Parikh matrices of arrays under Dejean array morphism

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Abstract. The Parikh matrix of a word $w$ over an ordered alphabet is an upper triangular matrix associated with the word $w$. The entries of this matrix above the main diagonal are numbers of certain subsequences, called subwords in the word $w$. Properties of Parikh matrices of morphic images of words under different types of morphisms have also been investigated. On the other hand, a rectangular picture array of symbols is an extension of the notion of a word to two dimensions. The concept of Parikh matrix was extended to such arrays by defining two kinds of matrices, called row and column Parikh matrices and several properties have been established. Here we introduce an array morphism, called Dejean array morphism defined on three symbols. This array morphism extends to two dimensions a word morphism due to Dejean. We derive formulae to count subwords in the rows of image arrays of rectangular binary picture arrays under Dejean array morphism, thus yielding the row Parikh matrix of the image array.

1. Introduction

In any communication, whether it is between human beings in terms of a natural language or between human beings and machines in terms of an artificial language, words are the central objects. Mathematically expressed, a finite word or simply a word, is a finite sequence of symbols. The topic of \textit{Combinatorics on words} [2, 6, 7] is an interesting area of research in the field of theoretical computer science and finds applications in a variety of problems arising in different fields such as linguistics, biology, computer science and many others. Parikh matrix of a word, which is an upper triangular matrix, is a comparatively recent notion, introduced in [8]. This notion has attracted the attention of researchers and there has been an intensive research in the topic on various aspects associated with words and Parikh matrices. The entries above the principal diagonal in this matrix provides information on the number of certain subwords (also called scattered subwords) in a word. The notion of morphism on words, which is a mapping that preserves concatenation operation, has played an important role in studying very many combinatorial properties of words. There are several kinds of string morphisms, introduced and investigated in the recent past, such as Istrail morphism considered by Atanasiu
Figure 1. An array over \{a, b\}

[1] and extensions of Thue morphism and the Fibonacci morphism considered in [5, 13]. On the other hand, a rectangular picture array of symbols is an extension of the notion of a word to two dimensions. An example array is shown in Fig. 1. The concept of Parikh matrix has been extended [14] to such arrays. In this paper, we introduce Dejean array morphism for two dimensional picture arrays over \{a < b\} and compute the row Parikh matrix by deriving formulae for counting the number of subwords in the rows for image arrays over \{a < b < c\} under Dejean array morphism.

2. Preliminaries

We now recall needed concepts and results [7, 8]. An ordered alphabet is a finite set of symbols, called alphabet, with an ordering on the symbols. For example, the alphabet \{a, b\} with an ordering \(a < b\) is an ordered alphabet. We write this ordered alphabet as \(\{a < b\}\). We will be dealing with ordered alphabets having two or three symbols. A word is a finite sequence of symbols taken from an alphabet. A subword (also called scattered subword) \(y\) of a word \(x\) is a subsequence of \(x\). We denote the number of subwords \(y\) in a word \(x\) is denoted by \(|x|_y\). For example, if \(x = ababaabba\), \(y = bba\), then \(|x|_y = 8\) if the ordered alphabet is \(\{a < b\}\).

The Parikh matrix of a word \(v\) over an ordered alphabet \(V\) was introduced by Mateescu et al. [8], extending the notion of Parikh vector [9, 10] of \(w\), whose components give the number of occurrences of each of the symbols of the alphabet in the word \(v\). Since we are mainly concerned with only a binary or a ternary ordered alphabet, Parikh matrix for words over such an alphabet is now recalled. We refer to [8], for a formal definition of the Parikh matrix of a word over any ordered alphabet. The notations \(\Sigma_2\) and \(\Sigma_3\) denote the binary ordered alphabet \(\{a < b\}\) and the ternary ordered alphabet \(\{a < b < c\}\) respectively, unless mentioned otherwise. The Parikh matrix \(M(w)\) of a binary word \(w\) over \(\Sigma_2\) is

\[
M(w) = \begin{pmatrix}
1 & |w|_a & |w|_{ab} \\
0 & 1 & |w|_b \\
0 & 0 & 1
\end{pmatrix}.
\]

The Parikh matrix \(M(w)\) of a ternary word \(w\) over \(\Sigma_3\) is

\[
M(w) = \begin{pmatrix}
1 & |v|_a & |v|_{ab} & |v|_{abc} \\
0 & 1 & |v|_b & |v|_{bc} \\
0 & 0 & 1 & |v|_c \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

In order to form the Parikh matrix of a word, there is an ingenious technique resulting from the definition of Parikh matrix given in [8]. For a word \(w\) over an ordered alphabet, we substitute from left to right, for each symbol \(w_i\) in \(w\), a corresponding matrix and perform multiplication of matrices from left to right, to yield the Parikh matrix of \(w\). In fact for a word \(w\) over the binary ordered alphabet \(\Sigma_2 = \{a < b\}\), for each \(a\) in \(w\), the 3\times3 triangular matrix \(M(a) = \begin{pmatrix}1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1\end{pmatrix}\).
is substituted and for each $b$ in $w$, the $3 \times 3$ triangular matrix $M(b) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ is substituted.

