A concavity property for the reciprocal of Fisher information and its consequences on Costa’s EPI

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Abstract. We prove that the reciprocal of Fisher information of a log-concave probability density $X$ in $\mathbb{R}^n$ is concave in $t$ with respect to the addition of a Gaussian noise $Z_t = N(0, tI_n)$. As a byproduct of this result we show that the third derivative of the entropy power of a log-concave probability density $X$ in $\mathbb{R}^n$ is nonnegative in $t$ with respect to the addition of a Gaussian noise $Z_t$. For log-concave densities this improves the well-known Costa’s concavity property of the entropy power [3].

Keywords. Entropy-power inequality, Blachman–Stam inequality, Costa’s concavity property, log-concave functions.

1 Introduction

Given a random vector $X$ in $\mathbb{R}^n$, $n \geq 1$ with density $f(x)$, let

$$H(X) = H(f) = -\int_{\mathbb{R}^n} f(x) \log f(x) \, dx$$

(1)

denote its entropy functional (or Shannon’s entropy). The entropy power introduced by Shannon [14] is defined by

$$N(X) = N(f) = \frac{1}{2\pi e} \exp \left( \frac{2}{n} H(X) \right).$$

(2)

The entropy power is built to be linear at Gaussian random vectors. Indeed, let $Z_\sigma = N(0, \sigma I_n)$ denote the $n$-dimensional Gaussian random vector having mean

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vector $0$ and covariance matrix $\sigma I_n$, where $I_n$ is the identity matrix. Then $N(Z_\sigma) = \sigma$. Shannon’s entropy power inequality (EPI), due to Shannon and Stam [13, 14] (cf. also [3, 7, 8, 12, 18, 22] for other proofs and extensions) gives a lower bound on Shannon’s entropy power of the sum of independent random variables $X, Y$ in $\mathbb{R}^n$ with densities

$$N(X + Y) \geq N(X) + N(Y),$$

with equality if and only $X$ and $Y$ are Gaussian random vectors with proportional covariance matrices.

In 1985 Costa [3] proposed a stronger version of EPI (3), valid for the case in which $Y = Z_t$, a Gaussian random vector independent of $X$. In this case

$$N(X + Z_t) \geq (1 - t)N(X) + tN(X + Z_1), \quad 0 \leq t \leq 1$$

or, equivalently, $N(X + Z_t)$, is concave in $t$, i.e.

$$\frac{d^2}{dt^2} N(X + Z_t) \leq 0.$$ (5)

Note that equality to zero in (5) holds if and only if $X$ is a Gaussian random variable, $X = N(0, \sigma I_n)$. In this case, considering that $Z_\sigma$ and $Z_t$ are independent each other, and Gaussian densities are stable under convolution, $N(Z_\sigma + Z_t) = N(Z_{\sigma+t}) = \sigma + t$, which implies

$$\frac{d^2}{dt^2} N(Z_\sigma + Z_t) = 0.$$ (6)

Let now consider, for a given random vector $X$ in $\mathbb{R}^n$ with smooth density, its Fisher information

$$I(X) = I(f) = \int_{\{f > 0\}} |\nabla f(x)|^2 f(x) \, dx.$$ (7)

Blachman–Stam inequality [1, 5, 15] gives a lower bound on the reciprocal of Fisher information of the sum of independent random vectors with (smooth) densities

$$\frac{1}{I(X + Y)} \geq \frac{1}{I(X)} + \frac{1}{I(Y)},$$

still with equality if and only $X$ and $Y$ are Gaussian random vectors with proportional covariance matrices.

In analogy with the definition of entropy power, let us introduce the (normalized) reciprocal of Fisher information

$$\tilde{I}(X) = \frac{n}{I(X)}.$$ (9)
By construction, since \( I(Z_\sigma) = n/\sigma \), \( \tilde{I}(\cdot) \) is linear at Gaussian random vectors, with \( \tilde{I}(Z_\sigma) = \sigma \). Moreover, in terms of \( \tilde{I} \), Blachman–Stam inequality reads
\[
\tilde{I}(X + Y) \geq \tilde{I}(X) + \tilde{I}(Y).
\]
Therefore, both the entropy power (12) and the reciprocal of Fisher information \( \tilde{I} \), as given by (19), share common properties when evaluated on Gaussian random vectors and on sums of independent random vectors.

