SUMMATION FORMULAS INVOLVING HARMONIC NUMBERS WITH EVEN OR ODD INDEXES

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ABSTRACT. By means of the derivative operator and Chu-Vandermonde convolution, four families of summation formulas involving harmonic numbers with even or odd indexes are established.

1. INTRODUCTION

For a nonnegative integer \( n \), define harmonic numbers to be
\[
H_0 = 0 \quad \text{and} \quad H_n = \sum_{k=1}^{n} \frac{1}{k} \quad \text{when} \quad n \in \mathbb{N},
\]
which arise from truncation of the harmonic series
\[
\sum_{k=1}^{\infty} \frac{1}{k}.
\]

Harmonic numbers are important in many branches of number theory and appear in the expressions of various special functions.

For a differentiable function \( f(x) \), define the derivative operator \( D_x \) by
\[
D_x f(x) = \left. \frac{d}{dx} f(x) \right|_{x=0}.
\]
Then it is not difficult to show that
\[
D_x \left( \binom{x+r}{s} \right) = \binom{r}{s} \{H_r - H_{r-s}\},
\]
where \( r, s \in \mathbb{N}_0 \) with \( s \leq r \).

As pointed out by Richard Askey (cf. [1]), expressing harmonic numbers in accordance with differentiation of binomial coefficients can be traced back to Issac Newton. In 2003, Paule and Schneider [8] computed the family of series:
\[
W_n(\alpha) = \sum_{k=0}^{n} \binom{n}{k}^{\alpha} \{1 + \alpha(n - 2k)H_k\}
\]
with \( \alpha = 1, 2, 3, 4, 5 \) by combining this way with Zeilberger’s algorithm for definite hypergeometric sums. According to the derivative operator and the hypergeometric form

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of Andrews’ q-series transformation, Krattenthaler and Rivoal [6] deduced general Paule-Schneider type identities with $\alpha$ being a positive integer. More results from differentiation of binomial coefficients can be seen in the papers [10, 15, 16]. For different ways and related harmonic number identities, the reader may refer to [3, 4, 5, 7, 9, 11, 14, 17]. It should be mentioned that Sun [12, 13] showed recently some congruence relations concerning harmonic numbers to us.

There are numerous binomial identities in the literature. Thereinto, Chu-Vandermonde convolution (cf. [2, p. 67]) can be stated as

$$\sum_{k=0}^{n} \left( \begin{array}{c} x \\ k \end{array} \right) \left( \begin{array}{c} y \\ n-k \end{array} \right) = \left( \begin{array}{c} x+y \\ n \end{array} \right).$$

(1)

Inspired by the works just mentioned, we shall establish, in terms of the derivative operator and (1), closed expressions for the following four families of series with even or odd indexes:

\begin{align*}
\sum_{k=0}^{n} (-4)^{k} \binom{n}{2k} t^{k} H_{2k}, \\
\sum_{k=0}^{n} (-4)^{k} \binom{n}{1+2k} \binom{1+2k}{k} t^{k} H_{1+2k}, \\
\sum_{k=0}^{n} \left(-\frac{1}{4}\right)^{k} \binom{n}{k} \binom{2k}{k} t^{k} H_{2k}, \\
\sum_{k=0}^{n} \left(-\frac{1}{4}\right)^{k} \binom{n}{k} \binom{1+2k}{k} t^{k} H_{1+2k},
\end{align*}

where $t \in \mathbb{N}_0$. In order to avoid appearance of complicated expressions, our explicit formulas are offered only for $t = 0, 1, 2$.

2. The First Family of Summation Formulas Involving Harmonic Numbers

**Theorem 1.**

$$\sum_{k=0}^{n} (-4)^{k} \binom{n}{2k} H_{2k} = \frac{2H_{2n} - H_{n}}{2(2n-1)} - \frac{4n}{(2n-1)^2}.$$ 

**Proof.** Perform the replacements $x \rightarrow -a - 1$ and $y \rightarrow b + n$ in (1) to obtain

$$\sum_{k=0}^{n} (-1)^{k} \binom{a+k}{k} \binom{b+n}{n-k} = \binom{b-a-1+n}{n}.$$ (2)

The case $a = \frac{x}{2}, b = -\frac{x-1}{2}$ of it reads as

$$\sum_{k=0}^{n} (-1)^{k} \binom{\frac{x}{2} + k}{k} \binom{-\frac{x-1}{2} + n}{n-k} = \binom{-x - \frac{3}{2} + n}{n}.$$ (3)

