UPPER AND LOWER BOUNDS FOR THE FIRST EIGENVALUE AND THE VOLUME ENTROPY OF NONCOMPACT KÄHLER MANIFOLDS

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Abstract. We find upper and lower bounds for the first eigenvalue and the volume entropy of a noncompact real analytic Kähler manifold, in terms of Calabi’s diastasis function and diastatic entropy, which are sharp in the case of the complex hyperbolic space. As a corollary we obtain explicit lower bounds for the first eigenvalue of the geodesic balls of an Hermitian symmetric space of noncompact type.

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1. Introduction and statement of the main results

The first eigenvalue \( \lambda_1(M, g) \) of the Laplace operator of a Riemannian manifold \((M, g)\) is one of the most natural Riemannian invariant and its estimation is a classical problem (see e.g. [11, 12]). When \( M \) is noncompact with or without boundary, we call the bottom of the spectrum or, with abuse of language, the first eigenvalue of \((M, g)\) the following Riemannian invariant

\[
\lambda_1(M, g) = \inf \{ \lambda_1(D) : D \in \mathcal{M}_0 \},
\]

where \( \mathcal{M}_0 \) is the set of compact domains \( D \) contained in the interior of \( M \) with regular boundary \( \partial D \) and \( \lambda_1(D) \) denotes the first eigenvalue of \( D \) i.e. the smallest \( \lambda \) satisfying

\[
\Delta u = \lambda u
\]

for some non-zero function \( u \) on \( M \) with \( u|_{\partial D} = 0 \). We are interested in finding upper and lower bound of the first eigenvalue in the case of noncompact Kähler manifolds.
(we refer to [2] and reference therein for the compact case). More precisely, we consider the case of real analytic Kähler manifolds \((M, g)\) which admit a globally defined Calabi’s diastasis function \(D_p : M \to \mathbb{R}\), the canonical Kähler potential defined by E. Calabi in his celebrated paper [10] (see next section for the definition of the diastasis and its main properties). Our first result is the following theorem, where we give a lower bound of \(\lambda_1 (M, g)\) in terms of \(D_p\).

**Theorem 1.1.** Let \((M, g)\) be a real analytic Kähler manifold of complex dimension \(n\) and let \(p \in M\) be a point for which the diastasis \(D_p : M \to \mathbb{R}\) is globally defined. Then

\[
\frac{4n^2}{\mathcal{X}(p)} \leq \lambda_1 (M, g),
\]

where \(\mathcal{X}(p) = \sup_q \|d_q D_p\|^2\). If \(\mathcal{X}(p) = \infty\) we set \(\frac{1}{\mathcal{X}(p)} = 0\).

When the manifold involved is an Hermitian symmetric space of noncompact type (HSSNT in the sequel) one obtains the following corollary of Theorem 1.1.

**Theorem 1.2.** Let \((\Omega, g_{hyp})\) be an \(n\)-dimensional irreducible HSSNT with holomorphic sectional curvature \(-4\) and rank \(r\). Then

\[
\frac{n^2}{r} \leq \lambda_1 (\Omega, g_{hyp}),
\]

which is an equality if and only if \((\Omega, g_{hyp})\) is the complex hyperbolic space. Moreover, if we denote by \(B^\Omega_p (t) \subset \Omega\) the geodesic ball of radius \(t\) and centre \(p\), we have

\[
\frac{n^2}{r \tanh^2 \left( \frac{t}{\sqrt{r}} \right)} \leq \lambda_1 (B^\Omega_p (t), g_{hyp}).
\]

As a consequence of Theorem 1.1 we get a lower bound of the volume entropy \(\text{Ent}_v (M, g)\) (see next section for details) i.e.

**Corollary 1.3.** Let \((M, g)\) be as in Theorem 1.1. Assume moreover that \((M, g)\) is complete with infinite volume. Then

\[
\frac{4 n}{\sqrt{\mathcal{X}(p)}} \leq \text{Ent}_v (M, g),
\]

where \(\text{Ent}_v (M, g)\) is defined by

\[
\text{Ent}_v (M, g) = \inf \left\{ c \in \mathbb{R}^+ : \int_M e^{-c \rho(p, x)} \, dv_g (x) < \infty \right\}.
\]

Inspired by the results obtained in [17] where the author joint with A. Loi studied the concept of diastatic exponential, defined by substituting in the usual exponential map the geodesic distance with the diastasis, it has been natural to give the

