ON THE INTEGRABLE HIERARCHIES
ASSOCIATED WITH $N = 2$ SUPER $W_n$ ALGEBRA

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Abstract

A new Lax operator is proposed from the viewpoint of constructing
the integrable hierarchies related with $N = 2$ super $W_n$ algebra. It is
shown that the Poisson algebra associated to the second Hamiltonian
structure for the resulted hierarchy contains the $N = 2$ super Virasoro
algebra as a proper subalgebra. The simplest cases are discussed in
detail. In particular, it is proved that the supersymmetric two-boson
hierarchy is one of $N = 2$ supersymmetric KdV hierarchies. Also, a
Lax operator is supplied for one of $N = 2$ supersymmetric Boussinesq
hierarchies.

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1 Introduction

Recently, $N = 2$ supersymmetric integrable hierarchies have attracted much attention (see [2] for example). It is believed that a full understanding of these hierarchies will shed light on the study of conformal field theory. Besides their physical relevance, these hierarchies are mathematically interesting. One of the appealing problems, which has not been explained, is why there exist three different $N = 2$ supersymmetric hierarchies.

One way to construct $N = 2$ super $W_n$ algebra is to use Gel’fand-Dickey bracket for an odd order super differential operator $L$ below. For the cases of $n = 2, 3$ and $4$, the resulted brackets are indeed the corresponding $N = 2$ super $W_n$ algebras respectively [4]-[6], although the situation in general is still a conjecture. We will argue that two Lax operators can be formed out of $L$. Among them, one is the one proposed by Inami and Kanno [6][7], the other one is new in general. For the new proposed one, our calculation shows that the Poisson algebra associated to the second Hamiltonian structure contains $N = 2$ super Virasoro algebra in proper. This is the evidence that the general case will lead to $N = 2$ super $W_n$ algebra. We will further show that this is indeed the case in the simplest examples.

The layout of the letter is as follows. We consider the general case in next section and calculate explicitly the $N = 2$ super Virasoro algebra. Section three is devoted to the simplest examples. In particular, we will show that the supersymmetric two-boson (or Kaup-Broer) hierarchy of Brunelli and Das [8] is equivalent to one of the $N = 2$ supersymmetric KdV hierarchies of Laberge and Matheiu [8]. In the next simplest case, our system is identical to so-called $N = 2$ supersymmetric Boussinesq equation whose Lax pair was not known before, so we provide a Lax operator for this system in $N = 1$ form.

2 $N = 2$ Super $W_n$ Algebra and Lax Operators

The general $N = 2$ super $W_n$ algebra is believed to be related with an odd order super differential operator [4]-[6]

$$L = D^{2n-1} + u_{2n-3}D^{2n-3} + \cdots + u_0,$$

where $D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial x}$ is the super derivative. Indeed, given the operator $L$, one may calculate the second Gel’fand-Dickey bracket from the Poisson
tensor

\[ \Theta : \Delta H \rightarrow (L \Delta H)_{\geq 0}L - L(\Delta HL)_{\geq 0} \]  

(2)

where \( \Delta H \) is properly parametrized (see [4] for example). We also use the standard notations for a given super pseudo-differential operator:

\[ A = \sum_{i \geq 0} a_i D^i + \sum_{i \leq -1} D^i a_i := \sum_{i \geq 0} A_i + \sum_{i \leq -1} A_i, \quad \text{sres} \ A = a_{-1}. \]

Since the operator \( L \) is lacking of term \( D^{2n-2} \), we have to modify the general tensor (2) so that it will give us a proper expression. The situation is much same as in the pure bosonic case and the reduced Poisson tensor is

\[ \hat{\mathcal{P}} : \Delta H \rightarrow (L \Delta H)_{\geq 0}L - L(\Delta HL)_{\geq 0} + (-1)^{\Delta H}[L, D^{-1}\text{sres}[L, \Delta H]], \]

where \( D^{-1} = \int dx d\vartheta \) denotes the super integration with \( z = (x, \vartheta) \) and \([,]\) means the graded commutator. The parity of \( \Delta H \) is indicated by \(|\Delta H|\).

That this approach indeed supplies the \( N = 2 \) super \( W \) algebra is worked out explicitly for the simplest three cases \( n = 2, n = 3 \) and \( n = 4 \) in [4]-[6][14], while the correctness of the general case is still an unproved conjecture.

