On the first Hochschild cohomology of admissible algebras *

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Abstract

Our aim in this paper is to investigate the first Hochschild cohomology of admissible algebras which can be seen as a generalization of basic algebras. For this purpose, we study differential operators on an admissible algebra. Firstly, differential operators from a path algebra to its quotient algebra as an admissible algebra are discussed. Based on this discussion, the first cohomology with admissible algebras as coefficient modules is characterized, including their dimension formula. Besides, for planar quivers, the \( k \)-linear bases of the first cohomology of acyclic complete monomial algebras and acyclic truncated quiver algebras are constructed over the field \( k \) of characteristic 0.

Keywords: quiver, admissible algebra, differential operators, Hochschild cohomology.

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1 Introduction

The Hochschild cohomology of algebras is invariant under Morita equivalence. Hence it is enough to consider basic connected algebras when the algebras are Artinian. Let \( \Gamma = (V,E) \) be a finite connected quiver where \( V \) (resp. \( E \)) is the set of vertices (resp. arrows) in \( \Gamma \). Let \( k \) be an arbitrary field and \( k\Gamma \) be the corresponding path algebra. Denote by \( R \) the two-sided ideal of \( k\Gamma \) generated by \( E \). Recall that an ideal \( I \) is called admissible if there exists \( m \geq 2 \) such that \( R^m \subseteq I \subseteq R^2 \) (See [2]). According to the Gabriel theorem, a finite dimensional basic \( k \)-algebra over an algebraically closed field \( k \) is in the form of \( k\Gamma/I \) for a finite quiver \( \Gamma \) and an admissible ideal \( I \).

An Artinian algebra is called a monomial algebra (see [3]) if it is isomorphic to a quotient \( k\Gamma/I \) of a path algebra \( k\Gamma \) for a finite quiver \( \Gamma \) and an idea \( I \) of \( k\Gamma \) generated by some paths in \( \Gamma \). In particular, denote by \( k^n\Gamma \) the ideal of \( k\Gamma \) generated by all paths of length \( n \). Then the monomial algebra \( k\Gamma/k^n\Gamma \) is called the \( n \)-truncated quiver algebra.

The study of Hochschild cohomology of quiver related algebras started with the paper of Happel in 1989 [11], who gave the dimensions of Hochschild cohomology of arbitrary orders of path algebras

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for acyclic quivers. Afterwards, there have been extensive studies on the Hochschild cohomology of quiver related algebras such as truncated quiver algebras, monomial algebras, schurian algebras and 2-nilpotent algebras [6][7][4][19][12][17][13][1][15]. In [11], a minimal projective resolution of a finite dimensional algebra $A$ over its enveloping algebra is described in terms of the combinatorics when the field $k$ is an algebraically closed field. In these papers listed above, the authors use this kind of projective resolution or its improving version to compute the Hochschild cohomology.

In [10], the authors applied an explicit and combinatorial method to study $HH^1(k\Gamma)$. In this paper, we improve the method in [10] to the case of algebras with relations in order to study the $HH^1(k\Gamma/I)$ where $k\Gamma/I$ is an admissible algebra. This way does not depend on projective resolution and the requirement of $k$ being an algebraically closed field. Using this method, we can obtain some structural results which were not arisen by the classical method in the above listed papers.

If $I \subseteq R^2$ holds for a two-sided ideal $I$, we call $k\Gamma/I$ an admissible algebra (see Definition 2.1). So finite-dimensional basic algebras are always admissible algebras. We will give Proposition 2.1 which shows admissible algebras, including basic algebras, possess the similar characterization of monomial algebras and truncated quiver algebras, although it is not graded. From this point of view, admissible algebra is motivated to unify and generalize basic algebra and monomial algebra.

In the following, we always assume that $k\Gamma/I$ is an admissible algebra. This paper includes three sections except for the introduction. In Section 2, we introduce the basic definitions which are used in this paper. In particular, we define the notion of an acyclic admissible algebra, which can be thought as a generalization of the notion of an acyclic quiver. A sufficient and necessary condition is obtained for a linear operator from $k\Gamma$ to $k\Gamma/I$ to be a differential operator. Next, we give a standard basis of $\text{Diff}(k\Gamma, k\Gamma/I)$.

In Section 3, we investigate $H^1(k\Gamma, k\Gamma/I)$. In Eq.(3.22), a dimension formula of $H^1(k\Gamma, k\Gamma/I)$ is given for a finite dimensional admissible algebra. Moreover, in Theorem 3.7, we construct a basis of $H^1(k\Gamma, k\Gamma/I)$ when $\Gamma$ is planar and $k\Gamma/I$ is an acyclic admissible algebra.

In Section 4, we characterize $HH^1(k\Gamma/I)$. In Eq.(4.31), we give the dimension formula of $HH^1(k\Gamma/I)$ for any finite dimensional admissible algebras $k\Gamma/I$. Moreover, we apply this method to complete monomial algebras and truncated quiver algebras. In Theorems 4.7 and 4.10, we construct $k$-linear bases of their first cohomology groups under certain conditions. The Hochschild cohomology of monomial algebras and truncated quiver algebras has been studied in [7][19][12][15][16][18]. Our results in Section 4 can be seen as the generalization of those corresponding conclusions in the listed references above. In the same section, two examples of admissible algebras are given which are not monomial algebras. Their first Hochschild cohomology is characterized using our theory.

2 $k$-linear basis of $\text{Diff}(k\Gamma, k\Gamma/I)$

We always assume $\Gamma = (V, E)$ is a finite connected quiver, where $V$ (resp. $E$) is the set of vertices (resp. arrows) in $\Gamma$. For a path $p$, denote its starting vertex by $t(p)$, called the tail of $p$, and the ending point by $h(p)$, called the head of $p$. For two paths $p$ and $q$, if $t(p) = t(q)$ and $h(p) = h(q)$, we say $p$ and $q$ to be parallel, denote as $p \parallel q$. Denote by $\mathcal{P} = \mathcal{P}_\Gamma$ the set of paths in a quiver $\Gamma$ including its vertices; denote by $\mathcal{P}_A$ the set of its acyclic paths. Trivially, $\Gamma$ is acyclic if and only if $\mathcal{P}_\Gamma \backslash V = \mathcal{P}_A$. Throughout this paper, we always assume quivers are finite and connected.
Definition 2.1. Suppose $\Gamma = (V, E)$ is a quiver, $I$ is a two-sided ideal of $k\Gamma$, we call the quotient algebra $k\Gamma/I$ an admissible algebra if $I \subseteq R^2$ where $R$ denotes the two-sided ideal of $k\Gamma$ generated by $E$.

Proposition 2.1. Suppose $k\Gamma/I$ is an admissible algebra, then there exists a subset $\mathcal{P}'$ of $\mathcal{P}$ such that $V \cup E \subseteq \mathcal{P}'$ and $\mathcal{P} = \{ x \in \mathcal{P} \}$ forms a basis of $k\Gamma/I$ for $x = I$

Proof. Let $X$ be a $k$-linear basis of $I$. Denotes by $\mathcal{P}_{\geq 2}$ the set of all paths of length $\geq 2$. Define

$$T := \{ Y \subseteq k\Gamma : Y \text{ is linearly independent in } k\Gamma \text{ satisfying } X \subseteq Y \subseteq X \cup \mathcal{P}_{\geq 2} \}.$$ 

$T$ becomes a partial set due to the order of inclusion between subsets of $k\Gamma$. It is easy to see $T \neq \emptyset$ and $T$ satisfies the upper bound condition of chains. So by the famous Zorn’s Lemma, $T$ has a maximal element, denoted by $Z$.

