Conformal Sigma Models with Anomalous Dimensions and Ricci Solitons

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Abstract

We present new non-Ricci-flat Kähler metrics with $U(N)$ and $O(N)$ isometries as target manifolds of superconformally invariant sigma models with an anomalous dimension. They are so-called Ricci solitons, special solutions to a Ricci-flow equation. These metrics explicitly contain the anomalous dimension and reduce to Ricci-flat Kähler metrics on the canonical line bundles over certain coset spaces in the limit of vanishing anomalous dimension.

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1 Introduction

String compactification with consistent non-trivial background is one of important subjects for long time. Sigma model approach gives a set of equations of motion for such backgrounds by vanishing condition on the beta function \[1\]. Calabi-Yau manifolds, Ricci-flat Kähler manifolds, are required as compactified manifolds with constant dilaton backgrounds. However no explicit metrics for compact Calabi-Yau manifold are known. On the other hand, if we allow non-compact manifolds, some explicit metrics for non-compact Calabi-Yau manifolds can be constructed \[2\]–\[11\]. Explicit solutions for non-Ricci-flat Kähler manifold with a non-trivial dilaton background were obtained by Kiritsis, Kounnas and Lust \[12\] as a generalization of two dimensional Euclidean black hole \[13\] \[14\].

We can consider nonlinear sigma models whose scalar fields have an anomalous dimension instead of a constant dilaton background, because they can be transformed to each other by a field redefinition for the scalar fields to absorb the dilaton field \[14\]. Using the Wilsonian renormalization group equation, the beta function of the \(N=(2,2)\) supersymmetric nonlinear sigma models consisting of the complex scalar fields \((\varphi^i, \varphi^{*i})\) with an anomalous dimension \(\gamma\) was obtained in \[15\] \[16\] \[17\] (see \[18\] for a review) as

\[
- \frac{\partial g_{ij}}{\partial t} = \beta(g_{ij}) = \frac{1}{2\pi} \left( \Gamma_{ijk} \Gamma^{ijk} + \varphi_i \partial_j \varphi^k + \varphi^{*i} \partial_j \varphi^{*k} + 2g_{ij} \right) \tag{1.1}
\]

with \(g_{ij}^{*}, R_{ij}^{*}\) and \(\Gamma_{ij}^{*}\) being the Kähler metric, the Ricci-form and the connection, respectively. Conformally invariant models are defined by the condition of vanishing beta function

\[
R_{ij}^{*} + 2\pi \gamma \left( \varphi^i g_{ij} + \varphi^{*i} g_{ij} + 2g_{ij} \right) = 0. \tag{1.2}
\]

A \(U(N)\)-invariant solution for this equation was obtained in \[16\]. It was shown to be equivalent to the \(U(N)\)-invariant solution with a dilaton background constructed in \[12\]. This is due to the equivalence of this model to the dilaton model. In this note we derive another \(U(N)\)-invariant solution and its extension to \(O(N)\).

Eq. (1.1) is a so-called Ricci-flow equation which has attracted much attention recently in mathematics (see \[20\] for a review by a physicist). In the Riemann manifold with a metric \(g_{\mu\nu}\) and some vector field \(\xi^\mu\), the general Ricci-flow equation is written as

\[
- \frac{\partial g_{\mu\nu}}{\partial t} = R_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu. \tag{1.3}
\]

If we take \(\xi^\mu = 2\pi \gamma (\varphi^i, \varphi^{*i})\) as a special case in the Kähler manifold, the Ricci-flow equation (1.3) reduces to Eq. (1.1). Solutions to \(\frac{\partial g_{\mu\nu}}{\partial t} = R_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0\) are called Ricci solitons and play a central role in classification of manifolds. Our new solutions presented in this note are Kähler Ricci solitons and we hope that these solutions are useful for classification of Kähler manifolds.

\[1\] To derive this equation, it was useful to expand the Lagrangian in terms of Kähler normal coordinates \[19\] which are natural extension of Riemann normal coordinates to Kähler manifolds.
2 \( U(N) \) Invariant Model

