Quantum phase transition and criticality in quasi one-dimensional spinless Dirac fermions

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We study quantum criticality of spinless fermions on the quasi one dimensional \( \pi \)-flux square lattice in cylinder geometry, by using the infinite density matrix renormalization group and abelian bosonization. For a series of the cylinder circumferences \( L_y = 4n + 2 = 2, 6, \ldots \) with the periodic boundary condition, there are quantum phase transitions from gapped Dirac fermion states to charge density wave (CDW) states. We find that the quantum phase transitions for such circumferences are continuous and belong to the \((1+1)\)-dimensional Ising universality class. On the other hand, when \( L_y = 4n = 4, 8, \cdots \), there are gapless Dirac fermions at the non-interacting point and the phase transition to the CDW state is Gaussian. Both of these two criticalities are described in a unified way by the bosonization. We clarify their intimate relationship and demonstrate that a central charge \( c = 1/2 \) Ising transition line arises as a critical state of an emergent Majorana fermion from the \( c = 2 \) Gaussian transition point.

I. INTRODUCTION

Criticality associated with a phase transition is one of the central issues in condensed matter physics. Various phase transitions have been established mainly for insulators which are well described by bosonic models such as Ising, XY, and Heisenberg models. However, phase transitions in metals where gapless fermions are coupled with bosons are rather poorly understood compared to insulators only with bosons. In such a system, fermions strongly affect low energy behaviors of the bosonic order parameters and consequently could change criticality of the phase transition. The critical bosonic fluctuations turn influence the fermions, and resulting non-Fermi liquid like behaviors are often observed in various systems.\(^1\)\(^2\)

The criticality depends on structures of fermionic excitations such as dimensionality of the Fermi surface and the number of fermion flavors (orbitals and spins). One of the simplest examples is the spinless fermions on a one-dimensional (1D) chain at half-filling with the nearest neighbor repulsive interaction \( V \), where the classical ground states for \( V \rightarrow \infty \) are the charge density wave (CDW) states.\(^5\)\(^6\) When one introduces fermionic hopping \( t \), there will be a Kosterlitz-Thouless phase transition to a Tomonaga-Luttinger liquid, which is distinct from the Ising transition in bosonic models such as the transverse Ising model. Quantum criticality in higher dimensional systems are also of great interest, and in this context, a semi-metallic system is an ideal platform to study interplay between fermions and bosons where the Fermi surface is a point. Indeed, critical behaviors of phase transitions in Dirac systems have been extensively studied, and the gapless Dirac excitations can lead to new criticalities such as chiral Ising, chiral XY, chiral Heisenberg universality classes.\(^7\)\(^22\) The critical exponents of these phase transitions have been evaluated accurately by several methods, e.g. analytical calculations and unbiased quantum Monte Carlo simulations. In these (semi)metallic systems, the gapless fermions play essential roles and the resulting quantum criticality is different from that in the corresponding purely bosonic system with gapped fermions.

These two criticalities are usually studied separately as distinct properties of metals and insulators. For example, the quantum phase transition from a gapless Dirac state to an antiferromagnetic state in a honeycomb lattice is described by \((2+1)D\) chiral Heisenberg universality class, while the one from a spin-orbit coupled gapped Dirac state to the antiferromagnetic state belongs to \(3D\) XY universality class.\(^3\)\(^\dagger\)\(^23\)\(^24\) Similarly, one can separately discuss two criticalities of phase transitions from a metal or a band insulator to an ordered state in general. However, such separate discussions would be somewhat subtle when the band gap is very small, and there will be crossover between fermionic criticality and bosonic criticality in a narrow gap system. Then, a natural question is how these two criticalities are connected along the critical line of the phase transition in an extended phase diagram including both metals and insulators (Fig. 1).

