QUANTUM KAC-MOODY ALGEBRAS
AND VERTEX REPRESENTATIONS

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Abstract. We introduce an affinization of the quantum Kac-Moody algebra associated to a symmetric generalized Cartan matrix. Based on the affinization, we construct a representation of the quantum Kac-Moody algebra by vertex operators from bosonic fields. We also obtain a combinatorial identity about Hall-Littlewood polynomials.

To appear in Lett. Math. Phys.

1. Introduction

Vertex representations of quantum affine algebras were first studied in \[2\] for irreducible level one representations of quantum affine algebras of both untwisted and twisted types. With the latest realizations of quantum affine algebras \(U_q(G_2^{(1)})\) \[3\] and \(U_q(F_4^{(1)})\) \[4\] all quantum affine algebras have been constructed explicitly at least for level one and some for other levels. An overview of the constructions in terms of quantum \(Z\)-algebras was given in \[5\] as well as the historic developments circled around various constructions.

In an abstract setting, Lusztig established the theory of highest weight modules for arbitrary symmetrizable quantum Kac-Moody algebras \[6\]. These are done by prescribing the actions of the Serre generators of the quantum Kac-Moody algebras \[7\] paralleled to the classical cases of \(q = 1\). On the other hand in the classical case of \(q = 1\), I. Frenkel \[8\] had constructed a vertex representation for the Kac-Moody Lie algebra associated to a symmetric generalized Cartan matrix in order to study root multiplicities of the hyperbolic Kac-Moody algebra.

In this paper we will give a quantum generalization of I. Frenkel’s construction for quantum Kac-Moody algebras. We first imbed the

\[1991\] Mathematics Subject Classification. Primary: 17B, Secondary: 05E.

Key words and phrases. Kac-Moody algebras, Quantum \(Z\)-operators, Vertex operators.

Research supported in part by NSA grant MDA904-97-1-0062.
quantum Kac-Moody algebra into an affinization of the quantum Kac-Moody algebra. Then we construct a (level one) vertex representation of the quantum affinization. Our construction gives a nontrivial combinatorial identity (cf. 5.4), which is interesting by itself. This identity was trivial in the classical case and was avoided by using usual vertex operator techniques in [4]. We proved the quantum Serre relations by a spiral method using the representation.

The paper is organized as follows. In section two we review the basic definition of quantum enveloping algebras associated to generalized Cartan matrices. In section three we introduce a quantum Heisenberg algebra associated to the root lattice of the Kac-Moody algebra and the representation is stated thereafter. In sections four and five we set out to prove that the construction gives a realization of the quantum Kac-Moody algebra. The proof of the Serre relations is obtained by showing a stronger identity of vertex operators, and this latter identity is equivalent to an identity of symmetric functions. Finally we remark about the connection of the combinatorial identity (5.4) with Hall-Littlewood polynomials.

2. Quantum Kac-Moody algebras and their affinizations

The notation of Kac-Moody algebras essentially follows [3]. Let \( g(A) \) be the Kac-Moody Lie algebra associated to the generalized Cartan matrix \( A = (a_{ij}) \), where \( a_{ii} = 2, a_{ij} = a_{ji} \leq 0 \) for \( i,j = 1, \cdots, l \). Let \( Q = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_l \) be the root lattice generated by the simple roots \( \alpha_i \), \( Q^\vee = \mathbb{Z}h_1 \oplus \cdots \oplus \mathbb{Z}h_l \) the coroot lattice generated by the coroots \( h_i \), and \( P^\vee = \mathbb{Z}\lambda_1 \oplus \cdots \oplus \mathbb{Z}\lambda_l \) the weight lattice generated by the fundamental weights \( \lambda_i \). The normalized invariant bilinear form \((\; | \; )\) on \( g(A) \) satisfies the following properties:

\[
(\alpha_i|\alpha_i) = 2, \quad (\alpha_i|\alpha_j) = a_{ij}.
\]

