Dissipative Collapse Through the Critical Point

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Abstract

A relativistic model of a heat conducting collapsing star, which includes thermal pre-relaxation processes, is presented. Particular attention is paid to the influence of a given parameter defined in terms of thermodynamic variables, on the outcome of evolution. Evaluation of the system when passing through a critical value of the aforesaid parameter, does not yield evidence of anomalous behaviour.

keywords: Self-gravitating systems, Relativistic stars: stability, Late stages of stellar evolution, Relativistic fluid dynamics

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1 Introduction

The behaviour of dissipative systems at the very moment when they depart from hydrostatic equilibrium has been recently studied [1, 2, 3].

In [1] it appears that a parameter \( \alpha \) formed by a specific combination of thermal conductivity coefficient \( \kappa \), relaxation time \( \tau \), temperature \( T \), proper energy density \( \rho \) and pressure \( p \),

\[
\alpha = \frac{\kappa T}{\tau (\rho + p)},
\]

may critically affect the evolution of the object. Specifically, it was shown that in the equation of motion of any fluid element, the inertial mass density term vanishes for \( \alpha = 1 \) (critical point) and is negative beyond that value.

In some cases (pure shear or bulk viscosity) [3], the critical point is well beyond the point where the causality requirements are violated and therefore forbidden.

In others (pure thermal conduction) [1, 2], later requirements are violated slightly below the critical point.

In the general case (heat conduction plus viscosity) it appears that causality may break down beyond the critical point [3].

However, it should be stressed that the critical point (as well as causality conditions) is obtained in the context of a linear perturbative scheme, where the system is evaluated immediately after leaving the equilibrium and therefore time derivative of radial velocity as well as dissipative variables are considered small quantities, such that only linear expressions of them are kept. On the other hand the vanishing, at the critical point, of the inertial mass density term, leading to accelerated fluid elements in the absence of total radial forces, might suggest that linear approximation is not longer reliable there [2, 3].

The question arises then, whether a physical system may reach the critical point without exhibiting a clear unphysical behavior or if, on the contrary, for any physical system reaching that point the march of physical variables becomes physically unacceptable as suggested by the linear approximation. In other words we want to see how the physical meaning of the above mentioned parameter \( \alpha \), as implied by the linear approximation, carries over to nonlinear regimes.

In order to elucidate this question we shall consider here an exact numerical model of an evolving dissipative star (without viscosity). The system is forced to evolve through the critical point, and fundamental variables are monitored to detect any anomalous behaviour. Modeling is performed by means of the HJR method [4], a brief résumé of which is given in next section.
The heat conduction equation is given in section 3, and the model is described in section 4. Finally, a brief analysis of our results is given in the last section.

2 The HJR method

We shall consider a non-static spherically symmetric distribution of matter which consists of fluid, which may be locally anisotropic, and heat flow (radiation). Assuming Bondi coordinates \[3, 5\], the metric takes the form

\[
ds^2 = e^{2\beta} \left[ \frac{V}{r} du^2 + 2du dr \right] - r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right), \tag{1}
\]

where \( u \) is a time like coordinate, \( r \) is the null coordinate and \( \theta \) and \( \phi \) are the usual angle coordinates. A generalization of the “mass aspect” defined by Bondi et al \[5\] can be introduced by means of function \( \tilde{m}(u, r) \)

\[
V = e^{2\beta} (r - 2\tilde{m}(u, r)), \tag{2}
\]

where \( \beta \) and \( V \) are functions of \( u \) and \( r \). Inside the fluid distribution, the stress-energy tensor can be written as -see \[7\] for details-

\[
T_{\mu\nu} = (\rho + P_\perp) U_\mu U_\nu - P_\perp g_{\mu\nu} + (P_r - P_\perp) \chi_\mu \chi_\nu + 2q_{(\mu} U_{\nu)}, \tag{3}
\]

where \( \rho, P_r, P_\perp \) are the energy density, radial pressure and tangential pressure respectively as measured by a Minkowskian observer in the Lagrangean frame, and \( \chi_\mu = -q_\mu/q \), being \( q = \sqrt{-q^\mu q_\mu} \) the heat flow. Using a Lorentz transformation in the radial direction and the coordinate transformation between Bondi coordinates and local Minkowskian coordinates

