Relating quantization and time evolution in the presence of
gauge symmetry

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Abstract

Up to now there is no definition or an example of a gauge-invariant quantum field theory
in four-dimensional space-time, with mathematical rigor and in the absence of approxima-
tions. This is mainly due to three features of the theory: 1) the canonical quantization is
not manifestly Lorentz invariant; 2) the Lagrangian is singular due to the gauge symmetry;
and 3) the gaussian measure is not gauge invariant. At least one of these features becomes
an obstacle to mathematical rigor in either the canonical or the path-integral quantization.

Using the fact that there is a wave-function associated to any probability distribution, we
study a class of statistical field theories in four-dimensional space-time where the (classical)
canonical coordinates when modified by the unitary time evolution, verify the canonical
commutation relations. We show that these statistical field theories have all the features of
a gauge-invariant quantum field theory in four-dimensional space-time. Thus the features
1), 2) and 3) are not obstacles to define the theory in our formalism.

1 Introduction

Schrödinger described quantization as the consequence of solving an eigenvalue problem for the
Hamiltonian [1]: in an infinite-dimensional linear space of functions, continuous and discrete
(i.e. quantized) energy spectra may coexist. Thus, from the very beginning there was a relation
between the time evolution (defined by the Hamiltonian) and the notion of quantization.

There is no doubt that the best known description of the experimental data collected so far is
based on a quantum theory [2]. However, the notion of quantization is not much clearer than it
was in 1926 [3]. In this paper we will change this status, proposing a simple and mathematically
meaningful definition of quantization. We start by addressing what quantization is not.

Quantization is not replacing Poisson bracket’s by canonical commutation relations. The
method of replacing Poisson bracket’s by canonical commutation relations can always be applied
(for analytical functions), it is called prequantization [4, 5]. However it doesn’t lead by itself to
useful results (hence the name prequantization). Of course, we can try to improve the method
so that it leads to useful results (this is the geometrical quantization program [4]), however we
end up with a definition of quantization which is so complex and arbitrary that it is not useful in practice, in particular in the presence of gauge symmetries.

Quantization is also not second quantization (based on a Fock-space), which relates a quantum description of a single-particle system to a quantum description of a many-particle system [5]. We can only apply second quantization to a quantum theory, hence the name “second”.

Quantization is also not computing the Feynman’s path integral, since we know that the Feynman’s path integral does not have the property (sigma-additivity), which allows computation of the integral by approximating the integrand [6], and thus it is not an integral. Of course as in prequantization, we can try to improve the path integral [7], however we are very far from a consistent definition of path integral which is useful in practice.

Quantization is also not a perturbative expansion or a lattice regularization. These two different approximations are useful and have a clear definition, but since we know that they are complementary [6] then neither of them can be used to define quantization.

Note that there is enough experimental evidence to conclude that all the methods above mentioned —namely prequantization, second quantization, Feynman’s path integral, perturbative expansion, lattice regularization— are related to the quantum phenomena and thus they are necessarily related with the definition of quantization. But we insist that it is also clear that none of them by itself can be used to define quantization.

There is a big conceptual problem with the notion of quantization: we are trying to relate a deterministic theory (classical mechanics) with a non-deterministic theory (quantum mechanics). From the point of view of (classical) information theory [8], the root of probabilities (i.e. non-determinism) is the absence of information. Statistical methods are required whenever we lack complete information about a system, as so often occurs when the system is complex [9]. Thus we can convert a deterministic theory to a statistical theory unambiguously (using trivial probability distributions); but we cannot convert a statistical theory into a deterministic theory unambiguously since we need new information.

On the other hand, the relation between quantum mechanics and a statistical theory (both are non-deterministic) is clear: the wave-function is a parametrization for any probability distribution [11]. It is a very useful parametrization because it allows us to represent a group of transformations using linear transformations on an hypersphere. Since these linear transformations have an intrinsically random nature, Quantum Mechanics is a generalization of Classical Mechanics (but not of probability theory, thanks to the wave-function’s collapse [11]).

