SEQUENTIAL ESTIMATION OF THE SQUARE OF THE RAYLEIGH PARAMETER

Mohamed Tahir

Department of Mathematics, University of Sharjah, P.O. Box 27272, Sharjah, United Arab Emirates

Received 2014-02-11; Revised 2014-03-02; 2014-06-19

ABSTRACT

The problem addressed is that of sequentially estimating the square of the parameter of the Rayleigh distribution, subject to a weighted squared loss plus cost of sampling. We propose a sequential procedure and provide a second-order asymptotic expansion for the incurred regret. It is seen that the asymptotic regret is negative for a range of values of the parameter.

Keywords: Anscombe’s Theorem, Excess Over the Stopping Boundary, Hölder’s Inequality, Regret, Sequential Procedure

1. INTRODUCTION

Let \( X_1, \ldots, X_n \) denote independent observations to be taken sequentially up to a predetermined stage \( n \) from the Rayleigh distribution with p.d.f:

\[
    f_{\theta}(x) = \begin{cases} 
    \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}} & \text{if } x > 0 \\
    0 & \text{if not,}
    \end{cases}
\]

where, \( \theta \) is an unknown positive number. It is desired to estimate \( \theta^2 \), subject to the loss function considered by (Chow and Yu, 1981; Martinsek, 1988) that is Equation 1:

\[
    L_{\alpha}(w_n, \theta^2) = a^2 \theta^{2\alpha-4} [w_n - \theta^2]^2 + n,
\]

where, \( a \) is a known positive number, determined by the cost of estimation relative to the cost of a single observation, \( \beta \geq 1 \) is a given number and \( w_n \) is an appropriate point estimate of \( \theta^2 \) (defined below). In practice, one might be interested in estimating \( \theta^2 \), subject to the loss function considered by (Chow and Yu, 1981; Martinsek, 1988) that is Equation 1:

\[
    L_{\alpha}(w_n, \theta^2) = a^2 \theta^{2\alpha-4} [w_n - \theta^2]^2 + n,
\]

where, \( \alpha > 0 \). It follows that the maximum likelihood estimator of \( \theta \) is:

\[
    \hat{\theta}_n = \sqrt{\frac{1}{2n} \sum_{i=1}^{n} X_i^2} = \sqrt{\bar{Y}_n},
\]

where, \( \bar{Y}_n = \frac{1}{n} \sum_{i=1}^{n} Y_i \) with \( Y_i = X_i^2 / 2 \), \( i = 1, \ldots, n \) and where the random variables \( Y_1, \ldots, Y_n \) are independent with common distribution the Exponential distribution with mean \( \mu_Y = \theta^2 \) and standard deviation \( \sigma_Y = \theta^2 \).

The risk incurred by estimating \( \theta^2 \) with \( W_n = \hat{\theta}_n^2 = \bar{Y}_n \) under the loss (1) is:

\[
    R_n(n) = a^2 \theta^{2\alpha-4} E[(\bar{Y}_n - \theta^2)^2] + n = \frac{a^2 \theta^{2\alpha}}{n} + n
\]

For any fixed value of \( \alpha > 0 \), this risk is minimized with respect to \( n \) by choosing \( n \) as the greatest integer less than or equal to \( n_\alpha = a\theta^{2\alpha} = a\sigma_Y^2 \); in which case, the minimum risk is Equation 2:

\[
    R_\alpha(n_\alpha) = 2n_\alpha = 2a\sigma_Y^2
\]
Since $n_a$ depends on the unknown value of $\theta$, there is no fixed-sample-size procedure that attains the minimum risk $R'_2$ in practice. Therefore, we propose to use the sequential procedure $(T, \bar{Y})$ which stops the sampling process after observing $Y_1, \ldots, Y_T$ and estimates $\theta$ by $W_T = \bar{Y}$, where Equation 3:

$$T = \inf \left\{ n \geq m_a : n > \frac{a}{1} \left( \frac{1}{\sigma} \sum_{i=1}^{n} (Y_i - \bar{Y})^2 \right)^{\beta/2} \right\}$$

(3)

with $m_a$ being a positive integer. Note that the standard deviation is biased for $\sigma$, instead of $W_n = \bar{Y}$, since $\sigma^2$ is also the standard deviation of $Y_i$.