\[
dt = e^\beta (\sqrt{\frac{V}{r}} du + \sqrt{\frac{r}{V}} dr), \tag{4}
\]

\[
dx = e^\beta \sqrt{\frac{r}{V}} dr, \tag{5}
\]

\[
\ dy = r d\theta, \tag{6}
\]

\[
\ dz = r \sin \theta d\phi, \tag{7}
\]

it is possible to express the components of the stress-energy tensor in Bondi coordinates, in terms of variables measured in the Minkowskian Lagrangean frame. In Bondi coordinates the four-velocity \( U_\mu \), and the heat flux vector \( q^\mu \) are given by

\[
U_\mu = e^\beta \left( \sqrt{\frac{V}{r}} \frac{1}{(1 - \omega^2)^{1/2}} , \sqrt{\frac{r}{V}} \left( \frac{1 - \omega}{1 + \omega} \right)^{1/2} , 0, 0 \right), \tag{8}
\]
and
\[ q^\mu = q e^{-\beta} \left( -\sqrt{\frac{r}{V}} \left( \frac{1 - \omega}{1 + \omega} \right)^{1/2}, \sqrt{\frac{V}{r (1 - \omega^2)}}^{1/2}, 0, 0 \right), \] (9)

where \( \omega \) is the velocity of matter as measured by the Minkowskian observer defined by (4)-(7).

At the outside of the fluid distribution the space-time is described by the Vaidya metric [8], which in Bondi coordinates is given by \( \beta = 0 \) and \( V = r - 2m(u) \).

The Einstein field equations, inside the matter distribution, can be written as [9]:
\[ \frac{1}{4\pi r(r - 2\tilde{m})} (-\tilde{m}_0 e^{-2\beta} + (1 - 2\tilde{m}/r)\tilde{m}_1) = \frac{1}{1 - \omega^2} (\rho + 2\omega q + P r \omega^2), \] (10)
\[ \frac{\tilde{m}_1}{4\pi r^2} = \tilde{\rho}, \] (11)
\[ \beta_1 \frac{r - 2\tilde{m}}{2\pi r^2} = \tilde{\rho} + \tilde{P}, \] (12)
\[ -\beta_0 e^{-2\beta} + \frac{1}{8\pi} (1 - 2\tilde{m}/r)(2\beta_{11} + 4\beta_1^2 - \beta_1/r) + \frac{3\beta_1(1 - 2\tilde{m}_1) - \tilde{m}_{11}}{8\pi r} = P_\perp, \] (13)

where subscripts 0 and 1 denote partial derivative with respect to \( u \) and \( r \) respectively, and the effective energy density
\[ \tilde{\rho} = \frac{1}{1 + \omega} (\rho - q(1 - \omega) - P r \omega), \] (14)

and the effective pressure
\[ \tilde{P} = \frac{1}{1 + \omega} (-\omega \rho - q(1 - \omega) + P r) \] (15)

are two auxiliary functions introduced in the HJR formalism [4, 9, 10] whose physical meaning becomes clear in the static case, in which they reduce to the energy density and radial pressure respectively.

Matching the Vaidya metric to the Bondi metric at the surface \( r = a \) of the fluid distribution implies – see [11] for details –
\[ \tilde{P}_a = -\omega_a \tilde{\rho}_a, \] (16)

which is equivalent to the well-known condition
\[ q_a = P r a, \] (17)

for radiative spheres [12], (subscript a indicates that the quantity is evaluated at the boundary surface).
The HJR method is based on a system of three ordinary differential equations for quantities evaluated on the boundary surface (surface equations), which will allow us to find the evolution of the physical quantities.

The dimensionless variables

\[
\begin{align*}
A & \equiv \frac{a}{\tilde{m}(0)} \\
M & \equiv \frac{\tilde{m}}{\tilde{m}(0)} \\
u & \equiv \frac{u}{\tilde{m}(0)} \\
F & \equiv 1 - \frac{2M}{A} \\
\Omega & \equiv \frac{1}{1 - \omega_a},
\end{align*}
\]

are defined to derive the surface equations, where \(\tilde{m}(0)\) is the initial total mass of the system. Then, using (18) and boundary conditions one obtains the first surface equation

\[
\dot{A} = F(\Omega - 1),
\]

which gives the evolution of the radius of the star, (where dot denotes derivative with respect to the dimensionless \(u\)).