\footnote{Notice that formula (5) make sense also in the Riemannian setting (see the discussion in Section 2).}
The following definition of *diastatic entropy*, obtained by replacing in (5) the geodesic distance with the diastasis function:

**Definition 1.4.** Let $(M, g)$ be a Kähler manifold with globally defined diastasis $\mathcal{D}_p : M \to \mathbb{R}$ centred in $p \in M$, the *diastatic entropy* in $p$ is defined by

$$
\text{Ent}_d (M, g) (p) = \inf \left\{ c \in \mathbb{R}^+ : \int_M e^{-c \mathcal{D}_p} \, dv_g < \infty \right\}.
$$

Our last result is represented by the following theorem:

**Theorem 1.5.** Let $(M, g)$ be a complete Kähler manifold with infinite volume and diastasis $\mathcal{D}_p$ globally defined for some point $p \in M$. Then

$$
\text{Ent}_v (M, g) \leq \text{Ent}_d (M, g) (p) \sqrt{\mathcal{X}(p)}.
$$

and

$$
\lambda_1 (M, g) \leq \frac{\text{Ent}_d^2 (M, g) (p)}{4} \mathcal{X}(p).
$$

The paper is organized as follows. In the next section, after recalling the definition and the main properties of Calabi’s diastasis, we show that our definition of volume entropy (6) extends the classical one. Moreover we show that Definition (6) of the diastatic entropy, essentially, does not depend on the point $p$ and we give lower and upper bounds for the volume entropy of an HSSNT in terms of its diastatic entropy (Corollary 2.3). In Section 3 we prove Theorem 1.1 and Theorem 1.5. In the last section we recall some standard facts about HSSNT and Hermitian positive Jordan triple systems, needed in the proof of Theorem 1.2.

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*2. Calabi’s diastasis function, volume and diastatic entropy*

**Calabi’s diastasis function.** We briefly recall the main properties of the diastasis function, defined by Calabi in his seminal paper [10], the key tool of this paper. Let $M$ be a complex manifold endowed with a real analytic Kähler metric $g$. A Kähler potential for $g$ is a real analytic function $\Phi : U \to \mathbb{R}$ defined in a neighborhood of a point $p$ such that $\omega = i \partial \bar{\partial} \Phi$, where $\omega$ is the Kähler form associated to $g$. The Kähler potential $\Phi$ is not unique: it is defined up to an addition with the real part of a holomorphic function. By duplicating the variables $z$ and $\bar{z}$ the potential $\Phi$ can be complex analytically continued to a function $\Phi$ defined in a neighborhood of the diagonal of $U \times \bar{U}$ (where $\bar{U}$ is the manifold conjugated to $U$). The diastasis
function centered in $p$ is the Kähler potential $D_p : W \to \mathbb{R}$ around $p$ defined by

$$D_p(q) = \bar{\Phi}(q, \bar{\Phi}) + \bar{\Phi}(p, \bar{\Phi}) - \bar{\Phi}(p, q) - \bar{\Phi}(q, p).$$

We say that $D_p$ is \textit{globally defined} when $W = M$. The basic properties of the diastasis function (see [10]) are the following:

(i) it is uniquely determined by the Kähler metric: it does not depend on the choice of the Kähler potential $\Phi$;

(ii) it is real valued in its domain of (real) analyticity;

(iii) it is symmetric in the sense that $D_p(q) = D_q(p)$;

(iv) it is equal to zero in the origin i.e. $D_p(p) = 0$;

(v) if $\rho_p(q)$ denotes the distance between $p \in M$ and $q$, then

$$D_p(q) = \rho_p^2(q) + O(\rho_p^4(q)) \text{ when } \rho_p(q) \to 0.$$

The \textbf{volume entropy}. For a compact riemannian manifold $(X, g)$ the classical definition of volume entropy is the following

$$\text{Ent}_{\text{vol}}(X, g) = \lim_{t \to \infty} \frac{1}{t} \log \text{Vol}(\bar{B}_p(t)), \quad (9)$$

where $\text{Vol}(\bar{B}_p(t))$ denotes the volume of the geodesic ball $\bar{B}_p(t) \subset \bar{X}$, of center $p$ and radius $t$, contained in the Riemannian universal covering $(\bar{X}, \bar{g})$ of $(X, g)$.

