Irrational Axions as a Solution of The Strong CP Problem in an Eternal Universe

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We exhibit a novel solution of the strong CP problem, which does not involve any massless particles. The low energy effective Lagrangian of our model involves a discrete spacetime independent axion field which can be thought of as a parameter labeling a dense set of $\theta$ vacua. In the full theory this parameter is seen to be dynamical, and the model seeks the state of lowest energy, which has $\theta_{\text{eff}} = 0$. The processes which mediate transitions between $\theta$ vacua involve heavy degrees of freedom and are very slow. Consequently, we do not know whether our model can solve the strong CP problem in a universe which has been cool for only a finite time. We present several speculations about the cosmological evolution of our model.
1. Introduction and a 1+1 Dimensional Example

There have been a number of mechanisms proposed for resolving the strong CP problem of Quantum Chromodynamics (QCD). Most of them are either ruled out by experiment or only marginally consistent with the combination of experimental data and conventional cosmology. In the present note we would like to present a new solution of the strong CP problem. We will exhibit a flat space-time quantum field theory in which the QCD $\theta$ parameter is “screened.” That is to say, although $\theta$ appears in the Lagrangian of the model, the observable correlation functions in the true ground state do not depend on it. In particular, they are all CP invariant.

We do not yet know whether our model provides a realistic solution of the strong CP problem in the world, as opposed to a model of a flat eternal universe. It has a large number of nearly degenerate metastable vacuum states, and it is not clear that the system will ever find its ground state in a universe which has been large only for a finite time. The details of the discussion may also depend on the mechanism that ensures that the cosmological constant vanishes. Since our understanding of these cosmological issues is not complete, we can only offer a few speculations about them. These will be presented in section III.

The idea for our solution of the strong CP problem is most easily demonstrated by first examining another theory with a $\theta$ parameter: 1+1 dimensional QED. The Abelian gauge group of the theory can be either the compact group $U(1)$ or its non-compact covering group $\mathbb{R}$. The choice between $\mathbb{R}$ and $U(1)$ has significant consequences. The charges of the matter fields in the $\mathbb{R}$ theory are not restricted but they have to be integers in the $U(1)$ theory. Another difference between these two theories is that unlike the $U(1)$ gauge theory, the $\mathbb{R}$ gauge theory on a compact two-dimensional parameter space has no $\theta$ dependence. On a non-compact parameter space both theories exhibit $\theta$ dependence which is interpreted as a background electric field originating from classical charges at infinity. This fact remains true even in the presence of dynamical matter fields whose charges are quantized.

Now consider the case of dynamical matter fields whose charges are not quantized. In particular, consider the Abelian Higgs model with Higgs field of charge one and a massive

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1 The single exception that we are aware of is the class of models in which discrete symmetries guarantee that the argument of the determinant of the quark mass matrix is zero at tree level.
fermion of irrational charge \( q \). It is particularly interesting to examine the limit in which the fermion mass is much larger than the scale defined by the Higgs model. Since the charges are not quantized, the gauge group must be \( \mathbb{R} \) and there is no \( \theta \) dependence on a compact parameter space. Perhaps more surprising is the fact that there is also no \( \theta \) dependence on a non-compact parameter space. The point is that any background electric field \( \theta \) can be screened by popping charges out of the vacuum. With the two charges, one and \( q \), every value of \( \theta = 2\pi(n + mq) \) with \( n \) and \( m \) integers can be screened. The screening process may cost a lot of energy (and be very slow) because the irrationally charged fermions are so heavy, but this will always be compensated by gaining the constant field energy over the infinite volume of space. Since the ground state energy is minimized when the external field is zero and since any value of \( \frac{\theta}{2\pi} \) can be arbitrarily closely approximated by a number of the form \( (n + mq) \), the ground state has vanishing background field regardless of the value of \( \theta \) in the Lagrangian. This is the essence of our solution of the strong CP problem. The bosonized version of this model can be immediately generalized to \( 3 + 1 \) dimensions. This will be done in the next section.