Let us consider the problem of constructing integrable hierarchies associated with this algebra. Since the \( L \) is an odd operator, it is not possible to construct integrable hierarchies directly with it. To have meaningful results, let us define two Lax operators via \( L \)

\[ L_1^{(n)} = \partial^n + v_{2n-2}D^{2n-2} + v_{2n-3}D^{2n-3} + \cdots + v_1 D, \]
\[ L_2^{(n-1)} = \partial^{n-1} + w_{2n-4}D^{2n-4} + w_{2n-5}D^{2n-5} + \cdots + w_0 + D^{-1}w_{-1}. \]

(3)

and \( L_1^{(n)} \) and \( L_2^{(n-1)} \) are related to \( L \) by

\[ L_1^{(n)} = LD, \quad L_2^{(n-1)} = D^{-1}L. \]

(4)

We notice the \( L_1^{(n)} \) is proposed first by Inami and Kanno in the context of \( W \) algebra[3][4], there it is shown that this Lax operator gives one of \( N = 2 \) supersymmetric hierarchies.

We concern here with the operator \( L_2^{(n-1)} \). Since the \( L_2^{(n-1)} \) is of type of constrained Modified KP[11], we may call it constrained super modified KP. The integrable hierarchy can be constructed by means of standard fractional power approach

\[ \frac{\partial}{\partial t_k} L_2^{(n-1)} = [\{ (L_2^{(n-1)})^{k/2} \}_{\geq 1}, L_2^{(n-1)}], \]

(5)
as for the Hamiltonian structures, we may use the results of Oevel and Strampp\cite{OevelStrampp}. The system (5) is indeed a bi-Hamiltonian system with the first Poisson tensor given by

\[ Q : \Delta H \rightarrow ([L_2^{(n-1)}, \Delta H])_{\geq -1}, \]  

(6)

and the second one reads as

\[ P : \Delta H \rightarrow (L_2^{(n-1)} \Delta H)_{\geq 0} L_2^{(n-1)} - L_2^{(n-1)} (\Delta H L_2^{(n-1)})_{\geq 0} + [L_2^{(n-1)}, (L_2^{(n-1)} \Delta H)_0] - D^{-1} (\text{res}[\Delta H, L_2^{(n-1)}]) L_2^{(n-1)} + [D^{-1} (\text{res}[\Delta H, L_2^{(n-1)}]), L_2^{(n-1)}], \]  

(7)

where \( \Delta H \) is parametrized as

\[ \Delta H = \frac{\delta H}{\delta w_{-1}} - \sum_{i=0}^{2n-4} (-D)^{-i-1} \frac{\delta H}{\delta w_i}, \]

with \( \frac{\delta H}{\delta w_{2n-1}} \) are bosonic and the rest of them fermionic.

Let us now calculate the subalgebra for the first two coefficients. To this end, we take \( \Delta H \) as

\[ \Delta H = \partial^{n+2} (D^{-1} \Lambda + X), \quad \Lambda = \frac{\delta H}{\delta w_{2n-4}} , \quad X = - \frac{\delta H}{\delta w_{2n-5}}, \]

calculating (7) and picking up the coefficients of \( \partial^{n-2} \) and \( \partial^{n-3} D \), we have

\[ \{ w_{2n-4}(z), w_{2n-4}(z') \} = (n(n-1) \partial D + (D w_{2n-4}) - 2 w_{2n-5}) \Delta(z - z'), \]
\[ \{ w_{2n-4}(z), w_{2n-5}(z') \} = - \left( \frac{n(n-1)}{2} \partial^2 + \partial w_{2n-4} - w_{2n-5} D \right) \Delta(z - z'), \]
\[ \{ w_{2n-5}(z), w_{2n-5}(z') \} = - (w_{2n-5} \partial + \partial w_{2n-5}) \Delta(z - z'), \]

(8)

where \( \Delta(z - z') \) is the super delta function. To see the connection with the \( N = 2 \) super Virasoro algebra, we perform the following invertible transformation

\[ J = u, \quad T = -\alpha + \frac{1}{2} (Du), \]
brings the algebra \( (8) \) to

\[
\{ J(z), J(z') \} = (n(n-1)\partial D + 2T)\Delta(z - z'),
\]

\[
\{ J(z), T(z') \} = (\partial J - \frac{1}{2}(DJ)D)\Delta(z - z'),
\]

\[
\{ T(z), T(z') \} = \left( \frac{1}{4}n(n-1)\partial^2 D + \frac{3}{2}T\partial + \frac{1}{2}(DT)D + T_x \right)\Delta(z - z'),
\]

which is nothing but the the \( N = 2 \) super Virasoro algebra in the \( N = 1 \) form (cf.\( \cite{6,5,4} \)). Thus, we see that the Poisson algebra related with the Hamiltonian structure has the \( N = 2 \) super Virasoro algebra as its a proper subalgebra. This observation leads us to a conjecture: the hierarchy constructed out of \( L_2^{(n-1)} \) is a hierarchy coincident with one of the hierarchies associated with \( N = 2 \) super \( W_n \) algebra.