We claim that $Z$ is linearly equivalent to $\mathcal{P}_{\geq 2}$. Otherwise, there exists $p \in \mathcal{P}_{\geq 2}$ such that $p$ cannot be linearly expressed by $Z$, then $Z \cup \{ p \}$ is linearly independent in $k\Gamma$, which contradicts to the maximal property of $Z$.

Since $Z$ is linearly equivalent to $\mathcal{P}_{\geq 2}$, it follows that $V \cup E \cup Z$ is linearly equivalent to $\mathcal{P} = V \cup E \cup \mathcal{P}_{\geq 2}$. By the definition of $T$, $Z \subseteq X \cup \mathcal{P}_{\geq 2}$ and $I$ is generated by $X$. Hence $V \cup E \cup (Z \setminus X)$ forms a basis of the complement space of $I$ in $k\Gamma$. It means that $\mathcal{P} = \{ x : x \in V \cup E \cup (Z \setminus X) \}$ forms a basis of $k\Gamma/I$. It is clear that $V \cup E \cup (Z \setminus X) \subseteq \mathcal{P}$ and it is the $\mathcal{P}'$ we want.

When $I \subseteq R^2$ is finite dimensional, we have an explicit way to determine the $\mathcal{P}'$. Concretely, suppose $\{x_1, x_2 \cdots x_m\}$ is a basis of $I$. Then there exists a finite subset $\{p_1, p_2 \cdots p_n\}$ of $\mathcal{P}$ such that $x_i$ can be expressed by the linear combinations of $p_j$. Suppose $x_i = \sum_{j=1}^{n} a_{ij}p_j$ for $i = 1, 2 \cdots m$, then we obtain a $m \times n$ matrix $A = (a_{ij})$. We can transform the matrix $A$ into a row-ladder matrix $B = (b_{ij})$ through only row transformations. Suppose $b_{i,e(i)}$ is the first nonzero number of the $i$-th row of $B$. Since $B$ is a row-ladder matrix, we have $c_i = c_k$ for $i \neq k$. Then $\{x_1, x_2 \cdots x_m\} \cup \{p_l|l \neq c_1, c_2 \cdots c_m\}$ is linearly equivalent to $\{p_1, p_2 \cdots p_n\}$. Hence $(\mathcal{P}\setminus\{p_1, p_2 \cdots p_n\}) \cup \{p_l|l \neq c_1, c_2 \cdots c_m\}$ is a basis of the complement space of $I$ in $k\Gamma$. Then the residue classes in $k\Gamma/I$ of all elements in this basis form a basis of $k\Gamma/I$.

On the other hand, in some special cases, e.g., when $k\Gamma/I$ is a monomial algebra, even if $I$ is not finite dimensional, the choice of $\mathcal{P}'$ is also given in the same way. If $k\Gamma/I$ is a monomial algebra and $I$ is generated by a set of paths of length $\geq 2$, the set of paths that do not belong to $I$ is just the $\mathcal{P}'$ required.

Definition 2.2. Let $A$ be a $k$-algebra and $M$ an $A$-bimodule. A differential operator (or say, derivation) from $A$ into $M$ is a $k$-linear map $D : A \rightarrow M$ such that

$$D(xy) = D(x)y + xD(y). \quad (2.1)$$

In particular, when $M = A$, this coincides with the differential operator of algebras.

Lemma 2.2. Suppose $D$ is a differential operator from $k\Gamma$ into $k\Gamma/I$. Then $D$ is determined by its action on the set $V$ of vertices of $\Gamma$ and the set $E$ of arrows of $\Gamma$. 

\section{Conclusion}
Lemma 2.3. Let $\Gamma$ be a quiver. Denote $kV$ (resp. $kE$) by the linear space spanned by the set $V$ of the vertices of $\Gamma$ (resp. the set $E$ of the arrows of $\Gamma$). Assume we have a pair of linear maps $D_0 : kV \rightarrow k\Gamma/I$ and $D_1 : kE \rightarrow k\Gamma/I$ satisfying that

$$D_0(x)x + xD_0(x) = D_0(x), \; x \in V; \quad (2.2)$$

$$D_0(x)y + xD_0(y) = 0, \; x, y \in V, \; x \neq y, \quad (2.3)$$

$$D_0(x)q + xD_1(q) = D_1(q), \; x \in V, \; q \in E, \; t(q) = x, \quad (2.4)$$

$$D_1(q)y + qD_0(y) = D_1(q), \; y \in V, \; q \in E, \; h(q) = y. \quad (2.5)$$

Then, the pair of linear maps $(D_0, D_1)$ can be uniquely extended to a differential operator $D : k\Gamma \rightarrow k\Gamma/I$ satisfying that

$$D(p) := \sum_{i=1}^{l} p_1 \cdots p_{i-1}D_1(p_i) \cdots p_l. \quad (2.6)$$

for any path $p = p_1p_2 \cdots p_{i-1}p_i \in E, 1 \leq i \leq l, l \geq 2$.

Proof. One only need to prove that $D$ is indeed a differential operator. For this, we need to check Eq. $(2.6)$ in the next four cases:

(a) $x, y \in V$; (b) $x \in V, y \in \mathcal{P}\setminus V$; (c) $x \in \mathcal{P}\setminus V, y \in V$; (d) $x, y \in \mathcal{P}\setminus V$.

However, the checking process is routine, so we omit it here. \qed

In the sequel, we always suppose $k\Gamma/I$ is an admissible algebra for the given ideal $I$ and the notations in Definition 2.1 are used. From Definition 2.1 and Proposition 2.1, there exists a basis of $k\Gamma/I$ which consists of residue classes of some paths including that of $V$ and $E$. Denote the fixed basis of $k\Gamma/I$ by $\mathcal{Q}$. Suppose $D : k\Gamma \rightarrow k\Gamma/I$ is a linear operator, then for any $p \in \mathcal{P}$, $D(p)$ is a unique combination of the basis $\mathcal{Q}$ of $k\Gamma/I$. Write this linear combination by

$$D(p) = \sum_{\overline{q} \in \mathcal{Q}} c_{\overline{q}}. \quad (2.7)$$

where all $c_{\overline{q}} \in k$. We will use this notation throughout this paper. As convention, for the empty set $\emptyset$, we say $\sum_{\overline{q} \in \emptyset} c_{\overline{q}} = 0$.

Lemma 2.4. Suppose $q_1, q_2 \in \mathcal{P}$, $q_1, q_2 \not\in I$ and $\overline{q_1} = \overline{q_2}$ in $k\Gamma/I$, then $t(q_1) = t(q_2), h(q_1) = h(q_2)$, i.e., $q_1 \parallel q_2$.

Proof. If $t(q_1) \neq t(q_2)$, then $\overline{q_1} = t(q_1)\overline{q_1} = t(q_1)\overline{q_2} = \overline{0}$, a contradiction, so $t(q_1) = t(q_2)$. Similarly, $h(q_1) = h(q_2)$. \qed

According to the Lemma above, for $\overline{p} \in \mathcal{Q}$, we can define $t(\overline{p}) := t(q)(\text{resp.} h(\overline{p}) := h(q))$ for any path $q \in \mathcal{P}$ satisfying $\overline{q} = \overline{p}$ in $k\Gamma/I$. For a path $s \in \mathcal{P}$ and $\overline{s} \in \mathcal{Q}$, if $t(s) = t(\overline{p})$ and $h(s) = h(\overline{p})$, we say $s$ and $\overline{p}$ to be parallel, denoted as $s \parallel \overline{p}$.

