MULTI-BUMP SOLUTIONS FOR A CLASS OF QUASILINEAR EQUATIONS ON $\mathbb{R}$

CLAUDIANOR O. ALVES
Unidade Acadêmica de Matemática e Estatística
Universidade Federal de Campina Grande
58109-9700, Campina Grande, PB, Brazil

OLÍMPIO H. MIYAGAKI
Departmento de Matemática, Universidade Federal de Juiz de Fora
36036-330, Juiz de Fora, MG, Brazil

SÉRIO H. M. SOARES
Departmento de Matemática
Instituto de Ciências Matemáticas e de Computação
Universidade de São Paulo, 13560-970, São Carlos, SP, Brazil

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Abstract. This paper is concerned with the existence of multi-bump solutions to a class of quasilinear Schrödinger equations in $\mathbb{R}$. The proof relies on variational methods and combines some arguments given by del Pino and Felmer, Ding and Tanaka, and Séré.

1. Introduction. Recently, there has been growing interested in the quasilinear Schrödinger equation

\[ i\varepsilon_t - W(x)z + z'' + |z|^{q-1}z + k(|z|^2)'z = 0, \quad x \in \mathbb{R}, \]

where $i^2 = -1$, $k, x, t \in \mathbb{R}$, $z' = \frac{\partial z}{\partial x}$ and $W : \mathbb{R} \to \mathbb{R}$ is a continuous function.

Knowledge of the solutions of

\[ v'' - (W(x) - w)v + |v|^{q-1}v + k(|v|^2)'v = 0, \quad x \in \mathbb{R}, \]

where $w \in \mathbb{R}$, has a great importance for studying standing wave solutions for (1), that is, solutions of the form $z(x, t) = e^{-i\varepsilon t}v(x)$.

As observed in [18], the quasilinear equation (1) with $k > 0$ has a great relevance because it appears in mathematical models associated with several physical phenomena such as in the study of plasma physics, see for example [23, 27], in the theory of superfluid film and in dissipative quantum mechanics, see for example [15, 16, 20].

Existence of positive solutions to quasilinear equation (2) has been established in [4, 5, 9, 22] and [25] via variational methods. Equations related to (2) in higher
dimensions on $\mathbb{R}^N$ ($N \geq 2$) have been also considered, we refer the reader to [10, 17, 18, 19, 21, 22, 24], and references therein.

In [5], Ambrosetti and Wang proved the existence of positive solutions for a large class of quasilinear elliptic equations, which includes the following problem

$$\begin{cases}
- v'' + (\lambda V(x) + 1) v - |v|^2 v = v^q, \quad x \in \mathbb{R}, \\
\quad v > 0 \text{ in } \mathbb{R}, \quad v \in H^1(\mathbb{R}),
\end{cases}$$

provided that the positive parameter $\lambda$ is sufficiently small, $q \geq 3$, $V$ is a real function satisfying certain hypotheses and the problem has a unique solution for $\lambda = 0$.

The discussion of existence of solutions for equations of the type

$$\begin{cases}
- \Delta v + (\lambda V(x) + 1) v = f(v), \quad v > 0 \text{ in } \mathbb{R}^N \ (N \geq 2), \\
\quad v \in H^1(\mathbb{R}^N),
\end{cases}$$

in terms of the parameter $\lambda$ has been extensively investigated in several papers. We refer the reader to [1, 2, 3, 6, 7, 8, 11, 13, 14], and references therein.

In [13], Ding and Tanaka have used some arguments developed by Séré [26] and del Pino and Felmer [12] to consider (3) for the case $f(v) = v^q$, with $1 < q < \frac{N+2}{N-2}$, and $V \geq 0$ is a continuous function such that $\Omega = \text{int} \left( V^{-1}(\{0\}) \right)$ is a non-empty bounded set $\Omega = V^{-1}(\{0\})$ and $\Omega$ has $m$ connected components $\Omega_j, j \in \{1, \ldots, m\}$, satisfying $d(\Omega_j, \Omega_i) > 0$ for $i \neq j$.

In [13] the existence of $2^m - 1$ solutions is proved assuming that $\lambda$ is large. Moreover, it is stated that for non-empty subset $\Gamma \subset \{1, \ldots, m\}$ there exists a multi-bump solution, that is, a positive solution $u_\lambda$ of (3) which converges, as $\lambda \to +\infty$, to a least energy solution of

$$\begin{cases}
- \Delta u + u = u^q, \quad x \in \Omega_\Gamma, \\
\quad u > 0, \text{ in } \Omega_\Gamma, \\
\quad u = 0, \text{ on } \partial \Omega_\Gamma.
\end{cases}$$

(4)

where $\Omega_\Gamma = \cup_{j \in \Gamma} \Omega_j$.

Existence of multi-bump solutions for problems involving the p-Laplacian operator and nonlinearities with critical growth for $N \geq 2$ has been dealt with in [1], [2] and [3].

Motivated by [5] and [13], we are interested in finding multi-bump solutions to (2) assuming that $W(x) = \lambda V(x) + 1 + w$ and $k = \frac{1}{2}$. More precisely, we consider the following quasilinear elliptic problem

$$\begin{cases}
- v'' + (\lambda V(x) + 1) v - \frac{1}{2} |v|^2 v = |v|^{q-1} v, \quad x \in \mathbb{R}, \\
\quad v > 0 \text{ in } \mathbb{R}, \quad v \in H^1(\mathbb{R}),
\end{cases}$$

(5)

where $q > 3$, the function $V$ verifies the condition $(V)$ and $\lambda$ is a positive parameter large enough.

The main difficulty in proving our result is the presence of the quasilinear term $\int_{\mathbb{R}} |v|^2 |v'|^2$ in the energy functional associated with the problem. Since this term is homogeneous of order 4 and non-convex, we have serious difficulties to show the Palais-Smale condition. Moreover, it is not clear that the characterization of the mountain pass values used in [13] can be used here, because the operator and the nonlinearity do not have the same degree of homogeneity. Thus, we modify the sets that appear in the minimax arguments explored in [13].

Our main result is the following
Theorem 1.1. Suppose that $q > 3$ and (V) holds. Then, for any non-empty subset $\Gamma \subset \{1, \ldots, m\}$, there exists $\lambda^* > 0$ such that, for $\lambda \geq \lambda^*$, the equation (5) has a family $\{u_\lambda\}$ of solutions such that, for any sequence $\lambda_n \to +\infty$, there exists a subsequence $\lambda_{n_i}$ such that $u_{\lambda_{n_i}}$ converges strongly in $H^1(\mathbb{R})$ to a function $u$ which satisfies $u(x) = 0$ for $x \not\in \Omega_{1'}$ and the restriction $u|_{\Omega_j}$, for $j \in \Gamma$, is a least energy solution of

$$
\begin{cases}
-u'' + u - \frac{1}{2}(|u|^2)''u = u^q, & x \in \Omega_j \\
u = 0 & \text{on } \partial\Omega_j.
\end{cases}
$$

Corollary 1.1. Under the assumptions of Theorem 1.1, there exists $\lambda^* > 0$ such that (5) has at least $2^m - 1$ solutions for all $\lambda \geq \lambda^*$.