Theories such as classical electrodynamics or more generally classical non-abelian gauge theories [12] involve a system of non-linear partial differential equations. It is a very hard problem to study in general the space of classical solutions of such systems. Even when a few solutions can be found, they may not be the ones that describe the physical system correctly. A consistent theory covering many cases only exists (at the moment) for systems of linear partial differential equations [14]. Thus to solve many non-linear deterministic theories we may not

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1E.g. the assumptions required by the deterministic models in reference [10] are new information.
2A well known example is the Navier-Stokes equation [13].
have better alternative (at the moment) than to consider them as a particular case of a statistical theory and apply linear quantum methods on its wave-function parametrization \[15, 16\]—then the building blocks of the overall deterministic theory are non-deterministic.

The non-commutativity of operators is thus intrinsic to any statistical theory. This saves us from the need to “deform” commutative algebras into non-commutative ones upon quantization. In our opinion, either the quantization of a classical theory or the classical limit of a quantum theory cannot go much beyond Koopman-von Neumann version of classical mechanics \[15\], i.e. a description of classical mechanics as a statistical theory (which is always possible, since a deterministic theory is a particular case of a statistical theory).

2 Lorentz covariance

A complete physical system is a free system. If we neglect gravity, the wave-function associated to the free system is a unitary representation of the Poincare group, regardless of the interactions occurring within the free system \[17\]. A priori we can restrict the system in such a way that there is a referential where its 3-momentum is null, i.e. the center-of-momentum. This restriction excludes a non-trivial system traveling at the speed of light (e.g. a photon with non-null energy). Then the dynamics determined by the Hamiltonian becomes linked with the time coordinate (for a photon this would not be the case \[18\]). Moreover, despite we do not know a priori the diagonal form of the Hamiltonian, we know that it is either continuous or discrete in the neighborhood of the eigenvalue 0 (in the center-of-momentum). If it is continuous then the zero energy has null measure. If it is discrete, we can modify the Hamiltonian adding an appropriate constant such that the zero energy is not one of the eigenvalues (this is equivalent to adding to the system a free massive particle with null momentum relative to the system). In any case, we can assume without loss of generality that our system is a superposition of massive free systems. Then, the Lorentz transformations become known and are given by the Wigner irreducible massive representations of the Poincare group \[17\]. If the Hamiltonian is bounded from below then the vacuum state is not Lorentz invariant, as it was already suggested \[19\].

In the center-of-momentum, the relevant group is not the Poincare group, but the little group of spatial rotations and the translation in time \[17\]. Thus the spatial and time coordinates of space-time, become separated. The fields are no longer representations of the Lorentz group, but only of the rotation group and the canonical commutation relations in no way are in conflict with the little group of spatial rotations. The relevant gauge symmetry is the one along the time component only, since the gauge symmetry along the spatial components does not commute with the 3-momentum.

Therefore and unlike what it is often claimed in the literature, it is false that quantization is incompatible with Lorentz covariance. The only restriction is that we need to treat differently systems which behave differently under the action of the Poincare group, e.g. we cannot use the same formalism for a system where there is a center-of-momentum and also for a system
travelling at the speed of light. Similar assumptions concerning the energy-momentum of the full system are also done in the Källén-Lehmann representation of a non-perturbative two-point correlation function, where it is assumed that the eigenvalues of the 3-momentum squared are not larger than those of the squared energy \[20]\[p203].

Note that a formalism based on a Lorentz scalar (such as the path integral or the Peierls bracket \[21]\) instead of an Hamiltonian) is already problematic at the classical level, since the dynamics is Lorentz covariant but not invariant \[22]. For instance, numerical simulations of (classical) general relativity use an Hamiltonian formalism \[22]. Moreover, the phenomenologically successful (but ill defined) path integral formalism based on the Lagrangian is in fact equivalent to a path integral based on the Hamiltonian \[23].

3 Quantization due to time evolution

In this section we will consider fields defined in a one dimensional time, neglecting the space dimensions. As we have seen in the previous section, in the center-of-momentum frame and in harmony with Lorentz covariance, the time is completely separated from space and plays a special role due to its association with the Hamiltonian. Therefore, the extension of the results of this section to fields defined in four dimensional space-time is straightforward.

