DERIVATIONS OF TENSOR PRODUCT OF ALGEBRAS

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Abstract. We prove a theorem about the derivation algebra of the tensor product of two algebras. As an application, we determine the derivation algebra of the fixed point algebra of the tensor product of two algebras, with respect to the tensor product of two finite order automorphisms of the involved algebras. These results generalize some well-known theorems in the literature.

Dedicated to Professor Bruce Allison on the occasion of his sixtieth birthday

0. Introduction

In 1969, R. E. Block [B] showed that the algebra of derivations of the tensor product of two algebras (satisfying certain finite dimensionality conditions) can be expressed in terms of the algebra of derivations and the centroid of each of the involved algebras. In 1986, G. Benkart and R. V. Moody [BM] used this (with a new proof) to establish several interesting results about the derivation algebra of the fixed points of the tensor product of two algebras, with respect to the tensor product of two finite order automorphisms of the involved algebras. They applied their results to determine the algebra of derivations of several important classes of infinite dimensional Lie algebras, including twisted and untwisted affine Kac–Moody Lie algebras [K], Virasoro algebras, and some subclasses of extended affine Lie algebras (for information about extended affine Lie algebras see [AABGP], [BGK] and [N2]).

All algebras we consider will be over a field $k$. We denote by $\mathcal{D}(A)$ and $C(A)$ the algebra of derivations and centroid of an algebra $A$, respectively. Let $A$ be a perfect algebra and $S$ be a commutative associative unital algebra. It is proved in [B, Theorem 7.1] and [BM, Theorem 1.1] that if $A$ is finite dimensional then

$$\mathcal{D}(A \otimes S) = \mathcal{D}(A) \otimes S \oplus C(A) \otimes \mathcal{D}(S).$$

(1)

Therefore any derivation $d \in \mathcal{D}(A \otimes S)$ can be represented as

$$d = \sum_{i \in I} d_i \otimes s_i + \sum_{j \in J} \gamma_j \otimes d'_j,$$

(2)

where $d_i$’s are in $\mathcal{D}(A)$, $d'_j$’s are in $\mathcal{D}(S)$, $\{s_i\}_{i \in I}$ is a basis of $S$ and $\{\gamma_j\}_{j \in J}$ is a basis of $C(A)$. Moreover, the $d_i$’s and $d'_j$’s are zero except for finitely many $i \in I$ and $j \in J$, respectively. It turns out that the main reason for using finite dimensionality in proving (1) is to show that the natural map $\psi : C(A) \otimes S \to C(A \otimes S)$ is an isomorphism (see (1.1) for the definition of $\psi$). However, one can show that under certain less restrictive

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conditions \( \psi \) remains an isomorphism. Indeed this holds if \( \mathcal{A} \) is finitely generated as an algebra over \( k \) or as a module over its centroid, or if \( \mathcal{A} \) is unital (see Lemma 1.2). When \( \psi \) is an isomorphism we are able to prove a generalization of (1) in the sense that each derivation \( d \) of \( \mathcal{A} \otimes S \) can be expressed in the form (2), where \( \{d_i\}_{i \in I} \) and \( \{d'_j\}_{j \in J} \) are summable families in \( \mathcal{D}(\mathcal{A}) \) and \( \mathcal{D}(S) \), respectively. Here by a summable family of endomorphisms \( \{d_i\}_{i \in I} \) of a vector space \( V \), we mean that for each \( v \in V \), \( f_i(v) = 0 \), except for finitely many \( i \in I \) (see Theorem 2.8).

Let \( \sigma_1 \) and \( \sigma_2 \) be period \( m \) automorphisms of \( \mathcal{A} \) and \( S \), respectively. Then \( \sigma_1, \sigma_2 \) and \( \sigma_1 \otimes \sigma_2 \) induce \( \mathbb{Z}_m \)-gradings on \( \mathcal{A}, S, \mathcal{A} \otimes S \) and also on the algebra of derivations and centroids of these algebras. Then [BM, Theorem 1.3] states that if \( \mathcal{A} \) is finite dimensional, \( k \) contains all \( m \)-th-roots of unity (\( p \nmid m \) if \( \text{char}(k) = p > 0 \)) and the homogeneous subspace of \( S \) of degree 1 contains a unit, then the restriction map

\[
\pi : (\mathcal{D}(\mathcal{A} \otimes S))_0 \to \mathcal{D}((\mathcal{A} \otimes S)_0)
\]

is an isomorphism. This has some very nice applications. We have been able to generalize this theorem to the extent that it holds for all algebras such that the map \( \psi \) is an isomorphism. Our approach to the proof is different from [BM] and it corrects an inaccuracy which occurs in the surjectivity part of Benkart-Moody’s proof (Remark 4.4).

Our interest in the algebra of derivations of tensor product of algebras arise from the study of so called iterated and multi loop algebras (see [ABP]). These algebras cover some very interesting classes of algebras including centerless affine Lie algebras and in general almost all centerless Lie tori [ABFP] (see [N1] for the definition of Lie tori). We are considering the algebra of derivations of iterated and multi-loop algebras in an ongoing project.

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1. Centroids and pfgc algebras

Throughout this work we fix a field \( k \) and two algebras \( \mathcal{A} \) and \( S \) over \( k \). All other algebras also will be over \( k \). The multiplication algebra of \( \mathcal{A} \), denoted \( \text{Mult}(\mathcal{A}) \) is the subalgebra of endomorphisms of \( \mathcal{A} \) over \( k \) generated by the identity element and left and right multiplication by elements of \( \mathcal{A} \). The centroid of \( \mathcal{A} \) is by definition the set of endomorphisms of \( \mathcal{A} \) which commute with all elements of \( \text{Mult}(\mathcal{A}) \). That is

\[
C(\mathcal{A}) = \{ \gamma \in \text{End}(\mathcal{A}) \mid \gamma(xy) = \gamma(x)y = x\gamma(y) \text{ for all } x, y \in \mathcal{A} \}.
\]

Clearly \( C(\mathcal{A}) \) is a unital subalgebra of \( \text{End}(\mathcal{A}) \). Then \( \mathcal{A} \) can be considered as a left \( C(\mathcal{A}) \)-module by \( \gamma \cdot a = \gamma(a) \), \( \gamma \in C(\mathcal{A}), a \in \mathcal{A} \).

An algebra \( \mathcal{A} \) is called perfect if \( \mathcal{A}\mathcal{A} = \mathcal{A} \). The centroid of any perfect algebra is commutative [J, Ch. X, § 1, Lemma 1]. Following [ABP] we call \( \mathcal{A} \) a pfgc algebra if
(i) $\mathcal{A} \neq 0$,
(ii) $\mathcal{A}$ is perfect,
(iii) $\mathcal{A}$ is finitely generated as a module over $C(\mathcal{A})$.

For $s \in S$ let $L_s$ denote the left multiplication by $s$, and consider the linear map
\[
\psi : C(\mathcal{A}) \otimes S \rightarrow C(\mathcal{A} \otimes S)
\]
\[
\gamma \otimes s \mapsto \gamma \otimes L_s.
\]

The following lemma indicates certain situations in which $\psi$ is an isomorphism of algebras. The proof of parts (i) and (iii) can be found in [ABP §2], where they are considering algebras over a ring. The proof of all four parts can be found in the recent work [BN] Proposition 2.19]. Since this is a central tool for the rest of this work and since its proof is short and simple in our setting, we provide the proof for the convenience of reader.

**Lemma 1.2.** Let $S$ be a unital commutative associative algebra. Suppose one of the following holds:
(i) $\mathcal{A}$ is perfect and $S$ is finite dimensional.
(ii) $\mathcal{A}$ is perfect and finitely generated over $k$.
(iii) $\mathcal{A}$ is a pfgc algebra.
(iv) $\mathcal{A}$ is unital.

Then the map $\psi$ defined by (1.1) is an isomorphism of associative algebras.

**Proof.** It is clear that $\psi$ is a homomorphism. Since $\mathcal{A}$ is perfect, $C(\mathcal{A})$ is commutative. Fix a basis $\{s_i\}_{i \in I}$ of $S$. Then any element $T$ of $C(\mathcal{A}) \otimes S$ can be written uniquely in the form $T = \sum_{i \in I} \gamma_i \otimes s_i$, $\gamma_i \in C(\mathcal{A})$. If $\psi(T) = 0$, then $\sum_{i \in I} \gamma_i(a) \otimes L_{s_i}(1) = 0$ for all $a \in A$. Thus $\gamma_i = 0$ for all $i \in I$ and so $\psi$ is 1-1.

To see $\psi$ is onto let $\Gamma \in C(\mathcal{A} \otimes S)$. Then for $a \in \mathcal{A}$, $\Gamma(a \otimes 1) = \sum_{i \in I} \gamma_i(a) \otimes s_i$, where $\gamma_i \in C(\mathcal{A})$ and $\gamma_i(a) = 0$ except for a finitely many $i \in I$. We show that under either of conditions (i)-(iv) in the statement there is a finite subset $I_0$ of $I$ such that
\[
\sum_{i \in I} \gamma_i(a) \otimes s_i = \sum_{i \in I_0} \gamma_i(a) \otimes s_i \quad \text{for all } a \in A. \tag{1.3}
\]
To see this take $I_0 = I$ if (i) holds. If $\mathcal{A}$, either as an algebra over $k$ or as a module over $C(\mathcal{A})$, is generated by elements $a_1, \ldots, a_m$, take
\[
I_0 = \bigcup_{j=1}^m \{ i \in I \mid \gamma_i(a_j) \neq 0 \}.
\]
Finally if $\mathcal{A}$ is unital, take $I_0 = \{ i \in I \mid \gamma_i(1) \neq 0 \}$. In either of these cases $I_0$ is finite and using the facts that $\gamma_i \in C(\mathcal{A})$ and $C(\mathcal{A})$ is commutative, it is easy to see that (1.3) holds.

Now $T := \sum_{i \in I} \gamma_i \otimes s_i$ is an element of $C(\mathcal{A}) \otimes S$ and for any $a, a' \in \mathcal{A}$ and $s \in S$
\[
(\sum_{i \in I_0} \gamma_i \otimes s_i)(aa' \otimes s) = (\sum_{i \in I_0} \gamma_i(a) \otimes s_i)(a' \otimes s) = \Gamma(a \otimes 1)(a' \otimes s) = \Gamma(aa' \otimes s),
\]
but $\mathcal{A}$ is perfect so $\psi(T) = \Gamma$ and we are done. \qed
2. Derivations

In this section we generalize a result of [B, Theorem 7.1], regarding the algebra of derivations of tensor product of two algebras (see also [BM, Theorem 1.1]).

