HARNACK TYPE INEQUALITY ON RIEMANNIAN MANIFOLDS OF DIMENSION 5.

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ABSTRACT. We give an estimate of type \( \sup \times \inf \) on Riemannian manifold of dimension 5 for the Yamabe equation.

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1. INTRODUCTION AND MAIN RESULTS

In this paper, we deal with the following Yamabe equation in dimension \( n = 5 \):

\[
- \Delta g u + \frac{n - 2}{4(n - 1)} R_g u = n(n - 2) u^{N-1}, \quad u > 0, \quad \text{and} \quad N = \frac{n + 2}{n - 2}.
\]  
(1)

Here, \( R_g \) is the scalar curvature.

The equation (1) was studied a lot, when \( M = \Omega \subset \mathbb{R}^n \) or \( M = \mathbb{S}_n \) see for example, [2-4], [11], [15]. In this case we have a \( \sup \times \inf \) inequality. The corresponding equation in two dimensions on open set \( \Omega \) of \( \mathbb{R}^2 \), is:

\[
- \Delta u = V(x) e^u,
\]  
(2)

The equation (2) was studied by many authors and we can find very important result about a priori estimates in [8], [9], [12], [16], and [19]. In particular in [9] we have the following interior estimate:

\[
\sup_K u \leq c = c(\inf_{\Omega} V, ||V||_{L^\infty(\Omega)}, \inf_{\Omega} u, K, \Omega).
\]

And, precisely, in [8], [12], [16], and [20], we have:

\[
C \sup_K u + \inf_{\Omega} u \leq c = c(\inf_{\Omega} V, ||V||_{L^\infty(\Omega)}, K, \Omega),
\]

and,

\[
\sup_K u + \inf_{\Omega} u \leq c = c(\inf_{\Omega} V, ||V||_{C^\alpha(\Omega)}, K, \Omega),
\]

where \( K \) is a compact subset of \( \Omega \), \( C \) is a positive constant which depends on \( \frac{\inf_{\Omega} V}{\sup_{\Omega} V} \), and, \( \alpha \in (0, 1] \). When \( \frac{4(n - 1)h}{n - 2} = R_g \) the scalar curvature, and \( M \) compact, the equation (1) is Yamabe equation. T. Aubin and R. Schoen have proved the existence of solution in this case, see
for example [1] and [14] for a complete and detailed summary. When \( M \) is a compact Riemannian manifold, there exist some compactness result for equation (1) see [18]. Li and Zhu see [18], proved that the energy is bounded and if we suppose \( M \) not diffeomorphic to the three sphere, the solutions are uniformly bounded. To have this result they use the positive mass theorem. Now, if we suppose \( M \) Riemannian manifold (not necessarily compact) Li and Zhang [17] proved that the product \( \sup \times \inf \) is bounded. Here we extend the result of [5]. Our proof is an extension Li-Zhang result in dimension 3, see [3] and [17], and, the moving-plane method is used to have this estimate.

We refer to Gidas-Ni-Nirenberg for the moving-plane method, see [13]. Also, we can see in [3, 6, 11, 16, 17, 10], some applications of this method, for example an uniqueness result. We refer to [7] for the uniqueness result on the sphere and in dimension 3. Here, we give an equality of type \( \sup \times \inf \) for the equation (1) in dimension 5. In dimension greater than 3 we have other type of estimates by using moving-plane method, see for example [3, 5]. There are other estimates of type \( \sup + \inf \) on complex Monge-Ampere equation on compact manifolds, see [20-21]. They consider, on compact Kahler manifold \((M, g)\), the following equation:

\[
\begin{cases}
(\omega + \partial \bar{\partial} \varphi)^n = e^{f-t \frac{1}{2}} \omega^n, \\
\omega + \partial \bar{\partial} \varphi > 0 \text{ on } M
\end{cases}
\]  

And, they prove some estimates of type \( \sup_M + m \inf_M \leq C \) or \( \sup_M + m \inf_M \geq C \) under the positivity of the first Chern class of \( M \). Here, we have,

**Theorem 1.1.** For all compact set \( K \) of \( M \), there is a positive constant \( c \), which depends only on, \( K, M, g \) such that:

\[
\left( \sup_K u \right)^{1/3} \times \inf_M u \leq c,
\]

for all \( u \) solution of (1).

