NONZERO-SUM DIFFERENTIAL GAME OF BACKWARD DOUBLY STOCHASTIC SYSTEMS WITH DELAY AND APPLICATIONS

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Abstract. This paper is concerned with a kind of nonzero-sum differential game of backward doubly stochastic system with delay, in which the state dynamics follows a delayed backward doubly stochastic differential equation (SDE). To deal with the above game problem, it is natural to involve the adjoint equation, which is a kind of anticipated forward doubly SDE. We give the existence and uniqueness of solutions to delayed backward doubly SDE and anticipated forward doubly SDE. We establish a necessary condition in the form of maximum principle with Pontryagin’s type for open-loop Nash equilibrium point of this type of game, and then give a verification theorem which is a sufficient condition for Nash equilibrium point. The theoretical results are applied to study a nonzero-sum differential game of linear-quadratic backward doubly stochastic system with delay.

1. Introduction. Game theory has been pervading the economic theory, and it attracts more and more research attention. It was firstly introduced by Von Neumann and Morgenstern [19]. Nash [12] made the fundamental contribution in non-cooperate games and gave the classical notion of Nash equilibrium point. In recent years, many articles on stochastic differential game problems driven by stochastic differential equations (SDEs) have appeared. Yu and Ji [27] studied the backward linear-quadratic (LQ) nonzero-sum stochastic differential game problem. Wang and
Yu [20, 21] discussed differential game problems for backward stochastic systems. Wang and Shi [22] studied a linear-quadratic game problem for stochastic Volterra integral equations. Shi et al. [17] discussed a linear-quadratic stochastic Stackelberg differential game. Du et al. [8] studied a linear-quadratic mean-field game of backward stochastic differential systems. Wei and Yu [23] discussed a kind of time-inconsistent recursive zero-sum stochastic differential game problem.

In order to provide a probabilistic interpretation for the solutions of a class of semilinear stochastic partial differential equations (SPDEs), Pardoux and Peng [13] introduced the following backward doubly stochastic differential equations (BDSDEs):

\[
y(t) = \zeta + \int_{t}^{T} F(s, y(s), z(s)) \, ds + \int_{t}^{T} G(s, y(s), z(s)) \, dW(s) - \int_{t}^{T} z(s) \, dB(s)
\]

\[0 \leq t \leq T.
\]

Note that the integral with respect to \(\{B(t)\}\) is a “backward Itô integral” and the integral with respect to \(\{W(t)\}\) is a standard forward Itô integral. These two types of integrals are particular cases of the Itô-Skorohod integral (see [13]). Pardoux and Peng [13] proved the existence and uniqueness of a solution for BDSDEs (1). Due to their important significance to SPDEs, the interest for BDSDEs grew a lot (see [1, 2, 10, 11, 15, 24, 29, 30, 31, 32]). It is well known that there may exist so called informal trading such as “insider trading” in the market (for more argumentations about this, see e.g. [3, 4] and references therein). That is, the players at the current time \(t\) possess extra information of the future developing of the market from \(t\) to \(T\) that is represented by \(\mathcal{F}_{B}^{T}_{t,T}\), as well as the accumulated information \(\mathcal{F}_{W}^{T}_{t}\) from 0 to \(t\).

In 2003, Peng and Shi [14] introduced a type of time-symmetric forward-backward stochastic differential equations, i.e., so-called fully coupled forward-backward doubly stochastic differential equations (FBDSDEs):

\[
p(t) = x + \int_{0}^{t} f(s, p(s), y(s), q(s), z(s)) \, ds - \int_{0}^{t} q(s) \, B(s)
\]

\[+ \int_{0}^{t} g(s, p(s), y(s), q(s), z(s)) \, W(s),
\]

\[y(t) = \Phi(y(T)) + \int_{t}^{T} F(s, p(s), y(s), q(s), z(s)) \, ds - \int_{t}^{T} z(s) \, B(s)
\]

\[+ \int_{t}^{T} G(s, p(s), y(s), q(s), z(s)) \, dW(s).
\]

In FBDSDEs (2), the forward equation is “forward” with respect to a standard stochastic integral \(\int_{t}^{T} dW(s)\), as well as “backward” with respect to a backward stochastic integral \(\int_{t}^{T} dB(s)\); the coupled “backward equation” is “forward” under the backward stochastic integral \(\int_{t}^{T} dB(s)\) and “backward” under the forward one. In other words, both the forward equation and the backward one are types of BDSDE (1) with different directions of stochastic integrals. So (2) provides a very general framework of fully coupled forward-backward stochastic systems. Peng and Shi [14] proved the existence and uniqueness of solutions to FBDSDEs (2) with arbitrarily fixed time duration under some monotone assumptions. Zhu et al. [33] and Zhu
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and Shi [34] have extended the results in [14] to different dimension and weakened the monotone assumptions. FBDSDEs can provide more extensive frameworks for stochastic Hamiltonian systems arising in stochastic optimal control problems (see [9, 18, 28, 35]).

However, the study of various natural and social phenomena shows that the future development of many processes depends not only on their present state but also on their previous history. Chen and Wu [5, 6] discussed an optimal control problem for the stochastic system described by stochastic differential equations with delay. Chen and Yu [7] studied the nonzero-sum stochastic differential game of stochastic differential delay equation (SDDE), Shi and Wang [16] discussed the nonzero-sum differential game of backward stochastic differential equation (BSDE) with time-delayed generator. Xu and Han [25] and Xu [26] introduced one kind of delayed doubly stochastic differential equations and discussed the maximum principle for this kind delayed doubly stochastic control systems.

In this paper, we study a kind of nonzero-sum differential game of backward doubly stochastic system with delay. Our work distinguishes itself from the above ones in the following aspects. First, we study the stochastic differential game of delayed backward doubly SDEs, which is a valuable supplement to stochastic differential game problems of backward doubly stochastic systems. Second, we introduce a kind of anticipated forward doubly SDEs, which is used to the adjoint equation of our game, we give the existence and uniqueness of solutions to this kind anticipated forward doubly SDEs. Third, we study an LQ game of backward doubly stochastic system with delay and get an explicit form of the unique equilibrium point. To the best of our knowledge, these kinds of results have not been found in existing works.

This paper is organized as follows. In Section 2, we give notations, statement of the problems and results of delayed backward doubly SDEs and anticipated forward doubly SDEs. In Section 3, we establish a necessary condition in the form of maximum principle with Pontryagin’s type for open-loop Nash equilibrium point of this type of game. In Section 4, we give a verification theorem which is a sufficient condition for Nash equilibrium point. Using the above results, we study a nonzero-sum differential game of linear-quadratic backward doubly stochastic system with delay and get the Nash equilibrium point for the game problem in Section 5. Finally, we end this paper with a concluding remark.

2. Preliminaries.

2.1. Notations and statement of the problems. Let \((\Omega, \mathcal{F}, P)\) be a probability space, and \([0, T]\) be a fixed arbitrarily large time duration throughout this paper. Suppose \(\{W(t); 0 \leq t \leq T\}\) and \(\{B(t); 0 \leq t \leq T\}\) be two mutually independent standard Brownian motions defined on \((\Omega, \mathcal{F}, P)\), with values in \(\mathbb{R}^d\) and \(\mathbb{R}^l\), respectively. Let \(\mathcal{N}\) denote the class of \(P\)-null elements of \(\mathcal{F}\). For each \(t \in [0, T]\), we define \(\mathcal{F}_t = \mathcal{F}^W_t \vee \mathcal{F}^B_{t,T}\), where \(\mathcal{F}^W_t = \mathcal{N} \vee \sigma \{W(r) - W(0); 0 \leq r \leq t\}\), \(\mathcal{F}^B_{t,T} = \mathcal{N} \vee \sigma \{B(T) - B(r); t \leq r \leq T\}\). Note that the collection \(\{\mathcal{F}_t, t \in [0, T]\}\) is neither increasing nor decreasing, and it does not produce a filtration. \(\mathbb{E}\) denotes the expectation on \((\Omega, \mathcal{F}, P)\). \(\mathbb{E}^{\mathcal{F}_t} := \mathbb{E}[\cdot | \mathcal{F}_t]\) denotes the conditional expectation under \(\mathcal{F}_t\). We use the usual inner product \(\langle \cdot, \cdot \rangle\) and Euclidean norm \(|\cdot|\) in \(\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^{m \times l}\) and \(\mathbb{R}^{n \times d}\). The notation \(\top\) appearing in the superscripts denotes the transpose of a matrix. All the equalities and inequalities mentioned in this paper are in the
sense of $dt \times dP$ almost surely on $[0, T] \times \Omega$. We introduce the following notations:

$$L^2(\mathcal{F}_T; \mathbb{R}^n) = \{ \xi : \xi \text{ is an } \mathbb{R}^n\text{-valued, } \mathcal{F}_T\text{-measurable random variable} \}$$

s.t. $E[|\xi|^2 < \infty]$,

$$L^2_2(s, r; \mathbb{R}^n) = \{ v(t), s \leq t \leq r : v(t) \text{ is an } \mathbb{R}^n\text{-valued, } \mathcal{F}_t \text{-measurable process} \}$$

s.t. $E \int_s^r |v(t)|^2 dt < \infty$.

We consider a controlled delayed backward doubly SDE

$$
\begin{cases}
-dy(t) = F(t, y(t), y_0(t), z(t), z_0(t), v_1(t), v_2(t)) dt - z(t) \; dW(t) \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad + G(t, y(t), y_0(t), z(t), z_0(t), v_1(t), v_2(t)) \; dB(t), \; t \in [0, T],
\end{cases}
$$

(3)

where $(y(\cdot), z(\cdot)) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}$ is the state process pair, $0 < \delta < T$ is a constant time delay parameter, and $y_0(t) = y(t - \delta), z_0(t) = z(t - \delta)$. Here, $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \times \mathbb{R}^{n \times d} \times \mathbb{R}^k \times \mathbb{R}^k \rightarrow \mathbb{R}^n, G : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \times \mathbb{R}^{n \times d} \times \mathbb{R}^k \times \mathbb{R}^k \rightarrow \mathbb{R}^{n \times l}$ are given functions, $\zeta$ is a given $\mathcal{F}_T$-measurable random variable and $(\phi(\cdot), \psi(\cdot)) \in L^2_2((-\delta, 0; \mathbb{R}^n) \times L^2_2((-\delta, 0; \mathbb{R}^{n \times d})$ is the initial path of $(y, z)$.