Let \( A \) be an algebra and \( B \) be a subalgebra of \( A \). By a \textit{derivation} from \( B \) into \( A \) we mean a \( k \)-linear map \( \delta : B \to A \) such that

\[
\delta(bb') = \delta(b)b' + b\delta(b') \quad \text{for all } b, b' \in B.
\]

Denote the space of all such derivations by \( D(B, A) \). This space is usually denoted by \( \text{Der}(B, A) \), however we are using the abbreviated notation \( D(B, A) \) since it will appear frequently. If \( A = B \), we simply write it as \( D(A) \).

Let \( S \) be a commutative, associative unital algebra. Then \( A \otimes S \) can be considered as a \( S \)-bimodule by \( s' \cdot (a \otimes s) = (a \otimes s) \cdot s' = a \otimes ss' \), for \( a \in A, s, s' \in S \). We note that this action associates with the product on \( A \otimes S \). Let \( D_S(A \otimes S) \) denote the subalgebra of \( D(A \otimes S) \) consisting of \( S \)-module derivations. So if \( d \in D(A \otimes S) \), then

\[
d(a \otimes ss') = s'd(a \otimes s) = d(a \otimes s)s'
\]

for all \( a \in A, s, s' \in S \).

We note that the map

\[
\tau : D_S(A \otimes S) \longrightarrow D(A \otimes 1, A \otimes S)
\]

\[
d \longmapsto d|_{A \otimes 1}
\]

is a vector space isomorphism. Therefore we can transfer the Lie algebra structure on \( D_S(A \otimes S) \) to \( D(A \otimes 1, A \otimes S) \) by

\[
[d_1, d_2] = \tau([\tau^{-1}(d_1), \tau^{-1}(d_2)]),
\]

\( d_1, d_2 \in D(A \otimes 1, A \otimes S) \). Then as Lie algebras

\[
D_S(A \otimes S) \cong D(A \otimes 1, A \otimes S).
\]

Using this isomorphism we identify \( D(A \otimes 1, A \otimes S) \) as a subalgebra of \( D(A \otimes S) \).

Finally we set

\[
D_{A \otimes 1}(A \otimes S) = \{ \delta \in D(A \otimes S) \mid \delta(A \otimes 1) = 0 \}.
\]

Clearly \( D_{A \otimes 1}(A \otimes S) \) is a Lie subalgebras of \( D(A \otimes S) \).

Lemma 2.3. Let \( A \) be an algebra and \( S \) be a commutative, associative unital algebra. Then

\[
D(A \otimes S) = D_S(A \otimes S) \oplus D_{A \otimes 1}(A \otimes S)
\]

where the sum on the right is the direct sum of vector spaces.

Proof. Let \( \delta \in D(A \otimes S) \) and define \( d \in \text{End}(A \otimes S) \) by

\[
d(a \otimes s) = \delta(a \otimes 1)s = s\delta(a \otimes 1), \quad a \in A, \ s \in S.
\]
To see that $d \in \mathcal{D}_S(A \otimes S)$, let $a, a' \in \mathcal{A}$ and $s, s' \in S$. Then
\[
d((a \otimes s)(a' \otimes s')) = \delta(aa' \otimes 1)ss' + (\delta(a \otimes 1)(a' \otimes 1))ss' + \delta(a \otimes 1)s(a' \otimes 1)s' + (a \otimes 1)s\delta(a' \otimes 1)s' = d(a \otimes s)(a' \otimes s') + d(a \otimes s)(a' \otimes s'),
\]
and
\[
d((a \otimes s)s') = d(a \otimes ss') = \delta(a \otimes 1)ss' = d(a \otimes s)s'.
\]
Clearly $d = \delta$ on $A \otimes 1$ and so $(\delta - d)(A \otimes 1) = 0$. So $\delta = d + (\delta - d)$ where $d \in \mathcal{D}_S(A \otimes S)$ and $\delta - d \in \mathcal{D}_{A \otimes 1}(A \otimes S)$. It is now easy to see that the sum is direct. \hfill \Box

Let $1$ denote the identity operator on $\mathcal{A}$ and consider $1 \otimes S$ as a subalgebra of $C(A) \otimes S$.

**Lemma 2.4.** Assume that $A$ is perfect and $S$ is commutative, associative and unital. If the map $\psi$ defined by (1.1) is an isomorphism then as vector spaces
\[
\mathcal{D}_{A \otimes 1}(A \otimes S) \cong \mathcal{D}(1 \otimes S, C(A) \otimes S).
\]
In particular, taking this as an identification, we may consider $\mathcal{D}(1 \otimes S, C(A) \otimes S)$ as a Lie subalgebra of $\mathcal{D}(A \otimes S)$.

**Proof.** Since $\mathcal{A}$ is perfect, $C(A) \otimes S$ is a commutative associative unital algebra. Using $\psi$ we define the linear map
\[
\Phi : \mathcal{D}(1 \otimes S, C(A) \otimes S) \longrightarrow \mathcal{D}_{A \otimes 1}(A \otimes S)
\]
by
\[
\Phi(d)(a \otimes s) = \psi(d(1 \otimes s))(a \otimes 1),
\]
for $d \in \mathcal{D}(1 \otimes S, C(A) \otimes S)$, $a \in \mathcal{A}$ and $s \in S$. Since $d(1 \otimes 1) = 0$ we have $\Phi(d)(a \otimes 1) = 0$.

We now show that $\Phi(d)$ is a derivation of $1 \otimes S$ into $C(A) \otimes S$. So let $a, a' \in A$ and $s, s' \in S$. Since $\psi(d(1 \otimes s))$ and $\psi(d(1 \otimes s'))$ are in $C(A) \otimes S$ we have
\[
\psi\psi(d(1 \otimes s)(aa' \otimes s')) = (\psi(d(1 \otimes s))(a \otimes 1))(a' \otimes s')
\]
and
\[
\psi\psi(d(1 \otimes s')(aa' \otimes s)) = (a \otimes s)(\psi(d(1 \otimes s'))(a' \otimes 1)).
\]
Therefore
\[
\Phi(d)(aa' \otimes ss') = \psi(d(1 \otimes ss'))(aa' \otimes 1) = [\psi\psi(d(1 \otimes s)(1 \otimes L_s') + (1 \otimes L_s)\psi(d(1 \otimes s'))(aa' \otimes 1)]
\]
\[
= \psi\psi(d(1 \otimes s)(aa' \otimes s')) + (1 \otimes L_s)(\psi(d(1 \otimes s'))(aa' \otimes 1))
\]
\[
= (\psi(d(1 \otimes s)(a \otimes 1))(a' \otimes s') + (a \otimes s)(\psi(d(1 \otimes s'))(a' \otimes 1))
\]
\[
= \Phi(d)(a \otimes s)(a' \otimes s') + (a \otimes s)\Phi(d)(a' \otimes s').
\]
Next we show that $\Phi$ is 1-1 and onto. For this we define an inverse map for $\Phi$ as follows. Set

$$\Phi' : D_{A \otimes 1}(A \otimes S) \rightarrow D(1 \otimes S, C(A) \otimes S),$$

$$d' \mapsto (\psi^{-1} \circ \text{add}' \circ \psi)|_{1 \otimes S}.$$  

We note that $[D(A \otimes S), C(A \otimes S)] \subseteq C(A \otimes S)$ and so $\Phi'$ is well-defined. First we show that $\Phi \circ \Phi'$ is the identity map on $D_{A \otimes 1}(A \otimes S)$. So let $d' \in D_{A \otimes 1}(A \otimes S)$, $a \in A$ and $s \in S$. Since $d'(A \otimes 1) = 0$ we have

$$(\text{add}' \circ \psi(1 \otimes s))(a \otimes 1) = [d', 1 \otimes L_s](a \otimes 1) = d'(a \otimes s).$$

Therefore

$$(\Phi \circ \Phi')(d')(a \otimes s) = \Phi(\psi^{-1} \circ \text{add}' \circ \psi)(a \otimes s)$$

$$= \psi((\psi^{-1} \circ \text{add}' \circ \psi)(1 \otimes s))(a \otimes 1)$$

$$= \psi(\psi^{-1}(d'(a \otimes s))) = d'(a \otimes s).$$

Finally, we show that $\Phi' \circ \Phi$ is the identity map on $D(1 \otimes S, C(A) \otimes S)$. Let $d \in D(1 \otimes S, C(A) \otimes S)$, $a \in A$ and $s, s' \in S$. Then

$$[\Phi(d), 1 \otimes L_s](a \otimes s') = \psi(d(1 \otimes ss'))(a \otimes 1) - (1 \otimes L_s)\psi(d(1 \otimes s'))(a \otimes 1)$$

$$= \psi(d(1 \otimes s)(1 \otimes s') + (1 \otimes s)d(1 \otimes s'))(a \otimes 1)$$

$$- (1 \otimes L_s)\psi(d(1 \otimes s'))(a \otimes 1)$$

$$= \psi d(1 \otimes s)(a \otimes s').$$

Thus

$$(\Phi' \circ \Phi)(d)(1 \otimes s) = (\psi^{-1} \circ \text{add}\Phi(d) \circ \psi)(1 \otimes s)$$

$$= (\psi^{-1}\text{add}\Phi(d))(1 \otimes L_s)$$

$$= \psi^{-1}(\psi(d)(1 \otimes s)) = d(1 \otimes s).$$

This completes the proof.  

