CARDINAL INVARIANTS CONNECTED WITH QUOTIENTS OF REAL FUNCTIONS

BY

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Abstract. We study cardinal invariants related to quotients in the case of the complement in $\mathbb{R}^R$ of families of continuous, quasi-continuous, cliquish and Darboux functions.

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1. Introduction

The letters $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{N}$ denote the real line, the set of rationals and the set of positive integers, respectively. The family of all functions from a set $X$ into $Y$ is denoted by $Y^X$. The word function denotes a mapping from $\mathbb{R}$ to $\mathbb{R}$ unless otherwise explicitly stated. For each set $A \subset \mathbb{R}$ the symbol $\chi_A$ denotes the characteristic function of $A$. We consider cardinals as ordinals not in one-to-one correspondence with the smaller ordinals. The symbol $\text{card} X$ stands for the cardinality of a set $X$. We write $c = \text{card} \mathbb{R}$. For each set $A \subset \mathbb{R}$ the symbols $\text{int} A$ and $\text{cl} A$ denote the interior and the closure of $A$, respectively. Let $\kappa$ be a cardinal number such that $\omega \leq \kappa \leq c$. We say that a set $A \subset \mathbb{R}$ is bilaterally $\kappa$-dense in itself if $\text{card}(A \cap I) = \kappa$ for every nondegenerate interval $I$ with $A \cap I \neq \emptyset$.

Let $f : \mathbb{R} \to \mathbb{R}$. The symbol $C(f)$ denotes the set of points of continuity of $f$. For each $y \in \mathbb{R}$ let $[f = y] = \{x \in \mathbb{R} : f(x) = y\}$. Similarly we define the symbols $[f \neq y]$, $[f > y]$ and $[f < y]$.

The following symbols denote respective classes of functions:
$\mathcal{D}$ – of all Darboux functions, i.e., $f \in \mathcal{D}$ iff it has the intermediate value property;

$\mathcal{C}$ – of all continuous functions;

$\mathcal{Q}$ – of all quasi-continuous functions in the sense of Kempisty [6], i.e., $f \in \mathcal{Q}$ iff for each $x \in \mathbb{R}$ there is a sequence $(x_n) \subset C(f)$ such that $x_n \to x$ and $f(x_n) \to f(x)$ (see, e.g., [4] or [5, Lemma 2]);

$\mathcal{C}_q$ – of all cliquish functions [16], i.e., $f \in \mathcal{C}_q$ iff $\text{cl} C(f) = \mathbb{R}$ (see, e.g., [14]).

The following cardinal function has been defined for families $A \subset \mathbb{R}$ (see [13]):

$$a(A) \overset{\text{df}}{=} \min \left( \{ \text{card} \mathcal{F} : \mathcal{F} \subset \mathbb{R} & \exists g \forall f \in \mathcal{F} (f + g \in A) \} \cup \{2^c\} \right).$$

The above value for different classes of real functions has been studied in several papers (see e.g. [2] and [3]).

The following cardinal function connected with quotients of functions has been defined for families $A \subset \mathbb{R}$ in [8] (compare also [7] and [12]):

$$q(A) \overset{\text{df}}{=} \min \left( \{ \text{card} \mathcal{F} : \mathcal{F} \subset A/A & \exists g \forall f \in \mathcal{F} (f/g \in A) \} \cup \{ \text{card} A/A \} \right).$$

where $A/A \overset{\text{df}}{=} \{ f/g : f, g \in A, g(x) \neq 0 \text{ for each } x \in \mathbb{R} \}$. In particular, were examined the values of $q(\mathcal{D})$ and $q(\mathcal{Q})$ (see [7, Theorem] and [8, Theorem 2.7]).

In the above definition, it is quite natural to restrict ourselves to subfamilies of $A/A$ only. Indeed, if there is a function $g$ such that both $f/g$ and $1/g$ are in $A$, then $f \in A/A$.

We denote the complement of a family $A \subset \mathbb{R}$ by $\neg A$.

