Noncommutative double scalar fields in FRW cosmology as cosmical oscillators

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Abstract
We investigate the effects caused by noncommutativity of the phase space generated by two scalar fields that have non-minimal conformal couplings to the background curvature in the FRW cosmology. We restrict deformation of the minisuperspace to noncommutativity between the scalar fields and between their canonical conjugate momenta. Then, the investigation is carried out by means of a comparative analysis of the mathematical properties (supplemented with some diagrams) of the time evolution of variables in a classical model and the wavefunction of the universe in a quantum perspective, both in the commutative and noncommutative frames. We find that the imposition of noncommutativity causes more ability in tuning time solutions of the scalar fields and, hence, has important implications in the evolution of the Universe. We find that the noncommutative parameter in the momenta sector is the only responsible parameter for the noncommutative effects in the spatially flat universes. A distinguishing feature of the noncommutative solutions of the scalar fields is that they can be simulated with the well-known three harmonic oscillators depending on three values of the spatial curvature, namely the free, forced and damped harmonic oscillators corresponding to the flat, closed and open universes, respectively. In this respect, we call them cosmical oscillators. In particular, in closed universes, when the noncommutative parameters are small, the cosmical oscillators have an analogous effect with the familiar beating effect in the sound phenomena. Some of the special solutions in the classical model and the allowed wavefunctions in the quantum model make bounds on the values of the noncommutative parameters. The existence of a non-zero constant potential (i.e. a cosmological constant) does not change time evolutions of the scalar fields, but modifies the scale factor. An interesting feature of the well-behaved solutions of the wavefunctions is that the functional form of the radial part is the same as the commutative ones within a given replacement of constants caused by the noncommutative parameters. Furthermore, the Noether theorem has been employed to explore the effects of noncommutativity on the underlying symmetries in the commutative frame. Two of the six Noether symmetries of
spatially flat universes, in general, are retained in the noncommutative case, and one out of the three in non-flat universes.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Scalar field theories have become generic playgrounds for building cosmological models related to particle physics [1], and more recently have played very important contributions in the other branches of cosmology, e.g., as a powerful tool for providing expanding accelerated universes. Also, they have key roles in some models of the early cosmological inflation [2], and are viable favorite candidates for dark matter [3]. Indeed, scalar field cosmological models have extensively been studied in the literature, see, e.g., [4] and references therein.

Unifying theories of interactions, such as the Kaluza–Klein, supergravity and superstring theories, usually predict non-minimal couplings between geometry of spacetime and scalar fields. These theories mostly are derived from effective actions of string theories [5] or from compactified Kaluza–Klein theories in four dimensions [6]. In both cases, resulting models have non-minimal couplings between gravity and one (or even more) scalar field. Actually, considering more than one scalar field has been viewed as an easier approach to study accelerating models. These cosmological ideas have also been employed in models of inflation to describe the early Universe [7] or an evolving scalar field known as quintessence [8]. Recently, double scalar–tensor theories have been applied in extended gravitational theories, and have given successful descriptions of inflationary scenarios [9, 10]. Besides, it has been shown [11] that the reduction of a five-dimensional Brans–Dicke gravity into four dimensions is equivalent to a 4-metric coupled to two scalar fields, which may account for the present accelerated expansion of the Universe. Also, effects of additional scalar fields plus independent exponential potentials have been considered in the literature [12].

In cosmological models, scalar fields usually present degrees of freedom and appear as dynamical variables of corresponding minisuperspaces. This point can be viewed as relevance of noncommutativity in these models. The proposal of noncommutativity concept between spacetime coordinates, in the first time, was introduced in 1947 [13]. Thereafter, a mathematical theory, nowadays known as the noncommutative geometry (NCG), has begun to take shape based on this concept since 1980 [14]. Noncommutativity among spacetime coordinates implies noncommutativity among fields as minisuperspace coordinates. Such a kind of noncommutativity has gotten more attention in the literature. In the last decade, study and investigation of physical theories in the noncommutative (NC) frames, such as the string and M-theory [15, 16], have caused a renewed interest on noncommutativity in the classical and quantum perspectives. In particular, a novel interest has been developed in studying the NC classical and quantum cosmologies, e.g. [17] and references therein. Also, deformation of the phase space structure has been viewed as an alternative path, in the context of cosmology, toward understanding quantum gravity [18]. The influence of noncommutativity has been examined by formulation of a version of the NC cosmology in which a deformation of the minisuperspace coordinates [19–22] or the minisuperspace momenta [23, 24] is required instead of the spacetime coordinates. From a qualitative point of view, noncommutativity in the minisuperspace coordinates (space sector) leads to general effects; however, a non-trivial
noncommutativity in the momentum sector introduces distinct and instructive effects in the behavior of dynamical variables.

In this work, we consider homogeneous and isotropic cosmological models based on a particular (multi)scalar–tensor gravity theory with two scalar fields that have non-minimal conformal couplings to the spacetime. The effects of noncommutativity on the phase space, generated by these fields plus the scale factor, are investigated. Our approach is to build a NC scenario via a deformation conveyed by the Moyal product [15] in a classical and a quantum perspective, where their cosmological implications are more appreciated in the classical one. The procedure of quantization is proceeded by the Wheeler–DeWitt (WD) equation for a wavefunction of the Universe by the Hamiltonian operator of the model via the generalized Dirac quantization. We introduce noncommutativity between the scalar fields and between their canonical conjugate momenta, and will pay more attention to the outcome of results. Here, we neglect noncommutativity between scalar fields with the scale factor; such effects can be found in, e.g., [22, 24]. Actually we treat the effects of noncommutativity by two parameters which are the NC parameters corresponding to the space and momentum sectors. Then, we analyze the mathematical properties of solutions and look for relations, including the ranges and values, among the NC parameters for which particular or allowed solutions exist.

The paper is organized as follows. In section 2, we specify our toy model and investigate the classical version within the commutative and NC frames. The quantum version of this model, by probing the universe wavefunctions, is considered in section 3, where the general properties of solutions in the commutative and NC frames are compared. In section 4, we employ the Noether theorem and explore the effects of noncommutativity on the underlying symmetries in the commutative frame. Conclusions are presented in the last section, and some necessary formulations have been furnished in the appendix.

2. The classical model

We consider a classical model consisting of a cosmological system presented by a four-dimensional action with two non-minimally scalar fields coupled to gravity in a Friedman–Robertson–Walker (FRW) universe. To specify the NC effects of the model, we first treat the commutative version and then consider the NC one in the following subsections.

2.1. The commutative phase space

A general action for non-minimally coupled double scalar fields can be written as [9]

\[ A = \int \sqrt{-g} \left\{ F(\phi, \psi) R - \frac{1}{2} g^{\mu\nu} [A(\phi, \psi) \nabla_\mu \phi \nabla_\nu \phi + B(\phi, \psi) \nabla_\mu \psi \nabla_\nu \psi] - V(\phi, \psi) \right\} d^4x, \tag{1} \]

where \( g \) is the determinant of the metric \( g_{\mu\nu} \), \( R \) is the Ricci scalar, \( F(\phi, \psi) \) and \( V(\phi, \psi) \) are coupling and (total) potential functions of scalar fields, respectively. Also, \( A(\phi, \psi) \) and \( B(\phi, \psi) \) are two typical functions coupled to the kinetic terms. We assume homogeneous scalar fields, that is, \( \phi = \phi(t) \) and \( \psi = \psi(t) \), in the following FRW metric:

\[ ds^2 = -N^2(t) \, dt^2 + a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 \, d\Omega^2 \right), \tag{2} \]

where \( N(t) \) is a lapse function, \( a(t) \) is a scale factor and \( k \) specifies spatial geometry of the universe.
We restrict our considerations to a non-interacting conformally scalar field model. This ansatz is obviously restrictive. However, a general reason for selecting such scalar fields is for simplicity. Indeed, it allows exact solutions in simple cases, as those will be discussed in this work and are rich enough to be useful as a probe for significant modifications that NCG introduces in the classical and quantum cosmologies. Thus in this case, one can set \( F(\phi, \psi) = f(\phi) + f(\psi) \) where \( f \) is a function of its argument as \( f(\chi) = 1/(4\kappa) - \xi \chi^2/2 \). Also, \( \kappa = 8\pi G/c^4 \) and \( \xi \) is a non-minimal coupling parameter with an arbitrary value that represents a direct coupling between the curvature and the scalar fields. The case \( \xi = 0 \) obviously is the minimally coupling situation; however in this work, we consider the conformal coupling case, i.e. \( \xi = 1/6 \), and employ the units \( \hbar = 1 = c \) and \( \kappa = 3 \) (i.e. \( G = 3/(8\pi) \)). We also, in general, consider the scalar fields that are not subject to any potential\(^1\), and assume more simple cases with \( \Lambda(\phi, \psi) = 1 = B(\phi, \psi) \). However, at the end of this section, we will investigate the particular case of non-zero constant values of the potential function (i.e. the cosmological constant).

