Repair rate lower bounds for distributed storage

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Abstract—One of the primary objectives of a distributed storage system is to reliably store a large amount \( dsize \) of source data for long durations using a large number \( N \) of unreliable storage nodes, each with capacity \( nsize \). The storage overhead \( \beta \) is the fraction of system capacity available beyond \( dsize \), i.e.,

\[
\beta = 1 - \frac{dsize}{N \cdot nsize},
\]

(1)

Storage nodes fail randomly over time and are replaced with initially empty nodes, and thus data is erased from the system at an average rate

\[
erate = \lambda \cdot N \cdot nsize,
\]

(2)

where \( 1/\lambda \) is the average lifetime of a node before failure. To maintain recoverability of the source data, a repairer continually reads data over a network from nodes at some average rate \( rrate \), and generates and writes data to nodes based on the read data.

The main result is that, for any repairer, if the source data is recoverable at each point in time then it must be the case that

\[
rrate \geq \frac{erate}{2 \cdot \beta}
\]

(3)

asymptotically as \( N \) goes to infinity and \( \beta \) goes to zero. Thus, Inequality (3) provides a fundamental lower bound on the average rate that any repairer needs to read data from the system in order to maintain recoverability of the source data.

Index Terms—distributed information systems, data storage systems, data warehouses, information science, lower bounds, information theory, information entropy, error compensation, mutual information, channel capacity, channel coding, time-varying channels, error correction codes, Reed-Solomon codes, network coding, signal to noise ratio, throughput, distributed algorithms, algorithm design and analysis, reliability, reliability engineering, reliability theory, fault tolerance, redundancy, robustness, failure analysis, equipment failure.

I. OVERVIEW

A distributed storage system generically consists of interconnected storage nodes, where each node can store a large quantity of data. We let \( N \) be the number of storage nodes in the system, where each node has \( nsize \) bits of storage capacity.

Commonly, distributed storage systems are built using relatively inexpensive and generally not completely reliable hardware. For example, nodes can go offline for periods of time (transient failure), in which case the data they store is temporarily unavailable, or permanently fail, in which case the data they store is permanently erased. Permanent failures are not uncommon, and transient failures are frequent.

Although it is often hard to accurately model failures, an independent failure model can provide insight into the strengths and weaknesses of a practical system, and can provide a first order approximation to how a practical system operates. In fact, one of the primary reasons practical storage systems are built using distributed infrastructure is so that failures of the infrastructure are as independent as possible.

In our model, each storage node permanently fails independently and randomly at rate \( \lambda \) at each point in time and is replaced with a new node initialized to zeroes when it fails, and thus bits are erased from the system at an average rate \( erate \) as defined in Equation (2).

A primary goal of a distributed storage system is to reliably store as much source data as possible for a long time, i.e., at each point in time the source data should be recoverable from the data stored in the system at that point in time. We let \( dsize \) be the size of the source data to be stored. To maintain recoverability of the source data, a repairer continually reads data over a network from nodes at some average rate \( rrate \), and generates and writes data to nodes based on the read data.

Distributed storage systems generally allocate a fraction of their capacity to storage overhead, which is used by the repairer to help maintain recoverability of source data as failures occur. The storage overhead \( \beta \) is the fraction of capacity available beyond the size of the source data, i.e., \( \beta \) is defined in Equation (1), and thus \( dsize = (1-\beta) \cdot N \cdot nsize \).

The main result is that, for any repairer, if the source data is recoverable at each point in time then it must be the case that

\[
rrate \geq \frac{erate}{2 \cdot \beta}
\]

expresses a fundamental trade-off between the repairer read rate and storage overhead as a function of the erasure rate.

A. Practical system parameters

An example of a practical system is one with \( N = 10^5 \) nodes, with \( nsize = 10^{10} \) bits of capacity at each node, thus \( N \cdot nsize = 10^{21} \) bits is the system capacity. The amount of storage needed by the repairer to store its programs and state generously is at most something like \( nsize = 10^{13} \) bits. Generally, \( nsize \gg vsize \gg N \). We assume \( nsize \geq N \) and \( vsize \ll N \cdot nsize \) in our bounds with respect to growing \( N \).

Practical values of \( \beta \) range from \( 2/3 \) (triplication) to \( 1/20 \) and smaller. In the example, \( dsize = (1-\beta) \cdot N \cdot nsize \approx 10^{21} \) bits. In practice nodes fail in a few years, e.g., \( 1/\lambda = 3 \) years.
For practical systems, source data is generally maintained at the granularity of objects, and erasure codes are used to generate redundant data for each object. When using a \((n, k, r)\) erasure code, each object is segmented into \(k\) source fragments, an encoder generates \(r = n - k\) repair fragments from the \(k\) source fragments, and each of these \(n = k + r\) fragments is stored at a different node. An erasure code is MDS (maximum distance separable) if the object can be recovered from any \(k\) of the \(n\) fragments.

B. Small code systems

Replication is an example of a trivial MDS erasure code, i.e., each fragment is a copy of the original object. For example, triplication can be thought of as using the simple \((3, 1, 2)\) erasure code, wherein the object can be recovered from any one of the three copies. Some practical distributed storage systems use triplication.

Reed-Solomon codes \([2], [3], [5]\) are MDS codes that are used in a variety of applications and are a popular choice for storage systems. For example, \([11]\) and \([9]\) use a \((9, 6, 3)\) Reed-Solomon code, and \([12]\) uses a \((14, 10, 4)\) Reed-Solomon code. These are examples of small code systems, i.e., systems that use small values of \(n, k\) and \(r\).

There are some issues that complicate the design of small code systems. For example, the data for each object is spread over a tiny fraction of the nodes, i.e., in a system of 100,000 nodes, triplication spreads the data for each object over only 3 nodes, and a \((14, 10, 4)\) Reed-Solomon code spreads the data for each object over only 14 nodes. Thus, an issue for a small code system is how to distribute the data for all the objects smoothly over all the nodes.

A typical approach is to assign each object to a placement group, where each placement group maps to \(n\) of the \(N\) nodes, which determines where the \(n\) fragments of the object are stored. An equal amount of object data should be assigned to each placement group, and an equal number of placement groups should map a fragment to each node. For small code systems, Ceph \([14]\) recommends \(\frac{100N}{n}\) placement groups, i.e., 100 placement groups map a fragment to each node. A placement group should avoid mapping fragments to nodes with correlated failures, e.g., to the same rack. Pairs of placement groups should avoid mapping fragments to the same pair of nodes. Placement groups are continually remapped as nodes fail and are added. These and other issues make the design of small code systems challenging.

Since a small number \(r + 1\) of failures can cause source data loss for small code systems, reactive repair is used, i.e., the repairer operates as quickly as practical to regenerate fragments lost from a node that permanently fails before another node fails, and typically reads \(k\) fragments to regenerate each lost fragment. Thus, the peak read rate is higher than the average read rate, and the average read rate is \(k\) times the failure erasure rate.

As highlighted in \([12]\), the read rate needed to maintain source data recoverability for small code systems can be substantial. Modifications of standard erasure codes have been designed for storage systems to reduce this rate, e.g., local reconstruction codes \([10], [12]\), and regenerating codes \([7], [8]\). Some versions of local reconstruction codes have been used in deployments, e.g., by Microsoft Azure.

C. Liquid systems

Another approach introduced in \([13]\) is liquid systems, which use erasure codes with large values of \(n, k\) and \(r\). For example, \(n = N\) and a fragment is assigned to each node for each object, i.e., only one placement group is used for all objects. The RaptorQ code \([4], [6]\) is an example of an erasure code that is suitable for a liquid system, since objects with large numbers of fragments can be encoded and decoded efficiently in linear time.

Typically \(r\) is large for a liquid system, thus source data is unrecoverable only when a large number of nodes fail. A liquid repairer is lazy, i.e., repair operates to slowly regenerate fragments erased from nodes that have permanently failed. The repairer reads \(k\) fragments for each object to regenerate around \(r\) fragments erased over time due to failures, and the peak read rate is close to the average read rate. The peak read rate for the liquid repairer described in \([13]\) is within a factor of two of the lower bounds on the read rate, and the peak read rate for the advanced liquid repairer described in \([19]\) asymptotically approaches the lower bounds.

II. Related work

The groundbreaking research of Dimakis et. al., described in \([7]\) and \([8]\), is closest to our work: An object-based distributed storage framework is introduced, and optimal tradeoffs between storage overhead and local-computation repairer read rate are proved with respect to repairing an individual object. \([7]\) and \([8]\) describe a framework, the types of repairers that fit into the framework, and lower bounds on these types of repairers within the framework, which are hereafter referred to as the Regenerating framework, Regenerating repairers, and Regenerating lower bounds, respectively.

The Regenerating framework was originally introduced to model repair of a single lost fragment, and is applicable to reactive repair of a single object. The Regenerating framework is parameterized by \(\{(n, k, d, \alpha, \gamma)\}: n\) is the number of fragments for the object (each stored at a different node); \(k\) is the number of fragments from which the object must be recoverable; \(d\) is the number of fragments used to generate a lost fragment at a new node when a node fails; \(\alpha\) is the fragment size; and \(\gamma\) is the total amount of data generated and read across the network to generate a fragment at a new node, i.e., \(\gamma/d\) is the amount of data generated from each of \(d\) fragments needed to generate a fragment at a new node. Regenerating lower bounds on the local-computation repairer read data rate prove necessary conditions on the Regenerating framework parameters used by any Regenerating repairer to ensure that an individual object remains recoverable when using reactive repair.

The Regenerating lower bounds were not originally designed to provide general lower bounds for a large system of nodes. Nevertheless, it is interesting to interpret the Regenerating framework parameterized by \(\{(n, k, d, \alpha, \gamma)\}\) in the context of a system so that the Regenerating lower bounds...
can be as closely compared as possible to the system level lower bounds proved in this paper. Let \( d_{\text{size}} \) be the amount of source data to be stored in the system. Then, \( n \) is set to the number of nodes \( N \) in the system, and \( \alpha \) is set to the storage capacity \( n_{\text{size}} \) of each node, and thus the storage overhead \( \beta \) is as shown in Equation (1). Since we want the best tradeoff possible between the amount of data \( \gamma \) read by the Regenerating repairer to replace each failed node and the storage overhead \( \beta \), we set \( k = d = N - 1 \). (At a general point in time a failed node is being replaced and there are only \( N - 1 \) available nodes.) Thus, at the system level we consider the Regenerating framework with parameters \( (n = N, k = N - 1, d = N - 1, \alpha = n_{\text{size}}, \gamma) \).

The Regenerating framework uses labeled acyclic directed graphs, where each directed edge is labeled with the amount of data transferred from the node at the tail to the node at the head of the edge, to represent the actions of Regenerating repairers, and it is the properties of these graphs that are used to prove the Regenerating lower bounds. The labeled acyclic graphs restrict the possible actions taken by Regenerating repairers as follows. Suppose a node with identifier \( id \) fails at time \( t \) and the next failure is at time \( t' > t \). A Regenerating repairer is restricted to the following actions between time \( t \) and \( t' \):

- For each node \( id' \) other than the failed node \( id \), the Regenerating repairer computes a function of the \( n_{\text{size}} \) bits stored at node \( id' \) to generate \( \frac{1}{N-1} \) bits, and transfers the \( \frac{1}{N-1} \) bits to the replacement node \( id \).
- From the \( \gamma \) bits received at the replacement node for \( id \) from the \( N-1 \) nodes other than node \( id \), the Regenerating repairer computes a function of the \( \gamma \) bits to generate the \( n_{\text{size}} \) bits to be stored at the replacement node for \( id \).

Thus, between time \( t \) and \( t' \), a fixed amount of data is transferred to the replacement node for \( id \) and no data is transferred to any other node; once another node fails at time \( t' \), no more data is transferred to the replacement node for \( id \) until it fails again and is replaced by another replacement node; an equal amount of data \( \frac{1}{N-1} \) is read and transferred from each of the \( N - 1 \) non-failing node to the replacement node for \( id \).

**Dimakis Lemma 2.1.** The following holds as \( N \) goes to infinity. For any Regenerating repairer parameterized by \( (N, N - 1, N - 1, n_{\text{size}}, \gamma) \), if

\[
\gamma < \frac{n_{\text{size}}}{2 \cdot \beta},
\]

then the source data cannot be reliably recovered at the end of any failure sequence with \( N - 1 \) distinct failures.