Remarks:

- The general formulae(8) are not valid in the simplest case \( n = 2 \). However we will see, in the next section, we still have \( N = 2 \) super Virasoro algebra;

- The validity of our conjecture will be shown in the cases with \( n = 2 \) and \( n = 3 \) by direct calculation.

3 Examples

We perform calculation in this section in the simplest cases and show that our conjecture made above is indeed valid in these concrete cases. For clarity, we will use the fields without index in Lax operators in the sequel.

3.1 KdV or Kaup-Broer Case

In the case \( n = 2 \), we have

\[
L_2^{(1)} = \partial + u + D^{-1}\alpha,
\]

and the simplest flow(\( t_2 \)) is

\[
u_t = u_{xx} + u^2 + 2(D\alpha)_x, \quad \alpha_t = -\alpha_{xx} + 2(\alpha u)_x.
\]
We remark here that the Lax operator $L_2^{(1)}$ is essentially the one proposed in\cite{3} for the supersymmetric two-boson(or Kaup-Broer) system. We will see that as a by-product of our analysis, supersymmetric two-boson hierarchy is equivalent to one of $N = 2$ supersymmetric KdV hierarchy.

The first a few Hamiltonians are

$$H_1 = \int \alpha \, dz, \quad H_2 = \int u\alpha \, dz, \quad H_3 = \int \alpha (u_x + u^2 + (D\alpha)) \, dz,$$

two Poisson tensors are easily calculated

$$Q = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 2D\partial + 2\alpha + (Du) & \partial^2 + \partial u + \alpha D \\ -\partial^2 + u\partial - D\alpha & \alpha \partial + \partial \alpha \end{pmatrix},$$

(12)

and the whole hierarchy is bi-Hamiltonian, in particular the $t_2$ flow\cite{11} can be written as

$$\begin{pmatrix} u \\ \alpha \end{pmatrix}_t = P \left( \begin{pmatrix} \delta H_2 \\ \delta \alpha \end{pmatrix} \right) = Q \left( \begin{pmatrix} \delta H_3 \\ \delta \alpha \end{pmatrix} \right).$$

By means of $L_2^{(1)} = D^{-1}L$ or

$$\partial + u + D^{-1}\alpha = D^{-1}(D\partial + \hat{u}D + \hat{\alpha})$$

we obtain

$$\hat{u} = u, \quad \hat{\alpha} = \alpha + (Du).$$

(13)

On the other hand, we may have a Poisson algebra from the Gel’fand-Dickey bracket for $L$, which in terms of Poisson tensor reads as

$$\hat{P} = \begin{pmatrix} 2D\partial + 2\alpha - (D\hat{u}) & -\partial^2 + \partial \hat{u} - \hat{\alpha}D \\ \partial^2 + \hat{u}\partial + D\hat{\alpha} & \hat{\alpha}\partial + \partial \hat{\alpha} \end{pmatrix},$$

(14)

a simple calculation shows that the invertible map\cite{13} is a Hamiltonian map between $P$ and $\hat{P}$.

It is well known that the following map

$$J = \hat{u}, \quad T = \hat{\alpha} - \frac{1}{2}(D\hat{u}),$$

(15)
converts the Poisson algebra defined by $\hat{P}$ to the $N = 2$ super Virasoro algebra with $n = 2$ in (9), so to bring the Poisson algebra inherited from $P$ to the $N = 2$ super Virasoro algebra, we need only compose the map (13) with (15). The composed map will bring our $Q$ and $P$ to two Hamiltonian operators of the $N = 2$ supersymmetric KdV system with $a = 4$ (see [9] also). This argument leads us to the conclusion that the supersymmetric two-boson hierarchy [3] is equivalent to the $N = 2$, $a = 4$ supersymmetric KdV hierarchy.

