Test Map and Discreteness Criteria for Subgroups in $PU(1, n; C)$

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Abstract

In this paper, we study the discreteness for non-elementary subgroup $G$ in $PU(1, n; C)$, under the assumption that $G$ satisfies Condition $A$. Mainly, we present that one can use a test map, which need not to be in $G$, to examine the discreteness of $G$, and also show that $G$ is discrete, if every two-loxodromic-generator subgroup of $G$ is discrete.

1. Introduction

The discreteness of Möbius groups is a fundamental problem, which have been investigated by a number of authors. In 1976, Jørgensen [13] proved a necessary condition for a non-elementary two generator subgroup of $SL(2, C)$ to be discrete, which is called Jørgensen’s inequality. By using this inequality, Jørgensen established the following famous result[14]:

**Theorem 1.1.** A non-elementary subgroup $G$ of $SL(2, C)$ is discrete if and only if all its two-generator subgroups are discrete.

This important result has become standard in literature and it indicates that the discreteness of a non-elementary Möbius group depends on the information of all its rank two subgroups. Furthermore, Gilman [15] and Isochenko [20] strengthened the above theorem, and showed that $G$ is discrete if every subgroup generated by two loxodromic elements is discrete. There are many further discussions about discreteness criteria in this direction. For more details, see the references[9,21,22,23].

In [1,7,16,18], the authors have discussed the generalization of Theorem 1.1 to higher dimensional hyperbolic space. Moreover, Fang and Nai [7] also obtained the following result:

**Theorem 1.2.** Let a non-elementary subgroup $G$ of $SL(2, \Gamma_n)$ satisfy condition $A$. Then $G$ is discrete if and only if two arbitrary loxodromic elements $f$ and $g$ in $G$ the group $\langle f, g \rangle$ is discrete.

In 2004, Chen min [19] showed that one can even use a fixed Möbius transformations as a test map to test the discreteness of a given Möbius group. More precisely,

**Theorem 1.3.** Let $G$ be an $n$-dimensional subgroup of Isom($H^n$), and $f$ be a non-trivial Möbius transformation. If for each $g \in G$, the group $\langle f, g \rangle$ is discrete, then $G$ is

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The result suggests that the discreteness is not a totally interior affair of the involved group, and this provides a new point of view to the discreteness problem.

In complex hyperbolic space, Kamiya [17] established a similar version of theorem 1.1 for finitely generated subgroups of $PU(1,n;C)$ as follows:

**Theorem 1.4.** Let $G$ be a non-elementary finitely generated subgroup of $PU(1,n;C)$, then $G$ is discrete if and only if $\langle f, g \rangle$ is discrete for any $f$ and $g$ in $G$.

In 2001, Dai B, Fang and Nai [6] proved that:

**Theorem 1.5.** Let $G$ be a non-elementary subgroup of $PU(1,n;C)$ with condition A, then $G$ is discrete if and only if $\langle f, g \rangle$ is discrete for any $f$ and $g$ in $G$.

Here, $G$ is said to satisfy **condition A** if it has no sequence $\{g_i\}$ of distinct elements of finite order such that $\text{Card}(\text{fix}(g_i)) = \infty$ and $g_i \to I$ as $i \to \infty$, where

$$\text{fix}(g_i) = \{x \in \partial H^n_C : g_i(x) = x\}.$$ 

In this paper, we continue to discuss the discreteness criteria for non-elementary subgroup $G$ in $PU(1,n;C)$, and we will acquire three conclusions under the assumption that $G$ satisfies Condition A. The first result is similar to Theorem 1.3, which primarily consider to use a parabolic or loxodromic element as a test map to examine the discreteness of $G$, but whether one can use a elliptic element remains a open problem. The next result is followed from the idea of Theorem 1.2, and it shows that $G$ is discrete, if each two-loxodromic-generator subgroup of $G$ is discrete. And the third conclusion strengthened the second result, for details, see the section 3.

