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Quantum entanglement for atoms coupling to fluctuating electromagnetic field in the cosmic string spacetime

Zhiming Huang

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Abstract
We investigate entanglement dynamics for two atoms coupling with fluctuating electromagnetic field in the cosmic string spacetime. We calculate the entanglement for different conditions. It is found that the entanglement behaviors are dependent on vacuum fluctuation, spacetime topology, two-atom separation and atomic polarization orientation. After a long time of evolution, entanglement would vanish, which means entanglement affected by electromagnetic fluctuation cannot maintain for a long time. For different spacetime topologies, entanglement presents different behaviors dependent on various parameters. When deficit angle parameter $\nu = 1$ and atom–string distance is toward infinity, the results in flat spacetime are recovered. When atoms keep close to the string, entanglement can be improved; specially, when two atoms locate on the string and have no polarization of axial direction, atoms are not affected by the electromagnetic fluctuation and entanglement can remain unchanged. When two-atom separation is relatively large, entanglement exhibits oscillation behavior as atom–string distance varies. This indicates that the existence of string profoundly modifies on the vacuum fluctuation and atom–field interaction. In addition, when two-atom separation is small, entanglement gains better improvement. Many parameters and conditions provide us with greater freedom to control the entanglement behaviors. In principle, this is useful to sense the cosmic string spacetime topology structure and property and discriminate different kinds of spacetime.

Keywords Quantum entanglement · Cosmic string spacetime · Electromagnetic field

1 Introduction
Quantum entanglement is most fundamental difference between quantum classical world and quantum world, which has many amazing potential applications,
such as quantum information processing and quantum computation [1,2]. However, the ineluctable interaction between quantum system and environment may damage quantum entanglement, such as atom–field interaction. Recently, the entanglement dynamics is investigated for atoms coupling to fluctuating scalar field or electromagnetic field in Minkowski spacetime, and the results show that entanglement presents behaviors of degradation, generation, revival and enhancement under the influence of vacuum fluctuation [3,4]. Lately entanglement dynamics for atoms affected by quantum field in curved spacetime are studied [5,6], and the results show that entanglement behaviors can be used to distinguish different spacetime backgrounds.

On the other hand, the cosmic string spacetime with nontrivial topological defect is a typical curved spacetime. The topological defect of cosmic string is originated from the evolution of the early universe due to the symmetry breaking phase transitions [7,8]. The cosmic string formed under certain conditions can be stable and can exist for a long time until now, which may provide us with information about particle physics and the early universe. However, due to its inherent complexity, it is not easy to understand the nature of the cosmic string. Cosmic string has energy momentum tensor, and it can induce some interesting gravitational effects, such as gravitational lensing effect [9,10] and gravitational Aharonov–Bohm effect [11,12]. For better understanding the nature of the cosmic string, some quantum processes have been investigated in the cosmic string spacetime, such as radiative properties [13–15], geometric phase [16,17] and Casimir effect [18].

Recently, the entanglement dynamics for two accelerating atoms interacting with a massless scalar field in the cosmic string spacetime have been investigated [19,20]. In comparison with massless scalar field, the fluctuating vacuum electromagnetic field is a more realistic model. In this work, we intend to analyze the entanglement behaviors for two identical atoms immersing in fluctuating electromagnetic field in the background of the cosmic string spacetime. We are intriguing in how the nontrivial spacetime topology affects the atom–field interaction and the entanglement behaviors, and whether entanglement behaviors can be exploited to detect the spacetime character and differentiate various spacetimes. Our exploration would offer new insight for the property of the topological defect universe and the quantum field theory of curved spacetime.

This paper is organized as follows. In Sect. 2, we firstly introduce interaction model between two two-level atoms and a quantum electromagnetic field, and the entanglement measure. And then in Sect. 3, we do relevant calculation of entanglement quantity and discuss the entanglement behaviors in different circumstances. Finally, we present a summary of this paper in Sect. 4. The natural units $\hbar = c = 1$ are adopted in this paper.
2 Preliminaries

Let us consider two identical two-level atoms weakly coupling with fluctuating electromagnetic field. The Hamiltonian of the whole system can be written as

\[ H = H_S + H_F + H_I. \]  

(1)

\( H_S \) is the Hamiltonian of two atoms, which takes the form

\[ H_S = H_S^{(1)} + H_S^{(2)}, \quad H_S^{(\alpha)} = \frac{\omega}{2} \sigma_3^{(\alpha)}, \quad (\alpha = 1, 2), \]  

