Abstract. We develop the theory of Fraïssé limits for classes of finite-dimensional multi-seminormed spaces, which are defined to be vector spaces equipped with a finite sequence of seminorms. We define a notion of a Fraïssé Fréchet space and we use the Fraïssé correspondence in this setting to obtain many examples of such spaces. This allows us to give a Fraïssé-theoretic construction of \((G^\omega, (\|\cdot\|_n)_{n<\omega})\), the separable Fréchet space of almost universal disposition for the class of all finite-dimensional Fréchet spaces with an infinite sequence of seminorms. We then identify and prove an approximate Ramsey property for various classes of finite-dimensional multi-seminormed spaces using known approximate Ramsey properties of normed spaces. A version of the Kechris-Pestov-Todorčević correspondence for approximately ultrahomogeneous Fréchet spaces is also established and is used to obtain new examples of extremely amenable groups. In particular, we show that the group of surjective linear seminorm-preserving isometries of \((G^\omega, (\|\cdot\|_n)_{n<\omega})\) is extremely amenable.

1. Introduction

The study of universal Banach spaces was initiated by Banach and Mazur \cite{1}, who showed that every separable Banach space embeds isometrically into \(C([0,1])\) (see also \cite{9} Corollary 12.14 or \cite{29} Theorem 8.7.2). Motivated by the Rotation Problem of Mazur, Gurarij \cite{14} constructed a separable Banach space \(G\), now known as the Gurarij space, which is universal for separable Banach spaces and which satisfies the following additional property: For every \(\varepsilon > 0\), every pair \(X \subseteq Y\) of finite-dimensional Banach spaces, and every isometric embedding \(f : X \to G\), there is an \(\varepsilon\)-isometric embedding \(\tilde{f} : Y \to G\) which extends \(f\). In other words, \(G\) is of almost universal disposition for the class of all finite-dimensional Banach spaces. Lusky \cite{23} later showed that the Gurarij space is unique up to isometry, in the sense that any other space which is of almost universal disposition for all finite-dimensional Banach spaces must be isometrically isomorphic to \(G\). (A simpler proof of the uniqueness and universality of the \(G\) can be found in \cite{20}.) As a consequence of the fact that \(G\) is of almost universal disposition, one can show that \(G\) is approximately ultrahomogeneous: For every finite-dimensional subspace \(X \subseteq G\), every \(\varepsilon > 0\) and every isometric embedding \(f : X \to G\), there is a surjective linear isometry \(g : G \to G\) such that \(\|g(x) - f(x)\| \leq \varepsilon\) for each \(x \in X\). For more information on the Mazur Rotation Problem and related concepts in the context of Banach spaces, we refer the reader to the recent survey \cite{7}.

Similar developments have taken place in the setting of Fréchet spaces: Mazur and Orlicz (see, e.g., \cite{27} p. 101]) showed that \(C(\mathbb{R})\) is homeomorphically universal for the class of all separable

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Fréchet spaces, in the sense that every Fréchet space admits a linear homeomorphism into $C(\mathbb{R})$. Moving toward the isometric theory of Fréchet spaces, Bargez, Kąkol and Kubiś [2] recently proved the existence of an analogue of the Gurarij space in the setting of Fréchet spaces equipped with a fundamental system of seminorms; they construct a separable Fréchet space $(G^\omega, (\|\cdot\|_n)_{n<\omega})$ with a suitable sequence of seminorms which is approximately ultrahomogeneous (with respect to isometric embeddings which preserve each given seminorm simultaneously) for the class of all finite-dimensional Fréchet spaces equipped with an infinite sequence of seminorms. A similar space is also constructed in [2] for the class of all finite-dimensional graded Fréchet spaces, i.e. Fréchet spaces equipped with an increasing sequence of seminorms.

Interest in universal spaces such as the Gurarij space is partially motivated by the study of their groups of linear isometries. In particular, the isometry group $\text{Iso}(G)$ of the Gurarij space has been an object of intense investigation in recent years (see, for instance, [4, 5]). One way to develop a better understanding of Polish groups such as $\text{Iso}(G)$ is via their topological dynamics; a relevant property here is that of extreme amenability. Recall that a topological group $G$ is extremely amenable if every continuous action of $G$ on a compact Hausdorff space $X$ has a common fixed point, i.e. a point $x \in X$ such that $g \cdot x = x$ for all $g \in G$. Many examples of such groups have been obtained using a link – known as the Keichis-Pestov-Todorčević (KPT) correspondence – between the Ramsey property and the extreme amenability of automorphism groups of ultrahomogeneous first-order structures. Examples of such structures come from Fraïssé theory, which provides a correspondence between countable ultrahomogeneous structures and certain classes of finitely-generated first-order structures, known as Fraïssé classes. Since Fraïssé’s original paper [11] (see also [16]), many other versions of the Fraïssé correspondence have been developed; one of the more prominent instances of this is the development the Fraïssé theory of metric structures (i.e. metric spaces with additional compatible structure) due to Ben Yaacov [6]. Other presentations of Fraïssé theory in the context of structures from functional analysis, including “non-commutative” structures, have been developed recently in [10, 22]. Naturally, these events led to the establishment of the KPT correspondence in more general settings, including Banach spaces [4, 10]. In fact, the first successful application of the KPT correspondence for metric structures developed in [25] was recently achieved in [4], where the authors show that $\text{Iso}(G)$ is extremely amenable by proving an approximate Ramsey property for the class of all finite-dimensional Banach spaces. Interestingly, the latter result makes crucial use of the Dual Ramsey Theorem of Graham and Rothschild [13]. In this paper we obtain analogous results in setting of Fréchet spaces and more generally for multi-seminormed spaces, i.e. vector spaces equipped with a finite or infinite sequence of seminorms. In particular, we develop Fraïssé theory for multi-seminormed spaces together with the relevant versions of the KPT correspondence and the approximate Ramsey property. A similar approach was used in [3] to study different classes of exact operator spaces, where a sequence of norms is also an essential part of the structure. In the present paper we provide a Fraïssé-theoretic proof of the existence of the spaces constructed in [2], which were originally defined using properties of a universal (Fraïssé) operator on $G$ originally considered in [12]. Such an operator can be seen as a “Gurarij” version of the Rota universal operator on the separable infinite-dimensional Hilbert space (for which we refer the reader to, e.g., [22, Section 4.1]). In addition to the known examples discussed above, we also obtain many new examples of Fréchet spaces with strong forms of approximate ultrahomogeneity,
including examples which naturally arise from various combinations of Hilbert spaces and $L_p$ spaces. As a result, we obtain many new examples of extremely amenable groups; in particular, we show that the group $\text{Iso}(G, (\| \cdot \|_n)_{n<\omega})$ of all surjective linear seminorm-preserving isometries of $(G, (\| \cdot \|_n)_{n<\omega})$ is extremely amenable.

The rest of the paper is organized as follows. In Section 2 we consider amalgamation and Fraïssé properties of finite-dimensional multi-seminormed spaces. In particular, we show that the class $\mathcal{M}_{<\omega}$ of all finite-dimensional vector spaces equipped with a finite sequence of seminorms is Fraïssé. Similarly, the classes consisting of all finite-dimensional vector spaces with a sequence of seminorms of length at most $n$ (for various $n \geq 1$) are shown to be Fraïssé as long as one restricts to separated multi-seminormed spaces, where a space is separated if the natural topology induced by the associated sequence of seminorms is a Hausdorff topology. More generally, we show that Fraïssé classes of finite-dimensional normed spaces naturally give rise to Fraïssé classes of finite-dimensional multi-seminormed spaces, and we use this to obtain many examples of such classes. In Section 3 we develop a notion of a Fraïssé Fréchet space and we obtain examples of such Fréchet spaces by taking Fraïssé limits of classes of certain finite-dimensional multi-seminormed spaces. We also show that the separable Fréchet spaces of almost universal disposition constructed in [2] can be realized as Fraïssé limits. Section 4 contains a proof of the analogue of the KPT correspondence for multi-seminormed spaces, which involves isolating a useful version of the approximate Ramsey property for such spaces. To conclude, we show how to transfer the approximate Ramsey property of classes of finite-dimensional normed spaces to the corresponding approximate Ramsey property for certain classes of multi-seminormed spaces, thus obtaining new examples of extremely amenable groups by way of the KPT correspondence.

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2. Fraïssé classes of finite-dimensional Fréchet spaces

We use standard set-theoretic notation; in particular, $\omega = \{0, 1, \ldots\}$ denotes the first countably infinite ordinal and $\mathbb{N}$ will denote $\omega \setminus \{0\}$. When convenient, an integer $n$ will be identified with the set $\{0, \ldots, n-1\}$ of its predecessors.

Let $X$ be a Fréchet space, i.e. a locally convex Hausdorff topological vector space which is completely metrizable via a translation-invariant metric. A fundamental system of seminorms on $X$ is a sequence of seminorms $(\| \cdot \|_{X,n})_{n<\alpha}$ on $X$, where $\alpha \leq \omega$, such that the sets

$$B_{n,\varepsilon}(x) = \{ y \in X : \max_{m<n} |x - y|_{X,m} < \varepsilon \}, \quad x \in X, n < \alpha, \varepsilon > 0$$

form a basis for the topology of $X$. It is a standard fact that every Fréchet space admits a (non-unique) fundamental system of seminorms and so we can view a Fréchet space as a multi-seminormed space of the above form, i.e. a (finite or infinite) tuple consisting of a topological vector space equipped with a sequence of seminorms. By definition, a fundamental system of seminorms is always separating: For every non-zero $x \in X$ there is a seminorm $\| \cdot \|_{X,n}, n < \alpha$ such that $\|x\|_{X,n} \neq 0$; in this case we also say that $X$ is separated, which happens precisely when the sequence of seminorms induces a Hausdorff topology on $X$. In this paper we will work more generally with multi-seminormed spaces.
of the form $X = (X, (\| \cdot \|_{X,n})_{n<\lambda_X})$, where $X$ is a vector space, $1 \leq \lambda_X \leq \omega$ is the length of $X$, which is defined as the length of the associated sequence of seminorms (which we always assume is non-empty), and where $(\| \cdot \|_{X,n})_{n<\lambda_X}$ is a (finite or infinite) sequence of seminorms on $X$. We will always conflate the above tuple $X$ with its underlying space $X$, and so we will always write $\lambda_X$ for the length when the sequence of seminorms is understood. We point out that in general we do not require a multi-seminormed space to be separated. With this in mind, we reserve the term “Fréchet space” for a separated multi-seminormed space $(X, (\| \cdot \|_{X,n})_{n<\lambda_X})$ such that the topology generated by the seminorms $\| \cdot \|_{X,n}$ is complete. It is clear that in this case the sequence of seminorms $(\| \cdot \|_{X,n})$ forms a fundamental system of seminorms on $X$. Finally, following the terminology in [2, 31], we say that a multi-seminormed space $(X, (\| \cdot \|_{X,n})_{n<\lambda_X})$ is graded if, for all $x \in X$, $\|x\|_{X,n} \leq \|x\|_{X,m}$ whenever $n \leq m < \lambda_X$. For more information about (graded) Fréchet spaces, we refer the reader to [15, 24, 31]. The reader is also referred to [21, 30] for more information about the general theory of multi-seminormed spaces (which are also sometimes called seminormed spaces or multinormed spaces in the literature).

Throughout this paper we will work with various subclasses of the class $\mathcal{M}$ of finite-dimensional multi-seminormed spaces equipped with a fundamental system of seminorms. Of particular interest will be the subclass $\mathcal{G}$ consisting of all graded $X \in \mathcal{M}$. Given any subclass $\mathcal{K} \subseteq \mathcal{M}$, and $\alpha < \omega$, we let $\mathcal{K}_{<\alpha}, \mathcal{K}_{\leq \alpha}$ and $\mathcal{K} = \alpha$ denote the collections of all $X \in \mathcal{K}$ such that $\lambda_X < \alpha$, $\lambda_X \leq \alpha$ and $\lambda_X = \alpha$, respectively. For any such $\mathcal{K}$ we also let $\mathcal{K}_{\text{sep}}$ denote the subclass of $\mathcal{K}$ consisting of all separated members of $\mathcal{K}$.

Given two multi-seminormed spaces $X$ and $Y$ and a linear mapping $f : X \to Y$, we say that a linear mapping $f : X \to Y$ is multi-bounded when $\lambda_X \leq \lambda_Y$ and when for every $m < \lambda_X$ one has

$$\|f\|_{\text{mb}} := \sup_{m<\lambda_X} \sup_{x \in X, \|x\|_{X,m} \leq 1} \|f(x)\|_{Y,m} < \infty,$$

and where as usual $\|f\|_{(X,\|\cdot\|_{X,m}),(Y,\|\cdot\|_{Y,m})}$ is the (semi)norm of $f$ defined as $\sup_{x \in X, \|x\|_{X,m} \leq 1} \|f(x)\|_{Y,m}$. When working with the operator norm, we will usually suppress the notation and omit the reference to the underlying spaces when this is clear from context. Observe that since we are dealing with seminorms, this quantity might be infinite, even when $X$ is finite-dimensional. A multi-isomorphism $f : X \to Y$ is linear bijection such that both $f$ and $f^{-1}$ are multi-bounded. Observe also that an isomorphism is a Lipschitz mapping, when considering the metrics associated to the fixed seminorms on $X$ and $Y$.

Much like in the class of normed spaces and others, we define a Banach-Mazur-like pseudometric, quantifying distances between isometric types. (The relevant notion of an isometry in this setting will be defined below.) We define

$$d_{\text{BM}}(X, Y) := \log \inf_f \|f\|_{\text{mb}} \|f^{-1}\|_{\text{mb}},$$

where $f$ runs over all multi-isomorphisms (if any) between $X$ and $Y$.

**Proposition 2.1.** $\mathcal{M}_{<\omega}$ and $\mathcal{G}_{<\omega}$ are $\sigma$-compact, and consequently separable.

**Proof.** $\mathcal{G}_{<\omega}$ is closed in $\mathcal{M}_{<\omega}$, so it suffices to prove that $\mathcal{M}_{<\omega}$ is $\sigma$-compact. As in the case of the Banach-Mazur pseudometric on the class of finite-dimensional normed spaces, it is not difficult to see that the closed $d_{\text{BM}}$-balls on $\mathcal{M}_{<\omega}$ are compact. But unlike for the normed space case, where for each
dimension \( k \) the diameter is finite (hence the Banach-Mazur compactum is compact), this is not the case for multi-seminormed spaces. Given \((X, (\| \cdot \|_{X,k})_{k < \lambda_X}) \in \mathcal{M}_{<\omega}\), let \( \overline{\pi}_X = (\alpha^X_s)_{s \leq \lambda_X} \) be defined for \( s \leq \lambda_X \) by \( \alpha^X_s := \dim \bigcap_{k \in s} \ker \| \cdot \|_{X,k} \) and \( \alpha^X_0 := \dim X \).

