LOCAL INDEX FORMULA AND TWISTED SPECTRAL TRIPLES

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ABSTRACT. We prove a local index formula for a class of twisted spectral triples of type III modeled on the transverse geometry of conformal foliations with locally constant transverse conformal factor. Compared with the earlier proofs in the untwisted case, the novel aspect resides in the fact that the twisted analogues of the JLO entire cocycle and of its retraction are no longer cocycles in their respective Connes bicomplexes. We show however that the passage to the infinite temperature limit, respectively the integration along the full temperature range against the Haar measure of the positive half-line, has the remarkable effect of curing in both cases the deviations from the cocycle and transgression identities.

Dedicated to Alain Connes, with admiration, friendship and gratitude

INTRODUCTION

The local-global principle, as epitomized by the Atiyah-Singer index theorem but in the larger operator theoretic framework, has played a pivotal role in Alain Connes' overarching design of the foundations of Noncommutative Geometry. Although I was too overwhelmed by his brilliant intellect and fantastic mathematical insight to fully realize it at the time, this very theme was in fact the subtext of our first mathematical conversation, in the autumn of 1978, while we were both visiting the Institute for Advanced Study. As his program advanced, the theme gradually evolved into a perennial context for a substantial part of our collaborative work and, last but not least, it became a pretext for a close, lifelong friendship. As a token of my deep appreciation, I thought it would befit the occasion to try to run anew the machinery that has emerged, this time in the presence of a twist.

The basic template for a space in the Connes program is encoded in the notion of spectral triple. In our recent joint paper [10] we showed that this notion can be adapted to include certain type III spaces by the simple device of incorporating in it a twisting automorphism of the algebra of coordinates. The paradigmatic examples of such spaces are those describing the transverse geometry of a foliation of codimension 1, or of a conformal foliation of arbitrary codimension. While it is always possible to associate

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a spectral triple of type II to any foliation by passing to the frame bundle of a complete transversal (cf. [8]), the very construction introduces a large number of additional parameters, which in turn makes the task of computing the characteristic classes quite formidable (see [9]). The twisting device on the other hand, whenever applicable, allows to bypass the extra step of geometric reduction to type II.

Since the primary effect of the twisting automorphism is the replacement of the bimodule of noncommutative differential forms of a spectral triple with a bimodule of twisted differential forms, one would have expected the characteristic classes of a twisted spectral triple to be captured by a twisted version of the Connes-Chern character, landing in twisted cyclic cohomology. Somewhat counterintuitively, it turned out that no cohomological twisting is actually needed, and that Connes’ original construction of the Chern character in noncommutative geometry [3] remains in fact operative in the twisted case as well. The natural question that arises is whether the Connes-Chern character of a twisted spectral triple can also be expressed in local terms as in [8], by means of a residue integral that eliminates all quantum infinitesimal perturbations of order strictly larger than 1.

In this paper we produce such a local index formula for a class of spectral triples twisted by scaling automorphisms, modeled on the geometry of a conformal foliation whose holonomy consists of germs of conformal transformations of $\mathbb{R}^n$. The proof is patterned on the strategy that evolved over a number of years of joint (and joyful) work leading up to the residue index formula [5, 6, 7, 8], with the important difference that the twisted counterpart of the JLO cocycle [18], which played a key role in our earlier proof (as well as in in Higson’s [16]) is no longer a cocycle.

The plan of the paper is as follows. In §1 we recall from [10] the basic definitions concerning twisted spectral triples and their characters. Extrapolating from the expression of local Hochschild cocycle in op.cit. we then make a straightforward Ansatz in §2, predicting the form of the twisted local formula for the Connes-Chern character. In §3 we test the Ansatz on “real-life” examples of twisted spectral triples occurring in conformal geometry. These are the spectral triples describing the transverse geometry of the conformal foliations whose holonomy is given by germs of Möbius transformations of $S^n$. We conclude that the Ansatz holds true if the holonomy is restricted to the parabolic subgroup preserving a point, or equivalently to the similarity transformations of $\mathbb{R}^n$.

The main results of the paper are proved in §4, where we establish the validity of the Ansatz for an abstract class of twisted spectral triples, modeled on the conformal foliations with locally constant transverse conformal factor. While the twisted entire cochain analogous to [18] is no longer a cocycle, passing to the infinite temperature limit has the remarkable effect of restoring the cocycle identity, and the resulting cocycle can be expressed
in terms of a residue integral as in \cite{8}. To show that this residue cocycle represents the Connes-Chern character, one needs to transit through a transgressed version, as in \cite{7}. In turn, the transgression process does yield a genuine cocycle because it involves integrating along the full temperature range, with respect to the Haar measure of $\mathbb{R}^+$, which miraculously cures again the deviation from the cocycle identity.

1. Twisted spectral triples and their characters

We begin by briefly reviewing the notion of twisted spectral triple of type III, introduced in \cite{10}, together with some of its basic properties.

1.1. Twisted spectral triple. A twisted spectral triple $(\mathcal{A}, \mathcal{H}, D, \sigma)$ consists of a local Banach $\ast$-algebra $\mathcal{A}$ represented in the Hilbert space $\mathcal{H}$ by bounded operators, an automorphism $\sigma \in \text{Aut}(\mathcal{A})$, and a self-adjoint unbounded operator $D$ such that, for any $a \in \mathcal{A},$

\begin{equation}
(1.1) \quad a(D^2 + 1)^{-\frac{1}{2}} \in \mathcal{K}(\mathcal{H}) \quad \text{(compact operators)};
\end{equation}

\begin{equation}
(1.2) \quad a(\text{Dom}D) \subset \text{Dom}D \quad \text{and} \quad [D, a]_\sigma := Da - \sigma(a)D \quad \text{is bounded};
\end{equation}

\begin{equation}
(1.3) \quad \sigma(a^*) = (\sigma^{-1}(a))^*.
\end{equation}

A $\mathbb{Z}_2$-graded (or even) $\sigma$-spectral triple has the additional datum of a grading operator

$$
\gamma = \gamma^* \in \mathcal{L}(\mathcal{H}), \quad \gamma^2 = I
$$

which anticommutes with $D$, and commutes with the action of $\mathcal{A}$. In case the algebra itself is $\mathbb{Z}_2$-graded, the commutators properties (including the twisted commutators) are understood in the graded sense. We shall be concerned with finitely-summable twisted spectral triples, \textit{i.e.} with those that satisfy a stronger version of (1.1), namely the $(p, \infty)$-summability condition

\begin{equation}
(1.4) \quad a(D^2 + 1)^{-\frac{1}{2}} \in \mathcal{L}^{(p, \infty)}(\mathcal{H}), \quad \forall a \in \mathcal{A},
\end{equation}

for some $1 \leq p < \infty$ of the same parity as the spectral triple. The notation is that of \cite{4, IV, 2.6},

\begin{equation}
(1.5) \quad \mathcal{L}^{(p, \infty)}(\mathcal{H}) = \{ T \in \mathcal{K}(\mathcal{H}) ; \quad \sum_{i=0}^{N} \mu_i(T) = O(N^{1 - \frac{1}{p}}) \}, \quad \text{if} \quad p > 1,
\end{equation}

\begin{equation}
(1.6) \quad \mathcal{L}^{(1, \infty)}(\mathcal{H}) = \{ T \in \mathcal{K}(\mathcal{H}) ; \quad \sum_{i=0}^{N} \mu_i(T) = O(\log N) \}.
\end{equation}
1.2. Graded double. As in the untwisted situation, there is a canonical way (cf. [3, Part I, §7]) to pass from an ungraded (or odd) twisted spectral triple \((A, \mathfrak{H}, D, \sigma)\) to a \(\mathbb{Z}_2\)-graded twisted one over the \(\mathbb{Z}_2\)-graded algebra \(A_{\text{gr}} = A \otimes C_1\).

Here \(C_1\) denotes the Clifford algebra \(\text{Cliff}(\mathbb{R}) \otimes \mathbb{C}\); its even part is \(C_1^+ = \mathbb{C}1\), with 1 the unit of \(C_1\), while the odd part is \(C_1^- = \mathbb{C} \epsilon\), with \(\epsilon^2 = 1\). The automorphism remains \(\sigma\), identified to \(\sigma \otimes \text{Id}\). One constructs an \(A_{\text{gr}}\)-module by first letting \(H_{1} = H_{1}^+ \oplus H_{1}^-\) be the \(\mathbb{Z}_2\)-graded Hilbert space with \(H_{1}^\pm = \mathbb{C}\) on which \(C_1\) acts via

\[
\lambda 1 + \mu \epsilon \mapsto \begin{pmatrix} \lambda & \mu \\ \mu & \lambda \end{pmatrix}, \quad \lambda, \mu \in \mathbb{C},
\]

and then taking

\(\mathfrak{H}_{\text{gr}} = \mathfrak{H} \otimes H_{1}\), with \(\mathfrak{H}_{\text{gr}}^\pm = \mathfrak{H} \otimes H_{1}^\pm\),

on which \(A_{\text{gr}}\) acts via the exterior tensor product representation; the corresponding grading operator is

\[
\gamma = \text{Id}_{\mathfrak{H}} \otimes \gamma_1, \quad \text{where} \quad \gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Finally, as operator one takes

\[
D_{\text{gr}} = D \otimes P_1, \quad \text{where} \quad P_1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} : \mathfrak{H}_1 \rightarrow \mathfrak{H}_1.
\]

1.3. Invertible double. When \(D\) is not invertible, we shall resort to the construction described in [3, Part I, §6] (akin to the passage from the Dirac operator on flat space to the Dirac Hamiltonian with mass) to canonically associate to \((A, \mathfrak{H}, D, \gamma, \sigma)\) a new \(\sigma\)-spectral triple \((A, \tilde{\mathfrak{H}}, \tilde{D}, \tilde{\gamma}, \sigma)\) with invertible operator, defined as follows. With \(\mathfrak{H}_1\) as above, one takes

\[
\tilde{\mathfrak{H}} = \mathfrak{H} \otimes \mathfrak{H}_1 \quad \text{(graded tensor product)}, \quad \tilde{\gamma} = \gamma \otimes \gamma_1,
\]

\[
\tilde{D} = D \otimes \text{Id} + \text{Id} \otimes F_1, \quad \text{where} \quad F_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : \mathfrak{H}_1 \rightarrow \mathfrak{H}_1.
\]

The algebra \(A\) is made to act on \(\tilde{\mathfrak{H}}\) via the representation

\[
a \in A \mapsto \tilde{a} := a \otimes e_1, \quad \text{where} \quad e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} : \mathfrak{H}_1 \rightarrow \mathfrak{H}_1.
\]

1.4. Lipschitz regularity. In [10] a twisted spectral triple \((A, \mathfrak{H}, D, \sigma)\) with invertible \(D\) was called \textit{Lipschitz regular} if it satisfies the additional condition

\[
(1.7) \quad \langle [D, a]_{\sigma} := |D| a - \sigma(a) |D| \rangle \text{ is bounded,} \quad \forall a \in A.
\]

Such a twisted spectral triple can be ‘untwisted’ by passage to its ‘phase’ operator \(F = D |D|^{-1}\). Indeed,

\[
[F, a] = |D|^{-1} ((D a - \sigma(a) D) - (|D| a - \sigma(a) |D|) F),
\]
which shows that these commutators are compact operators, of the same order of magnitude as $D^{-1}$. Thus, $(\tilde{\mathcal{F}}, F)$ is a Fredholm module over $\mathcal{A}$, defining a $K_*$-cycle over the norm closure $C^*$-algebra of $\mathcal{A}$. Moreover, if $(\mathcal{A}, \tilde{\mathcal{F}}, D, \sigma)$ is $(p, \infty)$-summable so is $(\tilde{\mathcal{F}}, F)$.

We extend the Lipschitz regularity condition to a general spectral triple $(\mathcal{A}, \tilde{\mathcal{F}}, D, \sigma)$ by requiring that its invertible double $(\mathcal{A}, \tilde{\mathcal{F}}, D, \tilde{\sigma}, \tilde{\gamma}, \sigma)$ be Lipschitz regular. Since

$$\tilde{F} := \bar{D} |\bar{D}|^{-1} = D(D^2 + 1)^{-\frac{1}{2}} \hat{\sigma} \otimes \text{Id} + (D^2 + 1)^{-\frac{1}{2}} \hat{\sigma} \otimes F_1,$$

and therefore

$$[\tilde{F}, a] = [D(D^2 + 1)^{-\frac{1}{2}}, a] \otimes e_{11} + (D^2 + 1)^{-\frac{1}{2}} a \otimes e_{21} - a(D^2 + 1)^{-\frac{1}{2}} \otimes e_{12},$$

Lipschitz regularity can be alternatively phrased as the requirement

$$[D(D^2 + 1)^{-\frac{1}{2}}, a] \in \mathcal{K}(\tilde{\mathcal{F}}), \quad \forall a \in \mathcal{A}. \quad (1.8)$$

In the $(p, \infty)$-summable case these commutators belong to the ideal $\mathcal{L}^{(p, \infty)}(\tilde{\mathcal{F}})$ of $\mathcal{K}(\tilde{\mathcal{F}})$.

1.5. **Connes-Chern character.** By resorting if necessary to the doubling procedures described in [12] and [14], we may assume without essential loss of generality that the $(p, \infty)$-summable twisted spectral triple $(\mathcal{A}, \tilde{\mathcal{F}}, D, \sigma)$ under consideration is $\mathbb{Z}_2$-graded and $D$ invertible. We shall often do so in the sequel without any further mention.

Let $(\mathcal{A}, \tilde{\mathcal{F}}, D, \sigma)$ be such a twisted spectral triple which is also Lipschitz-regular. Then, as remarked in [14], it gives rise to a ‘phase’ Fredholm module $(\tilde{\mathcal{F}}, F)$ over $\mathcal{A}$. In turn, the latter has a well-defined Connes-Chern character in cyclic cohomology, cf. [3], which up to a normalizing constant is represented by the cyclic cocycle

$$\tau^p_F(a_0, a_1, \ldots, a_p) := \text{Tr} (\gamma F [F, a_0] [F, a_1] \cdots [F, a_p]), \quad a_i \in \mathcal{A}. \quad (1.9)$$

In [10], we have actually shown that a $(p, \infty)$-summable twisted spectral triple as above admits a Connes-Chern character without assuming Lipschitz regularity. Indeed, we showed that the $(p + 1)$-linear form on $\mathcal{A}$

$$\tau^p_F(a_0, a_1, \ldots, a_p) := \text{Tr} (\gamma D^{-1}[D, a_0] \cdots D^{-1}[D, a_p]),$$

is a cyclic cocycle in $Z^p_1(\mathcal{A})$, by means of which one can recover the index pairing of $D$ with $K^*(\mathcal{A})$. More precisely, up to a universal constant factor $c_p$, Index$(\sigma(e) D e)$ is given by $\tau^p_F(e, \ldots, e)$, for any class $[e] \in K^0(\mathcal{A})$.

1.6. **Conformally perturbed spectral triple.** An instructive class of examples of twisted spectral triples arises from conformal-type perturbations of ordinary spectral triples. Let $(\mathcal{A}, \tilde{\mathcal{F}}, D)$ be a $(p, \infty)$-summable spectral triple, and let $h = h^* \in \mathcal{A}$. By setting

$$D_h = e^h D e^h, \quad \text{and} \quad \sigma_h(a) = e^{2h} a e^{-2h}, \quad \forall a \in \mathcal{A}$$
one easily sees that
\begin{equation}
D_h a - \sigma_h(a) D_h = e^h [D, \sigma_{h/2}(a)] e^h \in \mathcal{L}(\mathfrak{g}),
\end{equation}
thus giving rise to a twisted spectral triple \((\mathcal{A}, \mathfrak{g}, D_h, \sigma_h)\). Noting that
\begin{equation}
D_h^{-1} [D_h, a]_\sigma = e^{-h} D^{-1} [D, \sigma_{h/2}(a)] e^h,
\end{equation}
one obtains the identity
\begin{equation}
(\prod \mathfrak{g}) D_h^{-1} [D_h, a]_\sigma D_h^{-1} [D_h, a_1]_\sigma \cdots D_h^{-1} [D_h, a_p]_\sigma = \text{Tr} (\gamma D^{-1} [D, \sigma_{h/2}(a_0)] D^{-1} [D, \sigma_{h/2}(a_1)] \cdots D^{-1} [D, \sigma_{h/2}(a_p)]).
\end{equation}
The right hand side is a cyclic cocycle on \(\mathcal{A}\) that represents, for \(h = 1\) and up to normalization, the Connes-Chern character
\[
Ch^*(\mathcal{A}, \mathfrak{g}, D) \in \text{HC}^*(\mathcal{A})
\]
of \((\mathcal{A}, \mathfrak{g}, D)\) viewed as a Fredholm module, cf. [3, Part I, §6]. It follows that the left hand side, which is the cyclic cocycle obtained by composition with the inner automorphism \(\sigma_{h/2}\), determines the same class in periodic cyclic cohomology. This justifies regarding (1.13) as defining the periodic Connes-Chern character of the conformally perturbed spectral triple:
\begin{equation}
Ch^*(\mathcal{A}, \mathfrak{g}, D_h, \sigma_h) := Ch^*(\mathcal{A}, \mathfrak{g}, D) \in \text{HP}^*(\mathcal{A}).
\end{equation}

1.7. Local Hochschild cocycle. With the goal of extending the local index formula of [3] to twisted spectral triple, we looked in [10] for an analogue of Connes’ residue formula [4, IV.2.] for the Hochschild class of the Connes-Chern character.