(2)

where \( \sigma_i^{(1)} = \sigma_i \otimes \sigma_0, \sigma_i^{(2)} = \sigma_0 \otimes \sigma_i \), with \( \sigma_i (i = 1, 2, 3) \) being the Pauli operators and \( \sigma_0 \) being the \( 2 \times 2 \) identity operator, and \( \omega \) denotes the energy level spacing. \( H_F \) is the Hamiltonian of the electromagnetic field. \( H_I \) is the dipole interaction Hamiltonian between atoms and electromagnetic field, which has the form

\[ H_I(\tau) = -D^{(1)}(\tau) \cdot E(x^{(1)}(\tau)) - D^{(2)}(\tau) \cdot E(x^{(2)}(\tau)), \]  

(3)

where \( \tau \) is the proper time of the atoms, \( D^{(\alpha)}(\tau) (\alpha = 1, 2) \) is the electric-dipole moment of the atom, and \( E(x^{(\alpha)}(\tau)) \) denotes the electric field. In the interaction representation, the atomic dipole operators take the form \( D^{(\alpha)}(\tau) = d^{(\alpha)} \sigma e^{-i\phi} + d^{(\alpha)\ast} \sigma e^{i\phi} \) with \( d^{(\alpha)} = \langle 0 | D^{(\alpha)} | 1 \rangle \).

We assume initially there is no correlation between the atoms and the electromagnetic field. Then, the initial state of the total system can be written as \( \rho_{tot}(0) = \rho(0) \otimes |0\rangle \langle 0| \), where \( \rho(0) \) is the initial state of the atomic system and \( |0\rangle \) is the vacuum state of the electromagnetic field. In the weakly coupling limit, the evolution process of the atoms can be described by the Kossakowski–Lindblad master equation \([21–23]\)

\[ \frac{\partial \rho(\tau)}{\partial \tau} = -i[H_{eff}, \varphi(\phi)] + \mathcal{L}[\varphi(\phi)], \]  

(4)

where

\[ H_{eff} = H_S - \frac{1}{2} \sum_{\alpha, \beta=1}^2 \sum_{i,j=1}^3 H_{ij}^{(\alpha\beta)} \sigma_i^{(\alpha)} \sigma_j^{(\beta)}, \]  

(5)

and

\[ \mathcal{L}[\rho] = \frac{1}{2} \sum_{\alpha, \beta=1}^2 \sum_{i,j=1}^3 S_{ij}^{(\alpha\beta)} [2\sigma_j^{(\beta)} \rho \sigma_i^{(\alpha)} - \sigma_i^{(\alpha)} \sigma_j^{(\beta)} \rho - \rho \sigma_i^{(\alpha)} \sigma_j^{(\beta)}]. \]  

(6)

\( S_{ij}^{(\alpha\beta)} \) and \( H_{ij}^{(\alpha\beta)} \) are determined by the electromagnetic field correlation functions:

\[ G_{kl}^{(\alpha\beta)}(\tau - \tau') = \langle 0 | E_k(\tau, x^{(\alpha)}) E_l(\tau', x^{(\beta)}) | 0 \rangle. \]  

(7)
The Fourier transforms and Hilbert transforms of field correlation function are defined as

\begin{align}
G_{kl}(\lambda) & = \int_{-\infty}^{\infty} d\Delta e^{i\Delta \lambda} G_{kl}^{(\alpha\beta)}(\Delta), \\
K_{kl}(\lambda) & = \frac{P}{\pi i} \int_{-\infty}^{\infty} d\varpi \frac{G_{kl}^{(\alpha\beta)}(\varpi)}{\varpi - \lambda},
\end{align}

where \( P \) is the principal value. The Kossakowski matrix \( S_{ij}^{(\alpha\beta)} \) can be expressed analytically as

\begin{align}
S_{ij}^{(\alpha\beta)} & = A^{(\alpha\beta)} \delta_{ij} - i B^{(\alpha\beta)} \epsilon_{ijk} \delta_{3k} - A^{(\alpha\beta)} \delta_{3i} \delta_{3j},
\end{align}

where

\begin{align}
A^{(\alpha\beta)} & = \frac{1}{4} [ G^{(\alpha\beta)}(\omega) + G^{(\alpha\beta)}(-\omega) ], \\
B^{(\alpha\beta)} & = \frac{1}{4} [ G^{(\alpha\beta)}(\omega) - G^{(\alpha\beta)}(-\omega) ],
\end{align}

with

\begin{align}
G^{(\alpha\beta)}(\omega) & = \sum_{k,l=1}^{3} d^{(\alpha)}_{k} d^{(\beta)*}_{l} G_{kl}^{(\alpha\beta)}(\omega).
\end{align}

Similarly, \( H_{ij}^{(\alpha\beta)} \) can be obtained by replacing \( G_{ij}^{(\alpha\beta)} \) with \( K_{ij}^{(\alpha\beta)} \) in the above expressions.

