R-MATRIX PRESENTATION OF QUANTUM AFFINE ALGEBRA IN TYPE $A^{(2)}_{2n-1}$

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Abstract. In this paper, we give an RTT presentation of the twisted quantum affine algebra of type $A^{(2)}_{2n-1}$ and show that it is isomorphic to the Drinfeld new realization via the Gauss decomposition of the L-operators. This provides the first such presentation for twisted quantum affine algebras with nontrivial central element.

1. Introduction

Quantum groups are usually referred to two important classes: the quantum enveloping algebras introduced by Drinfeld and Jimbo as certain deformations of universal enveloping algebras of the Kac-Moody Lie algebras $[11, 22]$ and the Yangian algebras of Drinfeld $[11]$ associated with simple Lie algebras. In this first presentation, the quantum algebras are defined by generators and Serre relations associated with the simple Lie algebras and their generalizations. The second presentation was given by Faddeev-Reshetikhin-Tacktajan $[14]$ in the study of quantum inverse scattering method using the Yang-Baxter equation and the RTT relation. An important extension was given by Reshetikhin and Semenov-Tian-Shanski $[32]$ for the quantum current algebra with nontrivial central extension. It was shown by Drinfeld that all finite dimensional representations of the quantum affine algebras and the Yangian algebras are classified by the third presentation of the quantum algebra—Drinfeld’s new realization $[11]$. A detailed construction of the isomorphism between the first and the third presentations for the quantum affine algebras were given in $[2, 23]$ for the untwisted types and in $[8, 9]$ (see also $[27]$) in the twisted types.

The exact equivalence of the Drinfeld realization and the Drinfeld-Jimbo presentation for the quantum affine algebra was proved by Ding and Frenkel in 1993 with the help of the Faddev-Reshetikhin-Tacktajan’s RTT formalism. The same identification for the Yangian algebra was done later by Brundan and Kleshchev in 2005 $[1]$. The identification of the Yangian algebras of other classical types were proved recently by two groups of mathematicians: the identification of the RTT formulation and the Drinfeld realization by Jing-Liu-Molev $[24]$ and that of the RTT formulation and the Drinfeld’s J-presentation by Guay-Regelski-Wendlandt $[19]$. Similar equivalence between the Drinfeld realization and the FRT formalism in quantum (untwisted) affine cases have also been completed for all other classical types $[20, 25, 26]$ via the Gassian generators of the L-operators. The R-matrix formulation of quantum super loop algebras and the quantum loop algebra of $A^{(2)}_{2n-1}$ have been studied in $[30, 34]$ as well. However, the RTT

MSC (2010): Primary: 17B37; Secondary:
Keywords: Quantum affine algebras, matrix presentations, Drinfeld realization, symmetric functions.
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formulation of the twisted quantum affine algebras (with nontrivial central element) is still unclear.

In this paper, we use the R-matrix method to study the quantum twisted affine algebras by explicitly describing all Drinfeld currents in the most general situation. The method has the advantage of revealing the inner-relationship between the untwisted and twisted quantum affine algebras, and we have completed the grand picture of characterizing quantum affine algebras: Drinfeld-Jimbo, Drinfeld’s current algebra and RTT presentations. Among the three presentations, the RTT presentation can provide the direct comultiplication formulas for quantum root vectors, and a triangular decomposition mimicking the classical case.

2. Twisted Quantum Affine Algebra

2.1. Drinfeld-Jimbo and Drinfeld presentations. Let \( \mathfrak{g} \) be the finite dimensional complex simple Lie algebra \( \mathfrak{sl}_2 \) with the Chevalley generators
\[
e_i', f_i', h_i', \quad i = 1, \ldots, 2n - 1,
\]
and let \( \alpha_i', i \in I_0 = \{1, \ldots, 2n - 1\} \) be the the simple roots, and let \( A_0' = (A_{ij}') \) be the Cartan matrix defined by \( \langle \alpha_i' | \alpha_j' \rangle = 2A_{ij}' \), \( i, j \in I_0 \), where we normalize the invariant bilinear form such that \( \langle \alpha | \alpha \rangle = 4 \).

Let \( \sigma \) be the automorphism of \( \mathfrak{g} \) defined by \( \sigma(e_i') = e_{2n-i}' \), \( \sigma(f_i') = f_{2n-i}' \), which induces a diagram automorphism \( \sigma \) of the Dynkin diagram \( \Gamma_0 \) so that
\[
\sigma(\alpha_i') = \alpha_{2n-i}' = 1, \ldots, 2n - 1.
\]

It is well-known that the datum \( (\mathfrak{sl}_2, \sigma) \) gives rise to the twisted affine Lie algebra \( \widehat{\mathfrak{g}} \) of type \( A^{(2)}_{2n-1} \) with the affine Dynkin diagram \( \Gamma = \Gamma_0 \cup \{0\} \), where the vertices of \( \Gamma_0 \) consists of the \( \sigma \)-orbits of the vertices of \( \Gamma_0 \). More explicitly, \( \alpha_i = \frac{1}{2}(\alpha_i' + \alpha_{2n-i}') \), \( i \in I_0 = \{1, \ldots, n\} \cong I_0/\sigma \) are the simple roots for the invariant Lie subalgebra \( \mathfrak{g}_0 \). The induced invariant form on the root lattice \( \Gamma_0 \) is given by
\[
\langle \alpha_i | \alpha_j \rangle = \frac{1}{2} \left( \langle \alpha_i' | \alpha_j' \rangle + \langle \alpha_{\sigma(i)}' | \alpha_j' \rangle \right)
\]
where \( i, j \in I_0 \). Subsequently the finite Cartan matrix \( A_0 \) of \( \Gamma_0 \) is induced from the Cartan matrix \( A_0' \): for \( i, j \in I_0 \)
\[
A_{ij} = \frac{2\langle \alpha_i | \alpha_j \rangle}{\langle \alpha_i | \alpha_i \rangle} = \sum_{u \in \mathbb{Z}_2} A_{\sigma^u(i)j}' \sum_{u \in \mathbb{Z}_2} A_{\sigma^u(i)i}'.
\]

In this case, \( A_0 \) is of type \( C_n \).

Let \( \theta_0 \) be the highest weight of \( \mathfrak{g} \) as a \( \mathfrak{g}_0 \)-module. Then \( \theta_0 \) is the sum of the two highest roots, and thus \( \theta_0 = \alpha_1' + 2 \sum_{i=1}^{2n-2} \alpha_i' + \alpha_{2n-1}' = \alpha_1 + 2 \sum_{i=2}^{n} \alpha_i + \alpha_n \). Then \( \delta = \alpha_0 + \theta_0 \) is the canonical null root for the twisted affine Lie algebra \( A^{(2)}_{2n-1} \). It has the simple roots \( \alpha_i, i \in I = \{0, \ldots, n\} \), then \( \underline{2.2} \) holds for the affine root system.

Let \( d_i = \frac{1}{2}(\alpha_i | \alpha_i) \), then \( (d_0, d_1, \ldots, d_n) = (1, 1, \ldots, 1, 2) \) diagonalizes \( A \): \( d_i A_{ij} = d_j A_{ji} \).

Let \( q \) be a nonzero complex number, and set \( q_i = q^{d_i} \) for \( i = 0, \ldots, n \). We also need the standard \( q \)-numbers:
\[
[k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}
\]
and the Gaussian $q$-numbers: $[k]_q! = \prod_{s=1}^{k} [s]_q [k]_q = \frac{[k]_q!}{[q]_q [k-q]_q!}$.

The quantum affine algebra $U_q(\hat{\mathfrak{g}})$ is a unitary associative algebra over $\mathbb{C}(q)$ with generators $E_{\pm \alpha_i}$ and $k_i^{\pm 1}$ $(i = 0, 1, \ldots, n)$ subject to the defining relations:

\begin{align*}
k_i k_i^{-1} &= k_i^{-1} k_i = 1, \quad k_i k_j = k_j k_i, \\
k_i E_{\pm \alpha_j} k_i^{-1} &= q_i^{\pm A_{ij}} E_{\pm \alpha_j}, \quad [E_{\alpha_i}, E_{-\alpha_j}] = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}, \\
\prod_{r=0}^{1-A_{ij}} (-1)^r \left[ \frac{1 - A_{ij}}{r} \right] (E_{\pm \alpha_i})^r (E_{\pm \alpha_j})^{1-A_{ij}-r} &= 0, \quad \text{if } i \neq j.
\end{align*}

**Definition 2.1.** [12] The Drinfeld realization of the quantum affine algebra $U_q^{Dr}(\hat{\mathfrak{g}})$ of type $A_{2n-1}^{(2)}$ is the $\mathbb{C}(q)$ unitary associative algebra generated by $\tilde{x}_{i,m}^{\pm}, \tilde{a}_{i,l}, \tilde{k}_i^{\pm}$ and $q^{\pm c/2}$ for $i = 1, \ldots, 2n - 1$ and $m, l \in \mathbb{Z}$ with $l \neq 0$, subject to the following defining relations: the elements $q^{\pm c/2}$ are central,

\begin{align*}
\tilde{k}_{2n-i} &= \tilde{k}_i, \quad \tilde{a}_{2n-i,l} = (-1)^l \tilde{a}_{i,l}, \quad \tilde{x}_{2n-i,m}^\pm = (-1)^m \tilde{x}_{i,m}^\pm, \\
\tilde{k}_i \tilde{k}_i^{-1} &= 1 = \tilde{k}_i^{-1} \tilde{k}_i, \quad \tilde{k}_i \tilde{k}_j = \tilde{k}_j \tilde{k}_i, \\
\tilde{k}_i \tilde{x}_j^\pm \tilde{k}_i^{-1} &= q^{\pm \sum_{r \in \mathbb{Z}_2} (\tilde{a}_i, \tilde{a}_{\sigma^r(j)})} \tilde{x}_j^\pm, \quad \tilde{k}_i \tilde{a}_{j,l} = \tilde{a}_{i,l} \tilde{k}_i, \\
\tilde{a}_{i,k} \tilde{a}_{j,l} &= \delta_{k+l,0} \frac{1}{k} \left( \sum_{r \in \mathbb{Z}_2} \left[ k \tilde{A}_i, \tilde{A}_{\sigma^r(j)} \right] / d_i \right) q_i^{-kr} q^{k} - q^{-k} \\
\prod_{s \in \mathbb{Z}_2} (u - (-1)^s q^{(\tilde{a}_i, \tilde{a}_{\sigma^s(j)})} v) \tilde{x}_i^+(u) \tilde{x}_i^-(v) &= \prod_{s \in \mathbb{Z}_2} (u q^{(\tilde{a}_i, \tilde{a}_{\sigma^s(j)})} - (-1)^s v) \tilde{x}_i^+(v) \tilde{x}_i^-(u) \\
[\tilde{x}_i^+(k), \tilde{x}_j^-(l)] &= \sum_{s \in \mathbb{Z}_2} \frac{\delta_{\sigma^s(i), j} (-1)^m}{q_i - q_i^{-1}} (q^\frac{1-k}{2} c \tilde{\varphi}_i^+(k+l) - q^\frac{l-k}{2} c \tilde{\varphi}_i^-(k+l))
\end{align*}

where $\tilde{\varphi}_i^\pm(\pm m)$ $(m \in \mathbb{Z}_{\geq 0})$ are defined by

\begin{align*}
\sum_{m=0}^{\infty} \tilde{\varphi}_i^\pm(\pm m) z^m &= \tilde{k}_i^{\pm 1} \exp \left( \pm (q_i - q_i^{-1}) \sum_{k=1}^{\infty} \tilde{a}_{i,\pm k} z^k \right) \\
\text{Sym}_{u_1, u_2} \{ P_{ij}^+(u_1, u_2)(\tilde{x}_i^+(v)\tilde{x}_j^+(u_1)\tilde{x}_i^+(u_2) - [2]_q^{\tilde{A}_{ij}} \tilde{x}_i^+(u_1)\tilde{x}_i^+(v)\tilde{x}_j^+(u_2)} \\
+ \tilde{x}_i^+(u_1)\tilde{x}_j^+(u_2)\tilde{x}_j^+(v) \}_3 &= 0, \quad \text{if } \tilde{A}_{ij} = -1, \sigma(i) \neq j,
\end{align*}
where $d_{ij}$ and $P_{ij}^{\pm}$ are defined as follows: (1) if $\sigma(i) = i$, then $d_{ij} = \frac{1}{2}, P_{ij}^{\pm}(u_1, u_2) = 1$; (2) if $A_{ij}(i) = 0$ and $\sigma(j) \neq j$, then $d_{ij} = \frac{1}{2}, P_{ij}^{\pm}(u_1, u_2) = 1$; (3) if $A_{ij}(i) = 0$ and $\sigma(j) = j$, then $d_{ij} = \frac{1}{2}, P_{ij}^{\pm}(u_1, u_2) = \frac{u_1 + u_2}{u_1 u_2}$; (4) if $A_{ij}(i) = -1$, then $d_{ij} = \frac{1}{2}, P_{ij}^{\pm}(u_1, u_2) = u_1 q^{1/2} + u_2$.

\[
\text{Sym}_{u_1, u_2, u_3} \left\{ (q^{1/2} u_1^{\pm 1} - [2] q^{1/2} u_2^{\pm 1}) \bar{x}_i^{\pm}(u_1) \bar{x}_i^{\pm}(u_2) \bar{x}_i^{\pm}(u_3) \right\} = 0, \quad A_{ij}(i) = -1
\]

(12)

\[
\text{Sym}_{u_1, u_2, u_3} \left\{ (q^{-1/2} u_1^{\pm 1} - [2] q^{-1/2} u_2^{\pm 1}) \bar{x}_i^{\pm}(u_1) \bar{x}_i^{\pm}(u_2) \bar{x}_i^{\pm}(u_3) \right\} = 0, \quad A_{ij}(i) = -1.
\]

A simplified Drinfeld presentation of $U_{q}^{D_2}(\hat{\mathfrak{g}})$ in terms of $A_0$ datum was given in [9].

**Definition 2.2.** The Drinfeld realization of the quantum affine algebra of type $A_{2n-1}^{(2)}$ is the $\mathbb{C}(q)$ unital associative algebra generated by $x_{i,m}^{\pm}, a_{i,j}, k^\pm$ and $q^{c/2}$ for $i = 1, \ldots, n$ and $m, l \in \mathbb{Z}$ with $l \neq 0$, subject to the following defining relations: the elements $q^{c/2}$ are central,

\[
x_{n,2m+1}^\pm = 0, \quad a_{n,2m+1} = 0,
\]

(13)

\[
k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad q^{c/2} q^{-c/2} = q^{-c/2} q^{c/2} = 1,
\]

(14)

\[
k_i k_{ij} = k_{ij} k_i, \quad k_i a_{ij} = a_{ij} k_i, \quad k_i x_{j,m}^{\pm} k_i^{-1} = q_i A_{ij} x_{j,m}^{\pm}
\]

(15)

\[
[a_{i,m}, a_{j,l}] = \delta_{m,-l} \frac{d_{ij}[m A_{ij}/d_{ij}] q_i^{mc} - q_i^{-mc}}{m} q_i - q_i^{-1},
\]

(16)

\[
[a_{i,m}, x_{j,l}^\pm] = \pm \frac{d_{ij}[m A_{ij}/d_{ij}] q_i q^{\pm |m| c/2} x_{j,m+l}^{\pm}}{m} d_{ij} = \max\{d_i, d_j\}
\]

(17)

\[
x_{i,m}^\pm x_{j,l}^\pm - q_i A_{ij} x_{j,l}^\pm x_{i,m}^\pm + q_{i,j} A_{ij} x_{i,m}^\pm x_{j,l}^\pm - x_{j,l+d_{ij}}^\pm x_{i,m}^\pm
\]

(18)

\[
\sum_{\pi \in S_r} \sum_{l=0}^{r} (-1)^l \binom{r}{l} x_{i,s(1)}^\pm \ldots x_{i,s(l)}^\pm x_{j,m+s(n(l+1)}^\pm \ldots x_{i,s(n(r))}^\pm = 0, \quad i \neq j, \quad A_{ij} \in \{0, -1\},
\]

where $r = 1 - A_{ij}$.

\[
\sum_{\pi \in S_2} \left(q(x_{j,s}^{i,r_{(1)}} x_{j,s}^{i,r_{(2)}} + [2] q^{1/2} x_{j,s}^{i,r_{(1)}} x_{j,s}^{i,r_{(2)}} + x_{j,s}^{i,r_{(1)}} x_{j,s}^{i,r_{(2)}} + x_{j,s}^{i,r_{(1)}} x_{j,s}^{i,r_{(2)}}) + q^{-1}(x_{j,s}^{i,r_{(1)}} x_{j,s}^{i,r_{(2)}} + [2] q^{-1} x_{j,s}^{i,r_{(1)}} x_{j,s}^{i,r_{(2)}} + x_{j,s}^{i,r_{(1)}} x_{j,s}^{i,r_{(2)}} + x_{j,s}^{i,r_{(1)}} x_{j,s}^{i,r_{(2)}}) = 0
\]

for $A_{ij} = -2$. The elements $\varphi_{i,m}^+$ and $\varphi_{i,-m}^-$ with $m \in \mathbb{Z}_+$ are defined by

\[
\varphi_{i}^\pm(u) := \sum_{m=0}^{\infty} \varphi_{i,m}^\pm u^{m} = k_i^\pm \exp \left( \pm (q_i - q_i^{-1}) \sum_{s=1}^{\infty} a_{i,s,u}^\pm \right)
\]

(20)

whereas $\varphi_{i,m}^+ = \varphi_{i,-m}^- = 0$ for $m < 0$. 


2.2. Isomorphism between Drinfeld-Jimbo and Drinfeld presentations. By using the braid group action, the set of generators of the algebra $U_q(A_{2n-1}^{(2)})$ can be extended to the set of affine root vectors of the form $E_{\alpha + k\delta}, F_{\alpha + k\delta}, E_{(k\delta, i)}$ and $F_{(k\delta, i)}$, where $\alpha$ runs over the positive roots of $\mathfrak{sp}_{2n}$, $d_\alpha = (\alpha, \alpha)/2$ and $\delta$ is the basic imaginary root; see [2,3] for details. The root vectors are used in the explicit isomorphism between the Drinfeld–Jimbo presentation of the algebra $U_q(A_{2n-1}^{(2)})$ and the “new realization” of Drinfeld which goes back to [12], while detailed arguments were given by Beck [2]; see also [3, 27]. In particular, for the Drinfeld presentation of the algebra $U_q(\tilde{\mathfrak{g}}_N)$ given in the Introduction, we find that the isomorphism between these presentations is given by

$$
x_{i,kd}^+ \mapsto o(i)^k E_{\alpha_i + k\delta}, \quad x_{i,kd}^- \mapsto o(i)^k F_{\alpha_i + k\delta}, \quad k \geq 0,
$$

$$
x_{i,kd}^+ \mapsto -o(i)^k F_{\alpha_i + k\delta} k_i^{-1} q^{kd/2}, \quad x_{i,kd}^- \mapsto -o(i)^k q^{-kd/2} k_i E_{\alpha_i + k\delta}, \quad k > 0,
$$

$$
a_{i,kd} \mapsto o(i)^k q^{-kd/2} E_{(k\delta, i)}, \quad a_{i,kd} \mapsto o(i)^k F_{(k\delta, i)} q^{kd/2}, \quad k > 0,
$$

where $o : \{1, 2, \ldots, n\} \rightarrow \{\pm 1\}$ is a map such that $o(i) = -o(j)$ whenever $A_{ij} < 0$ and furthermore $o(i) = 1$ when $A_{ij} = -2$, thus $o(n) = -1$. The pattern of $o(i)$ is $((-1)^n, (-1)^{n-1}, \ldots, +, -1)$ or $o(i) = (-1)^{n+1-i}$.

2.3. Universal R-matrix. We recall the explicit formulas for the universal $R$-matrix for the algebra $U_q(\tilde{\mathfrak{g}})$ obtained by Khoroshkin and Tolstoy [28] and Damiani [6, 7].

Recall the Cartan matrix $A_0$ for $\mathfrak{sp}_{2n}$, the underlying simple Lie algebra in $A_{2n-1}^{(2)}$, defined by (2.2). The symmetrized Cartan matrix $B_0 = DA_0$ is given by $B_{ij} = (\alpha_i, \alpha_j), 1 \leq i, j \leq n$. Let $\tilde{B} = [\tilde{B}_{ij}]$ be the inverse matrix $B_0^{-1}$, then its entries $\tilde{B}$ are given by

$$
\tilde{B}_{ij} = \begin{cases} n/4 & \text{for } i = j = n, \\ j/2 & \text{for } i = n > j, \\ j & \text{for } n > i. \end{cases}
$$

In order to describe the explicit formula for universal $R$-matrix given in [6], we introduce the matrix $Z^k$ for $k \in \mathbb{Z}_{>0}$ as following:

$$
z^k_{ij} = \begin{cases} [n-i]^k [j]^k & \text{for } k \text{ is odd,} \\ \frac{[n]^k}{[i]^k} & \text{for } i = j = n, \text{ and } k \text{ is even,} \\ \frac{[2]^k}{[2]^k} [n-k]^k & \text{for } n = i > j, \text{ and } k \text{ is even,} \\ \frac{[2]^k}{[2]^k} [n-i]^k [j]^k & \text{for other cases.} \end{cases}
$$

Note that $z_{ij}^kd_j = z_{ij}^kd_i$, so $Z^k$ is symmetrizable. We adjoin the degree element $d$ to $U_q(A_{2n-1}^{(2)})$ to form the quantum affine algebra $\tilde{U}_q(A_{2n-1}^{(2)})$ with the relations:

$$[d, k_i] = 0, \quad [d, E_{\pm\alpha_i}] = \pm \delta_{i,0} E_{\pm\alpha_i}.
$$

For a formal variable $u$, define an automorphism $D_u$ of the algebra $\tilde{U}_q(A_{2n-1}^{(2)}) \otimes \mathbb{C}[u, u^{-1}]$ by

$$D_u(x_{i,k}^\pm) = u^{k}x_{i,k}^\pm, \quad D_u(a_{i,k}) = u^{k}a_{i,k},
$$

for $i = 1, \ldots, n$, and

$$D_u(x_{n,2k}^\pm) = u^{2k+1}x_{n,2k}^\pm, \quad D_u(a_{n,2k}) = u^{2k}a_{n,2k}, \quad D_u(k_i) = k_i, \quad D_u(d) = d.
The universal $R$-matrix of the quantum affine algebra is an element $\mathcal{R}$ in the completed tensor product $\tilde{U}_q(A^{(2)}_{2n-1}) \otimes \tilde{U}_q(A^{(2)}_{2n-1})$ (see Drinfeld [13]) satisfying certain conditions. The conditions imply that $\mathcal{R}$ is a solution of the Yang-Baxter equation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12},$$

over $\tilde{U}_q(A^{(2)}_{2n-1}) \otimes \tilde{U}_q(A^{(2)}_{2n-1})$. Here $\mathcal{R}_{ij}$ means that if $A \in V \otimes V$ then $A_{23} = I \otimes A \in V^{\otimes 3}$ etc. The explicit formula for $\mathcal{R}$ is given in the $h$-adic settings and we set $q = \exp(h) \in \mathbb{C}[[h]]$ and regard the quantum affine algebra over $\mathbb{C}[[h]]$. Introduce elements $h_1, \ldots, h_n$ by setting $k_i = \exp(h_i)$.

