CONTRACTION RATES FOR BAYESIAN INVERSE PROBLEMS

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Abstract. We prove a general lemma for deriving Contraction rates for linear inverse problems with nonparametric nonconjugate priors. We then apply it to Gaussian priors and obtain minimax rates in mildly ill-posed case. This includes, in particular, Meyer wavelet priors with Gaussian priors. The result for severely ill-posed problems is derived only when the prior is sufficiently smooth. The bases for which our method works is dense in the space of all bases in a sense which shall be described in this paper.

1. Introduction

Inverse problems can always be formulated as solutions of equations of the type $y = G(u)$ where $y$ is known (typically some measurements or data from an experiment) and we need to solve the equation to find $u$.[8, Chapter 1] The model which we shall investigate here consists of the statistical version of infinite dimensional linear inverse problems of this type. Specifically, we assume that $G$ is a linear, injective operator from Hilbert space $H_1$ to Hilbert space $H_2$ (both infinite dimensional). We treat $u$ as a random variable on $H_1$ and define $y$ by the following equation:

\[ y = G(u) + \frac{1}{\sqrt{n}} Z, \]

where $Z$ is the white noise on $H_2$: $Z \sim N(0, I)$. In a Bayesian formulation of the problem, also known as nonparametric (parametric) Bayesian framework when $u$ is infinite (finite) dimensional, we assume that $u$ is distributed according to a measure $\mu$ on $H_1$, to be considered as the prior measure. We further assume that $u$ and $Z$ are independent. Given the above assumptions, Bayes’ theorem gives the distribution of the conditional random variable $u$ given $y$, called the posterior distribution, to be denoted henceforth by $\mu_{yn}$. We will be interested in the properties of the posterior in the limit that $n \to \infty$.

In many practical applications, the inverse problem is ill-posed, in the sense that $u$ may not exist for given $y$, or it may not be unique, or $u$ may not depend continuously on $y$. In such cases, many methods of regularization are well developed for linear ill-posed inverse problems but are current topics of research for many nonlinear problems, see, e.g. [8] and references therein. Another approach in dealing with ill-posed inverse problems is the statistical approach similar to the one introduced in the first paragraph. Under very general assumptions on $G$, the posterior distribution for the conditional random variable $u$ given $y$ is continuous with respect to $y$ even when $G^{-1}$ is not continuous. (See [13, 4] for details, further discussions and references.)

Depending on the context, various properties and quantities related to the posterior distribution have been found to be of interest. A far from exhaustive list of recent studies includes credible sets of the posterior [9]; computational methods and finite dimensional approximations of equation (1) and of the posterior $\mu_{yn}$ [7]; convergence of MAP (maximum a posteriori) and CM (conditional mean) estimators. For an extensive bibliography, one may refer to [6].

Another major question in this context is that of consistency: as the number of observations goes to infinity (large data limit), or as the observations become more accurate (small data-noise limit), one expects the posterior density to approach Dirac delta supported on the true value of the unknown. We shall outline the details of both in the next section. The first result in large data limit is due to Doob (in 1949—reference needed), and have been investigated in many different contexts since then. There has been only a recent interest in the small data-noise limit and some of the papers that discuss this issue are [12, 9, 2, 10].

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1 Choosing $Z$ as white noise makes the target space large, as the white noise is supported on a much bigger space than $H_2$. We explain White noise in a little detail in section 2. Use of white noise is pretty common in literature and for us, it will reduce the number of eigenbases that we deal with.
some weak conditions, these large-data and small-data-noise limits are equivalent \([3]\) and this equivalence is exploited in \([12]\) and will also be used explicitly by us in this work.

Our main focus will be the small data-noise limit. We extend the class of priors \(\mu\) for which we can obtain the contraction rates at which the posterior concentrates around the true value. In \(2\), we shall note the particular technical aspects we have to deal with to get this extension. First, we give precise definitions for consistency and contraction rates in section \(2\), followed by our main result on contraction rates, Lemma \(2.5\), in section \(2.2\). Compared to the results obtained in \([12, 9, 2, 10]\), our conditions are more general. In the last section \(3\), we discuss some examples where we can apply our results.

2. CONSISTENCY AND CONTRACTION RATES

Consistency in Bayesian inverse problems primarily means how well the random posterior measure concentrates around the true solution,

- either as the number of observations grow to infinity,
- or, as the observations get more precise by way of the observational noise going to zero in an appropriate way.

First we shall see how the posterior looks in either case and then move onto the details of consistency in the second case. Below, we outline the bayesian setup and how it applies to our needs.

2.1. basic setup. A Bayesian setup consists of an abstract probability space \((\Omega, \mathcal{F}, \mathbb{P})\), with \(u\) and \(y\) random variables taking values in \(\mathcal{R}_1\) and \(\mathcal{R}_2\), respectively, where \(\mathcal{R}_1\) and \(\mathcal{R}_2\) are abstract spaces. Here \(y\) is referred to as the observed data and \(u\) as the unknown/parameter. One needs to find the conditional distribution of \(u\) given \(y\) when the joint distribution of \(u\) and \(y\), and marginal distributions of \(u\) and \(y\) are given. The marginal distribution of \(u\) is called the prior distribution (say \(\mu(u)\)). Writing \(p(u \mid y)\) as the required conditional density of \(u\) given \(y\), Bayes’ theorem translates to the following

\[
p(u \mid y) = \frac{p(u, y)}{\int_{\mathcal{H}_2} p(u, y) d\mu(u)}
\]

where \(p(u, y)\) is the joint density of \(u\) and \(y\). Basically, for fixed \(y\), \(p(u \mid y)\) is proportional to \(p(u, y)\) which in turn is proportional to \(p(y \mid u)\) for fixed \(u\). The densities are all taken with respect to appropriate reference measures.

