of the 2-cells. For a morphism \( F: X \to Y \) in \( \mathcal{E} \), we write \( F^{\text{op}}: Y \to X \) for the corresponding morphism in \( \mathcal{E}^{\text{op}} \).

**Equivalences and adjunctions in a bicategory.** We recall the notions of an equivalence and an adjunction in a bicategory \( \mathcal{E} \). A morphism \( F: X \to Y \) in a bicategory \( \mathcal{E} \) is said to be an equivalence if there exists a morphism \( U: Y \to X \) together with invertible 2-cells \( \alpha: G \circ F \to 1_X \) and \( \beta: F \circ U \to 1_Y \). We write \( X \simeq Y \) to indicate that \( X \) and \( Y \) are equivalent. An adjunction \( (F,U,\eta,\varepsilon): X \to Y \) in \( \mathcal{E} \) consists of morphisms \( F: X \to Y \) and \( U: Y \to X \) and 2-cells \( \eta: 1_X \to U \circ F \) and \( \varepsilon: F \circ U \to 1_X \) satisfying the triangular laws, expressed the commutative diagrams

\[
\begin{array}{ccc}
F & \xrightarrow{\eta U} & U \\
\downarrow F \circ \varepsilon & \downarrow & \downarrow 1_U \\
F \circ U & \xrightarrow{U \circ \eta} & U
\end{array}
\]

The morphism \( F \) is the left adjoint and the morphism \( U \) is the right adjoint; the 2-cell \( \eta \) is the unit of the adjunction and the 2-cell \( \varepsilon \) is the counit. Recall that the unit \( \eta \) of an adjunction \((F,U,\eta,\varepsilon)\) determines the counit \( \varepsilon \) and conversely. More precisely, \( \varepsilon: F \circ U \to 1_Y \) is the unique 2-cell such that \( (U \circ \varepsilon) \cdot (\eta \circ U) = 1_U \), and \( \eta: 1_X \to U \circ F \) is the unique 2-cell such that \( (\varepsilon \circ F) \cdot (F \circ \eta) = 1_F \). We often write \( F \dashv U \) to indicate the existence of an adjunction \((F,U,\eta,\varepsilon)\). An adjunction is called
a reflection (resp. coreflection) if its counit (resp. unit) is invertible. If \((F,U,\eta,\varepsilon) : X \to Y\) is an adjunction in \(E\), then \((U^{op}, F^{op}, \eta, \varepsilon) : X \to Y\) is an adjunction in the opposite bicategory \(E^{op}\).

**Homomorphisms.** For bicategories \(E\) and \(F\), we write \(\Phi : E \to F\) to indicate that \(\Phi\) is a homomorphism from \(E\) to \(F\). The homomorphisms from \(E\) to \(F\) are the objects of a bicategory \(\text{HOM}[E,F]\) whose morphisms are pseudo-natural transformations and 2-cells are modifications. A contravariant homomorphism \(\Phi : E \to F\) is defined to be a homomorphism \(\Phi : E^{op} \to F\). The canonical homomorphism
\[
E[-,-] : E^{op} \times E \to \text{CAT}
\]
takes a pair of objects \((X,Y)\) to the category \(E[X,Y]\). In particular, there is a covariant homomorphism \(E[\mathbb{K},-] : E \to \text{CAT}\) and a contravariant homomorphism \(E[-,\mathbb{K}] : E \to \text{CAT}\) for each object \(\mathbb{K} \in E\).

We say that a homomorphism \(\Phi : E \to F\) is full and faithful if for every \(X, Y \in E\) the functor
\[
\Phi_{X,Y} : E[X,Y] \to F[\Phi X, \Phi Y]
\]
is an equivalence of categories. We say that \(\Phi : E \to F\) is essentially surjective if for every object \(Y \in F\) there exists an object \(X \in E\) together with an equivalence \(\Phi X \simeq Y\). We say that \(\Phi : E \to F\) is an equivalence if it is full and faithful and essentially surjective. The coherence theorem for bicategories, asserts that every bicategory is equivalent to a 2-category \([46]\). Thanks to this result, it is possible to treat the horizontal composition in a bicategory as if it were strictly associative and unital, which we will often do in the following. We say that a homomorphism \(\Phi : E \to F\) is an inclusion if it is injective on objects and full and faithful. In this case, we will often write \(E \subseteq F\) and treat the action of \(\Phi\) on objects as if it were the identity.

A homomorphism \(\Phi : E \to F\) takes an adjunction \((F,U,\eta,\varepsilon) : X \to Y\) in \(E\) to an adjunction \((\Phi F, \Phi U, \Phi \eta, \Phi \varepsilon) : \Phi X \to \Phi Y\) in \(F\). Dually, a contravariant homomorphism \(\Phi : E \to F\) takes an adjunction \((F,U,\eta,\varepsilon) : X \to Y\) in \(E\) to an adjunction \((\Phi U, \Phi F, \Phi \eta, \Phi \varepsilon) : \Phi Y \to \Phi X\) in \(F\). For example, if \(\mathbb{K}\) is an object of \(E\), then the homomorphism \(E[\mathbb{K},-] : E \to \text{CAT}\) takes an adjunction \((F,U) : X \to Y\) in \(E\) to an adjunction \((E[\mathbb{K},F], E[\mathbb{K},U]) : E[\mathbb{K},X] \to E[\mathbb{K},Y]\) in \(\text{CAT}\). Dually, the contravariant homomorphism \(E[-,\mathbb{K}] : E \to \text{CAT}\) takes an adjunction \((F,U) : X \to Y\) in \(E\) to an adjunction \((E[U,\mathbb{K}], E[F,\mathbb{K}]) : E[Y,\mathbb{K}] \to E[X,\mathbb{K}]\).

**Prestacks.** By a prestack on a (locally small) bicategory \(E\) we mean a contravariant homomorphism \(\Phi : E \to \text{Cat}\). The bicategory of prestacks
\[
\mathbf{P}(E) = \text{def} \ \text{HOM}(E^{op}, \text{Cat})
\]
is a 2-category, since \(\text{Cat}\) is a 2-category. The Yoneda homomorphism
\[
y_{\mathbb{X}} : E \to \mathbf{P}(E)
\]
takes an object \(\mathbb{X} \in E\) to the prestack \(y_{\mathbb{X}}(\mathbb{X}) = \text{def} E[-,\mathbb{X}]\). The bicategorical Yoneda lemma asserts that if \(\Phi \in \mathbf{P}(E)\) and \(\mathbb{X} \in E\), then the functor
\[
\mathbf{P}(E)[y_{\mathbb{X}}, \Phi] \to \Phi(\mathbb{X})
\]
which takes a pseudo-natural transformation \(\alpha : y_{\mathbb{X}}(\mathbb{X}) \to \Phi\) to the object \(\alpha_{\mathbb{X}}(1_{\mathbb{X}}) \in \Phi(\mathbb{X})\) is an equivalence of categories. It follows from the bicategorical Yoneda lemma that the Yoneda homomorphism is full and faithful. We say that a prestack \(\Phi : E \to \text{Cat}\) on a bicategory \(E\) is representable if there exists an object \(\mathbb{X} \in E\) together with a pseudo-natural equivalence \(\alpha : y(\mathbb{X}) \to \Phi\). It follows from Yoneda lemma that \(\alpha\) is determined by the object \(\alpha_{\mathbb{X}} = \text{def} \alpha(1_{\mathbb{X}}) \in \Phi(\mathbb{X})\). We say that \(\Phi\) is represented by the pair \((\mathbb{X}, \alpha_{\mathbb{X}})\). Such a pair \((\mathbb{X}, \alpha_{\mathbb{X}})\) is unique up to equivalence of pairs when it exists.
Gabriel factorisation. Let us recall that every homomorphism $\Phi: \mathcal{E} \to \mathcal{F}$ admits a factorisation of the form

$$
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\Phi} & \mathcal{F} \\
\downarrow & & \downarrow \\
\mathcal{G} & \xrightarrow{\Delta} & \mathcal{F}
\end{array}
$$

(2.2)

where $\Gamma$ is essentially surjective and $\Delta$ is full and faithful. In fact, we may suppose that $\Gamma$ is the identity on objects, in which case we have a Gabriel factorization of $\Phi$. In order to obtain a Gabriel factorisation, the bicategory $\mathcal{G}$ is defined as having the same objects as $\mathcal{E}$ and letting,

$$
\mathcal{G}[X,Y] = \text{def} \mathcal{F}[\Phi X, \Phi Y].
$$

The composition law of $\mathcal{G}$ is defined via the composition law of $\mathcal{F}$ in the evident way. The homomorphism $\Gamma: \mathcal{E} \to \mathcal{G}$ is the identity on objects, while $\Gamma_{X,Y}: \mathcal{E}[X,Y] \to \mathcal{G}[X,Y]$ is defined to be $\Phi_{X,Y}: \mathcal{E}[X,Y] \to \mathcal{F}[\Phi X, \Phi Y]$. The homomorphism $\Delta: \mathcal{G} \to \mathcal{F}$ is defined on objects by letting $\Delta(X) = \text{def} \Phi(X)$, for $X \in \mathcal{E}$, while $\Delta_{X,Y}: \mathcal{G}[X,Y] \to \mathcal{F}[\Phi X, \Phi Y]$ is the identity functor.

**Example.** Let us consider the Gabriel factorization of the Yoneda homomorphism $y: \mathcal{E} \to \mathcal{P}(\mathcal{E})$,

$$
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{y} & \mathcal{P}(\mathcal{E}) \\
\downarrow & & \downarrow \\
\mathcal{G} & \xrightarrow{\Delta} & \mathcal{P}(\mathcal{E})
\end{array}
$$

The bicategory $\mathcal{G}$ is a 2-category, since the bicategory $\mathcal{P}(\mathcal{E})$ is a 2-category. Moreover, the homomorphism $\Gamma$ is an equivalence of bicategories, since the Yoneda homomorphism is full and faithful. Hence, the bicategory $\mathcal{E}$ is equivalent to a 2-category, giving a proof of the coherence theorem for bicategories [46].

There is a slight variation of the definitions given above which arises when we are given not only the homomorphism $\Phi: \mathcal{E} \to \mathcal{F}$ but also, for each $X, Y \in \mathcal{E}$, a category $\mathcal{G}[X,Y]$ and an equivalence

$$
\Delta_{X,Y}: \mathcal{G}[X,Y] \to \mathcal{F}[\Phi X, \Phi Y].
$$

(2.3)

In this case, we obtain again a Gabriel factorisation of $\Phi$. The bicategory $\mathcal{G}$ has again the same objects as $\mathcal{E}$ and its hom-categories are given by the given categories $\mathcal{G}[X,Y]$. The composition functors of $\mathcal{G}$ are determined (up to unique isomorphism) by requiring that the following diagram commutes up to natural isomorphism:

$$
\begin{array}{ccc}
\mathcal{G}[Y,Z] \times \mathcal{G}[X,Y] & \xrightarrow{(-)z(-)} & \mathcal{G}[X,Z] \\
\downarrow & \downarrow & \downarrow \\
\mathcal{F}[\Phi X, \Phi Z] \times \mathcal{F}[\Phi X, \Phi Y] & \xrightarrow{(-)\phi(-)} & \mathcal{F}[\Phi X, \Phi Z].
\end{array}
$$

Similarly, the identity morphism $1_X: X \to X$ on an object $X \in \mathcal{G}$, is determined (up to unique isomorphism) by requiring that there is an isomorphism $\Delta(1_X) \cong 1_{\Phi X}$. These associativity and unit isomorphisms can be defined in a similar way. The definition of the required homomorphism $\Delta: \mathcal{G} \to \mathcal{F}$ now follows easily. Its action on objects is given by mapping $X \in \mathcal{G}$ to $\Phi X \in \mathcal{F}$ and its action on hom-categories is given by the equivalences in (2.3), so that $\Delta$ is full and faithful by construction. The homomorphism $\Delta: \mathcal{E} \to \mathcal{G}$ can then be defined as the identity on objects, while its action on hom-categories is essentially determined by requiring that the
diagram in \( \square \) commutes up to pseudo-natural equivalence. We will illustrate this method of constructing bicategories in Section 3 and apply it again in Section 4 and Section 8.

**Adjunctions between bicategories.** If \( \mathcal{E} \) and \( \mathcal{F} \) are bicategories, then an adjunction (sometimes also referred to as a biadjunction) \( \theta : \Phi \dashv \Psi \) between two homomorphisms \( \Phi : \mathcal{E} \to \mathcal{F} \) and \( \Psi : \mathcal{F} \to \mathcal{E} \) is defined to be a pseudo-natural equivalence

\[
\theta : \mathcal{E}[X, Y] \simeq \mathcal{F}[\Phi X, \Psi Y].
\]

The homomorphism \( \Phi \) is said to be the left adjoint and the homomorphism \( \Psi \) to be the right adjoint. A homomorphism \( \Phi : \mathcal{E} \to \mathcal{F} \) has a right adjoint if and only if the prestack

\[
\mathcal{E}[-, Y] : \mathcal{E} \to \text{Cat}
\]

is representable for every object \( Y \in \mathcal{F} \). The counit of the adjunction is a pseudo-natural transformation \( \varepsilon : \Phi \circ \Psi \to \text{Id}_{\mathcal{F}} \) defined by letting \( \varepsilon_Y = \text{def} \theta(1_{\Psi Y}) \) for \( Y \in \mathcal{F} \). The unit of the adjunction is a pseudo-natural transformation \( \eta : \text{Id}_{\mathcal{E}} \to \Psi \circ \Phi \) defined by letting \( \eta_X = \text{def} \theta^{-1}(1_{\Phi X}) \) for \( X \in \mathcal{E} \), where \( \theta^{-1} \) is a quasi-inverse of \( \theta \). Either of the pseudo-natural transformations \( \eta \) and \( \varepsilon \) determine the adjunction \( \theta \).

**Cartesian, cocartesian and cartesian closed bicategories.** We recall the notion of cartesian bicategory. We say that an object \( \top \) in a bicategory \( \mathcal{E} \) is terminal if the category \( \mathcal{E}[C, \top] \) is equivalent to the terminal category for every object \( C \in \mathcal{E} \). A terminal object \( \top \in \mathcal{E} \) is unique up to equivalence when it exists. Given two objects \( X_1, X_2 \in \mathcal{E} \), we say that an object \( X \in \mathcal{E} \) equipped with two morphisms \( \pi_1 : X \to X_1 \) and \( \pi_2 : X \to X_2 \) is the cartesian product of \( X_1 \) and \( X_2 \) if the functor

\[
\pi : \mathcal{E}[C, X] \to \mathcal{E}[C, X_1] \times \mathcal{E}[C, X_2],
\]

defined by letting \( \pi(F) = \text{def} (\pi_1 \circ F, \pi_2 \circ F) \) is an equivalence of categories for every object \( C \in \mathcal{E} \). The cartesian product of the objects \( X_1 \) and \( X_2 \) is unique up to equivalence when it exists. In this case, we will denote it by \( X_1 \sqcap X_2 \) and refer to the morphisms \( \pi_k : X_1 \sqcap X_2 \to X_k \) \((k = 1, 2)\) as the projections. When every pair of objects in \( \mathcal{E} \) has a cartesian product, then the diagonal homomorphism \( \Delta_E : \mathcal{E} \to \mathcal{E} \times \mathcal{E} \) has a right adjoint,

\[
(\cdot) \sqcap (\cdot) : \mathcal{E} \times \mathcal{E} \to \mathcal{E},
\]

which associates to \((X_1, X_2)\) the cartesian product \( X_1 \sqcap X_2 \). We say that a bicategory \( \mathcal{E} \) with a terminal object is cartesian if every pair of objects in \( \mathcal{E} \) has a cartesian product. Dually, we say that a bicategory \( \mathcal{E} \) is cocartesian if the opposite bicategory \( \mathcal{E}^{\text{op}} \) is cartesian. We write \( \sqcup \) for the initial object and \( X_1 \sqcup X_2 \) for the coproduct of two objects \( X_1 \) and \( X_2 \), and refer to the morphisms \( \iota_k : X_k \to X_1 \sqcup X_2 \) \((k = 1, 2)\) as the inclusions.

We recall the notion of cartesian closed bicategory. Given objects \( X, Y \) of a cartesian bicategory \( \mathcal{E} \), we will say that an object \( \mathcal{E} \in \mathcal{E} \) equipped with a morphism \( \text{ev} : \mathcal{E} \sqcap X \to Y \) is the exponential of \( Y \) by \( X \) if the functor

\[
\mathcal{E}[K, X] \xrightarrow{(-) \sqcap X} \mathcal{E}[K \sqcap X, \mathcal{E} \sqcap X] \xrightarrow{\mathcal{E}[\mathcal{E} \sqcap X, \text{ev}]} \mathcal{E}[K \sqcap X, Y]
\]

is an equivalence of categories for every object \( K \in \mathcal{E} \). This condition means that the prestack

\[
\mathcal{E}[- \sqcap X, Y] : \mathcal{E}^{\text{op}} \to \text{Cat}
\]

is represented by the pair \((\mathcal{E}, \text{ev})\). The exponential of \( Y \) by \( X \) is unique up to equivalence when it exists and we denote it by \( Y^X \) or \([X, Y]\) and refer to the morphism

\[
\text{ev} : Y^X \sqcap X \to Y
\]

as the evaluation. We say that a cartesian bicategory \( \mathcal{E} \) is closed if the exponential \( Y^X \) exists for every \( X, Y \in \mathcal{E} \). A cartesian bicategory \( \mathcal{E} \) is closed if and only if, for every object \( X \in \mathcal{E} \), the
homomorphism \((-) \cap X: \mathcal{E} \to \mathcal{E}\) has a right adjoint \((-)^X: \mathcal{E} \to \mathcal{E}\). The resulting homomorphism mapping \((X,Y)\) to \(Y^X\) is contravariant in the first variable and covariant in the second.

**Monoidal bicategories.** A cartesian bicategory is an example of a symmetric monoidal bicategory, a notion that we limit ourselves to review in outline. First, recall that by definition, a monoidal bicategory is a tricategory with one object [31, 32]. We will not describe this notion here because of its complexity (see [19, 31, 32, 60] for details). It will suffice to say that a monoidal structure on bicategory \(\mathcal{E}\) is a 9-tuple

\[
(\otimes, I, \alpha_1, \alpha_2, \lambda_1, \lambda_2, \rho_1, \rho_2, \mu),
\]

where the tensor product

\[
(-) \otimes (-): \text{Bim}(\mathcal{E}) \times \text{Bim}(\mathcal{E}) \to \text{Bim}(\mathcal{E})
\]

is a homomorphism, \(\alpha_1, \lambda_1, \rho_1\) are pseudo natural (adjoint) equivalences and \(\alpha_2, \lambda_2, \rho_2, \mu\) are invertible modifications. More precisely,

\[
\alpha_1^1(X,Y,Z): (X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)
\]

is the 1-associativity constraint and the 2-associativity constraint \(\alpha_2^2(X,Y,Z,W)\) is a 2-cell fitting in the pentagon

\[
\begin{array}{ccc}
(X \otimes (Y \otimes Z)) \otimes W & \xrightarrow{\alpha^1} & (X \otimes Y) \otimes (Z \otimes W) \\
& \alpha^2 \otimes W & \alpha^1\\nX \otimes ((Y \otimes Z) \otimes W) & \xrightarrow{X \otimes \alpha^1} & X \otimes (Y \otimes (Z \otimes W)).
\end{array}
\]

The associativity constraints satisfy coherence conditions that we omit. We also omit the coherence conditions for the unit object \(I\) and its constraints \((\lambda_1, \rho_1, \lambda_2, \rho_2, \mu)\). A *symmetry structure* on a monoidal bicategory as above is a pseudo-natural (adjoint) equivalence

\[
\sigma_{X,Y}: X \otimes Y \simeq Y \otimes X
\]

together with certain higher dimensional constraints [33].

**3. \(\mathcal{V}\)-categories and presentable \(\mathcal{V}\)-categories**

Since our development focuses on enriched categories, it is convenient to recall some aspects of enriched category theory from [35]. Let \(\mathcal{V} = (\mathcal{V}, \otimes, I, [-,-])\) be a locally presentable symmetric monoidal closed category, which we shall consider fixed throughout this paper. If \(X\) is a small \(\mathcal{V}\)-category, we write \(X[x,y]\) or simply \([x,y]\) for the hom-object between two objects \(x,y \in X\). We write \(\text{Cat}_\mathcal{V}\) (resp. \(\text{CAT}_\mathcal{V}\)) for the 2-category of small (resp. locally small) \(\mathcal{V}\)-categories, \(\mathcal{V}\)-functors and \(\mathcal{V}\)-natural transformations. The category \(\text{Cat}_\mathcal{V}\) is complete and cocomplete. In particular, its terminal object is the \(\mathcal{V}\)-category \(I\) defined by letting \(\text{Obj}(I) = \{\ast\}\) and \(I[\ast,\ast] = \top\), where \(\top\) is the terminal object of \(\mathcal{V}\). The terminal object of \(\text{Cat}_\mathcal{V}\) is the \(\mathcal{V}\)-category \(I\) defined by letting \(\text{Obj}(I) = \{\ast\}\) and \(I[\ast,\ast] = \top\), where \(\top\) is the terminal object of \(\mathcal{V}\). The category \(\text{Cat}_\mathcal{V}\) has also a symmetric monoidal closed structure. We write \(X \otimes Y\) for the tensor
product two small \( \mathcal{V} \)-categories \( X \) and \( Y \). This is defined by letting \( \text{Obj}(X \otimes Y) = \text{def} \text{Obj}(X) \times \text{Obj}(Y) \) and

\[
(X \otimes Y)[(x, y), (x', y')] = \text{def} X[x, x'] \otimes \mathcal{V}[y, y'].
\]

Sometimes we write \( x \otimes y \in X \otimes Y \) instead of \( (x, y) \in X \otimes Y \). The unit object for this monoidal structure is the \( \mathcal{V} \)-category \( I \) defined by letting \( \text{Obj}(I) = \text{def} \{ * \} \) and \( I[*,*] = \text{def} I \). The hom-object \( [X, Y] \) is the \( \mathcal{V} \)-category of \( \mathcal{V} \)-functors from \( X \) to \( Y \) and \( \mathcal{V} \)-natural transformations. For \( \mathcal{V} \)-categories \( X, Y, Z, \) a \( \mathcal{V} \)-functor of two variables \( F : X \times Y \to Z \) is defined to be a \( \mathcal{V} \)-functor \( F : X \otimes Y \to Z \). The next definition recalls the notion of a (locally) presentable \( \mathcal{V} \)-category, with which we will work throughout the paper. The reader is invited to refer to [39] for further information about it.

**Definition 3.1.** We say that a \( \mathcal{V} \)-category \( \mathcal{E} \) is (locally) presentable if it is \( \mathcal{V} \)-cocomplete and its underlying ordinary category is (locally) presentable in the usual sense.

We write \( \text{PCAT}_\mathcal{V} \) for the 2-category of presentable \( \mathcal{V} \)-categories, cocontinuous \( \mathcal{V} \)-functors and \( \mathcal{V} \)-natural transformations. For example, the \( \mathcal{V} \)-category \( P(X) = \text{def} [X^{\text{op}}, \mathcal{V}] \) of presheaves on a small \( \mathcal{V} \)-category is presentable for every small \( \mathcal{V} \)-category \( X \). In particular, the terminal \( \mathcal{V} \)-category \( 1 \simeq P(0) \) is presentable, where \( 0 \) is the \( \mathcal{V} \)-category with no objects. For a small \( \mathcal{V} \)-category \( X \), we write \( y_X : X \to P(X) \) for the Yoneda \( \mathcal{V} \)-functor, which is defined by letting \( y_X(x) = \text{def} X[-, x] \), for \( x \in X \). By the enriched version of the Yoneda lemma, there is an isomorphism

\[
P(X)[y_X(x), A] \cong A(x),
\]

for every \( A \in P(X) \) and \( x \in X \). It follows that \( y_X \) is full and faithful; we will often regard it as an inclusion by writing \( x \) instead of \( y_X(x) \). If \( X \) is a small \( \mathcal{V} \)-category, then the \( \mathcal{V} \)-category \( P(X) \) is cocomplete and freely generated by \( X \). More precisely, the Yoneda functor \( y_X : X \to P(X) \) exhibits \( P(X) \) as the free cocompletion of \( X \). This means that if \( \mathcal{E} \) is a cocomplete \( \mathcal{V} \)-category, and \( \text{CCAT}_\mathcal{V}[P(X), \mathcal{E}] \) denotes the (large, locally small) \( \mathcal{V} \)-category of cocontinuous \( \mathcal{V} \)-functors from \( P(X) \) to \( \mathcal{E} \), then the restriction functor

\[
y_X^* : \text{CCAT}_\mathcal{V}[P(X), \mathcal{E}] \to [X, \mathcal{E}],
\]

defined by letting \( y_X^*(F) = \text{def} F \circ y_X \), is an equivalence of \( \mathcal{V} \)-categories. In particular, every \( \mathcal{V} \)-functor \( F : X \to \mathcal{E} \) admits a cocontinuous extension \( F_c : P(X) \to \mathcal{E} \) which is unique up to a unique \( \mathcal{V} \)-natural isomorphism,

\[
\xymatrix{X \ar[r]^{y_X} & P(X) \ar[d]^F \ar@{>->}[ld]_{F_c} \ar[r]^\cong & \mathcal{E}.}
\]

The \( \mathcal{V} \)-functor \( F_c \) is the left Kan extension of \( F \) along \( y_X \) and its action on \( A \in P(X) \) is given by the coend formula

\[
F_c(A) = \text{def} \int^{x \in X} F(x) \otimes A(x).
\]

The \( \mathcal{V} \)-functor \( F_c \) is left adjoint to the “singular \( \mathcal{V} \)-functor” \( F^* : \mathcal{E} \to P(X) \), given by letting

\[
F^*(y)(x) = \text{def} \mathcal{E}[F(x), y],
\]

for \( y \in \mathcal{E} \) and \( x \in X \). We write

\[
P : \text{Cat}_\mathcal{V} \to \text{PCAT}_\mathcal{V}
\]
for the homomorphism which takes a small \( \mathcal{V} \)-category \( \mathcal{X} \) to \( P(\mathcal{X}) \). If \( u: \mathcal{X} \to \mathcal{Y} \) is a \( \mathcal{V} \)-functor between small \( \mathcal{V} \)-categories, we define \( P(u) = \defeq u_!: P(\mathcal{X}) \to P(\mathcal{Y}) \), i.e. as the cocontinuous extension of \( y \circ u: \mathcal{X} \to P(\mathcal{Y}) \). Hence, the diagram

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{y} & P(\mathcal{X}) \\
\downarrow{u} & & \downarrow{u_!} \\
\mathcal{Y} & \xrightarrow{y} & P(\mathcal{Y})
\end{array}
\]

commutes up to a canonical isomorphism and, for every \( A \in P(\mathcal{X}) \) and \( y \in \mathcal{Y} \), we have

\[
u_!(A)(y) = \defeq \int_{x \in \mathcal{X}} \mathcal{X}[y, u(x)] \otimes A(x).
\] (3.4)

The functor \( u_!: P(\mathcal{X}) \to P(\mathcal{Y}) \) has a right adjoint \( u^*: P(\mathcal{Y}) \to P(\mathcal{X}) \) defined by letting

\[
u^*(B)(x) = \defeq B(u(x)),
\] (3.5)

for \( B \in P(\mathcal{Y}) \) and \( x \in \mathcal{X} \).

If \( \mathcal{E} \) is a presentable \( \mathcal{V} \)-category and \( \mathcal{C} \) is a cocomplete \( \mathcal{V} \)-category, then any cocontinuous \( \mathcal{V} \)-functor from \( \mathcal{E} \) to \( \mathcal{C} \) has a right adjoint. Because of this, products and coproducts in \( \text{PCAT}_\mathcal{V} \) are intimately related, as we now recall. First of all, the cartesian product \( \mathcal{E} = \prod_{k \in K} \mathcal{E}_k \) of a family of presentable \( \mathcal{V} \)-categories \( \mathcal{E}_k \) is presentable. Each projection \( \pi_k: \mathcal{E}_k \to \mathcal{E} \) has a left adjoint \( \iota_k: \mathcal{E} \to \mathcal{E}_k \) and the family \( (\iota_k \mid k \in K) \) is a coproduct diagram in \( \text{PCAT}_\mathcal{V} \). In particular, the terminal \( \mathcal{V} \)-category 1 is both initial and terminal in the bicategory \( \text{PCAT}_\mathcal{V} \).

**Lemma 3.2.** The homomorphism \( P: \text{Cat}_\mathcal{V} \to \text{PCAT}_\mathcal{V} \) preserves coproducts.

**Proof.** This follows from the universal property of \( P(\mathcal{X}) \). Indeed, let us consider a family of small \( \mathcal{V} \)-categories \( (\mathcal{X}_k \mid k \in K) \) and let \( \mathcal{X} = \bigsqcup_{k \in K} \mathcal{X}_k \) and let \( \iota_k: \mathcal{X}_k \to \mathcal{X} \) be the inclusion for \( k \in K \). We prove that the family of maps \( (\iota_k)_*: P(\mathcal{X}_k) \to P(\mathcal{X}) \) is a coproduct diagram in \( \text{PCAT}_\mathcal{V} \). For this, it suffices to show for every presentable \( \mathcal{V} \)-category \( \mathcal{E} \), the family of functors

\[
\text{PCAT}_\mathcal{V}[\iota_k, \mathcal{E}]: \text{PCAT}_\mathcal{V}[P(\mathcal{X}), \mathcal{E}] \to \text{PCAT}_\mathcal{V}[P(\mathcal{X}_k), \mathcal{E}] \quad (k \in K)
\]

is a product diagram in the 2-category \( \text{CAT} \). But this family is equivalent to the family of functors

\[
[\iota_k, \mathcal{E}]: [\mathcal{X}, \mathcal{E}] \to [\mathcal{X}_k, \mathcal{E}] \quad (k \in K)
\]

by the equivalence in \( \text{3.1} \). This proves the result, since the family \( \iota_k: \mathcal{X}_k \to \mathcal{X} \), for \( k \in K \), is a coproduct diagram also in the 2-category of locally small \( \mathcal{V} \)-categories. \( \square \)

**Remark 3.3.** If \( \mathcal{E} \) and \( \mathcal{F} \) are presentable \( \mathcal{V} \)-categories, then so is the \( \mathcal{V} \)-category \( \text{CCAT}_\mathcal{V}[\mathcal{E}, \mathcal{F}] \) of cocontinuous \( \mathcal{V} \)-functors from \( \mathcal{E} \) to \( \mathcal{F} \). This defines the hom-object of a symmetric monoidal closed structure on the 2-category \( \text{PCAT}_\mathcal{V} \). By definition, the completed tensor product \( \mathcal{E} \otimes \mathcal{F} \) of two presentable \( \mathcal{V} \)-categories \( \mathcal{E} \) and \( \mathcal{F} \) is a presentable \( \mathcal{V} \)-category equipped with a \( \mathcal{V} \)-functor in two variables from \( \mathcal{E} \times \mathcal{F} \) to \( \mathcal{E} \otimes \mathcal{F} \) that is \( \mathcal{V} \)-cocontinuous in each variable and universal with respect to that property. The unit object for the completed tensor product is \( \mathcal{V} \). If we consider the 2-categories \( \text{Cat}_\mathcal{V} \) and \( \text{PCAT}_\mathcal{V} \) as equipped with the symmetric monoidal structures, the homomorphism \( P: \text{Cat}_\mathcal{V} \to \text{PCAT}_\mathcal{V} \) is symmetric monoidal. Indeed, for \( \mathcal{X}, \mathcal{Y} \in \text{Cat}_\mathcal{V} \), we have a \( \mathcal{V} \)-functor of two variables

\[
\phi_{\mathcal{X}, \mathcal{Y}}: P(\mathcal{X}) \times P(\mathcal{Y}) \to P(\mathcal{X} \otimes \mathcal{Y})
\]
defined by letting \( \phi_{X,Y}(F,G)(x \otimes y) = \text{def} \ F(x) \otimes G(y) \). This \( \mathcal{V} \)-functor exhibits \( P(X \otimes Y) \) as the completed tensor product of \( P(X) \) and \( P(Y) \) and so we have an equivalence
\[
P(X) \hat{\otimes} P(Y) \simeq P(X \otimes Y)
\]

We conclude this section with a straightforward observation, which we state explicitly for future reference. Recall that if \( X \) is a small \( \mathcal{V} \)-category and \( \mathcal{E} \) is a locally small \( \mathcal{V} \)-category, then the \( \mathcal{V} \)-category \([X, \mathcal{E}]\) of \( \mathcal{V} \)-functors from \( X \) to \( \mathcal{E} \) is locally small.

**Proposition 3.4.** Let \( X, Y \) be a small \( \mathcal{V} \)-categories and \( \mathcal{E} \) be a locally small \( \mathcal{V} \)-category. The \( \mathcal{V} \)-functors
\[
\lambda^Y : [X \otimes Y, \mathcal{E}] \to [X, [Y, \mathcal{E}]], \quad \lambda^X : [X \otimes Y, \mathcal{E}] \to [Y, [X, \mathcal{E}]]
\]
defined by letting
\[
(\lambda^Y F)(x)(y) = \text{def} \ F(x, y), \quad (\lambda^X F)(y)(x) = \text{def} \ F(x, y),
\]
for \( F : X \otimes Y \to \mathcal{E} \), \( x \in X \) and \( y \in Y \), are equivalences of \( \mathcal{V} \)-categories. \( \square \)

4. Distributors

Let us recall the notion of a distributor (sometimes called bimodule or profunctor in the literature). See [16, Volume I, Chapter 7] and [9, 10, 47] for further information and [18] for applications of distributors in theoretical computer science.

**Definition 4.1.** Let \( X, Y \in \text{Cat}_\mathcal{V} \). A distributor \( F : X \to Y \) is a \( \mathcal{V} \)-functor \( F : Y^{op} \otimes X \to \mathcal{V} \).

For a distributor \( F : X \to Y \), we write \( F[y, x] \) for the result of applying \( F \) to \( (y, x) \in Y^{op} \otimes X \). Small \( \mathcal{V} \)-categories, distributors and \( \mathcal{V} \)-transformations form a bicategory, called the bicategory of distributors and denoted \( \text{Dist}_\mathcal{V} \), in which the hom-category of morphisms between two small \( \mathcal{V} \)-categories \( X \) and \( Y \) is defined by letting
\[
\text{Dist}_\mathcal{V}[X, Y] = \text{def} \ [Y^{op} \otimes X, \mathcal{V}] .
\]
The bicategory \( \text{Dist}_\mathcal{V} \) fits into a Gabriel factorisation of the form
\[
\begin{array}{ccc}
\text{Cat}_\mathcal{V} & \xrightarrow{P} & \text{PCAT}_\mathcal{V} \\
\downarrow (-) & \downarrow (-)^t & \\
\text{Dist}_\mathcal{V} & \xrightarrow{(-)^t} & \text{PCAT}_\mathcal{V}
\end{array}
\] (4.1)

The Gabriel factoriation essentially determines the composition operation and the unit morphisms of the bicategory \( \text{Dist}_\mathcal{V} \) and provides a proof that they satisfy the appropriate coherence conditions. We illustrate this fact since we will use the same method to define other bicategories in Section 6 and Section 8. First of all, for small \( \mathcal{V} \)-categories \( X, Y \), we define
\[
(-)^t : \text{Dist}_\mathcal{V}[X, Y] \to \text{PCAT}_\mathcal{V}[P(X), P(Y)]
\]
to be the composite of an equivalence given by Proposition 3.4 and the quasi-inverse of the equivalence in (4.1):
\[
\lambda : \text{CAT}_\mathcal{V}[X, P(Y)] \xrightarrow{(-)} \text{PCAT}_\mathcal{V}[P(X), P(Y)].
\]
It is convenient to express the effect of this functor by the following derivation:
\[
F : X \to Y
\]
\[
\lambda F : X \to P(Y)
\]
\[
F^t : P(X) \to P(Y).
\]
Explicitly, for \( A \in P(\mathcal{X}) \), we have
\[
F^\dagger(A)(y) = (\lambda F)_c(A)(y)
\]
\[
= \int_{x \in \mathcal{X}} (\lambda F)(x)(y) \otimes A(x)
\]
\[
= \int_{x \in \mathcal{X}} F[y,x] \otimes A(x).
\]

The functor in (4.2) is an equivalence of categories, since it is the composite of equivalences. Because of this, the composite \( G \circ F \) of two distributors \( F: \mathcal{X} \to \mathcal{Y} \) and \( G: \mathcal{Y} \to \mathcal{Z} \) is determined up to unique isomorphism by the requirement that there is an isomorphism
\[
\varphi_{F,G}: (G \circ F)^\dagger \to G^\dagger \circ F^\dagger.
\]

Thus, \( G \circ F: \mathcal{X} \to \mathcal{Z} \) can be defined by letting
\[
(G \circ F)[z,x] = \int_{y \in \mathcal{Y}} G[z,y] \otimes F[y,x].
\]

Similarly, the identity distributor \( \text{Id}_\mathcal{X}: \mathcal{X} \to \mathcal{X} \) is determined up to unique isomorphism by the requirement that there is an isomorphism
\[
\varphi_\mathcal{X}: (\text{Id}_\mathcal{X})^\dagger \to 1_{P(\mathcal{X})}.
\]

Thus, \( \text{Id}_\mathcal{X}: \mathcal{X} \to \mathcal{X} \) can be defined by letting
\[
\text{Id}_\mathcal{X}[x,y] = \text{def} \mathcal{X}[x,y].
\]

Using the same reasoning, it is possible to show that horizontal composition of distributors is functorial and associative up to coherent isomorphism, and that the identity morphisms provide two-sided units for this operation up to coherent isomorphism. For example, for distributors \( F: \mathcal{X} \to \mathcal{Y} \), \( G: \mathcal{Y} \to \mathcal{Z} \), \( H: \mathcal{Z} \to \mathcal{W} \), the associativity isomorphism
\[
\alpha_{F,G,H}: (H \circ G) \circ F \to H \circ (G \circ F)
\]
can be defined as the unique 2-cell such that the following diagram commutes (where we omit subscripts of the 2-cells to improve readability):
\[
\begin{array}{ccc}
(H \circ G) \circ F & \xrightarrow{\alpha} & H \circ (G \circ F) \\
\phi & & \phi \\
\downarrow & & \downarrow \\
(H \circ G)^\dagger \circ F^\dagger & \xrightarrow{H \circ \varphi} & H^\dagger \circ (G \circ F)^\dagger
\end{array}
\]

(4.6)

It follows that we can define a homomorphism \((\text{Id})^\dagger: \text{Dist}_\mathcal{V} \to \text{PCAT}_\mathcal{V}\) by letting
\[
\mathcal{X}^\dagger = \text{def} P(\mathcal{X})
\]

and taking its action on morphisms and 2-cells be defined by the functor in (4.2). The required isomorphisms expressing pseudo-functoriality are then given by the 2-cells in (4.3) and (4.5), which satisfy the required coherence conditions by the definition of the associativity and unit isomorphisms in Dist\(\mathcal{V}\), as done above. For example, the diagram in (4.6) states exactly one the coherence conditions. Furthermore, by construction, the homomorphism \((\text{Id})^\dagger: \text{Dist}_\mathcal{V} \to \text{PCAT}_\mathcal{V}\) is full and faithful, as required from the second part of a Gabriel factorisation.
Indeed, for $A \in P(X)$ and $y \in Y$, we have

$$(u_\bullet)^\dagger (A)(y) = \int_{x \in X} \mathbb{V}[y, u(x)] \otimes A(x) = u_\bullet(A)(y).$$

The distributor $u_\bullet : X \to Y$ has a right adjoint $u^\bullet : Y \to X$, which is defined by letting for $x \in X$, $y \in Y$,

$$u^\bullet[x, y] = \text{def } \mathbb{V}[u(x), y].$$

Indeed, we have that $u^\bullet \cong (u_\bullet)^\dagger$, since for $B \in P(Y)$ and $x \in X$, we have

$$(u^\bullet)^\dagger (B)(x) = \int_{y \in Y} \mathbb{V}[u(x), y] \otimes B(y) \cong B(u(x)) = u^\bullet(B)(x).$$

Since there is an adjunction $u \dashv u^\bullet$, we also have an adjunction $u_\bullet \dashv u^\bullet$. The components of its unit $\eta : \text{Id}_X \to u^\bullet \circ u_\bullet$ are the maps $u_{x, x'} : X[x, x'] \to \mathbb{V}[u(x), u(x')]$ given by $u$. The components of the counit $\varepsilon : u_\bullet \circ u^\bullet \to \text{Id}_Y$ are the canonical maps

$$\varepsilon_{y, y'} : \int_{x \in X} \mathbb{V}[y, u(x)] \otimes \mathbb{V}[u(x), y'] \to \mathbb{V}[y, y'].$$

For $u : X \to Y, v : X \to Y$ we have $(v \circ u) \cong v \circ u_\bullet$, and for $X \in \text{Cat}_\mathbb{V}$, we have $(1_X)_\bullet \cong 1_{P(X)}$. Therefore, there are canonical isomorphisms

$$(v \circ u)_\bullet \cong v_\bullet \circ u_\bullet, \quad (1_X)_\bullet \cong \text{Id}_X,$$

which necessarily satisfy the coherence conditions for a homomorphism $(\cdot)_\bullet : \text{Cat}_\mathbb{V} \to \text{Dist}_\mathbb{V}$.

Part (i) of the next lemma will be used to prove Theorem \ref{thm:main} while part (ii) will be used in the proof of Proposition \ref{prop:adjunction}.

**Lemma 4.2.** Let $F : X \to Y$ be a distributor.

(i) For all $\mathbb{V}$-functors $u : X' \to X$, $\lambda(F \circ u_\bullet) \cong \lambda(F) \circ u$.

