ON THE WELL-POSEDNESS OF THE FULL COMPRESSIBLE NAVIER-STOKES SYSTEM IN CRITICAL BESOV SPACES

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Abstract. We are concerned with the Cauchy problem of the full compressible Navier-Stokes equations satisfied by viscous and heat conducting fluids in $\mathbb{R}^n$. We focus on the so-called critical Besov regularity framework. In this setting, it is natural to consider initial densities $\rho_0$, velocity fields $u_0$ and temperatures $\theta_0$ with $a_0 := \rho_0 - 1 \in B^{\frac{1}{p}}_{p,1}$, $u_0 \in B^{\frac{1}{p}-1}_{p,1}$ and $\theta_0 \in B^{\frac{1}{p}-2}_{p,1}$. After recasting the whole system in Lagrangian coordinates, and working with the total energy along the flow rather than with the temperature, we discover that the system may be solved by means of Banach fixed point theorem in a critical functional framework whenever the space dimension is $n \geq 2$, and $1 < p < 2n$. Back to Eulerian coordinates, this allows to improve the range of $p$’s for which the system is locally well-posed, compared to [7].

1. Introduction

We consider the Cauchy problem of the following full compressible Navier-Stokes equations in $\mathbb{R}^n$, $n \geq 2$:

$$\begin{cases}
\partial_t \rho + \text{div}(\rho u) = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla P = \text{div} \tau, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\
\partial_t \left[ \rho \left( \frac{|u|^2}{2} + e \right) \right] + \text{div} \left[ u \left( \rho \left( \frac{|u|^2}{2} + e \right) + P \right) \right] = \text{div} (\tau \cdot u) - \text{div} q, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\
\rho(x,t) = (\rho_0, u_0, \theta_0), & x \in \mathbb{R}^n,
\end{cases}$$

(1.1)

where $\rho = \rho(t,x) \in \mathbb{R}_+$, $u = u(t,x) \in \mathbb{R}^n$ and $e = e(t,x) \in \mathbb{R}$ are the unknown functions, representing the fluid density, the velocity vector field and the internal energy per unit mass, respectively. We restrict ourselves to the case of Newtonian gases, namely we assume the viscous stress tensor $\tau$ to be given by

$$\tau := \lambda \text{div} u \text{ Id} + 2\mu D(u),$$

where $D(u)$ designates the deformation tensor defined by

$$D(u) := \frac{1}{2} (Du + \nabla u) \quad \text{with} \quad (Du)_{ij} := \partial_j u^i \quad \text{and} \quad (\nabla u)_{ij} := (\partial_i (Du))_{ij} = \partial_i u^j.$$
The given function $P$ represents the pressure depending on $\rho$ and $\theta$. In that paper, we restrict ourselves to the following pressure law:

$$P(\rho, \theta) := \pi_0(\rho) + \theta \pi_1(\rho),$$

where $\pi_0$ and $\pi_1$ are given smooth functions.

Important examples of such pressure laws are ideal fluids (for which $\pi_0(\rho) = 0$ and $\pi_1(\rho) = R\rho$ for some positive constant $R$), barotropic gases ($\pi_1(\rho) = 0$) and Van der Waal gases ($\pi_0(\rho) = -\alpha \rho^2$ and $\pi_1(\rho) = \beta \rho/(\gamma - \rho)$ for some positive constant $\alpha$, $\beta$, $\gamma$).

The boundary conditions at infinity are that $u$ and $\theta$ tend to 0, and that $\rho$ tends to some positive constant $\rho^*$. The exact meaning of the convergence will follow from the functional framework we shall work in. For simplicity, we assume $C_v = 1$ and $\rho^* = 1$ in all that follows. With no loss of generality, one can impose in addition that $\pi_0(1) = 0$.

1.1. Aim of the paper. Our main goal is to solve the full Navier-Stokes equations in the so-called critical regularity framework. This approach originates from a paper of Fujita-Kato [12] devoted to the well-posedness issue for the incompressible Navier-Stokes equations. In our context, the idea is to solve (1.1) in a functional space having the same invariance by time and space dilations as (1.1), namely $(\rho, u, \theta) \rightarrow (\rho_\nu, u_\nu, \theta_\nu)$ with

$$\rho_\nu(t, x) = \rho(\nu^2 t, \nu x), \quad u_\nu(t, x) = \nu u(\nu^2 t, \nu x) \quad \text{and} \quad \theta_\nu(t, x) = \nu^2 \theta(\nu^2 t, \nu x). \quad (1.2)$$

The above family of transforms does not quite leave (1.1) invariant (as $P$ has to be changed into $\nu^2 P$). Nevertheless, the pressure term is, to some extent, lower order, and it is thus suitable to address the solvability issue of the system in ‘critical’ spaces, that is in spaces with norm invariant for all $\nu > 0$ by the scaling transformation $(\rho, u, \theta) \rightarrow (\rho_\nu, u_\nu, \theta_\nu)$.

Following recent works dedicated to this issue (see e.g. [6, 7]), we here employ homogeneous Besov spaces with summation index 1. The main reasons why are that those spaces have nice embedding properties that fail to be true in e.g. Sobolev spaces, and are particularly well adapted to the study of systems related to the heat equation (which is the case here for the velocity and energy equations) as they allow to gain two full derivatives with respect to the data, after taking a $L^1$ norm in time (see Section 2 below).

Before giving more insight on our main result, let us recall the definition of Besov spaces with last index 1. Hereafter, we denote by $L^p$ ($1 \leq p \leq \infty$) standard Lebesgue spaces on $\mathbb{R}^n$, and by $\ell^p$ the set of sequences with summable $p$-th powers. Let $\{\phi_j\}_{j \in \mathbb{Z}}$ be a Littlewood-Paley dyadic decomposition. Namely, let $\phi \in \mathcal{S}$ be a non-negative radially symmetric function that satisfies

$$\text{supp} \hat{\phi} \subset \{ \xi \in \mathbb{R}^n; 2^{-1} < |\xi| < 2 \},$$

$$\hat{\phi}_j := \hat{\phi}(2^{-j} \xi) \quad (\forall j \in \mathbb{Z}) \quad \text{and} \quad \sum_{j \in \mathbb{Z}} \hat{\phi}_j(\xi) = 1 \quad (\forall \xi \neq 0).$$

We further set $\Phi(\xi) := 1 - \sum_{j \geq 1} \hat{\phi}_j(\xi)$ and $\dot{S}_m u := \Phi(2^{-m} \cdot) * u$, for $m \in \mathbb{Z}$.

**Definition 1** (Homogeneous Besov spaces). Let $\mathcal{S}'(\mathbb{R}^n)$ be the space of tempered distributions on $\mathbb{R}^n$. For $1 \leq p \leq \infty$ and $s \leq n/p$, we denote by $\dot{B}^s_{p,1}(\mathbb{R}^n)$ (or more simply
\( \hat{B}^{s}_{p,1} \) the space of tempered distributions \( u \) so that 1

\[
    u = \sum_{j \in \mathbb{Z}} \hat{\Delta}_j u \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n) \quad \text{with} \quad \hat{\Delta}_j u := \phi_j * u
\]

and

\[
    \|u\|_{\hat{B}^{s}_{p,1}} := \sum_{j \in \mathbb{Z}} 2^{js} \|\hat{\Delta}_j u\|_{L^p} < \infty.
\]

In this framework, it is clear that data \( \rho_0 = 1 + a_0, \ u_0 \) and \( \theta_0 \) corresponding to the scaling invariance (1.2) have to be taken as follows:

\[
    a_0 \in \hat{B}^{n}_{p,1}, \quad u_0 \in \hat{B}^{n-1}_{p,1} \quad \text{and} \quad \theta_0 \in \hat{B}^{n-2}_{p,1}.
\]

Let us recall that in the barotropic case, the critical Besov regularity was first considered by the latter author in a \( L^2 \) type framework to obtain a global solution [6] for small perturbations of a stable constant state \( (\rho^*,0) \) with \( \rho^* > 0 \). Since then, there have been a number of refinements as regards admissible exponents for the global existence (see [3,5] and the references therein). The local-in-time existence issue in the critical regularity framework with both large \( u_0 \) and \( a_0 \) (with \( \rho_0 \) bounded away from 0) has been addressed only in the barotropic case. The proof either involves the time-weighted norm or the frequency localization techniques (see [2,9] and [13] for their generalization). The slightly nonhomogeneous case (density close to some constant) is easier and has been investigated for the full Navier-Stokes equations as well in [7].

When solving (1.1) or its barotropic version bluntly, the main difficulty is that the system is only partially parabolic, owing to the mass conservation equation which is of hyperbolic type. This precludes any attempt to use the Banach fixed point theorem in a suitable space. As a matter of fact, existence may be proved either through compactness methods, or through a high norm uniform bounds / low norm stability estimates scheme, as in the case of quasilinear symmetric hyperbolic systems. Another drawback of this direct approach is that the loss of regularity in the stability estimates considerably restricts the set of data for which uniqueness may be proved (see Chap. 10 of [1] for more details).

Prompted by the recent paper dedicated to the compressible barotropic flow [10] or by the work in [11] concerning incompressible inhomogeneous fluids, we here aim at solving the full compressible system (1.1) in the Lagrangian coordinates. Let us emphasize that this approach has already been successfully applied in the case of smooth data (see e.g. [15,16,19,20]). We here want to perform it in the critical regularity framework.

The motivation behind introducing Lagrangian coordinates is to effectively eliminate the hyperbolic part of the system, given that the density equation becomes explicitly solvable once the flow of the velocity field has been determined. At the same time, the system for the velocity and energy in Lagrangian coordinates remains of parabolic type (at least for small enough time), and the Banach fixed point theorem turns out to be applicable for obtaining the existence and uniqueness of the solution in the same class of spaces as in the Eulerian framework. This is the key to improving the set of data leading to well-posedness, compared to [7].

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1See e.g [1] or [18] for more details on the Besov spaces.
1.2. **Notation.** Before introducing the Lagrangian system, let us list some notational conventions. Throughout the paper, we denote by $C$ a generic harmless ‘constant’ the value of which may vary from line to line. The notation $A \lesssim B$ means that $A \leq CB$. For a $C^1$ function $F : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$, we define $\text{div} F : \mathbb{R}^n \to \mathbb{R}^m$ by

$$(\text{div} F)^j := \sum_i \partial_i F_{ij}, \quad 1 \leq j \leq m.$$  

For $n \times n$ matrices $A = (A_{ij})_{1 \leq i,j \leq n}$ and $B = (B_{ij})_{1 \leq i,j \leq n}$, we define the trace product $A : B$ by

$$A : B = \text{tr}AB = \sum_{ij} A_{ij}B_{ji}.$$  

We denote by $\text{adj}(A)$ the adjugate matrix of $A$, i.e. the transpose of the cofactor matrix of $A$. Given some matrix $A$, we define the “twisted” deformation tensor and divergence operator (acting on vector fields $z$) by the formulae

$$D_A(z) := \frac{1}{2}(Dz \cdot A + {}^tA \cdot \nabla z),$$

$$\text{div}_A z := {}^tA \cdot \nabla z = Dz : A.$$  

The flow $X_u$ of the time dependent vector field $u$ is (formally) defined as the solution to

$$X_u(t,y) = y + \int_0^t u(\tau, X_u(\tau,y)) \, d\tau.$$  

We denote by $E$ the *total energy by unit volume* of the fluid, that is, remembering that $e = C_v\theta$ and that $C_v = 1$,

$$E := \rho\left(\frac{|u|^2}{2} + e\right) = \rho\left(\frac{|u|^2}{2} + \theta\right).$$

With the new set of unknowns $(\rho, u, E)$, the system (1.1) is converted to

$$\begin{aligned}
\partial_t \rho + \text{div} (\rho u) &= 0, \\
\partial_t (\rho u) - \text{div} \tau + \text{div} (\rho u \otimes u) + \nabla P &= 0, \\
\partial_t E + \text{div} (uE) - \text{div} \left[ k(\rho) \nabla \left(\frac{E}{\rho}\right) \right] + \tau \cdot u - k(\rho) \nabla \left(\frac{|u|^2}{2}\right) - u\pi_0(\rho) - u\left(\frac{E}{\rho} - \frac{|u|^2}{2}\right)\pi_1(\rho) &= 0.
\end{aligned}$$  

1.3. **Lagrangian coordinates.** Let $\bar{\rho}(t,y) := \rho(t,X_u(t,y))$, $\bar{u}(t,y) := u(t,X_u(t,y))$ and $\bar{E}(t,y) := E(t,X_u(t,y))$ denote the density, velocity and energy functions in Lagrangian coordinates. Setting $J = J_u := \det(DX_u)$ and $A = A_u := (DX_u)^{-1}$, it is shown in Appendix that System (1.5) recasts in

$$\begin{aligned}
\partial_t (J\bar{\rho}) &= 0, \\
\rho_0 \partial_t \bar{u} - \text{div} \left(\text{adj}(DX)(2\mu(\bar{\rho}))D_A\bar{u} + \lambda(\bar{\rho}) \text{div}_A \bar{u} - P(\bar{\rho}, \bar{E})\text{Id}\right) &= 0, \\
\partial_t (J\bar{E}) - \text{div} \left(\text{adj}(DX)(k(\bar{\rho})^tA \nabla (\frac{\bar{E}}{\bar{\rho}}) + \bar{\tau} \cdot \bar{u} - k(\bar{\rho})^tA \nabla (\frac{|\bar{u}|^2}{2}) - \bar{u}\bar{P}(\bar{\rho}, \bar{E})\right) &= 0, \\
(\bar{\rho}, \bar{u}, \bar{E})|_{t=0} &= (\rho_0, u_0, E_0).
\end{aligned}$$  