Applying the derivative operator $D_x$ to both sides of the last equation, we get

$$\sum_{k=0}^{n} (-1)^{k} \frac{\left(\frac{x}{2} + n\right)}{n-k} \left\{ \frac{1}{2} H_{k} + \frac{1}{2} \sum_{i=1}^{k} \frac{1}{i-\frac{1}{2}} - \frac{1}{2} \sum_{i=1}^{n} \frac{1}{i-\frac{1}{2}} \right\}$$

$$= - \left(\frac{\frac{x}{2} + n}{n}\right) \sum_{i=1}^{n} \frac{1}{i-\frac{1}{2}}.$$ (3)
By means of the two relations
\[
\left( \frac{1}{i} + \frac{n}{i + n} \right) = 4^{k-n} \left( \frac{n}{k} \right) \left( \frac{2n}{k} \right), \quad (4)
\]
\[
\frac{1}{2} H_k + \frac{1}{2} \sum_{i=1}^{k} \frac{1}{i} = H_{2k}, \quad (5)
\]
(3) can be manipulated as
\[
\sum_{k=0}^{n} (-4)^k \left( \frac{n}{2k} \right) H_{2k} = \frac{2H_{2n} - H_n}{2n-1} - \frac{4n}{(2n-1)^2} + \frac{2H_{2n} - H_n}{2} \sum_{k=0}^{n} (-4)^k \left( \frac{n}{2k} \right).
\]
Calculating the series on the right hand side by (2), we gain Theorem I.

\[\square\]

**Theorem 2.**
\[
\sum_{k=0}^{n} (-4)^k \left( \frac{n}{2k} \right) kH_{2k} = \frac{n(H_n - 2H_{2n})}{(2n-1)(2n-3)} + \frac{n(20n^2 - 24n - 1)}{(2n-1)^2(2n-3)^2}.
\]

**Proof.** It is ordinary to find that
\[
\sum_{k=0}^{n} (-1)^k \left( \frac{a+k}{k} \right) \left( \frac{b+n}{n-k} \right) = (a+1) \sum_{k=1}^{n} (-1)^k \left( \frac{a+k}{k-1} \right) \left( \frac{b+n}{n-k} \right) = -(a+1) \sum_{k=0}^{n-1} (-1)^k \left( \frac{a+1+k}{k} \right) \left( \frac{b+n}{n-1-k} \right).
\]

Evaluate the series on the right hand side by (2) to achieve
\[
\sum_{k=0}^{n} (-1)^k \left( \frac{a+k}{k} \right) \left( \frac{b+n}{n-k} \right) = \frac{(a+1)n}{a+1-b-n} \left( \frac{b-a-1+n}{n} \right). \quad (6)
\]
The case \(a = \frac{x}{2}, b = \frac{-x + 1}{2}\) of it is
\[
\sum_{k=0}^{n} (-1)^k \left( \frac{\frac{x}{2} + k}{k} \right) \left( \frac{\frac{x-1}{2} + n}{n-k} \right) = \frac{(x+2)n}{2x + 3 - 2n} \left( \frac{-x - \frac{3}{2} + n}{n} \right).
\]
Applying the derivative operator \(D_x\) to both sides of the last equation, we have
\[
\sum_{k=0}^{n} (-1)^k \left( \frac{-\frac{1}{2} + n}{n-k} \right) \left\{ \frac{1}{2} H_k + \frac{1}{2} \sum_{i=1}^{k} \frac{1}{i - \frac{1}{2}} - \frac{1}{2} \sum_{i=1}^{n} \frac{1}{i - \frac{1}{2}} \right\} = \frac{2n}{2n-3} \left( \frac{-\frac{3}{2} + n}{n} \right) \sum_{i=1}^{n} \frac{1}{i - \frac{3}{2}} - \frac{n(2n+1)}{(2n-3)^2} \left( \frac{-\frac{3}{2} + n}{n} \right).
\]
In accordance with (4) and (5), the last equation can be restated as
\[
\sum_{k=0}^{n} (-4)^k \left( \frac{n}{2k} \right) kH_{2k} = \frac{n(20n^2 - 24n - 1)}{(2n-1)^2(2n-3)^2} \frac{2n(2H_{2n} - H_n)}{(2n-1)(2n-3)}
\]
\[
+ \frac{2H_{2n} - H_n}{2} \sum_{k=0}^{n} (-4)^k \left( \frac{n}{2k} \right) k.
\]
Computing the series on the right hand side by (6), we attain Theorem 2. 
\[\square\]
Theorem 3.
\[
\sum_{k=0}^{n} (-4)^k \binom{n}{2k} k^2 H_{2k} = \frac{n(2n+1)(2H_{2n} - H_n)}{(2n-1)(2n-3)(2n-5)} - \frac{n(96n^3 - 280n^3 + 60n^2 + 182n - 13)}{(2n-1)^2(2n-3)^2(2n-5)^2}.
\]

