FROM ENRIQUES SURFACE TO ARTIN-MUMFORD COUNTEREXAMPLE

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Abstract. After an Introduction to the themes of Enriques surfaces and Rationality questions, the Artin-Mumford counterexample to Lüroth problem is revisited. A construction of it is given, which is related in an explicit way to the geometry of Enriques surfaces, more precisely to the special family of Reye congruences and their classical geometry.

1. Perspectives on Enriques surface and Rationality

This section serves as a non technical introduction to this paper and to a wider theme of investigation, in algebraic geometry and its history, we can well summarize by the key words Enriques surface and rationality of algebraic varieties. These words certainly represent central and related issues, typical of the Italian contribution to algebraic geometry: since the golden age of the classification of complex algebraic surfaces, by Castelnuovo and Enriques, to the present times. Their presence is therefore natural in the Proceedings of an INdAM workshop bearing the title The Italian contribution to Algebraic Geometry between tradition and future.

In particular, the names of Castelnuovo and Enriques are definitely related to Castelnuovo Criterion of Rationality and to the discovery of Enriques surfaces. As is well known these achievements, in the theory of complex algebraic surfaces and their birational classification, are strictly related and represent one of the culminating points, in the history of algebraic geometry and of the Italian contribution, during some fortunate years at the juncture of 19th and 20th centuries.

Without entering in technical issues, the discovery of Enriques surfaces was motivated, independently from the work of birational classification, by the search of counterexamples to a quite natural conjecture, concerning the rationality of an algebraic surface $S$. More precisely, it was somehow natural to expect that $S$ is rational if and only if its geometric genus $p_g(S)$ and its irregularity $q(S)$ are zero.

This led Enriques, in the long days of a summer vacation of 1894, to consider the family of sextic surfaces, in the complex projective space $\mathbb{P}^3$, having multiplicity two along the edges of a tetrahedron. He proved that a general such a sextic is non rational, though its geometric genus and...
irregularity are zero. This is a celebrated episode, see the correspondence Castelnuovo-Enriques [6] p. x and 111.

In this way an entire class of surfaces was discovered and nowadays these bear the name of Enriques surfaces. Actually any Enriques surface turns out to be birational to a sextic as above. Castelnuovo’s Criterion was proven in the same period: let $P_2(S)$ be the bigenus of $S$, then $S$ is rational iff

$$P_2(S) = q(S) = 0.$$ 

A general motivation for this paper is to stress that, along their more than centennial history, Enriques surfaces certainly did not stay confined in the classification of algebraic surfaces as a kind of special or exotic plant. Instead their original interaction with rationality problems in dimension two impressively started to extend to higher dimension, implicitly or explicitly, touching in particular the famous Lüroth’s problem.

Lüroth’s problem is the question of deciding whether an algebraic variety $V$ admitting rational parametric equations $f : \mathbb{C}^N \to V$ is also rational, that is, it admits birational parametric equations $g : \mathbb{C}^d \to V$, $d = \dim V$.

Lüroth’s theorem and Castelnuovo’s Criterion imply this property for curves and surfaces, but the problem was staying open for years in dimension $\geq 3$. This until the crucial 1971-1972, when three counterexamples came out in dimension 3: by Artin-Mumford, Clemens-Griffiths, Manin-Iskovskikh. This famous triple episode can be considered as the beginning of a ‘Modern Era’ in the domain of rationality problems, see [3]-1.3 and section 4.1.

Whatever it is, starting from Castelnuovo and Enriques and considering all the decades until the present days of Modern Era, the overlapping of perspectives, on the mentioned themes of Enriques surfaces and rationality problems, are often visible and always very interesting.

As an example, let us mention the debate on Lüroth’s problem, originated in the Fifties from Roth’s book [24], and the related Serre’s theorem that unirational implies simply connected, [25]. The threefold considered is a unirational sextic in $\mathbb{P}^4$ whose hyperplane sections are sextic Enriques surfaces. The discussion was about, possibly, proving its irrationality via some features of irrationality of its hyperplane sections, see [5], [23].

We cannot add more views and perspectives in this Introduction, but mentioning the extraordinary wave of breakthrough results from the very last years. This is true in particular for stable rationality of some threefolds and cohomological decomposition of their diagonal, see [29, 30, 31].

Coming to the contents of this paper, we revisit the counterexample of Artin-Mumford, putting in evidence the presence, sometimes behind the scene in the literature, of an Enriques surface $S$. $S$ is embedded in the Grassmannian $\mathbb{G}$ of lines of $\mathbb{P}^3$ and well known as a Reye congruence.

$S$ brings us in the middle of classical algebraic geometry: $S$ is obtained from a general 3-dimensional linear system $W$, a web, of quadric surfaces. Moreover the threefold to be considered is the double covering of $W$, 

$$f : \tilde{W} \to W, \quad 2$$
parametrizing the rulings of lines of the quadrics of $W$. It is easy to see that $\tilde{W}$ is unirational, see 4.7 and [5]. The branch surface of $f$ is a Cayley quartic symmetroid $\tilde{S} +$, parametrizing the singular quadrics of $W$. A natural desingularization $\tilde{W}'$ of $W$ is the blow up of $\tilde{W}$ at its ten singular points. Artin and Mumford prove that, for a smooth variety $V$, the torsion of $H^3(V, \mathbb{Z})$ is a birational invariant. Moreover they prove

$$H^3(\tilde{W}', \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z},$$

which implies the irrationality of $\tilde{W}'$, since $H^3(\mathbb{P}^3, \mathbb{Z}) = 0$. As is well known the Enriques surface $S$ has the same feature of irrationality:

$$H^3(S, \mathbb{Z}) \cong H_1(S, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}.$$ 

We describe, and use, the nice geometry offered by $S$ and by the Fano threefold $\tilde{W}$, to reconstruct explicitly, from the nonzero class of $H^1(S, \mathbb{Z})$, the nonzero class of $H^3(\tilde{W}', \mathbb{Z})$. In particular we profit of the special feature of the Fano surface of lines of $\tilde{W}$. Indeed this is split in two irreducible components birational to $S$, see 4.4 and [17].

**Remark 1.1.** For $t \geq 3$, let $W_t$ be a general linear system of dimension $\binom{t}{2}$ of quadrics of $\mathbb{P}^t$. Let $W_t^4 \subset W_t$ be the locus of quadrics of rank $\leq 4$. Then $\dim W_t^4 = 2t - 3$ and we have a double covering $f : \tilde{W}_t^4 \to W_t^4$, parametrizing the rulings of subspaces of maximal dimension of the quadrics of $W_t^4$. A special family of these linear systems is associated to Reye congruences as in [12]. In this case $W_t$ defines an embedding of a Reye congruence $S$ as a surface of class $(t, 3t - 2)$ in the Grassmannian $G_t$ of lines of $\mathbb{P}^t$. It seems that, like for $t = 3$, $S$ determines nonzero 2-torsion in $H^3(\tilde{W}_t^4, \mathbb{Z})$, $\tilde{W}_t^4$ being a desingularization of $\tilde{W}_t^4$. This will be possibly reconsidered elsewhere.

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2. **Webs of quadrics and Enriques surfaces**

Now we introduce the family of those Enriques surfaces bearing the name of Reye congruences.\(^2\) Let $G$ be the Grassmannian of lines of $\mathbb{P}^3$, a general member $S$ of this family is a smooth Enriques surface embedded in $G$ and such that its rational equivalence class in the Chow ring $\mathrm{CH}^* (G)$ is

$$(1) \quad 7\sigma_{2,0} + 3\sigma_{1,1} \quad \mathbb{Z}. \quad \mathbb{Z}.$$ 

Then $S$ has degree 10 and sectional genus 6 in the Plücker embedding of $G$. Moreover $O_S(1)$ is an example of Fano polarization of an Enriques surface, [11] 3.5. Notice that, in the period space of Enriques surfaces, the locus

\(^2\)See [13] 3.7 for a historical note.

\(^3\) $\sigma_{1,1}$, $(\sigma_{2,0})$, is the class of the set of lines in a plane, (through a point).
of Reye congruences is an irreducible divisor and coincides with the locus of periods of Enriques surfaces containing a smooth rational curve, \[22\]. The construction of these surfaces was given by Reye, \[24\]. It relies on projective methods and the beautiful geometry of webs of quadric surfaces.

