Optimal System and Invariant Solutions of a New AKNS Equation with Time-dependent Coefficients

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Abstract: The Lie point symmetries are reported by performing the Lie symmetry analysis to the Ablowitz-Kaup-Newell-Suger (AKNS) equation with time-dependent coefficients. In addition, the optimal system of one-dimensional subalgebras is constructed. Based on this optimal system, several categories of similarity reduction and some new invariant solutions for the equation are obtained, which include power series solutions and travelling and non-traveling wave solutions.

Keywords: AKNS equation with time-dependent coefficients; Lie symmetry analysis; Optimal system; Invariant solutions

1. Introduction

The term nonlinear partial differential equation (NLPDE) is broadly utilized as a model in order to represent actual phenomena that occur many science areas, particularly in plasma physics, optical fields, and fluid mechanics. It is well known that many physical phenomena are described by NLPDEs with variable coefficients in light of the fact that the vast majority of genuine nonlinear physical conditions have variable coefficients. On the one hand, many types of exact solutions have also been constructed to explain complex physical phenomena, such as solitary wave solutions [1], doubled Wronskian solutions [2], multiple rogue wave solutions [3], and localized excitation solutions [4]; on the other hand, many powerful methods have been developed to construct solutions of NLPDEs, such as the Hirota method [5–7], the generalized Darboux transformation [8–10], the extended tanh method [11,12], the generalized Jacobi elliptic functions technique [13], numerical method [14], and the Lie group method [15–17].

As well as we know, Lie symmetry analysis is a powerful and prolific method for constructing exact solutions for NLPDEs with constant variable [18–20]. Recently, the Lie symmetry analysis is extended to find exact solutions of fractional and variable coefficient NLPDEs, such as Time-Fractional Boussinesq-Burgers [21], Gardner equations [22], coupled short pulse equation [23] and so on [24–26].

Recently, Zhang et al. [27] studied the multi-soliton solutions of the following Ablowitz-Kaup-Newell-Suger (AKNS) equation

\[ q_i = \alpha_1(t)(q_{xx} - 6qr) + \alpha_2(t)(-q_{xx} + 2q^2r) + \alpha_3(t)q_x - \alpha_4(t)q, \]
\[ r_i = \alpha_1(t)(r_{xx} - 6rr_x) + \alpha_2(t)(r_{xx} - 2r^2q) + \alpha_3(t)r_x + \alpha_4(t)r, \]

which is a particular example at \( m = 3 \) of the generalized AKNS hierarchy...
\[
\left( \frac{q}{r} \right)_j = \sum_{i=0}^{m} \alpha_i(t)L\left( -\frac{q}{r} \right), \quad (m = 1,2,\ldots),
\]

where the recursive operator is being utilized, as follows
\[
L = \sigma \frac{\partial}{\partial t} + 2 \left( \frac{q}{r} \right) \frac{\partial}{\partial (r,q)}, \quad \sigma = \left( \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right), \quad \frac{\partial}{\partial x} = \frac{1}{2} \int_{-\infty}^{x} dx - \int_{x}^{\infty} dx.
\]

We note that system (1) includes a lot of famous NLPDEs as its special cases. For example, if \( \alpha_0(t) = \alpha_1(t) = \alpha_2(t) = 0, \quad \alpha_3(t) = -1 \) and \( r = 1 \), then system (1) is the KdV equation
\[
q_t + q_{xxx} + 6qq_x = 0.
\]

If \( \alpha_0(t) = \alpha_1(t) = \alpha_2(t) = 0, \quad \alpha_3(t) = -1 \) and \( r = -q \), then system (1) is the mKdV equation
\[
q_t + q_{xxx} + 6qq_x = 0.
\]

If \( \alpha_0(t) = \alpha_1(t) = 0, \quad \alpha_2(t) = i, \quad \alpha_3(t) = -1 \) and \( r = -q \), then system (1) is the mKdV-NLS equation
\[
q_t + q_{xxx} + 6qq_x + i(q_{xx} + 2q^3) = 0.
\]

If \( \alpha_0(t) = \alpha_1(t) = \alpha_3(t) = 0 \) and \( \alpha_2(t) = i \), then system (1) is the second order AKNS coupled system [28,29].
\[
i q_r = q_{xx} - 2q^3r,
\]

\[
i r_r = -r_{xx} + 2r^2 q.
\]

To our knowledge, the AKNS equation with time-dependent coefficients has not been studied via Lie symmetry analysis. The aim of the present paper is to construct optimal system and invariant solutions to (1) based on Lie point symmetries. The rest of this paper is organized, as follows. In Sect. 2, the Lie point symmetries of (1) are obtained by utilizing Lie symmetry analysis. In Sect. 3, we construct the optimal system of one-dimensional subalgebras of Lie algebra spanned by \( V_1 - V_3 \). In Sect. 4, several types of similarity reduction and some invariant solutions are discussed on the optimal system. In Secttion 5, we conclude this paper.

2. Symmetry Analysis

In this section, our aim is to obtain the symmetry algebra of the AKNS equation (1) while using the Lie symmetry analysis [15–17]. Suppose that the associated vector field of system (1) is as follows:
\[
V = \xi(t,x,q,r) \frac{\partial}{\partial x} + \eta(t,x,q,r) \frac{\partial}{\partial t} + Q(t,x,q,r) \frac{\partial}{\partial q} + R(t,x,q,r) \frac{\partial}{\partial r},
\]

where \( \xi(t,x,q,r), \eta(t,x,q,r), Q(t,x,q,r), \) and \( R(t,x,q,r) \) are unknown functions that need to be determined.

