Pieri-Type Formulas for the Nonsymmetric Macdonald Polynomials

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Abstract

In symmetric Macdonald polynomial theory the Pieri formula gives the branching coefficients for the product of the \( r \)th elementary symmetric function \( e_r(z) \) and the Macdonald polynomial \( P_\kappa(z) \). In this paper we give the nonsymmetric analogues for the cases \( r = 1 \) and \( r = n - 1 \). We do this by first deducing the decomposition for the product of any nonsymmetric Macdonald polynomial \( E_\eta(z) \) with \( z_i \) in terms of nonsymmetric Macdonald polynomials. As a corollary of finding the branching coefficients of \( e_1(z) E_\eta(z) \) we evaluate the generalised binomial coefficients \( \binom{\eta}{\nu} \) associated with the nonsymmetric Macdonald polynomials for \(|\eta| = |\nu| + 1\).

1 Introduction

The nonsymmetric Macdonald polynomials \( E_\eta := E_\eta(z;q,t) \) are polynomials of \( n \) variables \( z = (z_1, \ldots, z_n) \) having coefficients in the field \( \mathbb{Q}(q,t) \) of rational functions of the indeterminants \( q \) and \( t \). The compositions \( \eta := (\eta_1, \ldots, \eta_n) \) of non-negative integers parts \( \eta_i \) label these polynomials. The nonsymmetric Macdonald polynomials can be defined, up to normalisation, as the unique simultaneous eigenfunctions of the commuting operators

\[
Y_i = t^{-n+1} T_i \ldots T_{n-1} \omega T_1^{-1} \ldots T_{i-1}^{-1}, \quad (i = 1, \ldots, n)
\]

satisfying the eigenvalue equations

\[
Y_i E_\eta(z; q, t) = \bar{\eta}_i E_\eta(z; q, t).
\]

In (1) \( T_i \) denotes the Demazure-Lustig operator,

\[
T_i := t + \frac{tz_i - z_{i+1}}{z_i - z_{i+1}} (s_i - 1),
\]

while

\[
\omega := s_n \ldots s_1 \tau_1,
\]

where \( s_i \) is a transposition operator with the action on functions

\[
(s_i f)(z_1, \ldots, z_i, z_{i+1}, \ldots, z_n) := f(z_1, \ldots, z_{i+1}, z_i, \ldots, z_n).
\]
The operator \( \tau_i \) has the action on functions

\[
(\tau_i f)(z_1, ..., z_n) := f(z_1, ..., qz_i, ..., z_n)
\]

and so corresponds to a \( q \)-shift of the variable \( z_i \). The eigenvalue \( \overline{\eta}_i \) in (2) is given by

\[
\overline{\eta}_i := q^{\eta_i} t^{-\ell'_\eta(i)},
\]

where

\[
\ell'_\eta(i) := \# \{ j < i; \eta_j \geq \eta_i \} - \# \{ j > i; \eta_j > \eta_i \}.
\]

Nonsymmetric Macdonald polynomials are of the triangular form

\[
E_\eta(z; q, t) := z^\eta + \sum_{\nu \prec \eta} b_{\eta\nu} z^\nu,
\]

for coefficients \( b_{\eta\nu} \). The notation \( z^\eta \) denotes the monomial

\[
z^\eta := z_1^{\eta_1} ... z_n^{\eta_n}.
\]

In (7) the coefficient of \( z^\eta \) has been chosen to be unity as a normalisation. The ordering \( \prec \) is a partial ordering on compositions having the same modulus, where \( |\eta| := \sum_{i=1}^{n} \eta_i \) denotes the modulus of \( \eta \). The partial ordering is defined by

\[
\mu \prec \eta \text{ iff } \mu^+ < \eta^+ \text{ or in the case } \mu^+ = \eta^+, \mu \prec \eta
\]

where \( \eta^+ \) is the unique partition obtained by permuting the components of \( \eta \) and \( \mu < \eta \) iff \( \mu \neq \eta \) and \( \sum_{i=1}^{p} (\eta_i - \mu_i) \geq 0 \) for all \( 1 \leq p \leq n \).

Nonsymmetric Macdonald polynomials were first introduced in 1994 [11,2], six years after Macdonald’s paper [12] introducing what are now referred to as symmetric Macdonald polynomials \( P_\kappa(z; q, t) \). The symmetric Macdonald polynomials are indexed by partitions \( \kappa \) rather than compositions. The nonsymmetric Macdonald polynomials can be regarded as building blocks of their symmetric counterparts, as symmetrisation of \( E_\eta \) gives \( P_{\eta^+} \). The required symmetrisation operation is defined by

\[
U^+ := \sum_{\sigma \in S_n} T_\sigma,
\]

where \( S_n \) denotes the set of all permutations of \( \mathbb{N}^n \) and with \( \sigma := s_{i_1} ... s_{i_1} \) the operator \( T_\sigma \) is specified by

\[
T_\sigma := T_{i_1} ... T_{i_1},
\]

where \( T_i \) is defined by [3]. The symmetrising operator allows many fundamental properties of the symmetric Macdonald polynomials to be deduced as corollaries of the corresponding properties of the nonsymmetric Macdonald polynomials [14]. However, the converse does not apply, as some special properties of symmetric Macdonald polynomials
have no known nonsymmetric analogues. For example, the Pieri-type formula [13, Section VI.6]

\[ e_r (z) P_\kappa (z; q, t) = \sum_\lambda \psi_{\lambda/\kappa} P_\lambda (z; q, t) \]  

(8)
giving the explicit form of the branching coefficients \( \psi_{\lambda/\kappa} \) for the product of \( P_\kappa (z; q, t) \) with the \( r \)th elementary symmetric function

\[ e_r (z) = \sum_{1 \leq i_1 < ... < i_r \leq n} z_{i_1} ... z_{i_r} \]

has no known non-symmetric analogue. In (8) the sum is over \( \lambda \) such that \( \lambda/\kappa \) is a vertical \( m \)-strip.

Pieri-type formulas themselves have found recent applications in studies of certain vanishing properties of Macdonald polynomials at \( t^{k+1}q^{r-1} = 1 \) [3]. Furthermore, the dual of (8) has found application in the study of certain probabilistic models related to the Robinson-Schensted-Knuth correspondence [4].

It is the main objective of this paper to provide the explicit branching coefficients for the products \( z_i E_\eta (z; q, t) \), \( e_1 (z) E_\eta (z; q, t) \) and \( e_{n-1} (z) E_\eta (z; q, t) \) in terms of higher order nonsymmetric polynomials. The latter expansions, which in fact will be derived as corollaries of the first, are the nonsymmetric analogues of (8) for \( r = 1 \) and \( r = n - 1 \).