(ii) For all $\mathbb{V}$-functors $u : X' \to X$ and $v : Y' \to Y$, $(v^\bullet \circ F \circ u_\bullet)[y', x'] \cong F[y'(y'), u(x')]$. 
Proof. For (i), let \( x' \in X \). Then
\[
\lambda(F \circ u_\bullet)(x')(y) = (F \circ u_\bullet)[y, x']
\]
\[
= \int_{x \in X} F[y, x] \otimes u_\bullet[x, x']
\]
\[
= \int_{x \in X} F[y, x] \otimes X'[x, u(x')]
\]
\[
\cong F[y, u(x')]
\]
\[
= (\lambda(F) \circ u)(x')(y). \qed
\]
For (ii), let \( x' \in X' \) and \( y' \in Y' \). Then
\[
(v^* \circ F \circ u_\bullet)[y', x'] \cong \int_{x \in X} \int_{y \in Y} v^*[y', y] \otimes F[y, x] \otimes u_\bullet[x, x']
\]
\[
= \int_{x \in X} \int_{y \in Y} Y[v(y'), y] \otimes F(y, x) \otimes X[x, u(x')]
\]
\[
\cong F[v(y'), u(x')].
\]

Proposition 4.3. The bicategory \( \text{Dist}_V \) has coproducts and the homomorphisms
\[
(-)_\bullet : \text{Cat}_V \to \text{Dist}_V, \quad (-)^\uparrow : \text{Dist}_V \to \text{PCAT}_V
\]
preserve coproducts.

Proof. This follows from Lemma 3.2 and the fact that \( \text{Dist}_V \) fits into a Gabriel factorisation. \( \qed \)

Remark 4.4. Although we will not need it in the following, let us recall that the symmetric monoidal structure on \( \text{Cat}_V \) extends to a symmetric monoidal structure on \( \text{Dist}_V \), defined in the same way on objects. The tensor product \( F_1 \otimes F_2 : X_1 \otimes X_2 \to Y_1 \otimes Y_2 \) of two distributors \( F_1 : X_1 \to Y_1 \) and \( F_2 : X_2 \to Y_2 \) is defined by letting
\[
(F_1 \otimes F_2)[(y_1, y_2), (x_1, x_2)]:= F_1[y_1, x_1] \otimes F_2[y_2, x_2],
\]
for \( x_1 \in X_1, x_2 \in X_2, y_1 \in Y_1 \) and \( y_2 \in Y_2 \). This defines a symmetric monoidal structure on the bicategory \( \text{Dist}_V \). The monoidal unit \( \iota : I \to \text{Dist}_V \) is the same way on objects. The tensor product \( \text{Dist}_V \) on the bicategory \( \text{Dist}_V \). The monoidal unit \( \iota : I \to \text{Dist}_V \) is compact (also called rigid): the dual of a small \( V \)-category \( X \) is the opposite \( V \)-category \( X^{\text{op}} \). The counit \( \varepsilon : X^{\text{op}} \otimes X \to I \) is given by the hom-functor \( I^{\text{op}} \otimes X^{\text{op}} \otimes X = X^{\text{op}} \otimes X \to V \) and similarly for the unit \( \eta : I \to X \otimes X^{\text{op}} \). Here, \( I \) is the \( V \)-category giving the unit for the tensor product on \( \text{Cat}_V \), as defined in Section 3.

Remark 4.5 (The bicategory of matrices). In view of Section 10 and Section 11 it will be useful for to give an explicit description of the sub-bicategory of \( \text{Dist}_V \) spanned by sets, viewed as discrete \( V \)-categories. In order to do so, we need to recall the definition and some basic properties of the functor mapping an ordinary category to the free \( V \)-category on it. If \( I \) is the unit object of the monoidal category \( V \), then the functor \( V(I, -) : \text{Set} \to V \) has a left adjoint \( (-) \cdot I : \text{Set} \to V \) which associates to a set \( S \) the coproduct \( S \cdot I = \bigsqcup_S I \) of \( S \) copies of \( I \). This left adjoint functor is symmetric (strong) monoidal. Hence, for any pair of sets \( S \) and \( T \), we have an isomorphism
\[
(S \times T) \cdot I \cong (S \cdot I) \otimes (T \cdot I).
\]
A similar situation occurs for the functor \((-\cdot)\cdot I\): Set \(\to\) Cat\(_\mathcal{V}\) which takes a set \(S\) to the \(\mathcal{V}\)-category \(S \cdot I = \bigsqcup_S I\). The functor Und: Cat\(_\mathcal{V}\) \(\to\) Cat mapping a \(\mathcal{V}\)-category to its underlying category has also a left adjoint \((-\cdot)\cdot I\): Cat \(\to\) Cat\(_\mathcal{V}\) which associates to a category \(C\) the \(\mathcal{V}\)-category \(C \cdot I\) defined by letting \(\text{Obj}(C \cdot I) = \text{def} \text{Obj}(C)\) and \((C \cdot I)[x,y] = \text{def} C[x,y] \cdot I\). This left adjoint is symmetric (strong) monoidal. Hence, for every \(C,D \in\text{Cat}\), we have an isomorphism
\[
(C \times D) \cdot I \cong (C \cdot I) \otimes (D \cdot I).
\]
Recall that, for sets \(X\) and \(Y\), a matrix \(F: X \to Y\) is a functor \(F: Y \times X \to V\), i.e. a family of sets \(F(y,x)\), for \((y,x) \in Y \times X\). Sets, matrices and natural transformations form a bicategory, called the bicategory of matrices and denoted Mat\(_\mathcal{V}\), which can be identified with the full subbicategory of Dist\(_\mathcal{V}\) spanned by discrete \(\mathcal{V}\)-categories. Indeed, for a set \(X\), the discrete \(\mathcal{V}\)-category with set of objects \(X\) is the same thing as the free \(\mathcal{V}\)-category on the discrete category with set of objects \(X\), which we denote as \(X \cdot I\). Furthermore, for every pair of sets \(X\) and \(Y\), we have isomorphisms of categories
\[
\text{Mat}_{\mathcal{V}}[X,Y] = \text{CAT}[Y \times X, \mathcal{V}]
\cong \text{CAT}_{\mathcal{V}}[(Y \times X) \cdot I, \mathcal{V}]
\cong \text{CAT}_{\mathcal{V}}[(Y \cdot I) \otimes (X \cdot I), \mathcal{V}]
\cong \text{Dist}_{\mathcal{V}}[X \cdot I, Y \cdot I].
\]
The composition and identity morphisms in Mat\(_\mathcal{V}\) can be then defined so that we have a full and faithful homomorphism from Mat\(_\mathcal{V}\) to Dist\(_\mathcal{V}\). Given two matrices \(F: X \to Y\) and \(G: Y \to Z\), their composite \(G \circ F: X \to Z\) is defined so that there is an isomorphism \((G \circ F) \cdot I \cong (G \cdot I) \circ (F \cdot I)\). It follows that, for \(x \in X\) and \(z \in Z\), we can define
\[
(G \circ F)[z, x] = \text{def} \bigsqcup_{y \in Y} G[z, y] \otimes F[y, x].
\]
For a set \(X\), the identity matrix \(\text{Id}_X: X \to X\) is defined so that \(\text{Id}_X \cdot I \cong \text{Id}_X I\). Hence, for \(x, y \in X\), we can define
\[
\text{Id}_X[x, y] = \text{def} \begin{cases} I & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}
\]
where \(I\) and \(0\) denote the unit object and the initial object of \(\mathcal{V}\), respectively. These definitions determine an inclusion \(\text{Mat}_{\mathcal{V}} \subseteq \text{Dist}_{\mathcal{V}}\).

We conclude this section by defining the operation of composition of a distributor with a \(\mathcal{V}\)-functor, which will be useful in the discussion of composition of analytic functors in Section 11. If \(\mathcal{E}\) is a presentable \(\mathcal{V}\)-category, then we define the composite of a \(\mathcal{V}\)-distributor \(F: X \to Y\) with a \(\mathcal{V}\)-functor \(T: \mathcal{Y} \to \mathcal{E}\) as the \(\mathcal{V}\)-functor \(T \circ F: X \to \mathcal{E}\) defined by mapping \(x \in X\) to
\[
(T \circ F)(x) = \text{def} \bigsqcup_{y \in \mathcal{Y}} T(y) \otimes F(y, x). \tag{4.7}
\]

**Lemma 4.6.** Let \(\mathcal{X}, \mathcal{Y}\) be small \(\mathcal{V}\)-categories and \(\mathcal{E}\) a presentable \(\mathcal{V}\)-category. Let \(F: \mathcal{X} \to \mathcal{Y}\) be a distributor and \(T: \mathcal{Y} \to \mathcal{E}\) be a \(\mathcal{V}\)-functor. There is an isomorphism
\[
(T \circ F)_c \cong T_c \circ F^!,
\]
where \(T_c: P(\mathcal{Y}) \to \mathcal{E}\) is the cocontinuous extension of \(T: \mathcal{Y} \to \mathcal{E}\).
Proof. The functor $T_c \circ F^\dagger : P(\mathbb{X}) \to \mathcal{E}$ is cocontinuous and for every $x \in \mathbb{X}$ we have

$$(T_c \circ F^\dagger)(y_{\mathbb{X}}(x)) \cong T_c(\lambda F)(x) = \int_{y \in Y} T(y) \otimes \lambda(F)(x)(y) = \int_{y \in Y} T(y) \otimes F(y, x) = (T \circ F)(x).$$

Thus, $(T \circ F)_c \cong T_c \circ F^\dagger$ by the uniqueness up to unique isomorphism of the conontinuous extension of a functor. □

**Proposition 4.7.** Let $\mathbb{X}, \mathbb{Y}, \mathbb{Z} \in \text{Cat}_V$ and $\mathcal{E} \in \text{PCAT}_V$.

(i) For all distributors $F : \mathbb{X} \to \mathbb{Y}$, $G : \mathbb{Y} \to \mathbb{Z}$ and $V$-functors $T : \mathbb{Z} \to \mathcal{E}$, $(T \circ G) \circ F \cong T \circ (G \circ F)$.

(ii) For all $T : \mathbb{Z} \to \mathcal{E}$, $T \circ \text{Id}_\mathbb{Z} \cong T$.

**Proof.** For part (i), it suffices to show that we have $((T \circ G) \circ F)_c \cong T \circ (G \circ F)_c$. By Lemma 4.6 and the isomorphism in (4.3) we have

$$(T \circ G) \circ F \cong T \circ (G \circ F),$$

and Part (ii) follows by a similar reasoning. □

5. **Monoidal $V$-categories and $V$-rigs**

We suppose that the reader familiar with the notions of monoidal $V$-category, lax monoidal $V$-functor, and monoidal $V$-natural transformation. We limit ourselves to recalling that, for monoidal $V$-categories $(M, \otimes, e)$ and $(N, \otimes, e)$, a lax monoidal $V$-functor $F : M \to N$ is equipped with multiplication and a unit

$$\mu(x, y) : F(x) \otimes F(y) \to F(x \otimes y), \quad \eta : e \to F(e).$$

We say that $F$ is a monoidal $V$-functor if $\mu$ and $\eta$ are invertible. Recall also that a $V$-natural transformation between lax monoidal $V$-functors is monoidal if it respects the multiplication and unit. We write $\text{MonCat}^\text{lax}_V$ (resp. $\text{MonCat}_V$) for the 2-category of small monoidal $V$-categories, lax monoidal (resp. monodial) $V$-functors and monoidal $V$-natural transformations. If $M$ and $N$ are monoidal $V$-categories, then so is the $V$-category $M \otimes N$. This defines a symmetric monoidal structure on the categories $\text{MonCat}^\text{lax}_V$ and $\text{MonCat}_V$. The unit object is the $V$-category $I$ that is the unit for the tensor product on $\text{Cat}_V$, defined in Section 3. It is easy to verify that $I$ is initial in the bicategory $\text{MonCat}_V$, in the sense that for every $M \in \text{MonCat}_V$ we have an equivalence of categories $\text{MonCat}^\text{lax}_V[I, M] \simeq 1$, where 1 is the terminal category. The notion of a $V$-rig that we introduce in Definition 5.1 below is closely related to the notion of a 2-rig in [3] and that of a 2-ring in [21].

**Definition 5.1.** A $V$-rig is a monoidal closed presentable $V$-category.
A \V\-rig can be defined equivalently as a monoid in the monoidal bicategory \((\mathbf{PCAT}_\V, \otimes, \V)\) in an appropriately weak sense. If \(\mathcal{R}\) and \(\mathcal{S}\) are \V\-rigs, we say that a cocontinuous \V\-functor \(F: \mathcal{R} \to \mathcal{S}\) is a lax homomorphism (resp. homomorphism) of \V\-rigs if it is a lax monoidal (resp. monoidal) functor. We write \(\text{Rig}^\text{lax}_\V\) (resp. \(\text{Rig}_\V\)) for the 2-category of \V\-rigs, lax homomorphisms (resp. homomorphism) and monoidal natural transformations.

We need to recall some basic facts about Day’s convolution tensor product \([22, 23, 34]\). For a small monoidal \V\-category \(\mathcal{M} = (\mathcal{M}, \otimes, 0)\) and a \V\-rig \(\mathcal{R} = (\mathcal{R}, \circ, e)\), the \V\-category \([\mathcal{M}, \mathcal{R}]\) can be equipped with a monoidal structure, called the convolution product, making it into a \V\-rig. By definition, the convolution product \(A_1 \ast A_2\) of two \V\-functors \(A_1, A_2: \mathcal{M} \to \mathcal{R}\) is defined by letting, for \(x \in \mathcal{M}\),

\[
(A_1 \ast A_2)(x) = \text{def} \int_{x_1 \in \mathcal{M}} \int_{x_2 \in \mathcal{M}} A_1(x_1) \otimes A_2(x_2) \otimes \mathcal{M}[x_1 \otimes x_2, x].
\]

(5.1)

The unit object for the convolution product is the functor \(E = \text{def} \mathcal{M}(0, -) \otimes e\). An important case of the convolution tensor product is given by considering \V\-rigs of the form \(P(\mathcal{M}) = [\mathcal{M}^{\text{op}}, \V]\), where \(\mathcal{M} = (\mathcal{M}, \otimes, 0)\) is a small monoidal \V\-category. In this case, for \(A_1, A_2 \in P(\mathcal{M})\), \(x \in \mathcal{M}\), we have

\[
(A_1 \ast A_2)(x) = \int_{x_1 \in \mathcal{M}} \int_{x_2 \in \mathcal{M}} A_1(x_1) \otimes A_2(x_2) \otimes \mathcal{M}[x, x_1 \otimes x_2].
\]

The function mapping a small monoidal \V\-category \(\mathcal{M}\) to the \V\-rig \(P(\mathcal{M})\) extends to a homomorphism

\[
P: \text{MonCat}^\text{lax}_\V \to \text{Rig}^\text{lax}_\V.
\]

(5.2)

Indeed, for every small monoidal \V\-category \(\mathcal{M}\) the Yoneda embedding \(y_\mathcal{M}: \mathcal{M} \to P(\mathcal{M})\) becomes a monoidal functor and it exhibits \(P(\mathcal{M})\) as the free \V\-rig on \(\mathcal{M}\). More precisely, the restriction functor

\[
y_\mathcal{M}^*: \text{Rig}^\text{lax}_\V[P(\mathcal{M}), \mathcal{R}] \to \text{MONCAT}^\text{lax}_\V[\mathcal{M}, \mathcal{R}]
\]

(5.3)

along \(y_\mathcal{M}: \mathcal{M} \to P(\mathcal{M})\) is an equivalence of categories for any \V\-rig \(\mathcal{R}\). The inverse equivalence takes a lax monoidal \V\-functor \(F: \mathcal{M} \to \mathcal{R}\) to its cocontinuous extension \(F_\circ: P(\mathcal{M}) \to \mathcal{R}\), defined as in \([34]\), which can be equipped with a lax monoidal structure. Thus, the homomorphism in \((5.2)\) takes a lax monoidal \V\-functor \(u: \mathcal{M} \to \mathcal{N}\) to the lax homomorphism of rigs \(P(u) = \text{def} u: P(\mathcal{M}) \to P(\mathcal{N})\). All of the above restricts in an evident way to the 2-category \(\text{MonCat}_\V\) so as to give also a homomorphism

\[
P: \text{MonCat}_\V \to \text{Rig}_\V.
\]

(5.4)

Remark 5.2. If \(\mathcal{R}\) and \(\mathcal{S}\) are \V\-rigs, then so is the presentable \V\-category \(\mathcal{R} \hat{\otimes} \mathcal{S}\) discussed in Remark 3.3. This defines the tensor product of a symmetric monoidal closed structure on the 2-categories \(\text{Rig}^\text{lax}_\V\) and \(\text{Rig}_\V\), with unit the category \(\V\). Furthermore, \(\V\) is initial in the 2-category \(\text{Rig}_\V\), in the sense that for every \(\mathcal{R} \in \text{Rig}_\V\) we have an equivalence \(\text{Rig}_\V[I, \mathcal{R}] \simeq 1\). If we consider the 2-categories \(\text{MonCat}^\text{lax}_\V\) and \(\text{Rig}^\text{lax}_\V\) (resp. \(\text{MonCat}_\V\) and \(\text{Rig}_\V\)) as equipped with their symmetric monoidal structures, the homomorphism \(P: \text{MonCat}^\text{lax}_\V \to \text{Rig}^\text{lax}_\V\) (resp. \(P: \text{MonCat}_\V \to \text{Rig}_\V\)) is symmetric monoidal. Indeed, if \(\mathcal{M}\) and \(\mathcal{N}\) are small monoidal \V\-categories, then the equivalence of presentable categories

\[
\phi_{\mathcal{M}, \mathcal{N}}: P(\mathcal{M}) \hat{\otimes} P(\mathcal{N}) \to P(\mathcal{M} \otimes \mathcal{N})
\]

of Remark 3.3 is an equivalence of \V\-rigs.

The homomorphisms in \((5.2)\) and \((5.3)\) will be used in Section 5 to define the bicategories of lax monoidal distributors and of monoidal distributors, respectively, via a Gabriel factorisation. In order to do this, we establish some auxiliary results.
Lemma 5.3. Let $\mathcal{M}, \mathcal{N}$ be small monoidal $\mathcal{V}$-categories, and $\mathcal{R}$ be a $\mathcal{V}$-rig. The equivalences
\[ \lambda^\mathcal{M} : [\mathcal{M} \otimes \mathcal{N}, \mathcal{R}] \to [\mathcal{N}, [\mathcal{M}, \mathcal{R}]], \quad \lambda^\mathcal{N} : [\mathcal{M} \otimes \mathcal{N}, \mathcal{R}] \to [\mathcal{M}, [\mathcal{N}, \mathcal{R}]] \]
are monoidal.

Proof. Let $\lambda = \lambda^\mathcal{M}$. For $\mathcal{V}$-functors $A_1, A_2 : \mathcal{M} \otimes \mathcal{N} \to \mathcal{R}$, we construct a natural isomorphism
\[ \lambda(A_1) \ast \lambda(A_2) \cong \lambda(A_1 \ast A_2). \]
By definition, for $y \in \mathcal{N}$, we have
\[ (\lambda(A_1) \ast \lambda^\mathcal{M}(A_2))(y) = \int^{y_1 \in \mathcal{N}} \int^{y_2 \in \mathcal{N}} \lambda(A_1)(y_1) \ast \lambda(A_2)(y_2) \otimes [y_1 \oplus y_2, y]. \]
Hence, for $x \in \mathcal{M}$ and $y \in \mathcal{N}$, we have
\[ (\lambda(A_1) \ast \lambda^\mathcal{M}(A_2))(y)(x) = \int^{y_1 \in \mathcal{N}} \int^{y_2 \in \mathcal{N}} (\lambda(A_1)(y_1) \ast \lambda(A_2)(y_2))(x) \otimes [y_1 \oplus y_2, y]. \]
But, for $x \in \mathcal{M}$ and $y_1, y_2 \in \mathcal{N}$, we have
\[ (\lambda(A_1)(y_1) \ast \lambda(A_2)(y_2))(x) = \int^{x_1 \in \mathcal{M}} \int^{x_2 \in \mathcal{M}} \lambda(A_1)(y_1)(x_1) \ast \lambda(A_2)(y_2)(x_2) \otimes \mathcal{M}[x_1 \oplus x_2, x] \]
\[ \cong \int^{x_1 \in \mathcal{M}} \int^{x_2 \in \mathcal{M}} A_1(x_1, y_1) \ast A_2(x_2, y_2) \otimes \mathcal{M}[x_1 \oplus x_2, x]. \]
By substituting the right-hand side of (5.6) in the right-hand side of (5.5), it follows that
\[ (\lambda(A_1) \ast \lambda^\mathcal{M}(A_2))(y)(x) \]
\[ \cong \int^{y_1 \in \mathcal{N}} \int^{y_2 \in \mathcal{N}} \int^{x_1 \in \mathcal{M}} \int^{x_2 \in \mathcal{M}} A_1(x_1, y_1) \ast A_2(x_2, y_2) \otimes \mathcal{M}[x_1 \oplus x_2, x] \otimes [y_1 \oplus y_2, y] \]
\[ \cong \int^{(x_1, y_1) \in \mathcal{M} \otimes \mathcal{N}} \int^{(x_2, y_2) \in \mathcal{M} \otimes \mathcal{N}} A_1(x_1, y_1) \ast A_2(x_2, y_2) \otimes (\mathcal{M} \otimes \mathcal{N})[(x_1, y_1) \oplus (x_2, y_2), (x, y)] \]
\[ = (A_1 \ast A_2)(x, y) \]
\[ = \lambda^\mathcal{M}(A_1 \ast A_2)(y)(x), \]
as required. 

Let $\mathcal{M} = (\mathcal{M}, \oplus, 0)$ be a small monoidal $\mathcal{V}$-category and $\mathcal{R} = (\mathcal{R}, \circ, e)$ be a $\mathcal{V}$-rig. By the definition of the convolution product of $A_1, A_2 \in [\mathcal{M}, \mathcal{R}]$, as given in (5.1), there is a canonical map
\[ \text{can} : A_1(x_1) \ast A_2(x_2) \to (A_1 \ast A_2)(x_1 \oplus x_2). \]
If $A = (A, \mu, \eta)$ is a monoid object in $[\mathcal{M}, \mathcal{R}]$, then the composite
\[ A(x_1) \ast A(x_2) \xrightarrow{\text{can}} (A \ast A)(x_1 \oplus x_2) \xrightarrow{\mu(x_1 \oplus x_2)} A(x_1 \oplus x_2) \]
is a lax monoidal structure on $A$ with components $\mu(x_1, x_2) : A(x_1) \ast A(x_2) \to A(x_1 \oplus x_2)$ on $A$. This defines a functor $\rho : \text{Mon}[\mathcal{M}, \mathcal{R}] \to \text{MONCAT}_\mathcal{V}^{\text{lax}}[\mathcal{M}, \mathcal{R}]$, where $\text{Mon}[\mathcal{M}, \mathcal{R}]$ denotes the category of monoids in $\text{CAT}_\mathcal{V}[\mathcal{M}, \mathcal{R}]$ and $\text{MONCAT}_\mathcal{V}^{\text{lax}}[\mathcal{M}, \mathcal{R}]$ denotes the category of lax monoidal $\mathcal{V}$-functors from $\mathcal{M}$ to $\mathcal{R}$. The next lemma is essentially as in [22, Example 3.2.2] and [51, Proposition 22.1].

Lemma 5.4. The functor $\rho : \text{Mon}[\mathcal{M}, \mathcal{R}] \to \text{MONCAT}_\mathcal{V}^{\text{lax}}[\mathcal{M}, \mathcal{R}]$ is an equivalence of categories.
Proof. Let us describe an inverse to \( \rho \). Let \( \mu(x_1, x_2): A(x_1) \diamond A(x_2) \to A(x_1 \oplus x_2) \) be a lax monoidal structure on a functor \( A: M \to R \). We have

\[
(A \ast A)(x) = \int_{x_1 \in M} \int_{x_2 \in M} M(x_1 \oplus x_2, x) \otimes A(x_1) \diamond A(x_2)
\]

and the natural transformation \( \mu(x_1, x_2): A(x_1) \diamond A(x_2) \to A(x_1 \oplus x_2) \) induces a map

\[
(A \ast A)(x) \to \int_{x_1 \in M} \int_{x_2 \in M} M(x_1 \oplus x_2, x) \otimes A(x_1 \oplus x_2) \to A(x)
\]

which defines the multiplication \( \mu: A \ast A \to A \) of a monoid object \( (A, \mu, \eta) \) in \([M, R]\). It is easy to verify that this is an inverse to \( \rho \).

We can now extend Proposition 3.4 to categories of monoidal functors.

**Proposition 5.5.** The equivalences of categories

\[
\lambda^M: [M \otimes N, R] \to [N, [M, R]], \quad \lambda^N: [M \otimes N, R] \to \text{CAT}_V([M, N, R])
\]

restrict to equivalences of categories

\[
\lambda^M: \text{MONCAT}_{\text{lax}}^V[M \otimes N, R] \to \text{MONCAT}_{\text{lax}}^V[N, [M, R]],
\lambda^N: \text{MONCAT}_{\text{lax}}^V[M \otimes N, R] \to \text{MONCAT}_{\text{lax}}^V[M, [N, R]].
\]

**Proof.** We consider only \( \lambda^M \), since the argument for \( \lambda^N \) is analogous. First of all, observe that \( \lambda^M \) is monoidal by Lemma 5.3. It thus induces an equivalence between the category of monoids in the monoidal category \([M \otimes N, R]\) and the category of monoids in the monoidal category \([N, [M, R]]\). The claim then follows by Lemma 5.4. \( \square \)

**Corollary 5.6.** Let \( M, N \) be small monoidal \( V \)-categories. For every distributor \( F: M \to N \), there is a bijective correspondence between the lax monoidal structures on the \( V \)-functors

\[
F: N^{\text{op}} \otimes M \to V, \quad \lambda F: M \to P(N), \quad F^\dagger: P(M) \to P(N).
\]

**Proof.** By Proposition 5.5 and the equivalence in (5.3). \( \square \)

6. **Monoidal distributors**

The next definition introduces the notion of a lax monoidal distributor.

**Definition 6.1.** Let \( M, N \) be small monoidal \( V \)-categories. A lax monoidal distributor \( F: M \to N \) is a lax monoidal functor \( F: N^{\text{op}} \otimes M \to V \).

We now introduce the bicategory of lax monoidal distributors.

**Theorem 6.2.** Small monoidal \( V \)-categories, lax monoidal distributors and monoidal \( V \)-transformations form a bicategory, called the bicategory of lax monoidal distributors and denoted \( \text{MonDist}_{\text{lax}}^V \), which fits into a Gabriel factorisation diagram

\[
\begin{array}{ccc}
\text{MonCat}_{\text{lax}}^V & \xrightarrow{P} & \text{Rig}_{\text{lax}}^V \\
\downarrow (-)^* & & \downarrow (-)^\dagger \\
\text{MonDist}_{\text{lax}}^V & & \\
\end{array}
\]
Theorem 6.4. Small monoidal $\mathcal{V}$-categories, monoidal distributors and monoidal $\mathcal{V}$-transformations form a bicategory, called the bicategory of monoidal distributors and denoted $\text{MonDist}_{\mathcal{V}}$.

Proof. For small monoidal $\mathcal{V}$-categories $M$ and $N$, we define the hom-category of lax monoidal distributors from $M$ to $N$ by letting

$$\text{MonDist}^{\text{lax}}_{\mathcal{V}}[M, N] = \text{def} \ \text{MONCAT}^{\text{lax}}_{\mathcal{V}}[N^{\text{op}} \otimes M, \mathcal{V}].$$

Then, we define the functor

$$( - )^{\dagger} : \text{MonDist}^{\text{lax}}_{\mathcal{V}}[M, N] \to \text{Rig}^{\text{lax}}_{\mathcal{V}}[P(M), P(N)]$$

as the composite

$$\text{MonDist}^{\text{lax}}_{\mathcal{V}}[M, N] \xrightarrow{\lambda} \text{MONCAT}^{\text{lax}}_{\mathcal{V}}[M, P(N)] \xrightarrow{( - )_e^{\dagger}} \text{Rig}^{\text{lax}}_{\mathcal{V}}[P(M), P(N)].$$

The functor $\lambda$ is an equivalence by Proposition 5.5. Since $( - )_e$ is also an equivalence (being a quasi-inverse to composition with $y_M$), it follows that $( - )^{\dagger}$ is an equivalence, as required. The rest of the data necessary to have a bicategory is determined by the requirement to have a Gabriel factorisation. In particular, the second part of the Gabriel factorisation is then defined by mapping a small monoidal $\mathcal{V}$-category to $\mathcal{M}^{\dagger} = \text{def} (P(M)$, viewed as a $\mathcal{V}$-rig with the convolution monoidal structure. For the first part of the factorisation, we need to show that if $u : M \to N$ is a lax monoidal $\mathcal{V}$-functor, then the distributor $u_* : M \to N$ is lax monoidal. But the functor $u : P(M) \to P(N)$ is lax monoidal, since the functor $u : M \to N$ is lax monoidal. Corollary 5.6 implies that the distributor $u_*$ is lax monoidal, since we have $(u_*)^{\dagger} \cong u$. □

The composition operation of $\text{MonDist}^{\text{lax}}_{\mathcal{V}}$ is obtained as a restriction of the composition operation of $\text{Dist}_{\mathcal{V}}$. Indeed, for lax monoidal distributors $F : M \to N$ and $G : N \to P$, the composite distributor $G \circ F : M \to P$ is lax monoidal, since there is an isomorphism

$$(G \circ F)^{\dagger} \cong G^{\dagger} \circ F^{\dagger}$$

and lax monoidal $\mathcal{V}$-functors are closed under composition. Similarly, the identity morphism on a small monoidal $\mathcal{V}$-category is the identity distributor $\text{Id}_M : M \to M$, which is lax monoidal since the $\mathcal{V}$-functor of $M$ is a lax monoidal $\mathcal{V}$-functor. All of the above admits a restriction to the case of monoidal, rather than lax monoidal, $\mathcal{V}$-functors. In order to make the theory work out smoothly, however, it is appropriate to define the notion of a monoidal distributor as follows.

Definition 6.3. Let $M, N$ be small monoidal $\mathcal{V}$-categories. A monoidal distributor $F : M \to N$ is a monoidal distributor such that the lax monoidal functor $\lambda F : N \to P(M)$ is monoidal.

Let us point out that requiring a lax monoidal distributor $F : M \to N$ to be monoidal is not equivalent to requiring the lax monoidal $\mathcal{V}$-functor $F : N^{\text{op}} \otimes M \to \mathcal{V}$ to be monoidal. For example, consider the identity distributor $\text{Id}_M : M \to M$, which is given by the hom-functor $M(-, -) : M^{\text{op}} \otimes M \to \mathcal{V}$. This $\mathcal{V}$-functor is lax monoidal, but not monoidal. However, $\text{Id}_M : M \to M$ is a monoidal distributor since $\lambda(\text{Id}_M) : M \to P(M)$ is the Yoneda embedding $y_M : M \to P(M)$, which is monoidal. Note that, by Corollary 5.6, a lax monoidal distributor $F : M \to N$ is monoidal if and only if the lax monoidal functor $F^{\dagger} : P(N) \to P(M)$ is monoidal.

The next theorem defines the bicategory of monoidal distributors.

Theorem 6.4. Small monoidal $\mathcal{V}$-categories, monoidal distributors and monoidal $\mathcal{V}$-transformations form a bicategory, called the bicategory of monoidal distributors and denoted $\text{MonDist}_{\mathcal{V}}$. 
which fits in a Gabriel factorisation diagram

\[
\begin{array}{ccc}
\text{MonCat}_V & \xrightarrow{P} & \text{Rig}_V \\
\downarrow (-) \circ & & \downarrow (-)^t \\
\text{MonDist}_V & & \\
\end{array}
\]

Proof. For small monoidal \(V\)-categories \(M\) and \(N\), we define the category \(\text{MonDist}_V[M,N]\) as the full sub-category of \(\text{MonDist}_V^{\text{lax}}[M,N]\) spanned by monoidal distributors. The rest of the proof follows the pattern of the one of Theorem 6.2. In particular, the functor \(\lambda\) used in the proof of Theorem 6.2 restricts to an equivalence

\[
\lambda: \text{MonDist}_V[M,N] \to \text{Rig}_V[M,P(N)]
\]

by the very definition of the notion of a monoidal distributor. \(\square\)

Remark 6.5. The tensor product \(F_1 \otimes F_2: M_1 \otimes M_2 \to N_1 \otimes N_2\) of lax monoidal (resp. monoidal) distributors \(F_1: M_1 \to N_1\) and \(F_2: M_2 \to N_2\) is lax monoidal (resp. monoidal). The operation defines a symmetric monoidal structure on the bicategories \(\text{MonDist}_V^{\text{lax}}\) and \(\text{MonDist}_V\). Moreover, the homomorphisms in the Gabriel factorizations of Theorem 6.2 and Theorem 6.4 are symmetric monoidal. Let us also point out that the symmetric monoidal bicategory \(\text{MonDist}_V\) is compact: the dual of a monoidal \(V\)-category \(M\) is the opposite \(V\)-category \(M^{\text{op}}\). The counit \(\varepsilon: M^{\text{op}} \otimes M \to I\) is given by the hom-functor \(I^{\text{op}} \otimes M^{\text{op}} \otimes M = M^{\text{op}} \otimes M \to V\) and similarly for the unit \(\eta: I \to M \otimes M^{\text{op}}\). In contrast, the symmetric monoidal bicategory \(\text{MonDist}_V\) is not compact.

We conclude this section by restricting to the monoidal case the operation of composition of a distributor with a functor, defined in [4.7].

Proposition 6.6. Let \(R\) be a \(V\)-rig. Then the composite of a monoidal distributor \(F: M \to N\) with a monoidal functor \(T: N \to R\) is a monoidal functor \(T \circ F: M \to R\).

Proof. It suffices to show that \((T \circ F)_c: P(M) \to R\) is monoidal. The functor \(F^t: P(M) \to P(N)\) is monoidal, since the distributor \(F: M \to N\) is monoidal, and the functor \(T_c: P(N) \to R\) is monoidal, since \(T\) is monoidal. Hence, the functor \(T_c \circ F^t: P(M) \to R\) is monoidal. This proves the result, since we have \((T \circ F)_c \cong T_c \circ F^t\) by Lemma 4.6. \(\square\)

7. Symmetric monoidal \(V\)-categories and symmetric \(V\)-rigs

The aim of this section is to develop the counterpart for symmetric monoidal \(V\)-categories of the material in Section 5. Let us recall that, for symmetric monoidal \(V\)-categories \(A\) and \(B\), a lax monoidal \(V\)-functor \(F: A \to B\) is said to be symmetric if, for any pair of objects \(x \in A\) and \(y \in B\), the following square commutes

\[
\begin{array}{ccc}
F(x) \otimes F(y) & \xrightarrow{\mu(x,y)} & F(x \otimes y) \\
\downarrow {\sigma} & & \downarrow {F(\sigma)} \\
F(y) \otimes F(x) & \xrightarrow{\mu(y,x)} & F(y \otimes x),
\end{array}
\]

where we use \(\sigma\) to denote the symmetry isomorphism of both \(A\) and \(B\). We write \(\text{SMonCat}_V^{\text{lax}}\) (resp. \(\text{SMonCat}_V\)) for the 2-category of small symmetric monoidal \(V\)-categories, symmetric lax monoidal (resp. symmetric monoidal) \(V\)-functors and monoidal \(V\)-natural transformations. If \(A\)
and \( B \) are symmetric monoidal \( \mathcal{V} \)-categories, then so is the \( \mathcal{V} \)-category \( A \otimes B \). This operation defines a symmetric monoidal structure on the categories \( \text{SMonCat}_{\mathcal{V}}^{\text{lax}} \) and \( \text{SMonCat}_V \). The unit object is the \( \mathcal{V} \)-category \( \mathcal{I} \) giving the unit for the tensor product of \( \text{Cat}_V \), as defined in Section 3.

It is easy to verify that \( I \) is initial in the 2-category \( \text{SMonCat}_V \). If \( \mathcal{A} = (A, \oplus, 0, \sigma) \) is a symmetric monoidal category, then the interchange law

\[
(x_1 \oplus x_2) \oplus (y_1 \oplus y_2) \cong (x_1 \oplus y_1) \oplus (x_2 \oplus y_2)
\]

is a natural (symmetric) monoidal structure on the tensor product functor of \( \mathcal{A} \).

**Lemma 7.1.** The 2-category \( \text{SMonCat}_{\mathcal{V}} \) has finite coproducts. In particular, the coproduct of two small symmetric monoidal \( \mathcal{V} \)-categories \( A_1 \) and \( A_2 \) is their tensor product \( A_1 \otimes A_2 \).

**Proof.** Let \( A_1 = (A_1, \oplus, 0) \), \( A_2 = (A_2, \oplus, 0) \) be two symmetric monoidal categories. We define the functors \( t_1 : A_1 \to A_1 \otimes A_2 \) and \( t_2 : A_2 \to A_1 \otimes A_2 \) by letting \( t_1(x) = \text{def} (x, 0) \) and \( t_2(x) = \text{def} (0, x) \). We now have to show that the functor

\[
\pi : \text{SMonCat}_V[A_1 \otimes A_2, B] \to \text{SMonCat}_V[A_1, B] \times \text{SMonCat}_V[A_2, B],
\]

defined by letting \( \pi(F) = \text{def} (F \circ t_1, F \circ t_2) \), is an equivalence of categories for any symmetric monoidal \( \mathcal{V} \)-category \( B \). The tensor product functor \( \mu : B \otimes B \to B \) is symmetric monoidal, since the monoidal category \( \mathcal{B} \) is symmetric. Thus, if \( F_1 : A_1 \to B \) and \( F_2 : A_2 \to B \) are symmetric monoidal functors, then the functor \( \mu \circ (F_1 \otimes F_2) : A_1 \otimes A_2 \to B \) is symmetric monoidal. Hence, we obtain a functor

\[
\rho : \text{SMonCat}_V[A, B] \times \text{SMonCat}_V[A_2, B] \to \text{SMonCat}_V[A_1 \otimes A_2, B]
\]

defined by letting \( \rho(F_1, F_2) = \text{def} \mu \circ (F_1 \otimes F_2) \). The verification that the functors \( \pi \) and \( \rho \) are mutually pseudo-inverse is left to the reader. \( \square \)

**Definition 7.2.** A **symmetric** \( \mathcal{V} \)-**rig** is a symmetric monoidal closed presentable \( \mathcal{V} \)-category.

If \( \mathcal{R} \) and \( \mathcal{S} \) are symmetric \( \mathcal{V} \)-rings, we say that a lax homomorphism (resp. homomorphism) of \( \mathcal{V} \)-rings \( F : \mathcal{R} \to \mathcal{S} \) is a symmetric if it is symmetric as a lax monoidal (resp. monoidal) \( \mathcal{V} \)-functor. We write \( \text{SRig}^{\text{lax}}_V \) (resp. \( \text{SRig}_V \)) for the 2-category of symmetric \( \mathcal{V} \)-rings, symmetric lax homomorphisms (resp. symmetric homomorphisms) and monoidal \( \mathcal{V} \)-natural transformations.

The convolution tensor product extends in a natural way to the symmetric case \([23, 34]\). Indeed, if \( \mathcal{A} = (A, \oplus, 0, \sigma_A) \) is a small symmetric monoidal \( \mathcal{V} \)-category and \( \mathcal{R} = (\mathcal{R}, \circ, e, \sigma_\mathcal{R}) \) is symmetric \( \mathcal{V} \)-ring then the \( \mathcal{V} \)-rig \( [\mathcal{A}, \mathcal{R}] \) is symmetric. By definition, for \( F, G : \mathcal{A} \to \mathcal{R} \), the value at \( x \in \mathcal{A} \) of the symmetry isomorphism \( \sigma : F \ast G \to G \ast F \) is the coend over \( x_1, x_2 \in \mathcal{A} \) of the maps

\[
\sigma_{\mathcal{R}} \otimes [\sigma_A, x] : F(x_1) \circ G(x_2) \otimes [\mathcal{A}, x_1 \oplus x_2, x] \to G(x_2) \circ F(x_1) \otimes [\mathcal{A}, x_2 \oplus x_1, x].
\]

If \( \mathcal{A} \) is a small symmetric monoidal \( \mathcal{V} \)-category, then \( P(\mathcal{A}) = [\mathcal{A}^{\text{op}}, \mathcal{V}] \) is a symmetric \( \mathcal{V} \)-rig. The Yoneda functor \( y_{\mathcal{A}} : \mathcal{A} \to P(\mathcal{A}) \) is symmetric monoidal and exhibits \( P(\mathcal{A}) \) as the free symmetric \( \mathcal{V} \)-rig on \( \mathcal{A} \). More precisely, this means that the restriction functor

\[
y_{\mathcal{A}} : \text{SMonCat}^{\text{lax}}_{\mathcal{V}}[P(\mathcal{A}), \mathcal{R}] \to \text{SMonCat}^{\text{lax}}_{\mathcal{V}}[\mathcal{A}, \mathcal{R}]
\]

along the Yoneda functor \( y_{\mathcal{A}} : \mathcal{A} \to P(\mathcal{A}) \) is an equivalence of categories for any symmetric \( \mathcal{V} \)-rig \( \mathcal{R} \). The inverse equivalence takes a symmetric lax monoidal \( \mathcal{V} \)-functor \( F : \mathcal{A} \to \mathcal{R} \) to the functor \( F_{\mathcal{A}} : P(\mathcal{A}) \to \mathcal{R} \), which can be equipped with the structure of a lax homomorphism of \( \mathcal{V} \)-rings. We write

\[
P : \text{SMonCat}^{\text{lax}}_{\mathcal{V}} \to \text{SRig}^{\text{lax}}_{\mathcal{V}}
\]

for the homomorphism of bicategories which takes a symmetric lax monoidal functor \( u : \mathcal{A} \to \mathcal{B} \) to the lax symmetric homomorphism of symmetric \( \mathcal{V} \)-rings \( P(u) = \text{def} u_* : P(\mathcal{A}) \to P(\mathcal{B}) \). All of
the above restricts to symmetric monoidal $V$-functors and symmetric homomorphisms of $V$-rigs and so we obtain a homomorphism

$$P : \text{SMonCat}_V \to \text{SRig}_V.$$ 

**Remark 7.3.** If $\mathcal{R}$ and $\mathcal{S}$ are symmetric $V$-rigs, then so is their completed tensor product $\mathcal{R} \hat{\otimes} \mathcal{S}$ (cf. Remark 5.3). This defines a symmetric monoidal structure on the bicategories $\text{SRig}_V^{\text{lax}}$ and $\text{SRig}_V$. The unit object is the category $V$. If $A$ and $B$ are symmetric monoidal $V$-categories, then the canonical functor

$$P(A) \hat{\otimes} P(B) \to P(A \otimes B)$$

is an equivalence of symmetric $V$-rigs. This witnesses the fact that the homomorphisms of bicategories $P : \text{SMonCat}_V \to \text{SRig}_V$ and $P : \text{SMonCat}_V^{\text{lax}} \to \text{SRig}_V^{\text{lax}}$ are symmetric monoidal.