If $m_a$ in (3) is such that $\delta(a) \leq m_a = o(a)$ as $a \to \infty$ for some $\delta > 0$, then Equation 4:

$$E[\bar{Y}_n] = \mu + \frac{\beta}{a} \sigma^\beta + o(1)$$

(4)

As a $\theta \theta$, by Martinsek (1988), since the skewness of $Y_1$ is equal to 2. This shows that $\bar{Y}_n$ is biased large values of $a$. Thus, consider the biased-corrected estimator Equation 6:

$$\theta'_n = \bar{Y}_n + \frac{\beta}{a} n^{-\beta/2}$$

(5)

For $n \geq 1$, where $\beta > 1$. The regret of the sequential procedure $(T, \theta'_n)$ is defined as Equation 6:

$$r_T(T, \theta'_n) = E[L_n(T, \theta'_n)] - R'_2$$

(6)

where, $R'_2$ is as in (2). In this study we provide a second-order asymptotic expansion, as as $a \to \infty$, for $r_T(T, \theta'_n)$ and show that this regret is asymptotically negative if we choose $\theta < \theta < \sqrt[4]{4\beta - \delta} / (2.5\beta + T)$.

Starr and Woodroofe (1969) considered the case in which $X_1, X_2, \ldots$ are i.i.d. Normal random variables and showed that the regret of their procedure is $O(1)$. Then, Woodroofe (1977) showed that the regret is $0.5 + o(1)$ if $m_a \geq 4$. Martinsek (1983) extended Woodroofe’s result to the nonparametric case. Tahir (1989) proposed a class of bias-reduction estimators of the mean of the one-parameter exponential family and provided an asymptotic second-order lower bound for the regret. Kim and Han (2009) considered estimation of the scale parameter of the Rayleigh distribution under general progressive censoring. Mousa et al. (2005; Prakash, 2013) focused on Bayesian prediction and Bayesian estimation for Rayleigh models.

2. ASYMPTOTIC EXPANSION FOR THE REGRET OF THE SEQUENTIAL PROCEDURE

Rewrite the stopping time $T$ in (3) as Equation 7:

$$T = \inf \left\{ n \geq m_a : n > a \left( \frac{1}{\sigma} \sum_{i=1}^{n} (Y_i - \bar{Y})^2 \right)^{\beta/2} \right\}, \text{where} \left( Y_n - \bar{Y}_n \right)^{\beta/2} = \nu$$

(7)

And let $U_n = t(V_n)\nu^{1/2} - a$ denote the excess over the stopping boundary. Chang and Hsiung (1979) showed that the excess $U_n$ converges in distribution to a random variable $U$ as $a \to \infty$.

**Lemma 1**

Let $T$ be as in (3). Then $T \to 0$ w.p.1 as $a \to \infty$. Moreover:

$$E[T] = a + \nu - 1.375 + o(1)$$

As $\nu \to 0$, where $\nu = E[U]$ is the asymptotic mean of the excess over the boundary.

**Proof**

The first assertion follows from Lemma 1 of Chow and Robbins (1985). For the second assertion:

$$E[T] = a + \nu - 0.5 - \frac{3}{8\sigma^2} E\left[ (Y_i - \mu_i)^2 - \sigma_i^2 \right] + o(1)$$

$$= a + \nu - 0.5 - \frac{3}{8}(\kappa - 1) + o(1)$$

$$= a + \nu - 1.375 + o(1)$$

As $\nu \to 0$, by Chang and Hsiung (1979), using the fact that the kurtosis of $Y_1$ is $\kappa = \sigma_i^4 E(Y_i - \mu_i)^4 = 6$.

**Proposition 1**

Let $\theta'_n$ be defined by (5) and let $T$ be defined by (3) with $m_a$ being such that $\delta(a) \leq m_a = o(a)$ as $a \to \infty$ for some $\delta > 0$. Then, $E[\theta'_n] = \theta + o(1/a)$ as $a \to \infty$.

**Proof**

For $a > 0$ Equation 8:
The proposition follows by taking the limit as \( a \to \infty \) in (8) and using (4) and the fact that 
\[ E(T/a)^{-(1+\beta)} \to \sigma^\beta_\alpha \] as \( a \to \infty \) if \( \beta > 1 \), by the first assertion of Lemma 1 and (2.2) of Martinsek (1983).