In this study, we consider quasi 1D half-filled spinless fermions on a \( \pi \)-flux square lattice in cylinder geometry with the circumference \( L_y \), as a simple example for the quantum phase transition of \( Z_2 \) symmetry breaking. When the nearest neighbor repulsive interaction \( V \) is weak, there are Dirac fermions with a mass \( m \) due to the finite system size \( L_y \), for \( L_y = 2, 6, 10, \cdots \) under the periodic boundary condition along the \( y \)-direction, while there are gapless Dirac fermions at \( V = 0 \) for \( L_y = 4n = 4, 8, \cdots \). The system exhibits a staggered CDW ordered state for large \( V \). The quantum phase transition is studied with use of the infinite density matrix renormalization group (iDMRG)\(^25\)\(^26\) together with the recently developed scaling analysis.\(^27\) Then, we demonstrate that the quantum phase transition at a critical \( V = V_c > 0 \) between the gapped Dirac fermions and the CDW state is continuous, and the corresponding criticality is simply \((1+1)D\) Ising universality class. On the other hand, the iDMRG results suggest that the phase transition from the gapless Dirac state is smooth...
around $V = 0$, which turns out to be Gaussian. These two behaviors are well described within the bosonization approach in a unified manner, and a global phase diagram in the $V$-$m$ plane is discussed. We clarify their intimate relationship and demonstrate that the central charge $c = 1/2$ Ising transition line arises as a critical state of an emergent Majorana fermion from the $c = 2$ Gaussian transition point.

![Diagram of phase diagram](image)

FIG. 1. A schematic phase diagram including both insulating and (semi)metallic states. Generally, the blue and green phase transition lines and the red transition point would be characterized by different criticalities.

II. MODEL AND PHASE TRANSITION

A. Model

We consider spinless fermions on a $\pi$-flux square lattice at half-filling,

$$H = -\sum_{\langle i,j \rangle} t_{ij} c_i^\dagger c_j + V \sum_{\langle i,j \rangle} n_i n_j, \quad (1)$$

where $t_{ij} = t \, (-t)$ along the $x$-direction at even (odd) $y_i$ and $t_{ij} = t$ along the $y$-direction. $\langle i,j \rangle$ represents a pair of nearest neighbor sites (Fig. 2). We use the energy unit $t = 1$. The system size is $L_x \times L_y = \infty \times L_y$ with the periodic boundary condition for the $y$-direction otherwise specified. In 2D ($L_y = \infty$) at $V = 0$, this model has two Dirac points and there is a continuous quantum phase transition to a staggered CDW state at $V_c \simeq 1.30^{[2,12]}$. The criticality of the CDW phase transition belongs to the (2+1)D chiral Ising universality class, whose critical exponents are evaluated as $\beta \simeq 0.60 \pm 0.07$ and $\nu \simeq 0.79 \sim 0.80$ by the quantum Monte Carlo calculations$^{[2,12]}$.

For a finite $L_y > 2$, the single particle dispersion under the periodic boundary condition for the $y$-direction is given by

$$\varepsilon(k_x, k_y) = \pm \sqrt{(2t \cos k_x)^2 + (2t \cos k_y)^2}, \quad (2)$$

where $k_x$ takes continuum values and $k_y = 2\pi n/L_y, (n = 0, 1, \ldots, L_y/2 - 1)$. Similarly, $\varepsilon(k_x) = \pm \sqrt{(2t \cos k_x)^2 + t^2}$ for $L_y = 2$. Due to the discreteness of $k_y$, the dispersion is qualitatively different when $L_y = 4n = 4, 8, 12, \ldots$ and $L_y = 4n + 2 = 2, 6, 10, \ldots$; the gapless Dirac points exist for $L_y = 4n$, while the Dirac fermions are massive with the gap size $m \sim t_y/L_y$ for $L_y = 4n + 2$. $\varepsilon(k)$ is shown in Fig. 3 for $L_y = 8$ and $L_y = 10$ as an example.

![Diagram of dispersion relations](image)

FIG. 2. (a) A $L_y = 4 \pi$-flux square lattice. The hopping on the black bonds is $-t$ and that on the red bonds is $+t$, which gives a $\pi$-flux for each square plaquette. (b) Schematic picture of the staggered CDW order. The blue circles represent the fermion particle density.

![Diagram of dispersion relations](image)

FIG. 3. Single particle dispersion relations (a) for $L_y = 8$ and (b) for $L_y = 10$ under the periodic boundary condition in the $y$-direction.

To discuss effects of the interaction $V$, we use iDMRG for a system of cylinder geometry and abelian bosonization. The iDMRG allows a highly accurate calculation, and has been used extensively not only for one dimensional systems but also for two dimensional systems. One can directly describe a quantum phase transition of discrete symmetry in such an infinite length cylinder by using iDMRG. Later, we also perform bosonization analysis around $V = 0$ but with a twisted boundary condition for the $y$-direction, which enables us to discuss the gapped and gapless fermions on an equal footing.

