ON A REFINEMENT OF THE BIRCH AND SWINNERTON-DYER CONJECTURE IN POSITIVE CHARACTERISTIC

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Abstract. We formulate a refined version of the Birch and Swinnerton-Dyer conjecture for abelian varieties over global function fields. This refinement incorporates both families of congruences between the leading terms of Artin-Hasse-Weil $L$-series and also strong restrictions on the Galois structure of natural Selmer complexes and constitutes a precise analogue for abelian varieties over function fields of the equivariant Tamagawa number conjecture for abelian varieties over number fields. We then provide strong supporting evidence for this conjecture including giving a full proof, modulo only the assumed finiteness of Tate-Shafarevich groups, in an important class of examples.

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1. Introduction

1.1. Let $A$ be an abelian variety that is defined over a function field $K$ in one variable over a finite field of characteristic $p$.

In [28] Artin and Tate formulated a precise conjectural formula for the leading term at $s = 1$ of the Hasse-Weil $L$-series attached to $A$.

This formula constituted a natural ‘geometric’ analogue of the Birch and Swinnerton-Dyer Conjecture for abelian varieties over number fields and was subsequently verified unconditionally by Milne [21] in the case that $A$ is constant and by Ulmer [31] in certain other special cases. Further partial results have been obtained by many other authors and these efforts culminate in the main result of the seminal article of Kato and Trihan [17] which
shows that the conjecture is valid whenever there exists a prime $\ell$ such that the $\ell$-primary component of the Tate-Shafarevic group of $A$ over $K$ is finite.

In this article we now formulate, and provide strong evidence for, a refined version of this conjecture that also incorporates precise families of integral congruence relations between the leading terms at $s = 1$ of the Artin-Hasse-Weil $L$-series that are attached to $A$ and to irreducible complex characters (with open kernel) of the absolute Galois group of $K$. This conjecture is a precise analogue for abelian varieties over function fields of the equivariant Tamagawa number conjecture ('ETNC'), including the $p$-primary part, for the motive $h^1(A)(1)$ of abelian varieties $A$ over number fields.

To be a little more precise about our results we now fix a finite Galois extension $L$ of $K$ with group $G$.

Then, as a first step, we shall prove that the leading terms of the Artin-Hasse-Weil $L$-series that are attached to $A$ and to the irreducible complex characters of $G$ are interpolated by a canonical element of the Whitehead group $K_1(\mathbb{R}[G])$ of the group ring $\mathbb{R}[G]$. (This result is, a priori, far from clear and requires one to prove, in particular, that leading terms at irreducible symplectic characters are strictly positive.)

Our central conjecture is then a precise formula for the image of this element under the connecting homomorphism from $K_1(\mathbb{R}[G])$ to the relative algebraic $K_0$-group $K_0(\mathbb{Z}[G], \mathbb{R}[G])$ of the ring inclusion $\mathbb{Z}[G] \subset \mathbb{R}[G]$.

The conjectural formula involves a canonical Euler characteristic element that is constructed by combining a natural ‘Selmer complex’ of $G$-modules together with the classical Neron-Tate height pairing of $A$ over $L$. This Selmer complex is constructed from the flat cohomology of the torsion subgroup scheme of the Néron model of $A$ over the projective curve $X$ with function field $K$ and is both perfect over $\mathbb{Z}[G]$ and has cohomology groups that are closely related to the classical Mordell-Weil and Selmer groups of $A'$ and $A$ over $L$.

The formula also involves the Euler characteristic of an auxiliary perfect complex of $G$-modules that is constructed directly from the Zariski cohomology of an appropriate line bundle over $X$ and is necessary in order to compensate for certain choices of pro-$p$ subgroups that are made in the definition of the Selmer complex.

If $L = K$, then $K_0(\mathbb{Z}[G], \mathbb{R}[G])$ identifies with the quotient of the multiplicative group $\mathbb{R}^\times$ by $\{\pm 1\}$ and we check that in this case our conjecture recovers the classical Birch and Swinnerton-Dyer conjecture for $A$ over $K$.

In the general case, the conjecture incorporates both a family of explicit congruence relations between the leading terms of the Artin-Hasse-Weil $L$-series attached to $A$ and to characters of $G$ and also strong restrictions on the Galois structure of Selmer complexes (see Remarks 4.3 and 4.4).

As far as evidence is concerned, we are able to adapt the methods of [17] in order to prove that, whenever there exists a prime $\ell$ such that the $\ell$-primary component of the Tate-Shafarevic group of $A$ over $L$ is finite, then our conjecture is valid modulo a certain finite subgroup $T_{A,L/K}$ of $K_0(\mathbb{Z}[G], \mathbb{R}[G])$, the nature of which depends both on the reduction properties of $A$ and the ramification behaviour in $L/K$.

For example, if $A$ is semistable over $K$ and $L/K$ is tamely ramified, then $T_{A,L/K}$ vanishes and so we obtain a full verification of our conjecture in this case.
In the worst case the group $T_{A,L/K}$ coincides with the torsion subgroup of the subgroup $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$ of $K_0(\mathbb{Z}[G], \mathbb{R}[G])$ and our result essentially amounts to proving a version of the main result of [17] for the leading terms of the Artin-Hasse-Weil $L$-series attached to $A$ and to each character of $G$.

However, even the latter result is new and of interest since, for example, it both establishes the ‘order of vanishing’ part of the Birch and Swinnerton-Dyer conjecture for Artin-Hasse-Weil $L$-series and, in addition, plays a key role in a complementary article of the first two authors in which the conjecture formulated here is proved, modulo the standard finiteness hypothesis on Tate-Shafarevic groups, in the case of abelian varieties $A$ that are generically ordinary.

As a key step in the proof of our main result we shall combine Grothendieck’s description of Artin-Hasse-Weil $L$-series in terms of the action of Frobenius on $\ell$-adic cohomology (for some prime $\ell \neq p$) together with a result of Schneider concerning the Néron-Tate height-pairing to show that our conjectural formula naturally decomposes as a sum of ‘$\ell$-primary parts’ over all primes $\ell$.

It is thus of interest to note that in some related recent work Trihan and Vauclair [30] have adapted the approach of [17] in order to formulate and prove a natural main conjecture of ($p$-adic) non-commutative Iwasawa theory for $A$ relative to unramified $p$-adic Lie extensions of $K$ under the assumptions both that $A$ is semistable over $K$ and that certain Iwasawa-theoretic $\mu$-invariants vanish.

In addition, for each prime $\ell \neq p$, Witte [33] has used techniques of Waldhausen $K$-theory to deduce an analogue of the main conjecture of non-commutative Iwasawa theory for $\ell$-adic sheaves over arbitrary $p$-adic Lie extensions of $K$ from Grothendieck’s formula for the Zeta function of such sheaves.

It seems likely that these results can be combined with the descent techniques developed by Venjakob and the first author in [3] and the explicit interpretation of height pairings in terms of Bockstein homomorphisms that we use below to give an alternative, although rather less direct, proof of the $\ell$-primary part of our main result for any $\ell \neq p$ and of the $p$-primary part of our main result in the special case that $L/K$ is unramified and suitable $\mu$-invariants vanish.

However, even now, there are still no ideas as to how one could formulate a main conjecture of (non-commutative) Iwasawa theory for $A$ relative to any general class of ramified $p$-adic Lie extensions of $K$.

It is thus one of the main observations of the present article that the techniques developed by Kato and Trihan in [17] are essentially themselves sufficient to prove refined versions of the Birch and Swinnerton-Dyer conjecture without the necessity of developing the appropriate formalism of non-commutative Iwasawa theory (and hence without the need to assume the vanishing of relevant $\mu$-invariants).

This general philosophy also in fact underpins the complementary work of the first two authors regarding generically ordinary abelian varieties.

In a little more detail, the main contents of this article is as follows.

Firstly, in [2] we use the leading terms of the Artin-Hasse-Weil $L$-series attached to complex characters of $G$ to define a canonical element of $K_1(\mathbb{R}[G])$. Then, in [3] we define a natural family of ‘Selmer complexes’ of $G$-modules and establish some of its key properties.
In §4 we formulate our main conjecture and state the main evidence for this conjecture that we prove in later sections.

In §5 we prove certain useful preliminary results including a purely $K$-theoretic observation that plays a key role in several subsequent calculations. We also show that our conjecture is consistent in some important respects and use a result of Schneider to give a useful reformulation of the conjecture.

In §6 and §7 we investigate the syntomic cohomology complexes introduced by Kato and Trihan in [17], with a particular emphasis on understanding conditions under which these complexes can be shown to be perfect.

In §8 we analyse when certain morphisms of complexes that arise naturally in the theory are ‘semisimple’ (in the sense of Galois descent) and deduce, modulo the assumed finiteness of Tate-Shafarevich groups, the order of vanishing part of the Birch and Swinnerton-Dyer conjecture for Artin Hasse-Weil $L$-series.

Then, in §9 we combine the results established in earlier sections to prove our main results.

Finally, in an appendix to this article we show that the coherent cohomology over a ‘separated’ formal fs log scheme can be computed via the Čech resolution with respect to an affine Kummer-étale covering. (This result plays an important role in the arguments of §7 and, whilst it is surely well-known to experts, we have not been able to find a good reference for it.)

1.2. To end this introduction we fix certain notation and conventions that are to be used in the sequel.

We fix a prime number $p$ and a function field $K$ in one variable over a finite field of characteristic $p$. We write $X$ for the proper smooth connected curve over $\mathbb{F}_p$ that has function field $K$.

Let $A$ be an abelian variety over $K$. Let $U$ be a dense open subset of $X$ such that $A/K$ has good reduction on $U$. We write $\mathcal{A}$ for the Néron model of $A$ over $X$.

Let $F$ be a finite extension of $K$. Let $X_F$ denote the proper smooth curve over $\mathbb{F}_p$ that has function field $F$. We will denote the ‘base extension’ of an object $*$ over either $K$ or $X$ to that over $F$ of $X_F$ by a subscript $*_F$. For example $A_F$ and $U_F$ denote $A \times_K F$, $U \times_X X_F$ respectively. If there is no danger of confusion we often omit the subscript $F$.

The Pontryagin dual of an abelian group $M$ is denoted by $M^* := \text{Hom}(M, \mathbb{Q}/\mathbb{Z})$. We fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and for every prime $\ell$ an algebraic $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$. Furthermore, for every prime $\ell$, we fix an embedding of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}}_\ell$.

For each natural number $n$ the $n$-torsion subgroup of an abelian group $M$ is denoted by $M[n]$. The full torsion subgroup of $M$ is denoted by $M_{\text{tor}}$ and, for each prime $\ell$, the $\ell$-primary part of $M_{\text{tor}}$ is denoted by $M[\ell]$.

For a finite group $G$ we write $\text{Ir}(G)$ for the set of its irreducible complex valued characters and $\text{Ir}^*(G)$ for the subset of $\text{Ir}(G)$ comprising characters that are symplectic.

For any commutative ring $R$ we write $R[G]$ for the group ring of $G$ over $R$ and denote its centre by $\zeta(R[G])$. We identify $\zeta(\mathbb{C}[G])$ with $\prod_{\text{Ir}(G)} \mathbb{C}$ in the standard way.

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2. Leading Terms of Artin-Hasse-Weil $L$-series

For each $\chi$ in $\text{Ir}(G)$ we write $L_U^*(A, \chi, s)$ for the Artin-Hasse-Weil $L$-series of the pair $(A, \chi)$ that is truncated by removing the Euler factors for all places outside $U$.

We now show that there exists a canonical element of the Whitehead group $K_1(\mathbb{R}[G])$ that naturally interpolates the leading terms $L_U^*(A, \chi, 1)$ at $s = 1$ in the Taylor expansions of the functions $L_U(A, \chi, s)$ as $\chi$ ranges over $\text{Ir}(G)$.

This ‘$K$-theoretical leading term’ will then play a key role in the conjecture that we discuss in subsequent sections.

To define the element we use that fact that the algebra $\mathbb{R}[G]$ is semisimple and hence that the classical reduced norm construction induces a homomorphism $\text{Nrd}_{\mathbb{R}[G]}$ of abelian groups from $K_1(\mathbb{R}[G])$ to $\mathbb{C}^\times$.

**Theorem 2.1.** There exists a unique element $L_U^*(A_{L/K}, 1)$ of $K_1(\mathbb{R}[G])$ with the property that $\text{Nrd}_{\mathbb{R}[G]}(L_U^*(A_{L/K}, 1)) = L_U^*(A, 1)$ for all $\chi$ in $\text{Ir}(G)$.

**Proof.** Since the natural map $\mathbb{R}[G]^\times \to K_1(\mathbb{R}[G])$ is surjective, the Hasse-Schilling-Maass norm theorem implies both that $\text{Nrd}_{\mathbb{R}[G]}$ is injective and that its image is equal to the subgroup of $\prod_{\text{Ir}(G)} \mathbb{C}^\times$ comprising elements $(x_\chi)_\chi$ that satisfy both of the following conditions

\[
\begin{align*}
x_{\tau \cdot \chi} &= \tau(x_\chi), & \text{for all } \chi \in \text{Ir}(G), \text{ and } \\
x_\chi &\in \mathbb{R} \text{ and } x_\chi > 0, & \text{for all } \chi \in \text{Ir}(G),
\end{align*}
\]

where $\tau$ denotes complex conjugation. (For a proof of this result see [8, Th. (45.3)].)

The injectivity of $\text{Nrd}_{\mathbb{R}[G]}$ implies that there can be at most one element of $K_1(\mathbb{R}[G])$ with the stated property and to show that such an element exists it is enough to show that the element $(L_U^*(A, \chi, 1))_\chi$ of $\prod_{\text{Ir}(G)} \mathbb{C}^\times$ satisfies the above displayed conditions.

This fact is established in Proposition 2.2 below.

The following result extends an observation of Kato and Trihan from [17, Appendix].

**Proposition 2.2.** The following claims are valid for every $\chi$ in $\text{Ir}(G)$.

(i) $\tau(L_U^*(A, \chi, 1)) = L_U^*(A, \tau \circ \chi, 1)$, where $\tau$ denotes complex conjugation.

(ii) Write $\mathbb{F}_q$ and $\overline{\mathbb{F}_q}$ for the total field of constants of $K$ and $L$ respectively. Then if $\chi$ is both real valued and not inflated from a non-trivial one dimensional representation of $\text{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$, then the leading term $L_U^*(A, \chi, 1)$ is a strictly positive real number.

**Proof.** At the outset we fix a prime $\ell$ with $\ell \neq p$ and write $\mathbb{Q}_\ell^c$ for the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. We also fix an isomorphism $\mathbb{C} \cong \mathbb{C}_\ell$ that we suppress from the notation.

In particular, for each $\rho$ in $\text{Ir}(G)$ we fix a realisation $V_\rho$ of $\rho$ over $\mathbb{Q}_\ell^c$ and do not distinguish between it and the space $\mathbb{Q}_\ell^c \otimes_{\mathbb{Q}_\ell} V_\rho$.

Now for every $\rho$ in $\text{Ir}(G)$ Grothendieck [13] (see also the proof of [22, Chap. VI, Th. 13.3]) has proved that there is an equality of power series

\[
L_U(A, \rho, s) = \prod_{i=0}^{i=2} Q_{\rho,i}(q^{-s}) (-1)^{i+1},
\]

where $Q_{\rho,i}(q^{-s})$ are certain power series.

This fact is established in Proposition 2.2 below.

\[
Q_{\rho,i}(q^{-s}) = \sum_{\chi \in \text{Ir}(G)} \frac{(\chi, 1)^i}{\chi(q^{-s})}
\]

where $\chi(q^{-s})$ is the character $\chi$ evaluated at $q^{-s}$.

**Remark.** The results of this section are normally only valid for ''good'' primes $\ell$ for the field $K$. In the sequel we will always assume that the prime $\ell$ is good for $K$.

\[
\text{Ir}(G) = \{ \chi : \chi(\ell) = 1 \}
\]
where each $Q_{\rho,i}(u)$ is a polynomial in $u$ over $\mathbb{Q}^c$ that can be computed as

$$Q_{\rho,i}(u) := \det(1 - u \cdot \varphi_q | H^i_{\ell,t,c}(U^c, V_\ell(A))(-1) \otimes \mathbb{Q}_\ell V_\rho)$$

where $U^c$ denotes $U \times \mathbb{F}_q \mathbb{P}^1$, $V_\ell(A)$ the $\mathbb{Q}_\ell$-space spanned by the $\ell$-adic Tate module of $A$ and $\varphi_q$ the $q$-th power Frobenius map.

For each $i$ we set $d_{\rho,i} := \dim_{\mathbb{Q}_\ell}(H^i_{\ell,t,c}(U^c, V_\ell(A))(-1))$ and write the eigenvalues, counted with multiplicity, of $\varphi_q$ on this space as $\{\alpha_{ia}\}_{1 \leq a \leq d_{\rho,i}}$. Then, with $\{\beta_{\rho,b}\}_{1 \leq b \leq \rho(1)}$ denoting the eigenvalues, counted with multiplicity, of $\varphi_q$ on $V_\rho$, one has

$$Q_{\rho,i}(u) = \prod_{a=1}^{d_{\rho,i}} \prod_{b=1}^{\rho(1)} (1 - \alpha_{ia} \beta_{\rho,b} u). \quad (2)$$

Grothendieck’s result implies that the set $\{\alpha_{ia}\}_{1 \leq a \leq d_{\rho,a}}$ is a subset of $\mathbb{Q}^c$ that is stable under the action of any automorphism $\omega$ of $C$. For any such $\omega$ one therefore has

$$\omega(Q_{\rho,i}(u)) = \prod_{a=1}^{d_{\rho,i}} \prod_{b=1}^{\rho(1)} (1 - \alpha_{ij} \beta_{\rho,j} u) = \prod_{a=1}^{d_{\rho,i}} \prod_{b=1}^{\rho(1)} (1 - \alpha_{ij} \omega(\beta_{\rho,j}) u) = (Q_{\omega \circ \rho,i}(u)) \quad (3)$$

and so (1) implies that there is an equality of power series

$$\omega(L_U(A, \rho, s)) = L_U(A, \omega \circ \rho, s). \quad (4)$$

By applying this observation with $\omega = \tau$ we deduce that the orders of vanishing at $s = 1$ of the series $L_U(A, \rho, s)$ and $L_U(A, \tau \circ \rho, s)$ are equal and moreover that

$$\tau(L_\omega^+(A, \rho, 1)) = L_\omega^+(A, \tau \circ \rho, 1), \quad (5)$$

as required to prove claim (i).

To prove claim (ii) we assume that $\rho$ is real-valued and hence, by (3) with $\omega = \tau$, that each polynomial $Q_{\rho,i}(u)$ belongs to $\mathbb{R}[u]$.

Now, since the weight on $U$ of $V_\ell(A)(-1)$ is one, Deligne [9] has shown that $|\alpha_{ia}| < q^{-\frac{i+1}{2}}$ for all values of $i$ and $a$.

Further, as the space $H^2_{\ell,t,c}(U^c, V_\ell(A))(-1)$ is dual to $H^0(U^c, V_\ell(A^t))(-1)$ and the weight on $U$ of the representation $V_\ell(A^t)(1)$ is $-3$, one has $|\alpha_{2a}| = q^{\frac{3}{2}}$ for all $a$.

Since $|\beta_{\rho,b}| = 1$ for all $b$ these observations combine with (2) to imply that neither of the terms $Q_{\rho,0}(q^{-1})$ or $Q_{\rho,2}(q^{-1})$ vanishes.

In particular, if $m$ denotes the order of vanishing of $L_U(A, \chi, s)$ at $s = 1$, then one has

$$L_\omega^+(A, \chi, 1) = Q_{\rho,0}(q^{-1})^{-1} Q_{\rho,2}(q^{-1})^{-1} \cdot \lim_{s \to 1} (s-1)^{-m} Q_{\rho,1}(q^{-s}). \quad (5)$$

To prove that this quantity is a strictly positive real number we shall split it into a number of subproducts and show that each separate subproduct is a strictly positive real number. To do this we abbreviate each term $\alpha_{ia} \beta_{\rho,b}$ to $\alpha_{iab}$.

At the outset we note that if a term $\alpha_{iab}$ is not real, then (since $Q_{\rho,i}(u)$ belongs to $\mathbb{R}[u]$) there must exist indices $a'$ and $b'$ with either $a \neq a'$ or $b \neq b'$ and such that $\alpha_{ia'a'} = \tau(\alpha_{iab})$ and so the product $(1 - \alpha_{iab} q^{-1})^{-1} - (1 - \alpha_{iab'q^{-1}})$ is a strictly positive real number.
We need therefore only consider terms $\alpha_{iab}$ that are real and to do this we define for each $i \in \{0, 1, 2\}$ sets of indices

$$J_i' := \{(a, b) : 1 \leq a \leq d_{\rho,i}, 1 \leq b \leq \rho(1) \text{ and } \alpha_{iab} = q^{\frac{i+1}{2}}\} \subset J_i := \{(a, b) : 1 \leq a \leq d_{\rho,i}, 1 \leq b \leq \rho(1) \text{ and } \alpha_{iab} \in \mathbb{R}\}.$$ 

Now if either $i = 0$ and $(a, b) \in J_0$ or if $1 \leq i \leq 2$ and $(a, b) \in J_i \setminus J_i'$, then one checks easily that $(1 - \alpha_{iab}q^{-1}) > 0$.

Furthermore, one has $m = |J_1'|$ and

$$\lim_{s \to 1} (s - 1)^{-m} \prod_{(a, b) \in J_1'} (1 - \alpha_{1ab}q^{-s}) = (\lim_{s \to 1} (s - 1)^{-1}(1 - q^{1-s}))^m = (\log(q))^m > 0$$

is a strictly positive real number.

To prove (5) is strictly positive we are therefore reduced to showing that the product $\prod_{j \in J_1'} (1 - \alpha_{2ab}q^{-1}) = \prod_{j \in J_2'} (1 - q^{\frac{1}{2}})$ is positive, or equivalently that $|J_2'|$ is even.

To do this we set $\Delta := \text{Gal}(L/F_q/K^c_{/q})$ and recall that $H^{2}_{\text{et},c}(U^c, V_{\ell}(A)(\rho))(-1)$ is dual to the 1-twist of the space

$$H^0(U^c, V_{\ell}(A^1)(\rho^\vee)) = (V_{\ell}(A^1)(\rho^\vee))^{\text{Gal}(K^c/\mathbb{F}_q)} = ((V_{\ell}(A^1))^{\text{Gal}(K^c/\mathbb{F}_q)}(\rho^\vee))^\Delta \cong (V_{\ell}(B)(\rho^\vee))^\Delta = V_{\ell}(B) \otimes (\rho^\vee)^\Delta.$$

Here the first equality is obvious and the second is true because the restriction of $\rho$ to $\text{Gal}(K^c/\mathbb{F}_q)$ is trivial, $B$ is the $L/\mathbb{F}_q'$ trace of $A^1$ (which exists since $L/\mathbb{F}_q'$ is primary) and the last equality is true because $B$ is defined over $\mathbb{F}_q'$.