**Claim 2.1.1.** The classes of multi-seminormed spaces with finite distances (i.e. isomorphic classes) are exactly, given a sequence \( \tilde{\alpha} = (\alpha_s)_{s \in \mathbb{N}} \) of positive integers, the sets \( \mathcal{A}_{\tilde{\alpha}} := \{ X \in \mathcal{M}_{<\omega} : \tilde{\alpha}_X = \tilde{\alpha} \} \).

This fact easily implies that \( \mathcal{M}_{<\omega} \) is \( \sigma \)-compact. We prove now the claim. The sets \( \mathcal{A}_{\tilde{\alpha}} \) are closed under isomorphic images because if \( f : X \to Y \) is a multi-isomorphism, then \( \overline{\pi}_X = \overline{\pi}_Y \), because

\[
\frac{1}{1+\varepsilon} \| x \|_{X,m} \leq \| f(x) \|_{Y,m} \leq (1+\varepsilon) \| x \|_{X,m}
\]

for each \( x \in X \) and \( m < \lambda_X \). When \( \varepsilon = 0 \), we simply refer to such a mapping as a multi-isometric embedding; the collection of all such mappings \( f : X \to Y \) will be denoted \( \text{Emb}(X,Y) \). Note that, unlike in the setting of normed spaces, a seminorm-preserving mapping need not be injective. Also, a seminorm-preserving mapping \( f : X \to Y \) need not be continuous when \( \lambda_X < \lambda_Y \). A multi-isometry (resp. multi-\( \varepsilon \)-isometry) is a surjective multi-isometric embedding (resp. multi-\( \varepsilon \)-isometric embedding).

Below we will make use of the notion of a *modulus* relative to a set \( S \), i.e. a function \( \varpi : S \times [0, \infty[ \to [0, \infty[ \) such that, for every \( s \in S \), the function \( \varpi(s, \cdot) \) is increasing and continuous at \( 0 \) with value \( 0 \). In our case, \( S \) will be the product \( \mathbb{N} \times (\mathbb{N} \cup \{1\}) \).

**Definition 2.2.** Let \( \mathcal{K} \) be a class of finite-dimensional multi-seminormed spaces.

1. \( \mathcal{K} \) has the hereditary property (HP) if \( X \in \mathcal{K} \) whenever \( Y \in \mathcal{K} \) and \( \text{Emb}(X,Y) \neq \emptyset \).
2. \( \mathcal{K} \) has the joint embedding property (JEP) if for every \( X, Y \in \mathcal{K} \) there is \( Z \in \mathcal{K} \) such that \( \text{Emb}(X,Z) \) and \( \text{Emb}(Y,Z) \) are non-empty.
3. \( \mathcal{K} \) has the near amalgamation property (NAP) if for every \( \varepsilon > 0 \), every \( X, Y, Z \in \mathcal{K} \) and every pair of multi-isometric embeddings \( f_0 : X \to Y, f_1 : X \to Z \), there is \( W \in \mathcal{K} \) together with
multi-isometric embeddings \(g_0 : Y \to W, g_1 : Z \to W\) such that \(\|g_0 \circ f_0 - g_1 \circ f_1\|_n \leq \varepsilon\) for each \(n < \lambda_X\).

(4) \(\mathcal{K}\) is an amalgamation class with modulus of stability \(\varepsilon\) if \((\{0\}, \| \cdot \|) \in \mathcal{K}\) and for every \(d \in \mathbb{N}, l \in \mathbb{N} \cup \{\omega\}, \varepsilon > 0\) and \(\delta \geq 0\), every \(X, Y, Z \in \mathcal{K}\) such that \(\dim X = d\) and \(\lambda_X = l\), and every pair of multi-\(\delta\)-isometric embeddings \(f_0 : X \to Y, f_1 : X \to Z\), there is \(W \in \mathcal{K}\) together with multi-isometric embeddings \(g_0 : Y \to W, g_1 : Z \to W\) such that:

(i) \(\|g_0 \circ f_0 - g_1 \circ f_1\|_n \leq \varepsilon(d, l, \delta) + \varepsilon\) for each \(n < \lambda_X\).

(ii) \(W\) is separated.

(5) \(\mathcal{K}\) is Fraïssé if it is a hereditary \(d_{BM}\)-closed amalgamation class such that \(\mathcal{K} \subseteq \mathcal{M}_{\leq \omega}\).

Observe that every amalgamation class \(\mathcal{K}\) automatically has the JEP since one can simply amalgamate over the trivial normed space. We remark that condition (4)(ii) does not appear in the usual presentation of metric Fraïssé theory (and in particular the Fraïssé theory of Banach spaces developed in [10]). However, it appears to be necessary when working with arbitrary multi-normed spaces in order to guarantee injectivity of certain seminorm-preserving mappings encountered when constructing a Fraïssé limit. This condition will be trivially satisfied by all collections \(\mathcal{K}\) of interest. Conditions (4)(i) and (4)(ii) together imply that for any \(X\) belonging to an amalgamation class \(\mathcal{K}\), there is some separated \(Y \in \mathcal{K}\) such that \(\text{Emb}(X, Y) \neq \emptyset\). We will make use of this fact without reference.

We now proceed to show that the classes considered above are amalgamation classes. The following is a particular case of a more general result explained in Proposition 2.6, where it is shown that classes of multi-seminormed spaces defined from amalgamation classes of normed spaces are also amalgamation classes. We present the proof here since a version of the relevant construction (which is essentially a pushout in the category of multi-seminormed spaces) will be used in the sequel.

**Proposition 2.3.** For each \(\mathcal{K} \in \{\mathcal{M}, \mathcal{G}\}\) and each \(\alpha \leq \omega\), the classes \(\mathcal{K}_{\leq \alpha}^{\text{sep}}, \mathcal{K}_{< \alpha}^{\text{sep}}\) and \(\mathcal{K}_{\leq \alpha}^{\text{sep}}\) are amalgamation classes with modulus of stability \(\varepsilon(d, l, \delta) = 2\delta\). Furthermore, \(\mathcal{K}_{< \omega}\) is an amalgamation class with the same modulus of stability.

**Proof.** First we will show that \(\mathcal{K}_{\leq \alpha}^{\text{sep}}\) is an amalgamation class with modulus \(\varepsilon(d, l, \delta) = \delta\) with respect to \(\text{expansive}\) multi-\(\delta\)-isometric embeddings, i.e. multi-\(\delta\)-isometric embeddings \(f : X \to Y\) which additionally satisfy \(\|f(x)\|_{Y,n} \geq \|x\|_{X,n}\) for each \(x \in X\) and \(n < \lambda_X\). To this end, fix \(\varepsilon > 0, \delta \geq 0, d \in \mathbb{N}\) and \(l \in \mathbb{N} \cup \{\omega\}\). Let \(X, Y, Z \in \langle \mathcal{K} \rangle\) where \(\dim X = d\) and \(\lambda_X = l\), and fix expansive multi-\(\delta\)-isometric embeddings \(f : X \to Y\) and \(g : X \to Z\). We can suppose, without loss of generality, \(\lambda_Y \leq \lambda_Z\). Consider the sum \(Y \oplus Z\) together with the canonical inclusion mappings \(i : Y \to Y \oplus Z\) and \(j : Z \to Y \oplus Z\). Equip \(Y \oplus Z\) with a sequence \((\| \cdot \|_n)_{n < \lambda_Z}\) of seminorms in the following way:

(i) For each \(n \in [0, \lambda_X]\), let

\[
\|(y, z)\|_n := \inf\{\|u\|_{Y,n} + \|v\|_{Z,n} + (\delta + \varepsilon)\|x\|_{X,n} : x \in X, u \in Y, v \in Z, y = u + f(x), z = v - g(x)\}.
\]

(ii) Suppose \(n \in [\lambda_X, \lambda_Y]\). If \(\mathcal{K} \neq \mathcal{G}\), let \(\|(y, z)\|_n := \max\{\|y\|_{Y,n}, \|z\|_{Z,n}\}\). Otherwise, assuming inductively that \(\|(y, z)\|_{n-1}\) has been defined, let \(\|(y, z)\|_n := \max\{\|(y, z)\|_{n-1}, \|y\|_{Y,n}, \|z\|_{Z,n}\}\).

(iii) Suppose \(n \in [\lambda_Y, \lambda_Z]\). If \(\mathcal{K} \neq \mathcal{G}\), let \(\|(y, z)\|_n := \|z\|_{Z,n}\). Otherwise, assuming inductively that \(\|(y, z)\|_{n-1}\) has been defined, let \(\|(y, z)\|_n := \max\{\|(y, z)\|_{n-1}, \|z\|_{Z,n}\}\).
Note that the alternative in the last two parts of the above construction is needed to guarantee that \((\| \cdot \|)_n < \lambda_X\) is a graded sequence of seminorms whenever \(X, Y\) and \(Z\) are graded. Furthermore, \(Y \oplus Z\) is separated whenever \(Y\) and \(Z\) are both separated. In the case where we are working in \(K_{<\omega}\), the above space can be turned into a separated space by extending the above sequence of seminorms by a norm.

First we check that \(i\) is a multi-isometric embedding. Suppose first that \(n < \lambda_X\). Then for each \(y \in Y\), \(\| i(y) \|_n = \| y, 0 \|_n \leq \| y \|_{Y,n}\) by definition of \(\| \cdot \|_n\). On the other hand, given any \(x \in X, u \in Y\) and \(v \in Z\) such that \(y = u + f(x)\) and \(0 = v - g(x)\), we have

\[
\| u \|_{Y,n} + \| v \|_{Z,n} + (\delta + \varepsilon) \| x \|_{X,n} = \| u \|_{Y,n} + \| g(x) \|_{Z,n} + (\delta + \varepsilon) \| x \|_{X,n} \\
\geq \| u \|_{Y,n} + \| x \|_{X,n} + (\delta + \varepsilon) \| x \|_{X,n} \\
= \| u \|_{Y,n} + (1 + \delta + \varepsilon) \| x \|_{X,n} \\
\geq \| u \|_{Y,n} + \| f(x) \|_{Y,n} \geq \| u + f(x) \|_{Y,n} = \| y \|_{Y,n}
\]

and so \(\| i(y) \|_n \geq \| y \|_{Y,n}\). Thus \(i\) preserves the first \(\lambda_X\) seminorms. If \(\lambda_X \leq n \leq \lambda_Y\), then \(\| i(y) \|_n = \| y \|_{Y,n}\) by definition and so \(i\) is a multi-isometric embedding.

Next we check that \(j\) is a multi-isometric embedding. The case where \(n < \lambda_X\) is exactly as before. In the case where \(\lambda_X \leq n < \lambda_Y\), we have \(\| j(z) \|_n = \| z \|_{Z,n}\) by definition. Otherwise, \(n \geq \lambda_Y\). In the graded case we have

\[
\| j(z) \|_n = \max\{\| (0, z) \|_{X,\lambda_Y-1}, \| z \|_{Z,n}\} = \max\{\| z \|_{Z,\lambda_Y}, \| z \|_{Z,n}\} = \| z \|_{Z,n}
\]

since \(Z\) is graded. In the non-graded case we simply have \(\| j(z) \|_n = \| z \|_{Z,n}\) by definition, and so \(j\) is a multi-isometric embedding.

It remains to check \(\| i \circ f - j \circ g \| \leq \delta + \varepsilon\). To see this, note that for each \(x \in X\) and \(n < \lambda_X\) we have

\[
\| i(f(x)) - j(g(x)) \|_n = \| (f(x), -g(x)) \|_n \leq (\delta + \varepsilon) \| x \|_{X,n}
\]

by taking \(u = 0\) and \(v = 0\) in the definition of \(\| \cdot \|_n\). Thus \(i : Y \to Y \oplus Z\) and \(j : Z \to Y \oplus Z\) are the desired multi-isometric embeddings.

Now suppose that \(f\) and \(g\) are merely multi-\(\delta\)-isometric embeddings. Define a new sequence of seminorms \((\| \cdot \|_X)_n < \lambda_X\) on \(X\) by setting \(\| x \|_X = \frac{1}{1 + \delta} \| x \|_{X,n}\). Then \(f\) and \(g\) become expansive multi-\(\delta\)'-isometric embeddings from \(X\) (equipped with the new sequence of seminorms) into \(Y\) and \(Z\), respectively, where \(\delta' = 2\delta + \delta^2\). Thus we can apply the above argument to find \(W \in K_{<\alpha}^{sep}\) and multi-isometric embeddings \(i : Y \to W, j : Z \to W\) such that

\[
\| i(f(x)) - j(g(x)) \|_W \leq (\delta' + \varepsilon) \| x \|_X
\]

for all \(n < \lambda_X\) and \(x \in X\) such that \(\| x \|_X \leq 1\).

Now fix \(n < \lambda_X\) and \(x \in X\) such that \(\| x \|_X \leq 1\). Then \(\| x \|_X \leq \frac{1}{1 + \delta} \leq 1\) and so

\[
\| i(f(x)) - j(g(x)) \|_W \leq (\delta' + \varepsilon) \| x \|_X \leq \frac{\delta'}{1 + \delta} + \varepsilon \leq \frac{\delta + \delta(1 + \delta)}{1 + \delta} + \varepsilon \leq 2\delta + \varepsilon.
\]

This shows that \(K_{<\alpha}^{sep}\) is an amalgamation class with modulus \(\omega(d, l, \delta) = 2\delta\). In the case where \(K = G\), note that the space \(Y \oplus Z\) constructed above is graded whenever \(X, Y\) and \(Z\) are graded. This proofs for \(K_{<\alpha}^{sep}, K_{=\alpha}^{sep}\) and \(K_{<\omega}\) are similar. \(\square\)
Remark 2.4. The above proof can easily be adapted to prove amalgamation properties for classes of other “norm-like” structures from functional analysis. For instance, one can similarly show that the class of all finite-dimensional F-(semi)-normed spaces (as in [17]) is an amalgamation class with respect to $\delta$-isometric embeddings.

Another source of examples of amalgamation classes of multi-seminormed spaces will come from the following proposition, which will make use of known amalgamation properties of classes of normed spaces. (The relevant definitions are analogous to our case; the reader is referred to [10] for more details, examples and discussion.) The following is standard notation: Given a seminormed space $X = (X, \| \cdot \|)$, we denote by $X_{\| \cdot \|}$ the normed space $(X/\ker \| \cdot \|, \| \cdot \|)$ where $\|[x]\| := \|x\|$. Note that this norm is well-defined.