We now proceed to extend Proposition 5.3 to functor categories of symmetric lax monoidal $V$-functors. The first step is the following lemma, which is a counterpart of Lemma 5.3 for symmetric monoidal $V$-categories.

**Lemma 7.4.** Let $A, B$ be small symmetric monoidal $V$-categories and $\mathcal{R}$ be a symmetric $V$-rig. Then, the monoidal equivalences

$$\lambda^A : [A \otimes B, \mathcal{R}] \to [B, [A, \mathcal{R}]], \quad \lambda^B : [A \otimes B, \mathcal{R}] \to [A, [B, \mathcal{R}]]$$

are symmetric.

**Proof.** Similar to the proof of Lemma 5.3.

Let $A = (\mathcal{A}, \otimes, 0, \sigma_\mathcal{A})$ be a small symmetric monoidal $V$-category and $\mathcal{R} = (\mathcal{R}, \circ, e, \sigma_\mathcal{R})$ be a symmetric $V$-rig. As we have just seen, the $V$-rig $[A, \mathcal{R}]$ is symmetric. Now, for $F, G : A \to \mathcal{R}$, if

$$\sigma : F \otimes G \to G \circ F$$

is the symmetry isomorphism, then the following diagram commutes for every $x_1, x_2 \in A$,

$$\begin{align*}
F(x_1) \circ G(x_2) &\xrightarrow{\text{can}} (F \circ G)(x_1 \oplus x_2) \\
\sigma_\mathcal{R} &\downarrow \quad \downarrow \sigma_\mathcal{R} \\
G(x_2) \circ F(x_1) &\xrightarrow{\text{can}} (G \circ F)(x_2 \oplus x_1),
\end{align*}$$

(7.2)

where the maps labelled $\text{can}$ are as in (5.4). For a small symmetric monoidal $V$-category $A$ and a symmetric $V$-rig $\mathcal{R}$, we write $\text{CMon}[A, \mathcal{R}]$ for the category of commutative monoid objects in the symmetric $V$-rig $[A, \mathcal{R}]$, which is a full subcategory of the category $\text{Mon}[A, \mathcal{R}]$ of monoid objects in $[A, \mathcal{R}]$. The next lemma is a counterpart of Lemma 5.4 for symmetric monoidal $V$-categories.

**Lemma 7.5.** Let $A$ be a small symmetric monoidal $V$-category and $\mathcal{R}$ be a symmetric $V$-rig. Then, the equivalence of categories

$$\rho : \text{Mon}[A, \mathcal{R}] \to \text{MONCAT}_V^{\text{lax}}[A, \mathcal{R}]$$

restricts to an equivalence of categories

$$\text{CMon}[A, \mathcal{R}] \to \text{SMONCAT}_V^{\text{lax}}[A, \mathcal{R}].$$

**Proof.** Let us show that the functor $\rho$ takes a commutative monoid object $F = (F, \mu, \eta)$ in $[A, \mathcal{R}]$ to a symmetric lax monoidal functor. By definition, the lax monoidal structure $\mu(x_1, x_2) : F(x_1) \otimes F(x_2) \to F(x_1 \oplus x_2)$ on the functor $F : A \to \mathcal{V}$ is obtained by composing the maps

$$\begin{align*}
F(x_1) \otimes F(x_2) &\xrightarrow{\text{can}} (F \otimes F)(x_1 \oplus x_2) \xrightarrow{\mu(x_1 \oplus x_2)} F(x_1 \oplus x_2),
\end{align*}$$

(7.3)
Let us now consider the following diagram:

\[
(F \ast F)(x_1 \oplus x_2) \xrightarrow{\mu(x_1 \oplus x_2)} F(x_1 \oplus x_2) \\
\sigma(x_1 \oplus x_2) \\
(F \ast F)(x_1 \oplus x_2) \xrightarrow{\mu(x_1 \oplus x_2)} F(x_1 \oplus x_2) \\
(F \ast F)(\sigma_{x_1}) \\
(F \ast F)(x_2 \oplus x_1) \xrightarrow{\mu(x_2 \oplus x_1)} F(x_2 \oplus x_1).
\]

Its top square commutes since the product \(\mu: F \ast F \to F\) is commutative, while the bottom square commutes by naturality. It follows by composing that following square commutes,

\[
(F \ast F)(x_1 \oplus x_2) \xrightarrow{\mu(x_1 \oplus x_2)} F(x_1 \oplus x_2) \\
\sigma_{x_1} \\
(F \ast F)(\sigma_{x_1}) \\
(F \ast F)(x_2 \oplus x_1) \xrightarrow{\mu(x_2 \oplus x_1)} F(x_2 \oplus x_1).
\]

(7.4)

But the following square commutes, being an instance of the diagram in (7.2)

\[
F(x_1) \circ F(x_2) \xrightarrow{\text{can}} (F \ast F)(x_1 \oplus x_2) \\
\sigma_{x_1} \\
F(x_2) \circ F(x_1) \xrightarrow{\text{can}} (F \ast F)(x_2 \oplus x_1).
\]

(7.5)

If we compose horizontally the squares in (7.4) and (7.5), we obtain the following commutative square

\[
F(x_1) \circ F(x_2) \xrightarrow{\mu(x_1, x_2)} F(x_1 \oplus x_2) \\
\sigma_{x_1} \\
F(x_2) \circ F(x_1) \xrightarrow{\mu(x_2, x_1)} F(x_2 \oplus x_1).
\]

This shows that the lax monoidal structure in (7.3) is symmetric. \(\square\)

It is now possible to extend Proposition 5.5 to the symmetric case.

**Proposition 7.6.** The equivalences of categories

\[
\lambda^A: [A \otimes B, R] \to [B, [A, R]], \quad \lambda^B: [A \otimes B, R] \to [A, [B, R]]
\]

restrict to equivalences of categories

\[
\text{SMONCAT}^{\text{lax}}_\lambda[A \otimes B, R] \simeq \text{SMONCAT}^{\text{lax}}_\lambda[B, [A, R]], \\
\text{SMONCAT}^{\text{lax}}_\lambda[A \otimes B, R] \simeq \text{SMONCAT}^{\text{lax}}_\lambda[A, [B, R]].
\]

**Proof.** This follows from Lemma 7.4 and Lemma 7.5. \(\square\)

The next corollary is the counterpart of Corollary 5.6 in the symmetric monoidal case.
Corollary 7.7. Let \( A, B \) be small symmetric \( \mathcal{V} \)-categories. For every distributor \( F: A \to B \), the symmetric lax monoidal structures on the \( \mathcal{V} \)-functors

\[
F: \mathcal{V} \to \mathcal{V}, \quad \lambda F: A \to P(B), \quad F^\dagger: P(A) \to P(B)
\]

are in bijective correspondence.

Proof. The claim follows by Proposition 7.6 and the equivalence in (7.1). \( \square \)

8. Symmetric monoidal distributors

We begin by introducing the notion of a symmetric lax monoidal distributor.

Definition 8.1. Let \( A, B \) be small symmetric monoidal \( \mathcal{V} \)-categories. A symmetric lax monoidal distributor \( F: A \to B \) is a symmetric lax monoidal functor \( F: \mathcal{V} \to \mathcal{V} \) such that the hom-functor from \( A \) to \( B \) is symmetric lax monoidal. Furthermore, the identity distributor \( I \) is symmetric lax monoidal.

Theorem 8.2. Small symmetric monoidal \( \mathcal{V} \)-categories, symmetric lax monoidal distributors and monoidal \( \mathcal{V} \)-natural transformations form a bicategory, called the bicategory of symmetric lax monoidal distributors and denoted \( \text{SMonDist}^{\text{lax}} \), which fits in a Gabriel factorisation

\[
\begin{array}{ccc}
\text{SMonCat}^{\text{lax}} & \xrightarrow{P} & \text{SRig}^{\text{lax}} \\
\downarrow (-)_\bullet & & \downarrow (-)^\dagger \\
\text{SMonDist}^{\text{lax}} & \xleftarrow{\text{SRig}^{\text{lax}}} & \end{array}
\]

Proof. For small symmetric monoidal \( \mathcal{V} \)-categories \( A \) and \( B \) we define the hom-category of symmetric lax monoidal distributors from \( A \) to \( B \) by letting

\[
\text{SMonDist}^{\text{lax}}[A, B] \overset{\text{def}}{=} \text{SMONCAT}^{\text{lax}}[\mathcal{V} \to \mathcal{V}, A, B].
\]

With this definition, we have an equivalence of categories

\[
(\_\_)^\dagger: \text{SMonDist}^{\text{lax}}[A, B] \to \text{SRig}^{\text{lax}}[P(A), P(B)],
\]

which is defined as the composite of the following two equivalences:

\[
\begin{array}{ccc}
\text{SMonDist}^{\text{lax}}[A, B] & \xrightarrow{\lambda} & \text{SMONCAT}^{\text{lax}}[\mathcal{V} \to \mathcal{V}, A, P(B)] \\
\downarrow (-)^\circ & & \downarrow (-)^\circ \\
\text{SRig}^{\text{lax}}[P(A), P(B)] & \xleftarrow{\text{SRig}^{\text{lax}}} & \end{array}
\]

The rest of the data is determined by the requirement of having a Gabriel factorisation. In particular, the action of \( (\_\_)^\dagger \): \( \text{SMonDist}^{\text{lax}}[\mathcal{V} \to \mathcal{V}, A, B] \to \text{SRig}^{\text{lax}}[P(A), P(B)] \) on objects by letting \( \lambda \): \( A \to B \) is symmetric lax monoidal. By Corollary 7.7 the lax monoidal functor \( u: P(A) \to P(B) \) is symmetric, since the lax monoidal functor \( u: A \to B \) is symmetric. Hence the lax monoidal distributor \( u \) is symmetric, since we have \( (u\_\_)^\dagger \overset{\text{def}}{=} u \). \( \square \)

Note that the composition law and the identity morphisms are defined as in \( \text{Dist}_{\mathcal{V}} \). Indeed, for symmetric lax monoidal distributors \( F: A \to B \) and \( G: B \to C \), the distributor \( G \circ F: \mathcal{V} \to \mathcal{V} \) is symmetric lax monoidal, since \( (G \circ F)^\dagger \overset{\text{def}}{=} G^\dagger \circ F^\dagger \) and symmetric lax monoidal functors are closed under composition. Furthermore, the identity distributor \( \text{Id}_A: A \to A \) is symmetric lax monoidal, since the hom-functor from \( A \to A \) to \( \mathcal{V} \) is symmetric lax monoidal.

Definition 8.3. If \( A \) and \( B \) are small symmetric monoidal \( \mathcal{V} \)-categories, we say that a monoidal distributor \( F: A \to B \) is symmetric if the monoidal functor \( \lambda F: A \to P(B) \) is symmetric.
The next theorem introduces the bicategory of symmetric monoidal distributors.

**Theorem 8.4.** Small symmetric monoidal $\mathcal{V}$-categories, symmetric monoidal distributors and monoidal $\mathcal{V}$-transformations form a bicategory $\text{SMonDist}_\mathcal{V}$ which fits in a Gabriel factorisation

$$\begin{array}{ccc}
\text{SMonCat}_\mathcal{V} & \xrightarrow{P} & \text{SRig}_\mathcal{V} \\
\downarrow \Phi & & \downarrow \Phi^t \\
\text{SMonDist}_\mathcal{V} & \xrightarrow{(-)^t} & .
\end{array}$$

**Proof.** For small symmetric monoidal $\mathcal{V}$-categories $A$ and $B$, we define $\text{SMonDist}_\mathcal{V}[A, B]$ as the full sub-category of $\text{SMonDist}_\mathcal{V}^{\text{lax}}[A, B]$ spanned by symmetric monoidal distributors. In this way, the functor $\lambda$ in the proof of Theorem 8.2 restricts to an equivalence $\lambda: \text{SMonDist}_\mathcal{V}[A, B] \to \text{SRig}_\mathcal{V}[A, P(B)]$. The rest of the proof follows the usual pattern. $\square$

The composition law and the identity morphisms of $\text{SMonDist}_\mathcal{V}$ are defined as in $\text{Dist}_\mathcal{V}$, for reasons analogous to those given in the case of $\text{MonDist}_\mathcal{V}$ in Section 6.

**Remark 8.5.** The tensor product $F_1 \otimes F_2: A_1 \otimes A_2 \to B_1 \otimes B_2$ of symmetric lax monoidal (resp. symmetric monoidal) distributors $F_1: A_1 \to B_1$ and $F_2: A_2 \to B_2$ is symmetric lax monoidal (resp. symmetric monoidal). The operation defines a symmetric monoidal structure on the bicategories $\text{SMonDist}_\mathcal{V}^{\text{lax}}$ and $\text{SMonDist}_\mathcal{V}$. Moreover, the homomorphisms involved in the Gabriel factorizations of Theorem 8.2 and Theorem 8.4 are symmetric monoidal. The symmetric monoidal bicategory $\text{SMonDist}_\mathcal{V}^{\text{lax}}$ is compact: the dual of a symmetric monoidal $\mathcal{V}$-category $A$ is the opposite $\mathcal{V}$-category $A^{\text{op}}$. The counit $\varepsilon: A^{\text{op}} \otimes A \to I$ is given by the hom-functor $I^{\text{op}} \otimes A^{\text{op}} \otimes A = A^{\text{op}} \otimes A \to \mathcal{V}$ and similarly for the unit $\eta: I \to A \otimes A^{\text{op}}$. In contrast, the symmetric monoidal bicategory $\text{SMonDist}_\mathcal{V}$ is not compact.

We conclude the section by extending Proposition 6.6 to the symmetric case.

**Proposition 8.6.** Let $\mathcal{R}$ be a symmetric $\mathcal{V}$-rig. Then the composite of a symmetric monoidal distributor $F: A \to B$ with a symmetric monoidal functor $T: B \to \mathcal{R}$ is a symmetric monoidal functor $T \circ F: A \to \mathcal{R}$. $\square$

9. **Free symmetric monoidal $\mathcal{V}$-categories**

The forgetful 2-functor $U: \text{SMonCat}_\mathcal{V} \to \text{Cat}_\mathcal{V}$ has a left adjoint $S: \text{Cat}_\mathcal{V} \to \text{SMonCat}_\mathcal{V}$ which associates to a small $\mathcal{V}$-category $\mathcal{X}$ the symmetric monoidal $\mathcal{V}$-category $S(\mathcal{X})$ freely generated by $\mathcal{X}$ [13]. This $\mathcal{V}$-category is defined by letting

$$S(\mathcal{X}) = \text{def} \bigsqcup_{n \in \mathbb{N}} S^n(\mathcal{X}),$$

(9.1)

where $S^n(\mathcal{X})$ is the symmetric $n$-power of $\mathcal{X}$. More explicitly, observe that the $n$-th symmetric group $\Sigma_n$ acts naturally on the $\mathcal{V}$-category $\mathcal{X}^n$ with the right action defined by letting

$$\mathbf{e} \cdot \sigma = \text{def} (x_{\sigma 1}, \ldots, x_{\sigma n}),$$
for \( \mathcal{T} = (x_1, \ldots, x_n) \in X^n \) and \( \sigma \in \Sigma_n \). If we apply the Grothendieck construction to this right action, we obtain the symmetric \( n \)-power of \( X \),

\[
S^n(X) = \operatorname{def} \Sigma_n \int X^n.
\]

Explicitly, an object of \( S^n(X) \) is a sequence \( \mathcal{T} = (x_1, \ldots, x_n) \) of objects of \( X \), and the hom-object between \( \mathcal{T}, \mathcal{F} \in S^n(X) \) is defined by letting

\[
S^n(X)[\mathcal{T}, \mathcal{F}] = \operatorname{def} \bigsqcup_{\sigma \in \Sigma_n} X[x_1, y_{\sigma(1)}] \otimes \cdots \otimes X[x_n, y_{\sigma(n)}],
\]

where the coproduct on the right-hand side is taken in \( V \). The tensor product of \( \mathcal{T} \in S^m(X) \) and \( \mathcal{F} \in S^n(X) \) is the concatenation

\[
\mathcal{T} \oplus \mathcal{F} = \operatorname{def} (x_1, \ldots, x_m, y_1, \ldots, y_n).
\]

The symmetry \( \sigma_{\mathcal{T}, \mathcal{F}}: \mathcal{T} \oplus \mathcal{F} \to \mathcal{F} \oplus \mathcal{T} \) is the shuffle permutation swapping the first \( m \)-elements of the first sequence with the last \( m \)-elements of the second. The unit is the empty sequence \( e \). The inclusion \( V \)-functor

\[
i_X: X \to S(X),
\]

which takes \( x \in X \) to the one-element sequence \( (x) \in S^1(X) \), exhibits \( S(X) \) as the free symmetric monoidal \( V \)-category on \( X \). More precisely, for every symmetric monoidal \( V \)-category \( A = (A, \oplus, 0, \sigma) \) the restriction functor

\[
i_A^*: \text{SMonCat}_V[S(X), A] \to \text{Cat}_V[X, A],
\]

defined by letting \( i_A^*(v) = \operatorname{def} v \circ i_X \), is an equivalence of categories. It follows that every \( V \)-functor \( T: X \to A \) admits a symmetric monoidal extension \( T^e: S(X) \to A \) fitting in the diagram,

\[
\begin{array}{ccc}
X & \xrightarrow{i_X} & S(X) \\
\downarrow T & & \downarrow T^e \\
A, & & 
\end{array}
\]

and that \( T^e \) is unique up to unique isomorphism of symmetric monoidal \( V \)-functors. Explicitly, for \( \mathcal{T} = (x_1, \ldots, x_n) \in S(X) \), we have

\[
T^e(\mathcal{T}) = \operatorname{def} T(x_1) \oplus \cdots \oplus T(x_n).
\]

Our next theorem shows how the adjunction between \( \text{Cat}_V \) and \( \text{SMonCat}_V \) extends to an adjunction between \( \text{Dist}_V \) and \( \text{SMonDist}_V \).

**Theorem 9.1.** The forgetful homomorphism \( U: \text{SMonDist}_V \to \text{Dist}_V \) has a left adjoint

\[
S: \text{Dist}_V \to \text{SMonDist}_V.
\]

**Proof.** Let \( i = i_X: X \to S(X) \), so that we have a distributor \( i_*: X \to S(X) \). We need to show that the restriction functor

\[
i_*^*: \text{SMonDist}_V[S(X), A] \to \text{Dist}_V[X, A]
\]
defined by letting $\iota_\star(F) = \text{def} \ F \circ \iota_\star$ is an equivalence of categories for any symmetric monoidal $\mathcal{V}$-category $A$. The following diagram commutes up to isomorphism by part (i) of Lemma 4.2.

$$
\begin{array}{c}
\text{SMonDist}_\mathcal{V}[S(\mathcal{X}), A] \\
\downarrow \lambda \\
\text{SMonCat}_\mathcal{V}[S(\mathcal{X}), P(A)]
\end{array}
\xrightarrow{(-)\iota_\star} 
\begin{array}{c}
\text{Dist}_\mathcal{V}[\mathcal{X}, A] \\
\downarrow \lambda \\
\text{CAT}_\mathcal{V}[\mathcal{X}, P(A)].
\end{array}
$$

The bottom side of this diagram is an equivalence of categories. Hence also the top side, since the vertical sides are equivalences.

We give an explicit formula for the symmetric monoidal extension $F^e : S(\mathcal{X}) \to A$ of a distributor $F : \mathcal{X} \to A$, which fits in the diagram of $S$-distributors

$$
\begin{array}{c}
\mathcal{X} \\
\downarrow F \\
\mathcal{Y}
\end{array}
\xrightarrow{(\iota_\star \bullet)}
\begin{array}{c}
S(\mathcal{X}) \\
\downarrow F^e \\
A
\end{array}
$$

and is the unique such distributor up to a unique isomorphism of symmetric monoidal $\mathcal{V}$-distributors. By construction, the functor $\lambda(F^e) : S(\mathcal{X}) \to P(\mathcal{A})$ is the symmetric monoidal extension of the $\mathcal{V}$-functor $\lambda F : \mathcal{X} \to P(\mathcal{A})$. Thus, $\lambda(F^e) = (\lambda F)^e$ and it follows that

$$
F^e[y; \mathcal{X}] = \text{def} \int_{y_1 \in A} \cdots \int_{y_n \in A} F[y_1, x_1] \otimes \cdots \otimes F[y_n, x_n] \otimes A[y, y_1 \otimes \cdots \otimes y_n],
$$

(9.2)

for $\mathcal{X} = (x_1, \ldots, x_n) \in S^n(\mathcal{X})$ and $y \in A$. A special case of these definitions that will be of importance for our development arises by considering $\mathcal{A} = S(\mathcal{Y})$, where $\mathcal{Y}$ is a small $\mathcal{V}$-category. In this case, the symmetric monoidal extension $F^e : S(\mathcal{X}) \to S(\mathcal{Y})$ of a distributor $F : \mathcal{X} \to S(\mathcal{Y})$ is defined by letting

$$
F^e[\mathcal{Y}; \mathcal{X}] = \text{def} \int_{\mathcal{Y}_1 \in S(\mathcal{Y})} \cdots \int_{\mathcal{Y}_n \in S(\mathcal{Y})} F[\mathcal{Y}_1; x_1] \otimes \cdots \otimes F[\mathcal{Y}_n; x_n] \otimes S(\mathcal{Y})[\mathcal{Y}_1, \mathcal{Y}_1 \oplus \cdots \oplus \mathcal{Y}_n],
$$

(9.3)

for $\mathcal{X} = (x_1, \ldots, x_n) \in S^n(\mathcal{X})$ and $\mathcal{Y} \in S(\mathcal{Y})$.

It will be useful to describe the action of the homomorphism $S : \text{Dist}_\mathcal{V} \to \text{SMonDist}_\mathcal{V}$ on morphisms. For a distributor $F : \mathcal{X} \to \mathcal{Y}$, the symmetric monoidal distributor $S(F) : S(\mathcal{X}) \to S(\mathcal{Y})$ is defined by letting $S(F) = \text{def} \ (\iota_\star \bullet F)^e$ and therefore makes following diagram commute up to a canonical isomorphism

$$
\begin{array}{c}
\mathcal{X} \\
\downarrow F \\
\mathcal{Y}
\end{array}
\xrightarrow{\iota_\star}
\begin{array}{c}
S(\mathcal{X}) \\
\downarrow S(F) \\
S(\mathcal{Y}).
\end{array}
$$

Explicitly, for $\mathcal{X} = (x_1, \ldots, x_n)$ and $\mathcal{Y} = (y_1, \ldots, y_m)$, we have

$$
S(F)[\mathcal{Y}; \mathcal{X}] = \text{def} \begin{cases} 
\bigcup_{\sigma \in \Sigma_n} F[y_{\sigma(1)}, x_{\sigma(1)}] \otimes \cdots \otimes F[y_{\sigma(n)}, x_{\sigma(n)}], & \text{if } m = n, \\
0 & \text{otherwise},
\end{cases}
$$

We conclude this section by stating an important property of the interaction between tensor products and coproducts of $\mathcal{V}$-categories. For $\mathcal{X}, \mathcal{Y} \in \text{Cat}_\mathcal{V}$, define the $\mathcal{V}$-functor

$$
eq S(\mathcal{X}) \otimes S(\mathcal{Y}) \to S(\mathcal{X} \sqcup \mathcal{Y}),
$$

(9.4)
by letting $c_{\mathcal{X},\mathcal{Y}}(\mathcal{F} \otimes \mathcal{G}) = \text{def} \mathcal{F} \oplus \mathcal{G}$. The next proposition will be used in the proofs of Proposition \ref{prop:10.6} and Theorem \ref{thm:11.6}.

**Proposition 9.2.** For every $\mathcal{X}, \mathcal{Y} \in \text{Cat}_\mathcal{V}$, the $\mathcal{V}$-functor $c_{\mathcal{X},\mathcal{Y}}: S(\mathcal{X}) \otimes S(\mathcal{Y}) \to S(\mathcal{X} \sqcup \mathcal{Y})$ is a symmetric monoidal equivalence.

**Proof.** Let us write $c$ for $c_{\mathcal{X},\mathcal{Y}}$. It is easy to verify that $c: S(\mathcal{X}) \otimes S(\mathcal{Y}) \to S(\mathcal{X} \sqcup \mathcal{Y})$ is symmetric monoidal. Let us show that it is an equivalence. The 2-functor $S: \text{Cat}_\mathcal{V} \to \text{SMonCat}_\mathcal{V}$ preserves coproducts, since it is a left adjoint. Thus, if $\iota_1: \mathcal{X} \to \mathcal{X} \sqcup \mathcal{Y}$ and $\iota_2: \mathcal{Y} \to \mathcal{X} \sqcup \mathcal{Y}$ are the inclusions, then the functors $S(\iota_1)$ and $S(\iota_2)$ exhibit $S(\mathcal{X} \sqcup \mathcal{Y})$ as the coproduct of $S(\mathcal{X})$ and $S(\mathcal{Y})$. But the functors $\iota_1: S(\mathcal{X}) \to S(\mathcal{X}) \otimes S(\mathcal{Y})$ and $\iota_2: S(\mathcal{Y}) \to S(\mathcal{X}) \otimes S(\mathcal{Y})$ defined by letting $\iota_1(\mathcal{F}) = \text{def} \mathcal{F} \otimes e$ and $\iota_2(\mathcal{G}) = \text{def} e \otimes \mathcal{G}$, where we write $e$ for the empty sequence in both $S(\mathcal{X})$ and $S(\mathcal{Y})$, exhibit $S(\mathcal{X}) \otimes S(\mathcal{Y})$ as the coproduct of $S(\mathcal{X})$ and $S(\mathcal{Y})$ by Lemma \ref{lem:7.1}. It follows that the $\mathcal{V}$-functor $c: S(\mathcal{X}) \otimes S(\mathcal{Y}) \to S(\mathcal{X} \sqcup \mathcal{Y})$ is an equivalence, since the diagram

$$
\begin{array}{ccc}
S(\mathcal{X}) & \xrightarrow{\iota_1} & S(\mathcal{X}) \otimes S(\mathcal{Y}) & \xleftarrow{\iota_2} & S(\mathcal{Y}) \\
\downarrow{c} & & \downarrow{c} & & \downarrow{c} \\
S(\mathcal{X} \sqcup \mathcal{Y}) & \xrightarrow{S(\iota_1)} & S(\mathcal{X} \sqcup \mathcal{Y}) & \xleftarrow{S(\iota_2)} & S(\mathcal{X} \sqcup \mathcal{Y})
\end{array}
$$

commutes. \hfill \Box

10. $S$-distributors

We introduce the notion of an $S$-distributor.

**Definition 10.1.** Let $\mathcal{X}, \mathcal{Y}$ be small $\mathcal{V}$-categories. An $S$-distributor $F: \mathcal{X} \to \mathcal{Y}$ is a distributor $F: \mathcal{X} \to S(\mathcal{Y})$, i.e. a $\mathcal{V}$-functor $F: S(\mathcal{Y})^{\text{op}} \otimes \mathcal{X} \to \mathcal{V}$.

We write $F[\mathcal{G}; \mathcal{X}]$ for the result of applying an $S$-distributor $F: \mathcal{X} \to \mathcal{Y}$ to $(\mathcal{G}, \mathcal{X}) \in S(\mathcal{Y})^{\text{op}} \otimes \mathcal{X}$. The next theorem introduces the bicategory of $S$-distributors. Once again, its proof uses a Gabriel factorisation.

**Theorem 10.2.** Small $\mathcal{V}$-categories, $S$-distributors and $\mathcal{V}$-transformations form a bicategory, called the bicategory of $S$-distributors and denoted $\text{SMonDist}_\mathcal{V}$, which fits into a Gabriel factorisation diagram

$$
\begin{array}{ccc}
\text{Dist}_\mathcal{V} & \xrightarrow{S} & \text{SMonDist}_\mathcal{V} \\
\downarrow{\delta} & & \downarrow{(-)^*} \\
S\text{-Dist}_\mathcal{V} & \xrightarrow{(-)^*} & \text{SMonDist}_\mathcal{V}
\end{array}
$$

**Proof.** Let $\mathcal{X}, \mathcal{Y}$ be small $\mathcal{V}$-categories. We define the hom-category of $S$-distributors from $\mathcal{X}$ to $\mathcal{Y}$ by letting $S\text{-Dist}_\mathcal{V}[\mathcal{X}, \mathcal{Y}] = \text{def} \text{Dist}_\mathcal{V}[\mathcal{X}, S(\mathcal{Y})]$. We then define a functor

$$
(-)^*: S\text{-Dist}_\mathcal{V}[\mathcal{X}, \mathcal{Y}] \to \text{SMonDist}_\mathcal{V}[S(\mathcal{X}), S(\mathcal{Y})] \tag{10.1}
$$

by mapping an $S$-distributor $F: \mathcal{X} \to \mathcal{Y}$, given by a distributor $F: \mathcal{X} \to S(\mathcal{Y})$, to its symmetric monoidal extension $F^e: S(\mathcal{X}) \to S(\mathcal{Y})$, defined as in \ref{eq:11.3}. We represent the action of this functor by the derivation

$$
\begin{array}{c}
F: \mathcal{X} \to \mathcal{Y} \\
F^e: S(\mathcal{X}) \to S(\mathcal{Y})
\end{array}
$$
Given two $S$-distributors $F: \mathcal{X} \to \mathcal{Y}$ and $G: \mathcal{Y} \to \mathcal{Z}$, we define their composite $G \circ F: \mathcal{X} \to \mathcal{Z}$ so as to have an isomorphism $(G \circ F)^e \cong G^e \circ F^e$. For $\mathcal{X} \in S(\mathcal{X})$ and $\mathcal{Y} \in S(\mathcal{Z})$, we have

$$(G^e \circ F^e)[\mathcal{X}, \mathcal{Z}] = \int_{\mathcal{Y} \in S(\mathcal{Y})} G^e[\mathcal{X}, \mathcal{Y}] \otimes F^e[\mathcal{Y}, \mathcal{Z}],$$

and therefore, for $x \in \mathcal{X}$ and $\mathcal{Z} \in S(\mathcal{Z})$, we let

$$(G \circ F)[\mathcal{X}, x] = \text{def} \int_{\mathcal{Y} \in S(\mathcal{Y})} G^e[\mathcal{X}, \mathcal{Y}] \otimes F^e[\mathcal{Y}, x]. \quad (10.2)$$

Here, the distributor $G^e: S(\mathcal{Y}) \to S(\mathcal{Z})$ is defined via the formula in (10.3). The horizontal composition of $S$-distributors is functorial, coherently associative since the horizontal composition of the bicategory $S\text{-MonDist}_\mathcal{V}$ is so. The identity $S$-distributor $\text{Id}_\mathcal{X}: \mathcal{X} \to \mathcal{X}$ is determined in an analogous way as the composition operation; explicitly, we define it as the distributor $(\iota_\mathcal{X})^e: \mathcal{X} \to S(\mathcal{X})$, where $\iota_\mathcal{X}: \mathcal{X} \to S(\mathcal{X})$ is the inclusion $\mathcal{V}$-functor. Thus,

$$\text{Id}_\mathcal{X}[x, \ldots, x, x] = \text{def} S(\mathcal{X})[(x, \ldots, x), (x)] = \begin{cases} \mathcal{X}[x, x] & \text{if } n = 1, \\ 0 & \text{otherwise}. \end{cases} \quad (10.3)$$

This completes the definition of the bicategory $S\text{-Dist}_\mathcal{V}$. The homomorphism

$$(-)^e: S\text{-Dist}_\mathcal{V} \to S\text{-MonDist}_\mathcal{V}$$

is then defined as follows. Its action on objects is defined by letting $\mathcal{X}^e = \text{def} S(\mathcal{X})$, while its action on hom-categories is given by the functors in (11.1). We complete the definition of the required Gabriel factorisation by defining the homomorphism $\delta: \text{Dist}_\mathcal{V} \to S\text{-Dist}_\mathcal{V}$. Its action on object is the identity. Given a distributor $F: \mathcal{X} \to \mathcal{Y}$ we define the $S$-distributor $\delta(F): \mathcal{X} \to S(\mathcal{Y})$ by letting $\delta(F)[\mathcal{X}, \mathcal{Y}] = \text{def} \iota^e \circ F$, where $\iota: \mathcal{Y} \to S(\mathcal{Y})$ is the inclusion $\mathcal{V}$-functor. In this way, we have

$$S(F) \cong (\iota \circ F)^e = \delta(F)^e. \quad (10.4)$$

Explicitly, we have

$$\delta(F)[\mathcal{X}, \mathcal{Y}] = \begin{cases} F[y, x] & \text{if } \mathcal{Y} = (y) \in S(\mathcal{Y}), \\ 0 & \text{otherwise}. \end{cases}$$

For distributors $F: \mathcal{X} \to \mathcal{Y}$ and $G: \mathcal{Y} \to \mathcal{Z}$, the pseudo-functoriality isomorphism

$$\delta(G) \circ \delta(F) \cong \delta(G \circ F)$$

is obtained combining the fact that the functor in (10.1) is an equivalence with the existence of an isomorphism

$$\delta(G \circ F)^e \cong S(G \circ F)^e \cong S(G) \circ S(F)^e \cong \delta(G)^e \circ \delta(F)^e,$$

where we used the isomorphism in (10.4) twice. In a similar way one obtains an isomorphism $\delta(\text{Id}_\mathcal{X}) \cong \text{Id}_\mathcal{X}$ for every small $\mathcal{V}$-category $\mathcal{X}$.\hfill \Box

**Remark 10.3.** For every pair of small $\mathcal{V}$-categories $\mathcal{X}$, $\mathcal{Y}$, the following diagram commutes:

$$\begin{array}{ccc}
\text{Dist}_\mathcal{V}[\mathcal{X}, \mathcal{Y}] & \xrightarrow{\lambda} & \text{Cat}_\mathcal{V}[\mathcal{X}, PS(\mathcal{Y})] \\
\text{SMonDist}_\mathcal{V}[\mathcal{X}, S(\mathcal{Y})] & \xrightarrow{\lambda} & \text{SMonCat}_\mathcal{V}[S(\mathcal{X}), PS(\mathcal{Y})] \\
\xrightarrow{(-)^e} & & \xrightarrow{(-)^e}
\end{array}$$

Thus, for a distributor $F: \mathcal{X} \to \mathcal{Y}$, we write $\lambda F^e: S(\mathcal{X}) \to PS(\mathcal{Y})$ for the common value of the composite functors. Note that the functor $(\lambda F^e)_0: PS(\mathcal{X}) \to PS(\mathcal{Y})$ is a homomorphism of
symmetric \( \mathcal{V} \)-rigs. Symbolically,

\[
\begin{align*}
F^c : \mathcal{X} & \to \mathcal{Y} \text{ in } S\text{-Dist}_V \\
F^c : S(\mathcal{X}) & \to S(\mathcal{Y}) \text{ in } S\text{MonDist}_V \\
\lambda F^c : S(\mathcal{X}) & \to PS(\mathcal{Y}) \text{ in } S\text{MonCat}_V \\
(\lambda F^c)_c : PS(\mathcal{X}) & \to PS(\mathcal{Y}) \text{ in } S\text{Rig}_V.
\end{align*}
\]

**Lemma 10.4.** If \( \mathcal{X}_1 \) and \( \mathcal{X}_2 \) are small \( \mathcal{V} \)-categories and \( \iota_k : \mathcal{X}_k \to \mathcal{X}_1 \sqcup \mathcal{X}_2 \) is the \( k \)-th inclusion, then the symmetric monoidal distributors \( S(\iota_k)_* : S(\mathcal{X}_k) \to S(\mathcal{X}_1 \sqcup \mathcal{X}_2) \) for \( k = 1, 2 \) exhibit \( S(\mathcal{X}_1 \sqcup \mathcal{X}_2) \) as the coproduct of \( S(\mathcal{X}_1) \) and \( S(\mathcal{X}_2) \) in the bicategory \( S\text{MonDist}_V \).

**Proof.** The distributors \( (\iota_k)_* : \mathcal{X}_k \to \mathcal{X}_1 \sqcup \mathcal{X}_2 \) (for \( k = 1, 2 \)) exhibit \( \mathcal{X}_1 \sqcup \mathcal{X}_2 \) as the coproduct of \( \mathcal{X}_1 \) and \( \mathcal{X}_2 \) in the bicategory \( \text{Dist}_V \) by Proposition [10.3]. Hence the symmetric monoidal distributors \( S(\iota_k)_* = S((\iota_k)_*) : S(\mathcal{X}_k) \to S(\mathcal{X}_1 \sqcup \mathcal{X}_2) \) exhibits the coproduct of \( S(\mathcal{X}_1) \) and \( S(\mathcal{X}_2) \) in the bicategory \( S\text{MonDist}_V \), since the homomorphism \( S : \text{Dist}_V \to S\text{MonDist}_V \) is a left adjoint and hence preserves finite coproducts. \( \square \)

**Proposition 10.5.** The bicategory \( S\text{-Dist}_V \) has finite coproducts and the homomorphisms

\[
\delta : \text{Dist}_V \to S\text{-Dist}_V, \quad (-)^c : S\text{-Dist}_V \to S\text{MonDist}_V
\]

preserve finite coproducts.

**Proof.** This follows from Lemma [10.4] since the homomorphism \( S : \text{Dist}_V \to S\text{MonDist}_V \) preserves finite coproducts. \( \square \)

We establish an explicit formula for the coproduct homomorphism on \( S \)-distributors, which will be used in the proof of Theorem [10.3]. By definition, the coproduct of two \( S \)-distributors \( F_1 : \mathcal{X}_1 \to \mathcal{Y}_1 \) and \( F_2 : \mathcal{X}_2 \to \mathcal{Y}_2 \) is a \( S \)-distributor \( F_1 \sqcup_S F_2 : \mathcal{X}_1 \sqcup \mathcal{X}_2 \to \mathcal{Y}_1 \sqcup \mathcal{Y}_2 \) fitting in a commutative diagram of \( S \)-distributors,

\[
\begin{array}{ccc}
\mathcal{X}_1 & \xrightarrow{F_1} & \mathcal{Y}_1 \\
\downarrow{\delta(\iota_1)} & & \downarrow{\delta(\iota_1^\ast)} \\
\mathcal{X}_1 \sqcup \mathcal{X}_2 & \xrightarrow{F_1 \sqcup_F F_2} & \mathcal{Y}_1 \sqcup \mathcal{Y}_2 \\
\downarrow{\delta(\iota_2)} & & \downarrow{\delta(\iota_2^\ast)} \\
\mathcal{X}_2 & \xrightarrow{F_2} & \mathcal{Y}_2
\end{array}
\]

(10.5)

where \( \iota_k : \mathcal{X}_k \to \mathcal{X}_1 \sqcup \mathcal{X}_2 \) and \( j_k : \mathcal{Y}_k \to \mathcal{Y}_1 \sqcup \mathcal{Y}_2 \) are the inclusions.