3.2 Boussinesq Case

Now we work with the following Lax operator

$$L_2^{(2)} = \partial^2 + u\partial + \alpha D + v + D^{-1}\beta,$$

and the simplest flow ($t_2$) reads as

$$u_t = 2v_x, \quad \alpha_t = -2\beta_x,$$
$$v_t = v_{xx} + 2(D\beta)_x + uv_x + \beta(Du) + \alpha(Dv) + 2\alpha\beta,$$
$$\beta_t = -\beta_{xx} + (\beta u)_x + D(\beta\alpha).$$

The first a few Hamiltonians are

$$H_1 = -\int \alpha \, dz, \quad H_2 = \int \beta \, dz,$$
$$H_3 = \frac{1}{2} \int (\beta u - v\alpha - \frac{1}{2} u\alpha_x + \frac{1}{4} \alpha(D\alpha) + \frac{1}{4} u^2\alpha) \, dz, \quad H_4 = \int v\beta \, dz.$$

We now calculate the Hamiltonian structures for the related hierarchy. The first one is

$$Q = \begin{pmatrix}
0 & 0 & 0 & 2\partial \\
0 & 0 & -2\partial & 0 \\
0 & -2\partial & 2\partial D - (Du) + 2\alpha & \partial^2 + u\partial + \alpha D \\
2\partial & 0 & -\partial^2 + \partial u - D\alpha & 0
\end{pmatrix},$$

as for the second one, it is in rather complicated form and for clarity, we present it in the table 1 below:
Table 1: Matrix entries of operator $P$.

\[
\begin{aligned}
P_{11} &= 6\partial^2 - \partial u + \alpha D, \\
P_{12} &= -3\partial^2 - \partial u + \alpha D, \\
P_{13} &= 4u\partial^2 + 2(Du)\partial + 4u_x D + 2(Du)_x + (Dv) + 2\beta, \\
P_{14} &= 2\partial^3 + 2\partial u\partial + 2\partial D + 2\partial v + \beta D, \\
P_{21} &= 3\partial^2 - u\partial - D\alpha, \\
P_{22} &= -\alpha \partial - \partial D, \\
P_{23} &= -2\partial^3 - 2\partial^2 - 2\partial D - 2\partial D - 2\partial D - 2\partial D - 2\partial D - 2\partial D - 2\partial D - 2\partial D - 2\partial D - 2\partial D - 2\partial D, \\
P_{24} &= -2\partial^3 - 2\partial^2 - 2\partial D - 2\partial D - 2\partial D - 2\partial D - 2\partial D - 2\partial D, \\
P_{31} &= u\partial^2 + 2(Du)\partial + 2(Du)\partial + \alpha D + 2(Du)\partial + 2(Du)\partial + 2(Du)\partial + 2(Du)\partial + 2(Du)\partial, \\
P_{32} &= -2\partial^3 - 2\partial^2 - 2\partial D - 2\partial D - 2\partial D - 2\partial D - 2\partial D - 2\partial D, \\
P_{33} &= 2\partial^3 - 2\partial^2 - 2\partial D - 2\partial D - 2\partial D - 2\partial D - 2\partial D - 2\partial D, \\
P_{34} &= 2\partial^3 - 2\partial^2 - 2\partial D - 2\partial D - 2\partial D - 2\partial D, \\
P_{41} &= 2\partial^3 - 2\partial^2 - 2\partial D - 2\partial D - 2\partial D - 2\partial D, \\
P_{42} &= -2\partial^3 - 2\partial^2 - 2\partial D - 2\partial D - 2\partial D, \\
P_{43} &= 2\partial^3 - 2\partial^2 - 2\partial D - 2\partial D, \\
P_{44} &= 2\partial^3 - 2\partial^2 - 2\partial D, \\
\end{aligned}
\]

We notice that in particular the $t_2$-flow [17] can be written as

\[
u_t = P \frac{\delta H_2}{\delta u} = Q \frac{\delta H_4}{\delta u},
\]

where $u = (u, \alpha, v, \beta)$.

As we conjectured in last section, the Poisson algebra associated with $P$ should be $N = 2$ super $W_3$ algebra. To justify this, we recall that the algebra is constructed out of the operator

\[
\hat{L} = \partial^2 D - \partial u \partial D - \partial \partial D - \partial \partial D - \beta,
\]

using Gel’fand-Dickey second bracket. The above form is chosen for the comparison with [1]. The explicit expression is listed in the Table 2:
Table 2: Matrix entries of operator $\hat{P}$.