In 1996, Jordan [9] examined the values of $a(\neg A)$, where classes $A$ are chosen from the classes of Darboux-like functions. Notice that $a(\neg A)$ has the following interpretation:

$a(\neg A)$ is the smallest cardinality of a family $B \subset \mathbb{R}$ such that $A - B = \mathbb{R}$, where $A - B = \{ f - g : f \in A & g \in B \}$.

Similarly, the values of $q(\neg A)$, where $A$ is the family of peripherally continuous or closed graph functions, were studied in [10] and [11], respectively.

The purpose of this paper is to find the values of $q$ for the families $\neg \mathcal{C}$, $\neg \mathcal{Q}$, $\neg \mathcal{C}_q$, $\neg \mathcal{D}$. We obtained the following results:
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• \( q(\neg C) = q(\neg Q) = 2^\omega \) (see Theorem 2.7);

• \( \omega < q(\neg D) \leq 2^\omega \) (see Theorem 2.4);

• \( \omega < q(\neg C) < d \leq \omega \) (see Corollary 3.4 and Theorem 3.6).

Remark 1.1. From [10, Theorem 3.1.] we obtain

Corollary 1.2. If \( A \in \{C, Q, C_q\} \), then \( a(\neg A) = 2^\omega \).

Recall that the cardinal \( a(\neg D) \) was examined by Jordan [9, Theorem 8].

2. Families \( \neg D, \neg C, \neg Q \)

First, we will find the values of \( q \) for the families \( \neg D, \neg C, \neg Q \). We will need the following lemma.

Lemma 2.1. Let \( \kappa \) be a cardinal number such that \( \omega \leq \kappa \leq \omega \) and assume that \( \{A_\alpha : \alpha < \kappa\} \) is a family of non-empty and \( \kappa \)-dense in itself subsets of \( \mathbb{R} \). There exists a family \( \{A_{\alpha \beta} : \alpha, \beta < \kappa\} \) of pairwise disjoint sets such that for each \( \alpha, \beta < \kappa \) we have \( A_{\alpha \beta} \subset A_\alpha \subset \text{cl } A_{\alpha \beta} \).

Proof. Let \( \mathcal{A} \equiv \{(p, q, \alpha, \beta) : p, q \in \mathbb{Q}, \alpha, \beta < \kappa, (p, q) \cap A_\alpha \neq \emptyset\} \). Arrange all elements of \( \mathcal{A} \) in a sequence \( \{(p_\gamma, q_\gamma, \alpha_\gamma, \beta_\gamma) : \gamma < \kappa\} \). For each \( \gamma < \kappa \) we choose a point \( x_\gamma \in (p_\gamma, q_\gamma) \cap A_{\alpha_\gamma} \setminus \{x_\delta : \delta < \gamma\} \). For each \( \alpha, \beta < \kappa \) define \( A_{\alpha \beta} \equiv \{x_\gamma : \gamma < \kappa, \alpha_\gamma = \alpha, \beta_\gamma = \beta\} \). We will show that the family \( \{A_{\alpha \beta} : \alpha, \beta < \kappa\} \) has all required properties. Fix \( \alpha, \beta < \kappa \).

Clearly \( A_{\alpha \beta} \subset A_\alpha \).

Let \( p, q \in \mathbb{Q} \) and \( (p, q) \cap A_\alpha \neq \emptyset \). Then \( (p, q, \alpha, \beta) = (p_\gamma, q_\gamma, \alpha_\gamma, \beta_\gamma) \) for some \( \gamma < \kappa \). Since \( x_\gamma \in (p, q) \cap A_{\alpha \beta} \), we have \( (p, q) \cap A_{\alpha \beta} \neq \emptyset \). It follows that \( A_\alpha \subset \text{cl } A_{\alpha \beta} \).