**Proposition 10.6.** Given \( S \)-distributors \( F_1 : \mathcal{X}_1 \to \mathcal{Y}_1 \) and \( F_2 : \mathcal{X}_2 \to \mathcal{Y}_2 \), then

\[
(F_1 \sqcup_S F_2)^c[\bar{\mathfrak{g}}_1 \oplus \mathfrak{g}_2, \bar{\mathfrak{r}}_1 \oplus \bar{\mathfrak{r}}_2] \cong F_1^c[\bar{\mathfrak{g}}_1, \mathfrak{r}_1] \otimes F_2^c[\mathfrak{g}_2, \bar{\mathfrak{r}}_2]
\]

for \( \mathfrak{r}_1 \in S(\mathcal{X}_1), \mathfrak{r}_2 \in S(\mathcal{X}_2), \bar{\mathfrak{g}}_1 \in S(\mathcal{Y}_1) \) and \( \bar{\mathfrak{g}}_2 \in S(\mathcal{Y}_2) \).
Proof. The image of the diagram in (10.5) by the homomorphism \((-)^c : \text{S-Dist}_V \to \text{SMonDist}_V\) is a commutative diagram of symmetric monoidal distributors,

\[
\begin{array}{ccc}
S(X_1) & \xrightarrow{F_1^c} & S(Y_1) \\
S(i_1^\ast) \downarrow & & \downarrow S(j_1^\ast) \\
S(X_1 \sqcup X_2) & \xrightarrow{(F_1 \oplus F_2)^c} & S(Y_1 \sqcup Y_2) \\
S(i_2^\ast) \downarrow & & \downarrow S(j_2^\ast) \\
S(X_2) & \xrightarrow{F_2^c} & S(Y_2),
\end{array}
\]

from which we obtain the following commutative square of symmetric monoidal distributors,

\[
\begin{array}{ccc}
S(X_1) \otimes S(X_2) & \xrightarrow{F_1^c \otimes F_2^c} & S(Y_1) \otimes S(Y_2) \\
\downarrow c & & \downarrow c \\
S(X_1 \sqcup X_2) & \xrightarrow{(F_1 \oplus F_2)^c} & S(Y_1 \sqcup Y_2),
\end{array}
\]

where \(c : S(X_1) \otimes S(X_2) \to S(X_1 \sqcup X_2)\) is the \(V\)-functor of (11.4), which in this case is defined by letting \(c(x \otimes y) = \text{def} x_1 \oplus x_2\). We have \(c^c \circ c_\ast \cong \text{Id}_{S(Y_1) \otimes S(Y_2)}\), since \(c\) is an equivalence by Proposition 11.2. Hence the following diagram of distributors commutes

\[
\begin{array}{ccc}
S(X_1) \otimes S(X_2) & \xrightarrow{F_1^c \otimes F_2^c} & S(Y_1) \otimes S(Y_2) \\
\downarrow c & & \downarrow c^c \\
S(X_1 \sqcup X_2) & \xrightarrow{(F_1 \oplus F_2)^c} & S(Y_1 \sqcup Y_2).
\end{array}
\]

This proves the result, since

\[
(F_1^c \otimes F_2^c)[y_1 \otimes y_2, x_1 \otimes x_2] = \text{def} F_1^c[y_1, x_1] \otimes F_2^c[y_2, x_2]
\]

and, by part (ii) of Lemma 11.2, we have

\[
(c^c \circ (F_1 \oplus F_2)^c \circ c_\ast)[y_1 \otimes y_2, x_1 \otimes x_2] \cong (F_1 \oplus F_2)^c[y_1 \oplus y_2, x_1 \oplus x_2]. \tag*{\Box}
\]

Remark 10.7. We wish to give an explicit description of the full sub-bicategory of \(\text{S-Dist}_V\) spanned by sets, in analogy with what we did in Remark 4.5, where we showed how the bicategory of matrices \(\text{Mat}_V\) can be seen as the full sub-bicategory of the bicategory of distributors spanned by sets. If \(M\) is a symmetric monoidal category, then \(M \cdot 1\) has the structure of a symmetric monoidal \(V\)-category and the functor mapping \(M\) to \(M \cdot 1\) is left adjoint to the functor \(\text{Und} : \text{SMonCat}_V \to \text{SMonCat}\) mapping a symmetric monoidal \(V\)-category to its underlying symmetric monoidal category. We also have a natural isomorphism,

\[
S(C) \cdot 1 \cong S(C \cdot 1)
\]

since the diagram

\[
\begin{array}{ccc}
\text{SMonCat}_V & \xrightarrow{\text{Und}} & \text{SMonCat} \\
U \downarrow & & \downarrow U \\
\text{Cat}_V & \xrightarrow{\text{Und}} & \text{Cat}
\end{array}
\]
commutes and a composite of left adjoints is left adjoint to the composite. As a special case of the definitions in Section \[ \text{4} \] the free symmetric monoidal category \( S(X) \) on a set \( X \) admits the following direct description. For \( n \in \mathbb{N} \), let \( S^n(X) \) be the category whose objects are sequences \( \mathfrak{x} = (x_1, \ldots, x_n) \) of elements of \( X \) and whose morphisms \( \sigma: (x_1, \ldots, x_n) \to (x'_1, \ldots, x'_n) \) are permutations \( \sigma \in \Sigma_n \) such that \( x'_i = x_{\sigma(i)} \) for \( 1 \leq i \leq n \). We then let \( S(X) \) be the coproduct of the categories \( S_n(X) \), for \( n \in \mathbb{N} \),

\[
S(X) = \text{def} \bigsqcup_{n \in \mathbb{N}} S^n(X).
\]

For sets \( X \) and \( Y \), we define an \( S \)-matrix \( F: X \to Y \) to be a functor \( F: S(Y)^{\text{op}} \times X \to \mathcal{V} \). Sets, \( S \)-matrices and natural transformations form a bicategory \( S\text{-Mat}_\mathcal{V} \) which can be identified with the full sub-bicategory of the bicategory \( S\text{-Dist}_\mathcal{V} \) of \( S \)-distributors spanned by discrete \( \mathcal{V} \)-categories. Indeed, for sets \( X \) and \( Y \), we have the following chain of isomorphisms:

\[
S\text{-Mat}_\mathcal{V}[X,Y] = \text{Cat}[S(Y)^{\text{op}} \times X, \mathcal{V}]
\]

\[
\cong \text{Cat}_\mathcal{V}[(S(Y)^{\text{op}} \times X) \cdot I, \mathcal{V}]
\]

\[
\cong \text{Cat}_\mathcal{V}[(S(Y)^{\text{op}} \cdot I) \otimes (X \cdot I), \mathcal{V}]
\]

\[
\cong \text{Cat}_\mathcal{V}[(S(Y) \cdot I)^{\text{op}} \otimes (X \cdot I), \mathcal{V}]
\]

\[
\cong S\text{-Dist}_\mathcal{V}[X, I, Y \cdot I] .
\]

The composition operation and the identity morphisms of \( S\text{-Mat}_\mathcal{V} \) are determined by those of \( S\text{-Dist}_\mathcal{V} \) analogously to the way in which composition operation and the identity morphisms of \( \text{Mat}_\mathcal{V} \) are determined by those of \( \text{Dist}_\mathcal{V} \). We do not unfold these definitions, since in Section \[ \text{14} \] we will describe explicitly its opposite bicategory. Note that we obtain the following diagram of inclusions:

\[
\begin{array}{ccc}
\text{Mat}_\mathcal{V} & \longrightarrow & S\text{-Mat}_\mathcal{V} \\
\downarrow & & \downarrow \\
\text{Dist}_\mathcal{V} & \longrightarrow & S\text{-Dist}_\mathcal{V} .
\end{array}
\]

We conclude this section by defining the operation of composition of an \( S \)-distributor with a functor, in analogy with the definition of composition of a distributor with a functor, given in \[ \text{17} \]. Let \( \mathcal{R} \) be a symmetric \( \mathcal{V} \)-rig. Then the composite of an \( S \)-distributor \( F: \mathcal{X} \to \mathcal{Y} \) with a \( \mathcal{V} \)-functor \( T: \mathcal{Y} \to \mathcal{R} \) is the \( \mathcal{V} \)-functor \( T \circ F: \mathcal{X} \to \mathcal{R} \) defined by letting

\[
(T \circ F)(x) = \text{def} \int_{\mathfrak{y} \in S(\mathcal{Y})} T^e(\mathfrak{y}) \otimes F[\mathfrak{y}, x] , \tag{10.6}
\]

for \( x \in \mathcal{X} \).

**Lemma 10.8.** Let \( \mathcal{X}, \mathcal{Y} \) be small \( \mathcal{V} \)-categories and \( \mathcal{R} \) be a symmetric \( \mathcal{V} \)-rig. For every \( S \)-distributor \( \mathcal{F}: \mathcal{X} \to \mathcal{Y} \) and \( \mathcal{V} \)-functor \( T: \mathcal{Y} \to \mathcal{R} \), we have

\[
(T \circ \mathcal{F})^e \cong T^e \circ \mathcal{F}^e ,
\]

where

\[
(T^e \circ \mathcal{F}^e)(\mathfrak{x}) = \int_{\mathfrak{y} \in S(\mathcal{Y})} T^e(\mathfrak{y}) \otimes \mathcal{F}^e[\mathfrak{y}, \mathfrak{x}] .
\]

**Proof.** The functor \( T^e \circ \mathcal{F}^e: S(\mathcal{X}) \to \mathcal{R} \) is symmetric monoidal by Proposition \[ \text{8.6} \] since the distributor \( \mathcal{F}^e: S(\mathcal{X}) \to S(\mathcal{Y}) \) is symmetric monoidal and the \( \mathcal{V} \)-functor \( T^e: S(\mathcal{Y}) \to \mathcal{R} \) is symmetric.
monoidal. Moreover, if $\mathcal{F} = (x)$ with $x \in \mathcal{X}$, then

$$(T^e \circ F^e)((x)) = \int_{y \in S(\mathcal{Y})} T^e(y) \otimes F^e[y; (x)] = \int_{y \in S(\mathcal{Y})} T^e(y) \otimes F[y; x] = (T \circ F)(x).$$

Hence the symmetric monoidal functor $T^e \circ F^e : S(\mathcal{X}) \to \mathcal{R}$ is an extension of $T \circ F : \mathcal{X} \to \mathcal{R}$. This shows that $(T \circ F)^e \cong T^e \circ F^e$ by the uniqueness up to unique isomorphism of the extension. □

**Proposition 10.9.** Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be small $\mathcal{V}$-categories and $\mathcal{R}$ be a symmetric $\mathcal{V}$-rig.

(i) For all distributors $F : \mathcal{X} \to \mathcal{Y}$, $G : \mathcal{Y} \to \mathcal{Z}$ and $\mathcal{V}$-functors $T : \mathcal{Z} \to \mathcal{R}$, $(T \circ G) \circ F \cong T \circ (G \circ F)$.

(ii) For every $\mathcal{V}$-functor $T : \mathcal{X} \to \mathcal{R}$, $T \circ \text{Id}_\mathcal{X} \cong T$.

**Proof.** For part (i), it suffices to show that $((T \circ G) \circ F)^e \cong (T \circ (G \circ F))^e$. But, by Proposition 4.7 and Lemma 10.3, we have

$$(T \circ G)^e \cong T \circ G \circ F^e$$

$$\cong (T \circ G^e) \circ F^e$$

$$\cong T^e \circ (G^e \circ F^e)$$

$$\cong T^e \circ (G \circ F)^e.$$

The proof of part (ii) is analogous. □

**11. Symmetric sequences and analytic functors**

We begin by introducing the notion of a symmetric sequence between small $\mathcal{V}$-categories.

**Definition 11.1.** Let $\mathcal{X}, \mathcal{Y}$ be small $\mathcal{V}$-categories. A **categorical symmetric sequence** $F : \mathcal{X} \to \mathcal{Y}$ is an $S$-distributor from $\mathcal{Y}$ to $\mathcal{X}$, i.e. a $\mathcal{V}$-functor $F : S(\mathcal{X})^{op} \otimes \mathcal{Y} \to \mathcal{V}$.

If $F : \mathcal{X} \to \mathcal{Y}$ is a categorical symmetric sequence, then for a symmetric $\mathcal{V}$-rig $\mathcal{R} = (\mathcal{R}, \circ, e)$ we define its associated analytic functor $F : \mathcal{R}^\mathcal{X} \to \mathcal{R}^\mathcal{Y}$ by letting, for $T \in \mathcal{R}^\mathcal{X}$ and $y \in \mathcal{Y}$,

$$F(T)(y) = \int_{x \in S(\mathcal{X})} F(x; y) \otimes T^x,$$

where, for $x = (x_1, \ldots, x_n) \in S(\mathcal{X})$, $T^x = \int_{y \in \mathcal{Y}} T(x_1) \otimes \cdots \otimes T(x_n)$. We represent the correspondence between categorical symmetric sequences and analytic functors as follows:

$$F : \mathcal{X} \to \mathcal{Y}$$

$$\overline{F} : \mathcal{R}^\mathcal{X} \to \mathcal{R}^\mathcal{Y}.$$

**Example 11.2.** For $\mathcal{V} = \mathcal{R} = \text{Set}$ and $\mathcal{X} = \mathcal{Y} = 1$, where 1 is the terminal category, we obtain exactly the notion of an analytic functor introduced in [30]. Indeed, a symmetric sequence $F : 1 \to 1$ is the same thing as a functor $F : \text{Set} \to \text{Set}$. Here, $\text{Set} = S(1)$ is the category of natural numbers and permutations. The analytic functor $F : \text{Set} \to \text{Set}$ associated to such a symmetric sequence has the following form, already recalled in the introduction:

$$F(T) = \int_{n \in S} F[n] \otimes T^n.$$

See [3, 12, 35] for applications of the theory of analytic functors to combinatorics and [2] for recent work on categorical aspects of the theory.
Example 11.3. For $\mathcal{V} = \mathcal{R} = \text{Set}$, we obtain the notion of analytic functor between categories of covariant presheaves considered in [26]. In that context, the analytic functor $F: \text{Set}^X \to \text{Set}^Y$ of a symmetric sequence $F: \mathcal{X} \to \mathcal{Y}$, i.e. a functor $F: S(\mathcal{X})^\text{op} \otimes \mathcal{Y} \to \text{Set}$, is obtained as a left Kan extension, fitting in the diagram

$$
\begin{array}{ccc}
S(\mathcal{X})^\text{op} & \xrightarrow{\sigma_X} & \text{Set}^X \\
\downarrow{\lambda F} & & \downarrow{F} \\
\text{Set}^X & \xrightarrow{} & \text{Set}^Y,
\end{array}
$$

where $\sigma_X: S(\mathcal{X})^\text{op} \to \text{Set}^X$ is the functor defined by letting

$$\sigma_X(x_1, \ldots, x_n) = \bigcup_{1 \leq k \leq n} X[x_k, -].$$

Hence, for $T \in \text{Set}^X$ and $y \in \mathcal{Y}$, we have

$$F(T)(y) = \int_{\mathcal{X}} \lambda F(\overline{T})(y) \otimes [\sigma_X(\overline{T}), T]$$

$$= \bigcup_{n \in \mathbb{N}} \int_{\mathcal{X}} \bigcup_{\mathcal{X}} F[\overline{T}; y] \otimes \left[ \bigcup_{1 \leq k \leq n} X[x_k, -], T \right]$$

$$= \bigcup_{n \in \mathbb{N}} \int_{\mathcal{X}} \bigcup_{\mathcal{X}} F[\overline{T}; y] \otimes \prod_{1 \leq k \leq n} P(X)[X[x_k, -], T]$$

$$= \bigcup_{n \in \mathbb{N}} \int_{\mathcal{X}} \bigcup_{\mathcal{X}} F[\overline{T}; y] \otimes \prod_{1 \leq k \leq n} T(x_k)$$

$$= \int_{\mathcal{X}} F[\overline{T}; y] \otimes T^\overline{T},$$

where for $\overline{T} = (x_1, \ldots, x_n)$, we have $T^\overline{T} = T(x_1) \times \ldots \times T(x_n)$. Note how this construction of analytic functors as a left Kan extension does not carry over to the enriched setting.

Small $\mathcal{V}$-categories, categorical symmetric sequences and $\mathcal{V}$-natural transformations form a bicategory, which we call the bicategory of categorical symmetric sequences and denote as $\text{CatSym}_V$. This bicategory is defined as the opposite of the bicategory $\text{S-Dist}_V$ of $S$-distributors:

$$\text{CatSym}_V = \text{def} \ (\text{S-Dist}_V)^\text{op}.$$

In particular, for small $\mathcal{V}$-categories $\mathcal{X}$ and $\mathcal{Y}$, we have

$$\text{CatSym}_V[\mathcal{X}, \mathcal{Y}] = \text{S-Dist}_V[\mathcal{Y}, \mathcal{X}] = \text{Dist}_V[\mathcal{Y}, S(\mathcal{X})] = \text{CAT}_V[S(\mathcal{X})^\text{op} \otimes \mathcal{Y}, \mathcal{V}].$$

Our next theorem relates the composition of categorical symmetric sequences with the composition of analytic functors. In particular, it shows how the analytic functor associated to the composite of two categorical symmetric sequences is isomorphic to the composites of the analytic functors associated to the categorical symmetric sequences. We also show that the analytic functor associated to the identity categorical symmetric sequence is naturally isomorphic to the identity functor. This generalises Theorem 3.2 in [26] to the enriched setting.

Theorem 11.4. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be small $\mathcal{V}$-categories and $\mathcal{R}$ be a symmetric $\mathcal{V}$-rig.

---

1 The analytic functors studied in [26] were between categories of presheaves, but we prefer to consider covariant presheaves to match our earlier definitions.
(i) For every pair of categorical symmetric sequences $F: X \to Y$ and $G: Y \to Z$, there is a natural isomorphism with components
\[(G \circ F)(T) \cong G(F(T)).\]

(ii) There is a natural isomorphism with components
\[\text{Id}_X(T) \cong T.\]

Proof. For this proof, it is convenient to have some auxiliary notation: for a categorical symmetric sequence $F: X \to Y$, we write $F^\text{op}: Y \to X$ for the corresponding $S$-distributor. For part (i), let $T: X \to R$. We begin by showing that $F(T) = T \circ F^\text{op}$, where $T \circ F^\text{op}$ denotes the composite of the $S$-distributor $F^\text{op}: Y \to X$ with the $V$-functor $T: X \to R$, defined in (10.6). For $y \in X$, we have
\[F(T)(y) = \int_{x \in S(X)} F[x; y] \otimes T^x = \int_{x \in S(X)} F[x; y] \otimes T^x(y) = (T \circ F^\text{op})(y).\]

It follows by part (i) of Proposition 10.9 that we have
\[(G \circ F)(T) = T \circ (G \circ F)^\text{op}
\[= T \circ (F^\text{op} \circ G^\text{op})
\[\cong (T \circ F^\text{op}) \circ G^\text{op}
\[= F(T) \circ G^\text{op}
\[= G(F(T)).\]

For part (ii), if $T: X \to R$, then by part (ii) of Proposition 10.9 we have
\[\text{Id}_X(T) = T \circ \text{Id}_Y^\text{op}
\[\cong T.\]

We now wish to generalise to the enriched setting the main result of [26] and show that the bicategory of categorical symmetric sequences is cartesian closed. Existence of products in $\text{CatSym}_V$ follows easily by results in earlier sections. We state the result explicitly for emphasis.

Proposition 11.5. The bicategory $\text{CatSym}_V$ has finite products. In particular, the product of two small $V$-categories $X$ and $Y$ in $\text{CatSym}_V$ is given by their coproduct $X \sqcup Y$ in $\text{Cat}_V$.

Proof. This follows by Proposition 10.5 by duality, since the bicategory $S\text{-Dist}_V$ has finite coproducts and these are given by coproducts in $\text{Cat}_V$. \qed

The fact that coproducts in $\text{CatSym}_V$ are given by coproducts in $\text{Cat}_V$ can be seen intuitively by the following chain of equivalences:

\[
\text{CatSym}_V[Z, X] \times \text{CatSym}_V[Z, Y] = S\text{-Dist}_V[X, Z] \times S\text{-Dist}_V[Y, Z]
\[= \text{Dist}_V[X, S(Z)] \times \text{Dist}_V[Y, S(Z)]
\[\cong \text{Dist}_V[X \sqcup Y, S(Z)]
\[= S\text{-Dist}_V[X \sqcup Y, Z]
\[= \text{CatSym}_V[Z, X \sqcup Y].
\]
We now consider the definition of exponentials in $\text{CatSym}_\mathcal{V}$. For small $\mathcal{V}$-categories $X$ and $Y$, let us define

$$[X, Y] = \text{def} \ S(X)^{\text{op}} \otimes Y.$$  

Then for every a small $\mathcal{V}$-category $Z$, we have

$$\text{CatSym}_\mathcal{V}[Z, [X, Y]] = S\text{-Dist}_\mathcal{V}[S(X)^{\text{op}} \otimes Y, Z]$$

$$= \text{Dist}_\mathcal{V}[S(X)^{\text{op}} \otimes Y, S(Z)]$$

$$= \text{Cat}_\mathcal{V}[S(Z)^{\text{op}} \otimes S(X)^{\text{op}} \otimes Y, \mathcal{V}]$$

$$= \text{Dist}_\mathcal{V}[Y, S(Z) \otimes S(X)]$$

$$\simeq \text{Dist}_\mathcal{V}[Y, S(Z \sqcup X)]$$

$$= S\text{-Dist}_\mathcal{V}[Y, Z \sqcup X]$$

$$= \text{CatSym}_\mathcal{V}[Z \sqcap X, Y].$$

Let us consider the effect of this chain of equivalences on a categorical symmetric sequence $F: Z \to [X, Y]$, which is an is a distributor $F: Y \to S(Z) \otimes S(X)$. Let $c = c_{Z, X}: S(Z) \otimes S(X) \to S(Z \sqcup X)$ be the functor in (11.1), which in this case is given by $c(\Xi \oplus \Xi) = \text{def} \ \Xi \oplus \Xi$. The distributor $c_\bullet: S(Z) \otimes S(X) \to S(Z \sqcup X)$ is an equivalence, since the functor $c$ is an equivalence, as stated in Proposition 11.2. We then have

$$F: Z \to [X, Y] \text{ in } \text{CatSym}_\mathcal{V}$$

$$\xrightarrow{\text{ev}} Y \to S(Z) \otimes S(X) \text{ in } \text{Dist}_\mathcal{V}$$

$$c_\bullet \circ F: Y \to S(Z \sqcup X) \text{ in } \text{Dist}_\mathcal{V}$$

$$c_\bullet \circ F: Y \to Z \sqcup X \text{ in } S\text{-Dist}_\mathcal{V}$$

$$c_\bullet \circ F: Z \sqcap X \to Y \text{ in } \text{CatSym}_\mathcal{V}.$$  

By considering the particular case of $Z = [X, Y]$, we define the categorical symmetric sequence

$$\text{ev}: [X, Y] \sqcap X \to Y,$$

by letting $\text{ev} = \text{def} \ c_\bullet \circ \text{Id}$, where $\text{Id}: [X, Y] \to [X, Y]$ is the identity categorical symmetric sequence on $[X, Y]$. By definition, we have

$$\text{ev} = (c_\bullet \circ \text{Id}): Y \to (S(X)^{\text{op}} \otimes Y) \sqcup X$$

where $\text{Id}: S(X)^{\text{op}} \otimes Y \to S(X)^{\text{op}} \otimes Y$ is the identity $S$-distributor, as in (11.1), which in this case is given by the distributor $\text{Id}: S(X)^{\text{op}} \otimes Y \to S(X)^{\text{op}} \otimes Y$ defined by

$$\text{Id}(\Xi, \Xi^{\text{op}} \otimes y) = (\Xi, \Xi^{\text{op}} \otimes y) = \text{def} \ S(X)^{\text{op}} \otimes Y, \Xi \oplus y,$$

for $\Xi \in S(X)^{\text{op}} \otimes Y$, $\Xi \in S(X)$ and $y \in Y$. Hence,

$$\text{ev}[\Xi, y] = \int_{\Xi \in S(X)^{\text{op}} \otimes Y} \int_{\Xi \in S(X)} [\Xi, \Xi^{\text{op}} \otimes y] = \int_{\Xi \in S(X)} [\Xi, (\Xi^{\text{op}} \otimes y) \oplus \Xi], \text{ (11.1)}$$

for $\Xi \in S(X)^{\text{op}} \otimes Y \sqcup X$ and $y \in Y$. Our next theorem generalises to the enriched case the main result in [20].

**Theorem 11.6.** The bicategory $\text{CatSym}_\mathcal{V}$ is cartesian closed. More precisely, the analytic functor $\text{ev}: [X, Y] \sqcap X \to Y$ exhibits the exponential of $Y$ by $X$.

**Proof.** We have to show that the functor

$$\varepsilon: \text{CatSym}_\mathcal{V}[Z, [X, Y]] \to \text{CatSym}_\mathcal{V}[Z \sqcap X, Y]$$
defined by letting \( \varepsilon(F) = \text{def} \ ev \circ (F \cap X) \) is an equivalence of categories for every small \( \mathcal{V} \)-category \( Z \). By duality, this amounts to showing that the functor

\[
\varepsilon : S\text{-Dist}_V[S(X)^{\text{op}} \otimes Y, Z] \to S\text{-Dist}_V[Y, Z \sqcup X]
\]

defined by letting \( \varepsilon(F) = \text{def} \ (F \sqcup X) \circ ev \) is an equivalence of categories for every small \( \mathcal{V} \)-category \( Z \). Notice that \( \varepsilon \) has the form

\[
\varepsilon : \text{Dist}_V[Y, S(Z) \otimes S(X)] \to \text{Dist}_V[Y, S(Z \sqcup X)]
\]

since

\[
S\text{-Dist}_V[S(X)^{\text{op}} \otimes Y, Z] = \text{Cat}_V[S(Z)^{\text{op}} \otimes S(X)^{\text{op}} \otimes Y, V] = \text{Dist}_V[Y, S(Z) \otimes S(X)].
\]

If \( c : S(Z) \otimes S(X) \to S(Z \sqcup X) \) is the functor in (9.4), which in this case is defined by letting \( c(\overline{s} \otimes \overline{x}) = \text{def} \overline{s} \otimes \overline{x} \), let us show that we have \( \varepsilon(F) = c \circ F \)

\[
\begin{array}{ccc}
Y & \xrightarrow{F} & S(Z) \otimes S(X) \\
\downarrow{\varepsilon(F)} & & \downarrow c \\
S(Z \sqcup X)
\end{array}
\]

Observe that we have \( c \circ c^* \cong \text{Id}_{S(Z) \otimes S(X)} \), since the functor \( c \) is an equivalence of categories by Proposition 9.2. Hence it suffices to show that we have \( c^* \circ \varepsilon(F) = F \), as in the following diagram of distributors:

\[
\begin{array}{ccc}
Y & \xrightarrow{F} & S(Z) \otimes S(X) \\
\downarrow{\varepsilon(F)} & & \downarrow c^* \\
S(Z \sqcup X)
\end{array}
\]

In other words, it suffice to show that for \( \overline{s} \in S(Z) \), \( \overline{x} \in S(X) \) and \( y \in Y \) we have

\[
\varepsilon(F)[\overline{s} \oplus \overline{x}; y] = F[\overline{s} \oplus \overline{x}].
\]

By definition, \( \varepsilon(F) \) is a composite of \( S \)-distributors:

\[
\begin{array}{ccc}
Y & \xrightarrow{ev} & (S(X)^{\text{op}} \otimes Y) \sqcup X \\
\downarrow{\varepsilon(F)} & & \downarrow F \sqcup X \\
Z \sqcup X
\end{array}
\]

Thus,

\[
\varepsilon(F)[\overline{s} \oplus \overline{x}; y] = \int_{\overline{w} \in S(X)^{\text{op}} \otimes Y \sqcup X} (F \sqcup X)^{\text{op}}[\overline{s} \oplus \overline{x}, \overline{w}] \otimes ev(\overline{w}; y).
\]

It then follows from (11.1) that

\[
\varepsilon(F)[\overline{s} \oplus \overline{x}; y] = \int_{\overline{w} \in S(Y)} \int_{\overline{w} \in S(Y)} (F \sqcup Y)^{\text{op}}[\overline{s} \oplus \overline{x}, \overline{w}] \otimes [\overline{w}, (\overline{w}^{\text{op}} \otimes y) \oplus \overline{w}]
\]

\[
= \int_{\overline{w} \in S(Y)} (F \sqcup Y)^{\text{op}}[\overline{s} \oplus \overline{x}, (\overline{w}^{\text{op}} \otimes y) \oplus \overline{w}].
\]
But we have \((F \sqcup Y)[\xi \oplus \tau, (\tau^{\op} \otimes y) \oplus [v, \tau]] = F^e[\xi, \tau^{\op} \otimes y] \otimes [v, \tau] \) by Proposition \(10.6\). Thus,

\[
\varepsilon(F)(\xi \oplus \tau, y) = \int_{\tau \in S(Y)} F^e[\xi, \tau^{\op} \otimes y] \otimes [v, \tau] \\
= F^e[\xi, \tau^{\op} \otimes y] \\
= F^e[\xi \otimes \tau, y] \\
= F(\xi \otimes \tau, y). 
\]

This proves that \(\varepsilon(F) = c_\bullet \circ F\). Hence the functor mapping \(F\) to \(\varepsilon(F)\) is an equivalence, as required, since the distributor \(c_\bullet\) is an equivalence.

\(\square\)

**Remark 11.7.** We write \(\text{Sym}_V\) for the full sub-bicategory of \(\text{CatSym}_V\) spanned by sets, viewed as discrete categories. This bicategory can be defined simply as

\[\text{Sym}_V = \text{def} \ (S-\text{Mat}_V)^{\op},\]

where \(S-\text{Mat}_V\) is full sub-bicategory of \(S-\text{Dist}_V\) spanned by sets, as defined in Remark \(10.7\). We unfold some definitions since they will be useful in Section \(\section{14}\). The objects of \(\text{Sym}_V\) are sets and the hom-category between two sets \(X\) and \(Y\) is defined by

\[\text{Sym}_V[X, Y] = \text{def} \ S-\text{Mat}_V[Y, X] = \text{Cat}[S(X)^{\op} \times Y, V].\]

For sets \(X\) and \(Y\), we define a symmetric sequence \(F: X \to Y\) to be a functor \(F: S(X)^{\op} \times Y \to V\). For a symmetric \(V\)-rig \(R\), the extension \(F: R^X \to R^Y\) of such a symmetric sequence is given by

\[F(T)(y) = \text{def} \int_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \int_{(y_1, \ldots, y_m) \in S^m(Y)} F[x_1, \ldots, x_n; y] \otimes T(x_1) \otimes \ldots \otimes T(x_n). \quad (11.2)\]

It will be useful to have also an explicit description of the composition operation in \(\text{Sym}_V\). For sets \(X, Y, Z\) and symmetric sequences \(F: X \to Y, G: Y \to Z\), their composite \(G \circ F: X \to Z\) is given by

\[(G \circ F)(\tau, z) = \text{def} \bigcup_{m \in \mathbb{N}} \int_{(y_1, \ldots, y_m) \in S^m(Y)} G[y_1, \ldots, y_m; z] \otimes \\
\int_{\tau_1 \in S(X)} \ldots \int_{\tau_m \in S(X)} [\tau, \tau_1 \oplus \ldots \oplus \tau_m][F[\tau_1; y_1] \otimes \cdots \otimes F[\tau_m; y_m]]. \quad (11.3)\]

For a set \(X\), the identity symmetric sequence \(\text{Id}_X: X \to X\) is defined by letting

\[\text{Id}_X[\tau; x] = \begin{cases} 1 & \text{if } \tau = (x), \\ 0 & \text{otherwise.} \end{cases} \quad (11.4)\]

**Example 11.8.** Let \(S = S(1)\), the category of natural numbers and permutations. The monoidal structure on \(\text{Sym}_V[1, 1] \cong [S^{op}, V]\) given by the horizontal composition in \(\text{Sym}_V\) as defined in \((11.3)\) is exactly the substitution monoidal structure discussed in the introduction, which characterises the notion of a single-sorted operad, in the sense that monoids in \([S^{op}, V]\) with respect to this monoidal structure are exactly single-sorted operads \(\bullet\). Indeed, as a special case of the formula in \((11.3)\), we get

\[(G \circ F)[n] = \int_{m \in S} G[m] \otimes \int_{n_1 \in S} \cdots \int_{n_m \in S} [n, n_1 + \ldots + n_m] \otimes F[n_1] \otimes \cdots \otimes F[n_m]. \]
Similarly, the unit $J$ for the substitution tensor product is given by a special case of the formula in (11.4):

$$J[n] = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We will see in Section 12 that the horizontal composition of $\text{Sym}_V$ can be used to characterise the notion of an operad.

12. Monads, modules and bimodules

The construction of the bicategory of operads from of the bicategory of symmetric sequences, which we will discuss in Section 14, is an instance of a general construction, known as the bimodule construction, that is well-known in category theory. This section and the next one are devoted to reviewing this construction and presenting some auxiliary notions and results. For this section, let us consider a fixed bicategory $E$.

Definition 12.1. Let $X \in E$.

(i) A monad on $X$ is a triple $(A, \mu, \eta)$ consisting of a morphism $A : X \to X$, a 2-cell $\mu : A \circ A \to A$, called the multiplication of the monad, and a 2-cell $\eta : 1_X \to A$, called the unit of the monad, such that the following diagrams (expressing associativity and unit axioms) commute:

(ii) Let $A = (A, \mu_A, \eta_A)$ and $B = (B, \mu_B, \eta_B)$ be monads on $X$. A monad map from $A$ to $B$ is a 2-cell $\pi : A \to B$ such that the following diagrams commute:

For $X \in E$, we write $\text{Mon}(X)$ for the category of monads on $X$ and monad maps. Sometimes we will use $\mu$ and $\eta$ for the multiplication and the unit of different monads, whenever the context does not lead to confusion. Note that the notion of a monad is self-dual, in the sense that a monad in $E$ is the same thing as a monad in $E^{\text{op}}$. The category $\text{Mon}(X)$ can be defined equivalently as the category of monoids and monoid morphisms in the category $E[X,X]$, considered as a monoidal category with composition as tensor product and the identity morphism $1_X : X \to X$ as unit. Hence, a homomorphism $\Phi : E \to \mathcal{F}$ sends a monad $A : X \to X$ to a monad $\Phi(A) : \Phi(X) \to \Phi(X)$, since it induces a monoidal functor $\Phi_{X,X} : E[X,X] \to \mathcal{F}(\Phi(X), \Phi(X))$. Clearly, monads on a small $\mathcal{V}$-category in the 2-category $\text{Cat}_\mathcal{V}$ are $\mathcal{V}$-monads in the usual sense. We give some further examples below.

Example 12.2 (Monoids as monads). For a monoidal category $C$, monads in the bicategory $\Sigma(C)$ are monoids in $C$ [8, Section 5.4.1]. In particular, monads in $\Sigma(\text{Ab})$ are rings [45, Section VII.3].

Example 12.3 (Categories as monads). We recall from [8] that a monad on a set $X$ in the bicategory of matrices $\text{Mat}_\mathcal{V}$ of Section 3 is the same thing as a $\mathcal{V}$-category with $X$ as its set of objects. Indeed, if $A : X \to X$ is a monad, we can define a $\mathcal{V}$-category $\mathcal{X}$ with $\text{Obj}(\mathcal{X}) = X$ by
letting $X[x, y] = \def\ A[x, y]$, for $x, y \in X$, since the matrix $A: X \to X$ is a function $A: X \times X \to V$.

The composition operation and the identity morphisms of $X$ are given by the multiplication and unit of the monad, since they have components of the form

$$\mu_{x, z}: \bigsqcup_{y \in X} X[y, z] \times X[x, y] \to X[x, z], \quad \eta_x: I \to X[x, x].$$

The associativity and unit axioms for a monad, as stated in Definition 12.4, then reduce to the associativity and unit axioms for the composition operation in a $V$-category.

**Example 12.4** (Operads as monads). A monad on a set $X$ in $\text{Sym}_V$ is the same thing as an operad (by which we mean a symmetric many-sorted $V$-operad, which is the same thing as a symmetric $V$-multicategory), with $X$ as its set of sorts (or set of objects). This fact is an immediate generalisation of a result by Kelly [40] recalled in Remark 11.8 and therefore we limit ourselves to outline the proof. Let $A: X \to X$ be a monad in $\text{Sym}_V$, i.e. a symmetric sequence $A: X \to X$, given by a functor $A: S(X)^{op} \times X \to V$, equipped with a multiplication and a unit. We define an operad with set of objects $X$ as follows. First of all, for $x_1, \ldots, x_n, x \in X$, the object of operations with inputs of sorts $x_1, \ldots, x_n$ and output of sort $x$ to be $A[x_1, \ldots, x_n; x]$. The symmetric group actions required to have an operad follow from the functoriality of $A$, since we have a morphism

$$\sigma^*: X[x_{\sigma(1)}, \ldots, x_{\sigma(n)}; x] \to X[x_1, \ldots, x_n; x]$$

for each each permutation $\sigma \in \Sigma_n$ and $(x_1, \ldots, x_n, x) \in S_n(X)^{op} \times X$. By the definition of the composition operation in $\text{Sym}_V$, as given in (11.3), the multiplication $\mu: A \circ A \to A$ amounts to having a family of morphisms

$$\theta_{\overline{m}, \ldots, \overline{m}, x}: A[\overline{x}; x] \otimes A[\overline{x}_1; x_1] \otimes \ldots \otimes A[\overline{x}_m; x_m] \to A[\overline{x}_1 \oplus \ldots \oplus \overline{x}_m; x],$$

where $\overline{x} = (x_1, \ldots, x_m)$, which is natural in $\overline{x}, \overline{x}_1, \ldots, \overline{x}_m \in S(X)^{op}$ and satisfies the equivariance condition expressed by the commutativity of the following diagram:
where $\langle \sigma \rangle: \mathfrak{F}_{\sigma(1)} \oplus \ldots \oplus \mathfrak{F}_{\sigma(m)} \to \mathfrak{F}_1 \oplus \ldots \oplus \mathfrak{F}_m$ is the evident morphism in $S(X)^{\text{op}}$ induced by $\sigma$. It is common to represent maps $f: I \to A[x_1, \ldots, x_n; x]$ as corollas of the form

With this notation, composition operation may be represented diagrammatically as a grafting operation. For example, the composite represented by the following grafting diagram

is represented by

where $g \circ (f_1, f_2, f_3) = \theta(g, f_1, f_2, f_3)$. By the definition of the identity symmetric sequence in (11.4), the unit $\eta: \text{Id}_X \to A$ then amounts to having a morphism $1_x: I \to A[(x); x]$ for each $x \in X$. These give the identity operations of the operad. The associativity and unit axioms for a monad then correspond to the associativity and unit axioms for operads.

An operad $(X, A)$ determines, for every symmetric $\mathcal{V}$-rig $\mathcal{R}$, a monad $A: \mathcal{R}^X \to \mathcal{R}^X$, whose underlying functor is the analytic functor associated to the symmetric sequence $A: X \to X$, which is given by the formula

$A(T)(x) = \text{def} \int_{\mathfrak{F} \in S(X)} A[\mathfrak{F}, x] \otimes T^{\mathfrak{F}}$.

We write $\text{Alg}_{\mathcal{R}}(A)$ for the category of algebras and algebra morphisms for this monad, which is related to the category $\mathcal{V}^X$ by a monadic adjunction

$\text{Alg}_{\mathcal{R}}(A) \xrightarrow{F} \mathcal{R}^X$.

**Definition 12.5.** Let $A: \mathcal{K} \to \mathcal{K}$ be a monad in $\mathcal{E}$. Let $\mathcal{K} \in \mathcal{E}$. 
(i) A left $A$-module with domain $\mathbb{K}$ is a morphism $M : \mathbb{K} \to X$ equipped with a left $A$-action, i.e. a 2-cell $\lambda : A \circ M \to M$ such that the following diagrams commute:

$$
\begin{array}{ccc}
A \circ A \circ M & \xrightarrow{A \circ \lambda} & A \circ M \\
\mu \circ M & \downarrow & \downarrow \lambda \\
A \circ M & \xrightarrow{\lambda} & M,
\end{array}
\quad
\begin{array}{ccc}
M & \xrightarrow{\eta \circ M} & A \circ M \\
1_M & \downarrow & \downarrow \\
M & \xrightarrow{\lambda} & M.
\end{array}
$$

(ii) If $M$ and $M'$ are left $A$-modules with domain $\mathbb{K}$, then a left $A$-module map from $M$ to $M'$ is a 2-cell $f : M \to M'$ such that the following diagram commutes:

$$
\begin{array}{ccc}
A \circ M & \xrightarrow{A \circ f} & A \circ M' \\
\lambda & \downarrow & \downarrow \lambda' \\
M & \xrightarrow{f} & M'.
\end{array}
$$

We write $\mathcal{E}[\mathbb{K},X]^A$ for the category of left $A$-modules with domain $\mathbb{K}$ and left $A$-module maps.

**Example 12.6** (Left modules in a monoidal category). For a monoidal category $C = (C, \otimes, I)$, a left module over a monoid $A$, viewed as a monad in $\Sigma(C)$, is the same thing as an object $M \in C$ equipped with a left $A$-action, i.e. a morphism $\lambda : A \otimes M \to M$ satisfying associativity and unit axioms. In particular, we obtain the familiar notion of left modules over a ring when $C = \text{Ab}$.

**Example 12.7** (Left modules for categories). For a small $\mathcal{V}$-category $X$, viewed as a monad on $X = \text{def} \text{Obj}(\mathcal{X})$ in $\text{Mat}_\mathcal{V}$, left $\mathcal{X}$-modules are families of presheaves on $\mathcal{X}$, i.e. contravariant $\mathcal{V}$-functors from $\mathcal{X}^\text{op}$ to $\mathcal{V}$. Indeed, a left $\mathcal{X}$-module $M$ with domain $K$ is a matrix $M : K \to \text{Obj}(\mathcal{X})$, i.e. a functor $M : \text{Obj}(\mathcal{X}) \times K \to \mathcal{V}$, equipped with a natural transformation with components

$$
\lambda_{x,k} : \bigsqcup_{x' \in \text{Obj}(\mathcal{X})} \mathcal{X}[x,x'] \otimes M[x',k] \to M[x,k],
$$

satisfying associativity and unit axioms. It is immediate to see that this is the same thing as a family of $\mathcal{V}$-functors $M_k : \mathcal{X}^\text{op} \to \mathcal{V}$, for $k \in K$.

**Example 12.8** (Left modules for operads). Let us consider an operad, given as a monad $(X,A)$ in $\text{Sym}_\mathcal{V}$. A left $A$-module with domain $K$ consists of an symmetric sequence $M : K \to X$, i.e. a functor $M : S(K)^\text{op} \times X \to \mathcal{V}$ equipped with a left $A$-action $\lambda : A \circ M \to M$. By the definition of the composition operation in $\text{Sym}_\mathcal{V}$, as given in (11.3), such a left action amounts to having maps in $\mathcal{V}$ of the form

$$
M[\mathcal{T}_1; x_1] \otimes \ldots \otimes M[\mathcal{T}_m; x_m] \otimes A[x_1, \ldots, x_m; x] \to M[\mathcal{T}_1 \oplus \ldots \oplus \mathcal{T}_m; x],
$$

which satisfy associativity and unit axioms and an equivariance condition. When $K = \emptyset$, we have $S(\emptyset) \cong 1$ and therefore a left $A$-module with domain $\emptyset$ is a family of objects $M(x) \in \mathcal{V}$, for $x \in X$, equipped with maps in $\mathcal{V}$ of the form

$$
M(x_1) \otimes \ldots \otimes M(x_m) \otimes A[x_1, \ldots, x_m; x] \to M(x),
$$

satisfying the associativity and unit axioms and an equivariance condition. Such left modules and their left module maps are exactly algebras and algebra morphisms for $A$ in $\mathcal{V}$, in the sense of Example 11.4 and so we have $\text{Sym}_\mathcal{V}[\emptyset, X]^A = \text{Alg}_\mathcal{V}(A)$. Diagrammatically, if we represent a
map \( m: I \to M(x) \) as

then the left \( A \)-action can be seen, for example as acting as follows:

\[
\begin{array}{c}
m_1 \\
\downarrow m_2 \\
m_3
\end{array}
\]

\[
f: x \\
\downarrow f \\
n(x)
\]

\[
\begin{array}{c}
m_1 \\
\downarrow m_2 \\
m_3 \\
\end{array} \quad \mapsto \quad f \cdot (m_1, m_2, m_3)
\]

**Remark 12.9.** Let \( K \in \mathcal{E} \). The homomorphism \( \mathcal{E}[K, -]: \mathcal{E} \to \text{Cat} \) sends a monad \( A: \mathcal{X} \to \mathcal{X} \) to a monad \( \mathcal{E}[K, A]: \mathcal{E}[K, \mathcal{X}] \to \mathcal{E}[K, \mathcal{X}] \). Then, the category of left \( A \)-modules with domain \( K \) and left \( A \)-module maps is exactly the category of algebras and algebra morphisms for the monad \( \mathcal{E}[K, A] \). Therefore, we have an adjunction

\[
\mathcal{E}[K, \mathcal{X}]^A \xrightarrow{\text{forgetful}} \mathcal{E}[K, \mathcal{X}],
\]

where the right adjoint is the forgetful functor and the left adjoint takes a morphism \( M: K \to \mathcal{X} \) to the free left \( A \)-module on it, \( A \circ M: K \to \mathcal{X} \).