**2. Notations and Preliminary Results**

Throughout this paper, we will adopt the same notations and definitions as in [4,10,12]. Now we start by giving some general facts about $PU(1,n;C)$.

Let $C$ be the field of complex numbers, $V = V^{1,n}(C)(n \geq 1)$ denote the vector space $C^{n+1}$, together with the unitary structure defined by the Hermitian form

$$\Phi(z^*, w^*) = -\overline{z}_0^* w_0^* + \sum_{j=1}^{n} -\overline{z}_j^* w_j^*$$

for $z^* = (z_0^*, \ldots, z_n^*)$, $w^* = (w_0^*, \ldots, w_n^*) \in V$.

An automorphism $g$ of $V$, that is a linear bijection such that

$$\Phi(z^*, w^*) = \Phi(g(z^*), g(w^*))$$

for $z^*, w^* \in V$, will be called a unitary transformation. We denote the group consisting of all unitary transformation by $U(1,n;C)$. Let

$$V_0 = \{z^* \in V : \Phi(z^*, z^*) = 0\}, \quad V_- = \{z^* \in V : \Phi(z^*, z^*) < 0\}.$$
Set

\[ PU(1, n; C) = U(1, n; C)/(\text{center}). \]

It is obvious that \( V_0 \) and \( V_- \) are invariant under \( U(1, n; C) \). Set

\[ V^s = V_- \cup V_0 \setminus \{0\}. \]

Let \( P : V^s \to P(V^s) \) be the projection map defined by

\[ P(z_0^*, z_1^*, ..., z_n^*) = (z_1^*z_0^{-1}, ..., z_n^*z_0^{-1}). \]

We denote \( P(0, 1, ..., 0) \) by \( \infty \). We may identify \( P(V_-) \) with the Siegel domain

\[ H^n_C = \{ w = (w_1, w_2, ..., w_n) \in \mathbb{C}^n : \text{Re}(w_1) > \frac{1}{2} \sum_{j=2}^{n} |w_j|^2 \}. \]

An element of \( PU(1, n; C) \) acts on \( H^n_C \) and its boundary \( \partial H^n_C \). Denote \( H^n_C \cup \partial H^n_C \) by \( \overline{H^n_C} \). As in [4,12], a non-trivial element \( g \) in \( PU(1, n; C) \) is called

1) elliptic if it has a fixed point in \( H^n_C \);
2) parabolic if it has exactly one fixed point and the point lies on \( \partial H^n_C \);
3) loxodromic if it has exactly two fixed points and the points lie on \( \partial H^n_C \).

For a subgroup \( G \subset PU(1, n; C) \), the limit set \( L(G) \) of \( G \) is defined as

\[ L(G) = \overline{G(p)} \cap \partial H^n_C(p \in H^n_C). \]

The fixed point sets of \( f \in G \) and of \( G \) are

\[ \text{fix}(f) = \{ x \in \overline{H^n_C} : f(x) = x \}, \quad \text{fix}(G) = \bigcap_{f \in G} \text{fix}(f). \]

**Definition 2.1**[12]. A subgroup \( G \subset PU(1, n; C) \) is said to be non-elementary, if \( G \) contains two non-elliptic elements of infinite order with distinct fixed points, or \( G \) is said to be elementary.

**Definition 2.2**[12]. \( G_L = \{ g \in G : g(x) = x, \text{for any } x \in L(G) \} \).

**Definition 2.3**[12]. A subgroup \( G \subset PU(1, n; C) \) is said to be bounded torsion if there exists an integer number \( m \) such that for each \( g \in G \) has \( \text{ord}(g) \leq m \) or \( \text{ord}(g) = \infty \).

**Definition 2.4**[12]. Let \( X \) be subgroup of the vector space \( V \). The span of \( X \) denoted as \( \langle X \rangle \) is the smallest \( C \)-subspace containing \( X \). If \( X \) is a subset of \( H^n_C \), the span \( \langle X \rangle \) is defined by \( \langle X \rangle = P((P^{-1}(X))) \cap V_- \).