The organization of this paper is as follows: Section 2 sets up the variational framework and a modified functional is introduced which satisfies the Palais-Smale conditions. In Section 3 some technical lemmas and propositions related to existence of multi-bump solutions are proved. Finally, Section 4 offers a proof of the main theorem.

Notation: In this paper we use the following notations:

- $B_R(z)$ denotes the open interval with center at $z$ and radius $R$.
- In all the integrals we omit the symbol “$dx$”.
- $|u|_s = \left(\int_\mathbb{R} |u|^s\right)^{1/s}$ denotes the usual norm in $L^s$-space.
- $C$ denotes positive constants that may change from use to use, but depend only on quantities that are constant in the calculation.
- Given $\Theta \subset \mathbb{R}$ we write $\|\cdot\|_\Theta$, $|\cdot|_{r,\Theta}$, $|\cdot|_{\infty,\Theta}$ for the $H^1(\Theta)$, $L^r(\Theta)$, $L^\infty(\Theta)$-norm, respectively. When $\Theta = \mathbb{R}$, we simply write $\|\cdot\|$, $|\cdot|_r$, $|\cdot|_\infty$.

2. Preliminaries. In this section, we introduce the variational framework used to prove our results. Let

$$E = \left\{ u \in H^1(\mathbb{R}) : \int_\mathbb{R} V(x)|u|^2 < +\infty \right\}$$

endowed with the norm

$$\|u\|_\lambda = \left(\int_\mathbb{R} |u'|^2 + (\lambda V(x) + 1)|u|^2\right)^{1/2}.$$

A direct argument works to show that for each $\lambda \geq 0$, $(E, \|\cdot\|_\lambda)$ is a Hilbert space and that the embedding $E \hookrightarrow H^1(\mathbb{R})$ is continuous. The positive solutions of (5) are critical points of the functional $I : E \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2} \int_\mathbb{R} \left[ (1 + |u|^2)|u'|^2 + (\lambda V(x) + 1)|u|^2 \right] - \frac{1}{q+1} \int_\mathbb{R} (u^+)^{q+1}$$

where $u^+ = \max\{0, u\}$. The functional $I$ is well defined because of $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$.

For an open set $\Theta \subset \mathbb{R}$, we also write

$$E(\Theta) = \left\{ u \in H^1(\Theta) : \int_\Theta V(x)|u|^2 < \infty \right\}$$

and

$$\|u\|_{\lambda, \Theta} = \left(\int_\Theta |u'|^2 + (\lambda V(x) + 1)|u|^2\right)^{1/2}.$$

Since $V$ is nonnegative, we have

$$|u|_{2, \Theta} \leq \|u\|_{\lambda, \Theta} \text{ for all } u \in E.$$
The next result is a consequence of the previous considerations:

**Lemma 2.1.** For any open set \( \Theta \subset \mathbb{R} \), there exist \( \delta_0 > 0 \) and \( \nu_0 \in (0, 1/2) \) such that

\[
\delta_0 \|u\|_{L^2, \Theta}^2 \leq \|u\|_{L^2, \Theta}^2 - \nu_0 |u|_{L^2, \Theta}^2,
\]

for all \( u \in E(\Theta) \) and \( \lambda > 0 \).

### 2.1. Modified functional

In this subsection, based on the arguments used in [13], we introduce a modified functional which satisfies the Palais-Smale conditions.

Since we intend to find positive solutions, we define \( \nu_0 > 0 \) given by Lemma 2.1, let \( a \) be a positive real such that \( \frac{f(a)}{a} = \nu_0 \) and define the function

\[
\tilde{f}(t) = \begin{cases} 
  f(t), & \text{if } t \leq a \\
  \nu_0 t, & \text{if } t \geq a.
\end{cases}
\]

Now, we fix a non-empty subset \( \Gamma \subset \{1, \ldots, m\} \) and take the sets

\[
\Omega_\Gamma = \bigcup_{j \in \Gamma} \Omega_j \quad \text{and} \quad \Omega'_\Gamma = \bigcup_{j \in \Gamma} \Omega'_j
\]

where \( \Omega'_j \) is a bounded open set verifying

\[
\overline{\Omega_j} \subset \Omega'_j \quad \text{and} \quad \Omega_j \cap \Omega_i = \emptyset \; \forall \; i \neq j.
\]

Moreover, we consider the function

\[
g(x, t) = \chi_\Gamma(x)f(t) + (1 - \chi_\Gamma(x))\tilde{f}(t)
\]

where \( \chi_\Gamma \) denotes the characteristic function of the set \( \Omega'_\Gamma \). The function \( g \) is Carathéodory and, for \( x \in \mathbb{R} \), \( t \mapsto g(x, t) \) is a \( C^1 \) function satisfying

\[
\begin{align*}
(g_1) \quad & 0 < (q + 1)G(x, t) \leq g(x, t)t \quad \forall x \in \Omega'_\Gamma \quad \text{and} \quad t > 0 \\
(g_2) \quad & 2G(x, t) \leq g(x, t)t \leq \nu_0 t^2 \quad \forall x \not\in \Omega'_\Gamma \quad \text{and} \quad t \geq 0
\end{align*}
\]

where \( G(x, t) = \int_0^t g(x, \tau)d\tau \).

Finally, we define the modified functional \( \Phi_\lambda : E \to \mathbb{R} \) by

\[
\Phi_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}} [(1 + |u|^2)|u'|^2 + (\lambda V(x) + 1)|u|^2] - \int_{\mathbb{R}} G(x, u).
\]

Since \( q > 3 \) and \( V \) is nonnegative, \( \Phi_\lambda \in C^1(E, \mathbb{R}) \) with

\[
\Phi'_\lambda(u)v = \int_{\mathbb{R}} [u'v' + |u|^2u'v' + u|u'|^2v + (\lambda V(x) + 1)uv] - \int_{\mathbb{R}} g(x, u)v,
\]

and every critical point of \( \Phi_\lambda \) is a weak solution of

\[
- u'' + (\lambda V(x) + 1)u - \frac{1}{2}(|u|^2)'u = g(x, u), \; x \in \mathbb{R}. \tag{7}
\]

An important point that we would like to stress is the fact that if \( u \) is a positive weak solution of (7) with \( u(x) \leq a \) in \( \mathbb{R} \setminus \Omega'_\Gamma \), then \( u \) is a positive weak solution of (5).
2.2. (PS) condition. In this section we make use of the modification on the non-linearity $f$ in order to show that $\Phi_\lambda$ satisfies the Palais-Smale condition ((PS) for short).

In the following, we state a result whose proof follows the same arguments developed in [22, Lemma 2]. This result will be essential to study the behavior of the Palais-Smale sequences.

**Lemma 2.2.** For any $T > 0$, the functional $N : E \to \mathbb{R}$ given by

$$N(u) = \int_{-T}^{T} |u|^2 |u'|^2$$

is weakly sequentially lower semicontinuous, that is,

$$u_n \rightharpoonup u \text{ weakly in } E \Rightarrow \liminf_{n \to +\infty} N(u_n) \geq N(u).$$

**Lemma 2.3.** Let $(u_n) \subset E$ be a Palais-Smale sequence at the level $c \in \mathbb{R}$, that is,

$$\Phi_\lambda(u_n) \to c \text{ and } \Phi'_\lambda(u_n) \to 0$$

for some $c \in \mathbb{R}$. Then $(u_n)$ is bounded in $E$.