That such branching formulas can be derived is suggested by Jack polynomial theory. Jack polynomials \( E_\eta (z; \alpha) \) are the limit \( q = t^\alpha, q \to 1 \) of Macdonald polynomials. Marshall [15] derived the branching coefficients for the products \( z_i E_\eta (z; \alpha) \) and \( e_1 (z) E_\eta (z; \alpha) \) following a strategy of Knop and Sahi [8], which proceeds by exploiting the theory of interpolation Jack polynomials.

Similarly to the Jack case, the interpolation polynomials play a key role in deriving the branching coefficients for the product \( z_i E_\eta (z; q, t) \). The nonsymmetric interpolation Macdonald polynomials are denoted by \( E_\eta^* (z; q, t) \) and can be defined, up to normalisation, as the unique polynomial of degree \( \leq |\eta| \) satisfying

\[ E_\eta^* (\mu) = 0, \quad |\mu| \leq |\eta|, \mu \neq \eta \]  

(9)

and \( E_\eta^* (\overline{\eta}) \neq 0 \), where \( \overline{\eta} = (\overline{\eta}_1, ..., \overline{\eta}_n) \) with \( \overline{\eta}_j \) specified by (5). The interpolation Macdonald polynomials have a triangular expansion in terms of Macdonald polynomials

\[ E_\eta^* (z; q, t) = E_\eta (z; q, t) + \sum_{|\mu| \leq |\eta| \atop \mu \neq \eta} b_{\eta\mu} E_\mu (z; q, t) \],
for coefficients $b_{\eta \mu}$. Again, the leading coefficient has been chosen to be unity as a normalisation.

The overall strategy for finding the coefficients is to introduce a mapping $\Psi$ between $E_\eta$ and $E^*_\eta$ that can be used to intertwine the actions of multiplication by $z_i$ on $E_\eta$ and a certain operator $Z_i$ on $E^*_\eta$. Hence, by first determining an explicit form for the coefficients $c^{(i)}_{\lambda \eta}$ in the expansion

$$ Z_i E^*_\eta (z) = \sum_{\nu} c^{(i)}_{\lambda \eta} E^*_\lambda (z) $$

we can apply the mapping $\Psi$ to obtain an explicit form of the coefficients of $z_i E_\eta$ in terms of the $E_\lambda$ (Section 3). Using this result we can derive the explicit formula for the expansion of $e_1 (z) E_\eta$ (Section 4). The expansion of $e_{n-1} (z) E_\eta$ (Section 5) then follows from this using the identity $E_\eta (z^{-1}; q, t) = E_{-\eta^R} (z; q, t)$ [14], where $\eta^R := (\eta_n, ..., \eta_1)$.

The branching coefficients for $z_i E_\eta$, $e_1 (z) E_\eta$ and $e_{n-1} (z) E_\eta$ are given in Propositions 7, 8 and 10 respectively. As a consequence of finding $e_1 (z) E_\eta$ we are able to give an evaluation of the generalised binomial coefficients $^\eta_\nu$ associated with the nonsymmetric Macdonald polynomials for $|\eta| = |\nu| + 1$ (Section 8). This is given in Proposition 9.

In the final section we take the limit $t = q^{1/\alpha}$, $q \rightarrow 1$ of our result for $e_1 (z) E_\eta (z; q^{-1}, t^{-1})$ to reclaim the known expansion of $E_\eta (z; \alpha)$ in the theory of nonsymmetric Jack polynomials [15].

Note added: After completing this work, and posting it on the arXiv, correspondence was received from Ole Warnaar, pointing out a recent manuscript of Lascoux [10], available only on his website, containing results equivalent to our Propositions 7 and 8.

## 2 Hecke Operators and the Intertwining Formula

Hecke operators play an important role in interpolation Macdonald polynomial theory. They are realisations of the type-A Hecke algebra

$$ (H_i + 1) (H_i - t) = 0 $$

$$ H_i H_{i+1} H_i = H_{i+1} H_i H_{i+1}, \quad i = 2, ..., n - 2 $$

$$ H_i H_j = H_j H_i, \quad |i - j| > 1. $$

The Hecke operators of interest, $H_i$, are defined by

$$ H_i := \frac{(t - 1) z_i}{z_i - z_{i+1}} + \frac{z_i - tz_{i+1}}{z_i - z_{i+1}} s_i, $$

(12)
where $s_i$ is specified by (4). These Hecke operators appear in the eigenoperators of the interpolation Macdonald polynomials. The eigenoperators, which mutually commute are defined by

$$\Xi_i := z_i^{-1} + z_i^{-1} H_i \ldots H_{n-1} \Phi H_1 \ldots H_{i-1},$$

(13)

where

$$\Phi := (z_n - t^{-n+1}) \Delta$$

(14)

and

$$\Delta f(z_1, \ldots, z_n) = f\left(\frac{z_n}{q}, z_1, \ldots, z_{n-1}\right).$$

(15)

Explicitly, the operators $\Xi_i$ satisfy

$$\Xi_i E^*_\eta(z; q, t) = \eta_i^{-1} E^*_\eta(z; q, t),$$

(15)

where $\eta_i$ is given by (5). The algebraic relations (11) are invariant under the mapping $H_i \mapsto -H_i - 1 + t$. Hence, the operators $-\Pi_i$, where

$$\Pi_i := (t-1) z_{i+1} + \frac{z_i - tz_i + 1}{z_i - z_{i+1}},$$

are also realisations of the type-A Hecke algebra. These operators appear in the eigenoperator of $E^*_\eta(z; q^{-1}, t^{-1})$ according to

$$\xi_i^{-1} := \Pi_i \ldots \Pi_{n-1} \Delta H_1 \ldots H_{i-1}.$$ 

By observing

$$\Xi_i = \xi_i^{-1} + \text{degree lowering terms},$$

Knop [9] showed that the top homogeneous component of any interpolation Macdonald polynomial $E^*_\eta(z; q, t)$ is $E^*_\eta(z; q^{-1}, t^{-1})$. Hence, we can define an isomorphism $\Psi$ mapping each Macdonald polynomial $E^*_\eta(z; q^{-1}, t^{-1})$ to its corresponding interpolation polynomial $E^*_\eta(z; q, t)$,

$$\Psi E^*_\eta(z; q^{-1}, t^{-1}) = E^*_\eta(z; q, t).$$

(16)

From this isomorphism we are able to define the important intertwining formula, Eqn (18) below. This is due to Knop [9], however in the following an alternative proof is given.