**Proof** Inequality (16) of [8] implies that if

\[
\sum_{i=0}^{k-1} \min \left\{ \frac{(d-i) \cdot \gamma}{d}, \alpha \right\} < d_{\text{size}}
\]

for a Regenerating repairer then the source data cannot be reliably recovered at the end of any failure sequence with \( k \) distinct failures. With \( k = d = N - 1 \), \( \alpha = n_{\text{size}} \), and using Equation (1), we can rewrite Inequality (5) as

\[
\sum_{j=0}^{N-1} \min \left\{ \frac{j \cdot \gamma}{N}, n_{\text{size}} \right\} < (1 - \beta) \cdot N \cdot n_{\text{size}}.
\]

As \( N \) goes to infinity, we can approximate the sum by integration to yield:

\[
\int_{s=0}^{\infty} s \cdot \gamma \, ds + \int_{s=n_{\text{size}}}^{1} n_{\text{size}} \cdot ds < (1 - \beta) \cdot n_{\text{size}}.
\]

Simplifying yields Inequality (4).

Dimakis Lemma 2.1 is tight, i.e., [7], [8] describe Regenerating repairers with \( \gamma = \frac{n_{\text{size}}}{2 \cdot \beta} \) that maintain source data recoverability for periodic failure sequences.

Dimakis Lemma 2.1 holds for any Regenerating repairer and for any failure sequence with \( N - 1 \) distinct failures, even if the Regenerating repairer is provided the failures in advance. However, the following repairer maintains recoverability of the source data, uses storage overhead \( \beta = \frac{1}{\gamma} \), and reads \( n_{\text{size}} \) bits per failure for any failure sequence that is provided in advance. The source data of size \( d_{\text{size}} = (N - 1) \cdot n_{\text{size}} \) is stored on \( N - 1 \) nodes, and the remaining node is empty. Just before the next failure, the repairer copies all data from the node that is going to fail to the empty node, and the new node replacing the failed node becomes the empty node.

This repairer: (a) is not a Regenerating repairer; (b) violates Inequality (4) of Dimakis Lemma 2.1 and yet the source data can always be reliably recovered; (c) shows there is no non-trivial general lower bound if the failure sequence is provided to the repairer in advance. This shows that Dimakis Lemma 2.1 is not a lower bound that applies to all repairers, and that it is impossible to prove non-trivial general lower bounds if the failure sequence is provided to the repairer in advance.

The right-hand side of Inequality (4) of Dimakis Lemma 2.1 converges to the same value as the right-hand side of Inequality (30) of Poisson Failures Theorem 8.3 as \( N \) goes to infinity \( \beta \) goes to zero. The main differences between Dimakis Lemma 2.1 and Poisson Failures Theorem 8.3 are the generality of the repairers to which the lower bounds apply and the type of failure sequences used to prove the lower bounds.

Dimakis Lemma 2.1 applies to Regenerating repairers within the restrictions of the Regenerating framework, i.e., a Regenerating repairer that predictably reads a given amount of data from each node and transfers a predictable amount of data to a replacement node between failures. On the other hand, Poisson Failures Theorem 8.3 applies to any repairer, i.e., a repairer that can be completely unpredictable.

For a Regenerating repairer that doesn’t read enough data, the failure sequence that causes the source data to be unrecoverable in Dimakis Lemma 2.1 is an atypical failure sequence with \( N - 1 \) distinct failures. On the other hand, for a general repairer that doesn’t read enough data, the failure sequence that causes the source data to be unrecoverable in Poisson Failures Theorem 8.3 is a typical random failure sequence that is chosen independently of the repairer.

The following examples show that repairers for practical systems do not belong to the class of Regenerating repairers.
for which Dimakis Lemma 2.1 applies, and thus Dimakis Lemma 2.1 is not a lower bound on repairers in general.

As can be inferred from Section I-B a typical repairer for a small code system reads data from only a fraction of the $N$ nodes to replace the data on a failed node. Thus, repairers for small code systems are not Regenerating repairers for which the Regenerating lower bound of Dimakis Lemma 2.1 applies.

Liquid systems repairers [18], [19] transfer data incrementally to a replacement node over a constant fraction of $N$ failures after the node it replaces fails. Thus, repairers for liquid systems are not Regenerating repairers for which the Regenerating lower bound of Dimakis Lemma 2.1 applies.

III. SYSTEM MODEL

We introduce a model of distributed storage which is inspired by properties inherent and common to systems described in Section I. This model captures some of the essential features of any distributed storage system. All lower bounds are proved with respect to this system model.

There are a number of possible strategies beyond those outlined in Section I that could be used to implement a distributed storage system. One of our primary contributions is to provide fundamental lower bounds on the read rate needed to maintain source data recoverability for any distributed storage system, current or future, for a given storage overhead and failure rate. Appendix B provides details on how the system model introduced in this section applies to real systems.

A. Architecture

Figure 1 shows an architectural overview of the distributed storage model. A storer generates data from source data $x \in \{0, 1\}^{dsize}$ received from a source, and stores the generated data at nodes. In our model, the source data $X$ is randomly and uniformly chosen, where $X \in \mathcal{U} \{0, 1\}^{dsize}$ is a random variable and $\epsilon_{id}$ indicates randomly and uniformly chosen. Thus, $||X|| = H(X) = dsize$, where $||X||$ is the length of $X$ and $H(X)$ is the entropy of $X$.

Figure 2 shows the nodes of the distributed storage system, together with the network that connects each node to a repairer. Each of $N$ nodes $C_0, \ldots, C_{N-1}$ can store $nsize$ bits, and the capacity is $N \cdot nsize$.

As nodes fail and are replaced, a repairer continually reads data from the nodes, computes a function of the read data, and writes the computed data back to the nodes. The repairer tries to ensure that the source data can be recovered at any time from the data stored at the nodes.

As shown in Figure 1 after some amount of time $t$ passes, a recoverer reads data from the nodes to generate $x'$, which is provided to a destination, where $x'$ is reliably recovered if $x' = x$. The goal is to maximize the amount of time $t$ the recoverer can reliably recover $x$.

B. Failures

A failure sequence determines when and what nodes fail as time passes. A failure sequence is a combination of two sequences, a timing sequence

$$t_0 \leq t_1 \leq \cdots \leq t_i \cdots,$$

where for index $i$, $t_i$ is the time at which a node fails, and an identifier sequence

$$id_0, id_1, \ldots, id_i, \ldots,$$

where $id_i$ is the identifier of the node that fails at time $t_i$.

All $nsize$ bits stored at node $id_i$ are immediately erased at time $t_i$ when the node fails, where erasing a bit means setting its value to zero. This can be viewed as immediately replacing a failed node with a replacement node with storage initialized to zeroes. Thus, at each time there are $N$ nodes.

A primary objective of practical distributed storage architectures is to distribute the components of the system so that failures are as independent as possible. Poisson failure distributions are an idealization of this primary objective, and are often used to model and evaluate distributed storage systems in practice. For a Poisson failure distribution with rate $\lambda$, the time between when a node is initialized and when it fails is an independent exponential random variable with rate $\lambda$, i.e., $\frac{1}{\lambda}$ is the average lifetime of a node between when it is initialized and when it fails. Our main lower bounds in Section VIII are with respect to Poisson failure distributions.
C. Network

The model assumes there is a network interface between each node and the repairer over which all data from and to the node travels. One of the primary lower bound metrics is the amount of data that travels over interfaces from nodes to the repairer, which is counted as data read by the repairer. For the lower bounds, this is the only network traffic that is counted. All other data traffic within the repairer, i.e. data traffic internal to a distributed repairer, data traffic over an interface from the repairer to nodes, or any other data traffic that does not travel over an interface from a node to the repairer, is not counted for the lower bounds. It is assumed that the network is completely reliable, and that all data that travels over an interface from a node to the repairer is instantly available everywhere within the repairer.

D. Storer

A storer takes the source data \( x \) and generates and stores data at the nodes in a preprocessing step when the system is first initialized and before there are any failures. We assume that the storer can reliably recover \( x \) from the data stored at the nodes immediately after the preprocessing step finishes.

For simplicity, we view the storer preprocessing step as part of the repairer, and any data read during the storer preprocessing step is not counted in the lower bounds.

For the lower bounds, there are no assumptions about how the storer generates the stored data from the source data, i.e. no assumptions about any type of coding used, no assumptions about partitioning the source data into objects, etc. As an example, the source data can be encrypted, compressed, encoded using an error-correcting code or erasure code, replicated, or processed in any other way known or unknown to generate the stored data, and still the lower bounds hold. Analogous remarks hold for the repairers described next.

E. Repairer

A repairer for a system is a process that operates as follows. The identifier \( \text{id}_i \) is provided to a repairer at time \( t_i \), which alerts the repairer that all \( nsize \) bits stored on node \( \text{id}_i \) are lost at that time. As nodes fail and are replaced, the repairer reads data over interfaces from nodes, performs computations on the read data, and writes computed data over interfaces to nodes. A primary metric is the number of bits the repairer reads over interfaces from storage nodes.

Appendix [A] provides a detailed description of repairers, including local-computation repairers.

F. Recoverer

For any repairer \( R \) there is a recoverer \( A \) such that if the source data is \( x \) and the state at time \( t \) is \( S(t) \) when the repairer is \( R \) then \( A(S(t)) \) should be equal to \( x \).

Source data \( x \) is recoverable at time \( t \) with respect to repairer \( R \) and recoverer \( A \) if \( A(S(t)) = x \). Source data \( x \) is unrecoverable at time \( t \) with respect to repairer \( R \) and recoverer \( A \) if \( A(S(t)) \neq x \).

G. System State

At time \( t \), let \( V(t) \) be the bits stored in the global memory of the repairer, where \( nsize = ||V(t)|| \).

\[ C_0(t), \ldots, C_{N-1}(t) \]

be the bits stored at nodes \( 0, \ldots, N-1 \), respectively, where \( nsize = ||C_j(t)|| \) is the capacity of each node \( j \), and

\[ S(t) = \{ V(t), \{ C_0(t), \ldots, C_{N-1}(t) \} \} \]

is the global state of the system at time \( t \), where \( nsize = ||S(t)|| \). Thus,

\[ ssize = vsize + N \cdot nsize \]

is the size of the global system state at any time \( t \).

IV. GUIDE FOR LOWER BOUND PROOFS

The ultimate goal is to prove that Inequality (3) holds with respect to the Poisson failure distribution for any repairer.

In practice, the timing of failures, and the identity of which nodes fail, are not known in advance, and thus repairers must handle these uncertainties. A much simpler model for repairers to handle is a periodic failure sequence, i.e., the time between consecutive failures is a constant known to the repairer. Many of the lower bounds we prove hold for periodic failure sequences, and the only uncertainty is which nodes fail.

Lower bounds for \( \beta \geq \frac{1}{2} \) are not of great interest, since for \( \beta = \frac{1}{2} \), the repairer that maintains a duplicate copy of the source data succeeds in maintaining recoverability of the source data forever for periodic failure sequences. Furthermore, in practice, the interest is to decrease \( \beta \) as much as possible, and thus we hereafter restrict attention to \( \beta \leq \frac{1}{2} \).

Section [V] introduces the notion of a phase, where a phase is a failure sequence of a specified number \( M \) of distinct failures. Let \( rrate_i \) be the average read rate of a repairer over the first \( i \) failures of the phase for a periodic failure sequence. The overall idea of the lower bound proof is to show that, for any repairer, either there is an \( i \leq M \) such that \( rrate_i \) is above a lower bound rate, or at the end of the phase the source data is unrecoverable.

Section [V] shows that the system state at the end of the phase can be generated from \( D \), where \( D \) is the concatenation of the data read from nodes that fail in a phase before they fail and the data at nodes at the start of the phase that don’t fail in the phase. The crucial but simple Compression Lemma [5.3] and Compression Corollary [5.2] show that if \( ||D|| < dsize \) then the source data is unrecoverable from \( D \), where \( ||D|| \) is the length of \( D \). Since the system state at the end of the phase can be generated from \( D \), this implies that if \( ||D|| < dsize \) then the source data is unrecoverable from the system state at the end of the phase.

Section [V] introduces a restricted class of repairers, Equal-read repairers, that predictably read an equal amount of data from each node between failures. Equal-read repairers are introduced for two reasons: (1) they are similar to (but more general than) the Regenerating repairers discussed in Section [I] (2) based on the framework introduced in Section [V] the lower bound proofs for Equal-read repairers are easy and
straightforward. Equal-read Lemma 6.1 of Section VII shows that if the predictable read rate of the Equal-read repairer is below a lower bound then the source data is unrecoverable with very high probability at the end of the phase.