**Proof.** From (8),

$$\Phi_\lambda(u_n) - \frac{1}{q+1} \Phi'_\lambda(u_n)u_n = c + o_n(1) + \epsilon_n \|u_n\|_\lambda$$

where $\epsilon_n \to 0$ as $n \to +\infty$. From the definition of $g$ and condition $(g_1)$,

$$\left(\frac{1}{2} - \frac{1}{q+1}\right) \|u_n\|_\lambda^2 - \int_{\mathbb{R} \setminus \Omega'_T} [G(x, u_n) - \frac{1}{q+1} g(x, u_n)u_n] \leq c + o_n(1) + \epsilon_n \|u_n\|_\lambda.$$

From $(g_2)$,

$$G(x, s) - \frac{1}{q+1} g(x, s)s \leq \left(\frac{1}{2} - \frac{1}{q+1}\right) \nu_0 s^2, \quad \forall x \not\in \Omega'_T \text{ and } s \in \mathbb{R}$$

and consequently

$$\left(\frac{1}{2} - \frac{1}{q+1}\right) \left(\|u_n\|_\lambda^2 - \nu_0 \|u_n\|_\lambda^2\right) \leq c + o_n(1) + \epsilon_n \|u_n\|_\lambda,$$

which together with Lemma 2.1 yields

$$\delta_0 \left(\frac{1}{2} - \frac{1}{q+1}\right) \|u_n\|_\lambda^2 \leq c + o_n(1) + \epsilon_n \|u_n\|_\lambda.$$

Therefore,

$$\limsup_{n \to \infty} \|u_n\|_\lambda^2 \leq \delta_0^{-1} \left(\frac{1}{2} - \frac{1}{q+1}\right)^{-1} c.$$

Then there exists $K > 0$ such that $\|u_n\|_\lambda \leq K$ for all $n \in \mathbb{N}$.

**Proposition 2.1.** The functional $\Phi_\lambda$ satisfies the Palais-Smale condition at the level $c$, for every $c \in \mathbb{R}$, that is, any sequence satisfying (8) possesses a convergent subsequence in $E$.

**Proof.** Let $(u_n) \subset E$ be a Palais-Smale sequence at the level $c \in \mathbb{R}$. By Lemma 2.3, $(u_n)$ is bounded in $E$. Hence there exist $u \in E$ and a subsequence, still denoted by $(u_n)$, such that $(u_n)$ is weakly convergent to $u$ in $E$. 

Choose \( R > 0 \) such that \( \Omega'_R \subset B_R(0) \) and let \( \Psi_R \) be a cut-off function such that \( \Psi_R(x) = 1 \) for all \( x \in B_{2R}(0), \Psi_R(x) = 0 \) for all \( x \in B_R(0), 0 \leq \Psi_R(x) \leq 1, \) \( |\Psi'_R(x)| \leq \frac{C}{R} \) for all \( x \in \mathbb{R}. \) Since \((u_n)\) is a bounded Palais-Smale sequence,

\[
o_n(1) = \Phi'_A(u_n)(\Psi_R u_n) = \int_{\mathbb{R}} |u_n'|^2 \Psi_R + \int_{\mathbb{R}} u_n u'_n \Psi'_R + \int_{\mathbb{R}} (\lambda V(x) + 1)|u_n|^2 \Psi_R + 2 \int_{\mathbb{R}} |u_n|^2 \Psi_R |u_n'|^2 + \int_{\mathbb{R}} u_n^3 u'_n \Psi'_R - \int_{\mathbb{R}} g(x, u_n) u_n \Psi_R
\]

\[
\geq \int_{|x| \geq 2R} |u_n'|^2 + \int_{|x| \geq 2R} \lambda V(x) + 1 - \nu_0)|u_n|^2 + \int_{\mathbb{R}} u_n u'_n \Psi'_R + \int_{\mathbb{R}} u_n^3 u'_n \Psi'_R
\]

where in this last inequality we used \((g_2)\). On the other hand, since \((u_n)\) is bounded in \( E, \)

\[
\int_{\mathbb{R}} |u_n||u_n'||\Psi'| \leq C \int_{\mathbb{R}} |u_n||u_n'| \leq C \int_{\mathbb{R}} |u_n| |u_n'|^2 \leq \frac{C}{R}
\]

and

\[
\int_{\mathbb{R}} |u_n^3||u_n'||\Psi'_R | \leq C \int_{\mathbb{R}} |u_n^3||u_n'| \leq C \int_{\mathbb{R}} |u_n|^2 |u_n'|^2 \leq \frac{C}{R}
\]

for some constant \( C > 0. \) From \((9)-(11)\) and Lemma 2.1, there exists \( C > 0 \) such that

\[
\limsup_{n \to +\infty} \int_{|x| \geq 2R} (|u_n'|^2 + \lambda V(x) + 1)|u_n|^2) \leq \frac{C}{R}
\]

Observing that \( \Omega'_R \subset B_R(0), \) it follows from definition of \( g \) that

\[
g(x, u_n) u_n \leq C |u_n|^2, \quad \forall |x| \geq 2R.
\]

Thereby, this inequality and \((12)\) then show

\[
\int_{|x| \geq 2R} g(x, u_n) u_n = O(R^{-1})
\]

uniformly in \( n. \) Hence

\[
\int_{\mathbb{R}} g(x, u_n) u_n = \int_{-2R}^{2R} g(x, u_n) u_n + O(R^{-1})
\]

uniformly in \( n. \) Since \( E \) is compactly embedded in \( L^p_{loc}(\mathbb{R}) \) for \( 1 \leq p < +\infty, \)

\[
\int_{-2R}^{2R} g(x, u_n) u_n = \int_{-2R}^{2R} g(x, u) u + o_n(1).
\]

By \((13)\) and \((14)\), we get

\[
\lim_{n \to +\infty} \int_{\mathbb{R}} g(x, u_n) u_n = \int_{\mathbb{R}} g(x, u) u.
\]

On the other hand, since \( u_n \rightharpoonup u \) weakly in \( E, \)

\[
||u_n - u||^2_2 = \langle u_n, u_n \rangle - \langle u_n, u \rangle + o_n(1),
\]

where \( \langle \cdot, \cdot \rangle \) denotes the inner product in \( E.\)
It follows that
\[
\|u_n - u\|_\lambda^2 = \Phi_\lambda'(u_n)u_n - \Phi_\lambda(u_n)u - 2 \int_{\mathbb{R}} |u_n|^2 u_n' + \int_{\mathbb{R}} |u_n|^2 u_n u' + \int_{\mathbb{R}} u_n' u_n + \int_{\mathbb{R}} g(x, u_n)u_n - \int_{\mathbb{R}} g(x, u_n)u + o_n(1).
\]

