NOTE ON KHOVANOV LINK COHOMOLOGY

BOJAN GORNIK

Abstract. We extend Lee’s result on $sl(2)$ Khovanov cohomology of a link $L$ to the general $sl(n)$ case: a filtered chain complex $C(L)$ whose spectral sequence $E_2$ term equals Khovanov cohomology is exhibited. We also compute $C(L)$’s cohomology: it depends only on linking numbers of certain sublinks of $L$.

1. Introduction

In [3] Khovanov defines an invariant of a link $L \subseteq S^3$ which takes the form of a doubly graded abelian group, and generalizes the Jones polynomial in the sense that group’s Euler characteristic with respect to one of its gradings equals the Jones polynomial. Given a link $L$, definition of Khovanov’s invariant involves defining a graded chain complex $C(L)$ with boundary maps which preserve the grading and taking its cohomology. In [6] Lee shows that working over a field and modifying the boundary map by adding a certain part which lowers the grading, gives cohomology which is particularly simple - for knots it is always two dimensional. Lee’s result can also be interpreted as a construction of a filtered chain complex whose spectral sequence has $E_2$ term equal to Khovanov cohomology and converges to the simple form described above for knots.

Recently, Khovanov and Rozansky [4] introduced link cohomology which generalizes the earlier construction: for each $n \geq 2$ they define link cohomology $H_n(L)$ with Euler characteristic equal to polynomial $sl(n)$-quantum invariant $P_n(L)$; the $n = 2$ case is the original Jones polynomial construction.

The aim of this paper is to extend Lee’s result to the general $n \geq 2$ case. The main results are

**Theorem 1.** Let $L$ be a generic planar diagram of a link in $S^3$ and $n \geq 2$. There exists a filtered chain complex $C(L)$ whose associated spectral sequence has the $E_2$ term equal to $H_n(L)$.

and

**Theorem 2.** The dimension of $C(L)$’s (from Theorem 1) cohomology $\bar{H}(L)$ equals $n^l$ where $l$ is the number of $L$’s components. Moreover, to each mapping

$$\psi : \{\text{components of } L\} \to \Sigma_n,$$

we can assign an element $a_\psi \in \bar{H}(L)$ which lies in the cohomological degree

$$\sum_{(\varepsilon_1, \varepsilon_2) \in \Sigma_n \times \Sigma_n, \varepsilon_1 \neq \varepsilon_2} \text{lk}(\psi^{-1}(\varepsilon_1), \psi^{-1}(\varepsilon_2)).$$

All $a_\psi$’s generate $\bar{H}(L)$.

Here, $\Sigma_n$ is a set with $n$ distinct elements, to be defined later.
The paper is organized as follows: in section 2, we describe the construction of $\overline{C}(L)$. We rely heavily on reader’s familiarity with [4] - the construction closely mimics the Khovanov-Rozansky construction; essentially, the only difference is the replacement of the potential $w(x) = x^{n+1}$ corresponding to cohomology ring of $\mathbb{CP}^n$ with $w(x) = x^{n+1} - (n+1)\beta^n x$ corresponding to quantum cohomology ring of $\mathbb{CP}^n$. In section 3, we establish the relation between our $\overline{C}(L)$ and the Khovanov cohomology $H_n(L)$. Finally, in section 4, we compute $\overline{C}(L)$’s cohomology $H(L)$.

Let’s pause briefly, to establish some notation. We fix a generic planar diagram of a link $L \subseteq S^3$ - by abuse of notation we call it $L$ as well. Also, fix $n \geq 2$ and a nonzero scalar $\beta \in \mathbb{C}$. Although our constructions of various objects will depend on $n$ and $\beta$, the notation will not explicitly reflect that. For example, what we called $H_n(L)$ in this introduction, will be denoted simply as $H(L)$.

2. Construction of $\overline{C}(L)$

In this section we construct $\overline{C}(L)$. We will be cavalier about the details - for those, we refer the reader to [4].

Here’s the outline of the construction. Each crossing of $L$ can be resolved in two ways, depicted in Figure 1. We thus obtain $2^k$ resolutions, where $k$ is the number of crossings in $L$. To each resolution $\Gamma$ of $L$ we will assign a filtered $\mathbb{C}$-vector space $\overline{H}(\Gamma)$ (with some additional structure), and to a pair of resolutions $\Gamma_1$ and $\Gamma_2$, differing locally as in Figure 2, we will assign filtered maps of degree 1

$$\chi_0 : \overline{H}(\Gamma_0) \to \overline{H}(\Gamma_1), \quad \chi_1 : \overline{H}(\Gamma_0) \to \overline{H}(\Gamma_1).$$

With these objects the filtered complex $\overline{C}(L)$ can be formed in the same manner as in [4], the only difference being the interpretation of the $s$ operator as the filtration (and not grading) shift by $s$; note that filtration of $\overline{H}(\Gamma)$’s induces a filtration of $\overline{C}(L)$, and that the boundary mappings of $\overline{C}(L)$ respect this filtration, i.e. are mappings of filtered degree 0.

For future reference, we denote the corresponding objects in Khovanov-Rozansky construction with $H(\Gamma), \chi_0, \chi_1, C(L)$.

Now, we turn to the description of $\overline{H}(\Gamma)$ and maps $\chi_0, \chi_1$.