Now assume that \( A_{\alpha', \beta'} \cap A_{\alpha'', \beta''} \neq \emptyset \) for some \( \alpha', \beta', \alpha'', \beta'' < \kappa \). There are \( \gamma', \gamma'' < \kappa \) such that \( x_{\gamma'} = x_{\gamma''}, \alpha_{\gamma'} = \alpha', \beta_{\gamma'} = \beta', \alpha_{\gamma''} = \alpha'' \) and \( \beta_{\gamma''} = \beta'' \). Since the sequence \( \{x_\gamma : \gamma < \kappa\} \) is injective, we have \( \gamma' = \gamma'' \). Thus, \( \alpha' = \alpha_{\gamma'} = \alpha_{\gamma''} = \alpha'' \) and \( \beta' = \beta_{\gamma'} = \beta_{\gamma''} = \beta'' \), and the proof is complete. \( \square \)

Theorem 2.2. Let \( \{f_\alpha : \alpha < \omega\} \subset \mathbb{R} \setminus \{\chi_0\} \). There exists a function \( g : \mathbb{R} \to \mathbb{R} \setminus \{0\} \) such that \( f_\alpha / g \in \neg D \cap \neg Q \) for each \( \alpha < \omega \).
Proof. We can assume that the set \([f_\alpha \neq 0]\) is \(c\)-dense in itself for each \(\alpha < c\). Indeed, otherwise for each function \(g: \mathbb{R} \to \mathbb{R} \setminus \{0\}\) the set \([f_\alpha / g \neq 0] = [f_\alpha \neq 0] \cap \mathbb{R} \setminus \mathbb{Q}\) is not \(c\)-dense in itself and consequently \(f_\alpha / g \in \mathbb{R} \setminus \mathbb{Q}\).

By Lemma 2.1, there is a family \(\{A_{\alpha i} : \alpha < c, i \in \{0, 1\}\}\) of pairwise disjoint sets such that \(A_{\alpha i} \subset [f_\alpha \neq 0] \subset \text{cl} A_{\alpha i}\) for each \(\alpha < c\) and \(i \in \{0, 1\}\). Clearly we may assume that each set \(A_{\alpha i}\) is countable. Arrange all elements of the set \(\mathbb{R} \setminus \bigcup_{\alpha < c} (A_{\alpha 0} \cup A_{\alpha 1})\) in a sequence \(\{x_\beta : \beta < \kappa\}\). Now, by transfinite induction we will define a transfinite sequence of real numbers \(\{y_\alpha : \alpha < c\}\) and the function \(g: \mathbb{R} \to \mathbb{R} \setminus \{0\}\).

First, for each \(\alpha < c\) define \(K_\alpha \overset{\text{df}}{=} \bigcup_{\beta < \alpha} (A_{\beta 0} \cup A_{\beta 1}) \cup \{x_\beta : \beta < \min\{\alpha, \kappa\}\}\). One can easily see that \(\text{card} K_\alpha < c\) for each \(\alpha < c\).

Choose any real number \(y_\alpha\) such that

\[
(1) \quad y_\alpha \in \mathbb{R} \setminus ((f_\alpha / g)[K_\alpha] \cup \{0\})
\]

and define \(g\) on \(K_{\alpha+1} \setminus K_\alpha\) with

\[
(2) \quad y_\beta \not\in (f_\beta / g)[K_{\alpha+1} \setminus K_\alpha] \quad \text{for each } \beta \leq \alpha
\]

and \(A_{\beta 0} \subset [f_\beta / g < y_\alpha], A_{\beta 1} \subset [f_\beta / g > y_\alpha]\).

Fix \(\beta \leq \alpha\). Observe that, by conditions i) and (1), we obtain \(y_\beta \in \mathbb{R} \setminus ((f_\beta / g)[K_\alpha] \cup \{0\})\). Furthermore, by condition (2), we have \(y_\beta \in \mathbb{R} \setminus ((f_\beta / g)[K_{\alpha+1}] \cup \{0\})\). Clearly \(A_{\beta 0} \subset [f_\beta / g < y_\beta]\) and \(A_{\beta 1} \subset [f_\beta / g > y_\beta]\).

Then conditions i)--iii) hold for each \(\beta \leq \alpha\).

Fix \(\alpha < c\). First we will show that \(f_\alpha / g \in \mathbb{R} \setminus \mathbb{Q}\).