The Equal-read Lemma 6.1 lower bound for Equal-read repairers holds for any failure sequence with distinct failures, even if the failure sequence is known in advance to the Equal-read repairer. However, as outlined in Section II there is no non-trivial lower bound for general repairers if the failure sequence is known in advance. Thus, using random failure sequences for which the repairer cannot predict which future nodes will fail is key to proving general lower bounds.

Repaired actions can be unpredictable: A repairer may read different amounts of data from individual nodes between failures, and may read different amounts of data in aggregate from all nodes between different failures. The repaired actions can depend on the source data, which nodes have failed in the past, and the timing of past failures.

Section VII provides the technical core of the lower bound proofs that use random failure sequences with distinct failures to prove lower bounds on general repairers that may act unpredictably. The proof of Core Lemma 7.1 in Section VII is the most technically challenging proof in the paper. It shows that, for any repairer, when the identifier sequence consists of randomly chosen distinct identifiers within a phase of $M$ failures then there is only a tiny probability $\delta_i$ that $\|D_i\| \geq dsize - nsize$ when $rate_i$ is below a lower bound for each $i \leq M$. Supermartingale Theorem D.1 is used to prove Core Lemma 7.1, and may be of independent interest. Core Theorem 7.2 directly uses Core Lemma 7.1 to show that with very high probability there is an $i \leq M$ where the read rate $rate_i$ for the repairer up to the $i^{th}$ failure in the phase is above a lower bound or else the source data is unrecoverable at the end of the phase.

A phase terminates early if the repairer reads enough data from all nodes in a prefix of the phase, i.e., a phase is terminated at the first index $i \leq M$ where $rate_i$ is above a lower bound, and another phase is started at that point. The overall lower bounds are proved by stitching together consecutive phases. Thus, the lower bound holds over the failure sequence within each stitched together phase. There are some technical issues with stitching together phases. The actions of the repairers have an influence on when one phase ends and the next phase begins. Since the distinct failures within a phase depend on when the phase starts, the repairer has an influence on the failure sequence.

What we would like instead are lower bounds where the failure sequence is chosen completely independently of the repairer, which is achieved in Section VII. Distinct Failures Lemma 8.1 together with Uniform Failures Theorem 8.2 shows that the lower bounds for periodic failures shown in Core Theorem 7.2 still apply when the failures within a failure sequence are chosen uniformly at random and are no longer required to be distinct within a phase. Poisson Failures Theorem 8.3 the main result of this research, shows that the lower bounds of Uniform Failures Theorem 8.2 extend when the timing sequence of the failure sequence is Poisson distributed instead of being restricted to being periodic. Thus, Poisson Failures Theorem 8.3 shows that for any repairer the lower bounds apply when nodes fail independently according to a Poisson process.

V. EMULATING REPAIRERS IN PHASES

We prove lower bounds based on considering the actions of a repairer $R$, or local-computation repairer $R_j$, running in phases. Each phase considers a failure sequence with $M$ failures, where each of the $M$ failures within a phase are distinct, as described in more detail below.

For any $M \leq N$, we write

$$\langle id_0, \ldots, id_{M-1} \rangle$$

when all $M$ identifiers are distinct, i.e., $id_i \neq id_{i'}$ for $0 \leq i \neq i' \leq M - 1$, which we hereafter refer to as a distinct identifier sequence. We write

$$\langle id_0, \ldots, id_{j-1}, ID_j, \ldots, ID_{M-1} \rangle$$

when $\langle id_0, \ldots, id_{j-1} \rangle$ are distinct identifiers, random variable $ID_j$ is defined as

$$ID_j \in U \{0, \ldots, N - 1\} - \{id_0, \ldots, id_{j-1}\},$$

and for $i = j + 1, \ldots, M - 1$, random variable $ID_i$ is defined as

$$ID_i \in U \{0, \ldots, N - 1\} - \{id_0, \ldots, id_{j-1}, ID_j, \ldots, ID_{i-1}\},$$

where $\in U$ indicates randomly and uniformly chosen. Thus, $\langle id_0, \ldots, id_{j-1}, ID_j, \ldots, ID_{M-1} \rangle$ is a distribution on distinct identifier sequences.

A phase consists of executing $R$ on a failure sequence $(tseq, idseq)$, where

$$tseq = \{t_0, t_1, \ldots, t_{M-1}\}$$

is the timing sequence and

$$idseq = \langle id_0, id_1, \ldots, id_{M-1} \rangle$$

is the distinct identifier sequence that is revealed to $R$ as the phase progresses.

For $i \in \{0, \ldots, M - 1\}$, let

$$tseq_i = \{t_0, \ldots, t_i\}$$

be a prefix of $tseq$, and let

$$idseq_i = \langle id_0, \ldots, id_i \rangle$$

be a prefix of $idseq$.

Fix repairer $R$, recoverer $A$, timing sequence $tseq$ and identifier sequence $idseq$, and $x$. The variables defined below depend on these parameters, but to simplify notation this dependence is not explicitly expressed in the variable names.

For $i \in \{0, \ldots, M - 1\}$, $j \in \{0, \ldots, N - 1\}$, let $R_{i,j}$ be the data read by $R$ from node $j$ between $t_0$ and $t_i$ with respect to $x$, $R$, $(tseq_i, idseq_{i-1})$, and let $\text{size}_{i,j} = ||R_{i,j}||$ be the size (or length) of $R_{i,j}$. (If $R$ is a local-computation repairer, then $R_{i,j}$ is the the locally-computed bits read by $R$ over the interface from node $j$ between $t_0$ and $t_i$.) For
Let $i \in \{0, \ldots, M-1\}$, let $R_i$ be the data read from all nodes by $R$ between $t_0$ and $t_i$, and let

$$rsize_i = \sum_{j \in \{0, \ldots, N-1\}} rsize_{i,j} = ||R_i||$$

be the total amount of data read from all nodes by $R$ between $t_0$ and $t_i$.

Let

$$rfsize_i = ||R_{i, id}||,$$

where $R_{i, id}$ is the data read from the node $id_i$ between $t_0$ and the time $t_i$ of its failure with respect to $x$, $R$, $(tseq_i, idseq_i)$. Let

$$rfsize = \sum_{i=0}^{M-1} rfsize_i$$

be the total length of data read by $R$ in the phase from nodes that fail before their failure in the phase.

Before a phase begins, the storer generated and stored data at the nodes based on source data $x$, and the repairer $R$ has been executed with respect to a failure sequence up till time $t_0$, where $t_0$ is just before the time of the first failure of the phase at time $t_0$. We assume that the recoverer $A$ can recover source data $x$ from the state $S(t_0)$.

A. Compressed state $D$

For this subsection, we fix repairer $R$, recoverer $A$, timing sequence $tseq$ and identifier sequence $idseq$ and source data $x$. The variables defined below depend on these parameters, but to simplify notation this dependence is not explicitly expressed in the variable names.

We conceptually define two executions of a phase with respect to $x$, $R$, $A$, and $(tseq, idseq)$. The first execution runs $R$ normally from $t_0$ to $t_{M-1}$ starting system state $S(t_0)$ and ending in $S(t_{M-1})$, where $t_0$ is just before $t_0$ and $t_{M-1}$ is just after $t_{M-1}$. Thus, the failures at times $t_0$ and $t_{M-1}$ are within the phase, but $R$ does not read any bits before $t_0$ or after $t_{M-1}$ in the phase.

Let $RF$ be the concatenation of bits read by $R$ from nodes that fail before they fail in the phase, concatenated in the order they are read. Thus $RF$ contains all the bits of

$$\{R_{i, id} : i \in \{0, \ldots, M-1\}\}$$

and $rfsize = ||RF||$, but the order of the bits in $RF$ is defined by the order in which they are read by $R$.

Let

$$D = \{ V(t_0), C_j(t_0) : j \not\in idseq \}, RF \},$$

which we hereafter refer to as the compressed state with respect to $x$, $R$, and $(tseq, idseq)$, and thus

$$||D|| = vsize + (N-M) \cdot nsize + rfsize.$$  \hfill (6)

The second execution is an exact replay of the first execution, i.e., the repairer $R$ reads, computes, and writes exactly the same bits at the same times as in the first execution with respect to the failure sequence $(tseq, idseq)$ to arrive in the same final state $S(t_{M-1})$ as the first execution. However, the second execution uses the compressed state $D$ in place of $S(t_0)$ as the starting point of the execution. The initial global memory state of $R$ is set to $V(t_0)$ at time $t_0$. For all $j \not\in idseq$, the state of node $j$ is initialized to $C_j(t_0)$ at time $t_0$.

Initially at time $t_0$, $f : \{0, \ldots, N-1\} \to \{0, 1\}$ is set as:

$$f(j) = 0 \text{ for all } j \not\in idseq \text{ and } f(j) = 1 \text{ for all } j \in idseq.$$

Let $t$ be a time within the phase, i.e., $t_0 \leq t \leq t_{M-1}$.

Suppose at time $t$ that $R$ is to read bits over the interface from node $j$: if $f(j) = 0$ then the requested bits are read from $C_j(t)$ exactly the same as in the first execution; if $f(j) = 1$ then the requested bits are provided to $R$ from the next consecutive portion of $RF$ not yet provided to $R$, which, by the properties of $RF$, are guaranteed to be the bits read from node $j$ at time $t$ in the first execution.

Suppose at time $t$ that $R$ is to write bits to node $j$: if $f(j) = 0$ then the bits are written to $C_j(t)$ exactly the same as in the first execution; if $f(j) = 1$ then the write is skipped since whatever bits are subsequently read from node $j$ up till the time node $j$ fails are already part of $RF$.

For each $i \in \{0, \ldots, M-1\}$, $f(id_i)$ is reset to 0 and the state of node $id_i$, $C_{id_i}(t_i)$, is initialized to zeroes at time $t_i$.

If $R$ is a local-computation repairer instead of a repairer then when $R$ is to produce and read locally-computed bits over the interface from node $j$ at time $t$ and $f(j) = 0$ the requested bits are locally-computed by $R$ based also on $C_j(t)$.

It can be verified that the state of the system is $S(t_{M-1})$ at the end of the second execution, whether $R$ is a repairer or a local-computation repairer. Thus, $(S(t_{M-1})), (tseq, idseq))$ can be generated from $(D, (tseq, idseq))$ based on $R$.

B. Viewing $D$ as a cut in an acyclic graph

Similar to [7], [8], the compressed state $D$ can be viewed as a cut in an acyclic graph. An example of the acyclic graph is shown in Figure 3 where there are $N = 6$ storage nodes. The beginning of the phase is at the bottom, and going vertically up corresponds to time flowing forward. The leftmost vertical column is for the global memory $V$ of size $vsize$, and there is a vertical column for each of the $N$ storage nodes, each of size $nsize$. The bottom row of vertices corresponds to the system state $S(t_0)$ at the start of the phase; the second from the bottom row of vertices corresponds to the system state $S(t_{M-1})$; the top row of vertices corresponds to the system state $S(t_{M-1})$ at the end of the phase. The vertices in a storage node column correspond to the state of that storage node over time, where edges flowing out of the column correspond to data transfer out of the node, and edges flowing into the column correspond to data transfer into the node. Similar remarks hold for the global memory column.

The edges pointing vertically up are labeled with the capacity of the corresponding entity, i.e., $vsize$ is the capacity of $V$, and $nsize$ is the capacity of each storage node. The non-vertical edges that connect a first vertex to a second vertex correspond to a data transfer, where the label of the edge corresponds to the amount of data transferred.

In the example shown in Figure 3 $C_5$ fails at time $t_0$, $C_1$ fails slightly later, and $C_3$ fails at a slightly later time. Each
node that fails is replaced with an empty node, and thus there
is no edge from a vertex corresponding to a node just before
it fails to the vertex above corresponding to the replacement
node. Thus, $D$ includes all the data transferred along the edges
that emanate from the vertices in the columns corresponding
to $C_1$, $C_2$, and $C_3$ before their failures, where these edges
are shown in gray in Figure 3.

In the example shown in Figure 3 $C_0$, $C_2$ and $C_4$ do not fail
before the end of the phase. Thus, $D$ includes the $nsize$ bits
of data transferred along the vertical edges from the bottom
to the second from the bottom row for each column
corresponding to these storage nodes, where these edges
are shown in gray in Figure 3. In addition, $D$ includes the $vsize$
bits of data transferred along the vertical edge from the bottom
to the second from the bottom row for the first column
corresponding to global memory $V$.

The cut corresponding to $D$ is shown in Figure 3 as the curved gray line, where $||D||$ is the sum of the labels of the
edges crossing the cut from the vertices below the cut.

