Towards a Discrete Spacetime

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Abstract

A formalism is proposed to generate (the first step of) a discrete spacetime: spacetime with an inbuilt length scale. We follow the celebrated Landau theory of liquid - solid phase transition induced by Spontaneous Symmetry Breaking by a condensate whose Fourier transform has support at a non-zero momentum. The latter requirement is essential for breaking the translation symmetry. This, in turn, compels us to generalize Einstein action to higher derivative terms.

Is there a fundamental way to introduce a (short distance) length scale, eg. the Planck scale, in Einstein’s General Relativity (GR)? It can then serve as an effective theory of Quantum Gravity. So far the attempts have been along the lines of Non-Commutative (NC) geometry framework \cite{1} where one exploits Seiberg Witten map \cite{2} to incorporate NC corrections on GR. In the present paper we follow a completely different route taking cue from Condensed Matter Physics: the Landau theory of liquid-solid phase transition \cite{3}.

In a series of pioneering works by Alexander and McTague \cite{4} it was explicitly demonstrated how one is able to construct a discrete lattice, (where translation and rotation symmetries are lost), from a liquid phase, (where the symmetries are intact), through Spontaneous Symmetry Breaking (SSB) with a crucial additional requirement: the (Fourier transform of) non-zero VEV of condensate that minimizes the free energy must have support
at a non-vanishing momentum. The difference between the (inhomogeneous) density for solid and (constant) density of liquid acts as the order parameter. The free energy is expanded around the higher symmetry liquid phase and it should be mentioned that to form a proper crystal lattice one needs terms, at least of third and fourth order in the order parameter, in the expansion. This idea was later applied by Rabinovici et.al. [5] in string theory compactification. The beauty of the formalism lies in its universality and independence of details of the specific model in question. More explicitly [4, 5, 3], consider the density difference, $\rho(\vec{x}) = \rho_{\text{solid}}(\vec{x}) - \rho_{\text{liquid}}$, between (lower symmetry) solid phase $\rho_{\text{solid}}(\vec{x})$ and symmetric liquid phase with constant $\rho_{\text{liquid}}$ and take its Fourier decomposition, $\rho(\vec{x}) = \sum \rho(\vec{p}) \exp(i\vec{p}.\vec{x}) + h.c.$. Now, from the first non-trivial quadratic term in the effective action $L \sim \int d\vec{p} d\vec{p}_1 \rho(p_1) \rho(p_2) A(|\vec{p}_1|^2) \delta(\vec{p}_1 - \vec{p}_2)$, one needs to ascertain the value of $|\vec{p}_1|$ that will minimize $A(|\vec{p}_1|^2)$. If it is different from zero, $\rho(\vec{x})$ breaks translation invariance, but still rotation invariance is intact since $|\vec{p}_1|^2$ is involved. Reality of $\rho(\vec{x})$ demands $\rho(\vec{p}) = \rho(-\vec{p})$ and one gets $\rho(\vec{x}) \sim \cos(\vec{q}.\vec{x})$. The third and fourth order terms in $L$ introduces more structure to $\rho(\vec{x})$ that breaks rotational invariance as well [4, 5]. The order parameter can have tensorial structure as well. In our case where the “solidification” of spacetime is involved, the role of order parameter will be played by the linearized part of the metric $g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$.

Our longterm goal is to generate a spacetime lattice through SSB in the above way. However, in the present article our target is more modest: as a first step towards this objective, we intend to introduce a short distance scale ($\sim$ Planck length) in the GR action that will break the translation invariance, without affecting the rotational symmetry. Indeed, the extension of GR to NC spacetimes [1] also aims to achieve that but, compared to this ad-hoc procedure, our approach is much more basic and intuitive. We introduce the deformation directly in the metric fluctuation over the condensate, that plays the role of a (tensorial) order parameter in the continuum-discrete spacetime phase transition. At the same time it is very interesting and encouraging that our proposed form of the metric bears important
similarities with the NC extended metric of [1].

