The Kähler-Ricci flow on Kähler manifolds with 2 traceless bisectional curvature operator

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Dedicated to Professor W. Y. Ding for his 60’s birthday

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1 Introduction

In 1982, in a famous paper [10], R. Hamilton proved that in a 3 dimensional compact manifold, if the initial metric has positive Ricci curvature, then this positivity condition will be preserved under the Ricci flow. He showed that the underlying manifold must be diffeomorphic to the standard $S^3$ or its finite quotient. Following this paper, there are intensive active researches on Ricci flow, and many works are devoted to study when certain convex cones of curvature pinching condition is preserved by the Ricci flow. In [11], R. Hamilton proved that the positive curvature operator is preserved under the Ricci flow in all dimensions. H. Chen [5] further showed that, a weaker notion, that the sum of any two eigenvalues is positive, is preserved under the Ricci flow. In the Kähler setting, it is well known that the positive bisectional curvature is preserved under the Kähler Ricci flow through the work of S. Bando [1] for complex dimension $n = 3$, and
later N. Mok [15] for general dimension. Following the argument of N. Mok, in an unpublished work of Cao-Hamilton, they proved that the orthogonal bisectional curvature is preserved under the Kähler Ricci flow. There are other convex cones of curvature pinching conditions which are preserved, for instance [3] and [13]. A more complete reference on this topic can be found in [12].

Analyzing the evolution equation (2.4) of the Ricci tensor, it is somewhat unfortunate that the parabolic Laplacian of the Ricci tensor involves the full sectional curvature. It is then not surprising that we only know the positivity of Ricci tensor is preserved in real dimension 2 and 3 by the earlier work of R. Hamilton. A counter example to the possible extension of R. Hamilton’s result on Ricci tensor in high dimension seems to be difficult to construct. More recently, Ni Lei [14] constructed first counter example to this in Riemannian setting where the positivity of Ricci curvature is not preserved by the Ricci flow. Very recently, Dan Knopf [9] constructed similar counter example in the Kähler setting. However, both examples are in non-compact manifold. Therefore, it is still an open question whether or not positive Ricci curvature is preserved under the Ricci flow in the case of compact manifold. In particular, in the case of compact Kähler manifold, there might be some hope that some form of lower bound of Ricci curvature will be preserved in [6] where the first named author showed, along with other results\(^1\), that any metric with positive orthogonal bisectional curvature, even a negative lower bound of Ricci curvature, is preserved and improved under the Kähler Ricci flow.

In a compact Kähler manifold \(X\), the bisectional curvature tensor acts as a symmetric bilinear form on the space of \((1,1)\) form (which we will denote as \(\Lambda_{1,1}^1(X)\)). Furthermore, this action (of traceless part of the bisectional curvature) preserves the traceless part of this space (which we will denote as \(\Lambda_{0,1}^{1,1}(X)\)). In a recent paper by Phong-Sturm [19], they observed that, the condition that the sum of any two eigenvalues of the traceless bisectional curvature operator is positive, is preserved under the Kähler Ricci flow in complex dimension 2. Note that this condition is different from the condition used by H. Chen, even though the main idea of proof is very similar. The main theorem they proved in [19] is that, if this curvature condition hold, then positive Ricci curvature will be preserved under the Kähler Ricci flow in complex surface. The proof there is complicated, largely due to the fact that the action of curvature operator in \(\Lambda_{0,1}^{1,1}(X)\) is very complicated.

The 2-positive traceless bisectional curvature is certainly different to the popular notion of positive bisectional curvature. For instance, when this curvature condition holds, the Ricci curvature might not be positive. In [7] [8], the first named author and G. Tian studied the convergence of Kähler Ricci flow in Kähler Einstein manifold where the initial metric has positive bisectional curvature and showed that the Kähler Ricci flow must converges to the Fubni-Study metric exponentially over the flow. The present work can be viewed as a continuation of [7] [8] in the sense that the curvature condition is relaxed in some subtle way. However, this type of curvature condition was indeed studied first by H. Chen in [3]. Following [5] [19], we study

\(^1\)In [9], the first named author proved that, any irreducible Kähler manifold where the positive orthogonal bisectional curvature is preserved under the Kähler Ricci flow, must be biholomorphic to \(\mathbb{CP}^n\).
systematically geometrical properties of this 2-positive traceless bisectional curvature operator on any Kähler manifold. Our first result is:

**Theorem 1.1** Let \( X \) be a compact Kähler manifold of dimension \( n \geq 2 \). Along the Kähler Ricci flow, we have

1. If the initial metric has non-negative traceless bisectional curvature operator, then the evolved metrics also have non-negative traceless bisectional curvature operator. If it is positive at one point initially, then it is positive everywhere for all \( t > 0 \).

2. If the initial metric has 2-non-negative traceless bisectional curvature operator, then the evolved metrics also have 2-non-negative traceless bisectional curvature. If it is positive at one point initially, then it is positive everywhere for all \( t > 0 \).

Under either of these two conditions, the positivity of Ricci tensor is preserved under the Kähler Ricci flow.

The relation of 2-positive traceless bisectional curvature with the notion of positive orthogonal bisectional curvature is much more subtle. They are defined in a completely different manner and the action of bisectional curvature operator on the space of (1,1) forms is very complicated. It is hard to visualize what 2 positive traceless bisectional curvature really is. A somewhat surprising result we prove in this paper is that (c.f. Theorem 1.2 below) any Kähler metric which has 2-positive traceless bisectional curvature, must also have positive orthogonal bisectional curvature. The last part of the preceding theorem follows directly from the application of Hamilton’s maximal principle for tensors to the evolution equation of the Ricci tensor. Comparing to the main theorem in [19], our theorem is for all dimensional and our proof is simpler and more straightforward.

**Theorem 1.2** In a Kähler manifold with 2-non-negative traceless bisectional curvature operator, then the orthogonal bisectional curvature must be non-negative. If the scalar curvature is uniformly bounded, then the bisectional curvature is uniformly bounded. Moreover, if we assume that the traceless bisectional curvature operator is non-negative, then the sum of any two eigenvalues of the Ricci tensor is non-negative.

**Remark 1.3** The proof that the scalar curvature controls the full bisectional curvature is sophisticated, and somewhat lengthy. In the special case of complex surface, similar estimate is derived also in [20]. In an unpublished work of G. Perelman, the scalar curvature is uniformly bounded along the Kähler Ricci flow. Combining this with Theorem 1.2, we conclude that the full bisectional curvature is uniformly bounded over the Kähler Ricci flow when the initial metric has 2 positive traceless bisectional curvature.

Following Remark 1.3 and a general theorem on the Kähler Ricci flow (c.f. [21], [22]) we arrive the following
Corollary 1.4 Along the Kähler Ricci flow, if the initial metric has 2-positive traceless bisectional curvature operator, then the flow converges by sequence to some Kähler Ricci Soliton in the limit in the sense of Cheeger-Gromov.