Let $U_i$ be a nonempty convex subset of $\mathbb{R}^3$ and $v_i(\cdot)$ be the control process of player $i, i = 1, 2$. We denote by $U_i$ the set of $U_i$-valued control processes $v_i(0, T; \mathbb{R}^k)$ and it is called the admissible control set for player $i, i = 1, 2$. Each element of $U_i$ is called an (open-loop) admissible control for player $i, i = 1, 2$. And $\mathcal{U} = U_1 \times U_2$ is called the set of admissible controls for the two players.

We assume that

(H1) $F$ and $G$ are continuously differentiable in $(y, y_0, z, z_0, v_1, v_2)$. Moreover, the partial derivatives $F_{y_0}, F_{y_0 z}, F_{z_0}, F_{z_0 z}, F_{v_1}, F_{v_2}$ of $F$ with respect to $(y, y_0, z, z_0, v_1, v_2)$ are uniformly bounded, and the partial derivatives $G_{y_0}, G_{y_0 z}, G_{z_0}, G_{z_0 z}, G_{v_1}, G_{v_2}$ of $G$ with respect to $(y, y_0, z, z_0, v_1, v_2)$ are uniformly bounded.

The nonzero-sum stochastic differential game for the two players is that, besides ensuring to achieve the joint pre-given goal $\zeta$ at the terminal time $T$, the two players have their own benefits, which are described by the cost functional

$$J_i(v_1(\cdot), v_2(\cdot)) = E \left\{ \int_0^T l_i(t, y(t), y_0(t), z(t), z_0(t), v_1(t), v_2(t)) dt + \Phi_i(y(0)) \right\},$$

(4)

where $l_i : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \times \mathbb{R}^{n \times d} \times \mathbb{R}^k \times \mathbb{R}^k \rightarrow \mathbb{R}$ and $\Phi_i : \mathbb{R}^n \rightarrow \mathbb{R}$ are given functions, $i = 1, 2$, satisfying the condition

$$E \left\{ \int_0^T l_i(t, y(t), y_0(t), z(t), z_0(t), v_1(t), v_2(t)) dt + \Phi_i(y(0)) \right\} < \infty, \; i = 1, 2.$$  

(5)

We also assume

(H2) $l_i$ is continuously differentiable in $(y, y_0, z, z_0, v_1, v_2)$, its partial derivatives are continuous in $(y, y_0, z, z_0, v_1, v_2)$ and bounded by $c(1 + |y| + |y_0| + |z| + |z_0| + |v_1| + |v_2|)$. Moreover, $\Phi_i(y)$ is continuously differentiable in $y$ and $\Phi_{t y_i}(y)$ is bounded by $c(1 + |y|)$.

Suppose each player hopes to minimize her/his cost functional $J_i(v_1(\cdot), v_2(\cdot))$ by selecting an appropriate admissible control $v_i(\cdot)(i = 1, 2)$. Then the problem is to
find a pair of admissible controls \((u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2\) such that
\[
\begin{align*}
J_1(u_1(\cdot), u_2(\cdot)) &= \min_{v_1(\cdot) \in \mathcal{U}_1} J_1(v_1(\cdot), u_2(\cdot)), \\
J_2(u_1(\cdot), u_2(\cdot)) &= \min_{v_2(\cdot) \in \mathcal{U}_2} J_2(u_1(\cdot), v_2(\cdot)).
\end{align*}
\]
We call the problem above a backward non-zero sum stochastic differential game, where the word backward means that the game system is described by a delayed backward doubly SDE. For simplicity, we denote it by Problem (BNZ). If we can find an admissible controls \(u(\cdot) = (u_1(\cdot), u_2(\cdot))\) satisfying (6), then we call it an equilibrium point of Problem (BNZ) and denote the corresponding state trajectory by \((y(\cdot), z(\cdot)) = (y^u(\cdot), z^u(\cdot))\).

2.2. Results of delayed backward doubly SDEs. We consider a controlled delayed backward doubly SDE
\[
\begin{align*}
-dy(t) &= F(t, y(t), y_\delta(t), z(t), z_\delta(t)) \, dt + z(t) \, dB(t) \\
&\quad + G(t, y(t), y_\delta(t), z(t), z_\delta(t)) \, dB(t), \ t \in [0, T], \\
y(T) &= \zeta, \ y(t) = \phi(t), \ z(t) = \psi(t), \ t \in [-\delta, 0],
\end{align*}
\]
where \((y(\cdot), z(\cdot)) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}\) is the state process pair, \(0 < \delta < T\) is a constant time delay parameter, and \(y_\delta(t) = y(t - \delta), \ z_\delta(t) = z(t - \delta)\). Here, \(F : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \times \mathbb{R}^{n \times d} \to \mathbb{R}^n, \ G : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \times \mathbb{R}^{n \times d} \to \mathbb{R}^n\) are given functions, \(\zeta\) is a given \(\mathcal{F}_T\)-measurable random variable and \((\phi(\cdot), \psi(\cdot)) \in L^2_F(\Omega, \mathbb{R}^n) \times L^2_F(\Omega, \mathbb{R}^{n \times d})\) is the initial path of \((y, z)\).

We assume that
(A1) (a) \(\zeta \in L^2(\mathcal{F}_T; \mathbb{R}^n)\); 
(b) there exists a constant \(L > 0\), for all \(t \in [0, T]\), and for all nonnegative and integrable function \(\theta(\cdot)\), such that
\[
\int_t^T \theta(s - \delta) ds \leq \int_t^T L\theta(s) ds;
\]
(c) there exist constants \(C > 0\) and \(0 < \alpha < \frac{1}{1 + T}\), for any \((\omega, t) \in \Omega \times [0, T], y, y_\delta, y', y'_\delta, z, z_\delta, z', z'_\delta \in \mathbb{R}^n, z_\delta, z_\delta' \in \mathbb{R}^{n \times d}\), the functions \(F\) and \(G\) satisfy
\[
\begin{align*}
|F(t, y, y_\delta, z, z_\delta) - F(t, y', y'_\delta, z', z'_\delta)|^2 &\leq C(|y - y'|^2 + |y_\delta - y'_\delta|^2 + |z - z'|^2 + |z_\delta - z'_\delta|^2) ; \\
|G(t, y, y_\delta, z, z_\delta) - G(t, y', y'_\delta, z', z'_\delta)|^2 &\leq C(|y - y'|^2 + |y_\delta - y'_\delta|^2 + \alpha(|z - z'|^2 + |z_\delta - z'_\delta|^2)).
\end{align*}
\]
Then we have the following existence and uniqueness result of the delayed backward doubly SDE (7).

**Theorem 2.1.** Under the assumption (A1), for sufficiently small time delay \(\delta\), delayed backward doubly SDE (7) has a unique solution \((y(\cdot), z(\cdot)) \in L^2_F(\Omega, \mathbb{R}^n) \times L^2_F(\Omega, \mathbb{R}^{n \times d})\).

**Proof.** We first consider the simple backward doubly stochastic differential equation:
\[
\begin{align*}
-dy(t) &= F(t) \, dt + G(t) \, dB(t) - z(t) \, dB(t), \ t \in [0, T], \\
y(T) &= \zeta,
\end{align*}
\]
similar to Pardoux and Peng [13], by using the Itô’s martingale representation theorem and contraction mapping principle, there exists a unique pair \((y, z)\), which solves (8).

Let us define a space \(B^2 = L^2_T (\mathbb{R}^n) \times L^2_T (\mathbb{R}^n)\) and introduce a norm in it:
\[
\| (y(\cdot), z(\cdot)) \|_\beta = \left\{ \mathbb{E} \int_{-\delta}^{T} e^{-\beta t} (\gamma|y(t)|^2 + |z(t)|^2) dt \right\}^{1/2}, \beta > 0.
\]

Let
\[
\left\{ \begin{array}{ll}
-dy(t) = & F(t, Y(t), Y_\delta(t), Z(t), Z_\delta(t)) \ dt - z(t) \, \overleftarrow{d} \ W(t) \\
+ & G(t, Y(t), Y_\delta(t), Z(t), Z_\delta(t)) \, \overrightarrow{d} \ B(t), \ t \in [0, T], \\
y(T) = & \zeta, \ y(t) = \phi(t), \ z(t) = \psi(t), \ t \in [-\delta, 0].
\end{array} \right.
\]

Define a mapping \(h : B^2 \to B^2\), such that \(h[(Y(\cdot), Z(\cdot))] = (y(\cdot), z(\cdot)).\) So, if we can prove that \(h\) is a contraction mapping under the norm \(\| \cdot \|_\beta\), then the desired result can be obtained by the fixed-point theorem. For two arbitrary elements \((Y(\cdot), Z(\cdot)), (Y'(\cdot), Z'(\cdot)) \in B^2\), let
\[
(y(\cdot), z(\cdot)) = h[(Y(\cdot), Z(\cdot))], \ (y'(\cdot), z'(\cdot)) = h[(Y'(\cdot), Z'(\cdot))].
\]

Denote their difference by
\[
(y(\cdot), z(\cdot)) - (y'(\cdot), z'(\cdot)) = \hat{y}(\cdot), \hat{z}(\cdot), \ (Y(\cdot) - Y'(\cdot), Z(\cdot) - Z'(\cdot)), \ (\hat{Y}(\cdot), \hat{Z}(\cdot)) = (Y(\cdot) - Y'(\cdot), Z(\cdot) - Z'(\cdot)),
\]
and
\[
\hat{F}(t) = F(t, Y(t), Y_\delta(t), Z(t), Z_\delta(t)) - F(t, Y'(t), Y'_\delta(t), Z'(t), Z'_\delta(t)),
\]
\[
\hat{G}(t) = G(t, Y(t), Y_\delta(t), Z(t), Z_\delta(t)) - G(t, Y'(t), Y'_\delta(t), Z'(t), Z'_\delta(t)).
\]

Then \((\hat{y}(\cdot), \hat{z}(\cdot))\) satisfies equation
\[
\left\{ \begin{array}{ll}
-d\hat{y}(t) = & \hat{F}(t) \ dt + \hat{G}(t) \overrightarrow{d} \ B(t) - \hat{z}(t) \overleftarrow{d} \ W(t), \ t \in [0, T], \\
\hat{y}(T) = & 0, \ \hat{y}(t) = 0, \ \hat{z}(t) = 0, \ t \in [-\delta, 0].
\end{array} \right.
\]

For any \(\beta > 0\), applying Itô’s formula to \(e^{-\beta t} |\hat{y}(t)|^2\), we have
\[
\mathbb{E} \int_0^T e^{-\beta t} (\beta |\hat{y}(t)|^2 + |\hat{z}(t)|^2) dt \leq 2 \mathbb{E} \int_0^T e^{-\beta t} \hat{F}(t) \hat{y}(t) dt + \mathbb{E} \int_0^T e^{-\beta t} |\hat{G}(t)|^2 dt.
\]