**Proposition 1.** [9, Theorem 5.1] Define

$$Z_i := t^{-\binom{d}{2}} (z_i \Xi_i - 1) \Xi_1 \ldots \hat{\Xi}_i \ldots \Xi_n,$$

(17)

where the hat superscript on $\hat{\Xi}_i$ denotes the absence of $\Xi_i$ in the product of operators $\Pi_{j=1}^n \Xi_j$, and let $M$ be the operator which acts on the subspace of homogeneous polynomials of degree $d$ by multiplication with $q^{-\binom{d}{2}}$. With $\Psi$ as defined in (16) we have

$$Z_i \Psi M = \Psi M z_i.$$ 

(18)
Proof. First consider the action of $Z_i$ on $E^*_\eta (z; q, t)$. By the definition of $Z_i$ and commutativity of the $\Xi_i$ we have

$$Z_i E^*_\eta (z; q, t) = (z_i - \Xi_i^{-1}) t^{-\binom{z_i}{2}} \Xi_1 \cdots \Xi_n E^*_\eta (z; q, t). \quad (19)$$

Using (15), (5), then the identity $\Sigma_i t'_\eta (i) = \binom{n}{2}$ we can simplify (19) to

$$q^{-|\eta|}(z_i - \Xi^{-1}_i) E^*_\eta (z; q, t). \quad (20)$$

Since (20) vanishes for all $z = \lambda$, $|\lambda| \leq |\eta|$, due to (9), and has degree $|\eta| + 1$ we must have

$$Z_i E^*_\eta (z; q, t) = q^{-|\eta|} \sum_{\lambda: |\lambda| = |\eta| + 1} c^{(i)}_{\lambda \eta} E^*_\lambda (z; q, t), \quad (21)$$

for some coefficients $c^{(i)}_{\lambda \eta}$. Equating the leading terms of (20) and the right hand side of (21) gives

$$z_i E^*_\eta (z; q^{-1}, t^{-1}) = \sum_{\lambda: |\lambda| = |\eta| + 1} c^{(i)}_{\lambda \eta} E^*_\lambda (z; q^{-1}, t^{-1}). \quad (22)$$

Applying the action of $\Psi M$ to both sides of (22) and using (16) shows

$$\Psi M z_i E^*_\eta (z; q^{-1}, t^{-1}) = q^{-\binom{|\eta| + 1}{2}} \sum_{\lambda: |\lambda| = |\eta| + 1} c^{(i)}_{\lambda \eta} E^*_\lambda (z; q, t). \quad (23)$$

Using (21), the right hand side of (23) can be simplified to

$$q^{-\binom{|\eta|}{2}} Z_i E^*_\eta (z; q, t). \quad (24)$$

By recalling the action of $M$ and again using (16) we obtain

$$\Psi M z_i E^*_\eta (z; q^{-1}, t^{-1}) = Z_i \Psi M E^*_\eta (z; q^{-1}, t^{-1}). \quad (25)$$

Finally, since the $\{E^*_\eta \}$ form a basis for analytic functions in $\{z^\eta \}$ it follows that the intertwining property (18) holds generally.

Corollary 1. We have

$$z_i E^*_\eta (z; q^{-1}, t^{-1}) = q^{2|\eta|} \Psi^{-1} Z_i E^*_\eta (z; q, t). \quad (26)$$

Proof. Follows from (23) and (24).

3 The Product $z_i E^*_\eta$

The previous corollary indicates that the next step towards finding the decomposition of $z_i E^*_\eta$ is to determine an explicit formula for $Z_i E^*_\eta$. The latter can be deduced as a corollary of the following lemma, specifying the expansion of $(z_i \Xi_i - 1) f (z)$, where according to (13) $z_i \Xi_i - 1 := H_i \cdots H_{n-1} \Phi H_1 \cdots H_i$. 
Lemma 1. Let \( \tilde{Z}_i = H_i \ldots H_{n-1} \Phi H_1 \ldots H_{i-1} \). The action of \( \tilde{Z}_i \) on \( f(z) \) is given by

\[
\tilde{Z}_i f(z) = \sum_{I \subseteq \{1, \ldots, n\}} \prod_{i \in I} r_i^{(i)}(z) f(Iz).
\] (27)

Here the rational function \( r_i^{(i)}(z) \) can be expressed as

\[
r_i^{(i)}(z) = \chi_i^{(i)}(z) A_i(z) B_i(z)
\] (28)

where

\[
I = \{t_1, \ldots, t_s\}, 1 \leq t_1 < \ldots t_s \leq n,
\] (29)

\[
A_i(z) = \hat{a} \left( \frac{z_{t_1}}{q}, z_{t_1} \right) \prod_{u=1}^{s-1} \hat{a} \left( z_{t_u}, z_{t_{u+1}} \right)
\] (30)

\[
B_i(z) = (z_{t_s} - t^{-n+1}) \left( \prod_{j=1}^{t_s-1} \hat{b} \left( \frac{z_{t_s}}{q}, z_j \right) \right)
\times \left( \prod_{u=1}^{s} \prod_{j=t_u+1}^{t_{u+1}} \hat{b} (z_{t_u}, z_j) \right),
\] (31)

\[
\chi_i^{(i)}(z) = \begin{cases} 
\frac{1}{a(z_{t_{k-1}}, z_1)} & ; i = t_k, \ k = 2, \ldots, s \\
\frac{a(z_{t_{k-1}}, z_{t_k})}{a(z_{t_{k-1}}, z_1)} & ; i = t_1,
\end{cases}
\] (32)

and \( Iz \) is defined as

\[
(Iz)_i = \begin{cases} 
z_{t_u-1} & ; i = t_u, \text{ if } u = 2, \ldots, s \\
\frac{z_{t_{k-1}}}{q} & ; i = t_1 \\
z_i & ; i \notin I.
\end{cases}
\]

The quantities \( \hat{a}(x, y) \) and \( \hat{b}(x, y) \) are defined in (34) below.

Proof. Using (12) the action of \( H_i \) on \( f(z) \) can be expressed as

\[
H_i f(z) = \hat{a}(x, y) f(z) + \hat{b}(x, y) s_i f(z),
\]

where

\[
\hat{a}(x, y) := \frac{(t-1)x}{x-y}, \ \hat{b}(x, y) := \frac{x-ty}{x-y}.
\] (34)

Hence \( \tilde{Z}_i \) can be written as

\[
\left( \hat{a}(z_i, z_{i+1}) + \hat{b}(z_i, z_{i+1}) s_i \right) \ldots \left( \hat{a}(z_{n-1}, z_n) + \hat{b}(z_{n-1}, z_n) s_{n-1} \right)
\times \Phi \left( \hat{a}(z_1, z_2) + \hat{b}(z_1, z_2) s_1 \right) \ldots \left( \hat{a}(z_{i-1}, z_i) + \hat{b}(z_{i-1}, z_i) s_{i-1} \right).
\] (35)