We will work with the parameter-dependent $R$-matrix defined by

$$\mathcal{R}(u) = (D_u \otimes \text{id}) \mathcal{R} q^{\otimes d + d \otimes c}.$$

It satisfies the Yang–Baxter equation in the form

$$(2.23) \quad \mathcal{R}_{12}(u) \mathcal{R}_{13}(uvq^{-c}) \mathcal{R}_{23}(v) = \mathcal{R}_{23}(v) \mathcal{R}_{13}(uvq^{-c}) \mathcal{R}_{12}(u)$$

where $c_2 = 1 \otimes c \otimes 1$; cf. [16].

The $R$-matrix $\mathcal{R}(u)$ has the following triangular decomposition (cf. [7]):

$$(2.24) \quad \mathcal{R}(u) = \mathcal{R}^{>0}(u) \mathcal{R}^0(u) \mathcal{R}^{<0}(u),$$

where

$$\mathcal{R}^{>0}(u) = \prod_{\alpha \in \Delta_+} \prod_{k > 0} \exp(q_\alpha \left( (q_\alpha^{-1} - q_\alpha) D_u(E_{\alpha+kd_\alpha}) \otimes F_{\alpha+kd_\alpha} \right),$$

$$\mathcal{R}^{<0}(u) = T^{-1} \prod_{\alpha \in \Delta_+} \prod_{k > 0} \exp(q_\alpha \left( (q_\alpha^{-1} - q_\alpha) D_u(E_{-\alpha+kd_\alpha}) \otimes F_{-\alpha+kd_\alpha} \right) T$$

$$\mathcal{R}^0(u) = \exp\left( \sum_{k > 0} \sum_{i,j=1}^n \frac{1}{d_i} \frac{(q_i^{-1} - q_i)(q_j^{-1} - q_j)}{q^{-1} - q} \frac{k}{[k]_q} \xi_{ij}^{k} o(i)^k o(j)^k D_u(E_{kd_\delta,i}) \otimes F_{kd_\delta,j} \right) T,$$

where $T = \exp(-h \sum_{i,j} \tilde{B}_{ij} h_i \otimes h_j)$.

A straightforward calculation verifies the following formulas for the vector representation of the quantum affine algebra. As before, write $e_{ij} \in \text{End} \mathbb{C}^{2n}$ the standard matrix units. Here $V = \sum_{i=1}^n \mathbb{C} v_i \oplus \mathbb{C} v'_i$ and $e_{ij}v_k = \delta_{jk}e_i$ for $i, j \in \{1, \ldots, n, n', \ldots, 1\}$.

**Proposition 2.3.** The mappings $q^{\pm c/2} \mapsto 1$.

- $x_{ik}^+ \mapsto -q^{-i}k e_{i+1,i} + \zeta q^i k e_{i',(i+1)'},$  $x_{ik}^- \mapsto -q^{-i}k e_{i,i+1} + \zeta q^i k e_{(i+1)',i'},$
- $a_{ik} \mapsto \frac{[k]_q}{k} (q^{-i}k e_{i+1,i+1} - q^i k e_{ii}) + \zeta q^i k(e_{i,i'} - q^i e_{(i+1)',(i+1)'})$,
- $k_i \mapsto q(e_{i+1,i+1} + e_{i',i'}) + q^{-1}(e_{i,i} + e_{(i+1)',(i+1)'}) + \sum_{j \neq i,i',(i+1)'} e_{jj},$

for $i = 1, \ldots, n - 1$, and

- $x_{n,2k}^+ \mapsto -q^{-nk} e_{n+1,n},$  $x_{n,2k}^- \mapsto -q^{-nk} e_{n,n+1},$
- $a_{n,2k} \mapsto \frac{[k]_q}{k} (q^{-nk} q^{-k} e_{n+1,n+1} - q^k e_{nn})$,
- $k_n \mapsto q^2 e_{n+1,n+1} + q^{-2} e_{n,n} + \sum_{j \neq n,n+1} e_{jj},$

for $i = 1, n$. Linear functions $\alpha_0$ and $\beta_0$ are given by

$$\alpha_0(x_{ik}^+) = \alpha_0(x_{ik}^-) = \alpha_0(a_{ik}) = 0,$$$$

$$\beta_0(x_{ik}^+) = \beta_0(x_{ik}^-) = \beta_0(a_{ik}) = 0.$$
define a representation \( \pi_V : U_q(A^{(2)}_{2n-1}) \to \text{End} V \) of \( U_q(A^{(2)}_{2n-1}) \) on the vector space \( V = \mathbb{C}^{2n} \). □

It follows from the results of [16] that

\[
R(u) = (\pi_V \otimes \pi_V) R(u) = f(u) \overline{R}(u),
\]

where

\[
f(u) = \xi q^{-2} \prod_{r=0}^{\infty} \frac{(1 - u \xi^{2r})(1 - u q^{-2} \xi^{2r+1})(1 - u q^{2} \xi^{2r+1})(1 - u \xi^{2r+2})}{(1 - u \xi^{2r+1})(1 - u q^{2} \xi^{2r+2})(1 - u q^{-2} \xi^{2r})},
\]

\[
\overline{R}(u) = \sum_{i=1}^{2n} e_{ii} \otimes e_{ii} + \frac{u - 1}{qu - q^{-1}} \sum_{i \neq j} e_{ii} \otimes e_{jj} + \frac{q - q^{-1}}{qu - q^{-1}} \sum_{i > j, i \neq j'} e_{ij} \otimes e_{ji} + \frac{1}{(u - q^{-2}) (u - \xi)} \sum_{i,j=1}^{2n} a_{ij}(u) e_{i'j'} \otimes e_{ij},
\]

where \( \xi = -q^{-2n}, (1, 2, \ldots, 2n) = (n - 1, n - 2, \ldots, 0, 0, \ldots, -n + 1) \) and

\[
a_{ij}(u) = \begin{cases} (q^{-2} u - \xi)(u - 1), & i = j; \\ (q^{-2} - 1)(q^{2} - 1)(u - \xi) - \delta_{ij'}(u - \xi), & i < j; \\ (q^{-2} - 1)u(q^{2} - 1)(u - \xi) - \delta_{ij'}(u - \xi), & i > j. \end{cases}
\]

Introduce the \( L \)-operators in \( U_q(A^{(2)}_{2n-1}) \otimes \text{End} V \) by the formulas

\[
\tilde{L}^+ (u) = (\text{id} \otimes \pi_V) R_{21} (u q^{c/2}), \quad \tilde{L}^- (u) = (\text{id} \otimes \pi_V) R_{12} (u^{-1} q^{-c/2})^{-1}
\]

viewed as linear operator over the ring \( U_q(A^{(2)}_{2n-1}) \) or matrices with entries in \( U_q(A^{(2)}_{2n-1}) \).

The Yang-Baxter equation (2.23) immediately implies the following proposition:

**Proposition 2.4.** The following relations hold in \( U_q(A^{(2)}_{2n-1}) \otimes \text{End} V^\otimes 2 \) \((u_\pm = u q^{\pm c/2})\):

\[
R(u/v) \tilde{L}^+_i (u) \tilde{L}^+_j (v) = \tilde{L}^+_j (v) \tilde{L}^+_i (u) R(u/v),
\]

\[
R(u_+ / v_-) \tilde{L}^+_i (u) \tilde{L}^+_j (v) = \tilde{L}^+_j (v) \tilde{L}^+_i (u) R(u_- / v_+).
\]

3. **Extended quantum affine algebra** \( U_q^{\text{ext}}(A^{(2)}_{2n-1}) \)

3.1. **Extended twisted quantum affine algebras.** We will introduce the extended twisted quantum affine algebra \( U_q^{\text{ext}}(A^{(2)}_{2n-1}) \), which contains \( U_q(A^{(2)}_{2n-1}) \) as a subalgebra.

**Definition 3.1.** The extended quantum affine algebra \( U_q^{\text{ext}}(A^{(2)}_{2n-1}) \) is an associative algebra with generators \( X^{\pm}_{i,k}, h^{\pm}_{i,m}, h_{j,-m} \) and \( q^{c/2} \), where \( i = 1, \ldots, n, k \in \mathbb{Z}, j = 1, \ldots, n + 1 \) and \( m \in \mathbb{Z}_+ \).

The defining relations are written in terms of generating functions in a formal variable \( u \):

\[
X^{\pm}_i (u) = \sum_{k \in \mathbb{Z}} X^{\pm}_{i,k} u^{-k}, \quad h^{\pm}_i (u) = \sum_{m=0}^{\infty} h^{\pm}_{i,\pm m} u^{\mp m},
\]

and take the following form: The element \( q^{c/2} \) is central and invertible,

\[
h^{+}_{i,0} h^{-}_{i,0} = h^{+}_{i,0} h^{-}_{i,0} = h^{+}_{n,0} h^{-}_{n,0} = 1, \quad x^{\pm}_{n,2k+1} h^{\pm}_{n,2k+1} = h^{\pm}_{n,2k+1} = h^{\pm}_{n+1,2k+1} = 0,
\]

\[
h^{+}_i (u) h^{+}_j (v) = h^{+}_j (v) h^{+}_i (u),
\]
\begin{equation}
(3.2) \quad f\left(\frac{u_+ / v_-}{v_-} \pm 1\right) h_i^\pm(u) h_i^\mp(v) = f\left(\frac{u_- / v_+}{v_+} \pm 1\right) h_i^\mp(v) h_i^\pm(u),
\end{equation}

\begin{equation}
(3.3) \quad f\left(\frac{u_+ / v_-}{v_-} \pm 1\right) \frac{u_- - v_+}{q u_- - q^{-1} v_+} h_i^\pm(u) h_j^\mp(v) = f\left(\frac{u_- / v_+}{v_+} \pm 1\right) \frac{u_+ - v_-}{q u_+ - q^{-1} v_-} h_j^\mp(v) h_i^\pm(u)
\end{equation}

for \(i < j\) and \(i \neq n\), and

\begin{equation}
(3.4) \quad f\left(\frac{u_+ / v_-}{v_-} \pm 1\right) \frac{u_-^2 - v_+^2}{q^2 u_-^2 - q^{-2} v_+^2} h_n^\pm(u) h_{n+1}^\mp(v) = f\left(\frac{u_- / v_+}{v_+} \pm 1\right) \frac{u_+^2 - v_-^2}{q^2 u_+^2 - q^{-2} v_-^2} h_{n+1}^\mp(v) h_n^\pm(u).
\end{equation}

The relations involving \(h_i^\pm(u)\) and \(X_j^\pm(v)\) are

\[
h_i^\pm(u) X_j^+(v) h_i^\mp(u)^{-1} = \frac{u/v_+ - 1}{q(\epsilon,\alpha_j)u/v_+ - q^{-1}(\epsilon,\alpha_j)} X_j^+(v),
\]

\[
h_i^\pm(u)^{-1} X_j^-(v) h_i^\mp(u) = \frac{u/v_- - 1}{q(\epsilon,\alpha_j)u/v_- - q^{-1}(\epsilon,\alpha_j)} X_j^-(v)
\]

for \(i = 1, \ldots, n\), \(j = 1, \ldots, n-1\) together with

\[
h_i^\pm(u) X_n^+(v) h_i^\mp(u)^{-1} = \frac{(u/v_+)^2 - 1}{q(\epsilon,\alpha_j)(u/v_+)^2 - q^{-1}(\epsilon,\alpha_j)} X_n^+(v),
\]

\[
h_i^\pm(u)^{-1} X_n^-(v) h_i^\mp(u) = \frac{(u/v_-)^2 - 1}{q(\epsilon,\alpha_j)(u/v_-)^2 - q^{-1}(\epsilon,\alpha_j)} X_n^-(v)
\]

for \(i = 1, \ldots, n\) and

\[
h_{n+1}^\pm(u) X_n^+(v) h_{n+1}^\mp(u)^{-1} = \frac{(u/v_+)^2 - 1}{q^2(u/v_+)^2 - q^2} X_n^+(v),
\]

\[
h_{n+1}^\pm(u)^{-1} X_n^-(v) h_{n+1}^\mp(u) = \frac{(u/v_-)^2 - 1}{q^2(u/v_-)^2 - q^2} X_n^-(v)
\]

and

\[
h_{n+1}^\pm(u) X_{n-1}^+(v) h_{n+1}^\mp(u)^{-1} = \frac{q^{-1} u/v_+ + q}{u/v_+ + 1} X_{n-1}^+(v),
\]

\[
h_{n+1}^\pm(u)^{-1} X_{n-1}^-(v) h_{n+1}^\mp(u) = \frac{q^{-1} u/v_- + q}{u/v_- + 1} X_{n-1}^-(v),
\]

while

\[
h_{n+1}^\pm(u) X_i^+(v) = X_i^+(v) h_{n+1}^\pm(u), \quad h_{n+1}^\pm(u) X_i^-(v) = X_i^-(v) h_{n+1}^\pm(u),
\]

for \(1 \leq i \leq n-2\). For the relations involving \(X_i^\pm(u)\) we have

\[
(u - q^{\pm(\alpha_i,\alpha_j)} v) X_i^\pm(uq^i) X_j^\pm(vq^j) = (q^{\pm(\alpha_i,\alpha_j)} u - v) X_j^\pm(vq^j) X_i^\pm(uq^i)
\]

for \(i, j = 1, \ldots, n-1\);

\[
(u^2 - q^{\pm(\alpha_i,\alpha_n)} v^2) X_i^\pm(uq^i) X_n^\pm(vq^n) = (q^{\pm(\alpha_i,\alpha_n)} u^2 - v^2) X_n^\pm(vq^n) X_i^\pm(uq^i)
\]

for \(i = 1, \ldots, n\) and

\[
[X_i^\pm(u), X_j^\pm(v)] = \delta_{ij}(q_i - q_i^{-1}) \left( \delta(uq^i v^{-1} h_i^+(v_+)^{-1} h_{i+1}^-(v_+) - \delta(uq^i v h_i^+(u_+))^{-1} h_{i+1}^+(u_+) \right)
\]
for \( j \neq n \) and

\[
[X_i^+(u), X_n^-(v)] = \delta_{in}(q_n-q_n^{-1})\left(\delta\left(\frac{uq^{-c}}{v}\right)q_i h_i^- (v_+)^{-1} h_i^{-1}(v_+)-\delta\left(\frac{uq^{-c}}{v}\right) q_i^c h_i^+(u_+)^{-1} h_i^{+1}(u_+)\right)
\]

where the formal delta function \( \delta(u) = \sum_{r \in \mathbb{Z}} u^r \) and the Serre relations are

\[
\sum_{\pi \in S_r} \sum_{l=0}^{r} (-1)^l \left[ \begin{array}{c} r \\ l \end{array} \right] \sum_{i=0}^{\left\lfloor \frac{r}{2} \right\rfloor} (u_1^{\pi(i)}) \cdots (u_1^{\pi(i)}) X_i^\pm(u_1) \cdots X_i^\pm(u_1) X_i^\pm(u_1) X_i^\pm(u_1) = 0,
\]

which hold for all \( i \neq j \) and \( r = 1 - A_{ij} \). For \( A_{ij} = -1 \), the relation is

\[
\sum_{\sigma \in S_2} \sigma \left( (q^2 u_1 + u_2) (X_n^+(v) X_n^- (u_1) X_n^- (u_2)) - [2]_{q^2} X_{n-1}^+(u_1) X_{n-1}^+(u_2) + X_{n-1}^+(u_1) X_{n-1}^+(u_2) X_{n-1}^+(v) \right) = 0
\]

Introduce two formal series \( z^\pm(u) \) in \( u^{1} \) with coefficients in \( U_q^{\text{ext}}(A_{2n-1}^2) \) by \( (\xi = -q^{-2n}) \)

\[
z^\pm(u) = \prod_{i=1}^{n-1} h_i^+(u_+ q^{2i})^{-1} \prod_{i=1}^{n} h_i^+(u_+ q^{2i-2}) h_{n+1}^+(u).
\]

**Proposition 3.2.** The coefficients of \( z^\pm(u) \) are central elements of \( U_q^{\text{ext}}(A_{2n-1}^2) \).

**Proof.** We take \( z^+(u) \) to show the argument. By [3.1] we have \( z^+(u) h_j^+(v) = h_j^+(v) z^+(u) \) for all \( j = 1, \ldots, n+1 \). Also it follows from [3.1] that \( z^+(u) X_j^+(v) = X_j^+(v) z^+(u) \) for \( j = 1, \ldots, n \). We now check \( z^+(u) h_{n+1}^-(v) = h_{n+1}^-(v) z^+(u) \) in detail by using (2.26). By (3.2) we have

\[
z^+(u) h_{n+1}^-(v) = f(u_-/v_+) f(u_+/v_-)^{-1} \prod_{i=1}^{n-1} h_i^+(u_+ q^{2i})^{-1} \prod_{i=1}^{n} h_i^+(u_+ q^{2i-2}) h_{n+1}^+(v) h_{n+1}^+(u).
\]

Due to the commutativity (3.1), the last product can be rewritten as

\[
\prod_{i=1}^{n-2} h_i^+(u_+ q^{2i})^{-1} h_{i+1}^+(u_+ q^{2i}) h_{i-1}^+(u_-) h_{n+1}^+(v) h_{n+1}^+(u).
\]

Now (3.3) and (3.4) imply

\[
z^+(u) h_{n+1}^-(v) = f(u_-/v_+) f(u_+/v_-)^{-1} \frac{(q^{-2}u_-/v_+ - 1)(u_+/v_- - 1)}{(q^{-2}u_+/v_+ - 1)(u_-/v_+ - 1)} \times \prod_{i=1}^{n-2} h_i^+(u_+ q^{2i})^{-1} h_{i+1}^+(u_+ q^{2i}) h_{i-1}^+(u_+) h_{n+1}^+(v) h_{n+1}^+(u).
\]

Now, applying (3.3) again, we come to the relation

\[
z^+(u) h_{n+1}^-(v) = f(u_-/v_+) f(u_+/v_-) \frac{f(u_-/v_+)}{f(u_+/v_-) f(u_+/v_-)} \frac{(q^{-2}u_-/v_+ - 1)(u_+/v_- - 1)}{(q^{-2}u_+/v_+ - 1)(u_-/v_+ - 1)} \times \frac{u_-/v_+ - 1}{u_-/v_+ + 1} \frac{u_+/v_- - q^{-1}}{u_+/v_- + 1} h_{n+1}^+(v) z^+(u).
\]

Then \( z^+(u) h_{n+1}^-(v) = h_{n+1}^-(v) z^+(u) \) follows due to: \( f(u) f(u_+) = \frac{(\xi q^{-2})(1-u q^2)(1-u)}{q(1-u q^{-2})(1-u^2)} \). \( \square \)
Proposition 3.3. The map \( q^{c/2} \mapsto q^{c/2}, k_i \mapsto h_{0i}^+(h_{i+1,0}^+)^{-1} \) (1 \( \leq i \leq n - 1 \)), \( k_n \mapsto (h_{n,0}^+)^2 \) and 
\[ x_i^\pm(u) \mapsto (q_i - q_i^{-1})^{-1} X_i^\pm(uq_i^2), \quad 1 \leq i \leq n \]
\[ \varphi_i^+(u) \mapsto \begin{cases} h_{i+1}^+(uq_i^2) h_i^+(uq_i^2)^{-1}, & 1 \leq i \leq n - 1 \\ q_i^\pm h_{n+1}^+(uq_n^2) h_n^+(uq_n^2)^{-1}, & i = n \end{cases} \]
defines an embedding \( \varsigma : U_q(A_{2n-1}^{(2)}) \hookrightarrow U_q^{\text{ext}}(A_{2n-1}^{(2)}) \).

Proof. In terms of generating series, it is straightforward to verify that the map \( \varsigma \) defines a homomorphism from \( U_q(A_{2n-1}^{(2)}) \) to \( U_q^{\text{ext}}(A_{2n-1}^{(2)}) \). To show the injectivity, we will construct another homomorphism \( \rho : U_q^{\text{ext}}(A_{2n-1}^{(2)}) \rightarrow U_q(A_{2n-1}^{(2)}) \) such that the composition \( \rho \circ \varsigma \) is the identity homomorphism on \( U_q(A_{2n-1}^{(2)}) \). We extend \( U_q(A_{2n-1}^{(2)}) \) by adjoining the square roots \( k_n^{\pm 1/2} \) to \( U_q(A_{2n-1}^{(2)}) \) and keep the same notation for the extended algebras for the rest of the proof. First we can verify that the mapping \( \rho_1 : U_q^{\text{ext}}(A_{2n-1}^{(2)}) \rightarrow U_q^{\text{ext}}(A_{2n-1}^{(2)}) \) by
\[ X_j^\pm(u) \mapsto X_j^\pm(u), \]
(3.8)
\[ h_i^+(u) \mapsto \prod_{m=0}^\infty z^+(u\xi^{-2m-1})z^+(u\xi^{-2m-2})^{-1} h_i^+(u) \]
is an automorphism of \( U_q^{\text{ext}}(A_{2n-1}^{(2)}) \).