For example, if $x = abbaab$, the Parikh matrix $M(x)$ is the matrix product

$$M(x) = M(a)M(b)M(b)M(a)M(a)M(b) = \begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

Like wise, for the ternary ordered alphabet $\Sigma_3 = \{a < b < c\}$, with each of $a, b$ and $c$, $4 \times 4$ triangular matrices $M(a) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $M(b) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $M(c) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

are associated. Two words $x, y$ with Parikh matrices $M(x), M(y)$ are said to be $M$-equivalent (also called $M$-ambiguous or amiable) [8], denoted as $x \equiv_M y$, if $M(u) = M(v)$.

We now recall the notion of a morphism [7] on words as a mapping $\phi : \Gamma_1^* \rightarrow \Gamma_2^*$, where $\Gamma_1$ and $\Gamma_2$ are two alphabets, such that $\phi(u_1u_2) = \phi(u_1)\phi(u_2)$, for words $u_1, u_2$ over $\Gamma_1$.

3. Dejean Array morphism and Subwords in the Rows of Image Arrays

A number of morphisms have been introduced and investigated in different studies in the literature. Thue morphism [7] is an extensively investigated morphism in different contexts (see, for example, [11]). Dejean [4] introduced a morphism, which we call here as Dejean morphism, which has also been studied [3, 15] for its properties.

**Definition 1** [15] The Dejean morphism $d : \Sigma_3^+ \rightarrow \Sigma_3^+$ is defined by

$$d(a) = abacbacabacbacab, d(b) = beacbacabacbacab, d(c) = cabcbacabacbacab.$$ 

**Remark 1** Denoting $d(a), d(b), d(c)$ respectively by $\alpha, \beta, \gamma$, we have $|\alpha|_a = 6; |\alpha|_b = 6; |\alpha|_c = 7; |\beta|_a = 7; |\beta|_b = 6; |\beta|_c = 6; |\gamma|_a = 6; |\gamma|_b = 7; |\gamma|_c = 6; |\alpha|_ab = 18; |\alpha|_bc = 21; |\beta|_ab = 21; |\beta|_bc = 18; |\gamma|_ab = 21; |\gamma|_bc = 21; |\alpha|_abc = 46; |\beta|_abc = 31; |\gamma|_abc = 49.$

**Definition 2** The Dejean array morphism $D : \Sigma_3^{++} \rightarrow \Sigma_3^{++}$ is such that

$$D(a) = abacbacabacbacab, D(b) = beacbacabacbacab, D(c) = cabcbacabacbacab.$$ 

**Example 1** For the array $W = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ where $a < b$

$$D(W) = \begin{pmatrix} abacbacabacbacababacbacbacacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbac bacabacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacbacba

Formulae to count subwords in the ternary image words of binary words under Dejean morphism $d$ are known [15]. Analogous formulae for two related morphisms $d_1, d_2$ defined below, can be derived and are given below.
Lemma 1 [15] For $w \in \Sigma_2 = \{a < b\}$,

\[
\begin{align*}
|d(w)|_a &= 6|w| + |w|_a, |d(w)|_b = 6|w| + |w|_a \nonumber \\
|d(w)|_{ab} &= 21|w|^2 - 3p^2 - 6|w|_{ab}, |d(w)|_{bc} = 21|w|^2 - 3q^2 - 6|w|_{ab} \\
|d(w)|_{abc} &= 126(p^2 + q^2) - 95p - 77q + 273|w|_{ab} + 234|w|_{ba} + 252 \left( |w|_{aa} + |w|_{ab} + |w|_{ba} + |w|_{bb} \right) \\
&\quad + 216 \left( |w|_{ba} + |w|_{aab} + |w|_{abb} \right) + 294 \left( |w|_{ab} + |w|_{abb} \right)
\end{align*}
\]

