A Note on Global Optimization for Max-Plus Linear Systems

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Abstract This note further addresses the global optimization problem for max-plus linear systems considered in [Automatica 119 (2020) 109104]. Firstly, the operations between $\pm\infty$ and real numbers involved in the formulas of solving global optimization problems are explained explicitly. Secondly, the formula of the greatest lower bound and the criterion of solvability of globally optimal solutions are simplified. Thirdly, the criterion of uniqueness of globally optimal solutions and the set of all globally optimal solutions are presented.

Keywords Max-plus linear system, global optimization, optimal solutions

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1 Introduction

Max-plus linear systems can describe some nonlinear time-evolution systems with synchronization but no concurrency, such as flexible manufacturing systems, flow shop scheduling, traffic managements, communication networks (see, e.g., [1–3]). Many progresses have been made in control and optimization of max-plus linear systems (see, e.g., [4–19]).

Global optimization is to find the global minimizer of a function or a set of functions over a given set, which has been a basic tool in all areas of engineering, medicine, economics, and other sciences. The global optimization for max-plus linear systems has been considered in [20], whose objective function is a max-plus function and constraint function is a real function. This note is a further explanation and extension of the results in [20].

Some formulas for globally optimal solutions in [20] involves the operations between some special elements in max-plus algebra and real numbers. This note will explain these operations definitely and simplify the formulas. In addition, reference [20] neglects to consider the 0 coefficients in the constraint function when it discusses the uniqueness of globally optimal solutions. This note will present a necessary and sufficient condition for the uniqueness and construct the set of all globally optimal solutions.

2 Preliminaries

Let us introduce some basic definitions and notations from the max-plus algebra, which can be consulted for more details in [1–3].

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}^+$ be the set of positive integers. For $n \in \mathbb{N}^+$, denote by $\mathbb{N}_n$ the set $\{1, 2, \ldots, n\}$. Let $\mathbb{R}_\text{max}$ be the set $\mathbb{R} \cup \{-\infty\}$ with max and + as two binary operations $\oplus$ and $\otimes$, respectively, i.e., for $a, b \in \mathbb{R}_\text{max}$,

$a \oplus b = \max\{a, b\}$ and $a \otimes b = a + b$.

$(\mathbb{R}_\text{max}, \oplus, \otimes)$ is called the max-plus algebra, in which $-\infty$ is the zero element denoted by $\varepsilon$, and 0 is the identity element denoted by $e$. For $a, b \in \mathbb{R}_\text{max}$, $a \leq b$ if $a \oplus b = b$.

Let $\mathbb{R}_\text{max}^n$ and $\mathbb{R}_\text{max}^{m \times n}$ be the set of $n$-dimensional vectors and $m \times n$ matrices with entries in $\mathbb{R}_\text{max}$, respectively. The vectors and matrices are repre-
sented by bold-type letters. The addition and multiplication can be extended to the vector-matrix algebra over $\mathbb{R}_{\text{max}}$ by the direct analogy to the conventional linear algebra: For $A = (a_{ij})$, $B = (b_{ij}) \in \mathbb{R}_{\text{max}}^{m \times n}$, $(A \oplus B)_{ij} = a_{ij} \oplus b_{ij}$; for $A = (a_{ij}) \in \mathbb{R}_{\text{max}}^{m \times p}$ and $B = (b_{ij}) \in \mathbb{R}_{\text{max}}^{p \times n}$,

$$(A \oplus B)_{ij} = \bigoplus_{k=1}^{p} a_{ik} \oplus b_{kj}.$$ 

For $A, B \in \mathbb{R}_{\text{max}}^{m \times n}$, $A \leq B$ if $A \oplus B = B$. For $A, B \in \mathbb{R}_{\text{max}}^{m \times n}$ and $x, y \in \mathbb{R}_{\text{max}}^{n}$, if $A \leq B$ and $x \leq y$, then $A \otimes x \leq B \otimes y$.

For $A \in \mathbb{R}_{\text{max}}^{m \times n}$ and $x \in \mathbb{R}_{\text{max}}^{n}$, $F(x) = A \otimes x$ is called a max-plus function of type $(n, m)$. Denoted by $F_i(x)$ the $i$th component of $F(x)$. A max-plus system is a system that can be described by using max-plus functions.