(1.6)
Looking at the energy equation, it is thus natural to introduce the total energy along the flow defined by

$$\mathcal{K} := J\mathcal{E} = \rho_0(\theta + \frac{|\bar{u}|^2}{2}).$$  \hspace{1cm} (1.7)

We shall thus eventually consider the following system

$$\begin{cases}
\partial_t (J\bar{\rho}) = 0, \\
\rho_0 \partial_t \bar{\rho} - \text{div} \left[ \text{adj}(DX)(2\mu(\bar{\rho})D\bar{u} + \lambda(\bar{\rho}) \text{div}\bar{u} - \mathcal{P}(\bar{\rho}, \mathcal{K}) \text{Id}) \right] = 0, \\
\partial_t \mathcal{K} - \text{div} \left[ \text{adj}(DX)(k(\bar{\rho})^t A \nabla(\mathcal{K}_{\rho_0}) - k(\bar{\rho})^t A \nabla(\frac{|\bar{u}|^2}{2}) + \bar{\rho} \cdot \bar{u} - \bar{u} \mathcal{P}(\bar{\rho}, \mathcal{K})) \right] = 0,
\end{cases}$$  \hspace{1cm} (1.8)

where we have redefined the initial data \( K_0 \) as

$$K_0 := E_0 = \rho_0 \left( \theta_0 + \frac{|u_0|^2}{2} \right),$$  \hspace{1cm} (1.9)

and the pressure function \( \mathcal{P} \) as

$$\mathcal{P}(\bar{\rho}, \mathcal{K}) := \pi_0(\bar{\rho}) + \left( \frac{\mathcal{K}}{\rho_0} - \frac{|\bar{u}|^2}{2} \right) \pi_1(\bar{\rho}).$$

Let us finally emphasize that one may forget any reference to the initial Eulerian vector-field \( u \) by defining directly the “flow” \( X \) of \( \bar{\rho} \) by the formula

$$X(t, y) = y + \int_0^t \bar{\rho}(\tau, y) \, d\tau.$$  \hspace{1cm} (1.10)

1.4. Main results. We shall obtain the existence and uniqueness of a local-in-time solution \((\bar{\rho}, \bar{\rho}, \mathcal{K})\) for (1.8), with \( \bar{\rho} := \bar{\rho} - 1 \) in \( C([0, T]; \dot{B}^{\frac{\alpha}{p-1}}_{p,1}) \) and \( (\bar{\rho}, \mathcal{K}) \) in the space

$$E_p(T) := \left\{ (v, \psi) \left| \begin{array}{l}
v \in C([0, T]; \dot{B}^{\frac{\alpha}{p-1}}_{p,1}), \ \partial_t v, \nabla^2 v \in L^1(0, T; \dot{B}^{\frac{\alpha}{p-1}}_{p,1}) \\
\psi \in C([0, T]; \dot{B}^{\frac{\alpha}{p-2}}_{p,1}), \ \partial_t \psi, \nabla^2 \psi \in L^1(0, T; \dot{B}^{\frac{\alpha}{p-2}}_{p,1})
\end{array} \right. \right\}$$  \hspace{1cm} (1.11)

endowed with the norm

$$\| (v, \psi) \|_{E_p(T)} := \| v \|_{L^\infty_t (\dot{B}^{\frac{\alpha}{p-1}}_{p,1})} + \| \partial_t v, \nabla^2 v \|_{L^1_t (\dot{B}^{\frac{\alpha}{p-1}}_{p,1})} + \| \psi \|_{L^\infty_t (\dot{B}^{\frac{\alpha}{p-2}}_{p,1})} + \| \partial_t \psi, \nabla^2 \psi \|_{L^1_t (\dot{B}^{\frac{\alpha}{p-2}}_{p,1})}.$$  \hspace{1cm} (1.12)

It is easily checked that \( E_p(T) \) is critical in the meaning of (1.2).

Let us now state our main result.

**Theorem 1.1.** Let \( 1 < p < 2n \) and \( n \geq 2 \). Let \( u_0 \) be a vector field in \( \dot{B}^{\frac{\alpha}{p-1}}_{p,1} \) and \( K_0 \), a real valued function in \( \dot{B}^{\frac{\alpha}{p-2}}_{p,1} \). Assume that \( \rho_0 \) satisfies \( a_0 := (\rho_0 - 1) \in \dot{B}^{\frac{\alpha}{p-1}}_{p,1} \) and

$$\inf_x \rho_0(x) > 0.$$  \hspace{1cm} (1.12)

Then System (1.8) admits a unique local solution \((\bar{\rho}, \bar{\rho}, \mathcal{K})\) with \( \bar{\rho} \) bounded away from zero, \( \bar{\rho} := \bar{\rho} - 1 \) in \( C([0, T]; \dot{B}^{\frac{\alpha}{p-1}}_{p,1}) \) and \( (\bar{\rho}, \mathcal{K}) \) in \( E_p(T) \).

Moreover, the flow map \((a_0, u_0, K_0) \mapsto (\bar{\rho}, \bar{\rho}, \mathcal{K})\) is Lipschitz continuous from \( \dot{B}^{\frac{\alpha}{p-1}}_{p,1} \times \dot{B}^{\frac{\alpha}{p-2}}_{p,1} \times E_p(T) \) to \( C([0, T]; \dot{B}^{\frac{\alpha}{p-1}}_{p,1} \times \dot{B}^{\frac{\alpha}{p-2}}_{p,1} \times E_p(T)). \)

In Eulerian coordinates, the above theorem implies:
Theorem 1.2. Under the same assumptions as in Theorem 1.1, with in addition $n \geq 3$ and $1 < p < n$, System (1.1) has a unique local solution $(\rho, u, \theta)$ with $(u, \theta) \in E_p(T)$, $\rho$ bounded away from 0 and $\rho - 1 \in \mathcal{C}([0,T]; \dot{B}^\frac{n}{p - 1}_{p,1})$.

Remark 1.1. Because our techniques rely on Fourier analysis, the same statements hold true for periodic boundary conditions.

Remark 1.2. The equivalence between the Eulerian and the Lagrangian systems is provable only in the range $1 < p < n$ and if $n \geq 3$ (see Proposition 3.1 below), whence the stronger conditions on $p$ and $n$. Nevertheless the above statement improves the results of [7, 8] as regards uniqueness: there, the condition $p \leq 2n/3$ was required. Besides, only the case of small $a_0$ was considered.

In dimension $n = 2$, or if $n \leq p < 2n$, only partial results are available. First, in the critical functional framework, prescribing $(a_0, u_0, \theta_0)$ or $(a_0, u_0, E_0)$ is no longer equivalent since the product does not map $\dot{B}^\frac{n}{p - 1}_{p,1} \times \dot{B}^\frac{n}{p - 2}_{p,1}$ in $\dot{B}^\frac{n}{p - 2}_{p,1}$ any longer, and the data are interrelated through (1.7). Second, even if one chooses to work with $(a, u, E)$ rather than with $(a, u, \theta)$, having $(u, E)$ in $E_p(T)$ does not quite imply that $(\bar{u}, \bar{K})$ is in $E_p(T)$ (and the converse is false, too). Nevertheless, it is still possible to solve (1.5), see Corollary 3.2 for more details.

Remark 1.3. The restriction that $1 < p < n$ and $n \geq 3$ in Theorem 1.2 is consistent with the recent paper by Chen-Miao-Zhang [4]. There, the authors established the ill-posedness of the full compressible Navier-Stokes system in three dimension in the sense that the continuity of data-solution map fails at the origin in the critical Besov framework that we used, if $p > n$. In other words, up to the limit case $p = n$, Theorem 1.2 is optimal as regards the local well-posedness issue with unknowns $(\rho, u, \theta)$.

Remark 1.4. Different formulations are known for expressing the third equation of (1.1). Namely, the following quantities may be used to rewrite the energy equation: the temperature $\theta$, the total energy by unit mass $M = \frac{|u|^2}{2} + \theta$ and the total energy by unit volume $E = \rho(\frac{|u|^2}{2} + \theta)$. Those formulations are equivalent for smooth enough solutions. In the critical framework, working with the total energy along the flow $\bar{K}$ in Lagrangian coordinates allows to get the widest range of exponents.

1.5. Banach fixed point argument. We end this section with a quick presentation of the Banach fixed point argument that will enable us to prove Theorem 1.1. To simplify the notation, we drop the bars of the Lagrangian coordinates.

To start with, let us rewrite (1.8) as a system of parabolic equations with nonsmooth (but time independent) coefficients. Regarding the velocity equation, we proceed as in [10]. Next, we write the equation for $K$ as follows:

$$
\partial_t K - \text{div} \left( k(\rho_0) \nabla \left( \frac{K}{\rho_0} \right) \right) = \text{div} \left[ (k(J^{-1} \rho_0) \text{adj}(DX)^t A - k(\rho_0) \text{Id}) \nabla \left( \frac{K}{\rho_0} \right) \right] + k(J^{-1} \rho_0) \text{adj}(DX)^t A \nabla \left( \frac{|u|^2}{2} + \eta \cdot u - uP(J^{-1} \rho_0, K) \right).
$$

Denoting

$$
L_{\rho_0} u := \partial_t u - \rho_0^{-1} \text{div} \left( 2\mu(\rho_0) D(u) + \lambda(\rho_0) \text{div} u \text{Id} \right)
$$

and

$$
H_{\rho_0} K := \partial_t K - \text{div} \left( k(\rho_0) \nabla (\rho_0^{-1} K) \right),
$$

(1.13)
System (1.8) thus writes
\[
\begin{align*}
L_{\rho_0} u + \rho_0^{-1} \nabla (\rho_0^{-1} \pi_1 (\rho_0) K) &= \rho_0^{-1} \text{div} \left( I_1 (u, v) + I_2 (u, v) + I_3 (u, v) + I_4 (u, v) \right), \\
H_{\rho_0} K &= \text{div} \left( I_5 (v, \psi) + I_6 (v, \psi) + I_7 (v, \psi) + I_8 (v, \psi) \right),
\end{align*}
\]
with
\[
\begin{align*}
I_1 (v, w) &:= (\text{adj} (DX_v) - \text{Id}) \left( 2 \mu (J_v^{-1} \rho_0) D_{A_v} (w) + \lambda (J_v^{-1} \rho_0) \text{div}_{A_v} w \text{Id} \right), \\
I_2 (v, w) &:= 2 \mu (J_v^{-1} \rho_0) - \mu (\rho_0) ) D_{A_v} (w) + (\lambda (J_v^{-1} \rho_0) - \lambda (\rho_0) ) \text{div}_{A_v} w \text{Id}, \\
I_3 (v, w) &:= 2 \mu (\rho_0) (D_{A_v} (w) - D (w)) + \lambda (\rho_0) (\text{div}_{A_v} w - \text{div} w) \text{Id}, \\
I_4 (v, \psi) &:= - \text{adj} (DX_v) P (J_v^{-1} \rho_0, \psi) + \frac{\pi_1 (\rho_0)}{\rho_0} \psi \text{Id}, \\
I_5 (v, \psi) &:= (k (J_v^{-1} \rho_0) \text{adj} (DX_v) A_v - k (\rho_0) \text{Id}) \nabla \left( \frac{\psi}{\rho_0} \right), \\
I_6 (v, \psi) &:= k (J_v^{-1} \rho_0) \text{adj} (DX_v) A_v \nabla \left( \frac{|\psi|^2}{2} \right), \\
I_7 (v, \psi) &:= P (J_v^{-1} \rho_0, \psi) \text{adj} (DX_v) \cdot v, \\
I_8 (v, w) &:= \text{adj} (DX_v) (\lambda (J_v^{-1} \rho_0) \text{div}_{A_v} w \text{Id} + 2 \mu (J_v^{-1} \rho_0) D_{A_v} (w)) \cdot w.
\end{align*}
\]
In order to solve (1.8) locally, it suffices to show that the map
\[
\Phi : (v, \psi) \mapsto (u, K)
\]
with \((u, K)\) the solution to
\[
\begin{align*}
L_{\rho_0} u + \rho_0^{-1} \nabla (\rho_0^{-1} \pi_1 (\rho_0) K) &= \rho_0^{-1} \text{div} \left( I_1 (v, v) + I_2 (v, v) + I_3 (v, v) + I_4 (v, \psi) \right), \\
H_{\rho_0} K &= \text{div} \left( I_5 (v, \psi) + I_6 (v, \psi) + I_7 (v, \psi) + I_8 (v, v) \right),
\end{align*}
\]
has a fixed point in \(E_p (T)\) for small enough \(T\).

The rest of the paper unfolds as follows: in the second section, we establish the maximal regularity estimates for the linear parabolic system corresponding to the l.h.s. of (1.17). It turns out that completely decoupling the system into two parabolic equations for the velocity and energy will cause some loss of estimate: we ought to take into account the pressure term as the linear term of the system, which is not necessary in the barotropic case. In the third section, we shall prove Theorem 1.1 and Theorem 1.2 by combining the a priori estimate in the second section and Banach’s fixed point theorem. In the Appendix, we list some results concerning the Lagrangian coordinates and Besov spaces that may be found in the literature (see [1, 10, 11]).

