On $W_\infty$ Algebras, Gauge Equivalence of KP Hierarchies, Two-Boson Realizations and their KdV Reductions

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ABSTRACT

The gauge equivalence between basic KP hierarchies is discussed. The first two Hamiltonian structures for KP hierarchies leading to the linear and non-linear $W_\infty$ algebras are derived. The realization of the corresponding generators in terms of two boson currents is presented and it is shown to be related to many integrable models which are bi-Hamiltonian. We can also realize those generators by adding extra currents, coupled in a particular way, allowing for instance a description of multi-layered Benney equations or multi-component non-linear Schrödinger equation. In this case we can have a second Hamiltonian bracket structure which violates Jacobi identity. We consider the reduction to one-boson systems leading to KdV and mKdV hierarchies. A Miura transformation relating these two hierarchies is obtained by restricting gauge transformation between corresponding two-boson hierarchies. Connection to Drinfeld-Sokolov approach is also discussed in the $SL(2,\mathbb{R})$ gauge theory.

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1 Introduction

This is an expository account of results concerning various integrable systems described as KP hierarchies and a method of studying them by a symplectic gauge transformation. The method allows to introduce an equivalence principle between various KP hierarchies, which is useful in view of the growing number of integrable models entering recently the area of high energy physics (e.g. matrix models). In the process one encounters different realizations of various $\mathcal{W}$-infinity algebras in terms of two currents and new understanding of KdV hierarchies and corresponding soliton equations via symplectic reduction from two-boson KP hierarchies.

Section 2 introduces three basic KP hierarchies and the algebraic structure behind their construction. We point out that the Adler-Kostant-Symes (AKS)\footnote{\url{https://doi.org/10.1103/PhysRev.177.2426}} theory with a Poisson bracket structure defined in terms of the R-matrix\footnote{\url{https://doi.org/10.1016/0375-9601(80)90401-9}} Lie-Poisson bracket is the right setting to study these models and to define the two-boson restriction of full KP hierarchy. We also introduce the gauge transformation connecting basic flow equations.

In section 3 we present the hamiltonian structure of KP hierarchies defining both first and second Poisson structures. We also introduce here the fundamental two-boson KP hierarchy used later in section 4 to construct various realization of the area preserving diffeomorphisms. We go one step further in section 5 using two-boson KP hierarchy to present two-current realization of $\mathcal{W}_{1+\infty}$ algebra and the corresponding non-linear $\hat{\mathcal{W}}_\infty$ algebra.

Section 6 studies the concept of symplectic gauge equivalence between various two-boson KP hierarchies playing the role of generalized Miura transformations.\footnote{\url{https://doi.org/10.1007/BF01041818}} We emphasize the symplectic character of equivalence of KP$_{\ell=1}$ and KP and show how this feature explains the 2-boson representation of $\mathcal{W}_{1+\infty}$ and $\hat{\mathcal{W}}_\infty$ in terms of the Faà di Bruno polynomials.

In Section 7 we apply Dirac reduction scheme to two-boson KP hierarchies. We obtain in the process of reduction the standard “one-boson” KdV and mKdV hierarchies. On reduced manifold the gauge transformation connecting the two models takes the form of the Miura transformation.

The results of section 6 and 7 can be rephrased using the language of $SL(2,\mathbb{R})$ gauge theory to reveal connection between soliton equations and zero-curvature conditions. This is done in section 8. This formalism enables us to formulate the Dirac reduction to KdV systems in the language of Drinfeld-Sokolov reduction.

2 KP Hierarchies and their Algebraic Structure

Let us introduce the following general KP pseudo differential operator\footnote{\url{https://doi.org/10.1007/BF02099469}}\footnote{\url{https://doi.org/10.1007/BF02187791}}\footnote{\url{https://doi.org/10.1142/9789812385530_0005}}\footnote{\url{https://doi.org/10.1007/BF00167251}}\footnote{\url{https://doi.org/10.1007/BF02104783}}

\[ L \equiv u_{-2}D + \sum_{i=-1}^{\infty} u_i D^{-i-1} = u_{-2}D + u_{-1} + \sum_{i=0}^{\infty} u_i D^{-i-1} \]  

(2.1)

where

\[ D \equiv \frac{\partial}{\partial x} \]  

(2.2)

and functions $u_i(x,t)$ dependent on infinitely many variables $(x,t) = (x,t_1,t_2,\ldots)$. 

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In order to define $D^{-n}$, where $n$ is a positive integer, let us recall a Leibniz rule

$$D^n f = \sum_{\alpha=0}^{\infty} \frac{n(n-1)\ldots(n-\alpha+1)}{\alpha!} (\partial^\alpha f) D^{-n-\alpha}$$  \hspace{1cm} (2.3)$$

and let $n \rightarrow -n$ obtaining:

$$D^{-n} f = \sum_{\alpha=0}^{\infty} (-1)^\alpha \frac{(n+\alpha-1)!}{\alpha!(n-1)!} (\partial^\alpha f) D^{-n-\alpha}$$ \hspace{1cm} (2.4)

Other useful relations are:

$$\partial^n f = \sum_{\alpha=0}^{n} (-1)^\alpha \binom{n}{\alpha} D^{n-\alpha} f D^\alpha$$ \hspace{1cm} (2.5)$$

$$f D^n = \sum_{\alpha=0}^{n} (-1)^\alpha \binom{n}{\alpha} D^{n-\alpha} (\partial^\alpha f)$$ \hspace{1cm} (2.6)$$

There are three classes of integrable systems connected with the general object $L = u_{-2} D + u_{-1} + \sum_{i=0}^{\infty} u_i D^{-i-1}$ in (2.1). We label them by the parameter $\ell$ taking values 0, 1, 2 and define as follows [7, 8, 9, 10, 11]:

$\ell = 0$ : Lax operator $L$ with $u_{-2} = 1$ and $u_{-1} = 0$ (standard KP case)

$\ell = 1$ : Lax operator $L$ with $u_{-2} = 1$ and $u_{-1} \neq 0$ (first non-standard KP case)

$\ell = 2$ : Lax operator $L$ with the most general form as in (2.1) (second non-standard KP case)

It appears that existence of these three classes has origin in fundamental algebraic properties of a group of pseudo-differential operators on a circle. We will extract here few basic details of the algebraic formalism [7, 8, 11]. An object of interest is the Lie algebra $\mathcal{G}$ of pseudo-differential operators on a circle. An element of $\mathcal{G}$ is given by an arbitrary pseudo-differential operator $X = \sum_{i \geq -\infty} D^i X_i(x)$.

Origin of three models presented above can be traced to the fundamental fact that there exist precisely three decompositions of $\mathcal{G}$ into a linear sum of two subalgebras [7, 8, 11] (i.e. $\mathcal{G} = \mathcal{G}_+^{\ell} \oplus \mathcal{G}_-^{\ell}$, parametrized by the same index $\ell$ taking values $\ell = 0, 1, 2$):

$$\mathcal{G}_+^{\ell} = \{ X_{\geq\ell} = \sum_{i=\ell}^{\infty} D^i X_i(x) \} \hspace{1cm} ; \hspace{1cm} \mathcal{G}_-^{\ell} = \{ X_{<\ell} = \sum_{i=-\ell+1}^{\infty} D^{-i} X_{-i}(x) \}$$ \hspace{1cm} (2.7)

The dual spaces to subalgebras $\mathcal{G}_+^{\ell}$ are given by:

$$\mathcal{G}_+^{\ell,*} = \{ L_{<\ell} = \sum_{i=\ell+1}^{\infty} u_{-i}(x) D^{-i} \} \hspace{1cm} ; \hspace{1cm} \mathcal{G}_-^{\ell,*} = \{ L_{\geq\ell} = \sum_{i=-\ell}^{\infty} u_{i}(x) D^i \}$$ \hspace{1cm} (2.8)

Application of the general Adler-Kostant-Symes (AKS) [11] formalism results in all three decompositions giving rise to integrable models with flow equations allowing generalized Lax representations:

$$\frac{\partial L}{\partial t_n} = [ P_{\geq\ell}(L^n) , L ]$$ \hspace{1cm} (2.9)
where the projection $P_{\geq \ell}$ projects on terms $a_iD^i$ with $i \geq \ell$.

The basic structure of AKS construction is the R-matrix, which for all the above cases is defined as $R_\ell \equiv P^+_\ell - P^-_{\ell}$, where $P^\pm_\ell$ are projections on $G^\ell_{\pm}$. It follows from the general formalism that

$$[X,Y]_{R_\ell} = \frac{1}{2} [R_\ell X,Y] + \frac{1}{2} [X,R_\ell Y] = [X_{\geq \ell}, Y_{\geq \ell}] - [X_{< \ell}, Y_{< \ell}] \quad (2.10)$$

defines an additional (with respect to usual commutator) Lie structure on $G$ (see [11] and references therein). From the general relation for the $R$-coadjoint action of $G$ on its dual space we find that the infinitesimal shift along an $R$-coadjoint orbit $O(R_\ell)$ has the form:

$$\delta_{R_\ell} L = \text{ad}^*_{R_\ell} (X)L = [X_{\geq \ell}, L_{< \ell}]_{< \ell} - [X_{< \ell}, L_{\geq \ell}]_{\geq \ell}.$$ We now make contact with three special cases of the general Lax from (2.1) showing how they appear naturally in the AKS formalism sketched above.

### 2.1 Standard KP $\ell=0$ Hierarchy. Sato’s Theorem.

The KP $\ell=0$ model is defined in terms of the element of coadjoint orbit. The construction goes as follows. Note that $D \in G_{\ell=0}^\vee$ is invariant under the coadjoint action $\delta_{R_\ell} D = 0$. By adding to it the general elements of $G_{\ell=0}^\vee$ from (2.8) we arrive at the $R$-coadjoint orbit of the form $O(R_0) = \{L\}$ with

$$L \equiv D + \sum_{i=0}^{\infty} u_i(x,t)D^{-1-i} \quad (2.11)$$

According to (2.9) evolution of this standard KP hierarchy is governed by flows equations

$$\frac{\partial L}{\partial t_n} = [B_n^-, L] = [B_n^+, L] \quad n = 1, 2, \ldots \quad (2.12)$$

where we have introduced the potentials:

$$B_n^- \equiv -(L^n)_- \quad ; \quad B_n^+ \equiv (L^n)_+ \quad (2.13)$$

where the subscript (+) means taking the purely differential part of the corresponding operator.

One can establish by induction that

$$\frac{\partial L^m}{\partial t_n} = [B_n^-, L^m] \quad (2.14)$$

holds too. As a consequence we find $\partial B_m^- / \partial t_n = ([B^+_n, B^-_m])_-$ from which one derives the Zakharov-Shabat equations:

$$\frac{\partial B_m^-}{\partial t_n} - \frac{\partial B_n^-}{\partial t_m} = [B_n^-, B_m^-] \quad (2.15)$$

The following fundamental result is due to Sato:

**Theorem.** There exists a pseudo-differential operator $W$

$$W \equiv 1 + \sum_{i=1}^{\infty} w_i(x,t)D^{-i} \quad (2.16)$$
for which the following Sato equations are valid:

\[ L = W D W^{-1} \]  
\[ \frac{\partial W}{\partial t} = B_{-} W = B_{+} W - W D_{n} \]  

(2.17)

2.2 Nonstandard Hierarchies. Gauge Map to the KP Hierarchy.

We now turn our attention to KP\( \ell = 1 \) hierarchy. We first consider elements of \( G_{\ell = 1}^{\ell = 1} \) of the type \( L^{(1)} = D + u_{-1} + u_{0} D^{-1} \), which preserve their form under \( \delta_{R_{1}} L^{(1)} = a d_{R_{1}}(X) L^{(1)} \), spanning therefore a finite \( R_{1} \)-orbit. A complete Lax operator is obtained by adding \( L^{(1)} \) to the general element \( L_{-} \) of \( G^{\ell = 1}_{\ell = 1} \) obtaining

\[ L = L^{(1)} + L_{-} = D + u_{-1} + u_{0} D^{-1} + \sum_{i \geq 1} u_{i} D^{-i-1}. \]  

(2.18)

According to (2.9) this Lax operator satisfies:

\[ \frac{\partial L}{\partial t} = \left[ P_{\geq 1}(L_{n}) , L \right] \]  

(2.19)

As pointed out already by Sato (as referenced in [10]) there is a gauge transformation:

\[ K \equiv G^{-1} L G = D + \sum_{i=0}^{\infty} v_{i} D^{-i} ; \quad G \equiv \exp \left( - \int x u_{-1} dx \right) \]  

(2.20)

which removes the constant \( u_{-1} \) term and gives rise to the transformed Lax operator of the standard KP\( \ell = 0 \) form. In fact, this gauge transformed Lax operator satisfies the standard KP flow equation as well. This is a basic result, described by Kiso in [10]. The argument goes as follows.

Theorem. If \( L \) satisfies the KP\( \ell = 1 \) flow equation (2.19) then the gauge transformed Lax \( K \) satisfies the standard KP evolution equation (2.12): \( \partial K/\partial t_{n} = \left[ (K_{n})_{+} , K \right] \).

