Master integrals with one massive propagator
for the two-loop electroweak form factor

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Abstract
We compute the master integrals containing one massive propagator entering
the two-loop electroweak form factor, i.e. the process $\bar{f}f \to X$, where
$\bar{f}f$ is an on-shell massless fermion pair and $X$ is a singlet particle under
$SU(2)_L \times U(1)_Y$, such as a virtual gluon or an hypothetical $Z'$. The method
used is that of the differential equation in the evolution variable $x = -s/m^2$,
where $s$ is the c.m. energy squared and $m$ is the mass of the $W$ or $Z$ bosons
(assumed to be degenerate). The $1/\epsilon$ poles and the finite parts are com-
puted exactly in terms of one-dimensional harmonic polylogarithms of the
variable $x$, $H(w; x)$, with $\epsilon = 2 - D/2$ and $D$ the space-time dimension. We
present large-momentum expansions of the master integrals, i.e. expansions
for $|s| \gg m^2$, which are relevant for the study of infrared properties of the
Standard Model. We also derive small-momentum expansions of the master
integrals, i.e. expansions in the region $|s| \ll m^2$, related to the threshold
behaviour of the form factor (soft probe). Comparison with previous results
in the literature is performed finding complete agreement.

Key words: Feynman diagrams, Multi-loop calculations, Vertex diagrams,
Electroweak Sudakov
PACS: 11.15.Bt, 12.15.Lk

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‡This work was supported by the European Union under contract HPRN-CT-2000-00149


1 Introduction

Present day and planned high-energy experiments often require accurate perturbative calculations. While in quantum-chromodynamics (QCD) higher order computations are available for various processes, in the electroweak (EW) case progress is more difficult because of the presence of the large, non negligible, masses of $W$, $Z$ and Higgs bosons. A full calculation of two-loop electroweak corrections to a typical process such as $e^+e^-\rightarrow \mu^+\mu^-$ is still outside present-day technology. It involves indeed the computation of a large number of massive two-loop box diagrams having thresholds at $s=0$, $m^2$, $4m^2$ and $s=9m^2$, assuming $m \sim m_W \sim m_Z \sim m_H$. To have an explicit estimate of the size of two-loop electroweak corrections, we consider then the model process:

$$f(p_1) + \bar{f}(p_2) \rightarrow X(q),$$

where $f$ ($\bar{f}$) is a massless fermion (antifermion) on its mass-shell, $p_1^2 = 0$ ($p_2^2 = 0$), and $X$ is any particle without electromagnetic or weak charge, i.e. a singlet under the electroweak gauge group $SU(2)_L \times U(1)_Y$, such as a (time-like) gluon or an hypothetical $Z'$. Eq. also represents the simplest process containing electroweak double logarithms, considered for the first time, as far as we know, in [1]. Reaction is the electroweak analog of the QCD processes involving a color-neutral probe, such as Drell-Yan production of vector bosons or muon pairs: $q\bar{q}^{(e)} \rightarrow W, Z, \mu^+\mu^-$ or Higgs production by gluon-gluon fusion: $gg \rightarrow H$.

At two-loop level, the annihilation in Eq. involves the emission of a pair of virtual bosons among $\gamma, W, Z$ and $H$‘s. In the case of double photon emission, no mass is present in the loops and we recover the known results for the QED or QCD amplitudes. The latter, for dimensional reasons, are of the form:

$$(\text{massless amp.}) = G(\epsilon) \left(\frac{-s}{\mu^2}\right)^{-2\epsilon} s^{4-n_d} = a^{4-n_d} \left(\frac{\mu^2}{a}\right)^{2\epsilon} G(\epsilon) x^{4-n_d-2\epsilon},$$

where $s$ is the c.m. energy squared, $G(\epsilon)$ is a function having a Laurent expansion in $\epsilon = 2 - D/2$ (with $D$ the space-time dimension), $\mu$ is the mass-scale of Dimensional Regularization and $n_d$ is the number of scalar propagators. We have defined:

$$x = \frac{-s}{m^2},$$

where $m$ is the mass of the $W$, $Z$ or Higgs bosons. On the other hand, if $W$, $Z$ or $H$’s are also emitted, the dependence on $x$ is not factorized in a single power and the amplitude has the more general form

$$(\text{massive amp.}) = a^{4-n_d} \left(\frac{\mu^2}{a}\right)^{2\epsilon} G(x; \epsilon).$$

The amplitudes involving a single massive propagator, i.e. a single $W$ or $Z$ emission, have thresholds only at $s = 0$ and $s = m^2$. As a consequent this simple.

\footnote{The minus sign in the definition of $x$ is inserted for convenience.}
structure, these amplitudes can be exactly computed in terms of one-dimensional harmonic polylogarithms \[2, 3\] (a generalization of the Nielsen’s polylogarithms \[4, 5\]) of argument \(x, H(\vec{w}, x)\), with weight up to \(w = 4\) included. In this paper we present the evaluation of the master integrals entering the above-mentioned amplitudes, leaving to a future work the more complicated amplitudes with two and three massive propagators, involving also the threshold \(s = 4m^2\). Since external fermions are massless and are taken on their mass-shell, infrared divergences (soft and collinear) occur in the amplitudes. When an intermediate vector boson is radiated, its mass \(m\) acts as an infrared regulator and mass logarithms do appear in four dimensions, i.e. terms of the form \(\log^k(-s/m^2)\) with \(k\) up to two included. Infrared divergences related to photon emission instead are not regulated by any physical parameter. Our computation is done within the Dimensional Regularization scheme \[6\], in which the latter manifest themselves as poles in \(\frac{1}{\epsilon}\). These poles are unphysical and, in the physical cross-section, are canceled by the corresponding ones appearing in the real photon emission contributions or are factorized into QED structure functions. Ultraviolet divergences — related to short-distance behaviour and leading to additional \(1/\epsilon\) poles — are also regulated within Dimensional Regularization and can be subtracted with ordinary renormalization prescriptions, such as, for example, the \(\overline{MS}\) scheme.

The computation of the form factor is naturally divided in two steps.

The first one involves the reduction of the original tensor diagrams, given in Figs. 1 and 2 into a minimal set of scalar amplitudes, the so-called master integrals. This step is discussed in section 2 and involves in turn three different operations: a) the decomposition of tensor amplitudes into scalar amplitudes by means of projectors on invariant form factors, see section 2.1; b) the transformation to linearly independent scalar amplitudes by means of scalar product rotations, as discussed in section 2.2; c) the reduction of all the independent scalar amplitudes generated to a small subset of them — the so called master integrals — by means of the integration-by-parts identities \[7\] and other symmetry relations, as discussed in section 2.3.

The second step involves the explicit evaluation of the master integrals, which we perform in all the massive cases with the method of differential equations on the external kinematical invariants \[8\] \[9\] \[10\] \[11\] \[12\] \[13\] \[14\]. A set of special functions of the polilogarithm kind is needed to represent the pole and finite parts of the master integrals. We choose the basis of the (one-dimensional) harmonic polylogarithms \[2, 3\]. This step is described in section 3.

In section 4 we list the results for the pole and finite parts of the master integrals containing up to five denominators included. In section 5 we give the expressions of all the scalar integrals containing six denominators. The planar topologies are reducible while the crossed ladders — two amplitudes — are master integrals. We decided to list them together because all the six-denominator amplitudes are interesting by themselves for reference. We also compare with previous results in the literature. Section 6 is devoted to the large momentum expansion \(|s| \gg m^2\) of all the six-denominator amplitudes. As already said, this expansion is relevant for the study of asymptotic properties of multiple electroweak radiation. Section 7 deals with the small momentum expansion — i.e. the expansion for \(|s| \ll m^2\) — of all the
six-denominator amplitudes. In the conclusions we summarize the results obtained
and we discuss future perspectives. To not bore the reader with too many technical
details or formulas — which are anyway worth reporting — we added a few appen-
dices. Appendix \textbf{A} contains the expressions of all the denominators entering the
two-loop amplitudes. In appendix \textbf{B} we give the results of the one-loop amplitudes
which enter the factorized two-loop diagrams, i.e. the diagrams in which the two
loops do not have common propagators. Finally, in appendix \textbf{C} we list the results
for all the independent topologies of Figs. \textbf{5} to \textbf{7} which are not master integrals
but can be useful for general reference.

\section{The reduction to master integrals}

All the two-loop vertex diagrams containing six denominators with at most one
massive propagator are reported in Figs. \textbf{1} and \textbf{2}. The graphical conventions we
use are the following. A massless propagator is represented by a wavy internal
line, while a massive propagator is denoted by a straight segment. An external
line carrying the light-cone momentum $p_1$ or $p_2$ is represented by an external wavy
line, while the external particle $X$ carrying the general momentum $q$ is depicted as
a straight external line. This set of conventions is clearly a generalization of the
well-known QED ones. In Fig. \textbf{1} we consider the basic three topologies with the
various mass configurations:

- (a), (b), (c) and (d) are ladder diagrams, i.e. corrections to the hard vertex of
  the basic one-loop amplitude;

- (e) and (f) are crossed ladders — the non-planar topology;

- (g), (h) and (i) finally are the vertex-insertion diagrams, containing a correc-
tion to the vertices with momentum $p_1$ or $p_2$.

The number of internal lines with different momenta entering a diagram is an indica-
tor of the \textit{topology} of the diagram. The diagrams in Fig. \textbf{1} are \textbf{real} six-denominator
amplitudes and the related topologies are given in Fig. \textbf{4}.

In Fig. \textbf{2} the diagrams containing self-energy corrections to the internal lines of
the one-loop vertices are plotted. Note that, in contrast to Fig. \textbf{1} diagrams (a), (b),
(c) and (d) in Fig. \textbf{2} involve only five distinct denominators — one of them being
squared — as two internal lines carry the same momentum. These diagrams then,
do not represent \textbf{real} six-denominator topologies and are more naturally grouped
among the five-denominator ones. The topologies of diagrams (a), (b), (c) and (d)
are shown in Fig. \textbf{5} in (h), (e), (f) and (s) respectively. Diagram (e) of Fig. \textbf{2}
involves two internal lines with the same momenta but with different masses. It can
be expressed as a superposition of integrals belonging to topologies (s) and (t) of
\footnote{Actually, diagram (f) in Fig. \textbf{2} contains only five denominators, but we list it in this figure as it belongs to the same class of the six-denominator ones.}
Figure 1: two-loop vertex diagrams with up to one massive propagator. They involve the ladder, crossed-ladder and vertex-insertion topologies and are real six-denominator topologies (see text).
Figure 2: Two-loop vertex diagrams with up to one massive propagator involving a self-energy correction to the basic one-loop vertices. The related topologies are five and four denominator topologies (see text).
Fig. 5 by means of partial fractioning in the loop momentum $k_1$:

$$\frac{1}{k_1^2(k_1^2 + a)} = \frac{1}{a} \left[ \frac{1}{k_1^2} - \frac{1}{k_1^2 + a} \right].$$

(5)

The scalar product of two vectors is defined as: $a \cdot b = -a_0 b_0 + \vec{a} \cdot \vec{b}$\(^6\). In graphical form, the decomposition reads:

Finally, diagram (f) is actually a four-denominator topology, indicated in Fig. 6 (g).

To conclude, let us note that the topologies related to diagrams of Fig. 2 can be obtained by properly shrink one internal line (or two internal lines in the case of diagram (f)) among those in the diagrams of Fig. 4.

2.1 Decomposition of a tensor amplitude into scalar amplitudes

In this section we consider the decomposition of a general tensor amplitude among those shown in Figs. 1 and 2 into scalar amplitudes. Even though the decomposition is completely general, let us pick up, for clarity’s sake, a specific case: the ladder diagram with a mass on the external boson line (the diagram (b) in Fig. 1). Omitting an overall constant dependent on the couplings of the theory, this diagram involves a double integral over the loop momenta $k_1$ and $k_2$ and can be written, in all generality as:\(^7\)

$$I_{\mu
u ij...}(p_1, p_2) = \int \{d^Dk_1\} \{d^Dk_2\} \frac{N_{\mu
u ij...}(p_1, p_2, k_1, k_2; a)}{P_1 P_2 P_3 P_4 P_5 P_6},$$

(7)

where $\{d^Dk\}$ is the loop integration measure (see section 4). According to our routing of the loop momenta, we have defined the denominators:

$$P_1 = k_1^2 + a,$$

(8)

$$P_2 = (p_1 - k_1)^2,$$

(9)

$$P_3 = (p_2 + k_1)^2,$$

(10)

$$P_4 = k_2^2,$$

(11)

$$P_5 = (p_1 - k_1 + k_2)^2,$$

(12)

$$P_6 = (p_2 + k_1 - k_2)^2.$$

(13)

The numerator $N_{\mu
u ij...}$ depends on the spin of the particles entering the diagram and on the type of interaction; the indices $\mu \nu ij...$ denote collectively Lorentz indices,

\(^6\)Note a difference of sign with respect to the more common definition.

\(^7\)Depending on the gauge choice for vector particles, some of the denominators $P_i$ may actually appear to power two instead of one: the reduction scheme is clearly unchanged. We do not consider the non covariant gauges, leading to additional denominators and scalar products involving a constant gauge vector $e^\mu$. 

6

7
Dirac indices, etc. By using projectors on invariant form factors, we can always restrict ourselves to consider scalar numerators (see for example [15]). The latter typically involve a trace over spinor indices and a sum over the polarizations of the vector particles. Therefore, from now on, let us consider:

\[ T(p_1 \cdot p_2) = \int \{d^D k_1\} \{d^D k_2\} \frac{\mathcal{N}(S_1, S_2, S_3, S_4, S_5, S_6, S_7)}{P_1 P_2 P_3 P_4 P_5 P_6}, \]  

(14)

where we have explicitly indicated the dependence on the seven invariants, dependent on the loop momenta \( k_1 \) and \( k_2 \):

\[
S_1 = k_1^2,  \\
S_2 = k_2^2,  \\
S_3 = k_1 \cdot p_1,  \\
S_4 = k_1 \cdot p_2,  \\
S_5 = k_2 \cdot p_1,  \\
S_6 = k_2 \cdot p_2,  \\
S_7 = k_1 \cdot k_2.  
\]

(15)-(21)

The scalar numerator can be expanded in a power series of the invariants as:

\[ \mathcal{N}(S_1, S_2, S_3, S_4, S_5, S_6, S_7) = \sum_{l_1, \ldots, l_7 \geq 0} b(l_1, l_2, \ldots, l_7) S_1^{l_1} S_2^{l_2} S_3^{l_3} S_4^{l_4} S_5^{l_5} S_6^{l_6} S_7^{l_7}, \]  

(22)

where the (known) coefficients \( b(l_1, l_2, \ldots, l_7) \) also depend on the external invariant \( s \), the mass squared \( a = m^2 \) and the space-time dimension \( D \). With the typical gauge interactions, we expect the indices \( l_i \) to be restricted to \( l_i \leq 3 \).

### 2.2 Decomposition into independent scalar amplitudes

The scalar amplitudes in Eq. (14), with \( \mathcal{N} \) given by the expansion in Eq. (22) are not, in general, linearly independent on each other; we can indeed express six of the invariants listed in Eqs. (15–21) in terms of the denominators \( P_i \). The next step is the reduction to a set of linearly independent scalar amplitudes in order to progress in the evaluation of the tensor amplitude of Eq. (7).

In this work, we used two different reduction schemes of the scalar amplitudes, using consistency of the results as a crossed check. The first one is the scheme of the auxiliary denominator (or auxiliary diagram), which is essentially based on the reduction of all the amplitudes to scalar ones containing formally only denominators; the second one is the scheme of the shifts, in which the independent scalar amplitudes are integrals with a subset of the scalar products at the numerator.

In the following two sections we will recall briefly both schemes, paying more attention on the first one, not discussed often in the literature, and just sketching the second one, for which we refer to the bibliography. Our results are presented in section 4 in the framework of this second scheme.
2.2.1 Auxiliary-denominator scheme

As already said in the previous section, we aim at expressing the invariants listed in Eqs. (15–21) in terms of the denominators $P_i$ appearing in the diagram. Since there are seven invariants and only six denominators, we introduce a seventh auxiliary denominator, independent from the others, in such a way to form a basis. The choice is of course, not unique; let us take for example:

$$P_7 = (k_1 - k_2)^2.$$  \hfill (23)

We then have the linear relations:

$$S_i = \sum_{j=1}^{7} A_{ij} (P_j - a_j) \quad (i = 1, \ldots, 7),$$ \hfill (24)

where $a_j = \delta_j,1a$ and $A_{ij}$ an invertible $7 \times 7$ constant matrix. The non-zero matrix elements are given by:

1. $S_1 = P_1 - a,$ \hfill (25)
2. $S_2 = P_4,$ \hfill (26)
3. $S_3 = \frac{1}{2} (-P_2 + P_1 - a),$ \hfill (27)
4. $S_4 = \frac{1}{2} (+P_3 - P_1 + a),$ \hfill (28)
5. $S_5 = \frac{1}{2} (-P_7 + P_5 - P_2 + P_1 - a),$ \hfill (29)
6. $S_6 = \frac{1}{2} (+P_7 - P_6 + P_3 - P_1 + a),$ \hfill (30)
7. $S_7 = \frac{1}{2} (-P_7 + P_4 + P_1 - a).$ \hfill (31)

Let us note that one can construct a fictitious Feynman diagram having all the seven denominators, called the auxiliary diagram [16]:

$$\text{(aux. diag.)} = \int \frac{\{d^Dk_1\} \{d^Dk_2\}}{P_1 P_2 P_3 P_4 P_5 P_6 P_7}.$$ \hfill (32)

In our example, the auxiliary diagram is a planar double box with final momenta equal to the initial ones $p_1$ and $p_2,$ i.e. a four-point forward amplitude: see Fig. 3. In general, an auxiliary diagram has a number of independent denominators equal to the number of the invariants (seven in our case). The scalar numerator $N$ can now be expanded in powers of the basis “vectors” $\{P_i\}_{i=1,7}$ as:

$$N = \sum_{s_1, \ldots, s_7 \geq 0} c(s_1, s_2, \ldots, s_7) P_1^{s_1} P_2^{s_2} \cdots P_7^{s_7},$$

with the new coefficients $c(s_1, s_2, \ldots, s_7)$ being linear combinations of the old ones $b(l_1, l_2 \cdots, l_7).$ Inserting the above expansion in the expression (14) for the diagram,
we obtain:

\[
I(p_1 \cdot p_2) = \sum_{s_1, \ldots, s_7 \geq 0} c(s_1, \ldots, s_7) \int \frac{\{d^D k_1\} \{d^D k_2\}}{P_1^{1-s_1} P_2^{1-s_2} P_3^{1-s_3} P_4^{1-s_4} P_5^{1-s_5} P_6^{1-s_6} P_7^{s_7}}.
\]  

(33)

Eq. (33) is the main result of this section. It says that the general scalar diagram \(I\) is a linear combination — with known coefficients — of independent scalar amplitudes involving the original denominators \(\{P_i\}_{i=1}^{6}\) and an additional, fictitious, denominator \(P_7\). A compact notation for the auxiliary scalar amplitudes is the following:

\[
\text{Topo} \left( n_1, n_2, n_3, n_4, n_5, n_6; n_7 \right) = \int \frac{\{d^D k_1\} \{d^D k_2\}}{P_1^{n_1} P_2^{n_2} P_3^{n_3} P_4^{n_4} P_5^{n_5} P_6^{n_6} P_7^{n_7}},
\]  

(34)

according to which Eq. (33) is written as:

\[
I(p_1, p_2) = \sum_{n_1, \ldots, n_6 \leq 1, n_7 \leq 0} d(n_1, n_2, \ldots, n_7) \text{Topo} \left( n_1, n_2, n_3, n_4, n_5, n_6; n_7 \right).
\]  

(35)

A few remarks are in order. First, the auxiliary denominator \(P_7\), if it appears, it only does at the numerator: it cannot clearly be generated at the denominator by tensor decomposition. In other words, \(n_i = 1, 0, -1, -2, -3, \ldots\) for \(i = 1, \ldots, 6\) while \(n_7 = 0, -1, -2, -3, \ldots\) The second remark concerns the so-called topologies. As already anticipated, an important quantity for a scalar amplitude is the number of distinct denominators with positive indices, \(n_i > 0\), which we call \(t\):

\[
t \equiv \sum_{i=1}^{7} \theta(n_i),
\]  

(36)

with \(\theta(x)\) the unit step function defined in such a way that \(\theta(0) = 0\). The scalar amplitude with \(n_1 = 2, n_2 = \cdots = n_6 = 1\) and \(n_7 = -1\) for example has \(t = 6\), the maximal value for two-loop vertex functions. If a scalar amplitude \(\text{Topo} \left( n_1, n_2, n_3, n_4, n_5, n_6; n_7 \right)\) has \(n_i \leq 0\), the \(i\th\) denominator eventually appears
only at the numerator and the \( i\)th internal line is shrunk to a point: the diagram then has less than six internal lines. Here we give an example of a decomposition into auxiliary amplitudes with \( t = 6 \) and \( t = 5 \):

\[
\int \{d^D k_1\} \{d^D k_2\} \frac{S_7}{P_1 P_2 P_3 P_4 P_5 P_6} = -\frac{1}{2} \text{Topo}(1, 1, 1, 1, 1; -1) \\
- \frac{a}{2} \text{Topo}(1, 1, 1, 1, 1; 0) \\
+ \frac{1}{2} \text{Topo}(0, 1, 1, 1, 1; 0) \\
+ \frac{1}{2} \text{Topo}(1, 1, 1, 0, 1; 0).
\]

(37)

In general, a diagram \( I \) is decomposed into auxiliary amplitudes with decreasing \( t \) starting from \( t = 6 \) included, i.e. \( t = 6, 5, 4, 3^8 \).

### 2.2.2 Shift scheme

We described above the scheme of the auxiliary diagram to reduce a general scalar amplitude. Let us now present another scheme of reduction which has also been used in the past (see [17, 13, 14]). The main results of this work are presented in sections 4 and 5 according to this second scheme, so let us briefly overview it.

In essence, it uses shifts on the loop momenta in order to maximally simplify the structure of the sub-topologies. Let us consider a routing with two internal lines having the loop momenta \( k_1 \) and \( k_2 \), as for instance \( P_1 \) and \( P_4 \) in the discussion above. We choose for example to keep the invariant \( S_7 \) and we simplify, i.e. cancel, the factors \( S_i (i = 1, \ldots, 6) \) appearing at the numerator with the denominators by means of the formulas:

\[
S_1 = P_1 - a \\
S_2 = P_4 \\
S_3 = \frac{1}{2}(-P_2 + P_4 + a) \\
S_4 = \frac{1}{2}(+P_3 - P_1 + a) \\
S_5 = \frac{1}{2}(+P_5 - P_4 - P_2) + S_7 \\
S_6 = \frac{1}{2}(-P_6 + P_4 + P_3) - S_7.
\]

(38-43)

The simplification of a scalar product with the selected denominator brings to the appearance of sub-topologies in the case the denominator appears to power one. In principle, a diagram of topology number \( t \) can have \( t! \) different sub-topologies. In our case, however, some sub-topologies actually vanish because all the propagators, except possibly one, are massless. Furthermore, eventual discrete symmetries of the

\^8Amplitudes with \( t < 3 \) vanish, in our case, as they contain a massless tadpole.
diagram, such as an up-down symmetry, further reduce the number of independent sub-topologies.