We can obtain the following formula by basic inequality
\[
2 \mathbb{E} \int_0^T e^{-\beta t} |\hat{y}(t)|^2 \cdot \hat{F}(t) dt \leq \frac{\beta}{2} \mathbb{E} \int_0^T e^{-\beta t} |\hat{y}(t)|^2 dt + \frac{2}{\beta} \mathbb{E} \int_0^T e^{-\beta t} |\hat{F}(t)|^2 dt.
\]

From (A1), we have
\[
\mathbb{E} \int_0^T e^{-\beta t} |\hat{F}(t)|^2 dt \leq \mathbb{E} \int_0^T e^{-\beta t} C(|\hat{Y}(t)|^2 + |\hat{Y}_\delta(t)|^2 + |\hat{Z}(t)|^2 + |\hat{Z}_\delta(t)|^2) dt,
\]
and there exists a constant \(L > 0\), such that
\[
\int_0^T |\hat{Y}_\delta(t)|^2 dt \leq \int_0^T L|\hat{Y}(t)|^2 dt,
\]
\[
\int_0^T |\hat{Z}_\delta(t)|^2 dt \leq \int_0^T L|\hat{Z}(t)|^2 dt.
\]
Then the inequality (11) can be written as
\[
E \int_0^T e^{-\beta t} |\hat{F}(t)|^2 dt \leq E \int_0^T e^{-\beta t} C(|\hat{Y}(t)|^2 + L|\hat{Y}(t)|^2 + |\hat{Z}(t)|^2 + L|\hat{Z}(t)|^2) dt
\]
\[
\leq C(1 + L)E \int_0^T e^{-\beta t}|\hat{Y}(t)|^2 dt + C(1 + L)E \int_0^T e^{-\beta t}|\hat{Z}(t)|^2 dt.
\]
Similarly, we have
\[
E \int_0^T e^{-\beta t} |\tilde{G}(t)|^2 dt \leq E \int_0^T e^{-\beta t} C(|\hat{Y}(t)|^2 + |\hat{Z}(t)|^2) dt
\]
\[
\leq E \int_0^T e^{-\beta t}[C(|\hat{Y}(t)|^2 + L|\hat{Y}(t)|^2) + \alpha(|\hat{Z}(t)|^2 + L|\hat{Z}(t)|^2)] dt
\]
\[
\leq C(1 + L)E \int_0^T e^{-\beta t}|\hat{Y}(t)|^2 dt + \alpha(1 + L)E \int_0^T e^{-\beta t}|\hat{Z}(t)|^2 dt.
\]
Then we have
\[
E \int_0^T e^{-\beta t} \left( \frac{\beta}{2} |\tilde{g}(t)|^2 + |\tilde{z}(t)|^2 \right) dt
\]
\[
\leq C(1 + L)(\frac{2}{\beta} + 1)E \int_0^T e^{-\beta t}|\hat{Y}(t)|^2 dt + (1 + L)(\frac{2C}{\beta} + \alpha)E \int_0^T e^{-\beta t}|\hat{Z}(t)|^2 dt.
\]
(12)
We select
\[
\beta = \frac{4C(1 + L)}{1 - \alpha(1 + L)} + \frac{2C}{\alpha} + 4, \ \gamma = 2 + \frac{C}{\alpha}.
\]
Note that \(|\tilde{g}(t)|^2 = 0, |\tilde{z}(t)|^2 = 0, \) when \(t \in [-\delta, 0].\) Then the inequality (12) can be written as
\[
E \int_{-\delta}^T e^{-\beta t}(\gamma |\tilde{g}(t)|^2 + |\tilde{z}(t)|^2) dt \leq E \int_{-\delta}^T e^{-\beta t}(\gamma |\hat{Y}(t)|^2 + |\hat{Z}(t)|^2) dt.
\]
(13)
We obtain that \(h\) is a contraction mapping. Then it follows from the fixed-point theorem that the delayed backward doubly SDE (7) has a unique solution.

Using a similar method, we can get the following result.

**Theorem 2.2.** Let the assumption (A1) hold, for sufficiently small time delay \(\delta,\) we have the following estimate of the solution of delayed backward doubly SDE (7):

\[
E \left[ \sup_{0 \leq t \leq T} |y(t)|^2 + \int_0^T |z(t)|^2 dt \right]
\]
\[
\leq C E \left[ |\zeta|^2 + \int_0^T |F(t, 0, 0, 0, 0)|^2 dt + \int_0^T |G(t, 0, 0, 0, 0)|^2 dt \right],
\]
with some constant \(C > 0.\)

**2.3. Results of anticipated forward doubly SDEs.** We consider an anticipated forward doubly SDE
\[
\begin{cases}
dp(t) = f(t, p(t), p_{\delta+}(t), q(t), q_{\delta+}(t)) dt - q(t) \int \frac{d}{dt} B(t) \\
+ g(t, p(t), p_{\delta+}(t), q(t), q_{\delta+}(t)) \frac{d}{dt} W(t), \ t \in [0, T], \\
p(0) = x, \ p(t) = \xi(t), \ q(t) = \eta(t), \ t \in [T, T + \delta],
\end{cases}
\]
(14)
Proof. Let the assumption forward doubly SDE (14). Define a mapping $h$ (similar to Pardoux and Peng [13] and Peng and Shi [14], there exists a unique pair can prove that Denote their difference by $R(A2) (a)$ there exists a constant $L > 0$ and for all nonnegative and integrable function $\rho(\cdot)$, such that \[
abla \int_0^T \rho(s + \delta) ds \leq \int_0^{T+\delta} L \rho(s) ds; \]
(b) there exist constants $C > 0$ and $0 < \alpha < \frac{1}{1+\beta}$, for any $(\omega, t) \in \Omega \times [0, T]$, $p, p_{\delta_{+}}', p_{\delta_{+}'} \in \mathbb{R}^n, q, q_{\delta_{+}}, q', q_{\delta_{+}'} \in \mathbb{R}^{n \times l}$, the functions $f$ and $g$ satisfy \[
abla |f(t, p, p_{\delta_{+}}, q, q_{\delta_{+}}) - f(t, p', p_{\delta_{+}'}', q', q_{\delta_{+}'})|^2 \leq C(|p - p'|^2 + E^{\mathbb{P}}(|p_{\delta_{+}} - p_{\delta_{+}'}|^2) + |q - q'|^2 + E^{\mathbb{P}}(|q_{\delta_{+}} - q_{\delta_{+}'}|^2)); \]
\[
abla |g(t, p, p_{\delta_{+}}, q, q_{\delta_{+}}) - g(t, p', p_{\delta_{+}'}', q', q_{\delta_{+}'})|^2 \leq C(|p - p'|^2 + E^{\mathbb{P}}(|p_{\delta_{+}} - p_{\delta_{+}'}|^2)) + \alpha(|q - q'|^2 + E^{\mathbb{P}}(|q_{\delta_{+}} - q_{\delta_{+}'}|^2)). \]
Then we have the following existence and uniqueness result of the anticipated forward doubly SDE (14).

Theorem 2.3. Let the assumption (A2) hold, for sufficiently small anticipated time $\delta$, then anticipated forward doubly SDE (14) has a unique solution $(p(\cdot), q(\cdot)) \in L^2_T (0, T + \delta; \mathbb{R}^n) \times L^2_T (0, T + \delta; \mathbb{R}^{n \times l})$. 

Proof. We first consider the forward doubly stochastic differential equation: \[
\begin{cases} 
    dp(t) = f(t) dt + g(t) dW(t) - q(t) dB(t), \quad t \in [0, T], \\
    p(0) = x.
\end{cases}
\]
similar to Pardoux and Peng [13] and Peng and Shi [14], there exists a unique pair $(p, q)$, which solves (15).

Let us define a space $D^2 = L^2_T (0, T + \delta; \mathbb{R}^n) \times L^2_T (0, T + \delta; \mathbb{R}^{n \times l})$ and introduce a norm in it: \[
\|\begin{pmatrix} p(\cdot), q(\cdot) \end{pmatrix}\|_\beta = \left\{ \mathbb{E} \int_0^{T+\delta} e^{\beta t}(\gamma|p(t)|^2 + |q(t)|^2) dt \right\}^{1/2}, \quad \beta > 0.
\]
Let \[
\begin{cases} 
    dp(t) = f(t, P(t), P_{\delta_{+}}(t), Q(t), Q_{\delta_{+}}(t)) dt - q(t) dB(t) \\
    + g(t, P(t), P_{\delta_{+}}(t), Q(t), Q_{\delta_{+}}(t)) dW(t), \quad t \in [0, T], \\
    p(0) = x, \quad p(t) = \xi(t), \quad q(t) = \eta(t), \quad t \in [T, T + \delta].
\end{cases}
\]
Define a mapping $h : D^2 \to D^2$, such that $h((P(\cdot), Q(\cdot))) = (p(\cdot), q(\cdot))$. So, if we can prove that $h$ is a contraction mapping under the norm $\| \cdot \|_\beta$, then the desired result can be obtained by the fixed-point theorem. For two arbitrary elements $(P'(\cdot), Q'(\cdot)), (P''(\cdot), Q''(\cdot)) \in D^2$, let \[
(p(\cdot), q(\cdot)) = h((P(\cdot), Q(\cdot))), \quad (p'(\cdot), q'(\cdot)) = h((P'(\cdot), Q'(\cdot))), \quad (p''(\cdot), q''(\cdot)) = h((P''(\cdot), Q''(\cdot))).
\]
Denote their difference by \[
(p(\cdot), q(\cdot)) = (p(\cdot) - p'(\cdot), q(\cdot) - q'(\cdot)), \quad (\tilde{P}(\cdot), \tilde{Q}(\cdot)) = (P(\cdot) - P'(\cdot), P(\cdot) - P''(\cdot)).
\]
and
\[
\hat{f}(t) = f(t, P(t), P_{\delta}(t), Q(t), Q_{\delta}(t)) - f(t, P'(t), P'_{\delta}(t), Q'(t), Q'_{\delta}(t)),
\]
\[
\hat{g}(t) = g(t, P(t), P_{\delta}(t), Q(t), Q_{\delta}(t)) - g(t, P'(t), P'_{\delta}(t), Q'(t), Q'_{\delta}(t)).
\]
Then \((\hat{p}(t), \hat{q}(t))\) satisfies equation
\[
\begin{cases}
\frac{d\hat{p}(t)}{dt} = \hat{f}(t) dt + \hat{g}(t) \left( \int_t^T dW(t) - \hat{q}(t) \int_t^T dB(t) \right), & t \in [0, T], \\
\hat{p}(0) = 0, \hat{p}(t) = 0, \hat{q}(t) = 0, & t \in [T, T + \delta].
\end{cases}
\]
(17)