We recall that if \((\mathcal{A}, \mathfrak{g}, D)\) is an (even, invertible) \((p, \infty)\)-summable spectral triple satisfying the smoothness condition
\begin{equation}
\mathcal{A}, [D, \mathcal{A}] \subset \bigcap_{k>0} \text{Dom}(\delta^k), \quad \text{where} \quad \delta(T) := [[D], T],
\end{equation}
the Hochschild cohomology class \(I(Ch^*(\mathcal{A}, \mathfrak{g}, D)) \in HH^*(\mathcal{A})\) admits a local representation, given by the formula
\begin{equation}
\varepsilon_D(a_0, a_1, \ldots, a_p) := \text{Tr}_\omega (\gamma a_0 [D, a_1] \cdots [D, a_p] D^{-p}), \quad a_i \in \mathcal{A},
\end{equation}
which defines a Hochschild cocycle. Here \(\text{Tr}_\omega\) stands for a Dixmier trace (see [4, IV.2.\(\beta, \gamma\)]) on the ideal \(\mathcal{L}^{(1,\infty)}(\mathfrak{g})\). The local nature of the above formula stems from the fact that the Dixmier trace vanishes on the subideal
\[
\mathcal{L}^{(1,\infty)}_0(\mathfrak{g}) = \{ T \in \mathcal{K}(\mathfrak{g}) ; \sum_{i=0}^N \mu_i(T) = o(\log N) \},
\]
which contains the trace class operators, and in particular all smoothing operators on any closed manifold. Thus, \(\varepsilon_D(a_0, a_1, \ldots, a_p)\) depends only on the class of the operator \(a_0 [D, a_1] \cdots [D, a_p] D^{-p} \in \mathcal{L}^{(1,\infty)}(\mathfrak{g})\) modulo \(\mathcal{L}^{(1,\infty)}_0(\mathfrak{g})\), which plays the role of its symbol.
Using the identity
\[
[D, a] D^{-k} = D^{-k+1} (D^k a D^{-k} - D^{k-1} a D^{-k+1}), \quad \forall a \in \mathcal{A},
\]
one can successively move \(D^{-p}\) to the left and rewrite the cocycle \(\kappa_D\) in the form
\[
\kappa_D(a_0, a_1, \ldots, a_p) = \text{Tr}_\omega \left( \gamma a_0 (D a_1 D^{-1} - a_1) \cdots (D^p a_p D^{-p} - D^{p-1} a_p D^{-p+1}) \right).
\]

In the twisted case, taking a clue from (1.12), one is led to make the formal substitution
\[
D^k a D^{-k} \mapsto D^k \sigma^{-k}(a) D^{-k}, \quad \forall a \in \mathcal{A},
\]
and use the twisted version of (1.17), namely
\[
[D, \sigma^{-k}(a)] \sigma D^{-k} = D^{-k+1} (D^k \sigma^{-k}(a) D^{-k} - D^{k-1} \sigma^{-k+1}(a) D^{-k+1}),
\]
to reverse the process of distributing \(D^{-p}\) among the factors. Assuming that the domain condition which permits the above operation is fulfilled, one thus arrives at the expression
\[
\kappa_{D, \sigma}(a_0, a_1, \ldots, a_p) := \text{Tr}_\omega \left( \gamma a_0 [D, \sigma^{-1}(a_1)] \cdots [D, \sigma^{-p}(a_p)] D^{-p} \right).
\]
This was shown in [10] to be indeed a Hochschild \(p\)-cocycle, and it will be useful to reproduce the elementary calculation that validates this statement. It relies on two basic properties of twisted spectral triples. The first is the obvious fact that the \(\sigma\)-bracket with \(D\) satisfies the twisted derivation rule
\[
[D, ab]_\sigma = \sigma(a) [D, b]_\sigma + [D, a]_\sigma b, \quad a, b \in \mathcal{A}.
\]
The second is the observation that the positive linear functional on \(\mathcal{A}\),
\[
a \mapsto \text{Tr}_\omega(a \| D \|^{p}),
\]
is a \(\sigma^{-p}\)-trace on \(\mathcal{A}\), i.e. satisfies
\[
\text{Tr}_\omega(a b \| D \|^{-p}) = \text{Tr}_\omega(b \sigma^{-p}(a) \| D \|^{-p}), \quad \forall a, b \in \mathcal{A}.
\]
It is in fact a \(\sigma^{-p}\)-hypertrace, since for any \(T \in \mathcal{L}(\mathcal{H})\) one still has
\[
\text{Tr}_\omega(T \sigma^{-p}(a) \| D \|^{-p}) = \text{Tr}_\omega(a T \| D \|^{-p}).
\]
Making use of the Leibniz rule (1.21), one computes the Hochschild coboundary of $\kappa_{D,\sigma} \in Z^p(A, A^\ast)$ as follows:

$$b\kappa_{D,\sigma}(a_0, a_1, ..., a_{p+1}) =$$

$$= \sum_{i=0}^{p} (-1)^i \kappa_{D,\sigma}(a_0, ..., a_i a_{i+1}, ..., a_{p+1}) + (-1)^{p+1} \kappa_{D,\sigma}(a_{p+1} a_0, a_1, ..., a_p)$$

$$= \text{Tr}_\omega(\gamma a_0 a_1 [D, \sigma^{-1}(a_2)] \sigma \cdots [D, \sigma^{-p}(a_{p+1})] \sigma D^{-p})$$

$$- \text{Tr}_\omega(\gamma a_0 a_1 [D, \sigma^{-1}(a_2)] \sigma \cdots [D, \sigma^{-p}(a_{p+1})] \sigma D^{-p})$$

$$- \text{Tr}_\omega(\gamma a_0 [D, \sigma^{-1}(a_1)] \sigma^{-1}(a_2) \cdots [D, \sigma^{-p}(a_{p+1})] \sigma D^{-p}) + \ldots$$

$$+ (-1)^p \text{Tr}_\omega(\gamma a_0 [D, \sigma^{-1}(a_1)] \sigma \cdots$$

$$\ldots [D, \sigma^{-p+1}(a_{p-1})] \sigma \sigma^{-p+1}(a_p) [D, \sigma^{-p}(a_{p+1})] \sigma D^{-p})$$

$$+ (-1)^p \text{Tr}_\omega(\gamma a_0 [D, \sigma^{-1}(a_1)] \sigma$$

$$\cdots [D, \sigma^{-p+1}(a_{p-1})] \sigma [D, \sigma^{-p}(a_p)] \sigma \sigma^{-p}(a_{p+1}) D^{-p})$$

$$+ (-1)^{p+1} \text{Tr}_\omega(\gamma a_{p+1} a_0 [D, \sigma^{-1}(a_1)] \sigma \cdots [D, \sigma^{-p}(a_p)] \sigma D^{-p})$$

$$= 0.$$

The end result is 0 because the successive terms cancel in pairs, with the last two terms canceling each other thanks to the enhanced $\sigma^{-p}$-trace property Eq. (1.22).

2. Ansatz for a twisted local index formula

2.1. Local index formula for spectral triples. The local index formula that delivers in full the Connes-Chern character in cyclic cohomology was developed in [8], in terms of residue functionals that generalize Wodzicki’s noncommutative residue, and in the framework of an abstract pseudodifferential calculus which gives a precise meaning to the notion of symbol. We briefly recall the salient notions.

Let $(\mathcal{A}, \mathcal{F}, D)$ be a $(p, \infty)$-summable spectral triple (with $D$ invertible), which satisfies the smoothness condition (1.15). We denote by $\mathcal{B}$ the algebra generated by $\bigcup_{k \geq 0} \delta^k(\mathcal{A} + [D, \mathcal{A}])$, and also set $\mathcal{F}^\infty := \bigcap_{k \geq 0} \text{Dom}(|D|^k)$.

We now consider linear operators $P : \mathcal{F}^\infty \rightarrow \mathcal{F}^\infty$ that admit an expansion of the form

$$(2.1) \quad P \sim \sum_{k \geq 0} b_k |D|^{-s-k}, \quad \text{with} \quad b_k \in \mathcal{B}, \quad s \in \mathbb{C},$$
in the sense that
\[ P = \sum_{0 \leq k < N} b_k |D|^{s-k} \in B \cdot \text{OP}^{\Re s-N}, \quad \forall N > 0, \]
where
\[ R \in \text{OP}^r \iff |D|^{-r} R \in \bigcap_{k > 0} \text{Dom}(\delta^k), \]
and
\[ B \cdot \text{OP}^r := \left\{ \sum_j b_j R_j ; \quad b_j \in B, \quad R_j \in \text{OP}^r \right\}. \]

Thanks to the key commutation relation (see [8, Appendix B, Thm. B.1])
\begin{equation}
[D]^s b \sim \sum_{k \geq 0} \frac{s(s-1) \cdots (s-k+1)}{k!} \delta^k(b) |D|^{s-k}, \quad \forall b \in B, \quad s \in \mathbb{C},
\end{equation}
these operators form a filtered algebra.

Let now \( D(A, \tilde{\mu}, D) \) be the algebra generated by \( \bigcup_{k \geq 0} \nabla^k (A + [D, A]) \), where \( \nabla \) denotes the derivation \( \nabla(T) = [D^2, T] \). Its elements play the role of differential operators. They too have a natural order, determined by total power of \( \nabla \) involved in each monomial. Furthermore, the analogue of \((2.2)\) holds: for any \( q \)-th operator \( T \in D^q(A, \tilde{\mu}, D) \) and \( N > 0 \),
\begin{equation}
D^{2s} T - \sum_{0 \leq k < N} \frac{s(s-1) \cdots (s-k+1)}{k!} \nabla^k(T) D^{2(s-k)} \in \text{OP}^{2\Re s + q - N}.
\end{equation}

The intrinsic pseudodifferential calculus for the spectral triple is based on the algebra \( \Psi^\bullet(A, \tilde{\mu}, D) \), generated by the operators defined by \((2.1)\) together with the differential operators. It is a filtered algebra, and its quotient modulo the ideal of smoothing operators \( \Psi^{-\infty}(A, \tilde{\mu}, D) = \bigcap_{N \geq 0} B \cdot \text{OP}^{-N} \) gives the corresponding algebra of complete symbols \( \text{CS}^\bullet(A, \tilde{\mu}, D) := \Psi^\bullet(A, \tilde{\mu}, D)/\Psi^{-\infty}(A, \tilde{\mu}, D) \).

Underlying the setup for the local index formula is the essential assumption that the spectral triple admits a discrete dimension spectrum, to which all singularities of zeta functions associated to elements of \( B \) are confined; the postulated spectrum is a discrete subset \( \Sigma \subset \mathbb{C} \), such that the holomorphic functions
\begin{equation}
\zeta_b(z) = \text{Tr}(b |D|^{-z}), \quad \Re z > p, \quad b \in B,
\end{equation}
admit holomorphic extensions to \( \mathbb{C} \setminus \Sigma \). This requirement is supplemented by a technical condition stipulating that the functions \( \Gamma(z) \zeta_b(z) \) decay rapidly on finite vertical strips.

For the sake of convenience, we shall make the stronger assumption that the dimension spectrum is simple, i.e. \( \Sigma \) consists of simple poles. Then, for any
$P \in \Psi^N(\mathcal{A}, \mathcal{F}, D)$ the zeta function

\begin{equation}
\zeta_P(z) = \text{Tr}(P|D|^{-z}), \quad \Re z > p + N,
\end{equation}

can be meromorphically continued to $\mathbb{C}$, with simple poles in $\Sigma + N$. Furthermore, the residue functional

\begin{equation}
\int_D P := \text{Res}_{z=0} \zeta_P(2z), \quad P \in \Psi^*(\mathcal{A}, \mathcal{F}, D)
\end{equation}
is an (algebraic) trace. By its very construction it vanishes on $\Psi^*(\mathcal{A}, \mathcal{F}, D) \cap \mathcal{L}^1(\mathcal{F})$, and in particular it descends to a trace on $\mathcal{CS}^*(\mathcal{A}, \mathcal{F}, D)$.

The local index formula expresses the Connes-Chern character $Ch^*(\mathcal{A}, \mathcal{F}, D) \in HC^*(\mathcal{A})$ in terms of a cocycle in the bicomplex $\{CC(\mathcal{A}), b, B\}$, whose components are defined by means of the symbolic trace. In the (invertible) odd case, these components are as follows: for $q = 2\ell + 1, \quad \ell \in \mathbb{Z}^+$,

\begin{equation}
\tau^q_{\text{odd}}(a_0, \ldots, a_q) = \sqrt{2i} \sum_k c_{q,k} \int_D a_0 [D, a_1]^{(k_1)} \cdots [D, a_q]^{(k_q)} |D|^{-|k|-q},
\end{equation}

where

\begin{equation}
c_{q,k} = \frac{(-1)^{|k|}}{k_1! \cdots k_q! (k_1 + 1) \cdots (k_1 + \ldots + k_q + q)} \Gamma\left(|k| + \frac{q}{2}\right),
\end{equation}

where $k = (k_1, \ldots, k_q), \quad |k| = k_1 + \ldots + k_q$.

In the even case, with $q = 2\ell, \quad \ell \in \mathbb{Z}^+$,

\begin{equation}
\tau^q_{\text{ev}}(a) = \text{Res}_{z=0} \left(\Gamma(z) \text{Tr}(a|D|^{-2z})\right),
\end{equation}

\begin{equation}
\tau^q_{\text{ev}}(a_0, \ldots, a_q) = \sum_k c_{q,k} \int_D \gamma a_0 [D, a_1]^{(k_1)} \cdots [D, a_q]^{(k_q)} |D|^{-|k|-q}.
\end{equation}

Each component $\tau^q = \tau^q_{\text{ev/odd}}$ has finitely many nonzero summands, and $\tau^q \equiv 0$ for any $q > p$.

Since the expressions $\tau^q(a_0, \ldots, a_q)$ are unaffected by the scaling $D \mapsto tD, \quad t \in \mathbb{R}$, we can write them in terms of the scale-invariant operators

$\alpha^k(a) := D^k a D^{-k}, \quad a \in \mathcal{A}$.

Indeed, using the obvious identity

$[D, a]^{(k+1)} D^{-2k-3} = D^2 \left([D, a]^{(k)} D^{-2k-1}\right) D^{-2} - [D, a]^{(k)} D^{-2k-1}$,

one verifies by induction that for any $k \geq 0$,

$[D, a]^{(k)} D^{-2k-1} = \sum_{j=0}^k (-1)^j \binom{k}{j} \left(\alpha^{2(k-j)+1}(a) - \alpha^{2(k-j)}(a)\right)$.
Therefore, for any $\ell \in \mathbb{Z}$,

$$[D,a]^{(k)} D^{-\ell} = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left( \alpha^{2(k-j)+1}(a) - \alpha^{2(k-j)}(a) \right) D^{2k+1-\ell}$$

$$= D^{2k+1-\ell} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left( \alpha^{\ell-2j}(a) - \alpha^{\ell-2j-1}(a) \right).$$

With the abbreviated notation

$$\Sigma^{(k,\ell)}(a) := \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left( \alpha^{\ell-2j}(a) - \alpha^{\ell-2j-1}(a) \right),$$

the above equality takes the form

$$[D,a]^{(k)} D^{-\ell} = D^{2k+1-\ell} \Sigma^{(k,\ell)}(a).$$

Successive application of the identity (2.11) brings the components $\tau^q$ with $q > 0$ to the form

$$\tau^q(a_0, \ldots, a_q) = \sum_{k} c_{q,k} \int_{D} \gamma_{a_0} \Sigma^{(k_1,2k_1+1)}(a_1) \cdots \Sigma^{(k_q,2(k_1+\ldots+k_q)+1)}(a_q).$$

This formula covers the case of either parity, provided that in the odd case we define $\gamma := F = D|D|^{-1}$, and incorporate the factor $\sqrt{2i}$ in the expression (2.8) of the coefficients $c_{q,k}$ for $q$ odd.

2.2. Ansatz for the twisted case. Assume now that $(\mathcal{A}, \mathfrak{H}, D, \sigma)$ is a $(p, \infty)$-summable (invertible) twisted $\sigma$-spectral triple. The twisted analogue of the usual bimodule of gauge potentials or noncommutative differential forms is obviously the linear subspace $\Omega^1_{D,\sigma}(\mathcal{A}) \subset \mathcal{L}(\mathfrak{H})$ consisting of operators of the form

$$A = \sum_{i} a_i (D b_i - \sigma(b_i) D), \quad a_i, b_i \in \mathcal{A},$$

which is a bimodule for the action

$$a \cdot \omega \cdot b = \sigma(a) \omega b, \quad \forall a, b \in \mathcal{A}, \quad \forall \omega \in \Omega^1_{D,\sigma}(\mathcal{A}).$$

In the presence of the Lipschitz regularity axiom (1.8), one can similarly define a bimodule $[\Omega^1_{D,\sigma}(\mathcal{A}) \subset \mathcal{L}(\mathfrak{H})]$, by simply replacing $D$ with $|D|$. Furthermore, as noted before cf. (1.21), the map

$$a \mapsto d_\sigma(a) = D a - \sigma(a) D$$

is a $\sigma$-derivation of $\mathcal{A}$ with values in $\Omega^1_{D,\sigma}(\mathcal{A})$, and clearly, so is the map

$$a \mapsto \delta_\sigma(a) = |D| a - \sigma(a) |D|.$$

However, in order for the analogue of the smoothness condition (1.15) to make sense, one needs to postulate the existence of an extension of the automorphism $\sigma \in \text{Aut}(\mathcal{A})$, and consequently of the $\sigma$-derivation $\delta_\sigma$, to a
larger subalgebra of $L(H)$, which should contain $\Omega_{D,\sigma}^1(A)$ as well as its higher $\delta_\sigma$-iterations.