We employ Wootters concurrence [24] as the entanglement measure. For two-qubit X-shape states

\begin{align}
\rho_{X} & = \frac{1}{4} (I \otimes I + \sum_{i} c_{i} \sigma_{i} \otimes \sigma_{i} + c_{4} I \otimes \sigma_{3} + c_{5} \sigma_{3} \otimes I),
\end{align}

with \(-1 \leq c_{i} \leq 1\), the concurrence is given by

\begin{align}
E & = \max(0, F_{1}, F_{2}),
\end{align}

where \( F_{1} = \frac{1}{2} |c_{1} + c_{2}| - \sqrt{(c_{3} - c_{4} - c_{5} + 1)(c_{3} + c_{4} + c_{5} + 1)} \) and \( F_{2} = \frac{1}{2} |c_{1} - c_{2}| - \sqrt{(c_{3} + c_{4} - c_{5} - 1)(c_{3} - c_{4} + c_{5} - 1)} \).

### 3 Dynamics of quantum entanglement in the cosmic string spacetime

In this section, we examine the entanglement dynamics for two-atom system coupled to electromagnetic vacuum fluctuation in the cosmic string spacetime. Let us assume
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A straight and infinitely long comics string lies at the z-axis, the spacetime metric in the cylindrical coordinate \((t, r, \theta, z)\) is described as

\[
ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - dz^2,
\]

which looks like Minkowski spacetime metric, except that the azimuth angle \(\theta\) is restricted in the range \(0 \leq \theta < 2\pi/n\) with \(n = (1 - 4G\mu)^{-1}\) with \(G\) and \(\mu\) denoting the Newton gravitational constant and the string’s linear mass density, respectively. The metric is a simple solution of the Einstein equations. The metric describes a locally flat but not globally flat spacetime with a planar angle deficit \(\delta\theta = 8\pi G\mu\).

By some calculations and derivations, the operator of electromagnetic field in the cosmic string spacetime can be written as [13]

\[
A_\xi(x) = \int d\mu_j [c_\xi_j(t) f_\xi_j(x) + \text{H.c.}],
\]

where

\[
\int d\mu_j = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{n} d\kappa \int_{0}^{\infty} dk k, \tag{17}
\]

with \(j \equiv \{n, \kappa, k_\perp\}\), \(c_\xi_j(t) = c_\xi_j(0)e^{-i\omega_j t}\) is the annihilation operator of the field, and

\[
f_\xi_j(x) = \frac{1}{2\pi} \sqrt{\frac{\nu}{\omega_j}} e^{i\kappa z} e^{i\nu n \theta} J_{\nu+n+\xi}(k_\perp r),
\]

with \(\xi \in \{0, 1, -1, 3\}\), \(\kappa \in (-\infty, \infty)\), \(n \in \mathbb{Z}\), \(k_\perp \in [0, \infty)\), \(\omega_j = \sqrt{\kappa^2 + k_\perp^2}\) and \(J_{\nu+n+\xi}(k_\perp r)\) being the Bessel function. After some calculations, the correlation function can be expressed as [13]

\[
\langle 0 | E_k(x) E_l(x') | 0 \rangle = \delta_k \delta_l'\langle 0 | A_0(x) A_0(x') | 0 \rangle + \delta_0 \delta_l'\langle 0 | A_k(x) A_1(x') | 0 \rangle,
\]

where

\[
A_r(x) = \frac{1}{\sqrt{2}} \int d\mu_j [(c_{+j}(t) f_{+j}(x) + c_{-j}(t) f_{-j}(x)) + \text{H.c.}],
\]

\[
A_\theta(x) = -i r \frac{1}{\sqrt{2}} \int d\mu_j [(c_{+j}(t) f_{+j}(x) - c_{-j}(t) f_{-j}(x)) - \text{H.c.}],
\]

\[
A_{\xi, i}(x) = \int d\mu_j [c_{3j, 0j}(t) f_{0j}(x) + \text{H.c.}],
\]
The trajectories of two static atoms parallel to string can be expressed as

\[
t_1(\tau) = \tau, \quad \theta_1(\tau) = 0, \quad r_1(\tau) = r, \quad z_1(\tau) = 0,
\]
\[
t_2(\tau) = \tau, \quad \theta_2(\tau) = 0, \quad r_2(\tau) = r, \quad z_2(\tau) = L.
\]