We prepare an \( N \)-vector \( \vec{\phi} = (\phi^1, \cdots, \phi^N) \) belonging to the fundamental representation of \( U(N) \). Let us assume that the Kähler potential \( K \) is written as a function of the \( U(N) \)-invariant as

\[
K = K(X), \quad X \equiv \vec{\phi}^\dagger \vec{\phi}.
\]

Geometric quantities can be calculated [10], to yield

\[
g_{ij} = K' \delta_{ij} + K'' \phi^i \phi^j \quad (2.2)
\]

\[
R_{ij} \equiv -\delta_i \delta_j \text{tr} \log g_{kl} = -(N-1)K'' K' \left[ \frac{K''}{K'} + \frac{2K'' + K''' X}{K' + K'' X} \right] \delta_{ij} - \left( N - 1 \right) \left( \frac{K''}{K'} \right)^2 \frac{K''^2 + K''' X}{K' + K'' X} \right] \phi^i \phi^j, \quad (2.3)
\]

\[
g_{ij} \Gamma^l_{ik} = g_{ij} \delta_{kl} + K'' (\phi^k \delta_{ij} + \phi^i \delta_{kj}) + K''' \phi^k \phi^i \phi^j \quad (2.4)
\]

with the prime denoting a differentiation with respect to \( X \). Substituting these into Eq. (1.2), we obtain an ordinary differential equation, from a term proportional to \( \delta_{ij} \),

\[
(N-1)K'' K' + \frac{2K'' + K''' X}{K' + K'' X} a(K' + K'' X) = 0 \quad (2.5)
\]

with \( a \) a constant defined by the anomalous dimension as \( a \equiv -4\pi \gamma \), and the derivative of this equation with respect to \( X \) from a term proportional to \( \phi^i \phi^j \). The differential equation (2.5) can be integrated to give

\[
(K')^{N^{-1}} (K' + K'' X) = c e^{-aK' X} \quad (2.6)
\]

with \( c \) an integration constant. Defining the function

\[
F \equiv K' X, \quad (2.7)
\]

Eq. (2.6) can be rewritten as

\[
F^{N-1} F' X^{1-N} = c e^{-aF}. \quad (2.8)
\]

Again this can be integrated to give the algebraic equation

\[
e^{aF} \sum_{r=0}^{N-1} \frac{(-1)^r (N-1)! F^{(N-1)-r}}{(N-1-r)! a^{r+1}} = \frac{c}{N} X^N + b \quad (2.9)
\]

with \( b \) an integration constant. This reproduces the solution found in [12, 16] with a boundary condition \( F(0) = 0 \) which implies \( K(0) = \text{const.} \). It reduces to \( F = \frac{1}{a} \log(1 + aX) \) for \( N = 1 \) which defines the two dimensional Euclidean black hole [13, 14] for \( a > 0. \)
We now construct a new $U(N)$-invariant solution. To this end it is useful to define new coordinates by

$$\bar{\phi}^T = \sigma(1, z^T).$$

(2.10)

We label $z$ by the same indices $i, j, \cdots$ with $\phi$ in the following. Then the invariant can be rewritten as

$$X = \bar{\phi}^T \phi = |\sigma|^2 (1 + |z|^2) \equiv |\sigma|^2 Z.$$

(2.11)

It is useful to write down the metric in these coordinates as

$$g = \begin{pmatrix}
g_{\sigma\sigma} & g_{\sigma j^*} 
g_{i\sigma^*} & g_{ij^*}
go_{\sigma^*} & \sigma \partial^2 - K'Z + K''|\sigma|^2 Z^2, 
\sigma \partial^2 - K'\sigma^* \partial_j Z + K'' \sigma^* |\sigma|^2 Z \partial_j Z, 
\sigma \partial^2 - K' \partial_i \partial_j Z + K'' |\sigma|^4 \partial_i \partial_j Z.
\end{pmatrix}$$

(2.12)