Let

\[
K_i^{(i)} = s_i \ldots \hat{s}_{t_{r+1}} \ldots \hat{s}_{t_{r-1}} s_{n-1} \Delta s_1 \ldots \hat{s}_{t_{r}} \ldots \hat{s}_{t_{r-1}} s_{i-1}, \text{ for } i \in I,
\]

with
where \(1 \leq t_1 < \ldots < t_r = i < t_{r+1} < \ldots < t_s \leq n\), the hat superscript used as in Section 2 to denote the absence of the corresponding operators and \(I\) as defined in the statement of the result. It is clear that the expansion of \(\tilde{Z}_i\) will be of the form

\[
\tilde{Z}_i = \sum_{I \subseteq \{1, \ldots, n\}} r^{(i)}_I(z) K^{(i)}_I
\]

for coefficients \(r^{(i)}_I(z)\) involving \(\hat{a}(x,y)\) and \(\hat{b}(x,y)\). Further, it is easily verified that \(K^{(i)}_I f(z) = f(Iz)\). The coefficients \(r^{(i)}_I(z)\) are found by considering the individual terms in the expansion of (35). Due to the need to commute the transposition operators \(s_i\) through to the right the final formula is more simply obtained by expanding (35) termwise from the right. Inevitably, the exercise is rather tedious, however it can be structured somewhat by considering four disjoint classes of sets \(I\)

\[
I_1 = \{i\}, \\
I_2 = \{\ldots, i\}, \\
I_3 = \{i, \ldots\}, \\
I_4 = \{\ldots, i, \ldots\},
\]

which exhaust all possibilities. This cataloguing allows the coefficients of the corresponding four forms of \(K^{(i)}_I\) to be considered separately and the result is more easily observed. Explicitly, the four forms of \(K^{(i)}_I\) are

\[
K^{(i)}_{I_1} = s_i s_{n-1} \Delta s_1 \ldots s_{i-1}, \\
K^{(i)}_{I_2} = s_i s_{n-1} \Delta \hat{s}_{t_1} \ldots \hat{s}_{t_{r-1}} s_{i-1}, \\
K^{(i)}_{I_3} = s_i \hat{s}_{t_{r+1}} \ldots \hat{s}_{t_{s-1}} s_{n-1} \Delta s_1 \ldots s_{i-1}, \\
K^{(i)}_{I_4} = s_i \hat{s}_{t_{r+1}} \ldots \hat{s}_{t_{s-1}} s_{n-1} \Delta \hat{s}_{t_1} \ldots \hat{s}_{t_{r-1}} s_{i-1}.
\]

In relation to \(K_{I_2}, K_{I_4}\) the coefficient of \(s_1 \ldots \hat{s}_{t_1} \ldots \hat{s}_{t_{r-1}} \ldots s_{i-1}\) in the partial expansion, that is terms to the right of \(\Phi\) of (35) is

\[
\hat{a}(z_1, z_{t_1+1}) \prod_{u=1}^{r-2} \hat{a}(z_{t_u+1}, z_{t_{u+1}+1}) \times \prod_{j=1}^{t_1-1} \hat{b}(z_1, z_{j+1}) \prod_{u=1}^{r-1} \prod_{j=t_u+1}^{t_{u+1}-1} \hat{b}(z_{t_u+1}, z_{j+1}).
\]

Hence the coefficient of \(\Delta s_1 \ldots \hat{s}_{t_1} \ldots \hat{s}_{t_{r-1}} \ldots s_{i-1}\) will be

\[
(z_n - t^{-n+1}) \hat{a} \left( \frac{z_n}{q}, z_{t_1} \right) \prod_{u=1}^{r-2} \hat{a}(z_{t_u}, z_{t_{u+1}}) \times \prod_{j=1}^{t_1-1} \hat{b} \left( \frac{z_n}{q}, z_j \right) \prod_{u=1}^{r-1} \prod_{j=t_u+1}^{t_{u+1}-1} \hat{b}(z_{t_u}, z_{j+1}).
\]
Similarly, for $K_{I_1}, K_{I_3}$, the coefficient of $s_1\ldots s_{i-1}$ and $\Delta s_1\ldots s_{i-1}$ are
\[
\prod_{j=1}^{i-1} \hat{b}(z_1, z_{j+1})
\]
and
\[
(z_n - t^{-n+1}) \prod_{j=1}^{i-1} \hat{b} \left( \frac{z_n}{q}, z_j \right),
\]
respectively. The final $r_{I_j}(z)'s$ are found by continuing the expansion of (35) from the right and considering the four forms of $K_I$ separately. Thus we find that
\[
\begin{align*}
[r_{I_1}^{(i)}(z)] &= \frac{A_I(z) B_I(z)}{\hat{a} \left( \frac{z}{q}, z_i \right)} \\
[r_{I_2}^{(i)}(z)] &= \frac{A_I(z) B_I(z)}{\hat{a} \left( z_{t_{r-1}}, z_{t_r} \right)} \\
[r_{I_3}^{(i)}(z)] &= \frac{A_I(z) B_I(z)}{\hat{a} \left( \frac{z_{t_r}}{q}, z_{t_1} \right)} \\
[r_{I_4}^{(i)}(z)] &= \frac{A_I(z) B_I(z)}{\hat{a} \left( z_{t_{r-1}}, z_{t_r} \right)},
\end{align*}
\]
where $A_I(z)$ and $B_I(z)$ are defined by (30) and (31), respectively. After recalling the definition of $\chi^{(i)}_I$ given above, the sought explicit formula (28) follows.

**Corollary 2.** We have
\[
Z_i E_\eta^*(z) = q^{-|\eta|} \eta_i \sum_{I \subseteq \{1,\ldots,n\}} \sum_{i \in I} r_{I_1}^{(i)}(z) E_\eta^*(I z).
\]  
(36)

**Proof.** Follows after recalling from (17) that
\[
Z_i := t^{-\binom{z}{2}} (z_i \Xi_i - 1) \Xi_1 \ldots \Xi_i \ldots \Xi_n.
\]  
\]
Together Proposition 1 and Corollary 2 allow us to derive an initial expansion $z_i E_\eta(z; q^{-1}, t^{-1})$ in terms of the Macdonald polynomials of degree $|\eta| + 1$.

**Proposition 2.** We have
\[
z_i E_\eta(z; q^{-1}, t^{-1}) = \eta_i q^{-|\eta|} \sum_{|\lambda| = |\eta| + 1} \sum_{I \subseteq \{1,\ldots,n\}} \sum_{i \in I} \frac{r_{I_1}^{(i)}(\lambda)}{E_\lambda^*(\lambda)} E_\lambda^*(I \lambda) E_\lambda(z; q^{-1}, t^{-1}).
\]  
(37)
Proof. By the vanishing properties of $E^*_\eta$, (9), when the right hand sides of (21) and (36) are equated and evaluated at $z = \bar{\lambda}$ we obtain

$$c_{\lambda \eta}^{(i)} = \eta_i \sum_{I \subseteq \{1, \ldots, n\}} r(I) \frac{E^*_\eta(I \bar{\lambda})}{E^*_\lambda(I \bar{\lambda})}.$$ 

Substituting this back into (21) and applying Corollary II gives (37).