This notion of entropy is related with one of the main invariant for the dynamics of the geodesic flow of $(X, g)$, the topological entropy $\text{Ent}_{\text{top}}(X, g)$ of this flow. For every compact manifold $(X, g)$ A. Manning in [19] proved the inequality $\text{Ent}_{\text{vol}}(X, g) \leq \text{Ent}_{\text{top}}(X, g)$, which is an equality when the curvature is negative. We refer the reader to the paper [21] (see also [8] and [9]) of G. Besson, G. Courtois and S. Gallot for an overview about the volume entropy and for the proof of the celebrated minimal entropy theorem.

The next lemma show that the two definition of volume entropy (5) and (9) coincide in the following sense

$$\text{Ent}_{\text{vol}}(X, g) = \text{Ent}_{\text{v}}(\bar{X}, \bar{g}). \quad (10)$$

\textbf{Lemma 2.1.} Let $(X, g)$ be a complete $n$-dimensional Riemannian manifold with infinite volume. Denote by

$$L := \lim_{R \to +\infty} \inf \left( \frac{1}{R} \log \text{Vol}(B(x_0, R)) \right)$$

and

$$\overline{L} := \lim_{R \to +\infty} \sup \left( \frac{1}{R} \log \text{Vol}(B(x_0, R)) \right),$$



where \( B(x_0, R) \subset (X, g) \) is the geodesic ball of centre \( x_0 \) and radius \( R \). Then the two limits does not depend on \( x_0 \) and
\[
\underline{L} \leq \text{Ent}_v(X, g) \leq \overline{L}.
\]

**Proof.** The inferior limit does not depend on \( x_0 \), indeed set \( D = d(x_0, x_1) \) and \( R > D \), by the triangular inequality
\[
B(x_0, R - D) \subset B(x_1, R) \subset B(x_0, R + D).
\]
so
\[
\lim_{R \to +\infty} \inf \left( \frac{1}{R} \log (\text{Vol } B(x_1, R)) \right) \leq \lim_{R \to +\infty} \inf \left( \frac{1}{R} \log (\text{Vol } B(x_0, R + D)) \right)
\]
\[
= \lim_{R' \to +\infty} \inf \left( \frac{R'}{R' - D} \frac{1}{R'} \log (\text{Vol } B(x_0, R')) \right)
\]
\[
\leq \lim_{R' \to +\infty} \inf \left( \frac{1}{R} \log (\text{Vol } B(x_0, R')) \right).
\]
With an analogous argument one can prove the inequality in the other direction and the equality for the superior limit.

Assume \( \overline{L} < \infty \). By the definition of limit inferior and superior, for every \( \varepsilon > 0 \), there exists \( R_0(\varepsilon) \) such that, for \( R \geq R_0(\varepsilon) \),
\[
\underline{L} - \varepsilon \leq \left( \frac{1}{R} \log (\text{Vol } B(x_0, R)) \right) \leq \overline{L} + \varepsilon
\]
equivalently
\[
e^{(\underline{L} - \varepsilon)R} \leq \text{Vol } B(x_0, R) \leq e^{(\overline{L} + \varepsilon)R} \tag{11}
\]
Integrating by parts we obtain
\[
I := \int_M e^{-c \rho(p(x_0, x))} dv(x) = \int_0^\infty e^{-cr} \text{Vol}_{n-1}(S(x_0, r)) \, dr
\]
\[
= c \int_0^\infty e^{-cr} \text{Vol}(B(x_0, r)) \, dr,
\]
where in the last equality we use that \( \overline{L} < \infty \). Now, by (11)
\[
\int_{R_0(\varepsilon)}^\infty e^{(\underline{L} - \varepsilon)r} \, dr \leq \int_{R_0(\varepsilon)}^\infty e^{-cr} \text{Vol}(B(x_0, r)) \, dr \leq \int_{R_0(\varepsilon)}^\infty e^{-(e-L-\varepsilon)r} \, dr.
\]
By the second inequality one can deduce that if \( c > \overline{L} \) then \( I \) is convergent, i.e. \( \overline{L} \geq \text{Ent}_v \). On the other hand the first inequality imply that \( I \) is not convergent when \( c < \underline{L} \), that is \( \text{Ent}_v \geq \underline{L} \), as wished. \( \square \)

**The diastatic entropy.** The following proposition shows that the Definition of the diastatic entropy, essentially, does not depend on the point \( p \) chosen.
Proposition 2.2. Let \((M, g)\) be a Kähler manifold and let \(\mathcal{D}_p : M \to \mathbb{R}\) be globally defined for every \(p \in M\) and \(\sup_{p \in M} \mathcal{X}(p) < \infty\), then \(\text{Ent}_d (M, g) (p)\) does not depend on \(p \in M\).