2. A PRIORI ESTIMATES FOR LINEAR PARABOLIC SYSTEMS

We here aim at establishing well-posedness and a priori estimates for the linear part of (1.17), namely
\[
\begin{align*}
\partial_t u - \rho_0^{-1} \text{div} \left( 2 \mu (\rho_0) D (u) + \lambda (\rho_0) \text{div} u \text{Id} \right) + \rho_0^{-1} \nabla (\rho_0^{-1} \pi_1 (\rho_0) K) &= f, \\
\partial_t K - \text{div} \left( k (\rho_0) \nabla (\rho_0^{-1} K) \right) &= g.
\end{align*}
\]

The analysis of the first equation is based on results that have been established recently in [10] for the following Lamé system with nonsmooth coefficients:
\[
\partial_t u - 2 \text{div} (\mu D (u)) - b \nabla (\lambda \text{div} u) = f,
\]
(here both \( u \) and \( f \) are valued in \( \mathbb{R}^n \)) when the following uniform ellipticity condition is satisfied:

\[
\alpha := \min \left( \inf_{(t,x) \in [0,T] \times \mathbb{R}^n} (a \mu)(t,x), \inf_{(t,x) \in [0,T] \times \mathbb{R}^n} (2a \mu + b \lambda)(t,x) \right) > 0. \tag{2.3}
\]

**Proposition 2.1** ([10]). Let \( a, b, \lambda \) and \( \mu \) be bounded functions satisfying (2.3). Assume that \( a \nabla \mu, b \nabla \lambda, \mu \nabla a \) and \( \lambda \nabla b \) are in \( L^\infty(0,T; B_{p,1}^{\tilde{\mu}}) \) for some \( 1 < p < 2n \), and that there exist some constants \( \bar{a}, \bar{b}, \bar{\lambda} \) and \( \bar{\mu} \) satisfying

\[
2\bar{a} \bar{\mu} + \bar{b} \bar{\lambda} > 0 \text{ and } \bar{a} \bar{\mu} > 0,
\]

and such that \( a - \bar{a}, b - \bar{b}, \lambda - \bar{\lambda} \) and \( \mu - \bar{\mu} \) are in \( C([0,T]; \dot{B}_{p,1}^{\bar{\mu}}) \). Finally, suppose that

\[
\lim_{m \to +\infty} \| (\text{Id} - \dot{S}_m)(a \nabla \mu, b \nabla \lambda, \mu \nabla a, \lambda \nabla b) \|_{L^\infty_t(\dot{B}_{p,1}^{\bar{\mu}})} = 0.
\]

Then there exist two constants \( \eta \) and \( C \) such that if \( m \) is so large as to satisfy

\[
\min \left( \inf_{(t,x) \in [0,T] \times \mathbb{R}^n} \dot{S}_m(a \mu)(t,x), \inf_{(t,x) \in [0,T] \times \mathbb{R}^n} \dot{S}_m(2a \mu + b \lambda)(t,x) \right) \geq \frac{\alpha}{2}, \tag{2.4}
\]

\[
\| (\text{Id} - \dot{S}_m)(a \nabla \mu, b \nabla \lambda, \mu \nabla a, \lambda \nabla b) \|_{L^\infty_t(\dot{B}_{p,1}^{\bar{\mu}})} \leq \eta \alpha, \tag{2.5}
\]

then we have for all \( t \in [0,T] \),

\[
\| u \|_{L^\infty_t(\dot{B}_{p,1}^{\bar{\mu}})} + \alpha \| \nabla u \|_{L^1_t(\dot{B}_{p,1}^{\bar{\mu}})} \leq C \left( \| u_0 \|_{\dot{B}_{p,1}^{\bar{\mu}}} + \| f \|_{L^1_t(\dot{B}_{p,1}^{\bar{\mu}})} \right) \exp \left( \frac{C}{\alpha} \int_0^t \| \dot{S}_m(a \nabla \mu, b \nabla \lambda, \mu \nabla a, \lambda \nabla b) \|_{L^2(\dot{B}_{p,1}^{\bar{\mu}})}^2 \, d\tau \right).
\]

As the energy equation of (2.1) is of the following form:

\[
\partial_t u - \text{div} (k \nabla (cu)) = f, \tag{2.6}
\]

and thus does not quite enter in the framework of Proposition 2.1, we shall need the following statement.

**Proposition 2.2.** Let \( k \) be a bounded function such that there exists a constant \( \beta \) with \( k \geq \beta > 0 \). Assume that \( \nabla k \) and \( \nabla c \) are in \( L^\infty(0,T; \dot{B}_{p,1}^{\bar{\mu}}) \) for some \( 1 < p < \infty \), that

\[
\lim_{m \to +\infty} \| (\text{Id} - \dot{S}_m)(k \nabla c, c \nabla k) \|_{L^\infty_t(\dot{B}_{p,1}^{\bar{\mu}})} = 0,
\]

and that \( k - \bar{k} \) and \( c - \bar{c} \) are in \( C([0,T]; \dot{B}_{p,1}^{\bar{\mu}}) \) for some positive constants \( \bar{k} \) and \( \bar{c} \).

Then there exist two constants \( \eta \) and \( C \) such that if for some \( m \in \mathbb{Z} \) we have

\[
\inf_{(t,x) \in [0,T] \times \mathbb{R}^n} \dot{S}_m(k c)(t,x) \geq \frac{\beta}{2}, \tag{2.7}
\]

\[
\| (\text{Id} - \dot{S}_m)(k \nabla c, c \nabla k) \|_{L^\infty_t(\dot{B}_{p,1}^{\bar{\mu}})} \leq \eta \beta, \tag{2.8}
\]
then the solutions to (2.2) satisfy for all \( t \in [0, T] \),
\[
\|u\|_{L^p(B_{\rho,1}^s)} + \beta \|u\|_{L^1(B_{\rho,1}^{s+2})} \leq C \left( \|u_0\|_{B_{\rho,1}^s} + \|f\|_{L^1(B_{\rho,1}^{s})} \right) \exp \left( \frac{C}{\beta} \int_0^t \|\hat{S}_m(k\nabla c, c\nabla k)\|^2_{\hat{B}_{\rho,1}^s} \, d\tau \right)
\]
whenever \( s \) satisfies
\[
- \min\left( \frac{n}{p}, \frac{n}{p'} \right) - 1 < s \leq \frac{n}{p} - 2.
\]

**Proof.** We focus on the proof of a priori estimates. Existence follows from the continuity method as for Proposition 2.1 (see [10]).

First, we smooth out the coefficient \( kc \) according to the low frequency cut-off operator \( \hat{S}_m \), with \( m \in \mathbb{Z} \) to be determined later:
\[
\partial_t u - \text{div} (\hat{S}_m(kc) \nabla u) = f + \text{div}(k\nabla c \cdot u) + \text{div}((\text{Id} - \hat{S}_m)(kc) \nabla u).
\]

Next, applying Littlewood-Paley operator \( \hat{\Delta}_j \) to the above equation yields
\[
\partial_t u_j - \text{div} (\hat{S}_m(kc) \nabla u_j) = f_j + \text{div} \hat{\Delta}_j (\hat{S}_m(k\nabla c) \cdot u) + \text{div} \hat{\Delta}_j ((\text{Id} - \hat{S}_m)(k\nabla c) \cdot u) + \text{div} [\hat{\Delta}_j, \hat{S}_m(kc)] \nabla u + \text{div} \hat{\Delta}_j ((\text{Id} - \hat{S}_m)(kc) \nabla u).
\]

From energy arguments combined with the Bernstein-type inequality of the Appendix of [7], we get (formally)
\[
\frac{d}{dt} \|u_j\|_{L^p} + \beta 2^{2j} \|u_j\|_{L^p} \lesssim \|f_j\|_{L^p} + \|\text{div} \hat{\Delta}_j (\hat{S}_m(k\nabla c) \cdot u)\|_{L^p} + \|\text{div} \hat{\Delta}_j ((\text{Id} - \hat{S}_m)(k\nabla c) \cdot u)\|_{L^p} \leq \|u_0\|_{B_{\rho,1}^s} + \|f\|_{L^1(B_{\rho,1}^{s})}
\]
\[
+ \int_0^t \sum_j 2^{js} (\|\text{div} \hat{\Delta}_j ((\text{Id} - \hat{S}_m)(kc) \nabla u)\|_{L^p} + \|\text{div} \hat{\Delta}_j ((\text{Id} - \hat{S}_m)(k\nabla c) \cdot u)\|_{L^p}) \, d\tau.
\]

In the following computations, let us denote by \( (c_j)_{j \in \mathbb{Z}} \) a sequence belonging to the unit sphere of \( \ell^1(\mathbb{Z}) \). If \( -\min\left( \frac{n}{p}, \frac{n}{p'} \right) - 1 < s \leq \frac{n}{p} - 1 \) then we have by Proposition 4.1:
\[
\|\text{div} \hat{\Delta}_j ((\text{Id} - \hat{S}_m)(kc) \nabla u)\|_{L^p} \leq c_j 2^{-js} \|\text{div} \hat{S}_m(kc)\|_{\hat{B}_{\rho,1}^s} \|\nabla u\|_{\hat{B}_{\rho,1}^{s+1}}.
\]
If \( s \) satisfies \( -\min\left( \frac{n}{p}, \frac{n}{p'} \right) - 1 < s \leq \frac{n}{p} - 2 \) then
\[
\|\text{div} \hat{\Delta}_j ((\text{Id} - \hat{S}_m)(k\nabla c) \cdot u)\|_{L^p} \leq c_j 2^{-js} \|\text{div} \hat{S}_m(k\nabla c)\|_{\hat{B}_{\rho,1}^{s-1}} \|u\|_{B_{\rho,1}^{s+2}}.
\]
Consequently, the second line of the (2.10) may be absorbed by the l.h.s. if \( \eta \) has been chosen small enough in (2.8). Next, if \( s \) satisfies \( -\min\left( \frac{n}{p}, \frac{n}{p'} \right) - 1 < s \leq \frac{n}{p} - 1 \) then
\[
\|\text{div} \hat{\Delta}_j (\hat{S}_m(k\nabla c) \cdot u)\|_{L^p} \leq c_j 2^{-js} \|\hat{S}_m(k\nabla c)\|_{\hat{B}_{\rho,1}^s} \|u\|_{\hat{B}_{\rho,1}^{s+1}},
\]
Finally, for \( -\min\left(\frac{n}{p}, \frac{n}{p'}\right) - 1 < s \leq \frac{n}{p} - 1 \), we have by Proposition 4.3,
\[
\|\text{div} [\hat{\Delta}_j \hat{S}_m(kc)] \nabla u\|_{L^p} \leq c_j 2^{-js} \|\nabla \hat{S}_m(kc)\|_{B^s_{p,1}} \|\nabla u\|_{B^s_{p,1}}.
\]
Therefore by interpolation and Young’s inequality, we get for all \( \eta > 0 \),
\[
\|\text{div} [\hat{\Delta}_j \hat{S}_m(kc)] \nabla u\|_{L^p} + \|\text{div} \hat{\Delta}_j (\hat{S}_m(k\nabla c) \cdot u)\|_{L^p} \\
\leq c_j 2^{-js} \left( \frac{C}{\eta \beta} (\|\nabla \hat{S}_m(kc)\|^2_{B^s_{p,1}} + \|\hat{S}_m(k\nabla c)\|^2_{B^s_{p,1}}) \|u\|_{B^s_{p,1}} + \eta \beta \|u\|_{B^{s+2}_{p,1}} \right).
\]
It is now clear that taking \( \eta \) small enough completes the proof of the proposition. \( \square \)

Combining Propositions 2.1 and 2.2, one can now consider the following linear system:
\[
\begin{cases}
\partial_t u - \text{div} (2\mu D(u) + \lambda \text{div} \text{Id}) = f, \\
\partial_t K - \text{div} (k \nabla (cK)) = g,
\end{cases}
\quad \text{(2.11)}
\]

**Proposition 2.3.** Let \( 1 < p < 2n \). Let \( u_0 \in \dot{B}^{\frac{n}{p}}_{p,1} \), \( k_0 \in \dot{B}^{\frac{n}{p} - 2}_{p,1} \), \( f \in L^1(0,T; \dot{B}^{\frac{n}{p} - 1}_{p,1}) \) and \( g \in L^1(0,T; \dot{B}^{\frac{n}{p} - 2}_{p,1}) \). Let \( a, b, \lambda \) and \( \mu \) satisfy the assumptions of Proposition 2.1 and \( k \) and \( c \) satisfy those of Proposition 2.2 with \( s = \frac{n}{p} - 2 \). Assume that \( \pi \) belongs to the multiplier space \(^2\mathcal{M}(\dot{B}^\frac{n}{p}_{p,1}) \). Finally, suppose that
\[
\lim_{m \to +\infty} \left\| (\text{Id} - \hat{S}_m)(k\nabla c, c\nabla k, a\nabla \mu, b\nabla \lambda, \mu \nabla a, \lambda \nabla b) \right\|_{L^\infty(\dot{B}^{\frac{n}{p} - 1}_{p,1})} = 0.
\]
Then System (2.11) admits a unique solution \((u,K)\) with
\[
u \in C([0,T]; \dot{B}^{\frac{n}{p} - 1}_{p,1}) \cap L^1(0,T; \dot{B}^{\frac{n}{p} + 1}_{p,1}) \quad \text{and} \quad K \in C([0,T]; \dot{B}^{\frac{n}{p} - 2}_{p,1}) \cap L^1(0,T; \dot{B}^{\frac{n}{p}}_{p,1}).
\]
Besides, if \( m \) is large enough (as in Propositions 2.1 and 2.2) then \((u,K)\) fulfills for all \( t \in [0,T] \),
\[
\|K\|_{L^\infty_t(\dot{B}^{\frac{n}{p} - 2}_{p,1})} + \|\nabla K\|_{L^1_t(\dot{B}^{\frac{n}{p}}_{p,1})} \leq C \left( \|k_0\|_{\dot{B}^{\frac{n}{p} - 2}_{p,1}} + \|g\|_{L^1_t(\dot{B}^{\frac{n}{p}}_{p,1})} \right)
\times \exp \left( C \int_0^t \|\hat{S}_m(k\nabla c, c\nabla k)\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \right),
\]
\[
\|u\|_{L^\infty_t(\dot{B}^{\frac{n}{p} - 1}_{p,1})} + \alpha \|\nabla u\|_{L^1_t(\dot{B}^{\frac{n}{p} + 1}_{p,1})} \leq C \left( \|u_0\|_{\dot{B}^{\frac{n}{p} - 1}_{p,1}} + \|f\|_{L^1_t(\dot{B}^{\frac{n}{p} - 1}_{p,1})} \right)
\times \exp \left( C \int_0^t \|\hat{S}_m(a\nabla \mu, b\nabla \lambda, \mu \nabla a, \lambda \nabla b)\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \right).
\]

