THEORETICAL PEARLS

Blind graph rewriting systems

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Abstract

We consider a simple (probably, the simplest) structure for random access memory. This structure can be used to construct a universal system with nearly void processor, namely, we demonstrate that the processor of such a system may have empty instruction set, in a more strong manner than the existing ZISC (zero instruction set computer based on ideas for artificial neural networks) and NISC architecture (no instruction set computing). More precisely, the processor will be forbidden to analyze any information stored in the memory, the latter being the only state of such a machine. This particular paper is to cover an isolated aspect of the idea, specifically, to provide the logical operations embedded into a system without any built-in conditional statements.

1 Graph rewriting systems

Graph rewriting systems appeared to be essential when Wadsworth used sharing of \( \lambda \)-expressions, practically inventing what is nowadays called lazy evaluation (Wadsworth, 1971). One particular branch of further developments based on this idea, including "call-by-need" evaluation strategy for functional programming languages, was focused on optimal reduction, that is a reduction mechanism that uses optimal sharing to minimize the reduction steps to achieve normal form if any. Asymptotically by complexity of \( \lambda \)-expressions, optimal reduction is the best possible evaluation technique.

The first algorithm for optimal reduction was that by Lamping who formulated his results in a very special form of graph rewriting system (Lamping, 1990). Specifically, his system had such properties as strong confluence and locality, the latter being useful for pattern matching and tracking of redexes.

The idea behind graph rewriting systems similar to that by Lamping was caught by Lafont who generalized and described them as interaction systems (Lafont, 1990). The latter consist of a signature, that is, a set of agents that constitute interaction nets and rules for interaction between agents. Lafont introduced a simple language for interaction systems, and the Lamping algorithm can be defined using this language as well:

\[
\Sigma = \{ (\varepsilon, 0) \} \cup \{ (\triangle_i, 2) \mid i \in \mathbb{Z} \} \cup \{ (\cap_i, 1) \mid i \in \mathbb{Z} \} \cup \{ (\cap_i, 1) \mid i \in \mathbb{Z} \};
\]

\[
\forall (\alpha, i) \in \Sigma : \quad \varepsilon \vartriangleright \alpha[\varepsilon, \ldots, \varepsilon];
\]

\[
\forall i \in \mathbb{Z} : \quad \triangle_i[a, b] \vartriangleleft \triangle_i[a, b] \land \cap_i[a] \vartriangleleft \cap_i[a] \land \cap_i[a] \vartriangleleft \cap_i[a];
\]

\[
\forall i, j \in \mathbb{Z} : \quad \left( i \neq j \Rightarrow \triangle_i[\triangle_j(a, b), \triangle_j(c, d)] \vartriangleleft \triangle_j[\triangle_i(a, c), \triangle_i(b, d)]; \right)
\]
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∀ i, j ∈ Z : i ≠ j ⇒ \( \nabla_i [\triangle_{j+1}(a,b)] \bowtie \triangle_j [\nabla_i(a), \nabla_i(b)] \);
∀ i, j ∈ Z : i ≠ j ⇒ \( \nabla_i [\triangle_{j-1}(a,b)] \bowtie \triangle_j [\nabla_i(a), \nabla_i(b)] \).

Taking into account that some interaction systems are equivalent in the sense that one can simulate another, and for any graph rewriting system, there can be constructed an equivalent interaction system, Lafont considers the simplest interaction systems, and eventually finds an extremely simple universal interaction system of interaction combinators (Lafont, 1997):

\[
\Sigma = \{(\delta, 2), (\gamma, 2), (\varepsilon, 0)\};
\delta[x, y] \bowtie \delta[x, y], \quad \gamma[\delta(a, b), \delta(c, d)] \bowtie \delta[\gamma(a, c), \gamma(b, d)], \quad \gamma[x, y] \bowtie \gamma[y, x];
\varepsilon \bowtie \delta[\varepsilon, \varepsilon], \quad \varepsilon \bowtie \varepsilon, \quad \varepsilon \bowtie \gamma[\varepsilon, \varepsilon].
\]

Bechet went even farther, and managed to find a universal interaction system with only two agents (Bechet, 2001), thus even simpler than Lafont’s one. However, the price was that the rules had to be too complicated. He also stated a question which, as far as we know, still remains open: is it possible to find a universal system with two agents (Bechet, 2001), thus even simpler than Lafont’s one. However, the price was

There is obviously no way to simplify this signature, however, we can get back to graph rewriting systems, focusing on their implementations using random access memory heap, and try to apply methods similar to those leading to interaction combinators.

The rest of this paper is an attempt to find the simplest automata able to implement arbitrary graph rewriting system. As we will see, there exist automata with static transition function which in some sense does not rely on the current state. We believe this kind of evaluation could result in interesting forms of computation, for instance, based on RAM with CPU that performs one and the same chain of move instructions.