B. iDMRG calculations

1. Order parameter

In this section, the CDW quantum phase transition is investigated by iDMRG$^{[23,25]}$ with use of the open source code TenPy$^{[23,25]}$. We discuss the CDW order parameter associated with the $\mathbb{Z}_2$ symmetry breaking,

$$\Delta = \frac{1}{L_x L_y} \sum_i (-1)^{|i|} n_i, \quad (3)$$
where $L'_x$ is the unit period assumed in the iDMRG calculation. The summation is over $x = 1,2,\cdots,L'_x$ and $y = 1,2,\cdots,L_y$. We have performed calculations for various $L'_x$ and confirmed that the results are essentially independent of $L'_x$. Firstly, we show $|\Delta|$ for the massive case ($L_y = 4n+2$) and massless case ($L_y = 4n$) respectively in Fig.4. For the massive case $L_y = 2,6,10,14$, we find a clear quantum phase transition from the gapped Dirac state to the CDW state at the criticality within iDMRG. In the next part, we focus on the massive case and discuss its criticality within iDMRG.

![Image](image)

**FIG. 4.** The CDW order parameter $\Delta$ as a function of the interaction $V$ calculated by iDMRG with the periodic boundary condition for the $y$-direction. (a) $L_y = 4n+2 = 6,10,14$ with $\chi = 1000$ (red), 1600 (blue). For $L_y = 2, \chi = 100$ (red), 200 (blue). (b) $L_y = 4n = 4,8,12$ with $\chi = 1000$ (red), 1600 (blue). Note that the data for $L_y = 2,4$ with the different values of $\chi$ almost coincide in the present scale of the figures.

2. Finite correlation length scaling for $L_y = 4n+2$

The criticality of the phase transition for $L_y = 4n+2 = 2,6,10,\cdots$ is expected to be $(1+1)$D Ising universality class if it is continuous, because the CDW state breaks $\mathbb{Z}_2$ translation symmetry and there is no gapless Dirac fermions at $V = 0$ for these $L_y$. In order to examine the criticality numerically, we use the scaling ansatz recently developed for tensor network states in iPEPS15. Since the one-dimensional system size $L_x$ is infinite in iDMRG, criticality is controlled not by $L_x = \infty$ but by the correlation length $\xi_\chi$ in our calculations. The correlation length $\xi_\chi$ is computed from the second largest eigenvalue of the transfer matrix for a given bond dimension $\chi$, and $\xi_\chi$ characterizes finite bond dimension effects. One would naively expect that the system may exhibit the $(2+1)$D $\chi$-Ising criticality if $\xi_\chi \ll L_y$, while it shows $(1+1)$D bosonic Ising criticality if $\xi_\chi \gg L_y$. In the following, we focus only on the latter case with $\xi_\chi \gg L_y$.

The scaling ansatz for the ground state energy density is written as

$$E(g, h, \xi_\chi^{-1}) = b^{-2}E(b^{\nu}g, b^{\beta/\nu}h, b\xi_\chi^{-1}),$$

where $g = (V - V_c(L_y))/V_c(L_y)$ and $h$ is the conjugate field to $\Delta$. We have assumed the dynamical critical exponent is $z = 1$. The $L_y$-dependent critical points $V_c(L_y)$ are determined so that a scaling behavior of the order parameter Eqs. (5), (7) hold for larger $\chi$. We obtain $V_c(L_y = 2) \approx 2.8686, V_c(6) \approx 1.624$, and $V_c(10) \approx 1.50$ as will be discussed in the following. At the critical point $g = 0$, the CDW order parameter exhibits the scaling behaviors

$$\Delta(g = 0) \sim \xi_\chi^{-\beta/\nu},$$

$$\frac{\partial g}{\partial \Delta}(g = 0) \sim \xi_\chi^{1/\nu},$$

which are derived from the scaling ansatz Eqs. (4). From these two equations, we can determine the critical exponents $\beta$ and $\nu$.