Let \( r_n(T, \theta^\beta) \) be as in (6). Then Equation 9:

\[
\begin{align*}
&= \frac{a^{-(1+\beta)}}{T^{1-\beta}} a \beta \sigma^\beta_\alpha 
&+ \beta \sigma^\beta_\alpha \left[ \frac{a^{-(1+\beta)}}{T^{1-\beta}} a \left( \overline{Y}_r - \mu_r \right) \right] 
\end{align*}
\]

Lemma 2

Let \( T \) be defined by (3) with \( m^*_a \) being such that \( \delta(\sigma_\alpha) = o(\alpha) \) as \( \alpha \to \infty \) for some \( \delta > 0 \) and with \( \beta > 1 \). Then:

\[
E \left[ \frac{a^{-(1+\beta)}}{T^{1-\beta}} a \left( \overline{Y}_r - \mu_r \right) \right] = \frac{2(1-\beta)}{\beta \sigma^2_\alpha} \frac{\beta}{\sigma^2_\beta} + o(1)
\]

As \( \alpha \to \infty \).

Proof

First, observe that Equation 10:

\[
E \left[ \frac{a^{-(1+\beta)}}{T^{1-\beta}} a \left( \overline{Y}_r - \mu_r \right) \right] = \left( \frac{2(1-\beta)}{\beta \sigma^2_\alpha} \frac{\beta}{\sigma^2_\beta} + o(1) \right)
\]

For \( a > 0 \). Moreover Equation 11:

\[
aE[\overline{Y}_r - \mu_r] = \frac{\beta}{\sigma^2_\beta} + o(1)
\]

As \( a \to \infty \), by (4). Next, expand \( g(y) = y^{1/\beta} \) at \( y = \sigma^\beta_\beta \), substitute \( y = T/\alpha \) and multiply by \( a(\overline{Y}_r - \mu_r) \) to obtain Equation 12:

\[
\begin{align*}
&= \frac{a^{-(1+\beta)}}{T^{1-\beta}} \frac{1}{\sigma^2_\alpha} \left( \overline{Y}_r - \mu_r \right) 
&= \left( \frac{1}{\beta} - 1 \right) \frac{T^{1-\beta}}{\alpha} \frac{1}{\sigma^2_\alpha} \left( \overline{Y}_r - \mu_r \right)
\end{align*}
\]

where, \( T^* \) is a random variable such that \( \overline{Y}_r, \sigma^\beta_\alpha \sim |T/a - \sigma^\beta_\beta| \). Next, rewrite \( T \) in (3) as \( T = \inf\{n \geq m^*_a; n(\sigma_\beta/n)^{1/\beta} > a \} \), where, \( \sigma_\beta \) is as in (7) and let:

\[
U^* = \frac{T}{\sigma^\beta_\beta} - a
\]

Denote the excess over the stopping boundary. Expanding \( h(y) = y^{1/\beta} \) at \( y = \sigma^\beta_\beta \), substituting \( y = V_T/T \) and multiplying by \( T \) yields:

\[
\begin{align*}
&= T \left( \frac{V_T}{T} \right)^{1-\beta/\sigma^\beta_\alpha} - \frac{\lambda_T}{\sigma^\beta_\alpha} + \frac{\beta(\beta+2)}{8\lambda^2_T} \left( \frac{V_T - T\sigma^\beta_\alpha}{T} \right)^2
\end{align*}
\]

for \( a > 0 \), where \( \lambda_T \) is a random variable between \( V_T/T \) and \( \sigma^\beta_\beta \). Furthermore, write:

\[
V_T = \sum_{i=1}^\infty (Y_i - \mu_i)^2 - T(\overline{Y}_r - \mu_r)^2
\]

To obtain:

\[
U^* = \frac{T}{\sigma^\beta_\alpha} - a - \frac{\beta}{2\sigma^\beta_\alpha} (W_i - T\sigma^\beta_\alpha) 
+ \frac{\beta}{2\sigma^\beta_\alpha} T(\overline{Y}_r - \mu_r)^2 + \frac{\beta(\beta+2)}{8\lambda^2_T} \left( \frac{V_T - T\sigma^\beta_\alpha}{T} \right)^2
\]