Using a solution $F$ of the same equation (2.9) with a boundary condition $F(0) = \text{const.} \neq 0$ different from the previous one, the metric can be calculated in these coordinates as

$$ds^2 = c e^{-aF} F^{1-N} Z^N |\sigma|^{2N-2} d\sigma^2 + [c e^{-aF} F^{1-N} Z^{N-1} |\sigma|^{2N-2} \sigma^* \partial_j Z d\sigma d\sigma^{*j} + \text{c.c.}]
+ [F(Z^{-1} \partial_i \partial_j Z - Z^{-2} \partial_i Z \partial_j Z) + c e^{-aF} F^{1-N} Z^{N-2} |\sigma|^{2N} \partial_i Z \partial_j Z] d\sigma^i d\sigma^{*j}.$$

(2.13)

Due to that boundary condition, this has a coordinate singularity in the limit $\sigma \to 0$. We perform the coordinate transformation

$$\rho = \sigma^N / N$$

(2.14)

to remove this coordinate singularity. We thus obtain the final form

$$ds^2 = c e^{-aF} F^{1-N} Z^N |\rho|^2 + [c N e^{-aF} F^{1-N} Z^{N-1} \partial_j Z \rho^* d\rho d\sigma^{*j} + \text{c.c.}]
+ [F(Z^{-1} \partial_i \partial_j Z - Z^{-2} \partial_i Z \partial_j Z) + c N^2 e^{-aF} F^{1-N} Z^{N-2} |\rho|^2 \partial_i Z \partial_j Z] d\rho^i d\rho^{*j}$$

(2.15)

with $Z = 1 + |z|^2$.

The metric at the $\rho = 0$ surface

$$ds^2|_{\rho=0} = F(0)(Z^{-1} \partial_i \partial_j Z - Z^{-2} \partial_i Z \partial_j Z) dz^i dz^{*j} = F(0) \partial_i \partial_j \log(1 + |z|^2) dz^i dz^{*j}$$

(2.16)

is the Fubini-Study metric on $\mathbb{CP}^{N-1} \simeq SU(N)/[SU(N - 1) \times U(1)]$. Therefore the metric is the (canonical) line bundle over $\mathbb{CP}^{N-1}$, $C \otimes \mathbb{CP}^{N-1}$. In fact, if we take $a = 0$ we recover the Ricci-flat Kähler metric of Calabi [2] (see also [8]).
In the limit of the boundary condition constant $F(0)$ tending to zero, $\mathbb{C}P^{N-1}$ with the metric (2.16) shrinks and the whole metric (2.15) contains a singularity. It is an orbifold singularity in the orbifold $\mathbb{C}^N/\mathbb{Z}_N$ defined by the identification (2.14). Therefore $F(0)$ is a blow-up parameter for the orbifold singularity.

Since the asymptotic form of $F$ is $F \approx \frac{N}{a} \log aX$ for large $X$, the metric becomes asymptotically

$$
\begin{align*}
\frac{ds^2}{aX} &\simeq (N \log aX)^{1-N}Z^N|d\rho|^2 + \left[ \frac{cN}{aX}(N \log aX)^{1-N}Z^{N-1}\partial_j^*Z\rho^*d\rho d^*z^j + \text{c.c.} \right] \\
&\quad + \left[ \frac{N}{a} \log aX(Z^{-1}\partial_i\partial_j^*Z-Z^{-2}\partial_iZ\partial_j^*Z) + cN^2 \left( \frac{\log aX}{aX} \right)^{1-N}Z^{N-2}|\rho|^2\partial_iZ\partial_j^*Z \right] dz^i dz^j.
\end{align*}
$$

(2.17)

The difference between the previous solution [12, 16] and the present solution is locally just the boundary condition. For the previous case, they required regularity on $K$ at $X = 0$, $K(0) = \text{const.}$ (or $F(0) = 0$), and therefore $K = k_0 + k_1X + k_2X^2 + \cdots$. It is, however, not necessary for regularity on the metric as seen above. For our case the condition is $F(0) = \text{const.}$, and therefore $K = k_{-1} \log X + k_0 + k_1X + k_2X^2 + \cdots$ is not regular at $X \to 0$. As a result the topology is drastically changed. The previous solution has topology $\mathbb{C}^N$, but the present solution has topology $\mathbb{C} \times \mathbb{C}P^{N-1}$ blowing up the orbifold singularity in $\mathbb{C}^N/\mathbb{Z}_N$.