The bit values of $D$ determine the edges and the edge label
values in the acyclic graph, i.e., the edges and edge label
values in the acyclic graph depend on the bit values stored
at the vertices in the graph. This is unlike the acyclic graph
representation in [7], [8], where the edges and the edge label
values are independent of the bit values stored at the vertices
in the graph.

Although the acyclic graph visualization of $D$ provides
some good intuition, Section V-A provides the formal defini-
tion of $D$ and its properties.

C. Compression lemma

For this subsection, we fix repairer $R$, recoverer $A$, timing
sequence $tseq$ and identifier sequence $idseq$. The variables
defined below depend on these parameters, but to simplify
notation this dependence is not explicitly expressed in the
variable names.

The value of the source data $x$ is a variable in this
subsection. Random variable $X \leq u \{0,1\}^{dsize}$ is uniformly
distributed on the source data. We let

$$[x:D:x']$$

denote that source data $x$ is mapped before the start of the
phase to a value of $S(t_0)$ by $R$, which in turn is mapped
by $(tseq,idseq)$ to a value $D$ by the first execution of the
emulation of $R$, which is mapped to a value of $S(t_{M-1})$ by
$(tseq,idseq)$ by the second execution of the emulation of $R$,
which in turn is mapped by $A$ to $x'$.

Compression Lemma 5.1: Fix any repairer or local-
computation repairer $R$, recoverer $A$, timing sequence $tseq$,
and distinct identifier sequence $idseq$. Let $\ell \leq dsize$. Then,

$$\Pr_X \left[ [x:D:x'] \text{ s.t. } ||D|| \leq dsize - \ell \land X = x' \right] \leq 2^{-\ell+1}$$

Proof Fix $R$, $A$, and $(tseq,idseq)$. The size of the set

$$\{x' : \exists x \text{ s.t. } [x:D:x'] \land ||D|| \leq dsize - \ell\}$$
is at most $2^{dsize-\ell+1}$ since there are at most $2^{dsize-\ell+1}$ bit-
strings of size at most $dsize - \ell$ and any fixed value of $D$
maps to a unique value $x'$ in the second execution. Thus, there
are at most $2^{dsize-\ell+1}$ values for $x$ such that $x = x'$ when
$||D|| \leq dsize - \ell$. \hfill \square

D. Compression corollary

Let

$$o\text{size} = N \cdot nsize - dsize + vsize,$$ (7)

and let

$$F = \left[ \frac{o\text{size}}{nsize} \right]$$ (8)

be the minimal number of nodes so that $F \cdot nsize \geq o\text{size}$.

Let

$$\beta' = \frac{F}{N}$$ (9)

Note that

$$\beta \leq \beta' \leq \beta + \frac{vsize}{N \cdot nsize} + \frac{1}{N}$$

Generally, $\beta' \approx \beta$, e.g., for the practical system described in
Section IV-A $\beta' \leq \beta + 10^{-8} + 10^{-5}$.

Throughout the remainder of this section, Section VI and
Section VII we set

$$M = 2 \cdot F \leq N,$$ (10)
to be the number of failures in a phase, and thus from
Equation (9),

$$\beta' \leq \frac{1}{2 \cdot F}.$$ (11)

Note that the restriction $\beta' \leq 1/2$ is mild, since $\beta' \to 0$
is more interesting in practice than $\beta' \approx 1$.

Compression Corollary 5.2: Fix any repairer or local-
computation repairer $R$, recoverer $A$, timing sequence $tseq$,
and distinct identifier sequence $idseq$. Let $\ell \leq dsize$, and

$$X = \{x : r\text{size} \leq F \cdot nsize - \ell\}$$

where $r\text{size}$ is defined with respect to $x$, $R$, and $(tseq,idseq)$.

$$\Pr_X \left[ [X \in X] \land [A(S(t_{M-1})) = X] \right] \leq 2^{-\ell+1}.$$ (10)

Proof Follows from Compression Lemma 5.1 and Equations (6), (7), (8), (10).
Note that Compression Lemma 5.1 and Compression Corollary 5.2 rely upon the assumption that the source data is uniformly distributed, and all subsequent technical results rely on this assumption.

A natural relaxation of this assumption is that the source data has high min-entropy, where the min-entropy is the log base two of one over the probability of \( x \), where \( x \) is the most likely value for the source data. Thus the min-entropy of the source data is always at most \( dsize \).

Since the source data for practical systems is composed of many independent source objects, typically the min-entropy of the source data for a practical system is close to \( dsize \). It can be verified that all of the lower bounds hold if the min-entropy of the source data is universally substituted for \( dsize \).

VI. EQUAL-READ REPAIRER LOWER BOUND

This section introduces and proves a lower bound on a constrained repairer, which we hereafter call an Equal-read repairer, within the model introduced in Section III. An Equal-read repairer is in some ways similar to the Regenerating repairer of [7], [8] described in Section II, in the sense that between each consecutive failures an Equal-read repairer is constrained to read an equal amount \( \frac{N}{2} \) of data from each of the \( N \) nodes between consecutive failures, and thus \( \gamma \) is the total amount of data read from all \( N \) nodes between failures. Unlike a Regenerating repairer, an Equal-read repairer is not constrained in any other way, e.g., which data and the amount of data transferred to each node between failures is unconstrained, and there is no constraint on when data is transferred to nodes.

Equal-read Lemma 6.1: For any Equal-read repairer \( R \) any recoverer \( A \), for any timing sequence \( tseq \) and distinct identifier sequence \( idseq \), if

\[
\gamma \leq \frac{\text{nsize}}{2} \cdot \beta'
\]

then

\[
\Pr_X[A(S(t_{M-1}^i))] = X 
\leq 2^{-\frac{\text{nsize}}{2}+1}.
\]

Proof For any Equal-read repairer \( R \), the amount of data read from each node between failures is exactly \( \frac{\gamma}{N} \), independent of \( (tseq, idseq) \) and \( x \). Thus, for any \( x \),

\[
\text{rfsize} = \sum_{i=1}^{M-1} \frac{i \cdot \gamma}{N} 
\leq \frac{(M-1) \cdot M}{2} \cdot \frac{\text{nsize}}{2} \cdot \beta'.
\]

\[
\leq F \cdot \text{nsize} - \frac{\text{nsize}}{2},
\]

where Inequality (13) follows from Equations (9) and (10). The proof follows from applying Compression Corollary 5.2 where

\[
\mathcal{X} = \left\{ x : \text{rfsize} \leq F \cdot \text{nsize} - \frac{\text{nsize}}{2} \right\} = \{0, 1\}^{dsize}.
\]

VII. CORE LOWER BOUNDS

From Compression Corollary 5.2, a necessary condition for source data \( x \) to be reliably recoverable at the end of the phase is that repairer or local-computation repairer \( R \) must read a lot of data from nodes that fail during the phase, and \( R \) must read this data before the nodes fail.

On the other hand, \( R \) cannot predict which nodes are going to fail during a phase, and only a small fraction of the nodes fail during a phase. Thus, to ensure that enough data has been read from nodes that fail before the end of the phase, a larger amount of data must be read in aggregate from all the nodes.

Core Lemma 7.1: The primary technical contribution of this section, is used to prove Core Theorem 7.2 and all later results.

Core Lemma 7.1: Fix \( \epsilon_c > 0 \) and let

\[
\delta_c = M \cdot e^{-\frac{2 \cdot X}{M} + \epsilon_c}.
\]

For \( i = 1, \ldots, M-1 \), let

\[
\Gamma_i = (1 - \epsilon_c) \cdot \frac{i \cdot (N - \frac{i+1}{2}) \cdot \text{nsize}}{M - 1}.
\]

For any repairer or local-computation repairer, \( x, tseq, idseq \),

\[
\Pr_{idseq} \left[ \left\{ \text{rfsize} \leq \frac{\gamma M^{-1} \text{nsize}}{2} \right\} \cap \left\{ \text{rfsize} > (F-1) \cdot \text{nsize} \right\} \right] 
\leq \delta_c,
\]

where \( IDseq = \langle idseq, ID_1, \ldots, ID_{M-1} \rangle \).

Proof The proof can be found in Appendix C.

With the settings in Section I-A and \( \beta' = 0.1 \), \( \epsilon_c \leq 3 \cdot 10^{-7} \) when \( \epsilon_c = 0.1 \), and \( \delta_c \leq 10^{-35} \) when \( \epsilon_c = 0.2 \).

Core Theorem 7.2: Fix \( \epsilon_c \) with \( 0 \leq \epsilon_c \leq 1 \), and Equation (16) defines \( \delta_c \). For any repairer \( R \) and recoverer \( A \), for any fixed \( tseq, idseq \), with probability at most \( \delta_c + 2^{-nsize+1} \) with respect to \( X \) and \( IDseq = \langle idseq, ID_1, \ldots, ID_{M-1} \rangle \) the following two statements are both true:

(1) For all \( i \in \{1, \ldots, M-1\} \) the average number of bits read by the repairer between \( t_0 \) and \( t_i \) per each of the \( i \) failures is less than

\[
(1 - \epsilon_c) \cdot \frac{(1 - \beta') \cdot \text{nsize}}{2 \cdot \beta'}.
\]

(2) Source data \( X \) is recoverable at time \( t_{M-1}^i \), i.e.,

\[
A(S(t_{M-1}^i)) = X.
\]

Proof Let

\[
\mathcal{X} = \left\{ x : \text{rfsize} \leq (F-1) \cdot \text{nsize} \right\},
\]

where \( \text{rfsize} \) is defined with respect to \( x \), \( R \), and \( tseq \) and any \( idseq \), and \( \mathcal{X} = \{0, 1\}^{dsize} - \mathcal{X} \), i.e.,

\[
\bar{\mathcal{X}} = \left\{ x : \text{rfsize} > (F-1) \cdot \text{nsize} \right\}.
\]

The probability of (1) and (2) both being true with respect to \( X \) and \( IDseq \) is at most the sum of the following two probabilities with respect to \( X \) and \( IDseq \):

(a) The probability that (1) and (2) and \( X \in \mathcal{X} \) are all true.

This is at most the probability that (2) and \( X \in \mathcal{X} \) are
both true, which Compression Corollary 5.2 shows is at most $2^{-n_{\text{size}}+1}$.

(b) The probability that (1) and (2) and $X \in \bar{X}$ are all true. This is at most the probability that (1) and $X \in \bar{X}$ are both true. Note that
\[
\frac{\Gamma_i}{M} \geq \frac{\Gamma_{M-1}}{M} \geq (1 - \epsilon') \cdot \frac{(1 - \beta') \cdot n_{\text{size}}}{2 \cdot \beta'}
\]
for any $i \in \{1, \ldots, M-1\}$, where Equation 17 defines $\Gamma_i$. Thus, Core Lemma 7.1 shows that this probability is at most $\delta_e$.

Note that $2^{-n_{\text{size}}}$ is essentially zero in any practical setting. For example, $2^{-n_{\text{size}}} \leq 10^{-3} \cdot 10^{15}$ for the settings in Section 4A.

VIII. MAIN LOWER BOUNDS

In all previous sections, all the failures within a phase are distinct, and when a phase ends and a new phase begins depends on the actions of the repairer, and thus the analysis does not apply to failure sequences where the failures are independent of the repairer. This section extends the results to random and independent failure sequences.

Uniform Failures Theorem 8.2 in Section VIII-C proves lower bounds for any fixed timing sequence with respect to a uniform identifier sequence distribution. A uniform identifier sequence distribution with $M$ distinct failures can be generated as follows, where $F$ is defined in Equation 8, and $M$ is defined in Equation 10.

A. Uniform identifier sequence distribution within a phase

Let $Bseq = \{B(1), B(2), \ldots\}$ be a sequence of independently and uniformly distributed in $[0,1]$ random variables. For $i \geq 1$, define geometric random variable $G_i$ with respect to $Bseq$ as
\[
G_i = \arg\min_{j \geq 1} \left\{ B(j) \leq \frac{N - i}{N} \right\},
\]
and thus $E[G_i] = \frac{N}{N-1}$. Let
\[
Gseq = \{G_1, \ldots, G_{M-1}\}
\]
be a sequence of independent geometric random variables, each defined with respect to an independent $Bseq$ sequence. Let
\[
IDseq = \langle id_0, ID_1, \ldots, ID_{M-1} \rangle
\]
be a random distinct identifier sequence as described in Section 4. The uniform identifier sequence distribution $Useq$ for the phase can be generated as follows from $Gseq$ and $IDseq$. Let $GS_0 = 0$. For $i = 1, \ldots, M-1$, let
\[
GS_i = \sum_{j=1}^{i} G_j,
\]
For $i \in \{1, \ldots, M-1\}$, let
\[
U_{GS_i} = ID_i,
\]
and for $j = GS_{i-1} + 1, \ldots, GS_i - 1$, let
\[
U_j \in U \{id_0, ID_1, \ldots, ID_{i-1}\}.
\]
Then,
\[
Useq = \{id_0, U_1, U_2, \ldots, U_{GS_{M-1}}\}
\]
is a uniform identifier sequence distribution.

