Hamiltonian Structures for the Generalized Dispersionless KdV Hierarchy

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Abstract

We study from a Hamiltonian point of view the generalized dispersionless KdV hierarchy of equations. From the so called dispersionless Lax representation of these equations we obtain three compatible Hamiltonian structures. The second and third Hamiltonian structures are calculated directly from the r-matrix approach. Since the third structure is not related recursively with the first two ones the generalized dispersionless KdV hierarchy can be characterized as a truly tri-Hamiltonian system.

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1. Introduction

The representation of nonlinear integrable equations by a system of linear equations is due to Lax and have been studied extensively in the past [1-3]. An interesting class of nonlinear equations are the so called dispersionless Lax equations which are the quasi-classical limit of ordinary Lax equations. This quasi-classical limit corresponds to the solutions which slowly depend on the variables $x,t$. Let us take the KdV equation

$$4u_t = u_{xxx} + 6uu_x$$  \hspace{1cm} (1.1)

and after dropping out the dispersive term (or make the substitution $\frac{\partial}{\partial t} \to \epsilon \frac{\partial}{\partial t}, \frac{\partial}{\partial x} \to \epsilon \frac{\partial}{\partial x}$ and $\epsilon \to 0$) we end up with the equation

$$u_t = \frac{3}{2} uu_x$$ \hspace{1cm} (1.2)

This is the Riemann equation (also called inviscid Burgers', Hopf etc.). Solutions of (1.2) can be written through the implicit form [4]

$$u = f(x - ut)$$ \hspace{1cm} (1.3)

and this dependence gives rise to the breaking of the wave shape leading to a transition from conservative to dissipative behaviour [5]. As it is well known the balancing between the dispersive and nonlinear terms in (1.1) is responsible for the soliton solutions and the integrability of (1.1). What is interesting is that the evolution equation (1.2), at least before the breaking of its wave solutions, is a integrable Hamiltonian system much like the KdV system (1.1). In fact this behaviour was observed for the dispersionless equations describing an incompressible nonviscous fluid with a free surface in the approximation of long waves, the Benney’s equation [6]. A Hamiltonian structure for this equation was obtained and studied by Kupershmidt and Manin [7], Manin [8], Lebedev and Manin [9], and Lebedev [10]. As point out by Olver and Nutku [11] equation (1.2) has an infinite sequence of zero order (the highest order derivative which a $F(u, u_x, \ldots)$ depends) conserved charges

$$H_1 = \int dx \, u, \quad H_3 = \frac{1}{4} \int dx \, u^2, \quad H_5 = \frac{1}{8} \int dx \, u^3, \ldots$$ \hspace{1cm} (1.4)
and has three first-order Hamiltonian structures given by

\[
\begin{align*}
\mathcal{D}_1 &= 2\partial \\
\mathcal{D}_2 &= u\partial + \partial u \\
\mathcal{D}_3 &= u^2\partial + \partial u^2
\end{align*}
\]  

(1.5)

These three Hamiltonian operators are compatible in the Magri’s sense [12,13] making (1.2) a tri-Hamiltonian system, i.e., equation (1.2) can be written in three Hamiltonian forms

\[
\begin{align*}
\partial_t u &= \mathcal{D}_1 \frac{\delta H_5}{\delta u} = \mathcal{D}_2 \frac{\delta H_3}{\delta u} = \frac{3}{4} \mathcal{D}_3 \frac{\delta H_1}{\delta u}
\end{align*}
\]  

(1.6)

However, it is important to point out that since

\[
\mathcal{D}_3 \neq \mathcal{D}_2 \mathcal{D}_1^{-1} \mathcal{D}_2
\]  

(1.7)

the Hamiltonian operators are not trivially related. Therefore, the Hamiltonian structures (1.5) make the Riemann equation a truly tri-Hamiltonian system. This is to be contrasted with the Kupershmidt’s equations for the dispersive water wave equations [14]. In that situation the three Hamiltonian structures are related by a unique recursion operator.