The formula (37) can be improved by three simplifications. The first is to restrict the summation in (37) by removing a number of vanishing terms. For this we require the following two propositions, and associated definitions.

**Proposition 3.** Let $I = \{t_1, \ldots, t_s\}$ with $1 \leq t_1 < \ldots < t_s \leq n$ and $I \neq \emptyset$. We call $I$ comaximal with respect to $\lambda$ if:

1. $\lambda_j \neq \lambda_{t_u}$, $j = t_u + 1, \ldots, t_{u+1} - 1$, ($u = 1, \ldots, s$; $t_{s+1} = n + 1$);
2. $\lambda_j \neq \lambda_{t_s} - 1$, $j = 1, \ldots, t_1 - 1$;
3. $\lambda_{t_s} \neq 0$.

If $I$ is not comaximal with respect to $\lambda$ then $r(I) \frac{1}{E^*_\lambda(I \bar{\lambda})} = 0$.

**Proof.** If any one of the three conditions in the definition of $I$ comaximal with respect to $\lambda$ fail, then $\hat{B}_I(\bar{\lambda}) = 0$ and therefore $r(I) \frac{1}{E^*_\lambda(I \bar{\lambda})} = 0$.

**Proposition 4.** Let $I = \{t_1, \ldots, t_s\}$ with $1 \leq t_1 < \ldots < t_s \leq n$ and $I \neq \emptyset$. We call $I$ maximal with respect to $\lambda$ if

1. $\lambda_j \neq \lambda_{t_u}$, $j = t_u + 1, \ldots, t_u - 1$, ($u = 1, \ldots, s$; $t_0 := 0$);
2. $\lambda_j \neq \lambda_{t_s} + 1$, $j = t_s + 1, \ldots, n$.

Also define the composition $c_I(\lambda)$ for such a set $I$ by

$$(c_I(\lambda))_j = \begin{cases} 
\lambda_{t_k+1} & : j = t_k, \text{ if } k = 1, \ldots, s - 1 \\
\lambda_{t_s} + 1 & : j = t_s \\
\lambda_j & : j \notin I.
\end{cases}$$

Set $I$ is comaximal with respect to $\lambda$ if and there exists a composition $\nu$ such that $I$ is maximal with respect to $\nu$, $\lambda = c_I(\nu)$ and $\bar{I} \lambda = \bar{\nu}$.

**Proof.** Follows from the definitions.

It is shown in [9] that it is only these maximal subsets which give distinct compositions $\lambda$. Thus it is convenient to introduce the set $\mathbb{I}_\eta^I$ of maximal subsets

$$\mathbb{I}_\eta^I := \{I : I \text{ is maximal with respect to } \eta\}$$

and the corresponding set of compositions

$$\mathbb{J}_\eta^\lambda := \{\lambda : \lambda = c_I(\eta), \ I \in \mathbb{I}_\eta^I\}.$$
Corollary 3. If $I$ is comaximal with respect to $\lambda$ then $E^*_\eta(I\overline{\lambda}) \neq 0$ iff $I$ is maximal with respect to $\eta$.

Proof. Follows from Proposition 4 and the vanishing properties of $E^*_\eta(\overline{\eta})$. \hfill \Box

Using these results we can begin to simplify (37).

**Proposition 5.** We have

$$z_i E^*_\eta(z; q^{-1}, t^{-1}) = \pi_i \sum_{I \in J, i \in I \text{ such that } c_I(\eta) = \lambda} 
\frac{r^{(i)}_I(c_I(\eta)) E^*_\eta(\overline{\eta})}{E^*_{c_I(\eta)}(c_I(\eta))} E_{c_I(\eta)}(z; q^{-1}, t^{-1}).$$

(41)

Proof. Using Proposition 3 we can restrict the second summation of (37) to the sets $I$ that are comaximal with respect to $\lambda$. Proposition 4 allows us to restrict the sum further to sets $I$ that are maximal with respect to $\eta$ and hence to $\lambda$ of the form $\lambda = c_I(\eta)$, giving the required result. \hfill \Box

The second simplification is made by giving an evaluation formula for $E^*_\eta(\overline{\eta})$. The derivation draws upon areas of Macdonald polynomial theory not used elsewhere in this work. Hence to avoid a long deviation from the overall goal, the reader is referred to [17] for the details of such results.

**Proposition 6.** We have

$$E^*_\eta(\overline{\eta}) := k_\eta = \left( \prod_{i=1}^{n} \pi_i^2 \right) d'_{\eta}(q^{-1}, t^{-1}),$$

(42)

where

$$d'_{\eta}(q^{-1}, t^{-1}) := \prod_{(i,j) \in \text{diag}(\eta)} \left( 1 - q^{-a_\eta(s)-1} t^{-l_\eta(s)} \right),$$

(43)

where $\text{diag}(\eta) := \{(i,j) \in \mathbb{Z}_2^2, 1 \leq j \leq \eta_i\}$. The quantities $a_\eta(s)$ and $l_\eta(s)$ are the arm and leg length respectively and defined by

$$a_\eta(s) = \eta_i - j \text{ and } l_\eta(s) = \# \{k > i; j \leq \eta_k \leq \eta_i\} + \# \{k < i; j \leq \eta_k + 1 \leq \eta_i\}. \quad (44)$$

Proof. Use will be made of the operations $s_i$ and $\Phi$ defined to act on functions of $n$ variables by (11) and (14), with their actions now on compositions. The action of $s_i$ on $\eta$ is to exchange parts in positions $i$ and $i + 1$, while $\Phi$ acts on compositions according to

$$\Phi \eta := (\eta_2, ..., \eta_i, \eta_i + 1).$$

These operators can generate all compositions recursively, starting with $(0, ..., 0)$, and allow (42) to be proved inductively. Clearly, when $\eta = (0, ..., 0)$ we have $k_\eta = 1 =$
\( E^*_s(\eta; q, t) \), which establishes the base case. Assume for \( \eta \) general \( E^*_s(\eta) = k_\eta \). Our task is to deduce from this that

\[
E^*_{s_1\eta}(s_i\eta; q, t) = k_{s_1\eta}
\]

and

\[
E^*_{\Phi\eta}(\Phi\eta; q, t) = k_{\Phi\eta}.
\]

To show (45) we must consider the cases \( \eta_i < \eta_{i+1} \) and \( \eta_i > \eta_{i+1} \) separately. We begin with the case \( \eta_i < \eta_{i+1} \). To relate \( E^*_{s_i\eta}(s_i\eta) \) to \( E^*_\eta(\eta) \) we consider two different perspectives on the computation of \( H_i E^*_s(z) \). The first is found by recognising that \( H_i = T_i^{-1}[t^{-1}], \) where \( T_i^{-1} \) is the inverse of the Demazure-Lusztig operator \( T_i \) defined by (3). From (3) and the quadratic relation of (11) we have

\[
T_i^{-1} := t^{-1} - 1 + t^{-1}T_i.
\]