Thus, the formulation of a twisted version for the pseudodifferential calculus is not canonical. Ignoring this aspect for now, let us pretend that an adequate analogue $\Psi(A, 5, D, \sigma)$ of the algebra of pseudodifferential operators has already been constructed, and assume that the twisted $\sigma$-spectral triple $(A, 5, D, \sigma)$ admits a simple discrete dimension spectrum $\Sigma \subset \mathbb{C}$. We can then focus on finding an appropriate candidate for the local character cocycle. Denoting

$$\alpha_\sigma^k(a) := D^k \sigma^{-k}(a) D^{-k}, \quad a \in A,$$

and

$$\Sigma_\sigma^{(k, \ell)}(a) := \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left( \alpha_\sigma^{\ell-2j}(a) - \alpha_\sigma^{\ell-2j-1}(a) \right),$$

the analogues of the summands in Eq. (2.12) are the ‘residue integrals’

$$\int_D \gamma a_0 \Sigma_\sigma^{(k_1, 2k_1+1)}(a_1) \ldots \Sigma_\sigma^{(k_q, 2(k_1+\ldots+k_q)+q)}(a_q).$$

We next define the twisted version of the higher commutators as follows:

$$\left(a_\sigma^{(k)} \right) := \sum_{j=0}^{k} (-1)^j \binom{k}{j} D^{2(k-j)} \sigma^{2j}(a) D^{2j},$$

respectively

$$\left[D, a_\sigma^{(k)} \right] = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left( D^{2(k-j)+1} \sigma^{2j}(a) D^{2j} - D^{2(k-j)} \sigma^{2j+1}(a) D^{2j+1} \right).$$

**Remark 2.1.** Noting that

$$\left(a_\sigma^{(k+1)} \right) = D^2 \left(a_\sigma^{(k)} \right) - \left(\sigma^2(a)\right)_\sigma^{(k)} D^2,$$

$$\left[D, a_\sigma^{(k+1)} \right] = D^2 \left[D, a_\sigma^{(k)} \right] - \left[D, \sigma^2(a)\right]_\sigma^{(k)} D^2,$$

one could be tempted to regard the expressions (2.16), (2.17) as genuine iterated twisted commutators with $D^2$. However, that would not be correct, because there is no guarantee that if, for instance, $\left[D, a_\sigma^{(k)} \right] = 0$ then $\left[D, a_\sigma^{(k)} \right] = 0$ for all $k \geq 1$. A counterexample can be easily obtained in the setting of §3.1, using Eq. (3.12).

At any rate, with the above notation, we can now put (2.14) in a form similar to Eq. (2.9). Indeed, the counterpart of the identity (2.11) is

$$D^{2k+1-\ell} \Sigma_\sigma^{(k, \ell)}(a) = \left[D, \sigma^{-\ell}(a)\right]_\sigma^{(k)} D^{-\ell},$$

which we then employ to reverse the process by which the expression (2.12) was obtained from (2.7). Keeping the same notational conventions used in
(2.12), one thus arrives at the following Ansatz for the twisted version of the local character cocycle:

\[(2.18) \quad \sum_{k} c_{q,k} \int_{D} \gamma a_{0} |D, \sigma^{-2k_{1}-1}(a_{1})|^{(k_{1})_{\sigma}} \cdots [D, \sigma^{-2(k_{1}+\ldots+k_{q})-q}(a_{q})]^{(k_{q})_{\sigma}} |D|^{-2|k|-q} \cdot \]

There is an immediate obstruction for this formula to define a \((b,B)\)-cocycle, which arises from the \(B\)-coboundary of \(\tau_{1}\) odd. Indeed,

\[B \tau_{1}(a) = \sum_{k \geq 0} c_{1,k} \int_{D} F [D, \sigma^{-2k-1}(a)]^{(k)} D^{-2k-1} = \sum_{k \geq 0} c_{1,k} \int_{D} F \Sigma_{\sigma}^{(k,2k+1)}(a) \]

\[= \sum_{k \geq 0} c_{1,k} \sum_{j=0}^{k} (-1)^{j} \binom{k}{j} \left( \int_{D} F a_{\sigma}^{2(k-j)+1}(a) - \int_{D} F a_{\sigma}^{2(k-j)}(a) \right) \]

\[= \sum_{k \geq 0} c_{1,k} \sum_{j=0}^{k} (-1)^{j} \binom{k}{j} \left( \int_{D} F \sigma^{-2(k-j)-1}(a) - \int_{D} F \sigma^{-2(k-j)}(a) \right). \]

This expression vanishes if

\[\text{(2.19)} \quad \int_{D} F \sigma(a) = \int_{D} F a, \quad \forall a \in A, \]

or equivalently

\[\int_{D} [D, a]_{\sigma} |D|^{-1} = 0, \quad \forall a \in A, \]

As it will become apparent in the next section, Eq. (2.19) has something in common with the Selberg principle for orbital integrals of reductive Lie groups. We shall show later that for a special class of conformally twisted spectral triples there are no higher obstructions to the validity of the Ansatz.

### 3. Conformal geometry and twisted spectral triples

In order to shed some light on the nature and plausibility of the above setup for the Ansatz, we examine in this section some authentic examples of twisted spectral triples arising in conformal geometry.

#### 3.1. Transversely conformal spectral triple.

Let \(M\) be a smooth connected closed spin manifold of dimension \(n\). To each riemannian metric \(g\) on \(M\) one can canonically associate a Dirac operator \(D = D_{g}\) acting on the Hilbert space \(\mathcal{H} = \mathcal{H}_{g} := L^{2}(M, S_{g})\) of \(L^{2}\)-sections of the spin bundle \(S = S_{g}\), and thus a corresponding spectral triple \((C^{\infty}(M), \mathcal{H}, D)\) over the algebra \(C^{\infty}(M)\). Assume now that \(M\) is endowed with a conformal structure \([g]\), consisting of all riemannian metrics conformally equivalent to a given riemannian metric \(g\). Let \(SCO(M, [g])\) be the group of diffeomorphisms of \(M\) that preserve the conformal structure, the orientation and the
spin structure. It is a Lie group, and we denote by $G = SCO(M, [g])_0$ its connected component of the identity. We then form the discrete crossed product algebra $A_G = C^\infty(M) \rtimes G$. This algebra consists of finite sums of the form

$$a = \sum_{\phi} f_\phi v_\phi, \quad f_\phi \in C^\infty(M), \quad \phi \in G,$$

with the product rule determined by

$$v_\phi f = (f \circ \phi^{-1}) v_\psi, \quad v_\phi v_\psi = v_{\phi \psi}.$$

It can be represented by bounded linear operators on the Hilbert space $H = L^2(M, S)$ of $L^2$-sections of the spin bundle $S$, by letting a function $f \in C^\infty(M)$ act as the multiplication operator

$$\pi(f)(u) = fu, \quad u \in L^2(M, S),$$

and the diffeomorphisms $\phi \in G$ act as translation operators

$$\pi(v_\phi)(u) \equiv V_\phi(u) := \tilde{\phi} \circ u \circ \phi^{-1}, \quad u \in L^2(M, S),$$

where $\tilde{\phi}$ is the canonical lift of $\phi$ to an automorphism of $S$; such a lift is well-defined, not just modulo $\mathbb{Z}/2\mathbb{Z}$, for any $\phi \in SCO(M, [g])_0$. To make $G$ act by unitary operators, one needs to replace each operator $V_\phi^{-1}$, $\phi \in G$, by the operator

$$U_\phi^{-1}(u) = e^{-nh_\phi} V_\phi^{-1}(u) = e^{-nh_\phi} \tilde{\phi}^{-1} \circ u \circ \phi, \quad u \in L^2(M, S),$$

where $h_\phi \in C^\infty(M)$ is determined by the conformal factor via the equation

$$\phi^*(g) = e^{-4h_\phi} g.$$

Indeed, using the fact that the riemannian volume forms are related by the equality

$$\text{vol}_{\phi^*(g)} = e^{-2nh_\phi} \text{vol}_g,$$

and denoting the fiberwise norm by $|\cdot|$, one easily checks that $U_\phi^{-1}$ is unitary:

$$||U_\phi^{-1}(u)||^2 = \int_M e^{-2nh_\phi} |\tilde{\phi}^{-1}(u \circ \phi)|^2 \text{vol}_g = \int_M |\tilde{\phi}^{-1}(u \circ \phi)|^2 \phi^*(\text{vol}_g) = \int_M |u|^2 \text{vol}_g = ||u||^2, \quad \forall u \in L^2(M, S).$$

**Lemma 3.1.** For any $\phi \in G = SCO(M, [g])_0$ and with $D = \mathcal{D}_g$, one has

$$U_\phi^* \circ D \circ U_\phi = e^{h_\phi} \circ D \circ e^{h_\phi}.$$

**Proof.** Via the natural identification $\beta_\phi^*(g)$ corresponding to the change of metric, defined as in [2], the Dirac operator $\mathcal{D}_\phi^*(g)$ can be implemented as an operator

$$D_{\phi^*(g), g} = \left(\beta_\phi^*(g)^{-1} \circ \mathcal{D}_{\phi^*(g)} \circ \beta_\phi^*(g)\right).$$
acting on the sections of the bundle $S_g$. It is explicitly given by the formula
\begin{equation}
D_{\phi^*(g),g} = e^{(n+1)h} \mathcal{D}_g e^{(-n+1)h}.
\end{equation}
On the other hand, as differential operator,
\begin{equation}
D_{\phi^*(g),g} = \mathcal{V}_g^{-1} \circ \mathcal{D}_g \circ \mathcal{V}_g.
\end{equation}
Combining (3.7) and (3.8) one obtains
\begin{equation}
\mathcal{V}_g^{-1} \circ \mathcal{D}_g \circ \mathcal{V}_g = e^{(n+1)h} \mathcal{D}_g e^{(-n+1)h},
\end{equation}
or equivalently
\begin{equation}
e^{-nh} \circ \mathcal{V}_g^{-1} \circ \mathcal{D}_g \circ \mathcal{V}_g \circ e^{nh} = e^h \circ \mathcal{D}_g \circ e^h.
\end{equation}

Let $\sigma$ be the algebra automorphism of $A_G$ defined on generators by
\begin{equation}
\sigma(f \psi^{-1}) = e^{-2h} f \psi^{-1}, \quad f \in C^\infty(M), \quad \psi \in G.
\end{equation}

**Lemma 3.2.** The twisted commutators
\[ [D, \pi(a)]_\sigma := D \circ \pi(a) - \pi(\sigma(a)) \circ D, \quad a \in A_G, \]
are bounded.

**Proof.** It suffices to check the claimed property for $a = e^{-nh} \psi^{-1}$. In that case one has
\[ [D, \pi(a)]_\sigma = D \circ U_{\phi}^* - e^{-2h} \circ U_{\phi}^* \circ D = \left( D - e^{-2h} \circ U_{\phi}^* \circ D \circ U_{\phi} \right) \circ U_{\phi}^*. \]
In view of Eq. (3.6), it follows that
\begin{equation}
[D, U_{\phi}^*]_\sigma = \left( D - e^{-h} \circ D \circ e^h \right) U_{\phi}^* = -e^{-h} [D, e^h] U_{\phi}^* = -c(dh) U_{\phi}^*.
\end{equation}
For further reference, we note that as a consequence of (3.10) one has
\begin{equation}
[D, f U_{\phi}^*]_\sigma = c(df - f dh) U_{\phi}^*,
\end{equation}
and in particular
\begin{equation}
[D, e^h U_{\phi}^*]_\sigma = 0.
\end{equation}

**Proposition 3.3.** The algebra $A_G = C^\infty(M) \times G$, endowed with the automorphism $\sigma \in \text{Aut} A_G$ and the representation $\pi$ on the Hilbert space $\mathcal{F} = L^2(M, S)$, together with the Dirac operator $D = \mathcal{D}_g$, define an $(n, \infty)$-summable $\sigma$-spectral triple $(A_G, \mathcal{F}, D, \sigma)$, which moreover satisfies the strong Lipschitz-regularity property
\begin{equation}
|D|^{-t} \left( |D|^t a - \sigma^t(a) |D|^t \right) \in \mathcal{L}^{(n, \infty)}(\mathcal{F}), \quad \forall t \in \mathbb{R}.
\end{equation}
Lipschitz regularity, one notes that, when viewed as pseudodifferential operator, \( U_\phi^{*} \circ D \circ U_\phi \) has principal symbol

\[
\sigma(U_\phi^{*} \circ D \circ U_\phi)(x, \xi) = e^{2t_\phi c(\xi)}, \quad \xi \in T^*_x M
\]

where \( c(\xi) \in \text{End}(S_x) \) stands for the Clifford multiplication by \( \xi \). Furthermore, for any \( t \in \mathbb{R} \), the principal symbol of \( U_\phi^{*} \circ |D|^t \circ U_\phi \) is

\[
\sigma(U_\phi^{*} \circ |D|^t \circ U_\phi)(x, \xi) = e^{2t_\phi ||\xi||^t},
\]

since

\[
U_\phi^{*} \circ |D|^t \circ U_\phi = |U_\phi^{*} \circ D \circ U_\phi|^t.
\]

Now

\[
|D|^t \circ U_\phi^{*} - \sigma(U_\phi^{*}) \circ |D|^t = \left(|D|^t - e^{-2t_\phi} U_\phi^{*} \circ |D|^t \circ U_\phi\right) \circ U_\phi^{*},
\]

and by Eq. (3.15),

\[
\sigma_{pr} \left(|D|^t - e^{-2t_\phi} U_\phi^{*} \circ |D|^t \circ U_\phi\right) = ||\xi||^t - e^{-2t_\phi} e^{2t_\phi} ||\xi||^t = 0.
\]

Thus, the operator \( |D|^t - e^{-2t_\phi} U_\phi^{*} \circ |D|^t \circ U_\phi \) is pseudodifferential of order \( t - 1 \), hence its product by \( |D|^{-t} \) is of order \(-1\) and therefore in \( \mathcal{L}^{(n, \infty)} \). \( \square \)

**Remark 3.4.** In the same fashion, \( \mathbb{R}^n \) with its standard metric \( g_0 \), together with the flat Dirac operator \( D_0 = D_{\mathbb{R}^n} \), give rise to the \((n, \infty)\)-summable non-unital \( \sigma \)-spectral triple \((\mathcal{A}_G, \mathfrak{F}_0, D_0, \sigma)\), where \( G_0 = CO(\mathbb{R}^n, g_0) \), and \( \mathcal{A}_G = C^\infty_c(\mathbb{R}^n) \rtimes G_0 \).

According to the Ferrand-Obata theorem (cf. [12] for a complete proof), the conformal group \( CO(M, [g]) \) of a (not necessarily closed) manifold \( M \) of dimension \( n \geq 2 \) is inessential, i.e. reduces to the group of isometries for a metric in the conformal class \([g]\), except when \( M^n \) is conformally equivalent to the standard sphere \( S^n \) or to the standard Euclidean space \( \mathbb{R}^n \). Correspondingly, the only twisted spectral triples arising from the above construction which are not isomorphic to ordinary spectral triples are those associated to the \( n \)-sphere and to the flat \( n \)-space.

### 3.2. Transverse noncommutative residue.

With the same assumptions as in the preceding subsection, let \( \Psi^*(M; S) \) denote the algebra of classical pseudodifferential operators acting on the sections of the spin bundle. The group \( G \) acts on \( \Psi^*(M; S) \) in the natural fashion:

\[
\phi \cdot P := V_\phi P V_\phi^{-1}, \quad \phi \in G, \quad P \in \Psi^*(M; S).
\]

One can thus form the crossed product algebra \( \Psi^*(M; S) \rtimes G \). The representation \( \pi : \mathcal{A}_G \to \mathcal{L}(\mathfrak{F}) \) extends in a tautological manner to a representation
of the enlarged algebra $\Psi^\bullet(M;\mathcal{S}) \rtimes G$ by densely defined linear operators on $\mathcal{H} = L^2(M,\mathcal{S})$, which will still be denoted by $\pi$:

$$\pi(P v_\phi)(u) := P(V_\phi(u)) = P(\tilde{\phi} \circ u \circ \phi^{-1}), \quad u \in C^\infty(M,\mathcal{S}).$$

We set out to show that given any $\mathcal{P} \in \Psi^N(\mathcal{A}_G)$ the zeta function

$$\zeta_P(z) := \text{Tr}(\mathcal{P} |D|^{-z})^\times, \quad \Re z > n + N$$

can be meromorphically continued to the whole complex plane; by linearity, it suffices to take $\mathcal{P} = PV_\phi$, with $P \in \Psi^N(M;\mathcal{S})$ and $\phi \in G$.

