Dimensional Universality of Schauder Estimates Constants for Fourth Order Heat-Type Equations

Nicolò Cangiotti∗ & Lorenzo Marino†

Abstract

A new method to compute Schauder Estimates for multidimensional fourth order heat-type equations is proposed. In particular, we show how knowing Schauder or Sobolev estimates for the one-dimensional fourth order heat equation allows to derive their multidimensional analogs for equations with time inhomogeneous coefficients with the same constants as in the case of the one-dimensional heat equation. Our method relies on a merger between [37], where they actually showed the same result for the classical second order heat equation and [21], where a probabilistic construction of solutions for the fourth order heat equation is presented.

Keywords. Schauder Estimates, fourth order parabolic equation, Hölder norm.

MSC. Primary: 35K25, 35R45; Secondary: 60H30.

1 Introduction

The method presented in this work gives a different point of view to the theory of parabolic equations. Starting from the Schauder Estimates for the one-dimensional fourth order parabolic equation, we obtain its multidimensional analog with the same constants as in the one-dimensional case. Our study could be considered as the first attempt to reformulate the problem concerning Schauder Estimates by using two different approaches. As we shall cover more intensively in the sections below, one of the most important elements is the introduction of a suitable complex space. This complex structure allows us to exploit the imaginary direction to reach the fourth order. Stochastic integration plays an important role also in the second crucial point of the paper. Indeed, we will employ a recent idea based on Poisson processes and the finite difference method adapting it to our context.

∗Dipartimento di Matematica “F. Casorati”, Università di Pavia, Via Adolfo Ferrata 5, 27100 Pavia, Italy. E-mail Address: nicolo.cangiotti@unipv.it.
†Laboratoire de Modélisation Mathématique d’Evry (LaMME), Université d’Evry Val d’Essonne, 23, Boulevard de France 91037 Evry, France and Dipartimento di Matematica “F. Casorati”, Università di Pavia, Via Adolfo Ferrata 5, 27100 Pavia, Italy. E-mail Address: lorenzo.marino@univ-evry.fr.
We start giving our general framework. Fixed $k, d$ in $\mathbb{N}$ and a final time $T > 0$, we are interested in the following Cauchy Problem in $\mathbb{R}^d_T := [0, T] \times \mathbb{R}^d$:

$$
\begin{cases}
\partial_t u(t, x) = \sum_{|\gamma|=4} A^\gamma(t) D^\gamma u(t, x) + f(t, x), & \text{on } \mathbb{R}^d_T; \\
u(0, x) = 0, & \text{on } \mathbb{R}^d,
\end{cases}
$$

(1.1)

where $f : \mathbb{R}^d_T \to \mathbb{R}$ and for any $\gamma$ in $\mathbb{N}^d$ such that $|\gamma| := \gamma_1 + \cdots + \gamma_d = 4$, $A^\gamma : [0, T] \to \mathbb{R}$ is a bounded, Borel measurable function with suitable properties. More in details, it is supposed that the diffusion family $\{A^\gamma : |\gamma| = 4\}$ is uniformly elliptic at any fixed time $t$. Namely,

\[ \text{[UE]} \] there exists a constant $\eta > 0$ such that for any $t$ in $[0, T]$, it holds that

$$
\sum_{|\gamma|=4} A^\gamma(t) \xi^\gamma \geq \eta |\xi|^4, \quad \xi \in \mathbb{R}^d
$$

where $\xi^\gamma := \xi_1^{\gamma_1} \xi_2^{\gamma_2} \cdots \xi_d^{\gamma_d}$. Condition [UE] can be seen as a natural generalization of the classical uniform ellipticity for second order differential operators to our framework (see e.g. \cite{17, 20, 23, 33}).

Moreover, we require some integrability in time for the “trace” of $\{A^\gamma : |\gamma| = 4\}$. In particular,

\[ \text{[I]} \] for any $i$ in $[1,d]$, it holds that

$$
\int_0^T A^\gamma(t) dt < \infty \quad \text{if } \gamma = 4e_i,
$$

where $\{e_1, \ldots, e_d\}$ is the canonical orthonormal basis in $\mathbb{R}^d$.

**Mathematical Outline.** In the theory of parabolic equations the study of Sobolev and Schauder Estimates has a key role as evidenced by an extensive literature \cite{10, 18, 25, 26, 40, 41}. In particular, N. V. Krylov and E. Priola have proved in \cite{37} a fundamental result concerning the independence of these estimates on the dimension of the space variable. To be more precise, they considered the one-dimensional heat equation

$$
\partial_t u(t, x) = \partial_x^2 u(t, x) + f(t, x),
$$
in its integral form (with zero initial condition)

$$
u(t, x) = \int_0^t \left( \partial_x^2 u(s, x) + f(s, x) \right) ds,
$$

where $t \in [0, T]$ and $x \in \mathbb{R}$.

Let us introduce $B_b(0, T; C_c^\infty(\mathbb{R}))$ as the space of all the Borel bounded functions, such that $f(t, \cdot) \in C_0^\infty(\mathbb{R})$ for any $t$ in $[0, T]$, the $C^n(\mathbb{R})$-norms of $f(t, \cdot)$ are bounded on $(0, T)$ for any $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and the supports of $f(t, \cdot)$ belong to the same ball.

setting $\beta \in (0, 1)$ and $p \in (1, \infty)$, it is well-known \cite{34} that if $f \in B_b(0, T; C_c^\infty(\mathbb{R}))$, then the problem above has a solution $u(t, x)$ with the following properties:
(i) \( u(t, x) \) is continuous in \([0, T] \times \mathbb{R} \);

(ii) \( u(t, \cdot) \in C^{2+\beta}(\mathbb{R}) \), for any \( t \in [0, T] \), and

\[
\sup_{t \in [0, T]} \|u(t, \cdot)\|_{C^{2+\beta}(\mathbb{R})} \leq TC_\beta \sup_{t \in [0, T]} \|f(t, \cdot)\|_{C^\beta(\mathbb{R})},
\]

where \( C_\beta \) is a constant (depending on \( \beta \)). Moreover, there is a unique solution with properties (i) and (ii) and the following ones:

(iii) \[
\sup_{(t,x) \in \mathbb{R}_T} |u(t,x)| \leq T \sup_{(t,x) \in \mathbb{R}_T} |f(t,x)|;
\]

(iv) \[
\sup_{t \in [0, T]} \left[ \partial_x^2 u(t, \cdot) \right]_{C^\beta} \leq C_\beta \sup_{t \in [0, T]} [f(t, \cdot)]_{C^\beta};
\]

(v) \[
\left\| \partial_x^2 u \right\|_{L^p(\mathbb{R}_T)}^p \leq C_p \|f\|_{L^p(\mathbb{R}_T)}^p,
\]

where \( C_\beta \) and \( C_p \) are two constants and \( L^p \)-spaces are defined with respect to Lebesgue measure. Starting from the estimates above, Krylov and Priola showed how to derive their multidimensional analogs with coefficients depending only on the time variable, and with the same constants \( C_\beta \) and \( C_p \) as in the one-dimensional case. Their proof is based on the introduction of a Poisson stochastic process and the use of the finite difference method. We already emphasize that the method used in [37] cannot be trivially extended to our framework. In order to get higher-order Laplacians, it would be indeed necessary to iterate the reasoning exploiting the Poisson processes. This method, however, fails already for the order three since the position of the signs in the finite differences (independently on the center of the development) could be proven to be wrong (thus we cannot recover the analogous of Lemma 3.2 in [37] by the obvious extension).

The second main inspiration for this work is due to Funaki [21], who defined a complex stochastic process by composing two independent Brownian Motions \( \{B_t: t \geq 0\} \) and \( \{W_t: t \geq 0\} \) as

\[
\tilde{B}_t := \tilde{B}(t) = \begin{cases} 
B(W(t)), & \text{when } W(t) \geq 0; \\
iB(-W(t)), & \text{when } W(t) < 0.
\end{cases}
\]

Funaki proved that, under suitable conditions for a class of analytic initial datum \( u_0 \), the solution of the Cauchy Problem

\[
\begin{align*}
\frac{\partial}{\partial t} u(t, x) &= \frac{1}{8} \partial_x^4 u(t, x), & \text{on } [0, \infty) \times \mathbb{R}; \\
u(0, x) &= u_0(x), & \text{on } \mathbb{R},
\end{align*}
\]

has the following stochastic representation:

\[
u(t, x) = \mathbb{E} \left[u_0(x + \tilde{B}_t)\right].\]
To obtain such result, Funaki firstly defined the class $\mathcal{D}(\mathbb{R})$ of suitable real-valued functions $\phi(x)$ on $\mathbb{R}$, for which there exists an entire extension $\tilde{\phi}(z)$ on $\mathbb{C}$ such that $\tilde{\phi}(x) = \phi(x)$, for every $x \in \mathbb{R}$ and $\left(|\tilde{\phi}(z)| + |\partial_x \tilde{\phi}(z)| + |\partial_y \tilde{\phi}(z)|\right)e^{-h|z|^2}$ is bounded on $\mathbb{C}$ for any fixed $h > 0$, where $z = x + iy$. However, these conditions turn out to be too restrictive for our purpose. For this reason we shall show, in Section 3, a local version of Funaki's results, which actually allow us to use a more canonical space. Moreover, it is interesting to point out that the process $\tilde{B}_t$ in (1.2) gives rise to many convergence problems (as one can notice by writing explicitly the density function). This issues was analyzed and solved by considering stable-like processes with Brownian time in [29], among others. The iteration proposed by Funaki could appear as a non-canonical choice. However, our interest is to keep the Markovian structure and a relatively simple equation, which does not involve the derivatives of the initial datum (see, for instance, [3]). It is clear that a more suitable choice could be a process as $Y(t) = B_2(|B_1(t)|)$, but it is well known that it is not time homogeneous neither Markovian [47]. Many authors have studied the properties of iterated Brownian motion, which has become a fundamental tool in stochastic analysis, we suggest the interested reader to see, for instance, [2, 11, 32]. We remark that the complex construction aims to provide a powerful tool for our purpose, however for the stochastic results in the following, one could consider the real axis and the imaginary axis individually. It goes without saying that we shall highlight the points, where the complex analysis acts decisively.

Schauder-type Estimates have been recently considered by many authors in the framework of integro-partial differential equations [12, 27, 28, 38, 43, 42], however the interest around this kind of estimates for the higher-order parabolic equations is still present [6, 13, 14, 15, 48] and it fits in a wider tradition (see, e.g., [1, 7, 8, 24, 44]). This latter prompted us to investigate in such a direction. The main object of our research is the following fourth order Cauchy Problem:

$$\partial_t u(t, x) = \partial^4_x u(t, x) + f(t, x), \quad (1.3)$$

where we set the initial condition equal to zero.

Let us stress that the method proposed by Funaki induces the introduction of an additional variable, which alternatively acts both as a variable or as a parameter. Heuristically, this is the reason for the extension of our space in Section 3 in order to apply a similar argument. Moreover, we highlight that the usage of the parameters in the context of Schauder Estimates should be handled with care, as we explain in Section 6.

**Organization of Paper.** The paper is organized as follows. In Section 2 we fix the basic notations, defining the spaces and the tools, which we shall use in the following. In Section 3 we introduce the fourth order heat-type equation. Section 4 is devoted to the proofs of our main results. Direct applications of our method are given in Section 5. Finally, in Section 6 we summarize some suggestions for possible developments and further investigations.
2 Basic Notations and Main Results

We start recalling some useful notations we will need below. In the following, we shall denote by $C$ a generic positive constant whose precise value is irrelevant and that may change from line to line. We will explicitly specify any dependency that may occur.

Given a function $\phi: \mathbb{R}^d \to \mathbb{C}$ and a point $x$ in $\mathbb{R}^d$, we represent by $\partial_x \phi(x)$ the partial derivative along $x_i$ at a point $x$, when it exists. For a multi-index $\gamma = (\gamma_1, \ldots, \gamma_d)$ in $\mathbb{N}^d$, $D^\gamma \phi(x)$ represents the $\gamma$-th order derivative of $\phi$ at $x$, i.e.

$$D^\gamma \phi(x) := \partial_{x_1}^{\gamma_1} \cdots \partial_{x_d}^{\gamma_d} \phi(x).$$

We denote by $B_0(\mathbb{R}^d)$ the family of all the Borel measurable and bounded functions $\phi: \mathbb{R}^d \to \mathbb{C}$. It is a Banach space endowed with the supremum norm $\| \cdot \|_{\infty}$. Its closed subspace $C_0^\infty(\mathbb{R}^d)$ consists of all the smooth functions with compact support.

Fixed $m \in \mathbb{N}_0$ and $\beta \in (0, 1)$, we follow Krylov [33], denoting the usual Hölder seminorm of a function $\phi: \mathbb{R}^d \to \mathbb{C}$ as

$$[\phi]_{C^{m+\beta}} := \sup_{|\gamma| = m} \sup_{x,y \in \mathbb{R}^d} \frac{|D^\gamma \phi(x) - D^\gamma \phi(y)|}{|x - y|^\beta} \left( \frac{0}{0} := 0 \right).$$

Then, the Hölder space $C^{m+\beta}(\mathbb{R}^d)$ is the family of functions $\phi: \mathbb{R}^d \to \mathbb{C}$ such that $\phi$ and its derivatives up to order $m$ are bounded, continuous and the norm

$$\| \phi \|_{C^{m+\beta}} := \sum_{j=0}^m \sup_{|\gamma| = j} \| D^\gamma \phi \|_{\infty} + [\phi]_{C^{m+\beta}}$$

is finite.

We denote by $S(\mathbb{R}^d)$ the Schwartz space on $\mathbb{R}^d$ given by

$$S(\mathbb{R}^d) := \{ \phi \in C^\infty(\mathbb{R}^d): \forall \alpha, \gamma \in \mathbb{N}^d, \| x^\alpha D_x^{\gamma} \phi \|_{\infty} < \infty \}.$$

Equipped with the family of seminorms

$$\rho_m(\phi) = \sup_{|\alpha|, |\gamma| \leq m} \| x^\alpha D_x^{\gamma} \phi \|_{\infty}$$

indexed by $m$ non-negative integer, the space $S(\mathbb{R}^d)$ is a metrizable, Fréchet space. Moreover, $C_0^\infty(\mathbb{R}^d)$ is continuously and densely embedded in $S(\mathbb{R}^d)$.

Since we are working with evolution equations, the functions we consider will quite often depend on time, too. For this reason, we denote by $B_0([0,T]; S(\mathbb{R}^d))$ the space of all the Borel bounded functions $\phi: \mathbb{R}^d \to \mathbb{C}$ such that $\phi(t, \cdot)$ is in $S(\mathbb{R}^d)$ for any $t$ in $[0, T]$ and the seminorms $\rho_m(\phi(t, \cdot))$ are bounded in $t$. It is equipped with the “natural” family of seminorms $\{ \sup_{t \in [0,T]} \rho_m \}$.

It will be often convenient to exploit an additional semi-norm for functions in the standard Hölder space $C^{m+\beta}(\mathbb{R}^d)$ that “calculates” the Hölder regularity only along the one-dimensional subspaces of $\mathbb{R}^d$. Fixed $l$ in $\mathbb{R}^d$ and $\phi: \mathbb{R}^d \to \mathbb{C}$, we firstly denote the directional derivative of $\phi$ along $l$ as

$$D^\gamma_l \phi(x) := \sum_{|\gamma| = m} l^\gamma D^\gamma \phi(x),$$
where \( l^\gamma := l_1^{\gamma_1} l_2^{\gamma_2} \ldots l_d^{\gamma_d} \). We then define the seminorm \( [\phi]_{C^{m+\beta}} \) through
\[
[\phi]_{C^{m+\beta}} := \sup_{x_0 \in \mathbb{R}^d} \sup_{|l|=1} [D_l^m \phi(x_0 + l \cdot)]_{C^\beta[\mathbb{R}]}.
\] (2.1)

The following lemma shows that the two seminorms defined above are indeed equivalent if \( m = 0 \) while in the general case \( m > 0 \), the standard seminorm \( [\cdot]_{C^{m+\beta}} \) is controlled by the other one. Importantly, the constants appearing below are independent from \( d \).