**Lemma 2.5** (Lemma 2.1 of [3]). Suppose that \( f \) and \( g \in PU(1, n; C) \) generate a discrete and non-elementary group. Then

i) if \( f \) is parabolic or loxodromic, we have

\[ \max\{N(f), N(f, g)\} \geq 2 - \sqrt{3} \]
where \([f, g] = fgf^{-1}g^{-1}\) is the commutator of \(f\) and \(g\), \(N(f) = \|f - I\|\).

ii) if \(f\) is elliptic, we have

\[
\max\{N(f), N([f, g_i]): i = 1, 2, ..., n + 1\} \geq 2 - \sqrt{3}.
\]

**Lemma 2.6.** Let \(G\) be a non-elementary subgroup of \(PU(1, n; C)\) and let \(O_1\) and \(O_2\) be disjoint open sets both meeting \(L(G)\). Then there is a loxodromic \(g\) in \(G\) with a fixed point in \(O_1\) and a fixed point in \(O_2\).

**Proof.** First we recall that if \(f\) is loxodromic with an attractive fixed \(\alpha\) and a repulsive fixed point \(\beta\), then as \(n \to \infty\), \(f^n \to \alpha\) uniformly on each compact subgroup of \(\Pi^n_C - \{\beta\}\) and \(f^{-n} \to \beta\) uniformly on each compact subset of \(\Pi^n_C - \{\alpha\}\). The repulsive fixed point of \(f\) is the attractive fixed point of \(f^{-1}\).

Now consider \(G\), \(O_1\) and \(O_2\) as in the lemma. It follows that there is a loxodromic \(p\) with attractive fixed point in \(O_1\) and a loxodromic \(q\) with attractive fixed point in \(O_2\). Since \(G\) is non-elementary, there is a loxodromic \(f\) with attractive fixed point \(\alpha\) and repulsive fixed point \(\beta\), neither fixed by \(p\). Now choose and (then fix) some sufficiently large value of \(m\) so that

\[
g = p^mfp^{-m}
\]

has its attractive fixed point \(\alpha_1 = p^m\alpha\) and repulsive fixed point \(\beta_1 = p^m\beta\) in \(O_1\). Then choose (and fix) some sufficiently large value of \(r\) so that

\[
h = q^r
\]

maps \(\alpha_1\) into \(O_2\): put \(\alpha_2 = h(\alpha_1)\).

Next, construct open convex neighborhood \(E\) and \(K\) of \(\beta_1\) and \(\alpha_2\) with the properties

\[
\beta_1 \in E \subset \overline{E} \subset O_1
\]

\[
\alpha_2 \in K \subset \overline{K} \subset O_2.
\]

As \(\beta_1\) is not in \(\overline{K}\) we see that \(g^n \to \alpha_1\) uniformly on \(\overline{K}\) as \(n \to \infty\). As \(h^{-1}(K)\) is an open neighborhood of \(\alpha_1\) we see that for all sufficiently large \(n\),

\[
g^n(\overline{K}) \subset h^{-1}(K)
\]

and so

\[
 hg^n(\overline{K}) \subset K
\]

As \(h(\alpha_1)\) is not in \(\overline{E}\) so \(\alpha_1\) is not in \(h^{-1}(E)\) and so \(g^{-n} \to \beta_1\) uniformly on \(h^{-1}(E)\) as \(n \to \infty\). Thus for all sufficiently large \(n\),

\[
g^{-n}h^{-1}(E) \subset E
\]

Choose a value of \(n\) for which (2.1) and (2.2) hold. By Brouwer fixed point theorem, \(hg^n\) is loxodromic with a fixed point in \(K\): also, \(g^{-n}h^{-1}\), which is \((hg^n)^{-1}\), has a fixed point
in $E$, hence so does $hg^n$. By definition, $hg^n$ is not parabolic. According to Lemma 3.3.2 of [5], $hg^n$ is not elliptic either. So $hg^n$ is a loxodromic element with one fixed point in $O_1$ and the other in $O_2$. \qed

Lemma 2.7. Let $\{f_m\}$ be a sequence in $PU(1, n; C)$ converging to a loxodromic element $f$. Then $f_m$ is loxodromic for sufficiently large $m$.

