STRONGLY RIGID METRICS IN SPACES OF METRICS

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Abstract. A metric space is said to be strongly rigid if no positive distance is taken twice by the metric. In 1972, Janos proved that a separable metrizable space has a strongly rigid metric if and only if it is zero-dimensional. In this paper, we shall develop this result for the theory of space of metrics. For a strongly zero-dimensional metrizable space, we prove that the set of all strongly rigid metrics is dense in the space of metrics. Moreover, if the space is the union of countable compact subspaces, then that set is comeager. As its consequence, we show that for a strongly zero-dimensional metrizable space, the set of all metrics possessing no nontrivial (bijective) self-isometry is comeager in the space of metrics.

1. Introduction

1.1. Background. Let $X$ be a topological space, and $S$ a subset of $[0, \infty)$ with $0 \in S$. We denote by $\text{Met}(X; S)$ the set of all metrics on $X$ taking values in $S$ and generating the same topology of $X$. We also denote by $D_X$ the supremum metric on $\text{Met}(X; S)$; namely, $D_X(d, e) = \sup_{x, y \in X} |d(x, y) - e(x, y)|$. We often write $\text{Met}(X) = \text{Met}(X; [0, \infty))$. Remark that $D_X$ is a metric taking values in $[0, \infty]$. As is the case of ordinary metric spaces, we can introduce the topology on $\text{Met}(X)$ generated by open balls. In what follows, we consider that $\text{Met}(X)$ is equipped with this topology. The author [6, 7, 8, 10] proved the denseness and determined the Borel hierarchy of a subset of $\text{Met}(X)$ which can be represented as $\{ d \in \text{Met}(X) \mid d \text{ satisfies } P \}$ for some property $P$ on metric spaces under certain conditions. For example, in [8], the author proved that the set of all doubling metrics in $\text{Met}(X)$ is dense and $F_\sigma$ for every compact metrizable space $X$.

A metric $d$ on a set $X$ is said to be strongly rigid if for all $x, y, u, v \in X$, the relations $d(x, y) = d(u, v)$ and $d(x, y) \neq 0$ imply $\{x, y\} = \{u, v\}$.

In 1972, Janos [11] proved that a separable metric space $X$ is strongly 0-dimensional if and only if there exists a strongly rigid metric $d \in S$. 
The existence of strongly rigid metrics affected research on a characterization of the dimension of metrizable spaces using values of metrics (see, for example, [1], [12] and [1]).

A topological space $X$ is said to be strongly 0-dimensional if for all pair $A, B$ of disjoint closed subsets of $X$, there exists a clopen subset $V$ of $X$ such that $A \subset V$ and $V \cap B = \emptyset$. Such a space is sometimes said to be ultranormal.

In this paper, we develop the result on the existence of strongly rigid metrics for the theory of spaces of metrics. For a strongly zero-dimensional metrizable space, we prove that the set of all strongly rigid metrics is dense in the space of metrics. Moreover, if the space is $\sigma$-compact, then that set is $G_\delta$. As its consequence, we show that for a strongly zero-dimensional metrizable space, the set of all metrics possessing no nontrivial (bijective) self-isometry is comeager in the space of metrics.

1.2. Main results. The symbol “$c$” stands for the cardinality of the continuum. For a set $S$, we denote by $\text{Card}(S)$ the cardinality of $S$. A subset of a topological space is said to be $G_\delta$ if it is the intersection of countable open subsets.

Let $X$ be a metrizable space. We denote by $\text{LI}(X)$ the set of all metrics $d$ such that if $x, y, u, v \in X$ satisfies $x \neq y, u \neq v$, and ${x, y} \neq {u, v}$, then $d(x, y)$ and $d(u, v)$ are linearly independent over $\mathbb{Q}$. The following is our first result:

**Theorem 1.1.** Let $X$ be a strongly 0-dimensional metrizable space with $\text{Card}(X) \leq c$. Let $\epsilon \in (0, \infty)$ and $d \in \text{Met}(X)$. Then there exists $e \in \text{LI}(X)$ such that $D_X(d, e) \leq \epsilon$. Namely, the set $\text{LI}(X)$ is dense in $(\text{Met}(X), D_X)$. Moreover, if $X$ is completely metrizable, we can choose $e$ as a complete metric.

We denote by $\text{SR}(X)$ the set of all strongly rigid metrics in $\text{Met}(X)$. As a consequence of Theorem 1.1, we obtain our second result:

**Theorem 1.2.** Let $X$ be a strongly 0-dimensional metrizable space with $\text{Card}(X) \leq c$. Then the set $\text{SR}(X)$ is dense in $\text{Met}(X)$. Moreover, if $X$ is $\sigma$-compact, then $\text{SR}(X)$ is dense $G_\delta$ in $(\text{Met}(X), D_X)$.

**Remark 1.1.** Note that Theorem 1.2 is true even if $X$ is not locally compact. For example, Theorem 1.2 is true for $X = \mathbb{Q}$.

**Remark 1.2.** Our Theorem 1.2 can be considered as an analogue of Rouyer’s result that generic metric spaces in the Gromov–Hausdorff space are strongly rigid [20, Theorem 2]. Rouyer uses the term “totally anisometric” instead of “strongly rigid”.

We say that a metric $d$ on a set $X$ is said to be rigid if every bijective isometry $f$: $(X, d) \rightarrow (X, d)$ must be the identity map. We denote by $\text{R}(X)$ the set of all rigid metrics in $\text{Met}(X)$.
Let $X$ be a topological space. A subset $S$ of $X$ is said to be \textit{comeager} if $S$ contains a dense $G_δ$ subset of $X$. As an application of Theorem 1.1, we obtain our third result:

**Theorem 1.3.** Let $X$ be a strongly 0-dimensional metrizable space. If $X$ is $σ$-compact and satisfies $3 \leq \text{Card}(X) \leq c$, then the set $R(X)$ is comeager in $(\text{Met}(X), \mathcal{D}_X)$.

**Remark 1.3.** Our Theorem 1.3 can be considered as a 0-dimensional analogue of the result that the set of all rigid Riemannian (or pseudo-Riemannian) metrics is open dense in the space of Riemannian metrics with respect to the Whitney $C^∞$-topology (see [2] and [15]).

As a consequence of Theorem 1.2, we know the existence of strange metrics on $σ$-compact locally compact Hausdorff spaces.

**Theorem 1.4.** Let $X$ be a strongly zero-dimensional $σ$-compact locally compact Hausdorff space. Then there exists $d \in \text{Met}(X)$ such that for every $ξ \in X$, the map $F_ξ : X \to [0, ∞)$ defined by $F_ξ(x) = d(x, ξ)$ is a topological embedding.

The organization of this paper is as follows: In Section 2, we introduce a ubiquitously dense subset of a metrizable space, and we construct a strongly rigid discrete metric taking values in a given ubiquitously dense subset of $[0, ∞)$. In Section 3, we give a system of linearly independent real numbers over $\mathbb{Q}$ using a bijection between $\mathbb{Z}_{≥0}$ and $\mathbb{Q}_{≥0}$. The argument in this section is devoted to show Lemma 4.18, which is an important part of the proof of Theorem 1.1. In Section 4, we first construct a strongly rigid metric on the countable power $N(Ω)$ of the discrete space $Ω$ with $\text{Card}(Ω) = c$. Since every strongly 0-dimensional metrizable space $X$ with $\text{Card}(X) ≤ c$ can be topologically embedded into $N(Ω)$, we can obtain a strongly rigid metric on $X$. In Section 5, we prove Theorems 1.1, 1.2, 1.3, and 1.4.

## 2. A construction of strongly rigid discrete metrics

In this paper, we say that a metric is \textit{discrete} if it generates the discrete topology. We first give a construction of strongly rigid metrics on discrete spaces. We begin with the following basic proposition on the triangle inequality.

**Proposition 2.1.** Let $N_1, N_2, N_3 \in \mathbb{Z}_{≥1}$. We assume that $M_1, M_2, M_3 \in (0, ∞)$ satisfy $M_i \in (N_i + 2^{−N_i−1}, N_i + 2^{−N_i})$ for all $i \in \{1, 2, 3\}$. If $N_1 ≤ N_2 + N_3$, then we have $M_1 < M_2 + M_3$.

**Proof.** We may assume that $N_2 ≤ N_3$. If $N_1 < N_2 + N_3$, then we have $M_1 < N_1 + 1 ≤ N_2 + N_3 < M_2 + M_3$. 

In the case of $N_1 = N_2 + N_3$, we have

$$M_1 < N_1 + 2^{-N_1} = N_1 + 2^{-(N_2+N_3)} \leq N_1 + 2^{-N_3}$$

$$= N_1 + 2^{-N_3} - 1 + 2^{-N_3} \leq N_2 + N_3 + 2^{-N_2} - 1 + 2^{-N_3}$$

$$= (N_2 + 2^{-N_2}) + (N_3 + 2^{-N_3}) < M_2 + M_3.$$  

Thus, we conclude that $M_1 < M_2 + M_3$.  

**Remark 2.1.** A crucial point of Proposition 2.1 is that we can choose $M_i$ depending only on $N_i$.

**Definition 2.1.** Let $X$ be a metrizable space. We say that a subset $S$ of $X$ is *ubiquitously dense* if for every non-empty open subset $U$ of $X$, we have $\text{Card}(U \cap S) = \text{Card}(S)$.

Note that if there exists a ubiquitously dense subset $S$ of a metrizable space $X$ with $1 < \text{Card}(X)$, then $S$ should be infinite and the space $X$ has no isolated points.

In this paper, we use the set-theoretic representation of cardinal. For example, the relation $\alpha < \kappa$ means $\alpha \not= \kappa$ and $\alpha \in \kappa$, and we have $\kappa = \{ \alpha \mid \alpha < \kappa \}$. For more discussion, we refer the readers to [13].

**Theorem 2.2.** Let $X$ be a separable metrizable space with $1 < \text{Card}(X)$ and $S$ a ubiquitously dense subset of $X$. Put $\kappa = \text{Card}(S)$. Then there exists a family $\{A(\alpha)\}_{\alpha < \kappa}$ of subsets of $X$ such that

1. each $A(\alpha)$ is countable;
2. each $A(\alpha)$ is dense in $X$;
3. if $\alpha, \beta \in \kappa$ satisfy $\alpha \not= \beta$, then $A(\alpha) \cap A(\beta) = \emptyset$;
4. $S = \bigcup_{\alpha < \kappa} A(\alpha)$.