The second surface equation

\[
\dot{F} = \frac{2L + F(1 - F)(\Omega - 1)}{A},
\]

gives the evolution of the redshift at the surface -see \([9, 4, 10]\) for details. The luminosity, \(L\), as measured by an observer at rest at infinity reads

\[
L = -\dot{M} = \frac{E}{(1 + z_a)^2} = EF = \dot{E}(2\Omega - 1)F = 4\pi A^2 q_a (2\Omega - 1) F,
\]

where \(z_a\) refers to the boundary gravitational redshift, \(\dot{E}\) is the luminosity as seen by a comoving observer, and \(E\) is the luminosity measured by a non moving observer located on the surface.

The third surface equation is model dependent. For anisotropic fluids the relationship \((T^\mu_{\tau\mu})_a = 0\) can be written as

\[
\begin{align*}
\frac{\dot{F}}{F} + \frac{\dot{\Omega}}{\Omega} & - \frac{\dot{\rho}_a}{\rho_a} + F\Omega^2 \frac{\ddot{R}_{\perp a}}{\rho_a} - \frac{2}{A} F\Omega \frac{P_{\perp a}}{\rho_a} = \\
(1 - \Omega) \left[ 4\pi A \rho_a \frac{3\Omega - 1}{\Omega} - \frac{3 + F}{2A} + F\Omega \frac{\dot{\rho}_a}{\rho_a} + \frac{2F\Omega}{A\rho_a} (P_{\perp} - P_r)_a \right].
\end{align*}
\]

being

\[
\ddot{R}_{\perp a} = \ddot{P}_{\perp a} + \left( \frac{\ddot{P} + \ddot{\rho}}{1 - 2\tilde{m}/r} \right)_a \left( 4\pi r \frac{\ddot{P} + \ddot{m}}{r^2} \right)_a - \left( \frac{2}{r} (P_{\perp} - P_r) \right)_a.
\]

Expression (22) generalizes the Tolman-Oppenheimer-Volkov equation to the non-static radiative anisotropic case.
The HJR method [4] allows us to find nonstatic solutions of the Einstein equations from static ones. The algorithm, extended for anisotropic fluids, can be found in [10]. Nevertheless, for our purpose here, it is only necessary to find the evolution of $A$, $F$, $\Omega$, and $L$. Thus, the algorithm can be resumed as follows

1. Take a static but otherwise arbitrary interior solution of the Einstein equations for a spherically symmetric fluid distribution ("seed" solution)

\[ P_{st} = P(r), \quad \rho_{st} = \rho(r). \]  

(24)

2. Assume that the $r$-dependence of the effective quantities is the same as that of the energy density and radial pressure of the "seed" solution. Nevertheless, note that junction conditions in terms of effective variables, read as (16). This condition allows us to find out the relation between the $u$-dependence of $\tilde{\rho} \equiv \tilde{\rho}(u, r)$ and $\tilde{P} \equiv \tilde{P}(u, r)$.

3. If we have an expression for $\tilde{\rho}(u, r = a)$ and $\tilde{P}(u, r = a)$, it is possible to solve the system of surface equations (19), (20), and (22) for a given luminosity.

3 Heat conduction equation

As mentioned in the Introduction, the value of parameter $\alpha$, defined as

\[ \alpha = \frac{\kappa T}{\tau(\rho + p)} \]  

appears to be important in the evolution of radiating stars. In fact, as it is shown in [1, 2], if the system reaches the critical point ($\alpha = 1$), then the inertial mass term vanishes. Furthermore it can be shown that conditions ensuring stability and causality [13, 14] are violated at the critical point [2]. This strange result may have, at least, two different interpretations. On one hand, it may happen that, because of the fact that the inertial mass term vanishes at the critical point, a linear perturbative scheme fails at the critical point, invalidating thereby any restriction obtained from linear approximation (e.g. causality and stability conditions mentioned above). Alternatively, it may happen that the system is actually prevented from reaching the critical point, in which case it should exhibit some abnormal behaviour when approaching the critical point. The correct interpretation can be found by solving the Einstein field equations, together with the transport equation, without applying linear perturbation theory.
In order to study the behaviour of the system at the critical point, it is convenient to adopt a physical framework in which condition $\alpha = 1$ could be overtaken for reasonable values of physical quantities. A good candidate seems to be a collapsing neutron star in which the equilibrium is reached by means of a huge emission of neutrinos. The large temperature reached during the process of collapse would explain values of $\alpha$ greater than 1.