Definition 2.5. Let $\tilde{\mathcal{K}} := \{\mathcal{K}_n\}_{n<\alpha}$, $\alpha \leq \omega$, be a collection of classes $\mathcal{K}_n$ of normed spaces. We define $\langle \mathcal{K} \rangle_{<\alpha}$, $\langle \mathcal{K} \rangle_{\alpha}$, $\langle \mathcal{K} \rangle_{=\alpha}$ as the classes of separated multi-seminormed spaces $(X, (\| \cdot \|_n)_{n<\lambda_X}) \in \mathcal{M}_{<\alpha}$, $\mathcal{M}_{\leq \alpha}$ and in $\mathcal{M}_{=\alpha}$, respectively, such that $X_{\| \cdot \|_n} \in \mathcal{K}_n$ for all $n < \lambda_X$.

Proposition 2.6. Let $\{\mathcal{K}_n\}_{n<\alpha}$ be a sequence of families of finite-dimensional normed spaces.

(a) If each $\mathcal{K}_n$ is hereditary, then so are $\langle \mathcal{K} \rangle_{<\alpha}$ and $\langle \mathcal{K} \rangle_{=\alpha}$ for each $\alpha < \omega$.

(b) Suppose each $\mathcal{K}_n$ is an amalgamation class with modulus of stability $\varpi^\mathcal{K}_n : \mathbb{N} \to [0, \infty]$. Then for each $\alpha < \omega$ the class $\langle \mathcal{K} \rangle_{<\alpha}$ is also an amalgamation class with modulus of stability

$$\varpi(d, l, \delta) := \max_{n<l} \varpi^\mathcal{K}_n(d, \delta).$$

Furthermore, for each $n < \omega$ the classes $\langle \tilde{\mathcal{K}} \rangle_{\leq n}$ and $\langle \tilde{\mathcal{K}} \rangle_{= n}$ are amalgamation classes with the same modulus of stability.

Proof. The proof of (a) is relatively straightforward, so we leave the details to the reader. We only check (b) for $\langle \mathcal{K} \rangle_{<\alpha}$ since the proofs for the other classes are similar. Fix $\varepsilon > 0, \delta \geq 0$ together with integers $d, l$. Let $X, Y, Z \in \langle \mathcal{K} \rangle_{<\alpha}$ where $\dim X = d$ and $\lambda_X = l$, and fix multi-$\delta$-isometric embeddings $f_0 : X \to Y$ and $g_0 : X \to Z$. Suppose, without loss of generality, $\lambda_Y \leq \lambda_Z$. To construct the required element of $\langle \mathcal{K} \rangle_{<\alpha}$, we will first define a sequence $(W_i)_{i<\lambda_Z}$ of finite-dimensional normed spaces. For each integer $i < l$, define mappings $f^0_i : X_{\| \cdot \|_i} \to Y_{\| \cdot \|_i}$ and $g^0_i : X_{\| \cdot \|_i} \to Z_{\| \cdot \|_i}$, by setting

$$f^0_i([x]_i) = [f_0(x)]_i \text{ and } g^0_i([x]_i) = [g_0(x)]_i.$$ 

Then each $f^0_i$ and $g^0_i$ is a $\delta$-isometric embedding between elements of $\mathcal{K}_i$ and so, since $\mathcal{K}_i$ is an amalgamation class for each $i < n$, there are $W_i \in \mathcal{K}_i$ and isometric embeddings $f^1_i : Y_{\| \cdot \|_i} \to W_i, g^1_i : Z_{\| \cdot \|_i} \to W_i$ such that

$$\|f^1_i \circ f^0_i - g^1_i \circ g^0_i\|_{X_{\| \cdot \|_i}, W_i} \leq \varpi^\mathcal{K}_i(d, \delta) + \varepsilon.$$ 

For $i \in [l, \lambda_Y)$, let $W_i$ be obtained from an application of the JEP to the pair $Y_{\| \cdot \|_i}, Z_{\| \cdot \|_i}$; let $f^1_i$ and $g^1_i$, respectively, denote the corresponding multi-isometric embeddings. Finally, for $i \in (\lambda_Y, \lambda_Z)$, simply let $W_i = Z_{\| \cdot \|_i}$.

Let $W = \prod_{i<\lambda_Z} W_i$ and equip $W$ with a family of seminorms defined by setting

$$\|(w_0, \ldots, w_{\lambda_Z-1})\|_i = \|w_i\|_{W_i} \text{ for } i < \lambda_Z.$$
Observe that \( W \in \langle \mathcal{K} \rangle_{< \alpha} \) since \( W \) is separated and \( W \mid \| \cdot \|_i \cong W_i \) for each \( i < \lambda Z \). Define \( f_1 : Y \to W \) and \( g_1 : Z \to W \) by
\[
\begin{align*}
f_1(y) &= \left( f_1^0([y]_0), \ldots, f_1^\lambda y - 1([y]_{\lambda y - 1}), 0, \ldots, 0 \right), \\
g_1(z) &= \left( g_1^0([z]_0), \ldots, g_1^\lambda y - 1([z]_{\lambda y - 1}), [z]_{\lambda y}, \ldots, [z]_{\lambda y - 1} \right).
\end{align*}
\]
Then it is straightforward to check that \( f_1 \) and \( g_1 \) are multi-isometric embeddings which witness the amalgamation property for \( \mathcal{K} \) for the above parameters. Injectivity follows from the fact that \( Y \) and \( Z \) are separated.

Combining our two sources of examples of amalgamation classes yields the following list:

**Example 2.7.** (1) The class \( \mathcal{M} \subset \bar{\omega} \) of all finite-dimensional multi-seminormed spaces \( X \) such that \( \lambda_X < \omega \) is a Fraïssé class. In this case, observe that we do not need to restrict to separated multi-seminormed spaces since an arbitrary finite sequence of seminorms can always be extended by a norm. The same is true of the subclass \( \mathcal{G} \subset \bar{\omega} \) of all graded \( X \in \mathcal{M} \subset \bar{\omega} \).

(2) For each \( \alpha \leq \omega \), the classes \( \mathcal{M}^{\text{sep}}_{\leq \alpha}, \mathcal{M}^{\text{sep}}_{\leq \alpha} \) and \( \mathcal{M}^{\text{sep}}_{= \alpha} \) of all finite-dimensional separated multi-seminormed spaces such that \( \lambda_X < \alpha, \lambda_X \leq \alpha \) and \( \lambda_X = \alpha \), respectively, are amalgamation classes. Note, however, that these are not Fraïssé classes since it is possible for a non-separated multi-seminormed space to embed into a separated space according to our definition of a multi-isometric embedding. The same is true for the corresponding graded versions \( \mathcal{G}^{\text{sep}}_{\leq \alpha}, \mathcal{G}^{\text{sep}}_{\leq \alpha} \) and \( \mathcal{G}^{\text{sep}}_{= \alpha} \).

(3) For each infinite \( I \subseteq \omega \), let \( \mathcal{M}^I \subset \bar{\omega} \) denote the collection of all \( (X, (\| \cdot \|_n)_{n < \lambda_X}) \in \mathcal{M} \subset \bar{\omega} \) such that \( \| \cdot \|_I \) is a norm for each \( n \in I \cap [0, \lambda_X] \). Then \( \mathcal{M}^I \subset \bar{\omega} \) is a Fraïssé class. (This follows from the proof of Proposition 2.3.) The amalgamation classes \( \mathcal{M}^I_{< n} \) and \( \mathcal{M}^I_{= n} \) can be defined similarly.

(4) Let \( \mathcal{M}^H \subset \bar{\omega} \) denote the class of all finite-dimensional Fréchet-Hilbert spaces with a finite sequence of seminorms, where \( (X, (\| \cdot \|_n)_{n < \lambda_X}) \) is a Fréchet-Hilbert space if \( \| \cdot \|_n \) is a Hilbertian seminorm, i.e. a seminorm induced by a semi-inner product on \( X \). Equivalently, \( X \) is Fréchet-Hilbert if each quotient space \( X/\| \cdot \|_n \) is a Hilbert space. Then \( \mathcal{M}^H \subset \bar{\omega} \) is a Fraïssé class. (We refer the reader to [24] for more on Fréchet-Hilbert spaces and related concepts.)

(5) Given a sequence \( (p_n)_{n < \omega} \subseteq [1, \infty[ \) with \( p_n \notin \{4, 6, 8, \ldots\} \), consider the class \( \mathcal{M}^{(p_n)} \subset \bar{\omega} \) of all multi-seminormed spaces \( (X, (\| \cdot \|_n)_{n < \lambda_X}) \) such that for each \( n < \lambda_X \) the normed space \( X/\| \cdot \|_n \) can be isometrically embedded into \( L_{p_n}[0, 1] \). Then this is a Fraïssé class for any such sequence \( (p_n) \). This follows from the fact that \( \text{Age}(L_p[0, 1]) \) is a Fraïssé class of finite-dimensional normed spaces for each \( p \in [1, \infty[ \) such that \( p \notin \{4, 6, 8, \ldots\} \) (see [10]). In this case, the corresponding modulus of stability \( \varpi \) associated to \( \text{Age}(L_p[0, 1]) \) depends on the dimension for each \( p \), and hence so does the corresponding modulus of stability associated to \( \mathcal{M}^{(p_n)} \).

(6) In general, one can combine the various classes of normed spaces considered above to form new amalgamation classes of multi-seminormed spaces. For instance, given any sequence \( (E_n)_{n < \alpha} \) of Fraïssé Banach spaces, one can set \( \mathcal{K}_n = \text{Age}(E_n) \) for each \( n \) (noting that these are all Fraïssé classes of normed spaces) and form the associated class \( \langle \mathcal{K} \rangle_{\leq \alpha} \) of multi-seminormed spaces, where \( \mathcal{K} = (\mathcal{K}_n) \).

In all but the last two examples, the corresponding modulus of stability is always independent of both the “dimension” and the “length” parameters in the sense that modulus only depends on the third variable. In the setting of normed spaces, such classes have been studied in [22] and are called stable.
Fraïssé classes. Thus, stable Fraïssé classes of finite-dimensional normed spaces give rise to stable Fraïssé classes of finite-dimensional multi-semi-normed spaces. In the last two families of examples, however, it is unknown if the modulus of stability depends on the dimension. This observation inspires the following:

**Question 2.8.** Are there any examples of amalgamation classes of finite-dimensional seminormed spaces such that the corresponding modulus of stability depends on all three parameters?

3. **Fraïssé Fréchet spaces**

For a multi-semi-normed space $X$, let $\text{Iso}(X)$ be the group of multi-isometries $f : X \to X$ equipped with the topology generated by basic open sets of the form

$$\{g \in \text{Iso}(X) : \max_{m \leq n} \| (f - g) \|_m < \varepsilon \}$$

where $f \in \text{Iso}(X)$, $Y$ is a finite-dimensional linear subspace of $X$, $\varepsilon > 0$ and $n < \lambda_X$. This is the analogue of the strong operator topology in the setting of multi-semi-normed spaces; in particular, a sequence $(f_k)$ in $\text{Iso}(X)$ converges to $f$ if and only if $\|f_k(x) - f(x)\|_n$ converges to 0 as $k \to \infty$ for each $x \in X$ and each $n \in \mathbb{N}$. For the sake of brevity, we will simply refer to this topology as the topology of pointwise convergence on $\text{Iso}(X)$. We also equip $\text{Iso}(X)$ with its *left uniform structure*, i.e. the uniformity generated by entourages of the diagonal of the form

$$\{(f, g) \in \text{Iso}(X)^2 : f^{-1} \circ g \in U\}$$

where $U$ is a neighbourhood of the identity.

For a multi-semi-normed space $(E, (\| \cdot \|_{E,n})_{n \leq \omega})$, let $\text{Age}(E)$ denote the *age* of $E$, which is defined as the class of all finite-dimensional multi-semi-normed spaces of the form

$$(X, (\| \cdot \|_{E,n})_{n \leq \lambda_X})$$

where $X$ is a linear subspace of $E$ and $\lambda_X \leq \lambda_E$. (Warning: This is not the standard definition of the age, which is usually defined as the collection of all finitely-generated substructures of a given structure.) We also let $\text{Age}_\alpha(E) := (\text{Age}(E))_{\alpha} = \text{Age}(E) \cap M=\alpha$ for each $\alpha \leq \omega$. The collections $\text{Age}_\alpha(E)$ and $\text{Age}_\alpha(E)$ are defined similarly.

**Definition 3.1.** Let $(E, (\| \cdot \|_n)_{n \leq \omega})$ be a multi-semi-normed space and let $\mathcal{K}$ be a class of finite-dimensional multi-semi-normed spaces.

(a) $E$ is *$\mathcal{K}$-universal* if every $X \in \mathcal{K}$ embeds into $E$ via a multi-isometric embedding.

(b) $E$ is *approximately $\mathcal{K}$-ultrahomogeneous* if for every $\varepsilon > 0$, $X \in \mathcal{K}$ and $\gamma, \eta \in \text{Emb}(X, E)$, there is $g \in \text{Iso}(E)$ such that $\| g \circ \gamma - \eta \|_n \leq \varepsilon$ for each $n < \lambda_X$.

(c) $E$ is *Fraïssé with modulus of stability $\varpi$* if for every $d \in \mathbb{N}, l \in \mathbb{N} \cup \{\omega\}, \varepsilon > 0$ and $\delta \geq 0$, every $X \in \mathcal{K}$ and every $\gamma, \eta \in \text{Emb}_\delta(X, E)$, there is $g \in \text{Iso}(E)$ such that $\| g \circ \gamma - \eta \|_n \leq \varpi(d, l, \delta) + \varepsilon$ for each $n < l$.

When $\mathcal{K} = \text{Age}(E)$, we will occasionally omit the reference to $\mathcal{K}$ in the above definitions. We also omit the reference to the modulus of stability when working with a general Fraïssé Fréchet space.
In order to transfer properties of an amalgamation class to a relevant Fraïssé Fréchet space, we will need a slight modification of the given modulus. To this end, for a modulus
\[ \varpi : \mathbb{N} \times (\mathbb{N} \cup \{ \omega \}) \times [0, \infty] \rightarrow [0, \infty] \]
we define a new modulus
\[ \varpi^* (d, l, \delta) := \inf_{\delta' \geq \delta} \varpi(d, l, \delta'). \]
Note that \( \varpi^* \) is indeed a modulus which furthermore satisfies \( \varpi(d, l, \delta) \leq \varpi^*(d, l, \delta) \) for any \( d \in \mathbb{N} \) and \( l \in \mathbb{N} \cup \{ \omega \} \).

Our main goal in this section is to show that when \( \mathcal{K} \) is a Fraïssé class with modulus \( \varpi \), then there is a unique (up to a multi-isometry) separable, \( \mathcal{K} \)-universal and \( \mathcal{K} \)-Fraïssé Fréchet space with modulus \( \varpi^* \) whose finite-dimensional subspaces are precisely those from \( \mathcal{K} \). Note that any \( \mathcal{K} \)-Fraïssé space is automatically approximately \( \mathcal{K} \)-ultrahomogeneous. Furthermore, if \( E \) is Fraïssé, then \( \text{Age}_{< \omega} (E) \) is a Fraïssé class; the amalgamation property follows from the Fraïssé property together with the fact that any finite-dimensional subspace \( X \in \text{Age}_{< \omega} (E) \) is eventually separated by the sequence of seminorms from \( E \).