Let \( q \) be a generic complex number (that is, \( q \) is not a root of unity). The quantum Kac-Moody algebra is an associative algebra \( U_q = U_q(g(A)) \) over \( \mathbb{C}(q^{1/2}) \) generated by elements \( e_i, f_i, q^h, i = 1, \cdots, l, \)
$h \in P^* = \text{Hom}(P, \mathbb{Z})$ with the relations:

$$q^h = 1 \quad \text{for} \quad h = 0,$$

$$q^{h+h'} = q^h q^{h'} \quad \text{for} \quad h, h' \in P^*,$$

$$q^h e_i q^{-h} = q^{(h|\alpha_i)} e_i \quad \text{and} \quad q^h f_i q^{-h} = q^{-(h|\alpha_i)} f_i,$$

$$[e_i, f_j] = \delta_{ij} \frac{t_i - t_i^{-1}}{q - q^{-1}},$$

$$\sum_{k=0}^{1-a_{ij}} \frac{(-1)^k}{[k]![b-k]!} e_i^k e_j e_i^{b-k} = 0, \quad \sum_{k=0}^{b} \frac{(-1)^k}{[k]![b-k]!} f_i^k f_j f_i^{b-k} = 0 \quad \text{for} \quad i \neq j.$$

where $t_i = q^{h_i}$ and

$$[k] = \frac{q^k - q^{-k}}{q - q^{-1}}, \quad [k]! = [1][2] \cdots [k], \quad \text{and} \quad t_i = q^{h_i}.$$

The associative algebra $U_q$ has a Hopf algebra structure with the following comultiplication.

$$\Delta(q^h) = q^h \otimes q^h, \quad \Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i.$$

Generalizing the Drinfeld loop algebra realization of quantum affine algebras, we introduce the following associate algebra, which is a $q$-deformation of the affinization of Kac-Moody algebra.

The $q$-affinization algebra is an associative algebra $U_q(\widehat{g(A)})$ generated by elements $x_{in}^\pm$, $a_{ik}$, $K_i^\pm$, $q^d$ and $\gamma$ for $n \in \mathbb{Z}, k \in \mathbb{Z} - 0$ with the following defining relations:

(2.1) \hspace{1cm} [\gamma^{\pm 1/2}, u] = 0 \quad \text{for all} \quad u \in U,

(2.2) \hspace{1cm} [a_{ik}, a_{jl}] = \delta_{k+l,0} \frac{[(\alpha_i|\alpha_j)k]}{k} \gamma^k \gamma^{-k}

(2.3) \hspace{1cm} [a_{ik}, K_j^{\pm 1}] = [q^{d}, K_j^{\pm 1}] = 0,

(2.4) \hspace{1cm} q^d x_{ik}^\pm q^{-d} = q^k x_{ik}^\pm, \quad q^d a_{id} q^{-d} = q^l a_{il},

(2.5) \hspace{1cm} K_j x_{jk}^\pm K_i^{-1} = q^{-(\alpha_i|\alpha_j)} x_{jk}^\pm,

(2.6) \hspace{1cm} [a_{ik}, x_{jl}^\pm] = \pm \frac{[(\alpha_i|\alpha_j)k]}{k} \gamma^{\pm |k|/2} x_{j,k+l}^\pm.
\[-\frac{\alpha_l}{\alpha_j}+1 \prod_{r=0}^{\infty} (z-q^{\pm(\alpha_l/\alpha_j)+2r}w)x_i^+(z)x_j^+(w)\]
\[+ \prod_{r=0}^{\infty} (w-q^{\pm(\alpha_l/\alpha_j)+2r}z)x_j^+(w)x_i^+(z) = 0,\]
(2.7)

\[[x_i^+(z), x_j^-(w)] = \frac{\delta_{ij}}{q-q^{-1}} \left( \psi_i(w^{-1/2})\delta(w^{-1/2}) - \varphi_i(w^{-1/2})\delta(w^{-1}) \right)\]
(2.8)

where \[x_i^+(z) = \sum_{n\in\mathbb{Z}} x_{i,n} z^{-n-1}\], \[\psi_{im}\] and \[\varphi_{im} (m \in \mathbb{Z}_{\geq 0})\] are defined by

\[\sum_{m=0}^{\infty} \psi_{im} z^{-m} = K_i \exp \left( (q-q^{-1}) \sum_{k=1}^{\infty} a_{ik} z^{-k} \right),\]
(2.9)