After the cancellations between the \( S_i \) and \( P_j \) have been done, we end up with a set of subdiagrams in which the denominators \( P_1 \) and \( P_4 \) containing the loop momenta \( k_1 \) and \( k_2 \) may be absent. In this case, we perform shifts on the loop momenta to reproduce denominators containing again \( k_1 \) and \( k_2 \). For example, if \( P_1 \) is absent but \( P_2 \) is still present in the given amplitude, we may do the shift

\[
k_1 \rightarrow p_1 - k_1. \tag{44}
\]

If \( P_1 \) and \( P_2 \) are absent but \( P_3 \) is still present, we may do the shift \( k_1 \rightarrow k_1 - p_2 \), and so on. In general, we construct a table of shifts on the loop momenta. With this step, new denominators are generated. Under (44) for example,

\[
\begin{align*}
P_3 & \rightarrow (p_1 + p_2 - k_1)^2, \\
P_5 & \rightarrow (k_1 + k_2)^2, \\
P_6 & \rightarrow (p_1 + p_2 - k_1 - k_2)^2, \\
\end{align*}
\tag{45}
\]
denominators which were not initially present. We then extend the set of cancellation rules including those between the new denominators and the scalar products. The scalar product \( k_1 \cdot k_2 \) cancels against the denominator \( (k_1 + k_2)^2 \) — no arbitrariness — while \( (p_1 + p_2 - k_1)^2 \) may be cancelled for example with \( p_2 \cdot k_1 \) and \( (p_1 + p_2 - k_1 - k_2)^2 \) with \( p_2 \cdot k_2 \). The remaining scalar products are then \( p_1 \cdot k_1 \) and \( p_1 \cdot k_2 \). Depending on the topology number \( t \) of the diagram under consideration, we can re-express \( t \) of the 7 scalar products in terms of denominators and simplify such expressions with a power of the corresponding denominator. It follows that a diagram with topology number \( t \) can have at most \( 7 - t \) independent scalar products in the numerator. A six-denominator amplitude \( (t = 6) \), for example, has at most one scalar product, while a sunrise diagram \( (t = 3) \) has at most four distinct scalar products in the numerator. After all the second-step cancellations have been done, we go for a second step of shifts of the loop momenta, and so on. We repeat iteratively the cancellation-shift procedure until no more simplifications are feasible. The whole set of required denominators is listed in appendix A. In our computation we found 9 independent scalar amplitudes with six denominators, i.e. \( t = 6 \), (Fig. 4), 22 independent amplitudes with five denominators, (Fig. 5), 17 independent amplitudes with four denominators (Fig. 6) and 4 independent amplitudes with 3 denominators (Fig. 7). This makes a total of 44 independent scalar amplitudes.

Within this method, for a given topology, i.e. for a given set of denominators \( D_{i_1}D_{i_2}...D_{i_t} \) (see appendix A), the independent scalar integrals are of the form:

\[
\text{Topo}(n_1, \ldots, n_t; s_1, \ldots, s_\tau) = \int \left\{ d^D k_1 \right\} \left\{ d^D k_2 \right\} \frac{S_{s_1}^{i_1} \cdots S_{s_\tau}^{i_\tau}}{D_{i_1}^{n_1} \cdots D_{i_t}^{n_t}}, \tag{46}
\]

where \( \tau = 7 - t \) denotes the number of independent scalar products \( S_{i_1} \cdots S_{i_\tau} \). Note that \( s_1, \ldots, s_\tau \geq 0 \) while \( n_1, \ldots, n_t \geq 1 \).
Figure 4: The set of 9 independent 6-denominator topologies.

The advantages of the auxiliary diagram method are the simplicity and a shorter code for its implementation. The main advantage of the shift method is that a given topology appears in a unique form, so the CPU time needed is less.

Let us conclude this section with a general remark. A straightforward strategy to evaluate $\mathcal{I}$ is that of computing all the independent scalar amplitudes entering its decomposition. In simple calculations, this is probably the simplest thing to do. In more complicated cases, however, there may be a large number of such scalar amplitudes and the individual computation of all of them may become rather laborious. In these cases, it is more convenient to use a set of relations connecting different scalar amplitudes, and to compute directly only a subset of them. The latter is our strategy that is described in the next section.

### 2.3 Relation between independent scalar amplitudes: the integration by parts identities

As we have shown in the previous section, the task of evaluating a two-loop vertex diagram with an arbitrary scalar numerator $\mathcal{N}$ is shifted to that of computing a set of independent scalar amplitudes either containing an auxiliary denominator: $\text{Topo}(n_1, n_2, n_3, n_4, n_5, n_6; n_7)$, or containing a selected basis of denominators and scalar products: $\text{Topo}(n_1, \ldots, n_t; s_1, \ldots, s_7)$. Let us discuss the simpler case of the auxiliary diagram first. The allowed range of the indices are: $n_1, \ldots, n_6 \leq 1$ and $n_7 \leq 0$ (see Eq. (34)), but in practice, we expect the relevant $n_i$'s to be restricted to $n_i \geq -3$, which however gives still a rather large set of scalar amplitudes to compute. Relations among different scalar amplitudes are obtained by considering
Figure 5: The set of 22 independent 5-denominator topologies.
Figure 6: The set of 17 independent 4-denominator topologies.

Figure 7: The set of 4 independent 3-denominator topologies.
the so-called integration-by-parts (ibp) identities [7]:

\[ 0 = \int \{d^D k_1\} \{d^D k_2\} \frac{\partial}{\partial k_i^\mu} \{ P_1^{n_1} P_2^{n_2} P_3^{n_3} P_4^{n_4} P_5^{n_5} P_6^{n_6} P_7^{n_7} \}, \]  

with \( i = 1, 2 \) and \( v = k_1, k_2, p_1, p_2 \). The integral above vanishes because of the following argument. It is the space integral in \( D \) dimensions of the total divergence of the vector in curly brackets. It can then be transformed into the flux integral of this vector over a spherical surface with infinite radius, \( r = \infty \). For small enough dimension \( D \), the surface integral vanishes for \( r \to \infty \). It can then be defined to identically vanish for any \( D \). By explicitly performing the derivations in Eq. (47) and expressing the scalar products in terms of the assumed basis \( \{ P_i \}_{i=1,7} \) according to Eqs. (24), we obtain identities of the form:

\[ 0 = c \text{Topo}(n_1, n_2, n_3, n_4, n_5, n_6; n_7) + \sum_{i=1}^{7} n_i d_i \text{Topo}(n_1, \cdots, n_i+1, \cdots, n_7) \]
\[ + \sum_{i \neq j} n_i e_{ij} \text{Topo}(n_1, \cdots, n_i+1, \cdots, n_j-1, \cdots; n_7). \]  

For a given set of the indices \( \{ n_i \}_{i=1,7} \) there are eight of such identities; in general, however, they are not all linearly independent. Each identity contains, in general, three kinds of amplitudes:

1. the amplitude itself: \( \text{Topo}(n_1, n_2, n_3, n_4, n_5, n_6; n_7) \);

2. amplitudes with one of the indices increased by one, \( n_i \to n_i + 1 \) \( (i = 1, 7) \): \( \text{Topo}(n_1, \cdots, n_i+1, \cdots; n_7) \). These terms clearly originate from the derivation of the factor \( P_i^{-n_i} \to -n_i P_i^{-n_i-1} \). We pulled a factor \( n_i \) out of the coefficient \( d_i \) to point out that these terms are absent for \( n_i = 0 \). In other words, it is not possible to generate an absent denominator by differentiation. This implies that the ibp identities for a given amplitude with topological number \( t \) involve only amplitudes with smaller or equal topological number, \( t' \leq t \);

3. amplitudes with one index increased by one, \( n_i \to n_i + 1 \) and another index decreased by one, \( n_j \to n_j - 1 \): \( \text{Topo}(n_1, \cdots, n_i+1, \cdots, n_j-1, \cdots; n_7) \). These terms originate from the derivation of the factor \( P_i^{-n_i} \) and the cancellation of a power of \( P_j^{-n_j} \) by the invariants generated at the numerator. We pulled out a factor \( n_i \) also out of the coefficients \( e_{ij} \) to point out similar properties as those discussed in 2).

Other identities are obtained by considering discrete symmetries of the auxiliary diagram. The ladder topologies in (a), (b) and (c) of Fig. 4 for example have an up-down symmetry, resulting in the exchange of the fermion lines expressed by the relation:

\[ \text{Topo}(n_1, n_2, n_3, n_4, n_5, n_6; n_7) = \text{Topo}(n_1, n_3, n_2, n_4, n_6, n_5; n_7). \]  

(49)
The massless crossed ladder also has an up-down symmetry, which interchanges also the boson lines. By solving the above identities, it is possible to express a general scalar amplitude as a linear combination of a finite set of \( N_F \) basic integrals, called master integrals (MI's):

\[
\text{Topo}(n_1, n_2, n_3, n_4, n_5, n_6; n_7) = \sum_{i=1}^{N_F} c_i(n_1, n_2, n_3, n_4, n_5, n_6; n_7) F_i. \tag{50}
\]

In more formal terms, we may say that the ibp identities induce such a strong linear dependence between all the infinite scalar amplitudes to project them into a finite dimensional linear space. The \( F_i \)'s have the same role as the basis vectors in a linear space. The choice of the MI's is arbitrary, as it is the choice of the basis vectors; only their number \( N_F \) is fixed and equals the dimension of the space spanned by any choice of the \( F_i \)'s. Changing the set of MI's in the decomposition (50) is analogous to a change of basis. To give an explicit example, let us report the MI decomposition of the basic six-denominator amplitude of the ladder with a massive external boson in Fig. 1 (b), the example in the previous section:

\[
\text{Topo}(1, 1, 1, 1, 1, 1; 0) = 2 \left( \frac{2}{\epsilon^2} - \frac{7}{\epsilon} + 6 \right) \left( \frac{1}{x^2} + \frac{1}{x} + \frac{1}{1-x} \right) \times
\]

\[
\times \text{Topo}(0, 1, 0, 1, 0, 1; 0) + \left[ \left( \frac{1}{\epsilon^2} - \frac{4}{\epsilon} + 3 \right) \frac{1}{(1-x)^2} \right] \text{Topo}(1, 0, 0, 1, 1, 0; 0)
\]

\[
-2 \left( \frac{1}{\epsilon} - 1 \right) \left( \frac{1}{x} + \frac{1}{1-x} \right) [\text{Topo}(1, 0, 0, 1, 1, 0; 0) + \text{Topo}(1, 0, 0, 0, 1, 1; 0) - 2 \left( \frac{1}{\epsilon^2} - \frac{4}{\epsilon} + 4 \right) \text{Topo}(1, 1, 0, 1, 0, 1; 0)
\]

\[
- \left( \frac{1}{\epsilon} - 2 \right) \frac{1}{x} \text{Topo}(1, 0, 0, 1, 1, 1; 0) - \left[ \left( \frac{2}{\epsilon^2} - \frac{9}{\epsilon} + 9 \right) \frac{1}{(1-x)^2} \right. 
\]

\[
+ \left( \frac{2}{\epsilon^2} - \frac{11}{\epsilon} + 12 \right) \left( \frac{1}{x} + \frac{1}{1-x} \right) \text{Topo}(1, 0, 0, 1, 1, -1)
\]

\[
+ \left( \frac{1}{\epsilon} - 2 \right) \frac{1}{x} \text{Topo}(1, 1, 1, 0, 1, 1; 0) + \frac{2}{x} \text{Topo}(1, 1, 0, 1, 1, 1; 0), \tag{51}
\]

where we have taken for simplicity \( a = 1 \). In this case \( N_F = 8 \). We have chosen MI’s with a subset of the denominators with unitary indices: \( n_i = 1 \), the remaining denominators having vanishing indices: \( n_j = 0 \). For the topology represented by the amplitude Topo(1, 0, 0, 1, 1, 1; 0) this was not sufficient as two master integrals are involved. We added the amplitude with the auxiliary denominator brought to the numerator, i.e. with index \( n_7 = -1 \): Topo(1, 0, 0, 1, 1, 1; -1); we could have taken \( n_2 = -1, n_3 = 0, n_7 = 0 \) or \( n_2 = 0, n_3 = -1, n_7 = 0 \) as well. Still another possibility would have been to consider amplitudes with one of the denominators squared, such as for example Topo(2, 0, 0, 1, 1, 1; 0). In general, the choice of the MI’s is dictated by practical considerations. In some cases, for example, it may be useful to require the absence of ultraviolet or infrared singularities from the MI’s [18]; in
other cases, the set of MI’s can be chosen in such a way that the corresponding
system of differential equations become easier to solve (see next section).

Let us now discuss the methods to solve the ibp identities to arrive to the results
\[50\]. The ibp identities constitute a system of linear equations in which the un-
knowns are the scalar integrals themselves. The oldest approach, developed in the
original article on the ibp’s, involves a symbolic solution of the identities, treated
as recurrence equations on the indices \[7, 18\]. One introduces operators raising or
lowering one of the indices
\[\topo(n_1, \ldots, n_i, \ldots; n_7) = \topo(n_1, \ldots, n_i \pm 1, \ldots; n_7).\tag{52}\]
Each ibp equation involves the identity operator 1 and the raising operators \(i^+\) and
\(i^+j^-\). If one can combine the equations in such a way that an amplitude is written
in terms of amplitudes all containing lowering operators, then reduction to simpler
topologies has been achieved. The advantages of the symbolic approach are the
elegance and that its implementation does not require in general a lot of CPU time.
The main disadvantage is that a careful, case-by-case, inspection of the equations
is required. A more recent approach is that of replacing numerical values for the
indices and solve the resulting linear system \[13, 14\]. Let us describe this method
in practical terms. An initial linear system can be obtained by assigning to the
indices the values \(n_i = 0, 1\) for \(i = 1, \ldots, 6\) and \(n_7 = 0\). This is a set of \(8 \times 2^6 = 512\)
equations\(^9\). If these equations are not sufficient for a complete MI reduction, one
may solve a larger system in which one of the indices can also take the value \(n_i = -1\)
for \(i = 1, \ldots, 7\); this is a set of \((3 \times 2^5 \times 6 + 2^7) \times 8 = 5632\) equations. Let us stress
that all the equations with one index = -1 have to be considered simultaneously,
because the amplitudes involved are coupled by the \(i^+j^-\) operators; for example:
\[7^+6^- : \topo(1, 1, 1, 1, 1, 0; -1) \rightarrow \topo(1, 1, 1, 1, -1; 0).\tag{53}\]
One may also consider a system in which one of the indices takes the value \(n_i = 2\)
for \(i = 1, 6\); this amounts to \(3 \times 2^5 \times 6 \times 8 = 4608\) equations. If these systems are not
sufficiently large, one may consider equations with indices taking both the values 2
and \(-1\) and so on. The general idea is that, by going to larger systems, the number
of equations grows faster than the number of amplitudes, till a “critical mass” of
equations is reached, allowing the full reduction to MI’s\(^{10}\). The linear system
can be solved with the method of elimination of the variables. In general, one has
to devise a rule about which amplitudes in the system to solve first. Important
quantities for the scalar integrals are the sum of the positive indices,
\[d = \sum_{i=1}^{7} n_i \theta(n_i)\tag{54}\]
\(^9\)As already said, not all the equations in the system are linearly independent on each other and
some of them are actually trivial (i.e. \(0 = 0\)), such as for example those with five indices equal to
zero.
\(^{10}\)One of us (U.A.) wishes to thank S. Laporta for a discussion on this point.
and minus the sum of the negative indices,

\[ s = \sum_{i=1}^{7} -n_i \theta(-n_i). \]  

(55)

A convenient criterion is to solve for the amplitudes with the greatest \( t \), \( d \) and \( s \) first. The advantage of the numerical indices method is that it does not require a hand inspection of the equations — requiring patience and often a good mathematical intuition — and is therefore very general. The main disadvantage is that it may require a large CPU time and very long intermediate expressions may be generated because of the division by the coefficients of the unknowns amplitudes.

Let us now discuss the ibp identities in the framework of the second reduction scheme discussed in the previous section, the one involving independent scalar products at the numerator and loop momentum shifts. As with the auxiliary diagrams, the scalar integrals of Eq. (46) can be related by the ibp identities, which now read:

\[ \int \{d^Dk_1\}\{d^Dk_2\} \frac{\partial}{\partial k_i^\mu} \left\{ v^\mu S_{i_1}^{s_1} \cdots S_{i_t}^{s_t} \over D_{i_1}^{n_1} \cdots D_{i_t}^{n_t} \right\} = 0, \]  

(56)

where, as before, \( i = 1, 2 \) and \( v^\mu = k_1^\mu, k_2^\mu, p_1^\mu, p_2^\mu \). For a given set of indices \( \{s_i, n_j\} \) of the input amplitude, we find eight identities involving integrals with up to an additional power on a denominator and an additional power of a scalar product at the numerator. Sub-topologies with \( t - 1, t - 2, \ldots \) coming from simplifications between scalar products and corresponding denominators also appear. As in the auxiliary diagram case, discrete symmetries of various topologies generate additional identities to be combined with the ibp identities [14].

Concerning our problem, we can reduce the calculation of all the scalar integrals related to the topologies shown in Figs. 5, 6 and 7 — and therefore the calculation of the Feynman diagrams of Figs. 1 and 2 — to that of the 22 MI’s in Fig. 8. Four of them are the product of two one-loop MI’s. The picture of the diagram alone denotes the scalar integral — only denominators — while the picture of the diagram accompanied by a scalar product denotes the scalar integral with that scalar product at the numerator.

The algorithms explained in this section were implemented in a chain of programs written in the algebraic computer language FORM [19] and, in the case of the auxiliary-denominator scheme, also in Mathematica [20]. For the solution of the linear system of ibp identities, we used SOLVE [21]. This program uses the language C as a shell to manage a user-provided basic code written in FORM. External calls to Maple [22] are done in order to simplify the coefficients coming in intermediate steps.

3 Evaluation of the MI’s: the differential equations method

Let us consider a general amplitude Topo \((n_1, n_2, n_3, n_4, n_5, n_6; n_7)\) in the framework of the auxiliary diagram. As shown in the previous section, it can be decomposed,
Figure 8: The set of 22 Master Integrals. Four of them are the product of one-loop master integrals.
by means of the ibp identities, into a superposition of a set of primitive amplitudes called master integrals (MI’s). The latter cannot be computed with the ibp identities, which, in general, only allow for a reduction of the number of independent amplitudes to be individually computed. The MI’s have been computed with a variety of techniques over the years: Feynman parameters, dispersion relations, small momentum expansions, large momentum expansions, just to mention the more common ones. A technique developed over the last few years which is very efficient in our case is that of the differential equations in the external kinematical invariants \cite{8, 9, 10, 11, 12, 13, 14}. Let us then review the basics of this method. Once again, let us consider a specific example: the computation of the MI’s in Fig. 8 (k) in the auxiliary diagram scheme. Let us define:

\begin{align}
F_1(s, a, \epsilon) &= \text{Topo}(1, 0, 0, 1, 1; 0), \\
F_2(s, a, \epsilon) &= \text{Topo}(1, 0, 0, 1, 1; -1),
\end{align}

The above two MI’s appear in the decomposition of the six-denominator amplitude presented in the previous section (Fig. 8 (b)). They are functions of $s$ and $a$, as well as of the space-time dimension $D = 4 - 2\epsilon$. Let us take the derivative of $F_i$ with respect to $s$ at fixed $a$ and with respect to $a$ at fixed $s$:

\begin{align}
{s \frac{\partial}{\partial s} F_i(s, a, \epsilon)} &= p_1^\mu \frac{\partial}{\partial p_1^\mu} F_i(s, a, \epsilon) = p_2^\mu \frac{\partial}{\partial p_2^\mu} F_i(s, a, \epsilon), \\
{a \frac{\partial}{\partial a} F_i(s, a, \epsilon)}.
\end{align}

The derivatives with respect to the external momentum component $p_1^\mu$ or $p_2^\mu$ and with respect to the mass squared $a$ can be taken inside the loop integral, because of the properties of Dimensional Regularization. The resulting amplitudes can be reduced to combinations of MI’s according to the methods described in the previous section, so that:

\begin{align}
s \frac{\partial}{\partial s} F_i(s, a, \epsilon) &= \sum_j C_{ij}(s, a, \epsilon) F_j(s, a, \epsilon), \\
a \frac{\partial}{\partial a} F_i(s, a, \epsilon) &= \sum_j K_{ij}(s, a, \epsilon) F_j(s, a, \epsilon).
\end{align}

Note the double role played by the ibp identities in the calculation: they allow for the reduction of independent scalar amplitudes into combinations of master integrals and they also allow for the generation of the differential equations to be solved for the MI’s themselves.

