On the Transformation $T_{+m}$ Due Gray and Clark

Kacem Belghaba
Department of Mathematics, Faculty of Sciences, University of Oran-Senia, P.O Box 1524, Algeria

Abstract: We determine the values of the integer $m$ for which the parametric transformation $T_{+m}$ due Gray and Clark is well conditioned. This process of acceleration being quasilinear transformation, we use an adequate definition of the condition numbers that apply to the real sequences of linear convergence. The results obtained for this set are enough meaning.

Key words: Conditions numbers, linear convergence, logarithmic convergence, punctual conditioning, asymptotic conditioning.

INTRODUCTION

Nonlinear sequences transformations are generally used for solve the extrapolation of the limit or to accelerate slowly convergent sequences. One can also use them to sum the divergent series\cite{1,11}. But most of the time, the implementation of these transformations provides us with sequences whose terms are sullied with errors due to the limits imposed by the capacities of the computer on the one hand and by the truncation errors on the other hand.

To these errors, we can add the conditioning of the transformation which is used. It is thus significant to make a study of conditioning. We are interested in this work in the conditioning of the transformation $T_{+m}$ of Gray and Clark, which generalizes the famous $\Delta^2$-Aitken's process.

For that we use the definition\cite{4} of the number of conditioning when the transformation considered is quasi-linear. Thereafter, we will discuss the values of the integer $m$ for which we obtain the best conditioned transformation.

The first generalization of Aitken's $\Delta^2$ process is $T_{+m}$ transformation due to Gray\cite{8} and Clark. Let $m$ be a strictly positive integer and consider the sequence

$$T_{+m} = T_{+m}(S_n) = S_n - \frac{\Delta S_n}{\Delta S_{+m} - \Delta S_n} \cdot (S_{+m} - S_n)$$

where $(S_n)_{n \geq 0}$ is a sequence or an infinite series.

For $m = 1$, Aitken's $\Delta^2$ process is recovered. Acceleration results were given by the previous authors and by Streit\cite{14}. Let $x = (x_k)_k$ be a convergent sequence of a limit $x^*$ and let $\Phi$ be a mapping defined on $\mathbb{R}^{p+1}$, where $p \geq 2$. In the paper, we shall consider $\Phi$ as a function of $(p+1)$-variables $u_1, u_2, \ldots, u_p$.

The sequence transformation $\Phi : x \rightarrow y$ is defined by

$$y_i = \Phi(x_0, x_1, \ldots, x_{i+p}) = \Phi_i$$

Let us assume that $\Phi$ is function of class $C^1$ on an open subset $A$ of $\mathbb{R}^{p+1}$.

Definition 1: The transformation $\Phi$ is quasilinear if the following properties are satisfied

(a) A property of translativity:

$$\forall (u_0, u_1, \ldots, u_p) \in A \forall b \in \mathbb{R}^p :$$

$$\Phi(u_0 + b, u_1 + b, \ldots, u_p + b) = \Phi(u_0, u_1, \ldots, u_p) + b$$

(b) A property of homogeneity:

$$\forall (u_0, u_1, \ldots, u_p) \in A \forall a \in \mathbb{R}^p :$$

$$\Phi(a u_0, a u_1, \ldots, a u_p) = a \Phi(u_0, u_1, \ldots, u_p)$$

Most extrapolation processes are quasilinear transformations such as Aitken's $\Delta^2$ process\cite{14}, Wynn's epsilon-algorithm\cite{5,12} and more generally the E-algorithm\cite{7} (under certain assumption), the $\Theta$-algorithm of Brezinski\cite{6} and many other processes.

Now, we consider the mapping $\Phi$ defined on $\mathbb{R}^{m+2}$ by

Corresponding Author: Kacem Belghaba, Department of Mathematics, Faculty of Sciences, University Oran -Senia, P.B. 1524, Algeria
If we apply \( \Phi_m \) to 
\[
(S_n, S_{n+1}, \ldots, S_{n+m-1}, S_{n+m})
\]
we will obtain 
\[
\Phi_m (S_n, S_{n+1}, \ldots, S_{n+m-1}, S_{n+m}) = S_n - \frac{S_{n+1} - S_n}{S_{n+m} - S_{n+1}} \cdot (S_{n+m} - S_n) = S_n - \frac{S_{n+1} - S_n}{S_{n+m} - S_n} \cdot (S_{n+m} - S_n) = T_m(n)
\]

his map \( \Phi_m \) is associated with the \( T_m \) transformation.

