Influence of order enhancement of the Runge-Kutta method on low-order turbulence statistics of incompressible wall turbulence under a second-order central difference scheme

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Abstract. In this study, we attempt to understand the effects of the accuracy order of higher-order Runge-Kutta methods on the turbulence statistics of incompressible wall turbulence. A second-order central difference scheme is used to spatially discretize the governing equations, considering the widely used lower-order spatial discretization. The effects of the difference in the order of the Runge-Kutta scheme are investigated using a turbulent channel flow. The turbulence statistics obtained are compared with those reported in three previous studies. Based on our results, we infer that the rms values of the mean velocity, mean pressure, Reynolds stress, and velocity fluctuation of the turbulent channel flow are not affected by the difference in the accuracy order of the higher-order Runge-Kutta method. The fluctuation of the rms values of static pressure obtained by using the fourth-order Runge-Kutta method were consistent with those obtained in previous studies.

1. Introduction

Turbulent shear flow is widely observed in the field of fluid engineering. For example, wall turbulence, which is a type of turbulent shear flow, is widely observed in the flows related to applications of fluids engineering. Numerical analyses have been a useful method to model the phenomenon of wall turbulence. Fluctuations in the static pressure are an essential physical quantity in the turbulent shear flow and turbulence phenomena, and they can be accurately characterized using numerical analyses. In addition, the uncertainty associated with numerical simulations can be significantly reduced if the accuracy of the simulation is maintained. Numerical analyses can also characterize the fluctuation in the static pressure with a sufficient accuracy because of the negligible uncertainty in the numerical simulation. Time integration methods have a direct effect on the accuracy of numerical analysis, and the use of low accuracy order time integration methods may affect the turbulent flow field reproduced by numerical analysis.

Turbulent channel flow statistics have been shown with sufficient accuracy in previous studies [1-3]. In the numerical analysis of turbulence phenomena using the finite difference method, an appropriate discretization form that explicitly conserves turbulent kinetic energy of the flow has been
studied and developed [4] and has been widely used in studies conducted by the present research group (e.g., [5-7]). The improvement in the spatial resolution depending on the spatial discretization form has also been studied to improve the accuracy of numerical analysis of turbulence phenomena [8, 9]. A previous study has pointed out the need to maintain conservation laws, described by governing equations of the incompressible flow [10]. OpenFOAM, which is a fluid analysis solver that has been widely used in recent years, has been verified for the case of a turbulent channel flow [11]. In addition, the necessity of measuring the fluctuation of the static pressure to clarify the turbulence phenomenon has been discussed in various studies [12, 13].

As discussed above, improvement in the accuracy of numerical analysis has been achieved by improving spatial resolution. Nevertheless, application of the time integration method, which directly affects the time resolution, has been inconsistent. For analyzing uniform isotropic turbulence, the fourth-order Runge-Kutta method is has been commonly used in previous works. However, in the case of turbulent shear flows, lower-order Runge-Kutta and Adams-Bashforth methods have been used, other than the fourth-order Runge-Kutta method. It should be noted that the effect of the improvement of the temporal resolution on the total computational cost is significantly lower than that of the spatial resolution. In turbulence analysis, fluctuation of the static pressure is one of the essential quantities, which can effectively explain the turbulence phenomenon. Therefore, in this study, we attempt to clarify the characteristics of fluctuation of the static pressure using high-quality numerical analysis.

To use an economical scheme for analyzing the turbulent shear flow, we investigate the influence of the difference in the accuracy order of Runge-Kutta method on the turbulence statistics. Further, we investigate the effect of the accuracy order difference between third-order and fourth-order Runge-Kutta schemes [14, 15]. Turbulent channel flow is still simulated in a recent work [16]. Based on previous studies, a lower-order central difference method is used to discretize the governing equations in this study. To verify the results, turbulent channel flows with three previous databases are used. By using the mean flow and fluctuating flow fields of the turbulent channel flow, the effects of the difference in the accuracy order between third-order and fourth-order Runge-Kutta is investigated.

