ON RIGHT-ANGLED POLYHEDRA IN RIEMANNIAN 3-MANIFOLDS WITH NON-NEGATIVE SCALAR CURVATURE

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Abstract. We give a complete characterization of the combinatorial types of all the 3-dimensional right-angled mean curvature convex (or totally geodesic) polyhedra in Riemannian 3-manifolds with non-negative scalar curvature. This result can be considered as an analogue of Andreev’s theorem on right-angled polyhedra in the 3-dimensional hyperbolic space.

1. Introduction

A Riemannian manifold with corners is a smooth n-manifold with corners $W$ embedded as a polyhedral domain in a Riemannian n-manifold $(M, g)$ with the induced Riemannian metric $g(W) = g|_W$ (see [8, §1]). Typical example of Riemannian manifolds with corners are the intersections of finitely many domains with smooth mutually transversal boundaries in a Riemannian manifold (e.g. convex polytopes in Euclidean space $\mathbb{R}^n$).

Recall an $n$-dimensional (smooth) manifold with corners $Q$ is a Hausdorff space together with a maximal atlas of local charts onto open subsets of $\mathbb{R}^n_{\geq 0}$ such that the transitional functions are (diffeomorphisms) homeomorphisms which preserve the codimension of each point. Two smooth manifolds with corners $W$ and $W'$ are called smoothly equivalent if there exists a bijective map $\varphi : W \to W'$ such that both $\varphi$ and $\varphi^{-1}$ are smooth with respect to the local charts of $W$ and $W'$. A manifold with corners $Q$ is called nice if each codimension-$k$ face of $Q$ is contained in exactly $k$ different facets (codimension-one faces) of $Q$. The reader is referred to [8] for more definitions related to manifolds with corners.

We call a Riemannian manifold with corners $W$ a Riemannian polyhedron if it is smoothly equivalent to a simple convex polytope in an Euclidean space. Recall an $n$-dimensional convex polytope $P \subset \mathbb{R}^n$ is called simple (see [20]) if

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every vertex of $P$ is contained in exactly $n$ facets of $P$ (i.e. $P$ is nice manifold with corners). In addition, we have the following definitions for a Riemannian polyhedron:

- We call $W$ mean curvature convex if every facet of $W$ has non-negative mean curvature in the ambient Riemannian manifold (i.e. the variations of the volume of every codimension-one face $F$ of $W$ are non-positive under infinitesimal inward deformations of $F$). Especially when the proper faces of $W$ are all totally geodesic submanifolds, we call $W$ a totally geodesic Riemannian polyhedron.

- We call $W$ acute or non-obtuse if the dihedral angle function on every edge of $W$ ranges in $(0, \frac{\pi}{2})$ or $(0, \frac{\pi}{2}]$. Especially, we call $W$ right-angled if the dihedral angle function on every edge of $W$ is constantly $\frac{\pi}{2}$.

The primary interest of this paper is to study the 3-dimensional mean curvature convex Riemannian polyhedra with non-obtuse dihedral angles in a Riemannian 3-manifold with positive or non-negative scalar curvature.

For $n \geq 1$, let $\Delta^n$ and $[0,1]^n$ denote the standard $n$-simplex and $n$-cube in $\mathbb{R}^n$, respectively. The following are the main theorems of this paper.

**Theorem 1.1.** Let $W$ be a 3-dimensional mean curvature convex Riemannian polyhedron in a Riemannian 3-manifold $(M, g)$.

(a) If $W$ is non-obtuse and $(M, g)$ has positive scalar curvature, then it is necessary that $W$ is combinatorially equivalent to a simple convex polytope in $\mathbb{R}^3$ obtained from $\Delta^3$ by a sequence of vertex-cuts.

(b) If $W$ is non-obtuse and $(M, g)$ has non-negative scalar curvature, then it is necessary that $W$ is either combinatorially equivalent to $[0,1]^3$ or a polytope in the previous case.

Two simple convex polytopes are called combinatorially equivalent if there is a bijection between their faces that preserves the inclusion relation (see [20, p. 5]).