Right modules and right module maps are defined in a dual way to left modules and left modules map. We state the explicit definition below.

**Definition 12.10.** Let \( A: \mathcal{X} \to \mathcal{X} \) be a monad in \( \mathcal{E} \). Let \( K \in \mathcal{E} \).

(i) A **right \( A \)-module with codomain** \( K \) is a morphism \( M: \mathcal{X} \to K \) equipped with a right \( A \)-action \( \rho: M \circ A \to M \) such that the following diagrams commute:

\[
\begin{array}{ccc}
M \circ A \circ A & \xrightarrow{\rho \circ A} & M \circ A \\
\downarrow M_{\circ A} & & \downarrow 1_M \\
M \circ A & \xrightarrow{\rho} & M.
\end{array}
\]

(ii) If \( M \) and \( M' \) are two right \( A \)-modules with codomain \( K \), then a **right \( A \)-module map** from \( M \) to \( M' \) a 2-cell \( f: M \to M' \) such that the following diagram commutes:

\[
\begin{array}{ccc}
M \circ A & \xrightarrow{f \circ A} & M' \circ A \\
\downarrow \rho & & \downarrow \rho' \\
M & \xrightarrow{f} & M'.
\end{array}
\]

We write \( \mathcal{E}[\mathcal{X}, K]_A \) for the category of a right \( A \)-module with codomain \( K \) and right \( A \)-module maps.

**Example 12.11** (Right modules in a monoidal category). For a monoidal category \( \mathcal{C} = (\mathcal{C}, \otimes, I) \), a right module over a monoid \( A \), viewed as a monad in \( \Sigma(\mathcal{C}) \), is the same thing as an object \( M \in \mathcal{C} \) equipped with a right \( A \)-action \( \rho: M \otimes A \to M \). In particular, right modules in \( \text{Ab} \) are the same thing as right modules over a ring in the usual sense.
Example 12.12 (Right modules for categories). For a small $\mathcal{V}$-category $X$, viewed as a monad $(X, A)$ in $\text{Mat}_\mathcal{V}$, right $A$-modules are families of covariant $\mathcal{V}$-functors from $X$ to $\mathcal{V}$. Indeed, a right $A$-module with codomain $K$ is a matrix $M: X \to K$, i.e. a functor $M: K \times X \to \mathcal{V}$, equipped with a natural transformation with components

$$\rho_{k,x}: \bigoplus_{x' \in X} M[k, x'] \otimes A[x', x] \to M[k, x],$$

satisfying associativity and unit axioms. It is immediate to see that this is the same thing as a family of $\mathcal{V}$-functors $M_k: X \to \mathcal{V}$, for $k \in K$.

Example 12.13 (Right modules for operads). For a small $\mathcal{V}$-operad, viewed as a monad $(X, A)$ in $\text{Sym}_\mathcal{V}$, a right $A$-module with codomain $K$ consists of a symmetric sequence $M: X \to K$, i.e. a functor $M: \text{Sym}(X)^{\text{op}} \times K \to \mathcal{V}$ equipped with a right $X$-action $\rho: M \circ A \to M$. By the definition of composition in $\text{Sym}_\mathcal{V}$, as given in (11.3), such an action amounts to having maps in $\mathcal{V}$ of the form

$$A[\overline{x}_1; x_1] \otimes \ldots \otimes A[\overline{x}_m; x_m] \otimes M[x_1, \ldots, x_m; k] \to M[\overline{x}_1 \oplus \ldots \oplus \overline{x}_m; k]$$

which satisfy associativity and unit axioms, as well as an equivariance condition. Diagrammatically, using notation analogous to the one adopted above, we have, for example, that

![Diagram](https://example.com/diagram.png)

is mapped to

![Diagram](https://example.com/diagram.png)

When $X = 1$, these maps have the form

$$A[n_1] \otimes \ldots \otimes A[n_m] \otimes M_k(m) \to M_k(n_1 + \ldots + n_m),$$

where we write $M_k(n)$ for $M[n; k]$. These are $K$-indexed families of right $A$-modules for the operad, as usually defined in the literature (see, for example, [27, 37]).

Remark 12.14. Let $\mathcal{K} \in \mathcal{E}$. The homomorphism $\mathcal{E}[\mathcal{K}, -]: \mathcal{E} \to \text{Cat}$ sends a monad $A: X \to X$ to a monad $\mathcal{E}[\mathcal{K}, A]: \mathcal{E}[X, \mathcal{K}] \to \mathcal{E}[X, \mathcal{K}]$. Right $A$-modules with domain $\mathcal{K}$ and right $A$-module maps are the algebras and the algebra morphisms for the monad $\mathcal{E}[\mathcal{K}, A]$. Hence, we have an adjunction

$$\mathcal{E}[\mathcal{K}, \mathcal{X}]_A \quad \dashv \quad \mathcal{E}[\mathcal{K}, \mathcal{X}],$$
where the right adjoint is the forgetful functor and the left adjoint takes a morphism \( M : \mathcal{X} \to \mathcal{K} \) to the right \( A \)-module \( M \circ A : \mathcal{K} \to \mathcal{X} \).

Next, we define the notions of a bimodule and bimodule map, which will play a fundamental role throughout the rest of the paper. In particular, in Section 13 we will recall how (under appropriate assumptions on \( \mathcal{E} \)) monads, bimodules and bimodule maps form a bicategory.

**Definition 12.15.** Let \( A : \mathcal{X} \to \mathcal{K} \) and \( B : \mathcal{X} \to \mathcal{K} \) be monads in \( \mathcal{E} \).

(i) A \((B, A)\)-bimodule is a morphism \( M : \mathcal{X} \to \mathcal{Y} \) equipped with a left \( B \)-action \( \lambda : B \circ M \to M \) and a right \( A \)-action \( \rho : M \circ A \to M \) which commute with each other, in the sense that the following diagram commutes:

\[
\begin{array}{ccc}
B \circ M \circ A & \xrightarrow{\lambda \circ A} & M \circ A \\
\downarrow{B \circ \rho} & & \downarrow{\rho} \\
B \circ M & \xrightarrow{\lambda} & M
\end{array}
\]  

(12.1)

(ii) If \( M, M' : \mathcal{X} \to \mathcal{Y}' \) are \((B, A)\)-bimodules, then a bimodule map from \( M \) to \( M' \) is a 2-cell \( f : M \to M' \) that is a map of left \( B \)-modules and of right \( A \)-modules.

We write \( \mathcal{E}[\mathcal{X}, \mathcal{Y}]^B_A \) for the category of \((B, A)\)-bimodules and bimodule maps.

**Example 12.16** (Bimodules in a monoidal category). For a monoidal category \( \mathcal{C} = (\mathcal{C}, \otimes, I) \), bimodules in a \( \Sigma(\mathcal{C}) \) are the same thing as objects of \( \mathcal{C} \) equipped with a right action and a left action by a monoid which distribute over each other. In particular, bimodules in \( \Sigma(Ab) \) in the sense of the previous definition are the same thing as bimodules over a ring in the standard algebraic sense.

**Example 12.17** (Bimodules for categories). As well-known, bimodules in the bicategory \( \mathcal{Mat} \) are exactly distributors, in the sense of Definition 4.1. Indeed, for a small \( \mathcal{V} \)-category \( \mathcal{X} \) with set of objects \( X \) and a small \( \mathcal{V} \)-category \( \mathcal{Y} \) with set of objects \( Y \), a \((\mathcal{Y}, \mathcal{X})\)-bimodule is a function \( M : \mathcal{Y} \times \mathcal{X} \to \mathcal{V} \) equipped with natural transformations with components

\[
\rho_{y,x} : \bigoplus_{x' \in X} M[y, x'] \otimes \mathcal{X}[x', x] \to M[y, x], \quad \lambda_{x,y} : \bigoplus_{y' \in X} \mathcal{V}[y, y'] \otimes M[y', y] \to M[y, x],
\]

satisfying associativity and unit axioms. It is immediate to see that this is the same thing as a functor \( M : \mathcal{Y}^{op} \otimes \mathcal{X} \to \mathcal{V} \), i.e. a distributor \( M : \mathcal{X} \to \mathcal{Y} \).

**Example 12.18** (Operad bimodules). Bimodules in \( \text{Sym}_n \), are operad bimodules \([37]\), which we define explicitly below. Let us consider two operads \( \mathcal{X} = (X, A) \) and \( \mathcal{Y} = (Y, B) \). Then, an \((B, A)\)-bimodule consists of a symmetric sequence \( M : \mathcal{X} \to \mathcal{Y} \), i.e. a functor \( M : S(\mathcal{X})^{op} \times \mathcal{Y} \to \mathcal{V} \), equipped with a right \( A \)-action \( \rho : M \circ A \to M \) and a left \( B \)-action \( \lambda : B \circ M \to M \) satisfying the compatibility condition in (12.1). Explicitly, the right \( A \)-action amounts to having maps of the form

\[
A[\tau_1; x_1] \otimes \ldots \otimes A[\tau_m; x_m] \otimes M[x_1, \ldots, x_m; y] \to M[\tau_1 \oplus \ldots \oplus \tau_m; y],
\]

while the left \( B \)-action amounts to having maps of the form

\[
M[\tau_1; y_1] \otimes \ldots \otimes M[\tau_m; y_m] \otimes B[y_1, \ldots, y_m; y] \to M[\tau_1 \oplus \ldots \oplus \tau_m; y],
\]

all satisfying associativity, unit, compatibility and equivariance conditions. Bimodules for non-symmetric operads were defined in \([18]\) Definition 2.36.

**Proposition 12.19.** The forgetful functor \( U : \mathcal{E}[\mathcal{X}, \mathcal{Y}]^B_A \to \mathcal{E}[\mathcal{X}, \mathcal{Y}] \) is monadic.
Proof. Observe that the endofunctor $E[A, B] : E[X, Y] \to E[X, Y]$ has the structure of a monad with multiplication $\mu = E[\mu_A, \mu_B]$ and unit $\eta = E[\eta_A, \eta_B]$. A $(B, A)$-bimodule is the same thing as an $E[A, B]$-algebra, which is a morphism $M : X \to Y$, equipped with a 2-cell $\alpha : B \circ M \circ A \to M$ such that the following diagrams commute:

\[
\begin{array}{c}
B \circ B \circ M \circ A \circ A \xymatrix{ \ar[r]^-{B \circ \alpha} & B \circ M \circ A} \ar[rd]^-{\mu_B \circ M \circ \mu_A} \\
B \circ M \circ A \ar[ru]^-{\alpha} \ar[r] & M, \quad M \xymatrix{ \ar[r]^-{\eta_B \circ M \circ \eta_A} & B \circ M \circ A} \ar[ru]^-{\alpha}
\end{array}
\]

From the 2-cell $\alpha$ we obtain two actions $\lambda = \alpha \cdot (B \circ M \circ \eta_A)$ and $\rho = \alpha \cdot (\eta_B \circ M \circ A)$ which commute with each other. Conversely, from a commuting pair of actions $(\lambda, \rho)$ we obtain a 2-cell $\alpha$ which makes the required diagrams commute by letting $\alpha = \text{def} \cdot (\lambda \circ A) = \lambda \cdot (B \circ \rho)$, i.e. the common value of the composites in $[\lambda \cdot \eta_B]$.

Remark 12.20. The category $E[X, Y]^B_A$ is related to the categories $E[X, Y]^B$ and $E[X, Y]^A$ by the following commutative squares of monadic forgetful functors (all written $U$) and left adjoints (all written $F$):

\[
\begin{align*}
E[X, Y]^B_A \xrightarrow{U} E[X, Y]_A & \quad E[X, Y]^B_A \xrightarrow{F} E[X, Y]^B \\
E[X, Y]_A & \xrightarrow{U} E[X, Y] \quad E[X, Y]_A \xleftarrow{F} E[X, Y], \\
E[X, Y]^B_A & \xleftarrow{F} E[X, Y]^B_A \quad E[X, Y]^B_A \xrightarrow{U} E[X, Y]^B \\
E[X, Y]_A & \xleftarrow{F} E[X, Y] \quad E[X, Y]_A \xrightarrow{U} E[X, Y].
\end{align*}
\]

13. Bicategories of bimodules

We review how monads, bimodules and bimodule maps in a bicategory $E$ satisfying appropriate assumptions form a bicategory. This construction is well-known in category theory, see [13] [17] [28] [42] [63] for further information.

Definition 13.1.

(i) We say that $E$ is regular if for every $X, Y \in E$ the category $E[X, Y]$ has reflexive coequalizers and the horizontal composition functor of $E$ preserves coequalizers in each variable.

(ii) If $E$ and $F$ are regular bicategories, we say that a homomorphism $\Phi : E \to F$ is regular if for every $X, Y \in E$, the functor $\Phi_{X,Y} : E[X, Y] \to F[\Phi X, \Phi Y]$ preserves reflexive coequalizers.

If $E$ and $F$ are regular bicategories, we write $\text{REG}(E, F)$ for the full sub-bicategory of $\text{HOM}(E, F)$ whose objects are regular homomorphisms from $E$ to $F$.

From now until the end of the section, let $E$ be a fixed regular bicategory. Let $A : X \to X$, $B : Y \to Y$ be monads in $E$. For a left $A$-module $M : X \to X$ and a $(B, A)$-bimodule $F : X \to Y$, we define a left $B$-module $F \circ A M : X \to Y$ as follows. Its underlying morphism is defined by following reflexive coequalizer diagram:

\[
F \circ A \circ M \xrightarrow{\rho \circ M} F \circ M \xrightarrow{\eta} F \circ A M.
\]

(13.1)
The left $B$-action is determined by the universal property of coequalizers, as follows:

\[
\begin{array}{c}
B \circ F \circ A \circ M & \xrightarrow{B \circ \rho \circ M} & B \circ F \circ M & \xrightarrow{B \circ q} & B \circ F \circ A M \\
\lambda \circ A \circ M & \xrightarrow{B \circ \rho \circ \lambda} & B \circ F \circ \lambda & \xrightarrow{B \circ M} & B \circ F \\
F \circ A \circ M & \xrightarrow{\rho \circ M} & F \circ M & \xrightarrow{q} & F \circ A M \\
\lambda \circ M & \xrightarrow{\rho \circ A} & F \circ A M & \xrightarrow{\rho} & \\
\end{array}
\]

Here, the top row is a coequalizer diagram by the assumption that $E$ is regular, being obtained from the diagram in (13.1) by composition with $B$. The verification of the axioms for a left $A$-action uses the fact that the actions of $A$ and $B$ commute with each other and it is essentially straightforward. Furthermore, this definition can easily be shown to extend to a functor

\[- \circ_A (-) : E[[X, Y]^A \times E[[Y, Z]]^A \rightarrow E[[X, Z]^B.\]

Dually, for a $(B, A)$-bimodule $F : X \rightarrow Y$ and a right $B$-module $M : Y \rightarrow Z$, we define a right $A$-module $M \circ_B F : X \rightarrow Z$ as follows. Its underlying morphism is defined by following reflexive coequalizer diagram:

\[
M \circ B \circ F \xrightarrow{\rho \circ F} M \circ F \xrightarrow{q} M \circ_B F . \tag{13.2}
\]

The left $B$-action is determined by the universal property of coequalizers, as follows:

\[
\begin{array}{c}
M \circ B \circ F \circ A & \xrightarrow{M \circ \lambda \circ A} & M \circ F \circ A & \xrightarrow{q \circ A} & M \circ_B F \circ A \\
M \circ B \circ \rho & \xrightarrow{M \circ \rho \circ \lambda} & M \circ \rho & \xrightarrow{q} & M \circ_B F . \\
\end{array}
\]

In this way, we obtain a functor

\[- \circ_B (-) : E[[X, Y]^B \times E[[Y, Z]]^B \rightarrow E[[X, Z]^A.\]

Let us now assume to have monads $A : X \rightarrow A, B : Y \rightarrow Y$ and $C : Z \rightarrow Z$. A $(B, A)$-bimodule $F : X \rightarrow Y$ and a $(C, B)$-bimodule $G : Y \rightarrow Z$. If we consider $F$ as a left $B$-module and $G$ as a $(C, B)$-bimodule, the morphism given by the formula in (13.1) coincides with the morphism given by the formula in (13.2), applied considering $F$ as a $(B, A)$-bimodule and $G$ as a right $B$-module. Hence, this morphism $G \circ_B F : X \rightarrow Z$ is equipped with both a left $C$-action and a right $A$-action, which can be easily seen to commute with each other, thus giving us a $(C, A)$-bimodule $G \circ_B F : X \rightarrow Z$. Explicitly, the morphism $G \circ_B F$ is defined by the reflexive coequalizer

\[
G \circ B \circ F \xrightarrow{\rho \circ F} G \circ F \xrightarrow{q} G \circ_B F .
\]

Its right $A$-action is determined by the diagram

\[
\begin{array}{c}
G \circ B \circ F \circ A & \xrightarrow{\rho \circ F \circ \lambda} & G \circ F \circ A & \xrightarrow{q \circ A} & (G \circ_B F) \circ A \\
G \circ B \circ \rho & \xrightarrow{G \circ \rho \circ \lambda} & G \circ \rho & \xrightarrow{q} & G \circ_B F .
\end{array}
\]
while its left $C$-action is given by the diagram

\[
\begin{array}{ccc}
C \circ G \circ B \circ F & \xrightarrow{C \circ G \circ \lambda} & C \circ G \circ F \\
\downarrow \lambda \circ B \circ F & \quad & \downarrow \lambda \\
G \circ B \circ F & \xrightarrow{G \circ \lambda} & G \circ F
\end{array}
\]

\[
\begin{array}{ccc}
C \circ G \circ B \circ F & \xrightarrow{C \circ G \circ \lambda} & C \circ G \circ F \\
\downarrow \lambda \circ B \circ F & \quad & \downarrow \lambda \\
G \circ B \circ F & \xrightarrow{G \circ \lambda} & G \circ F
\end{array}
\]

In this way, we obtain a functor

\[
(-) \circ_B (-): \mathcal{E}[\mathcal{Y}, \mathcal{Z}]^B \times \mathcal{E}[\mathcal{X}, \mathcal{Y}]_A^B \to \mathcal{E}[\mathcal{X}, \mathcal{Z}]_A^C,
\]

which we sometimes call relative composition. This operation generalizes the circle over construction defined by Rezk [57, Section 2.3.10], which is called the relative composition product in [27, Section 5.1.5]. We can now define the bicategory $\text{Bim}(\mathcal{E})$. The objects of $\text{Bim}(\mathcal{E})$ are pairs of the form $(\mathcal{X}, A)$, where $\mathcal{X} \in \mathcal{E}$ and $A: \mathcal{X} \to \mathcal{X}$ is a monad. In order to simplify the notation, if we consider a pair $(\mathcal{X}, A)$ as an object of $\text{Bim}(\mathcal{E})$ we denote it as $\mathcal{X}/A$ and sometimes refer to it simply as a monad. For a pair of monads $\mathcal{X}/A$ and $\mathcal{Y}/B$, we then define

\[
\text{Bim}(\mathcal{E})[\mathcal{X}/A, \mathcal{Y}/B] = \text{def} \mathcal{E}[\mathcal{X}, \mathcal{Y}]_A^B.
\]

Hence, a morphism in $\text{Bim}(\mathcal{E})$ from $\mathcal{X}/A$ to $\mathcal{Y}/B$ is a $(B, A)$-bimodule $M: \mathcal{X}/A \to \mathcal{Y}/B$ and a 2-cell $f: M \to N$ in $\text{Bim}(\mathcal{E})$ is a bimodule map. The composition operation of $\text{Bim}(\mathcal{E})$ is then given by the relative composition of bimodules in (13.4). For an object $\mathcal{X}/A \in \text{Bim}(\mathcal{E})$, the identity bimodule $1_{\mathcal{X}/A}: \mathcal{X}/A \to \mathcal{X}/A$ is given by the morphism $A: \mathcal{X} \to \mathcal{X}$, viewed as an $(A, A)$-bimodule by taking the monad multiplication $\mu: A \circ A \to A$ as both the left and the right $A$-action. In order to complete the definition of the data of the bicategory $\text{Bim}(\mathcal{E})$, it remains to exhibit the associativity and unit isomorphisms. For the associativity isomorphisms, let us define the joint composition of three bimodules

\[
(V, D) \xrightarrow{L} (\mathcal{X}, A) \xrightarrow{M} (\mathcal{Y}, B) \xrightarrow{N} (\mathcal{Z}, C)
\]

as the colimit $N \circ_B M \circ_A L$ of a double (reflexive) graph

\[
\begin{array}{ccc}
N \circ B \circ M \circ A \circ L & \xrightarrow{N \circ B \circ M \circ L} & N \circ B \circ M \circ L \\
\downarrow & \quad & \downarrow \\
N \circ M \circ A \circ L & \xrightarrow{N \circ M \circ L} & N \circ M \circ L
\end{array}
\]

If the colimit is calculated horizontally and then vertically, we obtain $N \circ_B (M \circ_A L)$. If the colimit is calculated vertically and then horizontally, instead, we obtain $(N \circ B M) \circ_A L$. Thus, we have the required isomorphism $a_{L, M, N}: N \circ_B (M \circ_A L) \to (N \circ_B M) \circ_A L$. This isomorphism is the unique 2-cell $a$ fitting in the commutative diagram of canonical maps,

\[
\begin{array}{ccc}
N \circ (M \circ L) & \xrightarrow{(N \circ M) \circ L} & (N \circ M) \circ L \\
\downarrow & \quad & \downarrow \\
N \circ_B (M \circ_A L) & \xrightarrow{a} & (N \circ_B M) \circ_A L
\end{array}
\]
For the unit isomorphisms, observe that for $X/A, Y/B \in \text{Bim}(\mathcal{E})$ and $M: X/A \to Y/B$, we have an isomorphism $\ell_M: B \circ_B M \to M$ which fits in the diagram

\[
\begin{array}{ccc}
B \circ_B B \circ M & \xrightarrow{M \circ \mu} & B \circ M \\
\mu \circ A & \downarrow & \downarrow \\
M & \xrightarrow{\ell_M} & M
\end{array}
\]

since

\[
\begin{array}{ccc}
B \circ_B B \circ M & \xrightarrow{\eta \circ B \circ M} & B \circ M \\
\mu \circ M & \downarrow & \downarrow \\
M & \xrightarrow{\lambda} & M
\end{array}
\]

is a split fork. Dually, we have also an isomorphism $r_M: M \circ_A A \to M$ making the following diagram commute

\[
\begin{array}{ccc}
M \circ A \circ A & \xrightarrow{M \circ \mu} & M \circ A \\
\rho \circ A & \downarrow & \downarrow \\
M & \xrightarrow{r_M} & M
\end{array}
\]

since

\[
\begin{array}{ccc}
M \circ A \circ A & \xrightarrow{M \circ \lambda \circ \eta} & M \circ A \\
\rho \circ A & \downarrow & \downarrow \\
M & \xrightarrow{\rho} & M
\end{array}
\]

is a split fork. The verification of the coherence axioms for a bicategory is a straightforward diagram-chasing argument.

**Remark 13.2.** If the monad $B: Y \to Y$ is the identity $1_Y: Y \to Y$, then we have a canonical isomorphism $G \circ_{1_Y} F \cong G \circ F$, which we will consider as an equality for simplicity.

**Example 13.3** (Bimodules in a monoidal category). For a monoidal category $\mathcal{C}$ with reflexive coequalizers in which the tensor product preserves reflexive coequalizers, the bicategory $\text{Bim}(\mathcal{C})$ has monoids in $\mathcal{C}$ as objects, bimodules as morphisms and bimodule maps as 2-cells (see also [5] for a discussion of these notions). In particular, $\text{Bim}(\text{Ab})$ is the bicategory of rings, ring bimodules and bimodule maps. Given a $(B, A)$-bimodule $M$ and a $(C, B)$-bimodule $N$, where $A, B$ and $C$ are rings, their tensor product $N \otimes_B M$ fits in the following reflexive coequalizer in $\text{Ab}$:

\[
\begin{array}{ccc}
N \otimes B \otimes M & \xrightarrow{\rho \otimes M} & N \otimes M \\
\rho \otimes A & \downarrow & \downarrow \\
N \otimes_B M
\end{array}
\]

**Example 13.4** (Distributors). The bicategory of bimodules in the bicategory of matrices $\text{Mat}_\mathcal{V}$ is exactly the bicategory of distributors:

\[
\text{Dist}_\mathcal{V} = \text{Bim}(\text{Mat}_\mathcal{V}).
\]

Indeed, we have seen that small $\mathcal{V}$-categories, which are the objects of $\text{Dist}_\mathcal{V}$, are monads in $\text{Mat}_\mathcal{V}$, distributors $F: \mathcal{X} \to \mathcal{Y}$ are the same thing as $(\mathcal{Y}, \mathcal{X})$-bimodules, and $\mathcal{V}$-natural transformations between distributors are exactly a bimodule maps. Furthermore, composition and identities in $\text{Dist}_\mathcal{V}$ arise as special cases of the general definition of composition and identities in bicategories of bimodules, as direct calculations show.
In Section 14 we will show that the bicategory $\text{Sym}_V$ is regular and use this fact to organize operads, operad bimodules and operad bimodule maps into a bicategory.

**Remark 13.5.** For every regular bicategory $\mathcal{E}$, there is an isomorphism

$$\text{Bim}(\mathcal{E})^{op} \cong \text{Bim}(\mathcal{E}^{op}).$$

Recall that we write $F^{op}: \mathcal{Y} \to \mathcal{X}$ for the morphism in $\mathcal{E}^{op}$ associated to a morphism $F: \mathcal{X} \to \mathcal{Y}$ in $\mathcal{E}$. The required isomorphism sends an object $\mathcal{X}/A \in \text{Bim}(\mathcal{E})^{op}$ to the object $\mathcal{X}/A^{op} \in \text{Bim}(\mathcal{E}^{op})$. Given $\mathcal{X}/A, \mathcal{Y}/B \in \text{Bim}(\mathcal{E})$, if $M: \mathcal{X} \to \mathcal{Y}$ has a $(B, A)$-bimodule structure in $\mathcal{E}$, then $M^{op}: \mathcal{Y} \to \mathcal{X}$ has an $(A^{op}, B^{op})$-bimodule structure in $\mathcal{E}^{op}$ and so we have an isomorphism

$$\text{Bim}(\mathcal{E}^{op})[\mathcal{Y}/B^{op}, \mathcal{X}/A^{op}] \cong \text{Bim}(\mathcal{E})[\mathcal{X}/A, \mathcal{Y}/B] = (\text{Bim}(\mathcal{E}))^{op}[\mathcal{Y}/B, \mathcal{X}/A].$$

Moreover, if $N: \mathcal{Y} \to \mathcal{Z}$ has a $(C, B)$-bimodule structure, we have $(N \circ_B M)^{op} = M^{op} \circ_B N^{op}$.

**Remark 13.6.** For every regular homomorphism $\Phi: \mathcal{E} \to \mathcal{F}$, there is a homomorphism $\text{Bim}(\Phi): \text{Bim}(\mathcal{E}) \to \text{Bim}(\mathcal{F})$ defined by letting $\text{Bim}(\Phi)(\mathcal{X}/A) = \Phi(\mathcal{X})/\Phi(A)$ for $\mathcal{X}/A \in \text{Bim}(\mathcal{E})$, $\text{Bim}(\Phi)(M) = \Phi(M)$ for $M: \mathcal{X}/A \to \mathcal{X}/B$ and $\text{Bim}(\Phi)(\alpha) = \Phi(\alpha)$ for $\alpha: M \to N$. The condition that $\Phi$ is regular ensures that $\text{Bim}(\Phi): \text{Bim}(\mathcal{E}) \to \text{Bim}(\mathcal{F})$ preserves composition and identity morphisms up to coherent natural isomorphism. In this way, an inclusion $\mathcal{E} \subseteq \mathcal{F}$ determines an inclusion $\text{Bim}(\mathcal{E}) \subseteq \text{Bim}(\mathcal{F})$.

14. The bicategory of operads

The aim of this section is to prove that the bicategory $\text{Sym}_V$ of symmetric sequences, defined in Section 11, is regular. This allows us to show that operads, operad bimodules and operad bimodule forms a bicategory, called the bicategory of operads and denoted $\text{Opd}_V$, using the bimodule construction of Section 13. More precisely, we define

$$\text{Opd}_V = \text{def} \ Bim(\text{Sym}_V).$$

In particular, for operads $(X, A)$ and $(Y, B)$, we have

$$\text{Opd}_V[(X, A), (Y, B)] = \text{Sym}_V[X, Y]_A^B.$$

We will show that the bicategory $S\text{-Dist}_V$ of Section 10 is regular, from which the fact that $\text{Sym}_V$ follows immediately. Before doing this, we wish to define the analytic functors associated to an operad bimodule. Recall that for an operad $(X, A)$, we write $\text{Alg}_R(A)$ for its category of algebras and algebra morphisms in a symmetric $V$-ring $R$. Let $(X, A)$ and $(Y, B)$ be operads. Given an operad bimodule $F: (X, A) \to (Y, B)$, for a symmetric $V$-rig $R$ we define the *analytic functor*

$$\text{Alg}_R(F): \text{Alg}_R(A) \to \text{Alg}_R(B)$$

associated to $F$ as follows. For an $A$-algebra $M$, we let

$$\text{Alg}_R(F)(M) = \text{def} F \circ_A M,$$

where $F \circ_A M$ is given by the following reflexive coequalizer diagram

$$F \circ A \circ M \xrightarrow{\rho_M} F \circ M \xrightarrow{F \circ \lambda} F \circ_A M.$$
This object has a $B$-algebra structure given as in \(13.3\). This functor fits in the following commutative diagram

\[
\begin{array}{ccc}
\text{Alg}_R(A) & \xrightarrow{\text{Alg}_R(F)} & \text{Alg}_R(B) \\
\downarrow & & \downarrow \\
\mathcal{R}^X & \xrightarrow{F} & \mathcal{R}^Y,
\end{array}
\]

where the vertical arrows are the evident free algebra and forgetful functors, and \(F: \mathcal{R}^X \to \mathcal{R}^Y\) is the analytic functor associated to the symmetric sequence \(F: X \to Y\), defined in \((11.2)\). As we will see in Section \([15]\), examples of analytic functors between categories of algebras for operads include the restriction and extension functors associated to an operad morphism.

Let us now turn our attention to showing that \(S\)-\text{Dist}_\mathcal{V}\) is a regular bicategory. This generalises a corresponding fact for single-sorted symmetric sequences proved in \([57]\). We begin by recalling the notion of a sifted category and recall some basic facts about it. For further information, see \([1, \text{Chapter 3}]\).

**Definition 14.1.** We say that a small category \(K\) is **sifted** if the colimit functor

\[
\text{colim}_K : \text{Set}^K \to \text{Set}
\]

preserves finite products. A category \(K\) is said to be **cosifted** if the opposite category \(K^{\text{op}}\) is sifted.

A category \(K\) is sifted if and only if it is non-empty and the diagonal functor \(d_K : K \to K \times K\) is cofinal. Dually, \(K\) is cosifted if and only if it is non-empty and the diagonal functor \(d_K : K \to K \times K\) is coinitial. Let us say that a presheaf \(X : K^{\text{op}} \to \text{Set}\) is **connected** if \(\text{colim}_K X = 1\). Then a category \(K\) is cosifted if and only if it is non-empty and the cartesian product \(K(-, j) \times K(-, k)\) of representable presheaves is connected. A sifted (or cosifted) category is connected. In the statement of the next lemma, we write \(\Delta\) for the usual category of finite ordinals and monotone maps, and \(\Delta|_1\) for its full subcategory spanned by the objects \([0]\) and \([1]\).

**Lemma 14.2.** The categories \(\Delta\) and \(\Delta|_1\) are cosifted.

**Proof.** The colimit of a simplicial set \(X : \Delta^{\text{op}} \to \text{Set}\) is the set of its connected components. It is well known that the canonical map \(\pi_0(X \times Y) \to \pi_0(X) \times \pi_0(Y)\) is bijective for any pair of simplicial sets \(X\) and \(Y\). Moreover, \(\pi_0(\Delta[0]) = 1\). Similarly, a presheaf \(X : K^{\text{op}} \to \text{Set}\) is connected if its colimit is the set \(\pi_0(X)\) of its connected components. It is easy to verify that the canonical map \(\pi_0(X \times Y) \to \pi_0(X) \times \pi_0(Y)\) is bijective for any pair of reflexive graphs \(X\) and \(Y\). Moreover, \(\pi_0(\Delta[0]) = 1\). \( \square \)

**Remark 14.3.** Since a reflexive graph in \(\mathcal{E}\) is exactly a contravariant functor \(X : \Delta^{\text{op}}_1 \to \mathcal{E}\), reflexive coequalisers are sifted colimits.

We now establish some auxiliary facts which will allow us to establish that \(S\)-\text{Dist}_\mathcal{V}\) is regular. If \(K\) is a small category and \(\mathcal{R}\) is a symmetric \(\mathcal{V}\)-rig, then category \(\mathcal{R}^K\) of \(K\)-indexed diagrams in \(\mathcal{R}\) has a symmetric monoidal (closed) structure with the pointwise tensor product:

\[
(A \otimes B)(k) =_{\text{def}} A(k) \otimes B(k).
\]

The unit object for the pointwise tensor product is the constant diagram \(cI : K \to \mathcal{R}\) with value the unit object \(I \in \mathcal{R}\). If \(A, B \in \mathcal{R}^K\) then the canonical map

\[
\text{colim}_{(i,j) \in K \times K} A(i) \otimes B(j) \to \text{colim}_{i \in K} A(i) \otimes \text{colim}_{j \in K} B(j)
\]

(14.3)
is an isomorphism, since the tensor product functor of \( R \) is cocontinuous in each variable. If \( K \) is sifted, then the canonical map
\[
\text{colim}_{i \in K} A(i) \otimes B(i) \to \text{colim}_{(i,j) \in K \times K} A(i) \otimes B(j)
\]
is an isomorphism, since the diagonal \( d_K : K \to K \times K \) is cofinal.

**Proposition 14.4.** Let \( K \) be a small category and \( R \) a symmetric \( V \)-rig. If \( K \) is sifted, then the canonical map
\[
\text{colim}_{i \in K} A(i) \otimes B(i) \to \text{colim}_{(i,j) \in K \times K} A(i) \otimes B(j)
\]
is an isomorphism, since the diagonal \( d_K : K \to K \times K \) is cofinal.

**Proposition 14.4.** Let \( K \) be a small category and \( R \) a symmetric \( V \)-rig. If \( K \) is sifted, then
\[
\text{colim}_{K} : R^K \to R \text{ is a symmetric monoidal functor.}
\]

**Proof.** We saw above that the map in (14.3) is an isomorphism for any pair of diagrams \( A, B : K \to R \). Moreover, the canonical map \( \text{colim}_{K} \mathbb{C} \to I \) is an isomorphism, since \( K \) is connected. \( \square \)

**Corollary 14.5.** The \( n \)-fold tensor product functor \( T^n : R^n \to \mathbb{R} \) preserves sifted colimits for every \( n \geq 0 \).

**Proof.** If \( K \) is a sifted category, then the colimit functor \( \text{colim}_{K} : R^K \to R \) is monoidal by Proposition 14.4. It thus preserves \( n \)-fold tensor products, from which the claim follows. \( \square \)

Let \( W \) be a small \( V \)-category and \([W, R]\) be the \( V \)-category of \( V \)-functors from \( W \) to \( R \). If \( w \in W \), then the evaluation functor \( \text{ev}_w : [W, R] \to [W, R] \) defined by letting \( \text{ev}_w(F) = F(w) \) is cocontinuous. If \( \overline{w} = (w_1, \ldots, w_n) \in W^n \), let us put
\[
\text{ev}_{\overline{w}}(F) = F(w_1) \otimes \ldots \otimes F(w_n).
\]

**Lemma 14.6.** The functor \( \text{ev}_{\overline{w}} : [W, R] \to \mathbb{R} \) preserves sifted colimits for every \( \overline{w} \in W^n \).

**Proof.** The functor \( \text{ev}_{\overline{w}} \) is the composite of the functor \( \rho_{\overline{w}} : [W, R] \to R^n \) defined by letting \( \rho_{\overline{w}}(F) = (F(w_1), \ldots, F(w_n)) \) followed by the \( n \)-fold tensor product functor \( T^n : R^n \to \mathbb{R} \). The first functor is cocontinuous, while the second preserves sifted colimits by Corollary 14.5. \( \square \)

Recall from Section 10 that to an \( S \)-distributor \( F : X \to Y \), i.e. a distributor \( F : X \to S(Y) \), we associate a distributor \( F^e : S(X) \to S(Y) \) defined as in (9.3).

**Lemma 14.7.** For every pair of small \( V \)-categories \( X, Y \) the functor
\[
(-)^e : \text{Dist}_V[X, S(Y)] \to \text{Dist}_V[S(X), S(Y)]
\]
preserves sifted colimits.

**Proof.** The claim follows if we show that the functor
\[
(-)^e : \text{CAT}_V[X, PS(Y)] \to \text{CAT}[S(X), PS(Y)]
\]
preserves sifted colimits. But this is a consequence of Lemma 14.6 since for a \( V \)-functor \( F : X \to PS(Y) \), we have \( F^e(\pi) = ev_{\overline{w}}(F) \). \( \square \)

**Theorem 14.8.** The bicategory \( S \)-\text{Dist}_V is regular. In particular, the horizontal composition functors of \( S \)-\text{Dist}_V preserve sifted colimits in the first variable and are cocontinuous in the second variable.
Proof. The hom-categories of \( S\text{-Dist}_V \) clearly have reflexive coequalizers. Recall from [10.2] that for \( F \in S\text{-Dist}_V[\mathbb{X}, \mathbb{Y}] \) and \( G \in S\text{-Dist}_V[\mathbb{Y}, \mathbb{Z}] \) we have

\[
(G \circ F)[\zeta; x] = \int_{\mathbb{Y}} G^\varepsilon[\zeta; y] \otimes F[y; x] = (G^\varepsilon \circ F)[\zeta; x].
\]

But the composition functors in \( \text{Dist}_V \) are cocontinuous in each variable. Hence, the functor \( F \mapsto G \circ F \) is cocontinuous, while the functor \( G \mapsto G^\varepsilon \circ F \) preserves sifted colimits by Lemma 14.7. \( \Box \)

Corollary 14.9. The bicategories \( \text{CatSym}_V \) and \( \text{Sym}_V \) are regular.

Proof. Since a bicategory is regular if and only if its opposite is so, the claim that the bicategory \( \text{CatSym}_V \) is regular follows from Theorem 14.8, since \( \text{CatSym}_V \) is the opposite of \( S\text{-Dist}_V \), which is regular by Theorem 14.8. The bicategory \( \text{Sym}_V \) is regular since it is a full sub-bicategory of the regular bicategory \( \text{CatSym}_V \). \( \Box \)

15. Monad morphisms and bimodules

The aim of this section is to relate the bicategory of monads, bimodules and bimodule maps introduced in Section 13 with the bicategory of monads, monad morphisms and maps of monad morphisms defined as in [61]. This will be useful to provide examples of analytic functors between categories of operadic algebras in Section 16 and to prove that the bicategory \( \text{Opd}_V \) is cartesian closed in Section 19. We begin by recalling some definitions from [61], using a slightly different terminology.

Definition 15.1. Let \( A : \mathbb{X} \to \mathbb{X}, B : \mathbb{Y} \to \mathbb{Y} \) be monads in \( \mathcal{E} \).

(i) A lax monad morphism \((F, \phi) : (\mathbb{X}, A) \to (\mathbb{Y}, B)\) consists of a morphism \( F : \mathbb{X} \to \mathbb{Y} \) in \( \mathcal{E} \) and a 2-cell \( \phi : B \circ F \to F \circ A \) such that the following diagrams commute:

\[
\begin{array}{cccc}
B \circ B \circ F & \xrightarrow{B \circ \phi} & B \circ F \circ A & \xrightarrow{\phi \circ A} & F \circ A \circ A \\
\mu_B \circ F & \downarrow & F \circ \mu_A & \downarrow & F \circ \mu_A \\
B \circ F & \xrightarrow{\phi} & F \circ A, & & F \circ A.
\end{array}
\]

(ii) A map of lax monad morphisms \( f : (F, \phi) \to (F', \phi') \) is a 2-cell \( f : F \to F' \) in \( \mathcal{E} \) such that the following diagram commutes:

\[
\begin{array}{ccc}
B \circ F & \xrightarrow{\phi} & F \circ A \\
\downarrow & & \downarrow f \circ A \\
B \circ F' & \xrightarrow{\phi'} & F' \circ A.
\end{array}
\]

Following [61], we write \( \text{Mnd}(\mathcal{E}) \) for the bicategory whose objects are pairs \((\mathbb{X}, A)\), where \( \mathbb{X} \in \mathcal{E} \) and \( A : \mathbb{X} \to \mathbb{X} \) is a monad, morphisms are lax monad morphisms and 2-cells are maps of lax monad morphisms. The composition operation of morphisms in \( \text{Mnd}(\mathcal{E}) \) is defined in the following way: for monad morphisms \((F, \phi) : (\mathbb{X}, A) \to (\mathbb{Y}, B)\) and \((G, \psi) : (\mathbb{Y}, B) \to (\mathbb{Z}, C)\), their composite is given by the morphism \( G \circ F : \mathbb{X} \to \mathbb{Z} \) equipped with the 2-cell

\[
\begin{array}{ccc}
C \circ G \circ F & \xrightarrow{\psi \circ F} & G \circ B \circ F & \xrightarrow{G \circ \phi} & G \circ F \circ A.
\end{array}
\]
Note that a lax monad morphism \((F, \phi): (X, A) \to (Y, B)\) is an equivalence in \(\text{Mnd}(\mathcal{E})\) if and only if \(F: X \to Y\) is an equivalence in \(\mathcal{E}\) and \(\phi\) is invertible. Also note that a homomorphism \(\Phi: E \to F\) induces a homomorphism \(\text{Mnd}(\Phi): \text{Mnd}(\mathcal{E}) \to \text{Mnd}(\mathcal{F})\), defined in the evident way. As we show in Lemma 15.2 below, every lax monad morphism gives rise to a bimodule and every map of lax monad morphisms gives rise to a bimodule map.