2. Methods

In this study, an incompressible turbulent channel flow is analyzed and such a turbulent channel flow has been often used in previous studies to estimate the accuracy of a numerical analysis [8, 10]. Figure 1 shows a schematic of the turbulent channel flow used in this study. This turbulent channel flow is driven by a streamwise mean pressure gradient [1-3]. To focus on the shear stresses, the mean pressure gradient rather than the flow rate is maintained as a constant in this work. The characteristic velocity and length are the friction velocity, $u_f$ and the channel half-width, $\delta$. The coordinate system of this study is composed of the streamwise direction, the direction normal to the wall, and the span-wise direction. For the numerical analysis, the governing equations are the continuity equation for an incompressible fluid and the Navier-Stokes equation. These equations are made dimensionless using the friction Reynolds number, $Re_f = u_f \delta / \nu$, where $\nu$ is the kinematic viscosity.

A second-order central difference scheme studied by a previous work [4] is used to discretize the governing equations spatially. Here, the skew-symmetric form [4] rather than the divergence and convective discretization forms are used to discretize the convection terms. The second-order central difference scheme used in this study conserves the kinetic energy explicitly in the discretized equations as also shown in a previous study [4]. Therefore, the turbulence statistics of the turbulent channel flow is accurately reproduced in spite of using a lower-order central difference scheme. A previous work had developed higher-order central difference schemes [4], which could conserve the kinetic energy explicitly. However, keeping in view that second-order central difference schemes are widely used in common fluid solvers [11], we incorporate the same in our study to improve the applicability of the obtained results.
Figure 1. Schematic of the present turbulent channel flow. The coordinate system consists of the streamwise, transverse, and span-wise directions, $x$, $y$, and $z$, respectively.

Table 1. Coefficients of the Runge-Kutta schemes.

| Coefficients | Third-order Runge-Kutta by Williamson (1980) | Fourth-order Runge-Kutta by Carpenter and Kennedy (1998) |
|--------------|---------------------------------------------|----------------------------------------------------------|
| $A_1$        | 0                                           | 0                                                        |
| $A_2$        | $-\frac{5}{9}$                             | $-\frac{567301805773}{1357537059087}$                   |
| $A_3$        | $-\frac{153}{128}$                        | $-\frac{2404267990393}{2016746695238}$                 |
| $A_4$        | N/A                                         | $-\frac{3550918686646}{2091501179385}$                 |
| $A_5$        | N/A                                         | $-\frac{1275806237668}{842570457699}$                  |
| $B_1$        | $\frac{1}{3}$                             | $\frac{1432997174477}{9575080441755}$                  |
| $B_2$        | $\frac{15}{16}$                           | $\frac{5161836677717}{13612068292357}$                 |
| $B_3$        | $\frac{8}{15}$                            | $\frac{1720146321549}{2090206949498}$                  |
| $B_4$        | N/A                                         | $\frac{3134564353537}{4481467310338}$                   |
| $B_5$        | N/A                                         | $\frac{2277821191437}{14882151754819}$                 |

Table 2. Computational conditions.

| Studies                          | $Re$ | $L_x \times L_y \times L_z$ | $N_x \times N_y \times N_z$ |
|----------------------------------|------|-----------------------------|-----------------------------|
| Present                          | 395  | $2\pi \times 2 \times \pi$  | $256 \times 128 \times 256$ |
| Moser, Kim, and Mansour (1999)   | 395  | $2\pi \times 2 \times \pi$  | $256 \times 193 \times 192$ |
| Iwamoto, Suzuki, and Kasagi (2002)| 400  | $2.5\pi \times 2 \times \pi$ | $192 \times 257 \times 192$ |
| Abe, Kawamura, and Matsuo (2001) | 395  | $12.8 \times 2 \times 6.4$   | $512 \times 192 \times 512$ |

The improvement in spatial resolution is considered to be important in this study. In the past, we have used higher-order discretization schemes to solve the governing equations [7, 9] but second-order spatial discretization scheme has been widely used in numerical simulations, especially in engineering applications. Therefore, we focus on the use of the second-order scheme in this study. Further, lower-order turbulence statistics such as mean velocity, Reynolds stress, rms values, are estimated and can be obtained by the second-order scheme with sufficient accuracy. The use of higher-order discretization schemes is important when turbulence statistics are investigated on smaller scales. The two governing equations are analyzed using a fractional step method based on the Runge-Kutta schemes as described below. The pressure Poisson equation in the fractional stages is analyzed by the direct method using the fast Fourier transform.
Figure 2. Results of the velocity-magnitude iso-surface of the present turbulent channel flow using the instantaneous streamwise velocity fluctuation field, shown in (a) and (b), where the flow field is obtained using the fourth-order Runge-Kutta scheme. In these visualization results, red and blue isosurfaces are characterized by $(\tilde{u}^+ - U^+) = 0.1$ and $(\tilde{u}^+ - U^+) = -0.1$.