**Definition 1.2 (Vertex-Cut).** Let $P$ be an $n$-dimensional simple convex polytope in $\mathbb{R}^n$ and $v$ a vertex of $P$. Choose a plane $H$ in $\mathbb{R}^n$ such that $H$ separates $v$ from the other vertices of $P$. Let $H_\geq$ and $H_\leq$ be the two half spaces determined by $H$ and assume that $v$ belongs to $H_\geq$. Then $P \cap H_\geq$ is an $(n-1)$-simplex, and $P \cap H_\leq$ is a simple polytope, which we refer to as a vertex-cut of $P$. For example, a vertex-cut of $\Delta^3$ is combinatorially equivalent to $\Delta^2 \times [0,1]$ (the triangular prism).

The following question proposed in Gromov [8] is related to Theorem 1.1.

**Question** (Gromov [8 §1.7]): What are the possible combinatorial types of mean curvature convex Riemannian manifolds with corners $W$ with non-negative scalar curvature and acute dihedral angles at all edges?
Theorem 1.1 implies that if $W$ in the above question is a 3-dimensional acute Riemannian polyhedron with non-negative scalar curvature, the combinatorial type of $W$ must be one of the cases described in Theorem 1.1. But conversely, it is not clear whether we can actually construct an acute Riemannian polyhedron for every combinatorial type described in Theorem 1.1. In fact, the only example we know is the 3-simplex.

But if we only consider right-angled Riemannian polyhedra in Theorem 1.1, we can completely determine the combinatorial types of all the 3-dimensional right-angled mean curvature convex (or totally geodesic) Riemannian polyhedra in 3-manifolds with non-negative or positive scalar curvature.

**Theorem 1.3.** Let $P$ be a 3-dimensional simple convex polytope in $\mathbb{R}^3$.

(a) The combinatorial type of $P$ can be realized by a mean curvature convex (or totally geodesic) right-angled Riemannian polyhedron in a Riemannian 3-manifold $(M, g)$ with positive scalar curvature if and only if $P$ is combinatorially equivalent to a polytope that can be obtained from $\Delta^3$ by a sequence of vertex-cuts.

(b) If we replace the “positive scalar curvature” condition on $(M, g)$ in (a) by “non-negative scalar curvature”, then $P$ is combinatorially equivalent either to $[0,1]^3$ or a polytope in the previous case.

Note that the result in Theorem 1.3 still holds if the Riemannian 3-manifold $(M, g)$ in Theorem 1.3 is assumed to have positive (or non-negative) constant scalar curvature (see Corollary 3.5).

Theorem 1.3 can be considered as an analogue of Andreev’s description on right-angled totally geodesic polyhedra in 3-dimensional hyperbolic spaces $\mathbb{H}^3$. In 1970, E. M. Andreev ([1, 2]) published a classification of all three-dimensional compact totally geodesic hyperbolic polyhedra having non-obtuse dihedral angles (see [15] for a new proof). This classification is essential for proving Thurston’s Hyperbolization theorem for Haken 3-manifolds. Andreev’s description on the 3-dimensional hyperbolic polyhedra with non-obtuse dihedral angles consists of some conditions on the combinatorial structure of a polyhedron and a set of linear inequalities on the dihedral angles. Especially, Andreev’s theorem tells us that a simple convex 3-polytope $P$ in $\mathbb{R}^3$ can be realized as a right-angled totally geodesic hyperbolic polyhedron in $\mathbb{H}^3$ if and only if $P$ has no prismatic 3-circuits or prismatic 4-circuits (this result was also obtained by A. V. Pogorelov in [14]).

The paper is organized as follows. In Section 2, we first introduce the notion of real moment-angle manifold associated to a simple convex polytope. Then we quote some results from Wu-Yu [18] on when a real moment-angle manifold in dimension three can admit a Riemannian metric with positive or non-negative scalar curvature. In Section 3, give a proof of Theorem 1.1 and Theorem 1.3 using
the results from [13] and some arguments in Gromov [8] and Li-Mantoulidis [12] on smoothing singularities of a metric.