For any \(\beta > 0\), applying Itô’s formula to \(e^{\beta t}|\hat{p}(t)|^2\), we have
\[
\mathbb{E} \int_0^T e^{\beta t} |\hat{p}(t)|^2 + |\hat{q}(t)|^2 dt \leq 2\mathbb{E} \int_0^T e^{\beta t} \hat{p}(t) \cdot \hat{f}(t) dt + \mathbb{E} \int_0^T e^{\beta t} |\hat{q}(t)|^2 dt.
\]
We can obtain the following formula by basic inequality
\[
2\mathbb{E} \int_0^T e^{\beta t} \hat{p}(t) \cdot \hat{f}(t) dt \leq \frac{\beta}{2} \mathbb{E} \int_0^T e^{\beta t} |\hat{p}(t)|^2 dt + \frac{2}{\beta} \mathbb{E} \int_0^T e^{\beta t} |\hat{f}(t)|^2 dt.
\]
From (A2), we have
\[
\mathbb{E} \int_0^T e^{\beta t} |\hat{f}(t)|^2 dt \leq \mathbb{E} \int_0^T e^{\beta t} C(|\hat{P}(t)|^2 + |\hat{P}_{\delta}(t)|^2 + |\hat{Q}(t)|^2 + |\hat{Q}_{\delta}(t)|^2) dt,
\]
and there exists a constant \(L > 0\), such that
\[
\int_0^T |\hat{P}_{\delta}(t)|^2 dt \leq \int_0^{T+\delta} L |\hat{P}(t)|^2 dt,
\]
\[
\int_0^T |\hat{Q}_{\delta}(t)|^2 dt \leq \int_0^{T+\delta} L |\hat{Q}(t)|^2 dt.
\]
Then the inequality (18) can be written as
\[
\mathbb{E} \int_0^T e^{\beta t} |\hat{f}(t)|^2 dt \leq \mathbb{E} \int_0^{T+\delta} e^{\beta t} C(|\hat{P}(t)|^2 + L |\hat{P}(t)|^2 + |\hat{Q}(t)|^2 + L |\hat{Q}(t)|^2) dt
\]
\[
\leq C(1 + L) \mathbb{E} \int_0^{T+\delta} e^{\beta t} |\hat{P}(t)|^2 dt + C(1 + L) \mathbb{E} \int_0^{T+\delta} e^{\beta t} |\hat{Q}(t)|^2 dt.
\]
Similarly, we have
\[
\mathbb{E} \int_0^T e^{\beta t} |\hat{g}(t)|^2 dt \leq \mathbb{E} \int_0^T e^{\beta t} C(|\hat{P}(t)|^2 + |\hat{P}_{\delta}(t)|^2 + \alpha(|\hat{Q}(t)|^2 + |\hat{Q}_{\delta}(t)|^2)) dt
\]
\[
\leq \mathbb{E} \int_0^{T+\delta} e^{\beta t} C(|\hat{P}(t)|^2 + L |\hat{P}(t)|^2) + \alpha(|\hat{Q}(t)|^2 + L |\hat{Q}(t)|^2)) dt
\]
\[
\leq C(1 + L) \mathbb{E} \int_0^{T+\delta} e^{\beta t} |\hat{P}(t)|^2 dt + \alpha(1 + L) \mathbb{E} \int_0^{T+\delta} e^{\beta t} |\hat{Q}(t)|^2 dt.
\]
Then we have
\[
\mathbb{E} \int_0^T e^{\beta t} \left( \frac{\beta}{2} |\hat{p}(t)|^2 + |\hat{q}(t)|^2 \right) dt
\]
\[
\leq C(1 + L) \left( \frac{2}{\beta} + 1 \right) \mathbb{E} \int_0^{T+\delta} e^{\beta t} |\hat{P}(t)|^2 dt + (1 + L) \left( \frac{2C}{\beta} + \alpha \right) \mathbb{E} \int_0^{T+\delta} e^{\beta t} |\hat{Q}(t)|^2 dt.
\]
(19)
We select
\[ \beta = \frac{4C(1 + L)}{1 - \alpha(1 + L)} + \frac{2C}{\alpha} + 4, \quad \gamma = 2 + \frac{C}{\alpha}. \]
Note that \(|\hat{p}(t)|^2 = 0, |\hat{q}(t)|^2 = 0\), when \(t \in [T, T + \delta]\). Then the inequality (19) can be written as
\[ E \int_0^{T+\delta} e^{\beta t} (\gamma |\hat{p}(t)|^2 + |\hat{q}(t)|^2) dt \leq E \int_0^{T+\delta} e^{\beta t} (\gamma |\hat{P}(t)|^2 + |\hat{Q}(t)|^2) dt. \] (20)
We obtain that \(h\) is a a contraction mapping. Then it follows from the fixed-point theorem that the anticipated forward doubly SDE (14) has a unique solution. \(\square\)

Using a similar method, we can get the following result.

**Theorem 2.4.** Under the assumption (A2), for sufficiently small anticipated time \(\delta\), we have the following estimate of the solution of anticipated forward doubly SDE (14):
\[ E \left[ \sup_{0 \leq t \leq T} |p(t)|^2 + \int_0^T |q(t)|^2 dt \right] \leq C E \left[ \int_0^T |f(t, 0, 0, 0, 0)|^2 dt + \int_0^T |g(t, 0, 0, 0, 0)|^2 dt \right], \]
with some constant \(C > 0\).

3. **Necessary maximum principle.** For the convex admissible control set, the classical way to derive necessary optimality conditions is to use the convex perturbation method. Let \(u(\cdot) = (u_1(\cdot), u_2(\cdot))\) be an equilibrium point of Problem (BNZ) and \((y(\cdot), z(\cdot))\) be the corresponding optimal trajectory. Let \((v_1(\cdot), v_2(\cdot))\) be such that \((u_1(\cdot) + v_1(\cdot), u_2(\cdot) + v_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2\). Since \(\mathcal{U}_1\) and \(\mathcal{U}_2\) are convex, for any \(0 \leq \rho \leq 1\), \((u_1^\rho(\cdot), u_2^\rho(\cdot)) = (u_1(\cdot) + \rho v_1(\cdot), u_1(\cdot) + \rho v_2(\cdot))\) is also in \(\mathcal{U}_1 \times \mathcal{U}_2\). As illustrated before, we denote by \((y^{u_1^\rho}(\cdot), z^{u_1^\rho}(\cdot))\) and \((y^{u_2^\rho}(\cdot), z^{u_2^\rho}(\cdot))\) the corresponding state trajectories of game system (3) along with the controls \((u_1^\rho(\cdot), u_2^\rho(\cdot))\) and \((u_1(\cdot), u_2(\cdot))\).

For convenience, we use the following notations in this paper
\[
\begin{align*}
\varphi(t) &= \varphi(t, y(t), y_0(t), z(t), z_0(t), u_1(t), u_2(t)), \\
\varphi^\rho(t) &= \varphi(t, y^\rho(t), y_0^\rho(t), z^\rho(t), z_0^\rho(t), v_1(t), v_2(t)), \\
\varphi^{u_1^\rho}(t) &= \varphi(t, y^\rho(t), y_0^\rho(t), z^\rho(t), z_0^\rho(t), u_1^\rho(t), u_2(t)), \\
\varphi^{u_2^\rho}(t) &= \varphi(t, y^\rho(t), y_0^\rho(t), z^\rho(t), z_0^\rho(t), u_1(t), u_2^\rho(t)),
\end{align*}
\]
where \(\varphi\) denotes one of \(F, G, l, i = 1, 2\).

We introduce the following variational equation:
\[
\begin{aligned}
- &dy_1^i(t) = [F_p(t)y_1^i(t) + F_{y_1}(t)y_1^i(t) + F_z(t)z_1^i(t) + F_{z_1}(t)z_1^i(t) + F_{v_1}(t)v_1(t)]dt \\
&+ [G_p(t)y_1^i(t) + G_{y_1}(t)y_1^i(t) + G_z(t)z_1^i(t) + G_{z_1}(t)z_1^i(t) + G_{v_1}(t)v_1(t)]dt \\
&+ G_{v_1}(t)v_1(t) \int_0^t \beta B(t - z_1^i(t)) dW(t), \quad t \in [0, T],
\end{aligned}
\]
\[
y_1^i(T) = 0, \quad y_1^i(t) = 0, \quad z_1^i(t) = 0, \quad t \in [-\delta, 0], \quad (i = 1, 2).
\] (21)
By (H1), it is easy to know that (21) admits a unique solution \((y_1^1(t), z_1^1(t)) \in L^2_{\mathcal{F}}(-\delta, T; \mathbb{R}^n) \times L^2_{\mathcal{F}}(-\delta, T; \mathbb{R}^{n + d}), i = 1, 2\).
For $t \in [0, T]$, $\rho > 0$, we set

$$
\tilde{y}_i^\rho(t) = \frac{y_{u_i}(t) - y(t)}{\rho} - y_1(t),
$$

$$
\tilde{z}_i^\rho(t) = \frac{z_{u_i}(t) - z(t)}{\rho} - z_1(t), \ (i = 1, 2).
$$