From (15),
\[
\|u_n - u\|_\lambda^2 = -2 \int_{\mathbb{R}} |u_n|^2 |u_n'|^2 + \int_{\mathbb{R}} |u_n|^2 u_n' u' + \int_{\mathbb{R}} u_n |u_n'|^2 + o_n(1).
\]

Since $E$ is continuously embedded in $L^p(\mathbb{R})$ for $1 \leq p \leq +\infty$, for each $\delta > 0$ fixed, there exists $T > 0$ such that
\[
\|u_n - u\|_\lambda^2 \leq -2 \int_{-T}^{T} |u_n|^2 |u_n'|^2 + \int_{-T}^{T} |u_n|^2 u_n' u' + \int_{-T}^{T} u_n |u_n'|^2 + \int_{-T}^{T} u_n' u_n' + o_n(1).
\]

Writing
\[
\int_{-T}^{T} |u_n|^2 |u_n'|^2 - u^2 |u'|^2 = \int_{-T}^{T} |u_n|^2 - u^2 |u_n'|^2 + \int_{-T}^{T} u^2 |u_n'|^2 - \int_{-T}^{T} u^2 |u'|^2
\]
and
\[
\int_{-T}^{T} u_n |u_n'|^2 u_n - |u_n|^2 |u_n'|^2 = \int_{-T}^{T} u_n (u_n u_n' u_n', u_n', u_n') + o_n(1)
\]
we deduce that
\[
\int_{-T}^{T} |u_n|^2 u_n' u' = \int_{-T}^{T} u^2 |u'|^2 + o_n(1)
\]
and
\[
\int_{-T}^{T} u_n |u_n'|^2 u_n = \int_{-T}^{T} |u_n|^2 |u_n'|^2 + o_n(1),
\]
where the $o_n(1)$ terms were obtained using the converge $u_n \to u$ in $C(-T, T)$ and the boundedness of sequences $(u_n)$ in $L^\infty(-T, T)$ and $(u_n')$ in $L^2(-T, T)$. Therefore, we can conclude that
\[
\|u_n - u\|_\lambda^2 \leq -\int_{-T}^{T} |u_n|^2 |u_n'|^2 + \int_{-T}^{T} u^2 |u'|^2 + \frac{\delta}{4} + o_n(1).
\]

Applying Lemma 2.2, we get
\[
\lim_{n \to +\infty} \|u_n - u\|_\lambda^2 = 0.
\]

This completes the proof of Proposition 2.1.

In the following, we use a version of the notion of Palais-Smale sequence introduced in [13]. We say that $(u_n)$ is a $(PS)'$ sequence if

\[
\begin{cases}
    u_n \in E, \text{ for every } n, \\
    (\Phi_{\lambda_n}(u_n)) \text{ is bounded}, \\
    \|\Phi_{\lambda_n}(u_n)\|_{\lambda_n} \to 0, \text{ as } n \to +\infty, \\
    \lambda_n \to +\infty, \text{ as } n \to +\infty.
\end{cases}
\]

(PS)'}
Proposition 2.2. Let \((u_n)\) be a \((PS)'\) sequence. Then, up to subsequence, there exists \(u \in H^1(\mathbb{R})\) such that \((u_n)\) weakly converges to \(u\) in \(H^1(\mathbb{R})\). Moreover, 

\(i)\) \(u \equiv 0\) on \(\mathbb{R} \setminus \Omega\) and \(u\) is a nonnegative solution of 
\[
\begin{align*}
-u'' + \frac{1}{2}(|u|^2)'u &= u^3, & \text{in } \Omega_j \\
u &= 0 & \text{on } \partial \Omega_j
\end{align*}
\]

for each \(j \in \Gamma\).

\(ii)\) \(\|u_n - u\|_{\lambda_n} \to 0\), as \(n \to +\infty\), which implies \(u_n \to u\) in \(H^1(\mathbb{R})\). 

\(iii)\) \(u_n\) also satisfies 
\[
\lambda_n \int_{\mathbb{R}} V(x)|u_n|^2 \to 0, \quad \|u_n\|^2_{\lambda_n, \mathbb{R} \setminus \Omega_j} \to 0 \quad \text{and} \quad \|u_n\|^2_{\lambda_n, \Omega_j} \to \|u\|^2_{H^1(\Omega_j)} \quad \forall j \in \Gamma.
\]

Proof. As in the proof of Lemma 2.3, it follows that there exists \(K > 0\) such that 
\[
\|u_n\|^2_{\lambda_n} \leq K \forall n \in \mathbb{N}.
\]

In particular, \((u_n)\) is bounded in \(H^1(\mathbb{R})\). Thus, we can assume that 
\[
u_n \to u \quad \text{weakly in } H^1(\mathbb{R})
\]
and 
\[
u_n(x) \to u(x) \quad \text{a.e. in } \mathbb{R}
\]
for some \(u \in H^1(\mathbb{R})\).

In order to prove \(i)\), given \(k \in \mathbb{N}\), define the set 
\[
A_k = \{x \in \mathbb{R} : V(x) \geq \frac{1}{k}\}.
\]

Then 
\[
\int_{A_k} |u_n|^2 \leq \frac{k}{\lambda_n} \int_{\mathbb{R}} \lambda_n V(x)|u_n|^2 \leq \frac{k}{\lambda_n} \|u_n\|^2_{\lambda_n} \leq \frac{kK}{\lambda_n}.
\]

Combining this inequality with Fatou Lemma, we have 
\[
\int_{A_k} |u|^2 = 0 \quad \forall k \in \mathbb{N},
\]
leading to \(u = 0\) on \(A_k\) for every \(k \in \mathbb{N}\). Observing that 
\[
\mathbb{R} \setminus \overline{\Omega} = \left\{x \in \mathbb{R} : V(x) > 0\right\} = \bigcup_{k \in \mathbb{N}} A_k
\]
we can conclude that \(u = 0\) on \(\mathbb{R} \setminus \overline{\Omega}\). Hence \(u \in H^1_0(\Omega_j)\) for every \(j \in \{1, \ldots, m\}\) (recall that \(\Omega = \Omega_1 \cup \ldots \cup \Omega_m\)).

Arguing as in the proof of Proposition 2.1, for any \(R > 0\) sufficiently large, 
\[
\lim_{n \to +\infty} \sup_{|x| \geq 2R} \left(|u_n'|^2 + |u_n|^2\right) \leq \frac{C}{R}.
\]

Since \(u_n \to u\) weakly in \(H^1(\mathbb{R})\) and \(u = 0\) on \(\mathbb{R} \setminus \overline{\Omega}\), we have 
\[
\int_{\mathbb{R}} |u_n' - u'|^2 = \int_{\mathbb{R}} u_n'(u_n' - u') + o_n(1), \quad (16)
\]
\[
\int_{\mathbb{R}} (\lambda_n V(x) + 1)|u_n - u|^2 = \int_{\mathbb{R}} (\lambda_n V(x) + 1)u_n(u_n - u) + o_n(1) \quad (17)
\]
and 
\[
\int_{\mathbb{R}} g(x, u_n)u_n = \int_{\mathbb{R}} g(x, u_n)u + o_n(1) = \int_{\mathbb{R}} g(x, u)u + o_n(1). \quad (18)
\]
Combining (16)-(18),
\[ \| u_n - u \|_{H_0^1}^2 = -2 \int_{\Omega} |u_n|^2|u_n'|^2 + \int_{\Omega} |u_n|^2u_n'u' + \int_{\Omega} u_n|u_n'|^2u + o_n(1). \]
Repeating the same arguments explored in the proof of Proposition 2.1, the last equality implies that
\[ \| u_n - u \|_{H_0^1}^2 \to 0 \text{ as } n \to +\infty. \]
In particular, \( u_n \to u \) in \( H^1(\Omega) \), which gives that (i) holds.

