A Natural Dynamics for Bargaining on Exchange Networks

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Abstract

Bargaining networks model the behavior of a set of players that need to reach pairwise agreements for making profits. Nash bargaining solutions are special outcomes of such games that are both stable and balanced. Kleinberg and Tardos proved a sharp algorithmic characterization of such outcomes, but left open the problem of how the actual bargaining process converges to them. A partial answer was provided by Azar et al. who proposed a distributed algorithm for constructing Nash bargaining solutions, but without polynomial bounds on its convergence rate. In this paper, we introduce a simple and natural model for this process, and study its convergence rate to Nash bargaining solutions.

At each time step, each player proposes a deal to each of her neighbors. The proposal consists of a share of the potential profit in case of agreement. The share is chosen to be balanced in Nash’s sense as far as this is feasible (with respect to the current best alternatives for both players). We prove that, whenever the Nash bargaining solution is unique (and satisfies a positive gap condition) this dynamics converges to it in polynomial time.

Our analysis is based on an approximate decoupling phenomenon between the dynamics on different substructures of the network. This approach may be of general interest for the analysis of local algorithms on networks.
1 Introduction and main results

Exchange networks model social and economic relations among individuals under the premise that any relationship has a potential value for its partners. In a purely economic setting, one can imagine that each relation corresponds to a trading opportunity, and its value is the amount of money to be earned from the trade. A fascinating question in this context is that of how network structure influences the power balance between nodes (i.e., their earnings).

Controlled experiments [Wil99, LY+01, SW93] have been carried out by sociologists in a set-up that can be summarized as follows. A graph $G = (V,E)$ is defined, with positive weights $w_{ij} > 0$ associated to the edges $(i,j) \in E$. A player sits at each node of this network, and two players connected by edge $(i,j)$ can share a profit of $w_{ij}$ dollars if they agree to trade with each other. Each player can trade with at most one of her neighbors (this is called the 1-exchange rule), so that a set of valid trading pairs forms a matching $M$ in the graph $G$. It is often the case that players are provided information only about their immediate neighbors.

Network exchange theory studies the possible outcomes of such a process. While each instance admits a multitude of outcomes, special classes of outcomes are selected on the basis of ‘desirable’ properties. In this paper, we focus on ‘balanced outcomes’, a solution concept that dates back to Nash’s bargaining theory [Nas50], and was generalized in [Ro84, CY92, KT08].

A balanced outcome, or Nash bargaining (NB) solution, is a pair $(M,\gamma)$ where $M \subseteq E$ is a matching of $G$, and $\gamma = \{\gamma_i : i \in V\}$ is the vector of players’ profits. Clearly $\gamma_i \geq 0$, and $(i,j) \in M$ implies $\gamma_i + \gamma_j = w_{ij}$. Denote by $\partial_i$ the set of neighbors of node $i$ in $G$. The pair $(M,\gamma)$ is a NB solution if it satisfies the following requirements.

**Stability.** If player $i$ is trading with $j$, then she cannot earn more by simply changing her trading partner. Formally $\gamma_i + \gamma_j \geq w_{ij}$ for all $(i,j) \in E \setminus M$.

**Balance.** If player $i$ is trading with $j$, then the surplus of $i$ over his best alternative must be equal to the surplus of $j$ over his best alternative. Mathematically,

$$\gamma_i - \max_{k \in \partial_i \setminus j} (w_{ik} - \gamma_k) = \gamma_j - \max_{l \in \partial_j \setminus i} (w_{jl} - \gamma_l)$$  \hspace{1cm} (1.1)

for all $(i,j) \in M$.

It turns out that the interplay between the 1-exchange rule and the stability and balance conditions results in highly non-trivial predictions regarding the influence of network structure on individual earnings. Some of these predictions agree with experimental findings, but alternative predictive frameworks exist as well [SW93].

1.1 A natural dynamics

It is a fundamental open question whether NB solutions describe the outcomes of actual bargaining processes. The stream of controlled experiments on small networks will surely help to get an answer [Wil99]. On the other hand, an important step forward was achieved by Kleinberg and Tardos [KT08] who proved that NB solutions can be constructed in polynomial time.

However, even a superficial look at experimental conditions reveals that players cannot possibly run the algorithm described in [KT08]. There are two possibilities: Either there exists a realistic model for the bargaining dynamics that converges to NB solutions, or the solution concept has to be revised. For the former possibility, the underlying dynamics should satisfy the following requirements:

1. It should converge to NB solutions in polynomial time;
2. It should be *natural*.

While the first requirement is easy to define and motivate, the second one is more subtle but not less important. A few properties of a natural dynamics are the following ones: It should be...
local, i.e. involve limited information exchange along edges and processing at nodes; It should be time invariant, i.e. the players’ behavior should be the same/similar on identical local information at different times; It should be interpretable, i.e. the information exchanged along the edges should have a meaning for the players involved, and should be consistent with reasonable behavior for players.

In the model we propose, at each time $t$, each player sends a message to each of her neighbors. The message has the meaning of ‘best current alternative’. We denote the message from player $i$ to player $j$ by $\alpha_{i\rightarrow j}^t$. Player $i$ is telling player $j$ that she (player $i$) can currently earn $\alpha_{i\rightarrow j}^t$ elsewhere, if she chooses not to trade with $j$.

The vector of all such messages is denoted by $\alpha^t \in \mathbb{R}^{2|E|}$. Each agent $i$ makes an ‘offer’ to each of her neighbors, based on her own ‘best alternative’ and that of her neighbor. The offer from node $i$ to $j$ is denoted by $m_{i\rightarrow j}^t$ and computed according to

$$m_{i\rightarrow j}^t = (w_{ij} - \alpha_{i\rightarrow j}^t)_+ - \frac{1}{2}(w_{ij} - \alpha_{i\rightarrow j}^t - \alpha_{j\rightarrow i}^t)_+.$$  \hspace{1cm} (1.2)

It is easy to realize that this definition corresponds to the following policy: (i) An offer is always non-negative, and a positive offer is never larger than $w_{ij} - \alpha_{i\rightarrow j}^t$ (no player is interested in earning less than what is currently being offered); (ii) Subject to the above constraints, the surplus $(w_{ij} - \alpha_{i\rightarrow j}^t - \alpha_{j\rightarrow i}^t)$ (if non-negative) is shared equally. We denote by $\overline{m}^t \in \mathbb{R}^{2|E|}$ the vector of offers.

Notice that $\overline{m}$ is just a deterministic coordinate-by-coordinate function of $\alpha^t$. In the rest of the paper we shall describe the network status uniquely through the latter vector, and use $\overline{m}^t|\alpha^t$ to avoid ambiguity when required. Each node can estimate its potential earning based on the network status, using

$$\gamma_i^t \equiv \max_{k \in \partial i} m_{k\rightarrow i}^t, \hspace{1cm} (1.3)$$

the corresponding vector being denoted by $\gamma^t \in \mathbb{R}^{2|V|}$.

Messages are updated synchronously through the network, according to the rule

$$\alpha_{i\rightarrow j}^{t+1} = \kappa \max_{k \in \partial i \cup j} m_{k\rightarrow i}^t + (1 - \kappa) \alpha_{i\rightarrow j}^t. \hspace{1cm} (1.4)$$

Here $\kappa \in (0,1)$ is a ‘damping’ factor: $(1 - \kappa)$ can be thought of as the inertia on the part of the nodes to update their outgoing messages. The use of $\kappa < 1$ eliminates pathological behaviors related to synchronous updates (such as oscillations on even-length cycles). We expect that the use of asynchronous updates also eliminates such problems.

Throughout the paper we let $W \equiv \max_{(i,j) \in E} w_{ij}$. It is easy to see that this implies $\alpha^t \in [0, W]^{2|E|}$, $\overline{m}^t \in [0, W]^{2|E|}$ and $\gamma^t \in [0, W]^{|V|}$ at all times (unless the initial condition violates these bounds). Thus we call $\alpha$ a ‘valid’ message vector if $\alpha \in [0, W]^{2|E|}$.

### 1.2 Main results: Fixed point properties and convergence

Our first result is that fixed points of the update equations (1.2), (1.4) (hereafter referred to as ‘natural dynamics’) are indeed in correspondence with Nash bargaining solutions when such solutions exist. Recall the LP relaxation to the maximum weight matching problem

$$\text{maximize} \quad \sum_{(i,j) \in E} w_{ij} x_{ij},$$

subject to \hspace{1cm} \begin{align*}
\sum_{j \in \partial i} x_{ij} &\leq 1 \quad \forall i \in V, \\
x_{ij} &\geq 0 \quad \forall (i,j) \in E
\end{align*}$$

(1.5)
Theorem 1. Let $G$ be an instance for which the LP (1.5) admits a unique optimum, and this is integer. Let $(\alpha, m, \gamma)$ be a fixed point of the natural dynamics. Then $\gamma$ is the allocation of a Nash bargaining solution (i.e. there exists a matching $M$, such that the pair $(M, \gamma)$ is stable and balanced). Conversely, every Nash bargaining solution $(M, \gamma_{NB})$, corresponds to a unique fixed point of the natural dynamics with $\gamma = \gamma_{NB}$.

The natural dynamics appear to converge rapidly to a fixed point on all the cases we studied. In particular, we will prove this to be the case whenever the NB solution is unique, and under the assumption of a positive gap $\sigma > 0$. The definition of ‘gap’ is somewhat technical and is deferred to Section 1.4.1. There we will also argue that the conditions is generic. Further, thanks to [KT08], it can be checked efficiently. In the rest of the paper we use $C$ to represent any constant that is independent of the instance $G$, and $n, \sigma$ in particular.

Theorem 2. Let $G$ be an instance having unique Nash bargaining solution with gap $\sigma > 0$, and let $\gamma_{NB}$ denote the corresponding allocation. Then there exists $T^*(n, \sigma, \epsilon) = C n^7 \left( \frac{W}{\sigma} + \log \left( 1 + \frac{\sigma}{\epsilon} \right) \right)$, such that, for any initial condition with $\alpha^0 \in [0, W]^{2|E|}$, and any $t \geq T^*$, the natural dynamics yields earning estimates $\gamma^t$, with $|\gamma^t_i - \gamma_{NB,i}| \leq \epsilon$ for all $i \in V$.

It is worth stressing that the assumption of unique NB solution seems to be a weakness of our proof technique, rather than a necessary condition for fast convergence. In Section 3 we show indeed that the natural dynamics always converges on bipartite graphs if run from extremal initial conditions. On the other hand, the class of instances with unique NB solution already includes a large class of cases.

The previous theorem provides a convergence guarantee for the earnings $\gamma^t$ that players expect. Our last result shows the correct pairing among players also emerges from the dynamics.

Theorem 3. Under the hypotheses of Theorem 2 assume $t \geq T^*(n, \sigma, \sigma/3)$. Then, for each node $i \in V$ receiving non-zero offers, there exists a unique neighbor $P(i) \in \partial i$ such that the offer $m^i_{P(i) \rightarrow i}$ from $P(i)$ to $i$ is strictly larger than the offers $m^i_{l \rightarrow i}$ from other nodes $l \in \partial i \setminus P(i)$.

Further $j = P(i)$ if and only if $i = P(j)$. The pairs $(i, P(i))$ thus defined coincide with the ones in the unique Nash bargaining solution.

This theorem follows immediately from the proof of Theorem 2 as we will prove there that, for $t \geq T^*(n, \sigma, \epsilon)$, $|\alpha^t_{i,j} - \alpha_{i,j}| \leq \epsilon$, where $\alpha$ is the unique fixed point. For $\epsilon \leq \sigma/3$, this is sufficient to unambiguously determine pairings.

1.3 Related work

Following [Ro84, CY92], Kleinberg and Tardos [KT08] first considered balanced outcomes on general exchange networks and proved that a network $G$ admits a balanced outcome if and only if it admits a stable outcome. Further, the latter happens if and only if a linear programming relaxation of the maximum weight matching problem on $G$ admits an integral optimum.

The same paper describes a polynomial algorithm for constructing balanced outcomes. This is in turn based on the dynamic programming algorithm of Aspvall and Shiloach [AS] for solving systems of linear inequalities. Our convergence proof exploits the structural decomposition of the network that is produced by this algorithm.

Alternative solution concepts for bargaining on networks were studied in [CKK09].
Azar and co-authors [AB+09] first studied the question as to whether a balanced outcome can be produced by a local dynamics, and were able to answer positively. Their results left however two outstanding challenges: (I) The bound on the convergence time proved in [AB+09] is exponential in the network size, and therefore does not provide a solid justification for convergence to NB solutions in large networks; (II) The algorithm analyzed by these authors first selects a matching $M$ in $G$ [BSS05], corresponding to the pairing of players that trade. In a second phase the algorithm determines the profit of each player. While such an algorithm can be implemented in a distributed way, Azar et al. point out that it is not entirely realistic. Indeed the rules of the dynamics change after the matching is found. Further, if the pairing is established at the outset, the players lose their bargaining power.

The present paper aims at tackling these challenges.

### 1.4 Outline of the proof

We next describe the main steps in the proof of our convergence result, Theorem 2. This is based on the following strategy:

1. Prove that, under the positive gap condition, the natural dynamics on different substructures of the networks approximately decouples.

2. Analyze the dynamics on each structure by comparison with an appropriate random walk process.

An approach based on these two steps might be applicable to the analysis of a wide class of local algorithms on networks.

Step 1 above requires recalling the construction in [KT08], which we do next. Section 1.4.2 describes the main steps of the proof. The fixed point properties of the natural dynamics are summarized in Section 2, which outlines the proof of Theorem 1. A simple argument for convergence on bipartite graph is provided in Section 3, while the much more challenging case of general graphs is contained in Section 4 and the appendices.

#### 1.4.1 The KT construction and the gap of a solution

Let $G$ be an instance which admits at least one stable outcome, $M^*$ be the corresponding matching (recall that this is a maximum weight matching), and consider the Kleinberg-Tardos (KT) procedure for finding a NB solution [KT08]. Any NB solution $\gamma^*$ can be constructed by this procedure with appropriate choices at successive stages. At each stage, a linear program is solved with variables $\gamma_i$ attached to node $i$. The linear program maximizes the minimum ‘slack’ of all unmatched edges and nodes, whose values have not yet been set (the slack of edge $(i,j) \notin M$ is $\gamma_i + \gamma_j - w_{ij}$).

At the first stage, the set of nodes that remain unmatched (i.e. are not part of $M^*$) is found, if such nodes exist. Call the set of unmatched nodes $C_0$. After this, at successive stages of the KT procedure, a sequence of structures $C_1, C_2, \ldots, C_k$ characterizing the LP optimum are found. We call this the KT sequence. Each such structure is a pair $C_q = (V(C_q), E(C_q))$ with $V(C_q) \subseteq V$, $E(C_q) \subseteq E$. According to [KT08] $C_q$ belongs to one of four topologies: alternating path, blossom, bicycle, alternating cycle (Figure 1). The $q$-th linear program determines the value of $\gamma^*_i$ for $i \in V(C_q)$. Further, one has the

![Figure 1: Examples of basic structures: path, blossom, bicycle, and cycle (matched edges in bold).](image)
partition $E(C_q) = E_1(C_q) \cup E_2(C_q)$ with $E_1(C_q)$ consisting of all matching edges along which nodes in $V(C_q)$ trade, and $E_2(C_q)$ consists of edges $(i, j)$ such that some $i \in V(C_q)$ receives its second-best, positive offer from $j$.