Now, based on these assumptions, substituting metric (2) into action (1) yields the Lagrangian density

\[
\mathcal{L} = \left( k N a - \frac{a a^2}{N} \right) \left( 1 - \frac{\phi^2 + \psi^2}{2} \right) + (\phi \dot{\phi} + \psi \dot{\psi}) \frac{a^2 \dot{a}}{N} + \frac{a^3}{2N} (\dot{\phi}^2 + \dot{\psi}^2),
\]

where a total derivative term has been neglected. By rescaling the scalar fields as \( x = a\phi/\sqrt{2} \) and \( y = a\psi/\sqrt{2} \), Lagrangian (3) reads

\[
\mathcal{L} = k N a - \frac{a \dot{a}^2}{N} + \frac{a (\dot{x}^2 + \dot{y}^2)}{2N} - \frac{kN(x^2 + y^2)}{a}.
\]

Then, the corresponding Hamiltonian is

\[
\mathcal{H} = N \left[ -\frac{p_x^2}{4a} - \frac{p_y^2}{4a} + kN(a^2 + k(x^2 + y^2)) - ka + \frac{kN(x^2 + y^2)}{a} \right],
\]

where \( p_x, p_y \) and \( p_a \) are the corresponding canonical conjugate momenta. For the conformal time gauge selection, namely \( N = a \), one gets

\[
\mathcal{H} = -\frac{p_x^2}{4} - \frac{p_y^2}{4} - ka^2 + k(x^2 + y^2).
\]

Therefore, the Hamilton equations are

\[
\dot{a} = \{a, \mathcal{H}\} = -\frac{1}{2} p_a,
\]

\[
p_a = \{p_a, \mathcal{H}\} = 2ka,
\]

\[
x = \{x, \mathcal{H}\} = \frac{1}{2} p_x,
\]

\[
p_x = \{p_x, \mathcal{H}\} = -2kx,
\]

\[
y = \{y, \mathcal{H}\} = \frac{1}{2} p_y,
\]

\[
p_y = \{p_y, \mathcal{H}\} = -2ky.
\]

Solutions of equations (8), with the Hamiltonian constraint \( \mathcal{H} \approx 0 \), depend on the curvature index; actually their solutions are

\[
a(t) = A_1 \cos t + A_2 \sin t, \quad x(t) = B_1 \cos t + B_2 \sin t \quad \text{and} \quad y(t) = C_1 \cos t + C_2 \sin t,
\]

with constraint \( 2(A_1^2 + A_2^2) = \sum_{i=1}^2 (B_i^2 + C_i^2) \).

\(^1\) A vanishing (total) potential, \( V(\phi, \psi) \), can also be achieved by non-interacting (inner) potentials, e.g. \( V(\phi, \psi) = U(\phi) + W(\psi) = \Lambda + (-\Lambda) = 0 \), where \( \Lambda \) is the cosmological constant. Such a case is an important one in models with double scalar fields \([9]\).
\[
\begin{align*}
\alpha(t) &= A_5 t + A_6, \\
x(t) &= B_5 t + B_6, \\
y(t) &= C_5 t + C_6,
\end{align*}
\] with constraint \(A_5 = B_5 + C_5\),
\[
\begin{align*}
\alpha(t) &= A_3 e^t + A_4 e^{-t}, \\
x(t) &= B_3 e^t + B_4 e^{-t}, \\
y(t) &= C_3 e^t + C_4 e^{-t},
\end{align*}
\] with constraint \(2A_3 A_4 = B_3^2 + C_3^2\),
\[
\begin{align*}
k &= -1 : \\
\alpha(t) &= A_3 e^t + A_4 e^{-t}, \\
x(t) &= B_3 e^t + B_4 e^{-t}, \\
y(t) &= C_3 e^t + C_4 e^{-t},
\end{align*}
\] with constraint \(2A_3 A_4 = B_3^2 + C_3^2\),
\[
\begin{align*}
k &= 0 : \\
\alpha(t) &= A_3 t + A_4, \\
x(t) &= B_3 t + B_4, \\
y(t) &= C_3 t + C_4,
\end{align*}
\] where the \(A_i, B_i,\) and \(C_i\) are constants of integration. As it is obvious, if one of the scalar fields vanishes and/or be equal to each other, i.e. \(\phi = \psi(\equiv \chi/\sqrt{2})\), all equations will lead to those derived in the case of one scalar field in our previous work [24].

We will compare these solutions with their NC analogs in the next section.

2.2. The NC phase space

In the classical physics, noncommutativity is achieved by replacing ordinary product with the so-called Moyal product, shown by the \(*\) notation. This associative product on 2\(^l\)-dimensional phase space is defined between two arbitrary functions of general phase space variables, namely \(\xi^a = (q^i, p^j)\) for \(i, j = 1, \ldots, l\), as [15]
\[
(f * g)(\xi) = \exp \left[ \frac{1}{2} \alpha_{ab} \left( a^{(1)} b^{(2)} \right) \right] f(\xi) g(\xi) \bigg|_{\xi_1 = \xi_2 = \xi},
\] such that the symplectic structure is
\[
(\alpha_{ab}) = \begin{pmatrix} \theta_{ij} & \delta_{ij} + \sigma_{ij} \\ -\delta_{ij} - \sigma_{ij} & \beta_{ij} \end{pmatrix},
\] where \(a, b = 1, 2, \ldots, 2l\). The matrix elements \(\alpha_{ab}\) are assumed to be real, \(\theta_{ij}\) and \(\beta_{ij}\) are antisymmetric and \(\sigma_{ij}\) (which can be written as a combination of \(\theta_{ij}\) and \(\beta_{ij}\)) is symmetric. The modified (or the \(\alpha\)-star deformed) Poisson brackets are defined, in terms of the Moyal product, as
\[
\{f, g\}_a = f * g - g * f.
\] Hence, the modified Poisson brackets of the phase space variables are
\[
\begin{align*}
\{q_i, q_j\}_a &= \theta_{ij}, \\
\{q_i, p_j\}_a &= \delta_{ij} + \sigma_{ij} \\
\{p_i, p_j\}_a &= \beta_{ij}.
\end{align*}
\] For classical evolution, one starts by proposing that the NC nature of the minisuperspace variables can be encoded in a deformation of the Poisson structure. The main advantage of this approach is that the Hamiltonian does not need any modification. Hence, for our model, we require that the algebra of the minisuperspace variables obeys relations (15) in terms of the Moyal product defined in (12). However, to introduce such deformations, it is more convenient to consider a linear and non-canonical transformation on the classical phase space as (see e.g. [25, 26])
\[
\begin{align*}
x'_i &= x_i - \frac{1}{2} \theta_{ij} p^j \\
p'_i &= p_i + \frac{1}{2} \beta_{ij} x^j.
\end{align*}
\] Then, if the variables of classical phase space obey the usual Poisson brackets, i.e. \(\{x_i, x_j\}_0 = 0\), \(\{p_i, p_j\}_0 = \delta_{ij}\), the primed variables will yield
\[
\begin{align*}
\{x'_i, x'_j\} &= \theta_{ij}, \\
\{x'_i, p'_j\} &= \delta_{ij} + \sigma_{ij} \\
\{p'_i, p'_j\} &= \beta_{ij},
\end{align*}
\] with \(\sigma_{ij} = -\theta_{kl}(\beta_{ij} \delta_{kl}/4\). Consequently, to consider the noncommutativity effects, one can work with the ordinary Poisson brackets (17) instead of the \(\alpha\)-star deformed Poisson brackets (15). Indeed, transformation (16) allows an extension of the commutative classical mechanics to the NC one. In the geometrical language, the usual Poisson brackets are mapped onto the modified Poisson brackets through transformation (16) and viceversa. It is evident that, for a
compatible extension, transformation (16) must be invertible, and this imposes a condition on
the NC parameters which we will specify for our model in below.

Furthermore, let \( \mathcal{H} = \mathcal{H}(x_i, p_i) \) be the Hamiltonian of a system including the commutative
variables; we shift the canonical variables through (16) and assume that the functional form
of the Hamiltonian in the NC case is still the same as the commutative one, i.e.

\[ \mathcal{H}_{nc} \equiv \mathcal{H}(x'_i, p'_i) = \mathcal{H}(x_i - \frac{1}{2} \theta_{ij} p^i, p_i + \frac{1}{2} \beta_{ij} x^i). \]  

(18)

This function is also defined on the commutative space and, obviously, the equations of motion
for unprimed variables are \( \dot{x}^i = \frac{\partial \mathcal{H}_{nc}}{\partial p_i} \) and \( \dot{p}_i = -\frac{\partial \mathcal{H}_{nc}}{\partial x_i} \). Evidently, the effects due
to the noncommutativity arise by terms including the parameters \( \theta_{ij} \) and \( \beta_{ij} \).