From definition and (2.19) we find by direct calculation and use of (2.19) that:

\[ \frac{\partial K}{\partial t_{n}} = \frac{\partial G^{-1} L G}{\partial t_{n}} = \left[ G^{-1} P_{\geq 1}(L_{n}) G - G^{-1} \frac{\partial G}{\partial t_{n}} , K \right] \]  

(2.21)

Using the definition \( P_{\geq 1}(L_{n}) = L_{n} - \sum_{j \leq 0}(L_{n})_{j} \) we find \( \left[ P_{\geq 1}(L_{n}) , L \right] = - \left[ \sum_{j \leq 0}(L_{n})_{j} , L \right] \) and consequently

\[ \frac{\partial u_{-1}}{\partial t_{n}} = \left( \frac{\partial L}{\partial t_{n}} \right)_{0} = - \left( \left[ \sum_{j \leq 0}(L_{n})_{j} , L \right] \right)_{0} = \partial L_{0}^{n} \]  

(2.22)

This last relation leads to

\[ \frac{\partial G}{\partial t_{n}} = - \int^{x} \frac{\partial u_{-1}}{\partial t_{n}} dx G = - \int^{x} \partial L_{0}^{n} dx G = - L_{0}^{n} G \]  

(2.23)
Hence, we obtain

\[ G^{-1} P_{\geq 1}(L^n) G - G^{-1} \frac{\partial G}{\partial t_n} = G^{-1} \left( L^n - \sum_{j \leq 0} (L^n)_j \right) G - G^{-1} \frac{\partial G}{\partial t_n} \]

\[ = K^n - G^{-1} \sum_{j \leq -1} (L^n)_j G = (K^n)_+ \]  

(2.24)

Inserting this in (2.21) we arrive at the standard KP evolution equation (2.12).

Finally, let us make few comments on the general case of \( \ell = 2 \) of KP hierarchy. Here elements of \( G^2_{-*} \) of the form \( L^{(2)} = u_{-2} D + u_{-1} + u_0 D^{-1} + u_1 D^{-2} \) span an invariant subspace under \( \delta_{R_2} L^{(2)} = ad_{R_2}^* (X) L^{(2)} \). Defining the complete Lax operator as \( L = L^{(2)} + L_- = u_{-2} D + u_{-1} + u_0 D^{-1} + u_1 D^{-2} + \sum_{i \geq 2} u_i D^{-i-1} \) we find (2.1) in its most general form.

3 Hamiltonian Structure of KP Hierarchies

3.1 The First Hamiltonian Structure of KP Hierarchies

Let us go back to the definitions (2.7) and (2.8) of relevant subspaces of \( G \) and \( G^* \). Using the Adler trace one defines an invariant, non-degenerate bilinear form:

\[ \langle L | X \rangle \equiv \text{Tr} (LX) = \int dx \text{Res} (LX) \]  

(3.1)

where the residue (Res) means the coefficient of \( D^{-1} \). The above Tr has the cyclic property.

We focus on a case of KP_{\ell=1} with the Lax operator (2.18). Let

\[ Q = \sum_{n=-1}^{\infty} D^n q_n \quad ; \quad V = \sum_{n=-1}^{\infty} D^n v_n \]  

(3.2)

be objects in \( G = G^{\ell=1}_{+} \oplus G^{\ell=1}_{-} \) dual to \( L \) from (2.18) (the fact that \( Q, V \) are truncated algebra elements will not alter the generality of our discussion). Inserting \( Q \) into the definition (3.1) yields

\[ \langle L | Q \rangle = \sum_{n=-1}^{\infty} \int dx \ u_n(x) q_n(x) \]  

(3.3)

as can easily be seen from

\[ \langle L | Q \rangle = \sum_{n=-1}^{\infty} \sum_{i=-1}^{\infty} \int dx \text{Res} \ u_i(x) D^{-i-1+n} q_n(x) \]  

(3.4)

which, because of (2.4), gets the contribution from \( i = n \) terms only.

Recall now the definition of R-bracket given in (2.10) for the \( \ell = 1 \) case. It turns out that the first Hamiltonian structure coincides with a first Gelfand-Dickey [6, 12] bracket:

\[ \{ \langle L | Q \rangle, \langle L | V \rangle \}_{\ell=1} \equiv \langle L | [Q, V]_{\ell=1} \rangle \]  

(3.5)
with
\[ [Q, V]_{\ell=1} \equiv \sum_{n,m=1}^{\infty} [D^n q_n, D^m v_m] - [q_0 + D^{-1} q_{-1}, v_0 + D^{-1} v_{-1}] \] (3.6)

Using (2.6) it follows that the first term in (3.6) is given by
\[ \sum_{n,m=1}^{\infty} \sum_{\alpha=0}^{m} (-1)^{\alpha} \binom{m}{\alpha} D^{n+m-\alpha} q_n^{(\alpha)} v_{m} - \sum_{n,m=1}^{\infty} \sum_{\alpha=0}^{m} (-1)^{\alpha} \binom{n}{\alpha} D^{n+m-\alpha} q_n v_{m}^{(\alpha)} \] (3.7)

with
\[ q_n^{(\alpha)} \equiv \frac{\partial^\alpha q_n}{\partial x^\alpha} ; \quad v_n^{(\alpha)} \equiv \frac{\partial^\alpha v_n}{\partial x^\alpha} \] (3.8)

while the relevant terms from the second term in (3.6) are \(-v_0 q'_0 D^{-2} + v'_0 q_{-1} D^{-2}\). The calculation of \(\langle L | [Q, V]_{\ell=1} \rangle\) gives after integration by parts:
\[ \langle L | [Q, V]_{\ell=1} \rangle = \sum_{n,m=0}^{\infty} \int dx dy \sum_{\alpha=0}^{m} \binom{m}{\alpha} \frac{\partial^\alpha}{\partial x^\alpha} (u_{n+m-\alpha}(x) \delta(x-y)) q_n(x) v_{m}(y) - \]
\[ \sum_{n,m=0}^{\infty} \int dx dy \sum_{\alpha=0}^{n} (-1)^{\alpha} u_{n+m-\alpha}(x) (\frac{\partial^\alpha}{\partial x^\alpha} \delta(x-y)) q_n(x) v_{m}(y) \]
\[ + \int dx dy (\delta'(x-y)) (v_{-1}(y) q_0(x) - v_0(y) q_{-1}(x)) \] (3.9)

Using (3.3) in the l.r.s. of (3.3) and comparing the coefficients of \(q_n(x)v_m(y)\) we obtain from (3.3)
\[ \{\tilde{u}_n(x), \tilde{u}_m(y)\}_{\ell=1} = \{u_{n+1}(x), u_{m+1}(y)\}_{\ell=1} = \Omega_{nm}^{(1)}(\tilde{u}(x)) \delta(x-y) \quad n, m = 0, 1, \ldots \] (3.10)
\[ \{u_0(x), u_m(y)\}_{\ell=1} = \{u_{-1}(x), u_{m}(y)\}_{\ell=1} = 0 \] (3.11)
\[ \{u_0(x), u_{-1}(y)\}_{\ell=1} = \delta'(x-y) \quad \{u_0(x), u_0(y)\}_{\ell=1} = \{u_{-1}(x), u_{-1}(y)\}_{\ell=1} = 0 \] (3.12)

where we have introduced for convenience \(\tilde{u}_n(x) = u_{n+1}(x)\) and where
\[ \Omega_{nm}^{(\ell)}(u(x)) \equiv -\sum_{k=0}^{n+\ell} (-1)^k \binom{n+\ell}{k} u_{n+m+\ell-k}(x) D_x^k + \sum_{k=0}^{m+\ell} \binom{m+\ell}{k} D_x^k u_{n+m+\ell-k}(x) \] (3.13)

We see therefore that the pair \((u_0, u_{-1})\) decouples from the rest of algebra. This result will soon become very crucial for our discussion of two-boson systems.

Let us now turn our attention to the case of standard KP hierarchy with \(\ell = 0\). Here construction of the first Gelfand-Dickey bracket is based on the commutator:
\[ [Q, V]_{\ell=0} \equiv \sum_{n,m=0}^{\infty} [D^n q_n, D^m v_m] - [D^{-1} q_{-1}, D^{-1} v_{-1}] \] (3.14)

Using (2.6) it follows that the first term in (3.14) is given by
\[ \sum_{n,m=0}^{\infty} \sum_{\alpha=0}^{m} (-1)^{\alpha} \binom{m}{\alpha} D^{n+m-\alpha} q_n^{(\alpha)} v_{m} - \sum_{n,m=0}^{\infty} \sum_{\alpha=0}^{m} (-1)^{\alpha} \binom{n}{\alpha} D^{n+m-\alpha} q_n v_{m}^{(\alpha)} \] (3.15)
We can now calculate \( \langle L \mid [Q, V]_{t=0} \rangle \). Although in this case one should consequently take \( L \) Lax operator as given by (2.11) we will keep using \( L \) from (2.13) anticipating that anyway the extra term \( u_{-1} \) will automatically decouple from the remaining variables. This time only the first term of (3.14) contributes, giving after integration by parts:

\[
\langle L \mid [Q, V]_{t=0} \rangle = \sum_{n,m=0}^{\infty} \int dxdy \sum_{\alpha=0}^{m} \left( \frac{m}{\alpha} \right) \partial_x^\alpha (u_{n+m-\alpha}(x) \delta(x-y)) q_n(x)v_m(y) - \\
\sum_{n,m=0}^{\infty} \int dxdy \sum_{\alpha=0}^{n} \left( \frac{1}{\alpha} \right) (-1)^\alpha u_{n+m-\alpha}(x) (\partial_x^\alpha \delta(x-y)) q_n(x)v_m(y)
\]

(3.16)

Using (3.3) in the l.r.s. of (3.5) and comparing the coefficients of \( q_n(x)v_m(y) \) we obtain from (3.16)

\[
\{u_n(x), u_m(y)\}_{t=0} = \Omega_{nm}^{(0)}(u(x)) \delta(x-y)
\]

(3.17)

\[
\{u_{-1}(x), u_{-1}(y)\}_{t=1} = \{u_{-1}(x), u_{m}(y)\}_{t=0} = 0
\]

(3.18)

We see therefore that \( u_{-1} \), in fact decouples from the rest of the algebra (3.17). The algebra (3.17) was first derived by Watanabe [13] (see also [8, 14]). Let us note that for \( n = m = 1 \), (3.17) gives the Virasoro algebra without central term. By measuring the spin of the fields \( u_n(x) \) through the Virasoro field \( u_1(x) \) (which has spin 1) we see that \( u_0(x) \) has spin 2, \( u_2(x) \) has spin 3 and so on. Therefore (3.17) describes the \( W_{1+\infty} \) algebra without central term.

### 3.2 The Second Hamiltonian Structure

The second Gelfand-Dickey [8] bracket is given by

\[
\{\langle L \mid Q \rangle, \langle L \mid V \rangle\}_2 = \text{Tr} (L(QL)_+V - (LQ)_+LV) = \text{Tr} ((LQ)_-LV - L(QL)_-V)
\]

(3.19)

where the subscripts \( \pm \) denote the parts of the pseudo-differential operator containing non-negative and negative powers of \( D \) and where we continue to use the Lax operator \( L \) from (2.18).

By performing explicit calculation of (3.19) we obtain [4, 3]

\[
\{u_n(x), u_m(y)\}_{2}^{GD} = \Omega_{nm}^{(1)}(u(x)) \delta(x-y)
\]

(3.20)

\[
+ \sum_{i=0}^{m-1} \sum_{k=1}^{m-i-1} \binom{m-i-1}{k} u_i(x) D_x^k u_{m+n-i-k-1}(x)
\]

\[
- \sum_{k=1}^{n} (-1)^k \binom{n}{k} u_{n+i-k}(x) D_x^k u_{m-i-1}(x) \delta(x-y)
\]

\[
- \sum_{i=0}^{m-1} \sum_{k=0}^{m-i-1} \sum_{l=1}^{n} (-1)^k \binom{n}{k} \binom{m-i-1}{l} u_{n+i-k}(x) D_x^{k+l} u_{m-i-l-1}(x) \delta(x-y)
\]

for \( m, n \geq 0 \) together with

\[
\{u_n(x), u_{-1}(y)\}_2 = - \sum_{k=1}^{n} (-1)^k \binom{n}{k} u_{n-k}(x) D_x^k \delta(x-y)
\]

(3.21)
\[ \{u_\ell(x), u_m(y)\}_2 = \sum_{k=1}^{m-1} \binom{m}{k} D_k^x u_{m-k}(x) \delta(x-y) \]  
\[ \{u_\ell(x), u_\ell(y)\}_2 = -\delta'(x-y) \]  

We then see that \( u_\ell(x) \) couples to itself and to other fields \( u_n(x) \) for \( n \neq 0 \). Recall however that by the appropriate gauge transformation (2.20) we were able to remove the constant \( u_\ell \) term casting the transformed Lax operator into the standard KP \( t=0 \) form. One suspects therefore that one can always impose the condition \( u_\ell = 0 \). In view of (3.23) \( u_\ell = 0 \) represents a second class constraint and thus we consider the Dirac bracket [1):

\[ \{ u_n(x), u_m(y) \}_2^D = \{ u_n(x), u_m(y) \}_2 - \int \int dz_1 dz_2 \{ u_n(x), u_{-1}(z_1) \}_2 \{ u_{-1}(z_1), u_{-1}(z_2) \}_2^{-1} \{ u_{-1}(z_2), u_m(y) \}_2 \]  

which leads, with the use of eqs. (3.20),(3.21),(3.22),(3.23) to

\[ \{ u_n(x), u_m(y) \}_2^D = \{ u_n(x), u_m(y) \}_2 - \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} (-1)^{n-i} \binom{n}{i} \binom{m}{j} u_i(x) D_x^{m+n-i-j} u_j(y) \delta(x-y) \]  

We recognize in the second term on the r.h.s of (3.25) a Drinfeld-Sokolov bracket. Notice that it satisfies the Jacobi identity due to the fact that (3.19) or (3.20) also do it. The algebra in (3.25) describes a nonlinear \( \mathbb{W}_\infty \) algebra since its lowest spin generator \( u_0 \) satisfies the Virasoro algebra.