**Proof.** Notice that $\aleph_0 \leq \kappa$. By $\kappa \times \aleph_0 = \kappa$, there exists a (strictly) linear order $\prec$ such that $(\kappa \times \mathbb{Z}_{\geq 0}, \prec)$ and $(\kappa, \prec)$ are isomorphic to each other as ordered sets. Since $X$ is separable, there exists a countable open base $\{U_i\}_{i \in \mathbb{Z}_{\geq 0}}$ of $X$. Using transfinite induction, we shall define a sequence $\{q(\alpha, i)\}_{(\alpha, i) \in \kappa \times \mathbb{Z}_{\geq 0}}$ such that

(A) if $(\alpha, i), (\beta, j) \in \kappa \times \mathbb{Z}_{\geq 0}$ satisfy $(\alpha, i) \not= (\beta, j)$, then $q(\alpha, i) \not= q(\beta, j)$;

(B) for all $(\alpha, i) \in \kappa \times \mathbb{Z}_{\geq 0}$, we have $q(\alpha, i) \in U_i \cap S$.

We assume that we have already obtained a sequence $\{q(\alpha, i)\}_{(\alpha, i) \prec (\theta, n)}$ satisfying the conditions [A] and [B] for the set $\{(\alpha, i) \mid (\alpha, i) \prec (\theta, n)\}$ instead of $\kappa \times \mathbb{Z}_{\geq 0}$. We shall construct $q(\theta, n) \in X$. Put $B = \{q(\alpha, i) \mid (\alpha, i) \prec (\theta, n)\}$. Since $(\kappa \times \mathbb{Z}_{\geq 0}, \prec)$ is isomorphic to $(\kappa, \prec)$, we have $\text{Card}(B) < \kappa$. Thus, from $\text{Card}(U_n \cap S) = \kappa$, it follows that $(U_n \cap S) \setminus B \not= \emptyset$. Take $q(\theta, n) \in (U_n \cap S) \setminus B$. Then $\{q(\theta, n)\} \cup B$ satisfies the conditions (A) and (B) for the set $\{\theta, n\} \cup \{(\alpha, i) \mid (\alpha, i) \prec (\theta, n)\}$ instead of $\kappa \times \mathbb{Z}_{\geq 0}$. Therefore, by transfinite induction, we obtain a sequence $\{q(\alpha, i)\}_{(\alpha, i) \in \kappa \times \mathbb{Z}_{\geq 0}}$ satisfying the condition (A) and (B).
For each $\alpha \in \kappa$, we define $B(\alpha) = \{ q(\alpha, i) \mid i \in \mathbb{Z}_{\geq 0} \}$. Then the conditions (A) and (B) imply that the family $\{ B(\alpha) \}_{\alpha < \kappa}$ satisfies the following conditions:

(a) each $B(\alpha)$ is a subset of $S$;
(b) if $\alpha, \beta \in c$ satisfy $\alpha \neq \beta$, then $B(\alpha) \cap B(\beta) = \emptyset$;
(c) each $B(\alpha)$ is countable and dense in $X$.

We put $\tau = \text{Card}(S \setminus \bigcup_{\alpha < \kappa} B(\alpha))$ and we take a bijection $\xi : \tau \to S \setminus \bigcup_{\alpha < \kappa} B(\alpha)$. Notice that $\tau \leq \kappa$. For each $\alpha < \kappa$, we define $A(\alpha)$ by $A(\alpha) = B(\alpha) \cup \{ \xi(\alpha) \}$ if $\alpha \tau$; otherwise, $A(\alpha) = B(\alpha)$. Then the family $\{ A(\alpha) \}_{\alpha < \kappa}$ is a desired one. \hfill \Box

The following lemma is identical with [10, Proposition 2.5], which is related to metric-preserving functions.

**Lemma 2.3.** Let $X$ be a discrete topological space. Let $\eta \in (0, \infty)$ and $d \in \text{Met}(X)$. Then, there exists a metric $e \in \text{Met}(X; \eta \cdot \mathbb{Z})$ such that $\mathcal{D}_X(d, e) \leq \eta$ and $\eta \leq e(x, y)$ for all distinct $x, y \in X$.

For a set $X$, we define $[X]^2$ by $[X]^2 = \{ \{ x, y \} \mid x, y \in X, x \neq y \}$. Note that if $X$ is infinite, we have $\text{Card}(X) = \text{Card}([X]^2)$. A metric $d$ on a set $X$ is said to be uniformly discrete if there exists $c \in (0, \infty)$ such that $c < d(x, y)$ for all distinct $x, y \in X$.

**Theorem 2.4.** Let $S$ be a ubiquitously dense subset of $[0, \infty)$ with $0 \in S$ and $\kappa = \text{Card}(S)$. Let $X$ be a discrete space with $\text{Card}(X) \leq \kappa$ and $d \in \text{Met}(X)$. If $\varepsilon \in (0, \infty)$, then there exists a metric $e \in \text{Met}(X; S)$ such that

1. we have $\mathcal{D}_X(d, e) \leq \varepsilon$;
2. the metric $e$ is uniformly discrete;
3. we have $e(x, y) < e(x, z) + e(z, y)$ for all distinct $x, y, z \in X$;
4. the metric $e$ is strongly rigid.

**Proof.** We put $\eta = \varepsilon/2$ and $T = \eta^{-1} \cdot S = \{ \eta^{-1} s \mid s \in S \}$. Note that $T$ is ubiquitously dense in $[0, \infty)$ and $0 \in T$.

Lemma 2.3 guarantees the existence of a metric $h \in \text{Met}(X; \eta \cdot \mathbb{Z}_{\geq 0})$ such that $\mathcal{D}_X(h, e) \leq \eta$. Put $u = \eta^{-1} \cdot h \in \text{Met}(X; \mathbb{Z}_{\geq 0})$. Due to Theorem 2.2, we can take a mutually disjoint dense decomposition $\{ A(\alpha) \}_{\alpha < \kappa}$ of $T$. Put $\tau = \text{Card}(X)$. Then $\tau \leq \kappa$. We take a bijection $\varphi : \tau \to [X]^2$. We represent $\varphi(\alpha) = (x_\alpha, y_\alpha)$ and $\varphi^{-1}(\{ x, y \})$. For each $\alpha \in \tau$, we put $N_\alpha = u(x_\alpha, y_\alpha) \in \mathbb{Z}_{\geq 1}$ and take $w(\alpha) \in (N_\alpha + 2^{-N_\alpha}, N_\alpha + 2^{-N_\alpha}) \cap A(\alpha)$. Since each $A(\alpha)$ is dense in $[0, \infty)$, the existence of $w(\alpha)$ is always guaranteed.

We define a function $v : X^2 \to [0, \infty)$ by $v(x, y) = w(\theta(x, y))$ if $x \neq y$; otherwise, $v(x, x) = 0$. According to Proposition 2.1, the function $v : X^2 \to [0, \infty)$ satisfies the triangle inequality. Since $1 + 2^{-2} \leq v(x, y)$ for all distinct $x, y \in X$, the metric $v$ generates the discrete topology on $X$, namely, $v \in \text{Met}(X; T)$. Notice that $\mathcal{D}_X(u, v) \leq 1$. 


Put \( e = \eta \cdot v \). Since \( v \) is in \( \text{Met}(X) \), so is \( e \). By \( v \in \text{Met}(X; T) \) and \( T = \eta^{-1} \cdot S \), we have \( e \in \text{Met}(X; S) \).

Using \( h = \eta \cdot u, e = \eta \cdot v, \text{D}_X(d, h) \leq \eta, \) and \( \text{D}_X(u, v) \leq 1 \), we obtain

\[
\text{D}_X(d, e) \leq \text{D}_X(d, h) + \text{D}_X(h, e) = \text{D}_X(d, h) + \text{D}_X(\eta \cdot u, \eta \cdot v)
\]

\[
\leq \eta + \eta \text{D}_X(u, v) \leq 2 \eta = \varepsilon.
\]

This implies the condition (1).

Since \( 1 + 2^{-2} \leq v(x, y) \) for all distinct \( x, y \in X \), we observe that \( e(= \eta \cdot v) \) is uniformly discrete. This means that the condition (2) is true.

From Proposition 2.1, it follows that \( e(x, y) < e(x, z) + e(z, y) \) for all distinct \( x, y, z \in X \). This proves the condition (3).

We shall prove \( e \) is strongly rigid. Take \( \{ x, y \}, \{ a, b \} \in [X]^2 \) with \( \{ x, y \} \neq \{ a, b \} \). Then \( \theta_{\{ x,y \}} \neq \theta_{\{ a,b \}} \) and \( A(\theta_{\{ x,y \}}) \cap A(\theta_{\{ a,b \}}) = \emptyset \). In particular, we have \( w(\theta_{\{ x,y \}}) \neq w(\theta_{\{ a,b \}}) \), and hence \( e(x, y) \neq e(a, b) \). Namely, the metric \( e \) is strongly rigid. This implies that \( e \) satisfies the condition (4). □

Remark 2.2. In the case where \( X \) is finite, Theorem 2.4 gives a new proof of [20, Lemma 3] stating that the set of finite metric spaces which are totally anisometric (strongly rigid) and without collinear points is dense in the Gromov–Hausdorff space.

3. A SYSTEM OF LINEARLY INDEPENDENT NUMBERS OVER \( \mathbb{Q} \)

In this subsection, we give a system yielding real numbers which are linearly independent over \( \mathbb{Q} \).

Definition 3.1. Let \( \alpha = \{ a_i \}_{i \in \mathbb{Z}_{\geq 0}} \) be a summable sequence of positive real numbers. Let \( Q: \mathbb{Z}_{\geq 0} \to \mathbb{Q}_{\geq 0} \) be a bijection. For a non-empty subset \( B \) of \( \mathbb{Q}_{\geq 0} \), we define

\[
\Sigma_Q[\alpha, B] = \sum_{Q(i) \in B} a_i.
\]

If \( B = \emptyset \), we also define \( \Sigma_Q[\alpha, \emptyset] = 0 \).

Definition 3.2. We denote by \( Q \) the set of all \( F: \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0} \) such that \( F \) is strictly increasing and satisfies

\[
\lim_{n \to \infty} (F(n + 1) - F(n)) = \infty,
\]

and

\[
\lim_{n \to \infty} \sum_{m=n+1}^{\infty} 2^{F(n)-F(m)} = 0.
\]

The author is inspired by [5] (see also [17] and [16]) with respect to a construction of linearly independent real numbers over \( \mathbb{Q} \) in the following proposition:
Proposition 3.1. Let $F \in Q$. We define a sequence $\lambda = \{\lambda_i\}_{i \in \mathbb{Z}_+}$ by $\lambda_i = 2^{-F(i)}$. Let $k \in \mathbb{Z}_+ \geq 0$ and $\{P_i\}_{i=0}^k$ a family of subsets of $Q_{\geq 0}$. Let $S$ be a subset of $Q_{\geq 0}$. We assume that there exist $a, b_0, \ldots, b_k$ in $[0, \infty)$ such that

(I) $a < b_i$ for all $i \in \{0, \ldots, k\}$;

(II) if $i \neq j$, then $b_i \neq b_j$;

(III) we have $S \cap P_i = [a, b_i) \cap Q_{\geq 0}$ for all $i \in \{0, \ldots, k\}$.