2.1.1. large data limit. The model here consists of a measure space \((X, \mathcal{H})\) and the model space \(\mathcal{P}\) of a class of probability measure on \((X, \mathcal{H})\). A prior distribution \(\Pi\) is given on \(\mathcal{P}\). A sequence of random samples \(X_i\) is taken from some probability distribution \(P\) in \(\mathcal{P}\). Then the posterior distribution on \(\Pi\) given \(\{X_1, X_2, ..., X_n\}\) is calculated and its rate of convergence to the true distribution \(P\) is found. Here, \(\mathcal{R}_1\) and \(\mathcal{R}_2\) from last paragraph are \(\mathcal{P}\) and \(X^n\) respectively. It is easy now to deduce the posterior distribution from the Bayes’ theorem above -

\[
\Pi_n(B \mid X_1, X_2, ..., X_n) = \frac{\int_{B} \Pi^0 P(X_i) d\Pi_n}{\int_{\mathcal{P}} \Pi^0 P(X_i) d\Pi_n}
\]

2.1.2. small noise limit. We get back to equation 1. \(\mathcal{H}_1\) and \(\mathcal{H}_2\) are Hilbert spaces with countably infinite basis. \(Z\) is distributed according to the white noise. White noise on a Hilbert space \(\mathcal{H}\) is the probability measure gotten as follows. Pick any orthonormal basis \(\{e_i\}\) of \(\mathcal{H}\). Equip each one dimensional span of \(e_i\) (Let call it \(R_i\)) with the standard Gaussian \(N(0, 1)\) and consider the product measure on \(\Pi R_i\). This is the White noise defined on \(\mathcal{H}\). Contrary to what it looks like, this measure is independent of the choice of the basis \(\{e_i\}\). Infact, we can check that \(\text{cov}(Z_x, Z_y) = \langle x, y \rangle\). Further, if the same construction is done with a different basis, the new product space, say \(\Pi R’_i\), is the same as \(\Pi R_i\).

This makes \(y\) belong to \(\Pi R_i\). The operator \(G\) besides being linear and injective,shall be a map with a singular value decomposition or SVD(That is, \(G^T G\) will have an eigenbasis) and discontinuous inverse. This implies that 0 is a limit point of eigenvalues of \(G^T G\), say \(\rho_k^2\). The model is then called ill posed. Two important cases of ill posedness which we shall deal with are stated here. The model is called severely illposed if the eigenvalues of \(G^T G\) satisfy the condition-

\[
C_1(1 + k^2)^{-\alpha_0} e^{-2C_0k^2} \leq \rho_k^2 \leq C_2(1 + k^2)^{-\alpha_1} e^{-2C_0k^2}.
\]
It is called Mildly illposed if

\[ C_1(1 + k^2)^{-\alpha/2} \leq \rho_k^2 \leq C_2(1 + k^2)^{-\alpha/2}. \]

The prior, say \( \mu \) is defined in a fashion similar to that of white noise. We pick a basis \( \{ \phi_i \} \) of \( \mathcal{H}_1 \), define a measure on its one dimensional span(\( R_i \)) and take the product measure. Unlike white noise, generally the measures defined like this depends on choice of \( \{ \phi_i \} \). Also, the measure may actually be supported on \( \mathcal{H}_1 \) itself. For example, if we define the Gaussian \( N(0, \lambda^2) \) on \( R_i \) where \( \sum \lambda_i^2 \) is summable, the prior is then supported on \( \mathcal{H}_1 \). We shall deal with such cases only. Now, for the form of posterior distribution here, we again use the Bayes’ theorem. First, we calculate the density \( p(y|u) \) with respect to the white noise measure. For this, we consider white noise on the basis of \( G^T G \), say \( \{ \epsilon_k \} \). Also, let the eigenvalues of \( G^T G \) be \( \{ \rho_k^2 \} \). The density will then just be the product of all the one dimensional densities. Thus, we have

\[
p(y|u) = \exp \left( n \sum_k \rho_k y_k u_k - n/2 \sum \rho_k^2 u_k^2 \right).
\]

Hence, for \( p(u|y) \), the density of posterior with respect to the prior \( \mu \), we have

\[
p(u|y) = \frac{\exp \left( n \sum_k \rho_k y_k u_k - n/2 \sum \rho_k^2 u_k^2 \right)}{\int_{\mathcal{H}_1} \exp \left( n \sum_k \rho_k y_k u_k - n/2 \sum \rho_k^2 u_k^2 \right) du}.
\]

See ([4] section 3.1.3.2) for details. The posterior distribution \( \mu_n^\mu \) then becomes

\[
\mu_n^\mu(B) = \int_B \exp \left( n \sum_k \rho_k y_k u_k - n/2 \sum \rho_k^2 u_k^2 \right) du.
\]

We restrict our attention to the second type of consistency, that of the small observational noise limit. Next, we define this notion of consistency and contraction rates.

2.2. Contraction rates for infinite dimensions. First, we deal with the notion of consistency. As discussed earlier, we want to find if the posterior concentrates around the true solution. We start with the posterior measure of complements of neighbourhoods given by \( B(\epsilon) = \{ u : ||u - u_0|| \leq \epsilon \} \) of the true solution. It is given by the \( y \) dependent function \( \mu_n^\mu(B^r(\epsilon)) \). For fixed \( n \), given the true solution \( u_0 \), \( y \) is the random variable given by \( G(u_0) + \frac{Z}{\sqrt{n}} \). This makes \( \mu_n^\mu(B^r(\epsilon)) \) into a random variable(keep in mind that \( y \) is also dependent on \( n \)). We now have a sequence of random variables \( \{ \mu_n^\mu(B^r(\epsilon)) \} \) and we can talk about its convergence. In this paper, we shall be concerned with convergence in probability and we say that the posterior is consistent if the above sequence of random variables converge to 0. Assume now that we replace \( \epsilon \) with a sequence \( \epsilon_n \). If the random variables still converge to 0, then \( \epsilon_n \) is a rate of contraction.

Example 2.1. We consider a finite dimensional example here, i.e \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) are \( R^p \) and \( R^q \) respectively with \( q \geq p \). \( G \) is again injective with a singular value decomposition. Since target spaces are finite dimensional, this makes \( G^{-1} \) continuous on the image of \( G \). The contraction rate we get here is \( \frac{1}{\sqrt{n}} \).