If vector field (2) generates a symmetry of system of Eq. (1), then \( V \) must satisfy the symmetry condition
\[
pr^{(1)}V(\Delta_1) \bigg|_{\Delta_1} = 0,
\]
\[
pr^{(1)}V(\Delta_2) \bigg|_{\Delta_2} = 0,
\]

where \( \Delta_1 = \alpha_1(t)(q_{xx} - 6qq_r) + \alpha_2(t)(-q_{xx} + 2q^2 r) + \alpha_3(t)q_x - \alpha_5(t)q - q_t, \)
\( \Delta_2 = \alpha_1(t)(r_{xx} - 6qr_r) + \alpha_2(t)(r_{xx} - 2r^2 q) + \alpha_3(t)r_x + \alpha_5(t)r - r_t. \)

The infinitesimals \( \xi, \eta, Q \) and \( R \) must satisfy the following invariant conditions
\[
Q' = \alpha_1'(t)\eta(q_{xx} - 6q_{qq}) + \alpha_2(t)(Q_{xx} - 6Qq_r - 6qRq_x - 6qRq_r).
\]
\[ R' = \alpha'_t(t) \eta(r_{xxx} - 6qrr_r - 6qR_r - 6qrR_x) + \alpha_x(t)(R_{xxx} - 4rR_q - 2r^2Q) + \alpha'_z(t)(r_{xx} - 2r^2q) + \alpha_z(t)(R_{xx} - 4rR_q - 2r^2Q) + \alpha'_r(t)\eta r_q + \alpha_r(t)R, \quad (3) \]

where

\[ R' = D_x(R - \xi r_q - \eta r_q) + \xi r_{xx} + \xi r_{rr}, \]
\[ R'' = D_x(R - \xi r_q - \eta r_q) + \xi r_{xx} + \xi r_{rr}, \]
\[ R^{xxx} = D_{xxx}(R - \xi r_q - \eta r_q) + \xi r_{xxx} + \xi r_{xxx}, \]
\[ R^{xxx} = D_{xxx}(R - \xi r_q - \eta r_q) + \xi r_{xxx} + \xi r_{xxx}, \]
\[ Q' = D_x(Q - \xi q_x - \eta q_x) + \xi q_{xx} + \xi q_{rr}, \]
\[ Q'' = D_x(Q - \xi q_x - \eta q_x) + \xi q_{xx} + \xi q_{rr}, \]
\[ Q'' = D_{xxx}(Q - \xi q_x - \eta q_x) + \xi q_{xxx} + \xi q_{xxx}, \]
\[ Q^{xxx} = D_{xxx}(Q - \xi q_x - \eta q_x) + \xi q_{xxx} + \xi q_{xxx}. \quad (4) \]

Substituting (4) into system (3), we obtain a large number of determining equations

\[ \xi_j = 0, \quad \xi_x = 0, \quad Q = 0, \quad Q_{qq} = 0, \quad R = 0, \quad R_{rr} = 0, \]
\[ \alpha_i \eta + \alpha_i \eta - \alpha_i \xi = 0, \quad \alpha_i \eta + \alpha_i \eta - \alpha_i \xi = 0, \quad \alpha_i \eta + \alpha_i \eta - 3\alpha_i \xi = 0, \]
\[ \alpha_i \xi_q r - \alpha_i \eta q r - \alpha_i \xi q r - \alpha_i q R - \alpha_i r Q = 0, \]
\[ \alpha_i \xi_q^2 r + \alpha_i \eta q^2 r + \alpha_i q^2 R - \alpha_i Q q^2 r + 2\alpha_i q r Q = 0, \]
\[ \alpha_i \eta q + \alpha_i q Q - \alpha_i q Q + \alpha_i Q = 0, \]
\[ \alpha_i \eta r + \alpha_i q r - \alpha_i r R + \alpha_i R - R_i = 0. \quad (5) \]

Solving the system, one can get

\[ \xi = c_1 x + c_2, \quad \eta = \frac{1}{\alpha_3} \left( 3c_1 \int \alpha_3 dt + c_1 \right), \]
\[ Q = \left( -\frac{3c_1 \alpha_0}{\alpha_3} \int \alpha_3 dt - \frac{c_1 \alpha_0}{\alpha_3} - c_1 \right) q, \quad R = \left( \frac{3c_1 \alpha_0}{\alpha_3} \int \alpha_3 dt + \frac{c_1 \alpha_0}{\alpha_3} - c_1 \right) r, \quad (6) \]

where \( c_1, \ c_2, \) and \( c_3 \) are arbitrary constants, and two coefficient functions \( \alpha_1 \) and \( \alpha_2 \) are determined by

\[ \eta \alpha_1 + \eta \alpha_1 - c_1 \alpha_1 = 0, \quad \eta \alpha_2 + \eta \alpha_2 - 2c_2 = 0. \quad (7) \]

The Lie algebra of infinitesimal symmetries of system (1) is generated by the three vector fields:

\[ V_1 = x \frac{\partial}{\partial x} + \left( \frac{3}{\alpha_3} \int \alpha_3 dt \right) \frac{\partial}{\partial t} - \left( \frac{3\alpha_0}{\alpha_3} \int \alpha_3 dt + 1 \right) q \frac{\partial}{\partial q} + \left( \frac{3\alpha_0}{\alpha_3} \int \alpha_3 dt - 1 \right) r \frac{\partial}{\partial r}, \]
\[ V_2 = \frac{\partial}{\partial x}, \]
\[ V_3 = \frac{1}{\alpha_3} \frac{\partial}{\partial t} - \left( \frac{\alpha_0}{\alpha_3} q \right) \frac{\partial}{\partial q} + \left( \frac{\alpha_0}{\alpha_3} r \right) \frac{\partial}{\partial r}. \quad (8) \]

Table 1 presents the commutator table.