Let \(x \in \mathbb{R}\). Since \(\bigcup_{\beta < \alpha} K_\beta = \mathbb{R}\) and the sequence \(\{K_\beta : \beta < c\}\) is increasing, we have \(x \in K_\beta\) for some \(\alpha < \beta < c\). By condition i), we have \(y_\alpha \not\in (f_\alpha / g)[K_\beta]\). It follows that \((f_\alpha / g)(x) \neq y_\alpha\) and, consequently, \([f_\alpha / g = y_\alpha] = \emptyset\). Moreover \([f_\alpha / g < y_\alpha] \neq \emptyset \neq [f_\alpha / g > y_\alpha]\) (see conditions ii) and iii)). This implies that \(f_\alpha / g \in \mathbb{R} \setminus \mathbb{Q}\).

Now, we will show that \(f_\alpha / g \in \mathbb{R} \setminus \mathbb{Q}\). Let \(x_0 \in [f_\alpha \neq 0] = [f_\alpha / g \neq 0]\). Using the fact that \([f_\alpha / g \neq 0] \subset \text{cl} A_{\alpha 0}\) and condition ii) we get that

\[
\liminf_{x \to x_0} (f_\alpha / g)(x) \leq y_\alpha.
\]

Similarly, using the fact that \([f_\alpha / g \neq 0] \subset \]
cl $A_{\alpha 1}$ and condition iii) we get that $\limsup_{x \to x_0} (f_\alpha/g)(x) \geq y_\alpha$. Furthermore, since $(f_\alpha/g)(x_0) \neq y_\alpha$, we have $x_0 \notin C(f_\alpha/g)$. Hence $C(f_\alpha/g) \subset \{f_\alpha/g = 0\}$. Thus, $f_\alpha/g \notin \neg Q$, and the proof is complete. □

Corollary 2.3. If $A \in \{-\mathcal{C}, -\mathcal{D}, -\mathcal{Q}\}$, then $A/\mathcal{A} = \mathbb{R}^\mathbb{R} \setminus \{x_\emptyset\}$.

Proof. Clearly $x_\emptyset \notin A/\mathcal{A}$. Let $f \in \mathbb{R}^\mathbb{R} \setminus \{x_\emptyset\}$. Then, by Theorem 2.2, there is a function $g$ such that $f/g \in \mathcal{A}$ and $1/g \in \mathcal{A}$. Thus $f = (f/g)/(1/g) \in A/\mathcal{A}$. □

Theorem 2.4. If $A \in \{-\mathcal{C}, -\mathcal{D}, -\mathcal{Q}\}$, then $c < q(A) \leq 2^c$.

Proof. The inequality $q(A) > c$ follows from Theorem 2.2 and Corollary 2.3. Let $F \overset{\text{def}}{=} (\mathbb{R} \setminus \{0\})^\mathbb{R}$. Clearly $F \subset A/\mathcal{A}$ (see Corollary 2.3). If $g: \mathbb{R} \to \mathbb{R} \setminus \{0\}$, then evidently $g \in F$ and $g/g = x_\mathbb{R} \notin A$. Consequently $q(A) \leq \text{card} F = 2^c$.

Using Theorem 2.4 we obtain

Theorem 2.5. If $c^+ = 2^c$, then $q(-\mathcal{D}) = 2^c$.

The following problem is open

Problem 1. Can the equality $q(-\mathcal{D}) = 2^c$ be proved in ZFC?

Now, we will show that $q(-\mathcal{C}) = q(-\mathcal{Q}) = 2^c$. We start with a useful lemma [10, Lemma 3.4.]:

Lemma 2.6. Let $A \subset \mathbb{R}$ and $\text{card} A = c$. If $\mathcal{F} \subset (\mathbb{R} \setminus \{0\})^\mathbb{R}$ and $\text{card} \mathcal{F} < 2^c$, there is a function $g: A \to \mathbb{R} \setminus \{0\}$ such that the function $f/g: A \to \mathbb{R} \setminus \{0\}$ is unbounded for each $f \in \mathcal{F}$.