**Lemma 15.2.**

(i) If \((F, \phi): (X, A) \to (Y, B)\) is a lax monad morphism, then the morphism \(F \circ A: X \to Y\) has the structure of a \((B, A)\)-bimodule.

(ii) If \(f: (F, \phi) \to (F', \phi')\) is a map of lax monad morphisms, then \(f \circ A: F \circ A \to F' \circ A\) is a bimodule map.

**Proof.** The morphism \(F \circ A: X \to Y\) has the structure of a free right \(A\)-module with the right action \(\rho = F \circ \mu_A: (F \circ A) \circ A \to F \circ A\). The left action \(\lambda: B \circ (F \circ A) \to F \circ A\) of the monad \(B\) is defined to be the composite

\[
B \circ F \circ A \xrightarrow{\phi \circ A} F \circ A \circ A \xrightarrow{F \circ \mu_A} F \circ A.
\]

The proof the required properties is a straightforward diagram-chasing argument. \(\Box\)

Lemma 15.2 allows us to define a homomorphism \(R: \text{Mnd}(\mathcal{E}) \to \text{Bim}(\mathcal{E})\) as follows. Its action on objects is the identity. For a monad morphism \((F, \phi): (X, A) \to (Y, B)\), we define \(R(F, \phi): X/A \to Y/B\) to be the \((B, A)\)-bimodule with underlying morphism \(F \circ A: X \to Y\), as in the proof of Lemma 15.2. Given a monad 2-cell \(f: (F, \phi) \to (F', \phi')\), we define \(R(f): F \circ A \to F' \circ A\) to be \(f \circ A: F \circ A \to F' \circ A\), which is a bimodule map by part (ii) of Lemma 15.2. The remaining data necessary to define a homomorphism can be derived easily. In particular, for lax monad morphisms \((F, \phi): (X, A) \to (Y, B)\) and \((G, \psi): (Y, B) \to (Z, C)\), we have an isomorphism

\[
R(G, \psi) \circ_B R(F, \phi) \cong R(G \circ F, (G \circ \phi) \cdot (\psi \circ F)),
\]

since

\[
R(G, \psi) \circ_B R(F, \phi) = (G \circ B) \circ_B (F \circ A) \cong G \circ F \circ A = R(G \circ F).
\]

Recall that for a bicategory \(\mathcal{E}\), we write \(\mathcal{E}^{\text{op}}\) for the bicategory obtained from \(\mathcal{E}\) by formally reversing the direction of morphisms, but not that of 2-cells. Since \((\text{Bim}(\mathcal{E}^{\text{op}}))^{\text{op}} = \text{Bim}(\mathcal{E})\), Lemma 15.2 admits a dual, which we state explicitly below since it will be useful in the following. We begin by recalling from [61] an explicit description of the morphisms and 2-cells in \((\text{Mnd}(\mathcal{E}^{\text{op}}))^{\text{op}}\).

**Definition 15.3.** Let \(A: X \to X, B: Y \to Y\) be monads in \(\mathcal{E}\).

(i) An oplax monad morphism \((F, \phi): (X, A) \to (Y, B)\) consists of a morphism \(F: X \to Y\) in \(\mathcal{E}\) and a 2-cell \(\psi: F \circ A \to B \circ F\) such that the following diagrams commute:

\[
\begin{align*}
F \circ A \circ A & \xrightarrow{\psi \circ A} B \circ F \circ A \\
F \circ A & \xrightarrow{\mu_B \circ F} B \circ F
\end{align*}
\]

\[
\begin{align*}
F \circ A & \xrightarrow{\phi \circ A} B \circ F \circ A \\
F \circ A & \xrightarrow{\eta_B \circ F} B \circ F
\end{align*}
\]
(ii) A map of oplax monad morphisms \( f: (F, \psi) \to (F', \psi') \) is a 2-cell \( f: F \to F' \) in \( \mathcal{E} \) such that the following diagram commutes:

\[
\begin{array}{ccc}
F \circ A & \xrightarrow{\psi} & B \circ F \\
\downarrow f \circ A & & \downarrow B \circ f \\
F' \circ A & \xrightarrow{\psi'} & B \circ F'.
\end{array}
\]

The bicategory \( \text{Mnd}(\mathcal{E}^{op})^{op} \) has the same objects as \( \text{Mnd}(\mathcal{E}) \), oplax monad morphisms as morphisms and maps of oplax monad morphisms as 2-cells. We now state the dual of Lemma 15.2.

**Lemma 15.4.**

(i) If \( (F, \psi): (X, A) \to (Y, B) \) is an oplax monad morphism in a bicategory \( \mathcal{E} \), then the morphism \( B \circ F: X \to Y \) has the structure of a \( (B, A) \)-bimodule.

(ii) If \( f: (F, \psi) \to (F', \psi') \) is a map of oplax monad morphisms, then \( B \circ f: B \circ F \to B \circ F' \) is a bimodule map.

**Proof.** The morphism \( B \circ F: X \to Y \) has the structure of a free left \( B \)-module with the left action \( \mu \circ F: B \circ B \circ F \to B \circ B \). The right \( A \)-action is defined to be the composite

\[
B \circ F \circ A \xrightarrow{B \circ \psi} F \circ A \xrightarrow{\mu \circ A} B \circ F.
\]

As for Lemma 15.2, we omit the details of the verification. \( \square \)

By Lemma 15.4, it is possible to define a homomorphism \( L: (\text{Mnd}(\mathcal{E}^{op})^{op}) \to \text{Bim}(\mathcal{E}) \), in complete analogy with the way in which we defined the homomorphism \( R: \text{Mnd}(\mathcal{E}) \to \text{Bim}(\mathcal{E}) \) using Lemma 15.2. We omit the details, which are straightforward. The following remark will be useful in the proofs of Proposition 19.1 and Theorem 19.2.

**Remark 15.5.** Let \( \Phi, \Psi: \mathcal{E} \to \mathcal{F} \) be homomorphisms and let \( F: \Phi \to \Psi \) be a pseudo-natural transformation. If \( A: \mathcal{X} \to \mathcal{X} \) is a monad in \( \mathcal{E} \), we have monads \( \Phi(A): \Phi(\mathcal{X}) \to \Phi(\mathcal{X}) \) and \( \Psi(A): \Psi(\mathcal{X}) \to \Psi(\mathcal{X}) \) in \( \mathcal{F} \). Then, the morphism \( F_\mathcal{X}: \Phi(\mathcal{X}) \to \Psi(\mathcal{X}) \) and the pseudo-naturality 2-cell

\[
\Phi(\mathcal{X}) \xrightarrow{F_\mathcal{X}} \Psi(\mathcal{X}) \\
\Phi(A) \xrightarrow{\Phi(f_A)} \Psi(A) \\
\Phi(\mathcal{X}) \xrightarrow{F_\mathcal{X}} \Psi(\mathcal{X})
\]

give us a lax monad morphism

\( (F_\mathcal{X}, f_A): (\Phi(\mathcal{X}), \Phi(A)) \to (\Psi(\mathcal{X}), \Psi(A)). \)

Since the 2-cell \( f_A \) is invertible, we also have an oplax monad morphism

\( (F_\mathcal{X}, f_A^{-1}): (\Phi(\mathcal{X}), \Phi(A)) \to (\Psi(\mathcal{X}), \Psi(A)). \)

We now show how, under appropriate assumptions, Lemma 15.2 and Lemma 15.4 allow us to construct an adjunction in \( \text{Bim}(\mathcal{E}) \) from an adjunction in \( \mathcal{E} \). Let us begin by recalling that if \( (F, G): \mathcal{X} \to \mathcal{Y} \) is an adjunction in \( \mathcal{E} \), then a monad \( B: \mathcal{Y} \to \mathcal{Y} \) induces a monad \( B': \mathcal{X} \to \mathcal{X} \), where \( B' = \text{def} G \circ B \circ F \), with multiplication defined as the composite

\[
G \circ B \circ F \circ G \circ B \circ F \xrightarrow{G \circ B \circ F \circ B \circ F} G \circ B \circ B \circ F \xrightarrow{G \circ B \circ F} G \circ B \circ F,
\]
and unit $\eta': 1_X \to G \circ B \circ F$ defined as the composite

$$1_X \xrightarrow{\eta} G \circ F \xrightarrow{G \eta_B \circ F} G \circ B \circ F.$$ 

**Theorem 15.6.** Let $\mathcal{E}$ be a regular bicategory. Let $A: \mathcal{X} \to \mathcal{X}, B: \mathcal{Y} \to \mathcal{Y}$ be monads in $\mathcal{E}$ and let $(F, G, \eta, \varepsilon): \mathcal{X} \to \mathcal{Y}$ be an adjunction in $\mathcal{E}$. Then, a monad map $\xi: A \to G \circ B \circ F$ determines an adjunction $(F', G', \eta', \varepsilon') : (\mathcal{X}, A) \to (\mathcal{Y}, B)$ in $\text{Bim}(\mathcal{E})$.

**Proof.** Given a monad map (in the sense of Definition 15.4) $\xi: A \to G \circ B \circ F$, we define the 2-cell $\psi: F \circ A \to B \circ F$ as the composite

$$F \circ A \xrightarrow{F \eta} F \circ G \circ B \circ F \xrightarrow{\varepsilon \circ B \circ F} B \circ F.$$ 

A standard diagram-chasing argument shows that $(F, \psi): (\mathcal{X}, A) \to (\mathcal{Y}, B)$ is an oplax monad morphism. Similarly, we define the 2-cell $\phi: A \circ G \to G \circ B$ as the composite

$$A \circ G \xrightarrow{\varepsilon \circ G} G \circ B \circ F \circ G \xrightarrow{G \varepsilon_B \circ G} G \circ B$$

and obtain a lax monad morphism $(G, \phi): (\mathcal{Y}, B) \to (\mathcal{X}, A)$. We define $F': (\mathcal{X}, A) \to (\mathcal{Y}, B)$ to be the $(B, A)$-bimodule associated to the oplax monad morphism $(F, \psi): (\mathcal{X}, A) \to (\mathcal{Y}, B)$, as in Lemma 15.4. Explicitly, the morphism $F' \overset{\text{def}}{=} B \circ F$ is equipped with the left $B$-action

$$\lambda: B \circ F' \to F'$$

given by $\mu \circ F: B \circ B \circ F \to B \circ F$ and the right $A$-action $\rho: F' \circ A \to F'$ given by the composite

$$(\mu_B \circ F) \cdot (B \circ \psi): B \circ F \circ A \to B \circ F.$$ 

Similarly, the right adjoint $G': (\mathcal{Y}, B) \to (\mathcal{X}, A)$ is the $(A, B)$-bimodule associated to the lax monad morphism $(G, \phi): (\mathcal{Y}, B) \to (\mathcal{X}, A)$, as in Lemma 15.2. Explicitly, $G' \overset{\text{def}}{=} G \circ B$ is equipped with the left $A$-action

$$\lambda: A \circ G' \to G'$$

and the right $A$-action $\rho: G' \circ A \to G'$. 

In order to define the unit of the adjunction $\eta': 1_{(\mathcal{X}, A)} \to G' \circ B$ $F'$, observe that by the definition of the relative composition we have

$$G' \circ_B F' \cong G \circ B \circ F.$$ 

Hence, we define $\eta'$ to be the monoid map $\xi: A \to G \circ B \circ F$. The counit $\varepsilon': F' \circ_A G' \to 1_{(\mathcal{Y}, B)}$ is obtained via the universal property of coequalizers, via the diagram

$$F' \circ A \circ G' \xrightarrow{\varepsilon'} F' \circ G' \xrightarrow{F' \circ_A G'} F' \circ A \circ G'$$

where $\sigma$ is the composite

$$B \circ F \circ G \circ B \xrightarrow{B \varepsilon_B} B \circ B \xrightarrow{B \mu} B.$$
Unfolding the relevant definitions, the triangular laws amount to the commutativity of the diagrams

\[
\begin{align*}
B \circ F \circ_A A &\xrightarrow{B \circ F \circ_A \xi} B \circ F \circ_A G \circ B \circ F \\
&\xrightarrow{\ell_{B \circ F}} B \circ F, \\
\end{align*}
\]

and

\[
\begin{align*}
A \circ_A G \circ B &\xrightarrow{\xi \circ_A G \circ B} G \circ B \circ F \circ_A G \circ B \\
&\xrightarrow{r_{G \circ B}} G \circ B, \\
\end{align*}
\]

where \( \ell_{B \circ F} \) and \( r_{G \circ B} \) are the unit isomorphisms of \( \text{Bim}(\xi) \), defined as in (13.5) and (13.6). In both cases, the required commutativity follows by the universal property defining the relative composition operation and in particular the definition of the unit isomorphisms. \( \square \)

16. Extension and restriction as analytic functors

The aim of this section is to show how the familiar restriction functor and extension functor associated to an operad morphism are analytic functors in the sense of Section 14. This will follow by an application of Theorem 15.6. Let us begin by recalling some definitions. Let \( \mathcal{X} = (X,A) \) and \( \mathcal{Y} = (Y,B) \) be operads, viewed as monads in the bicategory \( \text{Sym}_\mathcal{V} \). Thus, \( X, Y \) are sets and \( A : X \to X, B : Y \to Y \) are symmetric sequences equipped with multiplication and unit. Let us also fix an operad morphism \( (u, \xi) : \mathcal{X} \to \mathcal{Y} \), which consists of a function \( u : X \to Y \) and a monoid morphism \( \xi : A \to B' \), where \( B' : X \to X \) is the monad whose underlying symmetric sequence is defined by letting

\[ B'[x_1, \ldots, x_n; x] = \text{def} \ B[ux_1, \ldots, ux_n; ux]. \quad (16.1) \]

It will be useful to define the symmetric sequence \( u^\circ : X \to Y \) and \( u_\circ : Y \to X \) by letting

\[ u^\circ[x_1, \ldots, x_n; y] = \text{def} \ B[ux_1, \ldots, ux_n; y] , \quad u_\circ[y_1, \ldots, y_n; x] = \text{def} \ B[y_1, \ldots, y_n; ux]. \quad (16.2) \]

We wish to show that \( u_\circ \) and \( u^\circ \) form an adjunction in \( \text{Sym}_\mathcal{V} \). We make some preliminary observations. As a special case of the corresponding facts for \( \mathcal{V} \)-functors and distributors, recalled in Section 13, the function \( u : X \to Y \) determines an adjunction \((u_\circ, u^\circ) : X \to Y \) in \( \text{Mat}_\mathcal{V} \). The homomorphism \( \delta : \text{Mat}_\mathcal{V} \to \text{S-Mat}_\mathcal{V} \) takes this adjunction to an adjunction \((\delta(u_\circ), \delta(u^\circ)) : X \to Y \) in \( \text{S-Mat}_\mathcal{V} \), which gives us an adjunction \((\delta(u^\circ), \delta(u_\circ)) : X \to Y \) in \( \text{Sym}_\mathcal{V} \) by duality. Explicitly, the symmetric sequence \( \delta(u^\circ) : X \to Y \) and \( \delta(u_\circ) : Y \to X \) are defined by letting

\[
\delta(u^\circ)[\mathcal{F}; y] = \text{def} \begin{cases} 
I & \text{if } \mathcal{F} = (x) \text{ and } ux = y, \\
0 & \text{otherwise,}
\end{cases} \quad \delta(u_\circ)[\mathcal{G}; x] = \text{def} \begin{cases} 
I & \text{if } \mathcal{G} = (ux), \\
0 & \text{otherwise.}
\end{cases}
\]

Here, \( I \) and \( 0 \) denote the unit and the initial object of \( \mathcal{V} \), respectively. After these preliminary observations, we state and prove a lemma which relates these symmetric sequences with those defined in (16.2).

**Lemma 16.1.** There are isomorphisms

(i) \( u^\circ \cong B \circ \delta(u^\circ) \),

(ii) \( u_\circ \cong \delta(u_\circ) \circ B \).
**Proof.** For the proof, we write $u[\tau]$ for $(u_1, \ldots, u_n)$, where $\tau = (x_1, \ldots, x_n)$. For (i), observe that by the definition of composition in $\text{Sym}_\mathcal{V}$, we have

$$(B \circ \delta(u^*))[\tau; y] = \bigsqcup_{n \in \mathbb{N}} \int_{(y_1, \ldots, y_n) \in S^n(Y)} \int_{\tau_1 \in S(X)} \cdots \int_{\tau_n \in S(X)} \delta(u^*)[\tau_1; y_1] \otimes \cdots \otimes \delta(u^*)[\tau_n; y_n] \otimes S(X)[\tau, \tau_1 \oplus \cdots \oplus \tau_n] \otimes B[y_1, \ldots, y_n; y].$$

By the definition of $\delta(u^*)$, the right-hand side is isomorphic to

$$\bigsqcup_{n \in \mathbb{N}} \int_{x_1 \in X} \cdots \int_{x_m \in X} S(X)[\tau, (x_1, \ldots, x_m)] \otimes B[u_1, \ldots, u_m; y].$$

This, in turn, is isomorphic to $B[u[\tau]; y]$, as required. The proof of (ii) is similar.

The next lemma gives an alternative description of the monad $B': X \to X$ defined in (16.1).

**Lemma 16.2.** There is an isomorphism $B' \cong \delta(u_*) \circ B \circ \delta(u^*)$.

**Proof.** By part (i) of Lemma 16.1, it is sufficient to exhibit an isomorphism $B' \cong \delta(u_*) \circ u^\circ$. We have

$$(\delta(u_*) \circ u^\circ)[\tau; x] = \bigsqcup_{n \in \mathbb{N}} \int_{(y_1, \ldots, y_n) \in S^n(Y)} \int_{\tau_1 \in S(X)} \cdots \int_{\tau_n \in S(X)} u^\circ[\tau_1; y_1] \otimes \cdots \otimes u^\circ[\tau_n; y_n] \otimes S(X)[\tau, \tau_1 \oplus \cdots \oplus \tau_n] \otimes \delta(u^*)[y_1, \ldots, y_n; x].$$

By the definition of $\delta(u_*)$, the left-hand side is isomorphic to

$$\int_{\tau \in S(X)} u^\circ[\rho'; u x] \otimes S(X)[\tau, \tau'],$$

which is isomorphic to $u^\circ[\tau; u x]$. But, by definition of $u^\circ$ and $B'$, we have $u^\circ[\tau; u x] = B'[\tau; x]$. 

**Lemma 16.3.** The symmetric sequence $u^\circ: X \to Y$ has the structure of a $(B, A)$-bimodule, the symmetric sequence $u_\circ: Y \to X$ has the structure of an $(A, B)$-bimodule and the resulting operad bimodules form an adjunction $(u^\circ, u_\circ): (X, A) \to (Y, B)$ in the bicategory $\text{Opd}_\mathcal{V}$.

**Proof.** First of all, observe that we have an adjunction $(\delta(u^*), \delta(u_*)): X \to Y$ in the bicategory $\text{Sym}_\mathcal{V}$. Secondly, the monoid morphism $\xi: A \to B'$ determines, by Lemma 16.2 a monoid morphism $\xi': A \to \delta(u_* \circ B \circ \delta(u^*))$. By Proposition 15.6 it follows that we have an adjunction

$$(B \circ \delta(u^*), \delta(u_*) \circ B): (X, A) \to (Y, B)$$

in $\text{Opd}_\mathcal{V}$. But, by Lemma 16.1 the symmetric sequences $B \circ \delta(u^*)$ and $\delta(u_*) \circ B$ are isomorphic to the symmetric sequences $u^\circ$ and $u_\circ$, which therefore inherit a bimodule structure so as to give us the required adjunction.

We can apply Lemma 16.3 to give a general version of the restriction and extension functors between categories of algebras for operads. We continue to consider a fixed operad morphism $(u, \xi): (X, A) \to (Y, B)$. For a set $K$, we define the restriction functor $u^*: [K, X]^A \to [K, Y]^B$ as follows. For a left $B$-module $N: K \to Y$, we define the left $A$-module $u^*(N): K \to X$ by letting

$$u^*(N)[k_1, \ldots, k_n; x] =_{\text{def}} N[k_1, \ldots, k_n; u(x)].$$

**Theorem 16.4.** For every operad morphism $(u, \xi): X \to Y$, the functor $u^*: [K, X]^A \to [K, Y]^B$ is an analytic functor and it has a left adjoint $u_*: [K, Y]^B \to [K, X]^A$, which is also an analytic functor.
Proof. We show that \( u^* \) is the analytic functor associated to the operad bimodule \( u_o : (Y, B) \to (X, A) \). By the formula in \(14.2\), this amounts to showing that we have a natural isomorphism

\[
u^*(N) \cong u_o \circ_B N,
\]

for every left \( B \)-module \( N : K \to Y \). By part (ii) of Lemma \(16.1\) and the unit isomorphism of \( \text{Opd}_Y \), we have \( u_o \circ_B N \cong \delta(u_o) \circ b \circ_B N \cong \delta(u_o) \circ N \). Hence, it suffices to exhibit an isomorphism \( u^*(N) \cong \delta(u_o) \circ N \), which can be done using calculations similar to those in the proofs of Lemma \(16.1\) and Lemma \(16.2\). By the isomorphism in \(16.3\) and Lemma \(16.3\), it follows that we can define the left adjoint \( u_l : [K,Y] \to [K,X] \) as the analytic functor associated to the operad bimodule \( u^o : (X, A) \to (Y, B) \). Explicitly, for a left \( A \)-module \( M : K \to X \), we have

\[
u_l(M) = \text{def} \ u^o \circ_A M.
\]

The required adjointness \( u_l \dashv u^* \) now follows immediately from the adjointness \( u^o \dashv u_o \) proved in Lemma \(16.3\). \(\square\)

17. Formal theory of monads in regular bicategories

The aim of this section is to develop some aspects of the formal theory of monads in the setting of a regular bicategory. We begin by reviewing some notions and results from \([61]\), adapting them from the setting of 2-category to that of a bicategory, and then focus on the peculiar aspects that arise in regular bicategories. Let \( \mathcal{E} \) be a fixed bicategory. Recall from Section \(12\) that, for a monad \( A : \mathcal{X} \to \mathcal{X} \) in \( \mathcal{E} \) and \( \mathcal{K} \in \mathcal{E} \), we write \( \mathcal{E}[\mathcal{K}, \mathcal{X}]^A \) for the category of left \( A \)-modules with domain \( \mathcal{K} \) and left \( A \)-module maps. The category \( \mathcal{E}[\mathcal{K}, \mathcal{X}]^A \) depends pseudo-functorially on \( \mathcal{K} \) and so we obtain a prestack \( \mathcal{E}[-, \mathcal{X}]^A : \mathcal{E}^{\text{op}} \to \text{Cat} \).

**Definition 17.1.** An Eilenberg-Moore object for a monad \( A : \mathcal{X} \to \mathcal{X} \) in \( \mathcal{E} \) is a representing object for the prestack \( \mathcal{E}[-, \mathcal{X}]^A : \mathcal{E}^{\text{op}} \to \text{Cat} \).

Concretely, an Eilenberg-Moore object for a monad \( A : \mathcal{X} \to \mathcal{X} \) consists of an object \( \mathcal{X}^A \in \mathcal{E} \) and a morphism \( U : \mathcal{X}^A \to \mathcal{X} \) equipped with a left \( A \)-action \( \lambda : A \circ U \to U \), which is universal in the following sense: the functor

\[
\mathcal{E}[\mathcal{K}, \mathcal{X}]^A \to \mathcal{E}[\mathcal{K}, \mathcal{X}]
\]

which takes a morphism \( N : \mathcal{K} \to \mathcal{X}^A \) to the morphism \( U \circ N : \mathcal{K} \to \mathcal{X} \) equipped with the left action \( \lambda \circ N : A \circ U \circ N \to U \circ N \) is an equivalence of categories for every object \( \mathcal{K} \in \mathcal{E} \). In particular, for any left \( A \)-module \( M : \mathcal{K} \to \mathcal{X} \) there exists a morphism \( N : \mathcal{K} \to \mathcal{X}^A \) together with an invertible 2-cell \( \alpha : M \to U \circ N \) such that the following square commutes,

\[
\begin{array}{ccc}
A \circ M & \xleftarrow{A \circ \alpha} & A \circ U \circ N \\
\downarrow & & \downarrow \lambda \circ N \\
M & \xrightarrow{\alpha} & U \circ N.
\end{array}
\]

In the following, we will adopt a slight abuse of language and refer to either the object \( \mathcal{X}^A \) or the morphism \( U : \mathcal{X}^A \to \mathcal{X} \) as the Eilenberg-Moore object for a monad \( A : \mathcal{X} \to \mathcal{X} \). Note that the universal property characterising an Eilenberg-Moore object considered here is weaker than introduced in \([61]\), even in a 2-category, since we require the functor in \(17.1\) to be an equivalence rather than an isomorphism. In particular, an Eilenberg-Moore object for a monad, as defined here, is unique up to equivalence rather than up to isomorphism as in \([61]\).

**Proposition 17.2.** If \( U : \mathcal{X}^A \to \mathcal{X} \) is an Eilenberg-Moore object for a monad \( A : \mathcal{X} \to \mathcal{X} \), then the functor \( \mathcal{E}[\mathcal{K}, U] : \mathcal{E}[\mathcal{K}, \mathcal{X}^A] \to \mathcal{E}[\mathcal{K}, \mathcal{X}] \) is monadic for every \( \mathcal{K} \in \mathcal{E} \).
Proof. We have a commutative diagram of functors

\[
\begin{array}{ccc}
\mathcal{E}[\mathbb{K}, X^A] & \xrightarrow{\sim} & \mathcal{E}[\mathbb{K}, X]^A, \\
\downarrow & & \downarrow \\
\mathcal{E}[\mathbb{K}, U] & \xrightarrow{\sim} & \mathcal{E}[\mathbb{K}, X],
\end{array}
\]

where the vertical arrow is the evident forgetful functor, which is monadic by construction. Hence also the functor \(\mathcal{E}[\mathbb{K}, U]\), since it is the composite of a monadic functor and an equivalence of categories. □

The next proposition is a version of [61, Theorem 2] in the context of bicategories. We omit the proof.

Proposition 17.3. Every Eilenberg-Moore object \(U : X \to X\) for a monad \(A : X \to X\) has a left adjoint \(F : X \to X\) and the monad map \(\pi : A \to U \circ F\) given by the composite

\[
A \xrightarrow{\alpha} A \circ U \circ F \xrightarrow{\lambda} U \circ F
\]
is invertible. □

Let \(A : X \to X\) be a monad in \(\mathcal{E}\). Recall from Section 12 that we write \(\mathcal{E}[\mathbb{K}, X]^A\) for the category of right \(A\)-modules with domain \(\mathbb{K}\) and right \(A\)-module maps between them. The category \(\mathcal{E}[\mathbb{K}, X]_A\) depends pseudo-functorially on the object \(\mathbb{K}\) and so we obtain a prestack \(\mathcal{E}[X, -]_A : \mathcal{E}^{op} \to \text{Cat}\).

Definition 17.4. A Kleisli object for a monad \(A : X \to X\) in a bicategory \(\mathcal{E}\) is a representing object for the prestack \(\mathcal{E}[X, -]_A : \mathcal{E}^{op} \to \text{Cat}\).

Concretely, a Kleisli object for a monad \(A : X \to X\) consists of an object \(X_A \in \mathcal{E}\) and a morphism \(F : X \to X_A\) equipped with a right \(A\)-action \(\rho : F \circ A \to F\) which is universal in the following sense: the functor

\[
\mathcal{E}[F, \mathbb{K}] : \mathcal{E}[X, \mathbb{K}] \to \mathcal{E}[X, \mathbb{K}]_A
\]
which takes a morphism \(N : X_A \to \mathbb{K}\) to the morphism \(N \circ F_A : X \to \mathbb{K}\) equipped with the right action \(N \circ F \circ A \to N \circ F\) is an equivalence of categories for every \(\mathbb{K} \in \mathcal{E}\). In particular, for any right \(A\)-module \(M : X \to \mathbb{K}\) there exists a morphism \(N : X_A \to \mathbb{K}\) together with an invertible 2-cell \(\alpha : M \to N \circ F\) such that the following square commutes:

\[
\begin{array}{ccc}
M \circ A & \xrightarrow{\alpha \circ A} & N \circ F \circ A \\
\rho \downarrow & & \downarrow N \circ \rho \\
M & \xrightarrow{\alpha} & N \circ F.
\end{array}
\]

Observe that a right \(A\)-module \(F : X \to X_A\) is a Kleisli object for a monad \(A : X \to X\) if an only if the left \(A^{op}\)-module \(F^{op} : X_A \to X\) in \(\mathcal{E}^{op}\) is an Eilenberg-Moore object for the monad \(A^{op} : X \to X\) in \(\mathcal{E}^{op}\).

Proposition 17.5. If \(F : X \to X_A\) is a Kleisli object for a monad \(A : X \to X\), then the functor \(\mathcal{E}[F, \mathbb{K}] : \mathcal{E}[X, \mathbb{K}] \to \mathcal{E}[X, \mathbb{K}]\) is monadic for every object \(\mathbb{K} \in \mathcal{E}\).

Proof. This follows by duality from Proposition 17.2. □
Proposition 17.6. Every Kleisli object \( F : X_A \to X \) for a monad \( A : X \to X \) has a right adjoint \( U : X \to X_A \) and the monad map \( \pi : A \to U \circ F \) given by the composite
\[
A \xrightarrow{\eta A} U \circ F \circ A \xrightarrow{U \circ \rho} U \circ F
\]
is invertible.

Proof. This follows by duality from Proposition 17.3. \( \square \)

Definition 17.7. We say that an adjunction \((F, U, \eta, \varepsilon) : X \to Y \) in \( E \) is

(i) monadic if the morphism \( U : Y \to X \), equipped with the left action by the monad \( U \circ F \), is an Eilenberg-Moore object for the monad \( U \circ F : X \to X \),

(ii) opmonadic if the morphism \( F : Y \to X \), equipped with the right action of the monad \( U \circ F \), is a Kleisli object for the monad \( U \circ F : X \to X \),

(iii) bimonadic if it is both monadic and opmonadic.

In the 2-category \( \text{Cat} \), an adjunction is monadic in the sense of Definition 17.7 if and only if it is monadic in the usual sense [5, Section 3.3]. An adjunction \((F, U) : X \to Y \) in a bicategory \( \mathcal{E} \) is monadic if and only if the adjunction \((\mathcal{E}[K, F], \mathcal{E}[K, G]) : \mathcal{E}[K, X] \to \mathcal{E}[K, Y] \) is monadic in \( \text{Cat} \) for every \( K \in \mathcal{E} \). By Proposition 17.3, the adjunction \((F, U) : X \to X^A \) associated to an Eilenberg-Moore object is monadic. Dually, by Proposition 17.6, the adjunction \((F, U) : X \to X^A \) associated to a Kleisli object is opmonadic. Observe that an adjunction \((F, U, \eta, \varepsilon) \) is opmonadic in \( \mathcal{E} \) if and only if the opposite adjunction \((U^\text{op}, F^\text{op}, \eta, \varepsilon) \) is monadic in \( \mathcal{E}^\text{op} \). Consequently, the notion of a bimonadic adjunction is self-dual, in the sense that an adjunction \((F, U, \eta, \varepsilon) : X \to Y \) in \( \mathcal{E} \) is bimonadic if and only if the opposite adjunction \((U^\text{op}, F^\text{op}, \eta, \varepsilon) : X \to Y \) in \( \mathcal{E}^\text{op} \) is bimonadic.

Definition 17.8. We will say that an adjunction \((F, U, \eta, \varepsilon) : X \to Y \) in \( \mathcal{E} \) is effective if the fork
\[
F \circ U \circ F \circ U \xrightarrow{\varepsilon F \circ U} F \circ U \xrightarrow{\varepsilon} 1_Y
\]
is a coequaliser diagram.

The notion of an effective adjunction is self-dual, in the sense that an adjunction \((F, U, \eta, \varepsilon) \) is effective in \( \mathcal{E} \) if and only if the opposite adjunction \((U^\text{op}, F^\text{op}, \eta, \varepsilon) \) is effective in \( \mathcal{E}^\text{op} \). Effective adjunctions have been studied extensively in category theory (see [41] and references therein for further information).

Proposition 17.9.

(i) A monadic adjunction is effective.

(ii) An opmonadic adjunction is effective.

Proof. For part (i), let us show that a monadic adjunction \((F, U, \eta, \varepsilon) : X \to Y \) is effective. The adjunction
\[
\mathcal{E}[K, X] \xrightarrow{\varepsilon[K,F]} \mathcal{E}[K, Y]
\]
is monadic for every \( K \in \mathcal{E} \), since \( U : Y \to X \) is an Eilenberg-Moore object for the monad \( U \circ F : X \to X \). This is true in particular in the case where \( K = Y \). Let us now show that the fork
\[
F \circ U \circ F \circ U \xrightarrow{\varepsilon F \circ U} F \circ U \xrightarrow{\varepsilon} 1_Y
\]
is a coequaliser diagram.
is a coequaliser diagram. But the image of the fork in (17.2) by the functor $U \circ (-)$ is a split coequaliser:

\[
\begin{array}{ccc}
\eta \circ \eta & \circ U \circ F \circ U & \circ U \\
\eta \circ \eta & \circ U \circ F \circ U & \circ U \\
\end{array}
\]

By Beck’s monadicity theorem, which we can apply because the adjunction $E[X, F] \dashv E[Y, U]$ is monadic, the fork in (17.2) is a coequaliser diagram. Part (ii) follows by duality. \qed

Theorem 17.10. For an adjunction $(F, U, \eta, \varepsilon) : X \rightarrow Y$ in a regular bicategory $E$, the conditions of being effective, monadic, opmonadic and bimonadic are equivalent.

Proof. We already saw in Proposition 17.9 that every monadic adjunction is effective. Conversely, let us show that if $E$ is regular, then every effective adjunction $(F, U, \eta, \varepsilon) : X \rightarrow Y$ in $E$ is monadic. For this, we show that the adjunction

\[
E[X, X] \xrightarrow{E[X, F]} E[X, Y]
\]

is monadic for every $K \in E$. We apply the Crude Monadicity Theorem. The category $E[X, Y]$ has reflexive coequalisers and the functor $E[X, U] : E[X, Y] \rightarrow E[X, X]$ preserves them by the hypothesis on $E$. Hence it remains to show that the functor $E[X, U]$ is conservative. Let $f : M \rightarrow N$ be a 2-cell in $E[X, Y]$ and suppose that the 2-cell $U \circ f : U \circ M \rightarrow U \circ N$ is invertible. Let us show that $f$ is invertible. Consider the following commutative diagram,

\[
\begin{array}{ccc}
F \circ U \circ F \circ U \circ M & \xrightarrow{coF \circ U \circ f} & F \circ U \circ M \\
\downarrow{coF \circ U \circ f} & & \downarrow{coM} \\
F \circ U \circ F \circ U \circ N & \xrightarrow{coF \circ U \circ N} & F \circ U \circ N \\
\end{array}
\]

The top fork of the diagram is a coequaliser diagram, since the adjunction $F \dashv U$ is effective and the functor $E[M, Y] : E[Y, Y] \rightarrow E[K, Y]$ preserves reflexive coequalisers. Similarly, the bottom fork of the diagram is a coequaliser diagram. But the left and middle vertical cells of the diagram are invertible, since the 2-cell $U \circ f$ is invertible. It follows that $f$ is invertible. We have proved that the adjunction $F \dashv U$ in $E$ is monadic. It follows by duality that the adjunction $U^{op} \dashv F^{op}$ in $E^{op}$ is monadic if and only if it is effective. But the adjunction $U^{op} \dashv F^{op}$ is monadic in $E^{op}$ if and only if the adjunction $F \dashv U$ is monadic in $E$. Moreover, the adjunction $U^{op} \dashv F^{op}$ is effective in $E^{op}$ if and only if the adjunction $F \dashv U$ is effective. This proves that the adjunction $F \dashv U$ is monadic if and only if it is effective. \qed

Corollary 17.11. Let $E$ be a regular bicategory and $(F, U) : X \rightarrow Y$ be an adjunction in $E$. The following conditions are equivalent:

(i) the adjunction $(F, U) : X \rightarrow Y$ is effective,
(ii) the morphism $U: Y \to X$ is an Eilenberg-Moore object for the monad $U \circ F: \mathcal{X} \to \mathcal{X}$,
(iii) the morphism $F: X \to Y$ is a Kleisli object for the monad $U \circ F: \mathcal{X} \to \mathcal{X}$. □

Corollary 17.12. Let $\mathcal{E}$ be a regular bicategory. If $U: \mathcal{X}^A \to \mathcal{X}$ is an Eilenberg-Moore object for
a monad $A: \mathcal{X} \to \mathcal{X}$, then its left adjoint $F: \mathcal{X} \to \mathcal{X}^A$ is a Kleisli object for $A$. Conversely, if
$F: \mathcal{X} \to \mathcal{X}_A$ is a Kleisli object for $A$, then its right adjoint $U: \mathcal{X}_A \to \mathcal{X}$ is an Eilenberg-Moore
object for $A$. □

Note that Corollary 17.12 implies that Kleisli objects and Eilenberg-Moore objects coincide in a regular bicategory. The next proposition involves the notion of a regular homomorphism between regular bicategories introduced in Definition 13.1.

Proposition 17.13. Let $\mathcal{E}$ and $\mathcal{F}$ be regular bicategories. A regular homomorphism $\Phi: \mathcal{E} \to \mathcal{F}$
preserves monadic (and hence also opmonadic, bimonadic and effective) adjunctions. It thus
preserves Eilenberg-Moore (and hence also Kleisli) objects.

Proof. Let us show that $\Phi$ preserves effective adjunctions. If $(F,U,\eta,\varepsilon): \mathcal{X} \to \mathcal{Y}$ is an effective
adjunction in $\mathcal{E}$, let us show that the adjunction $(\Phi F, \Phi U, \Phi \eta, \Phi \varepsilon): \Phi \mathcal{X} \to \Phi \mathcal{X}$ is effective. The
fork
$$F \circ U \circ F \circ U \xrightarrow{\varepsilon \circ F \circ \varepsilon} F \circ U \xrightarrow{\varepsilon} 1_{\mathcal{Y}}$$

is a reflexive coequaliser diagram in $\mathcal{E}[\mathcal{Y}, \mathcal{Y}]$, since the adjunction $(F,U,\eta,\varepsilon)$ is effective. Hence
also the fork
$$\Phi F \circ \Phi U \circ \Phi F \circ \Phi U \xrightarrow{\Phi \varepsilon \circ \Phi F \circ \Phi \varepsilon} \Phi F \circ \Phi U \xrightarrow{\Phi \varepsilon} 1_{\Phi \mathcal{Y}}$$
is a reflexive coequalizer, since the functor $\Phi_{X,Y}: \mathcal{E}[\mathcal{Y}, \mathcal{Y}] \to \mathcal{F}[\mathcal{Y}, \mathcal{Y}]$ preserves reflexive co-
equalisers for every $Y \in \mathcal{E}$. This proves that $\Phi$ preserves effective adjunctions. It then follows
from Theorem 17.10 that $\Phi$ preserves monadic, opmonadic and bimonadic adjunctions. It thus
preserves Eilenberg-Moore and Kleisli objects by Corollary 17.11. □

18. Eilenberg-Moore objects in bicategories of bimodules

For every regular bicategory $\mathcal{E}$ there is a homomorphism

$$J_\mathcal{E}: \mathcal{E} \to \text{Bim}(\mathcal{E})$$

which maps an object $X \in \mathcal{E}$ to the identity monad $X/1_X$. The action of $J_\mathcal{E}$ on morphisms and
2-cells is evident and hence we do not spell it out. We conclude this section by introducing the
appropriate notion of morphism between regular bicategories and make an observation about the
functorial character of the bimodule construction.

Proposition 18.1. Let $\mathcal{E}$ be a regular bicategory. The bicategory $\text{Bim}(\mathcal{E})$ is regular and the
homomorphism $J_\mathcal{E}: \mathcal{E} \to \text{Bim}(\mathcal{E})$ is full and faithful and proper.

Proof. Let us show that $J_\mathcal{E}: \mathcal{E} \to \text{Bim}(\mathcal{E})$ is fully faithful. A morphism $M: X \to Y$ in $\mathcal{E}[X,Y]$ has
a unique structure of $(1_Y,1_X)$-bimodule, and every 2-cell $f: M \to N$ in $\mathcal{E}[X,Y]$ is a map of $(1_Y,1_X)$-bimodules. This shows that the functor

$$J_{X,Y}: \mathcal{E}[X,Y] \to \text{Bim}(\mathcal{E})[X/1_X,Y/1_Y]$$
is an isomorphism of categories for every pair of objects $X,Y \in \mathcal{E}$.