Eqs. (61) and (62) can be simplified changing variables from $s$ and $a$ to $x = -s/a$ and $a$. This triangular change of variables implies that:

\begin{align}
x \frac{\partial}{\partial x} F_i(x, a, \epsilon) &= s \frac{\partial}{\partial s} F_i(s, a, \epsilon), \\
a \frac{\partial}{\partial a} F_i(x, a, \epsilon) &= s \frac{\partial}{\partial s} F_i(s, a, \epsilon) + a \frac{\partial}{\partial a} F_i(s, a, \epsilon) = \frac{d_i}{2} F_i(x, a, \epsilon).
\end{align}
The second equation holds because of the dimensional relation
\[ F_i(x, a, \epsilon) = a^{d_i/2} f_i(x, \epsilon), \]  
(65)

where \( d_i \) is the mass dimension of the MI: \( d_i/2 = D - \sum_{k=1}^{7} n_k^{(i)} \). In the new variables, the scale \( a \) represents an over-all scale of the amplitudes and its evolution equation is consequently trivial. We therefore concentrate on the \( x \)-evolution equation only and we set \( a = 1 \) from now on in this section:

\[ \frac{\partial}{\partial x} F_i(x, \epsilon) = \sum_{j=1}^{N_F} A_{ij}(x, \epsilon) F_j(x, \epsilon). \]  
(66)

In our case, by performing the above steps, one obtains the following linear system of coupled ordinary differential equations with variable coefficients:

\[
\begin{align*}
\frac{dF_1}{dx}(x, \epsilon) &= -\frac{1}{x} F_1(x, \epsilon) - (2 - 3\epsilon) \left( \frac{1}{x} + \frac{1}{1-x} \right) F_2(x, \epsilon) \\
&\quad - (1 - \epsilon) \left( \frac{1}{x} + \frac{1}{1-x} \right) \left[ F_3(\epsilon) + F_4(x, \epsilon) \right], \\
\frac{dF_2}{dx}(x, \epsilon) &= - \left[ (1 - \epsilon) \frac{1}{x} + (2 - 3\epsilon) \frac{1}{1-x} \right] F_2(x, \epsilon) \\
&\quad - (1 - \epsilon) \left( \frac{1}{x} + \frac{1}{1-x} \right) F_3(\epsilon) - (1 - \epsilon) \frac{1}{1-x} F_4(x, \epsilon),
\end{align*}
\]  
(67-68)

where we defined:

\[
\begin{align*}
F_3(\epsilon) &= \text{Topo}(1, 0, 0, 1, 1, 0; 0), \\
F_4(x, \epsilon) &= \text{Topo}(1, 0, 0, 0, 1, 1; 0).
\end{align*}
\]  
(69-70)

\( F_3 \) is represented in Fig. \( \mathbb{S} \) (u); it is a sunrise diagram with one massive line, evaluated at a light-cone external momentum, equivalent to the null momentum. \( F_3 \) is then effectively a vacuum amplitude, so we dropped the dependence on \( x \). It is obtained from Fig. \( \mathbb{S} \) (k) by shrinking one of the internal lines having an endpoint in the hard vertex. \( F_4 \) is represented in Fig. \( \mathbb{S} \) (v) and is the product of a massless bubble and a tadpole; it is obtained from Fig. \( \mathbb{S} \) (k) by shrinking the massless line in the bubble. Note that there are no other non-vanishing sub-topologies. We consider the sub-topologies \( F_3 \) and \( F_4 \) as known amplitudes\(^{\text{11}} \), so that the system of Eqs. (67) and (68) is schematically written as:

\[ \frac{\partial}{\partial x} F_i(x, \epsilon) = \sum_{j=1}^{n_F} A_{ij}(x, \epsilon) F_j(x, \epsilon) + \Omega_i(x; \epsilon), \]  
(71)

where the \( \Omega_i(x, \epsilon) \) are known functions constituting the non-homogeneous part of the system. Let us note that the coefficients of the homogeneous terms have a regular expansion around \( \epsilon = 0 \), involving, in this case, only two terms:

\[ A_{ij}(x, \epsilon) = A_{ij}^{(0)}(x) + \epsilon A_{ij}^{(1)}(x). \]  
(72)

\(^{\text{11}}\)Their expressions are given respectively in Eqs. (121–125) (where we put \( F_3 = F^{(4)} \)) ans Eqs. (128–132) (where \( F_4 = F^{(5)} \)).
A great simplification is offered by the fact that the system is triangular in our basis, because the equation for $dF_2/dx$ depends only on $F_2$ and not on $F_1$: $A_{21}(x, \epsilon) = 0$. Let us then consider this equation first. We substitute the explicit expression for the expansion of $\Omega_2$ in powers of $\epsilon$ and the symbolic expansion for $F_2$, up to the desired order $n$:

$$
\Omega_2(x, \epsilon) = \sum_{j=-2}^{n} \epsilon^j \Omega_2^{(j)}(x) + \mathcal{O}(\epsilon^{n+1}),
$$

$$
F_2(x, \epsilon) = \sum_{j=-2}^{n} \epsilon^j F_2^{(j)}(x) + \mathcal{O}(\epsilon^{n+1}).
$$

The double pole — the leading singularity — obeys a differential equation of the form

$$
\frac{dF_2^{(-2)}}{dx}(x) = A_{22}^{(0)}(x)F_2^{(-2)}(x) + \Omega^{(-2)}(x)
$$

where

$$
A_{22}^{(0)}(x) = -\frac{1}{x} - \frac{2}{1-x}.
$$

The associated homogeneous equation,

$$
\frac{d\omega_2}{dx}(x) = A_{22}^{(0)}(x)\omega_2(x)
$$

is solved by separation of variables to give:

$$
\omega_2(x) = \exp \left[ \int_{x}^{x} A_{22}^{(0)}(t)dt \right] = \frac{(1-x)^2}{x},
$$

with an overall constant omitted in the last member. The double pole coefficient is then computed with the method of of the variation of the constants:

$$
F_2^{(-2)}(x) = \omega_2(x) \int_{x}^{x} K_2(t)\Omega^{(-2)}(t)dt,
$$

where the primitive involves an arbitrary constant to be fixed imposing initial conditions, and where the kernel $K_2$ is given by

$$
K_2(t) = \frac{1}{\omega_2(t)} = \frac{1}{1-t} + \frac{1}{(1-t)^2}.
$$

Higher orders are computed through the recursive relation in $\epsilon$:

$$
\frac{dF_2^{(i)}}{dx} = A_{22}^{(0)}(x)F_2^{(i)}(x) + A_{22}^{(1)}(x)F_2^{(i-1)}(x) + \Omega^{(i)}(x).
$$

The solution of the above equation is again obtained with the method of variation of constants:

$$
F_2^{(i)}(x) = \omega_2(x) \int_{x}^{x} K_2(t)\tilde{\Omega}^{(i)}(t)dt,
$$

22
where
\[
\tilde{\Omega}^{(i)}(t) = \Omega^{(i)}(t) + A^{(1)}_{22}(t)F^{(i-1)}_2(t)
\] (83)
is the inhomogeneous term redefined to include the amplitude of previous order in the \(\epsilon\) expansion. Note that the kernel \(K_2\) is the same for each order in \(\epsilon\).

The full evaluation of \(F_2\) requires, as we already said, an initial condition on the \(x\) evolution. The latter can often be derived by studying the behaviour of the MI close to a threshold or a pseudothreshold. In our case, let us consider the point \(x = 1\): it is a pseudothreshold as it corresponds to \(s = -m^2 < 0\) and therefore is not a singular point for the amplitude, so that:
\[
\lim_{x \to 1} (1 - x) \frac{dF_2(x, \epsilon)}{dx} = 0.
\] (84)
By imposing the above condition to the differential equation, an initial value for the amplitude in \(x = 1\) is obtained:
\[
(2 - 3\epsilon)F_2(1; \epsilon) + (1 - \epsilon)F_3(\epsilon) + (1 - \epsilon)F_4(1; \epsilon) = 0.
\] (85)
Once \(F_2\) has been explicitly computed, one replaces the result in the first equation and solves it for \(F_1\) in the same way.

In general, however, the system is not triangular and the above method is not directly applicable. One can try a transformation to a new set of MI's,
\[
F_1, F_2 \to F'_1, F'_2,
\] (86)
in order to have the system in a triangular form. A triangularization to order \(\epsilon\) is actually sufficient, i.e. one for which \(A^{(0)}_{21} = 0\) but \(A^{(1)}_{21} \neq 0\).

The MI’s can be expressed in terms of the one-dimensional harmonic polylogarithms (HPL’s), the latter having the following recursive integral definition
\[
H(1; x) = -\log(1 - x)
\] (87)
\[
H(0, x) = \log(x)
\] (88)
\[
H(-1; x) = \log(1 + x)
\] (89)
and
\[
H(a, \bar{w}; x) = \int_0^x f(a; y)H(\bar{w}; y)dy,
\] (90)
where the basic weight functions are defined as:
\[
f(1; x) = \frac{1}{1 - x}
\] (91)
\[
f(0; x) = \frac{1}{x}
\] (92)
\[
f(-1; x) = \frac{1}{1 + x}.
\] (93)

\[12\text{In more complicated cases, it may happen that a triangular form of the system cannot be reached, and the coupled system of } n_F \text{ equations must be solved. This is equivalent to the solution of a differential equation of order } n_F \text{ for anyone of the MI’s, for which no general algorithm exists. In our problem, the number of MI’s for a given topology is at most } n_F = 2, \text{ corresponding to a second-order differential equation. Moreover, it turned out that the solution of the associated homogeneous system was rather simple, so that the complete solution could be found with the method of the variation of arbitrary constants.}\]
For a complete discussion of HPL’s and their properties we refer to [2]; for their numerical evaluation see [3].

As an "empirical" rule, we found that it was sufficient to stop the $\epsilon$ expansion beyond the order involving weight $w = 4$ HPL’s.

The massless MI’s cannot be computed with the differential equation method and a more traditional method as that of Feynman parameters or dispersion relations has to be used. As discussed in the introduction, they are indeed of the form:

\[
(\text{massless amp.}) = a^{4-n_d} \left( \frac{\mu^2}{a} \right)^{2\epsilon} G(\epsilon) x^{4-n_d-2\epsilon}
\]

(94)

Since the $x$-dependence is factorized, the $x$ evolution gives only a trivial, dimensional, information, analogous to the $a$ evolution. The non-trivial quantity to compute is $G(\epsilon)$. In other words, all the information is contained in the initial data — for example the amplitude at $x = 1$ — that can be extracted only by direct integration.

For a graphical summary of the whole method discussed in sections 2 and 3, see the flowchart in Fig. 9.

4 Results for the master integrals

In this section we present the results of our computation of the MI’s involving up to five denominators included, which constitute a necessary input for the calculation of the two-loop vertex diagrams in Figs. 1 and 2. They are expanded in a Laurent series in

\[
\epsilon = 2 - D/2,
\]

(95)

up to the required order in $\epsilon$. The coefficients of the series are expressed in terms of HPL’s [2, 3] of the variable $x$, defined as:

\[
x = \frac{q^2}{a},
\]

(96)

where $q = p_1 + p_2$, and $a = m^2$. We denote by $\mu$ the mass scale of the Dimensional Regularization (DR) — the so-called unit of mass. The explicit values of the denominators $D_i$ are given in appendix A. We work in Minkowski space and we normalize the loop measure as:

\[
\{d^D k\} = \frac{d^D k}{i\pi^{D/2} \Gamma(3 - \frac{D}{2})} = \frac{d^{(4-2\epsilon)} k}{i\pi^{(2-\epsilon)} \Gamma(1 + \epsilon)}.
\]

(97)

This definition makes the expression of the one-loop tadpole — the simplest of all loop diagrams — particularly simple (see appendix B).

\[\text{Note that with our definition of the scalar product } s = -q^2.\]
Figure 9: Flowchart of the method used for the reduction to the MI's and their evaluation.
4.1 Topology $t = 3$

\[ = \mu^{2(4-D)} \int \left\{ d^Dk_1 \right\} \left\{ d^Dk_2 \right\} \frac{1}{\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_{11}} \]

\[ = \left( \frac{\mu^2}{a} \right)^{2e} \sum_{i=-2}^{2} \epsilon_i F_i^{(1)} + \mathcal{O} \left( \epsilon^3 \right), \quad (99) \]

where:

\[ F_{-2}^{(1)} \frac{a}{a} = 0, \quad (100) \]

\[ F_{-1}^{(1)} \frac{a}{a} = - \frac{x}{4}, \quad (101) \]

\[ F_0^{(1)} \frac{a}{a} = -x \left[ \frac{13}{8} - \frac{1}{2} H(0, x) \right], \quad (102) \]

\[ F_1^{(1)} \frac{a}{a} = -x \left[ \frac{115}{4} - 2\zeta(2) - 13H(0, x) + 4H(0, 0, x) \right], \quad (103) \]

\[ F_2^{(1)} \frac{a}{a} = -x \left[ \frac{865}{16} - \frac{13\zeta(2)}{2} - 5\zeta(3) - \left( \frac{115}{4} - 2\zeta(2) \right) H(0, x) \right. \]

\[ \left. + 13H(0, 0, x) - 4H(0, 0, 0, x) \right], \quad (104) \]

The above amplitude is easily computed with Feynman parameters or with an iteration of the general formula for a massless bubble given in [7].

\[ = \mu^{2(4-D)} \int \left\{ d^Dk_1 \right\} \left\{ d^Dk_2 \right\} \frac{1}{\mathcal{D}_2 \mathcal{D}_{11} \mathcal{D}_{12}} \]

\[ = \left( \frac{\mu^2}{a} \right)^{2e} \sum_{i=-2}^{2} \epsilon_i F_i^{(2)} + \mathcal{O} \left( \epsilon^3 \right), \quad (106) \]

\[ \langle k_1 \cdot k_2 \rangle = \mu^{2(4-D)} \int \left\{ d^Dk_1 \right\} \left\{ d^Dk_2 \right\} \frac{k_1 \cdot k_2}{\mathcal{D}_2 \mathcal{D}_{11} \mathcal{D}_{12}} \]

\[ = \left( \frac{\mu^2}{a} \right)^{2e} \sum_{i=-2}^{2} \epsilon_i F_i^{(3)} + \mathcal{O} \left( \epsilon^3 \right), \quad (108) \]

where:

\[ F_{-2}^{(2)} \frac{a}{a} = - \frac{1}{2}, \quad (109) \]

\[ F_{-1}^{(2)} \frac{a}{a} = - \frac{3}{2} - \frac{x}{4}, \quad (110) \]
\[
\begin{align*}
\frac{F_0^{(2)}}{a} &= -3 - \zeta(2) - \frac{13}{8} x - \frac{1}{2} \left[ \frac{1}{x} - x \right] H(-1, x) + H(0, -1, x), \quad (111) \\
\frac{F_1^{(2)}}{a} &= -\frac{15}{4} - 3\zeta(2) + \zeta(3) - \frac{x}{16} [115 + 8\zeta(2)] - \frac{1}{4} \left( \frac{1}{x} - x \right) [13H(-1, x) \\
&\quad - 8H(-1, -1, x)] + \frac{1}{2} \left[ 6 + 2x - \frac{1}{x} \right] H(0, -1, x) \\
&\quad - 4H(0, -1, -1, x) + H(0, 0, -1, -1, x), \\
\frac{F_2^{(2)}}{a} &= \frac{21}{8} - 6\zeta(2) - \frac{9\zeta^2(2)}{5} + 3\zeta(3) - \left[ \frac{865}{32} + \frac{13\zeta(2)}{4} - \frac{\zeta(3)}{2} \right] x \\
&\quad - \frac{1}{8} [115 + 8\zeta(2)] \left[ \frac{1}{x} - x \right] H(-1, x) + 13 \left[ \frac{1}{x} - x \right] H(-1, -1, x) \\
&\quad + \frac{1}{4} \left[ 24 + 8\zeta(2) + 26x - \frac{13}{x} \right] H(0, -1, x) - \left[ \frac{1}{x} - x \right] \times \\
&\quad \times [8H(-1, -1, -1, x) - 3H(-1, 0, -1, x)] + \left[ 3 + x - \frac{1}{2x} \right] \times \\
&\quad \times [H(0, 0, -1, x) - 4H(0, -1, -1, x)] + 16H(0, -1, -1, -1, x) \\
&\quad - 6H(0, -1, 0, -1, x) - 4H(0, 0, -1, -1, x) \\
&\quad + H(0, 0, 0, -1, x)], \\
\frac{F_0^{(3)}}{a^2} &= -\frac{1}{4}, \\
\frac{F_1^{(3)}}{a^2} &= -\frac{3}{4} - \frac{x}{4} - \frac{x^2}{24}, \\
\frac{F_0^{(3)}}{a^2} &= -\frac{19}{12} - \frac{\zeta(2)}{2} - \frac{35x}{24} - \frac{13x^2}{48} + \frac{1}{12} \left[ 3 - \frac{2}{x} + 6x + x^2 \right] H(-1, x) \\
&\quad + \frac{1}{2} H(0, 0, -1, x), \\
\frac{F_1^{(3)}}{a^2} &= -\frac{5}{2} - \frac{3\zeta(2)}{2} + \frac{\zeta(3)}{2} - \left[ \frac{95}{16} + \frac{\zeta(2)}{2} \right] x - \left[ \frac{115}{96} + \frac{\zeta(2)}{12} \right] x^2 \\
&\quad + \left[ \frac{35}{24} + \frac{37x}{12} + \frac{13x^2}{24} - \frac{13}{12x} \right] H(-1, x) + \left[ \frac{7}{4} + x + \frac{x^2}{6} \right] \times \\
&\quad - \frac{1}{6x} H(0, -1, x) - \left[ 1 + 2x + \frac{x^2}{3} - \frac{2}{3x} \right] H(-1, -1, x) \\
&\quad - 2H(0, -1, -1, x) + \frac{1}{2} H(0, 0, -1, x), \\
\frac{F_2^{(3)}}{a^2} &= -\frac{41}{24} - \frac{19\zeta(2)}{6} - \frac{9\zeta^2(2)}{10} + \frac{3\zeta(3)}{2} - \frac{1}{12} \left[ 35\zeta(2) - 6\zeta(3) \\
&\quad + \frac{2015}{8} \right] x - \frac{1}{24} \left[ 13\zeta(2) - 2\zeta(3) + \frac{865}{8} \right] x^2 + \frac{1}{2} \left[ \frac{95}{8} + \zeta(2) \\
&\quad + \left( \frac{105}{4} + 2\zeta(2) \right) x + \left( \frac{115}{24} + \frac{\zeta(2)}{3} \right) x^2 - \left( \frac{115}{12} + \frac{2\zeta(2)}{3} \right) \frac{1}{x} \right] \times
\end{align*}
\]
× \( H(-1, x) + \left[ \frac{37}{8} + \zeta(2) + 6x + \frac{13x^2}{12} - \frac{13}{12x} \right] H(0, -1, x) \)

\[-\frac{1}{6} \left[ 35 + 74x + 13x^2 - \frac{26}{x} \right] H(-1, -1, x) - \frac{1}{6} \left[ 3 + 6x \right.\]

\[+ x^2 - \frac{2}{x} \right] \left[ 3H(-1, 0, -1, x) - 8H(-1, -1, -1, x) \right] \]

\[+ \frac{1}{12} \left[ 21 + 12x + 2x^2 - \frac{2}{x} \right] \left[ H(0, 0, -1, x) - 4H(0, -1, -1, x) \right] \]

\[+ \frac{1}{2} \left[ H(0, 0, 0, -1, x) - 6H(0, -1, 0, -1, x) - 4H(0, 0, -1, -1, x) \right.\]

\[+ 16H(0, -1, -1, -1, x) \right] . \]  

\[
\begin{align*}
\frac{F^{(4)}}{a} &= -\frac{1}{2}, \\
\frac{F^{(4)}}{a} &= -\frac{3}{2}, \\
\frac{F^{(4)}}{a} &= -\frac{7}{2} - \zeta(2), \\
\frac{F^{(4)}}{a} &= -\frac{15}{2} - 3\zeta(2) + \zeta(3), \\
\frac{F^{(4)}}{a} &= -\frac{31}{2} - 7\zeta(2) - \frac{9\zeta^2(2)}{5} + 3\zeta(3).
\end{align*}
\]

The above amplitude is easily computed with Feynman parameters.

\[
\begin{align*}
\frac{F^{(5)}}{a} &= -1,
\end{align*}
\]

\[
\begin{align*}
\frac{F^{(5)}}{a} &= -1,
\end{align*}
\]

\[
\begin{align*}
\frac{F^{(5)}}{a} &= -1,
\end{align*}
\]
\[
\begin{align*}
\frac{F_{-2}^{(5)}}{a} &= -3 + H(0, x), \\
\frac{F_{-1}^{(5)}}{a} &= -7 + \zeta(2) + 3H(0, x) - H(0, 0, x), \\
\frac{F_{0}^{(5)}}{a} &= -15 + 3\zeta(2) + 2\zeta(3) + (7 - \zeta(2))H(0, x) - 3H(0, 0, x) \\
&\quad + H(0, 0, 0, x), \\
\frac{F_{1}^{(5)}}{a} &= -31 + 7\zeta(2) + \frac{9}{10}\zeta^2(2) + 6\zeta(3) + (15 - 3\zeta(2) - 2\zeta(3))H(0, x) \\
&\quad - (7 - \zeta(2))H(0, 0, x) + 3H(0, 0, 0, x) - H(0, 0, 0, 0, x). \\
\end{align*}
\]

4.2 Topology \( t = 4 \)

\[
\begin{align*}
\frac{\mu^2(4-D)}{a} &= \int \{d^Dk_1\} \{d^Dk_2\} \frac{1}{D_2D_3D_9D_12}, \\
&= \left( \frac{\mu^2}{a} \right)^2 \sum_{i=-2}^2 \epsilon^i F_i^{(6)} + \mathcal{O} (\epsilon^3),
\end{align*}
\]

where:

\[
\begin{align*}
F_{-2}^{(6)} &= \frac{1}{2}, \\
F_{-1}^{(6)} &= \frac{5}{2} - \left[ 1 + \frac{1}{x} \right] H(-1, x), \\
F_0^{(6)} &= \frac{19}{2} + \zeta(2) - H(0, x) - \left[ 1 + \frac{1}{x} \right] [4H(-1, x) - 2H(-1, -1, x) \\
&\quad - H(-1, 0, x) + H(0, -1, x)], \\
F_1^{(6)} &= \frac{65}{2} + 3\zeta(2) - \zeta(3) - 7H(0, x) - 12 \left[ 1 + \frac{1}{x} \right] H(-1, x) + 2H(0, 0, x) \\
&\quad + \left[ 1 + \frac{1}{x} \right] [4H(-1, 0, x) - 4H(0, -1, x) + 8H(-1, -1, x) \\
&\quad - 4H(-1, 1, -1, x) - 2H(-1, -1, 0, x) + 2H(-1, 0, -1, x) \\
&\quad - 2H(-1, 0, 0, x) + 2H(0, -1, -1, x) + H(0, -1, 0, x) \\
&\quad - H(0, 0, -1, x) \\
&\quad - H(0, 0, 0, x)], \\
F_2^{(6)} &= \frac{211}{2} + 5\zeta(2) + \frac{9\zeta^2(2)}{5} - 9\zeta(3) - (33 - 2\zeta(2))H(0, x) \\
&\quad - 2(16 - 3\zeta(3)) \left[ 1 + \frac{1}{x} \right] H(-1, x) + 14H(0, 0, x) - 4H(0, 0, 0, x) \\
&\quad + \left[ 1 + \frac{1}{x} \right] [2(6 - \zeta(2))H(-1, 0, x) - 12H(0, -1, x) + 24H(-1, -1, x) \\
&\quad - 8H(-1, 1, -1, x) - 8H(-1, 0, -1, x) - 8H(-1, 0, 0, x) \\
&\quad - 8H(0, -1, -1, x) + 4H(0, -1, 0, x) - 4H(0, 0, -1, x) \\
&\quad - H(0, 0, -1, x). \\
\end{align*}
\]
\[-16H(-1, -1, -1, -1, x) + 8H(-1, -1, -1, 1, x) + 4H(-1, -1, -1, 0, x) - 4H(-1, -1, 0, -1, x) + 4H(-1, -1, 0, 0, x) - 4H(-1, 0, -1, -1, x) - 2H(-1, 0, -1, 0, x) + 2H(-1, 0, 0, -1, x) + 4H(-1, 0, 0, 0, x) - 4H(0, -1, -1, 1, x) - 2H(0, -1, -1, 0, x) + 2H(0, -1, 0, -1, x) - 2H(0, -1, 0, 0, x) + 2H(0, 0, -1, -1, x) + H(0, 0, -1, 0, x) - H(0, 0, 0, -1, x)] \cdot \tag{139}
}

\[
\begin{align*}
\text{where:} \\
F^{(7)}_{-2} &= 1, \\
F^{(7)}_{-1} &= 4 - 2H(0, x), \\
F^{(7)}_0 &= 12 - 2\zeta(2) - 8H(0, x) + 4H(0, 0, x), \\
F^{(7)}_1 &= 32 - 8\zeta(2) - 4\zeta(3) - 4[6 - \zeta(2)]H(0, x) + 16H(0, 0, x) - 8H(0, 0, 0, x), \\
F^{(7)}_2 &= 80 - 24\zeta(2) - \frac{4\zeta^2(2)}{5} - 16\zeta(3) - 8[8 - 2\zeta(2) - \zeta(3)]H(0, x) + 8[6 - \zeta(2)]H(0, 0, x) - 32H(0, 0, 0, x) + 16H(0, 0, 0, 0, x). \tag{146}
\end{align*}
\]