Now, in our model, the following notations are adopted:

\[
\begin{align*}
(a^1, a^2, a^3) &= (a, x, y) \\
(p^1, p^2, p^3) &= (p_a, p_x, p_y).
\end{align*}
\]

In this work, we assume that the only non-zero NC parameters are \( \theta^{23} = \theta \neq 0 \) and \( \beta^{23} = 4\beta \geq 0 \);

\[
\text{hence } \sigma^{22} = \sigma^{33} = \sigma^{\beta \beta} = 0.
\]

(19)

and equations of motion are

\[
\dot{a} = \{ a, \mathcal{H}_{nc} \} = -\frac{1}{2} p_a, \quad \dot{p}_a = \{ p_a, \mathcal{H}_{nc} \} = 2k a.
\]

(22)

and

\[
\begin{align*}
\dot{x} &= \{ x, \mathcal{H}_{nc} \} = \frac{1}{2} (1 + k \theta^2) p_x + (k \theta + \beta) y, \\
p_x &= \{ p_x, \mathcal{H}_{nc} \} = -2(k + \beta^2) x + (k \theta + \beta) p_y, \\
\dot{y} &= \{ y, \mathcal{H}_{nc} \} = \frac{1}{2} (1 + k \theta^2) p_y - (k \theta + \beta) x, \\
p_y &= \{ p_y, \mathcal{H}_{nc} \} = -2(k + \beta^2) y - (k \theta + \beta) p_x.
\end{align*}
\]

(23)

Equations (23) show that, in general, the motion equations of the scalar fields are coupled in
the NC case, and these equations reduce to the commutative equations (8) for \( \theta = 0 = \beta \),
as expected. Also, in the purposed model, the noncommutativity does not affect the time
dependence of the scale factor and its solution is the same as the commutative case.

The motion equations of the scalar fields, after eliminating momenta variables in (23), are

\[
\begin{align*}
\ddot{x} &= -k(1 - \theta \beta)^2 x + 2(k \theta + \beta) \dot{y} \\
\ddot{y} &= -k(1 - \theta \beta)^2 y - 2(k \theta + \beta) \dot{x},
\end{align*}
\]

(24)

with the Hamiltonian constraint

\[
x^2 + y^2 = \text{constant} \quad \text{if} \quad 1 + k \theta^2 = 0
\]

or

\[
(x^2 + y^2) + k(1 - \theta \beta)^2 (x^2 + y^2) = \text{constant} \quad \text{if} \quad 1 + k \theta^2 \neq 0.
\]

2 Note that, noncommutativity between \( x \) and \( y \) while they commute with the scale factor is completely consistent
with noncommutativity between the original scalar fields, \( \phi \) and \( \psi \), while they also commute with the scale factor.

3 The condition \( 1 + k \theta^2 = 0 \) is possible only when \( k = -1 \), and hence \( \theta = 1 \).
Spacetime geometries with \( k = 0, 1 \) only satisfy the latter constraint for any \( \theta \), while \( k = -1 \) geometry fulfills the former constraint with \( \theta = 1 \) (and consequently, because of the inversion condition, \( \beta \neq 1 \)), and the latter one with \( \theta \neq 1 \). In general, solutions of (24) depend on the sign of a quantity defined as \( \Delta \equiv (1 + k \theta^2)(\beta^2 + k) \).

For \( k = 0, 1 \), the sign of \( \Delta \) is always positive, and thus, the real solution of equations (24) can be written as

\[
k = 0, 1 : \begin{cases}x(t) = A \cos \omega_1 t + B \cos \omega_2 t \\
y(t) = A \sin \omega_1 t + B \sin \omega_2 t,
\end{cases}
\]

where \( A \) and \( B \) are constants of integration subject to the corresponding Hamiltonian constraint, and

\[
\omega_1 \equiv \sqrt{\Delta} - k \theta - \beta \quad \text{and} \quad \omega_2 \equiv -\sqrt{\Delta} - k \theta - \beta.
\]

Obviously, the corresponding terms in (25) are in the \( \pi/2 \) phase difference.

In the case of \( k = 0 \), the NC parameter \( \beta \) does not actually appear in equations (24), and hence, in a spatially flat FRW universe, the scalar fields motion equations are affected only by the NC parameter \( \beta \). Besides, one gets \( \omega_1 = 0 \) and \( \omega_2 = -2 \beta \), and thus solutions are similar to the motion of a free harmonic oscillator. However, the time dependence of solutions are modified from linear in the commutative case to periodic in the NC case with the period of \( \pi/\beta \) which allows the model to be adjusted more easily with the observational data.

For \( k = 1 \) geometry, the commutative solutions (9) are simple harmonics with the period of \( 2\pi \), whereas, the NC solutions (25) in general are not, though they still oscillate between two limits. Incidentally, solutions (25) are symmetric with respect to the NC parameters, and results do not vary by replacement of their roles. Besides, as \( \omega_1 \neq \omega_2 \), each solution of (25) can be considered as a forced (driven) harmonic oscillator, such that if the ratio of \( \omega_2/\omega_1 \) is a rational fraction, then solutions will be periodic (as in the Lissajous figures) with angular frequency given by the greatest common divisor of \( \omega_1 \) and \( \omega_2 \). Otherwise, solutions are non-periodic and never repeat themselves. A general condition that picks periodic solutions has been obtained in the appendix (supplemented with a diagram). Meanwhile, in the following special example, we provide a better insight about this situation.

Without loss of generality, let us consider the special case \( \beta = 0 \), with \( A = -B \equiv D/2 \), for which solutions (25) read

\[
\begin{cases}x(t) = D \sin(\theta t) \sin(\sqrt{1 + \theta^2} t) \\
y(t) = D \cos(\theta t) \sin(\sqrt{1 + \theta^2} t),
\end{cases}
\]

with initial conditions \( x(0) = 0 = y(0), \dot{x}(0) = 0 \) and \( \dot{y}(0) = D \sqrt{1 + \theta^2} \). The condition for the periodic solutions is

\[
\frac{\omega_2}{\omega_1} = -\frac{\sqrt{1 + \theta^2 + \theta}}{\sqrt{1 + \theta^2 - \theta}} = -\frac{n_2}{n_1},
\]

where \( n_1 \) and \( n_2 \) are positive integers and obviously \( n_2 > n_1 \). By solving this equation in terms of \( \theta \), one gets

\[
\theta = \frac{1}{2}(n_2 - n_1)/\sqrt{n_1 n_2} = \frac{1}{2}(\sqrt{n_2/n_1} - \sqrt{n_1/n_2}).
\]

This relation provides different values of the \( \theta \) parameter in terms of two integers, for which solutions (27) can be periodic. It is constructive to plot \( \theta \) in terms of \( r \equiv n_2/n_1 \) as a continuous quantity\(^4\), and this has been illustrated in figure 1 for the range \( 1 \leq r < 80 \). As it is obvious,

\footnote{Note that \( n_2/n_1 \) is a rational number greater than unity, indeed, \( r = 1 \) yields \( \theta = 0 \) which resumes the commutative case.}

7
Figure 1. The NC parameter $\theta$, in the $k = 1$ case with $\beta = 0$, as a function of $r \equiv n_2/n_1$. The points whose $r$ are rational fractions correspond to the periodic scalar fields, and other points correspond to the non-periodic ones.

$\frac{d\theta}{dr}$ is always positive; thus the function $\theta(r)$ changes monotonically with $r$. Also, when $r$ has very large values, the rate $\frac{d\theta}{dr}$ approximately decreases as $1/\sqrt{r}$. For the smallest allowed values of integers, i.e. $n_2 = 2$ and $n_1 = 1$, we get the smallest value of $\theta$ parameter, i.e. $\theta_{\text{min}} = 1/2$ for the periodic solutions. In solutions (27), the terms $D \sin(\theta t)$ and $D \cos(\theta t)$ are envelopes of the corresponding curves. These types of oscillations, due to the periodic variation of the amplitude, are usually called beats. The well-known example of such oscillations is the simple harmonic one, where a driven force causes mechanical beats. Also, this situation can be simulated in the acoustical systems that produce a sound effect, known as beating [27], when $|\omega_2| - |\omega_1| = 2\theta$ is sufficiently small and the terms including $\theta^2$ in (27) are ignored. As a sample illustration of the beating effect for these cosmical oscillators, we have graphed a plot of $x(t)$, equations (27), with its envelopes $\pm D \sin(\theta t)$, in figure 2(right) for the numerical values $\theta = 0.1$ and $D = 1$. As the figure illustrates, beats are more obvious when the envelopes are drawn.