Substituting the atomic trajectories into field correlation function (19), for \(\alpha = \beta\), we obtain

\[
G^{(11)}_{11}(\tau - \tau') = G^{(22)}_{11}(\tau - \tau') = \frac{v}{8\pi^2} \int d\mu_j e^{-i\omega_j \Delta \tau} \times \left[ \frac{\omega_j}{2} \left( J_{|v|+1}(k_r) + J_{|v|-1}(k_r) \right) - \frac{1}{\omega_j} \left( \frac{\partial J_{|v|}(k_r)}{\partial r} \right)^2 \right],
\]
\[
G^{(11)}_{22}(\tau - \tau') = G^{(22)}_{22}(\tau - \tau') = \frac{v^2r}{8\pi^2} \int d\mu_j e^{-i\omega_j \Delta \tau} \times \left[ \frac{\omega_j}{2} \left( J_{|v|+1}(\kappa r) + J_{|v|-1}(\kappa r) \right) - \frac{1}{\omega_j} \left( \frac{\partial J_{|v|}(\kappa r)}{\partial r} \right)^2 \right],
\]
\[
G^{(11)}_{33}(\tau - \tau') = G^{(22)}_{33}(\tau - \tau') = \frac{v}{8\pi^2} \int d\mu_j e^{-i\omega_j \Delta \tau} \frac{k_r^2}{\omega_j} J_{|v|}(\kappa r),
\]

where \(\Delta \tau = \tau - \tau'\). For \(\alpha \neq \beta\), we obtain

\[
G^{(12)}_{11}(\tau - \tau') = G^{(21)}_{11}(\tau - \tau') = \frac{v}{8\pi^2} \int d\mu_j \cos(\kappa L) e^{-i\omega_j \Delta \tau} \times \left[ \frac{\omega_j}{2} \left( J_{|v|+1}(k_r) + J_{|v|-1}(k_r) \right) - \frac{1}{\omega_j} \left( \frac{\partial J_{|v|}(k_r)}{\partial r} \right)^2 \right],
\]
\[
G^{(12)}_{22}(\tau - \tau') = G^{(21)}_{22}(\tau - \tau') = \frac{v^2r}{8\pi^2} \int d\mu_j \cos(\kappa L) e^{-i\omega_j \Delta \tau} \times \left[ \frac{\omega_j}{2} \left( J_{|v|+1}(\kappa r) + J_{|v|-1}(\kappa r) \right) - \frac{1}{\omega_j} \left( \frac{\partial J_{|v|}(\kappa r)}{\partial r} \right)^2 \right],
\]
\[
G^{(12)}_{33}(\tau - \tau') = G^{(21)}_{33}(\tau - \tau') = \frac{v}{8\pi^2} \int d\mu_j \cos(\kappa L) e^{-i\omega_j \Delta \tau} \frac{k_r^2}{\omega_j} J_{|v|}(\kappa r),
\]
\[
G^{(12)}_{13}(\tau - \tau') = -G^{(21)}_{13}(\tau - \tau') = -G^{(12)}_{31}(\tau - \tau') = G^{(21)}_{31}(\tau - \tau') = \frac{v}{8\pi^2} \int d\mu_j \sin(\kappa L) e^{-i\omega_j \Delta \tau} \frac{k_r^2}{\omega_j} J_{|v|}(\kappa r) \frac{\partial J_{|v|}(\kappa r)}{\partial r}.
\]

The rest of terms are zero. According to the residue theorem, the Fourier transforms (8) of the two point functions can be obtained. For \(\alpha = \beta\), we obtain

\[
G^{(11)}_{ij}(\omega) = G^{(22)}_{ij}(\omega) = \frac{\omega^3}{3\pi} f^{(11)}_{ij},
\]

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where

\[
\begin{align*}
&f^{(11)}_{11} = \frac{3\nu}{4} \sum_{n=-\infty}^{\infty} \int_{0}^{1} d\eta \frac{\eta^2 J_{|vn|-1}(\omega \eta) J_{|vn|+1}(\omega \eta) + (2 - \eta^2) J_{|vn+1|}^2(\omega \eta)}{\sqrt{1 - \eta^2}}, \\
&f^{(11)}_{22} = \frac{3\nu}{4} \sum_{n=-\infty}^{\infty} \int_{0}^{1} d\eta \frac{\eta \eta^2 J_{|vn|-1}(\omega \eta) J_{|vn|+1}(\omega \eta) + (2 - \eta^2) J_{|vn+1|}^2(\omega \eta)}{\sqrt{1 - \eta^2}}, \\
&f^{(11)}_{33} = \frac{3\nu}{2} \sum_{n=-\infty}^{\infty} \int_{0}^{1} d\eta \frac{\eta^3 J_{|vn|}(\omega \eta)}{\sqrt{1 - \eta^2}},
\end{align*}
\]

(27)

and the rest of components are zero. For \(\alpha \neq \beta\),

\[
G_{ij}^{(\alpha\beta)}(\omega) = \frac{\omega^3}{3\pi} g_{ij}^{(\alpha\beta)},
\]