\[\sum_{m=0}^{\infty} \varphi_{i,-m} z^m = K^{-1}_i \exp \left( -(q-q^{-1}) \sum_{k=1}^{\infty} a_{i,-k} z^k \right),\]
(2.10)

\[\sum_{r=0,\sigma \in S_m} (-1)^r \left[ \begin{array}{c} m \\ r \end{array} \right] \sigma. x_{i}^+(z_1) \cdots x_{i}^+(z_r) \cdot x_{j}^+(w) x_{i}^+(z_{r+1}) \cdots x_{i}^+(z_m) = 0,\]
(2.11)

where the symmetric group \(S_m\) acts on \(z_i\)’s by permuting their indices.

**Theorem 2.1.** The mapping \(e_i \mapsto x_{i,0}^+, f_i \mapsto x_{i,0}^-, t_i \mapsto K_i\) defines an imbedding of the quantum Kac-Moody algebra \(U_q(g(A))\) into the quantum affinization algebra \(U_q(q(A))\).

**Proof.** It is easy to see that \(X_{i,0}^\pm\) and \(K_i\) satisfy the relations of \(e_i, f_i, t_i\). The only thing needs explanation is the Serre relation. This follows from (2.11) by taking the constant coefficient of \(w, z_1, \cdots, z_m\) and noting the invariance of each term under \(S_m\).

\[\square\]

### 3. Fock space representations

The quantum Heisenberg algebra is the associative algebra generated by \(a_i(n)\) and the central element \(\gamma^{\pm1}\) satisfying the relations (2.2). We will consider the level one \(q\)-Heisenberg algebra, that is \(\gamma = q\). Let \(a_i(m) (i = 1, \cdots, l, m \in \mathbb{Z})\) be the operators satisfying:

\[[a_i(m), a_j(n)] = \delta_{m+n,0} \frac{[\alpha_i|\alpha_j]m}{m} [m].\]
(3.1)

where we take the limit of \(m \mapsto 0\) if the mode \(m = 0\).
We define the Fock module $\mathcal{F}$ as the tensor product of the symmetric algebra generated by $a_i(-n)\ (n \in \mathbb{N})$ and the twisted group algebra $\mathbb{C}\{Q\}$ generated by $e^\alpha, \alpha \in Q$ subject to relation:

$$e^{\alpha_1}e^{\alpha_2} = (-1)^{(\alpha_1|\alpha_2)}e^{\alpha_2}e^{\alpha_1}.$$  

Let the element 1 be the vacuum state with the actions of $\alpha_i(n)$:

$$a_i(n).1 = 0 \quad (n > 0).$$

The element $a_i(0)$ act as differential operators by

$$a_i(0)e^\alpha = (\alpha_i|\alpha)e^\alpha.$$  

Clearly this defines a representation of the $q$-Heisenberg algebra for $\gamma = q$.

As usual we define the normal product as the ordered product by moving annihilation operators $a_i(n), a_i(0), a_i(0)\ (n > 0)$ to the left.

Let us introduce the following vertex operators.

$$X_{i}^{\pm}(z) = \exp(\pm \sum_{n=1}^{\infty} \frac{a_i(-n)}{[n]}q^{\mp \frac{n}{2}}z^{n})\exp(\mp \sum_{n=1}^{\infty} \frac{a_i(n)}{[n]}q^{\mp \frac{n}{2}}z^{-n})e^{\alpha_i}z^{\mp a_i(0)}$$

$$= \sum_{n \in \mathbb{Z}} X_i^{\pm}(n)z^{-n-1}.$$  

**Theorem 3.1.** The space $\mathcal{F}$ is a $U_q(\widehat{\mathfrak{g}}(\mathfrak{A}))$-module of level one under the action defined by $\gamma \mapsto q, K_i \mapsto q^{a_i(0)}, a_{im} \mapsto a_i(m), q^d \mapsto q^d$, and

$$x_{i,n}^{\pm} \mapsto X_i^{\pm}(n).$$  

The proof of the theorem will occupy this and next two sections. We start by recalling the $q$-binomial functions $(1 - x)^n_q$ introduced in [3]:

$$\frac{(1 - x)^n_q}{(q^{n+1}x; q^2)_{\infty}} = (q^{-a+1}x; q^2)^n_q.$$  

In particular for $a \in \mathbb{N}$ we have:

$$(1 - x)^a_q = (1 - q^{-a+1}x)(1 - q^{-a+3}x) \cdots (1 - q^{a-1}x) = (q^{-a+1}x; q^2)_a.$$  