Using a wave-function, we can parametrize the probability distribution for a field in time. The linear space generated by all wave-functions is a Fock space \[16\]. The Fock space has the properties of a continuous tensor product of Fock-spaces corresponding to fields defined in infinitesimal time-intervals, i.e.

\[
\varphi(t)dt.
\]

The parallel transport in time will not only advance the time-intervals forward, but it will modify the wave functions corresponding to each time interval accordingly to an Hamiltonian which plays here the role of a covariant derivative. With abuse of language, we can describe the situation as a continuous tensor product of initial-value (i.e. Cauchy) problems, instead of just one initial value problem as in non-relativistic Quantum Mechanics. If the Hamiltonian is trivial, then the parallel transport will only advance the time-intervals forward, without any other modification to the Fock space.

We have the self-adjoint position \(x(t)\) and momentum \(p(t)\) operators, verifying the Weyl relations.

\[
e^i\int dt f(t)x(t)e^{-i}\int ds g(s)p(s) = e^{-i}\int dt f(t)g(t)e^i\int ds g(s)p(s) e^{-i}\int dt f(t)x(t)
\]

where \(f, g\) are real functions.

The Stone-von Neumann theorem implies that the Weyl relations uniquely define the unitary operators \(e^i\int dt f(t)x(t)\) and \(e^{-i}\int ds g(s)p(s)\) up to a unitary transformation.

Thus, we can assume without loss of generality that the momentum and position operator satisfy the canonical commutation relations:
\[ [p(t), x(\tau)] = i\delta(t - \tau) \]  

(2)

We introduce now the procedure of quantization due to unitary time evolution. Time evolution transforms a sequence of time-ordered operators \[24\] (which commute algebraically but the time-ordering is non-commutative) into a sequence of (algebraically) non-commuting operators acting on a single slice of time of the wave-function. In particular, the canonical commutation relations of position and momentum are reproduced for \(\epsilon\) sufficiently small.

This method of quantization is inspired by the Source formalism of Schwinger \[25\] which is itself both an alternative to and inspired by the Feynman’s path integral, where time-ordering \[24, 26\] plays a key role. At the technical level, we use the Fock-space parametrization of a statistical field theory \[16\] to implement the concept of time-ordering \[24, 26\] consistently.

We can define a unitary translation operator as

\[
T(a) e^{i \int dt f(t)x(t)} T^\dagger(a) = e^{i \int dt f(t-a)x(t)} \]

acting on the momentum operator in an analogous way. We can express

\[
T(a) = e^{i a^2 \int dt \partial_t x(t) - x(t) \partial_t p(t)}
\]

We now consider the Trotter exponential product approximation \[27\], verifying for small \(\epsilon\):

\[
e^{i \epsilon A} e^{i \epsilon B} = e^{i \epsilon (A+B) - \frac{\epsilon^2}{2}[A,B] + iO(\epsilon^3)}
\]

(3)

This is a good approximation since it works for unbounded operators \(A,B\).

Let now \(\epsilon = \frac{1}{n}\) with \(n\) arbitrarily large. Then,

\[
e^{i \epsilon A} e^{i \epsilon B} e^{-i \epsilon A} = (e^{i \epsilon A} e^{i \epsilon B} e^{-i \epsilon A})^n = e^{iB - \epsilon[A,B] + iO(\epsilon^2)}
\]

(4)

Therefore, for small enough \(\epsilon\)

\[
e^{i \epsilon} \int d\tau p^2(\tau) e^{i \int dt f(t)x(t)} e^{-i \epsilon} \int d\tau p^2(\tau) = e^{i \int dt f(t)(x(t) + \epsilon p(t))}
\]

(5)

\[
T(\epsilon) e^{i \epsilon} \int d\tau p^2(\tau) e^{i \int dt f(t)x(t-\epsilon)} e^{-i \epsilon} \int d\tau p^2(\tau) T^\dagger(\epsilon) = e^{i \int dt f(t)(x(t) + \epsilon p(t))}
\]

(6)

Now we need a definition of covariant derivative in time of the position operator \(x\), consistent with the fact that only \(F(a) = U(a) e^{i \int dt f(t)x(t-a)} U^\dagger(a)\) (where \(U(a)\) is the parallel transport) is bounded while \(x\) is unbounded. If we would be dealing with a commutative algebra, then the natural definition would be

\[
\lim_{\epsilon \to 0} F^\dagger(0)(F^\dagger(\epsilon))\dagger
\]

(7)