We outline the proof below. We start by estimating the expectation of \( \mu_n^\mu \{ ||u - u_0|| > \epsilon_n \} \) with respect to the distribution of \( (Z/\sqrt{n})^u \). We denote it, in short, by \( E_{u_0}[\mu_n^\mu \{ ||u - u_0|| > \epsilon_n \}] \)

\[
E_{u_0}[\mu_n^\mu \{ ||u - u_0|| > \epsilon_n \}] = E_{u_0}[I_{\{||y-G(u_0)||>r_n\}} \mu_n^\mu \{ ||u - u_0|| > \epsilon_n \} + E_{u_0}[I_{\{||y-G(u_0)||\leq r_n\}} \mu_n^\mu \{ ||u - u_0|| > \epsilon_n \}]
\]

The first term on the RHS is bounded from above by \( K_1 \frac{\sigma}{\sqrt{mn}} \exp(-\frac{r_n^2}{2}) \). For any \( r_n = \frac{f(n)}{\sqrt{n}} \) where \( f(n) \) goes to infinity, this bound goes to 0 as \( n \) goes to infinity.

For the second term, it can be bounded from above by \( K_2 \mu_n^G(u_0) \{ ||u - u_0|| > \frac{\epsilon_n}{2} \} \) when \( r_n \leq \frac{\epsilon_n}{2} \).

\[
\mu_n^G(u_0) \{ ||u - u_0|| > \epsilon_n/2 \} = \frac{1}{Z_n^G(u_0)} \int_{||u-u_0||>\epsilon_n} \exp(-\frac{n}{2} ||G(u-u_0)||^2) du_0
\]

The numerator here can be bounded from above by \( \exp(-\frac{n}{2} (K_3 \epsilon_n)^2) \) where \( K_3 \) is the Lipshitz constant of \( G^{-1} \) in the cameron martin norm of the prior.

The denominator can be bounded from below by \( \exp(-\frac{n}{2} (\frac{\epsilon_n}{2})^2) \mu_0 \{ ||u - u_0|| \leq \frac{\epsilon_n}{2} \} \) Where \( K_4 \) is the Lipshitz constant of \( G \).
Hence,

\[ \mu_n^{G(u_0)} \{ \|u - u_0\| > \epsilon_n/2 \} \leq \frac{\exp(-n\epsilon_n^2)}{\mu_0 \{ \|u - u_0\| \leq \frac{Kn}{2K_4} \}} \]

Thus we have,

\[ E_{u_0} \{ \mu_n^G \{ \|u - u_0\| > \epsilon_n \} \} \leq K_1 \frac{\sigma}{\sqrt{n}r_n} \exp(-\frac{r_n^2 n}{2\sigma^2}) + \frac{\exp(-n\epsilon_n^2)}{\mu_0 \{ \|u - u_0\| \leq \frac{Kn}{2K_4} \epsilon_n \}} \]

Now, note that denominator of the second term in RHS is bounded (above and below) asymptotically by a polynomial of order \( p \) in \( \epsilon_n \). Next, if we put \( r_n = \sqrt{\log \frac{n}{\epsilon_n}} \) and \( \epsilon_n = \sqrt{\log \frac{n}{\epsilon_n}} \), both the terms in RHS go to 0.

Thus, the rate of convergence \( \epsilon_n \) is \( \sqrt{\log \frac{n}{\epsilon_n}} \).

Thus as we see in the example above, in the finite dimensional setup, a nondegenerate linear \( G \) ensures well-posedness of the inverse problem and makes the problem of contraction rates easy. The same clearly does not hold for infinite dimensional case. We shall now outline the work done for contraction rates in infinite dimensions. In [9], the authors deal with mildly ill posed case 4. They first calculate the mean and covariance operator of the posterior distribution (which is Gaussian) and bound \( \mu_n^G(B^c(\epsilon)) \) above using Markov inequality and then take the expectation of the resulting random variable. At this point, they use the assumption that \( \phi_k = e_k \) where \( \{ \phi_k \} \) is basis on which the prior is defined and \( \{ e_k \} \) is the eigenbasis of \( G^T G \) (as described before). Using the assumption, they show that the required expectation of \( \mu_n^G(B^c(\epsilon)) \) equals the sum of a series and then they estimate the value of the sum to get contraction rates. In [2], the authors deal with severely ill posed problems 3. They use the above mentioned method to get up to showing that the expectation equals the sum of a series but estimating the sum required more subtle methods. As we can see, in both these papers, the assumption that \( \phi_k = e_k \) was used and prior was assumed to be Gaussian.

The above method is ill suited for cases where \( e_k \) and \( \phi_k \) are different. It is because in this case, the integrations involved above would lead to too many cross terms. In [1], the authors try a different method to determine the rate of contraction for Gaussian priors (even the noise maybe Gaussian measures, and not just white noise) when \( G \) may not have an SVD. They find the contraction rates under some technical functional analytic assumptions relating covariance operator of the Gaussian prior and the linear operator \( G \). The method works when \( G \) is restricted to small variations to mildly ill posed problems.

As far as we know, other than [1], only [12] discusses the case where \( \{ e_k \} \) is not the same as \( \{ \phi_k \} \). Furthermore, in [12], the author deals with priors that may not be Gaussian. In his main lemma [12, Lemma 2.1], he reduces finding contraction rates to verifying certain estimates. The main assumption there is that \( \phi_k \) is independent of all but finitely many \( e_k \)'s. For his lemma, Ray uses the idea in [5] of getting test functions satisfying certain properties which then implies certain contraction rates. However, the author discusses the Gaussian case for \( e_k = \phi_k \) only. We shall prove the lemma under slightly more general conditions and apply it to get contraction rates for inverse problems including that for Gaussian priors even when \( \{ e_k \} \) is not the same as \( \{ \phi_k \} \). To this end we shall define a notion of closeness of the 2 eigenbasis. \( \{ \phi_k \} \) is said to be close to \( \{ e_k \} \) iff

\[ \sum_k \left( \frac{\langle \phi_j, e_k \rangle}{\rho_k} \right)^2 < \infty \text{ for all } j. \]

We prove our main result (Lemma 2.5) the condition that \( \{ e_k \} \) and \( \{ \phi_k \} \) are close in the sense defined above. Further, to get the contraction rates for Gaussian case, we cannot verify the conditions in the lemma by directly using the methods in [12] or [10]. However, we use the standard transformation of measure equations from [11] to make the methods used in [12] amenable to our needs.

We now state out main assumptions under which we prove the main contraction lemma 2.5.