Let us make a comment about the quasiclassical counterparts of the algebraic structures we studied above. Making in (3.17) the substitution \( \partial_x \to h \partial_x \) and setting \( h \to 0 \), we obtain the “classical limit” of \( \mathbb{W}_{1+\infty} \) algebra:

\[ \{ \omega_n(x), \omega_m(y) \}_1 = (n \omega_{m+n-1}(x) D_x + m D_x \omega_{m+n-1}(x)) \delta(x-y) \]  

which is the area preserving diffeomorphism algebra called \( \mathfrak{w}_{1+\infty} \) algebra. Using the same trick in equation (3.25) we obtain the non-linear \( \mathbb{W}_\infty \) algebra, namely:

\[ \{ \omega_n(x), \omega_m(y) \}_1 = ((n+1) \omega_{m+n}(x) D_x + (m+1) D_x \omega_{m+n}(x)) \delta(x-y) + 2nm \omega_{n-1}(x) D_x \omega_{m-1}(x) \delta(x-y) \]  

3.3 Hamiltonian Structure and KP equation

The first bracket (3.17) and the second bracket (3.25) lead to an Hamiltonian description of the KP flow equation (2.12). The Hamiltonians are defined in terms of \( L \) from (2.11) as

\[ H_m = \frac{1}{m} \text{Tr} \ L^m = \frac{1}{m} \int \text{Res} \ L^m \]  

and satisfy

\[ \frac{\partial L}{\partial t_m} = [(L^m)_+, L] = \{ L, H_{m+1} \}_1 = \{ L, H_m \}_2^D \]
Equivalently, we have
\[ \frac{\partial u_n}{\partial t_m} = \{u_n, H_{m+1}\}_1 = \{u_n, H_m\}^D_2 \]
(3.30)

From (3.28) we find:
\[ H_1 = \int u_0(x) \, dx \quad ; \quad H_2 = \int u_1(x) \, dx \]
\[ H_3 = \int (u_2(x) + u_0^2) \, dx \quad ; \quad H_4 = \int (u_3(x) + 3u_1(x)u_0(x)) \, dx \]
\[ H_5 = \int \left( u_4 + 4u_0u_2 + 2u_1^2 + 2u_0^3 + u_0u_0'' - 2u_1u_0' \right) \, dx \]
(3.31)

Expressions (3.30) and (3.31) give after the use of (3.10) and (3.25) [5]:
\[ \frac{\partial u_n}{\partial x} = \{u_n, H_2\}_1 = \{u_n, H_1\}^D_2 \]
\[ \frac{\partial u_n}{\partial y} = \frac{\partial u_n}{\partial t_2} = \{u_n, H_3\}_1 = \{u_n, H_2\}^D_2 = \frac{\partial^2 u_n}{\partial x^2} + 2 \frac{\partial u_n+1}{\partial x} - 2 \sum_{k=1}^{n} (-1)^k \binom{n}{k} u_{n-k} \frac{\partial^k u_0}{\partial x^k} \]
\[ \frac{\partial u_0}{\partial t} = \frac{\partial u_0}{\partial t_3} = \{u_0, H_4\}_1 = \{u_0, H_3\}^D_2 = \frac{\partial^3 u_0}{\partial x^3} + 6 \frac{\partial u_0}{\partial x} + 3 \left( \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial u_2}{\partial x} \right) \]
(3.32)

where we have denoted \( t_1 = x, t_2 = y \) and \( t_3 = t \). Using the first two equations we obtain from the third equation in (3.32):
\[ \frac{\partial}{\partial x} \left( \frac{\partial u_0}{\partial t} - \frac{1}{4} \frac{\partial^3 u_0}{\partial x^3} - 3 \frac{\partial u_0}{\partial x} \right) = \frac{3}{4} \frac{\partial^2 u_0}{\partial y^2} \]
(3.33)

which is the KP (Kadomtsev-Petviashvili) equation. Let us mention that if we constrain our Lax operator (3.1) by setting \( u_1 = u_2 = ... = 0 \), then the third eqn (3.32) gives the KdV equation for \( u_0 \). This will be more explicitly seen in terms of the two current realization of the KP-hierarchy.

### 3.4 Two-Boson KP Hierarchy

Here we go back to KP\(_{t=1}\) and make the following crucial observation. Consider truncated elements of \( G_1^* \) of the type \( L_j^{(1)} = D + u_0 + u_1 D^{-1} = D - J + JD^{-1} \), where we have introduced two Bose currents \( (J, \bar{J}) \) to create fit with notation used in [13]. Recall that under the coadjoint action \( \delta_{R_i} L_j^{(1)} = ad_{R_i}(X)L_j^{(1)} \) this finite Lax operator maintains its form, i.e. the two-boson Lax operators span an \( R_1 \)-orbit of finite functional dimension 2. This observation, already present in [4] clarifies status of two-boson \( (J, \bar{J}) \) system as a consistent restriction of the full \( KP_{t=1} \)-hierarchy understood as an orbit model. Note that there are only two possibilities for the invariant \( R_1 \)-orbit; the two-boson system and the full KP\(_{t=1}\) system (in a quasiclassical limit situation is much richer with any number of fields defining invariant subspace). A Poisson bracket obtained as Lie-Poisson \( R \)-bracket in (3.12) yields the first bracket structure of the two-boson \( (J, \bar{J}) \) system:
\[ \{J(x), J(y)\}_1 = -\delta'(x - y) \]
\[ \{J(x), J(y)\}_1 = \{J(x), J(y)\}_1 = 0 \]
(3.34)
The higher bracket structures have been investigated in [8, 15]. One finds the following second bracket structure:

\[
\{\bar{J}(x), J(y)\}_2 = J(x)\delta'(x - y) - h\delta''(x - y)
\]

\[
\{\bar{J}(x), \bar{J}(y)\}_2 = 2\bar{J}(x)\delta'(x - y) + \bar{J}'(x)\delta(x - y)
\]

\[
\{J(x), J(y)\}_2 = c\delta'(x - y)
\]

(3.35)

Consistency check based on Lenard relation forces the deformation parameters \(c, h\) to take values \(c = 2, h = 1\).

The three lowest Hamiltonian functions are:

\[
H_{J1} = \int \bar{J} \quad H_{J2} = \int -\bar{J}J \quad H_{J3} = \int (\bar{J}J^2 + \bar{J}J' + \bar{J}^2)
\]

(3.36)

For the general Hamiltonian matrix structure \(P_i\) we have

\[
\frac{\partial}{\partial t_r} \left( J \right) = P_{Ji} \left( \frac{\delta H_{Jr+2-i}/\delta J}{\delta H_{Jr+2-i}/\delta \bar{J}} \right) = P_{J1} \left( \frac{\delta H_{r+1}/\delta J}{\delta H_{r+1}/\delta \bar{J}} \right) = P_{J2} \left( \frac{\delta H_r/\delta J}{\delta H_r/\delta \bar{J}} \right)
\]

(3.37)

Among the multi-Hamiltonian structures only \(P_{J1}\) and \(P_{J2}\) are independent. All other matrices \(P_{Ji}, i = 3, 4, \ldots\) are related to \(P_{J2}\) through \(P_{Ji} = (P_{J2}(P_{J1})^{-1})^i - 2\) involving the so-called recurrence matrix \(P_{J2}(P_{J1})^{-1}\). The explicit form of first and second local Hamiltonian structures corresponding to (3.34) and (3.35) with \(c = 2\) and \(h = 1\) is:

\[
P_{J1} = \begin{pmatrix} 0 & -D \\ -D & 0 \end{pmatrix}, \quad P_{J2} = \begin{pmatrix} 2D & D^2 + DJ \\ -D^2 + JD & DJ + JD \end{pmatrix}
\]

(3.38)

Taking \(r = 2\) in (3.37) we especially get the Boussinesq equation:

\[
J_{t_2} = \{ J, H_{J2} \}_2 = \{ J, H_{J3} \}_1 - hJ'' - \left( J^2 \right)' - 2\bar{J}'
\]

\[
\bar{J}_{t_2} = \{ J, \bar{J}_{J2} \}_2 = \{ J, H_{J3} \}_1 hJ'' - 2 \left( \bar{J}J \right)'
\]

(3.39)

where we re-introduced \(h\) as a deformation parameter. In the dispersiveless limit \(h \to 0\) taken in (3.39) we obtain the classical dispersiveless long wave equations (Benney equations) [17, 8].

4 Realization of \(\omega_{1+\infty}\) and \(\hat{\omega}_{\infty}\) in Terms of Currents - Applications to Benney Equations and WZNW

4.1 The Benney Equations.

Consider the following realization of the area-preserving generators:

\[
\omega_n = (-1)^n \bar{J}(x)J(x)^n
\]

(4.1)

\(n = 0, 1, 2, \ldots\) where the currents \(J\) and \(\bar{J}\) satisfy the first bracket structure (3.34). It is not difficult to verify that (1.1) satisfy (3.26) after use is made of (3.34).
Taking $H_3$

$$H_3 = \int (\omega_2(y) + \omega_0^2(y))dy = \int (\bar{J}(y)J^2(y) + J^2(y))dy \quad (4.2)$$

as the evolution generator in time for our system with respect to the first bracket, yields:

$$\frac{dJ(x)}{dt} = \underbrace{\frac{dJ(x)}{dt_2}}_{=\{J(x), H_3\}_1} = -(J^2(x))' - 2\bar{J}'(x)$$

$$\frac{d\bar{J}(x)}{dt} = \underbrace{\frac{d\bar{J}(x)}{dt_2}}_{=\{\bar{J}(x), H_3\}_1} = -2(\bar{J}(x)J(x))' \quad (4.3)$$

where $J'(x) = \frac{dJ(x)}{dx}$, etc.

The non-linear deformation of the area preserving diffeomorphism algebra (3.24) can be generated by (4.1) where now $J$ and $\bar{J}$ satisfy the second-bracket structure (3.35) with the deformation parameter $h$ set to zero. Recalling (3.30) it follows using (3.35) that

$$\frac{dJ(x)}{dt} = \underbrace{\{J(x), H_2\}_2}_{=\{J(x), H_2\}_3} = -(J^2(x))' - 2\bar{J}'(x)$$

$$\frac{d\bar{J}(x)}{dt} = \underbrace{\{\bar{J}(x), H_2\}}_{=\{\bar{J}(x), H_2\}_2} = -2(\bar{J}(x)J(x))' \quad (4.4)$$

where $H_2 = -\int \bar{J}(y)J(y)dy$. Equations (4.3) or (4.4) are known to be the Benney equations of hydrodynamics [17, 18, 19].

In this way the Benney equations are related to the classical KP-hierarchy where the first Poisson bracket structure is defined by $w_{1+\infty}$ while the second bracket is given by the non-linear extension of $w_\infty$.

In an attempt to generalize (1.1) let us consider $N$ copies of currents $J$ and $\bar{J}$ and define

$$\omega_n(x) = (-1)^n \sum_{k=1}^{N} J_k(x)(J_k(x))^n \quad (4.5)$$

The Lax equations for $\omega_0$ and $\omega_1$ with respect to $t_2$ and $t_3$ take the form:

$$\frac{d\omega_0(x)}{dt_2} = 2\omega_1'(x)$$

$$\frac{d\omega_1(x)}{dt_2} = 2\omega_2'(x) + (\omega_0^2(x))' \quad (4.6)$$

and

$$\frac{d\omega_0(x)}{dt_3} = 3\omega_2'(x) + 6\omega_0\omega_0'$$

$$\frac{d\omega_1(x)}{dt_3} = 3\omega_3'(x) + 6(\omega_0(x)\omega_1(x))' \quad (4.7)$$

The first two equations (4.6) are compatible with

$$\frac{dJ_k(x)}{dt_2} = -(J_k^2(x))' - 2\bar{J}'(x)$$

$$\frac{d\bar{J}_k(x)}{dt_2} = -2(\bar{J}_k(x)J_k(x))' \quad (4.8)$$
where

\[ \bar{J}(x) = \sum_{k=1}^{N} \bar{J}_k(x) \]  

Equations (4.9) describe coupled multi-layered Benney equations studied by Zakharov [18]. On the other hand the second set of equations (4.7), is compatible with two sets of flows for \( J_k \) and \( \bar{J}_k \), namely:

\[
\frac{dJ_k(x)}{dt_3} = (J_k(x)^3)' + 3(J_k(x) \sum_l \bar{J}_l(x))' + 3 \sum_l (\bar{J}_l(x)J_l(x))'
\]

\[
\frac{d\bar{J}_k(x)}{dt_3} = 3(J_k(x)J_k^2(x))' + 3 \sum_l (\bar{J}_l(x)\bar{J}_k(x))'
\]  (4.10)

and

\[
\frac{dJ_k(x)}{dt_3} = (J_k(x)^3)' + 2(J_k(x) \sum_l \bar{J}_l(x))' + 4 \sum_l (\bar{J}_l(x)J_l(x))'
\]

\[
\frac{d\bar{J}_k(x)}{dt_3} = 3(\bar{J}_k(x)J_k^2(x))' + 4\bar{J}_k(x) \sum_l (\bar{J}_l(x))' + 2(\bar{J}_k(x))' \sum_l \bar{J}_l(x)
\]  (4.11)

We can now define a first bracket structure in terms of \( J_k \) and \( \bar{J}_k \) as

\[
\{ \bar{J}_k(x), J_l(y) \}_1 = -\delta'(x - y) \delta_{kl}
\]

\[
\{ J(x)_k, J_l(y) \}_1 = \{ \bar{J}_k(x), \bar{J}_l(y) \}_1 = 0
\]  (4.12)

Therefore (1.3) generates under the first bracket the algebra \( w_{1+\infty} \otimes w_{1+\infty} \otimes \ldots \).

The corresponding Hamiltonian equations of motion:

\[
\frac{dJ_k(x)}{dt_r} = \{ J_k(x), H_{r+1} \}_1
\]

\[
\frac{d\bar{J}_k(x)}{dt_r} = \{ \bar{J}_k(x), H_{r+1} \}_1
\]  (4.13)

reproduce for \( r = 2, 3 \) the flow equations (4.8) and (4.10), respectively.