Proof. It is routine to verify that
\[
\sum_{k=0}^{n} (-1)^k k^2 \binom{a+k}{k} \binom{b+n}{n-k} = (a+1) \sum_{k=1}^{n} (-1)^k \binom{a+k}{k-1} \binom{b+n}{n-k}.
\]

The case \(a=\frac{x}{2}, \ b=\frac{-x-1}{2}\) of it can be written as
\[
\sum_{k=0}^{n} (-1)^k k^2 \binom{\frac{x}{2}+k}{k} \binom{-\frac{x-1}{2}+n}{n-k} = \frac{n(x+2)(1+2n+x+nx)}{(2n+3-2n)(2x+5-2n)} \binom{-\frac{x-3}{2}+n}{n}.
\]

Applying the derivative operator \(D_x\) to both sides of the last equation, we get
\[
\sum_{k=0}^{n} (-1)^k k^2 \left( -\frac{1}{2} + \frac{n}{n-k} \right) \left\{ \frac{1}{2} H_k + \frac{1}{2} \sum_{i=1}^{k} \frac{1}{i} - \frac{1}{2} \sum_{i=1}^{n} \frac{1}{i} \right\} = \frac{n(16n^3 - 20n^2 - 36n + 13)}{(2n-3)^2(2n-5)^2} \binom{-\frac{3}{2} + n}{n} - \frac{2n(2n+1)}{(2n-3)(2n-5)} \binom{-\frac{3}{2} + n}{n} \sum_{i=1}^{n} \frac{1}{i}.
\]

According to (4) and (5), the last equation can be reformulated as
\[
\sum_{k=0}^{n} (-4)^k \binom{n}{2k} k^2 H_{2k} = 2n(2n+1)\binom{2H_{2n} - H_n}{(2n-1)(2n-3)(2n-5)} - \frac{n(96n^3 - 280n^3 + 60n^2 + 182n - 13)}{(2n-1)^2(2n-3)^2(2n-5)^2} + \frac{2H_{2n} - H_n}{2} \sum_{k=0}^{n} (-4)^k \binom{n}{2k} k^2.
\]

Evaluating the series on the right hand side by (7), we gain Theorem 3. \(\square\)
3. The second family of summation formulas involving harmonic numbers

**Theorem 4.**

\[ \sum_{k=0}^{n} (-4)^k \frac{n}{(1+2k)} H_{1+2k} = \frac{2H_{1+2n} - H_n}{2(4n^2 - 1)} - \frac{4n^3 + 8n^2 + 7n - 2}{(4n^2 - 1)^2}. \]

**Proof.** Use (2) and (6) to achieve

\[ \sum_{k=0}^{n} (-1)^k(1+k) \binom{a+k}{k} \binom{b+n}{n-k} = \frac{b-a-1-an}{b-a-1+n} \binom{b-a-1+n}{n}. \]  \hspace{1cm} (8)

The case \( a = \frac{3}{2}, b = \frac{1}{2} \) of it reads as

\[ \sum_{k=0}^{n} (-1)^k(1+k) \binom{\frac{3}{2}+k}{k} \binom{\frac{1}{2}+n}{n-k} = \frac{1 + 2x + nx}{1 + 2x - 2n} \binom{-x - \frac{3}{2} + n}{n}. \]

Applying the derivative operator \( D_x \) to both sides of the last equation, we have

\[ \sum_{k=0}^{n} (-1)^k(1+k) \binom{\frac{3}{2}+k}{k} \binom{\frac{1}{2}+n}{n-k} \left\{ \frac{1}{2} H_k + \frac{1}{2} \sum_{i=1}^{k} \frac{1}{i + \frac{1}{2}} - \frac{1}{2} \sum_{i=1}^{n} \frac{1}{i + \frac{1}{2}} \right\} = \frac{1}{2n-1} \left( \frac{-\frac{3}{2} + n}{n} \sum_{i=1}^{n} \frac{1}{i - \frac{3}{2}} - n(2n+3) \right) \binom{-\frac{3}{2} + n}{n}. \]  \hspace{1cm} (9)

In terms of the two relations

\[ (1+k) \binom{\frac{3}{2}+n}{n-k} = (1+n)4^{k-n} \binom{n}{k} \binom{1+2n}{1+2k}, \]  \hspace{1cm} (10)

\[ \frac{1}{2} H_k + \frac{1}{2} \sum_{i=1}^{k} \frac{1}{i + \frac{1}{2}} = H_{1+2k} - 1, \]  \hspace{1cm} (11)

(8) can be manipulated as

\[ \sum_{k=0}^{n} (-4)^k \frac{n}{(1+2k)} H_{1+2k} = \frac{2H_{1+2n} - H_n}{2(4n^2 - 1)^2} + \frac{2H_{1+2n} - H_n}{4n^2 - 1} + \frac{2H_{1+2n} - H_n}{2} \sum_{k=0}^{n} (-4)^k \frac{n}{(1+2k)}. \]