We concentrate on these related topics, recovering a general picture.

\[\text{2.1. Webs of quadrics.}\]

Let \(E\) be a 4-dimensional vector space, we fix the notation

\[\mathbb{P}^3 := \mathbb{P}(E)\]

and define the Plücker embedding of the Grassmannian of lines of \(\mathbb{P}^3\) by

\[G \subset \mathbb{P}^5^{-},\]

where \(\mathbb{P}^5^{-} := \mathbb{P}(\wedge^2 E)\). In particular \(G\) is a smooth quadric hypersurface. Then we consider \(E \otimes E\) and its standard direct sum decomposition

\[E \otimes E = \wedge^2 E \oplus \text{Sym}^2 E,\]

via the eigenspaces of the involution exchanging the factors of \(E \otimes E\). The induced involution on \(\mathbb{P}^{15} := \mathbb{P}(E \otimes E)\) will be denoted by

\[\iota : \mathbb{P}^{15} \to \mathbb{P}^{15}.\]

We also put \(\mathbb{P}^{9+} := \mathbb{P}(\text{Sym}^2 E)\) and observe that the set of fixed points of \(\iota\) is \(\mathbb{P}^5^{-} \cup \mathbb{P}^{9+}\). Then we define the commutative diagram

\[\begin{array}{ccc}
G & \leftarrow & \mathbb{P}^3 \times \mathbb{P}^3 \\
\downarrow & & \downarrow \\
\mathbb{P}^5^{-} & \leftarrow & \mathbb{P}^{15} \\
& & \rightarrow \mathbb{P}^{9+}.
\end{array}\]

Here \(S\) is the set of points defined by symmetric tensor \(a \otimes b + b \otimes a\) having rank \(\leq 2\). Notice that \(S\) is biregular to the quotient \(\mathbb{P}^3 \times \mathbb{P}^3 / \langle \iota \rangle\). The vertical arrows are the natural inclusions. Moreover \(\lambda_+\) and \(\lambda_-\) are the natural linear projections onto the spaces \(\mathbb{P}^{9+}\) and \(\mathbb{P}^5^{-}\). The restrictions of \(\lambda_-\) and \(\lambda_+\) to \(\mathbb{P}^3 \times \mathbb{P}^3\) admit an elementary description. Let \((x, y) \in \mathbb{P}^3 \times \mathbb{P}^3\) and let \(\ell \in G\), where \(\ell\) is the line through \(x, y\) if \(x \neq y\), then we have

\[\lambda_-(x, y) = \ell \in G \quad \text{and} \quad \lambda_+(x, y) = x + y \in S.\]

\[\text{Definition 2.1.} \quad T^r \text{ is the set of points in } \mathbb{P}^{15} \text{ defined by tensors } t \in E \otimes E \text{ having rank } \leq r. \text{ In particular } T^2 \text{ is } \mathbb{P}^3 \times \mathbb{P}^3 \text{ and } T^1 \text{ is its diagonal.}\]

From the loci \(T_r\) we have of course the rank stratification

\[T^1 \subset T^2 \subset T^3 \subset \mathbb{P}^{15}.\]

Now we pass to the dual space \(\mathbb{P}^{15*}\) of bilinear forms, to be denoted by

\[\mathbb{B} := \mathbb{P}(E^* \otimes E^*).\]
Let \( b \in \mathcal{B} \) then \( b^\perp \subset \mathbb{P}^{15} \) will denote its orthogonal hyperplane. Moreover let \( W \subset \mathcal{B} \) then the orthogonal subspace of \( W \) is by definition

\[
W^\perp := \bigcap_{b \in W} b^\perp.
\]

Let \( W \) be a subspace of dimension \( c \) then \( \text{codim } W^\perp = c + 1 \) and we have

\[
|\mathcal{I}_{W^\perp}(1)| = W,
\]

where we reserve the notation \( \mathcal{I}_{W^\perp} \) to the ideal sheaf of \( W^\perp \) in \( \mathbb{P}^{15} \). Notice that \( \mathcal{B} \) contains the subspace \( Q := \mathbb{P}(\text{Sym}^2 E^*) \) and that this is just the space of quadrics of \( \mathbb{P}^3 \). We denote its rank stratification by

\[
Q^1 \subset Q^2 \subset Q^3 \subset Q.
\]

Finally we come to web of quadrics of \( \mathbb{P}^3 \). We use the traditional word \textit{web} for a 3-dimensional linear system of divisors of a variety. For short we will use the word \textit{web of quadrics} for a web of quadric surfaces of \( \mathbb{P}^3 \). Let

\[
W \subset Q \subset \mathcal{B}
\]

be a general web of quadrics, then \( W \) naturally defines an Enriques surface which is a Reye congruence, [15] 7. We construct it as follows. We have

\[
W = \mathbb{P}(V),
\]

where \( V \subset \text{Sym}^2 E^* \) is a general 4-dimensional space. Notice that \( \iota^* \) is the identity on \( V = H^0(\mathcal{I}_{W^\perp}(1)) \), therefore the 5-dimensional subspace

\[
W^\perp \subset \mathbb{P}^{15}
\]

satisfies \( \iota(W^\perp) = W^\perp \). Now observe that the set of fixed points of the involution \( \iota|W^\perp \) is the disjoint union of the space \( \mathbb{P}^5^- \subset W^\perp \) and of

\[
W^\perp^+ := W^\perp \cap \mathbb{P}^9^+.
\]

Therefore \( \mathbb{P}^5^- \cup W^\perp^+ \) generates \( W^\perp \) and \( \text{codim } W^\perp^+ = 4 \) in \( \mathbb{P}^9^+ \), so that

\[
\lambda^*_+ (W^\perp^+) = W^\perp.
\]

This implies that the family of spaces \( W^\perp^+ \) coincides with the Grassmannian of codimension 4 spaces in \( \mathbb{P}^{9^+} \). Notice also that the family of spaces \( W^\perp \) is the family of codimension 4 spaces of \( \mathbb{P}^{15} \) containing \( \mathbb{P}^5^- \). Now consider

\[
\lambda_+|T_2 : T_2 \to S
\]

as in [73]. This is the quotient map of \( \iota|T_2 \) and a finite double cover branched on \( \text{Sing } S \), that is, on the image of the diagonal \( T_1 \) of \( T_2 = \mathbb{P}^3 \times \mathbb{P}^3 \).
2.2. Reye congruences of lines.

**Definition 2.2.** Given a web $W$ let us fix the notation
\begin{equation}
S_+ := S \cdot W^\perp, \quad \bar{S} := T_2 \cdot W^\perp.
\end{equation}

We assume that $W$ is general, so that $W^\perp$ is disjoint from $\text{Sing } S$ and transversal to the map $\lambda_+$. By Bertini theorem, $\bar{S}_+$ and $\bar{S}$ are smooth, irreducible surfaces and linear sections of $T_2$ and $S$. It is also clear that
\begin{equation}
\iota|\bar{S} : \bar{S} \to \bar{S}.
\end{equation}
is a fixed point free involution. Indeed $\iota^*$ is the identity on $V$ and $\bar{S}$ is disjoint from the set $T_1$ of fixed points of $\iota$. Hence $\lambda_+: T_2 \to S$ restricts to an étale double covering $\lambda_+|\bar{S} : \bar{S} \to S_+$. To complete the picture, we consider the rational map $\lambda_- : P^{15} \to P^5$ and its restriction $\lambda_-|\bar{S}$, we fix
\begin{equation}
S := \lambda_-(\bar{S}).
\end{equation}

Theorem 2.1. The surfaces $S$ and $S_+$ are embeddings, of degree 10 and sectional genus 6, of the same Enriques surface $\bar{S}/\langle \iota|\bar{S} \rangle$. Moreover one has
\begin{equation}
\omega_{S_+}(1) \cong \mathcal{O}_S(1).
\end{equation}

**Proof.** $\bar{S}$ is a K3 surface, endowed with the fixed point free involution $\iota|\bar{S}$ and embedded in $T_2 = P^3 \times P^3$ as a complete intersection of four elements of $|\mathcal{O}_{P^3 \times P^3}(1,1)|$. This follows by its construction and adjunction formula. Now $S$ and $S_+$ are embeddings of the same smooth surface $\bar{S}/\langle \iota|\bar{S} \rangle$. Since $\iota|\bar{S}$ is fixed point free and $\bar{S}$ is a smooth K3 surface, then $\bar{S}/\langle \iota \rangle$ is a smooth Enriques surface. On the other hand the K3 surface $\bar{S}$ is embedded in $W^\perp$ by $\mathcal{O}_{\bar{S}}(1) := \mathcal{O}_{P^3 \times P^3}(1,1) \otimes \mathcal{O}_{\bar{S}}$, a polarization of genus 11 and degree 20. Moreover the involution $\iota|\bar{S}$ acts on $H^0(\mathcal{O}_{\bar{S}}(1))$ and its quotient map $\lambda_+|\bar{S} : \bar{S} \to S_+$ is the double cover defined by the canonical sheaf $\omega_{S_+}$ and $\lambda_+^* \mathcal{O}_{S_+}(1) \cong \mathcal{O}_{\bar{S}}(1)$. Then it follows that we have the decomposition
\begin{equation}
H^0(\mathcal{O}_{\bar{S}}(1)) = \lambda_+^* H^0(\mathcal{O}_{S_+}(1)) \oplus \lambda_+^* H^0(\omega_{S_+}(1)).
\end{equation}
The summands respectively are the 6-dimensional +1 and −1 eigenspaces of $(\iota|\bar{S})^*$. This implies the last part of the statement, we omit some details. □

Clearly $\lambda_-|\bar{S} : \bar{S} \to P^5$ factors through $\iota|\bar{S}$ and defines the embedding
\begin{equation}
S \subset \mathcal{G} \subset P^5.
\end{equation}
Both $S$ and $S_+$ are examples of Fano models of an Enriques surface. $S$ is known as a Reye congruences of lines. A general Fano model is projectively normal, hence it is not contained in a quadric, cfr. [11], 3.5. Instead $S$ is contained in the smooth quadric $\mathcal{G}$. This is the special feature of this family. Each $\bar{S}$ is endowed with a special rank two vector bundle, namely the restriction of the universal bundle of $\mathcal{G}$. We will use it to introduce the classical construction of $S$, see [14], 24 and [15]-7.
2.3. Further notation.