For \( a > 0 \), where \( W_i = \sum_{i=1}^\infty (Y_i - \mu_i)^2 \). It follows easily that

Equation 13:

\[
\begin{align*}
&= \frac{T}{\sigma^\beta_\alpha} = \frac{\sigma^\beta_\alpha}{a} (U^* - \xi^\beta_\alpha) + \frac{\beta}{2a\sigma^\beta_\alpha} (W_i - T\sigma^\beta_\alpha)
\end{align*}
\]

For \( a > 0 \), where:

\[
\xi^\beta_\alpha = \frac{\beta}{2\sigma^\beta_\alpha} T(\overline{Y}_r - \mu_r)^2 + \frac{\beta(\beta+2)}{8\lambda^2_T} \left( \frac{V_T - T\sigma^\beta_\alpha}{T} \right)^2
\]

Substituting (13) in (12) yields Equation 14:
(\frac{a^{1+\beta}}{T^{1+\beta}} - \frac{1}{\sigma_\varepsilon}}^\varepsilon (Y_t - \mu_t)
\]

\begin{align*}
&= \left(1 - \frac{1}{\beta}\right) \sigma_\varepsilon^2 T (U_t - \xi_t) (Y_t - \mu_t) \\
&+ \left(1 - \frac{1}{\beta}\right) \frac{\beta}{2\alpha_\varepsilon} r_t^{1+\beta} (W_t - T \sigma_\varepsilon^2) (Y_t - \mu_t) \\
&= \left(1 - \frac{1}{\beta}\right) \sigma_\varepsilon^2 I_1(a) + \frac{1 - \beta}{2\alpha_\varepsilon} I_2(a),
\end{align*}

\tag{14}

Say. Let \(S_n = Y_1 + \ldots + Y_n\), \(n \geq 1\). Then Equation 15:

\[E[I^2(a)] = E \left[ \frac{T^{1+\beta}}{T^{1+\beta}} (U_t - \xi_t) (S_n - \mu_t T) \right] = \frac{\sigma_\varepsilon^2}{\sqrt{a^2\alpha_\varepsilon^2}} E \left[ \frac{(U_t - \xi_t) a T^{1+\beta}}{\sqrt{a^2\alpha_\varepsilon^2}} (S_n - \mu_t T) \right]
\]

\[\leq \frac{\sigma_\varepsilon^2}{\sqrt{a}} \sqrt{E[(U_t - \xi_t)^2]} \leq T^{1+\beta} \left( \frac{a^2}{\sqrt{a^2\alpha_\varepsilon^2}} \right)^2 \rightarrow 0
\]

as \(a \rightarrow \infty\), by Hölder’s inequality, the fact that \(T \rightarrow \sigma_\varepsilon^2 \left[ |T| - \sigma_\varepsilon^2 |T| \right] \rightarrow 0\) w.p.1 since \(T / a \rightarrow \sigma_\varepsilon^2\), \(S_n - \mu_t T\) converges in distribution to a Standard Normal random variable by Anscombe’s theorem, the facts that \(E[U_t] \rightarrow E[U_1] < \infty\) and \(E[\xi_t^2] = O(1) \rightarrow \infty\) and (2.3), (2.8) and (2.9) of Martinsek (1983). To evaluate \(E[I^2(a)]\), observe that Equation 16:

\[I_1(a) = 2a^2 \sigma_\varepsilon^2 T^{1+\beta} (W_t - T \sigma_\varepsilon^2) (S_n - \mu_t T) \frac{\alpha_\varepsilon^2}{a^2\alpha_\varepsilon^2}
\]

\[= 2a^2 \sigma_\varepsilon^2 \left( T^{1+\beta} - \frac{1}{\alpha_\varepsilon^2} \right) (W_t - T \sigma_\varepsilon^2) (S_n - \mu_t T) \frac{\alpha_\varepsilon^2}{a^2\alpha_\varepsilon^2}
\]

\[\rightarrow 2a^2 \sigma_\varepsilon^2 T^{1+\beta} (2\xi_t)^2
\]

\[-2a^2 \sigma_\varepsilon^2 \left( T^{1+\beta} - \frac{1}{\alpha_\varepsilon^2} \right) (W_t - T \sigma_\varepsilon^2)^2
\]

\[-2a^2 \sigma_\varepsilon^2 \left( T^{1+\beta} - \frac{1}{\alpha_\varepsilon^2} \right) (S_n - \mu_t T)^2
\]

as in distribution \(\rightarrow 2a^2 \sigma_\varepsilon^2 T^{1+\beta} (2\xi_t)^2
\]