The $\gamma$ values for nodes on the limiting structure are uniquely determined if the structure is an alternating path, blossom or bicycle\footnote{In [KT08] it is claimed that the $\gamma$ values ‘may not be fully determined’ also in the case of bicycles. However it is not hard to prove that this is not the case.}. In case of an alternating cycle there is one degree of freedom – setting a value $\gamma_i^*$ for one node $i \in C_q$ fully determines the values at the other nodes.

We emphasize that, within the present definition, $C_q$ is not necessarily a subgraph of $G$, in that it might contain an edge $(i, j)$ but not both its endpoints. On the other hand, $V(C_q)$ is always subset of the endpoints of $E(C_q)$. We denote by $V_{ext}(C_q) \supseteq V(C_q)$ the set of nodes formed by all the endpoints of edges in $E(C_q)$.

For all nodes $i \in V(C_q)$ the second best offer is equal to $\gamma_i^* - \sigma_q$, where $\sigma_q$ is the slack of $C_q$. Therefore

$$
\gamma_i^* + \gamma_j^* - w_{ij} = \begin{cases} 
0 & \text{if } (i, j) \in E_1(C_q), \\
\sigma_q & \text{if } (i, j) \in E_2(C_q)
\end{cases} \quad (1.7)
$$

The slacks form an increasing sequence ($\sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_k$). We say that $\alpha^*$ has a gap $\sigma$ if

$$
\sigma \leq \min \{ \sigma_1; \sigma_2 - \sigma_1; \ldots; \sigma_k - \sigma_{k-1} \}, \quad (1.8)
$$

and if for each edge $(i, j)$ such that $i, j \in V_{ext}(C_q)$ and $(i, j) \notin E(C_q)$,

$$
\gamma_i^* + \gamma_j^* - w_{ij} \geq \sigma_q + \sigma. \quad (1.9)
$$

It is possible to prove that the positive gap condition is generic in the following sense. The set of all instances such that the NB solution is unique can be regarded as a subset $G \subseteq [0, W]^{|E|}$ ($W$ being the maximum edge weight). It turns out that $G$ has dimension $|E|$ (i.e. the class of instances having unique NB solution is large) and that the subset of instances with gap $\sigma > 0$ is both open and dense in $G$.

1.4.2 Convergence

We begin with a basic property. Throughout the paper, for a vector $x$, we let $||x||_\infty \equiv \max_i |x_i|$.

**Lemma 1.** The natural dynamics never leads to an expansion in the sup-norm. More precisely, for any two initial message vectors $\alpha^0$ and $\beta^0$, we have

$$
||\alpha^1 - \beta^1||_\infty \leq ||\alpha^0 - \beta^0||_\infty. \quad (1.10)
$$

The proof of this fact is elementary and deferred to Appendix [B.1]. In view of this lemma, it is natural to consider the unique (by assumption) fixed point $\alpha^*$ and consider the cost function $U_G(\alpha) = ||\alpha - \alpha^*||_\infty$.

In order to prove Theorem \footnote{In [KT08] it is claimed that the $\gamma$ values ‘may not be fully determined’ also in the case of bicycles. However it is not hard to prove that this is not the case.} it is sufficient to show that this cost decreases by a non-negligible amount in a polynomial number of iterations. In particular, it is sufficient to prove the following.

**Theorem 4.** Let $G$ be an instance having unique solution $\alpha^*$ with gap $\sigma$ and consider an initial condition $\alpha^0$ with $U_G(\alpha^0) \leq \Delta$. Then there exists $C < \infty$ such that for all $t \geq C n^7$,

$$
U_G(\alpha^t) \leq \Delta - \min(\sigma, \Delta)/2. \quad (1.11)
$$
Theorem 4 is proved by analyzing sequentially each separate structure in the KT sequence. The key remark is that such a separate analysis is possible because the convergence of each structure decouples from all subsequent structures on a scale smaller than $\sigma$.

Given a subgraph $F = (V_F, E_F) \subseteq G$, we let
\[
U_{G,F}(\alpha) = \max_{(i,j) \in E_F} |\alpha_{i,j} - \alpha^*_{i,j}|. \tag{1.12}
\]
Note that $U_{G,F}(\alpha)$ depends on $F$ only through its edge set $E_F$. Thus, there will be no ambiguity in the notation $U_{G,C}(\alpha)$. We let $G$ be the directed multi-graph with the same vertex set as $G$ and all edges obtained by directing the edges in $G$ (hence for each $(i,j)$ in $G$ we have two directed edges $i \to j$ and $j \to i$ in $G$). With an abuse of notation, we shall write $U_{G,F}(\alpha)$ also when $F$ is a directed subgraph of $G$. We denote by $G_{q,t} \in \{1,2,\ldots,k+1\}$ the directed graph including all directed edges between vertices in $V(C_0) \cup V(C_1) \cup \cdots \cup V(C_q)$ and all directed edges with the first endpoint in this set of vertices. We also let $G_0$ be the empty graph.

**Definition 1.** For $\Delta > 0$, $\delta \leq \min(\Delta, \sigma)$, we define $B_{\Delta}(q, \delta, t)$ to be the condition
\[
U_{G,G_q}(\alpha^t) \leq \Delta - \delta, \quad U_{G}(\alpha^t) \leq \Delta. \tag{1.13}
\]
In the following, we will sometimes drop the subscript $\Delta$.

We can now state the key lemma for analyzing the convergence of structures in the KT sequence.

**Lemma 2.** Let $G$ be an instance having unique solution with gap $\sigma$. Consider an initial condition $\alpha^0$ such that $B_{\Delta}(q, \delta, 0)$ holds for some $\Delta > 0$, $\delta \leq \min(\Delta, \sigma)$ and for some $q \in \{0,1,\ldots,k\}$. Then there exists $t_*(n) = C_n^0$ such that $B_{\Delta}(q + 1, \delta(1 - (5n)^{-1}), t)$ holds for all $t \geq t_*(n)$.

The proof of this lemma is based on a case-by-case analysis of the possible topologies of $C_q$ and can be found in Sections 4, B.2, B.3, B.4. Using the lemma to prove Theorem 3 and hence Theorem 2 is immediate.

**Proof (Theorem 4).** We know that $k \leq n$. Start with $\alpha^0$ such that $U_{G,G_q}(\alpha^0) \leq \Delta$ at $t = 0$, i.e. $B_{\Delta}(q, \delta, 0)$. Define $\delta_0 = \min(\sigma, \Delta)$, $\delta_1 = \delta_0(1 - (5n)^{-1})$, ..., $\delta_k = \delta_0(1 - (5n)^{-1})^k$.

We know that $B(0, \delta_0, 0)$ holds by assumption. We deduce from Lemma 2 that $B(1, \delta_1, t_*)$ holds. Proceeding inductively, it follows that $B(k + 1, \delta_k, (k + 1)t_*)$ holds. Now, we only need to show $\delta_k \geq \frac{1}{2}\delta_0$, which follows from $(1 - (5n)^{-1})^n \geq 1/2$ for all $n \geq 1$. \qed

## 2 Fixed point properties: Proof of Theorem 1

The dual problem to (1.5) is
\[
\text{minimize} \quad \sum_{i \in V} y_i, \\
\text{subject to} \quad y_i + y_j \geq w_{ij} \quad \forall (i,j) \in E, \quad y_i \geq 0 \quad \forall i \in V \tag{2.1}
\]
A feasible point $x$ for LP (1.5) is called half-integral if for all $e \in E$, $x_e \in \{0,1,\frac{1}{2}\}$. It is well known that problem (1.5) always has an optimum $x^*$ that is half-integral. We also denote the subset of edges $e$ in $E$ with $x_e^* \in \{\frac{1}{2},1\}$ by $M^*$ and say $M^*$ is a half-integral matching.

In this paper we consider only those problems that have unique $x^*$, that has greater weight than any other corner $x$ of the primal polytope by at least $\epsilon > 0$, i.e. $\sum_{e \in E} w_e x_e^* - \sum_{e \in E} w_e x_e \geq \epsilon$ for all half-integral solutions $x \neq x^*$. We call such LP an $\epsilon$-pointed problem. Further, if $x^*$ is, in fact, integral, we say that the LP relaxation is $\epsilon$-tight. We will simply use pointed and tight, whenever the value of $\epsilon > 0$ is immaterial (let us stress that, according to this terminology, tight implies pointed).

We call $e \in E$ a 1-solid edge if $x_e^* = 1$, a $\frac{1}{2}$-solid edge if $x_e^* = \frac{1}{2}$, and a non-solid edge if $x_e^* = 0.$
Proof of Theorem 1 From fixed points to NB solutions. The direct part follows from the following set of fixed point properties, which hold for pointed problems. The proofs of these properties are given in Appendix A. Throughout $(\alpha, m, \gamma)$ is a fixed point of the dynamics $(1.2), (1.4)$ (with $\gamma$ given by $(1.3)$).

1. Two players $(i,j) \in E$ are called partners if $\gamma_i + \gamma_j = w_{ij}$. Then the following are equivalent: (a) $i$ and $j$ are partners, (b) $w_{ij} - \alpha_{i,j} - \alpha_{j,i} \geq 0$, (c) $\gamma_i = m_{j \to i}$ and $\gamma_j = m_{i \to j}$.

2. Let $P(i)$ be the set of all partners of $i$. Then the following are equivalent: (a) $P(i) = \{j\}$, (b) $P(j) = \{i\}$, (c) $w_{ij} - \alpha_{i,j} - \alpha_{j,i} > 0$.

3. We say that $(i,j)$ is a weak-dotted edge if $w_{ij} - \alpha_{i,j} - \alpha_{j,i} = 0$, a strong-dotted edge if $w_{ij} - \alpha_{i,j} - \alpha_{j,i} > 0$, and a non-dotted edge otherwise. If $i$ has no adjacent dotted edges, then $\gamma_i = 0$.

4. $\gamma$ is an optimum solution for the dual LP $(2.1)$ and $m_{i \to j} = (w_{ij} - \gamma_i)_+$ holds for all $(i,j) \in E$.

5. The balance property $(1.1)$, holds at every edge $(i,j) \in E$ (with both sides being non-negative).

6. An edge is 1-solid ($\frac{1}{2}$-solid) iff it is strongly (weakly) dotted.

Proof of Theorem 1 direct implication. Assume that the LP $(1.5)$ is tight. Then, by property 6, the set of strong-dotted edges form the unique maximum weight matching $M^*$. By properties 4 and 5 the pair $(M^*, \gamma)$ is stable and balanced and thus forms a NB solution.

Remark: Notice that properties 1-6 hold under the weaker condition that the problem $(1.5)$ is pointed. In this more general setting, fixed point correspond to dual optima satisfying the unmatched balance property $(1.1)$.

Proof of Theorem 1 From NB solutions to fixed points.

Proof of Theorem 1 converse implication. Consider any NB solution $(M, \gamma_{NB})$. Construct a corresponding FP as follows. Set $m_{i \to j} = (w_{ij} - \gamma_{NB,i})_+$ for all $(i,j) \in E$. Compute $\alpha$ using $\alpha_{i,j} = \max_{k \in \partial(i)} m_{k \to i}$. We claim that this is a FP and that the corresponding $\gamma_{NB}$ is $\gamma$ . To prove that we are at a fixed point, we imagine updated offers $m_{\text{upd}}$ based on $\alpha$, and show $m_{\text{upd}} = m$.

Consider a matching edge $(i,j) \in M$. We know that $\gamma_{NB,i} + \gamma_{NB,j} = w_{ij}$. Also stability and balance tell us $\gamma_{NB,i} - \max_{k \in \partial(i)} (w_{ik} - \gamma_{NB,k})_+ = \gamma_{NB,j} - \max_{l \in \partial(j)} (w_{jl} - \gamma_{NB,l})_+$, and both sides are non-negative. Hence, $\gamma_{NB,i} - \alpha_{i,j} = \gamma_{NB,j} - \alpha_{j,i} \geq 0$. Therefore $\alpha_{i,j} + \alpha_{j,i} \leq w_{ij}$.

$$m_{\text{upd}} = \left( \frac{w_{ij} - \alpha_{i,j} + \alpha_{j,i}}{2} \right) = \left( \frac{w_{ij} - \gamma_{NB,i} + \gamma_{NB,j}}{2} \right) = \gamma_{NB,j} - \gamma_{NB,i} = m_{i \to j}. \quad (2.2)$$

By symmetry, we also have $m_{\text{upd}}_{j \to i} = \gamma_{NB,j} = m_{j \to i}$. Hence, the offers remain unchanged. Moreover, $\gamma_i = \max (m_{j \to i}, \alpha_{i,j}) = \gamma_{NB,i}$. Similarly, $\gamma_j = \gamma_{NB,j}$. Now consider $(i,j) \notin M$. We have $\gamma_{NB,i} + \gamma_{NB,j} \geq w_{ij}$ and, $\gamma_{NB,i} = \max_{k \in \partial(i)} (w_{ik} - \gamma_{NB,k})_+ = \alpha_{i,j}$. Similar equation holds for $\gamma_{NB,j}$. The validity of this identity can be checked individually in the cases when $i \in M$ and $i \notin M$. Hence, $\alpha_{i,j} + \alpha_{j,i} \leq w_{ij}$.

This leads to $m_{\text{upd}} = (w_{ij} - \alpha_{i,j})_+ = (w_{ij} - \gamma_{NB,i})_+ = m_{i \to j}$. By symmetry, we know that both the offers remain unchanged.

Finally, we show $\gamma = \gamma_{NB}$. For all $(i,j) \in M$, we already found that $m_{i \to j} = \gamma_{NB}$ and vice versa. For any edge $(ij) \notin M$, we know $m_{i \to j} = (w_{ij} - \gamma_{NB,i})_+ \leq \gamma_{NB,j}$. This immediately leads to $\gamma = \gamma_{NB}$. It is worth noting that making use of the uniqueness of LP optimum, we can further show that $M = M^*, \gamma_i = m_{j \to i} > 0$ iff $(ij) \in M$ i.e. the fixed point reconstructs the pairing $M = M^*$.
3 An easier case: bipartite graphs

Bipartite graphs are natural graphs to consider, since they include the case of ‘buyers’ and ‘sellers’ negotiating with each other. This case is substantially easier because of a natural partial ordering on message vectors, and in particular, on the Nash bargaining solutions. This ordering loosely corresponds to the perceived strength or negotiating power of buyers relative to sellers. Further, there are ‘extremal’ NB solutions with respect to this partial ordering, which we call the buyer/seller dominant solution. We show that if the dynamics starts with a sufficiently buyer/seller dominant state, then it always converges to one of these extreme NB solutions.

Unfortunately, the partial ordering does not extend to general graphs, and a much more technical analysis is required in that case. Nevertheless, the bipartite graph case suggests that the natural dynamics indeed converges even in instances with multiple NB solutions.