We can realize (3.27) through (4.3) if we assume a second bracket structure to be:

\[
\{ \bar{J}_k(x), J_l(y) \}_2 = J_l(x)\delta_{kl} \delta'(x - y)
\]

\[
\{ \bar{J}_k(x), \bar{J}_l(y) \}_2 = 2\bar{J}_l(x)\delta_{kl} \delta'(x - y) + \bar{J}_l(x)\delta_{kl} \delta(x - y)
\]

\[
\{ J_k(x), J_l(y) \}_2 = 2 \delta'(x - y)
\]  (4.14)

where the r.h.s. of the last equation is independent of \( k \) and \( l \). We should point out at this stage that the above algebraic structure violates the Jacobi identity. However, it is satisfied at the level of \( \omega' \)s defined in (4.3). This second bracket can also be extended to the Hamiltonian framework as follows:

\[
\frac{dJ_k(x)}{dt_r} = \{ J_k(x), H_r \}_2
\]

\[
\frac{d\bar{J}_k(x)}{dt_r} = \{ \bar{J}_k(x), H_r \}_2
\]  (4.15)
Taking $r = 2, 3$ we reproduce after using the algebra (4.14) the flow equations (4.8) and (4.11), respectively. The system is therefore not bi-hamiltonian since the hierarchies of equations (4.8) relative to brackets 1 and 2 are different.

4.2 The Current Algebra of WZNW Model

The ordinary WZNW model associated to a Lie group $G$ possesses two commuting chiral copies of the current algebra:

$$\{ J_a(x), J_b(y) \} = f_{ab}^c J_c(x) \delta(x - y) + kg_{ab} \delta'(x - y)$$

(4.16)

where $f_{ab}^c$ are the structure constants of the Lie algebra $G$ of $G$, and $g_{ab}$ is the Killing form of $G$. The two chiral components of the energy momentum tensor are of the Sugawara form

$$T(x) = \sum_{a,b=1}^{\text{dim}G} g_{ab} J_a(x) J_b(x)$$

(4.17)

where $g_{ab}$ is the inverse of the Killing form. Such tensor satisfies the Virasoro algebra with vanishing central term

$$\{ T(x), T(y) \} = 2T(x)\delta'(x - y) + T'(x)\delta(x - y)$$

(4.18)

The currents are spin one primary fields:

$$\{ T(x), J_a(y) \} = J_a(x)\delta'(x - y)$$

(4.19)

Suppose now one has a self-commuting current $J$, $\{ J(x), J(y) \} = 0$. For the non compact WZNW model this current can, for instance, be the one associated to a step operator $J(E_{\alpha})$ for any root $\alpha$ of $G$. One then sees that the system $(T, J)$ generates an algebra isomorphic to (3.35) with the difference that now the last bracket relation is zero. One can construct out of them the quantities $w_n(x) \equiv T(x)J^{n-2}$ satisfying the area preserving diffeomorphism algebra, i.e.

$$\{ w_n(x), w_m(y) \} = (n + m - 2) w_{n+m-2}(x)\delta'(x - y) + (m - 1) (w_{n+m-2}(x))' \delta(x - y)$$

(4.20)

By taking now a $U(1)$ subalgebra:

$$\{ J(x), J(y) \} = kg J \delta'(x - y)$$

(4.21)

one sees that the $(T, J)$ system now generates an algebra which is isomorphic to (3.35) where now the last bracket is different from zero. The quantities $w_n$ introduced above will then generate a deformed (nonlinear) area preserving diffeomorphism algebra, i.e.

$$\{ w_n(x), w_m(y) \} = (n + m - 2) w_{n+m-2}(x)\delta'(x - y) + (m - 1) (w_{n+m-2}(x))' \delta(x - y)$$

$$- kg J (n - 2) (m - 2) w_{n-1}(x) \partial_x(w_{m-1}(x) \delta(x - y))$$

(4.22)
5 A Bose Construction of KP Hierarchy and Faá di Bruno Polynomials

In order to provide a two current realization of $W_{1+\infty}$ and the corresponding non-linear $\hat{W}_{\infty}$ algebras we shall now give an introduction to the Faá di Bruno polynomials.

5.1 Two-Boson KP Hierarchy and Faá di Bruno Polynomials

We now construct the KP hierarchy in terms of a pair of Bose currents $J$ and $\bar{J}$. We propose the following Lax components $W_n$:

$$W_n(x) = (-1)^n \bar{J}(x) P_n(J(x))$$ (5.1)

given in terms of the Faá di Bruno polynomials

$$P_n(J) = (D + J)^n \cdot 1 = e^{-\Phi} \partial^n e^\Phi$$ (5.2)

where $J(x) = \Phi'(x)$ and $\partial^n = \frac{\partial^n}{\partial x^n}$. One easily recognizes in (5.1) deformations of the area preserving generators (4.1).

We associate to (5.1) the Lax operator given by

$$L = D + J \frac{1}{D + J} = D + \sum_{n=0}^{\infty} W_n(x) D^{-1-n}$$ (5.3)

Faá di Bruno Polynomials and Their Properties.

The technical analysis we are about to present relies rather heavily on properties of Faá di Bruno polynomials defined by

$$P_n(J) \equiv e^{-\Phi} \partial^n e^\Phi$$ (5.4)

It follows from (5.4) that the generating functional for the Faá di Bruno polynomials is given by

$$\exp \left\{ \sum_{k=1}^{\infty} e^k J^{(k-1)}/k! \right\} = \sum_{k=0}^{\infty} P_k(J) e^k/k!$$ (5.5)

which follows from

$$e^{\Phi(x+\epsilon) - \Phi(x)} = \exp \left\{ \sum_{k=1}^{\infty} \frac{e^k}{k!} J^{(k-1)} \right\}$$ (5.6)

Consider now $f = \phi'_1$, $g = \phi'_2$ and:

$$\sum_{k=0}^{n} \binom{n}{k} P_k(f) P_{n-k}(g) = \sum_{k=0}^{n} \binom{n}{k} e^{-\phi_1} \partial^n e^{\phi_1} e^{-\phi_2} \partial^{n-k} e^{\phi_2} e^{\phi_2}$$

$$= e^{-(\phi_1 + \phi_2)} \sum_{k=0}^{n} \binom{n}{k} \partial^k e^{\phi_1} \partial^{n-k} e^{\phi_2}$$

$$= e^{-(\phi_1 + \phi_2)} \partial^n e^{\phi_1+\phi_2}$$ (5.7)
where we had use Leibniz rule in the last step. Therefore we find the identity
\[
    P_n(f + g) = \sum_{k=0}^{n} \binom{n}{k} P_k(f) P_{n-k}(g), \quad n = 0, 1, \ldots
\]  

(5.8)

From (5.4) it follows the recurrence relation
\[
    \partial P_n = P_{n+1} - J P_n
\]

(5.9)

which suggests the alternative expression for the Faà di Bruno polynomials:
\[
    P_n(J) = (D + J)^n 1
\]

(5.10)

satisfying (5.9). The lowest order polynomials are
\[
    P_0 = 1 ; \quad P_1 = J ; \quad P_2 = J^2 + J' ; \quad P_3 = J^3 + 3JJ' + J'' \quad \text{etc}
\]

(5.11)

Using Leibniz rule we also have
\[
    (D + J)^n = e^{-\Phi} D^n e^\Phi = \sum_{l=0}^{n} \binom{n}{l} e^{-\Phi} \partial^{n-l} e^\Phi D^l = \sum_{l=0}^{n} \binom{n}{l} P_{n-l}(J) D^l
\]

(5.12)

Using equation (2.3) we also have
\[
    P_n(J) = e^{-\Phi} \partial^n e^\Phi = \sum_{l=0}^{n} (-1)^l \binom{n}{l} e^{-\Phi} D^{n-l} e^\Phi D^l = \sum_{l=0}^{n} (-1)^l \binom{n}{l} (D + J)^{n-l} D^l
\]

(5.13)

With the help of Leibniz rule we also obtain
\[
    (D - J)^n = e^{\Phi} D^n e^{-\Phi} = \sum_{l=0}^{n} (-1)^{n-l} \binom{n}{l} D^l (\partial^{n-l} e^\Phi) e^{-\Phi} = \sum_{l=0}^{n} (-1)^{n-l} \binom{n}{l} D^l P_{n-l}(J)
\]

(5.14)

Similarly we get
\[
    P_n(J) (D - J)^m = \partial^m e^\Phi D^m e^{-\Phi} = \sum_{l=0}^{m} (-1)^{m-l} \binom{m}{l} D^l e^{-\Phi} \partial^{m+n-l} e^\Phi = \sum_{l=0}^{m} (-1)^{m-l} \binom{m}{l} D^l P_{m+n-l}(J)
\]

(5.15)

From which we get
\[
    P_n(J) \sum_{k=0}^{m} (-1)^k \binom{m}{k} D^k P_{m-k}(J) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} D^k P_{n+m-k}(J)
\]

(5.16)

Again it follows from the Leibniz rule:
\[
    D^n = D^n (e^\Phi e^{-\Phi}) = \sum_{l=0}^{n} \binom{n}{l} e^{-\Phi} (\partial^{n-l} e^\Phi) e^\Phi D^l e^{-\Phi}
\]

\[
    = \sum_{l=0}^{n} \binom{n}{l} P_l(J) (D - J)^{n-l}
\]

(5.17)

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and using (5.1) we find

\[ D^n = \left(e^{-\Phi} e^\Phi\right) D^n = \sum_{l=0}^{n} (-1)^l \binom{n}{l} e^{-\Phi} D^{n-l} e^\Phi \partial_y e^\Phi \]

\[ = \sum_{l=0}^{n} (-1)^l \binom{n}{l} (D + J)^{n-l} P_l(J) \]  

From (2.4) it follows

\[ e^{-\Phi} D^{-1} e^\Phi = \sum_{l=0}^{\infty} (-1)^l e^{-\Phi} \partial_y e^\Phi D^{-l-1} = \sum_{l=0}^{\infty} (-1)^l P_l(J) D^{-l-1} \]  

The r.h.s. is precisely \((D + J)^{-1}\). Therefore

\[ (D + J)^{-1} = e^{-\Phi} D^{-1} e^\Phi \]  

as expected from (5.12).

### 5.2 The Algebraic Structure of Two-Boson KP Hierarchy

Armed with the above technical details we proceed here to calculate the algebra satisfied by the generators (5.1) in case when the canonical variables \(J\) and \(\bar{J}\) satisfy the bracket (3.34). As a consequence of (3.34) we clearly find:

\[ \{ \bar{J}(x) , e^{\pm \Phi(y)} \} = \mp \delta(x - y) e^{\pm \Phi(y)} \]  

The advantage of using the exponential representation (5.4) is that it makes relatively easy to calculate brackets between generators \(W_n(x) = (-1)^n J(x) e^{-\Phi} \partial^y e^\Phi\)

\[ \{ W_n(x) , W_m(y) \}_1 = (-1)^{n+m} \bar{J}(y) e^{-\Phi(y)} \partial_y^m (\delta(x - y) \partial_y^n e^\Phi(y)) \]

\[ - (-1)^{n+m} \bar{J}(x) e^{-\Phi(x)} \partial_x^n (\delta(x - y) \partial_x^m e^\Phi(x)) \]

\[ = \sum_{k=0}^{m} (-1)^{n+m+k} \binom{m}{k} \bar{J}(y) e^{-\Phi(y)} \partial_y^{m+n-k} e^\Phi(y) \partial_x^k \delta(x - y) \]

\[ - \sum_{k=0}^{n} (-1)^{n+m+k} \binom{n}{k} \bar{J}(x) e^{-\Phi(x)} \partial_x^{m+n-k} e^\Phi(x) \partial_y^k \delta(x - y) \]

One recognizes in (5.22) the first Gelfand Dickey structure written in the Watanabe form (3.17)

We will now show how to generate the second bracket structure from the representation given by (5.1). This time the algebra of \(J\) and \(\bar{J}\) is given by (3.35). From (3.35) it is possible to show that

\[ \{ P_n(x) , P_m(y) \}_2 = -c \left[ \sum_{l=1}^{n} \sum_{p=1}^{m} (-1)^p \binom{n}{l} \binom{m}{p} P_{n-l}(x) P_{m-p}(y) \partial_x^{l+p-1} \right] \delta(x - y) \]  

\[ (5.23) \]
We obtain therefore the total second bracket for the generators in (5.1) as the sums of linear and non linear terms

\[ \{W_n(x) , W_m(y)\}_2 = \Omega_{nm}^{(1)}(x,y) \delta(x-y) + c \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} (-1)^{n-1}(n \atop i)(m \atop j) W_i(x) D_{x}^{n-i} W_j(x) \delta(x-y) \]

for \( n, m \geq 0 \). The first term is the linear part which coincides with the linear part of (3.25).

The result (5.24) appears to be surprising since we recognize in the non linear part of it only the Drinfeld-Sokolov structure from (3.25) multiplied by \( c \), while the non linear part of the second Gelfand-Dickey bracket from (5.24) appears to be missing. However one can show that

\[ \{W_n(x) , W_m(y)\}_{2}^{GD}_{\text{nonlinear}} = \{W_n(x) , W_m(y)\}^{DS} \]

That is, the non linear part of the second Gelfand-Dickey bracket is exactly equal to the Drinfeld-Sokolov bracket. As a consequence of that we can rewrite relation (5.24) for the choice \( c = 2 \) as

\[ \{W_n(x) , W_m(y)\}_2 = \{W_n(x) , W_m(y)\}^{GD}_2 + \{W_N(x) , W_m(y)\}^{DS} \]

and therefore reproduce the form (3.25).