\[
\begin{align*}
\text{where:} \\
F^{(8)}_{-2} &= 1, \\
F^{(8)}_{-1} &= 4 - H(0, x) - \left[1 + \frac{1}{x}\right]H(-1, x), \\
F^{(8)}_0 &= 12 - \zeta(2) - 4H(0, x) - 4\left[1 + \frac{1}{x}\right]H(-1, x) + H(0, 0, x) + \left[1 + \frac{1}{x}\right][H(-1, 0, x) + 2H(-1, 1, x)]. \tag{151}
\end{align*}
\]
\[ F_{1}^{(8)} = 32 - 4\zeta(2) - 2\zeta(3) - [12 - \zeta(2)]H(0, x) - [12 - \zeta(2)]\left[1 + \frac{1}{x}\right] \times \]
\[ \times H(-1, x) + 4H(0, 0, x) + \left[1 + \frac{1}{x}\right][4H(-1, 0, x) + 8H(-1, -1, x)] \]
\[ - H(0, 0, x) - \left[1 + \frac{1}{x}\right][H(-1, 0, 0, x) + 2H(-1, -1, 0, x)] \]
\[ + 4H(-1, -1, -1, 1, x), \] (152)
\[ F_{2}^{(8)} = 80 - 12\zeta(2) - \frac{9\zeta^2(2)}{10} - 8\zeta(3) - [32 - 4\zeta(2) - 2\zeta(3)]H(0, x) \]
\[ + \left(1 + \frac{1}{x}\right)H(-1, x) + [12 - \zeta(2)]\left\{H(0, 0, x) + \left(1 + \frac{1}{x}\right) \times \right\} \]
\[ \times [2H(-1, 0, x) + 2H(-1, -1, 1, x)] \} - 4H(0, 0, 0, x) \]
\[ - \left[1 + \frac{1}{x}\right][8H(-1, -1, 0, x) + 4H(-1, 0, 0, x) + 16H(-1, -1, -1, 1, x)] \]
\[ - H(-1, 0, 0, x) - 4H(-1, -1, -1, 0, x) - 2H(-1, -1, 0, 0, x) \]
\[ - 8H(-1, -1, -1, -1, 1, x)] + H(0, 0, 0, 0, x). \] (153)

\[ \mu^{2(4-D)} \int \{d^{D}k_{1}\}\{d^{D}k_{2}\} \frac{1}{D_{2}D_{7}D_{8}D_{12}} \] (154)
\[ = \left(\frac{\mu^{2}}{a}\right)^{2e} \sum_{i=-2}^{1} \epsilon^{i} F_{i}^{(9)} + \mathcal{O} \left( \epsilon^{2} \right), \] (155)
\[ (p_{2} \cdot k_{1}) = \mu^{2(4-D)} \int \{d^{D}k_{1}\}\{d^{D}k_{2}\} \frac{p_{2} \cdot k_{1}}{D_{2}D_{7}D_{8}D_{12}} \] (156)
\[ = \left(\frac{\mu^{2}}{a}\right)^{2e} \sum_{i=-2}^{1} \epsilon^{i} F_{i}^{(10)} + \mathcal{O} \left( \epsilon^{2} \right), \] (157)

where:
\[ F_{-2}^{(9)} = \frac{1}{2}, \] (158)
\[ F_{-1}^{(9)} = \frac{5}{2} - H(0, x), \] (159)
\[ F_{0}^{(9)} = \frac{19}{2} - 2\zeta(2) - 5H(0, x) + H(0, 0, x) - \left[1 - \frac{1}{x}\right] H(1, 0, x) \]
\[ + \frac{1}{x} H(0, 1, 0, x), \] (160)
\[ F_{1}^{(9)} = \frac{65}{2} - 10\zeta(2) - \zeta(3) - [19 - \zeta(2)]H(0, x) - 3\zeta(2) \left[1 - \frac{1}{x}\right] H(1, x) \]
\[ + 5H(0, 0, x) + \frac{3\zeta(2)}{x} H(0, 1, x) - 5 \left[1 - \frac{1}{x}\right] H(1, 0, x) \]
\[-H(0, 0, 0, x) + \frac{2}{x} H(0, 1, 0, x) - \left[ 1 - \frac{1}{x} \right] H(0, 1, 0, x)\]
\[-H(1, 0, 0, x) + 3 H(1, 1, 0, x)] + \frac{1}{x} [H(0, 0, 1, 0, x) - H(0, 1, 0, x) + 3 H(0, 1, 1, 0, x)],

\[
\frac{F_{-2}^{(10)}}{a} = \frac{x}{16},
\]
\[
\frac{F_{-1}^{(10)}}{a} = \frac{x}{32} [9 - 4 H(0, x)],
\]
\[
\frac{F_{0}^{(10)}}{a} = \frac{5}{8} + \frac{x}{64} [63 - 16 \zeta(2)] - \frac{3}{16} [2 + 3 x] H(0, x) + \frac{x}{8} H(0, 0, x) - \frac{1}{8} \left[ x - \frac{1}{x} \right] H(1, 0, x) + \frac{1}{4 x} H(0, 1, 0, x),
\]
\[
\frac{F_{1}^{(10)}}{a} = \frac{57}{16} + \frac{405 x}{128} - \frac{9}{8} (1 + x) \zeta(2) - \frac{x}{8} \zeta(3) - \frac{1}{32} [70 + 63 x] - 4 \zeta(2) x H(0, x) - \frac{3 \zeta(2)}{8} \left[ x - \frac{1}{x} \right] H(1, 0, x) + \frac{3}{16} [2 + 3 x] H(0, 0, x) - \frac{1}{16} \left[ 2 + 9 x - \frac{11}{x} \right] H(1, 0, x) + \frac{3 \zeta(2)}{4 x} H(0, 1, 0, x) - \frac{1}{8} H(0, 0, 0, x) + \frac{1}{2 x} H(0, 1, 0, x) - \frac{1}{8} \left[ x - \frac{1}{x} \right] [H(0, 1, 0, x) - H(1, 0, 0, x)] + 3 H(1, 1, 0, x) + \frac{1}{4 x} [H(0, 0, 1, 0, x) - H(0, 1, 0, 0, x)] + 3 H(0, 1, 1, 0, x)].
\]

\[
\mu^{2(4-D)} \int \{d^D k_1\} \{d^D k_2\} \frac{1}{D_1 D_2 D_7 D_8} = \left( \frac{\mu^2}{a} \right)^{2e} \sum_{i=-2}^{2} \epsilon^i F_{i}^{(11)} + O(\epsilon^3),
\]

where:
\[
F_{-2}^{(11)} = \frac{1}{2},
\]
\[
F_{-1}^{(11)} = \frac{5}{2} - H(0, x),
\]
\[
F_{0}^{(11)} = \frac{19}{2} - 5 H(0, x) + 2 H(0, 0, x),
\]
\[
F_{1}^{(11)} = \frac{65}{2} - 4 \zeta(3) - 19 H(0, x) + 10 H(0, 0, x) - 4 H(0, 0, 0, x),
\]
\[
F_{2}^{(11)} = \frac{211}{2} - \frac{12}{5} \zeta^2(2) - 20 \zeta(3) - [65 - 8 \zeta(3)] H(0, x) + 38 H(0, 0, x) - 20 H(0, 0, 0, x) + 8 H(0, 0, 0, 0, x).
\]
As discussed in section 3, the above amplitude, being massless, cannot be calculated by means of the differential equations technique. It is easily computed with Feynman parameters.

\[ \mu^{2(4-D)} \int \{d^D k_1\}\{d^D k_2\} \frac{1}{D_2D_4D_8D_{12}} \]

\[ = \left( \frac{\mu^2}{a} \right)^{2\epsilon} \left[ \sum_{i=-2}^{2} \epsilon^i F_i^{(12)} + \mathcal{O}(\epsilon^3) \right], \]

where:

\[
F_{-2}^{(12)} = \frac{1}{2},
\]

\[
F_{-1}^{(12)} = \frac{3}{2},
\]

\[
F_0^{(12)} = \frac{5}{2} + \zeta(2) + H(0, x) + \left[ 1 - \frac{1}{x} \right] H(1, 0, x),
\]

\[
F_1^{(12)} = \frac{-1}{2} + 5\zeta(2) - \zeta(3) + 7H(0, x) + 2\zeta(2) \left[ 1 - \frac{1}{x} \right] H(1, x)
- 2H(0, 0, x) + \left[ 1 - \frac{1}{x} \right] [4H(1, 0, x) + H(0, 1, 0, x)]
- 2H(1, 0, 0, x) + 2H(1, 1, 0, x)] ,
\]

\[
F_2^{(12)} = \frac{-51}{2} + 19\zeta(2) + \frac{9\zeta^2(2)}{5} + \zeta(3) + [33 - 2\zeta(2)]H(0, x) + 4[\zeta(3)
+ 2\zeta(2)] \left[ 1 - \frac{1}{x} \right] H(1, x) - 14H(0, 0, x) + 2\zeta(2) \left[ 1 - \frac{1}{x} \right] H(0, 1, x)
+ 2[6 - \zeta(2)] \left[ 1 - \frac{1}{x} \right] H(1, 0, x) + 4\zeta(2) \left[ 1 - \frac{1}{x} \right] H(1, 1, x)
+ 4H(0, 0, 0, x) + \left[ 1 - \frac{1}{x} \right] [4H(0, 1, 0, x) - 8H(1, 0, 0, x)]
+ 8H(1, 1, 0, x) + H(0, 0, 1, 0, x) - 2H(0, 1, 0, 0, x)
+ 2H(0, 1, 1, 0, x) + 4H(1, 0, 0, 0, x) + 2H(1, 0, 1, 0, x)
- 4H(1, 1, 0, 0, x) + 4H(1, 1, 1, 0, x)].
\]

### 4.3 Topology \( t = 5 \)

\[ \mu^{2(4-D)} \int \{d^D k_1\}\{d^D k_2\} \frac{1}{D_1D_4D_7D_8D_{13}} \]

\[ = \left( \frac{\mu^2}{a} \right)^{2\epsilon} \left[ \sum_{i=-1}^{0} \epsilon^i F_i^{(13)} + \mathcal{O}(\epsilon^4) \right], \]
where:
\[
aF_{-1}^{(13)} = \frac{1}{x} [H(0, 0, -1, x) + H(0, 1, 0, x)],
\]
\[
aF_0^{(13)} = \frac{1}{x} [3\zeta(2)H(0, 1, x) - 4H(0, 0, -1, -1, x) + H(0, 0, 0, -1, x)
+ H(0, 0, 1, 0, x) - H(0, 1, 0, 0, x)] + 3H(0, 1, 1, 0, x)].
\]

\[
\mathcal{A} = \mu^2(4 - D) \int \{d^Dk_1\} \{d^Dk_2\} \frac{1}{D_2^1D_4^1D_7^1D_8^1D_{12}^1}
\]
\[
= \left(\frac{\mu^2}{a}\right)^{2\epsilon} F_0^{(14)} + \mathcal{O}(\epsilon),
\]

where:
\[
aF_0^{(14)} = \frac{1}{x} [\zeta(2)H(0, 1, x) + H(0, 1, 0, 0, x) + H(0, 1, 1, 0, x)].
\]

\[
\mathcal{A} = \mu^2(4 - D) \int \{d^Dk_1\} \{d^Dk_2\} \frac{1}{D_1^1D_2^1D_7^1D_8^1D_{15}^1}
\]
\[
= \left(\frac{\mu^2}{a}\right)^{2\epsilon} F_0^{(15)} + \mathcal{O}(\epsilon),
\]

\[
(k_1 \cdot k_2) = \mu^2(4 - D) \int \{d^Dk_1\} \{d^Dk_2\} \frac{k_1 \cdot k_2}{D_1^1D_2^1D_7^1D_8^1D_{15}^1}
\]
\[
= \left(\frac{\mu^2}{a}\right)^{2\epsilon} \sum_{i=-2}^1 \epsilon_i^{(16)} + \mathcal{O}(\epsilon^2),
\]

where:
\[
aF_0^{(15)} = \frac{1}{x} [2\zeta(2)H(0, -1, x) - H(0, -1, 0, -1, x) + H(0, -1, 0, 0, x)],
\]
\[
aF_{-2}^{(16)} = \frac{1}{2},
\]
\[
aF_{-1}^{(16)} = 2 - H(0, x),
\]
\[
aF_0^{(16)} = 7 - 3\zeta(2) - \frac{5}{2}H(0, x) - (2 - \zeta(2)) \left[1 + \frac{1}{x}\right] H(-1, x)
+ \frac{1}{2} H(0, 0, x) + \frac{1}{2} \left[3 - \frac{1}{x}\right] H(0, -1, x) + \frac{1}{2} \left[1 + \frac{1}{x}\right] H(-1, 0, 0, x)
- H(-1, 0, -1, x)],
\]
\[
aF_1^{(16)} = 24 - \frac{17}{2} \zeta(2) + \frac{\zeta(3)}{2} - \frac{1}{2} [7 - 3\zeta(2)] H(0, x) - \left(13 - 2\zeta(2)
\right).
\[
\begin{align*}
&+ \frac{3}{2} \zeta(3) \left[ 1 + \frac{1}{x} \right] H(-1, x) - \frac{\zeta(2)}{2} \left[ 1 + \frac{1}{x} \right] H(-1, 0, x) \\
&+ 5H(0, -1, x) + \left( \zeta(2) - \frac{7}{2} \right) \left[ 1 + \frac{1}{x} \right] H(0, -1, x) \\
&+ 2(4 - \zeta(2)) \left[ 1 + \frac{1}{x} \right] H(-1, -1, x) + \frac{1}{2} H(0, 0, 0, x) \\
&- 8H(0, -1, -1, x) + 2H(0, 0, -1, x) + \frac{1}{2} \left[ 4H(0, 0, 0, x) - \frac{1}{x} \right] \\
&- H(0, 0, -1, x) - 2H(-1, 0, -1, x) + 2H(-1, 0, 0, x) \\
&+ 2H(-1, -1, 0, -1, x) - 2H(-1, -1, 0, 0, x) + 4H(-1, 0, -1, -1, x) \\
&- H(-1, 0, 0, -1, x) - 3H(-1, 0, 0, 0, x) - H(0, -1, 0, -1, x) \\
&+ H(0, -1, 0, 0, x)]. \tag{195}
\end{align*}
\]

The above topology is the only 5-denominator one having 2 MF's.

\[
\begin{align*}
&\left[ \begin{array}{c}
\end{array} \right] = \mu^{2(4-D)} \int \{ d^D k_1 \} \{ d^D k_2 \} \frac{1}{D_1 D_2 D_6 D_{11} D_{15}} \\
&\left[ \begin{array}{c}
\end{array} \right] = \left( \frac{\mu^2}{a} \right)^2 \sum_{i=-1}^{0} \epsilon^i F^{(17)}_i + O(\epsilon, x) \tag{197},
\end{align*}
\]

where:

\[
\begin{align*}
aF^{(17)}_{-1} &= \frac{1}{x} H(0, 1, 0, x) \tag{198}, \\
aF^{(17)}_{0} &= \frac{1}{x} \left[ 2\zeta(2)H(0, 1, x) + H(0, 0, 0, x) - 2H(0, 1, 0, 0, x) \\
&+ 2H(0, 1, 1, 0, x) \right] \tag{199}.
\end{align*}
\]

\[
\begin{align*}
&\left[ \begin{array}{c}
\end{array} \right] = \mu^{2(4-D)} \int \{ d^D k_1 \} \{ d^D k_2 \} \frac{1}{D_2 D_4 D_6 D_{10} D_{12}} \\
&\left[ \begin{array}{c}
\end{array} \right] = \left( \frac{\mu^2}{a} \right)^2 \sum_{i=-1}^{1} \epsilon^i F^{(18)}_i + O(\epsilon^2) \tag{201},
\end{align*}
\]

where:

\[
\begin{align*}
aF^{(18)}_{-1} &= -\frac{1}{x} H(1, 0, x) \tag{202}, \\
aF^{(18)}_{0} &= -\frac{1}{x} \left[ \zeta(2)H(1, x) + 2H(1, 0, x) - 3H(1, 0, 0, x) + H(1, 1, 0, x) \right], \tag{203} \\
aF^{(18)}_{1} &= -\frac{1}{x} \left[ 2(\zeta(2) + \zeta(3))H(1, x) + \zeta(2)H(1, 1, x) + (4 - 3\zeta(2))H(1, 0, x) \\
&- 6H(1, 0, 0, x) + 2H(1, 1, 0, x) + 7H(1, 0, 0, 0, x) \\
&- 3H(1, 1, 0, 0, x) + H(1, 1, 1, 0, x) \right]. \tag{204}
\end{align*}
\]

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The above amplitude has a simple ultraviolet pole coming from the massless bubble.

\[
\begin{align*}
\text{bubble} & \quad = \mu^{2(4-D)} \left\{ \frac{1}{D_1 D_3 D_9 D_{11} D_{13}} \right\} \\
& = \left( \frac{\mu^2}{a} \right)^2 \sum_{i=0}^{2} \epsilon^i F^{(19)}_i + \mathcal{O}(\epsilon^2),
\end{align*}
\]

where:

\[
\begin{align*}
aF^{(19)}_0 & = \frac{2}{x} \left[ \zeta(2)H(1, x) - H(0, 0, -1, x) + H(1, 0, 0, x) \\
& \quad - 2H(1, 0, -1, x) \right], \\
aF^{(19)}_1 & = \frac{2}{x} \left[ (2\zeta(2) - \zeta(3))H(1, x) + \zeta(2)(H(1, 0, x) + H(1, 1, x)) \\
& \quad - 2H(0, 0, -1, x) - 4H(1, 0, -1, x) + 2H(1, 0, 0, x) \\
& \quad + 4H(0, 0, 0, -1, x) - H(0, 0, 0, 0, x) + 8H(1, 0, 0, 0, x) + 8H(1, 0, 0, 0, x) \\
& \quad + 3H(1, 0, 0, 0, x) - H(1, 0, 0, 0, x) - 2H(1, 1, 0, 0, x) \\
& \quad + H(1, 1, 0, 0, x) \right].
\end{align*}
\]

The finite part of the above amplitude is also given in [23].

\[
\begin{align*}
\text{bubble} & \quad = \mu^{2(4-D)} \left\{ \frac{1}{D_2 D_3 D_4 D_5 D_{12}} \right\} \\
& = \left( \frac{\mu^2}{a} \right)^2 \sum_{i=-1}^{1} \epsilon^i F^{(20)}_i + \mathcal{O}(\epsilon^2),
\end{align*}
\]

where:

\[
\begin{align*}
aF^{(20)}_{-1} & = -\frac{1}{x}H(1, 0, x), \\
aF^{(20)}_0 & = -\frac{1}{x} \left[ (2\zeta(2) + 3\zeta(3))H(1, x) + \zeta(2)H(0, 1, x) + 4H(1, 0, x) \\
& \quad + (2\zeta(2) + 3\zeta(3))H(1, x) + H(0, 0, 0, x) + 2H(0, 1, 0, x) \\
& \quad - 2H(0, 1, 0, x) - H(0, 1, 0, x) + 4H(1, 0, 0, x) \\
& \quad + H(1, 0, 0, x) - 2H(1, 1, 0, x) + H(1, 1, 0, x) \right], \\
aF^{(20)}_1 & = -\frac{1}{x} \left[ (2\zeta(2) + 3\zeta(3))H(1, x) + \zeta(2)(H(1, 0, 0, x) + 2H(1, 1, 0, x) \\
& \quad + H(1, 0, 0, x) + H(1, 1, 0, x) + 2H(1, 1, 0, x) \\
& \quad - 2H(1, 1, 0, x) + H(1, 1, 0, x) \right].
\end{align*}
\]

The above amplitude has a simple ultraviolet pole coming from the nested bubble.
5 Six-denominator diagrams

In this section we present the results of all the six-denominator diagrams with up to one massive line entering the EW form factor. The ladder and vertex insertion diagrams — the planar topologies — are all reducible, while both the crossed ladders — the non-planar topologies — are MI’s, i.e. cannot be reduced to simpler topologies by using the ibp identities. Even though these diagrams have a different “diagrammatic status”, we present them together because they are the most important ones and may be used for reference. Series representations for the pole and finite parts of the ladder and crossed ladder diagrams have been given in \[23, 24\].

The authors compute a large number of terms in a small momentum expansion of the diagrams and then fit the results to an assumed basis of functions and transcendent constants. In the above paper, formulas for the resummation of the series in terms of Nielsen polylogarithms \[4, 5\] are also reported. By using them, we could check that the pole parts of our results agree with theirs. Finally, the leading logarithms of all the six denominator amplitudes have been computed in \[25\] by means of a pole approximation for the vector boson propagators; we are in agreement with them also.