### 3 $O(N)$ Invariant Model

Let us generalize the solution obtained in the last section to a $O(N)$-invariant solution. We prepare an $N$-vector $\vec{\phi} = (\phi^A, \cdots, \phi^N)$ again and put a constraint

$$
\vec{\phi}^2 = \sum_{A=1}^N (\phi^A)^2 = 0
$$

(3.1)

to define a conifold [3]. It is convenient to rewrite this constraint as

$$
\vec{\phi}^T J \vec{\phi} = 0,
$$

(3.2)

with $J$ the rank-2 $O(N)$ invariant tensor. The constraint (3.1) can be solved as [21, 22]

$$
\vec{\phi}^T = \sigma \left( 1, z^i, -\frac{1}{2}z^2 \right).
$$

(3.3)

Ricci-flat metrics on conifolds with the singularity deformed by $\vec{\phi}^2 = r$ were constructed in [3] for $N = 3$ and [4] for general $N$. Ricci-flat metrics on conifolds without deformation was constructed in
which is still regular by an integration constant. Our solution here is a non-Ricci-flat deformation of the latter one.

The $O(N)$-invariant can be written as

$$X \equiv \phi^+ \phi = |\sigma|^2 \left(1 + |z|^2 + \frac{1}{4} |z|^2 \right) \equiv |\sigma|^2 Z.$$  \hspace{1cm} (3.4)

The expression of the metric is the same as the metric (2.12) but with $X$ and $Z$ in Eq. (3.4). Components of the connection are calculated as

$$R_{\sigma \sigma^*} = -L^{-1} L' Z - L^{-2} [L L'' - (L')^2] |\sigma|^2 Z^2,$$

$$R_{\sigma j} = -L^{-1} L' \sigma^* \partial_j Z - L^{-2} [L L'' - (L')^2] |\sigma|^1 Z \partial_j Z,$$

$$R_{ij} = -L^{-1} L' |\sigma|^2 \partial_i \partial_j Z - L^{-2} [L L'' - (L')^2] |\sigma|^4 Z \partial_i Z \partial_j Z$$ \hspace{1cm} (3.5)

with $L \equiv (K')^{N-2}(K'' X^2 + K' X)$. Components of the connection are

$$g_{\sigma \sigma^*, i} = K'' |\sigma|^2 Z \partial_i Z + K''' |\sigma|^4 Z \partial_i Z,$$

$$g_{\sigma \sigma^*, i} = K' \partial_i Z + K'' |\sigma|^2 Z \partial_i Z + K''' |\sigma|^4 Z \partial_i Z,$$

$$g_{ij, \sigma^*} = K' \sigma^* \partial_i \partial_j Z + K'' (2 |\sigma|^2 Z \partial_i \partial_j Z + |\sigma|^2 \partial_i \partial_j Z \partial_j Z) + K''' |\sigma|^4 Z \partial_i Z \partial_j Z,$$

$$g_{ij, k^*} = K' |\sigma|^2 \partial_i \partial_j \partial_k Z + K'' |\sigma|^4 (\partial_k Z \partial_i \partial_j Z + \partial_i Z \partial_j Z \partial_k Z + \partial_j Z \partial_i \partial_k Z + \partial_i \partial_j \partial_k Z)$$

$$+ K''' |\sigma|^6 \partial_i Z \partial_j Z \partial_k Z.$$ \hspace{1cm} (3.6)

Substituting all these quantities to Eq. (3.2), we get

$$(\log L)' = -a (K' X)'.$$ \hspace{1cm} (3.7)