In order to complete the proof of (ii), for any \( j \in \Gamma \) and \( \phi \in C_0^\infty(\Omega_j) \), we have
\[ \Phi_{\lambda_n}(u_n)\phi = o_n(1), \]
which implies
\[ \int_{\Omega_j} u_n'\phi' + \int_{\Omega_j} u_n|u_n'|^2 \phi + \int_{\Omega_j} |u_n|^2u_n'\phi' + \int_{\Omega_j} u_n\phi - \int_{\Omega_j} (u_n^+)^q \phi = o_n(1) \]
where we have used that \( V(x) = 0 \) on \( \Omega_j \) and the definition of \( g \). By using the convergence \( u_n \to u \) in \( H^1(\Omega_j) \), the fact that \( u \in H^1_0(\Omega_j) \), and the density of \( C_0^\infty(\Omega_j) \) in \( H^1_0(\Omega_j) \), we get
\[ \int_{\Omega_j} u'\phi' + \int_{\Omega_j} |u|^2 \phi + \int_{\Omega_j} u^2\phi' + \int_{\Omega_j} u\phi - \int_{\Omega_j} (u^+)^q \phi = 0, \quad \forall \phi \in H^1_0(\Omega_j). \]
Choosing \( \phi = u^- = \min\{u, 0\} \) as a test function in (19), we find
\[ \int_{\Omega_j} |u^-|^2 - 2 \int_{\Omega_j} |u^-|^2(u^-)'^2 + \int_{\Omega_j} |u^-|^2 = 0 \]
which yields \( u^- = 0 \) in \( H^1_0(\Omega_j) \) for all \( j \in \Gamma \), that is, \( u(x) \geq 0 \) for all \( x \in \Omega_j \) and \( j \in \Gamma \). Thus, \( u \) is a nonnegative weak solution of
\[ \begin{cases}
   -u'' + u - \frac{1}{2}(|u|^2)'u = u^q, & x \in \Omega_j \\
   u = 0 & \text{on } \partial \Omega_j.
\end{cases} \]

For (iii), observing that \( u = 0 \) on \( \mathbb{R} \setminus \overline{\Omega} \) and \( V(x) = 0 \) on \( \overline{\Omega} \), it follows that
\[ \lambda_n \int_{\mathbb{R}} V(x)|u_n|^2 = \lambda_n \int_{\mathbb{R}} V(x)|u_n - u|^2 \leq \| u_n - u \|_{H^1_0(\Omega_j)}^2 \to 0, \]
\[ \| u_n \|_{\lambda_n, \mathbb{R} \setminus \Omega}^2 \leq \| u_n - u \|^2 + \lambda_n \int_{\mathbb{R}} V(x)|u_n - u|^2 \to 0, \]
and
\[ \| u_n \|_{\lambda_n, \Omega_j}^2 = \| u_n \|^2_{H^1(\Omega_j)} + \lambda_n \int_{\Omega_j} V(x)|u_n|^2 \to \| u \|^2_{H^1(\Omega_j)}. \]

\[ \square \]

3. The existence of multi-bump solutions. We start this section with the definition of the functionals \( I_j : H^1_0(\Omega_j) \to \mathbb{R} \) and \( \Phi_{\lambda,j} : H^1(\Omega_j) \to \mathbb{R} \) given by
\[ I_j(u) = \frac{1}{2} \int_{\Omega_j} \left( (1 + |u|^2)|u'|^2 + |u|^2 \right) - \frac{1}{q + 1} \int_{\Omega_j} (u^+)^{q+1} \]
and
\[ \Phi_{\lambda,j}(u) = \frac{1}{2} \int_{\Omega_j} \left( (1 + |u|^2)|u'|^2 + (\lambda V(x) + 1)|u|^2 \right) - \frac{1}{q + 1} \int_{\Omega_j} (u^+)^{q+1} \]
whose critical points are respectively associated with the weak solutions of the problems

\[
\begin{aligned}
&-u'' + u - \frac{1}{2}(|u|^2)'u = (u^+)^q, \quad x \in \Omega_j \\
&u = 0 \quad \text{on } \partial \Omega_j,
\end{aligned}
\]

and

\[
\begin{aligned}
&-u'' + (\lambda V(x) + 1)u - \frac{1}{2}(|u|^2)'u = (u^+)^q, \quad x \in \Omega_j' \\
&u' = 0 \quad \text{on } \partial \Omega_j'.
\end{aligned}
\]

Note the Neumann boundary condition in the latter boundary value problem.

Since \( q > 3 \), a familiar argument shows that these functionals satisfy the hypotheses of the mountain pass theorem, hence there are values \( c_j \) and \( c_{\lambda,j} \) given by

\[
c_j = \inf_{\gamma \in \Gamma_j} \max_{t \in [0,1]} I_j(\gamma(t)), \quad c_{\lambda,j} = \inf_{\gamma \in \Gamma_{\lambda,j}} \max_{t \in [0,1]} \Phi_{\lambda,j}(\gamma(t))
\]

where

\[
\Gamma_j = \{ \gamma \in C([0,1], H^1_0(\Omega_j)) : \gamma(0) = 0, \ I_j(\gamma(1)) < 0 \}
\]

and

\[
\Gamma_{\lambda,j} = \{ \gamma \in C([0,1], H^1(\Omega_j')) : \gamma(0) = 0, \ \Phi_{\lambda,j}(\gamma(1)) < 0 \}.
\]

Moreover, since \( I_j \) and \( \Phi_{\lambda,j} \) satisfy the Palais-Smale condition, there exist nonnegative functions \( w_j \in H^1_0(\Omega_j) \) and \( w_{\lambda,j} \in H^1(\Omega_j') \) such that

\[
I_j(w_j) = c_j, \quad I'_j(w_{\lambda,j}) = 0
\]

and

\[
\Phi_{\lambda,j}(w_{\lambda,j}) = c_{\lambda,j}, \quad \Phi'_{\lambda,j}(w_{\lambda,j}) = 0.
\]

Since \( \Omega_j \subset \Omega_j' \), it is easy to verify that \( c_{\lambda,j} \leq c_j \). We also observe that there exists \( \eta > 0 \), independent of \( \lambda \) and \( j \) such that

\[
\int_{\Omega_j'} |w_{\lambda,j}|^{q+1} \geq \eta.
\]

Effectively, this follows from \( \Phi'_{\lambda,j}(w_{\lambda,j}) = 0 \) and the Sobolev imbedding \( H^1(\Omega_j') \hookrightarrow L^{q+1}(\Omega_j') \).