We have the following

**Lemma 3.1.** Let assumptions (H1) and (H2) hold. Then, for $i = 1, 2$,

$$
\lim_{\rho \to 0} \sup_{0 \leq t \leq T} \mathbb{E}|\tilde{y}_i^\rho(t)|^2 = 0, \quad (22)
$$

$$
\lim_{\rho \to 0} \int_0^T |\tilde{z}_i^\rho(t)|^2 dt = 0. \quad (23)
$$

**Proof.** For $i = 1$, we have

$$
\begin{cases}
-\rho \tilde{y}_1^\rho(t) = & \int_0^t \left(F_{u_1}(t) - F(t)\right) - F_y(t) y_1(t) - F_z(t) z_1(t) - F_{u_1}(t) g_1^{1, \delta}(t) \\
& - \rho \int_0^t \left(G_{u_1}(t) - D(t)\right) - G_y(t) y_1(t) - G_z(t) z_1(t) - G_{u_1}(t) g_1^{1, \delta}(t) \\
& + \frac{1}{\rho} \left(G_{u_1}(t) - D(t)\right) - G_y(t) y_1(t) - G_z(t) z_1(t) - G_{u_1}(t) g_1^{1, \delta}(t) \\
& - \rho \int_0^t \left(G_{u_1}(t) - D(t)\right) - G_y(t) y_1(t) - G_z(t) z_1(t) - G_{u_1}(t) g_1^{1, \delta}(t) \\
& \tilde{y}_1^\rho(T) = 0, \quad \tilde{y}_1^\rho(t) = 0, \quad \tilde{z}_1^\rho(t) = 0, \quad t \in [-\delta, 0],
\end{cases}
$$

or

$$
\begin{cases}
-\rho \tilde{y}_1^\rho(t) = & \left[A_{u_1}(t) \tilde{y}_1(t) + A_{u_1}(t) \tilde{y}_1(t) + B_{u_1}(t) \tilde{y}_1(t) + B_{u_1}(t) \tilde{y}_1(t) + \Xi_{u_1}(t)\right] dt \\
& + \left[C_{u_1}(t) \tilde{y}_1(t) + C_{u_1}(t) \tilde{y}_1(t) + D_{u_1}(t) \tilde{y}_1(t) + D_{u_1}(t) \tilde{y}_1(t) + \Xi_{u_1}(t)\right] dt \\
& \tilde{y}_1^\rho(T) = 0, \quad \tilde{y}_1^\rho(t) = 0, \quad \tilde{z}_1^\rho(t) = 0, \quad t \in [-\delta, 0],
\end{cases}
$$

where we denote

$$
\begin{align*}
&A_{u_1}(t) = \int_0^1 F_y(t, y(t) + \lambda(y_{u_1}(t) - y(t)), z_{u_1}(t), u_1(t), u_2(t)) d\lambda, \\
&B_{u_1}(t) = \int_0^1 F_z(t, y(t) + \lambda(z_{u_1}(t) - z(t)), y_{u_1}(t), z_{u_1}(t), u_1(t), u_2(t)) d\lambda, \\
&\tilde{A}_{u_1}(t) = \int_0^1 F_y(t, y(t) + \lambda(y_{u_1}(t) - y(t)), z_{u_1}(t), u_1(t), u_2(t)) d\lambda, \\
&\tilde{B}_{u_1}(t) = \int_0^1 F_z(t, y(t) + \lambda(z_{u_1}(t) - z(t)), y_{u_1}(t), z_{u_1}(t), u_1(t), u_2(t)) d\lambda, \\
&C_{u_1}(t) = \int_0^1 G_y(t, y(t) + \lambda(y_{u_1}(t) - y(t)), z_{u_1}(t), y_{u_1}(t), z_{u_1}(t), u_1(t), u_2(t)) d\lambda, \\
&D_{u_1}(t) = \int_0^1 G_z(t, y(t) + \lambda(z_{u_1}(t) - z(t)), y_{u_1}(t), z_{u_1}(t), u_1(t), u_2(t)) d\lambda, \\
&\tilde{C}_{u_1}(t) = \int_0^1 G_y(t, y(t) + \lambda(y_{u_1}(t) - y(t)), z_{u_1}(t), y_{u_1}(t), z_{u_1}(t), u_1(t), u_2(t)) d\lambda,
\end{align*}
$$
Lemma 3.2. From this and Lemma 3.1, we have the following variational inequality.

For

\[ D_1(t) = \int_0^1 G_{x_2}(t, y(t), z(t), y_0(t), z_0(t) + \lambda(z_3^w(t) - z_3(t)), u_3^w(t), u_2(t))d\lambda, \]

\[ E(t) = \int_0^1 [F_{x_1}(t, u(t)) + F_{x_2}(t, u(t)) - F_{x_1}(t)]v_1(t)d\lambda \]

Similarly, we have

\[ A_1(t) = [A_1^w(t) - F_{x_1}(t)]y_1^w(t) + [A_1^w(t) - F_{x_1}(t)]z_1^w(t) \]

\[ \Xi_1(t) = \int_0^1 [G_{x_1}(t, y(t), z(t), y_0(t), z_0(t) + \rho\lambda v_1(t), u_2(t)) - F_{v_1}(t)]\lambda_1(t)d\lambda \]

\[ \Xi_2(t) = \int_0^1 [G_{x_1}(t, y(t), z(t), y_0(t), z_0(t) + \rho\lambda v_1(t), u_2(t)) - G_{v_1}(t)]v_1(t)d\lambda \]

Applying Itô’s formula to \( \Xi_1(t) \) on \([0, t] \), by virtue of (H1) we get

\[ E(\Xi_1(t)) + \frac{1}{2} \mathbb{E} \int_0^t |\Xi_1(s)|^2 ds \]

\[ \leq C_0 \mathbb{E} \int_0^t |\Xi_1(s)|^2 ds + \frac{1}{2} \mathbb{E} \int_0^t |\Xi_2(s)|^2 ds + C_1(\mathbb{E} \int_0^t (|\Xi_1(s)|^2 + |\Xi_2(s)|^2) ds). \]

By Gronwall’s inequality, we easily obtain the desired result. Similarly, we can show that the conclusion holds for \( i = 2 \).

Since \((u_1(\cdot), u_2(\cdot))\) is an equilibrium point of Problem (BNZ), then

\[ \rho^{-1} [J_1(u_1^w(\cdot), u_2(\cdot)) - J_1(u_2(\cdot), u_2(\cdot))] \geq 0, \quad (24) \]

\[ \rho^{-1} [J_2(u_1(\cdot), u_2^w(\cdot)) - J_2(u_1(\cdot), u_2(\cdot))] \geq 0. \quad (25) \]

From this and Lemma 3.1, we have the following variational inequality.

Lemma 3.2. Let the assumptions (H1) and (H2) hold. Then

\[ \mathbb{E} \int_0^T \left[ l_{1y}(t)y_1^w(t) + l_{1y}(t)y_1(t) + l_{1z}(t)z_1^w(t) + l_{1z}(t)z_1(t) + l_{1v}(t)v_1(t) \right] dt \]

\[ + \mathbb{E}[\Phi_1(y(0))]y_1^w(0) \geq 0, \quad (26) \]

Proof. For \( i = 1 \), from (22), we derive

\[ \rho^{-1} \mathbb{E}[\Phi_1(y_1^w(0)) - \Phi_1(y(0))] \]

\[ = \rho^{-1} \mathbb{E} \int_0^1 \Phi_1(y(0) + \lambda(y_1^w(0) - y(0)))(y_1^w(0) - y(0))d\lambda \]

\[ \to \mathbb{E}[\Phi_1(y(0))]y_1^w(0), \quad \rho \to 0. \]

Similarly, we have

\[ \rho^{-1} \left\{ \mathbb{E} \int_0^T [l_{1y}^w(t) - l_1(t)]dt \right\} \]

\[ \to \mathbb{E} \int_0^T [l_{1y}(t)y_1^w(t) + l_{1z}(t)z_1^w(t) + l_{1y}(t)y_1(t) + l_{1z}(t)z_1(t) + l_{1v}(t)v_1(t)] dt, \quad \rho \to 0. \]

Let \( \rho \to 0 \) in (24), then it follows that, for \( i = 1 \), (26) holds. Similarly, we can show that the conclusion holds for \( i = 2 \).
We define the Hamiltonian function \( H_i : [0,T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \times \mathbb{R}^{n \times d} \times \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \rightarrow \mathbb{R}^i \), \( i = 1, 2 \) as follows:

\[
H_i(t, y, y_\delta, z, z_\delta, v_1, v_2, p_i, q_i) \\
= - \langle F(t, y, y_\delta, z, z_\delta, v_1, v_2), p_i \rangle - \langle G(t, y, y_\delta, z, z_\delta, v_1, v_2), q_i \rangle \\
+ l_i(t, y, y_\delta, z, z_\delta, v_1, v_2), \quad i = 1, 2.
\]

(27)

Remark 1. We can see the above adjoint equation (28) is a linear anticipated forward doubly SDE. The existence and uniqueness of the solution for the equation (28) can be guaranteed by Theorem 2.3.

Starting from the variational inequality (26), we can now state necessary optimality conditions.

**Theorem 3.3** (Necessary maximum principle). Suppose (H1) and (H2) hold, and \((u_1(\cdot), u_2(\cdot))\) is an equilibrium point of Problem (BNZ) and \((y(\cdot), z(\cdot))\) is the corresponding state trajectory. Then we have

\[
\langle H_{1v_1}(t, \Theta(t), u_1(t), u_2(t), p_1(t), q_1(t)), v_1 - u_1(t) \rangle \geq 0,
\]

\[
\langle H_{2v_2}(t, \Theta(t), u_1(t), u_2(t), p_2(t), q_2(t)), v_2 - u_2(t) \rangle \geq 0,
\]

hold for any \((v_1, v_2) \in U_1 \times U_2, a.e.a.s.,\) where \(\Theta(t) = (y(t), y_\delta(t), z(t), z_\delta(t))\) and \((p_1(\cdot), q_1(\cdot))\)(i = 1, 2) is the solution of the adjoint equation (28).

Proof. For \( i = 1 \). Applying Itô’s formula to \(\langle p_1(t), y_1^i(t)\rangle\), we obtain

\[
E\langle \Phi(y(0)), y_1^i(0) \rangle \\
= - E \int_0^T \langle p_1(t), F_y(t)y_1^i(t) + F_z(t)z_1^i(t) + F_{y\delta}(t)y_1^i(t) \rangle \\
+ F_{z\delta}(t)z_1^i(t) + F_{v1}(t)v_1(t) \rangle dt \\
+ E \int_0^T \langle F_y^\top(t)p_1(t) + G_y^\top(t)q_1(t) - l_{1y}(t) \rangle \\
+ E \int_0^T \langle F_z^\top(t)p_1(t) + G_z^\top(t)q_1(t) - l_{1z}(t) \rangle dt \\
- E \int_0^T \langle q_1(t), G_y(t)y_1^i(t) + G_z(t)z_1^i(t) + G_{y\delta}(t)y_1^i(t) + G_{z\delta}(t)z_1^i(t) + G_{v1}(t)v_1(t) \rangle dt \\
+ E \int_0^T \langle F_z^\top(t)p_1(t) + G_z^\top(t)q_1(t) - l_{1z}(t) \rangle dt.
\]
Noticing the initial and terminal conditions, we derive
\[
E \int_0^T [(p_1(t), F_{y_1}(t)y_{1\delta}(t)) - \langle E^{F_1}[F_{y_1}^T(t + \delta)p_1(t + \delta)], y_{1}(t) \rangle]dt
= E \int_0^T \langle p_1(t), F_{y_1}(t)y_{1\delta}(t) \rangle dt - E \int_0^{T + \delta} \langle F_{y_1}^T(t)p_1(t), y_{1\delta}(t) \rangle dt
= E \int_0^T \langle p_1(t), F_{y_1}(t)y_{1\delta}(t) \rangle dt - E \int_T^{T + \delta} \langle F_{y_1}^T(t)p_1(t), y_{1\delta}(t) \rangle dt
= 0.
\]
Similarly, we have
\[
E \int_0^T [(q_1(t), G_{y_1}(t)y_{1\delta}(t)) - \langle E^{F_1}[G_{y_1}^T(t + \delta)q_1(t + \delta)], y_{1}(t) \rangle]dt = 0,
E \int_0^T [(q_1(t), G_{y_1}(t)y_{1\delta}(t)) - \langle E^{F_1}[G_{y_1}^T(t + \delta)q_1(t + \delta)], y_{1}(t) \rangle]dt = 0,
E \int_0^T [(q_1(t), G_{y_1}(t)y_{1\delta}(t)) - \langle E^{F_1}[G_{y_1}^T(t + \delta)q_1(t + \delta)], y_{1}(t) \rangle]dt = 0.
\]
Then, we get
\[
E \int_0^T \left[ l_{1y}(t)y_{1}(t) + l_{1z}(t)z_{1}(t) + l_{1y_1}(t)y_{1\delta}(t) + l_{1z_1}(t)z_{1\delta}(t) + l_{1v_1}(t)v_1(t) \right] dt
+ E\langle \Phi_{y_1}(y(0)), y_{1}(0) \rangle
= E \int_0^T \langle -F_{v_1}^T(t)p_1(t) - G_{v_1}^T(t)q_1(t) + l_{1v_1}(t), v_1(t) \rangle dt.
\]
From Lemma 3.2, we have
\[
E \int_0^T \langle H_{1v_1}(t, \Theta(t), u_1(t), u_2(t), p_1(t), q_1(t)), v_1(t) \rangle dt \geq 0.
\]
This implies that
\[
E \langle H_{1v_1}(t, \Theta(t), u_1(t), u_2(t), p_1(t), q_1(t)), v_1 - u_1(t) \rangle \geq 0, \forall v_1 \in U_1.
\] (29)