To state our next result we need to introduce the notion of a summable family of endomorphisms on a vector space. A family $\{f_i\}_{i \in I}$ of endomorphisms of a $k$-vector space $V$ is called summable on $V$, if for each $v \in V$, $f_i(v) = 0$ except for finitely many $i \in I$. If $\{f_i\}_{i \in I}$ is summable, then we define $\sum_{i \in I} f_i \in \text{End}(V)$ by

$$\sum_{i \in I} f_i(v) = \sum_{i \in I} f_i(v) \quad (v \in V).$$

Now let $V$ and $W$ be two vector spaces over $k$ and let $E_V$ and $E_W$ be subspaces of $\text{End}(V)$ and $\text{End}(W)$ respectively. If $\{f_i\}_{i \in I} \subseteq E_V$ is summable on $V$ and $\{g_i\}_{i \in I}$ is any family in $E_W$, then $\{f_i \otimes g_i\}_{i \in I}$ is summable on $V \otimes W$. Therefore $\sum_{i \in I} f_i \otimes g_i \in \text{End}(V \otimes W)$. We set

$$E_V \overset{\text{sup}}{\otimes} E_W := \{\sum_{i \in I} f_i \otimes g_i \mid \{f_i\}_{i \in I} \subseteq E_V \text{ summable on } V, \{g_i\}_{i \in I} \subseteq E_W\}.$$  

(The arrow points toward the subspace which the summable families belong to.) Let $\{h_j\}_{j \in J}$ be a fixed basis of $E_W$ and $\sum_{i \in I} f_i \otimes g_i \in E_V \overset{\text{sup}}{\otimes} E_W$. Let $g_i = \sum_{j \in J} a_{ij} h_j$, where
\[ \alpha^i_j \in k \text{ and set } \bar{f}_j = \sum_{i \in I} \alpha^i_j f_i. \] Then it is not difficult to show that \( \{ \bar{f}_j \}_{j \in J} \subseteq E_V \) is summable on \( V \) and
\[
\sum_{i \in I} f_i \otimes g_i = \sum_{j \in J} \bar{f}_j \otimes h_j.
\]

Therefore
\[
E_V \otimes E_W = \{ \sum_{j \in J} \bar{f}_j \otimes h_j | \{ \bar{f}_j \}_{j \in J} \subseteq E_V \text{ summable on } V \}.
\]

Clearly \( E_V \otimes E_W \) is a subspace of \( \text{End}(V \otimes W) \). Similarly, if \( \{ f_i \}_{i \in I} \) is a basis of \( E_V \), we can define
\[
E_V \otimes E_W := \{ \sum_{i \in I} f_i \otimes g_i | \{ g_i \}_{i \in I} \subseteq E_W \text{ summable on } W \}.
\]

We are now ready to state our next lemma. Note that when \( S \) is commutative and associative, we may identify it with a subspace of \( \text{End}(S) \) through left (or right) multiplication.

**Lemma 2.6.** Let \( A \) be perfect and \( S \) be commutative associative and unital. Then \( \mathcal{D}(A) \rightleftharpoons S \) and \( C(A) \rightleftharpoons \mathcal{D}(S) \) are subalgebras of \( \mathcal{D}(A \otimes S) \). Moreover, if \( \{ s_i \}_{i \in I} \) is a basis of \( S \) and \( \{ \gamma_j \}_{j \in J} \) is a basis of \( C(A) \) and \( d \in \mathcal{D}(A \otimes 1, A \otimes S), d' \in \mathcal{D}(1 \otimes S, C(A) \otimes S) \), then for \( a \in A, s \in S \),
\[
d(a \otimes 1) = \sum_{i \in I} d_i(a) \otimes s_i \quad \text{and} \quad d'(1 \otimes s) = \sum_{j \in J} \gamma_j \otimes d'_j(s),
\]
where \( \{ d_i \}_{i \in I} \) and \( \{ d'_j \}_{j \in J} \) are summable families in \( \mathcal{D}(A) \) and \( \mathcal{D}(S) \), respectively. In particular,
\[
\mathcal{D}(A \otimes 1, A \otimes S) \cong \mathcal{D}(A) \rightleftharpoons S \quad \text{and} \quad \mathcal{D}(1 \otimes S, C(A) \otimes S) \cong C(A) \rightleftharpoons \mathcal{D}(S),
\]
under the assignments
\[
d \mapsto \sum_{i \in I} d_i \otimes s_i \quad \text{and} \quad d' \mapsto \sum_{j \in J} \gamma_j \otimes d'_j,
\]
respectively.

**Proof.** We first show that \( \mathcal{D}(A) \rightleftharpoons S \) is a subalgebra of \( \mathcal{D}(A \otimes S) \). The result for \( C(A) \rightleftharpoons \mathcal{D}(S) \) follows by symmetry. Clearly \( \mathcal{D}(A) \rightleftharpoons S \) is a subspace of \( \mathcal{D}(A \otimes S) \). Now let \( \{ s_i \}_{i \in I} \) be a basis of \( S \) and let \( \{ d_i \}_{i \in I} \) and \( \{ d'_i \}_{i \in I} \) be two summable families in \( \mathcal{D}(A) \). Let \( d = \sum_{i \in I} d_i \otimes s_i \) and \( d' = \sum_{i \in I} d'_i \otimes s_i \). Then for \( a \in A \) and \( s \in S \), we have
\[
[d, d'](a \otimes s) = \sum_{i,j} [d_j, d'_j](a) \otimes s_i s_j.
\]
Let \( s_i s_j = \sum_i \alpha^i_{j,i} s_i \). Since \( \{ d_i \} \) and \( \{ d'_i \} \) are summable, we have for each \( t \) that
\[
\Delta_t = \sum_{i,j} \alpha^i_{j,i} [d_j, d'_j] \text{ is a well-defined element of } \mathcal{D}(A). \text{ Then}
\]
\[
[d, d'](a \otimes s) = \sum_t \Delta_t (a) \otimes s_i.
\]
So we are done if we show that \( \{ \Delta_t \}_{t \in I} \) is summable. Let \( a \in \mathcal{A} \) and set

\[
\begin{align*}
I_1 &= \{ i \in I \mid d'_i(a) \neq 0 \}, \\
I_2 &= \{ i \in I \mid d_i(a) \neq 0 \}, \\
I_3 &= \{ i \in I \mid d_i(d'_j(a)) \neq 0, \ j \in I_1 \cup I_2 \}, \\
I_4 &= \{ i \in I \mid d'_i(d_j(a)) \neq 0, \ j \in I_1 \cup I_2 \}, \\
I_0 &= \{ t \in I \mid \alpha_{i,j} \neq 0, \ i,j \in I_1 \cup I_2 \cup I_3 \cup I_4 \}.
\end{align*}
\]

Now \( I_0 \) is finite and if \( t \not\in I_0 \), then for each \( i,j \), either \( \alpha_{i,j} = 0 \) or \( [d_j, d'_i](a) = 0 \). This shows that \( \Delta_t(a) = 0 \) if \( t \not\in I_0 \). Thus \( \{ \Delta_t \}_{t \in I} \) is summable. This completes the proof of the first statement. The proof of the second statement is straightforward.

Recall that the differential centroid of an algebra \( \mathcal{A} \), denoted by \( dC(\mathcal{A}) \), is by definition the centralizer of \( D(\mathcal{A}) \) in \( C(\mathcal{A}) \), that is

\[ dC(\mathcal{A}) = \{ \gamma \in C(\mathcal{A}) \mid [\gamma, D(\mathcal{A})] = 0 \}. \]

We may consider \( \mathcal{A} \) also as a \( dC(\mathcal{A}) \)-module.

**Lemma 2.7.** Let \( \mathcal{A} \) be perfect and \( S \) be commutative associative and unital. Assume one of the following holds.

(a) \( S \) is finite dimensional,
(b) \( \mathcal{A} \) is finitely generated,
(c) \( \mathcal{A} \) is a pfgc algebra and \( [D(\mathcal{A}), C(\mathcal{A})] = 0 \),
(d) \( \mathcal{A} \) is finitely generated as a module over its differential centroid.

Then

\[ D(\mathcal{A}) \otimes S = D(\mathcal{A}) \otimes S \quad \text{and} \quad S \otimes D(\mathcal{A}) = S \otimes D(\mathcal{A}). \]

**Proof.** By symmetry, we only need to prove the first equality. We must show that under either of conditions (a)-(d), any summable family in \( D(\mathcal{A}) \) is finite. This is clear in the case (a). For the cases (b)-(d) let \( \{a_1, \ldots, a_m\} \) be a set of generators for \( \mathcal{A} \) either as an algebra, as a module over \( C(\mathcal{A}) \) or as a module over \( dC(\mathcal{A}) \), respectively. Let \( \{d_i\}_{i \in I} \) be an summable family in \( D(\mathcal{A}) \) and set

\[ I_0 = \bigcup_{j=1}^m \{ i \in I \mid d_i(a_j) \neq 0 \}. \]

Then \( I_0 \) is finite. Let \( i \in I \setminus I_0 \). For the case (b) it is clear that \( d_i(\mathcal{A}) = 0 \) and so \( \{d_i\}_{i \in I} \) is finite. For the case (c) we have

\[ d_i(\mathcal{A}) = \sum_{j=1}^m d_i(C(\mathcal{A}) \cdot a_j) = \sum_{j=1}^m [d_i, C(\mathcal{A})](a_j) + \sum_{j=1}^m C(\mathcal{A})d_i(a_j) = 0. \]

Argument for the case (d) is exactly the same as (c), replacing the role of \( C(\mathcal{A}) \) by \( dC(\mathcal{A}) \).

The following theorem summarizes the results of this section.
Theorem 2.8. Let $A$ and $S$ be algebras such that $A$ is perfect and $S$ is commutative
associative and unital. Assume that the map $\psi$ defined by (1.1) is an isomorphism. Then
\begin{equation}
\mathcal{D}(A \otimes S) = \mathcal{D}(A) \otimes \mathcal{D}(S) \oplus C(A) \otimes \mathcal{D}(S). \tag{2.9}
\end{equation}
In particular, if either of the following is satisfied
\begin{enumerate}[(i)]
  \item $S$ is finite dimensional,
  \item $A$ is finitely generated,
  \item $A$ is a pfge algebra,
  \item $A$ is unital,
\end{enumerate}
then (2.9) holds. Finally, if one of the following is satisfied:
\begin{enumerate}[(i)]
  \item $A$ or $S$ is finite dimensional.
  \item $C(A)$ is finite dimensional or $S$ is finitely generated over $k$, and either of the following holds:
    \begin{enumerate}[(a)]
      \item $A$ is finitely generated over $k$,
      \item $A$ is finitely generated as a $C(A)$-module and $[\mathcal{D}(A), C(A)] = 0$,
    \end{enumerate}
  \end{enumerate}
then
\begin{equation}
\mathcal{D}(A \otimes S) = \mathcal{D}(A) \otimes S \oplus C(A) \otimes \mathcal{D}(S). \tag{2.10}
\end{equation}
Proof. The first statement follows from (2.2) and Lemmas 2.3, 2.4, 2.6. The second statement then is clear from Lemma 1.2. Finally, the last statement follows from Lemma 2.7 and the previous statements.

Remark 2.11. Theorem 2.8 is a generalization of [B, Theorem 7.1] and [BM, Theorem 1.1]. In fact when $A$ is finite dimensional Theorem 2.8 coincides with those of [B] and [BM] (the work of [B] contains more results when $A$ is unital).