**Proposition 1:** The transformation \( T_m \) of Gray and Clark is a quasilinear transformation.

**Proof:** For one justification, one can refer to the proof given in [3].

In this work we are interested only in stationary processes defined in the sense of Ortega [9] and Rheinboldt. Let \( \Phi \) be the transformation defined before by

\[
y_i = \Phi(x_1, x_{i+1}, \ldots, x_{i+p}) \quad i = 0, 1, 2, \ldots
\]

We are concerned with the way by which \( y_i \) is changed when we perturb the numbers \( x_1, x_{i+1}, \ldots, x_{i+p} \). We denote by \( \delta x_q \) the perturbation on the term \( x_q \), \( x \in \mathbb{N} \). The resulting perturbation on the term \( y_i \) is given by

\[
\delta y_i = \delta x_1 \frac{\partial \Phi}{\partial x_1} (\xi_i) + \delta x_{i+1} \frac{\partial \Phi}{\partial x_{i+1}} (\xi_i) + \ldots + \delta x_{i+p} \frac{\partial \Phi}{\partial x_{i+p}} (\xi_i)
\]

where

\[
\xi_i = (x_i + \theta_1 \delta x_i, x_{i+1}, \ldots, x_{i+p})
\]

\( 0 < \theta_j < 1, \ j = 0, 1, \ldots \)

It follows that

\[
\left| \delta y_i \right| \leq \max \left| \delta x_i \right| \left| \frac{\partial \Phi}{\partial x_i} \right| \sum_{j=0}^{p} \left| \frac{\partial \Phi}{\partial x_j} \right| (\xi_i)
\]

Setting \( X_i = (x_1, x_{i+1}, \ldots, x_{i+p}) \in \mathbb{R}^{p+1} \), \( i = 0, 1, 2, \ldots \), and evaluating the partial derivatives at the point \( X_i \) rather than \( \xi_i \), we can give the following definition.

**Definition 2:** The punctual condition number \( i \)th step of \( \Phi \) for a sequence \( x \) is given by

\[
C_i(\Phi, x) = \sum_{j=0}^{p} \left| \frac{\partial \Phi}{\partial x_j} \right| (X_i) \quad i = 0, 1, 2, \ldots
\]

This number is in fact the factor of amplification of the errors made on the term \( x_i \).

In the case where the sequence of the punctual condition numbers converges, then we can define the asymptotic conditioning of the transformation \( \Phi \), applied to the sequence \( x \), by

\[
C_x(\Phi, x) = \lim_{i \to \infty} C_i(\Phi, x)
\]

Denoting the transformation defined on \( \mathbb{R}^{m+2} \) by \( \Phi_m \), we have

\[
\Phi_m (u_0, u_1, \ldots, u_m, u_{m+1}) = u_0 - \frac{u_{m+1} - u_0}{u_{m+1} - u_1 + u_0} (u_m - u_0)
\]

For \( m = 1 \), we find

\[
\Phi_1(u_0, u_1) = \frac{u_0 u_1 - u_1^2}{u_0 - 2u_1 + u_0}
\]

This is the transformation that defines the Aitken’s \( \Delta^2 \) process.

For \( m = 2 \), the transformation \( T_2 \) is

\[
y_{i+2} = \frac{S_{n+2} - S_{n+1} \Delta S_n}{\Delta S_{n+2} - \Delta S_n}
\]

and the \( \Phi_2 \) associated transformation is

\[
\Phi_2(u_0, u_1, u_2, u_3) = \frac{u_2 (u_3 - u_2) - u_3 (u_2 - u_1) - u_0 u_2 + u_1 u_3}{u_3 - u_2 + u_1 - u_0}
\]

**Calculus of partial derivatives**

\[
\frac{\partial \Phi_m}{\partial u_0} (u_0, u_1, u_2, u_3) = \left( \frac{u_3 - u_1}{u_1 - u_0} \right)^2
\]

\[
\frac{\partial \Phi_m}{\partial u_1} (u_0, u_1, u_2, u_3) = \left( \frac{u_3 - u_1}{u_1 - u_0} \right)^2
\]

\[
\frac{\partial \Phi_m}{\partial u_2} (u_0, u_1, u_2, u_3) = \left( \frac{u_3 - u_1}{u_1 - u_0} \right)^2
\]

\[
\frac{\partial \Phi_m}{\partial u_3} (u_0, u_1, u_2, u_3) = \left( \frac{u_3 - u_1}{u_1 - u_0} \right)^2
\]
It is easy to verify that
\[ \sum_{i=0}^{n} \partial_{\Phi_i}(u_0, u_1, u_2, u_3) = 1. \]

This is a general result. The quasilinear transformations verify this property.

**Definition 3:** We call LIN the set of convergent sequences that is the set of convergent sequences \( x = (x_n)_n \) such that \( \exists \rho \neq 0, \rho \in ]-1, 1[ \) such that
\[ \lim_{n \to \infty} \frac{x_{n+1}-x^*}{x_n-x^*} = \rho \]

If \( \rho \in ]-1, 0[ \), \( x \) is an alternating sequence. We denote this set by LIN^−.