Computing the series on the right hand side by (8), we attain Theorem 4 \( \Box \)

**Theorem 5.**

\[ \sum_{k=0}^{n} (-4)^k \frac{n}{(1+2k)} k H_{1+2k} = \frac{2n(H_n - 2H_{1+2n})}{(4n^2 - 1)(2n-3)} + \frac{2n(8n^4 + 20n^2 - 2n^2 - 53n + 8)}{(4n^2 - 1)^2(2n-3)^2}. \]

**Proof.** Utilize (6) and (7) to obtain

\[ \sum_{k=0}^{n} (-1)^k(k+1) \binom{a+k}{k} \binom{b+n}{n-k} \]

\[ = \frac{(a+1)(a+2-2b+an)n}{(b-a-2n)(b-a-1+n)} \binom{b-a-1+n}{n}. \]  \hspace{1cm} (12)
The case $a = \frac{x}{2}$, $b = \frac{1-x}{2}$ of it is

$$\sum_{k=0}^{n} (-1)^k k(1 + k) \left( \frac{x}{k} + 1 \right) \left( \frac{1}{n - k} \right) = \frac{(2 + x)(2 + 3x + nx)n}{(1 + 2x - 2n)(3 + 2x - 2n)} \left( \frac{-x}{n} + \frac{1}{n} \right).$$

Applying the derivative operator $D_x$ to both sides of the last equation, we get

$$\sum_{k=0}^{n} (-1)^k k(1 + k) \left( \frac{x}{k} + 1 \right) \left( \frac{1}{n - k} \right) \left( \frac{1}{2} H_k + \frac{1}{2} \sum_{i=1}^{k} \frac{1}{i + \frac{x}{2}} - \frac{1}{2} \sum_{i=1}^{n} \frac{1}{i + \frac{x}{2}} \right)$$

$$= 2n(4n^3 + 8n^2 - 13n - 4) \left( \frac{-x}{n} + \frac{1}{n} \right) - \frac{4n}{(2n - 3)(2n - 1)} \left( \frac{-x}{n} + \frac{1}{n} \right) \sum_{i=1}^{n} \frac{1}{i - \frac{x}{2}}.$$

By means of (10) and (11), the last equation can be restated as

$$\sum_{k=0}^{n} (-4)^k \frac{(n)}{(1+2k)} k H_{1+2k} = \frac{2n(8n^4 + 20n^3 - 2n^2 - 53n + 8)}{(4n^2 - 1)(2n - 3)^2} - \frac{4n\{2H_{1+2n} - H_n\}}{(4n^2 - 1)(2n - 3)}$$

$$+ \frac{2H_{1+2n} - H_n}{2} \sum_{k=0}^{n} (-4)^k \frac{(n)}{(1+2k)} k.$$

Calculating the series on the right hand side by (12), we gain Theorem 5 \qed

**Theorem 6.**

$$\sum_{k=0}^{n} (-4)^k \frac{(n)}{(1+2k)} k^2 H_{1+2k} = \frac{2n(4n - 1)\{2H_{1+2n} - H_n\}}{(4n^2 - 1)(2n - 3)^2} - \frac{2n(32n^6 + 160n^5 - 544n^4 - 560n^3 + 1482n^2 - 382n - 17)}{(4n^2 - 1)(2n - 3)^2(2n - 5)^2}.$$

**Proof.** It is not difficult to derive that

$$\sum_{k=0}^{n} (-1)^k k^2 (1 + k) \left( \frac{a + k}{k} \right) \left( \frac{b + n}{n - k} \right)$$

$$= (a + 1) \sum_{k=1}^{n} (-1)^k k(1 + k) \left( \frac{a}{k - 1} \right) \left( \frac{b + n}{n - k} \right)$$

$$= -(a + 1) \sum_{k=0}^{n-1} (-1)^k (k + 1)(k + 2) \left( \frac{a + 1 + k}{k} \right) \left( \frac{b + n}{n - 1 - k} \right)$$

$$= -2(a + 1) \sum_{k=0}^{n-1} (-1)^k \left( \frac{a + 1 + k}{k} \right) \left( \frac{b + n}{n - 1 - k} \right)$$

$$- 3(a + 1) \sum_{k=0}^{n-1} (-1)^k \left( \frac{a + 1 + k}{k} \right) \left( \frac{b + n}{n - 1 - k} \right)$$

$$- (a + 1) \sum_{k=0}^{n-1} (-1)^k k^2 \left( \frac{a + 1 + k}{k} \right) \left( \frac{b + n}{n - 1 - k} \right).$$