In order to prove the result, we need however to define firstly the \( \gamma \)-th order finite difference operator \( \delta_{h,\gamma} \), for any \( h > 0 \) and any multi-index \( \gamma \). Firstly, we recall that the first finite difference \( \delta_{h,1} \phi \) for a function \( \phi : \mathbb{R}^d \to \mathbb{C} \) of step \( h \) along \( x_i \) is given by
\[
\delta_{h,1} \phi(x) = \frac{\phi(x + he_i) - \phi(x)}{h}, \quad x \in \mathbb{R}^d.
\]

We can then extend the above construction for a multi-index \( \gamma \) in \( \mathbb{N}^d \), defining the function \( \delta_{h,\gamma} \phi(x) \) through
\[
\delta_{h,\gamma} \phi(x) = \delta_{h,1} \delta_{h,2} \ldots \delta_{h,d} \phi(x),
\]
where \( \delta_{h,i} \) is the \( \gamma_i \)-th times composition of \( \delta_{h,i} \) with itself.

**Lemma 2.1.** Fixed \( m \) in \( \mathbb{N} \) and \( \beta \) in \( (0,1) \), there exists a constant \( C := C(m, \beta) \) such that for any \( \phi : \mathbb{R}^d \to \mathbb{C} \), it holds that
\[
[\phi]_{\tilde{C}^\beta} = [\phi]_{C^\beta};
\]
\[
[\phi]_{C^{m+\beta}} \leq C[\phi]_{\tilde{C}^{m+\beta}}.
\]

**Proof.** We are going to follow the proof of Lemma 3.3 in [36], adapting it to our fourth order framework. Given \( \phi : \mathbb{R}^d \to \mathbb{C} \), it is not difficult to check that
\[
[\phi]_{\tilde{C}^\beta} = [\phi]_{C^\beta}.
\]

In order to show the other control, we fix \( k \) in \( \mathbb{N} \) and consider a multi-index \( \gamma \) in \( \mathbb{N}^d \) such that \( |\gamma| = m \). Let us then denote by \( T_{x_0} : \mathbb{R}^d \to \mathbb{R} \) the \( m \)-th order Taylor polynomial of \( \phi \) centered at some point \( x_0 \) in \( \mathbb{R}^d \). Fixed a unitary vector \( l \) in \( \mathbb{R}^d \) and \( t \) in \( \mathbb{R} \), we can use the mean value theorem to show that
\[
|\phi(x_0 + tl) - T_{x_0}(x_0 + tl)| = \frac{t^m}{m!} |D_l^m \phi(x_0 + \lambda tl) - D_l^m \phi(x_0)| \leq [\phi]_{\tilde{C}^{m+\beta}} \frac{r^{m+\beta}}{m!},
\]
for some \( \lambda \) in \( [0, 1] \). It follows in particular that for any \( r > 0 \) and any \( x_0 \) in \( \mathbb{R}^d \),
\[
|\phi(x) - T_{x_0}(x)| \leq [\phi]_{\tilde{C}^{m+\beta}} \frac{r^{m+\beta}}{k!}, \quad x \in B(x_0, r). \] (2.2)

Let us fix now \( h > 0 \). We can exploit iteratively that
\[
\delta_{h,i} \phi(x) = \partial_{x_i} \phi(x + \lambda h e_i), \quad i \in [1, d]
\]
for some \( \lambda \) in \([0, 1]\), to show that

\[
|D^\gamma \phi(x) - \delta_{h, \gamma} D^\gamma \phi(x)| = |D^\gamma \phi(x) - D^\gamma \phi(x + h)| \leq [\phi]_{C^{m+\beta}} (ml)^\beta h^\beta,
\]

for some \( l \in \mathbb{R}^d \) such that \(|l| \leq 1\). Fixed \( x, x_0 \) in \( \mathbb{R}^d \), we choose \( r = |x - x_0| + mh \) in (2.2) so that \( x + h\gamma' \) and \( x_0 + h\gamma' \) are in \( B(x_0, r) \) for any multi-index \( \gamma' \) such that \(|\gamma'| \leq m\). Noticing that \( \delta_{h, \gamma} T_{x_0} \) is constant since it is the \( m \)-th order finite difference operator of a \( m \)-th order polynomial, we can write that

\[
|D^\gamma \phi(x) - D^\gamma \phi(x_0)| = |D^\gamma \phi(x) - \delta_{h, \gamma} D^\gamma \phi(x)| + |D^\gamma \phi(x_0) - \delta_{h, \gamma} D^\gamma \phi(x_0)|
\]

\[
+ |\delta_{h, \gamma} D^\gamma \phi(x) - \delta_{h, \gamma} D^\gamma \phi(x_0)|
\]

\[
\leq 2[\phi]_{C^{m+\beta}} (ml)^\beta h^\beta + |\delta_{h, \gamma} [D^\gamma \phi - T_{x_0}](x)|
\]

\[
+ |\delta_{h, \gamma} [D^\gamma \phi - T_{x_0}](x_0)|
\]

\[
\leq 2(ml)^\beta [\phi]_{C^{m+\beta}} h^\beta + \frac{2^{m+1}}{m!} [\phi]_{\bar{C}^{m+\beta}} r^{m+\beta} h^m.
\]

Choosing now \( h = \epsilon|x - x_0| \) for some \( \epsilon > 0 \), it follows that

\[
\frac{|D^\gamma \phi(x) - D^\gamma \phi(x_0)|}{|x - x_0|^\beta} \leq 2\epsilon^\beta (ml)^\beta [\phi]_{C^{m+\beta}} + \frac{2^{m+1}(1 + \epsilon)^m h}{(ml)\epsilon^m} [\phi]_{\bar{C}^{m+\beta}}.
\]

We can finally choose \( \epsilon > 0 \) such that

\[
2\epsilon^\beta (ml)^\beta = \frac{1}{2}.
\]

Taking the supremum over \( \gamma \), we have concluded the proof.

Since the drift \( f \) we consider will be taken in the space \( B_b\left(0, T; \mathcal{S}(\mathbb{R}^d)\right) \), it should be clear that we cannot hope to find a classical solution of Cauchy Problem (1.1), due to the low regularity in time of \( f \). Instead, the Cauchy problem will be understood in an integral form, as in [37]. Namely, a function \( u: \mathbb{R}^d_T \to \mathbb{C} \) is a solution of (1.1) if

\[
u(t, x) = \int_0^t \left[ \sum_{|\gamma| = 1} A^\gamma(s) D^\gamma u(s, x) + f(s, x) \right] ds, \quad (t, x) \in \mathbb{R}^d_T.
\]

We can now present our main result.

**Theorem 2.2.** Let \( \beta \in (0, 1) \), \( p \geq 1 \) and \( f \) in \( B_b\left(0, T; \mathcal{S}(\mathbb{R}^d)\right) \). Then, there exists a unique bounded, continuous solution \( u: \mathbb{R}^d_T \to \mathbb{C} \) of Cauchy Problem (1.1). Moreover, there exist two positive constants \( C_\beta := C(\beta, \eta), C_p := C_p(p, \eta) \) such that

\[
\sup_{(t, x) \in \mathbb{R}^d_T} |u(t, x)| \leq T \sup_{(t, x) \in \mathbb{R}^d_T} |f(t, x)|;
\]

\[
\sup_{t \in [0, T]} [u(t, \cdot)]_{C^{4+\beta}} \leq C_\beta \sup_{t \in [0, T]} [f(t, \cdot)]_{C^\beta};
\]

\[
\sup_{|l| = 1} \left\| D^l u \right\|_{L^p(\mathbb{R}^d_T)} \leq C_p \left\| f \right\|_{L^p(\mathbb{R}^d_T)}.
\]
Even if the space $S(\mathbb{R}^d)$ is not a “standard” option in the context of Schauder Estimates, our choice is mainly motivated by our approach based on the merger of the Funaki’s method [21] and the construction due to Krylov and Priola [37]. Indeed, we point out that the most natural space given by the intersection between $C_\infty^c(\mathbb{R}^d)$, as used in [37], and the $d$-dimensional Funaki space $D(\mathbb{R}^d)$ leads to the (useless) set composed only by the zero function. Additionally, we underline that the results proved in [37] by considering the space $B_b\left(0, T; C_\infty^c(\mathbb{R}^d)\right)$ can be recovered in a similar way for our space $B_b\left(0, T; S(\mathbb{R}^d)\right)$.

3 Fourth Order Heat-Type Equation

Here, we briefly present our method of proof by working on an easier example, say without diffusion coefficients and with a time-independent drift. In particular, we are interested in the fourth order heat-type equation on $\mathbb{R}$ given by:

$$\begin{aligned}
&\begin{cases}
\partial_t u(t, x) = \frac{1}{8} \partial^4_x u(t, x) + f(x), & \text{on } \mathbb{R}; \\
u(0, x) = 0, & \text{on } \mathbb{R},
\end{cases}
\end{aligned}$$

for some function $f: \mathbb{R} \to \mathbb{C}$ with suitable properties. It is easy to check that it corresponds to our original dynamics (1.1) taking $d = 1$, and $A(t) = 1$.

Let us consider the family $C_b^\omega(\mathbb{R}^d)$ of all the bounded real-analytic functions $\phi: \mathbb{R}^d \to \mathbb{C}$. One important feature of such functions is the possibility to extend them to a subset of $\mathbb{C}^d$ while preserving their properties. Indeed, it is well-known that any real-analytic function $\phi$ admits a subset $D$ of $\mathbb{C}^d$ containing the real space $\mathbb{R}^d$ and a unique holomorphic function $\overline{\phi}$ on $D$ such that $\overline{\phi} = \phi$ on $\mathbb{R}^d$. Exploiting the continuity of the extension $\overline{\phi}$ and the boundedness of $\phi$, it is then easy to check that $\overline{\phi}$ is bounded on a (possibly smaller) open subset $D$. Choosing again a smaller open subset if needed, we will always assume that $\overline{\phi}$ is bounded continuous on $\overline{D}$, the topological closure of $D$ in $\mathbb{C}^d$, and that the border $\partial D$ has the following property:

[B] for any $x_0$ in $\partial D$, there exists a ball $B$ such that $B \cap \overline{D} = \{x_0\}$.

We will denote by $D_\phi$ such a domain of holomorphy for $\overline{\phi}$. These properties are necessary for the following well-known result we state below for further references. It can be obtained, for example, from Chapter 4.4 in [39] and Theorem 2.2.3 in [19]. We recall again that, as we have stated in Section 2 this complex construction does not attend in the proof of the classical (stochastic) results we are going to use, indeed we can work by taking real and imaginary part separately.

**Theorem 3.1.** Let $D$ be a domain on $\mathbb{R}^d$ such that [B] holds and $u_0$ a bounded, continuous function on $\overline{D}$. Then the unique bounded, continuous solution of

$$\begin{aligned}
&\begin{cases}
\partial_t u(t, x) = \frac{1}{2} \partial^2_x u(t, x), & \text{on } (0, \infty) \times D; \\
u(0, x) = u_0(x), & \text{on } D; \\
u(t, x) = u_0(x), & \text{on } [0, \infty) \times \partial D,
\end{cases}
\end{aligned}$$

is $u(t, x) = u_0(x)$ for all $t \geq 0$ and $x \in \overline{D}$. Therefore, the solution of the Cauchy problem is equal to the initial condition on $\overline{D}$.

8
is given by

\[ u(t, x) := \mathbb{E}\left[u_0(x + B_{t \wedge \tau_x})\right], \quad (t, x) \in [0, \infty) \times D, \]

where \( \{B_t\}_{t \geq 0} \) is a standard Brownian Motion on \( \mathbb{R}^d \) and \( \tau_x = \inf_{t > 0}\{x + B_t \notin D\} \).

Thanks to the above result, we have been able to show existence and uniqueness of solutions for the time-reversible heat equation:

\[
\begin{aligned}
&\partial_t u(t, x) = \frac{1}{2} \partial_x^2 u(t, x), \quad \text{on } \mathbb{R}_0 \times \mathbb{R}; \\
&u(0, x) = u_0(x), \quad \text{on } \mathbb{R},
\end{aligned}
\]

when the function \( u_0 \) lies in \( C^\infty_c(\mathbb{R}^d) \). Above, we have denoted \( \mathbb{R}_0 := \mathbb{R} \setminus \{0\} \).

Before stating it, we need however to introduce the notion of Funaki extension of a Brownian Motion, as firstly constructed in [21]. Given a real-valued Brownian Motion \( \{B_t\}_{t \geq 0} \), the Funaki extension of \( B_t \) is the complex-valued process \( \tilde{B}_t \) given by

\[
\tilde{B}_t := \begin{cases} 
B_t & \text{if } t \geq 0; \\
ib & \text{if } t < 0.
\end{cases}
\]

**Lemma 3.2.** Let \( u_0 \) be in \( C^\infty_c(\mathbb{R}) \). Then, there exists a bounded, continuous solution \( u: \mathbb{R}^2 \rightarrow \mathbb{C} \) of Cauchy problem (3.2) given by

\[ u(t, x) := \mathbb{E}\left[u_0(x + \tilde{B}_{t \wedge \tau_x})\right], \quad (t, x) \in \mathbb{R}^2 \]

where \( \{\tilde{B}_t\}_{t \geq 0} \) is a standard real Brownian Motion and \( \tau_x := \sup_{t < 0}\{x + \tilde{B}_t \notin D_{u_0}\} \).

**Proof.** Let us assume for the moment that \( t \geq 0 \). It then follows immediately that \( \tilde{B}_t = B_t \) and \( \tau_x < t \) for any \( x \in \mathbb{R} \). By standard results (see e.g. [19]), we can then conclude immediately that the function

\[ u(t, x) = \mathbb{E}\left[u_0(x + B_t)\right] \]

is the unique bounded continuous solution of Cauchy Problem (3.2).

We can now fix \( t < 0 \). Noticing that \( \sup_{t < 0}\{x + iB_{t \wedge \tau_x^t} \notin D_{u_0}\} = -\inf_{t > 0}\{x + iB_t \notin D_{u_0}\} \), we start rewriting the function \( u \) as:

\[ u(t, x) = \mathbb{E}\left[u_0(x + iB_{t \wedge \tau_x^t})\right], \]

where \( \tau_x^t = \inf_{t > 0}\{x + iB_t \notin D_{u_0}\} \). Then, we set \( \tau_x^t = \inf_{t > 0}\{z + iB_t \notin D_{u_0}\} \) and \( U: (-\infty, 0) \times \mathbb{C} \rightarrow \mathbb{C} \) given by

\[ U(t, z) := \mathbb{E}\left[u_0(z + iB_{t \wedge \tau_x^t})\right]. \]

Denoting \( z = x + iy \), we have that

\[ \partial_t U(t, z) = -\partial_y \mathbb{E}\left[u_0(x + i(y + B_{s \wedge \tau_x^t}))\right]_{s=-t} = -\partial^2_y U(t, z), \]

for any \( (t, z) \in (-\infty, 0) \times D_{u_0} \), where in the last step, we used Theorem 3.1. Recalling that \( u_0 \) is holomorphic on \( D_{u_0} \), we can now use the Cauchy-Riemann equations to conclude

\[ \partial_t U(t, z) = -\partial^2_y U(t, z) = \partial^2_z U(t, z), \]

for any \( (t, z) \) in \( (-\infty, 0) \times D_{u_0} \). Taking \( y = 0 \) above, we have finished the proof. \( \square \)
It is well-known (see e.g. [6, 14, 15]) that if $f$ is in $S(\mathbb{R})$, then there exists a unique bounded, continuous solution $u: \mathbb{R}_T \to \mathbb{C}$ of Cauchy Problem (3.1). Moreover, we know that for any $\beta$ in $(0, 1)$ and any $p \geq 1$, there exist two positive constants $C_\beta$, $C_p$ such that

$$\sup_{(t,x)\in \mathbb{R}_T} |u(t, x)| \leq T \sup_{x\in \mathbb{R}} |f(x)|;$$  \hfill (3.4)

$$\sup_{t\in [0,T]} [u(t, \cdot)]_{C^{4+\beta}} \leq C_\beta [f]_{C^\beta};$$  \hfill (3.5)

$$\|\partial_x^4 u\|_{L^p(\mathbb{R}_T)} \leq C_p \|f\|_{L^p(\mathbb{R})}.$$  \hfill (3.6)

The main purpose of this section is to extend the above controls to solutions of the same Equation (3.1) on $\mathbb{R}_T^2$, without changing the value of the two constants $C_\beta$, $C_p$. It would be then clear that iterating the same reasoning multiple times, we could show the independence of the constants from $d$ as appearing in Theorem 2.2.