Note that $GS_0, GS_1, \ldots, GS_{M-1}$, i.e., the $M$ indices of the identifiers in $Useq$ that are distinct from all the previous identifiers, are random variables defined in terms of $Gseq$. Thus, $Gseq$ determines the distinct failure indices in a phase. For $i = 1, \ldots, M-1$, let $\hat{T}_i = t_{GS_i}$ be the time of the $i$th distinct failure beyond the initial failure. This defines a timing sequence
\[
\hat{T}seq = \{\hat{t}_0, \hat{T}_1, \ldots, \hat{T}_{M-1}\},
\]
which is determined by $(tseq, Gseq)$ and is independent of $IDseq$.

The expected number of failures in a phase until there are $i$ distinct failures beyond the initial failure is
\[
E[GS_i] = \sum_{j=1}^{i} \frac{N}{N-j}.
\]
For $0 \leq \zeta < 1$,
\[
\ln(\zeta) = \ln \left( \frac{1}{1-\zeta} \right).
\]
For $0 \leq \zeta < 1$,
\[
\sum_{j=0}^{\zeta N-1} \frac{1}{N-j} < \ln(\zeta \cdot N) < \sum_{j=1}^{\zeta N} \frac{1}{N-j}.
\]
Setting $\zeta = 2 \cdot \beta'$, and using Equations 8, 10, 21, 23.
\[
E[GS_{M-1}] \leq \ln(2 \cdot \beta') \cdot N.
\]
Note that as $\zeta \to 0$, $\ln(\zeta) \to \zeta$.

Thus, $E[GS_{M-1}] \to 2 \cdot \beta' \cdot N$ as $\beta' \to 0$.

A phase proceeds as follows with respect to source data $x$ and failure sequence $(\hat{T}seq, IDseq)$, where Equation 17 defines $\Gamma_i$. For $i = 1, \ldots, M-1$, $R$ is executed up till time $\hat{T}_i$. If $rsize_i \geq \Gamma_i$ with respect to $x$, $(\hat{t}_0, \hat{T}_1, \ldots, \hat{T}_i)$, $\{id_0, ID_1, \ldots, ID_{i-1}\}$ then the phase ends at time $\hat{T}_i$. If the phase doesn’t end in the above process then $rsize_i < \Gamma_i$ for $i = 1, \ldots, M-1$, and the phase ends at time $\hat{T}_{M-1}$.

B. Distinct failures lemma

The condition $rsize_i \geq \Gamma_i$ ensures that that the amount of data read by $R$ up till time $\hat{T}_{GS_i}$ in a phase is at least $\frac{\Gamma_i}{2 \cdot \beta'}$. However, a lower bound on $\frac{\Gamma_i}{GS_i}$ is needed, since $GS_i$ is the total number of failures. The issue is that $GS_i$ is a random variable that can be highly variable relative to $i$ and can depend on $R$, and thus $\frac{\Gamma_i}{GS_i}$ can be highly variable and can be influenced by $\hat{T}seq$. Thus, it is difficult to provide lower bounds when considering only a single phase.

To circumvent these issues, we stitch phases together into a sequence of phases, and argue that $R$ must read a lot
of data per failure over a sequence of phases that covers a large enough number of distinct failures. Distinct Failures Lemma 8.1 below proves that if we stitch together enough phases then we can ensure that, independent of the actions of \( R \), with high probability the total number of failures aggregated over the phases is close to the expected number of failures relative to the number of distinct failures.

The phases can be stitched together as follows. Let 
\[
t_{\text{seq}} = \{t_0, t_1, \ldots, t_i, \ldots\}
\]
be a timing sequence and let 
\[
U_{\text{seq}} = \{i \cdot \text{id}_0, U_1, U_2, \ldots, U_i, \ldots\}
\]
be a uniform identifier sequence distribution. Run \( R \) on \((t_{\text{seq}}, U_{\text{seq}})\) until there are \( M \) distinct failures in total within the phases, and then continuing running \( R \) until the phase that is underway when there are \( M \) distinct failures in total completes. Let \( Y \) be the total number of distinct failures in this process, where \( M \leq Y < 2 \cdot M \), and let \( Y' \) be the total number of failures in this process, where \( Y' \geq Y \). Both \( Y \) and \( Y' \) are random variables that are determined by \( R \) and \((t_{\text{seq}}, U_{\text{seq}})\).

Let 
\[
M' = M \cdot \frac{\ln(2 \cdot \beta')}{2 \cdot \beta'} = \ln(2 \cdot \beta') \cdot N.
\]

From Equation (24), \( M' \rightarrow M \) as \( \beta' \rightarrow 0 \).

For \( 0 \leq \zeta < 1 \), define
\[
\ln(\zeta) = \zeta - \ln(1 + \zeta).
\]

Note that as \( \zeta \rightarrow 0 \),
\[
\ln(\zeta) \rightarrow \zeta^2/2.
\]

**Distinct Failures Lemma 8.1:** Fix \( \beta' < 1/2 \). Fix \( \epsilon_d > 0 \) and let
\[
\delta_d = M \cdot \frac{e^{-2 \beta' \cdot (1-2 \beta') \cdot N \cdot \ln(1+\epsilon_d)}}{1 + \epsilon_d}.
\]

For any repairer \( R \) and recoverer \( A \), for any \( x \) and \( t_{\text{seq}} \), with probability at least \( 1 - \delta_d \) with respect to \( U_{\text{seq}} \),
\[
Y \geq \frac{2 \cdot \beta'}{(1 + \epsilon_d) \cdot \ln(2 \cdot \beta')} \cdot Y',
\]
and
\[
Y' \leq (1 + \epsilon_d) \cdot 2 \cdot M'.
\]

**Proof** The proof can be found in Appendix E.

\[\] \[\]

**C. Uniform failures lower bound**

**Uniform Failures Theorem 8.2:** Fix \( \beta' < 1/2 \). Let \( \epsilon_u \) and \( \delta_u \) be as defined in Core Lemma 7.1 and let \( \epsilon_d \) and \( \delta_d \) be as defined in Distinct Failures Lemma 8.1 and let
\[
\delta_u = \delta_d + 2 \cdot M \cdot (\delta_u + 2^{-nsize}).
\]

For any repairer \( R \) and recoverer \( A \), for any fixed \( t_{\text{seq}} \), at least one of the following two statements is true with probability at least \( 1 - \delta_u \) with respect to \( X \) and \( U_{\text{seq}} \):

(1) There is an \( m \leq (1 + \epsilon_d) \cdot 2 \cdot M' \) such that the average number of bits read by the repairer between \( t^*_0 \) and \( t^*_m \) per each of the \( m \) failures is at least
\[
\frac{(1 - \epsilon_u) \cdot (1 - \beta') \cdot nsize}{(1 + \epsilon_d) \cdot \ln(2 \cdot \beta')},
\]
(2) Source data \( X \) is unrecoverable by \( A \) at time \( t^*_m \).

**Proof** From Distinct Failures Lemma 8.1 there is a sequence of phases that ends with \( Y' \leq (1 + \epsilon_d) \cdot 2 \cdot M' \) failures where the number of distinct failures \( Y \) is at least Equation (28) with probability at least \( 1 - \delta_d \). From Core Theorem 7.2 with respect to all \( x \) and \((t_{\text{seq}}, U_{\text{seq}}, ID_{\text{seq}})\) and using a union bound over at most \( 2 \cdot M \) phases in the sequence of phases, the average number of bits read by \( R \) between \( t^*_0 \) and \( t^*_Y \), per each distinct failure is at least Equation (18) with probability at least \( 1 - 2 \cdot M \cdot (\delta_u + 2^{-nsize}) \). Thus, overall the two statements hold with probability at least \( 1 - \delta_u \).

Since the end of one sequence of phases can be the beginning of the next sequence of phases, it follows that the average number of bits read by the repairer per failure must satisfy Equation (29) over the entire lifetime of the system for which the source data is recoverable.

Equation (29) holds independent of the timing sequence. Thus, if there are a lot of failures over a period of time then the read rate over this period of time must necessarily be high, whereas if there are fewer failures over a period of time then the read rate over this period of time can be lower. Automatic adjustments of the read rate as the failure rate fluctuates is one of the key contributions of the algorithms described in \[18\], which shows that there are algorithms that can match the lower bounds of Uniform Failures Theorem 8.2 even for a fluctuating timing sequence.

\[\]

D. Poisson failures lower bound

Uniform Failures Theorem 8.2 expresses lower bounds in terms of the average number of bits read per failure. Poisson Failures Theorem 8.3 presented in this section, instead expresses the lower bounds in terms of a read rate. The primary additional technical component needed to prove Poisson Failures Theorem 8.3 is a concentration in probability result: the number of failures for a Poisson failure distribution with rate \( \lambda \) over a suitably long period of time is relatively close to the expected number of failures with high probability. Poisson Failures Lemma 8.4 below proves that if we stitch together enough phases, with high probability the total number of failures aggregated over the phases is close to the expected number of failures relative to the number of distinct failures.

The Poisson failure distribution with rate \( \lambda \) can be generated as follows. For \( i \geq 1 \), let \( Q_i \) be an independent exponential random variable with rate \( \lambda \cdot N \), and let
\[
Q_{\text{seq}} = \{Q_1, \ldots, Q_i, \ldots\}.
\]

For \( i \geq 1 \), let
\[
T_i = t_0 + \sum_{j=1}^{i} Q_j,
\]
and let
\[
T_{\text{seq}} = \{t_0, T_1, \ldots, T_i, \ldots\}.
\]

---

\[\]

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For $i \geq 1$, let $U_i$ be an independent random variable that is uniformly distributed in $\{0, \ldots, N - 1\}$, and let

$$Useq = \{iU_0, U_1, \ldots, U_i, \ldots\}.$$  

Then, $(Tseq, Useq)$ is a random failure sequence with respect to the Poisson failure distribution with rate $\lambda$.

Capacity is erased from the system at a rate

$$erate = \lambda \cdot N \cdot nsize$$

with respect to the Poisson failure distribution with rate $\lambda$.

**Poisson Failures Theorem 8.3:** Fix $\beta' < 1/2$. Let $\epsilon_c$ be as defined in Core Lemma 7.1, $\epsilon_d$ be as defined in Distinct Failures Lemma 8.1, and $\delta_a$ be as defined in Uniform Failures Theorem 8.2. Let $\epsilon > 0$, let

$$\delta = \delta_a + (1 + \epsilon_d) \cdot 2 \cdot M' \cdot \frac{e^{-M\cdot\text{Ind}(\epsilon)}}{1 + \epsilon},$$

and

$$\Delta = (1 + \epsilon_d) \cdot (1 + \epsilon) \cdot 2 \cdot \frac{\ln(2 \cdot \beta')}{\lambda}. $$

For any repairer $R$ and recoverer $A$, for any starting time $t_0$, at least one of the following two statements is true with probability at least $1 - \delta$ with respect to a a Poisson failure distribution with rate $\lambda$:

1. There is a $t \leq t_0 + \Delta$ such that the average rate $rrate$ the repairer reads bits between $t_0$ and $t$ satisfies

$$rrate \geq \frac{(1 - \epsilon_c)}{(1 + \epsilon_d) \cdot (1 + \epsilon) \cdot \ln(2 \cdot \beta')} \cdot \text{erate}. \quad (30)$$

2. Source data $X$ is unrecoverable by $A$ at time $t_0 + \Delta$.

**Proof** From Uniform Failures Theorem 8.2 there is a sequence of phases that ends with $m \leq (1 + \epsilon_d) \cdot 2 \cdot M'$ failures where the number of distinct failures is provided by Equation (29) with probability at least $1 - \delta_a$. Since there are at least $M$ distinct failures in the process, $m \geq M$.

For each $\ell$ between $M$ and $(1 + \epsilon_d) \cdot 2 \cdot M'$, when

$$\delta'(\ell) = \frac{e^{-\ell \cdot \text{Ind}(\epsilon)}}{1 + \epsilon},$$

it follows from Theorem 5.1 of [17] that

$$\Pr \left[ \sum_{i=1}^{\ell} Q_i \geq (1 + \epsilon) \cdot \frac{\ell}{\lambda \cdot N} \right] \leq \delta'(\ell).$$

Using a union bound, it follows that with probability at least

$$1 - (1 + \epsilon_d) \cdot 2 \cdot M' \cdot \delta'(M),$$

$$\sum_{i=1}^{m} Q_i < (1 + \epsilon) \cdot \frac{m}{\lambda \cdot N}. $$

Thus, the time $t = t_0 + \hat{t}$ when there are $m$ failures in the process satisfies

$$\hat{t} \leq \frac{(1 + \epsilon) \cdot m}{\lambda \cdot N}. \quad (31)$$

with probability at least $1 - (1 + \epsilon_d) \cdot 2 \cdot M' \cdot \delta'(M)$.