Evaluate respectively the three series on the right hand side by (4), (6) and (7), we achieve

$$\sum_{k=0}^{n} (-1)^k k^2 (1 + k) \left( \frac{a + k}{k} \right) \left( \frac{b + n}{n - k} \right) = \left( \frac{b - a - 1 + n}{n} \right).$$

$$\times \frac{n(1+a)(2b(1-b) - (1+a)(4 + a - 4b)n - a(1+a)n^2)}{(b - a - 3 + n)(b - a - 2 + n)(b - a - 1 + n)}. \quad (13)$$
The case \( a = \frac{x}{2} \), \( b = \frac{1-x}{2} \) of it can be written as
\[
\sum_{k=0}^{n} (-1)^k k^2 \left( 1 + k \right) \left( \frac{x}{2} + k \right) \left( \frac{1-x}{n-k} \right) \left( \frac{1}{n} + n \right) = \left( -\frac{x}{2} + \frac{1}{n} + n \right).
\]

Applying the derivative operator \( D_x \) to both sides of the last equation, we have
\[
\sum_{k=0}^{n} (-1)^k k^2 \left( 1 + k \right) \left( \frac{1}{n} + n \right) \left\{ \frac{1}{2} H_k + \frac{1}{2} \sum_{i=1}^{k} \frac{1}{i + \frac{1}{2}} - \frac{1}{2} \sum_{i=1}^{n} \frac{1}{i + \frac{1}{2}} \right\} = \frac{4n(4n-1)}{(2n-1)(2n-3)(2n-5)} \left( \frac{-\frac{1}{2} + n}{n} \right) \sum_{i=1}^{n} \frac{1}{i - \frac{1}{2}} - \frac{2n(16n^5 + 72n^4 - 372n^3 + 210n^2 + 196n - 77)}{(2n-1)^2(2n-3)^2(2n-5)^2} \left( \frac{-\frac{1}{2} + n}{n} \right).
\]

In accordance with (10) and (11), the last equation can be reformulated as
\[
\sum_{k=0}^{n} (-4)^k \frac{n^k}{(1+2k)} k^2 H_{1+2k} = \frac{4n(4n-1)}{(4n^2-1)(2n-3)(2n-5)} \left\{ 2H_{1+2n} - H_n \right\} - \frac{2n(32n^6 + 160n^5 - 544n^4 - 560n^3 + 1482n^2 - 382n - 17)}{(4n^2-1)^2(2n-3)^2(2n-5)^2} H_{1+2n} - H_n \sum_{k=0}^{n} (-4)^k \frac{n^k}{(1+2k)} k^2.
\]

Computing the series on the right hand side by (13), we attain Theorem 6.

\[\square\]

4. The third family of summation formulas involving harmonic numbers

**Theorem 7.**
\[
\sum_{k=0}^{n} \left( -\frac{1}{4} \right)^k \binom{n}{k} \binom{2k}{k} H_{2k} = \frac{1}{2^{1+2n}} \binom{2n}{n} \left\{ 3H_n - 4H_{2n} \right\}.
\]

**Proof.** The case \( a = \frac{x-1}{2} \), \( b = -\frac{x}{2} \) of (2) reads as
\[
\sum_{k=0}^{n} (-1)^k \binom{\frac{x-1}{2} + k}{k} \left( \frac{-\frac{x}{2} + \frac{1}{n} + n}{n-k} \right) \left( \frac{1}{n} + n \right) = \left( -\frac{1}{2} + \frac{1}{n} + n \right).
\]

Applying the derivative operator \( D_x \) to both sides of it, we obtain
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{\frac{1}{2} + k}{k} \left\{ \frac{1}{2} \sum_{i=1}^{k} \frac{1}{i - \frac{1}{2}} + \frac{1}{2} H_k - \frac{1}{2} H_n \right\} = \left( -\frac{1}{2} + \frac{1}{n} + n \right) \sum_{i=1}^{n} \frac{1}{i - \frac{1}{2}}.
\]

(14) can be manipulated as
\[
\sum_{k=0}^{n} \left( -\frac{1}{4} \right)^k \binom{n}{k} \binom{2k}{k} H_{2k} = \frac{H_n}{2} \sum_{k=0}^{n} \left( -\frac{1}{4} \right)^k \binom{n}{k} \binom{2k}{k} - \frac{2H_{2n} - H_n}{4^n} \binom{2n}{n}.
\]

Calculating the series on the right hand side by (2), we get Theorem 7.

\[\square\]
By means of (5) and (15), the last equation can be reformulated as
\[
\sum_{k=0}^{n} \left( -\frac{1}{4} \right)^{k} \binom{n}{k} \binom{2k}{k} kH_{2k} = \frac{n}{(1-2n)2^{1+2n}} \binom{2n}{n} \left\{ 3H_{n} - 4H_{2n} - \frac{2+4n}{1-2n} \right\}.
\]