If $G$ is inessential, $\phi$ is an isometry and the statement can be proved via the Mellin transform and heat kernel asymptotics (see [17, §6.3]). A more direct proof, given in [11, Thm. 7.7.5], relies on the stationary phase method (cf. e.g. [17, Thm. 7.7.1]) and applied to a phase function whose expression in local charts $U$ covering a tubular neighborhood of the fixed point set $M_\phi$ is of the form

$$f(x,\xi) = \langle x - \phi(x),\xi \rangle, \quad x \in U, \quad \xi \in \mathbb{R}^n, ||\xi|| = 1.$$

By restriction to the fibers of the normal bundle to $M_\phi$, this function gives rise to a family of fiberwise phase functions, each having a single non-degenerate stationary point. Using the stationary phase for this family, it is shown in [11, Prop. 2.4] that the zeta function $\zeta_{V_\phi}P$ has a meromorphic extension to $\mathbb{C}$ whose poles are at most simple and located at the points $z_k = N + n_\phi - k, \quad k \in \mathbb{Z}^+$, where $n_\phi = \dim M_\phi$.

In the sphere case, by Liouville’s theorem the group of conformal automorphisms $CO(S^n,[g])$ coincides with the group $M(n) \cong PO(n + 1,1)$ of Möbius transformations in dimension $n$, and $G = SCO(S^n,[g])$ is its connected component. Now if $\phi \in G$ is elliptic, i.e. conjugate to an element in the maximal compact subgroup $O(n + 1)$, by replacing $D$ in formula (3.18) with a conjugate by a unitary operator we can reduce to the isometric case. The non-elliptic diffeomorphisms $\phi \in M(n)$ fall into two classes: hyperbolic and parabolic (see [19, §2]).

A hyperbolic transformation $\phi \in M(n)$ has two distinct fixed points, say $x^+$ and $x^-$, and its tangent map at each of these points $d\phi_{x^\pm}: T_{x^\pm}S^n \to T_{x^\pm}S^n$ is represented by an element of $O(n) \times \mathbb{R}^+$, with multiplier $\mu^\pm$, with $\mu > 1$. Because of the nontrivial multiplier, the phase function (3.19) has no critical points away from the zero section of the cotangent bundle $T^*S^n$. The stationary phase principle, in its most basic form (cf. [17, Thm. 7.7.1]) and utilized in the same manner as in [11], implies then that the zeta function $\zeta_{V_\phi}P$ extends to an entire function.

A parabolic transformation $\phi \in M(n)$ has a single fixed point $x_0 \in S^n$, however $\det(\text{Id} - d\phi_{x_0}) = 0$. Accordingly, the phase function has 0 as its only critical value, and the corresponding critical set is

$$C_\phi = \{(x_0,\xi) | \xi \in \mathbb{R}^k, \quad d\phi_{x_0}(\xi) = \xi\}.$$
The generalized stationary phase gives an asymptotic expansion
(3.21)
\[
\int_{||\xi||=1} e^{ir(x-\phi(x),\xi)} a(x,\xi) d^{n-1} \xi \sim \sum_{\alpha} \sum_{j=0}^{\infty} \sum_{k=0}^{2n-1} \delta_{j,k}(a) r^{\alpha-j} \log^k r,
\]
where \(\alpha = \frac{1}{2} \dim C_\phi - n\) and the distributions \(\delta_{j,k}\) are supported in \(C_\phi\).

Following the same line of arguments as in [11], but using stereographic coordinates instead of normal coordinates, one obtains the desired meromorphic continuation of the zeta function \(\zeta_{V_\phi} P\), with (at most) simple poles at the points
\[
z_k = N + \frac{1}{2} \dim C_\phi - k, \quad \text{where} \quad k \in \mathbb{Z}^+.
\]

We summarize the conclusion of the preceding discussion as follows.

**Theorem 3.5.** For any \(P \in \Psi^N(M^n; S)\) and any \(\phi \in G\), the associated zeta function \(\zeta_{P, V_\phi}\) has a meromorphic extension to \(\mathbb{C}\). Moreover,

1. if \(\phi \in G\) is elliptic, then the poles of \(\zeta_{V_\phi} P\) are at most simple and are located at the points \(z_k = N + \dim M_\phi - k, \quad k \in \mathbb{Z}^+\);
2. if \(\phi \in G\) is hyperbolic, then \(\zeta_{V_\phi} P\) is entire;
3. if \(\phi \in G\) is parabolic, then the poles of \(\zeta_{V_\phi} P\) are at most simple and are located at the points \(z_k = N + \frac{1}{2} \dim C_\phi - k, \quad k \in \mathbb{Z}^+\).

This provides the transverse noncommutative residue functional
\[
\int_D P = \text{Res}_{z=0} \zeta_P(2z), \quad P \in \Psi^\bullet(M; S) \rtimes G,
\]
which satisfies a property analogous to the Selberg Principle.

**Corollary 3.6.** For any hyperbolic transformation \(\phi \in G\) and any \(P \in \Psi^N(S^n; S)\),
\[
\int_D P V_\phi = 0.
\]

As another consequence, one can explicitly compute the candidate for the Hochschild character given by Eq. (1.20), and thus directly verify that it gives the expected result.

**Proposition 3.7.** The local Hochschild cocycle of the transversely conformal \(\sigma\)-spectral triple \((\mathcal{A}_G, \mathcal{S}_g, \mathcal{D}_g, \sigma)\) associated to a closed spin manifold \(M^n\) modulo the conformal group \(G = \text{SCO}(M, [g])\) is a cyclic cocycle whose periodic cyclic cohomology class coincides with the transverse fundamental class \([M/G]\).
Proof. Let $a^k = f_k U_{\phi_k}^* \in A_G$, $k = 0, 1, \ldots, n$. The integrand in the formula (1.20),
$$a^0 [D, \sigma^{-1}(a^1)]_\sigma \cdots [D, \sigma^{-n}(a^n)]_\sigma |D|^{-n},$$
can be put in the form $PV_\phi$ with $P \in \Psi^{-n}(S^n; S)$ and $\phi^{-1} = \phi_n \circ \cdots \circ \phi_0$. It follows from Prop. 3.5, specialized to the case when $N = -n$, that the zeta function $\zeta_{PV_\phi}(z)$ has no pole at $z = 0$. Therefore
$$\int_D PV_\phi = 0, \quad \text{unless } \phi = \text{Id},$$
i.e. the cocycle (1.20) is localized at the identity. Employing Getzler’s symbol calculus for asymptotic operators as indicated in [8, Remark II.1], and using the expression (3.11) of the twisted commutators, the Hochschild cocycle (1.20) can be explicitly computed. The end result is a cyclic cocycle, which is easily seen to differ by a coboundary from the standard transverse fundamental cocycle (comp. [10, Thm 3.11])
$$\tau_{M/G}(f_0 U_{\phi_0}^*, \ldots, f_n U_{\phi_n}^*) = \begin{cases} \int_M f_0 d(f_1 \circ \phi_0) \wedge \cdots \wedge d(f_n \circ \phi_{n-1} \circ \cdots \circ \phi_0), & \text{if } \phi_n \circ \cdots \circ \phi_0 = \text{Id}; \\
0, & \text{otherwise}. \end{cases}$$

Remark 3.8. If $M$ is closed and $G$ inessential, hence compact, the corresponding $\sigma$-spectral triple is a conformal perturbation, cf. (1.6) of an equivariant spectral triple. Its Connes-Chern character, given by (1.14), can be explicitly computed from the local index formula of [8] by employing an equivariant version of the Getzler symbol calculus, or the equivariant heat kernel techniques in [1].

3.3. Transverse similarities. Endow $\mathbb{R}^n$ with the Euclidean metric $g_0 = \sum_{i=1}^n dx^i \otimes dx^i$. The group $G = \text{Sim}(n)$ of conformal (or similarity) transformations of the Euclidean $n$-space is generated by rotations, translations by vectors $y \in \mathbb{R}^n$,
$$\tau_y(x) = x - y, \quad \forall x \in \mathbb{R}^n,$$
and homotheties $\rho_\lambda, \lambda > 0$,
$$\rho_\lambda(x) = \lambda^{-1}x, \quad \forall x \in \mathbb{R}^n.$$
The only non-isometries are the homotheties,
$$\rho_\lambda^* g_0 = \lambda^{-2} g_0,$$
i.e. in the notation of (3.1) the corresponding conformal factor is
$$e^{-4b_{\rho_\lambda}}(x) = \lambda^{-2}, \quad \forall x \in \mathbb{R}^n.$$
With $\mathcal{A}_G := C_c^\infty(\mathbb{R}^n) \rtimes G$, the definition \([3.3]\) of the automorphism $\sigma \in \text{Aut}\mathcal{A}_G$ specializes to
\[
\sigma(f U_\phi) = \mu(\phi) f U_\phi, \quad \phi \in G,
\]
where $\mu : G \to \mathbb{R}^+$ is the character determined by the multiplier of the similarity transformation:
\[
\mu(\phi) = 1 \quad \text{if} \quad \phi \in O(n), \quad \mu(\tau_y) = 1, \forall y \in \mathbb{R}^n, \quad \text{and} \quad \mu(\rho_\lambda) = \lambda, \forall \lambda > 0.
\]
Also, the covariance relation \((3.6)\) becomes
\[
U_\phi^{-1} \circ D \circ U_\phi = \mu(\phi) D, \quad \phi \in G.
\]

Let $\Psi_c^* (\mathbb{R}^n; S)$ denote the algebra of classical pseudodifferential operators with $x$-compact support. Since the conformal factors are constant, one can easily extend $\sigma$ to an automorphism of $\Psi_c^* (\mathbb{R}^n; S) \rtimes G$, by simply setting
\[
\sigma(P U_\phi) = \mu(\phi) P U_\phi, \quad \phi \in G, \quad P \in \Psi_c^* (\mathbb{R}^n, S).
\]
Indeed,
\[
(3.25) \quad \sigma(P U_\phi) \cdot \sigma(Q U_\psi) = (\mu(\phi) P U_\phi) \cdot (\mu(\psi) Q U_\psi) = \mu(\phi \psi) P \cdot (U_\phi QU_\psi^{-1}) U_{\phi \psi} = \sigma(P \cdot (U_\phi QU_\psi^{-1}) U_{\phi \psi}) = \sigma(P U_\phi \cdot Q U_\psi).
\]

**Proposition 3.9.** The residue functional $\int_D : \Psi_c^* (\mathbb{R}^n; S) \rtimes G \to \mathbb{C}$ is a $\sigma$-invariant trace.

**Proof.** The $\sigma$-invariance of the residue is a consequence of Corollary \([3.6]\) (Selberg Principle). Indeed, let $P = PU_\phi \in \Psi_c^* (\mathbb{R}^n; S) \rtimes G$. If $\mu(\phi) \neq 1$ then $\phi$ has a unique fixed point $x_0 \in \mathbb{R}^n$, at which $d_\phi x_0 = \mu(\phi) \text{Id}$. Thus, there are no fixed points on $T^* \mathbb{R}^n$, hence the zeta function $\zeta_P(z)$ is entire (see \([3.2]\)). On the other hand, if $\mu(\phi) = 1$ then $\sigma(P) = P$.

To prove that the residue functional is a trace, let $PU_\phi$, $QU_\psi \in \Psi_c^* (\mathbb{R}^n; S) \rtimes G$. One has
\[
\text{Tr}(PU_\phi QU_\psi |D|^{-z}) = \text{Tr}(QU_\psi |D|^{-z} PU_\phi) = \text{Tr}(QU_\psi P |D|^{-z} U_\phi) + \text{Tr}(QU_\psi |D|^{-z} \rho_\lambda U_\phi)
\]
Using the identity Eq. \([2.2]\) to express the commutator $[|D|^{-z}, P]$, one sees that
\[
\text{Res}_{z=0} \text{Tr}(QU_\psi |D|^{-z}, P] U_\phi) = 0,
\]
hence,
\[
\int_D PU_\phi QU_\psi = \text{Res}_{z=0} \text{Tr}(QU_\psi P |D|^{-z} U_\phi) = \text{Res}_{z=0} \text{Tr}(QU_\psi PU_\phi U_\phi^{-1} |D|^{-z} U_\phi).
\]
By Eq. \(3.23\), \[ U_\phi^{-1}|D|^{-z}U_\phi = \mu(\phi)^{-z}|D|^{-z}, \]
therefore
\[
\int_D PU_\phi QU_\phi = \text{Res}_{z=0} (\mu(\phi)^{-z} \text{Tr}(QU_\phi PU_\phi |D|^{-z})) = \int_D QU_\phi PU_\phi.
\]

**Remark 3.10.** One can explicitly verify in this specific case that the cochain \(2.18\) of the Ansatz does satisfy the cocycle identity
\[
b \tau_\sigma^{q-1} + B \tau_\sigma^{q+1} = 0.
\]
Indeed the direct, albeit lengthy, computations by which the cocycle identity is checked in the beginning of the proof in [8, Theorem II.1] can be reproduced almost verbatim. Once the commutator with \(D^2\) is substituted by the twisted commutator
\[
(3.26) \quad \nabla_\sigma(\mathcal{P}) = D^2 \mathcal{P} - \sigma^2(\mathcal{P}) D^2,
\]
and usual iterated Leibniz rule is replaced with its twisted version
\[
(3.27) \quad \nabla_\sigma^m(\mathcal{P}_1 \mathcal{P}_2 \cdots \mathcal{P}_q) = \sum_{m_1 + \cdots + m_q = m} \frac{m!}{m_1! \cdots m_q!} \nabla_\sigma^{m_1}(\sigma^{2(m_2 + \cdots + m_q)}(\mathcal{P}_1)) \nabla_\sigma^{m_2}(\sigma^{2(m_3 + \cdots + m_q)}(\mathcal{P}_2)) \cdots \nabla_\sigma^{m_q}(\mathcal{P}_q),
\]
the “integration by parts” property, which is repeatedly used in those calculations, becomes a consequence of Proposition [3.9]. As a simple illustration,
\[
\int_D \nabla_\sigma(\mathcal{P}) D^{-2} = \int_D D^2 \mathcal{P} D^{-2} - \int_D \sigma^2(\mathcal{P}) = \int_D \mathcal{P} - \int_D \sigma^2(\mathcal{P}) = 0.
\]
A different approach, which provides a complete verification of the Ansatz in greater generality, makes the object of the section that follows.

## 4. Twisting by scaling automorphisms and the local index formula

By abstracting the essential features of the preceding example, we define in this section a general class of spectral triples twisted by scaling automorphisms, for which we shall prove the validity of the Ansatz in its entirety.

### 4.1. Scaling automorphisms

Motivated by the Euclidean similarities in \(3.3\) we introduce the following abstract version of a spectral triple twisted by similarities.

**Definition 4.1.** Let \((\mathcal{A}, \mathfrak{H}, D)\) be a spectral triple over the (non-unital) involutive algebra \(\mathcal{A}\). A scaling automorphism of \((\mathcal{A}, \mathfrak{H}, D)\) is defined by a unitary operator \(U \in \mathcal{U}(\mathfrak{H})\) such that
\[
(4.1) \quad U \mathcal{A} U^* = \mathcal{A}, \quad \text{and} \quad U D U^* = \mu(U) D, \quad \text{with} \quad \mu(U) > 0.
\]
Scaling automorphisms form a group $\text{Sim}(A, H, D)$, endowed by construction with a scaling character $\mu : \text{Sim}(A, H, D) \to \mathbb{R}^+$. Its subgroup $\text{Ker}\mu$ consists of the isometries of $(A, H, D)$, and we will be denoted $\text{Isom}(A, H, D)$.

For the clarity of the exposition it will be convenient to assume $D$ invertible. This can always be achieved by passage to the invertible double, cf. §1.3, $\tilde{D} = D \otimes \text{Id} + \text{Id} \otimes F_1$.

However, in doing so the similarity condition (4.1) cannot be exactly reproduced. Instead, it takes the modified form

$$(4.2) \quad U \tilde{D} U^* = \mu(U) \tilde{D} \otimes \text{Id} + (1 - \mu(U)) \text{Id} \otimes F_1.$$ 

We will explain at the end of the paper the minor modifications need to handle the perturbed similarity condition.

In the remainder of the paper we fix a group of scaling automorphisms $G \subset \text{Sim}(A, H, D)$, and let $A_G = A \rtimes G$. We shall also denote by $G_0$ the subgroup of isometries in $G$.

**Proposition 4.2.** The formula

$$(4.3) \quad \sigma(a U) = \mu(U)^{-1} a U, \quad \forall U \in G, a \in A,$$

defines an automorphism $\sigma : A_G \to A_G$, and $(A_G, \tilde{H}, D, \sigma)$ is a $\sigma$-spectral triple. Moreover,

$$(4.4) \quad [D, a U]_{\sigma} = [D, a] U.$$ 

**Proof.** Indeed, one has for any monomials $a U, b V \in A_G = A \rtimes G$,

$$\sigma(a U) \sigma(b V) = (\mu(U)^{-1} a U) (\mu(V)^{-1} b V) = \mu(UV)^{-1} a (U b U^*) U V = \sigma(a U(b)) U V = \sigma(a U b V).$$

Furthermore, in view of (4.1),

$$[D, a U]_{\sigma} = [D, a] U + a \left( D - \mu(U)^{-1} U D U^* \right) U = [D, a] U \in \mathcal{L}(\tilde{H}).$$ 

The resulting $\sigma$-spectral triple $(A_G, \tilde{H}, D, \sigma)$ will be called twisted by scaling automorphisms. With the goal of establishing the validity of the Ansatz for twisted spectral triples of this form, we start with the assumption that the base spectral triple $(A, \tilde{H}, D)$ is $(p, \infty)$-summable, and satisfies the smoothness condition (1.15). We recall that the corresponding algebra of pseudo-differential operators $\Psi^*(A, \tilde{H}, D)$ is $\mathbb{Z}$-filtered (by the order) and $\mathbb{Z}_2$-graded (even/odd), and that it includes the subalgebra of differential operators $D(A, \tilde{H}, D)$.