In Fig. we show $\xi_\chi$-dependence of $\Delta$ and $\partial g/\partial \Delta$ for $L_y = 2$. First of all, the quantum phase transition is continuous since the scaling behaviors hold up to large $\chi$, although a discontinuous transition was potentially possible. The critical interaction strength is obtained as $V_c(L_y = 2) = 2.8686$ from the figure. The critical behaviors of $\Delta$ are in good agreement with those of $(1+1)$D Ising universality class with $\beta = 0.125, \nu = 1$, as we have expected. Similarly, we show $\xi_\chi$-dependence of $\Delta$ and $\partial g/\partial \Delta$ for $L_y = 6$ in Fig.7. The critical interaction is evaluated as $V_c(L_y = 6) = 1.624$. Although there is some signature for dimensional crossover from $(2+1)$D chiral Ising universality class for small $\xi_\chi \ll L_y$, the true criticality close to the critical point $g = 0$ belongs to the $(1+1)$D Ising universality class. For $L_y = 10$, however, it is difficult to explicitly demonstrate the critical behavior of the $(1+1)$D Ising universality class as shown in Fig.7 because of the heavy finite $\chi$ effects. Here, we used $\chi$ up to 2400, and the critical interaction is estimated to be $V_c(L_y = 10) \sim 1.50$. We think that the critical behavior of the $(1+1)$D Ising universality class will be reproduced for sufficiently large $\chi$ similarly to the cases for $L_y = 2,6$.

To further confirm the critical behaviors of the $(1+1)$D Ising universality class, in Fig.8 we show the scaling plot

$$\Delta_\chi^{\beta/\nu} = M(g\xi_\chi^{1/\nu}),$$

where $M$ is a scaling function. Here, we have used only the data for $\xi_\chi > L_y$ to avoid effects of the dimensional crossover. All the data collapse into a single curve in each system size $L_y = 2,6$, which gives a cross check
for the Ising universality class of the CDW phase transition. Finally, Fig. 6 shows the entanglement entropy $S$ for bipartitioning the infinite one dimensional chain in the iDMRG calculation into two half-infinite chains. In such bipartitioning, the entanglement entropy at the critical point is characterized by the central charge $c$ of the underlying conformal field theory and is given by

$$S = \frac{c}{6} \ln \xi + S_0,$$  

where $S_0$ is a constant. In the present system, the calculated $S$ at the critical point is well fitted by this formula with $c = 1/2$, which means that the corresponding conformal field theory is the $c = 1/2$ Ising theory in agreement with the critical behavior of the order parameter $\Delta$.

FIG. 5. The scaling plots of the CDW order parameter for $L_y = 2$. The correlation length $\xi$ is denoted as $\xi$ for simplicity. (a) The scaling plot Eq. (5), and the black line is $\Delta \sim \xi^{-\beta/\nu}$ with $\beta = 0.125, \nu = 1$. (b) The scaling plot Eq. (6), and the black line is $\partial_y \Delta/\Delta \sim \xi^{1/\nu}$ with $\nu = 1$. The $g$-derivative is approximated by $\partial_y \Delta(V) = V_c[\Delta(V+\delta V)-\Delta(V-\delta V)]/\delta V$ with $\delta V = 0.0001$. The bond dimension is used up to $\chi \lesssim 200$.

FIG. 6. The scaling plots of the CDW order parameter for $L_y = 6$. (a) The scaling plot Eq. (5) and (b) Eq. (6). The black lines are the same as in Fig. 5 while the $g$-derivative is approximated with $\delta V = 0.001$. The bond dimension is used up to $\chi \lesssim 2800$.

C. Bosonization and global phase diagram

In this section, we discuss the relationship between the CDW phase transitions from gapless and gapped Dirac states within the bosonization approach. Our primary purpose is to find an effective theory description for the iDMRG calculation results. To discuss the gapless and gapped states on an equal footing, we introduce the twisted boundary condition with the twist angle $\theta$ for the $y$-direction, or equivalently insert a flux $\theta$ along the cylinder. When $\theta = 0$ the boundary condition is realized and the non-interacting Dirac fermions are gapless for $L_y = 4n$. The band gap in Eq. (2) is tuned by the twisting angle $\theta$ since the allowed discrete $k_y = (2\pi n + \theta)/L_y$ points for given finite $L_y$ changes as $\theta$ is varied. For example in $L_y = 4n$ case, the band gap becomes maximum at $\theta = \pi$, for which there is a CDW phase transition from a gapped Dirac state whose criticality is (1+1)D Ising universality class. In this way, one can smoothly connect the two extreme cases, the gapless Dirac semimetal and maximally gapped Dirac band insulator, for fixed system size $L_y$.

We firstly consider the non-interacting excitation spectra in the $\pi$-flux cylinder for example with a fixed $L_y = \ldots$

FIG. 7. The scaling plots of the CDW order parameter for $L_y = 10$. (a) The scaling plot Eq. (5) and (b) Eq. (6). The black lines are the same as in Fig. 5 while the $g$-derivative is approximated with $\delta V = 0.005$. The bond dimension is used up to $\chi \lesssim 2400$.