For convenience we will also use the following $q$-analogue of binomial functions:

$$\frac{(z - w)^a_q}{(z^a_q)} = (1 - \frac{w}{z})^a_qz^a.$$  

(3.3)
Under our symmetric $q$-numbers, the $q$-binomial theorem becomes:

$$\sum_{r=0}^{n} \binom{n}{r} (-z)^r = (q^{1-n} z ; q^2)_n.$$  \hspace{1cm} (3.4)

In particular we have

$$\sum_{r=0}^{n} \binom{n}{r} (-q^i)^r = 0, \quad \text{for } i = 1 - n, 3 - n, \ldots, n - 1.$$  \hspace{1cm} (3.5)

We now prove the theorem by checking that the action satisfies the relations of the $q$-affinization algebra. It is clear that the relations (2.1-2.5) are true by the construction. Relation (2.6) follows from the definition of $X_{i}^\pm(z)$. So we only need to show the relations (2.7-2.8) and the Serre relation (2.11).

The operator product expansions (OPE) for $X_{i}^\pm(z)$ are computed as follows (cf. [8]):

$$X_{i}^\pm(z)X_{j}^\mp(w) =: X_{i}^\pm(z)X_{j}^\mp(w) : (1 - q^{\frac{1}{2}} \frac{w}{z})^{(\alpha_i|\alpha_j)} z^{(\alpha_i|\alpha_j)},$$
$$X_{i}^\pm(z)X_{j}^\mp(w) =: X_{i}^\pm(z)X_{j}^\mp(w) : (1 - w) \frac{1}{q^2} z^{-(\alpha_i|\alpha_j)}.$$  \hspace{1cm} (3.6)

where the contraction functions like $(1 - q^{\frac{1}{2}} \frac{w}{z})^{(\alpha_i|\alpha_j)} z^{(\alpha_i|\alpha_j)}$ are understood as power series in $w/z$. Using the notation (3.3) we can combine the OPE into a self-suggested form:

$$X_{i}^\pm(z)X_{j}^{\prime}^\mp(w) =: X_{i}^\pm(z)X_{j}^{\prime}^\mp(w) : (z - q^{(\epsilon+\epsilon')/2} w) \frac{1}{q^2}^{(\alpha_i|\alpha_j)},$$
$$: X_{i}^\pm(z)X_{j}^{\prime}^\mp(w) =: (-1)^{(\alpha_i|\alpha_j)} : X_{i}^\pm(z)X_{j}^{\prime}^\mp(w) :.$$  \hspace{1cm} (3.7)

The operator product expansions imply immediately the commutation relations (2.7). We will find out that there exist more detailed commutation relations between $X_{i}^\pm(z)$ and $X_{j}^\pm(w)$.

The case of $i = j$ in the relations (2.8) follows from the case of $U_q(\hat{sl}_2)$ [8]. When $i \neq j$ we have

$$[X_{i}^\pm(z), X_{j}^{\mp}(w)] =: X_{i}^\pm(z)X_{j}^{\mp}(w) : (1 - \frac{w}{z})^{(\alpha_i|\alpha_j)} z^{-(\alpha_i|\alpha_j)}$$
$$(-1)^{(\alpha_i|\alpha_j)} (1 - \frac{z}{w})^{(\alpha_i|\alpha_j)} w^{-(\alpha_i|\alpha_j)}$$
$$= 0$$

since the contraction functions $(z - w)^{(\alpha_i|\alpha_j)} = (-1)^{(\alpha_i|\alpha_j)} (w - z)^{(\alpha_i|\alpha_j)}$ as $q$-polynomials in $z,w$ (cf. [3,3]).
4. Q-DIFFERENCE OPERATORS AND COMMUTATION RELATIONS

The commutation relations of operators $X_i^\pm(z)$ and $X_j^\pm(w)$ can be improved by using $q$-difference operators.