**Table 1.** Table of Lie brackets.
3. Optimal System of Subalgebras

In present work, we shall construct the optimal system of one-dimensional subalgebra of the Lie algebra $L_3$ for AKNS equation (1) by the method proposed in [19, 30, 31].

An arbitrary operator $V \in L_3$ is written in the form

$$V = l^1 V'_1 + l^2 V'_2 + l^3 V'_3.$$  (9)

The following generators are used in order to find the linear transformations of the vector $l = (l^1, l^2, l^3)$,

$$E_i = c_{ij}^r l^j \frac{\partial}{\partial l^i}, \quad i = 1, 2, 3, \quad (10)$$

where $c_{ij}^r$ is defined by $[V_i, V_j] = c_{ij}^r V_r$. According to Eq. (10) and Table 1, $E_1$, $E_2$, and $E_3$ are

$$E_1 = -l^2 \frac{\partial}{\partial l^1} - 3l^3 \frac{\partial}{\partial l^2},$$
$$E_2 = l^1 \frac{\partial}{\partial l^1},$$
$$E_3 = 3l^1 \frac{\partial}{\partial l^3}. \quad (11)$$

For the generators $E_1$, $E_2$, and $E_3$, the Lie equations with parameters $a_1$, $a_2$, and $a_3$ with the initial condition $\left.\vec{t}\right|_{a_i=0} = l$, $i = 1, 2, 3$ are written as

$$\frac{dl^1}{da_1} = 0, \quad \frac{dl^2}{da_1} = -l^2, \quad \frac{dl^3}{da_1} = -3l^3, \quad (12)$$
$$\frac{dl^1}{da_2} = 0, \quad \frac{dl^2}{da_2} = -l^2, \quad \frac{dl^3}{da_2} = 0, \quad (13)$$
$$\frac{dl^1}{da_3} = 0, \quad \frac{dl^2}{da_3} = 0, \quad \frac{dl^3}{da_3} = 2l^1. \quad (14)$$

The solutions of Equations (12)–(14) provide the transformation

$$T_1: \quad T^1 = l^1, \quad T^2 = e^{-a_2}l^2, \quad T^3 = e^{-a_3}l^3, \quad (15)$$
$$T_2: \quad T^1 = l^1, \quad T^2 = a_2 l^1 + l^2, \quad T^3 = l^3, \quad (16)$$
$$T_3: \quad T^1 = l^1, \quad T^2 = l^2, \quad T^3 = 2a_3 l^1 + l^3. \quad (17)$$

The method of constructing an optimal system needs a simplification of the vector $l = (l^1, l^2, l^3)$ by means of the transformation $T_1 - T_3$. Our aim is to find the simplest representative of each class of similar vectors (18). The construction will be carried out under the following cases.

Case 1. $l^1 \neq 0$

By taking $a_2 = -\frac{l^2}{l^1}$ in the transformation $T_2$, $a_3 = -\frac{l^3}{2l^1}$ in the transformation $T_3$, we obtain $T^2 = 0, T^3 = 0$. Thus, vector (18) can be reduced to the form
This case gives the operator:

\[ V_1. \]

Case 2. \( l^1 = 0 \)

2.1. \( l^2 \neq 0 \)

The vector (18) can be reduced to the form

\[ l = (0, l^2, l^3). \] (20)

Using all of the possible combinations, this case give rise to following operators:

\[ V_2, \ V_2 + V_3, \ V_2 - V_3. \]

2.2. \( l^2 = 0 \)

The vector (22) is reduced to the form

\[ l = (0, 0, l^3). \] (21)

Thus, we have the operator

\[ V_3. \]

**Theorem 1.** The optimal system of one-dimensional subalgebras of the Lie algebra is spanned by \( V_1, V_2, V_3 \) of Eq. (1), as given by

\[ V_1, \ V_2, \ V_3, \ V_2 + V_3, \ V_2 - V_3. \] (22)

4. Symmetry Reductions and Exact Solutions

By virtue of the optimal system (22), we will deal with the similarity reductions and group invariant solutions to the AKNS equation with time-dependent coefficients.

4.1. Solutions through \( V_1 \)

The characteristic equations of the generator \( V_1 \) can be written as

\[
\frac{dx}{x} = \frac{dt}{\frac{3}{\alpha_3} \int \alpha_3 dt} = \frac{dq}{-\left(\frac{3\alpha_0}{\alpha_3} \int \alpha_3 dt + 1\right) q} = \frac{dr}{\left(\frac{3\alpha_0}{\alpha_3} \int \alpha_3 dt - 1\right) r}.
\] (23)