Below we will need to make use of an inductive limit construction for spaces with a finite sequence of seminorms. Suppose \( (X_n, I_n)_{n \in \omega} \) is a sequence such that:

1. \( X_n \in \mathcal{M}_{< \omega} \) for each \( n \).
2. \( \lambda_{X_n} \downarrow \omega \) is non-decreasing and converges to a fixed \( \lambda \leq \omega \).
3. \( I_n \in \text{Emb}(X_n, X_{n+1}) \) for each \( n \).

We define the inductive limit \( \lim_n (X_n, I_n) \) as follows: First, for each \( m \leq n \), define \( I_{m,n} \in \text{Emb}(X_m, X_n) \) by setting \( I_{m,m} = \text{id}_{X_m} \) and \( I_{m,n+1} = I_n \circ I_{m,n} \). Then let \( V \) be the linear subspace of the product space \( \prod_n X_n \) defined by declaring \( (x_n)_{n \in \omega} \in V \) if and only if there is some \( m \) such that \( x_n = I_{m,n} (x_m) \) for all \( n \geq m \). For each \( k < \lambda \), let \( N_k \) be the linear subspace of \( V \) consisting of all \( (x_n) \) such that there is \( m \) such that \( k < \lambda_{X_m} \) and \( \|x_n\|_k = 0 \) for all \( n \geq m \). Let \( N = \bigcap_{k < \omega} N_k \) and let \( V_0 = V/N \).

Define a sequence of seminorms \( (\| \cdot \|_k)_{k < \lambda} \) on \( V_0 \) by
\[ \| (x_n)_n + N \|_k = \| x_m \|_{X_m, k} \]
where \( m \) is the least integer such that \( k < \lambda_{X_m} \) and \( x_n = I_{m,n} (x_m) \) for all \( n \geq m \). Note that these seminorms are well-defined and they form an increasing sequence whenever each \( X_n \) is graded since each \( I_n \) is a multi-isometric embedding.

To take a completion, we proceed as in the case of normed spaces. To this end, define an equivalence relation \( \sim \) on the space of all Cauchy sequences in \( V_0 \) by declaring
\[ (\sigma_n) \sim (\tau_n) \iff (\forall k < \lambda) \lim_n \| \sigma_n - \tau_n \|_k = 0, \]
where a sequence \( (\sigma_n) \subseteq V_0 \) is Cauchy if it is Cauchy with respect to each seminorm \( \| \cdot \|_k \) on \( V_0 \). Let \( \lim_n (X_n, I_n) \) denote the resulting quotient space equipped with the sequence of seminorms of length \( \lambda \) given by
\[ \| (\sigma_n) \|_{\lambda, k} = \lim_n \| \sigma_n \|_k \quad \text{for each} \quad k < \lambda. \]
Observe that \( (\lim_n (X_n, I_n), (\| \cdot \|_{\lambda, k})_{k < \lambda}) \) is a complete, separated multi-seminormed space, and so it is a Fréchet space. Given \( x \in V_0 \), let \( \mathcal{C}(x) \) denote the equivalence class (in \( \lim_n (X_n, I_n) \)) of the Cauchy sequence \( (x_n) \).
sequence with constant value $x$. Then for each $m$ the canonical mapping

$$I_m^{(\infty)} : X_m \to \lim_{n \to \infty} (X_n, I_n) : x \mapsto \mathcal{C}(\{(0, \ldots, 0, x, I_{m,m+1}(x), I_{m,m+2}(x), \ldots)\})$$

is a seminorm-preserving linear mapping. When $X_m$ is separated, it is routine to check that $I_m^{(\infty)}$ is injective and hence a multi-isometric embedding. Note that union $\bigcup_n I_n^{(\infty)}(X_n)$ is dense in $\lim_n (X_n, I_n)$, in the sense that for every $[\sigma] \in \lim_n (X_n, I_n)$, there is a sequence $(\sigma_m)$ belonging to the union which converges to $[\sigma]$ with respect to each $\|\cdot\|_k$. In particular, such a sequence converges to $[\sigma]$ with respect to the pseudometric induced by $\max_{1<\kappa} \|\cdot\|_k$ for any given $k < \lambda$. From now on, whenever we are working with an inductive limit of separated spaces $X_n$, we will identify $X_n$ with its image $I_n^{(\infty)}(X_n)$ in $\lim_n (X_n, I_n)$. In this way, each $I_n : X_n \to X_{n+1}$ becomes the corresponding inclusion mapping, so that $(X_n)$ is an increasing sequence of finite-dimensional subspaces of $\lim_n (X_n, I_n)$ and $X_n$ is equipped with the first $\lambda_{X_n}$ seminorms induced by the inductive limit. Furthermore, $\bigcup_n X_n$ is dense in the inductive limit.

We will need the following consequence of the amalgamation property, the proof of which can be found in [10], Lemma 2.32.

**Lemma 3.2.** Suppose $\mathcal{K}$ is an amalgamation class with modulus $\varpi$. Fix $\varepsilon > 0$, $\Delta \subseteq \mathbb{R}^+$ finite and $\mathcal{A} \cup \{Y\} \subseteq \mathcal{K}$ finite. Then there is some $Z \in \mathcal{K}$ and $I \in \text{Emb}(Y, Z)$ such that for every $X \in \mathcal{A}$, every $\delta \in \Delta$ and every $\gamma, \eta \in \text{Emb}_\delta(X, Y)$, there is $J \in \text{Emb}(Y, Z)$ such that

$$\max_{\lambda < \lambda_X} \| I \circ \gamma - J \circ \eta \|_1 \leq \varpi (\dim X, \lambda_X, \delta) + \varepsilon.$$ 

Before proceeding, we require one more piece of notation. Given two classes $\mathcal{K}$ and $\mathcal{K}'$ of finite-dimensional multi-seminormed spaces, write $\mathcal{K} \leq \mathcal{K}'$ when for every $X \in \mathcal{K}$ there are $Y \in \mathcal{K}'$ and a multi-isometry of $X$ onto $Y$, and write $\mathcal{K} \equiv \mathcal{K}'$ when $\mathcal{K} \leq \mathcal{K}' \leq \mathcal{K}$.

**Theorem 3.3.** If $\mathcal{K}$ is an amalgamation class with modulus $\varpi$, then there is a separable $\mathcal{K}$-Fraïssé Fréchet space $E$ with modulus $\varpi^*$ such that $\mathcal{K} \leq \text{Age}_{\leq \omega}(E)$. Furthermore, if $\mathcal{K}$ is a Fraïssé class then $\text{Age}_{\leq \omega}(E) \equiv \mathcal{K}$.

**Proof.** Let $\{Z_n\}_{n<\omega}$ be a countable $d_{BM}$-dense subset of $\mathcal{K}$. Fix an enumeration $(\delta_n)$ of $\mathbb{Q} \cap [0,1]$ such that $\delta_0 = 0$. Using Lemma 3.2, we find a sequence $(X_n, I_n)_{n<\omega}$ of separated spaces $X_n \in \mathcal{K}$ and multi-isometric embeddings $I_n \in \text{Emb}(X_n, X_{n+1})$ with the following properties:

(a) Let $\lambda_\mathcal{K} := \sup_{X \in \mathcal{K}} \lambda_X$.

(i) If $\lambda_\mathcal{K} = \omega$, then $\lambda_{X_n} \geq n$ for all $n$.

(ii) If $\lambda_\mathcal{K} < \omega$, then $\lambda_{X_n} = \lambda_\mathcal{K}$ for all $n$.

(b) For every $k \leq n$, every $X \in \{Z_j\}_{j \leq n} \cup \{X_j\}_{j \leq n}$ and every $\gamma, \eta \in \text{Emb}_\delta(X, X_n)$, there is $J \in \text{Emb}(X_n, X_{n+1})$ such that

$$\max_{\lambda < \lambda_X} \| I_n \circ \gamma - J \circ \eta \|_1 \leq \varpi (\dim X, \lambda_X, 2^{-n}) + 2^{-n}.$$ 

(c) $\text{Emb}(Z_m, X_n) \neq \emptyset$ for each pair $m < n$.

Note that (c) can be arranged by applying the JEP of $\mathcal{K}$. Let $E = \lim_n (X_n, I_n)$. For simplicity, we assume from now on that $(X_n)$ is an increasing sequence of subspaces of $E$, $I_n$ is the inclusion mapping.
Letting and so an application of the triangle inequality then yields

On the other hand, we also have

Then for every $s, t < \omega$ and every we have

\[ \| J_{s+t}(x) - J_s(x) \|_l \leq \sum_{0 \leq i \leq t-1} \| J_{s+i+1}(x) - J_{s+i}(x) \|_l \leq \sum_{0 \leq i < t-1} 3 \cdot 2^{-(n+2s+1+i)} \| x \|_l. \]
In particular, this implies that for each \( x \in \bigcup_{n<\omega} X_n \) the sequence \((J_s(x))_s\) is Cauchy with respect to each seminorm \( \| \cdot \|_{E,l} \) for \( l < \lambda E \); indeed, given \( l < \lambda E \), simply choose a large enough \( N \) such that \( l < \lambda X_n \) and \( x \in X_n \) for all \( n > N \), noting that inequality (1) holds for all such \( n \). Thus \((J_s)_{s<\omega}\) is pointwise Cauchy in \( E \), so by completeness we can define a linear mapping \( J : \bigcup_{n<\omega} X_n \to E \) by setting \( J(x) = \lim_{s \to \infty} J_s(x) \) where \( k \) is least such that \( x \in X_k \); we then extend \( J \) to a mapping \( J : E \to E \). Note that \( J \) is a seminorm-preserving linear mapping. To see that \( J \) is a bijection, we define as before a seminorm-preserving linear mapping \( L : \bigcup_{n<\omega} X_n \to E \) by setting \( L(y) = \lim_{s \to \infty} L_s(y) \) where \( k \) is least such that \( y \in X_k \), and we extend it to a mapping \( L : E \to E \). Then, since \( E \) is separated, (v) and (vi) imply \( L \circ J = J \circ L = \operatorname{id}_E \). Thus \( J \) and \( L \) are both multi-isometries. Finally, note that for each \( l < \lambda X \) we have

\[
\|J \circ \eta - \gamma\|_l \leq \varpi(\dim X, \lambda X, \delta_j) + \sum_{0 \leq i \leq 2s} 2^{-(n+i)} \leq \varpi(\dim X, \lambda X, \delta_j) + 2^{-(n-1)}.
\]

Taking the limit as \( s \to \infty \) we see that \( \|J \circ \eta - \gamma\|_l \leq \varpi(\dim X, \lambda X, \delta_j) + 2^{-(n-1)} \) and so

\[
\|J \circ \eta - \gamma\|_l \leq \|J \circ \eta - J \circ \tilde{\eta} \circ \theta\|_l + \|J \circ \tilde{\eta} \circ \theta - \gamma\|_l + \|\gamma \circ \theta - \gamma\|_l \\
\leq \frac{\varepsilon}{3} + (\varpi(\dim X, \lambda X, \delta_j) + 2^{-(n-1)})\|\theta\|_l + \frac{\varepsilon}{3} \\
\leq \varpi(\dim X, \lambda X, \delta_j) + \varepsilon \leq \varpi(\dim X, \lambda X, \delta') + \varepsilon.
\]

Thus \( E \) is \( K \)-Fraïssé with modulus \( \varpi^* \).

\[\Box\]

**Claim 3.3.2.** \( K \leq \operatorname{Age}_{<\omega}(E) \). Furthermore, \( \operatorname{Age}_{<\omega}(E) \leq K \) whenever \( K \) is Fraïssé.

**Proof of Claim.** Fix \( X \in K \). Since \( K \) is an amalgamation class, \( X \) isometrically embeds into a separated space \( X' \). Thus we can assume without loss of generality that \( X \) itself is separated. Now, find a decreasing positive sequence \((\delta_n)_n\) such that \( \varpi(\dim X, \lambda X, \delta_n) \leq 2^{-n} \) for each \( n \). Using the definition of the sequence \( \{Z_n\} \) together with property (c), for each \( n \) we find some \( \gamma_n \in \operatorname{Emb}_{\delta_n}(X,E) \). Since \( E \) is \( K \)-Fraïssé, for each \( n \) we can choose \( g_n \in \operatorname{Iso}(E) \) such that \( \|g_n \circ \gamma_{n+1} - \gamma_n\|_l \leq 2^{-(n-1)} \) (where we take \( \varepsilon = 2^{-n} \)) for each \( l < \lambda X \). Define a sequence \( (\eta_n)_n \) of multi-\( \delta_n \)-isometric embeddings of \( X \) into \( E \) by setting \( \eta_0 = \gamma_0 \) and

\[
\eta_{n+1} = g_0 \circ \cdots \circ g_n \circ \gamma_{n+1} \text{ for each } n > 0.
\]

Then \( (\eta_n)_n \) is pointwise Cauchy, since

\[
\|\eta_{n+k} - \eta_n\|_l \leq \sum_{j=n}^{n+k-1} 2^{-(j-1)} \leq 2^{-(n-2)}
\]

and so the limit \( \eta : X \to E \) is a multi-isometric embedding; injectivity follows from the assumption that \( X \) is separated.

Finally, note that the construction of \( E \) implies \( \operatorname{Age}_{<\omega}(E) \leq \overline{K}^{BM} \), where the latter collection is the \( d_{BM} \)-closure of \( K \) in \( M_{<\omega} \). Thus, if \( K \) is a Fraïssé class, i.e. a \( d_{BM} \)-closed amalgamation class with the hereditary property, then the latter collection is precisely \( K \). Thus \( \operatorname{Age}_{<\omega}(E) \leq K \) and so \( \operatorname{Age}_{<\omega}(E) \equiv K \) in this case.

\[\Box\]

This completes the proof of the two claims and hence of the existence of a separable, \( K \)-universal, \( K \)-Fraïssé Fréchet space.
We will henceforth refer to the space \( E \) constructed in Theorem \ref{thm:fraissé-limit} as the Fraïssé limit of the class \( K \) and we will denote it by \( \text{Flim}(K) \). Next we show that such a space is unique whenever \( K \) is a Fraïssé class. More generally, we have:

**Proposition 3.4.** Suppose \( E \) and \( F \) are separable approximately ultrahomogeneous Fréchet spaces such that \( \lambda_E = \lambda_F \) and \( \text{Age}_{<\omega}(E) \equiv \text{Age}_{<\omega}(F) \). Then \( E \) and \( F \) are multi-isometric.