In the proof of the following theorem we shall use methods of the proof of [10, Theorem 3.5.].

Theorem 2.7. If $A \in \{-\mathcal{C}, -\mathcal{Q}\}$, then $q(A) = 2^c$.

Proof. The inequality $q(A) \leq 2^c$ follows from Theorem 2.4. Now, we will show that $q(A) \geq 2^c$.

Let $\mathcal{F} \subset A/\mathcal{A} = \mathbb{R}^\mathbb{R} \setminus \{x_\emptyset\}$ (c.f. Corollary 2.3) and assume that $\text{card} \mathcal{F} < 2^c$. It is enough to show that there is a function $g: \mathbb{R} \to \mathbb{R} \setminus \{0\}$ such that $f/g \in \mathcal{A}$ for each function $f \in \mathcal{F}$. First recall that, if $f \in \mathcal{F}$ and $\text{cl}[f = 0] \neq \mathbb{R}$, then $f/g \in \mathcal{A}$ for each $g: \mathbb{R} \to \mathbb{R} \setminus \{0\}$. So we may assume that $\text{cl}[f = 0] \neq \mathbb{R}$ for each $f \in \mathcal{F}$.
Choose a partition \( \{ S_\alpha : \alpha < c \} \) of \( \mathbb{R} \) into pairwise disjoint \( c \)-dense sets and let \( \{ I_\alpha : \alpha < c \} \) be an enumeration of the open intervals in \( \mathbb{R} \). Let \( A_\alpha \overset{\text{df}}{=} S_\alpha \cap I_\alpha \). Note that \( \text{card} \ A_\alpha = c \) and \( A_\alpha \cap A_\beta = \emptyset \) for \( \alpha < \beta < c \). Fix \( \alpha < c \). Let \( \mathcal{F}_\alpha \overset{\text{df}}{=} \{ f \mid A_\alpha : f \in \mathcal{F} \& A_\alpha \subset [f \neq 0] \} \). Evidently \( \text{card} \mathcal{F}_\alpha < 2^c \).

By Lemma 2.6 there is some \( g_\alpha : A_\alpha \to \mathbb{R} \setminus \{0\} \) such that \( f/g_\alpha \) is not bounded on \( A_\alpha \) for every \( f \in \mathcal{F} \) with \( A_\alpha \subset [f \neq 0] \). Let \( g : \mathbb{R} \to \mathbb{R} \setminus \{0\} \) extend \( \bigcup \{ g_\alpha : \alpha < c \} \). Observe that for each \( f \in \mathcal{F} \) there is a nondegenerate interval \( I_f \) such that \( C((f/g)|I_f) = \emptyset \). Thus, \( f/g \in A \) for every \( f \in \mathcal{F} \) and the proof is complete. \( \square \)

3. The family \( \neg C_q \)

Now, we will examine the value of \( q(\neg C_q) \).

Denote by \( \mathcal{B} \) the family of all functions \( f \in \mathbb{R}^\mathbb{R} \) such that the set \( [f \neq 0] \) is not nowhere dense, i.e. \( \text{int} \text{cl}[f \neq 0] \neq \emptyset \). We start with a simple proposition.

**Proposition 3.1.** \( \neg C_q/\neg C_q \subset \mathcal{B} \).

**Proof.** Assume that there is a function \( f \in \neg C_q/\neg C_q \setminus \mathcal{B} \). Let \( f = g/h \), where \( g, h \in \neg C_q \). Since \( f \notin \mathcal{B} \) and \( [f \neq 0] = [g \neq 0] \), we have \( \text{cl} \text{int}[g = 0] = \mathbb{R} \). Consequently \( g \in C_q \), an impossibility. \( \square \)

**Theorem 3.2.** Let \( f_1, f_2, \ldots \in \mathcal{B} \). There exists a function \( g : \mathbb{R} \to \mathbb{R} \setminus \{0\} \) such that \( f_n/g \in \neg C_q \), for each \( n \in \mathbb{N} \).