(28)

where

\[
\begin{align*}
g^{(12)}_{11} &= g^{(21)}_{11} = \frac{3\nu}{4} \sum_{n=-\infty}^{\infty} \int_{0}^{1} d\eta \frac{\cos(\omega L \sqrt{1 - \eta^2}) \eta}{\sqrt{1 - \eta^2} J_{|vn|-1}(\omega \eta) J_{|vn|+1}(\omega \eta) + (2 - \eta^2) J_{|vn+1|}^2(\omega \eta)}, \\
g^{(12)}_{22} &= g^{(21)}_{22} = \frac{3\nu}{4} \sum_{n=-\infty}^{\infty} \int_{0}^{1} d\eta \frac{\cos(\omega L \sqrt{1 - \eta^2}) \eta}{\sqrt{1 - \eta^2} J_{|vn|-1}(\omega \eta) J_{|vn|+1}(\omega \eta) + (2 - \eta^2) J_{|vn+1|}^2(\omega \eta)}, \\
g^{(12)}_{33} &= g^{(21)}_{33} = \frac{3\nu}{2} \sum_{n=-\infty}^{\infty} \int_{0}^{1} d\eta \frac{\cos(\omega L \sqrt{1 - \eta^2}) \eta^3 J_{|vn|}(\omega \eta)}{\sqrt{1 - \eta^2}}, \\
g^{(12)}_{13} &= g^{(21)}_{31} = -g^{(21)}_{31} = -g^{(21)}_{13} = \frac{3\nu}{4} \sum_{n=-\infty}^{\infty} \int_{0}^{1} d\eta \frac{\cos(\omega L \sqrt{1 - \eta^2}) \eta^2 J_{|vn|}(\omega \eta)}{\sqrt{1 - \eta^2}}, \\
&\times [J_{|vn|-1}(\omega \eta) - J_{|vn|+1}(\omega \eta)],
\end{align*}
\]

(29)

and the other terms are zero. For obtaining the above results, we have utilized the following properties of Bessel function \([25]\)

\[
\begin{align*}
\sum_{n=-\infty}^{\infty} J_{|vn+1|}^2(x) &= \sum_{n=-\infty}^{\infty} J_{|vn-1|}^2(x), \\
\sum_{n=-\infty}^{\infty} J_{|vn+1|}^2(x) + \sum_{n=-\infty}^{\infty} J_{|vn-1|}^2(x) &= 2 \sum_{n=-\infty}^{\infty} J_{|vn+1|}^2(x), \quad (v \geq 1) \\
J_{n-1}(x) + J_{n+1}(x) &= \frac{2n}{x} J_n(x), \quad J_{n-1}(x) - J_{n+1}(x) = 2 \frac{dJ_n(x)}{dx}.
\end{align*}
\]

(30)
According to Eqs. (11) and (12), we can rewrite Eq. (10) as

\[ S_{ij}^{(11)} = A \delta_{ij} - i A \epsilon_{ijk} \delta_{3k} - A \delta_{3i} \delta_{3j}, \]

\[ S_{ij}^{(22)} = B \delta_{ij} - i B \epsilon_{ijk} \delta_{3k} - B \delta_{3i} \delta_{3j}, \]

\[ S_{ij}^{(12)} = S_{ij}^{(21)} = C \delta_{ij} - i C \epsilon_{ijk} \delta_{3k} - C \delta_{3i} \delta_{3j}, \]  

(31)

with

\[ A = \frac{\Gamma}{4} \sum_{i=1}^{3} f_{ii}^{(11)} |\hat{d}_{i}^{(1)}|^2, \]

\[ B = \frac{\Gamma}{4} \sum_{i=1}^{3} f_{ii}^{(11)} |\hat{d}_{i}^{(2)}|^2, \]

\[ C = \frac{\Gamma}{4} \sum_{i} \sum_{j} g_{ij}^{(12)} \hat{d}_{i}^{(1)} \hat{d}_{i}^{(2)*}, \]  

(32)

where \( \Gamma = \omega^3 |d|^2/3\pi \) is the atomic spontaneous emission rate, and \( \hat{d}_{i}^{(\alpha)} = d_{i}^{(\alpha)}/|d| \) is a unit vector with \( d_{i}^{(\alpha)} \) being the electric dipoles of the atoms. We assume that \( |d^{(1)}| = |d^{(2)}| = |d| \).