To be completely coherent with the inductive reasoning presented in Proposition 4.7, we should have started with estimates with respect to the alternative semi-norm $[\cdot]_{\tilde{C}_4^{4+\beta}}$ as defined in (2.1). It is however easy to check that the two semi-norms indeed coincide in the one-dimensional case, even for $k \neq 0$ (see Lemma 2.1).

Let us fix a function $f$ in $S(\mathbb{R}^2) \cap C^\omega_b(\mathbb{R}^2)$. Such a space is non-empty, since it contains at least $e^{-|x|^2}$. We will denote from this point further by $D_f$ the domain of holomorphy of $f$, as explained at the beginning of this section.

Keeping in mind the construction of solutions for the fourth order heat equation given by Theorem 1 in [21], we start extending $f$ on $\mathbb{R}^3$ by considering the unique bounded, continuous solution $F: \mathbb{R}^3 \to \mathbb{C}$ of the following Cauchy Problem:

$$\begin{cases}
\partial_s F(s, x, y) = \frac{1}{2} \partial_y^2 F(s, x, y), & \text{on } \mathbb{R}_0 \times \mathbb{R}^2; \\
F(0, x, y) = f(x, y), & \text{on } \mathbb{R}^3.
\end{cases}$$  \hfill (3.7)

A probabilistic representation of the function $F$ can obtained from Lemma 3.2 reasoning at any fixed $x$ in $\mathbb{R}$:

$$F(s, x, y) := \mathbb{E}\left[ f(x, y + \tilde{B}_{s\vee \tau^s_y}) \right], \quad (s, x, y) \in \mathbb{R}^3$$

where $\{B_t\}_{t\geq 0}$ is a standard real Brownian Motion, $\tau^s_y := \sup_{t<0} \{ y + \tilde{B}_t \in \partial D^x_f \}$ and $D^x_f$ is the $x$-section of $D_f$.

Exploiting that in this setup, integration and translation preserve holomorphy, it is not difficult to check that the function $F(s, \cdot, \cdot)$ remains real-analytic, at any fixed $s$ in $\mathbb{R}$. Moreover, its domain of holomorphy coincides with $D_f$ and it is independent from $s$.

Reasoning at any fixed $(s, y)$ in $\mathbb{R}^2$, we can now solve the following equation

$$\partial_t u(t, s, x, y) = \frac{1}{8} \partial_x^4 u(t, s, x, y) + F(s, x, y), \quad \text{on } \mathbb{R}_T^2, \quad (s, y) \in \mathbb{R}^2$$  \hfill (3.8)

with zero initial condition. By the above argument, there exists a unique bounded, continuous solution $u: \mathbb{R}_T^2 \to \mathbb{R}$ of Cauchy Problem (3.8) depending on $s$ and $y$ as
parameters such that Schauder Estimates (3.4)-(3.6) hold again at any fixed \((s, y)\) in \(\mathbb{R}^2\), when we replace \(u(t, x)\) and \(f(x)\) with \(u(t, s, x, y)\) and \(F(s, x, y)\).

The next step of our method consists in adding a Laplacian in \(s\) to Equation (3.8). Afterwards, we will exploit the dynamics of \(F\) given in (3.7) to change it to a bi-Laplacian with respect to \(y\).

Even if the reasoning in [37] were tailored for second order parabolic equations, we point out that the arguments therein are general enough to be used in our context, too. Indeed, there are no particular restriction on the class of differential operators defining the problem studied by Krylov and Priola. Following the same reasoning (with respect to \(s\)) used in Section 2 of [37], we then obtain the next result.

**Proposition 3.3.** There exists a unique bounded, continuous solution \(v: \mathbb{R}^3 \rightarrow \mathbb{R}\) of the following Cauchy Problem:

\[
\begin{aligned}
\partial_t v(t, s, x, y) &= \frac{1}{8} \partial_x^4 v(t, s, x, y) + \frac{1}{2} \partial_s^2 v(t, s, x, y) + F(s, x, y), \\
v(0, s, x, y) &= 0,
\end{aligned}
\tag{3.9}
\]

Moreover, at any fixed \(s \in \mathbb{R}\), it holds that

\[
\sup_{(t, x, y) \in \mathbb{R}_t^2} |v(t, s, x, y)| \leq T \sup_{(x, y) \in \mathbb{R}^2} |F(s, x, y)|;
\]

\[
\sup_{(t, y) \in \mathbb{R}_t} [\partial_x^4 v(t, s, \cdot, y)]_{C^3} \leq C_\beta \sup_{y \in \mathbb{R}^3} [F(s, \cdot, y)]_{C^3};
\]

\[
\|\partial_x^4 v(\cdot, s, \cdot, \cdot)\|_{L^p(\mathbb{R}^2)} \leq C_p \|F(s, \cdot, \cdot)\|_{L^p(\mathbb{R}^3)}.
\]

At first sight, it seems quite “unnatural” to state the controls above at any fixed \(s\), especially since they also hold uniformly in \(s\). This is indeed necessary because we will conclude taking \(s = 0\) in order to move back to the original drift \(f\) (since \(F(0, x, y) = f(x, y)\)).

The next step is to show that \(v(\cdot, s, \cdot, \cdot)\) is indeed a solution of the fourth order heat Equation (3.1) at any fixed \(s \in \mathbb{R}\). We will follow again Funaki [21] exploiting the ad hoc construction of the drift \(F\) and the auxiliary coordinate \(s\) we have introduced with it. We start considering the unique bounded, continuous solution \(G: \mathbb{R}^4 \rightarrow \mathbb{C}\) of

\[
\begin{aligned}
\partial_p G(s, p, x, y) &= \frac{1}{2} \partial_p^2 G(s, p, x, y), \\
G(s, 0, x, y) &= F(s, x, y),
\end{aligned}
\tag{3.10}
\]

By Lemma 3.2 with respect to \((p, x)\), we can write such a function \(G\) as:

\[
G(s, p, x, y) := \mathbb{E}[\tilde{F}(s, x + \tilde{M}_p \land \tau^p_{1/y}, y)], \quad (s, p, x, y) \in \mathbb{R}^4
\tag{3.11}
\]

where \(\{M_p\}_{p \geq 0}\) is a standard real Brownian Motion, \(\tau^p_x := \sup_{p < 0} \{x + \tilde{M}_p \in \partial D^y_f\}\) and \(D^y_f\) is the \(y\)-section of \(D_f\).
Proposition 3.4. The unique bounded, continuous solution $v$ of Cauchy Problem (3.9) has the following form:

$$v(t, s, x, y) = \int_0^t \mathbb{E} \left[ G \left( s + W_r^1, W_r^2, x, y \right) \right] dr, \quad (t, s, x, y) \in \mathbb{R}_+^3,$$  \hspace{1cm} (3.12)

where $\{(W_t^1, W_t^2)\}_{t \geq 0}$ is a standard Brownian Motion on $\mathbb{R}^2$.

Proof. Bearing in mind the “standard” Brownian representation of solutions to the heat equation, we introduce the auxiliary function $w : \mathbb{R}_+^4 \to \mathbb{C}$ given by

$$w(t, s, s', x, y) := \int_0^t \mathbb{E} \left[ G \left( s + W_r^1, s' + W_r^2, x, y \right) \right] dr, \quad (t, s, s', x, y) \in \mathbb{R}_+^4.$$  

By the standard Feynman-Kac formula (see, e.g., Sect. 7.4 in [46]) using $t$ as time variable and $(s, s')$ as space variables, the function $w$ is now solution of

$$\begin{align*}
\partial_t w(t, s, s', x, y) &= \frac{1}{2} \partial_s^2 w(t, s, s', x, y) + \frac{1}{2} \partial_{s'}^2 w(t, s, s', x, y) + G(s, s', x, y), \quad \text{on } \mathbb{R}_+^4; \\
w(0, s, s', x, y) &= 0, \quad \text{on } \mathbb{R}^4.
\end{align*}$$

In order to calculate explicitly $\partial_s^2 w$, we then exploit Equation (3.10) to show that

$$\partial_s G \left( s + W_r^1, s' + W_r^2, x, y \right) = \partial_p G \left( s + W_r^1, p, x, y \right) \big|_{p = s' + W_r^2} = \frac{1}{2} \partial_s^2 G \left( s + W_r^1, s' + W_r^2, x, y \right).$$

Hence, $\partial_s w(t, s, s', x, y) = \frac{1}{2} \partial_s^2 w(t, s, s', x, y)$. It finally holds that

$$\begin{align*}
\partial_t w(t, s, s', x, y) &= \frac{1}{2} \partial_s^2 w(t, s, s', x, y) + \frac{1}{2} \partial_{s'}^2 w(t, s, s', x, y) + G(s, s', x, y) \quad \text{on } \mathbb{R}_+^4; \\
w(0, s, s', x, y) &= 0 \quad \text{on } \mathbb{R}^4.
\end{align*}$$

Since $G(s, 0, x, y) = F(s, x, y)$ and $w(t, s, 0, x, y) = v(t, s, x, y)$ as wanted, we can conclude taking $s' = 0$ in the above equation.

We have just presented a Feynman-Kac-type formula that allows us to express the solution $v$ of Cauchy Problem (3.9) in terms of the drift $F$ (via the intermediate function $G$). Taking in mind Representation (3.11), it is now easy to check from Equation (3.7) that

$$\partial_s G(s, p, x, y) = \frac{1}{2} \partial_y^2 G(s, p, x, y), \quad (s, p, x, y) \in \mathbb{R}^4.$$  

Using now the representation (3.12), it follows immediately that the function $v$ is also the unique bounded, continuous solution of

$$\begin{align*}
\partial_t v(t, s, x, y) &= \frac{1}{8} \partial_x^4 v(t, s, x, y) + \frac{1}{8} \partial_y^4 v(t, s, x, y) + F(s, x, y) \quad \text{on } \mathbb{R}_+^3; \\
v(0, s, x, y) &= 0 \quad \text{on } \mathbb{R}_+^3.
\end{align*}$$

Taking $s = 0$ so that $F(0, x, y) = f(x, y)$, we can now denote, for simplicity, $v(t, x, y) := v(t, 0, x, y)$. The next result now follows immediately.
Proposition 3.5. There exists a unique bounded, continuous solution \( v : \mathbb{R}^2_T \rightarrow \mathbb{C} \) of

\[
\begin{cases}
\partial_t v(t, z) = \frac{1}{8} \Delta_z^2 v(t, z) + f(z), & \text{on } \mathbb{R}^2_T; \\
v(0, z) = 0, & \text{on } \mathbb{R}^2,
\end{cases}
\]

where \( \Delta_z^2 = \partial_z^4 + \partial_y^4 \) and \( z := (t, y) \) in \( \mathbb{R}^2 \). Moreover, it holds that

\[
\sup_{(t, z) \in \mathbb{R}^2_T} |v(t, z)| \leq T \sup_{z \in \mathbb{R}^2} |f(z)|;
\]

\[
\sup_{(t, y) \in \mathbb{R}_T} [\partial_x^4 v(t, \cdot, y)]_{C^\beta} \leq C_\beta \sup_{y \in \mathbb{R}} [f(\cdot, y)]_{C^\beta};
\]

\[
\|\partial_x^4 v\|_{L^p(\mathbb{R}_T^2)} \leq C_p \|f\|_{L^p(\mathbb{R}^2)}.
\]

To conclude, it remains to show that the above controls hold with respect to the derivatives in \( y \), too. We will exploit here the associated semi-norm \([\,]\)_{\tilde{C}^{4+\beta}} and then, the controls given in Lemma 2.1.

We start fixing a unitary vector \( l \) in \( \mathbb{R}^2 \) and then consider the orthogonal transformation on \( \mathbb{R}^2 \) such that \( S e_1 = l \), where we are using the classical representation matrix for linear transformations. Denoting now for simplicity \( w(t, z) := v(t, Sz) \) and \( g(z) := f(Sz) \) for any \((t, z) \in \mathbb{R}^2_T\), we use that the bi-Laplacian \( \Delta_z^2 \) is rotationally invariant to show that

\[
\begin{cases}
\partial_t w(t, z) = \Delta_z^2 w(t, z) + g(z), & \text{on } \mathbb{R}^2_T; \\
w(0, z) = 0, & \text{on } \mathbb{R}^2.
\end{cases}
\]

Hence, Proposition 3.5 can be applied for the solution \( w \), showing in particular that

\[
\sup_{(t, y) \in \mathbb{R}_T} [\partial_x^4 w(t, \cdot, y)]_{C^\beta} \leq C_\beta \sup_{y \in \mathbb{R}} [g(\cdot, y)]_{C^\beta}.
\]

On the one hand, it holds for any fixed \( t \) in \([0, T]\), that

\[
\sup_{y \in \mathbb{R}} [g(\cdot, y)]_{C^\beta} = \sup_{x, x', y \in \mathbb{R}} \frac{|g(x, y) - g(x', y)|}{|x - x'|^\beta} = \sup_{x, x', y \in \mathbb{R}} \frac{|f(S(x, y)) - f(S(x', y))|}{|x - x'|^\beta}.
\]

\[
= \sup_{x, x', y \in \mathbb{R}} \frac{|f(x(S e_1 + y(S e_2)) - f(x'(S e_1) + y(S e_2))|}{|x - x'|^\beta}.
\]

\[
= \sup_{x \in \mathbb{R}^2} \sup_{x \in \mathbb{R}^2} \frac{|f(x_0 + x l) - f(x_0 + x' l)|}{|x - x'|^\beta} = [f]_{\tilde{C}^\beta}.
\]

On the other hand, we can use

\[
\partial_x^4 w(t, x, y) = \sum_{|\alpha| = 4} \Gamma^\alpha D^\alpha v(t, S(x, y)) = D^\alpha v(t, S(x, y))
\]

for any \((t, x, y) \in \mathbb{R}^2_T\), to show similarly that

\[
\sup_{(t, y) \in \mathbb{R}_T} [\partial_x^4 w(t, \cdot, y)]_{C^\beta} = \sup_{t \in [0, T]} [w(t, \cdot)]_{\tilde{C}^{4+\beta}}.
\]
In order to prove the control in $\|\cdot\|_{L^p}$, we exploit instead that the Jacobian of an orthogonal transformation is unitary and again Equation (3.17) to write that

$$\int_0^T \int_{\mathbb{R}^2} |D_4^l u(t, z)|^p \, dz \, dt \leq C_p^p \int_{\mathbb{R}^2} |f(z)|^p \, dz,$$

(3.19)

for any unitary vector $l$ in $\mathbb{R}^2$.