The universal bundle over $G$ is $p : U \rightarrow G$, then $U_\ell \subset E$ and we define
\begin{equation}
(25) \quad U_\ell := \mathbb{P}(U_\ell) \subset \mathbb{P}^3,
\end{equation}
for each $\ell \in G$. This is the line in $\mathbb{P}^3$ defining the point $\ell \in G$. After some tradition, we say that a $\ell$ is a ray of $G$. We will also use the $\mathbb{P}^3$-bundle
\begin{equation}
(26) \quad \pi : \mathbb{P}(U \otimes U) \rightarrow G,
\end{equation}
whose fibre at $\ell$ is $\mathbb{P}^3_\ell := \mathbb{P}(U_\ell \otimes U_\ell)$. Let $\tau : \mathbb{P}(U \otimes U) \rightarrow \mathbb{P}^{15}$ be its tautological map, then $\tau_\ell : \mathbb{P}^3_\ell \rightarrow \mathbb{P}^{15}$ is the linear embedding defined by the inclusion of $U_\ell \otimes U_\ell \subset E \otimes E$. Moreover $\tau$ is a morphism birational onto its image. We identify $\mathbb{P}^3_\ell$ to its image by $\tau_\ell$ so that $\mathbb{P}^3_\ell \subset \mathbb{P}^{15}$. Since we have
\begin{equation}
(27) \quad U_\ell \otimes U_\ell = \wedge^2 U_\ell \oplus \text{Sym}^2 U,
\end{equation}
then $\mathbb{P}^3_\ell$ is $\iota$-invariant. Its point $\ell = \mathbb{P}(\wedge^2 U_\ell)$ and its plane $Q_\ell := \mathbb{P}(\text{Sym}^2 U_\ell)$ are the set of fixed points of $\iota|_{\mathbb{P}^3_\ell}$. We observe that $T_2 \cap \mathbb{P}^3_\ell$ is the quadric $U_\ell \times U_\ell$ and $T_1 \cap \mathbb{P}^3_\ell$ its diagonal. Finally, the next diagram will be useful:
\begin{equation}
(28) \quad \begin{array}{ccc}
\mathbb{P}^{15} & \xleftarrow{\tau} & \mathbb{P}(U \otimes U) \\
\uparrow & & \uparrow \\
T_2 & \xleftarrow{\delta} & D \xrightarrow{\pi} G
\end{array}
\end{equation}
Its vertical arrows are the natural inclusions and $D$ is the projectivized set of points defined in $\mathbb{P}(U \otimes U)$ by the decomposable tensors in $U \otimes U$.

3. REYE CONGRUENCES AND SYMMETROIDS

3.1. The classical construction.

It is now useful to revisit in modern terms Reye’s classical construction of the surface $S$, cfr. [24, 14, 10, 15]. Let $W$ be a general web of quadrics, defining as above a smooth $K3$ surface $\tilde{S}$ and the embeddings $S \subset G$ and $S_+ = \tilde{S} \cdot W$ of $S$. We assume $W = \mathbb{P}(V)$, then $V$ is a space of quadratic forms on $E$ and we have the natural inclusions
\begin{equation}
(29) \quad V \subset H^0(\mathcal{O}_{\mathbb{P}^1}(2)) \subset H^0(\text{Sym}^2 U^*).
\end{equation}
The evaluation of global sections of $\text{Sym}^2 U^*$ defines a morphism
\begin{equation}
(30) \quad e : \mathcal{O}_G \otimes V \rightarrow \text{Sym}^2 U^*,
\end{equation}
of vector bundles of ranks 4 and 3. Counting dimensions, the degeneracy scheme of $e$ is a surface, provided it is proper and non empty.

**Theorem 3.1.** The degeneracy scheme of $e$ is the Enriques surface $S$ and its rational equivalence class is $7\sigma_{1,1} + 3\sigma_{2,0}$ in $\text{CH}^*(G)$.

**Proof.** Assume the degeneracy scheme $S_e$ of $e$ is proper. Then, computing its class in the Chow ring of $G$, we obtain that $[S_e] = 7\sigma_{1,1} + 3\sigma_{2,0}$ in
Hence $\deg S_e = \deg S = 10$ and the equality $\text{Supp } S_e = S$ implies the statement. To show the equality consider $\ell \in G$ and the fibrewise map
\[(31)\]
$$e_\ell : V \to H^0(\mathcal{O}_{U_\ell}(2)).$$

This is the restriction map to the line $U_\ell \subset \mathbb{P}^3$, defined as above. Equivalently, keeping our previous identifications, we can assume that the curve
\[(32)\]
$$\Delta_\ell \subset U_\ell \times U_\ell \subset \mathbb{P}^3 \subset \mathbb{P}^{15}$$

is the diagonal embedding of $U_\ell$, and that $V \subset H^0(\mathcal{O}_{\mathbb{P}^3}(1))$. Then $e_\ell$ is the restriction map $V \to H^0(\mathcal{O}_{\Delta_\ell}(1))$. Moreover we know that $V$ has a basis $a_1, a_2, a_3, s$ so that $i^*a_j = -a_j$ and $i^*s = s$, where $s$ is zero on $\Delta_\ell$. Hence $e_\ell$ degenerates iff its rank is two. Equivalently $V \to H^0(\mathcal{O}_{\mathbb{P}^3}(1))$ defines a pencil of planes in $\mathbb{P}^3_\ell$ and its base line intersects $U_\ell \times U_\ell$ in two distinct points $x,y \in \tilde{S}$ such that $y = \iota(x)$. This implies $\text{Supp } S_e = \lambda_S = S$. \qed

The proof reveals the main geometric feature of $S$, we have:
\[(33)\]
$$S = \{ \ell \in G \mid \dim(V \cap H^0(I_\ell(2)) = 2) \}.$$ 

In other words $\ell \in S$ iff two quadrics of $W$ contain $\ell$, that is, $\ell$ is in the base locus of a pencil of quadrics of $W$. We can conclude as follows.

**Definition 3.1.** The Reye congruence of $W$ is the degeneracy scheme $S$ of the previous morphism $e$, provided $S$ is proper.

**Theorem 3.2.** The Reye congruence of $W$ is the family of lines of $\mathbb{P}^3$ which are in the base locus of a pencil of quadrics contained in $W$.

### 3.2. Order and class of $S$. 

Following the classical language the order of a surface $Y \subset G$ is the number $a$ of rays of $Y$ passing through a general point and the class is the number $b$ of rays of $Y$ in a general plane. Then one has $[Y] = a\sigma_{1,1} + b\sigma_{2,0}$ in $CH^*(G)$. Of course these notions naturally extend for surfaces in any Grassmannian of lines. Let us motivate geometrically the equality

$$[S] = 7\sigma_{1,1} + 3\sigma_{2,0} \in CH^*(G).$$

The coefficient 7 means that exactly seven rays of $S$ contain a general point $o \in \mathbb{P}^3$. Consider the net $W_o \subset W$ of quadrics through $o$, then its base locus is a set of eight distinct points $\{o,o_1\ldots o_7\}$. It is easy to see that the lines $\overline{o_1\ldots o_7}$ are the seven rays of the family passing through $o$.

The coefficient 3 means that exactly three rays of $S$ are contained in a general plane $P$. The web $W$ restricts on $P$ to a web $W_P$ of conics. How many lines in $P$ are fixed component of a pencil of conics of $W_P$? The answer is classical: $W_P$ is generated by four double lines, supported on lines in general position. These define a complete quadrilateral. One can check that its three diagonals are precisely the lines with the required property.
3.3. The quartic symmetroid.

Now we concentrate further on $W$, considering the quartic surface

$\tilde{S} + = \mathbb{Q}^3 \cdot W$

parametrizing the singular quadrics of a general $W$. Clearly such a surface is defined, in the projective space $W$, by the determinant of a symmetric $4 \times 4$ matrix of linear forms. For this reason it bears the following name.

**Definition 3.2.** A quartic symmetroid is a surface $\tilde{S} +$, constructed as above from a general web of quadrics $W$.