FIG. 8. The scaling plot of the CDW order parameter $\Delta$ for (a) $L_y = 2$ and (b) $L_y = 6$. The critical exponents used are those for the (1+1)D Ising universality class $\beta = 0.125, \nu = 1$.

FIG. 9. The entanglement entropy $S$ for (a) $L_y = 2$ and (b) $L_y = 6$. The black lines are $S = (c/6) \ln \xi + S_0$ with the central charge $c = 1/2$. 
4π under the periodic boundary condition as shown in Fig. 3(a), and focus only on the gapless Dirac fermion branches and neglect other gapped bands. There are two pairs of linear dispersions with positive and negative velocities around $k_x = \pm \pi/2$. If we introduce a twist angle $\theta$, a band gap $m(\theta)$ will be induced in the pre-existing gapless Dirac bands. The two branches can be reproduced by an effective two leg ladder model

$$H_{\text{eff}} = \sum_{s=1,2} \sum_i -t_s c^\dagger_{i,s} c_{i+1,s} - t_\perp \sum_i c^\dagger_i c_2 + (\text{h.c.})$$

$$+ \tilde{U} \sum_i n_i n_i + \tilde{V} \sum_{s=1,2} \sum_i n_i n_i + t_{ss} = 1,$$  

where $t_{ss} = (-1)^{s+1}t, t_{\perp} = 2t|\sin \theta|, \tilde{U} = 2V/L_y, \tilde{V} = V/L_y$. A similar effective model was studied before in the context of carbon nanotubes. By using the transformation $c_{i,s} \rightarrow c_{i,s}, c_{2} \rightarrow (-1)c_{2}$, the Hamiltonian is rewritten into the familiar form with an additional staggered hybridization term $(-1)^s t_{\perp}$.

$$H_{\text{eff}} \rightarrow \sum_{s=1,2} \sum_i -t c^\dagger_{i,s} c_{i+1,s} - t_\perp (-1)^i c^\dagger_i c_2 + (\text{h.c.})$$

$$+ \tilde{U} \sum_i n_i n_i + \tilde{V} \sum_{s=1,2} \sum_i n_i n_i,$$

where hopping along the chain is $t$ for both $s = 1, 2$.

The fermion operators are approximated around the Fermi point $k_F = \pm \pi/2a$ as $\psi_s(x) = e^{-ik_Fx}\psi\lambda_s(x) + e^{ik_Fx}\psi\lambda_s(x)$ with $\psi\lambda_s(x) = \eta_s e^{-i(r\phi_s - \theta_\perp)}/\sqrt{2\pi a}$, where $a$ is the lattice constant and $\eta_s$ is the Klein factor. The bosonic phase operators satisfy the commutation relations

$$[\phi_s(x), \partial_{x'} \phi_s(x')] = i\pi \delta_{ss'} \delta(x - x').$$

Furthermore, we introduce new fields $\phi_{0, \pi} = (\phi_1 + \phi_2)/\sqrt{2}$ for convenience. Then the Hamiltonian is bosonized into

$$H_{\text{eff}} = H_{\text{kin}} + H_{\text{int}},$$

$$H_{\text{kin}} = \sum_{k=0, \pi} \frac{v_k}{2\pi} \int dx [K^{-1}_k(\partial \phi_k)^2 + K_k(\partial \theta_k)^2],$$

$$H_{\text{int}} = \int dx [g_1 \cos \sqrt{8} \phi_0 + g_2 \cos \sqrt{8} \phi_\pi$$

$$+ g_3 \cos \sqrt{8} \phi_0 \cos \sqrt{8} \phi_\pi + g_4 \cos \sqrt{8} \phi_0 \sin \sqrt{8} \theta_\pi],$$

where $g_1 = -\tilde{U}/2\pi^2 a, g_2 = \tilde{V}/2\pi^2 a, g_3 = \tilde{V}/\pi a, g_4 = 2t_{\perp}/\pi a$. For small $\tilde{U}, \tilde{V}$, the parameters are given by $v_0 = \nu_F/K_0, v_\pi = \nu_F/K_\pi$, and

$$K_0^{-1} = \sqrt{1 + \frac{a}{\nu_F}(\tilde{U} + \tilde{V})} \simeq 1 + \frac{a}{2\nu_F}(\tilde{U} + \tilde{V}),$$