Then the $(k + 2)$-many numbers $\Sigma_Q[\lambda, P_0], \ldots, \Sigma_Q[\lambda, P_k]$, and 1 are linearly independent over $Q$.

Proof. We may assume that $b_i < b_{i+1}$ for all $i \in \{0, \ldots, k-1\}$. For every $i \in \{0, \ldots, k\}$, we define $r(i) = \Sigma_Q[\lambda, P_i]$. To prove the proposition, we assume that integers $c_0, \ldots, c_k, c_{k+1} \in \mathbb{Z}$ satisfy

$$(3.1) \quad (c_0 \cdot r(0) + \cdots + c_k \cdot r(k) + c_{k+1} \cdot 1 = 0).$$

We first prove $c_k = 0$. Since $S \cap P_k = [a, b_k) \cap Q$ and $b_{k-1} < b_k$, we observe that the set $P_k \setminus \bigcup_{i=0}^{k-1} P_i$ is infinite. Thus, by $F \in Q$, we can take a sufficient large $n \in \mathbb{Z}_{\geq 0}$ such that

1. $(|c_0| + \cdots + |c_k|) \sum_{m=n+1}^{\infty} 2^{F(n)-F(m)} < 1.$

2. $n \in P_k$

3. we have $n \not\in \bigcup_{i=0}^{k-1} P_i$.

4. $|c_k| < 2^{F(n)-F(n-1)}$.

Put

$$I = c_0 \sum_{j \in P_k \cap [0, n]} 2^{F(n)-F(j)} + \cdots + c_k \sum_{j \in P_k \cap [0, n]} 2^{F(n)-F(j)} + c_{k+1} 2^{F(n)}.$$ 

and

$$J = c_0 \sum_{j \in P_k \cap (n, \infty)} 2^{F(n)-F(j)} + \cdots + c_k \sum_{j \in P_k \cap (n, \infty)} 2^{F(n)-F(j)}.$$ 

Then the equality (3.1) implies

$$I + J = 2^{F(n)} \times (c_0 r(0) + \cdots + c_k r(k) + c_{k+1}) = 0.$$

By [1] we have $|J| < 1$. Note that $I \in \mathbb{Z}$ (if $j \leq n$, then $F(n) - F(j) \geq 0$). By $J = -I$, we obtain $J \in \mathbb{Z}$. Combining $J \in \mathbb{Z}$ and $|J| < 1$, we conclude that $J = 0$. Thus, we also have $I = 0$.

Since for all $j \leq n$, we have $2^{F(n) - F(j)}$ can be divided by $2^{F(n) - F(n-1)}$. Thus, by $n \in P_k$, we have

$$(3.2) \quad \sum_{j \in P_k \cap [0, n]} 2^{F(n)-F(j)} = 1 + L_k \cdot 2^{F(n)-F(n-1)}$$

for some $L_k \in \mathbb{Z}$. Due to [3] for all $i$ with $i < k$, we have $n \not\in P_i$. Hence

$$(3.3) \quad \sum_{j \in P_k \cap [0, n]} 2^{F(n)-F(j)} = L_i \cdot 2^{F(n)-F(n-1)}$$
for some $L_i \in \mathbb{Z}$. Since $c_{k+1}$ is an integer, and since $F(n) - F(n-1) < F(n)$,

$$\tag{3.4} c_{k+1}2^{F(n)} = L_{k+1} \cdot 2^{F(n)-F(n-1)}$$

for some $L_{k+1} \in \mathbb{Z}$ (of cause, $L_{k+1} = c_{k+1}2^{F(n-1)}$).

From $I = 0$, and the equalities (3.2), (3.3), and (3.4), it follows that

$$c_k = M \cdot 2^{F(n)} - F(n-1)$$

for some $M \in \mathbb{Z}$. By $|c_k| < 2^{F(n)-F(n-1)}$ (the condition (4)), we conclude that $M = 0$. Thus $c_k = 0$. Using the same argument, by induction, we obtain $c_k = c_{k-1} = \cdots = c_0 = 0$. Hence $c_{k+1} = 0$. This means that the numbers $\Sigma_{Q}[\lambda, P_0], \ldots, \Sigma_{Q}[\lambda, P_k]$ and 1 are linearly independent over $\mathbb{Q}$.

\[\square\]

4. A CONSTRUCTION OF STRONGLY RIGID METRICS

In this subsection, we construct a strongly rigid metric on a given strongly 0-dimensional metrizable space.

The following proposition is deduced from the existence and uniqueness of the binary representation of a real number with infinitely many digits which are 1. We omit the proof.

**Proposition 4.1.** Let $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ be an injective map and define a sequence $\lambda = \{\lambda_i\}_{i \in \mathbb{Z}_{\geq 0}}$ by $\lambda_i = 2^{-f(i)}$. Fix a bijection $Q : \mathbb{Z}_{\geq 0} \to \mathbb{Q}_{\geq 0}$. Let $S$ and $T$ be infinite or empty subsets of $\mathbb{Q}_{\geq 0}$. If $\Sigma_{Q}[\lambda, S] = \Sigma_{Q}[\lambda, T]$, then we have $S = T$.

**Definition 4.1.** Let $k \in \mathbb{Z}_0$. We define $F_k : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ by $F_k(n) = 2^n + k$. Remark that $F_k \in \mathbb{Q}$. We also define a sequence $\zeta_{(k)} = \{\zeta_{(k),i}\}_{i \in \mathbb{Z}_{\geq 0}}$ by $\zeta_{(k),i} = 2^{-F_k(i)}$.

**Definition 4.2.** In what follows, we fix the discrete space $\Omega$ with $\text{Card}(\Omega) = c$. We define $N(\Omega) = \Omega^{\mathbb{Z}_{\geq 0}}$. We consider that the set $N(\Omega)$ is always equipped with the product topology.

Remark that the space $N(\Omega)$ is sometimes called the Baire space of weight $c$ (see [19] or [21]).

Our first purpose is to construct strongly rigid metrics on $\Omega$ and $N(\Omega)$. For this purpose, we utilize strongly rigid semi-metrics.

**Definition 4.3.** Let $S$ be a subset of $[0, \infty)$. We say that a map $r : X \times X \to [0, \infty)$ is an $S$-semi-metric on $X$ if

1. for all $x, y \in X$, we have $r(x, y) = r(y, x)$;
2. for all $x, y \in X$, we have $r(x, y) = 0$ if and only if $x = y$;
3. for all distinct $x, y \in X$, we have $r(x, y) \in S$.

The strong rigidity of an $S$-semi-metric is defined in the same way of ordinary metrics.
Note that semi-metrics are not assumed to satisfy the triangle inequality.

**Proposition 4.2.** Let $S$ be a subset $(0, \infty)$ with $\text{Card}(S) = \mathfrak{c}$. Then there exists a strongly rigid $S$-semi-metric $r$ on $\Omega$.

**Proof.** Since $\text{Card}([\Omega]^2) = \mathfrak{c}$, we can take a bijection $\phi: [\Omega]^2 \to S$. We define $r(x, y) = \phi(\{x, y\})$ if $x \neq y$, otherwise, $r(x, x) = 0$. Then $r$ is a desired one. □

**Definition 4.4.** Let $Q: \mathbb{Z}_{\geq 0} \to \mathbb{Q}_{\geq 0}$ be a bijection. We define $\mu(m) = \min Q^{-1}(\{m, m + 1\} \cap \mathbb{Q})$. We say that the map $Q$ satisfies the property (M) if

1. for all $m \in \mathbb{Z}_{\geq 0}$, we have $Q(\mu(m)) = m$;
2. for all $m \in \mathbb{Z}_{\geq 0}$, we have $\mu(m) < \mu(m + 1)$.

Notice that $\mu(0) = 0$.

**Lemma 4.3.** There exists a bijection $Q: \mathbb{Z}_{\geq 0} \to \mathbb{Q}_{\geq 0}$ satisfying the property (M).

**Proof.** Take a mutually disjoint family \(\{A_i\}_{i \in \mathbb{Z}_{\geq 0}}\) of infinite subsets of $\mathbb{Z}_{\geq 0}$ satisfying that $\bigcup_{i \in \mathbb{Z}_{\geq 0}} A_i = \mathbb{Z}_{\geq 0}$ and $\min A_i < \min A_{i+1}$ for all $i \in \mathbb{Z}_{\geq 0}$. For each $i \in \mathbb{Z}_{\geq 0}$, we take a bijection $\theta_i: A_i \to [i, i + 1) \cap \mathbb{Q}$ with $\theta_i(\min A_i) = i$. Gluing them together, we obtain a bijection $\mathbb{Z}_{\geq 0} \to \mathbb{Q}_{\geq 0}$ with the property (M). □

**Remark 4.1.** In what follows, based on Lemma 4.3, we fix the bijection $Q: \mathbb{Z}_{\geq 0} \to \mathbb{Q}_{\geq 0}$ with the property (M).

**Definition 4.5.** Fix $k \in \mathbb{Z}_{\geq 0}$. Let $m \in \mathbb{Z}_{\geq 0}$ and $r$ an $S$-semi-metric on $\Omega$. We define $H_{m, r(x, y)} = \{m, r(x, y)\} \cap \mathbb{Q}$. Note that if $x = y$, the set $H_{m, r(x, y)}$ is always empty. We also define $[r]_{k, m}: \Omega \times \Omega \to [0, \infty)$ by

$$[r]_{k, m}(x, y) = \Sigma Q(\zeta(k), H_{m, r(x, y)}].$$

**Remark 4.2.** Under the same assumptions in Definition 4.5, we notice that $H_{0, r(x, x)} = [0, 0) \cap \mathbb{Q} = \emptyset$ for all $x \in \Omega$.

**Lemma 4.4.** Fix $k \in \mathbb{Z}_{\geq 0}$. Let $m \in \mathbb{Z}_{\geq 0}$ and $S$ a subset of $(m, m + 1)$. Let $X$ be a discrete space with $\text{Card}(X) \leq \mathfrak{c}$. If $r$ is a strongly rigid $S$-semi-metric on $\Omega$, then the following statements are true:

1. We have $[r]_{k, m}(x, y) \in (\zeta(k), \mu(m); 2\zeta(k), \mu(m))$ for all distinct $x, y \in X$.
2. The function $[r]_{k, m}$ is a metric on $\Omega$ and we have $[r]_{k, m}(x, y) \in \text{Met}(\Omega)$.
3. The metric $[r]_{k, m}$ is strongly rigid.