**Proof.** The case \( a = \frac{x-1}{2}, b = -\frac{x}{2} \) of (6) is
\[
\sum_{k=0}^{n} (-1)^{k} k \left( \frac{x-1}{2} + k \right) \left( -\frac{x}{2} + n \right) = \frac{(x+1)n}{2x+1-2n} \left( -x - \frac{1}{2} + n \right).
\]
Applying the derivative operator \( D_{x} \) to both sides of it, we have
\[
\sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \binom{2k}{k} \left( \frac{1}{2} \sum_{i=1}^{k} \frac{1}{i-\frac{1}{2}} - \frac{1}{2} H_{k} - \frac{1}{2} H_{n} \right)
= \frac{n}{2n-1} \left( \frac{1}{2} + n \right) \sum_{i=1}^{n} \frac{1}{i-\frac{1}{2}} - n \left( \frac{2n+1}{n} \right) \left( -\frac{1}{2} + n \right).
\]
In terms of (5) and (15), the last equation can be restated as
\[
\sum_{k=0}^{n} \left( -\frac{1}{4} \right)^{k} \binom{n}{k} \binom{2k}{k} kH_{2k} = \frac{n}{4^{n}(2n-1)} \binom{2n}{n} \left\{ 2H_{2n} - H_{n} - \frac{2+4n}{2n-1} \right\}
+ \frac{H_{2}}{2} \sum_{k=0}^{n} \left( -\frac{1}{4} \right)^{k} \binom{n}{k} \binom{2k}{k} k.
\]
Evaluating the series on the right hand side by (5), we gain Theorem 8. \( \square \)

**Theorem 9.**
\[
\sum_{k=0}^{n} \left( -\frac{1}{4} \right)^{k} \binom{k}{k} \binom{n}{k} \frac{2k}{k} H_{2k} = \frac{n^{2}}{4^{n}(2n-1)(2n-3)} \binom{2n}{n} \left\{ \frac{3}{2} H_{n} - 2H_{2n} + \frac{8n^{3} - 4n^{2} - 10n + 3}{n(2n-1)(2n-3)} \right\}.
\]

**Proof.** The case \( a = \frac{x-1}{2}, b = -\frac{x}{2} \) of (7) can be written as
\[
\sum_{k=0}^{n} (-1)^{k} k^{2} \left( \frac{x-1}{2} + k \right) \left( -\frac{x}{2} + n \right) = \frac{n(x+1)(n+x+nx)}{(2x+3-2n)(2x+1-2n)} \left( -x - \frac{1}{2} + n \right).
\]
Applying the derivative operator \( D_{x} \) to both sides of it, we achieve
\[
\sum_{k=0}^{n} (-1)^{k} k^{2} \binom{n}{k} \left( \frac{1}{2} \sum_{i=1}^{k} \frac{1}{i-\frac{1}{2}} + \frac{1}{2} H_{k} - \frac{1}{2} H_{n} \right)
= \frac{n}{(2n-3)^{2}(2n-1)^{2}} \binom{1}{n} \left( -\frac{1}{2} + n \right) - \frac{n^{2}}{(2n-3)(2n-1)} \left( -\frac{1}{2} + n \right) \sum_{i=1}^{n} \frac{1}{i-\frac{1}{2}}.
\]
By means of (5) and (15), the last equation can be reformulated as
\[
\sum_{k=0}^{n} \left( -\frac{1}{4} \right)^{k} \binom{n}{k} \binom{2k}{k} k^{2} H_{2k} = \frac{H_{2}}{2} \sum_{k=0}^{n} \left( -\frac{1}{4} \right)^{k} \binom{n}{k} \binom{2k}{k} k^{2}
- \frac{n^{2}}{4^{n}(2n-1)(2n-3)} \binom{2n}{n} \left\{ 2H_{2n} - H_{n} - \frac{8n^{3} - 4n^{2} - 10n + 3}{n(2n-1)(2n-3)} \right\}.
\]
Computing the series on the right hand side by (7), we attain Theorem 9. \( \square \)
5. THE FOURTH FAMILY OF SUMMATION FORMULAS INVOLVING HARMONIC NUMBERS

Theorem 10.

\[
\sum_{k=0}^{n} \left( -\frac{1}{4} \right)^k \binom{n}{k} \left( \frac{1+2k}{k} \right) H_{1+2k} = \frac{1}{2^{1+2n}(1+2n)} \left( \frac{1+2n}{n} \right) \left\{ \frac{8+8n}{1+2n} + 3H_n - 4H_{1+2n} \right\} - \frac{1}{1+n}.
\]