2. Irrational Discrete Axions

Consider a model consisting of an axion, \( a \), coupled to two nonabelian gauge fields in the standard manner. The Euclidean Lagrangian has the form

\[
\mathcal{L} = \frac{f^2}{2}(\partial a)^2 + \frac{1}{4g_1^2}F_1^2 + \frac{1}{4g_2^2}F_2^2 + i(a + \theta_1)Q_1 + i(qa + \theta_2)Q_2
\]  (2.1)

where the \( Q_i \) are the topological charge densities of the two gauge fields, \( f \) is the axion decay constant and \( q \) is a dimensionless parameter. The second gauge group is supposed to represent QCD, while the first is another confining theory with a much larger confinement scale, \( \Lambda_1 \gg \Lambda_2 = \Lambda_{QCD} \). We assume that the fermions coupled to these two gauge theories have no global symmetries that could be used to rotate away the topological terms in the Lagrangian. Clearly, without loss of generality we can use the shift symmetry of the axion to set \( \theta_1 = 0 \). The conventional wisdom is that this leaves us with no freedom to change the value of \( \theta_2 \) and the model (2.1) has strong CP violation. However, a closer examination

\[\text{The fundamental result that is needed to prove this is that the error in a rational approximant to any number can be made to vanish like one over the square of the denominator in the approximant} \ \text{[3].}\]
shows that after $\theta_1$ is set to zero we still have the freedom to shift $a$ by $2\pi n$ for integer $n$. If the parameter $q$ is irrational, we can use this freedom to set $\theta_2$ arbitrarily close to any desired value. Therefore, the theory based on the Lagrangian (2.1) is independent of both $\theta_1$ and $\theta_2$ and is equivalent to

$$\mathcal{L} = \frac{f^2}{2} (\partial a)^2 + \frac{1}{4g_1^2} F_1^2 + \frac{1}{4g_2^2} F_2^2 + ia(Q_1 + qQ_2) \ .$$

(2.2)

Unlike the conventional wisdom, one axion field $a$ can remove more than one $\theta$ parameter.

What we have just argued is that the theory based on (2.1) is equivalent to that based on (2.2). The latter is obviously CP invariant, if the axion field $a$ is defined to change sign under a CP transformation. What we would like to show now is that CP is not spontaneously broken and therefore all correlation functions are CP invariant. To show that we first integrate out all the non-zero modes of $a$. The result of this Gaussian integral is

$$\mathcal{L}_{\text{eff}} = \frac{1}{4g_1^2} F_1^2(x) + \frac{1}{4g_2^2} F_2^2(x) + \frac{q}{f^2} \int dy Q_1(x)G(x, y)Q_2(y) + ia_0(Q_1(x) + qQ_2(x)) \tag{2.3}$$

where $G(x, y)$ is the propagator of $a$ with the zero mode, $a_0$, removed. Now, integrating over the gauge fields we find the effective potential $V_{\text{eff}}(a_0)$. Following the argument in [3], we see that the point $a_0 = 0$ is a global minimum of $V_{\text{eff}}$. The integrand of the gauge field functional integral is positive for $a_0 = 0$ and carries a phase for any other value of $a_0$. The partition function is therefore maximized, and the energy minimized, for the CP conserving vacuum $a_0 = 0$.