When the ratio of $\omega_2$ and $\omega_1$ is not a rational fraction, solutions are non-periodic but, as periodic cases, their behaviors still depend on the values of the NC parameters. For example, if one constructs a plot with numerous or a few relative extremum in a given time interval, then iterative drawings will indicate that the separation between the high and low points increases when the NC parameters tend to smaller values. This property is intensified for values less than unity, which is illustrated in figure 3 for $y(t)$ in equations (25) with constants $A = 2$ and $B = 1$. From this point of view, the NC solutions have particular preference with respect to the corresponding commutative ones.
Figure 2. The NC case for $k = 1$: periodic field (left) with $\theta = 2 = \beta$, beating effect (right) with $\theta = 0.1$ and $\beta = 0$ including the cosmical oscillator (solid line) and envelops (dashed lines).

Figure 3. Non-periodic field in the NC case for $k = 1$: with $\theta = 2$ and $\beta = 0.1$ (solid line) and with $\theta = 0.2$ and $\beta = 0.01$ (dashed line).

For $k = -1$ geometries, one has $\Delta = (1 - \theta^2)(\beta^2 - 1)$ that can be either positive, or negative or zero depending on different choices of the NC parameters, that is, when $(\theta > 1$ and $\beta < 1)$ or $(\theta < 1$ and $\beta > 1)$, $\Delta$ is positive and hence one gets solutions of type (25). However, in this case, the NC parameters have upper and lower bounds, and there is more
restriction on finding the periodic solutions than in \( k = 1 \) geometry. On the other hand, when \( \Delta \) is negative and \( \theta \neq \beta \), the real solutions of equations (24) are

\[
\begin{align*}
\dot{x}(t) &= (C \sinh \sqrt{-\Delta}t + D \cosh \sqrt{-\Delta}t) \cos(\theta - \beta)t, \\
\dot{y}(t) &= (C \sinh \sqrt{-\Delta}t + D \cosh \sqrt{-\Delta}t) \sin(\theta - \beta)t.
\end{align*}
\]

(28)

where \( C \) and \( D \) are constants of integration subject to the corresponding Hamiltonian constraint. Solutions (28) are constrained to the condition that both NC parameters simultaneously have a lower bound, namely \( \theta \) and \( \beta > 1 \), or an upper bound, \( \theta \) and \( \beta < 1 \). Also, in the case \( k = 1 \), oscillations of the scalar fields in (28) have the phase difference of \( \pi/2 \). These solutions, in contrast to their commutative analogs, are oscillating with a time hyperbolic amplitude that depends on the NC parameters and gives enough room for better adjustments. For instance, one can change the time interval between two successive zero points of the scalar fields, for the interval is \( \pi/|\theta - \beta| \). Besides, increasing and decreasing amplitudes in (28) with respect to the time depend on initial conditions. As an example, if one chooses the initial condition \( C = -D \), then solutions of (28) will oscillate with decreasing amplitudes. Such solutions are similar to the damped harmonic oscillators with amplitudes proportional to \( \exp(-\sqrt{-\Delta}t) \) as envelopes. The main characters of such an oscillator, namely the time decay and the natural frequency, can be described in terms of the NC parameters as \( \tau = 1/\sqrt{-\Delta} \) and \( \omega_0 = \sqrt{\omega_1 \omega_2} = |1 - \theta \beta| \), respectively. Another interesting point is that, if one assumes \( \theta \) and \( \beta > 1 \), then solutions will damp quickly. Inversely, to possess more lately damped oscillations, the best choice of the NC parameters in the range \( \theta \) and \( \beta < 1 \) is when one assumes \( \theta \to 1^- \) and \( \beta \to 0 \), in which each of the scalar fields has a maximum number of oscillations before the complete damping occurs. The diagram of such an oscillation is plotted in figure 4 for the \( x(t) \) of solutions (28) with \( D = 1 \) for numerical values \( (\theta = 1.5 \) and \( \beta = 1.05) \) as quick damping, and for \( (\theta = 0.999 \) and \( \beta = 0.001) \) as late damping.

When \( \theta = \beta \) in \( k = -1 \) geometry, solutions of (24) are decoupled hyperbolic time functions with the coefficient \( (1 - \theta^2) \) in the exponent, which is the only difference between the NC and commutative solutions. The case \( \Delta = 0 \) is also possible when \( \theta = 1 \) or when \( \beta = 1 \), where solutions (28) are again periodic with the period of \( 2\pi/|\beta - 1| \) or \( 2\pi/|\theta - 1| \), respectively. Note that, the case \( k = -1 \) with \( \theta = 1 \) resembles the constraint condition \( 1 + k\theta^2 = 0 \) with arbitrary \( \beta \), and however for \( \beta = 1 \) one gets trivial constant solutions, but this value is not allowed.

**Non-zero constant potential**

Let us obtain the equations of motion for the scale factor when one considers a non-zero constant value for the potential function (i.e. a cosmological constant) in action (1). Thus, suppose the potential function is \( V(\phi, \psi) = \Lambda \). In a manner similar to the free-potential case, one gets the Hamiltonian

\[
H = -\frac{p_n^2}{4} + \frac{p_x^2 + p_y^2}{4} - ka^2 + k(x^2 + y^2) + \Lambda a^4,
\]

(29)
in the commutative case, and

\[
H_{nc} = -\frac{p_n^2}{4} + \frac{1 + k\theta^2}{4}(p_x^2 + p_y^2) - ka^2 + (k + \beta^2)(x^2 + y^2) + (k\theta + \beta)(yp_x - xp_y) + \Lambda a^4,
\]

(30)
in the NC one. It is obvious that the Hamiltonian equations for the scalar fields, derived from (29) and (30), are the same as those obtained in the corresponding free-potential case, namely equations (8) and (23), respectively. But, the Hamiltonian equations of the scale factor, for both the commutative and NC cases, are modified in the same manner as

\[
a = -\frac{p_n}{2} \quad \text{and} \quad p_n = 2ka - 4\Lambda a^3.
\]

(31)
Figure 4. Damping scalar field in the NC case $k = -1$: late damping with $\theta = 0.999$ and $\beta = 0.001$ (solid line), and quick damping with $\theta = 1.5$ and $\beta = 1.05$ (dashed line).

Eliminating the momentum variable gives

$$\ddot{a} = 2\Lambda a^3 - ka. \quad (32)$$

By integrating, one gets

$$\dot{a}^2 = \Lambda a^4 - ka^2 + v_0^2, \quad (33)$$

where $v_0^2$ is an integration constant. Solution of equation (33) can be written in terms of the Jacobi elliptic functions, that is,

$$a(t) = \frac{v_0}{\alpha} \text{sn}(\alpha t, m), \quad (34)$$

where the initial condition $a(0) = 0$,

$$\alpha^2 = \frac{1}{2} \left(k + \sqrt{k^2 - 4\Lambda v_0^2}\right) \quad \text{and} \quad m = \left[\left(\alpha^2 / \Lambda v_0^2\right) - 1\right]^{-1/2}. \quad (35)$$

The Jacobi function, on the right-hand side of (34), behaves as a sine function when $\alpha$ is a real and $m$ is either a real or a pure imaginary number. Therefore, for negative potentials in an arbitrary geometry, and also for positive potentials in the $k = 1$ geometry when $4\Lambda v_0^2 \lesssim 1$, the scale factor behaves periodically as a sinusoidal function. On the other hand, for positive potentials in $k = 0, -1$ geometries, and in the $k = 1$ geometry when $4\Lambda v_0^2 > 1$, the Jacobi function has a complex value, and hence these situations are not allowed.

In the limit $\Lambda \to 0$ for $k = 1$, one gets $m \to 0$, $\alpha^2 \to 1$ and $\text{sn}(t, 0)$ behaves as $\sin(t)$. Hence, we get back to the free-potential solution (9), as expected. However, in such a limit,

5 For the properties of the elliptic functions, see, e.g., [28].
for $k = -1, 0$, as $\alpha \to 0$, solution (34) is not valid, and hence one must consider equation (33). Now, by regarding the limit $\Lambda \to 0$ in equation (33) for $k = -1, 0$, and solving the resulted equation, one again gets the free-potential solutions (10) and (11), respectively.

3. The quantum model

Although the effects of NC cosmology are mostly desired and important in classical approaches, but still it is instructive to investigate its quantum counterparts. In particular, if a universe has been commenced by a big bang or from a very very tiny scale, then it would be plausible that quantum behaviors should have significant implications in its evolution and cosmology. Thus, in this section we proceed to quantize the cosmological model given by action (1) for free-potential, such that the canonical quantization of the phase space leads to the WD equation, $\hat{H}\Psi = 0$, where $\hat{H}$ is the Hamiltonian operator and $\Psi$ is a wavefunction of the universe [29]. We employ the usual canonical transition from classical to quantum mechanics via the generalized Dirac quantization of the Poisson brackets to quantum commutators, i.e. $[\cdot, \cdot] \to -i \{\cdot, \cdot\}$. Then, as the classical approach, we investigate the general properties of the wavefunction in the commutative and NC frames of the quantum model in the following subsections.