Now, let \( F \) be an arbitrary element of the \( \sigma \)-algebra \( \mathcal{F}_t \). And set
\[
w(t) = v1_F + u(t)1_{t}F.
\]
It is obvious that \( w(t) \) is an admissible control.

Applying the inequality (29) with \( w(t) \), we get
\[
E[1_F(H_{1v_1}(t, \Theta(t), u_1(t), u_2(t), p_1(t), q_1(t)), v_1 - u_1(t))] \geq 0, \forall F \in \mathcal{F}_t,
\]
which implies that
\[
E[\langle H_{1v_1}(t, \Theta(t), u_1(t), u_2(t), p_1(t), q_1(t)), v_1 - u_1(t) \rangle|\mathcal{F}_t] \geq 0.
\]
The quantity inside the conditional expectation is \( \mathcal{F}_t \)-measurable, and thus the result follows immediately. Proceeding in the same way as the above proof, we can show that the other inequality holds for any \( v_2 \in U_2 \). \( \square \)
4. **Sufficient maximum principle.** In this section, we investigate a sufficient maximum principle for Problem (BNZ). Let \( (y(t), z(t), u_1(t), u_2(t)) \) be a quintuple satisfying (3) and suppose there exists a solution \((p_i(t), q_i(t))\) of the corresponding adjoint forward SDE (28). We assume that

(H3) For \( i = 1, 2, \) for all \( t \in [0,T] \), \( H_i(t, y, y_t, z, z_t, v_1, v_2, p_i, q_i) \) is convex in \((y, y_t, z, z_t, v_1, v_2)\), and \( \Phi_i(y) \) is convex in \( y \).

**Theorem 4.1 (Sufficient maximum principle).** Let assumptions (H1)-(H3) hold, and the following conditions hold

\[
H_i(t, \Theta(t), u_1(t), u_2(t), p_1(t), q_1(t)) = \min_{v_1 \in U_1} H_i(t, \Theta(t), v_1(t), u_2(t), p_1(t), q_1(t)),
\]

(30)

\[
H_2(t, \Theta(t), u_1(t), u_2(t), p_2(t), q_2(t)) = \min_{v_2 \in U_2} H_2(t, \Theta(t), u_1(t), v_2(t), p_2(t), q_2(t)).
\]

(31)

Then \((u_1(\cdot), u_2(\cdot))\) is an equilibrium point of Problem (BNZ).

**Proof.** For any \( v_1(\cdot) \in U_1 \), we consider

\[
J_1(v_1(\cdot), u_2(\cdot)) = J_1(u_1(\cdot), u_2(\cdot))
\]

\[
= \mathbb{E} \int_0^T \left[ l_1(t, \Theta(t), v_1(t), u_2(t)) - l_1(t, \Theta(t), u_1(t), u_2(t)) \right] dt
\]

\[
+ \mathbb{E} \left[ \Phi_1(y^{v_1}(0)) - \Phi_1(y(0)) \right].
\]

(32)

Now applying Itô’s formula to \( \langle p_1(t), y^{v_1}(t) - y(t) \rangle \) on \([0,T]\), we get

\[
\mathbb{E} \langle \Phi_1(y(0)), y^{v_1}(0) - y(0) \rangle
\]

\[
= - \mathbb{E} \int_0^T \langle y^{v_1}(t) - y(t), H_{1y}(t, \Theta(t), u_1(t), u_2(t), p_1(t), q_1(t)) \rangle dt
\]

\[
+ \mathbb{E} \int_0^T \langle F(t, \Theta(t), v_1(t), u_2(t)) - F(t, \Theta(t), u_1(t), u_2(t)) \rangle dt
\]

\[
- \mathbb{E} \int_0^T \langle z^{v_1}(t) - z(t), H_{1z}(t, \Theta(t), u_1(t), u_2(t), p_1(t), q_1(t)) \rangle dt
\]

\[
+ \mathbb{E} \int_0^T \langle G(t, \Theta(t), v_1(t), u_2(t)) - G(t, \Theta(t), u_1(t), u_2(t)) \rangle dt.
\]

Since \( \Phi_1 \) is convex, we have

\[
\Phi_1(y^{v_1}(0)) - \Phi_1(y(0)) \geq \langle \Phi_1(y(0)), y^{v_1}(0) - y(0) \rangle.
\]
Then, we have

\[ J_1(v_1(\cdot), u_2(\cdot)) - J_1(u_1(\cdot), u_2(\cdot)) \]

\[ \geq \mathbb{E} \int_0^T \left[ H_1(t, \Theta(t), v_1(t), u_2(t), p_1(t), q_1(t)) - H_1(t, \Theta(t), u_1(t), u_2(t), p_1(t), q_1(t)) \right] dt \]

\[ - \mathbb{E} \int_0^T (y^{v_1}(t) - y(t), H_{1y_1}(t, \Theta(t), u_1(t), u_2(t), p_1(t), q_1(t))) dt \]

\[ + \mathbb{E} \mathbb{P} \left[ H_{1y_1}(t + \delta, \Theta(t + \delta), u_1(t + \delta), u_2(t + \delta), p_1(t + \delta), q_1(t + \delta)) \right] dt \]

\[ - \mathbb{E} \int_0^T (z^{v_1}(t) - z(t), H_{1z_1}(t, \Theta(t), u_1(t), u_2(t), p_1(t), q_1(t))) dt \]

\[ + \mathbb{E} \mathbb{P} \left[ H_{1z_1}(t + \delta, \Theta(t + \delta), u_1(t + \delta), u_2(t + \delta), p_1(t + \delta), q_1(t + \delta)) \right] dt. \]

By the virtue of convexity of \( H_1 \) with respect to \( (y, z, z_\delta, v_1, v_2) \), we obtain

\[ H_1(t, \Theta(t), v_1(t), u_2(t), p_1(t), q_1(t)) - H_1(t, \Theta(t), u_1(t), u_2(t), p_1(t), q_1(t)) \]

\[ \geq (y^{v_1}(t) - y(t), H_{1y_1}(t, \Theta(t), u_1(t), u_2(t), p_1(t), q_1(t))) \]

\[ + (z^{v_1}(t) - z(t), H_{1z_1}(t, \Theta(t), u_1(t), u_2(t), p_1(t), q_1(t))) \]

\[ + (y^{v_1}(t) - y(t), H_{1y_1}(t, \Theta(t), u_1(t), u_2(t), p_1(t), q_1(t))) \]

\[ + (z^{v_1}(t) - z(t), H_{1z_1}(t, \Theta(t), u_1(t), u_2(t), p_1(t), q_1(t))) \]

\[ + (v_1(t) - u_1(t), H_{1v_1}(t, \Theta(t), u_1(t), u_2(t), p_1(t), q_1(t))). \]

Noticing the fact that

\[ \mathbb{E} \int_0^T (y^{v_1}(t) - y(t), H_{1y_1}(t, \Theta(t), u_1(t), u_2(t), p_1(t), q_1(t))) dt \]

\[ - \mathbb{E} \int_0^T (y^{v_1}(t) - y(t), \mathbb{E} \mathbb{P} \left[ H_{1y_1}(t + \delta, \Theta(t + \delta), u_1(t + \delta), u_2(t + \delta), p_1(t + \delta), q_1(t + \delta)) \right] dt) \]

\[ = \mathbb{E} \int_0^T (y^{v_1}(t) - y(t), H_{1y_1}(t, \Theta(t), u_1(t), u_2(t), p_1(t), q_1(t))) dt \]

\[ - \mathbb{E} \int_0^{T + \delta} (y^{v_1}(t) - y(t), H_{1y_1}(t, \Theta(t), u_1(t), u_2(t), p_1(t), q_1(t))) dt \]

\[ = \mathbb{E} \int_0^T (y^{v_1}(t) - y(t), H_{1y_1}(t, \Theta(t), u_1(t), u_2(t), p_1(t), q_1(t))) dt \]

\[ - \mathbb{E} \int_0^{T + \delta} (y^{v_1}(t) - y(t), H_{1y_1}(t, \Theta(t), u_1(t), u_2(t), p_1(t), q_1(t))) dt \]

\[ = 0. \]