**Proof.** For each \( n \in \mathbb{N} \) choose an open interval \( I_n \subset \text{cl}[f_n \neq 0] \). By Lemma 2.1, there is a family \( \{ A_{nk} : n \in \mathbb{N}, k \in \{0,1\} \} \) of pairwise disjoint sets such that \( A_{nk} \subset I_n \cap [f_n \neq 0] \subset \text{cl} A_{nk} \) for each \( n \in \mathbb{N} \) and \( k \in \{0,1\} \). Define a function \( g : \mathbb{R} \to \mathbb{R} \setminus \{0\} \) by formula

\[
g(x) = \begin{cases} (-1)^k f_n(x), & \text{if } x \in A_{nk}, \ n \in \mathbb{N}, \ k \in \{0,1\}, \\ 1, & \text{otherwise.} \end{cases}
\]

Fix \( n \in \mathbb{N} \). Notice that for each \( k \in \{0,1\} \), \( I_n \subset \text{cl} I_n = \text{cl}(I_n \cap \text{cl}[f_n \neq 0]) \subset \text{cl} \text{cl} A_{nk} = \text{cl} A_{nk} \). Moreover \( f_n/g = 1 \) on \( A_{n0} \) and \( f_n/g = -1 \) on \( A_{n1} \). Consequently \( C(f_n/g) \cap I_n = \emptyset \) and \( f_n/g \in \neg C_q \). \( \square \)

The next corollary follows from Proposition 3.1 and Theorem 3.2 (see also the proof of Corollary 2.3).
Corollary 3.3. \( -C_q/\neg C_q = \mathcal{B} \).

From Corollary 3.3 and Theorem 3.2 we obtain the following

**Corollary 3.4.** \( q(-C_q) > \omega \).

Now, we need the following cardinal: \( \mathfrak{d} \stackrel{\text{df}}{=} \min \{ \text{card} \ F : F \subset \omega^\omega, \forall g \in \omega^\omega, \exists f \in F \ (g \leq f) \} \). It is well-known that \( \omega < \mathfrak{d} \leq \mathfrak{c} \) and it is consistent with ZFC that \( \mathfrak{d} = \omega_1 \) and \( \mathfrak{c} = \omega_2 \) (see [1] or [15]).

Recall also the following Lemma [11, Lemma 2.6]

**Lemma 3.5.** There exists a family \( F \subset \mathbb{R}^\mathbb{R} \) of cardinality \( \mathfrak{d} \) such that:

a) \( [f \neq 0] = \mathbb{Q} \) for each function \( f \in F \),

b) for each function \( g : \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\} \) there exists a function \( f \in F \) such that \( \lim_{t \to x} (f/g)(t) = 0 \) for each \( x \in \mathbb{R} \).

**Theorem 3.6.** \( q(-C_q) \leq \mathfrak{d} \).

**Proof.** Let \( F \) be a family of functions defined in Lemma 3.5. By Corollary 3.3 and condition a) of Lemma 3.5, we have \( F \subset -C_q/\neg C_q \).

Fix \( g : \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\} \). By condition b) of Lemma 3.5, there exists a function \( f \in F \) such that \( \lim_{t \to x} (f/g)(t) = 0 \) for each \( x \in \mathbb{R} \). Moreover by condition a) of Lemma 3.5 we have \( [f/g = 0] = \mathbb{R} \setminus \mathbb{Q} \). Consequently \( \mathcal{C}(f/g) = \mathbb{R} \setminus \mathbb{Q} \) and \( f/g \in C_q \). It follows that \( q(-C_q) \leq \text{card} \ F = \mathfrak{d} \). \( \square \)

Using Corollary 3.4, Theorems 3.6 and the inequality \( \mathfrak{d} \leq \mathfrak{c} \), we conclude that

**Theorem 3.7.** The Continuum Hypothesis implies \( q(-C_q) = \mathfrak{c} \).

The following problem is open:

**Problem 2.** Can the inequality \( q(-C_q) < \mathfrak{d} \) be a consequence of ZFC?

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