By setting \( \varphi_i^+(u) = k_i^\pm 1 \varphi_i^+(u) \) and
\[ h_0^+(u) = \prod_{m=0}^{n-1} \prod_{j=1}^{i-1} \varphi_j^+(u\xi^{-2m,q^j})^{-1} \varphi_j^+(u\xi^{-2m-1,q^j}) \varphi_j^+(u\xi^{-2m-2,q^j-1}) \varphi_j^+(u\xi^{-2m-2,q^j}) \]
\[ \times \prod_{m=0}^{n-1} \varphi_n^+(-u\xi^{-2m,q^n})^{-1} \varphi_n^+(-u\xi^{-2m-1,q^n}) \]
we define the map \( \rho_2 : U_q^{\text{ext}}(A_{2n-1}^{(2)}) \rightarrow U_q(A_{2n-1}^{(2)}) \) by
\[ X_i^\pm(u) \mapsto (q_i - q_i^{-1}) x_i^\pm(uq^{-i}), \ i = 1, \cdots, n, \]
while for \( 1 \leq i \leq n - 1 \)
\[ h_i^+(u) \mapsto h_0^+(u) \prod_{j=1}^{i-1} \varphi_j^+(uq^{-j}) \times \prod_{j=i}^{n-1} k_j^\pm k_n^\pm 1/2, \]
\[ h_n^+(u) \mapsto h_0^+(u) \prod_{j=1}^{n-1} \varphi_j^+(uq^{-j}) \times k_n^\pm 1/2 q_i^c, \]
\[ h_{n+1}^+(u) \mapsto h_0^+(u) \prod_{j=1}^{n-1} \varphi_j^+(uq^{-j}) \varphi_n^+(uq^{-n}) \times k_n^\pm 1/2. \]

Direct calculation shows that the map \( \rho_2 \) defines a homomorphism from \( U_q^{\text{ext}}(A_{2n-1}^{(2)}) \) to \( U_q(A_{2n-1}^{(2)}) \). Furthermore, it is easy to check \( (\rho_2 \circ \rho_1) \circ \varsigma \) is the identity map on \( U_q(A_{2n-1}^{(2)}) \) by the formulas of \( z^\pm(u) \). \( \square \)
3.2. \textbf{L-operators in }$U_q^{\text{ext}}(A^{(2)}_{2n-1})$. Proposition 3.3 implies the following elements belong to $U_q^{\text{ext}}(A^{(2)}_{2n-1}) \otimes \text{End} V$:

$\mathbf{L}^\pm(u) := \mathbf{L}^\pm(u) \prod_{m=0}^\infty z^\pm(u \xi^{-2m}) z^\pm(u \xi^{-2m-2})^{-1}$.  

Although the entries of $\mathbf{L}^\pm(u)$ contain a completion of $ZU_q^{\text{ext}}(A^{(2)}_{2n-1})$, it turns out that the coefficients of the series in $u^{\pm 1}$ actually belong to $U_q^{\text{ext}}(A^{(2)}_{2n-1})$ (cf. proof of Thm. 3.3). The following result is an immediate consequence of Prop. 2.4.

\textbf{Proposition 3.4.} \textit{$\mathbf{L}^\pm(u)$ satisfy the following relations in $U_q^{\text{ext}}(A^{(2)}_{2n-1}) \otimes \text{End} V \otimes 2$ :}

$R(u/v)\mathbf{L}^+_u\mathbf{L}^+_v = \mathbf{L}^+_u\mathbf{L}^+_v R(u/v)$,

$R(u_+/v_-)\mathbf{L}^+_u\mathbf{L}^+_v = \mathbf{L}^+_u\mathbf{L}^+_v R(u_-/v_+)$,

The triangular decomposition (2.23) of the universal R-matrix $\mathbf{R}(u)$ and the construction of $\mathbf{L}^\pm(u)$ imply the following corresponding decomposition of $\mathbf{L}^\pm(u)$:

$\mathbf{L}^\pm(u) = F^\pm(u)H^\pm(u)E^\pm(u)$,

where

$F^+(u) = (\text{id} \otimes \pi_V) \mathbf{R}^{>0}_{21}(u_+)$,  

$E^+(u) = M^{-1}(\text{id} \otimes \pi_V) \mathbf{R}^{<0}_{21}(u_+)$,

$H^+(u) = (\text{id} \otimes \pi_V) \mathbf{R}^{0}_{21}(u_+)M \times \prod_{m=0}^\infty z^+(u \xi^{-2m-1}) z^+(u \xi^{-2m-2})^{-1}$,

$F^-(u) = (\text{id} \otimes \pi_V) \mathbf{R}^{<0}(1/u_+)^{-1} M$,  

$E^-(u) = (\text{id} \otimes \pi_V) \mathbf{R}^{>0}(1/u_+)^{-1}$,

$H^-(u) = M^{-1}(\text{id} \otimes \pi_V) \mathbf{R}^{0}(1/u_+)^{-1} \times \prod_{m=0}^\infty z^-(u \xi^{-2m-1}) z^-(u \xi^{-2m-2})^{-1}$,

where $M = \sum_{i \neq n} 1 \otimes e_{ii} + q^{-c} \otimes e_{mn}$.

We define the following generating series for the Drinfeld generators $x^\pm_{i,k}$ of $U_q(A^{(2)}_{2n-1})$:

$x^+_{i,k}(u)^{\geq 0} = \sum_{k \geq 0} x^+_{i,-k} u^k$,  

$x^+_{i,k}(u)^{> 0} = \sum_{k > 0} x^+_{i,-k} u^k$,  

$x^+_{i,k}(u)^{< 0} = \sum_{k > 0} x^+_{i,k} u^{-k}$,  

$x^+_{i,k}(u)^{\leq 0} = \sum_{k \geq 0} x^+_{i,k} u^{-k}$.

For $i = 1, \ldots, n - 1$, set

$f^+_{i}(u) = (q_i - q_i^{-1}) x^+_{i}(u_+ q^{-i})^{\geq 0}$,  

$e^+_{i}(u) = (q_i - q_i^{-1}) x^+_{i}(u_- q^{-i})^{> 0}$,

$f^-_{i}(u) = (q_i^{-1} - q_i) x^-_{i}(u_- q^{-i})^{< 0}$,  

$e^-_{i}(u) = (q_i^{-1} - q_i) x^-_{i}(u_+ q^{-i})^{\leq 0}$

and

$f^+_n(u) = (q_n - q_n^{-1}) u_+ x^-_{n}(u_+ q^{-n})^{\geq 0}$,  

$e^+_n(u) = (q_n - q_n^{-1}) (u_-)^{-1} x^+_n(u_- q^{-n})^{> 0}$,

$f^-_n(u) = (q_n^{-1} - q_n) u_- x^-_n(u_- q^{-n})^{< 0}$,  

$e^-_n(u) = (q_n^{-1} - q_n) u_+ x^+_n(u_+ q^{-n})^{\leq 0}$.

We will need the following result, which is similarly proved as in \cite[Lemma 5.3]{25}.  


Lemma 3.5. The image \((\text{id} \otimes \pi_V)(T_{21})\) is the diagonal matrix

\[
\text{diag}
\begin{bmatrix}
\prod_{a=1}^{n-1} k_a k_n^{1/2}, \prod_{a=2}^{n-1} k_a k_n^{1/2}, \ldots, k_n^{1/2}, k_n^{-1/2}, k_{n-1}^{-1/2}, \ldots, \prod_{a=1}^{n-1} k_a^{-1} k_n^{-1/2}
\end{bmatrix}
\]

Theorem 3.6. In \(U_q^{\text{ext}}(A_{2n-1}^{(2)})\), we have

\[
F^\pm(u) =
\begin{bmatrix}
1 & f_1^+(u) & 1 & \cdots & \cdots & 1 \\
& 1 & f_2^+(u) & 1 & \cdots & \cdots & 1 \\
& & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & & 1 & e_n^+(u) & \cdots & 1 \\
& & & & 1 & -e_2^+(u \xi q^4) & \cdots & 1 \\
& & & & & 1 & -e_1^+(u \xi q^2) & 1 \\
& & & & & & \cdots & 1 \\
& & & & & & & \cdots \\
& & & & & & & & 1 \\
\end{bmatrix}
\]

\[
E^\pm(u) =
\begin{bmatrix}
1 & e_1^+(u) & e_2^+(u) & \cdots & \cdots & 1 \\
& 1 & e_2^+(u) & \cdots & \cdots & 1 \\
& & \ddots & \ddots & \ddots & \ddots \\
& & & 1 & e_n^+(u) & \cdots & 1 \\
& & & & 1 & -e_2^+(u \xi q^4) & \cdots & 1 \\
& & & & & 1 & -e_1^+(u \xi q^2) & 1 \\
& & & & & & \cdots & 1 \\
& & & & & & & \cdots \\
& & & & & & & & 1 \\
\end{bmatrix}
\]

and

\[
H^\pm(u) = \text{diag}[h_1^\pm(u), \ldots, h_n^\pm(u), z_i^\pm[1](u) h_n^\pm(u \xi[1])^{-1}, \ldots, z_i^\pm[n](u) h_1^\pm(u \xi[n])^{-1}].
\]

Proof. In order to study the \((i+1, i)\)-entry in \(F^+(u)\), we only need to evaluate the image of the product

\[
\prod_{k \geq 0} \exp_{q_i}((q_i^{-1} - q_i) F_{\alpha_i + k \delta_i} \otimes D_{u_+}(E_{\alpha_i + k \delta_i}))
\]

for simple roots \(\alpha_i\) \((1 \leq i \leq n)\). In the following let \(t_i = q_i - q_i^{-1}\). By the isomorphism of Sec. 2.2, the above is rewritten as

\[
\prod_{k \geq 0} \exp_{q_i}(-t_i x_i^{-} \otimes D_{u_+}(x_i^{+})).
\]

Suppose \(i \leq n - 1\), then \(d_i = 1\). By Proposition 2.3 it follows that

\[
(3.13) \quad (\text{id} \otimes \pi_V) \prod_{k \geq 0} \exp_{q_i}(-t_i (u_+)^{k d_i} x_i^{-} \otimes x_i^{+})
\]

\[
= \prod_{k \geq 0} \exp_{q_i}(t_i (u_+ q_i^{-1})^k x_i^{-} \otimes e_{i+1,i} - t_i (u_+ \xi q_i^k) x_i^{-} \otimes e_{i+1,i}).
\]
Expanding the q-exponent, we can write this expression in the form
\[ 1 + t_i \sum_{k \geq 0} x_{i-k}^- (u+q^{-i})^k \otimes e_{i+1, i} - t_i \sum_{k \geq 0} x_{i-k}^- (u+\xi q^i)^k \otimes e_{i', (i+1)'} \]
\[ = 1 + t_i x_i^- (u+q^{-i})^\geq \otimes e_{i+1, i} - t_i x_i^- (u+\xi q^i)^\geq \otimes e_{i', (i+1)'} \]
which coincides with \( 1 + f_i^1(u) \otimes e_{i+1, i} - f_i^1(u\xi q^2) \otimes e_{i', (i+1)'} \), as required. Now we consider the case \( i = n \), so \( d_n = 2 \) and \( D_u(x_{n, 2k}) = u^{2k+1} x_{n, 2k} \), it follows from Prop. 2.3 that
\[ (id \otimes \pi_V) \prod_{k \geq 0} \exp_{q_n} (-t_n x_{n, -kd_n} \otimes D_{u+}(x_{n, kd_n}^+)) \]
\[ = \prod_{k \geq 0} \exp_{q_n} (t_n u_+ x_{n, -2k}^- (u+q^{-n})^{2k} \otimes e_{n+1, n}) \]
Expanding the q-exponent, we can write this expression in the form
\[ 1 + t_n \sum_{k \geq 0} u_+ x_{n, -2k}^- (u+q^{-n})^{2k} \otimes e_{n+1, n} = 1 + t_n u_+ x_{n}^- (u+q^{-n})^{\geq} \otimes e_{n+1, n} \]
which coincides with \( 1 + f_n^+(u) \otimes e_{n+1, n} \) as required.

For the \((i, i+1)-th\) entry in \( E^+(u) \), we first evaluate the image of the product
\[ T_{21}^{-1} \prod_{k \geq 0} \exp_{q_i} (-t_i F_{-\alpha_i, k\delta} \otimes D_{u+}(E_{-\alpha_i, k\delta})) T_{21} \]
\[ = -T_{21}^{-1} \prod_{k \geq 0} \exp_{q_i} (-t_i q^{-k\delta} x_{i-k}^+ \otimes D_{u+}(q^{k\delta} c_{k_i}^{-1} x_{i-k}^-)) T_{21}. \]
For \( i = 1, \ldots, n-1, d_i = 1 \) and \( D_u(x_{i, 1}^-) = u^k x_{i, k}^- \), by the action of \( \pi_V \) (Prop. 2.3) we have
\[ (id \otimes \pi_V) \prod_{k \geq 0} \exp_{q_i} (-t_i q^{-k\delta} x_{i-k}^+ \otimes D_{u+}(q^{k\delta} c_{k_i}^{-1} x_{i-k}^-)) \]
\[ = \prod_{k > 0} \exp_{q_i} (q_i t_i u^k x_{i-k}^+ \otimes (q^{-ik} e_{i,i+1} - \xi^{i} q^{ik} e_{(i+1)'i'})) \]
\[ = 1 + q_i t_i \sum_{k \geq 0} x_{i-k}^- (u^{-q^{-i}})^k \otimes e_{i+1} - q_i t_i \sum_{k \geq 0} x_{i-k}^+ (u^{-\xi q^i})^k \otimes e_{(i+1)'i'}. \]
Then by Lemma 2.5 we have for \( i < n-1 \),
\[ (id \otimes \pi_V) (T_{21}^{-1} \prod_{k \geq 0} \exp_{q_i} (-t_i q^{-k\delta} x_{i-k}^+ \otimes D_{u+}(q^{k\delta} c_{k_i}^{-1} x_{i-k}^-)) T_{21}) \]
\[ = 1 + q_i t_i \sum_{k > 0} \left( \prod_{j=i}^{n-1} k_{j+1}^{-1/2} x_{i-k}^+ \prod_{j=i}^{n-1} k_{j+1}^{-1/2} \right) (u^{-q^{-i}})^k \otimes e_{i,i+1} \]
\[ - q_i t_i \sum_{k \geq 0} \left( \prod_{j=i+1}^{n-1} k_{j}^{-1/2} x_{i-k}^+ \prod_{j=i+1}^{n-1} k_{j}^{-1/2} \right) (u^{-\xi q^i})^k \otimes e_{(i+1)'i'}. \]
By using the fact \( k_i x_{j,k}^+ c_{k_i}^{-1} x_{j,k}^+ = q_i^{\pm A_{ij}} x_{j,k}^+ \), the last equation is equal to
\[ = 1 + t_i x_i^- (u^{-q^{-i}})^{\geq} \otimes e_{i,i+1} - t_i x_i^+ (u^{-\xi q^i})^{\geq} \otimes e_{(i+1)'i'} \]
\[ = 1 + e_i^+(u) \otimes e_{i,i+1} - e_i^+(u\xi q^{2i}) \otimes e_{(i+1)'i'}. \]
Thus, we get the \((i, i + 1)\)-th entry of \(E^+(z)\) is \(e_i^+(z)\), and the corresponding \(((i + 1)', i')\)-th entry is \(-e_i^+(z\xi q^{2n})\).

For \(i = n - 1\), \(q_{n-1} = q\), so we have

\[
(id \otimes \pi_V) \left( T_{21}^{-1} \exp_q \left( -tq^{-nc}x_{n-1,-k}^{+k_{n-1}} \otimes D_{u+} (q^{kc}k_{n-1}^{-1}x_{n-1,k}^-) \right) T_{21} \right)
= 1 + qt \sum_{k \geq 0} \left( k_{n-1}^{-1/2}x_{n-1,-k}^{+k_{n-1}^{1/2}} \right) (u_-q^{-(n-1)})^k \otimes e_{n-1,n}
- qt \sum_{k \geq 0} \left( k_{n-1}^{1/2}x_{n-1,-k}^{-k_{n-1}^{1/2}} \right) (u_-\xi q^{n-1})^k \otimes e_{n',(n-1)'}.
\]

By using the fact \(k_ix_{j,k}^{-1} = q_i^{A_{ij}}x_{j,k}^{\pm}\), we have the last equation is simplified to

\[
= 1 + tx_{n-1}^+(u_-q^{-(n-1)})^0 \otimes e_{n-1,n} - tx_{n-1}^+(u_-\xi q^{n-1})^0 \otimes e_{n',(n-1)'}
= 1 + e_{n-1}^+(z) \otimes e_{n-1,n} - e_{n-1}^+(z\xi q^{2(n-1)}) \otimes e_{n',(n-1)'}.
\]

Thus the \((n-1, n)\)-th and \((n', (n-1)')\)-th entries of \(E^+(z)\) are \(e_{n-1}^+(z)\) and \(-e_{n-1}^+(z\xi q^{2(n-1)})\).

For \(i = n\), by the action of \(\pi_V\) (see Proposition 2.3) we have

\[
(id \otimes \pi_V) \prod_{k > 0} \exp_{q_n} \left( -tq^{-nc}x_{n,-k}^{+k_n} \otimes D_{u+} (q^{kc}k_{n}^{-1}x_{n,k}) \right)
= (id \otimes \pi_V) \prod_{k > 0} \exp_{q_n} \left( -tq^{-n}x_{n,-2k}^{+2k} \right) \otimes e_{n,n+1}
= 1 + qnt_n \sum_{k > 0} (u_-q^{-n})^2k \otimes e_{n,n+1}
\]

Then by Lemma 3.3 and also \(k_ix_{j,k}^{-1} = q_i^{A_{ij}}x_{j,k}^{\pm}\), we have that

\[
(id \otimes \pi_V) \left( T_{21}^{-1} \prod_{k > 0} \exp_{q_n} \left( -tq^{-nc}x_{n,-k}^{+k_n} \otimes D_{u+} (q^{kc}k_{n}^{-1}x_{n,k}) \right) T_{21} \right)
= 1 + qnt_n \sum_{k > 0} \left( k_{n}^{-1/2}x_{n,-2k}^{+k_{n}^{1/2}} \right) (u_-q^{-n})^2k \otimes e_{n,n+1}
= 1 + nt_n(u_+)^{-1}x_{n}^+(u_-q^{-n})^0 \otimes e_{n,n+1} = 1 + q^{-n}e_{n}^+(z) \otimes e_{n,n+1}.
\]

Thus, we get the \((n, n+1)\)-th entry of \(E^+(z)\) is \(e_{n}^+(z)\).

Now we consider \(H^+(u)\), in which we only treat the \((n, n)\)- and \((n', n')\)- entries. By definition and the action of \(\pi_V\), the \((n, n)\)-entry of \((id \otimes \pi_V)R_{21}^0(u_+)^{n,n+1}\) is

\[
\exp \left[ \sum_{k > 0} \sum_{j = 1}^{n-1} -t_jq^{-nk}z_{n-1,j} a_{j,-k} \right] \exp \left[ \sum_{k > 0} \sum_{j = 1}^{n-1} t_j(-1)^k q^{-(n-1)2k} z_{nj} a_{j,-2k} \right]
\cdot \exp \left[ \sum_{k > 0} -tn(2k + 2k)^2 a_{n-2k}z_{n,n} \right]
\]
Due to (2.22) for $z^k_{ij}$, the above can be rewritten as (note $ξ = -q^{-2n}$)
\[(3.15) \quad \exp\left[\sum_{k>0}^{n-1} \sum_{j=1}^{\infty} t_j \frac{q^{jk} - q^{-jk}}{1 + ξ^{-k}a_{j,-k}u^k}\right] \exp\left[\sum_{k>0}^{m} t_n \frac{(-q^n)^{2k}}{1 + ξ^{-2k}a_{n,-2k}u^{2k}}\right]
\]
Expanding the fractions into power series, we can write the expression as
\[
\exp\left(\sum_{k>0}^{n-1} \sum_{j=1}^{\infty} (-1)^{m} t_j ξ^{-mk}(q^{jk} - q^{-jk})u^k a_{j,-k}\right)
\]
\[\times \exp\left(\sum_{k>0}^{m} (-1)^{m} t_n ξ^{-2mk}(-q^n)^{2k}u^{2k}a_{n,-2k}\right),
\]
which becomes the following by using $ϕ_i(u)$ (see (2.20)):
\[
\prod_{m=0}^{\infty} \prod_{j=1}^{n-1} \frac{V_j(uξ^{-2m}q^j)^{-1}V_j(uξ^{-2m}q^{-j})}{V_j(uξ^{-2m-1}q^j)V_j(uξ^{-2m-1}q^{-j})^{-1}}
\]
\[\times \prod_{m=0}^{\infty} \frac{V_n(-uξ^{-2m}q^n)^{-1}V_n(-uξ^{-2m+1}q^n)}{V_n(-uξq^n)^{-1}}.
\]
Setting $\overline{h}_i^+(u) = t_i^{-1}h_i^+(u)$ with $t_i = h_{i,0}^+$, it follows from Proposition 3.3 that the above is
\[
\prod_{m=0}^{\infty} \prod_{j=1}^{n-1} \overline{h}_j^+(uξ^{-2m}q^{2j})\overline{h}_j^+(uξ^{-2m-1}q^{2j}) \times \prod_{m=0}^{\infty} \prod_{j=1}^{n} \overline{h}_j^+(uξ^{-2m}q^{-2j})^{-1}\overline{h}_j^+(uξ^{-2m-1}q^{-2j})^{-1}
\]
\[\times \prod_{m=0}^{\infty} \overline{h}_{n+1}^+(uξ^{-2m-1})^{-1}\overline{h}_{n+1}^+(uξ^{-2m-2}) × \overline{h}_n^+(u)q^c.
\]
One then invokes (3.17) for $z^\pm(u)$ to conclude that
\[
\exp\left(\sum_{k>0}^{n} t_j \tilde{B}_{1,j}(q^k)u^k a_{j,-k}\right) = \prod_{m=0}^{\infty} z^+(uξ^{-2m-1})^{-1}z^+(uξ^{-2m-2}) \times \overline{h}_n^+(u)q^c.
\]
In particular, the coefficients of $u^n$ in the infinite product (occurring in (3.9)) belong to the algebra $U^\text{ext}_q(\mathfrak{sp}_{2n})$.