2.1. Definition of $\overline{H}(\Gamma)$.

Fix a resolution $\Gamma$. We will define $\overline{H}(\Gamma)$ as the cohomology of a certain 2-periodic complex $\overline{M}(\Gamma)$. We proceed to define it.

First, place marks on thin edges of $\Gamma$ so that each thin edge has at least one mark and introduce $R$, the ring of polynomials with $\mathbb{C}$ coefficients in (commuting) variables $\{x_i \mid i \in I\}$.
where $I$ is the set of all marks. Introduce grading on $R$ by declaring each formal variable $x_i$ to be of degree 2. Then, to each arc between two neighboring marks $x, y$, oriented from $x$ to $y$, assign a factorization

$$R \xrightarrow{\pi_{xy} - (n+1)\beta_n} R \{1 - n\} \xrightarrow{x-y} R,$$

where $\pi_{xy} = (x^{n+1} - y^{n+1})/(x - y)$. To each thick edge (see Figure 2) assign the tensor product of factorizations

$$R \xrightarrow{u_1 - (n+1)\beta_n} R \{-n\} \xrightarrow{x_1 + x_2 - x_3 - x_4} R \{-1\}$$

and

$$R \xrightarrow{u_2} R \{3 - n\} \xrightarrow{x_1 x_2 - x_3 x_4} R.$$

Here

$$u_1 = u_1(x_1, x_2, x_3, x_4) = \frac{g(x_1 + x_2, x_1 x_2) - g(x_1 + x_4, x_1 x_2)}{x_1 + x_2 - x_3 - x_4},$$

$$u_2 = u_2(x_1, x_2, x_3, x_4) = \frac{g(x_3 + x_4, x_1 x_2) - g(x_3 + x_4, x_3 x_4)}{x_1 x_2 - x_3 x_4},$$

and $g(z, w)$ is the unique two-variable polynomial for which

$$g(x + y, xy) = x^{n+1} + y^{n+1}.$$

Finally, the (graded) 2-complex $\overline{M}(\Gamma)$ is defined as the tensor product (over $R$) of these factorizations.

Note that the boundary maps of $\overline{M}(\Gamma)$ are a sum of two parts: one with degree $(n + 1)$ - call it $d_0, d_1$ - and another, corresponding to the $\beta^n$ part of factorization maps, with degree $(-n + 1)$ - call it $d'_0, d'_1$

$$\overline{M}^0(\Gamma) \xrightarrow{d_0 + d'_0} \overline{M}^1(\Gamma) \xrightarrow{d_1 + d'_1} \overline{M}^0(\Gamma).$$

Therefore, the grading of $\overline{M}(\Gamma)$ does not induce a grading on its cohomology, but only a filtration. We define the filtration $F$ of cohomology of $\overline{M}(\Gamma)$ as the one induced by the following filtration $F'$ of $\overline{M}(\Gamma)$:

$$F^{sk} \overline{M}(\Gamma) = \bigoplus_{i \leq k} \overline{M}_i(\Gamma)$$
where $\overline{M}_i(\Gamma)$ is the degree $i$ direct summand of $\overline{M}(\Gamma)$. Thus, the following is a definition of a filtered $\mathbb{C}$-vector space:

**Definition 2.1.** Let $p(\Gamma)$ be the mod 2 number of circles in the modification of $\Gamma$, obtained by replacing all thick edge neighborhoods with the 0-resolution, i.e. by performing the transformation from right to left in Figure 2. Then,

$$\overline{p}(\Gamma) := H^p(\Gamma)(\overline{M}(\Gamma)).$$

Although we will make no use of this fact, it may be worth mentioning that, while $\overline{M}(\Gamma)$'s $\mathbb{Z}$ grading does not, its mod 2$n$ reduction grading does descend to $\overline{H}(\Gamma)$. Therefore, $\overline{H}(\Gamma)$ comes with a $\mathbb{Z}_{2n}$ grading as well. This is a consequence of the fact that the degrees of $d_0, d_1$ are equal to those of $d'_0, d'_1$ mod 2$n$.

2.2. **Definition of $\overline{\alpha}_0, \overline{\alpha}_1$.** The definition of maps $\overline{\alpha}_0, \overline{\alpha}_1$ is also analogous to the Khovanov-Rozansky construction. Consider local resolutions $\Gamma_0, \Gamma_1$ as in Figure 2. The factorization $M_1$ associated to $\Gamma_0$ is the tensor product of factorizations, associated to the two arcs$^1$:

$$\begin{bmatrix} R \\ R\{2-2n\} \end{bmatrix} \xrightarrow{p_0} \begin{bmatrix} R\{1-n\} \\ R\{1-n\} \end{bmatrix} \xrightarrow{p_1} \begin{bmatrix} R \\ R\{2-2n\} \end{bmatrix}$$

where

$$p_0 = \begin{bmatrix} \pi_{13} - (n + 1)\beta^n & x_2 - x_4 \\ \pi_{24} - (n + 1)\beta^n & x_3 - x_1 \end{bmatrix}, \quad p_1 = \begin{bmatrix} x_1 - x_3 & x_2 - x_4 \\ (n + 1)\beta^n & -\pi_{13} + (n + 1)\beta^n \end{bmatrix},$$