Let us now show that the bicategory $\text{Bim}(\mathcal{E})$ is regular. First of all, recall that the category $\mathcal{E}[X,Y]$ has reflexive coequalisers by the assumption that $\mathcal{E}$ is regular. Moreover, the
monad \( B \circ (-) \circ A : \mathcal{E}[\mathcal{X}, \mathcal{Y}] \rightarrow \mathcal{E}[\mathcal{X}, \mathcal{Y}] \) preserves reflexive coequalisers since it is defined by composition. By Proposition \([12.19]\) the category \( \mathcal{E}[\mathcal{X}, \mathcal{Y}]^B \) has reflexive coequalisers and that the forgetful functor \( \mathcal{E}[\mathcal{X}, \mathcal{Y}]^B \rightarrow \mathcal{E}[\mathcal{X}, \mathcal{Y}] \) preserves and reflects reflexive coequalisers. Let us now show that the horizontal composition functors of \( \text{Bim}(\mathcal{E}) \) preserve coequalizers on the left. For this we have to show that the functor \( N \circ_B (-) : [\mathcal{X}/A, \mathcal{Y}/B] \rightarrow [\mathcal{X}/A, \mathcal{Z}/C] \) preserves reflexive coequalisers for every morphism \( N : \mathcal{Y}/B \rightarrow \mathcal{Z}/C \) in \( \text{Bim}(\mathcal{E}) \). The following square commutes

\[
\begin{array}{ccc}
\mathcal{E}[\mathcal{X}, \mathcal{Y}]^B_A & \xrightarrow{N \circ_B (-)} & \mathcal{E}[\mathcal{X}, \mathcal{Z}]_A^C \\
\downarrow U_1 & & \downarrow U_2 \\
\mathcal{E}[\mathcal{X}, \mathcal{Y}]^B & \xrightarrow{N \circ_B (-)} & \mathcal{E}[\mathcal{X}, \mathcal{Z}]
\end{array}
\]

by definition of the functor \( N \circ_B (-) \), where \( U_1 \) and \( U_2 \) are the forgetful functors. Moreover, the functors \( U_1 \) and \( U_2 \) preserve and reflect reflexive coequalizers. Hence, it suffices to show that the composite

\[
U_2(N \circ_B (-)) : \mathcal{E}[\mathcal{X}, \mathcal{Y}]^B \rightarrow \mathcal{E}[\mathcal{X}, \mathcal{Z}]
\]

preserves reflexive coequalisers. But for this it suffices to show that the functor

\[
N \circ_B (-) : \mathcal{E}[\mathcal{X}, \mathcal{Y}] \rightarrow \mathcal{E}[\mathcal{X}, \mathcal{Z}]
\]

preserves reflexives coequalisers, since the square commutes and the functor \( U_1 \) preserves reflexive coequalizers. For every \( M \in \mathcal{E}[\mathcal{X}, \mathcal{Y}]^B \) we have a coequaliser diagram

\[
\begin{array}{ccc}
\mathcal{E}[\mathcal{X}, \mathcal{Y}]^B_A & \xrightarrow{\rho_M} & \mathcal{E}[\mathcal{X}, \mathcal{Z}]_A^C \\
\downarrow N \circ_B (-) & & \downarrow N \circ_B (-) \\
\mathcal{E}[\mathcal{X}, \mathcal{Y}]^B & \xrightarrow{N \circ_B (-)} & \mathcal{E}[\mathcal{X}, \mathcal{Z}]
\end{array}
\]

(18.1)

in the category \( \mathcal{E}[\mathcal{X}, \mathcal{Z}] \), where \( \rho_N \) is the right action of \( B \) on \( N \) and \( \lambda_M \) is the left action of \( B \) on \( M \). The functors

\[
N \circ_B (-) : [\mathcal{X}, \mathcal{Y}] \rightarrow [\mathcal{X}, \mathcal{Z}] , \quad N \circ (-) : [\mathcal{X}, \mathcal{Y}] \rightarrow [\mathcal{X}, \mathcal{Z}]
\]

preserve reflexive coequalisers, since the bicategory \( \mathcal{E} \) is regular. Hence also their composite with the forgetful functor \( [\mathcal{X}, \mathcal{Y}]^B \rightarrow [\mathcal{X}, \mathcal{Y}] \). This shows that the functor \( N \circ_B (-) \) is a colimit of functors preserving reflexive coequalisers. It follows that the functor \( N \circ_B (-) \) preserves reflexive coequalisers, since colimits commute with colimits. We have proved that the horizontal composition functors of \( \text{Bim}(\mathcal{E}) \) preserve coequalizers on the left. It follows by the duality of Remark \([13.5]\) that the horizontal composition functors of \( \text{Bim}(\mathcal{E}) \) preserve coequalizers also on the right, and hence \( \mathcal{E} \) is regular.

It remains to show that the homomorphism \( J_\mathcal{E} : \mathcal{E} \rightarrow \text{Bim}(\mathcal{E}) \) is proper. But \( J_\mathcal{E} \) preserves local reflexive coequalisers, since the functor \( J_{\mathcal{X}, \mathcal{Y}} : \mathcal{E}[\mathcal{X}, \mathcal{Y}] \rightarrow \text{Bim}(\mathcal{E})[\mathcal{X}/1_X, \mathcal{Y}/1_Y] \) is an equivalence of categories for every pair of objects \( \mathcal{X}, \mathcal{Y} \in \mathcal{E} \). \( \square \)

Proposition \([13.5]\) implies that \( J_\mathcal{E} : \mathcal{E} \rightarrow \text{Bim}(\mathcal{E}) \) can be regarded as an inclusion \( \mathcal{E} \subseteq \text{Bim}(\mathcal{E}) \). Because of this, in the following we will identify an object \( \mathcal{X} \in \mathcal{E} \) with the object \( \mathcal{X}/1_X \) of \( \text{Bim}(\mathcal{E}) \) and the category \( \mathcal{E}[\mathcal{X}, \mathcal{Y}] \) with the category \( \text{Bim}(\mathcal{E})[\mathcal{X}/1_X, \mathcal{Y}/1_Y] \) for every pair \( \mathcal{X}, \mathcal{Y} \in \mathcal{E} \). Our next goal is to establish that \( \text{Bim}(\mathcal{E}) \) is Eilenberg-Moore complete. We begin with two observations about the relationship between monads in \( \mathcal{E} \) and in \( \text{Bim}(\mathcal{E}) \).

**Proposition 18.2.** Let \( \mathcal{E} \) a regular bicategory.

(i) An adjunction in \( \mathcal{E} \) is monadic in \( \mathcal{E} \) if and only if it is monadic in \( \text{Bim}(\mathcal{E}) \).

(ii) A morphism \( U : \mathcal{Y} \rightarrow \mathcal{X} \) in \( \mathcal{E} \) is an Eilenberg-Moore object for a monad \( A : \mathcal{X} \rightarrow \mathcal{X} \) if and only if it is an Eilenberg-Moore object for \( A : \mathcal{X}/1_X \rightarrow \mathcal{X}/1_X \) in \( \text{Bim}(\mathcal{E}) \).
Proof. Let us show that an adjunction $\mathcal{E}$ is monadic if and only if it is monadic in $\text{Bim}(\mathcal{E})$. But an adjunction in $\mathcal{E}$ is effective if and only if it is effective in $\text{Bim}(\mathcal{E})$, since the functor $\text{J}_{X,Y}: \mathcal{E}[Y,Y] \to \text{Bim}(\mathcal{E})[Y,Y]$ is an equivalence of categories (it is actually an isomorphism of categories). The result then follows from Theorem 17.10 since $\mathcal{E}$ and $\text{Bim}(\mathcal{E})$ are regular. Corollary [17.11] implies that a morphism $U: Y \to X$ in $\mathcal{E}$ is an Eilenberg-Moore object for a monad $A: X \to X$ if and only if it is an Eilenberg-Moore object for $A: X/1_X \to X/1_X$ in $\text{Bim}(\mathcal{E})$. \hfill \Box

Let $A: X \to X$ be a monad in a regular bicategory $\mathcal{E}$. Then the morphism $A: X \to X$ has the structure of a left $A$-module $F: X \to X/A$ and of a right $A$-module $U: X/A \to X$. We wish to show that we have an adjunction $\langle F, U \rangle$ in $\text{Bim}(\mathcal{E})$. Since $U \circ A F = A \circ A \cong A$ and $F \circ U = A \circ A$, we define the unit $\eta: 1_X \to U \circ A F$ to be $\eta_A: 1_X \to A$ and the counit $\varepsilon: F \circ U \to 1_{X/A}$ to be $\mu_A: A \circ A \to A$. To prove that we have an adjunction, we need to show that the triangular identities hold. This amounts to proving that

$$(A \circ_A \mu_A) \cdot (\eta_A \circ A) = 1_A, \quad (\mu_A \circ_A A) \cdot (A \circ_A \eta_A) = 1_A.$$ 

But we have $A \circ_A \mu_A = A$, since the 2-cell $\mu_A: A \circ A \to A$ is a map of left $A$-modules. Thus,

$$(A \circ_A \mu_A) \cdot (\eta_A \circ A) = \mu_A \cdot (\eta_A \circ A) = 1_A,$$

since $\eta_A$ is a unit for the multiplication $\mu_A$. Dually, we have $\mu_A \circ_A A = A$, since the 2-cell $\mu_A: A \circ A \to A$ is a map of right $A$-modules. Thus,

$$(\mu_A \circ_A A) \cdot (A \circ \eta) = \mu_A \cdot (A \circ \eta_A) = 1_A,$$

since $\eta_A$ is a unit for the multiplication $\mu_A$.

Lemma 18.3. Let $\mathcal{E}$ be a regular bicategory. Let $A: X \to X$ be a monad in $\text{Bim}(\mathcal{E})$. Then the adjunction $(F, U): X \to X/A$ in $\text{Bim}(\mathcal{E})$ described above is monadic and the monad $U \circ A F: X \to X$ is isomorphic to $A: X \to X$. Hence, $U: X/A \to X$ is an Eilenberg-Moore object for $A$.

Proof. Let us begin by verifying that the monad $U \circ A F$ is isomorphic to $A$. Obviously, $U \circ A F = A \circ A A = A$. The unit of the monad $U \circ A F$ is defined to be the unit $\eta$ of the adjunction $F \circ U$. But we have $\eta = \eta_A$ by definition. The multiplication of the monad $U \circ A F$ is defined to be the 2-cell $U \circ A \varepsilon \circ A F$. But we have

$$U \circ A \varepsilon \circ A F = A \circ A \mu_A \circ A A = \mu_A,$$

since $\mu_A: A \circ A \to A$ is a map of $(A, A)$-bimodules. Let us now show that the adjunction $(F, U, \eta, \varepsilon)$ is monadic. By Proposition 18.3 the bicategory $\text{Bim}(\mathcal{E})$ is regular and therefore, by Theorem 17.10 it suffices to show that the adjunction is effective. For this we have to show that the fork

$$\begin{array}{ccc}
F \circ U \circ A F & \xrightarrow{\varepsilon} & F \circ U \\
\cong & \downarrow & \downarrow \\
F \circ U \circ A F & \xrightarrow{\varepsilon} & 1_{X/A}
\end{array}$$

is a coequaliser diagram in the category of $(A, A)$-bimodules. But this fork is isomorphic to

$$\begin{array}{ccc}
A \circ A & \xrightarrow{\mu_A \circ A} & A \\
\cong & \downarrow & \downarrow \\
A \circ A & \xrightarrow{\mu_A} & A
\end{array}$$

since $F = U = A$, $A \circ_A A \cong A$ and $\varepsilon = \mu_A$. Let us show that the fork in (18.2) is a coequaliser diagram. But its image by the forgetful functor $U: \mathcal{E}[X, X]^A \to \mathcal{E}[X, X]$ splits in the
category \( E[X,X] \), as we have the following diagram

\[
\begin{array}{ccc}
A \circ A & \xrightarrow{\mu_A \circ A} & A \\
\downarrow{A \circ \mu_A} & \swarrow{\eta_A \circ A} & \\
A \circ A \circ A & & A \circ A.
\end{array}
\]

This shows that the fork in \( \textbf{18.2} \) is a coequaliser diagram, since the functor \( U \) is monadic, as observed in Proposition \( \textbf{12.19} \). \( \square \)

**Remark 18.4.** Let \( \text{Un}_E : \text{Mnd}(E) \to E \) be the homomorphism mapping a monad \((X,A)\) to its underlying object \(X\). For a monad \((X,A)\) in \(E\), the bimodule \(U : X/A \to X\) of Lemma \( \textbf{18.3} \) can be viewed as a morphism

\[
U^A : R(X,A) \to (J_E \circ \text{Un}_E)(X,A),
\]

in \( \text{Bim}(E) \), where \( R_E : \text{Mnd}(E) \to \text{Bim}(E) \) is the homomorphism defined via Lemma \( \textbf{15.2} \). The family of morphisms \( U^A : X/A \to X \), for \((X,A) \in \text{Mnd}(E)\), can then be seen as the components of a pseudo-natural transformation fitting in the diagram

\[
\begin{array}{ccc}
\text{Mnd}(E) & \xrightarrow{\text{Un}_E} & E \\
\downarrow{R_E} & \downarrow{J_E} & \\
\text{Bim}(E).
\end{array}
\]

For a monad morphism \((F,\phi) : (X,A) \to (Y,B)\), the required pseudo-naturality 2-cell, which should fit in the diagram

\[
\begin{array}{ccc}
X/A & \xrightarrow{U^A} & X/1_X \\
\downarrow{R(F)} & \downarrow{\phi \circ_F} & \downarrow{F} \\
Y/B & \xrightarrow{U^B} & Y/1_Y,
\end{array}
\]

is given by the following chain of isomorphisms and equalities:

\[
F \circ U^A = F \circ A \cong B \circ_B F \circ A = B \circ_B R(F) = U^B \circ_B R(F).
\]

Lemma \( \textbf{18.3} \) shows that every monad in \( E \) admits an Eilenberg-Moore object in \( \text{Bim}(E) \). Below, we show that in fact every monad in \( \text{Bim}(E) \) has an Eilenberg-Moore object in \( \text{Bim}(E) \). In order to do this, we need some preliminary observations. Let \((A,\mu_A,\eta_A)\) be a monad on \(X \in E\). Then, for a monad \((B,\mu_B,\eta_B)\) on \(X\) and a map of monads \(\pi : A \to B\), the morphism \(B : X \to X\) has the structure of an \((A,A)\)-bimodule, which we denote as \(B_A : X/A \to X/A\). The left action \(\lambda : A \circ B \to B\) and the right action \(\rho : B \circ A \to B\) are defined by the following diagrams

\[
\begin{array}{ccc}
A \circ B & \xrightarrow{\pi \circ B} & B \circ B \\
\downarrow{\mu_B} & \downarrow{B \circ \pi} & \downarrow{B \circ A} \\
B & & B.
\end{array}
\]

Furthermore, the bimodule \(B_A : X/A \to X/A\) has the structure of a monad in \(\text{Bim}(E)\) with the multiplication \(\mu^B : B \circ_A B \to B\) defined by the following commutative diagram, determined by
the universal property of $B \circ A$ B, as follows:

$$B \circ A \circ B \xrightarrow{\rho \circ B} B \circ B \xrightarrow{\eta \circ B} B \circ B$$

The unit of the monad $B_A; \mathcal{X}/A \to \mathcal{X}/A$ is the 2-cell $\pi: A \to B$. This defines a functor $(-)_A: A^{\mathcal{E}}(\mathcal{X}) \to \text{MonBim}(\mathcal{E})(\mathcal{X}/A)$.

**Lemma 18.5.** For every monad $(A, \mu_A, \eta_A)$ in a regular category $\mathcal{E}$, the functor $(-)_A: A^{\mathcal{E}}(\mathcal{X}) \to \text{MonBim}(\mathcal{E})(\mathcal{X}/A)$ is essentially surjective.

**Proof.** Let $(E, \mu, \eta)$ be a monad on $\mathcal{X}/A$ in Bim($\mathcal{E}$). Thus, we have a morphism $E; \mathcal{X} \to \mathcal{X}$ equipped with the structure of an $(A, A)$-bimodule together with bimodule maps $\mu: E \circ A E \to E$ and $\eta: A \to E$ satisfying the monad axioms. We define a monad $(B, \mu_B, \eta_B)$ on $\mathcal{X}$ as follows.

The morphism $B; \mathcal{X} \to \mathcal{X}$ is given by $E; \mathcal{X} \to \mathcal{X}$ itself. The multiplication $\mu_B; B \circ B \to B$ is obtained by composing the canonical map $q: E \circ E \to E \circ A E$ with $\mu: E \circ A E \to E$, and the unit $\eta_B: 1_B \to B$ is defined to be the composite of the unit $\eta_A; 1_A \to A$ of the monad $A$ with the unit $\eta; A \to E$ of the monad $E$. It is easy to verify that the monad axioms are satisfied.

We can also define a monad map $\pi: A \to B$ by letting $\pi = \text{def } \eta$. It is now immediate to check that $B_A \cong E$, as required. \qed

Let $\mathcal{X} \in \mathcal{E}$. Let $A = (A, \mu_A, \eta_A), B = (B, \mu_B, \eta_B)$ be monads on $\mathcal{X}$. If $\pi: A \to B$ is a map of monads, then the morphism $B; \mathcal{X} \to \mathcal{X}$ has the structure of both an $(A, B)$-bimodule and a $(B, A)$-bimodule. We will denote these bimodules by $U: \mathcal{X}/B \to \mathcal{X}/A$ and $F; \mathcal{X}/A \to \mathcal{X}/B$, respectively. We wish to show that these morphisms are adjoint. In order to do so, let us define the unit of the adjunction $\eta; \text{Id}_{\mathcal{X}/A} \to U \circ B F$ as the composite of $\pi: A \to B$ with the isomorphism $B \cong B \circ B$. We then define the counit of the adjunction $\varepsilon; F \circ A U \to \text{Id}_{\mathcal{X}/B}$ as the multiplication $\mu^B; B \circ A B \to B$ of the monad $B_A; \mathcal{X}/A \to \mathcal{X}/A$ defined above. It remains to verify the triangular laws, which in this case amounts to verifying the commutativity of the following diagrams:

For the diagram on the left-hand side, observe that $\mu^B \circ B = \mu^B$, since $\mu^B: B \circ A B \to B$ is a map of right $B$-modules. Thus,

$$(\mu^B \circ B) \cdot (B \circ A \pi) = \mu^B \cdot (B \circ A \pi) = 1_B,$$

since $\pi$ is the unit of the monad $B_A; \mathcal{X}/A \to \mathcal{X}/A$. For the diagram on the right-hand side, dually, we have $B \circ B \mu^B = \mu^B$, since $\mu^B: B \circ A B \to B$ is a map of left $B$-modules. Thus,

$$(U \circ B \mu^B) \cdot (\pi \circ A U) = \mu^B \cdot (\pi \circ A B) = 1_B = 1_U,$$

since $\pi$ is the unit of the monad $B_A$. We have therefore proved that $(F, U, \eta, \varepsilon): \mathcal{X}/A \to \mathcal{Y}/B$ is an adjunction.
**Proposition 18.6.** The adjunction $(F,U,\eta,\varepsilon): \mathcal{X}/A \to \mathcal{X}/B$ defined above is monadic and the monad $U \circ F: \mathcal{X}/A \to \mathcal{X}/A$ is isomorphic to the monad $B_A: \mathcal{X}/A \to \mathcal{X}/A$. Hence, the bimodule $U: \mathcal{X}/B \to \mathcal{X}/A$ is an Eilenberg-Moore object for the monad $B_A: \mathcal{X}/A \to \mathcal{X}/A$.

**Proof.** Let us show that the adjunction is monadic. By Proposition [18.1] the bicategory $\text{Bim}(\mathcal{E})$ is regular and therefore, by Theorem [17.10] it suffices to show that the adjunction is effective. For this we have to show that the fork

$$F \circ A \circ B \circ A F \circ A U \xrightarrow{\mu^B \circ B \circ F \circ A U} F \circ A U \xrightarrow{\mu^B} B$$

is a coequaliser diagram in the category of $(B,B)$-bimodules. But the fork in (18.3) is isomorphic to the fork

$$B \circ A B \circ A B \xrightarrow{\mu^B \circ A B} B \circ A B \xrightarrow{\mu^B} B,$$

since $F = U = B$ and $B \circ B = B$. But the image of the fork in (18.3) under the forgetful functor $U: \mathcal{E}[X, X]/B \to \mathcal{E}[X, X]$ splits in the category $\mathcal{E}[X, X]$.

Finally, let us show that the monad $U \circ F: \mathcal{X}/A \to \mathcal{X}/A$ is isomorphic to $B_A: \mathcal{X}/A \to \mathcal{X}/A$. First of all, we have

$$U \circ F = B \circ B = B.$$

Since the functor $U$ is monadic, as observed in Proposition [12.19] this shows that the fork in (18.3) is a coequaliser diagram. We have proved that the adjunction $(F,U,\eta,\varepsilon)$ is monadic.

**Definition 18.7.** We say that a bicategory $\mathcal{E}$ is Eilenberg-Moore complete (resp. Kleisli complete) if every monad in $\mathcal{E}$ admits an Eilenberg-Moore object (resp. Kleisli object).

For example, the 2-category Cat is both Eilenberg-Moore complete and Kleisli complete. A bicategory $\mathcal{E}$ is Eilenberg-Moore complete if and only if its opposite $\mathcal{E}^{op}$ is Kleisli complete. Since Eilenberg-Moore and Kleisli objects coincide in a regular bicategory by Corollary [17.12], a regular bicategory is Eilenberg-Moore complete if and only if it is Kleisli complete.

**Theorem 18.8.** For any regular category $\mathcal{E}$, the bicategory $\text{Bim}(\mathcal{E})$ is Eilenberg-Moore complete.

**Proof.** Let us show that every monad $E = (E, \mu, \eta)$ over an object $X/A \in \text{Bim}(\mathcal{E})$ admits an Eilenberg-Moore object. By Lemma [18.5] we have $E \cong B_A$ for a monad $(B_A, \mu_B, \eta_B)$ on $X$ and a map of monads $\pi: A \to B$ in $\text{Mon}(X)$. It then follows from Proposition [18.6] that the bimodule $U: \mathcal{X}/B \to \mathcal{X}/A$ defined above is an Eilenberg-Moore object for the monad $B_A: \mathcal{X}/A \to \mathcal{X}/A$.

**Proposition 18.9.** A regular bicategory $\mathcal{E}$ is Eilenberg-Moore complete if and only if the homomorphism $J_\mathcal{E}: \mathcal{E} \to \text{Bim}(\mathcal{E})$ is an equivalence.
Proof. If $J_E$ is an equivalence, then $\mathcal{E}$ is Eilenberg-Moore complete because $\text{Bim}(\mathcal{E})$ is Eilenberg-Moore complete, as proved in Theorem 18.8. Conversely, let us assume that $\mathcal{E}$ is Eilenberg-Moore complete and show that $J_E$ is an equivalence. Recall that $J_E$ is full and faithful by Proposition 18.1. Hence it suffices to show that $J_E$ is essentially surjective. For this we have to show that every object $\mathbb{X}/A \in \text{Bim}(\mathcal{E})$ is equivalent to an object of $\mathcal{E}$. The monad $(\mathbb{X}, A)$ admits an Eilenberg-Moore object $U: \mathbb{X}^A \to \mathbb{X}$ in $\mathcal{E}$, since $\mathcal{E}$ is Eilenberg-Moore complete by hypothesis. This Eilenberg-Moore object is also an Eilenberg-Moore object in $\text{Bim}(\mathcal{E})$ by Proposition 18.2. Hence we have an equivalence $\mathbb{X}^A \simeq \mathbb{X}/A$ in $\text{Bim}(\mathcal{E})$, since any two Eilenberg-Moore objects for a monad are equivalent. □

Recall from Section 13 that, for regular bicategories $\mathcal{E}$ and $\mathcal{F}$, we write $\text{REG}[\mathcal{E}, \mathcal{F}]$ for the full sub-bicategory of $\text{HOM}[\mathcal{E}, \mathcal{F}]$ whose objects are regular homomorphisms. Clearly, the composite of two regular homomorphisms is proper.

**Definition 18.10.** Given a regular bicategory $\mathcal{E}$ and a regular and Eilenberg-Moore complete bicategory $\mathcal{E}'$, we say that a regular homomorphism $J: \mathcal{E} \to \mathcal{E}'$ exhibits $\mathcal{E}'$ as the Eilenberg-Moore completion of $\mathcal{E}$ as a regular bicategory if the homomorphism

$$( - ) \circ J: \text{REG}[\mathcal{E}', \mathcal{F}] \to \text{REG}[\mathcal{E}, \mathcal{F}]$$

is a biequivalence for any regular and Eilenberg-Moore complete bicategory $\mathcal{F}$.

It follows from this definition that such an Eilenberg-Moore completion of a regular bicategory, if it exists, is unique up to biequivalence. If $\mathcal{E}$ a regular bicategory, then the bicategory $\text{Bim}(\mathcal{E})$ is regular by Theorem 18.1 and Eilenberg-Moore complete by Theorem 18.8 and the homomorphism $J_E: \mathcal{E} \to \text{Bim}(\mathcal{E})$ is regular by Proposition 18.1. The next theorem is essentially a special case of the results in [17, 63] and [28]. But we state it explicitly for future reference and prove it in Appendix B.

**Theorem 18.11.** For a regular bicategory $\mathcal{E}$, the homomorphism $J_E: \mathcal{E} \to \text{Bim}(\mathcal{E})$ exhibits $\text{Bim}(\mathcal{E})$ as the Eilenberg-Moore completion of $\mathcal{E}$ as a regular bicategory.

**Proof.** See Appendix B □

**Remark 18.12.** As shown in [44], for a 2-category $\mathcal{E}$, not necessarily regular, it is possible to define its Eilenberg-Moore completion $\text{EM}(\mathcal{E})$, which comes equipped with a 2-functor $I_\mathcal{E}: \mathcal{E} \to \text{EM}(\mathcal{E})$ satisfying a suitable universal property. The definitions of $\text{EM}(\mathcal{E})$ and $I_\mathcal{E}: \mathcal{E} \to \text{EM}(\mathcal{E})$ make sense also when $\mathcal{E}$ is a bicategory, in which case $\text{EM}(\mathcal{E})$ is also a bicategory and $I_\mathcal{E}$ is a homomorphism. We can then relate $\text{EM}(\mathcal{E})$ and $\text{Bim}(\mathcal{E})$ via a homomorphims $\Gamma: \text{EM}(\mathcal{E}) \to \text{Bim}(\mathcal{E})$, defined below, which makes the following diagram commute:

$$\begin{array}{ccc}
\mathcal{E} & \xrightarrow{1_{\mathcal{E}}} & \text{EM}(\mathcal{E}) \\
\downarrow J_{\mathcal{E}} & & \downarrow \Gamma \\
\text{Bim}(\mathcal{E}) & & \end{array}$$

For a bicategory $\mathcal{E}$, the objects and the morphisms of $\text{EM}(\mathcal{E})$ are the same as those of the bicategory $\text{Mnd}(\mathcal{E})$ recalled in Section 15. Given morphisms $(M, \phi), (G, \psi): (\mathbb{X}, A) \to (\mathbb{Y}, B)$, a 2-cell $f: (M, \phi) \to (M', \phi')$ in $\text{EM}(\mathcal{E})$, instead, is a 2-cell $f: M \to M' \circ A$ making the following
diagram commute:

\[
\begin{array}{c}
B \circ M \xrightarrow{\phi} M \circ A \xrightarrow{f \circ A} M' \circ A \circ A \\
\downarrow B \circ f \\
B \circ M' \circ A \xrightarrow{\phi' \circ A} M' \circ A \circ A \xrightarrow{M' \circ f} M' \circ A.
\end{array}
\]

The homomorphism \( \Gamma: EM(\mathcal{E}) \to Bim(\mathcal{E}) \) is defined exactly as the homomorphism \( R: \text{Mnd}(\mathcal{E}) \to Bim(\mathcal{E}) \) of Section 15 on objects and morphisms. For a 2-cell \( f: (M, \phi) \to (M', \phi') \) in \( EM(\mathcal{E}) \), we define \( \Gamma(f): M \circ A \to M' \circ A \) as the composite

\[
M \circ A \xrightarrow{f \circ A} M' \circ A \circ A \xrightarrow{M' \circ \mu A} M' \circ A.
\]

The commutativity of the required diagrams follows easily.

**Proposition 18.13.** Let \( \mathcal{E} \subseteq \mathcal{F} \) be an inclusion of regular bicategories. If every object of \( \mathcal{F} \) is an Eilenberg-Moore object (or, equivalently, a Kleisli object) for a monad in \( \mathcal{E} \), then the induced inclusion \( Bim(\mathcal{E}) \subseteq Bim(\mathcal{F}) \) is an equivalence.

**Proof.** We have the following diagram:

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{1_{\mathcal{E}}} & \mathcal{F} \\
\downarrow J_{\mathcal{E}} & & \downarrow J_{\mathcal{F}} \\
Bim(\mathcal{E}) & \xrightarrow{J} & Bim(\mathcal{F}).
\end{array}
\]

We show that \( Bim(\mathcal{E}) \subseteq Bim(\mathcal{F}) \) is essentially surjective. Let \((Y, B) \in Bim(\mathcal{F})\). By the hypothesis, \( Y \in \mathcal{F} \) is an Eilenberg-Moore object for a monad \( \mathcal{A}: \mathcal{X} \to \mathcal{X} \) in \( \mathcal{E} \). By Proposition 18.1, the homomorphism \( J_{\mathcal{F}}: \mathcal{F} \to Bim(\mathcal{F}) \) is regular and so, by Proposition 17.15, it preserves Eilenberg-Moore objects. Hence, \((Y, 1_Y) \in Bim(\mathcal{F})\) is an Eilenberg-Moore object for \( \mathcal{A}: (\mathcal{X}, 1_\mathcal{X}) \to (\mathcal{X}, 1_\mathcal{X}) \) in \( Bim(\mathcal{F}) \). But also \((\mathcal{X}, \mathcal{A}) \in Bim(\mathcal{E})\) is an Eilenberg-Moore object for the same monad by Lemma 18.3 and so it is also an Eilenberg-Moore object for it in \( Bim(\mathcal{F}) \). Therefore, there is an equivalence \( Y \simeq (\mathcal{X}, \mathcal{A}) \) in \( Bim(\mathcal{F}) \) and so there is also an equivalence \((Y, B) \simeq ((\mathcal{X}, \mathcal{A}), E)\) in \( Bim(\mathcal{F}) \) for some monad \( E: (\mathcal{X}, \mathcal{A}) \to (\mathcal{X}, \mathcal{A}) \) in \( Bim(\mathcal{E}) \). By Lemma 18.5, \( E \) must have the form \( B'_A: (\mathcal{X}, \mathcal{A}) \to (\mathcal{X}, \mathcal{A}) \) for some monad \( B': \mathcal{X} \to \mathcal{X} \) and monad map \( \pi: \mathcal{A} \to B' \). By Proposition 18.6, an Eilenberg-Moore object for \( E \) is given by \((\mathcal{X}, B')\), which is therefore equivalent to \((Y, B)\). Since \((\mathcal{X}, B') \in Bim(\mathcal{E})\), we have the required essential surjectivity. \( \square \)

Proposition 18.13 implies the known fact that the inclusion \( Bim(\mathcal{E}) \subseteq Bim(Bim(\mathcal{E})) \) is an equivalence \( \mathbb{L}^{7} \). In particular, we have that \( \text{Dist}_V \subseteq Bim(\text{Dist}_V) \) is an equivalence. Recall from Section 11 that we defined the bicategory of analytic functors as the opposite of the bicategory of \( S \)-\( \text{Dist}_V \), by letting \( \text{CatSym}_V \overset{\text{def}}{=} \text{Dist}_V^{op} \) and the bicategory of operads by letting \( \text{Opd}_V \overset{\text{def}}{=} \text{Bim}(S \text{-Mat}_V^{op}) \). Thus, the inclusion \( S \text{-Mat}_V^{op} \subseteq S \text{-Dist}_V^{op} \) induces an inclusion \( \text{Opd}_V \subseteq Bim(\text{CatSym}_V) \).

**Theorem 18.14.** The inclusion \( \text{Opd}_V \subseteq Bim(\text{CatSym}_V) \) is an equivalence.

**Proof.** By duality, it is sufficient to prove that the inclusion \( Bim(S \text{-Mat}_V) \subseteq Bim(S \text{-Dist}_V) \) is an equivalence. In order to do so, we apply Proposition 18.13 and show that every object of \( S \text{-Dist}_V \) is a Kleisli object for a monad in \( S \text{-Mat}_V \). Let \( \mathcal{X} \) be a small \( \mathcal{V} \)-category. Since monads in \( S \text{-Mat}_V \) are operads, we can regard \( \mathcal{X} \) also as a monad in \( S \text{-Mat}_V \). So, we show that \( \mathcal{X} \), viewed
as an object of $S\text{-Dist}_{\mathcal{V}}$, is a Kleisli object for $\mathcal{X}$, viewed as a monad in $S\text{-Mat}_{\mathcal{V}}$. In order to do so, we begin by defining a right $\mathcal{X}$-module with domain $\text{Obj}(\mathcal{X})$,

$$F: \text{Obj}(\mathcal{X}) \to \mathcal{X},$$

in $S\text{-Dist}_{\mathcal{V}}$. The $\mathcal{V}$-functor $F: S(\mathcal{X})^{\text{op}} \otimes \text{Obj}(\mathcal{X}) \to \mathcal{V}$ is defined by letting

$$F[\tau, x'] = \begin{cases} \mathcal{X}[x, x'] & \text{if } \tau = (x) \text{ for some } x \in \mathcal{X} \\ 0 & \text{otherwise.} \end{cases}$$

The right $\mathcal{X}$-action is then defined by the composition operation of $\mathcal{X}$ in the evident way. In order to show that $F: \text{Obj}(\mathcal{X}) \to \mathcal{X}$ is the required Kleisli object, we need to show that, for every small $\mathcal{V}$-category $\mathcal{K}$, the functor

$$S\text{-Dist}_{\mathcal{V}}[\mathcal{X}, \mathcal{K}] \to S\text{-Dist}_{\mathcal{V}}[\text{Obj}(\mathcal{X}), \mathcal{K}]_{\mathcal{X}},$$

defined by composition with $F$, is an equivalence of categories, where $S\text{-Dist}_{\mathcal{V}}[\text{Obj}(\mathcal{X}), \mathcal{K}]_{\mathcal{X}}$ denotes the category of right $\mathcal{X}$-modules with codomain $\mathcal{K}$. In order to see this, observe that to give a right $\mathcal{X}$-action on an $S$-distributor $M: \text{Obj}(\mathcal{X}) \to \mathcal{K}$, i.e. a $\mathcal{V}$-functor $M: S(\mathcal{X})^{\text{op}} \otimes \text{Obj}(\mathcal{X}) \to \mathcal{V}$, is the same thing as extending $M$ to a $\mathcal{V}$-functor $M': S(\mathcal{X})^{\text{op}} \otimes \mathcal{X} \to \mathcal{V}$. □

19. Cartesian closed bicategories of bimodules

We show that if a regular bicategory $\mathcal{E}$ is cartesian closed, then so is the bicategory $\text{Bim}(\mathcal{E})$. We then apply this result to prove that the bicategory of operads $\text{Opd}_{\mathcal{V}}$ is cartesian closed. We begin by considering the cartesian structure.

**Proposition 19.1.** Let $\mathcal{E}$ be a regular bicategory. If $\mathcal{E}$ is cartesian, then so is $\text{Bim}(\mathcal{E})$.

**Proof.** Let us first verify that a terminal object $\top$ in $\mathcal{E}$ remains a terminal object in $\text{Bim}(\mathcal{E})$. For this, we have to show that the category $\mathcal{E}[\mathcal{X}/A, \top]$ is equivalent to the terminal category for every object $\mathcal{X}/A \in \text{Bim}(\mathcal{E})$. By definition, $\mathcal{E}[\mathcal{X}/A, \top] = \mathcal{E}[\mathcal{X}, \top]_{\mathcal{A}}$ is the category of algebras of the monad $\mathcal{E}[A, \top]$ acting on the category $\mathcal{E}[\mathcal{X}, \top]$. But the monad $\mathcal{E}[A, \top]$ is isomorphic to the identity monad, since every morphism in $\mathcal{E}[\mathcal{X}, \top]$ is invertible. It follows that the category $\mathcal{E}[\mathcal{X}/A, \top]_{\mathcal{A}}$ is equivalent to the category $\mathcal{E}[\mathcal{X}, \top]$. Hence the category $\mathcal{E}[\mathcal{X}/A, \top]$ is equivalent to the terminal category.

Let us now show that the category $\text{Bim}(\mathcal{E})$ admits binary cartesian products. The cartesian product homomorphism $(-) \times (-): \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ takes a monad $(B_1, B_2): (\mathcal{Y}_1, \mathcal{Y}_2) \to (\mathcal{Y}_1, \mathcal{Y}_2)$ in $\mathcal{E} \times \mathcal{E}$ to a monad $B_1 \times B_2: \mathcal{Y}_1 \times \mathcal{Y}_2 \to \mathcal{Y}_1 \times \mathcal{Y}_2$ in $\mathcal{E}$. We will prove that

$$\mathcal{Y}_1/B_1 \times \mathcal{Y}_2/B_2 = (\mathcal{Y}_1 \times \mathcal{Y}_2)/(B_1 \times B_2),$$

i.e. that the product of $\mathcal{Y}_1/B_1$ and $\mathcal{Y}_2/B_2$ in $\text{Bim}(\mathcal{E})$ is given by $(\mathcal{Y}_1 \times \mathcal{Y}_2)/(B_1 \times B_2)$. The projections $\pi_1: \mathcal{Y}_1 \times \mathcal{Y}_2 \to \mathcal{Y}_1$ and $\pi_2: \mathcal{Y}_1 \times \mathcal{Y}_2 \to \mathcal{Y}_2$ are components of pseudo-natural transformations. Hence the left hand square in the following diagrams commute up to a canonical isomorphism $\sigma_1: B_1 \circ \pi_1 \cong \pi_1 \circ (B_1 \times B_2)$ and the right hand square up to a canonical isomorphism $\sigma_2: B_2 \circ \pi_2 \cong \pi_2 \circ (B_1 \times B_2)$:

$$\begin{array}{ccc}
\mathcal{Y}_1 & \xrightarrow{\pi_1} & \mathcal{Y}_1 \times \mathcal{Y}_2 \\
B_1 \downarrow & & \downarrow B_1 \times B_2 \\
\mathcal{Y}_1 & \xrightarrow{\pi_1} & \mathcal{Y}_1 \times \mathcal{Y}_2
\end{array} \quad \begin{array}{ccc}
\mathcal{Y}_1 \times \mathcal{Y}_2 & \xrightarrow{\pi_2} & \mathcal{Y}_2 \\
B_2 \downarrow & & \downarrow B_2 \\
\mathcal{Y}_1 \times \mathcal{Y}_2 & \xrightarrow{\pi_2} & \mathcal{Y}_2
\end{array}
$$

Moreover, $(\pi_1, \sigma_1): (\mathcal{Y}_1 \times \mathcal{Y}_2, B_1 \times B_2) \to (\mathcal{Y}_1, B_1)$ and $(\pi_2, \sigma_2): (\mathcal{Y}_1 \times \mathcal{Y}_2, B_1 \times B_2) \to (\mathcal{Y}_2, B_2)$ are monad morphisms by Remark 15.3. It follows by Lemma 15.4 that the morphism

$$\overline{\pi}_1 = \text{def } \pi_1 \circ (B_1 \times B_2): \mathcal{Y} \to \mathcal{Y}_1$$
has the structure of a \((B_1, B_1 \times B_2)\)-bimodule and that the morphism
\[
\tilde{\pi}_2 = \text{def } \pi_2 \circ (B_1 \times B_2) : Y \to Y_2
\]
has the structure of a \((B_2, B_1 \times B_2)\)-bimodule. Let us put \(Y = \text{def } Y_1 \times Y_2\) and \(B = \text{def } B_1 \times B_2\) and show that the object \(Y/B \in \text{Bim}(E)\) equipped with the morphisms \(\tilde{\pi}_1 : Y/B \to Y_1/B_1\) and \(\tilde{\pi}_2 : Y/B \to Y_2/B_2\) is the cartesian product of the objects \(Y_1/B_1\) and \(Y_2/B_2\). For this we have to show that the functor
\[
(\tilde{\pi}_1, \tilde{\pi}_2) \circ_B (-) : \mathcal{E}[X,Y]_{B1 \times B2} \to \mathcal{E}[X, Y_1]_{B1} \times \mathcal{E}[X, Y_2]_{B2}
\]
defined by letting \((\tilde{\pi}_1, \tilde{\pi}_2) \circ_B M = (\tilde{\pi}_1 \circ_B M, \tilde{\pi}_2 \circ_B M)\) is an equivalence of categories for every object \(X/A \in \text{Bim}(E)\). The equivalence of categories
\[
(\pi_1, \pi_2) \circ (-) = (\mathcal{E}[X, \pi_1], \mathcal{E}[X, \pi_2]) : \mathcal{E}[X, Y] \to \mathcal{E}[X, Y_1] \times \mathcal{E}[X, Y_2]
\]
is the component associated to the triple \((X, Y_1, Y_2) \in \mathcal{E}^{op} \times \mathcal{E} \times \mathcal{E}\) of a pseudo-natural transformation. By Remark \([15.3]\), it induces a lax monad morphism
\[
(\mathcal{E}[X, Y_1 \times Y_2], \mathcal{E}[A, B_1 \times B_2]) \to (\mathcal{E}[X, Y_1], \mathcal{E}[A, B_1]) \times (\mathcal{E}[X, Y_2], \mathcal{E}[A, B_2])
\]
in \(\text{Cat}\). This is an equivalence in the 2-category \(\text{Mnd}(\text{Cat})\) since the functor \((\mathcal{E}[X, \pi_1], \mathcal{E}[X, \pi_2])\) is an equivalence. Hence the induced functor
\[
\mathcal{E}[X, Y]_{B1 \times B2} \to \mathcal{E}[X, Y_1]_{B1} \times \mathcal{E}[X, Y_2]_{B2}
\]
which takes a \(M \in \mathcal{E}[X, Y]_{B1 \times B2}\) to \((\pi_1 \circ M, \pi_2 \circ M)\), is an equivalence of categories. But we have
\[
\tilde{\pi}_1 \circ_B M \equiv \pi_1 \circ B \equiv \pi_1 \circ M \quad \text{and} \quad \tilde{\pi}_2 \circ_B M \equiv \pi_2 \circ B \equiv \pi_2 \circ M.
\]
Thus, \((\tilde{\pi}_1 \circ_B M, \tilde{\pi}_2 \circ_B M) \equiv (\pi_1 \circ M, \pi_2 \circ M)\). This shows that the functor in \((19.2)\) is an equivalence of categories. \(\square\)

**Theorem 19.2.** Let \(\mathcal{E}\) be a regular bicategory. If \(\mathcal{E}\) is cartesian closed, then so is \(\text{Bim}(\mathcal{E})\).