It follows, for instance from Magri’s theorem [12,13], that the Hamiltonians (1.4) are in involution with respect to any of the three Poisson brackets

\[
\{H_n, H_m\}_i = \int dx \frac{\delta H_n}{\delta u} \frac{\delta H_m}{\delta u} = 0, \quad i = 1, 2, 3, \ldots
\]  

(1.8)

Whence, (1.2) is an integrable Hamiltonian system.

We can associate with (1.2) a Lax representation using the Chen et al approach [15]. The Lax pair is

\[
\frac{\partial L}{\partial t} = [L, B]
\]  

(1.9)

where $L$ is the recursion operator obtained from (1.5)

\[
L = \mathcal{D}_2 \mathcal{D}_1^{-1} = u + \partial u \partial^{-1}
\]  

(1.10)
and $B$ is the Fréchet derivative obtained through a linearization of the Riemann equation (1.2)
\[ v_t = \frac{3}{2} \frac{d}{d\epsilon} (u + \epsilon v)(u + \epsilon v)_x \bigg|_{\epsilon=0} = \frac{3}{2} \partial u v = Bv \] (1.11)

Also, there is another Lax representation (1.9) with
\[ L = \partial + u \]
\[ B = \frac{3}{4} u^2 \] (1.12)

Although we have Lax representations for the Riemann equation we do not know how to use them for an inverse scattering problem or use the pseudo-differential operator algebra to obtain its Hamiltonian structures through the Gelfand-Dickey approach [3]. However, Lebedev [10] has noticed that for the case of Benney’s equation an alternative Lax representation is possible. This Lax representation is called dispersionless Lax equation and was also considered by Krichever [16] in his studies about topological minimal models. In this paper we will use this dispersionless Lax representation and the algebraic setup behind it to derive the Hamiltonian structures of the Riemann equation and the hierarchy of equations which contains it as the first nontrivial equation. This paper is organized as follows. In section 2 we introduce the generalized dispersionless KdV hierarchy of equations and we show that the Riemann equation belongs to it. In section 3 we obtain the first Hamiltonian structure. The second and third Hamiltonian structures for the generalized dispersionless KdV hierarchy are obtained in section 4 by the r-matrix method. We present our conclusions in section 5.

2. Dispersionless KdV Hierarchy

The dispersionless KdV equation (1.2) can be obtained directly, bypassing the dispersionless limit of (1.1). Let $E_n$ be the polynomial of degree $n$ in $p$ [16]
\[ E_n = p^n + u_{-1}p^{n-1} + u_0 p^{n-2} + \cdots + u_{n-2} = \sum_{i=0}^{n} u_{n-i-2} p^i \] (2.1)
where $u_{-2} = 1$ and the polynomial coefficients $u_i$ are functions of the variable $x$ and various time variables $t_k$ ($k = 1, 2, 3, \ldots$). We denote $A_+$ and $A_-$ the parts of the Laurent polynomial $A$ containing nonnegative and negative powers of $p$ respectively. The generalized
dispersionless KdV hierarchy is given by the Lax equation

\[ \frac{\partial E_n}{\partial t_k} = \{(E_n^{k/n})^+, E_n\} = \{E_n, (E_n^{k/n})^-\} \]  \hspace{1cm} (2.2)

where the bracket is defined \([9,10]\) to be

\[ \{A, B\} = \partial_p A \partial_x B - \partial_p B \partial_x A \]  \hspace{1cm} (2.3)

and \(E_n^{k/n}\) is the \(k\)th power of the Laurent polynomial \(E_n^{1/n}\) satisfying \((E_n^{1/n})^n = E_n\).

First, let us denote the degree of a Laurent polynomial of the form \(A = \sum_i a_i p^i\) as

\[ \text{deg } A \equiv \text{maximum } i \text{ for which } a_i \neq 0 \]  \hspace{1cm} (2.4)

and observe that the bracket reduces the degree of a polynomial by one unit. Therefore,

\[ \text{deg } \{A, B\} = \text{deg } A + \text{deg } B - 1 \]  \hspace{1cm} (2.5)

From (2.2) we have

\[ \text{deg } \frac{\partial E_n}{\partial t_k} = \text{deg } \{E_n, (E_n^{k/n})^-\} \leq n - 1 - 1 = n - 2 \]  \hspace{1cm} (2.6)

and when we compare with (2.1) we conclude that

\[ \frac{\partial u_{-1}}{\partial t_k} = 0 \]  \hspace{1cm} (2.7)

for any \(k\).