Taking the known result [1]

\[
T_i^{-1}E^*_\eta(z; q, t) = \frac{t^{-1} - 1}{1 - \delta_{i,\eta}(q, t)}E^*_\eta(z; q, t) + E^*_{s_i\eta}(z; q, t), \text{ when } \eta_i < \eta_{i+1},
\]

where \( \delta_{i,\eta} := \eta_i/\eta_{i+1} \), and replacing \( (q, t) \) by \( (q^{-1}, t^{-1}) \) allows us to apply the mapping \( \Psi \) (10) to both sides of the equation. Making use of the fact that \( \Psi \) commutes with \( H_i \) [9] Section 5] we obtain

\[
H_i E^*_\eta(z; q, t) = \frac{t - 1}{1 - \delta_{i,\eta}(q^{-1}, t^{-1})}E^*_\eta(z; q, t) + E^*_{s_i\eta}(z; q, t). \tag{48}
\]

The second perspective is obtained directly from definition (12) which gives

\[
H_i E^*_\eta(z; q, t) = \frac{(t - 1)z_i}{z_i - z_{i+1}}E^*_\eta(z; q, t) + \frac{z_i - tz_{i+1}}{z_i - z_{i+1}}E^*_\eta(s_i; q, t). \tag{49}
\]

Equating the right hand sides of (48) and (49) and evaluating at \( z = s_i\eta \) we obtain

\[
\frac{1 - t\delta_{i,\eta}^{-1}(q^{-1}, t^{-1})}{1 - \delta_{i,\eta}^{-1}(q^{-1}, t^{-1})} = \frac{E^*_{s_i\eta}(s_i\eta)}{E^*_\eta(\eta)}.
\]

Since for \( \eta_i < \eta_{i+1} \) [7],

\[
\frac{d'_{s_i\eta}(q, t)}{d'_{\eta}(q, t)} = \frac{1 - \delta_{i,\eta}^{-1}(q, t)}{1 - t^{-1}\delta_{i,\eta}(q, t)},
\]

we have

\[
\frac{k_{s_i\eta}}{k_{\eta}} = \frac{d'_{s_i\eta}(q^{-1}, t^{-1})}{d'_{\eta}(q^{-1}, t^{-1})} = \frac{E^*_{s_i\eta}(s_i\eta)}{E^*_\eta(\eta)}.
\]

Hence \( E^*_\eta(\eta) = k_{\eta} \) implies \( E^*_{s_i\eta}(s_i\eta) = k_{s_i\eta} \). The case where \( \eta_i > \eta_{i+1} \) is proven similarly.

The first step to showing (16) is to consider the vanishing properties of \( (\Phi E^*_\eta)(z) \). By Knop [9] Corollary 3.3 if \( |\lambda| \leq |\eta| \) then \( (\Phi E^*_\eta)(\lambda) \) is a linear combination of \( E^*_\eta(\eta) \) for
We make our final improvement to the formula for $z_i E_\eta(z; q^{-1}, t^{-1})$ by simplifying the coefficient $\overline{\eta}_i\overline{r}^{(i)}_I\left(\overline{c}_I(\eta)\right)$. 

Corollary 4. We have

$$z_i E_\eta(z; q^{-1}, t^{-1}) = \overline{\eta}_i \sum_{I \subseteq I^+_{\lambda,1}, \sum_{i \in I} c_i(\eta) = \lambda} \frac{r^{(i)}_I\left(\overline{c}_I(\eta)\right)}{k_{\overline{c}_I(\eta)}} E_{c_I(\eta)}(z; q^{-1}, t^{-1}). \quad (53)$$
Proposition 7. Let
\[ \tilde{B}_I (z) = \prod_{u=1}^{s} \prod_{j=t_{u-1}+1}^{t_u-1} \tilde{b}(z_{t_u}, z_j) \prod_{j=t_s+1}^{n} \tilde{b}(q z_{t_1}, z_j) \times (q z_{t_1} - t^{-n+1}), \quad t_0 := 0 \] (54)
and
\[ \tilde{\chi}_I^{(i)} (z) := \left\{ \begin{array}{ll} \frac{a(z_i, z_{t_{i+k+1}})}{z_i} & ; i = t_k, \ k = 1, \ldots, s - 1 \\
\frac{a(z_i, q z_{t_1})}{z_i} & ; i = t_s, \end{array} \right. \]
where \( I = \{ t_1, \ldots, t_s \} \subseteq \{ 1, \ldots, n \} \), with \( 1 \leq t_1 < \cdots < t_s \leq n \) and \( I \neq \emptyset \). We have
\[ z_i E_\eta (z; q^{-1}, t^{-1}) = \sum_{I \in \mathcal{I}_\eta, \ i \in I \atop c_I (\eta) = \lambda} \tilde{\chi}_I^{(i)} (\eta) A_I (\eta) \tilde{B}_I (\eta) k_{c_I (\eta)} E_{c_I (\eta)} (z; q^{-1}, t^{-1}). \] (55)

Proof. It can be seen that for \( I \) maximal with respect to \( \eta \) we have \( \tilde{\chi}_I^{(i)} (\eta) = \tilde{\eta}_i \tilde{\chi}_I^{(i)} (c_I (\eta)) \) and \( B_I (\tilde{c}_I (\eta)) = \tilde{B}_I (\eta) \). Since \( A_I (\tilde{c}_I (\eta)) = A_I (\eta) \), it follows that
\[ \tilde{\eta}_i r_I^{(i)} (\tilde{c}_I (\eta)) = \tilde{\chi}_I^{(i)} (\eta) A_I (\eta) \tilde{B}_I (\eta). \] (56)
By substituting (56) into (53) we arrive at our final decomposition (55).

4 The Pieri-type Formula for \( r = 1 \) and the Generalised Binomial Coefficient

The second major result of the paper is to determine the nonsymmetric analogue of the Pieri-type formula (58) for \( r = 1 \). This formula gives the branching coefficients of Macdonald polynomials of degree \( |\eta| + 1 \) in the expansion of \( e_1 (z) = z_1 + \ldots + z_n \) times \( E_\eta (z; q^{-1}, t^{-1}) \). These coefficients can be derived as a consequence of Proposition 7.