3.1. The commutative quantum cosmology

As usual, the operator form of Hamiltonian (7) can be acquired by the replacements $p_a \to -i\partial_a$, $p_x \to -i\partial_x$ and $p_y \to -i\partial_y$. Assuming a particular factor ordering, the corresponding WD equation is

$$\left[ \frac{\partial^2}{\partial a^2} - \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + 4k(x^2 + y^2 - a^2) \right] \Psi(a, x, y) = 0.$$  (36)

In terms of the polar coordinates

$$x = \rho \cos \varphi \quad \text{and} \quad y = \rho \sin \varphi,$$  (37)

equation (36) reads

$$\left[ \frac{\partial^2}{\partial a^2} - 4ka^2 - \left( \frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial \rho^2} \right) \right] \Psi(a, \rho, \varphi) = 0.$$  (38)

Let us choose a product ansatz as a solution of equation (38), namely

$$\Psi(a, \rho, \varphi) = A(a)B(\rho)e^{2i\nu},$$  (39)

where $\nu$ is a real constant. By substituting (39) into equation (38), one gets

$$A'' + 4(\mu - ka^2)A = 0$$  (40)

and

$$\rho^2 B'' + \rho B' + 4(\mu \rho^2 - k\rho^4 - \nu^2)B = 0,$$  (41)

where the prime denotes ordinary derivative with respect to the argument and $\mu$ is a constant of separation. Solutions of equations (40) and (41) for real values of $\mu$, corresponding to the curvature index and boundary conditions, are

$$k = 0: \begin{cases} A_\mu(a) = C_1 \sin(2\sqrt{\mu}a) + C_2 \cos(2\sqrt{\mu}a) \\ B_\mu(\rho) = D_1 J_{2\nu}(2\sqrt{\mu}\rho) + D_2 Y_{2\nu}(2\sqrt{\mu}\rho) \end{cases}$$  (42)
and
\begin{equation}
\begin{aligned}
    k = -1, 1 : \quad & A_\mu(a) = a^{-1/2} \left[ C_3 M_\mu^{-1/2}(2\sqrt{k} a^2) + C_4 W_\mu^{-1/2}(2\sqrt{k} a^2) \right] \\
    & B_{\mu\nu}(\rho) = \rho^{-1} \left[ D_3 M_{\mu\nu}^{-1/2}(2\sqrt{k} \rho^2) + D_4 W_{\mu\nu}^{-1/2}(2\sqrt{k} \rho^2) \right],
\end{aligned}
\end{equation}

where \( J_{2\nu} \) and \( Y_{2\nu} \) are respectively the Bessel functions of the first and second kind, \( M_{\eta,\lambda} \) and \( W_{\eta,\lambda} \) are the Whittaker functions and their indices, see again [28].

For positive (zero) curvature, the Whittaker (Bessel) term \( M_{\eta,\lambda} (Y_{2\nu}) \), in the classically forbidden region, is divergent\(^6\). Thus, one can discard these terms and write the well-defined eigenfunctions of equation (38) as

\[
\psi_{\mu\nu}(a, \rho) = A_\mu(a) B_{\mu\nu}(\rho) \\
\propto \begin{cases} 
    [C_1 \sin(2\sqrt{k}a) + C_2 \cos(2\sqrt{k}a)] J_{2\nu}(2\sqrt{k} \rho) & \text{for } k = 0, \\
    (\rho \sqrt{a})^{-1} W_{2\nu}^{-1/2}(2a^2) W_{2\nu}^{-1/2}(2\rho^2) & \text{for } k = 1, \\
    (\rho \sqrt{a})^{-1} M_{2\nu}^{-1/2}(2a^2) M_{2\nu}^{-1/2}(2\rho^2) & \text{for } k = -1,
\end{cases}
\]

where in \( k = -1 \), for simplicity, we have written solution (44) only in terms of the Whittaker function \( M_{\eta,\lambda} \). The wave packet corresponding to (44) is

\[
\Psi(a, \rho, \varphi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E_\mu E_\nu \psi_{\mu\nu} \ e^{2i\nu \varphi} \ d\mu \ d\nu.
\]

where \( E_\mu \) can be taken [16, 19] to be a shifted Gaussian weight function with constants \( b \) and \( c \) as in \( \exp[-b(\mu - c)^2] \), and a similar expression for \( E_\nu \).

3.2. The NC quantum cosmology

It is well known [26] that, in the NC quantum mechanics, the original phase space and its symplectic structure are modified. That is, for the NC proposal of quantum cosmology, we assume that operators (variables) of the FRW minisuperspace obey a kind of deformed Heisenberg algebra like the ones in the NC quantum mechanics as [26]

\[
[\hat{x}_i, \hat{p}_j] = i\theta_{ij}, \quad [\hat{x}_i, \hat{p}_j] = i(\delta_{ij} + \sigma_{ij}) \quad \text{and} \quad [\hat{p}_i, \hat{p}_j] = i\beta_{ij}.
\]

The notations and definitions are the same as in the NC classical model. This kind of extended noncommutativity maintains symmetry between the canonical operators, and yields the usual Heisenberg algebra in the limit \( \theta_{ij} \) and \( \beta_{ij} \to 0 \). As usual, this deformation can be redefined in terms of noncommutativity of minisuperspace functions with the Moyal product defined in (12). Thus, the corresponding NC WD equation can be written by replacing the operator product, in equation (36), with the star product, namely \( \hat{H} \ast \Psi = 0 \).

However, it is possible to reformulate equations in terms of the commutative operators with the ordinary product of functions if the new operators are introduced, again as our previous assumption

\[
\hat{x}' = \hat{x} - \frac{\theta}{2} \hat{p}_y, \quad \hat{p}'_x = \hat{p}_x + 2\beta \hat{y}, \quad \hat{y}' = \hat{y} + \frac{\theta}{2} \hat{p}_x \quad \text{and} \quad \hat{p}'_y = \hat{p}_y - 2\beta \hat{x}.
\]

Clearly, if unprimed operators obey the usual Heisenberg commutators, then the non-zero primed operators will obey the deformed Heisenberg commutators (46) in the form

\[
[\hat{x}', \hat{y}'] = i\theta, \quad [\hat{x}', \hat{p}'_x] = i(1 + \theta \beta) = [\hat{y}', \hat{p}'_y] \quad \text{and} \quad [\hat{p}'_x, \hat{p}'_y] = 4i\beta.
\]

\(^6\) For the properties of the Whittaker and Bessel functions and their indices, see again [28].
Therefore, operator transformation (47) can be regarded as a generalization of the usual quantum mechanics to the NC one. On the other hand, the inverse transformation of (47) are
\[
\hat{x} = \eta \left( \hat{x}' + \frac{\theta}{2} \hat{p}'_y \right), \quad \hat{p}_x = \eta (\hat{p}'_x - 2\beta \hat{y}'),
\]
\[
\hat{y} = \eta \left( \hat{y}' - \frac{\theta}{2} \hat{p}'_x \right) \quad \text{and} \quad \hat{p}_y = \eta (\hat{p}'_y + 2\beta \hat{x}'),
\]
where \( \eta \equiv 1/(1-\theta \beta) \). Consequently, one can go from the usual commutators to the deformed ones and vice versa provided that again \( \theta \beta \neq 1 \). As a result, the original equation, employing the new operators, reads [30]
\[
\hat{H} (\hat{x}_i, \hat{p}_j) \ast \Psi = \hat{H} (\hat{x}_i - \frac{1}{2} \theta ij \hat{p}^j, \hat{p}_i + \frac{1}{2} \beta ij \hat{x}^j) \Psi = 0. \tag{48}
\]
Hence, the noncommutative WD equation corresponding to the NC Hamiltonian (21) is
\[
\left[ \partial_x^2 - 4ka^2 - (1 + k\theta^2) (\partial_x^2 + \partial_y^2) \right. \left. + 4i (k\theta + \beta_x) (x \partial_y - y \partial_x) \right.
\]
\[
\left. + 4(k + \beta^2)(x^2 + y^2) \right] \Psi_{nc}(a, x, y) = 0, \tag{49}
\]
which in the polar coordinates (37) yields
\[
\left[ \partial_\rho^2 + \frac{2v}{\rho} \right. \left. - \frac{k + \beta^2}{1 + k\theta^2} \rho^2 \right] \Psi_{nc}(a, \rho, \varphi) = 0. \tag{50}
\]
When \( 1 + k\theta^2 \neq 0 \), by using the ansatz (39), equation (50) is separable to
\[
A'' + 4(\mu - ka^2)A = 0, \tag{51}
\]
and
\[
\rho^2 B'' + \rho B' + 4 \left[ \frac{\mu + 2v(k\theta + \beta)}{1 + k\theta^2} \rho^2 - \frac{k + \beta^2}{1 + k\theta^2} \rho^2 - \nu^2 \right] B = 0, \tag{52}
\]
where \( \mu \) is a constant of separation. The scale factor part of the wavefunction, equation (51), is similar to its commutative analog (40), as expected. The radial part of the wavefunction, equation (52), first reduces to its commutative analog, equation (41), when \( \theta = 0 = \beta \), as again expected. Secondly, in the case \( k = 0 \) with \( \beta = 0 \), equation (52) once again reduces to equation (41) even if the NC parameter \( \theta \) does exist. However, in \( k = 0 \), solutions of equation (52) do not depend on the NC parameter \( \theta \) at all. Namely in a flat FRW universe, the \( \beta \) parameter is the only responsible parameter for the NC effects. These properties are common with the classical model.