2. Real moment-angle manifolds

Suppose \( P \) is an \( n \)-dimensional simple convex polytope in Euclidean space \( \mathbb{R}^n \). Let \( \mathcal{F}(P) = \{ F_1, \ldots, F_m \} \) be the set of all facets of \( P \). Let \( \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} \) and let \( e_1, \ldots, e_m \) be a basis of \( (\mathbb{Z}_2)^m \). Define a function \( \lambda_0 : \mathcal{F}(P) \to (\mathbb{Z}_2)^m \) by

\[
\lambda_0(F_i) = e_i, \quad 1 \leq i \leq m.
\]

For any proper face \( f \) of \( P \), let \( G_f \) denote the subgroup of \( (\mathbb{Z}_2)^m \) generated by the set \( \{ \lambda_0(F_i) \mid f \subset F_i \} \). For any point \( p \in P \), let \( f(p) \) denote the unique face of \( P \) that contains \( p \) in its relative interior. In [7, Construction 4.1], the real moment-angle manifold \( \mathbb{R}Z_P \) of \( P \) is a closed orientable \( n \)-manifold defined by the following quotient construction

\[
\mathbb{R}Z_P := P \times (\mathbb{Z}_2)^m / \sim
\]

where \( (p, g) \sim (p', g') \) if and only if \( p = p' \) and \( g^{-1}g' \in G_{f(p)} \). So at every vertex of \( P \), \( 2^n \) copies of \( P \) are glued together so that locally they look like the \( 2^n \) cones of \( \mathbb{R}^n \) bounded by the \( n \) coordinate hyperplanes meeting at the origin.

There is a canonical action of \( (\mathbb{Z}_2)^m \) on \( \mathbb{R}Z_P \) defined by

\[
g' \cdot [(p, g)] = [(p, g' + g)], \quad \forall p \in P, \forall g, g' \in (\mathbb{Z}_2)^m,
\]

whose orbit space can be identified with \( P \). Let \( \Theta_P : \mathbb{R}Z_P \to P \) be the orbit map. Note that each facet \( F \) of \( P \) is also a simple convex polytope and, \( \Theta_P^{-1}(F) \) is a disjoint union of several copies of \( \mathbb{R}Z_F \) embedded in \( \mathbb{R}Z_P \) with trivial normal bundles.

Real moment-angle manifolds are important objects to study in toric topology. The reader is referred to [7, 4] and [5] for more information of the topological and geometric properties of real moment-angle manifolds.

Remark 2.1. The construction of \( \mathbb{R}Z_P \) in [2] makes perfect sense for any smooth nice manifold with corners \( Q \), denoted by \( \mathbb{R}Z_Q \). The topology of such generalized spaces are studied in a recent paper Yu [19].

In addition, we can realize \( \mathbb{R}Z_P \) as a non-degenerate intersection of \( m - n \) real quadrics (quadric hypersurfaces) in \( \mathbb{R}^m \), which induces a \( (\mathbb{Z}_2)^m \)-invariant smooth structure on \( \mathbb{R}Z_P \) (see [5] §6)). Consider a presentation of \( P \) as follows:

\[
P = P(A, b) = \{ x \in \mathbb{R}^n \mid \langle a_i, x \rangle + b_i \geq 0, \quad i = 1, \ldots, m \}
\]

where \( A = (a_1, \ldots, a_m) \) is an \( n \times m \) real matrix. Since \( P \) has a vertex, the rank of \( A \) is equal to \( n \). Define a map

\[
i_{A,b} : \mathbb{R}^n \to \mathbb{R}^m, \quad i_{A,b}(x) = A'x + b
\]
where \( b = (b_1, \ldots, b_m) \in \mathbb{R}^m \). So the map \( i_{A,b} \) embeds \( P \) into the positive cone \( \mathbb{R}^m_{\geq 0} = \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_i \geq 0, i = 1, \ldots, m\} \). We can define a space \( \mathbb{R}Z_{A,b} \) by the following commutative diagram

\[
\begin{array}{ccc}
\mathbb{R}Z_{A,b} & \xrightarrow{i_Z} & \mathbb{R}^m \\
\downarrow & & \downarrow \mu \\
P & \xrightarrow{i_{A,b}} & \mathbb{R}^m_{\geq 0}
\end{array}
\]