5.1 Planar topologies

\[
\begin{align*}
\mu^{2(4-D)} \int \{d^D k_1\} \{d^D k_2\} \frac{1}{D_1 D_2 D_3 D_4 D_5 D_6} & = (214) \\
= \left( \frac{\mu^2}{a} \right)^{2\varepsilon} \sum_{i=-4}^{0} \varepsilon^i A_i + \mathcal{O}(\varepsilon),
\end{align*}
\]

where:

\[
\begin{align*}
a^2 A_{-4} & = \frac{1}{4x^2}, & (216) \\
a^2 A_{-3} & = -\frac{1}{2x^2} H(0, x), & (217) \\
a^2 A_{-2} & = \frac{1}{x^2} \left[ \zeta(2) + H(0, 0, x) \right], & (218) \\
a^2 A_{-1} & = \frac{1}{x^2} \left[ 5\zeta(3) - 2\zeta(2) H(0, 0) - 2H(0, 0, 0, x) \right], & (219) \\
a^2 A_0 & = \frac{1}{x^2} \left[ \frac{11\zeta^2(2)}{5} - 10\zeta(3) H(0, 0) + 4\zeta(2) H(0, 0, 0) + 4H(0, 0, 0, 0, x) \right]. & (220)
\end{align*}
\]

The above diagram has also been computed in \[26\] using Feynman parameters and in \[27\] using dispersion relations; we computed it by means of the reduction to simpler (massless) amplitudes, the latter computed with Feynman parameters. We are in complete agreement with these references. The leading singularity of this
diagram is a pole in $\epsilon$ of fourth order or, equivalently, a forth power in $\log(x)$ in the finite part for $x \to 0$. We have indeed a soft and a collinear singularity for each loop, each one contributing by a simple pole to the amplitude.

$$= \mu^{2(4-D)} \int \{d^D k_1\} \{d^D k_2\} \frac{1}{D_2 D_4 D_5 D_7 D_8 D_{12}}$$

$$= \left(\frac{\mu^2}{a}\right)^{2\epsilon} B_0 + O(\epsilon), \quad (222)$$

where:

$$a^2 B_0 = \frac{1}{x^2} \left[ 6\zeta(3)H(1, x) - 2\zeta(2)H(1, 1, x) + H(1, 0, 1, 0, x) - 2H(1, 1, 0, 0, x) - 2H(1, 1, 0, 0, x) \right]. \quad (223)$$

As expected on the basis of one’s physical intuition, the above diagram is infrared finite. The massive vector boson cannot indeed propagate for large distances and the same is true for the photon, which is “trapped” inside the external loop.

$$= \mu^{2(4-D)} \int \{d^D k_1\} \{d^D k_2\} \frac{1}{D_1 D_4 D_5 D_7 D_8 D_{13}}$$

$$= \left(\frac{\mu^2}{a}\right)^{2\epsilon} \sum_{i=-2}^{0} \epsilon^i C_i + O(\epsilon), \quad (225)$$

where:

$$a^2 C_{-2} = -\frac{1}{x^2} H(1, 0, x), \quad (226)$$

$$a^2 C_{-1} = -\frac{1}{x^2} \left[ \zeta(2)H(1, x) + 4H(1, 0, -1, x) + 3H(1, 1, 0, x) - 3H(1, 0, 0, x) \right], \quad (227)$$

$$a^2 C_0 = -\frac{1}{x^2} \left[ 2\zeta(3)H(1, x) + 7\zeta(2)H(1, 1, x) - 3\zeta(2)H(1, 0, x) - 16H(1, 0, -1, x) + 6H(1, 0, 0, -1, x) + 7H(1, 0, 0, 0, x) + 3H(1, 0, 1, 0, x) + 4H(1, 1, -1, x) - 5H(1, 1, 0, 0, x) + 9H(1, 1, 1, 0, x) \right]. \quad (228)$$

Unlike Eq. (221), in the above diagram the photon is not trapped by the massive line and can propagate to asymptotic times. As a consequence, we have a double infrared pole, having a similar dynamical origin to the one in the massless one-loop vertex.
\[ = \mu^{2(4-D)} \int \{d^Dk_1\} \{d^Dk_2\} \frac{1}{D_1D_2D_3D_4D_5D_6D_7D_8D_15} \]
\[ = \left( \frac{\mu^2}{a} \right)^2 \sum_{i=-2}^0 \epsilon^i E_i + \mathcal{O}(\epsilon) , \]

where:

\[ a^2E_{-2} = \frac{1}{2} \left[ \frac{1}{x} - \frac{1}{(1+x)} \right] \left[ \zeta(2) + H(0,-1,x) \right] , \]
\[ a^2E_{-1} = \frac{1}{2} \left[ \frac{1}{x} - \frac{1}{(1+x)} \right] \left[ \zeta(3) - 2\zeta(2)(H(0,x) + H(-1,x)) - 2H(-1,0,-1,x) - 2H(0,-1,0,0) + H(0,0,-1,x) \right] , \]
\[ a^2E_0 = \left[ \frac{1}{x} - \frac{1}{(1+x)} \right] \left\{ -\frac{\zeta^2(2)}{10} - \zeta(3)[H(0,x) + H(-1,x)] + \zeta(2)[2H(0,0,x) + 2H(-1,-1,x) + 2H(-1,0,x)] - 3H(0,-1,x) \right\} , \]

The above diagram has also been computed in [26] using Feynman parameters and in [27] using dispersion relations; we have computed it by means of the reduction
to simpler (massless) amplitudes, the latter computed with Feynman parameters in section 3. We are in complete agreement with the previous results. The massless vertex correction has at most a triple pole in $\epsilon$ and is therefore less singular than the massless ladder or crossed ladder (see later). As well known, the vertex correction is indeed neglected in the double logarithmic analysis of QED form factors.

$$\int \{d^D k_1\} \{d^D k_2\} \frac{1}{D_1 D_2 D_3 D_4 D_5 D_{17}}$$

$$= \left( \frac{\mu^2}{a} \right)^{2\epsilon} \sum_{i=-3}^{0} \epsilon^i I_i + \mathcal{O}(\epsilon), \quad (240)$$

where:

$$a^2 I_{-3} = -\frac{3}{4 x}, \quad (242)$$

$$a^2 I_{-2} = -\frac{1}{2 x^2} [(1 + x) H(-1, x) - x H(0, x)], \quad (243)$$

$$a^2 I_{-1} = \frac{1}{x^2} \left[ \zeta(2) x - 2 x H(0, -1, x) + 2(1 + x) H(-1, -1, x) \right], \quad (244)$$

$$a^2 I_0 = \frac{1}{x^2} \left\{ \zeta(2)(1 + x) H(-1, x) - \zeta(2) x H(0, x) - x H(0, 0, 0, x) + 2 x [4 H(0, -1, -1, x) - H(0, 0, -1, x)] + (1 + x) [H(-1, 0, 0, x) + 2 H(-1, 0, -1, x) - 8 H(-1, -1, -1, x)] \right\}. \quad (245)$$

$$\int \{d^D k_1\} \{d^D k_2\} \frac{1}{D_3 D_4 D_5 D_6 D_{12}}$$

$$= \left( \frac{\mu^2}{a} \right)^{2\epsilon} \sum_{i=-2}^{0} \epsilon^i J_i + \mathcal{O}(\epsilon), \quad (247)$$

where:

$$a^2 J_{-2} = \frac{\zeta(2)}{2 x}, \quad (248)$$

$$a^2 J_{-1} = \frac{1}{2 x} \left\{ \zeta(3) - 2 \zeta(2) (H(0, x) + H(1, x)) - 2 H(0, 1, 0, x) - 2 H(1, 1, 0, x) \right\}, \quad (249)$$

$$a^2 J_0 = -\frac{1}{x} \left\{ \frac{\zeta^2(2)}{10} + \zeta(3) [H(0, x) + H(1, x)] - \zeta(2) [2 H(0, 0, x) - 3 H(0, 1, x) + 2 H(1, 0, x) - 3 H(1, 1, x) - H(0, 0, 1, 0, x) + 2 H(0, 1, 0, x) - 3 H(0, 1, 1, 0, x) - H(1, 0, 1, 0, x) + 2 H(1, 1, 0, x) - 3 H(1, 1, 1, 0, x)] \right\}. \quad (250)$$
\begin{align*}
\Phi &= \mu^{2(4-D)} \int \{d^D k_1\} \{d^D k_2\} \frac{1}{D_1 D_2 D_4^2 D_5 D_7} \\
&= \left( \frac{\mu^2}{a} \right)^{2\epsilon} \sum_{i=-2}^{0} e^i K_i + O(\epsilon), \quad (251) \\
\end{align*}

where:

\begin{align*}
a^2 K_{-2} &= \frac{3}{2x^2}, \\
a^2 K_{-1} &= \frac{3}{2x^2} \left[ 1 - 2H(0, x) \right], \\
a^2 K_0 &= \frac{3}{2x^2} \left[ 3 - 2\zeta(2) - 2H(0, x) + 4H(0, 0, x) \right]. \quad (255) \\
\end{align*}

\begin{align*}
\Phi &= \mu^{2(4-D)} \int \{d^D k_1\} \{d^D k_2\} \frac{1}{D_1 D_2 D_4^2 D_5 D_7 D_{13}} \\
&= \left( \frac{\mu^2}{a} \right)^{2\epsilon} \sum_{i=-2}^{0} e^i L_i + O(\epsilon), \quad (257) \\
\end{align*}

where:

\begin{align*}
a^2 L_{-2} &= -\frac{1}{2x} \left[ 1 - \frac{4}{x} \right], \\
a^2 L_{-1} &= \frac{1}{2x} \left[ \frac{5}{x} + \left( 1 - \frac{4}{x} \right) H(0, x) - \left( 1 + \frac{2}{x} + \frac{1}{x^2} \right) H(-1, x) \right], \\
a^2 L_0 &= \frac{1}{4x} \left[ 17 \right] + 2\zeta(2) \left( 1 - \frac{4}{x} \right) + \left( 3 - \frac{8}{x} \right) H(0, x) - 3 \left( 1 + \frac{4}{x} \right) \\
&\quad + \frac{3}{x^2} \left( 1 + \frac{4}{x} \right) H(-1, x) - 2 \left( 1 + \frac{4}{x} \right) H(0, 0, x) - 6H(0, -1, x) \\
&\quad + 8 \left( 1 + \frac{2}{x} + \frac{1}{x^2} \right) H(-1, -1, x). \quad (260) \\
\end{align*}

\begin{align*}
\Phi &= \mu^{2(4-D)} \int \{d^D k_1\} \{d^D k_2\} \frac{1}{D_2 D_4^2 D_5 D_7 D_{12}} \\
&= \left( \frac{\mu^2}{a} \right)^{2\epsilon} \sum_{i=-2}^{0} e^i M_i + O(\epsilon), \quad (262) \\
\end{align*}

where:

\begin{align*}
a^2 M_{-2} &= -\frac{1}{2x}, \quad (263) \\
\end{align*}
\[ a^2 M_{-1} = -\frac{1}{2x} + \left(\frac{1}{x} + \frac{1}{(1-x)}\right) H(0, x), \]  
(264)

\[ a^2 M_0 = -\frac{1}{2x} (7 - 2\zeta(2)) + \left(\frac{1}{x} + \frac{1}{(1-x)}\right) \left[2\zeta(2) + H(0, x) - 2H(0, 0, x) + 2H(1, 0, x)\right]. \]  
(265)

\[ \Rightarrow \mu^{2(4-D)} \int \{d^D k_1\} \{d^D k_2\} \frac{1}{D_1^2 D_2 D_3 D_4 D_5} \]  
(266)

\[ = \left(\frac{\mu^2}{a}\right)^{2\epsilon} \sum_{i=-2}^{0} e^i N_i + O(\epsilon), \]  
(267)

where:

\[ a^2 N_{-2} = -\frac{3}{4x^2}, \]  
(268)

\[ a^2 N_{-1} = \frac{3}{2x^2} H(0, x), \]  
(269)

\[ a^2 N_0 = -\frac{3}{x^2} (1 + H(0, 0, x)). \]  
(270)

\[ = \mu^{2(4-D)} \int \{d^D k_1\} \{d^D k_2\} \frac{1}{D_1^2 D_2 D_3 D_4 D_5 D_{12}} \]  
(271)

\[ = \left(\frac{\mu^2}{a}\right)^{2\epsilon} \sum_{i=-3}^{0} e^i O_i + O(\epsilon), \]  
(272)

where:

\[ a^2 O_{-3} = \frac{1}{4x}, \]  
(273)

\[ a^2 O_{-2} = \frac{1}{2x} [1 - H(0, x)], \]  
(274)

\[ a^2 O_{-1} = \frac{1}{x} [1 - H(0, x) + H(0, 0, x) - H(1, 0, x)], \]  
(275)

\[ a^2 O_0 = \frac{1}{x} [2 - 2\zeta(3) - 2H(0, x) + \zeta(2)H(1, x) + 2H(0, 0, x) + 2H(1, 0, x) - 2H(1, 0, x) + H(1, 1, 0, x)]. \]  
(276)

\[ = \mu^{2(4-D)} \int \{d^D k_1\} \{d^D k_2\} \frac{1}{D_1^2 D_4 D_5 D_{13}} \]  
(277)

\[ = \left(\frac{\mu^2}{a}\right)^{2\epsilon} \sum_{i=-2}^{0} e^i P_i + O(\epsilon), \]  
(278)
where:

\[ a_{P-2} = \frac{2}{x^2}, \]  
\[ a_{P-1} = -\frac{2}{x^2}H(0, x), \]  
\[ a_{P0} = \frac{2}{x^2}[1 - \zeta(2) + H(0, 0, x)]. \]  

The above amplitude is actually a 5 denominator one; we include it here just for practical convenience.

### 5.2 Crossed ladders

The massless crossed ladder is a MI, i.e. it cannot be reduced to simpler (massless) topologies by means of ibp identities. Its direct computation is rather involved so we report the result as taken from ref. [26]. We checked Gonsalves result for the crossed ladder with Eq. (3.4) in [27]. If we include in the latter equation the dimensional factor \( x^{-2-2\epsilon} \) (in our notation), a complete agreement between the two computations is found\(^{14}\).

\[
\begin{align*}
\text{\includegraphics[width=1in]{crossed_ladder.png}} & = \mu^{2(4-D)} \int \{d^D k_1\} \{d^D k_2\} \frac{1}{D_1 D_2 D_3 D_4 D_6 D_{11}} \\
& = \left( \frac{\mu^2}{a} \right)^{2\epsilon} \sum_{i=-4}^{0} \epsilon^i Q_i + O(\epsilon),
\end{align*}
\]

where:

\[ a^2 Q_{-4} = \frac{1}{x^2}, \]  
\[ a^2 Q_{-3} = -\frac{2}{x^2}H(0, x), \]  
\[ a^2 Q_{-2} = -\frac{1}{x^2}[7\zeta(2) - 4H(0, 0, x)], \]  
\[ a^2 Q_{-1} = -\frac{1}{x^2}[27\zeta(3) - 14\zeta(2)H(0, x) + 8H(0, 0, 0, x)], \]  
\[ a^2 Q_0 = -\frac{1}{x^2} \left[ \frac{57\zeta^2(2)}{5} - 54\zeta(3)H(0, x) + 28\zeta(2)H(0, 0, x) - 16H(0, 0, 0, x) \right]. \]  

Note that the above diagram is the only one having a fourth order (infrared) pole in \( \epsilon \) together, as we have seen, with the planar ladder.

\(^{14}\)Note that the definition of \( \epsilon \) in the above cited Van Neerven paper is different from ours: \( \epsilon_{VN} = -2\epsilon \).
The coefficients not explicitly given are zero. and powers of \( L \) where:

\[
\text{The exact expression of the massless diagrams involves only an overall power of } \frac{1}{x} \text{, i.e. the expansion in powers of } \frac{1}{x} \text{ up to the order } 1/x^4 \text{ included. As already noted in the introduction, these expansions are relevant for the study of the structure of the infrared logarithms coming from multiple emission. In this section we give the asymptotic expansions of the six denominator scalar}
\]

\[
\begin{align*}
\mathcal{E}_2 &= \mu^{2(4-D)} \int \{d^D k_1\} \{d^D k_2\} \frac{1}{D_1 D_2 D_3 D_6 D_{14} D_{15}} \\
&= \left(\frac{\mu^2}{a}\right)^{2\epsilon} \sum_{i=-2}^{0} \epsilon^i R_i + \mathcal{O}(\epsilon),
\end{align*}
\]

where:

\[
\begin{align*}
a^2 R_{-2} &= \left[ \frac{1}{2} \left( \frac{1}{x} + \frac{1}{1-x} \right) \right] [\zeta(2) - 2H(0, -1, x)], \\
a^2 R_{-1} &= \left[ \frac{1}{2} \left( \frac{1}{x} + \frac{1}{1-x} \right) \right] [\zeta(3) - 2\zeta(2)(H(0, x) - 2H(1, x)) \\
&\quad + 8H(0, -1, -1, x) - 4H(0, 0, -1, x) + 2H(0, 1, 0, x) \\
&\quad - 8H(1, 0, -1, x)], \\
a^2 R_0 &= \left[ \frac{1}{2} \left( \frac{1}{x} + \frac{1}{1-x} \right) \right] \left\{ -\frac{\zeta^2(2)}{10} - \zeta(3)[H(0, x) - 2H(1, x)] \\
&\quad + \zeta(2)[2H(0, 0, x) + 8H(1, 1, x) - 4H(1, 0, x) - 6H(0, -1, x) \\
&\quad + 7H(0, 1, x)] - 16H(0, -1, -1, x) + 8H(0, -1, 0, -1, x) \\
&\quad - 2H(0, -1, 0, 0, x) + 8H(0, 0, -1, -1, x) - 2H(0, 0, 0, -1, x) \\
&\quad + H(0, 0, 1, 0, x) - 8H(0, 1, 0, -1, x) - H(0, 1, 0, 0, x) \\
&\quad + 3H(0, 1, 1, 0, x) + 16H(1, 0, -1, -1, x) - 8H(1, 0, 0, -1, x) \\
&\quad + 4H(1, 0, 1, 0, x) - 16H(1, 1, 0, -1, x) \right\}.
\end{align*}
\]

The differential equation for the above amplitude is a rather lengthy expression, as it involves fourteen subtopologies.

6 Large momentum expansion, \( |s| \gg m^2 \)

In this section we give the asymptotic expansions of the six denominator scalar integrals, i.e. the expansion in powers of \( 1/x \) and \( L = \log(x) \) up to the order \( 1/x^4 \) included. As already noted in the introduction, these expansions are relevant for the study of the structure of the infrared logarithms coming from multiple emission. The exact expression of the massless diagrams involves only an overall power of \( x \) and powers of \( L \), so their expansion coincides with the former and is not duplicated. The coefficients not explicitly given are zero.

\[
\begin{align*}
\mathcal{E}_2 &= \mu^{2(4-D)} \int \{d^D k_1\} \{d^D k_2\} \frac{1}{D_1 D_2 D_3 D_6 D_{14} D_{15}} \\
&= \left(\frac{\mu^2}{a}\right)^{2\epsilon} \sum_{j=-2}^{0} \epsilon^j B_{-j} + \mathcal{O}(\epsilon),
\end{align*}
\]

where:

\[
\begin{align*}
a^2 B_{-2} &= \frac{37\zeta^2(2)}{5} - 4\zeta(3)L + 2\zeta(2)L^2 + \frac{1}{24}L^4,
\end{align*}
\]
\[ a^2B_{-3}^0 = -10 - 2\zeta(2) + 4\zeta(3) - 4[1 + \zeta(2)]L - \frac{1}{2}L^2 - \frac{1}{6}L^3, \quad (296) \]
\[ a^2B_{-4}^0 = -\frac{43}{8} + \frac{\zeta(2)}{2} + 2\zeta(3) - \left[ \frac{5}{2} + 2\zeta(2) \right]L - \frac{1}{8}L^2 - \frac{1}{12}L^3. \quad (297) \]

\[ \left( \frac{\mu^2}{a} \right)^2 \sum_{j=2}^{4} \frac{1}{x^j} \left[ \sum_{i=-2}^{0} e^{iC_{i,j}} \right] + \mathcal{O} \left( \frac{1}{x^5} \right), \quad (298) \]

where:

\[ a^2C_{-2}^{-2} = 2\zeta(2) + \frac{1}{2}L^2, \quad (299) \]
\[ a^2C_{-2}^{-1} = 3\zeta(3) - \zeta(2)L - \frac{1}{3}L^2, \quad (300) \]
\[ a^2C_{-2}^{0} = \frac{61\zeta^2(2)}{10} - 11\zeta(3)L + 2\zeta(2)L^2 + \frac{1}{6}L^4, \quad (301) \]
\[ a^2C_{-3}^{-2} = -1 - L, \quad (302) \]
\[ a^2C_{-3}^{-1} = \zeta(2) - L + L^2, \quad (303) \]
\[ a^2C_{-3}^{0} = -26 + 3\zeta(2) + 11\zeta(3) - 2[3 + 2\zeta(2)]L + 3L^2 - \frac{2}{3}L^3, \quad (304) \]
\[ a^2C_{-4}^{-2} = -\frac{1}{4} - \frac{1}{2}L, \quad (305) \]
\[ a^2C_{-4}^{-1} = \frac{-5}{4} + \frac{\zeta(2)}{2} - \frac{7}{4}L + \frac{1}{2}L^2, \quad (306) \]
\[ a^2C_{-4}^{0} = \frac{-117}{8} + \frac{17\zeta(2)}{4} + \frac{11\zeta(3)}{2} - \left[ \frac{5}{2} + 2\zeta(2) \right]L + \frac{13}{4}L^2 - \frac{1}{3}L^3, \quad (307) \]

The expansion above is in agreement with that obtained in [28] by using a kinematic region decomposition of the integrand.

\[ \left( \frac{\mu^2}{a} \right)^2 \sum_{j=2}^{4} \frac{1}{x^j} \left[ \sum_{i=-2}^{0} e^{iE_{i,j}} \right] + \mathcal{O} \left( \frac{1}{x^5} \right), \quad (308) \]

where:

\[ a^2E_{-2}^{-2} = \zeta(2) + \frac{1}{4}L^2, \quad (309) \]
\[ a^2E_{-2}^{-1} = \frac{5\zeta(3)}{2} - \frac{3\zeta(2)}{2}L - \frac{1}{4}L^3, \quad (310) \]
\[ a^2E_{-2}^{0} = \frac{7\zeta^2(2)}{4} - 8\zeta(3)L + \frac{7\zeta(2)}{4}L^2 + \frac{7}{48}L^4, \quad (311) \]
\[ a^2E_{-3}^{-2} = -\frac{1}{2} - \zeta(2) - \frac{1}{4}L^2, \quad (312) \]
\[ a^2E_{-3}^{-1} = \frac{1}{2} - 2\zeta(2) - \frac{5\zeta(3)}{2} + \frac{3\zeta(2)}{2}L - \frac{1}{2}L^2 + \frac{1}{4}L^3, \quad (313) \]
\[ a^2E_{-3}^{0} = \frac{-17}{2} + 2\zeta(2) - \frac{7\zeta^2(2)}{4} - 5\zeta(3) - 4L + 3\zeta(2)L + 8\zeta(3)L \]
\[-\frac{7\zeta(2)}{4} L^2 + \frac{1}{2} L^3 - \frac{7}{48} L^4,\]  

\[a^2 E_{-4}^2 = \frac{5}{8} + \zeta(2) + \frac{1}{4} L^2,\]  

\[a^2 E_{-4}^{-1} = -\frac{1}{16} + 3\zeta(2) + \frac{5\zeta(3)}{2} - \frac{3\zeta(2)}{2} L + \frac{3}{4} L^2 - \frac{1}{4} L^3,\]  

\[a^2 E_{-4}^0 = \frac{297}{32} - \frac{\zeta(2)}{2} + \frac{7\zeta^2(2)}{4} - \frac{15\zeta(3)}{2} + \frac{11}{2} L - \frac{9\zeta(2)}{2} L - 8\zeta(3) L \]
\[+ \frac{1}{2} L^2 + \frac{7\zeta(2)}{4} L^2 - \frac{3}{4} L^3 + \frac{7}{48} L^4,\]  

\[= \left(\frac{\mu^2}{a}\right)^2 \sum_{j=1}^{4} \frac{1}{x^j} \left[ \sum_{i=-3}^{0} \epsilon^i I_{-j}^i \right] + \mathcal{O} \left( \frac{1}{x^5} \right),\]  

where:

\[a^2 I_{-1}^{-3} = -\frac{3}{4},\]  

\[a^2 I_{-1}^{-1} = -\zeta(2),\]  

\[a^2 I_{-1}^{0} = 4\zeta(3),\]  

\[a^2 I_{-2}^{-2} = -\frac{1}{2} - \frac{1}{2} L,\]  

\[a^2 I_{-2}^{-1} = 2 + 2L + L^2,\]  

\[a^2 I_{-2}^{0} = -5 + 3\zeta(2) - 4\zeta(3) - 5L + 3\zeta(2)L - \frac{5}{2} L^2 - \frac{5}{6} L^3,\]  

\[a^2 I_{-3}^{-3} = -\frac{1}{4},\]  

\[a^2 I_{-3}^{-1} = \frac{1}{2} + L,\]  

\[a^2 I_{-3}^{0} = \frac{21}{8} + \frac{3\zeta(2)}{2} + \frac{1}{4} L - \frac{5}{4} L^2,\]  

\[a^2 I_{-4}^{-2} = \frac{1}{12},\]  

\[a^2 I_{-4}^{-1} = \frac{2}{9} - \frac{1}{3} L,\]  

\[a^2 I_{-4}^{0} = -\frac{211}{216} - \frac{\zeta(2)}{2} - \frac{47}{36} L + \frac{5}{12} L^2,\]  