We partition the nodes on a bipartite graph \( G = (V_B, V_S) \), representing buyers and sellers respectively. Given message vectors \( \alpha \) and \( \beta \), we define a partial ordering as \( \alpha \succeq \beta \) if inequalities \( \alpha_i \wedge \beta_{i'} \geq \beta_i \wedge \beta_{i'} \) and \( \alpha_i \wedge \beta_{i'} \leq \beta_{i'} \wedge \beta_{i} \) hold for all \( (i, j') \in E \), \( i \in V_B, j' \in V_S \).

**Lemma 3.** For a bipartite graph \( G \), if \( \alpha^0 \succeq \beta^0 \) then \( \alpha^t \succeq \beta^t \) for all \( t \geq 0 \) i.e. the dynamics preserves partial ordering.

**Proof.** \( m_{i \rightarrow j'} \) is non-increasing in \( \alpha_i \wedge \beta_{i'} \) and non-decreasing in \( \alpha_{i'} \wedge \beta_i \) leading to \( m_{i \rightarrow j'} |_{\alpha^0} \leq m_{i \rightarrow j'} |_{\beta^0} \).

Similarly, \( m_{j' \rightarrow i} |_{\alpha^0} \geq m_{j' \rightarrow i} |_{\beta^0} \). It follows that \( \alpha^1 \succeq \beta^1 \). Induction completes the proof.

**Corollary 2.** Assume \( \alpha^0 \) to be such that \( \alpha^0 \succeq \alpha^1 \). Then \( \lim_{t \rightarrow \infty} \alpha^t = \alpha^* \) for some NB solution \( \alpha^* \).

**Proof.** Since \( \alpha^0 \succeq \alpha^1 \), using Lemma 3 we have \( \alpha^t \succeq \alpha^{t+1} \) for all \( t \). We know that \( \alpha \in [0, W]^{|E|} \) for all valid message vectors \( \alpha \), so the monotone sequence \( \{\alpha^t\}_{t \geq 0} \) converges to a fixed point of the dynamics (as the update rules are continuous in \( \alpha^t \)).

Define the message vectors \( \alpha^\text{top} \) and \( \alpha^\text{bot} \) by: \( \alpha^\text{top}_{i \wedge j'} = W \), \( \alpha^\text{top}_{i' \wedge j} = 0 \) and \( \alpha^\text{bot}_{i \wedge j'} = 0 \), \( \alpha^\text{bot}_{i' \wedge j} = W \) for all \( (i, j') \in E \). Therefore, \( \alpha^\text{top} \succeq \alpha \succeq \alpha^\text{bot} \) for any valid \( \alpha \in [0, W]^{|E|} \).

If \( \alpha^0 = \alpha^\text{top} \) then \( \alpha^0 \succeq \alpha^1 \). It follows from Corollary 2 that \( \alpha^t \) converges, say to \( \alpha^{*, \text{up}} \). Consider any other fixed point (i.e. NB solution) \( \alpha^* \). \( \alpha^0 \succeq \alpha^* \Rightarrow \alpha^t \succeq \alpha^* \forall t \geq 0 \). Hence, \( \alpha^{*, \text{up}} \succeq \alpha^* \).

Thus, there exists a ‘maximal’ fixed point with respect to the partial ordering. It is buyer optimal in a strong sense – Each buyer earns the most possible at \( \alpha^{*, \text{up}} \) among all NB solutions, and each seller earns the least possible. With a similar argument using \( \alpha^\text{bot}_{i \wedge j'} \), we can show the existence of a ‘minimal’ fixed point \( \alpha^{*, \text{bot}} \) that is seller optimal.

Although the simple monotonicity argument does not imply any bound on the convergence rate, we expect that the methods developed for general graphs (next Section) can be used to prove polynomial convergence in the present case.

4 General convergence: Proof of Lemma 2

The condition \( B(q, \delta, t) \) from Section 1.4.1 means that at a fixed step \( t \) of the natural dynamics the deviation of \( \alpha^t \) from the unique fixed point \( \alpha^* \) is at most \( \Delta \) on all edges of \( E \), and is at most \( \Delta - \delta \) on all edges of \( \gamma_q \), or \( U_G(\alpha^t) \leq \Delta, U_{G, \gamma_q}(\alpha^t) \leq \Delta - \delta \).

The goal now is to show that if we start with an initial condition that satisfies \( B(q, \delta, 0) \) then after \( Cn^6 \) iterations of the natural dynamics we reach the condition \( B(q + 1, \delta(1 - (5n)^{-1}), t) \).
First we state the following useful properties of the condition $B(q, \delta, t)$ with respect to the natural dynamics. Their proofs are given in [3.1] The first lemma gives an upper bound on offers to nodes in $V(C_q)$ from outside $C_q$. This allows us to restrict the analysis to messages on $C_q$.

**Lemma 6.** Assume that $P$ conditions. The simplified dynamics on $V$ has a unique fixed point $\hat{\alpha}$ in $V$. Its proof is given in B.2 based on the analysis of a random walk on a finite tuned boundary and initial conditions. But first we state a crucial result on the convergence of the $M$ alternate edges included in $V$. Their proofs are given in B.1. The first lemma gives an upper bound on offers to nodes.

**Lemma 2** for $q = 0$ follows directly from Lemma 6 with $\epsilon = \delta/(5n)$. For $q > 0$, the Lemma [6] reduces our problem to bounding $U_{G,C_q}(\alpha^t)$. The proof is divided into three cases depending on the structure $C_q$: path, blossom, bicycle. Next we discuss the main steps in the case of a path. The blossom and bicycle require a more delicate analysis that uses the same ideas. These cases are deferred to Appendix sections [3.3] [3.4] in the interest of space.

### 4.1 Proof of Lemma 2: Path

We prove the bound $U_{G,C_q}(\alpha^t) \leq \Delta - \delta(1 - (10n)^{-1})$ in three steps:

**Step 1:** Define two different dynamics on the path that are simpler to analyze, we call these ‘bounding processes’. Each bounding process has a unique fixed point and converges rapidly to it.

**Step 2:** Show that the message vector $\alpha^t$ of the natural dynamics on this path is bounded from above and below (sandwiched) by these two bounding processes.

**Step 3:** Show that the fixed points of the bounding processes have deviations $\Delta - \delta$ from $\alpha^*$.

Hence, the natural dynamics quickly achieves $U_{G,C_q}(\alpha^t) \leq \Delta - \delta(1 - (10n)^{-1})$.

**Bounding processes.** Define a path as $P = (V_P, E_P)$ with $V_P = \{0, 1, \ldots, \ell\}$, and $E_P = \{(i, i+1) : i = 0, \ldots, \ell - 1\}$ with positive weights on the edges, and $\ell \geq 1$. Let $M_P$ be a matching on $P$ with alternate edges included in $M_P$. We define the simplified dynamics on $P$ wrt matching $M_P$, boundary conditions $\{b^t_L\}_{t \geq 0}, \{b^t_R\}_{t \geq 0}$ (arbitrary real numbers) by

$$
\forall (i,j) \in E_P, \quad \tilde{m}^t_{i,j} = \begin{cases} \frac{1}{2}(w_{i,j} - \tilde{\alpha}^t_{i,j} + \tilde{\alpha}^t_{j,i}) & \text{if } (i,j) \in M_P, \\ w_{i,j} - \alpha^t_{i,j} & \text{otherwise.} \end{cases} \tag{4.1}
$$

$$
\tilde{\alpha}^t_{0\setminus 1} = \kappa b^t_L + (1 - \kappa)\tilde{\alpha}^t_{0\setminus 1}, \quad \tilde{\alpha}^t_{\ell \setminus \ell - 1} = \kappa b^t_R + (1 - \kappa)\tilde{\alpha}^t_{\ell \setminus \ell - 1},
$$

$$
\tilde{\alpha}^t_{i\setminus i+1} = \kappa \tilde{m}^t_{i-1\setminus i} + (1 - \kappa)\tilde{\alpha}^t_{i\setminus i+1}, \quad \tilde{\alpha}^t_{i\set\setminus i-1} = \kappa \tilde{m}^t_{i+1\setminus i} + (1 - \kappa)\tilde{\alpha}^t_{i\set\setminus i-1}, \quad i = 1, 2, \ldots, \ell - 1
$$

for all $t \geq 0$.

Our bounding processes will be defined later as instances of this simplified dynamics with specially tuned boundary and initial conditions. But first we state a crucial result on the convergence of the simplified dynamics. Its proof is given in [3.2] based on the analysis of a random walk on a finite length segment.

**Lemma 7.** Consider $P$ with $\ell \geq 1$. Let $b^t_L = b_L, b^t_R = b_R, t \geq 0$ be arbitrary constant boundary conditions. The simplified dynamics on $P$ wrt the given boundary conditions and matching $M_P$ has a unique fixed point $\tilde{\alpha}$ satisfying $\tilde{\alpha}^t_{0\setminus 1} = b^t_L, \tilde{\alpha}^t_{\ell \set\setminus \ell - 1} = b^t_R$. Moreover, for any initial condition $\tilde{\alpha}^0$,

$$
\|\tilde{\alpha}^t - \tilde{\alpha}^*\|_\infty \leq Ct^2 \exp\left(-\frac{t}{Ct^2}\right)\|\tilde{\alpha}^0 - \tilde{\alpha}^*\|_\infty, \quad C < \infty. \tag{4.2}
$$
As a consequence, the error term will be smaller than any desired inverse polynomial factor in time $O(n^2 \log n)$.

Now we define the bounding processes. Suppose $C_q$ is a path $P$. Let $M^*_P$ be the restriction of $M^*$ to $P$. Assume $\Delta > 0$, $\delta \leq \min(\sigma, \Delta)$ and $s \in \{+1, -1\}$. The $(s, \Delta, \delta)$-bounding process on $P$ with respect to the simplified dynamics on $P$ is the sequence of message assignments $\{A^t\}_{t \geq 0}$ on $P$ produced by the bounding processes on $P$ with respect to the matching $M^*_P$, the constant boundary conditions $b_L = s(\Delta - \delta)$, $b_R = s(-1)\delta (\Delta - \delta)$ with the initial condition (for $i \in \{0, \ldots, \ell - 1\}$) $A^0_{\pi(i+1)} = \alpha^*_{\pi(i+1)} + s(-1)i\Delta$, $A^0_{\pi(i+1)} = \alpha^*_{\pi(i+1)} - s(-1)i\Delta$. When convenient, we include the sign $s$ explicitly in the notation for the bounding process as $\{A^t(s)\}_{t \geq 0}$.

As per Lemma 4, the bounding process has a unique fixed point to which it converges fast. Call this fixed point $\hat{A}^t$. It is easy to see that for all $i \in \{0, 1, \ldots, \ell - 1\}$: $\hat{A}^t_{\pi(i+1)} = \beta_{\pi(i+1)} + s(-1)i(\Delta - \delta)$, $\hat{A}^t_{\pi(i+1)} = \beta_{\pi(i+1)} - s(-1)i(\Delta - \delta)$.

The following lemma establishes that the simplified dynamics and natural dynamics produce the same results when acting on $\hat{A}^t$, except when $\hat{A}^t_{i \rightarrow j} < 0$ for some $(i, j) \in P$. This property is critical in proving that $\hat{A}^t$ sandwichs the natural dynamics (Lemma 9). The proof is in Appendix B.2.

**Lemma 8.** Let $\{A^t\}_{t \geq 0}$ be the $(s, \Delta, \delta)$-bounding process on $P$. Then for any $t \geq 0$, we have $A^t_{ij} + A^t_{ji} - w_{ij} \leq 0$ for $(i, j) \in M^*_P$ and $A^t_{ij} + A^t_{ji} - w_{ij} \geq 0$ for $(i, j) \notin M^*_P$.

**Sandwiching the message vector.** Given two message assignments $\alpha$ and $\beta$ on $P$, we shall say that $\alpha$ dominates $\beta$ (and write $\alpha \succeq \beta$) if

- for $i$ even, $\alpha_{\pi(i+1)} \geq \beta_{\pi(i+1)}$, $\alpha_{\pi(i+1)} \leq \beta_{\pi(i+1)}$
- for $i$ odd, $\alpha_{\pi(i+1)} \leq \beta_{\pi(i+1)}$, $\alpha_{\pi(i+1)} \geq \beta_{\pi(i+1)}$

The natural dynamics (and the simplified dynamics) on $P$ preserve this ordering (cf. Appendix B.2).

Note that $A^0(-) \leq A^0_P \leq A^0(\cdot)$ by our definition of the bounding processes, where $A^0_P$ denotes the restriction of $\alpha$ to $P$. The next lemma shows that this sandwich property continues to hold for all $t \geq 0$. The proof is given in Appendix B.2.

**Lemma 9.** Let $G$ be an instance admitting a unique NB solution with gap $\sigma$, and assume its KT sequence to be $(C_0, C_1, C_2, \ldots, C_k)$, with $C_q, q \in \{1, \ldots, k\}$ a path. Let $P = (V_{ext}(C_q), E(C_q))$. Further assume $B_\alpha(q, \delta, 0)$ holds. If we denote by $\{A^t(s)\}_{t \geq 0}$ the $(s, \Delta, \delta)$ bounding process on $P$, then for any $t \geq 0$:

$$\hat{A}^t(-) \approx A^t_P \approx \hat{A}^t(\cdot).$$

**Proof (Lemma 9: Path).** From Lemma 9 we have

$$||\hat{A}^t - \alpha^*||_{\infty} \leq \max \left\{ ||\hat{A}^t(\cdot) - \alpha^*||_{\infty}, ||\hat{A}^t(-) - \alpha^*||_{\infty} \right\}.$$

We know that $\ell \leq n$. Since $||\hat{A}^t - \alpha^*||_{\infty} = \Delta - \delta$ by Lemma 7 applied to the bounding processes we can show the right hand side becomes smaller than $\Delta - (1 - \frac{1}{10n})\Delta$ for all $t \geq c_P n^3$ for some $c_P$ finite. Finally, we use Lemma 4 with $\epsilon = \frac{4}{10n}$ to show that $B(q + 1, \delta(1 - (5n)^{-1}), t)$ holds for all $t \geq t_P, s$, where $t_P, s \leq Cn^6$ as required.

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A Proofs of fixed point properties

In this section we state and prove the fixed point properties that were used for the proof of Theorem 1 in Section 2.

Lemma 10. $\gamma$ satisfies the constraints of the dual problem (2.1).