In particular, if $(\rho^\vee)^\Delta$ vanishes, then $|J_2'| = d_2 = 0$ is even and we are done.

We claim now that $(\rho^\vee)^\Delta$ does indeed vanish unless $\rho$ is trivial. To show this we note that $\Delta$ identifies with a normal subgroup of $G$ in such a way that the quotient is isomorphic to the cyclic group $H := \text{Gal}(\mathbb{F}_q'/\mathbb{F}_q)$.

Thus, if $\eta$ is any irreducible subrepresentation of $\text{res}_G^\Delta(\rho^\vee)$, then Clifford’s theorem (cf. [N, Th. 11.1(i)]) implies that $\text{res}_G^\Delta(\rho^\vee)$ is direct sum of conjugates of $\eta$ and hence that $\text{res}_G^\Delta(\rho^\vee)$ does not vanish if and only if $\eta$ is trivial.

It follows that $\text{res}_G^\Delta(\rho^\vee)$ does not vanish if and only if $\text{res}_G^\Delta(\rho^\vee)$ is trivial and this happens if and only if $\rho^\vee$, and hence also $\rho$ itself, is inflated from a representation of $H$.

Hence, since we have assumed that $\rho$ is both irreducible and not inflated from a non-trivial representation of $H$, the representation $\text{res}_G^\Delta(\rho^\vee)$ does not vanish if and only if $\rho$ is the trivial representation of $G$.

We have now verified the assertion of claim (ii) for all but the trivial representation of $G$ and in this case the claimed result is proved by Kato and Trihan in [17, Appendix]. 

\[\square\]

3. Arithmetic complexes

In this section we construct certain canonical complexes of Galois modules whose Euler characteristics will occur in the formulation of our refined Birch and Swinnerton-Dyer conjecture.
In the sequel, for any noetherian ring $R$ we shall write $D^{\text{perf}}(R)$ for the full triangulated subcategory of the derived category $D(R)$ of (left) $R$-modules comprising complexes that are ‘perfect’ (that is, isomorphic in $D(R)$ to a bounded complex of finitely generated projective $R$-modules).

3.1. Selmer groups. The Tate-Shafarevich group and, for any natural number $n$, the $n$-torsion Selmer group of $A$ over any finite extension $F$ of $K$ are respectively defined to be the kernels

$$\text{III}(A/F) := \ker(H^1(F, A) \to \bigoplus_v H^1(F_v, A))$$

and

$$\text{Sel}_n(A/F) := \ker(H^1(F, A[n]) \to \bigoplus_v H^1(F_v, A)).$$

Here the groups $H^1(F, A), H^1(F, A[n])$ and $H^1(F_v, A)$ denote Galois cohomology and in both cases $v$ runs over all places of $F$ and the arrow denotes the natural diagonal restriction map.

One then defines Selmer groups of $A$ over $F$ via the natural limits

$$\text{Sel}_{Q/\mathbb{Z}}(A/F) := \varprojlim_n \text{Sel}_n(A/F) \quad \text{and} \quad \text{Sel}_{\hat{\mathbb{Z}}}(A/F) := \varprojlim_n \text{Sel}_n(A/F)$$

and, for convenience, we write $X(A/F)$ for the Pontryagin dual of $\text{Sel}_{Q/\mathbb{Z}}(A/F)$.

Remark 3.1. We make much use in the sequel of the fact that the above definitions lead naturally to canonical exact sequences

$$0 \to A(F) \otimes_{\mathbb{Z}} \hat{\mathbb{Z}} \to \text{Sel}_{\hat{\mathbb{Z}}}(A/F) \to \varprojlim_n \text{III}(A/F)[n] \to 0$$

and

$$0 \to (\text{III}(A/F)_{\text{tors}})^{\vee} \to X(A/F) \to \text{Hom}_{\mathbb{Z}}(A(F), \hat{\mathbb{Z}}) \to 0.$$  

3.2. Arithmetic cohomology. For each place $v$ of $F$ outside $U_F$ we fix a pro-$p$ open subgroup $V_v$ of $A(F_v)$ and denote the family $(V_v)_{v \not\in U_F}$ by $V_{U_F}$, or more simply by either $V_F$ or $V$ when the context is clear.

We then follow Kato and Trihan [17] in defining the ‘arithmetic cohomology’ complex $R\Gamma_{ar,V}(U_F, \mathcal{A}_{\text{tors}})$ to be the mapping fibre of the morphism

$$R\Gamma\hat{\mathbb{H}}(U_F, \mathcal{A}_{\text{tors}}) \oplus (\bigoplus_{v \not\in U_F} V_v \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z})[-1] \xrightarrow{(\kappa_1, \kappa_2)} \bigoplus_{v \not\in U_F} R\Gamma\hat{\mathbb{H}}(F_v, \mathcal{A}_{\text{tors}}).$$

Here $\kappa_1$ denotes the natural diagonal localisation morphism in flat cohomology and $\kappa_2$ the restriction of the morphism

$$(A(F_v) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z})[-1] \to \bigoplus_{v \not\in U_F} R\Gamma\hat{\mathbb{H}}(F_v, \mathcal{A}_{\text{tors}})$$

that is obtained by applying $- \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$ to the morphism $A(F_v)[-1] \to R\Gamma\hat{\mathbb{H}}(F_v, \varprojlim_n A[n])$ in $D(\mathbb{Z}[G])$ induced by the fact that $H^1_{\hat{\mathbb{H}}}(F_v, \varprojlim_n A[n])$ vanishes whilst $A(F_v)$ is canonically isomorphic to a submodule of $H^1_{\hat{\mathbb{H}}}(F_v, \varprojlim_n A[n])$. 

Lemma 3.4. There exists a divisor submodule $W$ in (6) implies that it identifies with the compactly support ed ´etale cohomology complex $Pro$.

Remark 3.3. For any prime $w$ each place for our purposes we need to make an appropriate choice of the family (4) by repeating the proof of [17, Lem. 6.2(2)].

For this reason we only sketch the proof, following closely the argument of [17, §6].

To do this we fix a finite Galois extension $L/K$ and set $G := \text{Gal}(L/K)$. We let $X_L$ be the proper smooth curve with function field $L$, and let $U_L \subset X_L$ be the preimage of $U$ (and we will later `shrink' $U$ so that $L/K$ is unramified at places in $U$). For any place $w$ of $L$ we write $G_w$ for its decomposition subgroup in $G$.

We write $A'$ for the Néron model of $A_L$ over $X_L$, and $A_{X_L}$ for the pull back of $A$. 

Lemma 3.4. There exists a divisor $E = \sum_{v \notin U} n(v)v$ on $X$ with supp$(E) = X \setminus U$ and for each place $w \notin U$, a $G_w$-stable pro-$p$ open subgroup $V'_w$ of $A_L(L_w)$ and an open $O_v[G_w]$-submodule $W'_w$ of Lie($A_L(L_w)$) that satisfy all of the following properties.

1. For $w \notin U_K$, we have $\mathcal{A}(m^{2n(v)e(w|v)}_w) \subset V'_w \subset A_{X_L}(m^{n(v)e(w|v)}_w)$, where $e(w|v)$ is the ramification index for $L_w/K_v$.

2. For $w \notin U_L$, we have Lie($\mathcal{A}'$)($m^{2n(v)e(w|v)}_w$) $\subset W'_w \subset$ Lie($A_{X_L}$($m^{n(v)e(w|v)}_w$)).

3. For $w \notin U_L$, the canonical isomorphism $\mathcal{A}'(m^{n(v)e(w|v)}_w)/\mathcal{A}'(m^{2n(v)e(w|v)}_w) \cong$ Lie($\mathcal{A}'$)($m^{n(v)e(w|v)}_w$)/Lie($\mathcal{A}'$)($m^{2n(v)e(w|v)}_w$))

sends the image of $V'_w$ to $W'_w$.

4. For each place $v$ outside $U$ the products $\prod_{w|v} V'_w$ and $\prod_{w|v} W'_w$ are stable under the action of $G$ and for each natural number $i$ the associated cohomology groups $H^i(G, \prod_{w|v} V'_w)$ and $H^i(G, \prod_{w|v} W'_w)$ vanish.

Proof. This result is only a slight adaptation of [17] Lem. 6.4] (see Remark 33 below). For this reason we only sketch the proof, following closely the argument of [17, §6].

The key point is that it suffices to construct a divisor $E$ and family of subgroups $\{W'_v\}_v$ with the properties stated in Lemma 3.3 since the family $\{W'_v\}_v$ uniquely determines the family $\{V'_v\}_v$ by property (3) and then the latter family can be shown to satisfy property (4) by repeating the proof of [17] Lem. 6.2(2)].

Now, by the argument of [17] Lem. 6.2(1)], for each place $w$ of $L$ outside $U_L$ there exists a constant $c(w)$ such that for any integer $n \geq 0$ there exists an $G_w$-stable $O_v$-lattice $W'_w$ of

Proposition 3.2. The complex $C^\ast_V := R\Gamma_{ar,V}(U_F, \mathcal{A}_{tors})$ is acyclic outside degrees 0, 1 and 2. In addition, there exists a canonical exact sequence

$$0 \to H^0(C^\ast_V) \to A(F)_{tors} \to \bigoplus_{v \notin U_F} A(F_v)/V_v \to H^1(C^\ast_V) \to \text{Sel}_{\mathbb{Q}/\mathbb{Z}}(A/F) \to 0,$$

and a canonical isomorphism $H^2(C^\ast_V) \cong \text{Hom}_{\mathbb{Z}}(\text{Sel}_\mathbb{Z}(A^1/F), \mathbb{Q}/\mathbb{Z})$. 

Proof. This is proved in [17] §2.5].
Proposition 3.7. The following claims are valid.

Let $\text{Lie}(A_L)(\mathcal{O}_w)$ such that both

$$\text{Lie}(A_L)(m_{w^{-e(w)v^2}+c(v)}) \subset W_w' \subset \text{Lie}(A_L)(m_{w^{-e(w)v}})$$

and the group $H^i(G_w, W_w')$ vanishes for all $i \geq 1$.

By the argument of [17], Lem. 3.3, we may in addition assume that the subgroups $W_w'$ satisfy property (2), at least provided that $n(v)$ is sufficiently large.

To ensure that the product $\prod_{w \notin U_L} W_w'$ is stable under the action of $G$, we first fix a place $w$ over each $v \notin U$ and a subgroup $W_w'$ that has property (2) and is such that $H^i(G_w, W_w')$ vanishes for all $i \geq 1$.

For each $\sigma$ in $G$, we then set $W_{\sigma(w)} := \sigma(W_w') \subset \text{Lie}(A_L)(\mathcal{O}_{\sigma(w)})$ (which, we note, only depends on $\sigma(w)$). Then the collection of subgroups $\{W_w'\}_{w \notin U_L}$ clearly has both of the properties (2) and (1). □

Remark 3.5. Lemma 3.3 only differs from [17], Lem. 6.4, in that we require each group $W_w'$ to be an open $\mathcal{O}_w[G_w]$-submodule of $\text{Lie}(A)'(\mathcal{O}_w)$ rather than an open $\mathcal{O}_w$-submodule as in loc. cit. In fact, in [17], Lem. 6.2(1), it is claimed that $W_w'$ can be chosen as an $\mathcal{O}_w$-sublattice of $\text{Lie}(A_L)(\mathcal{O}_w)$, but the indicated proof seems only to show that it can be chosen as a $G_w$-stable $\mathcal{O}_w$-lattice.

Remark 3.6. The proof of Lemma 3.3 shows that for any place $v$ of $K$ that is both unramified in $L$ and of good reduction for $A$, the subgroup $V_w'$ can be chosen as $A(m_v)$.

3.4. Selmer complexes. For each place $w$ outside $U_L$ we now fix a choice of subgroups $V_w'$ as in Lemma 3.3. For any place $v$ outside $U$ we then define a group

$$V_v := \left(\prod_{w \mid v} V_w'\right)^G$$

and we denote the associated families of subgroups $(V_w')_{w \notin U_L}$ and $(V_v)_{v \notin U}$ by $V_L$ and $V_K$ (or, occasionally just $V$) respectively.

In the following result we use these subgroups to construct a canonical ‘Selmer complex’ $\text{SC}_{V_L}(A, L/K)$ that will play a key role in the formulation of our refined Birch and Swinnerton-Dyer conjecture.

We also use the $G$-module $X_{\mathbb{Z}}(A/F)$ that is defined as the pre-image of $\text{Hom}_{\mathbb{Z}}(A(F), \mathbb{Z})$ under the natural surjective homomorphism $X(A/F) \to \text{Hom}_{\mathbb{Z}}(A(F), \mathbb{Z})$ (see remark 3.1).

Proposition 3.7. The following claims are valid.

(i) $R\Gamma_{ar,V_L}(U_L, A_{\text{tors}})'[-2]$ is an object of $D^{\text{perf}}(\hat{\mathbb{Z}}[G])$ that is acyclic outside degrees 0, 1 and 2.

(ii) If the groups $\text{III}(A/L)$ and $\text{III}(A'/L)$ are both finite, then there exists a complex $\text{SC}_{V_L} := \text{SC}_{V_L}(A, L/K)$ in $D^{\text{perf}}(\hat{\mathbb{Z}}[G])$ that is acyclic outside degrees 0, 1 and 2, is unique up to isomorphisms in $D^{\text{perf}}(\hat{\mathbb{Z}}[G])$ that induce the identity map in all degrees of cohomology and also has both of the following properties:

(a) One has $H^0(\text{SC}_{V_L}) = A'(L)$, $H^1(\text{SC}_{V_L})$ contains $X_{\mathbb{Z}}(A/L)$ as a submodule of finite index and $H^2(\text{SC}_{V_L})$ is finite.

(b) There exists a canonical isomorphism in $D^{\text{perf}}(\hat{\mathbb{Z}}[G])$ of the form $\hat{\mathbb{Z}}\otimes_{\mathbb{Z}} SC_{V_L} \cong R\Gamma_{ar,V_L}(U_L, A_{\text{tors}})'[-2]$. 

Proof. For each subgroup $J$ of $G$ we set $C_{V,J}^\text{ar} := R\Gamma_{\text{ar},V,J}(U_{L,J},A_{\text{tors}})$ and we abbreviate $C_{V,J}^\text{ar}$ to $C_V^\text{ar}$ when $J$ is the trivial subgroup.

Then, since $H^i(C_{V,J}^\text{ar}[-2]) = H^2-i(C_{V,J}^\text{ar})^*$ in all degrees $i$, the result of Proposition 3.2 implies that each complex $C_{V,J}^\text{ar}$ is acyclic in all degrees outside 0, 1 and 2 and that its cohomology is finitely generated over $\hat{\mathbb{Z}}$ in all degrees.

By a standard criterion, it follows that $C_V^\text{ar}$, and hence also $C_{V,J}^\text{ar}[-2]$, belongs to $D_{\text{perf}}(\hat{\mathbb{Z}}[G])$, and so claim (i) is valid, if for every subgroup $J$ of $G$ there is an isomorphism in $D(\hat{\mathbb{Z}})$ of the form $\hat{\mathbb{Z}} \otimes_{\mathbb{Z}[J]} C_{V,J}^\text{ar} \cong C_{V,J}^\text{ar}$.

In view of the natural isomorphisms $\hat{\mathbb{Z}} \otimes_{\mathbb{Z}[J]} C_V^\text{ar} \cong R\text{Hom}_{\mathbb{Z}[J]}(\mathbb{Z}, C_V^\text{ar})$ we are therefore reduced to showing the existence of isomorphisms in $D(\hat{\mathbb{Z}})$ of the form

\begin{equation}
R\text{Hom}_{\mathbb{Z}[J]}(\mathbb{Z}, R\Gamma_{\text{ar},V,J}(U_{L,J},A_{\text{tors}})) \cong R\Gamma_{\text{ar},V,J}(U_{L,J},A_{\text{tors}})
\end{equation}

and this is proved by Kato and Trihan in [17, Lem. 6.1].

Turning to claim (ii), we note that claim (i) combines with the general result of Lemma 3.8 below to imply it suffices to show that, under the stated hypotheses, the group $H^0(C_V^\text{ar})^*$ is naturally isomorphic to $\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} A^t(L)$, the group $H^2(C_V^\text{ar})^*$ is finite and there exists a finitely generated $G$-module $M^1$ that contains $X_\mathbb{Z}(A/L)$ as a submodule of finite index and is such that there is a canonical isomorphism $\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} M^1 \cong H^1(C_V^\text{ar})^*$ of $\hat{\mathbb{Z}}[G]$-modules.

In this direction, the exact sequence in Proposition 3.2 implies directly that $H^0(C_V^\text{ar})^*$ is finite.

In addition, since the limit $\varprojlim_n H^0(A^t/L)[n]$ vanishes under the assumption $\text{III}(A^t/L)$ is finite, the displayed isomorphism in Proposition 3.2 combines with the first exact sequence in Remark 3.1 to give a canonical isomorphism

\[ H^2(C_V^\text{ar})^* \cong (\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} A^t(L))^* = \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} A^t(L) \]

of the required form.

Next we note that, since $\text{III}(A/L)$ is assumed to be finite, the second exact sequence in Remark 3.1 implies $X_\mathbb{Z}(A/L)$ is finitely generated.

Thus, if we write $Y$ for the (finite) cokernel of the map $A(L)_{\text{tors}} \to \bigoplus_{v \notin U_L} A(L_v)/V_v$ that occurs in Proposition 3.2, then the natural map of finite groups

\[ \text{Ext}_G^1(Y^*, X_\mathbb{Z}(A/L)) = \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \text{Ext}_G^1(Y^*, X_\mathbb{Z}(A/L)) \to \text{Ext}_G^1(\hat{\mathbb{Z}}[G], Y^*, X(A/L)) \]

is bijective and so there exists an exact commutative diagram of $G$-modules

\[
\begin{array}{ccccccc}
0 & \longrightarrow & X_\mathbb{Z}(A/L) & \longrightarrow & M & \longrightarrow & Y^* & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & X(A/L) & \longrightarrow & H^2-i(C_V^\text{ar})^* & \longrightarrow & Y^* & \longrightarrow & 0 \\
\end{array}
\]

in which the first vertical arrow is the natural inclusion, and so induces an isomorphism $\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} X_\mathbb{Z}(A/L) \cong X(A/L)$, and the lower row is induced by the Pontryagin dual of the long exact sequence in Proposition 3.2.

In particular, from the upper row of the above diagram we can deduce that $M$ is finitely generated and hence is a suitable choice for the module $M^1$ that we seek. \qed
We recall that a $G$-module is said to be ‘cohomologically-trivial’, or ‘c-t’ for short in the sequel, if for every integer $i$ and every subgroup $J$ of $G$ the Tate cohomology group $H^i(J, M)$ vanishes.

**Lemma 3.8.** Let $\hat{C}$ be a cohomologically-bounded complex of $\mathbb{Z}[G]$-modules and assume to be given in each degree $j$ a finitely generated $G$-module $M^j$ and an isomorphism of $\mathbb{Z}[G]$-modules of the form $\iota_j : \mathbb{Z} \otimes_{\mathbb{Z}} M^j \cong H^j(\hat{C})$.

Then there exists an object $C$ of $D(\mathbb{Z}[G])$ with all of the following properties:

(i) $H^j(C) = M^j$ for all $j$.

(ii) There exists an isomorphism $\iota : \mathbb{Z} \otimes_{\mathbb{Z}} C \cong \hat{C}$ in $D(\mathbb{Z}[G])$ for which in each degree $j$ one has $H^j(\iota) = \iota_j$.

(iii) $C$ belongs to $D^{\text{perf}}(\mathbb{Z}[G])$ if and only if $\hat{C}$ belongs to $D^{\text{perf}}(\mathbb{Z}[G])$.

Any such complex $C$ is unique to within an isomorphism $\kappa$ in $D(\mathbb{Z}[G])$ for which $H^j(\kappa)$ is the identity automorphism of $M^j$ in each degree $j$.

**Proof.** We prove this by induction on the number of non-zero cohomology groups of $\hat{C}$.

If there is only one non-zero such group, in degree $d$ say, then $\hat{C}$ is isomorphic in $D(\mathbb{Z}[G])$ to $(\mathbb{Z} \otimes_{\mathbb{Z}} M^d)[-d]$ and we write $C$ for the complex $M^d[-d]$ in $D(\mathbb{Z}[G])$.

In this case claim (i) is clear and claim (ii) is true with $\iota$ the morphism induced by $\iota_d$. Finally, in this case $C$ belongs to $D^{\text{perf}}(\mathbb{Z}[G])$ if and only if $M^d$ is c-t and $\hat{C}$ belongs to $D^{\text{perf}}(\mathbb{Z}[G])$ if and only if $\hat{M}^d$ is c-t. This implies claim (iii) since a finitely generated $G$-module $N$ is c-t if and only if $\mathbb{Z} \otimes_{\mathbb{Z}} N$ is c-t as a consequence of the fact that in each degree $i$ and for each subgroup $J$ of $G$ the natural map $H^i(J, N) \to H^i(J, \mathbb{Z} \otimes_{\mathbb{Z}} N)$ is bijective.

To deal with the general case we assume $\hat{C}$ is not acyclic and write $d$ for the unique integer such that $H^d(\hat{C}) \neq 0$ and $H^i(\hat{C}) = 0$ for all $i > d$. We then abbreviate the truncated complexes $\tau_{d-1} \hat{C}$ and $\tau_d \hat{C}$ to $\hat{C}_1$ and $\hat{C}_2$ and recall that there is a canonical exact triangle in $D(\mathbb{Z}[G])$ of the form $\hat{C}_1 \to \hat{C} \to \hat{C}_2 \xrightarrow{\theta} \hat{C}_1[1]$. We note that this triangle induces an isomorphism $\kappa$ in $D(\mathbb{Z}[G])$ between $\hat{C}$ and $\text{Cone}(\theta)[-1]$, where we write $\text{Cone}(\alpha)$ for the mapping cone of a morphism $\alpha$.