3. Gradings induced by automorphisms

In this section we discuss the gradations induced by finite order automorphisms. We assume that $k$ contains a primitive $m^{th}$-root of unity. Starting from two period $m$ automorphisms $\sigma_1$ and $\sigma_2$ of $A$ and $S$, respectively, we consider the various $\mathbb{Z}_m$-gradations they induce and investigate the relation between these gradations.

Let $\mathbb{Z}_m$ be the group of integers congruent to $m$ and let $i : \mathbb{Z} \rightarrow \mathbb{Z}_m$ be the canonical map. Recall that a $\mathbb{Z}_m$-grading of the algebra $A$ is an indexed family $\{A_i\}_{i \in \mathbb{Z}_m}$ of subspaces of $A$ so that $A = \bigoplus_{i \in \mathbb{Z}_m} A_i$ and $A_i A_j \subseteq A_{i+j}$ for $i, j \in \mathbb{Z}_m$. Let $\sigma$ be an automorphism of $A$ of period $m$, and let $A = \sum_{i \in \mathbb{Z}_m} A_i$ be the corresponding gradation on $A$ where
\begin{align*}
A_i &= \{a \in A \mid \sigma(a) = \omega^i a\},
\end{align*}
$\omega$ a primitive $m^{th}$-root of unity. The automorphism $\sigma$ extends to an automorphism $\sigma^*$ of $\text{End}(A)$ of period $m$, by
\begin{align*}
\sigma^*(T) &= \sigma T \sigma^{-1}.
\end{align*}
This in turn induces a $\mathbb{Z}_m$-grading $\text{End}(A) = \sum_{i \in \mathbb{Z}_m} \text{End}(A)_i$, where
\[
\text{End}(A)_i = \{ T \in \text{End}(A) \mid T(A_j) \subseteq A_{i+j} \text{ for all } j \in \mathbb{Z}_m \}.
\]

By restriction $\sigma^*$ induces two automorphisms on $\mathcal{D}(A)$ and $C(A)$, of period $m$, and so we also have the $\mathbb{Z}_m$-gradings $\mathcal{D}(A) = \sum_{i \in \mathbb{Z}_m} \mathcal{D}(A)_i$ and $C(A) = \sum_{i \in \mathbb{Z}_m} C(A)_i$ where
\[
\mathcal{D}(A)_i = \text{End}(A)_i \cap \mathcal{D}(A) \quad \text{and} \quad C(A)_i = \text{End}(A)_i \cap C(A).
\]

Next let $\sigma_1$ and $\sigma_2$ be automorphisms of period $m$ of algebras $A$ and $S$, respectively. Then $\sigma = \sigma_1 \otimes \sigma_2$ is a period $m$ automorphism of $A \otimes S$. So $A \otimes S = \sum_{i \in \mathbb{Z}_m} (A \otimes S)_i$ where $(A \otimes S)_i = \sum_{j \in \mathbb{Z}_m} A_{i-j} \otimes S_j$. Also, by restriction, $\sigma^*$ induces automorphisms of $\mathcal{D}_S(A \otimes S)$ and $\mathcal{D}_{A \otimes 1}(A \otimes S)$.

**Lemma 3.1.** Assume that the field $k$ contains all $m^{th}$-roots of unity for some integer $m$, $A$ is perfect, $S$ is commutative associative and unital, and that the map $\psi$ defined by $\mathfrak{L}A$ is an isomorphism. Let $\sigma_1$ and $\sigma_2$ be period $m$ automorphisms of $A$ and $S$, respectively. Then with respect to the $\mathbb{Z}_m$-gradings induced from $\sigma^*$, $\sigma_1^*$ and $\sigma_2^*$ we have
\[
\mathcal{D}(A \otimes S) = \sum_{j \in \mathbb{Z}_m} \mathcal{D}(A \otimes S)_j = \sum_{j \in \mathbb{Z}_m} (\mathcal{D}(A) \otimes S)_j \oplus (\mathcal{D}(A) \otimes (A \otimes S))_j,
\]
where
\[
(\mathcal{D}(A) \otimes S)_j = \sum_{k \in \mathbb{Z}_m} \mathcal{D}(A)_k \otimes S_{j-k} \quad \text{and} \quad (\mathcal{D}(A) \otimes (A \otimes S))_j = \sum_{k \in \mathbb{Z}_m} C(A)_k \otimes (A \otimes S)_{j-k}.
\]

**Proof.** The first two equalities in the statement follows from Lemma 2.8 and 2.10 of Theorem 2.8 where under the identification made in Section 2 we have
\[
(\mathcal{D}(A) \otimes S)_j = \{ d \in \mathcal{D}(A) \otimes S \mid d(A_i \otimes 1) \subseteq (A \otimes S)_{i+j} \text{ for all } i \in \mathbb{Z}_m \},
\]
and
\[
(\mathcal{D}(A) \otimes (A \otimes S))_j = \{ d \in C(A) \otimes (A \otimes S) \mid d(1 \otimes S_i) \subseteq (C(A) \otimes S)_{i+j} \text{ for all } i \in \mathbb{Z}_m \}.
\]

We continue the proof by proving the third equality, the fourth equality follows by symmetry. From 3.2 it is clear that the right hand side of this equality is a subset of the left hand side. To see the reverse inclusion, let $d \in (\mathcal{D}(A) \otimes S)_j$ and let $\{ s_i \}_{i \in I}$ be a basis of $S$ consisting of homogeneous elements with respect to the $\mathbb{Z}_m$-grading on $S$. For $k \in \mathbb{Z}_m$ let $I_k = \{ i \in I \mid s_i \in S_k \}$. Now $d = \sum_{i \in I} d_i \otimes s_i$ where $\{ d_i \}_{i \in I}$ is an summable family in $\mathcal{D}(A)$. For $k \in \mathbb{Z}_m$ define
\[
d_i^k = \begin{cases} d_i & \text{if } i \in I_k \\ 0 & \text{otherwise}. \end{cases}
\]

Then $\{ d_i^k \}_{i \in I}$ is an summable family in $\mathcal{D}(A)$. Moreover, we have $d = \sum_{k \in \mathbb{Z}_m} d^k$ where $d^k = \sum_{i \in I} d_i^k \otimes s_i$. So it is enough to show that for each $k \in \mathbb{Z}_m$,
\[
d^k \in \sum_{k \in \mathbb{Z}_m} \mathcal{D}(A)_k \otimes S_{j-k}.
\]
Therefore without loss of generality we may assume that 
\[ d = \sum_{i \in I} d_{i}^{k,i} \otimes s_{i} \] 
for some \( k \in \mathbb{Z}_{m} \). This means that \( d_{i}^{k,i} \otimes s_{i} \in \mathcal{D}(A) \otimes S_{k} \) for all \( i \in I \). Next, with respect to the \( \mathbb{Z}_{m} \)-grading on \( \mathcal{D}(A) \) we have 
\[ d = \sum_{i \in \mathbb{Z}_{m}} d_{i}^{k,i} \otimes s_{i}, \] 
where for each \( i \) \in \( \mathbb{Z}_{m} \), the family \( \{ d_{i}^{k,i} \}_{i \in I} \) is summable in \( \mathcal{D}(A)_{\bar{i}} \) and \( d_{i}^{k,i} \otimes s_{i} \in \mathcal{D}(A)_{\bar{i}} \otimes S_{k} \) for all \( i \in I \). So again without loss of generality we may assume that 
\[ d = \sum_{i \in I} d_{i}^{k,i} \otimes s_{i}, \] 
where for each \( \bar{i} \in \mathbb{Z}_{m} \), the family \( \{ d_{i}^{k,i} \}_{i \in I} \) is summable in \( \mathcal{D}(A)_{\bar{i}} \) and \( d_{i}^{k,i} \otimes s_{i} \in \mathcal{D}(A)_{\bar{i}} \otimes S_{k} \). Now since \( d \in (\mathcal{D}(A) \otimes S)_{\bar{j}} \), we have from (4.2) that for each \( \bar{i} \in \mathbb{Z}_{m} \), \( d(A_{\bar{i}} \otimes 1) \subseteq (A \otimes S)_{\bar{i}+\bar{j}} \). But for each \( i \), 
\[ (d_{i}^{k,i} \otimes s_{i})(A_{\bar{i}} \otimes 1) \subseteq A_{\bar{i}+1} \otimes S_{k} \subseteq (A \otimes S)_{k+\bar{i}+\bar{j}}, \]
and so \( k + \bar{i} = \bar{j} \). So for each \( i \), \( d_{i}^{k,i} \otimes s_{i} \in \mathcal{D}(A)_{\bar{i}} \otimes S_{j-\bar{i}} \) and we are done. \( \Box \)

4. The interaction of fixed points and derivations

This section contains the main result of this work (Theorem 4.1) which is a generalization of Theorem 1.3 of the very interesting article [BM]. The proof of Theorem 4.1 in part corrects the proof of [BM] Theorem 1.3 (see Remark 4.4).

**Theorem 4.1.** Let \( A \) and \( S \) be algebras over \( k \) where \( k \) contains all \( m^{th} \)-roots of unity for some integer \( m \). If \( \text{char}(k) = p > 0 \) assume that \( p \nmid m \). Assume \( A \) and \( S \) satisfy the followings:

(i) \( A \) is perfect,
(ii) \( S \) is commutative associative and unital,
(iii) \( \sigma_{1} \in \text{Aut}(A) \), \( \sigma_{2} \in \text{Aut}(S) \), \( \sigma^{m}_{1} = 1 \), and \( \sigma^{m}_{2} = 1 \),
(iv) For some unit \( q \in \mathbb{Z}_{m} \), there is a unit \( u \) in \( S_{q} \).
(v) The map \( \psi \) defined by (4.1) is an isomorphism.

If \( (A \otimes S)_{0} \) denotes the fixed points of \( A \otimes S \) with respect to \( \sigma := \sigma_{1} \otimes \sigma_{2} \) then the restriction map
\[
\pi : (\mathcal{D}(A \otimes S))_{0} \longrightarrow \mathcal{D}(A \otimes S)_{0}
\]
\[ D \longrightarrow D_{\vert_{(A \otimes S)_{0}}} \quad \text{(4.2)} \]
is an isomorphism. In particular,
\[
\mathcal{D}(A \otimes S)_{0} \cong \sum_{i \in \mathbb{Z}_{m}} \mathcal{D}(A)_{\bar{i}} \otimes S_{-\bar{i}} \oplus C(A)_{\bar{i}} \otimes \mathcal{D}(S)_{-\bar{i}}. \quad \text{(4.3)}
\]

Before starting the proof we make an important remark followed by an example.