For a trivial parallel transport, we would get as required:

\[
\lim_{\epsilon \to 0} F^\dagger(0)(F^\dagger(\epsilon))\dagger = \lim_{\epsilon \to 0} e^{i \int dt f(t) \frac{x(t) - x(t-\epsilon)}{\epsilon}}
\]

(8)
But since we are dealing with a non-commutative algebra, we need to use the Trotter exponential product approximation formula, to define the exponential version of the covariant derivative:

$$\lim_{\epsilon \to 0} \lim_{n \to \infty} (e^{\frac{1}{n}\epsilon}(0)(e^{\frac{1}{n}\epsilon}))^n$$

(9)

And so for the parallel transport $U(\epsilon) = T(\epsilon)e^{i\epsilon \int d\tau p^2(\tau) + V(x(\tau))}$ where $V(x)$ is a potential only dependent on the position operator, the exponential version of the covariant derivative of the position operator $x$ is:

\[
\lim_{\epsilon \to 0} \lim_{n \to \infty} (e^{\frac{1}{n}\epsilon}(0)(e^{\frac{1}{n}\epsilon}))^n = (e^{i\frac{1}{n} \int dt f(t)x(t)}U(\epsilon)e^{-i\frac{1}{n} \int dt f(t)x(t-\epsilon)}U^\dagger(\epsilon))^n = e^{i \int dt f(t)p(t)}
\]

(10)

The result is the exponential of the momentum operator, which verifies the Weyl relations with respect to the exponential of the position operator. With some abuse of language, we can say that for this type of time-evolution (and thus for this type of Hamiltonian $p^2 + V(x)$, which is most common in Quantum Mechanics), the covariant derivative of the position operator is the momentum operator. Thus the quantization (i.e. the Weyl relations) may appear in a statistical field theory due to a particular time evolution, which is characteristic of quantum phenomena.

4 Singular Lagrangian

Now we will address one important problem about quantization in the presence of gauge symmetries. How do we handle the singularity of the Lagrangian due to the gauge symmetry? This will lead to the non-uniqueness of the solution to the Euler-Lagrange equations, given initial conditions [28–30]. Both Dirac’s canonical quantization and Feynman’s path-integral, are different ways to produce a unique evolution of the wave-function. In both ways, the quantization of gauge theories is so complex that a mathematical definition of quantization is not possible.

But if in classical field theory, the solution to the Euler-Lagrange equations is not necessarily unique, why should the solution to the Euler-Lagrange equations be unique in statistical field theory? Since the wave-function is a parametrization of any probability distribution, such statistical field theory has a wave-function associated.

The Euler-Lagrange equations can always be expressed as the definition of the time-derivative of a sub-set of variables (such definition is dependent only on the variables defining the phase-space) [28], plus constraints on the variables of the phase-space. This may have the cost of introducing more variables and equations, but it is always possible to reduce the equations to at most first derivatives in time.

These Euler-Lagrange equations uniquely determine an Hamiltonian and a corresponding
time-evolution, in the following way: the Hamiltonian is defined such that its commutator with the variables (or with the momentum operators corresponding to the variables) is equal to the non-trivial side of the Euler-Lagrange equations\(^3\). This will produce all Euler-Lagrange equations, including the constraints. Such Hamiltonian will usually have a similar form to a Lagrangian, with the constraints produced at the cost of variables whose time derivative is missing from the Hamiltonian. In the region of phase space where the constraints (primary and secondary [28]) are null, the time evolution leaves invariant the variables (and the corresponding momentum) whose time derivative is missing from the Hamiltonian—these variables play the role of Lagrangian multipliers in such relevant region of phase space.

Note that since the phase space is composed by variables (fields) which are functions of (space-)time, then the Lagrange multipliers are arbitrary\(^4\) functions of time and will appear in the equations of motion with the correct time corresponding to the equation of motion. Thus the solution to the Euler-Lagrange equations, given initial conditions, is not unique.

This Hamiltonian will not yet produce phenomena with quantum characteristics, since the Euler-Lagrange equations commute with the variables defining the phase-space. However, since we can define a gauge theory using a wave-function without gauge-fixing, we may now consider more general Hamiltonians such that its commutator with the variables does not commute with the variables defining the phase-space (e.g. as in the previous section, note that the full Hamiltonian includes the translation operator \(p(t)\frac{\partial}{\partial t}x(t)\)). The only requirement is that the Hamiltonian be gauge invariant, then the unitary operator responsible for the parallel transport in time commutes with the time-dependent gauge transformation and thus the scattering matrix is gauge-invariant.