Proof. Since offers $m_{i \rightarrow j}$ are by definition non-negative therefore for all $v \in V$ we have $\gamma_v \geq 0$. So we only need to show $\gamma_i + \gamma_j \geq w_{ij}$ for any edge $(ij) \in E$. It is to see that $\gamma_i \geq \alpha_{i \setminus j}$ and $\gamma_j \geq \alpha_{i \setminus j}$. Moreover, if $\alpha_{i \setminus j} + \alpha_{i \setminus j} \geq w_{ij}$ then $\gamma_i + \gamma_j \geq w_{ij}$ holds and we are done. Otherwise, for $\alpha_{i \setminus j} + \alpha_{i \setminus j} < w_{ij}$ we have $m_{i \rightarrow j} = \frac{w_{ij} - \alpha_{i \setminus j} - \alpha_{i \setminus j}}{2}$ and $m_{j \rightarrow i} = \frac{w_{ij} - \alpha_{i \setminus j} + \alpha_{i \setminus j}}{2}$ which gives $\gamma_i + \gamma_j \geq m_{i \rightarrow j} + m_{j \rightarrow i} = w_{ij}$. \qed

Recall that for any $(ij) \in E$, we say that $i$ and $j$ are ‘partners’ if $\gamma_i + \gamma_j = w_{ij}$ and $P(i)$ denotes the partners of node $i$. In other words $P(i) = \{j : j \in \partial i, \gamma_i + \gamma_j = w_{ij}\}$.

Lemma 11. The following are equivalent:

(a) $i$ and $j$ are partners,

(b) $w_{ij} - \alpha_{i \setminus j} - \alpha_{j \setminus i} \geq 0$,

(c) $\gamma_i = m_{j \rightarrow i}$ and $\gamma_j = m_{i \rightarrow j}$.

Moreover, if $\gamma_i = m_{j \rightarrow i}$ and $\gamma_j > m_{i \rightarrow j}$ then $\gamma_i = 0$.

Proof. We will prove $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$.

$(a) \Rightarrow (b)$: Since $\gamma_i \geq \alpha_{i \setminus j}$ and $\gamma_j \geq \alpha_{j \setminus i}$ always holds then $w_{ij} = \gamma_i + \gamma_j \geq \alpha_{i \setminus j} + \alpha_{j \setminus i}$.

$(b) \Rightarrow (c)$: If $w_{ij} - \alpha_{i \setminus j} - \alpha_{j \setminus i} \geq 0$ then $(w_{ij} - \alpha_{i \setminus j} - \alpha_{j \setminus i})/2 \geq \gamma_{j \setminus i}$. But $m_{i \rightarrow j} = (w_{ij} - \alpha_{i \setminus j} + \alpha_{j \setminus i})/2$ therefore $\gamma_j = m_{i \rightarrow j}$. The argument for $\gamma_i = m_{j \rightarrow i}$ is similar.

$(c) \Rightarrow (a)$: If $\gamma_i \geq m_{j \rightarrow i}$ then $m_{i \rightarrow j} = (w_{ij} - \alpha_{i \setminus j} + \alpha_{j \setminus i})/2$ and $m_{j \rightarrow i} = (w_{ij} - \alpha_{i \setminus j} + \alpha_{j \setminus i})/2$ which gives $\gamma_i + \gamma_j = m_{i \rightarrow j} + m_{j \rightarrow i} = w_{ij}$ and we are done. Otherwise, we have $\gamma_i + \gamma_j = m_{i \rightarrow j} + m_{j \rightarrow i} \leq (w_{ij} - \alpha_{i \setminus j})_+ + (w_{ij} - \alpha_{j \setminus i})_+ < \max \left( (w_{ij} - \alpha_{i \setminus j})_+, (w_{ij} - \alpha_{j \setminus i})_+, 2w_{ij} - \alpha_{i \setminus j} - \alpha_{j \setminus i} \right) \leq w_{ij}$ which contradicts Lemma 10 that $\gamma$ satisfies the constraints of the dual problem (2.1).

Finally, we need to show that $\gamma_i = m_{j \rightarrow i}$ and $\gamma_j > m_{i \rightarrow j}$ give $\gamma_i = 0$. First note that by equivalence of $(b)$ and $(c)$ we should have $w_{ij} < \alpha_{i \setminus j} + \alpha_{j \setminus i}$. On the other hand $\alpha_{i \setminus j} \leq \gamma_i = m_{j \rightarrow i} \leq (w_{ij} - \alpha_{j \setminus i})_+$. Now if $w_{ij} - \alpha_{j \setminus i} > 0$ we get $\alpha_{i \setminus j} \leq w_{ij} - \alpha_{j \setminus i}$ which is a contradiction. Therefore $\gamma_i = (w_{ij} - \alpha_{j \setminus i})_+ = 0$. \qed

Lemma 12. The following are equivalent:

(a) $P(i) = \{j\}$,

(b) $P(j) = \{i\}$,

(c) $w_{ij} - \alpha_{i \setminus j} - \alpha_{j \setminus i} > 0$.

Proof. $(a) \Rightarrow (b) \Rightarrow (c)$: $(a)$ means $\alpha_{i \setminus j} < m_{j \rightarrow i}$. It also follows that $m_{j \rightarrow i} > 0$ or $(w_{ij} - \alpha_{j \setminus i})_+ = w_{ij} - \alpha_{j \setminus i}$. Therefore, $m_{j \rightarrow i} \leq (w_{ij} - \alpha_{j \setminus i})_+ = w_{ij} - \alpha_{j \setminus i}$ which gives $w_{ij} - \alpha_{i \setminus j} - \alpha_{j \setminus i} > 0$ or $(c)$. From this we can explicitly write $m_{i \rightarrow j} = (w_{ij} - \alpha_{i \setminus j} + \alpha_{j \setminus i})/2$ which is strictly bigger than $\alpha_{j \setminus i}$. Hence we obtain $(b)$.

By symmetry $(b) \Rightarrow (c) \Rightarrow (c)$. This finishes the proof. \qed
Recall that \((ij)\) is a weak-dotted edge if \(w_{ij} - \alpha_{i\setminus j} - \alpha_{j\setminus i} = 0\), a strong-dotted edge if \(w_{ij} - \alpha_{i\setminus j} - \alpha_{j\setminus i} > 0\), and a non-dotted edge otherwise. Basically, for any dotted edge \((ij)\) we have \(j \in P(i)\) and \(i \in P(j)\).

**Corollary 3.** One corollary of Lemmas 11-12 is that strong-dotted edges are only adjacent to non-dotted edges. Also each weak-dotted edge is adjacent to at least one weak-dotted edge at each end.

**Lemma 13.** If \(i\) has no adjacent dotted edges, then \(\gamma_i = 0\)

*Proof.* Assume that the largest offer to \(i\) comes from \(j\). Therefore, \(\alpha_{i\setminus j} \leq m_{j\rightarrow i} \leq (w_{ij} - \alpha_{j\setminus i})_+\). Now if \(w_{ij} - \alpha_{j\setminus i} > 0\) then \(\alpha_{i\setminus j} \leq w_{ij} - \alpha_{j\setminus i}\) or \((ij)\) is dotted edge which is impossible. Thus, \(w_{ij} - \alpha_{j\setminus i} = 0\) and \(\gamma_i = 0\). \(\square\)

**Lemma 14.** The following are equivalent:

(a) \(\alpha_{i\setminus j} = \gamma_i\),

(b) \(w_{ij} - \alpha_{i\setminus j} - \alpha_{j\setminus i} \leq 0\),

(c) \(m_{i\rightarrow j} = (w_{ij} - \alpha_{i\setminus j})_+\).

*Proof.* (a) \(\Rightarrow\) (b): Follows from Lemma 12 since \(\alpha_{i\setminus j} = \gamma_i\) gives \(|P(i)| > 1\).

(b) \(\Rightarrow\) (c): Follows from the definition of \(m_{i\rightarrow j}\).

(c) \(\Rightarrow\) (a): From \(m_{i\rightarrow j} = (w_{ij} - \alpha_{i\setminus j})_+\) we have \(w_{ij} - \alpha_{i\setminus j} - \alpha_{j\setminus i} \leq 0\). Therefore, \(m_{i\rightarrow j} = (w_{ij} - \alpha_{i\setminus j})_+ \leq \max[|w_{ij} - \alpha_{j\setminus i}|, 0] \leq \alpha_{i\setminus j}\). \(\square\)

Note that (c) is symmetric in \(i\) and \(j\), so (a) and (b) can be transformed by interchanging \(i\) and \(j\).

**Corollary 4.** \(\alpha_{i\setminus j} = \gamma_i\) iff \(\alpha_{j\setminus i} = \gamma_j\)

**Lemma 15.** \(m_{i\rightarrow j} = (w_{ij} - \gamma_i)_+\) holds \(\forall (ij) \in E\)

*Proof.* If \(w_{ij} - \alpha_{i\setminus j} - \alpha_{j\setminus i} \leq 0\) then the result follows from Lemma 14. Otherwise, \((ij)\) is strongly dotted and \(\gamma_i = m_{j\rightarrow i} = (w_{ij} - \alpha_{j\setminus i} + \alpha_{i\setminus j})/2, \gamma_j = m_{i\rightarrow j} = (w_{ij} - \alpha_{i\setminus j} + \alpha_{j\setminus i})/2\). From here we can explicitly calculate \(w_{ij} - \gamma_i = (w_{ij} - \alpha_{i\setminus j} + \alpha_{j\setminus i})/2 = m_{i\rightarrow j}\). \(\square\)

**Lemma 16.** The unmatched balance property, equation (1.1), holds at every edge \((ij) \in E\), and both sides of the equation are non-negative.

*Proof.* In light of lemma 15, (1.1) can be rewritten at a fixed point as

\[\gamma_i - \alpha_{i\setminus j} = \gamma_j - \alpha_{j\setminus i}\] (A.1)

which is easy to verify. The case \(w_{ij} - \alpha_{i\setminus j} - \alpha_{j\setminus i} \leq 0\) leads to both sides of Eq. (A.1) being 0 by Corollary 4. The other case \(w_{ij} - \alpha_{i\setminus j} - \alpha_{j\setminus i} > 0\) leads to

\[m_{i\rightarrow j} - \alpha_{j\setminus i} = m_{j\rightarrow i} - \alpha_{i\setminus j} = \frac{w_{ij} - \alpha_{i\setminus j} - \alpha_{j\setminus i}}{2}\] (A.2)

Clearly, we have \(\gamma_i = m_{j\rightarrow i}\) and \(\gamma_j = m_{i\rightarrow j}\). So Eq. (A.1) holds. \(\square\)

Next lemmas show that dotted edges are in correspondence with the solid edges that were defined in Section 2.
Lemma 17. A non-solid edge cannot be a dotted edge, weak or strong.

Before proving the lemma let us define alternating paths. A path \( P = (i_1, i_2, \ldots, i_k) \) in \( G \) is called alternating path if: (a) There exist a partition of edges of \( P \) into two sets \( A, B \) such that either \( A \subset M^* \) or \( B \subset M^* \). Moreover \( A \) (\( B \)) consists of all odd (even) edges; i.e. \( A = \{(i_1, i_2), (i_3, i_4), \ldots \} \) \((B = \{(i_2, i_3), (i_4, i_5), \ldots \})\). (b) The path \( P \) might intersect itself or even repeat its own edges but no edge is repeated immediately. That is, for any \( 1 \leq r \leq k - 2 \) : \( i_r \neq i_{r+1} \) and \( i_r \neq i_{r+2} \). \( P \) is called an alternating cycle if \( i_1 = i_k \).

Also, consider \( x^* \) and \( y^* \) that are optimum solutions of the LP and its dual, \((1.5)\) and \((2.1)\). The complementary slackness conditions (see \( [Sch03] \)) for more details) state that for all \( v \in V \), \( y_v^*(\sum_{e \in \partial v} x_e^* - 1) = 1 \) and for all \( e = (ij) \in E \), \( x_e^*(y_v^* + y_j^* - w_{ij}) = 0 \). Therefore, for all solid edges the equality \( y_v^* + y_j^* = w_{ij} \) holds. Moreover, any node \( v \in V \) is adjacent to a solid edge iff \( y_v^* > 0 \).

Proof of Lemma \([17]\) We need to consider two cases:

Case (I). Assume that the optimum LP solution \( x^* \) is integer (having a tight LP).

The idea of the proof is that if there exist a non-solid edge which is dotted, we use a similar analysis to \([BB+07]\) to construct an alternating path consisting of dotted and solid edges that leads to creation of at least two optimal solutions to LP \((1.5)\). This contradicts with uniqueness assumption on the optimum solution of LP.

Now assume the contrary: take \((i_1, i_2)\) that is a non-solid edge but it is dotted. Consider an endpoint of \((i_1, i_2)\). For example take \(i_2\). Either there is a solid edge attached to \(i_2\) or not. If there is not, we stop. Otherwise, assume \((i_2, i_3)\) is a solid edge. Using Lemma \([13]\) either \(\gamma_{i_3} = 0\) or there is a dotted edge connected to \(i_3\). But if this dotted edge is \((i_2, i_3)\) then \(P(i_2) \supset \{i_1, i_3\}\). Therefore, by Lemma \([12]\) there has to be another dotted edge \((i_3, i_4)\) connected to \(i_3\). Now, depending on whether \(i_4\) has (has not) an adjacent solid edge we continue (stop) the construction. Similar procedure could be done by starting at \(i_1\) instead of \(i_2\). Therefore, we obtain an alternating path \( P = (i_{-k}, \ldots, i_{-1}, i_0, i_1, i_2, \ldots, i_{k}) \) with all odd edges being dotted and all even edges being solid. Using the same argument as in \([BB+07]\) one can show that one of the following four scenarios occur.

- **Path:** Before \( P \) intersects itself, both end-points of the path stop. Either the last edge is solid (then \(\gamma_v = 0 \) for the last node) or the last edge is a dotted edge. Now consider a new solution \( x' \) to LP \((1.5)\) by \( x'_e = x_e^* \) if \( e \notin P \) and \( x'_e = 1 - x_e^* \) if \( e \in P \). It is easy to see that \( x' \) is a feasible LP solution at all points \( v \notin P \) and also for internal vertices of \( P \). The only nontrivial case is when \( v = i_{-k} \) (or \( v = i_0 \)) and the edge \((i_{-k}, i_{-k+1})\) (or \((i_{k-1}, i_k)\)) is dotted. In both of these cases, by construction \( y_v^* = 0 \) which means no solid edge is attached to \( v \) outside of \( P \) so making any change inside of \( P \) is safe. Now denote the weight of all solid (dotted) edges of \( P \) by \( w(P_{\text{solid}}) \) \((w(P_{\text{dotted}}))\). Hence, \(\sum_{e \in E} w_e x_e^* - \sum_{e \in E} w_e x'_e = w(P_{\text{solid}}) - w(P_{\text{dotted}})\).

But \(w(P_{\text{dotted}}) = \sum_{e \in P} \gamma_v\). Moreover, from Lemma \([10]\) \(\gamma\) is dual feasible which gives \(w(P_{\text{solid}}) \leq \sum_{e \in P} \gamma_v\). We are using the fact that if there is a solid edge at an endpoint of \( P \) the \(\gamma\) of the endpoint should be 0. Now \(A.3\) reduces to \(w_e x_e^* - \sum_{e \in E} w_e x'_e \leq 0\). This contradicts the tightness of LP relaxation \((1.5)\) since \(x'_e \neq x_e^* \) holds at least for \( e = (i_1, i_2)\).