**Remark 4.4.** (i) When \( A \) is finite dimensional, condition (v) of Theorem 4.1 is automatically satisfied (see Lemma 4.2). In this case Theorem 4.1 is identical to [BM] Theorem 1.3.

(ii) While checking the proof of [BM] Theorem 1.3, we realized an inaccuracy which occurs in the surjectivity part of the proof. In fact the proof is based on the claim that
the restriction map is an isomorphism. To show that \( \pi \) is surjective, the authors consider \( d \in \mathcal{D}(A \otimes S)_{\bar{0}} \) and extend it to an element \( D \in \text{End}(A \otimes S) \) as follows: Let \( \bar{i} \in \mathbb{Z}_m \) and \( 0 \leq s < m \). Since \( \bar{q} \) is unit in \( \mathbb{Z}_m \) there is a unique \( 0 \leq r < m \) such that \( \bar{s} = \bar{q} \bar{r} \) in \( \mathbb{Z}_m \). Then for \( x_{\bar{i}} \in A_{\bar{i}} \) and \( b_{-\bar{i}+\bar{s}} \in S_{-\bar{i}+\bar{s}} \), define
\[
D(x_{\bar{i}} \otimes b_{-\bar{i}+\bar{s}}) = (1 \otimes L_u) d(x_{\bar{i}} \otimes u^{-r} b_{-\bar{i}+\bar{s}}) = u^r d(x_{\bar{i}} \otimes u^{-r} b_{-\bar{i}+\bar{s}}). \tag{4.5}
\]
They claim that \( D \in (\mathcal{D}(A \otimes S))_{\bar{0}} \). Unfortunately this is not true, we have provided a counterexample in Example 4.6. In Lemma 4.18 by assuming that \( d \in \mathcal{D}(A \otimes S)_{\bar{0}} \) has an extension \( D \in (\mathcal{D}(A \otimes S))_{\bar{0}} \), we extract the right formula for \( D \) in terms of \( d \). Then it takes quite a bit of non-straightforward work to show that this formula really provides a derivation (see Lemmas 4.18 and 4.19). Finally, we should mention that our approach to the proof of 4.1 is different form [BM].

**Example 4.6.** Let \( A = k1 \) and let \( S = k[z^{\pm}] \) be the algebra of Laurent polynomials in variable \( z \). Let \( \sigma_1 = 1 \) and \( \sigma_2(z^n) = w^n z^n \) where \( w = e^{2\pi \sqrt{-1}/4} \). Then \( A = A_{\bar{0}} \) and \( S = \oplus_{i \in \mathbb{Z}_4} S_i \) where \( S_i = z^i \mathbb{Z} \). Then \( 1 \otimes dz/dz \in (\mathcal{D}(A \otimes S))_{\bar{0}} \). So \( d := (1 \otimes dz/dz)_{(i \in \mathbb{Z}_4)_0} \in \mathcal{D}(A \otimes S)_{\bar{0}} \). Let \( D \in \text{End}(A \otimes S) \) be as in 4.13, where \( \bar{q} = 1 \) and \( u = z \in \mathbb{Z}_4 \). Now \( 1 \otimes z^5 \in A_{\bar{0}} \otimes S_{\bar{0}+\bar{s}} \) where \( \bar{s} = 1 \). So \( r = 1 \), is the unique integer with \( 0 \leq r < 4 \) such that \( \bar{r} \bar{q} = \bar{s} \). Then
\[
D(1 \otimes z^5) = z^1 d(1 \otimes z^{-1} z^5) = 4(1 \otimes z^5).
\]
A similar computation shows that
\[
D(1 \otimes z^3) = z^3 d(1 \otimes z^{-3} z^3) = 0 \quad \text{and} \quad D(1 \otimes z^2) = z^2 d(1 \otimes z^{-2} z^2) = 0.
\]
Thus \( D \) is not a derivation.

To proceed with the proof of the theorem we need a few lemmas.

For \( i \in \mathbb{Z}_m \) let \( \epsilon(i) \) be the unique preimage of \( i \) in \( \{0, 1, \ldots, m - 1\} \), under the map \( \epsilon : \mathbb{Z} \rightarrow \mathbb{Z}_m \). Then
\[
\epsilon(i + \bar{j}) = \begin{cases} 
\epsilon(i) + \epsilon(j) & \text{if } \epsilon(i) + \epsilon(j) < m \\
\epsilon(i) + \epsilon(j) - m & \text{if } \epsilon(i) + \epsilon(j) \geq m.
\end{cases}
\]
As it is mentioned in Remark 4.3, the expression \( (iv) \) defined by [BM] is not the right way of extending an element \( d \in \mathcal{D}(A \otimes S)_{\bar{0}} \) to an element \( D \in \mathcal{D}(A \otimes S) \). In the following lemma we analyze what would be the right way of extending \( d \). Before that we note that if \( \bar{q} \) and \( u \) are as in Theorem 4.4 and \( \bar{q} \bar{1} = \bar{1} \) then \( u \in S_{\bar{q}} \) if and only if \( \bar{u} := u^{\epsilon(\bar{q})} \in S_{\bar{1}} \). Thus, condition (iv) of Theorem 4.4 is equivalent to the condition:
\[
(iv)' \quad S_{\bar{1}} \text{ contains a unit.}
\]
From now on and for the sake of simplicity we work with \( (iv)' \) instead of \( (iv) \).

**Lemma 4.7.** Under the conditions of Theorem 4.4 (with \( (iv)' \) in place of \( (iv) \)) let \( d \in \mathcal{D}(A \otimes S)_{\bar{0}} \) be extended to an element \( D \in (\mathcal{D}(A \otimes S))_{\bar{0}} \). Let \( a_{\bar{i}} \in A_{\bar{i}} \) and \( b_{-\bar{i}+\bar{s}} \in S_{-\bar{i}+\bar{s}} \). Then the followings hold:
(a) For any integer \( n \) such that \( n^{-1} \) makes sense we have
\[
\begin{align*}
D(a_i \otimes b_{-i+\tilde{s}}) &= d(a_i \otimes u^{-\epsilon(i)}b_{-i+\tilde{s}})u^{(\tilde{s})} \\
&= \epsilon(\tilde{s})u^{-\epsilon(i)+nm}d(a_i \otimes u^{-\epsilon(i)})b_{-i+\tilde{s}}.
\end{align*}
\] (4.8)

In particular if \( \text{char}(k) = 0 \), this holds for any nonzero integer \( n \).

(b) If \( \text{char}(k) = p \), then
\[
D(a_i \otimes b_{-i+\tilde{s}}) = d(a_i \otimes u^{-pr}b_{-i+\tilde{s}})u^{pr},
\]
where \( r = \epsilon(\tilde{s}p^{-1}) \).

**Proof.** (a) Let \( d \) and \( D \) be as in the statement. By Lemma 3.1, \( D = D_1 + D_2 \) where
\[
D_1 \in \sum_{j \in \mathbb{Z}_m} D(A)_j \otimes S_{-j} \quad \text{and} \quad D_2 \in \sum_{j \in \mathbb{Z}_m} C(A)_j \otimes D(S)_{-j}.
\]
Therefore we can write \( D_1 = \sum_{i \in I} d_i \otimes s_i \) and \( D_2 = \sum_{i \in J} \gamma_i \otimes d'_i \), where for each \( i \), \( d_i \otimes s_i \in D(A)_j \otimes S_{-j} \), for some \( j \in \mathbb{Z}_m \) and similarly \( \gamma_i \otimes d'_i \in C(A)_j \otimes D(S)_{-j} \) for some \( j \in \mathbb{Z}_m \). Let \( x_{-i} = u^{-\epsilon(i)}b_{-i+\tilde{s}} \). Then \( x_{-i} \in S_{-j} \) and
\[
\begin{align*}
D(a_i \otimes b_{-i+\tilde{s}}) &= D_1(a_i \otimes u^{(\tilde{s})}x_{-i}) + D_2(a_i \otimes u^{(\tilde{s})}x_{-i}) \\
&= D_1(a_i \otimes x_{-i})u^{(\tilde{s})} + D_2(a_i \otimes x_{-i})u^{(\tilde{s})},
\end{align*}
\]
Since \( D(a_i \otimes x_{-i}) = d(a_i \otimes x_{-i}) \), we obtain
\[
D(a_i \otimes b_{-i+\tilde{s}}) = d(a_i \otimes x_{-i})u^{(\tilde{s})} + D_2(a_i \otimes u^{(\tilde{s})})x_{-i}.
\] (4.9)

Now let \( n \) be an integer such that \( n^{-1} \) makes sense (if \( \text{char}(k) = 0 \), \( n \) could be any nonzero integer, and if \( \text{char}(k) = p \), \( n \) could be any integer with \((n, p) = 1\)), then for any \( t \in \mathbb{Z} \),
\[
D_2(a_i \otimes u^t) = tu^{t-1}D_2(a_i \otimes u) = tn^{-1}u^{t-n}D_2(a_i \otimes u^n).
\] (4.10)

Also,
\[
d(a_i \otimes u^{-\epsilon(i)}u^{nm}) = D(a_i \otimes u^{-\epsilon(i)+nm}) \\
= D(a_i \otimes u^{-\epsilon(i)})u^{nm} + D_2(a_i \otimes u^{nm})u^{-\epsilon(i)} \\
= d(a_i \otimes u^{-\epsilon(i)})u^{nm} + D_2(a_i \otimes u^{nm})u^{-\epsilon(i)}.
\]

Thus
\[
D_2(a_i \otimes u^{nm}) = d(a_i \otimes u^{-\epsilon(i)+nm})u^{(\tilde{s})} - d(a_i \otimes u^{-\epsilon(i)})u^{nm+\epsilon(i)}.
\] (4.11)

Then from (4.10) and (1.11) we have
\[
D_2(a_i \otimes u^{(\tilde{s})}) \\
= \epsilon(\tilde{s})u^{-\epsilon(i)-nm}d(a_i \otimes u^{nm}) \\
= \epsilon(\tilde{s})(nm)^{-1}u^{(\tilde{s})-nm}d(a_i \otimes u^{nm}) \\
= \epsilon(\tilde{s})(nm)^{-1}(u^{(\tilde{s})-nm}d(a_i \otimes u^{-\epsilon(i)}) - d(a_i \otimes u^{-\epsilon(i)})u^{nm+\epsilon(i)}) \\
= \epsilon(\tilde{s})(nm)^{-1}(u^{(\tilde{s})+\epsilon(i)-nm}d(a_i \otimes u^{-\epsilon(i)}) - u^{(\tilde{s})+\epsilon(i)}d(a_i \otimes u^{-\epsilon(i)})).
\]
Replacing this in \((4.4)\) we obtain
\[
D(a^i \otimes u^{(s)} x_{-i}) = d(a^i \otimes x_{-i}) u^{(s)}
+ \epsilon(s)(nm)^{-1} u^{(i) + (s)} [u^{-nm} d(a^i \otimes u^{-e(i) + nm}) - d(a^i \otimes u^{-e(i)})] x_{-i}.
\]

Replacing \(x_{-i}\) with \(u^{-\epsilon(s)} b_{-i+\bar{s}}\) we obtain
\[
D(a^i \otimes b_{-i+\bar{s}}) = d(a^i \otimes u^{-\epsilon(s)} b_{-i+\bar{s}}) u^{(s)}
+ \epsilon(s)(nm)^{-1} u^{(i)} [u^{-nm} d(a^i \otimes u^{-e(i) + nm}) - d(a^i \otimes u^{-e(i)})] b_{-i+\bar{s}},
\]
where \(n\) is any integer such that \(n^{-1}\) makes sense.