\[\left(\frac{\mu^2}{a}\right)^2 \sum_{j=1}^{4} \frac{1}{x^j} \left[ \sum_{i=-3}^{0} \epsilon^i J_{-j}^i \right] + \mathcal{O} \left( \frac{1}{x^5} \right),\]  

where:

\[a^2 J_{-1}^{-2} = \frac{\zeta(2)}{2},\]  

\[a^2 J_{-1}^{-1} = -\frac{\zeta(3)}{2}.\]
\[ a^2 J_{-1}^0 = \frac{6\zeta^2(2)}{5}, \]  
\[ a^2 J_{-2}^1 = 1 + \zeta(2) + L + \frac{1}{2}L^2, \]  
\[ a^2 J_{-2}^0 = -4 + \zeta(2) - 4\zeta(3) - 4L + \zeta(2)L - 2L^2 - \frac{2}{3}L^3, \]  
\[ a^2 J_{-3}^1 = -\frac{5}{8} + \frac{\zeta(2)}{2} - \frac{1}{4}L + \frac{1}{4}L^2, \]  
\[ a^2 J_{-3}^0 = \frac{13}{8} + \frac{7\zeta(2)}{4} - 2\zeta(3) + \frac{9}{4}L + \frac{\zeta(2)}{2}L + \frac{3}{4}L^2 - \frac{1}{3}L^3, \]  
\[ a^2 J_{-4}^1 = -\frac{59}{108} + \frac{\zeta(2)}{3} - \frac{7}{18}L + \frac{1}{6}L^2, \]  
\[ a^2 J_{-4}^0 = \frac{67}{648} + \frac{29\zeta(2)}{18} - \frac{4\zeta(3)}{3} + \frac{155}{108}L + \frac{\zeta(2)}{3}L + \frac{37}{36}L^2 - \frac{2}{9}L^3, \]  
\[ \frac{1}{B5} = \left( \frac{\mu^2}{a} \right)^2 \sum_{j=1}^{4} \frac{1}{x^j} \left[ \sum_{i=-2}^{0} \epsilon^i L_{-j}^i \right] + \mathcal{O} \left( \frac{1}{x^5} \right), \]  
where:

\[ a^2 L_{-1}^2 = -\frac{1}{2}, \]  
\[ a^2 L_{-1}^0 = -\zeta(2), \]  
\[ a^2 L_{-2}^2 = 2, \]  
\[ a^2 L_{-2}^1 = 2 - 3L, \]  
\[ a^2 L_{-2}^0 = 5 - 2\zeta(2) - 3L + 3L^2, \]  
\[ a^2 L_{-3}^1 = -\frac{3}{4} - \frac{1}{2}L, \]  
\[ a^2 L_{-3}^0 = -2 + \frac{3}{4}L + L^2, \]  
\[ a^2 L_{-4}^1 = -\frac{1}{6}, \]  
\[ a^2 L_{-4}^0 = \frac{1}{6} + \frac{2}{3}L, \]  
\[ \frac{1}{B6} = \left( \frac{\mu^2}{a} \right)^2 \sum_{j=1}^{4} \frac{1}{x^j} \left[ \sum_{i=-2}^{0} \epsilon^i M_{-j}^i \right] + \mathcal{O} \left( \frac{1}{x^5} \right), \]  
where:

\[ a^2 M_{-1}^2 = -\frac{1}{2}, \]  
\[ a^2 M_{-1}^1 = -\frac{1}{2}, \]  
\[ a^2 M_{-1}^0 = -\frac{3}{2} - \zeta(2), \]
\[ a^2 M_{-2} = -L, \]
\[ a^2 M_{-2}^0 = 2\zeta(2) - L + 2L^2, \]
\[ a^2 M_{-3} = -L, \]
\[ a^2 M_{-3}^0 = 2 + 2\zeta(2) - 3L + 2L^2, \]
\[ a^2 M_{-4} = -L, \]
\[ a^2 M_{-4}^0 = \frac{5}{2} + 2\zeta(2) - 4L + 2L^2, \]
\[ = \left( \frac{\mu}{a} \right)^2 \sum_{j=1}^{4} \frac{1}{x^j} \left[ \sum_{i=-3}^{0} \epsilon^i O_{-j}^i \right] + \mathcal{O} \left( \frac{1}{x^5} \right), \]

where:
\[ a^{2}O_{-1} = \frac{1}{4}, \]
\[ a^{2}O_{-1}^0 = \frac{1}{2} - \frac{1}{2}L, \]
\[ a^{2}O_{-1}^{-1} = 1 - 2\zeta(2) - L, \]
\[ a^{2}O_{-1}^{-2} = 2 - 4\zeta(2) - \zeta(3) - 2L - \zeta(2)L, \]
\[ a^{2}O_{-2}^{-1} = 1 + L, \]
\[ a^{2}O_{-2}^{-2} = -1 - \zeta(2) - L - \frac{3}{2}L^2, \]
\[ a^{2}O_{-3}^{-1} = \frac{1}{4} + \frac{1}{2}L, \]
\[ a^{2}O_{-3}^{0} = \frac{7}{8} - \frac{\zeta(2)}{2} + \frac{3}{4}L - \frac{3}{4}L^2, \]
\[ a^{2}O_{-4}^{-1} = \frac{1}{9} + \frac{1}{3}L, \]
\[ a^{2}O_{-4}^{0} = \frac{25}{36} - \frac{\zeta(2)}{3} + \frac{5}{6}L - \frac{1}{2}L^2, \]

\[ = \left( \frac{\mu}{a} \right)^2 \sum_{j=2}^{4} \frac{1}{x^j} \left[ \sum_{i=-2}^{0} \epsilon^i R_{-j}^i \right] + \mathcal{O} \left( \frac{1}{x^5} \right), \]

where:
\[ a^{2}R_{-2} = \frac{\zeta(2)}{2} + \frac{1}{2}L^2, \]
\[ a^{2}R_{-2}^0 = -\frac{21\zeta(3)}{2} + 3\zeta(3)L - \frac{5}{6}L^3, \]
\[ a^{2}R_{-3}^{0} = -\frac{163\zeta^2(2)}{10} + 36\zeta(3)L - \frac{19\zeta(2)}{2}L^2 + \frac{13}{24}L^4, \]
\[ a^{2}R_{-3}^{-1} = -1 + \frac{\zeta(2)}{2} + \frac{1}{2}L^2, \]
\[ a^{2}R_{-3}^{-2} = 8 + 2\zeta(2) - \frac{21\zeta(3)}{2} + 9L + 3\zeta(2)L + 2L^2 - \frac{5}{6}L^3, \]

\[ 48 \]
\[ a^2 R_{-3}^0 = 34 + 15\zeta(2) - \frac{163}{10} \zeta^2(2) - 42\zeta(3) - 2L + 12\zeta(2)L + 36\zeta(3)L - 13L^2 - \frac{19\zeta(2)}{2}L^2 - \frac{10}{3}L^3 + \frac{13}{24}L^4, \] (378)

\[ a^2 R_{-4}^2 = -\frac{3}{4} + \frac{\zeta(2)}{2} + \frac{1}{2}L^2, \] (379)

\[ a^2 R_{-4}^{-1} = \frac{15}{2} + 3\zeta(2) - \frac{21\zeta(3)}{2} + \frac{37}{4}L + 3\zeta(2)L + 3L^2 - \frac{5}{6}L^3, \] (380)

\[ a^2 R_{-4}^0 = \frac{453}{8} + \frac{87\zeta(2)}{4} - \frac{163\zeta^2(2)}{10} - 63\zeta(3) + \frac{31}{2}L + 18\zeta(2)L + 36\zeta(3)L - \frac{39}{4}L^2 - \frac{19\zeta(2)}{2}L^2 - 5L^3 + \frac{13}{24}L^4. \] (381)

### 7 Small momentum expansions, \(|s| \ll m^2\)

In this section we present the small momentum expansions of the six-denominator diagrams, i.e. expansions in powers of \(x\) and \(L = \log(x)\) up to first order in \(x\) included. As in the previous section, we do not consider the massless scalar integrals since their expansion is trivial. The coefficients not explicitly given are zero.

\[ \begin{align*}
\begin{array}{c}
\text{Diagram 1}
\end{array}
&= \left( \frac{\mu^2}{a} \right)^{2e} \sum_{j=-1}^{1} x^j B_j^0 + O(x^2),
\end{align*} \] (382)

where:

\[ a^2 B_{-1}^0 = 6\zeta(3), \] (383)

\[ a^2 B_0^0 = -3 - \zeta(2) + 3\zeta(3) + 2L - \frac{1}{2}L^2, \] (384)

\[ a^2 B_1^0 = -\frac{17}{12} - \zeta(2) + 2\zeta(3) + \frac{5}{4}L - \frac{1}{2}L^2, \] (385)

\[ \begin{align*}
\begin{array}{c}
\text{Diagram 2}
\end{array}
&= \left( \frac{\mu^2}{a} \right)^{2e} \sum_{j=-1}^{1} x^j \left[ \sum_{i=-2}^{0} c^i C^i_j \right] + O(x^2),
\end{align*} \] (386)

where:

\[ a^2 C_{-1}^{-2} = 1 - L, \] (387)

\[ a^2 C_{-1}^{-1} = 3 - \zeta(2) - 3L + \frac{3}{2}L^2, \] (388)

\[ a^2 C_{-1}^0 = 7 - 3\zeta(2) - 2\zeta(3) - 7L + 3\zeta(2)L + \frac{7}{2}L^2 - \frac{7}{6}L^3, \] (389)

\[ a^2 C_{-2}^0 = \frac{1}{4} - \frac{1}{2}L, \] (390)

\[ a^2 C_{-1}^1 = \frac{5}{8} - \frac{\zeta(2)}{2} - \frac{9}{4}L + \frac{3}{4}L^2, \] (391)
\[
\begin{align*}
a^2C_0 &= \frac{89}{16} - \frac{17\zeta(2)}{4} - \zeta(3) - \frac{49}{8}L + \frac{3\zeta(2)}{2}L + \frac{17}{8}L^2 - \frac{7}{12}L^3, \\
ad &= 16 - 17\zeta(2) - 49L + 3\zeta(2)L + \frac{17}{8}L^2 - \frac{7}{12}L^3, \\
\end{align*}
\]  
(392)

\[
\begin{align*}
a^2C_{-1} &= \frac{1}{9} - \frac{1}{3}L, \\
\end{align*}
\]  
(393)

\[
\begin{align*}
a^2C_1 &= \frac{31}{36} - \frac{\zeta(2)}{3} - \frac{11}{6}L + \frac{1}{2}L^2, \\
\end{align*}
\]  
(394)

\[
\begin{align*}
a^2C_0 &= \frac{4709}{648} - \frac{23\zeta(2)}{6} - \frac{2\zeta(3)}{3} - \frac{160}{27}L + \zeta(2)L + \frac{59}{36}L^2 - \frac{7}{18}L^3, \\
\end{align*}
\]  
(395)

where:

\[
\begin{align*}
a^2E_{-1} &= \frac{\zeta(2)}{2}, \\
\end{align*}
\]  
(397)

\[
\begin{align*}
a^2E_{-1} &= \frac{\zeta(3)}{2} - \zeta(2)L, \\
\end{align*}
\]  
(398)

\[
\begin{align*}
a^2E_0 &= -\frac{\zeta^2(2)}{10} - \zeta(3)L + \zeta(2)L^2, \\
\end{align*}
\]  
(399)

\[
\begin{align*}
a^2E_{-2} &= \frac{1}{2} - \frac{\zeta(2)}{2}, \\
\end{align*}
\]  
(400)

\[
\begin{align*}
a^2E_{-1} &= \frac{5}{2} - \zeta(2) - \frac{\zeta(3)}{2} - L + \zeta(2) L, \\
\end{align*}
\]  
(401)

\[
\begin{align*}
a^2E_0 &= -\frac{13}{2} + 5\zeta(2) + \frac{1}{10}\zeta^2(2) - \zeta(3) - 3L + 2\zeta(2) L + \zeta(3) L + \frac{1}{2}L^2, \\
\end{align*}
\]  
(402)

\[
\begin{align*}
a^2E_{-2} &= -\frac{5}{8} + \frac{\zeta(2)}{2}, \\
\end{align*}
\]  
(403)

\[
\begin{align*}
a^2E_{-1} &= -\frac{53}{16} + \frac{3\zeta(2)}{2} + \frac{\zeta(3)}{2} + \frac{5}{4}L - \zeta(2)L, \\
&\quad -\zeta(2)L^2, \\
\end{align*}
\]  
(404)

\[
\begin{align*}
a^2E_0 &= -\frac{333}{32} + \frac{29\zeta(2)}{4} - \frac{\zeta^2(2)}{10} + \frac{3\zeta(3)}{2} - \frac{39}{8}L - 3\zeta(2)L - \zeta(3)L \\
&\quad -\frac{5}{8}L^2 + \zeta(2)L^2, \\
\end{align*}
\]  
(406)

\[
\begin{align*}
\frac{\mu^2}{a} = \left( \sum_{j=-1}^{0} x^j \right)^2 + O(x^2), \\
\end{align*}
\]  
(396)

where:

\[
\begin{align*}
a^2L_{-3} &= 3 \frac{3}{4}, \\
\end{align*}
\]  
(408)

\[
\begin{align*}
a^2L_{-2} &= -\frac{1}{2} + \frac{1}{2}L, \\
\end{align*}
\]  
(409)

\[
\begin{align*}
a^2L_{-1} &= \zeta(2), \\
\end{align*}
\]  
(410)
\[ a^2 I_{-1}^0 = 1 + \zeta(2) - L - \zeta(2)L + \frac{1}{2} L^2 - \frac{1}{6} L^3, \quad (411) \]
\[ a^2 I_{-2}^0 = -\frac{1}{4}, \quad (412) \]
\[ a^2 I_{-1}^{-1} = -1, \quad (413) \]
\[ a^2 I_0^0 = \frac{1}{8} + \frac{\zeta(2)}{2} - \frac{3}{4} L + \frac{1}{4} L^2, \quad (414) \]
\[ a^2 I_1^{-1} = \frac{1}{12}, \quad (415) \]
\[ a^2 I_{-1}^1 = -\frac{1}{2}, \quad (416) \]
\[ a^2 I_0^1 = \frac{215}{216} - \frac{\zeta(2)}{6} + \frac{5}{36} L - \frac{1}{12} L^2, \quad (417) \]
\[ \frac{\beta}{BD} = \left( \frac{\mu^2}{a} \right)^2 e^x \sum_{j=-1}^{1} x^j \left[ \sum_{i=-2}^{0} \epsilon^i J_j^i \right] + O(x^2), \quad (418) \]

where:

\[ a^2 J_{-1}^0 = \frac{\zeta(2)}{2}, \quad (419) \]
\[ a^2 J_{-1}^{-1} = \frac{\zeta(3)}{2} - \zeta(2)L, \quad (420) \]
\[ a^2 J_0^0 = -\frac{\zeta(2)}{10} - \zeta(3)L + \zeta(2) L^2, \quad (421) \]
\[ a^2 J_{0}^{-1} = 2 - \zeta(2) - L, \quad (422) \]
\[ a^2 J_0^1 = 9 - 5\zeta(2) - \zeta(3) - 5L + 2\zeta(2)L + L^2, \quad (423) \]
\[ a^2 J_1^{-1} = 1 - \frac{\zeta(2)}{2} - \frac{3}{4} L, \quad (424) \]
\[ a^2 J_1^0 = \frac{81}{16} - \frac{11\zeta(2)}{4} - \frac{\zeta(3)}{2} - \frac{27}{8} L + \zeta(2)L + \frac{3}{4} L^2, \quad (425) \]
\[ \frac{\beta}{BE} = \left( \frac{\mu^2}{a} \right)^2 e^x \sum_{j=-2}^{1} x^j \left[ \sum_{i=-2}^{0} \epsilon^i L_j^i \right] + O(x^2), \quad (426) \]

where:

\[ a^2 L_{-2}^{-1} = 2, \quad (427) \]
\[ a^2 L_{-2}^{-1} = 2 - 2L, \quad (428) \]
\[ a^2 L_{-2}^0 = 2 - 2\zeta(2) - 2L + L^2, \quad (429) \]
\[ a^2 L_{-1}^{-2} = -\frac{1}{2}, \quad (430) \]
\[ a^2 L_{-1}^{-1} = -\frac{3}{4} + \frac{1}{2} L, \quad (431) \]
\[ a^2 L_{-1}^0 = -\frac{7}{8} + \zeta(2) - \frac{3}{4} L - \frac{1}{4} L^2, \quad (432) \]
\[ a^2 L_0^{-1} = -\frac{1}{6}, \quad (433) \]
\[ a^2 L_0^0 = -\frac{1}{2}, \quad (434) \]
\[ a^2 L_1^{-1} = \frac{1}{24}, \quad (435) \]
\[ a^2 L_1^0 = \frac{11}{48}, \quad (436) \]

\[ \left( \frac{\mu^2}{a} \right)^{2\epsilon} \sum_{j=-1}^{1} x^j \left[ \sum_{i=-2}^{0} \epsilon_i M_j^i \right] + O(x^2), \quad (437) \]

where:

\[ a^2 M_{-1}^{-2} = -\frac{1}{2}, \quad (438) \]
\[ a^2 M_{-1}^{-1} = -\frac{1}{2} + L, \quad (439) \]
\[ a^2 M_{-1}^0 = -\frac{3}{2} + \zeta(2) + L - L^2, \quad (440) \]
\[ a^2 M_{-1}^{-1} = L, \quad (441) \]
\[ a^2 M_0^0 = -2 + 2\zeta(2) + 3L - L^2, \quad (442) \]
\[ a^2 M_1^{-1} = L, \quad (443) \]
\[ a^2 M_0^0 = -\frac{5}{2} + 2\zeta(2) + 4L - L^2, \quad (444) \]

\[ \left( \frac{\mu^2}{a} \right)^{2\epsilon} \sum_{j=-1}^{1} x^j \left[ \sum_{i=-3}^{0} \epsilon_i O_j^i \right] + O(x^2), \quad (445) \]

where:

\[ a^2 O_{-1}^{-3} = \frac{1}{4}, \quad (446) \]
\[ a^2 O_{-1}^{-2} = \frac{1}{2} - \frac{1}{2}L, \quad (447) \]
\[ a^2 O_{-1}^{-1} = 1 - L + \frac{1}{2}L^2, \quad (448) \]
\[ a^2 O_{-1}^0 = 2 - 2\zeta(3) - 2L + L^2 - \frac{1}{3}L^3, \quad (449) \]
\[ a^2 O_{-1}^{-1} = -1 + L, \quad (450) \]
\[ a^2 O_0^0 = -6 + \zeta(2) + 5L - L^2, \quad (451) \]
\[ a^2 O_1^{-1} = -\frac{1}{4} + \frac{1}{2}L, \quad (452) \]
\[ a^2 O_1^0 = -\frac{7}{4} + \frac{\zeta(2)}{2} + \frac{9}{4}L - \frac{1}{2}L^2. \quad (453) \]

\[ \left( \frac{\mu^2}{a} \right)^{2\epsilon} \sum_{j=-1}^{1} x^j \left[ \sum_{i=-2}^{0} \epsilon_i R_j^i \right] + O(x^2), \quad (454) \]
where:

\begin{align*}
a^2 R_{-1}^{-2} &= \frac{\zeta(2)}{2}, \quad (455) \\
a^2 R_{-1}^{-1} &= \frac{\zeta(3)}{2} - \zeta(2)L, \quad (456) \\
a^2 R_{-1}^{0} &= -\frac{\zeta^2(2)}{10} - \zeta(3)L + \zeta(2)L^2, \quad (457) \\
a^2 R_{0}^{-2} &= -1 + \frac{\zeta(2)}{2}, \quad (458) \\
a^2 R_{0}^{-1} &= -4 + 2\zeta(2) + \frac{\zeta(3)}{2} + L - \zeta(2)L, \quad (459) \\
a^2 R_{0}^{0} &= -14 + 5\zeta(2) - \frac{\zeta^2(2)}{10} + 2\zeta(3) + 7L - 4\zeta(2)L - \zeta(3)L - \frac{3}{2}L^2, \quad (460) \\
a^2 R_{1}^{-2} &= -\frac{3}{4} + \frac{\zeta(2)}{2}, \quad (461) \\
a^2 R_{1}^{-1} &= -5 + 3\zeta(2) + \frac{\zeta(3)}{2} + \frac{5}{4}L - \zeta(2)L, \quad (462) \\
&\quad + \zeta(2)L^2, \quad (463) \\
a^2 R_{1}^{0} &= -\frac{187}{8} + \frac{53\zeta(2)}{4} - \frac{\zeta^2(2)}{10} + 3\zeta(3) + \frac{77}{8}L - 6\zeta(2)L - \zeta(3)L \\
&\quad - \frac{11}{8}L^2 + \zeta(2)L^2, \quad (464) \\
\end{align*}

8 Conclusions

We have presented an exact evaluation of the master integrals with at most one massive propagator entering the electroweak form factor. Due to the rather simple threshold structure, we have succeeded in obtaining exact representations of all the master integrals in terms of one-dimensional harmonic polylogarithms of \(x\). By an expansion of the harmonic polylogarithms for large and small values of the argument, we could directly obtain the large and small momentum expansions of the master integrals. We checked our results with previous ones present in the literature finding complete agreement \[23, 26, 27, 25, 28\]. The remaining master integrals, containing two or three massive propagators, have thresholds not only in \(s = 0\) and \(s = m^2\) but also in \(s = 4m^2\) and their exact evaluation requires probably the introduction of new classes of special functions.

9 Acknowledgement

We wish to thank E. Remiddi for discussions and for use of the C program SOLVE to solve the linear systems generated by the ibp identities.

U.A. wishes to thank T. Gehrmann for discussions about two-loop computations and D. Comelli, M. Ciafaloni and P. Ciafaloni for discussions about the ew form factor.
R.B. wishes to thank the Fondazione Della Riccia for supporting his permanence at CERN, the Theory Division of CERN for the hospitality during a large portion of this work, and the Università di Roma “La Sapienza” for hospitality during the final part of this work.