Comparing equation (52) with the commutative analog (41) shows that the functional form of the radial part of the wavefunction (and hence, the whole wavefunction) is the same as the commutative ones provided that the coefficients \( \mu \) and \( k \) in equation (41) are replaced by
\[
\mu \rightarrow \mu + 2v(k\theta + \beta) \quad \text{and} \quad k \rightarrow \frac{k + \beta^2}{1 + k\theta^2}. \tag{53}
\]
Therefore, for \( k = 0, 1 \) and \( k = -1 \) with \( \theta \neq 1 \), the NC eigenfunctions are in terms of the Whittaker functions similar to the commutative solutions (44). However, in the case \( k = -1 \), there are bounds on the NC parameters. For a better illustration, let us write (53) as
\[
\mu \rightarrow \sigma \equiv \frac{\mu + 2v(\beta - \theta)}{1 - \theta^2} \quad \text{and} \quad k \rightarrow K \equiv \frac{1 - \beta^2}{1 - \theta^2}. \tag{54}
\]
As will be seen, this condition should automatically be satisfied when the wavefunction is well behaved.
Since this modified curvature index, $K$, can have positive, negative and zero values, the solutions again are similar to the commutative cases (42) and (43), where $K = 0$ also requires $\beta = 1$. But, for $K \neq 0$, the NC parameters must satisfy inequalities depending on the sign of $K$. Namely if $K < 0$, then there will be $(\theta$ and $\beta < 1)$ or $(\theta$ and $\beta > 1)$, and if $K > 0$, then $(\theta > 1$ and $\beta < 1)$ or $(\theta < 1$ and $\beta > 1)$. For instance, by choosing inequality $(\theta$ and $\beta < 1)$ when $K < 0$, the corresponding wave packet can be written in the form

$$\Psi_{nc}(a, \rho, \phi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E_{\mu} E_{\nu}(\rho \sqrt{a})^{-1} M_{-1/4} \left(2i a^2\right) M_{-1/4} \left(2\sqrt{K} \rho^2\right) e^{2i\nu} d\mu d\nu,$$

(54)

where the lower limits, in contrast to the commutative wave packet (45), are bounded by $\mu + 2\nu(\beta - \theta) \geq 0$.

In the case $1 + 2k = 0$, namely when $k = -1$ and $\theta = 1$ (hence $\beta \neq 1$), with the separation of variables $\Psi_{nc} = A(a)B(\rho, \phi)$, equation (50) reduces to two differential equations. One equation is similar to equation (51), and the other one is

$$[i(1 - \beta)\partial_\phi + (1 - \beta^2)\rho^2]B(\rho, \phi) = C,$$

(55)

where $C$ is a separation constant. As, in general, $C$ has a non-zero value, equation (55) does not have a suitable solution for $\beta = 1$ (which itself is not allowed). For $\beta \neq 1$, one easily shows that its solution can be written as

$$B(\rho, \phi) = F(\rho) e^{i(1+\beta)\rho^2} + B_0/\rho^2,$$

where $F$ is an arbitrary function and $B_0 = C/(1 - \beta^2)$. As this function is not well behaved when $\rho$ tends to zero, it is not allowed.

4. The Noether symmetries

In this section, we employ the Noether theorem and explore the effects of noncommutativity on the underlying symmetries in the commutative frame. For this purpose, following [9, 31], one can define the Noether symmetry as a vector field, say $X$, on the tangent space of the phase space. In our model, it can be, in general, as

$$X = A \frac{\partial}{\partial a} + B \frac{\partial}{\partial x} + C \frac{\partial}{\partial y} + \frac{dA}{dt} \frac{\partial}{\partial \dot{a}} + \frac{dB}{dt} \frac{\partial}{\partial \dot{x}} + \frac{dC}{dt} \frac{\partial}{\partial \dot{y}},$$

(56)

such that the Lie derivative of the Lagrangian with respect to this vector field vanishes, i.e.

$$L_X \mathcal{L} = 0.$$

(57)

For simplicity, we assume the unknown functions $A$, $B$ and $C$ to be linear in terms of the configuration variables $a$, $x$ and $y$. The time derivative $d/dt$ represents the Lie derivative along the dynamical vector fields, which in our model is $d/dt = \dot{a}\partial/\partial a + \dot{x}\partial/\partial x + \dot{y}\partial/\partial y$.

Now, in order to obtain constants of motion, let us rewrite equation (57) as

$$\left( A \frac{\partial \mathcal{L}}{\partial a} + \frac{dA}{dt} \frac{\partial \mathcal{L}}{\partial \dot{a}} \right) + \left( B \frac{\partial \mathcal{L}}{\partial x} + \frac{dB}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} \right) + \left( C \frac{\partial \mathcal{L}}{\partial y} + \frac{dC}{dt} \frac{\partial \mathcal{L}}{\partial \dot{y}} \right) = 0.$$

(58)

By employing the Lagrange equation $\partial \mathcal{L}/\partial q = d\rho_q/dt$, it reads
which yields
\[
d\frac{d}{dt}(A_{pa} + B_{px} + C_{py}) = 0.
\] (60)
Therefore, the constants of motion are
\[
Q \equiv A_{pa} + B_{px} + C_{py},
\] (61)
for different unknown functions of \(A\), \(B\) and \(C\). To obtain these functions, one can employ equation (58) or equation (59) which is more suitable in the Hamiltonian formalism. To manage this, one can write equation (59) in terms of the Poisson bracket,
\[
\{, H\} = \frac{d}{dt}, as
\]
\[
A\{pa, H\} + B\{px, H\} + C\{py, H\} + \left[ \frac{\partial A}{\partial a} (a, H) + \frac{\partial A}{\partial x} (x, H) + \frac{\partial A}{\partial y} (y, H) \right] p_a
\]
\[
+ \left[ \frac{\partial B}{\partial a} (a, H) + \frac{\partial B}{\partial x} (x, H) + \frac{\partial B}{\partial y} (y, H) \right] p_x
\]
\[
+ \left[ \frac{\partial C}{\partial a} (a, H) + \frac{\partial C}{\partial x} (x, H) + \frac{\partial C}{\partial y} (y, H) \right] p_y = 0.
\] (62)
This equation, in general, gives quadratic polynomials in terms of the momenta with coefficients being partial derivatives of \(A\), \(B\) and \(C\) with respect to the configuration variables. Hence, the expression identically is equal to zero if and only if these coefficients vanish, which lead to a system of partial differential equations for \(A\), \(B\) and \(C\).

In the following subsections, we obtain such symmetries for the model in the commutative and NC cases.

4.1. Symmetries in the commutative frame

In this case the Hamiltonian is given by relation (7); hence by substituting the corresponding Poisson brackets into equation (62), one gets
\[
2k(aA - xB - yC) + \frac{1}{2} \left( -\frac{\partial A}{\partial a} p_a^2 + \frac{\partial B}{\partial x} p_x^2 + \frac{\partial C}{\partial y} p_y^2 \right) + \frac{1}{2} \left( \frac{\partial A}{\partial x} - \frac{\partial B}{\partial a} \right) p_a p_x
\]
\[
+ \frac{1}{2} \left( \frac{\partial A}{\partial y} - \frac{\partial C}{\partial a} \right) p_a p_y + \frac{1}{2} \left( \frac{\partial B}{\partial y} + \frac{\partial C}{\partial x} \right) p_x p_y = 0.
\] (63)
Let us first treat spatially non-flat geometry, \(k \neq 0\), for which equation (63) leads to
\[
aA - xB - yC = 0,
\]
\[
\frac{\partial A}{\partial a} = \frac{\partial B}{\partial x} = \frac{\partial C}{\partial y} = 0.
\] (64)
A general solution of constraints (64) is
\[
A = c_1 x + c_2 y, \quad B = c_1 a - c_3 y \quad and \quad C = c_2 a + c_3 x,
\] (65)
where \(c_i\) are three constants of integration. For an easier representation of the commutation relations between symmetric vectors, let us change \(x \rightarrow -x\) and \(y \rightarrow -y\) in solution (65). Thus by (61), three independent constants of motion are
\[
Q_1 = ap_x - xp_a, \quad Q_2 = ap_y - yp_a \quad \text{and} \quad Q_3 = yp_x - xp_y, \tag{66}
\]

which are the well-known angular momenta about the configuration variables. The corresponding symmetric vectors are

\[
X_1 = a \frac{\partial}{\partial x} - \frac{\partial}{\partial a} + \dot{a} \frac{\partial}{\partial \dot{x}} - \dot{x} \frac{\partial}{\partial \dot{a}},
\]

\[
X_2 = a \frac{\partial}{\partial y} - \frac{\partial}{\partial a} + \dot{a} \frac{\partial}{\partial \dot{y}} - \dot{y} \frac{\partial}{\partial \dot{a}}, \tag{67}
\]

\[
X_3 = y \frac{\partial}{\partial x} - \frac{\partial}{\partial y} + \dot{y} \frac{\partial}{\partial \dot{x}} - \dot{x} \frac{\partial}{\partial \dot{y}},
\]

which satisfy the Lie algebra \([X_i, X_j] = \varepsilon_{ijk}X_k\), where \(\varepsilon_{ijk}\) is the Levi-Civita tensor.