Now, for general Laurent polynomials of the form

\[ A = \sum_{i=-\infty}^{+\infty} a_i(x) p^i \]  \hspace{1cm} (2.8)

we define the residue as the coefficient of the \(p^{-1}\) term

\[ \text{Res } A = a_{-1} \]  \hspace{1cm} (2.9)

and the Adler trace \([17]\) as

\[ \text{Tr } A = \int dx \text{ Res } A \]  \hspace{1cm} (2.10)
For general Laurent polynomials $A = \sum_{i=-\infty}^{+\infty} a_ip^i$ and $B = \sum_{i=-\infty}^{+\infty} b_ip^i$ it is straightforward to show that

$$\text{Res}\{A, B\} = \sum_{i=-\infty}^{+\infty} i(a_ib_{-i})'$$

which implies

$$\text{Tr}\{A, B\} = 0$$

Also, it follows easily from (2.12) the useful property

$$\text{Tr}(A\{B, C\}) = \text{Tr}(B\{C, A\})$$

Let us note from (2.2) that we can write

$$\frac{\partial E_{m/n}}{\partial t_k} = \{(E_{k/n}^{\ell/n})+, E_{m/n}^{\ell/n}\}$$

for an arbitrary integer $m$. Taking the trace of (2.14) and after using (2.12) we obtain

$$\frac{\partial}{\partial t_k} \text{Tr}(E_{m/n}^{\ell/n}) = 0$$

Thus, we define the conserved charges as

$$H_m = \frac{n}{m} \text{Tr}(E_{m/n}^{\ell/n}), \quad m = 1, 2, 3 \ldots$$

Now let us show that the flows given by (2.2) commute. Let be

$$\frac{\partial^2 E_n}{\partial \tau \partial t_k} = \frac{\partial}{\partial \tau} \{(E_{k/n}^{\ell/n})+, E_n\}$$

where we have used (2.14). Now using the Jacobi identity

$$\frac{\partial^2 E_n}{\partial \tau \partial t_k} = \{(E_{\ell/n}^{\ell/n})+, E_n^{\ell/n} (E_{k/n}^{\ell/n})+, E_n\} + \{(E_{\ell/n}^{\ell/n})+, (E_{\ell/n}^{k/n})+, E_n\} + \{(E_{\ell/n}^{\ell/n})+, (E_{\ell/n}^{k/n})+, E_n\}$$

(2.18)
Therefore, the hierarchy of equations (2.2) has an infinite number of conserved laws (2.16) and an infinite number of commuting flows and can be formally be considered integrable.

Let us illustrate these results for (2.1) with \( n = 2 \) and \( u_{-1} = 0 \). We obtain for \( E_2 \equiv E \) and \( u_0 \equiv u \)

\[
E = p^2 + u
\]

\[
E^{1/2} = p + \frac{1}{2} up^{-1} - \frac{1}{8} u^2 p^{-3} + \frac{1}{16} u^3 p^{-5} - \frac{5}{128} u^4 p^{-7} + \ldots
\]

\[
E^{3/2} = p^3 + \frac{3}{2} up + \frac{3}{8} u^2 p^{-1} + \ldots
\]

\[
E^{5/2} = p^5 + \frac{5}{2} up^3 + \frac{15}{8} u^2 p + \frac{5}{16} u^3 p^{-1} + \ldots
\]

\[
E^{7/2} = p^7 + \frac{7}{2} up^5 + \frac{35}{8} u^2 p^3 + \frac{35}{16} u^3 p + \frac{35}{128} u^4 p^{-1} + \ldots
\]

\[
: \quad (2.19)
\]