A  Propagators

In this appendix we list all the denominators entering the two-loop integrals computed in this paper (see section 2.2).

\[ D_1 = k_1^2, \]
\[ D_2 = k_2^2, \]
\[ D_3 = (k_1 + k_2)^2, \]
\[ D_4 = (p_1 - k_1)^2, \]
\[ D_5 = (p_2 + k_1)^2, \]
\[ D_6 = (p_2 - k_2)^2, \]
\[ D_7 = (p_1 - k_1 + k_2)^2, \]
\[ D_8 = (p_2 + k_1 - k_2)^2, \]
\[ D_9 = (p_1 + p_2 - k_1)^2, \]
\[ D_{10} = (p_1 + p_2 - k_2)^2, \]
\[ D_{11} = (p_1 + p_2 - k_1 + k_2)^2, \]
\[ D_{12} = k_1^2 + a, \]
\[ D_{13} = k_2^2 + a, \]
\[ D_{14} = (k_1 + k_2)^2 + a, \]
\[ D_{15} = (p_1 - k_1)^2 + a, \]
\[ D_{16} = (p_2 + k_1)^2 + a, \]
\[ D_{17} = (p_2 - k_2)^2 + a, \]
\[ D_{18} = (p_1 - k_1 + k_2)^2 + a, \]
\[ D_{19} = (p_2 + k_1 - k_2)^2 + a, \]
\[ D_{20} = (p_1 + p_2 - k_1)^2 + a, \]
\[ D_{21} = (p_1 + p_2 - k_2)^2 + a, \]
\[ D_{22} = (p_1 + p_2 - k_1 - k_2)^2 + a. \]

B  One-loop amplitudes

In this appendix we present the results for the one-loop diagrams which are necessary for the computation of the factorized two-loop amplitudes. We have recomputed them with the method of differential equations described in the main body of the paper in terms of HPL’s. In some cases, we found that it was necessary to compute the one-loop amplitudes up to the third order in \( \epsilon \).
\[ B.1 \text{ Tadpole} \]

\[
\begin{align*}
&= \mu^{(4-D)} \int \{d^D k\} \frac{1}{k^2 + a} \\
&= \left( \frac{\mu^2}{a} \right)^\varepsilon \sum_{i=-1}^{3} \epsilon^i A_i + \mathcal{O}(\epsilon^4), \tag{487}
\end{align*}
\]

where:

\[
\frac{A_i}{a} = -1. \tag{488}
\]

\[ B.2 \text{ Bubbles} \]

\[
\begin{align*}
&= \mu^{(4-D)} \int \{d^D k\} \frac{1}{k^2 (p - k)^2} \\
&= \left( \frac{\mu^2}{a} \right)^\varepsilon \sum_{i=-1}^{3} \epsilon^i E_i + \mathcal{O}(\epsilon^4), \tag{489}
\end{align*}
\]

where:

\[
\begin{align*}
E_{-1} &= 1, \tag{490} \\
E_0 &= 2 - H(0, x), \tag{491} \\
E_1 &= 4 - \zeta(2) - 2H(0, x) + H(0, 0, x), \tag{492} \\
E_2 &= 8 - 2(\zeta(2) + \zeta(3)) + (\zeta(2) - 4)H(0, x) + 2H(0, 0, x) - H(0, 0, 0, x), \tag{493} \\
E_3 &= 16 - 4\zeta(2) - \frac{9}{10} \zeta^2(2) - 4\zeta(3) - 8H(0, x) + 2(\zeta(2) + \zeta(3))H(0, x) + (4 - \zeta(2))H(0, 0, x) - 2H(0, 0, 0, x) + H(0, 0, 0, 0, x), \tag{494}
\end{align*}
\]

The above amplitude — the simplest massless one — is easily computed with a Feynman parameter or with the general formula in the appendix of [7].

\[
\begin{align*}
&= \mu^{(4-D)} \int \{d^D k\} \frac{1}{k^2 [(p - k)^2 + a]} \\
&= \left( \frac{\mu^2}{a} \right)^\varepsilon \sum_{i=-1}^{2} \epsilon^i F_i + \mathcal{O}(\epsilon^3), \tag{495}
\end{align*}
\]
where:

\[ F_{-1} = 1, \]
\[ F_0 = 2 - \left[ 1 + \frac{1}{x} \right] H(-1, x), \]
\[ F_1 = 4 - \left[ 1 + \frac{1}{x} \right] \left[ 2H(-1, x) + H(0, -1, x) - 2H(-1, -1, x) \right], \]
\[ F_2 = 8 - \left[ 1 + \frac{1}{x} \right] \left[ 4H(-1, x) + 2H(0, -1, x) - 4H(-1, -1, x) + 4H(0, -1, -1, x) - 2H(0, -1, -1, x) + H(0, 0, -1, x) \right], \]
\[ F_3 = 16 + \left[ 1 + \frac{1}{x} \right] \left[ -8H(-1, x) + 8H(-1, -1, x) - 4H(0, -1, x) - 8H(-1, -1, -1, x) + 4H(0, -1, -1, x) + 2H(-1, 0, 0, -1, x) - 4H(0, -1, -1, x) + 2H(0, -1, 0, -1, x) + 2H(0, 0, -1, -1, x) - H(0, 0, 0, -1, x) \right], \]

\[ \frac{B}{L} = \mu ^{4-D} \int \{ d^Dk \} \frac{1}{k^2 + a} \]
\[ = \left( \frac{\mu ^2}{a} \right) ^{\epsilon} \sum_{i=-1}^{3} \epsilon ^1 G_i + \mathcal{O} \left( \epsilon ^4 \right), \]

where:

\[ G_i = 1. \]

B.3 Vertices

\[ = \mu ^{4-D} \int \{ d^Dk \} \frac{1}{k^2 \left[ (p_1 - k)^2 + a \right] (p_2 + k)^2} \]
\[ = \frac{1}{a(1 + x)} \left[ \frac{1 - 2\epsilon}{\epsilon} \left( \frac{\mu ^2}{a} \right) ^{\epsilon} \sum_{i=-1}^{2} \epsilon ^2 L_i + \mathcal{O} \left( \epsilon ^3 \right) \right] \]
where:

\[ a_{L-1} = -\frac{1}{x}H(-1, x), \tag{505} \]
\[ a_L = \frac{1}{x}[2H(-1, -1, x) - H(0, -1, x)], \tag{506} \]
\[ a_L^1 = \frac{1}{x}[2H(-1, 0, -1, x) + 2H(0, -1, -1, x) - 4H(-1, -1, -1, x)
- H(0, 0, -1, x)], \tag{507} \]
\[ a_L^2 = \frac{1}{x}[8H(-1, -1, -1, x) - 4H(-1, -1, 0, -1, x)
- 4H(-1, 0, -1, -1, x) + 2H(-1, 0, 0, -1, x)
- 4H(0, -1, -1, -1, x) + 2H(0, -1, 0, -1, x)
+ 2H(0, 0, -1, -1, x) - H(0, 0, 0, -1, x)]. \tag{508} \]

The above amplitude is not a master integral.

\[ \frac{\mu^{(4-D)}}{ \langle \overrightarrow{B} \rangle } \int \left\{ d^Dk \right\} \frac{1}{[k^2 + a](p_1 - k)^2(p_2 + k)^2} \]
\[ = \left( \frac{\mu^2}{a} \right)^\epsilon \sum_{i=0}^{\epsilon} \epsilon^i M_i + \mathcal{O}(\epsilon^3), \tag{509} \]

where:

\[ a_{M0} = -\frac{1}{x}H(1, 0, x), \tag{510} \]
\[ a_{M1} = -\frac{1}{x}[\zeta(2)H(1, x) + H(0, 1, 0, x) - H(1, 0, 0, x) + H(1, 1, 0, x)], \tag{511} \]
\[ a_{M2} = \frac{1}{x}[-2\zeta(3)H(1, x) - \zeta(2)(H(0, 1, x) - H(1, 0, x) + H(1, 1, x))
- H(0, 0, 1, 0, x) + H(0, 1, 0, x) - H(0, 1, 1, 0, x) - H(1, 0, 0, x)
- H(1, 0, 1, 0, x) + H(1, 1, 0, x) - H(1, 1, 1, 0, x)], \tag{512} \]

The above amplitude does not have infrared \(1/\epsilon\) poles because a non-vanishing mass in the boson line acts as an infrared cutoff on the transverse momenta: \(k_t > m\).

\[ \frac{\mu^{(4-D)}}{ \langle \overrightarrow{B} \rangle } \int \left\{ d^Dk \right\} \frac{1}{k^2(p_1 - k)^2(p_2 + k)^2} \]
\[ = \frac{1 - 2\epsilon}{\epsilon} \frac{1}{ax} \left( \frac{\mu^2}{a} \right)^\epsilon \sum_{i=-2}^{\epsilon} \epsilon^i N_i + \mathcal{O}(\epsilon^3), \tag{513} \]

\[ \frac{\mu^{(4-D)}}{ \langle \overrightarrow{B} \rangle } \int \left\{ d^Dk \right\} \frac{1}{k^2(p_1 - k)^2(p_2 + k)^2} \]
\[ = \left( \frac{\mu^2}{a} \right)^\epsilon \sum_{i=-2}^{\epsilon} \epsilon^i N_i + \mathcal{O}(\epsilon^3), \tag{514} \]
where:
\[
aN_{-2} = \frac{1}{x}, \quad (515)
\]
\[
aN_{-1} = -\frac{1}{x}H(0, x), \quad (516)
\]
\[
aN_0 = -\frac{1}{x}[\zeta(2) - H(0, 0, x)], \quad (517)
\]
\[
aN_1 = -\frac{1}{x}[2\zeta(3) - \zeta(2)H(0, x) + H(0, 0, 0, x)], \quad (518)
\]
\[
aN_2 = -\frac{1}{x}\left[\frac{9}{10}\zeta^2(2) - 2\zeta(3)H(0, x) + \zeta(2)H(0, 0, x) - H(0, 0, 0, x)\right]. \quad (519)
\]

C Independent two-loop amplitudes

In this appendix we give the expressions of the reducible diagrams of figures 5, 6 and 7.

\[
\begin{align*}
\begin{array}{c}
\includegraphics{figure5} \\
\end{array}
\end{align*}
\]
\[
= \mu^{2(4-D)} \int \{d^Dk_1\} \{d^Dk_2\} \frac{1}{D_1 D_2 D_9 D_{11}}
\]
\[
= \left(\frac{\mu^2}{a}\right)^2e^2 \sum_{i=-2}^2 \epsilon^i R_i^{(1)} + O(\epsilon^3),
\]
\[
(521)
\]

where:
\[
R_{i-2}^{(1)} = \frac{1}{2}, \quad (522)
\]
\[
R_{i-1}^{(1)} = \frac{5}{2} - H(0, x), \quad (523)
\]
\[
R_0^{(1)} = \frac{19}{2} - \zeta(2) - 5H(0, x) + 2H(0, 0, x), \quad (524)
\]
\[
R_1^{(1)} = \frac{65}{2} - 5\zeta(2) - 5\zeta(3) - [19 - 2\zeta(2)]H(0, x) + 10H(0, 0, x) - 4H(0, 0, 0, x), \quad (525)
\]
\[
R_2^{(1)} = \frac{211}{2} - 19\zeta(2) - \frac{11\zeta^2(2)}{5} - 25\zeta(3) - [65 - 10\zeta(2)]H(0, x) - 10\zeta(3)H(0, x) + [38 - 4\zeta(2)]H(0, 0, x) - 20H(0, 0, 0, x) + 8H(0, 0, 0, x), \quad (526)
\]

\[
\begin{align*}
\begin{array}{c}
\includegraphics{figure6} \\
\end{array}
\end{align*}
\]
\[
= \mu^{2(4-D)} \int \{d^Dk_1\} \{d^Dk_2\} \frac{1}{D_1 D_9 D_{11} D_{13}}
\]
\[
= \left(\frac{\mu^2}{a}\right)^2e^2 \sum_{i=-2}^2 \epsilon^i R_i^{(2)} + O(\epsilon^3),
\]
\[
(528)
\]
where:

\[ R^{(2)}_{-2} = \frac{1}{2} \],

\[ R^{(2)}_{-1} = \frac{5}{2} - H(0, x) \],

\[ R^{(2)}_{0} = \frac{19}{2} - 2\zeta(2) - 3H(0, x) - 2 \left[ 1 + \frac{1}{x} \right] H(-1, x) + H(0, 0, x) \]
\[ + \left[ 1 - \frac{1}{x} \right] H(0, -1, x) \],

\[ R^{(2)}_{1} = \frac{65}{2} - 4\zeta(2) - \zeta(3) - \left[ 7 - \zeta(2) \right] H(0, x) - 12 \left[ 1 + \frac{1}{x} \right] H(-1, x) \]
\[ + 3H(0, 0, x) - \left[ 1 + \frac{5}{x} \right] H(0, -1, x) + 8 \left[ 1 + \frac{1}{x} \right] H(-1, -1, x) \]
\[ - H(0, 0, 0, x) + \left[ 1 - \frac{1}{x} \right] \left[ H(0, 0, -1, x) - 4H(0, -1, -1, x) \right] \],

\[ R^{(2)}_{2} = \frac{211}{2} - 2\zeta(2) - \frac{27\zeta^2(2)}{10} - 5\zeta(3) - \left[ 15 - 3\zeta(2) - 2\zeta(3) \right] H(0, x) \]
\[ - \left[ 50 + 4\zeta(2) \right] \left[ 1 + \frac{1}{x} \right] H(-1, x) + \left[ 7 - \zeta(2) \right] H(0, 0, x) - \left[ 17 - 2\zeta(2) \right] \]
\[ + \left( \frac{19 + 2\zeta(2)}{x} \right) H(0, -1, x) + 48 \left[ 1 + \frac{1}{x} \right] H(-1, -1, x) \]
\[ - 3H(0, 0, 0, x) - \left[ 1 + \frac{5}{x} \right] \left[ H(0, 0, -1, x) - 4H(0, -1, -1, x) \right] + \left[ 1 \right] \]
\[ \left[ 1 + \frac{1}{x} \right] \left[ 12H(-1, 0, 0, x) - 32H(-1, -1, -1, x) \right] + \left[ 1 \right] \]
\[ \left[ 1 + \frac{1}{x} \right] \left[ H(0, 0, 0, 0, x) - 6H(0, -1, 0, -1, x) \right] \]
\[ + 16H(0, -1, -1, -1, x) - 4H(0, 0, -1, -1, x) \]
\[ + H(0, 0, 0, 0, x) \],

\[ \mu^{2(4-D)} \left\{ \int d^D k_1 \{ d^D k_2 \} \frac{1}{D_2 D_3 D_12} \right\} \]
\[ = \left( \frac{\mu^2}{a} \right)^{2\epsilon} \sum_{i=-2}^{1} \epsilon^i R^{(3)}_i + O(\epsilon^2) \].

where:

\[ R^{(3)}_{-2} = \frac{1}{2} \],

\[ R^{(3)}_{-1} = \frac{5}{2} - \left[ 1 + \frac{1}{x} \right] H(-1, x) \],

\[ R^{(3)}_{0} = \frac{19}{2} + \zeta(2) - \left[ 1 + \frac{1}{x} \right] \left[ 5H(-1, x) + H(0, -1, x) - 4H(-1, -1, x) \right] \].
\[ R_{1}^{(3)} = \frac{65}{2} + 5\zeta(2) - \zeta(3) - [19 + 2\zeta(2)] \left[ 1 + \frac{1}{x} \right] H(-1, x) \]

\[ -10 \left[ 1 + \frac{1}{x} \right] [H(0, -1, x) - 2H(-1, -1, x)] + 5H(0, -1, x) \]

\[ - \left[ 1 + \frac{1}{x} \right] [H(0, 0, -1, x) - 4H(0, -1, -1, x) - 6H(-1, 0, -1, x) + 16H(-1, -1, -1, x)] + 4H(0, -1, -1, x) - H(0, 0, -1, x) , \] (539)

\[ = \mu^{2(4-D)} \int \left\{ d^D k_1 \right\} \left\{ d^D k_2 \right\} \frac{1}{D_2 D_5 D_8 D_{12}} \] (540)

\[ = \left( \frac{\mu^2}{a} \right)^{2e} \sum_{i=-2}^{2} \epsilon^i R_i^{(4)} + \mathcal{O}(\epsilon^3) , \] (541)

where:

\[ R_{-2}^{(4)} = \frac{1}{2} , \] (542)

\[ R_{-1}^{(4)} = \frac{3}{2} , \] (543)

\[ R_{0}^{(4)} = \frac{7}{2} + \zeta(2) , \] (544)

\[ R_{1}^{(4)} = \frac{15}{2} + 3\zeta(2) - \zeta(3) , \] (545)

\[ R_{2}^{(4)} = \frac{31}{2} + 7\zeta(2) + \frac{9\zeta^2(2)}{5} - 3\zeta(3) , \] (546)

\[ = \mu^{2(4-D)} \int \left\{ d^D k_1 \right\} \left\{ d^D k_2 \right\} \frac{1}{D_1 D_5 D_8 D_{13}} \] (547)

\[ = \left( \frac{\mu^2}{a} \right)^{2e} \sum_{i=-2}^{2} \epsilon^i R_i^{(5)} + \mathcal{O}(\epsilon^3) , \] (548)

where:

\[ R_{-2}^{(5)} = -\frac{1}{2} , \] (549)

\[ R_{-1}^{(5)} = -\frac{1}{2} , \] (550)

\[ R_{0}^{(5)} = -\frac{1}{2} - \zeta(2) , \] (551)

\[ R_{1}^{(5)} = -\frac{1}{2} - \zeta(2) + \zeta(3) , \] (552)

\[ R_{2}^{(5)} = -\frac{1}{2} - \zeta(2) - \frac{9\zeta^2(2)}{5} + \zeta(3) , \] (553)

60
\[ = \mu^{2(4-D)} \int \{d^D k_1\} \{d^D k_2\} \frac{1}{D_1 D_2 D_7 D_{15}} \]
\[ = (\frac{\mu^2}{a})^{2\epsilon} \sum_{i=-2}^{2} \epsilon^i R_1^{(6)} + \mathcal{O} (\epsilon^3) , \] (554)

where:

\[ R_{-2}^{(6)} = \frac{1}{2} , \] (556)
\[ R_{-1}^{(6)} = \frac{3}{2} , \] (557)
\[ R_0^{(6)} = \frac{7}{2} + \zeta(2) , \] (558)
\[ R_1^{(6)} = \frac{15}{2} + 3\zeta(2) - \zeta(3) , \] (559)
\[ R_2^{(6)} = \frac{31}{2} + 7\zeta(2) + \frac{9\zeta^2(2)}{5} - 3\zeta(3) . \] (560)

\[ = \mu^{2(4-D)} \int \{d^D k_1\} \{d^D k_2\} \frac{1}{D_2 D_4 D_{10} D_{12}} \]
\[ = (\frac{\mu^2}{a})^{2\epsilon} \sum_{i=-2}^{2} \epsilon^i R_1^{(7)} + \mathcal{O} (\epsilon^3) , \] (561)

where:

\[ R_{-2}^{(7)} = 1 , \] (563)
\[ R_{-1}^{(7)} = 3 - H(0, x) , \] (564)
\[ R_0^{(7)} = 7 - \zeta(2) - 3H(0, x) + H(0, 0, x) , \] (565)
\[ R_1^{(7)} = 15 - 3\zeta(2) - 2\zeta(3) - [7 - \zeta(2)]H(0, x) \]
\[ + 3H(0, 0, x) - H(0, 0, 0, x) , \] (566)
\[ R_2^{(7)} = 31 - 7\zeta(2) - \frac{9\zeta^2(2)}{10} - 2\zeta(3) - [15 - 3\zeta(2) - 2\zeta(3)]H(0, x) \]
\[ + 7H(0, 0, x) - 3H(0, 0, 0, x) + H(0, 0, 0, 0, x) , \] (567)

\[ = \mu^{2(4-D)} \int \{d^D k_1\} \{d^D k_2\} \frac{1}{D_1 D_4 D_8 D_{13}} \]
\[ = (\frac{\mu^2}{a})^{2\epsilon} \sum_{i=-2}^{1} \epsilon^i R_1^{(8)} + \mathcal{O} (\epsilon^2) , \] (568)
where:

\[
    R_{-2}^{(s)} = -\frac{1}{2},
\]

\[
    R_{-1}^{(s)} = -\frac{5}{2} + \left[ 1 + \frac{1}{x} \right] H(-1, x) + \frac{1}{x} H(0, -1, x),
\]

\[
    R_{0}^{(s)} = -\frac{19}{2} - \zeta(2) + \left[ 1 + \frac{1}{x} \right] [5H(-1, x) - 4H(-1, -1, x) + 3H(0, -1, x)]
    - H(0, -1, x) - \frac{1}{x} [4H(0, -1, -1, x) - H(0, 0, -1, x)],
\]

\[
    R_{1}^{(s)} = -\frac{65}{2} - 5\zeta(2) + \zeta(3) + \left[ (19 + 2\zeta(2)) \left( 1 + \frac{1}{x} \right) \right] H(-1, x)
    - 20 \left[ 1 + \frac{1}{x} \right] H(-1, -1, x) + \left[ 10 + (9 + 2\zeta(2)) \frac{1}{x} \right] H(0, -1, x)
    + \left[ 1 + \frac{1}{x} \right] [3H(0, 0, -1, x) + 16H(-1, -1, -1, x) - 6H(-1, 0, -1, x)]
    - 12H(0, -1, -1, x) - H(0, 0, -1, x) + 4H(0, -1, -1, x)
    + \frac{1}{x} [16H(0, -1, -1, -1, x) - 6H(0, -1, 0, -1, x)]
    - 4H(0, 0, -1, -1, x) + H(0, 0, 0, -1, x)].
\]  

where:

\[
    R_{-2}^{(g)} = \frac{1}{2},
\]

\[
    R_{-1}^{(g)} = \frac{3}{2},
\]

\[
    R_{0}^{(g)} = \frac{5}{2} + \zeta(2) + \left[ 1 + \frac{1}{x} \right] H(-1, x)
    - H(0, -1, x),
\]

\[
    R_{1}^{(g)} = -\frac{1}{2} + 3\zeta(2) - \zeta(3) + \left[ 1 + \frac{1}{x} \right] [7H(-1, x) - 4H(-1, -1, x)]
    + H(0, -1, x)] - 3H(0, -1, x) + 4H(0, -1, -1, x)
    - H(0, 0, -1, -1, x),
\]

\[
    R_{2}^{(g)} = -\frac{51}{2} + 5\zeta(2) - 3\zeta(3) + \frac{9\zeta^2(2)}{5} + \left[ (33 + 2\zeta(2)) \left( 1 + \frac{1}{x} \right) \right] H(-1, x)
    - 28 \left( 1 + \frac{1}{x} \right) H(-1, -1, x) + \left[ 2 - 2\zeta(2) + \frac{7}{x} \right] H(0, -1, x)
\]

\[
    \mu^{2(4-D)} \int \{d^Dk_1 \} \{d^Dk_2 \} \frac{1}{D_1 D_2 D_8 D_{15}} = \left( \frac{\mu^2}{a} \right)^{2e} \sum_{i=-2}^{2} e^i R_{i}^{(g)} + O(\varepsilon^3),
\]
\[ 
\begin{align*}
+ \left[ 1 + \frac{1}{x} \right] \left[ 16H(-1, -1, -1, x) - 6H(-1, 0, -1, x) \right] \\
+ \left[ 2 - \frac{1}{x} \right] \left[ 4H(0, -1, -1, x) - H(0, 0, -1, x) \right] \\
- 16H(0, -1, -1, -1, x) + 6H(0, -1, 0, -1, x) \\
+ 4H(0, 0, -1, -1, x) - H(0, 0, 0, -1, x).
\end{align*}
\]

\[ 
\begin{align*}
\mathcal{J} &= \mu^{2(4-D)} \int \left\{ d^Dk_1 \right\} \left\{ d^Dk_2 \right\} \frac{1}{D_1 D_2 D_5 D_7} \\
&= \left( \frac{\mu^2}{a} \right)^{2\epsilon} \sum_{i=-2}^{2} \epsilon^i R^{(10)}_i + \mathcal{O} \left( \epsilon^3 \right),
\end{align*}
\]

where:

\[ R^{(10)}_{-2} = -\frac{1}{2}, \]  
(582)