**Lemma 3.1.** Let \( (\lambda_n) \) be a real sequence such that \( \lambda_n \to +\infty \) as \( n \to +\infty \). For any \( j \in \Gamma \), the corresponding critical levels \( c_{\lambda_n,j} \) satisfy

\[
\lim_{n \to +\infty} c_{\lambda_n,j} = c_j.
\]

**Proof.** For \( j \in \Gamma \), define \( w_n = w_{\lambda_n,j} \). As in the proof of Lemma 2.3, (25) and \( c_{\lambda_n,j} \leq c_j \) imply \( ||w_n||_{\lambda_n, \Omega_j} \) is bounded in \( \mathbb{R} \). Employing similar arguments used in the proof of Proposition 2.2, there exists \( \tilde{w}_j \in H^1(\Omega_j') \) such that

\[
\begin{aligned}
w_n &\to \tilde{w}_j \quad \text{in } H^1(\Omega_j'), \quad \text{as } n \to +\infty \\
||w_n - \tilde{w}_j||_{\lambda_n,j} &\to 0, \quad \text{as } n \to +\infty \\
w_n &\to \tilde{w}_j \quad \text{in } L^{q+1}(\Omega_j'),
\end{aligned}
\]

\[
\int_{\Omega_j} |\tilde{w}_j|^{q+1} \geq \eta
\]

and

\[
\tilde{w}_j = 0 \quad \text{on } \Omega_j' \setminus \Omega_j,
\]

(28)
Now observing that 
\[ \Phi'_{\lambda,n,j}(w_n)\phi = 0, \ \forall \phi \in H^1(\Omega_j), \]
it follows that 
\[ \Phi'_{\lambda,n,j}(w_n)\phi = 0, \ \forall \phi \in C^\infty_0(\Omega_j). \tag{29} \]

Taking \( n \to +\infty \) in (29) and using (27), we obtain 
\[ I'_j(\bar{w}_j)\phi = 0, \ \forall \phi \in C^\infty_0(\Omega_j). \]

Hence, \( \bar{w}_j \neq 0 \) and \( I'_j(\bar{w}_j) = 0 \). Since (28) leads to \( \bar{w}_j \in H^1_0(\Omega_j) \), we can conclude that 
\[ c_j \leq I_j(\bar{w}_j) = \Phi_{\lambda,n,j}(w_n) + o_n(1) = c_{\lambda,n,j} + o_n(1) \leq c_j + o_n(1). \]

Therefore, \( c_{\lambda,n,j} \to c_j \). \( \square \)

3.1. Special minimax values of \( \Phi_\lambda \). By continuity of \( I_j \), there exists \( R > 1 \) such that 
\[ \left| I_j(\frac{1}{R}w_j) \right| < \frac{c_j}{2} \text{ and } |I_j(Rw_j) - c_j| \geq 1 \]
for all \( j \in \Gamma \). By using the definition of \( c_j \), we can verify that 
\[ \max_{t \in [\frac{1}{R}, 1]} I_j(tRw_j) = c_j \ \forall j \in \Gamma. \]

Recalling that \( \Gamma \subset \{1, \ldots, m\} \), it can be assumed that \( \Gamma = \{1, \ldots, s\} \) for some \( s \leq m \). Define 
\[ \gamma_0(t_1, \ldots, t_s) = \sum_{j=1}^s t_jRw_j, \ \forall (t_1, \ldots, t_s) \in [\frac{1}{R^2}, 1]^s \]
\[ \Gamma_* = \left\{ \gamma \in C([\frac{1}{R^2}, 1]^s, E) : \gamma = \gamma_0 \text{ on } \partial([\frac{1}{R^2}, 1]^s) \right\}, \]
and 
\[ b_{\lambda, \Gamma} = \inf_{\gamma \in \Gamma_*} \max_{(t_1, \ldots, t_s) \in [\frac{1}{R^2}, 1]^s} \Phi_\lambda(\gamma(t_1, \ldots, t_s)). \tag{32} \]

We observe that \( \gamma_0 \in \Gamma_* \), thus \( \Gamma_* \neq \emptyset \) and \( b_{\lambda, \Gamma} \) is well defined.

**Lemma 3.2.** For every \( \gamma \in \Gamma_* \) there is \((t^{1*}_1, \ldots, t^{1*}_s) \in [\frac{1}{R^2}, 1]^s \) such that for all \( j \in \Gamma \):
\[ \Phi'_{\lambda,j}(\gamma(t^{1*}_1, \ldots, t^{1*}_s))\gamma(t^{1*}_1, \ldots, t^{1*}_s) = 0. \]

**Proof.** For \( \gamma \in \Gamma_* \), define the map \( \tilde{\gamma} : [\frac{1}{R^2}, 1]^s \to \mathbb{R}^s \) by 
\[ \tilde{\gamma}(t_1, \ldots, t_s) = (\Phi'_{\lambda,1}(\gamma(t_1, \ldots, t_s))\gamma(t_1, \ldots, t_s), \ldots, \Phi'_{\lambda,s}(\gamma(t_1, \ldots, t_s))\gamma(t_1, \ldots, t_s)). \]

Then, \( \tilde{\gamma} = (\Phi'_{\lambda,1}(\gamma_0), \ldots, \Phi'_{\lambda,s}(\gamma_0)) \) on \( \partial([\frac{1}{R^2}, 1]^s) \). We also observe that 
\[ \Phi'_{\lambda,j}(\gamma_0(t_1, \ldots, t_s))\gamma_0(t_1, \ldots, t_s) \neq 0 \]
if \( t_j \in \{0, 1\} \), for all \( j \in \Gamma \), hence \((0, \ldots, 0) \not\in \tilde{\gamma}(\partial([\frac{1}{R^2}, 1]^s)) \). By using the fact that \( I'_j(w_j/R)|w_j/R| > 0 \) and \( I'_j(Rw_j)|Rw_j| < 0 \) (cf. [5]) and properties of the topological degree, we conclude that 
\[ \text{deg}(\tilde{\gamma}, (\frac{1}{R^2}, 1)^s, (0, \ldots, 0)) = (-1)^s \neq 0. \]

Thus, there exists \((t^{1*}_1, \ldots, t^{1*}_s) \in (\frac{1}{R^2}, 1)^s \) such that 
\[ \Phi'_{\lambda,j}(\gamma(t^{1*}_1, \ldots, t^{1*}_s))\gamma(t^{1*}_1, \ldots, t^{1*}_s) = 0 \ \forall j \in \Gamma. \] \( \square \)
The next proposition is crucial to prove Theorem 1.1. It is established an important relation between $c_{\lambda,j}$ and $b_{\lambda,\Gamma}$. In the sequel we denote by $c_{\Gamma}$ the number $c_{\Gamma} = \sum_{j=1}^{s} c_{j}$.

**Proposition 3.1.**

a) $\sum_{j=1}^{s} c_{\lambda,j} \leq b_{\lambda,\Gamma} \leq c_{\Gamma}$.

b) $b_{\lambda,\Gamma} \to c_{\Gamma}$ as $\lambda \to +\infty$.

c) For any $\gamma \in \Gamma_*$, $\Phi_{\lambda}(\gamma(t_1, \ldots, t_s)) < c_{\Gamma}$ for all $\lambda > 0$ and $(t_1, \ldots, t_s) \in \partial([\frac{1}{n^{2}}, 1]^s)$. 