| $P_{11}$ | $P_{12}$ |
|--------|--------|
| $6\partial D + (D\dot{u}) - 2\dot{\alpha}$ | $-3\partial^2 - \partial \dot{u} - \dot{\alpha} D$ |
| $P_{13}$ | $P_{14}$ |
| $-3\partial^2 D - 3\partial D\dot{u} + \partial \dot{\alpha} + (D\dot{v}) - 2\dot{\beta}$ | $2\partial^2 + 2\partial^2 \dot{u} - 2\partial D\dot{\alpha} - 2\partial \dot{v} + \dot{\beta} D$ |
| $P_{21}$ | $P_{22}$ |
| $3\partial^2 - \dot{u} \partial - D\dot{\alpha}$ | $-\dot{\alpha} \partial - \partial \dot{\alpha}$ |
| $P_{23}$ | $P_{24}$ |
| $-\dot{\alpha}^3 - \partial^2 \dot{u} + \partial D\dot{\alpha} - 2\dot{v} \partial - \dot{v} \partial_x - D\dot{\beta}$ | $-3\dot{\beta} \partial - 2\dot{\beta}_x$ |
| $P_{31}$ | $P_{32}$ |
| $3\partial^2 D - 3\dot{u} \partial D - \dot{\alpha} \partial + (D\dot{v}) - 2\dot{\beta}$ | $-\dot{\alpha}^3 + \dot{u} \partial^2 + \dot{\alpha} \partial D - 2\dot{v} \partial + \dot{\beta} D - \dot{v}_x$ |
| $P_{33}$ | $P_{34}$ |
| $-2\partial^3 D - 2\partial^2 D\dot{u} + 2\dot{u} \partial^2 D + \dot{\alpha} \partial^2 + \partial^2 \dot{\alpha} + 2\dot{v} \partial D + 2\dot{u} \partial D\dot{u} - \dot{u} \partial \dot{\alpha}$ | $\partial^4 + \dot{\alpha} \partial^3 - \partial^2 D\dot{\alpha} - \dot{\alpha} \partial^2 D - \dot{u} \partial^2 \dot{u} - \partial^2 \dot{v} + \dot{u} \partial D\dot{\alpha} - \dot{\beta} \partial D$ |
| $P_{41}$ | $P_{42}$ |
| $2\partial^3 - 2\dot{u} \partial^2 - 2\dot{\alpha} \partial D - 2\dot{v} \partial - D\dot{\beta}$ | $-\dot{\beta}_x - 3\dot{\beta} \partial$ |
| $P_{43}$ | $P_{44}$ |
| $-\partial^4 - \partial^3 \dot{u} + \dot{u} \partial^3 + \partial^2 D\dot{\alpha} + \dot{\alpha} \partial^2 D + \dot{u} \partial^2 \dot{u} + \dot{v} \partial^2 - \dot{u} \partial D\dot{\alpha} + \dot{\alpha} \partial D\dot{u}$ | $\dot{u} \partial \dot{\beta} + \dot{\beta} \partial \dot{u} - 2\dot{\beta}_x \partial + \dot{\alpha} D\dot{\beta} - \dot{\beta} D\dot{\alpha} - \dot{\beta}_x \partial$ |

Through the relationship between $L_2^{(2)}$ and $L$, we have a change of coordinates

\[ \dot{u} = -u, \quad \dot{\alpha} = \alpha - (Du), \quad \dot{v} = -v - (D\alpha), \quad \dot{\beta} = -\beta - (Dv), \]

(19)

this is obviously an invertible change of variables. By laborious but straightforward computation, one can check that the above map is a Hamiltonian map between $P$ and $\hat{P}$.

It is known that suitable combinations of hatted variables lead to primary
fields, namely

\[ J = \hat{u}, \]
\[ T = \hat{\alpha} - \frac{1}{2}(D\hat{u}), \]
\[ W_2 = \hat{v} - \frac{1}{3}(D\hat{\alpha}) - \frac{1}{3}\hat{u}_x + \frac{2}{9}\hat{u}^2, \]
\[ W_\frac{2}{3} = \hat{\beta} - \frac{1}{2}(D\hat{v}) - \frac{1}{2}\hat{\alpha}_x + \frac{1}{6}(D\hat{u})_x + \frac{4}{9}\hat{u}\hat{\alpha} - \frac{2}{9}\hat{u}(D\hat{u}), \]

thus the composition of (19) and (20) will give us the correct formulation for unhatted variables.

Above argument reveals that the hierarchy we constructed from \( L_2^{(2)} \) is a hierarchy associated with \( N = 2 \) super \( W_3 \) algebra. We notice that these hierarchies were constructed by Yung\[13\] and Pichugin \textit{et al}\[1\] independently. Those authors presented three possible integrable \( N = 2 \) supersymmetric Boussinesq systems. It is further shown one of these hierarchies has two local Hamiltonian operators\[12\][1]. Our hierarchy is equivalent to this system. To see this, we notice that the first Hamiltonian structure for this hierarchy is obtained by a simple shift of the field \( W_2 \), which corresponds a shift of \( v \) in our coordinates. However this simple shift precisely gives us the first Poisson tensor \( Q \). Since two compatible Hamiltonian operators determines a hierarchy uniquely, we conclude that these hierarchies are indeed same. Another point we can now make is the non-reducible of the extended Boussinesq system to the classical Boussinesq system. This should be clear from the form our Lax operator.

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