**Proof.** Recursively define increasing sequences \((X_n)\) and \((Y_n)\) of elements of \( \text{Age}(E) \) and \( \text{Age}(F) \), respectively, sequences \((k_n)\) and \((l_n)\) of integers, and sequences of multi-isometric embeddings \( (\gamma_n : X_n \to Y_n) \) and \( (\eta_n : Y_n \to X_{n+1}) \) such that the following conditions hold:

(i) \((k_n)\) and \((l_n)\) are non-decreasing and converge to \( \lambda_E = \lambda_F \).

(ii) \( X_n \in \text{Age}_{k_n}(E) \) and \( Y_n \in \text{Age}_{l_n}(F) \).

(iii) \( \|\eta_n \circ \gamma_n - \text{id}_{X_n}\|_{E,m} \leq 2^{-n} \) for all \( m < k_n \).

(iv) \( \|\gamma_{n+1} \circ \eta_n - \text{id}_{Y_n}\|_{F,m} \leq 2^{-n} \) for all \( m < l_n \).

(v) \( \bigcup_{n<\omega} X_n \) and \( \bigcup_{n<\omega} Y_n \) are dense in \( E \) and \( F \), respectively.

To start, let \( X_0 = Y_0 = \{0\}, \gamma_0 = 0 \) and \( k_0 = 1 \). Now assume we have defined \( X_n, Y_n, \gamma_n, \eta_{n-1}, k_n \) and \( l_n \) for \( n \geq 0 \). Using the fact that \( \text{Age}_{<\omega}(E) \equiv \text{Age}_{<\omega}(F) \), fix \( \theta \in \text{Emb}(Y_n, E) \). Since \( E \) is approximately ultrahomogeneous, we can find \( g \in \text{Iso}(E) \) such that

\[
\|g \circ \theta \circ \gamma_n - \text{id}_{X_n}\|_{E,m} \leq 2^{-n} \text{ for every } m < k_n.
\]

Let \( \eta_n := g \circ \theta \in \text{Emb}(Y_n, E) \), let \( k_{n+1} = \min\{l_{n+1}, \lambda_E\} \) and let \( X_{n+1} \) be a finite-dimensional subspace of \( E \) containing \( X_n \cup \eta_n(Y_n) \), appropriately enlarged so that condition (v) will eventually hold, and equipped with the first \( k_{n+1} \) seminorms from \( E \). The construction of \( Y_{n+1}, \gamma_{n+1} : Y_n \to X_{n+1} \) and \( l_{n+1} \) is similar. This completes the inductive construction.

Now, note that for each \( x \in \bigcup_{n<\omega} X_n \) the sequence \((\gamma_n(x))_n\) is Cauchy with respect to each seminorm \( \|\cdot\|_{F,m} \). Given \( m < \omega \), we can choose a large enough \( N \) such that \( m < \lambda_{X_n} \) and \( x \in X_n \) for all \( n > N \).

Then \( \|\gamma_{n+1}(x) - \gamma_n(x)\|_{F,m} \leq 2^{-(n-1)} \), which implies that the sequence \((\gamma_n)_n\) is pointwise Cauchy with respect to \( \|\cdot\|_{F,m} \). Thus \((\gamma_n)_n\) is pointwise Cauchy in \( F \), so by completeness we can define a linear mapping \( \gamma : \bigcup_{n<\omega} X_n \to F \) by setting \( \gamma(x) = \lim_{n \to k} \gamma_n(x) \) where \( k \) is least such that \( x \in X_k \).

Similarly, we can define a multi-isometric embedding \( \eta : \bigcup_{n<\omega} Y_n \to E \) by \( \eta(y) = \lim_{n \to k} \eta_n(y) \) where \( k \) is least such that \( y \in Y_k \), and we extend it to a mapping \( \eta : F \to E \). Then, since \( E \) and \( F \) are both separated, (iii) and (iv) imply \( \gamma \circ \eta = \text{id}_F \) and \( \eta \circ \gamma = \text{id}_E \), and so \( \gamma \) is a multi-isometry. \( \square \)

**Corollary 3.5** (Fraïssé correspondence). Let \( K \) be a class of finite-dimensional multi-seminormed spaces such that \( K \subseteq \mathcal{M}_{<\omega} \). The following are equivalent:

1. \( K \) is a Fraïssé class.

2. \( K \equiv \text{Age}_{<\omega}(E) \) for a unique separable Fraïssé Fréchet space \( E = \text{Flim}(K) \).

**Proof.** Suppose \( K \) is a Fraïssé class. By the previous two results, \( \text{Flim}(K) \) exists and is unique. By construction, we have \( K \equiv \text{Age}_{<\omega}(\text{Flim}(K)) \). To prove the other direction of the corollary, assume \( E \) is a separable Fraïssé Fréchet space such that \( K \equiv \text{Age}_{<\omega}(E) \). Then \( K \) is a hereditary, \( d_{BM} \)-closed class. Furthermore, the fact that \( E \) is separated implies that the class of separated elements of \( \text{Age}_{<\omega}(E) \) is cofinal in \( \text{Age}_{<\omega} \). Indeed, given \( (X, (\|\cdot\|_n)_{n<m}) \in \text{Age}_{<\omega}(E) \), we can use the fact that \( X \) is finite-dimensional to find a sufficiently large \( N \) such that \( (X, (\|\cdot\|_n)_{n<N}) \) becomes a separated subspace of
Example 3.6. (1) Let $G^\omega$ be the product of countably many copies of the Gurarij space $G$. In [2] it is shown that there is a sequence $(\| \cdot \|_n)_{n<\omega}$ of seminorms on $G^\omega$ such that $(G^\omega, (\| \cdot \|_n)_{n<\omega})$ is a separable graded Fréchet space which is Fraïssé and universal for the class of all finite-dimensional graded multi-seminormed spaces with an infinite sequence of seminorms. It is also shown that there is a sequence $(\| \cdot \|'_n)_{n<\omega}$ of seminorms such that $(G^\omega, (\| \cdot \|'_n)_{n<\omega})$ is a separable Fréchet space which is Fraïssé and universal for the class of all finite-dimensional multi-seminormed spaces with an infinite sequence of seminorms. (The authors of [2] do not use Fraïssé-theoretic terminology; however, this follows from [2, Proposition 4.1] and [2, Proposition 5.5], respectively.) Below we will show that these two spaces can be obtained as Fraïssé limits of the classes $G_{<\omega}$ and $M_{<\omega}$, respectively.

(2) For each $n \geq 1$, the spaces $\text{Flim}(M_{<\omega}^{\text{sep}})$ and $\text{Flim}(G_{<\omega}^{\text{sep}})$ can be seen as separated $n$-seminormed versions of the spaces considered in the previous example. (See Example 2.7 for the relevant notation.) Note that $\text{Age}_{<\omega}(\text{Flim}(M_{<\omega}^{\text{sep}}))$ is strictly larger than $M_{<\omega}^{\text{sep}}$, since the former collection contains non-separated multi-seminormed spaces. An analogous fact holds for $\text{Flim}(G_{<\omega}^{\text{sep}})$.

(3) Let $E$ be the space $(G^\omega, (\| \cdot \|_n)_{n<\omega})$ considered in Example 3.6(1). For each $k \in \mathbb{N}$, let $E_k$ be the multi-seminormed space $(G^\omega, (\| \cdot \|_n)_{n<k})$ obtained by truncating the associated sequence of seminorms. Then using the properties of $(G^\omega, (\| \cdot \|_n)_{n<\omega})$ it follows that $E_k$ is universal and Fraïssé for $M_{<k}$. An interesting special case occurs when $n = 1$, since $E_1$ can be seen as a seminormed version of the Gurarij space. In fact, if we let $\| \cdot \|$ be the seminorm on $E_1$, then the quotient $E_1/\ker \| \cdot \|$ equipped with the corresponding quotient norm is separable, universal and approximately ultrahomogeneous for the class of all finite-dimensional Banach spaces, and so it is isometric to the Gurarij space. Note that the spaces $E_k$ are not separated, and so it is unclear if they are unique up to isometry.

(4) $E = \text{Flim}(M_{<\omega}^{\text{H}})$ is the unique separable Fréchet-Hilbert space which is $M_{<\omega}^{\text{H}}$-Fraïssé. To see that $E$ is indeed Fréchet-Hilbert, observe that each quotient space $E_{1\| \cdot \|}$ (where $\| \cdot \|$ is a seminorm belonging to the sequence of seminorms associated to $E$) is approximately ultrahomogeneous for the class of all finite-dimensional Hilbert spaces, and hence is isometric to a Hilbert space.

(5) For each sequence $(p_n) \subseteq [1, \infty[$ with $p_n \notin \{4, 6, 8, \ldots \}$, $\text{Flim}(M_{<\omega}^{(p_n)})$ is a Fraïssé Fréchet space. In particular, if $p_n = p$ for each $n$, then $\text{Flim}(M_{<\omega}^{p})$ can be seen as a Fréchet-$L_p$-space, which is the $L_p$ analogue of the space considered in the previous example.

To conclude this section, we will use Proposition 3.4 to show that the spaces considered in Example 3.6(1) can be obtained as Fraïssé limits; we will make use of the fact that $E$ is universal and approximately ultrahomogeneous for the class of all finite-dimensional (graded) multi-seminormed spaces with an infinite sequence of seminorms. From now until the end of the section, $E$ will denote one of these two spaces. Before proceeding, we need some new terminology: Given $n < \omega$ and two multi-seminormed spaces $X$ and $Y$ such that $\lambda_X = \lambda_Y = \omega$, an \textit{multi-isometric $n$-embedding} from $X$ to $Y$ is an injective linear mapping $f : X \to Y$ such that $\|f(x)\|_{Y,m} = \|x\|_{X,m}$ for each $x \in X$ and each $m < n$.

The following is a version of the near amalgamation property in the context of multi-isometric $n$-embeddings for a fixed $n$. 

Lemma 3.7. Suppose $X, Y$ and $Z$ are finite-dimensional separated multi-seminormed spaces with $\lambda_X = \lambda_Y = \lambda_Z = \omega$ and $f : X \to Y$, $g : X \to Z$ are multi-isometric $n$-embeddings for a fixed $n < \omega$. Then for every $\varepsilon > 0$ there is a finite-dimensional multi-seminormed space $W$ with $\lambda_W = \omega$ and multi-isometric embeddings $i_Y : Y \to W$ and $i_Z : Z \to W$ such that $\|i_Y \circ f - i_Z \circ g\|_m \leq \varepsilon$ for all $m < n$. Furthermore, $W \in G$ whenever $X, Y, Z \in G$.

Proof. Fix all parameters and consider the sum $Y \oplus Z$ together with the canonical inclusion mappings $i_Y : Y \to Y \oplus Z$ and $i_Z : Z \to Y \oplus Z$. Equip $Y \oplus Z$ with a sequence $(\| \cdot \|_m)_{m < \omega}$ of seminorms defined by declaring

$$
\|(y, z)\|_m := \inf \{ \|u\|_{Y, m} + \|v\|_{Z, m} + \varepsilon \|x\|_{X, m} : x \in X, u \in Y, v \in Z, y = u + f(x), z = v - g(x) \}
$$

for each $m < n$, and $\|(y, z)\|_m := \|y\|_{Y, m} + \|z\|_{Z, m}$ for each $m \geq n$. Let $W$ be the sum $Y \oplus Z$ equipped with this sequence of seminorms. Note that $W$ is a graded multi-seminormed space whenever $X, Y$ and $Z$ are graded. As in the proof of Lemma 2.3, it is straightforward to check that the inclusion mappings $i_Y : Y \to Y \oplus Z$ and $i_Z : Z \to Y \oplus Z$ are multi-isometric embeddings which satisfy $\|i_Y \circ f - i_Z \circ g\|_m \leq \varepsilon$ for all $m < n$. \hfill \Box

Lemma 3.8. Let $X$ be a finite-dimensional subspace of $E$ such that $\lambda_X = \omega$. For every $\varepsilon > 0$, $n \in \mathbb{N}$, finite-dimensional multi-seminormed space $Y$ and multi-isometric $n$-embeddings $\eta : X \to E$ and $\gamma : X \to Y$, there is a multi-isometric embedding $f : Y \to E$ such that $\|f \circ \gamma - \eta\|_m \leq \varepsilon$ for each $m < n$.

Proof. Apply the previous lemma to $\varepsilon/2$, $\gamma : X \to Y$ and $\eta : X \to Z := \eta(X)$ to find the corresponding $W, i_Y$ and $i_Z$. Since $E$ is approximately ultrahomogeneous, there is $g \in \text{Iso}(E)$ such that $\|g(i_Y(z)) - z\|_m \leq \frac{\varepsilon}{2} \|z\|_m$ for all $z \in Z$ and $m \in \mathbb{N}$. Let $f = g \upharpoonright i_Y$. Then for each $x \in X$ and $m < n$ we have

$$
\|f(\gamma(x)) - \eta(x)\|_m \leq \|g(i_Y(\gamma(x))) - g(i_Z(\eta(x)))\|_m + \|g(i_Z(\eta(x))) - \eta(x)\|_m \\
\leq \|g(i_Y(\gamma(x)) - i_Z(\eta(x)))\|_m + \frac{\varepsilon}{2} \|\eta(x)\|_m \\
\leq \frac{\varepsilon}{2} \|x\|_m + \frac{\varepsilon}{2} \|x\|_m \leq \varepsilon \|x\|_m.
$$

Thus $f$ is the desired multi-isometric embedding. \hfill \Box

Corollary 3.9. $(G^\omega, (\| \cdot \|_n)_{n < \omega})$ and $(G^\omega, (\| \cdot \|_n)_{n < \omega})$ are approximately ultrahomogeneous for $G_{< \omega}$ and $\mathcal{M}_{< \omega}$, respectively. In particular, $(G^\omega, (\| \cdot \|_n)_{n < \omega}) = \text{Flim}(G_{< \omega})$ and $(G^\omega, (\| \cdot \|'_n)_{n < \omega}) = \text{Flim}(\mathcal{M}_{< \omega})$.