This theorem extend to the dimension 5 the result of Li and Zhang, see [17]. Here, we use the method of Li and Zhang in [17]. Also, we extend a result of [5].

**Corollary 1.2.** For all compact set \( K \) of \( M \) there is a positive constant \( c \), such that:

\[
\sup_K u \leq c = c(g, m, K, M) \quad \text{if} \quad \inf_M u \geq m > 0,
\]

for all \( u \) solution of (1).

2. PROOF OF THE THEOREMS

**Proof of theorem 1.1:** We want to prove that

\[
e^3 (\max_{B(0, \varepsilon)} u)^{1/3} \times \min_{B(0, 4 \varepsilon)} u \leq c = c(M, g).
\]  

(4)
We argue by contradiction and we assume that
\[
(\max_{B(0, \epsilon_k)} u_k)^{1/3} \times \min_{B(0,4\epsilon_k)} u_k \geq k\epsilon_k^{-3}.
\]  
(5)

**Step 1: The blow-up analysis.** The blow-up analysis gives us: For some \(\bar{x}_k \in B(0, \epsilon_k)\), \(u_k(\bar{x}_k) = \max_{B(0,\epsilon_k)} u_k\), and, from the hypothesis,
\[
u_k(\bar{x}_k)^{4/9} \epsilon_k \to +\infty.
\]

By a standard selection process, we can find \(x_k \in B(\bar{x}_k, \epsilon_k/2)\) and \(\sigma_k \in (0, \epsilon_k/4)\) satisfying,
\[
\begin{align*}
  u_k(x_k)^{4/9} &\to +\infty, & (6) \\
  u_k(x_k) &\geq u_k(\bar{x}_k), & (7) \\
  \text{and, } u_k(x) &\leq C u_k(x_k), \text{ in } B(x_k, \sigma_k), & (8)
\end{align*}
\]
where \(C\) is some universal constant. It follows from above (5), (7) that
\[
(\max_{B(x_k)} u_k)^{1/3} \times (\min_{\partial B(x_k, 2\epsilon_k)} u_k)^{3/2} \geq (u_k(\bar{x}_k))^{1/3} \times (\min_{B(0,4\epsilon_k)} u_k)\epsilon_k^3 \geq k \to +\infty.
\]  
(9)

We use \(\{z^1, \ldots, z^n\}\) to denote some geodesic normal coordinates centered at \(x_k\) (we use the exponential map). In the geodesic normal coordinates, \(g = g_{ij}(z)dz^idz^j\),
\[
g_{ij}(z) - \delta_{ij} = O(r^2), \quad g := \det(g_{ij}(z)) = 1 + O(r^2), \quad h(z) = O(1),
\]  
(10)
where \(r = |z|\). Thus,
\[
\Delta_y u = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j u) = \Delta u + b_i \partial_i u + d_{ij} \partial_{ij} u,
\]
where
\[
b_j = O(r), \quad d_{ij} = O(r^2)
\]  
(11)

We have a new function
\[
v_k(y) = M_k^{-1} u_k(M_k^{-2/(n-2)} y) \quad \text{for } |y| \leq 3\epsilon_k M_k^{2/(n-2)}
\]
where \(M_k = u_k(0)\). From (6) and (9) we have
\[
\begin{align*}
  \Delta v_k + \tilde{b}_i \partial_i v_k + \tilde{d}_{ij} \partial_{ij} v_k - \tilde{c} v_k + v_k^{N-1} &\quad = 0 \quad \text{for } |y| \leq 3\epsilon_k M_k^{2/(n-2)} \\
  v_k(0) &\quad = 1 \\
  v_k(y) &\quad \leq C_1 \quad \text{for } |y| \leq \sigma_k M_k^{2/(n-2)} \\
  \lim_{|y|\to\infty} \min_{|y|=2\epsilon_k M_k^{2/(n-2)}} (v_k(y)|y|^3) &\quad = +\infty.
\end{align*}
\]  
(12)
where \(C_1\) is a universal constant and
\[
\tilde{b}_i(y) = M_k^{-2/(n-2)} b_i(M_k^{-2/(n-2)} y), \quad \tilde{d}_{ij}(y) = d_{ij}(M_k^{-2/(n-2)} y)
\]  
(13)
and,
\[
\tilde{c}(y) = M_k^{-4/(n-2)} h(M_k^{-2/(n-2)} y).
\]  
(14)