From Uniform Failures Theorem 8.2 and combining Equations (29) and (31), it follows that with probability at least $1 - \delta$ the rate at which the repairer reads data between $t_0$ and $t$ is at least as large as the right-hand side of Inequality (30) or else the source data is unrecoverable at time $t = t_0 + \hat{t}$, and thus unrecoverable at time $t_0 + \Delta \geq t_0 + \hat{t}$.

From Equation (27), $\delta_c$ shrinks exponentially fast as $N$ goes to infinity for fixed $\epsilon > 0$, $\delta_d$ shrinks exponentially fast as $N$ goes to infinity for fixed $\epsilon_d > 0$, and thus $\delta$ shrinks exponentially fast as $N$ goes to infinity for fixed $\epsilon_c > 0$, $\epsilon_d > 0$, and $\epsilon > 0$.

Since $\epsilon_c > 0$, $\epsilon_d > 0$, and $\epsilon > 0$ can be arbitrarily small constants as $N$ goes to infinity, the Inequality (30) lower bound on $rrate$ in Poisson Failures Theorem 8.3 approaches

$$rrate \geq \frac{1 - \beta'}{\ln(2 \cdot \beta')} \cdot \text{erate}.$$ \quad (32)

as $N$ goes to infinity. Since the end of one interval can be the beginning of the next interval, it follows that $rrate$ must also satisfy Equation (32) over the entire lifetime of the system. From Equation (24) and Inequality (32), as $N$ goes to infinity and $\beta$ goes to 0, the Inequality (30) lower bound on $rrate$ in Poisson Failures Theorem 8.3 approaches Inequality (3).

### E. Distributed storage source data capacity

Our distributed storage model and results are inspired by Shannon’s communication model [1]. For a system with capacity $N \cdot nsize$, we define the source data capacity to be the amount $dsize$ of source data that can be reliably stored for long periods of time by the system.

Based on Equations (1) and (2), Inequality (3) can be expressed as

$$dsize \leq \left( 1 - \frac{erate}{2 \cdot \text{erate}} \right) \cdot N \cdot nsize.$$ \quad (33)

asymptotically as $N$ and $\text{erate}$ approach infinity. Inequality (33) expresses a fundamental lower bound on the source data capacity as a function of the system capacity, the erasure rate and the read rate of the repairer. The paper [18] shows that storage source data capacity asymptotically approaching the right-hand side of Inequality (33) can be achieved as $N$ and $\text{erate}$ approach infinity, and thus

$$dsize \approx \left( 1 - \frac{erate}{2 \cdot \text{erate}} \right) \cdot N \cdot nsize$$

expresses a fundamental source data capacity limit as a function of the system capacity, the erasure rate and the read rate of the repairer as $N$ and $\text{erate}$ approach infinity.

### IX. Future work

There are many ways to extend this research, accounting for practical issues in storage system deployments.

Failures in deployed systems can happen at a variable rate that is not known a priori. For example, a new batch of nodes introduced into a deployment may have failure rates that are dramatically different than previous batches. The paper [18] introduces repair algorithms that automatically adjust to fluctuating failure rates.

Both time and spatial failure correlation is common in deployed systems. Failures in different parts of the system are not
completely independent, e.g., racks of nodes fail concurrently, entire data centers go offline, power and cooling units fail, node outages occur due to rolling system maintenance and software updates, etc. All of these events introduce complicated correlations between failures of the different components of the system.

Intermittent failures are common in deployed systems, accounting for a vast majority (e.g., 90%) of failures. In the case of an intermittent node failure, the data stored at the node is lost for the duration of the failure, but after some period of time the data stored on the node is available again once the node recovers (the period of time can be variable, e.g., ranging from a few seconds to days). Intermittent failures can also affect entire data centers, a rack of nodes, etc.

Repairing fragments temporarily unavailable due to transient failures wastes network resources. Thus, a timer is typically set to trigger a fixed amount of time after a node fails (e.g., 15 minutes), and the node is declared permanently failed and scheduled for repair if it has not recovered within the trigger time. Setting the trigger time can be tricky for a small code system; a short trigger time can lead to unnecessary repair, whereas a long trigger time can reduce reliability. The paper [18] provides simulations that highlight the impact of setting the trigger time for different systems.

Data can silently be corrupted or lost without any notification to the repairer; the only mechanism by which a repairer may become aware of such corruption or loss of data is by attempting to read the data, i.e., data scrubbing. (The data is typically stored with strong checksums, so that the corruption or loss of data becomes evident to the repairer when an attempt to read the data is made.) For example, the talk [16] reports that read traffic due to scrubbing can be greater than all other read data traffic combined. The paper [18] provides simulations that highlight the impact of silent corruption on different systems.

There can be a delay between when a node permanently fails and when a replacement node is added. For example, in many cases adding nodes is performed by robots, or by manual intervention, and nodes are added in batches instead of individually.

It is important in many systems to distribute the repair evenly throughout the nodes and the network, instead of having a centralized repairer. This is important to avoid CPU and network hotspots. The algorithms described in [18] distributed the repair traffic smoothly among all nodes of the system. The more advanced algorithms described in [19] can be modified to distribute the repair traffic smoothly among all nodes of the system. Based on this, it can be seen that distributed versions of the lower bounds and upper bounds asymptotically converge as the storage overhead approaches zero.

Network topology is an important consideration in deployments, for example when objects are geo-distributed to multiple data centers. In these deployments, the available network bandwidth between different nodes may vary dramatically, e.g., there may be abundant bandwidth available between nodes within the same data center, but limited bandwidth available between nodes in different data centers. The paper [15] addresses these issues, and the papers [10], [11] introduce some erasure codes that may be used in solutions to these issues. An example of such a deployment is described in [13].

Enhancing the distributed storage model by incorporating the elements described above into the model and providing an analysis can be of value in understanding fundamental tradeoffs for practical systems.

X. Conclusions

We introduce a mathematical model of distributed storage that captures some of the relevant features of practical systems, and prove tight lower bounds on the tradeoff between the repairer read rate and the storage overhead as a function of the erasure rate. Our hope is that the model and bounds will be helpful in understanding and designing practical distributed storage systems.

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Michael Luby earned a BSc in Applied Math from MIT and a PhD in Theoretical Computer Science from UC Berkeley. He founded Digital Fountain Inc. in 1999 and served as CTO until acquired by Qualcomm Inc. in 2009, where he was a VP of Technology through late 2018. He is currently a VP of Technology at Fountain Inc. in 1999 and served as CTO until acquired by Qualcomm Inc. in 2009, where he was a VP of Technology through late 2018. He is currently a member of the National Academy of Engineering and is an IEEE Fellow and an ACM Fellow.

This section provides a full description of a repairer, filling in the details of the brief description provided in Section III-E.

A repairer $R$ can be viewed as a process that ensures that the source data reads when data generated from the source data is stored at the unreliable nodes. A repairer for a system operates as follows. The identifier $id_i$ is provided to repairer $R$ at time $t_i$, which alerts the repairer that all $n_{size}$ bits stored on node $id_i$ are lost at that time. As nodes fail and are replaced, the repairer reads data over interfaces from nodes, performs computations on the read data, and writes computed data over interfaces to nodes. A primary metric is the number of bits the repairer reads over interfaces from storage nodes.

At time $t$, let $V(t)$ be the bits stored in the global memory of $R$, where $n_{size} = |V(t)|$.

Let $fseq(t) = (tseq, idseq)$ be the failure sequence up to time $t$, where $tseq = \{t_0, \ldots, t\}$, $idseq = \{id_0, \ldots, id_t\}$, and $\ell = \text{arg max}_i\{t_i \leq t\}$. The repairer $R$ has access to $fseq(t)$ at time $t$.

The actions of $R$ at time $t$ are determined by $(t, V(t), fseq(t))$. If a node $j$ fails at time $t$ then $R$ is notified at time $t$ that node $j$ failed and $fseq(t)$ is updated. If $R$ reads data over the interface from node $j$ at time $t$ (when to read from node $j$ is determined by $(t, V(t), fseq(t))$) then the amount and location of the data read from $C_j(t)$ is determined by $(t, V(t), fseq(t))$. The repairer receives the data and computes the data over the interface from node $j$ in response to a request initiated at time $t$ is assumed to be instantaneous available, i.e. all of the requested data is available at time $t$ over the interface from node $j$.

$V(t)$ can be used by the repairer to store the programs the repairer executes, store information from the past, temporarily store data read from nodes, perform computations on read data, temporarily store computed data before it is written to nodes, and generally store any information the repairer needs immediate access to that is not stored at the nodes. The distinction between $V(t)$ and the node is that $V(t)$ is persistent memory (not subject to any type of failure in the model) and available globally to $R$ (there is no read or write cost for accessing $V(t)$). $R$ can also store such information at the nodes, but this information is subject to loss due to possible failures.

Repairers are allowed to use an unbounded amount of computation, since computation time is not a metric of interest in the lower bounds. The granularity of how much data is read or written in one step is unconstrained, e.g. one bit or Terrabytes of data can be read over an interface from a node during a read step, and the lower bounds still hold. The granularity of the timing of read and write steps is also unconstrained, e.g. there may be a read step each nanosecond, or every twenty minutes.

Local-computation repairers, inspired by [7] and [8], are more powerful than repairers. The motivation for the local-computation repairer model is that a node often has CPUs, memory and storage, and often the impact of traffic between storage and memory at a node is much less than the impact of traffic over the interface from the node to the system. Thus, an
arbitrary amount of data may be accessed locally from storage into local memory at a node, local CPUs may compute and store in the local memory a much smaller amount of data from the data accessed into local memory, and it is the much smaller amount of data computed by the CPUs that is sent over the interface from the node to the system. The model does not count the data accessed from storage into local memory, it only counts the data in the local memory that is read by the system over the interface from the node.

Formally, for a local-computation repairer, when data is to be read over the interface from node \( j \) initiated at time \( t \) (when to read from node \( j \) is determined by \( (t, V(t), fseq(t)) \)), a copy of the entire global memory of the local-computation repairer is assumed to be instantaneously available in the local memory at node \( j \) at no cost. As the local computation at node \( j \) progresses, the copy may evolve to be different than the global memory of the local-computation repairer at time \( t \), but the only information the local-computation repairer potentially receives about any changes to the copy in the local memory is from the locally computed data read by the local-computation repairer over the interface from node \( j \). The locally computed data is generated based on \( (t, V(t), fseq(t), C_j(t)) \), and then the locally computed data is read by the local-computation repairer over the interface from node \( j \). The local computational power at node \( j \) and the throughput of the interface at node \( j \) are assumed to be unlimited, and thus the locally computed data requested at time \( t \) by the local-computation repairer is available instantly at time \( t \) over the interface from node \( j \).

Thus the data read over the interface from node \( j \) when the request for the data is initiated at time \( t \) is determined by \( (t, V(t), fseq(t), C_j(t)) \). In this model only the locally computed data is counted as data read over the interface from node \( j \); the data accessed from storage at node \( j \) to produce the locally computed data (which could be all of \( C_j(t) \)) is not counted.

For example, in the extreme a local-computation repairer could locally access all data stored at a node to produce 1 KB of locally computed data, and then only the 1 KB of locally computed data is read over the interface from the node. In this example, only 1 KB of data is counted towards data read by the local-computation repairer. Thus there is a significant cost to this generalization that is not counted in the amount of data read from nodes by the repairer. These issues are discussed in more detail in Section II.

A repairer is a special case of a local-computation repairer: a repairer is simply a local-computation repairer where the data accessed from storage at the node is directly sent over the interface from the node to the repairer.

A repairer may employ a randomized algorithm, which could be modeled by augmenting the repairer with random and independently chosen bits. However, since the repairer is deterministic for a fixed setting of the random bits and the lower bounds hold for any deterministic repairer, the same lower bounds hold for any randomized repairer. Thus, we describe lower bounds only for deterministic repairers, noting that all the lower bound results immediately carry over to randomized repairers.