### 5.3 Examples; Two loop WZNW, Conformal Toda Theories and \( W_\infty \) algebra

Let us consider the WZNW model associated to a Kac-Moody algebra whose conserved currents satisfy the two loop Kac-Moody algebra [20]:

\[
\begin{align*}
[J_a^m(x),J_b^n(y)] &= i f^{c}_{ab} J_c^{m+n}(x) \delta(x-y) + kg_{ab} \partial_x \delta(x-y) \delta_{m,-n} + J^C(x) \delta(x-y) g_{ab} m \delta_{m,-n} \\
[J^D(x),J_a^m(y)] &= m J_a^m(y) \delta(x-y) \\
[J^C(x),J^D(y)] &= k \partial_x \delta(x-y) \\
[J^C(x),J_a^m(y)] &= 0
\end{align*}
\]

In this case the Sugawara energy-momentum tensor is

\[ T(x) = \frac{1}{2} \sum_{a,b=1}^{d_{\text{im}}} \sum_{m \in \mathbb{Z}} g^{ab} J_a^m(x) J_b^{-m}(x) + J^D(x) J^C(x) \]

It is easy to show that \( T(x) \) and \( J^C(x) \) reproduce the algebra (3.35) with \( h = c = 0 \) showing that the two-loop WZNW has the \( W_\infty \) structure. In fact, this model possesses a bigger symmetry. In order to show this let us consider the modified energy-momentum tensor given as

\[ U(x) = T(x) + \frac{\partial}{\partial x} (2 J_\delta(x) + h J^D(x) + \frac{2 \hat{\delta}^2}{h} J^C(x)) \]

where \( J_\delta(x) = k Tr(\hat{g}^{-1}(x) \partial \hat{g}(x) \hat{\delta} H) \), \( \hat{\delta} = \frac{1}{2} \sum_{\alpha > 0} \frac{\alpha}{\alpha} \) and \( \hat{g} \) denotes a group element generated by exponentiation of the Kac-Moody algebra (see [20] for further details), \( h \) is the
Coxeter number of the underlying Lie algebra. It can be shown that the operators $U(x)$ given in (5.29) and $J^C(x)$ satisfy precisely the structure (3.33) with $c = 0$ and the deformation parameter $\hbar$ being the Coxeter number. In this case we have $W_\infty$ algebra as the symmetry of the problem [15]. In this last example it is known that the modification of the energy-momentum tensor allows the Hamiltonian reduction of the two-loop WZNW model to the Conformal Affine Toda model. This model, inherits in turn the same symmetry structure $W_\infty$ in contrast to the usual Conformal Toda models which has lost invariance under $w_\infty$.

6 “Gauge” Equivalence between various KP Hierarchies

The fundamental result of [11] was the proof that all three hierarchies labelled by $\ell = 0, 1, 2$ are “gauge” equivalent via generalized Miura transformations. Here we focus on the link between two KP$_{\ell=0}$ and KP$_{\ell=1}$ systems discussed above. Reference [11] presented a gauge transformation between those two systems, which mapped the underlying first bracket structures into each other. Explicitly this map is provided by $G$ from (2.20) where it mapped $L \rightarrow K$ with $K$ being in the KP hierarchy. We have seen that this map connects various KP flow equations. In [11] it was shown that the map transforms two different first bracket structures into each other. Take namely vectors $Q, V$ from (3.2) and notice that only $Q \geq 0, V \geq 0$ contribute to $\langle K \mid Q \rangle$ and $\langle K \mid V \rangle$. Proof given in [11] verified the validity of the following identity:

$$\langle K \mid [Q \geq 0, V \geq 0] \rangle = \{\langle K \mid Q \rangle, \langle K \mid V \rangle\}^{\ell=0}_{\ell=1} = \{\langle G^{-1}L_G \mid Q \rangle, \langle G^{-1}L_G \mid V \rangle\}^{\ell=1}_{\ell=0}$$

where in the last expression the bracket was understood according to definition in KP$_{\ell=1}$ [11]. The proof is based on the definition of the first bracket structure as a Lie-Poisson $R$-bracket for functions $F, H \in C^\infty(G^*, \mathbb{R})$:

$$\{F, H\}_R(L) = \langle L \mid [\nabla F(L), \nabla H(L)]_R \rangle$$

where the gradient $\nabla F : G^* \rightarrow G$ is defined by the standard formula given in [1, 11]. It is in this sense KP$_{\ell=0}$ and KP$_{\ell=1}$ are called “gauge” equivalent.

As we will see the mapping (2.20) transforming the Poisson bracket structure of KP into that of KP$_{\ell=1}$ and vice versa deserves a name of the generalized Miura transformation.

As the first application let us connect the two-boson KP hierarchy Lax operator $L^{(1)}_J$ with the Lax operator (3.3) expressed in terms of Faà di Bruno polynomials. Consider namely the gauge transformation between KP$_{\ell=1}$ and KP$_{\ell=0}$ generated by $\Phi$ such that $\Phi' = J$:

$$L_J = e^{-\Phi}L^{(1)}_J e^\Phi = D + J(D + J)^{-1} = D + \sum_{n=0}^{\infty} (-1)^n J P_n(J) D^{-1-n}$$

where $P_n(J) = (D + J)^n \cdot 1$ are as before the Faà di Bruno polynomials. As a corollary of the symplectic character of the “gauge” transformation used in (3.3), we conclude that $u_n = (-1)^n J P_n(J)$ belonging to the Lax operator of KP$_{\ell=0}$ must satisfy the Poisson-bracket $W_{1+\infty}$.
algebra described by the form $\Omega^{(0)}$ according to [3.17] [8], [15]. This represents an elegant
version of the technical proof given in section 4. It is possible to introduce a deformation
parameter into the Faà di Bruno representation of $\mathcal{W}_{1+\infty}$ algebra by redefining $u_n$ to $u_n(h) = (-1)^n \tilde{J}(hD + J)^n \cdot 1$ [15]. Now the semiclassical limit is simply obtained by taking $h \to 0$ in $u_n(h)$ and yields the generators of $\mathcal{W}_{1+\infty}$ algebra.

We now will discuss gauge equivalence between various two-boson hierarchies.

6.1 Gauge Equivalence of Various Two-Boson KP Hierarchies.

The Non-Linear Schroedinger Hierarchy. The non-linear Schroedinger (NLS) system is a
constrained KP system described by:

$$L_{\text{NLS}} = D + \bar{\psi}D^{-1}\psi$$

(6.4)

with the following (non-trivial) flow equations at the lowest level:

$$\frac{d}{dt_2} \left( \bar{\psi} \right) = \left( \bar{\psi}'' + 2\psi\bar{\psi}' - \bar{\psi}' \psi' - 2\bar{\psi}\psi\psi' \right)$$

$$\frac{d}{dt_3} \left( \psi \right) = \left( \psi''' + 6\bar{\psi}\psi\psi' \right)$$

(6.5)

These first equations of the NLS hierarchy can be reproduced by the Hamiltonian approach.

The first Hamiltonians functions obtained via usual definitions are

$$H_1 = \int \bar{\psi}\psi$$

$$H_2 = \int \bar{\psi}'\psi$$

$$H_3 = \int \left( \bar{\psi}'\psi^2 - \bar{\psi}'\psi' \right)$$

(6.6)

while the first two bracket structures are given by:

$$\{\bar{\psi}(x), \psi(y)\}_1 = \delta(x - y)$$

$$\{\bar{\psi}(x), \bar{\psi}(y)\}_1 = \{\psi(x), \psi(y)\}_1 = 0$$

(6.7)

and

$$\{\psi(x), \psi(y)\}_2 = -2\psi(x)\partial_x^{-1}\psi(x)\delta(x - y)$$

$$\{\bar{\psi}(x), \bar{\psi}(y)\}_2 = \delta'(x - y) + 2\psi(x)\partial_x^{-1}\bar{\psi}(x)\delta(x - y)$$

$$\{\bar{\psi}(x), \bar{\psi}(y)\}_2 = -2\bar{\psi}(x)\partial_x^{-1}\bar{\psi}(x)\delta(x - y)$$

(6.8)

The last structure can also be rewritten as:

$$\{\psi(x), \bar{\psi}(y)\}_2 = -\bar{\psi}(x)\bar{\psi}(y)\epsilon(x - y)$$

$$\{\psi(x), \psi(y)\}_2 = \delta'(x - y) + \psi(x)\bar{\psi}(y)\epsilon(x - y)$$

$$\{\bar{\psi}(x), \bar{\psi}(y)\}_2 = -\bar{\psi}(x)\bar{\psi}(y)\epsilon(x - y)$$

(6.9)

where $\epsilon(x - y)$ is the sign function. It is easy to see that the formulas (6.8) and (6.9) satisfy
the Jacobi identity. One can show that NLS hierarchy for independent $\psi$ and $\bar{\psi}$ is equivalent
to Faà di Bruno two-boson KP. The proof is based, in the spirit of [11], on establishing gauge
transformation between two hierarchies (see also [21, 22, 23, 24]). We show now the argument to illustrate the power of gauge transformation argument in the KP setting. Consider

\[ L_{NLS} \rightarrow G^{-1}L_{NLS}G = G^{-1}DG + G^{-1}\bar{\psi} \bar{D}^{-1}\psi G \]  

(6.10)

Take \( G^{-1} = \psi \), which leads to:

\[ G^{-1}L_{NLS}G = \psi D\psi^{-1} + \bar{\psi} \psi D^{-1} = D + \psi (\psi^{-1})' + \bar{\psi} \psi D^{-1} \]  

(6.11)

Clearly the gauge transformed \( L_{NLS} \) is an element of KP\(_1\) hierarchy and it is therefore natural to introduce now new variables such that

\[ \bar{J}(x) = \bar{\psi}(x)\psi(x) \]

\[ J(x) = \frac{\psi'(x)}{\psi(x)} \]  

(6.12)

and the inverse relation being \( \psi = \exp(\int J) \) and \( \bar{\psi} = \bar{J}\exp(-\int J) \). Since we now have established a gauge equivalence between two hierarchies it is clear that the first bracket structure in (6.7) leads to (3.34) with the generators of a linear \( W_{1+\infty} \) algebra being

\[ W_n(x) = (-1)^n\bar{\psi}(x)\psi^{(n)}(x) \]  

(6.13)

where \( \psi^{(n)}(x) = \frac{d^n\psi}{dx^n} \), while

\[ P_n(J(x)) = \frac{\psi^{(n)}(x)}{\psi(x)} \]  

(6.14)

The second bracket structure in (6.8) leads to (3.35) with \( c = 2, h = 1 \) and correspondingly non-linear \( \hat{W}_\infty \). If we only took the linear structure in (6.8) (i.e. \( \{ \bar{\psi}(x), \psi(y) \} = \delta'(x-y) \)) we would have induced (3.35) in its “un-deformed” form with the last equation of (3.35) being zero, corresponding to \( \Omega^{(1)} \) or \( W_\infty \).

We can also generalize above construction by adding \( N \) independent copies of currents:

\[ W_n(x) = (-1)^n\sum_{k=1}^{N} \bar{J}_k(x)P_n(J_k(x)) = (-1)^n\sum_{k=1}^{N} \bar{\psi}_k(x)\psi^{(n)}_k(x) \]  

(6.15)

The first bracket given by

\[ \{ \bar{\psi}_k(x), \psi_l(y) \}_1 = \delta_{kl}\delta(x-y) \]

\[ \{ \bar{\psi}_k(x), \bar{\psi}_l(y) \}_1 = \{ \psi_k(x), \psi_l(y) \}_1 = 0 \]  

(6.16)

The second bracket structure generating the non linear \( \hat{W}_\infty \) can be obtained through the algebra:

\[ \{ \psi_k(x), \psi_l(y) \}_2 = -\psi_k(x)\psi_l(y)\epsilon(x-y) \]

\[ \{ \psi_k(x), \bar{\psi}_l(y) \}_2 = \delta_{kl}\delta'(x-y) + \psi_k(x)\psi_l(y)\epsilon(x-y) \]  

\[ \{ \bar{\psi}_k(x), \bar{\psi}_l(y) \}_2 = -\bar{\psi}_k(x)\bar{\psi}_l(y)\epsilon(x-y) \]  

(6.17)
and through the definition

\[ J_k(x) \equiv \frac{\psi'_k(x)}{\psi_k(x)}, \quad \bar{J}_k(x) \equiv \bar{\psi}_k(x)\psi_k(x) \]  

(6.18)

which reproduce the algebra (3.35) with an anomaly term in the first relation, i.e.

\[
\begin{align*}
\{ \bar{J}_k(x), J_l(y) \}_2 &= J_l(x)\delta_{kl}\delta'(x-y) - \delta''_{kl}\delta(x-y) \\
\{ \bar{J}_k(x), \bar{J}_l(y) \}_2 &= 2\bar{J}_l(x)\delta_{kl}\delta'(x-y) + J_l'(x)\delta_{kl}\delta(x-y) \\
\{ J_k(x), J_l(y) \}_2 &= 2\delta'(x-y)
\end{align*}
\]  

(6.19)

As in the discussion of the multi-layered Benney equations the above algebra does not satisfy the Jacobi identity. Correspondingly the brackets for the fields \( \psi \)'s and \( \bar{\psi} \)'s given in (6.17) will violate Jacobi as well.

The brackets (6.16) and (6.17) lead to a system of coupled non-linear Schroedinger equations [25]:

\[
\begin{align*}
\frac{d\psi_m(x)}{dt} &= \{ \psi_m(x), H_3 \}_1 = \{ \psi_m(x), H_2 \}_2 = -\psi''_m(x) - 2\psi_m(x) \sum_{k=1}^{N} \bar{\psi}_k(x)\psi_k(x) \\
\frac{d\bar{\psi}_m(x)}{dt} &= \{ \bar{\psi}_m(x), H_3 \}_1 = \{ \bar{\psi}_m(x), H_2 \}_2 = \bar{\psi}''_m(x) + 2\bar{\psi}_m(x) \sum_{k=1}^{N} \bar{\psi}_k(x)\psi_k(x)
\end{align*}
\]  

(6.20)

The hierarchies of these equations relative to brackets 1 and 2 are different.