Proof. Denoted \(\mathcal{X} = \sup_p \sqrt{\mathcal{X}(p)} = \sup_{p,q} \|d_q \mathcal{D}_p\|\), for every \(p, q, x \in M\) we have that
\[
|\mathcal{D}_p (x) - \mathcal{D}_q (x)| = |\mathcal{D}_x (p) - \mathcal{D}_x (q)| \leq \mathcal{X} \rho (p, q).
\]
So
\[
e^{-c \mathcal{X} \rho(p,q)} \int e^{-c \mathcal{D}_p(x)} \leq \int e^{-c \mathcal{D}_q(x)} \leq e^{c \mathcal{X} \rho(p,q)} \int e^{-c \mathcal{D}_p(x)}
\]
Therefore \(\int e^{-c \mathcal{D}_p(x)} < +\infty\) if and only if \(\int e^{-c \mathcal{D}_q(x)} < +\infty\).

\(\square\)

In [20] the author show that \(\text{Ent}_d (M, g) (p)\) and the balanced condition on \(g\) are deeply linked (the notion of balancedness has been defined by S. K. Donaldson [13] for the compact case and then extended to the noncompact case by C. Arezzo and A. Loi [3]). In the same paper one can also find the computation of the diastatic entropy for every homogeneous domain in terms of Piatetskii-Shapiro constants [20, (8)]. In particular, for any irreducible HSSNT, we have
\[
\text{Ent}_d (\Omega, g_{\text{hyp}}) = \gamma - 1,
\]
where \(\gamma\) is the genus of \(\Omega\). If \(g_B\) denotes the Bergman metric on \(\Omega\), the genus is defined by \(g_B = \gamma g_{\text{hyp}}\). This formula should be compared with the expression of the volume entropy (see [21])
\[
\text{Ent}_v (\Omega, g_{\text{hyp}}) = 2 \sqrt{\sum_{j=1}^{\frac{r}{2}} (b + 1 + a (r - j))^2},
\]
in terms of the invariants \(r, a\) and \(b\) associated to \(\Omega\) (see [21, Table 1] for a description of these invariants). Recalling that \(\gamma = b + 2 + a (r - 1)\) (see [20]) we obtain the following result, (which is in accordance with (7), since, by (24) below, \(\mathcal{X}(p) = 4\) r

Corollary 2.3. Let \((\Omega, g_{\text{hyp}})\) be an \(n\)-dimensional irreducible HSSNT with holomorphic sectional curvature \(-4\) and rank \(r\). We have
\[
2 \text{Ent}_d (\Omega, g_{\text{hyp}}) \leq \text{Ent}_v (\Omega, g_{\text{hyp}}) \leq 2 \sqrt{r} \text{Ent}_d (\Omega, g_{\text{hyp}}),
\]
where the equalities are attained only for \(r = 1\) or \(a = 0\) i.e. when \(\Omega\) is the complex hyperbolic disc.

3. Proof of Theorems 1.1 and 1.5
The main ingredient in the proof of Theorems 1.1 is the following lemma.
Lemma 3.1. (Barta’s Lemma, [3]) Let \( \phi : M \rightarrow \mathbb{R} \) be a positive function such that \( \Delta \phi \geq \lambda \phi \), then
\[ \lambda_1 (M) \geq \lambda. \]