This classical surface is well known. Let $W \subset \mathbb{Q}$ be transversal to the quartic hypersurface $\mathbb{Q}^3 \subset \mathbb{Q}$ and to its singular locus $\mathbb{Q}^2$. Then, counting dimensions and degrees, we have $\mathbb{Q}^1 \cap W = \emptyset$ and, moreover, the set

$\operatorname{Sing} \tilde{S} + = \mathbb{Q}^2 \cap \tilde{S} +$

consists of ten ordinary modes. Actually $\tilde{S} +$ is a birational projective model of $\tilde{S}$, as we are going to see. To this purpose let us fix on $\mathbb{P}^3 \times \mathbb{P}^3$ coordinates

$(x, y) := (x_1 : x_2 : x_3 : x_4) \times (y_1 : y_2 : y_3 : y_4),$

so that $\tilde{S}$ is defined by the four symmetric bilinear equations

$\sum_{1 \leq i,j \leq 4} q^{[k]}_{ij} x_i y_j = 0, \quad k = 1 \ldots 4.$

The set of quadratic forms $q^{[k]} := \sum q^{[k]}_{ij} x_i x_j$ is a basis for the vector space $V$ such that $W = \mathbb{P}(V)$. From now on we set $z := (z_1 : z_2 : z_3 : z_4)$ and

$q_z := z_1 q^{[1]} + z_2 q^{[2]} + z_3 q^{[3]} + z_4 q^{[4]},$

denoting by $Q_z$ the quadric in $\mathbb{P}^3$ defined by $q_z$. Then we compute that

$\frac{\partial q_z}{\partial x_j} = z_1 q^{[1]}_j + z_2 q^{[2]}_j + z_3 q^{[3]}_j + z_4 q^{[4]}_j, \quad j = 1 \ldots 4,$

where we put $q^{[k]}_j := \sum_{1 \leq i \leq 4} q^{[k]}_{ij} x_i$. Clearly the four bilinear equations

$\frac{\partial q_z}{\partial x_j} = 0, \quad j = 1 \ldots 4,$

define in the product $\mathbb{P}^3 \times W$ the incidence correspondence

$\Xi := \{(x, z) \in \mathbb{P}^3 \times W \mid x \in \operatorname{Sing} Q_z\}.$

This is the universal singular locus over the family of quadrics $W$. Let

$p_x : \Xi \rightarrow \mathbb{P}^3 \text{ and } p_z : \Xi \rightarrow W,$

be the projections of $\Xi$ in the factors of $\mathbb{P}^3 \times W$. Obviously $p_z(\Xi)$ is the quartic symmetroid $\tilde{S} +$, defined by the symmetric determinant

$\det(z_1 q^{[1]}_j + \cdots + z_4 q^{[4]}_j).$
Let $\tilde{S}_x = p_x(\Xi)$ then, eliminating $(z_1 : z_2 : z_3 : z_4)$ from the equations of $\Xi$, the equation of $\tilde{S}_x$ is a quartic form in $(x_1 : x_2 : x_3 : x_4)$, namely

$$\tilde{S}_x = \{\det (q_i^k) = 0\}.$$  

Now consider $\tilde{S}_v \subset \P^3 \times \P^3$ and its equations in $(x,y)$. Let $\tilde{p}_x : \tilde{S} \to \P^3$ be the first projection and $\tilde{S}_v = \tilde{p}_x(\tilde{S})$. Eliminating $y$, one computes that

$$\tilde{S}_v = \{\det (q_i^k) = 0\}.$$  

Then $\tilde{S}_x = \tilde{S}_v$ and hence the surfaces $\tilde{S}_v$, $\tilde{S}$, $\tilde{S}_+$ are birational projective models of the same symmetroid $\tilde{S}_+$. Moreover let $\tilde{p}_y : \tilde{S} \to \P^3$ be the second projection and $\tilde{S}_y = p_y(\tilde{S})$, then $\tilde{S}_x$ and $\tilde{S}_y$ are projectively isomorphic. This follows because $\iota(\tilde{S}) = \tilde{S}$ and hence $\tilde{p}_y = \tilde{p}_x \circ \iota$. We keep the notation $\tilde{S}_v$ for $\tilde{S}_x$ and $\tilde{S}_y$. In particular $\tilde{S}_v$ is the birational image of $p_z : \tilde{S}_+ \to \P^3$. The next theorem summarizes our discussion and implements the picture.

**Theorem 3.3.** The $K3$ surface $\tilde{S}$ in $\P^3 \times \P^3$ and the quartic symmetroid $\tilde{S}_+$ in $W$ are birational to the quartic surface $\tilde{S}_v$. Moreover this surface is the locus in $\P^3$ of the singular points of the singular quadrics of $W$.

Occasionally $\tilde{S}_v$ is said to be the Steinerian of $\tilde{S}_+$, after some tradition, \[15\] 7.2. It is now the time to recall some facts on quartic double solids.

### 3.4. Quartic double solids.

To begin we recall that a **quartic double solid** is a finite double cover

$$f : X \to \P^3$$

whose branch scheme is a quartic surface $B \subset \P^3$. See \[8\], \[28\] and \[7\] for new results and update. **We assume that no line is in $B$ and that $\text{Sing} B$ is a finite set of ordinary double points.** Let us consider the blowing up

$$\sigma : \P^3' \to \P^3,$$

of $\text{Sing} B$, then a desingularization of $X$ is provided by the base change

$$X' \xrightarrow{f'} \P^3'.$$

Then $f'$ is the finite double cover branched on the strict transform of $B$ by $\sigma$. This is a smooth, minimal model of $B$ embedded in $\P^3'$, we denote by

$$B' \subset \P^3'.$$

The line geometry of $\P^3$ strongly influences the geometry of $X$. Consider indeed the universal line $U = \{(x, \ell) \in \P^3 \times \G | x \in \U(\ell)\}$ and its projections

$$\P^3 \xleftarrow{t} U \xrightarrow{u} \G.$$
Then \( u \) is the projective universal bundle and \( t \) its tautological morphism. The quartic surface \( B \) clearly defines a rational section
\[(51) \quad s : G \to \mathbb{P}(\text{Sym}^4 U^*),\]
sending the ray \( \ell \in \mathbb{G} \) to the intersection divisor \( U_\ell \cdot B \in |O_{U_\ell}(4)| \).

**Definition 3.3.** \( s : G \to \mathbb{P}(\text{Sym}^4 U^*) \) is the section defined by \( B \).

Since \( B \) does not contain lines, the rational map \( s \) is a morphism. Let
\[ \mathbb{P}(\text{Sym}^2 U^*) \subset \mathbb{P}(\text{Sym}^4 U^*) \]
be the embedding defined by the squaring map. By definition this means
any point \( d \in |O_{U_\ell}(2)| = \mathbb{P}(\text{Sym}^2 U^*)_\ell \) is embedded as the point
\[ 2d \in |O_{U_\ell}(4)| = \mathbb{P}(\text{Sym}^4 U^*)_\ell. \]

We use this embedding to define the family of bitangent lines to \( B \).

**Definition 3.4.** \( \mathbb{F}(B) \) is the pull-back of \( \mathbb{P}(\text{Sym}^2 U^*) \) by \( s \).

Clearly, \( \text{Supp} \mathbb{F}(B) \) is the set of bitangent lines to \( B \). The structure of \( \mathbb{F}(B) \) is known, see [28] 3 and [13] 3.4. We summarize as follows.

**Theorem 3.4.** If \( B \) is general \( \mathbb{F}(B) \) is a smooth integral surface and
\[ [\mathbb{F}(B)] = 12\sigma_{1,1} + 28\sigma_{2,0} \in CH^*(\mathbb{G}). \]

In the classical language \( \mathbb{F}(B) \) has order 12 and class 28. These numbers are easily explained: 28 is the number of bitangent lines to a general plane section of \( B \), which is a smooth plane quartic. Instead 12 is the number of ordinary nodes of the branch curve of a general projection \( B \to \mathbb{P}^2 \).

**Definition 3.5.** \( \mathbb{F}(B) \) is the congruence of bitangent lines of \( B \).

The surprising case of \( \mathbb{F}(B), B \) a quartic symmetroid, will be discussed in detail. Let us fix our notation for the natural involutions of \( X \) and \( X' \).

**Definition 3.6.** We respectively denote by \( j' : X' \to X' \) and \( j : X \to X \) the biregular involutions induced by \( f' : X' \to \mathbb{P}^3 \) and by \( f : X \to \mathbb{P}^3 \).

Finally we introduce the Fano surface of lines of \( X' \). Let \( \ell \in \mathbb{F}(B) \) then the line \( U_\ell \) is bitangent to \( B \). Moreover, for \( \ell \) general in \( \mathbb{F}(B) \), it is also true that \( U_\ell \cap \text{Sing} B = \emptyset \). Assuming this the curve \( f'^* U_\ell \) splits as follows:
\[(52) \quad f'^* U_\ell = R'_{\ell,+} + R'_{\ell,-}, \]
where \( R'_{\ell,+}, R'_{\ell,-} \) are biregular to \( \mathbb{P}^1 \) and exchanged by the involution \( j' \). They belong to the Hilbert scheme of curves of arithmetic genus 0 and of degree 1 for \( (f' \circ \sigma)^* \mathcal{O}_{\mathbb{P}^3}(1) \). This is a well known connected surface and the family of curves \( R'_{\ell,+}, R'_{\ell,-} \) is open and dense in it. We denote it by \( \mathbb{F}(X') \).

**Definition 3.7.** \( \mathbb{F}(X') \) is the Fano surface of lines of \( X' \).
The surface $\mathbb{F}(X')$ is *smooth and irreducible* for a general $B$. \cite{28,8,13}. We point out that $j' : X' \to X'$ defines a biregular involution
\begin{equation}
    j'^* : \mathbb{F}(X') \to \mathbb{F}(X').
\end{equation}
As above let $\sigma : \mathbb{P}^3 \to \mathbb{P}^3$ be the blow up of Sing $B$ and let $G'$ the Hilbert scheme of the pull-back by $\sigma$ of a general line of $\mathbb{P}^3$. Then the push-forward of cycles by the blowing up $\sigma$ defines a natural birational morphism
\begin{equation}
    \sigma_* : G' \to G.
\end{equation}
Let $\mathbb{F}(B) \to G$ be the inclusion map, then the base change by $\sigma_*$
\begin{equation}
\begin{array}{ccc}
\mathbb{F}(B') & \longrightarrow & G' \\
\downarrow & & \downarrow \\
\mathbb{F}(B) & \longrightarrow & G
\end{array}
\end{equation}
defines the surface $\mathbb{F}(B')$ in $G'$. If Sing $B = \emptyset$ this is $\mathbb{F}(B)$, we only mention that $\mathbb{F}(B') \to \mathbb{F}(B)$ is the normalization map if $B$ is general with Sing $B \neq \emptyset$. Moreover the push-forward of cycles defines the next commutative diagram, where $f'_*$ generically concides with the quotient map of $j'^*$:
\begin{equation}
\begin{array}{ccc}
\mathbb{F}(X') & \longrightarrow & \mathbb{F}(B') \\
\downarrow & & \downarrow \\
\mathbb{F}(X) & \longrightarrow & \mathbb{F}(B)
\end{array}
\end{equation}
To conclude we recall the following theorem.