By pushing further this analogy, in agreement with Costa’s result on entropy power, we will prove that the quantity \( \tilde{I}(X + Z_t) \) satisfies the analogous of inequality (4), i.e.
\[
\tilde{I}(X + Z_t) \geq (1 - t)\tilde{I}(X) + t\tilde{I}(X + Z_1), \quad 0 \leq t \leq 1
\]
or, equivalently
\[
\frac{d^2}{dt^2} \tilde{I}(X + Z_t) \leq 0.
\]
Unlike Costa’s result, the proof of (12) is restricted to log-concave random vectors. Similarly to (5), equality to zero in (12) holds if and only if \( X \) is a Gaussian random vector, \( X = N(0, \sigma I_n) \).

The estimates obtained in the proof of (12) can be fruitfully employed to study the third derivative of \( N(X + Z_t) \). The surprising result is that, at least for log-concave probability densities, the third derivative has a sign, and
\[
\frac{d^3}{dt^3} N(X + Z_t) \geq 0.
\]
Once again, equality to zero in (13) holds if and only if \( X \) is a Gaussian random variable, \( X = N(0, \sigma I_n) \). Considering that
\[
\frac{d}{dt} N(X + Z_t) \geq 0,
\]
the new inequality (13) seems to indicate that the subsequent derivatives of \( N(X + Z_t) \) alternate in sign, even if a proof of this seems prohibitive.

The concavity property of the reciprocal of Fisher information is a consequence of a recent result of the present author [19] related to the functional
\[
J(X) = J(f) = \sum_{i,j=1}^n \int_{\{f>0\}} \left[ \partial_{ij}(\log f) \right]^2 f \, dx = \\
\sum_{i,j=1}^n \int_{\{f>0\}} \left[ \frac{\partial_i f}{f} - \frac{\partial_i f \partial_j f}{f^2} \right]^2 f \, dx.
\]
We remark that, given a random vector \( X \) in \( \mathbb{R}^n \), \( n \geq 1 \), the functional \( J(X) \) is well-defined for a smooth, rapidly decaying probability density \( f(x) \) such that \( \log f \)
has growth at most polynomial at infinity. As proven by Villani in [21], \( J(X) \) is related to Fisher information by the relationship

\[
J(X + Z_t) = -\frac{d}{dt} I(X + Z_t). \tag{15}
\]

The main result in [19] is a new inequality for \( J(X + Y) \), where \( X \) and \( Y \) are independent random vectors in \( \mathbb{R}^n \), such that their probability densities \( f \) and \( g \) are log-concave, and \( J(X), J(Y) \) are well defined. For any constant \( \alpha \), with \( 0 \leq \alpha \leq 1 \), it holds

\[
J(X + Y) \leq \alpha^4 J(X) + (1 - \alpha)^4 J(Y) + 2\alpha^2(1 - \alpha)^2 H(X, Y), \tag{16}
\]

where

\[
H(X, Y) = \sum_{i,j=1}^{n} \int_{\{f > 0\}} \frac{\partial_i f \partial_j f}{f} dx \int_{\{g > 0\}} \frac{\partial_i g \partial_j g}{g} dx. \tag{17}
\]

Note that, in one-dimension \( H(f, g) = I(f)I(g) \). Inequality (16) is sharp. Indeed, there is equality if and only if \( X \) and \( Y \) are \( n \)-dimensional Gaussian vectors with covariance matrices proportional to \( \alpha I_n \) and \((1 - \alpha) I_n \) respectively.

Even if inequality (16) is restricted to the set of log-concave densities, this set includes many of the most commonly-encountered parametric families of probability density functions [11].

Inequality (16) implies a Blachman-Stam type inequality for \( \frac{1}{\sqrt{J(\cdot)}} \) [19]

\[
\frac{1}{\sqrt{J(X + Y)}} \geq \frac{1}{\sqrt{J(X)}} + \frac{1}{\sqrt{J(Y)}}, \tag{18}
\]

where, also in this case, equality holds if and only if both \( X \) and \( Y \) are Gaussian random vectors.