Figure 3. Transverse profiles of streamwise mean velocity shown in (a) and static mean pressure shown in (b), where these quantities are normalized using the friction velocity. In the figures, results obtained by three previous works are also provided as reference.

Third-order and fourth-order Runge-Kutta methods [14, 15] are used in the present work. The low-storage form of a Runge-Kutta method proposed by Williamson [14] is used as the third-order Runge-Kutta method. The values of the two primary coefficients of this third-order Runge-Kutta method, $A_j$ and $B_j$, for $j$ stage, are shown in Table 1. This study attempts to investigate the effect on turbulence statistics of different schemes, i.e., third- and fourth-order Runge-Kutta schemes. The lower-storage Runge-Kutta scheme developed by Carpenter and Kennedy is used as the fourth-order Runge-Kutta method [15]. Table 1 also shows the coefficients for the fourth-order Runge-Kutta method.

The friction Reynolds number is set to 395 in the present analysis. This value is either same or nearly equal to that of the three previous studies [1-3] and is often used in numerical analysis. Table 2 compares the value of the friction Reynolds number of this study with that of the previous studies. The size of the computational domain in this study is set to $L_x \times L_y \times L_z = 2\pi \times 2 \times \pi$ and similarly, the number of grid points in this study is set to $N_x \times N_y \times N_z = 256 \times 128 \times 256$. The non-uniform grid spacing condition is set in the direction normal to the wall. Table 2 compares the size of the
computational domain and the number of grid points in this study with those of previous studies. As shown in the table, the conditions of the computational domain size and the grid points are sufficient and appropriate to investigate the influence of the difference in the two Runge-Kutta schemes. The instantaneous flow field obtained using RK4 is set as an initial flow field for the case applying RK3. The ensemble average time is determined after the initial transients using the time series of friction velocity. The ensemble time to obtain statistics was decided using the computational domain size for the streamwise direction and was found to be sufficiently large. All turbulence statistics obtained in the submitted work are verified to converge by checking third-order turbulence statistics, such as $\langle w^3 \rangle$, which is analytically zero, where $w$ is the span-wise velocity fluctuation and $\langle \rangle$ denotes the ensemble average.

3. Results and Discussion

In the results and discussion section, this study first visualizes the instantaneous flow field. Figure 2 shows the visualization results for the higher and lower velocity regions compared with the streamwise mean velocity. Here, high and low velocity regions are characterized by $u^\prime (= \bar{u} - U) = 0.1$ and $u^\prime (= \bar{u} - U) = -0.1$, respectively. As shown in Figure 2 (a), higher and lower velocity regions are arranged in stripes near the smooth wall as high-speed and low-speed streaks, respectively. These are general turbulent structures found near the smooth surface. Figures 2(a) and 2(b) depict the typical turbulent structures in wall turbulence. These results show that the instantaneous flow field of the turbulent channel flow is correctly reproduced in this study.

Then, this study approaches the purpose of this research using turbulence statistics. Figure 3(a) shows a profile of the streamwise mean velocity. The mean velocity in the streamwise direction is made dimensionless by using the friction velocity. The direction normal to the wall is set to be dimensionless by the viscous scale. As shown in the figure, the present profiles of the streamwise mean velocities are in good agreement with the profiles shown in the previous study. In addition, the mean velocity profiles obtained by the third-order and fourth-order Runge-Kutta methods agree well with each other. Therefore, we can infer that the difference between the third and fourth-order Runge-Kutta methods has little effect on the mean velocity profile. The streamwise mean velocity shown in the figure is made dimensionless using the friction velocity. The agreement between the profiles by the third and fourth-order Runge-Kutta methods also suggests that this difference does not affect the drag coefficient of the wall.