**Proof.** It suffices to first solve the second equation of (2.11) according to Proposition 2.2, then look at \( u \) as the solution to
\[
\partial_t u - \text{div} (2\mu D(u) + \lambda \text{div} \text{Id}) = f - \alpha \nabla (\pi K).
\]
Given the assumptions on \( a \) and \( \pi \), and the fact that \( K \) is in \( L^1(0,T; \dot{B}^{\frac{n}{p}}_{p,1}) \), we see that \( u \) may be constructed according to Proposition 2.1. \( \square \)

\(^2\) The multiplier space \( \mathcal{M}(\dot{B}^n_{p,1}) \) is the set of all functions \( f \in \dot{B}^n_{p,1} \) such that \( \|f\|_{\mathcal{M}(\dot{B}^n_{p,1})} := \sup_{\|h\|_{\dot{B}^n_{p,1}} = 1} \|hf\|_{\dot{B}^n_{p,1}} < \infty. \)
3. Proof of the main theorem

Let \((u_L, K_L)\) be the solution to the linear system corresponding to the l.h.s. of (1.14) with \(\rho = 1\), namely

\[
L_1 u_L + \pi_1(1) \nabla K_L = 0, \quad u_L|_{t=0} = u_0, \\
H_1 K_L = 0, \quad K_L|_{t=0} = K_0.
\]

3.1. The fixed point scheme. We claim that the Banach fixed point theorem applies to the map \(\Phi\) defined in (1.16) in some closed ball \(B_{E_p(T)}((u_L, \theta_L), R)\) with suitably small \(T\) and \(R\).

To justify our claim, we set \(\tilde{u} := u - u_L\) and \(\tilde{K} := K - K_L\), and observe that solving (1.17) for some given \((v, S) \in E_p(T)\) is equivalent to solving

\[
\begin{cases}
L_{\rho_0} \tilde{u} + \rho_0^{-1} \nabla (\rho_0^{-1} \pi_1(\rho_0) \tilde{K}) = \rho_0^{-1} \text{div} (I_1(v, v) + I_2(v, v) + I_3(v, v) + I_4(v, \psi)) \\
+ (L_1 - L_{\rho_0}) u_L - \rho_0^{-1} \nabla (\rho_0^{-1} \pi_1(\rho_0) K_L) + \nabla (\pi_1(1) K_L), \\
H_{\rho_0} \tilde{K} = \text{div} (I_5(v, \psi) + I_6(v, \psi) + I_7(v, \psi) + I_8(v, v)) + (H_1 - H_{\rho_0}) K_L.
\end{cases}
\]

From the definition of the space \(\tilde{B}_{p,1}^\infty\) (which involves a convergent series) and the fact that it embeds in the set of bounded continuous functions, it is clear that there exists some \(m \in \mathbb{Z}\) so that

\[
\min \left( \inf_{x \in \mathbb{R}^n} \hat{S}_m \left( \frac{\mu(\rho_0)}{\rho_0} \right), \inf_{x \in \mathbb{R}^n} \hat{S}_m \left( \frac{\lambda(\rho_0)}{\rho_0} \right), \inf_{x \in \mathbb{R}^n} \hat{S}_m \left( \frac{k(\rho_0)}{\rho_0} \right) \right) \geq \frac{\max(\alpha, \beta)}{2},
\]

\[
\| (\text{Id} - \hat{S}_m) \left( \frac{\mu(\rho_0)}{\rho_0} \nabla \rho_0, \frac{\mu'(\rho_0)}{\rho_0} \nabla \rho_0, \frac{\lambda(\rho_0)}{\rho_0} \nabla \rho_0, \frac{\lambda'(\rho_0)}{\rho_0} \nabla \rho_0, \frac{k(\rho_0)}{\rho_0} \nabla \rho_0 \right) \|_{L^\infty(B_{p,1}^\infty)} \leq \eta \min(\alpha, \beta).
\]

Therefore, in order to solve the above system by means of Proposition 2.3, it suffices to check that the r.h.s. of the first and second equations are in \(L^1(0, T; \tilde{B}_{p,1}^{\infty})\) and \(L^1(0, T; \tilde{B}_{p,1}^{\infty-2})\), respectively.

First step: Stability of the ball \(\tilde{B}_{E_p(T)}((u_L, K_L), R)\) for suitably small \(T\) and \(R\).

From now on, we assume that for a small enough \(\bar{c}\), we have

\[
\| Dv \|_{L^1_T(B_{p,1}^{\infty})} \leq \bar{c}. \tag{3.1}
\]

Proposition 2.3 and the definition of the multiplier space \(\mathcal{M}(\tilde{B}_{p,1}^{\infty-1})\) ensure that

\[
\| (\tilde{u}, \tilde{K})\|_{E_p(T)} \leq C \varepsilon \rho_0 m T \left( \| (L_1 - L_{\rho_0}) u_L \|_{L^1_T(B_{p,1}^{\infty})} + \| (H_1 - H_{\rho_0}) K_L \|_{L^1_T(B_{p,1}^{\infty})} \right) \left( \| \rho_0^{-1} \pi_1(\rho_0) - \pi_1(1) \|_{L^1_T(B_{p,1}^{\infty})} \right) \left( \| \rho_0^{-1} \|_{\mathcal{M}(B_{p,1}^{\infty})} \| I_1(v, v) + I_2(v, v) + I_3(v, v) + I_4(v, \psi) \|_{L^1_T(B_{p,1}^{\infty})} \right) + \| I_5(v, \psi) + I_6(v, \psi) + I_7(v, \psi) + I_8(v, v) \|_{L^1_T(B_{p,1}^{\infty-1})} \right). \tag{3.2}
\]
We may confirm that \( \rho_0^{-1} \) belongs to \( \mathcal{M}(\tilde{B}_{p,1}^{\#-1}) \) by the product estimate:

\[
\|\rho_0^{-1} h\|_{\tilde{B}_{p,1}^{\#-1}} \leq \left( \frac{a_0}{1 + a_0} - 1 \right) h\|_{\tilde{B}_{p,1}^{\#-1}} \leq (\|a_0\|_{\tilde{B}_{p,1}^{\#}} + 1) \|h\|_{\tilde{B}_{p,1}^{\#-1}}.
\]

Likewise,

\[
\|(L_1 - L_{\rho_0})uL\|_{L^1(B_{p,1}^{\#})} \leq (\|a_0\|_{\tilde{B}_{p,1}^{\#}} + 1) \|a_0\|_{\tilde{B}_{p,1}^{\#}} \|DuL\|_{L^1(B_{p,1}^{\#})}, \tag{3.3}
\]

\[
|(\rho_0^{-1} \pi_1(\rho_0) - \pi_1(1)K_L)\|_{L^1(B_{p,1}^{\#})} \leq \|a_0\|_{\tilde{B}_{p,1}^{\#}} \|K_L\|_{L^1(B_{p,1}^{\#})} \tag{3.4}
\]

and

\[
|(H_1 - H_{\rho_0})K_L\|_{L^1(B_{p,1}^{\#})} \leq (\|a_0\|_{\tilde{B}_{p,1}^{\#}} + 1)^2 \|a_0\|_{\tilde{B}_{p,1}^{\#}} \|K_L\|_{L^1(B_{p,1}^{\#})}. \tag{3.5}
\]

**Estimate of I_1, I_2, I_3**: Terms \( I_1, I_2 \) and \( I_3 \) have been estimated in [10] as follows:

\[
\|I_j(v, w)\|_{L^1(B_{p,1}^{\#})} \leq (\|a_0\|_{\tilde{B}_{p,1}^{\#}} + 1) \|Dv\|_{L^1(B_{p,1}^{\#})} \|Dw\|_{L^1(B_{p,1}^{\#})} \quad \text{for} \quad j = 1, 2, 3.
\]

**Estimate of I_4**: Let us recall that the pressure is given by

\[
P(J_v^{-1} \rho_0, \psi) = \pi_0(J_v^{-1} \rho_0) + \left( \frac{1}{\rho_0} \psi - \frac{|\psi|^2}{2} \right) \pi_1(J_v^{-1} \rho_0)
\]

so that \( I_4 \) can be written as

\[
I_4(v, \psi) = -\text{adj}(DX_v)\pi_0(J_v^{-1} \rho_0) - \left( \text{adj}(DX_v) \frac{\pi_1(J_v^{-1} \rho_0)}{\rho_0} - \frac{\pi_1(\rho_0)}{\rho_0} \text{Id} \right) \psi
\]

\[
\quad \quad \quad \quad + \text{adj}(DX_v) \pi_1(J_v^{-1} \rho_0) \frac{|\psi|^2}{2}.
\]

Let us notice that

\[
J_v^{-1} \rho_0 - 1 = (J_v^{-1} - 1)(a_0 + 1) + a_0.
\]

Hence, taking advantage of (3.1) and of the results of the appendix,

\[
\|J_v^{-1} \rho_0 - 1\|_{L^\infty(B_{p,1}^{\#})} \lesssim (\|a_0\|_{\tilde{B}_{p,1}^{\#}} + 1).
\]

Next, we have

\[
\frac{\pi_1(J_v^{-1} \rho_0)}{\rho_0} = (\pi_1(J_v^{-1} \rho_0) - \pi_1(1 + 1)(1 - \frac{a_0}{a_0 + 1})).
\]

Therefore, using again (3.1) together with composition estimates yields

\[
\|\frac{\pi_1(J_v^{-1} \rho_0)}{\rho_0}\|_{\mathcal{M}(L^\infty(B_{p,1}^{\#}))} \lesssim (\|a_0\|_{\tilde{B}_{p,1}^{\#}} + 1)^2.
\]

Since

\[
\left( \text{adj}(DX_v) \frac{\pi_1(J_v^{-1} \rho_0)}{\rho_0} - \frac{\pi_1(\rho_0)}{\rho_0} \text{Id} \right) \psi
\]

\[
= (\text{adj}(DX_v) - \text{Id}) \frac{\pi_1(J_v^{-1} \rho_0)}{\rho_0} \psi + \left( \frac{\pi_1(J_v^{-1} \rho_0)}{\rho_0} - \frac{\pi_1(\rho_0)}{\rho_0} \right) \psi \text{Id}
\]

and

\[
\frac{\pi_1(J_v^{-1} \rho_0)}{\rho_0} - \frac{\pi_1(\rho_0)}{\rho_0} = (\pi_1(J_v^{-1} \rho_0) - \pi_1(\rho_0))(1 - \frac{a_0}{a_0 + 1}),
\]
we conclude that
\[ \| (\text{adj}(DX_v) \frac{\pi_1(J_v^{-1}\rho_0)}{\rho_0} - \frac{\pi_1(\rho_0)}{\rho_0}) \psi \text{Id} \|_{L^1_t(B_{p,1})} \lesssim (\|a_0\|_{B_{p,1}} + 1)^2 \|Dv\|_{L^1_t(B_{p,1})} \|\psi\|_{L^1_t(B_{p,1})}. \]

Finally, by Proposition 4.1 and 4.2,
\[ \| \frac{|v|^2}{2} \pi_1(J_v^{-1}\rho_0) \|_{L^1_t(B_{p,1})} \lesssim (1 + \|a_0\|_{B_{p,1}}) \|v\|^2_{L^2_t(B_{p,1})}. \]

Therefore, using the hypothesis that \( \pi_0(1) = 0 \), we have
\[ \| I_4(v, \psi) \|_{L^1_t(\mathcal{H}_{p,1})} \lesssim T \|\pi_0(J_v^{-1}\rho_0)\|_{L^\infty_t(\mathcal{H}_{p,1})} + \| (\text{adj}(DX_v) \frac{\pi_1(J_v^{-1}\rho_0)}{\rho_0} - \frac{\pi_1(\rho_0)}{\rho_0} \text{Id}) \psi \|_{L^1_t(B_{p,1})} + \| \frac{|v|^2}{2} \pi_1(J_v^{-1}\rho_0) \|_{L^1_t(B_{p,1})} \lesssim (\|a_0\|_{B_{p,1}} + 1)(T + (\|a_0\|_{B_{p,1}} + 1)\|\psi\|_{L^1_t(B_{p,1})} \|Dv\|_{L^1_t(B_{p,1})} + \|v\|^2_{L^2_t(B_{p,1})}). \]

**Estimate of \( I_5 \)**: We can write the term \( I_5 \) as
\[ I_5(v, \psi) = (k(J_v^{-1}\rho_0) \text{adj}(DX_v) A_v - k(\rho_0) \text{Id}) \nabla(\frac{\psi}{\rho_0}) 
= ((k(J_v^{-1}\rho_0) - k(\rho_0))(\text{Id} + (\text{adj}(DX_v) A_v - \text{Id})) + k(\rho_0)(\text{adj}(DX_v) A_v - \text{Id})) \nabla(\frac{\psi}{\rho_0}). \]