$$\pi_{ij} = (x_{i}^{n+1} - x_{j}^{n+1})/(x_i - x_j).$$

Similarly, the factorization $M_2$ associated to $\Gamma_1$ is

$$\begin{bmatrix} R\{-1\} \\ R\{3-2n\} \end{bmatrix} \xrightarrow{q_0} \begin{bmatrix} R\{-n\} \\ R\{2-n\} \end{bmatrix} \xrightarrow{q_1} \begin{bmatrix} R\{-1\} \\ R\{3-2n\} \end{bmatrix}$$

with

$$q_0 = \begin{bmatrix} u_1 - (n + 1)\beta^n & x_1x_2 - x_3x_4 \\ u_2 & x_3 + x_4 - x_1 - x_2 \end{bmatrix}, \quad q_1 = \begin{bmatrix} x_1 + x_2 - x_3 - x_4 & x_1x_2 - x_3x_4 \\ u_2 & -u_1 + (n + 1)\beta^n \end{bmatrix}.$$  

We define $\overline{\chi}_0 : \overline{H}(\Gamma_0) \rightarrow \overline{H}(\Gamma_1)$ to be the map induced by the following homomorphism of factorizations $M_1 \rightarrow M_2$:

$$U_0 : (M_1)^0 \rightarrow (M_2)^0, U_1 : (M_1)^1 \rightarrow (M_2)^1,$$

$$U_0 = \begin{bmatrix} x_3 - x_2 \\ a_1 \end{bmatrix}, \quad U_1 = \begin{bmatrix} x_3 - x_2 \\ -1 \quad 1 \end{bmatrix},$$

$$a_1 = -u_2 + (u_1 + x_1u_2 - \pi_{24})/(x_1 - x_3).$$

The map $\overline{\chi}_1 : \overline{H}(\Gamma_1) \rightarrow \overline{H}(\Gamma_0)$ is induced by the following homomorphism of factorizations $M_2 \rightarrow M_1$:

$$V_0 : (M_2)^0 \rightarrow (M_1)^0, V_1 : (M_2)^1 \rightarrow (M_1)^1,$$

$$V_0 = \begin{bmatrix} 1 & 0 \quad x_3 - x_2 \end{bmatrix}, \quad V_1 = \begin{bmatrix} 1 & x_2 \\ 1 \quad x_3 \end{bmatrix}.$$  

With these definitions one easily checks

$$\overline{\alpha}_0\overline{\chi}_1 = m(x_1) - m(x_4), \quad \overline{\chi}_1\overline{\alpha}_0 = m(x_1) - m(x_4),$$

$^1$Note that the corresponding formulas in [4] have indices 3 and 4 exchanged.
where $m(x_i)$ denotes the endomorphism of $\overline{\mathcal{P}}(\Gamma_{0,1})$ induced by the multiplication by $x_i$, endomorphism of $\overline{M}(\Gamma_{0,1})$.

2.3. The algebra $\overline{R}(\Gamma)$; $\overline{\mathcal{P}}(\Gamma)$ as an $\overline{R}(\Gamma)$ module. It will be useful for our purposes to introduce a certain (complex) algebra $\overline{R}(\Gamma)$ associated with a resolution $\Gamma$, and the $\overline{R}(\Gamma)$ module structure of $\overline{\mathcal{P}}(\Gamma)$. This algebra is the analogue of $R(\Gamma)$ introduced in [5], section 6.

Definition 2.2. The complex algebra $\overline{R}(\Gamma)$ is defined to be $R/(a, b)$, where we view $R$, the polynomial ring in variables corresponding to all marks, as a complex algebra, and $a, b$ are all polynomials that appear in factorizations used to define $\overline{M}(\Gamma)$.

It immediately follows, that $\overline{\mathcal{P}}(\Gamma)$ is an $\overline{R}(\Gamma)$ module, since multiplication by any polynomial in $(a, b)$ induces a nullhomotopic endomorphism of $\overline{M}(\Gamma)$ (see [4], Proposition 2). Moreover, the following holds:

Proposition 2.3. As an $\overline{R}(\Gamma)$ module, $\overline{\mathcal{P}}(\Gamma)$ is free of rank one.

Proof. This is a consequence of Proposition 2.2 and Theorems 3, 4.

In conclusion, we give a more explicit description of $\overline{R}(\Gamma)$.

Proposition 2.4. The algebra $\overline{R}(\Gamma)$ is spanned by generators $X_e$, where $e$ runs over all thin edges of the resolution $\Gamma$, subject to the following relations: for every closed circle $i$ in $\Gamma$, we have

$$X_i^n = \beta^n,$$

and for every thick edge in $\Gamma$ the generators $X_1, X_2, X_3, X_4$ (numbered as in Figure 2) satisfy

$$X_1 + X_2 - X_3 - X_4 = X_1X_2 - X_3X_4 = u_1(X_1, X_2, X_3, X_4) - (n + 1)\beta^n = u_2(X_1, X_2, X_3, X_4) = 0.$$

Proof. Since $x_i - x_j$ is in $(a, b)$ for all marks $i, j$ that are adjacent and lie on the same thin edge, we can retain only one mark per thin edge. Also, for a closed circle with one mark $i$, the expression $\pi_i - (n + 1)\beta^n = (n + 1)(x_i^n - \beta^n)$ is in $(a, b)$.