Denote

$$\mathcal{Q}_A := \{ \overline{p} \in \mathcal{Q} | t(\overline{q}) \neq h(\overline{q}) \} \quad \text{and} \quad \mathcal{Q}_C := \{ \overline{p} \in \mathcal{Q} | t(\overline{q}) = h(\overline{q}) \}.$$ 

Moreover $k\mathcal{Q}_A(\text{resp.} k\mathcal{Q}_C)$ denotes the subspace of $k\Gamma/I$ generated by $\mathcal{Q}_A(\text{resp.} \mathcal{Q}_C)$. Clearly, as $k$-linear spaces, $k\Gamma/I = k\mathcal{Q}_A \oplus k\mathcal{Q}_C$. 

\[ \text{4} \]
Definition 2.3. Using the above notations, an admissible algebra \( k\Gamma/I \) is called **acyclic** if
\[
\mathcal{D}C\{\mathcal{D}v \in V\} = \emptyset.
\]

It is easy to see from this definition that
(i) The fact whether the given \( k\Gamma/I \) is acyclic is independent with the choice of \( \mathcal{D} \).
(ii) If the quiver \( \Gamma \) is acyclic, then \( k\Gamma/I \) is acyclic; the converse is not true in general.
(iii) If \( k\Gamma/I \) is acyclic, then it is finite dimensional; the converse is not true, e.g., \( k\Gamma/k^n\Gamma \) if \( \Gamma \) is a loop for \( n \geq 2 \).

Proposition 2.5. Let \( D : k\Gamma \rightarrow k\Gamma/I \) be a \( k \)-linear operator.

(i) If \( D \) is a differential operator, then
\[
D(v) = \sum_{\mathcal{D}q \in \mathcal{D}, h(\mathcal{D}q) = v, t(\mathcal{D}q) \neq v} c^{\mathcal{D}q}_{\mathcal{D}v} + \sum_{\mathcal{D}q \in \mathcal{D}, h(\mathcal{D}q) = v, t(\mathcal{D}q) \neq v} c^{\mathcal{D}q}_{\mathcal{D}v}, \tag{2.8}
\]

(ii) Conversely, assume the linear map \( D \) from \( kV \oplus kE \) to \( k\Gamma/I \) satisfies Eqs. (2.8), (2.9), (2.10), then \( D \) can be uniquely extended linearly to a differential operator as Eq. (2.6).

Proof. (i) For a given \( v \in V \), since \( vv = v \), we have
\[
D(v) = D(vv) = D(v)v + vD(v).
\]
So by the direct computation, we can get
\[
D(v) = \sum_{\mathcal{D}q \in \mathcal{D}, h(\mathcal{D}q) = v} c^{\mathcal{D}q}_{\mathcal{D}v} + \sum_{\mathcal{D}q \in \mathcal{D}, t(\mathcal{D}q) = v} c^{\mathcal{D}q}_{\mathcal{D}v}. \tag{2.11}
\]
Moreover,
\[
D(v) = D(v)v + vD(v) = (D(v)v + vD(v))v + vD(v) = D(v)v + vD(v)v + vD(v),
\]
so we have \( vD(v)v = 0 \). That means \( \sum_{\mathcal{D}q \in \mathcal{D}, t(\mathcal{D}q) = h(\mathcal{D}q)} c^{\mathcal{D}q}_{\mathcal{D}v} = 0 \). So we get Eq. (2.8).

Also, for a given \( p \in E \), we have
\[
D(p) = D(t(p)ph(p))
= D(t(p))ph(p) + t(p)D(p)h(p) + t(p)pD(h(p))
= D(t(p))p + \sum_{\mathcal{D}q \in \mathcal{D}, \mathcal{D}q \parallel p} c^{\mathcal{D}q}_{\mathcal{D}p} + pD(h(p))
\]
Since \( t(p), h(p) \in V \), by Eq. (2.8), we can easily get Eq. (2.9).

Let \( x, y \in V, \ x \neq y \). By (2.8),

\[
D(xy) = D(x)y + xD(y) = \sum_{\gamma \in \mathcal{P}, t(\gamma) = x, h(\gamma) = y} c_{\gamma}^x \gamma + \sum_{\gamma \in \mathcal{P}, t(\gamma) = x, h(\gamma) = y} c_{\gamma}^y \gamma
\]

But, \( D(xy) = 0 \) since \( xy = 0 \). So,

\[
\sum_{\gamma \in \mathcal{P}, t(\gamma) = x, h(\gamma) = y} (c_{\gamma}^x + c_{\gamma}^y) \gamma = 0.
\]

For a path \( \gamma \in \mathcal{P} \) such that \( t(\gamma) = h(\gamma) \), substituting \( x \) and \( y \) respectively with \( t(\gamma) \) and \( h(\gamma) \), we get Eq. (2.10).

(ii) We only need to verify the conditions of Lemma 2.3 are satisfied. Because the process is straightforward, we leave it to the readers.

Next, we apply Proposition 2.5 to display a standard basis of differential operators from \( k\Gamma \) to \( k\Gamma/\mathcal{I} \), for any admissible algebra \( k\Gamma/\mathcal{I} \).

**Proposition 2.6. (Differential operator \( D_{r,\gamma} \).)** For a quiver \( \Gamma = (V, E) \), let \( r \in E \) and \( s \in \mathcal{P} \) with \( r \parallel s \). Define the \( k \)-linear operator \( D_{r,\gamma} : kV \oplus kE \to k\Gamma/\mathcal{I} \) satisfying

\[
D_{r,\gamma}(p) = \begin{cases} 
\gamma, & p = r \text{ for } p \in E, \\
0, & p \neq r \text{ for } p \in E \cup V,
\end{cases}
\]

Then, the conditions of Lemma (2.3) are satisfied and thus, \( D_{r,\gamma} \) can be uniquely extended to a differential operator from \( k\Gamma \) to \( k\Gamma/\mathcal{I} \), denoted still by \( D_{r,\gamma} \) for convenience.

**Proof.** Eqs. (2.2), (2.3), Eq. (2.4) and Eq. (2.5) can be checked easily by the definition of \( D_{r,\gamma} \).

For a given \( s \in \mathcal{P} \), we have the corresponding inner differential operator:

\[
D_{s} : k\Gamma \to k\Gamma/\mathcal{I}, \quad D_{s}(q) = \overline{sq} - \overline{qs}, \quad \forall q \in \mathcal{P}.
\]

**Theorem 2.7.** Let \( \Gamma = (V, E) \) be a quiver and \( \mathcal{I} \) be an ideal such that \( k\Gamma/\mathcal{I} \) is an admissible algebra. Then the set

\[
\mathfrak{B} = \mathfrak{B}_1 \cup \mathfrak{B}_2
\]

is a basis of the \( k \)-linear space of differential operators from \( k\Gamma \) to \( k\Gamma/\mathcal{I} \), where

\[
\mathfrak{B}_1 := \{ D_{s} | \overline{s} \in \mathcal{A} \}, \quad \mathfrak{B}_2 := \{ D_{r,\gamma} | r \in E, \overline{s} \in \mathcal{P}, r \parallel \overline{s} \}.
\]

**Proof.** We only need to verify that the operators in \( \mathfrak{B} \) are linearly independent and any differential operators can be generated \( k \)-linearly by \( \mathfrak{B} \).