There is an elegant way of interpreting the masslessness of photon: they are the Goldstone bosons induced by SSB of diffeomorphism invariance [6]. We, in particular, extend the work of Kraus and Tomboulis [7] by taking into account higher derivative terms in the action, $R^\mu\nu R_\mu\nu$ and $(\eta^{\mu\nu}R_{\mu\nu})^2$. Apart from the conventional $(\eta^{\mu\nu}R_{\mu\nu})$-term, the above are needed to ensure that the kinetic term is minimized for a non-zero momentum. All the subtle arguments of [7], regarding the degrees of freedom count after SSB that yields the massless physical graviton (to lowest order) remains applicable in our model as well since the higher order terms respect the same symmetries as GR [8]. The only possible ambiguity can arise from the fact that the higher derivative massive spin two states can bring in negative energy states [8]. However, once again, falling back to the argument of [7], these massive modes possibly do not affect the low energy physics, at least for length scales large compared to Planck length [8] (the parameter introduced by us in the present work).

We start with the higher derivative lagrangian $L$

$$L = \int d^4x \mathcal{L} = \int d^4x [\sqrt{g} \left( R + \alpha R^2 + \beta R^\mu_\nu R^\mu_\nu \right) + V]$$ (1)

where $g_{\mu\nu}, R_{\mu\nu}, R$ are respectively the metric tensor, Ricci tensor and Ricci scalar. $\alpha$ and $\beta$ are numerical parameters. $V$ is some suitable symmetry breaking potential that can appear e.g. from effective one-loop matter coupling [7].

Throughout we will work in weak field approximation with $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$, $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. Considering terms up to second order in $h$ (i.e. neglecting terms of $O(h^3)$ and higher) $\mathcal{L}$ in (1) reduces to

$$\mathcal{L} = \frac{1}{4} h^\square h - \frac{1}{4} h^\rho_\sigma \square h^\rho_\sigma + \frac{1}{2} h^{\mu\rho} \partial_\mu \partial_\rho h^\nu_\sigma - \frac{1}{2} h^{\mu\nu} \partial_\mu \partial_\nu h - \alpha h^\square h^2 + 2\alpha h^{\lambda\mu} \partial_\lambda \partial_\mu \partial_\nu \partial_\rho \partial_\sigma h^\rho_\sigma - \frac{1}{2} h^{\rho\sigma} \partial_\rho \partial_\sigma h^\mu_\nu + \beta h^{\mu\sigma} \partial_\mu \partial_\nu \partial_\sigma \partial_\lambda \partial_\mu \partial_\nu \partial_\sigma h^\rho_\sigma - \frac{1}{2} h^{\mu\rho} \partial_\rho \partial_\sigma h^\lambda_\nu - \frac{1}{2} h^{\mu\nu} \partial_\lambda \partial_\mu \partial_\nu \partial_\sigma h^\rho_\sigma + V.$$ (2)

We have used the notation $h = \eta^{\mu\nu} h_{\mu\nu}$, $\square = \partial_\mu \partial^\mu$. Furthermore, if we put $\alpha = -\frac{\beta}{2}$ in (1), it
takes the following form

\begin{equation}
\mathcal{L} = \frac{1}{4} h \square (1 - 2\alpha \Box) h - \frac{1}{2} h^\mu\nu \partial_\mu \partial_\nu (1 - 2\alpha \Box) h - \frac{1}{4} h^\rho\sigma \square (1 - 2\alpha \Box) h_{\rho\sigma} + \frac{1}{2} h^\rho\sigma \partial_\mu \partial_\rho (1 - 2\alpha \Box) h^\mu + V.
\end{equation}

(3)

The structure is essentially same as the Einstein gravity only with the operator \( \sim -2\alpha \Box \) appended.

One can still use the harmonic gauge \( \partial_\mu h^\mu = \frac{1}{2} \partial^\rho h \), to get

\begin{equation}
\mathcal{L} = \frac{1}{8} h (1 - 2\alpha \Box) \Box h - \frac{1}{4} h^\rho\sigma (1 - 2\alpha \Box) \Box h_{\rho\sigma} + V.
\end{equation}

(4)

But we will not consider the last form [7].

To consider SSB effects, let us consider fluctuations \( \tilde{h}_{\mu\nu} \) above the condensate:

\begin{equation}
g_{\mu\nu} = \eta_{\mu\nu} + \tilde{h}_{\mu\nu} = \eta_{\mu\nu} + <h_{\mu\nu}> \cos(q.x) + \tilde{h}_{\mu\nu}(x),
\end{equation}

(5)

where \(< >\) represents vacuum expectation value and \(q_\mu\) is a non-zero vector with \(q^2 = -1/(4\alpha)\). This is the crucial extension of [7] that comes from minimization of the kinetic energy part in (4), thanks to the higher derivative terms. The specific form, \(\cos(q.x)\) comes from symmetry arguments while treating \(\tilde{h}_{\mu\nu}\) as the order parameter (as explained briefly, for details see [4, 5]). The normalization is fixed from minimizing the kinetic part of the action, \(-k^2(1 + 2\alpha k^2)\) in (4) below and identifying \(k^\mu \equiv q^\mu\). Since we would like to interpret \(q^\mu\) as the Planck momentum this requires \(\alpha\) to be small indicating that effect of the higher derivative terms in the action are small. It is now clear that to make a full fledged spacetime “crystal” one requires third and fourth terms in \(h_{\mu\nu}\), (as explained in [4, 5] for liquid-solid transition), in the action which will considerably complicate the model.

