Some exact controllability results for the linear KdV equation and uniform controllability in the zero-dispersion limit

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In this paper, we deal with controllability properties of linear and nonlinear Korteweg-de Vries equations in a bounded interval. The main part of this paper is a result of uniform controllability of a linear KdV equation in the limit of zero-dispersion. Moreover, we establish a result of null controllability for the linear equation via the left Dirichlet boundary condition, and of exact controllability via both Dirichlet boundary conditions. As a consequence, we obtain some local exact controllability results for the nonlinear KdV equation.

We will present two types of results:

- Mainly, we are interested in how the cost of the null controllability of a linear Korteweg-de Vries (KdV) evolves as the dispersive term is brought to 0+ (Theorem 0.1). In the case of the vanishing viscosity limit (that is, when a dissipative term is considered rather than a dispersive one), this problem has been studied in [1] and [3].

- Next, we consider the problem of exact controllability for this equation, when the dispersion coefficient is fixed (Theorems 0.3 and 0.4). Such results yield results of local exact controllability for the usual (nonlinear) KdV equation (Theorems 0.5 and 0.6). The controllability of the KdV equation has already been studied in several papers, see in particular [4].

Let us be more specific on the problem under view. Let $T > 0$ be a given final time. Our system is the following one:

$$
\begin{cases}
y_t + \nu y_{xxx} + (My)_x = 0 \\
y_{x=0} = v_1, y_{x=1} = v_2, y_{x=1} = v_3 \\
y_{t=0} = y_0
\end{cases}
\quad \text{in } (0, T) \times (0, 1),$$

where \(\nu\) is a positive dispersion coefficient, \(M = M(t, x)\) is a transport coefficient (constant in the main problem), \(v_i\) \((i = 1, 2, 3)\) are time-dependent functions which constitute the controls of our system. Observe that the classical KdV equation corresponds to \(M(t, x) = 1 + \frac{2}{3}(t, x)\).

The principal result which we consider in this paper is the problem of uniform controllability of equation (1) (where \(M\) is a constant) as the dispersion parameter tends to 0+. Of course, one can hope to reach such a property only when the limit system (obtained by setting \(\nu = 0\) in (1)) is controllable. In this situation, this means \(M \neq 0\) and the time of controllability \(T\) is greater than \(1/|M|\). Moreover, we consider a time of controllability which is of the form \(K_0/|M|\), but our proof does not apply for any \(K_0 > 1\) (such a limitation appears also in the case of vanishing viscosity, see [1, 3]). Our result is the following.

**Theorem 0.1** There exists a positive constant \(K_0\) such that for any negative constant \(M\), there exists \(\nu_0 > 0\) such that for any \(T \geq K_0/|M|\), any \(y_0 \in W^{1,\infty}(0, 1)\) and any \(\nu \in (0, \nu_0)\), there exist \(v_1^\nu, v_2^\nu, v_3^\nu \in L^2(0, T)\) such that the solution \(y \in L^2((0, T) \times (0, 1)) \cap C^{0}([0, T]; H^{-1}(0, 1))\) of (1) satisfies \(y_{|t=T} = 0\) in \((0, 1)\) and moreover the controls are uniform in \(\nu\) in the sense that

$$
\|v_1^\nu\|_{L^2(0, T)} + \|v_2^\nu\|_{L^2(0, T)} + \|v_3^\nu\|_{L^2(0, T)} \leq K_1 \|y_0\|_{W^{1,\infty}(0, 1)},
$$

for a constant \(K_1 > 0\) independent of \(\nu\) and \(y_0\).

**Remark 0.2** As far as we know, the question of uniform (local exact) controllability of the KdV equation (5) in the limit \(\nu \to 0^+\), is an open problem. In the case of a vanishing viscosity limit for Burgers equation, such a result was established in [2].