Now, let us derive the master equation (4) in the interaction representation. Any two-qubit states can be represented with Pauli operators

\[ \rho = \frac{1}{4} \sum_{i=0}^{3} \sum_{j=0}^{3} p_{i,j}(\tau) \sigma_i \otimes \sigma_j. \]  

(33)

Here, we suppose two atoms initially are prepared in X-shape state, and combining Eq. (33) with Eq. (4), we can obtain the nontrivial coupled differential equations

\[ p'_{0,0}(\tau) = 0, \]

\[ p'_{1,1}(\tau) = 2[(A + B)p_{1,1}(\tau) + C p_{0,3}(\tau) + C (p_{3,0}(\tau) + 2 p_{3,3}(\tau))], \]

\[ p'_{0,3}(\tau) = -2(2B p_{0,0}(\tau) + 2 B p_{0,3}(\tau) + C (p_{1,1}(\tau) - p_{2,2}(\tau))), \]

\[ p'_{2,2}(\tau) = -2(A + B) p_{2,2}(\tau) - 2C (p_{0,3}(\tau) + p_{3,0}(\tau)) - 4 C p_{3,3}(\tau), \]

\[ p'_{3,0}(\tau) = -2(2 A p_{0,0}(\tau) + 2 A p_{3,0}(\tau) + C p_{1,1}(\tau) - C p_{2,2}(\tau)), \]

\[ p'_{3,3}(\tau) = -4(A p_{0,3}(\tau) + B p_{3,0}(\tau)) - 4(A + B) p_{3,3}(\tau) + 4 C (p_{1,1}(\tau) - p_{2,2}(\tau)). \]  

(35)
From above equations, we know that the X-shape evolution state is obtained. With Eq. (14), we can get concurrence

\[ E = \max[0, \frac{1}{2}|p_{1,1}(\tau) + p_{2,2}(\tau)| - T_1, \frac{1}{2}|p_{1,1}(\tau) - p_{2,2}(\tau)| - T_2], \]

with \( T_1 = \frac{1}{2}\sqrt{(1 + p_{0,3}(\tau) - p_{3,0}(\tau) - p_{3,3}(\tau))(1 - p_{0,3}(\tau) + p_{3,0}(\tau) - p_{3,3}(\tau))} \)
and \( T_2 = \frac{2}{\sqrt{2}}\sqrt{(1 - p_{0,3}(\tau) - p_{3,0}(\tau) + p_{3,3}(\tau))(1 + p_{0,3}(\tau)p_{3,0}(\tau) + p_{3,3}(\tau))}. \)

Firstly, let us examine the case of equilibrium state that can be obtained by setting change rate of Eq. (35) to be zero provided that two atoms are not very close to each other and not very near the string. The linear equations can be solved and get \( p_{1,1}(\infty) = p_{2,2}(\infty) = 0, p_{0,3}(\infty) = p_{3,0}(\infty) = -1, p_{3,3}(\infty) = 1. \) From these solutions, we know that the state after a long-time evolution is a separable state \( |\rangle \) which is independent of two-atom distance, atom–string distance, atom polarization and deficit angle parameter. This implies that two atoms cannot get entangled for a long evolution time whatever the initial state of two atom is. In the following discussion, we choose the Werner state \( p|\phi\rangle\langle\phi| + (1 - p)\frac{1}{2} \) \( |\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \) [26] as the atoms initial state. Werner state is a X-shape entangled state when \( p > \frac{1}{2}. \) Here, we set \( p = \frac{2}{3}. \)

When \( \nu = 1, \) the metric (15) describes a flat spacetime absent of the cosmic string. According to the following properties of Bessel function [25]

\[ \sum_{n=-\infty}^{\infty} J_{|n|}^2(x) = 1, \quad \sum_{n=-\infty}^{\infty} J_{|n|+1}(x)J_{|n|-1}(x) = 0, \]

we have

\[ f_{11}^{(11)} = f_{22}^{(11)} = f_{33}^{(11)} = 1, \]
\[ g_{11}^{(12)} = g_{22}^{(12)} = \frac{3[(L^2\omega^2 - 1)\sin(L\omega) + L\omega\cos(L\omega)]}{2L^3\omega^3}, \]
\[ g_{33}^{(12)} = \frac{3[\sin(L\omega) - L\omega\cos(L\omega)]}{L^3\omega^3}, \]
\[ g_{13}^{(12)} = 0. \]

From above equations, we know that the results of Minkowski spacetime are recovered [27].

From Fig. 1, it can be seen that in both spacetimes, there exists time gap where entanglement vanishes and entanglement lifetime becomes longer when atomic separation becomes smaller. However, compared with the free Minkowski spacetime, entanglement in cosmic string spacetime decays more slowly in the beginning and revives later. Entanglement lifetime in cosmic string spacetime is longer than Minkowski spacetime when two-atom distances in both spacetimes are the same.