Similar results are also proved by Jacob -Phong [20] in special case of complex Kähler surfaces.

2 Basic Kähler geometry

2.1 Setup of notations

Let $X$ be an $n$-dimensional compact Kähler manifold. A Kähler metric can be given by its Kähler form $\omega$ on $X$. In local coordinates $z_1, \cdots, z_n$, this $\omega$ is of the form

$$
\omega = \sqrt{-1} \sum_{i,j=1}^{n} g_{ij} dz^i \wedge d\bar{z}^j > 0,
$$

where $\{g_{ij}\}$ is a positive definite Hermitian matrix function. The Kähler condition requires that $\omega$ is a closed positive $(1,1)$-form. In other words, the following holds

$$
\frac{\partial g_{ik}}{\partial z^j} = \frac{\partial g_{ij}}{\partial z^k} \quad \forall \ i, j, k = 1, 2, \cdots, n.
$$

The Kähler metric corresponding to $\omega$ is given by

$$
\sqrt{-1} \sum_{i=1}^{n} g_{i\bar{j}} \, dz^i \otimes d\bar{z}^j.
$$

For simplicity, in the following, we will often denote by $\omega$ the corresponding Kähler metric. The Kähler class of $\omega$ is its cohomology class $[\omega]$ in $H^2(X, \mathbb{R})$. By the Hodge theorem, any other Kähler metric in the same Kähler class is of the form

$$
\omega_\varphi = \omega + \sqrt{-1} \sum_{i,j=1}^{n} \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j} > 0
$$

for some real valued function $\varphi$ on $X$. The functional space in which we are interested (often referred as the space of Kähler potentials) is

$$
\mathcal{P}(X, \omega) = \{ \varphi \mid \omega_\varphi = \omega + \sqrt{-1} \partial \bar{\partial} \varphi > 0 \text{ on } X \}.
$$

Given a Kähler metric $\omega$, its volume form is

$$
\omega^n = \frac{1}{n!} (\sqrt{-1})^n \det (g_{ij}) \, dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n.
$$

Its Christoffel symbols are given by

$$
\Gamma_{ij}^k = \sum_{l=1}^{n} g^{kl} \frac{\partial g_{ij}}{\partial z^l} \quad \text{and} \quad \Gamma_{ij}^{k\bar{l}} = \sum_{l=1}^{n} g^{k\bar{l}} \frac{\partial g_{ij}}{\partial z^l}, \quad \forall \ i, j, k = 1, 2, \cdots, n.
$$
The curvature tensor is

\[ R_{ijkl} = -\frac{\partial^2 g_{ij}}{\partial z^k \partial z^l} + \sum_{p,q=1}^{n} g^{pq} \frac{\partial g_{ip}}{\partial z^k} \frac{\partial g_{jq}}{\partial z^l}, \quad \forall i,j,k,l = 1,2,\ldots n. \]

We say that \( \omega \) is of nonnegative bisectional curvature if

\[ R_{ijkl} v^i w^j w^k w^l \geq 0 \]

for all non-zero vectors \( v \) and \( w \) in the holomorphic tangent bundle of \( X \). The bisectional curvature and the curvature tensor can be mutually determined. The Ricci curvature of \( \omega \) is locally given by

\[ R_{ij} = -\frac{\partial^2 \log \det(g_{kl})}{\partial z_i \partial \bar{z}_j}. \]

So its Ricci curvature form is

\[ \text{Ric}(\omega) = \sqrt{-1} \sum_{i,j=1}^{n} R_{ij}(\omega) d z^i \wedge d \bar{z}^j = -\sqrt{-1} \partial \partial \log \det(g_{kl}). \]

It is a real, closed (1,1)-form. Recall that \([\omega]\) is called a canonical Kähler class if this Ricci form is cohomologous to \( \lambda \omega \), for some constant \( \lambda \). In our setting, we require \( \lambda = 1 \).

### 2.2 The Kähler Ricci flow

Now we assume that the first Chern class \( c_1(X) \) is positive. The normalized Ricci flow (c.f. [10] and [11]) on a Kähler manifold \( X \) is of the form

\[ \frac{\partial g_{ij}}{\partial t} = g_{ij} - R_{ij}, \quad \forall i, j = 1,2,\ldots,n. \tag{2.1} \]

If we choose the initial Kähler metric \( \omega \) with \( c_1(X) \) as its Kähler class. The flow (2.1) preserves the Kähler class \([\omega]\). It follows that on the level of Kähler potentials, the Ricci flow becomes

\[ \frac{\partial \varphi}{\partial t} = \log \frac{\omega^{\varphi}}{\omega^n} + \varphi - h_\omega, \tag{2.2} \]

where \( h_\omega \) is defined by

\[ \text{Ric}(\omega) - \omega = \sqrt{-1} \partial \partial h_\omega, \quad \text{and} \quad \int_X (e^{h_\omega} - 1) \omega^n = 0. \]

Then the evolution equation for bisectional curvature is

\[ \frac{\partial}{\partial t} R_{ijkl} = \triangle R_{ijkl} + R_{ijpq} R_{kl}^{pq} - R_{ij} R_{kpq}^{pq} - R_{ij} R_{kpq}^{pq} + R_{ij} R_{kpq}^{pq} + R_{ij} R_{kpq}^{pq} + R_{ij} R_{kpq}^{pq} \]

\[ -\frac{1}{2} \left( R_{ijkl} + R_{ij} R_{kl} + R_{ij} R_{kl} + R_{ij} R_{kl} + R_{ij} R_{kl} \right). \tag{2.3} \]
The evolution equation for Ricci curvature and scalar curvature are
\[
\frac{\partial R_{ij}}{\partial t} = \triangle R_{ij} + R_{ipq}R_{jp} - R_{i\tilde{p}R_{p}}, \quad (2.4)
\]
\[
\frac{\partial R}{\partial t} = \triangle R + R_{ij}R_{ji} - R. \quad (2.5)
\]

For direct computations and using the evolved frames, we can obtain the following evolution equations for the bisectional curvature:
\[
\frac{\partial R_{ijkl}}{\partial t} = \triangle R_{ijkl} - R_{ijkl} + R_{ijm\tilde{n}R_{m\tilde{n}k\tilde{l}}} - R_{i\tilde{m}k\tilde{n}R_{m\tilde{n}j\tilde{l}}} + R_{i\tilde{m}n\tilde{l}R_{m\tilde{n}k\tilde{l}}} \quad (2.6)
\]

As usual, the flow equation (2.1) or (2.2) is referred as the Kähler Ricci flow on X. It is proved by Cao [2], who followed Yau’s celebrated work [23], that the Kähler Ricci flow exists globally for any smooth initial Kähler metric.

3 The traceless bisectional curvature operator

3.1 Definition and the evolution equations

In Riemannian geometry, the curvature tensor for Riemannian metric can always be decomposed orthogonally into three parts: \( Rm = W + V + U \), where \( W \) is the Weyl tensor and \( V, U \) are the traceless Ricci part and the scalar curvature part respectively. The decomposition for Kähler case is slight different. The bisectional curvature tensor can also be decomposed into orthogonal parts as well.