**Step 1. \( \mathfrak{B} \) is linearly independent.** Suppose there are \( c_{\gamma}, c_{r,\gamma} \in k \) such that

\[
\sum_{\gamma \in \mathcal{P}, t(\gamma) \neq t(\overline{\gamma})} c_{\gamma} D_{\gamma} + \sum_{r \in E, \overline{s} \in \mathcal{P}, r \parallel \overline{s}} c_{r,\gamma} D_{r,\gamma} = 0.
\]
Then for any given $\overline{p_0} \in \mathcal{D}, h(\overline{p_0}) \neq t(\overline{p_0})$, by the definition of $D_{\overline{p}}$ and $D_{r, \overline{p}}$, we have

$$0 = \sum_{\overline{p} \in \mathcal{D}, h(\overline{p}) \neq h(\overline{p_0})} c_{\overline{p}} D_{\overline{p}}(h(\overline{p_0})) + \sum_{r \in E, \overline{p} \in \mathcal{D}, r \parallel \overline{p}} c_{r} D_{r, \overline{p}}(h(\overline{p_0}))$$

$$= \sum_{\overline{p} \in \mathcal{D}, h(\overline{p}) \neq h(\overline{p_0})} c_{\overline{p}} (ph(\overline{p_0}) - h(\overline{p_0})p) + 0$$

$$= \sum_{\overline{p} \in \mathcal{D}, h(\overline{p}) \neq h(\overline{p_0})} c_{\overline{p}} \overline{p} - \sum_{\overline{p} \in \mathcal{D}, h(\overline{p}) = h(\overline{p_0})} c_{\overline{p}} \overline{p}.$$

In the last formula above, $\overline{p} \neq \overline{p}$ always holds. Thus, their coefficients are all zero. In particular, $c_{\overline{p}} = 0$ for any $\overline{p_0} \in \mathcal{D}$ with $h(\overline{p_0}) \neq t(\overline{p_0})$.

Thus, from (2.16), we get that

$$\sum_{r \in E, \overline{p} \in \mathcal{D}, r \parallel \overline{p}} c_{r} D_{r, \overline{p}} = 0.$$

Further, for any given $r_0 \in E, \overline{p} \in \mathcal{D}$ with $\parallel r_0$, we have:

$$\sum_{r \in E, \overline{p} \in \mathcal{D}, r \parallel \overline{p}} c_{r} D_{r, \overline{p}}(r_0) = 0 \implies \sum_{\overline{p} \in \mathcal{D}, r_0 \parallel \overline{p}} c_{r_0, \overline{p}} = 0.$$

It follows that $c_{r_0, \overline{p}} = 0$ for any $r_0 \in E, \overline{p} \in \mathcal{D}$ with $r \parallel \overline{p}$.

Hence, $\mathcal{B}$ is $k$-linearly independent.

**Step 2.** $\mathcal{B}$ is the set of $k$-linear generators. Let $D : k\Gamma \to k\Gamma/I$ be any differential operator. Then for $v \in V$ and $p \in E$, by Eqs (2.8), (2.9) and (2.10) we have

$$D(v) = \sum_{\overline{q} \in \mathcal{D}, h(\overline{q}) \neq t(\overline{q}) = v} c_{\overline{q}} \overline{q} + \sum_{\overline{q} \in \mathcal{D}, t(\overline{q}) = v} c_{\overline{q}} \overline{q}.$$  (2.17)

$$D(p) = -\sum_{\overline{q} \in \mathcal{D}, h(\overline{q}) \neq t(\overline{q}) = p} c_{\overline{q}} \overline{q} + \sum_{\overline{q} \in \mathcal{D}, t(\overline{q}) = p} c_{\overline{q}} \overline{q} + \sum_{\overline{q} \in \mathcal{D}, h(\overline{q}) \neq t(\overline{q}) = h(p)} c_{\overline{q}} \overline{q}.$$  (2.18)

We claim that $D$ agrees with the differential operator $\overline{D}$ defined by the linear combination

$$\overline{D} = -\sum_{\overline{q} \in \mathcal{D}, h(\overline{q}) \neq t(\overline{q})} c_{\overline{q}} \overline{q} D_{\overline{q}} + \sum_{r \in E, \overline{p} \in \mathcal{D}, r \parallel \overline{p}} c_{r} D_{r, \overline{p}},$$  (2.19)

where $c_{\overline{q}}$ and $c_{\overline{q}}$ come from Eqs (2.17) and (2.18). Any path in $\mathcal{D}$ is either a vertex or a product of arrows. Thus by the product rule of differential operators, to show $D = \overline{D}$, we only need to verify that $D(q) = \overline{D}(q)$ for each $q = v \in V$ and $q = p \in E$. The verification is straightforward, so we omit it.

We call the set $\mathcal{B}$ in Theorem 2.7 the standard basis of the $k$-linear space $\text{Diff}(k\Gamma, k\Gamma/I)$ generated by all differential operators from $k\Gamma$ to $k\Gamma/I$.

From this theorem, we get $\text{Diff}(k\Gamma, k\Gamma/I) = \mathcal{D}_1 \oplus \mathcal{D}_2$, where $\mathcal{D}_1$ is the $k$-linear space generated by $\mathcal{B}_i$ for $i = 1, 2$ in (2.15).

For any $p \in E$, $D_{\overline{p}, \overline{r}} \in \mathcal{B}_2$ is called arrow differential operator from $k\Gamma$ to $k\Gamma/I$. Let $\mathcal{B}_E := \{D_{\overline{p}, \overline{r}}|p \in E\}$ and $\mathcal{D}_E := k\mathcal{B}_E$ is called the space of arrow differential operators.

**Remark 2.1.** When $r \in E$ is a loop of $\Gamma$, i.e., $t(r) = h(r)$, then $D_{r, h(r)} \in \mathcal{B}_2$. 

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3 $H^1(k\Gamma, k\Gamma/I)$ for an admissible algebra $k\Gamma/I$

**Proposition 3.1.** Let $q \in \mathcal{P}$ be such that $h(q) = t(q) = v_0$. We have

\[
D_\tau = \sum_{p \in E, t(p) = v_0} D_{p, \tau} - \sum_{r \in E, h(r) = v_0} D_{r, \tau}.
\] (3.20)

**Proof.** Note that the both sides of (3.20) are $k$-linearly generated by differential operators. So, by the product formula of differential operators, we only need to verify that the both sides always agree when they act on the elements of $V$ and $E$. Since the computation is direct, we omit it here.

**Remark 3.1.** For $v \in V$, it is clear that $t(v) = h(v) = v$. From Proposition 3.1, we have

\[
D_\tau = \sum_{p \in E, t(p) = v} D_{p, \tau} - \sum_{r \in E, h(r) = v} D_{r, \tau}.
\] (3.21)

We call $D_\tau$ the **vertex differential operator** from $k\Gamma$ to $k\Gamma/I$. Let $\mathcal{D}_V$ denote the linear space spanned by $\{D_\tau | v \in V\}$, called the **space of vertex differential operators**. It is clear that $\mathcal{D}_V$ is a subspace of $\mathcal{D}_E$.

**Lemma 3.2.** Let $p \in \mathcal{P}$, then $\mathcal{P}$ is always in the $k$-subspace $k\{\tau \in \mathcal{D} | \parallel_\tau p\}$ generated by $\parallel_\tau p$.