Proof. We only prove the result for $G_{< \omega}$; the proof for $\mathcal{M}_{< \omega}$ is identical. Fix $X \in G_{< \omega}$ together with multi-isometric embeddings $\gamma, \eta : X \to E$. Extend the sequence of seminorms $(\| \cdot \|_{X, n})_{n < \lambda_X}$ to an infinite sequence in the natural way by declaring $\| \cdot \|_{X, m} = \| \cdot \|_{X, \lambda_X - 1}$ for all $m \geq \lambda_X$. Then $\gamma$ and $\eta$ become multi-isometric $\lambda_X$-embeddings. Let $Y = \gamma(X) \subseteq E$ be equipped with the sequence of seminorms from $E$ and apply the previous lemma to find a multi-isometric embedding $f : Y \to E$ such that $\|f(\gamma(x)) - \eta(x)\|_m \leq \frac{\varepsilon}{2} \|x\|_m$ for each $x \in X$ and $m < \lambda_X$. By the approximate ultrahomogeneity of $E$, there is $g \in \text{Iso}(E)$ such that $\|g(y) - f(y)\|_m \leq \frac{\varepsilon}{2} \|y\|_m$ for each $y \in Y$ and $m \in \mathbb{N}$. Then

$$
\|g(\gamma(x)) - \eta(x)\|_m \leq \|g(\gamma(x)) - f(\gamma(x))\|_m + \|f(\gamma(x)) - \eta(x)\|_m \leq \varepsilon \|x\|_m
$$

for each $x \in X$ and $m < \lambda_X$, as required. \hfill \Box
4. THE APPROXIMATE RAMSEY PROPERTY

As in [4] or [10], we can characterize the extreme amenability of the group of multi-isometries of certain Fréchet spaces in terms of an approximate Ramsey property. Before stating the relevant Ramsey properties, we need some terminology. Given $r \in \mathbb{N}$, an $r$-colouring of a set $X$ is simply a mapping $X \to r$. If $X$ is equipped with a finite sequence of seminorms $(\| \cdot \|_m)_{m \leq \lambda X}$ and $n \leq \lambda X$, an $n$-continuous colouring of $X$ is a mapping $c : X \to [0,1]$ such that

$$|c(x) - c(y)| \leq \max_{m < n} \|x - y\|_m \text{ for all } x, y \in X.$$ 

Thus, an $n$-continuous colouring is simply a mapping of $X$ into $[0,1]$ which is 1-Lipschitz with respect to the pseudometric induced by $\max_{m < n} \cdot \| \cdot \|_m$.

**Definition 4.1.** Let $\mathcal{K}$ be a collection of finite-dimensional multi-seminormed spaces.

(a) $\mathcal{K}$ has the discrete approximate Ramsey property (discrete ARP) if for every $X, Y \in \mathcal{K}$, $r \in \mathbb{N}$ and $\varepsilon > 0$ there is $Z \in \mathcal{K}$ such that every $r$-colouring of $\text{Emb}(X, Z)$ $\varepsilon$-stabilizes on a set of the form $\gamma \circ \text{Emb}(X, Y)$ for some $\gamma \in \text{Emb}(Y, Z)$, i.e. there is $i < r$ such that $\gamma \circ \text{Emb}(X, Y)$ is contained in the set

$$(c^{-1}\{i\})_\varepsilon := \{\xi \in \text{Emb}(X, Z) : \exists \eta \ c(\eta) = i \text{ and } \|\xi - \eta\|_m \leq \varepsilon \text{ for all } m < \lambda X\},$$

where the $m$th seminorm $\cdot \| \cdot \|_m$ on the space of embeddings $\gamma : X \to Z$ is defined by setting $\|\gamma\|_m := \sup\{\|\gamma(x)\|_m : x \in X, \|x\|_m = 1\}$. In this case we will also say that $\gamma \circ \text{Emb}(X, Y)$ is $\varepsilon$-monochromatic.

(b) $\mathcal{K}$ has the continuous approximate Ramsey property (continuous ARP) if for every $X \in \mathcal{K}_{<\omega}, Y \in \mathcal{K}$ and $\varepsilon > 0$ there is $Z \in \mathcal{K}$ such that for every $\lambda X$-continuous colouring $c$ of $\text{Emb}(X, Z)$ there is $\gamma \in \text{Emb}(Y, Z)$ such that the oscillation of $c$ on $\gamma \circ \text{Emb}(X, Y)$, defined as

$$\text{osc}(c \upharpoonright \gamma \circ \text{Emb}(X, Y)) := \sup\{|c(\gamma \circ \eta) - c(\gamma \circ \eta')| : \eta, \eta' \in \text{Emb}(X, Y)\},$$

is at most $\varepsilon$. In this case we will say that $c$ $\varepsilon$-stabilizes on $\gamma \circ \text{Emb}(X, Y)$.

We will abbreviate the conclusions of the discrete and continuous ARP by writing $Z \to^\varepsilon (Y)_X$ and $Z \to^\varepsilon (Y)^X$, respectively. Exactly as in [4], it turns out that these two notions are equivalent.

**Lemma 4.2.** Let $\mathcal{K} \subseteq \mathcal{M}_{<\omega}$. Then $\mathcal{K}$ satisfies the discrete ARP if and only if it satisfies the continuous ARP.

**Proof.** Suppose first that $\mathcal{K}$ has the discrete ARP. Fix $X, Y \in \mathcal{K}$ and $\varepsilon > 0$. Let $D \subseteq [0,1]$ be a finite $\varepsilon$-dense set. Apply the discrete ARP with $|D|$-many colours to find $Z \in \mathcal{K}$ such that $Z \to^\varepsilon (Y)^X_{|D}$. We claim that $Z$ witnesses the continuous ARP for the above parameters. Indeed, given a $\lambda X$-continuous colouring $c : \text{Emb}(X, Z) \to [0,1]$, define a $|D|$-colouring $\tilde{c} : \text{Emb}(X, Z) \to D$ by the condition $|c(\varphi) - c(\varphi')| \leq \varepsilon$ for every $\varphi \in \text{Emb}(X, Z)$. Then there is $\gamma \in \text{Emb}(Y, Z)$ such that $\tilde{c} \varepsilon$-stabilizes on $\gamma \circ \text{Emb}(X, Y)$. It follows that $c$ $4\varepsilon$-stabilizes on $\gamma \circ \text{Emb}(X, Y)$.

Now suppose $\mathcal{K}$ has the continuous ARP. We prove that $\mathcal{K}$ has the discrete ARP by induction on $r$, the number of colours. When $r = 1$ this is trivial, so suppose inductively that $\mathcal{K}$ satisfies the discrete ARP for $r$-colourings. Fix $X, Y \in \mathcal{K}$ and $\varepsilon > 0$. By the inductive hypothesis, there is $Z_0 \in \mathcal{K}$ such that $Z_0 \to^\varepsilon (Y)^X_0$. Since $\mathcal{K}$ has the continuous ARP, there is $Z \in \mathcal{K}$ such that $Z \to^\varepsilon (Z_0)^X$. We
claim that \( Z \) witnesses the discrete ARP for the parameters \( X, Y, \varepsilon \) and \( r + 1 \). Indeed, fix a colouring \( c : \Emb(X, Z) \to r + 1 \) and define \( \tilde{c} : \Emb(X, Z) \to [0, 1] \) by setting

\[
\tilde{c}(\varphi) = \min \left\{ 1, \inf_{\psi \in c^{-1}(r)} \max_{m<\lambda_X} \| \varphi - \psi \|_m \right\}.
\]

It is routine to check that \( \tilde{c} \) is an \( \lambda_X \)-continuous colouring, so there is \( \gamma \in \Emb(Z_0, Z) \) such that \( \tilde{c} \varepsilon \)-stabilizes on \( \gamma \circ \Emb(X, Z_0) \). If there is some \( \varphi \in \Emb(X, Z_0) \) such that \( c(\gamma \circ \varphi) = r \), then \( \gamma \circ \Emb(X, Z_0) \subseteq (c^{-1}(r))_\varepsilon \) and so we are done since then \( c \varepsilon \)-stabilizes on \( \gamma_0 \circ \Emb(X, Y) \) for any choice of \( \gamma_0 \in \Emb(Y, Z_0) \). If no such \( \varphi \) exists, we can define an \( r \)-colouring \( d \) of \( \Emb(X, Z_0) \) by setting \( d(\varphi) = c(\gamma \circ \varphi) \). By definition of \( Z_0 \), there is \( \gamma_0 \in \Emb(Y, Z_0) \) such that \( d \varepsilon \)-stabilizes on \( \gamma_0 \circ \Emb(X, Y) \). It then follows that \( c \varepsilon \)-stabilizes on \( \gamma_0 \circ \Emb(X, Y) \).

The next lemma is a particular instance of a more general phenomenon which rephrases the Ramsey property of an age in terms of its limit. The proof is similar to that of \([25, \text{Proposition 3.4}]\). Recall that \( \Age_n(E) \) denotes the collection of all finite-dimensional linear subspaces \( X \subseteq E \) equipped with the first \( n \) seminorms from \( E \).

**Lemma 4.3.** Suppose \( E \) is an approximately ultrahomogeneous multi-seminormed space. Then the collection \( \Age_n(E) \) has the continuous ARP if and only if for every \( X, Y \in \Age_n(E), \varepsilon > 0 \) and \( \lambda_X \)-continuous colouring \( c \) of \( \Emb(X, E) \), there is \( \gamma \in \Emb(Y, E) \) such that \( \text{osc}(c \upharpoonright \gamma \circ \Emb(X, Y)) \leq \varepsilon \).

**Proof.** The left-to-right direction is straightforward and will not be used in what follows, so we will only show the right-to-left direction. We will prove the contrapositive. First, let \( H = \{\eta_1, \ldots, \eta_k\} \) be a finite \( \varepsilon/3 \)-dense subset of \( \Emb(X, Y) \), where the latter set is equipped with the pseudometric induced by \( \max_{m<\lambda_X} \| . \|_m \). We will need the following claim, the proof of which is routine.

**Claim 4.3.1.** Suppose there is \( Z \in \Age_n(E) \) such that for every \( \lambda_X \)-continuous colouring of \( \Emb(X, Z) \), there is \( \gamma \in \Emb(Y, Z) \) such that \( \text{osc}(c \upharpoonright \gamma \circ \Emb(X, Z)) \leq \varepsilon/3 \). Then \( Z \to \varepsilon (Y)^X \).

Now, if \( \Age_n(E) \) does not have the continuous ARP, then there are \( X, Y \in \Age_n(E) \) and \( \varepsilon > 0 \) such that no \( Z \in \Age_n(E) \) witnesses \( Z \to \varepsilon (Y)^X \). Thus, by the Claim, for each such \( Z \) we can fix a bad \( \lambda_X \)-continuous colouring \( c_Z \) such that \( \text{osc}(c_Z \upharpoonright \gamma \circ H) \geq \varepsilon \) for any choice of \( \gamma \in \Emb(Y, Z) \). Fix an ultrafilter \( \mathcal{U} \) on \( \Age_n(E) \) such that

\[
\{ W \in \Age_n(E) : V \subseteq W \} \in \mathcal{U} \text{ for each } V \in \Age_n(E).
\]

Define a mapping \( c : \Emb(X, E) \to [0, 1] \) by setting \( c(\gamma) = \lim_{\mathcal{U}} c_Z(\gamma) \). Note that the ultralimit exists (by boundedness) and is well-defined since \( \{ W \in \Age_n(E) : \gamma(X) \subseteq W \} \in \mathcal{U} \). Furthermore, \( c \) is an \( \lambda_X \)-continuous colouring. We claim that \( c \) is a bad colouring of \( \Emb(X, E) \). To this end, take any \( \rho \in \Emb(Y, E) \) and note that \( \{ W \in \Age_n(E) : \rho(Y) \subseteq W \} \in \mathcal{U} \). Furthermore, for any such \( \rho \) we have \( \rho \in \Emb(Y, W) \) and so by our initial hypothesis we know

\[
| c_W(\rho \circ \eta_i) - c_W(\rho \circ \eta_j) | > \varepsilon \text{ for some } i, j \in \{1, \ldots, k\}.
\]

Since \( \mathcal{U} \) is an ultrafilter, it follows that there are \( i, j \in \{1, \ldots, k\} \) such that

\[
\{ W \in \Age_n(E) : | c_W(\rho \circ \eta_i) - c_W(\rho \circ \eta_j) | > \varepsilon \} \in \mathcal{U}.
\]
It then follows that \(|c(\rho \circ \eta_i) - c(\rho \circ \eta_j)| > \varepsilon\). Since \(\rho\) was arbitrary, we see that \(c\) is a bad colouring of \(\text{Emb}(X, E)\).

The following is the KPT correspondence for multi-seminormed spaces (cf. [18, 25]).

**Theorem 4.4** (Kechrion-Pestov-Todorčević correspondence). Suppose \(E\) is an infinite-dimensional multi-seminormed space which is approximately ultrahomogeneous. The following are equivalent:

- (i) \(\text{Age}_n(E)\) has the ARP for each \(n \in \mathbb{N}\) such that \(1 \leq n \leq \lambda_E\).
- (ii) \(\text{Iso}(E)\) is extremely amenable when endowed with the topology of pointwise convergence.

**Proof.** (i) \(\rightarrow\) (ii): Fix an \(\text{Iso}(E)\)-flow \(\text{Iso}(E) \curvearrowright K\) for \(K\) compact, \(\varepsilon > 0, p \in K\), an entourage \(U\) and a finite \(F \subseteq \text{Iso}(E)\). By one of the well-known characterizations of extreme amenability (see [26] or [10, Claim 5.11.2]) it is enough to find \(g \in \text{Iso}(E)\) such that \(F \cdot (g \cdot p)\) is \(U\)-small, i.e. such that

\[
(f_0 \cdot (g \cdot p), f_1 \cdot (g \cdot p)) \in U \text{ for each } f_0, f_1 \in F.
\]

Before proceeding, we will define a directed family of pseudometrics which generate the left uniformity of \(\text{Iso}(E)\): For each \(n \in \mathbb{N}\) such that \(n \leq \lambda_E\) and each finite-dimensional \(X \subseteq E\), define a pseudometric \(d^\gamma_X\) on \(\text{Iso}(E)\) by

\[
d^\gamma_X(g, h) = \max_{m < n} \| g \mid X - h \mid X \|_m.
\]

We can assume without loss of generality that all entourages are symmetric. Fix an entourage \(V\) such that \(V \circ V \circ V \circ V \subseteq U\). Since the mapping \(\text{Iso}(E) \rightarrow K : g \mapsto g^{-1} \cdot p\) is left uniformly continuous (see, e.g., [26, Lemma 2.1.5]), there are \(n, X \subseteq E\) and \(\delta > 0\) such that

\[
d^\gamma_X(g, h) \leq \delta \implies (g^{-1} \cdot p, h^{-1} \cdot p) \in V.
\]

Equip \(X\) with the first \(n\) seminorms induced from \(E\), so that \(X \in \text{Age}_n(E)\). Let \(Y\) be any member of \(\text{Age}_n(E)\) containing \(\bigcup \{g(X) : g^{-1} \in F \cup \{\text{id}\}\}\). Fix a finite set \(\{x_i\}_{i < r} \subseteq K\) such that \(K \subseteq \bigcup_{i < r} V[x_i]\), where \(V[x] = \{y \in K : (x, y) \in V\}\). Apply the approximate Ramsey property of \(\text{Age}(E)\) to the parameters \(n, X, Y, \delta/3\) and \(r\) to obtain \(Z \in \text{Age}_n(E)\) such that

\[
Z \rightarrow^{\delta/3,n} (Y)^r_X.
\]