Similarly, we have

\[ \mathbb{E} \int_0^T (z^{v_1}(t) - z(t), H_{1z_1}(t, \Theta(t), u_1(t), u_2(t), p_1(t), q_1(t))) dt \]

\[ - \mathbb{E} \int_0^T (z^{v_1}(t) - z(t), \mathbb{E} \mathbb{P} \left[ H_{1z_1}(t + \delta, \Theta(t + \delta), u_1(t + \delta), u_2(t + \delta), p_1(t + \delta), q_1(t + \delta)) \right] dt) \]

\[ = 0. \]
Then, we get
\[ J_1(v_1(\cdot), u_2(\cdot)) - J_1(u_1(\cdot), u_2(\cdot)) \geq \mathbb{E} \int_0^T \langle H_{v_1}(t, \Theta(t), u_1(t), u_2(t), p_1(t), q_1(t)), v_1(t) - u_1(t) \rangle dt. \]
Finally, by the necessary optimality condition (30), we obtain
\[ J_1(v_1(\cdot), u_2(\cdot)) - J_1(u_1(\cdot), u_2(\cdot)) \geq 0. \]
Then it implies that
\[ J_1(u_1(\cdot), u_2(\cdot)) = \min_{v_1(\cdot) \in U_1} J_1(v_1(\cdot), u_2(\cdot)). \]
In the same way
\[ J_2(u_1(\cdot), u_2(\cdot)) = \min_{v_2(\cdot) \in U_2} J_2(u_1(\cdot), v_2(\cdot)). \]
Hence, we draw the desired conclusion. \( \square \)

5. Application in LQ Case with delay. In this section, we work out an example of the nonzero-sum differential game problem of LQ doubly stochastic systems with delay to illustrate the application of the theoretical results. For notational simplification, we assume \( d = l = 1 \). The control system is
\[
\begin{cases}
-dy(t) = [Ay(t) + \bar{A}(t)y_\delta(t) + B(t)z(t) + \bar{B}(t)z_\delta(t) + E_1(t)v_1(t) + E_2(t)v_2(t)]dt \\
\quad + [C(t)y(t) + \bar{C}(t)y_\delta(t) + D(t)z(t) + \bar{D}(t)z_\delta(t)]t \bar{d}B(t) \\
- z(t) = \bar{d}W(t), \ t \in [0, T], \\
y(T) = \zeta, \ y(t) = \phi(t), \ z(t) = \psi(t), \ t \in [-\delta, 0],
\end{cases}
\]
where \((\phi(\cdot), \psi(\cdot)) \in L_2^\infty(-\delta, T; \mathbb{R}^n) \times L_2^\infty(-\delta, T; \mathbb{R}^n)\) is the initial path of \((y, z)\), \(A(\cdot), \bar{A}(\cdot), B(\cdot), \bar{B}(\cdot), C(\cdot), \bar{C}(\cdot), D(\cdot), \bar{D}(\cdot)\) are \( n \times n \) bounded matrices, \(v^1(t)\) and \(v^2(t), \ t \in [0, T], \) are two admissible control processes, i.e., \( F_t \)-adapted square-integrable processes taking values in \( \mathbb{R}^k \). \( E_1(\cdot) \) and \( E_2(\cdot) \) are \( n \times k \) bounded matrices. We denote \( J_1(v(\cdot)) \) and \( J_2(v(\cdot)) \), \( v(\cdot) = (v_1(\cdot), v_2(\cdot)) \), which are the cost functionals corresponding to the players 1 and 2
\[
J_i(v(\cdot)) = \frac{1}{2} \mathbb{E} \left\{ \int_0^T [(M_i(t)y(t), y(t)) + (R_i(t)z(t), z(t)) + (N_i(t)v_i(t), v_i(t))] dt + (Q_iy(0), y(0)) \right\}, \ i = 1, 2,
\]
where \( M_i(\cdot), R_i(\cdot), Q_i, i = 1, 2, \) are \( n \times n \) nonnegative symmetric bounded matrices, and \( N_i(\cdot), i = 1, 2, \) are \( k \times k \) positive symmetric bounded matrices and the inverse \( N_i^{-1}(\cdot), i = 1, 2, \) are also bounded. Our task is to find \((u_1(\cdot), u_2(\cdot)) \in \mathbb{R}^k \times \mathbb{R}^k\) such that
\[
\begin{cases}
J_1(u_1(\cdot), u_2(\cdot)) = \min_{v_1(\cdot) \in U_1} J_1(v_1(\cdot), u_2(\cdot)), \\
J_2(u_1(\cdot), u_2(\cdot)) = \min_{v_2(\cdot) \in U_2} J_2(u_1(\cdot), v_2(\cdot)).
\end{cases}
\]
We need the following assumption
(H4) \( E_i(t)N_i^{-1}(t)E_i^\top(t)S(t) = S(t)E_i(t)N_i^{-1}(t)E_i^\top(t), \)
where \( S(t) = A(t), \bar{A}(t), B(t), \bar{B}(t), C(t), \bar{C}(t), D(t), \bar{D}(t), \) and \( i = 1, 2. \) Then from the necessary maximum principle (Theorem 3.3), we have the explicit form of a Nash equilibrium point for this game problem.
Theorem 5.1. The mapping
\[ (u_1(t), u_2(t)) = (-N_1^{-1}(t)E_1^T(t)p_1(t), -N_2^{-1}(t)E_2^T(t)p_2(t)), \ t \in [0,T], \]  
(36)
is one Nash equilibrium point for the above game problem, where \((y(t), z(t), p_1(t), q_1(t)), i = 1, 2,\) is the solution of the following different dimensional general FBDSDE:

\[
\begin{aligned}
-dy(t) &= [A(t)y(t) + \bar{A}(t)y_\delta(t) + B(t)z(t) + B(t)z_\delta(t) \\
&- E_1(t)N_1^{-1}(t)E_1^T(t)p_1(t) - E_2(t)N_2^{-1}(t)E_2^T(t)p_2(t)]dt \\
+ [C(t)y(t) + \bar{C}(t)y_\delta(t) + D(t)z(t) + D(t)z_\delta(t)]d\bar{B}(t) \\
- z(t)\ dW(t), \ t \in [0,T],
\end{aligned}
\]

\[

dp_i(t) = \left\{ \begin{array}{ll}
A^T(t)p_i(t) + E^{F_i}[A^T(t)p_{i\delta}(t)] + C^T(t)q_i(t) + E^{F_i}[C^T(t)q_{i\delta}(t)] \\
+ M_i(t)y(t) \right\}dt \\
+ \left\{ \begin{array}{ll}
B^T(t)p_i(t) + E^{F_i}[B^T(t)p_{i\delta}(t)] + D^T(t)q_i(t) + E^{F_i}[D^T(t)q_{i\delta}(t)] \\
+ R_i(t)z(t)) \right\}d\bar{W}(t) - q_i(t)\ d\bar{B}(t), \ t \in [0,T], \ i = 1, 2,
\end{array} \right.
\]

\[
y(T) = \zeta, \ y(t) = \phi(t), \ z(t) = \psi(t), \ t \in [-\delta,0], \]

\[
p_i(0) = -Q_iy(0), \ p_i(t) = 0, \ q_i(t) = 0, \ t \in [T,T+\delta], \ i = 1, 2.
\]

(37)

Proof. We first prove the existence of the solution of equation (37). We consider the following FBDSDE:

\[
\begin{aligned}
-dY(t) &= [A(t)Y(t) + B(t)Z(t) + \bar{A}(t)Y_\delta(t) + \bar{B}(t)Z_\delta(t) - P(t)]dt \\
&+ [C(t)Y(t) + D(t)Z(t) + \bar{C}(t)Y_\delta(t) + \bar{D}(t)Z_\delta(t)]d\bar{B}(t) \\
- Z(t)\ dW(t), \ t \in [0,T],
\end{aligned}
\]

\[

dP(t) = \left\{ \begin{array}{ll}
A^T(t)P(t) + C^T(t)Q(t) + E^{F_i}[A^T(\delta)P_{i\delta}(t)] + E^{F_i}[\bar{C}(t)Q_{\delta}(t)] \\
+ [E_1(t)N_1^{-1}(t)E_1^T(t)M_1(t) + E_2(t)N_2^{-1}(t)E_2^T(t)M_2(t)]Y(t) \right\}dt \\
+ \left\{ \begin{array}{ll}
B^T(t)P(t) + D^T(t)Q(t) + E^{F_i}[\bar{B}(t)P_{i\delta}(t) + \bar{D}(t)Q_{\delta}(t)] \\
+ [E_1(t)N_1^{-1}(t)E_1^T(t)R_1(t) + E_2(t)N_2^{-1}(t)E_2^T(t)R_2(t)]Z(t) \right\}d\bar{W}(t) \\
- Q(t)\ d\bar{B}(t), \ t \in [0,T],
\end{array} \right.
\]

\[
Y(T) = \zeta, \ Y(t) = \phi(t), \ Z(t) = \psi(t), \ t \in [-\delta,0], \]

\[
P(0) = [E_1(0)N_1^{-1}(0)E_1^T(0)Q_1 + E_2(0)N_2^{-1}(0)E_2^T(0)Q_2]Y(0),
\]

\[
P(t) = 0, \ t \in [T,T+\delta], \ Q(t) = 0, \ t \in [T,T+\delta].
\]

(38)

From the assumption (H4), we notice that if \((y(t), p_1(t), p_2(t), z(t), q_1(t), q_2(t))\) is a solution of equation (37), \((Y(t), P(t), Z(t), Q(t))\) solves the FBDSDE (38), here

\[
Y(t) = y(t), \ P(t) = E_1(t)N_1^{-1}(t)E_1^T(t)p_1(t) + E_2(t)N_2^{-1}(t)E_2^T(t)p_2(t),
\]

\[
Z(t) = z(t), \ Q(t) = E_1(t)N_1^{-1}(t)E_1^T(t)q_1(t) + E_2(t)N_2^{-1}(t)E_2^T(t)q_2(t).
\]

On the other hand, if \((Y(t), P(t), Z(t), Q(t))\) solves the FBDSDE (38), we can let \(y(t) = Y(t)\) and \(z(t) = Z(t)\), from the existence and uniqueness result of anticipated forward doubly SDE (see Theorem 2.3), we can get the solution \((p_1(t), q_1(t))\) and
(p_2(t), q_2(t)) of the following anticipated forward doubly SDE:

\[
\begin{align*}
    dp_1(t) &= \{ A^\top(t)p_1(t) + C^\top(t)q_1(t) + \mathbb{E}^{\mathbb{F}_t}[\bar{A}_{\delta+}^\top(t)p_{\delta+}(t)] \} dt + \mathbb{E}^{\mathbb{F}_t}[\bar{C}_{\delta+}^\top(t)q_{\delta+}(t)] dt - q_1(t) \int dB(t) \\
    dp_2(t) &= \{ A^\top(t)p_2(t) + C^\top(t)q_2(t) + \mathbb{E}^{\mathbb{F}_t}[\bar{A}_{\delta+}^\top(t)p_{\delta+}(t)] \} dt + \mathbb{E}^{\mathbb{F}_t}[\bar{C}_{\delta+}^\top(t)q_{\delta+}(t)] dt - q_2(t) \int dB(t) \\
    dq_1(t) &= \{ B^\top(t)p_1(t) + D^\top(t)q_1(t) + \mathbb{E}^{\mathbb{F}_t}[\bar{B}_{\delta+}^\top(t)p_{\delta+}(t)] \} dt + \mathbb{E}^{\mathbb{F}_t}[\bar{D}_{\delta+}^\top(t)q_{\delta+}(t)] + R_1(t)z(t) \int dW(t), \quad t \in [0, T], \\
    dq_2(t) &= \{ B^\top(t)p_2(t) + D^\top(t)q_2(t) + \mathbb{E}^{\mathbb{F}_t}[\bar{B}_{\delta+}^\top(t)p_{\delta+}(t)] \} dt + \mathbb{E}^{\mathbb{F}_t}[\bar{D}_{\delta+}^\top(t)q_{\delta+}(t)] + R_2(t)z(t) \int dW(t), \quad t \in [0, T], \\
    p_1(0) &= Q_1y(0), \quad p_2(0) = Q_2y(0), \\
    q_1(t) &= 0, \quad q_2(t) = 0, \quad \forall t \in [T, T + \delta].
\end{align*}
\]