Now, since $H^j(C_1) = H^j(\hat{C}_1)$ for $j < d$ and $H^j(C_1) = 0$ for all $j \geq d$, the inductive hypothesis implies the existence of $C_1$ in $D(\mathbb{Z}[G])$ and an isomorphism $\iota_1 : \mathbb{Z} \otimes_{\mathbb{Z}} C_1 \cong \hat{C}_1$ in $D(\mathbb{Z}[G])$ such that in each degree $j$ with $j < d$ one has $H^j(\iota_1) = \iota_1$.

In addition, since $\hat{C}_2$ is acyclic outside degree $d$, the argument given above shows the existence of a complex $C_2$ in $D(\mathbb{Z}[G])$ and an isomorphism $\iota_2 : \mathbb{Z} \otimes_{\mathbb{Z}} C_2 \cong \hat{C}_2$ in $D(\mathbb{Z}[G])$ with $H^d(\iota_2) = \iota_d$.

Next we recall that the group $\text{Hom}_{D(\mathbb{Z}[G])}(\hat{C}_2, \hat{C}_1[1])$ is equal to $H^0(\text{RHom}_{\mathbb{Z}[G]}(\hat{C}_2, \hat{C}_1[1]))$ and so can be computed by using the spectral sequence

$$E_2^{p,q} = \prod_{a \in \mathbb{Z}} \text{Ext}_G^p(H^a(\hat{C}_2), H^{a+q}(\hat{C}_1[1])) \Rightarrow H^{p+q}(\text{RHom}_{\mathbb{Z}[G]}(C_2, C_1[1]))$$

constructed by Verdier [32, III, 4.6.10]. We also note that there is no degree in which the complexes $\hat{C}_2$ and $\hat{C}_1[1]$ have cohomology groups that are both non-zero and that any group of the form $\text{Ext}_G^p(-,-)$ vanishes for $p < 0$ and is torsion for $p > 0$. Given these facts,
the above spectral sequence implies that Hom_{D(\tilde{\mathbb{Z}}[G])}(\mathcal{C}_2, \mathcal{C}_1[1]) is finite and hence that the diagonal localisation map Hom_{D(\mathbb{Z}[G])}(C_2, C_1[1]) \to Hom_{D(\tilde{\mathbb{Z}}[G])}(\mathcal{C}_2, \mathcal{C}_1[1]) is bijective.

We write \( \theta \) for the pre-image of \( \hat{\theta} \) under the latter isomorphism and claim that the mapping fibre \( C := \text{Cone}(\theta)[-1] \) has all of the required properties.

Firstly, this definition implies directly that \( H^j(C) \) is equal to \( H^j(C_1) \) if \( j < d \) and to \( H^j(C_2) \) if \( j \geq d \), and so claim (i) follows immediately from the given properties of \( C_1 \) and \( C_2 \). The definition also implies directly that \( \tilde{\mathbb{Z}} \otimes \mathbb{Z} C \) is isomorphic to \( \text{Cone}(\hat{\theta})[-1] \) and hence that \( \kappa \) induces an isomorphism in \( D(\tilde{\mathbb{Z}}[G]) \) between \( \tilde{\mathbb{Z}} \otimes \mathbb{Z} C \) and \( \hat{C} \) with the property described in claim (ii).

To prove claim (iii) it suffices to check that \( C \) belongs to \( D^{\text{perf}}(\mathbb{Z}[G]) \) if \( \hat{C} \) belongs to \( D^{\text{perf}}(\tilde{\mathbb{Z}}[G]) \). To do this we can assume, by a standard resolution argument, that \( C \) is a bounded complex of finitely generated \( G \)-modules in which all but the first (non-zero) module, \( M \) say, is free. If we then also assume that the complex \( \hat{C} \) is isomorphic in \( D(\tilde{\mathbb{Z}}[G]) \) to a bounded complex of finitely generated projective \( \tilde{\mathbb{Z}}[G] \)-modules \( Q \), then there exists a quasi-isomorphism \( \pi : \tilde{\mathbb{Z}} \otimes \mathbb{Z} C \cong Q \) of complexes of \( \mathbb{Z}[G] \)-modules.

Now, since all terms of \( \tilde{\mathbb{Z}} \otimes \mathbb{Z} C \) and \( Q \) are free \( \tilde{\mathbb{Z}}[G] \)-modules, except possibly for \( \tilde{\mathbb{Z}} \otimes \mathbb{Z} M \), the acyclicity of \( \text{Cone}(\pi) \) implies that the \( \tilde{\mathbb{Z}}[G] \)-module \( \tilde{\mathbb{Z}} \otimes \mathbb{Z} M \) is c-t. This in turn implies that \( M \) is c-t and hence that the complex \( C \) belongs to \( D^{\text{perf}}(\mathbb{Z}[G]) \), as claimed. \( \square \)

3.5. Coherent cohomology. The Selmer complex that is constructed in Proposition 3.7(ii) depends on the choice of subgroups \( V_L \). We shall therefore need to introduce an auxiliary perfect complex that will be used to compensate for this dependence in the formulation of our conjecture.

To do this we use for each place \( w \) outside \( U \) the subgroup \( W'_w \) that corresponds in Lemma 3.4 to the subgroup \( V'_w \) fixed at the beginning of 3.4. For any place \( v \) outside \( U \) we then set

\[ W_v := (\prod_{w|v} W'_w)^G \]

and we denote the associated families of subgroups \( (W'_w)_{w \notin U} \) and \( (W_v)_{v \notin U} \) by \( W_L \) and \( W_K \) respectively.

We then define \( L_v \) to be the coherent \( \mathcal{O}_X \)-submodule of \( \text{Lie}(A) \) that extends \( \text{Lie}(A|_U) \) and is such that \( L_v = W_v \subset \text{Lie}(A)(\mathcal{O}_v) \) for each \( v \notin U \).

We similarly define \( L_L \) to be the \( G \)-equivariant coherent \( \mathcal{O}_X \)-submodule of \( \pi_*\text{Lie}(A_X)_L \) with \( L_{L,v} = \prod_{w|v} W'_w \) for each \( v \notin U \), where we write \( \pi : X_L \to X \) for the natural projection.

Lemma 3.9. The complex \( R\Gamma(X, L_L)^* \) belongs to \( D^{\text{perf}}(\mathbb{F}_p[G]) \), and hence to \( D^{\text{perf}}(\mathbb{Z}_p[G]) \).

Proof. For each subgroup \( J \) of \( G \) the complex \( R\Gamma(X, (L_L)^J)^* \) is represented by a complex of finite \( \mathbb{F}_p[G] \)-modules that is acyclic outside degrees 0 and 1.

By the same argument as used to prove Proposition 3.7(i) we are therefore reduced to proving that for each \( J \) there is a natural isomorphism in \( D(\mathbb{F}_p) \) of the form

\[ R\text{Hom}_{\mathbb{F}_p[G]}(\mathbb{F}_p, R\Gamma(X, L_L)) \cong R\Gamma(X, (L_L)^J) \]
and this is proved by Kato and Trihan in \cite[p. 585]{17}.

**Remark 3.10.** In view of Remark \[3.5\] we have here defined $L_L$ to be a $F_p[G]$-equivariant vector bundle over $X$ rather than a vector bundle over $X_L$, as in \cite[§ 6.5]{17}. This means that various arguments in loc. cit. that rely on the ‘geometric $p$-adic cohomology theory’ over $X_L$ and will be referred to in later sections must in our case be carried out over $X$ by using the relevant push-forward constructions. This, however, is a routine difference that we do not dwell on.

4. **Statements of the conjecture and main results**

In this section we formulate our refinement of the Birch and Swinnerton-Dyer Conjecture, establish some basic properties of the conjecture and state the main supporting evidence for it that we will obtain in the rest of the article.

4.1. **Relative $K$-theory.** Before stating our conjecture we quickly review relevant aspects of relative algebraic $K$-theory.

For a Dedekind domain $R$ with field of fractions $F$, an $R$-order $\mathfrak{A}$ in a finite dimensional separable $F$-algebra $A$ and a field extension $E$ of $F$ we set $A_E := E \otimes_F A$.

4.1.1. We use the relative algebraic $K_0$-group $K_0(\mathfrak{A}, A_E)$ of the ring inclusion $\mathfrak{A} \subset A_E$, as described explicitly in terms of generators and relations by Swan in \cite[p. 215]{27}.

We recall that for any extension field $E'$ of $E$ there exists an exact commutative diagram

\[
\begin{array}{cccc}
K_1(\mathfrak{A}) & \longrightarrow & K_1(A_{E'}) & \longrightarrow & K_0(\mathfrak{A}, A_{E'}) & \longrightarrow & K_0(\mathfrak{A}) \\
\downarrow & & \downarrow & & \downarrow & & \\
K_1(\mathfrak{A}) & \longrightarrow & K_1(A_E) & \longrightarrow & K_0(\mathfrak{A}, A_E) & \longrightarrow & K_0(\mathfrak{A})
\end{array}
\]

in which the upper and lower rows are the respective long exact sequences in relative $K$-theory of the inclusions $\mathfrak{A} \subset A_E$ and $\mathfrak{A} \subset A_{E'}$ and both of the vertical arrows are injective and induced by the inclusion $A_E \subseteq A_{E'}$. (For more details see \cite[Th. 15.5]{27}.)

In particular, if $R = \mathbb{Z}$ and for each prime $\ell$ we set $\mathfrak{A}_\ell := \mathbb{Z}_\ell \otimes_{\mathbb{Z}} \mathfrak{A}$ and $A_\ell := \mathbb{Q}_\ell \otimes_{\mathbb{Q}} A$, then we can regard each group $K_0(\mathfrak{A}_\ell, A_\ell)$ as a subgroup of $K_0(\mathfrak{A}, A)$ by means of the canonical composite homomorphism

\[
\bigoplus_{\ell} K_0(\mathfrak{A}_\ell, A_\ell) \cong K_0(\mathfrak{A}, A) \subset K_0(\mathfrak{A}, A_{\mathbb{R}}),
\]

where $\ell$ runs over all primes, the isomorphism is as described in the discussion following \cite[(49.12)]{3} and the inclusion is induced by the relevant case of $\ell'$.

For each element $x$ of $K_0(\mathfrak{A}, A)$ we write $(x_\ell)_\ell$ for its image in $\bigoplus_{\ell} K_0(\mathfrak{A}_\ell, A_\ell)$ under the isomorphism in \([10]\).

4.1.2. We shall also use the description of $K_0(\mathfrak{A}, A_E)$ in terms of the formalism of ‘non-abelian determinants’ that is given by Fukaya and Kato in \cite[§1]{12}.

We recall, in particular, that any pair comprising an object $C$ of $D_{\text{perf}}(\mathfrak{A})$ and a morphism $h$ of (non-abelian) determinants $\text{Det}_{A_E}(E \otimes_R C) \to \text{Det}_{A_E}(0)$ gives rise to a canonical element of $K_0(\mathfrak{A}, A_E)$ that we shall denote by $\chi_{\mathfrak{A}}(C, h)$. 
If an object $C$ of $D^\text{perf}(\mathfrak{A})$ is such that $E \otimes_R C$ is acyclic, then the canonical morphism of determinants $\det_{A_E}(E \otimes_R C) \to \det_{A_E}(0)$ gives an element of $K_0(\mathfrak{A}, A_E)$ that we shall denote by $\chi_\mathfrak{A}(C, 0).

There are two standard properties of such elements that are established by Fukaya and Kato in loc. cit. and will be frequently used (without explicit comment) in the sequel.

Firstly, if $C_1 \to C_2 \to C_3 \to C_1[1]$ is any exact triangle in $D^\text{perf}(\mathfrak{A})$ such that the complex $F \otimes_R C_3$ is acyclic, then any morphism $h : \det_{A_E}(E \otimes_R C_1) \to \det_{A_E}(0)$ combines with the given triangle to induce a morphism $h' : \det_{A_E}(E \otimes_R C_2) \to \det_{A_E}(0)$ for which one has

$$\chi_\mathfrak{A}(C_2, h') = \chi_\mathfrak{A}(C_1, h) + \chi_\mathfrak{A}(C_3, 0).$$

Secondly, if $h$ and $h'$ are any two morphisms $\det_{A_E}(E \otimes_R C) \to \det_{A_E}(0)$, then one has

$$\chi_\mathfrak{A}(C, h') = \chi_\mathfrak{A}(C, h) + \partial_{\mathfrak{A}, E}(h' \circ h^{-1})$$

where the composite $h' \circ h^{-1} : \det_{A_E}(0) \to \det_{A_E}(0)$ is regarded as an element of $K_1(A_E)$ in the natural way.

For convenience, we shall often abbreviate the notation $\chi_{\mathbb{Z}[G]}(C, h)$ to $\chi_G(C, h)$ and, when $E$ is clear from context, also write $\partial_G$ and $\partial'_G$ in place of $\partial_{\mathbb{Z}[G], E}$ and $\partial'_{\mathbb{Z}[G], E}$.

### 4.2. The refined Birch and Swinnerton-Dyer Conjecture.

#### 4.2.1. In the sequel we write

$$h_{A, L}^{\text{NT}} : A(L) \times A'(L) \to \mathbb{R}$$

for the classical Néron-Tate height-pairing for $A$ over $L$.

This pairing is non-degenerate and hence, assuming $\text{III}(A_L)$ to be finite, combines with the properties of the Selmer complex $SC_{V_L}(A, L/K)$ established in Proposition 3.7(ii) to induce a canonical isomorphism of $\mathbb{R}[G]$-modules

$$h_{A, L}^{\text{NT}, \det} : \det_{\mathbb{R}[G]}(\mathbb{R} \otimes_{\mathbb{Z}} SC_{V_L}(A, L/K)) \cong \det_{\mathbb{R}[G]}(0).$$

In particular, since $SC_{V_L}(A, L/K)$ belongs to $D^\text{perf}(\mathbb{Z}[G])$, we obtain an element of $K_0(\mathbb{Z}[G], \mathbb{R}[G])$ by setting

$$\chi_G^{\text{BSD}}(A, V_L) := \chi_G(SC_{V_L}(A, L/K), h_{A, L}^{\text{NT}, \det}).$$

In a similar way, since the complex $\Gamma'(X_L, \mathcal{L}_L)^*$ considered in Lemma 3.3 belong to $D^\text{perf}(\mathbb{F}_p[G])$, and hence is such that $\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma'(X_L, \mathcal{L}_L)^*$ is acyclic, we obtain an element that lies in the image of the natural homomorphism

$$K_0(\mathbb{F}_p[G]) \to K_0(\mathbb{Z}[G], \mathbb{Q}[G]) \subset K_0(\mathbb{Z}[G], \mathbb{R}[G])$$

by setting

$$\chi_G^{\text{coh}}(A, V_L) := \chi_G(\Gamma'(X_L, \mathcal{L}_L)^*, 0).$$

Finally, for each prime $\ell$ we shall define (via Proposition 8.1(i) and the equality (42)) a canonical integer $a_\ell = a_{A, L, \ell}$ in $\{0, 1\}$. We then define an element of $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$ of order dividing two by setting

$$\chi_G^{\text{sgn}}(A) := \sum_\ell \partial_G(Q(\langle(-1)^{a_\ell}, \mathbb{Q} \cdot A(L)\rangle)_\ell),$$
where \( \ell \) runs over all prime divisors of \(|G|\).

4.2.2. We can now state our refined version of the Birch and Swinnerton-Dyer Conjecture for \( A \) over \( L \).

In this conjecture we write \( r_{an}(\chi) \) for the order of vanishing at \( s = 1 \) of the series \( L(A, U, \chi, s) \) and set

\[
    r_{\text{alg}}(\chi) := \chi(1)^{-1} \cdot \dim_C(e_\chi(\mathbb{C} \otimes \mathbb{Z} A^i(F))).
\]

We also write \( \tilde{\chi} \) for the contragredient of \( \chi \) and use the ‘leading term’ element \( L_U^*(A_{L/K}, 1) \) of \( K_1(\mathbb{R}[G]) \) that is defined in Theorem \((2.1)\).

**Conjecture 4.1.** The following claims are valid.

(i) For each character \( \chi \) in \( \text{Irr}(G) \) one has \( r_{an}(\chi) = r_{\text{alg}}(\tilde{\chi}) \).

(ii) The group \( \text{III}(A/L) \) is finite.

(iii) If \( U \) is any dense open subset of \( X \) at each point of which both \( L/K \) is unramified and \( A/K \) has good reduction, and \( V_L = V_{UL} \) is any family of groups chosen as in \((\ref{5.4})\) then in \( K_0(\mathbb{Z}[G], \mathbb{R}[G]) \) one has

\[
    \partial_G(L_U^*(A_{L/K}, 1)) = \chi_G^{\text{BSD}}(A, V_L) - \chi_G^{\text{coh}}(A, V_L) + \chi_G^{\text{sgn}}(A).
\]

**Remark 4.2.** If \( L = K \), then \( K_0(\mathbb{Z}[G], \mathbb{R}[G]) \) identifies with the multiplicative group \( \mathbb{R}^\times /\{\pm 1\} \) and in Proposition \((\ref{5.2})\) below we shall show that this case of Conjecture 4.1 recovers the classical Birch and Swinnerton-Dyer conjecture for \( A \). In \((\ref{5.2})\) we also show that the validity of Conjecture 4.1 is independent of the choices of the family of subgroups \( V_L \) and of the open set \( U \).

**Remark 4.3.** The formulation of Conjecture 4.1(iii) in terms of relative algebraic \( K \)-theory is motivated by the general formalism of ‘equivariant Tamagawa number conjectures’ that is discussed in \([2]\). If, for each prime \( \ell \), we fix an isomorphism of fields \( \mathbb{C} \cong \mathbb{C}_\ell \), then the long exact sequence of relative \( K \)-theory of the inclusion \( \mathbb{Z}_\ell[G] \subseteq \mathbb{C}_\ell[G] \) implies that Conjecture 4.1(iii) determines the image of \( (L_U^*(A, \chi, 1))_{\chi \in \text{Irr}(G)} \) in \( \zeta(\mathbb{C}_\ell[G])^\times \) modulo the subgroup \( \text{Nrd}_{\mathbb{C}_\ell[G]}(K_1(\mathbb{Z}_\ell[G])) \). In view of the description of \( \text{Nrd}_{\mathbb{C}_\ell[G]}(K_1(\mathbb{Z}_\ell[G])) \) obtained by the second author in \([15]\), this means that Conjecture 4.1 incorporates families of explicit congruence relations between the leading terms \( L_U^*(A, \chi, 1) \) for varying \( \chi \) in \( \text{Irr}(G) \).

**Remark 4.4.** The equality of Conjecture 4.1(iii) also constitutes a strong restriction on the Galois structure of the Selmer complex \( SC_{V_L}(A, L/K) \). The following result makes this restriction explicit in a special case.

**Proposition 4.5.** If \( G \) is a group of \( p \)-power order, then Conjecture 4.1(iii) implies that \( SC_{V_L}(A, L/K) \) is quasi-isomorphic to a bounded complex of finitely generated free \( G \)-modules.

**Proof.** The exactness of the rows in diagram \((\ref{4.3})\) implies that \( \partial_G(L_U^*(A, L/K, 1)) \) lies in the kernel of the connecting homomorphism \( \partial_G^*: K_0(\mathbb{Z}[G], \mathbb{R}[G]) \to K_0(\mathbb{Z}[G]) \).

We now assume that \( G \) has \( p \)-power order. In this case, every finite projective \( \mathbb{F}_p[G] \)-module is free and this implies that the image of the homomorphism \((\ref{12})\) belongs to \( \ker(\partial_G^*) \). In particular, one has \( \partial_G^*(\chi_G(R\Gamma(X_L, L_L^*)^x, 0)) = 0 \).

In addition, in this case the result of Lemma \((\ref{4.6})\) below implies that \( \chi_{\text{sgn}}^G(A) \) is equal to \( \partial_G(\chi_G(R\Gamma(X_L, L_L^*)^x, 0)) = 0 \).

In this case the result of Lemma \((\ref{4.6})\) below implies that \( \chi_{\text{sgn}}^G(A) \) is equal to \( \partial_G(\chi_G(R\Gamma(X_L, L_L^*)^x, 0)) = 0 \) and hence belongs to \( \ker(\partial_G^*) \).
Putting these facts together with the equality in Conjecture 4.1(iii) one finds that the Euler characteristic of $SC_{V_{L}}(A, L/K)$ in $K_{0}(\mathbb{Z}[G])$ vanishes and this implies, by a standard resolution argument, that $SC_{V_{L}}(A, L/K)$ is quasi-isomorphic to a bounded complex of finitely generated free $G$-modules.

\begin{lemma}
If $\ell$ is any prime that does not divide $\#G$, then $\tilde{c}_{G, \mathbb{Q}}(\langle -1, \mathbb{Q} \cdot A(L) \rangle)_{\ell}$ vanishes.
\end{lemma}

\begin{proof}
If $\ell$ does not divide $\#G$, then the $\mathbb{Z}_{\ell}$-order $\mathbb{Z}_{\ell}[G]$ is maximal and so $\mathbb{Q}_{\ell} \otimes_{\mathbb{Z}} A^{i}(L)$ has a full sublattice that is a projective $\mathbb{Z}[G]$-module. This implies $\langle -1, \mathbb{Q}_{\ell} \otimes_{\mathbb{Z}} A(L) \rangle$ belongs to the image of the natural map $K_{1}(\mathbb{Z}_{\ell}[G]) \to K_{1}(\mathbb{Q}_{\ell}[G])$ and hence that the element $\tilde{c}_{G, \mathbb{Q}}(\langle -1, \mathbb{Q} \cdot A(L) \rangle)_{\ell} = \tilde{c}_{\mathbb{Z}_{\ell}[G], \mathbb{Q}_{\ell}}(\langle -1, \mathbb{Q}_{\ell} \otimes_{\mathbb{Z}} A(L) \rangle)$ vanishes as a consequence of the long exact sequence of relative $K$-theory.
\end{proof}

4.3. **The main results.** In order to state our main result we must define the finite subgroup $T_{A, L/K}$ of $K_{0}(\mathbb{Z}[G], \mathbb{R}[G])$ that was discussed in the introduction.

If $\Xi$ is a quotient of a subgroup $\Delta$ of a finite group $\Gamma$, then we write $\pi_{\Xi}^{\Gamma}$ for the composite of inertia groups in $H'$ of each place in $K'$.

We set $\pi_{K'} := \pi_{H'/P'}^{G'}$ and then define

\begin{equation}
T_{A, L/K} := \bigcap_{K'} \pi_{K'}^{G'}(\ker(\pi_{K'})_{\text{tor}})
\end{equation}

where the intersection runs over all possible choices of $K'$. We regard this group as a subgroup of $K_{0}(\mathbb{Z}[G], \mathbb{R}[G])$ via the natural embeddings

$K_{0}(\mathbb{Z}_{p}[G], \mathbb{Q}_{p}[G]) \subset K_{0}(\mathbb{Z}[G], \mathbb{Q}[G]) \subset K_{0}(\mathbb{Z}[G], \mathbb{R}[G])$.