- **Cycle:** \( P \) intersects itself and will contain an even cycle \(C_{2s} \). This case can be handled very similar to the path by defining \( x'_e = x_e^* \) if \( e \notin C_{2s} \) and \( x'_e = 1 - x_e^* \) if \( e \in C_{2s} \). The proof is even simpler since the extra check for the boundary condition is not necessary.
• **Blossom**: $P$ intersects itself and will contain an odd cycle $C_{2s+1}$ with a path (stem) $P'$ attached to the cycle at point $u$. In this case let $x'_e = x^*_e$ if $e \notin P' \cup C_{2s+1}$, and $x'_e = 1 - x^*_e$ if $e \in P'$, and $x'_e = \frac{1}{2}$ if $e \in C_{2s+1}$. From here, we drop the subindex $2s + 1$ to simplify the notation. Since the cycle has odd length, both neighbors of $u$ in $C$ have to be dotted. Therefore,

$$
\sum_{e \in E} w_e x^*_e - \sum_{e \in E} w_e x'_e = w(P'_\text{solid}) + w(C_{\text{solid}}) - w(P'_\text{dotted}) - \frac{w(C_{\text{dotted}}) + w(C_{\text{solid}})}{2} \tag{A.4}
$$

$$
x'_e = w(P'_\text{solid}) + \frac{w(C_{\text{solid}})}{2} - w(P'_\text{dotted}) - \frac{w(C_{\text{dotted}})}{2} \tag{A.5}
$$

$$
\leq \sum_{v \in P'} \gamma_v + \sum_{v \in C} \gamma_v - \gamma_u - \left( \sum_{v \in P'} \gamma_v - \gamma_u \right) - \sum_{v \in C} \gamma_v + \gamma_u \leq 0, \tag{A.6}
$$

which is again a contradiction.

• **Bicycle**: $P$ intersects itself at least twice and will contain two odd cycles $C_{2s+1}$ and $C'_{2s'+1}$ with a path (stem) $P'$ that is connecting them. Very similar to Blossom, let $x'_e = x^*_e$ if $e \notin P' \cup C \cup C'$, $x'_e = 1 - x^*_e$ if $e \in P'$, and $x'_e = \frac{1}{2}$ if $e \in C \cup C'$. The proof follows similar to the case of blossom.

**Case (II)**. Assume that the optimum LP solution $x^*$ is not necessarily integer.

Everything is similar to Case (I) but the algebraic treatments are slightly different. Some edges $e$ in $P$ can be $\frac{1}{2}$-solid ($x^*_e = \frac{1}{2}$). In particular some of the odd edges (dotted edges) of $P$ can now be $\frac{1}{2}$-solid. But the subset of $\frac{1}{2}$-solid edges of $P$ can be only sub-paths of odd length in $P$. On each such sub-path defining $x' = 1 - x^*$ means we are not affecting $x^*$. Therefore, all of the algebraic calculations should be considered on those sub-paths of $P$ that have no $\frac{1}{2}$-solid edge which means both of their boundary edges are dotted.

• **Path**: Define $x'$ as in Case (I). Using the discussion above, let $P^{(1)}, \ldots, P^{(r)}$ be disjoint sub-paths of $P$ that have no $\frac{1}{2}$-solid edge. Thus,

$$
\sum_{e \in E} w_e x^*_e - \sum_{e \in E} w_e x'_e = \sum_{i=1}^r \left[ w(P^{(i)}_{\text{solid}}) - w(P^{(i)}_{\text{dotted}}) \right].
$$

Since in each $P^{(i)}$ the two boundary edges are dotted, $w(P^{(i)}_{\text{solid}}) \leq \sum_{v \in P^{(i)}} \gamma_v$ and $\sum_{v \in P^{(i)}} \gamma_v = w(P^{(i)}_{\text{dotted}})$. The rest can be done as in Case (I).

• **Cycle, Blossom, Bicycle**: These cases can be done using the same method of breaking the path and cycles into sub-paths $P^{(i)}$ and following the case of path.

\[\square\]

**Lemma 18.** Every 1-solid edge is a strong-dotted edge. Also, every $\frac{1}{2}$-solid edge is a weak-dotted edge.

*Proof.* From Lemma 17 it follows that the set of dotted edges is a subset of the solid edges. In particular, no node can be adjacent to more than two dotted edges. Now using Lemma 12 the set of dotted edges is a disjoint union of isolated edges (strongly dotted) and cycles (weakly dotted edges). If we define a $x'$ to be zero on all non-dotted edges and $x'_e = 1$ when $e$ is strongly dotted and $x'_e = \frac{1}{2}$ for weakly dotted ones. Then clearly $x'$ is feasible to (1.5). On the other hand using Lemma 13 we have $\sum_{e \in E} w_e x'_e = \sum_{v \in V} \gamma_v$. But $\gamma$ is dual feasible which means $\sum_{v \in V} \gamma_v \geq \sum_{v \in V} y_v = \sum_{e \in E} w_e x^*_e$ which shows that $x'$ is also an optimum solution to (1.5). By uniqueness assumption on $x^*$ we obtain the desired result. \[\square\]
Lemma 19. $\gamma$ is an optimum for the dual problem.

Proof. Lemma 10 guarantees feasibility. Optimality follows from lemmas 13, 17 and 18.

A.1 Fixed points and the KT sequence

Lemma 20. Let $G$ be an instance admitting a NB solution $\alpha^*$ with gap $\sigma$, and KT sequence $(C_0, C_1, C_2, \ldots, C_k)$. Denote by $M^*$ the max weight matching of $G$. Then for any $q \in \{1, \ldots, k\}$ and any edge $(i, j) \in E(C_q)$,

$$\alpha^*_{i,j} + \alpha^*_{j,i} - w_{ij} = \begin{cases} -2\sigma_q & \text{if } (i, j) \in E_1(C_q), \\ \sigma_q & \text{if } (i, j) \in E_2(C_q) \end{cases}$$

(A.7)

Further, for any edge $(i, j) \not\in C_q$ for any $q$, s.t. $i \in C_q(i)$, $j \in C_q(j)$, we have

$$\alpha^*_{i,j} + \alpha^*_{j,i} - w_{ij} \geq \max(\sigma_{q(i)}, \sigma_{q(j)}).$$

(A.8)

Proof. Consider $(i, j) \in E(C_q)$.

Suppose $(i, j) \in M^*$, i.e. $(i, j) \in E_1(C_q)$. We know that $i, j \in V(C_q)$. Hence the node slacks for each of $i$ and $j$ are $\sigma_q$. Hence, $\gamma^*_{i,j} - \alpha^*_{i,j} = \gamma^*_{j,i} - \alpha^*_{j,i} = \sigma_q$. Also, $\gamma^*_{i} + \gamma^*_{j} = w_{ij}$. Hence

$$\alpha^*_{i,j} + \alpha^*_{j,i} - w_{ij} = -2\sigma_q.$$

Suppose $(i, j) \not\in M^*$, i.e. $(i, j) \in E_2(C_q)$. We know that $\alpha^*_{i,j} = \gamma^*_{i}$, $\alpha^*_{j,i} = \gamma^*_{j}$. (A.7) now yields

$$\alpha^*_{i,j} + \alpha^*_{j,i} - w_{ij} = \sigma_q$$

as required.

Now consider any edge $(i, j) \not\in C_q$ for any $q$. Consider $m^*_{j \rightarrow i}$. We must have $m^*_{j \rightarrow i} \leq \gamma^*_{i} - \sigma_{q(i)} = \alpha^*_{j,i} - \sigma_{q(i)}$. We now that $\alpha^*_{i,j} + \alpha^*_{j,i} - w_{ij} \geq 0$ so $m^*_{j \rightarrow i} \geq w_{ij} - \alpha^*_{j,i}$. Combining, we have $\alpha^*_{i,j} + \alpha^*_{j,i} - w_{ij} \geq \sigma_{q(i)}$. Similarly, $\alpha^*_{i,j} + \alpha^*_{j,i} - w_{ij} \geq \sigma_{q(j)}$. This completes the proof.

Note: If we include the condition given by Eq. (1.9) in the definition of $\sigma$, we have the stronger condition $\alpha^*_{i,j} + \alpha^*_{j,i} - w_{ij} \geq \sigma_{q(i)} + \sigma$ when $q(i) = q(j) \geq 1$.

B Proof of convergence lemmas

In this section we first prove some basic properties of the natural dynamics in Section B.1 Then in Sections B.2 B.3 the lemmas that are used for the proofs of convergence on basic structures (path, blossom, and bicycle) are proved.

Throughout these sections, we say that an alternating path or blossom is anchored [KT08] at its degree-1 node(s). Note that $V_{\text{ext}}(C_q) V(C_q)$ may contain anchored node(s), but no other nodes.

B.1 Basic properties

Claim 5. Consider message vectors $\alpha$ and $\beta$. Suppose, for $(ij) \in E$, $|\alpha_{i,j} - \beta_{i,j}| \leq \Delta$ and $|\alpha_{j,i} - \beta_{j,i}| \leq \Delta$. We have

$$|m_{i \rightarrow j}^{\alpha} - m_{i \rightarrow j}^{\beta}| \leq \Delta$$

(B.1)

where $m_{i \rightarrow j}^{\alpha}$ ($m_{i \rightarrow j}^{\beta}$) refers to the offers corresponding to message vector $\alpha$ ($\beta$).

Proof. This follows from definition 1.2 $m_{i \rightarrow j} = f(\alpha_{i,j}, \alpha_{j,i})$, where $f(x, y) : \mathbb{R}^2_+ \rightarrow \mathbb{R}_+$ is defined as

$$f(x, y) = \begin{cases} \frac{w_{ij} + x + y}{2} & x + y \leq w_{ij} \\ (w_{ij} - x)_+ & \text{otherwise}. \end{cases}$$

(B.2)
It is easy to check that $f$ is continuous everywhere in $\mathbb{R}^2_+$. Also, it is differentiable except in $\{(x, y) \in \mathbb{R}^2_+: x + y = w_{ij} \text{ or } x = w_{ij}\}$, and satisfies $\|\nabla f\|_1 = |\frac{\partial f}{\partial x}| + |\frac{\partial f}{\partial y}| \leq 1$ Hence, $f$ is Lipschitz continuous in the $L^\infty$ norm, with Lipschitz constant 1, leading to (1.1).

Proof of Lemma 7

Use Claim 5 with $\Delta = ||\alpha^0 - \beta^0||_\infty$ for every offer $m_{i,j}^0$ in the graph. The result follows from the update rule (1.4).

\[ \text{Proof of Lemma 5.} \]

Note that $\psi^{\infty}$ follows from the update rule (1.4).

Proof of Lemma 4.

By Eq. (A.8), we know that $\psi$ is continuous everywhere in $\mathbb{R}^2_+$, so there is $\|\psi\| = \alpha^0$ as required.
Case (ii): \( l \not\in V(G_{Q+1}) \)

Lemma 4 is applicable (same as Case (iii) in the proof of Lemma 4). Hence,

\[ m_{l \rightarrow i}^0 \leq \psi_i^* + \Delta - \delta \]  

(B.5)

With (B.4) and (B.5) established, we now check that \( \forall l \in \partial i \)

\[ |\alpha_{i,l}^1 - \alpha_{i,l}^*| \leq \Delta - \delta \]  

(B.6)

\[ \alpha_{i,lp}^* = \psi_i^* \]. We verify separately for each of the cases \( |S| = 0 \) and \( |S| > 0 \) that

\[ |\max_{j \in \partial i \setminus lp} m_{j \rightarrow i}^0 - \psi_j^*| \leq \Delta - \delta \]  

(B.7)

Combine with \( |\alpha_{i,lp}^0 - \alpha_{i,lp}^*| \leq \Delta - \delta \) to get (B.7) for \( l = ip \).

If \( l \in \partial i \setminus ip \), \( \alpha_{i,l}^* = \gamma_i^* \). We know that

\[ |m_{ip \rightarrow i}^0 - m_{ip \rightarrow i}^*| = |m_{ip \rightarrow i}^0 - \gamma_i^*| \leq (\Delta - \delta) \]  

(B.8)

Combining (B.7) and (B.8) leads to

\[ |\max_{j \in \partial i \setminus l} m_{j \rightarrow i}^0 - \gamma_j^*| \leq \Delta - \delta \]  

(B.9)

and hence to (B.6) for \( l \neq ip \).

Hence, B(\( Q + 1, \delta, 1 \)) holds as required. Induction on \( Q \) completes the proof. \( \square \)

Proof of Lemma 6. By Lemma 5, we know that B(\( q, \delta, t \)) holds for all \( t \geq 0 \).

Consider \( q = 0 \). Take any \( i \in V(C_0) \). By Lemma 5 for any \( ip \in \partial i \) we know that \( m_{ip \rightarrow i}^t \leq (\Delta - \delta) \).

Hence, \( \max_{i \in \partial i} m_{ip \rightarrow i}^t \leq (\Delta - \delta) \) for any \( j \in \partial i \), \( \forall t \geq 0 \). The result follows from \( \alpha_{i,lp}^0 \leq \Delta \) and the update rule [1.4].

Consider \( q > 0 \). Take any \( i \in V(C_q) \). We know that \((i, ip) \in M^* \) for some \( ip \in V(C_q) \). It follows from the assumption that \( U_{C,C_q}^t(\alpha_i^t) \leq \Delta - \delta \) that \( m_{ip \rightarrow i}^t - \gamma_i^* \leq (\Delta - \delta) \).

For any \( j \in \partial i \setminus ip \), we know that \( m_{j \rightarrow i}^t \leq m_{j \rightarrow i}^* + \Delta \leq \gamma_j^* + \Delta \leq \gamma_i^* + \Delta - \delta \). Hence, \( \max_{j \in \partial i \setminus l} m_{j \rightarrow i}^t - \alpha_{i,l}^* \leq (\Delta - \delta) \) for any \( l \in \partial i \setminus ip \). The result follows. \( \square \)

B.2 Path convergence

Definition 7. Recall the path structures from Section 4.1.

1. We use \( P_\ell \) to denote a path \( 0 - 1 - \ldots - \ell \).

2. We call a matching \( M_P \) on a path \( P \) with alternate edges included a ‘valid path matching’.

Lemma 21. Let \( P = P_\ell \), \( \ell \geq 1 \). Suppose we apply arbitrary boundary conditions \( \{b^t_i\}_{t \geq 0}, \{b^t_p\}_{t \geq 0} \). Consider the simplified dynamics on \( P_\ell \) wrt given boundary conditions, a valid path matching \( M_P \) and two different initial conditions \( \widehat{\alpha}^0 \) and \( \widehat{\beta}^0 \). We have,

\[ ||\hat{\alpha}^t - \hat{\beta}^t||_{\infty} \leq C\ell \exp\left(-\frac{t}{C\ell^2}\right) ||\alpha^0 - \beta^0||_{\infty}, \]  

(B.10)

for some \( C > 0 \). Also,

\[ ||\hat{\alpha}^t - \hat{\beta}^t||_{\infty} \leq ||\alpha^0 - \beta^0||_{\infty} \]  

(B.11)
Proof. Consider the difference $\Delta t^t = \Delta^t - \beta^t$. We know that $\|\Delta^0\|_\infty = \|\beta^0\|_\infty$. We bound $\Delta^t$ by the ‘mass’ in an appropriately defined random walk on the path.