Note that
\[ \text{adj}(DX_v) A_v - \text{Id} = (\text{adj}(DX_v) - \text{Id})(A_v - \text{Id}) + (\text{adj}(DX_v) - \text{Id}) + (A_v - \text{Id}), \]
and hence, we have according to Proposition 4.5
\[ \| \text{adj}(DX_v) A_v - \text{Id} \|_{L^\infty_t(\mathcal{H}_{p,1})} \lesssim \| \text{adj}(DX_v) - \text{Id} \|_{L^\infty_t(\mathcal{H}_{p,1})} \| A_v - \text{Id} \|_{L^\infty_t(B_{p,1})} + \| \text{adj}(DX_v) - \text{Id} \|_{L^\infty_t(B_{p,1})} \| A_v - \text{Id} \|_{L^\infty_t(B_{p,1})} \lesssim \|Dv\|^2_{L^1_t(B_{p,1})} + 2\|Dv\|_{L^1_t(B_{p,1})} \| A_v - \text{Id} \|_{L^\infty_t(B_{p,1})}. \]

Next, we have
\[ k(J_v^{-1}\rho_0) - k(\rho_0) = \int_0^1 k'((J_v^{-1} - 1)a_0\tau + a_0 + 1)d\tau \times (J_v^{-1} - 1)a_0. \]

Hence thanks to Propositions 4.1 and 4.2, and to (3.1), we have
\[ \| k(J_v^{-1}\rho_0) - k(\rho_0) \|_{L^\infty_t(B_{p,1})} \lesssim (\|J_v^{-1} - 1\|a_0 + a_0\|L^\infty_t(B_{p,1}) + 1)(\|J_v^{-1} - 1\|a_0\|L^\infty_t(B_{p,1}) \lesssim (\|a_0\|_{B_{p,1}} + 1)\|a_0\|_{B_{p,1}} \|Dv\|_{L^1_t(B_{p,1})}. \]

Note also that
\[ \| k(\rho_0) \|_{M(B_{p,1})} \lesssim \|a_0\|_{B_{p,1}} + 1. \]
Therefore,
\[ \|I_5(v, \psi)\|_{L^1_t(B_{p,1}^{\frac{n}{p}-1})} \lesssim \|k(J^{-1} \rho_0) - k(\rho_0)\|_{L^1_t(B_{p,1}^{\frac{n}{p}-1})} + \|\text{adj}(DX_v) A_v \nabla (\frac{\psi}{\rho_0})\|_{L^1_t(B_{p,1}^{\frac{n}{p}-1})} \]
\[ + \|k(J^{-1} \rho_0) - k(\rho_0)\|_{L^\infty_t(B_{p,1}^{\frac{n}{p}})} \|\text{adj}(DX_v) A_v\|_{L^\infty_t(B_{p,1}^{\frac{n}{p}})} \|\nabla (\frac{\psi}{\rho_0})\|_{L^1_t(B_{p,1}^{\frac{n}{p}-1})} \]
\[ + \|k(J^{-1} \rho_0) - k(\rho_0)\|_{L^\infty_t(\mathcal{M}(B_{p,1}^{\frac{n}{p}-1}))} \|\nabla (\frac{\psi}{\rho_0})\|_{L^1_t(B_{p,1}^{\frac{n}{p}-1})} \]
\[ \lesssim (1 + \|a_0\|_{B_{p,1}^{\frac{n}{p}}})^3 \|Dv\|_{L^1_t(B_{p,1}^{\frac{n}{p}+1})} \|\psi\|_{L^1_t(B_{p,1}^{\frac{n}{p}+1})}. \]

**Estimate of \(I_6\):** Owing to (3.1) and to Proposition 4.5, we have
\[ \|k(J^{-1} \rho_0)\|_{L^\infty_t(\mathcal{M}(B_{p,1}^{\frac{n}{p}-1}))} \lesssim \|a_0\|_{B_{p,1}^{\frac{n}{p}}} + 1. \]

Therefore,
\[ \|I_6(v, \psi)\|_{L^1_t(B_{p,1}^{\frac{n}{p}-1})} \lesssim \|k(J^{-1} \rho_0)\|_{L^\infty_t(\mathcal{M}(B_{p,1}^{\frac{n}{p}-1}))} \|\nabla (\frac{\psi}{\rho_0})\|_{L^1_t(B_{p,1}^{\frac{n}{p}-1})} \]
\[ \lesssim (1 + \|a_0\|_{B_{p,1}^{\frac{n}{p}}}) \|\nabla (\frac{\psi}{\rho_0})\|_{L^1_t(B_{p,1}^{\frac{n}{p}-1})}. \]

**Estimate of \(I_7\):** Recall that \(P(J^{-1} \rho_0, K) = \pi_0(J^{-1} \rho_0) + (\frac{1}{\rho_0} \psi - \frac{|v|^2}{2}) \pi_1(J^{-1} \rho_0)\). Hence
\[ I_7(v, \psi) = \text{adj}(DX_v) \left( v\pi_0(J^{-1} \rho_0) + v(\frac{1}{\rho_0} \psi - \frac{|v|^2}{2}) \pi_1(J^{-1} \rho_0) \right). \]

We already proved that if (3.1) is satisfied and \(1 \leq p < 2n\) then
\[ \|\text{adj}(DX_v)\|_{L^\infty_t(\mathcal{M}(B_{p,1}^{\frac{n}{p}-1}))} \lesssim \|Dv\|_{L^1_t(B_{p,1}^{\frac{n}{p}})} + 1, \]
\[ \|\rho_0^{-1}\|_{L^\infty_t(\mathcal{M}(B_{p,1}^{\frac{n}{p}-1}))} \lesssim \|a_0\|_{B_{p,1}^{\frac{n}{p}}} + 1, \]
and
\[ \|\pi_1(J^{-1} \rho_0)\|_{L^\infty_t(\mathcal{M}(B_{p,1}^{\frac{n}{p}-1}))} \lesssim \|a_0\|_{B_{p,1}^{\frac{n}{p}}} + 1. \]

Therefore, by Proposition 4.1, we have
\[ \|I_7(v, \psi)\|_{L^1_t(B_{p,1}^{\frac{n}{p}-1})} \lesssim \|v\pi_0(J^{-1} \rho_0)\|_{L^1_t(B_{p,1}^{\frac{n}{p}-1})} + (\|a_0\|_{B_{p,1}^{\frac{n}{p}}} + 1) \|v\|_{L^1_t(B_{p,1}^{\frac{n}{p}})} \]
\[ + (\|a_0\|_{B_{p,1}^{\frac{n}{p}}} + 1) \|v\|_{L^1_t(B_{p,1}^{\frac{n}{p}-1})} \]
\[ \lesssim \|\pi_0(J^{-1} \rho_0)\|_{L^\infty_t(B_{p,1}^{\frac{n}{p}})} \|v\|_{L^\infty_t(B_{p,1}^{\frac{n}{p}-1})} + (\|a_0\|_{B_{p,1}^{\frac{n}{p}}} + 1) \|v\|_{L^2_t(B_{p,1}^{\frac{n}{p}})} \|\psi\|_{L^2_t(B_{p,1}^{\frac{n}{p}-1})} \]
\[ + (\|a_0\|_{B_{p,1}^{\frac{n}{p}}} + 1) \|v\|_{L^2_t(B_{p,1}^{\frac{n}{p}-1})} \|v\|_{L^2_t(B_{p,1}^{\frac{n}{p}-1})} \]
\[ \lesssim (\|a_0\|_{B_{p,1}^{\frac{n}{p}}} + 1) \|v\|_{L^\infty_t(B_{p,1}^{\frac{n}{p}-1})} + \|v\|_{L^2_t(B_{p,1}^{\frac{n}{p}})} \|\psi\|_{L^2_t(B_{p,1}^{\frac{n}{p}-1})} \]
\[ + \|v\|_{L^2_t(B_{p,1}^{\frac{n}{p}-1})} \|v\|_{L^2_t(B_{p,1}^{\frac{n}{p}-1})}. \]

**Estimate of \(I_8\):** Recall that
\[ I_8(v, w) = \text{adj}(DX_v)(\lambda(J^{-1} \rho_0)^t A_v : \nabla w \text{ Id} + \mu(J^{-1} \rho_0)(Dw \cdot A_v + t A_v \cdot \nabla w)) \cdot w. \]
From the previous computations, we know that for any smooth enough function $z$,
\[
\|\text{adj}(DX_v)z(J_v^{-1} \rho_0)A_x\|_{L^\infty_T(\mathcal{M}(\mathbb{B}_{p,1}^{\frac{n}{p}}))} \lesssim 1 + \|a_0\|_{\mathbb{B}_{p,1}}.
\]
Therefore, we have by Proposition 4.1
\[
\|I_8(v, \psi)\|_{L^1_T(\mathbb{B}_{p,1}^{\frac{n}{p}})} \lesssim (1 + \|a_0\|_{\mathbb{B}_{p,1}})\|v\|_{L^\infty_T(\mathbb{B}_{p,1}^{\frac{n}{p}})}^2.
\]
In summary, we have that
\[
\|I_1(v, v) + I_2(v, v) + I_3(v, v) + I_4(v, \psi)\|_{L^1_T(\mathbb{B}_{p,1}^{\frac{n}{p}})} \lesssim (\|a_0\|_{\mathbb{B}_{p,1}} + 1)^2
\]
\[
\times (T + \|Dv\|_{L^1_T(\mathbb{B}_{p,1}^{\frac{n}{p}})}^2 + \|a_0\|_{\mathbb{B}_{p,1}} \|v\|_{L^2_T(\mathbb{B}_{p,1}^{\frac{n}{p}})}^2 + \|\psi\|_{L^1_T(\mathbb{B}_{p,1}^{\frac{n}{p}})} \|Dv\|_{L^1_T(\mathbb{B}_{p,1}^{\frac{n}{p}})}).
\]
and that
\[
\|I_5(v, \psi) + I_6(v, \psi) + I_7(v, \psi) + I_8(v, \psi)\|_{L^1_T(\mathbb{B}_{p,1}^{\frac{n}{p}})} \lesssim (\|a_0\|_{\mathbb{B}_{p,1}} + 1)^3 (T \|v\|_{L^\infty_T(\mathbb{B}_{p,1}^{\frac{n}{p}})} + \|Dv\|_{L^1_T(\mathbb{B}_{p,1}^{\frac{n}{p}})} \|\psi\|_{L^1_T(\mathbb{B}_{p,1}^{\frac{n}{p}})}
\]
\[
+ (\|v\|_{L^\infty_T(\mathbb{B}_{p,1}^{\frac{n}{p}})} + 1) \|v\|_{L^2_T(\mathbb{B}_{p,1}^{\frac{n}{p}})}^2 + \|\psi\|_{L^2_T(\mathbb{B}_{p,1}^{\frac{n}{p}})} \|\psi\|_{L^2_T(\mathbb{B}_{p,1}^{\frac{n}{p}})}).
\]
Plugging inequalities (3.6) and (3.7) into (3.2), we obtain
\[
\|(\tilde{u}, \tilde{K})\|_{E_p(T)} \leq Ce^{C_{p,m}T}(\|a_0\|_{\mathbb{B}_{p,1}} + 1)^3 \left(\|Du_L\|_{L^1_T(\mathbb{B}_{p,1}^{\frac{n}{p}})} + \|K_L\|_{L^1_T(\mathbb{B}_{p,1}^{\frac{n}{p}})}
\]
\[
+ T(\|u_0\|_{\mathbb{B}_{p,1}^{\frac{n}{p}}}) + \|Du_L\|_{L^2_T(\mathbb{B}_{p,1}^{\frac{n}{p}})}^2 + R^2
\]
\[
+ (\|Du_L\|_{L^1_T(\mathbb{B}_{p,1}^{\frac{n}{p}})} + R)\|K_L\|_{L^1_T(\mathbb{B}_{p,1}^{\frac{n}{p}})} + R)
\]
\[
+ (\|u_0\|_{\mathbb{B}_{p,1}^{\frac{n}{p}}}) + \|Du_L\|_{L^2_T(\mathbb{B}_{p,1}^{\frac{n}{p}})}^2 + R^2
\]
\[
+ (\|u_0\|_{\mathbb{B}_{p,1}^{\frac{n}{p}}}) + \|Du_L\|_{L^2_T(\mathbb{B}_{p,1}^{\frac{n}{p}})}^2 + \|K_L\|_{L^2_T(\mathbb{B}_{p,1}^{\frac{n}{p}})} + R)
\]
\[
\leq Ce^{C_{p,m}T}(\|a_0\|_{\mathbb{B}_{p,1}} + 1)^3 (\|u_0\|_{\mathbb{B}_{p,1}^{\frac{n}{p}}}) + R\left(\|Du_L\|_{L^1_T(\mathbb{B}_{p,1}^{\frac{n}{p}})} + \|K_L\|_{L^1_T(\mathbb{B}_{p,1}^{\frac{n}{p}})}
\]
\[
+ T + \|Du_L\|_{L^2_T(\mathbb{B}_{p,1}^{\frac{n}{p}})}^2 + \|K_L\|_{L^2_T(\mathbb{B}_{p,1}^{\frac{n}{p}})}^2 + \|u_0\|_{\mathbb{B}_{p,1}^{\frac{n}{p}}}^2 + \|K_L\|_{L^2_T(\mathbb{B}_{p,1}^{\frac{n}{p}})}^2 + R^2.
\]
We first choose $R$ so that for a small enough constant $\eta$,
\begin{equation}
2C(\|a_0\|_{B_{p,1}^\infty} + 1)^3(\|u_0\|_{B_{p,1}^\infty} + 1)R \leq \eta
\end{equation}
and take $T$ so that
\begin{equation}
C_{\rho_0,m}T \leq \log 2, \quad T \leq R^2, \quad \|Du_L\|_{L_1^1(B_{p,1})^d} \leq R,
\end{equation}
\begin{equation}
\|K_L\|_{L_1^1(B_{p,1})} \leq R, \quad \|u_L\|_{L_1^2(B_{p,1})^d} \leq R, \quad \|K_L\|_{L_1^2(B_{p,1})^d} \leq R,
\end{equation}
then we may conclude that $\Phi$ is a self-map on the ball $\bar{B}_{E_p(T)}((u_L, K_L), R)$.