We can now state the main evidence that we shall offer in support of Conjecture 4.1

\begin{theorem}
If the $\ell$-primary component of $\text{III}(A/L)$ is finite for some prime $\ell$, then the following claims are also valid.

(i) Claims (i) and (ii) of Conjecture 4.1 are valid.

(ii) The equality in Conjecture 4.1(iii) is valid modulo the finite subgroup $T_{A, L/K}$ of $K_{0}(\mathbb{Z}[G], \mathbb{R}[G])$.
\end{theorem}

\begin{remark}
It is proved by Kato and Trihan in [17] that $\text{III}(A/L)$ is finite if and only if at least one of its $\ell$-primary components is finite. Thus, under the hypotheses of Theorem 4.7 we can (and do) assume, without further comment, that $\text{III}(A/L)$ is finite (and hence that Conjecture 4.1(ii) is valid).
\end{remark}

\begin{remark}
The main result that we prove is, in principle, stronger than Theorem 4.7 but is more technical to state. For more details see Remark 9.4.
\end{remark}
Remark 4.10. In a subsequent article, the first two authors will provide additional evidence in support of Conjecture 4.1 in the setting of generically ordinary abelian varieties.

In special cases it is possible to describe $T_{A,L/K}$ explicitly and hence to make Theorem 4.7(ii) much more concrete.

For example, if $A$ is semistable over $K$ (so one can take $K' = K$, and hence $G' = G$, in the above) and $L/K$ is tamely ramified (so $P'$ is trivial), then $T_{A,L/K}$ vanishes and so Theorem 4.7 has the following more explicit consequence.

**Corollary 4.11.** Assume that $A$ is semistable, that $L/K$ is tamely ramified and that some $\ell$-primary component of $\Sha(A/L)$ is finite. Then Conjecture 4.1 is unconditionally valid.

This result gives the first verification, modulo only the assumed finiteness of Tate-Shafarevich groups, of a refined version of the Birch-Swinnerton-Dyer conjecture in the context of ramified extensions.

5. Preliminary results

In this section we first prove a purely algebraic result that is important for several subsequent arguments.

We then verify that the statement of Conjecture 4.1 is consistent in certain key respects (as promised in Remark 4.2).

Finally we use a result of Schneider to give a reinterpretation of the conjecture that plays an essential role in the proof of Theorem 4.7.

5.1. A result in $K$-theory. The following purely algebraic observation will underpin the proof of several subsequent results.

**Proposition 5.1.** Let $R$ be a Dedekind domain with field of fractions $F$ and $A$ an $R$-order in a finite dimensional semisimple $F$-algebra $A$.

We suppose to be given exact triangles in $D^{\text{perf}}(A)$ of the form

\[(14) \quad C_0 \to C_1 \xrightarrow{\theta} C_2 \to C_0[1] \quad \text{and} \quad C_0 \to C_1 \xrightarrow{\phi} C_2 \to C_0[1]\]

that satisfy all of the following conditions.

(a) In each degree $i$ there are natural identifications $F \otimes_R H^i(C_1) = F \otimes_R H^i(C_2)$, with respect to which one has

(b) the composite tautological homomorphism of $A$-modules

\[F \otimes_R \ker(H^i(\theta)) \subseteq F \otimes_R H^i(C_1) = F \otimes_R H^i(C_2) \to F \otimes_R \cok(H^i(\theta))\]

is bijective, and

(c) the map $H^i(\phi)$ induces the identity homomorphism on $F \otimes_R H^i(C_1) = F \otimes_R H^i(C_2)$.

Then the following claims are valid.

(i) The bijectivity of the maps in (b) combines with the first triangle in (14) to induce a canonical morphism $\tau_0 : \text{Det}_A(F \otimes_R C_0) \cong \text{Det}_A(0)$ of (non-abelian) determinants.

(ii) In each degree $i$ the homomorphism $H^i(\theta)$ induces an automorphism $H^i(\theta)^\circ$ of any $A$-equivariant complement to $F \otimes_R \ker(H^i(\theta))$ in $F \otimes_R H^i(C_1)$ in such a way that $\text{Nrd}_A(H^i(\theta)^\circ)$ is independent of the choice of complement.
(iii) The complex $F \otimes_R C_\phi$ is acyclic.

(iv) In $K_0(\mathfrak{A}, A)$ one has

$$\chi_{\mathfrak{A}}(C_\theta, \tau_0) - \chi_{\mathfrak{A}}(C_\phi, 0) = c_{\mathfrak{A}, F} \left( \prod_{i \in \mathbb{Z}} (H^i(\theta)_F)^{(-1)^i} \right),$$

where we identify each automorphism $H^i(\theta)_F$ with the associated element of $K_1(A)$.

Proof. If $M$ denotes either an $R$-module or a complex of $R$-modules, then we abbreviate $F \otimes_R M$ to $M_F$.

To construct a morphism $\tau_0$ as in claim (i) we note first that the long exact cohomology sequence of the left hand exact triangle in (14) gives in each degree $i$ a short exact sequence of $\mathfrak{A}$-modules $0 \to \text{cok}(H^{i-1}(\theta)) \to H^i(C_\theta) \to \text{ker}(H^i(\theta)) \to 0$.

Then, upon tensoring these exact sequences with $F$ (over $R$), applying the determinant functor $\text{Det}_A$ and then taking account of the isomorphisms given in (b) one obtains isomorphisms of (non-abelian) determinants

$$\text{Det}_A(H^i(C_\theta)_F) \cong \text{Det}_A(\text{cok}(H^{i-1}(\theta))_F) \cdot \text{Det}_A(\text{ker}(H^i(\theta))_F) \cong \text{Det}_A(\text{cok}(H^{i-1}(\theta))_F) \cdot \text{Det}_A(\text{cok}(H^i(\theta))_F).$$

We then define the morphism $\tau_0$ in claim (i) to be the composite

$$\text{Det}_A((C_\theta)_F) \cong \prod_{i \in \mathbb{Z}} \text{Det}_A(H^i(C_\theta)_F)^{(-1)^i} \cong \prod_{i \in \mathbb{Z}} [\text{Det}_A(\text{cok}(H^{i-1}(\theta))_F) \cdot \text{Det}_A(\text{cok}(H^i(\theta))_F)]^{(-1)^i} \cong \prod_{i \in \mathbb{Z}} [\text{Det}_A(\text{cok}(H^i(\theta))_F)]^{-1} \cdot \text{Det}_A(\text{cok}(H^i(\theta))_F)]^{(-1)^i} \cong \prod_{i \in \mathbb{Z}} \text{Det}_A(0)^{(-1)^i} \cong \text{Det}_A(0).$$

Here the first map is the canonical ‘passage to cohomology’ map, the second is induced by the maps $[13]$ in each degree $i$, the third by the obvious rearrangement of terms and the fourth from the canonical morphisms $\text{Det}_A(\text{cok}(H^i(\theta))_F)^{-1} \cdot \text{Det}_A(\text{cok}(H^i(\theta))_F) \cong \text{Det}_A(0)$.

Claim (ii) is a straightforward consequence of the condition (b) and claim (iii) follows directly upon combining the long exact cohomology sequence of the second triangle in (14) with the condition (c).

Finally, to prove claim (iv) we fix bounded complexes of finitely generated projective $\mathfrak{A}$-modules $P_1$ and $P_2$ that are respectively isomorphic in $D(\mathfrak{A})$ to $C_1$ and $C_2$. Then the morphisms $\theta$ and $\phi$ are represented by morphisms of complexes of $\mathfrak{A}$-modules of the form $\theta': P_1 \to P_2$ and $\phi': P_1 \to P_2$.

The key to our argument is then to consider the exact triangle

$$C_\theta \oplus \text{Cone}(\phi') \xrightarrow{(\kappa, \text{id})} P_1 \oplus \text{Cone}(\phi') \xrightarrow{(\kappa', 0)} \text{Cyl}(\theta') \to (C_\theta \oplus \text{Cone}(\phi'))[1]$$

in $D(\mathfrak{A})$ where $\kappa$ is the morphism $C_\theta \to P_1$ induced by the first triangle in (14) and $\kappa'$ the morphism $P_1 \to \text{Cyl}(\theta')$ induced by $\theta'$ and the natural quasi-isomorphism $\text{Cyl}(\theta') \cong P_2$. 

We note first that this triangle satisfies the analogues of conditions (a) and (b) (with $C_1$, $C_2$ and $\theta$ replaced by $P_1 \oplus \text{Cone}(\phi')$, $Cyl(\theta')$ and $(\kappa', 0)$) and, in addition, that in each degree $i$ one has $(P_1 \oplus \text{Cone}(\phi'))^i = Cyl(\theta')^i$.

Further, the acyclicity of $F \otimes_R C_\phi$ is an immediate consequence of assumption (c) and implies that
\[
\chi_\mathfrak{A}(C_\phi \oplus \text{Cone}(\phi'), \tau_\theta) = \chi_\mathfrak{A}(C_\phi, \tau_\theta) + \chi_\mathfrak{A}(C_\phi[1], 0) = \chi_\mathfrak{A}(C_\phi, \tau_\theta) - \chi_\mathfrak{A}(C_\phi, 0),
\]
where the second equality is true because $\text{Cone}(\phi')$ is isomorphic to $C_\phi[1]$.

In particular, after replacing the first triangle in (14) by (16), we are reduced to proving that if $C_1$ and $C_2$ are represented by bounded complexes of finitely generated projective $\mathfrak{A}$-modules $P_1$ and $P_2$ with $P_1^i = P_2^i$ in each degree $i$, then the conditions (a), (b) and (c) combine to imply an equality
\[
\chi_\mathfrak{A}(C_\phi, \tau_\theta) = \delta_\mathfrak{A}(\prod_{i \in \mathbb{Z}} \ker \text{Nrd}_A(H^i(\theta)_F)^{(-1)^i}),
\]
where $\delta_\mathfrak{A}$ denotes the composite $\hat{\delta}_\mathfrak{A} F \circ (\text{Nrd}_A)^{-1} : \text{im(}\text{Nrd}_A) \rightarrow K_0(\mathfrak{A}, A)$.

To do this we note first that, under these conditions, an easy downward induction on $i$ (using hypothesis (c)) implies that in each degree $i$ the $F$-spaces spanned by the groups of boundaries $B^i(P_1)$ and $B^i(P_2)$ have the same dimension.

If necessary, we can then also change $\theta$ by a homotopy (without changing conditions (b)) in order to ensure that, in each degree $i$, the restriction of $\theta^{i+1}$ is injective on $B^i(P_1)$ and hence induces an isomorphism $F \otimes_R B^i(P_1) \cong F \otimes_R B^i(P_2)$ (for details of such an argument see, for example, the proof of [5 Lem. 7.10]).

Having made these constructions, one can then simply mimic the argument of [1 Prop. 3.1] in order to prove the required equality (17) by using induction on the number of non-zero terms in $P_1$. 

\[\square\]

5.2. Consistency checks.

**Proposition 5.2.** If $L = K$, then Conjecture [4,7] recovers the classical Birch and Swinnerton-Dyer conjecture for $A$.

**Proof.** We assume $\text{III}(A_K)$ is finite and abbreviate $\text{SC}_{V_K}(A, K/K)$ to $\text{SC}_{V_K}$.

Now, if $L = K$, then $G$ is the trivial group $\text{id}$ and $K_{0}(\mathbb{Z}[G], \mathbb{R}[G])$ identifies with the multiplicative group $\mathbb{R}^\times/\{\pm 1\}$. In addition, upon unwinding the definition of euler characteristic one finds that, with respect to the latter identification, there is an equality
\[
\chi_{\text{BSD}}(A, V_K) = \text{disc}(h_{A,K}^{\text{NT}}) \cdot \prod_{i \in \mathbb{Z}} \#(H^i(\text{SC}_{V_K})_{\text{tor}})^{(-1)^{i+1}} \pmod{\pm 1}
\]
where $\text{disc}(h_{A,K}^{\text{NT}})$ denotes the discriminant of the pairing $h_{A,K}^{\text{NT}}$.

To compute the above product we write $\theta$ for the natural map $A(K)_{\text{tor}} \rightarrow \bigoplus_{v \notin U} A(K_v)/V_v$. Then, from Propositions [3,2] and [3,7] one finds that there are equalities $H^0(\text{SC}_{V_K}) = A^i(K)$ and $H^2(\text{SC}_{V_K}) = \ker(\theta)^*$ and a short exact sequence of the form
\[
0 \rightarrow X_Z(A/K) \rightarrow H^1(\text{SC}_{V_K}) \rightarrow \text{cok}(\theta)^* \rightarrow 0.
\]
Upon combining these observations with the natural exact sequences
\[ 0 \to \ker(\theta) \to A(K)_{\text{tor}} \to \bigoplus_{v \notin U} A(K_v)/V_v \to \coker(\theta) \to 0 \]
and
\[ 0 \to \Theta(A/K)^* \to X_\mathbb{Z}(A/K) \to \text{Hom}_\mathbb{Z}(A(K), \mathbb{Z}) \to 0 \]
one computes that
\[ \text{disc}(h_{A,K}^{\text{NT}}) : \prod_{i \in \mathbb{Z}} \#(H^i(\mathcal{C}_V)_{\text{tor}})^{-1} = \frac{\# \Theta(A_K)^{\text{NT}}}{\# A(K)_{\text{tor}} \# A^i(K)_{\text{tor}}} \prod_{v \notin U} [A(K_v) : V_v]. \tag{19} \]

On the other hand, from \[17\, 3.7.3\], one finds that
\[ \chi_{\text{id}}^{\text{coh}}(V_K, K/K) = \frac{\# H^1(X, L)}{\# H^0(X, L)} = \frac{\prod_{v \notin U} [A(K_v) : V_v]}{\text{vol}([\prod_{v \notin U} A(K_v)])} \pmod{\pm 1}, \tag{20} \]
where the ‘volume term’ here is as defined in \[17\, \S 1.7\].

Thus, since \( \chi_{\text{id}}^{\text{str}}(A) \) is clearly trivial, the expressions \[18\], \[19\] and \[20\] combine to show the equality in Conjecture \[4.1\, \text{(iii)}\) is equivalent to an equality
\[ L^*_{U}(A, 1) = \pm \frac{\# \Theta(A_K)^{\text{NT}}}{\# A(K)_{\text{tor}} \# A^i(K)_{\text{tor}}} \text{vol}([\prod_{v \notin U} A(K_v)]). \]

Since \( L^*_{U}(A, 1) \) is known to be a strictly positive real number (by Proposition \[2.2\, \text{(ii)}\]), this equality is precisely the form of the Birch and Swinnerton-Dyer Conjecture that is discussed in \[17\, \S 1.8\].\]

**Proposition 5.3.** The validity of Conjecture \[4.1\, \text{(iii)}\) is independent of the choice of the family of subgroups \( V_L \).

**Proof.** It is clearly enough to show that the difference \( \chi_G^{\text{BSD}}(A, V_L) - \chi_G^{\text{coh}}(A, V_L) \) is independent of the choice of \( V_L \).

In addition, it suffices to consider replacing \( V_L \) by a family of subgroups \( V'_L = (V'_w)_{w \notin U} \) that satisfies \( V'_w \subseteq V_w \) for all \( w \notin U \).

In this case, the definition of the complexes \( \mathcal{C}_V(L, L'/K) \) and \( \mathcal{C}_V(L, L'/K) \) via the (dual of the) mapping fibre of the respective morphisms \[6\] leads naturally to an exact triangle in \( D^\text{perf}(\mathbb{Z}[G]) \) of the form
\[ \mathcal{C}_V(L, L/K) \to \mathcal{C}_V(L, L/K) \to Q^*_1[0] \to \]
with \( Q_1 := \bigoplus_{w \notin U} (V_w/V'_w) \), and hence to an equality in \( K_0(\mathbb{Z}[G], \mathbb{R}[G]) \)
\[ \chi_G^{\text{BSD}}(A, V_L) - \chi_G^{\text{BSD}}(A, V'_L) = \chi_G(Q^*_1[0], 0). \tag{21} \]

On the other hand, if \( L'_L \) and \( L_L \) are the coherent sheaves that correspond (as in \[3.5\]) to the collections \( V'_L \) and \( V_L \) respectively, then there is a natural short exact sequence
\[ 0 \to L' \to L \to Q_2 \to 0 \] with \( Q_2 := \bigoplus_{w \notin U} W_w/W'_w \). This sequence gives rise to an exact triangle in \( D^\text{perf}(\mathbb{F}_p[G]) \) of the form
\[ R\Gamma(X, L'_L)^* \to R\Gamma(X, L_L)^* \to Q^*_2[0] \to \]
and hence to an equality
\[
(22) \quad \chi_{\text{coh}}^G(A, V_L) - \chi_{\text{coh}}^G(A, V_L') = \chi_{\mathbb{Z}_p[G]}(Q_2^*, 0).
\]

Now, given the explicit construction of the groups $V_w$ and $V_w'$ from $W_w$ and $W'_w$, it is straightforward to show that, for both $i = 1$ and $i = 2$ there exists a (finite length) decreasing filtration $(Q_{i,j})_{j \geq 0}$ of the finite $\mathbb{Z}_p[G]$-module $Q_i$ such that each module $Q_{i,j}$ is cohomologically-trivial for $G$ and the graded modules $\text{gr}(Q_i) := \bigoplus_{j \geq 0} (Q_{i,j}/Q_{i,j+1})$ are both cohomologically-trivial for $G$ and mutually isomorphic.

By a standard d\'evissage argument this implies that
\[
\chi_{\mathbb{Z}_p[G]}(Q_1^*, 0) = \chi_{\mathbb{Z}_p[G]}(\text{gr}(Q_1)^*, 0) = \chi_{\mathbb{Z}_p[G]}(\text{gr}(Q_2)^*, 0) = \chi_{\mathbb{Z}_p[G]}(Q_2^*, 0)
\]
and this combines with (21) and (22) to imply the required result. \qed

**Proposition 5.4.** The validity of Conjecture 4.4(iii) is independent of the choice of open set $U$.

**Proof.** It suffices to fix $v_0$ in $U$ and consider the effect of replacing $U$ by the set $U' := U \setminus \{v_0\}$.

We fix a family $V_L = (V_w)_{w \not\equiv U}$ of subgroups as in Lemma 3.4 and assume, following Remark 3.6, that for each place $w$ above $v_0$ one has $V_w' = A(m_w)$. We also write $V_L'$ for the associated family $(V_w')_{w \not\equiv U'}$.

Then it is enough to prove that in $K_0(\mathbb{Z}[G], \mathbb{Z}[G])$ one has
\[
(23) \quad \partial_{\mathbb{Z}_p[G]}(E_{v_0}) = (\chi_{\text{BSD}}^G(A, V_L) - \chi_{\text{BSD}}^G(A, V_L')) - (\chi_{\text{coh}}^G(A, V_L') - \chi_{\text{coh}}^G(A, V_L))
\]
where we set $E_{v_0} := L_{v_0}(A, L/K, 1) \cdot L_{v_0}(A, L/K, 1)^{-1}$ and abbreviate $SC_{v_0}(A, L/K)$ and $SC_{v_0}(A, L/K)$ to $SC(U)$ and $SC(U')$ respectively.

Now $E_{v_0}$ belongs to the subgroup $K_1(\mathcal{Q}[G])$ of $K_1(\mathbb{R}[G])$ and can be computed as the inverse of the evaluation at $u = 1$ of the following element
\[
\text{Nrd}_{\mathbb{Q}_p[G]}(1 - u^{\text{deg}(v_0)} \varphi_p^{\text{deg}(m)}) : T_{\ell}(A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_\ell[G] \to H_{\text{crys}}^0(k(v_0)/\mathbb{Z}_p, D_{v_0}) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p[G].
\]
Here $\ell$ is any choice of prime different from $p$ and $\varphi_p$ is the geometric $p$-th power Frobenius map on $T_{\ell}(A)$, the endomorphism $\varphi$ is such that $\varphi \varphi$ is induced by the crystalline Frobenius on the fibre $D_{v_0}$ at $v_0$ of the covariant Dieudonné crystal $D$ and the equality follows from the result [19] Theorem 1 of Katz and Messing.

In addition, our assumption that $V_w' = A(m_w)$ for places $w$ above $v_0$ implies that there are exact triangles in $D_{\text{perf}}(\mathbb{Z}[G])$ of the form
\[
\left\{ \begin{array}{l}
SC(U) \to SC(U') \to \bigoplus_{w \mid v_0} A(k(w))[{-1}] \to \\
R\Gamma(X, \mathcal{L}_L)^* \to R\Gamma(X, \mathcal{L}_L')^* \to \bigoplus_{w \mid v_0} \text{Lie}(A)(k(w))[0] \to,
\end{array} \right.
\]
where $\mathcal{L}_L$ and $\mathcal{L}_L'$ are the coherent sheaves associated to the families $V_L$ and $V_L'$ (so that $\mathcal{L}_{v_0} = \text{Lie}(A)(\mathcal{O}_{v_0})$).

These triangles in turn imply that there are equalities in $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$
\[
\left\{ \begin{array}{l}
\chi_{\text{BSD}}^G(A, V_L) - \chi_{\text{BSD}}^G(A, V_L') = \chi_{\mathbb{Z}_p[G]}(\bigoplus_{w \mid v_0} A(k(w))^*[-1], 0) \\
\chi_{\text{coh}}^G(A, V_L') - \chi_{\text{coh}}^G(A, V_L) = \chi_{\mathbb{Z}_p[G]}(\bigoplus_{w \mid v_0} \text{Lie}(A)(k(w))[-1], 0).
\end{array} \right.
\]
To prove the required equality (23) it is thus enough to show that for each prime $\ell \neq p$ one has

$$\delta_{G,\ell}(\text{Nrd}_{\mathbb{Q}_p[G]}(1 - \varphi_{\deg(v_0)} : T_{\ell}(A) \otimes \mathbb{Q}_p[G])) = \chi_{\mathbb{Z}_p[G]}(\bigoplus_{w|v_0} A(k(w))\{\ell\}^*[-1], 0)$$

and in addition that

$$\delta_{G,p}(\text{Nrd}_{\mathbb{Q}_p[G]}(1 - \varphi_{\deg(v_0)} : H^0_{\text{cris}}(k(v_0), \mathcal{D}) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p[G]))$$

$$= \chi_{\mathbb{Z}_p[G]}(\bigoplus_{w|v_0} A(k(w))\{p\}^*[-1], 0) - \chi_{\mathbb{Z}_p[G]}(\bigoplus_{w|v_0} \text{Lie}(A)(k(w))[-1], 0),$$

where, for each prime $q$, we write $\delta_{G,q}$ for the composite homomorphism

$$\delta_{\mathbb{Z}_p[G],\mathbb{Q}_q} \circ (\text{Nrd}_{\mathbb{Q}_p[G]})^{-1} : \zeta(\mathbb{Q}_q[G])^\times \to K_0(\mathbb{Z}_q[G], \mathbb{Q}_q[G]).$$

If $\ell \neq p$, then the complex $R\Gamma(k(v_0), T_{\ell}(A) \otimes \mathbb{Z}[G]) \cong \bigoplus_{w|v_0} R\Gamma(k(w), T_{\ell}(A))$ is acyclic outside degree one and has cohomology $\bigoplus_{w|v_0} A(k(w))\{\ell\}^*$ in that degree. This gives rise to a short exact sequence of $\mathbb{Z}_p[G]$-modules

$$0 \to T_{\ell}(A) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[G] \xrightarrow{1 - \varphi_{\deg(v_0)}} T_{\ell}(A) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[G] \to \bigoplus_{w|v_0} A(k(w))\{\ell\}^* \to 0$$

which leads directly to the equality (24).