(b) Let \(r\) be as in the statement, then
\[
D_2(a^i \otimes u^{-pr} b_{-i+\bar{s}}) = D_2(a^i \otimes u^{-pr}) b_{-i+\bar{s}} + D_2(a^i \otimes b_{-i+\bar{s}}) u^{-pr}
= -pru^{-pr-1} D_2(a^i \otimes u) b_{-i+\bar{s}} + D_2(a^i \otimes b_{-i+\bar{s}}) u^{-pr}
= D_2(a^i \otimes b_{-i+\bar{s}}) u^{-pr}.
\]
Thus
\[
d(a^i \otimes u^{-pr} b_{-i+\bar{s}}) = D_1(a^i \otimes b_{-i+\bar{s}}) u^{-pr} + D_2(a^i \otimes b_{-i+\bar{s}}) u^{-pr}
= D(a^i \otimes b_{-i+\bar{s}}) u^{pr}.
\]

\[\square\]

**Corollary 4.12.** Under the conditions of Theorem \(4.7\) the map \(\pi\) is injective.

**Proof.** Let \(D \in (\mathcal{D}(A \otimes S))\) and \(d := \pi(D) = 0\). By Lemma \(4.7\) \(D = 0\) is the unique extension of \(d\). \[\square\]

**Lemma 4.13.** Let \(A\) and \(S\) be algebras which satisfy conditions (i)-(iv) of Theorem \(4.7\) (with (iv)' in place of (iv)). Let \(d \in \mathcal{D}(A \otimes S)\), \(i, n \in \mathbb{Z}\) and \(a \in A_i\). Then the following formulas hold:
\[
u^{-nm} d(a \otimes u^{-i+nm}) + u^{nm} d(a \otimes u^{-i-nm}) = 2d(a \otimes u^{-i}), \tag{4.14}
\]
\[
u^{-nm} d(a \otimes u^{-i+nm}) + nu^{-m} d(a \otimes u^{-i-m}) = (1 + n)d(a \otimes u^{-i}), \tag{4.15}
\]
\[
u^{-nm} d(a \otimes u^{-i-nm}) - nu^{-m} d(a \otimes u^{-i+m}) = (1 - n)d(a \otimes u^{-i}). \tag{4.16}
\]
In particular, for \(a_i \in A_i\) and \(a_j \in A_j\),
\[
u^{\epsilon(i+j)} [u^{-m} d(a_i a_j \otimes u^{-e(i+j)+m}) - d(a_i a_j \otimes u^{-e(i+j)})]
= u^{\epsilon(i)+\epsilon(j)} [u^{-m} d(a_i a_j \otimes u^{-e(i) - e(j)+m}) - d(a_i a_j \otimes u^{-e(i)+\epsilon(j)})]. \tag{4.17}
\]
Proof. We use induction on \( n \geq 0 \) to prove (4.14). Since \( A \) is perfect, without loss of generality we may assume that \( a = xy \) where \( x \in A_j \) and \( y \in A_{i-j} \) for some \( j \in \mathbb{Z}_m \). Clearly formula (4.14) holds for \( n = 0 \). To see it for \( n = 1 \) note that

\[
d(a \otimes u^{m-i})u^{-m} + d(a \otimes u^{-m-i})u^m
= d(x \otimes u^{-j})(y \otimes u^{-i+j}) + (x \otimes u^{-j})d(y \otimes u^{-i+j})
\]

Next assuming \( n \geq 2 \), we have (using induction hypothesis)

\[
d(a \otimes u^{n-m})u^{-nm} + d(a \otimes u^{-n-m})u^m
\]

Using induction, we first prove formula (4.15) for \( n \geq 0 \). It clearly holds for \( n = 0 \) and it holds for \( n = 1 \) by (4.14). So we may assume \( n \geq 2 \). Then using induction hypothesis we have

\[
d(a \otimes u^{m-i})u^{-m} + d(a \otimes u^{-m-i})u^m
= d(x \otimes u^{-j+m})(y \otimes u^{-i+j-m}) + (x \otimes u^{-j-m})d(y \otimes u^{-i+j+m})
\]

Thus (4.15) holds for \( n \geq 0 \). To get (4.15) for \( n \leq 0 \), subtract (4.15) for \( n \geq 0 \) from (4.14).
Next we prove (4.16). From (4.14) we have
\[ nu^{-m}d(a \otimes u^{-i+m}) + nu^{m}d(a \otimes u^{-i-m}) = 2nd(a \otimes u^{-i}). \]
Now subtracting this from (4.16) we get (4.17). Finally we show (4.18). If \( \epsilon(i + j) = \epsilon(i) + \epsilon(j) \), there is nothing to be proved. If \( \epsilon(i + j) = \epsilon(i) + \epsilon(j) - m \), replace \( a \) with \( a_i a_j \) and \( i \) with \( \epsilon(i) + \epsilon(j) \) in (4.16) for \( n = 2 \) to get (4.17).

**Lemma 4.18.** Under the conditions (i)-(iv) of Theorem 4.1 (with (iv)' in place of (iv)), let \( d \in D((A \otimes S)_0) \), \( i, j, s, t \in \mathbb{Z} \), \( a_i \in A_i \), \( a_j \in A_j \), \( b_{-i+s} \in S_{-i+s} \) and \( b_{-j+i} \in S_{-j+i} \). Then
\[
[u^{-m+s}d(a_i \otimes u^{-s+m}b_{-i+s}) - u^{s}d(a_i \otimes u^{-s}b_{-i+s})](a_j \otimes b_{-j+i})
\]
\[= (a_i \otimes b_{-i+s})[u^{-m+t}d(a_j \otimes u^{t+m}b_{-j+i}) - u^{t}d(a_j \otimes u^{-t}b_{-j+i})]. \quad (4.19)
\]
In particular,
\[
(a_i \otimes u^{-i})[u^{-m+j}d(a_j \otimes u^{-j+m}) - d(a_j \otimes u^{-j})u^{j}]u^{-j}
\]
\[= [u^{-m+i}d(a_i \otimes u^{-i+m}) - d(a_i \otimes u^{-i})u^{i}]u^{-i}(a_j \otimes u^{-j}), \quad (4.20)
\]
and
\[
(a_i \otimes 1)[u^{-m+i}d(a_j \otimes u^{-t+m}b_{-j+i}) - d(a_j \otimes u^{-t}b_{-j+i})u^{i}]
\]
\[= [u^{-m+i}d(a_i \otimes u^{-i+m}) - d(a_i \otimes u^{-i})u^{i}](a_j \otimes b_{-j+i}). \quad (4.21)
\]

**Proof.** For the sake of simplicity, we put \( b_1 = b_{-i+s} \) and \( b_2 = b_{-j+i} \). Using the fact that \( d \) is a derivation, we compute
\[ M := d(a_i a_j \otimes u^{-s-t+m}b_1 b_2)u^{s+t} - d(a_i a_j \otimes u^{-s-t}b_1 b_2)u^{s+t} \]
in the following two ways:
\[ M = u^{-m+s+t}d(a_i \otimes u^{-s+m}b_1)(a_j \otimes u^{-t}b_2) + (a_i \otimes u^{-s+m}b_1)d(a_j \otimes u^{-t}b_2)]
\[-u^{s+t}[d(a_i \otimes u^{-s}b_1)(a_j \otimes u^{-t}b_2) + (a_i \otimes u^{-s}b_1)d(a_j \otimes u^{-t}b_2)]
\]= \[u^{-m+s}d(a_i \otimes u^{-s+m}b_{-i+s})(a_j \otimes b_{-j+i}) - u^{s}d(a_i \otimes u^{-s}b_{-i+s})(a_j \otimes b_{-j+i})]
\]
and similarly
\[ M = u^{-m+s+t}[d(a_i \otimes u^{-s}b_1)(a_j \otimes u^{-t}b_2) + (a_i \otimes u^{-s}b_1)d(a_j \otimes u^{-t}b_2)]
\[-u^{s+t}[d(a_i \otimes u^{-s}b_1)(a_j \otimes u^{-t}b_2) + (a_i \otimes u^{-s}b_1)d(a_j \otimes u^{-t}b_2)]
\]= \[u^{-m+t}d(a_i \otimes u^{-t+m}b_{-j+i}) - u^{t}d(a_i \otimes u^{-t}b_{-j+i})]
\]
Now comparing the result of the above computations for \( M \) we get (4.18).

Next substitute \( s = 0 \), \( b_{-i+s} = u^{-i} \), \( t = 0 \) and \( b_{-j+i} = u^{-j} \) in (4.19) to get (4.20). To get (4.21) apply \( s = i \) and \( b_{-i+i} = 1 \) to (4.19).