**Proof.** Suppose $\mathcal{P} = \sum_{\tau \in \mathcal{D}} c_\tau \mathcal{P}$, then $t(p)h(p) = \sum_{\tau \in \mathcal{D}} c_\tau t(p)h(p) = \sum_{\parallel_\tau p} c_\tau.$

**Corollary 3.3.** Let $q \in \mathcal{P}$ be such that $h(q) = t(q)$. Then $D_{\tau} \subset k\mathcal{B}_2 = \mathcal{D}_2$.

**Proof.** For $r \in E, r \parallel s \in \mathcal{P}$; by Lemma 3.2, suppose $\mathcal{T} = \sum_{\parallel_\tau p} c_\tau$, and it is clear that $D_{r, \tau} = \sum_{\parallel_\tau p} c_\tau D_{r, \tau}$, then use Proposition 3.1.

**Remark 3.2.** For $\tau \in \mathcal{D}, t(\tau) = h(\tau)$, from Theorem 2.1 and Corollary 3.3, we know that $D_{\tau} \subset k\mathcal{B}_2 = \mathcal{D}_2$, but not in $k\mathcal{B}_1 = \mathcal{D}_1$. Denote $\mathcal{D}_C := k\{D_{\tau} | \parallel_\tau \in \mathcal{D}, t(\tau) = h(\tau)\}$. Then $\mathcal{D}_C \subset \mathcal{D}_2$ and $\mathcal{D}_C \cap \mathcal{D}_1 = 0$.

Denote by $\text{Inn-Diff}(k\Gamma, k\Gamma/I)$ the linear space consisting of inner differential operators from $k\Gamma$ to $k\Gamma/I$. Then, $\text{Inn-Diff}(k\Gamma, k\Gamma/I) = \mathcal{D}_1 + \mathcal{D}_C$. Thus, we have

\[
H^1(k\Gamma, k\Gamma/I) = \frac{\text{Diff}(k\Gamma, k\Gamma/I)}{\text{Inn-Diff}(k\Gamma, k\Gamma/I)} = \frac{\mathcal{D}_1 + \mathcal{D}_2}{\mathcal{D}_1 + \mathcal{D}_C} \cong \mathcal{D}_2/(\mathcal{D}_2 \cap \mathcal{D}_C) \cong \mathcal{D}_2/\mathcal{D}_C.
\]

Since the basis of $k\Gamma/I$ given in Proposition 2.1 contains the residue classes of $V$ and $E$, we can see that the center of $k\Gamma/I$ as $k\Gamma$-bimodule and the center of $k\Gamma/I$ as an algebra are the same, denoted by $Z(k\Gamma/I)$.
Proposition 3.4. Let $k\Gamma/I$ be a finite dimensional admissible algebra, then

$$\dim_k H^1(k\Gamma,k\Gamma/I) = |\mathfrak{B}_2| + \dim_k Z(k\Gamma/I) - |\mathfrak{D}_C|. \tag{3.22}$$

Proof. By the discussion above, $\dim_k HH^1(k\Gamma,k\Gamma/I) = |\mathfrak{B}_2| - \dim_k \mathfrak{D}_C$. And

$$\mathfrak{D}_1 \oplus \mathfrak{D}_C = \text{Inn-Diff}(k\Gamma,k\Gamma/I) \cong (k\Gamma/I)/Z(k\Gamma/I) \cong (k\mathfrak{D}_C \oplus k\mathfrak{D}_A)/Z(k\Gamma/I) \cong k\mathfrak{D}_C/(Z(k\Gamma/I)) \oplus k\mathfrak{D}_A \cong k\mathfrak{D}_C/(Z(k\Gamma/I)) \oplus \mathfrak{D}_1,$$

where the first isomorphism is assured by Eq. (2.13), the second and fourth isomorphisms are trivial, the third is because of the facts that $Z(k\Gamma/I) \subseteq k\mathfrak{D}_C$ and $Z(k\Gamma/I) \cap k\mathfrak{D}_A = 0$. So $\mathfrak{D}_C \cong k\mathfrak{D}_C/Z(k\Gamma/I)$ as $k$-linear spaces, it follows that

$$\dim_k H^1(k\Gamma,k\Gamma/I) = \dim_k \mathfrak{D}_2 - \dim_k \mathfrak{D}_C = |\mathfrak{B}_2| + \dim_k Z(k\Gamma/I) - |\mathfrak{D}_C|. \tag{3.23}$$

If $k\Gamma/I$ is acyclic, then $Z(k\Gamma/I) \cong k$ and $|\mathfrak{D}_C| = |V|$. Thus, we have

Corollary 3.5. If $k\Gamma/I$ is an acyclic admissible algebra (in particular, if $\Gamma$ is an acyclic quiver), then

$$\dim_k H^1(k\Gamma,k\Gamma/I) = |\mathfrak{B}_2| + 1 - |V|. \tag{3.24}$$

On the other hand, when $\Gamma$ is a planar quiver and $k\Gamma/I$ is an acyclic admissible algebra, we can apply the approach of [10] to give a basis of $HH^1(k\Gamma,k\Gamma/I)$. A planar quiver is a quiver with a fixed embedding into the plane $\mathbb{R}^2$. The set $F$ of faces of a planar quiver $\Gamma$ is the set of connected component of $\mathbb{R}^2\setminus\Gamma$.

We will need the famous Euler formula on planar graph, see [5][9], which states that for any finite connected planar graph (which can be thought as the underlying graph of a quiver $\Gamma$), we have

$$|V| - |E| + |F| = 2. \tag{3.25}$$

For each face of $\Gamma$, its boundary is called a **primitive cycle**. Let $p_0$ denote the boundary of the unique unbounded face $f_0$ of $\Gamma$. Let $\Gamma_F$ denote the set of primitive cycles of $\Gamma$ and $\Gamma_F^- := \Gamma_F \setminus p_0$. Then clearly, the set $\Gamma_F$ of primitive cycles of $\Gamma$ is in bijection with the set $F$ of the faces of $\Gamma$. So $|F| = |\Gamma_F|$.

For a face $f \in F$, denote $p_f$ the corresponding primitive cycle of $f$. Suppose $p_f$ is comprised of an ordered list arrows $p_1, \cdots, p_s \in E$, define an operator from $k\Gamma$ to $k\Gamma/I$

$$D_{p_f} := \pm D_{p_1, p_1} \pm \cdots \pm D_{p_s, p_s}, \tag{3.26}$$

where a $\pm D_{p_i, p_i}$ is $+D_{p_i, p_i}$ if $p_i$ is in clockwise direction when viewed from the interior of the face of $p_f$ and is $-D_{p_i, p_i}$ otherwise. We call $D_{p_f}$ a **face differential operator** from $k\Gamma$ to $k\Gamma/I$. Let $\mathfrak{D}_F$ denote the linear space spanned by $\{D_p|p \in \Gamma_F\}$, called the **space of face differential operators**.

The next lemma is similar to Theorem 4.9 in [10].
Lemma 3.6. Let $\Gamma$ be a planar quiver with the ground field $k$ of characteristic 0, then

(a) $\dim \mathcal{D}_V = |V| - 1$;
(b) $\dim \mathcal{D}_P = |F| - 1 = |\Gamma_E|$;
(c) $\mathcal{D}_V$ and $\mathcal{D}_P$ are linearly disjoint subspaces of $\mathcal{D}_E$.