We can see that for \(|y| \leq 3\epsilon_k M_k^{2/(n-2)}\),
\[
|\tilde{b}_i(y)| \leq C M_k^{-4/(n-2)} |y|; \quad |\tilde{d}_{ij}(y)| \leq C M_k^{-4/(n-2)} |y|^2; \quad |\tilde{c}(y)| \leq C M_k^{-4/(n-2)}
\]  
(15)
where $C$ depends on $n, M, g$.

It follows from (12), (13), (14), (15) and the elliptic estimates, that, along a subsequence, $v_k$ converges in $C^2$ norm on any compact subset of $\mathbb{R}^2$ to a positive function $U$ satisfying

$$
\begin{align*}
\Delta U + U^{N-1} &= 0, \text{ in } \mathbb{R}^n, \quad \text{with } N = \frac{n + 2}{n - 2} \\
U(0) = 1, \quad 0 < U \leq C.
\end{align*}
$$

(16)

In the case where $C = 1$, by a result of Caffarelli-Gidas-Spruck, see [10], we have:

$$
U(y) = (1 + |y|^2)^{-\frac{(n-2)}{2}},
$$

(17)

But, here we do not need this result.

Now, we need a precision in the previous estimates, we take a conformal change of metric such that, the Ricci tensor vanish,

$$
R_{j;p} = 0.
$$

(18)

We have by the expressions for $g$ and $g_{ij}$, as in the paper of Li-Zhang,

$$
\begin{align*}
b_j &= O(r^2), \quad R = O(r), \quad d_{ij} = -\frac{1}{3} R_{ipqj} y^p y^q + O(1).
\end{align*}
$$

(19)

Thus,

$$
|c| \leq C|y|M_k^{-2}, \quad |\bar{b}_i| \leq C|y|^2 M_k^{-2},
$$

(20)

and,

$$
\bar{d}_{ij} = -\frac{1}{3} M_k^{-4/3} R_{ipqj} y^p y^q + O(1) M_k^{-2} |y|^3.
$$

(21)

As, in the paper of Li-Zhang, we have:

$$
v_k(y) \geq C|y|^{-3}, \quad 1 \leq |y| \leq 2 \epsilon_k M_k^{2/3}.
$$

(22)

with $C > 0$.

For $x \in \mathbb{R}^2$ and $\lambda > 0$, let,
\[ v_k^{\lambda,x}(y) := \frac{\lambda}{|y-x|} \left( x + \frac{\lambda^2(y-x)}{|y-x|^2} \right), \]
\[ (23) \]
denote the Kelvin transformation of \( v_k \) with respect to the ball centered at \( x \) and of radius \( \lambda \).

We want to compare for fixed \( x, v_k \) and \( v_k^{\lambda,x} \). For simplicity we assume \( x = 0 \). We have:

\[ v_k^\lambda(y) := \frac{\lambda}{|y|} v_k(y^\lambda), \quad \text{with} \quad y^\lambda = \frac{\lambda^2 y}{|y|^2}. \]

For \( \lambda > 0 \), we set,

\[ \Sigma_\lambda = B(0, \epsilon_k M_k^2) - \overline{B}(0, \lambda). \]