Set
\[
S_{ij} = R_{ij} - \frac{1}{n} Rg_{ij} = R^0_{ij}, \quad (3.7)
\]
\[
S_{abcd} = R_{abcd} - \frac{1}{n} (S_{abg_{cd}} + S_{cdg_{ab}}) - \frac{1}{n^2} Rg_{ab}g_{cd}. \quad (3.8)
\]

As in the Riemannian case, the "Weyl" part \( S_{abcd} \) is also trace free:
\[
S_{abcd} = S_{cdab}, \quad g^{ab}S_{abcd} = 0.
\]

As in the previous subsection, under some evolved moving frame, we can rewrite the evolution equation for curvature as below

**Proposition 3.1** Along the Kähler Ricci flow the evolution equation related the traceless bisectional curvature operator are as follows:
\[
\frac{\partial R}{\partial t} = \triangle R - R + \frac{1}{n} R^2 + S_{a\tilde{b}S_{b\tilde{a}}} \quad (3.9)
\]
\[
\frac{\partial S_{ab}}{\partial t} = \triangle S_{ab} + \frac{1}{n} (R - n) S_{ab} + S_{abij}S_{ji} \quad (3.10)
\]
\[
\frac{\partial S_{abcd}}{\partial t} = \triangle S_{abcd} - S_{abcd} + S_{abij}S_{fica} + S_{a\tilde{m}jd}S_{ld\tilde{c}j} - S_{a\tilde{c}j}S_{b\tilde{j}d} + \frac{1}{n} S_{ab}S_{cd} \quad (3.11)
\]
The bisectional curvature operator can be viewed as a symmetric operator on the space of real $(1,1)$ forms $\Lambda^{1,1}(X)$. For any pair of $(1,1)$ forms $\eta, \tau$, the action of the bisectional curvature is:

$$R(\eta, \tau) = R_{ijkl} \eta_{ab} \tau_{cd} g^{ab} g^{cd} g^{ij} g^{kl}.$$  

If we decompose the space $\Lambda^{1,1}(X)$ into the line consists of the multiple of the Kähler form and its orthogonal complementatory subspace $\Lambda_0^{1,1}(X)$, then the action of $S_{ijkl}$ preserves $\Lambda_0^{1,1}(X)$. Denote the action of $S_{ijkl}$ by $S$. In some special basis, we will use $M$ to denote the matrix of the operator $S$. We often referred $S$ as the traceless bisectional curvature operator. Moreover, there is a nice decomposition formula for the bisectional curvature operator in $\Lambda^{1,1}(X)$:

$$\begin{pmatrix}
R & Ric^0 \\
Ric^0 & S
\end{pmatrix}.$$  \tag{3.12}

If the action of $S$ in $\Lambda_0^{1,1}(X)$, is non-negative, then we call the underlying Kähler metric has a non-negative traceless bisectional curvature operator. If the action of $S$ in $\Lambda_0^{1,1}(X)$ has a property that the sum of any two eigenvalues is non-negative, then we call the underlying Kähler metric has a 2-non-negative traceless bisectional curvature operator.

### 3.2 Geometric properties of the traceless bisectional curvature operator

In this subsection, we derive some geometric properties of the traceless bisectional curvature operator. First, in any local coordinate, after fixing an frame such that the metric tensor at the origin is identity matrix. There is a natural orthonormal basis for $\Lambda_0^{1,1}(X)$ at the origin point (here $i, j = 1, 2, \cdots n$):

$$\{\sqrt{-1}dz^i \wedge d\bar{z}^j, dz^i \wedge d\bar{z}^j - dz^j \wedge d\bar{z}^i, \sqrt{-1}(dz^i \wedge d\bar{z}^j + dz^j \wedge d\bar{z}^i)\}.$$

Note that the space of traceless real $(1,1)$ form is spanned by

$$\{\sqrt{-1}(dz^i \wedge d\bar{z}^j - dz^j \wedge d\bar{z}^i), dz^i \wedge d\bar{z}^j - dz^j \wedge d\bar{z}^i, \sqrt{-1}(dz^i \wedge d\bar{z}^j + dz^j \wedge d\bar{z}^i)\}.$$

For convenience, we use the following spaces in this paper:

**Definition 3.2**: The space $\Lambda_0^{1,1}(X)$ is locally spanned by the following elements:

- $A^{ij} = dz^i \wedge d\bar{z}^j - dz^j \wedge d\bar{z}^i$,
- $B^{ij} = dz^i \wedge d\bar{z}^j + dz^j \wedge d\bar{z}^i$,
- $C^{ij} = -\sqrt{-1}(dz^i \wedge d\bar{z}^j - dz^j \wedge d\bar{z}^i)$

and the space $\Lambda^{1,1}(X)$ is locally spanned by the following elements:

- $a^{ii} = 2dz^i \wedge d\bar{z}^i$,
- $B^{ij} = dz^i \wedge d\bar{z}^j + dz^j \wedge d\bar{z}^i$,
- $C^{ij} = -\sqrt{-1}(dz^i \wedge d\bar{z}^j - dz^j \wedge d\bar{z}^i)$
where \( i, j = 1, 2 \cdots n \). Our definition differs from the space of the traceless (1,1) form because we want the eigenvalues of \( S \) to be positive for Fubni-Study metric.

**Proposition 3.3** If the traceless bisectional curvature operator is 2-nonnegative, then the orthogonal bisectional curvature is nonnegative. If the traceless bisectional curvature operator is nonnegative, then we have the following inequalities:

\[
R_{\bar{i}\bar{i}} + R_{\bar{j}\bar{j}} \geq 2R_{\bar{i}\bar{j}} \geq 0, \quad R_{\bar{i}\bar{i}} + R_{\bar{j}\bar{j}} \geq 0
\]

for any \( i \neq j \).

**Proof**. (1) If \( A \) is a symmetric matrix and the sum of two lowest eigenvalues of \( A \) is nonnegative, then \( A_{ii} + A_{jj} \geq 0 \). To see this, assume \( m_1 \leq m_2 \leq \cdots \leq m_n \) are the eigenvalues of \( A \), then we have

\[
m_1 + m_2 = \inf \{ A(x, x) + A(y, y) | |x| = |y| = 1, x \perp y \} \geq 0,
\]

so we have

\[
A_{ii} + A_{jj} = A(e_i, e_i) + A(e_j, e_j) \geq 0
\]

where \( \{e_i\} \) are the standard basis of \( \mathbb{R}^n \).