Firstly, we write update equations for $\Delta$. We distinguish between the two ‘end’ edges that are at the ends of the path and the other ‘interior’ edges. For an interior non-matching edge $(i, i+1) \notin M_P$,

$$\Delta_{i,i+1}^{t+1} = \kappa \left( \frac{-\Delta_{i-1,i}^t + \Delta_{i,i-1}^t}{2} \right) + (1 - \kappa)\Delta_{i,i+1}^t$$

$$\Delta_{i+1,i}^{t+1} = \kappa \left( \frac{-\Delta_{i+2,i+1}^t + \Delta_{i+1,i+2}^t}{2} \right) + (1 - \kappa)\Delta_{i+1,i}^t \tag{B.12}$$

For an interior matching edge $(i, i+1) \in M_P$

$$\Delta_{i,i+1}^{t+1} = -\kappa\Delta_{i-1,i}^t + (1 - \kappa)\Delta_{i,i+1}^t$$

$$\Delta_{i+1,i}^{t+1} = -\kappa\Delta_{i+2,i+1}^t + (1 - \kappa)\Delta_{i+1,i}^t \tag{B.13}$$

For the end edges, we have $\Delta_{0,1}^{t+1} = \kappa\Delta_{0,1}^t$, $\Delta_{\ell-1,\ell}^{t+1} = \kappa\Delta_{\ell-1,\ell}^t \forall t \geq 0$. In the other direction, i.e. for $\Delta_{1,0}^{t+1}$ and $\Delta_{\ell-1,\ell}^{t+1}$ the relevant update rule from Eqs. (B.12) and (B.13) is applicable, depending on the type of edge.

Now, define $\rho^t_{i,i+1} = \rho^t_{i+1,i} = \|\Delta^0\|_\infty \forall i \in \{0, 1, \ldots, \ell - 1\}$. Update $\rho$ as follows. For an interior non-matching edge $(i, i+1) \notin M_P$,

$$\rho_{i,i+1}^{t+1} = \kappa \left( \frac{\rho_{i-1,i}^t + \rho_{i,i-1}^t}{2} \right) + (1 - \kappa)\rho_{i,i+1}^t$$

$$\rho_{i+1,i}^{t+1} = \kappa \left( \frac{\rho_{i+2,i+1}^t + \rho_{i+1,i+2}^t}{2} \right) + (1 - \kappa)\rho_{i+1,i}^t \tag{B.14}$$

For an interior matching edge $(i, i+1) \in M_P$

$$\rho_{i,i+1}^{t+1} = \kappa\rho_{i-1,i}^t + (1 - \kappa)\rho_{i,i+1}^t$$

$$\rho_{i+1,i}^{t+1} = \kappa\rho_{i+2,i+1}^t + (1 - \kappa)\rho_{i+1,i}^t \tag{B.15}$$

For the end edges, we define $\rho_{0,1}^{t+1} = \kappa\rho_{0,1}^t$, $\rho_{\ell-1,\ell}^{t+1} = \kappa\rho_{\ell-1,\ell}^t \forall t \geq 0$. In the other direction, i.e. for $\rho_{0,1}^{t+1}$ and $\rho_{\ell-1,\ell}^{t+1}$ the relevant update rule from Eqs. (B.14) and (B.15) is applicable, depending on the type of edge.

Claim 8. $\rho_{i,i+1}^t \geq |\Delta_{i,i+1}^t|$, $\rho_{i+1,i}^t \geq |\Delta_{i+1,i}^t| \forall i \in \{0, 1, \ldots, \ell - 1\}$ and $t \geq 0$

Proof. Clearly, the claim is true at $t = 0$. An elementary proof by induction can be constructed based on $|x + y| \leq |x| + |y|$. □

In light of claim 8 it is sufficient to prove that $\|\rho^t\|_\infty \leq C\ell \exp(-t/C\ell^2)\|\Delta^0\|_\infty$. Now, for a normalization $Z = 2\ell\|\Delta^0\|_\infty$, $\rho^t/Z$ can be thought of as the probability distribution of a particle performing the following random walk. The particle is positioned on an edge $\{1, \ldots, \ell\}$ and has a direction (right or left). It starts at a uniformly random edge, with uniformly random direction. If the edge is non-matching, the particle keeps its direction and moves along it to the next edge, with probability $\kappa$. If the edge is matching, with probability $\kappa/2$ it moves forward maintaining direction,
with probability $\kappa/2$ it reverses direction and moves forward one step, and with probability $(1 - \kappa)$ retains its status. The particle is disappears when crossing any of the ends of the path.

Notice that the particle position, when watched on matching edges, is a lazy random walk (i.e. a simple random walk that keeps its position with probability bounded away from 0 and 1), killed at 0 and $\ell'$, with $\ell' \leq (\ell + 1)/2$. Hence $\|\rho_i\|_{\infty}/Z$ is not larger than the survival probability of such random walk, which is bounded as in the claim.

It is easy to check that Claim \ref{claim:fixed-point} holds also for the simplified dynamics. \hfill \square

**Proof of Lemma 7.** This follows directly from (B.10) in Lemma 21. \hfill \square

**Lemma 22.** Let $G$ be an instance and $P \subseteq G$ be a path in $G$. Consider two initial conditions $\alpha^0$, $\beta^0$ on $G$, such that $\hat{\alpha}^0_{\partial P} = \hat{\beta}^0_{\partial P}$ (they coincide outside $P$) and $\alpha^0_P \geq \beta^0_P$ (on $P$). Then, after one iteration $\alpha^1_P \geq \beta^1_P$.

**Proof of Lemma 22.** Consider a node $i$ on $P$ with $i$ even. We compare the offers received by $i$ under $\alpha$ and $\beta$ at time $t$. Clearly, all offers coming in from $G - P$ are identical. If $i > 0$ and $i$ even, by condition (4.4) and the fact that $m_{i-1} - i$ is non-increasing in $\alpha_{i-1}$ and non-decreasing in $\alpha_{i-1}$, we know that $m_{i-1} - i \alpha^0 \geq m_{i-1} - i \beta^0$. Hence, $\alpha_{i+1}^1 \geq \beta_{i+1}^1$ as required. We can argue similarly for all other types of messages on $P$. \hfill \square

**Lemma 23.** Consider the simplified dynamics on a path instance $G = P_t$ wrt a valid path matching $\hat{\alpha}^t$. Let $\hat{\beta}^t$ denote the message vector resulting at time $t \geq 0$ with initialization $\hat{\alpha}^0$ and boundary conditions $\{b_{1L}^t\}_{t \geq 0}, \{b_{2L}^t\}_{t \geq 0}$. Similarly, let $\hat{\beta}^t$ denote the message vector resulting at time $t \geq 0$ with initialization $\hat{\beta}^0$ and boundary conditions $\{b_{1L}^t\}_{t \geq 0}, \{b_{2L}^t\}_{t \geq 0}$. If $\hat{\alpha}^0 \geq \beta^0$ and

$$b_{1L}^t \geq b_{2L}^t, \quad (-1)^t b_{1R}^t \geq (-1)^t b_{2R}^t, \quad \forall t \geq 0$$

then $\hat{\alpha}^t \geq \hat{\beta}^t \forall t \geq 0$.

**Proof of Lemma 23.** We show that $\hat{\alpha}^t \geq \hat{\beta}^t$. This follows for ‘internal’ messages from the fact that $\hat{m}_{i \to j}$ defined in (4.1) is monotonic non-increasing in $\hat{\alpha}_{i \to j}$ and non-decreasing in $\hat{\alpha}_{i \to j}$. $b_{1L}^0 > b_{2L}^0$ ensures that $\hat{\alpha}_{0,1} \geq \hat{\beta}_{0,1}^1$ and similarly for the other boundary message.

Induction on $t$ completes the proof of the quoted result. \hfill \square

Let $\hat{m}^t(s)$ denote the offers made along edges of $P$ under the bounding process $\hat{A}^t(s), t \geq 0$.

As discussed in Section 4.1 we have the following unique fixed point for the bounding process:

$$A^*_{i+1} = \alpha^*_{i+1} + s(-1)^i(\Delta - \delta), \quad A^*_{i+1} = \alpha^*_{i+1} - s(-1)^i(\Delta - \delta),$$

for all $i \in \{0, 1, \ldots, \ell - 1\}$.

Clearly, $\hat{A}^*(+) \leq \hat{A}^0(+)$. Hence, by Lemma 23 with $b_{1L}^t = b_{2L}^t$, we know that $\hat{A}^*(+) \leq \hat{A}^t(+) \forall t \geq 0$. Also, from (B.11) we have $\|\hat{A}^t - \hat{A}^*\|_{\infty} \leq \|\hat{A}^0 - \hat{A}^*\|_{\infty} = \delta$ leading to $\hat{A}^t(+) \leq \hat{A}^0(+)$. Combining, for all $t \geq 0$ we have

$$\hat{A}^*(+) \leq \hat{A}^t(+) \leq \hat{A}^0(+) \quad \ldots (B.17)$$

**Lemma 24.** Let $\{\hat{A}^t\}_{t \geq 0}$ be the $(s, \Delta, \delta)$-bounding process on $P$. There exists a finite $c_{\text{raw}}$ such that for any $\epsilon \in (0, \delta)$, for $t > c_{\text{raw}}/2^1(1 + \log(\delta/\epsilon))$,

$$\|\hat{A}^t - \alpha^*\|_{\infty} \leq \Delta - \delta + \epsilon \quad \ldots (B.18)$$
\textit{Proof.} $||A^0 - A^*||_{\infty} = \delta$. Hence, by Lemma 21 a finite $c$ exists s.t.

$$||A^t - A^*||_{\infty} \leq \epsilon \quad \forall t \geq c_{RW} \ell^{2.1} (1 + \log(\delta/\epsilon)).$$

(B.19)

Also, $||A^* - \tilde{\alpha}^*||_{\infty} = \Delta - \delta$. The result follows.

\textit{Proof of Lemma 8.} It follows from (B.17) that for any $(i, j) \in E_P$ and all $t \geq 0$,

$$|(A_{i,j}^t + A_{j,i}^t) - (\alpha_{i,j}^* + \alpha_{j,i}^*)|.$$  \hspace{1cm} (B.20)

Combining Eqs. (B.20), and (A.9), and noting that $\delta \leq \alpha \leq \sigma I$ we obtain our result.

\textit{Proof of Lemma 9.} We proceed by induction. Clearly, Eq. (4.5) holds at $t = 0$. Suppose it holds at time $t$. We prove $\alpha_{i,j}^{t+1} \leq \alpha_{i,j}^{t+1}(+)$. We obtain from (B.17) and (B.21) that $\tilde{\alpha}_{i,j}^t$ is $A^t$ after thresholding it to keep it in the valid range $[0, W]^{2|E|}$. Clearly,

$$\alpha_{i,j}^t \preceq \tilde{\alpha}_{i,j}^t(+) \preceq A_i^t(+) \hspace{1cm} (B.21)$$

In words, the right inequality says that thresholding can only reduce the positively bounding components of $A_i^t$ and only increase the negatively bounding components. Let $\beta^t = (\hat{\alpha}_{i,j}^t - E_{\cdot i}, \tilde{\alpha}_{i,j}^t)$. Note that $\beta^t$ is a valid message vector. It follows from (B.17) and (B.21) that $B(q, \delta)$ (cf. definition 6) holds for $\beta^t$. We let $\beta^t$ evolve for one time step under the original dynamics. Let the resulting messages on the path be $B^{t+1}(+)$. By Lemma 22 and the first inequality in (B.21), we know that $B^{t+1}(+) \succeq \alpha_{i,j}^{t+1}$. We show that $\alpha_{i,j}^{t+1}(+) \succeq B^{t+1}(+)$. Consider a node $i$ on $P$ with $i$ even, $0 < i < \ell$. We show that $\alpha_{i,i+1}^{t+1} \preceq \alpha_{i,i+1}^{t+1}$. Firstly, we know that $\tilde{\alpha}_{i-1\rightarrow i}(+) \geq m_{i-1\rightarrow i} + (\Delta - \delta)$ from the first inequality in (B.17). We have,

$$\tilde{\alpha}_{i-1\rightarrow i}(+) \overset{(a)}{=} w_{i-1,i} - A_{i-1\setminus i}^t - \left( \frac{w_{i-1,i} - A_{i-1\setminus i}^t - A_{i\setminus i}^t}{2} \right) +$$

$$\overset{(b)}{=} \left( w_{i-1,i} - A_{i-1\setminus i}^t \right) - \left( \frac{w_{i-1,i} - A_{i-1\setminus i}^t - A_{i\setminus i}^t}{2} \right) +$$

$$\overset{(c)}{=} \left( w_{i-1,i} - \tilde{A}_{i-1\setminus i}^t \right) - \left( \frac{w_{i-1,i} - \tilde{A}_{i-1\setminus i}^t - \tilde{A}_{i\setminus i}^t}{2} \right) +$$

$$\overset{(d)}{=} m_{i-1\rightarrow i}(+) \hspace{1cm} (B.21)$$

(a) follows from Lemma 6 (b) follows from $\tilde{\alpha}_{i-1\rightarrow i}(+) > 0 \Rightarrow w_{i-1,i} - A_{i-1\setminus i}^t > 0$. (c) holds since $\tilde{A}_{i-1\setminus i}^t \geq A_{i-1\setminus i}^t, \tilde{A}_{i\setminus i-1} \leq A_{i\setminus i-1}$. (d) follows since $A_{i\setminus i-1}^t \geq \alpha_{i\setminus i-1}^t \geq 0$. From Lemma 4 for all $j \in \partial i \setminus i - 1$ we have,

$$m_{j\rightarrow i}(\gamma_i) \leq \gamma_i^* - \sigma_q - \delta + \Delta \leq m_{i-1\rightarrow i} + \Delta - \delta \leq \tilde{m}_{i-1\rightarrow i}(+)$$

Thus, $\tilde{m}_{i-1\rightarrow i}(+) \geq \max_{j \in \partial i \setminus i+1} m_{j\rightarrow i}(\gamma_i^*)$. Also, $A_{i\setminus i+1}^t \geq \tilde{A}_{i\setminus i+1}^t = \beta_{i\setminus i+1}^t$. Hence, $A_{i\setminus i+1}^t \geq B_{i\setminus i+1}^t$. 

Now consider a node $i$ on $P$ with $i$ odd, $0 < i < \ell$. We show that $A_{i,i+1}^{t+1} \leq B_{i,i+1}^{t+1}$. Under the simplified dynamics, in light of Lemma 4 we know that

$$
\tilde{m}_{i-1}^t \rightarrow = w(i,i) - A_{i,i-1}^t - \left( \frac{w(i,i-1) - A_{i-1,i}^t - A_{i,i-1}^t}{2} \right) + \\
\leq (w(i,i-1) - \tilde{A}_{i-1,i}^t - \tilde{A}_{i,i-1}^t) + \left( \frac{w(i,i-1) - \tilde{A}_{i-1,i}^t - \tilde{A}_{i,i-1}^t}{2} \right) + \\
= m_{i-1}^t |B^t| + \\
\leq \max_{j \in \partial i} m_{j \rightarrow i}^t |B^t|
$$

where we used $\tilde{A}_{i-1,i}^t \leq A_{i-1,i}^t$, $\tilde{A}_{i,i-1}^t \geq A_{i,i-1}^t$. Also, $A_{i,i+1}^t \leq \tilde{A}_{i,i+1}^t = \beta_{i,i+1}^t$. Hence, $A_{i,i+1}^{t+1} \leq B_{i,i+1}^{t+1}$.