**Second step : Contraction estimate.**

We set $u_j := \Phi_1(v_j, \psi_j), \ K_j := \Phi_2(v_j, \psi_j)$ for $j = 1, 2$, and $\delta u := u_2 - u_1$ and $\delta K := K_2 - K_1$. To simplify the notation, we set $X_i := X_{v_i}, \ A_i := A_{v_i}$ and $J_i := J_{v_i}$.

In order to prove that $\Phi$ is contractive, it is just a matter of applying Proposition 2.3 to the system fulfilled by $(\delta u, \delta K)$, namely
\begin{equation}
\begin{cases}
L_{\rho_0}\delta u + \rho_0^{-1}\nabla(\rho_0^{-1} \pi_1(\rho_0)\delta K) \\
\quad = \rho_0^{-1}\text{div}\left(\sum_{j=1}^3(I_j(v_2, v_2) - I_j(v_1, v_1)) + (I_4(v_2, \psi_2) - I_4(v_1, \psi_1))\right),
\end{cases}
\end{equation}
\begin{equation}
H_{\rho_0}\delta K = \text{div}\left(\sum_{j=1}^3(I_j(v_2, v_2) - I_j(v_1, \psi_1)) + (I_8(v_2, v_2) - I_8(v_1, v_1))\right).
\end{equation}

Taking advantage of the computations in [10], we get for $j = 1, 2, 3$,
\begin{equation}
\|I_j(v_2, v_2) - I_j(v_1, v_1)\|_{L_1^1(B_{p,1})} \leq C_{\rho_0}(\|Dv_1, Dv_2\|_{L_1^1(B_{p,1})} \|D\delta v\|_{L_1^1(B_{p,1})}).
\end{equation}

Concerning the pressure term, a straightforward calculation based on Proposition 4.6 ensures that for some constant $C_{\rho_0}$ depending only on $\rho_0, n$ and $p$,
\begin{equation}
\|I_4(v_2, \psi_2) - I_4(v_1, \psi_1)\|_{L_1^1(B_{p,1})} \leq C_{\rho_0}(\|Dv_1, Dv_2\|_{L_1^1(B_{p,1})} \|\delta \psi\|_{E_p(T)}).
\end{equation}

Indeed:
\begin{equation}
I_4(v, \psi) = -\text{adj}(DX_v)\pi_0(J_{v^{-1}}\rho_0) - (\text{adj}(DX_v)\pi_1(J_{v^{-1}}\rho_0) - \pi_1(\rho_0)\psi)\text{Id}
\end{equation}
\begin{equation}
+ \text{adj}(DX_v)\pi_1(J_{v^{-1}}\rho_0)\|\psi\|_2^2.
\end{equation}

Hence
\begin{equation}
I_4(v_2, \psi_2) - I_4(v_1, \psi_1) = -\left(\text{adj}(DX_2)\pi_0(J_{v^{-1}}\rho_0) - \text{adj}(DX_1)\pi_0(J_{v^{-1}}\rho_0)\right)
\end{equation}
\begin{equation}
- \left(\text{adj}(DX_2)\pi_1(J_{v^{-1}}\rho_0)\psi_2 - \text{adj}(DX_1)\pi_1(J_{v^{-1}}\rho_0)\psi_1\right) + \frac{\pi_1(\rho_0)}{\rho_0}\delta \psi \text{Id}
\end{equation}
\begin{equation}
+ \frac{1}{2}\left(\text{adj}(DX_2)\pi_1(J_{v^{-1}}\rho_0) - \text{adj}(DX_1)\pi_1(J_{v^{-1}}\rho_0)\right)\|v_1\|^2 + \frac{1}{2}\text{adj}(DX_2)\pi_1(J_{v^{-1}}\rho_0)\delta \psi \cdot (v_2 + v_1).
\end{equation}
Now we have, for the first term of the above equality,
\[
\|\text{adj}(DX_2)\pi_0(J_2^{-1}\rho_0) - \text{adj}(DX_1)\pi_0(J_1^{-1}\rho_0)\|_{L^1_p(B^2_{p,1})} \\
\lesssim \|\text{adj}(DX_2)(\pi_0(J_2^{-1}\rho_0) - \pi_0(J_1^{-1}\rho_0))\|_{L^1_p(B^2_{p,1})} \\
+ \|\text{adj}(DX_2) - \text{adj}(DX_1)\|_{L^1_p(B^2_{p,1})} \|\pi_0(J_1^{-1}\rho_0)\|_{L^1_p(B^2_{p,1})} \\
\leq C_{\rho_0} T\|(Dv_1, Dv_2)\|_{L^1_p(B^2_{p,1})} \|D\delta v\|_{L^1_p(B^2_{p,1})}.
\]

For the second and third terms, it is easily obtained that
\[
\|
\|\text{adj}(DX_2)\pi_1(J_2^{-1}\rho_0)\psi_2 - \text{adj}(DX_1)\pi_1(J_1^{-1}\rho_0)\psi_1\|_{L^1_p(B^2_{p,1})} \\
\lesssim \|
\|\text{adj}(DX_2)\pi_1(J_2^{-1}\rho_0)\psi_2 - \text{adj}(DX_1)\pi_1(J_2^{-1}\rho_0)\psi_2\|_{L^1_p(B^2_{p,1})} \\
+ \|\text{adj}(DX_1)(\pi_1(J_2^{-1}\rho_0) - \pi_1(J_1^{-1}\rho_0))\psi_2\|_{L^1_p(B^2_{p,1})} \\
+ \|\text{adj}(DX_1)(\pi_1(J_2^{-1}\rho_0) - \pi_1(J_1^{-1}\rho_0)\text{Id})\delta\psi\|_{L^1_p(B^2_{p,1})} \\
\leq C_{\rho_0} \|(Dv_1, Dv_2, \psi_2)\|_{L^1_p(B^2_{p,1})} \|(D\delta v, \delta\psi)\|_{L^1_p(B^2_{p,1})}.
\]

For the last terms, in the same manner as above, we may check that
\[
\|\text{adj}(DX_2)\pi_1(J_2^{-1}\rho_0)\psi_2 - \text{adj}(DX_1)\pi_1(J_1^{-1}\rho_0)\psi_2\|_{L^1_p(B^2_{p,1})} \\
\leq C_{\rho_0} \|D\delta v\|_{L^1_p(B^2_{p,1})} \|v_1\|_{L^2_p(B^2_{p,1})} \\
\]
\[
\|\text{adj}(DX_2)\pi_1(J_2^{-1}\rho_0)\delta v \cdot (v_2 + v_1)\|_{L^1_p(B^2_{p,1})} \\
\leq C_{\rho_0} \|\delta v\|_{L^1_p(B^2_{p,1})} \|v_1 + v_2\|_{L^2_p(B^2_{p,1})}.
\]

Finally, to handle the terms $I_5$, $I_6$, $I_7$ and $I_8$, we use the following decompositions:
\[
I_5(v_2, \psi_2) - I_5(v_1, \psi_1) = (k(J_2^{-1}\rho_0) - k(J_1^{-1}\rho_0))\text{adj}(DX_2)^tA_2\nabla(\frac{\psi_2}{\rho_0}) \\
+ k(J_2^{-1}\rho_0)(\text{adj}(DX_2) - \text{adj}(DX_1))^tA_2\nabla(\frac{\psi_2}{\rho_0}) \\
+ k(J_1^{-1}\rho_0)\text{adj}(DX_1)^t(A_2 - A_1)\nabla(\frac{\psi_2}{\rho_0}) \\
+ (k(J_1^{-1}\rho_0)\text{adj}(DX_1)^tA_1 - \text{Id})\nabla(\frac{\delta\psi}{\rho_0}),
\]
\[
I_6(v_2, \psi_2) - I_6(v_1, \psi_1) = (k(J_2^{-1}\rho_0) - k(J_1^{-1}\rho_0))\text{adj}(DX_2)^tA_2\nabla(\frac{|v_2|^2}{2}) \\
+ k(J_2^{-1}\rho_0)(\text{adj}(DX_2) - \text{adj}(DX_1))^tA_2\nabla(\frac{|v_2|^2}{2}) \\
+ k(J_1^{-1}\rho_0)\text{adj}(DX_1)^t(A_2 - A_1)\nabla(\frac{|v_2|^2}{2}) \\
+ (k(J_1^{-1}\rho_0)\text{adj}(DX_1)^tA_1 - \text{Id})\nabla(\frac{|v_1|^2}{2} - \frac{|v_2|^2}{2}),
\]
\[ I_7(v_2, \psi_2) - I_7(v_1, \psi_1) = (\text{adj}(DX_2)v_2 - \text{adj}(DX_1)v_1)(\pi_0(J_2^{-1} \rho_0) + \frac{\psi_2}{\rho_0} - \frac{|v_2|^2}{2})\pi_1(J_2^{-1} \rho_0) \]
\[ + \text{adj}(DX_1)v_1(\pi_0(J_2^{-1} \rho_0) - \pi_0(J_1^{-1} \rho_0)) \]
\[ + \text{adj}(DX_1)v_1((\frac{\psi_2}{\rho_0} - \frac{|v_2|^2}{2})\pi_1(J_2^{-1} \rho_0) - (\frac{\psi_1}{\rho_0} - \frac{|v_1|^2}{2})\pi_1(J_1^{-1} \rho_0)), \]
\[ I_8(v_2, \psi_2) - I_8(v_1, \psi_1) = \text{adj}(DX_2)(\lambda(J_2^{-1} \rho_0)\text{div}A_2 v_2 \text{Id} + 2\mu(J_2^{-1} \rho_0)D_{A_2}(v_2)) \cdot v_2 \]
\[ - \text{adj}(DX_1)(\lambda(J_1^{-1} \rho_0)\text{div}A_1 v_1 \text{Id} + 2\mu(J_1^{-1} \rho_0)D_{A_1}(v_1)) \cdot v_1. \]

Then using Proposition 4.6, it is easy to see that for \( j = 5, 6, 7, 8 \), we have
\[ \|I_j(v_2, \psi_2) - I_j(v_1, \psi_1)\|_{L^3_T(\tilde{B}_{p,1}^{\frac{2}{n}})} \leq C\rho_0(\|Dv_1, Dv_2, \psi_1, \psi_2\|_{E_p(T)})(\|v\|, \|\psi\|)_{E_p(T)} \quad (3.11) \]

Proposition 2.3 gives us that
\[ \|\hat{\delta}v, \hat{\delta}K\|_{E_p(T)} \leq Ce^{C_p,mT} \left(\left|\sum_{j=1}^{3} (I_j(v_2, \psi_2) - I_j(v_1, \psi_1))\right|_{L^1_T(\tilde{B}_{p,1}^{\frac{2}{n}})} + \left|\sum_{j=5}^{7} (I_j(v_2, \psi_2) - I_j(v_1, \psi_1))\right|_{L^1_T(\tilde{B}_{p,1}^{\frac{2}{n}})}\right) \]
\[ \leq Ce^{C_p,mT} \left|\|v_1, v_2, \psi_1, \psi_2\|_{E_p(T)}\right|\|\hat{\delta}v, \hat{\delta}K\|_{E_p(T)}. \]

Given that \( v_j, \psi_j \in \tilde{B}_{E_p(T)}((u_L, K_L), R) \ (j = 1, 2) \), taking \( \eta, T \) and \( R \) smaller as the case may be, we end up with
\[ \|\hat{\delta}v, \hat{\delta}K\|_{E_p(T)} \leq \frac{1}{2}\|\hat{\delta}v, \hat{\delta}K\|_{E_p(T)}. \quad (3.12) \]

One can thus conclude that \( \Phi \) admits a unique fixed point in \( \tilde{B}_{E_p(T)}((u_L, K_L), R) \).

Third step: Regularity of the density.

Granted with the above velocity field \( u \) in \( E_p(T) \), we set \( \rho := J_u^{-1} \rho_0 \). By construction, the triplet \( (\rho, u, K) \) satisfies (1.8). In order to prove that \( a = \rho - 1 \) is in \( C([0,T]; \tilde{B}_{p,1}^{\frac{2}{n}}) \), we use the fact that
\[ a = (J_u^{-1} - 1)a_0 + a_0. \]

Given Proposition 4.5, and the fact that \( Du \in L^1(0,T; \tilde{B}_{p,1}^{\frac{2}{n}}) \), it is clear that \( (J_u^{-1} - 1) \) belongs to \( C([0,T]; \tilde{B}_{p,1}^{\frac{2}{n}}) \). Hence \( a \) belongs to \( C([0,T]; \tilde{B}_{p,1}^{\frac{2}{n}}) \), too. Because \( \tilde{B}_{p,1}^{\frac{2}{n}} \) is continuously embedded in \( L^\infty \), Condition \( \inf_x \rho > 0 \) is fulfilled on \([0,T]\) (taking smaller \( T \) if needed).