### 6.2 Quadratic Two-Boson KP Hierarchy and Generalized Miura Transformation

**Quadratic Two-Boson KP Hierarchy.** Here we call quadratic two-boson KP hierarchy the construction presented by Wu and Yu [26, 27] in order to realize \( \hat{W}_\infty \) as a hidden current algebra in the 2d \( SL(2, \mathbb{R})/U(1) \) coset model. Construction is based on the pseudo-differential operator:

\[ L_j = D + j(D - j - \bar{j})^{-1}j \]  

(6.21)

Let us discuss the Hamiltonian structure first. The three lowest Hamiltonian functions are:

\[
\begin{align*}
H_{j1} &= \int j \bar{j} \, dx \\
H_{j2} &= \int \left( -j' \bar{j} + j^2 \bar{j} + j \bar{j}^2 \right) \, dx \\
H_{j3} &= \int \left( j'' \bar{j} - 3j' \bar{j} - 2j' \bar{j}^2 - j \bar{j} j' + j^3 \bar{j} + 3j^2 \bar{j}^2 + j \bar{j}^3 \right) \, dx
\end{align*}
\]  

(6.22, 6.23, 6.24)

Among the Hamiltonian structures only second and third are local and are given by

\[
\begin{align*}
\{ j(x), \bar{j}(y) \}_2 &= \delta'(x-y) \\
\{ j(x), j(y) \}_2 &= \{ \bar{j}(x), \bar{j}(y) \}_2 = 0
\end{align*}
\]  

(6.25)
and
\[
\{ j(x), j(y) \}_3 = (j(x) + j(x)) \delta'(x - y) + j'(x) \delta(x - y) - \delta''(x - y) \quad (6.26)
\]
\[
\{ j(x), j(y) \}_3 = 2j(x) \delta'(x - y) + j'(x) \delta(x - y)
\]
\[
\{ j(x), j(y) \}_3 = 2j(x) \delta'(x - y) + j'(x) \delta(x - y)
\]

**Proposition.** The Hamiltonian structure corresponding to the Lax operator \( L_j \) in (6.21) is invariant under the following two transformations:

\[
j \to \bar{j} - \frac{j'}{j} \quad \bar{j} \to j
\]
\[
\bar{j} \to j + \frac{j'}{j} \quad j \to \bar{j}
\]

**Proof.** One verifies relatively easily that both bracket structures

\[
P_{j2} = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}, \quad P_{j3} = \begin{pmatrix} D_j + jD & -D^2 + D_j + jD \\ D^2 + jD + D\bar{j} & D\bar{j} + jD \end{pmatrix}
\]

(6.29)
corresponding to (6.25) and (6.26) are invariant under the transformations (6.27) and (6.28). Since \( P_1 = P_2P_3^{-1}P_2 \), a recurrence matrix \( P_2(P_1)^{-1} \) and all remaining higher hamiltonian structures must therefore remain invariant under (6.27) and (6.28). This in principle completes the proof. One can also verify directly that all three Hamiltonians (6.22), (6.23), (6.24) are invariant under (6.27) and (6.28) too. Hence we conclude that the Lax operators given by

\[
L_j = D + j \left( D - j - \bar{j} + \frac{j'}{j} \right)^{-1} \left( \bar{j} - \frac{j'}{j} \right)
\]
(6.30)
and

\[
\bar{L}_j = D + \left( j + \frac{j'}{j} \right) \left( D - j - \bar{j} - \frac{j'}{j} \right)^{-1} \bar{j}
\]
(6.31)
lead to the same Hamiltonian functions as (6.21).

**Gauge Equivalence between Faá di Bruno and Quadratic Two-Boson Hierarchies. Generalized Miura Map.**

We apply on \( L_j \) from (6.21) the gauge transformation \( \exp\left( \phi + \bar{\phi} \right) \) with result

\[
L_j \to \exp(-\phi - \bar{\phi}) L_j \exp(\phi + \bar{\phi}) = D + j + \bar{j} + jD^{-1} \bar{j}
\]
(6.32)

which is already an object in KP1 hierarchy. Acting furthermore with the gauge transformation \( \exp(-\ln j) \) we obtain finally the object in the Faá di Bruno hierarchy.

\[
L_j \to \exp(\ln j) L_j \exp(-\ln j) = D + j + \bar{j} + j(j^{-1})' + \bar{j}D^{-1} = D - J + \bar{J}D^{-1}
\]
(6.33)

where we have introduced

\[
J = -j - \bar{j} + \frac{j'}{j}
\]
\[
\bar{J} = \bar{j} \frac{j'}{j}
\]
(6.34)
One can now verify explicitly that with the bracket structure given (6.26) variables defined in (6.34) satisfy the second bracket structure of Faá di Bruno hierarchy (3.35) with \( c = 2 \) and \( h = 1 \) leading as shown in [15] to \( \hat{W}_\infty \). This is a short proof for the quadratic two-boson KP hierarchy [26] system realizing \( \hat{W}_\infty \). We now have obtained a Miura transform for two-boson hierarchies in form of (6.34) which generalizes the usual Miura transformation between the one-bose KdV and mKdV structures (as shown below).

Of course the higher hamiltonian structures of quadratic two-boson hierarchy are being mapped by (6.34) to their counterparts in Faá di Bruno hierarchy. This is also true for the Hamiltonian functions as one can see comparing (3.36) to (6.22), (6.23), (6.24).

Let us go back to the alternative expression (6.30) for the quadratic two-boson hierarchy. It can be rewritten under multiplication by 1 = \( \gamma \gamma \) from the right and left as follows

\[
L_j = 1 \left( \frac{1}{L_j} \right) = \frac{1}{\gamma \gamma} \gamma \gamma - \frac{1}{\gamma \gamma} \gamma \gamma = D + (j - j)^{-1} (j j - j')
\]

Next step is to gauge transform (6.35) from KP to KP \(_1\) hierarchy by acting with gauge transformation generated by \( \exp(\phi + \bar{\phi}) \) obtaining

\[
L_j \sim \exp\left(-\phi - \bar{\phi}\right) L_j \exp\left(\phi + \bar{\phi}\right) = D + j + \bar{j} + D^{-1}(j j - j')
\]

which is of the form of the Faá di Bruno hierarchy (up to conjugation) with \( J = -j - \bar{j} \) and \( \bar{J} = j j - j' \), which is equal to what was done in [15] chapter 3.3. Note that under (6.28) this is transformed into \( J = -j - \bar{j} - \frac{j'}{2} \) and \( \bar{J} = j j \) differing from (6.34) by a conjugation \( \gamma \leftrightarrow \bar{\gamma} \).

Similarly for (6.31) we find

\[
L_j = j^{-1} j L_j j^{-1} j = D + (j j + j') (D - j - j)^{-1}
\]

The same transformation as in (6.36) gives

\[
L_j \sim \exp\left(-\phi - \bar{\phi}\right) L_j \exp\left(\phi + \bar{\phi}\right) = D + j + \bar{j} + (j j + j') D^{-1}
\]

producing KP \(_1\) object with \( J = -j - \bar{j} \) and \( \bar{J} = j j + j' \). This time under (6.27) these variables are transformed into \( J = -j - \bar{j} + \frac{j'}{2} \) and \( \bar{J} = j j \) being precisely a transformation from (6.34).

We see that because of (6.27) and (6.28) there is an ambiguity in the possible form of generalized Miura transformation and (6.34) can appear also in other forms all of them connecting the Poisson bracket structure of Faá di Bruno hierarchy with the corresponding Poisson bracket structure of the quadratic two-boson hierarchy.

**Quadratic KP Hierarchy and the NLS Systems.** We note first that the NLS system is also gauge equivalent to quadratic KP hierarchy if we make in (6.11) a substitution \( \tilde{\psi} = j \exp(\phi + \bar{\phi}) \) and \( \psi = \exp(-\phi - \bar{\phi}) j \) or inversely \( \psi' / \psi = -j - j' / j \) and \( \tilde{\psi} \psi = \tilde{j} j \). This relation takes the following simple form in terms of the gauge transformation acting on the Lax operator \( L_j = D + j + \bar{j} + j D^{-1} j \):

\[
e^{(\phi + \bar{\phi})} \left(D + j + \bar{j} + j D^{-1} j\right) e^{-(\phi + \bar{\phi})} = D + \tilde{\psi} D^{-1} \psi
\]

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Changing the gauge function from \( \exp(\phi + \tilde{\phi}) \) to \( \exp(-\phi + \tilde{\phi}) \) we establish link with so-called derivative Non-Linear Schrödinger (dNLS) system:

\[
e^{(-\phi + \tilde{\phi})} \left( D + j + \bar{j} + \bar{j} D^{-1} j \right) e^{(\phi - \tilde{\phi})} = D + 2j + \bar{j} e^{(-\phi + \tilde{\phi})} D^{-1} j e^{(\phi - \tilde{\phi})} = D + 2r q + \left( rq^2 + q' \right) D^{-1} r \quad (6.40)
\]

where we have introduced new variables \[24\]:

\[
j(x) = q(x) r(x) \\
\bar{j}(x) = q(x) r(x) + \frac{q'(x)}{q(x)} = j(x) + \frac{q'(x)}{q(x)} \quad (6.41)
\]

The Hamiltonian \( H_{j1} \) is written in terms of these variables as

\[
H_{j1} = \int (q^2(y) r^2(y) + q'(y) r(y)) dy \quad (6.42)
\]

We now propose the following algebraic structure

\[
\{q(x), r(y)\}_3 = \delta'(x - y) \\
\{q(x), q(y)\}_3 = \{r(x), r(y)\}_3 = 0 \quad (6.43)
\]

Using the definitions \[(6.41)\] we find exactly the third bracket structure given in eq. \[(6.26)\].

The corresponding equations of motion

\[
\dot{q}(x) = \{q(x), H_{j1}\}_3 = q'''(x) + 2(q^2(x) r(x))' \\
\dot{r}(x) = \{r(x), H_{j1}\}_3 = -r'''(x) + 2(r^2(x) q(x))' \quad (6.44)
\]

correspond to the derivative NLS equations described in \[28\].

Define now, according to Kaup and Newell \[28\]

\[
R(x) = q(x) e^{\mu(x)} \\
Q(x) = r(x) e^{-\mu(x)} \\
\mu(x) = \int_{-\infty}^{x} q(y) r(y) dy \quad (6.45)
\]

It then follows that

\[
H_{j1} = \int R' Q \\
H_{j2} = -\int (R' Q' - R' R Q^2) \quad (6.46)
\]

which coincides with expressions \[(42b)\] and \[(42c)\] of \[28\] up to factors \( i \).

The second bracket \[(6.20)\] can be realized by

\[
\{q(x), q(y)\}_2 = q(x) q(y) \epsilon(x - y) \\
\{r(x), q(y)\}_2 = -\delta(x - y) - r(x) q(y) \epsilon(x - y) \\
\{q(x), r(y)\}_2 = \delta(x - y) - r(y) q(x) \epsilon(x - y) \\
\{r(x), r(y)\}_2 = r(x) r(y) \epsilon(x - y) \quad (6.47)
\]

The equations \( \dot{q}(x) = \{q(x), H_{j2}\}_2 \) and \( \dot{r}(x) = \{r(x), H_{j2}\}_2 \) reproduce \[(6.44)\] showing that the system is indeed bi-hamiltonian.
7 Reduction to One-boson KdV Systems

We here apply the Dirac reduction scheme to obtain one-boson hierarchies from two-boson hierarchies. The general feature will be a transformation of some two-boson Hamiltonian equations of motion

$$\frac{\partial \mathcal{O}}{\partial t_r} = \{ \mathcal{O}, H_r \}$$  \hfill (7.1)

(where \( \mathcal{O} \) denote original degrees of freedom) to one-boson Hamiltonian system according to the Dirac scheme:

$$\frac{\partial X}{\partial t_r} = \{ X, H_r^{\text{Dirac}} \}$$  \hfill (7.2)

with \( X \) denoting a surviving one-boson degree of freedom. Another general feature would be that reduction would take the Lax operator from KP\(_1\) hierarchy to the conventional KP hierarchy.