The required inequalities for $A_{i,i+1}^{t+1}$, can be proved similarly.

This leaves us with checking that $B_{0,1}^{t+1} \leq A_{0,1}^{t+1}$ and $(-1)^t A_{\ell,\ell-1}^{t+1} \geq (-1)^t B_{\ell,\ell-1}^{t+1}$. We prove the second inequality and the first one follows from repeating the analysis for the $\ell$ even case.

If $\ell \notin V(G_q) \Rightarrow \ell \in G_q$ and the result follows from Lemma 5 applied to the initial state $\beta^t$ for one step along with (B.17). Else, $\ell \in V(G_q) \Rightarrow (\ell, \ell - 1) \in M^*, \gamma_\ell^* = \sigma_q, \alpha_{\ell,\ell-1}^* = 0$.

Case (i): \ell odd

Follows from $A_{\ell,\ell-1}^{t+1} \leq 0 \leq B_{\ell,\ell-1}^{t+1}$.

Case (ii): \ell even

From Lemma 4 we obtain $m_{j \rightarrow \ell} B^{t} \leq (\Delta - \delta) \forall j \in \partial \ell \setminus 1$. Also, $\beta_{\ell,\ell-1}^t = \tilde{A}_{\ell,\ell-1}^t \leq A_{\ell,\ell-1}^t$. $B_{\ell,\ell-1}^{t+1} \leq A_{\ell,\ell-1}^{t+1}$ follows.

$$A_{\ell,\ell-1}^{t+1} \leq \alpha_{\ell,\ell-1}^t \leq \beta_{\ell,\ell-1}^t \leq \tilde{A}_{\ell,\ell-1}^t$$

follows similarly. Induction on $t$ completes the proof. \hfill \Box

We also generalize our analysis to the case for use in analysis the cases of blossom and bicycle. Suppose $P$ is a subgraph of $(V_{ext}(C_q), E(C_q))$, with the subset $V_P$ of nodes in $V_{ext}$ appropriately given the aliases $0, 1, \ldots, \ell$. We define the bounding processes as in Section 4.1 on $P$, using the matching $M_P^*$ that is the restriction of $M^*$ to $P$. We prove the following refinement of Lemma 6.

**Lemma 25.** Let $G$ be an instance admitting a unique NB solution with gap $\sigma$, and assume its KT sequence to be $(C_0, C_1, C_2, \ldots, C_k)$. Let path $P = P_\ell$ be a subgraph of $(V_{ext}(C_q), E(C_q))$ for some $q \in \{1, \ldots, k\}$, such that $\partial P = \{(i, j) : i \in V_{ext}(C_q), E(C_q)\}$ for some $q \in \{1, \ldots, k\}$, such that $\partial P = \{(i, j) : i \in V_{ext}(C_q), E(C_q)\}$ is identical to $\partial P = \{(i, j) : i \in V_{ext}(C_q) \setminus V(P), j \in V(P), (i, j) \in E(C_q)\}$ i.e. $P$ touches the rest of $C_q$ only at its ends. Assume $\mathbb{B}_\Delta(q, \delta, 0)$ holds. Also, suppose that

$$U_{G, \partial P}(\alpha^t) \leq \Delta - \delta' \quad \forall t \geq 0$$

for $\delta' \in [0, \delta]$. If we denote by $\{A^t(s)\}_{t \geq 0}$ the $(s, \Delta, \delta')$ bounding process on $P$, then for any $t \geq 0$:

$$A^t(-) \leq \alpha^t_P \leq A^t(+) \quad \text{ (B.22)}$$

**Proof.** $\mathbb{B}_\Delta(q, \delta, 0) \Rightarrow \mathbb{B}_\Delta(q, \delta', 0)$. The proof here is almost identical to the one for Lemma 4 with $\delta'$ replacing $\delta$. We use induction on $t$. The required inequalities at $t + 1$ for internal messages are proved exactly as before. This leaves us with the boundary messages.
We must show $A_{0^11}^{t+1}(-) \leq \alpha_{01}^{t+1} \leq A_{01}^{t+1}(+)$. $A_{01}^{t}(-) \leq \alpha_{01}^{t} \leq A_{01}^{t}(+)$ holds by hypothesis. Thus it suffices to show that

$$\max_{j \in \partial 01} m_{j \to 01}^j - \alpha_{01}^* \leq \Delta - \delta'$$

(B.23)

If $0 \notin V(C_q) \Rightarrow 0 \in G_q$ and the result follows from Lemma 5 applied to the initial state $\alpha^t$ for one step along with (B.17). Else, $0 \in V(C_q) \Rightarrow 0 \in M^*$. Two cases arise.

Case (i): $\alpha_{01}^* > 0$

In this case, there exists a $(v, 0) \in \partial P$ for some $v \in V_{ext}(C_q) \setminus P$ s.t. $m_{v \to 0} = \alpha_{01}^* \geq \gamma_0^* - \sigma_q$. $m_{v \to 0} \geq m_{v \to 0} - (\Delta - \delta')$ yields the lower bound in (B.23).

For any $(j, 0) \in \partial P$, we have $m_{j \to 0} \leq m_{j \to 0} + \Delta - \delta' \leq \alpha_{01}^* + \Delta - \delta'$. For any $(j, 0) \notin \partial P \cup \{(1, 0)\}$, Lemma 4 yields $m_{j \to 0} \leq (\gamma_0^* - \sigma_q - \Delta + \delta)_+ \leq \alpha_{01}^* + \Delta - \delta'$. Thus we have the required upper bound.

Case (ii): $\alpha_{01}^* = 0$

In this case there is no $(v, 0) \in \partial P$, and we know that $(0, 1) \in M^*$ and $\gamma_0^* = \sigma_q$. The lower bound is trivial in this case. Lemma 4 leads to $m_{j \to 0} \leq (\Delta - \sigma)_+, \forall j \in \partial 01$. The upper bound follows. The inequality

$$(-1)\ell A_{\ell \ell - 1}^{\ell + 1}(-) \leq (-1)\ell A_{\ell \ell - 1}^{\ell + 1} \leq (-1)\ell A_{\ell \ell - 1}^{\ell + 1}(+)$$

follows similarly.

\[ \square \]

### B.3 Blossom convergence

The blossom consists of a ‘stem’ and a ‘cycle’. There is a node shared by the stem and the cycle with its matching partner in the stem; call this node 0. Starting at 0 and going down the stem to its anchored end, we label the next node 1, the one after that 2 and so on up to $(\ell_s)_s$, where $\ell_s$ is the number of edges in the stem. We denote by $P_s$ the path $0 - 1_s - 2_s - \ldots - (\ell_s)_s$. The matching of interest in $P_s$ is always $M_{P_s}^* = \{(0, 1_s), (2_s, 3_s), \ldots \}$ with every alternate edge matched. Similarly, starting at 0 and going around the cycle in an arbitrary direction, we label the next node 1, the one after that 2, and so on up to $(\ell_c - 1)$, after which we return to 0. Here $\ell_c$ is the (odd) number of edges in the cycle. The cycle can be thought of as a path that is closed on itself i.e. the two ends of the path are the same. Denote by $P_c$ this path $0_c - 1_c - 2_c - \ldots - (\ell_c - 1)_c - (\ell_c)_c$, where 0, and $\ell_c$ are copies of the node 0. The matching of interest in $P_c$ is $M_{P_c}^* = \{(i_c, (i + 1)_c) : 1 \leq i \leq (\ell_c - 2), i \text{ odd}\}$. Note that both end edges $(0_c, 1_c)$ and $((\ell_c - 1)_c, (\ell_c)_c)$ i.e. the edges $(0, 1_c)$ and $((\ell_c - 1)_c, 0)$ in the blossom are unmatched.

Let $G$ be an instance admitting a unique NB solution $\alpha^*$ with gap $\sigma$. Suppose $C_q$ corresponds to a blossom i.e. $(V_{ext}(C_q), E(C_q))$ is a blossom with stem $P_s$ and cycle mapped to $P_c$. Nodes in $V_{ext}$ are appropriately given the aliases $0, 1_s, \ldots, (\ell_s)_s, 1_c, 2_c, \ldots, (\ell_c)_c$. 0c and $\ell_c$ are copies of node 0.

We develop a new bounding process for the path $P_c$ and show that this bounding process ‘contracts’ reducing the error on $P_c$. The fact that the cycle is odd plays a critical role in this contraction. Next, the analysis from Section 4.1 is used to show reduction of the error in $P_s$. We iterate the reduction in error in the stem and cycle to show that the overall error in the blossom reduces.

In the rest of this section, we sometimes drop the subscript $c$ e.g. Node 1 refers to node 1c, since the new results developed are for $P_c$ and not $P_s$.

Let $M_{P_c}^*$ be the restriction of $M^*$ to $P_c$. We know that $\ell = \ell_c$ is odd. Assume $\Delta > 0, \delta \leq \min(\sigma, \Delta), s \in \{+1, -1\}$ and $\Sigma = (\Sigma_+, \Sigma_-)$ be a partition of $\mathbb{N} \cup \{0\}$ (i.e. $\Sigma_+ \cup \Sigma_- = \mathbb{N} \cup \{0\}$, $\Sigma_+ \cap \Sigma_- = \emptyset$). Let $\delta_c, \delta_s \in (0, \delta]$. The $(s, \Delta, \delta, \Sigma, \delta_c, \delta_s)$-bounding process on $P_c$ is the sequence of
message assignments \( \{A^t_i\}_{i \geq 0} \) produced by the simplified dynamics on \( P_c \) wrt \( M^*_{P_c} \), and the boundary conditions

\[
\begin{align*}
    b_L^t &= A^*_{0\setminus 1}(s) + s\, \delta_s, & t &\in \Xi_+, \\
    b_R^t &= A^*_{\ell \setminus \ell - 1}(s), & t &\in \Xi_+,
\end{align*}
\]

\[
\begin{align*}
    b_L^t &= A^*_{0\setminus 1}(s), & t &\in \Xi_+, \\
    b_R^t &= A^*_{\ell \setminus \ell - 1}(s) - s\delta_s, & t &\in \Xi_-.
\end{align*}
\]

with the initial condition (for \( i \in \{0, \ldots, \ell - 1\} \))

\[
A^0_{i\setminus i + 1} = A^*_{i\setminus i + 1}(s) + s(-1)^i\delta_c, \quad A^0_{i+1\setminus i} = A^*_{i+1\setminus i}(s) - s(-1)^i\delta_c,
\]

where \( A^*(s) \) is given by Eq. (B.16).

The fact that only one of the boundary conditions is different from \( A^* \) at each time is instrumental in the ‘contraction’ of this bounding process. The odd length of the cycle is crucial in ensuring that such a bounding process, with a ‘helpful’ boundary condition at at least one end, sandwiches the actual dynamics (cf. Lemma 29).

**Lemma 26.** Let \( \{A^t_i\}_{i \geq 0} \) be the \( (s, \Delta, \delta, \Xi, \delta, \delta_b) \)-bounding process on \( P_c \) wrt \( M^*_{P_c} \) for some \( \delta \leq \min(\sigma, \Delta) \) and \( \delta_c, \delta_b \in (0, \delta] \). Then for arbitrary \( \Xi = (\Xi_+, \Xi_-) \), \( \exists c_1 \) finite and \( c_2 > 0 \) s.t. for any \( \epsilon > 0 \),

\[
\begin{align*}
    ||A^t_i - A^*||_\infty &\leq \delta_s + \epsilon \\
    \max(|A^t_{i\setminus 0} - A^*_{i\setminus 0}|, |A^t_{\ell - 1\setminus \ell} - A^*_{\ell - 1\setminus \ell}|) &\leq \delta_s \left(1 - \frac{c_2}{\ell^2}\right) + \epsilon
\end{align*}
\]

hold \( \forall t \geq c_1 \ell^{2.1}[\log(1 + \delta_c/\epsilon)] \).

**Proof.** After taking differences with respect to \( A^* \), \( \tilde{\Delta}^t = A^t - A^* \), the initialization becomes

\[
\tilde{\Delta}^0_{i\setminus i + 1} = s(-1)^i\delta_c, \quad \tilde{\Delta}^0_{i+1\setminus i} = -s(-1)^i\delta_c, \quad \text{for } i \in \{0, \ldots, \ell - 1\}
\]

the boundary conditions

\[
\begin{align*}
    \tilde{b}_L^t &= s\delta_s, & t &\in \Xi_+, \\
    \tilde{b}_R^t &= 0, & t &\in \Xi_+,
\end{align*}
\]

\[
\begin{align*}
    \tilde{b}_L^t &= 0, & t &\in \Xi_-, \\
    \tilde{b}_R^t &= -s\delta_s, & t &\in \Xi_-
\end{align*}
\]

and the dynamics reduces to (B.12), (B.13) for internal edges, and

\[
\begin{align*}
    \tilde{\Delta}^{t+1}_{0\setminus 1} &= \kappa \tilde{b}_L^t + (1 - \kappa)\tilde{\Delta}^t_{0\setminus 1} \\
    \tilde{\Delta}^{t+1}_{\ell\setminus \ell - 1} &= \kappa \tilde{b}_R^t + (1 - \kappa)\tilde{\Delta}^t_{\ell\setminus \ell - 1}
\end{align*}
\]

for boundary edges, for \( t \geq 0 \).

Thus, in order to prove Lemma 26 it suffices to prove the Lemma 27 which is then combined with the contraction result in Lemma 7.

**Lemma 27.** Consider a path \( P_\ell \) (with no edge weights) for \( \ell \) odd, a valid path matching \( M_P \) and arbitrary partition \( \Xi = (\Xi_+, \Xi_-) \) of \( \mathbb{N} \cup \{0\} \). Consider the dynamics on the vector \( \rho \) defined by (B.14) and (B.15) for interior messages and

\[
\begin{align*}
    \rho^{t+1}_{0\setminus 1} &= \kappa \tilde{b}_L^t + (1 - \kappa)\rho^t_{0\setminus 1}, \\
    \rho^{t+1}_{\ell\setminus \ell - 1} &= \kappa \tilde{b}_R^t + (1 - \kappa)\rho^t_{\ell\setminus \ell - 1},
\end{align*}
\]

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for boundary messages, with
\[ b^t_L = 1, \quad b^t_R = 0 \] if \( t \in \mathcal{T}_+ \),
\[ b^t_L = 0, \quad b^t_R = 1 \] if \( t \in \mathcal{T}_- \).