**Proof.** The internal hom homomorphism \((-)^{\text{op}} : \mathcal{E}^{op} \times \mathcal{E} \to \mathcal{E}\) takes a monad \((B, C)\) on the object \((Y, Z) \in \mathcal{E}^{op} \times \mathcal{E}\) to a monad \(C^B\) on the object \(Z^Y \in \mathcal{E}\). We will prove that
\[
(Z/C)^{Y/B} = Z^Y/C^B,
\]
i.e. that the exponential of \((Z/C)\) by \(Y/B\) in the bicategory \(\text{Bim}(\mathcal{E})\) is given by \(Z^Y/C^B\). If ev : \(Y^Z \times Y \to Z\) is the evaluation then the adjunction
\[
\theta : \mathcal{E}[X, Z^Y] \to \mathcal{E}[X \times Y, Z]
\]
is defined by letting \(\theta(M) = \text{def } \text{ev} \circ (M \times Y)\) for every \(X \in \mathcal{E}\) and \(M : X \to Z^Y\). If \((A, B, C)\) is a monad on the object \((X, Y, Z) \in \mathcal{E}^{op} \times \mathcal{E} \times \mathcal{E}\), then by Remark \([15.3]\) \(\theta\) induces a lax monad morphism
\[
(\mathcal{E}[X, Z^Y], \mathcal{E}[A, C^B]) \to (\mathcal{E}[X \times Y, Z], \mathcal{E}[A \times B, C]),
\]
since it is a component of a pseudo-natural transformation. This lax monad morphism induces an equivalence between the categories of algebras over these monads
\[
\tilde{\theta} : \mathcal{E}[X, Z^Y]_{A \times C} \to \mathcal{E}[X \times Y, Z]_{A \times B}
\]
since \(\theta\) is an equivalence. By definition, we have a commutative square
\[
\begin{array}{ccc}
\mathcal{E}[X, Z^Y] & \xrightarrow{\tilde{\theta}} & \mathcal{E}[X \times Y, Z] \\
\theta \downarrow & & \downarrow \\
\mathcal{E}[X, Z] & \xrightarrow{\theta} & \mathcal{E}[X \times Y, Z],
\end{array}
\]
in which the vertical arrows are forgetful functors. Note that
\[ \mathcal{E}[X, Z^Y]^{C_B}_A = \text{Bim}(\mathcal{E})[X/A, Z^Y/C^B] \]
and
\[ \mathcal{E}[X \times Y, Z]^{C_B}_{A \times B} = \text{Bim}(\mathcal{E})[X/A \times Y/B, Z/C]. \]

Let us show that the equivalence
\[ \tilde{\theta}: \text{Bim}(\mathcal{E})[X/A, Z^Y/C^B] \to \text{Bim}(\mathcal{E})[X/A \times Y/B, Z/C] \]
is natural in \( X/A \in \text{Bim}(\mathcal{E}) \). But \( \tilde{\theta} \) is natural if and only if it is of the form
\[ \tilde{\theta}(M) = \tilde{\epsilon} \circ_{C^B} (M \times Y/B) \]
for some morphism \( \tilde{\epsilon}: Z^Y/C^B \times Y/B \to Z/C \) in the bicategory \( \text{Bim}(\mathcal{E}) \). If we apply this formula to the case \( M = 1_{Z^Y} = C^B \), we obtain that
\[ \tilde{\epsilon} = \tilde{\theta}(C^B \times B) = \text{ev} \circ (C^B \times B). \]

Conversely, let us define \( \tilde{\epsilon}: Z^Y \times Y \to Z \) by letting
\[ \tilde{\epsilon} = \text{def} \text{ ev} \circ (C^B \times B). \]

Let us show that the morphism \( \tilde{\epsilon} \) so defined has the structure of a \((C, C^B \times B)\)-bimodule. Note that \( C^B \times B = (C^Y \times Y) \circ (Z^B \times B) \) and that the monad \( C^Y \times Y \) commutes with the monad \( Z^B \times B \), since we have
\[ C^Y \circ Z^B = C^B = Z^B \circ C^Y \]
by functoriality. The morphism \( \text{ev}: Z^Y \times Y \to Z \) is component of a pseudo-natural transformation in \( Z \in \mathcal{E} \). By Remark \[15.5\] it defines a lax monad morphism \( (\text{ev}, \alpha): (Z^Y \times Y, C^Y \times Y) \to (Z, C) \) and it follows by Lemma \[15.2\] that the morphism \( \text{ev} \circ (C^Y \times Y) \) has the structure of a \((C, C^Y \times Y)\)-bimodule. Hence the morphism
\[ \tilde{\epsilon} = \text{def} \text{ ev} \circ (C^B \times B) = \text{ev} \circ (C^Y \times Y) \circ (Z^B \times B) \]
has the structure of a \((C, C^B \times B)\)-bimodule, since the monad \( Z^B \times B \) commutes with the monad \( C^Y \times Y \). For every \( M \in \mathcal{E}[X, Z^Y]^{C_B}_A \) we have
\[ \tilde{\theta}(M) = \text{ev} \circ (M \times Y) = \text{ev} \circ (C^B \times B) \circ (C^B \times B) (M \times Y) = \tilde{\epsilon} \circ (C^B \times B) (M \times Y). \]

This shows that the equivalence \( \tilde{\theta} \) is natural and hence that \( (Z/C)^{Y/B} = Z^Y/C^B \). \( \square \)

We conclude with the following theorem, which combines most of the ideas discussed in the paper.

**Theorem 19.3.** The bicategory \( \text{Opd}_V \) is cartesian closed.

**Proof.** Recall that the bicategory \( \text{CatSym}_V \) is cartesian closed by Theorem \[11.0\] and so the associated bicategory of bimodules \( \text{Bim}(\text{CatSym}_V) \) is cartesian closed by Theorem \[19.2\]. The result follows since, as stated in Theorem \[18.14\], \( \text{Opd}_V \) is equivalent to \( \text{Bim}(\text{CatSym}_V) \). \( \square \)
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Appendix A. A compendium of bicategorical definitions

Definition. A bicategory \( \mathcal{E} \) consists of the data in (i)-(vi) subject to the axioms in (vii)-(viii), below.

(i) A class \( \text{Obj}(\mathcal{E}) \) of objects. We will write simply \( X \in \mathcal{E} \) instead of \( X \in \text{Obj}(\mathcal{E}) \).

(ii) For every \( X, Y \in \mathcal{E} \), a category \( \mathcal{E}[X, Y] \). An object of \( \mathcal{E}[X, Y] \) is called a 1-cell or a morphism and denoted \( A : X \to Y \), while a morphism \( f : A \to B \) of \( \mathcal{E}[X, Y] \) is called a 2-cell.

(iii) For every \( X, Y, Z \in \mathcal{E} \), a functor \( (\cdot)^{-1} \circ (\cdot) : \mathcal{E}(Y, Z) \times \mathcal{E}(X, Y) \to \mathcal{E}(X, Z) \)

which associates a morphism \( B \circ A : X \to Z \) to a pair of morphisms \( A : X \to Y, B : Y \to Z \)
and a 2-cell \( f \circ g : B \circ A \to B' \circ A' \) to a pair of 2-cells \( f : A \to A' : X \to Y \) and \( g : B \to B' : Y \to Z \).

(iv) For every \( X \in \mathcal{E} \), a morphism \( 1_X : X \to X \).

(v) A natural isomorphism with components

\[
\alpha_{A,B,C} : (C \circ B) \circ A \to C \circ (B \circ A)
\]

for \( A : X \to Y, B : Y \to Z \) and \( C : Z \to U \).

(vi) Two natural isomorphisms with components

\[
\lambda_A : 1_Y \circ A \to A, \quad \rho_A : A \circ 1_X \to A
\]

for \( A : X \to Y \).
(vii) For all $A: X \to Y$, $B: Y \to Z$, $C: Z \to U$ and $D: U \to V$, the following diagram commutes:

\[ ((D \circ C) \circ B) \circ A \]

\[ (D \circ (C \circ B)) \circ A \]

\[ D \circ ((C \circ B) \circ A) \]

\[ D \circ (C \circ (B \circ A)) \].

(viii) For all $A: X \to Y$ and $B: Y \to Z$, the following diagram commutes:

\[ (B \circ 1_Y) \circ A \]

\[ B \circ (1_Y \circ A) \]

\[ B \circ A \].

**Definition.** A homomorphism $\Phi: \mathcal{E} \to \mathcal{F}$ of bicategories consists of the data in (i)-(iv) subject to axioms (v)-(vi) below.

(i) A function $\Phi: \text{Obj}(\mathcal{E}) \to \text{Obj}(\mathcal{F})$.

(ii) For every $X, Y \in \mathcal{E}$, a functor

\[ \Phi_{X,Y} : \mathcal{E}(X, Y) \to \mathcal{F}(\Phi X, \Phi Y) \].

Below, we write $\Phi$ instead of $\Phi_{X,Y}$.

(iii) A natural isomorphism with components

\[ \phi_{B,A} : \Phi(B \circ A) \to \Phi(B) \circ \Phi(A) \],

for $A: X \to Y$ and $B: Y \to Z$.

(iv) For each $X \in \mathcal{E}$, an isomorphism $\iota_X : \Phi(1_X) \to 1_{\Phi X}$.

(v) For every $A: X \to Y$, $B: Y \to Z$ and $C: Z \to U$, the following diagram commutes:

\[ \Phi((C \circ B) \circ A) \]

\[ \Phi(C \circ B) \circ \Phi(A) \]

\[ \Phi(C) \circ \Phi(B \circ A) \]

\[ (\Phi(C) \circ \Phi(B)) \circ \Phi(A) \]

\[ \Phi(C) \circ (\Phi(B) \circ \Phi(A)) \].
(vi) For every $A: X \to Y$, the following diagrams commute:

![Diagrams](image_url)

**Definition.** A *pseudo-natural transformation* $P: \Phi \to \Psi$ between two homomorphism $E \to F$ consists of the data in (i)-(ii) subject to the axioms (iii)-(iv) below.

(i) For each $X \in E$, a morphism $P(X): \Phi X \to \Psi X$

(ii) For each morphism $A: X \to Y$, an invertible 2-cell

![Diagram](image_url)

(iii) For every $A: X \to Y$ and $B: Y \to Z$, we have

![Diagram](image_url)

(iv) For every $X \in E$, 

![Diagram](image_url)
**Definition.** Let \( P, Q : \Phi \to \Psi \) be pseudo-natural transformations. A modification \( \sigma : P \to Q \) consists of a family of 2-cells \( \sigma_X : P(X) \to Q(X) \) such that

\[
\begin{array}{c}
\Phi X \\
\Phi A \\
\Phi Y
\end{array}
\quad \begin{array}{c}
\bigtriangleup \\
\bigtriangleup \\
\bigtriangleup
\end{array}
\quad \begin{array}{c}
\Psi X \\
\Psi A \\
\Psi Y
\end{array}
\quad \begin{array}{c}
\sigma_X \\
\sigma_Y
\end{array}
\quad \begin{array}{c}
P(X) \\

\end{array}
\quad \begin{array}{c}
P(Y) \\

\end{array}
\quad \begin{array}{c}
P(X) \\

\end{array}
\quad \begin{array}{c}
P(Y) \\

\end{array}
\quad \begin{array}{c}
\Psi X \\
\Psi A \\
\Psi Y
\end{array}
\quad \begin{array}{c}
\bigtriangleup \\
\bigtriangleup \\
\bigtriangleup
\end{array}
\quad \begin{array}{c}
\sigma_Y \\
\sigma_X
\end{array}
\quad \begin{array}{c}
\bigtriangleup \\
\bigtriangleup \\
\bigtriangleup
\end{array}
\quad \begin{array}{c}
\bigtriangleup \\
\bigtriangleup \\
\bigtriangleup
\end{array}
\quad \begin{array}{c}
\bigtriangleup \\
\bigtriangleup \\
\bigtriangleup
\end{array}
\quad \begin{array}{c}
\bigtriangleup \\
\bigtriangleup \\
\bigtriangleup
\end{array}
\end{array}
\]

**Appendix B. Proof of Theorem 18.11**

In order to prove Theorem 18.11 we need to recall some facts about the formal theory of monads. Let us consider a fixed bicategory \( \mathcal{E} \), which for the moment we do not need to suppose being regular. The inclusion homomorphism \( \text{Inc} : \mathcal{E} \to \text{Mnd}(\mathcal{E}) \) takes an object \( X \) to the pair \((X, 1_X)\), where \( 1_X \) is the identity monad on \( X \).

As proved in [61] in the context of 2-categories, the bicategory \( \mathcal{E} \) admits Eilenberg-Moore objects if and only if the inclusion homomorphism \( \text{Inc} : \mathcal{E} \to \text{Mnd}(\mathcal{E}) \) has a right adjoint \( \text{EM} : \text{Mnd}(\mathcal{E}) \to \mathcal{E} \).

It will be useful to give an explicit description of it. The homomorphism \( \text{EM} : \text{Mnd}(\mathcal{E}) \to \mathcal{E} \) takes a monad \((X, A)\) to the Eilenberg-Moore object \( X^A \) and the counit of the adjunction is the monad morphism \( (U^A, \lambda^A) : (X^A, 1_{X^A}) \to (X, A) \), where \( \lambda^A : A \circ U^A \to U^A \) is the left action of \( A \) on \( U^A : X^A \to X \) that is part of the structure of an Eilenberg-Moore object. The image by \( \text{EM} \) of a lax monad morphism \( (M, \phi) : (X, A) \to (Y, B) \) is constructed as follows: the 2-cell

\[
\begin{array}{c}
B \circ M \circ U^A \\
\phi \circ U^A
\end{array}
\quad \begin{array}{c}
M \circ A \circ U^A \\
M \circ \lambda^A
\end{array}
\quad \begin{array}{c}
M \circ U^A
\end{array}
\]

is a left action of the monad \( B \) on \( M \circ U^A \). There is then a morphism \( M' : X^A \to Y^B \) together with an isomorphism of left \( B \)-modules

\[
\begin{array}{c}
X^A \\
M'^M
\end{array}
\quad \begin{array}{c}
U^A \\
\sigma_M
\end{array}
\quad \begin{array}{c}
X \\
M
\end{array}
\quad \begin{array}{c}
Y^B \\
U^B
\end{array}
\]

By definition, \( \text{EM}(M, \phi) = \text{def} M' \). Finally, let us describe the image by \( \text{EM} \) of a monad 2-cell \( \alpha : (M_1, \phi_1) \to (M_2, \phi_2) \). By construction, \( \text{EM}(\alpha) \) is the unique 2-cell \( \alpha' : M'_1 \to M'_2 \) such that
the following cylinder commutes:

Let us now recall the statement of Theorem 18.11 and give its proof.

**Theorem B.1.** If $\mathcal{E}$ is a regular bicategory, then the homomorphism

\[ J_{\mathcal{E}} : \mathcal{E} \to \text{Bim}(\mathcal{E}) \]

exhibits $\text{Bim}(\mathcal{E})$ as an Eilenberg-Moore completion of $\mathcal{E}$ as a regular bicategory.

**Proof.** We will prove that the homomorphism

\[ (-) \circ J_{\mathcal{E}} : \text{HOM}^p(\text{Bim}(\mathcal{E}), \mathcal{F}) \to \text{HOM}^p(\mathcal{E}, \mathcal{F}) \]  
(B.1)

is a surjective equivalence for any Eilenberg-Moore complete regular bicategory $\mathcal{F}$.

Let us first show that the homomorphism is surjective on objects. The homomorphism $J_{\mathcal{F}} : \mathcal{F} \to \text{Bim}(\mathcal{F})$ is an equivalence by Proposition 18.9, since the bicategory $\mathcal{F}$ is Eilenberg-Moore complete. Hence there is a homomorphism $\Theta : \text{Bim}(\mathcal{F}) \to \mathcal{F}$ together with a pseudo-natural transformation $E : \text{Id}_\mathcal{F} \to \Theta \circ J_{\mathcal{F}}$, whose components are equivalences in $\mathcal{F}$. Let us show that the pair $(\Theta, E)$ can be chosen so that the pseudo-natural transformation $\Theta$ is the identity. The homomorphism $\Theta$ is constructed by first choosing an Eilenberg-Moore object $U^A : \mathcal{X}^A \to \mathcal{X}$ in $\mathcal{F}$ for each object $\mathcal{X}/A \in \text{Bim}(\mathcal{F})$ and letting $\Theta(\mathcal{X}/A) = \mathcal{X}^A$, and then by choosing for each morphism $M : \mathcal{X}/A \to \mathcal{Y}/B$ in $\text{Bim}(\mathcal{F})$, a morphism $\Theta(M) : \mathcal{X}^A \to \mathcal{Y}^B$ in $\mathcal{F}$ together with an isomorphism of left $B$-modules $\sigma_M : R^B \circ \Theta(M) \to M \circ_A R^A$

The value of $\Theta$ on 2-cells is determined by these choices afterward. More precisely, if $\alpha : M \to N$ is a 2-cell, then $\Theta(\alpha) : \Theta(M) \to \Theta(N)$ is the unique 2-cell such that the following equality of pasting diagrams holds:
If $A = 1_X$, we can choose $X^A = X$ and $U^A = 1_X$. And if $M: X/1_X \to Y/1_Y$ we can choose $\Theta(M) = M$ and $\sigma_M$ to be the identity 2-cell. It follows from these choices that we have $\Theta(\alpha) = \alpha$ for every $\alpha: M \to N$. Thus, $\Theta \circ J_F = \text{Id}_F$.

Let us now show that for every regular homomorphism $\Phi: E \to F$ can be extended as a regular homomorphism $\Phi'$: $\text{Bim}(E) \to F$. As observed in Remark 13.6, the homomorphism $\Phi: E \to F$ has a natural extension $\text{Bim}(\Phi): \text{Bim}(E) \to \text{Bim}(F)$. Then, the following square commutes by construction

$$
\begin{array}{ccc}
\text{E} & \xrightarrow{\Phi} & \text{F} \\
\text{Bim}(E) & \xrightarrow{\text{Bim}(\Phi)} & \text{Bim}(F).
\end{array}
$$

It is easy to verify that the homomorphism $\text{Bim}(\Phi)$ is proper. Let us put $\Phi' = \Theta \circ \text{Bim}(\Phi)$. Then we have

$$
\Phi' \circ J_E = \Theta \circ \text{Bim}(\Phi) \circ J_E = \Theta \circ J_E \circ \Phi = \text{Id}_F \circ \Phi = \Phi.
$$

We have proved that the homomorphism in (B.1) is surjective on objects. For any regular homomorphism $\Phi: \text{Bim}(E) \to F$, let us define $\Phi|_E = \text{def} \Phi \circ J_E$. For a pair of regular homomorphisms $\Phi, \Psi: \text{Bim}(E) \to F$, we can then define the restriction functor

$$
\text{Res}(\Phi, \Psi): [\Phi, \Psi] \to [\Phi|_E, \Psi|_E]
$$

in the evident way. It remains to show that the functor $\text{Res}(\Phi, \Psi)$ is an equivalence of categories surjective on objects. We will prove that the functor $\text{Res}(\Phi, \Psi)$ is surjective on objects in Lemma 13.7 and that it is full and faithful in Lemma 13.7.

We need to prove a few intermediate results. We first recall the bicategory of 1-cells $E^{(1)}$ of a bicategory $E$. The bicategory $E^{(1)}$ is equipped with a universal pseudo-natural transformation $U: s_0 \to t_0$ between two homomorphisms

$$
\begin{array}{ccc}
E^{(1)} & \xrightarrow{s_0} & E.
\end{array}
$$

The universality of $U$ means that for any bicategory $F$ and any pseudo-natural transformation $M: \Phi \to \Psi$ between a pair of homomorphisms $\Phi, \Psi: F \to E$, there exists a unique homomorphism $\Theta: F \to E^{(1)}$ such that $s_0 \circ \Theta = \Phi$, $t_0 \circ \Theta = \Psi$ and $U \Theta = M$. By construction, an object of $E^{(1)}$ is a 1-cell $M: X_0 \to X_1$ in the bicategory $E$. A 1-cell $M \to N$ of $E^{(1)}$ is a pseudo-commutative square

$$
\begin{array}{ccc}
X_0 & \xrightarrow{E_0} & Y_0 \\
\downarrow & \alpha & \downarrow \ \\
X_1 & \xrightarrow{E_1} & Y_1
\end{array}
$$

defined by a triple $(E_0, E_1, \alpha)$, where $\alpha: E_1 \circ M \to N \circ E_0$ is an isomorphism. Composition of 1-cells is defined by pasting squares. If $(F_0, F_1, \beta): M \to N$ is another pseudo-commutative
square then a 2-cell \((E_0, E_1, \alpha) \to (F_0, F_1, \beta)\) in \(\mathcal{E}(1)\) is a cylinder

![Diagram](image)

defined by a pair of 2-cells \(\gamma_0: E_0 \to F_0, \gamma_1: E_1 \to F_1\) such that the following square commutes,

\[
\begin{array}{ccc}
E_1 \circ M & \xrightarrow{\gamma_1 \circ M} & F_1 \circ M \\
\downarrow \alpha & & \downarrow \beta \\
N \circ E_0 & \xrightarrow{N \circ \gamma_0} & N \circ F_0
\end{array}
\]

Vertical composition of 2-cells in \(\mathcal{E}(1)\) is defined component-wise. Observe that the source and target homomorphisms

\[
\mathcal{E}(1) \xrightarrow{s_0 \circ t_0} \mathcal{E}
\]

are preserving composition strictly. The pseudo-natural transformation \(U: s_0 \to t_0\) takes the object \(M: \mathcal{X}_0 \to \mathcal{X}_1\) of \(\mathcal{E}(1)\) to the 1-cell \(M: s_0(M) \to t_0(M)\) of \(\mathcal{E}\), and it takes the 1-cell \((E_0, E_1, \alpha): M \to N\) to the isomorphism \(\alpha: E_1 \circ M \to N \circ E_0\).

**Lemma B.2.** If \(\mathcal{E}\) is regular, then so is its bicategory of 1-cells \(\mathcal{E}(1)\).

**Proof.** If \(M: \mathcal{X}_0 \to \mathcal{X}_1\) and \(N: \mathcal{Y}_0 \to \mathcal{Y}_1\), then the bicategory \(\mathcal{E}(1)[M,N]\) is defined by a pseudo-pullback square

\[
\begin{array}{ccc}
\mathcal{E}(1)[M,N] & \xrightarrow{\gamma_1 \circ M} & \mathcal{E}[X_0, Y_0] \\
\downarrow \alpha & & \downarrow \beta \\
\mathcal{E}[\mathcal{X}_1, \mathcal{Y}_1] & \xrightarrow{\gamma \circ (-)} & \mathcal{E}[\mathcal{X}_0, \mathcal{Y}_1]
\end{array}
\]

It follows that the bicategory \(\mathcal{E}(1)[M,N]\) admits reflexive coequalisers, since the functors \(N \circ (-)\) and \((-) \circ M\) are preserving reflexive coequalisers. Moreover, the functor

\(\omega, \tau): \mathcal{E}(1)[M,N] \to \mathcal{E}[\mathcal{X}_0, \mathcal{Y}_0] \times \mathcal{E}[\mathcal{X}_1, \mathcal{Y}_1]\)

preserves and reflects reflexive coequalisers. It follows that the composition functor

\((-) \circ (-): \mathcal{E}(1)[N,P] \times \mathcal{E}(1)[M,N] \to \mathcal{E}(1)[M,P]\)

preserves reflexive coequalisers in each variable. Hence the bicategory \(\mathcal{E}(1)\) is regular. 

A monad on an object \(M: \mathcal{X}_0 \to \mathcal{X}_1\) of the bicategory \(\mathcal{E}(1)\) is a 6-tuple

\[
A = (A_0, \mu_0, \eta_0, A_1, \mu_1, \eta_1, \alpha),
\]
where \( A_0 = (A_0, \mu_0, \eta_0) \) is a monad on \( X_0 \), \( A_1 = (A_1, \mu_1, \eta_1) \) is a monad on \( X_1 \) and \( \alpha : A_1 \circ M \to M \circ A_0 \) is an isomorphism satisfying the coherence conditions expressed by the diagrams:

\[
\begin{array}{ccc}
A_1 \circ A_1 \circ M & \xrightarrow{A_1 \circ \alpha} & A_1 \circ M \circ A_0 \\
\alpha & \Downarrow & M \circ \alpha \\
A_1 \circ M & \xrightarrow{\alpha} & M \circ A_0 \\
\end{array}
\]

It follows that we have a morphism \( (M, \alpha) : (X_0, A_0) \to (X_1, A_1) \) in the bicategory \( \text{Mnd}(\mathcal{E}) \), defined in Section \[15\].

**Lemma B.3.** If \( \mathcal{E} \) is Eilenberg-Moore complete, then so is its bicategory of 1-cells \( \mathcal{E}^{(1)} \).

**Proof.** Let us show that every monad in \( \mathcal{E}^{(1)} \) admits an Eilenberg-Moore object. We will use the homomorphism \( \text{EM} : \text{Mnd}(\mathcal{E}) \to \mathcal{E} \). If \( A = (A_0, \mu_0, \eta_0, A_1, \mu_1, \eta_1, \alpha) \) is a monad on an object \( M : X_0 \to X_1 \) of the bicategory \( \mathcal{E}^{(1)} \), then \( (M, \alpha) : (X_0, A_0) \to (X_1, A_1) \) is a morphism in \( \text{Mnd}(\mathcal{E}) \). Let \( U_0 : X_0^A \to X_0 \) be an Eilenberg-Moore object for \( A_0 \) and \( U_1 : X_1^A \to X_1 \) be an Eilenberg-Moore object for \( A_1 \). There is then a morphism \( M^A = \text{def} \ \text{EM}(M, \alpha) : X_0^A \to X_1^A \) in the bicategory \( \mathcal{E} \) together with an isomorphism of left \( A_1 \)-modules \( \sigma : U_1 \circ M^A \to M \circ U_0 \),

\[
\xymatrix{
X_0^A \ar[r]^{M^A} & X_1^A \\
X_0 \ar[u]^{U_0} \ar[r]_{\sigma} & X_1 \ar[u]_{U_1}
}
\]

It is easy to verify that the morphism \( (U_0, U_1, \sigma) : M^A \to M \) in \( \mathcal{E}^{(1)} \) is an Eilenberg-Moore object for the monad \( A \). \[\square\]

**Lemma B.4.** The restriction functor \( \text{Res}(\Phi, \Psi) : [\Phi, \Psi] \to [\Phi \mid \mathcal{E}, \Psi \mid \mathcal{E}] \) is surjective on objects for any pair of regular homomorphisms \( \Phi, \Psi : \text{Bim}(\mathcal{E}) \to \mathcal{F} \).

**Proof.** Let us show that any pseudo-natural transformation \( M : \Phi \Rightarrow \Psi \) admits an extension \( M' : \Phi \Rightarrow \Psi \). The pseudo-natural transformation \( M \) is defining a homomorphism \( M : \mathcal{E} \to \mathcal{F}^{(1)} \). The homomorphism \( M \) is proper, since the homomorphisms \( s_0(M) = \Phi \mathcal{E} \) and \( t_0(M) = \Psi \mathcal{E} \) are proper. It follows by the first part of the proof of Theorem \[14,1\] that the homomorphism \( M \) can be extended as a regular homomorphism \( M' : \text{Bim}(\mathcal{E}) \to \mathcal{F}^{(1)} \). It remains to show that \( M' \) can be chosen so that \( s_0(M') = \Phi \) and \( t_0(M') = \Psi \).

\[
\xymatrix{
\mathcal{E} \ar[r]^{M} & \mathcal{F}^{(1)} \\
\text{Bim}(\mathcal{E}) \ar[u]^{M'} \ar[r]_{(s_0, t_0)} & \mathcal{F} \times \mathcal{F} \ar[u]_{(\Phi, \Psi)}
}
\]

The homomorphism \( M : \mathcal{E} \to \mathcal{F}^{(1)} \) sends \( X \in \mathcal{E} \) to a morphism \( M(X) : \Phi(X) \to \Psi(X) \) in \( \mathcal{F} \) and a monad \( A \) on \( X \) in \( \mathcal{E} \), to a monad \( M(A) \) on \( (M(X)) \) in \( \mathcal{F}^{(1)} \). Thus,

\[
M(A) = (\Phi(A), \Phi(\mu), \Phi(\eta), \Psi(A), \Psi(\mu), \Psi(\eta), \alpha),
\]

where \( \Phi(A) = (\Phi(A), \Phi(\mu), \Phi(\eta)) \) is a monad on \( \Phi(X) \), \( \Psi(A) = (\Psi(A), \Psi(\mu), \Psi(\eta)) \) is a monad on \( \Psi(X) \) and \( \alpha \) is an invertible 2-cell

\[
\alpha : \Psi(A) \circ M(X) \to M(X) \circ \Phi(A)
\]
defining a monad morphism \( (M(X), \alpha): (\Phi(X), \Phi(A)) \to (\Psi(X), \Psi(A)) \),

\[
\begin{array}{ccc}
\Phi(X) & \xrightarrow{\Phi(A)} & \Phi(X) \\
\downarrow M(X) & \Downarrow \alpha & \downarrow M(X) \\
\Psi(X) & \xrightarrow{\Psi(A)} & \Psi(X)
\end{array}
\]

The morphism \( A: X/A \to X/1_X \) in \( \text{Bim}(\mathcal{E}) \) is an Eilenberg-Moore object for the monad \( A: X \to X \) by Lemma 18.3. Hence, the morphism \( \Phi(A): \Phi(X/A) \to \Phi(X) \) is an Eilenberg-Moore object for the monad \( \Phi(A) \) on the object \( \Phi(X) \), since the homomorphism \( \Phi \) is proper. Similarly, the morphism \( \Psi(A): \Psi(X/A) \to \Psi(X) \) is an Eilenberg-Moore object for the monad \( \Psi(A) \), since the homomorphism \( \Psi \) is proper. The homomorphism \( M': \text{Bim}(\mathcal{E}) \to F(1) \) can be constructed by choosing for each \( X/A \in \text{Bim}(\mathcal{E}) \) a morphism \( M'(X/A): \Phi(X/A) \to \Psi(X/A) \) together with an isomorphism of left \( \Psi(A) \)-modules \( \sigma_A: \Psi(A) \circ M(X/A) \to M(X) \circ \Phi(A) \),

\[
\begin{array}{ccc}
\Phi(X/A) & \xrightarrow{M'(X/A)} & \Psi(X/A) \\
\Phi(A) & \Downarrow \psi \sigma_A & \Psi(A) \\
\Phi(X) & \xrightarrow{M(X)} & \Psi(X)
\end{array}
\]

We then have \( s_0(M') = \Phi \) and \( t_0(M') = \Psi \). If \( A = 1_X \), we can choose \( M(X/A) = M(X) \) and \( \sigma_A = \text{Id} \). In which case we have \( M'_{|E} = M \), as required.

We now recall the definition of the bicategory of 2-cells \( \mathcal{E}^{(2)} \) of the bicategory \( \mathcal{E} \). The bicategory \( \mathcal{E}^{(2)} \) is equipped with a universal modification \( \alpha: s_1 \to t_1 \) between a pair of pseudo-natural transformations

\[
\begin{array}{ccc}
\mathcal{E}^{(2)} & \xrightarrow{s_1} & \mathcal{E}^{(1)} \\
\downarrow t_1 & & \downarrow t_1
\end{array}
\]

such that

\[
s_0 s_1 = s_0 t_1 , \quad t_0 s_1 = t_0 t_1 .
\]

By construction, an object of \( \mathcal{E}^{(2)} \) is a 2-cell \( \alpha: M_0 \to M_1 \) of the bicategory \( \mathcal{E} \). We refer to \( \alpha \) as a disk and represent it as follows:

\[
\begin{array}{c}
\alpha \\
\downarrow M_0 \xrightarrow{\alpha} M_1 \\
\downarrow X_0 \xrightarrow{X_0} X_1
\end{array}
\]

If \( \beta: N_0 \to N_1: Y_0 \to Y_1 \) is another disk, then a 1-cell \( (E_0, E_1, \gamma_0, \gamma_1): \alpha \to \beta \) in \( \mathcal{E}^{(2)} \) is a 4-tuple where \( \gamma_0: E_1 \circ M_0 \to N_0 \circ E_0 \) and \( \gamma_1: E_1 \circ M_1 \to N_1 \circ E_0 \) are invertible 2-cells such that the following square commutes,

\[
\begin{array}{ccc}
E_1 \circ M_0 & \xrightarrow{\gamma_0} & N_0 \circ E_0 \\
E_1 \circ M_0 \downarrow E_1 \circ M_1 & & \downarrow \beta \circ E_0 \\
E_1 \circ M_1 & \xrightarrow{\gamma_1} & N_1 \circ E_0
\end{array}
\]
We represent such a 2-cell by a cylinder

Composition of 1-cells in $E^{(2)}$ is defined by pasting cylinders. A 2-cell

$((\sigma_0, \sigma_1): (E_0, E_1, \gamma_0, \gamma_1) \to (F_0, F_1, \delta_0, \delta_1))$

in $E^{(2)}$ is a pair of 2-cells $\varepsilon_0: E_0 \to F_0$ and $\varepsilon_1: E_1 \to F_1$ such that the following two squares commute:

\[
\begin{array}{c}
E_1 \circ M_0 \xrightarrow{\gamma_0} N_0 \circ E_0 \\
\varepsilon_1 \circ M_0 \downarrow \downarrow \downarrow \varepsilon_0 \circ \gamma_0 \\
F_1 \circ M_0 \xrightarrow{\delta_0} N_0 \circ F_0,
\end{array}
\]

\[
\begin{array}{c}
E_1 \circ M_1 \xrightarrow{\gamma_1} N_1 \circ E_0 \\
\varepsilon_1 \circ M_1 \downarrow \downarrow \downarrow \varepsilon_0 \circ \gamma_1 \\
F_1 \circ M_1 \xrightarrow{\delta_1} N_1 \circ F_0.
\end{array}
\]

We represent such a 2-cell as a “deformation” of cylinders

Composition of 2-cells in $E^{(1)}$ is defined component-wise. The source and target homomorphisms

are preserving composition strictly.

**Lemma B.5.** If $E$ is regular, then so is its bicategory of 2-cells $E^{(2)}$

**Proof.** Similar to the proof of Lemma B.2. $\square$

A monad on an object $\gamma: M \to N: X_0 \to X_1$ of the bicategory $E^{(2)}$ is an 8-tuple

$A = (A_0, \mu_0, \eta_0, A_1, \mu_1, \eta_1, \alpha, \beta)$,

where $(A_0, \mu_0, \eta_0)$ is a monad on $X_0$, $(A_1, \mu_1, \eta_1)$ is a monad on $X_1$, $\alpha: A_1 \circ M \to M \circ A_0$ and $\beta: A_1 \circ N \to N \circ A_0$ are invertible 2-cells such that $(M, \alpha)$ and $(N, \beta)$ are monad morphisms.
and the following diagram commutes,

\[
\begin{array}{c}
A_1 \circ M \xrightarrow{\alpha} M \circ A_0 \\
\downarrow \quad \downarrow \\
A_1 \circ N \xrightarrow{\beta} N \circ A_0.
\end{array}
\]

It follows that we have a monad 2-cell \( \gamma: (M, \alpha) \to (N, \beta) \) in \( \text{Mnd}(\mathcal{E}) \).

**Lemma B.6.** If \( \mathcal{E} \) is Eilenberg-Moore complete, then so is its bicategory of 2-cells \( \mathcal{E}^{(2)} \).

**Proof.** Let us show that every monad in \( \mathcal{E}^{(2)} \) admits an Eilenberg-Moore object. We will use the homomorphism \( \text{EM}: \text{Mnd}(\mathcal{E}) \to \mathcal{E} \). Let \( A = (A_0, \mu_0, \eta_0, A_1, \mu_1, \eta_1, \alpha, \beta) \) be a monad on an object \( \gamma: M \to N \) of \( \mathcal{E}^{(2)} \), where \( MN: X_0 \to X_1 \). Then, \( \alpha: A_1 \circ M \to M \circ A_0 \) determines a monad morphism \( (M, \alpha): A_0 \to A_1 \) in \( \text{Mnd}(\mathcal{E}) \), \( \beta: A_1 \circ N \to N \circ A_0 \) determines a monad morphism \( (N, \beta): A_0 \to A_1 \), and \( \gamma: M \to N \) determines a monad 2-cell \( (M, \alpha) \to (N, \beta) \).

Let \( U_0: X_0^0 \to X_0 \) be an Eilenberg-Moore object for \( A_0 \) and \( U_1: X_1^0 \to X_1 \) be an Eilenberg-Moore object for \( A_1 \). There then is a morphism \( M^A: X_0^0 \to X_1^1 \) in the bicategory \( \mathcal{E} \), defined by letting \( M^A \overset{\text{def}}{=} \text{EM}(M, \alpha) \), together with an isomorphism of left \( A_1 \)-modules

\[
\begin{array}{c}
X_0^0 \xrightarrow{U_0} X_1^1 \xrightarrow{\psi \sigma_M} X_1.
\end{array}
\]

Similarly, there is a morphism \( N^A: X_0^0 \to X_1^1 \) in \( \mathcal{E} \) together with an isomorphism of left \( A_1 \)-modules

\[
\begin{array}{c}
X_0^0 \xrightarrow{R_0} X_1^1 \xrightarrow{\psi \sigma_N} X_1.
\end{array}
\]

If we define \( \gamma^A: M^A \to N^A \) by letting \( \gamma^A \overset{\text{def}}{=} \text{EM}(\gamma) \), then the following square commutes

\[
\begin{array}{c}
U_1 \circ M^A \xrightarrow{\sigma M} M \circ R_0 \\
\downarrow \quad \downarrow \\
R_1 \circ \gamma^A \xrightarrow{\gamma \circ R_0} N \circ R_0.
\end{array}
\]
This means that the following cylinder in $\mathcal{E}$ commutes:

$$
\begin{array}{c}
\Phi(X/A) \\
M(X/A)
\end{array}
\begin{array}{c}
\Phi(A) \\
\Psi(X/A)
\end{array}
\begin{array}{c}
\Phi(X) \\
M(X)
\end{array}
\begin{array}{c}
\Phi(A) \\
\Psi(X)
\end{array}
\begin{array}{c}
\Phi(X/A) \\
M(X/A)
\end{array}
\begin{array}{c}
\Phi(A) \\
\Psi(X/A)
\end{array}
\begin{array}{c}
\Phi(X) \\
M(X)
\end{array}
\begin{array}{c}
\Phi(A) \\
\Psi(X)
\end{array}

It is easy to verify that the morphism $(U_0, U_1, \sigma_M, \sigma_N): \gamma^A \to \gamma$ in $\mathcal{E}^{(2)}$ is an Eilenberg-Moore object for the monad $A$. □

**Lemma B.7.** The restriction functor $\text{Res}(\Phi, \Psi): [\Phi, \Psi] \to [\Phi|_{\mathcal{E}}, \Psi|_{\mathcal{E}}]$ is full and faithful for any pair of regular homomorphisms $\Phi, \Psi: \text{Bim}(\mathcal{E}) \to \mathcal{F}$.

**Proof.** If $M, N: \Phi \to \Psi$ are pseudo-natural transformations, let us show that every modification $\alpha: M|_{\mathcal{E}} \to N|_{\mathcal{E}}$ admits a unique extension $\alpha': M \to N$. For every $X/A \in \mathcal{E}$, we have an open cylinder,

$$
\begin{array}{c}
\Phi(X/A) \\
M(X/A)
\end{array}
\begin{array}{c}
\Phi(A) \\
\Psi(X/A)
\end{array}
\begin{array}{c}
\Phi(X) \\
M(X)
\end{array}
\begin{array}{c}
\Phi(A) \\
\Psi(X)
\end{array}
\begin{array}{c}
\Phi(X/A) \\
M(X/A)
\end{array}
\begin{array}{c}
\Phi(A) \\
\Psi(X/A)
\end{array}
\begin{array}{c}
\Phi(X) \\
M(X)
\end{array}
\begin{array}{c}
\Phi(A) \\
\Psi(X)
\end{array}

with back and front faces given by isomorphisms

$$
\sigma_M: \Psi(A) \circ M(X/A) \simeq M(X) \circ \Phi(A), \quad \sigma_N: \Psi(A) \circ N(X/A) \simeq N(X) \circ \Phi(A).
$$

The morphism $\Phi(A): \Phi(X/A) \to \Phi(X)$ is an Eilenberg-Moore object for $\Phi(A)$ and the morphism $\Psi(A): \Psi(X/A) \to \Psi(X)$ is an Eilenberg-Moore object for $\Psi(A)$. The 2-cell $\alpha(X): M(X) \to N(X)$ is a monad 2-cell. It follows that there is a unique 2-cell $\alpha'(X/A): M(X/A) \to N(X/A)$ such that the following square commutes

$$
\begin{array}{c}
\Psi(A) \circ M(X/A) \\
\Psi(A) \circ N(X/A)
\end{array}
\begin{array}{c}
\sigma_M \\
\sigma_N
\end{array}
\begin{array}{c}
M(X) \circ \Phi(A) \\
N(X) \circ \Phi(A)
\end{array}
\begin{array}{c}
\Psi(A) \circ \Phi(X) \\
\Psi(A)
\end{array}
\begin{array}{c}
\Phi(A) \\
\Phi(X)
\end{array}
$$

We obtain a full cylinder

$$
\begin{array}{c}
\Phi(X/A) \\
M(X/A)
\end{array}
\begin{array}{c}
\Phi(A) \\
\Psi(X/A)
\end{array}
\begin{array}{c}
\Phi(X) \\
M(X)
\end{array}
\begin{array}{c}
\Phi(A) \\
\Psi(X)
\end{array}
\begin{array}{c}
\Phi(X/A) \\
M(X/A)
\end{array}
\begin{array}{c}
\Phi(A) \\
\Psi(X/A)
\end{array}
\begin{array}{c}
\Phi(X) \\
M(X)
\end{array}
\begin{array}{c}
\Phi(A) \\
\Psi(X)
\end{array}
$$
This defines the extension $\alpha': M \to N$ of the modification $\alpha: M_{\mathit{cf}} \to N_{\mathit{cf}}$. The uniqueness of $\alpha'$ is clear from the construction. □

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