**Proposition 4.3.** With the above notation and hypotheses,
(1) the action of $G$ extends to an action by automorphisms on $\Psi(A, H, D)$,
(4.5) $P \mapsto U \triangleright P := UPU^*$, $\forall P \in \Psi(A, H, D)$, $U \in G$,
which respects both the order filtration and the even/odd grading;
(2) the automorphism $\sigma \in \text{Aut}(A_G)$ extends to an automorphism $\sigma$ of the crossed product algebra $\Psi(A_G, H, D) := \Psi(A, H, D) \ltimes G$, by setting
(4.6) $\sigma(PU) = \mu(U)^{-1}PU$, $\forall U \in G$, $P \in \Psi(A, H, D)$,
(3) the twisted commutators by $D$, $|D|$ and $D^2$ define twisted derivations $d_\sigma$, $\delta_\sigma$, resp. $\nabla_\sigma$, of the algebra $\Psi(A_G, H, D)$.

Proof. The condition (4.1) ensures that $D(\Psi(A, H, D))$ remains invariant under conjugation by $U \in G$, and also implies that
(4.7) $U |D|^z U^* = \mu(U)^z |D|^z$, $\forall z \in \mathbb{C}$.

The verification of the other claims is straightforward. $\square$

We now add the extended simple dimension spectrum hypothesis: there exists a discrete set $\Sigma_G \subset \mathbb{C}$, such that the holomorphic functions
(4.8) $\zeta_B(z) = \text{Tr}(B |D|^{-z})$, $\Re z > p$, $\forall B \in B_G := B \rtimes G$,
admit meromorphic extensions to $\mathbb{C}$ with simple poles in $\Sigma_G$, and the functions $\Gamma(z) \zeta_B(z)$ decay rapidly on finite vertical strips.

The proof of Proposition 3.9 applies verbatim and shows that residue functional
$$\int_D \mathcal{P} := \text{Res}_{z=0} \zeta_B(2z), \quad \mathcal{P} \in \Psi(A_G, H, D)$$
is automatically a trace. We require it to be $\sigma$-invariant:
(4.9) $\int_D \sigma(\mathcal{P}) = \int_D \mathcal{P}$.

This axiom de facto enforces the Selberg Principle, since it implies
(4.10) $\int_D P U = 0$, if $\mu(U) \neq 1$, $P \in \Psi(A, H, D)$, $U \in G$;
in particular, the residue functional is necessarily supported on $\Psi(A_{G_0}, H, D)$, where $G_0 := \text{Isom}(A, H, D)$.

4.2. Twisted JLO brackets. We define the twisted JLO bracket of order $q$ as the $q + 1$-linear form on $\Psi(A_G, H, D)$ which for $\alpha_0, \ldots, \alpha_q \in \Psi(A, H, D)$ and $U_0, \ldots, U_q \in G$ has the expression
(4.11) $\langle \alpha_0 U_0^*, \ldots, \alpha_q U_q^* \rangle_D = \int_{\Delta_q} \text{Tr}(\gamma \alpha_0 U_0^* e^{-s_0 \mu(U_0) D^2} \alpha_1 U_1^* e^{-s_1 \mu(U_0 U_1) D^2} \ldots \alpha_q U_q^* e^{-s_q \mu(U_0 \cdots U_q) D^2})$, $\mu(U) \neq 1$,
where the integration is over the $q$-simplex
\[
\Delta_q := \{ s = (s_0, \ldots, s_q) \in \mathbb{R}^{q+1} \mid s_j \geq 0, \quad s_0 + \ldots + s_q = 1 \}.
\]

Throughout the rest of this subsection, we shall assume that $\alpha_0, \ldots, \alpha_q$ are polynomial expressions in $D$ and the elements of $\mathcal{A}$, $[D, \mathcal{A}]$, and are homogeneous in $\lambda$ as when $D$ is replaced by $\lambda D$. Given a JLO bracket as in (4.11), for any $\varepsilon > 0$ we denote by $\langle \alpha_0 U_0^*, \ldots, \alpha_q U_q^* \rangle_D(\varepsilon)$ the expression obtained by replacing every $D$ occurring in each $\alpha_0, \ldots, \alpha_q$ by $\varepsilon^{1/2} D$. Equivalently,
\[
\langle \alpha_0 U_0^*, \ldots, \alpha_q U_q^* \rangle_D(\varepsilon) = \varepsilon^{m/2} \langle \alpha_0 U_0^*, \ldots, \alpha_q U_q^* \rangle_{\varepsilon^{1/2} D},
\]
where $m$ is the total degree in $\lambda$ after replacing every $D$ by $\lambda D$ in the product $\alpha_0 \cdots \alpha_q$. As before, by $U_0, \ldots, U_q$ we denote arbitrary elements in $G$.

Proposition 4.4. Let $\alpha_0 \in \mathcal{A}$, and $\alpha_1, \ldots, \alpha_q \in [D, \mathcal{A}]$. There is an asymptotic expansion of the form
\[
\langle \alpha_0 U_0^*, \ldots, \alpha_q U_q^* \rangle_D(\varepsilon) \sim_{\varepsilon \downarrow 0} \sum_{j \in J} \left( c_j + c_j' \log \varepsilon \right) \varepsilon^{-\rho_j} + O(1),
\]
with $\rho_0, \ldots, \rho_m$ a finite set of points in the half-plane $\Re z \geq \frac{q}{2}$.

Proof. Moving all the unitaries $U_i$ to the rightmost position,
\[
\begin{align*}
\text{Tr} & \left( \gamma \alpha_0 U_0^* e^{-s_0 \mu (U_0)^2 D^2} U_0 (U_0^* U_0^*) \alpha_1 (U_1 U_0) U_0^* U_1^* e^{-s_1 \mu (U_0 U_1)^2 D^2} (U_1 U_0) \cdots \right. \\
& \quad \left. (U_0^* \cdots U_{q-1}^*) \alpha_q (U_{q-1} \cdots U_0) (U_0^* \cdots U_q^*) e^{-s_q \mu (U_0 \cdots U_q)^2 D^2} (U_q \cdots U_0) U_0^* \cdots U_q^* \right),
\end{align*}
\]
the twisted JLO bracket relative to $\varepsilon^{1/2} D$ takes the form
\[
\langle \alpha_0 U_0^*, \ldots, \alpha_q U_q^* \rangle_{\varepsilon^{1/2} D} = \\
\int_{\Delta_q} \text{Tr} \left( \gamma \alpha_0 e^{-s_0 \varepsilon D^2} \alpha_1' e^{-(s_2 - s_1) \varepsilon D^2} \cdots \alpha_q' e^{-(s_q - s_{q-1}) \varepsilon D^2} U_0^* \cdots U_q^* \right),
\]
where $\alpha_1' = U_0^* U_1^* \triangleright \alpha_1, \ldots, \alpha_q' = U_0^* \cdots U_{q-1}^* \triangleright \alpha_q$.

We next use the expansion
\[
e^{-\varepsilon D^2} \alpha \sim_{\varepsilon \downarrow 0} \sum_{n=0}^{\infty} \frac{(-1)^n \varepsilon^n}{n!} \nabla^n (\alpha) e^{-\varepsilon D^2}, \quad \alpha \in D(\mathcal{A}, \mathfrak{g}, D),
\]
which is the heat operator analogue of the expansion (2.3) (cf. also (4.21) infra, for its twisted version), to move the heat operators in Eq. (4.14) to
the right and bring them all in the last position. One obtains

\begin{equation}
(4.16) \quad \langle \alpha_0 U_0^*, \ldots, \alpha_q U_q^* \rangle_{\varepsilon^1/2 D} \sim_{\varepsilon \downarrow 0} \frac{1}{q!} \sum_{N \geq 0} \sum_{n_1 + \ldots + n_q = N} (-1)^N \varepsilon^N \cdot \\
\int_{0 \leq s_1 \leq \ldots \leq s_q \leq 1} \mathcal{I}_{n_1}^{s_1} \ldots \mathcal{I}_{n_q}^{s_q} \cdot \text{Tr}(\gamma \alpha_0 \nabla^{n_1}(\alpha'_1) \ldots \nabla^{n_q}(\alpha'_q)e^{-\varepsilon D^2} U_0^* \ldots U_q^*)
\end{equation}

\begin{equation}
= \frac{1}{q!} \sum_{N \geq 0} \sum_{n_1 + \ldots + n_q = N} (-1)^N \varepsilon^N \cdot \\
\cdot \text{Tr}(\gamma \alpha_0 \nabla^{n_1}(\alpha'_1) \ldots \nabla^{n_q}(\alpha'_q)e^{-\varepsilon D^2} U_0^* \ldots U_q^*).
\end{equation}

The \textit{extended simple dimension spectrum} hypothesis ensures that the zeta functions

\[ \zeta_N(z) = \text{Tr}(\gamma U_0^* \ldots U_q^* \alpha_0 \nabla^{n_1}(\alpha'_1) \ldots \nabla^{n_q}(\alpha'_q)|D|^{-2\varepsilon-2N}), \quad \Re z > \frac{P}{2}, \]

have meromorphic continuation with simple poles. One has

\[ \zeta_N(z) = \frac{1}{\Gamma(z + N)} \int_0^\infty t^{z+N-1} \text{Tr}(\gamma U_0^* \ldots U_q^* \alpha_0 \nabla^{n_1}(\alpha'_1) \ldots \nabla^{n_q}(\alpha'_q)e^{-tD^2}) dt. \]

Proceeding as in the proof of [8, Theorem II.1], one establishes by means of the inverse Mellin transform the existence of an asymptotic expansion

\begin{equation}
(4.17) \quad \varepsilon^{N+\frac{q}{2}} \text{Tr}(\gamma U_0^* \ldots U_q^* \alpha_0 \nabla^{n_1}(\alpha'_1) \ldots \nabla^{n_q}(\alpha'_q)e^{-\varepsilon D^2}) \sim_{\varepsilon \downarrow 0} \\
\sum_j (c_{N,j} + c'_{N,j} \log \varepsilon) \varepsilon^{\frac{q}{2} - \rho_{N,j}} + O(1),
\end{equation}

where the exponents \( \rho_{N,j} \) are the real parts of the poles of \( \zeta_N(z) \) in the half-plane \( \Re z > \frac{q}{2} \).

**Definition 4.5.** We define the constant term \( \langle \alpha_0 U_0^*, \ldots, \alpha_q U_q^* \rangle_D |_{0} \) as the finite part \( \text{F} \langle \alpha_0 U_0^*, \ldots, \alpha_q U_q^* \rangle_D |_{0} \) in the asymptotic expansion \( \langle \alpha_0 U_0^*, \ldots, \alpha_q U_q^* \rangle_D |_{\varepsilon} \); it is given by the coefficient \( c_0 \) when \( \rho_0 = \frac{q}{2} \), and is 0 otherwise.

**Proposition 4.6.** The constant term \( \langle \alpha_0 U_0^*, \ldots, \alpha_q U_q^* \rangle_D |_{0} \) satisfies

\begin{equation}
(4.18) \quad \langle \alpha_0 U_0^*, \ldots, \alpha_q U_q^* \rangle_D |_{0} = 0, \quad \text{unless} \quad \mu(U_0 \ldots U_q) = 1;
\end{equation}

\begin{equation}
(4.19) \quad \langle \alpha_0 U_0^*, \ldots, \alpha_q U_q^* \rangle_D |_{0} = \langle \alpha_0 U_0^*, \ldots, \alpha_q U_q^* \rangle_D |_{0}.
\end{equation}

**Proof.** Up to a numerical factor, \( \langle \alpha_0 U_0^*, \ldots, \alpha_q U_q^* \rangle_D |_{0} \) coincides with the residue \( \text{Res}_{z=0} \zeta_N \). In view of the axiom (4.10),

\[ \text{if} \quad \mu(U_0 \ldots U_q) \neq 1 \quad \text{then} \quad \text{Res}_{z=0} \zeta_N = 0, \quad \forall N \geq 0. \]

This proves the property (4.18), which in turn readily implies (4.19). \( \square \)
In order to compute the constant term, we shall employ the elementary Duhamel-type commutator formula

\[(4.20) \quad -(\beta - \alpha)\lambda^2 D^2 A = A e^{-(\beta - \alpha)\lambda^2 D^2} = \int_{\alpha}^{\beta} e^{-(s - \alpha)\lambda^2 D^2} \lambda^2 (D^2 A - \mu^2 AD^2) e^{-(\beta - s)\lambda^2\mu^2 D^2} ds,\]

where \( A \in \mathcal{A}_G, \lambda, \mu > 0 \) and \([\alpha, \beta] \subset \mathbb{R}\). It is obtained by integrating the identity

\[d(s) e^{-(s-\alpha)D^2} A e^{-(\beta-s)\mu^2 D^2} = -e^{-(s-\alpha)D^2}(D^2 A - \mu^2 AD^2)e^{-(\beta-s)\mu^2 D^2}\]

and then replacing \( D \) by \( \lambda D \).

By iterating \((4.20)\), and using the abbreviation \( \nabla_\mu(A) = D^2 A - \mu^2 AD^2 \), one obtains for any \( N \in \mathbb{N} \),

\[(4.21) \quad e^{-t^2 D^2} A = \sum_{k=0}^{N-1} \frac{(-1)^k t^{2k}}{k!} \nabla_\mu^k(A) e^{-t^2\mu^2 D^2} + R_N(D, A, \mu, t) . \]

The remainder is given by the formula

\[(4.22) \quad R_N(D, A, \mu, t) = (-1)^N t^{2N} \int_{\Delta_N} e^{-s_1 t^2 D^2} \nabla_\mu^N(A) e^{-(1-s_1) t^2 \mu^2 D^2} ds_1 \cdots ds_N . \]

Using the finite-summability assumption, it is easy to estimate the above expression and thus show that Eq. \((4.21)\) does provide an asymptotic expansion as \( t \downarrow 0 \).

Applying this expansion for a twisted bracket as in Proposition 4.3 one obtains

\[
\langle \alpha_0 U_0^*, \ldots, \alpha_q U_q^* \rangle_{tD} = \int_{\Delta_q} \text{Tr}(\gamma \alpha_0 U_0^* e^{-s_1 \mu(U_0)^2 t^2 D^2} \alpha_1 U_1^* e^{-(s_2-s_1) \mu(U_0 U_1)^2 t^2 D^2} \ldots e^{-(s_q-s_{q-1}) \mu(U_0 \cdots U_{q-1})^2 t^2 D^2} \alpha_q U_q^* e^{-(1-s_q) \mu(U_0 \cdots U_q)^2 t^2 D^2}) \sim_{t \downarrow 0} \sum_{N \geq 0} (-1)^N t^{2N} \sum_{n_1 + \ldots + n_q = N} \frac{\mu(U_0)^{2(n_1 + \ldots + n_q)} \ldots \mu(U_{q-1})^{2n_q}}{n_1! \ldots n_q!}. \]

\[
\text{Tr}(\gamma \alpha_0 U_0^* \nabla_\sigma^{n_1} (\alpha_1 U_1^*) \ldots \nabla_\sigma^{n_q} (\alpha_q U_q^*) e^{-\mu(U_0 \cdots U_q)^2 t^2 D^2}) \int_{0 \leq s_1 \leq \ldots \leq s_q \leq 1} s_1^{n_1} \cdots s_q^{n_q} \]

\[
= \sum_{N \geq 0} (-1)^N t^{2N} \sum_{n_1 + \ldots + n_q = N} \frac{\mu(U_0)^{2(n_1 + \ldots + n_q)} \ldots \mu(U_{q-1})^{2n_q}}{n_1! \ldots n_q!(n_1 + 1) \ldots (n_1 + \ldots n_q + q)}. \]

\[
\text{Tr}(\gamma \alpha_0 U_0^* \nabla_\sigma^{n_1} (\alpha_1 U_1^*) \ldots \nabla_\sigma^{n_q} (\alpha_q U_q^*) e^{-t^2 D^2}). \]

In view of (4.18), we may assume \( \mu(U_0 \cdots U_q)^{-2(n_1+\ldots+n_q)} = 1 \); multiplying by \( \mu(U_0 \cdots U_q)^{-2(n_1+\ldots+n_q)} \), we can continue by

\[
\sum_{N \geq 0} (-1)^N t^{2N} \sum_{n_1+\ldots+n_q = N} \frac{\mu(U_1)^{-2n_1} \cdots \mu(U_q)^{-2(n_1+\ldots+n_q)}}{n_1! \cdots n_q!(n_1+1) \cdots (n_1+\ldots+n_q+q)} \cdot 
\text{Tr} \left( \gamma_0 U_0^* U_1^* \cdots U_q^* \sigma^{-2n_1}(\sigma^{-2n_1}(\sigma^{-2n_1}(\sigma^{-2n_1}(\alpha_0 U_0^*) \cdots \nabla^{n_q}_\sigma(\alpha_q U_q^*) e^{-t^2D^2}) \right)
\]

Comparing with the expansion obtained in the proof of Proposition 4.4, and converting the result into a residue via the Mellin transform, we arrive at the following conclusion.