Proposition 8. We have
\[ e_1 (z) E_\eta (z; q^{-1}, t^{-1}) = \sum_{I \in \mathcal{I}_\eta} a_{\eta, c_I (\eta)} E_{c_I (\eta)} (z; q^{-1}, t^{-1}), \] (57)
where \( a_{\eta, c_I (\eta)} \) is defined by
\[ a_{\eta, c_I (\eta)} := - \frac{(q - 1) d_\eta (q^{-1}, t^{-1}) A_I (\eta) \tilde{B}_I (\eta)}{q^{\min(I)+1} (t - 1) d_{c_I (\eta)} (q^{-1}, t^{-1})}. \] (58)

Proof. Summing (53) over all \( i \) and then reversing the order of summation gives
\[ e_1 (z) E_\eta (z; q^{-1}, t^{-1}) = \sum_{I \in \mathcal{I}_\eta} \sum_{i \in I} \tilde{\chi}_I^{(i)} (\eta) A_I (\eta) \tilde{B}_I (\eta) k_{c_I (\eta)} \times E_{c_I (\eta)} (z; q^{-1}, t^{-1}). \] (59)
We have
\[ \sum_{i \in I} \tilde{\chi}_{I}^{( \nu )} ( \eta ) = \frac{7 \min ( l ) ( 1 - q )}{( t - 1 )} \]  
(60)
and
\[ \frac{k_{\eta}}{k_{c_{I} \eta}} = \frac{d'_{\eta} ( q^{-1}, t^{-1} )}{q^{2 \min ( l )} + 1} t^{l_{\eta} ( \min ( l ) )} d''_{c_{I} \eta} ( q^{-1}, t^{-1} ). \]  
(61)
Substituting (60) and (61) into (59) gives the required result.

On obtaining the Pieri-type formula for \( r = 1 \) we are naturally lead to deducing an explicit formula for the generalised binomial coefficient \( \binom{\nu}{\eta}_{q, t} \) when \( |\nu| = |\eta| + 1 \). Generalised binomial coefficients appear in the theory of Macdonald polynomials. We define nonsymmetric \( q \)-binomial coefficients \( \binom{\nu}{\eta}_{q, t} \) according to the generating function formula
\[ E_{\eta} ( z; q^{-1}, t^{-1} ) \prod_{i=1}^{n} \frac{1}{( z_{i}; q )_{\infty}} = \sum_{\nu} \binom{\nu}{\eta}_{q, t} t^{l(\nu) - l(\eta)} d'_{\eta} ( q, t ) d'_{\nu} ( q, t ) E_{\nu} ( z; q^{-1}, t^{-1} ), \]  
(62)
where \( (z_{i}; q)_{\infty} \) is the Pockhammer symbol and is defined as
\[ (u; q)_{\infty} := \prod_{j=1}^{\infty} (1 - uq^{j-1}) \]  
(63)
and \( l(\eta) := \sum_{s \in \eta} l_{\eta}(s) \). Unlike the classical binomial coefficients
\[ \binom{l}{p} := \frac{l!}{(l-p)!p!} \]  
(64)
there is no known explicit formula for \( \binom{\nu}{\eta}_{q, t} \). However, by restricting our attention to the monomials of degree 1 in the expansion of \( \prod_{i=1}^{n} \frac{1}{( z_{i}; q )_{\infty}} \) we are able to use Proposition 8 to deduce an explicit formula for \( \binom{\nu}{\eta}_{q, t} \) when \( |\nu| = |\eta| + 1 \).

**Proposition 9.** Suppose \( |\nu| = |\eta| + 1 \). Then
\[ \binom{\nu}{\eta}_{q, t} = - \frac{A_{f} ( \eta ) \tilde{B}_{f} ( \eta )}{( t - 1 )}, \]  
(65)
where \( \nu = c_{I} ( \eta ) \). If there is no such \( I \) such that \( \nu = c_{I} ( \eta ) \) then \( \binom{\nu}{\eta}_{q, t} = 0 \).

**Proof.** Using (63) and the identity \( \frac{1}{1-u} = 1 + u + u^{2} + \ldots \) we can simplify \( \prod_{i=1}^{n} \frac{1}{( z_{i}; q )_{\infty}} \) to
\[ 1 + \frac{1}{1 - q} e_{1} ( z ) + \text{higher order terms}. \]  
(66)
Equating terms of degree $|\eta| + 1$ in (62) gives

$$e_1(z) E_\eta (z; q^{-1}, t^{-1}) = \sum_{|\nu| = |\eta| + 1} \binom{\nu}{\eta}_{q,t} \frac{t^{l(\nu) - l(\eta)} (1 - q) d'_\nu(q,t)}{d'_\eta(q,t)} E_\nu (z; q^{-1}, t^{-1})$$

(this equation can also be deduced from [10, Eq. (16)]). Comparison with (58) gives

$$\binom{\nu}{\eta}_{q,t} = d'_\eta(q^{-1}, t^{-1}) d'_\nu(q^{-1}, t^{-1}) q^{\min(I)} (t - 1) t^{l(\nu) - l(\eta)} (67)$$

Since

$$\frac{1 - x}{1 - x^{-1}} = -x,$$

we have

$$\frac{d'_\mu(q,t)}{d'_\mu(q^{-1}, t^{-1})} = \prod_{s \geq \mu} (-q^{a(\mu(s)) + 1} t^{l(\mu)}) = (-1)^{|\mu|} q^{\sum_{s \geq \mu} (a(\mu(s)) + 1)} t^{l(\mu)} (68)$$

The final result is obtained by appropriately substituting (68) into (67) and noting that

$$(-1)^{|\nu| - |\eta|} = -1$$ while

$$q^{\sum_{s \geq \nu} (a(\nu(s)) + 1)} q^{\sum_{s \geq \eta} (a(\eta(s)) + 1)} = q^{\min(I)} + 1.$$

A viewpoint of the classical binomial coefficients is that they are a ratio of evaluations of the one variable interpolation polynomial

$$f_p(x) := x(x - 1)...(x - p + 1),$$

explicitly

$$\binom{\nu}{p} = \frac{f_p(l)}{f_p(p)}.$$

Similarly in the multivariable nonsymmetric Macdonald polynomial theory the generalised binomial coefficient (in particular (65)) satisfy [16]

$$\binom{\nu}{\eta}_{q,t} = \frac{E^*_\eta(\tau)}{E^*_\eta(\eta)} (69)$$

Comparing (69) with (65) and making use of (42) gives a new evaluation formula for $E^*_\eta(\tau)$ where $|\nu| = |\eta| + 1$.

**Corollary 5.** Suppose $|\nu| = |\eta| + 1$. Then

$$E^*_\eta(\tau) = -\frac{A_I(\tau) \tilde{B}_I(\tau)}{(t - 1)} \left( \prod_{i=1}^{n} \tau_i \right) d'_\eta(q^{-1}, t^{-1})$$

where $\nu = c_I(\eta)$. If there is no such $I$ such that $\nu = c_I(\eta)$ then $E^*_\eta(\tau) = 0$.  

16
5 The Pieri-type Formula for \( r = n - 1 \)

In this section we give our last Pieri-type formula, the nonsymmetric analogue of \([8]\) for \( r = n - 1 \). The result can be derived almost immediately from the expansion of \( e_1(z) E_\eta(z) \) using the identity \([14]\)

\[
E_\eta(z^{-1}; q, t) = E_{\eta^R}(z; q, t),
\]

where \( \eta^R := (\eta_n, ..., \eta_1) \).