KdV Hierarchy. Consider the Dirac constraint: \( \Theta = J = 0 \) for system in (6.3). First let us discuss the resulting Dirac bracket structure. We find:

$$\{ \bar{J}(x), \bar{J}(y) \}^2 = \{ \bar{J}(x), \bar{J}(y) \}^2 - \int dz dz' \{ \bar{J}(x), \Theta(z) \}^2 \{ \Theta(z), \Theta(z') \}^{-1} \{ \Theta(z'), \bar{J}(y) \}$$  \hfill (7.3)

which yields

$$\{ \bar{J}(x), \bar{J}(y) \}^2 = 2 \bar{J}(x) \delta'(x-y) + \bar{J}'(x) \delta(x-y) + 1 \frac{1}{2} \delta'''(x-y)$$  \hfill (7.4)

The reduced Lax operator looks now as:

$$l_J = D + \bar{J}D^{-1}$$  \hfill (7.5)

and the corresponding (non-zero) lowest Hamiltonian functions \( H_r^{KdV} \equiv \text{Tr} l_J^r / r \) are

$$H_1^{KdV} = \int \bar{J} \, dx \quad ; \quad H_3^{KdV} = \int \bar{J}^2 \, dx \quad ; \quad H_5^{KdV} = \int (2 \bar{J}' + \bar{J}''') \, dx$$  \hfill (7.6)

Moreover one checks that the flow equation:

$$\frac{\delta l_J}{\delta t_r} = \left[ (l_J^r)^{+}, l_J \right]$$  \hfill (7.7)

gives

$$\delta \bar{J}/\delta t_1 = \bar{J}' \quad ; \quad \delta \bar{J}/\delta t_3 = \bar{J}''' + 6 \bar{J} \bar{J}'$$  \hfill (7.8)

where the second equation reproduces the famous KdV equation. This equation can also be obtained by inserting \( X = \bar{J} \) and \( H_3^{KdV} \) into (7.2).

mKdV Hierarchy. Now consider the quadratic two-boson hierarchy with Lax given in (6.21), (6.30) or (6.33). We choose as a Dirac constraint: \( \theta = j + \bar{j} = 0 \). The resulting Dirac bracket structure is:

$$\{ j(x), j(y) \}^2 = -\int dz dz' \{ j(x), \theta(z) \}^2 \{ \theta(z), \theta(z') \}^{-1} \{ \theta(z'), j(y) \} = -\frac{1}{2} \delta'(x-y)$$  \hfill (7.9)
and the reduced Lax operator is:

\[ l_j = D - j D^{-1} J = D + \sum_{n=0}^{\infty} (-1)^{n+1} j^{(n)} D^{-1-n} \]  

(7.10)

Note that imposing the constraint \( \theta = 0 \) on the equivalent Lax operators from (6.30) and (6.31) respectively, we get:

\[ l_j = L_j |_{\theta=0} = D + \left( D + \frac{j'}{j} \right)^{-1} \left( -j - \frac{j'}{j} \right) = D + D^{-1} \left( -j^2 - j' \right) \]  

(7.11)

\[ l_j = L_j |_{\theta=0} = D - \left( j + \frac{j'}{j} \right) \left( D - \frac{j'}{j} \right)^{-1} j = D + \left( -j^2 - j' \right) D^{-1} \]  

(7.12)

The last equalities in (7.11) and (7.12) were obtained using the trick of multiplying \( l_j \) by \( 1 = \frac{1}{j} - 1 \). Obviously we could express everywhere \( j \) by \( \bar{j} \) hence the one-boson system must be invariant under transformation \( j \leftrightarrow -j \). The flow equations calculated as in (7.7) are

\[ \frac{dj}{dt_1} = j' \quad ; \quad \frac{dj}{dt_2} = 0 \quad ; \quad \frac{dj}{dt_3} = j''' + 6j^2(j)' \]  

(7.13)

Hence the flow equation for \( dj/dt_3 \) is the mKdV equation. Moreover the mKdV equation could also be obtained from Hamiltonian \( H_3 \) defined in a standard way:

\[ H_1^{\text{mKdV}} = -\int j^2 \, dx \quad ; \quad H_3^{\text{mKdV}} = \int \left( j^4 - j j'' \right) \, dx \quad ; \quad H_5^{\text{mKdV}} = -\int \left( 2j^6 + 10j^2(j')^2 + j j^{(IV)} \right) \, dx \]  

(7.14)

(and zero for even indices). Because of existence of symmetry described in (5.27) (and (5.28)) we could equivalently impose the constraints \( \theta_1 = j + \bar{j} - j'/j = 0 \) or alternatively \( \theta_2 = j + \bar{j} + j'/\bar{j} = 0 \) without changing the Dirac bracket structure and the constraint manifold. Imposing \( \theta_1 = 0 \) on the Lax operator in (6.21) we get

\[ l_j = D + \left( -j + \frac{j'}{j} \right) \left( D - \frac{j'}{j} \right)^{-1} j = D + \left( -j^2 + j' \right) D^{-1} \]  

(7.15)

The last equality was obtained by the trick of multiplying left hand side by \( 1 = j^{-1} j \) from both sides. Taking however the equivalent Lax operator as given in (5.30) we get automatically again

\[ l_j = L_j |_{\theta=0} = D - j D^{-1} j \]  

(7.16)

Hence the mKdV hierarchy is given in terms of three alternative and equivalent Lax operators given in (7.10), (7.11) and (7.15). Especially the mKdV Hamiltonians (including those in (7.14)) are invariant under transformation \( j \leftrightarrow -j \).

**Miura Map.** Take now the generalized Miura transformation (5.34) and impose the Dirac constraint \( J = -j - \bar{j} + j'/j = 0 \). As a result we get the conventional Miura map:

\[ \bar{J} |_{j=0} = j \left( -j + \frac{j'}{j} \right) = -j^2 + j' \]  

(7.17)
It is easy to find via Dirac procedure that \( j \) satisfies the bracket

\[
\{j(x), j(y)\}_2^D = -\int dzdz'\{j(x), j(z)\}_2\{j(z), j(z')\}_2^{-1}\{j(z'), j(y)\}_2 = -\frac{1}{2} \delta'(x - y)
\]  

(7.18)

which is perfectly consistent with \( J = -j^2 + j' \) satisfying the bracket (7.3).

Especially we see that all Hamiltonians from (7.6) go to Hamiltonians in (7.14) under \( J \to -j^2 \pm j' \).

**Bi-Hamiltonian Structure of KdV Hierarchy** The evolution equation (7.2) specified to the constraint manifold \( J = 0 \) results in

\[
\frac{\partial J}{\partial t} \bigg|_{j=0} = \{ J, H_{KdV}^r \}_2^D = \left( D\bar{J} + \bar{J}D + \frac{1}{2}D^3 \right) \frac{\delta H_r}{\delta J} \bigg|_{J=0}
\]  

(7.19)

in which one recognizes the second Hamiltonian structure of KdV hierarchy. To recover the first Hamiltonian structure of KdV hierarchy we recall that from (3.37) we have for the Faà di Bruno hierarchy:

\[
\frac{\partial}{\partial t} \left( J \frac{\delta H_{r+1}}{\delta J} \right) = P_{J1} \left( \frac{\delta H_{r+1}}{\delta J} \right) = P_{J2} \left( \frac{\delta H_r}{\delta J} \right)
\]  

(7.20)

where in the last identity we used \( P_{J1} \) and \( P_{J2} \) from (3.38). Let us now take \( r \) odd so \( H_{r+1} \to 0 \) for \( J \to 0 \). We find from (7.20) using \( P_{J1} \) that

\[
\frac{\partial J}{\partial t} \bigg|_{J=0} = -D\frac{\delta H_{r+1}}{\delta J} \bigg|_{J=0}
\]  

(7.21)

On the other hand using both \( P_{J1} \) and \( P_{J2} \) to calculate \( \partial J/\partial t_{r+1} \) we find in the general case

\[
2D\frac{\delta H_{r+1}}{\delta J} + \left( D^2 + D\bar{J} \right) \frac{\delta H_{r+1}}{\delta J} = -D\frac{\delta H_{r+2}}{\delta J}
\]  

(7.22)

However in the limit \( J \to 0 \) since \( H_{r+1} \to 0 \) we have also \( \delta H_{r+1}/\delta J \to 0 \). Note however that we can not claim that also \( \delta H_{r+1}/\delta J \to 0 \) follows in this case. Therefore summarizing we find

\[
\frac{\partial J}{\partial t} \bigg|_{J=0} = -D\frac{\delta H_{r+1}}{\delta J} \bigg|_{J=0} = \frac{1}{2}D\frac{\delta H_{r+2}}{\delta J} \bigg|_{J=0}
\]  

(7.23)

which reproduces well-known result about bi-Hamiltonian structure of KdV (see also [8]). Equation (7.23) can be also treated as a recurrence relation which proves that the system defined by Lax given in (7.5) is indeed KdV system to all orders of Hamiltonian function.

One can now find the bi-Hamiltonian structure for the case of mKdV. First we recall a formula [29]:

\[
(D \mp 2j) D (D \pm 2j) = 2 \left( \frac{1}{2}D^3 + (-j^2 \pm j')D + D(-j^2 \pm j') \right)
\]  

(7.24)
From Miura transformation we find [29]

\[
\frac{\delta H_{mKdV}^r}{\delta J} = \frac{D\bar{J} \delta H_{rKdV}^d}{\delta J} = (-D - 2J) \frac{\delta H_{KdV}^d}{\delta J} \quad (7.25)
\]

We therefore have:

\[
\frac{\delta H_{mKdV}^{r+2}}{\delta J} = (-D - 2J) \frac{\delta H_{r+2KdV}^d}{\delta J}
\]

\[
= (-D - 2J) D^{-1} 2 \left( \frac{1}{2} D^3 + (-j^2 + j') D + D(-j^2 + j') \right) \frac{\delta H_{rKdV}^d}{\delta J}
\]

\[
= (-D - 2J) D^{-1} (D - 2J) D (D + 2J) (-D - 2J)^{-1} \frac{\delta H_{mKdV}^d}{\delta J}
\]

\[
= (D + 2J) D^{-1} (D - 2J) D \frac{\delta H_{rKdV}^d}{\delta J} = \left( D - 4J D^{-1} \right) D \frac{\delta H_{rKdV}^d}{\delta J} \quad (7.26)
\]

where we used both (7.24) and (7.24). Relation (7.26) reveals a bi-Hamiltonian (but non-local) structure of mKdV hierarchy and can be rewritten in a more simple way as Lenard’s recursion relation:

\[
D \frac{\delta H_{mKdV}^d}{\delta J} = \left( D^3 - 4D J D^{-1} \right) D \frac{\delta H_{rKdV}^d}{\delta J} \quad (7.27)
\]

**Schwarzian-KdV Hierarchy.** Here few remarks are given about Schwarzian-KdV (S-KdV) hierarchy. We start be discussion of invariance of the Miura map \( \bar{J} = -j^2 + j' = -(\phi')^2 + \phi'' \) where as before \( \phi' = j \). Invariance of \( \bar{J} \) under some transformation \( \delta \) results in

\[
\delta \left( -(\phi')^2 + \phi'' \right) = 0 \quad \rightarrow \quad \delta \phi'' = 2\phi' \delta \phi' \quad (7.28)
\]

Solution of (7.28) takes therefore a simple form

\[
\delta \phi' = \delta J = \epsilon^{-1} \exp(2\phi) \quad (7.29)
\]

or

\[
\delta \phi = \frac{\epsilon^0}{2} + \int \epsilon^{-1} \exp(2\phi) \quad (7.30)
\]

where \( \epsilon^0 \) and \( \epsilon^{-1} \) are some arbitrary constants. Introduce now the function \( f \) such that \( f'' = \exp(2\phi) \) and which is connected to \( j \) through the Cole-Hopf type of transformation

\[
j = \phi' = \frac{1}{2} \frac{f''}{f'} \quad (7.31)
\]

We find that (7.30) corresponds to \( sl_2 \) transformation

\[
\delta f = \epsilon^1 + \epsilon^0 f + \epsilon^{-1} f^2 \quad (7.32)
\]

and leaves \( \bar{J} = S(f)/2 \) invariant, where \( S(f) \) is a Schwarzian.
It is known that (7.31) relates the mKdV hierarchy to the S-KdV hierarchy with equation $f_t/f' = S(f)$. We are using Weiss nomenclature [30], [31] is using the name of KdV-singularity hierarchy. Hence according to (7.31) we will be interested in one-boson Lax operator of the form

$$L = D - \frac{1}{2} \frac{f''}{f'} D^{-1} \frac{1}{2} \frac{f''}{f'}$$

(7.33)

There are many ways of promoting this operator to two-boson system. If we consider a very simple choice

$$L = D + \frac{1}{2} \frac{f''}{f'} + j + \frac{f''}{f'} D^{-1} j$$

(7.34)

the second bracket structure is

$$\{ j(x), f(y) \} = -2 f'(x) D^{-1} \delta(x - y)$$

(7.35)

Another choice could be

$$L = D + \frac{f''}{f'} + 2 \rho + \left( \frac{f''}{f'} + \rho \right) D^{-1} \rho$$

(7.36)

leads to (7.33) under constraint $\frac{f''}{f'} + 2 \rho = 0$. Defining $\rho = v'$ we can now make contact with quadratic KP hierarchy by defining a map:

$$\bar{\rho} = v' + \frac{f''}{f'} ; \quad \rho = v'$$

(7.37)

Of course the ambiguity of (6.27) allows equally well the map:

$$\bar{\rho} = v' + \frac{f''}{f'} - \frac{v''}{v'} ; \quad \rho = v'$$

(7.38)

The structure in (7.37) has a non-local bracket structure equivalent to structure in (6.26). We find easily e.g.

$$\{ v(x), f(y) \} = D_x^{-1} f'(x) D_x^{-1} \delta(x - y)$$

(7.39)

8 Two-boson KP Hierarchies in Terms of a SL(2) Gauge Theory

8.1 Zero Curvature Condition and Soliton Equations.

We first establish a connection between a typical two-boson Lax operator $L$ of KP$_{\ell=1}$ hierarchy characterized by three functions $A, B, C$ and a component $\mathcal{A}$ of $SL(2, \mathbb{R})$ Lie algebra valued gauge field. We express this connection in terms of the following equivalence relation:

$$L = D + A + B D^{-1} C \sim \mathcal{A} = \begin{pmatrix} -\frac{1}{2} A & -C \\ B & \frac{1}{2} A \end{pmatrix}$$

(8.1)
Under the gauge transformation applied on \(L\) the above equivalence takes the following form

\[
L' = e^{-\chi}Le^{\chi} = D + (A + \partial\chi) + (e^{-\chi}B)D^{-1}(Ce^{\chi}) \sim \mathcal{A}' = \begin{pmatrix}
-\frac{1}{2}(A + \partial\chi) & (Ce^{\chi}) \\
(e^{-\chi}B) & \frac{1}{2}(A + \partial\chi)
\end{pmatrix}
\]

(8.2)

We note that by the above equivalence principle a gauge transformation among Lax operators of \(\text{KP}_{\ell=1}\) corresponds to the \(\text{SL}(2, \mathbb{R})\) gauge transformation:

\[
\mathcal{A}' = gA_{g}^{-1} + g\partial g^{-1}
\]

(8.3)

induced by the following diagonal \(2 \times 2\)-real unimodular matrix:

\[
g \equiv \begin{pmatrix}
e^{\chi/2} & 0 \\
0 & e^{-\chi/2}
\end{pmatrix}
\]

(8.4)

One easily verifies that the following three gauge configurations

\[
\begin{pmatrix}
-\frac{1}{2}A & -C \\
B & \frac{1}{2}A
\end{pmatrix} \sim \begin{pmatrix}
-\frac{1}{2}(A + B'/B) & -BC \\
1 & \frac{1}{2}(A + B'/B)
\end{pmatrix} \sim \begin{pmatrix}
-\frac{1}{2}(A - C'/C) & -1 \\
BC & \frac{1}{2}(A - C'/C)
\end{pmatrix}
\]

(8.5)

are gauge equivalent with gauge functions

\[
g_B = \begin{pmatrix}
B^{\frac{1}{2}} & 0 \\
0 & B^{-\frac{1}{2}}
\end{pmatrix} ; \quad g_C = \begin{pmatrix}
C^{-\frac{1}{2}} & 0 \\
0 & C^{\frac{1}{2}}
\end{pmatrix}
\]

(8.6)

generating connections between the first and the second and the first and the third gauge connection of equation (8.5).