From

\[
\frac{\partial E}{\partial t_k} = \{(E^{k/2})_+, E\}
\]  (2.20)

we get the hierarchy of equations

\[
\frac{\partial u}{\partial t_1} = u_x
\]

\[
\frac{\partial u}{\partial t_3} = \frac{3}{2} uu_x
\]

\[
\frac{\partial u}{\partial t_5} = \frac{15}{8} u^2 u_x
\]

\[
\frac{\partial u}{\partial t_7} = \frac{35}{16} u^3 u_x
\]

\[
: \quad (2.21)
\]

The conserved charges from (2.16) are

\[
H_m = \frac{2}{m} \text{Tr}(E^{m/2})
\]  (2.22)
and results in

\[ H_1 = \int dx \, u \]
\[ H_3 = \frac{1}{4} \int dx \, u^2 \]
\[ H_5 = \frac{1}{8} \int dx \, u^3 \]
\[ H_7 = \frac{5}{64} \int dx \, u^4 \]

(2.23)

Thus, from (2.21) we get the Riemann equation as the first nonlinear equation in the hierarchy. Also, the charges (2.23) are exactly the Hamiltonians (1.4). Then we will call (2.20) and (2.2) the dispersionless KdV (Riemann) hierarchy and the generalized dispersionless KdV (Riemann) hierarchy, respectively.

3. First Hamiltonian Structure

Now we will derive in a systematic way the first Hamiltonian structure associated with the generalized dispersionless KdV hierarchy of equations (2.2). Particularly, as an example we will obtain the first Hamiltonian structure in (1.5) for the Riemann equation. Here we will follow the Drinfeld and Sokolov approach [18] as in [19] for the usual KdV hierarchy.

Let us observe that

\[(E_n^{k/n})^- = E_n^{k/n} - (E_n^{k/n})_+ = \sum_{i=1}^{n-1} \text{Res} \left( E_n^{k/n} p^{i-1} \right) p^{-i} + \mathcal{O}(p^{-n})\]

(3.1)

and even though \((E_n^{k/n})^-\) has an infinite number of terms the only ones that will contribute to give dynamical equations are the ones up to degree \(p^{-n+1}\). Since

\[ p^i = \int dx \frac{\delta E_n}{\delta u_{n-i-2}} \]

(3.2)

we rewrite (3.1) as

\[ (E_n^{k/n})^- = \sum_{i=1}^{n-1} \text{Res} \left( E_n^{k/n} \int dx \frac{\delta E_n}{\delta u_{n-i-1}} \right) p^{-i} + \mathcal{O}(p^{-n}) \]

(3.3)

\[ = \sum_{i=1}^{n-1} \frac{\delta H_{n+k}}{\delta u_{n-i-1}} p^{-i} + \mathcal{O}(p^{-n}) \]
where $H_{n+k}$ is given by (2.16).

Using (3.3) in (2.2) we get

$$\frac{\partial E_n}{\partial t_k} = \{E_n, \sum_{i=1}^{n-1} \frac{\delta H_{n+k}}{\delta u_{n-i-1}} p^{-i} \}$$

(3.4)

since the terms of order $O(p^{-n})$ in (3.3) do not contribute. Let us introduce the dual to $E_n$

$$Q = \sum_{i=1}^{n} q_{n-i-1} p^{-i}$$

(3.5)

where the $q$’s are assumed to be independent of the $u$’s. This yields a linear functional

$$\text{Tr } E_n Q = \int dx \sum_{i=0}^{n-1} u_{n-i-2} q_{n-i-2}$$

(3.6)

Also, let us note that

$$\sum_{i=1}^{n-1} \frac{\delta H_{n+k}}{\delta u_{n-i-1}} p^{-i} = \sum_{i=1}^{n-1} \int dy \frac{\delta H_{n+k}}{\delta u_{n-i-1}(y)} V_i(x, y)$$

(3.7)

where

$$V_i(x, y) \equiv \delta(x - y)p^{-i}$$

(3.8)

and which gives

$$\text{Tr } E_n V_i(x, y) = u_{n-i-1}(y)$$

(3.9)