Furthermore, Lemma 3.5 implies that the $(n, n)$-entry of the matrix $(\text{id} \otimes π_V)(T_{21})$ equals $k_{n/2} = t_n$, so the $(n, n)$-entry of the matrix $H^+(u)$ is $h_{n}^+(u)$.

By definition and recalling the action of $π_V$, we have $(n', n')$- entry of $(\text{id} \otimes π_V)R_{21}^0(u_+)$ is
\[
\exp\left[\sum_{k>0}^{n-1} t_j ξ^{nk}z_{n-1,j}^{-1}a_{j,-k}u^k\right] \times \exp\left[\sum_{k>0}^{n} t_j (-1)^k q^{-(n+1)2k}z_{nj}^{2k}a_{j,-2k}u^{2k}\right]
\]
\[\times \exp\left[\sum_{k>0}^{n} t_n ξ^{-2k}(-(-1)^k q^{2nk}z_{n-1,n}^{-2k} + q^{-(n+1)2k}z_{mn}^{2k}) a_{n,-2k}u^{2k}\right]
\]
Recalling the meaning of $z_{ij}^k$, (3.16) is rewritten as

$$
(3.17) \quad \exp \left[ \sum_{k>0} \sum_{j=1}^{n-1} t_j \frac{q^{jk} - q^{-jk}}{1 + \xi^{-k} a_{j,-k}} u^k \right] \exp \left[ \sum_{k>0} t_n \frac{(-q^{-n})^{2k}}{1 + \xi^{-2k} a_{n,-2k}} u^{2k} \right]
$$

Expanding the fractions to power series, the above can be written as

$$
\exp \left( \sum_{k>0} \sum_{j=1}^{n-1} \sum_{m=0}^{\infty} (-1)^m t_j \xi^{-mk} (q^{jk} - q^{-jk}) u^j a_{j,-k} \right) \times \exp \left( \sum_{k>0} \sum_{m=0}^{\infty} (-1)^m t_n \xi^{-mk} (-q^{-n})^{2k} u^{2k} a_{n,-2k} \right),
$$

which is converted into the following by using the series $\varpi_i^-(u)$:

$$
\prod_{m=0}^{\infty} \prod_{j=1}^{n-1} \varpi_j^- (u \xi^{-2m} q^j)^{-1} \varpi_j^- (u \xi^{-2m} q^{-j}) \varpi_j^- (u \xi^{-2m-1} q^j) \varpi_j^- (u \xi^{-2m-1} q^{-j})^{-1} \times \prod_{m=0}^{\infty} \varpi_n^- (u \xi^{-2m} q^{-n}) \varpi_n^- (u \xi^{-2m-1} q^{-n})^{-1},
$$

and this is exactly (by Prop. 3.3):

$$
\prod_{m=0}^{\infty} \prod_{j=1}^{n-1} \overline{h}_j^+ (u \xi^{-2m} q^{2j}) \overline{h}_j^+ (u \xi^{-2m} q^{-2j})^{-1} \times \prod_{m=0}^{\infty} \prod_{j=1}^{n} \overline{h}_j^+ (u \xi^{-2m} q^{2j-2}) \overline{h}_j^+ (u \xi^{-2m-1} q^{2j-2}) \times \prod_{m=0}^{\infty} \overline{h}_{n+1}^+ (u \xi^{-2m-1})^{-1} \overline{h}_{n+1}^+ (u \xi^{-2m-2}) \times \overline{h}_{n+1}^+(u).
$$

Finally using the formula (3.7) of $z^{\pm}(u)$, we get that

$$
\exp \left( \sum_{k>0} \sum_{j=1}^{n} t_j \tilde{B}_{1j}(q^k) u^k a_{j,-k} \right) = \prod_{m=0}^{\infty} z^+(u \xi^{-2m-1})^{-1} z^+(u \xi^{-2m-2}) \times \overline{h}_{n+1}^+(u).
$$

As a consequence, the coefficients of $u^n$ in the infinite product (also (3.9)) is an element in the algebra $U_q^{\text{ext}}(\widehat{sp}_{2n})$.

Moreover, Lemma 3.5 implies that the $(n, n)$-entry of the matrix $(\text{id} \otimes \pi_V)(T_{21})$ is $k_n^{-1/2} = t_n^{-1}$, hence the $(n, n)$-entry of the matrix $H^+(u)$ is

$$
h_{n+1}^+(u) = z^{[1]}(u) h_{n}^+(u \xi^{[1]})^{-1}.
$$

\[ \square \]

4. R-matrix realization of $U_q(A_{2n-1}^{(2)})$

Recall that two related R-matrices have been defined in (2.25). Accordingly two R-matrix algebras $U(R)$ and $\tilde{U}(\overline{R})$ are defined in this section.
4.1. The algebras $U(R)$ and $U(\overline R)$.

**Definition 4.1.** The associative algebra $U(R)$ over $\mathbb{C}(q)$ is generated by an invertible central element $q^{c/2}$ and elements $l_{ij}^+[\mp m]$, $1 \leq i, j \leq 2n$, subject to the following defining relations:

\[ l_{ij}[0] = l_{ji}[0] = 0 \quad \text{for} \quad i < j, \quad l_{nn}^+[0] l_{n+1,n+1}^-[0] = 1 \quad \text{and} \quad l_{ii}^+[0] l_{ii}^-[0] = l_{ii}^-[0] l_{ii}^+[0] = 1, \]

and the remaining relations are written in matrix forms as follows. Let

\[ L^+(u) = \sum_{i,j=1}^{2n} l_{ij}^+(u) \otimes e_{ij} \in U(R)[[u,u^{-1}]] \otimes \text{End} \mathbb{C}^{2n}. \]

where the entries $l_{ij}^+(u) = \sum_{m=0}^{\infty} l_{ij}^+[m] u^{k+m}$ are $U(R)$-valued formal power series. Consider the tensor product algebra $U(R) \otimes \text{End} \mathbb{C}^{2n} \otimes \text{End} \mathbb{C}^{2n}$ and set

\[ L_1^+(u) = \sum_{i,j=1}^{2n} e_{ij} \otimes 1 \otimes l_{ij}^+(u) \quad \text{and} \quad L_2^+(u) = \sum_{i,j=1}^{2n} 1 \otimes e_{ij} \otimes l_{ij}^+(u). \]

Then the remaining defining relations take the form

\[ R(u/v) L_1^+(u) L_2^+(v) = L_2^+(v) L_1^+(u) R(u/v), \]

\[ R(uq^c/v) L_1^+(u) L_2^+(v) = L_2^+(v) L_1^+(u) R(uq^{-c}/v). \]

Similarly, the algebra $U(\overline R)$ over $\mathbb{C}(q)$ is generated by an invertible central element $q^{c/2}$ and elements $\ell_{ij}^+[\mp m]$ with $1 \leq i, j \leq 2n$ and $m \in \mathbb{Z}_+$ such that

\[ \ell_{ij}^+[0] = \ell_{ji}^+[0] = 0 \quad \text{for} \quad i < j, \quad \ell_{nn}^+[0] \ell_{n+1,n+1}^-[0] = 1 \quad \text{and} \quad \ell_{ii}^+[0] \ell_{ii}^-[0] = \ell_{ii}^-[0] \ell_{ii}^+[0] = 1. \]

The remaining defining relations of $U(\overline R)$ take the form

\[ \overline R(u/v) L_1^+(u) L_2^+(v) = L_2^+(v) L_1^+(u) \overline R(u/v), \]

\[ \overline R(uq^c/v) L_1^+(u) L_2^+(v) = L_2^+(v) L_1^+(u) \overline R(uq^{-c}/v), \]

where $\mathcal{L}^+(u)$ are defined similarly as in (4.1)-(4.2).

**Remark 4.2.** The R-matrix $\overline R(u)$ obeys the unitarity property: $\overline R_{12}(u)\overline R_{21}(1/u) = 1$, which implies that relation (4.6) can be written in the equivalent form

\[ \overline R(uq^{-c}/v) L_1^+(u) L_2^+(v) = L_2^+(v) L_1^+(u) \overline R(uq^c/v). \]

**Remark 4.3.** The defining relations satisfied by the series $\ell_{ij}^+(u)$ with $1 \leq i, j \leq n$ coincide with those for the quantum affine algebra $U_q(\mathfrak{gl}_n)$ in [10].

The algebras $U(R)$ and $U(\overline R)$ are related via a Heisenberg algebra as follows. Introduce the Heisenberg algebra $\mathcal{H}_q(n)$ with generators $q^c$ and $\beta_r$, $r \in \mathbb{Z} \setminus \{0\}$ subject to the defining relations

\[ [\beta_r, \beta_s] = \delta_{r,-s} \alpha_r, \quad r \geq 1, \]

and $q^c$ is central and invertible, where the scalars $\alpha_r$ are defined by the expansion

\[ \exp \sum_{r=1}^{\infty} \alpha_r u^r = \frac{g(u^q)}{g(uq^c)}. \]
So we have the identity
\[ g(u q^r/v) \exp \sum_{s=1}^{\infty} \beta_s v^{-s} = g(u q^{-r}/v) \exp \sum_{s=1}^{\infty} \beta_s v^{-s} \exp \sum_{r=1}^{\infty} \beta_r u^r. \]

**Proposition 4.4.** The mappings
\[ L^+(u) \mapsto \exp \sum_{r=1}^{\infty} \beta_r u^{-r} \cdot L^+(u), \quad L^-(u) \mapsto \exp \sum_{r=1}^{\infty} \beta_r u^{-r} \cdot L^-(u), \]
define a homomorphism \( U(R) \to \mathcal{H}_q(n) \otimes_{\mathbb{C}[q^r, q^{-r}]} U(R). \)

Let \( t \) be the matrix transposition of \( \text{End} \mathbb{C}^{2n} \) given by \( e_{ij}^t = e_{j',i'} \), and can be lifted to a partial transpose on tensor products of \( \text{End} \mathbb{C}^{2n} \): the partial transpose \( t_a \) refers to applying \( t \) to the \( a \)-th copy in the tensor product. The following crossing symmetry relations are satisfied by the \( R \)-matrices:
\[ \overline{R}(u)D_1 \overline{R}(u \xi)^{11} D_1^{-1} = \frac{(u - q^2)(u \xi - 1)}{(1 - u)(1 - u \xi q^2)}, \]
\[ R(u)D_1 R(u \xi)^{11} D_1^{-1} = \xi^2 q^{-2}, \]
where the diagonal matrix \( D = \text{diag}\{q^1, \ldots, q^{2n}\} \) and the meaning of the subscripts is the same as in (4.12).

**Proposition 4.5.** In the algebras \( U(R) \) and \( U(\overline{R}) \) we have the relations
\[ DL^\pm(u \xi)^t D^{-1} L^\pm(u) = L^\pm(u) DL^\pm(u \xi)^t D^{-1} = z^\pm(u) 1, \]
and
\[ DL^\pm(u \xi)^t D^{-1} L^\pm(u) = L^\pm(u) DL^\pm(u \xi)^t D^{-1} = z^\pm(u) 1, \]
for certain series \( z^\pm(u) \) and \( z^\pm(u) \) with coefficients in the respective algebra.

**Proof.** The proof is similar for both cases, so we consider the algebra \( U(\overline{R}) \). Multiply both sides of (4.5) by \( u/v - \xi \) and set \( u/v = \xi \) to get
\[ Q L^\pm_1(u \xi)L^\pm_2(u) = L^\pm_2(u)L^\pm_1(u \xi) Q, \]
where \( Q = \sum_{i,j=1}^{2n} q^{-1} e_{i',j'} \otimes e_{ij} = D_1^{-1} P^{11} D_1 \) and \( P = \sum_{i,j=1}^{2n} e_{ij} \otimes e_{ji} \). Therefore, (4.13) takes the form
\[ P^{11} D_1 L^\pm_1(u \xi)L^\pm_2(u) = L^\pm_2(u)L^\pm_1(u \xi) D_1^{-1} P^{11}. \]

The image of the operator \( P^{11} \) in \( \text{End} (\mathbb{C}^N)^{\otimes 2} \) is one-dimensional, so that each side of this equality must be equal to \( P^{11} \) times a certain series \( z^\pm(u) \) with coefficients in \( U(\overline{R}) \). Observe that \( P^{11} D_1 = P^{11} D_2^{-1} \) and \( P^{11} L^\pm_1(u \xi) = P^{11} L^\pm_2(u \xi)^t \) and so we get
\[ P^{11} D_2 L^\pm_2(u \xi)^t D_2^{-1} L^\pm_2(u) = L^\pm_2(u)L_2^\pm(u \xi)L_2^\pm(u \xi)^t D_2^{-1} P^{11} = z^\pm(u) P^{11}. \]
The required relations now follow by taking trace of the first copy of \( \text{End} \mathbb{C}^N \). \( \square \)

**Proposition 4.6.** All coefficients of the series \( z^+(u) \) and \( z^-(u) \) belong to the center of the algebra \( U(R) \).
Proof. We will verify that \( z^+(u) \) commutes with all series \( L^-_{ij}(v) \); the remaining cases follow by similar or simpler arguments. By the defining relations (4.1) we can write
\[
D_1 L^+_1(u \xi)^t D_1^{-1} L^+_1(u) L^-_1(v) = D_1 L^+_1(u \xi)^t D_1^{-1} R(uq^{-c}/v)^{-1} L^-_2(v) L^+_1(u) R(uq^{-c}/v).
\]
By (4.10) the right hand side equals
\[
\xi^{-2} q^2 D_1 L^+_1(u \xi)^t R(u \xi q^{-c}/v)^{t_1} L^-_2(v) D_1^{-1} L^+_1(u) R(uq^{-c}/v).
\]
Applying the partial transposition \( t_1 \) to both sides in (4.14) we get the relation
\[
L^+_1(u \xi)^t R(u \xi q^{-c}/v)^{t_1} L^-_2(v) = L^-_2(v) R(u \xi q^{-c}/v)^{t_1} L^+_1(u \xi)^t.
\]
Hence, using (4.10) and (4.11) we obtain
\[
z^+(u)L^-_1(v) = D_1 L^+_1(u \xi)^t D_1^{-1} L^+_1(u)L^-_1(v)
= \xi^{-2} q^2 L^-_2(v) D_1 R(u \xi q^{-c}/v)^{t_1} D_1^{-1} L^+_1(u \xi)^t D_1^{-1} L^+_1(u) R(uq^{-c}/v)
= \xi^{-2} q^2 L^-_2(v) D_1 R(u \xi q^{-c}/v)^{t_1} D_1^{-1} z^+(u) R(uq^{-c}/v) = L^-_2(v) z^+(u),
\]
as required. \( \square \)

Remark 4.7. The crossing symmetry (4.10) of the \( R \)-matrix \( R(u) \) is essential for Proposition 4.6. Although the coefficients of the series \( \delta^+(u) \) and \( \delta^-(u) \) are central in the respective subalgebras of \( U(\overline{R}) \) generated by the coefficients of the series \( t^+_1(u) \) and \( t^-_1(u) \), they are not central in the entire algebra \( U(\overline{R}) \). We will give the explicit formulas of \( \delta^+(u) \) and \( \delta^-(u) \) in Section 6. \( \square \)

4.2. Quasi-determinant and quantum minors. Let \( A = [a_{ij}] \) be an \( N \times N \) matrix over a ring with 1. Denote by \( A^{ij} \) the matrix obtained from \( A \) by deleting the \( i \)-th row and \( j \)-th column. Suppose that the matrix \( A^{ij} \) is invertible. The \( ij \)-th quasi-determinant of \( A \) is defined by the formula
\[
|A|_{ij} = a_{ij} - r_i^j (A^{ij})^{-1} c_j^i,
\]
where \( r_i^j \) is the row matrix obtained from the \( i \)-th row of \( A \) by deleting the element \( a_{ij} \), and \( c_j^i \) is the column matrix obtained from the \( j \)-th column of \( A \) by deleting the element \( a_{ij} \); see [17].

In particular, the four quasi-determinants of a \( 2 \times 2 \) matrix \( A \) are
\[
|A|_{11} = a_{11} - a_{12} a_{21}^{-1} a_{22}, \quad |A|_{12} = a_{12} - a_{11} a_{21}^{-1} a_{22},
|A|_{21} = a_{21} - a_{22} a_{11}^{-1} a_{12}, \quad |A|_{22} = a_{22} - a_{21} a_{11}^{-1} a_{12}.
\]

The quasi-determinant \( |A|_{ij} \) is also denoted by boxing the entry \( a_{ij} \),
\[
|A|_{ij} = \begin{vmatrix} a_{11} & \ldots & a_{1j} & \ldots & a_{1N} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \ldots & a_{ij} & \ldots & a_{iN} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{N1} & \ldots & a_{Nj} & \ldots & a_{NN} \end{vmatrix}.
\]

Note that elements of the tensor product algebra \( U(\overline{R}) \otimes \text{End}(\mathbb{C}^N)^{\otimes m} \) can be viewed as operators on the space \( (\mathbb{C}^N)^{\otimes m} \) with coefficients in \( U(\overline{R}) \). Accordingly, for such an element
\[
X = \sum_{a_i b_i} X_{b_1 \ldots b_m}^{a_1 \ldots a_m} \otimes e_{a_1 b_1} \otimes \ldots \otimes e_{a_m b_m}
\]
where $e_{ij} = |i\rangle\langle j|$, so the minor
\begin{equation}
X^{a_1\ldots a_m}_{b_1\ldots b_m} = \langle a_1, \ldots, a_m \mid X \mid b_1, \ldots, b_m \rangle
\end{equation}
and the tensors $X \mid b_1, \ldots, b_m \rangle$ and $\langle a_1, \ldots, a_m \mid X$ are defined as usual.

Consider the algebra $U(\mathbb{R})$ and for any $2 \leq i, j \leq 2'$ introduce the quasideterminant
\begin{equation}
s_{ij}^{\pm}(u) = \left| \begin{array}{cc} \ell_{11}^{\pm}(u) & \ell_{1j}^{\pm}(u) \\ \ell_{1j}^{\pm}(u) & \ell_{jj}^{\pm}(u) \end{array} \right| = \ell_{11}^{\pm}(u) - \ell_{1j}^{\pm}(u) \ell_{jj}^{\pm}(u) - 1 \ell_{1j}^{\pm}(u).
\end{equation}

Let the power series $\ell^{\pm a_1 a_2}_{b_1 b_2}(u)$ (quantum minors) with coefficients in $U(\mathbb{R})$ be defined by
\begin{equation}
\ell^{\pm a_1 a_2}_{b_1 b_2}(u) = \langle a_1, a_2 \mid \hat{R}(q^{-1}) \mathcal{L}_1^{\pm}(u) \mathcal{L}_2^{\pm}(u u_q^{-1}) \mid b_1, b_2 \rangle,
\end{equation}
where $a_i, b_i \in \{1, \ldots, N\}$ and we set
\begin{equation}
\hat{R}(u) = \frac{u q - q^{-1}}{u - 1} R(u).
\end{equation}

The following symmetry properties are straightforward to verify.

**Lemma 4.8.** (i) If $a_1 \neq a_2$ and $a_1 < a_2$ then $\ell^{\pm a_1 a_2}_{b_1 b_2}(u) = -q^{-1} \ell^{\pm a_2 a_1}_{b_1 b_2}(u)$.

(ii) If $b_1 \neq b_2$ and $b_1 < b_2$ then $\ell^{\pm a_1 a_2}_{b_1 b_2}(u) = -q \ell^{\pm a_2 a_1}_{b_2 b_1}(u)$.

**Lemma 4.9.** For any $2 \leq i, j \leq 2'$ we have
\begin{equation}
s_{ij}^{\pm}(u) = \ell_{11}^{\pm}(u q^{-1}) - 1 \ell_{1j}^{\pm}(u q^{-1}).
\end{equation}
Moreover,
\begin{equation}
[\ell_{11}^{\pm}(u), \ell_{1j}^{\pm}(v)] = 0
\end{equation}
and
\begin{equation}
\frac{q^{-1}u_{\mp} - q v_{\mp}}{u_{\mp} - v_{\mp}} \ell_{11}^{\pm}(u) \ell_{1j}^{\pm}(v) = \frac{q^{-1}u_{\pm} - q v_{\pm}}{u_{\pm} - v_{\pm}} \ell_{11}^{\pm}(v) \ell_{1j}^{\pm}(u).
\end{equation}

The following consequences of Lemma 4.9 are easily seen: for $2 \leq i, j \leq 2'$ we have
\begin{equation}
[\ell_{11}^{\pm}(u), \ell_{ij}^{\pm}(v)] = 0
\end{equation}
and
\begin{equation}
\frac{u_{\mp} - v_{\mp}}{q u_{\mp} - q^{-1} v_{\mp}} \ell_{11}^{\pm}(u) \ell_{ij}^{\pm}(v) = \frac{u_{\pm} - v_{\pm}}{q u_{\pm} - q^{-1} v_{\pm}} \ell_{1j}^{\pm}(v) \ell_{11}^{\pm}(u).
\end{equation}

### 4.3. Homomorphism from $U(\mathbb{R}[n-1])$ to $U(\mathbb{R}[n])$

Now we establish a relation between the algebras $U(\mathbb{R})$ associated with the Lie algebras $\mathfrak{g}_{N-2}$ and $\mathfrak{g}_N$. Let us denote by $\mathbb{R}^{[n]}$ the R-matrix associated with rank $n$. Consider the algebra $U(\mathbb{R}^{[n-1]})$ and let the indices of the generators $\ell_{ij}^{\pm}[\mp m]$ range over the sets $2 \leq i, j \leq 2'$ and $m = 0, 1, \ldots$, where $i' = N - i + 1$, as before.