(13a)

$$K_\pi^{-1} = \sqrt{1 + \frac{a}{\nu_F}(\tilde{U} + \tilde{V})} \simeq 1 + \frac{a}{2\nu_F}(\tilde{U} + \tilde{V}),$$

(13b)

where $v_F = 2t$ is the Fermi velocity of the periodic boundary condition. The scaling dimensions of the operators are easily read off as

$$[g_1] = 2K_0 \simeq 2 - \frac{a}{\nu_F}(\tilde{U} + \tilde{V}),$$

(14a)

$$[g_2] = 2K_\pi \simeq 2 - \frac{a}{\nu_F}(\tilde{U} + \tilde{V}),$$

(14b)

$$[g_3] = 2K_0 + 2K_\pi \simeq 4 - \frac{8a}{\nu_F} \tilde{V},$$

(14c)

$$[g_4] = \frac{K_0}{2} + \frac{1}{2K_\pi} \simeq 1 - \frac{a}{2\nu_F} \tilde{U}. $$

(14d)

We first consider the case with $t_{\perp} = 0$, or equivalently $g_4 = 0$. Then, the most relevant term is the $g_1$-term, and the $\phi_0$-field gets pinned to $\langle \phi_0 \rangle = 0$ because of the strong coupling $g_1 \rightarrow -\infty$. The remaining $g_2, g_3$-terms will have the same functional form, $\cos \sqrt{8} \phi_\pi$, and be renormalized to $g_2, g_3 \rightarrow \infty$. Therefore, both of the two fields $\phi_0, \phi_\pi$ become gapped as long as $V > 0$, and the phase transition is a Gaussian transition from the $c = 2$ two-flavor gapless Dirac state to the fully gapped CDW state. This is consistent with the iDMRG calculation where the CDW order parameter is non-zero for very small $V$ when $L_y = 4n = 4, 8, \ldots$ under the periodic boundary condition.

Next, we consider a very small $0 < t_{\perp} < V$, for which the renormalized parameters still satisfy $|g_4| \ll |g_1|$ down to some energy scale under the renormalization group. In this energy scale, $\phi_\pi$-field is nearly locked as $\langle \phi_0 \rangle \approx 0$ and the low energy physics is described by the $\phi_\pi$-field only,

$$H_{\text{eff}} \simeq \frac{\nu_\pi}{2\pi} \int dx[K^{-1}_\pi(\partial \phi_\pi)^2 + K_\pi(\partial \theta_\pi)^2]$$

$$+ \int dx[g_{23} \cos \sqrt{8} \phi_\pi + g_4 \sin \sqrt{2} \theta_\pi],$$

(15)

where $g_{23} = g_2 + g_3$ and we have used the approximation $\langle \cos \sqrt{8} \phi_0 \rangle \approx 0$. Note that the parameters in Eq. (15) should be regarded as renormalized ones under the renormalization group flow down to the above mentioned energy scale. In this Hamiltonian, the $g_{23}$-term favors the CDW state while the $g_4$-term leads to the band insulator, and this competition can lead to a gapless state when these two perturbations cancel each other. The resulting gapless state is described by the $c = 1/2$ Majorana fermions, which corresponds to the criticality of the CDW phase transition from the band gapped Dirac state discussed in the previous section. To see this, we focus on a fine-tuned state where the two perturbation terms are maximally competing having the same scaling dimensions, $[g_{23}] = [g_4]$, namely

$$2K_\pi = \frac{1}{2K_\pi} \Rightarrow K_\pi = \frac{1}{2}.$$  

(16)

By redefining the boson fields as $\phi'_\pi = \phi_\pi/\sqrt{K_\pi}, \theta'_\pi = \sqrt{K_\pi} \theta_\pi - \pi/4$ with $K_\pi = 1/2$, the Hamiltonian is rewritten...
ten as

$$H_{\text{eff}} = \frac{v_x}{2\pi} \int dx [(\partial \phi_x^\prime)^2 + (\partial \theta_y^\prime)^2] + \int dx [g_{23} \cos 2\phi_x^\prime + g_4 \cos 2\theta_y^\prime]. \quad (17)$$