**Theorem 3.5.** Let $X$ be a general quartic double solid then the map
\[ f'_* : \mathbb{F}(X') \to \mathbb{F}(B') \]
is an étale double covering of smooth regular surfaces of general type.

See the seminal papers \cite{8,28} and \cite{13} for some recent new results.

**Remark 3.1.** Clearly, for a general $B$, the morphism is defined by a non zero 2-torsion element of Pic $\mathbb{F}(B')$. Then $\mathbb{F}(B')$ is not simply connected. However it is regular and of general type. Coming to a quartic symmetroid $\tilde{S} +$, the surface $\mathbb{F}(\tilde{S} +)$ shows up as a very interesting limit of a general $\mathbb{F}(B)$. We will see that it is singular and normalizes to the Reye congruence $S$.

4. **The Artin-Mumford counterexample revisited**

4.1. *Artin-Mumford double solids.*

The study of the rationality problem for quartic double solids, with its related issues, plays a very important role, historically and not only. The unirationality of a quartic double solid is known since longtime, cfr. \cite{3}-4, \cite{24}-10, 11. Its irrationality, when it is branched on a quartic symmetroid, was proven in 1972 by Artin and Mumford, \cite{2}. This result is one of the three first counterexamples to Lüroth problem in dimension 3, appearing
simultaneously in 1971 - 1972 and relying on different methods. The other examples also rely on very famous results: the proof of the irrationality of a smooth cubic threefold, by Clemens and Griffiths, and the irrationality of a smooth quartic threefold, proven by Manin and Iskovskih. Nowadays the rationality problem for quartic double solids is settled, by application of similar methods and further work, at least as follows, cfr. [7].

**Theorem 4.1.** Let $f : X \to \mathbb{P}^3$ be a quartic double solid and let $m = | \text{Sing } B |$. Then $X$ is irrational if $0 \leq m \leq 6$ and rational if $m \geq 11$.

Nevertheless this matter is a not exhausted field of top interest for several reasons. Though there is no space for further digression, let us mention once more the results on non stable rationality, and non decomposition of the diagonal in $H^*(X, \mathbb{Z})$, for a very general quartic double solid [29, 30].

**Definition 4.1.** We say that a quartic double solid is an Artin-Mumford double solid if its branch surface is a general quartic symmetroid $\tilde{S}_+.

To treat Artin-Mumford double solids let us fix our notation as follows: as above $W \subset Q$ is a general web in the space $Q$ of quadric surfaces of $\mathbb{P}^3$. Then $W$ is a 3-dimensional subspace and $\tilde{S}_+ = W \cdot Q^3$. We denote by

\[ (57) \quad f : \tilde{W} \to W \]

the finite double covering of $W$ branched on $\tilde{S}_+$. The next diagram defines, exactly as in (48) of the previous section, a desingularization $\tilde{W}'$ of $\tilde{W}$:

\[ (58) \quad \tilde{W}' \xrightarrow{f'} W' \xrightarrow{\sigma} \tilde{W} \xrightarrow{f} W. \]

We will say, with a slight abuse, that the morphism $f' : \tilde{W}' \to W'$ is the desingularization of $f : \tilde{W} \to W$ and that $\tilde{W}'$ is the smooth model of $\tilde{W}$.

In the above diagram, constructing an Artin-Mumford double solid, the implicit presence of an Enriques surface is clear. $\tilde{S}$ is indeed the minimal desingularization of $\tilde{S}_+$ and an étale double covering of the Reye congruence $S$. The existence of $S$ is not mentioned in Artin-Mumford paper [2]. However the irrationality of $\tilde{W}$ follows there from the same irrationality feature of $S$, namely the presence of non zero torsion in the third cohomology group. At first it is shown in [2] that the torsion subgroup of it is birational invariant, for any smooth projective variety. Then the next theorem is proven.

**Theorem 4.2.** The torsion of $H^3(\tilde{W}', \mathbb{Z})$ is non trivial.

The proof relies on the notion of Brauer group and on some Severi-Brauer varieties, conic bundles in this case, related to $\tilde{W}$. Since $H^3(\mathbb{P}^3, \mathbb{Z}) = 0$, the irrationality of $W$ follows. Moreover, as we will see, $\tilde{W}$ is unirational. Then $\tilde{W}$ is a counterexample to Lüroth problem. Since the above torsion group is a stably rational invariant, $\tilde{W}$ is not stably rational as well, [29].
Actually both $S$ and $\tilde{W}'$ have a non zero torsion subgroup in the third cohomology and this is, in both cases, $\mathbb{Z}/2\mathbb{Z}$. The existence of $S$ is variably used and considered in the literature on Artin-Mumford double solids. The same is even more true for an explicit and geometric description of the relation between $H^3(S, \mathbb{Z})$ and $H^3(\tilde{W}', \mathbb{Z})$. As far as we know, the surface $S$ was used at first by Beauville in [3]-9, to this respect, as follows.

Let $\tilde{G}$ be the blow up of $G$ at $S$, one constructs very geometrically a dominant morphism $\tilde{\phi} : \tilde{G} \to \tilde{W}$. Since $G$ is rational, $\tilde{W}$ is unirational. Moreover $H^*(S, \mathbb{Z})$ is a summand of $H^*(\tilde{G}, \mathbb{Z})$ and hence $\tilde{\phi}$ naturally defines a homomorphism $h : H^* (\tilde{W}', \mathbb{Z}) \to H^* (S, \mathbb{Z})$. Using it, a proof of the previous theorem is given by application of cohomological methods, see [3] p.30. In particular $h$ restricts to an isomorphism between the torsion subgroup of $H^3(\tilde{W}', \mathbb{Z})$ and $H^3(S, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. See also [20]-4.

Next we introduce some geometry linking $S$ and $\tilde{W}'$. From it a quite explicit description of the torsion of $H_3(\tilde{W}', \mathbb{Z})$, via $H_1(S, \mathbb{Z})$, will follow. Let $U|S$ be the universal line $U \to G$ restricted over $S$. The description essentially relies on a rational map $\nu : U|S \to \tilde{W}'$ embedding a general fibre of $U|S$ as a line of $\tilde{W}'$, that is, an element of the Fano surface $\mathbb{F}(\tilde{W}')$. Then a 'cylinder map' $c : H_1(S, \mathbb{Z}) \to H_3(\tilde{W}', \mathbb{Z})$ is induced by $\nu$. As we will see $c$ is injective. Hence $H_1(S, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ injects in $H_3(\tilde{W}', \mathbb{Z})$ and, by Poincaré duality, in $H^3(\tilde{W}', \mathbb{Z})$. Actually $c$ is an isomorphism.

Let us also point out that $U|S$ defines a morphism onto the surface of bitangent lines to $\tilde{S}_+$, sending $\ell \in S$ to the pencil $I_\ell$ of all quadrics of $W$ through $U_\ell$. This is a bitangent line to $\tilde{S}_+$. We denote this morphism by

$$\nu : S \to \mathbb{F}(\tilde{S}_+)$$

where $\mathbb{F}(\tilde{S}_+)$ is defined like $B$ in diagram (56). Differently from a general quartic, $\mathbb{F}(\tilde{S}_+)$ is non normal and birational to the Reye congruence $S$. We outline here a description of $\mathbb{F}(\tilde{S}_+)$. However, during our work, we became aware of the beautiful description already performed by Ferretti, [16], [17]-3.

Remark 4.1. The latter one relates $\mathbb{F}(\tilde{S}_+)$ to the theory of EPW-sextics, when the corresponding hyperkähler fourfold is the Hilbert scheme $\tilde{S}^{[2]}$ of two points of $\tilde{S}$. The surface $\mathbb{F}(\tilde{S}_+)$ is birational to the locus of fixed points of the natural involution induced on $\tilde{S}^{[2]}$ by the quartic $\tilde{S}_+$, [17]-3.1.1.

4.2. The congruence of bitangent lines $\mathbb{F}(\tilde{S}_+)$. As in [12] let $Q$ be the space of quadrics of $\mathbb{P}^3$, we assume that $W \subset Q$ is a general web. Then $\tilde{S}_+$ is a general quartic symmetroid. Since now

$$G_Q \subset \mathbb{P}^{44}$$

is the Plücker embedding of the Grassmannian of lines of $Q$, then $G_Q$ is the family of all pencils of quadrics of $\mathbb{P}^3$. The space of its orbits, under the action of $\text{Aut} \mathbb{P}^3$, classifies these pencils up to projective equivalence. The
classification goes back at least to Corrado Segre, [26, 27]. See [19], [1]-7 for some recent revisiting. It follows from the classification that the locus of pencils which are bitangent lines to the quartic discriminant $Q^3 \subset \mathbb{Q}$, is an irreducible subvariety of codimension two. Its description is classical and very well known: a general element $P$ of $F(Q^3)$ is a pencil of quadrics whose base scheme is a complete intersection of two quadrics

$$L \cup C \subset \mathbb{P}^3,$$

where $L$ is a line and $C$ is a smooth, rational normal cubic curve. Moreover

$$L \cap C = \{v_1, v_2\} = \text{Sing } L \cup C$$

where $v_1, v_2$ are ordinary nodes of $L \cup C$. Notice that $P$ is generated by two quadrics $Q_1$ and $Q_2$ of rank 3, respectively singular at $v_1$ and $v_2$. These are the tangency points of $P$ to $Q^3$ and the two singular quadrics of the pencil.