**Proposition 4.7.** The constant term has the expression

\[
\langle \alpha_0 U_0^*, \ldots, \alpha_q U_q^* \rangle_D |_0 = \frac{1}{q!} \sum_{n_1, \ldots, n_q \geq 0} (-1)^{n_1+\ldots+n_q} \frac{\Gamma(n_1+\ldots+n_q+q)}{n_1! \cdots n_q!(n_1+1) \cdots (n_1+\ldots+n_q+q)} \cdot 
\int_D \gamma_0 U_0^* \nabla^{n_1}_\sigma(\sigma^{-2n_1}(\sigma^{-2n_1}(\sigma^{-2n_1}(\alpha_0 U_0^*) \cdots \nabla^{n_q}_\sigma(\sigma^{-2n_1+\ldots+n_q}(\alpha_q U_q^*)) e^{-t^2D^2}) \right) |D|^{-2(n_1+\ldots+n_q)-q}.
\]

4.3. **The cocycle identity.** To compute coboundaries of the twisted brackets in the cyclic bicomplex, one needs to establish identities similar to those satisfied by the usual JLO brackets, cf. [13 Lemma 2.2].

**Lemma 4.8.** With the same notation as in the preceding subsection, one has

\[
\langle \alpha_0 U_0^*, \ldots, \alpha_q U_q^* \rangle_D = \sum_{k=0}^q \langle \alpha_0 U_0^*, \ldots, 1, \alpha_k U_k^*, \ldots, a_q U_q^* \rangle_D.
\]
Lemma 4.9. Proceedings as in [15] loc. cit., one writes

$$\langle \alpha_0 U_0^*, \ldots , \alpha_q U_q^* \rangle_D = \int_0^1 \left\langle \alpha_0 U_0^*, \ldots , \alpha_q U_q^* \right\rangle_D ds =$$

$$= \int_0^1 ds \int_{0 \leq s_1 \leq \ldots \leq s_q \leq 1} \text{Tr} \left( \gamma \alpha_0 U_0^* e^{-s_1 \mu(U_0)^2 D^2} \alpha_1 U_1^* \cdot e^{-(s_2-s_1) \mu(U_0^2 U_1)^2 D^2} \ldots \alpha_q U_q^* e^{-(1-s_q) \mu(U_0 \ldots U_q)^2 D^2} \right) =$$

$$= \int_{0 \leq s_1 \leq \ldots \leq s_q \leq 1} \text{Tr} \left( \gamma \alpha_0 U_0^* e^{-s_1 \mu(U_0)^2 D^2} \cdot 1 \cdot e^{-(s_1-s_2) \mu(U_0)^2 D^2} \alpha_1 U_1^* \cdot e^{-(s_2-s_1) \mu(U_0 U_1)^2 D^2} \ldots \alpha_q U_q^* e^{-(1-s_q) \mu(U_0 \ldots U_q)^2 D^2} \right) +$$

$$+ \int_{0 \leq s_1 \leq s_2 \leq \ldots \leq s_q \leq 1} \text{Tr} \left( \gamma \alpha_0 U_0^* e^{-s_1 \mu(U_0)^2 D^2} \alpha_1 U_1^* e^{-(s_1-s_2) \mu(U_0 U_1)^2 D^2} \alpha_2 U_2^* \cdot 1 \cdot e^{-(s_2-s_3) \mu(U_0 U_1 U_2)^2 D^2} \ldots \alpha_q U_q^* e^{-(1-s_q) \mu(U_0 \ldots U_q)^2 D^2} \right) + \ldots$$

$$\ldots + \int_{0 \leq s_1 \leq \ldots \leq s_q \leq 1} \text{Tr} \left( \gamma \alpha_0 U_0^* e^{-s_1 \mu(U_0)^2 D^2} \cdots \alpha_q U_q^* \cdot e^{-(s_q-s_{q-1}) \mu(U_0 \ldots U_q)^2 D^2} \cdot 1 \cdot e^{-(1-s_q) \mu(U_0 \ldots U_q)^2 D^2} \right).$$

Lemma 4.9. For $j = 1, \ldots, q - 1$, one has

$$\langle \alpha_0 U_0^*, \ldots , \alpha_{j-1} U_{j-1}^* \cdot \alpha_j U_j^*, \alpha_{j+1} U_{j+1}^*, \ldots , \alpha_q U_q^* \rangle_D$$

$$- \langle \alpha_0 U_0^*, \ldots , \alpha_{j-1} U_{j-1}^* \cdot \alpha_j U_j^* \cdot \alpha_{j+1} U_{j+1}^*, \ldots , \alpha_q U_q^* \rangle_D$$

$$= \langle \sigma^2(\alpha_0 U_0^*), \ldots , \sigma^2(\alpha_{j-1} U_{j-1}^*) \rangle_D [D^2, \alpha_j U_j^*]_{\sigma} \langle \alpha_{j+1} U_{j+1}^*, \ldots , \alpha_q U_q^* \rangle_D.$$

Proof. Making use of the commutator formula [47, 20], one can write

$$\langle \alpha_0 U_0^*, \ldots , \alpha_{j-1} U_{j-1}^* \cdot \alpha_j U_j^* \cdot \alpha_{j+1} U_{j+1}^*, \ldots , \alpha_q U_q^* \rangle_D - \langle \alpha_0 U_0^*, \ldots , \alpha_{j-1} U_{j-1}^* \rangle_D$$

$$\alpha_j U_j^* \cdot \alpha_{j+1} U_{j+1}^*, \ldots , \alpha_q U_q^* \rangle_D$$

$$= \int_{\Delta_q} \text{Tr} \left( \gamma \alpha_0 U_0^* e^{-s_1 \mu(U_0)^2 D^2} \cdots \alpha_j U_j^* \cdot \alpha_{j+1} U_{j+1}^* \cdots \alpha_q U_q^* \right)$$

$$\cdot e^{-(s_j-s_{j-1}) \mu(U_0 \cdots U_{j-1})^2 D^2} \cdot \mu(U_0 \cdots U_{j-1})^2 [D^2, \alpha_j U_j^*]_{\sigma} e^{-(s_{j+1}-s_j) \mu(U_0 \cdots U_j)^2 D^2} ds_j \right).$$

$$\alpha_{j+1} U_{j+1}^* \cdots \alpha_q U_q^* e^{-(1-s_q) \mu(U_0 \cdots U_q)^2 D^2}$$

$$= \int_{\Delta_q} \text{Tr} \left( \gamma \sigma^2(\alpha_0 U_0^* e^{-s_1 \mu(U_0)^2 D^2} \cdots \sigma^2(\alpha_{j-1} U_{j-1}^*) \right.$$

$$\left. \left( \int_{t_{j-1}}^{t_{j+1}} e^{-(s_{j+1}-s_j) \mu(U_0 \cdots U_j)^2 D^2} \cdot \mu(U_0 \cdots U_j)^2 [D^2, \alpha_j U_j^*]_{\sigma} e^{-(s_j-s_{j-1}) \mu(U_0 \cdots U_{j-1})^2 D^2} ds_j \right) \cdot \

\alpha_{j+1} U_{j+1}^* \cdots \alpha_q U_q^* e^{-(1-s_q) \mu(U_0 \cdots U_q)^2 D^2} \right).$$

In contrast with the untwisted case, the cyclic symmetry property is no longer exactly satisfied. It only subsists in a weaker form.
Lemma 4.10. With $m$ denoting the degree in $D$ of the product $\alpha_0 \cdots \alpha_q$, one has
\begin{equation}
\langle \alpha_0 U^*_0, \ldots, \alpha_q U^*_q \rangle_D |_0 = \langle \alpha_1 U^*_1, \ldots, \alpha_q U^*_q, \sigma^{-m}(\alpha_0 U^*_0) \rangle_D |_0.
\end{equation}
Moreover, if $\mu(U_0 \cdots U_q) = 1$, then
\begin{equation}
\langle \alpha_0 U^*_0, \ldots, \alpha_q U^*_q \rangle_D(\varepsilon) = \langle \alpha_1 U^*_1, \ldots, \alpha_q U^*_q, \sigma^{-m}(\alpha_0 U^*_0) \rangle_D(\mu(U_0)^2 \varepsilon).
\end{equation}

Proof. Indeed,
\begin{align*}
\langle \alpha_0 U^*_0, \ldots, \alpha_q U^*_q \rangle_D(\mu(U_0)^{-2} \varepsilon) &= \mu(U_0)^{-m} \varepsilon^\frac{\bar{m}^2}{2} \int_{\Delta_q} \text{Tr}(\gamma \alpha_0 U^*_0 e^{-s_{q+1} \varepsilon D^2} \alpha_1 U^*_1 e^{-s_1 \mu(U_1)^2} \varepsilon D^2) \ldots \\
&\ldots \alpha_q U^*_q e^{-s_q \mu(U_1 \cdots U_q)^2} \varepsilon D^2 \rangle = \mu(U_0)^{-m} \varepsilon^\frac{\bar{m}^2}{2} \int_{\Delta_q} \text{Tr}(\gamma \alpha_1 U^*_1 e^{-s_1 \mu(U_1)^2} \varepsilon D^2) \ldots \\
&\ldots \alpha_q U^*_q e^{-s_q \mu(U_1 \cdots U_q)^2} \varepsilon D^2 \rangle \\
&= \varepsilon^\frac{\bar{m}^2}{2} \int_{\Delta_q} \text{Tr}(\gamma \alpha_1 U^*_1 e^{-s_1 \mu(U_1)^2} \varepsilon D^2) \ldots \alpha_q U^*_q e^{-s_q \mu(U_1 \cdots U_q)^2} \varepsilon D^2 \rangle,
\end{align*}
which under the assumption $\mu(U_0 \cdots U_q) = 1$ equals
\begin{align*}
&= \varepsilon^\frac{\bar{m}^2}{2} \int_{\Delta_q} \text{Tr}(\gamma \alpha_1 U^*_1 e^{-s_1 \mu(U_1)^2} \varepsilon D^2) \ldots \alpha_q U^*_q e^{-s_q \mu(U_1 \cdots U_q)^2} \varepsilon D^2 \rangle \sigma^{-m}(\alpha_0 U^*_0) \mu(U_0)^2 \varepsilon D^2, \\
&\text{This proves (4.24), and also implies the equality of the their constant terms.}
\end{align*}
On the other hand, if $\mu(U_0 \cdots U_q) \neq 1$, then both sides of (4.24) vanish, cf. (4.18).

We now introduce the twisted version of the JLO cocycles by defining, for any $q + 1$ elements $A_0, \ldots, A_q \in \mathcal{A}_G$,
\begin{equation}
J^q(D)(A_0, \ldots, A_q) = \langle A_0, [D, \sigma^{-1}(A_1)]_\sigma, \ldots, [D, \sigma^{-q}(A_q)]_\sigma \rangle_D.
\end{equation}
The collection $\{J^q(D)\}_{q=0, 2, 4, \ldots}$, resp. $\{J^q(D)\}_{q=1, 3, 5, \ldots}$, is a cochain in the entire cyclic cohomology bicomplex of $\mathcal{A}_G$ but, because of the failure of cyclic symmetry pointed out above, this cochain is not a cocycle. Instead, we form the 1-parameter family
\begin{equation}
J^q(\varepsilon^{1/2} D)(A_0, \ldots, A_q) := \varepsilon^\frac{\bar{m}}{2} \langle A_0, [D, \sigma^{-1}(A_1)]_\sigma, \ldots, [D, \sigma^{-q}(A_q)]_\sigma \rangle_{\varepsilon^{1/2} D},
\end{equation}
where $\varepsilon \in \mathbb{R}^+$, and passing to the constant term we define
\begin{equation}
\mathfrak{J}^q(D)(A_0, \ldots, A_q) := \langle A_0, [D, \sigma^{-1}(A_1)]_\sigma, \ldots, [D, \sigma^{-q}(A_q)]_\sigma \rangle_D |_0.
\end{equation}
According to Proposition 4.11 it has the explicit form predicted by the Ansatz
\begin{align}
\mathfrak{J}^q(D)(A_0, \ldots, A_q) &= \sum_k e_{q,k} \int_D \gamma A_0 [D, \sigma^{-2k_1-1}(A_1)](k_1) \ldots \\
&\ldots [D, \sigma^{-2(k_1 + \ldots + k_q) - q}(A_q)](k_q) |D|^{-2|k|-q}.
\end{align}
which in particular implies that

\[(4.29) \quad \mathfrak{J}(D) = 0, \quad \text{for any } \quad q > p; \]

also, by Proposition 4.4

\[(4.30) \quad \mathfrak{J}(D)(a_0 U_0^*, \ldots, a_q U_q^*) = 0, \quad \text{if } \mu(U_0^* \cdots U_q^*) \neq 1. \]

Thus, \{\mathfrak{J}_q(D)\}_{q=0,2,4,\ldots}, resp. \{\mathfrak{J}_q(D)\}_{q=1,3,5,\ldots}, defines a cochain in the \((b,B)\)-bicomplex of \(A_G\), which is supported on the conjugacy classes from \(G_0\).

**Theorem 4.11.** The cochain \(\mathfrak{J}(D)\) satisfies the cocycle identity

\[(4.31) \quad b\mathfrak{J}^{-1}(D)(a_0 U_0^*, \ldots, a_q U_q^*) + B\mathfrak{J}^{q+1}(D)(a_0 U_0^*, \ldots, a_q U_q^*) = 0. \]

**Proof.** The first stage of the proof will consist in establishing the identity

\[(4.32) \quad b\mathfrak{J}^{-1}(D)(a_0 U_0^*, \ldots, a_q U_q^*) = \sum_{j=1}^{q} (-1)^{j-1} \langle \sigma(a_0 U_0^*), \ldots, [D, \sigma^{-(j-2)}(a_{j-1} U_{j-1}^*)]_{\sigma}, [D, \sigma^{-j}(a_j U_j^*)]_{\sigma}, [D, \sigma^{-j-1}(a_{j+1} U_{j+1}^*)]_{\sigma}, \ldots, [D, \sigma^{-q}(a_q U_q^*)]_{\sigma} \rangle_D \mid_0. \]

To this end, we compute

\[
b \mathfrak{J}^{-1}(D)(a_0 U_0^*, \ldots, a_q U_q^*) = \langle a_0 U_0^* \cdot a_1 U_1^*, \ldots, [D, \sigma^{-(q-1)}(a_q U_q^*)]_{\sigma} \rangle_D + \\
+ \sum_{j=1}^{q-1} (-1)^j \langle a_0 U_0^*, \ldots, [D, \sigma^{-j}(a_j U_j^* \cdot a_{j+1} U_{j+1}^*)]_{\sigma}, \ldots \rangle_D + \\
+ (-1)^q \langle a_q U_q^* \cdot a_0 U_0^*, \ldots, [D, \sigma^{-(q-1)}(a_{q-1} U_{q-1}^*)]_{\sigma} \rangle_D \\
= \langle a_0 U_0^* \cdot a_1 U_1^*, \ldots, [D, \sigma^{-(q-1)}(a_q U_q^*)]_{\sigma} \rangle_D + \\
- \langle a_0 U_0^*, a_1 U_1^* \cdot [D, \sigma^{-1}(a_2 U_2^*)]_{\sigma}, \ldots, [D, \sigma^{-(q-1)}(a_q U_q^*)]_{\sigma} \rangle_D + \\
+ \sum_{j=2}^{q-1} (-1)^{j-1} \langle a_0 U_0^*, \ldots, [D, \sigma^{-(j-1)}(a_{j-1} U_{j-1}^*)]_{\sigma} \cdot \sigma^{-j}(a_j U_j^*), \ldots \rangle_D + \\
+ \sum_{j=2}^{q-1} (-1)^j \langle a_0 U_0^*, \ldots, \sigma^{-j}(a_j U_j^*) \cdot [D, \sigma^{-j}(a_{j+1} U_{j+1}^*)]_{\sigma}, \ldots \rangle_D + \\
+ (-1)^{q-1} \langle a_0 U_0^*, [D, \sigma^{-1}(a_1 U_1^*)]_{\sigma}, \ldots, [D, \sigma^{-(q-1)}(a_q U_q^*)]_{\sigma} \rangle_D \\
+ (-1)^q \langle a_q U_q^* \cdot a_0 U_0^*, \ldots, [D, \sigma^{-(q-1)}(a_{q-1} U_{q-1}^*)]_{\sigma} \rangle_D. \]
which by Lemma 4.9 is equal to
\[ \langle \sigma^2(a_0 U_0^*), [D^2, a_1 U_1^*]_\sigma, \ldots, [D, \sigma^{-(q-1)}(a_q U_q^*)]_\sigma \rangle_D + \]
\[ + \sum_{j=2}^{q-1} (-1)^{j-1} \langle \sigma^2(a_0 U_0^*), \ldots, [D, \sigma^{-(j-3)}(a_{j-1} U_{j-1}^*)]_\sigma, [D^2, \sigma^{-(j-1)}(a_j U_j^*)]_\sigma, \]
\[ [D, \sigma^{-j}(a_{j+1} U_{j+1}^*)]_\sigma, \ldots, [D, \sigma^{-(q-1)}(a_q U_q^*)]_\sigma \rangle_D \]
\[ + (-1)^{(q-1)} \langle a_0 U_0^*, [D, \sigma^{-1}(a_1 U_1^*)]_\sigma, \]
\[ \ldots, [D, \sigma^{-(q-1)}(a_{q-1} U_{q-1}^*)]_\sigma \rangle_D \]
\[ + (-1)^q \langle a_q U_q^* \cdot a_0 U_0^*, \ldots, [D, \sigma^{-(q-1)}(a_{q-1} U_{q-1}^*)]_\sigma \rangle_D. \]