We let

\[
    \bar{P}(t) = E_1(t)N_1^{-1}(t)E_1^\top(t)p_1(t) + E_2(t)N_2^{-1}(t)E_2^\top(t)p_2(t), \\
    \bar{Q}(t) = E_1(t)N_1^{-1}(t)E_1^\top(t)q_1(t) + E_2(t)N_2^{-1}(t)E_2^\top(t)q_2(t).
\]

Then we get

\[
\begin{align*}
    d\bar{P}(t) &= \{ A^\top(t)\bar{P}(t) + C^\top(t)\bar{Q}(t) + \mathbb{E}^{\mathbb{F}_t}[\bar{A}_{\delta+}^\top(t)\bar{P}_{\delta+}(t) + \bar{C}_{\delta+}^\top(t)\bar{Q}_{\delta+}(t)] \} dt + \{ E_1(t)N_1^{-1}(t)E_1^\top(t)M_1(t) + E_2(t)N_2^{-1}(t)E_2^\top(t)M_2(t) \} dt + \{ B^\top(t)\bar{P}(t) + D^\top(t)\bar{Q}(t) + \mathbb{E}^{\mathbb{F}_t}[\bar{B}_{\delta+}^\top(t)\bar{P}_{\delta+}(t) + \bar{D}_{\delta+}^\top(t)\bar{Q}_{\delta+}(t)] \} dt \\
    &\quad + \{ E_1(t)N_1^{-1}(t)E_1^\top(t)R_1(t) + E_2(t)N_2^{-1}(t)E_2^\top(t)R_2(t) \} Z(t) \int dW(t) \\
    \bar{P}(0) &= [E_1(0)N_1^{-1}(0)E_1^\top(0)Q_1 + E_2(0)N_2^{-1}(0)E_2^\top(0)Q_2]Y(0), \\
    \bar{Q}(t) &= 0, \quad \forall t \in [T, T + \delta].
\end{align*}
\]

For fixed (Y(t), Z(t)), because of the uniqueness of the BDSDE, we have

\[
\begin{align*}
    P(t) &= \bar{P}(t) = E_1(t)N_1^{-1}(t)E_1^\top(t)p_1(t) + E_2(t)N_2^{-1}(t)E_2^\top(t)p_2(t), \\
    Q(t) &= \bar{Q}(t) = E_1(t)N_1^{-1}(t)E_1^\top(t)q_1(t) + E_2(t)N_2^{-1}(t)E_2^\top(t)q_2(t).
\end{align*}
\]

Then (Y(t), P(t), Z(t), Q(t)) is the unique solution of FBDSDE (38) and \((y(t), p_1(t), p_2(t), z(t), q_1(t), q_2(t))\) is a solution of equation (37). Moreover, the uniqueness of the FBDSDE (37) is equivalent to the uniqueness of the FBDSDE (38).

Now we try to prove \((u_1(\cdot), u_2(\cdot))\) is a Nash equilibrium point for our nonzero-sum game problem. We only prove

\[
J_1(u_1(\cdot), u_2(\cdot)) \leq J_1(v_1(\cdot), u_2(\cdot)), \quad v_1(\cdot) \in \mathbb{R}^k.
\]

It is similar to get the other inequality of (35). \((y^{v_1}(t), z^{v_1}(t))\) denotes the solution of the system:
Applying Itô’s formula to conditions:

\[
\begin{align*}
-dy_i(t) &= [A(t)y_i(t) + B(t)z_i(t) + \hat{A}(t)y_i^\delta(t) + \hat{B}(t)z_i^\delta(t) + E_1(t)v_1(t) \\
& \quad + E_2(t)u_2(t)]dt \\
& \quad + [C(t)y_i(t) + D(t)z_i(t) + \hat{C}(t)y_i^\delta(t) + \hat{D}(t)z_i^\delta(t)]dW(t) \\
\end{align*}
\]

(39)

\[y_i^\delta(T) = \zeta, \quad y_i^\delta(t) = \phi(t), \quad z_i^\delta(t) = \psi(t), \quad t \in [-\delta, 0],\]

then

\[
J_1(v_1(\cdot), u_2(\cdot)) - J_1(u_1(\cdot), u_2(\cdot)) = \frac{1}{2} \left( \mathbb{E} \int_0^T \left[ (M_1(t)y_i(t), y_i(t)) - (R_1(t)z_i^\delta(t), z_i^\delta(t)) \\
\right.ight.
\]

\[
\left. \left. - (R_1(t)z_i(t), z_i(t)) + (N_1(t)v_1(t), v_1(t)) - (N_1(t)u_1(t), u_1(t)) \right] dt \\
+ \mathbb{E} \left[ (Q_1y_i(0), y_i^\delta(0)) - (Q_1y(0), y(0)) \right] \right) \geq \mathbb{E} \int_0^T \left[ (M_1(t)y(t), y_i(t) - y(t)) + (R_1(t)z(t), z_i^\delta(t) - z(t)) \\
+ (N_1(t)u(t), v_i(t) - u_i(t)) \right] dt + \mathbb{E} [(Q_1y(0), y_i^\delta(0) - y(0))].
\]

Applying Itô’s formula to \((y_i^\delta(t) - y(t), p_1(t))\), we have

\[
\mathbb{E} (y_i^\delta(0) - y(0), Q_1y(0))
\]

\[
= \mathbb{E} \int_0^T \left[ (y_i^\delta(t) - y(t), M_1(t)y(t)) - (z_i^\delta(t) - z(t), R_1(t)z(t)) \\
+ (v_1(t) - u_1(t), E_1^T(t)p_1(t)) \right] dt.
\]

In fact, the above result is due to the following virtue of the initial and terminal conditions:

\[
\begin{align*}
\mathbb{E} \int_0^T \{ \langle \tilde{A}(t)y_i^\delta(t) - y_i(t), p_1(t) \rangle - \langle \mathbb{E}^F_t[\tilde{A}_1^T(t)p_{1\delta}(t)], y_i^\delta(t) - y(t) \rangle \} dt \\
= \mathbb{E} \int_0^T \langle \tilde{A}(t)y_i^\delta(t) - y_i(t), p_1(t) \rangle dt - \mathbb{E} \int_{t+\delta}^{t+\delta} \langle \tilde{A}^T(t)y_i^\delta(t) - y_i(t), p_1(t) \rangle dt \\
= \mathbb{E} \int_{t-\delta}^{t-\delta} \langle \tilde{A}(t)y_i^\delta(t) - y_i(t), p_1(t) \rangle dt - \mathbb{E} \int_{t-\delta}^{t+\delta} \langle \tilde{A}_1^T(t)y_i^\delta(t) - y_i(t), p_1(t) \rangle dt \\
= 0,
\end{align*}
\]

and

\[
\begin{align*}
\mathbb{E} \int_0^T \{ \langle \hat{B}(t)z_i^\delta(t) - z_i(t), p_1(t) \rangle - \langle \mathbb{E}^F_t[\hat{B}_1^T(t)p_{1\delta}(t)], z_i^\delta(t) - z(t) \rangle \} dt = 0, \\
\mathbb{E} \int_0^T \{ \langle \hat{C}(t)z_i^\delta(t) - z_i(t), q_1(t) \rangle - \langle \mathbb{E}^F_t[\hat{C}_1^T(t)q_{1\delta}(t)], y_i^\delta(t) - y(t) \rangle \} dt = 0, \\
\mathbb{E} \int_0^T \{ \langle \hat{D}(t)z_i^\delta(t) - z_i(t), q_1(t) \rangle - \langle \mathbb{E}^F_t[\hat{D}_1^T(t)z_{1\delta}(t)], z_i^\delta(t) - z(t) \rangle \} dt = 0.
\end{align*}
\]
As $N_1(t)$ is positive, and $R_1(t)$ and $Q_1$ are nonnegative, noting that $u_1(t) = -N_1^{-1}(t)E_1^T(t)p_1(t)$, we have

$$J_1(v_1(\cdot), u_2(\cdot)) - J_1(u_1(\cdot), u_2(\cdot)) \geq E \int_0^T [\langle N_1(t)u_1(t), v_1(t) - u_1(t) \rangle + \langle v_1(t) - u_1(t), E_1^T(t)p_1(t) \rangle] dt \geq 0.$$ 

So $(u_1(t), u_2(t)) = (-N_1^{-1}(t)E_1^T(t)p_1(t), -N_2^{-1}(t)E_2^T(t)p_2(t))$ is a Nash equilibrium point for our LQ doubly stochastic game with delay.

6. Conclusion. In this paper, we have discussed one kind of stochastic differential game of backward doubly stochastic system with delay. More specially, based on convex variational method and the introduction of anticipated forward doubly SDE as adjoint process, we established a necessary condition and a sufficient condition for the equilibrium point of the game. The theoretical results established here are applied to solve a LQ case. A unique equilibrium point of the LQ game is obtained explicitly. Since LQ models are usually applied to describe some economic and financial phenomena, we hope that the LQ game theory of BDSDEs has a broad range of applications in these fields. To the authors’ knowledge, our paper is the first attempt to study delayed backward doubly SDEs, anticipated forward doubly SDEs and apply them to nonzero-sum differential game problem. In our LQ game problem, both the state equation and the adjoint equation are fully coupled, then constitute a kind of linear FBDSDE, in which the forward equation is anticipated forward doubly SDE and the backward equation is delayed backward doubly SDEs. We will try to investigate the theory and applications of this kind of general fully coupled FBDSDE in future.

Although we focus on this kind of game, we are also able to develop some results for optimal control of BDSDEs, for example Han et al. [9], Shi and Zhu [18], Xu and Han [25, 26], Zhang and Shi [28], Zhu and Shi [35].

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