2.3. Contraction Lemma. As mentioned earlier, contraction rate for the posterior measure is a way of quantifying concentration of the posterior measure around the true value. One of the principal requirement for this is that the true solution has enough probability mass around it, which here requires the following:
First, we show the existence of such test functions. We will show later that existence of such test functions implies that the rate of posterior contraction is

\begin{equation}
\mu\{u : \|G(u) - G(u_0)\|_2 \leq \epsilon_n\} \geq e^{-Cn\epsilon_n^2}.
\end{equation}

Additionally, we also need to ensure that, with high probability (under the prior measure), elements in $\mathcal{H}_1$ are well approximated by the finite dimensional projections. More precisely,

**Assumption 2.3.** we assume that there exist constants $\xi_n$ and a sequence of positive integers $k_n$ satisfying $k_n < Rn\epsilon_n^2$ for some $R > 0$ and $\max\{\xi_n, k_n^{-1}\} \to 0$ as $n \to \infty$, such that writing $P_m$ as the projection onto the subspace spanned by $\varphi_1, \ldots, \varphi_m$ we must have

\begin{equation}
\mu\{u : \|P_{k_n}(u) - u\|_1 > C_2\xi_n\} \leq e^{-(C+4)n\epsilon_n^2}.
\end{equation}

As stated earlier, our approach works when the basis of linear operator and the basis on which prior is defined are close. We define this closeness precisely in the following assumption

**Assumption 2.4.** We state the assumption in two parts since it will be easier to check. As before, let $\{\varphi_k\}$ be the basis on which the prior $\mu$ is defined. Further, we have $\{e_k, \rho_k^2\}$ as the eigenbasis of $G^T G$. we assume that

\begin{equation}
\sum_i \left(\frac{\langle \varphi_j, e_i \rangle}{\rho_i}\right)^2 < \infty.
\end{equation}

The above assumption allows us to define $g_n$ (for example, by Schwartz’s inequality)

\begin{equation}
g_n^2 := \max_{\|h\|_{\mathcal{H}_1} = 1} \sum_{j=1}^{\infty} \frac{\langle h, e_j \rangle^2}{\rho_j}.
\end{equation}

We shall require that $\sqrt{g_n} \leq C_1\xi_n/\epsilon_n$.

Clearly, this is a weaker condition than assumption 1 along with condition 2.6 in lemma 2.1 in [12].

With the above assumptions, we shall now state our main result concerning the contraction rates of posterior measure $\mu_n^n$ given by the following formula:

\[
\mu_n^n(B) = \frac{\int_B \exp \left( n \sum_k \rho_k y_k u_k - n/2 \|G(u)\|_2^2 \right) d\mu}{\int_{\mathcal{H}_1} \exp \left( n \sum_k \rho_k y_k u_k - n/2 \|G(u)\|_2^2 \right) d\mu}
\]

**Lemma 2.5** (Contraction lemma). Consider the model given by (1), together with the assumptions 2.2-2.4 stated above. Also, let $u_0$ be such that $\|P_{k_n}(u_0) - u_0\|_1 = O(\xi_n)$ then, $(\mu_n^n(u : \|u - u_0\| > M\xi_n)) \to 0$ in probability.

**Remark 2.6.** Note that this result is identical to the lemma 2.1 of [12] but with much weaker condition. As we will see, the method of proof is also very similar.

It was shown in [5] that the posterior contraction rates are closely related to the existence of certain test functions. In particular, we need to prove the existence of sequence of test functions $\phi_n : \mathcal{H}_2 \to \mathbb{R}$ for $n \geq 1$ such that

\[
\sup_{\{u : \|P_{k_n}(u) - u\|_1 \leq C_2\xi_n, \|u - u_0\|_1 \geq M\xi_n\}} \mathbb{E}_u(1 - \phi_n) \leq e^{-(C+4)n\epsilon_n^2} \\
\mathbb{E}_{u_0}(\phi_n) \to 0
\]

We will show later that existence of such test functions implies that the rate of posterior contraction is $\{\xi_n\}$. First, we show the existence of such test functions.
Proof of Lemma 2.5: We shall prove that even under the modified assumptions mentioned above, the sequence of test functions $\phi_n$, proposed in equation (4.3) of [12], do satisfy the following conditions:

\[
\sup_{\{u: P_n(u) - u\leq C_2, \|u-u_0\|_1 \geq M\xi_n\}} E_u(1 - \phi_n) \leq e^{-(C+4)ne_n^2}
\]

\[
E_{u_0}(\phi_n) \to 0
\]

After showing that the proposed test functions satisfy the above conditions, we shall use methods similar to that in [5] to conclude that $\xi_n$ is the required contraction rate.

We shall begin with defining $\tilde{\varphi}_k = \sum_i \frac{\varphi_{k,i}e_i}{\rho_i}$ where $\varphi_{k,i} = G_{e_i}$. Writing $P_\varphi$ as the projection onto $\tilde{\varphi}$, define $\tilde{Z}_k = P\tilde{\varphi}_k Z$.

Then consider the test functions

\[
\phi_n(Y) = 1\{\|u_n - u_0\|_1 \geq M_0\xi_n\}
\]

where, $u_n = P_{k_n}u + \frac{1}{\sqrt{n}} \sum_{k=1}^{k_n} \tilde{Z}_k \varphi_k$. $u_n$ here are basically estimators of $u$. We shall begin by estimating how $u_n$ is distributed around its mean. To that end, we shall be using the Borell’s inequality, we have

\[
e^{-\frac{x^2}{2\sigma^2}} \geq \mathbb{P}(\|u_n - \mathbb{E}u_n\|_1 - \mathbb{E}\|u_n - \mathbb{E}u_n\|_1 \geq x),
\]

where $\sigma$ is given as

\[
\sigma^2 = \sup_{h \in B_0} \mathbb{E} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{k_n} \tilde{Z}_k \langle h, \varphi_k \rangle_1 \right)^2 = \sup_{h \in B_0} K(h)^2.
\]

To get our estimates on $u_n$, we need to now estimate $\sigma$ and $\mathbb{E}\|u_n - \mathbb{E}u_n\|_1$ By Jensen’s inequality, we have

\[
\mathbb{E}\|u_n - \mathbb{E}u_n\|_1^2 = \mathbb{E} \left( \frac{k_n}{n} \sum_{i=1}^{k_n} \tilde{Z}_k \langle h, \varphi_k \rangle_1 \right)^2 \leq 1/n \sum_{i=1}^{k_n} \mathbb{E} \tilde{Z}_k^2 = 1/n \sum_{i=1}^{k_n} \|\tilde{\varphi}_k\|^2
\]

\[
\leq \frac{k_n}{n} \sup_{h \in H_{k_n} \cap B_0} \left( \sum_{i} \left( \frac{\langle h, e_i \rangle}{\rho_i} \right)^2 \right) = \frac{k_n}{n} g_{k_n}
\]

Here $B_0$ is a countable dense set in the unit ball of $H_1$.