We assume the evolution of the heat flow to be governed by the Maxwell-Cattaneo transport equation,

$$\tau h^\mu_\nu \dot{q}^\nu + q^\mu = \kappa h^{\mu\nu} \left( T_{\nu,\nu} - T_{\nu} \right), \quad (26)$$

where

$$\dot{U}_{\nu} = U^{\alpha} U_{\nu,\alpha}, \quad (27)$$

$$\dot{q}_{\nu} = U^{\alpha} q_{\nu,\alpha}, \quad (28)$$

and $\kappa$ and $\tau$ denote the thermal conductivity coefficient and relaxation time respectively. Evaluating (26) in the surface, the transport equation can be written as

$$\tau q_a + q_a \sqrt{F(2\Omega - 1)} =$$

$$\kappa_a \left[ T_a - T_{1a} F(2\Omega - 1) - T_a \left( \frac{\Omega(1 - F)}{2A} + \frac{L}{AF} + \frac{\Omega}{2\Omega - 1} \right) \right]. \quad (29)$$

The thermal conductivity coefficient for a mixture of matter and radiation is given by [15]

$$\kappa = \frac{4}{3} b T^3 \tau_{\text{col}}, \quad (30)$$

where $b = 7N_{\nu} a/8$ for neutrinos, being $a$ the radiation constant, $N_{\nu}$ the number of neutrino flavors and $\tau_{\text{col}}$ the matter-neutrinos time collision.

On the other hand, the luminosity perceived by a comoving observer located on the surface can be connected with the effective temperature, $T_{\text{eff}}$, as

$$\hat{E} = \frac{L}{F(2\Omega - 1)} = 4\pi A^2 q_a = \left[ 4\pi r^2 \sigma T_{\text{eff}}^4 \right]_{r=a}, \quad (31)$$

where $\sigma = a/4$ is the Steffan-Boltzman radiation constant. The concept of effective temperature has been widely used in theory of stellar atmospheres [16, p.586], [17, p.70], and [18, p.295]. The effective temperature can be connected to the material temperature by means of

$$T_a^4 = \frac{1}{2} \left[ T_{\text{eff}}^4 \right]_{r=a}. \quad (32)$$
Thus, from (31) and (32), the dimensionless heat flow at the surface is

\[ q_a = \frac{2\zeta \xi^2}{2\Omega - 1} T_a^4, \tag{33} \]

where \( \zeta = \sigma M_\odot^2 \simeq 3.4097 \times 10^{-54} \text{ K}^{-4} \), and \( \xi = M_\odot / M_\odot \). In the HJR formalism the initial mass is normalized to unity. Thus, every term in (29) is dimensionless and \( \kappa \) must be expressed as

\[ \kappa = \frac{14}{3} \zeta T^3 \xi^2 N_{\nu} \tau_{\text{col}}. \tag{34} \]

On the other hand, evaluating (25) at the surface, and by means of the definition of (14), (17) and (34), it takes the form

\[ \alpha = \frac{14 y^4 \xi^2 N_{\nu} (2\Omega - 1)}{\frac{1}{3} (\dot{\rho}_a (2\Omega - 1)^2 + 4y^4 \xi^2 \Omega)}, \tag{35} \]

where we have assumed \( \tau \sim \tau_{\text{col}} \), and the parameter \( y \) has been defined as

\[ y^4 = \zeta T_a^4. \tag{36} \]