Note that for any $X_e$ it holds

$$X_e^n = \beta^n.$$

This is evident for thin edges $e$ that correspond to closed circles. For other thin edges this follows from thick edge relations:

$$(n + 1)(x_i^n - \beta^n) = \partial_i u_1(X_1 + X_2 - X_3 - X_4) + (u_1 - (n + 1)\beta^n)\partial_i(X_1 + X_2 - X_3 - X_4) + \partial_i u_2(X_1X_2 - X_3X_4) + u_2\partial_i(X_1X_2 - X_3X_4) = 0,$$

where $\partial_i$ is the (formal) partial derivative with respect to $X_i$ ($i = 1, 2, 3, 4$).

2.4. The polynomial $P_n(\Gamma)$, admissible states of $\Gamma$ and $\overline{R}(\Gamma)$. The goal of this subsection is to establish a simple relation between $\overline{R}(\Gamma)$ and the $sl(n)$ polynomial quantum invariant $P_n(\Gamma)$.

First, let’s digress briefly to define $P_n(\Gamma) \in \mathbb{Z}[q, q^{-1}]$, a polynomial in variable $q$ with positive integer coefficients, associated to a resolution $\Gamma$. Extending the definition of $P_n$ to links, by the rule in Figure 3, we obtain a polynomial invariant of links; in particular, $P_2$ is the Jones polynomial, and $P_n$’s are specializations of the HOMFLY polynomial.

For our purposes, the following description of $P_n(\Gamma)$ will be most useful (see [2], Example 1.16). By $\Sigma_n$ we denote the set of $n$ complex roots of the equation $x^n - 1 = 0$. Let $e(\Gamma)$
stand for the set of all thin edges of the resolution $\Gamma$. Then, an assignment of elements in $\Sigma_n$ to all thin edges of $\Gamma$

$$\varphi : e(\Gamma) \to \Sigma_n$$

is called a state of $\Gamma$. A state $\varphi$ is called an admissible state if at each thick edge the $\varphi$ assignments of neighboring thin edges are either of type 1 or 2 (see the top half of Figure 6).

The set of all states is called $S'(\Gamma)$, and the set of all admissible states is $S(\Gamma)$. Now,

$$P_n(\Gamma) = \sum_{\varphi \in S(\Gamma)} q^{\alpha(\varphi)},$$

where $\alpha(\varphi)$ is an integer whose precise form will not interest us - it can be found in [2].

It is our goal to find a basis indexed by admissible states of $\Gamma$ for $C$-vector space $R(\Gamma)$.

We start with the following construction of idempotents $Q_\varphi$:

**Proposition 2.5.** Given a state $\varphi$, we define

$$Q_\varphi := \prod_{e \in e(\Gamma)} \frac{1}{n} \left( 1 + \frac{X_e}{\beta \varphi(e)} + \frac{X_e^2}{\beta^2 \varphi(e)^2} + \cdots + \frac{X_e^{n-1}}{\beta^{n-1} \varphi(e)^{n-1}} \right) \in R(\Gamma).$$

It holds

$$Q_\varphi^2 = Q_\varphi, \quad \forall \varphi \in S'(\Gamma),$$

$$Q_\varphi Q_{\varphi^2} = 0, \quad \forall \varphi_1, \varphi_2 \in S'(\Gamma), \, \varphi_1 \neq \varphi_2,$$

$$\sum_{\varphi \in S'(\Gamma)} Q_\varphi = 1.$$

**Proof.** A straightforward calculation using $X_e^n = \beta^n$. $\square$

It is a matter of simple algebra to check the validity of

**Lemma 2.6.** Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \Sigma_n$. Then

$$\lambda_1 \lambda_2 - \lambda_3 - \lambda_4 = 0, \quad \lambda_1 \lambda_2 - \lambda_3 \lambda_4 = 0,$$

$$u_1(\beta \lambda_1, \beta \lambda_2, \beta \lambda_3, \beta \lambda_4) = (n+1) \beta^n, \quad u_2(\beta \lambda_1, \beta \lambda_2, \beta \lambda_3, \beta \lambda_4) = 0$$

if and only if

$$\{\lambda_1, \lambda_2\} = \{\lambda_3, \lambda_4\}, \quad \lambda_1 \neq \lambda_2.$$

Note that (6) is precisely the condition of admissibility of a state from the top half of Figure 6. We now state the main result of this subsection, a structure theorem for $R(\Gamma)$:

$$= q^{n-1} \bigg\mid - q^n$$

$$= q^{1-n} \bigg\mid - q^n$$

**Figure 3.** Behavior of $P_n$ with respect to resolutions.
Theorem 3. For non-admissible states \( \varphi \in S'(\Gamma) - S(\Gamma) \) it holds
\[
Q_\varphi = 0.
\]

For admissible states \( \varphi \in S(\Gamma) \), however, we have
\[
0 \neq CQ_\varphi = \mathcal{R}(\Gamma)Q_\varphi \quad \implies \quad \dim \mathcal{R}(\Gamma)Q_\varphi = 1.
\]
Therefore, we have a direct sum decomposition of \( \mathbb{C} \)-algebras
\[
\mathcal{R}(\Gamma) = \bigoplus_{\varphi \in S(\Gamma)} \mathcal{C}Q_\varphi.
\]

In particular,
\[
\dim \mathcal{R}(\Gamma) = P_n(\Gamma)_{|q = 1}.
\]