By integrating this we find that $F = K' X$ satisfies the equation similar to the $U(N)$ case,

$$F^{N-2} F' X^{3-N} = c e^{-a F}$$ \hspace{1cm} (3.8)

with an integration constant $c$. Again this can be integrated to yield

$$e^{a F} \sum_{r=0}^{N-2} (-1)^r \frac{(N-2)!}{(N-2-r)!} a^{r+1} = \frac{c}{N-2} X^{N-2} + b$$ \hspace{1cm} (3.9)

with an integration constant $b$. The coordinate transformation

$$\rho = \frac{\sigma^{N-2}}{N-2}$$ \hspace{1cm} (3.10)

is needed to remove the coordinate singularity. We thus obtain the final form of the metric

$$ds^2 = c e^{-a F} F^{2-N} Z^{2} |d\rho|^2 + [c(N-2) e^{-a F} F^{2-N} Z^{N-3} \partial_j Z \rho d\rho dz^s] + c.c.$$  

$$+ [F(Z^{-1} \partial_i \partial_j Z - Z^{-2} \partial_i Z \partial_j Z) + c(N-2)^2 e^{-a F} \partial_i \partial_j Z |dz^i dz^j]$$ \hspace{1cm} (3.11)
with \( Z = 1 + |z|^2 + \frac{1}{4} |\vec{z}|^2 \).

The metric at the \( \rho = 0 \) surface is
\[
 ds^2|_{\rho=0} = F(0)(Z^{-1}\partial_i\partial_j^*Z - Z^{-2}\partial_i\partial_jZ)d\vec{z}^i d\vec{z}^*j = F(0)\partial_i\partial_j^* \log \left( 1 + |z|^2 + \frac{1}{4} |\vec{z}|^2 \right) d\vec{z}^i d\vec{z}^*j. \tag{3.12}
\]
This is the metric on the quadric surface \( Q^{N-2} \simeq SO(N)/[SO(N-2) \times U(1)] \). The metric (3.11) is thus the (canonical) line bundle over \( Q^{N-2} \). The Ricci-flat metric on a conifold \([6, 8]\) is obtained by taking \( a = 0 \) in the metric (3.11). Our metric is non-Ricci-flat deformation of that conifold.

The asymptotic form of the metric (3.11) is
\[
 ds^2 \simeq \frac{N-2}{a} \log aX \left( (N-2)X \log aX \right)^{2-N} \left( \frac{a}{2} \log aX \right)^2 Z^{N-3} \partial_j^* \partial_i \rho d\rho d\vec{z}^i d\vec{z}^*j + c.c. \right)
\]
\[
 + c(N-2)^2 \left\{ (N-2) X \log aX \right\}^{2-N} \left( \frac{a}{2} \log aX \right)^2 \partial_i \partial_j Z \partial_i Z \partial_j Z \right) d\vec{z}^i d\vec{z}^*j. \tag{3.13}
\]

4 Conclusion

We have given two new metrics (2.15) and (3.11) with \( U(N) \) and \( O(N) \) symmetries, respectively as solutions of Eq. (1.2) for target spaces of conformally invariant sigma models with an anomalous dimension \( \gamma \). They are canonical line bundles over the projective space \( CP^{N-1} \) and the quadric surface \( Q^{N-2} \), respectively. These metrics contain the anomalous dimension explicitly through the parameter \( a = -4\pi \gamma \). In the limit of vanishing anomalous dimension \( a \to 0 \), they reduce to those for Calabi-Yau manifolds \([6, 8]\). Generalization to other base coset spaces \([8, 9]\) is straightforward. These new solutions give examples of Kähler Ricci solitons, singular solutions to the Ricci-flow equation (1.3).

A paper \([23]\) recently posted to ArXiv has overlap with the present work. They have obtained explicit solutions with a dilaton background which also reduce to the same Calabi-Yau manifolds \([6, 10]\) with the present work. Therefore solutions in the present paper and those in \([23]\) would be equivalent to each other by some field redefinition.

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