Now, it is possible to write \( y \) in terms of \( \alpha \) by means of (35)

\[ y^4 = \frac{3\ddot{\rho}_a (2\Omega - 1)^2 \alpha}{2\xi^2 \Omega (7N_{\nu} (2\Omega - 1) - 6\alpha)}, \tag{37} \]

and by substitution of this expression and (36), into (33)

\[ q_a = \frac{3\ddot{\rho}_a (2\Omega - 1) \alpha}{\Omega (7N_{\nu} (2\Omega - 1) - 6\alpha)}. \tag{38} \]

After some elementary algebra, expression (29) can be written in terms of \( \alpha \) instead of \( y \) and \( q_a \)

\[ \frac{y}{y} F(2\Omega - 1) = \frac{\dot{\Omega}}{2\Omega - 1} - \left( \frac{6}{\Phi + 6\alpha} - 1 \right) + \frac{\Omega(F - 1)}{2A} - \frac{L}{AF} - \frac{3\sqrt{F}}{7\tau_{\text{col}} N_{\nu} \sqrt{2\Omega - 1}} + \left( \frac{\dot{\rho}_a}{\rho_a} + \frac{\dot{\alpha}}{\alpha} \left[ 1 + \frac{6\alpha}{\Phi} \right] + \frac{\dot{\Omega}}{\Omega \Phi(2\Omega - 1)} \right) \left[ \frac{1}{4} - \frac{3}{\Phi + 6\alpha} \right], \tag{39} \]

where \( \Phi = 7N_{\nu}(2\Omega - 1) - 6\alpha \).

Our purpose is to discern what anomalous effects (if any) can take place when the system overtakes the critical point. Thus, it seems reasonable to impose a profile for \( \alpha \), and study the evolution of the system. The Maxwell-Cattaneo transport equation (39) allows us to find the temperature gradient in the surface for a given \( \alpha(u) \). At first glance, expression (39) does not seem to present anomalous behaviour in the critical point. Nevertheless, it is necessary to study the complete evolution of the system to confirm this suspicion. The method that we shall use can be summarized as follows
1. Impose a profile for $\alpha$. According to (25), $\alpha$ must vanish initially if the system departs from equilibrium ($T_a(u = 0) \sim 0$).

2. For a given $\alpha$, it is possible to find the luminosity $L$ by means of (31) and (38).

3. Follow the HJR method, outlined above, to solve the system of surface equations.

4. Once $A$, $F$, and $\Omega$ are found it is possible to find the temperature gradient by means of (39). Assuming that cooling by absorption and emission of neutrinos is the responsible to drive the star to a new equilibrium state, the mean collision time $\tau_{col}$ can be roughly expressed as

$$\tau_{col} \sim \frac{A M_\odot \xi ^{3/8}}{\rho \sqrt{Y y^2}}. \quad (40)$$

where $A = 10^9 \text{K}^{3/2} \text{m}^{-1}$, and $Y$ stands for the electron fraction.

4 The model

In order to study the evolution of the system beyond the critical point we adopt the Gokhroo & Mehra-type solution [19] as the ”seed” solution. This solution corresponds to an anisotropic fluid with inhomogeneous energy density, and it can take account of the large density and pressure gradients close to surface [7].

It can be shown [7] that the effective energy density and effective pressure are given for this model by

$$\tilde{\rho} = \rho_c \frac{K(u)}{K_o} \left( 1 - K(u) \frac{r^2}{A^2} \right), \quad (41)$$

and

$$\tilde{P} = \lambda \rho_c \frac{K(u)}{K_o} \left( 1 - 2 \frac{\tilde{m}}{r} \right) \left( 1 - G(u) \frac{r^2}{A^2} \right)^n, \quad (42)$$

where $n \geq 1$, and the central energy density in the static case $\rho_c$ is given by

$$\rho_c = \frac{15}{4 \pi A_o (5 - 3 K_o)} \quad (43)$$

If we define

$$\gamma = \frac{8 \pi \rho_c}{3}, \quad (44)$$
then functions $K(u)$, and $G(u)$ are

$$K(u) = \begin{cases} \frac{5}{6} \left[ 1 + \sqrt{1 - \left( \frac{12Ko}{5\gamma} \right) \left( \frac{1 - F}{A^2} \right)} \right] & \text{if } Ko > \frac{5}{6} \\ \frac{5}{6} \left[ 1 - \sqrt{1 - \left( \frac{12Ko}{5\gamma} \right) \left( \frac{1 - F}{A^2} \right)} \right] & \text{if } Ko < \frac{5}{6} \end{cases}, \quad (45)$$

and

$$G(u) = 1 - \left[ \frac{(1 - \Omega)(1 - K)}{F\Omega\lambda} \right]^{1/n}. \quad (46)$$