Lemma 2 Let $d_1$ and $d_2$ be morphisms on words such that $d_1(a) = bcaabacacbacabcb$, $d_1(b) = cabcbacabacbacacba$, and $d_2(a) = abcabcabcabacacba$, $d_2(b) = abcabcabcabacacba$. Then for $w \in \Sigma_2 = \{a < b\}$

\[
\begin{align*}
|d_1(w)|_a &= 6|w| + |w|_a, |d_1(w)|_b = 6|w| + |w|_b, |d_1(w)|_c = 6|w| \\
|d_1(w)|_{ab} &= 21|w|^2 - 6pq + 13|w|_{ab}, |d_1(w)|_{bc} = 21|w|^2 - 3p^2 - 6|w|_{ab} \\
|d_1(w)|_{abc} &= 126(p^2 + q^2) - 95p - 77q + 273|w|_{ab} + 234|w|_{ba} + 252 \left( |w|_{aa} + |w|_{ab} + |w|_{ba} + |w|_{bb} \right) \\
&\quad + 216 \left( |w|_{ba} + |w|_{aab} + |w|_{abb} \right) + 294 \left( |w|_{ab} + |w|_{abb} \right)
\end{align*}
\]

Subramanian et al. [12] introduced a weak-ratio property relating to words. Two words $u, v$ over $\Sigma = \{a < b < c\}$ are said to satisfy the weak-ratio property, written $u \sim_{wr} v$, if $\frac{|u|_a}{|u|_b} = \frac{|v|_a}{|v|_b} = \frac{|v|_c}{|v|_b} = \alpha$, for some $\alpha \neq 0$. Weak-ratio property of binary words $u, v$ over $\{a < b\}$ is similarly defined. This notion has been extended to arrays in [14]. Given two arrays $M_1, M_2$ over $\Sigma_3 = \{a < b < c\}$, the arrays $M_1, M_2$ are said to satisfy the weak-ratio property, written $M_1 \sim_{wr} M_2$ if $\frac{|M_1|_a}{|M_2|_a} = \frac{|M_1|_b}{|M_2|_b} = \frac{|M_1|_c}{|M_2|_c} = \alpha$, for some $\alpha \neq 0$, where $|A|_x$ is the number of symbols $x$ in $A$.

Theorem 1 For non-empty arrays $X$ and $Y$ over $\Sigma_2 = \{a < b\}$, we have $D(X) \sim_{wr} D(Y)$ whenever $X \sim_{wr} Y$ where $D$ is the Dejean array morphism.

Proof. Let $X$ and $Y$ be two arrays of sizes $m \times n$ and $k \times l$ respectively, so that $X \sim_{wr} Y$. Then $|X|_a = \delta |Y|_a$ and $|X|_b = \delta |Y|_b$ for some $\delta \neq 0$. Let $v_1, v_2, v_3, \ldots, v_m$ and $w_1, w_2, w_3, \ldots, w_k$ respectively be the words in the successive rows of the arrays $X$ and $Y$. Then $|v_i| = n, |w_j| = l$ for $1 \leq i \leq m, 1 \leq j \leq k$. Let $|v_i|_a = p_i, |v_i|_b = q_i$ for $1 \leq i \leq m$ and $|w_j|_a = r_j, |w_j|_b = s_j$ for $1 \leq j \leq k$. In the image array $D(X)$ of $X$ under $D$, each $D(v_i)$ yields three consecutive rows of words in $D(X)$ and let these words be $x_i, y_i$ and $z_i$ by the application of $D$ on $v_i$. Hence $|D(X)|_a = \sum_{i=1}^m |D(v_i)|_a = 19 \sum_{i=1}^m |v_i|_a + |v_i|_b = 19 |X|_a + |X|_b$. Similarly, $|D(X)|_b = |D(X)|_c = 19 |X|_a + |X|_b$. Since $|X|_a = \delta |Y|_a$ and $|X|_b = \delta |Y|_b$, we obtain $|D(X)|_a = \delta |D(Y)|_a, |D(X)|_b = \delta |D(Y)|_b, |D(X)|_c = \delta |D(Y)|_c$. Thus $D(X) \sim_{wr} D(Y)$. 

4
Theorem 2 Let \( w_1, w_2, w_3, \ldots, w_m \) be the consecutive rows of an \( m \times n \) array \( W \) over \( \Sigma_2 = \{a < b\} \). Then \( |D(w)|_a = 19mn = |D(w)|_b = |D(w)|_c \) where \( D \) is the Dejean array morphism.

Proof. Each row \( w_i \) in \( W \) produces three consecutive rows in \( D(W) \) and these are obtained by applying the three string morphisms, namely, \( d \) and the two similar morphisms \( d_1 \) and \( d_2 \) as defined in Lemma 2. Now, from Lemma 2, we have

\[
|D(w)|_a = \sum_{i=1}^{m} \{|d(w_i)|_a + |d_1(w_i)|_a + |d_2(w_i)|_a\}
\]

\[
= \sum_{i=1}^{m} \{6|w_i| + |w_i|_b + 6|w_i| + |w_i|_a + 6|w_i|\} = \sum_{i=1}^{m} 19|w_i| = 19mn
\]

In a similar manner, we can prove that \( |D(w)|_b = |D(w)|_c = 19mn \).