Let us recall three main examples of the two-boson \(\text{KP}_{\ell=1}\) hierarchies with their corresponding components of the \(sl(2, \mathbb{R})\) connection:

\[
L_{NLS} = D + \bar{\psi}D^{-1}\psi \sim A_{NLS} = \begin{pmatrix}
0 & -\psi \\
\bar{\psi} & 0
\end{pmatrix}
\]

(8.7)

\[
L_j = D + J + \bar{J}D^{-1}J \sim A_j = \begin{pmatrix}
-\frac{1}{2}(J + \bar{J}) & -J \\
\bar{J} & \frac{1}{2}(J + \bar{J})
\end{pmatrix}
\]

(8.8)

\[
L_{\bar{J}} = D - J + \bar{J}D^{-1} \sim A_{\bar{J}} = \begin{pmatrix}
\frac{1}{2}J & -1 \\
\bar{J} & -\frac{1}{2}J
\end{pmatrix}
\]

(8.9)

Defining the element of \(\text{SL}(2, \mathbb{R})\):

\[
g_\psi = \begin{pmatrix}
\psi^{-\frac{1}{2}} & 0 \\
0 & \psi^{\frac{1}{2}}
\end{pmatrix}
\]

(8.10)

we are able to transform \(A_{NLS}\) to the form of \(A_j\):

\[
A'_{NLS} = g_\psi A_{NLS}g_\psi^{-1} + g_\psi \partial g_\psi^{-1} = \begin{pmatrix}
\frac{1}{2}\partial_x \psi/\psi & -1 \\
\psi \partial_x \psi/\psi & -\frac{1}{2}\partial_x \psi/\psi
\end{pmatrix}
\]

(8.11)
Similarly applying gauge transformation generated by
\[ g_j = \begin{pmatrix} e^{-\frac{1}{2}(\phi + \bar{\phi})} & 0 \\ 0 & e^{\frac{1}{2}(\phi + \bar{\phi})} \end{pmatrix} \] (8.12)
to \( A_j \) we get
\[ A'_j = g_j A_j g_j^{-1} + g_j \partial g_j^{-1} = \begin{pmatrix} 0 & -e^{-\phi - \bar{\phi}} j \\ j e^{\phi + \bar{\phi}} & 0 \end{pmatrix} \] (8.13)
the gauge field component belonging to NLS hierarchy.

We now put the \( A \) component in the complete the \( sl(2, \mathbb{R}) \) connection \( (A, B) \) satisfying the zero curvature condition \([32]\):
\[ \partial_x B - \partial_t A + [B, A] = 0 \] (8.14)
where we have introduced the other component of the \( sl(2, \mathbb{R}) \) gauge field:
\[ B = \begin{pmatrix} B^0 & B^+ \\ B^- & -B^0 \end{pmatrix} \] (8.15)
In components \([8.14]\) reads:
\[ \partial_x B^0 - \psi B^- - \bar{\psi} B^+ = 0 \] (8.16)
\[ \partial_x B^+ + \partial_t \psi + 2\psi B^0 = 0 \] (8.17)
\[ \partial_x B^- - \partial_t \bar{\psi} + 2\bar{\psi} B^0 = 0 \] (8.18)
Taking
\[ B_{NLS} = \begin{pmatrix} \bar{\psi} \psi & \partial_x \psi \\ \partial_x \bar{\psi} & -\bar{\psi} \psi \end{pmatrix} \] (8.19)
we see that \([8.16]\) is satisfied automatically while \((8.17)\) and \((8.18)\) yield equations of NLS hierarchy \([33]\):
\[ \partial_t \psi = -\partial_x^2 \psi - 2\psi^2 \bar{\psi} \quad ; \quad \partial_t \bar{\psi} = \partial_x^2 \bar{\psi} + 2\psi \bar{\psi}^2 \] (8.20)
It is interesting at this point to comment on connection between NLS system and the Heisenberg Model.

Since \( A_{NLS} \) and \( B_{NLS} \) satisfy the zero curvature equation \((8.14)\) it is natural to represent them (locally) as pure gauge configurations \([34]\):
\[ A_{NLS} = g^{-1} g_x \]
\[ B_{NLS} = g^{-1} g_t \] (8.21)
Introduce now the traceless matrix:
\[ S = g \sigma_3 g^{-1} \] (8.22)
which has the property \( S^2 = 1 \). It can easily be shown that:
\[ S S_x = -S_x S = -2 g_x g^{-1} \] (8.23)
and consequently
\[ g^{-1}[S, S_{xx}]g = -4 \frac{d}{dx} A_{NLS} \tag{8.24} \]

On the other hand, taking time derivative of \( S \) we find
\[ g^{-1} S \dot{t} g = [B_{NLS}, \sigma_3] \tag{8.25} \]

Using explicit form of \( A_{NLS} \) and \( B_{NLS} \) given above in (8.7) and (8.19), we arrive at
\[ S_t = \frac{1}{2} [S, S_{xx}] \tag{8.26} \]

which describes the isotropic Heisenberg ferromagnet model.

Applying the gauge transformation (8.10) to \( B_{NLS} \) we obtain
\[ B_J = \begin{pmatrix} \bar{J} + \frac{1}{2} \partial_t \Phi & J \\ \partial_x \bar{J} - J \bar{J} & -\bar{J} - \frac{1}{2} \partial_t \Phi \end{pmatrix} \tag{8.27} \]

which when inserted into (8.14) (together with \( A_J \)) yields the Bussinesq equations.

### 8.2 Reduction to the KdV Systems.

It is also easy in this framework to discuss the reduction of KP systems to KdV systems. The link is obtained by putting \( j + \bar{j} = 0 \) and \( J = 0 \) in (8.8) and (8.9) getting
\[ A_{mKdV} = \begin{pmatrix} 0 & -j \\ -j & 0 \end{pmatrix} ; \quad A_{KdV} = \begin{pmatrix} 0 & -1 \\ J & 0 \end{pmatrix} \tag{8.28} \]

Solutions to zero curvature equation involving gauge fields components of type given in (8.28) (with spectral parameter \( \lambda \) instead of zeros) have been discussed in [32]. We recall the main points of this discussion. Let us first take \( A_{mKdV} \) modified by adding to it the spectral parameter \( \lambda \sigma_3 \). After inserting it and matrix (8.13) into the zero-curvature equation (8.14) we obtain
\[ \begin{align*}
\partial_x B^0 - j b_- &= 0 \tag{8.29} \\
\partial_x b_- - 4j B^0 - 2\lambda b_+ &= 0 \tag{8.30} \\
\partial_x b_+ + 2 \partial_t J - 2\lambda b_- &= 0 \tag{8.31}
\end{align*} \]

where for convenience we have introduced \( b_\pm \equiv B^\pm \mp B^- \). We now eliminate \( b_\pm \) in terms of \( B^0 \) using (8.29) and (8.30). First we introduce the function \( C(j, \lambda) \) such that
\[ b_- = \partial_x B^0 / j = \lambda C' \tag{8.32} \]

Hence we find that \( B^0 = \lambda D^{-1} j C' \) and from (8.30) we get \( b_+ = -2j D^{-1} j C' + C''/2 \). Inserting these quantities into (8.31) we arrive at
\[ \partial_t J = \left( D_j D^{-1} j C' - \frac{1}{4} C'' \right) + \lambda^2 C' \tag{8.33} \]
Expanding $C$ in $\lambda$ as in $C = \sum_{k=0}^{n}(\lambda^2)^{n-k}C_k$ we obtain from (8.33):

$$
DC_k = \left(\frac{1}{4}D^3 - DfD^{-1}jD\right)C_{k+1}, \quad k = 0, 1, \ldots, n - 1
$$

$$
\partial_tJ = \left(DfD^{-1}jD - \frac{1}{4}D^3\right)C_n
$$

(8.34)

We clearly recognize in (8.34) the bi-hamiltonian structure of mKdV equation. Moreover putting $C_n = j$ we recover the mKdV equation $\partial_tJ = DfD^{-1}jD - \frac{1}{4}D^3j/4 = 3j^2j'/2-j''/4$.

We also recover the mKdV equation behind the equation (8.35).

Also here expanding $B^+$ in powers of $\lambda$ reveals both bi-Hamiltonian structure and KdV equation behind the equation (8.35).

8.3 Drinfeld-Sokolov reduction of Two-boson Hierarchies to KdV

As we have seen above we can associate to each two-boson Lax operator a $sl_2$ matrix according to (8.1), in such a way that the gauge transformation of the Lax operator $L' = e^{-x}Le^x$ corresponds to the $sl_2$ gauge transformation of $sl_2$ connection $A' = gAg^{-1} + gdg^{-1}$ with diagonal $2 \times 2$-real unimodular matrix (8.4).

Let us now take the special example of Faá di Bruno hierarchy represented by $A_J$ from (8.9). Important point is that there is a residual gauge transformation generated by

$$
g_0 \equiv \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}
$$

(8.36)

which preserves the form of $A_J$ under

$$
A' = g_0^{-1}Ag_0 + g_0^{-1}dg_0 = \begin{pmatrix} \frac{1}{2}J - \gamma & \gamma' \\ \bar{J} - \gamma J + \gamma^2 + \gamma & \frac{1}{2}J + \gamma \end{pmatrix}
$$

(8.37)

(8.37)

It is useful at this point to explain what is happening using the Drinfeld-Sokolov formalism [85]. Consider space of first order differential operators with coefficients being $2 \times 2$ matrices:

$$
M_\mathcal{E} = \left\{ L^{(1)} = D - \mathcal{E} + \omega \mid \mathcal{E} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \omega = \begin{pmatrix} \omega_{11} & 0 \\ \omega_{21} & \omega_{22} \end{pmatrix} \right\}
$$

(8.38)

and the group

$$
\Gamma \equiv \left\{ \Gamma \mid \Gamma \equiv \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \right\}
$$

(8.39)

acting on $M_\mathcal{E}$ according to

$$
\Gamma^{-1}(D - \mathcal{E} + \omega)\Gamma = D - \mathcal{E} + \omega'
$$

(8.40)
with
\[ \omega' = \begin{pmatrix} \omega_{11} - \gamma & 0 \\ \omega_{21} - \gamma(\omega_{22} - \omega_{11}) + \gamma^2 + \gamma' & \omega_{22} + \gamma \end{pmatrix} \] (8.41)

In the spirit of Hamiltonian Drinfeld-Sokolov reduction consider the quotient space \( M_{\text{red}} = M_\mathcal{E}/\Gamma \). There exist a convenient realization of \( M_{\text{red}} \) in terms of second order differential operators with scalar coefficients. The procedure to obtain it is as follows. Consider the relation
\[ L^{(1)}(\psi_1, \psi_2) = 0 \] (8.42)
Eliminating \( \psi_2 \) from this equation we arrive at
\[ L^{(2)} \psi_1 = 0 \] with
\[ L^{(2)} = \alpha \left( L^{(1)} \right) = D^2 + (\omega_{11} + \omega_{22})D + \omega_{21} + \omega_{11}\omega_{22} + \omega'_{11} \] (8.43)

Because \( \alpha \left( \Gamma^{-1}L^{(1)}\Gamma \right) = \alpha \left( L^{(1)} \right) \) the space of second order differential operators from (8.43) parameterizes the quotient space \( M_{\text{red}} \).

Consider now the special case of two-boson KP hierarchy:
\[ \omega = \begin{pmatrix} \omega_{11} & 0 \\ \omega_{21} & \omega_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}J & 0 \\ \bar{J} - \frac{1}{4}J^2 & -\frac{1}{2}J' \end{pmatrix} \] (8.44)

Take \( \Gamma \) with \( \gamma = \frac{1}{2}J \) so the transformed \( \omega \) matrix
\[ \omega' = \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix} = \begin{pmatrix} \bar{J} - \frac{1}{4}J^2 + \frac{1}{2}J' & 0 \\ 0 & 0 \end{pmatrix} \] (8.45)
has diagonal elements equal to zero. It means that the corresponding Lax operator is:
\[ L^{(1)} = D + uD^{-1} \quad ; \quad u = \bar{J} - \frac{1}{4}J^2 + \frac{1}{2}J' \] (8.46)
One can check that with \( (\bar{J}, J) \) satisfying the second Poisson bracket (3.35) with \( c = 2, h = 1 \), \( u \) satisfies the Virasoro algebra:
\[ \{ u(x), u(y) \} = 2u(x) \delta'(x - y) + u'(x) \delta(x - y) + \frac{1}{2} \delta''(x - y) \] (8.47)
We also note that with \( \omega \) like in (8.44) the second-order differential operator (8.43) becomes a typical KdV operator \( L^{(2)} = D^2 + u \). Hence \( \omega' \) from (8.43) or first order Lax \( L^{(1)} \) from (8.46) represent just the special gauge choice on \( M_\mathcal{E} \) equivalent to the KdV Lax operator. This shows Drinfeld-Sokolov reduction as an alternative to the Dirac reduction of two-boson hierarchy to KdV hierarchy.

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