where \( \mu(x_1, \ldots, x_m) = (x_1^2, \ldots, x_m^2) \). Clearly \( (\mathbb{Z}_2)^m \) acts on \( \mathbb{R}Z_{A,b} \) with quotient space \( P \) and \( i_Z \) is a \( (\mathbb{Z}_2)^m \)-equivariant embedding. It is easy to see that \( \mathbb{R}Z_{A,b} \) is homeomorphic to \( \mathbb{R}Z_P \). In addition, the image of \( \mathbb{R}^n \) under \( i_{A,b} \) is an affine plane of dimension \( n \) in \( \mathbb{R}^m \), which we can specify by \( m - n \) linear equations:

\[
i_{A,b}(\mathbb{R}^n) = \{ y \in \mathbb{R}^m \mid y = A^t x + b, x \in \mathbb{R}^n \} = \{ y \in \mathbb{R}^m \mid \Gamma y = \Gamma b \}
\]

where \( \Gamma = (\gamma_{jk}) \) is an \((m - n) \times m\) matrix of rank \( m - n \) so that \( \Gamma A^t = 0 \). In other words, the rows of \( \Gamma \) form a basis of all the linear relations among \( a_1, \ldots, a_m \). Then we can write the image of \( \mathbb{R}Z_{A,b} \) under \( i_Z \) explicitly as the common zeros of \( m - n \) real quadratic equations in \( \mathbb{R}^m \):

\[
i_Z(\mathbb{R}Z_{A,b}) = \{ (y_1, \ldots, y_m)^t \in \mathbb{R}^m \mid \sum_{k=1}^m \gamma_{jk} y_k^2 = \sum_{k=1}^m \gamma_{jk} b_k, 1 \leq j \leq m - n \}.
\]

The above intersection of real quadrics is non-degenerate (i.e. the gradients of these quadrics are linearly independent everywhere in their intersection). This implies that \( \mathbb{R}Z_{A,b} \) is embedded as an \( n \)-dimensional smooth submanifold in \( \mathbb{R}^m \) where \( (\mathbb{Z}_2)^m \) acts smoothly. So \( P \) can embedded as a Riemannian polyhedron in \( i_Z(\mathbb{R}Z_{A,b}) \) with the induced metric from \( \mathbb{R}^m \).

The theorem below tells us when a 3-dimensional real moment-angle manifold can admit a Riemannian metric with non-negative or positive scalar curvature.

**Theorem 2.2** (Proposition 4.8 and Corollary 4.10 in [18]).

Let \( P \) be a 3-dimensional simple convex polytope in \( \mathbb{R}^3 \) with \( m \) facets.

(a) \( \mathbb{R}Z_P \) admits a Riemannian metric with non-negative scalar curvature if and only if \( P \) is combinatorially equivalent either to \([0,1]^3\) or a polytope that can be obtained from \( \Delta^3 \) by a sequence of vertex cuts.

(b) \( \mathbb{R}Z_P \) admits a Riemannian metric with positive scalar curvature if and only if \( P \) is combinatorially equivalent to a polytope that can be obtained from \( \Delta^3 \) by a sequence of vertex cuts.
Moreover, if $\mathbb{R}Z_P$ admits a Riemannian metric with positive (or non-negative) scalar curvature, we can choose such a Riemannian metric to be invariant with respect to the canonical $(\mathbb{Z}_2)^n$-action on $\mathbb{R}Z_P$.

3. Proof of Theorem 1.1 and Theorem 1.3

Let us first introduce some notions that are used in the following proof.

Definition 3.1 (see Definition 1.3 in [12]). A uniformly Euclidean (or $L^\infty$) metric on a closed manifold $M$ is a measurable section of $\text{Sym}_2(T^*M)$ that satisfies

$$L^{-1}g_0 \leq g \leq Lg_0$$

almost everywhere on $M$ for some Riemannian metric $g_0$ on $M$ and some $L > 0$.