More globally $F(Q^3)$ is parametrized by the smooth correspondence

$$\mathcal{G} := \{(\ell, P) \in \mathbb{G} \times \mathbb{G}_Q \mid U_\ell \text{ is in the base scheme of } P\}.$$ 

Indeed consider the natural projections

$$\mathbb{G} \xrightarrow{u_\mathcal{G}} \mathcal{G} \xrightarrow{t_\mathcal{G}} F(Q^3),$$

then $u_\mathcal{G} : \mathcal{G} \rightarrow \mathbb{G}$ is smooth and a Grassmann bundle whose fibre at $\ell$ is the Grassmannian of the pencils of quadrics containing $U_\ell$. Moreover it is easy to see that $t_\mathcal{G} : \mathcal{G} \rightarrow \mathbb{G}_Q$ is a birational morphism onto its image $F(Q^3)$.

Clearly $t_\mathcal{G}$ is biregular over each $P \in F(Q^3)$ such that $P$ is a general pencil as above. Notice also that, for any $P \in F(Q^3)$, the fibre of $t_\mathcal{G}$ at $P$ is a scheme supported on the points $(\ell, P) \in \mathbb{G} \times \{P\}$ such that $U_\ell$ is in the base scheme of $P$. This remark is the starting point to describe $\text{Sing } F(Q^3)$.

**Theorem 4.3.** $\text{Sing } F(Q^3)$ is irreducible of codimension 3 in $\mathbb{G}_Q$. Let $P$ be general in $\text{Sing } F(Q^3)$, then $P$ is an non normal ordinary double point and the base scheme $B_P$ of $P$ contains a conic $D = L \cup L'$ of rank 2.

Actually the pairs $(L, P), (L', P) \in \mathcal{G}$ correspond to the branches of the node $P \in F(Q^3)$. Then such a general singular point $P$ is a pencil whose base scheme is a complete intersection $D \cup D'$, being a smooth conic. This implies that $P$ contains a quadric $Q$ of rank 2, namely $Q \in \mathbb{Q}^2$ is the union of the planes supporting $D$ and $D'$. In particular let us consider

$$\Sigma_{Q^3} \subset \mathbb{G}_Q,$$

the locus of all pencils $P$ intersecting $Q^2$, then the next theorem follows.

**Theorem 4.4.** $\text{Sing } F(Q^3) = F(Q^3) \cdot \Sigma_{Q^2}.$

We use these properties for a basic description of the surface $F(W)$. Let

$$\mathcal{G}_W \subset \mathbb{G}_Q$$
be the Grassmannian of lines of $W$, then $G_W$ is a Schubert variety and a smooth 4-dimensional quadric in $G_Q$. According to the classification of pencils of quadrics, the space $F(Q^3)$ is quasi-homogeneous for the action of $\text{Aut} \mathbb{P}^3$, then union of finitely many orbits $\Gamma$. By transversality of general translate, we can assume that $G_W$ is transversal to each $\Gamma$. Let $O \subset F(Q^3)$ be the family of pencils $P \in F(Q^3)$ which are general as above, it is easy to see that $O$ is irreducible and the unique orbit which is open in $F(Q^3)$. By transversality of general translate again, $O \cap G_W$ is a smooth, irreducible open set of the surface $F(\tilde{S}_+)$. From Segre classification for pencils $P$ of $F(Q^3)$ one can see the complement of $O \cap F(\tilde{S}_+)$ in $F(\tilde{S}_+)$. This is a union of curves in $F(\tilde{S}_+)$. For it we fix our notation and description as follows:

$$F(\tilde{S}_+) - (O \cap F(\tilde{S}_+)) = N \cup D.$$  

(1) The singular curve $N := \text{Sing} F(\tilde{S}^+)$. The above properties imply that $N$ is the family of bitangent lines intersecting $\text{Sing} \tilde{S}_+$. Therefore we have $N = \cup_{o \in \text{Sing} \tilde{S}_+} N_o$, where the curve $N_o$ is

$$N_o := \{P \in F(\tilde{S}_+) \mid o \in P\}.$$  

The curve $N_o$ lies in the plane $\mathbb{P}^2 \subset G_W$ parametrizing all rays passing through $o$. As is well known $N_o$ is union of two smooth cubics intersecting at nine points, three of which are on a smooth conic. Moreover $N_o$ is the discriminant curve of the conic bundle structure naturally defined by

$$\pi_o \circ f' : \tilde{W}^+ \longrightarrow \mathbb{P}^2,$$

where $\pi_o : W \rightarrow \mathbb{P}^2$ is the linear projection of center $o$, cfr. [15]-7, [20]-4.

(2) The curve $D$ of hyperflex bitangent lines. We say that $P$ is a hyperflex tangent line if $P \cdot \tilde{S}_+$ is a unique point of $P$, with multiplicity 4. We omit more details on the well known curve $D$, since we will not use it.

In what follows we will concentrate on two important and elementary peculiarities of Artin-Mumford double solids, which are determinant for the existence of a $\mathbb{Z}/2\mathbb{Z}$ torsion subgroup of $H^3(\tilde{W}', \mathbb{Z})$. These are:

- A rational map $\psi : G \longrightarrow W$ factorizing as follows

$$\tilde{G} \xrightarrow{\tilde{\phi}} \tilde{W} \xrightarrow{f} W,$$

where $\beta : \tilde{G} \rightarrow G$ is the blowing up of $S$ and $\psi \circ \beta$ is a morphism. A remarkable fact is that $f$ is the Stein factorization of $\psi \circ \beta$.

- A birational morphism $n : S \rightarrow F(\tilde{S}_+)$ and its birational lifting $\tilde{n} : S \rightarrow F(\tilde{W}')$, 

onto its image $F^+ \subset F(\tilde{W}')$. Remarkably the surface $F(\tilde{W}')$ splits as

$$F(\tilde{W}') = F^+ \cup F^-,$$

with $F^+, F^-$ birational to $S$ and exchanged by the map $j^*$, see $4.4$

4.3. The rational map $\psi : G \rightarrow W$.

Let $W$ be a general web as above, we consider the rational map

$$\psi : G \rightarrow W,$$

sending a general $\ell \in G$ to the unique quadric $Q \in W$ through $U_\ell$. We recall that the Reye congruence $S$ of $W$ is by definition the degeneracy scheme of the map of vector bundles in $(30)$. This easily implies that $S$ is the indeterminacy scheme of $\psi$. To have a resolution of it we study the graph

$$\tilde{G} := \{ (\ell, Q) \in G \times W \mid U_\ell \subset Q \},$$

of $\psi$. We also consider its two natural projections

$$\mathbb{G} \xleftarrow{\beta} \tilde{G} \xrightarrow{\phi} W.$$

From the description of $S$ in $(32)$ it follows that $\beta$ is the contraction to $G$ of a $\mathbb{P}^1$-bundle over $S$. Then $\beta$ is the blowing up of $G$ at $S$. Notice also that the linear system of the quadrics of $W$ through $U_\ell$ is the fibre $\beta^*(\ell)$ of $\beta$. For the exceptional divisor of $\beta$ we fix the notation

$$\mathbb{I} := \{ (\ell, Q) \in G \times W \mid \dim \beta^*(\ell) = 1 \}.$$

Clearly the birational morphism $\beta$ restricts on $\mathbb{I}$ to the $\mathbb{P}^1$-bundle

$$\beta|\mathbb{I} : \mathbb{I} \rightarrow S$$

and, at $\ell \in S$, its fibre $\mathbb{I}_\ell$ is the pencil of quadrics of $W$ through $U_\ell$. Let $w \in W$ then its corresponding quadric embedded in $\mathbb{P}^3$ will be denoted by

$$Q_w \subset \mathbb{P}^3.$$

Now we consider the projection map $\phi : \tilde{G} \rightarrow W$. This is a morphism whose general fibre is not connected. Indeed let $w \in W$ then we have

$$\phi^*(w) = \tilde{G} \cdot (G \times \{w\}).$$

The equality just says that $\phi^*(w)$ is the Hilbert scheme of lines of the quadric surface $Q_w$. Then the proof of the next theorem is elementary.

**Theorem 4.5.** For a general $w \in W$ the two rulings of lines of $Q_w$ are the connected components of $\phi^*(w)$. For any fibre $\phi^*(w)$ can be as follows:

(i) rank $Q_w = 4$: disjoint union of two smooth conics,

(ii) rank $Q_w = 3$: a smooth conic having multiplicity 2,

(iii) rank $Q_w = 2$: union of two planes $P_1, P_2$. $P_1 \cap P_2 = \{ \text{one point} \}$.