4 Proof of Main Theorem

This section is entirely devoted to the proof of our main result in its general formulation. At first, in Subsection 4.1, we present an extension of the Funaki’s approach as introduced in the previous sections. It will constitute a fundamental tool in the proof of our main Theorem 2.2 that will be exposed in the subsequent Section 4.2.

We shortly introduce here the tensorial formalism we are going to use in this section. Fixed $N$ in $\mathbb{N}$, let $\{x_i\}_{i \in [1,N]}$, $\{y_j\}_{j \in [1,N]}$ be two copies of the “standard” orthonormal bases in $\mathbb{R}^N$. We can write the generic tensor product $x \otimes y$ between two elements $x, y \in \mathbb{R}^N$ as follows:

$$x \otimes y = \left( \sum_{i=1}^N \alpha_i x_i \right) \otimes \left( \sum_{j=1}^N \beta_j y_j \right) = \sum_{i=1}^N \sum_{j=1}^N \alpha_i \beta_j (x_i \otimes y_j), \quad \alpha_i, \beta_j \in \mathbb{R}.$$

We notice that $\mathbb{R}^N \otimes \mathbb{R}^N$ is actually a vector space with dimension $N^2$. Thus, we can write the generic element $T$ in $\mathbb{R}^N \otimes \mathbb{R}^N$ by considering the base $x_i \otimes y_j$ as:

$$T = \sum_{i,j=1}^N T_{ij} (x_i \otimes y_j), \quad T_{ij} \in \mathbb{R}.$$

We can now iterate this procedure considering, for a fixed $m$ in $\mathbb{N}$, the $m$-th tensor power of the vector space $\mathbb{R}^N$ as the $m$-fold tensor product of $\mathbb{R}^N$ with itself, i.e.

$$\left( \mathbb{R}^N \right)^{\otimes m} := \mathbb{R}^N \otimes \cdots \otimes \mathbb{R}^N$$

$m$ times.

It is now clear that any tensor of order $m$ in $\left( \mathbb{R}^N \right)^{\otimes m}$ can be represented as

$$T = (T_{i_1,\ldots,i_m}).$$

Given two tensors $T$ and $U$ in $\left( \mathbb{R}^N \right)^{\otimes m}$, we define the tensor product between $T$ and $U$ as a tensor in $\left( \mathbb{R}^N \right)^{\otimes 2m}$ given by

$$T \otimes U = (T_{i_1,\ldots,i_m}) \otimes (U_{j_1,\ldots,j_m}) = (T_{i_1,\ldots,i_m} U_{h_{m+1},\ldots,h_{2m}}) = (T \otimes U)_{i_1,\ldots,i_{2m}}.$$

It will be useful for our scope to introduce also the notion of symmetric tensor. We say that $T \in \left( \mathbb{R}^N \right)^{\otimes m}$ is symmetric if

$$(T_{\sigma(1)\sigma(2)\ldots\sigma(m)}) = (T_{i_1i_2\ldots i_m})$$
holds for every permutation $\sigma : [1, m] \to [1, m]$.
Finally, we shall denote by $\text{Tr}(\cdot, \cdot)$ the following bilinear form:

$$\text{Tr}(T, U) := \sum_{i_1, \ldots, i_m = 1}^M T_{i_1 \ldots i_m} U_{i_1 \ldots i_m}, \quad T, U \in \left(\mathbb{R}^M\right)^{\otimes m}.$$

With a little abuse of notation we shall call the operator $\text{Tr}(\cdot, \cdot)$ the trace. Let us mention the fact that when $T$ is a symmetric, $m$-th order tensor, we can recover the standard notion of trace (here denoted by $\tilde{\text{Tr}}(\cdot)$):

$$\tilde{\text{Tr}}(T) \equiv \text{Tr}(T, \mathbb{I}_m),$$

where $\mathbb{I}_m$ is the identity element in $\left(\mathbb{R}^N\right)^{\otimes m}$ given by

$$(\mathbb{I}_m)_{i_1 \ldots i_m} := \begin{cases} 1, & \text{if } i_1 = \cdots = i_m; \\ 0, & \text{otherwise}. \end{cases} \quad (4.1)$$

It is easy to check from the definition of trace that for any $T, U$ tensors of order $m$, the following property holds:

$$\text{Tr} \left( T, U \text{Tr}(T, U) \right) = \text{Tr}(T \otimes T, U \otimes U). \quad (4.2)$$

We will exploit, in Proposition 4.8 below, a useful decomposition result for a particular class of tensors, which can be actually considered as a natural extension of the matrix case studied in [35]. For a more technical and detailed treatise of the argument, as well as the most recent developments, we suggest the interested reader to see [5, 9].

**Lemma 4.1.** Fixed $\epsilon > 0$ and $M > 0$, let $\Gamma$ be a closed subset of $\mathcal{S}(\epsilon, M)$, the family of all the symmetric tensors $T$ in $\left(\mathbb{R}^d\right)^{\otimes 2m}$ such that $\text{Tr}(\mathbb{I}_{2m}, T) < M$ and

$$\text{Tr}(T, \xi^{\otimes 2m}) > \epsilon |\xi|^{2m}, \quad \xi \in \mathbb{R}^d \setminus \{0\}.$$

Then, there exist $n \in \mathbb{N}$, $Q_1, \ldots, Q_n$ symmetric matrices and $\nu_1, \ldots, \nu_n$ real-analytical functions from $\mathcal{S}(\epsilon, M)$ to $\mathbb{R}$ such that for any $T$ in $\Gamma$, it holds that

$$T = \sum_{j=1}^n \nu_j(T) Q_j^{\otimes m}.$$  

We highlight in addition that above we have identified the matrix $Q$ with the corresponding 2-tensor in $\left(\mathbb{R}^d\right)^{\otimes 2}$, so that $Q^{\otimes 2}$ is a 4-th tensor. From this point further, we will always assume this convention valid.

The introduction of the tensorial notation allows us to reformulate our problem in a more coherent way. Namely, we are given a time-dependent tensor $A : [0, T] \to \left(\mathbb{R}^d\right)^{\otimes 4}$ of fourth order such that

- $A(t)$ is symmetric at any fixed $t$ in $[0, T]$;
• it holds that
\[ \int_0^T \text{Tr}(I, A(t)) \, dt < \infty; \]

• there exists \( \eta > 0 \) such that
\[ \text{Tr}(A(t), \xi^4) \geq \eta |\xi|^4, \quad (t, \xi) \in \mathbb{R}^d. \]

Given a smooth function \( \phi : \mathbb{R}^d \to \mathbb{R} \) and a point \( x \) in \( \mathbb{R}^d \), we denote, for notational convenience, the fourth derivative tensor \( D^4_x \phi(x) \) of \( \phi \) as
\[ (D^4_x \phi(x))_{i_1, \ldots, i_4} := \partial_{x_{i_1}} \cdots \partial_{x_{i_4}} \phi(x). \]

Our main goal is to show the well-posedness and the associated Schauder Estimates of Theorem 2.2 for the following Cauchy Problem:
\[
\begin{aligned}
\partial_t u(t, x) &= \text{Tr}(A(t), D^4_x u(t, x)) + f(t, x), \quad \text{on } \mathbb{R}^d; \\
\left. u \right|_{t=0} &= 0, \quad \text{on } \mathbb{R}^d, \quad (4.3)
\end{aligned}
\]
where \( f \) is a function in \( B_b(0, T; S(\mathbb{R}^d)) \).

### 4.1 Funaki Method Extended

We present here a generalization of the results firstly appeared in [21] and then modified by us in Section 3. In particular, we prove in Theorem 4.3 a Feynman-Kac-type formula for solutions of the following Cauchy Problem:
\[
\begin{aligned}
\partial_t u(t, x) &= \frac{1}{8} \text{Tr}(Q \otimes 2, D^4_x u(t, x)) + f(t, x), \quad \text{on } \mathbb{R}^d; \\
\left. u \right|_{t=0} &= 0, \quad \text{on } \mathbb{R}^d, \quad (4.4)
\end{aligned}
\]
where \( Q \) is a non-negative matrix in \( \mathbb{R}^d \otimes \mathbb{R}^d \) and \( f : \mathbb{R}^d \to \mathbb{R} \) is a function with suitable properties.

The proof of this result will present some of the main tools, such as the introduction of auxiliary coordinates (see e.g. Equation (4.11)), that will appear fundamental in the proof of Theorem 2.2.

We highlight in addition that the following results have been stated here in a form useful for our scopes. Indeed, they can be easily adapted to much more general settings.

The Feynman-Kac-type formula we want to establish is based on a new stochastic object: the multi-valued, iterated Brownian diffusion. Let us recall some useful notations. Generalizing (3.3), we recall that the Funaki extension of a stochastic process \( \{X_t\}_{t \geq 0} \) is a complex process \( \{\bar{X}_t\}_{t \in \mathbb{R}} \) given by
\[ \bar{X}_t := \begin{cases} X_t & \text{if } t \geq 0 \\ iX_{-t} & \text{if } t < 0. \end{cases} \quad (4.5) \]

An analogous definition for multi-valued processes can be then obtained reasoning component-wise.
Let us consider now a time-dependent matrix \( \sigma : [0, \infty) \to \mathbb{R}^d \otimes \mathbb{R}^d \) such that \( \sigma(t) \) is non-negative definite for any \( t \) in \( [0, \infty) \). We define the diffusion \( X_t \) associated to \( \sigma \) as the solution, starting at zero, of the following stochastic differential equation (in its integral form):

\[
X_t = \int_0^t \sigma(s) dM_s,
\]

where \( \{M_t\}_{t \geq 0} \) is a Brownian Motion on \( \mathbb{R}^d \). The function \( \sigma \) is usually called the diffusion coefficient of \( \{X_t\}_{t \geq 0} \).

**Lemma 4.2.** Let \( \phi \) be in \( C^\omega_b(\mathbb{R}^d) \) and \( \{X_t\}_{t \geq 0} \) a \( d \)-dimensional diffusion associated with \( Q \). Then, the function \( \Phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C} \) given by

\[
\Phi(p, x) := \mathbb{E}\left[\phi(x + \bar{X}_{p \wedge \tau_x})\right], \quad (p, x) \in \mathbb{R} \times \mathbb{R}^d
\]

where \( \tau_x = \sup_{p < 0} \{x + \bar{X}_t \notin D_\phi\} \), is a solution of the following Cauchy Problem

\[
\begin{aligned}
\partial_p \Phi(p, x) &= \frac{1}{2} \text{Tr} \left( Q, D_x^2 \Phi(p, x) \right), \quad \text{on } \mathbb{R}_0 \times \mathbb{R}^d; \\
\Phi(0, x) &= \phi(x), \quad \text{on } \mathbb{R}^d. 
\end{aligned}
\]

**Proof.** For \( p > 0 \), it is trivial to check the result using standard results (cf. [19]), since \( \bar{X}_{p \wedge \tau_x} = X_p \). So, let us fix \( p < 0 \). Then, it holds that

\[
\Phi(p, x) = \mathbb{E}\left[\phi \left( x + iX_{-p \wedge \tau'_x}\right) \right], \quad x \in \mathbb{R}^d,
\]

where we have denoted \( \tau'_x = \inf_{p > 0} \{x + iX_t \notin D_\phi\} \). Using the complex extension, it is then possible to extend \( \Phi(p, \cdot) \) to \( D_\phi \subseteq \mathbb{C}^d \) through

\[
\Phi(p, z) := \mathbb{E}\left[\phi \left( z + iX_{-p \wedge \tau'_x}\right) \right],
\]

with the extension \( \tau'_z = \inf_{p > 0} \{z + iX_t \notin D_\phi\} \). Denoting \( z = x + iy \) for \( x, y \) in \( \mathbb{R}^d \), we can now show that

\[
\partial_p \Phi(p, z) = -\partial_y \mathbb{E}\left[\phi \left( x + i(y + X_{r \wedge \tau'_x})\right) \right]_{r=-p}
\]

\[
= -\frac{1}{2} \text{Tr} \left( Q, D_x^2 \mathbb{E}\left[\phi \left( x + i(y + X_{-p \wedge \tau'_x})\right)\right] \right),
\]

where in the last step we used the standard Brownian representation for solutions of the Heat equation with respect to \( y \) (and at any fixed \( x \)). Reasoning component by component for the tensors inside the trace, we then apply Cauchy-Riemann equations in order to write that

\[
\partial_p \Phi(p, z) = \frac{1}{2} \text{Tr} \left( Q, D_x^2 \mathbb{E}\left[\phi \left( x + i(y + X_{-p \wedge \tau'_x})\right)\right] \right) = \frac{1}{2} \text{Tr} \left( Q, D_x^2 \Phi(p, z) \right),
\]

for any \( (p, z) \) in \( (-\infty, 0) \times D_\phi \). Taking \( y = 0 \), we have concluded. \( \square \)
Thanks to the above lemma, we can now show our Feynman-Kac-type formula, exploiting the iterated diffusions approach. Let us start defining the space \( B_0 \left( 0, \infty; C^\omega_b(\mathbb{R}^d) \right) \) composed by all the Borel bounded functions \( f : \mathbb{R}^d_T \to \mathbb{C} \), such that \( f(t, \cdot) \) belongs to \( C^\omega_b(\mathbb{R}^d) \) and there exists an open subset \( D \) of \( \mathbb{C}^d \) that is contained in all the associated domains of holomorphy \( D_f(t, \cdot) \). For notational convenience, we will denote such a minimal domain of holomorphy \( D \) as \( D_f \). Moreover, we will assume without loss of generality that \( D_f \) satisfies again condition \( [B] \) in Theorem 3.1.

**Theorem 4.3.** Let us assume that:

- \( f \) in \( B_0 \left( 0, \infty; C^\omega_b(\mathbb{R}^d) \right) \);
- \( \{X_t\}_{t \geq 0} \) is a \( d \)-dimensional diffusion associated with \( Q \);
- \( \{W_t\}_{t \geq 0} \) is a real Brownian Motion that is independent from \( \{X_t\}_{t \geq 0} \).

Then, the function \( u : \mathbb{R}^d \to \mathbb{C} \) defined by

\[
 u(t, x) := \int_0^t \mathbb{E} \left[ J(t - r, x + X_{W_r \wedge \tau_x}) \right] dr,
\]

where \( \tau_x := \sup_{t < 0} \{x + X_t \notin D_f\} \), is the unique bounded, continuous solution of Cauchy Problem (4.4).

**Proof.** It is straightforward to check that the function \( u \) given in (4.8) is continuous and bounded. Furthermore, it is well-known (see e.g. [22]) that uniqueness of solutions holds in this class of functions for linear, higher order parabolic problems. Hence, it remains to show that \( u \) is indeed a solution of Equation (4.4).