Proof. To prove the first statement, pick \( \varphi \in S'(\Gamma) - S(\Gamma) \) and choose a thick edge where the admissibility condition (Figure 0) is violated. Call the neighboring thin edges \( e_1, e_2, e_3, e_4 \) and the variables corresponding to these thin edges \( X_1, X_2, X_3, X_4 \). Since \( X_\varphi Q_\varphi = \beta \varphi(e)Q_\varphi \) for all thin edges of \( \Gamma \), we have
\[
0 = (X_1 + X_2 - X_3 - X_4)Q_\varphi = \beta(\varphi(e_1) + \varphi(e_2) - \varphi(e_3) - \varphi(e_4))Q_\varphi,
\]
\[
0 = (X_1X_2 - X_3X_4)Q_\varphi = \beta^2(\varphi(e_1)\varphi(e_2) - \varphi(e_3)\varphi(e_4))Q_\varphi,
\]
\[
0 = (u_1(X_1, X_2, X_3, X_4) - (n + 1)\beta^n)Q_\varphi = (u_1(\beta\varphi(e_1), \beta\varphi(e_2), \beta\varphi(e_3), \beta\varphi(e_4)) - (n + 1)\beta^n)Q_\varphi,
\]
\[
0 = u_2(X_1, X_2, X_3, X_4)Q_\varphi = u_2(\beta\varphi(e_1), \beta\varphi(e_2), \beta\varphi(e_3), \beta\varphi(e_4))Q_\varphi.
\]

According to Lemma 2.6, one of the rightmost expressions is a nontrivial multiple of \( Q_\varphi \), implying \( Q_\varphi = 0 \).

To prove the second statement, we first observe that \( Q_\varphi \neq 0 \). For, if \( Q_\varphi \) were 0, its representative in \( C[X_e | e \in \epsilon(\Gamma)] \) would be in the ideal generated by all relations from Proposition 2.4. But all these relations, according to Lemma 2.6, give 0 when evaluated at \( X_\varphi = \beta \varphi(e) \), whereas the expression in [3] clearly evaluates to 1 at \( X_\varphi = \beta \varphi(e) \).

Introduce \( P(\Gamma) = C[X_e | e \in \epsilon(\Gamma)]/(X_\varphi^n = \beta^n | e \in \epsilon(\Gamma)) \) and elements \( Q_\varphi' \in P(\Gamma) \) given by the same expression as in [5]. Clearly, proposition 2.5 holds for elements \( Q_\varphi' \) as well and they form a basis for \( P(\Gamma) \). Therefore, \( CQ_\varphi' = P(\Gamma)Q_\varphi' \) and the same relation holds in \( \mathcal{R}(\Gamma) \) which is a quotient of \( P(\Gamma) \).

The last equality is a consequence of 4.4. \( \square \)

3. Relation to Khovanov-Rozansky construction

First, we briefly recall Khovanov-Rozansky’s construction of the complex \( C(L) \) and state some of its properties that are relevant for us. Then, we give a proof of Theorem 4.4.

3.1. Khovanov-Rozansky construction. Simply put, objects \( M(\Gamma), H(\Gamma), \chi_{0,1}, C(L) \) are defined as in subsections 2.1 and 2.2 with \( \beta = 0 \). Note that matrices \( \chi_{0,1}, \chi_{1,1} \) don’t change. Also, the underlying graded \( \mathbb{C} \)-spaces of 2-complexes \( M(\Gamma) \) and \( \overline{M}(\Gamma) \) are equal, the difference is in the boundary maps; in particular, omitting \( d_0', d_1' \) in \( \overline{M}(\Gamma) \) we obtain precisely \( M(\Gamma) \). Since \( d_0, d_1 \) are homogeneous (of degree \((n + 1)\)), the \( M(\Gamma) \)-cohomology \( H(\Gamma) \) is a graded space. We summarize:
Proposition 3.1. The 2-complex \( M(\Gamma) \)
\[
\begin{array}{c}
\overline{M}'(\Gamma) \xrightarrow{-d_0} \overline{M}^1(\Gamma) \xrightarrow{-d_1} \overline{M}^0(\Gamma)
\end{array}
\]
has nontrivial cohomology only in cohomological degree \( p(\Gamma) \), the mod 2 number of circles in the modification of \( \Gamma \), obtained by replacing all thick edge neighborhoods with the 0-resolution, ie. by performing the transformation from right to left in Figure 4 on all thick edges. This cohomology, by definition, is \( H(\Gamma) \) and its \((q-)\)graded dimension is \( P_n(\Gamma) \).

Proof. See [1]. □

3.2. Proof of Theorem 1. We start with the observation, that the filtered dimension of \( \overline{P}(\Gamma) \) and the graded dimension of \( H(\Gamma) \) are equal.

Proposition 3.2. Let \( k \in \mathbb{Z} \). If \( H_k(\Gamma) \) stands for the degree \( k \) direct summand of graded space \( H(\Gamma) \), there exists an isomorphism of \( \mathbb{C} \)-vector spaces
\[
\Phi_{k,\Gamma} : H_k(\Gamma) \rightarrow F^k\overline{P}(\Gamma)/F^{k-1}\overline{P}(\Gamma).
\]
In particular, the filtered dimension of \( \overline{P}(\Gamma) \) is \( P_n(\Gamma) \).