If \( x \in ]0, 1[ \), \( x \) is a monotonic sequence and we denote this set by LIN^+.

**Proposition 2:** If \( x = (x_n)_n \) is a sequence of the set LIN, then
\[ \lim_{n \to \infty} \frac{x_{n+1}-x^*}{x_n-x^*} = \rho \Rightarrow \lim_{n \to \infty} \frac{x_{n+2}-x_{n+1}}{x_{n+1}-x_n} = \rho \]

**Proof**
\[
\begin{align*}
\frac{x_{n+1}-x^*}{x_n-x^*} &= \frac{(x_{n+2}-x^*)/(x_{n+1}-x^*) - 1}{(x_n-x^*)/(x_{n-1}-x^*) - 1} \\
&= \lim_{n \to \infty} \frac{x_{n+1}-x^*}{x_n-x^*} \lim_{n \to \infty} \frac{(x_{n+2}-x^*)/(x_{n+1}-x^*) - 1}{(x_{n+1}-x^*)/(x_n-x^*) - 1} \\
&= \rho \left( \frac{\rho-1}{\rho-1} \right) = \rho
\end{align*}
\]

**Definition 4:** Let \( x = (x_n) \) be a convergent sequence of a limit \( x^* \). Then a sequence of transformation \( T \) that
\[ y_n = T(x_n) \]
is said to be regular if the transformed sequence is converging to the same limit \( x^* \).

**Proposition 3:** On the set LIN, the quasilinear \( T_m \) of Gray and Clark is a regular transformation.
We apply the transformation \( \Phi_m \) to terms \( x_n, x_{n+1}, \ldots, x_{n+m}, x_{n+m+1} \) of the sequence \( (x_n)_n \in \text{LIN} \). The partial derivatives evaluated at the points \( x_n, x_{n+1}, \ldots, x_{n+m}, x_{n+m+1} \) are respectively

\[
\frac{\partial \Phi_m}{\partial u_0}(x_n, \ldots, x_{n+m+1}) = \frac{(x_n - x_{n+1})}{(x_{n+1} - x_n)} \frac{(x_{n+m+1} - x_{n+m})}{(x_{n+m} - x_{n+m+1})} \frac{(x_{n+m+1} - x_n)}{(x_n - x_{n+1})}
\]

\[
\frac{\partial \Phi_m}{\partial u_i}(x_n, \ldots, x_{n+m+1}) = \frac{(x_{n+m+1} - x_{n+m})}{(x_{n+m} - x_{n+m+1})} \frac{(x_{n+m+1} - x_n)}{(x_n - x_{n+1})}
\]

\[
\frac{\partial \Phi_m}{\partial u_2}(x_n, \ldots, x_{n+m+1}) = \frac{0}{0} \quad \frac{\partial \Phi_m}{\partial u_{m+1}}(x_n, \ldots, x_{n+m+1}) = 0
\]

Using \( \rho_n = \frac{x_{n+2} - x_{n+1}}{x_{n+1} - x_n} \), we can formulate theses partial derivatives as follows

\[
\frac{\partial \Phi_m}{\partial u_0}(x_n, \ldots, x_{n+m+1}) = \frac{\prod_{i=1}^m \rho_{n+i}}{\prod_{i=1}^m \rho_{n+i+1}} \frac{\sum_{j=0}^m \prod_{i=1}^m \rho_{n+i+1}}{\prod_{i=1}^m \rho_{n+i+1}} \frac{1}{\prod_{i=1}^m \rho_{n+i+1}}
\]

\[
\frac{\partial \Phi_m}{\partial u_i}(x_n, \ldots, x_{n+m+1}) = -\frac{\prod_{i=1}^m \rho_{n+i+1}}{\prod_{i=1}^m \rho_{n+i+1}} \frac{\sum_{j=0}^m \prod_{i=1}^m \rho_{n+i+1}}{\prod_{i=1}^m \rho_{n+i+1}} \frac{1}{\prod_{i=1}^m \rho_{n+i+1}}
\]

\[
\frac{\partial \Phi_m}{\partial u_2}(x_n, \ldots, x_{n+m+1}) = 0 \quad \frac{\partial \Phi_m}{\partial u_{m+1}}(x_n, \ldots, x_{n+m+1}) = 0
\]

The punctual conditioning of the sequence \( x = (x_n)_n \) is

\[
C_n(\Phi_m, x) = \sum_{i=0}^m \left| \frac{\partial \Phi_m}{\partial u_i}(x_n, \ldots, x_{n+m+1}) \right| \quad n = 0, 1, 2, \ldots
\]