We have thus from (3.4) using (3.9) and (2.13)

$$\frac{\partial}{\partial t_k} \text{Tr } E_n Q = \sum_{i=1}^{n-1} \int dy \text{Tr } E_n \{V_i(x, y), Q(x)\} \frac{\delta H_{n+k}}{\delta u_{n-i-1}(y)}$$

(3.10)

If we want to write this equation in Hamiltonian form as

$$\frac{\partial}{\partial t_k} \text{Tr } E_n Q = \{\text{Tr } E_n Q, H_{n+k}\}_1$$

$$= \sum_{i=1}^{n-1} \int dy \{\text{Tr } E_n Q, \text{Tr } E_n V_i\}_1 \frac{\delta H_{n+k}}{\delta u_{n-i-1}(y)}$$

(3.11)

(where we have used (3.9)) we get after comparing it with (3.10)

$$\{\text{Tr } E_n Q, \text{Tr } E_n V_i\}_1 = \text{Tr } E_n \{V_i, Q\}$$

(3.12)
In this way, the dispersionless Lax equation (2.2) can be written in Hamiltonian form with respect to the first Poisson bracket

$$\{\text{Tr} E_n Q, \text{Tr} E_n V\}_1 = \text{Tr} E_n \{V, Q\}$$

for any dual $Q$ and $V$ relative to $E_n$. As an example let be

$$E_2 \equiv E = p^2 + u_{-1}p + u_0$$

with the duals

$$V = v_{-1} p^{-2} + v_0 p^{-1}$$
$$Q = q_{-1} p^{-2} + q_0 p^{-1}$$

We get that

$$\text{Tr} E \{V, Q\} = 2 \int dx q_0 v_0'$$

and

$$\text{Tr} EQ = \int dx (u_{-1}q_{-1} + u_0q_0)$$
$$\text{Tr} EV = \int dx (u_{-1}v_{-1} + u_0v_0)$$

Now, (3.13) yields

$$\{u_{-1}(x), u_{-1}(y)\}_1 = 0$$
$$\{u_{-1}(x), u_0(y)\}_1 = 0$$
$$\{u_0(x), u_0(y)\}_1 = 2 \partial \delta(x - y)$$

and constraining (3.14) to $E = p^2 + u$, where $u \equiv u_0$, we must set $u_{-1} = 0$. We finally get from (3.18)

$$\{u(x), u(y)\}_1 = 2 \partial \delta(x - y) = D_1 \delta(x - y)$$

which is the first Hamiltonian structure for the Riemann equation (1.2).

We now could try to write the dispersionless Lax equation in other Hamiltonian forms. However, in the next section we will use the algebraic structure behind the dispersionless hierarchy and apply the r-matrix formalism to obtain the other Hamiltonian structures.
4. Second and Third Hamiltonian Structures: r-Matrix Approach

It is well known by now that the so called first Hamiltonian structure of integrable models is the sympletic structure of Kostant-Kirillov [20] on the orbits of the coadjoint representation of Lie groups [17,21]. For dispersionless equations given by the Lax representation (2.2) the corresponding Lie algebra is given by the associative algebra of Laurent polynomials endowed with the bracket (2.3). The Hamiltonian structure (3.13) can also be obtained from this result. For Benney’s equation this interpretation was already given by Lebedev [10] and Lebedev and Manin [9].

Semenov-Tian-Shansky [22] has shown that the multi-Hamiltonian nature of integrable equations could be explained in terms of the so called r-matrix. Let g be an abstract associative algebra with a non-degenerate trace form \( \text{Tr}: g \to \mathbb{R} \). In this way we can identify \( g \) with its dual \( g^\ast \) by

\[
\langle A|B \rangle = \text{Tr} AB \tag{4.1}
\]