**APPENDIX B
APPLYING THE LOWER BOUNDS TO REAL SYSTEMS**

The description of the model makes some very unrealistic assumptions about how real systems operate in practice. However, it is these assumptions that ensure that the lower bounds apply to all real systems. Consider a real system where nodes fail randomly as in the model, but also portions of the network intermittently fail, network bandwidth availability is limited and varying between different parts of the system, memory is not completely reliable, multiple distributed semi-autonomous processes are interacting with different sets of nodes, responses are not immediate to data requests over node interfaces, processes are not immediately notified when nodes fail, notification of node failure is not global, nodes are not immediately replaced, computational resources are limited, etc. We describe an omniscient agent acting with respect to the model in the role of the repairer, where the agent emulates the processes and behaviors of the real system. This shows the lower bounds also apply to the real system.

In the model, nodes that fail are immediately replaced and the agent is immediately notified. In the real system, a failed node may not be replaced immediately. Thus, to emulate the real system, the agent disallows any response to a request to read or write data to a failed node from a process until the time when the node would have been replaced in the real system.

In the real system, notifications of node failures may not be instantaneous, and only some processes may be notified. Thus, the agent only notifies the appropriate processes of node failures when they would have been notified in the real system.

In the model, the agent receives an immediate and complete response to a request for data over an interface to a node. In the real system, interfaces can have a limited amount of bandwidth, and there can be delays in delivering responses to requests for data by processes over a node interface due to computational limits or other constraints. Thus, the agent delivers data to requesting processes with the delays and at the speed of the real system.

In the real system, only a small portion of the global memory state may be available in the local memory of a node when local-computation repair is used. Thus, the agent may only need a small portion of the global memory at the node to emulate a local-computation repairer of the real system.

In the model, the agent acting as a repairer has one global memory. In the real system, repair may be implemented by a distributed set of processes \( R_1, \ldots, R_i \) executing concurrent reads and writes over node interfaces, each with their own private memory \( V_i(t), \ldots, V_i(t) \) at time \( t \). The agent \( R \) can emulate \( R_1, \ldots, R_i \) as follows. The global memory of \( R \) is

\[
V(t) = \{V_1(t), \ldots, V_i(t)\}.
\]

If processes \( R_i \) and \( R_j \) send bits between their local memories at time \( t \) then these same bits are copied between \( V_i(t) \) and \( V_j(t) \) by \( R \) at time \( t \). The movement of data between the local memories of the processes that the agent is emulating is at no cost. Thus, the lower bound on the amount of data read over interfaces from nodes by \( R \) in the model is a lower
bound on the amount of data read over interfaces from nodes by R_1, ..., R_L.

In the model, the agent has a single interface with each node. In the real system, a node can have multiple interfaces. These multiple interfaces are considered as one logical interface by the agent when counting the amount of data traveling over interfaces from the agent to the node, and the agent delivers data traveling over the multiple interfaces to the appropriate requesting processes of the emulated real system.

The count of data traffic for the lower bounds is conservative, i.e. the amount of data that travels over interfaces from nodes to the agent is a lower bound on the amount of data traveling over the network in the real system.

Thus, a real system, whether it is perfectly architected and has non-failing infinite network bandwidth, zero computational delays, instant node failure notification, or whether it is more realistic as described above, can be emulated by the agent in the model as described above. Since the lower bounds apply to the agent with respect to the model, the lower bounds also apply to any real system.

**APPENDIX C**

**PROOF OF CORE LEMMA**

**Proof** Fix x, τseq and id_0. The parameterization with respect to x and τseq are implicit in the remainder of the proof. Fix η = (F − 1) · nsize. We first prove that for any repairer or local-computation repairer R there is a repairer or local-computation repairer R′ such that

\[
\text{Pr} \left[ \left( \bigvee_{i=1}^{M-1} \text{rsize}_i < \Gamma_i \right) \land \left( \text{rsize}' > \eta \right) \right] \leq \text{Pr} \left[ \text{rsize} > \eta \right]
\]

and

\[
\text{Pr} \left[ \left( \bigvee_{i=1}^{M-1} \text{rsize}_i < \Gamma_i \right) \right] = 1
\]

with respect to \(\{id_0, ID_1, ..., ID_{M-1}\}\), and where rsize′_i and rsize′ are defined with respect to R′ and rsize_i and rsize are defined with respect to R.

Let predicate P be defined as follows on input \(\{id_0, ..., id_{M-1}\}\).

\[P \text{ is true} \iff \bigvee_{i=1}^{M-1} \text{rsize}_i < \Gamma_i\]

with respect to \(\{id_0, ..., id_{M-1}\}\).

R acts the same way as R′ with respect to \(\{id_0, ..., id_{M-1}\}\) for which P is true, thus rsize_i = rsize′_i for i = 1, ..., M − 1, and rsize = rsize′, with respect to \(\{id_0, ..., id_{M-1}\}\) for which P is true.

Fix \(\{id_0, ..., id_{M-1}\}\) for which P is false, let

\[\ell = \arg\min_{i=1,...,M-1} \{\text{rsize}_i \geq \Gamma_i\}\]

with respect to \(\{id_0, ..., id_{M-1}\}\). R acts the same way up till time t_ℓ−1, but doesn’t read data from nodes after t_ℓ−1, with respect to \(\{id_0, ..., id_{M-1}\}\). Thus, rsize_i = rsize′_i for i = 1, ..., ℓ − 1, rsize_i = rsize′_ℓ−1 for i = ℓ, ..., M − 1, with respect to \(\{id_0, ..., id_{M-1}\}\). From this, \(\bigvee_{i=1}^{M-1} \text{rsize}_i < \bigvee_{i=1}^{M-1} \text{rsize}_i \leq \Gamma_i\) with respect to any \(\{id_0, ..., id_{M-1}\}\) for which P is false. Thus, condition [35] holds for repairer R.

Since rsize = rsize′ with respect to all \(\{id_0, ..., id_{M-1}\}\) for which P is true, it follows that

\[
\text{Pr} \left[ \left( \bigvee_{i=1}^{M-1} \text{rsize}_i < \Gamma_i \right) \land \left( \text{rsize}' > \eta \right) \right] = \text{Pr} \left[ P = \text{true} \land \text{rsize} > \eta \right] \leq \text{Pr} \left[ \text{rsize} > \eta \right]
\]

with respect to \(\{id_0, ID_1, ..., ID_{M-1}\}\), thus Inequality [34] holds.

The rest of the proof bounds Pr [rsize > η] for repairer or local-computation repairer R, which provides the bound on Inequality [34]. It can be verified that

\[
E[rsize_i] = \frac{rsize_i - \sum_{\ell=1}^{i-1} rsize_\ell}{N - i}
\]

with respect to \(\{id_0, ..., id_{i-1}, ID_i\}\). Let

\[
\rho = \frac{(1 - \epsilon \cdot nsize)}{2 \cdot F - 1},
\]

\[
\tau_i = \sum_{\ell=1}^{i} \ell \cdot \left(\frac{(i + 1)}{2}\right).
\]

If

\[
\sum_{\ell=1}^{i-1} rsize_\ell \geq \tau_{i-1} \cdot \rho
\]

with respect to \(\{id_0, ..., id_{i-1}\}\) then

\[
E[rsize_i] \leq i \cdot \rho
\]

with respect to \(\{id_0, ..., id_{i-1}, ID_i\}\). This follows from Equation [36], Condition [35], Inequality [37], and because

\[
\Gamma_i = \tau_{i-1} : \rho = \frac{i \cdot \rho}{N - i}
\]

Define z_0 = 0, and for i = 1, ..., M − 1,

\[
Z_i = z_{i-1} + rsize_i - i \cdot \rho = \sum_{\ell=1}^{i} rsize_\ell - \tau_i \cdot \rho
\]

with respect to \(\{id_0, ..., id_{i-1}, ID_i\}\), and define z_i similarly with respect to \(\{id_0, ..., id_{i-1}, id_i\}\). It can be verified that

\[
\tau_{2 \cdot F - 1} \cdot \rho = F \cdot nsize - \epsilon \cdot F \cdot nsize = \eta - (\epsilon \cdot F - 1) \cdot nsize,
\]

thus

\[
\text{Pr} \left[ Z_{2 \cdot F - 1} > (\epsilon \cdot F - 1) \cdot nsize \right] = \text{Pr} \left[ \text{rsize} > \eta \right]
\]

with respect to \(\{id_0, ID_1, ..., ID_{M-1}\}\).

It can be verified that

\[
|z_i - z_{i-1}| \leq nsize
\]

with respect to all \(\{id_0, ..., id_{i-1}, id_i\}\). Also, Equation [39] and Inequalities [37] and [38] imply that if z_{i-1} ≥ 0 then

\[
E[Z_i] = z_{i-1}
\]

with respect to \(\{id_0, ..., id_{i-1}, ID_i\}\). Thus, z_0, Z_1, ..., Z_{2 \cdot F - 1} with respect to \(\{id_0, ID_1, ..., ID_{M-1}\}\) satisfies the conditions of Supermartingale Theorem [D.1] of Appendix [D] with n = M − 1, c = nsize, and α = (ε \cdot F − 1) \cdot nsize. Thus, from Supermartingale
Theorem D.1 and Equation (40), it can be verified that
\[ \Pr \left[ rf_{size} > \eta \right] \leq \delta, \]
with respect to \( \langle id_0, ID_1, \ldots, ID_{M-1} \rangle \). The lemma follows from Inequality (34).

\section*{APPENDIX D
SUPERMARTINGALE THEOREM D.1}

We provide a probability bound used in the proof of Core Lemma 7.1 that may be of independent interest. Any improvement to this bound provides an immediate improvement to Core Lemma 7.1. We generalize previous notation.

\textbf{Supermartingale Theorem D.1:} Let \( z_0, z_1, \ldots, z_n \) be a random sequence of real-values defined with respect to another random sequence \( \{id_0, ID_1, \ldots, ID_n\} \), such that \( z_0 = 0 \) and the following conditions are satisfied for \( i = 1, \ldots, n \):
- \( z_i \) is determined by \( \{id_0, id_1, \ldots, id_i\} \).
- \( z_i \) is determined with respect to all \( \{id_0, \ldots, id_{i-1}, id_i\} \).
- if \( z_{i-1} > 0 \) then \( E[Z_i] \leq z_{i-1} \) with respect to \( \{id_0, \ldots, id_{i-1}, ID_i\} \).

Then, for any \( \alpha > 0 \),
\[ \Pr \left[ Z_n > \alpha + c \right] \leq n \cdot e^{\frac{\alpha^2}{n \cdot c^2}} \]
with respect to \( \{id_0, \ldots, id_\ell, \ldots, id_n\} \).

\textbf{Proof} For each \( \ell = 1, \ldots, n \), let predicate \( P_\ell \) be defined as follows on input \( \{id_0, \ldots, id_\ell, \ldots, id_n\} \), with \( i \in \{\ell, \ldots, n\} \).

\[ P_\ell \text{ is true } \iff z_{\ell-1} \leq 0 \land z_\ell > 0 \]
with respect to \( \{id_0, \ldots, id_\ell, \ldots, id_n\} \).

For each \( \ell = 1, \ldots, n \) and each \( \{id_0, \ldots, id_\ell\} \) such that \( P_\ell \) is true, define a sequence as follows.
- \( \delta_\ell^{\ell, \{id_0, \ldots, id_\ell\}} = z_\ell \) with respect to \( \{id_0, \ldots, id_\ell\} \).
- For \( i = \ell + 1, \ldots, n \),
\[ Z_i^{\ell, \{id_0, \ldots, id_\ell\}} = Z_i \text{ if } z_{i-1}^{\ell, \{id_0, \ldots, id_\ell\}} > 0 \]
\[ Z_i^{\ell, \{id_0, \ldots, id_\ell\}} = z_{i-1}^{\ell, \{id_0, \ldots, id_\ell\}} \text{ if } z_{i-1}^{\ell, \{id_0, \ldots, id_\ell\}} \leq 0 \]
with respect to \( \{id_0, \ldots, id_\ell, id_{\ell+1}, \ldots, id_{i-1}, ID_i\} \).

It can be verified that, for all \( \{id_0, \ldots, id_\ell, id_{\ell+1}, \ldots, id_i\} \),
\[ z_i^{\ell, \{id_0, \ldots, id_\ell\}} - z_{i-1}^{\ell, \{id_0, \ldots, id_\ell\}} \leq c \]
with respect to \( \{id_0, \ldots, id_\ell, id_{\ell+1}, \ldots, id_i\} \).