At this point we pass to the constant terms and use Eq. (4.23) for the last two terms, to replace them by the sum
\[ (-1)^{(q-1)} \langle [D, \sigma^{-1}(a_1 U_1^*)]_\sigma, \]
\[ \ldots, [D, \sigma^{-(q-1)}(a_{q-1} U_{q-1}^*)]_\sigma, [D, \sigma^{-(q-1)}(a_q U_q^*)]_\sigma, \sigma^{-(q-1)}(a_q U_q^*) \cdot \sigma^{-(q-1)}(a_0 U_0^*) \rangle_D \]
\[ = (-1)^{(q-1)} \langle [D, \sigma(a_1 U_1^*)]_\sigma, \ldots, [D, \sigma^{-(q-3)}(a_{q-1} U_{q-1}^*)]_\sigma, [D^2, \sigma^{-(q-1)}(a_q U_q^*)]_\sigma, \sigma^{-(q-1)}(a_q U_q^*) \rangle_D \]
\[ = (-1)^{(q-1)} \langle \sigma^2(a_0 U_0^*), [D^2, \sigma(a_1 U_1^*)]_\sigma, \ldots, [D, \sigma^{-(q-3)}(a_{q-1} U_{q-1}^*)]_\sigma, [D^2, \sigma^{-(q-1)}(a_q U_q^*)]_\sigma \rangle_D \]
\[ B_{q-1}(D)(a_0 U_0^*, \ldots, a_q U_q^*) = \sum_{j=1}^{q} (-1)^{j-1} \langle \sigma^2(a_0 U_0^*), \ldots, [D, \sigma^{-(j-3)}(a_{j-1} U_{j-1}^*)]_\sigma, [D^2, [D, \sigma^{-(j-1)}(a_j U_j^*)]_\sigma]_\sigma, \]
\[ [D, \sigma^{-j}(a_{j+1} U_{j+1}^*)]_\sigma, \ldots, [D, \sigma^{-(q-1)}(a_q U_q^*)]_\sigma \rangle_D \]
\[ B_{q+1}(D)(a_0 U_0^*, \ldots, a_k U_k^*, a_{k+1} U_{k+1}^*, \ldots, a_q U_q^*) = \sum_{k=0}^{q} (-1)^k q_{q+1}(D)(1, a_k U_k^*, a_{k+1} U_{k+1}^*, \ldots, a_q U_q^*). \]
satisfies the identity

\[(4.33) \mathcal{B}^{q+1}(D)(a_0 U_0^*, \ldots, a_q U_q^*) = \langle [D, \sigma^{-1}(a_0 U_0^*)]_\sigma, \ldots, [D, \sigma^{-(q+1)}(a_q U_q^*)]_\sigma \rangle_D |_0.\]

To this end, we note that by Lemma 4.8

\[
\langle [D, \sigma^{-1}(a_0 U_0^*)]_\sigma, \ldots, [D, \sigma^{-(q+1)}(a_q U_q^*)]_\sigma \rangle_D = \\
= \sum_{k=0}^{q} \langle [D, \sigma^{-1}(a_0 U_0^*)]_\sigma, \ldots, 1, [D, \sigma^{-(k+1)}(a_k U_k^*)]_\sigma, \ldots, [D, \sigma^{-(q+1)}(a_q U_q^*)]_\sigma \rangle_D |_0.
\]

Passing to the constant term, we apply Eq. (4.23) \(k\)-times to the \(k\)-th term of the sum and rewrite it in the form

\[
\langle [D, \sigma^{-1}(a_0 U_0^*)]_\sigma, \ldots, 1, [D, \sigma^{-(k+1)}(a_k U_k^*)]_\sigma, \ldots, [D, \sigma^{-(q+1)}(a_q U_q^*)]_\sigma \rangle_D |_0 \\
= (-1)^{kq} \langle 1, [D, \sigma^{-(k+1)}(a_k U_k^*)]_\sigma, \ldots, [D, \sigma^{-(q+1)}(a_q U_q^*)]_\sigma, [D, \sigma^{-(q+2)}(a_0 U_0^*)]_\sigma, \\
\ldots, [D, \sigma^{-(q+k+1)}(a_{k-1} U_{k-1}^*)]_\sigma \rangle_D |_0.
\]

Summing up, one obtains

\[
\langle [D, \sigma^{-1}(a_0 U_0^*)]_\sigma, \ldots, [D, \sigma^{-(q+1)}(a_q U_q^*)]_\sigma \rangle_D |_0 \\
= \sum_{k=0}^{q} (-1)^{kq} \langle 1, [D, \sigma^{-(k+1)}(a_k U_k^*)]_\sigma, \ldots, [D, \sigma^{-(q+1)}(a_q U_q^*)]_\sigma, [D, \sigma^{-(q+2)}(a_0 U_0^*)]_\sigma, \\
\ldots, [D, \sigma^{-(q+k+1)}(a_{k-1} U_{k-1}^*)]_\sigma \rangle_D |_0 \\
= \sum_{k=0}^{q} (-1)^{kq} \mathcal{B}^{q+1}(D)(1, a_k U_k^*, \ldots, a_q U_q^*, a_0 U_0^*, \ldots, a_{k-1} U_{k-1}^*),
\]

which proves Eq. (4.33).

To relate the two identities satisfied by the coboundary operators, we use the generalized Leibniz rule \(3.27\) and write

\[
[D, a_0 U_0^* e^{-s_0 \mu(U_0)^2 D^2} \cdots [D, \sigma^{-q}(a_q U_q^*)]_\sigma e^{-s_q \mu(U_0 \cdots U_q)^2 D^2}] = \\
[D, a_0 U_0^*]_\sigma e^{-s_0 \mu(U_0)^2 D^2} \cdots [D, \sigma^{-q}(a_q U_q^*)]_\sigma e^{-s_q \mu(U_0 \cdots U_q)^2 D^2} + \\
\sigma(a_0 U_0^*) e^{-s_0 \mu(U_0)^2 D^2} [D^2, \sigma^{-1}(a_1 U_1^*)]_\sigma e^{-s_1 \mu(U_0 U_1)^2 D^2} \cdots \\
\ldots [D, \sigma^{-q}(a_q U_q^*)]_\sigma e^{-s_q \mu(U_0 \cdots U_q)^2 D^2} + \ldots + \\
(-1)^{q-1} \sigma(a_0 U_0^*) e^{-s_0 \mu(U_0)^2 D^2} [D, a_1 U_1^*]_\sigma e^{-s_1 \mu(U_0 U_1)^2 D^2} \cdots \\
\ldots [D^2, \sigma^{-q}(a_q U_q^*)]_\sigma e^{-s_q \mu(U_0 \cdots U_q)^2 D^2}.
\]
Since, in view of the Selberg property (4.10), \( \int_D \) vanishes on twisted graded commutators, one obtains

\[
- \langle [D, a_0 U^*_0], [D, \sigma^{-1}(a_1 U^*_1)] \rangle_{\sigma}, \cdots, [D, \sigma^{-q}(a_q U^*_q)] \rangle_D \mid_0 =
\]

\[
= \sum_{j=1}^{q} (-1)^{j-1} \langle \sigma(a_0 U^*_0), \cdots, [D, \sigma^{-(j-2)}(a_{j-1} U^*_j)] \rangle_{\sigma}, [D^2, \sigma^{-j}(a_j U^*_j)] \rangle_{\sigma}, [D, \sigma^{-j-1}(a_{j+1} U^*_j)] \rangle_{\sigma}, \cdots, [D, \sigma^{-q}(a_q U^*_q)] \rangle_D \mid_0.
\]

We can now rewrite Eq. (4.32) in the form

\[
b^{q-1}(D)(a_0 U^*_0, \ldots, a_q U^*_q) =
\]

\[
- \langle [D, a_0 U^*_0], [D, \sigma^{-1}(a_1 U^*_1)] \rangle_{\sigma}, \cdots, [D, \sigma^{-q}(a_q U^*_q)] \rangle_D \mid_0,
\]

Using once more the invariance property (4.19) and comparing with Eq. (4.33), one obtains the desired cocycle identity. \( \square \)

### 4.4. Transgression and proof of the Ansatz.

In view of the the property (4.30), we can restrict our considerations to the \((b, B)\)-subcomplex \(CC^*_G(A_G)\) of cochains supported by the conjugacy classes in \(G_0\). Far from being a mere convenience, this restriction is actually essential for the validity of the ensuing calculations.

We denote by \(\iota(V)\) the twisted contraction operator on \(CC^*(\Psi_G)\),

\[
\iota_{\sigma}(V)(A_0, \ldots, A_q)_D =
\]

\[
= \sum_{k=0}^{q} (-1)^{\#A_0 + \cdots + \#A_k} \#V \langle \sigma^2(A_0), \ldots, \sigma^2(A_k), V, A_{k+1}, \ldots, A_q \rangle_D,
\]

where \(\#A\) stands for the degree of \(A\), and extend it to cochain-valued functions by setting

\[
\iota_{\sigma}(V)(A_0, \ldots, A_q)_D(\varepsilon) := \iota(V)(\langle A_0, \ldots, A_q \rangle_D(\varepsilon)), \quad \varepsilon \in \mathbb{R}^+.
\]

In what follows, we shall denote by \(\tau \mapsto D_\tau\) one of the following two families of operators \(D_t = tD, t \in \mathbb{R}^+\) and \(D_u = D|D|^{-u}, u \in [0, 1]\), and will denote by \(\hat{D}\) the corresponding derivative. In each case, we define the cochains \(\mathcal{F}(D_{\tau}, V) \in CC^*_G(A_G)\) by the formula

\[
(4.34) \mathcal{F}(D_{\tau}, V)(a_0 U^*_0, \ldots, a_q U^*_q) =
\]

\[
= \iota_{\sigma}(V)(a_0 U^*_0, [D_\tau, \sigma^{-1}(a_1 U^*_1)] \rangle_{\sigma}, \cdots, [D_\tau, \sigma^{-q}(a_q U^*_q)] \rangle_{D_\tau},
\]

where \(V\) will be either the (odd) operator \(\hat{D}\) or the (even) operator \([D_{\tau}, \hat{D}_{\tau}]\).

We would like to evaluate the expression

\[
(4.35) \frac{d}{d\tau} J^q(D_{\tau}) + b\mathcal{F}^{q-1}(D_{\tau}, \hat{D}_{\tau}) + B\mathcal{F}^{q+1}(D_{\tau}, \hat{D}_{\tau}),
\]
which vanishes in the untwisted case (cf. e.g. [14] Prop. 10.12). The derivative
\[
\frac{d}{d\tau} J^q(\tau)(a_0 U^*_0, \ldots, a_q U^*_q) = \int_{\Delta_q} \frac{d}{d\tau} \text{Tr} \left( \gamma a_0 U^*_0 e^{-s_1 \mu(U_0)^2 D^2_\tau} [D_\tau, \sigma^{-1}(a_1 U^*_1)]_\sigma \right.
\]
\[\left. e^{-(s_2 - s_1) \mu(U_0 U_1)^2 D^2_\tau} \ldots [D_\tau, \sigma^{-q}(a_q U^*_q)]_\sigma e^{-(1-s_q) \mu(U_0 \ldots U_q)^2 D^2_\tau} \right).
\]

splits into two sums of terms. The first sum simply consists of the derivatives of the twisted commutators
(4.36)
\[
K^q(\tau)(a_0 U^*_0, \ldots, a_q U^*_q) := \sum_{j=1}^q \langle a_0 U^*_0, \ldots, [D_\tau, \sigma^{-j}(a_j U^*_j)]_\sigma, \rangle_{D_\tau}.
\]

To evaluate the second sum one relies, as in the standard case, on the Duhamel formula
\[
\frac{d}{d\tau} e^{-D^2_\tau} = - \int_0^1 e^{-s D^2_\tau} [D_\tau, \dot{D}_\tau] e^{-(1-s) D^2_\tau} ds.
\]

By applying it in the form
(4.37)
\[
\frac{d}{d\tau} e^{-(s_{j+1} - s_j) \mu U^2 D^2_\tau} = - \mu^2 \int_{s_j}^{s_{j+1}} e^{-(s-s_j) \mu U^2 D^2_\tau} [D_\tau, \dot{D}_\tau] e^{-(s_{j+1} - s) \mu U^2 D^2_\tau} ds,
\]

one obtains
\[
\sum_{j=0}^q \int_{\Delta_q} \text{Tr} \left( \gamma \cdots [D_\tau, \sigma^{-j}(a_j U^*_j)]_\sigma \frac{d}{d\tau} e^{-(s_{j+1} - s_j) \mu(U_0 \ldots U_j)^2 D^2_\tau} \ldots \right) = 0
\]

\[
= - \sum_{j=0}^q \int_{\Delta_q} \text{Tr} \left( \gamma \cdots [D_\tau, \sigma^{-j}(a_j U^*_j)]_\sigma \right.
\]
\[\left. e^{-(s_s - s_j) \mu(U_0 \ldots U_j)^2 D^2_\tau} [D_\tau, \dot{D}_\tau] e^{-(s_{j+1} - s) \mu(U_0 \ldots U_j)^2 D^2_\tau} \ldots \right) = 0
\]

\[
= - \sum_{j=0}^q \int_{\Delta_q} \langle \sigma^2(a_0 U^*_0), \ldots, [D_\tau, \sigma^{-j-2}(a_j U^*_j)]_\sigma, [D_\tau, \dot{D}_\tau], \ldots \rangle_{D_\tau}
\]
\[\ldots, [D_\tau, \sigma^{-q}(a_q U^*_q)]_\sigma \rangle_{D_\tau} = - \mathcal{I}^q(D_\tau, [D_\tau, \dot{D}_\tau])(a_0 U^*_0, \ldots, a_q U^*_q).
\]

This gives the identity
(4.38)
\[
\frac{d}{d\tau} J^q(\tau) = K^q(\tau) - \mathcal{I}^q(D_\tau, [D_\tau, \dot{D}_\tau]).
\]

On the other hand, in order to evaluate the coboundary of \( \mathcal{I}^q(D_\tau, \dot{D}_\tau) \), as in the proof of Theorem 4.11 we apply the Leibniz rule to the integrand for the expression of \( \mathcal{I}^q(D_\tau, \dot{D}_\tau)(a_0 U^*_0, \ldots, a_q U^*_q) \). By abuse of notation, we write this bracket operation in the form
\[
[D_\tau, \mathcal{I}^q(D_\tau, \dot{D}_\tau)(a_0 U^*_0, \ldots, a_q U^*_q)]_\sigma,
\]
and compute it as follows

\[ [D_\tau, \iota_\sigma(\hat{D}_\tau)](a_0 U_0^q, [D_\tau, \sigma^{-1}(a_1 U_1^q)]_\sigma, \ldots, [D_\tau, \sigma^{-q}(a_q U_q^q)]_\sigma)_{D_\tau} = \]

\[ = [D_\tau, (\sigma^2(a_0 U_0^q)]_\sigma, [D_\tau, \sigma^{-1}(a_1 U_1^q)]_\sigma, \ldots, [D_\tau, \sigma^{-q}(a_q U_q^q)]_\sigma)_{D_\tau} + \ldots \]
\[ = ([D_\tau, \sigma^2(a_0 U_0^q)]_\sigma, [D_\tau, [D_\sigma(a_1 U_1^q)]_\sigma, \ldots, [D_\tau, \sigma^{-q}(a_q U_q^q)]_\sigma)_{D_\tau} + \ldots \]

and they closely resemble those appearing in \( B\mathcal{F}^{q+1}(D_\tau, \hat{D}_\tau) \). The remaining terms are of the form

\[ \langle \sigma^4(a_0 U_0^q), \ldots, [D_\tau, \sigma^{-q}(a_q U_q^q)]_\sigma \rangle_{D_\tau}, \]

or

\[ \langle \sigma^4(a_0 U_0^q), \ldots, [D_\tau, \sigma^{-q}(a_q U_q^q)]_\sigma \rangle_{D_\tau}, \]

and they match those occurring in \( b\mathcal{F}^{q-1}(D_\tau, \hat{D}_\tau) + K_q \), plus terms which contain \([D_\tau, \hat{D}_\tau] \) and account for \( \mathcal{F}^{q-1}(D_\tau, [D_\tau, \hat{D}_\tau]) \).