We next note that, by Kato-Trihan [17, 5.14.6], for each $w$ dividing $v_0$ the complex $A(k(w))\{p\}[-1]$ identifies with $R\Gamma(k(w), \mathcal{S}_{D_w})$, where $\mathcal{S}_{D_w}$ is the syntomic complex over $k(w)$ (obtained as a fiber of the syntomic complex over $U$), and hence that there is an exact triangle in $D_{\text{perf}}(\mathbb{Z}_p[G])$ of the form

$$\bigoplus_{w|v_0} A(k(w))\{p\}[-1] \to R\Gamma_{\text{cris}}(k(v_0)/\mathbb{Z}_p, D^0_{v_0}) \otimes_{\mathbb{Z}_p[G]} \xrightarrow{1 - \varphi_{\deg(v_0)}} R\Gamma_{\text{cris}}(k(v_0)/\mathbb{Z}_p, D_{v_0}) \otimes_{\mathbb{Z}_p[G]} \to .$$

There is also an exact triangle

$$\bigoplus_{w|v_0} \text{Lie}(A)(k(w))[-1] \to R\Gamma_{\text{cris}}(k(v_0)/\mathbb{Z}_p, D^0_{v_0}) \otimes_{\mathbb{Z}_p[G]} \xrightarrow{1} R\Gamma_{\text{cris}}(k(v_0)/\mathbb{Z}_p, D_{v_0}) \otimes_{\mathbb{Z}_p[G]} \to .$$

The required equality (25) now follows directly upon applying Proposition 5.1 with $R = \mathbb{Z}_p[G]$ and the triangles in (14) taken to be the above two triangles. (The latter triangles are easily seen to satisfy the hypotheses of Proposition 5.1 since the modules $A(k(w))\{p\}$ and $\text{Lie}(A)(k(w))$ are finite.)

**Remark 5.5.** The results of Propositions 5.3 and 5.4 will play a key role in later arguments. In Proposition 9.2 below we will also establish a further consistency property of Conjecture 4.1 with respect to changes of field extension $L/K$.

### 5.3. A reformulation.

In this section we establish a useful reformulation of the equality in Conjecture 4.1(iii).

In [25, p. 509] Schneider shows that the pairing $h^{\text{NT}}_{A,L}$ can be factored in the form

$$h^{\text{NT}}_{A,L} = -\log(p) \cdot h_{A,L}$$

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for a certain non-degenerate skew-symmetric bilinear form $h_{A,L} : A(L) \times A^t(L) \to \mathbb{Q}$.

We write
\[ h_{A,L}^\text{det} : \text{Det}_{\mathbb{Q}[G]}(\mathbb{Q} \otimes_{\mathbb{Z}} SC_{V_L}(A, L/K)) \cong \text{Det}_{\mathbb{Q}[G]}(0). \]

for the induced isomorphism and then define an element of $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$ by setting
\[ \chi_{G,Q}^{\text{BSD}}(A, V_L) := \chi_G(\text{SC}_{V_L}(A, L/K), h_{A,L}^\text{det}). \]

For each $\chi$ in $\text{Ir}(G)$ we define a function of the $t := p^{-s}$ by setting
\[ Z_U(A, \chi, t) := L_U(A, \chi, s) \]
and we write $Z_U^*(A, \chi, p^{-1})$ for the leading term of this function at $t = p^{-1}$.

**Proposition 5.6.** The following claims are valid.

(i) There exists a unique element $Z_U^*(A_{L/K}, p^{-1})$ of $K_1(\mathbb{Q}[G])$ with the property that
\[ \text{Nrd}_{\mathbb{Q}[G]}(Z_U^*(A_{L/K}, p^{-1}))_{\chi} = Z_U^*(A, \chi, p^{-1}) \]
for all $\chi$ in $\text{Ir}(G)$.

(ii) If claims (i) and (ii) of Conjecture 4.1 are valid, then the equality in claim (iii) of Conjecture 4.1 is valid if and only if in $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$ one has
\[ \hat{\epsilon}_G(Z_U^*(A_{L/K}, p^{-1})) = \chi_{G,Q}^{\text{BSD}}(A, V_L) - \chi_G^{\text{coh}}(A, V_L) + \chi_G^{\text{sgn}}(A). \]

**Proof.** The argument of Proposition 2.2 implies, via (1) and (3), that $\omega(z_{U}^*(A, \chi, q^{-1})) = Z_U^*(A, \omega \circ \chi, q^{-1})$ for all $\chi$ and all automorphisms $\omega$ of $\mathbb{C}$, and hence that the element $(Z_U^*(A, \chi, s)_{\chi})$ of $\zeta(\mathbb{C}[G])^\times = \prod_{\chi} \mathbb{C} \times$ belongs to the subgroup $\zeta(\mathbb{Q}[G])^\times$.

Claim (i) now follows from the Hasse-Schilling-Maass Norm Theorem and the fact that the same proof also shows that $Z_U^*(A, \chi, s)$ is a strictly positive real number for all $\chi$ in $\text{Ir}^s(G)$.

To prove claim (ii) we set $r_\chi := s_{\text{an}}(\chi)$ and $r_{\chi}^\prime := s_{\text{alg}}(\chi)$. Then the order of vanishing of $Z_U(A, \chi, t)$ at $t = p^{-1}$ is equal to $r_\chi$ and hence, since
\[ (t - p^{-1}) = \frac{p^{-s} - p^{-1}}{s - 1}, \]
the leading term $Z_U^*(A, \chi, p^{-1})$ is equal to
\[ \lim_{t\to p^{-1}} (t - p^{-1})^{-r_\chi} Z_U(A, \chi, t) = \left( \frac{d}{ds} p^{-s} \right)_{s=1}^{r_\chi} \cdot \lim_{s\to 1} (s - 1)^{-r_\chi} L_U(A, \chi, s) \]
\[ = (-\log(p))^{-r_\chi} \cdot L_U^*(A, \chi, 1). \]

Thus, writing $\varepsilon_{L/K}$ for the unique element of $K_1(\mathbb{R}[G])$ with $\text{Nrd}_{\mathbb{R}[G]}(\varepsilon_{L/K})_{\chi} = (-\log(p))^{-r_\chi}$ for all $\chi$ in $\text{Ir}(G)$, one has
\[ Z_U^*(A, L/K, p^{-1}) = \varepsilon_{L/K} \cdot L_U^*(A, L/K, 1). \]

On the other hand, the equality (25) implies that
\[ \chi_{G,Q}^{\text{BSD}}(A, V_L) = \chi_G^{\text{BSD}}(A, V_L) + \hat{\epsilon}_G(\varepsilon_{L/K}^\prime) \]
where $\varepsilon_{L/K}^\prime$ is the element of $K_1(\mathbb{R}[G])$ that is represented by the automorphism of the $\mathbb{R}[G]$-module $\mathbb{R} \otimes_{\mathbb{Z}} H^1(\text{SC}_{V_L}(A, L/K)) = \mathbb{R} \otimes_{\mathbb{Z}} A^t(L)$ given by multiplication by $-\log(p)^{-1}$.
Given the last two displayed formulas, the claimed equivalence will follow if one can show that the assumed validity of Conjecture 4.1(i) implies \( \varepsilon'_{L/K} = \varepsilon_{L/K} \). But this is true since, in this case, an explicit computation of reduced norms shows that for \( \chi \) in \( \text{Ir}(G) \) one has
\[
\text{Nrd}_{R[G]}(\varepsilon'_{L/K}) \chi = (-\log(p))^{-r} = (-\log(p))^{-r(x)} = \text{Nrd}_{R[G]}(\varepsilon_{L/K}) \chi.
\]

\[\square\]

6. Syntomic cohomology

In this section we recall relevant facts concerning the complexes of syntomic cohomology with compact supports that are constructed by Kato and Trihan in [17].

At the outset we fix a finite Galois extension \( K' \) of \( K \) over which \( A \otimes K K' \) is semistable at all places, write \( L' \) for the compositum of \( L \) and \( K' \) and set \( G' := \text{Gal}(L'/K) \). Taking advantage of Proposition 5.4 we shrink \( U \) (if necessary) in order to assume that no point on \( U \) ramifies on \( L'/K \).

Since \( A \) is semistable over \( K' \), the construction in [17] §4.8] gives a Dieudonné crystal \( D_{\log}(A_{K'}) \) on \( \text{Crys}((X_{K'})^\log \mathbb{Z}_p) \). We denote this crystal by \( D \) and \( D^{(0)} \) for the kernel of the surjective morphism \( D \to i_{(X_{K'})^\log \mathbb{Z}_p,*}(\text{Lie}(D_{K'})) \) of sheaves in \( ((X_{K'})^\log \mathbb{Z}_p)_{\text{crys}} \) that is described at the beginning of [17, §5.5].

We fix a divisor \( E \) on \( X_{K'} \) with \( \text{Supp}(E) = X_{K'} - U_{K'} \), and we let \( D(-E) \) and \( D(-E)^{(0)} \) respectively denote the twist of \( D \) and \( D^{(0)} \) by \(-E\).

We also fix a Galois extension of fields \( F'/F \) with \( K' \subseteq F \subseteq F' \subseteq L' \) and set \( Q := \text{Gal}(F'/F) \).

Then, with \( N \) denoting either \( F' \) or \( F \) we set \( A_N := A \otimes K N \) and write \( X_N \) and \( U_N \) for the integral closures of \( X \) and \( U \) in \( N \) and \( A_N/X_N \) for the Neron model of \( A_N \) over \( N \). Since \( A \) is semistable over \( K' \), we have \( A_N^\text{sh} \cong A_{K'}^\text{sh} \times_{X_{K'}} X_N \).

Let \( \pi_N : X_N \to X_{K'} \) be the natural map. Then we write \( D(-E)_N := \pi_N^* \pi_{N,*}(D(-E)) \) and \( D(-E)_N^{(0)} := \pi_N^* \pi_{N,*}(D(-E)^{(0)}) \). Note that by construction of \( D_{\log} \) we have \( \pi_N^* (D(-E)) \cong D_{\log}(A_N)(-\pi_N^* E) \), and we have the following short exact sequence of sheaves in \( ((X_{K'})^\log \mathbb{Z}_p)_{\text{crys}} \):
\[
0 \to D(-E)_N^{(0)} \to D(-E)_N \to i_{(X_{K'})^\log \mathbb{Z}_p,*}(\pi_{N,*}\text{Lie}(D_N))(-d_{N/K',E}) \to 0,
\]
where \( d_{N/K'} := [N : K'] \).

Then for any locally-free \( \mathcal{O}_{X_{K'}}[Q] \)-submodule \( \mathcal{L}' \) of \( \pi_{N,*}\text{Lie}(D_{F'}) \) with
\[
(\pi_{N,*}\text{Lie}(A_{F'}))(-2d_{F'/K',E}) \subset \mathcal{L}' \subset (\pi_{N,*}\text{Lie}(A_{F'}))(-d_{F'/K',E}),
\]
Kato and Trihan construct (in [17, §5.12]) \( \mathcal{O}_{(K')}[Q] \)-submodules \( D(-E)^{(0)}_{F'} \) and \( D(-E)^{(0)}_{F'} \) of \( D(-E)_{F'} \) for which there are canonical morphisms of complexes of étale sheaves on \( X_{K'} \)
\[
Ru_{*}(D(-E)^{(0)}_{F'}) \text{ and } Ru_{*}(D(-E)^{(0)}_{F'}) \text{ are canonical morphisms of complexes of étale sheaves on } X_{K'}.
\]

where \( u : ((X_{K'})^\log \mathbb{Z}_p)_{\text{crys}} \to X_{K',\text{ét}} \) is the natural morphism of topoi and \( X_{K'} \) is endowed with the log structure associated to the divisor \( X_{K'} - U_{K'} \). They then define the syntomic
Lemma 6.1. Write $\pi^\ast v' E = \sum_{w' \notin U_{K'}} n(w')w'$, with $n(w') \geq 1$ for each $w'$. For each $w \notin U_{K'}$, write $V_w$ for the unique subgroup of $\prod_{w' | w} \mathcal{A}_{F'}(O_{w'})$ with

$$\prod_{w' | w} \mathcal{A}_{F'}(m_{n(w')}^{2n(w')}) \subseteq V_w \subset \prod_{w' | w} \mathcal{A}_{F'}(m_{n(w')}^{2n(w')})$$

and whose image in

$$\prod_{w' | w} \mathcal{A}_{F'}(m_{n(w')}^{2n(w')})/\mathcal{A}_{F'}(m_{n(w')}^{2n(w')}) \cong \prod_{w' | w} \text{Lie}(\mathcal{A}_{F'})(m_{n(w')}^{2n(w')}/\text{Lie}(\mathcal{A}_{F'})(m_{n(w')}^{2n(w')}))$$

coincides with the image of $\mathcal{L'}(O_w)$. Write $V_{F'}$ for the family $(V_w)_{w'}$.

Then, if $M$ is any intermediate field of $K'/K$ over which $F'$ is Galois, there are canonical exact triangles in $D(Z_p[\text{Gal}(F'/M)])$ of the form

$$P_{F'} \xrightarrow{1-\varphi} I_{F'} \xrightarrow{\theta} R\Gamma_{ar, V_{F'}}(U_{F'}, A_{\text{tor}})^{\ast}[2] \xrightarrow{\theta'} (\tau_{\leq 2} P_{F'})[1]$$

and

$$P_{F'} \xrightarrow{1} I_{F'} \xrightarrow{\theta} R\Gamma(X_{K'}, \mathcal{L'})^{\ast}[2] \xrightarrow{(\tau_{\leq 2} P_{F'})[1]}.$$

Proof. From [17] Prop. 5.13 one has a natural isomorphisms in $D(Z_p[\text{Gal}(F'/M)])$ of the form

$$R\Gamma(X_{K'}, \mathcal{S}_{D_{F'}}^{(E, C)}) \otimes^L Q_p/Z_p \cong R\Gamma_{ar, V_{F'}}(U_{F'}, A_{\text{tor}})^{\ast}.$$

(Though in loc. cit. this was proved when $\mathcal{L}'$ is an $O_{X_{F'}}$-submodule of $\text{Lie}(\mathcal{A}_{F'})$, the proof works in our setting because we have

$$R\Gamma(X_{K'}, Ru_{\ast} D(-E)_{F'}) \cong R\Gamma_{cryst}(X_{F'}^{\ast}/Z_p, D_{\log}(A_{F'})(-\pi^\ast v' E))$$

and the same holds for the étale cohomology of $Ru_{\ast} D(-E)_{F'}^{(0)}$ and the coherent cohomology of $\pi^\ast v' \text{Lie}(\mathcal{A}_{F'})$.)

Given this, the triangle (28) is obtained by applying the functor $\tau_{\leq 2} R\Gamma(X_{K'}, - \otimes^L Q_p/Z_p)$ to the exact triangle (of complexes of sheaves) that results from the definition of $\mathcal{S}_{D_{F'}^{(E, C)}}$ as the mapping fibre of (27).

The exact triangle (29) results in a similar way by using the canonical exact triangle

$$\mathcal{L}' \rightarrow Ru_{\ast} (D(-E)_{F'}^{(0)}) \otimes^L Q_p/Z_p \rightarrow Ru_{\ast} (D(-E)_{F'}^{(\mathcal{L})}) \otimes^L Q_p/Z_p$$
described by Kato and Trihan in [17, §6.7].

The complexes $I_{F'}$ and $P_{F'}$ are not known, in general, to belong to $D^{\text{perf}}(\mathbb{Z}_p[\text{Gal}(F'/F)])$ and hence, for our purposes, we adapt the triangles $\text{28}$ and $\text{29}$, as per the following result.

**Proposition 6.2.** Let $M$ is any intermediate field of $K'/K$ over which $F'$ is Galois. Let $J$ be a subgroup of $\text{Gal}(F'/M)$ and $\mathfrak{N}$ an order in $\mathbb{Q}_p[J]$ that contains $\mathbb{Z}_p[J]$ and is such that the complex $\tau_{\geq -1}(\mathfrak{N} \otimes_{\mathbb{Z}_p[J]} I_{F'})$ can be represented by a bounded complex of projective $\mathfrak{N}$-modules.

Then the triangles $\text{28}$ and $\text{29}$ induce exact triangles in $D^{\text{perf}}(\mathfrak{N})$ of the form

$$
\tau_{\geq -1}(\mathfrak{N} \otimes_{\mathbb{Z}_p[J]} P_{F'}) \overset{1}{\longrightarrow} \tau_{\geq -1}(\mathfrak{N} \otimes_{\mathbb{Z}_p[J]} I_{F'}) \rightarrow \mathfrak{N} \otimes_{\mathbb{Z}_p[J]} R\Gamma_{\text{ar},V,F'}(U_{F'}, A_{\text{tor}})^{\text{perf}}[2] \rightarrow
$$

and

$$
\tau_{\geq -1}(\mathfrak{N} \otimes_{\mathbb{Z}_p[J]} P_{F'}) \overset{1}{\longrightarrow} \tau_{\geq -1}(\mathfrak{N} \otimes_{\mathbb{Z}_p[J]} I_{F'}) \rightarrow \mathfrak{N} \otimes_{\mathbb{Z}_p[J]} R\Gamma(X_{F'}, \mathcal{L}')^*[-2] \rightarrow
$$

**Proof.** The results of Proposition 3.7(i) and Lemma 3.9 imply that both of the complexes $R\Gamma_{\text{ar},V,F'}(U_{F'}, A_{\text{tor}})^{\text{perf}}[2]$ and $R\Gamma(X_{F'}, \mathcal{L}')^*[-2]$ belong to $D^{\text{perf}}(\mathbb{Z}_p[J])$ and are acyclic outside degrees 0 and 1 and 2.

This implies that both $\mathfrak{N} \otimes_{\mathbb{Z}_p[J]} R\Gamma_{\text{ar},V,F'}(U_{F'}, A_{\text{tor}})^{\text{perf}}[2]$ and $\mathfrak{N} \otimes_{\mathbb{Z}_p[J]} R\Gamma(X_{F'}, \mathcal{L}')^*[-2]$ belong to $D^{\text{perf}}(\mathfrak{N})$ and are isomorphic to their respective truncations in degrees at least $-1$.

We therefore obtain exact triangles in $D(\mathfrak{N})$ of the stated form by applying the exact functor $\tau_{\geq -1}(\mathfrak{N} \otimes_{\mathbb{Z}_p[J]} -)$ to the triangles $\text{28}$ and $\text{29}$.

To prove that these triangles belong to $D^{\text{perf}}(\mathfrak{N})$ it is thus enough to prove that the complex $C := \tau_{\geq -1}(\mathfrak{N} \otimes_{\mathbb{Z}_p[J]} I_{F'})$ belongs to $D^{\text{perf}}(\mathfrak{N})$.

But, by [17, Prop. 5.15(i)], the cohomology groups of $C$ are finitely generated $\mathfrak{N}$-modules. Hence, since $C$ is assumed to be represented by a bounded complex of projective $\mathfrak{N}$-modules, this implies, by a standard construction, that $C$ belongs to $D^{\text{perf}}(\mathfrak{N})$, as required.

Since $\tau_{\geq -1}(\mathfrak{N} \otimes_{\mathbb{Z}_p[J]} I_{F'})$ is acyclic outside finitely many degrees the stated condition in Proposition 6.2 is automatically satisfied if the order $\mathfrak{N}$ is hereditary (and hence if it is maximal) by [8, Th. (26.12)].

With Theorem 4.7 in mind, in the next section we will show that, under suitable conditions on $A_M$ and $F'/M$ it can be satisfied by orders that are not maximal.

Then, in §6 we shall study in greater detail the long exact cohomology sequences of the exact triangles in Proposition 6.2.

### 7. Crystalline cohomology and tame ramification

In this section we continue to use the general notation of §6. We also assume that the extension $F'/F$ is tamely ramified and write $\pi : X_{F'} \to X_F$ for the corresponding cover of smooth projective curves. For brevity we shall usually abbreviate $X_F$ and $U_F$ to $X$ and $U$ respectively.
We fix a log structure on $X_F$ associated to the divisor $X_F - U_F$, write $X^2_F$ for the associated log scheme and note that the natural map $\pi^2 : X^2_F \to X^2_F$ is Kummer-étale (in the sense of \cite[Def. 2.13]{logDef}.

We write $u : (X^2_F/\mathbb{Z}_p)_{\text{crys}} \to X_F,\text{ét}$ and $u' : (X^2_F/\mathbb{Z}_p)_{\text{crys}} \to (X_F)_{\text{ét}}$ for the natural morphism of topos.

In this section we shall construct certain complexes of étale $\mathbb{Z}_p\{Q\}$-modules that represent $Ru_{\text{ét}}D(-E)_{(0)}$ and $Ru_{\text{ét}}'\pi_{\text{crys}*}D(-E)_{(0)}$ and are useful for the proof of Theorem 4.7.  

7.1. Digression on log de Rham complexes. The main result of this section is the following general observation concerning crystalline sheaves.

**Proposition 7.1.** Let $E$ be a locally free crystal of $\mathcal{O}_{(F)}$-modules (with $\mathcal{O}_{(F)} := \mathcal{O}_{X^2_F/\mathbb{Z}_p}$).

(i) There exists a bounded below complex $C(\pi^2_*E)$ of torsion free $\mathbb{Z}_p\{Q\}$-modules that has both of the following properties.