**Proof.** We first prove the statement \([1]\). Take distinct $x, y \in \Omega$. By the property (M), we have $Q(\mu(m)) = m$ and $m \in [m, d(x, y)) \cap \mathbb{Q} = H_{m, r(x, y)}$. Thus we obtain

$$\zeta(k, \mu(m)) \leq \Sigma Q(\zeta(k), H_{m, r(x, y)}] = [r]_{k, m}(x, y).$$
Since $\mu(m)$ is the minimal number of the set $Q^{-1}(\{m, m+1\} \cap \mathbb{Q})$ (see Definition 4.4), and since $F_k$ is strictly increasing, we have

$$\Sigma_Q(\zeta(k), H_{m,r(x,y)}) = \sum_{i \in Q^{-1}(\{m,m+1\} \cap \mathbb{Q})} 2^{-F_k(i)} \leq \sum_{F_k(\mu(m)) \leq i} 2^{-i} = 2 \cdot 2^{-F_k(\mu(m))} = 2\zeta(k)\mu(m).$$

Hence $\Sigma_Q(\zeta(k), H_{m,r(x,y)}) \leq 2\zeta(k,\mu(m))$. This implies the statement (1).

We next prove the statement (2). If $x, y \in \Omega$ satisfies $x = y$, we have $H_{m,r(x,y)} = \emptyset$. Thus $[r]_{k,m}(x,y) = \Sigma_Q(\zeta(k), H_{m,r(x,y)}) = 0$. To show that $[r]_{k,m}$ satisfies the triangle inequality, we take distinct $x, y, z \in \Omega$. According to the statement (1) we have

$$[r]_{k,m}(x,y) \leq 2\zeta(k,\mu(m)) = \zeta(k,\mu(m)) + \zeta(k,\mu(m)) < [r]_{k,m}(x,z) + [r]_{k,m}(z,y).$$

Thus, the function $[r]_{k,m}$ satisfies the triangle inequality. According to the statement (1) again, the metric $[r]_{k,m}$ is uniformly discrete, and hence $[r]_{k,m} \in \text{Met}(\Omega)$. This finishes the proof of (2).

We shall prove the statement (3). We assume that $x, y, u, v \in X$ satisfy $0 < [r]_{k,m}(x,y)$ and $[r]_{k,m}(x,y) = [r]_{k,m}(u,v)$. Then we have $\Sigma_Q(\zeta(k), H_{m,r(x,y)}) = \Sigma_Q(\zeta(k), H_{m,r(u,v)})$. By Proposition 4.5, we obtain $H_{m,r(x,y)} = H_{m,r(u,v)}$. Namely, we have $[m, r(x,y)) \cap \mathbb{Q} = [m, r(u,v)) \cap \mathbb{Q}$, and hence $r(x,y) = r(u,v)$. Since $r$ is strongly rigid, we conclude that $\{x, y\} = \{u, v\}$. This means that $d$ is strongly rigid. Then the statement (3) is true. This completes the proof of the lemma. \qed

**Definition 4.6.** Let $S$ be a dense subset of $[0, \infty)$. We say that a family $\{r_i\}_{i \in \mathbb{Z}_{\geq 0}}$ is an $S$-gauge system on $\Omega$ if each $r_i$ is a strongly rigid $(S \cap (i, i+1))$-semi-metric on $\Omega$.

**Proposition 4.5.** Let $S$ be a ubiquitously dense subset in $[0, \infty)$ with $\text{Card}(S) = \mathfrak{c}$. Then, there exists an $S$-gauge system $\{r_i\}_{i \in \mathbb{Z}_{\geq 0}}$ on $\Omega$.

**Proof.** Since $S$ is ubiquitously dense, we have $\text{Card}(S \cap (i, i+1)) = \mathfrak{c}$ for all $i \in \mathbb{Z}_{\geq 0}$. Then, by Proposition 4.2, there exists a strongly rigid $(S \cap (i, i+1))$-semi-metric $r_i$ on $\Omega$. Thus, the sequence $\{r_i\}_{i \in \mathbb{Z}_{\geq 0}}$ is an $S$-gauge system on $\Omega$. \qed

**Definition 4.7.** Fix $k \in \mathbb{Z}_{\geq 0}$. Let $S$ be a ubiquitously dense subset of $[0, \infty)$ and $R = \{r_i\}_{i \in \mathbb{Z}_{\geq 0}}$ an $S$-gauge system on $\Omega$. In this case, $H_{m,r_m(x_m,y_m)} \cap H_{m',r_{m'}(x_{m'},y_{m'})} = \emptyset$ for all distinct $m, m' \in \mathbb{Z}_{\geq 0}$ and for all $x = (x_i)_{i \in \mathbb{Z}_{\geq 0}}$ and $y = (y_i)_{i \in \mathbb{Z}_{\geq 0}}$ in $N(\Omega)$. We define

$$I_{R,x,y} = \prod_{m \in \mathbb{Z}_{\geq 0}} H_{m,r_m(x_m,y_m)}.$$ 

We also define a function $[R]_k: N(\Omega) \times N(\Omega) \to [0, \infty)$ by

$$[R]_k(x,y) = \Sigma_Q(\zeta(k), I_{R,x,y}).$$
Under the same assumptions in Definition 4.7, notice that we have

$$[R]_k(x, y) = \sum_{i=0}^{\infty} [r_m]_{k,m}(x_m, y_m),$$

where $x = (x_i)_{i \in \mathbb{Z}_{\geq 0}}$ and $y = (y_i)_{i \in \mathbb{Z}_{\geq 0}}$. Since $\zeta(k)$ is summable, we have $[R]_k(x, y) < \infty$ for all $x, y \in N(\Omega)$.

**Lemma 4.6.** Fix $k \in \mathbb{Z}_{\geq 0}$. Let $S$ be a ubiquitously dense subset of $[0, \infty)$ and $R = \{r_i\}_{i \in \mathbb{Z}_{\geq 0}}$ an $S$-gauge system on $\Omega$. Let $m \in \mathbb{Z}_{\geq 0}$ and $x = (x_i)_{i \in \mathbb{Z}_{\geq 0}}, y = (y_i)_{i \in \mathbb{Z}_{\geq 0}} \in N(\Omega)$. Then the following statements hold:

(A) If $x, y \in X$ satisfies $[R]_k(x, y) \leq \zeta(k, \mu(m))$, then we have $x_i = y_i$ for all $i \in \{0, \ldots, m\}$.

(B) If $x_i = y_i$ for all $i \in \{0, \ldots, m\}$, then we have $[R]_k(x, y) \leq 4\zeta(k, \mu(m+1))$.

**Proof.** We first prove (A). For the sake of contradiction, we suppose that there exists $i \in \{0, \ldots, m\}$ such that $x_i \neq y_i$ and $[R]_k(x, y) \leq \zeta(k, \mu(m))$. Since $Q$ satisfies the property (M), we have $\mu(i) \leq \mu(m)$, and hence $\zeta(k, \mu(m)) \leq \zeta(k, \mu(i))$. By $[R]_k(x, y) = \sum_{i=0}^{\infty} [r_m]_{k,m}(x_m, y_m)$ and by $\zeta(k, \mu(m)) \leq [r_m]_{k,i}(x_i, y_i)$ (see the statement (1) in Lemma 4.4), we have $\zeta(k, \mu(m)) \leq [R]_k(x, y)$. This is a contradiction. Therefore the statement (A) is true.

We next prove (B). If $x_i = y_i$ for all $i \in \{0, \ldots, m\}$, we have $[R]_k(x, y) = \sum_{i=m+1}^{\infty} [r_m]_{k,m}(x_m, y_m)$. According to the statement (1) in Lemma 4.4, for each $i \in \mathbb{Z}_{\geq m+1}$, we have $[r_m]_{k,m}(x_m, y_m) < 2\zeta(k, \mu(i))$. Since $[R]_k(x, y) = \sum_{i=0}^{\infty} [r_m]_{k,m}(x_m, y_m)$, and since $\sum_{i=m+1}^{\infty} \zeta(k, \mu(i)) \leq 2\zeta(k, \mu(m+1))$, we obtain $[R]_k(x, y) \leq 4\zeta(k, \mu(m+1))$. This proves the statement (B).

**Proposition 4.7.** Fix $k \in \mathbb{Z}_{\geq 0}$. Let $S$ be a ubiquitously dense subset of $[0, \infty)$ and $R = \{r_i\}_{i \in \mathbb{Z}_{\geq 0}}$ an $S$-gauge system on $\Omega$. Then the function $[R]_k: N(\Omega) \times N(\Omega) \to [0, \infty)$ satisfies the following statements:

1. We have $[R]_k \in \text{Met}(N(\Omega))$.
2. We have $\text{diam}([R]_k(N(\Omega))) \leq 2 \cdot 2^{-F_k(0)} (= 2^{-k})$.
3. The metric $[R]_k$ is complete.

**Proof.** We first prove the statement (1). Since each $[r_i],k,i$ generates the same topology on $\Omega$, the statements (A) and (B) in Lemma 4.6 imply that $[R]_k$ generates the same topology on $N(\Omega)$ (recall that $N(\Omega)$ is equipped with the product topology).

We next prove the statement (2). For all $x, y \in N(\Omega)$, we have

$$[R]_k(x, y) \leq \sum_{i \in \mathbb{Z}_{\geq 0}} \zeta(k, \mathbb{Q}_{\geq 0}) = \sum_{i \in \mathbb{Z}_{\geq 0}} 2^{-F_k(i)} \leq \sum_{F_k(0) \leq j} 2^{-j} = 2 \cdot 2^{-F_k(0)}.$$

Since $F_k(0) = 1 + k$, and $[R]_k(x, y) \leq 2 \cdot 2^{-F_k(0)}$, the statement (2) is true.
To prove the statement \([3]\), we take an arbitrary Cauchy sequence \(\{a(n)\}_{n \in \mathbb{Z}_{\geq 0}}\) in \(N(\Omega)\). We put \(a(n) = \{a(n, i)\}_{i \in \mathbb{Z}_{\geq 0}}\) for all \(i \in \mathbb{Z}_{\geq 0}\) and \(n \in \mathbb{Z}_{\geq 0}\). Due to the statement \([A]\) in Lemma 4.6, for each \(i \in \mathbb{Z}_{\geq 0}\), there exists \(N_i\) such that \(a(n, i) = a(n + 1, i)\) for all \(n \in \mathbb{Z}_{\geq 0}\) with \(N_i \leq n\). We define \(b_i = a(N_i, i)\). Then the point \(b = (b_i)_{i \in \mathbb{Z}_{\geq 0}} \in N(\Omega)\) is a limit of \(\{a(n)\}_{n \in \mathbb{Z}_{\geq 0}}\). Thus, the metric \([R]_k\) is complete. This finishes the proof of the proposition.

\[\square\]

**Remark 4.3.** The metric \([R]_k\) in Definition 4.7 is not strongly rigid. Indeed, if we take \(a, b \in \Omega\) with \(a \neq b\), and define \(x, y, u, v \in N(\Omega)\) by \(x = (a, a, a, \ldots), y = (a, b, b, \ldots), u = (b, a, a, \ldots),\) and \(v = (b, b, b, \ldots)\), then we have \(\{x, y\} \neq \{u, v\}\), and \([R]_{\{x, y\}}(x, y) = [R]_{\{u, v\}}(u, v)\) for a fixed integer \(k \in \mathbb{Z}_{\geq 0}\), and for every \(S\)-gauge system \(R\) on \(\Omega\).