Definition 3.2 (see [3]). Let $M$ be a smooth $n$-manifold and $\Sigma \subset M$ be an embedded codimension-2 submanifold. Near any point $p \in \Sigma$, we can find a local chart $(y^1, y^2, x^1, \cdots, x^{n-2})$ in which $\Sigma$ is given by $y^1 = y^2 = 0$. We then introduce an associated transversal polar coordinate system $(\rho, \theta, x^1, \cdots, x^{n-2})$ where $y^1 = \rho \cos \theta$, $y^2 = \rho \sin \theta$.

A Riemannian edge-cone metric $g$ of cone angle $2\pi \beta$ on $(M, \Sigma)$ is a smooth Riemannian metric on $M \setminus \Sigma$ which, for some $\varepsilon > 0$, can be expressed as

$$g = \overline{g} + \rho^{1+\varepsilon} h$$

where $h$ is a $C^2$-smooth section of $\text{Sym}_2(T^*M)$, and

- $u_j dx^j$ is a smooth 1-form on $\Sigma$.
- $\omega_{jk} dx^j dx^k$ is a smooth metric on $\Sigma$.

Definition 3.3. Let $M$ be a compact smooth manifold. The Yamabe invariant (or $\sigma$-invariant) of $M$ is:

$$\sigma(M) := \sup \{ \mathcal{Y}(M, [g_0]) \mid [g_0] \text{ is a conformal class of metrics on } M \},$$

where

$$\mathcal{Y}(M, [g_0]) := \inf \left\{ \int_M S(g) d\text{Vol}_g \mid g \in [g_0], \text{Vol}_g(M) = 1 \right\}$$

and $S(g)$ denotes the scalar curvature of a Riemannian metric $g$. The sign of $\sigma(M)$ determines the so-called Yamabe type of $M$ (see [10, 16]).

Proof of Theorem 1.1

Let $P$ be a simple convex 3-polytope in $\mathbb{R}^3$ that is smoothly equivalent to $W$. The real moment-angle manifold $\mathbb{R}Z_W$ of $W$ (see Remark 2.1) is equivariantly homeomorphic to $\mathbb{R}Z_P$. Then since every closed 3-manifold has a unique smooth structure, $\mathbb{R}Z_W$ is equivariantly diffeomorphic to $\mathbb{R}Z_P$. 
Let $F_1, \cdots, F_m$ be all the facets of $W$. Let $\Theta_W : \mathbb{R}Z_W \to W$ be the orbit map of the canonical $(\mathbb{Z}_2)^m$-action on $\mathbb{R}Z_W$. So $\Theta_W^{-1}(F_i)$ are compact 2-manifolds that intersect transversely in $\mathbb{R}Z_W$. By the argument in Gromov [8, §2.1], the metric $g(W) = g|_W$ induces a unique $(\mathbb{Z}_2)^m$-invariant metric $\tilde{g}$ on $\mathbb{R}Z_W$ with the stratified singular set
\[
\text{sing}(\tilde{g}) = \bigcup_{1 \leq i \leq m} \Theta_W^{-1}(F_i) = \tilde{F} \cup \tilde{E} \cup \tilde{V}
\]
where $\tilde{F}$, $\tilde{E}$ and $\tilde{V}$ denote the union of cells of dimension 2, 1 and 0, respectively in $\text{sing}(\tilde{g})$.

(a) Suppose $(M, g)$ has positive scalar curvature. Since $W$ is mean curvature convex and non-obtuse, it is argued in [8, §2.1] that the metric $\tilde{g}$ on $\mathbb{R}Z_W$ can be deformed to a (Riemannian) smooth metric $\tilde{g}_{\text{reg}}$ with everywhere positive scalar curvature except when $W$ has the following three properties

(i) $W$ is right-angled.
(ii) The mean curvature of every facet of $W \subset M$ is equal to zero.
(iii) The scalar curvature of $(W, g(W))$ is everywhere zero.

Here since the scalar curvature of $(W, g(W))$ is positive, it does not meet (iii). Therefore, $\tilde{g}$ can be deformed to a smooth metric with positive scalar curvature on $\mathbb{R}Z_W \cong \mathbb{R}Z_P$. Then by Theorem 2.2 (b), $P$ is combinatorially equivalent to a polytope that can be obtained from $\Delta^3$ by a sequence of vertex-cuts.