Proof. (a) Denote $\gamma_0 = |V|$. Since $\sigma = \sum_{i=1}^{\gamma_0} \tau_i$ is the identity of $k\Gamma/I$, which clearly lies in the center of $k\Gamma/I$, we have

$$D_\sigma = \sum_{i=1}^{\gamma_0} D_{\tau_i} = 0. \quad (3.27)$$

So $\dim \mathcal{D}_V \leq \gamma_0 - 1$. We next prove that $\dim \mathcal{D}_V \geq \gamma_0 - 1$. We may assume that $\gamma_0 \geq 2$.

We claim that any $\gamma_0 - 1$ elements of $\{D_{\tau_i} | i = 1, \ldots, \gamma_0\}$ is linearly independent. In fact, suppose $\sum_{i=1}^{\gamma_0-1} a_i D_{\tau_i} = 0$, where $a_i \in k$, which means that $\sum_{i=1}^{\gamma_0-1} a_i v_i$ is in the center of $k\Gamma/I$. Since $\Gamma$ is connected, let the vertex $v_{\gamma_0}$ be connected to $v_i$ by an arrow $p$ for $i \neq \gamma_0$. We may assume that $t(p) = v_i$ and $h(p) = v_{\gamma_0}$. We have

$$a_i p = (\sum_{i=1}^{\gamma_0-1} a_i v_i)p = \sum_{i=1}^{\gamma_0-1} a_i v_i = 0.$$

so $a_i = 0$. Note that $\Gamma$ is connected, we can repeat this process to get $a_j = 0$ for any $j$.

(b) Let $|F| = \gamma_2$. Through simple observation of planar quiver, we can see that if $p \in E$ is in the boundary, then it is at most in the boundary of two primitive cycles. Note that if $p \in E$ is in the boundary of two primitive cycles $P_1$ and $P_2$, then the sign of $D_{\tau_i} D_{P_1}$ in $D_{P_1}$ and $D_{P_2}$ are opposite. If $p \in E$ is in the boundary of only one primitive cycle $P$, then $D_{\tau_i} D_{P}$ occurs twice in $D_{\tau_i}$ with opposite sign. Thus we have

$$\sum_{j=0}^{\gamma_2-1} D_{P_j} = 0, \quad (3.28)$$

where $P_0$ denotes the primit cycle corresponding to $f_0$ as above. So $\dim \mathcal{D}_P \leq |F| - 1$.

We next prove that $\dim \mathcal{D}_P \geq |F| - 1$. We may assume that $|F| \geq 2$. Suppose

$$\sum_{j=1}^{\gamma_2-1} b_j D_{P_j} = 0, \quad (3.29)$$

where $b_j \in k$. If $p \in E$ is in the boundary of $P_0$ and $P_j$ for $j > 0$, then $\bar{0} = \sum_{j=1}^{\gamma_2-1} b_j D_{P_j}(p) = \pm b_j p$. So we have $b_j = 0$. This means that if $P_j$ and $P_0$ have a common $p \in E$ in their boundary, then $b_j = 0$. Replace $P_0$ with $P_j$, and repeat this process. Since the quiver is connected, we can get $b_j = 0$ for any $j > 0$.

(c) From [10] and Theorem 2.7, we know that $\mathcal{B}_E := \{D_{\tau_i} | p \in E\}$ and $\mathcal{B}_E := \{D_{\tau_i} | p \in E\}$ are $k$-linearly independent sets in $\text{Diff}(k\Gamma)$ and $\text{Diff}(k\Gamma, k\Gamma/I)$ respectively. Based on this, $D_{\tau_i}$ and $D_{P_j}$ can be linearly expressed by using $\mathcal{B}_E$, as well as $D_{v_i}$ and $D_{e_j}$ by using $\mathcal{B}_E$ in [10]. Under this correspondence, referring to Theorem 4.9 in [10] in the same process, we obtain that $\mathcal{D}_V$ and $\mathcal{D}_P$ are linearly disjoint subspaces of $\mathcal{D}_E$. \qed
By this lemma, \( \mathcal{B}_P := \{ D_p | p \in \Gamma_P \} \) is a basis of \( \mathcal{D}_P \).

**Theorem 3.7.** Let \( \Gamma \) be a planar quiver and \( k\Gamma/I \) be an acyclic admissible algebra with the ground field \( k \) of characteristic 0. Then the union set

\[
(\mathcal{B}_2 \setminus \mathcal{B}_E) \cup \mathcal{B}_P
\]

is a basis of \( H^1(k\Gamma, k\Gamma/I) \).

**Proof.** By the Euler formula and Lemma 3.6 we can get \( \mathcal{D}_E = \mathcal{D}_V \oplus \mathcal{D}_P \). Because \( k\Gamma/I \) is acyclic, we have \( \mathcal{D}_C = \mathcal{D}_V \), then

\[
H^1(k\Gamma, k\Gamma/I) \cong \mathcal{D}_2/\mathcal{D}_C
\]
\[
\cong (\mathcal{D}_E \oplus k\{ \mathcal{B}_2 \setminus \mathcal{B}_E \})/\mathcal{D}_V
\]
\[
\cong \mathcal{D}_P \oplus k\{ \mathcal{B}_2 \setminus \mathcal{B}_E \}
\]

\( \square \)

4 \( HH^1(k\Gamma/I) \) for an admissible algebra \( k\Gamma/I \)

**Lemma 4.1.** A differential operator of \( k\Gamma/I \) can induce naturally a differential operator from \( k\Gamma \) to \( k\Gamma/I \). Conversely, a differential operator \( D \) from \( k\Gamma \) to \( k\Gamma/I \) satisfying \( D(I) = \mathbf{0} \) can induce a differential operator of \( k\Gamma/I \).

**Proof.** Denote \( p \) the canonical map from \( k\Gamma \) to \( k\Gamma/I \). Given a differential operator \( D \) of \( k\Gamma/I \), we claim that the composition \( Dp \) is a differential operator from \( k\Gamma \) to \( k\Gamma/I \). Note that the canonical map from \( k\Gamma \) to \( k\Gamma/I \) is an algebra homomorphism, it can be directly verified. The converse result can be shown directly, too. \( \square \)

For a differential operator \( D \) from \( k\Gamma \) to \( k\Gamma/I \) satisfying \( D(I) = \mathbf{0} \), we denote \( \overline{D} \) the induced differential operator on \( k\Gamma/I \). Write

\[
\mathfrak{F}(I) := \{ D | D \in \text{Diff}(k\Gamma, k\Gamma/I), \ D(I) = \mathbf{0} \}, \quad \mathfrak{F}_i(I) := \{ D | D \in \mathfrak{D}_i, \ D(I) = \mathbf{0} \} \text{ for } i = 1, 2.
\]

It is clear that \( D_s(I) = \mathbf{0} \) for \( s \in \mathcal{P} \). So \( \mathfrak{F}_1(I) = \mathfrak{D}_1 \) and \( \mathfrak{F}(I) = \mathfrak{D}_1 \oplus \mathfrak{F}_2(I) \).