(2) Since the matrix of \( S \) is the same as the matrix of curvature operator \( Rm \) when acting on the space \( \Lambda^1_0(X) \), so

\[
R(B^{ij}, B^{ij}) + R(C^{ij}, C^{ij}) \geq 0,
\]

then simplify the above formula we have

\[
R_{\bar{i}\bar{j}} \geq 0, \quad \forall i \neq j.
\]

(3) If the traceless bisectional curvature operator is nonnegative, Since the matrices of \( S \) and \( Rm \) are the same under the basis of \( \{A^{ii}, B^{ij}, C^{ij}\} \), then \( R(A^{ii}, A^{ii}) \geq 0 \) and

\[
\begin{bmatrix}
R(A^{ii}, A^{ii}) & R(A^{ii}, A^{ij}) \\
R(A^{ii}, A^{ij}) & R(A^{ij}, A^{ij})
\end{bmatrix} \geq 0.
\]

So

\[
R(A^{ii}, A^{ii}) + R(A^{ij}, A^{ij}) \geq 2\sqrt{R(A^{ii}, A^{ii})R(A^{ij}, A^{ij})} \geq 2R(A^{ii}, A^{ij}).
\]

Then we have

\[
R(A^{ij}, A^{ij}) = R(A^{ii}, A^{ii}) + R(A^{ij}, A^{ij}) - 2R(A^{ii}, A^{ij}) \geq 0.
\]

i.e. \( R(A^{ij}, A^{ij}) = R_{\bar{i}\bar{i}} + R_{\bar{j}\bar{j}} - 2R_{\bar{i}\bar{j}} \geq 0 \). Thus

\[
R_{\bar{i}\bar{i}} + R_{\bar{j}\bar{j}} = R_{\bar{i}\bar{i}} + \sum_{\alpha \neq i} R_{\alpha\bar{i}\bar{i}} + R_{\bar{j}\bar{j}} + \sum_{\beta \neq j} R_{\beta\bar{j}\bar{j}}
\]

\[
\geq 2R_{\bar{i}\bar{j}} + \sum_{\alpha \neq i} R_{\alpha\bar{i}\bar{i}} + \sum_{\beta \neq j} R_{\beta\bar{j}\bar{j}}
\]

\[
\geq 0.
\]

where \( i \neq j \). This finishes the proof of the proposition.
4 Proof of Theorem 1.2

We follow notations in the previous section. Note that proposition 3.3 already show the first and last parts of Theorem 1.2. We need a technical lemma first.

Lemma 4.1 If \( M \) is a symmetric \( m \times m \) matrix and satisfies

1. \( \sum_i M_{ij} = 0, \quad (\forall 1 \leq j \leq m); \)
2. \( M_{ij} + M_{(i+1)(j+1)} - M_{i(i+1)} - M_{(i+1)j} = 0, \quad (\forall 1 \leq i, j \leq m - 1); \)

then \( M = 0 \).

Proof. From (1), we know

\[
\sum_i M_{ii} + 2 \sum_{i<j} M_{ij} = 0. \tag{4.13}
\]

And from (2), we have \( M_{ij} + M_{kl} - M_{il} - M_{jk} = 0 \). So \( M_{ii} + M_{jj} - 2M_{ij} = 0 (\forall i < j) \). Then we have

\[
(n - 1) \sum_i M_{ii} - 2 \sum_{i<j} M_{ij} = 0. \tag{4.14}
\]

Thus from (4.13) and (4.14) we have

\[
\sum_i M_{ii} = 0, \quad \sum_{i<j} M_{ij} = 0. \tag{4.15}
\]

From (4.13) and \( M_{ij} = M_{1i} + M_{1j} - M_{11} \), we have \( \sum_{1<i<j}(M_{1i} + M_{1j} - M_{11}) = 0 \). Thus

\[
(m - 2) \sum_{i>1} M_{1i} - \frac{(m - 1)(m - 2)}{2} M_{11} = 0.
\]

Since \( \sum_{i>1} M_{1i} + M_{11} = 0 \), we have \( M_{11} = 0 \) if \( m \geq 3 \). Now \( M_{ij} = M_{1i} + M_{1j} \), from (1) we have

\[
0 = \sum_j M_{ij} = \sum_j (M_{1i} + M_{1j}) = mM_{1i} + \sum_j M_{1j} = mM_{1i}.
\]

So \( M_{1i} = 0 \), then \( M_{ij} = 0 \). We can easily verify that the lemma still holds when \( m = 2 \). This finishes the proof of the lemma.

Now we are ready to give a proof of Theorem 1.2.

Proof. We divide the proof into three parts.

1. In this part, we want to prove that the trace of \( S \) is bounded by the scalar curvature. If the traceless bisectional curvature operator is nonnegative, the proof is easy. In this case, Proposition 3.3 implies that

\[
R_{iii} + R_{jjjj} \geq 2R_{ijjj} \geq 0, \quad \forall i \neq j. \tag{4.16}
\]
Let $M$ be the matrix expression of $S$ acting on $\Lambda^{1,1}_0(X)$ with respect to the basis (c.f. Definition 3.2). The trace of $M$ is

$$\text{tr}(M) = \sum_{2 \leq j \leq n} R(A^{1j}, A^{1j}) + \sum_{i < j} (R(B^{ij}, B^{ij}) + R(C^{ij}, C^{ij}))$$

$$= \sum_{2 \leq j \leq n} (R_{1111} + R_{jjjj} - 2R_{11jj}) + 4 \sum_{i < j} R_{ijj}$$

$$= R + (n - 2)R_{1111} + 2 \sum_{2 \leq i < j} R_{iijj}.$$  \hspace{1cm} (4.17)

From (4.16) (4.17), when $n \geq 4$

$$\text{tr}(M) \leq R + (n - 2)R_{1111} + \frac{n - 2}{2} \sum_{2 \leq i < j} 2R_{iijj}$$

$$\leq R + (n - 2) \sum_{1 \leq k \leq n} R_{kkkk}$$

$$\leq (n - 1)R.$$  \hspace{1cm} (4.18)

We can easily check that the above inequality still holds when $n = 2, 3$. In other words, the trace of $S$ can be bounded by the scalar curvature.