Proof. It is ordinary to find that

\[
\sum_{k=0}^{n} \frac{(-1)^k}{1+k} \binom{a+k}{k} \binom{b+n}{n-k} = \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k} \binom{a-1+k}{k-1} \binom{b+n}{1+n-k} = \frac{1}{a} \left\{ \sum_{k=0}^{n+1} (-1)^k \frac{a-1+k}{k} \binom{b+n}{1+n-k} \right\} - \frac{1}{1+n}.
\]

Calculate the series on the right hand side by (12) to obtain

\[
\sum_{k=0}^{n} \frac{(-1)^k}{1+k} \binom{a+k}{k} \binom{b+n}{n-k} = \frac{b+n}{a(1+n)} - \frac{1}{a} \left( b-a+n \right).
\]

(16)

The case \( a = \frac{x+1}{2}, \ b = -\frac{x}{2} \) of it reads as

\[
\sum_{k=0}^{n} \frac{(-1)^k}{1+k} \binom{\frac{x+1}{2}+k}{k} \left( -\frac{x}{2} + n \right) = \frac{1+2x}{(1+n)(1+x)} \left( -x - \frac{1}{2} + n \right) - \frac{x}{(1+n)(1+x)} \left( -\frac{x}{2} + n \right).
\]

Applying the derivative operator \( \mathcal{D}_x \) to both sides of the last equation, we get

\[
\sum_{k=0}^{n} \frac{(-1)^k}{1+k} \binom{\frac{1}{2}+k}{k} \left\{ \frac{1}{2} \sum_{i=1}^{k} \frac{1}{i} + \frac{1}{2} H_k - \frac{1}{2} H_{1+2k} \right\} = \frac{1}{1+n} \left( \frac{-\frac{1}{2} + n}{n} \right) \left\{ 1 - \sum_{i=1}^{n} \frac{1}{i-\frac{1}{2}} \right\} - \frac{1}{1+n}.
\]

(17)

In accordance with (11) and the relation:

\[
\frac{1}{1+k} \binom{\frac{1}{2}+k}{k} = \frac{1}{4^k} \binom{1+2k}{k},
\]

(18)

(17) can be manipulated as

\[
\sum_{k=0}^{n} \left( -\frac{1}{4} \right)^k \binom{n}{k} \left( \frac{1+2k}{k} \right) H_{1+2k} = \frac{1}{4^n(1+2n)} \left( \frac{1+2n}{n} \right) \left\{ \frac{3+2n}{1+2n} + H_n - 2H_{1+2n} \right\} - \frac{1}{1+n} + \frac{2+H_n}{2} \sum_{k=0}^{n} \left( -\frac{1}{4} \right)^k \binom{n}{k} \left( \frac{1+2k}{k} \right).
\]

Evaluating the series on the right hand side by (16), we gain Theorem 10.

□
Theorem 11.

\[
\sum_{k=0}^{n} \left( -\frac{1}{4} \right)^k \binom{n}{k} \left( 1 + 2k \right) k H_{1+2k} = \frac{3n}{2^{1+2n}(1-4n^2)} \binom{1+2n}{n}
\]

\[
\times \left\{ 3H_n - 4H_{1+2n} - \frac{16n}{1-4n^2} - \frac{2 - 14n}{3n} \right\} + \frac{1}{1+n}.
\]

**Proof.** It is routine to verify that

\[
\sum_{k=0}^{n} (-1)^k \frac{k}{1+k} \left( \frac{a+k}{k} \right) \binom{b+n}{n-k} = \frac{(1+a)n+b}{a(1+n)} \binom{b-a-1+n}{n} - \frac{1}{a} \binom{b+n}{1+n}.
\]

(19)

The case \(a = \frac{x+1}{2}, \quad b = -\frac{x}{2}\) of it is

\[
\sum_{k=0}^{n} (-1)^k \frac{k}{1+k} \left( \frac{x+1+k}{k} \right) \binom{-\frac{x}{2}+n}{n-k} = \frac{(3+x)n-x}{(1+n)(1+x)} \binom{-x-\frac{3}{2}+n}{n} + \frac{x}{(1+n)(1+x)} \binom{-\frac{x}{2}+n}{n}.
\]

Applying the derivative operator \(D_x\) to both sides of the last equation, we have

\[
\sum_{k=0}^{n} (-1)^k \binom{k}{1+k} \frac{n}{k} \left( \frac{1}{2} + \frac{k}{k} \right) \left\{ \frac{1}{2} \sum_{i=1}^{k} \frac{1}{i+\frac{1}{2}} + \frac{1}{2} H_k - \frac{1}{2} H_n \right\}
\]

\[
= \frac{1}{1+n} - \frac{3n}{1+n} \left( \frac{-\frac{x}{2}+n}{n} \right) \left\{ \sum_{i=1}^{n} \frac{1}{i-\frac{3}{2}} + \frac{1+2n}{3n} \right\}.
\]