This Hamiltonian is called the self-dual sine-Gordon model and has been studied extensively. Since the scaling dimensions of both $g_{23}, g_4$-terms are 1, one can refermionize them by using a spinless fermion operator $\psi(x) \simeq \eta_x e^{-i(r \phi_x^\prime - \theta_y^\prime)} / \sqrt{2\pi a}$ as

$$\cos 2\phi_x^\prime = -i\pi a[\psi_R^\dagger \psi_L - \psi_L^\dagger \psi_R], \quad (18a)$$

$$\cos 2\theta_y^\prime = -i\pi a[\psi_R^\dagger \psi_L^\prime - \psi_L^\dagger \psi_R^\prime]. \quad (18b)$$

Therefore the self-dual sine-Gordon model is mapped to a free spinless fermion model with mass terms,

$$H_{\text{eff}} = \int dx -iv_x[\psi_R^\dagger \partial \psi_R - \psi_L^\dagger \partial \psi_L] - im_{23}[\psi_R^\dagger \psi_L^\prime - \psi_L^\dagger \psi_R^\prime] - im_4[\psi_R^\dagger \psi_L^\prime - \psi_L^\dagger \psi_R^\prime], \quad (19)$$

where $m_{23} = \pi a g_{23}, m_4 = \pi a g_4$. Then we introduce Majorana fermions $\gamma^\dagger = (\psi + \psi^\dagger) / \sqrt{2}, \gamma^2 = (\psi - \psi^\dagger) / \sqrt{2}$ to write the Hamiltonian in the Majorana basis,

$$H_{\text{eff}} = \sum_{a=1,2} \int dx -i\frac{v_x}{2} \gamma_R^a \partial \gamma_R^a - \gamma_L^a \partial \gamma_L^a - im_{23} \gamma_R^a \gamma_L^a, \quad (20)$$

where $m_{23} = m_{23} + m_4, m_{23} = m_{23} - m_4$. Clearly, only one Majorana fermion $\gamma_1$ is gapless and the other one $\gamma_1$ is gapped along the special line given by $m_{23} = m_4$ in the $V-t_\perp$ plane. (Note that we have assumed $t_\perp > 0$ and thus $m_{23} \neq 0$ in this study.) This emergent gapless Majorana fermions describe the $c = 1/2$ conformal field theory which is the critical theory for the CDW phase transition from the band gapped Dirac state studied in the previous section. Physically, the Majorana fermions correspond to domain walls of the CDW order.

We have shown within the bosonization how the fermionic criticality at the Gaussian transition is connected to the bosonic criticality at the Ising transition. These discussions are summarized in the global phase diagram shown in Fig. 10. We expect that competition between the band gap and interaction would be important also for higher dimensions. For example in spinless fermions on the two dimensional $\pi$-flux square lattice, there is a CDW quantum phase transition with $2+1$D chiral Ising criticality at $V = V_\zeta > 0$ from the gapless Dirac semimetal, while a transition from the gapped Dirac insulator is expected to show 3D Ising criticality if it is continuous. The two phase transitions would be connected in a non-trivial way, and the familiar 3D Ising criticality might be understood as a critical state of an emergent object from the $2+1$D chiral Ising critical point. Further studies are necessary to develop theoretical understanding of these issues.

### III. SUMMARY AND DISCUSSION

We have studied the CDW quantum phase transition and its criticality in spinless fermions on the quasi one dimensional $\pi$-flux square lattice, by using iDMRG and bosonization. We find that the phase transition from a Dirac band insulator is continuous and its universality class is $(1+1)$D Ising with the central charge $c = 1/2$ when $L_y = 4n + 2, n = 0, 2, \cdots$ under the periodic boundary condition, while that from a Dirac semimetal is Gaussian with $c = 2$ when $L_y = 4n, n = 0, 2, \cdots$. By introducing the twisted boundary condition, we discussed how the fermionic criticality of the Gaussian transition in the gapless Dirac semimetal is connected to the bosonic criticality of the Ising transition in the gapped Dirac band insulator. The global phase diagram was discussed, where the $c = 2$ critical point is connected to the $c = 1/2$ critical line. The resulting $c = 1/2$ critical line arises from the competition between the band mass and the density interaction leading to the CDW gap, and is described by the emergent Majorana fermions which is regarded as a fractionalized object. This could give a new insight for a comprehensive understanding of phase transitions in both metals and insulators. Our results could provide a basis to understand higher dimensional systems, and also may be directly relevant for the artificially created $\pi$-flux systems in cold atoms with the synthetic magnetic field.

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