If the traceless bisectional curvature operator is 2-nonnegative, we need to bound $R_{1111}$ and $R_{iijj}$ from (4.17). Choose the basis $\{A^{ij}, B^{ij}, C^{ij}\}$ of the space $\Lambda^{1,1}_0(X)$ as in Subsection 3.2. Then

$$R(A^{ii}, A^{ii}) + R(A^{ij}, A^{ij}) \geq 0,$$

i.e.,

$$R_{11ii} + R_{11jj} \leq R_{1111} + \frac{1}{2}R_{iiii} + \frac{1}{2}R_{jjjj}.$$  \hspace{1cm} (4.19)

Similarly, if we choose the basis $\{A^{kj}, B^{ij}, C^{ij}\}$, we have

$$R_{iijj} + R_{kkkk} \leq R_{iiii} + \frac{1}{2}R_{jjjj} + \frac{1}{2}R_{kkkk},$$

$$R_{jjii} + R_{jjkk} \leq R_{jjjj} + \frac{1}{2}R_{iiii} + \frac{1}{2}R_{kkkk},$$

$$R_{kkii} + R_{kkjj} \leq R_{kkkk} + \frac{1}{2}R_{jjjj} + \frac{1}{2}R_{iiii}.$$ \hspace{1cm} (4.18)

Consequently,

$$\sum_{k \leq l} R_{kkll} \leq \sum_{k} R_{kkkk}.$$  \hspace{1cm} (4.19)

Therefore,

$$R = \sum_{k} R_{kkkk} + 2 \sum_{k < l} R_{kkll} \leq 3 \sum_{k} R_{kkkk} \leq 3R.$$  \hspace{1cm} (4.20)

The last inequality follows from $R_{kkll} \geq 0 (\forall k \neq l)$. Consequently,

$$\frac{R}{3} \leq \sum_{k} R_{kkkk} \leq R.$$
Since all the orthogonal bisectional curvature is nonnegative, all of them are uniformly bounded by the scalar curvature. Let us assume that the holomorphic sectional curvature satisfies the following inequality

$$R_{1111} \leq R_{2222} \leq \cdots \leq R_{nnnn}.$$ 

We claim that $R_{3333} \geq 0$. To see this, if $R_{3333} < 0$, then

$$R_{1111} \leq R_{2222} < 0.$$ 

Consequently, we have

$$M_{11} = R_{1111} + R_{2222} - 2R_{1122} < 0,$$

$$M_{22} = R_{1111} + R_{3333} - 2R_{1133} < 0.$$ 

Thus, $M_{11} + M_{22} < 0$, which is a contradiction! Consequently, we have

$$R_{3333} \geq 0.$$ 

We want to again divide into three cases for discussions.

**Case 1.** If $R_{1111} \geq 0$, then

$$0 \leq R_{1111} \leq \sum_k R_{kkkk} \leq R.$$ 

**Case 2.** If $R_{1111} \leq R_{2222} < 0$. Since $M$ is 2-nonnegative and from (4.18), we have

$$2R_{1111} + R_{2222} + R_{3333} \geq 0,$$

$$2R_{2222} + R_{1111} + R_{3333} \geq 0.$$ 

Thus

$$|R_{1111}| + |R_{2222}| \leq \frac{2}{3}R_{3333}.$$ 

Therefore,

$$R \geq R_{3333} - (|R_{1111}| + |R_{2222}|) \geq \frac{1}{3}R_{3333}. \quad (4.19)$$ 

Moreover, from $M_{11} + M_{22} \geq 0$, We have

$$M_{11} = R_{1111} + R_{2222} - 2R_{1122} < 0,$$

$$M_{22} = R_{1111} + R_{3333} - 2R_{1133} > 0.$$ 

Consequently,

$$2R_{1111} + R_{3333} \geq 2R_{1111} + R_{2222} + R_{3333} \geq 2(R_{1122} + R_{1133}) \geq 0. \quad (4.20)$$ 

Thus,

$$|R_{1111}| \leq \frac{1}{2}R_{3333}. \quad (4.21)$$ 

Combining inequalities (4.19) and (4.21), we have

$$|R_{1111}| \leq \frac{R}{6}.$$ 

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Case 3. If \( R_{1111} < 0 \leq R_{2222} \), and if \( M_{11} = R_{1111} + R_{2222} - 2R_{1122} \geq 0 \), then

\[
|R_{1111}| \leq R_{2222}.
\]

From the proof of case 2, we know \( |R_{1111}| \leq C(n)R \). If \( M_{11} < 0 \), then \( M_{22} \geq 0 \). So we have

\[
|R_{1111}| \leq R_{3333}.
\]

Thus, we can bound \( |R_{1111}| \) by the scalar curvature in this case.

In summary, we can always bound \( \text{tr}(M) \) by the scalar curvature \( R \) if the bisectional curvature is 2-nonnegative.

(2) In this part, we want to prove that every \( S_{ijkl} \) can be uniquely represented by the entries of the matrix \( M \). This is equivalent to say that \( S_{ijkl} = 0 \) if \( M = 0 \). Assume \( M = 0 \).

(Calculate \( S_{iiii}, S_{ijij} \)) Assume \( T_{ij} = S_{ijjj}, T = (T_{ij}) \). We want to find \( T \) from \( M \). Note that

\[
T_{ij} + T_{kl} - T_{il} - T_{kj} = S(dz^i \wedge dz^j - dz^k \wedge dz^l, dz^j \wedge dz^j - dz^j \wedge dz^j)
\]

\[
= S(A^{1k} - A^{1i}, A^{1l} - A^{1j})
\]

\[
= 0
\]

since \( S(A^{1k}, A^{1l}) = 0 \). Thus, the following equations hold:

\[
T_{ij} + T_{(i+1)(j+1)} - T_{i(j+1)} - T_{(i+1)j} = 0
\]

\[
\sum_i T_{ij} = 0, \forall j = 1, 2 \cdots n
\]

From Lemma 4.1, we have \( T = 0 \), i.e.

\[
S_{iiii} = S_{ijij} = 0. \tag{4.22}
\]

(Calculate other \( S_{ijkl} \)) Since \( dz^i \wedge dz^j = \frac{1}{2}(B^{ij} + \sqrt{-1}C^{ij}) \), we have

\[
S_{ijij} = 0 (\forall i \neq j). \tag{4.23}
\]

From \( S(B^{ij}, B^{ik}) = 0, S(C^{ij}, C^{ik}) = 0 \) and \( S(B^{ij}, C^{ik}) = 0, S(C^{ij}, B^{ik}) = 0 \), where \( i \neq j, j \neq k, k \neq i \), we have \( S_{ijik} + S_{jiki} = 0, S_{ijik} = S_{jiki} \). Consequently,

\[
S_{ijik} = S_{jiki} = 0. \tag{4.24}
\]

From \( S(A^{ij}, B^{kl}) = 0, S(A^{ij}, C^{kl}) = 0 \), where \( i \neq j, k \neq l \), we have \( S_{iikl} = S_{jijkl} \). Since \( \sum_i S_{iikl} = 0 \), we have

\[
S_{iikl} = 0. \tag{4.25}
\]

Let \( l = i \), we have

\[
S_{iikl} = 0. \tag{4.26}
\]

From \( 4.25 \) and \( S(A^{ij}, B^{ij}) + \sqrt{-1}S(A^{ij}, C^{ij}) = 0 \) where \( i \neq j \), we have

\[
S_{ijij} = S_{jjij} = 0. \tag{4.27}
\]
From $S(B^{ij}, B^{ij}) = 0$ where $i \neq j$, we have
\[ S_{i\bar{j}j\bar{i}} = 0. \] (4.28)

From (4.23) and $S(B^{ij}, B^{kl}) = 0$ where $i \neq j, j \neq k, k \neq l, l \neq i$, we have
\[ S_{i\bar{j}k\bar{l}} = 0. \] (4.29)

From (4.22)-(4.29), we know
\[ S_{i\bar{j}k\bar{l}} = 0. \]

(3) Finally we prove that the bisectional curvature is bounded by the scalar curvature. Let $m_1 \leq m_2 \leq \cdots \leq m_s (s = n^2 - 1)$ be the eigenvalues of the matrix $M$. If $M$ is positive, then all the eigenvalues can be bounded by the scalar curvature. Otherwise since $M$ is 2-positive, the eigenvalues must satisfy the following inequalities:
\[ m_1 \leq 0 \leq m_2 \leq \cdots \leq m_s, \quad m_1 + m_2 > 0. \]

Then, $m_3 < \text{tr}(M)$ and $|m_1| < m_2 \leq m_3 < \text{tr}(M)$. In other words, all the eigenvalues can be bounded by the scalar curvature.