Proof. For the sake of completeness we will give a proof of this lemma. Given \( f \in C_0^1 (M) \), we can write it as \( f = u \phi \), where \( u \in C_0^1 (M) \). We have
\[ \| df \|^2 = \phi^2 \| du \|^2 + u^2 \| d\phi \|^2 + \frac{1}{2} \langle d\phi^2, du^2 \rangle. \]
As \( \text{div} (u^2 d\phi^2) = - \text{tr} (\nabla (u^2 d\phi^2)) = u^2 \Delta \phi^2 - \langle du^2, d\phi^2 \rangle \), we get
\[ \| df \|^2 = \phi^2 \| du \|^2 + u^2 \phi \Delta \phi - \frac{1}{2} \text{div} (u^2 d\phi^2). \]
where we used that \( \frac{1}{2} \Delta \phi^2 = \phi \Delta \phi - \| d\phi \|^2 \). Since \( u^2 d\phi^2 \) and \( df \) have compact support we have \( \int_M \text{div} (u^2 d\phi^2) = 0 \) and \( \int_M \| df \|^2 < \infty \), therefore
\[ \int_M \| df \|^2 = \int_M (\phi^2 \| du \|^2 + u^2 \phi \Delta \phi) \]
\[ \geq \int_M u^2 \phi \Delta \phi = \lambda \int_M u^2 \phi^2 = \lambda \int_M f^2. \]
Then we have shown that
\[ \lambda \leq \frac{\int_M \| df \|^2}{\int_M |f|^2}, \]
for every \( f \in C_0^1 (M) \). The inequality \( \lambda_1 (M) \geq \lambda \) is a consequence of the equality
\[ \lambda_1 (M, g) = \inf_{f \in \mathcal{H}_0^1 (M)} \frac{\int_M \| df \|^2}{\int_M |f|^2}, \quad (15) \]
where \( \mathcal{H}_0^1 (M, g) \) denotes the completion of the space of the \( C^1 \)-differentiable function with compact support contained in the interior of \( M \) with respect the norm \( \| f \|_{1,2} = \sqrt{\int_M |f|^2 + \int_M |df|^2} \) (see [2] Lemma D.II.3) for a proof. \[ \square \]

Proof of Theorem 1.1. Set \( \phi := e^{-c \mathcal{D}_p} \). We have:
\[ \Delta \phi = - \text{tr} (\nabla d\phi) = c \left( \text{tr} (\nabla dD_p) - c \| dD_p \|^2 \right) \phi, \]
by the identity \( \nabla dD_p (v, v) + \nabla dD_p (Jv, Jv) = 4g(v, v) \) we get
\[ \Delta \phi \geq c \left( 4n - c \mathcal{X}(p) \right) \phi, \]
then by Barta’s Lemma we conclude
\[ \lambda_1 \geq \max_{c \in \mathbb{R}} \left[ c \left( 4n - c \mathcal{X}(q) \right) \right] = \frac{4n^2}{\mathcal{X}(p)} \]
and this ends the proof of Theorem 1.1.

Proof of Theorem 1.5. Since $D_{p}(p) = 0$, for every $x \in M$ we have

$$D_{p}(x) = D_{p}(x) - D_{p}(p) \leq \sup_{z \in M} \|d_{z}D_{p}\| \rho_{p}(x) \leq \sqrt{\mathcal{X}(p)} \rho_{p}(x),$$

so

$$\int_{M} e^{-c \mathcal{X}(p)} \rho_{p}(x) \leq \int_{M} e^{-c D_{p}(x)}.$$

We deduce that if $c \sqrt{\mathcal{X}(p)} \leq \text{Ent}_{v}(M, g)$ then $c \leq \text{Ent}_{d}(M, g)$. In particular, taking $c = \frac{\text{Ent}_{v}(M, g)}{\sqrt{\mathcal{X}(p)}}$, we obtain (7).

Now set $f(z) := e^{-c \rho_{p}(z)}$ is not hard to prove that $f \in \mathcal{H}^{1}_{0}(M)$ for every $c > \frac{\text{Ent}_{v}(M, g)}{2}$. Substituting in (15) we get

$$\lambda_{1} \leq \frac{\int_{M} ||df||^{2}}{\int_{M} |f|^{2}} = c^{2}$$

where in the last equality we use that $||d\rho_{p}||^{2} = 1$. As $c$ approach to $\frac{\text{Ent}_{v}(M, g)}{2}$ we obtain the following inequality

$$\lambda_{1}(M, g) \leq \frac{\text{Ent}_{v}^{2}(M, g)}{4} \quad (16)$$

(which is an equality when $(M, g) = (\Omega, g_{\text{hyp}})$, see [4, Appendix C] for a proof). Finally combining (7) with (16) we obtain (8).

Remark 3.2. Observe that by the same argument used to prove (16) one can obtain an alternative proof of inequality (8) by setting $f(z) := e^{-c D_{p}(z)}$.

4. Hermitian positive triple system and proof of Theorem 1.2

We refer the reader to [22] (see also [18]) for more details on Hermitian symmetric spaces of noncompact type and Hermitian positive Jordan triple systems (from now on HPJTS).