Theorem 3 Let \( w_1, w_2, w_3, \ldots, w_m \) be the consecutive rows of an \( m \times n \) array \( W \) over \( \Sigma_2 = \{a < b\} \). Then

\[
|D(W)|_{ab} = |D(W)|_{bc} = 60mn^2 + \sum_{i=1}^{m} |w_i|_{ab}
\]

where \( D \) is the Dejean array morphism.

Proof. Since each row \( w_i \) in \( W \) produces three consecutive rows in \( D(W) \) on applying the three string morphisms, \( d, d_1 \) and \( d_2 \) as defined in Lemma 2, we have

\[
|D(W)|_{ab} = \sum_{i=1}^{m} \left[ 21|w_i|^2 - 3|w_i|^2 - 6|w_i|_a \right] + \sum_{i=1}^{m} \left[ 21|w_i|^2 - 6|w_i|_a|w_i|_b + 13|w_i|_{ab} \right]
\]

\[
+ \sum_{i=1}^{m} \left[ 21|w_i|^2 - 3|w_i|^2 - 6|w_i|_a \right] = 60mn^2 + \sum_{i=1}^{m} |w_i|_{ab}
\]

\[
|D(W)|_{bc} = \sum_{i=1}^{m} \left[ 21|w_i|^2 - 3|w_i|^2 - 6|w_i|_b \right] + \sum_{i=1}^{m} \left[ 21|w_i|^2 - 3|w_i|^2 - 6|w_i|_a \right]
\]

\[
+ \sum_{i=1}^{m} \left[ 21|w_i|^2 - 6|w_i|_a|w_i|_b + 13|w_i|_{ab} \right] = 60mn^2 + \sum_{i=1}^{m} |w_i|_{ab}
\]

Theorem 4 Let \( w_1, w_2, w_3, \ldots, w_m \) be the consecutive rows of an \( m \times n \) array \( W \) over \( \Sigma_2 = \{a < b\} \). Then

\[
|D(W)|_{abc} = \sum_{i=1}^{m} \left[ 378(p_i^2 + q_i^2) + 762(|w_i|_{ab} + |w_i|_{ba}) + 756(|w_i|_{aaa} + |w_i|_{bbb}) \right]
\]

\[
+ 762(|w_i|_{aab} + |w_i|_{aba} + |w_i|_{ab} + |w_i|_{bab} + |w_i|_{bba}) - 252mn
\]

where \( D \) is the Dejean array morphism.

Proof. As in the proofs of Theorem 2 and Theorem 3, we have from Lemma 2,

\[
|D(W)|_{abc} = \sum_{i=1}^{m} \{|d(w_i)|_{abc} + |d_1(w_i)|_{abc} + |d_2(w_i)|_{abc}\}
\]

\[
= \sum_{i=1}^{m} \left[ 378(p_i^2 + q_i^2) + 762(|w_i|_{ab} + |w_i|_{ba}) + 756(|w_i|_{aaa} + |w_i|_{bbb}) \right]
\]

\[
+ 762(|w_i|_{aab} + |w_i|_{aba} + |w_i|_{ab} + |w_i|_{bab} + |w_i|_{bba}) - 252mn
\]
Remark 2 For an array $X$ over $\Sigma_2 = \{a < b\}$, the row Parikh matrix of the image array $D(X)$ can now be formed from the array $X$, since we can compute the quantities $|D(X)|_a, |D(X)|_b, |D(X)|_c, |D(X)|_{ab}, |D(X)|_{bc}, |D(X)|_{abc}$ using the formulae derived in Theorems 2, 3, 4. As an illustration, for the array $D(W)$ in Example 1, the row Parikh matrix of $D(W)$ is

$$
\begin{pmatrix}
1 & 76 & 481 & 2028 \\
0 & 1 & 76 & 481 \\
0 & 0 & 1 & 76 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

which can also be verified by direct computation.

4. Conclusion
We have defined Dejean array morphism and developed formulae for computing the row Parikh matrix of the image array under Dejean array morphism $D$. By the definition of $D$, for an $m \times n$ array $X$, the image array $D(X)$ has $3m$ rows and $57n$ columns by the definition of Dejean array morphism. Analogous to the formulae developed for counting subwords in the rows of the image array $D(X)$, we can also derive formulae for counting subwords in the columns and thus obtain column Parikh matrix of the image array $D(X)$. The problem of ambiguity of a picture array in the sense of two picture arrays having the same row Parikh matrix or having the same column Parikh matrix will be of interest to examine and this is for future work.

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