As \(a \rightarrow \infty\), by Anscombe’s theorem and the fact that \(T \otimes \sigma_\varepsilon^2\) w.p.1 as \(a \rightarrow \infty\) where \(Z\) is a random variable having the Standard Normal distribution. Thus Equation 17:

\[E[I^2(a)] = 4a^2 \sigma_\varepsilon^2 T^{1+\beta} + o(1)
\]

As \(a \rightarrow \infty\), by (16) and (2.3) and (2.4) of Martinsek (1983). Taking expectation in (14) and using (15) and (17) yields Equation 18:

\[E \left[ \frac{a^{1+\beta}}{T^{1+\beta}} - \frac{1}{\sigma_\varepsilon^2} (\widetilde{Y}_t - \mu_\varepsilon) \right] = \frac{2(1 - \beta)}{\sigma_\varepsilon^2} + o(1)
\]

\[\rightarrow \infty\] The lemma follows by taking the limit, as \(a \rightarrow \infty\), in (10) and using (11) and (18).

Theorem 1. Let \(T\) be defined by (3) with \(m_0\) being such that \(\delta a \leq m_0 = o(a)\) as \(a \rightarrow \infty\) for some \(\delta > 0\) and \(\beta > 1\). Let the regret of the biased-corrected procedure \((T, \mu_t)\) be as in (6). Then:

\[r_\varepsilon(T, \theta) = 3.25 \beta^2 + \beta - \frac{4\beta(\beta - 1)}{\theta} + o(1)
\]

As \(a \theta\).

Proof

First Equation 19:

\[r_\varepsilon(T, \widetilde{Y}_t) = E[a^2 \sigma_\varepsilon^2 (\widetilde{Y}_t - \mu_\varepsilon)^2 + T]
\]

\[-2a^2 \sigma_\varepsilon^2 = 5.25 \beta^2 + \beta + o(1)
\]

As \(a \rightarrow \infty\) if \(\delta > 1\), by Martinsek (1988). Next, take the limit, as \(a \rightarrow \infty\), in (9) and use (19), Lemma 2 and the fact that:

\[E \left[ \frac{a^{1+\beta}}{T^{1+\beta}} \right] = \frac{1}{\sigma_\varepsilon^2} + o(1)
\]

as \(a \rightarrow \infty\) if \(\delta > 1\), by the first assertion of Lemma 1 and (2.2) of Martinsek (1983), to complete the proof.

3. NEGATIVE ASYMPTOTIC REGRET

Theorem 1 shows that the biased-corrected procedure \((T, \mu_t)\) has a lower asymptotic regret than the procedure \((T, \mu_t)\). Also, the asymptotic regret of the procedure \((T, \mu_t)\) is negative if Equation 20:

\[0 < \theta < \frac{4\beta - 4}{3.25 \beta + 1} = \theta_\beta
\]

\(278\) JMSS
Table 1. Asymptotic regret for various choices of $\beta>1$ and $0<\theta<\theta_0$ (see (20))