**Theorem 4.10.** The maps $q^{\pm c/2} \mapsto q^{\pm c/2}$ and
\begin{equation}
\ell_{ij}^{\pm}(u) \mapsto \left| \begin{array}{cc} \ell_{11}^{\pm}(u) & \ell_{1j}^{\pm}(u) \\ \ell_{1j}^{\pm}(u) & \ell_{jj}^{\pm}(u) \end{array} \right|, \quad 2 \leq i, j \leq 2',
\end{equation}
define a homomorphism $U(\mathbb{R}^{[n-1]}) \to U(\mathbb{R}^{[n]})$. 


The following result can be checked directly.

**Lemma 4.11.** For $2 \leq i, j \leq 2'$, we have

\begin{equation}
\hat{R}_{12}^{[n]}(q^{-2})\hat{R}_{34}^{[n]}(q^{-2})\hat{R}_{13}^{[n]}(aq^{-2})\hat{R}_{24}^{[n]}(a)\hat{R}_{13}^{[n]}(a)\hat{R}_{23}^{[n]}(aq^{2})|1, i, 1, j\rangle
\end{equation}

\[\frac{aq^{-1} - q}{aq - q^{-1}}\hat{R}_{12}^{[n]}(q^{-2})\hat{R}_{34}^{[n]}(q^{-2})\hat{R}_{24}^{[n]}(a)\hat{R}_{13}^{[n]}(a)|1, i, 1, j\rangle
\]

and

\begin{equation}
\langle 1, i, 1, j |\hat{R}_{23}^{[n]}(aq^{2})\hat{R}_{13}^{[n]}(a)\hat{R}_{24}^{[n]}(a)\hat{R}_{14}^{[n]}(aq^{-2})\hat{R}_{12}^{[n]}(q^{-2})\hat{R}_{34}^{[n]}(q^{-2})
\end{equation}

\[\frac{aq^{-1} - q}{aq - q^{-1}}\langle 1, i, 1, j |\hat{R}_{24}^{[n-1]}(a)\hat{R}_{12}^{[n]}(q^{-2})\hat{R}_{34}^{[n]}(q^{-2}).
\]

**Proof.** To prove the theorem, introduce the matrices

\[\Gamma_{i,j}^{\pm}(u) = \sum_{i,j=2}^{2'} e_{ij} \otimes \ell_{ij}^{\pm}(u) \in \text{End} \mathbb{C}^N \otimes U(\hat{R}^{[n]}).
\]

Our next step is to verify that the following relations hold in the algebra $U(\hat{R}^{[n]})$:

\[\hat{R}_{23}^{[n-1]}(u/v)\Gamma_{1,2}^{\pm}(u)\Gamma_{2,1}^{\pm}(v) = \Gamma_{2,1}^{\pm}(v)\Gamma_{1,2}^{\pm}(u)\hat{R}_{23}^{[n-1]}(u/v),
\]

\[\frac{q^{-1}u_+ - qv_-}{qu_+ - q^{-1}v_-}\hat{R}_{12}^{[n]}(uq^{2}/v)\Gamma_{1,2}^{\pm}(u)\Gamma_{2,1}^{\pm}(v) = \frac{q^{-1}u_- - qv_+}{qu_- - q^{-1}v_+}\Gamma_{2,1}^{\pm}(v)\Gamma_{1,2}^{\pm}(u)\hat{R}_{12}^{[n]}(uq^{-2}/v).
\]

The calculations are quite similar in both cases so we only give details for the first relation. The Yang–Baxter equation and the defining relations for the algebra $U(\hat{R}^{[n]})$ give

\[\hat{R}_{23}^{[n]}(u/v)\hat{R}_{13}^{[n]}(u/v)\hat{R}_{24}^{[n]}(u/v)\hat{R}_{14}^{[n]}(u/vq)\hat{R}_{12}^{[n]}(q^{-2})\mathcal{L}_{1,2}^{\pm}(u)\mathcal{L}_{2,1}^{\pm}(u)\hat{R}_{34}^{[n]}(q^{-2})\mathcal{L}_{3,4}^{\pm}(v)\mathcal{L}_{4,3}^{\pm}(vq^{2})
\]

\[\hat{R}_{34}^{[n]}(q^{-2})\mathcal{L}_{3,4}^{\pm}(v)\mathcal{L}_{4,3}^{\pm}(vq^{2})\hat{R}_{12}^{[n]}(q^{-2})\mathcal{L}_{1,2}^{\pm}(u)\mathcal{L}_{2,1}^{\pm}(u)\hat{R}_{34}^{[n]}(q^{-2})\mathcal{L}_{3,4}^{\pm}(v)\mathcal{L}_{4,3}^{\pm}(vq^{2}).
\]

Hence, assuming that $2 \leq i, j, k, l \leq 2'$ and applying (4.24) and (4.25) we get

\[\langle 1, k, 1, l |\hat{R}_{24}^{[n-1]}(u/v)\hat{R}_{12}^{[n]}(q^{-2})\mathcal{L}_{1,2}^{\pm}(u)\mathcal{L}_{2,1}^{\pm}(u)\hat{R}_{34}^{[n]}(q^{-2})\mathcal{L}_{3,4}^{\pm}(v)\mathcal{L}_{4,3}^{\pm}(vq^{2})|1, i, 1, j\rangle
\]

\[= \langle 1, k, 1, l |\hat{R}_{34}^{[n]}(q^{-2})\mathcal{L}_{3,4}^{\pm}(v)\mathcal{L}_{4,3}^{\pm}(vq^{2})\hat{R}_{12}^{[n]}(q^{-2})\mathcal{L}_{1,2}^{\pm}(u)\mathcal{L}_{2,1}^{\pm}(u)\hat{R}_{24}^{[n]}(u/vq^{2})\hat{R}_{14}^{[n]}(u/v)\hat{R}_{13}^{[n]}(u/v)\hat{R}_{23}^{[n]}(u/vq^{2}).
\]

which is equivalent to

\[\hat{R}_{24}^{[n-1]}(u/v)\Gamma_{2,1}^{\pm}(u)\Gamma_{1,2}^{\pm}(v) = \Gamma_{1,2}^{\pm}(v)\Gamma_{2,1}^{\pm}(u)\hat{R}_{24}^{[n-1]}(u/v),
\]

as required. Finally, set

\[S_{i,j}^{\pm}(u) = \sum_{2 \leq i, j \leq 2'} e_{ij} \otimes s_{ij}^{\pm}(u).
\]

By Lemma 4.9

\[S_{i,j}^{\pm}(u) = \ell_{11}^{\pm}(uq^{-2})^{-1} \Gamma^{\pm}(uq^{-2})
\]
and
\[
\frac{q^{-1}u_\pm - q^v_\pm}{u_\mp - v_\mp} \ell_1^+(u) \Gamma^\mp(v) = \frac{q^{-1}u_\pm - q^v_\pm}{u_\mp - v_\mp} \Gamma^\mp(v) \ell_1^+(u).
\]
The above relations for the matrices \(\Gamma^\pm(u)\) imply
\[
\overline{R}^{[n-1]}(u/v)S^+_2(v)S^+_2(u) = S^+_2(v)S^+_2(u)\overline{R}^{[n-1]}(u/v),
\]
\[
\overline{R}^{[n-1]}(uq^{\pm c}/v)S^+_2(v)S^+_2(u) = S^+_2(v)S^+_2(u)\overline{R}^{[n-1]}(uq^{\mp c}/v),
\]
which completes the proof. \(\square\)

The following generalizes Theorem 4.10 by using the Sylvester theorem for quasideterminants \cite{Li,20}; cf. the proof of the Yangian algebra \cite[Thm 3.7]{BGV}. Fix a positive integer \(m\) such that \(m < n\). Suppose that the generators \(\ell_{ij}(u)\) of the algebra \(U(\overline{R}^{[n-m]})\) are labelled by the indices \(m + 1 \leq i, j \leq (m+1)\)' with \((m+1)' = N - i + 1\) as before.

**Theorem 4.12.** For \(m \leq n - 1\), the map
\[
\ell_{ij}(u) \mapsto \begin{vmatrix} \ell_{11}^+(u) & \cdots & \ell_{1m}^+(u) & \ell_{ij}^+(u) \\ \vdots & \ddots & \vdots & \vdots \\ \ell_{m1}^+(u) & \cdots & \ell_{mm}^+(u) & \ell_{mj}^+(u) \\ \ell_{11}^+(u) & \cdots & \ell_{im}^+(u) & \ell_{ij}^+(u) \end{vmatrix}, \quad m + 1 \leq i, j \leq (m+1)',
\]
defines a homomorphism \(\varphi_m^+ : U(\overline{R}^{[n-m]}) \rightarrow U(\overline{R}^{[n]})\). \(\square\)

All the maps \((4.26)\) are compatible in the following sense, as seen by using the Sylvester theorem for quasideterminants. Write the maps \(\varphi_m^+ = \psi_m^{(n)}\) for \(U(\overline{R}^{[n]})\), and for each admissible \(l\) we have the corresponding homomorphism
\[
\psi_m^{(n-l)} : U(\overline{R}^{[n-l-m]}) \rightarrow U(\overline{R}^{[n-l]})
\]
given by \((4.26)\). Then we have the equation:
\[
(4.27) \quad \psi_l^{(n)} \circ \psi_m^{(n-l)} = \psi_{l+m}^{(n)}.
\]

Let \(\{a_1, \ldots, a_k\}\) and \(\{b_1, \ldots, b_k\}\) be subsets of \(\{1, \ldots, N\}\), where \(a_1 < a_2 < \cdots < a_k\) and \(b_1 < b_2 < \cdots < b_k\) such that \(a_i \neq a'_j\) and \(b_i \neq b'_j\) for all \(i, j\). Introduce the type \(A\) quantum minors as the matrix elements \((1.15)\):
\[
\ell_{b_1, \ldots, b_k}^{\pm a_1, \ldots, a_k}(u) = \langle a_1, \ldots, a_k \mid \hat{R}_{k-1,k}(\hat{R}_{k-2,k}\hat{R}_{k-2,k-1})\ldots(\hat{R}_{1,k}\ldots\hat{R}_{1,2}) \times \mathcal{L}_1^\pm(u)\mathcal{L}_2^\pm(uq^2)\ldots\mathcal{L}_k^\pm(uq^{2k-2}) \mid b_1, \ldots, b_k\rangle,
\]
where \(\hat{R}_{ij} = \hat{R}_{ij}(q^{2(i-j)})\). Explicitly we have:
\[
\ell_{b_1, \ldots, b_k}^{\pm a_1, \ldots, a_k}(u) = \sum_{\sigma \in \mathfrak{S}_k} (-q)^{-l(\sigma)} \ell_{a_{\sigma(1)}b_1}(u) \cdots \ell_{a_{\sigma(k)}b_k}(uq^{2k-2})
\]
\[
= \sum_{\sigma \in \mathfrak{S}_k} (-q)^{l(\sigma)} \ell_{a_kb_{\sigma(k)}}(uq^{2k-2}) \cdots \ell_{a_1b_{\sigma(1)}}(u),
\]
where \(l(\sigma)\) is the number of inversions of the permutation \(\sigma \in \mathfrak{S}_k\). The assumption on the indices \(a_i\) and \(b_i\) implies that the quantum minors satisfy the same form as those for the quantum
affine algebra $U_q(\mathfrak{gl}_n)$. By applying $R$-matrix calculations (cf. [21], [29], [31 Ch. 1]), we have that for $1 \leq i, j \leq k$

$$[\ell^\pm_{a_i,b_j}(u), \ell^\pm_{a_1,a_2,\ldots,a_k}(v)] = 0,$$

$$\prod_{a=1}^{k-1} \frac{u_\pm q_{a_i,b_j}^{-k}}{u_\pm q_{1-k}} - v_\pm q_{a_i,b_j}^{-k} = \prod_{a=1}^{k-1} \frac{u_\pm q_{a_i,b_j}^{-k}}{u_\pm q_{1-k}} - v_\pm q_{a_i,b_j}^{-k} \ell^\pm_{a_1,a_2,\ldots,a_k}(v) \ell^\pm_{a_i,b_j}(u).$$

**Corollary 4.13.** Under the assumptions of Theorem 4.12 we have

$$[\ell^\pm_{ab}(u), \varphi^\pm_m(\ell^\mp_{ij}(v))] = 0,$$

$$\frac{u_\pm - v_\pm}{qu_\pm - q^{-1}v_\pm} \ell^\pm_{ab}(u) \varphi^\pm_m(\ell^\mp_{ij}(v)) = \frac{u_\pm - v_\pm}{qu_\pm - q^{-1}v_\pm} \varphi^\pm_m(\ell^\mp_{ij}(v)) \ell^\pm_{ab}(u),$$

for all $1 \leq a, b \leq m$ and $m + 1 \leq i, j \leq (m + 1)'$.

**Proof.** Both formulas are verified by the fact that the quasideterminants and quantum minors satisfy the following type A relations: (cf. similar ones for the Yangian [21 Sec. 3].

$$\begin{vmatrix}
\ell^\pm_{11}(u) & \ldots & \ell^\pm_{1m}(u) & \ell^\pm_{ij}(u) \\
\ldots & \ldots & \ldots & \ldots \\
\ell^\pm_{m1}(u) & \ldots & \ell^\pm_{mm}(u) & \ell^\pm_{mj}(u) \\
\ell^\pm_{i1}(u) & \ldots & \ell^\pm_{im}(u) & \ell^\pm_{ij}(u)
\end{vmatrix} = \ell^\pm_{1\ldots m} (u q_{-2m})^{-1} \cdot \ell^\pm_{1\ldots m} (u q_{-2m}).$$

\[ \square \]

5. GAUSS DECOMPOSITION

5.1. Gaussian generators. It is known that the matrix $L^\pm(u)$ of generators for $U(R^{\infty})$ admits a unique **Gauss decomposition**. Namely, $L^\pm(u)$ can be uniquely factored as

$$L^\pm(u) = F^\pm(u) H^\pm(u) E^\pm(u).$$

where

$$F^\pm(u) = \begin{bmatrix}
1 & 0 & \ldots & 0 \\
0 & f^\pm_{21}(u) & 1 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
f^\pm_{2n1}(u) & f^\pm_{2n2}(u) & \ldots & 1
\end{bmatrix}, \quad E^\pm(u) = \begin{bmatrix}
1 & e^\pm_{12}(u) & \ldots & e^\pm_{12n}(u) \\
0 & 1 & \ldots & e^\pm_{12n}(u) \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1
\end{bmatrix},$$

and $H^\pm(u) = \text{diag} [h^\pm_1(u), \ldots, h^\pm_{2n}(u)]$. Moreover, the entries can be expressed in terms of quasideterminants. as follows [17, 18]:

$$h^\pm_i(u) = \begin{bmatrix}
\ell^\pm_{ii1}(u) & \ldots & \ell^\pm_{i1}(u) & \ell^\pm_{i1}(u) \\
\vdots & \ddots & \vdots & \vdots \\
\ell^\pm_{i1}(u) & \ldots & \ell^\pm_{i1}(u) & \ell^\pm_{i1}(u) \\
\ell^\pm_{i1}(u) & \ldots & \ell^\pm_{i1}(u) & \ell^\pm_{i1}(u)
\end{bmatrix}, \quad i = 1, \ldots, 2n,
\begin{equation}
\begin{split}
e_{ij}^\pm(u) &= h_i^\pm(u)^{-1} \begin{vmatrix}
    l_{11}^\pm(u) & \cdots & l_{i-1}^\pm(u) & l_{ij}^\pm(u) \\
    \vdots & \ddots & \vdots & \vdots \\
    l_{i-11}^\pm(u) & \cdots & l_{i-1i-1}^\pm(u) & l_{i-1j}^\pm(u) \\
    l_{ij}^\pm(u) & \cdots & l_{i-1i-1}^\pm(u) & l_{ji}^\pm(u) \\
\end{vmatrix}
\end{split}
\tag{5.3}
\end{equation}

and
\begin{equation}
\begin{split}
f_{ij}^\pm(u) &= \begin{vmatrix}
    l_{11}^\pm(u) & \cdots & l_{i-1}^\pm(u) & l_{ij}^\pm(u) \\
    \vdots & \ddots & \vdots & \vdots \\
    l_{i-11}^\pm(u) & \cdots & l_{i-1i-1}^\pm(u) & l_{i-1j}^\pm(u) \\
    l_{ij}^\pm(u) & \cdots & l_{i-1i-1}^\pm(u) & l_{ji}^\pm(u) \\
\end{vmatrix} h_i^\pm(u)^{-1}
\end{split}
\tag{5.4}
\end{equation}

for $1 \leq i < j \leq 2n$. The same formulas hold for the entries of the respective triangular matrices $F^\pm(u)$ and $E^\pm(u)$ and the diagonal matrices
\[
H^\pm(u) = \text{diag} [h_1^\pm(u), \ldots, h_{2n}^\pm(u)]
\]
in terms of the formal series $l_{ij}^\pm(u)$ via the Gauss decomposition
\[
L^\pm(u) = F^\pm(u) H^\pm(u) E^\pm(u)
\]
for the algebra $U(R^{[n]})$. We will denote by $e_{ij}^\pm(u)$ and $f_{ij}^\pm(u)$ the entries of the respective matrices $E^\pm(u)$ and $F^\pm(u)$ for $i < j$.

The following is immediate from the Gaussian generators.

**Proposition 5.1.** Under the homomorphism $U(R) \to \mathcal{H}_q(n) \otimes_{\mathbb{C}[q, q^{-1}]} U(R)$ provided by Proposition 4.4, we have

\[
e_{ij}^\pm(u) \mapsto e_{ij}^\pm(u),
\]
\[
f_{ij}^\pm(u) \mapsto f_{ij}^\pm(u),
\]
\[
h_i^\pm(u) \mapsto \exp \sum_{k=1}^{\infty} \beta_{\pm k} u^{\pm k} \cdot h_i^\pm(u).
\]

5.2. Images of the generators under the homomorphism $\psi_m$. For $0 \leq m < n$, we use the superscript $[n - m]$ to indicate the submatrices with rows and columns labelled by $m+1, m+2, \ldots, (m+1)'$. Thus we set

\[
F^{[n-m]}(u) = \begin{bmatrix}
    1 & 0 & \cdots & 0 \\
    f_{m+2m+1}^\pm(u) & 1 & \cdots & 0 \\
    \vdots & \ddots & \ddots & \vdots \\
    f_{(m+1)'m+1}^\pm(u) & \cdots & f_{(m+1)'(m+2)'}^\pm(u) & 1
\end{bmatrix},
\]
\[
E^{[n-m]}(u) = \begin{bmatrix}
    1 & e_{m+1m+2}^\pm(u) & \cdots & e_{m+1(m+1)'}^\pm(u) \\
    0 & 1 & \cdots & \vdots \\
    \vdots & \ddots & \ddots & e_{(m+2)'(m+1)'}^\pm(u) \\
    0 & 0 & \cdots & 1
\end{bmatrix}
\]

\[\text{24}\]
and $\mathcal{H}^{n-m}(u) = \text{diag} [h_{m+1}^\pm(u), \ldots, h_{(m+1)'}^\pm(u)]$. Also we set that

$$\mathcal{L}^{n-m}(u) = \mathcal{F}^{n-m}(u) \mathcal{H}^{n-m}(u) \mathcal{E}^{n-m}(u) = (\ell_{ij}^{n-m}(u)).$$

The following result can be easily checked by the same argument as in the Yangian case (cf. [24, Prop. 4.1]):

**Proposition 5.2.** The series $\ell_{ij}^{n-m}(u)$ coincides with the image of the generator series $\ell_{ij}^j(u)$ of the extended quantum affine algebra $U(R^{n-m})$ under the homomorphism (4.26):

$$\ell_{ij}^{n-m}(u) = \psi_m(\ell_{ij}^j(u)), \quad m + 1 \leq i, j \leq (m + 1)'.$$

The following can be shown by using Proposition 5.2.

**Corollary 5.3.** The following relations hold in $U(R^{n})$:

$$\mathcal{T}^{n-m}_{12}[(u/v)] \mathcal{L}^{n-m} zu \mathcal{L}^{n-m} vu = \mathcal{L}^{n-m} vu \mathcal{L}^{n-m} zu \mathcal{T}^{n-m}_{12} (u/v),$$

(5.5)

$$\mathcal{T}^{n-m}_{12}[(u_+/v_-)] \mathcal{L}^{n-m} zu \mathcal{L}^{n-m} vu = \mathcal{L}^{n-m} vu \mathcal{L}^{n-m} zu \mathcal{T}^{n-m}_{12} (u_+/v_-).$$

(5.6)

The following proposition is the same as in the untwisted classical cases (BCD) [24, 26].