Proof. Let $x$ and $y$ be the attractive and repulsive fixed point of $f$, respectively. We have
\[
\lim_{j \to \infty} \lim_{m \to \infty} (f_m)^j(p) = \lim_{j \to \infty} f^j(p) = x, \\
\lim_{j \to \infty} \lim_{m \to \infty} (f_m)^{-j}(p) = \lim_{j \to \infty} f^{-j}(p) = y,
\]
for all $p$ in $\overline{H^n_C \setminus \{y\}}$ and $\overline{H^n_C \setminus \{x\}}$, respectively.

Let $U, V$ be two open convex neighborhood of $x$ and $y$ in $\overline{H^n_C}$ such that $U \cap V = \emptyset$.

Then for all sufficiently large $j, m$,
\[
(f_m)^j(U) \subset U, \quad (f_m)^{-j}(V) \subset V.
\]

Brouwer fixed point theorem tells us that, for all sufficiently large $j, m$, $(f_m)^j$ has one fixed point in $U$ and another in $V$. Hence $(f_m)^j$ is not parabolic. By lemma 3.3.2 in [5], $(f_m)^j$ is not elliptic either. Therefore all these $(f_m)^j$ are loxodromic. So $f_m$ is loxodromic for sufficiently large $m$. In fact, for the purpose of a contradiction, suppose $f_m$ is parabolic or elliptic. If $f_m$ is parabolic, then $f_m$ has exactly one fixed point in $\partial H^n_C$ and has $(f_m)^j$, that is, $(f_m)^j$ is parabolic, this is a contradiction. If $f_m$ is elliptic. Then $f_m$ has a fixed point in $H^n_C$ and has $(f_m)^j$, that is, $(f_m)^j$ is elliptic, also a contradiction. Consequently, $f_m$ must be loxodromic for sufficiently large $m$. \qed

We know that if $G \subset PU(1, n; C)$ is non-elementary then there must exist infinitely many loxodromic elements in $G$. Let $h \in G$ be some loxodromic element and let $x_0$ and $y_0$ be its distinct fixed points. Set
\[
G(x_0, y_0) = \{ f \in G : \{x_0, y_0\} \subset \text{fix}(f) \}.
\]

We also need the following lemma, which is a direct consequence of Lemma 2.2 in [12].

Lemma 2.8. Suppose a non-elementary subgroup $G$ of $PU(1, n; C)$ be discrete, then $G(x_0, y_0)$ is a bounded torsion.

3. Discreteness Criteria for Subgroups of $PU(1, n; C)$

In this section, we will state our principal results. Above all, we will introduce the first discreteness criterion for subgroups of $PU(1, n; C)$ by using a test map which need not to be in $G$.
Theorem 3.1. Let $G$ be a non-elementary subgroup of $PU(1, n; C)$ with condition $A$, and $h$ be a non-trivial element. If each $\langle h, g \rangle$ is discrete ($g \in G$), then $G$ is discrete.

Proof. Let $U_i \subset \prod C(i = 1, 2, 3)$ be disjoint open sets both meeting $L(G)$, and $h$ does not fix any point in $U_1$. By lemma 2.6, we can find loxodromic elements $f_i (i = 1, 2, 3)$ in $G$ which have the following properties:

(i) $f_1$ has its both attractive and repelling fixed points in $U_1$.

(ii) $f_i$ has its attractive fixed point in $U_i$ and repelling fixed point in $U_1$ for $i = 2, 3$.

Then there is an integer $k$ such that $f_i^k (fix(h)) \subset U_i (i = 1, 2, 3)$.