**Proof.** From (30)-(32), since $\gamma_0 \in \Gamma_*$ and $\Omega'_i \cap \Omega'_j = \emptyset$ for $i \neq j$,

$$b_{\lambda,\Gamma} \leq \max_{(t_1, \ldots, t_s) \in [\frac{1}{n^{2}}, 1]^s} \Phi_{\lambda}(\gamma_0(t_1, \ldots, t_s)) = \max_{(t_1, \ldots, t_s) \in [\frac{1}{n^{2}}, 1]^s} \sum_{j=1}^{s} I_j(t_j Rw_j) = c_{\Gamma}. \quad (33)$$

Now, recalling that $c_{\lambda,j}$ can be characterized by

$$c_{\lambda,j} = \inf \{ \Phi_{\lambda,j}(u) : u \neq 0 \text{ and } \Phi'_{\lambda,j}(u)u = 0 \},$$

setting $(t_1^*, \ldots, t_s^*)$ given by Lemma 3.2, we get

$$c_{\lambda,j} \leq \Phi_{\lambda,j}(\gamma(t_1^*, \ldots, t_s^*))$$

for all $j \in \Gamma$. Thus,

$$\sum_{j=1}^{s} c_{\lambda,j} \leq \sum_{j=1}^{s} \Phi_{\lambda,j}(\gamma(t_1^*, \ldots, t_s^*)) \leq \Phi_{\lambda}(\gamma(t_1^*, \ldots, t_s^*)) \leq \max_{(t_1, \ldots, t_s) \in [\frac{1}{n^{2}}, 1]^s} \Phi_{\lambda}(\gamma(t_1, \ldots, t_s)), \quad (34)$$

where we have used that $\Phi_{\lambda}(u) \geq 0$ for all $u \in H^1(\mathbb{R} \setminus \Omega'_\Gamma)$. Hence from definition of $b_{\lambda,\Gamma}$, we have

$$\sum_{j=1}^{s} c_{\lambda,j} \leq b_{\lambda,\Gamma}. \quad (34)$$

By (33) and (34) the proof of a) is finished.

For b), from Lemma 3.1 and a), it follows that $b_{\lambda,\Gamma} \to c_{\Gamma}$ as $\lambda \to +\infty$.

For c), since $\gamma = \gamma_0$ on $\partial([\frac{1}{n^{2}}, 1]^s)$ for all $\gamma \in \Gamma_*$, it follows that

$$\Phi_{\lambda}(\gamma(t_1, \ldots, t_s)) = \Phi_{\lambda}(\gamma_0(t_1, \ldots, t_s)) = \sum_{j=1}^{s} I_j(t_j Rw_j)$$

for all $(t_1, \ldots, t_s) \in \partial([\frac{1}{n^{2}}, 1]^s)$. Recalling that there exists $j_0 \in \{1, \ldots, m\}$ such that $t_{j_0} R \neq 1$, we have $I_{j_0}(t_{j_0} Rw_{j_0}) < c_{j_0}$, and therefore

$$\Phi_{\lambda}(\gamma(t_1, \ldots, t_s)) < \sum_{j=1}^{s} c_{j} = c_{\Gamma},$$

this implies c) and the proof is completed. \qed
4. Conclusion. In this section we establish the proof of Theorem 1.1. The goal is to find a positive solution \( u_\lambda \) which is close to a least energy solution in each \( \Omega_j, j \in \Gamma \), provided \( \lambda \) is sufficiently large. To this end, the key ingredients are the following propositions.

Hereafter,
\[
M = 1 + \sum_{j=1}^{m} \left( \frac{1}{2} - \frac{1}{q + 1} \right)^{-1} c_j
\]
and
\[
\overline{B}_{M+1}(0) = \{ u \in E : \|u\|_{\lambda} \leq M + 1 \}.
\]
Moreover, given \( \lambda, \mu > 0 \) define
\[
A^\lambda_\mu = \{ u \in \overline{B}_{M+1}(0) : \|u\|_{\lambda, \Gamma, \Omega_1^c} \leq \mu, |\Phi_{\lambda,j}(u) - c_j| \leq \mu, \forall j \in \Gamma \}
\]
and
\[
\Phi^c_{\lambda} = \{ u \in E : \Phi(u) \leq c_\Gamma \}.
\]

Perhaps it is appropriate at this point to note that the use of the ball in the definition of the set \( A^\lambda_\mu \) permits us to adapt the arguments exploit by Ding and Tanaka to prove Proposition 4.2 in [13]. We recall that we modify the sets that appear in the minimax arguments employed in [13].

We observe that \( u = \sum_{j=1}^{m} u_j \in A^\lambda_\mu \cap \Phi^c_{\lambda} \), showing that \( A^\lambda_\mu \cap \Phi^c_{\lambda} \neq \emptyset \). Now we fix \( 0 < \mu < \min_j \frac{c_j}{2} \). We have the following uniform estimate of \( \|\Phi'_{\lambda}(u)\|_{\lambda}^* \) on the annulus \( (A^{\lambda}_2 \setminus A^{\lambda}_1) \cap \Phi^c_{\lambda} \):

**Proposition 4.1.** There exist \( \sigma_0 > 0 \) and \( \lambda^* > 0 \) such that
\[
\|\Phi'_{\lambda}(u)\|_{\lambda}^* \geq \sigma_0
\]
for all \( \lambda > \lambda^* \) and \( u \in (A^{\lambda}_2 \setminus A^{\lambda}_1) \cap \Phi^c_{\lambda} \).

**Proof.** Suppose by contradiction that there exist \( \lambda_n \to +\infty \) and \( u_n \in (A^{\lambda_2}_n \setminus A^{\lambda_1}_n) \cap \Phi^c_{\lambda_n} \) such that \( \|\Phi'_{\lambda_n}(u_n)\|_{\lambda_n}^* \to 0 \). Since \( u_n \in A^{\lambda_2}_n \), \( \Phi_{\lambda_n}(u_n) \) is bounded, and then we can assume \( \Phi_{\lambda_n}(u_n) \to c \in (-\infty, c_\Gamma) \). Hence \( (u_n) \) is a \((PS)'
 sequence, and by Proposition 2.2, we can assume that
\[
\begin{align*}
  u_n &\to u \text{ in } H^1(\mathbb{R}), \\
  \lambda_n \int_{\mathbb{R}} V(x)|u_n|^2 &\to 0 \tag{35} \\
  \|u_n\|_{\lambda_n, \mathbb{R} \setminus \Omega_\Gamma}^2 &\to 0. \tag{37}
\end{align*}
\]
Since \( c_j \) is the least energy level of \( I_j \), we have two possibilities:
(i) \( I_j(u|_{\Omega_j}) = c_j \) for all \( j \in \Gamma \).
(ii) \( I_{j_0}(u|_{\Omega_{j_0}}) = 0 \) for some \( j_0 \in \Gamma \), that is, \( u|_{\Omega_{j_0}} = 0 \).