**Proposition 5.4.** Suppose that $m + 1 \leq j, k, l \leq (m + 1)'$ and $j \neq l'$. Then the following relations hold in $U(R^{n})$: if $j = l$ then

$$\ell_{mj}^j(u) \ell_{kl}^{n-m}(v) = \frac{qu_+ - q^{-1}v_+}{u_+ - v_+} \ell_{kl}^{n-m}(v) \ell_{mj}^j(u) - \frac{q - q^{-1}}{u_+ - v_+} \ell_{kl}^{n-m}(v) \ell_{mj}^j(u) \ell_{mj}^j(v),$$

$$\ell_{mj}^j(u) \ell_{kl}^{n-m}(v) = \frac{qu - q^{-1}v}{u - v} \ell_{kl}^{n-m}(v) \ell_{mj}^j(u) - \frac{q - q^{-1}}{u - v} \ell_{kl}^{n-m}(v) \ell_{mj}^j(u) \ell_{mj}^j(v);$$

if $j < l$ then

$$[\ell_{mj}^j(u), \ell_{kl}^{n-m}(v)] = \frac{(q - q^{-1})v_+}{u_+ - v_+} \ell_{kl}^{n-m}(v) \ell_{mj}^j(u) - \frac{(q - q^{-1})u_+}{u_+ - v_+} \ell_{kl}^{n-m}(v) \ell_{mj}^j(u),$$

$$[\ell_{mj}^j(u), \ell_{kl}^{n-m}(v)] = \frac{(q - q^{-1})v}{u - v} \ell_{kl}^{n-m}(v) \ell_{mj}^j(u) - \frac{(q - q^{-1})u}{u - v} \ell_{kl}^{n-m}(v) \ell_{mj}^j(u);$$

if $j > l$ then

$$[\ell_{mj}^j(u), \ell_{kl}^{n-m}(v)] = \frac{(q - q^{-1})u_+}{u_+ - v_+} \ell_{kl}^{n-m}(v) \ell_{mj}^j(u) - \ell_{kl}^{n-m}(v) \ell_{mj}^j(u) \ell_{mj}^j(v),$$

$$[\ell_{mj}^j(u), \ell_{kl}^{n-m}(v)] = \frac{(q - q^{-1})u}{u - v} \ell_{kl}^{n-m}(v) \ell_{mj}^j(u) - \ell_{kl}^{n-m}(v) \ell_{mj}^j(u) \ell_{mj}^j(v).$$

Similar arguments prove the following counterpart of Proposition 5.4 involving the generator series $f_{ij}^{n-m}(u)$.  

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[24]: Reference to a specific page or section in the source text is not possible as the full context is not provided. It is assumed that the reference is to a standard text on quantum affine algebras or a related field.
Proposition 5.5. Suppose that \( m + 1 \leq j, k, l \leq (m + 1)' \) and \( j \neq k' \). Then the following relations hold in \( U(R[n]) \): if \( j = k \) then

\[
\ell_{jm}^+(u)\ell_{jl}^{[n-m]}(v) = \frac{u-v}{qu-q^{-1}v} \ell_{jl}^{[n-m]}(v)\ell_{jm}^+(u) + \frac{(q-q^{-1})v}{qu-q^{-1}v} \ell_{jm}^+(v)\ell_{jl}^{[n-m]}(v),
\]

if \( j < k \) then

\[
[f_{jm}(u), \hat{e}_{kl}^{[n-m]}(v)] = \left( \frac{q-q^{-1}}{u-v} \right)v\ell_{km}^+(v)\ell_{jl}^{[n-m]}(v) - \left( \frac{q-q^{-1}}{u-v} \right)v\ell_{jm}^+(v)\ell_{kl}^{[n-m]}(v),
\]

if \( j > k \) then

\[
[f_{jm}(u), \hat{e}_{kl}^{[n-m]}(v)] = \left( \frac{q-q^{-1}}{u-v} \right)v\ell_{km}^+(v)\ell_{jl}^{[n-m]}(v) - \left( \frac{q-q^{-1}}{u-v} \right)v\ell_{jm}^+(v)\ell_{kl}^{[n-m]}(v).
\]

6. Isomorphism between \( U_q^{ext}(A^{(2)}_{2n-1}) \) and \( U(R[n]) \)

In this section, we derive the commuting relations among Gaussian generators in \( U(R[n]) \) and \( U(R[n]) \). By Proposition 5.1 only the case of \( U(R[n]) \) needs to be considered.

To do so, we will frequently employ the following Laurent series with coefficients in the respective algebras \( U(R[n]) \):

(6.1) \( X_i^+(u) = e_{ii+1}^+(u_+) - e_{ii+1}^-(u_-) \), \( X_i^-(u) = f_{ii+1,i}^+(u_-) - f_{ii+1,i}^-(u_+) \)

for \( i = 1, \ldots, n-1 \) and

(6.2) \( X_n^+(u) = u(e_{nn+1}^+(u_+) - e_{nn+1}^-(u_-)) \)

(6.3) \( X_n^-(u) = \frac{1}{u}(f_{nn+1,n}^+(u_-) - f_{nn+1,n}^-(u_+)) \)

and then \( X_i^\pm(u) \) are similarly defined for \( U(R[n]) \).

6.1. Type A relations. Due to the observation made in Remark 4.3 and the quasideterminant formulas (4.2), (4.3) and (4.4), some of the relations between the Gaussian generators follow from those for the quantum affine algebra \( U_q(\widehat{gl}_n) \); see [10]. To reproduce them, set

\[
\mathcal{L}^A_{\pm}(u) = \sum_{i,j=1}^n e_{ij} \otimes \ell_{ij}^\pm(u)
\]

and consider the \( R \)-matrix used in [10]:

(6.4) \( R_A(u) = \sum_{i=1}^n e_{ii} \otimes e_{ii} + \frac{u-1}{qu-q^{-1}} \sum_{i \neq j} e_{ii} \otimes e_{jj} \)
Comparing this with the $R$-matrix \[^{227}\] , we immediately see the relations in the algebra $U(\mathbb{R}^{[n]})$:

\[ R_A(u/w)\mathcal{L}_1^\pm(u)\mathcal{L}_2^\pm(v) = \mathcal{L}_2^\pm(v)\mathcal{L}_1^\pm(u)R_A(u/v), \]

\[ R_A(uq^c/v)\mathcal{L}_1^\pm(u)\mathcal{L}_2^\pm(v) = \mathcal{L}_2^\pm(v)\mathcal{L}_1^\pm(u)R_A(uq^{-c}/v). \]

Therefore we get the following relations for the Gaussian generators (cf. \[^{10}\]), where we use the notation similar to (6.1).

**Proposition 6.1.** In the algebra $U(\mathbb{R}^{[n]})$ we have

\[ h_i^\pm(u)h_j^\pm(v) = h_j^\pm(v)h_i^\pm(u), \quad \text{for } 1 \leq i, j \leq n. \]

Moreover,

\[ \frac{u_\pm - v_\pm}{qu_\pm - q^{-1}v_\pm}h_i^\pm(u)h_j^\pm(v) = \frac{u_\pm - v_\pm}{qu_\pm - q^{-1}v_\pm}h_j^\pm(v)h_i^\pm(u), \quad \text{for } 1 \leq i < j \leq n. \]

and

\[ (u - q^{\pm(\alpha_i, \alpha_j)})\mathcal{X}_i^\pm(uq^i)\mathcal{X}_j^\pm(vq^j) = (q^{\pm(\alpha_i, \alpha_j)}u - v)\mathcal{X}_j^\pm(vq^j)\mathcal{X}_i^\pm(uq^i), \]

\[ [\mathcal{X}_i^\pm(u), \mathcal{X}_j^\pm(v)] = \delta_{ij}(q - q^{-1})\left(\delta(uq^{-c}/v)h_i^-(v_+)^{-1}h_i^+(v_+) - \delta(uq^{c}/v)h_i^+(u_+)^{-1}h_i^+(u_+)\right) \]

for $1 \leq i, j < n$, together with the Serre relations

\[ \sum_{\pi \in S_r} \sum_{l=0}^{r} (-1)^l \left[ \begin{array}{c} r \\ l \end{array} \right] q_i^l \mathcal{X}_i^\pm(u_{\pi(1)}) \cdots \mathcal{X}_i^\pm(u_{\pi(l+1)}) \mathcal{X}_j^\pm(u_{\pi(l+1)}) \cdots \mathcal{X}_i^\pm(u_{\pi(r)}) = 0, \]

for all $1 \leq i \neq j < n$, here $r = 1 - A_{ij}$. \(\square\)

**Remark 6.2.** Consider the inverse matrices $\mathcal{L}^\pm(u^{-1}) = [\ell_{ij}^\pm(u)^{-1}]_{i,j=1,\ldots,2n}$. By the defining relations (4.5) and (4.6), we have

\[ \mathcal{L}_1^\pm(u)^{-1} \mathcal{L}_2^\pm(v)^{-1} R_{\mathbb{R}^{[n]}}(u/v) = R_{\mathbb{R}^{[n]}}(u/v)^{-1} \mathcal{L}_2^\pm(v)^{-1} \mathcal{L}_1^\pm(u)^{-1}, \]

\[ \mathcal{L}_2^\pm(v)^{-1} \mathcal{L}_1^\pm(u)^{-1} R_{\mathbb{R}^{[n]}}(uq^{-c}/v) = R_{\mathbb{R}^{[n]}}(uq^{-c}/v)^{-1} \mathcal{L}_1^\pm(u)^{-1} \mathcal{L}_2^\pm(v)^{-1}. \]

So we can obtain another family of generators of the algebra $U(\mathbb{R}^{[n]})$ satisfying the defining relations of $U_q(\hat{\mathfrak{gl}}_n)$. Namely, the same relations are also satisfied by the coefficients of the series $\ell_{ij}^\pm(u)'$ with $i, j = n', \ldots, 1'$. In particular, by taking the inverse matrices, we get a Gauss decomposition for the matrix $[\ell_{ij}^\pm(u)']_{i,j=n',\ldots,1'}$ from the Gauss decomposition of the matrix $\mathcal{L}^\pm(u)$. \(\square\)
6.2. Relations for the last root generators. In this subsection, we will study the relations among the last root generators in $U(\mathcal{R})$: $\mathcal{R}_n^\pm(u)$, $h_n^\pm(u)$ and $h_{n+1}^\pm(u)$. By Corollary 5.3, we have

\begin{equation}
\mathcal{R}_{12}^{(1)}(u/v) \mathcal{L}_1^{\pm[1]}(u) \mathcal{L}_2^{\mp[1]}(v) = \mathcal{L}_2^{\mp[1]}(v) \mathcal{L}_1^{\pm[1]}(u) \mathcal{R}_{12}^{(1)}(u/v),
\end{equation}

\begin{equation}
\mathcal{R}_{12}^{(1)}(u_\pm/v_\mp) \mathcal{L}_1^{\pm[1]}(u) \mathcal{L}_2^{\mp[1]}(v) = \mathcal{L}_2^{\mp[1]}(v) \mathcal{L}_1^{\pm[1]}(u) \mathcal{R}_{12}^{(1)}(u_\mp/v_\pm),
\end{equation}

where

\begin{equation}
\mathcal{R}^{[1]}(u) = \sum_{i=n}^{n+1} e_{ii} \otimes e_{ii} + \frac{u^2 - 1}{(u^2 - q^{-4})q^2} \sum_{i \neq j} e_{ii} \otimes e_{jj} + \frac{1 - q^{-4}}{u^2 - q^{-4}} \sum_{i \neq j} e_{ij} \otimes e_{ji}.
\end{equation}

**Proposition 6.3.** In $U(\mathcal{R})$, the following relations hold for $i, j = n, n + 1$

\begin{equation}
h_i^\pm(u) h_j^\mp(v) = h_j^\mp(v) h_i^\pm(u),
\end{equation}

\begin{equation}
h_i^\pm(u) h_i^\mp(v) = h_i^\mp(v) h_i^\pm(u),
\end{equation}

\begin{equation}
\frac{(u_\pm/v_\mp)^2 - 1}{q^2(u_\pm/v_\mp)^2 - q^{-2}} h_n^\pm(u) h_{n+1}^\mp(v) = \frac{(u_\mp/v_\pm)^2 - 1}{q^2(u_\mp/v_\pm)^2 - q^{-2}} h_{n+1}^\pm(v) h_n^\mp(u).
\end{equation}

\begin{equation}
h_n^\pm(u) \mathcal{X}_n^\pm(v) = \frac{(u/v_\mp)^2 - 1}{q^2(u/v_\mp)^2 - q^{-2}} \mathcal{X}_n^\pm(v) h_n^\pm(u),
\end{equation}

\begin{equation}
\mathcal{X}_n^-(v) h_n^\pm(u) = \frac{(u/v_\mp)^2 - 1}{q^2(u/v_\mp)^2 - q^{-2}} h_n^\pm(u) \mathcal{X}_n^-(v).
\end{equation}

\begin{equation}
h_{n+1}^\pm(u) \mathcal{X}_n^\pm(v) = \frac{(u/v_\mp)^2 - 1}{q^{-2}(u/v_\mp)^2 - q^2} \mathcal{X}_n^\pm(v) h_{n+1}^\pm(u),
\end{equation}

\begin{equation}
\mathcal{X}_n^-(v) h_{n+1}^\pm(u) = \frac{(u/v_\mp)^2 - 1}{q^{-2}(u/v_\mp)^2 - q^2} h_{n+1}^\pm(u) \mathcal{X}_n^-(v).
\end{equation}

\begin{equation}
\mathcal{X}_n^-(u) \mathcal{X}_n^+(v) = \frac{q^2(u/v)^2 - q^{-2}}{q^{-2}(u/v)^2 - q^2} \mathcal{X}_n^+(v) \mathcal{X}_n^+(u),
\end{equation}

\begin{equation}
\mathcal{X}_n^-(u) \mathcal{X}_n^-(v) = \frac{q^{-2}(u/v)^2 - q^2}{q^2(u/v)^2 - q^{-2}} \mathcal{X}_n^-(v) \mathcal{X}_n^-(u).
\end{equation}

**Proof.** We only prove (6.12), (6.13), (6.15) and (6.17), as the other relations can be treated similarly.

Relation (6.8) can be rewritten as:

\begin{equation}
\mathcal{L}_2^{\mp[1]}(v)^{-1} \mathcal{R}_{12}^{(1)}(u_\pm/v_\mp) \mathcal{L}_1^{\pm[1]}(u) = \mathcal{L}_1^{\pm[1]}(u) \mathcal{R}_{12}^{(1)}(u_\mp/v_\pm) \mathcal{L}_2^{\mp[1]}(v)^{-1}.
\end{equation}
Thus, we get

\begin{equation}
(6.20) \quad \frac{(u_\pm/v_\pm)^2 - 1}{q^2(u_\mp/v_\mp)^2 - q^2} & \ell_{n,n}^{\pm}(u) = \frac{(u_\pm/v_\pm)^2 - 1}{q^2(u_\mp/v_\mp)^2 - q^2} \ell_{n,n+1}^{\pm}(v)' = \frac{(u_\pm/v_\pm)^2 - 1}{q^2(u_\mp/v_\mp)^2 - q^2} \ell_{n,n+1}^{\pm}(v)' \ell_{n,n}^{\pm}(u),
\end{equation}

which is equivalent to the following form written in Gaussian generators:

\begin{equation}
(6.21) \quad \ell_{n,n}^{\pm}(u) = \ell_{n,n}^{\pm}(v) \ell_{n,n+1}^{\pm}(v) \ell_{n,n+1}^{\pm}(v),
\end{equation}

Thus, by the inveritability of \( \ell_{n,n}^{\pm}(v) \), we have

\begin{equation}
(6.22) \quad \ell_{n,n}^{\pm}(u) = \ell_{n,n+1}^{\pm}(u) \ell_{n,n+1}^{\pm}(u),
\end{equation}

Similarly, we also have

\begin{equation}
(6.23) \quad \ell_{n,n+1}^{\pm}(u) \ell_{n,n+1}^{\pm}(u) = \ell_{n,n+1}^{\pm}(v) \ell_{n,n+1}^{\pm}(v) \ell_{n,n+1}^{\pm}(v),
\end{equation}

Thus, \( (6.13) \) follows by using the definition of \( X^{+}(u) \).

Note that \( (6.8) \) has the following equivalent form:

\begin{equation}
(6.24) \quad \ell_{n,n+1}^{\pm}(u) \ell_{n,n+1}^{\pm}(v) = \ell_{n,n+1}^{\pm}(v) \ell_{n,n+1}^{\pm}(v),
\end{equation}

so we have that

\begin{equation}
(6.25) \quad \ell_{n,n+1}^{\pm}(u) \ell_{n,n+1}^{\pm}(v) = \ell_{n,n+1}^{\pm}(v) \ell_{n,n+1}^{\pm}(v),
\end{equation}

and

\begin{equation}
(6.26) \quad \ell_{n,n+1}^{\pm}(u) \ell_{n,n+1}^{\pm}(v) = \ell_{n,n+1}^{\pm}(v) \ell_{n,n+1}^{\pm}(v),
\end{equation}

which are equivalent to the following Gaussian generator form:

\begin{equation}
(6.27) \quad \ell_{n,n+1}^{\pm}(u) \ell_{n,n+1}^{\pm}(v) = \ell_{n,n+1}^{\pm}(v) \ell_{n,n+1}^{\pm}(v),
\end{equation}

Then we get that

\begin{equation}
(6.28) \quad \ell_{n,n+1}^{\pm}(u) \ell_{n,n+1}^{\pm}(v) = \ell_{n,n+1}^{\pm}(v) \ell_{n,n+1}^{\pm}(v),
\end{equation}

Thus, (6.12) follows by using the Gauss decomposition of \( \mathcal{L}^{\pm}(u) \).
Similarly, we also have
\[ h_{n+1}^+ (u) e_{n,n+1}^+ (v) = \frac{(u/v)^2 - 1}{q^2 (u/v)^2 - q^2} e_{n,n+1}^+ (v) h_{n+1}^+ (u) + \frac{(q^2 - 2)^2 u/v}{q^2 (u/v)^2 - q^2} h_{n+1}^+ (u) e_{n,n+1}^+ (v). \]

Therefore, by the definition of \( \mathcal{X}_n^+ (u) \) we have proved (6.15).

From (6.8), we have \( \ell_{n,n+1}^+ (u) \ell_{n,n+1}^+ (v) = \ell_{n,n+1}^+ (v) \ell_{n,n+1}^+ (u) \). Then using (6.22), we have
\[
\ell_{n,n+1}^+ (u) \ell_{n,n+1}^+ (v) - \frac{q^2 (u/v_±)^2 - q^2}{q^2 (u/v_±)^2 - q^2} \ell_{n,n+1}^± (v) \ell_{n,n+1}^± (u)
= \frac{(q^2 - 2)^2 u/v}{q^2 (u/v_±)^2 - q^2} \ell_{n,n+1}^± (v)^2 - \frac{(q^2 - 2)^2 u/v}{q^2 (u/v_±)^2 - q^2} \ell_{n,n+1}^± (v)^2.
\]

Similarly, we have
\[
\ell_{n,n+1}^± (u) \ell_{n,n+1}^± (v) - \frac{q^2 (u/v)^2 - q^2}{q^2 (u/v)^2 - q^2} \ell_{n,n+1}^± (v) \ell_{n,n+1}^± (u)
= \frac{(q^2 - 2)^2 u/v}{q^2 (u/v)^2 - q^2} \ell_{n,n+1}^± (v)^2 - \frac{(q^2 - 2)^2 u/v}{q^2 (u/v)^2 - q^2} \ell_{n,n+1}^± (v)^2.
\]

Thus, (6.17) follows by the definition of \( \mathcal{X}_n^+ (u) \). \( \square \)

**Proposition 6.4.** In \( U(\mathbb{R}) \), we have
\[
[\mathcal{X}_n^+ (u), \mathcal{X}_n^-(v)] = (q_n - q_n^{-1}) \left\{ \delta ((u_-/v_-)^2 q^2 h_{n+1}^+ (v) h_n^- (v)^{-1} - \delta ((u_+/v_+)^2 q^2 h_{n+1}^+ (u) h_n^- (u)^{-1} \right\}
\]

**Proof.** It follows from (6.8) that
\[
\frac{(u_+/v_-)^2 - 1}{q^2 (u_+/v_-)^2 - q^2} \ell_{n,n+1}^+ (u) \ell_{n,n+1}^+ (v) + \frac{(q^2 - 2)^2 u_+/v_-}{q^2 (u_+/v_-)^2 - q^2} \ell_{n,n}^+ (u) \ell_{n,n+1}^+ (v)
= \frac{(u_-/v_+)^2 - 1}{q^2 (u_-/v_+)^2 - q^2} \ell_{n,n}^+ (v) \ell_{n,n+1}^+ (u) + \frac{(q^2 - 2)^2 u_-/v_+}{q^2 (u_-/v_+)^2 - q^2} \ell_{n,n}^+ (v) \ell_{n,n+1}^+ (u)
\]

The Gauss decomposition of \( \mathcal{L}^{[\pm]} (u) \) implies that the right hand side of (6.25) can be written as
\[
\ell_{n,n+1}^+ (v) \left( \frac{(u_-/v_+)^2 - 1}{q^2 (u_-/v_+)^2 - q^2} \ell_{n,n}^+ (u) \ell_{n,n+1}^+ (v) + \frac{(q^2 - 2)^2 u_-/v_+}{q^2 (u_-/v_+)^2 - q^2} \ell_{n,n}^+ (v) \ell_{n,n+1}^+ (u) \right)
+ \frac{(q^2 - 2)^2 u_-/v_+}{q^2 (u_-/v_+)^2 - q^2} h_{n+1}^+ (v) h_n^+ (u).
\]

Note that
\[
\ell_{n,n+1}^+ (u) \ell_{n,n}^+ (v) = \left( \frac{(u_-/v_+)^2 - 1}{q^2 (u_-/v_+)^2 - q^2} \ell_{n,n}^+ (u) \ell_{n,n+1}^+ (v) + \frac{(q^2 - 2)^2 u_-/v_+}{q^2 (u_-/v_+)^2 - q^2} \ell_{n,n}^+ (v) \ell_{n,n+1}^+ (u) \right),
\]
so the right hand side of (6.25) equals to
\[
\ell_{n,n+1}^+ (v) h_n^+ (u) e_{n,n+1}^+ (u) + \frac{(q^2 - 2)^2 u_-/v_+}{q^2 (u_-/v_+)^2 - q^2} h_{n+1}^+ (v) h_n^+ (u).
\]
and the relations between \( f_{n+1,n}^- (v) \) and \( h_n^+ (u) \)
\[
 f_{n+1,n}^- (v) h_n^+ (u) = \frac{(u_+/v_+)^2 - 1}{q^2(u_+/v_+)^2 - q^{-2}} h_n^+ (u) f_{n+1,n}^- (v) + \frac{(q^2 - q^{-2}) u_+/v_-}{q^2(u_+/v_+)^2 - q^{-2}} f_{n+1,n}^- (u) h_n^+ (u),
\]

brings the right hand side of (6.25) to
\[
\frac{(u_+/v_-)^2 - 1}{q^2(u_+/v_-)^2 - q^{-2}} h_n^+ (u) f_{n+1,n}^- (v) + \frac{(q^2 - q^{-2}) u_+/v_-}{q^2(u_+/v_-)^2 - q^{-2}} f_{n+1,n}^- (u) h_n^+ (v)
\]