Let us examine the theory in more detail in order to understand the physical mechanism for resolving the strong CP problem. We set $\theta_1 = 0$ and integrate out the heavy gauge degrees of freedom. This fixes the VEV of the axion to $2\pi n$ for some integer $n$ and the axion acquires a mass of order $\frac{\Lambda_i^3}{f}$. Typically, one ignores the integer $n$ and sets it to zero. Then the low energy theory includes the light gauge fields (QCD), no axion and a $\theta$ parameter equal to $\theta_2$. This is essentially the argument that one axion can remove only one $\theta$ parameter. However, the general argument in the previous paragraphs shows that when $q$ is irrational the theory cannot depend on $\theta_2$. The reason for that is that the integer $n$ cannot be ignored. After integrating out the massive gauge fields, the theory has an infinite number of degenerate ground states labeled by $n$. The dynamics of the light gauge fields breaks this degeneracy. When $f \gg \Lambda_i$ the term with the propagator in (2.3)
is small and can be neglected. In this approximation the effective potential $V_{\text{eff}}(a_0)$ has the form

$$V_{\text{eff}}(a) = \Lambda_1^4 E_1(a) + \Lambda_2^4 E_2(qa + \theta_2)$$

where we have dropped the subscript of $a$. The functions $E_i$ are both periodic with period $2\pi$ and have their minimum when the argument vanishes. Assuming that $E_i$ are continuous at $0 \mod 2\pi$ we can minimize the total effective potential by making both of the arguments of the $E_i$ as close as possible to multiples of $2\pi$. Thus

$$a \approx 2\pi n \quad ; \quad qa + \theta_2 \approx 2\pi m$$

These two equations are compatible iff

$$\theta \approx 2\pi (m - qn)$$

These are of course just the equations that we discussed in the $1 + 1$ dimensional Higgs model. If $q$ is irrational, then we can satisfy this condition with arbitrary precision by appropriate choice of $n, m$.

Our mechanism has an obvious generalization to the case of several gauge groups all coupled to the axion $a$ through some coefficients $q_i$. If all these coefficients $q_i$ are relatively irrational, there is an infinite set of vacua where $(q_i\langle a \rangle + \theta_i) \mod 2\pi < \epsilon_i$ for any $\epsilon_i$. It might however be important for cosmological reasons to note that the fraction of vacua satisfying these inequalities goes like $\prod_i \epsilon_i$ when all the $\epsilon_i$ are small. Thus if the dynamics of the universe randomly chooses between all possible metastable states of the system, the probability of not seeing any low energy CP violation goes rapidly to zero as the number of gauge groups is increased.

The failure of the decoupling theorem for this model stems from two separate sources. First of all, the high energy theory has an infinite set of degenerate vacua, and the degeneracy is broken by QCD. Equally importantly, we are discussing the ground state of the model, which means that we are willing to wait an arbitrarily long time for the system to settle down. The processes by which the system moves from one of the almost degenerate metastable states with $\theta_{QCD} \neq 0$, to the true vacuum will be very slow. For the purposes of discussing local physics over finite time intervals the decoupling theorem is valid.

An equivalent way of thinking about the model is the following. We can include in our low energy effective Lagrangian a discrete field $n$ which is independent of the coordinates,
This field represents the value of $\langle a \rangle / 2\pi$ and thus labels the almost degenerate ground states. It is crucial that by varying the value of $n$ every value of $\theta_{QCD} = (q\langle a \rangle + \theta_2) \mod 2\pi$ can be approximated arbitrarily well. Therefore, the sum in the functional integral of the low energy theory over $n$ can be replaced by an integral over a single continuous variable $\theta_{QCD}$ which is independent of $x$. Therefore, our theory looks like ordinary QCD with one more integration variable $\theta_{QCD}$. The standard problem with such a theory is that ordinarily different values of $\theta_{QCD}$ correspond to different superselection sectors. There are no physical processes or local operators which communicate between these different sectors and therefore $\theta_{QCD}$ should not be integrated over. The novelty in our theory is that $\theta_{QCD}$ does not label different superselection sectors. The high energy theory makes the barrier between these sectors finite and allows transitions between them.

As with the standard axion, $\theta_{QCD}$ is a field which is integrated over and can relax to zero. In our case, though, only the zero momentum mode of the field $n$ and therefore also only the zero momentum mode of $\theta_{QCD}$ exists. Hence, we do not have a massless or light axion.