Since all the eigenvalues of the matrix $M$ are bounded by $R$, all of its entries of $M$ are also bounded by the scalar curvature function. From (2), every $S_{i\bar{j}k\bar{l}}$ is bounded by $R$, i.e. $|S_{i\bar{j}k\bar{l}}| < C(n) R$. Thus $|S_{kl}| \leq C(n) R$. From the definition of $S$ (c.f. (3.8), we have
\[ |R_{i\bar{j}k\bar{l}}| \leq C(n) R. \]

5 Proof of Theorem 1.1

In this section, we are ready to prove Theorem 1.1. Note that in [11], the positive curvature operator is preserved and in [5] the 2-positive curvature operator is preserved along the Ricci flow, one can also see both from [12]. Our proof here is similar to theirs. First recall R. Hamilton’s lemma:

Lemma 5.1 [11]. Suppose $\partial f / \partial t = \Delta f + \phi(f)$. Let $s(f)$ be a concave function on the bundle invariant under parallel translation whose level curves $s(f) \leq c$ are preserved by the ODE $df/dt = \phi(f)$. Then the inequality $s(f) \leq c$ is preserved by the PDE for any constant $c$. Furthermore if $s(f) < c$ at one point at time $t = 0$, then $s(f) < c$ everywhere on $X$ for all $t > 0$.

Now we begin to prove Theorem 1.1.

Proof . (1) Define
\[ [\phi^\lambda, \phi^\mu]_{ab} = \phi_\alpha^\lambda \phi^\mu_a - \phi_\alpha^\mu \phi^\lambda_a = C_\rho^\lambda \phi^\rho_a. \]
By calculation we have the following relations:

\[
\begin{align*}
[A^{ij}, B^{ij}] &= 2\sqrt{-1}C^{ij}, [A^{ij}, B^{ik}] = \sqrt{-1}C^{ik}, [A^{ij}, B^{jk}] = \sqrt{-1}C^{kj}, \\
[B^{ij}, C^{ij}] &= 2\sqrt{-1}A^{ij}, [B^{ij}, C^{ik}] = -\sqrt{-1}B^{ik}, [B^{ij}, C^{jk}] = -\sqrt{-1}B^{jk}, \\
[C^{ij}, A^{ij}] &= 2\sqrt{-1}B^{ij}, [C^{ij}, A^{ik}] = \sqrt{-1}B^{ii}, [C^{ij}, A^{jk}] = -\sqrt{-1}B^{ij}, \\
[B^{ij}, B^{ik}] &= -\sqrt{-1}C^{ij}, [C^{ij}, C^{ik}] = -\sqrt{-1}C^{jk}.
\end{align*}
\]

where \(i \neq j, j \neq k, k \neq i\), and other Lie brackets are zero. Note that all \(C^\lambda_\mu\) are zeros or pure imaginary numbers.

Define

\[
S_{a\bar{m}n\bar{d}}S_{mb\bar{c}n} = S_{a\bar{m}n\bar{c}}S_{mb\bar{d}} = M_{\alpha\beta}\phi_{\alpha m}\phi_{\beta n} - M_{\alpha\beta}\phi_{\alpha n}\phi_{\beta m}M_{\gamma\delta}\phi_{\gamma mb}\phi_{\delta nd}
= M_{\alpha\beta}\phi_{\alpha m}\phi_{\gamma nb}(\phi_{\beta n}\phi_{\delta m} - \phi_{\gamma m}\phi_{\beta nd})
= M_{\alpha\beta}\phi_{\alpha m}\phi_{\beta n}\phi_{\gamma m}\phi_{\delta n}C^{\delta\beta\gamma}_{\alpha \rho}
= -\frac{1}{2}C^\alpha_\beta C^\beta_\gamma M_{\alpha\beta}\phi_{\gamma ab}\phi_{\delta cd}.
\]

Define

\[
M^\#_{\mu \rho} = C^\alpha_\beta C^\beta_\gamma C_{\rho \alpha}M_{\gamma \delta},
\]

then we have

\[
S_{a\bar{m}n\bar{d}}S_{mb\bar{c}n} - S_{a\bar{m}n\bar{c}}S_{mb\bar{d}} = -\frac{1}{2}M^\#_{\mu \rho}\phi_{\mu \alpha}\phi_{\rho \beta}
\]

and

\[
\frac{\partial M}{\partial t} = -M + M^2 - \frac{1}{2}M^\# + \frac{1}{n}T.
\]

Now we have the following lemma:

**Lemma 5.2** If all \(C^\alpha_\gamma\) are real and \(M \geq 0\), then \(M^\# \geq 0\).

**Proof.** Without loss of generality, we may choose a basis \(\{\phi^\alpha\}\) which diagonalizes \(M\), so that \(M_{\alpha\beta} = \delta_{\alpha\beta}M_{\alpha\alpha}\). For any \(v = v^\alpha\phi^\alpha\), we have

\[
M^\#(v, v) = (v^\alpha C^\alpha_\beta)(v^\beta C^\beta_\gamma)M_{\alpha\beta}M_{\gamma i} = (v^\alpha C^\alpha_\beta)^2 M_{aa}M_{ii} \geq 0.
\]

The lemma is then proved.

Now we return to the proof of Theorem 1.1 again. Since in our case all \(C^\lambda_\mu\) are zero or pure imaginary numbers, then \(M^\# \leq 0\) if \(M \geq 0\). Since \(T\) is always non-negative, we have

\[
\frac{\partial M}{\partial t} = -M + M^2 - \frac{1}{2}M^\# + \frac{1}{n}T \geq 0.
\]

when \(M = 0\). Note that \(M \geq 0\) is convex and \(M(0) \geq 0\), we have \(M(t) \geq 0\) for all \(t > 0\). In other words, the nonnegative traceless bisectional curvature operator is preserved. From Lemma
5.1, if \( M \) is positive at one point at time \( t = 0 \), then \( M \) is positive everywhere for all time \( t > 0 \).