2 Primitive graph operations

In order to construct the simplest automata, we first define a set of their possible states as

\[
S(M) = \{(f_0, f_1) | f_0, f_1 : M \rightarrow M\};
\]

where \( M \) is a finite set. Each element of \( S(M) \) can be considered as a finite directed graph with the set of nodes \( M \), which has exactly two arrows from each node, the arrows being labeled 0 and 1. Let us take a look at the simplest operations on those graphs:

\[
e[b_0 b_1 \ldots b_n := a_1 \ldots a_m] : S(M) \rightarrow S(M),
\]

where \( e \in M \) and \( \forall i : a_i, b_i \in \{0, 1\} \). We will require

\[
e[b_0 b_1 \ldots b_n := a_1 \ldots a_m](f_0, f_1) = (g_0, g_1)
\]

to have certain properties. Namely, if

\[
a = f_{a_1} (\ldots f_{a_n}(e) \ldots), \quad b = f_{b_1} (\ldots f_{b_n}(e) \ldots),
\]

\( a \) must be equal to \( g_{b_0}(b) \), and \( b \) must be the only point where \( (g_0, g_1) \) differs from \( (f_0, f_1) \):

\[
a = g_{b_0}(b);
\]

\[
i \neq b_0 \Rightarrow \forall x \in M : g_i(x) = f_i(x);
\]

\[
\forall x \in M : x \neq b \Rightarrow g_{b_0}(x) = f_{b_0}(x).
\]
We will take the liberty to illustrate these primitive graph operations by its implementa-
tion in the C programming language:

```c
struct node {
    struct node *left, *right;
} state[MEMSIZE];

void op(struct node *element)
{
    element->left->right = element->right->left->left;
}
```

Here, if every structure’s fields all point to nodes in the array itself, the state corresponds
to an element of $S(M)$, $|M|$ being equal to the array size. Then, calling the function basically maps the array from one state to another, so it directly implements $e[01 := 100]$, $e$ corresponding to the function’s argument.

### 3 Embedding logical operations

One can notice that the graphs of the introduced type are similar to the internal representation of S-expressions, or lists, in the LISP programming language. The only difference is that our graphs do not contain any atoms, or data, thus being a pure recursive version of S-expressions.

An arbitrary graph rewriting rule defined for the elements of $S(M)$ will result in a graph rewriting system with only one type of nodes, each one having exactly two outgoing arrows. The latter makes the introduced structure relevant to the open question by Bechet mentioned in the beginning of this paper, once we find a composition

$$T = e[b_0b_1 \ldots b_n := a_1 \ldots a_m] \circ \ldots \circ e[y_0y_1 \ldots y_q := x_1 \ldots x_p]$$

that makes our system universal as its only graph rewriting rule.

But first, let us have at least conditional statements embedded. We may use the same approach conditional statements $BMN$ are implemented in $\lambda$-calculus: for true $B = \lambda x.\lambda y.x$, the expression is $M$; for false $B = \lambda x.\lambda y.y$, the expression is $N$. So, let $M$ be embedded into some subgraph starting at $m = f_0(f_0(e))$, $N$—at $n = f_1(f_0(e))$, and a boolean value—at $b = f_1(e)$ so that $f_1(f_0(b)) = f_0(b)$ for true and $f_1(f_0(b)) = f_1(b)$ otherwise. Then, a composition of primitive graph operations applied to this graph results in a graph

$$(g_0,g_1) = (e[001 := 00] \circ e[011 := 10])(f_0,f_1)$$

that has $m$ or $n$ at $g_0(g_1(g_0(g_1(e))))$ for true and false, respectively.

Using the similar methods, the reader can easily produce logical “and,” logical “or,” negation and so on, producing embedded version of nearly every computable function. Assuming $M$ to be unbounded, one can also prove Turing-completeness of such a system, by simulating any other universal system, like interaction combinators, or a universal Turing machine.

Finally, let us notice that for any $e[b_0b_1 \ldots b_n := a_1 \ldots a_m]$, the graph $(f_0,f_1)$ is a fixed point if and only if $f_{b_0}(f_{b_1}( \ldots f_{b_n}(e)(\ldots))) = f_{a_1}( \ldots f_{a_m}(e)(\ldots))$. This also provides an obvious way to construct fixed points for any composition $T$ of primitive graph operations. And
since a finite-state machine with the set of states $S(M)$ and transition function $T$ practically stops at the first fixed point reached from the initial state, this way we are able to simulate a system halt.

4 Possible applications

Let encoding $c : \Lambda \leftrightarrow S(M)$ for $\lambda$-expressions be compatible with $T$ in the sense that

$$\exists n \in \mathbb{N} : c^{-1}(T^n(c(P))) = Q \land T(c(Q)) = c(Q),$$

where $P$ is a $\lambda$-expression, and $Q$ is its normal form. Once the minimal value of $n$ is proportional to the number of steps on optimal reduction from $P$ to $Q$ asymptotically by the complexity of $P$, transition $T$ makes the corresponding finite-state machine practically the simplest implementation of optimal reduction. Currently, our primary goal is to answer the question whether there exists such a composition with compatible encoding.

But although we mainly pursue the simplest automata that would implement optimal reduction of $\lambda$-expressions, the “blind” rewriting systems considered in this paper might have some other possibly useful applications as well.

With respect to composition, the primitive graph operations defined above obviously generate a quite simple but still unusual algebraic structure which, as far as we know, does not directly correspond to any well-known mathematical structure. If this is the case, one could probably be interested in analyzing the generated structure. Otherwise, an attempt to find such a correspondence might deserve a separate research.

We believe that the ideas described in this paper could be interesting from the view point of actual implementing graph reduction for functional programming languages. Moreover, simplicity and low-level transparency of the structures discussed above might possibly make these ideas a fruitful direction for computer hardware significantly differing from the existing.

One can also consider input/output techniques for such a system, for instance, using some ideas behind TTA, transport triggered architecture. Namely, output can be implemented by introducing side effects of accessing particular nodes within its memory. In turn, input may be done by transition against $i \neq e$ initiated from outside.

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