Next, we consider the problem of controllability of (1) for fixed \(\nu\). We obtain the following two results in that case:

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Theorem 0.3 Let $M$ be a constant and $\nu > 0$ be fixed. Then, for any $y_0 \in H^{-1}(0, 1)$, there exists $v_1 \in L^2(0, T)$ such that the solution $y \in L^2((0, T) \times (0, 1)) \cap C^0([0, T]; H^{-1}(0, 1))$ of (I) with $v_2 = v_3 = 0$ satisfies $y_{|t=T} = 0$ in $(0, 1)$. Moreover, for any $\nu \in (0, 1)$, there exists $C^* > 0$ such that

$$\|v_1\|^2_{L^2(0,T)} \leq C^* \|y_0\|^2_{H^{-1}(0,1)}.$$  

(2)

Besides, for $\nu$ suitably small (in terms of $M$ only), one can estimate $C^*$ as follows:

$$C^* = \exp \left\{ \frac{\tilde{C} |M|^{1/2}}{\nu^{1/2}} \left( 1 + \frac{1}{T^{1/2}} |M|^{1/2} \right) \right\},$$  

(3)

where $\tilde{C} > 0$ is a constant independent of $M$, $\nu$ and $y_0$.

Theorem 0.4 Let $M$ be a constant and $\nu > 0$ be fixed. Then, for any $y_0$, $y_1 \in L^2(0, 1)$, there exist $v_1$ and $v_2$ in $L^2(0, T)$ such that the solution $y \in L^2((0, T) \times (0, 1)) \cap C^0([0, T]; H^{-1}(0, 1))$ of (I) with $v_3 = 0$ satisfies $y_{|t=T} = y_1$ in $(0, 1)$.

As a natural consequence of an exact controllability result for the linearized system, one can usually prove a local exact controllability result for the nonlinear system. The first one (Theorem 0.5) is a result of local exact controllability on trajectories where the control acts upon the left Dirichlet boundary condition, while the second one (Theorem 0.6) is a result of local exact controllability via both Dirichlet conditions.

Theorem 0.5 Let $\nu > 0$ be fixed. For $\tilde{\gamma}_0 \in L^2(0, 1)$, we consider $\overline{\gamma} \in L^\infty(0, T; L^2(0, 1)) \cap L^2(0, T; H_0^1(0, 1))$ the solution of

$$\begin{cases}
\overline{\gamma}_t + \overline{\gamma}_x + \nabla \overline{\gamma} + \nu \overline{\gamma}_{xxx} = 0 & \text{in } (0, T) \times (0, 1), \\
\overline{\gamma}|_{x=0} = 0, \overline{\gamma}|_{x=1} = 0, \overline{\gamma}_x|_{x=1} = 0 & \text{in } (0, T), \\
\overline{\gamma}|_{t=0} = \overline{\gamma}_0 & \text{in } (0, 1).
\end{cases}$$

(4)

Then, there exists $\gamma > 0$ such that for any $y_0 \in H^1(0, 1)$ satisfying $\|y_0 - \overline{\gamma}_0\|_{L^2(0,1)} \leq \gamma$, there exists $v_1 \in H^{1/2-\varepsilon}(0, T)$ for any $\varepsilon > 0$, such that the solution $y \in L^2((0, T); H^1(0, 1)) \cap C^0([0, T]; L^2(0, 1))$ of

$$\begin{cases}
y_t + y_x + \nu y_{xx} + \nu y_{xxx} = 0 & \text{in } (0, T) \times (0, 1), \\
y|_{x=0} = v_1, y|_{x=1} = 0, y_x|_{x=1} = 0 & \text{in } (0, T), \\
y|_{t=0} = y_0 & \text{in } (0, 1),
\end{cases}$$

satisfies $y_{|t=T} = \overline{\gamma}|_{t=T}$ in $(0, 1)$.

Theorem 0.6 Let $\nu > 0$ be fixed. There exists $\mu > 0$ such that for any $y_0, y_1 \in L^2(0, 1)$ satisfying

$$\|y_0\|_{L^2(0,1)} + \|y_1\|_{L^2(0,1)} < \mu,$$

(6)

there exists $v_1, v_2 \in L^2(0, T)$ such that the solution $y \in L^2((0, T) \times (0, 1)) \cap C^0([0, T]; H^{-1}(0, 1))$ of

$$\begin{cases}
y_t + y_x + \nu y_{xx} + \nu y_{xxx} = 0 & \text{in } (0, T) \times (0, 1), \\
y|_{x=0} = v_1, y|_{x=1} = v_2, y_x|_{x=1} = 0 & \text{in } (0, T), \\
y|_{t=0} = y_0 & \text{in } (0, 1),
\end{cases}$$

(7)

satisfies $y_{|t=T} = y_1$ in $(0, 1)$.

References

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