The asymptotic condition numbers is well defined since the denominator is different from zero. Since \( \rho = \lim_{n \to \infty} \rho_n \) it follows that

\[
\lim_{n \to \infty} \frac{\partial \Phi_m}{\partial u_0}(x_n, \ldots, x_{n+m+1}) = \rho^{m+1} \left(1 + \rho + \rho^2 + \ldots + \rho^m \right)
\]

\[
\lim_{n \to \infty} \frac{\partial \Phi_m}{\partial u_i}(x_n, \ldots, x_{n+m+1}) = \rho^m \left(1 + \rho + \rho^2 + \ldots + \rho^m \right)
\]

\[
\lim_{n \to \infty} \frac{\partial \Phi_m}{\partial u_2}(x_n, \ldots, x_{n+m+1}) = 0 \quad \lim_{n \to \infty} \frac{\partial \Phi_m}{\partial u_{m+1}}(x_n, \ldots, x_{n+m+1}) = 0
\]

\[
\lim_{n \to \infty} \frac{\partial \Phi_m}{\partial u_0}(x_n, \ldots, x_{n+m+1}) = -\rho \left(1 + \rho + \rho^2 + \ldots + \rho^m \right)
\]

\[
\lim_{n \to \infty} \frac{\partial \Phi_m}{\partial u_i}(x_n, \ldots, x_{n+m+1}) = 0 \quad \lim_{n \to \infty} \frac{\partial \Phi_m}{\partial u_{m+1}}(x_n, \ldots, x_{n+m+1}) = 0
\]

\[
\lim_{n \to \infty} \frac{\partial \Phi_m}{\partial u_2}(x_n, \ldots, x_{n+m+1}) = \rho^m \left(1 + \rho + \rho^2 + \ldots + \rho^m \right)
\]

\[
\lim_{n \to \infty} \frac{\partial \Phi_m}{\partial u_{m+1}}(x_n, \ldots, x_{n+m+1}) = \rho^m \left(1 + \rho + \rho^2 + \ldots + \rho^m \right)
\]

**Proposition 4:** According to the sign of \( \rho \) and the parity of \( m \), the signs of the partial derivatives are given below

\[
m \text{ odd; } \rho < 0 \quad \frac{\partial \Phi_m}{\partial u_0} > 0 \quad \frac{\partial \Phi_m}{\partial u_i} > 0 \quad \frac{\partial \Phi_m}{\partial u_{m+1}} > 0 \quad \frac{\partial \Phi_m}{\partial u_{m+1}} > 0
\]

\[
m \text{ even; } \rho > 0 \quad \frac{\partial \Phi_m}{\partial u_0} > 0 \quad \frac{\partial \Phi_m}{\partial u_i} < 0 \quad \frac{\partial \Phi_m}{\partial u_{m+1}} < 0 \quad \frac{\partial \Phi_m}{\partial u_{m+1}} > 0
\]
Concluding Results: According to the sign of $\rho$ and the parity of $m$, we obtain the results:

- $\rho < 0$, $m$ odd, $C_n(\Phi_m, x) = 1$ (well conditioned)
- $\rho < 0$, $m$ even, $C_n(\Phi_m, x) = \frac{1 + \rho^m}{1 - \rho^m}$
- $\rho > 0$, $m$ odd or even, $C_n(\Phi_m, x) = \frac{(1 + \rho)(1 + \rho^m)}{(1 - \rho)(1 - \rho^m)}$

Numerical applications

In the first table we give the initial sequence's terms computed with Digits := 30, Digits := 9 and Digits := 6.

If we consider the calculus with Digits := 30 as accurate, then that done with Digits := 9 or := 6 will be done with roundoff error which will be a source of a small error. This error will be propagated in the terms of the transformed sequence. The values of transformed sequence are given in the second table. In some cases, we have chosen to study the transformation $T_{m+1}$ for the values $m = 1$

Example: Sequences of linear convergence

$S_n = \sum_{k=0}^{n} (-0.98)^k$, $S = \sum_{k=0}^{\infty} (-0.98)^k$

The terms of the sequence $S = (S_n)$ approach to the limit $S$. Relativement à Digits := 30, the terms $S_n$ are calculated for Digits := 9, with eight exact digits and with five exact digits for digits := 6. We remark the roundoff errors generated to ninth digit of initial sequence terms $S_n$ are reproduced also to ninth digit of transformed sequence $W_n = T_{m+1}$. This prove the good conditioning of the transformation $T_{m+1}$. This strengthens the theoretical results that we have obtained with $\rho = (-0.98) < 0$ and $m = 1$ (odd).

CONCLUSION

In this study and within sight of the results obtained, we can say that the transformation of Gray and Clark is well conditioned for $m$ odd on the set of sequences to alternating convergence whereas for the other cases the asymptotic conditioning is expressed according to the integer $m$. We can plot the graph of the asymptotic conditioning.

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