Define a colouring \(c : \text{Emb}(X, Z) \rightarrow r\) by first choosing, for each \(\gamma \in \text{Emb}(X, Z)\), some \(g_{\gamma} \in \text{Iso}(E)\) such that \(\max_{m < n} \| g_{\gamma} \mid X - \gamma \|_m \leq \delta/3\); such a choice is possible by the approximate ultrahomogeneity of \(E\). Then define \(c(\gamma) = i\) if \(i < r\) is the least index such that \(g_{\gamma}^{-1} \cdot p \in V[x_i]\). By definition of \(Z\) there are \(\rho \in \text{Emb}(Y, Z)\) and \(i < r\) such that

\[
\rho \circ \text{Emb}(X, Y) \subseteq \langle c^{-1}\{i\}\rangle^{\delta/3,n}.
\]

In particular, this implies that for each \(\eta \in \text{Emb}(X, Y)\) there is \(h_{\eta} \in \text{Iso}(E)\) such that \(\max_{m < n} \| \rho \circ \eta - h_{\eta} \mid X \|_m \leq 2\delta/3\) and \(h_{\eta}^{-1} \cdot p \in V[x_i]\). Choose \(g \in \text{Iso}(E)\) such that \(\max_{m < n} \| g \mid Y - \rho \|_m \leq \delta/3\). Now, given \(f_0, f_1 \in F\), let \(\eta_j := f_j^{-1} \mid X\) for \(j = 0, 1\) and note \(\eta_j \in \text{Emb}(X, Y)\). Then for each \(j = 0, 1\),

\[
d^\gamma_X(g \circ f_j^{-1}, h_{\eta_j}) = \max_{m < n} \| g \circ \eta_j - h_{\eta_j} \mid X \|_m
\]

\[
\leq \max_{m < n} \| g \circ \eta_j - \rho \circ \eta_j \|_m + \max_{m < n} \| \rho \circ \eta_j - h_{\eta_j} \mid X \|_m
\]

\[
\leq \delta/3 + 2\delta/3 = \delta.
\]
By choice of δ, this implies \((f_j \circ g^{-1} \cdot p, h^{-1}_j \cdot p) \in V\) for each \(j\). Since \((x_i, h^{-1}_j \cdot p) \in V\) for each \(j\), our choice of \(V\) implies \((f_0 \circ g^{-1} \cdot p, f_1 \circ g^{-1} \cdot p) \in U\). Thus \(F \cdot (g^{-1} \cdot p)\) is \(U\)-small.

\((ii) \rightarrow (i)\): To prove the ARP, we use the previous two lemmata together with the following characterization of extreme amenability in terms of the family of pseudometrics defined above. (See \([25]\) or \([23\) Proposition 3.9].)

\((\ast)\) \(\text{Iso}(E)\) is extremely amenable if, and only if, for every finite \(F \subseteq \text{Iso}(E), \varepsilon > 0, X \subseteq E, n \in [1, \lambda_E] \cap \mathbb{N}\) and 1-Lipschitz map \(f : (\text{Iso}(E), d^n_X) \rightarrow [0, 1]\), there is \(g \in \text{Iso}(E)\) such that \(\text{osc}(f \mid g \circ F) \leq \varepsilon\).

Fix \(X, Y \in \text{Age}_n(E)\) together with \(\varepsilon > 0\) and an \(n\)-continuous colouring \(c\) of \(\text{Emb}(X, E)\). Let \(H = \{\eta_1, \ldots, \eta_k\}\) be a finite \(\varepsilon\)-dense subset of \(\text{Emb}(X, Y)\) where the latter set is equipped with the pseudometric induced by \(\max_{m < n} \| \cdot \|_m\). Apply the approximate ultrahomogeneity of \(\text{Emb}(X, Y)\) to find \(F = \{g_1, \ldots, g_k\} \subseteq \text{Iso}(E)\) such that \(\|g_i \mid X - \eta_i\|_m \leq \varepsilon\) for all \(m < n\) and all \(i \leq k\). Let \(\tilde{c} : (\text{Iso}(E), d^n_X) \rightarrow [0, 1]\) be the 1-Lipschitz mapping defined by \(\tilde{c}(g) = c(g \mid X)\) and use \((\ast)\) to find \(g \in \text{Iso}(E)\) such that \(\text{osc}(\tilde{c} \mid g \cdot F) \leq \varepsilon\). Then for any \(i, j \leq k\), by the triangle inequality the term \(|c(g \circ \eta_i) - c(g \circ \eta_j)|\) is bounded above by

\[|c(g \circ \eta_i) - c(g \circ \eta_j)| \leq |c(g \circ \eta_i \mid X) + |c(g \circ g_i \mid X) - c(g \circ g_j \mid X)| + |c(g \circ g_j \mid X) - c(g \circ \eta_j)|.\]

Since \(c\) is an \(n\)-continuous colouring, the first term is bounded above by

\[\max_{m < n} \| g \circ \eta_i - g \circ \eta_j \mid X\|_m = \max_{m < n} \| \eta_i - \eta_j \mid X\|_m \leq \varepsilon\]

by our choice of \(g_i\). Similarly, the third term is bounded above by \(\varepsilon\). To bound the second term, note that

\[|c(g \circ g_i \mid X) - c(g \circ g_j \mid X)| = |\tilde{c}(g \circ g_i) - \tilde{c}(g \circ g_j)| \leq \varepsilon\]

by our choice of \(g\). Thus, if we let \(\gamma = g \mid Y\), we see that the oscillation of \(c\) on \(\gamma \circ H\) is bounded by \(3\varepsilon\). Then, using the definition of the \(\eta_i\), it follows that \(\text{osc}(\gamma \circ \text{Emb}(X, Y)) \leq 5\varepsilon\). \(\square\)

Our next goal is to show that various classes of finite-dimensional multi-seminormed spaces have the ARP, which will allow us to apply the KPT correspondence in certain instances to obtain examples of extremely amenable multi-isometry groups. As in the case of the amalgamation property, our main source of examples of such classes will come from classes of finite-dimensional normed spaces which are known to have the ARP in the context of normed spaces. The key lemma is the following:

**Lemma 4.5.** Suppose \(K_1, \ldots, K_n\) are classes of finite-dimensional normed spaces with the ARP. Let \(X_1, \ldots, X_n, Y_1, \ldots, Y_n\) be finite-dimensional seminormed spaces such that \((X_i)_{\| \cdot \|}\) and \((Y_i)_{\| \cdot \|}\) belong to \(K_i\) for each \(i \leq n\). Let \(\varepsilon > 0\) and \(r < \omega\). Then there are finite-dimensional seminormed spaces \(Z_1, \ldots, Z_n\) such \((Z_i)_{\| \cdot \|} \in K_i\) for each \(i\) and, for every colouring \(e : \prod_{j=1}^n \text{Emb}(X_j, Z_j) \rightarrow r\), there are \(\rho_j \in \text{Emb}(Y_j, Z_j), j = 1, \ldots, n\), such that

\[\prod_{j=1}^n \rho_j \circ \text{Emb}(X_j, Y_j) \text{ is } \varepsilon\text{-monochromatic}.\]

The proof will involve a standard strategy for obtaining product Ramsey properties. First we will need:
Lemma 4.6. Suppose \( \mathcal{K} \) is a class of finite-dimensional normed spaces with the ARP. For any finite-dimensional seminormed spaces \( X \) and \( Y \) such that \( X \| \cdot \| \) and \( Y \| \cdot \| \) belong to \( \mathcal{K} \), any \( \varepsilon > 0 \) and \( r \in \mathbb{N} \), there is a finite-dimensional seminormed space \( Z \) such that \( Z \| \cdot \| \in \mathcal{K} \) and every colouring \( c : \text{Emb}(X, Z) \to r \) \( \varepsilon \)-stabilizes on a set of the form \( \rho \circ \text{Emb}(X, Y) \) for \( \rho \in \text{Emb}(Y, Z) \).

Proof. Let \( X, Y \) be finite-dimensional seminormed spaces, \( \varepsilon > 0 \) and \( r \in \mathbb{N} \). Let \( \tilde{X} = X/\ker \| \cdot \|_X \) and \( \tilde{Y} = Y/\ker \| \cdot \|_Y \) be equipped with the norms \( \| [x] \| = \| x \|_X \) and \( \| [y] \| = \| y \|_Y \) respectively. By the ARP of \( \mathcal{K} \), there is a finite-dimensional normed space \( Z \in \mathcal{K} \) such that

\[
Z \to \tilde{Y}^X.
\]

Consider the product \( Z \times Z \) equipped with the seminorm \( \| (z_1, z_2) \| := \| z_1 \|_Z \). We claim that this space witnesses the ARP for the parameters \( X, Y, \varepsilon, r \). Note that \( Z \times Z \in \mathcal{K}_\| \cdot \| \) since the associated quotient is isomorphic to \( Z \). Now, fix a colouring \( c : \text{Emb}(X, Z) \to r \) and define \( \tilde{c} : \text{Emb}(\tilde{X}, Z) \to r \) by \( \tilde{c}(\gamma) = c(\gamma \circ \pi_X) \) where \( \pi_X : X \to \tilde{X} \) is the canonical surjection. By definition of \( Z \), there is \( \rho \in \text{Emb}(\tilde{Y}, Z) \) such that

\[
\rho \circ \text{Emb}(\tilde{X}, \tilde{Y}) \subseteq (\tilde{c}^{-1}\{i\})_{\varepsilon} \text{ for some } i < r.
\]

Let \( \check{\rho} : \tilde{Y} \to Z \times Z \) be defined by setting \( \check{\rho}(y) = (\rho(y), 0) \). We will show

\[
(\check{\rho} \circ \pi_Y) \circ \text{Emb}(X, Y) \subseteq (c^{-1}\{i\})_{\varepsilon}.
\]

To this end, fix \( \eta \in \text{Emb}(X, Y) \) and define a mapping \( \varphi : \tilde{X} \to \tilde{Y} \) by \( \varphi([x]) = \pi_Y(\eta(x)) \); it is easy to check that \( \varphi \) is a well-defined multi-isometric embedding which satisfies \( \varphi \circ \pi_X = \pi_Y \circ \eta \). Then by definition of \( \rho \) there is \( \theta \in \tilde{c}^{-1}\{i\} \) such that \( \| \rho \circ \varphi - \theta \| \leq \varepsilon \). Let \( \check{\theta} : \tilde{X} \to Z \times Z \) be given by \( \check{\theta}(x) = (\theta(x), 0) \). The situation is summarized by the following diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{\pi_X} & \tilde{X} \\
\downarrow{\eta} & \circ & \downarrow{\varphi} \\
Y & \xrightarrow{\pi_Y} & \tilde{Y} \\
\end{array}
\begin{array}{ccc}
& \searrow{\delta} & \\
& \downarrow{\check{\rho}} & \\
& \check{\theta} & \searrow \\
& \check{\eta} & \\
& \rho \circ \varphi & \searrow \\
& \rho \circ \theta & \searrow \\
& \rho \circ \pi_X & \searrow \\
& \rho \circ \eta & \searrow \\
& \rho \circ (\varphi \circ \eta) & \searrow \\
& \rho \circ (\varphi \circ \pi_X) & \searrow \\
\end{array}
\]

Then \( c(\theta \circ \pi_X) = i \) and, since

\[
(\rho \circ \pi_Y) \circ \eta = \rho \circ (\pi_Y \circ \eta) = \rho \circ (\varphi \circ \pi_X),
\]

we have

\[
\| (\rho \circ \pi_Y) \circ \eta - \varphi \circ \pi_X \| = \| (\rho \circ \varphi) \circ \pi_X - \theta \circ \pi_X \| \leq \| \rho \circ \varphi - \theta \| \leq \varepsilon.
\]

Thus \( \rho \circ \pi_Y \) is the desired embedding. \( \square \)

Proof of Lemma 4.3. The proof is by induction on \( n \). The case \( n = 1 \) follows from the previous lemma, so fix all parameters and suppose that we can find seminormed spaces \( Z_1, \ldots, Z_n \) such that every \( r \)-colouring of \( \prod_{j=1}^n \text{Emb}(X_j, Z_j) \) has a \( \varepsilon/2 \)-monochromatic set of the form \( \prod_{j=1}^n \rho_j \circ \text{Emb}(X_j, Y_j) \). Let \( D \) be a finite \( \varepsilon/2 \)-dense subset of \( \prod_{j=1}^n \text{Emb}(X_j, Z_j) \), where the seminorm on the product is the maximum of the given seminorms. Apply Lemma 4.6 to \( X_{n+1} \) and \( Y_{n+1} \) to find a seminormed space \( Z \) that works for the error \( \varepsilon/2 \), and the number of colours being the cardinality of \( D_r \). We claim that
$Z_1, \ldots, Z_n, Z$ works. For suppose that $c : \prod_{j=1}^{n+1} \text{Emb}(X_j, Z_j) \to r$. We have the induced colouring 
\[
\widehat{c} : \text{Emb}(X_{n+1}, Z) \to \partial r,
\]
\[
\widehat{c}(\xi)(\eta_1, \ldots, \eta_n) := c(\eta_1, \ldots, \eta_n, \xi) \in r \text{ for every } (\eta_1, \ldots, \eta_n) \in D.
\]
By the choice of $Z$ there is some $\rho \in \text{Emb}(Y_{n+1}, Z)$ such that $\rho \circ \text{Emb}(X_{n+1}, Y_{n+1})$ is $\varepsilon/2$-monochromatic for $\widehat{c}$ with colour $\theta \in D^r$. The mapping $\theta : D \to r$ defines an $r$-colouring $\widehat{\theta} : \prod_{j=1}^n \text{Emb}(X_j, Z_j) \to r$ by $\varepsilon/2$-proximity. For each $j = 1, \ldots, n$, let $\rho_j \in \text{Emb}(Y_j, Z_j)$ be such that $\prod_{j=1}^n \rho_j \circ \text{Emb}(X_j, Y_j)$ is $\varepsilon/2$-monochromatic for $\widehat{\theta}$ with colour $s \in r$. Then the set
\[
\left( \prod_{j=1}^n \rho_j \circ \text{Emb}(X_j, Y_j) \right) \times (\rho \circ \text{Emb}(X_{n+1}, Y_{n+1})) \subseteq \prod_{j=1}^{n+1} \text{Emb}(X_j, Z_j)
\]
is $\varepsilon$-monochromatic for $c$ with colour $s$: Let $(\gamma_j)_{j=1}^{n+1} \in \prod_{j=1}^{n+1} \text{Emb}(X_j, Y_j)$. There is some $(\mu_j)_{j=1}^n \in \prod_{j=1}^n \text{Emb}(X_j, Z_j)$ such that $\widehat{\theta}(\mu_j)_{j=1}^n = s$ and $(\mu_j)_{j=1}^n$ is $\varepsilon/2$-close to $(\rho_j \circ \gamma_j)_{j=1}^n$. Choose $(\pi_j)_{j=1}^n \in D$ that is $\varepsilon/2$-close to $(\mu_j)_{j=1}^n$ such that $\theta(\pi_j)_{j=1}^n = s$. Let $\mu \in \text{Emb}(X, Z)$ be such that $\mu$ is $\varepsilon/2$-close to $\rho \circ \gamma$ and $\widehat{c}(\mu) = \theta$. This last equality means by definition that
\[
s = \theta((\mu_j)_{j=1}^n) = \widehat{c}(\mu)((\mu_j)_{j=1}^n) = c(\mu_1, \ldots, \mu_n).
\]
Finally, observe that $\mu$ is $\varepsilon/2$-close to $\rho$, and each $\mu_j$ is $\varepsilon$-close to $\rho_j \circ \gamma_j$.