Proof. Without loss of generality, we may assume \( p(\Gamma) = 1 \). Then, the 2-complex \( M(\Gamma) \) has trivial cohomology in degree 0, and cohomology in degree 1 equals \( H(\Gamma) \). Now, let’s define the map
\[
\phi : (\ker d_1)_k \to F^k\overline{P}(\Gamma)/F^{k-1}\overline{P}(\Gamma)
\]
where \( (\ker d_1)_k \) is the degree \( k \) direct summand of \( (\ker d_1) \). Pick \( \alpha \in (\ker d_1)_k \) and define
\[
\phi(\alpha) = \alpha + \alpha_1 + \alpha_2 + \ldots,
\]
where \( \alpha_i \in (M^1)_{k-2ni} \) is of degree \( k - 2ni \). Here’s how we define \( \alpha_1, \alpha_2, \ldots \). We need
\[
(d_1 + d_1')(\alpha + \alpha_1 + \alpha_2 + \ldots) = 0, \text{ or, equivalently, (consider the grading on } M^1)
\]
\[
d_1\alpha = 0, \quad d_1'\alpha + d_1\alpha_1 = 0, \quad d_1\alpha_1 + d_1\alpha_2 = 0, \ldots.
\]
Since \( d_0d_1\alpha = -d_0d_1\alpha = 0 \) (we used the fact that \( \overline{M} \) is a 2-complex) and \( \ker d_0 = \text{im } d_1 \), we see that there exists \( \alpha_1 \) such that \( d_1(-\alpha) = d_1'\alpha \).

In the next step we construct \( \alpha_2 \). Since \( d_0d_1\alpha_1 = -d_0d_1\alpha_1 = d_0d_1\alpha = 0 \), we again (due to \( \ker d_0 = \text{im } d_1 \)) obtain \( \alpha_2 \) such that \( d_1(-\alpha_2) = d_1'\alpha_1 \).

We continue with this algorithm, which eventually terminates since \( M \) doesn’t have arbitrarily low degrees.

First, we have to check that our definition is a good one, ie. that the procedure gives an element in \( F^k\overline{P}(\Gamma)/F^{k-1}\overline{P}(\Gamma) \) that does not depend on choices of \( \alpha, \alpha_2, \ldots \). Suppose an alternative set of numbers \( \alpha_1, \alpha_2', \ldots \) would also do the job. Then
\[
(d_1 + d_1')(\alpha_1 - \alpha_1') + (\alpha_2 - \alpha_2') + \ldots = 0
\]
implies what we need, since we’re quotienting out \( F^{k-1}\overline{P}(\Gamma) \), and the above element lies there - in fact it lies in \( F^{k-2n}\overline{P}(\Gamma) \).

Second, we note that \( \phi \) is surjective: take an \( M \)-representative of an element \( \alpha \in F^k\overline{P}(\Gamma)/F^{k-1}\overline{P}(\Gamma) \), strip away its part that lies in degrees strictly lower than \( k \), and what remains maps to \( \alpha \) via \( \phi \).

It is easy to check \( \ker \phi = (\text{im } d_0)_k \) - we omit the details.

Therefore \( \Phi_{k,\Gamma} \), the mapping \( (\ker d_1)_k/(\text{im } d_0)_k \rightarrow F^k\overline{P}(\Gamma)/F^{k-1}\overline{P}(\Gamma) \) induced by \( \phi \), is an isomorphism. □
Remark: We will not prove or need it, but it can be shown that cohomology of $M(\Gamma)$ in cohomological degree $(p(\Gamma) + 1) \mod 2$ is trivial.

Proof. (of Theorem 1) We have to show that the $E_1$ term of the spectral sequence, associated to filtered chain complex $C(L)$, is isomorphic to the chain complex $C(L)$. The $E_1$ term is a direct sum of spaces $F^k H(\Gamma)/F^{k-1} H(\Gamma)$ where $\Gamma$ runs over all resolutions of $L$. The boundary maps of $E_1$ are induced by $\chi_0, \chi_1$. In view of Proposition 3.2, we will have shown the statement of the theorem, once we show that maps $\chi_0, \chi_1$ (i.e. the boundary maps of $C(L)$), via the identification from Proposition 3.2, correspond to $\chi_0, \chi_1$ as maps of $F^k H/F^{k-1} H$:

\[
\begin{array}{c}
H_k(\Gamma_0) \xrightarrow{\Phi_{k,\Gamma_0}} F^k H(\Gamma_0)/F^{k-1} H(\Gamma_0) \\
\chi_0 \quad \xrightarrow{= \chi_0} \\
H_{k+1}(\Gamma_1) \xrightarrow{\Phi_{k+1,\Gamma_1}} F^{k+1} H(\Gamma_1)/F^k H(\Gamma_1) \\
\chi_1 \quad \xrightarrow{= \chi_1}
\end{array}
\]

This is a straightforward consequence of the definition of $\Phi$, and of the fact that maps $\chi_0$ and $\chi_0$ (and similarly for $\chi_1, \chi_1$), as maps of the graded $C$-space underlying chain complexes $M, M$, are equal and homogeneous (of degree 1).