(b) When $(M, g)$ has non-negative scalar curvature, we can check that the metric $\tilde{g}$ on $\mathbb{R}Z_W$ satisfies the conditions in [12, Theorem 1.7].

- At any point of $\tilde{F}$, the sum of mean curvatures of $\tilde{F}$ computed with respect to the two unit normals as outward unit normals in $\tilde{g}$ is non-negative. This is because $W$ is assumed to be mean curvature convex. It is known that the presence of such kind of codimension-one singularities does not affect the Yamabe type of $M$ (see [13, 17]).
- $\tilde{g}$ is an edge-cone metric along each component of $\tilde{E}$ with cone angle $\leq 2\pi$. Indeed, here we can choose $h \equiv 0$ in Definition 3.2 (thus the value of $\varepsilon$ is irrelevant).
- $\tilde{g}$ is $L^\infty$ across the vertices in $\tilde{V}$. Indeed, the quotient metric obtained by gluing eight copies of $W$ at a vertex (see Figure 1) is locally equivalent to a metric $g_{\alpha,\beta}$ on a 3-ball (see Figure 2) defined as follows.

We define the conic metric with cone index $(\pi\alpha, 2\pi\beta) \in \mathbb{R}_+^2$ on the 3-ball in Figure 2 to be (in spherical coordinates $(\rho, \theta, \varphi)$):
\[
g_{\alpha,\beta} = d\rho^2 + \alpha^2 \rho^2 d\varphi^2 + \beta^2 \rho^2 \sin^2(\alpha\varphi) d\theta^2
\]
where $(x_1, x_2, x_3) \in \mathbb{R}^3$ with $x_1 = \rho \cos \varphi, x_2 = \rho \sin \varphi \cos \theta, x_3 = \rho \sin \varphi \sin \theta$. 

Figure 1. Gluing eight copies of a simple polytope at a vertex

Figure 2. The conic metric with cone index $(\pi \alpha, 2\pi \beta)$ on a 3-ball

If $\alpha = \beta = 1$, $g_{1,1} = d\rho^2 + \rho^2 d\varphi^2 + \rho^2 \sin^2 \varphi d\theta^2$ is the standard Euclidean metric. For $0 < \alpha, \beta \leq 1$, we clearly have $\alpha^2 \beta^2 g_{1,1} \leq g_{\alpha,\beta} < g_{1,1}$, which implies that $g_{\alpha,\beta}$ is uniformly Euclidean on the 3-ball.

By the above properties of $\tilde{g}$, we can deduce that

- If the Yamabe invariant of $\mathbb{R} \mathcal{Z}_W$ is non-positive, then [12, Theorem 1.7] implies that there exists a Ricci-flat Riemannian metric on $\mathbb{R} \mathcal{Z}_W$.
- Otherwise if the Yamabe invariant of $\mathbb{R} \mathcal{Z}_W$ is positive, then $\mathbb{R} \mathcal{Z}_W$ admits a Riemannian metric with positive scalar curvature (see [10, 16]).
So in either case $\mathbb{R}Z_P \cong \mathbb{R}Z_W$ admits a Riemannian metric with non-negative scalar curvature. Then by Theorem 2.2(a), $P$ is combinatorially equivalent either to $[0, 1]^3$ or a polytope that can be obtained from $\Delta^3$ by a sequence of vertex-cuts.

\begin{proof}

\begin{remark}

The definition of the conic metric $g_{\alpha, \beta}$ on a 3-ball in $\mathbb{R}^3$ can be generalized to arbitrarily high dimensions. We define the \textit{conic metric with cone index} $\Lambda = (\pi \alpha_1, \ldots, \pi \alpha_{n-2}, 2\pi \beta) \in \mathbb{R}^{n-1}_{>0}$ on the $n$-ball $B^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ $(n \geq 2)$ to be