Solving these equations yields the three similarity variables

\[
\xi = x \left(\int \alpha_3 dt\right)^{-\frac{1}{3}}, \quad q = e^{-\left[\int \alpha_3 dt\right]^\frac{1}{3}} F(\xi), \quad r = e^{\int \alpha_3 dt\left[\int \alpha_3 dt\right]^\frac{1}{3}} H(\xi),
\] (24)

and solving the constrained conditions (7), we get

\[
\alpha_1 = \frac{1}{3} k_1 \alpha_3 \left(\int \alpha_3 dt\right)^\frac{1}{3}, \quad \alpha_2 = \frac{1}{3} k_2 \alpha_3 \left(\int \alpha_3 dt\right)^\frac{1}{3},
\]

where \( k_1 \) and \( k_2 \) are arbitrary constants and the AKNS equation (1) is reduced to the following nonlinear coupled ordinary differential equations (ODEs):

\[
-\frac{1}{3} F - \frac{1}{3} \xi F' = F'' - 6FHF' - \frac{k_2}{3} F^3 + \frac{2}{3} k_2 H^2 F + \frac{k_2}{3} F',
\]

\[
-\frac{1}{3} H - \frac{1}{3} \xi H' = H'' - 6HH' + \frac{k_2}{3} H^3 F + \frac{2}{3} k_2 H^2 F + \frac{k_2}{3} H'.
\] (25)

The solution for (25) in a power series can be found in the form [32]
Substituting (26) into (25), we get
\[ -\frac{1}{3} A_0 - \frac{1}{3} \sum_{n=1}^{\infty} A_n \xi^n - \frac{1}{3} \sum_{n=1}^{\infty} A_{n-1} \xi^n = 6 A_3 + \sum_{n=1}^{\infty} (n+3)(n+2)(n+1) A_{n+3} \xi^n - 6 A_0 A_B_0, \]
\[ -6 \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} (n-k+1) A_n A_{k+1} B_{n-k} \xi^n - \frac{2k_2}{3} A_2 - \frac{k_3}{3} \sum_{n=1}^{\infty} (n+2)(n+1) A_{n+2} \xi^n + \frac{k_2}{3} A_2 + \frac{k_3}{3} \sum_{n=1}^{\infty} (n+1) A_{n+1} \xi^n, \]
\[ + \frac{2k_2}{3} A_0 B_0 + \frac{k_3}{3} \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} A_n A_{k+1} B_{n-k} \xi^n. \]  

Now from (27), comparing coefficients, for \( n = 0 \), we get
\[ A_3 = \frac{1}{18} \left( 18 A_0 A_B_0 + 2k_2 A_2 - 2k_2 A_0 B_0 - k_1 A_1 - A_0 \right), \]
\[ B_3 = \frac{1}{18} \left( 18 A_0 A_B_0 - 2k_2 B_2 + 2k_2 A_0 B_0 - k_1 B_1 - B_0 \right). \]  

Generally, for \( n \geq 1 \), we obtain
\[ A_{n+3} = \frac{1}{(n+3)(n+2)(n+1)} \left[ -\frac{1}{3} A_n - \frac{1}{3} A_{n-1} + 6 \sum_{k=0}^{n-1} (n-k+1) A_n A_{k+1} B_{n-k} + \frac{k_2}{3} (n+2)(n+1) A_{n+2} \right], \]
\[ B_{n+3} = \frac{1}{(n+3)(n+2)(n+1)} \left[ -\frac{1}{3} B_n - \frac{1}{3} B_{n-1} + 6 \sum_{k=0}^{n-1} (n-k+1) A_n B_{k+1} B_{n-k} + \frac{k_2}{3} (n+2)(n+1) B_{n+2} \right]. \]  

From (27) and (28), we can get all of the coefficients \( A_n, B_n \) \( n \geq 3 \) of the power series (25). Substituting (28), (29) into (26) and using similarity transformations (24), we can obtain the solutions of system (1).

4.2. Solutions through \( V_2 \)

The similarity variables of this generator are
\[ \xi = t, \quad q = F(\xi), \quad r = H(\xi), \]  
and solving the constrained conditions (7), we get \( \alpha_0, \alpha_2 \) are arbitrary functions of \( t \).

These reduce the system (1) to the following nonlinear coupled ODEs:
\[ F' = 2\alpha_2 F^2 H - \alpha_0 F, \]
\[ H' = -2\alpha_2 H^2 F + \alpha_0 H. \]  

Solving Eq. (31) and using the similarity transformations (30), we obtain the solution of system (1) is
\[ q = e^{i \left( \int \frac{1}{2(\alpha_2 - \alpha_0)} \right) dt}, \]
4.3. Solutions through $V_3$

The similarity variables of this generator are

$$\xi = x, \quad q = e^{-\int \alpha_0 dt} F(\xi), \quad r = e^{\int \alpha_0 dt} H(\xi),$$

and solving the constrained conditions (7), we get

$$\alpha_1 = k_1 \alpha_2, \quad \alpha_2 = k_2 \alpha_3,$$

where $k_1$ and $k_2$ are arbitrary constants, and the AKNS equation (1) is reduced to the following nonlinear coupled ODEs:

$$F'' - 6FH' - k_2 F'' + 2k_2 F^2 H + k_1 F' = 0,$$

$$H'' - 6H'F'' + k_2 H'' - 2k_2 H^2 + k_1 H' = 0. \quad (34)$$

To obtain the solutions of the reduction (34), we shall use the $G'/G$ method, as described in [20,33]. Assume that the solution of (34) is given in a polynomial form, as follows:

$$F = \sum_{i=0}^{m} A_i \left( \frac{G'}{G} \right)^i, \quad H = \sum_{i=0}^{n} B_i \left( \frac{G'}{G} \right)^i. \quad (35)$$

By balancing highest order derivative term and nonlinear term in (34), we get $m = n = 1$ and $G = G(\xi)$ satisfies second-order linear ordinary differential equation (LODE)

$$G'' + \lambda G' + \mu G = 0. \quad (36)$$

Substituting (35) into (34) and equating coefficients of $\left( \frac{G'}{G} \right)$ to 0, we obtain an algebraic system of equations in $A_0, A_1, B_0,$ and $B_1$. With the help of Maple, we obtain

$$\lambda = \frac{A_1 B_0^2 + \mu}{A_1 B_0}, \quad A_0 = \frac{\mu}{B_0}, \quad B_1 = \frac{1}{A_0}, \quad k_i = \frac{2\mu A_1^2 B_0^2 + \mu A_1 B_0 k_2 - A_1^2 B_0^2 - A_1^2 B_0^2 k_2 - \mu^2}{A_1^2 B_0^2}, \quad (37)$$

where $A_1, B_0, k_2,$ and $\mu$ are the arbitrary constants.