Clearly $\phi^*(w)$ has type (iii) iff $w \in \text{Sing } \tilde{S}_+$ and (ii) iff $w \in \tilde{S}_+ - \text{Sing } \tilde{S}_+$. The type is (i) and the fibre is not connected iff $w \in \mathbb{P}^3 - \tilde{S}_+$.

Passing to the Stein factorization of $\phi$ we have the diagram

$$\begin{array}{ccc}
\tilde{G} & \xrightarrow{\phi} & \tilde{W} & \xrightarrow{f} & W, \\
\end{array}$$
where $φ = \tilde{φ} \circ f$ and $f$ is a finite double covering. It is clear that

$$f : \tilde{W} \to W$$

is branched on the symmetroid $\tilde{S}_+^+$ of $W$ and defines the Artin-Mumford double solid $\tilde{W}$. Let $w \in W$ then the fibre $f^*(w)$ is finite of length two. To denote the points of $\text{Supp } f^*(w)$ we fix the following convention.

$$\text{Supp } f^*(w) := \{ w^+, w^- \},$$

moreover $w^+ = w^-$ iff $w \in \tilde{S}_+$. Clearly $w^+$ and $w^-$ label the rulings of lines of the quadric $Q_w$ if its rank is $\geq 3$. Let us roughly summarize as follows.

**Remark 4.2.** $\tilde{W}$ is a parameter space for pairs $(w, w^+), (w, w^-)$, where $w \in W$ and $w^+ , w^-$ are the rulings of $Q_w$. $f$ is the natural forgetful map.

Finally we see in (77) that a rational variety, namely $\tilde{G}$, dominates $\tilde{W}$. Hence $\tilde{W}$ is unirational and the next theorem follows, cfr. [4]-6.

**Theorem 4.6.** $\tilde{W}'$ is unirational.

**Remark 4.3.** Notably, the properties of a blowing up imply

$$H^*(\tilde{G}, \mathbb{Z}) \cong H^*(G, \mathbb{Z}) \oplus \sigma^*(H^*(S, \mathbb{Z})),$$

where $\sigma : \tilde{G} \to G$ is the blowing up of $G$ at $S$. That has its importance.

4.4. $S$ and the Fano surface $\mathbb{F}(\tilde{W}')$.

Now we want to see that $S$ strongly interacts with the Fano surface $\mathbb{F}(\tilde{W}')$. Actually this is union of two irreducible components, birational to $S$. We begin with the Grassmannian $G_W \subset G_{\mathbb{P}^3}$ of lines of $W$ and fix the notation

$$\delta : S \to G_W$$

for the following rational map. For $\ell \in S$ let $\mathbb{I}_\ell \subset W$ be the linear system of quadrics through $U_\ell$. For each $\ell \in S$ this is a pencil, defining a point of $G_W$. By definition this is $\delta(\ell)$ and, moreover, $\delta : S \to G_W$ is a morphism.

**Theorem 4.7.** $\delta : S \to G_W$ is birational onto its image, moreover we have

$$\delta(S) = \mathbb{F}(\tilde{S}_+).$$

**Proof.** Recall that $W$ is always general, so that $G_W$ is transversal to $\mathbb{F}(\mathbb{P}^3)$. Now consider the orbit $O \subset \mathbb{F}(\mathbb{P}^3)$ as in [66]. As observed this is the unique irreducible open orbit under the action of $\text{Aut } \mathbb{P}^3$ on $\mathbb{F}(\mathbb{P}^3)$. Moreover $O \cap \mathbb{F}(\tilde{S}_+)$ is a smooth, irreducible open set of the surface $\mathbb{F}(\tilde{S}_+)$. Finally a point $P$ of it is a pencil whose base scheme $B_P$ is $L \cup C \subset \mathbb{P}^3$, where $C$ is a smooth rational normal cubic and $L$ is a line. Let $\ell \in G$ be the point defined by $L$, then we have $P = \mathbb{I}_\ell$ and $\delta(\ell) = P$. Since the only line in $B_P$ is $L$, $\delta^{-1}(P) = \{ \ell \} \subset S$. Moving $P$ along the mentioned open set of $\mathbb{F}(\tilde{S}_+)$ it follows that $\delta : S \to \mathbb{F}(\tilde{S}_+)$ is invertible and dominant. \[\Box\]
4.5. The counterexample of Artin-Mumford.

Finally we want to show that $H^3(\tilde{W}', \mathbb{Z})$ has a nonzero 2-torsion element, reconstructing it from the nonzero element of $H_1(S, \mathbb{Z})$. Let $\pi_1(S)$ be the fundamental group of $S$ and $\hat{S} \subset S$ a non empty Zariski open set, we recall that the inclusion $i : \hat{S} \to S$ defines a surjective homomorphism

$$i_* : \pi_1(\hat{S}) \to \pi_1(S).$$

This is well known for any complex, irreducible algebraic variety, [18] 0.3. Since we have $H_1(S, \mathbb{Z}) = \pi_1(S) = \mathbb{Z}/2\mathbb{Z}$, then the next property follows.

**Proposition 4.8.** For the generator $[\gamma]$ of $H_1(S, \mathbb{Z})$ one can choose $\gamma$ so that its image is in $\hat{S}$. Then $\gamma$ defines a nonzero 2-torsion element of $H_1(\hat{S}, \mathbb{Z})$.

After this remark we consider the exceptional divisor $I$ of the blowing up of $G$ at $S$. As in (74) this is a $\mathbb{P}_1$-bundle $\beta : I \to S$. Let $I_\ell$ be its fibre at $\ell \in S$, then $I_\ell$ sits in $\{\ell\} \times W = W$ as the pencil of quadrics

$$I_\ell = \{Q \in W \mid Q \text{ contains the line of } \mathbb{P}^3 \text{ defined by } \ell\}.$$

We already mentioned that the surface $\mathbb{F}(\tilde{S}_+)$ of bitangent lines to the quartic symmetroid $\tilde{S}_+$, just coincides with the family of pencils

$$\{I_\ell \subset W, \ell \in S\}.$$

As a surface in the Grassmannian $G_W$ of lines of $W$, this is the birational image of the morphism $\delta : S \to G_W$, sending $\ell$ to $I_\ell$. It is a singular model of $S$ having class $12\sigma_{20} + 28\sigma_{11}$ in $G_W$. Let $f : \tilde{W} \to W$ be the finite double cover branched on $\tilde{S}_+$. We also know that the fibre of $f$ at $Q \in W$ is bijective to the set of connected components of the family of lines of $Q$.

**Proposition 4.9.** Let $\ell \in S$ then its inverse image $f^{-1}(I_\ell)$ is union of two or one irreducible components, biregular to $I_\ell$ via $f$.

**Proof.** Let $Q \in I_\ell$ then the connected components of its family of lines are distinguished by the property of either containing the point $\ell$ or not. This easily implies that $f^{-1}(I_\ell)$ is the union prescribed by the statement. □

Let $w \in W$, we keep our notation $Q_w$ for the corresponding quadric in $\mathbb{P}^3$. For the irreducible components of $f^{-1}(I_\ell)$ we will have the notation

$$f^{-1}(I_\ell) := R_\ell^+ \cup R_\ell^-,$$

where $R_\ell^+$ is the ruling of $\ell$ in $Q_w$. Notice that no line is in the symmetroid of a general $W$. Then in this case we have $R_\ell^+ \neq R_\ell^-, \forall \ell \in S$. The above equality implies that the Enriques surface $S$ parametrizes an irreducible component of the Fano surface $\mathbb{F}(W)$ of lines $\tilde{W}$. Let $j : \tilde{W} \to \tilde{W}$ be the involution associated to $f$, then the next property easily follows.

**Proposition 4.10.** $\mathbb{F}(\tilde{W})$ is union of two irreducible components, birational to $S$ and exchanged by the involution $j^* : \mathbb{F}(\tilde{W}) \to \mathbb{F}(\tilde{W})$. 19
Remark 4.4 (A Wirtinger construction). To close on $F(\tilde{W})$, the following summary and concluding remark are due. Let $F^\pm := \{ R^\pm_\ell, \ell \in S \}$ then
\begin{equation}
F(\tilde{W}) = F^+ \cup F^-,
\end{equation}
where $F^+$ and $F^-$ are biregular to the surface $F(S_+)$ of bitangent lines to the quartic symmetroid $S_+$. Clearly its normalization is the disjoint union $S' \vee S''$ of two copies of $S$ and $j^*$ lifts to the standard involution
\[ j^* : S' \vee S'' \to S' \vee S'', \]
defined by exchanging $S'$ with $S''$ via the identity map of $S$. Moreover the pair $(F(\tilde{W}), j^*)$ is constructed from the pair $(S' \vee S'', j^*)$ by taking a suitable quotient surface $F(\tilde{W}) = S' \vee S''/ \sim$ of $S' \vee S''$. Let us consider a flat family
\[ \{ f_t : X_t \to \mathbb{P}^2, t \in T \}, \]
deforming $f : \tilde{W} \to W$, of quartic double solids. Then the corresponding family $\{ j^*_t : F(X_t) \to F(X_t), t \in T \}$ of fixed point free involutions is a deformation of $j^* : F^+ \cup F^- \to F^+ \cup F^-$. A similar deformation, for curves with a fixed point free involution, is known as a Wirtinger construction. \[ \square \]

We can think of a point of $\mathbb{I}$ as a pair $(\ell, w) \in S \times W$ such that $Q_w$ contains the line $U_\ell$. Moreover we can think of a point of $\tilde{W}$ as a pair $(w, r) \in W \times \tilde{W}$, where $r$ is a connected component of the family of lines in $Q_w$.