Thus, the system of surface equations for this model is

$$\dot{A} = F(\Omega - 1),$$

$$\dot{F} = \frac{1}{A} \left[ 2L + F(1 - F)(\Omega - 1) \right],$$

$$\dot{\Omega} = -\frac{F}{F\Omega} + \frac{K(1 - 2K)}{K(1 - K)}\Omega + \frac{4K\lambda L\Omega^2}{3\gamma KA^3(2\Omega - 1)(1 - K)} + \Omega(1 - \Omega)\Lambda, \quad (49)$$

where

$$\Lambda = \frac{3\gamma K}{2Ko} A(1 - K) \left( \frac{3\Omega - 1}{\Omega} \right) - 3 + \frac{F}{2A} + \frac{2F\Omega}{A(1 - K)}(\Psi - K),$$

$$\Psi = \frac{3}{10Ko} \lambda K^2 (1 - G)^n$$

$$+ \frac{A^2}{2F} \left[ \frac{3\gamma K}{2Ko} \lambda^2 F^2 (1 - G)^{2n} - \frac{2nG^2}{A^2} F^2 (1 - G)^{n-1} \right] + \frac{\gamma}{2} \left( 1 - \frac{3K}{5} \right) \frac{K(1 - K)}{Ko}. \quad (50)$$

The mean collision time (40) can be rewritten in terms of $\Omega$, $\alpha$ and $\tilde{\rho}$ as

$$\tau_{col} = \frac{AM_\odot}{\sqrt{\Phi}(\Phi + 3\alpha)} \left( \frac{\Omega\Phi}{\tilde{\rho}_a} \right)^{11/8} \left( \frac{\xi}{2\Omega - 1} \right)^{7/4} \left[ \frac{2\xi}{3\alpha} \right]^{3/8}, \quad (52)$$

whereas the luminosity can be found from (31) and (38)

$$L = \frac{12\pi A^2 \tilde{\rho}_a F(2\Omega - 1)^2\alpha}{\Omega\Phi}. \quad (53)$$

Thus, the temperature gradient in the surface can be found by means of (52), (53) and (39) for a given $N_\nu$.

The system of equations (47-49) and (39) has been solved for two set of initial values:
1. Model with initial values \( A_o = A(u = 0) = 6, \Omega_o = \Omega(u = 0) = 1, n = 1, K_o = 0.999, \lambda = 1/3, N_o = 3 \) and \( \lambda' = 0.2 \). This initial configuration corresponds to a star initially at rest, with a radius of 8862 meters, a central density equal to \( 1.70 \times 10^{15} \text{ g cm}^{-3} \), and a surface density equal to \( 1.70 \times 10^{12} \text{ g cm}^{-3} \). Its initial mass is \( 1M_\odot \), and its surface redshift is close to 0.225. We assume the center of the star composed by a highly relativistic Fermi gas. Thus, \( \lambda = 1/3 \). The profile for \( \alpha \) has been assumed as a gaussian centered in \( u = u_{\text{peak}} \), i.e.

\[
\alpha = \alpha_{\text{max}} \exp \left( -\frac{1}{2} \left( \frac{u - u_{\text{peak}}}{\Delta} \right)^2 \right).
\]

We have taken \( u_{\text{peak}} = 100 \) (\( \sim 0.49 \text{ msec} \)) and \( \Delta = 13 \) (\( \sim 0.064 \text{ msec} \)). For this values the temperature changes considerably along 1 msec and \( \alpha(0) \sim 0 \) as is demanded by condition \( T(0) \sim 0 \). We have considered three profiles with \( \alpha_{\text{max}} = 0.9, 1 \) and 1.1 (figure 1).