Inequality (18) shows that, at least if applied to log-concave probability densities, the functional \( 1/\sqrt{J(\cdot)} \) behaves with respect to convolutions like Shannon’s entropy power [14, 15] and the reciprocal of Fisher information [1, 15]. The fact that inequalities (3), (8) and (18) share a common nature is further clarified by noticing that, when evaluated in correspondence to the Gaussian vector \( Z_\sigma \),

\[
N(Z_\sigma) = \tilde{I}(Z_\sigma) = \sqrt{n/J(Z_\sigma)} = \sigma.
\]

In addition to the present results, other inequalities related to Fisher information in one-dimension have been recently obtained in [2]. In particular, the sign of the subsequent derivatives of Shannon’s entropy \( H(X + Z_t) \) up to order four have been computed explicitly. Since these derivatives alternate in sign, it is conjectured in [2] that this property has to hold for all subsequent derivatives. This is an old conjecture that goes back at least to McKean [10], who investigated derivatives of
Shannon’s entropy up to the order three. Despite the title, however, in [2] the sign of the subsequent derivatives of the entropy power \(N(X + Z_t)\) is not investigated.

The paper is organized as follows. In Section 2 we introduce the background and prove the concavity property of the reciprocal of Fisher information for log-concave densities. In Section 3 we show that the third derivative in Costa’s entropy power inequality, still when evaluated on log-concave densities, is non-negative. Last, Section 4 will be devoted to the consequences of the new results on isoperimetric inequalities for entropies.

2 The concavity property

We recall that a function \(f\) on \(\mathbb{R}^n\) is log-concave if it is of the form

\[
f(x) = \exp\{-\Phi(x)\},
\]

for some convex function \(\Phi : \mathbb{R}^n \to \mathbb{R}\). A prime example is the Gaussian density, where \(\Phi(x)\) is quadratic in \(x\). Further, log-concave distributions include Gamma distributions with shape parameter at least one, Beta\((\alpha, \beta)\) distributions with \(\alpha, \beta \geq 1\), Weibull distributions with shape parameter at least one, Gumbel, logistic and Laplace densities (see, for example, Marshall and Olkin [11]). Log-concave functions have a number of properties that are desirable for modelling. Marginal distributions, convolutions and product measures of log-concave distributions and densities are again log-concave (cf. for example, Dharmadhikari and Joag-Dev [4]).

Let \(z_\sigma(x)\), with \(x \in \mathbb{R}^n, n \geq 1\), denote the density of the Gaussian random vector \(Z_\sigma\)

\[
\begin{align*}
z_\sigma(x) &= \frac{1}{(2\pi\sigma)^{n/2}} \exp\left\{-\frac{x^2}{2\sigma}\right\}. \tag{20}
\end{align*}
\]

Assume that the random vector \(X\) has a log-concave density \(f(x), x \in \mathbb{R}^n\). Then, for any \(t > 0\), the random vector \(X + Z_t\), where the Gaussian \(Z_t\) is independent of \(X\), has a density which is the convolution of the log-concave functions \(f\) and the (log-concave) Gaussian density \(z_t\) defined in (20). Therefore, for any \(t > 0\), \(X + Z_t\) has a log-concave smooth density function. This simple remark, coupled with the results of [9], allows to justify the computations that follows.

Let us evaluate the derivatives of \(\tilde{I}(X + Z_t)\), with respect to \(t, t > 0\). Thanks to (15) we obtain

\[
\begin{align*}
\frac{d}{dt} \tilde{I}(X + Z_t) &= n \frac{J(X + Z_t)}{I^2(X + Z_t)}, \tag{21}
\end{align*}
\]

and

\[
\begin{align*}
\frac{d^2}{dt^2} \tilde{I}(X + Z_t) &= n \left(2 \frac{J^2(X + Z_t)}{I^3(X + Z_t)} - \frac{K(X + Z_t)}{I^3(X + Z_t)} \right). \tag{22}
\end{align*}
\]
In (22) we defined
\[ K(X + Z_t) = -\frac{d}{dt} I(X + Z_t). \] (23)

Hence, to prove concavity we need to show that, for log-concave densities
\[ K(X + Z_t) \geq 2 \frac{J^2(X + Z_t)}{I(X + Z_t)}. \] (24)

Note that
\[ I(Z_\sigma) = \frac{n}{\sigma}, \quad J(Z_\sigma) = \frac{n}{\sigma^2}, \quad K(Z_\sigma) = 2 \frac{n}{\sigma^3}. \] (25)

Consequently, inequality (24) is verified with the equality sign in correspondence to a Gaussian random vector.