Let $f(x)$ be a function. The $q$-difference operator $D_q(x)$ is defined by

$$D_q(x; q)f(x) = \frac{f(xq) - f(xq^{-1})}{(q - q^{-1})x}.$$  

We will simply write $D_q$ if the variable is clear from the context.

**Lemma 4.1.** $D_q(1 - x)_{q^2}^n = -[n](1 - x)^{n-1}$.

**Proof.** We compute directly that

$$D_q(1 - x)_{q^2}^n = \frac{1}{(q - q^{-1})x}((q^{-n+2}x; q^2)_n - (q^n x; q^2)_n)$$

$$= \frac{1}{(q - q^{-1})x}(q^{-n+2}x; q^2)_{n-1}(1 - q^n x - 1 + q^n x)$$

$$= -[n](q^{-n+2}x; q^2)_{n-1}$$

$\square$

The following result will be useful for what follows.

**Lemma 4.2.**

$$D_q^n f(x) = \frac{1}{(q - q^{-1})^{n}x^n} \sum_{i=0}^{n} (-1)^i q^{(n-1)(-n+2i)/2} \left[ \begin{array}{c} n \\ i \end{array} \right] f(q^{n-2i}x).$$

**Proof.** By induction on $n$ we have

$$D_q^n f(x) = \frac{D_q^{n-1} f(qx) - D_q^{n-1} f(q^{-1}x)}{(q - q^{-1})x}$$

$$= \frac{1}{(q - q^{-1})^{n}x^n} \left( q^{-n+1} \sum_{i=0}^{n-1} (-1)^i q^{(n-2)(-n+1+2i)/2} \left[ \begin{array}{c} n - 1 \\ i \end{array} \right] f(q^{n-2i}x) \right)$$

$$+ \sum_{i=1}^{n-1} (-1)^i q^{-i(n-1)} \left( q^{-i} \left[ \begin{array}{c} n - 1 \\ i \end{array} \right] + q^{-i} \left[ \begin{array}{c} n - 1 \\ i - 1 \end{array} \right] \right) f(q^{n-2i}x)$$

$$= \frac{1}{(q - q^{-1})^{n}x^n} \sum_{i=0}^{n} (-1)^i q^{-(n+1)(-n+2i)/2} \left[ \begin{array}{c} n \\ i \end{array} \right] f(q^{n-2i}x).$$
Theorem (4.3). We only need to show the case of +. For proof.

Theorem 4.3. Let $m = 1 - (\alpha_i | \alpha_j)$. On the Fock space $V$ we have the following commutation relation:

\[
X^\pm(z)X^\pm_i(w) - X^\pm_i(z)X^\pm(w) = 0, \quad \text{if} \ (\alpha_i | \alpha_j) = 0;
\]

\[
(z - q^{\pm(\alpha_i | \alpha_j)}w)X^\pm_i(z)X^\pm_j(w) + (w - q^{\pm(\alpha_i | \alpha_j)}z)X^\pm_i(w)X^\pm_j(z) = 0,
\]

\[
(z - q^{\pm(\alpha_i | \alpha_j)}w)X^\pm_i(z)X^\pm_j(w) + (w - q^{\pm(\alpha_i | \alpha_j)}z)X^\pm_j(w)X^\pm_i(z)
\]

\[
= X^\pm_i(z)X^\pm_j(w) : \frac{1}{(q - q^{-1}) \cdots (q^{m-2} - q^{-m+2})} \sum_{r=0}^{m-2} q^{-(m-2)+r(m-3)} \left[ m - 2 \choose r \right] \delta(q^{m-2-2r}w/z), \quad \text{if} \ (\alpha_i | \alpha_j) \leq -2.
\]