Last step: Uniqueness and continuity of the flow map.

We now consider two triplets \( (\rho_{01}, u_{01}, K_{01}) \) and \( (\rho_{02}, u_{02}, K_{02}) \) of data fulfilling the assumptions of Theorem 1.1 and we denote by \((\rho_1, u_1, K_1)\) and \((\rho_2, u_2, K_2)\) two solutions with \((a_1, u_1, K_1)\) and \((a_2, u_2, K_2)\) in \( E_p(T) \) corresponding to those data. Let \( M_{\rho_0}K \:= \)}
\( \rho_0^{-1} \nabla (\rho_0^{-1} \pi_1 (\rho_0^{-1}) K) \). Making difference of the two equations corresponding to \((\rho_1, u_1)\) and \((\rho_2, u_2)\), we have

\[
L_{\rho_2} u_2 - L_{\rho_1} u_1 = L_{\rho_2} \delta u + (L_{\rho_2} - L_{\rho_1}) u_2
\]

and

\[
K_{\rho_2} K_2 - K_{\rho_1} K_1 = M_{\rho_2} \delta K + (M_{\rho_2} - M_{\rho_1}) K_2.
\]

Setting \( \delta u := u_2 - u_1 \) and \( \delta K := K_2 - K_1 \), we thus get

\[
L_{\rho_1} \delta u + M_{\rho_1} \delta K = (L_{\rho_1} - L_{\rho_2})(u_2) + (M_{\rho_1} - M_{\rho_2}) K_2
\]

\[
+ (\rho_{02})^{-1} \text{div} \left( \sum_{j=1}^{3} (I_j^2(u_2, u_2) - I_j^2(u_1, u_1)) + (I_j^2(u_2, \psi_2) - I_j^2(u_1, \psi_1)) \right)
\]

\[
+ ((\rho_{02})^{-1} - (\rho_{01})^{-1}) \text{div} \left( \sum_{j=1}^{3} I_j^2(u_1, u_1) + I_j^2(u_1, \psi_1) \right)
\]

\[
+ (\rho_{01})^{-1} \text{div} \left( \sum_{j=1}^{3} (I_j^2 - I_j^1)(u_1, u_1) + (I_j^2 - I_j^1)(u_1, \psi_1) \right),
\]

where \( I_j^i \) \((j = 1, ..., 5)\) correspond to the quantities that have been defined previously in (1.15) with density \( \rho_{0i} \) for \( i = 1, 2 \):

\[
I_1^i(v, w) := (\text{adj}(DX_v) - Id)(\mu(J_v^{-1} \rho_{0i})(Dw A_v + \partial A_v \nabla w)
\]

\[
+ \lambda(J_v^{-1} \rho_{0i})(\partial A_v : \nabla w) \text{Id})
\]

\[
I_2^i(v, w) := (\mu(J_v^{-1} \rho_{0i}) - \mu(\rho_{0i}))(Dw \cdot A_v + \partial A_v \cdot \nabla w)
\]

\[
+ (\lambda(J_v^{-1} \rho_{0i}) - \lambda(\rho_{0i}))(\partial A_v : \nabla w) \text{Id})
\]

\[
I_3^i(v, w) := \mu(\rho_{0i})(Dw(A_v - \text{Id}) + \partial A_v - \text{Id}) \nabla w) + \lambda(\rho_{0i})(\partial A_v - \text{Id}) : \nabla w) \text{Id}
\]

\[
I_4^i(v, w) := -\text{adj}(DX_v)(J_v^{-1} \rho_{0i}, \psi),
\]

\[
I_5^i(v, w) := (k(J_v^{-1} \rho_{0i}) \text{adj}(DX_v) \partial A_v - k(\rho_{0i}) \text{Id}) \nabla \left( \frac{\psi}{\rho_{0i}} \right)
\]

\[
I_6^i(v, w) := k(J_v^{-1} \rho_{0i}) \text{adj}(DX_v) \partial A_v \nabla \left( \frac{|v|^2}{2} \right)
\]

\[
I_7^i(v, w) := \text{adj}(DX_v) v D(\rho_{0i}^{-1}) \psi,
\]

and \( I_8^i(v, w) := \text{adj}(DX_v)(\lambda(J_v^{-1} \rho_{0i}) \text{div} A_v w \text{ Id} + 2\mu(J_v^{-1} \rho_{0i}) D_{A_v}(w) \cdot w) \cdot w \).

The proof is carried out by applying Proposition 2.3 to (3.13) and using Proposition 4.6 to estimate each term on the left-hand side of (3.13), exactly as in the second step.

Assuming that \( \delta u_0 \), \( \delta u_0 \) and \( \delta K_0 \) are small enough, a bootstrap argument will provide us with, for small enough \( t \),

\[
\| (\delta u, \delta K) \|_{E_p(t)} \leq C_{\rho_{01}, \rho_{02}} (\| \delta u_0 \|_{B_p^{\frac{n}{p+1}}} + \| \delta u_0 \|_{B_p^{\frac{n}{p-1}}} + \| \delta K_0 \|_{B_p^{\frac{n}{p-2}}}).
\]
Regarding the density, we have
\[ \delta \alpha = J^{-1}_{u_1} \delta u_0 + (J^{-1}_{u_2} - J^{-1}_{u_1}) a_{02}. \]
Hence for all \( t \in [0, T] \),
\[ \| \delta \alpha (t) \|_{B_{p,1}^m} \leq C (1 + \| D \delta u \|_{L^1_t (B_{p,1}^m)} ) (\| \delta u_0 \|_{B_{p,1}^m} + \| \delta u_0 \|_{B_{p,1}^m}). \]
Therefore, we may conclude to both uniqueness and continuity of the data-solution map on a small enough time interval. Iterating the proof will yield uniqueness on the whole time interval \([0, T]\).

3.2. Proof of Theorem 1.2. Finally, we consider the possibility of reverting back the solution obtained in the Lagrangian coordinates to that in the Eulerian coordinates. Theorem 1.2 is a corollary of the following proposition which states that, under the restriction \( 1 < p < n \) and \( n \geq 3 \), Systems (1.1) and (1.8) (and consequently (1.5) and (1.6) as well) are equivalent in our functional framework.

**Proposition 3.1.** Let \( 1 < p < n \) with \( n \geq 3 \), and \((\rho, u, \theta)\) be a solution to (1.1) with \( \rho - 1 \in C([0, T]; B_{p,1}^n) \), \((u, \theta) \in E_p(T)\) and, for small enough \( c \),
\[ \int_0^T \| \nabla u \|_{B_{p,1}^m} \leq c. \tag{3.14} \]
Let \( X \) be the flow of \( u \) defined in (1.3) and \( E \), the total energy by unit volume defined in (1.4). Then after defining the triplet \((\bar{\rho}, \bar{\mu}, \bar{E}) := (\rho \circ X, u \circ X, E \circ X)\) and \( \bar{K} \) as in (1.7), the triplet \((\bar{\rho}, \bar{\mu}, \bar{K})\) belongs to the same functional space as \((\rho, u, \theta)\) and satisfies (1.8).

Conversely, if \( \bar{\rho} - 1 \in C([0, T]; B_{p,1}^n) \), \((\bar{\mu}, \bar{K}) \in E_p(T)\), and \((\bar{\rho}, \bar{\mu}, \bar{K})\) satisfies (1.8) and, for a small enough constant \( c \),
\[ \int_0^T \| \nabla \bar{\mu} \|_{B_{p,1}^m} \leq c \tag{3.15} \]
then the map \( X \) defined in (1.10) is a \( C^1 \) (and in fact locally \( B_{p,1}^{n+1} \)) diffeomorphism over \( \mathbb{R}^n \) and after having defined \( \bar{E} := J^{-1} \bar{K} \), \((\rho, u, E) := (\bar{\rho} \circ X^{-1}, \bar{\mu} \circ X^{-1}, \bar{E} \circ X^{-1})\) and \( \theta := \frac{\bar{E}}{\bar{\rho}} - \frac{|\bar{u}|^2}{\bar{\rho}} \), the triplet \((\rho, u, \theta)\) has the same regularity as \((\bar{\rho}, \bar{\mu}, \bar{K})\) and satisfies (1.1).

**Proof.** For a solution \((\rho, u, \theta)\) to (1.1) with the above properties, the definition of \( X \) in (1.3) implies that \( DX - \text{Id} \in C([0, T]; B_{p,1}^n) \). In addition, having defined \( E \) as in (1.4), Proposition 4.7 ensures that \((\bar{\rho}, \bar{\mu}, \bar{E})\) lies in the same functional space as \((\rho, u, \theta)\), and Proposition 4.5 ensures that \( A - \text{Id}, \text{adj}(DX) - \text{Id} \) and \( J^{\pm 1} - 1 \) are in \( C([0, T]; \bar{B}_{p,1}^n) \). After performing the change of variable, let us define \( \bar{K} := JE \); then it is clear by \( J^{-1} - 1 \in C([0, T]; \bar{B}_{p,1}^n) \) and the product laws that \( K \) also lies in the same space as \( E \) provided that \( 1 < p < n \) and \( n \geq 3 \) (see Proposition 4.1). So eventually, under this latter condition, \((\bar{\rho}, \bar{\mu}, \bar{K})\) fulfills (1.8) and belongs to \((1 + C([0, T]; \bar{B}_{p,1}^n)) \times E_p(T)\).

Conversely, let us assume that we are given some solution \((\bar{\rho}, \bar{\mu}, \bar{K})\) to (1.8) with \( \bar{\rho} \in C([0, T]; \bar{B}_{p,1}^n) \) and \((\bar{\mu}, \bar{K}) \in E_p(T)\).

Then one may prove that, under Condition (3.15), the “flow” \( X(t, \cdot) \) of \((\bar{\mu}, \bar{K})\) defined by (1.10) is a \( C^1 \)-diffeomorphism over \( \mathbb{R}^n \) (see [10] and [11]), and satisfies \( DX - \text{Id} \in \)
We follow the above steps from backward: first we define \( E := J^{-1}K \), then clearly \( E \) satisfies (1.6) under \( 1 < p < n \) and \( n \geq 3 \). Now, one may perform the change of variables
\[
(\rho, u, E) := (\tilde{\rho} \circ X^{-1}, \tilde{\pi} \circ X^{-1}, \tilde{E} \circ X^{-1})
\]
and set \( \theta := \frac{\tilde{E} \circ X^{-1}}{\tilde{\rho} \circ X^{-1}} \) to confirm that \((\rho, u, \theta)\) is indeed a solution to (1.5). Proposition 4.7 ensures that \((\rho, u, \theta)\) has the desired regularity. \(\square\)

**Proof of Theorem 1.2.** We consider data \((\rho_0, u_0, \theta_0)\) with \(\rho_0\) bounded away from 0, \((\rho_0 - 1, 0, 0) \in \dot{B}^{\frac{2}{p}-2}_{p,1}, u_0 \in \dot{B}^{\frac{n}{p}-1}_{p,1} \) and \(\theta_0 \in \dot{B}^{\frac{n}{p}-2}_{p,1} \). Defining \(K_0\) according to (1.9) and observing that \(n \geq 3\) and \(1 \leq p < n\) implies \(K_0 \in \dot{B}^{\frac{2}{p}-2}_{p,1}\). Then Theorem 1.1 provides a local solution \((\bar{\rho}, \bar{\pi}, \bar{K})\) to System (1.8) with \(\bar{\rho} \in \mathcal{C}([0,T]; \dot{B}^{\frac{n}{p}}_{p,1})\) and \((\bar{\pi}, \bar{K}) \in E_p(T)\). If \(T\) is small enough then (3.14) is satisfied so Proposition 3.1 ensures that
\[
(\rho, u, \theta) := (\bar{\rho} \circ X^{-1}, \bar{\pi} \circ X^{-1}, \frac{\bar{E} \circ X^{-1}}{\bar{\rho} \circ X^{-1}} - \frac{[\bar{\pi} \circ X^{-1}]^2}{2})
\]
is a solution of (1.1) in the desired functional space.

To prove uniqueness, we consider two Eulerian solutions \((\rho_1, u_1, \theta_1)\) and \((\rho_2, u_2, \theta_2)\) corresponding to the same data \((\rho_0, u_0, \theta_0)\). We then rewrite the system in the form of (1.5) as before and perform the Lagrangian change of variables (pertaining to the flow of \(u_1\) and \(u_2\) respectively). The obtained triplets \((\bar{\rho}_1, \bar{u}_1, \bar{\theta}_1)\) and \((\bar{\rho}_2, \bar{u}_2, \bar{\theta}_2)\) (where \(\bar{K}_j := J_\delta \bar{E}_j\) with \(E_j\) as before) are in \((1 + \mathcal{C}([0,T]; \dot{B}^{\frac{n}{p}}_{p,1})) \times E_p(T)\), and both satisfy (1.8) with the same \((\bar{\rho}_0, \bar{u}_0, K_0)\) (with \(K_0\) defined as in (1.9)). Hence they coincide, as a consequence of the uniqueness part of Theorem 1.1. \(\square\)

We conclude this section with a short discussion about the cases \(n = 2\), or \(n \geq 3\) and \(n \leq p < 2n\).

As already pointed out in the introduction, owing to the product laws (see Proposition 4.1), it is no longer possible to deduce that \(K_0\) (or \(E_0\)) is in \(\dot{B}^{\frac{2}{p}-2}_{p,1}\) from the hypothesis that \(a_0 \in \dot{B}^{\frac{2}{p}}_{p,1}\), \(u_0 \in \dot{B}^{\frac{n}{p}-1}_{p,1}\) and \(\theta_0 \in \dot{B}^{\frac{n}{p}-2}_{p,1}\). Therefore it is suitable to look at the equivalence between the Lagrangian Navier-Stokes equations (1.8), and the Eulerian Navier-Stokes equations written in terms of \((\rho, u, E)\) (namely (1.5)), rather than in terms of \((\rho, u, \theta)\).