From Fig. 2a, it can be found that bigger value of deficit angle parameter and entanglement lifetime becomes shorter, and entanglement decreases more rapidly at
Fig. 1 Entanglement as function of $\Gamma \tau$ with different atomic separations for $\omega r = 1/2$, and two atoms in cosmic string spacetime and Minkowski spacetime are polarized along isotropic direction ($\hat{d}^{(1)} = \hat{d}^{(2)} = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$).

Fig. 2 Entanglement as function of $\Gamma \tau$ with different values of deficit angle parameter for $\omega L = 1/2$ and $\omega r = 1/2$, a two atoms are polarized along isotropic direction ($\hat{d}^{(1)} = \hat{d}^{(2)} = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$), b an atom is polarized along radial direction ($\hat{d}^{(1)} = (1, 0, 0)$), and the other atom is polarized along tangential direction ($\hat{d}^{(2)} = (0, 1, 0)$).

first, but sudden birth happens earlier and the gap becomes narrower. From Fig. 2b, we can see when an atom is polarized along the positive radial direction and the other atom along the positive tangential direction, entanglement has different behaviors. Entanglement shows a monotone decrease with evolution time, and entanglement and its lifetime do not decrease monotonously with increasing value of deficit angle parameter.

When $\nu > 1$ and $r \to 0$ (two atoms lay on the string), according to the property of Bessel function

$$ J_x(0) = \begin{cases} 0 & x > 0 \\ 1 & x = 0 \end{cases}, \tag{39} $$

one obtains
Fig. 3 Entanglement as function of $\Gamma \tau$ with different atom–string distances for $\omega L = 1/2$ and $\nu = 2$, a two atoms are polarized along isotropic direction ($\hat{d}^{(1)} = \hat{d}^{(2)} = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$), b an atom is polarized along radial direction ($\hat{d}^{(1)} = (1, 0, 0)$), and the other atom is polarized along tangential direction ($\hat{d}^{(2)} = (0, 1, 0)$)

$$f_{11}^{(11)} = f_{22}^{(11)} = 0, \quad f_{33}^{(11)} = \nu,$$

$$g_{11}^{(12)} = g_{22}^{(12)} = 0, \quad g_{33}^{(12)} = 3\nu[\sin(L\omega) - L\omega \cos(L\omega)]/L^3\omega^3,$$

$$g_{13}^{(12)} = 0.$$  \hfill (40)

According to above equations, we get

$$A = \frac{\Gamma \nu}{4} |\hat{d}_3^{(1)}|^2,$$

$$B = \frac{\Gamma \nu}{4} |\hat{d}_3^{(2)}|^2,$$

$$C = \frac{3\Gamma \nu[\sin(L\omega) - L\omega \cos(L\omega)]}{4L^3\omega^3} \hat{d}_3^{(1)} \hat{d}_3^{(2)}. \hfill (41)$$

From above equations, we know that when there are not atoms polarized along axial direction, the system does not affect by the electromagnetic fluctuation and the system seems like a closed system.

Figure 3 shows that when two atoms relatively approach the string, entanglement decays more slowly and maintains more longer time; as Eq. (41) reveals, entanglement is freezing when atoms are not polarized along axial direction (see Fig. 3b). When atom–string distance is relatively large (see Fig. 4), entanglement exhibits oscillations as atom–string distance varies. Particularly when atoms are far away from the string ($r \to \infty$), Eq. (38) can be exactly obtained, which means the results in Minkowski are restored because the atoms are not affected by the cosmic string for this case. Entanglement has different behaviors for different polarization directions.

From Fig. 5, it can be observed that when two-atom distance becomes smaller, entanglement declines more slowly in the beginning, the dark period of entanglement becomes shorter and entanglement can persist in longer time; particularly, when two-atom distance is relatively big, entanglement revival no longer occurs. In this sense, we
can say small atomic separation can protect entanglement. When an atom is polarized along radial direction and the other atom is polarized along tangential direction, from Eq. (32), we know that entanglement evolution has nothing to do with interatomic distance.

4 Conclusion

In this work, we have analyzed the entanglement behaviors for two atoms weakly coupled with a bath of fluctuating electromagnetic field in the cosmic string spacetime. We find that entanglement behaviors depend on polarization direction of the atoms, interatomic separation, atom–string distance and value of deficit angle parameter. It is shown that the asymptotic equilibrium state is a pure separable state, which indicates entanglement can not persevere in for a long evolution time. When deficit angle parameter $\nu = 1$ and the atoms are far away from the string, the results reduce to that of Minkowski spacetime. For different values of deficit angle parameter, entanglement has different behaviors depending on various parameters. When atoms get
close to the string, entanglement can be efficiently protected; in particular, when atoms lie on the string and are not polarized along axial direction, entanglement is frozen, which is similar to the boundary effect [27]. When atom–string distance is relatively large, entanglement presents oscillating behaviors as atom–string distance changes. When two-atom separation is small, entanglement has better performance. Various parameters give us greater freedom to steer entanglement behaviors, and this would be profitable for us detecting the structure and property of the cosmic string spacetime and distinguishing different spacetimes.