Now comparing both sides of (6.25), we get
\[
\frac{(u_+/v_-)^2 - 1}{q^2(u_+/v_-)^2 - q^{-2}} h_{n+1}^+ (u) [e_{n,n+1}^+, f_{n+1,n}^- (v)] h_{n+1}^- (v)
\]

Using (6.12) and the invertibility of \( h_n^+ (u) \), we have
\[
[e_{n,n+1}^+, f_{n+1,n}^- (v)] = \frac{(q^2 - q^{-2}) u_+/v_+ - h_{n+1}^- (v) h_{n+1}^+ (u)}{(u_+/v_+)^2 - 1} h_{n+1}^- (v) h_{n+1}^+ (u) - \frac{(q^2 - q^{-2}) u_+/v_- h_{n+1}^- (u) h_{n+1}^+ (u)}{(u_+/v_-)^2 - 1}
\]

Similarly, we can have
\[
[e_{n,n+1}^+, f_{n+1,n}^- (v)] = \frac{(q^2 - q^{-2}) u_+/v_+ - h_{n+1}^- (v) h_{n+1}^+ (v)}{(u_+/v_+)^2 - 1} h_{n+1}^- (v) h_{n+1}^+ (v) - \frac{(q^2 - q^{-2}) u_+/v_- h_{n+1}^- (u) h_{n+1}^+ (u)}{(u_+/v_-)^2 - 1}
\]

Then using the definition of \( X_n^+ (u) \), we have proved (6.24). \( \square \)

6.3. Formulas of \( j^\pm (z) \). Write the relation (4.11) in the form
\[
D L^\pm (u \xi)^t D^{-1} = L^\pm (u)^{-1} j_n^\pm (u).
\]

Using the Gauss decomposition of \( L^\pm (z) \) and taking the \((2n,2n)\)-th entry on both sides of (6.26), we have
\[
h_{i+1}^\pm (u \xi) = h_i^\pm (u)^{-1} j_n^\pm (u).
\]

**Proposition 6.5.** In \( U( R[n]) \), we have the following relations hold for \( i = 1, \ldots, n \):
\[
e_{(i+1), i+1}^\pm (u) = e_{i,i+1}^\pm (u \xi q^{2i}),
\]

\[
f_{(i+1), i+1}^\pm (u) = f_{i+1,i}^\pm (u \xi q^{2i}).
\]

**Proof.** For \( 1 \leq i \leq n - 1 \), by Proposition 5.2 and 1.1 we have
\[
L^{[n-i+1]^\pm (u)^{-1} j^{[n-i+1]^\pm (u)} = D^{[n-i+1]} L^{[n-i+1]^\pm (u \xi q^{2i-2}) (D^{[n-i+1]})^{-1}},
\]

where
\[
D^{[n-i+1]} = \text{diag}(q^{2i}, \ldots, q^{2n}).
\]
Taking the $(i',i')$-th element on both sides of (6.30), we have

\begin{equation}
\hat{h}^\pm_i(uq^{2i-2}) = h^\pm_i(u)^{-1}z^{-[n-i+1]\pm}(u).
\end{equation}

By taking the $((i+1)',i')$-th entry of (6.30), we obtain

\begin{equation}
-\epsilon^\pm_{(i+1)',i}(u)\hat{h}^\pm_i(u)^{-1}z^{-[n-i+1]\pm}(u) = q\hat{h}^\pm_i(uq^{2i-2})\epsilon^\pm_{i,i+1}(uq^{2i-2}).
\end{equation}

Using (6.31), we have

\begin{equation}
-\epsilon^\pm_{(i+1)',i}(u)\hat{h}^\pm_i(uq^{2i-2}) = q\hat{h}^\pm_i(uq^{2i-2})\epsilon^\pm_{i,i+1}(uq^{2i-2}).
\end{equation}

Furthermore, we have

\begin{equation}
q\hat{h}^\pm_i(u)\epsilon^\pm_{i,i+1}(u) = \epsilon^\pm_{i,i+1}(uq^{2i})\hat{h}^\pm_i(u),
\end{equation}

which is the result of \[10\].

Thus, the equation (6.32) is equivalent to

\begin{equation}
\epsilon^\pm_{(i+1)',i}(u)\hat{h}^\pm_i(uq^{2i-2}) = \epsilon^\pm_{i,i+1}(uq^{2i})\hat{h}^\pm_i(uq^{2i-2}).
\end{equation}

The invertibility of $\hat{h}^\pm_i(uq^{2i-2})$ proves (6.28). Other relations are proved similarly. \[\square\]

**Proposition 6.6.** In $U(\overline{R}^n)$, we have that

\begin{equation}
\epsilon^\pm_{(n+1)',n}(u) = -\epsilon^\pm_{n,n+1}(-u),
\end{equation}

\begin{equation}
f^\pm_{n'(n+1)'}(u) = -f^\pm_{n+1,n}(-u),
\end{equation}

Thus, we have

\begin{equation}
\lambda^\pm_{n,2m+1} = 0.
\end{equation}

Now we are in a position to give an explicit formula of $\bar{z}^{[n]\pm}(z)$ in terms of $\bar{h}^\pm_i(z)$.

**Proposition 6.7.** In $U(\overline{R}^n)$, we have

\begin{equation}
z^{[n]\pm}(u) = \prod_{i=1}^{n-1} h^\pm_i(uq^{2i})^{-1}\prod_{i=1}^{n} h^\pm_i(uq^{2i-2})h^\pm_{n+1}(u).
\end{equation}

**Proof.** Taking the $(2',2')$-th entry on both sides of (6.30) and expressing the entries of $L^{[n]\pm}(u)$ in terms of the Gauss generators, we have

\begin{equation}
h^\pm_{2'}(uq) + f^\pm_{21}(uq)h^\pm_1(uq)e^\pm_{12}(uq) = (h^\pm_2(u)^{-1} + \epsilon^\pm_{21,1'}(u)h^\pm_1(u)^{-1}f^\pm_{12,2'}(u))z^{[n]\pm}(u).
\end{equation}

Since $z^{[n]\pm}(u)$ are central in the subalgebras generated by $L^{[n]\pm}(z)$, using (6.27), we can rewrite the above equation as

\begin{equation}
h^\pm_{2'}(u^{-1}z^{[n]\pm}(u)) = h^\pm_2(uq) + f^\pm_{21}(uq)h^\pm_1(uq)e^\pm_{12}(uq) - \epsilon^\pm_{21,1'}(u)h^\pm_1(uq)f^\pm_{12,2'}(u).
\end{equation}

Now apply Proposition 6.5 to obtain

\begin{equation}
h^\pm_{2'}(u^{-1}z^{[n]\pm}(u)) = h^\pm_2(uq) + f^\pm_{21}(uq)h^\pm_1(uq)e^\pm_{12}(uq) - \epsilon^\pm_{12}(uq^2)h^\pm_1(uq)f^\pm_{21}(uq).
\end{equation}

Since

\begin{equation}
h^\pm_1(u)e^\pm_{12}(u) = q^{-1}e^\pm_{12}(uq^2)h^\pm_1(u), \ h^\pm_1(u)f^\pm_{21}(uq^2) = q^{-1}f^\pm_{21}(uq)h^\pm_1(u)
\end{equation}

by the results of \[10\], we have

\begin{equation}
h^\pm_{2'}(u^{-1}z^{[n]\pm}(u)) = h^\pm_2(uq) - q^{-1}[e^\pm_{12}(uq^2), f^\pm_{21}(uq)]h^\pm_1(uq).
\end{equation}
Again using the results of \[10\]
\[
\left[ x_{12}^\pm(u), f_{21}^\pm(v) \right] = \frac{u(q-q^{-1})}{u-v}(h_2^\pm(v)h_1^\pm(v)^{-1} - h_2^\pm(u)h_1^\pm(u)^{-1}),
\]
we get
\[
h_2^\pm(u)^{-1} \zeta^{[n]}(u) = h_2^\pm(u\xi q^2)h_1^\pm(u\xi q^2)^{-1} h_1^\pm(u\xi).
\]
Since \( \zeta^{[n-1]}(u) = h_2^\pm(u)h_2^\pm(u\xi q^2) \), we get a recurrence relation
\[
\zeta^{[n]}(u) = h_1^\pm(u\xi q^2)^{-1} h_1^\pm(u\xi)\zeta^{[n-1]}(u).
\]
Then inductively we prove the proposition.

Similarly, we can get the same formula for \( \zeta^\pm(u) \in U(R^{[n]}) \).

**Proposition 6.8.** In \( U(R^{[n]}) \), we have
\[
\zeta^{[n]}(u) = \prod_{i=1}^{n-1} h_i^\pm(u\xi q^{2i})^{-1} \prod_{i=1}^{n} h_i^\pm(u\xi q^{2i-2})h_{n+1}^\pm(u).
\]

### 6.4. Other relations.

**Proposition 6.9.** In \( U(R^{[n]}) \), we have
\[
h_i^\pm(u)h_{n+1}^\pm(v) = h_{n+1}^\pm(v)h_i^\pm(u), \ i = 1, 2, \ldots, n,
\]
\[
\frac{u_\pm/v_\mp - 1}{q u_\mp/v_\pm - q^{-1}} h_i^\pm(u)h_{n+1}^\pm(v) = \frac{u_\mp/v_\pm - 1}{q u_\pm/v_\mp - q^{-1}} h_{n+1}^\pm(v)h_i^\pm(u), \ i = 1, 2, \ldots, n - 1,
\]
\[
h_{n+1}^\pm(u)\mathcal{X}_j^+(v) = \mathcal{X}_j^-(v)h_{n+1}^\pm(u), \ j \leq n - 2,
\]
\[
h_{n+1}^\pm(u)\mathcal{X}_j^-(v) = \mathcal{X}_j^+(v)h_{n+1}^\pm(u), \ j \leq n - 2.
\]
\[
h_{n+1}^\pm(u)^{-1}\mathcal{X}_{n-1}^+(v)h_{n+1}^\pm(u) = \frac{u/v_\pm + 1}{q^{-1}u/v_\pm + q}\mathcal{X}_{n-1}^+(v)
\]
\[
h_{n+1}^\pm(u)\mathcal{X}_{n-1}^-(v)h_{n+1}^\pm(u)^{-1} = \frac{u/v_\pm + 1}{q^{-1}u/v_\pm + q}\mathcal{X}_{n-1}^-(v)
\]
\[
h_i^\pm(u)\mathcal{X}_i^+(v) = \mathcal{X}_i^+(v)h_i^\pm(u), \ i \leq n - 1
\]
\[
h_i^\pm(u)\mathcal{X}_i^-(v) = \mathcal{X}_i^+(v)h_i^\pm(u), \ j \leq n - 1,
\]
\[
\mathcal{X}_i^+(u)\mathcal{X}_i^-(v) = \mathcal{X}_i^+(v)\mathcal{X}_i^-(u), \ i < n - 1,
\]
Proof. We only check (6.41)–(6.46) and (6.48), as the others are trivial or can be treated similarly.

It follows from Corollary 4.13 that
\[ \frac{u_{\pm}/v_{\mp}}{qu_{\pm}/v_{\pm} - q^{-1}} b_{n+1}^{\pm}(u) \left( b_{n+1}^{\mp}(v) + f_{n+1,n}(v)b_{n}^{\mp}(v) e_{n,n+1}^{\mp}(v) \right) = \]
\[ \frac{u_{\pm}/v_{\mp}}{qu_{\pm}/v_{\pm} - q^{-1}} b_{n+1}^{\pm}(v) \left( f_{n+1,n}(v)b_{n}^{\mp}(v) e_{n,n+1}^{\mp}(v) \right) b_{n+1}^{\pm}(u), \]
and
\[ \frac{u_{\pm}/v_{\mp}}{qu_{\pm}/v_{\pm} - q^{-1}} b_{i}^{\pm}(u) f_{n+1,n}^{\mp}(v)b_{n}^{\mp}(v) = \frac{u_{\pm}/v_{\mp}}{qu_{\pm}/v_{\pm} - q^{-1}} f_{n+1,n}^{\mp}(v)b_{n}^{\mp}(v) b_{i}^{\pm}(u), \]
for \( i = 1, \ldots, n - 1 \). Thus, the left hand side of (6.49) equals to
\[ \frac{u_{\pm}/v_{\mp}}{qu_{\pm}/v_{\pm} - q^{-1}} b_{n+1}^{\pm}(u) b_{n}^{\mp}(v) + \frac{u_{\pm}/v_{\mp}}{qu_{\pm}/v_{\pm} - q^{-1}} f_{n+1,n}^{\mp}(v)b_{n}^{\mp}(v) b_{i}^{\pm}(u), \]
Again by Corollary 4.13 it follows that
\[ \frac{u_{\pm}/v_{\mp}}{qu_{\pm}/v_{\pm} - q^{-1}} b_{i}^{\pm}(u) b_{n}^{\mp}(v) = \frac{u_{\pm}/v_{\mp}}{qu_{\pm}/v_{\pm} - q^{-1}} b_{n}^{\mp}(v) b_{i}^{\pm}(u), \quad i = 1, 2, \ldots, n - 1. \]
Then the left hand side of (6.49) can be written as
\[ \frac{u_{\pm}/v_{\mp}}{qu_{\pm}/v_{\pm} - q^{-1}} b_{i}^{\pm}(u) b_{n+1}^{\pm}(v) + \frac{u_{\pm}/v_{\mp}}{qu_{\pm}/v_{\pm} - q^{-1}} f_{n+1,n}^{\mp}(v)b_{n}^{\mp}(v) e_{n,n+1}^{\mp}(v) \]
Also by Corollary 4.13 we have that
\[ \frac{u_{\pm}/v_{\mp}}{qu_{\pm}/v_{\pm} - q^{-1}} b_{i}^{\pm}(u) b_{n}^{\mp}(v) e_{n,n+1}^{\mp}(v) = \frac{u_{\pm}/v_{\mp}}{qu_{\pm}/v_{\pm} - q^{-1}} b_{n}^{\mp}(v) e_{n,n+1}^{\mp}(v) b_{i}^{\pm}(u), \quad i = 1, 2, \ldots, n - 1, \]
Finally, we get the left hand side of (6.49) as the following form:
\[ \frac{u_{\pm}/v_{\mp}}{qu_{\pm}/v_{\pm} - q^{-1}} b_{i}^{\pm}(u) b_{n+1}^{\pm}(v) + \frac{u_{\pm}/v_{\mp}}{qu_{\pm}/v_{\pm} - q^{-1}} f_{n+1,n}^{\mp}(v)b_{n}^{\mp}(v) e_{n,n+1}^{\mp}(v) b_{i}^{\pm}(u) \]
which then proves (6.41).

It follows from Remark 6.2 and 10 that
\[ b_{n}^{\pm}(u) e_{\nu_{\nu',(i-1)'}(v)}^{\pm} = e_{\nu_{\nu',(i-1)'}(v)}^{\pm} b_{n}^{\pm}(u), \quad b_{n}^{\pm}(u) e_{\nu_{\nu',(i-1)'}(v)}^{\pm} = e_{\nu_{\nu',(i-1)'}(v)}^{\pm} b_{n}^{\pm}(u), \]
\[ b_{n}^{\pm}(u)^{-1} e_{\nu_{\nu',(n-1)'}(v)}^{\pm} b_{n}^{\pm}(u) = \frac{qu/v - q^{-1}}{u/v - 1} e_{\nu_{\nu',(n-1)'}(v)}^{\pm} - \frac{q - q^{-1}}{u/v - 1} e_{\nu_{\nu',(n-1)'}(v)}^{\pm}, \]
\[ b_{n}^{\pm}(u)^{-1} e_{\nu_{\nu',(n-1)'}(v)}^{\pm} b_{n}^{\pm}(u) = \frac{qu/v - q^{-1}}{u/v - 1} e_{\nu_{\nu',(n-1)'}(v)}^{\pm} - \frac{q - q^{-1}}{u/v - 1} e_{\nu_{\nu',(n-1)'}(v)}^{\pm}, \]
They can be rewritten as follows (using Prop. 6.5):
\[ b_{n}^{\pm}(u) e_{\nu_{\nu',(i-1)'}(v)}^{\pm} = e_{\nu_{\nu',(i-1)'}(v)}^{\pm} b_{n}^{\pm}(u), \quad b_{n}^{\pm}(u) e_{\nu_{\nu',(i-1)'}(v)}^{\pm} = e_{\nu_{\nu',(i-1)'}(v)}^{\pm} b_{n}^{\pm}(u), \]
\[ b_{n}^{\pm}(u)^{-1} e_{\nu_{\nu',(n-1)'}(v)}^{\pm} b_{n}^{\pm}(u) = \frac{qu/v - q^{-1}}{u/v - 1} e_{\nu_{\nu',(n-1)'}(v)}^{\pm} - \frac{q - q^{-1}}{u/v - 1} e_{\nu_{\nu',(n-1)'}(v)}^{\pm}, \]
\[ b_{n}^{\pm}(u)^{-1} e_{\nu_{\nu',(n-1)'}(v)}^{\pm} b_{n}^{\pm}(u) = \frac{qu/v - q^{-1}}{u/v - 1} e_{\nu_{\nu',(n-1)'}(v)}^{\pm} - \frac{q - q^{-1}}{u/v - 1} e_{\nu_{\nu',(n-1)'}(v)}^{\pm}. \]
Furthermore, Corollary 4.13 also gives that

\[
\begin{align*}
\mathbf{b}_{n'}^±(u)^{-1}e_{n-1,n}^±(v)\mathbf{b}_{n'}^±(u) &= \frac{q u/v - q^{-1}}{u/v - 1}e_{n-1,n}^±(v) - \frac{q - q^{-1}}{u/v - 1}e_{n-1,n}^±(u), \\
\mathbf{b}_{n}^±(u)^{-1}e_{n-1,n}^±(v)\mathbf{b}_{n}^±(u) &= \frac{q u/v - q^{-1}}{u/v - 1}e_{n-1,n}^±(v) - \frac{q - q^{-1}}{u/v - 1}e_{n-1,n}^±(u),
\end{align*}
\]

Then (6.42) - (6.45) follow by using the definitions of \( \mathcal{X}_i^±(u) \).

It follows from Corollary 4.13 that for \( i < n \)

\[
\begin{align*}
\frac{u_±/v_± - 1}{q u_±/v_± - q - 1}b_1^±(u)b_1^±(v)c_{n,n+1}^±(v) &= \frac{u_±/v_± - 1}{q u_±/v_± - q - 1}b_1^±(v)c_{n,n+1}^±(v)b_1^±(u).
\end{align*}
\]

Furthermore, Corollary 4.13 also gives that

\[
\begin{align*}
\frac{u_±/v_± - 1}{q u_±/v_± - q - 1}b_i^±(u)b_1^±(v) &= \frac{u_±/v_± - 1}{q u_±/v_± - q - 1}b_1^±(v)b_i^±(u).
\end{align*}
\]

Thus, we get

\[
\begin{align*}
b_i^±(u)c_{n,n+1}^±(v) &= c_{n,n+1}^±(v)b_i^±(u).
\end{align*}
\]

Similarly, we have \( b_i^±(u)c_{n,n+1}^±(v) = c_{n,n+1}^±(v)b_i^±(u) \). Then by the definition of \( \mathcal{X}_i^±(u) \) we can prove (6.46).

Next by Corollary 4.13 it follows that for \( i < n - 1 \)

\[
\begin{align*}
\frac{u_±/v_± - 1}{q u_±/v_± - q - 1}b_i^±(u)c_{i,i+1}^±(v)b_i^±(v) &= \frac{u_±/v_± - 1}{q u_±/v_± - q - 1}b_i^±(v)c_{i,i+1}^±(v)b_i^±(u)c_{i,i+1}^±(v),
\end{align*}
\]

(6.60)

\[
\begin{align*}
\frac{u_±/v_± - 1}{q u_±/v_± - q - 1}b_i^±(u)c_{i,i+1}^±(v)b_i^±(v) &= \frac{u_±/v_± - 1}{q u_±/v_± - q - 1}b_i^±(v)b_i^±(u)c_{i,i+1}^±(v).
\end{align*}
\]

Then (6.59) can be written as

\[
\begin{align*}
b_i^±(v)b_i^±(u)c_{i,i+1}^±(v)c_{n,n+1}^±(v) &= b_i^±(v)c_{n,n+1}^±(v)b_i^±(u)c_{i,i+1}^±(v).
\end{align*}
\]

Using the above equation and the fact \([b_i^±(u), c_{n,n+1}^±(v)] = 0 \) (see [10]), we have

\[
\begin{align*}
e_{i,i+1}^±(u)c_{n,n+1}^±(v) &= c_{n,n+1}^±(v)e_{i,i+1}^±(u).
\end{align*}
\]

Similarly, we have

\[
\begin{align*}
e_{i,i+1}^±(v)c_{n,n+1}^±(v) &= c_{n,n+1}^±(v)e_{i,i+1}^±(u).
\end{align*}
\]

Using these, we get \( \mathcal{X}_i^±(u)\mathcal{X}_n^±(v) = \mathcal{X}_i^±(v)\mathcal{X}_n^±(u) \) easily. Similarly, we can get \( \mathcal{X}_i^-(u)\mathcal{X}_n^-(v) = \mathcal{X}_i^-(v)\mathcal{X}_n^-(u) \).
Proposition 6.10. In $U(T^{[a]})$, we have the following relation hold:

\[(u^2q^2 - v^2)\mathcal{X}^+_{n-1}(uq^{-1})\mathcal{X}^+_{n}(vq^n) = (u^2 - q^2v^2)\mathcal{X}^+_{n}(vq^n)\mathcal{X}^+_{n-1}(uq^{-1}).\]

\[(u^2q^2 - v^2)^{-1}\mathcal{X}^-_{n-1}(uq^{-1})\mathcal{X}^-_{n}(vq^n) = (u^2 - q^2v^2)^{-1}\mathcal{X}^-_{n}(vq^n)\mathcal{X}^-_{n-1}(uq^{-1}).\]

Proof. Here we only prove (6.61), as (6.62) can be treated similarly.