$$g_{\Lambda} = d\rho^2 + \alpha_1^2 \rho^2 d\varphi_1^2 + \alpha_2^2 \rho^2 \sin^2(\alpha_1 \varphi_1) d\varphi_2^2 + \cdots + \alpha_{n-2}^2 \rho^2 \sin^2(\alpha_1 \varphi_1) \cdots \sin^2(\alpha_{n-3} \varphi_{n-3}) d\varphi_{n-2}^2 + \beta^2 \rho^2 \sin^2(\alpha_1 \varphi_1) \cdots \sin^2(\alpha_{n-2} \varphi_{n-2}) d\theta^2$$

where $(\rho, \varphi_1, \ldots, \varphi_{n-2}, \theta)$ is the $n$-dimensional spherical coordinates

$$
    \begin{align*}
        x_1 &= \rho \cos(\varphi_1) \\
        x_2 &= \rho \sin(\varphi_1) \cos(\varphi_2) \\
        &\vdots \\
        x_{n-1} &= \rho \sin(\varphi_1) \cdots \sin(\varphi_{n-2}) \cos \theta \\
        x_n &= \rho \sin(\varphi_1) \cdots \sin(\varphi_{n-2}) \sin \theta
    \end{align*}
$$

with $0 \leq \varphi_1, \ldots, \varphi_{n-2} \leq \pi$, $0 \leq \theta < 2\pi$ for $(x_1, \ldots, x_n) \in B^n \subset \mathbb{R}^n$. This definition should be useful to study general singularities in a metric.

\begin{proof}

The standard 3-cube $[0, 1]^3$ is a right-angled convex polytope in $\mathbb{R}^3$. So by Theorem 1.1 we only need to verify that any convex polytope $P \subset \mathbb{R}^3$ obtained from $\Delta^3$ by a sequence of vertex-cuts can be embedded as a right-angled totally geodesic Riemannian polyhedron in a Riemannian 3-manifold with positive scalar curvature.

Let $F_1, \ldots, F_m$ be all the facets of such a polytope $P$. By Theorem 2.2 there exists a $(\mathbb{Z}_2)^m$-invariant Riemannian metric $g_0$ with positive scalar curvature on the real moment-angle manifold $\mathbb{R}Z_P$. Note that $P$ is embedded in $\mathbb{R}Z_P$ as the fundamental domain of the canonical $(\mathbb{Z}_2)^m$-action, which is bounded by the submanifolds $\Theta^{-1}_P(F_i)$, $1 \leq i \leq m$. Moreover, by definition each $\Theta^{-1}_P(F_i)$ is the fixed point set of the generator $e_i \in (\mathbb{Z}_2)^m$ under the canonical $(\mathbb{Z}_2)^m$-action. It is a standard fact that every connected component of the fixed point set (with the induce Riemannian metric) of an isometry on a Riemannian manifold is a totally geodesic submanifold (see [11, Theorem 1.10.15]). So each $\Theta^{-1}_P(F_i)$ consists of totally geodesic submanifolds of $(\mathbb{R}Z_P, g_0)$. Moreover, the dihedral angles between

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any components of $\Theta_P^{-1}(F_i)$ and $\Theta_P^{-1}(F_j)$ (whenever they intersect) are always equal to $\pi/2$ since the $(\mathbb{Z}_2)^m$-action on $(\mathbb{R}Z_P, g_0)$ is isometric. So $P$ is embedded as a right-angled totally geodesic Riemannian polyhedron in $(\mathbb{R}Z_P, g_0)$. The theorem is proved. □

**Corollary 3.5.** The result in Theorem 1.3 still holds if the Riemannian 3-manifold $(M, g)$ is assumed to have positive (or non-negative) constant scalar curvature.

**Proof.** By the solution of the equivariant Yamabe problem in Hebey-Vaugon [9], if $\mathbb{R}Z_P$ of a simple convex 3-polytope $P$ admits a $(\mathbb{Z}_2)^m$-invariant Riemannian metric $g_0$ with positive (or non-negative) scalar curvature, then there exists a $(\mathbb{Z}_2)^m$-invariant Riemannian metric $\overline{g}_0$ conformal to $g_0$ on $\mathbb{R}Z_P$ which has positive (or non-negative) constant scalar curvature. So we can prove the corollary by the same argument for $(\mathbb{R}Z_P, \overline{g}_0)$ as the proof of Theorem 1.3. □

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