Substituting (37) into (35) and using similarity transformations (33), we obtain three types of solution of system (1), as follows:

When $\lambda^2 - 4 \mu > 0$,

$$q = e^{\int \alpha_0 dt} \left[ \frac{A_1}{2} \sqrt{\lambda^2 - 4 \mu} \times \left( \frac{C_1 \cosh \left( \frac{1}{2} \sqrt{\lambda^2 - 4 \mu} x \right) + C_2 \sinh \left( \frac{1}{2} \sqrt{\lambda^2 - 4 \mu} x \right)}{C_1 \sinh \left( \frac{1}{2} \sqrt{\lambda^2 - 4 \mu} x \right) + C_2 \cosh \left( \frac{1}{2} \sqrt{\lambda^2 - 4 \mu} x \right)} \right) - \frac{A_1 \lambda}{2 A_0} + \frac{2 \mu A_1}{A_0} \right],$$

$$r = e^{\int \alpha_0 dt} \left[ \frac{1}{2} \frac{A_1}{2} \frac{A_1}{2} \sqrt{\lambda^2 - 4 \mu} \times \left( \frac{C_1 \cosh \left( \frac{1}{2} \sqrt{\lambda^2 - 4 \mu} x \right) + C_2 \sinh \left( \frac{1}{2} \sqrt{\lambda^2 - 4 \mu} x \right)}{C_1 \sinh \left( \frac{1}{2} \sqrt{\lambda^2 - 4 \mu} x \right) + C_2 \cosh \left( \frac{1}{2} \sqrt{\lambda^2 - 4 \mu} x \right)} \right) - \frac{A_1 \lambda}{2 A_0} + \frac{2 \mu A_1}{A_0} \right] \right), \quad (38)$$

where $A_1, C_1, C_2, \lambda,$ and $\mu$ are arbitrary constants and

$$k_i = 2 \left( 6 \mu \lambda^2 + 4 \mu \lambda \kappa_2 - 8 \mu^2 - \lambda^4 - k_2 \lambda^3 \right) \pm \left( 4 \mu \lambda + 2 \mu k_2 - k_2 \lambda^3 - \lambda^3 \right) \sqrt{\lambda^2 - 4 \mu}.$$

When we take $A_1 = 1, C_1 = 2, C_2 = 1, \lambda = 3, \mu = 1$ and $\alpha_0 = \tan t$, the values of $q$ and $r$ are as illustrated in Figure 1, below.
Figure 1: (a). Spatial structure of the exact solution \( q \) of (38) for Eq. (1), with the parameters as \( A_1 = 1, \ C_1 = 2, \ C_2 = 1, \ \lambda = 3, \ \mu = 1, \ \text{and} \ \alpha_0 = \tan t \). (b). Spatial structure of the exact solution \( r \) of (38), in which the parameters are the same as (a).

When \( \lambda^2 - 4\mu < 0 \),

\[
q = e^{-\int_{\alpha_0}^{t} dt} \left\{ \frac{A_1}{2} \sqrt{4\mu - \lambda^2} \times \left( C_1 \cos \left( \frac{1}{2} \sqrt{4\mu - \lambda^2} x \right) - C_2 \sin \left( \frac{1}{2} \sqrt{4\mu - \lambda^2} x \right) \right) - \frac{A_1}{2} \lambda + \frac{2\mu A_1}{\lambda \pm i \sqrt{4\mu - \lambda^2}} \right\},
\]

\[
r = e^{-\int_{\alpha_0}^{t} dt} \left\{ \frac{1}{2} A_1 \sqrt{4\mu - \lambda^2} \times \left( C_1 \cos \left( \frac{1}{2} \sqrt{4\mu - \lambda^2} x \right) - C_2 \sin \left( \frac{1}{2} \sqrt{4\mu - \lambda^2} x \right) \right) - \frac{\lambda \pm i \sqrt{4\mu - \lambda^2}}{2A_1} \right\},
\]

(39)

where \( A_1, \ C_1, \ C_2, \ \lambda, \) and \( \mu \) are arbitrary constants and

\[
k_1 = 2 \frac{4k_2 \mu \lambda + 6 \mu \lambda^2 - k_2 \lambda^3 - \lambda^4 - 8 \mu^2 \pm (2k_2 \mu + 4 \mu k_2 \lambda - k_2 \lambda^2 - \lambda^4) i \sqrt{4\mu - \lambda^2}}{(\lambda \pm i \sqrt{4\mu - \lambda^2})^2}.
\]

When \( \lambda^2 - 4\mu = 0 \),

\[
q = e^{-\int_{\alpha_0}^{t} dt} \left\{ A_1 \left( -\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 x} \right) + \frac{2\mu A_1}{\lambda} \right\},
\]

\[
r = e^{-\int_{\alpha_0}^{t} dt} \left\{ \frac{1}{A_1} \left( -\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 x} \right) + \frac{\lambda}{2A_1} \right\},
\]

(40)

where \( A_1, \ C_1, \ C_2, \ \lambda, \) and \( \mu \) are arbitrary constants and \( k_1 = 0 \).