We claim that both of them do not occur. In fact, if (i) holds, from (35)-(37) \( u_n \in A^{\lambda_2}_n \) for large \( n \), which contradicts \( u_n \in (A^{\lambda_2}_n \setminus A^{\lambda_1}_n) \). Now, if (ii) occurs, from (35)-(37) again,
\[
|\Phi_{\lambda_n,j_0}(u_n) - c_{j_0}| \to c_{j_0} \geq 3\mu
\]
which is a contradiction with the fact that \( u_n \in (A^{\lambda_2}_n \setminus A^{\lambda_1}_n) \). This completes the proof of Proposition 4.1. \( \square \)

In the following proposition, we have to suppose a stronger condition on \( \mu \) than previously, for instance, \( \mu \leq \min\{\min_j \frac{c_j}{2}, 1\} \).
Proposition 4.2. Let \( \mu \) be a fixed number as above. Then \( \Phi_\lambda \) possesses a critical point \( u_\lambda \in A^\lambda_\mu \cap \Phi^\mu_\lambda \), for every \( \lambda \) large enough.

Proof. Arguing by contradiction, we suppose that for some large \( \lambda \geq \lambda^* \) the functional \( \Phi_\lambda \) does not have a critical point in \( A^\lambda_\mu \cap \Phi^\mu_\lambda \), where \( \lambda^* \) is given by Proposition 4.1. Hence, from Proposition 2.1, there exists \( d_\lambda > 0 \) such that \( \|\Phi'_\lambda(u)\|_{\lambda^*} \geq d_\lambda \) for all \( u \in A^\lambda_\mu \cap \Phi^\mu_\lambda \). By Proposition 4.1, \( \|\Phi'_\lambda(u)\|_{\lambda^*} \geq \sigma_0 \) for all \( u \in (A^\lambda_{2\mu} \setminus A^\lambda_\mu) \cap \Phi^\mu_\lambda \).

Now, consider \( \xi : E \rightarrow \mathbb{R} \) and \( H : \Phi^\mu_\lambda \rightarrow E \) locally Lipschitz continuous functions defined by

\[
\xi(u) = 1 \text{ for } u \in A^\lambda_{2\mu}, \quad \xi(u) = 0 \text{ for } u \not\in A^\lambda_{2\mu}, \quad 0 \leq \xi \leq 1 \text{ for } u \in E
\]

and

\[
H(u) = \begin{cases} -\xi(u)\|Y(u)\|_{\lambda^*}^{-1}Y(u), & u \in A^\lambda_{2\mu}, \\ 0, & u \not\in A^\lambda_{2\mu} \end{cases}
\]

where \( Y \) is a pseudo-gradient vector field for \( \Phi_\lambda \) on \( \mathcal{M} = \{u \in E : \Phi'_\lambda(u) \neq 0\} \). Thus, \( \|H(u)\|_{\lambda^*} \leq 1 \) for all \( \lambda \geq \lambda^* \) and \( u \in \Phi^\mu_\lambda \).

Consider the deformation flow \( \eta : [0, +\infty) \times \Phi^\mu_\lambda \rightarrow \Phi^\mu_\lambda \) given by

\[
\begin{cases} \frac{d\eta}{dt} = H(\eta(t)) \\ \eta(0, u) = u \in \Phi^\mu_\lambda \end{cases}
\]

Observing that there exists a constant \( K > 0 \) such that

\[
|\Phi_{\lambda,j}(u) - \Phi_{\lambda,j}(v)| \leq K\|u - v\|_{\lambda, \Omega_f^j},
\]

for every \( u, v \in \overline{\mathcal{B}}_{M+1}(0) \) and \( j \in \Gamma \), we can use the same kind of reasoning used on pp. 133–134 of [13] to obtain positive numbers \( T = T(\lambda) \) and \( \epsilon^* \) independent of \( \lambda \) such that

\[
\gamma^*(t_1, \ldots, t_s) = \eta(T, \gamma_0(t_1, \ldots, t_s)) \in \Gamma^*_t
\]

and

\[
\max_{(t_1, \ldots, t_s) \in [1/R^2, 1]^s} \Phi_\lambda (\gamma^*(t_1, \ldots, t_s)) \leq c_T - \epsilon^*.
\]

It should be noted that to achieve (38), one needs \( \gamma_0(\partial[1/R^2, 1]^s) \subset E \setminus A^\lambda_{2\mu} \). This is proved assuming that \( \mu \leq \min\{\min_j c^*_j, 1\} \).

From (38) and (39), we have \( b_{\lambda, \Gamma} \leq c_\gamma - \epsilon^* \). This contradicts Proposition 3.1 if \( \lambda \) is large enough. \( \square \)

4.1. Proof of Theorem 1.1. Now we are ready to establish the proof of Theorem 1.1. By Proposition 4.2, there is a family \( \{u_\lambda\} \) of positive solutions to equation (7), such that for fixed \( \mu > 0 \) there exists \( \lambda^* > 0 \) so that

\[
\|u_\lambda\|_{\lambda, \mathbb{R} \setminus \Omega_f^\gamma} \leq \mu \quad \forall \lambda \geq \lambda^*.
\]

Hence, for \( \mu \) sufficiently small, the last inequality together with Sobolev imbedding lead to

\[
|u_\lambda|_{\infty, \mathbb{R} \setminus \Omega_f^\gamma} \leq \mu \quad \forall \lambda \geq \lambda^*.
\]

showing that \( u_\lambda \) is a positive solution to equation (5).

Taking sequences \( \lambda_n \rightarrow +\infty \) and \( \mu_n \rightarrow 0 \), then the sequence \( \{u_{\lambda_n}\} \) verifies

\[
\Phi'_{\lambda_n}(u_{\lambda_n}) = 0, \quad \forall n \in \mathbb{N}, \quad \text{and} \quad \Phi_{\lambda_n,j}(u_{\lambda_n}) \rightarrow c_j, \quad \text{as} \quad n \rightarrow +\infty, \quad \forall j \in \Gamma,
\]

that is, \( \{u_{\lambda_n}\} \) be a \((PS')\) sequence. Then, from Proposition 2.2,

\[
\|u_{\lambda_n}\|_{\lambda_n, \mathbb{R} \setminus \Omega_f^\gamma} \rightarrow 0, \quad u_{\lambda_n} \rightarrow u \text{ in } H^1(\mathbb{R}) \quad \text{and} \quad u_{\lambda_n} \rightarrow 0 \text{ in } H^1(\mathbb{R} \setminus \Omega_f^\gamma),
\]

respectively.
as $n \to +\infty$. Thus $u \in H^1_0(\Omega_I)$ and satisfies $I_j'(u) = 0$ and $I_j(u) = c_j$ for all $j \in \Gamma$, that is, $u|_{\Omega_I}$ is a weak solution to (22) for all $j \in \Gamma$, and the proof of Theorem 1.1 is complete.

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E-mail address: coalves@dme.ufcg.edu.br
E-mail address: olimpio.hiroshi@ufjf.edu.br
E-mail address: monari@icmc.usp.br