2. Model initially in slow contraction with \( A_o = 5 \) (\( \sim 7838 \text{ meters} \)), \( \Omega_o = 0.99926 \) (\( \vert \omega \vert \ll 1 \)), \( n = 1, K_o = 0.999, \lambda = 1/3, N_o = 3 \) and \( \lambda' = 0.3 \). These values correspond to \( \rho_c \approx 2.94 \times 10^{15} \text{ g cm}^{-3} \) and \( \rho_a \approx 2.94 \times 10^{12} \text{ g cm}^{-3} \). The profile for \( \alpha \) has been assumed as

\[
\alpha = 2 - \exp \left( -\frac{1}{2} \left( \frac{u}{\Delta} \right)^2 \right),
\]

and

\[
\alpha = 1.
\]

As in the previous case \( \Delta = 13 \) (\( \sim 0.064 \text{ msec} \)) (figure 2).

5 Discussion

As mentioned before, our purpose here is to clarify the physical significance (if any) of the critical point in the evolution of a dissipative self-gravitating system.

In particular we want to find out if at the critical point the system, described by the full theory (without any approximation), exhibits an anomalous behaviour, as might suggest a linear approximation approach.

The three profiles of \( \alpha \) considered are displayed in figure 1. In one case the system never reaches the critical point, in other case it is at the critical point at some moment of its evolution, and in the third
case the system goes beyond the critical point before returning to equilibrium.

Figures (3)-(6) show the evolution of temperature, temperature gradient, radius and surface velocity for the three different profiles of $\alpha$.

It is apparent from these figures that even though increasing values of $\alpha$ are associated with more unstable configurations (in the sense of faster collapse), nothing strange seems to happens at or beyond the critical point. Figure (7), showing the evolution of the ratio of the neutrino mean free path to the radius of the star, indicates that the diffusion approximation is valid during most part of the emission process.

To reinforce this conclusion we have performed another numerical simulation with the profiles of $\alpha$ given in figure (2). The evolution of relevant physical variables displayed in figures (8)-(11), confirms the conclusion emerging from figures (3)-(6). The validity of diffusion approximation is corroborated in fig.(12).

To conclude, we may say that the increasing of instability (in the sense mentioned above) with higher values of $\alpha$, seems to confirm the decreasing of the inertial mass density factor, predicted in the linear approximation.

However, nothing dramatic appears at the critical point in the exact modeling, indicating that the later approximation is not reliable at (or close to) the critical point, as suggested before 2, 3.

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Figure captions

Figure 1.- Profiles of $\alpha$ given by expression (54). The values of $\alpha_{\text{max}}$ are 0.9, 1 and 1.1.

Figure 2.- Profiles of $\alpha$ for the model initially in slow contraction (55) and (56).

Figure 3.- Temperature evolution corresponding to three profiles of $\alpha$ shown in figure 1.

Figure 4.- Temperature gradient for the model initially at rest.

Figure 5.- Evolution of the radius for the model initially at rest.

Figure 6.- Evolution of the surface velocity for the model initially at rest.

Figure 7.- Mean free path of neutrinos in the first model.

Figure 8.- Temperature evolution for the model initially in slow contraction. The profiles of $\alpha$ corresponding to this model are shown in figure 2. The curve labeled with $\alpha = 1$ corresponds to the profile (56), whereas the other one corresponds to the profile of $\alpha$ given by expression (54).

Figure 9.- Temperature gradient for the model initially in slow contraction.

Figure 10.- Evolution of the radius for the model initially in slow contraction.

Figure 11.- Evolution of the surface velocity for the model initially in slow contraction.

Figure 12.- Same as figure 7 for the model initially in slow contraction.
$T$ (x $10^{12}$ K) vs. Time (msec)

- $\alpha_{\text{max}} = 1.1$
- $\alpha_{\text{max}} = 1.0$
- $\alpha_{\text{max}} = 0.9$
\[ a_{\text{max}} = 0.9 \]
\[ a_{\text{max}} = 1.0 \]
\[ a_{\text{max}} = 1.1 \]
$T_1 \times 10^{10}$ K m$^{-1}$ vs. Time (msec) for $\alpha = 1$. The graph shows a decreasing trend with time.
\[ \frac{\lambda}{A^\circ} = 1 \]