The boundary condition, \( (12) \), become:

\[ \lim_{k \to +\infty} \min_{|y| = \epsilon_k M_k^{4/9}} (v_k(y)|y|^3) = \lim_{k \to +\infty} \min_{|y| = 2\epsilon_k M_k^{4/9}} (v_k(y)|y|^3) = +\infty. \]
\[ (24) \]
As in the paper of Li-Zhang, we have:

\[ \Delta w_\lambda + \bar{b}_i \partial_i w_\lambda + \bar{d}_{ij} \partial_{ij} w_\lambda - \bar{c} w_\lambda + \frac{(n+2)}{(n-2)} \xi^{4/(n-2)} w_\lambda = E_\lambda \text{ in } \Sigma_\lambda. \]
\[ (25) \]
where \( \xi \) stay between \( v_k \) and \( v_k^\lambda \). Here,

\[ E_\lambda = -\bar{b}_i \partial_i v_k^\lambda - \bar{d}_{ij} \partial_{ij} v_k^\lambda + \bar{c} v_k^\lambda - E_1, \]

with,

\[ E_1(y) = - \left( \frac{\lambda}{|y|} \right)^{n+2} \left( \bar{b}_i(y^\lambda) \partial_i v_k(y^\lambda) + \bar{d}_{ij}(y^\lambda) \partial_{ij} v_k(y^\lambda) - \bar{c}(y^\lambda) v_k(y^\lambda) \right). \]
\[ (26) \]

**Lemma 2.1.** We have,

\[ |E_\lambda| \leq C_1 \lambda |y|^{-1} M_k^{-2} + C_2 \lambda^3 |y|^{-3} M_k^{-4/3}. \]
\[ (27) \]
Proof: as in the paper of Li-Zhang, we have a nonlinear term $E_\lambda$ with the following property,

$$|E_\lambda| \leq C_1 \lambda^3 M_k^{-2} |y|^{-2} + C_2 \lambda^4 M_k^{-4/3} |y|^{-4} \leq C_1 \lambda |y|^{-1} M_k^{-2} + C_2 \lambda^3 |y|^{-3} M_k^{-4/3}.$$ 

Next, we need an auxiliary function which correct the nonlinear term. Here we take the following auxiliary function:

$$h_\lambda = -C_1 \lambda M_k^{-2} (|y| - \lambda) - C_2 \lambda^2 M_k^{-4/3} \left( \left( 1 - \left( \frac{\lambda}{|y|} \right)^3 \right) - \left( 1 - \left( \frac{\lambda}{|y|} \right) \right) \right), \quad (28)$$

we have,

$$h_\lambda \leq 0, \quad (29)$$

$$\Delta h_\lambda = -C_1 \lambda |y|^{-1} M_k^{-2} - C_2 \lambda^3 |y|^{-3} M_k^{-4/3}, \quad (30)$$

and, thus,

$$\Delta h_\lambda + |E_\lambda| \leq 0.$$ 

As in the paper of Li-Zhang, we can prove the following lemma:

**Lemma 2.2.** We have,

$$w_\lambda + h_\lambda > 0, \text{ in } \Sigma \forall 0 < \lambda \leq \lambda_1. \quad (31)$$

Before to prove the lemma, note that, here, we consider the fact that,

$$\lambda \leq |y| \leq \epsilon_k M_k^{4/9} \leq \epsilon_k M_k^{2/3}. \quad (32)$$

And, as in the paper of Li-Zhang, we need the estimate [22]:

$$v_k(y) \geq C |y|^{-3}, \quad 1 \leq |y| \leq 2 \epsilon_k M_k^{2/3}.$$ 

with $C > 0$.

**Proof:**
**Step 1:** There exists $\lambda_0 > 0$ independent of $k$ such the assertion of the lemma holds for all $0 < \lambda < \lambda_0$.

To see this, we write:

$$w_\lambda = v_k(y) - v^\lambda_k(y) = |y|^{-3/2}(|y|^{3/2}v_k(y) - |y^\lambda|^{3/2}v_k(y^\lambda)).$$

Let, in polar coordinates,

$$f(r, \theta) = r^{3/2}v(r, \theta).$$

By the properties of $v_k$, there exist $r_0 > 0$ and $C > 0$ independent of $k$ such that:

$$\partial_r f(r, \theta) > Cr^{1/2}, \text{ for } 0 < r < r_0.$$ 

Thus,

$$w_\lambda(y) = |y|^{-3/2}(f(|y|, y/|y|) - f(|y^\lambda|, y/|y|)) = |y|^{-3/2}\int_{|y^\lambda|}^{|y|} \partial_r f(r, y/|y|)dr >$$

$$> C'||y|^{-3/2}(|y|^{3/2} - |y^\lambda|^{3/2}) > C''(|y| - \lambda) \text{ for } 0 < \lambda < |y| < r_0,$$

with, $C'', C'' > 0$.