For \(k = 0\), it is clear from equation (63) that the symmetries can be obtained from the last two constraints of (64). Thus, the corresponding solution is

\[
A = d_1x + d_2y + d_3, \quad B = d_1a + d_4y + d_5 \quad \text{and} \quad C = d_2a - d_4x + d_6, \tag{68}
\]

where \(d_i\) are six constants of integration. Therefore, the six independent constants of motion are

\[
Q_1 = p_a, \quad Q_2 = p_x, \quad Q_3 = p_y, \quad Q_4 = yp_x - xp_y, \quad Q_5 = yp_a + ap_y \quad \text{and} \quad Q_6 = xp_a + ap_x. \tag{69}
\]

Incidentally, in this case, a glance at Hamiltonian (7) shows that all configuration variables are cyclic and consequently their corresponding momenta are constants of motion, i.e. \(Q_1\) to \(Q_3\). In addition, Hamiltonian (7) for \(k = 0\) (actually the Lagrangian \(\dot{x}^2 + \dot{y}^2 - \dot{a}^2\)) is invariant under rotation about the \(a\)-axis. Thus, the angular momentum about this axis is conserved, i.e. \(Q_4\).

The corresponding symmetric vectors, in the flat geometry, are

\[
X_1 = \frac{\partial}{\partial a}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial y},
\]

\[
X_4 = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} + \dot{y} \frac{\partial}{\partial \dot{x}} - \dot{x} \frac{\partial}{\partial \dot{y}},
\]

\[
X_5 = y \frac{\partial}{\partial a} + a \frac{\partial}{\partial y} + \dot{y} \frac{\partial}{\partial \dot{a}} + \dot{a} \frac{\partial}{\partial \dot{y}},
\]

\[
X_6 = x \frac{\partial}{\partial a} + a \frac{\partial}{\partial x} + \dot{x} \frac{\partial}{\partial \dot{a}} + \dot{a} \frac{\partial}{\partial \dot{x}},
\]

which satisfy

\[
[X_1, X_2] = 0, \quad [X_1, X_3] = 0, \quad [X_1, X_4] = 0, \quad [X_1, X_5] = 0, \quad [X_1, X_6] = 0, \quad [X_2, X_3] = 0, \quad [X_2, X_4] = -X_3, \quad [X_2, X_5] = 0, \quad [X_2, X_6] = X_1, \quad [X_3, X_4] = X_2, \quad [X_3, X_5] = X_1, \quad [X_3, X_6] = 0, \quad [X_4, X_5] = -X_6, \quad [X_4, X_6] = X_5, \quad [X_5, X_6] = X_4. \tag{70}
\]

4.2. Symmetries in the NC frame

Now, let us find out which of the above symmetries survive in the NC case. Here, the Hamiltonian is given by relation (21), and the required Poisson brackets are given by
if $\theta$ is $k=18$ and a special value of the number of symmetries to 1 for yields the constants of motion and symmetric vectors (66) and (67). However, for which is an especial case of commutative solution (65) with initial conditions $d=\text{constant}$ with the symmetric vector $X_a=\text{constant}$, which corresponds to the symmetric vector $X_a=\text{constant}$ with the trivial Lie algebra $[X_a, X_b]=0$. This case also restricts commutative solution (68) with initial conditions $d_1=d_2=d_3=d_4=0$. Therefore, in the $k\theta+\beta\neq0$ case, the noncommutativity reduces the number of symmetries to 1 for $k\neq0$ and 2 for $k=0$.

When $k\theta+\beta\neq0$, by putting each coefficient in equation (71) equal to zero, one gets

$$2kA - (k + \beta^2)(xB + yC) + \frac{1}{2} \left[ \frac{\partial A}{\partial a} p_a^2 + \frac{1}{2} (1 + k\theta^2) \left( \frac{\partial B}{\partial x} p_x + \frac{\partial C}{\partial y} p_y \right) \right]$$

$$+ \frac{1}{2} \left[ (1 + k\theta^2) \frac{\partial A}{\partial x} - \frac{\partial B}{\partial a} \right] p_a p_x + \frac{1}{2} \left[ (1 + k\theta^2) \frac{\partial A}{\partial y} = \frac{\partial C}{\partial a} \right] p_a p_y$$

$$+ \frac{1}{2} (1 + k\theta^2) \left( \frac{\partial B}{\partial y} + \frac{\partial C}{\partial x} \right) p_x p_y + (k\theta + \beta) \left[ \left( \frac{\partial A}{\partial x} - \frac{x}{\partial y} \right) p_a \right]$$

$$+ \left( \frac{\partial B}{\partial x} - \frac{x}{\partial y} - C \right) p_x + \left( \frac{\partial C}{\partial x} - \frac{x}{\partial y} + B \right) p_y \right] = 0. \quad (71)$$

Obviously, equation (71) yields extra restrictions on $A$, $B$, and $C$ with respect to the commutative case (63).

When $k\theta+\beta\neq0$, by putting each coefficient in equation (71) equal to zero, one gets

$$2kA - (k + \beta^2)(xB + yC) = 0,$$

$$\frac{\partial A}{\partial a} = \frac{\partial B}{\partial x} = \frac{\partial C}{\partial y} = 0,$$

$$(1 + k\theta^2) \frac{\partial A}{\partial x} - \frac{\partial B}{\partial a} = (1 + k\theta^2) \frac{\partial A}{\partial y} - \frac{\partial C}{\partial a} = \frac{\partial B}{\partial y} + \frac{\partial C}{\partial x} = 0,$$

$$\frac{\partial A}{\partial x} - \frac{x}{\partial y} = y \frac{\partial C}{\partial x} - x \frac{\partial C}{\partial y} + B = 0. \quad (72)$$

Hence, one obtains the solution

$$B = h_0 y, \quad C = -h_0 x \quad (73)$$

and

$$A = \begin{cases} A_0 & \text{for } k = 0 \\ 0 & \text{for } k \neq 0, \end{cases} \quad (74)$$

where $h_0$ and $A_0$ are integration constants. Thus, when $k \neq 0$, the only constant of motion is

$$Q_{nc} = yp_x - xp_y, \quad (75)$$

with the symmetric vector

$$X_{nc} = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} + \lambda \frac{\partial}{\partial \lambda}, \quad (76)$$

which is an especial case of commutative solution (65) with initial conditions $c_1 = 0 = c_2$.

For $k = 0$ (with $\beta \neq 0$), in addition to $Q_{nc}$, we have another constant of motion, namely $Q_a = p_a$, which corresponds to the symmetric vector $X_a = \partial/\partial a$ with the trivial Lie algebra $[X_a, X_b]=0$. This case also restricts commutative solution (68) with initial conditions $d_1 = d_2 = d_3 = d_4 = 0$. Therefore, in the $k\theta+\beta\neq0$ case, the noncommutativity reduces the number of symmetries to 1 for $k \neq 0$ and 2 for $k = 0$.

When $k\theta+\beta=0$, the last row of constraints in (72) is omitted. In this case, one choice is $k = -1$ and $\theta = \beta$, where a non-trivial (i.e. $\theta = \beta \neq 0$) linear solution exists if and only if $\theta = \sqrt{2} = \beta$. Hence, by the sign change of functions, i.e. $B \rightarrow -B$ and $C \rightarrow -C$ (or equivalently $A \rightarrow -A$), the solution is the same as the commutative solution (65), which again yields the constants of motion and symmetric vectors (66) and (67). However, for $k = -1$ and a special value of $\theta = 1 = \beta$, it gives one solution (75), but this case is not allowed

$^8$ The equality $k\theta+\beta=0$ is possible when ($k = -1$ and $\theta = \beta$) and or ($k = 0 = \beta$ and any value of $\theta$).
due to the inversion condition $\theta \beta \neq 1$. For another choice $k = 0 = \beta$ with any value of $\theta$, it gives the commutative solution (68), that is, even though the noncommutativity is still present, the number of symmetries and constants of motion do not change with respect to the corresponding commutative case.

Note that, in all cases, irrespective of whether the noncommutativity exists or not, and for any value of the curvature index, the angular momentum about the $a$-axis is conserved, as expected. Besides, in the above considerations, we have not, in general, specified numerical values of the NC parameters.