We start considering the function \( F : \mathbb{R}^{d+1}_\infty \to \mathbb{C} \) given by

\[
 F(t, p, x) := \mathbb{E} \left[ J(t, x + X_{W_r \wedge \tau_x}) \right]
\]

By Lemma 4.2 for \( \phi = f(t, \cdot) \), we know that

\[
 \begin{cases}
 \partial_p F(t, p, x) = \frac{1}{2} \text{Tr} \left( Q, D_x^2 F(t, p, x) \right) & \text{on } [0, \infty) \times \mathbb{R}_0 \times \mathbb{R}^d; \\
 F(t, 0, x) = f(t, x) & \text{on } \mathbb{R}^d._\infty.
\end{cases}
\]

Keeping in mind the independence between \( \{W_t\}_{t \geq 0} \) and \( \{X_t\}_{t \geq 0} \), Equation (4.9) allows us to rewrite Representation (4.8) of \( u \) as

\[
 u(t, x) = \int_0^t \mathbb{E} \left[ F(t - r, W_r, x) \right] dr, \quad (t, x) \in \mathbb{R}^d._\infty.
\]

We can then extend \( u \) to \( \mathbb{R}^{d+1}_\infty \) considering the auxiliary function \( U : \mathbb{R}^{d+1}_\infty \to \mathbb{R} \) given by

\[
 U(t, s, x) = \int_0^t \mathbb{E} \left[ F(t - r, s + W_r, x) \right] dr.
\]

Recalling the “usual” Feynman-Kac representation (using \( s \) as the space coordinate), we notice that

\[
 \begin{cases}
 \partial_t U(t, s, x) = \frac{1}{2} \partial_s^2 U(t, s, x) + F(t, s, x), & \text{on } \mathbb{R}^{d+1}_\infty; \\
 U(0, s, x) = U_0(s, x), & \text{on } \mathbb{R}^{d+1}.
\end{cases}
\]
In order to find an explicit representation for $\partial_s U$, we now exploit (4.10) to write that

$$\partial_s U(t, s, x) = \partial_s \int_0^t E [F(t - r, s + W_r, x)] dr = \frac{1}{2} \text{Tr} \left( Q, D^2_x \int_0^t E [F(t - r, s + W_r, x)] dr \right) = \frac{1}{2} \text{Tr} \left( Q, D^2_x U(t, s, x) \right).$$

It then follows immediately that

$$\partial_t U(t, s, x) = \frac{1}{2} \partial^2_s U(t, s, x) + F(t, s, x) = \frac{1}{4} \text{Tr} \left( Q, D^2_x \partial_s U(t, s, x) \right) + F(t, s, x) = \frac{1}{8} \text{Tr} \left( Q^{\otimes 2}, D^2_x U(t, s, x) \right) + F(t, s, x),$$

where, in the last step, we exploited the trace property in (4.2). Moreover, we have that $U(0, s, x) = U_0(s, x)$. Taking $s = 0$, we find that $u$ is indeed a solution of Equation (4.4).

We conclude this subsection presenting two generalizations of Theorem 4.3. They cannot be derived directly from the Feynman-Kac formula we have just proven but instead they share, even if with different modifications, the same proving techniques of Theorem 4.3.

We start presenting an extension of Lemma 4.2 where we consider a version of Cauchy Problem (4.4) that allows the presence of multi-valued “time” coordinates $t = (t_1, \ldots, t_n)$. More in details, let us fix an integer $n$ in $\mathbb{N}$. We are then interested to show a Brownian representation formula as (4.8), for solutions of the following Cauchy Problem:

$$\begin{cases}
\partial_{p_i} \Phi(p, x) = \frac{1}{2} \text{Tr} \left( Q_1, D^2_x \Phi(p, x) \right), & \text{on } \mathbb{R}_0^n \times \mathbb{R}^d; \\
\vdots \\
\partial_{p_n} \Phi(p, x) = \frac{1}{2} \text{Tr} \left( Q_n, D^2_x \Phi(p, x) \right), & \text{on } \mathbb{R}_0^n \times \mathbb{R}^d; \\
\Phi(0, x) = \phi(x), & \text{on } \mathbb{R}^d,
\end{cases} \tag{4.13}$$

where $\mathbb{R}_0^n := \mathbb{R}^n \setminus \{0\}$ and $Q_1, \ldots, Q_n$ are non-negative definite matrices in $\mathbb{R}^d \otimes \mathbb{R}^d$.

**Lemma 4.4.** Let $\phi$ be in $C^\omega_b(\mathbb{R}^d)$ and $\{X^i_t\}_{t \geq 0}$ independent $d$-dimensional diffusions associated with $Q_j$, for any $j \in \llbracket 1, n \rrbracket$. Then, the function $\Phi: \mathbb{R}^{d+n} \to \mathbb{C}$ defined by

$$\Phi(p, x) := \mathbb{E} \left[ \phi \left( x + \sum_{j=1}^n X^j_{\tau_{x,j}} \right) \right], \quad (p, x) \in \mathbb{R}^{d+n},$$

where $\tau_{x,j} := \sup_{t < 0}\{ x + \bar{X}^j_t \notin D_\phi \}$, is a solution of the Cauchy Problem (4.13).
Proof. Clearly, the function $\Phi$ defined above satisfies $\Phi(0, x) = \phi(x)$. Then, let us introduce by iteration a finite family $\{\Phi_j : j \in [0, n]\}$ of real functions on $\mathbb{R}^{d+j}$. Fixed $\Phi_0(x) = \phi(x)$, we define $\Phi_j : \mathbb{R}^{d+j} \to \mathbb{R}$ as the solution of

$$
\begin{align*}
\left\{
\begin{array}{ll}
\partial_t \Phi_j(p_1, \ldots, p_j, x) = \frac{1}{2} \text{Tr} \left( Q_j, D^2_{x} \Phi_j(p_1, \ldots, p_j, x) \right), & \text{on } \mathbb{R}^{d+j}; \\
\Phi_j(p_1, \ldots, p_{j-1}, 0, x) = \Phi_{j-1}(p_1, \ldots, p_{j-1}, x), & \text{on } \mathbb{R}^{d+j-1},
\end{array}
\right.
\end{align*}
$$

(4.14)
given by Lemma 4.2, for any fixed $(p_1, \ldots, p_{j-1})$. Fixed $j \in [1, n]$, we now exploit the Feynman-Kac-type formula (4.8) for $\Phi_k$ for $k$ in $[j+1, n]$ and the independence between the diffusions $\{X^k_p\}_{p \geq 0}$, to check that

$$
\Phi(p, x) = \mathbb{E} \left[ \Phi_j \left( p_1, \ldots, p_j, x + \sum_{k=j+1}^{n} \bar{X}^k_{p_k \vee \tau_{x,k}} \right) \right], \quad (p, x) \in \mathbb{R}^{d+n}
$$

if $j \neq n$ and $\Phi(p, x) = \Phi_n(p, x)$. Using Equation (4.14), we can finally conclude that

$$
\partial_{p_j} \Phi(t, x) = \frac{1}{2} \text{Tr} \left( Q_j, D^2_{x} \Phi(p, x) \right),
$$

for any $(p, x)$ in $\mathbb{R}^{d+n}$ and $j$ in $[1, n]$. \qed

We now extend Theorem 4.3 in order to consider Cauchy Problems with multiple space derivatives of the form:

$$
\begin{align*}
\left\{
\begin{array}{ll}
\partial_t u(t, s, x) = \frac{1}{8} \Delta_x u(t, s, x) + \frac{1}{2} \sum_{j=1}^{n} \sigma_j^2(t) \partial_{s_j}^2 u(t, s, x) + f(t, s, x), & \text{on } \mathbb{R}^{d+n}; \\
u(0, s, x) = 0, & \text{on } \mathbb{R}^{d+n},
\end{array}
\right.
\end{align*}
$$

(4.15)

where $s = (s_1, \ldots, s_n)$ in $\mathbb{R}^n$ and $\sigma_1, \ldots, \sigma_n$ are non-negative time-dependent real functions. For notational simplicity, we denote by $\sigma$ the diagonal time-dependent matrix in $\mathbb{R}^n \otimes \mathbb{R}^n$ given by $\sigma = \text{diag}(\sigma_1, \ldots, \sigma_n)$.

**Corollary 4.5.** Let us assume that:

- $f : \mathbb{R}^{d+n}_\infty \to \mathbb{C}$ is bounded, continuous and such that $f(\cdot, s, \cdot)$ is in $B_b(0, T; A(\mathbb{R}^d))$ at any fixed $s$ in $\mathbb{R}^n$ and $D_f := D_{f(\cdot, s, \cdot)}$ is independent from $s$;
- $\{Y_t\}_{t \geq 0}$ is a $n$-dimensional diffusion associated with $\sigma$;
- for any $j$ in $[1, d]$, $\{X^j_t\}_{t \geq 0}$ is a $d$-dimensional diffusion associated with $e_j \otimes e_j$;
- $\{W_t\}_{t \geq 0}$ is a Brownian Motion on $\mathbb{R}^d$ such that $\{(Y_t, X^1_t, \ldots, X^d_t, W_t)\}_{t \geq 0}$ is an independent process on $\mathbb{R}^{n+d^2+d}$.

Then the function $u : \mathbb{R}^d_\infty \to \mathbb{C}$ defined by

$$
u(t, s, x) = \int_0^t \mathbb{E} \left[ f \left( t-r, s+Y_r, x + \sum_{j=1}^{d} \bar{X}^j_{r \vee \tau_{x,j}} \right) \right] dr,$$

(4.16)

where $\tau_{x,j} := \sup_{t<0} \{x + \bar{X}^j_t \notin D_j\}$, is the unique bounded, continuous solution of Cauchy Problem (4.15).
Proof. As said already at the beginning of the proof of Theorem 4.3, it is easy to check that, under our assumptions, the function \( u \) given in (4.16) is continuous and bounded. Moreover, uniqueness of solutions holds in this class of functions for linear, higher order parabolic problems (cf. [22]).

To conclude, we need to show that \( u \) is indeed a solution of Equation (4.15).

We start considering the function \( F : \mathbb{R}^{2d+n}_\infty \to \mathbb{C} \) given by

\[
F(t, s, p, x) := \mathbb{E} \left[ \tilde{f}(t, s, x + \sum_{j=1}^{d} \tilde{X}_{p_j \lor r_{x,j}}^j) \right], \quad (t, s, p, x) \in \mathbb{R}^{2d+n}_\infty. \tag{4.17}
\]

By Lemma 4.4 for \( n = d \), \( \phi = f(t, s, \cdot) \) and matrices \( Q_j = e_j \otimes e_j \), we know that

\[
\begin{aligned}
\partial_{p_i} F(t, p, s, x) &= \frac{1}{2} \partial^2_{x_1} F(t, p, s, x), & \text{on } \mathbb{R}_0^d \times \mathbb{R}_\infty^{d+n}; \\
& \vdots \\
\partial_{p_d} F(t, p, s, x) &= \frac{1}{2} \partial^2_{x_d} F(t, p, s, x), & \text{on } \mathbb{R}_0^d \times \mathbb{R}_\infty^{d+n}; \\
F(t, s, 0, x) &= f(t, s, x), & \text{on } \mathbb{R}_\infty^{d+n}.
\end{aligned} \tag{4.18}
\]

Keeping in mind the independence between \( \{W_t\}_{t \geq 0} \) and \( \{(Y_t, B_t)\}_{t \geq 0} \), Equation (4.17) allows us to rewrite Representation (4.16) of \( u \) as

\[
u(t, s) = \int_0^t \mathbb{E} \left[ F(t - r, s + Y_r, W_r, x) \right] \, dr, \quad (t, s, x) \in \mathbb{R}_\infty^{2d+n}.
\]

We can then extend \( u \) to \( \mathbb{R}^{2d+n}_\infty \) considering the auxiliary function \( U : \mathbb{R}^{2d+n}_\infty \to \mathbb{C} \) given by

\[
U(t, s, s', x) = \int_0^t \mathbb{E} \left[ F(t - r, s + Y_r, s' + W_r, x) \right] \, dr. \tag{4.19}
\]

Recalling the “usual” Feynman-Kac representation (using \((s, s') \) as space coordinates), we notice that

\[
\partial_t U(t, s, s', x) = \frac{1}{2} \Delta_x U(t, s, s', x) + \frac{1}{2} \sum_{j=1}^{n} \sigma_j(t) \partial^2_{s_j} U(t, s, s', x) + F(t, s, s', x), \tag{4.20}
\]

with initial condition \( U(0, s, s', x) = 0 \). In order to find an explicit representation for \( \partial_{s_k}^k U \), we now exploit (4.18) to write, for any \( k \) in \([1, d]\), that

\[
\partial_{s_k}^k U(t, s, s', x) = \partial_{s_k} \int_0^t \mathbb{E} \left[ F(t - r, s + Y_r, s' + W_r, x) \right] \, dr \\
= \int_0^t \mathbb{E} \left[ \partial_{s_k} F(t - r, s + Y_r, p, x) \right] \bigg|_{p = s' + W_r} \, dr \\
= \frac{1}{2} \partial^2_{s_k} \int_0^t \mathbb{E} \left[ F(t - r, s + Y_r, s' + W_r, x) \right] \, dr = \frac{1}{2} \partial^2_{s_k} U(t, s, s', x).
\]

It then follows immediately that

\[
\partial_t U(t, s, s', x) = \frac{1}{2} \sum_{k=1}^d \partial^2_{s_k} U(t, s, x) + \frac{1}{2} \sum_{j=1}^n \sigma_j(t) \partial^2_{s_j} U(t, s, s', x) + F(t, s, x) \\
= \frac{1}{8} \sum_{k=1}^d \Delta_x U(t, s, s', x) + \sum_{j=1}^n \sigma_j(t) \partial^2_{s_j} U(t, s, s', x) + F(t, s, x) \\
= \frac{1}{8} \Delta_x U(t, s, s', x) + \sum_{k=1}^d \sigma^2(t) \partial^2_{s_k} U(t, s, s', x) + F(t, s, x).
\]
Moreover, we have that \( U(0, s, s', x) = 0 \). Taking \( s' = 0 \), we find that \( u \) is indeed a solution of Equation (4.16) \( \square \)

### 4.2 Proof of Theorem 2.2

Here, we finally present the whole proof of Theorem 2.2. We recall again that such a result is based on the two approaches due to Funaki \([21]\) and Krylov and Priola \([37]\). For notational convenience, we set

\[
B_b\left(0, T; A(\mathbb{R}^d)\right) := B_b\left(0, T; S(\mathbb{R}^d)\right) \cap B_b\left(0, T; C_b^\infty(\mathbb{R}^d)\right).
\]

The main motivation behind the introduction of such a space is that \( B_b\left(0, T; A(\mathbb{R}^d)\right) \) is dense in \( B_b\left(0, T; S(\mathbb{R}^d)\right) \), when equipped with the same family of seminorms. Indeed, the space \( S(\mathbb{R}^d) \cap C_b^\infty(\mathbb{R}^d) \) is dense in \( S(\mathbb{R}^d) \), since it contains all the Hermite functions; See e.g. \([49]\). For any function \( f \) in \( B_b\left(0, T; S(\mathbb{R}^d)\right) \), one can then construct the sequence of functions \( \{f_n\}_{n \in \mathbb{N}} \) in \( B_b\left(0, T; A(\mathbb{R}^d)\right) \) given by

\[
f_n(t, x) := \sum_{j=1}^{2^n} f_{j,n}(x) 1_{[r_{j-1}, r_j)}(t), \quad (t, x) \in \mathbb{R}^d,
\]

where \( \{r_j := T; j2^{-n} : j = 0, \ldots, 2^n\} \) are the dyadic partitions of \([0, T]\) and \( \{f_{j,n}\}_{n \in \mathbb{N}} \) is the approximating sequence in \( A(\mathbb{R}^d) \) of \( f(t^n_{j-1}, x) \).

Thanks to this density argument, it makes sense to rewrite Theorem 2.2 in terms of the new space \( B_b\left(0, T; A(\mathbb{R}^d)\right) \). The main result of this section is given by the following theorem.