Since we have fields dependent on (space-)time, the time evolution of a set of fields may be dependent on another field which is part of the phase-space and it is left invariant by the time-evolution. In this way, the Hamiltonian for the theory defined in (space-)time is unique and when we consider only a slice of time, then the Hamiltonian becomes dependent on a field external to the phase-space and thus the time-evolution is not unique. For a single slice of time, by choosing the field external to the phase-space upon which the Hamiltonian depends, we are fixing the gauge.

In conclusion, we can define in a rigorous way a statistical field theory with a gauge-invariant time evolution which verifies all the requirements of a Quantum Field Theory.

We emphasize that to make numerical predictions from such well defined statistical field theory, the best methods are probably the same standard methods of Quantum Field Theory (e.g. perturbation theory based on Feynman diagrams, lattice simulations) and the numerical predictions may be the same. These methods are not necessarily easy (e.g. deriving the secondary constraints) and dubious approximations may be involved.

What has changed is the fact that we are applying these standard methods to a theory which

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\(^3\)By non-trivial side we mean that if we write the Euler-Lagrange equations as \(expr = 0\) then the commutator of the Hamiltonian with some variable or momentum operator should be equal to \(expr\).

\(^4\)They are not completely arbitrary, since the constraints must be null.
is well defined mathematically, unlike what happens nowadays with Quantum Gauge Field Theory. This has obvious advantages, e.g. to solve foundational problems. One foundational problem will be solved already in the next section.

5 Gauge-variant gaussian measure

One of the features of the Feynman’s path integral is that it has a flat (i.e. Lebesgue like) measure which is thus gauge-invariant. Yet, it proved that in rigor such infinite-dimensional Lebesgue measure cannot exist. Thus there will never be something like a gauge-invariant vacuum state, because such state requires a gaussian measure which is gauge-variant.

In our formalism, this is irrelevant since the gauge-invariant Hamiltonian (which includes the translation operator $p(t)\partial_t x(t)$) is not bounded from below. Thus, we are always dealing with a situation where there is no vacuum-state corresponding to the Hamiltonian. By choosing a wave-function we are also choosing a gauge and a time slice and if we evaluate the Hamiltonian corresponding to that gauge and to one time slice then it may be bounded from below and a vacuum state exists. However, there is no problem with gauge-invariance now since the gauge and time slice was fixed by our choice of state. The vacuum state defined in a single slice of time does not necessarily break the local gauge symmetry along the time component (it does break along the spatial component, but as we said in a previous section this symmetry is treated differently than the gauge symmetry along time in our formalism), since this symmetry acts on the vacuum state as a global symmetry.

After we evaluate the Hamiltonian in a single time slice, we can apply the path-integral and then a lattice regularization. This will be done in a fixed gauge. With a lattice regularization, there appears a Haar measure for the gauge group. So, we can do an average of our results over the gauge group. Since the time-evolution is gauge-invariant, this will not change the result. Note that when averaging over the gauge group we need to average the operators of the correlation function, the gauge-fixing term and the gaussian measure of the vacuum state. When the vacuum state conserves the gauge-symmetry, the results coincide with the ones we would obtain by the standard lattice methods, since the gauge-fixing terms are considered as part of the operators appearing in the correlation function. However, when the vacuum state breaks the gauge-symmetry then the standard lattice methods do not produce the same results as the gauge-fixed version, since the path-integral assumes that the vacuum of the canonical formalism conserves the gauge symmetry [31, 32].

The above implies that gauge-variant asymptotic states are perfectly fine, even at the non-perturbative level. Therefore, there is no valid argument based on gauge-symmetry which leads to a non-perturbative correction to the perturbative calculations in the context of the Higgs mechanism. Note that this does not imply that the mean-field approximation always works, but the non-perturbative problems of the mean-field approximation are not exclusive to the local gauge-symmetry (e.g. the mean-field approximation also breaks global symmetries in the
two-Higgs-doublet model).

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