5. Conclusions

We have carried out an investigation for the role of NCG in cosmological scenarios, based on a four-dimensional free-potential (multi)scalar–tensor action of gravity, by introducing a NC deformation in the minisuperspace variables. The phase space is generated by two non-interacting conformal scalar fields plus the scale factor with their canonical conjugate momenta. The scalar fields are non-minimally coupled to geometry whose background is the FRW metric, where the conformal time gauge evolutions have been studied. The noncommutativity has been introduced only between the scalar fields and between their canonical conjugate momenta via two NC parameters $\theta$ and $\beta$, respectively. The investigation has been carried out for this toy model by means of a comparative mathematical analysis of the time evolution of the dynamical variables in the classical level and of the wavefunction of the universe in the quantum perspective, both in the commutative and NC frames. We have paid more attention to the outcome of results and have looked for the relations, including ranges and values, among the NC parameters for which particular or allowed solutions exist.

In general, we have shown by this toy model that the purposed noncommutativity has important implications in the evolution of the universe, however, does not affect the time dependence of the scale factor, i.e. its solution is the same as the commutative case, as expected. Also, we have found that one of the particularity of the NC parameter in the momenta sector, i.e. $\beta$, is in the spatially flat FRW universe, where it is the only responsible parameter for the NC effects in the classical and quantum frames.

In the classical model, exact solutions have been obtained. One of the important aspects of the NC solutions is that they can be regulated with both NC parameters. For example, these parameters can be employed to adjust the time dependence of solutions with the experimental or observational data. A distinguished feature of the noncommutativity effects, which we call a cosmical oscillator, is that the scalar fields behave similar to (or can be simulated with) the three most important harmonic oscillators depending on three values of the spatial curvature. These are the free, forced and damped harmonic oscillators corresponding to the flat, closed and open universes, respectively. In the flat universes, the time dependence of solutions are modified from linear in the commutative case to periodic in the NC frame. In the closed universes, if the ratio of frequencies of the scalar fields is a rational fraction, then solutions will be periodic. This condition restricts the NC parameters. When this ratio is not a rational fraction, the solutions are non-periodic but their behaviors still depend on the values of the NC parameters. A plot with numerous (or a few) relative extremum in a given time interval indicates that the separation between the high and low points increases when the NC parameters tend to smaller values, and this property is intensified for values less than unity. From this point of view, the NC solutions have particular preference with respect to the corresponding commutative ones. Furthermore, in the $k = 1$ case, the solutions are symmetric with respect to the NC parameters, and the results do not vary by replacement of their roles. Also, when the NC parameters are small, the cosmical oscillators have analogous effects with
the familiar beating effects in the sound phenomena. For a better view on this situation, an example has been illustrated in the text. In the open universes, there are upper and lower bounds on the NC parameters. The quick and late damping of this case can be adjusted by the NC parameters. For example, for $\theta \to 1^-$ and $\beta \to 0$, the scalar fields have maximum number of oscillations before the complete damping occurs.

We have also shown that the existence of a non-zero constant value of the potential function (i.e. the cosmological constant) does not change the time evolutions of the scalar fields, but it modifies the time dependence of the scale factor in a same manner for both the commutative and NC frames. Indeed, we have obtained that in all allowed conditions the scale factor behaves as a sinusoidal function.

In the quantum model, the exact solutions for the wavefunctions of the universe have also been obtained. The scale factor part of the wavefunction is similar to its commutative analog, as expected. One still expects that when the noncommutativity effects are turned on in the quantum scenario, they should introduce significant modifications in the scalar fields. However, an interesting feature of the well-behaved solutions is that the functional form of the radial part of the NC wavefunction is the same as the commutative ones within the given replacements of constants caused by the NC parameters, although these replacements in turn may cause drastic effects. For example, the curvature index is modified, and in open universes, the allowed NC wavefunctions impose bounds on the NC parameters.

Finally, we have employed the Noether theorem and have explored the effects of noncommutativity on the underlying symmetries in the commutative frame. We have shown that for spatially flat universes, there are six Noether symmetries, and, in general, only two of them are retained in the NC case. In the special case $k = 0 = \beta$, all symmetries survive regardless of the existence of the $\theta$ parameter. For non-flat universes, there are three Noether symmetries in the commutative case, one of which is retained in the NC frame. However, in open universes, when the NC parameters have the values $\theta = \sqrt{2} = \beta$, the number and general form of symmetries do not change with respect to the commutative frame. The only difference is related to the sign of symmetries. Conservation of the angular momentum about the scale factor axis is a common face between the commutative and NC cases, as expected in the purposed noncommutativity.

We should emphasize that the scalar field solutions are given after rescalings of the original fields, relations (4), and in the conformal time. Consequently, the functional forms of the solutions are not as simple when translated to the original fields. However, any detection should be performed in a reference frame, and there are debates on the physical frame in the cosmological contexts. Besides, the model is to be taken as a toy model only that still provides a valuable contribution to the assessment of the implications of the NC geometrical deformations of the phase space upon the dynamics of the cosmological model envisaged. However, a more realistic NC cosmological model may be achieved when one also involves the noncommutativity of the scalar fields with the scale factor, where the value of its time derivative in the form of the Hubble parameter can be determined from observations.

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**Appendix.**

We first obtain a general relation among the NC parameters when the solutions (25) are periodic. Then, we treat a specific periodic solution as an example.
As mentioned in the text when \( k = 1 \), solutions (25) play as the forced oscillators, and the existence of periodic solutions is provided when the ratio \( \omega_2/\omega_1 \) is a rational fraction. In this case, the \( \Delta \) parameter, by its definition, is \( \Delta = (\theta + \beta)^2 + (\theta \beta - 1)^2 \); hence by the transformation reversibility constraint \( \theta \beta \neq 1 \), one always has \( \sqrt{\Delta} > \theta + \beta \). Therefore, by definitions (26), we can take \( \omega_1 = cn_1 \) and \( \omega_2 = -cn_2 \), where \( n_1 \) and \( n_2 \) are two positive integers with \( n_2 > n_1 \) and \( c \) is an arbitrary non-zero positive constant. Hence, again, definitions (26) give

\[
\theta + \beta = \frac{c}{2}(n_2 - n_1) \quad \text{and} \quad \theta \beta = 1 \pm c \sqrt{n_1 n_2} \geq 0. \quad (A.1)
\]

These relations impose a firm restriction on the values of the NC parameters (or equivalently on allowed positive integers). Indeed, the condition for periodic scalar fields is that, the values of the NC parameters must be such that there exist two positive integers satisfying relations (A.1). As \( \theta \) and \( \beta \) are real roots of the quadratic equation \( X^2 - (\theta + \beta)X + \theta \beta = 0 \), relations (A.1) lead to

\[
(n_2 - n_1)^2 \geq \left(16/c^2 \right)(1 \pm c \sqrt{n_1 n_2}). \quad (A.2)
\]

The above result implies that if one finds a pair of integers satisfying inequality (A.2), then the values of the NC parameters will be calculated from equations (A.1).

Let us treat a specific example for the above considerations. Consider the equality in relation (A.2), which yields \( \theta = \beta \), which also by the constraint \( \theta \beta \neq 1 \) gives \( \theta \neq 1 \) (note that we have assumed \( \theta \geq 0 \)). Also, by definitions (26), we have \( \omega_1 = (1-\theta)^2 \) and \( \omega_2 = -(1+\theta)^2 \). Thus, by imposing the periodic condition as

\[
-\frac{\omega_2}{\omega_1} = \left(\frac{1+\theta}{1-\theta}\right)^2 \quad \text{is a rational fraction} \equiv m^2 > 1, \quad (A.3)
\]

it leads to

\[
\theta = \frac{m+1}{m-1} > 1 \quad \text{or} \quad \theta = \frac{m-1}{m+1} < 1, \quad (A.4)
\]

where \( m \) is a real number greater than 1.\(^9\) In the limit \( m \to 1 \), one solution of (A.4) yields \( \theta \to 0 \), as expected; however, the other one gives infinity that is not physically accepted. Also, in the limit \( m \to \infty \), we have \( \theta \to 1 \), which, in general, may be viewed as a transition from the forced harmonic oscillator in the \( k = 1 \) geometry to the simple one in the \( k = 0 \) case. In other words, when the value of \( m \) slowly gets bigger and bigger, then the driven force is gradually removed away from the cosmical oscillators. The time evolution of scalar field \( x(t) \), equation (25), is depicted in figure 2 (left) for the numerical value \( m = 3 \) corresponding to \( \theta = 2 \), with superposition constants \( A = 2 \) and \( B = 1 \). The period of solutions is the least common multiple of distinct periods, namely \( 2\pi/(1-\theta)^2 \) and \( 2\pi/(1+\theta)^2 \).

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\(^9\) Clearly, the value \( m = 1 \) gives the commutative case.