**Theorem 4.6.** Let \( \beta \in (0, 1) \), \( p \geq 1 \) and \( f \) in \( B_b\left(0, T; A(\mathbb{R}^d)\right) \). Then, there exists a unique bounded, continuous solution \( u : \mathbb{R}^d_T \to \mathbb{C} \) of Cauchy Problem (1.1). Moreover, there exist two positive constants \( C_\beta := C(\beta, \eta) \), \( C_p := C_p(p, \eta) \) such that

\[
\begin{align*}
\sup_{(t, x) \in \mathbb{R}^d_T} |u(t, x)| & \leq T \sup_{(t, x) \in \mathbb{R}^d_T} |f(t, x)|; \quad (4.21) \\
\sup_{t \in [0, T]} |u(t, \cdot)|_{C^{4+\beta}} & \leq C_\beta \sup_{t \in [0, T]} |f(t, \cdot)|_{C^\beta}; \quad (4.22) \\
\sup_{|\lambda| = 1} \|D^\lambda u\|_{L^p(\mathbb{R}^d_T)} & \leq C_p \|f\|_{L^p(\mathbb{R}^d_T)}. \quad (4.23)
\end{align*}
\]

Once Theorem 4.6 has been showed, the proof of Theorem 2.2 follows immediately.

**Proof of Theorem 2.2.** We are going to follow an approximating procedure, exploiting the density of \( B_b\left(0, T; A(\mathbb{R}^d)\right) \) in the space \( B_b\left(0, T; S(\mathbb{R}^d)\right) \). Indeed, given a function \( f \) in \( B_b\left(0, T; S(\mathbb{R}^d)\right) \), we know that there exists a sequence \( \{f_n\}_{n \in \mathbb{N}} \) in \( B_b\left(0, T; A(\mathbb{R}^d)\right) \) such that \( f_n \to f \) with respect to the topology generated by the seminorms \( \{\sup_{t \in \mathbb{R}^d_T} \beta_m : m \in \mathbb{N}\} \).
Moreover, at any fixed $n$ in $\mathbb{N}$, we can apply Theorem 4.6 for $f_n$. Thus, there exists a unique bounded, continuous function $u_n : \mathbb{R}^d_T \to \mathbb{C}$ such that
\[
u_n(t, x) = \int_0^T \left[ \sum_{|\gamma|=4} A^\gamma(s) D^\gamma u_n(s, x) + f_n(s, x) \right] ds, \quad (t, x) \in \mathbb{R}^d_T. \tag{4.24}
\]
Moreover, there exist two positive constants $C_\beta, C_p$ independent from $n$ such that
\[
\sup_{(t, x) \in \mathbb{R}^d_T} |\nu_n(t, x)| \leq T \sup_{(t, x) \in \mathbb{R}^d_T} |f_n(t, x)|; \tag{4.25}
\]
\[
\sup_{t \in [0, T]} [\nu_n(t, \cdot)]_{C^{4+\beta}} \leq C_\beta \sup_{t \in [0, T]} [f_n(t, \cdot)]_{C^{\beta}}; \tag{4.26}
\]
\[
\sup_{|\gamma|=4} \left\| D^\gamma \nu_n \right\|_{L^p(\mathbb{R}^d_T)} \leq C_p \left\| f_n \right\|_{L^p(\mathbb{R}^d_T)}. \tag{4.27}
\]
It now follows from the Arzelà-Ascoli Theorem that there exists a subsequence $\{u_{n_j}\}_{j \in \mathbb{N}}$ that converges to a function $u$ with respect to the compact-open topology on $\mathbb{R}^d_T$, along with all its derivatives of any order. Moreover, $u$ is bounded continuous and, passing to the limit in (4.24), it holds that
\[
u(t, x) = \int_0^T \left[ \sum_{|\gamma|=4} A^\gamma(s) D^\gamma u(s, x) + f(s, x) \right] ds, \quad (t, x) \in \mathbb{R}^d_T.
\]
Exploiting that the convergence in $S(\mathbb{R}^d)$ is stronger than the convergence with respect to the norms $\|\cdot\|_{\infty}$, $[\cdot]_{C^\beta}$ or $\|\cdot\|_{L^p(\mathbb{R}^d)}$, the Schauder Estimates (4.25)-(4.27) for $u$ and $f$ follow immediately.

The starting point for the proof of Theorem 4.6 is given by the following proposition.

**Proposition 4.7.** The assertions of Theorem 4.6 holds true if
\[
A^\gamma(t) := \begin{cases} 1 & \text{if } \gamma \in \{4e_i : i \in \{1, d\}\}; \\ 0 & \text{otherwise.} \end{cases}
\]

**Proof.** Since we will heavily rely on the results in Subsection 4.1, it is useful to translate Cauchy Problem (1.1) into the tensorial formulation presented at the beginning of Section 4. It corresponds to Equation (4.3) with $A(t) = \mathbb{I}_4$, where $\mathbb{I}_4$ was defined in (4.1). We highlight that, in this case, it holds that
\[
\text{Tr} \left( \mathbb{I}_4, D^4_x \phi(x) \right) = \Delta^2_x \phi(x).
\]

We will follow a reasoning by induction on $d$. Let us assume that the result holds for some dimension $d$ and we will repeat the construction of Section 3 in order to prove it for $d + 1$.

More in details, let us consider a function $f$ in $B_0 \left( 0, T; \mathbb{A}(\mathbb{R}^{d+1}) \right)$. Keeping in mind the proving techniques presented in Section 4.1, we extend $f$ considering the solution $F : \mathbb{R}^{d+2}_T \to \mathbb{C}$ of
\[
\begin{align*}
\partial_s F(t, s, x, y) &= \partial^2_x F(t, s, x, y), \quad \text{on } [0, T] \times \mathbb{R}_0 \times \mathbb{R}^{d+1}; \\
F(t, 0, x, y) &= f(t, x, y), \quad \text{on } \mathbb{R}^{d+1}. \tag{4.28}
\end{align*}
\]
as given in Theorem 4.3 (with $Q = 2$). In particular, we have that

$$F(t, s, x, y) := \mathbb{E} \left[ f(t, x, y + 2\tilde{B}_{s\sigma^+}^y) \right], \quad \tau^y_T := \sup_{t \leq 0} \left\{ y + \tilde{B}_t \notin D_f^x \right\} \quad (4.29)$$

where $\{B_t\}_{t \geq 0}$ is real Brownian Motion and $D_f^x$ is the $x$-section of $D_f$.

From Equation (4.29), we then notice that $F(\cdot, s, \cdot)$ is in $B_0(0, T; \mathcal{A}(\mathbb{R}^{d+1}))$ for any fixed $s$ in $\mathbb{R}$ and that $D_F := D_{F(\cdot, s, \cdot)}$ is independent from $s$. Indeed, it is immediate to check that $F(\cdot, s, \cdot)$ is measurable and bounded as a function in $(t, x, y)$.

To show instead that $F(t, s, \cdot, \cdot)$ is real-analytic and its domain of holomorphy does not depend on $(t, s)$, we start considering its complex extension $F(t, s, z, z')$ defined by

$$F(t, s, z, z') := \mathbb{E} \left[ f(t, z, z' + 2\tilde{B}_{s\sigma^+}^z) \right], \quad z \in \mathbb{C}^d, z' \in \mathbb{C},$$

where, as above, $\tau^z_T := \sup_{t \leq 0} \{ z' + \tilde{B}_t \notin D_f^{z'} \}$ and $D_f^{z'}$ is the $z'$-section of $D_f$ in $\mathbb{C}^{d+1}$.

To prove that $F(t, s, \cdot, \cdot)$ is holomorphic on $D_f$, we now exploit Morera’s Theorem (cf. [4]). This will imply that $D_F \subseteq D_f$, too.

Let us fix a closed ball $B$ in $D_f$. It then holds, for any fixed $(t, s)$ in $\mathbb{R}_T$, that

$$\int_{\partial B} F(t, s, z, z') \, dz \, dz' = \int_{\partial B} \mathbb{E} \left[ f(t, z, z' + \tilde{B}_{s\sigma^+}^z) \right] \, dz \, dz' = \mathbb{E} \left[ \int_{\partial B} f(t, z, z' + \tilde{B}_{s\sigma^+}^z) \, dz \, dz' \right] = 0$$

where in the last passage, we used that $f(t, z, z' + \tilde{B}_{s\sigma^+}^z)$ is holomorphic on $D_f$. By Morera’s Theorem, we can thus conclude that $F(t, s, \cdot, \cdot)$ is holomorphic on $D_f$.

By inductive hypothesis, we know that there exists a unique bounded, continuous solution $u: \mathbb{R}^{d+2}_T \to \mathbb{C}$ of Cauchy Problem

$$\begin{cases}
\partial_t u(t, s, x, y) = \Delta^2_x u(t, s, x, y) + F(t, s, x, y), & \text{on } \mathbb{R}^{d+2}_T; \\
u(0, s, x, y) = 0, & \text{on } \mathbb{R}^{d+2},
\end{cases}$$

and that the following estimates hold

$$\sup_{(t, x, y) \in \mathbb{R}^{d+1}_T} |u(t, s, x, y)| \leq T \sup_{(t, x, y) \in \mathbb{R}^{d+1}_T} |F(t, s, x, y)|; \quad \text{(4.30)}$$

$$\sup_{(t, y) \in \mathbb{R}_T} [u(t, s, \cdot, y)]_{C^{4+\beta}} \leq C_\beta \sup_{(t, y) \in \mathbb{R}_T} [F(t, s, \cdot, y)]_{C^{\beta}}; \quad \text{(4.31)}$$

$$\sup_{|l| = 1} \|D^l_t u(t, s, \cdot, y)(x)\|_{L^p(\mathbb{R}^{d+1}_T)} \leq C_p \|F(t, s, x, y)\|_{L^p(\mathbb{R}^{d+1})}. \quad \text{(4.32)}$$

Using the same arguments appearing in [37], Lemma 3.2, it is now possible to show the existence of a unique bounded, continuous solution $v: \mathbb{R}^{d+2}_T \to \mathbb{R}$ of Cauchy Problem:

$$\begin{cases}
\partial_t v(t, s, x, y) = \Delta^2_x v(t, s, x, y) + \partial^2_x v(t, s, x, y) + F(t, s, x, y), & \text{on } \mathbb{R}^{d+2}_T; \\
v(0, s, x, y) = 0, & \text{on } \mathbb{R}^{d+2}. \quad \text{(4.33)}
\end{cases}$$
Moreover, the function \( v \) satisfies again Estimates (4.30)-(4.32) above. Using now Corollary 4.5 with space coordinate \((s, x)\) in \( \mathbb{R}^{d+1} \) at any fixed \( y \) in \( \mathbb{R} \), we know that the solution \( v \) has the following form:

\[
v(t, s, x, y) = \int_0^t \mathbb{E} \left[ \mathcal{F}(t-r, s+2W^1_r, x+8\tilde{B}_{W^2_{\tau^y_r}} y) \right] dr,
\]

where \( \tau^y_x := \sup_{t<0} \{ x + 8\tilde{B}_t \notin D_y \} \), \( D^y_\tau \) is the \( y \)-section of \( D_f \) and \( \{(W^1_t, W^2_t)\}_{t \geq 0} \), \( \{B_t\}_{t \geq 0} \) are two independent Brownian Motions on \( \mathbb{R}^2 \) and \( \mathbb{R}^d \), respectively.

Recalling the dynamics of \( F \) given in Equation (4.28), we can now show that

\[
\partial_s v(t, s, x, y) = \int_0^t \mathbb{E} \left[ \partial_s \mathcal{F}(t-r, s+2W^1_r, x+8\tilde{B}_{W^2_{\tau^y_r}} y) \right] dr = \partial^2_s v(t, s, x, y).
\]

Hence, it follows from Equation (4.33) that \( v \) is also solution of the following Cauchy Problem:

\[
\begin{aligned}
\left\{ \begin{array}{ll}
\partial_t v(t, s, z) &= \Delta^2 v(t, s, z) + F(t, s, z), & \text{on } \mathbb{R}^{d+2}_T, \\
v(0, s, z) &= 0, & \text{on } \mathbb{R}^{d+2},
\end{array} \right.
\]

where \( z = (x, y) \) in \( \mathbb{R}^{d+1} \). Taking \( s = 0 \) above, we then obtain that the function \( w(t, z) := v(t, 0, z) \) is the unique bounded, continuous solution of the following dynamics:

\[
\begin{aligned}
\left\{ \begin{array}{ll}
\partial_t w(t, z) &= \Delta^2 w(t, z) + f(t, z), & \text{on } \mathbb{R}^{d+1}_T, \\
w(0, z) &= 0, & \text{on } \mathbb{R}^{d+1},
\end{array} \right.
\]

and it holds that

\[
\sup_{(t, z) \in \mathbb{R}^{d+1}_T} |w(t, z)| \leq T \sup_{(t, z) \in \mathbb{R}^{d+1}_T} |f(t, z)|;
\]

\[
\sup_{(t, y) \in \mathbb{R}^T} \left[ w(t, \cdot, y) \right]_{C^{4+\beta}} \leq C_\beta \sup_{(t, y) \in \mathbb{R}^T} \left[ f(t, \cdot, y) \right]_{C^\beta};
\]

\[
\sup_{||l||=1} \left\| D^l_t w(t, \cdot, y) \right\|_{L^p(\mathbb{R}^{d+1}_T)} \leq C_p \left\| f(t, x, y) \right\|_{L^p(\mathbb{R}^{d+1})}.
\]

We can finally use the invariance of the bi-Laplacian \( \Delta^2 \) and the same arguments appeared in (3.13)-(3.19), to show that the above estimates hold with respect to the derivatives in \( y \), too. The reasoning by induction is now concluded.

In order to prove the estimates with respect to the standard Hölder norm \([\cdot]_{C^{4+\beta}}\), it is enough to apply Lemma 2.1. \( \square \)

Proposition 4.7 represents the first step of our proof. We now proceed by establishing Schauder Estimates for linear combinations of \( A^\gamma \).

**Proposition 4.8.** The assertions of Theorem 4.6 holds true if, in addition, \( \eta > 1 \) and \( A^\gamma : [0, T] \rightarrow \mathbb{R} \) is continuous for any \( \gamma \) in \( \mathbb{N}^d \) such that \( |\gamma| = 4 \).

**Proof.** In the same way as above, we start rewriting our problem by using the tensorial formalism introduced at the beginning of Section 4. We are interested in Cauchy Problem (4.3) for some continuous, symmetric \( A : [0, T] \rightarrow \left( \mathbb{R}^d \right)^{\otimes 4} \) such that

\[
\text{Tr} \left( A(t), \xi^{\otimes 4} \right) > (1 + \epsilon) |\xi|^4, \quad (t, \xi) \in \mathbb{R}^d_T
\]
for some $\epsilon > 0$, since $\eta$ is bigger than one.
Let us now consider another time-dependent tensor $\hat{A}(t)$ of order 4 given by

$$\hat{A}(t) := A(t) - I_4, \quad t \in [0, T].$$

It is easy to check that $\hat{A}(t)$ is again symmetric, continuous in time and such that

$$\text{Tr}(\hat{A}(t), \xi^{\otimes 4}) > \epsilon |\xi|^4, \quad (t, \xi) \in \mathbb{R}^d_T.$$

Since it is continuous in time, it follows immediately that

$$\text{Tr}(I_4, \hat{A}(t)) < M, \quad t \in [0, T],$$

for some positive constant $M$. We then have that $\{\hat{A}(t): t \in [0, T]\}$ is a closed subset of $\mathcal{E}(\epsilon, M)$ as defined in Lemma 4.1.

Applying now Lemma 4.1 to $\hat{A}$, we notice that we can decompose $A$ as:

$$A(t) = I_4 + \sum_{j=1}^{n} \sigma_j^2(t)Q_j^{\otimes 2}, \quad (4.34)$$

where $\sigma_1, \ldots, \sigma_n$ from $[0, T]$ to $\mathbb{R}$ and some non-negative matrices $Q_1, \ldots, Q_n$.

As above, we extend $f$ to $\mathbb{R}^{d+n}_T$ and some non-negative matrices $Q_1, \ldots, Q_n$.