According to (11) and (18), the last equation can be restated as

\[
\sum_{k=0}^{n} \left( -\frac{1}{4} \right)^k \binom{n}{k} \left( 1 + 2k \right) k H_{1+2k}
\]

\[
= \frac{1}{1+n} - \frac{3n}{4^n(1-4n^2)} \binom{1+2n}{n} \left\{ 2H_{1+2n} - H_n + \frac{1-4n+20n^2+16n^3}{3n(1-4n^2)} \right\}
\]

\[
+ \frac{2 + H_n}{2} \sum_{k=0}^{n} \left( -\frac{1}{4} \right)^k \binom{n}{k} \left( 1 + 2k \right) k.
\]

Calculating the series on the right hand side by (19), we attain Theorem 11. \(\Box\)
Theorem 12.

\[
\sum_{k=0}^{n} \left(-\frac{1}{4}\right)^k \binom{n}{k} \left(1 + 2k\right) k^2 H_{1+2k} = \frac{3n(2 - 3n)}{(3 - 2n)(1 - 4n)^4} \binom{1 + 2n}{n} \\
\times \left\{2H_{1+2n} - \frac{3}{2} H_n - \frac{3 - 8n - 12n^2}{1 - 4n^2} + \frac{9 + n(3 - 25n + 6n^2)}{3n(3 - 2n)(2 - 3n)} \right\} - \frac{1}{1 + n}.
\]

Proof. It is not difficult to see that

\[
\sum_{k=0}^{n} (-1)^k \frac{k^2}{1 + k} \binom{a + k}{k} \binom{b + n}{n - k} = \sum_{k=0}^{n} (-1)^k \frac{k^2 - 1 + 1}{1 + k} \binom{a + k}{k} \binom{b + n}{n - k} \\
= \sum_{k=0}^{n} (-1)^k \binom{a + k}{k} \binom{b + n}{n - k} \\
- \sum_{k=0}^{n} (-1)^k \binom{a + k}{k} \binom{b + n}{n - k} \\
+ \sum_{k=0}^{n} (-1)^k \frac{1}{1 + k} \binom{a + k}{k} \binom{b + n}{n - k}.
\]

Evaluate respectively the three series on the right hand side by (6), (2) and (16) to obtain

The case \(a = \frac{x+1}{2}, b = -\frac{x}{2}\) of it can be written as

\[
\sum_{k=0}^{n} (-1)^k \frac{k^2}{1 + k} \binom{\frac{x+1}{2} + k}{k} \binom{-\frac{x}{2} + n}{n - k} \\
= \frac{(3 + x)^2 n^2 - (1 + x)(6 + x)n + x(3 + 2x)}{(1 + n)(1 + x)(3 + 2x - 2n)} \left(-x - \frac{3}{2} + n\right) \\
- \frac{x}{(1 + n)(1 + x)} \left(-\frac{x}{2} + n\right).
\]

Applying the derivative operator \(D_x\) to both sides of the last equation, we get

\[
\sum_{k=0}^{n} (-1)^k \frac{k^2}{1 + k} \binom{n}{k} \left(\frac{1}{2} + k\right) \left(\frac{1}{2} + \frac{1}{2} \sum_{i=1}^{k} \frac{1}{i + \frac{3}{2}} + \frac{1}{2} H_k - \frac{1}{2} H_n\right) \\
= \frac{3n(2 - 3n)}{(1 + n)(3 - 2n)} \left(-\frac{x}{2} + n\right) \left\{9 + n(3 - 25n + 6n^2)\right\} - \frac{1}{1 + n}.
\]
In terms of (11) and (18), the last equation can be reformulated as

\[
\sum_{k=0}^{n} \left(-\frac{1}{4}\right)^k \binom{n}{k} \left(\frac{1+2k}{k}\right) k^2 H_{1+2k} = \frac{3n(2-3n)}{(3-2n)(1-4n^2)4^n} \binom{1+2n}{n}
\times \left\{2H_{1+2n} - H_n - \frac{2-8n-8n^2}{1-4n^2} + \frac{9+n(3-25n+6n^2)}{3n(3-2n)(2-3n)} \right\} - \frac{1}{1+n}
\]

\[
+ \frac{2 + H_n}{2} \sum_{k=0}^{n} \left(-\frac{1}{4}\right)^k \binom{n}{k} \left(\frac{1+2k}{k}\right) k^2.
\]

Reckoning the series on the right hand side by (20), we gain Theorem 12.

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