4. Computation of $\overline{H}(L)$

In this section we give a proof of Theorem 2. We start with a structure theorem for $H(\Gamma)$, which establishes a basis for $H(\Gamma)$ indexed by admissible states of $\Gamma$. Recall that $H(\Gamma)$ is an $R(\Gamma)$ module; by abuse of notation, the endomorphism of $H(\Gamma)$ induced by the action of $Y \in R(\Gamma)$ will also be denoted by $Y$; also, we adopt the notation introduced in subsection 2.4.

**Theorem 4.** For each admissible state $\varphi \in S(\Gamma)$ the $C$-space $Q_{\varphi} \overline{H}(\Gamma)$ is one-dimensional and

\[
\overline{H}(\Gamma) = \bigoplus_{\varphi \in S(\Gamma)} Q_{\varphi} \overline{H}(\Gamma).
\]

For all $x \in \overline{H}(\Gamma)$ we have

\[
x \in Q_{\varphi} \overline{H}(\Gamma) \iff X_\varphi x = \beta \varphi(e)x, \quad \forall e \in e(\Gamma).
\]

Proof. The implication $\Rightarrow$ of (8) is an obvious consequence of the definition of $Q_{\varphi}$; the implication $\Leftarrow$ follows from $x = Q_{\varphi} x$. Also, the direct sum decomposition (7) is a consequence of Proposition 2.3 and $Q_{\varphi} = 0$ for non-admissible states (Theorem 3).

To finish the proof, we have to show the spaces $Q_{\varphi} \overline{H}(\Gamma)$ for admissible states $\varphi$ are one-dimensional. Since the number of admissible states of $\Gamma$ is $P_n(\Gamma)/_{q=1}$, and the complex dimension of $\overline{H}(\Gamma)$ equals $P_n(\Gamma)/_{q=1}$ (Proposition 3.2), the decomposition (7) implies that it suffices to show

\[
\dim_C Q_{\varphi} \overline{H}(\Gamma) \geq 1.
\]

To that end, given an admissible state $\varphi$, we will construct a nonzero element of $Q_{\varphi} \overline{H}(\Gamma)$.

First, we digress briefly to extend our definition of $\overline{H}(\Gamma)$ to extended resolutions $\Gamma'$. They are resolutions, obtained by resolving a planar diagram of a link in one of the ways from Figure 1 or as in Figure 5. To an extended resolution $\Gamma'$ we associate a 2-complex $M(\Gamma')$ in
the same fashion as in subsection 2.1, we only have to add the description of the factorization, associated with the new resolution (the left part of Figure 4). It is the tensor product of factorizations associated to arcs $x_3x_1$ and $x_4x_2$, that is the tensor product of

$$R \xrightarrow{\pi_{13}-(n+1)/\beta} R\{1-n\} \xrightarrow{x_3-x_1} R,$$

and

$$R \xrightarrow{\pi_{24}-(n+1)/\beta} R\{1-n\} \xrightarrow{x_4-x_2} R,$$

where $\pi_{ij} = (x_i^{n+1} - x_j^{n+1})/(x_i - x_j)$.

We also define maps $\xi_0 : \mathcal{T}(\Gamma_1') \to \mathcal{T}(\Gamma_1), \xi_1 : \mathcal{T}(\Gamma_1) \to \mathcal{T}(\Gamma_1')$ (Figure 4). To do that, we use matrices $U_0, U_1, V_1, V_2$ from subsection 2.2 with the roles of $x_3$ of $x_4$ exchanged. It therefore holds

$$(10) \quad \xi_0 \xi_1 = m(x_1) - m(x_3), \quad \xi_1 \xi_0 = m(x_1) - m(x_3).$$

We are now ready to produce a nonzero element $a_\varphi \in \mathcal{Q}(\mathcal{T}(\Gamma))$. At each thick edge determine whether the state $\varphi$ is of type 1 or 2 (Figure 5). If it is type 1, replace that thick edge with a 0-resolution (the right to left trasformation of Figure 2); if it is type 2, replace that thick edge with the new resolution (the right to left transformation of Figure 4).

We thus obtain an extended resolution $\Gamma'$ which has only crossings of the new type. The extended resolution $\Gamma'$ is a collection of circles (intersecting themselves at new type crossings) $C_i, i = 1, \ldots, k$. The state $\varphi$ induces a state $\varphi'$ of $\Gamma'$ in an obvious way; $\varphi'$ has the property that it evaluates to the same number at all edges of any circle $C_i$, call that number $\varphi_i$. Note

Figure 4. The maps $\xi_0, \xi_1$.

Figure 5. The extended resolution.
that $\varphi_i \neq \varphi_j$ for any two circles $C_i, C_j$ that either intersect or share, one are each, the two arcs that resulted in performing the $0$-resolution at $\Gamma$’s thick edge of type 1.