(a) In each degree $i$ there is an isomorphism $\mathbb{Z}_p\{Q\}$-modules

$$C^i(\pi^2_*E) \cong \text{Ind}^{O}_Q(C^i(\pi^2_*E)^{\,\pi}),$$

where $e$ denotes the identity element of $Q$.

(b) For each normal subgroup $J$ of $Q$ there is an isomorphism in $D(\mathbb{Z}_p\{Q/J\})$

$$\text{Hom}_{\mathbb{Z}_p\{J\}}(\mathbb{Z}_p, C(\pi^2_*E)) \cong R\Gamma_{\text{crys}}(X^2_{F,J}/\mathbb{Z}_p, \pi^2_J E),$$

where $\pi^2_J : X^2_{F,J} \to X^2_F$ is the natural projection.

(ii) If there is a short exact sequence of sheaves $0 \to E^0 \to E \to i^*_{X^2_F/\mathbb{Z}_p} F \to 0$ for a vector bundle $F$ on $X_F$, then claim (i) is also true with $E$ replaced by $E^0$.

7.1.1. As preparation for the proof of this result we start with the following technical result.

**Lemma 7.2.** There exists a formal scheme $\tilde{X}^2_F$ over $\mathbb{Z}_p$ that is a smooth lift of $X^2_F$. Furthermore, for any Kummer-étale covering $X^2_{F'} \to X^2_F$, there exists a finite Kummer-étale covering $\tilde{\pi}^2 : \tilde{X}^2_{F'} \to \tilde{X}^2_F$ that lifts $\pi^2 : X^2_{F'} \to X^2_F$.

**Proof.** This lemma is obtained from the infinitesimal deformation theory for smooth log schemes (cf. \cite[Prop. 3.14]{logDef}). More precisely, if $X^2_{F,n}$ is a (flat) lift of $X_F$ over $\mathbb{Z}_p/p^n$, then it is easy to see that $X^2_{F,n}$ is log smooth over $\mathbb{Z}_p/p^n$ (where $\mathbb{Z}_p/p^n$ is given the trivial log structure). To see this, one applies Kato’s criterion \cite[Th. 3.5]{logDef}. By \cite[Prop. 3.14(4)]{logDef}, the obstruction class for lifting $X^2_{F,n}$ over $\mathbb{Z}_p/p^{n+1}$ lies in $H^2(X_F, \omega_{X^2_p}^\vee) = 0$, where $\omega_{X^2_p}$ is the sheaf of differentials with log poles at $X_F - U_F$. We write $\tilde{X}^2_F$ for the natural inverse limit $\varprojlim_n X^2_{F,n}$.

Since the sheaf of relative log differentials $\omega_{X^2_{F,n}/X^2_F}$ is trivial, it follows that the finite Kummer-étale covering $\pi^2 : X^2_{F'} \to X^2_F$ canonically lifts to $\tilde{\pi}^2 : \tilde{X}^2_{F',n} \to \tilde{X}^2_{F,n}$ (cf. \cite[Prop. 3.14]{logDef}). This produces the desired finite Kummer-étale covering $\tilde{\pi}^2 : \tilde{X}^2_{F'} \to \tilde{X}^2_F$. $\square$
We use this lemma to obtain some complexes representing of $Ru_*\mathcal{E}$ and $Ru'_*(\pi^+\mathcal{E})$ for a locally free crystal $\mathcal{E}$ of $\mathcal{O}_{(F^\prime)}$-modules. Given such $\mathcal{E}$, we obtain a vector bundle $\mathcal{E}_{X_{p,F}}$ that is equipped with an integrable connection with log poles $\nabla : \mathcal{E}_{X_{p,F}} \rightarrow \mathcal{E}_{X_{p,F}} \otimes_{\mathcal{O}_{X_{p,F}}} \omega_{X_{p,F}}$.

Furthermore, since $X_{p,F}^+ \hookrightarrow X_{F}^+$ is a good embedding in the sense of [17, §5.6], it follows that $\mathcal{E}$ is functorially determined by $(\mathcal{E}_{X_{F}^+}, \nabla)$ by [16, Th. 6.2]. The same holds for any locally free crystal $\mathcal{E}'$ of $\mathcal{O}_{(F^\prime)}$-modules, and the associated vector bundle with integrable connection with log poles $(\mathcal{E}'_{X_{p,F}^+}, \nabla)$.

Recall that the map $\tilde{\pi} : X_{F,n} \rightarrow X_{F}$ is flat and we have $\tilde{\pi}^*\omega_{X_{F}^+} \xrightarrow{\sim} \omega_{X_{F,n}^+}$ by [16, Prop. 3.12], so we can define pull back and push forward by $\tilde{\pi}$ for vector bundles with connection with log poles (just as the unramified case).

Furthermore, by unwinding the proof of [16, Th. 6.2], one can see that the construction $\mathcal{E} \mapsto (\mathcal{E}_{X_{F}^+}, \nabla)$ (and the same construction for $\mathcal{E}'$) respects the pull back and push forward by $\pi^+$ so that one has both $((\pi^+\mathcal{E})_{X_{F}^+}, \nabla) = \tilde{\pi}^*(\mathcal{E}_{X_{F}^+}, \nabla)$ and $((\pi^+_p\mathcal{E}')_{X_{F}^+}, \nabla) = \tilde{\pi}^*(\mathcal{E}'_{X_{F}^+}, \nabla)$.

In particular, both $(\pi^+\mathcal{E})_{X_{p,F}^+}$ and $(\pi^+_p\mathcal{E}')_{X_{F}^+}$ have natural horizontal actions of $Q$.

Let $X_{F,n}$ denote the closed subscheme of $X_{F}$ cut out by the ideal generated by $p^n$. Then a coherent $\mathcal{O}_{X_{F,n}}$-module $\mathcal{F}_n$ can be seen as a torsion étale sheaf on $X_F$, for any étale morphism $f : Y \rightarrow X_{F,n}$ we have $\mathcal{F}_n(Y) := \Gamma(Y, f^*\mathcal{F}_n)$. Similarly, any coherent $\mathcal{O}_{X_{F,n}}$-module $\mathcal{F}$ be can viewed as a $\mathbb{Z}_p$-étale sheaf on $X_F$; namely, the inverse system of torsion étale sheaves $\{\mathcal{F}|_{X_n}\}$.

Now, for any locally free crystal $\mathcal{E}$ of $\mathcal{O}_{(F^\prime)}$-module, the complex $Ru_*\mathcal{E}$ can be computed via the complex of $\mathbb{Z}_p$-étale sheaves on $X_F$ given by $\mathcal{E}_{X_{p,F}} \xrightarrow{\nabla} \mathcal{E}_{X_{p,F}} \otimes_{\mathcal{O}_{X_{p,F}}} \omega_{X_{p,F}}$, where the first term is placed in degree zero (cf. [17, §5.6]). One also obtains a similar expression for $Ru'_*(\pi^+\mathcal{E})$ as a complex of ‘$\mathbb{Z}_p[Q]$-étale sheaves’ on $X_{F^\prime}$.

Given a short exact sequence

$$0 \rightarrow \mathcal{E}^0 \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0,$$

where $\mathcal{F}$ is a vector bundle on $X_F$ viewed as a log crystalline sheaf, we have a short exact sequence $0 \rightarrow Ru_*\mathcal{E}^0 \rightarrow Ru_*\mathcal{E} \rightarrow \mathcal{F} \rightarrow 0$, where $\mathcal{F}$ is viewed as a torsion étale sheaf on $X_F$. Therefore, we may express

$$Ru_*\mathcal{E}^0 = [\mathcal{E}_{X_{p,F}}^0 \xrightarrow{\nabla} \mathcal{E}_{X_{p,F}}^0 \otimes_{\mathcal{O}_{X_{p,F}}} \omega_{X_{p,F}}],$$

where $\mathcal{E}_{X_{p,F}}^0$ denotes the kernel of $\mathcal{E}_{X_{p,F}}$.

**Remark 7.3.** Note that $\tilde{X}_{F}^+$ can be obtained as a $p$-adic completion of a proper smooth log scheme $\tilde{X}_{F}$ over $\mathbb{Z}_p$, where the underlying scheme $\tilde{X}_F$ is a smooth lift of $X_F$ and the log structure is given by relative divisor $\tilde{Z} \subset \tilde{X}_F$ smoothly lifting $Z := |X_F - U_F|$.

---

1 It suffices to verify the flatness at the formal neighbourhood of any closed point. And by Abhyankar’s lemma (cf. [11, A.11]), the map of completed local rings induced by $\tilde{\pi}$ is of the form $W(\mathbb{F}_q)[[t]] \rightarrow W(\mathbb{F}_q[[t^{1/e}]])$ for some $e$ not divisible by $p$. 

Let us now give some examples of $Ru_* \mathcal{E}$ for some $\mathcal{E}$. When $\mathcal{E} = \mathcal{O}_{(F)}$ then $Ru_* \mathcal{O}_{(F)}$ is the log de Rham complex of $\mathcal{X}_F^\dagger$; that is, the $p$-adic completion of the de Rham complex of $\tilde{X}_F$ with log poles along $\hat{Z}$.

Given any divisor $E$ of $X_F$ supported in $Z$, one obtains a rank one locally free crystal of $\mathcal{O}_{(F)}$ modules $\mathcal{E} := \mathcal{O}_{(F)}(E)$.

Let us now describe $Ru_* \mathcal{O}_{(F)}(E)$. Viewing $\mathcal{X}_F^\dagger$ as the $p$-adic completion of the log scheme $\tilde{X}_F$ with divisorial log structure associated to $\hat{Z}$, we can find a relative divisor $\tilde{E}$ of $\tilde{X}_F$ that lifts $E$ and is supported in $\hat{Z}$, Then from the definition of $\mathcal{O}_{(F)}(E)$ (cf. [17] § 5.12), one can check that

$$Ru_* \mathcal{O}_{(F)}(E) = [\mathcal{O}_{\tilde{X}_F}(\tilde{E}) \supseteq \mathcal{O}_{\tilde{X}_F}(\tilde{E}) \otimes \omega_{\tilde{X}_F}] \otimes \mathcal{O}_{\tilde{X}_F} \mathcal{O}_{X_F},$$

where $\nabla$ is induced by the universal derivation $d: \mathcal{O}_{\tilde{X}_F} \rightarrow \omega_{\tilde{X}_F} = \Omega_{\tilde{X}_F}(\log \hat{Z})$. (Here, $\nabla$ is well defined since $\tilde{E}$ is supported in $\hat{Z}$, where $\omega_{\tilde{X}_F}$ is allowed to have log poles.)

7.1.2. We are now ready to prove Proposition 7.1. We shall, for brevity, only prove claim (ii) since this is directly relevant to the proof of Theorem 1.7 and claim (i) can be proved by exactly the same argument.

Our strategy is to use Proposition [A.7] to construct a complex $C(\pi^{\dagger,*} \mathcal{E})$ of induced $\mathbb{Z}_p[Q]$-modules that represents $R\Gamma_{\text{crys}}(X^\dagger_F/\mathbb{Z}_p, \pi^{\dagger,*} \mathcal{E})$ in such a way that $C(\mathcal{E}) := C(\pi^{\dagger,*} \mathcal{E})^Q$ is naturally isomorphic in $D(\mathbb{Z}_p)$ to $R\Gamma_{\text{crys}}(X^\dagger_F/\mathbb{Z}_p, \mathcal{E})$. (Since each term of $C(\pi^{\dagger,*} \mathcal{E})$ is an induced $\mathbb{Z}_p[Q]$-module, the complex $C(\mathcal{E})$ of term-wise $Q$-invariants of $C(\pi^{\dagger,*} \mathcal{E})$ represents $R \text{Hom}_{\mathbb{Z}_p[Q]}(\mathbb{Z}_p, C(\pi^{\dagger,*} \mathcal{E})))$.)

We recall that $R\Gamma_{\text{crys}}(X^\dagger_F/\mathbb{Z}_p, \mathcal{E})$ identifies with $R\Gamma_{\text{crist}}(X_F, Ru_* \mathcal{E})$ and that $Ru_* \mathcal{E}$ is equal to the complex $\mathcal{E}^{\dagger}_\mathcal{X}_F \supseteq \nabla \mathcal{X}_F^\dagger \otimes \mathcal{O}_{\mathcal{X}_F} \text{\hat{\omega}_{\mathcal{X}_F}}$. In particular, since all the terms of $Ru_* \mathcal{E}$ are ‘coherent $\mathcal{O}_{\mathcal{X}_F}$-modules’, we can compute $R\Gamma_{\text{crist}}(X_F, Ru_* \mathcal{E})$ via Zariski topology on $\mathcal{X}_F$ (viewing $Ru_* \mathcal{E}$ as a complex of coherent $\mathcal{O}_{\mathcal{X}_F}$-modules with additive differential). Note that the same properties hold for $Ru_*^{\dagger}(\pi^{\dagger,*} \mathcal{E})$ as well.

We now choose the disjoint union of some $Q$-stable finite affine open covering $\mathcal{U}_F^\dagger$ of $X^\dagger_F$, and regard it as a Kummer-étale covering of $X^\dagger_F$. We then let $C(\mathcal{E})$ denote the total complex associated to the Čech resolution of $Ru_* \mathcal{E}$ with respect to $\mathcal{U}_F^\dagger$. Similarly, we let $C(\pi^{\dagger,*} \mathcal{E})$ denote the total complex associated to the Čech resolution of $Ru'_* (\pi^{\dagger,*} \mathcal{E})$ with respect to the Kummer-étale covering $\mathcal{U}_F^\dagger \times \mathcal{X}_F^\dagger$ of $X^\dagger_F$, which is a complex of $\mathbb{Z}_p[Q]$-modules where the $Q$-action is induced from the $Q$-action on $\mathcal{X}_F$. Then, by Proposition [A.7] we know that $C(\mathcal{E})$ is isomorphic in $D(\mathbb{Z}_p)$ to $R\Gamma_{\text{crys}}(X^\dagger_F/\mathbb{Z}_p, \mathcal{E})$ and that $C(\pi^{\dagger,*} \mathcal{E})$ is isomorphic in $D(\mathbb{Z}_p[Q])$ to $R\Gamma_{\text{crys}}(X^\dagger_F/\mathbb{Z}_p, \pi^{\dagger,*} \mathcal{E})$.

In addition, one has $\mathcal{U}_F^\dagger \times \mathcal{X}_F^\dagger \cong \mathcal{U}_F^\dagger \times Q$ and so in each degree $i$ there is an isomorphism of $\mathbb{Z}_p[Q]$-modules

$$C^i(\pi^{\dagger,*} \mathcal{E}) \cong \text{Hom}_{\mathbb{Z}_p[Q]}(\mathbb{Z}_p[Q], C^i(\mathcal{E})) = \text{Ind}_{\{\epsilon\}}C^i(\mathcal{E}),$$
where $C^i(\pi^{\sharp,*}\mathcal{E}^0)$ and $C^i(\mathcal{E}^0)$ denote the $i$-th term of $C(\pi^{\sharp,*}\mathcal{E}^0)$ and $C(\mathcal{E}^0)$, respectively.

(Indeed, we have that each term of $Ru^*_{\mathcal{E}^0}(\pi^{\sharp,*}\mathcal{E}^0)$ is obtained by the pull back of the terms of $Ru_{\mathcal{E}^0}$ as coherent sheaves, using the isomorphism $\hat{\pi}^*\hat{\otimes}_{\hat{X}^\mathcal{E}} \to \hat{\otimes}_{\hat{X}^\mathcal{E}}$ obtained in \[16\].)

Prop. 3.12.] Therefore, we have $C(\mathcal{E}^0) = C(\pi^{\sharp,*}\mathcal{E}^0)_Q$. (To see that the Čech differentials on both sides match, we note that the Čech resolution $C(\pi^{\sharp,*}\mathcal{E}^0)$ is constructed with respect to the pull back $\tilde{U}_F \times_{X_F} \mathcal{X}_{F'}$ of the Kummer-étale covering $\tilde{U}_F$ of $X_F$, which was used for constructing the Čech resolution $C(\mathcal{E}^0)$.)

It remains to show that for any subgroup $J$ of $Q$ the complex $C(\pi^{\sharp,*}\mathcal{E}^0)_J$ represents $R\Gamma_{\text{crys}}(X^\mathcal{E}_{F'/J}/\mathbb{Z}_p, \pi^{\sharp,*}\mathcal{E}^0)$. Note that we have

$$\tilde{U}_F \times_{X_F} \mathcal{X}^\mathcal{E}_{F'/J} = \tilde{U}_F \times_{X_F} (\mathcal{X}^\mathcal{E}_{F'}/J) \cong \tilde{U}_F \times (Q/J),$$

So it follows that $C(\pi^{\sharp,*}\mathcal{E}^0)_J$ is the total complex of the Čech resolution of $Ru_{F'/J,*}(\pi^{\sharp,*}\mathcal{E}^0)$ with respect to the Kummer-étale covering $\tilde{U}_F \times_{X_F} \mathcal{X}^\mathcal{E}_{F'/J}$ of $\mathcal{X}^\mathcal{E}_{F'/J}$, and so $C(\pi^{\sharp,*}\mathcal{E}^0)_J$ represents $R\Gamma_{\text{crys}}(X^\mathcal{E}_{F'/J}/\mathbb{Z}_p, \pi^{\sharp,*}\mathcal{E}^0)$ by Proposition \[7.1\].

This completes the proof of Proposition \[7.1\].

7.2. The complex $I_{F'}$. The following consequence of Proposition \[7.1\] regarding the complex $I_{F'}$ constructed in \[6\] will play an important role in the proof of Theorem \[4.7\].

**Proposition 7.4.** If the extension $F'/F$ is tamely ramified, then $I_{F'}$ lies in $D^{\text{perf}}(\mathbb{Z}_p[Q])$ and is acyclic in all degrees outside 0, 1 and 2.

**Proof.** Throughout this proof we use the notation introduced at the beginning of \[6\] with $K' = F$. By applying Proposition \[7.1\] to $\mathcal{E}^0 = D(-E)^0_{F'}$, we obtain a complex of torsion-free induced $\mathbb{Z}_p[Q]$-modules $C_{F'}$, representing

$$R\Gamma_{\text{crys}}(X^\mathcal{E}_{F'}/\mathbb{Z}_p, D(-E)^0_{F'}(Q)) = R\Gamma_{\text{crys}}(X^\mathcal{E}_{F'}/\mathbb{Z}_p, \pi^{\sharp,*}(D(-E)^0_{F'}(Q)))$$

such that for any subgroup $J$ of $Q$ the complex $C_{F'}^J$ represents $R\Gamma_{\text{crys}}(X^\mathcal{E}_{F'}/\mathbb{Z}_p, D(-E)^0_{F'/J}(Q))$. In particular, in each degree $i$ there is an isomorphism of $\mathbb{Z}_p[Q]$-modules $C_{F'}^i \cong \text{Hom}_{\mathbb{Z}_p}[\mathbb{Z}_p[Q], (C_{F'}^i)^Q]$.

Since $(C_{F'}^i)^Q$ is $\mathbb{Z}_p$-flat in all degrees $i$, for any normal subgroup $J$ of $Q$ there is an isomorphism in $D(\mathbb{Z}_p[Q/J])$

$$I_{F'/J} \cong (C_{F'}^J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^*$$

where the complexes on the right hand side are defined by the term-wise operations.

If we set $I_{F'}^i := (C_{F'}^i \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^*$ and $I_F^q := ((C_{F'}^q)^Q \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^*$ for any $i$, then we have

$$I_{F'} \cong \mathbb{Z}_p[Q] \otimes_{\mathbb{Z}_p} I_{F'},$$

which is a flat $\mathbb{Z}_p[Q]$-module. Therefore for any subgroup $J$ of $Q$ the derived coinvariants $\mathbb{Z}_p \otimes_{\mathbb{Z}_p[Q]} I_{F'}$ can be represented by the following complex defined by term-wise operations:

$$Z_p \otimes_{\mathbb{Z}_p[Q]} (C_{F'}^i \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^* \cong ((C_{F'}^i)^J \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p)^*.$$

This implies, in particular, that $Z_p \otimes_{\mathbb{Z}_p[Q]} I_{F'}$ is isomorphic in $D(\mathbb{Z}_p[Q/J])$ to $I_{F'/J}$.

Thus, since each complex $I_{F'/J}$ is acyclic outside degrees 0, 1 and 2 and each cohomology group of $I_{F'}$ is finitely generated over $\mathbb{Z}_p$, a standard argument (as already used at the
beginning of the proof of Proposition 3.7 implies that $I_{F'}$ belongs to $D_{\text{perf}}(\mathbb{Z}_p[Q])$, as claimed.

8. Crystalline cohomology, semisimplicity and vanishing orders

As further preparation for the proof of Theorem 1.7 in this section we establish a link between the long exact cohomology sequences of the exact triangles constructed in Lemma 6.1 and the rational height pairing of Schneider and then use it to study the orders of vanishing of Artin-Hasse-Weil $L$-series.

Throughout we use the notation of Lemma 6.1 and also set $Q_M := \text{Gal}(F'/M)$. For any $\mathbb{Z}_p$-module $Y$ we abbreviate $Q_p \otimes \mathbb{Z}_p Y$ to $Y_{Q_p}$.

8.1. Height pairings and semisimplicity. At the outset we recall that, by the general discussion given at the beginning of [17, §4.3], for each intermediate field $M$ of $L'/K$ the Dieudonné crystal on $\text{Crys}(U_M)$ that is associated to the $p$-divisible group over $U$ arising from $A$ comes from an overconvergent $F$-isocrystal on $U_M$ that we shall denote by $D'_M = D'(A_M)$.

We further recall that, by Le Stum and Trihan [20, Prop. 4.2] (see also [29, Prop. 1.5]), there are natural identifications

$$Q_p \otimes \mathbb{Z}_p I_{F'} = Q_p \otimes \mathbb{Z}_p P_{F'} = R \Gamma_{\text{rig}, c}(U_{F'}, D_{F'})$$

with respect to which the morphism 1 in the exact triangle (29) corresponds to the identity endomorphism on $R \Gamma_{\text{rig}, c}(U_{F'}, D_{F'})$.

Upon combining these identifications with the long exact cohomology sequence of the exact triangle (28) we obtain a composite homomorphism

$$\beta_{A,F',p} : Q_p \otimes \mathbb{Z} A^i(F') \xrightarrow{H^0(\hat{\varphi})} H^1(P_{F'}) \otimes_{Q_p} = H^1(I_{F'}) \otimes_{Q_p} H^1(\hat{\varphi}) \to Q_p \otimes \mathbb{Z} \text{Hom}_\mathbb{Z}(A(F'), \mathbb{Z}).$$

We also write

$$h_{A,F',p,*} : Q_p \otimes \mathbb{Z} A^i(F') \to Q_p \otimes \mathbb{Z} \text{Hom}_\mathbb{Z}(A(F'), \mathbb{Z})$$

for the isomorphism of $Q_p[Q_M]$-modules that is induced by the algebraic height pairing $h_{A,F'}$ that occurs in \(\text{5.3}\).