With respect to \( \{id_0, \ldots, id_\ell, id_{\ell+1}, \ldots, id_i\} \):
Equations (41) and (42) imply that \( z_i^{\ell, \{id_0, \ldots, id_\ell\}} = z_{i-1}^{\ell, \{id_0, \ldots, id_\ell\}} \) if \( z_{i-1}^{\ell, \{id_0, \ldots, id_\ell\}} > 0 \), and since \( E[Z_i] \leq z_{i-1}^{\ell, \{id_0, \ldots, id_\ell\}} \) if \( z_{i-1} > 0 \), it follows that
\[ E \left[ z_i^{\ell, \{id_0, \ldots, id_\ell\}} \right] \leq z_{i-1}^{\ell, \{id_0, \ldots, id_\ell\}} \]
if \( z_{i-1}^{\ell, \{id_0, \ldots, id_\ell\}} > 0 \). From Equation (42),
\[ E \left[ z_i^{\ell, \{id_0, \ldots, id_\ell\}} \right] = z_{i-1}^{\ell, \{id_0, \ldots, id_\ell\}} \]
if \( z_{i-1}^{\ell, \{id_0, \ldots, id_\ell\}} \leq 0 \). Thus,
\[ E \left[ z_i^{\ell, \{id_0, \ldots, id_\ell\}} \right] \leq z_{i-1}^{\ell, \{id_0, \ldots, id_\ell\}} \]
(44)
with respect to \( \{id_0, \ldots, id_\ell, id_{\ell+1}, \ldots, id_{i-1}, ID_i\} \).

From Equations (43) and (44), for \( \ell = 1, \ldots, n \), for all \( \{id_0, \ldots, id_\ell\} \) such that \( P_\ell \) is true,
\[ z_\ell^{\ell, \{id_0, \ldots, id_\ell\}} \leq z_{\ell-1}^{\ell, \{id_0, \ldots, id_\ell\}} \]
with respect to \( \{id_0, \ldots, id_\ell, ID_{\ell+1}, \ldots, ID_n\} \) is a supermartingale. Thus, from the Azuma’s inequality,
\[ \Pr \left[ Z_n^{\ell, \{id_0, \ldots, id_\ell\}} - z_\ell^{\ell, \{id_0, \ldots, id_\ell\}} > \alpha \right] \leq e^{\frac{\alpha^2}{n \cdot c^2}} \]
(45)
with respect to \( \{id_0, \ldots, id_\ell, ID_{\ell+1}, \ldots, ID_n\} \). It can be verified that \( z_\ell^{\ell, \{id_0, \ldots, id_\ell\}} \leq c \) if \( P_\ell \) is true for \( \{id_0, \ldots, id_\ell\} \), thus
\[ \Pr \left[ Z_n^{\ell, \{id_0, \ldots, id_\ell\}} - z_\ell^{\ell, \{id_0, \ldots, id_\ell\}} > \alpha + c \right] \leq \Pr \left[ Z_n^{\ell, \{id_0, \ldots, id_\ell\}} - z_\ell^{\ell, \{id_0, \ldots, id_\ell\}} > \alpha \right] \]
(46)
with respect to \( \{id_0, \ldots, id_\ell, ID_{\ell+1}, \ldots, ID_n\} \).

It can be verified that, for any \( \{id_0, \ldots, id_n\} \),
\[ z_n > \alpha + c \iff \exists_{\ell=1}^n P_\ell \text{ is true } \land z_\ell^{\ell, \{id_0, \ldots, id_\ell\}} > \alpha + c \]
with respect to \( \{id_0, \ldots, id_n\} \). From Equation (47) it follows that
\[ \Pr \left[ Z_n > \alpha + c \right] \leq \sum_{\ell=1}^n \Pr \left[ P_\ell \text{ is true } \land z_\ell^{\ell, \{id_0, ID_1, \ldots, ID_\ell\}} > \alpha + c \right] \]
(48)
with respect to \( \{id_0, ID_1, \ldots, ID_n\} \). From Inequalities (48), (49), it follows that
\[ \Pr \left[ Z_n > \alpha + c \right] \leq \sum_{\ell=1}^n e^{\frac{\alpha^2}{n \cdot c^2}} \leq n \cdot e^{\frac{\alpha^2}{n \cdot c^2}} \]
(50)
with respect to \( \{id_0, ID_1, \ldots, ID_n\} \).

\section*{APPENDIX E
PROOF OF DISTINCT FAILURES LEMMA 8.1}

\textbf{Proof} For now, fix \( \ell \) with \( M \leq \ell \leq 2 \cdot M \). Run \( R \) with respect to \( x, tseq \) and \( Useq \) until the aggregate number of distinct failures in the phases is exactly \( \ell \) (which may occur in the middle of an uncompleted phase). Let \( p \) be the number of phases including the last possibly partially completed phase. For \( j = 1, \ldots, p \), let \( d_j \) be the number of distinct failures in phase \( j \). As described in Section VII-A let
\[ Gseq = \{G_1, \ldots, G_{d_1}, G_2, \ldots, G_{d_2}, \ldots, G_p, \ldots, G_p \} \]
be the independent geometric random variables used in the \( p \) phases, where \( G_i \) is the same as \( G_i \) defined in Equation (20). Note that \( \sum_{j=1}^p d_j = \ell \), and
\[ Y'' = \sum_{j=1}^p \sum_{i=1}^{d_j} G_i \]
is the number of failures in the sequence of phases.
The random variables in $G_{\text{seq}}$ depend on the history of the process. For example, $G_i^j$ is used to determine the index of the next distinct failure after there are $i-1$ distinct failures in a phase $j$ that has not yet terminated. The geometric random variable in the sequence after $G_i^j$ depends on the actions of $R$ up till the time of the next distinct failure in phase $j$, where the actions of $R$ may depend on the evolving value of $G_i^j$ during this time. If the actions of $R$ cause phase $j$ not to terminate after there are $i$ distinct failures in phase $j$ then the next geometric random variable in the sequence is $G_{i+1}^j$, whereas if the actions of $R$ cause phase $j$ to terminate after there are $i$ distinct failures in phase $j$ then the next geometric random variable in the sequence is $G_i^{j+1}$.

However, once which geometric random variable to use next is determined within $G_{\text{seq}}$, the value of the determined geometric random variable is chosen independently of all previous history of the process, i.e., independent of the previous geometric random variables and their values in $G_{\text{seq}}$, and independently of $R$. Thus, $G_{\text{seq}}$ is a sequence of independent random variables, but which random variables are in $G_{\text{seq}}$ depends on the process.

Let $\hat{p} = \lceil \frac{\ell}{M-1} \rceil$, for $j = 1, \ldots, \hat{p} - 1$ let $\hat{d}_j = M - 1$, let $d_{\hat{p}} = \ell - (M - 1) \cdot (\hat{p} - 1)$. For $j = 1, \ldots, \hat{p}$, $i = 1, \ldots, \hat{d}_j$, let

$$B_{\text{seq}}^i = \{ B_i^1(1), B_i^1(2), \ldots \}$$

be a sequence of independently and uniformly distributed in $[0, 1]$ random variables, and let

$$B_{\text{seq}} = \{ B_{\text{seq}}^i : j = 1, \ldots, \hat{p}, i = 1, \ldots, \hat{d}_j \}$$

be a sequence of $\ell$ such sequences. Let

$$\hat{G}_{\text{seq}} = \{ \hat{G}_1^1, \ldots, \hat{G}_1^{\hat{d}_1}, \hat{G}_2^2, \ldots, \hat{G}_2^{\hat{d}_2}, \ldots, \hat{G}_{\hat{p}}^\hat{p}, \ldots, \hat{G}_{\hat{p}}^{\hat{d}_{\hat{p}}} \}$$

where $\hat{G}_i^j$ is calculated using $B_{\text{seq}}^i$ as described in Section VIII-A and defined in Equation (20).

As the sequence $G_{\text{seq}} = \{ G_{\text{seq}}^1, G_{\text{seq}}^2, G_{\text{seq}}^\hat{p} \}$ of $\ell$ geometric random variables defined by the process above is being generated, after index $i$ of phase $j$ has been determined, $G_i^j$ can be matched with an unmatched $G_i^j$ of $\hat{G}_{\text{seq}}$, where

$$i' = \arg \min \left\{ i' : \exists j' \text{ s.t. } \hat{G}_{i'}^{j'} \text{ is unmatched} \right\},$$

and thus $G_i^j$ is matched with $\hat{G}_i^{j'}$ where $i' \geq i$. There is always a match because, for all $i$,

$$\left\{ G_i^j \in G_{\text{seq}} \text{ s.t. } i' \geq i \right\} \leq \left\{ \hat{G}_i^{j'} \in \hat{G}_{\text{seq}} \text{ s.t. } i' \geq i \right\}.$$  

This holds independent of which random variables are added to $G_{\text{seq}}$ by the process. Only a prefix of $G_{\text{seq}}$ is known at the time of each match, but all of $\hat{G}_{\text{seq}}$ is known a priori.

If $G_i^j$ is matched to $\hat{G}_i^{j'}$ then the value of $G_i^j$ can be calculated as described in Section VIII-A and defined in Equation (20) using the same $B_{\text{seq}}^i$, as is used to calculate $\hat{G}_i^{j'}$. From $i' \geq i$ it follows that $\hat{G}_i^{j'} \geq G_i^j$ for all possible values of the random variables in $B_{\text{seq}}^{j'}$. Thus, $G_{\text{seq}}$ is determined by $B_{\text{seq}}$, $G_{\text{seq}}$ is determined by $R$, $t_{\text{seq}}$ and $B_{\text{seq}}$, and, for any $R$, $x$ and $t_{\text{seq}}$, for any positive $\eta$,

$$\Pr_{B_{\text{seq}} \text{seq}}[Y'' \geq \eta] \leq \Pr_{B_{\text{seq}} \text{seq}} \left[ \sum_{j=1}^{\hat{p}} \sum_{i=1}^{\hat{d}_j} \hat{G}_i^j \geq \eta \right]. \tag{51}$$

It can be verified from Equation (23) that

$$\mathbb{E} \left[ \sum_{j=1}^{\hat{p}} \sum_{i=1}^{\hat{d}_j} \hat{G}_i^j \right] \leq \frac{\ln(2 \cdot \beta')}{2 \cdot \beta'} \cdot \ell. \tag{52}$$

Let

$$\delta'_d = \frac{e^{-2 \cdot \beta' \cdot (1-2 \cdot \beta') \cdot N \cdot \ln(\epsilon_d)}}{1 + \epsilon_d}. \tag{53}$$

From Inequality (52), since the lefthand sum in Inequality (52) is over $\ell \geq M = 2 \cdot \beta' \cdot N$ geometric random variables, Theorem 2.1 of [17] implies that

$$\Pr_{B_{\text{seq}} \text{seq}} \left[ \sum_{j=1}^{\hat{p}} \sum_{i=1}^{\hat{d}_j} \hat{G}_i^j \geq (1 + \epsilon_d) \cdot \frac{\ln(2 \cdot \beta')}{2 \cdot \beta'} \cdot \ell \right] \leq \delta'_d,$$

and from Inequalities (51) and (53) it follows that, for any $R$, $x$ and $t_{\text{seq}},$

$$\Pr_{B_{\text{seq}} \text{seq}} \left[ Y'' \geq (1 + \epsilon_d) \cdot \frac{\ln(2 \cdot \beta')}{2 \cdot \beta'} \cdot \ell \right] \leq \delta'_d. \tag{54}$$

Now consider the process described just prior to the statement of Distinct Failures Lemma 8.1, where random variable $Y$ is the number of distinct failures in the sequence of phases and random variable $Y''$ is the number of overall failures. Let $\delta_d = M \cdot \delta'_d$. Since there are at most $M$ possible values for $Y$, Inequality (54) and a union bound show that, for any $R$, $x$ and $t_{\text{seq}},$

$$\Pr_{\text{seq}} \left[ Y' \geq (1 + \epsilon_d) \cdot \frac{\ln(2 \cdot \beta')}{2 \cdot \beta'} \cdot \ell \right] \leq \delta_d,$$

and thus

$$\Pr_{\text{seq}} \left[ Y \geq \frac{2 \cdot \beta'}{(1 + \epsilon_d) \cdot \ln(2 \cdot \beta')} \cdot Y' \right] \geq 1 - \delta_d.$$  

This also shows that

$$\Pr_{\text{seq}} \left[ Y \leq (1 + \epsilon_d) \cdot \frac{2 \cdot \beta'}{\ln(2 \cdot \beta')} \cdot M' \right] \geq 1 - \delta_d,$$

since $Y \leq 2 \cdot M = 2 \cdot M' \cdot \frac{2 \cdot \beta'}{\ln(2 \cdot \beta')}$ from Equation (25).