Now, note the $2$-complex $\overline{\mathcal{M}}(\Gamma')$ is equal to the tensor product of $2$-complexes (not just factorizations) $\overline{\mathcal{M}}(\Gamma')$, associated to circles $C_i$. The same must hold for the cohomology of $\overline{\mathcal{M}}(\Gamma')$: $\overline{\mathcal{H}}(\Gamma')$ is the tensor product of cohomologies of $\overline{\mathcal{M}}(\Gamma')$, each of them is obviously equal to $C[X_1]/(X_i^n = \beta^n)$. Therefore, we can choose a nonzero element in $a_\varphi \in \overline{\mathcal{H}}(\Gamma')$ that has $X_e a_\varphi = \beta \varphi'(e) a_\varphi$ for all thin edges $e$ of $\Gamma'$ - it is

$$a_\varphi = \Pi_{i=1}^k (1 + X_i/(\beta \varphi_i) + X_i^2/(\beta \varphi_i)^2 + \ldots + X_i^{n-1}/(\beta \varphi_i)^{n-1})/n.$$  

We define

$$a_\varphi := \Pi_i \xi_0 (e_1) \Pi_j \xi_0 (e_2) a_\varphi,$$

where the first product extends over thick edges $e_1$ of $\Gamma$ of type 1, and the second product over thick edges $e_2$ of $\Gamma$ of type 2. First, since multiplication by edge variables commutes with maps $\xi_i$, we have $X_e a_\varphi = \beta \varphi(e) a_\varphi$ and therefore $a_\varphi = Q_\varphi a_\varphi \in Q_\varphi \overline{\mathcal{H}}(\Gamma)$.

Also, we claim $a_\varphi$ is nonzero. This is because

$$\Pi_i \xi_0 (e_1) \Pi_j \xi_0 (e_2) a_\varphi =$$

$$= \Pi_i \xi_0 (e_1) \Pi_j \xi_0 (e_2) \xi_0 (e_2) a_\varphi =$$

$$= \Pi_i (\varphi_{i_1} - \varphi_{j_1}) \Pi_j (\varphi_{i_2} - \varphi_{j_2}) a_\varphi$$

is a nonzero multiple of nonzero element $a_\varphi$. We used Equations (3), (10) and the remark, made above, about a sufficient condition for $\varphi_i \neq \varphi_j$.

For each $\varphi \in S(\Gamma)$, choose a nonzero element of $Q_\varphi \overline{\mathcal{H}}(\Gamma)$ and call it $a_{\varphi, \Gamma}$. According to the previous theorem, $C a_{\varphi, \Gamma} = Q_\varphi \overline{\mathcal{H}}(\Gamma)$ and all vectors $a_{\varphi, \Gamma}$ as $\phi$ runs over all admissible states of $\Gamma$ form a basis for $\overline{\mathcal{H}}(\Gamma)$.

The boundary maps of $\overline{\mathcal{L}}(L)$ are $\overline{\chi}_0, \overline{\chi}_1$. It is easy to determine the action of these maps on basis vectors $a_\varphi$’s. Let $\Gamma_0, \Gamma_1$ be two resolutions, differing locally as in Figure 2. For any mark $i$ of $L$ we have

$$m(x_i) \overline{\chi}_1 a_{\varphi, \Gamma_1} = \overline{\chi}_1 m(x_i) a_{\varphi, \Gamma_1} = \varphi(e_i) \overline{\chi}_1 a_{\varphi, \Gamma_1}.$$  

This implies that for any admissible state $\varphi \in S(\Gamma_1)$, the element $\overline{\chi}_1 a_{\varphi, \Gamma_1}$ is a multiple of $a_{\varphi', \Gamma_0}$, where $\varphi'$ is the state of $\Gamma_0$, obtained by retaining the same values of $\varphi$ at all marks. For states $\varphi$ of type 2, the induced state $\varphi'$ is not well defined, since the two marks lying on one of the two arcs of $\Gamma_0$, evaluate to different values. Therefore, for type 2 states $\varphi$ we have $\overline{\chi}_1 a_{\varphi, \Gamma_1} = 0$. For type 1 states $\varphi$, however, $\varphi'$ is an admissible state and we have $\overline{\chi}_1 a_{\varphi, \Gamma_1} = c a_{\varphi', \Gamma_0}$. Here, $c$ must be nonzero since

$$c \overline{\chi}_0 a_{\varphi', \Gamma_0} = \overline{\chi}_0 \overline{\chi}_1 a_{\varphi, \Gamma_1} = (\varphi(\text{top, left}) - \varphi(\text{bottom, right})) a_{\varphi, \Gamma_1}$$

is clearly nonzero.

A similar calculation can be performed for $\overline{\chi}_0$. The behavior of boundary maps on basis elements is summarized in Figure 8.

We can now prove Theorem 2.

Proof. (of Theorem 2)

As a $C$-vector space, $\overline{\mathcal{L}}(L)$ is spanned by elements $a_{\varphi, \Gamma}$ for all resolutions $\Gamma$ of $L$, and all admissible states of these resolutions. From the boundary map behavior (Figure 8), it is clear that the states that survive in the cohomology $\overline{\mathcal{H}}(L)$ are precisely those, which are of type 2 at all crossings resolved to a $1$-resolution, and of type 4 at all crossings resolved.
to a 0-resolution. All such states clearly evaluate to the same value at every edge in any component of \( L \), so are induced by a map

\[
\psi : \{ \text{components of } L \} \to \Sigma_n.
\]

Also, each such map \( \psi \) defines precisely one admissible state: the one of a resolution, obtained by resolving to 0-resolution all crossings at which \( \psi \)-values of the two strands are equal, and resolving to 1-resolution all crossings at which \( \psi \)-values of the two strands differ.

To determine the cohomological degree of the state/resolution determined by some \( \psi \), recall that only 1-resolutions contribute to the cohomological degree: the contribution is 1 for positive crossings and \(-1\) for negative crossings.

\[\square\]

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