Since $\mathbb{E}(\tilde{Z}_r \tilde{Z}_s) = \langle \tilde{\varphi}_r, \tilde{\varphi}_s \rangle$, we have

\[
n\mathbb{E}K(h)^2 = \left\| \sum_{k=1}^{k_n} \langle h, \varphi_k \rangle_1 \tilde{\varphi}_k \right\|^2_2
\]

\[
= \sum_{i} \left( \frac{\left\langle \sum_{k=1}^{k_n} \langle h, \varphi_k \rangle \varphi_k, e_i \right\rangle}{\rho_i} \right)^2
\]

\[
= \sum_{i} \left( \frac{\langle P_{k_n}(h), e_i \rangle}{\rho_i} \right)^2
\]

(Where $P_{k_n}(h)$ is the projection of $h$ on subspace generated by $\{\varphi_1, ..., \varphi_{k_n}\}$) Hence, we need to maximise the above wrt to $h$ belonging to the subspace generated by $\{\varphi_1, ..., \varphi_{k_n}\}$, say $H_{k_n}$. Thus,

\[
\sigma^2 = \sup_{h \in B_0} K(h)^2 = 1/n \sup_{h \in B_0} \left( \sum_{i} \left( \frac{\langle P_{k_n}(h), e_i \rangle}{\rho_i} \right)^2 \right)
\]

\[
= 1/n \sup_{h \in H_{k_n} \cap B_0} \left( \sum_{i} \left( \frac{\langle h, e_i \rangle}{\rho_i} \right)^2 \right) = \frac{g_{k_n}}{n}
\]

Hence, we have $\sigma^2 \leq \frac{2g_{k_n}}{n}$ and from before $\mathbb{E}\|u_n - \mathbb{E}u_n\|_1 \leq \sqrt{\frac{2g_{k_n}}{n}}$. Substituting in Borell’s inequality (4.1 in [12]) and putting $x = 2L\epsilon_n^2 g_{k_n}$, we have

\[
\mathbb{P}\left(\|u_n - \mathbb{E}u_n\|_1 \geq M\epsilon_n \sqrt{g_{k_n}}\right) \leq \exp(-Ln\epsilon_n^2)
\]
It is analogous to 4.2 in [12] with \(\sqrt{g_{k_n}}\) replacing \(1/\delta_{k_n}\). We can now show that the test functions satisfy the appropriate criterion by following Ray’s proof line by line. Next, for the sake of completion, we shall show how the existence of test functions imply the contraction result. Test functions, like here, are mostly characteristic functions of sets. Intuitively speaking, we can see that existence of test function satisfying the required conditions ensures that the support of probability measure given by distribution of noise centred \(G(u_0)\) is separated from support of those centred inside \(\{u : \|P_{k_n}(u) - u\|_1 \leq C_2 \zeta_n, \|u - u_0\| \leq \zeta_n\}\). From the form of posterior distribution, intuitively it seems that \(\mu_n^u\{u : \|u - u_0\| \geq \zeta_n\}\) takes small values around \(G(u_0)\) in the product space supporting the noise. This also happens to be where the noise shifted by by \(G(u_0)\) is mostly concentrated. Hence, it seems plausible that \(\zeta_n\) is the rate of contraction. These ideas shall be made precise in the following proof.

**Claim 2.7.** Assume we have test functions

\[
\sup_{\{u : \|P_{k_n}(u) - u\|_1 \leq C_2 \zeta_n, \|u - u_0\| \geq M \zeta_n\}} E_u(1 - \phi_n) \leq e^{-(C+4)n\epsilon_n^2}, E_{u_0}(\phi_n) \to 0
\]

then we have

\[
\mu_n^u(B) = \int_B \exp(n \sum_k \rho_k y_k u_k - n/2 \|G(u)\|_2^2) d\mu \to 0
\]

in probability \(\mathbb{P}_{u_0}\). Here, \(B = \{u : \|u - u_0\| \geq M \zeta_n\}\).

**Proof.** We start by trying to get estimates on \(E_{u_0}\mu_n^u(B)\). We do this by dividing the integral in 2 parts, one on the support of test functions \(\phi_n\) and one on its complement and use the properties of the test functions.

\[
E_{u_0}\mu_n^u(B)\phi_n \leq E_{u_0}\phi_n \to 0
\]

Put \(u' = u - u_0\). Let \(W_n^u(B)\) denote the numerator of \(\mu_n^u(B)\).

\[
E_{u_0}W_n^u(B)(1 - \phi_n) = E_{u_0} \int_B (1 - \phi_n) \exp \left( n \sum_k \rho_k y_k u_k - n/2 \|G(u)\|_2^2 \right) d\mu
\]

\[
= E_{u_0} \int_{-u_0} (1 - \phi_n) \exp \left( n \sum_k \rho_k y_k u_k - n/2 \|G(u')\|_2^2 \right) d\mu(u')
\]

\[
= \int_{-u_0} E_{u_0}(1 - \phi_n) \exp(n \sum_k \rho_k y_k u_k' - n/2 \|G(u')\|_2^2) d\mu(u')
\]

\[
\leq Ke^{-(C+4)n\epsilon_n^2}
\]