One might ask whether our model suffers from the $U(1)$ problem when massless quarks are coupled to it. Then it has an axial $U(1)$ symmetry under which the coordinate $\theta_{QCD}$ is shifted by a constant. However, since $\theta_{QCD}$ is independent of $x$, it does not have a conjugate momentum and no charge generates this symmetry. From the high energy point of view this follows from the fact that nearby values of $\theta_{QCD}$ are associated with far separated points in field space. Therefore, when the symmetry is broken there is no Goldstone boson and there is no $U(1)$ problem. A simple toy model which exhibits such a behavior is based on the Lagrangian

$$\mathcal{L}_{toy} = \frac{f_a^2}{2} \partial_\mu \eta(x) \partial^\mu \eta(x) + V(\eta(x) + \theta_{QCD})$$

(2.7)

where $\theta_{QCD}$ is an integration variable in the functional integral. The field $\eta(x)$ plays the role of the would be light boson of the $U(1)$ problem. The theory (2.7) is invariant under the broken $U(1)$ symmetry $\eta(x) \rightarrow \eta(x) + \alpha; \theta_{QCD} \rightarrow \theta_{QCD} - \alpha$ and the potential $V$ can be arbitrary. We can use the $U(1)$ symmetry to set $\theta_{QCD} = 0$ in the Lagrangian and then the integral over $\theta_{QCD}$ factorizes. Clearly, the resulting theory does not have a massless $\eta$ particle.

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3 It should be stressed that at the level of the low energy theory, this field has no dynamics.
Technically, the violation of current algebra “theorems” relating the $U(1)$ Goldstone boson to $\theta$ dependence in QCD comes about in our model because the two point function of topological charge density is discontinuous at zero momenta. The discontinuity comes from intermediate metastable vacuum states, which are exactly stable in the low energy theory. We repeat that in the low energy theory we sum over superselection sectors. This is “forbidden” by the “axioms” of field theory, but in our model the high energy sector provides a dynamical rationale for summing over these sectors. The full theory satisfies all relevant axioms.

3. Cosmological Speculations

The model that we have discussed so far would solve the strong CP problem in an eternal flat world in the absence of gravitation. In the real world, an expanding universe which has undoubtedly been at a temperature below the QCD scale for only a finite time, one must ask whether the system we have described will ever find its true ground state. The answer to this question may involve very complicated, perhaps chaotic or spin glass-like dynamics, and/or be connected to other deep cosmological puzzles. At the present time we have no clear picture of how the model behaves. We will therefore simply suggest some possible scenarios and leave a more serious discussion for future work.

The first scenario is the simplest, and adheres most closely to the standard discussion of axion cosmology. It should be applicable for at least some range of parameters in our model. In this standard scenario we assume that the axion field is sufficiently weakly coupled that after inflation it simply falls into one of its classical vacua over regions much larger than the entire universe visible to us today. It is easy to argue that when $\theta_{\text{eff}} = (qa + \theta_2) \mod 2\pi$ is very small, the fraction of vacua with $\theta < \theta_{\text{eff}}$ is linear in $\theta_{\text{eff}}$, so there is only one chance in $10^{-9}$ that any given region has $\theta$ small enough to be compatible with the current experimental bound on the neutron electric dipole moment. All is not lost however if we make the further assumption that the cosmological constant vanishes at the true minimum of the axion potential. We emphasize that we have no idea why this is so, but that this is the standard assumption about the cosmological constant.