Suppose that $G$ satisfies the conditions of the theorem, but $G$ is not discrete. Then we can find a sequence $\{g_j\}$ of distinct element in $G$ such that $g_j \rightarrow I$ as $j \rightarrow \infty$. Thus we have

$$\max\{N(g_j), N([g_j, (f_i^k h f_i^{-k})^p]) : p = 1, 2, ..., n + 1\} \rightarrow 0.$$  

Since all groups $\langle g_j, f_i^k h f_i^{-k} \rangle = f_i^k (f_i^{-k} g_j f_i^k, h) f_i^{-k}$ are discrete by the assumption. In view of lemma 2.5, we get that each $\langle g_j, f_i^k h f_i^{-k} \rangle$ is elementary for large $j$. Because $G$ satisfies Condition $A$ and $\langle g_j, f_i^k h f_i^{-k} \rangle$ is discrete, we also have $Card(fix(g_j)) \leq 2$ for $j \rightarrow \infty$.

(a) $h$ is parabolic. Let $a$ be the fixed point of $h$. We have

$$L(g_j, f_i^k h f_i^{-k}) = fix(f_i^k h f_i^{-k}) = \{f_i^k(a)\} \quad (i = 1, 2, 3)$$

and each $g_j$ fixes $f_i^k(a) \in U_i, \bigcap U_i = \emptyset \quad (i = 1, 2, 3)$, this implies that $g_j$ has three distinct fixed points, but $Card(fix(g_j)) \leq 2$ for $j \rightarrow \infty$, this is a contradiction.

(b) $h$ is loxodromic. Assume $a$ and $b$ are the fixed points of $h$. We have

$$L((g_j, f_i^k h f_i^{-k})) = fix(f_i^k h f_i^{-k}) = \{f_i^k(a), f_i^k(b)\} \quad (i = 1, 2, 3).$$

g_j either fixes both $f_i^k(a)$ and $f_i^k(b)$ or interchanges them for sufficiently large $j$. Without loss of generality, we may assume that for each $j$, $g_j$ interchanges $f_i^k(a)$ and $f_i^k(b)$. So it follows that $g_j$ certainly fix both $f_i^k(a)$ and $f_i^k(b) \quad (i = 2, 3)$. Since $f_i^k(a) \in U_i, f_i^k(b) \in U_1 \quad (i = 2, 3)$ and $U_1 \cap U_2 \cap U_3 = \emptyset$, it is clear that $g_j$ have at least three distinct fixed points. But $Card(fix(g_j)) \leq 2$, as $j \rightarrow \infty$. This again leads to a contradiction. We complete the proof of the theorem.

Corollary 3.2. Let $G$ be a non-elementary subgroup of $PU(1, n; C)$, and $h \in G$ be a parabolic or loxodromic element. Then $G$ is discrete if and only if for every element $g(\neq h)$ in $G$ the group $\langle h, g \rangle$ is discrete.

Theorem 3.3. Let a non-elementary subgroup $G$ of $PU(1, n; C)$ satisfy condition $A$. Then $G$ is discrete if and only if for two arbitrary loxodromic element $f$ and $g$ in $G$ the group $\langle f, g \rangle$ is discrete.

Proof. The necessity is obvious, we only need to prove the sufficiency. Suppose that every two-loxodromic-generator subgroup of $G$ is discrete and yet $G$ is not discrete. Then
there is a distinct sequence \( \{g_j\} \subset G \) converging to the identity. Our aim is to reach a contradiction.

As \( G \) is non-elementary, there definitely exists a loxodromic element \( h \) in \( G \). Since \( g_jh \to h \) as \( j \to \infty \), it follows from Lemma 2.7 that \( g_jh \) is loxodromic for sufficiently large \( j \). We may assume that for each \( j \), \( g_jh \) is loxodromic. Since \( h \) and \( g_jh \) are loxodromic, by the assumption, we know that \( \langle h, g_jh \rangle = \langle h, g_j \rangle \) is discrete. Because \( G \) satisfies Condition \( A \) and \( \langle h, g_j \rangle \) is discrete, we obtain that \( \text{Card}(\text{fix}(g_j)) \leq 2 \) for sufficiently large \( j \).