Such a function $F$ is given in Lemma 4.4 for $\hat{Q}_j = 2Q_j$. As noticed in the proof of the previous Theorem (cf. arguments below Equation (4.29)), the function $F(\cdot, s, \cdot)$ is in $C^4(B_0(0, T; A(\mathbb{R}^d)))$ at any fixed $s$ in $\mathbb{R}^n$. Moreover, $D_F := D_{F(t, s, \cdot)}$ is independent from $s$.

Thanks to Proposition 4.7, we already know that there exists a unique bounded, continuous solution $u: \mathbb{R}^{d+2}_T \to C$ of Cauchy Problem

$$\begin{cases}
\partial_t u(t, s, x) = \Delta_2^2 u(t, s, x) + F(t, s, x), & \text{on } \mathbb{R}^{d+n}_T; \\
u(0, s, x) = 0, & \text{on } \mathbb{R}^{d+n}_T,
\end{cases} \quad (4.35)$$

and that, at any fixed $s \in \mathbb{R}^n$, it satisfies the following estimates:

$$\sup_{(t, x) \in \mathbb{R}^d_T} |u(t, s, x)| \leq T \sup_{(t, x) \in \mathbb{R}^d_T} |F(t, s, x)|; \quad (4.36)$$

$$\sup_{t \in [0, T]} [u(t, s, \cdot)]_{C^{4, \alpha}} \leq C_\alpha \sup_{t \in [0, T]} [F(t, s, \cdot)]_{\tilde{C}^{4, \alpha}}; \quad (4.37)$$

$$\sup_{\beta=1} \| D_4^\beta u(t, s, x) \|_{L^p(\mathbb{R}^d_T)} \leq C_p \| F(t, s, x) \|_{L^p(\mathbb{R}^d_T)}. \quad (4.38)$$
The same arguments appearing in Lemma 3.4 of [37] can be applied in order to show the existence of a unique bounded, continuous solution $v: \mathbb{R}^{d+n} \to \mathbb{C}$ of:

$$\begin{cases}
\partial_t v(t, s, x) = \Delta_x^2 v(t, s, x) + \sum_{j=1}^n \sigma_j^2 \partial_{x_j} v(t, s, x) + F(t, s, x); \\
v(0, s, x) = 0.
\end{cases} \quad (4.39)$$

Moreover, Schauder Estimates (4.36)-(4.38) still hold (with $u$ replaced by $v$). Considering $(s, x)$ as space coordinate, we can then exploit Corollary 4.5 to obtain the following representation for $v$:

$$v(t, s, x) = \int_0^t \mathbb{E} \left[ F(t - r, s + Y_r, x + 8\tilde{B}_{W_r} \vee \tau_x) \right] dr, \quad \tau_x := \sup_{t < 0} \{ x + 8\tilde{B}_t \notin D_F \},$$

where $\{Y_t\}_{t \geq 0}$ is a $n$-dimensional diffusion associated to the matrix $\sigma = \text{diag}(\sigma_1, \ldots, \sigma_n)$, $\{B_t\}_{t \geq 0}$ and $\{W_t\}_{t \geq 0}$ are two Brownian Motions on $\mathbb{R}^d$ and $\mathbb{R}$, respectively. Moreover, all the involved processes are assumed to be independent.

Thanks to Equation (4.35), we can finally show that for any $j$ in $[1, n]$, it holds that

$$\partial_{s_j} v(t, s, x) = \int_0^t \mathbb{E} \left[ \partial_{s_j} F(t - r, s + W_r, x + 8\tilde{B}_{W_r} \vee \tau_x) \right] dr = \text{Tr} \left( Q_j, D_x^2 v(t, s, x) \right).$$

Hence, it follows from Equation (4.39) (and the trace property in (4.2)) that $v$ also solves the following equation:

$$\begin{cases}
\partial_t v(t, s, x) = \Delta_x^2 v(t, s, x) + \sum_{j=1}^n \sigma_j^2(t) \text{Tr} \left( Q_j^{\sigma_2}, D_x^2 v(t, s, x) \right) + F(t, s, x) & \text{on } \mathbb{R}^{d+n}_T; \\
v(0, s, x) = 0 & \text{on } \mathbb{R}^{d+n}.
\end{cases}$$

Recalling decomposition (4.34), we can easily conclude the proof taking $s = 0$. \qed

We can finally show the last step of our construction and give a proof of Theorem 4.6.

**Proof of Theorem 4.6.** By assumptions, we consider a family $\{A^\gamma : |\gamma| = 4\}$ of real coefficients on $[0, T]$ such that assumptions $[UE]$ and $[I]$ hold. Dividing by $\eta$ if necessary, we can assume without loss of generality that $\eta = 1$.

With a view to mollifying the coefficients, we start extending any $A^\gamma$ to $\mathbb{R}$ defining $A^\gamma(t) = A^\gamma(T/2)$ if $t < 0$ or $t > T$. We can then consider the family $\{A^\gamma_n\}_{n \in \mathbb{N}}$ of approximated coefficients given by

$$A^\gamma_n(t) := \left( 1 + \frac{1}{n} \right) A^\gamma \ast \rho_n(t), \quad t \in [0, T],$$

where $\{\rho_n\}_{n \in \mathbb{N}}$ is a standard mollifier on $\mathbb{R}$. It is now easy to check that the family $\{A^\gamma_n : |\gamma| = 4\}$ satisfies the hypothesis of Proposition 4.8, at any fixed $n$ in $\mathbb{N}$. Hence, there exists a unique bounded, continuous function $u_n: \mathbb{R}^d_T \to \mathbb{R}$ such that

$$u_n(t, x) = \int_0^t \left[ \sum_{|\gamma|=4} A^\gamma_n(s) D_x^2 u_n(s, x) + f(s, x) \right] ds, \quad (4.40)$$

27
and the Schauder Estimates (4.21)-(4.23) hold for any $u_n$.

The next step is to let $n$ go to infinity, in order to bring back the original coefficients. We start noticing from Equation (4.40) that the sequence $\{u_n\}_{n \in \mathbb{N}}$ is uniformly Lipschitz on any compact subset of $\mathbb{R}^d$. Similar arguments work also for the sequence of derivatives of any order of $u_n$ with respect to $x$. Arzelà-Ascoli Theorem then implies the existence of a subsequence $\{u_n\}_{n \in \mathbb{N}}$ which converges to a function $u: \mathbb{R}^d_T \to \mathbb{R}$, uniformly on any compact set of $\mathbb{R}^d_T$.

Passing to the limit, as $n \to \infty$, in Equation (4.40) or in Schauder Estimates (4.21)-(4.23) for any $u_n$, the result follows immediately.

5 Applications

We present here some useful applications of our method, particularly of Theorem 2.2, to other different settings. In the first part, we will focus on the elliptic framework while an additional non-linearity in time for the parabolic Cauchy Problem will be considered at the end of the section.

Let us start showing how the well-posedness and associated Schauder Estimates presented in Theorem 2.2 for the parabolic framework can be adapted to the elliptic case.

**Proposition 5.1.** Let $\beta$ be in $(0,1)$, $p \geq 1$, $g$ in $\mathcal{S}(\mathbb{R}^d)$ and let us assume that the coefficients $A^\gamma$ are independent from $t$. Then, there exists a unique bounded, continuous solution $u: \mathbb{R}^d \to \mathbb{C}$ of Elliptic Equation:

$$\sum_{|\gamma|=4} A^\gamma D^\gamma_x u(x) = g(x), \quad \text{on } \mathbb{R}^d. \quad (5.1)$$

Moreover, it holds that

$$[u]_{C^{4+\beta}} \leq C_\beta [g]_{C^3};$$

$$\sup_{|l|=1} \|D^l_x u\|_{L^p(\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\mathbb{R}^d)},$$

where $C_\beta := C(\beta, \eta)$, $C_p := C_p(p, \eta)$ are the same constants appearing in Theorem 2.2.

**Proof.** Under our regularity assumptions, the existence of a bounded, continuous solution of Equation (5.1) is well-known; See e.g. [31]. We start with the issue of uniqueness. Fixed a final time $T > 0$, let us consider a bounded, continuous solution $u: \mathbb{R}^d \to \mathbb{C}$ of Equation (5.1). It is easy to check that the function $v: \mathbb{R}^d_T \to \mathbb{C}$ given by $v(t,x) := u(x)t/T$ is then a bounded, continuous solution of the following Cauchy Problem:

$$\begin{cases}
\partial_t v(t,x) = \sum_{|\gamma|=4} A^\gamma D^\gamma_x v(t,x) + f(t,x), \quad \text{on } \mathbb{R}^d_T; \\
v(0,x) = 0, \quad \text{on } \mathbb{R}^d,
\end{cases}$$

where $f(t,x) = u(x)/T - g(x)t/T$. Thus, the uniqueness of $u$ follows immediately from the uniqueness of solutions in Theorem 2.2. Moreover, we know that $v$ satisfies
Schauder Estimates (4.21)-(4.23). Hence, we have that
\[
[u]_{C^{4+\beta}} = \sup_{t \in [0,T]} [v(t,\cdot)]_{C^{4+\beta}} \leq C_\beta \sup_{t \in [0,T]} [f(t,\cdot)]_{C^\beta} = C_\beta \left( \frac{[u]_{C^\beta}}{T} - [g]_{C^\beta} \right).
\]
We can now let \(T\) go to infinity in the equation above. Recalling that \(C_\beta\) is independent from \(T\), we conclude that
\[
[u]_{C^{4+\beta}} \leq C_\beta [g]_{C^\beta}.
\]
In order to prove the estimates in \(L^p\)-spaces, we fix a unitary \(l\) in \(\mathbb{R}^d\) and we start noticing that
\[
\|D_t^4 v\|_{L^p(\mathbb{R}^d)}^p = \int_0^T \left( \frac{t}{T} \right)^p \int_{\mathbb{R}^d} |D_t^4 u(x)|^p \, dx \, dt = \frac{1}{p+1} \|D_t^4 u\|_{L^p(\mathbb{R}^d)}^p.
\]
Using Schauder Estimates (4.23) for \(v\), we then show that
\[
\|D_t^4 u\|_{L^p(\mathbb{R}^d)}^p = \frac{p+1}{T} \|D_t^4 v\|_{L^p(\mathbb{R}^d)}^p \leq C_p \frac{p+1}{T} \|f\|_{L^p(\mathbb{R}^d)}^p \leq C_p \frac{p+1}{T} \int_0^T \int_{\mathbb{R}^d} \left| u(x) - g(x) \right|^p \, dx \, dt
\]
\[
= C_p (p+1) \int_0^T \int_{\mathbb{R}^d} \left| u(x) - sg(x) \right|^p \, dx \, ds,
\]
where, in the last step, we applied the change of variables \(s = t/T\). Letting \(T\) go to zero above, we can conclude that
\[
\|D_t^4 u\|_{L^p(\mathbb{R}^d)}^p \leq C_p \|g\|_{L^p(\mathbb{R}^d)}^p \int_0^1 (p+1) s^p \, ds = C_p \|g\|_{L^p(\mathbb{R}^d)}^p.
\]

In the following, we go back to the parabolic Cauchy problem (1.1) with the purpose of adding a non-linearity to the dynamics, even if only dependent in time. Namely, we are interested in Cauchy problems driven by operators of the following form:
\[
L_t \phi(t,x) := \sum_{|\gamma|=4} A^\gamma(t) D_\gamma \phi(x) + \langle F(t), D_x \phi(x) \rangle - c(t) \phi(x), \quad (t,x) \in \mathbb{R}^d_T, \tag{5.2}
\]
where \(c: [0,T] \to [0,\infty)\) and \(F: [0,T] \to \mathbb{R}^d\) are two functions. For any sufficiently regular function \(\phi: [0,T] \to \mathbb{R}\), we are going to denote
\[
\mathcal{T} \phi(t,x) := e^{-\int_0^t c(s) \, ds} \phi(t, x + \int_0^t F(s) \, ds), \quad (t,x) \in \mathbb{R}^d_T. \tag{5.3}
\]
We will see in the next result that the “operator” \(\mathcal{T}\) transforms solutions of the Cauchy Problem (1.1) to solutions of the Cauchy Problem driven by \(L_t\), even if for a modified drift \(\mathcal{T} f\). Similar results in the framework of integro-partial differential equations are given in [43].
Lemma 5.2. Let $u_0$ be in $S(\mathbb{R}^d)$, $f$ in $B_{b}\left(0,T;S(\mathbb{R}^d)\right)$ and $c$, $F$ in $C_{b}([0,T])$. Then, a function $u: \mathbb{R}^d_T \to \mathbb{C}$ is a solution of Cauchy Problem (1.1) if and only if the function $v: \mathbb{R}^d_T \to \mathbb{C}$ given by $v(t,x) = \mathcal{T}u(t,x)$ is a solution of the following Cauchy Problem:

$$
\begin{align*}
\partial_t u(t,x) &= L_t u(t,x) + \mathcal{F}(t,x), \quad \text{on } \mathbb{R}^d_T; \\
u(0,x) &= u_0(x),
\end{align*}
$$

(5.4)

In particular, there exists a unique bounded, continuous solution $v: \mathbb{R}^d_T \to \mathbb{C}$ of Cauchy Problem (5.4).

Proof. Given a bounded, continuous solution $u$ of Cauchy Problem (1.1), we are going to show that the function $v := \mathcal{T}u$ given in (5.3) is indeed a bounded, continuous solution of Cauchy Problem (5.4). The inverse implication can be obtained analogously and for this reason, we will not show it here.

Let us assume for the moment that $u$ and $v$ are smooth in time. Then, a direct calculation shows that

$$
\partial_t v(t,x) = e^{-\tilde{c}(t)} \left[ \partial_t u(t,x - \tilde{F}(t)) + \langle F(t), D_x u(t,x - \tilde{F}(t)) \rangle - c(t) u(t,x - \tilde{F}(t)) \right],
$$

where we have denoted for simplicity

$$
\tilde{c}(t) := \int_0^t c(s) \, ds \quad \text{and} \quad \tilde{F}(t) := \int_0^t F(s) \, ds.
$$

(5.5)

Using that $u$ is an integral solution of (1.1) and exploiting a translation, it follows immediately that

$$
v(t,x) = \int_0^t \left[ \sum_{|\gamma|=4} A^\gamma(s) D_x^\gamma v(s,x) + \langle F(s), D_x v(s,x) \rangle - c(s) v(s,x) + f(s,x) \right] \, ds.
$$

The general result, when $u$, $v$ are only continuous in time, can be obtained through mollification, employing the stability given by the Schauder Estimates (4.21)-(4.23). It is straightforward to check that the function $\mathcal{T}u$ is bounded and continuous. $\square$

Thanks to the previous lemma, we are now able to show the Schauder Estimates for the solution $v$ of the Cauchy Problem (5.4). More importantly, we do it without changing the constants $C_\beta$, $C_p$ appearing in Theorem 2.2.

Proposition 5.3. Under the same assumptions of Lemma 5.2, let $\beta$ be in $(0,1)$, $p \geq 1$.