**Proposition 10.** Define

\[
\eta + (i^n) = (\eta_1 + i, ..., \eta_n + i),
\]

and

\[
\eta' := \eta - (\min(\eta)^n).
\]

We have

\[
e_{n-1}(z) E_\eta(z; q^{-1}, t^{-1}) = \sum_{I \in \mathcal{J}_I'} a_{\nu,c_I(\nu)} E_{\lambda+(\min(\eta)^n)}(z; q^{-1}, t^{-1}),
\]

where \( a_{\nu,c_I(\nu)} \) is defined by \([22]\),

\[
\nu = (-\eta' + (\max(\eta')^n))^R \quad \text{and} \quad \lambda = -c_I(\nu)^R + ((\max(\nu) + 1)^n).
\]

**Proof.** By Proposition \([8]\) we have

\[
e_1(z) E_{\nu}(z; q^{-1}, t^{-1}) = \sum_{I \in \mathcal{J}_I'} a_{\nu,c_I(\nu)} E_{c_I(\nu)}(z; q^{-1}, t^{-1}).
\]

Substituting \( z \) for \( z^{-1} \) and using \([70]\) we obtain

\[
e_1(z^{-1}) E_{-\nu^R}(z; q^{-1}, t^{-1}) = \sum_{I \in \mathcal{J}_I'} a_{\nu,c_I(\nu)} E_{-c_I(\nu)^R}(z; q^{-1}, t^{-1}).
\]

Multiplying both sides by \( z_1...z_n \) and using the identity \( z_1...z_n E_\eta(z) = E_\eta+1^n(z) \) \([14]\) we have

\[
e_{n-1}(z) E_{-\nu^R}(z; q^{-1}, t^{-1}) = \sum_{I \in \mathcal{J}_I'} a_{\nu,c_I(\nu)} E_{-c_I(\nu)^R+1^n}(z; q^{-1}, t^{-1}).
\]

Since \( \nu = (-\eta' + (\max(\eta')^n))^R \) we have \( \eta' = -\nu^R + (\max(\nu)^n) \), and hence, multiplying both sides of \([73]\) by \( (z_1...z_n)^{\max(\nu)} \) gives

\[
e_{n-1}(z) E_{\eta'}(z; q^{-1}, t^{-1}) = \sum_{I \in \mathcal{J}_I'} a_{\nu,c_I(\nu)} E_{\lambda}(z; q^{-1}, t^{-1}),
\]

where \( \lambda \) is defined in \([72]\). The final decomposition \([71]\) is now obtained by multiplying both sides of \([74]\) by \( (z_1...z_n)^{\min(\eta)} \). \( \square \)
The Classical Limit

The classical limit in Macdonald polynomial theory refers to setting \( t = q^{1/\alpha} \) and taking \( q \to 1 \). In particular

\[
\lim_{t=q^{1/\alpha}, \ q \to 1} E_\eta (z; q, t) = E_\eta (z; \alpha)
\]

where \( E_\eta (z; \alpha) \) is the nonsymmetric Jack polynomial (for an account of the latter see e.g. [7]). As remarked in the introduction, the expansion of the product \( e_1 (z) E_\eta (z; \alpha) \) in terms of \( \{E_\lambda (z; \alpha)\} \) has been given by Marshall [15]. We will conclude our study by taking the classical limit of Proposition 8. First we recall the result of [15].

**Proposition 11.** We have

\[
e_1 (z) E_\eta (z; \alpha) = \sum_{I \in J} a_{\eta, c_I (\eta)}^\alpha E_{c_I (\eta)} (z; \alpha),
\]

where

\[
a_{\eta, c_I (\eta)}^\alpha = \frac{-\alpha^2 d_{\alpha, \eta}^\prime (z)}{d_{\alpha, c_I (\eta)}^\prime},
\]

The quantities in (75) are specified by

\[
A_{\alpha, I} (z) = a (z_{t_s} - 1, z_{t_1}) \prod_{u=1}^{s-1} a (z_{t_u}, z_{t_{u+1}})
\]

\[
\tilde{B}_{\alpha, I} (z) = \prod_{u=1}^{s} \prod_{j=t_{u-1}+1}^{t_u} b (z_{t_u}, z_j) \prod_{j=t_s}^{n} b (z_{t_1} + 1, z_j)
\]

\[
	imes (z_{t_1} + 1 + \frac{n-1}{\alpha}), \ t_0 := 0
\]

with

\[
a (x, y) := \frac{1}{\alpha (x - y)}, \ b (x, y) := \frac{x - y - \frac{1}{\alpha}}{x - y}
\]

and

\[
d_{\alpha, \eta}^\prime := \prod_{(i, j) \in \eta} (\alpha (a (i, j) + 1) + l (i, j)),
\]

where \( a (i, j) \) and \( l (i, j) \) are defined by (44) and \( I \) by (29).

**Proof.** Our task is to show that

\[
\lim_{t=q^{1/\alpha}, \ q \to 1} a_{\eta, c_I (\eta)} = a_{\eta, c_I (\eta)}^\alpha.
\]

Comparing (43) with (79), it is immediate that

\[
\lim_{t=q^{1/\alpha}, \ q \to 1} \frac{(q-1) d_{\eta}^\prime (q^{-1}, t^{-1})}{q^{\min(I)+1} (t-1) d_{c_I (\eta)}^\prime (q^{-1}, t^{-1})} = \alpha^2 \frac{d_{\eta}^\prime}{d_{\alpha, c_I (\eta)}^\prime}.
\]
To proceed further, note from (34) and (78) that
\[
\lim_{t=q^{1/\alpha}, \, q \rightarrow 1} a\left(q^m t^n, q^{m'} t^{n'}\right) = a\left(\frac{m}{\alpha} + n, \frac{m'}{\alpha} + n'\right)
\]

and
\[
\lim_{t=q^{1/\alpha}, \, q \rightarrow 1} b\left(q^m t^n, q^{m'} t^{n'}\right) = b\left(\frac{m}{\alpha} + n, \frac{m'}{\alpha} + n'\right).
\]
Using this a term-by-term comparison of (30) and (54) with (76) and (77) also allows us to conclude that
\[
\lim_{t=q^{1/\alpha}, \, q \rightarrow 1} A_I (\overline{\eta}) = A_{\alpha,I} \left(\frac{\overline{\eta}}{\alpha}\right)
\]
and
\[
\lim_{t=q^{1/\alpha}, \, q \rightarrow 1} \tilde{B}_I (\overline{\eta}) = \tilde{B}_{\alpha,I} \left(\frac{\overline{\eta}}{\alpha}\right).
\]
This establishes (80), thus exhibiting Proposition 11 as a corollary of Proposition 8.

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