Proof. We only need to show the case of +. For $m = 1 - (\alpha_i | \alpha_j) \geq 2$ it follows from (3.1-3.7) that

\[
(z - q^{\pm(\alpha_i | \alpha_j)}w)X^\pm_i(z)X^\pm_j(w) + (w - q^{\pm(\alpha_i | \alpha_j)}z)X^\pm_j(w)X^\pm_i(z)
\]

\[
= X^\pm_i(z)X^\pm_j(w) : \frac{1}{(z - q^{3m}w)(z - q^{5m}w) \cdots (z - q^{m-3}w)}
\]

\[
+ \frac{(-1)^{m-2}}{(w - q^{3m}z)(w - q^{5m}z) \cdots (w - q^{m-3}z)}
\]

\[
= X^\pm_i(z)X^\pm_j(w) : \frac{1}{[m-2]!z^{m-2}} \left( D_q^{m-2}(w/z)(1 - z/w) + z/w \right)
\]

\[
= X^\pm_i(z)X^\pm_j(w) : \frac{1}{(q - q^{-1}) \cdots (q^{m-2} - q^{-m+2})} \sum_{r=0}^{m-2} (-1)^r q^{(m-3)(-m+2+2r)/2}
\]

\[
\cdot \left[ m - 2 \choose r \right] \delta(q^{m-2-2r}w/z)
\]

\[
= \frac{1}{(q - q^{-1})^{m-2}[m-2]} \sum_{r=0}^{m-2} q^{-(m-2)+r(m-3)} X^\pm_i z^{(\alpha_i + \alpha_j) + (z) \delta(q^{m-2-2r}w/z)}.
\]

It is clear that the commutation relation (2.7) is a consequence of Theorem (4.3).
5. Quantum Serre relations

The proof of Serre relations shows the power of our vertex representation. As an application we obtain a non-trivial combinatorial identity.

**Lemma 5.1.** Suppose that there exists a root $\alpha_k$ such that $(\alpha_k|\alpha_i) = 0$ and $(\alpha_k|\alpha_j) \neq 0$, that is, $\alpha_k$ joins with $\alpha_j$ but not with $\alpha_i$. Then the $q$-affinized Serre relation (2.11) associated with $a_{ij}$ is equivalent to the following identity:

\[
\sum_{r=0, \sigma \in S_m} (-1)^r \left[ \binom{m}{r} \sigma \cdot x_i^+(z_1) \cdots x_i^+(z_r) x_j^+(z_{r+1}) \cdots x_i^+(z_m) = 0 \right.
\]

for some $l \in \mathbb{Z}$ and $\sigma$ acts on indices of $z_i$'s.

**Proof.** The relation (5.1) is a special case of (2.11), so we only need to show the converse direction. Assume that (5.1) holds for some $l \in \mathbb{Z}$.

Taking commutator with $a_k(s)$ to (5.1) using (2.6), we see immediately by induction that

\[
\sum_{r=0, \sigma \in S_m} (-1)^r \left[ \binom{m}{r} \sigma \cdot x_i^+(z_1) \cdots x_i^+(z_r) x_j^+(z_{r+1}) \cdots x_i^+(z_m) = 0 \right.
\]

for any $l \in \mathbb{Z}$. These identities are all the $w$-coefficients of (2.11). \qed

We now use the quantum vertex operator calculus [5] to prove the Serre relations (2.11).

Repeatedly using (3.7) and Wick’s theorem we have for $m = 1 - (\alpha_i|\alpha_j) \geq 1$ (we only consider the + -case):

\[
\sum_{r=0}^{m} \sum_{\sigma \in S_m} \sigma \cdot \left[ \binom{m}{r} X_i^+(z_1) \cdots X_j^+(z_r) X_i^+(z_{r+1}) \cdots X_i(z_m) \right] = \sum_{r=0}^{m} \sum_{\sigma \in S_m} \sigma \cdot \left[ \binom{m}{r} X_i^+(z_1) \cdots X_j^+(z_r) X_i^+(z_{r+1}) \cdots X_i(z_m) : \right.
\]

\[
\prod_{i<j} (z_i - q^{-1} z_j)^2 q^2 \prod_{k=1}^{r} (z_k - q^{-1} w)^{1-m} q^2 \prod_{k=r+1}^{m} (w - q^{-1} z_k)^{1-m} = 0
\]

where the symmetric group $S_m$ acts on the indices of $z_i$'s: $\sigma(z_i) = z_{\sigma(i)}$. 