We have used Fubini’s theorem above. Let \(D = \mu\{u : \|G(u) - G(u_0)\|_2 \leq \epsilon_n\}\) and \(\nu\) be a probability measure on \(D\). Then we have,

\[
P_{u_0} \left( \left( \int_D \sum_k (\rho_k u_k y_k - 1/2 \rho_k^2 u_k^2) d\nu < -(1 + C)\epsilon^2 \right) \right)
\]

\[
P_{u_0} \left( \int_D \sum_k (\rho_k u_k Z_k + 1/2 \rho_k^2 u_k^2 - 1/2 \rho_k^2 (u_k - u_{0,k})^2) d\nu < -(1 + C)\epsilon^2 \right)
\]

\[
P_{u_0} \left( \int_D \sum_k \rho_k u_k Z_k d\nu < -C\epsilon^2 - 1/2 \rho_k^2 u_{0,k}^2 \right)
\]
Here, \( Z_k = y_k - \rho_k u_{0,k} \). In particular, \( \mathbb{E}_{u_0}(Z_i Z_k) = \delta_{rs} \). Now, using Chebyshev’s and Jensen’s inequality successively, we have

\[
\mathbb{P}_{u_0} \left( \int_D \sum_k \rho_k u_k Z_k d\nu < -C \varepsilon^2 - 1/2 \rho_k^2 u_{0,k}^2 \right) \leq \frac{\int_D \mathbb{E}_{u_0} \left( \sum_k \rho_k u_k Z_k \right)^2 d\nu}{(-C \varepsilon^2 - 1/2 \rho_k^2 u_{0,k}^2)^2} = \int_D \sum_k \rho_k^2 u_{0,k}^2 d\nu \frac{1}{n(-C \varepsilon^2 - 1/2 \rho_k^2 u_{0,k}^2)^2}
\]

(10)

Using, (9) and (10), we can now see that if we restrict the integral in the denominator of the expression for \( \mu_n^B(B) \) to \( D \), the expression approaches 0 in probability (\( \mathbb{P}_{u_0} \)), hence \( \mu_n^B(B) \) approaches 0 in probability. \( \square \)

Next, we shall use this result to get contraction result for linear illposed inverse problems.

### 3. Examples: Gaussian priors

The requirement \( \varepsilon_n \sqrt{g_{\nu_n}} < C \varepsilon_n \) is the main constraint in application of the above lemma. Depending on the prior, this condition dictates the kind of leeway we can get with the basis we define the prior on. For example, with Gaussian priors, we shall need \( g_{\nu_n} < \frac{C}{\varepsilon_n^2} \).

We shall now estimate the values of \( \varepsilon_n, \xi_n \) and \( k_n \). We shall just show the details for mildly ill posed case (4). The severely ill posed case (3) can be done in exactly the same way. Though, like in [12], we get the results for severely ill posed case only when prior is very smooth. First, we shall check the condition

\[
\mu \{ u : \| G(u) - G(u_0) \|_2 \leq \varepsilon_n \} \geq e^{-C n \varepsilon^2}.
\]

We will begin by showing that under certain conditions, \( \mu \) is equivalent to \( \nu \), a Gaussian measure whose covariance operator has the eigenbasis \( \varepsilon_k \). Then we can just use the estimates in [12]. Let the prior \( \mu \) have eigenpair \( \{ \varphi_k, (1 + k^2)^{-1/2 - \delta} \} \). We choose \( \nu \) such that the eigenpair of its covariance operator (say \( G_1 \)) is given by \( \{ \varepsilon_k, (1 + k^2)^{-1/2 - \delta} \} \). If we assume that \( I - G_1^{-1/2} R G_1^{-1/2} = S \) is trace class where \( R \) is the covariance operator of \( \mu \). Then \( \mu \) and \( \nu \) are equivalent measures and we have-

\[
d\mu \over d\nu (x) = det [I - S]^{-1/2} \exp \left( \left\{ -\frac{1}{2} \cdot S(I - S)^{-1} G_1^{-1/2} (x), G_1^{-1/2} (x) \right\} \right).
\]

Thus, we have

\[
\mu \{ u : \| G(u) - G(u_0) \|_2 \leq \varepsilon_n \}
= det [I - S]^{-1/2} \int \{ u : \| G(u) - G(u_0) \|_2 \leq \varepsilon_n \} \exp \left( \left\{ -\frac{1}{2} \cdot S(I - S)^{-1} G_1^{-1/2} (x), G_1^{-1/2} (x) \right\} \right) d\nu
\geq K \int \{ u : \| G(u) - G(u_0) \|_2 \leq \varepsilon_n \} \exp \left( -\frac{1}{2} K_1 \left\| G_1^{-1/2} (x) \right\|^2 \right) d\nu
\]

Since \( (I - S)^{-1} \) is continuous.

\[
\int \{ u : \| G(u) - G(u_0) \|_2 \leq \varepsilon_n \} \exp \left( -\frac{1}{2} K_1 \left\| G_1^{-1/2} (x) \right\|^2 \right) d\nu
= \int \{ u : \| G(u) - G(u_0) \|_2 \leq \varepsilon_n \} \exp \left( -\frac{1}{2} K_1 \sum (1 + k^2)^{\delta/2 + 1/2} x_k^2 \right) d\nu
= K_2 \nu \{ u : \| G(u) - G(u_0) \|_2 \leq \varepsilon_n \}
\]

Here \( \nu \) is a centered Gaussian measure given by eigenpair \( \{ \varepsilon_k, \frac{(1 + k^2)^{-1/2 - \delta/2}}{K_1 + 1} \} \).

Next, we have that

\[
\nu \{ u : \| G(u) - G(u_0) \|_2 \leq \varepsilon_n \} \geq K_3 \nu \{ u : \| G(u) - G(u_0) \|_2 \leq \varepsilon_n \}
\]

Since

\[
\int \{ u : \| G(u) - G(u_0) \|_2 \leq \varepsilon_n \} = 0
\]
and \( \nu \) is equivalent to \( \nu' \). So, all the calculations in [12] go through as they are for \( \mu \{ u : \| G(u) - G(u_0) \|_2 \leq \epsilon_n \} \) and hence estimating \( \epsilon_n \). Next, we estimate \( \mu \{ u : \| P_{k_n}(u) - u \|_1 > C_2 \xi_n \} \) as in [12]. This cylindrical set is oriented according to the prior measure, so we do not need to go through the previous calculations and use [12](page 2539, proof of proposition 3.4).