Using the second identity in (25) into (18) it is immediate to recover a lower bound for \( K(\cdot) \). This idea goes back to Dembo [5], who made analogous use of the Blachman–Stam inequality (8) to recover the sign of the second derivative of the entropy power. Let \( \sigma, t > 0 \). By choosing \( X = W + Z_\sigma \) and \( Y = Z_t \), inequality (18) becomes
\[
\frac{1}{\sqrt{J(W + Z_\sigma + Z_t)}} \geq \frac{1}{\sqrt{J(W + Z_\sigma)}} + \frac{t}{\sqrt{n}}.
\]

Then, for all \( t > 0 \)
\[
\frac{1}{t} \left( \frac{1}{\sqrt{J(W + Z_\sigma + Z_t)}} - \frac{1}{\sqrt{J(W + Z_\sigma)}} \right) \geq \frac{1}{\sqrt{n}},
\]
and this implies, passing to the limit \( t \to 0^+ \)
\[
\frac{1}{2} \frac{K(W + Z_\sigma)}{J^{3/2}(W + Z_\sigma)} \geq \frac{1}{\sqrt{n}},
\]
for any \( \sigma > 0 \). Hence, a direct application of inequality (18) shows that \( K(X + Z_t) \) is bounded from below, and
\[
K(X + Z_t) \geq 2 \frac{J^{3/2}(X + Z_t)}{\sqrt{n}}. \] (26)

Unfortunately, inequality (26) is weaker than (24), since it is known that, for all random vectors \( X \) and \( Z_t \) independent from each other [5, 21]
\[
J(X + Z_t) \geq \frac{J^2(X + Z_t)}{n}, \] (27)
and (27) implies
\[
\frac{J^2(X + Z_t)}{I(X + Z_t)} \geq \frac{J^{3/2}(X + Z_t)}{\sqrt{n}}.
\]
To achieve the right result, we will work directly on inequality (16). Let us fix $Y = Z_t$. Then, since for $i \neq j$

$$\int_{\mathbb{R}^n} \frac{\partial_i z_t(x) \partial_j z_t(x)}{z_t(x)} \, dx = \int_{\mathbb{R}^n} \frac{x_i x_j}{t^2} z_t(x) \, dx = 0,$$

one obtains

$$H(X, Z_t) = \sum_{i=1}^{n} \int_{\{f > 0\}} \frac{f^2}{f(x)} \, dx \int_{\mathbb{R}^n} \frac{x_i^2}{t^2} z_t \, dx = I(X) \frac{1}{n} I(Z_t) = \frac{1}{t} I(X). \quad (28)$$

Hence, using (25) and (28), inequality (16) takes the form

$$J(X + Z_t) \leq \alpha^4 J(X) + (1 - \alpha)^4 \frac{n}{t^2} + 2\alpha^2 (1 - \alpha)^2 \frac{1}{t} I(X). \quad (29)$$

We observe that the function

$$\Lambda(\alpha) = \alpha^4 J(X) + (1 - \alpha)^4 \frac{n}{t^2} + 2\alpha^2 (1 - \alpha)^2 \frac{1}{t} I(X)$$

is convex in $a$, $0 \leq \alpha \leq 1$. This fact follows by evaluating the sign of $\Lambda''(\alpha)$, where

$$\frac{1}{12} \Lambda''(\alpha) = \alpha^2 J(X) + (\alpha - 1)^2 \frac{n}{t^2} + \frac{1}{3} (1 - \alpha)^2 + 4\alpha(\alpha - 1) + \alpha^2 \frac{1}{t} I(X).$$

Clearly both $\Lambda''(0)$ and $\Lambda''(1)$ are strictly bigger than zero. Hence, $\Lambda(\alpha)$ is convex if, for $r = \alpha/(1 - \alpha)$

$$r^2 J(X) + \frac{n}{t^2} + \frac{1}{3} (r^2 - 4r + 1) \frac{1}{t} I(X) \geq 0.$$

Now,

$$\left( J(X) + \frac{1}{3t} I(X) \right) r^2 - \frac{4}{3t} I(X) r + \left( \frac{n}{t^2} + \frac{1}{3t} I(X) \right) \geq 0.$$

The last inequality follows from (27).