Definitions and notations. An Hermitian Jordan triple system is a pair $(M, \{., ., \})$, where $M$ is a complex vector space and $\{., ., \}$ is a map

$$\{., ., \} : M \times M \times M \rightarrow M$$

$$(u, v, w) \mapsto \{u, v, w\}$$

which is $\mathbb{C}$-bilinear and symmetric in $u$ and $w$, $\mathbb{C}$-antilinear in $v$ and such that the following Jordan identity holds:

$$\{x, y, \{u, v, w\}\} - \{u, v, \{x, y, w\}\} = \{\{x, y, u\}, v, w\} - \{u, \{v, x, y\}, w\}.$$
For \( x, y, z \in \mathcal{M} \) consider the following operators
\[
T(x, y) z = \{x, y, z\} \\
Q(x, z) y = \{x, y, z\} \\
Q(x, x) = 2Q(x) \\
B(x, y) = \text{id}_\mathcal{M} - T(x, y) + Q(x) Q(y).
\]
The operators \( B(x, y) \) and \( T(x, y) \) are \( \mathbb{C} \)-linear and the operator \( Q(x) \) is \( \mathbb{C} \)-antilinear. \( B(x, y) \) is called the Bergman operator. An Hermitian Jordan triple system is called \textit{positive} if the Hermitian form
\[
(u \mid v) = \text{tr} T(u, v)
\]
is positive definite. An element \( c \in \mathcal{M} \) is called \textit{tripotent} if \( \{c, c, c\} = 2c \). Two tripotents \( c_1 \) and \( c_2 \) are called \textit{(strongly) orthogonal} if \( T(c_1, c_2) = 0 \).

**HSSNT associated to HPJTS.** M. Koecher ([14], [15]) discovered that to every HPJTS \( (\mathcal{M}, \{\cdot, \cdot, \cdot\}) \) one can associate an HSSNT, in its realization as a bounded symmetric domain \( \Omega \) centered at the origin \( 0 \in \mathcal{M} \). The domain \( \Omega \) is defined as the connected component containing the origin of the set of all \( u \in \mathcal{M} \) such that \( B(u, u) \) is positive definite with respect to the Hermitian form \( (u \mid v) = \text{tr} T(u, v) \).

\textit{We will always consider such a domain in its (unique up to linear isomorphism) circled realization.} Suppose that \( \mathcal{M} \) is \textit{simple} (i.e. \( \Omega \) is irreducible). The flat form \( \omega_0 \) is defined by
\[
\omega_0 = -\frac{i}{2\gamma} \partial \bar{\partial} (z \mid z),
\]
where \( \gamma \) is the genus of \( \Omega \). If \( (z_1, \ldots, z_n) \) are orthonormal coordinates for the Hermitian product \( (u \mid v) \), then
\[
\omega_0 = \frac{i}{2\gamma} \sum_{m=1}^{d} dz_m \wedge d\bar{z}_m.
\]
The reproducing kernel \( K_\Omega \) of \( \Omega \), with respect \( \omega_0 \) is given by
\[
(K_\Omega (z, \bar{z}))^{-1} = C \text{ det } B(z, z),
\]
where \( C = \int_\Omega \frac{\omega^n}{n!} \). When \( \Omega \) is irreducible,
\[
\omega_{\text{hyp}} = -\frac{i}{2\gamma} \partial \bar{\partial} \log \text{ det } B.
\]
is the \textit{hyperbolic} form on \( \Omega \), with the associated hyperbolic metric \( g_{\text{hyp}} \) (whose holomorphic sectional curvature is \(-4\)). The HSSNT associated to \( \mathcal{M} \) is \( (\Omega, g_{\text{hyp}}) \).

Moreover the hyperbolic form is related to the flat form by
\[
\omega_{\text{hyp}}(z)(u, v) = \omega_0(B(z, z)^{-1}u, v).
\]
The HPJTS \((\mathcal{M}, \{\cdot, \cdot\})\) can be recovered by its associated HSSNT \(\Omega\) by defining \(\mathcal{M} = T_0\Omega\) (the tangent space to the origin of \(\Omega\)) and
\[
\{u, v, w\} = \frac{1}{2} (R_0(u, v) w + J_0 R_0(u, J_0 v) w),
\]
where \(R_0\) (resp. \(J_0\)) is the curvature tensor of the Bergman metric (resp. the complex structure) of \(\Omega\) evaluated at the origin. For more informations on the correspondence between HPJTS and HSSNT we refer also to p. 85 in Satake’s book [23].