**Proposition 8.1.** If III$(A/F')$ is finite, then the following claims are valid.

(i) One has $\beta_{A,F',p} = (\pm 1)^{a_{A,F',p}} \times h_{A,F',p,*}$ for a computable integer $a_{A,F',p}$ in $\{0,1\}$.

(ii) The homomorphisms $H^i(\hat{\varphi})_{Q_p}$ is bijective for all $i \neq 1$.

(iii) The $Q_p[Q_M]$-module $\ker(H^1(\hat{\varphi})_{Q_p})$ is naturally isomorphic to $Q_p \otimes A^i(F')$.

(iv) The composite map $\ker(H^1(\hat{\varphi})_{Q_p}) \subseteq H^1(P_{F'})_{Q_p} = H^1(I_{F'})_{Q_p} \to \text{cok}(H^1(\hat{\varphi})_{Q_p})$ is bijective.

**Proof.** Write $\mathcal{C}$ for the quotient of the category of $\mathbb{Z}_p[Q_M]$-modules by the category of finite $\mathbb{Z}_p[Q_M]$-modules.

Then, since III$(A_{F'})$ is assumed to be finite, the (non-degenerate) pairing $h_{A,F'}$ induces an isomorphism in $\mathcal{C}$ of the form

$$A(F') \otimes_{\mathbb{Z}} Q_p/\mathbb{Z}_p \to \text{Hom}_\mathbb{Z}(A^i(F'), \mathbb{Q}_p/\mathbb{Z}_p).$$
Next we set $C' := R\Gamma_{\ar, V'}(U_{F'}, A_{\text{tot}})$. Then, since the kernel of the homomorphism $H^1(C') \to \text{Sel}_{Q/Z}(A_{F'})$ in Proposition 3.2 is finite the natural map $A(F') \otimes_{\Z} Q/Z \to \text{Sel}_{Q/Z}(A_{F'})$ factors through a map $A(F') \otimes_{\Z} Q/Z \to H^1(C')$ in $\mathcal{C}$. This homomorphism then gives rise to a composite homomorphism in $\mathcal{C}$ of the form

\begin{equation}
A(F') \otimes_{\Z} Q/Z \to H^1(C_i'_{p}) \to H^1(I^*_{F'}) \xrightarrow{1} H^1(P^*_{F'}) \to H^2(C_i'_{p}) \to \text{Hom}_{\Z}(\text{Sel}_{Q}(A^i), Q_p/Z_p) \to \text{Hom}_{\Z}(A^i(F'), Q_p/Z_p),
\end{equation}

where the second and fourth maps are induced by the exact triangle \([28]\) and the fifth by Proposition 3.2.

To prove claim (i) it is sufficient, after taking Pontryagin duals, to show that the morphisms \([22]\) and \([33]\) in $\mathcal{C}$ coincide up to a computable sign and this is precisely what is established by the argument of Kato and Trihan in [17, 3.3.6.2].

To prove the other claims we note that the long exact cohomology sequence of the exact triangle \([28]\) combines with the descriptions in Proposition 3.1 ii) to imply that $H^i(\hat{\phi})_Q$ is bijective for all $i \notin \{0, 1\}$, that ker($H^1(\hat{\phi})_Q$) and cok($H^2(\hat{\phi})_Q$) vanish and that there are exact sequences of $Q_p[Q_M]$-modules

\begin{equation}
\begin{cases}
0 \to \text{cok}(H^0(\hat{\phi}))_Q \xrightarrow{H^0(\theta)} Q_p \otimes A^i(F') \xrightarrow{H^0(\phi')} \ker(H^1(\hat{\phi}))_Q \to 0,
0 \to \text{cok}(H^1(\hat{\phi}))_Q \xrightarrow{H^1(\theta)} Q_p \otimes \text{Hom}_{\Z}(A^i(F'), \Z) \xrightarrow{H^1(\phi')} \ker(H^2(\hat{\phi}))_Q \to 0.
\end{cases}
\end{equation}

Now, since $h^*_{A,F'}$ is bijective, claim (i) implies the same is true of the map $\beta_{A,F'}$ and this fact combines with the above exact sequences to imply that the spaces cok($H^0(\hat{\phi})_Q$) and ker($H^2(\hat{\phi})_Q$) vanish, as required to complete the proof of claim (ii), and hence that the upper sequence in \([34]\) gives an isomorphism of the sort required by claim (iii).

Finally, claim (iv) is true because the bijectivity of $\beta_{A,F'}$ combines with the upper sequence in \([34]\) to imply ker($H^1(\hat{\phi})_Q$) is disjoint from ker($H^1(\theta)$) whilst the lower sequence in \([33]\) implies that ker($H^1(\theta)$) is equal to im($H^1(\hat{\phi})_Q$).

\[\square\]

8.2. Orders of vanishing and leading terms. We now derive from Proposition 8.1 the following result about the order of vanishing $r_{A,M}(\chi)$ at $t = p^{-1}$ of the functions $Z_{U_M}(A_M, \chi, t)$ that are defined in \([5.3]\) for each character $\chi$ in $\text{Ir}(Q_M)$.

We fix (and do not in the sequel explicitly indicate) an isomorphism of fields $\mathbb{C} \cong \mathbb{C}_p$ and hence do not distinguish between $\text{Ir}(Q_M)$ and the set of irreducible $\mathbb{C}_p$-valued characters of $Q_M$.

In particular, for $\chi$ in $\text{Ir}(Q_M)$ we may then fix a representation $Q_M \to \text{Aut}_{\mathbb{C}_p}(V_\chi)$ (that we also denote by $\chi$) of character $\chi$, where $V_\chi$ is a finite dimensional vector space over $\mathbb{C}_p$.

If $R$ denotes either $\mathbb{Z}_p[Q_M]$ or $\mathbb{Q}_p[Q_M]$, then for each finitely generated $R$-module $W$ and each $\chi$ in $\text{Ir}(Q_M)$ we define a $\mathbb{C}_p$-vector space by setting

$$W^\chi := \text{Hom}_{\mathbb{C}_p[Q_M]}(V_\chi, \mathbb{C}_p[Q_M] \otimes_R W).$$

**Theorem 8.2.** For each $\chi$ in $\text{Ir}(Q_M)$ the following claims are valid.

(i) $r_{A,M}(\chi) = \dim_{\mathbb{C}_p}(\mathbb{Z}_p \otimes_{\mathbb{Z}} A^i(F')^\chi) = \chi(1)^{-1} \cdot \dim_{\mathbb{C}}(c_\chi(\mathbb{C} \otimes_{\mathbb{Z}} A^i(F'))).$
(ii) In each degree $i$ the homomorphism $H^i(1 - \varphi)$ induces an automorphism $H^i(1 - \varphi)_\chi$ of any fixed complement to $\ker(H^i(1 - \varphi)_\chi)$ in $H^i_{\text{rig, c}}(U_{F'}', D^i_{\text{c}})^\chi$.

(iii) $Z^s_{U_M}(A_M, \chi, p^{-1}) = \prod_{i=0}^{i=2} \det(H^i(1 - \varphi)_\chi)^{(1 - 1)^{+1}}$.

Proof. We write $D^i_{M}(\chi)$ for the twist of $D^i_{M}$ by the representation $\chi$. Then it is shown by Etesse and Le Stum in [10] that there is an identity of functions

$$(35) \quad Z_{U_M}(A_M, \chi, p^{-1}) = L_{U_M}(A_M, \chi, s + 1) = \prod_{i=0}^{i=2} \det(1 - \varphi \cdot t) H^i_{M}(\chi))^{-1)^{+1}}.$$  

where for each $i$ we set $H^i_{M}(\chi) := H^i_{\text{rig, c}}(U_M, D^i_{M}(\chi))$

Now, by Lemma 8.3 below, the space $H^i_{M}(\chi)$ is naturally isomorphic to

$$H^i_{F'} \otimes_{\mathbb{Q}_p[Q_M]} V_\chi \cong (H^i_{F'})^\chi$$

with $H^i_{F'} := H^i_{\text{rig, c}}(U_{F'}, D^i_{F'})$.

In particular, from Proposition 8.1(ii) we deduce that, setting $1 - \varphi = \hat{\varphi}$, the endomorphism $H^i(\varphi)$ is invertible on $H^i_{M}(\chi)$ for both $i = 0$ and $i = 2$. This fact then combines with the identity (35) to imply that $r\cdot A_M(\chi)$ is equal to the dimension over $C_p$ of the kernel of $\hat{\varphi}$ on $H^i_{M}(\chi) \cong (H^i_{F'})^\chi$ and hence to $\dim_{C_p}(\ker(\hat{\varphi}_{F'})^\chi)$, where we write $\hat{\varphi}_{F'}$ for the endomorphism induced by $\hat{\varphi}$ on $H^i_{F'}$.

To prove claim (i) it now only remains to note Proposition 8.1 implies that

$$\dim_{C_p}(\ker(\hat{\varphi}_{F'})^\chi) = \dim_{C_p}(\mathbb{Z}_p \otimes_{\mathbb{Z}} A^i(F'))^\chi) = (1 - 1)^{-1} \cdot \dim_C(e_\chi(\mathbb{C} \otimes_{\mathbb{Z}} A^i(F))).$$

Claim (ii) follows directly from Proposition 8.1(iv) and the fact (noted above) that $H^i(\varphi)$ is invertible on $H^i_{M}(\chi)$ for $i = 0$ and $i = 2$.

Finally, claim (iii) follows by combining claims (i) and (ii) with the identity (35). □

Lemma 8.3. For every representation $\chi : Q_M \to \text{Aut}_{\mathbb{C}_p}(V_\chi)$ as above, there is a natural isomorphism $H^i_{\text{rig, c}}(U_{F'}, D^i_{F'}) \otimes_{\mathbb{Q}_p[Q_M]} V_\chi \cong H^i_{\text{rig, c}}(U_M, D^i_{M}(\chi))$, for any $i$.

Proof. With $\pi : X_{F'} \to X_M$ denoting the natural morphism, there is a canonical isomorphism $\pi^* D^i(A_M) \cong D^i(A_{F'})$ and so in the above formula we can replace $D^i_{F'}$ by $\pi^* D^i_{M}(A)$.

Next we observe that for any $i$ Poicaré duality gives a canonical isomorphism

$$H^i_{\text{rig, c}}(U_{F'}, \pi^* D^i_{M}) \cong \text{Hom}_{\mathbb{Q}_p}(H^i_{\text{rig, c}}(U_{F'}, (\pi^* D^i_{M})^\vee), \mathbb{Q}_p),$$

so and it is enough to prove that

$$(H^i_{\text{rig, c}}(U_{F'}, \pi^* D^i_{M})^\vee) \cong (H^i_{\text{rig, c}}(U_{F'}, (\pi^* D^i_{M})^\vee) \otimes_{\mathbb{Q}_p} V_\chi)^{Q_M} \cong H^i_{\text{rig, c}}(U_M, (D^i_{M}(\chi))^\vee).$$

Recall that $D^i_{M}(\chi)$ is defined to be the unique overconvergent $F$-isocrystal (up to unique isomorphism) such that we have a natural $Q_M$-equivariant isomorphism $\pi^*(D^i_{M}(\chi))^\vee \cong (\pi^* D^i_{M})^\vee \otimes_{\mathbb{Q}_p} V_\chi$, where we give the diagonal $Q_M$-action on the right hand side. (The unique existence of such $D^i_{M}(\chi)$ is a consequence of étale descent theory of overconvergent
$F$-isocrystals over curves; cf. [7 Prop. 1.3]. Therefore the above isomorphism follows from the natural isomorphism

$$H^i_{\mathrm{rig}}(U_M, (D_M^\dagger(\chi))^\vee) \overset{\sim}{\to} H^i_{\mathrm{rig}}(U_{F'}, \pi^*(D_M^\dagger(\chi))^\vee))^{Q^M}$$

for each $i$, which was proved in [29] Prop. 4.6].

\hfill \square

9. Proof of the main result

In this section we use results from earlier sections to obtain a proof of Theorem 4.7.

At the outset we note that Theorem 4.7(i) is proved by Theorem 8.2(i) and that Remark 4.8 allows us to assume that $\mathbb{III}(A/L)$ is finite.

We therefore focus on establishing the validity of the equality in Conjecture 4.1(iii).

For convenience, for each Galois extension $F'/M$ (as in Proposition 9.1) we define an element of $K_0(\mathbb{Z}[Q_M], \mathbb{Q}[Q_M])$ by setting

$$\chi(A, F'/M) := \partial Q_M(Z_{U_M}(A, M, F'/M, p^{-1})) - \chi_{Q_M, Q}(A_M, V_{F'}) + \chi_{Q_M}^{\mathbb{coh}}(A_M, V_{F'}) - \chi_{Q_M, Q}^{\mathbb{sgn}}(A_M).$$

9.1. A first reduction step. For a finite group $\Gamma$, a prime $\ell$ and $x$ in $K_0(\mathbb{Z}[\Gamma], \mathbb{Q}[\Gamma])$ we write $x_\ell$ for the image of $x$ in $K_0(\mathbb{Z}[\Gamma], \mathbb{Q}_\ell[\Gamma])$ under the decomposition (10).

Proposition 9.1. Assume $\mathbb{III}(A/L)$ is finite. Then the statement of Theorem 4.7 is valid if and only if the following conditions are satisfied.

(i) If $\mathfrak{M}_p$ is any maximal $\mathbb{Z}_p$-order in $\mathbb{Q}_p[G']$ that contains $\mathbb{Z}_p[G']$, then $\chi(A, L/K)_p$ belongs to the kernel of the homomorphism $K_0(\mathbb{Z}[\mathfrak{M}_p], \mathbb{Q}[\mathfrak{M}_p]) \to K_0(\mathfrak{M}_p, \mathbb{Q}_p[G'])$.

(ii) Write $P'$ for the normal subgroup of $H' := \mathrm{Gal}(L'/K')$ that is generated by the Sylow $p$-subgroups of the inertia groups of all places that ramify in $L'/K'$. Then $\chi(A, (L')^{P'/K'})_p$ vanishes.

(iii) For each prime $\ell \neq p$ one has

$$\partial_{G, Q}(Z_{U}(A, L/K, p^{-1}))_\ell = \chi_{G, Q}^{\mathbb{coh}}(A, V_L)_\ell - \chi_{G, Q}^{\mathbb{sgn}}(A)_\ell.$$

Proof. It suffices to check that the stated conditions are equivalent to the validity of the equality in Conjecture 4.1(iii).

Thus, after taking account of Proposition 5.6, the decomposition (10) combines with the explicit definition of $T_{A, L/K}$ to reduce us to showing that $\chi(A, L/K)_\ell$ vanishes if $\ell \neq p$ and that $\chi(A, L/K)_p$ is the image under $\pi^G_{G'}$ of an element of $K_0(\mathbb{Z}_p[G'], \mathbb{Q}_p[G'])$ that is both torsion and belongs to $\ker(\pi^G_{H'/P'})$.

Now, by Proposition 9.2 below, one has $\chi(A, L/K)_p = \pi^G_{G'}(\chi(A, L'/K)_p)$ and so in the case $\ell = p$ it is enough to show $\chi(A, L'/K)_p$ has finite order and $\pi^G_{H'/P'}(\chi(A, L'/K)_p)$ vanishes.

To see that these conditions are respectively equivalent to the conditions stated in claims (i) and (ii) it suffices to note that the kernel of the scalar extension homomorphism $K_0(\mathbb{Z}_p[G'], \mathbb{Q}_p[G']) \to K_0(\mathfrak{M}_p, \mathbb{Q}_p[G'])$ is the torsion subgroup of $K_0(\mathbb{Z}_p[G'], \mathbb{Q}_p[G'])$ (by [2 §4.5, Lem. 11(d)]) and that $\pi^G_{H'/P'}(\chi(A, L'/K)_p) = \chi(A, (L')^{P'/K'})_p$ by Proposition 9.2.
Finally we note that if \( \ell \neq p \), then \( \chi^\text{coh}_G(A, V_L) \ell \) vanishes and so it is clear that the vanishing of the image in \( K_0(\mathbb{Z}[\ell], \mathbb{Q}[\ell]) \) of the equality in Conjecture 4.1(iii) is equivalent to the equality in claim (iii). \( \square \)

Before stating the next result we note that if \( J \) is a normal subgroup of a subgroup \( H \) of \( G \), and we set \( Q := H/J \), then there is a natural commutative diagram

\[
\begin{array}{ccc}
K_1(\mathbb{Q}[G]) & \xrightarrow{\theta_{G,H}} & K_1(\mathbb{Q}[H]) \\
\bar{\partial}_{G,Q} & & \bar{\partial}_{H,Q} \\
K_0(\mathbb{Z}[G], \mathbb{Q}[G]) & \xrightarrow{\theta_{G,H}} & K_0(\mathbb{Z}[H], \mathbb{Q}[H]) & \xrightarrow{\theta_{H,Q}} & K_0(\mathbb{Z}[Q], \mathbb{Q}[Q])
\end{array}
\]

where \( \theta_{G,H} \) and \( \theta_{H,Q} \) are the natural restriction and coinflation homomorphisms.

**Proposition 9.2.** If \( J \) is a normal subgroup of a subgroup \( H \) of \( G \), then the composite homomorphism \( \theta_{H,Q}^0 \circ \theta_{G,H}^0 \) sends \( \chi(A, L/K)^p \) to \( \chi(A_{L^J}, L^J/L^H)^p \).

**Proof.** We set \( \theta_{H,Q} := \theta_{H,Q}^0 \circ \theta_{G,H}^i \); \( E := L^H \) and \( F := L^J \). At the outset we note that, by a standard argument using the Artin formalism of \( L \)-functions, one finds that \( \theta_{G,Q}(Z_U^*(A, L/K, p^{-1})) = Z^*(A, F/E, p^{-1}) \) and so the commutative diagram (36) implies

\[
\theta_{G,Q}(\bar{\partial}_{G,Q}(Z_U^*(A, L/K, p^{-1}))) = \bar{\partial}_{Q,Q}(Z^*_U(A, F/E, p^{-1})).
\]

It is also clear that \( \theta_{G,Q}(\langle -1, Q \cdot A(L) \rangle) = \langle -1, Q \cdot A(F) \rangle \) and, given this, an explicit comparison of the equalities in Proposition 8.1(i) with \( F' \) equal to \( L \) and \( F \) implies

\[
\theta_{G,Q}^0(\chi^\text{sgn}(A, L/K)^p) = \chi^\text{sgn}(A, F/E)^p.
\]

To proceed we write \( \pi, \pi' \) and \( \pi'' \) for the natural morphisms \( X_L \to X, X_L \to X_E \) and \( X_E \to X \). We fix families of subgroups \( V_L \) and \( W_L \) for the extension \( L/K \) as in (8.1) (the choice of which is, following Proposition 7.3 unimportant) and write \( \mathcal{L}_L \) for the associated coherent \( \mathcal{O}_{X}[G] \)-submodule of \( \pi^*_s \text{Lie}(A_{X_L}) \). In the same way we fix families of subgroups \( V'_L \) and \( W'_L \) for the extension \( E/L \) and write \( \mathcal{L}'_L \) for the associated coherent \( \mathcal{O}_{X_E}[H] \)-submodule of \( \pi^*_s \text{Lie}(A_{X_L}) \).

We assume, as we may, that \( V_L \subseteq V'_L \), and hence also \( W_L \subseteq W'_L \). This implies that there are exact triangles in \( D^\text{perf}(\mathbb{Z}[H]) \) of the form

\[
\text{SC}_V(L, L/K) \to \text{SC}_V'(A_E, L/E) \to (V'_L/V_L)[0] \to
\]

and

\[
R\Gamma(X, \mathcal{L}_L) \to R\Gamma(X_E, \mathcal{L}'_L) \to (W'_L/W_L)[0] \to,
\]

where, in the latter case, we have used the fact that the complexes \( R\Gamma(X, \pi'_s \mathcal{L}'_L) \) and \( R\Gamma(X_E, \mathcal{L}'_L) \) are canonically isomorphic since \( \pi''_s \) is exact. These triangles in turn give rise to equalities in \( K_0(\mathbb{Z}[H], \mathbb{Q}[H]) \)

\[
\theta_{G,H}^0(\chi^\text{BSD}_{G,\mathbb{Q}}(A, V_L)) - \chi^\text{coh}_G(A, V_L) = \chi^\text{BSD}_{H,Q}(A, V'_L) - \chi^\text{coh}_{Z^p[H]}((V'_L/V_L)[0], 0)
\]

\[
- (\chi^\text{coh}_{H,Q}(A, V'_L) - \chi^\text{coh}_{Z^p[H]}((W'_L/W_L)[0], 0))
\]

\[
= \chi^\text{BSD}_{H,Q}(A, V'_L) - \chi^\text{coh}_{H,Q}(A, V'_L).
\]
where the last equality is valid since \( \chi_{F_p[H]}((V'_L/V_L)[0], 0) = \chi_{F_p[H]}((W'_L/W_L)[0], 0) \) (by the same argument as used in the proof of Proposition 5.3).

Upon combining the equalities (37), (38) and (39) one finds that the proof is reduced to showing that there are equalities

\[
\begin{align*}
\theta^0_{H,Q}(\chi^\text{BSD}_{H,Q}(A_E, V'_L)) &= \chi^\text{BSD}_{Q,Q}(A_E, (V'_L)^J), \\
\theta^0_{H,Q}(\chi^\text{coh}_{H,Q}(A_E, V'_L)) &= \chi^\text{coh}_{Q,Q}(A_E, (V'_L)^J).
\end{align*}
\]

These equalities follow directly from the isomorphisms in \( D^\text{perf}(Z[Q]) \)

\[
\begin{align*}
Z[Q] \otimes_{\smile[H]} SC_{V'_L}(A_E, L/E) &\cong SC_{(V'_L)^J}(A_E, F/E), \\
Z[Q] \otimes_{\smile[Q]} R\Gamma(X_E, E'_{L'})^* &\cong R\Gamma(X_E, (E'_{L'})^J)^*,
\end{align*}
\]

that are respectively used in the proofs of Proposition 3.7 and Lemma 3.9. \( \square \)

9.2. The case \( \ell = p \). In this section we verify that the conditions of Proposition 9.1(i) and (ii) are satisfied.

The key observation we shall use is provided by the following result.