(2) We want to prove that the 2-nonnegative traceless bisectional curvature operator is preserved along the Kähler Ricci flow. Let us assume that the eigenvalues of the traceless bisectional curvature operator on \( \Lambda \) are \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \), where \( m = n^2 - 1 \). From (4.1)-(4.2), we have

\[
\frac{d}{dt}(\lambda_1 + \lambda_2) \geq \frac{d}{dt}(M_{11} + M_{22}) \geq -(\lambda_1 + \lambda_2) + (\lambda_1^2 + \lambda_2^2) - \frac{1}{2} \sum_{p,q}((C_1^{pq})^2 + (C_2^{pq})^2)\lambda_p\lambda_q.
\]  

(4.3)

Note that the right side

\[
\frac{1}{2} \sum_{p,q}((C_1^{pq})^2 + (C_2^{pq})^2)\lambda_p\lambda_q = \sum_{p < q}((C_1^{pq})^2 + (C_2^{pq})^2)\lambda_p\lambda_q
\]

\[
= \sum_{q \geq 3}(C_2^{qq})^2(\lambda_1 + \lambda_2)\lambda_q + \sum_{p \geq 3}((C_1^{pq})^2 + (C_2^{pq})^2)\lambda_p\lambda_q.
\]

Note that \( \lambda_m \geq \cdots \geq \lambda_2 \geq 0 \). If \( \lambda_1 + \lambda_2 = 0 \), then the right side of (4.3) is nonnegative. Since \( \lambda_1 + \lambda_2 \) is a concave function on \( X \), \( \lambda_1 + \lambda_2 \geq 0 \) is preserved. From Lemma 5.1, if \( \lambda_1 + \lambda_2 > 0 \) is positive at one point at time \( t = 0 \), then \( \lambda_1 + \lambda_2 > 0 \) is positive everywhere for all time \( t > 0 \).

(3) Now we prove the last part of Theorem 1.1. If the traceless bisectional curvature positive is non-negative or 2-non-negative, Theorem 1.2, implies that

\[
R_{i\bar{j}j} \geq 0, \forall i \neq j.
\]

Let us assume initially the Ricci curvature is positive and after finite time \( t_0 > 0 \), at some point \( p \in X \) \( R_{i\bar{j}} \) vanishes at least at one direction. For convenience, set this direction as \( \frac{\partial}{\partial z_i} \) and diagonalize the Ricci curvature at this point. Then

\[
\frac{\partial R_{1\bar{1}1}}{\partial t} \big|_{t_0} \geq R_{1\bar{1}1}J_{i\bar{j}}R_{i\bar{j}} - R_{1\bar{1}1}R_{1\bar{1}1} = \sum_{j=2}^{n} R_{1\bar{1}1}J_{i\bar{j}}R_{i\bar{j}} \geq 0.
\]

By Hamilton’s maximum principle for tensors, this is enough to show that the positivity of Ricci curvature is preserved under the condition.

6 Proof of Corollary 1.4

We only need to prove the Kähler Ricci flow convergence by sequences to some Kähler Ricci soliton when the bisectional curvature is uniformly bounded from Theorem 1.2. In [21], she proved that \( \tau \)-flow converges by sequence to some Ricci soliton when the curvature operator and the diameter are uniformly bounded. In [22], she proved that the Kähler Ricci flow converges by sequence to some Kähler Ricci soliton except a set of isolated points on any complex compact Kähler surface without any curvature assumptions. First let us recall Perelman’s no local collapsing theorem:
Definition 6.1 [16]. Let $g_{ij}(t)$ be a smooth solution to the Ricci flow $(g_{ij})_t$ on $[0, T)$ on a Riemannian manifold $X$ of dimension $n$. We say that $g_{ij}(t)$ is locally collapsing at $T$, if there is a sequence of times $t_k \to T$ and a sequence of metric balls $B_k = B_k(p_k, r_k)$ at times $t_k$, such that $\frac{r_k^2}{t_k}$ is bounded, $|Rm|(g_{ij})(t_k) \leq r_k^{-2}$ in $B_k$ and $\frac{\text{Vol}(B_k)}{r_k^n} \to 0$.

Lemma 6.2 [16]. If $X$ is closed and $T < \infty$, then $g_{ij}(t)$ is not locally collapsing at $T$.

Now we begin to prove Corollary 1.4.

Proof. We divide the proof into two parts.
(1) First we are ready to prove that the injectivity radii have a uniformly positive lower bound along the Kähler Ricci flow. If the traceless bisectional curvature operator is 2-nonnegative and the scalar curvature is bounded along the flow, Theorem 1.2 implies that the curvature tensor is uniformly bounded.

Claim 6.3 The injectivity radius has a uniformly positive lower bound along the flow.

Proof. Let $(X, g_{ij})$ be the Kähler Ricci flow. Fix $T > 0$. Now we re-scale the metric

$$
\tilde{g}_{ij}(s) = (T - s)g_{ij}(-\log(\frac{T - s}{T})), \quad s \in [0, T). \tag{6.30}
$$

Then, $\tilde{g}_{ij}(s)$ is a solution with finite maximal existence interval to the unnormalized Kähler Ricci flow $\frac{\partial \tilde{g}_{ij}}{\partial s} = -R_{ij}$. Lemma 6.2 implies that $(X, \tilde{g}_{ij}(s))$ is not locally collapsing. In other words, for any sequence of times $s_k \to T$, any sequence of metric balls $B_k = B_k(x_k, r_k)$ at times $s_k$, such that $\frac{r_k^2}{s_k}$ is bounded and $|Rm| (\tilde{g}_{ij})(s_k) \leq r_k^{-2}$ in $B_k$, there exists a constant $\delta > 0$ such that

$$
\frac{\text{Vol}(B_k)}{r_k^{2n}} \geq \delta. \tag{6.31}
$$

Since $|Rm|(g_{ij}(t))$ is uniformly bounded along the Kähler Ricci flow, for the un-normalized flow, we have

$$
|Rm| (\tilde{g}_{ij}(s)) \leq \frac{C}{T - s}. \tag{6.32}
$$

We claim that there exists a constant $\epsilon > 0$ such that $\text{inj}(\tilde{g}(s)) \geq \sqrt{T - s} \epsilon$. We prove this by contradiction. If there exist a sequence of times $s_k \to T$, such that

$$
\frac{\text{inj}(\tilde{g}(s_k))}{\sqrt{T - s_k}} \to 0.
$$

We re-scale the metric

$$
h(s_k) = \frac{1}{T - s_k} \tilde{g}(s_k).
$$

Let $r_k^2 = T - s_k$, then

$$
|Rm|(h(s_k)) \leq C, \text{inj}(h(s_k)) \to 0. \tag{6.32}
$$
From (6.31), we have
\[ \text{Vol}(B_{h(s_k)}(x_k,1)) \geq \delta. \] (6.33)
Then, (6.32) (6.33) contradict with J.Cheeger’s injectivity radius estimate (c.f. [18]). Thus, we have \( \text{inj}(\tilde{g}(s)) \geq \sqrt{T-s} \epsilon \). Together with (6.30), we have
\[ \text{inj}(g(t)) \geq \epsilon > 0. \]

Claim 6.4 The diameter has a uniformly upper bound along the flow.