**Lemma 4.22.** Under the conditions of Theorem 4.1 (with (iv)' in place of (iv)), the map \( \pi \) defined by (4.22) is surjective.
Proof. Let \( d \in \mathcal{D}(\mathcal{A} \otimes S) \). Define \( D \in \text{End}(\mathcal{A} \otimes S) \) by

\[
D(a_i \otimes b_{-\overline{i} + \overline{s}}) = d(a_i \otimes u^{-\epsilon(\overline{s})}b_{-\overline{i} + \overline{s}})u^{(\overline{s})}
+ \epsilon(\overline{s})m^{-1}u^{(\overline{i})}[u^{-m}d(a_i \otimes u^{-\epsilon(\overline{i})} + m) - d(a_i \otimes u^{-\epsilon(\overline{i})})]b_{-\overline{i} + \overline{s}},
\]

where \( a_i \in A_i \) and \( b_{-\overline{i} + \overline{s}} \in S_{-\overline{i} + \overline{s}} \). We are done if we show that \( D \) is a derivation of \( \mathcal{A} \otimes S \). Let \( a_i \in A_i, a_j \in A_j, b_1 = b_{-\overline{i} + \overline{s}} \in S_{-\overline{i} + \overline{s}} \) and \( b_2 = b_{-\overline{j} + \overline{l}} \in S_{-\overline{j} + \overline{l}} \). Then by the definition of \( D \) we have

\[
A : = D(a_i a_j \otimes b_1 b_2) = d(a_i a_j \otimes u^{-\epsilon(\overline{s})}b_1 b_2)u^{(\overline{s})}
+ \epsilon(\overline{s} + \overline{l})m^{-1}u^{(\overline{i} + \overline{j})}[u^{-m}d(a_i a_j \otimes u^{-\epsilon(\overline{i} + \overline{j})} + m) - d(a_i a_j \otimes u^{-\epsilon(\overline{i} + \overline{j})})]b_1 b_2.
\]

By (4.23) we can change \( \epsilon(\overline{i} + \overline{j}) \) to \( \epsilon(\overline{i}) + \epsilon(\overline{j}) \) in the above expression, then we obtain

\[
A = d(a_i a_j \otimes u^{-\epsilon(\overline{s})}b_1 b_2)u^{(\overline{s})}
+ \epsilon(\overline{s} + \overline{l})m^{-1}u^{(\overline{i} + \overline{j})}[u^{-m}d(a_i a_j \otimes u^{-\epsilon(\overline{i})} - \epsilon(\overline{j})) + m) - d(a_i a_j \otimes u^{-\epsilon(\overline{i})} - \epsilon(\overline{j}))]b_1 b_2.
\]

We divide the argument to the cases \( \epsilon(\overline{s} + \overline{l}) = \epsilon(\overline{s}) + \epsilon(\overline{l}) \) and \( \epsilon(\overline{s} + \overline{l}) = \epsilon(\overline{s}) + \epsilon(\overline{l}) - m \).

Case \( \epsilon(\overline{s} + \overline{l}) = \epsilon(\overline{s}) + \epsilon(\overline{l}) \): We have from (4.23)

\[
A = [d(a_i \otimes u^{-\epsilon(\overline{s})}b_1)(a_j \otimes u^{-\epsilon(\overline{l})}b_2) + (a_i \otimes u^{-\epsilon(\overline{s})}b_1)d(a_j \otimes u^{-\epsilon(\overline{l})}b_2)]u^{(\overline{s}) + \epsilon(\overline{l})}
+ (\epsilon(\overline{s}) + \epsilon(\overline{l}))m^{-1}u^{(\overline{i} + \overline{j})}[u^{-m}d(a_i \otimes u^{-\epsilon(\overline{i})} - \epsilon(\overline{j})) + m) - d(a_i \otimes u^{-\epsilon(\overline{i})} - \epsilon(\overline{j}))]b_1 b_2
- \epsilon(\overline{s} + \overline{l})m^{-1}u^{(\overline{i} + \overline{j})}[d(a_i \otimes u^{-\epsilon(\overline{i})})(a_j \otimes u^{-\epsilon(\overline{j})})
+ (a_i \otimes u^{-\epsilon(\overline{i})})d(a_j \otimes u^{-\epsilon(\overline{j})})]b_1 b_2
+ (a_i \otimes u^{-\epsilon(\overline{i})})d(a_j \otimes u^{-\epsilon(\overline{j})})]b_1 b_2
= d(a_i \otimes u^{-\epsilon(\overline{s})}b_1)u^{(\overline{s})}(a_j \otimes b_2) + (a_i \otimes b_1)d(a_j \otimes u^{-\epsilon(\overline{l})}b_2)u^{(\overline{l})}
+ (\epsilon(\overline{s}))m^{-1}(a_i \otimes b_1)[d(a_j \otimes u^{-\epsilon(\overline{j})} + m)u^{(\overline{j})} - m - d(a_j \otimes u^{-\epsilon(\overline{j})})]b_1 b_2
+ (\epsilon(\overline{l}))m^{-1}(a_i \otimes b_1)[d(a_j \otimes u^{-\epsilon(\overline{j})} + m)u^{(\overline{j})} - m - d(a_j \otimes u^{-\epsilon(\overline{j})})]b_1 b_2.
\]

On the other hand

\[
B : = D(a_i \otimes b_1)(a_j \otimes b_2) + (a_i \otimes b_1)D(a_j \otimes b_2)
= (d(a_i \otimes u^{-\epsilon(\overline{s})}b_1)u^{(\overline{s})} + \epsilon(\overline{s})m^{-1}u^{(\overline{i})}[u^{-m}d(a_i \otimes u^{-\epsilon(\overline{i})} + m)
- d(a_i \otimes u^{-\epsilon(\overline{i})})]b_1)(a_j \otimes b_2)
+ (a_i \otimes b_1)(d(a_j \otimes u^{-\epsilon(\overline{l})}b_2)u^{(\overline{l})} + \epsilon(\overline{l})m^{-1}u^{(\overline{j})}[u^{-m}d(a_j \otimes u^{-\epsilon(\overline{j})} + m)
- d(a_j \otimes u^{-\epsilon(\overline{j})})]b_1)(a_j \otimes b_2)
\]

\[
= d(a_i \otimes u^{-\epsilon(\overline{s})}b_1)u^{(\overline{s})}(a_j \otimes b_2) + (a_i \otimes b_1)d(a_j \otimes u^{-\epsilon(\overline{l})}b_2)u^{(\overline{l})}
+ (\epsilon(\overline{s}))m^{-1}(a_i \otimes b_1)[u^{(\overline{i})-m}d(a_i \otimes u^{-\epsilon(\overline{i})} + m) - d(a_i \otimes u^{-\epsilon(\overline{i})})]b_1(a_j \otimes b_2)
+ (\epsilon(\overline{l}))m^{-1}(a_i \otimes b_1)[u^{(\overline{i})-m}d(a_j \otimes u^{-\epsilon(\overline{j})} + m) - d(a_j \otimes u^{-\epsilon(\overline{j})})]b_2.
\]
We must show $A = B$. But from the above computations we see that $A = B$ if and only if
\[
\epsilon(s) m^{-1}(a_i \otimes b_1)[d(a_j \otimes u^{-\epsilon(i)+m}) u^{\epsilon(i)+m} - d(a_j \otimes u^{-\epsilon(i)}) u^{\epsilon(i)}] b_2
\]
\[
= \epsilon(s) m^{-1}[d(a_i \otimes u^{-\epsilon(i)+m}) u^{\epsilon(i)+m} - d(a_i \otimes u^{-\epsilon(i)}) u^{\epsilon(i)}] b_1(a_j \otimes b_2).
\]
But this holds if and only if
\[
\epsilon(s) m^{-1}(a_i \otimes u^{-\epsilon(i)}) [d(a_j \otimes u^{-\epsilon(j)+m}) u^{\epsilon(j)+m} - d(a_j \otimes u^{-\epsilon(j)}) u^{\epsilon(j)}] b_1\]
\[
= \epsilon(s) m^{-1}[d(a_i \otimes u^{-\epsilon(i)+m}) u^{\epsilon(i)+m} - d(a_i \otimes u^{-\epsilon(i)}) u^{\epsilon(i)}] (a_j \otimes u^{-\epsilon(j)}) b_1 b_2.
\]
Now this holds by \((4.21)\) of Lemma 4.18.

Case $\epsilon(s + t) = \epsilon(s) + \epsilon(t) - m$: In this case we have from \((4.22)\) that
\[
A = d(a_i a_j \otimes u^{-\epsilon(i)-\epsilon(j)+m} b_1 b_2) u^{\epsilon(i)+\epsilon(j)} + (\epsilon(s) + \epsilon(t) - m) m^{-1}[d(a_i a_j \otimes u^{-\epsilon(i)-\epsilon(j)+m}) u^{\epsilon(i)+\epsilon(j)} + d(a_i a_j \otimes u^{-\epsilon(i)-\epsilon(j)+m})]
\]
\[
= D(a_i \otimes b_1)(a_j \otimes b_2) + (a_i \otimes b_1) D(a_j \otimes b_2) + M - N
\]
where
\[
M = d(a_i a_j \otimes u^{-\epsilon(i)-\epsilon(j)+m} b_1 b_2) u^{\epsilon(i)+\epsilon(j)} - d(a_i a_j \otimes u^{-\epsilon(i)-\epsilon(j)+m}) u^{\epsilon(i)+\epsilon(j)}
\]
and
\[
N = u^{\epsilon(i)+\epsilon(j)}[u^{-m} d(a_i a_j \otimes u^{-\epsilon(i)-\epsilon(j)+m}) - (a_i a_j \otimes u^{-\epsilon(i)-\epsilon(j)+m})] b_1 b_2.
\]
So we are done if we show that $M - N = 0$. We have seen in the proof of Lemma 4.18 that
\[
M = (a_i \otimes b_1)[d(a_j \otimes u^{-\epsilon(i)+m} b_2) u^{\epsilon(i)+m} - d(a_j \otimes u^{-\epsilon(i)+m}) u^{\epsilon(i)}]
\]
Also we have
\[
N = d(a_i \otimes u^{-\epsilon(i)+m}) u^{\epsilon(i)+m} b_1(a_j \otimes b_2) + (a_i \otimes b_1) d(a_j \otimes u^{-\epsilon(i)}) u^{\epsilon(i)} b_2
\]
\[
- d(a_i \otimes u^{-\epsilon(i)}) u^{\epsilon(i)} b_1(a_j \otimes b_2) - (a_i \otimes b_1) d(a_j \otimes u^{-\epsilon(i)}) u^{\epsilon(i)} b_2
\]
\[
= [d(a_i \otimes u^{-\epsilon(i)+m}) u^{\epsilon(i)+m} - d(a_i \otimes u^{-\epsilon(i)}) u^{\epsilon(i)}] b_1(a_j \otimes b_2).
\]
Now by acting $b_1$ to the both sides of equality \((4.21)\) of Lemma 4.18, we see that $M = N$. This completes the proof.