It follows that,

$$w_\lambda + h_\lambda \geq (C'' - o(1))(|y| - \lambda), \text{ for } 0 < \lambda < |y| < r_0,$$  \hspace{1cm} (33)

Now, for

$$r_0 \leq |y| \leq \epsilon_k M_k^{4/9} \leq \epsilon_k M_k^{2/3}.$$

we have by the definition of $h_\lambda$, and, as in the paper of Li-Zhang, we need the estimate [22]:

$$v_k(y) \geq C|y|^{-3}, \text{ 1} \leq |y| \leq 2\epsilon_k M_k^{2/3}.$$

to have,
\[ |h_\lambda| < \frac{1}{2} v_k(y). \]

Thus, as in the paper of Li-Zhang,

\[ w_\lambda + h_\lambda > 0, \text{ for } 0 < r_0 < |y| < 2\epsilon_k M_k^{4/9}. \]  \hspace{1cm} (34)

**Step 2:** Set,

\[ \bar{\lambda}^k = \sup \{ 0 < \lambda \leq \lambda_1, w_\mu + h_\mu \geq 0, \text{ in } \Sigma_\mu \forall 0 < \mu \leq \lambda \}, \]  \hspace{1cm} (35)

We claim that, \( \bar{\lambda}^k = \lambda_1 \).

In order to apply the maximum principle and the Hopf lemma, we need to prove that:

\[ (\Delta + \bar{b}_i \partial_i + \bar{d}_{ij} \partial_{ij} - \bar{c})(w_\lambda + h_\lambda) \leq 0 \text{ in } \Sigma_\lambda \]  \hspace{1cm} (36)

In other words, we need to prove that:

\[ \Delta h_\lambda + \bar{b}_i \partial_i h_\lambda + \bar{d}_{ij} \partial_{ij} h_\lambda - \bar{c}h_\lambda + E_\lambda \leq 0 \text{ in } \Sigma_\lambda. \]  \hspace{1cm} (37)

First note that, \( h_\lambda < 0 \). Here, we consider the fact that,

\[ \lambda \leq |y| \leq \epsilon_k M_k^{4/9} \leq \epsilon_k M_k^{2/3}. \]

We have,

\[ |\bar{c}| \leq C |y| M_k^{-2}. \]

Thus,

\[ |y| |\bar{c}h_\lambda| \leq C_1 M_k^{-4} \lambda |y|^2 (|y| - \lambda) + C_2 M_k^{-10/3} \lambda |y|^2 \leq o(1) M_k^{-2} \lambda, \]

which we can write as,

\[ |\bar{c}h_\lambda| \leq C_1 M_k^{-2} \lambda |y|^{-1}. \]  \hspace{1cm} (38)

We have,
Thus, \[ |\bar{b}_i| \leq C |y|^2 M_k^{-2}, \]

Thus,

\[ |\bar{b}_i \partial_i h_\lambda| \leq C_1 M_k^{-4} \lambda |y|^2 + C_2 M_k^{-10/3} (\lambda^5 |y|^{-2} + \lambda^3), \]

\[ |y| C_1 M_k^{-4} \lambda |y|^2 = o(1) M_k^{-2} \lambda, \]

which we can write as,

\[ C_1 M_k^{-4} \lambda |y|^2 = o(1) M_k^{-2} \lambda |y|^{-1}. \] (39)

and,

\[ |y|^3 C_2 M_k^{-10/3} \lambda^5 |y|^{-2} = C_2 M_k^{-10/3} \lambda^3 |y| = o(1) M_k^{-4/3} \lambda^3, \]

which we can write as,

\[ C_2 M_k^{-10/3} \lambda^5 |y|^{-2} = o(1) M_k^{-4/3} \lambda^3 |y|^{-3}. \] (40)

and,

\[ |y|^3 C_2 M_k^{-10/3} \lambda^3 = o(1) M_k^{-2} \lambda^3, \]

which we can write as,

\[ C_2 M_k^{-10/3} \lambda^3 = o(1) M_k^{-4/3} \lambda^3 |y|^{-3}. \] (41)