Proof. To see this, we assume that there are \( N \) points \( p_1, p_2, ..., p_N \) such that
\[ \text{dist}_{g(t)}(p_i, p_j) \geq 2\epsilon, \quad \forall 1 \leq i \neq j \leq N \]
where \( \epsilon > 0 \) is the uniformly lower bound on the injectivity radius from Claim 6.3. Hence, the balls \( B_{g(t)}(p_i, \epsilon) \) are embedded and pairwisely disjoint. Since the curvature operator is uniformly bounded and from the volume comparison theorem,
\[ V \geq \sum_{i=1}^{N} \text{Vol}(B_{g(t)}(p_i, \epsilon)) \geq NC\epsilon^{2n}. \]
Since the volume \( V \) is fixed along the flow, \( N \) is bounded from above. Consequently the diameter has a uniformly upper bound along the flow.

(2) Now we return to the proof of Corollary 1.4. Since we have uniformly bounds on curvature tensor and uniformly lower bound on the injectivity radius, by Hamilton’s compactness theorem, for every \( t_k \to \infty \) as \( k \to \infty \), there exists a subsequence such that \( (X, g(t_k + t)) \) converges to \( (X, h(t)) \), in the sense that there exist diffeomorphisms \( \phi_k : X \to X \), such that \( \phi_k^*g(t_k + t) \) converge uniformly together with their covariant derivatives to metrics \( h(t) \) on any compact subsets. For every sequence of times \( t_k \to \infty \), there exists a subsequence, such that the \( (X, g(t_k + t)) \) converges to a Kähler Ricci soliton as \( k \to \infty \). Now we outline that process (c.f. [21][22]).

Corresponding to the Kähler Ricci flow, Perelman’s functional:
\[ W(g, f, \frac{1}{2}) = (2\pi)^{-n} \int_X (|\nabla f|^2 + R + f - 2n)e^{-f}dV_g. \]
One can show that \( \mu(g, \frac{1}{2}) = \inf\{W(g, f, \frac{1}{2})|f \text{ satisfies } \int_X (2\pi)^{-n}e^{-f} = 1\} \) can be achieved by a smooth function \( f(t) \) such that \( \mu(g, \frac{1}{2}) = W(g, f, \frac{1}{2}) \). Moreover,
\[ 2\Delta f - |\nabla f|^2 + R + f - 2n = \mu(g, \frac{1}{2}). \]
Then, we have the following results:
(i) \( \lim_{i \to \infty} \mu(g_i, \frac{1}{2}) = \mu(h, \frac{1}{2}) \), so that \( \lim_{i \to \infty} \mu(g_i, \frac{1}{2}) \) is finite;
(ii) If \( \mu(g_k, \frac{1}{2}) = W(g, f_k, \frac{1}{2}) \) and \( \mu(h, \frac{1}{2}) = W(h, f, \frac{1}{2}) \), then \( f_k \to f \) in \( C^{2, \alpha} \) norm;
(iii) For any $t$, we can find $f_t$ such that $\mu(g(t), \frac{1}{2}) = W(g(t), f_t, \frac{1}{2})$. If we flow $f_t$ backward, we will get a function $f_t(s)$. Let $u_t = e^{-f_t}$. Then, $u_t(s)$ satisfies the following equation

$$
\begin{align*}
\frac{\partial}{\partial s} u_t(s) &= -\Delta u_t(s) + (n - R)u_t(s) \\
u_t(t) &= e^{-f_t}
\end{align*}
$$

Then, for every $A > 0$, there exist a uniformly constant $C$ depending on $A$, such that $|u_t(s)|_{C^{2,\alpha}} \leq C$ for every $t \geq A, s \in [t - A, t]$.

Now in our case, fix $A > 0$ and assume $g_{ij} - R_{ij} = \partial_i \partial_j u$,

$$
\mu(g(t_i + A), \frac{1}{2}) - \mu(g(t_i), \frac{1}{2}) \geq W(g(t_i + A), f_{t_i + A}, \frac{1}{2}) - W(g(t_i), f_{t_i + A}(t_i), \frac{1}{2})
$$

$$
= (2\pi)^{-n} \int_0^A |\nabla \tilde{\nabla}(f_{t_i + A} - u)|^2(t_i + s)dV_{g(t_i + s)}.
$$

From (i), we have $\lim_{t \to -\infty} |\nabla \tilde{\nabla}(f_{t_i + A} - u)|(t_i + s) = 0$ for all most $x \in X$ and $s \in [0, A]$. Since curvature is bounded, $u(t_i + s) \to \bar{u}(s)$ in $C^{2,\alpha}$ norm for any $s \in [0, A]$. From (iii) we can assume $u_{t_i + A}(t_i + s) \to \bar{u}_1(s)$. So we have

$$\nabla \tilde{\nabla} (\bar{f} - \bar{u}) = 0,$$

where $\bar{u}_1 = e^{-\bar{f}}$. Then, $\bar{u}(s) = \bar{f}(s)$ since both of them satisfy the same integral condition

$$
\int_X e^{-\bar{f}} dV_s = \int_X e^{-\bar{u}} dV_s = (2\pi)^n.
$$

Since $\bar{f}, \bar{u}$ satisfy the following equations

$$
\frac{\partial}{\partial s} \bar{f}(s) = -\Delta \bar{f} + |\nabla \bar{f}|^2 - R + n,
$$

$$
\frac{\partial}{\partial s} \bar{u}(s) = \Delta \bar{u} + \bar{u} + a(s),
$$

$$\Delta \bar{u} = n - R,$$

we have

$$\Delta \bar{u} - |\nabla \bar{u}|^2 + \bar{u} = -a(s).$$

Thus $a(s) = -(2\pi)^{-n} \int_X \bar{u} e^{-\bar{u}} dV_s$. Then, we can show that $a(s)$ is a constant. Note that

$$\mu(h(t), \frac{1}{2}) = (2\pi)^{-n} \int_X (|\nabla \bar{u}|^2 - \Delta \bar{u} + \bar{u} - n)e^{-\bar{u}} dV_{g(t)} = -a(t) - n.$$

Since there exists a finite $\lim_{t \to \infty} \mu(g(t), \frac{1}{2})$, then $\mu(h(t), 1/2) = \mu(h(s), 1/2)$ for all $s, t \in [0, A]$. Therefore, $a(t)$ is a constant. Note that

$$\frac{\partial}{\partial t} \int_X (\bar{u} + a) dV_{h(t)} = -\int_X |\nabla \nabla \bar{u}|^2 dV_{h(t)}$$

and

$$\frac{\partial}{\partial t} \int_X \bar{u} dV_{h(t)} = \int_X (|\nabla \bar{u}|^2 + \bar{u} \Delta \bar{u}) dV_{h(t)} = 0.$$
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