According to Lemma 2.5 and the assumption \( g_j \to I \) as \( j \to \infty \), we have that \( \langle h, g_j \rangle \) is discrete and elementary for sufficiently large \( j \). Since \( h \) is loxodromic, we have \( g_j \) either fixes the fixed points of \( h \) or exchanges them as \( j \to \infty \). As \( G \) is non-elementary, there exist another two loxodromic elements \( f_1, f_2 \) such that \( h \cap f_1 \cap f_2 = \emptyset \). For the above mentioned reason, it is not difficult to deduce that \( \langle f_i, g_j \rangle (i = 1, 2) \) is discrete elementary and \( g_j \) either fixes the fixed points of \( f_i (i = 1, 2) \) or interchanges them for enough large \( j \).

Without loss of generality, we may assume that each \( g_j \) exchanges the fixed points of \( h \), so \( g_j \) necessarily fixes each fixed point of \( f_i (i = 1, 2) \). However, \( f_1 \) and \( f_2 \) have no common fixed points, thus \( \text{Card}(\text{fix}(g_j)) = 4 \). This is a contradiction with \( \text{Card}(\text{fix}(g_j)) \leq 2 \) as \( j \to \infty \). Up to now, we complete the proof of the theorem.

\[ \square \]

Let \( h \in G \) be some loxodromic element and let \( x_0 \) and \( y_0 \) be its distinct fixed points. We now use \( G(x_0, y_0) \) to strengthen theorem 3.3 as follows.

**Theorem 3.4.** Suppose that \( G \) in \( PU(1, n; C) \) is a non-elementary subgroup, then \( G \) is discrete if and only if

1. \( G(x_0, y_0) \) satisfy condition A;
2. every two-loxodromic-generator subgroup is discrete.

**Proof.** In order to prove necessity, it suffices to show that \( G(x_0, y_0) \) has bounded torsion if \( G \) is discrete. By lemma 2.8, we know that \( G(x_0, y_0) \) has bounded torsion. Since a group with bounded torsion satisfies Condition \( A \), we directly deduce the conclusion.

Now we prove sufficiency. Suppose that \( G \) is not discrete although every subgroup generated by two loxodromic elements is discrete. Thus there is an infinite sequence \( \{g_j\} \) of distinct elements such that \( g_j \to I \) as \( j \to \infty \). We derive a contradiction as follows.

Let \( h \in G(x_0, y_0) \) be a loxodromic element. Since \( g_jh \to h \) as \( j \to \infty \), we get that \( \langle h, g_j \rangle = \langle h, g_jh \rangle \) is discrete for enough large \( j \), according to Lemma 2.7 and the assumption in Theorem. As the sequence \( \{g_j\} \) converges to the identity, we have

\[ \max\{N(g_j), N([g_j, h^i]) : i = 1, 2, ..., n + 1\} \to 0. \]

Thus by Lemma 2.5, there exists \( J \) such that \( \langle g_j, h \rangle \) is discrete and elementary when \( j > J \). Since \( h \) is a loxodromic element, \( g_j \) fix or interchange the two fixed points of \( h \) when \( j > J \), namely \( g_j\{x_0, y_0\} = \{x_0, y_0\} \). Therefore \( g_j^2 \in G(x_0, y_0) \) as \( j > J \). Since \( G(x_0, y_0) \) satisfies Condition \( A \) and \( \langle h, g_j^2 \rangle \) is discrete, we gain that \( \text{Card}(\text{fix}(g_j^2)) \leq 2 \) for
sufficiently large \( j \). As \( G \) is non-elementary, there exists another two loxodromic elements \( f_1 \) and \( f_2 \) such that \( f_i \) \((i = 1, 2)\) and \( h \) have no common fixed points. We can also acquire that \( Card(fix(g_j)) = 4 \) as \( j \to \infty \), for reason see the proof of theorem 3.3. So \( g_j^2 \) have at least four fixed points as \( j \to \infty \), this is a contradiction. We complete its proof of the last theorem. \( \square \)

**References**

[1] Abikoffw and Hass A, Nondiscrete groups of hyperbolic motions. Bull.London Math.Soc, 22 (1990), 233-238.