Indeed, the \( b \)-coboundary of \( \mathcal{F}^q(D_\tau, \hat{D}_\tau) \), we write

\[ b\mathcal{F}^{q-1}(D_\tau, \hat{D}_\tau)(a_0 U_0^q, \ldots, a_q U_q^q) = \]

\[ = \iota_\sigma(\hat{D}_\tau)](a_0 U_0^q, a_1 U_1^q, [D_\tau, \sigma^{-1}(a_2 U_2^q)]_\sigma, \ldots, [D_\tau, \sigma^{-q+1}(a_q U_q^q)]_\sigma)_{D_\tau} + \]

\[ \sum_{j=2}^{q-1} (-1)^{j-1} \iota_\sigma(\hat{D}_\tau)](a_0 U_0^q, \ldots, [D_\tau, \sigma^{-q+1}(a_q U_q^q)]_\sigma)_{D_\tau} + \]

\[ + \sum_{j=2}^{q-1} (-1)^{j} \iota_\sigma(\hat{D}_\tau)](a_0 U_0^q, \ldots, [D_\tau, \sigma^{-q+1}(a_q U_q^q)]_\sigma)_{D_\tau} + \]

\[ + (-1)^{q-1} \iota_\sigma(\hat{D}_\tau)](a_0 U_0^q, a_1 U_1^q, \ldots, [D_\tau, \sigma^{-q+1}(a_q U_q^q)]_\sigma)_{D_\tau} \]
Let us take a closer look at the first two terms, and expand \( \iota_\sigma(\mathring{D}_\tau) \). One has

\[
\iota_\sigma(\mathring{D}_\tau)(a_0 U_0^*, a_1 U_1^*, [D_\tau, \sigma^{-1}(a_2 U_2^*)]_\sigma, \ldots, [D_\tau, \sigma^{-q+1}(a_q U_q^*)]_\sigma)_{D_\tau} \\
- \iota_\sigma(\mathring{D}_\tau)(a_0 U_0^*, a_1 U_1^* \cdot [D_\tau, \sigma^{-1}(a_2 U_2^*)]_\sigma, \ldots, [D_\tau, \sigma^{-(q-1)}(a_q U_q^*)]_\sigma)_{D_\tau}
\]

\[
= \langle \sigma^2(a_0 U_0^* \cdot a_1 U_1^*), \mathring{D}_\tau, [D_\tau, \sigma^{-1}(a_2 U_2^*)]_\sigma, \ldots, [D_\tau, \sigma^{-q+1}(a_q U_q^*)]_\sigma \rangle_{D_\tau} \\
- \langle \sigma^2(a_0 U_0^*), \mathring{D}_\tau, a_1 U_1^* \cdot [D_\tau, \sigma^{-1}(a_2 U_2^*)]_\sigma, \ldots, [D_\tau, \sigma^{-(q-1)}(a_q U_q^*)]_\sigma \rangle_{D_\tau}
\]

\[
+ \sum_{k=2}^q (-1)^k \langle \sigma^2(a_0 U_0^*, a_1 U_1^*), [D_\tau, \sigma(a_2 U_2^*)]_\sigma, \ldots, [D_\tau, \sigma^{-(k-3)}(a_k U_k^*)]_\sigma \rangle_{D_\tau} \\
- \sum_{k=2}^q (-1)^k \langle \sigma^2(a_0 U_0^*), \sigma^2(a_1 U_1^*), [D_\tau, \sigma(a_2 U_2^*)]_\sigma, \ldots, [D_\tau, \sigma^{-(k-3)}(a_k U_k^*)]_\sigma \rangle_{D_\tau}
\]

\[
= \langle \sigma^2(a_0 U_0^*, a_1 U_1^*), \mathring{D}_\tau, [D_\tau, \sigma^{-1}(a_2 U_2^*)]_\sigma, \ldots, [D_\tau, \sigma^{-q+1}(a_q U_q^*)]_\sigma \rangle_{D_\tau} \\
- \langle \sigma^2(a_0 U_0^*), \mathring{D}_\tau, a_1 U_1^* \cdot [D_\tau, \sigma^{-1}(a_2 U_2^*)]_\sigma, \ldots, [D_\tau, \sigma^{-(q-1)}(a_q U_q^*)]_\sigma \rangle_{D_\tau}
\]

\[
+ \sum_{k=2}^q (-1)^k \langle \sigma^2(a_0 U_0^*), [D_\tau, \sigma^2(a_1 U_1^*)]_\sigma, [D_\tau, \sigma(a_2 U_2^*)]_\sigma, \ldots, [D_\tau, \sigma^{-(k-3)}(a_k U_k^*)]_\sigma \rangle_{D_\tau}
\]

where we have used Lemma 4.9 after the first pair of terms.

We now look at pairs of terms indexed by the same \( j = 2, \ldots, q - 1 \),

\[
\iota_\sigma(\mathring{D}_\tau)(a_0 U_0^*, \ldots, [D_\tau, \sigma^{-(j-1)}(a_j U_j^*)]_\sigma, \ldots, [D_\tau, \sigma^{-(j-3)}(a_{j-1} U_{j-1}^*)]_\sigma)_{D_\tau} \\
- \iota_\sigma(\mathring{D}_\tau)(a_0 U_0^*, \ldots, \sigma^{-(j-1)}(a_j U_j^*) \cdot [D_\tau, \sigma^{-(j-3)}(a_{j-1} U_{j-1}^*)]_\sigma, \ldots)_{D_\tau} =
\]

\[
= \sum_{k \leq j-2} (-1)^k \langle \sigma^4(a_0 U_0^*), \ldots, \mathring{D}_\tau, [D_\tau, \sigma^{-(j-3)}(a_{j-1} U_{j-1}^*)]_\sigma, \ldots, [D_\tau, \sigma^{-(j-1)}(a_j U_j^*)]_\sigma, \ldots \rangle_{D_\tau}
\]

\[
+ (-1)^{j-1} \langle \sigma^2(a_0 U_0^*), \ldots, [D_\tau, \sigma^{-(j-3)}(a_{j-1} U_{j-1}^*)]_\sigma, \ldots \rangle_{D_\tau} \\
- (-1)^{j-1} \langle \sigma^2(a_0 U_0^*), \ldots, \sigma^{-(j-3)}(a_{j-1} U_{j-1}^*) \cdot [D_\tau, \sigma^{-(j-3)}(a_j U_j^*)]_\sigma, \ldots \rangle_{D_\tau}
\]

\[
+ \sum_{k \geq j} (-1)^k \langle \sigma^4(a_0 U_0^*), \ldots, [D_\tau, \sigma^{-(j-5)}(a_{j-1} U_{j-1}^*)]_\sigma, [D_\tau^2, \sigma^{-(j-3)}(a_j U_j^*)]_\sigma, \ldots \rangle_{D_\tau}.
\]
where we have again applied Lemma 4.9. We focus on the last two terms,

\[
\begin{align*}
\iota(\bar{D}_\tau)&\langle q_0U_0^*, [D_\tau, \sigma^{-1}(a_1U_1^*)]\rangle_{\sigma}, \ldots, [D_\tau, \sigma^{-(q-1)}(a_{q-1}U_{q-1}^*)]\rangle_{\sigma} \cdot \sigma^{-(q-1)}(a_qU_q^*) D_\tau \\
&- \iota(\bar{D}_\tau)(q_0U_0^* \cdot a_0U_0^*) \cdot \sigma^{-q}(a_{q-1}U_{q-1}^*) D_\tau \\
&= \langle \sigma^2(a_0U_0^*), \bar{D}_\tau, [D_\tau, \sigma^{-1}(a_1U_1^*)]\rangle_{\sigma}, \ldots, [D_\tau, \sigma^{-(q-1)}(a_{q-1}U_{q-1}^*)]\rangle_{\sigma} \cdot \sigma^{-(q-1)}(a_qU_q^*) D_\tau \\
&- \langle \sigma^2(a_qU_q^* \cdot a_0U_0^*), \bar{D}_\tau, [D_\tau, \sigma^{-(q-1)}(a_{q-1}U_{q-1}^*)]\rangle_{\sigma} D_\tau + \ldots
\end{align*}
\]

because one has to use Eq. (4.21) to prepare them for the application of Lemma 4.9 as follows:

\[
= (-1)^{(q-1)} \langle \bar{D}_\tau, [D_\tau, \sigma^{-1}(a_1U_1^*)]\rangle_{\sigma}, \ldots, [D_\tau, \sigma^{-(q-1)}(a_{q-1}U_{q-1}^*)]\rangle_{\sigma} \cdot \sigma^{-(q-1)}(a_qU_q^*) D_\tau \\
+ (-1)^q \langle \bar{D}_\tau, [D_\tau, \sigma^{-1}(a_1U_1^*)]\rangle_{\sigma}, \ldots, [D_\tau, \sigma^{-(q-1)}(a_{q-1}U_{q-1}^*)]\rangle_{\sigma}, \sigma^{-(q-1)}(a_qU_q^*) D_\tau.
\]

In doing so, we have rescaled the operator \(D_\tau\). This kind of rescaling, which appears every time we need to make cyclic rearrangements, prevents the expression \(4.35\) from vanishing.

However, in the special case of the scaling family \(\tau = t \mapsto D_t = tD\), one can integrate from 0 to \(\infty\), replacing the ordinary integral near 0 with its finite part as in [2, §4]. Also, since \(D\) is invertible, the proof of Lemma 2 in op.cit. can be easily replicated to produce the necessary estimates for the behavior of \(J^*(tD)\) and \(X^*(tD, D)\) as \(t \to \infty\). After integrating all the expressions involved in the above calculation, the mismatching disappears and all cancelations that take place in the untwisted case do occur in this case too. Indeed, taking as an example the first of the two terms above, one has

\[
Pf_0 \int_\varepsilon^\infty \langle a_0U_0^*, D_t[D, \sigma^{-1}(a_1U_1^*)]\rangle_{\sigma}, \ldots, [D_t, \sigma^{-(q-1)}(a_{q-1}U_{q-1}^*)]\rangle_{\sigma} \cdot \sigma^{-(q-1)}(a_qU_q^*) \frac{dt}{t}.
\]

By Eq. (4.21) this equals

\[
= Pf_0 \int_\varepsilon^\infty \langle D, [D, \sigma^{-1}(a_1U_1^*)]\rangle_{\sigma}, \ldots, [D, \sigma^{-(q-1)}(a_{q-1}U_{q-1}^*)]\rangle_{\sigma} \cdot \sigma^{-(q-1)}(a_qU_q^*) D(t) \frac{dt}{t}.
\]
which after the substitution $t \mapsto \mu(U_0)^{-2} t$ becomes
\[
\begin{align*}
= \text{Pf}_0 \int_{\mu(U_0)^2 \varepsilon}^{\infty} & \langle D, [D, \sigma^{-1}(a_1 U_1^*)] \rangle t, \\
& \cdots, [D, \sigma^{-(q-1)}(a_{q-1} U_{q-1}^*)] \sigma \cdot \sigma^{-(q-1)}(a_q U_q^*) \rangle_D t \frac{dt}{t}.
\end{align*}
\]

In this way one obtains the following transgression formula:

**Lemma 4.12.** For any $q \geq 0$, one has
\[
(4.39) \quad \text{Pf}_0 J^q(tD) - \lim_{t \to \infty} J^q(tD) = b \left( \text{Pf}_0 \int_{\varepsilon}^{\infty} \chi^{q-1}(tD, D) dt \right) + B \left( \text{Pf}_0 \int_{\varepsilon}^{\infty} \chi^{q+1}(tD, D) dt \right).
\]

On the other hand, with virtually identical arguments as in the proof of [7, Proposition 2], one establishes the similar vanishing result:
\[
(4.40) \quad \lim_{t \to \infty} J^q(tD) = 0.
\]

In particular, for $q = p$ (summability dimension), applying the $b$-boundary to Eq. (4.39) and using the cocycle identity (4.31) together with the vanishing property (4.29), one obtains
\[
\begin{align*}
b B \left( \text{Pf}_0 \int_{\varepsilon}^{\infty} \chi^{p+1}(tD, D) dt \right) &= b \chi_p(D) = -B \chi^{p+2}(D) = 0.
\end{align*}
\]

This shows that
\[
(4.41) \quad \chi^p(D) := B \left( \text{Pf}_0 \int_{\varepsilon}^{\infty} \chi^{p+1}(tD, D) dt \right)
\]
is a cyclic cocycle.

**Lemma 4.13.** The $(b, B)$-cocycle $\chi^*(D)$ is cohomologous to the cyclic cocycle $\chi^p(D)$.

**Proof.** In view of Eqs. (4.39) and (4.40), the difference between the two cochains is a total coboundary in the periodic cyclic complex:
\[
(4.42) \quad \chi^*(D) - \chi^p(D) = (b + B) \left( \text{Pf}_0 \int_{\varepsilon}^{\infty} \chi^*(tD, D) dt \right).
\]

We are now ready to conclude the proof of the Ansatz for spectral triples twisted by scaling automorphisms.

**Theorem 4.14.** The periodic cyclic cohomology class in $HP^*(A_G)$ of the cocycle $\chi^*(D)$ coincides with the Connes-Chern character $Ch^*(A_G, S, D)$. 

Proof. The strategy for the proof remains the same as in [7, 8], and relies on employing the family \( D_u = D|D|^{-u}, u \in [0,1] \) in order to construct a homotopy between the cocycle \( \mathcal{P}(D) \) and the global cocycle \( \tau_F^p \). Using the fact that each \( D_u \) defines its own spectral triple twisted by scaling automorphisms (with character \( \mu^{1-u} \)), and with similar analytic estimates and algebraic manipulations as above, one establishes the analogue of [8, Proposition 3] in the form

\[
\mathcal{P}(D_0) - \mathcal{P}(D_1) = (b+B) \left( \int_0^1 \text{Pf}_0 \int_\varepsilon^\infty \mathbb{I}^*(tD_u, D_u) \, dt \, du \right).
\]

Since \( D_1 = F \) and \( F^2 = \text{Id} \), the cyclic cocycle \( \mathcal{P}(D_1) \) can be easily seen to coincide, up to the constant factor \( \frac{\Gamma(\frac{p+1}{2})}{2\pi^p} \), with the very cocycle \( \tau_F^p \) (cf. Eq. (1.9)) that defines the Connes-Chern character. \( \square \)

4.5. The non-invertible case. As noted after Definition 4.1, the passage from \((A, \tilde{\mathcal{F}}, D)\) to the invertible double \((\tilde{A}, \tilde{\mathcal{F}}, \tilde{D})\) necessitates the replacement of the exact similarity condition (4.1) by the perturbed version (4.2). This does affect the twisted commutators, but only up to higher order in the asymptotic expansion. More precisely,

\[(4.43) \quad [\varepsilon^{\frac{1}{2}}\tilde{D}, \tilde{a}U^*]_* = [\varepsilon^{\frac{1}{2}}\tilde{D}, \tilde{a}] U^* + \varepsilon^{\frac{1}{2}} a U^* \triangleright e_1 F_1.\]

At the same time,

\[\tilde{D}^2 = (D^2 + \text{Id}) \triangleright \text{Id}, \quad \text{hence} \quad U \tilde{D}^2 U^* = (\mu(U)^2 D^2 + \text{Id}) \triangleright \text{Id}.\]

Retracing the arguments leading to the expansion Eq. (4.17), one sees that the constant term remains unaffected by the perturbation.

Alternatively, one could proceed as in [16, §6.1], and add a compact operator ‘mass’ to the Dirac Hamiltonian. Specifically, in the construction of the invertible double, one takes

\[\tilde{D} = D \triangleright \text{Id} + K \triangleright F_1,\]

where \( K \in \text{OP}^{-\infty} \) is a smoothing operator such that

\[[K, D] = [D, K] \quad \text{and} \quad D^2 + K^2 \quad \text{is invertible.}\]

The similarity condition is again perturbed, this time by a smoothing operator. Since the residue integral factors through the complete symbols, the constant term remains of the same form as in Proposition 4.7 only with \( D \) replaced by \( \tilde{D} \).
4.6. Application to foliations with transverse similarity structure.

We conclude by briefly indicating how one can use the above result in order to compute the index pairing for the leaf space of a foliation with transverse similarity structure.

A codimension $n$ foliation $\mathcal{F}$ of a $N$-dimensional manifold $V$ has a transverse similarity structure if there exist an open cover $\{U_i\}_{i \in I}$ of $V$ and a family $\{h_i : U_i \to \mathbb{R}^n\}_{i \in I}$ of submersions such that $\mathcal{F}|_{U_i} = \{h_i^{-1}(y); y \in h_i(U_i)\}$ and the covering transformations $g_{ij} : h_i|_{U_i \cap U_j} \to h_j|_{U_j \cap U_j}$ are given by similarities in $\text{Sim}(n)$. Concrete examples of such foliations can be found in [20], where for the case $n = N - 1$ all nonsingular flows which admit a closed transversal (satisfying an additional property) are in fact classified. When $N = 3$ the notion of a transverse similarity structure to a nonsingular flow coincides with that of a complex affine structure, treated in [13] without the requirement for the existence of a closed transversal.

Given a foliation $\mathcal{F}$ with a transverse similarity structure, let $\mathcal{G}$ denote the smooth étale groupoid associated to a complete transversal $M$ and let $\mathcal{A}_G = C_c^\infty(\mathcal{G})$ (see [4, II, §§8-10]). The Dirac operator $D$ on $M$ defines a spectral triple twisted by similarities over the algebra $\mathcal{A}_G$, whose Connes-Chern character is given by the cocycle $\mathfrak{T}(D) \in CC^*(\mathcal{A}_G)$. On the other hand, let $P$ be a proper $\mathcal{G}$-manifold [4, II, §10] with compact quotient $P/\mathcal{G}$, and let $D$ be a $\mathcal{G}$-invariant elliptic differential operator on $P$. By a construction explained in [6, §5] for discrete groups and in [4, III, 7.γ] for étale groupoids, one associates to $D$ a well-defined K-theory class

$$\text{Ind}(D) \in K_*(\mathcal{A}_G \otimes R),$$

where $R$ is the algebra of infinite matrices with rapidly decaying entries. In the even case, this class can be represented by a difference idempotent

$$E_D - E_0 \in M_k(\mathcal{A}_G \otimes R), \quad k >> 0.$$

The index pairing between the K-homology class of $D$ and the K-theory class of $\text{Ind}(D)$ is then computed by the pairing of their explicitly expressed Chern characters:

$$< \mathfrak{T}(D), ch_*(E_D - E_0) > = < D, \text{Ind}(D) > \in \mathbb{Z}.$$

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