Given $A = (a_{ij}) \in \mathbb{R}_{\text{max}}^{m \times n}$ and $b = (b_i) \in \mathbb{R}_{\text{max}}^{m}$, a system of max-plus linear equations is defined as

$$A \otimes x = b,$$  

where $x = (x_j)$ is considered in $\mathbb{R}_{\text{max}} = \mathbb{R}_{\text{max}}(= \mathbb{R}_{\text{max}} \cup \{+\infty\})$, rather than in $\mathbb{R}_{\text{max}}$. System (1) is said to be solvable if there exists $\tilde{x} \in \mathbb{R}_{\text{max}}^n$ such that $A \otimes \tilde{x} = b$, and $\tilde{x}$ is called a solution of system (1). For $\tilde{x} \in \mathbb{R}_{\text{max}}^n$, $\tilde{x}$ is called a subsolution of system (1) if $A \otimes \tilde{x} \leq b$.

**Definition 1.** [20] Problem (3) is said to be solvable if there exists $\tilde{x} \in \mathbb{X}$ such that $F(\tilde{x}) \geq F(x)$ for any $x \in \mathbb{X}$, and $\tilde{x}$ is called a globally optimal solution of problem (3). Otherwise, problem (3) is said to be unsolvable.

Note that, for problem (3), an $\varepsilon$ row in $A$ has no effect on globally optimal solutions. In addition, if the $j_0$th column of $A$ is $\varepsilon$ and meanwhile $k_{j_0} = 0$, then $x_{j_0}$ disappears from both the objective function and the constraint function of problem (3). Consequently, problem (3) becomes into the optimization of variables $\{x_j \mid j \neq j_0\}$. Without loss of generality, let us make the following assumptions:

1) $A$ is row $\mathbb{R}$-astic, i.e., $A$ has no $\varepsilon$ row.
2) $k_j \neq 0$ if the $j$th column of $A$ is $\varepsilon$.

**Theorem 1.** [20] For problem (3), let $b = (b_i) \in \mathbb{R}_{\text{max}}^m$ be defined by

$$b_i = \frac{\sum_{j=1}^{n} k_{ij} a_{ij} + c}{\sum_{j=1}^{n} k_{ij}}, \quad i \in \mathbb{N}_m.$$  

Then, $b$ is the greatest lower bound of problem (3):

1) $F(x) \geq b$ for any $x \in \mathbb{X}$;
2) if $y$ is a lower bound of problem (3), then $y \leq b$.

The calculation of (4) may refer to some operations between elements $\pm \infty$ in $\mathbb{R}_{\text{max}}$ and real numbers in $\mathbb{R}$. To compute the greatest lower bound by (4), one needs to make the following statements:

For $r, p \in \mathbb{R}$ and $p > 0$,

$$\max\{r, -\infty\} = r, \quad \max\{r, +\infty\} = +\infty,$$

$$r + (-\infty) = -\infty, \quad r + (+\infty) = +\infty,$$

$$r - (-\infty) = +\infty, \quad r - (+\infty) = -\infty,$$

$$(-\infty) - r = -\infty, \quad (+\infty) - r = +\infty,$$

$$p \times (-\infty) = -\infty, \quad p \times (+\infty) = +\infty,$$

$$0 \times (-\infty) = 0, \quad 0 \times (+\infty) = 0,$$

$$(+\infty) + (-\infty) = -\infty, \quad (-\infty) - (-\infty) = +\infty.$$  

Considering that 0 coefficients in the constraint function have no function in calculating the greatest lower bound, formula (4) can equivalently be simplified as

$$b_i = \frac{\sum_{j \in \mathcal{J}} k_{ij} a_{ij} + c}{\sum_{j \in \mathcal{J}} k_{ij}}, \quad i \in \mathbb{N}_m,$$  

where $\mathcal{J} = \{j \in \mathbb{N}_n \mid k_j > 0\}$. The formula above avoids the multiplication $\times$ between $0$ and infinity elements in $\mathbb{R}_{\text{max}}$.