Thus,

\[ |\bar{b}_i \partial_i h_\lambda| \leq o(1) M_k^{-2} \lambda |y|^{-1} + o(1) M_k^{-4/3} \lambda^3 |y|^{-3}. \] (42)

We have,

\[ |\bar{d}_{ij}| \leq |y|^2 M_k^{-4/3}, \]

Thus,
Thus,

\[ |\bar{d}_{ij}\partial_{i}h_{\lambda}| \leq \lambda|y|^{-1}M_{k}^{-10/3} + o(1)M_{k}^{-8/3}\lambda^{5}|y|^{-3} + o(1)M_{k}^{-8/3}\lambda^{3}|y|^{-1}, \]

Finally,

\[ |\bar{d}_{ij}\partial_{i}h_{\lambda}| \leq o(1)\lambda|y|^{-1}M_{k}^{-2} + o(1)M_{k}^{-4/3}\lambda^{3}|y|^{-3} + o(1)M_{k}^{-2}\lambda|y|^{-1}. \]  \hspace{1cm} (43)

Finally, we have,

\[ \Delta h_{\lambda} + \bar{b}_{i}\partial_{i}h_{\lambda} + \bar{d}_{ij}\partial_{ij}h_{\lambda} - \bar{c}h_{\lambda} + E_{\lambda} \leq 0 \text{ in } \Sigma_{\lambda}, \]

And, thus (36),

\[ (\Delta + \bar{b}_{i}\partial_{i} + \bar{d}_{ij}\partial_{ij} - \bar{c})(w_{\lambda} + h_{\lambda}) \leq 0 \text{ in } \Sigma_{\lambda}. \]

Also, we have from the boundary condition and the definition of \( v_{\lambda}^{k} \) and \( h_{\lambda} \), we have:

\[ |h_{\lambda}(y)| + v_{\lambda}^{k}(y) \leq \frac{C(\lambda_{1})}{|y|^{3}}, \quad \forall \ |y| = \epsilon_{k}M_{k}^{4/9}, \]  \hspace{1cm} (45)

and, thus,

\[ w_{\tilde{\lambda}}^{k}(y) + h_{\tilde{\lambda}}^{k}(y) > 0 \quad \forall \ |y| = \epsilon_{k}M_{k}^{4/9}, \]  \hspace{1cm} (46)

We can use the maximum principle and the Hopf lemma to have:

\[ w_{\tilde{\lambda}}^{k} + h_{\tilde{\lambda}}^{k} > 0, \text{ in } \Sigma_{\lambda}, \]  \hspace{1cm} (47)

and,

\[ \frac{\partial}{\partial \nu}(w_{\tilde{\lambda}}^{k} + h_{\tilde{\lambda}}^{k}) > 0, \text{ in } \Sigma_{\lambda}. \]  \hspace{1cm} (48)
From the previous estimates we conclude that $\bar{\lambda}_k = \lambda_1$ and the lemma is proved.

Given any $\lambda > 0$, since the sequence $v_k$ converges to $U$ and $h_{\lambda_k}$ converges to 0 on any compact subset of $\mathbb{R}^2$, we have:

$$U(y) \geq U^\lambda(y), \quad \forall \ |y| \geq \lambda, \quad \forall \ 0 < \lambda < \lambda_1. \quad (49)$$

Since $\lambda_1 > 0$ is arbitrary, and since we can apply the same argument to compare $v_k$ and $v_k^{\lambda,x}$, we have:

$$U(y) \geq U^{\lambda,x}(y), \quad \forall \ |y - x| \geq \lambda > 0. \quad (50)$$

Thus implies that $U$ is a constant which is a contradiction.

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