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References

1. Nilsen, M.A., Chuang, I.L.: Quantum Computation and Quantum Information. Cambridge University Press, Cambridge, UK (2000)
2. Horodecki, R., Horodecki, P., Horodecki, M., Horodecki, K.: Quantum entanglement. Rev. Mod. Phys. 81, 865 (2009)
3. Hu, J.W., Yu, H.W.: Entanglement dynamics for uniformly accelerated two-level atoms. Phys. Rev. A 91, 012327 (2015)
4. Yang, Y.Q., Hu, J.W., Yu, H.W.: Entanglement dynamics for uniformly accelerated two-level atoms coupled with electromagnetic vacuum fluctuations. Phys. Rev. A 94, 032337 (2016)
5. Hu, J.W., Yu, H.W.: Quantum entanglement generation in de Sitter spacetime. Phys. Rev. D 88, 104003 (2013)
6. Huang, Z.M., Tian, Z.H.: Dynamics of quantum entanglement in de Sitter spacetime and thermal Minkowski spacetime. Nucl. Phys. B 923, 458 (2017)
7. Copeland, E.J., Pogosian, L., Vachaspati, T.: Seeking string theory in the cosmos. Class. Quant. Grav. 28, 204009 (2011)
8. Hindmarsh, M.: Signals of inflationary models with cosmic strings. Prog. Theor. Phys. Suppl. 190, 197 (2011)
9. Vilenkin, A.: Cosmic strings as gravitational lenses. Astrophys. J. 282, L51 (1984)
10. Gott, J.R.: Gravitational lensing effects of vacuum strings-exact solutions. Astrophys. J. 288, 422 (1985)
11. Ford, L.H., Vilenkin, A.: A gravitational analogue of the Aharonov-Bohm effect. J. Phys. A Math. Gen. 14, 2353 (1981)
12. Bezerra, V.B.: Gravitational analogs of the Aharonov-Bohm effect. J. Math. Phys. 30, 2895 (1989)
13. Cai, H., Yu, H., Zhou, W.: Spontaneous excitation of a static atom in a thermal bath in cosmic string spacetime. Phys. Rev. D 92, 084062 (2015)
14. Zhou, W., Yu, H.: Spontaneous excitation of a uniformly accelerated atom in the cosmic string spacetime. Phys. Rev. D 93, 084028 (2016)
15. Cai, H., Ren, Z.: Radiative processes of two entangled atoms in cosmic string spacetime. Class. Quantum Grav. 35, 025016 (2018)
16. Bakke, K., Nascimento, J.R., Furtado, C.: Geometric phase for a neutral particle in the presence of a topological defect. Phys. Rev. D 78, 064012 (2008)
17. Cai, H., Ren, Z.: Geometric phase for a static two-level atom in cosmic string spacetime. Class. Quantum Grav. 35, 105014 (2018)
18. Bezerra de Mello, E.R., Saharian, A.A., Kh Grigoryan, A.: Casimir effect for parallel metallic plates in cosmic string spacetime. J. Phys. A Math. Theor. 45, 374011 (2012)
19. Huang, Z.M.: Quantum entanglement of nontrivial spacetime topology. Eur. Phys. J. C 80, 131 (2020)
20. He, P., et al.: Entanglement dynamics for static two-level atoms in cosmic string spacetime. Eur. Phys. J. C 80, 134 (2020)
21. Gorini, V., Kossakowski, A., Sudarshan, E.C.G.: Completely positive dynamical semigroups of N-level systems. J. Math. Phys. 17, 821 (1976)
22. Lindblad, G.: On the generators of quantum dynamical semigroups. Commun. Math. Phys. 48, 119 (1976)
23. Breuer, H.-P., Petruccione, F.: The Theory of Open Quantum Systems. Oxford University Press, Oxford (2002)
24. Wootters, W.K.: Entanglement of formation of an arbitrary state of two qubits. Phys. Rev. Lett. 80, 2245 (1998)
25. Andrews, G.E., Askey, R., Roy, R.: Special Functions. Cambridge University Press, Cambridge (1999)
26. Werner, R.F.: Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model. Phys. Rev. A 40, 4277 (1989)
27. Huang, Z.M.: Behaviors of quantum correlation for atoms coupled with fluctuating electromagnetic field with a perfectly reflecting boundary. Quantum Inf. Process. 18, 149 (2019)

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