Given these assumptions, the cosmology of our model is fairly standard until temperatures of order $T_c \sim \theta_1^4 \Lambda_{QCD}$. In particular, since the axion potential is of order $\Lambda_{H}^4$, the
energy density is not dominated by nonrelativistic axions which overclose the universe. Instead, cosmology follows the standard Robertson Walker scenario until the universe reaches temperatures of order $T_c$. At this point, the universe becomes cosmological constant dominated, with a cosmological constant $\sim \theta^2 \Lambda^4_{QCD}$. Weinberg has shown that a positive cosmological constant greater than about $10^3$ times the present observational limit will prevent the formation of galaxies. Thus, the only regions in which galaxy formation can take place are those in which

$$\theta^2 \Lambda^4_{QCD} \leq 10^{-9} eV^4 \quad (3.1)$$

Since $\Lambda_{QCD} \sim 10^8 eV$, regions containing galaxies have $\theta \leq 10^{-20}$. In other words, conventional inflationary axion dynamics, coupled with the standard assumption that the cosmological constant vanishes at the absolute minimum of the potential, implies that the only regions in our model universe which contain galaxies are those with $\theta$ much smaller than the bound from the neutron electric dipole moment. This seems to us to be a reasonably attractive resolution of the strong CP problem. Perhaps its greatest defect is that only about $1/30$ of the metastable domains containing galaxies will have a cosmological constant consistent with the present limit (despite the fact that we have fine tuned the true ground state energy to zero).

We are not at all sure that our model behaves as we have described in the previous paragraph. Our alternative scenarios are harder to analyze and none seem to work very well. They involve the assumption that the presently observable piece of the universe consists of multiple domains in which the effective value of $\theta$ is different. One must then analyze the dynamics of these domains as the temperature falls below the QCD scale. Those with very small values of $\theta$ are energetically favored, and begin to expand relative to the others. On the other hand, those with higher values of $\theta$ become dominated by their cosmological constants and expand exponentially. There are possible contributions to the energy density from domain walls, and nonrelativistic axion gases. The situation is made more complicated by the bizarre nature of the potential. States that are close in energy are far away in field space. Even a single classical variable with such a potential has

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4 At least, it is possible to choose a wide range of parameters for which the axion lifetime is short enough that this problem does not arise.

5 Here we assume that the bubbles are larger than their critical size when the QCD temperature is reached. The critical size is quite large because the surface tension in the domain walls is determined by the heavy scale $\Lambda_H$. 

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chaotic behavior and we are dealing with a field theory full of such degrees of freedom. At the present time we believe that the expansion of large $\theta$ domains due to their cosmological constants is the dominant effect. It is hard to see how a universe built in such a manner could resemble our own. Nonetheless, we feel that these complicated scenarios should be understood more fully. The wild speculation that the axion domains in such a chaotic system might have something to do with the foamlike large scale structure that has been recently observed \[5\] is immensely attractive. Indeed, because of the scarcity of states with small $\theta$ one might imagine that the most probable multiple domain configurations with galaxies would have some domains just above and some just below the Weinberg bound. It is amusing to speculate that the famous voids in Bootes and other parts of the sky are regions in which the effective cosmological constant was too large to allow for galaxy formation. For lack of talent and insight, we will have to leave such cosmic fantasies for a future publication.

Another speculative application of the irrational axion idea is to an anthropic solution of the cosmological constant problem.\[6\] Imagine that in some version of supergravity it is natural for the cosmological constant to be at the SUSY breaking scale $M$. Now consider the SUSY version of the model of this paper with both gauge groups also at the SUSY breaking scale. The total effective potential is $M^4(K + F(a))$. $F(a)$ has a set of minima with energies that fill an interval of order 1 densely. Thus there are many vacua in which $K$ is cancelled to an accuracy sufficient to allow galaxy formation, and in a fraction $10^{-3}$ of those, the cosmological constant is as small as that observed in our universe. A typical state with small cosmological constant will not be sufficiently metastable to serve as a model for our universe, however some fraction of these vacua will be.

4. Conclusions

In this paper we provide an example of a flat space field theory which solves the strong CP problem without massless particles of any kind. The model violates the usual decoupling theorems in an interesting way. The low energy theory has a discrete global variable, $n$, labeling a set of quasi degenerate vacua. The dynamics that allows these vacua to transform into one another and settle down into the true ground state cannot be

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\[6\] Some time ago, L. Abbott \[6\] suggested a scheme for cancelling the cosmological constant also involving axions. Unlike the present proposal, it was necessary to introduce an extremely small energy scale, and the low energy theory contained a light particle.
understood without appealing to the high energy theory. The decoupling theorems are still valid in the weak sense that local dynamics in each metastable ground state is described completely in terms of the low energy Lagrangian. The discrete variable does allow us to evade the usual argument connecting the strong CP problem to the \( U(1) \) problem.