\[ R^{(10)}_{-1} = -\frac{5}{2} + H(0, x), \]  
(583)

\[ R^{(10)}_{0} = -\frac{19}{2} + \zeta(2) + 5H(0, x) - 2H(0, 0, x), \]  
(584)

\[ R^{(10)}_{1} = -\frac{65}{2} + 5\zeta(2) + 3\zeta(3) + \left[ 19 - 2\zeta(2) \right] H(0, x) - 10H(0, 0, x) \\
+ 4H(0, 0, 0, x), \]  
(585)

\[ R^{(10)}_{2} = -\frac{211}{2} + 19\zeta(2) + \frac{11\zeta^2(2)}{5} + 25\zeta(3) + \left[ 65 - 10(\zeta(2) \\
+ \zeta(3)) \right] H(0, x) - \left[ 38 - 4\zeta(2) \right] H(0, 0, x) + 20H(0, 0, 0, x) \\
- 8H(0, 0, 0, 0, x). \]  
(586)

\[ \mathcal{Q} = \mu^{2(4-D)} \int \left\{ d^Dk_1 \right\} \left\{ d^Dk_2 \right\} \frac{1}{D_1 D_4 D_5 D_{13}} \\
= \left( \frac{\mu^2}{a} \right)^{2\epsilon} \sum_{i=-3}^{1} \epsilon^i R^{(11)}_i + \mathcal{O} \left( \epsilon^2 \right),
\]

where:

\[ R^{(11)}_{-3} = -\frac{1}{x}, \]  
(589)

\[ R^{(11)}_{-2} = -\frac{1}{x} \left[ 1 - H(0, x) \right], \]  
(590)

\[ R^{(11)}_{-1} = -\frac{1}{x} \left[ 1 - \zeta(2) - H(0, x) + H(0, 0, x) \right], \]  
(591)

\[ R^{(11)}_{0} = -\frac{1}{x} \left[ 1 - \zeta(2) - 2\zeta(3) - (1 - \zeta(2)) H(0, x) + H(0, 0, x) \right]. \]
\[
R_{1}^{(11)} = -\frac{1}{x} \left[ 1 - \zeta(2) - 2\zeta(3) - \frac{9\zeta^2(2)}{10} - (1 - \zeta(2) - 2\zeta(3)) H(0, x) \\
+ (1 - \zeta(2)) H(0, 0, x) - H(0, 0, 0, x) + H(0, 0, 0, 0, x) \right].
\]

\[
\begin{split}
= & \mu^{2(4-D)} \int \{ d^D k_1 \} \{ d^D k_2 \} \frac{1}{D_1 D_2 D_3 D_9 D_{11}} \\
= & \left( \frac{\mu^2}{a} \right)^{2\epsilon} \sum_{i=0}^{1} \epsilon^i R_{i}^{(12)} + O(\epsilon^2),
\end{split}
\]

where:

\[
aR_{0}^{(12)} = \frac{6\zeta(3)}{x},
\]

\[
aR_{1}^{(12)} = \frac{1}{x} \left[ \frac{18}{5} \epsilon(2) + 12\zeta(3) - 12\zeta(3) H(0, x) \right].
\]

\[
\begin{split}
= & \mu^{2(4-D)} \int \{ d^D k_1 \} \{ d^D k_2 \} \frac{1}{D_2 D_3 D_9 D_{11} D_{12}} \\
= & \left( \frac{\mu^2}{a} \right)^{2\epsilon} \sum_{i=0}^{1} \epsilon^i R_{i}^{(13)} + O(\epsilon^2),
\end{split}
\]

where:

\[
aR_{0}^{(13)} = \frac{1}{x} \left[ \zeta(2) H(-1, x) \\
+ 2H(-1, 0, -1, x) + H(-1, 0, 0, x) + 2H(0, -1, -1, x) \\
- H(0, -1, 0, x) \right],
\]

\[
aR_{1}^{(13)} = \frac{1}{x} \left[ 2\zeta(2) H(-1, x) - 2\zeta(2) H(0, -1, x) + \zeta(2) H(-1, 0, x) \\
- 2\zeta(2) H(-1, -1, x) - 4H(-1, 0, -1, x) + 2H(-1, 0, 0, x) \\
+ 4H(0, -1, -1, x) - 2H(0, -1, 0, x) + 4H(-1, -1, 0, -1, x) \\
- 2H(-1, -1, 0, 0, x) + 4H(-1, 0, -1, -1, x) + 2H(-1, 0, 0, 0, x) \\
- 2H(-1, 0, 0, -1, x) - 3H(-1, 0, 0, 0, x) - 12H(0, -1, 0, -1, x) \\
+ 2H(0, -1, 0, 0, x) + 4H(0, -1, 0, -1, x) + 2H(0, 0, -1, 0, x) \\
+ 2H(0, 0, 0, -1, x) + H(0, 0, -1, 0, x) \right],
\]

\[
\begin{split}
= & \mu^{2(4-D)} \int \{ d^D k_1 \} \{ d^D k_2 \} \frac{1}{D_1 D_2 D_5 D_9 D_{17}} \\
= & \left( \frac{\mu^2}{a} \right)^{2\epsilon} \sum_{i=-2}^{1} \epsilon^i R_{i}^{(14)} + O(\epsilon^2),
\end{split}
\]
where:

\[
aR_{-2}^{(14)} = \frac{1}{2},
\]

\[
aR_{-1}^{(14)} = \frac{1}{2},
\]

\[
aR_0^{(14)} = \frac{1}{2} + \zeta(2),
\]

\[
aR_1^{(14)} = \frac{1}{2} + \zeta(2) - \zeta(3),
\]

\[
\begin{align*}
\mu^2 (4-D) 
&\quad \int \{ d^D k_1 \} \{ d^D k_2 \} \frac{1}{D_1 D_2 D_5 D_7 D_8} \\
&\quad = \left( \frac{\mu^2}{a} \right)^{2 \epsilon} \sum_{i=-2}^{0} \epsilon^i R_i^{(15)} + \mathcal{O} (\epsilon),
\end{align*}
\]

where:

\[
aR_{-2}^{(15)} = \frac{\zeta(2)}{2x},
\]

\[
aR_{-1}^{(15)} = \frac{1}{2x} [\zeta(3) - 2\zeta(2) H(0, x)],
\]

\[
aR_0^{(15)} = -\frac{1}{x} \left[ \frac{\zeta^2(2)}{10} + \zeta(3) H(0, x) - 2\zeta(2) H(0, 0, x) \right],
\]

\[
\begin{align*}
\mu^2 (4-D) 
&\quad \int \{ d^D k_1 \} \{ d^D k_2 \} \frac{1}{D_1 D_4 D_5 D_8 D_{13}} \\
&\quad = \left( \frac{\mu^2}{a} \right)^{2 \epsilon} \sum_{i=-3}^{1} \epsilon^i R_i^{(16)} + \mathcal{O} (\epsilon^2),
\end{align*}
\]

where:

\[
aR_{-3}^{(16)} = \frac{1}{x},
\]

\[
aR_{-2}^{(16)} = \frac{1}{x} [1 - H(0, x)],
\]

\[
aR_{-1}^{(16)} = \frac{1}{x} \left[ 2 - \zeta(2) - H(0, x) - \left( 1 + \frac{1}{x} \right) H(-1, x) + H(0, 0, x) + H(0, -1, x) \right],
\]

\[
aR_0^{(16)} = \frac{1}{x} \left[ 4 - \zeta(2) - 2\zeta(3) - (1 - \zeta(2)) H(0, x) - 3 \left( 1 + \frac{1}{x} \right) H(-1, x) + H(0, 0, x) + 4 \left( 1 + \frac{1}{x} \right) H(-1, -1, x) - H(0, -1, x) \right]
\]
\[
aR_{1}^{(16)} = \left[ -H(0, 0, 0, x) + H(0, 0, -1, x) - 4H(0, -1, -1, x) \right], \quad (618)
\]

\[
aR_{1}^{(16)} = \frac{1}{x} \left[ 8 + \zeta(2) - \frac{9\zeta^2(2)}{10} - 2\zeta(3) - (1 - \zeta(2) - 2\zeta(3))H(0, x)
\right.
\]
\[
- (7 + 2\zeta(2)) \left( 1 + \frac{1}{x} \right) H(-1, x) + (1 - \zeta(2))H(0, 0, x) - (5
\]
\[
-2\zeta(2))H(0, -1, x) + 12 \left( 1 + \frac{1}{x} \right) H(-1, -1, x) - H(0, 0, 0, x)
\]
\[
H(0, 0, -1, x) + 4H(0, -1, -1, x) + \left( 1 + \frac{1}{x} \right) [6H(-1, 0, -1, x)
\]
\[
-16H(-1, -1, -1, x)] + H(0, 0, 0, 0, x) + H(0, 0, 0, x) - 6H(0, -1, 0, -1, x)
\]
\[
+ 4H(0, 0, -1, -1, x) \right], \quad (619)
\]

\[
\left[ \begin{array}{c}
\mu^2 (4-D) \int \left\{ d^D k_1 \right\} \left\{ d^D k_2 \right\} \frac{1}{D_2 D_4 D_5 D_8 D_{12}} \\
= \left( \frac{\mu^2}{a} \right)^{2e} \sum_{i=-1}^{1} e^i R_{i}^{(17)} + \mathcal{O} \left\{ \epsilon^2 \right\},
\end{array} \right.
\]

where:

\[
aR_{-1}^{(17)} = -\frac{1}{x} H(1, 0, x), \quad (622)
\]

\[
aR_{0}^{(17)} = -\frac{1}{x} \left[ 2\zeta(2)H(1, x) + 2H(1, 0, x) + H(0, 1, 0, x)
\right.
\]
\[
-2H(1, 0, 0, x) + 2H(1, 1, 0, x) \right], \quad (623)
\]

\[
aR_{1}^{(17)} = -\frac{1}{x} \left[ 4(\zeta(2) + \zeta(3))H(1, x) + 2\zeta(2)H(0, 1, x) + 2(2 - \zeta(2))H(1, 0, x)
\right.
\]
\[
+ 4\zeta(2)H(1, 1, x) + 2H(0, 1, 0, x) - 4H(1, 0, 0, x)
\]
\[
+ 4H(1, 1, 0, x) + H(0, 0, 1, 0, x) - 2H(0, 0, 0, 0, x)
\]
\[
+ 2H(0, 1, 0, x) + 4H(1, 0, 0, 0, x) + 2H(1, 0, 1, 0, x)
\]
\[
- 4H(1, 1, 0, 0, x) + 4H(1, 1, 1, 0, x) \right], \quad (624)
\]

\[
\left[ \begin{array}{c}
\mu^2 (4-D) \int \left\{ d^D k_1 \right\} \left\{ d^D k_2 \right\} \frac{1}{D_1 D_2 D_4 D_5 D_8 D_{15}} \\
= \left( \frac{\mu^2}{a} \right)^{2e} \sum_{i=-2}^{0} e^i R_{i}^{(18)} + \mathcal{O} \left\{ \epsilon \right\},
\end{array} \right.
\]

where:

\[
aR_{-2}^{(18)} = -\frac{1}{2x} H(-1, x), \quad (627)
\]

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\[ aR^{(18)}_{-1} = -\frac{1}{2x} [2H(-1, x) - 4H(-1, -1, x) + H(0, -1, x)], \]  
\[ aR^{(18)}_{0} = -\frac{1}{2x} [2(2 + \zeta(2))H(-1, x) + 2H(0, -1, x) - 8H(-1, -1, x) + 16H(-1, -1, x) - 6H(-1, 0, -1, x) - 4H(0, -1, -1, x) + H(0, 0, -1, x)], \]  
\[ = \mu^{2(4-D)} \int \{d^D k_1 \} \{d^D k_2 \} \frac{1}{D_1 D_2 D_4 D_5 D_8} \]  
\[ = \left( \frac{\mu^2}{a} \right)^{2x} \sum_{i=-3}^{1} e^{i R^{(19)}_i} + \mathcal{O}(\epsilon^2), \]  
where:

\[ aR^{(19)}_{-3} = \frac{1}{2x}, \]  
\[ aR^{(19)}_{-2} = \frac{1}{x} [1 - H(0, x)], \]  
\[ aR^{(19)}_{-1} = \frac{1}{x} [2 - 2\zeta(2) - 2H(0, x) + 2H(0, 0, x)], \]  
\[ aR^{(19)}_{0} = \frac{1}{x} [4 - 2\zeta(2) - 5\zeta(3) - 2(2 - \zeta(2))H(0, x) + 4H(0, 0, x) - 4H(0, 0, 0, x)], \]  
\[ aR^{(19)}_{1} = \frac{1}{x} \left[ 8 - 4\zeta(2) - \frac{11\zeta^2(2)}{5} - 10\zeta(3) - 2(4 - 2\zeta(2) - 5\zeta(3))H(0, x) + 4(2 - \zeta(2))H(0, 0, x) - 8H(0, 0, 0, x) + 8H(0, 0, 0, 0, x) \right], \]  
\[ = \mu^{2(4-D)} \int \{d^D k_1 \} \{d^D k_2 \} \frac{1}{D_1 D_2 D_4 D_5 D_7 D_8} \]  
\[ = \left( \frac{\mu^2}{a} \right)^{2x} \sum_{i=-2}^{1} e^{i R^{(20)}_i} + \mathcal{O}(\epsilon^2), \]  
where:

\[ aR^{(20)}_{-2} = -\frac{1}{x} H(1, x), \]  
\[ aR^{(20)}_{-1} = -\frac{1}{x} [2H(-1, x) - 2H(-1, -1, x) - H(-1, 0, x) + H(0, -1, x)], \]  
\[ aR^{(20)}_{0} = \frac{1}{x} [-4H(-1, x) + 4H(-1, -1, x) + 2H(-1, 0, x) - 2H(0, -1, x) + 4H(-1, -1, x) - 2H(-1, 0, -1, x) - 2H(-1, 0, 0, x)]. \]
\[-2H(-1, 0, 0, x) + 2H(0, -1, -1, x) + H(0, -1, 0, x)
- H(0, 0, -1, x)] ,
\\
aR^{(20)}_1 = \frac{1}{32x}[(8 - 6\zeta(3))H(-1, x) + 4H(0, -1, x) + 2\zeta(2)H(-1, 0, x)
- 4H(-1, 0, x) - 8H(-1, -1, x) + 4H(-1, 0, 0, x)
- 4H(-1, 0, -1, x) - 4H(0, -1, -1, x) + 2H(0, 0, -1, x)
- 2H(0, -1, 0, x) + 4H(-1, -1, 0, x) + 8H(-1, -1, -1, x)
- 8H(-1, -1, -1, -1, x) - 4H(-1, -1, 0, 0, x)
+ 4H(-1, -1, 0, -1, x) - 4H(-1, 0, 0, 0, x)
+ 4H(-1, 0, -1, -1, x) + 2H(-1, 0, -1, 0, x)
- 2H(-1, 0, 0, -1, x) - 4H(-1, 0, 0, 0, x)
+ 4H(0, -1, -1, -1, x) + 2H(0, -1, 0, -1, x)
- 2H(0, -1, 0, -1, x) + 2H(0, -1, 0, 0, x)
- 2H(0, 0, -1, -1, x) - H(0, 0, -1, 0, x)
+ H(0, 0, 0, -1, x)] ,
\\
\\= \mu^{2(4-D)} \int \left\{ d^D R_1^{(21)} \right\} \left\{ d^D R_2^{(21)} \right\} \frac{1}{D_1 D_2 D_3 D_6 D_{15}}
\\= \left( \frac{\mu^2}{a} \right)^{2x} \sum_{i=-2}^{0} e^{iR_i^{(21)}} + \mathcal{O}(\epsilon) ,
\\
where:
\\aR^{(21)}_{-2} = \frac{1}{2x} H(0, -1, x) ,
\\aR^{(21)}_{-1} = \frac{1}{2x} [H(0, 0, -1, x) - 4H(0, -1, -1, x)] ,
\\aR^{(21)}_0 = \frac{1}{2x} [2\zeta(2)H(0, -1, x) + H(0, 0, 0, -1, x) - 4H(0, 0, -1, -1, x)
- 6H(0, -1, 0, -1, x) + 16H(0, -1, -1, -1, x)] ,
\\
\\= \mu^{2(4-D)} \int \left\{ d^D k_1 \right\} \left\{ d^D k_2 \right\} \frac{1}{D_1 D_2 D_3 D_4 D_6}
\\= \left( \frac{\mu^2}{a} \right)^{2x} \sum_{i=-4}^{0} e^{iR_i^{(22)}} + \mathcal{O}(\epsilon) ,
\\
where:
\\aR^{(22)}_{-4} = \frac{1}{4x} ,
\\
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\[ aR_{-3}^{(22)} = -\frac{1}{2x} H(0, x), \quad (651) \]
\[ aR_{-2}^{(22)} = -\frac{1}{2x} [\zeta(2) - 2H(0, 0, x)], \quad (652) \]
\[ aR_{-1}^{(22)} = -\frac{1}{2x} [5\zeta(3) - 2\zeta(2)H(0, x) + 4H(0, 0, 0, x)], \quad (653) \]
\[ aR_0^{(22)} = -\frac{1}{x} \left[ \frac{11\zeta^2(2)}{10} - 5\zeta(3)H(0, x) + 2\zeta(2)H(0, 0, x) \right. \]
\[ \left. -4H(0, 0, 0, x) \right], \quad (654) \]

\[ = \mu^{2(4-D)} \int \{d^Dk_1\} \{d^Dk_2\} \frac{1}{D_1D_2D_3D_4D_5} \quad (655) \]
\[ = \left( \frac{\mu^2}{a} \right)^{2\epsilon} \sum_{i=-3}^{1} \epsilon^i R_i^{(23)} + \mathcal{O}(\epsilon^2), \quad (656) \]

where:

\[ aR_{-3}^{(23)} = \frac{1}{4x}, \quad (657) \]
\[ aR_{-2}^{(23)} = \frac{1}{2x} [1 - H(0, x)], \quad (658) \]
\[ aR_{-1}^{(23)} = \frac{1}{x} [1 - H(0, x) + H(0, 0, x)], \quad (659) \]
\[ aR_0^{(23)} = \frac{2}{x} [1 - \zeta(3) - H(0, x) + H(0, 0, x) - H(0, 0, 0, x)], \quad (660) \]
\[ aR_1^{(23)} = \frac{4}{x} \left[ 1 - \zeta(3) - \frac{3\zeta^2(2)}{10} - (1 - \zeta(3))H(0, x) + H(0, 0, x) \right. \]
\[ \left. -H(0, 0, 0, x) + H(0, 0, 0, 0, x) \right], \quad (661) \]

\[ = \mu^{2(4-D)} \int \{d^Dk_1\} \{d^Dk_2\} \frac{1}{D_1D_2D_3D_4D_5D_10} \quad (662) \]
\[ = \left( \frac{\mu^2}{a} \right)^{2\epsilon} \sum_{i=-3}^{1} \epsilon^i R_i^{(24)} + \mathcal{O}(\epsilon^2), \quad (663) \]

where:

\[ aR_{-3}^{(24)} = \frac{1}{x}, \quad (664) \]
\[ aR_{-2}^{(24)} = \frac{2}{x} [1 - H(0, x)], \quad (665) \]
\[ aR_{-1}^{(24)} = \frac{2}{x} [2 - \zeta(2) - 2H(0, x) + 2H(0, 0, x)], \quad (666) \]
\[ aR_{0}^{(24)} = \frac{4}{x} \left[ 2 - \zeta(2) - \zeta(3) - (2 - \zeta(2))H(0,x) + 2H(0,0,x) \right. \\
\left. -2H(0,0,0,x) \right], \quad (667) \]
\[ aR_{1}^{(24)} = \frac{4}{x} \left[ 4 - 2\zeta(2) - \frac{\zeta^2(2)}{5} - 2\zeta(3) - 2(2 - \zeta(2) - \zeta(3))H(0,x) \\
+2(2 - \zeta(2))H(0,0,x) - 4H(0,0,0,x) + 4H(0,0,0,0,x) \right], \quad (668) \]
\[
\begin{align*}
\mu^{2(4-D)} \int \{d^Dk_1\} \{d^Dk_2\} \frac{1}{D_1D_2D_3D_4D_5D_6D_7D_8D_9D_{10}D_{11}} \\
= \left( \frac{\mu^2}{a} \right)^{2\epsilon} \sum_{i=-2}^{1} \epsilon^i R_i^{(25)} + \mathcal{O} (\epsilon^2), \quad (670)
\end{align*}
\]
where:
\[
\begin{align*}
aR_{-2}^{(25)} &= -\frac{1}{x}H(-1,x), \quad (671) \\
aR_{-1}^{(25)} &= -\frac{1}{x} \left[ 2H(-1,x) + H(-1,0,x) - 2H(-1,-1,x) \right], \quad (672) \\
aR_{0}^{(25)} &= -\frac{1}{x} \left[ (4 - \zeta(2))H(-1,x) - 2H(-1,0,x) - 4H(-1,-1,x) \\
&\quad +4H(-1,-1,0,x) + 2H(-1,0,0,x) + H(-1,0,0,x) \right], \quad (673) \\
aR_{1}^{(25)} &= -\frac{1}{x} \left\{ 2(4 - \zeta(2) - \zeta(3))H(-1,x) - (4 - \zeta(2))[H(-1,0,x) \\
&\quad -2(4 - \zeta(2))H(-1,-1,x)] + 2H(-1,0,x,0,x) + 4H(-1,-1,0,x) \\
&\quad +8H(-1,-1,0,x) - H(-1,0,0,0,x) - 2H(-1,-1,0,x) \\
&\quad -8H(-1,-1,-1,0,x) \right\}, \quad (674)
\end{align*}
\]
\[
\begin{align*}
\mu^{2(4-D)} \int \{d^Dk_1\} \{d^Dk_2\} \frac{1}{D_1D_2D_3D_4D_5D_6D_7} \\
= \left( \frac{\mu^2}{a} \right)^{2\epsilon} \sum_{i=-3}^{1} \epsilon^i R_i^{(26)} + \mathcal{O} (\epsilon^2), \quad (676)
\end{align*}
\]
where:
\[
\begin{align*}
aR_{-3}^{(26)} &= \frac{1}{x}, \quad (677) \\
aR_{-2}^{(26)} &= \frac{1}{x} \left[ 1 - H(0,x) \right], \quad (678) \\
aR_{-1}^{(26)} &= \frac{1}{x} \left[ 1 - \zeta(2) - H(0,x) + H(0,0,x) \right], \quad (679)
\end{align*}
\]
\begin{equation}
\begin{aligned}
aR_{0}^{(26)} &= \frac{1}{x} \left[ 1 - \zeta(2) - 2\zeta(3) - (1 - \zeta(2))H(0, x) + H(0, 0, x) \right. \\
&- \left. H(0, 0, 0, x) \right], \\
aR_{1}^{(26)} &= \frac{1}{x} \left[ 1 - \zeta(2) - 2\zeta(3) - \frac{9\zeta^2(2)}{10} - (1 - \zeta(2) - 2\zeta(3))H(0, x) \\
&+ (1 - \zeta(2))H(0, 0, x) - H(0, 0, 0, x) + H(0, 0, 0, x) \right].
\end{aligned}
\end{equation}

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