For the next proposition, recall the definition of the classes $\langle \mathcal{K} \rangle_{= \kappa}$ from Definition 2.5.

**Proposition 4.7.** Let $\mathcal{K} = \{ \mathcal{K}_n \}_{n<\omega}$ be a collection of finite-dimensional normed spaces such that each $\mathcal{K}_n$ has the ARP. For every $n \in \mathbb{N}$, $X, Y \in \langle \mathcal{K} \rangle_{= \kappa}$, $r \in \mathbb{N}$, and every $\varepsilon > 0$ there is $Z \in \langle \mathcal{K} \rangle$ such that every $r$-colouring of $\text{Emb}(X, Z)$ has an $\varepsilon$-monochromatic set of the form $\rho \circ \text{Emb}(X, Y)$ for some $\rho \in \text{Emb}(Y, Z)$. Furthermore, if $\mathcal{K}_n = \mathcal{K}$ for each $n$ and $\mathcal{K}$ is closed under $\ell_\infty$-sums, then $Z$ can be chosen to be graded when $X$ and $Y$ are graded.

**Proof.** Fix all parameters and apply the previous lemma to $(X, \| \cdot \|_{j=1}^n)$, $(Y, \| \cdot \|_{j=1}^n)$, $\varepsilon$ and $r$ to find the corresponding $Z_1, \ldots, Z_n$. Let $Z := \prod_{j=1}^n Z_j$, and for each $j = 1, \ldots, n$, let
\[
\| (z_1, \ldots, z_n) \|_j := \| z_j \|_{Z_j}.
\]
Note that $Z \in \langle \mathcal{K} \rangle_{= \kappa}$ by construction. We claim that $Z$ witnesses the ARP for the given parameters. For suppose that $c : \text{Emb}(X, Z) \to r$ is a colouring. Given a sequence $\gamma = (\gamma_j)_{j=1}^n \in \prod_{j=1}^n \text{Emb}((X, \| \cdot \|_j), Z_j)$, define a mapping $F(\gamma) : X \to Z$ by
\[
F(\gamma)(x) = (\gamma_1(x), \ldots, \gamma_n(x)).
\]
Observe that $F(\gamma) \in \text{Emb}(X, Z)$ since each $\gamma_j$ is a multi-isometric embedding. Using this mapping, define an induced colouring $\widehat{c} : \prod_{j=1}^n \text{Emb}(X, \| \cdot \|_j) \to r$, by $\widehat{c}(\gamma) = c(F(\gamma))$ where $\gamma = (\gamma_j)_{j=1}^n$. By definition of each $Z_j$, there are $\rho_j \in \text{Emb}((Y, \| \cdot \|_j), Z_j)$ such that
\[
\prod_{j=1}^n \rho_j \circ \text{Emb}((X, \| \cdot \|_j), (Y_j, \| \cdot \|_j)) \text{ is } \varepsilon\text{-monochromatic.}
\]
Define $\rho \in \text{Emb}(Y, Z)$ by $\rho(y) = (\rho_1(y), \ldots, \rho_n(y))$. Note that $\rho \circ \eta = F((\rho_j \circ \eta_j)_j)$ and so $c(\rho \circ \eta) = \widehat{c}((\rho_j \circ \eta_j)_j)$. In particular, it follows that $\rho \circ \text{Emb}(X, Y)$ is $\varepsilon$-monochromatic.
In the case where $K_n = K$ for all $n$ and $K$ is closed under $\ell_\infty$-sums, we work with the same underlying space $Z$ but we instead equip it with the sequence of seminorms given by

$$
\|z_1, \ldots, z_n\|_j := \max_{i \leq j} \|z_i\|_Z.
$$

Then $Z_{\|\cdot\|_j}$ is multi-isometric to the $\ell_\infty$-sum of $Z_i$, $i \leq j$, and so $Z \in \langle \mathcal{K} \rangle_{= n}$ by our additional assumption on $K$. The rest of the proof is identical to that of the general case.

By appealing to the various known approximate Ramsey properties of classes of finite-dimensional normed spaces as considered in [4, 10], we can apply Theorem 4.4 and Proposition 4.7 together to obtain the following result. The notation used below corresponds to that of Example 2.7.

**Theorem 4.8.** The following groups are extremely amenable when equipped with the topology of pointwise convergence:

1. $\text{Iso}(G^\omega, (\|\cdot\|_{n < \alpha})$ and $\text{Iso}(G^{\omega'}, (\|\cdot\|_{n < \alpha})$ for each $\alpha \leq \omega$. In particular, the multi-isometry group of the separable (graded) Fréchet space of almost universal disposition for the class $\mathcal{M}_\omega$ (resp. $\mathcal{G}_\omega$) is extremely amenable.

2. $\text{Iso}(\text{Flim}(\mathcal{M}_{=\omega}))$.

3. $\text{Iso}(\text{Flim}(\mathcal{M}_{=\omega}(p_n)))$ for any sequence $(p_n) \subseteq [1, \infty[ \text{ with } p_n \notin \{4, 6, 8, \ldots \}$.

The extreme amenability of the groups $\text{Iso}(G^{\omega}, (\|\cdot\|_{n < \omega})$ and $\text{Iso}(G^{\omega'}, (\|\cdot\|_{n < \omega})$ should naturally be compared to the extreme amenability of $\text{Iso}(G)$. The following important questions remain open:

**Question 4.9.** Are $\text{Iso}(G^{\omega}, (\|\cdot\|_{n < \omega})$ and $\text{Iso}(G^{\omega'}, (\|\cdot\|_{n < \omega})$ topologically distinguishable from $\text{Iso}(G)$? Are $\text{Iso}(G^{\omega}, (\|\cdot\|_{n < \omega})$ and $\text{Iso}(G^{\omega'}, (\|\cdot\|_{n < \omega})$ universal Polish groups?

5. Concluding remarks

Throughout this paper we have adopted a very particular viewpoint in order to develop Fraïssé theory for Fréchet spaces. Specifically, we have been working with the category of multi-seminormed spaces with morphisms given by seminorm-preserving linear mappings. A natural question is whether or not a similar theory can be developed for more relaxed categories which capture the notion of a Fréchet space. For instance, the following general problem remains open:

**Problem 5.1.** Develop a theory of metric Fréchet spaces, i.e. pairs of the form $(X, d)$ where $X$ is a vector space and $d$ is a translation-invariant metric inducing a Fréchet topology on $X$.

In the above setting, one may be able to use more general machinery in order to develop a Fraïssé theory for such spaces, e.g. as in [6, 19]. It is unclear if the use of such machinery is possible in the setting of multi-seminormed spaces, since in general one needs to work with an arbitrarily large (finite) number of seminorms.

One specific corollary of our construction is that the spaces $\text{Flim}((\mathcal{M}_{<\omega})$ and $\text{Flim}((\mathcal{G}_{<\omega})$ are – in addition to their defining properties – Fraïssé for the classes $\mathcal{M}_\omega$ and $\mathcal{G}_\omega$, respectively, by virtue of being multi-isometric to the spaces constructed in [2]. Since the construction of the Fraïssé limit makes use of the fact that the elements of a given Fraïssé class are finitely-seminormed, it is not clear that
the aforementioned property must be true for general Fraïssé Fréchet spaces. More precisely, given a class $\mathcal{K} \subseteq \mathcal{M}_{<\omega}$, one can define a class $\mathcal{K}_\omega \subseteq \mathcal{M}_\omega$ by declaring that $(X, (\| \cdot \|_n)_{n<\omega}) \in \mathcal{K}_\omega$ exactly when $(X, (\| \cdot \|_m)_{m<\omega}) \in \mathcal{K}$ for all $m < \omega$.

This motivates the following:

**Problem 5.2.** If $E$ is a $\mathcal{K}$-Fraïssé Fréchet space, is $E$ necessarily $\mathcal{K}_\omega$-Fraïssé?

The proof that $G_\omega$ is $G_\omega$-Fraïssé (with an appropriate sequence of seminorms) from [2] makes use of the existence of a universal operator $\pi : G \to G$ constructed in [12]. This operator can essentially be seen as a Fraïssé operator in a precise sense (see, e.g., [22]). The operator constructed in [12] is the Gurarij analogue of Rota’s universal operator on $\ell_2$, as in [8, 28]. Thus, in order to shed light on a possible affirmative answer to Problem 5.2 one may need to construct similar operators for an arbitrary Fraïssé Banach space in place of $G$.

Finally, we mention that throughout the paper we have not assumed any condition on the continuity of the relevant mappings between multi-seminormed spaces. Since seminorm-preserving mappings (as defined above) need not be continuous, the following general problem presents itself:

**Problem 5.3.** For various classes $\mathcal{K}$ of finite-dimensional multi-seminormed spaces, study the notions of $\mathcal{K}$-universality and $\mathcal{K}$-Fraïssé where the embeddings are in addition assumed to be continuous. In particular, is there a version of the Fraïssé theory for multi-seminormed spaces developed above in which the associated embeddings are also required to be continuous?

**References**

[1] S. Banach. *Théorie des opérations linéaires*. Chelsea Publishing Co., New York, 1955.

[2] C. Bargetz, J. Kąkol, and W. Kubiś. A separable Fréchet space of almost universal disposition. *J. Funct. Anal.*, 272(5):1876–1891, 2017.

[3] D. Bartošová, J. Lopez-Abad, M. Lupini, and B. Mbombo. The Ramsey properties for Operator spaces and non-commutative Choquet simplices. Preprint, arXiv:2006.04799, 2020.

[4] D. Bartošová, J. Lopez-Abad, M. Lupini, and B. Mbombo. The Ramsey property for Banach spaces and Choquet simplices. *J. Eur. Math. Soc.*, to appear.

[5] I. Ben Yaacov. The linear isometry group of the Gurarij space is universal. *Proc. Amer. Math. Soc.*, 142(7):2459–2467, 2014.

[6] I. Ben Yaacov. Fraïssé limits of metric structures. *J. Symb. Log.*, 80(1):100–115, 2015.

[7] F. Cabello Sánchez, V. Ferenczi, and B. Randrianantoanina. On Mazur rotations problem and its multidimensional versions. Preprint, arXiv:2012.08344.

[8] S. R. Caradus. Universal operators and invariant subspaces. *Proc. Amer. Math. Soc.*, 23:526–527, 1969.

[9] N. L. Carothers. *A short course on Banach space theory*, volume 64 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2005.

[10] V. Ferenczi, J. Lopez-Abad, B. Mbombo, and S. Todorcevic. Amalgamation and Ramsey properties of $L_p$ spaces. *Adv. Math.*, 369:107190, 76, 2020.

[11] R. Fraïssé. Sur l’extension aux relations de quelques propriétés des ordres. *Ann. Sci. École Norm. Sup.* (3), 71:363–388, 1954.

[12] J. Garbulińska-Węgrzyn and W. Kubiś. A universal operator on the Gurarij space. *J. Operator Theory*, 73(1):143–158, 2015.

[13] R. L. Graham and B. L. Rothschild. Ramsey’s theorem for $n$-parameter sets. *Trans. Amer. Math. Soc.*, 159:257–292, 1971.
[14] V. I. Gurari˘ı. Spaces of universal placement, isotropic spaces and a problem of Mazur on rotations of Banach spaces. *Sibirsk. Mat. Ž.*, 7:1002–1013, 1966.

[15] R. S. Hamilton. The inverse function theorem of Nash and Moser. *Bull. Amer. Math. Soc. (N.S.)*, 7(1):65–222, 1982.

[16] W. Hodges. *A shorter model theory*. Cambridge University Press, Cambridge, 1997.

[17] N. J. Kalton. Universal spaces and universal bases in metric linear spaces. *Studia Math.*, 61(2):161–191, 1977.

[18] A. S. Kechris, V. G. Pestov, and S. Todorcevic. Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups. *Geom. Funct. Anal.*, 15(1):106–189, 2005.

[19] W. Kubiś. Metric-enriched categories and approximate Fraïssé limits. *Preprint*, arXiv:1210.6506.

[20] W. Kubiś and S. Solecki. A proof of uniqueness of the Gurari˘ı space. *Israel J. Math.*, 195(1):449–456, 2013.

[21] S. S. Kutateladze. *Fundamentals of functional analysis*, volume 12 of *Kluwer Texts in the Mathematical Sciences*. Kluwer Academic Publishers Group, Dordrecht, 1996. Translated from the second (1995) edition.

[22] M. Lupini. Fraïssé limits in functional analysis. *Adv. Math.*, 338:93–174, 2018.

[23] W. Lusky. The Gurarij spaces are unique. *Arch. Math. (Basel)*, 27(6):627–635, 1976.

[24] R. Meise and D. Vogt. *Introduction to functional analysis*, volume 2 of *Oxford Graduate Texts in Mathematics*. The Clarendon Press, Oxford University Press, New York, 1997. Translated from the German by M. S. Ramanujan and revised by the authors.

[25] J. Melleray and T. Tsankov. Extremely amenable groups via continuous logic. *Preprint*, arXiv:1708.01317.

[26] V. Pestov. *Dynamics of infinite-dimensional groups*, volume 40 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2006. The Ramsey-Dvoretzky-Milman phenomenon, Revised edition of its Dynamics of infinite-dimensional groups and Ramsey-type phenomena [Inst. Mat. Pura. Apl. (IMPA), Rio de Janeiro, 2005; MR2164572].

[27] S. Rolewicz. *Metric linear spaces*. PWN-Polish Scientific Publishers, Warsaw, 1972. Monografie Matematyczne, Tom. 56. [Mathematical Monographs, Vol. 56].

[28] G.-C. Rota. On models for linear operators. *Comm. Pure Appl. Math.*, 13:469–472, 1960.

[29] Z. Semadeni. *Banach spaces of continuous functions. Vol. I*. PWN—Polish Scientific Publishers, Warsaw, 1971. Monografie Matematyczne, Tom 55.

[30] J. Simon. *Banach, Fréchet, Hilbert and Neumann spaces*. Mathematics and Statistics Series. ISTE, London; John Wiley & Sons, Inc., Hoboken, NJ, 2017. Analysis for PDEs set. Vol. 1.

[31] D. Vogt. Operators between Fréchet spaces. Analysis Conference Manila, 1987.

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