To obtain a strongly rigid metric on \(N(\Omega)\), we construct a topological embedding \(\Psi_F: N(\Omega) \to N(\Omega)\) such that \(\Psi_F(x)\) contains information of all finite prefixes of \(x = (x_i)_{i \in \mathbb{Z}_{\geq 0}} \in N(\Omega)\).

**Definition 4.8.** For \(n \in \mathbb{Z}_{\geq 0}\), and for a point \(x = (x_i)_{i \in \mathbb{Z}_{\geq 0}} \in N(\Omega)\), we denote by \(\pi_n(x) \in \Omega^{n+1}\) the point \((x_0, x_1, \ldots, x_n)\). For each \(i \in \mathbb{Z}_{\geq 0}\), we take an injective map \(f_i: \Omega^{i+1} \to \Omega\). Since \(\text{Card}(\Omega) = \mathfrak{c}\) and \(\mathfrak{c}^{i+1} = \mathfrak{c}\), the injective map \(f_i: \Omega^{i+1} \to \Omega\) always exists. Put \(\mathcal{F} = \{f_i\}_{i \in \mathbb{Z}_{\geq 0}}\). We define a map \(\Psi_F: N(\Omega) \to N(\Omega)\) as follows: The \(i\)-th entry \(y_i\) of \(\Psi_F(x)\) is defined by

\[
y_i = \begin{cases} 
  x_n & \text{if } i = 2n; \\
  f_n(\pi_n(x)) & \text{if } i = 2n + 1,
\end{cases}
\]

where \(x = (x_i)_{i \in \mathbb{Z}_{\geq 0}}\). In what follows, we fix the family \(\mathcal{F} = \{f_i\}_{i \in \mathbb{Z}_{\geq 0}}\).

**Lemma 4.8.** The map \(\Psi_F: N(\Omega) \to N(\Omega)\) is a topological embedding and the image of \(\Psi_F\) is closed in \(N(\Omega)\).

**Proof.** By the definition of \(\Psi_F\), we observe that \(\Psi_F\) is a topological embedding. To prove that \(\Psi_F(N(\Omega))\) is closed, we take a sequence \(\{x_i\}_{i \in \mathbb{Z}_{\geq 0}} \in \Psi_F(N(\Omega))\) such that \(x_i \to a\) as \(i \to \infty\) for some point \(a = (a_i)_{i \in \mathbb{Z}_{\geq 0}} \in N(\Omega)\). We define \(b = (b_i)_{i \in \mathbb{Z}_{\geq 0}} \in N(\Omega)\) by \(b_i = a_{2i}\). Then \(\Psi_F(b) = a\) and hence \(a \in \Psi_F(N(\Omega))\). Thus, the set \(\Psi_F(N(\Omega))\) is closed in \(N(\Omega)\).

Since for all \(x = (x_i)_{i \in \mathbb{Z}_{\geq 0}}, y = (y_i)_{i \in \mathbb{Z}_{\geq 0}} \in N(\Omega)\) we have \(x = y\) if and only if \(x_i = y_i\) for all \(i \in \mathbb{Z}_{\geq 0}\), we obtain the following lemma:

**Lemma 4.9.** Let \(l \in \mathbb{Z}_{\geq 0}\) and \(p(0), \ldots, p(l) \in N(\Omega)\). If \(p(0), \ldots, p(l)\) are mutually distinct, then there exists \(N \in \mathbb{Z}_{\geq 0}\) such that for all \(n \in \mathbb{Z}_{\geq 0}\) with \(N < n\), the points \(\pi_n(p(0)), \ldots, \pi_n(p(l))\) are mutually distinct.

As a consequence of Lemma 4.9 we obtain:

**Lemma 4.10.** Let \(l \in \mathbb{Z}_{\geq 0}\) and \(x(0), y(0), x(1), y(1), \ldots, x(l), y(l) \in N(\Omega)\). If \(x(s) \neq y(s)\) for all \(s \in \{0, \ldots, l\}\), and if \(\{x(s), y(s)\} \neq \)
{x(t), y(t)} for all distinct s, t ∈ {0, . . . , l}, then there exists N such that for all n ∈ Z_≥0 with N < n,

1. we have \{π_n(x(s)), π_n(y(s))\} \neq \{π_n(x(t)), π_n(y(t))\} for all distinct s, t ∈ {0, . . . , l},
2. we have \{π_n(x(s))\} \neq π_n(y(s)) for all s ∈ {0, . . . , l}.

**Proof.** Put \{x(s)\}_{s \in \mathbb{Z}_≥0} and \{y(s)\}_{s \in \mathbb{Z}_≥0}. For each s ∈ {0, . . . , l}, we define \(u(s) = (u_i(s))_{i \in \mathbb{Z}_≥0}\) and \(v(s) = (v_i(s))_{i \in \mathbb{Z}_≥0}\) by

\[
u_i(s) = \begin{cases} x_j(s) & \text{if } i = 2j; \\ y_j(s) & \text{if } i = 2j + 1, \end{cases}
\]

and

\[
u_i(s) = \begin{cases} y_j(s) & \text{if } i = 2j; \\ x_j(s) & \text{if } i = 2j + 1. \end{cases}
\]

The inequality \(x(s) \neq y(s)\) implies \(u(s) \neq v(s)\). For all distinct \(s, t \in \{0, . . . , l\}\), by the assumption that \(\{x(s), y(s)\} \neq \{x(t), y(t)\}\), the points \(u(s), v(s), u(t), v(t)\) are mutually distinct. Therefore the points \(u(0), v(0), . . . , u(l), v(l)\) are mutually distinct. According to Lemma \[\ref{lem:rigid-metric}\] there exists \(M \in \mathbb{Z}_≥0\) such that for all \(m \in \mathbb{Z}_≥0\) with \(M < m\), the points \(π_m(u(0)), π_m(v(0)), . . . , π_m(u(l)), π_m(v(l))\) are mutually distinct. Take \(N \in \mathbb{Z}_≥0\) with \(M \leq 2N\). Then, by the definition of \(u(s)\) and \(v(s)\), the integer \(N\) satisfies the two conditions stated in the proposition. \[\square\]

Using the metric \([R]_k\) and the embedding \(Ψ_F\), we construct a strongly rigid metric on \(N(Ω)\).

**Definition 4.9.** Let \(S\) be a subset of \([0, ∞)\) and let \(R = \{r_i\}_{i \in \mathbb{Z}_≥0}\) be an \(S\)-gauge system on \(Ω\). We define

\([R]_k(x, y) = [R]_k(Ψ_F(x), Ψ_F(y)).\]

Note that we have

\([R]_k(x, y) = \Sigma_Q[ζ(k), I_{R,Ψ_F(x),Ψ_F(y)}].\]

For a metric space \((X, d)\), and for a subset \(A\) of \(X\), we denote by \(\text{diam}_d(A)\) the diameter of \(A\) with respect to \(d\). The following theorem plays an important role to prove our main results.

**Theorem 4.11.** Fix \(k \in \mathbb{Z}_≥0\). Let \(S\) be a ubiquitously dense subset of \([0, ∞)\), and \(R = \{r_i\}_{i \in \mathbb{Z}_≥0}\) an \(S\)-gauge system on \(Ω\). Then the metric \([R]_k: N(Ω) × N(Ω) → [0, ∞)\) satisfies the following statements:

1. We have \([R]_k \in \text{Met}(N(Ω)).
2. We have \(\text{diam}_{[R]_k}(N(Ω)) ≤ 2 \cdot 2^{-F(0)} = 2^{-k}).
3. The metric \([R]_k\) is complete.
4. The metric \([R]_k\) is strongly rigid.
Lemma 4.12. There exists a family \( c \) (but the successor of \( \mathbb{C} \)) such that each \( \mathbb{C} \) \( \mathbb{K} \) satisfies the three conditions stated in the lemma. This finishes the proof of the theorem.

Proof. Since \( \Psi F : N(\Omega) \to N(\Omega) \) is a topological embedding and its image is closed (see Lemma 4.3), the statements \([1]\) and \([3]\) in Proposition 4.7 implies that \([1]\) and \([3]\) in the theorem are true.

By \([2]\) in Proposition 4.7, the statement \([2]\) is true.

We now prove the statement \([4]\) in the theorem. Take \( x, y, u, v \in N(\Omega) \) such that \( x \neq y, u \neq v \), and \( \{x, y\} \neq \{u, v\} \). Due to Lemma 4.10, we can take \( n \in \mathbb{Z}_{\geq 0} \) such that \( \{\pi_n(x), \pi_n(y)\} \neq \{\pi_u(n), \pi_n(v)\} \).

Thus, using the injectivity of \( f_n \), we have
\[
\{f_n(\pi_n(x)), f_n(\pi_n(y))\} \neq \{f_n(\pi_n(u)), f_n(\pi_n(v))\}.
\]

Thus, we have
\[
r_{2n+1}(f_n(\pi_n(x)), f_n(\pi_n(y))) \neq r_{2n+1}(f_n(\pi_n(u)), f_n(\pi_n(v))).
\]

This inequality implies that
\[
H_{2n+1,r_{2n+1}}(f_n(\pi_n(x)), f_n(\pi_n(y))) \neq H_{2n+1,r_{2n+1}}(f_n(\pi_n(u)), f_n(\pi_n(v))).
\]

Therefore, we obtain
\[
I_{R, \Psi F(x), \Psi F(y)} \neq I_{R, \Psi F(x), \Psi F(y)}.
\]

According to Proposition 4.7, we notice that \([R]_k(x, y) \neq [R]_k(u, v)\). This means that the metric \([R]_k\) is strongly rigid, and the statement \([4]\) is true. This completes the proof of the theorem.

Definition 4.10. We denote by \( \text{Suc}(c) \) the set \( c \cup \{c\} \). This is nothing but the successor of \( c \) as an ordinal.

Lemma 4.12. There exists a family \( \{K(\alpha)\}_{\alpha \in \text{Suc}(c)} \) satisfying that

1. each \( K(\alpha) \) is a subset of \([0, \infty)\);
2. each \( K(\alpha) \) is ubiquitously dense in \([0, \infty)\) and \( \text{Card}(K(\alpha)) = c \);
3. if \( \alpha, \beta \in \text{Suc}(c) \) satisfy \( \alpha \neq \beta \), then \( K(\alpha) \cap K(\beta) = \emptyset \).