4.4. Solutions through \( V_2 + V_3 \)

The similarity variables of this generator are

\[
\xi = \int \alpha_1 dt - x, \quad q = e^{-\int_{\alpha_0}^{t} dt} F(\xi), \quad r = e^{-\int_{\alpha_0}^{t} dt} H(\xi),
\]

(41)

and solving the constrained conditions (7), we get

\[
\alpha_1 = k_1 \alpha_1, \quad \alpha_2 = k_2 \alpha_2,
\]

where \( k_1 \) and \( k_2 \) are arbitrary constants, and the AKNS equation (1) is reduced to the following nonlinear coupled ODEs:

\[
F' = -F''' + 6FH^2 - k_2 F'' + 2k_2 F^2 H - k_1 F',
\]
\[ H' = -H'' + 6FHH' + k_1H'' - 2k_2H^2F - k_1H'. \] (42)

We shall use the simplest equation method described in [34] to obtain the solutions of reduction (42). Let us consider the solutions of (42), as
\[ F = \sum_{i=0}^{n} A_i \phi^i(\xi), \quad H = \sum_{i=0}^{n} B_i \phi^i(\xi). \] (43)

By balancing highest order derivative term and nonlinear term in (42), we get \( m = n = 1 \) and \( \phi(\xi) \) satisfies the Riccati equation
\[ \phi'(\xi) = a\phi^2(\xi) + b\phi(\xi) + c. \] (44)

The solutions of (44) can be written as
\[ \phi(\xi) = -\frac{b}{2a} - \frac{\theta}{2a} \tanh\left(\frac{1}{2}\theta(\xi + C)\right), \] (45)

and
\[ \phi(\xi) = -\frac{b}{2a} - \frac{\theta}{2a} \tanh\left(\frac{1}{2}\theta\xi\right) + \frac{\sec h\left(\frac{\theta\xi}{2}\right)}{C \cosh\left(\frac{\theta\xi}{2}\right) - \frac{2a}{\theta} \sinh\left(\frac{\theta\xi}{2}\right)}, \] (46)

where \( \theta^2 = b^2 - 4ac \).

Substituting (43) into (42), an algebraic system of equations in \( A_0, A_1, B_0, \) and \( B_1 \) can be obtained by equating the coefficients of the functions \( \phi'(\xi) \) to zero. With the aid of Maple, solution to this system can be obtained, as follows:
\[ A_0 = \frac{ac}{B_0}, \quad A_1 = \frac{(b \pm \sqrt{\theta^2})a}{2B_0}, \quad B_1 = -\frac{2B_0a}{b \pm \sqrt{\theta^2}}, \quad k_2 = -\theta^2 - k_1 - 1 \pm \sqrt{\theta^2}, \] (47)

where \( B_0, k_1, a, b, \) and \( c \) are arbitrary constants.

Substituting (47) into (43) and using similarity transformations (41), we obtain a set of solutions of system (1) are
\[ q = e^{-[a_{\alpha_{dt}}] \left[ \frac{ac}{B_0} + \frac{(b + \theta)a}{2B_0} \left[ -\frac{b}{2a} - \frac{\theta}{2a} \tanh\left(\frac{1}{2}\theta(\xi + C)\right) \right] \right]}, \]
\[ r = e^{[a_{\alpha_{dt}}] \left[ B_0 + \frac{2B_0a}{b + \theta} \left[ -\frac{b}{2a} - \frac{\theta}{2a} \tanh\left(\frac{1}{2}\theta(\xi + C)\right) \right] \right]}, \] (48)

and
\[ q = e^{-[a_{\alpha_{dt}}] \left[ \frac{ac}{B_0} + \frac{(b + \theta)a}{2B_0} \left[ -\frac{b}{2a} - \frac{\theta}{2a} \tanh\left(\frac{1}{2}\theta\xi\right) + \frac{\sec h\left(\frac{\theta\xi}{2}\right)}{C \cosh\left(\frac{\theta\xi}{2}\right) - \frac{2a}{\theta} \sinh\left(\frac{\theta\xi}{2}\right)} \right] \right]}, \]
\[ r = e^{[a_{\alpha_{dt}}] \left[ B_0 + \frac{2B_0a}{b + \theta} \left[ -\frac{b}{2a} - \frac{\theta}{2a} \tanh\left(\frac{1}{2}\theta\xi\right) + \frac{\sec h\left(\frac{\theta\xi}{2}\right)}{C \cosh\left(\frac{\theta\xi}{2}\right) - \frac{2a}{\theta} \sinh\left(\frac{\theta\xi}{2}\right)} \right] \right]}, \] (49)

where \( \xi = \int \alpha_{\alpha} dt - x. \)

We can choose different values of \( \alpha_{\alpha} \) in solution (48) in order to construct travelling and non-travelling wave solutions of Eq. (1). Figure 2 depicts the travelling wave solution, which is
obtained by taking $\alpha_i = 1$. Figure 3 displays the non-travelling wave solution by selecting $\alpha_i = \cos t$. Other parameters are selected as $B_0 = -3$, $a = 1$, $b = 3$, $c = 1$, and $\alpha_0 = 0.05$.