**Proof of Theorem 4.1** By Corollary 4.11 and Lemma 4.22 the map $\pi$ is an isomorphism. The last part of the statement follows from Lemma 4.1. This completes the proof of theorem.
Corollary 4.24. Let $A$ and $S$ be algebras over $k$ where $k$ contains all $m^{th}$-roots of unity for some integer $m$. If $\text{char}(k)=p>0$ assume that $p
mid m$. Assume $A$ and $S$ satisfy the followings:

(i) $A$ is perfect and finitely generated,
(ii) $S$ is commutative associative unital and finitely generated,
(iii) $\sigma_1 \in \text{Aut}(A)$, $\sigma_2 \in \text{Aut}(S)$, $\sigma_1^m = 1$, and $\sigma_2^m = 1$,
(iv) For some unit $g \in \mathbb{Z}_m$, there is a unit $u$ in $S_g$.

Then

$$
\mathcal{D}((A \otimes S)_g) \cong \sum_{i \in \mathbb{Z}_m} \mathcal{D}(A)_i \otimes S_{-\bar{i}} \oplus C(A)_i \otimes \mathcal{D}(S)_{-\bar{i}}.
$$

Proof. By Lemma 1.2, condition (v) of Theorem 4.1 is satisfied and so (4.3) holds. Now the result follows from Theorem 2.8. \hfill \Box

Remark 4.25. (i) Corollary 4.24 holds (with the same proof) replacing condition (i) with one of the followings:

- $A$ is a pfge algebra with $[C(A), \mathcal{D}(A)] = 0$.
- $A$ is perfect and finitely generated as a module over $dC(A)$.

(ii) Condition (iv) of Theorem 4.1 (or Corollary 4.24) can never be satisfied if $\sigma_2 = \text{id}$. However if $\sigma_1 = \text{id}$ and $\sigma_2 = \text{id}$, this theorem (or corollary) holds by Theorem 2.8.

Under the conditions of Theorem 4.1 let $\varphi := \pi^{-1}$. Then Lemma 1.7 gives the exact formula for $\varphi$, namely for $d \in \mathcal{D}((A \otimes S)_g)$, $a_i \in A_i$ and $b_{-\bar{i}+\bar{s}} \in S_{-\bar{i}+\bar{s}}$,

$$
\varphi(d)(a_i \otimes b_{-\bar{i}+\bar{s}}) = d(a_i \otimes u^{-\sigma(\bar{s})}b_{-\bar{i}+\bar{s}})u^{\sigma(\bar{s})} + \epsilon(\bar{s})(m)^{-1}u^{\delta(\bar{i})}[u^{-m}d(a_i \otimes u^{-\epsilon(\bar{i})+m}) - d(a_i \otimes u^{-\epsilon(\bar{i})})]b_{-\bar{i}+\bar{s}}.
$$

(4.26)

In Example 4.6 we saw that the map suggested in [BM, Theorem 1.7] does not provide an inverse map for $\pi$. In the following example we see how our inverse map $\varphi$ works for that particular example. We also explain how it works for the most studied example, namely twisted affine Lie algebras.

Example 4.27. (i) In Example 4.6 we have $A = k 1, S = k[z \pm 1], m = 4, \bar{q} = 1$ and $u = z$. Let $d = 1 \otimes zd/dz$. We compute $\varphi(d)(1 \otimes z^2)$ from (4.26) with $\bar{i} = 0, \bar{s} = 2, a_0 = 1, b_{-\bar{i}+\bar{s}} = z^2, \epsilon(\bar{i}) = 0$ and $\epsilon(\bar{s}) = 2$. Then

$$
\varphi(d)(1 \otimes z^2) = d(1 \otimes z^{-2}z^2)z^2 + 2(4)^{-1}z^0[d(1 \otimes z^{-0+4}) - d(1 \otimes z^0)]z^2
$$

$$
= 0 + 2(4)^{-1}[4(1 \otimes 1) + 0]z^2
$$

$$
= 2(1 \otimes z^2).
$$

Replacing 2 with 3 in the above computations, gives $\varphi(1 \otimes z^3) = 3(1 \otimes z^2)$. Finally, to compute $\varphi(1 \otimes z^5)$, we note that $z^5 \in S_1 = S_{0+1}$, so $\bar{s} = 1$ and $\epsilon(\bar{s}) = 1$. Therefore

$$
\varphi(d)(1 \otimes z^5) = d(1 \otimes z^{-1}z^5)z^1 + (4)^{-1}z^0[d(1 \otimes z^{-0+4}) - d(1 \otimes z^0)]z^5
$$

$$
= 4(1 \otimes z^5) + (4)^{-1}[4(1 \otimes 1) + 0]z^5
$$

$$
= 5z^5.
$$
Thus 
\[ \varphi(d)(1 \otimes z^2 z^3) = (1 \otimes z^2) \varphi(d)(1 \otimes z^3) + \varphi(d)(1 \otimes z^2)(1 \otimes z^3). \]

(ii) Let \( A = k, S = k[z^{\pm 1}], \sigma_1 = 1 \) and \( \sigma_2(z^n) = \omega^{-n} z^n. \) Then \( (\mathcal{D}(S))_0 = \text{span}_k \{ z^{nm+1} d/dz \mid n \in \mathbb{Z} \} \) and \( \mathcal{D}(S)_0 = \text{span}_k \{ t^{n+1} d/dt \mid n \in \mathbb{Z} \}, \) where \( t = z^m. \)

Now \[ \mathcal{D}(S)_i = \text{span}_k \{ z^{nm} d/dz \mid n \in \mathbb{Z} \} \]

and \( \mathcal{D}(S)_0 = \text{span}_k \{ z^{nm+1} d/dz \mid n \in \mathbb{Z} \}. \)

Then with respect to the \( \mathcal{D}(S)_i \) gives the explicit formula for the isomorphism \( \mathcal{D}(S)_0 \cong (\mathcal{D}(S))_0. \) Namely if \( d = t^{n+1} d/dt, j \in \mathbb{Z} \) and \( \eta(j) := m^{-1}(j + \epsilon(-d)), \) then identifying \( 1 \otimes a \) with \( a, \) we obtain

\[ \varphi(d)(z^j) = d(z^{m\eta(j)} z^{-\epsilon(-j)} + \epsilon(-j) m^{-1} z^m d(z^{-m}) - d(1)) z^j \]

\[ = d(t^{m\eta(j)} z^{-\epsilon(-j)} + \epsilon(-j) m^{-1} d(t^{-1}) z^{m+k} \]

\[ = m^{-1} j z^{nm+j} \]

\[ = m^{-1} z^{nm+1} d/dz(z^j). \]

So \( \varphi \) takes \( t^{n+1} d/dt \) to \( m^{-1} z^{nm+1} d/dz. \)

(iii) Let \( \text{char}(k) = 0 \) and assume that \( \omega \) is a \( m \)-th primitive root of unity in \( k. \) Let \( g \) be a twisted affine Kac–Moody Lie algebra over \( k. \) Then \( g \) can be realized as the fixed points of a loop algebra \( A \otimes S, \) under a finite order automorphism \( \sigma = \sigma_1 \otimes \sigma_2, \) where

- \( A \) is a finite dimensional simple Lie algebra,
- \( S = k[z^{\pm 1}] \) is the algebra of Laurent polynomials in \( z, \)
- \( \sigma_1 \) is a period \( m \) automorphism of \( A \) and \( \sigma_2(z^n) = \omega^{-n} z^n. \)

Then with respect to the \( \mathbb{Z}_m \)-grading on \( A \otimes S, \) we have \( g = (A \otimes S)_0. \) We also have \( S_1 = z^{-i} k[z^{\pm 1}], C(A) = C(A)_0 = k \) and as above \( \mathcal{D}(S)_0 = \text{span}_k \{ z^{nm+1} d/dz \mid n \in \mathbb{Z} \}. \)

Therefore, by Theorem 4.4

\[ \mathcal{D}(g) = \mathcal{D}((A \otimes S)_0) = (\mathcal{D}(A \otimes S))_0 = \sum_{i \in \mathbb{Z}_m} \mathcal{D}(A)_i \otimes S_{-i} \oplus 1 \otimes (\mathcal{D}(S))_0 \]

\[ = \sum_{i \in \mathbb{Z}_m} \text{ad} A_i \otimes z^{-i} k[z^{\pm m}] \oplus 1 \otimes (\mathcal{D}(S))_0. \]

Now it is easy to see that for any \( a \in A_i \) and \( n \in \mathbb{Z}, \) \( \varphi(\text{ad}(a) \otimes z^{nm-i}) = \text{ad}(a) \otimes z^{nm-i} \)

and \( \varphi(1 \otimes z^{nm+1} d/dz) = (1 \otimes z^{nm+1} d/dz). \)

REFERENCES

[AABGP] B. Allison, S. Azam, S. Berman, Y. Gao and A. Pianzola, Extended affine Lie algebras and their root systems, Mem. Amer. Math. Soc. 126 (603), 1997.

[ABFP] B. Allison, S. Berman, J. Faulkner, and A. Pianzola, Realization of graded-simple algebras as loop algebras, (in preparation).

[ABP] B. Allison, S. Berman and A. Pianzola, Iterated loop algebras, arXiv:math.RT/0502225 v1, (2005).

[BM] G. Benkart and R.Y. Moody, Derivations, central extensions, and affine Lie algebras, Algebras Groups Geom. 3 (1986), 456–492.

[BN] G. Benkart and E. Neher, The centroid of extended affine and root graded Lie algebras, arXiv:math.RT/0502501 (2005), 1–35.

[BGK] S. Berman, Y. Gao, Y. Krylyuk, Quantum tori and the structure of elliptic quasi-simple Lie algebras, J. Funct. Anal. 135 (1996), 339–389.
[B] R.E. Block, *Determinations of the differentiably simple rings with a minimal ideal*, Ann. of Math. (2) 90 (1969), 433–459.

[J] N. Jacobson, *Lie algebras*, Dover, New York, 1979.

[K] V.G. Kac, *Infinite dimensional Lie algebras*, Third edition, Cambridge University Press, Cambridge, 1990.

[N1] E. Neher, *Lie tori*, C. R. Math. Rep. Acad. Sci. Canada Vol. 26, (3) (2004), 84–89.

[N2] E. Neher, *Extended affine Lie algebras*, C. R. Math. Rep. Acad. Sci. Canada Vol. 26, (3) (2004), 90–96.

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