Proof. The set \([0, \infty)\) is ubiquitously dense in \([0, \infty)\) and \( \text{Card}([0, \infty)) = c \). Thus, by Theorem 2.2 we obtain a mutually disjoint decomposition \( \{T(\alpha)\}_{\alpha < c} \) of \([0, \infty)\) such that each \( T(\alpha) \) is countable and dense in \([0, \infty)\). Since \( \text{Card}(\text{Suc}(c)) = c \), and since \( c \times c = c \) as cardinals, we can take a bijection \( \phi : \text{Suc}(c) \to \text{Suc}(c) \times c \), where \( \text{Suc}(c) \times c \) stands for the product as sets. Put \( \phi(\alpha) = (\theta(\alpha), \lambda(\alpha)) \). For each \( \alpha \in \text{Suc}(c) \), we define \( K(\alpha) = \bigcup_{\beta \in \text{Suc}(c), \theta(\beta) = \alpha} T(\beta) \). Then the family \( \{K(\alpha)\}_{\alpha \in \text{Suc}(c)} \) satisfies the three conditions stated in the lemma. This finishes the proof.

Definition 4.11. Fix \( k \in \mathbb{Z}_{\geq 0} \). In what follows, we fix a family \( \{K(\alpha)\}_{\alpha \in \text{Suc}(c)} \) stated in Lemma 4.12. For each \( \alpha \in \text{Suc}(c) \), we also fix a \( K(\alpha) \)-gauge system \( R(\alpha) = \{r_{i,\alpha}\}_{i \in \mathbb{Z}_{\geq 0}} \) (see Proposition 4.5).

For each \( \alpha \in \text{Suc}(c) \), we define
\[
F_{k,\alpha} = \{\|R(\alpha)\|_k(x, y) \mid x \neq y, x, y \in X\},
\]
and
\[
G_{k,\alpha} = \{0\} \cup F_{k,\alpha}.
\]
Namely, the set $G_{k,\alpha}$ is the image of the metric $[R(\alpha)]_k$ on $N(\Omega)$.

In a similar way to the proof of (4) in Theorem 4.11, we obtain Lemmas 4.13 and 4.14.

**Lemma 4.13.** Fix $k \in \mathbb{Z}_{\geq 0}$. If $\alpha, \beta \in \text{Suc}(c)$ satisfy $\alpha \neq \beta$, then $F_{k,\alpha} \cap F_{k,\beta} = \emptyset$.

**Proof.** For the sake of contradiction, we suppose that there exists $\alpha, \beta \in \text{Suc}(c)$ with $\alpha \neq \beta$ and $F_{k,\alpha} \cap F_{k,\beta} \neq \emptyset$. Take $d \in F_{k,\alpha} \cap F_{k,\beta}$ and put $d = [R(\alpha)]_k(x, y)$ and $d = [R(\beta)]_k(u, v)$, where $x, y, u, v \in N(\Omega)$, and $x \neq y$ and $u \neq v$. Take $i \in \mathbb{Z}_{\geq 0}$ such that $x_i \neq y_i$. Then $r_{i,\alpha}(x_i, y_i) \in K(\alpha) \setminus \{0\}$ and $r_{i,\beta}(u_i, v_i) \in K(\beta) \cup \{0\}$. From $K(\alpha) \cap K(\beta) = \emptyset$, it follows that $r_{i,\alpha}(x_i, y_i) \neq r_{i,\beta}(u_i, v_i)$. This implies that $H_i \cap r_{i,\alpha}(x, y) \neq H_i \cap r_{i,\beta}(u, v)$ and hence $I_{R(\alpha)} \gamma(x, y) \neq I_{R(\beta)} \gamma(u, v)$. According to Proposition 3.1, we obtain $[R(\alpha)]_k(x, y) \neq [R(\beta)]_k(u, v)$. This contradicts $d = [R(\alpha)]_k(x, y)$ and $d = [R(\beta)]_k(u, v)$. Therefore we conclude that $F_{k,\alpha} \cap F_{k,\beta} = \emptyset$ for all distinct $\alpha, \beta \in \text{Suc}(c)$.

A subset $S$ of $\mathbb{R}$ is said to be **linearly independent over** $\mathbb{Q}$ if all finite distinct elements in $S$ are linearly independent over $\mathbb{Q}$.

**Lemma 4.14.** Fix $k \in \mathbb{Z}_{\geq 0}$. Let $s, t \in \mathbb{Z}_{\geq 0}$. Let $\alpha_0, \ldots, \alpha_s \in \text{Suc}(c)$. For each $i \in \{0, \ldots, s\}$, we take $(t + 1)$-many arbitrary distinct elements $d_{i,0}, \ldots, d_{i,t} \in F_{k,\alpha_i}$. Then the set

$$
\{1\} \cup \{d_{i,j} | i \in \{0, \ldots, s\}, j \in \{0, \ldots, t\}\}
$$

is linearly independent over $\mathbb{Q}$.

**Proof.** Put $d_{i,j} = [R(\alpha_i)]_k(x(i,j), y(i,j))$, where $x(i,j), y(i,j) \in N(\Omega)$. Note that $x(i,j) \neq y(i,j)$. According to Lemma 4.10, we can take a sufficient large integer $n \in \mathbb{Z}_{\geq 0}$ such that for each $i \in \{0, \ldots, s\}$, for all distinct $j, j' \in \{0, \ldots, t\}$, we obtain the inequality

$$
\{\pi_n(x(i,j)), \pi_n(y(i,j))\} \neq \{\pi_n(x(i,j')), \pi_n(y(i,j'))\}.
$$

We put $r(i,j) = r_{2n+1,\alpha_i}(f_n(\pi_n(x(i,j))), f_n(\pi_n(y(i,j))))$. Then, for a fixed number $i \in \{0, \ldots, s\}$, for all distinct $j, j' \in \{0, \ldots, t\}$, we have $r(i,j) \neq r(i,j')$. Since $r(i,j) \in K(\alpha_i)$ for all $j \in \{0, \ldots, t\}$, and since $K(\alpha_i) \cap K(\alpha_i') = \emptyset$ for all distinct $i, i' \in \{0, \ldots, s\}$, we obtain $r(i,j) \neq r(i',j')$ for all distinct $(i,j), (i',j')$.

We put $I_{i,j} = I_{R(\alpha_i)} \gamma(x(i,j)), \gamma(y(i,j))$, and $S = [2n+1, 2n+2] \cap \mathbb{Q}$. Then $S \cap I_{i,j} = [2n+1, r(i,j)] \cap \mathbb{Q}$. Thus, by $d_{i,j} = \Sigma_{Q(\zeta(k), I_{i,j})}$, the numbers $d_{i,j} (i \in \{0, \ldots, s\}, j \in \{0, \ldots, t\})$ satisfy the assumptions of Proposition 3.1. Therefore, due to Proposition 3.1, we conclude that the set

$$
\{1\} \cup \{d_{i,j} | i \in \{0, \ldots, s\}, j \in \{0, \ldots, t\}\}
$$

is linearly independent over $\mathbb{Q}$.

\[\square\]
Corollary 4.15. Fix $k \in \mathbb{Z}_{\geq 0}$. Then the set $\{1\} \cup \bigcup_{\alpha \in \text{Suc}(c)} F_{k,\alpha}$ is linearly independent over $\mathbb{Q}$.

Definition 4.12. Fix $k \in \mathbb{Z}_{\geq 0}$. Recall that in Definition 4.11, we construct the family $\{F_{k,\alpha}\}_{\alpha \in \text{Suc}(c)}$, where $\text{Suc}(c) = c \cup \{c\}$. Using $c \times \mathbb{N} = c$, we can represent

$$F_{k,c} = \{s_{k,\alpha,i} \mid \alpha \in c, i \in \mathbb{Z}_{\geq 0}\}.$$  

We assume that if $s_{k,a,i} = s_{k,b,j}$, then $(\alpha, i) = (\beta, j)$. For each $(\alpha, i) \in c \times \mathbb{Z}_{\geq 0}$, we take $q_{k,\alpha,i} \in \mathbb{Q}_{>0}$ such that $q_{k,\alpha,i} \cdot s_{k,\alpha,i} \leq 2^{-i}$. We fix a bijection $P : \mathbb{Z}_{\geq 0} \to \mathbb{Q}_{>0}$. We define set $A_k$ and $X_k$ by

$$A_k = \{P(i) + q_{k,\alpha,i} s_{k,\alpha,i} \mid \alpha < c, i \in \mathbb{Z}_{\geq 0}\},$$

and

$$X_k = \{0\} \cup A_k,$$

respectively. In what follows, we no longer use the fact that the set $F_{k,c}$ is defined as a set of values of a metric on $N(\Omega)$. We rather use the property that the union of $F_{k,c}$ and $\{1\} \cup \bigcup_{\alpha < c} F_{k,\alpha}$ is linearly independent over $\mathbb{Q}$ (see Corollary 4.15).

Lemma 4.16. Fix $k \in \mathbb{Z}_{\geq 0}$. Then the set $X_k$ is ubiquitously dense in $[0, \infty)$ and $\text{Card}(X_k) = c$.

Proof. It suffices to show that $A_k$ is ubiquitously dense in $[0, \infty)$. Take $x \in [0, \infty)$ and $\varepsilon \in (0, \infty)$. Since $\mathbb{Q}_{>0}$ is dense in $[0, \infty)$, we can take $n \in \mathbb{Z}_{\geq 0}$ such that $|P(n) - x| < \varepsilon/2$ and $2^{-n} \leq \varepsilon/2$. Then, for all $\alpha < c$, we have

$$|P(n) + q_{k,\alpha,n} s_{k,\alpha,n} - x| \leq |P(n) - x| + |q_{k,\alpha,n} s_{k,\alpha,n}| < \varepsilon/2 + 2^{-n} \leq \varepsilon.$$  

Thus, the set $A_k$ is ubiquitously dense in $[0, \infty)$.

By Lemma 4.13 and Corollary 4.15, and by the definitions of $A_k$ and $F_{k,\alpha}$, we obtain:

Proposition 4.17. Fix $k \in \mathbb{Z}_{\geq 0}$. Then the following statements are true:

(1) For all $\alpha \in \text{Suc}(c)$, we have $A_k \cap F_{k,\alpha} = \emptyset$;

(2) The set $A_k \cup \bigcup_{\alpha < c} F_{k,\alpha}$ is linearly independent over $\mathbb{Q}$.

For two sets $A$ and $B$, we denote by $A \ominus B = (A \setminus B) \cup (B \setminus A)$. Namely, the set $A \ominus B$ is the symmetric difference of $A$ and $B$.

Lemma 4.18. Fix $k \in \mathbb{Z}_{\geq 0}$. Let $\alpha, \beta, \bar{\alpha}, \bar{\beta} \in c$ with $\alpha \neq \beta$ and $\bar{\alpha} \neq \bar{\beta}$.