We then have

\[
\mu \{ u : \| P_{k_n}(u) - u \|_1 \geq L' \left( k_n^{-\delta} + \sqrt{nk_n^{-\delta - 1/2}} \right) \} \leq e^{L \nu_n^2}.
\]

For mildly ill-posed problems, we take \( k_n = n^{-\frac{1}{2 + 2\alpha}} \). Therefore,

\[
\mu \{ u : \| P_{k_n}(u) - u \|_1 \geq L'' n^{-\frac{\min(\gamma, \delta)}{2 + 2\alpha}} \} \leq e^{L \nu_n^2}.
\]

Note that the above choice is different than that of Ray [12](Ray’s choice does not work when \( \gamma < \delta \) and was intimated to him). Now, we have for the last condition

\[
\| P_{k_n}(f_0) - f_0 \|_1 \leq R_1 k_n^{-\gamma} \leq (n \epsilon_n^2)^{-\gamma}.
\]

Thus, if we just put \( \xi_n = n^{-\frac{\min(\gamma, \delta)}{2 + 2\alpha + 1}} \), we are done. As pointed out earlier, \( \epsilon_n \sqrt{k_n} < C_1 \xi_n \) further imposes the condition that \( g_n \) has to be of the order of \( n^\alpha \). We shall now consider the conditions on the Gaussian prior and linear operator under which we can make the above analysis work and then see if any useful examples fit the bill.

As before, we take \( \{ \epsilon_k, (1 + k^2)^{-\alpha/2} \} \) to be the eigenpair of \( G^T G \) and \( R \) to be the covariance operator of the prior \( \mu \) with eigenpair \( \{ \phi_k, (1 + k^2)^{-1/2 - \delta} \} \). As before, we need to compare it with the Gaussian measure whose covariant operator, say \( G_1 \) has the eigenbasis \( \{ \epsilon_k, (1 + k^2)^{-1/2 - \delta} \} \). From what we have done till now, it is clear that following two conditions suffice for our method to work -

**Assumption 3.1.**

- \( I - G_{1}^{-1/2} R G_{1}^{-1/2} \) is trace class.
- \( g_n \leq C n^\alpha \) for all \( n \) and some constant \( C \).

For fixed \( G \), let \( R = TG_1 T^{-1} \) where \( T \) is a unitary operator. There is a unique correspondence between \( R \) and \( T \). Also note that \( \phi_k = T \epsilon_k \) and for fixed \( n \), \( g_n \) depends only on \( T \)(and not on \( \epsilon_k \)). With this in mind, we denote \( g_n \) by \( g_n^T \). Also, note that given the eigenvalues of \( G^T G \) and \( R \), the two conditions above are conditions on \( T \) alone. Using the corresponding unitary operators, we can talk about the space of bases \( \{ \phi_k : \phi_k = T \epsilon_k, T \in U(H_1) \} \). This correspondence gives the space of bases a group structure inherited from \( U(H_1) \). For ease of calculation, we now replace \( g_n \) in 3.1 with \( h_n \) defined as -

\[
h_r := \max_{\| h \|_{H_1} = 1} \sum_{j=1}^{\infty} \left( \frac{|\langle h, \epsilon_j \rangle|}{\rho_j} \right), \quad h \in \text{span}\{\varphi_1, \ldots, \varphi_r\}
\]

As with \( g_n \), \( h_n \) depends on \( T \) thus, we shall denote \( h_n \) by \( h_n^T \). We shall now show that if \( T_1 \) and \( T_2 \) satisfy the modified condition 3.1, then so does \( T_1 T_2 \). This will show that such \( T \)(and hence the corresponding bases \( \{ \phi_k \} \) form a group.

**Claim 3.2.** \( I - G_{1}^{-1/2} T_2^{-1} T_1^{-1} G_1 T_2 G_1^{-1/2} \) is trace class given that \( I - G_{1}^{-1/2} T_1^{-1} G_1 T_1 G_{1}^{-1/2} \) and \( I - G_{1}^{-1/2} T_2^{-1} G_1 T_2 G_{1}^{-1/2} \) are trace class.

**Proof.**

\[
I - G_{1}^{-1/2} T_2^{-1} T_1^{-1} G_1 T_1 G_{1}^{-1/2} = I - A_2 A_1 A_2 A_1^T = I - A_2 A_2^T + A_2 (I - A_1 A_1^T) A_2^T
\]

Where \( A_i = G_{1}^{-1/2} T_i^{-1} G_1^{-1/2} \). So, we need to show that \( A_2 (I - A_1 A_1^T) A_2^T \) is trace class. This will hold if \( A_2 \) (equivalently \( A_2^T \)) is continuous. It is indeed true as we can see as follows. \( I - A_2 A_2^T \) is trace class and self adjoint, therefore has as eigenbasis \( \{ h_k, \beta_k \} \) with \( \beta_k \to 0 \). Thus, \( A_2 A_2^T \) has eigenbasis \( \{ h_k, \gamma_k \} \) with \( \gamma_k^2 < N^2 \) for some \( N \) independent of \( k \). Now, let \( \| x \| = 1 \)

\[
\langle A_2^T x, A_2^T x \rangle = \langle A_2 A_2^T x, x \rangle = \sum_k \gamma_k^2 x_k^2 \leq N
\]

Hence, \( A_2^T \) is continuous and consequently \( T_1 T_2 \) is ”close” to \( I \). 

For the second condition -

\[ \square \]
Claim 3.3. Assume \( h_{n_1}^{T_1} \leq C_1 n^\alpha \) and \( h_{n_2}^{T_2} \leq C_2 n^\alpha \). Then, we have \( h_{n_1}^{T_1 T_2} \leq C_1 C_2 n^\alpha \).

Proof. Let \( x \in \text{span}\{T_1 T_2 e_1, \ldots, T_1 T_2 e_n\} \). Define \( \phi_i \equiv T_2 e_i \).

\[
\sum_i \left( \frac{\langle x, e_i \rangle}{\rho_i} \right) = \sum_i \left( \frac{\sum_j \langle x, \phi_j \rangle \langle \phi_j, e_i \rangle}{\rho_i} \right) \leq \sum_{i,j} \frac{\langle x, \phi_j \rangle \langle \phi_j, e_i \rangle}{\rho_i}
\]

We sum wrt \( i \) first. Note that \( \sum_i \left( \frac{\langle \phi_i, e_i \rangle}{\rho_i} \right) \leq C_2 j_\alpha \). Hence, we have -

\[
\sum_{i,j} \frac{\langle x, \phi_j \rangle \langle \phi_j, e_i \rangle}{\rho_i} \leq C_2 \sum_j \langle x, \phi_j \rangle j_\alpha \leq C_1 C_2 n^\alpha
\]

Thus we have, \( h_{n_1}^{T_1 T_2} \leq C_1 C_2 n^\alpha \). \( \square \)

We will now get to specific examples and in the process, show that the group above is dense in \( U(\mathcal{H}_1) \) under strong operator topology.