On the negative side, the basic Lagrangian of our model is nonrenormalizable, and it is easy to show that it cannot be the effective theory of any renormalizable field theory. In addition we have not been able to find a string compactification which leads to an axion with such irrational couplings. Thus, the fundamental basis for our model remains obscure. A possible origin for the irrational couplings central to our model may be found in the novel nonperturbative behavior of string theory that has been pointed out by Shenker \[7\]. He argued that intrinsically stringy nonperturbative effects will behave as \( e^{-\frac{1}{g}} \) instead of the \( e^{-\frac{1}{g^2}} \) characteristic of field theory, and has speculated that this behavior could be understood in terms of “instantons of continuous topological charge” \[8\]. If such instantons indeed exist in string theory, they might provide the irrational axion couplings that we require.

The general features of our mechanism may be applicable to other fine tuning problems in particle physics. Its fundamental characteristic is that it allows us to turn couplings into dynamical variables without invoking massless particles\[7\]. One need only have a high energy field with a discrete set of degenerate ground states whose integer label is irrationally related to a parameter in the low energy Lagrangian (so that the parameter can be given a dense set of values by appropriate choice of the integer). Low energy dynamics resolves the degeneracy and high energy processes mediate the transitions between states with different values of the effective coupling. We have already described a crude version of how such a mechanism might help us to understand the cosmological constant problem. It is to be hoped that one can do better than this. In string theory, our mechanism might help to resolve the problem of determining the string coupling. Conventionally it is said that any potential which allows the fine structure constant to be weak, and does not force it to vary significantly over geological time scales, implies the existence of a scalar particle of very small mass \[9\]. Since string theory also determines the couplings of this particle to be about gravitational strength, it is ruled out by astrophysical considerations. We now envisage the possibility of generating an “irrational” potential for the string coupling, which could determine it (or allow it to be set as an initial condition for our part of the

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\[7\] Or wormholes!
universe) without requiring any massless particles. This exciting idea is also left for future work.

We do not know whether we have found a solution to the strong CP problem in the real world. Conventional assumptions about axion dynamics in an inflationary cosmology, and about the value of the cosmological constant, lead to a correlation between the existence of galaxies in the observable part of the universe and the fact that $\theta$ is so small. Alternative assumptions about the spatial configuration of the axion field in our universe might lead to an explanation of foam like large scale structure, and great voids. At present, these two pictures do not seem compatible with each other, and the second probably leads to a highly inhomogeneous universe. However, our present understanding of the cosmology of this model is such that we can still hope for the best of all possible worlds.

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References

[1] A. Nelson, Phys.Lett. 136B (1984) 387; S. Barr, Phys. Rev. Lett. 53 (1984) 329, Phys. Rev. D30 (1984) 1805; R. Mohapatra and K. Babu, Phys. Rev. D41 (1990) 1286
[2] I. Niven, Numbers: Rational and Irrational, Random House, New York, p.94
[3] C. Vafa and E. Witten, Phys. Rev. Lett. 53 (1984) 535.
[4] S. Weinberg, Phys. Rev. Lett. 59 (1987) 2607.
[5] H. Rood, Ann. Rev. Astron. Astrophys. 26 (1988) 245, and references cited therein.
[6] L. Abbott, Phys. Lett. 150B (1985) 427.
[7] S. Shenker, in the Proc. of the Cargese meeting, Random Surfaces, Quantum Gravity and Strings, (1990).
[8] S. Shenker, Private Communication.
[9] M. Dine and N. Seiberg, Phys. Lett. 162B (1985) 299.