[2] A.F.Beardon, The Geometry of Discrete Groups. Graduate Text in Mathematics, Vol.91, Springer, Berlin, 1983.

[3] Cao W S and Xang X T, discreteness criteria and algebraic convergence theorem for subgroups in \( PU(1, n; C) \). Proc. Japan Acad. 82, Sec. A (2006), No. 3, 49-52.

[4] Cao W S and Wang X T, Geometric characterizations for subgroups of \( (PU(1, n; C)) \). Northeast.Math.J, 21(1)(2005), 45-53.

[5] Chen S and Greenberg L, Hyperbolic spaces. Contributions to analysis(New York: Academic Press)(1974)pp. 49-87.

[6] Dai B, Fang A and Nai B, Discreteness criteria for subgroups in complex hyperbolic space. Proc. Japan Acad, 77 (2001), 168-172.

[7] Fang A and Nai B, On the discreteness and convergence in \( n \)-dimensional Möbius groups. J.London Math.Soc, 61 (2000), 761-773.

[8] F.W.Gehring and G.J.Martin, Discrete quasiconformal groups I. Proc.London.Math.Soc, (3)55 (1987), 331-358.

[9] G.Rosenberger, Minimal generating systems of a subgroup of \( SL(2, C) \). Proc.Edinburgh Math.Soc, (2) 31 (1988), 261-265.

[10] Goldman W M, Complex hyperbolic geometry(Oxford: Oxford University Press), 1999.

[11] Gerhard Rosenberger, Some remarks on paper of C.Doyle and D.James on subgroups of \( SL(2, R) \). Illinois Journal of Mathematics, 28 (1984), 348-351.

[12] H.Wang and Y.P.Jiang, Discreteness criteria in \( PU(1, n; C) \). Indian Acad.Sci, 120 (2010), No.2, 243-248.

[13] Jørgensen T, On discrete groups of Möbius transformations. Am.J.Math. 98 (1976), 739-749.
[14] Jørgensen T, A note on subgroup of $SL(2, C)$. Quart.J.Math.Oxford. 33 (1982), 325-332.

[15] J.Gilman, Inequalities in discrete subgroups of $PSL(2, R)$. Canad.J.Math. 40 (1988), 115-130.

[16] Jiang Y P, On the discreteness of Möbius groups of high dimensions. Math.Proc.Cambridge Philos.Soc, 136 (2004), 547-555.

[17] Kamiya S, Chordal and matrix norms of unitary transformations. First Korean-Japanese Colloquium on finite or infinite dimensional complex analysis (eds) JKajiwara, H Kazama and K H Shon(1993)pp. 121-125.

[18] Martin G J, On discrete Möbius groups in all dimensions. Acta Math, 163 (1989), 253-289.

[19] M Chen, Discreteness and convergence of Möbius groups. Geom. Dedicata, 104 (2004), 61-69.

[20] N.A.Isokenko, Systems of generators of subgroups of $PSL(2, C)$. Siberian Math.J. 31 (1990), 162-165.

[21] P.Tukia and X.Wang, discreteness of subgroups of $SL(2, C)$ containing elliptic elements. Math.Scand, 91 (2002), 214-220.

[22] S.H.Yang, The test maps and discrete groups in $SL(2, C)$. Osaka J.Math, 46 (2009), 403-409.

[23] S.H.Yang, On the discreteness criterion in $SL(2, C)$. Math.Z, 255 (2007), 227-230.

[24] X.T.Wang, L.L.Li and W.S.Cao, Discreteness criteria for Möbius groups action on $T^d$. Israel Jouranal of Mathematics, 150 (2005), 357-368.

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