3 Global optimization

Consider the global optimization problem

$$\min_{x \in \mathbb{X}} F(x),$$  

where the optimization variable is $x = (x_j) \in \mathbb{R}^n$; the objective function is $F(x) = A \otimes x$, in which $A = (a_{ij}) \in \mathbb{R}_{\text{max}}^{m \times n}$; the constraint set is

$$\mathbb{X} = \left\{x \mid \sum_{j=1}^{n} k_j x_j = c, \ k_j \geq 0, \ k_j \text{ are not all 0}\right\},$$

in which $c \in \mathbb{R}$ is a constant.
4 Optimal solutions

Let us provide two necessary conditions for the existence of globally optimal solutions.

**Lemma 2.** If problem (3) is solvable, then
1) the greatest lower bound \( b \) given in (5) is finite;
2) \( x^*(A, b) \) is finite.

**Proof.** 1) Suppose that there exists \( i_0 \in \mathbb{N}_m \) such that \( b_{i_0} = \varepsilon \). Since \( A \) is row \( \mathbb{R} \)-astic, there exists \( j_0 \in \mathbb{N}_n \) such that \( a_{i_0j_0} \neq \varepsilon \). Let \( \tilde{x} = (\tilde{x}_j) \) be a globally optimal solution of problem (3). Then,
\[
\varepsilon = b_{i_0} = F_{i_0}(\tilde{x}) = \max_{j \in \mathbb{N}_n} \{a_{i_0j} + \tilde{x}_j\} \geq a_{i_0j_0} + \tilde{x}_{j_0}.
\]
This implies that \( \tilde{x}_{j_0} = \varepsilon \), and hence \( \tilde{x} \notin \mathcal{X} \). This contradiction indicates that \( b \) is finite.

2) Since \( b \) is finite, \( x^*_j(A, b) \neq \varepsilon \) for any \( j \in \mathbb{N}_n \). Suppose that there exists \( j_0 \in \mathbb{N}_n \) such that
\[
x^*_j(A, b) = \min_{i \in \mathbb{N}_m} \{a_{ij} - a_{ij_0}\} = +\infty.
\]
This implies that \( a_{ij_0} = \varepsilon \) for any \( i \in \mathbb{N}_m \). Assumption 2) ensures that \( k_{j_0} > 0 \), i.e., \( j_0 \notin \mathcal{J} \). According to (5), \( b = \varepsilon \), and hence problem (3) is unsolvable. This contradiction indicates that \( x^*(A, b) \) is finite. \( \square \)

It can be inferred from the lemma above that if
\[
\sum_{j \in \mathcal{J}} k_j x^*_j(A, b) = \sum_{j=1}^n k_j x^*_j(A, b) = c, \tag{6}
\]
then \( x^*(A, b) \) is finite and \( x^*(A, b) \in \mathcal{X} \). The criteria of solvability given in [20] can be then represented as follows.

**Theorem 2.** [20] For problem (3), the following statements are equivalent:
1) Problem (3) is solvable.
2) Equation (6) holds.
3) \( x^*(A, b) \) is a globally optimal solution.

Let us see a numerical example of global optimization problem that contains \( \varepsilon \) coefficients in the objective function and 0 coefficients in the constraint function.

**Example 1.** Consider the global optimization problem
\[
\min_{x \in \mathcal{X}} F(x), \tag{7}
\]
where the objective function is \( F(x) = A \otimes x \),
\[
A = \begin{pmatrix}
1 & 2 & -2 \\
-1 & e & \varepsilon \\
e & 1 & 3
\end{pmatrix},
\]
and the constraint set is \( \mathcal{X} = \{x | 2x_1 + x_2 = 2\} \). By (5), \( b = (2 \ e \ 1)^T \). By (2), \( x^*(A, b) = (1 \ e \ -2)^T \). Then, \( 2x^*_1(A, b) + x^*_2(A, b) = 2 = c \). It follows from Theorem 2 that problem (7) is solvable, and \( (1 \ e \ -2)^T \) is a globally optimal solution.

If the constraint set of problem (7) is replaced by \( \mathcal{X} = \{x | 2x_1 + x_2 + x_3 = 2\} \), then \( b_2 = \varepsilon \). It can be obtained from Lemma 2 that such a problem has no globally optimal solution.

Note that, the globally optimal solution introduced in Definition 1 may be not unique. According to [20, Theorem 3], if problem (3) is solvable, then \( k_j > 0 \) for any \( j \in \mathbb{N}_n \) is a sufficient condition for the uniqueness of globally optimal solutions. In fact, such a condition is also necessary.