**Lemma 4.2.** \( \mathfrak{F}(I) \cong \text{Diff}(k\Gamma/I) \) as \( k \)-linear spaces.

**Proof.** The map from \( \mathfrak{F}(I) \) to \( \text{Diff}(k\Gamma/I) \) is as follows,

\[
\mathfrak{F}(I) \mapsto \text{Diff}(k\Gamma/I), \quad D \mapsto \overline{D}.
\]

The proof of Lemma 4.1 assures the map from \( \mathfrak{F}(I) \) to \( \text{Diff}(k\Gamma/I) \) is surjective. As for the injectivity, suppose \( D_1, D_2 \in \mathfrak{F}(I) \) and \( D_1 \neq D_2 \), so according to Lemma 2.2 there exists a path \( p \in V \cup E \) such that \( D_1(p) \neq D_2(p) \). Since \( \mathbf{0} \neq \mathbf{p} \in k\Gamma/I, \overline{D}_1(p) \neq \overline{D}_2(p) \). \( \square \)
By this lemma, we can think $\text{Diff}(k\Gamma/I)$ is a $k$-subspace of $\text{Diff}(k\Gamma, k\Gamma/I)$.

From Lemma 4.2, we have

$$HH^1(k\Gamma/I) \cong \mathfrak{g}(I)/(\mathfrak{D}_1 \oplus \mathcal{D}_C) \cong (\mathfrak{D}_1 \oplus \mathfrak{H}_2(I))/(\mathfrak{D}_1 \oplus \mathcal{D}_C) \cong \mathfrak{H}_2(I)/\mathcal{D}_C$$

(4.30)

as linear spaces. This means that $HH^1(k\Gamma/I)$ can be embedded into $H^1(k\Gamma, k\Gamma/I) \cong \mathfrak{D}_2/\mathcal{D}_C$. Moreover, we have the next proposition.

**Proposition 4.3.** Suppose $k\Gamma/I$ is a finite dimensional admissible algebra, then

$$\dim_k HH^1(k\Gamma/I) = \dim_k \mathfrak{H}_2(I) + \dim_k Z(k\Gamma/I) - |\mathcal{I}_C|.$$  

(4.31)

**Proof.** Note that $k\{\mathcal{I}_C\}/(Z(k\Gamma/I)) \cong \mathcal{D}_C$ as linear spaces. By Equation 4.31 we have

$$\dim_k HH^1(k\Gamma/I) = \dim_k \mathfrak{H}_2(I) - \dim_k \mathcal{D}_C = \dim_k \mathfrak{H}_2(I) + \dim_k Z(k\Gamma/I) - |\mathcal{I}_C|.\]

\[\square\]

**Corollary 4.4.** If $k\Gamma/I$ is an acyclic admissible algebra (in particular, if $\Gamma$ is an acyclic quiver), then

$$\dim_k HH^1(k\Gamma/I) = \dim_k \mathfrak{H}_2(I) + 1 - |V|.$$  

(4.32)

If $k\Gamma/I$ is an acyclic admissible algebra, we have a standard procedure to compute $\dim_k \mathfrak{H}_2(I)$. First note that for a differential operator $D$ from $k\Gamma$ to $k\Gamma/I$, $D(I) = \mathfrak{I}$ if and only if $D(r_i) = \mathfrak{I}$ where $\{r_1, \ldots, r_1, \ldots r_n\}$ is a minimal set of generators of $I$. This property follows easily from the Leibnitz rule of differential operators. Since $k\Gamma/I$ is acyclic, $|\mathfrak{B}_2|$ is finite for $\mathfrak{B}_2$ as given in Theorem 2.7. Suppose $\sum_{D, \mathfrak{I} \in \mathfrak{B}_2} c_{r, \mathfrak{I}} D_{r, \mathfrak{I}}(r_i) = \mathfrak{I}$ for $i = 1, \ldots, n$. This means that the coefficients $c_{r, \mathfrak{I}}$ satisfy the system of these homogeneous linear equations. So $\dim \mathfrak{H}_2(I)$ is equal to the dimension of the solution space of the system of homogeneous linear equations.

Now we give two examples of admissible algebras that are not monomial algebras nor truncated quiver algebras, and characterize their first Hochschild cohomology.

**Example 4.1.** Let $\Gamma = (V, E)$ be the quiver

$$\begin{align*}
\begin{array}{c}
\alpha_1 \\
\beta_1
\end{array}
\end{align*}$$

and $I =$\(\langle\alpha_1\alpha_2 - \beta_1\beta_2\rangle\).

In this case, $\mathfrak{B}_2 = \{D_{\alpha_1, \alpha_2}, D_{\alpha_2, \alpha_2}, D_{\beta_1, \beta_1}, D_{\beta_2, \beta_2}\}$. So we have

$$\dim_k H^1(k\Gamma, k\Gamma/I) = |\mathfrak{B}_2| + \dim_k Z(k\Gamma/I) - |\mathcal{I}_C| = 4 + 1 - 4 = 1.$$

Suppose that

$$(aD_{\alpha_1, \alpha_1} + bD_{\alpha_2, \alpha_2} + cD_{\beta_1, \beta_1} + dD_{\beta_2, \beta_2})(\alpha_1\alpha_2 - \beta_1\beta_2) = (a + b - c - d)\alpha_1\beta_1 = \mathfrak{I},$$

then we get $a + b - c - d = 0$. Hence $\dim_k \mathfrak{H}_2(I) = 3$ and $\dim_k HH^1(k\Gamma/I) = 0$. 

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Example 4.2. Let $\Gamma$ be the quiver having one vertex with two loops, equivalently, $k\Gamma = k[x, y]$. Suppose the ideal $I = \langle xy - yx \rangle$. Then $k\Gamma/I = k[x, y]$.

In this case, $B_1 = \emptyset$ and $B_2 = \{D_{x,x}^m y^n, D_{y,y}^m x^n | m,n \geq 0\}$ are the basis of $\text{Diff}(k[x, y], k[x, y])$, where $x^m y^n$ means the multiplication in $k[x, y]$.

Since $I = (x - y)$ we get the required result.

Moreover, note that $D_{x,x}^m y^n(x y - y x) = 0$ and $D_{y,y}^m x^n(x y - y x) = 0$. Thus we obtain the basis of $H^1(k[x, y])$ to be

$$\{D_{x,x}^m y^n, D_{y,x}^m y^n | m,n \geq 0\}.$$ 

Similarly we can obtain the first Hochschild cohomology for $k[x_1, x_2, \ldots, x_n]$.

Assume $k\Gamma/I$ is a monomial algebra. The residue classes of paths that do not belong to $I$ form a basis of $k\Gamma/I$. For convenience, we also denote by $\mathcal{D}$ the basis of $k\Gamma/I$ when $k\Gamma/I$ is a monomial algebra.

Definition 4.1. A monomial algebra $k\Gamma/I$ is called complete if for any parallel paths $p, p'$ in $\Gamma$, $p \in I$ implies $p' \in I$.

Proposition 4.5. Suppose $k\Gamma/I$ is a complete monomial algebra with $I \subseteq R^2$. Then the following set

$$\overline{B} = \overline{B}_1 \cup \overline{B}_2$$

is a basis of $\text{Diff}(k\Gamma/I)$, where

$$\overline{B}_1 := \{D_{r,T} | r \in \mathcal{D}, h(T) \neq t(T)\}, \quad \overline{B}_2 := \{D_{r,T} | r \in E, T \in \mathcal{D}, r \parallel T\}.$$  

Proof. Since $k\Gamma/I$ is complete, we have $D_{r,T}(p) = \overline{0}$ for any $D_{r,T} \in \overline{B}_2$, where $p$ is any path in $I$. Then $\mathfrak{B}_2(I) = \mathfrak{D}_2$. It follows that $\text{Diff}(k\Gamma, k\Gamma/I) \cong \text{Diff}(k\Gamma/I)$ as $k$-linear spaces. Thus due to Theorem 2.7 the result follows.