**Spectral decomposition and polar coordinates.** Let \(\mathcal{M}\) be a HPJTS. Each element \(z \in \mathcal{M}\) has a unique spectral decomposition
\[
z = \lambda_1 c_1 + \cdots + \lambda_s c_s \quad (0 < \lambda_1 < \cdots < \lambda_s),
\]
where \((c_1, \ldots, c_s)\) is a sequence of pairwise orthogonal tripotents and the \(\lambda_j\) are real number called eigenvalues of \(z\). The integer \(s = \text{rk}(z)\) is called rank of \(z\). For every \(z \in \mathcal{M}\) let \(\max\{\cdot\}\) denote the largest eigenvalue of \(z\), then \(\max\{\cdot\}\) is a norm on \(\mathcal{M}\) called the spectral norm. The HSSNT \(\Omega \subset \mathcal{M}\) associated to \(\mathcal{M}\) is the open unit ball in \(\mathcal{M}\) centered at the origin (with respect the spectral norm), i.e.,
\[
\Omega = \{z = \sum_{j=1}^{s} \lambda_j c_j \mid \max\{z\} = \max_{j} \{\lambda_j\} < 1\}.
\]
The rank of \(\mathcal{M}\) is \(\text{rk}(\mathcal{M}) = \max\{\text{rk}(z) \mid z \in \mathcal{M}\}\), moreover \(\text{rk}(\mathcal{M}) = \text{rk}(\Omega) =: r\).
The elements \(z\) such that \(\text{rk}(z) = r\) are called regular. If \(z \in \mathcal{M}\) is regular, with spectral decomposition
\[
z = \lambda_1 e_1 + \cdots + \lambda_r e_r \quad (\lambda_1 > \cdots > \lambda_r > 0),
\]
then \((e_1, \ldots, e_r)\) is a (Jordan) frame of \(\mathcal{M}\).

The set \(F\) of frames (also called Fürstenberg-Satake boundary of \(\Omega\)) is a compact manifold. The map \(F : \{\lambda_1 > \cdots > \lambda_r > 0\} \times F \to \mathcal{M}_{\text{reg}}\) defined by:
\[
((\lambda_1, \ldots, \lambda_r), (c_1, \ldots, c_r)) \mapsto \sum_{j} \lambda_j c_j
\]
is a diffeomorphism onto the open dense set \(\mathcal{M}_{\text{reg}}\) of regular elements of \(\mathcal{M}\), moreover its restriction
\[
\{1 > \lambda_1 > \cdots > \lambda_r > 0\} \times F \to \Omega_{\text{reg}}
\]
is a diffeomorphism onto the (open dense) set \(\Omega_{\text{reg}}\) of regular elements of \(\Omega\). This map plays the same role as polar coordinates in rank one.

**Proof of Theorem 1.2** By using the rotational symmetries of \(\Omega \subset \mathcal{M}\) one can show that the diastasis function at the origin \(D^\text{hyp}_0 : M \to \mathbb{R}\), associated to \(g_{\text{hyp}}\), is
globally defined and reads as

$$D_{0}^{\text{hyp}}(z) = \frac{1}{\gamma} \log \left( C K_{\Omega}(z, \overline{z}) \right),$$

where $C = \int_{\Omega} \omega_{n}^{\Omega}$, (see [16] for a proof and further results on Calabi’s function for HSSNT).

Let $z \in \Omega$ be a regular point and let $z = \sum_{j=1}^{r} \lambda_{j} c_{j}$ be its expression in polar coordinates. We have that (see [22] for a proof)

$$B(z, z) c_{j} = (1 - \lambda_{j}^{2})^{2} c_{j}, \quad j = 1, \ldots, r, \quad (21)$$

$$\det B(z, z) = \prod_{j=1}^{r} (1 - \lambda_{j}^{2})^{\gamma},$$

Thus (17) yields the expression of the diastasis with respect the coordinates (20),

$$D_{0}^{\text{hyp}}(z) = - \frac{1}{\gamma} \log \det B(z, z) = - \log \prod_{j=1}^{r} (1 - \lambda_{j}^{2})$$

so

$$d_{z} D_{0}^{\text{hyp}} = \sum_{j=1}^{r} \frac{2 \lambda_{j}}{1 - \lambda_{j}^{2}} d\lambda_{j}. \quad (22)$$