In this new setting, one can mimic the proofs of Proposition 3.1 and Theorem 1.2. The only difference concerns the regularity issue when making the change from \(K\) to \(\bar{E} := J^{-1}K\) (or conversely). Indeed, from \(J^{\pm 1} - 1 \in \mathcal{C}([0,T]; \dot{B}^{\frac{2}{p}-2}_{p,1})\) and \(E \in \mathcal{C}([0,T]; \dot{B}^{\frac{n}{p}-2}_{p,1})\), it is no longer possible to deduce that \(E\) is in \(\mathcal{C}([0,T]; \dot{B}^{\frac{n}{p}-2}_{p,1})\), because Condition \(n/p - 2 > -\min(n/p,n/p')\) in Proposition 4.1 is not fulfilled. At the same time, arguing by interpolation, we see that the solution \((\bar{\pi}, \bar{\rho}, \bar{K})\) constructed in Theorem 1.1 is such that
\[
\bar{K} \in L^{\frac{1}{1-\delta}}(\dot{B}^{\frac{n}{p}-2\delta}_{p,1}) \quad \text{for all} \quad \delta \in [0,1].
\]
As \(J^{\pm 1} - 1 \in \mathcal{C}([0,T]; \dot{B}^{\frac{2}{p}-2}_{p,1})\), we conclude that \(E \in L^{\frac{1}{1-\delta}}(\dot{B}^{\frac{n}{p}-2\delta}_{p,1})\) whenever \(n \geq 2\), \(\delta < 1\) and \(p < n\min(2,1/\delta)\). Then it is easy to conclude to the following corollary:

**Corollary 3.2.** *Under the assumptions of Theorem 1.1 with \(n = 2\) and \(1 < p < 4\), or \(n \geq 3\) and \(n \leq p < 2n\), System (1.5) has a unique local solution \((\rho, u, E)\) with*
\[ \rho - 1 \in C([0, T]; \dot{B}_{p,1}^{-1}), \quad u \in C([0, T]; \dot{B}_{p,1}^{-1}) \cap L^1(0, T; \dot{B}_{p,1}^{+1}) \quad \text{and} \quad E \in L^{\frac{n}{n+p-2\delta}}(0, T; \dot{B}_{p,1}^{-2\delta}) \]
for all \( \delta \in [0, n/p) \).

4. Appendix

This section is devoted to presenting some technical results that have been used repeatedly in the paper. In the first paragraph, we recall basic nonlinear estimates involving Besov norms. Next, we state estimates for the flow. Finally, we give some details on how (1.8) may be derived from (1.5).

4.1. Estimates for product, composition and commutators. For the proofs of the following propositions, see [1, 10, 11, 17] and the references therein.

Proposition 4.1. Let \( \nu \geq 0 \) and \(-\min\left(\frac{n}{p}, \frac{n}{p'}\right) < \sigma \leq \frac{n}{p} - \nu\). The following product law holds:

\[ \|uv\|_{\dot{B}_{p,1}^\sigma} \leq C \|u\|_{\dot{B}_{p,1}^{\frac{n}{p} - \nu}} \|v\|_{\dot{B}_{p,1}^{\sigma + \nu}}. \]

Proposition 4.2. Let \( F : I \to \mathbb{R} \) be a smooth function (with \( I \) an open interval of \( \mathbb{R} \) containing 0) vanishing at 0. Then for any \( s > 0 \), \( 1 \leq p \leq \infty \) and interval \( J \) compactly supported in \( I \) there exists a constant \( C \) such that

\[ \|F(a)\|_{\dot{B}_{p,1}^s} \leq C \|a\|_{\dot{B}_{p,1}^s} \]

for any \( a \in \dot{B}_{p,1}^s \) with values in \( J \). In addition, if \( a_1 \) and \( a_2 \) are two such functions and \( s = \frac{n}{p} \) then we have

\[ \|F(a_2) - F(a_1)\|_{\dot{B}_{p,1}^s} \leq C \|a_2 - a_1\|_{\dot{B}_{p,1}^s}. \]

Proposition 4.3. Assume that \( \sigma, \nu \) and \( p \) are such that

\[ 1 \leq p \leq \infty, \quad 0 \leq \nu \leq \frac{n}{p} \quad \text{and} \quad -\min\left(\frac{n}{p}, \frac{n}{p'}\right) - 1 < \sigma \leq \frac{n}{p} - \nu. \quad (4.1) \]

Then there exists a constant \( C \) depending only on \( \sigma, \nu, p \) and \( n \) such that for all \( k \in \{1, ..., n\} \), we have for some sequence \( (c_j)_{j \in \mathbb{Z}} \) with \( \|c_j\|_{\ell^1} = 1 \)

\[ \|\partial_k [a, \phi_j] w\|_{L^p} \leq C c_j 2^{-j\sigma} \|\nabla a\|_{\dot{B}_{p,1}^{\frac{n}{p} - \nu}} \|v\|_{\dot{B}_{p,1}^{\sigma + \nu}} \quad \text{for all } j \in \mathbb{Z}. \]

Proposition 4.4. Let \( A(D) \) be a Fourier multiplier of degree 0. Then the following estimate holds.

\[ \|[A(D), q] w\|_{\dot{B}_{p,1}^{\sigma + 1}} \leq C \|\nabla q\|_{\dot{B}_{p,1}^{\frac{n}{p} - \nu}} \|w\|_{\dot{B}_{p,1}^{\sigma + \nu}}, \]

whenever

\[ 1 \leq p \leq \infty, \quad \nu \geq 0 \quad \text{and} \quad -\min\left(\frac{n}{p}, \frac{n}{p'}\right) - 1 < \sigma \leq \frac{n}{p} - \nu. \quad (4.2) \]
4.2. Estimates of flow. We here recall flow estimates that have been proved in [10,11].

**Proposition 4.5.** Let $1 \leq p < \infty$ and $v \in E_p(T)$. There exists a positive constant $\tilde{c}$ (independent of $T$) such that if

$$\left\| Dv \right\|_{L^p(B_{p,1}^s)} \lesssim \left\| D\bar{\pi} \right\|_{L^2(\tilde{B}_{p,1}^s)}^{1},$$

then for all $t \in [0,T]$, we have

$$\left\| \text{Id} - \text{adj}(DX_v(t)) \right\|_{B_{p,1}^s} \lesssim \left\| D\bar{\pi} \right\|_{L^1(\tilde{B}_{p,1}^s)}^{1},$$

$$\left\| \text{Id} - A_v(t) \right\|_{B_{p,1}^s} \lesssim \left\| D\bar{\pi} \right\|_{L^1(\tilde{B}_{p,1}^s)}^{1},$$

$$\left\| J_v^{\pm 1}(t) - 1 \right\|_{B_{p,1}^s} \lesssim \left\| D\bar{\pi} \right\|_{L^1(\tilde{B}_{p,1}^s)}^{1}.$$ 

Furthermore, if $\bar{w}$ is a vector field such that $\bar{w} \in L^1(0,T; \tilde{B}_{p,1}^s)$, then

$$\left\| \text{adj}(DX_v)D_{A_v}(\bar{w}) - D(\bar{w}) \right\|_{B_{p,1}^s} \lesssim \left\| D\bar{\pi} \right\|_{L^2(\tilde{B}_{p,1}^s)} \left\| D\bar{\pi} \right\|_{L^2(\tilde{B}_{p,1}^s)},$$

$$\left\| \text{adj}(DX_v)\text{div}_{A_v}(\bar{w}) - \text{div} \bar{w} \right\|_{B_{p,1}^s} \lesssim \left\| D\bar{\pi} \right\|_{L^2(\tilde{B}_{p,1}^s)} \left\| D\bar{\pi} \right\|_{L^2(\tilde{B}_{p,1}^s)}.$$ 

**Proposition 4.6.** Let $1 \leq p < \infty$ and $\bar{v}_1$ and $\bar{v}_2 \in E_p(T)$ satisfying

$$\left\| Dv \right\|_{L^p(B_{p,1}^s)} \lesssim \left\| D\bar{\pi} \right\|_{L^2(\tilde{B}_{p,1}^s)}^{1}$$

and $\delta v := \bar{v}_2 - \bar{v}_1$. Then for all $t \in [0,T]$, we have

$$\left\| A_{v_2}(t) - A_{v_1}(t) \right\|_{B_{p,1}^s} \lesssim \left\| D\delta v \right\|_{L^1(\tilde{B}_{p,1}^s)}^{1},$$

$$\left\| \text{adj}(DX_{v_2}(t)) - \text{adj}(DX_{v_1}(t)) \right\|_{B_{p,1}^s} \lesssim \left\| D\delta v \right\|_{L^1(\tilde{B}_{p,1}^s)}^{1},$$

$$\left\| J_{v_2}^{\pm 1}(t) - J_{v_1}^{\pm 1}(t) \right\|_{B_{p,1}^s} \lesssim \left\| D\delta v \right\|_{L^1(\tilde{B}_{p,1}^s)}^{1}.$$ 

4.3. Lagrangean coordinates. Let $X$ be a $C^1$-diffeomorphism over $\mathbb{R}^n$. For a vector-valued function $H : \mathbb{R}^n \to \mathbb{R}^m$, denote $\bar{H}(y) := H(x)$ with $x = X(y)$. The chain rule states that

$$D_y \bar{H}(y) = D_x H(X(y)) \cdot D_y X(y) \quad \text{and} \quad \nabla y \bar{H}(y) = \nabla y X(y) \cdot \nabla x H(X(y)).$$

Hence, setting $A(y) = (D_y X(y))^{-1} = D_x X^{-1}(X(y))$, we have

$$D_x H(X(y)) = D_y \bar{H}(y) \cdot A(y) \quad \text{and} \quad \nabla x H(X(y)) = A(y) \cdot \nabla y \bar{H}(y).$$

**Proposition 4.7 ([10][11]).** Let $X$ be a globally bi-Lipschitz diffeomorphism of $\mathbb{R}^n$ and $(s,p)$ with $1 \leq p < \infty$ and $-\frac{n}{p} < s \leq \frac{n}{p}$. Then a $a \circ X$ is a self-map over $B_{p,1}^s$ in the following cases:

1. $s \in (0,1)$,
2. $s \in (-1,0]$ and $J_{X^{-1}}$ is in the multiplier space $\mathcal{M}(\tilde{B}_{p,1}^s)$,
3. $s \geq 1$ and $(DX - \text{Id}) \in B_{p,1}^s$. 
Proposition 4.8 ([10][11]). Let $K$ be a $C^1$-scalar function over $\mathbb{R}^n$ and $H$ be a $C^1$-vector field. If $X$ is a $C^1$-diffeomorphism such that $J := \det(D_yX) > 0$, then
\[
\nabla_x K = J^{-1} \div_y (\text{adj}(D_yX)K), \\
\div_x H = J^{-1} \div_y (\text{adj}(D_yX)H),
\]
where $\text{adj}(D_yX)$ is the adjugate matrix of $D_yX$.

From the above proposition, we infer the following relations:
\[
\Delta_x u = J^{-1} \div_y (\text{adj}(D_yX)\nabla_x u) \\
\nabla_x \div_x u = J^{-1} \div_y (\text{adj}(D_yX)\nabla_x u) \\
\nabla_x P = J^{-1} \div_y (\text{adj}(D_yX)P).
\]

Lemma 4.9. Let $z : [0,T] \times \mathbb{R}^n \to \mathbb{R}^m$ and $X : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ be differentiable functions with, in addition, $X(t) : \mathbb{R}^n \to \mathbb{R}^n$ being a $C^1$ diffeomorphism for all $t \in \mathbb{R}$. Then the following relation holds:
\[
\partial_t(Jz) = J(\partial_t z + \div_x(u)).
\]

Proof. The proof is based on the following Jacobi formula:
\[
\frac{d}{dt} \det A = \det A \text{tr} \left( A^{-1} \frac{dA}{dt} \right)
\]
that holds true whenever $A : [0,T] \to M_n(\mathbb{R})$ is differentiable and $A(t)$ is invertible for all $t \in [0,T]$.

Now, applying Jacobi formula to $A(t) = D_yX(y,t)$, and using Leibniz rule, we discover that
\[
\partial_t(Jz) = (\partial_t J)z + J\partial_t z \\
= J\text{tr} \left( (D_yX)^{-1} \frac{dD_yX}{dt} \right) z + J\partial_t z.
\]

Since
\[
\frac{dD_yX}{dt} = D_y \frac{dX}{dt} = D_yu = \nabla_x u \cdot D_yX,
\]
we thus have
\[
\partial_t(Jz) = J \nabla_x u + J(\partial_t z + D_x z),
\]
whence the desired equality. \qed

Applying the above lemma to $z = \rho$, $z = \rho u$ or $z = E$, we thus get
\[
(\partial_t \rho + \div_x (\rho u)) = J^{-1} \partial_t (J\rho), \\
(\partial_t (\rho u) + \div_x (\rho u \otimes u)) = J^{-1} \partial_t (J\rho u), \\
(\partial_t E) + \div_x (uE) = J^{-1} \partial_t (JE).
\]

From those three relations, it is now clear that if $(\rho, u, E)$ satisfies (1.5) then $(\rho, u, E)$ fulfills (1.6).

Remark 4.1. Integrating against test functions, it is possible to considerably weaken the assumptions on $z$. 

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