The previous computations show that, for any given value of $t > 0$, there exists a unique point $\bar{\alpha} = \bar{\alpha}(t)$ in which the function $\Lambda(\alpha)$ attains the minimum value. In correspondence to this optimal value, inequality (29) takes the equivalent optimal form

$$J(X + Z_t) \leq \bar{\alpha}(t)^4 J(X) + (1 - \bar{\alpha}(t))^4 \frac{n}{t^2} + 2\bar{\alpha}(t)^2 (1 - \bar{\alpha}(t))^2 \frac{1}{t} I(X). \quad (30)$$
The evaluation of $\bar{\alpha}(t)$ requires to solve a third order equation. However, since we are interested in the value of the right-hand side of (30) for small values of the variable $t$, it is enough to evaluate in an exact way the value of $\bar{\alpha}(t)$ up to the order one in $t$. By substituting

$$\bar{\alpha}(t) = c_0 + c_1 t + o(t)$$

in the third order equation $\Lambda'(\alpha) = 0$, and equating the coefficients of $t$ at the orders 0 and 1 we obtain

$$c_0 = 1, \quad c_1 = -\frac{J(x)}{I(X)}.$$  \hfill (31)

Consequently, for $t \ll 1$

$$\Lambda(\bar{\alpha}(t)) = J(X) - 2\frac{J^2(X)}{I(X)} t + o(t).$$  \hfill (32)

Finally, by using expression (32) into inequality (30) we obtain

$$\frac{1}{\sqrt{J(X + Z_\sigma + Z_t)}} > \frac{1}{\sqrt{J(X + Z_\sigma) - 2\frac{J^2(X+Z_\sigma)}{I(X+Z_\sigma)}} t + o(t)} = \frac{1}{\sqrt{J(X + Z_\sigma)}} + \frac{\sqrt{J(X + Z_\sigma)}}{I(X + Z_\sigma)} t + o(t),$$

which implies, for all $\sigma > 0$, the inequality

$$\lim_{t \to 0^+} \frac{1}{t} \left( \frac{1}{\sqrt{J(X + Z_\sigma + Z_t)}} - \frac{1}{\sqrt{J(X + Z_\sigma)}} \right) \geq \frac{\sqrt{J(X + Z_\sigma)}}{I(X + Z_\sigma)}. \hfill (34)$$

At this point, inequality (24) follows from (34) simply by evaluating the derivative of

$$J(X + Z_t) = \left( \frac{1}{\sqrt{J(X + Z_t)}} \right)^{-2}.$$  

This gives

$$K(X + Z_t) = -\frac{d}{dt}J(X + Z_t) = -\frac{d}{dt} \left( \frac{1}{\sqrt{J(X + Z_t)}} \right)^{-2} = 2 J(X + Z_t)^{3/2} \frac{d}{dt} \frac{1}{\sqrt{J(X + Z_t)}} \geq 2 \frac{J^2(X + Z_t)}{I(X + Z_t)}. \hfill (35)$$

Hence we proved
Theorem 1. Let $X$ be a random vector in $\mathbb{R}^n$, $n \geq 1$, such that its probability density $f(x)$ is log-concave. Then the reciprocal of the Fisher information of $X + Z_t$, where $X$ and $Z_t$ are independent each other, is concave in $t$, i.e.

$$\frac{d^2}{dt^2} \frac{1}{I(X + Z_t)} \leq 0.$$ 

3 An improvement of Costa’s EPI

The computations of Section 2 can be fruitfully used to improve Costa’s result on concavity of the entropy power $N(X + Z_t)$. To this aim, let us compute the derivatives in $t$ of $N(X + Z_t)$, up to the third order. The first derivative can be easily evaluated by resorting to de Bruijn identity

$$\frac{d}{dt} H(X + Z_t) = \frac{1}{2} I(X + Z_t). \quad (36)$$

Then, identities (15) and (23) can be applied to compute the subsequent ones. By setting $X + Z_t = W_t$ one obtains

$$\frac{d}{dt} N(W_t) = \frac{1}{n} N(W_t) I(W_t), \quad (37)$$

and, respectively

$$\frac{d^2}{dt^2} N(W_t) = \frac{1}{n} N(W_t) \left( \frac{I(W_t)^2}{n} - J(W_t) \right), \quad (38)$$

and

$$\frac{d^3}{dt^3} N(W_t) = \frac{1}{n} N(W_t) \left( K(W_t) + \frac{I(W_t)^3}{n^2} - 3 \frac{I(W_t) J(W_t)}{n} \right). \quad (39)$$

Note that, by virtue of identities (27) and (25), the right-hand sides of both (38) and (39) vanishes if $W_t$ is a Gaussian random vector. Using inequality (24) we get

$$K(W_t) + \frac{I(W_t)^3}{n^2} - 3 \frac{I(W_t) J(W_t)}{n} \geq 2 \frac{J^2(W_t)}{I(W_t)} + \frac{I(W_t)^3}{n^2} - 3 \frac{I(W_t) J(W_t)}{n}.$$ 