In terms of \(\tilde{h}_{\mu\nu}\) [5] \(\mathcal{L}\) becomes

\begin{equation}
\mathcal{L} = \frac{1}{4} \eta^{\alpha\beta} \eta^{\lambda\sigma} \tilde{h}_{\alpha\beta} \square (1 - 2\alpha \Box) \tilde{h}_{\lambda\sigma} - \frac{1}{2} \eta^{\mu\sigma} \eta^{\nu\lambda} \tilde{h}_{\lambda\sigma} \eta^{\alpha\beta} \partial_\mu \partial_\nu (1 - 2\alpha \Box) \tilde{h}_{\alpha\beta}
\end{equation}

\begin{equation}
- \frac{1}{4} \eta^{\mu\sigma} \eta^{\nu\lambda} \tilde{h}_{\lambda\sigma} \square (1 - 2\alpha \Box) \tilde{h}_{\mu\nu} + \frac{1}{2} \eta^{\rho\alpha} \eta^{\sigma\beta} \tilde{h}_{\alpha\beta} \eta^{\mu\lambda} \partial_\mu \partial_\rho (1 - 2\alpha \Box) \tilde{h}_{\lambda\sigma}\]

\begin{equation}
+ \cos(q.x) \left[ \frac{1}{4} \eta^{\alpha\beta} \eta^{\lambda\sigma} < \tilde{h}_{\alpha\beta} > \square (1 - 2\alpha \Box) \tilde{h}_{\lambda\sigma} \right]
\end{equation}

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\end{equation}

\begin{equation}
- \frac{1}{4} \eta^{\mu\sigma} \eta^{\nu\lambda} \tilde{h}_{\lambda\sigma} \square (1 - 2\alpha \Box) \tilde{h}_{\mu\nu} + \frac{1}{2} \eta^{\rho\alpha} \eta^{\sigma\beta} \tilde{h}_{\alpha\beta} \eta^{\mu\lambda} \partial_\mu \partial_\rho (1 - 2\alpha \Box) \tilde{h}_{\lambda\sigma}\]

\begin{equation}
+ \cos(q.x) \left[ \frac{1}{4} \eta^{\alpha\beta} \eta^{\lambda\sigma} < \tilde{h}_{\alpha\beta} > \square (1 - 2\alpha \Box) \tilde{h}_{\lambda\sigma} \right]
\end{equation}
\[-\frac{1}{2} \eta^{\mu\sigma} \eta^{\nu\lambda} \langle h_{\lambda\sigma} \rangle > \eta^{\alpha\beta} \partial_{\mu} \partial_{\nu} (1 - 2 \alpha \Box) \tilde{h}_{\alpha\beta} \]

\[-\frac{1}{4} \eta^{\mu\sigma} \eta^{\nu\lambda} \langle h_{\lambda\sigma} \rangle > \Box (1 - 2 \alpha \Box) \tilde{h}_{\mu\nu} \frac{1}{2} \eta^{\sigma\beta} \langle h_{\alpha\beta} \rangle > \eta^{\mu\lambda} \partial_{\mu} \partial_{\rho} (1 - 2 \alpha \Box) \tilde{h}_{\lambda\sigma} \]

\[-\frac{1}{4} \eta^{\alpha\beta} \langle h_{\alpha\beta} \rangle > q^2 (1 + 2 \alpha q^2) \eta^{\lambda\sigma} \]

\[+ \frac{1}{2} \eta^{\mu\sigma} \eta^{\rho\beta} \tilde{h}_{\lambda\sigma} q_\mu q_\nu (1 + 2 \alpha q^2) \eta^{\alpha\beta} \langle h_{\alpha\beta} \rangle > \]

\[+ \frac{1}{4} \eta^{\mu\sigma} \eta^{\rho\beta} \tilde{h}_{\lambda\sigma} q_\mu q_\nu (1 + 2 \alpha q^2) \eta^{\alpha\beta} \langle h_{\alpha\beta} \rangle >] + V. \quad (6)\]

We have dropped the constant terms in (6).