Definition 4.2. The fundamental morphisms are the maps
\begin{equation}
v^+ : \mathbb{I} \to \tilde{W}, \ v^- : \mathbb{I} \to \tilde{W}
\end{equation}
defined as follows: $v^+(w, \ell) := (w, w^+)$, $w^+$ being the connected component of $\ell$ in the family of lines of $Q_w$. Moreover the map $v^-$ is $j^* \circ v^+$.

Theorem 4.11. The fundamental morphisms have degree 6.

Proof. We prove the statement for $v^+$, the case of $v^-$ is similar. The degree of $v^+$ is the number of pairs $(\ell', w')$ such that $w = w'$ and $w^+ = r$, where $(w, r) = v^+(\ell, w)$ and $(\ell, w)$ is general in $\tilde{W}$. Then we can assume $\text{rank} Q_w = 4$ and $\ell$ general in $S$. To compute this number consider the restriction of $W$ to $Q_w$: this is a net $N$ in $|O_{Q_w}(2)| := \mathbb{P}^8$. Hence $N$ is a plane in $\mathbb{P}^8$ and the degree of $v^+$ is the number of reducible elements of $N$, containing a line of the ruling of $\ell$. Let $|L|$ be such a ruling and $H \in |O_{Q_w}(1)|$, we have $|L| := \mathbb{P}^1$ and $|2H - L| := \mathbb{P}^5$ is a linear system of rational cubics. Let
\[ s : \mathbb{P}^1 \times \mathbb{P}^5 \to \mathbb{P}^8, \]
be the sum map, sending $(L, C) \in \mathbb{P}^1 \times \mathbb{P}^5$ to $L + C \in |2H| = \mathbb{P}^8$. We claim that $s^*N$ is finite. Then its length is the degree of $v^+$ and coincides with the degree of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^5$, which is six. To prove our claim, assume $s^*N$ is not finite. Then each element of $|L|$ is a point of the Enriques surface $S \subset G$ and the union of these points is a conic. Since $\ell$ is general in $S$ then $S$ is uniruled: a contradiction. \[ \square \]
Finally let $\sigma' : \tilde{W}' \to \tilde{W}$ be the natural desingularization obtained, as in [58], by blowing up the set of ten nodes $\text{Sing} \tilde{W}$. Then the diagram

\begin{equation}
S \xleftarrow{\beta} I \xrightarrow{\nu^+} W \xleftarrow{\sigma'} \tilde{W}'
\end{equation}

defines a 'cylinder map'

\begin{equation}
c : H_1(S, \mathbb{Z}) \to H_3(\tilde{W}', \mathbb{Z}),
\end{equation}

where $c := \sigma'^* \circ \nu_+^* \circ \beta^*$. Now let us consider the non empty Zariski open set $\hat{S} \subset S$ of points $\ell \in S$ satisfying the following conditions:

(a) $\nu^+ : I \to \tilde{W}$ is finite over $\nu^+(I_\ell)$ and generically unramified at $I_\ell$,

(b) $\nu^+(I_\ell) \cap \text{Sing} \tilde{W} = \emptyset$, that is, rank $Q_w \geq 3, \forall (w, r) \in \nu^+(I_\ell)$,

(c) $\delta(\ell) \notin \text{Sing} \delta(S)$, where the map $\delta : S \to \mathbb{G}_W$ sends $\ell$ to $I_\ell$.

Let $[\gamma] \in H_1(S, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ be the generator of this group. By proposition 1.3 we can choose $\gamma$ so that its image is in the Zariski open set $\hat{S}$. Now, in the euclidean topology, $I$ is homeomorphic to $S \times \mathbb{P}^1$ and we have

\begin{equation}
H_3(I, \mathbb{Z}) \cong H_1(S, \mathbb{Z}) \otimes H_2(\mathbb{P}^1, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}.
\end{equation}

Indeed, one has $H_1(\mathbb{P}^1, \mathbb{Z}) = H_3(\mathbb{P}^1, \mathbb{Z}) = 0$. Moreover, by Poincaré duality, one has $H_3(S, \mathbb{Z}) \cong H^1(S, \mathbb{Z})$. Furthermore $H^1(V, \mathbb{Z})$ has no torsion for any smooth, projective variety $V$: see [29] proof of lemma 2.9 or deduce this property from the exponential sequence of $V$. Since the first Betti number of $S$ is zero, it follows $H^1(S, \mathbb{Z}) = 0$. Then, applying Künneth formulae to $I$, the above isomorphism follows. Now let us consider on $I$ the 3-cycle

$$\beta^* \gamma := I_\gamma.$$ 

Clearly $I_\gamma$ defines the 2-torsion class generating $H_3(I, \mathbb{Z})$ and the 3-cycle

$$T_\gamma := \nu_+^*(I_\gamma).$$

By (b) $T_\gamma$ is a 3-cycle of $\tilde{W} \setminus \text{Sing} \tilde{W}$, moreover $\sigma'$ is biregular over the set $\tilde{W} \setminus \text{Sing} \tilde{W}$. Then consider the 3-cycle $T'_\gamma := \sigma'^*(T_\gamma)$ of $\tilde{W}'$ and its class

\begin{equation}
\tau' = c([\gamma]) \in H_3(\tilde{W}', \mathbb{Z}).
\end{equation}

This is a 2-torsion element, we can now conclude via the following result.

**Theorem 4.12.** $\tau'$ is a nonzero element.

**Proof.** It suffices to show that the image of $\tau'$ by $(\nu^{**} \circ \sigma'_*)$ is nonzero in $H_3(I, \mathbb{Z})$. This is equivalent to show that

$$(\nu^{**} \circ \sigma'_*) \circ (\tau') = v^{**}(\tau') = [I_\gamma].$$

Since $\sigma'$ is biregular over $\tilde{W} \setminus \text{Sing} \tilde{W}$ the first equality is immediate. Let us prove the second one. For the class $[\gamma]$ of $H_1(\hat{S}, \mathbb{Z})$ we can even assume that the map $\gamma : (0, 1) \to \hat{S}$ is a real analytic embedding. We have

$$v^{**}T'_\gamma = mI_\gamma + Z,$$

where $Z$ is the image by $i_*$ of a cycle of $I - I_\gamma$ and $i : (I - I_\gamma) \to I$ is the inclusion. Let $\Gamma := \gamma([0, 1])$ and $\ell \in \Gamma$ then, by assumption (a) on $\gamma$, the
morphism $v^+$ is finite over $v^+(\mathbb{I}_t)$ and generically unramified along $\mathbb{I}_t$. This implies $m = 1$ and that $Z$ is a 3-cycle. Now let $t \in S - \Gamma$, we claim that $v^+(\mathbb{I}_t)$ is not in $v^+(\mathbb{I}_\gamma)$. Hence no summand of $Z$ is a pull-back of a cycle by $\beta : \mathbb{I} \to S$. Let $H_3(\mathbb{I}, \mathbb{Z}) \cong \bigoplus_{a+b=3} H_a(S, \mathbb{Z}) \otimes H_b(\mathbb{P}^1, \mathbb{Z})$ be the Künneth isomorphism, then no summand of $Z$ defines a class in $H_1(S, \mathbb{Z}) \otimes H_2(\mathbb{P}^1, \mathbb{Z})$. This implies that the class of $Z$ is zero and hence the theorem follows. To complete the proof we show the above claim. Let $t \in S$, in what follows $W_t \subset W$ is the pencil of quadrics defined by $t$, that is, the image of $v^+(\mathbb{I}_t)$ by $f : \tilde{W} \to W$. Now let $t \in S - \Gamma$, then, by our assumption (c), $\Gamma$ is in $\delta(S) - \operatorname{Sing} \delta(S)$, hence $W_t$ is not in the family $\{W_\ell, \ell \in \Gamma\}$. Since $W_t$, $W_\ell$ are lines they intersect in at most one point and the same is true for $v^+(\mathbb{I}_t)$, $v^+(\mathbb{I}_\ell)$. Let $W_\gamma = \bigcup_{\ell \in \Gamma} W_\ell$, this implies by (a) that the intersection of $\mathbb{I}_t$ and $v^+ - 1(W_\gamma)$ has real dimension $\leq 1$. Hence $v^+(\mathbb{I}_t)$ is not in $v^+(\mathbb{I}_\gamma)$.

Since $H^3(\tilde{W}', \mathbb{Z})$ admits nonzero torsion and this is a birational invariant then $\tilde{W}$ is not rational. Since it is unirational, it is a counterexample to Lüroth problem, see [2].

So far, we have performed our goal of explicitly reconstructing the 2-torsion cohomology of an Artin-Mumford double solid $\tilde{W}$ from that of its Enriques surface $S$. The 2-torsion group $H_1(S, \mathbb{Z})$ embeds in $H^3(\tilde{W}', \mathbb{Z})$ via the map $c : H_1(S, \mathbb{Z}) \to H^3(\tilde{W}', \mathbb{Z})$ and Poincaré duality.

This offers a partially new proof of Artin-Mumford counterexample to Lüroth problem, based on some geometry of Enriques surfaces, more precisely of Reye congruences of lines.

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