Assume the initial condition to be \( \rho^0 = 0 \). Then for all \( t \geq 0 \) we have
\[ \rho^t_{i,j} \leq 1 \quad \forall (i,j) \in P, \tag{B.38} \]
\[ \max(\rho^t_{0,1}, \rho^t_{-1,0}) \leq 1 - \frac{c_2}{\ell^3}. \tag{B.39} \]

for some \( c_2 > 0 \) not dependent on \( \ell \).

**Proof.** Equation (B.38) follows immediately by induction using equations (B.14) and (B.15).

In order to prove Eq. (B.39), recall the random walk interpretation of the equations (B.14) and (B.15) provided in the proof of Lemma 12. With the new boundary conditions, \( \rho^t_{i,j} \) is the expected number of particles on edge \((i,j)\), and directed from \( i \) to \( j \), when the path system is initially empty (since \( \rho^0 = 0 \)) and a particle is injected with probability \( \kappa \) at times \( t \in \mathcal{T}_+ \) on edge \((0,1)\) (directed to the right), and at times \( t \in \mathcal{T}_+ \) on edge \((\ell, \ell - 1)\) (directed to the left).

By invariance of the random walk process under translation, we can consider an extension \( \mathcal{T}^{(-\infty)} = (\mathcal{T}_+^{(-\infty)}, \mathcal{T}_-^{(-\infty)}) \) of the above partition, and focus on the case \( t \to \infty \), which we will do hereafter.

Since the random walk process is left unchanged under left-right inversion of the path (\( \ell \) is odd), we have \( \rho^t_{-1,\ell} = \rho^t_{1,0} \), where \( \rho^t_{i,j} \) is the expected number of particles on edge \((i,j)\) under boundary condition that inverts \( \mathcal{T}_+^{(-\infty)} \) and \( \mathcal{T}_-^{(-\infty)} \). It is therefore sufficient to bound
\[ \max(\rho^t_{0,1}, \rho^t_{1,0}) = \rho^0_{1,0} + \rho^0_{0,1} - \min(\rho^0_{1,0}, \rho^0_{0,1}). \tag{B.40} \]

Notice that, by linearity, \( \rho^0_{1,0} + \rho^0_{0,1} \) is the expected number of particles on edge \((0,1)\) (directed towards 0) when a particle is injected at each time \( -1, -2, \ldots \) at both ends with probability \( \kappa \). It is easy to check that the stationary distribution is, in this case, of 1 expected particle per edge, yielding \( \rho^0_{1,0} + \rho^0_{0,1} = 1 \).

We are left with the task of showing that \( \min(\rho^0_{1,0}, \rho^0_{0,1}) \geq c_2/\ell^3 \). Again by monotonicity with respect to the particle injection, it is sufficient to show that there exist at least one time \( t < 0 \) such that the following two facts are true: (1) A particle injected at time \( t \) on edge \((0,1)\) is on edge \((1,0)\) at time 0 with probability larger than \( c_2/\ell^3 \); (2) A particle injected at time \( t \) on edge \((\ell, \ell - 1)\) is on edge \((1,0)\) at time 0 with probability larger than \( c_2/\ell^3 \).

Recall that the process when watched on matching edges, is a lazy random walk. Therefore, we reduced our problem to proving the following elementary fact about the gambler’s ruin model: There exists a time \( t \) such that both probabilities that the gambler has one dollar and that he has \( m' - 1 \) dollars \((m' \text{ being the maximum})\) are at least \( c_2/(m')^3 \). It is easy to check that this is the case, for some \( t = \Theta((m')^2) \) (see, for example, [Sp05]). Note that this analysis, which holds for \( \ell \geq \ell_0 \) for some \( \ell_0 < \infty \), suffices – for any fixed value of \( \ell \), we know that \( \exists t < 0 \) such that a particle injected at time \( t \) on edge \((0,1)\) is on edge \((1,0)\) with non-zero probability AND a particle injected at time \( t \) on edge \((\ell, \ell - 1)\) is on edge \((1,0)\) with non-zero probability.
Lemma 28. Let \( \{A^t\}_{t \geq 0} \) be the \((s, \Delta, \delta, \tau, \delta_c, \delta_s)\)-bounding process on \( P_c \) wrt \( M^*_\ell \) for some \( \delta \leq \min(\sigma, \Delta) \) and \( \delta_c, \delta_s \in (0, \delta] \). Then for any \( t \), and for any \((i, j) \in E_P\)
\[
\begin{align*}
A^t_{i,j} + A^t_{j,i} - w_{ij} &\leq 0 & \text{if } (i, j) \in M^*_\ell, \\
A^t_{i,j} + A^t_{j,i} - w_{ij} &\geq 0 & \text{if } (i, j) \notin M^*_\ell.
\end{align*}
\]

Proof. Consider \( s = (+) \). We will use Lemma 23 to bound \( A^t \) from both sides. Let \( \{b^t_i\}, \{b^t_j\} \) denote the boundary conditions for \( \{A^t(+)\} \) as defined in (B.24), (B.25). Let \( b^t_{s,0} = A^*_{0,1} \), \( b^t_{s,0} = A^*_{\ell,1} \).

Let \( b^t_{s,0} = A^*_{0,1} + \delta_c \), \( b^t_{s,0} = A^*_{\ell,1} - \delta_c \). (Recall that \( \ell \) is odd.) Clearly, \( A^0 \) is a fixed point with respect to the constant boundary conditions \( b^t_{s,0}, b^t_{s,0} \). We have,
\[
b^t_{s,0} \leq b^t_{s,0} \leq b^t_{s,0} \quad \forall t \geq 0
\]

Also, \( A^* \leq A^0 \). Using Lemma 23 twice we have
\[
A^*(+) \leq A^t(+) \leq A^0(+) \quad \forall t \geq 0
\]

Recall \( \delta_c \leq \delta \). The rest of the proof mirrors that of Lemma 8.

A similar proof can be constructed for \( s = (-) \).

□

Lemma 29. Let \( G \) be an instance admitting a unique NB solution with gap \( \sigma \), and assume its KT sequence to be \((C_0, C_1, C_2, \ldots, C_k)\), with \( C_q, q \in \{1, \ldots, k\} \) a blossom mapped to stem \( P_s \) and cycle \( P_c \). Suppose \( B_\Delta(q, \sigma, 0) \) holds for some \( \delta \leq \min(\sigma, \Delta) \). Also, assume
\[
\begin{align*}
U_{G,P_s}(A^t) &\leq \Delta - \delta + \delta_s \quad \forall t \geq 0 \\
U_{G,P_c}(A^0) &\leq \Delta - \delta + \delta_c
\end{align*}
\]

for \( \delta_c, \delta_s \in (0, \delta] \). Define
\[
\begin{align*}
\tau &\equiv \{ t : t \in \mathbb{N} \cup \{0\}, |m^t_{s,0} - m^*_{s,0}| \leq 0 \} \\
\tau &\equiv \{ t : t \in \mathbb{N} \cup \{0\}, |m^t_{s,0} - m^*_{s,0}| \leq 0 \} \\
\tau &\equiv (\tau_+ \cap \tau_-)
\end{align*}
\]

Denote by \( \{A^t(s)\}_{t \geq 0} \) the \((s, \Delta, \delta, \tau, \delta_c, \delta_s)\) bounding process on \( P_c \). Then for any \( t \geq 0 \):
\[
A^t(-) \leq A^*_P \leq A^t(+).
\]

Proof. The proof is by induction on \( t \). (B.46) holds at \( t = 0 \) by assumptions. Suppose it holds at \( t \). The proof of the required inequalities at \( t + 1 \) for internal messages mirrors the proof of Lemma 9.

We must show \( A^{t+1}(-) \leq A^{t+1}_{0,1} \leq A^{t+1}(+) \). By hypothesis, \( A^{t+1}_{0,1}(-) \leq A^{t+1}_{0,1} \leq A^{t+1}_{0,1}(+) \). From Claim 5 we know that
\[
|m^t_{s,0} - m^*_{s,0}| \leq \Delta - \delta + \delta_s
\]

Case (i): \( m^t_{s,0} \geq m^*_{s,0} \) i.e. \( t \in \tau_+ \)

It suffices to show that
\[
\max_{j \in \partial_0 \setminus 1} m^t_{j,0} \in \left[ A^0_{0,1} - (\Delta - \delta), A^0_{0,1} + (\Delta - \delta) + \delta_s \right]
\]
The lower bound follows from $m_{1,s}^t - m_{1,s}^0 = \alpha_{0,1}^t \geq \alpha_{0,1}^0 - (\Delta - \delta)$. We have

$$m_{j,0} \leq m_{j,0}^* + \Delta \leq m_{1,s}^* - \sigma_q + \Delta \leq \alpha_{0,1}^0 + (\Delta - \delta), \quad \forall \ j \in \partial 0 \backslash 1_s \quad (B.49)$$

(B.47) and (B.49) give the upper bound.

It suffices to show that

$$\max_{j \in \partial 0 \backslash 1_s} m_{j,0} \in [\alpha_{0,1}^0 - (\Delta - \delta) - \delta, \alpha_{0,1}^0 + (\Delta - \delta)]$$

(B.50)

The lower bound follows from (B.47). The upper bound follows from (B.49) and $m_{1,s}^t - m_{1,s}^0 = \alpha_{0,1}^t$.

Case (ii): $m_{1,s}^t < m_{1,s}^0$ i.e. $t \in \mathbb{T}_-$

We show that $\alpha_{0,1}^t$ is such that $\alpha_{0,1}^* \geq \alpha_{0,1}^t$.

We have

$$A^{t+1}_0(-) \geq \alpha^{t+1}_0 \geq A^{t+1}_0(+) \quad (+) \ follows \ similarly.$$

Induction on $t$ completes the proof. □

**Lemma 30.** Let $G$ be an instance admitting a unique NB solution $\alpha^*$. Suppose $C_q$ corresponds to a blossom and suppose the $B(q, \delta, 0)$ holds for some $\delta \leq \min(\sigma, \Delta)$. We have

$$U_{G,P}(\alpha^t) \leq \Delta - \delta + \frac{\delta}{10n} \quad \forall \ t \geq Cn^6 \quad (B.51)$$

for some $C$ finite.

**Proof.** Define $\delta_s(0) = \delta_b(0) = \delta$. For $i = 1, 2, \ldots, N$ define

$$\delta_b(i) = \delta \left(1 - \frac{c_2}{4n^3}\right)^{i-1} \left(1 - \frac{c_2}{2n^3}\right) \quad (B.52)$$

$$\delta_s(i) = \delta \left(1 - \frac{c_2}{4n^3}\right)^{i} \quad (B.53)$$

where $N$ is such that $\delta_s(N) < \frac{\delta}{10n} \leq \delta_s(N - 1)$. Clearly, $N \leq c_5n^{3.1}$ for some finite $c_5$.

We show that there exists $c_3, c_4$ finite such that $\forall i \in \{1, 2, \ldots, N\}$

$$\max(|\alpha^t_{1,0} - \alpha^*_1|, |\alpha^t_{(\ell-1)0} - \alpha^*_{(\ell-1)0}|) \leq \Delta - \delta + \delta_b(i) \quad \forall \ t \geq (ic_3 + (i - 1)c_4)n^{2.2} \quad (B.54)$$

$$U_{G,P}(\alpha^t) \leq \Delta - \delta + \delta_s(i) \quad \forall \ t \geq i(c_3 + c_4)n^{2.2} \quad (B.55)$$

where $c_2$ from Lemma 29.

We prove (B.54), (B.55) by induction on the iteration number $i$, defining $c_3$ and $c_4$ along the way. Clearly, (B.55) holds for $i = 0$. Suppose it holds for $i$. The iteration $i + 1$ consists of two phases.

Phase I:

Use Lemma 29 and Lemma 26 with $\delta_s = \delta_s(i), \delta_c = \delta, \epsilon = \frac{\delta}{10n} \cdot \frac{c_2}{2m^3}$. We have

$$\max(|\alpha^t_{1,0} - \alpha^*_1|, |\alpha^t_{(\ell-1)0} - \alpha^*_{(\ell-1)0}|) \leq \Delta - \delta + \delta_b(i)(1 - \frac{c_2}{n^3}) + \frac{\delta}{10n} \cdot \frac{c_2}{2n^3} \quad (B.56)$$

$$\leq \Delta - \delta + \delta_b(i)(1 - \frac{c_2}{2n^4}) \quad (B.57)$$

holds after additional time $c_3n^{2.2}$, where $c_3$ does not depend on $i$. Hence, (B.54) holds for $i + 1$.

Phase II:

Use Lemma 23 on $P_s$ with $\delta' = \delta - \delta_b(i + 1)$, and Lemma 24 with $\epsilon = \frac{\delta}{10n} \cdot \frac{c_2}{2m}$ on the $(s, \Delta, \delta')$-bounding process to show that (B.55) holds for $i + 1$. The constant $c_4$ is chosen such that $c_4n^{2.2} \geq c_{RW}n^{2.1}(1 + \log(\delta'/\epsilon)) \geq c_{RW}n^{2.1}(1 + \log(\delta'/\epsilon))$, since $\delta' \leq \delta$. □
**Proof (Lemma 2: Blossom).** We use Lemma 30 to bound the error on $P_c$. Next we use Lemma 29 and Lemma 26 with $\delta_s = \delta_s(N), \delta_c = \delta, \epsilon = \frac{\delta}{20n}$ to show that

$$U_{G,P_c}(\alpha^t) \leq \Delta - \delta + \frac{3\delta}{20n}$$

(B.58)

after additional time $c_5n^{2.2}$ for finite $c_5$. Finally, we use Lemma 6 with $\epsilon = \frac{\delta}{20n}$. This finally gives that $B(q + 1, \delta(1 - (5n)^{-1}), t)$ holds for all $t \geq Cn^6$ as required.

**B.4 Bicycle convergence**

An almost identical argument works for the bicycle as for the blossom. We define $P_s$ as the ‘rod’ or ‘frame’ of the bicycle, joining the two cycles mapped to $P_{c1}$ and $P_{c2}$, analogously to the definitions of $P_s$ and $P_c$ for the blossom. Lemma 30 can be shown to hold for the bicycle as well, with an almost identical procedure. In this case also, we have an iterative descent, alternating between the ‘rod’ and the two cycles (which descend simultaneously), as per

$$U_{G,\partial P_s}(\alpha^t) \leq \Delta - \delta + \delta_b(i) \quad \forall t \geq i(c_3 + (i - 1)c_4)n^{2.2}$$

(B.59)

$$U_{G,P_s}(\alpha^t) \leq \Delta - \delta + \delta_s(i) \quad \forall t \geq i(c_3 + c_4)n^{2.2}$$

(B.60)

where $\partial P = \{(i, j) : i \in V_{\text{ext}}(C_q) \setminus V(P), j \in V(P), (i, j) \in E(C_q)\}$. Phase I works by using Lemmas 29 and Lemma 26 simultaneously for both $P_{c1}$ and $P_{c2}$. Phase II works exactly as before.

**Lemma 3: Bicycle.** We first use Lemma 30. Next, we use Lemma 29 and Lemma 26 simultaneously for each of $P_{c1}$ and $P_{c2}$. Finally, we use Lemma 6. Note that we thus prove an identical bound on the descent time of a bicycle as for a blossom.