Varying (6) with respect to \( \tilde{h}_{\alpha\beta} \) we obtain the following equation of motion

\[\frac{1}{2} (1 - 2 \alpha \Box) [\eta^{\alpha\beta} (\Box \tilde{h} - \partial^\mu \partial^\nu \tilde{h}_{\mu\nu}) - \partial^\alpha \partial^\beta \tilde{h} - \Box \tilde{h}_{\alpha\beta} + \partial^\alpha \partial^\mu \tilde{h}_\mu^\beta + \partial^\beta \partial^\mu \tilde{h}_\mu^\alpha] \]

\[+ \frac{1}{2} (1 + 2 \alpha q^2) \cos(q.x) [\eta^{\alpha\beta} (-q^2 < h > + q_\mu q_\nu < h_{\mu\nu} >) \]

\[+ q_\alpha q_\beta < h > + q^2 < h_{\alpha\beta} > - q_\alpha q_\mu < h_{\mu\beta} > - q_\beta q_\mu < h_{\mu\alpha} >] = 0. \quad (7)\]

We will not consider the effects of the potential term any further since it will lead to mass terms and higher order interaction terms [7].

Notice that the second set of terms in (6) is independent of the field \( \tilde{h}_{\mu\nu} \) and hence can be treated as a source term, \( O(\tilde{h}^0) \), in order to solve (6) for \( \tilde{h}_{\mu\nu} \). The source is symmetric and conserved but is not of the perfect fluid form. But to be a realistic source it should satisfy, at the least, the Weak Energy Condition \( t_\mu t_\nu \tilde{h}_{\mu\nu} \geq 0 \), for arbitrary time-like \( t_\mu \) (see eg. [9]). In the present case this will lead to restricting the numerical values of \( < h_{\mu\nu} > \) which is perfectly justified. Also it might be interesting to study the properties of this source as a relativistic imperfect fluid (see eg. [10]).

The equation is trivially solved by

\[\tilde{h}_{\alpha\beta} = \cos(q.x) < h_{\alpha\beta} >. \quad (8)\]

Thus in the weak field limit we obtain

\[g_{\mu\nu} = \eta_{\mu\nu} + \cos(q.x) < h_{\mu\nu} >. \quad (9)\]
It is now straightforward to construct the Ricci tensor and scalar in an explicit way:

\[
R_{\mu\nu} = -\frac{1}{2} (q_\sigma q_\nu < h^\mu_\sigma > + q_\sigma q_\mu < h^\nu_\sigma > - q_\mu q_\nu < h > - q^2 < h_{\mu\nu} > ) \cos(q.x),
\]

\[
R = \eta^{\mu\nu} R_{\mu\nu} = -(q_\mu q_\nu < h^{\mu\nu} > - q^2 < h > ) \cos(q.x).
\] (10)

Clearly the condensate induces an effective curvature. Depending on the choice of parameters as well as the physically relevant form of curvature, it is possible to put restrictions on \( \cos(q.x) \). This can be translated to an effective minimum length scale.

Our framework, though considered from an entirely different perspective, compares favorably with the NC-extended metric [1] in some basic features, provided one relates \( \alpha \) with NC parameter.

I) In NC gravity NC effects generate a source term. In SSB scenario the condensate induces a source, in the form of an exotic fluid.

II) It has been established in various frameworks and diverse types of noncommutative structures that the NC corrections start from quadratic order of the NC parameter. There are generically no NC correction in linear order in the NC parameter. This is true in the SSB formalism as well since \( \cos(q.x) \approx 1 + (q.x)^2 + \ldots \). However it is probably not justified to treat \( q.x \) as an expansion parameter unless \( x \) is very small.

To conclude, we have shown a way in which higher derivative terms, extending the Einstein action, can be used to construct a discrete spacetime where translation and rotation symmetries can be broken. In the present work we have studied the first step towards this goal: introduction of a length scale in the metric. This modification generates an imperfect fluid like source term that is reminiscent of noncommutative extensions of gravity models [1]. One has to consider in more detail the important issue of negative energy that plagues higher derivative gravity theories since, (although “numerically” small), it plays an essential role in our scheme. The properties of the exotic fluid that appears in our model and it effects on the dynamics of test particles need to be discussed.

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