Figure 3(b) shows the mean pressure profile in the direction normal to the wall. The value of this mean pressure corresponds to a deviation from the given streamwise linear profile of the mean pressure, required to drive the turbulent channel flow. The mean pressure profile obtained by Abe et al. [3] is not used in the figure because this profile is not included in their results. As shown in the figure, the mean pressure profiles obtained using the third-order and fourth-order Runge-Kutta methods agree well with the results obtained from the previous studies. Further, we infer that the difference between the profiles obtained using the third-order and fourth-order Runge-Kutta methods are not significant. Therefore, the difference in the schemes has little effect on the mean pressure profile. In a turbulent channel flow, the momentum conservation law in the mean flow is given as follows: $\langle v^2 \rangle + P = 0$, where $v$ is the instantaneous velocity fluctuation in the direction normal to the wall. In this analysis, the momentum conservation law described by the Navier-Stokes equation is well established by the discretization scheme studied in the previous work. Therefore, the results of the mean pressure profile shown in the figure suggest that the $\langle v^2 \rangle$ profile agrees well with the results obtained from the third-order and fourth-order Runge-Kutta methods.

The Reynolds equation describing the mean velocity components of the mean flow includes the Reynolds stress, $\langle u'v' \rangle$. Figure 4 shows the Reynolds stress profile. Here, $v$ / $\delta$ rather than $y^+$ is used because of the symmetry of the Reynolds stress profile. As shown in the figure, the profile of the Reynolds stress obtained from the third-order and fourth-order Runge-Kutta methods is in agreement with the results obtained in the previous studies. In addition, Figure 4 shows that the difference between the Runge-Kutta methods does not affect the Reynolds stress value.
Figure 4. Transverse profiles of the Reynolds and total stresses, which is given by the sum of the Reynolds and viscous stresses. Here, the total stress profile is analytically derived by mean flow characteristics of the turbulent channel flow.

Figure 5. Transverse profiles of the rms values of three velocity fluctuation components, \((\langle u^2 \rangle)^{1/2}\), \((\langle v^2 \rangle)^{1/2}\), and \((\langle w^2 \rangle)^{1/2}\).

In addition to the mean flow velocity, excellent reproduction of Reynolds stress indicates that the mean flow field is accurately reproduced in this study. The total stress obtained by adding Reynolds stress, \((\langle uv \rangle)^+\), and viscous stress, \((1/Re) \cdot dU/dy^+\), is given analytically as follows: \((\langle uv \rangle)^+ + (1/Re) \cdot dU/dy^+ = 1 - (1/Re) \cdot y^+ = 1 - y/\delta\), where \(\delta = Re\). The profile of the total stress obtained in this study is shown in Figure 4 and it is in good agreement with the analytically obtained profile.

Figure 5 shows the rms profiles of velocity fluctuations. Here rms values of the streamwise, transverse, and span-wise velocity fluctuations are \((\langle u^2 \rangle)^{1/2}\), \((\langle v^2 \rangle)^{1/2}\), and \((\langle w^2 \rangle)^{1/2}\), respectively. The mean pressure profile shown above suggests that the \((\langle v^2 \rangle)\) profiles are adequately reproduced in the present simulation. As shown in the figure, the rms profiles of velocity fluctuations obtained using this analysis are in good agreement with those obtained in earlier works. In addition, the rms profiles of velocity fluctuations are in excellent agreement with the profiles obtained from the third-order and fourth-order Runge-Kutta methods. The present profiles of velocity fluctuation rms have sufficiently high symmetry. This symmetry indicates that the numerical code of this study is accurately constructed. From the results described above, we can state that the analysis of this study effectively reproduces the velocity fluctuation field as well as the mean flow velocity field.
Figure 6. Transverse profiles of the rms value of static pressure fluctuation. $\langle p^2 \rangle^{1/2}$.

Figure 7. Results of the conservation error of kinetic energy obtained using the periodic inviscid flow. The conservation error of kinetic energy obtained using RK3 is significantly larger than that due to RK4.