If $x \in \mathbb{G}_{k,\alpha}$, $a \in \mathbb{G}_{k,\bar{\alpha}}$, $y \in \mathbb{G}_{k,\beta}$, $b \in \mathbb{G}_{k,\bar{\beta}}$, and $z, c \in X_k$. If $\{x, y, z\} \neq \{a, b, c\}$, $x + y + z \neq 0$, and $a + b + c \neq 0$, then the numbers $x + y + z$ and $a + b + c$ are linearly independent over $\mathbb{Q}$.

Proof. We first prove the following claim:

- There exists non-zero $r$ such that $r \in \{x, y, z\} \ominus \{a, b, c\}$.
By \( \{x, y, z\} \neq \{a, b, c\} \), we observe that \( \{x, y, z\} \cap \{a, b, c\} \neq \emptyset \). If \( 0 \notin \{x, y, z\} \cap \{a, b, c\} \), then any element in this set satisfies the condition. If \( 0 \in \{x, y, z\} \cap \{a, b, c\} \), then \( 0 \in \{x, y, z\} \) or \( 0 \in \{a, b, c\} \). We may assume that \( 0 \notin \{a, b, c\} \). Thus, \( 0 \notin \{x, y, z\} \). Combining Lemma 4.16 and Theorem 2.4, we obtain the lemma.

**Remark** that it can happen that some of \( u \), \( v \), and \( w \) are non-zero numbers, and since all the three numbers \( x, y, z \) are non-zero, there exists non-zero \( r \in \{x, y, z\} \). This finishes the proof of the claim.

To prove the linear independence of \( x + y + z \) and \( a + b + c \) over \( \mathbb{Q} \), we assume that integers \( h_0 \) and \( h_1 \) satisfy

\[
(4.1) \quad h_0(x + y + z) + h_1(a + b + c) = 0.
\]

We put \( A = \{x, y, z\} \cap (0, \infty) \) and \( B = \{a, b, c\} \cap (0, \infty) \). We also put \( u = \sum_{l \in A \cap B} l, v = \sum_{l \in B \setminus A} l \), and \( w = \sum_{l \in A \setminus B} l \). Then the equality \((4.1)\) implies that

\[
(4.2) \quad h_0u + h_1v + (h_0 + h_1)w = 0.
\]

Remark that it can happen that some of \( u, v, w \) are 0. Using the claim explained above, we have \( u \neq 0 \) or \( v \neq 0 \). We may assume that \( u \neq 0 \). Put \( C = \{u, v, w\} \cap (0, \infty) \). Then \( u \in C \). According to Proposition 4.17 the set \( A \cup B \) is linearly independent over \( \mathbb{Q} \). Since \( A \setminus B \), \( B \setminus A \), and \( A \cap B \) are mutually disjoint subsets of \( A \cup B \), by the definitions of \( u, v, \) and \( w \), the set \( C \) is linearly independent over \( \mathbb{Q} \). Thus, from \((4.2)\) and and \( u \neq 0 \), it follows that \( h_0 = 0 \). The equality \((4.1)\) implies \( h_1(a + b + c) = 0 \). Since \( a + b + c \neq 0 \), we obtain \( h_1 = 0 \). Thus, we conclude that \( h_0 = h_1 = 0 \). Hence \( x + y + z \) and \( a + b + c \) are linearly independent over \( \mathbb{Q} \). 

\[ \square \]

**Lemma 4.19.** Fix \( k \in \mathbb{Z}_{\geq 0} \). Let \( X \) be a discrete space with \( \text{Card}(X) \leq c \). Let \( d \in \text{Met}(X) \) and \( \epsilon \in (0, \infty) \). Then there exists a strongly rigid uniformly discrete metric \( e \in \text{Met}(X; \mathbb{X}_k) \) with \( D_X(d, e) \leq \epsilon \).

**Proof.** Combining Lemma 4.16 and Theorem 2.3 we obtain the lemma.

\[ \square \]

The following is deduced from [19, Theorem 3.1, Chapter 7]. The latter part is deduced from [21, Theorem 2].

**Theorem 4.20.** If \( X \) is a strongly 0-dimensional metrizable space with \( \text{Card}(X) \leq c \), then \( X \) can be topologically embedded into \( N(\Omega) \). Moreover, if \( X \) is completely metrizable, the space \( X \) is homeomorphic to a closed subset of \( N(\Omega) \).

**Lemma 4.21.** Fix \( k \in \mathbb{Z}_{\geq 0} \). Let \( X \) be a strongly 0-dimensional metrizable space with \( \text{Card}(X) \leq c \). Let \( \alpha \in c \). Then there exists a strongly rigid metric \( e \in \text{Met}(X; G_{k, \alpha}) \) such that \( \text{diam}_e(X) \leq 2^{-k} \). Moreover, if \( X \) is completely metrizable, we can choose \( e \) as a complete metric.
4.11, the metric \( e \) metrizable, we can choose \( h \) as a closed map. Since \([R]_k\) is complete, in this case, so is the metric \( e \).

The following was first stated in [14].

**Corollary 4.22.** Let \( X \) be a strongly 0-dimensional metrizable space. Then the inequality \( \text{Card}(X) \leq c \) holds if and only if there exists a strongly rigid metric \( d \in \text{Met}(X) \).

## 5. Proofs of Main Results

The following proposition can be found in [10] Proposition 2.1] (See also [8 Proposition 3.1]).

**Proposition 5.1.** Let \((X, d)\) be a metric space. Let \( I \) be a set and \( \{B_i\}_{i \in I} \) a covering of \( X \) consisting of mutually disjoint clopen subsets. Let \( P = \{p_i\}_{i \in I} \) be points with \( p_i \in B_i \) and \( \{e_i\}_{i \in I} \) a set of metrics such that \( e_i \in \text{Met}(B_i) \). Let \( h \) be a metric on \( P \) generating the discrete topology on \( P \). We define a function \( D : X^2 \to [0, \infty) \) by

\[
D(x, y) = \begin{cases} 
  e_i(x, y) & \text{if } x, y \in B_i; \\
  e_i(x, p_i) + h(p_i, p_j) + e_j(p_j, y) & \text{if } x \in B_i \text{ and } y \in B_j.
\end{cases}
\]

Then \( D \in \text{Met}(X) \) and \( D|_{B_i} = e_i \) for all \( i \in I \). Moreover, if for every \( i \in I \) we have \( \text{diam}_d(B_i) \leq \epsilon \) and \( \text{diam}_{e_i}(B_i) \leq \epsilon \), then \( D_X(D, d) \leq 4\epsilon + D_P(d|_{p_2}, h) \).

**Proposition 5.2.** Under the same assumptions in Proposition 5.1, if \( h \) is uniformly discrete and each \( e_i \) is complete, then the metric \( D \) is complete.

**Proof.** Take \( c \in (0, \infty) \) such that \( c < h(p_i, p_j) \) for all distinct \( i, j \in I \). By the definition of \( D \), if \( x \in B_i \) and \( y \in B_j \) and \( i \neq j \), we have \( c < D(x, y) \). Take a Cauchy sequence \( \{x_n\}_{n \in \mathbb{Z}_{\geq 0}} \) in \((X, D)\) and take a sufficient large number \( N \in \mathbb{Z}_{\geq 0} \) such that for all \( n, m > N \), we have \( D(x_n, x_m) < c \). Then we observe that there exists \( i \in I \) satisfying that \( \{x_n \mid N < n \} \subset B_i \). Since \( D|_{B_i} = e_i \) and \( e_i \) is complete, the sequence \( \{x_i\}_{i \in \mathbb{Z}_{\geq 0}} \) has a limit point. Therefore we conclude that \((X, D)\) is complete. \( \square \)

We now prove Theorem 4.11.

**Proof of Theorem 4.11** Let \( X \) be a strongly 0-dimensional metrizable space with \( \text{Card}(X) \leq c \). Let \( d \in \text{Met}(X) \) and \( \epsilon \in (0, \infty) \).

Put \( \eta = \epsilon/5 \). Take \( k \in \mathbb{Z}_{\geq 0} \) such that \( 2^{-k} \leq \eta \). Since \( X \) is paracompact and strongly zero-dimensional, we can take a mutually disjoint
open cover \( \{O_\alpha\}_{\alpha<\tau} \) of \( X \) with \( \text{diam}_d(O_\alpha) \leq \varepsilon \), where \( \tau \leq \iota \) (see [3 Proposition 1.2 and Corollary 1.4]).

For each \( \alpha < \tau \), we take \( p_\alpha \in O_\alpha \). Put \( P = \{ p_\alpha \mid \alpha \in \tau \} \). Then \( P \) is a discrete space and \( d|_P \) is a discrete metric on \( P \). Lemma 4.19 guarantees the existence of a strongly rigid uniformly discrete metric \( h \in \text{Met}(P; X_\kappa) \) with \( \mathcal{D}_P(\rho_{d|_P}, h) \leq \eta \). Applying Lemma 4.21 to \( O_i \), we obtain a strongly rigid metric \( e_\alpha \in \text{Met}(O_\alpha; G_{k,\alpha}) \) with \( \text{diam}_e(O_\alpha) \leq 2^{-k} \leq \eta \). We define a metric \( e \) by

\[
e(x, y) = \begin{cases} 
  e_\alpha(x, y) & \text{if } x, y \in B_\alpha; \\
  e_\alpha(x, p_\alpha) + h(p_\alpha, p_\beta) + e_\beta(p_\beta, y) & \text{if } x \in B_\alpha \text{ and } y \in B_\beta.
\end{cases}
\]

Applying Proposition 5.1 to \( \{O_\alpha\}_{\alpha<\tau} \), \( P \), \( \{e_\alpha\}_{\alpha<\tau} \), \( h \), and \( \eta \), we obtain \( e \in \text{Met}(X) \) and \( \mathcal{D}(d, e) \leq 5\eta = \varepsilon \).

We shall prove \( e \in \text{LI}(X) \). From the definition of \( e \), and Lemma 4.18 it follows that for all \( x, y, u, v \in X \) with \( x \neq y, u \neq v \), and \( \{x, y\} \neq \{u, v\} \), the numbers \( e(x, y) \) and \( e(u, v) \) are linearly independent over \( \mathbb{Q} \). Hence we conclude that \( e \in \text{LI}(X) \).

To prove the latter part, we assume that \( X \) is completely metrizable. Since each \( O_\alpha \) is clopen in \( X \), the set \( O_\alpha \) is completely metrizable. By the latter part of Lemma 4.21 we can choose each \( e_\alpha \) as a complete metric. Since \( h \) is uniformly discrete, Proposition 5.2 implies that \( e \) is a complete metric. This finishes the proof of Theorem 1.1. \( \square \)

The proof of the following proposition is analogous with that of [20 Theorem 2].