Also, we can use the natural Lie algebra structure obtained by \([A, B] = AB - BA\) on \( g \). The linear mapping \( R: g \to g \) is a classical r-matrix on \( g \) whenever the modified bracket

\[
[A, B]_R = [RA, B] + [A, RB] \tag{4.2}
\]

satisfies the Jacobi identity [22]. This gives us a second Lie algebra structure on \( g \). The bracket (4.2) satisfies the Jacobi identity if the modified Yang-Baxter equation holds. The important result for us is that the new Lie product endows \( g = g^\ast \) with new Poisson structures. The first one is

\[
\{\text{Tr} EQ, \text{Tr} EV\}_2 = \frac{1}{2} \text{Tr} E \left( [Q, R(EV)] + [RQE, V] \right) \tag{4.3}
\]

where \( Q \) and \( V \) are duals to \( E \). This expression was given by Semenov-Tian-Shansky (formula (22) in [22]) and it is the analog of the second structure of Gelfand-Dickey [3]. In references [23] and [24] a third Poisson structure was introduced

\[
\{\text{Tr} EQ, \text{Tr} EV\}_3 = \frac{1}{2} \text{Tr} E \left( [Q, R(EVE)] + [R(EQE), V] \right) \tag{4.4}
\]
and it was shown that the Kostant-Kirillov structure (which gives (3.13) in our problem), (4.3) and (4.4) form a compatible tri-Hamiltonian system, i.e., the three structures are compatible in Magri’s sense.

For Lie algebras that can be written in the form

\[ g = g_+ \oplus g_- \]  

(4.5)

the r-matrix on \( g \) is given by

\[ R = P_+ - P_- \]  

(4.6)

where \( P_{\pm}g = g_{\pm} \) are the projections onto the subalgebras. For our particular case of dispersionless equations it is clear that

\[ g_+ = \left\{ A_+ = \sum_{i=0}^{\infty} a_i(x)p^i \right\} \]  

(4.7)

\[ g_- = \left\{ A_- = \sum_{i=1}^{\infty} a_{-i}(x)p^{-i} \right\} \]

with trace given by (2.6) and bracket given by (2.3). So, the Poisson brackets (4.3) and (4.4) assume the form

\[ \{ \text{Tr} E_n Q, \text{Tr} E_n V \}_2 = \]  

\[ = \frac{1}{2} \text{Tr} E_n \left( \{ Q, (E_n V)_+ \} - \{ Q, (E_n V)_- \} + \{(E_n Q)_+, V\} - \{(E_n Q)_-, V\} \right) \]  

(4.8a)

\[ \{ \text{Tr} E_n Q, \text{Tr} E_n V \}_3 = \]  

\[ = \frac{1}{2} \text{Tr} E_n \left( \{ Q, (E_n^2 V)_+ \} - \{ Q, (E_n^2 V)_- \} + \{(E_n^2 Q)_+, V\} - \{(E_n^2 Q)_-, V\} \right) \]  

(4.8b)

Let us again use the Riemann equation as an example. Using (3.14) and (3.15) we get, after straightforward algebra, the following Poisson brackets from (4.8)

\[ \{ u_{-1}(x), u_{-1}(y) \}_2 = -2\partial\delta(x-y) \]

\[ \{ u_{-1}(x), u_0(y) \}_2 = -\partial u_{-1}\delta(x-y) \]  

(4.9a)

\[ \{ u_0(x), u_0(y) \}_2 = (u_0\partial + \partial u_0 - u_{-1}\partial u_{-1}) \delta(x-y) \]

and

\[ \{ u_{-1}(x), u_{-1}(y) \}_3 = -2 (u_0\partial + \partial u_0) \delta(x-y) \]  

\[ \{ u_{-1}(x), u_0(y) \}_3 = -(2\partial u_0 u_{-1} + u_{-1}^2 \partial u_{-1}) \delta(x-y) \]  