Pulling out the normal product: \( X_j^+(x)X_i^+(z_1) \cdots X_i^+(z_m) \): and the anti-symmetric factor (under \( S_m \) action):

\[
\prod_{i<j}(z_i - z_j) \prod_{i=1}^m (w - z_i)^{2-m} \prod_{i=1}^m (w - q^{1-m}z_i)^{-1}(z_i - q^{1-n}w)
\]

and using (3.9) we find that the Serre relation is equivalent to the following identity:

\[
\sum_{\sigma \in S_m} \sum_{r=0}^m (-1)^{l(\sigma)} \begin{bmatrix} m \\ r \end{bmatrix} (w - q^{1-m}z_1) \cdots (w - q^{1-m}z_r) \cdot (z_{r+1} - q^{1-m}w) \cdots (z_m - q^{1-m}w) \prod_{i<j}(z_i - q^{-2}z_j) = 0
\]

Lemma (5.1) says that we only need to show this identity for a special component under the condition that there exists a root \( \alpha_k \) joining with \( \alpha_j \) but not with \( \alpha_i \). The constant coefficient of \( w \) in (5.4) is

\[
\sum_{\sigma \in S_m} \sum_{r=0}^m (-1)^{l(\sigma)} \begin{bmatrix} m \\ r \end{bmatrix} (-q^{1-m})^r z_1 \cdots z_m \prod_{i<j}(z_i - q^{-2}z_j)
\]

\[
= z_1 \cdots z_m (q^{-2m+2}, q^2)^m \sum_{\sigma \in S_m} (-1)^{l(\sigma)} \prod_{i<j}(z_i - q^{-2}z_j) = 0
\]

where we used the \( q \)-binomial theorem (3.4).

Thus we have shown that our construction satisfies the Serre relation under the special condition on \( \alpha_i \) and \( \alpha_j \), which in turn shows that the combinatorial identity (5.4) holds under this condition of \( \alpha_i \) and \( \alpha_j \). Note that the combinatorial identity depends only on \( (\alpha_i | \alpha_j) \), hence it is true in general. Then we use identity (5.4) to remove the restriction on \( \alpha_i \) and \( \alpha_j \). Hence the Serre relation is proved in general, and our construction indeed gives a representation of the \( q \)-affined Kac-Moody algebra, and in turn a representation of the quantum Kac-Moody algebra.

6. Combinatorial Identity

In this paper we give a realization of the quantum Kac-Moody algebra associated to the symmetric generalized Cartan matrix. As a by-product we obtain the combinatorial identity (5.4), which seems to be new. The identity can be rewritten into the following form:
Theorem 6.1. For any natural number \( m \) we have:

\[
\sum_{\sigma \in S_m} \sum_{r=0}^{m} \sigma \left[ \begin{array}{c} m \\ r \end{array} \right] (w - q^{m-1}z_1) \cdots (w - q^{m-1}z_r) \\
(z_{r+1} - q^{m-1}w) \cdots (z_m - q^{m-1}w) \prod_{i<j} \frac{z_i - q^2z_j}{z_i - z_j} = 0
\]

Note that this gives an entirely non-trivial relation for Hall-Littlewood polynomials defined in [11]:

\[
P_\lambda(z; q^2) = \frac{1}{v_\lambda(q)} \sum_{\sigma \in S_m} \sigma(z_1^{\lambda_1} \cdots z_m^{\lambda_m}) \prod_{i<j} \frac{z_i - q^2z_j}{z_i - z_j},
\]

where \( \lambda \) is a partition of \( m \), and \( v_\lambda(q) = q^{\sum (\lambda_i)} \prod_i [\lambda_i]! \). The collection of \( \{P_\lambda| \lambda \text{ partitions of } m\} \) forms a \( \mathbb{Z} \)-basis for the ring of symmetric functions of degree \( m \). With some effort the notion of \( P_\lambda \) can be extended to \( m \)-tuples. Our identity gives a relation for all \( P_\lambda \) when \( \lambda \) are not necessary partitions.

Although we proved the identity (6.1) by our representation of the quantum Kac-Moody algebra, it would be interesting to prove this combinatorial identity independently from quantum groups, which will be treated elsewhere.

The method used in this paper can be generalized to higher level cases using the quantum \( Z \)-algebra method (cf. [8]).

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