First, we try reflections. Let \( P \) be the hyperplane perpendicular to the unit vector \( v = \sum v_i e_i \). The second basis(say \( \phi_i \) is formed by reflecting \( e_i \) wrt \( P \). Let this reflect be our unitary transform \( T \).

\[
T(x) = x - 2 \langle x, v \rangle v
\]

Then we have \( R = TGT \) since \( T^2 = I \). Now, we need to see for what values of \( v \) that is \( \{ v_i \} \), \( I - G^{-1/2} R G^{-1/2} \) is trace class.

\[
I - G^{-1/2} R G^{-1/2} = I - G^{-1/2} T G T G^{-1/2} = A + A^t - AA^t
\]

Where \( A = 2G^{-1/2} S G^{1/2} \) and \( S(x) = \langle x, v \rangle v \). So, if \( A \) is trace class, so is \( A^t \) and \( AA^t \) and our required condition is satisfied.

\[
\text{trace}A = \sum_i \langle |A| e_i, e_i \rangle \leq \sum_i |\langle A| e_i \rangle| = \sum_i \|A e_i\| = \sum_i \langle A e_i, A e_i \rangle^{1/2}
\]

Here, \( |A| \) is called the absolute value of \( A \) and is given by \( |A| = (A^t A)^{1/2} \).

\[
A e_i = \lambda_i v_i \sum_j \frac{v_j}{\lambda_j} e_j
\]

Hence,

\[
\sum_i \langle A e_i, A e_i \rangle^{1/2} = \sum_i \lambda_i v_i \left( \sum_j \left( \frac{v_j}{\lambda_j} \right)^2 \right)^{1/2}
\]

Hence, we have our desired result if \( v \) is such that \( \left( \sum_j \left( \frac{v_j}{\lambda_j} \right)^2 \right) \leq \infty \). For the second condition -

\[
\sum_i \left( \frac{\langle x, e_i \rangle}{\rho_i} \right) = \sum_i \left( \frac{\sum_j \langle x, \phi_j \rangle \langle \phi_j, e_i \rangle}{\rho_i} \right) \leq \sum_j \langle x, \phi_j \rangle \sum_i \frac{\langle \phi_j, e_i \rangle}{\rho_i} \leq \sum_j \left| \langle x, \phi_j \rangle \right| \left( \frac{1}{\rho_j} - 2v_j \sum_i \frac{v_i}{\rho_i} \right) \leq \sum_j \left| \langle x, \phi_j \rangle \right| + 2|v_j \langle x, \phi_j \rangle| \sum_i \frac{v_i}{\rho_i}.
\]
Hence, if \( \sum_{i} |a_i| \leq C \), then we have \( h_n \leq C n^{\alpha} \) for some \( C \).

Since any finite dimensional rotation can be generated by a finite sequence of reflections which are also finite dimensional (i.e., corresponding \( v \) is such that \( v_i = 0 \) except but finitely many \( i \)), finite dimensional rotations also satisfy the conditions 3.1. The subgroup of finite dimensional rotations is dense in the group \( U(\mathcal{H}_1) \) in strong operator topology. Also, note that we can take \( v \) such that \( v \) does not belong to \( RH_1 \) but satisfies conditions 3.1. Reflection about this \( v \) gives rise to a prior basis which satisfies conditions stated above but not in [1] (assumption 3.1). Further, we note that we can do a Gaussian wavelet analysis with Meyer wavelet basis (Uniform wavelet analysis was done in [12] section 3.3) as the Meyer wavelet basis satisfies the required conditions. We skip the details of Meyer wavelets here (refer to [12]).

For the first condition, we shall use some Gaussian measure theory. It can be shown that if Gaussian measures \( \nu \) and \( \mu \) with eigenpairs \( \{ e_k, \rho_k^2 \} \) and \( \{ \phi_k, \rho_k^2 \} \) are such that they are absolutely continuous with respect to each other and have a finite density at 0, then \( I - G_1^{-1/2} R G_1^{-1/2} \) is trace class where \( G_1 \) and \( R \) are the corresponding covariance operators. Once again, using the fact that \( \phi_j \in \text{span}\{ e_1, ..., e_{2j} \} \), it is not difficult to see that \( \nu \) is absolutely continuous with respect to \( \mu' \) with finite density at 0 where \( \mu' \) has the eigenpair \( \{ e_k, \rho^2_k \} \). \( \mu' \) is clearly absolutely continuous with respect to \( \mu \) with finite density at 0. By transitivity, \( \nu \) and \( \mu \) satisfy the required properties, hence \( I - G_1^{-1/2} R G_1^{-1/2} \) is trace class. Thus, Meyer wavelet basis with Gaussian prior is amenable to our analysis.

Next, we try operators of type \( \exp(A) \) where \( A \) is an operator satisfying \( A = -A^T \) (lets call them skew-symmetric operators as in finite dimensions) and \( \exp A = \sum_{i=0}^{\infty} \frac{A^i}{i!} \). Though we do not have specific example of a useful basis which fits this example but still mention this since it forms a pretty large class of unitary operators and may find some use. Here, we once again try to get conditions on \( A \) so that \( I - R^{-1/2} \exp(A) R \exp(-A) R^{-1/2} \) is trace class. Using Baker-Campbell-Hausdorff formula, followed by induction (we use invariance of trace under cyclic permutation), we have

\[
\text{trace} \left( I - R^{-1/2} \exp(tA) R \exp(-tA) R^{-1/2} \right) \leq \sum_{i=1}^{\infty} \frac{1}{i!} \text{trace} \left( (B + B^T)^i \right)
\]

Here, \( B = R^{1/2} A R^{-1/2} \). Hence it is sufficient to check that \( B + B^T \) is trace class.

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