**Proposition 9.3.** If the data \( F', J \) and \( \mathfrak{N}_p \) are as in Proposition 6.2, then \( \chi(A, F'/((F')^J)p \) belongs to the kernel of the natural homomorphism \( K_0(\mathfrak{N}_p, Q_p[J]) \rightarrow K_0(\mathfrak{N}_p, Q_p[J]) \).

**Proof.** The exact triangles in Proposition 6.2 lie in \( D^\text{perf}(\mathfrak{N}_p) \) and satisfy all of the hypotheses of Proposition 5.1. In fact the only condition that is not straightforward to check is (b) and this is proved in Proposition 8.2(iii).

Set \( M := (F')^J \). Then, by applying the latter result in this case, and taking account of Proposition 8.1(i), one finds that the image of \( \chi(A, F'/M)_p \) in \( K_0(\mathfrak{N}_p, Q_p[J]) \) coincides with the image in \( K_0(\mathfrak{N}_p, Q_p[J]) \) of \( \partial_{I,Q}(Z_{\text{unr}}(A, F'/M, p^{-1}))_p - \partial_{Z_p[J],Q_p}(H^1(1 - \varphi)_{\mathfrak{N}_p}) \).

It is thus enough to note that the latter element vanishes as a consequence of Theorem 8.2(ii). \( \square \)

Turning to the conditions of Proposition 9.1 we note that \( \mathfrak{N}'_p \) is regular and hence satisfies the conditions of Proposition 6.2 (with \( J = G' \)). Given this, the above result implies, directly, that the condition of Proposition 9.1(i) is satisfied.

Next we note that the field \( F' := (L')'p \) is a tamely ramified Galois extension of \( K' \) and hence that Proposition 7.4 implies the conditions of Proposition 6.2 are satisfied by the data \( J = \text{Gal}(F'/K') \) and \( \mathfrak{N}_p = Z_p[J] \). In this case, therefore, Proposition 9.3 implies that \( \chi(A, (F')'p/K')_p \) vanishes, and hence that the condition of Proposition 9.1(ii) is satisfied.

**Remark 9.4.** The above argument shows that we actually prove Theorem 4.7(ii) with \( T_{A, L/K} \) replaced by the potentially smaller group obtained by replacing the term \( \ker(\pi_{K'}) \) in (33) by the kernel of the natural composite homomorphism

\[
K_0(\mathfrak{N}_p[G'], Q_p[G']) \rightarrow K_0(\mathfrak{N}_p[Q_M], Q_p[Q_M]) \rightarrow K_0(\mathfrak{N}_p, Q_p[Q_M]),
\]

where \( \mathfrak{N}_p \) is any order that satisfies the hypotheses of Proposition 6.2. We recall that the latter condition is automatically satisfied if \( \mathfrak{N}_p \) is hereditary but, aside from this, finding other interesting, and explicit, examples of such orders beyond those that are used in the above argument seems to be difficult.
9.3. The case $\ell \neq p$. In this section we verify that the condition of Proposition 9.1(iii) is satisfied.

To do this we fix a prime $\ell \neq p$, write $T_\ell(A)$ for the $\ell$-adic Tate module of $A$ and set $V_\ell(A) = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} T_\ell(A)$. We write $\mathbb{F}_p^c$ for its algebraic closure of $\mathbb{F}_p$ and $\varphi_p$ for the Frobenius automorphism at $p$ and set $U_\ell^c := U_\ell \times_{\mathbb{F}_p} \mathbb{F}_p^c$.

The key fact that we shall use is that for each $\chi$ in $\text{Ir}(G)$ the identity (11) combines with the Poincare duality theorem (as stated, for example, in [22, Chap. VI, Cor. 11.2]), to give an identity of functions

$$Z_U(A, \chi, p^{-1}t) = \prod_{i \in \mathbb{Z}} \det(1 - \varphi_p^{-1} \cdot p^{-1}t : H^i_{\text{ét}, c}(U_\ell^c, V_\ell(A))(\chi))^{(-1)^{i+1}}$$

$$= \prod_{i \in \mathbb{Z}} \det(1 - \varphi_p \cdot t : H^i_{\text{ét}}(U_\ell^c, V_\ell(A))(\chi))^{(-1)^{i+1}}.$$

Set $\text{SC}_\ell := \mathbb{Z}_\ell \otimes_{\mathbb{Z}} \text{SC}_{V_\ell}(A, L/K)$. Then the result of Proposition 3.7(ii)(b) combines with Remark [3,3] and Artin-Verdier duality to imply there are natural isomorphisms

$$\text{SC}_\ell \cong \mathbb{Z}_\ell \otimes_{\mathbb{Z}} R\Gamma_{\text{ét}, V_\ell}(U_L, A_{\text{tors}})^* [2] \cong R\Gamma_{\text{ét}, c}(U_L, A_{\text{tors}})^* [2] \cong R\Gamma_{\text{ét}}(U_L, T_\ell(A))$$

and hence also a natural exact triangle in $D^{\text{perf}}(\mathbb{Z}_\ell[G])$ of the form

$$\text{SC}_\ell \rightarrow R\Gamma_{\text{ét}}(U_L^c, T_\ell(A)) \xrightarrow{1 - \varphi_p} R\Gamma_{\text{ét}}(U_L^c, T_\ell(A)) \rightarrow \text{SC}_\ell[1].$$

We consider the composite homomorphism

$$\beta_{A, L, \ell} : \mathbb{Q}_\ell \otimes_{\mathbb{Z}} A^i(L) \cong H^0(\text{SC}_\ell)_{\mathbb{Q}_\ell} \rightarrow H^0_{\text{ét}}(U_L, V_\ell(A)) \rightarrow H^1(\text{SC}_\ell)_{\mathbb{Q}_\ell} \cong \mathbb{Q}_\ell \otimes_{\mathbb{Z}} \text{Hom}_{\mathbb{Z}}(A(L), \mathbb{Z}),$$

where the isomorphisms are from Proposition 3.7(ii)(a) and the other maps are induced by the long exact cohomology sequence of (11).

Then it is shown by Schneider in [25] (and also noted at the beginning of [17, §6.8]) that there exists a computable integer $a_{A, L, \ell} \in \{0, 1\}$ such that

$$\beta_{A, L, \ell} = (-1)^{a_{A, L, \ell}} \cdot h_{A, L, \ell,*}$$

where $h_{A, L, \ell,*}$ is the isomorphism $\mathbb{Q}_\ell \otimes_{\mathbb{Z}} A^i(L) \cong \mathbb{Q}_\ell \otimes_{\mathbb{Z}} \text{Hom}_{\mathbb{Z}}(A(L), \mathbb{Z})$ induced by the height pairing $h_{A, L}$.

Taken in conjunction with the same argument used in Proposition 8.1 this observation implies firstly that the endomorphism $H^i(1 - \varphi_p)_{\mathbb{Q}_\ell}$ is bijective for $i = 1$, secondly that (11) satisfies all of the hypotheses that are made in Proposition 8.1 (with $\mathfrak{F} = \mathbb{Z}_\ell[G]$) regarding the left hand triangle in (14), and thirdly (in view of (40)) that for each $\chi$ the order of vanishing of $Z_U(A, \chi, t)$ at $t = p^{-1}$ is equal to the dimension over $\mathbb{C}_\ell$ of the kernel of the endomorphism of

$$H^1_{\text{ét}}(U_\ell^c, V_\ell(A))(\chi) \cong \text{Hom}_{\mathbb{C}_\ell[G]}(V_\chi, H^1_{\text{ét}}(U_\ell^c, V_\ell(A)))$$

that is induced by $H^1(1 - \varphi_p)$. 


By applying Proposition 5.1 with the left and right hand triangles in (14) taken to be (41) and the zero triangle respectively we can therefore deduce that
\[
\iota_{G,\ell}^{\text{BSD}}(A, V) + \chi_G^{\text{sgn}}(A) = \chi_{Z\ell}(SC, \tau_{1-\varphi_p}) + \widehat{\beta}_{\ell}(\beta_{A,\ell} \circ h_{A,\ell}^{-1})
\]
Here the first equality follows directly from the definition of \(\chi_{\text{sgn}}(A, L/K)\) in terms of the integer \(a_{A,\ell}\), the equality (42) and the result of Lemma 4.6. In addition, the fourth equality follows from (40) (and the isomorphism (43)) and the fifth directly from the definition of the term \(Z_{U}^{\ast}(A, L/K, p^{-1})\).

This completes the proof that the condition of Proposition 9.1(iii) is satisfied and hence also completes the proof of Theorem 4.7.

**Appendix A. Kummer-étale descent for coherent cohomology**

In this appendix, we show that the coherent cohomology over a ‘separated’ formal fs log scheme can be computed via the Čech resolution with respect to an affine Kummer-étale covering (not necessarily a Zariski open covering). Whilst this result seems to be well known to experts, we have not been able to locate a good reference for it in the literature.

**A.1. Fs log schemes and their fibre products.** The main purpose of this section is to review the construction of fibre products for fs log schemes, which we need for the sheaf theory on Kummer-étale sites and the construction of Čech complexes. We will briefly recall some definitions of monoids and log schemes needed for the construction of fibre products. We do not give a complete review of basic definitions on monoids and log geometry but rather refer readers to [16] and [23] for basic definitions in log geometry and to [24] for a more comprehensive reference.

Recall that a (always commutative) monoid \(P\) is said to be fine if it is finitely generated and the natural map \(P \to P^\text{gp}\) is injective (where \(P^\text{gp}\) is the commutative group obtained by adjoining the inverse of each element of \(P\)). A fine monoid \(P\) is said to be saturated if for any \(\alpha \in P\), we have \(\alpha^n \in P\) for some \(n > 0\) if and only if \(\alpha \in P\). By a fs monoid, we mean a fine and saturated monoid.

For each monoid \(P\) we define a saturation \(P^\text{sat} := \{\alpha \in P^\text{gp}; \alpha^n \in P\text{ for some }n > 0\}\).

**Lemma A.1.** If \(P\) is finitely generated, then the monoid \(P^\text{sat}\) is fs.

**Proof.** It suffices to show that \(P^\text{sat}\) is finitely generated, which is a direct consequence of Gordon’s Lemma (cf. [24] Cor. 2.3.20)).

A log scheme \(X^\ast\) is called fs (i.e., fine and saturated) if étale locally on the underlying scheme \(X\), the log structure is generated by a map of monoids \(P \to O_X\) where \(P\) is a
Let $X^\sharp$ and $Y^\sharp$ be fs log schemes over $S^\sharp$ (with underlying schemes denoted as $X$, $Y$, and $S$). We want to construct a fs log scheme $X^\sharp \times_{S^\sharp} Y^\sharp$ satisfying the universal property of fibre product (cf. [24, Cor. 2.4.16]).

By replacing the formal log schemes with suitable étale coverings, we choose charts $P \to O_X$, $Q \to O_Y$ and $M \to O_S$ defining the log structures (where $P$, $Q$ and $M$ are fine saturated monoids, viewed as constant sheaves), such that there exist maps $M \to P, Q$ giving rise to the structure morphism $X^\sharp, Y^\sharp \to S^\sharp$. (The existence of such local charts follows from [16, Lem. 2.10].)

The most natural candidate is to endow $X^\sharp \times_{S^\sharp} Y^\sharp$ with the log structure associated to the chart $P \amalg M \to O_{X^\sharp \times_{S^\sharp} Y^\sharp}$, where $P \amalg M$ is the amalgamated sum of monoids. But this may not always work as $P \amalg M$ may not be fine nor saturated.

Writing $P \amalg_{\text{sat}} M$ for the saturation of $P \amalg M$ we can define the following fs log scheme

$$X^\sharp \times_{S^\sharp} Y^\sharp := (X \times_S Y) \times_{\text{Spec} \mathbb{Z}[P \amalg_{\text{sat}} M]} \text{Spec} \mathbb{Z}[P \amalg_{\text{sat}} M]$$

with the log structure given by the chart $P \amalg_{\text{sat}} M \to O_{X^\sharp \times_{S^\sharp} Y^\sharp}$ naturally extending $P \amalg_M Q \to O_{X \times_S Y}$. By glueing this étale-local construction, we obtain the fibre products for any fs log schemes. We repeat this construction to obtain fibre products of formal fs log schemes.

Note that this notion of fibre product may not be compatible with fibre products of (formal) schemes without log structure, as we can see from the explicit étale-local construction.

On the other hand, we have the following lemma.

**Lemma A.2.** The underlying scheme for $X^\sharp \times_{S^\sharp} Y^\sharp$ is finite over $X \times_S Y$. The same holds for formal fs log schemes.

**Proof.** The claim is étale-local on $X \times_S Y$, so we may replace $X$, $Y$ and $S$ by some étale neighbourhood so that there exists suitable charts as above. Then it suffices to show that $\mathbb{Z}[P \amalg_{\text{sat}} M]$ is a finite algebra over $\mathbb{Z}[P \amalg_M Q]$, which follows from Gordon’s lemma (cf. Lemma A.1). □

**Remark A.3.** To give a concrete example in which the underlying scheme for $X^\sharp \times_{S^\sharp} Y^\sharp$ differs from $X \times_S Y$ we fix a finite Galois Kummer-étale cover $\pi : X_L^\sharp \to X^\sharp$ of group $G$. In this case one has $X_L^\sharp \times_{X^\sharp} X_L^\sharp \cong G \times X_L^\sharp$ whereas $X_L \times_X X_L \cong G \times X_L$ only if $\pi$ is unramified.

The following corollary of lemma A.2 will be used later.

**Corollary A.4.** Let $X^\sharp$ be a fs log scheme, such that the underlying scheme is separated. Let $U^\sharp$ and $U'^\sharp$ be fs log schemes over $X^\sharp$, such that the underlying schemes $U$ and $U'$ are affine. Then $U^\sharp \times_{X^\sharp} U'^\sharp$ is also affine. The same holds for formal fs log schemes.
Proof. Under the hypotheses, the scheme \( U \times_X U' \) is affine, which follows from the cartesian diagram below:

\[
\begin{array}{ccc}
U \times_X U' & \rightarrow & U \times U' \\
\downarrow & & \downarrow \\
X' & \rightarrow & X \times X,
\end{array}
\]

Now by lemma A.2, the underlying scheme of \( U' \times_X U' \) is finite over an affine scheme \( U \times_X U' \). This proves the corollary.

□

A.2. Čech-to-derived functor spectral sequence for Kummer-étale cohomology.

For a log formal scheme \( \mathfrak{X} \), we write \( \mathfrak{X}^\xi \) for the associated Kummer-étale site (as per \[23, Def. 2.13\]).

We quickly recall the definition of Čech complex and Čech-to-derived functor spectral sequences in this setting.

**Definition A.5.** Let \( \mathfrak{U}^\xi \) be an Kummer-étale covering of \( \mathfrak{X}^\xi \) (i.e., the structure morphism \( \mathfrak{U}^\xi \rightarrow \mathfrak{X}^\xi \) is Kummer-étale and surjective), and let \( \mathcal{F} \) be a sheaf of abelian groups on the Kummer-étale site \( \mathfrak{X}^\xi \) két. Then we can form a Čech complex

\[
C^\ast(\mathfrak{U}^\xi, \mathcal{F}) := \left[ \Gamma(\mathfrak{U}^\xi, \mathcal{F}) \rightarrow \Gamma(\mathfrak{U}^\xi \times_{\mathfrak{X}^\xi} \mathfrak{U}^\xi, \mathcal{F}) \rightarrow \Gamma(\mathfrak{U}^\xi \times_{\mathfrak{X}^\xi} \mathfrak{U}^\xi \times_{\mathfrak{X}^\xi} \mathfrak{U}^\xi, \mathcal{F}) \rightarrow \cdots \right],
\]

with differentials defined in a standard way.

(The usual definition of Čech complexes for the case without log structure, cf. \[22, Ch. III, §2\], formally goes through.) For any bounded-below complexes \( \mathcal{F}^\ast \), we define the Čech complex \( C^\ast(\mathfrak{U}^\xi, \mathcal{F}^\ast) \) as the total complex of the double complex obtained from Čech complex of each term of \( \mathcal{F}^\ast \).

Whilst the Čech complex \( C^\ast(\mathfrak{U}^\xi, \mathcal{F}) \) does not necessarily represent \( R\Gamma(\mathfrak{X}^\xi \) két, \( \mathcal{F} \)), there exists a natural ‘Čech-to-derived functor spectral sequence’

\[
E_{i,0}^{i,j} : H^j(\mathfrak{U}^\xi_{i, két}, \mathcal{F}) \Rightarrow H^{i+j}(\mathfrak{X}^\xi \) két, \( \mathcal{F} \)),
\]

where \( \mathfrak{U}^\xi_{i, két} \) is the \((i+1)\)-fold self fibre product of \( \mathfrak{U}^\xi \) over \( \mathfrak{X}^\xi \). One way to read off this spectral sequence from the literature is via the technique of cohomological descent for (simplicial) topoi associated to the Kummer-étale sites \( \mathfrak{U}^\xi \) két and \( \mathfrak{X}^\xi \) két (cf. SGA 4\texttt{II}, Exp. Vbis. \[14\]). Indeed, since it admits a local section, \( \mathfrak{U}^\xi \rightarrow \mathfrak{X}^\xi \) is a ‘morphism of universal cohomological descent’ by \[loc. cit., Prop. (3.3.1)\] and so the above spectral sequence is just a special case of the descent spectral sequence from \[loc. cit., Prop. (2.5.5)\]).

**Remark A.6.** The complex \((E_{i,0}^{i,j}, d^{i,0})\) coincides with \( C^\ast(\mathfrak{U}^\xi, \mathcal{F}) \) and so the above spectral sequence implies \( C^\ast(\mathfrak{U}^\xi, \mathcal{F}) = R\Gamma(\mathfrak{X}^\xi \) két, \( \mathcal{F} \) if \( E_{i,0}^{i,j} \) vanishes for all \( j \geq 0 \) and \( i \geq 0 \).

A.3. Quasi-coherent cohomology. We first recall Kummer-étale descent theory for quasi-coherent sheaves.

Let \( \mathfrak{X}^\xi \) be a log formal scheme over \( \mathbb{Z}/p^n \) for some \( n \) and \( \mathcal{F} \) a quasi-coherent \( \mathcal{O}_\mathfrak{X} \)-module. Then, by Kato’s unpublished result (cf. \[23, Prop. 2.19\]) the presheaf \( \mathfrak{U}^\xi \in \mathfrak{X}^\xi \) két \( \rightarrow \Gamma(\mathfrak{U}, \mathcal{F}_\mathfrak{U}) \) is a sheaf on \( \mathfrak{X}^\xi \) két, where \( \mathcal{F}_\mathfrak{U} \) denotes the pull back of \( \mathcal{F} \) via the structure morphism \( \mathfrak{U} \rightarrow \mathfrak{X} \).
of the underlying formal schemes. We use the same notation $\mathcal{F}$ to denote the Kummer-étale sheaf associated to a quasi-coherent sheaf $\mathcal{F}$.

Similarly, if $\mathfrak{X}^i$ be a log formal scheme over $\text{Spf} \, \mathbb{Z}_p$, we can associate, to a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, a $\mathbb{Z}_p$-sheaf $\mathcal{F}$, which is the projective system of $\mathbb{Z}/p^n$-sheaves $\{\mathcal{F} \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^n\}$.

Now, we are interested in $C^* (\mathcal{U}^i, \mathcal{F})$ when $\mathcal{F}$ is a vector bundle on $\mathfrak{X}$ (viewed as a Kummer-étale sheaf), while $\mathcal{U}^i$ remains an Kummer-étale covering of $\mathfrak{X}^i$.

**Proposition A.7.** Let $\mathfrak{X}^i$ be a quasi-compact formal fs log scheme over $\text{Spf} \, R$ (for some noetherian adic ring $R$, with trivial log structure), and assume that $\mathfrak{X}$ is separated. Then for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ there is a natural isomorphism $R\Gamma (\mathfrak{X}^i_{k\acute{e}t}, \mathcal{F}) \cong R\Gamma (\mathfrak{X}, \mathcal{F})$.

Furthermore, for any Kummer-étale covering $\mathcal{U}^i \to \mathfrak{X}^i$ where $\mathcal{U}$ is affine, the Čech complex $C^* (\mathcal{U}^i, \mathcal{F})$ represents $R\Gamma (\mathfrak{X}, \mathcal{F})$.

The same holds if we replace $\mathcal{F}$ with a bounded-below complex $\mathcal{F}^*$ of quasi-coherent sheaves of $\mathcal{O}_X$-modules, such that the differential maps $d^i : \mathcal{F}^i \to \mathcal{F}^{i+1}$ are additive morphisms of Kummer-étale sheaves.

**Proof.** By standard argument with hypercohomology spectral sequences, the claim for $\mathcal{F}^*$ can be reduced to $\mathcal{F}$.

Let us first assume that $\mathfrak{X}$ is affine. Then by [23, Prop. 3.27] (and the theorem on formal functions), we have $R\Gamma (\mathfrak{X}^i_{k\acute{e}t}, \mathcal{O}_X) = \Gamma (\mathfrak{X}, \mathcal{O}_X)$. Now, by resolving $\mathcal{F}$ with free $\mathcal{O}_X$-modules, we obtain $R\Gamma (\mathfrak{X}^i_{k\acute{e}t}, \mathcal{F}) = \Gamma (\mathfrak{X}, \mathcal{F})$.

Choose a Kummer-étale covering $\mathcal{U}^i \to \mathfrak{X}^i$ with $\mathcal{U}$ affine. Then Corollary A.4 implies

$$\mathcal{U}^i := \underbrace{\mathcal{U}^i \times_{\mathfrak{X}^i} \cdots \times_{\mathfrak{X}^i} \mathcal{U}^i}_{i+1 \text{ times}}$$

has an affine underlying formal scheme. Therefore, by the Čech-to-derived spectral sequence argument it follows that $C^* (\mathcal{U}^i, \mathcal{F})$ represents $R\Gamma (\mathfrak{X}^i_{k\acute{e}t}, \mathcal{F})$ (cf. Remark A.6). Now if we choose $\mathcal{U}^i$ to be the disjoint union of finite affine open covering of $\mathfrak{X}$ (with the natural log structure induced from $\mathfrak{X}^i$), then $C^* (\mathcal{U}^i, \mathcal{F})$ represents $R\Gamma (\mathfrak{X}, \mathcal{F})$, as claimed.

**Remark A.8.** We apply Proposition A.7 to the log de-Rham complex $\mathcal{F}^*$, where the maps $d^i : \mathcal{F}^i \to \mathcal{F}^{i+1}$ are not $\mathcal{O}_X$-linear but are additive morphisms of Kummer-étale sheaves.

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