| $b$  | $q_0$  | $q$  | Asymptotic regret       |
|------|--------|------|-------------------------|
| 1.5  | 0.836  | 0.2  | -6686.1880000           |
| 1.5  | 0.836  | 0.3  | -4106.4138000           |
| 1.5  | 0.836  | 0.4  | -723.6093800            |
| 1.5  | 0.836  | 0.5  | -183.1875000            |
| 1.5  | 0.836  | 0.7  | -16.6870790             |
| 2.0  | 0.901  | 0.2  | -124985.0000000         |
| 2.0  | 0.901  | 0.3  | -10958.9370000          |
| 2.0  | 0.901  | 0.4  | -1938.1250000           |
| 2.0  | 0.901  | 0.5  | -497.0000000            |
| 2.0  | 0.901  | 0.8  | -15.5175780             |
| 2.0  | 0.901  | 0.9  | -0.0534114              |
| 5.0  | 0.988  | 0.2  | -109653.1200000         |
| 5.0  | 0.988  | 0.3  | -19445.0000000          |
| 5.0  | 0.988  | 0.6  | -1628.4276000           |
| 5.0  | 0.988  | 0.7  | -593.7387800            |
| 5.0  | 0.988  | 0.9  | -64.2841140             |
| 10   | 1.012  | 0.2  | -5624665.0000000        |
| 10   | 1.012  | 0.4  | -87555.6250000          |
| 10   | 1.012  | 0.7  | -2724.9495000           |
| 10   | 1.012  | 0.8  | -1038.2910000           |
| 10   | 1.012  | 0.9  | -342.4035100            |
| 15   | 1.020  | 0.2  | -13124254.0000000       |
| 15   | 1.020  | 0.3  | -1151517.1000000        |
| 15   | 1.020  | 0.5  | -53013.7500000          |
| 15   | 1.020  | 0.7  | -6393.6322000           |
| 15   | 1.020  | 0.9  | -834.3582000            |
| 15   | 1.020  | 1.0  | -93.7500000             |

This means that for the values of $\theta$ in the interval $(0, \theta_0)$ with $\beta>1$, the sequential procedure $(\hat{T}, \hat{\theta})$ performs better, for large values of $a$, than the best fixed-sample-size procedure $(a, \hat{\theta})$, where $n_a$ is the greatest integer less than or equal to $\frac{a}{\theta} \beta^2$ (see Table 1).

4. CONCLUSION

We have proposed a sequential procedure for estimating the square of the shape parameter of the Rayleigh distribution and provided a second-order asymptotic expansion for the incurred regret. It is seen that the proposed procedure performs better than the best fixed-sample-size procedure if the shape parameter lies in a specific subinterval of the positive real numbers.

For future research, it would be worth considering Bayesian sequential estimation of a function of the shape parameter of the Rayleigh distribution, in which the focus will be on finding a sequential procedure and approximating the Bayes regret, as well as comparing the proposed procedure with existing procedures.

5. REFERENCES

Chang, I.S. and C.A. Hsiung, 1979. Approximations to the expected sample size of certain sequential procedures. Proceedings of the Conference on, Recent Developments in Statistical Methods and Applications, (CRDS’79), Taipei, Institute of Mathematic Academy Sinica, pp: 71-82.

Chow, Y.S. and H. Robbins, 1985. On the asymptotic theory of fixed-width sequential confidence intervals for the mean. Annal. Math. Stat., 36: 457-462. DOI: 10.1007/978-1-4612-5110-1_19

Chow, Y.S. and K.F. Yu, 1981. The performance of a sequential procedure for the estimation of the mean. Annal. Stat., 9: 189-198.

Kim, C. and K. Han, 2009. Estimation of the scale parameter of the Rayleigh distribution under general progressive censoring. J. Korean Stat. Society, 38: 239-246. DOI: 10.1016/j.jkss.2008.10.005

Martinsek, A.T., 1983. Second order approximation to the risk of a sequential procedure. Annal. Stat., 11: 827-836.
Martinsek, A.T., 1988. Negative regret, optional stopping and the elimination of outliers. J. Am. Stat. Assoc., 83: 160-163.

Mousa, M.A. M. Ali and S.A. Al-Sagheer, 2005. Bayesian prediction for progressively Type-II censored data from Rayleigh model. Commun. Stat. Theory Meth., 34: 2353-2361. DOI: 10.1080/03610920500313767

Prakash, G., 2013. Bayes estimation in the inverse Rayleigh model. Electron. J. App. Stat. Anal., 6: 67-83. DOI: 10.1285/i20705948v6n1p67.

Starr, N. and M. Woodroofe, 1969. Remarks on sequential point estimation. Proc. Nat. Acad. Sci. USA., 63: 285-288.

Tahir, M., 1989. An asymptotic lower bound for the local minimax regret in sequential point estimation. Anal. Stat., 17: 1335-1346.

Woodroofe, M., 1977. Second order approximations for sequential point and interval estimation. Anal. Stat., 5: 984-995.