By [13], [21] and [22], we see that $g_{\text{hyp}} \left( \frac{\partial}{\partial \lambda_{j}}, \frac{\partial}{\partial x_{k}} \right) = \delta_{jk} (1 - \lambda_{j}^{2})$, we conclude that

$$\|d_{z} D_{0}^{\text{hyp}}\|^{2} = \sum_{j=1}^{r} \frac{4 \lambda_{j}^{2}}{(1 - \lambda_{j}^{2})^{2}} \|d\lambda_{j}\|^{2} = 4 \sum_{j=1}^{r} \lambda_{j}^{2}. \quad (23)$$

In polar coordinates of $\Omega$, the expression of the distance from the origin, with respect $g_{\text{hyp}}$, is $\rho_{\text{hyp}}(0, z) = \sqrt{\sum_{j=1}^{r} \arctanh^{2}(\lambda_{j})}$ (see [21] (8)). Assume now that $z \in B_{0}^{t}(t)$, i.e. that $\sqrt{\sum_{j=1}^{r} \arctanh^{2}(\lambda_{j})} < t$. By the concavity of the function $\arctanh^{2}\left(\sqrt{\cdot}\right)$ we get:

$$\sqrt{r} \arctanh \left( \frac{\sqrt{\sum_{j=1}^{r} \lambda_{j}^{2}}}{\sqrt{r}} \right) \leq \sqrt{\sum_{j=1}^{r} \arctanh^{2}(\lambda_{j})} < t,$$

thus

$$\sum_{j=1}^{r} \lambda_{j}^{2} \leq r \tanh^{2}\left( \frac{t}{\sqrt{r}} \right).$$

Substituting the previous inequality in (23) we get

$$\chi(0) = \sup_{z \in B_{0}^{t}(t)} \|d_{z} D_{0}\|^{2} \leq 4 r \tanh^{2}\left( \frac{t}{\sqrt{r}} \right). \quad (24)$$
Finally Theorem 1.1 yields,

\[ \frac{n^2}{r \tanh^2 \left( \frac{t}{r} \right)} \leq \lambda_1 \left( B^\Omega_p(t), g_{\text{hyp}} \right), \]

for any \( p \in \Omega \). This shows the validity of (3).

As \( t \) tend to infinity we see that \( X(0) = \frac{n^2}{r} \), so substituting this equality in (1) we obtain (2). Recalling that \( n = r \left( b + 1 + \frac{a^2}{2} (r - 1) \right) \), by (13), one can prove that

\[ \text{Ent}_v(\Omega, g_{\text{hyp}}) = \frac{2 n}{\sqrt{r}} \sqrt{1 + \frac{a^2 r^2}{12 n^2} (r^2 - 1)}, \]

therefore the equality in (2) is attained if and only if \( r = 1 \) or \( a = 0 \) i.e. when \( \Omega \) is the complex hyperbolic disc and this concludes the proof of Theorem 1.2.

\[ \square \]

**Remark 4.1.** Consider \( B^c(t) = \{ x \in C^n : \|z\| < t \} \) the ball of \( C^n \) of radius \( t \) with the induced euclidean metric \( g_e \). The diastasis \( D_0(z) = \|z\|^2 \). By Theorem 1.1 we get

\[ \frac{n^2}{t^2} \leq \lambda_1 \left( B^c(t), g_e \right), \]

which is in accordance with the celebrated J. Cheeger inequality \( \frac{h^2(M, g)}{4} \leq \lambda_1 (M, g) \), where \( h(M, g) \) is the so called Cheeger constant, defined as the minimum of \( \frac{\text{Vol}_{n-1}(\partial D)}{\text{Vol}(D)} \) among the compact domain \( D \) of a Riemannian manifold \( (M, g) \) and the isoperimetric inequality give us \( h \left( B^c(t), g_e \right) = \frac{\text{Vol}_{n-1}(\partial B^c(t), g_e)}{\text{Vol}(B^c(t), g_e)} = \frac{2 n}{t} \).

It is also worth pointing out that X. Wang in a recent paper [24] showed that for any HSSNT \( \lambda_1 (\Omega, g_{\text{hyp}}) = \frac{h^2(\Omega, g_{\text{hyp}})}{4} \). Nevertheless, in general, the computation of the Cheeger constant is a difficult matter and we will study the link between the diastasis function and the Cheeger constant in a forthcoming paper.

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