As described above, the difference between the third-order and fourth-order Runge-Kutta methods has little effect on the turbulence statistics related to the mean and velocity fluctuations. Figure 6(a) shows profiles of the static pressure fluctuation rms. The magnitude order of the static pressure fluctuation rms is same as that of the turbulent kinetic energy. In addition, the value of the static pressure fluctuation rms obtained by Abe et al. is not shown in the figure because it is not demonstrated in Abe et al. As shown in the figure, the profiles obtained by the fourth-order Runge-Kutta method agree well with those obtained in the previous studies. On the other hand, the profile obtained by the third-order Runge-Kutta method deviates from the other profiles. This deviation is
more clearly depicted in Figure 6(b). Further, it shows that the fourth-order Runge-Kutta method is more appropriate to analyze the static pressure fluctuation rms accurately. It is generally challenging to obtain static pressure fluctuation rms using experimental measurements in fluid engineering research [12]. Hence, in such situations, an accurate numerical analysis can appropriately investigate its value, as shown in the figure. In contrast to previous studies that give reference results [1,2], a finite difference method is used in this study to discretize the governing equations. Unlike the spectral method for obtaining the reference results, finite difference method can analyze flow fields with complex boundary conditions. The present study has estimated results for the flow field. In studies related to the thermal fields [17], budgets of temperature variance and turbulent heat flux, which include the diffusion terms related to static pressure fluctuation, are often calculated. As discussed in the present study, static pressure fluctuation is more sensitive to the improvement in the Runge-Kutta scheme compared to other fundamental quantities. The diffusion terms, including the static pressure fluctuation, can be influenced by the affected static pressure fluctuation.

When a turbulence analysis is performed using the finite difference scheme, it is necessary to minimize the conservation error of the kinetic energy, as noted in previous studies [4]. A set of numerical results performed using direct simulation on the anisotropic steady incompressible turbulence by our research group show that the conservation error of kinetic energy significantly affects the static pressure fluctuation rather than the velocity field. The conservation error of the kinetic energy in this study depends on the accuracy order of the Runge-Kutta method and it is investigated in the present analysis using an inviscid periodic flow [4] in which the kinetic energy is analytically constant. Figure 7 shows the conservation error of kinetic energy as a function of time increment. The size of the computational domain is set to $(2\pi)^3$ and the number of grid points is set to $16^3$. The initial flow field is a normal random number vector field that satisfies the continuity equation. The conservation error of the kinetic energy is mainly caused by the numerical dissipation of the time integration method. The RK3 method reveals a larger conservation error of the kinetic energy using the inviscid periodic flow as compared to RK4; it can be larger than the order of single-precision machine epsilon. This conservation error of kinetic energy due to RK3 method may be non-negligible if the time taken to obtain turbulence statistics is sufficiently large.

4. Conclusions

An incompressible turbulent flow is numerically analyzed, using a lower-order central difference method in this study. The effects of the accuracy order of the time integration Runge-Kutta methods on turbulent statistics were investigated. The low-order central difference method used in this study explicitly conserves the turbulent kinetic energy of the flow field. By solving the pressure Poisson equation using the direct method and incorporating the fractional steps, the conservation laws of the governing equations are established with high accuracy. Low-storage types are used as the third-order and fourth-order Runge-Kutta method in this study. Turbulent channel flow is used to solve the purpose of this study. The present value of the friction Reynolds number is set to 395 based on previous studies. Using the turbulence statistics of the turbulent channel flow, the results of this study are in good agreement with the results of three previous works. In addition, the numerical conditions used are comparable to those provided in previous studies. Therefore, the turbulent channel flow is accurately reproduced in this study.

In this study, the instantaneous velocity fluctuation field was first analyzed. The visualization results of the instantaneous velocity fluctuation field show that a typical turbulent channel flow was reproduced. Furthermore, the effects of the difference in the accuracy order of the Runge-Kutta method on the rms values of the mean flow velocity, mean static pressure, Reynolds stress, and velocity fluctuation were investigated. The difference in the accuracy order did not affect these turbulence statistics significantly. The results obtained are in good agreement with those of three previous studies. Additionally, the influence of accuracy order was investigated using the rms value of static pressure fluctuation. In contrast to the turbulent statistics, the rms values of the static pressure is affected by the accuracy order of the Runge-Kutta method; the rms values of the static pressure
obtained via the fourth-order Runge-Kutta method were in good agreement with the values reported by previous works. It is difficult to experimentally measure the rms values of static pressure fluctuation in turbulent statistics. As shown in this study, the use of the fourth-order Runge-Kutta method can increase the usefulness of numerical analyses, as compared to the third-order Runge-Kutta method.

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