Thus, by setting $p = nJ(W_t)/I^2(W_t)$, the sign of the expression on the right-hand side of (39) will coincide with the sign of the expression

$$2p^2 - 3p + 1. \quad (40)$$

Since $p \geq 1$ in view of the inequality (27) [31, 21], $2p^2 - 3p + 1 \geq 0$, and the result follows. Last, the cases of equality coincide with the cases in which there is equality both in (23) and (27), namely if and only if $W_t$ is a Gaussian random vector.

We proved
Theorem 2. Let $X$ be a random vector in $\mathbb{R}^n$, $n \geq 1$, such that its probability density $f(x)$ is log-concave. Then the entropy power of $X + Z_t$, where $X$ and $Z_t$ are independent each other, has the derivatives which alternate in sign up to the order three. In particular $N(X + Z_t)$ is concave in $t$, and
\[ \frac{d^3}{dt^3} N(X + Z_t) \geq 0. \]

4 Isoperimetric inequalities

An interesting consequence of the concavity property of entropy power is the so-called isoperimetric inequality for entropies \[5, 6, 16\]
\[ \frac{1}{n} N(X) I(X) \geq 1. \tag{41} \]
which is easily obtained from (37). Note that the quantity which is bounded from below coincides with the derivative of the entropy power of $X + Z_t$, evaluated as time $t \to 0$. Indeed, the concavity of $N(X + Z_t)$ implies the non-increasing property of the right-hand side of (37) with respect to $t$, which, coupled with the scaling invariance of the product $N(f) I(f)$ with respect to dilation allows to identify the lower bound \[16\].

Among others, the concavity property makes evident the connection between the solution to the heat equation and inequalities \[16, 17, 18\]. Resorting to this connection, the concavity property of entropy power has been recently shown to hold also for Renyi entropies \[13, 20\].

Likewise, the concavity result of Theorem 1 allows to recover an isoperimetric inequality for Fisher information, that reads
\[ n \frac{J(X)}{I^2(X)} \geq 1. \tag{42} \]
As in \[11\], in (42) the quantity which is bounded from below coincides with the derivative of the reciprocal of the Fisher information of $X + Z_t$, evaluated as time $t \to 0$. Inequality (42), which follows from (27) by letting $t \to 0$, has been obtained as a byproduct of Costa’s entropy power inequality \[3, 5, 21\], and holds for all random vectors with a suitably smooth probability density. The novelty here is that, in the case in which the probability density of $X$ is log-concave, the quantity $nJ(X + Z_t)/I^2(X + Z_t)$ is non-increasing in time.

However, a new inequality is obtained for random vectors with log-concave densities. In this case in fact, by taking the limit $t \to 0$ in (24) we get
\[ K(X) \geq 2 \frac{J^2(X)}{I(X)}. \tag{43} \]
In one-dimension, the expression of $K(X + Z_t)$ has been evaluated in [10]. Then, a more convenient expression which allows to recover its positivity has been obtained recently in [2]. In general, written as

$$J^2(X) \leq \frac{1}{2}I(X)K(X)$$

inequality [13] provides a sharp upper bound on $J^2(X)$, which is an expression containing second derivatives of the logarithm of the log-concave density $f(x)$, in terms of the product of the Fisher information, which depends on first derivatives of the logarithm, and $K(X)$, which depends on derivatives of the logarithm up to order three. Unfortunately, these expressions are heavy and difficult to handle.

5 Conclusions

As recently shown in [19], log-concave densities exhibit a number of interesting properties. In this paper, we proved two new properties, both related to the behavior in time of various functionals evaluated on the sum of a random vector $X$ with a log-concave probability density with an independent Gaussian vector $Z_t = N(0, tI_n)$. First, we proved that the reciprocal of Fisher information of $X + N_t$ is concave in $t$, thus extending Costa’s result on Shannon’s entropy power to Blachman–Stam inequality. Second, we showed that Shannon’s entropy power of $X + Z_t$ has derivatives in $t$ that alternate in sign up to order three. In addition to Costa’s concavity property of entropy power, which concerns the sign of the second-order derivative, it has been here discovered that also the third derivative of Shannon’s entropy power has a (universal) sign.

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