**Theorem 3.** Let problem (3) be solvable. Problem (3) has a unique globally optimal solution if and only if \( k_j > 0 \) for any \( j \in \mathbb{N}_n \).

**Proof.** The sufficiency can be obtained from the proof of [20, Theorem 3]. Let us now prove the necessity. Since problem (3) is solvable, equation (6) holds. Suppose that there exists \( j_0 \in \mathbb{N}_n \) such that \( k_{j_0} = 0 \), i.e., \( j_0 \notin \mathcal{J} \). Let \( \tilde{x} = (\tilde{x}_j) \in \mathbb{R}^n \) be defined by
\[
\tilde{x}_j = \begin{cases}
\varepsilon, & j = j_0; \\
x^*_j(A, b), & j \neq j_0.
\end{cases} \tag{8}
\]
It can be known from Lemma 2 that \( x^*(A, b) \) is finite. Then, \( \tilde{x} < x^*(A, b) \) and
\[
\sum_{j \in \mathcal{J}} k_j \tilde{x}_j = \sum_{j \in \mathcal{J}} k_j x^*_j(A, b) = c,
\]
i.e., \( \tilde{x} \in \mathcal{X} \). By Theorem 1, \( F(\tilde{x}) \geq b \). In addition,
\[
F(\tilde{x}) \leq F(x^*(A, b)) = A \otimes x^*(A, b) \leq b.
\]
Hence, \( F(\tilde{x}) = b \). This implies that \( \tilde{x} \) is also a globally optimal solution, which contradicts with the uniqueness. Hence, \( k_j > 0 \) for any \( j \in \mathbb{N}_n \). \( \square \)

The proof of Theorem 3 is constructive and formula (8) can be extended to find all globally optimal solutions as follows.

**Theorem 4.** If problem (3) is solvable, then the set of all globally optimal solutions is
\[
\mathcal{S} = \left\{ \tilde{x} \left| \begin{array}{c}
\tilde{x}_j \leq x^*_j(A, b) \text{ if } k_j = 0; \\
\tilde{x}_j = x^*_j(A, b) \text{ if } k_j > 0.
\end{array} \right. \right\}.
\]
Proof. Since problem (3) is solvable, equation (6) holds. On the one hand, for any \( \tilde{x} \in S \),
\[
\sum_{j \in J} k_j \tilde{x}_j = \sum_{j \in J} k_j x_j^* (A, b) = c,
\]
i.e., \( \tilde{x} \in X \). By Theorem 1, \( F(\tilde{x}) \geq b \). In addition,
\[
F(\tilde{x}) \leq F(x^* (A, b)) = A \otimes x^* (A, b) \leq b.
\]
Hence, \( F(\tilde{x}) = b \). This implies that \( \tilde{x} \) is a globally optimal solution. On the other hand, let \( \bar{x} \) be a globally optimal solution. Then, \( \bar{x} \in X \) and \( F(\bar{x}) = A \otimes \bar{x} = b \). By Lemma 1, \( \bar{x} \leq x^* (A, b) \). Suppose that there exists \( j_0 \in J \) such that \( \tilde{x}_{j_0} < x_{j_0}^* (A, b) \). Then,
\[
c = \sum_{j \in J \setminus \{ j_0 \}} k_j \tilde{x}_j + k_{j_0} \tilde{x}_{j_0}
< \sum_{j \in J \setminus \{ j_0 \}} k_j x_j^* (A, b) + k_{j_0} x_{j_0}^* (A, b)
= \sum_{j \in J} k_j x_j^* (A, b) = c.
\]
This contradiction implies that \( \tilde{x}_j = x_j^* (A, b) \) for any \( j \in J \), and so \( \tilde{x} \in S \). Hence, \( S \) is the set of all globally optimal solutions. \( \square \)

Owing to the theorem above, \( x^* (A, b) \) is referred to as the greatest globally optimal solution of problem (3) if equation (6) holds.

Example 2. Find all globally optimal solutions of problem (7). It has been calculated in Example 1 that \( x^* (A, b) = (1 \ 0 \ -2)^T \). By Theorem 4, the set of all globally optimal solutions is
\[
S = \{ (1 \ 0 \ a)^T \mid a \leq -2 \}.
\]

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