When we take $B_0 = -3$, $a = 1$, $b = 3$, $c = 1$, $\alpha_0 = -\sin t$, and $\alpha_i = t$ in solution (49), the shapes of non-travelling wave solutions of Eq. (1) are displayed in Figure 4.
Figure 4: (a). Spatial structure of the exact solution $q$ of (49) for Eq. (1), with the parameters as $B_0 = -3, \ a = 1, \ b = 3, \ c = 1, \ \alpha_0 = -\sin t$, and $\alpha_1 = t$. (b). Spatial structure of the exact solution $r$ of (49), in which the parameters are the same as (a).

4.5. Solutions through $V_2 - V_3$

The similarity variables of this generator are

$$\xi = \int \alpha_0 dt + x, \ q = e^{-\int \alpha_0 dt} F(\xi), \ r = e^{\int \alpha_0 dt} H(\xi),$$

and solving the constrained conditions (7), we get

$$\alpha_1 = k_2 \alpha_1, \ \alpha_2 = k_2 \alpha_2,$$

where $k_1$ and $k_2$ are arbitrary constants, and the AKNS equation (1) is reduced to the following nonlinear coupled ODEs:

$$F' = F''' - 6FHF' - k_1 F'' + 2k_2 F^2 H + k_1 F',$$

$$H' = H''' - 6FHH' + k_1 H'' - 2k_2 H^2 F + k_1 H'.$$

(51)

We shall use the simplest equation method to obtain the solutions of reduction (51) [34]. For the Bernoulli equation

$$\phi'(\xi) = a\phi(\xi)^2 + b\phi(\xi),$$

we use the following solution

$$\phi(\xi) = b \left\{ \frac{\cosh[b(\xi + C)] + \sinh[b(\xi + C)]}{1 - a \cosh[b(\xi + C)] - a \sinh[b(\xi + C)]} \right\}.$$

The balancing procedure gives $m = n = 1$ and the solutions of (51), as

$$F = A_0 + A_1 \phi, \ H = B_0 + B_1 \phi.$$  

(53)

Substitution of (53) into (51) yields

$$A_0 = 0, \ B_0 = \frac{ab}{A_1}, \ B_1 = \frac{a^2}{A_1}, \ k_1 = -b^2 + bk_2 + 1,$$

(54)

where $A_1, k_2, a,$ and $b$ are arbitrary constants.

Substituting (54) into (53) and using the similarity transformations (50), we obtain the solution of system (1), as

$$q = e^{-\int \alpha_0 dt} A_1 b \left\{ \frac{\cosh[b(\xi + C)] + \sinh[b(\xi + C)]}{1 - a \cosh[b(\xi + C)] - a \sinh[b(\xi + C)]} \right\},$$

where $A_1, k_2, a,$ and $b$ are arbitrary constants.
where $\xi = \int \alpha \, dt + x$.

Figure 5 illustrates the travelling wave solutions of Eq. (1) by taking $\alpha = 1$, $A = 1$, $a = -1$, $b = 1$, $C = 0$, and $\alpha_0 = \sin t$ in Eq. (55). Figure 6 portrays the non-travelling wave solutions of Eq. (1) by setting $\alpha = t$, and the other parameters are the same as those in Figure 5.
5. Conclusion

In summary, by performing the Lie symmetry analysis on the AKNS equation (1), Lie point symmetries of the AKNS equation are discussed. Moreover, we construct the optimal system of one-dimensional subalgebras of Lie algebra spanned by $V_1 - V_1$. Five types of similarity reduction are presented by using the optimal system. Meanwhile, some new exact solutions, such as power series solutions and travelling and non-travelling wave solutions are obtained for system (1).

It is easy to see that the obtained invariant solutions include coefficient functions $\alpha_0$ and $\alpha_1$, which provide enough freedom for us to construct travelling and non-travelling wave solutions for the AKNS equation (1). This paper shows that the Lie symmetry analysis method is an effective mathematical tool for constructing travelling and non-travelling wave solutions of some other nonlinear PDEs with variable coefficients.