It follows from Corollary [5.3] that

\[
\frac{u_\pm/v_\pm - 1}{qu_\pm/v_\pm - q^{-1}}\ell^+_{n-1,n}(u)\ell^+_{n,n+1}(v) + \frac{(q - q^{-1})u_\pm/v_\pm - 1}{qu_\pm/v_\pm - q^{-1}}\ell^+_{n,n+1}(u)\ell^+_{n-1,n+1}(v) = 0.
\]

\[
\frac{u_\pm/v_\pm - 1}{qu_\pm/v_\pm - q^{-1}}\ell^+_{n,n+1}(v)\ell^+_{n-1,n}(u) + \frac{(q - q^{-1})u_\pm/v_\pm - 1}{qu_\pm/v_\pm - q^{-1}}\ell^+_{n-1,n}(v)\ell^+_{n,n+1}(u) = 0.
\]

Plug the Gauss decomposition of $\ell^+_{n,n+1}(v)$ into the left-hand side of (6.63):

\[
\frac{u_\pm/v_\pm - 1}{qu_\pm/v_\pm - q^{-1}}\ell^+_{n-1,n}(u)\ell^+_{n,n+1}(v) + \frac{u_\pm/v_\pm - 1}{qu_\pm/v_\pm - q^{-1}}\ell^+_{n,n+1}(u)\ell^+_{n-1,n+1}(v)
+ \frac{(q - q^{-1})u_\pm/v_\pm - 1}{qu_\pm/v_\pm - q^{-1}}\ell^+_{n,n+1}(v)\ell^+_{n-1,n+1}(u) = 0.
\]

Recalling the the commuting relations between $\ell^+_{n,n-1}(u)$ and $\ell^+_{n,n-1}(v)$:

\[
\frac{u_\pm/v_\pm - 1}{qu_\pm/v_\pm - q^{-1}}\ell^+_{n,n-1}(u)\ell^+_{n,n+1}(v) + \frac{(q - q^{-1})u_\pm/v_\pm - 1}{qu_\pm/v_\pm - q^{-1}}\ell^+_{n,n+1}(u)\ell^+_{n-1,n+1}(v) = 0.
\]

we see the left hand side of (6.63) is equal to

\[
\frac{u_\pm/v_\pm - 1}{qu_\pm/v_\pm - q^{-1}}\ell^+_{n-1,n}(u)\ell^+_{n,n+1}(v) + \frac{u_\pm/v_\pm - 1}{qu_\pm/v_\pm - q^{-1}}\ell^+_{n,n+1}(u)\ell^+_{n-1,n+1}(v)
+ \frac{(q - q^{-1})u_\pm/v_\pm - 1}{qu_\pm/v_\pm - q^{-1}}\ell^+_{n,n+1}(v)\ell^+_{n-1,n+1}(u) = 0.
\]

which is equivalent to the following with the help of the Gauss decomposition of $L^\pm_{[2]}(u)$:

\[
\frac{u_\pm/v_\pm - 1}{qu_\pm/v_\pm - q^{-1}}\ell^+_{n-1,n}(u)\ell^+_{n,n+1}(v) + \frac{u_\pm/v_\pm - 1}{qu_\pm/v_\pm - q^{-1}}\ell^+_{n,n+1}(u)\ell^+_{n-1,n+1}(v)
+ \frac{(q - q^{-1})u_\pm/v_\pm - 1}{qu_\pm/v_\pm - q^{-1}}\ell^+_{n,n+1}(v)\ell^+_{n-1,n+1}(u) = 0.
\]
Then using the relations between $\ell_{n-1,n}^{\pm}(u)$ and $\ell_{n-1,n}^{\mp}(v)$:

$$
\ell_{n-1,n}^{\pm}(u)\ell_{n-1,n}^{\mp}(v) = \frac{u_\pm - v_\pm}{qu_\pm - q^{-1}v_\pm}\ell_{n-1,1}^{\pm}(v)\ell_{n-1,1}^{\mp}(u) \pm \frac{(q - q^{-1})u_\pm/v_\pm}{qu_\pm/v_\pm - q^{-1}}\ell_{1,n}^{\pm}(v)\ell_{1,n}^{\mp}(u),
$$

we can get the left hand side of (6.63) equals to

$$
\frac{u_\pm/v_\pm - 1}{qu_\pm/v_\mp - q^{-1}}\ell_{n-1,n}^{\pm}(u)\ell_{n-1,n}^{\mp}(v)e_{n,n+1}^{\mp}(v) + \frac{(q - q^{-1})u_\pm/v_\pm}{qu_\pm/v_\mp - q^{-1}}\ell_{n-1,n}^{\pm}(u)\ell_{n-1,n}^{\mp}(v)
$$

+ \ell_{n-1,n}^{\mp}(v)e_{n,n+1}^{\pm}(u)\ell_{n-1,n}^{\mp}(v).
$$

Next by the relations between $\ell_{n-1,n}^{\pm}(u)$ and $\ell_{n-1,n}^{\mp}(v)$:

$$
\ell_{n-1,n}^{\pm}(u)\ell_{n-1,n}^{\mp}(v) = \frac{(q - q^{-1})q^{-3}(u_\pm/v_\pm - 1)}{(qu_\pm/v_\pm - q^{-1})(u_\pm/v_\pm + q^{-1})}\ell_{n-1,n}^{\pm}(v)\ell_{n-1,n}^{\mp}(u) + \frac{(q - q^{-1})(q^{-4} + 1)u_\pm/v_\pm}{(qu_\pm/v_\pm - q^{-1})(u_\pm/v_\pm + q^{-3})}\ell_{n-1,n}^{\pm}(v)\ell_{n-1,n+1}(u)
$$

+ \frac{q^{-2}(u_\pm/v_\pm - 1)(qu_\pm/v_\pm + q^{-1})}{(qu_\pm/v_\pm - q^{-1})(u_\pm/v_\pm + q^{-4})}\ell_{n-1,n}^{\pm}(v)\ell_{n-1,n}^{\mp}(u) - \frac{q^{-1}(q - q^{-1})(u_\pm/v_\pm - 1)u_\pm/v_\pm}{(qu_\pm/v_\mp - q^{-1})(u_\pm/v_\pm + q^{-4})}\ell_{n-1,n}^{\pm}(v)\ell_{n-1,n-1}(u),
$$

the equation (6.63) is seen as the following:

$$
\frac{u_\pm/v_\pm - 1}{qu_\pm/v_\mp - q^{-1}}\ell_{n-1,n}^{\pm}(u)\ell_{n-1,n}^{\mp}(v)e_{n,n+1}^{\mp}(v) + \frac{(q - q^{-1})u_\pm/v_\pm}{qu_\pm/v_\mp - q^{-1}}\ell_{n-1,n}^{\pm}(v)\ell_{n-1,n}^{\mp}(u)
$$

+ \frac{(q - q^{-1})q^{-3}(u_\pm/v_\pm - 1)}{(qu_\pm/v_\pm - q^{-1})(u_\pm/v_\pm + q^{-1})}\ell_{n-1,n}^{\pm}(v)\ell_{n-1,n}^{\mp}(u) + \frac{(q - q^{-1})(q^{-4} + 1)u_\pm/v_\pm}{(qu_\pm/v_\pm - q^{-1})(u_\pm/v_\pm + q^{-3})}\ell_{n-1,n}^{\pm}(v)\ell_{n-1,n+1}(u)
$$

+ \frac{q^{-2}(u_\pm/v_\pm - 1)(qu_\pm/v_\pm + q^{-1})}{(qu_\pm/v_\pm - q^{-1})(u_\pm/v_\pm + q^{-4})}\ell_{n-1,n}^{\pm}(v)\ell_{n-1,n}^{\mp}(u) - \frac{q^{-1}(q - q^{-1})(u_\pm/v_\pm - 1)u_\pm/v_\pm}{(qu_\pm/v_\mp - q^{-1})(u_\pm/v_\pm + q^{-4})}\ell_{n-1,n}^{\pm}(v)\ell_{n-1,n-1}(u),
$$

By (5.7), we derive the following relations between $e_{n-1,n}^{\pm}(u)$ and $h_{1}^{\mp}(v)$:

$$
e_{n-1,n}^{\pm}(u)h_{1}^{\mp}(v) = \frac{qu_\pm/v_\pm - q^{-1}}{u_\pm/v_\pm - 1}h_{1}^{\mp}(v)\ell_{n-1,n}^{\pm}(u) - \frac{(q - q^{-1})u_\pm/v_\pm}{u_\pm/v_\pm - 1}h_{1}^{\mp}(v)e_{n-1,n}^{\mp}(v)
$$

Furthermore by the relation between $h_{n-1}^{\pm}(u)$ and $h_{1}^{\mp}(v)$:

$$
\frac{u_\pm/v_\pm - 1}{qu_\pm/v_\mp - q^{-1}}h_{n-1}^{\pm}(u)h_{1}^{\mp}(v) = \frac{u_\pm/v_\pm - 1}{qu_\pm/v_\mp - q^{-1}}h_{1}^{\mp}(v)h_{n-1}^{\pm}(u)
$$

we have

$$
\frac{u_\pm/v_\pm - 1}{qu_\pm/v_\mp - q^{-1}}\ell_{n-1,n}^{\pm}(u)h_{1}^{\mp}(v) = h_{1}^{\mp}(v)h_{n-1}^{\pm}(u)\left(e_{n-1,n}^{\pm}(u) - \frac{(q - q^{-1})u_\pm/v_\pm}{qu_\pm/v_\pm - q^{-1}}e_{n-1,n}^{\mp}(v)\right).
$$
Applying (6.67) to (6.66), we get
\[ (6.69) \]
Since we also have the following relation
\[ (6.68) \]
By the similar process, we have the relation between \( h_{n-1}(u) \) and \( e_{n,n+2}(v) \):

\[ (6.67) \]
Applying (6.67) to (6.69), we get
\[ (6.68) \]
Similarly, we obtain that
\[ (6.69) \]
Then we can prove (6.61). □

The following follows immediately from Proposition 6.10 and (6.5).

**Proposition 6.11.** In $U(R)$, we have

$$
\sum_{\sigma \in S_2} \sigma \left( (q^2 u_1 + u_2) (\mathcal{X}_n^\pm(v) \mathcal{X}_{n-1}^\pm(u_1) \mathcal{X}_{n-1}^\pm(u_2) \right)
\right) = 0
$$

(6.70)

**Proposition 6.12.** In $U(R^{[n]})$, we have that

$$
[\mathcal{X}_i^+, \mathcal{X}_n^-] = 0, \text{ for } i < n.
$$

(6.71)

**Proof.** For $i \leq n - 2$, it follows from Corollary 4.13 that

$$
u = \frac{u_\pm - v_\pm - q}{qu_\pm/v_\pm - q_n^{-1}} h_i^\pm(u) c_{i,i+1}^\pm(u) f_{n+1,n}^\pm(v) h_{n}^\pm(v) = \frac{u_\pm - v_\pm - q}{qu_\pm/v_\pm - q_n^{-1}} f_{n+1,n}^\pm(v) h_{n}^\pm(v) c_{i,i+1}^\pm(u).
$$

Again by Corollary 4.13 we have

$$
u = \frac{u_\pm - v_\pm - q}{qu_\pm/v_\pm - q_n^{-1}} f_{n+1,n}^\pm(v) h_{n}^\pm(v) c_{i,i+1}^\pm(u).
$$

Therefore

$$
h_i^\pm(u) c_{i,i+1}^\pm(u) f_{n+1,n}^\pm(v) h_{n}^\pm(v) = h_i^\pm(u) f_{n+1,n}^\pm(v) h_{n}^\pm(v) c_{i,i+1}^\pm(u).
$$

Furthermore, since $[h_i^\pm(v), c_{i,i+1}^\pm(u)] = 0$ (see result in [10]), we can prove

(6.72)

$$
[c_{i,i+1}^\pm(u), f_{n+1,n}^\pm(v)] = 0,
$$

for $i = 1, \ldots, n - 2$. Similarly, we have

(6.73)

$$
[c_{i,i+1}^\pm(u), f_{n+1,n}^\pm(v)] = 0,
$$

for $i = 1, \ldots, n - 2$. Thus, for $i = 1, \ldots, n - 2$ we have $[\mathcal{X}_i^+, \mathcal{X}_n^-] = 0$.

For $i = n - 1$, by (6.7) we have

$$
c_{i,n-1,n}^\pm(u) f_{n+1,n}^\pm(v) h_{n}^\pm(v) c_{i-1,n-1,n}^\pm(u) = \frac{qu_\pm/v_\pm - q_n^{-1}}{u_\pm/v_\pm - q_n^{-1}} f_{n+1,n}^\pm(v) h_{n}^\pm(v) c_{i-1,n-1,n}^\pm(u)
$$

and

$$
c_{i,n-1,n}^\pm(u) h_{n}^\pm(v) = \frac{qu_\pm/v_\pm - q_n^{-1}}{u_\pm/v_\pm - q_n^{-1}} h_{n}^\pm(v) c_{i-1,n-1,n}^\pm(u) - \frac{qu_\pm/v_\pm - q_n^{-1}}{u_\pm/v_\pm - q_n^{-1}} h_{n}^\pm(v) c_{i-1,n-1,n}^\pm(v).
$$

Then, by the invertibility of $h_i^\pm(v)$ we can prove $[c_{i,n-1,n}^\pm(u), f_{n+1,n}^\pm(v)] = 0$. Similarly, we have $[c_{i,n-1,n}^\pm(u), f_{n+1,n}^\pm(v)] = 0$. Thus we have $[\mathcal{X}_{n-1}^+, \mathcal{X}_n^-] = 0$. □

**Theorem 6.13.** The following relations between the Gaussian generators hold in the algebra $U(R^{[n]})$. For the relations involving $h_i^\pm(u)$ we have

(6.74)

$$
h_i^\pm(u) h_j^\pm(v) = h_j^\pm(v) h_i^\pm(u), \quad h_i^\pm(u) h_j^\pm(v) = h_j^\pm(v) h_i^\pm(u);
$$

(6.75)

$$
\frac{u_\pm - v_\pm}{qu_\pm - q_n^{-1}v_\pm} h_i^\pm(u) h_j^\pm(v) = \frac{u_\pm - v_\pm}{qu_\pm - q_n^{-1}v_\pm} h_j^\pm(u) h_i^\pm(v), \quad i < j, i \neq n;
$$

for $i, j = 1, \ldots, n$.
\[(6.76) \quad \frac{(u_\pm/v_\pm)^2 - 1}{q^2(u_\pm/v_\pm)^2 - q^{-2}} b_{n+1}^\pm(u) b_{n+1}^\mp(v) = \frac{(u_\pm/v_\pm)^2 - 1}{q^2(u_\pm/v_\pm)^2 - q^{-2}} h_{n+1}^\pm(v) h_{n+1}^\mp(u).\]

The relations involving \(b_{i+}^\pm(u)\) and \(X_{j+}^\pm(v)\) are

\[
b_{i+}^\pm(u) X_{j+}^\pm(v) b_{i+}^\pm(u)^{-1} = \frac{u/v_\pm - 1}{q^{(\epsilon, \alpha_j)}(u/v_\pm - q^{-1})} X_{j+}^\pm(v),
\]

\[
b_{i+}^\pm(u)^{-1} X_{j+}^\mp(v) b_{i+}^\pm(u) = \frac{u/v_\pm - 1}{q^{(\epsilon, \alpha_j)}(u/v_\pm - q^{-1})} X_{j+}^\mp(v),
\]

for \(i = 1, \ldots, n, \ j = 1, \ldots, n - 1\) together with

\[
b_{i+}^\pm(u) X_{n+}^\pm(v) b_{i+}^\pm(u)^{-1} = \frac{(u/v_\pm)^2 - 1}{q^{(\epsilon, \alpha_j)}(u/v_\pm)^2 - q^{-1}} X_{n+}^\pm(v),
\]

\[
b_{i+}^\pm(u)^{-1} X_{n+}^\mp(v) b_{i+}^\pm(u) = \frac{(u/v_\pm)^2 - 1}{q^{(\epsilon, \alpha_j)}(u/v_\pm)^2 - q^{-1}} X_{n+}^\mp(v)
\]

for \(i = 1, \ldots, n\) and

\[
b_{n+1}^\pm(u) X_{n+}^\pm(v) b_{n+1}^\pm(u)^{-1} = \frac{(u/v_\pm)^2 - 1}{q^{-2}(u/v_\pm)^2 - q^{-2}} X_{n+}^\pm(v),
\]

\[
b_{n+1}^\pm(u)^{-1} X_{n+}^\mp(v) b_{n+1}^\pm(u) = \frac{(u/v_\pm)^2 - 1}{q^{-2}(u/v_\pm)^2 - q^{-2}} X_{n+}^\mp(v)
\]

and

\[
b_{n+1}^\pm(u) X_{n-1}^\pm(v) b_{n+1}^\pm(u)^{-1} = \frac{q^{-1}u/v_\pm + q}{u/v_\pm + 1} X_{n-1}^\pm(v),
\]

\[
b_{n+1}^\pm(u)^{-1} X_{n-1}^\mp(v) b_{n+1}^\pm(u) = \frac{q^{-1}u/v_\pm + q}{u/v_\pm + 1} X_{n-1}^\mp(v)
\]

while

\[
b_{n+1}^\pm(u) X_{i+}^\pm(v) = X_{i+}^\pm(v) b_{n+1}^\pm(u),
\]

\[
b_{n+1}^\pm(u) X_{i-}^\pm(v) = X_{i-}^\pm(v) b_{n+1}^\pm(u),
\]

for \(1 \leq i \leq n - 2\). For the relations involving \(X_{i+}^\pm(u)\) we have

\[(u - q^{(\alpha_i, \alpha_j)}v) X_{i+}^\pm(uq^i) X_{j+}^\pm(vq^j) = (q^{(\alpha_i, \alpha_j)}u - v) X_{j+}^\pm(vq^j) X_{i+}^\pm(uq^i)\]

for \(i, j = 1, \ldots, n - 1\);

\[(u^2 - q^{(\alpha_i, \alpha_i)}v^2) X_{i+}^\pm(uq^i) X_{n+}^\pm(vq^n) = (q^{(\alpha_i, \alpha_i)}u^2 - v^2) X_{n+}^\pm(vq^n) X_{i+}^\pm(uq^i)\]

for \(i = 1, \ldots, n\) and

\[
[X_{i+}^\pm(u), X_{j-}^\pm(v)] = \delta_{ij}(q_i - q_n^{-1}) \left( \delta(uq^{-c}/v) h_{i-1}^\pm(v_+)^{-1} h_{i+1}^\pm(v_+) - \delta(uq^{c}/v) h_{i+1}^\pm(u_+)^{-1} h_{i+1}^\pm(u_+) \right)
\]

for \(j \neq n\) and

\[
[X_{i+}^\pm(u), X_{n-}^\pm(v)] = \delta_{in}(q_n - q_n^{-1}) \left( \delta((uq^{-c}/v)^2) q^c h_{i-1}^\pm(v_+)^{-1} h_{i+1}^\pm(v_+) - \delta((uq^{c}/v)^2) q^{-c} h_{i+1}^\pm(u_+)^{-1} h_{i+1}^\pm(u_+) \right)
\]
together with the Serre relations

\begin{equation}
\sum_{\pi \in \mathcal{C}, \prod_{i=0}^{r} (-1)^i \chi_{q}^{\pm}(u_{\pi(i)}) \chi_{q}^{\pm}(v) \chi_{q}^{\pm}(u_{\pi(i+1)}) \ldots \chi_{q}^{\pm}(u_{\pi(r)}) = 0,
\end{equation}

which hold for all \( i \neq j \) and we set \( r = 1 - A_{ij} \) for \( A_{ij} = -1 \), and

\begin{equation}
\sum_{\sigma \in \mathcal{S}_2} \sigma\left((q^2 u_1 + u_2) \chi_n^{\pm}(v) \chi_{n-1}^{\pm}(u_1) \chi_{n-1}^{\pm}(u_2) - [2]_{q^2} \chi_{n-1}^{\pm}(u_1) \chi_{n}^{\pm}(v) \chi_{n-1}^{\pm}(u_2) + \chi_{n-1}^{\pm}(u_1) \chi_{n-1}^{\pm}(u_2) \chi_{n}^{\pm}(v) \right) = 0.
\end{equation}

Finally we arrive at our main result.

**Theorem 6.14.** The mapping \( DR : U_q^{ext}(A_{2n-1}^{(2)}) \rightarrow U_q(R) : q^{c/2} \mapsto q^{c/2} \) and

\[ X^{\pm}(u) \mapsto X^{\pm}(u), \quad h^{\pm}(u) \mapsto h^{\pm}(u), \]

for \( j = 1, \ldots, n \), \( i = 1, \ldots, n+1 \), defines an isomorphism \( U_q^{ext}(A_{2n-1}^{(2)}) \rightarrow U_q(R) \).

**Proof.** The relations between \( U(R) \) and \( U(\mathcal{R}) \) and Theorem 6.13 imply the mapping \( DR \) is a surjective homomorphism. On the other hand Proposition 3.4 implies a homomorphism

\[ RD : U(R) \rightarrow U_q^{ext}(A_{2n-1}^{(2)}) \]

\[ L^{\pm}(u) \mapsto L^{\pm}(u). \]

Furthermore, we have \( DR \circ RD = ID \) by using Theorem 3.6. Thus the homomorphism \( DR \) is also injective.

\[ \square \]

**Acknowledgments**

The work is partially supported by the Simons Foundation (grant no. 523868) and the National Natural Science Foundation of China (grant no. 11531004).

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