**Proposition 5.3.** If \( X \) is a strongly 0-dimensional \( \sigma \)-compact metrizable space, then the set \( \text{SR}(X) \) is \( G_\delta \) in \( \text{Met}(X) \).

**Proof.** Take a sequence \( \{K_i\}_{i \in \mathbb{Z}_{\geq 0}} \) of compact subsets of \( X \) such that \( X = \bigcup_{i \in \mathbb{Z}_{\geq 0}} K_i \) and \( K_i \subset K_{i+1} \) for all \( i \in \mathbb{Z}_{\geq 0} \). For \( n, m \in \mathbb{Z}_{\geq 0} \), we denote by \( F_{n,m} \) the set of all \( d \in \text{Met}(X) \) such that there exists \( x, y, u, v \in K_m \) with

1. \( d(x, y) = d(u, v) \geq 2^{-m} \);
2. \( d(x, u) + d(y, v) \geq 2^{-m} \);
3. \( d(x, v) + d(u, y) \geq 2^{-m} \).

We now show that \( L_{n,m} \) is a closed set of \( \text{Met}(X) \). Take a sequence \( \{e_i\}_{i \in \mathbb{Z}_{\geq 0}} \) in \( L_{n,m} \) and take \( d \in \text{Met}(X) \) such that \( e_i \to d \) as \( i \to \infty \). We shall show \( d \in L_{n,m} \). By extracting a subsequence using the compactness of \( K_n \) if necessary, we may assume that there exist sequences \( \{x_i\}_{i \in \mathbb{Z}_{\geq 0}}, \{y_i\}_{i \in \mathbb{Z}_{\geq 0}}, \{u_i\}_{i \in \mathbb{Z}_{\geq 0}}, \) and \( \{v_i\}_{i \in \mathbb{Z}_{\geq 0}} \) in \( K_n \), and points \( x, y, z, w \in K_n \), such that

1. \( e_i(x_i, y_i) = e_i(u_i, v_i) \geq 2^{-m} \) for all \( i \in \mathbb{Z}_{\geq 0} \);
2. \( e_i(x_i, u_i) + e_i(y_i, v_i) \geq 2^{-m} \) for all \( i \in \mathbb{Z}_{\geq 0} \);
3. \( e_i(x_i, v_i) + e_i(y_i, u_i) \geq 2^{-m} \) for all \( i \in \mathbb{Z}_{\geq 0} \);
4. \( x_i \to x, y_i \to y, u_i \to u, \) and \( v_i \to v \) as \( i \to \infty \).
Since \( d \) and \( e_i \) generate the same topology of \( X \) and since \( e_i \to d \) as \( i \to \infty \), if \( p \in \{x, y, u, v\} \) and \( q \in \{x, y, u, v\} \), then we have \( e_i(p_i, q_i) \to d(p, q) \) as \( i \to \infty \). Thus, we obtain \( d(x, y) = d(u, v) \geq 2^{-m} \) and \( d(x, u) + d(y, v) \geq 2^{-m} \) and \( d(x, v) + d(y, u) \geq 2^{-m} \). Therefore \( d \in L_{n, m} \), and hence \( L_{n, m} \) is closed. Put \( G_{n, m} = \text{Met}(X) \setminus L_{n, m} \). Then each \( G_{n, m} \) is open in \( \text{Met}(X) \), and we obtain
\[
\text{SR}(X) = \bigcap_{n, m \in \mathbb{Z}_{\geq 0}} G_{n, m}
\]
This proves the proposition.

\[\square\]

**Proof of Theorem 1.3** Let \( X \) be a strongly 0-dimensional metrizable space with \( \text{Card}(X) \leq c \).

Due to Proposition 5.3 we only need to prove that \( \text{SR}(X) \) is dense in \( \text{Met}(X) \). We now show \( \text{LI}(X) \subset \text{SR}(X) \). Take \( d \in \text{LI}(X) \). Take \( x, y, u, v \in X \) with \( x \neq y \) and \( u \neq v \), and \( \{x, y\} \neq \{u, v\} \). Then \( d(x, y) \) and \( d(u, v) \) are linearly independent over \( \mathbb{Q} \). In particular, we obtain \( d(x, y) \neq d(u, v) \), and hence \( d \in \text{SR}(X) \). Thus, we have \( \text{LI}(X) \subset \text{SR}(X) \). According to Theorem 1.1 we observe that \( \text{SR}(X) \) is dense in \( \text{Met}(X) \). This complete the proof of Theorem 1.2

\[\square\]

**Proof of Theorem 1.4** Let \( X \) be a strongly 0-dimensional metrizable space. Assume that \( X \) is \( \sigma \)-compact and satisfies \( 3 \leq \text{Card}(X) \leq c \).

We only need to prove \( \text{SR}(X) \subset \text{R}(X) \). Take \( d \in \text{SR}(X) \), and let \( f: (X, d) \to (X, d) \) be a bijective isometry. Take arbitrary \( x \in X \), and take two points \( y, z \in X \) with \( \text{Card}({x, y, z}) = 3 \). Since \( d(x, y) = d(f(x), f(y)) \) and \( d(x, z) = d(f(x), f(z)) \), we have \( \{x, y\} = \{f(x), f(y)\} \) and \( \{x, z\} = \{f(x), f(z)\} \). Then we obtain \( f(x) \in \{x, y\} \cap \{x, z\} = \{x\} \), and hence \( f(x) = x \). Since \( x \) is arbitrary, we conclude that \( f \) is the identity map, which implies that \( \text{SR}(X) \subset \text{R}(X) \). This finishes the proof of Theorem 1.3

\[\square\]

Before proving Theorem 1.4 we introduce some notions. Let \( X \) and \( Y \) be topological space. A continuous map \( f: X \to Y \) is said to be proper if for every compact subset of \( Y \), the set \( f^{-1}(K) \) is compact. For a metric space \((X, d)\), the metric \( d \) is proper if all closed balls in \((X, d)\) are compact. Note that \( d \) is proper if and only if a map \( x \mapsto d(x, p) \) is a proper map for all \( p \in X \). We denote by \( B(x, r; d) \) the closed ball of \((X, d)\) centered at \( x \) with radius \( r \).

**Proof of Theorem 1.4** Let \( X \) be a strongly 0-dimensional \( \sigma \)-compact locally compact Hausdorff space. Remark that \( X \) is metrizable.

Let \( P \) be the set of all proper metrics in \( \text{Met}(X) \). Since \( X \) is \( \sigma \)-compact locally compact, we have \( P \neq \emptyset \) (see for example \[9\]). We now show that \( P \) is open in \( \text{Met}(X) \). Take \( d \in P \) and take \( e \in \text{Met}(X) \) such that \( D_X(d, e) \leq 1 \). In this setting, we notice that \( B(x, r; e) \subset \)
\(B(x, r + 1; d)\) for all \(x \in X\) and \(r \in (0, \infty)\). Since \(d\) is proper, so is \(e\). Thus, the set \(P\) is open in \(\text{Met}(X)\).

Due to Theorem \(\ref{thm:propermap}\), the set \(\text{SR}(X)\) is dense in \(\text{Met}(X)\). Since \(P\) is non-empty and open, we obtain \(P \cap \text{SR}(X) \neq \emptyset\), and we can take a member from this set, say \(d\). Recall that for each \(\xi \in X\), the map \(F_\xi: X \to [0, \infty)\) is defined as \(F_\xi(x) = d(x, \xi)\). Fix \(\xi \in X\). Next we verify that \(F_\xi\) is a topological embedding. Since \(d\) is strongly rigid, the map \(F_\xi\) is injective. According to \(d \in P\), the map \(F_\xi\) is proper as a map. From the fact that every proper map into a metrizable space is a closed map (see for instance \([\ref{ref:propermap}]\)), it follows that \(F_\xi\) is a closed map. Therefore we conclude that \(F_\xi\) is a topological embedding. This finishes the proof of Theorem \(\ref{thm:topembedding}\). \(\square\)

**References**

1. K. A. Broughan, *A metric characterizing Čech dimension zero*, Proc. Amer. Math. Soc. **39** (1973), 437–440.
2. D. G. Ebin, *The manifold of Riemannian metrics*, Global Analysis (Proc. Sympos. Pure Math., Vol. XV, Berkeley, Calif., 1968), Amer. Math. Soc., Providence, R.I., 1970, pp. 11–40.
3. R. L. Ellis, *Extending continuous functions on zero-dimensional spaces*, Math. Ann. **186** (1970), no. 2, 114–122.
4. Y. Hattori, *Congruence and dimension of nonseparable metric spaces*, Proc. Amer. Math. Soc. **108** (1990), no. 4, 1103–1105.
5. F. G. Dorais (https://mathoverflow.net/users/2000/françois-g-dorais), *explicit big linearly independent sets*, MathOverflow, URL:https://mathoverflow.net/q/23206 (version: 2017-04-13).
6. Y. Ishiki, *An interpolation of metrics and spaces of metrics*, (2020), preprint, arXiv:2003.13277.
7. , *An embedding, an extension, and an interpolation of ultrametrics*, p-Adic Numbers Ultrametric Anal. Appl. **13** (2021), no. 2, 117–147.
8. , *On dense subsets in spaces of metrics*, (2021), preprint arXiv:2104.12450, to appear in Colloq. Math.
9. , *Extending proper metrics*, (2022), preprint arXiv:2207.12905, to appear in Topology Appl.
10. , *On comeager sets of metrics whose ranges are disconnected*, (2022), preprint arXiv:2207.12765, to appear in Topology Appl.
11. L. Janos, *A metric characterization of zero-dimensional spaces*, Proc. Amer. Math. Soc. **31** (1972), 268–270.
12. L. Janos and H. Martin, *Metric characterizations of dimension for separable metric spaces*, Proc. Amer. Math. Soc. **70** (1978), no. 2, 209–212.
13. T. Jech, *Set theory*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, The third millennium edition, revised and expanded.
14. H. W. Martin, *Strongly rigid metrics and zero dimensionality*, Proc. Amer. Math. Soc. **67** (1977), no. 1, 157–161.
15. P. Mounoud, *Metrics without isometries are generic*, Monatsh. Math. **176** (2015), no. 4, 603–606.
16. J. Mycielski, *Independent sets in topological algebras*, Fund. Math. **55** (1964), 139–147.
17. J. v. Neumann, *Ein System algebraisch unabhängiger Zahlen*, Math. Ann. **99** (1928), no. 1, 134–141.
18. R. S. Palais, *When proper maps are closed*, Proc. Amer. Math. Soc. 24 (1970), 835–836.
19. A. R. Pears, *Dimension theory of general spaces*, Cambridge University Press, Cambridge, England-New York-Melbourne, 1975.
20. J. Rouyer, *Generic properties of compact metric spaces*, Topology Appl. 158 (2011), no. 16, 2140–2147.
21. A. H. Stone, *Non-separable Borel sets*, Rozprawy Mat. 28 (1962), 41.

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