(4.9b)

\[ \{ u_0(x), u_0(y) \}_3 = (u_0^2\partial + \partial u_0^2 - u_{-1}\partial u_0 u_{-1} - u_0 u_{-1}\partial u_{-1}) \delta(x-y) \]
From the first Poisson bracket (3.18) we see that $u_{-1}$ decouples from $u_0$. However, in the second and third brackets (4.9) $u_{-1}$ is coupled to itself and to $u_0$. From $\{u_{-1}(x), u_{-1}(y)\}$ in (4.9) it follows that $u_{-1} = 0$ corresponds to a second class constraint and we have to use the Dirac reduction (see [24] for example). We then obtain

$$\{u(x), u(y)\}_2 = (u\partial + \partial u)\delta(x - y) = D_2\delta(x - y) \quad (4.10a)$$

$$\{u(x), u(y)\}_3 = (u^2\partial + \partial u^2)\delta(x - y) = D_3\delta(x - y) \quad (4.10b)$$

where we have set $u \equiv u_0$. These are exactly the Hamiltonian structures in (1.5).

Finally, the $k$th flow in the generalized dispersionless KdV hierarchy (2.2) can be written in Hamiltonian form as

$$\frac{\partial u}{\partial t_k} = D_1 \frac{\delta H_{k+n}}{\delta u} = D_2 \frac{\delta H_k}{\delta u} = \frac{k(k-2)}{(k-1)^2} D_3 \frac{\delta H_{k-n}}{\delta u} \quad (4.11)$$

where $k > 1$ and $u = (u_{-1}, u_0, \ldots, u_{n-2})$. The Hamiltonians $H_n$ are given by (2.16) and the Hamiltonian structures $D_1$, $D_2$ and $D_3$ can be obtained from the Poisson brackets (3.13), (4.8a) and (4.8b) respectively. For $n = 2$ we obtain from (4.11) the dispersionless KdV hierarchy of equations (2.21).

5. Conclusions

We have studied the Hamiltonian structures of the generalized dispersionless KdV hierarchy of equations. We have obtained the second and third Hamiltonian structures directly from the r-matrix approach following the results of Semenov-Tian-Shansky [22], Lin and Parmentier [23] and Oevel and Ragnisco [24]. We have illustrated our main results through the Riemann equation (1.2). However, as was discussed by Olver and Nutku the Riemann equation has an additional Hamiltonian structure. Namely, the third-order Hamiltonian

$$\mathcal{E} = \partial \frac{1}{u_x} \partial \frac{1}{u_x} \partial \quad (5.1)$$

allows us to write the dispersionless KdV hierarchy of equations in Hamiltonian form. This Hamiltonian structure is only compatible with the $D_1$ structure. So, the Riemann equation and the higher order equations (2.21) are quadri-Hamiltonian systems. Consequently, this
property should be valid for the whole generalized KdV hierarchy of equations. We did
not succeeded in deriving a Poisson bracket, yielding (5.1) for the Riemann equation, from
the r-matrix approach. This is an interesting problem and is under investigation.

With the results obtained in this paper we can study the higher Hamiltonian structures
of other interesting dispersionless systems. One of them is the so called Benney [6] system
of equations. This system has a dispersionless Lax representation much like the generalized
dispersionless KdV hierarchy of equations. For instance, the classical dispersionless long
wave equation
\begin{align*}
  u_t + uu_x + h_x &= 0 \\
  h_t + (uh)_x &= 0
\end{align*}
\tag{5.2}

can be derived from the Benney’s system. It is not difficult to check that it has a simple
dispersionless nonstandard Lax representation
\begin{align*}
  \frac{\partial E}{\partial t} &= \{E, (E^2)_{\geq 1}\} \\
  \text{where} & \\
  E &= p + \frac{1}{2} u + \frac{1}{4} hp^{-1}
\end{align*}
\tag{5.3}

and the bracket is given by (2.3). Here \((E^2)_{\geq 1}\) stands for the purely nonnegative (without
\(p^0\) terms) part of the Laurent polynomial obtained from \(E^2\). For dispersive systems the
nonstandard Lax representation was introduced by Kupershmidt in [14] and the general-
ization of the Gelfand-Dikii brackets was performed in [25]. The derivation of the Poisson
brackets for equations with nonstandard dispersionless Lax representation is an interesting
and relevant problem and is also under investigation.

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