Corollary 4.6. Suppose $k\Gamma/I$ is an acyclic complete monomial algebra with $I \subseteq R^2$. Then

$$\dim_k H^1(k\Gamma/I) = |\overline{B}_2| + 1 - |V|.$$  

Proof. By the proof of Proposition 4.5 $\dim_k \mathfrak{B}_2(I) = \dim_k \mathfrak{D}_2 = \dim_k \overline{B}_2 = |\overline{B}_2|$. By Corollary 4.4 we get the required result.

In [15], the author gave a characterization of the first Hochschild cohomology of an acyclic complete monomial algebra through a projective resolution. However, its $k$-linear basis has not been constructed, so far. Here, we want to reach this aim in our method.

Theorem 4.7. Let $\Gamma$ be a planar quiver, $k\Gamma/I$ be an acyclic complete monomial algebra with $I \subseteq R^2$ over the field $k$ of characteristic 0. Then the union set

$$(\overline{B}_2 \setminus \mathfrak{D}_E) \cup \mathfrak{F}_P$$

is a basis of $H^1(k\Gamma/I)$, where $\mathfrak{B}_E = \{D_{p,T} | p \in E\}$ and $\mathfrak{B}_P = \{D_{p,D} | p \in V \setminus E\}$. 

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Proof. By Eq. (1.30) and $\mathcal{H}_2(I) = \mathcal{D}_2$, we have $HH^1(k\Gamma/I) \cong H^1(k\Gamma, k\Gamma/I)$ in this case. So from Theorem 3.7, we can directly get this theorem. \hfill \qed

For a truncated quiver algebra $k\Gamma/k^n\Gamma$ with $n \geq 2$, we can give a standard basis of $\text{Diff}(k\Gamma/k^n\Gamma)$. $k\Gamma/k^n\Gamma$ has the basis formed by the residue classes of the paths of length $\leq n - 1$, denoted also by $\mathcal{D}$.

**Proposition 4.8.** Let $\Gamma = (V, E)$ be a quiver and the field $k$ be of characteristic 0. A basis of $\text{Diff}(k\Gamma/k^n\Gamma)$ for any truncated quiver algebra $k\Gamma/k^n\Gamma$ with $n \geq 2$ is given by the set

$$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$$

where

$$\mathcal{B}_1 := \{ \overline{D}_r | r \in \mathcal{D}, h(\pi) \neq t(\pi) \}, \quad \mathcal{B}_2 := \{ \overline{D}_{r, \pi} | r \in E, \pi \in \mathcal{D}, s \notin V, r \parallel \pi \}. \tag{4.37}$$

Proof. It is clear that $\overline{D}_r(k^n\Gamma) = \emptyset$ for $\pi \in \mathcal{D}, h(\pi) \neq t(\pi)$ and $\overline{D}_{r, \pi}(k^n\Gamma) = \emptyset$ for $r \in E, \pi \in \mathcal{D}, s \notin V, r \parallel \pi$. Note that when $r$ is a loop of $\Gamma$, $\overline{D}_{r, h(r)}(k\Gamma/k^n\Gamma)$, but $\overline{D}_{r, h(r)}(r^n) = nr^{n-1} \neq \emptyset$. Moreover, for all loops $r_1, \ldots, r_s$ of $\Gamma$ and $c_1, \ldots, c_s$ not all 0, we claim that $\sum c_i \overline{D}_{r_i, h(r_i)}(k^n\Gamma) \neq \emptyset$. Without loss of generality, we can assume $c_1 \neq 0$. So we have

$$\sum c_i \overline{D}_{r_i, h(r_i)}(r^n_1) = nc_1 r_1^{n-1} \neq \emptyset.$$ 

Then by Theorem 2.7, the union set

$$\{ \overline{D}_r | r \in \mathcal{D}, h(\pi) \neq t(\pi) \} \cup \{ \overline{D}_{r, \pi} | r \in E, \pi \in \mathcal{D}, s \notin V, r \parallel \pi \}$$

forms a basis of the linear space $\mathcal{H}_2(k^n\Gamma)$ for $I = k^n\Gamma$. By Lemma 4.2, we have

$$\mathcal{H}_2(k^n\Gamma) \cong \text{Diff}(k\Gamma/k^n\Gamma).$$

Note the map from $\mathcal{H}_2(k^n\Gamma)$ to $\text{Diff}(k\Gamma/k^n\Gamma)$ in Lemma 4.2 we can see that the union set $\overline{\mathcal{B}} = \overline{\mathcal{B}}_1 \cup \overline{\mathcal{B}}_2$ is a $k$-linear basis of $\text{Diff}(k\Gamma/k^n\Gamma)$.

Thus $\text{Diff}(k\Gamma/k^n\Gamma) = \overline{\mathcal{B}}_1 \oplus \overline{\mathcal{B}}_2$, where $\overline{\mathcal{B}}_i$ is the $k$-linear space generated by $\overline{\mathcal{B}}_i$ for $i = 1, 2$.

**Corollary 4.9.** Let $\Gamma = (V, E)$ be a quiver and the field $k$ be of characteristic 0. Then

$$\text{dim}_k HH^1(k\Gamma/k^n\Gamma) = |\mathcal{B}_2| + \text{dim}_k Z(k\Gamma/k^n\Gamma) - |\mathcal{D}C|.$$ 

Proof. By the proof of Proposition 4.8 and the definition of $\mathcal{H}_2(I)$, we can see that $\{ \overline{D}_r | r \in E, \pi \in \mathcal{D}, s \notin V, r \parallel \pi \}$ is a basis of $\mathcal{H}_2(k^n\Gamma)$ for $I = k^n\Gamma$. And by Proposition 4.3, $\overline{\mathcal{B}}_2 := \{ \overline{D}_r | r \in E, \pi \in \mathcal{D}, s \notin V, r \parallel \pi \}$. Then by Proposition 4.3 and the correspondence between $D_r, \pi$ and $\overline{D}_r, \pi$ for each pair $(r, \pi)$, we get the required result. \hfill \qed

This corollary has indeed been given as Theorem 1 in [12] and Theorem 2 in [18]. The method we obtain it here is different with that in [12] and [18].

Moreover, when $k\Gamma/k^n\Gamma$ is acyclic, we can get a basis of $HH^1(k\Gamma/k^n\Gamma)$ as in Theorem 4.7.

**Theorem 4.10.** Let $\Gamma$ be a planar quiver, $k\Gamma/k^n\Gamma$ for $n \geq 2$ be acyclic over the field $k$ of characteristic 0, then the union set

$$\mathcal{B}_2 \cup \mathcal{B}_p$$

is a basis of $HH^1(k\Gamma/k^n\Gamma)$, where $\mathcal{B}_E = \{ \overline{D}_{p, \pi} | p \in E \}$ and $\mathcal{B}_P = \{ \overline{D}_p | p \in \mathcal{P} \}$.

Proof. Since $k\Gamma/I$ is acyclic, $\mathcal{D}_C = \mathcal{D}_V$. By Lemma 4.2 $\mathcal{B}_C \cong \mathcal{B}_C$, $\mathcal{B}_E \cong \mathcal{D}_E$, $\mathcal{B}_P \cong \mathcal{D}_P$. So the result can be obtained in the same way as the proof of Theorem 3.7. \hfill \qed
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