Then, the unique bounded, continuous solution $v: \mathbb{R}^d_T \to \mathbb{C}$ of Cauchy Problem (5.4) satisfies

$$
\sup_{(t,x) \in \mathbb{R}^d_T} |u(t,x)| \leq T \sup_{(t,x) \in \mathbb{R}^d_T} |f(t,x)|;
$$

(5.6)

$$
\sup_{t \in [0,T]} \left[ u(t,\cdot) \right]_{C^{4+\beta}} \leq C_\beta \sup_{t \in [0,T]} \left[ f(t,\cdot) \right]_{C^\beta};
$$

(5.7)

$$
\sup_{|l|=1} \left\| D^l u \right\|_{L^p(\mathbb{R}^d_T)} \leq C_p e^{\tilde{c}(T)} \|f\|_{L^p(\mathbb{R}^d_T)},
$$

(5.8)

where $C_\beta := C(\beta,\eta,k)$, $C_p := C_p(p,\eta,k)$ are the same constants appearing in Theorem 2.2, and $\tilde{c}$ is defined in (5.5).
Proof. We will exploit the same notation (5.5) used in the previous lemma. By Lemma 5.2, we know that if $v$ is a bounded, continuous solution of Cauchy Problem (5.4), then the function

$$u(t, x) := e^{	ilde{c}(t)}v(t, x - \tilde{F}(t))$$

is the unique bounded, continuous solution of Cauchy Problem (1.1) with $\tilde{f}$ instead of $f$, where

$$\tilde{f}(t, x) := e^{\tilde{c}(t)}f(t, x - \tilde{F}(t)), \quad (t, x) \in \mathbb{R}^d_T.$$ Moreover, we have that $\tilde{f}$ is in $B_b\left(0, T; C^\infty_c(\mathbb{R}^d)\right)$.

Using now the invariance of the Hölder norm under translations, we can show that

$$\left\|e^{\tilde{c}(t)}v(t, \cdot - \tilde{F}(t))\right\|_{C^\alpha+\beta} \leq C_\beta \sup_{s \in [0,t]} \left\|e^{\tilde{c}(s)}f(s, \cdot - \tilde{F}(t))\right\|_{C^\beta},$$

Using now the invariance of the Hölder norm under translations, we can show that

$$\left\|v(t, \cdot)\right\|_{C^\alpha+\beta} \leq C_\beta e^{-\tilde{c}(t)} \sup_{s \in [0,t]} \left\|e^{\tilde{c}(s)}f(s, \cdot)\right\|_{C^\beta} \leq C_\beta \sup_{s \in [0,t]} \left\|f(s, \cdot)\right\|_{C^\beta};$$

where in the last step we exploited that $\tilde{c}(t)$ is non-decreasing. Taking the supremum with respect to $t$ on both sides of the above inequality, we obtain the second inequality.

For the last estimate, we are going to use the following isomorphism:

$$L^p(\mathbb{R}^d_T) = L^p([0, T], \mathbb{R}^d) \simeq L^p\left([0, T], L^p(\mathbb{R}^d)\right). \quad (5.9)$$

By Theorem 2.2 and Equation (5.9), we have that

$$\int_0^T e^{\tilde{c}(t)p} \left\|D_t^i v(t, \cdot - \tilde{F}(t))\right\|^p_{L^p(\mathbb{R}^d)} dt \leq C_p \int_0^T e^{\tilde{c}(t)p} \left\|f(t, \cdot - \tilde{F}(t))\right\|^p_{L^p(\mathbb{R}^d)} dt.$$

We notice now that, by a standard change of variables we can get rid of the $\tilde{F}(t)$. Thanks to Hölder inequality we obtain:

$$\int_0^T e^{\tilde{c}(t)p} \left\|D_t^i v(t, \cdot)\right\|^p_{L^p(\mathbb{R}^d)} dt \leq C_p \int_0^T e^{\tilde{c}(t)p} \left\|f(t, \cdot)\right\|^p_{L^p(\mathbb{R}^d)} dt \leq C_p e^{\tilde{c}(T)p} \int_0^T \left\|f(t, \cdot)\right\|^p_{L^p(\mathbb{R}^d)} dt.$$

Using the fact that $e^{\tilde{c}(t)} \geq 1$ for all $t \in [0, T]$, we have that

$$\int_0^T \left\|D_t^i v(t, \cdot)\right\|^p_{L^p(\mathbb{R}^d)} dt \leq C_p e^{\tilde{c}(T)p} \int_0^T \left\|f(t, \cdot)\right\|^p_{L^p(\mathbb{R}^d)} dt.$$ We have finally concluded the proof:

$$\left\|D_t^i v(t, \cdot)\right\|_{L^p(\mathbb{R}^d)} \leq C_p e^{\tilde{c}(T)p} \left\|f(t, \cdot)\right\|_{L^p(\mathbb{R}^d)}.$$
6 Further Investigations

Starting from two different approaches, we have obtained multidimensional Schauder Estimates for the fourth order heat-type equation. The key role here is due to the method developed by Krylov and Priola joint with the probabilistic construction of the solution of a higher-order parabolic differential equations proposed by Funaki. It is remarkable notice that there are many possible interesting issues, which should be investigated in that direction.

The most natural generalization of this work is the extension of our method to all the orders. This improvement presents several difficulties linked to the use of the complex extension. In particular, the direct use of our approach leads to a combinatorial interaction between our complex variables, which should be analyzed individually with respect to their stopping time. In the fourth-order case, these complications disappear thanks to the symmetry of the problem. In this sense, this work could be considered as the first step of a deeper analysis around the Schauder Estimates for higher-order parabolic equations.

Another interesting developments is linked to the following construction. Let us introduce a suitable parabolic Hölder space as done in [33]. In this framework one can provide a new formulation for the fourth-order problem. In particular, it is possible to prove the equivalence of the original issue with the following system of “standard” heat equations:

\[
\begin{align*}
\partial_t w(t, s, x) &= \frac{1}{2} \partial_x^2 w(t, s, x) + F(t, s, x); \\
w(0, s, x) &= 0; \\
\partial_s w(t, s, x) &= \frac{1}{2} \partial_x^2 w(t, s, x); \\
w(t, 0, x) &= \mathcal{F}(t, x).
\end{align*}
\]

Here we have denoted

\[
\mathcal{F}(t, x) = \int_0^t \mathbb{E}[F(t - r, W_r, x)] \, dr.
\]

Now, if we look only at the first two equations in (6.1), it is well-known (see e.g. [33]) that

\[
[w(\cdot, \cdot, x)]_{1+\gamma/4, 2+\gamma/2} \leq C[F(\cdot, \cdot, x)]_{\gamma/4, \gamma/2}
\]

holds at any (fixed) \(x\) in \(\mathbb{R}\). On the other hand, the last two equations in (6.1) can be considered to show that

\[
[w(t, \cdot, \cdot)]_{\beta, \beta} \leq C[\mathcal{F}(t, \cdot)]_{\beta}
\]

at any (fixed) \(t > 0\). If we assume that the regularity with respect to the parameters remains invariant between the source and the solution and we then “compose” the Schauder Estimates, we cannot reach the “expected” Hölder exponents. Indeed, we can obtain only an Hölder regularity of order \(2 + \gamma\) with respect to \(x\).

Moreover, one can investigate, by using our construction, a class of degenerate Ornstein-Uhlenbeck operators in \(\mathbb{R}^d\), in the spirit of [16]. Another opportunity is given by the Funaki’s approach, which could be used to give a new point of view for the study of
fractional Brownian Motion (following the work started in [12]). Finally, a more in-depth analysis of the method proposed in this work could be useful to provide Schauder Estimates in the case of the Schrödinger equation; in fact, an exhaustive study about that seems only at the beginning (see, e.g., [30, 45]). These suggestive developments represent the next steps of our research.

References

[1] S. Albeverio, N. Cangiotti, and S. Mazzucchi. Generalized Feynman path integrals and applications to higher-order heat-type equations. *Expositiones Mathematicae*, 36(3–4):406–429, 2018.

[2] H. Allouba. Brownian-time processes: The PDE connection II and the corresponding Feynman-Kac formula. *Transactions of the American Mathematical Society*, 354(11):4627–4637, 2002.

[3] H. Allouba and W. Zheng. Brownian-time processes: the PDE connection and the half-derivative generator. *Annals of Probability*, 29(4):1780–1795, 2001.

[4] C. Berenstein, D.-C. Chang, D. Pascuas, and L. Zalcman. Variations on the theorem of Morera. In *The Madison Symposium on Complex Analysis (Madison, WI, 1991)*, volume 137 of *Contemporary Mathematics*, pages 63–78. American Mathematical Society, Providence, RI, 1992.

[5] A. Bernardi and D. Taufer. Waring, tangential and cactus decompositions. *Journal de Mathématiques Pures et Appliquées*, 143:1–30, 2020.

[6] S. Boccia. Schauder estimates for solutions of higher-order parabolic systems. *Methods and Applications of Analysis*, 20(1):47–67, 2013.

[7] S. Bonaccorsi, M. D’Ovidio, and S. Mazzucchi. Probabilistic representation formula for the solution of fractional high-order heat-type equations. *Journal of Evolution Equations*, 11 2016.

[8] S. Bonaccorsi and S. Mazzucchi. High order heat-type equations and random walks on the complex plane. *Stochastic Processes and their Applications*, 125(2):797–818, 2015.

[9] J. Brachat, P. Comon, B. Mourrain, and E. Tsigaridas. Symmetric tensor decomposition. *Linear Algebra and its Applications*, 433(11):1851–1872, 2010.

[10] M. Bramanti and L. Brandolini. Schauder estimates for parabolic nondivergence operators of Hörmander type. *Journal of Differential Equations*, 234(1):177 – 245, 2007.

[11] K. Burdzy. Some path properties of iterated Brownian motion. In *Seminar on Stochastic Processes, 1992 (Seattle, WA, 1992)*, volume 33 of *Progress in Probability*, pages 67–87. Birkhäuser Boston, Boston, MA, 1993.
P.-É. Chaudru de Raynal, S. Menozzi, and E. Priola. Schauder estimates for drifted fractional operators in the supercritical case. *Journal of Functional Analysis*, 278(8):108425, 57, 2020.

H. Dong and D. Kim. Higher order elliptic and parabolic systems with variably partially BMO coefficients in regular and irregular domains. *Journal of Functional Analysis*, 261(11):3279 – 3327, 2011.

H. Dong and D. Kim. On the $L_p$-solvability of higher order parabolic and elliptic systems with BMO coefficients. *Archive for Rational Mechanics and Analysis*, 199(3):889–941, 2011.

H. Dong and H. Zhang. Schauder estimates for higher-order parabolic systems with time irregular coefficients. *Calculus of Variations and Partial Differential Equations*, 54(1):47 – 74, 2015.

M. Bramanti, G. Cupini, E. Lanconelli et al. Global $L^p$ estimates for degenerate Ornstein-Uhlenbeck operators. *Mathematische Zeitschrift*, 266:789 – 816, 2010.

L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

P. M. N. Feehan and C. A. Pop. Schauder a priori estimates and regularity of solutions to boundary-degenerate elliptic linear second-order partial differential equations. *Journal of Differential Equations*, 256(3):895–956, 2014.

Mark Freidlin. *Functional integration and partial differential equations*, volume 109 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1985.

A. Friedman. *Partial differential equations of parabolic type*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964.

T. Funaki. Probabilistic construction of the solution of some higher order parabolic differential equation. *Proceedings of the Japan Academy, Ser. A, Mathematical Sciences*, 55(5):176–179, 1979.

I. M. Gel’fand and G. E. Shilov. *Generalized Functions, Volume 3: Theory of Differential Equations*. AMS Chelsea Publishing. American Mathematical Society, 2016.

M. Giaquinta. *Introduction to regularity theory for nonlinear elliptic systems*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1993.

K. Górska, A. Horzela, K. A. Penson, and G. Dattoli. The higher-order heat-type equations via signed Lévy stable and generalized Airy functions. *Journal of Physics. A. Mathematical and Theoretical*, 46(42):425001, 16, 2013.

C. E. Gutiérrez and E. Lanconelli. Schauder estimates for sub-elliptic equations. 9, 707 (2009). *Journal of Evolution Equations*, 9(707), 2009.

Q. Han. On the Schauder estimates of solutions to parabolic equations. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze*, Ser. 4, 27(1):1–26, 1998.
[27] Z. Hao, Z. Wang, and M. Wu. Schauder’s estimates for nonlocal equations with singular Lévy measures. Preprint: https://arxiv.org/abs/2002.09887v1, 2020.

[28] Z. Hao, M. Wu, and X. Zhang. Schauder estimates for nonlocal kinetic equations and applications. *Journal de Mathématiques Pures et Appliquées*, 140:139 – 184, 2020.

[29] K. J. Hochberg and E. Orsingher. Composition of stochastic processes governed by higher-order parabolic and hyperbolic equations. *Journal of Theoretical Probability*, 9:511–532, 1996.

[30] M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, and N. Nadirashvili. Interior Hölder estimates for solutions of Schrödinger equations and the regularity of nodal sets. *Communications in Partial Differential Equations*, 20(7–8):1241–1273, 1995.

[31] F. John. *Partial Differential Equations*. Applied mathematical sciences. Springer-Verlag, 1991.

[32] D. Khoshnevisan and T. M. Lewis. A uniform modulus results for iterated Brownian motion. *Annales de l’Institut Henri Poincaré*, 32(3):349–359, 1996.

[33] N. V. Krylov. *Lectures on Elliptic and Parabolic Equations in Hölder Spaces*. Graduate studies in mathematics. American Mathematical Society, 1996.

[34] N. V. Krylov. *Lectures on Elliptic and Parabolic Equations in Sobolev Spaces*. Graduate Studies in Mathematics, Graduate Studies in Mathema. American Mathematical Society, 2008.

[35] N. V. Krylov. On factorizations of smooth nonnegative matrix-values functions and on smooth functions with values in polyhedra. *Applied Mathematics and Optimization*, 58:373—392, 2008.

[36] N. V. Krylov and E. Priola. Elliptic and parabolic second-order PDEs with growing coefficients. *Communications in Partial Differential Equations*, 35:1–22, 2010.

[37] N. V. Krylov and E. Priola. Poisson stochastic process and basic Schauder and Sobolev estimates in the theory of parabolic equations. *Archive for Rational Mechanics and Analysis*, 225:1089–1126, 2017.

[38] F. Kühn. Schauder estimates for equations associated with Lévy generators. *Integral Equations Operator Theory*, 91(2):10–21, 2019.

[39] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural’ceva. *Linear and quasilinear equations of parabolic type*. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1968. Translated from the Russian by S. Smith.

[40] L. Lorenzi. Schauder estimates for degenerate elliptic and parabolic problems with unbounded coefficients in $\mathbb{R}^n$. *Differential Integral Equations*, 18(5):531–566, 2005.

[41] A. Lunardi. Schauder estimates for a class of degenerate elliptic and parabolic operators with unbounded coefficients in $\mathbb{R}^n$. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze*, Ser. 4, 24(1):133–164, 1997.
[42] L. Marino. Schauder estimates for degenerate stable Kolmogorov equations. *Bulletin des Sciences Mathématiques*, 162:102885, 2020.

[43] L. Marino. Schauder estimates for degenerate Lévy Ornstein-Uhlenbeck operators. *Journal of Mathematical Analysis and Applications*, 500(1):125168, 2021.

[44] S. Mazzucchi. Probabilistic representations for the solution of higher order differential equations. *International Journal of Partial Differential Equations*, 2013.

[45] G. Metafune, D. Pallara, and A. Rhandi. Kernel estimates for Schrödinger operators. *J. evol. equ.*, 6:433–457, 2006.

[46] P. Mörters and Y. Peres. *Brownian motion*, volume 30 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge, 2010. With an appendix by Oded Schramm and Wendelin Werner.

[47] E. Orsingher and X. Zhao. Iterated processes and their applications to higher order differential equations. *Acta Mathematica Sinica, English Series*, 15(2):173–180, 1999.

[48] L. Simon. Schauder estimates by scaling. *Calculus of Variations and Partial Differential Equations*, 5:391–407, 1997.

[49] A. H. Zemanian. Orthonormal series expansions of certain distributions and distributional transform calculus. *Journal of Mathematical Analysis and Applications*, 14:263–275, 1966.