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References

1. Kudryashov, N.A. Solitary wave solutions of hierarchy with non-local nonlinearity. *Appl. Math. Lett.* **2020**, *103*, 106155.
2. Chen, K.; Zhang, D.Z. Solutions of the nonlocal nonlinear Schrödinger hierarchy via reduction. *Appl. Math. Lett.* **2018**, *75*, 82–88.
3. Cui, W.Y.; Zha, Q. Multiple rogue wave and breather solutions for the (3+1)-dimensional KPI equation. *Appl. Math. Comput.* **2018**, *76*, 1099–1107.
4. Xu, G.X. Painlevé analysis, lump-kink solutions and localized excitation solutions for the (3+1)-dimensional Boiti-Leon-Manna-Pempinelli equation. *Appl. Math. Comput.* **2018**, *75*, 82–88.
5. Xie, X.Y.; Tian, B.; Sun, W.R.; Sun, Y. Rogue-wave solutions for the Kundu-Eckhaus equation with variable coefficients in an optical fiber. *Nonlinear Dyn.* **2015**, *81*, 1349–1354.
6. Wu, X.F.; Hua, G.S.; Ma, Z.Y. Evolution of optical solitary waves in a generalized nonlinear Schrödinger equation with variable coefficients. *Nonlinear Dyn.* **2012**, *70*, 2259–2267.
7. Lù, X.; Zhu, H.W.; Meng, X.H.; Yang, Z.C.; Tian, B. Soliton solutions and a Bäcklund transformation for a generalized nonlinear Schrödinger equation with variable coefficients from optical fiber communications. *J. Math. Anal. Appl.* **2007**, *336*, 1305–1315.
8. Zhang, H.Q.; Yuan, S.S. Dark soliton solutions of the defocusing Hirota equation by the binary Darboux transformation. *Nonlinear Dyn.* **2017**, *89*, 531–538.
9. Kudryavtsev, A.G. Nonlocal Darboux transformation of the two-dimensional stationary Schrödinger equation and its relation to the Moutard transformation. *Theor. Math. Phys.* **2016**, *187*, 455–462.
10. Su, J.J.; Gao, Y.T.; Ding, C.C. Darboux transformations and rogue wave solutions of a generalized AB system for the geophysical flows. *Appl. Math. Lett.* **2019**, *88*, 201–208.
11. Yomba, E. Construction of new soliton-like solutions for the (2+1) dimensional KdV equation with variable coefficients. *Chaos Solitons Fractals* **2004**, *21*, 75–79.
12. Adem, A.R. Symbolic computation on exact solutions of a coupled Kadomtsev-Petviashvili equation: Lie symmetry analysis and extended tanh method. *Appl. Math. Comput.* **2017**, *74*, 1897–1902.
13. Hong, B.; Lu, D. New Jacobi elliptic function-like solutions for the general KdV equation with variable coefficients. *Math. Comput. Model.* **2012**, *55*, 1594–1600.
14. Kuzenov, V.V.; Ryzhkov, S.V. Numerical modeling of laser target compression in an external magnetic field. *Math. Models Comput. Simul.* **2018**, *10*, 255–264.
15. Bluman, G.W.; Anco, S.C. *Symmetry and Integration Methods for Differential Equations*; Springer: New York, NY, USA, 2002.
16. Olver, P.J. *Applications of Lie Groups to Differential Equations*; Springer: Berlin/Heidelberg, Germany, 1993.
17. Ovsiannikov, L.V. *Group Analysis of Differential Equations*; Academic Press: New York, NY, USA, 1982.
18. Wang, G.W.; Liu, Y.X.; Han, S.X.; Wang, H.; Su, X. Generalized symmetries and mCK method analysis of the (2+1)-Dimensional coupled Burgers equations. *Symmetry* 2019, 11, 1473.
19. Abdulwahhab, M.A. Optimal system and exact solutions for the generalized system of 2-dimensional Burgers equations with infinite Reynolds number. *Commun. Nonlinear Sci. Numer. Simul.* 2015, 20, 98–112.
20. Kaur, L.; Wazwaz, A.M. Painlevé analysis and invariant solutions of generalized fifth-order nonlinear integrable equation. *Nonlinear Dyn.* 2018, 94, 2469–2477.
21. Shi, D.D.; Zhang, Y.F.; Liu, W.H.; Liu, J.G. Some exact solutions and conservation laws of the coupled Time-Fractional Boussinesq-Burgers system. *Symmetry* 2019, 11, 77.
22. Liu, H.; Li, J.; Liu, L. Painlevé analysis, Lie symmetries, and exact solutions for the time-dependent coefficients Gardner equations. *Nonlinear Dyn.* 2009, 59, 497–502.
23. Gupta, R.K.; Kumar, V.; Jiwarl, R. Exact and numerical solutions of coupled short pulse equation with time-dependent coefficients. *Nonlinear Dyn.* 2014, 79, 455–464.
24. Liu, N. Similarity reduction and explicit solutions for the variable-coefficient coupled Burger’s equations. *Appl. Math. Comput.* 2010, 217, 4178–4185.
25. Kumar, R.; Gupta, R.K.; Bhatia, S.S. Invariant solutions of variable coefficients generalized Gardner equation. *Nonlinear Dyn.* 2016, 83, 2103–2111.
26. Singh, K.; Gupta, R.K.; Kumar, S. Benjamin-Bona-Mahony (BBM) equation with variable coefficients: Similarity reductions and Painlevé analysis. *Appl. Math. Comput.* 2011, 217, 7021–7027.
27. Zhang, S.; Gao, X. Exact N-soliton solutions and dynamics of a new AKNS equation with time-dependent coefficients. *Nonlinear Dyn.* 2016, 83, 1043–1052.
28. Chen, D.Y.; Zhang, D.J.; Bi, J.B. New double Wronskian solutions of the AKNS equation. *Sci. China Ser. A Math.* 2008, 51, 55–69.
29. Chen, K.; Deng, X.; Lou, S.Y.; Zhang, D.J. Solutions of local and nonlocal equations reduced from the AKNS hierarchy. *arXiv* 2018, arXiv: 1710.10479v2.
30. Grigoriev, Y.N.; Meleshko, S.V.; Ibragimov, N.K.; Kovalev, V.F. *Symmetry of Integro-Differential Equations: With Applications in Mechanics and Plasma Physics*; Springer: New York, NY, USA, 2010.
31. Arrigo, D.J. *Symmetry Analysis of Differential Equations*; Wiley: New York, NY, USA, 2015.
32. Zhao, Z.L.; Han, B. Lie symmetry analysis of the Heisenberg equation. *Commun. Nonlinear Sci. Numer. Simul.* 2017, 45, 220–234.
33. Wang, M.L.; Li, X.; Zhang, J. The \( \frac{G'}{G} \)-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. *Phys. Lett. A* 2008, 372, 417–423.
34. Kudryashov, N.A. Simplest equation method to look for exact solutions of nonlinear differential equations. *Chaos Solitons Fractals* 2005, 24, 1217–1231.