Some considerations on topologies of infinite dimensional unitary coadjoint orbits.∗

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Abstract

The topology of the embedding of the coadjoint orbits of the unitary group $U(\mathcal{H})$ of an infinite dimensional complex Hilbert space $\mathcal{H}$, as canonically determined subsets of the B-space $\mathfrak{T}_s$ of symmetric trace class operators, is investigated. The space $\mathfrak{T}_s$ is identified with the B-space predual of the Lie-algebra $\mathcal{L}(\mathcal{H})_s$ of the Lie group $U(\mathcal{H})$. It is proved, that orbits consisting of symmetric operators with finite rank are (regularly embedded) closed submanifolds of $\mathfrak{T}_s$. An alternative method of proving this fact is given for the “one-dimensional” orbit, i.e. for the projective Hilbert space $P(\mathcal{H})$. Also a technical assertion concerning existence of simply related decompositions into one-dimensional projections of two unitary equivalent (orthogonal) projections in ‘generic relative position’ is formulated, proved, and illustrated.

1 Introduction

Mathematical formalism of non-Einstein-relativistic quantum mechanics (QM) is traditionally based on separable complex Hilbert space $\mathcal{H}$, and on closely connected ob-

∗This is a revised version of the preceding versions of this paper. The main difference consists of the correction of a wrong assertion of the Lemma 3.1 based on an erroneous assumption. The error was discovered in [12, Sec. 5].
jects: The $C^*$-algebra of bounded operators $\mathcal{L}(\mathcal{H})$, the $\sigma(\mathcal{L}(\mathcal{H}),\mathcal{L}(\mathcal{H})_*)$-continuous (with $\mathcal{L}(\mathcal{H})_* := \mathcal{T} := L^1(\mathcal{H}) :=$ the trace-class operators on $\mathcal{H}$) linear functionals on $\mathcal{L}(\mathcal{H})$ (identified with $\nu \in \mathcal{T}$), and the group of $\ast-$automorphisms $\ast$-Aut $(\mathcal{L}(\mathcal{H}))$ of $\mathcal{L}(\mathcal{H})$ (acting on linear functionals by the transposed maps). Dynamics (i.e. time evolution) and symmetries of physical systems are described by subgroups of the automorphism group of $\mathcal{L}(\mathcal{H})$. Since each automorphism $\alpha \in \ast$-Aut $(\mathcal{L}(\mathcal{H}))$ of the $C^*$-algebra $\mathcal{L}(\mathcal{H})$ is inner, it is described by a unitary operator ($\mathcal{U}(\mathcal{H})$ is the set of all unitary elements of $\mathcal{L}(\mathcal{H})$) $u_\alpha \in \mathcal{U}(\mathcal{H}) : \alpha(B) = u_\alpha B u_\alpha^*$ ($u_\alpha$ is determined by $\alpha$ up to a numerical factor). In physics, symmetries and dynamics are modelled by Lie groups $G$. In the traditional linear QM, Lie groups are represented by strongly continuous unitary (or projective) representations $g \in G \mapsto U(g) \in \mathcal{U}(\mathcal{H})$, hence linear dynamics is described by one parameter unitary groups $U_t \equiv \exp(\mathbf{-}itH)$, with a selfadjoint operator $H$ on (a dense domain of) $\mathcal{H}$. Since physically interesting objects describing “states” are not vectors $x \in \mathcal{H}$, but to the vectors $x$ corresponding one-dimensional projectors $P_x$ onto the subspaces of $\mathcal{H}$ containing $x$, as well as their convex combinations (so called mixed states described by ‘density matrices’ $\rho := \sum_j \lambda_j P_{x_j} \in \mathcal{S}_* := \{\rho \in \mathcal{T}_* : \rho \geq 0, Tr\rho = 1\} \subset \mathcal{T}_*$) the physically interesting orbits of actions of the considered groups (resp. their representations) are orbits of the coadjoint action of $\mathcal{U}(\mathcal{H})$ (and of its subgroups), considered as a Lie group (see below).

In a more general (also nonlinear) setting, cf. e.g. [3], symmetries and dynamics in such an “extended quantum mechanics” (EQM) can be described by unitary cocycles $(g;\rho) \in G \times \mathcal{S}_* \mapsto u_Q(g,\rho) \in \mathcal{U}(\mathcal{H}))$, $u_Q(g_1 \cdot g_2,\rho) = u_Q(g_1,\phi^Q_{g_2}(\rho)) \cdot u_Q(g_2,\rho)$, acting on the set $\mathcal{S}_*$ of all ‘density matrices’, again by means of the coadjoint action of $\mathcal{U}(\mathcal{H})$, i.e. $\phi^Q : (g;\rho) \in G \times \mathcal{S}_* \mapsto \phi^Q_g(\rho) := u_Q(g,\rho) \cdot \rho \cdot u_Q(g,\rho)^{-1}$ (here $Q$ is, in the case of one-parameter group $G := \mathbb{R}$, a Hamiltonian function given on a Poisson manifold, specifying the cocycle $u_Q$). These actions leave the orbits of the coadjoint representation $Ad^\ast(\mathcal{U}(\mathcal{H}))$ going through $\rho \in \mathcal{S}_*$ again invariant, leaving invariant also the whole set $\mathcal{S}_*$. This EQM is a general scheme of theories including Hamilton classical mechanics (CM), linear QM, and also various versions of nonlinear QM, and also other in physics used theoretical schemes as, e.g., various nonlinear “approximations” to QM (e.g., the time dependent Hartree-Fock theory).

These remarks have to stress that the coadjoint orbits of the Lie group (see below) $\mathcal{U}(\mathcal{H})$ going through symmetric trace class operators are important mathematical objects in physical description of a rather large scale of “processes”.

As I have learned from a discussion with colleagues Anatol Odzijewicz and Tudor Ratiu, there is an “innocently looking” question connected with a work with coadjoint action of Lie groups, which is far not trivial in the general case. It is the question in
which way the homogeneous spaces $G/G_\rho$ of a Lie group $G$ with their natural analytic manifold structure (with $G_\rho$ being the stability subgroup of $G$ at the point $\rho$), specifically their coadjoint orbits, are included into the topological spaces where the group acts. In more specific terms the question is, whether the injective inclusion is an immersion and homeomorphism of the analytic manifold $G/G_\rho$ onto a submanifold of the space $\mathcal{T}$ on which the group $G$ acts. E.g., an orbit $\mathcal{O}$ of a specific action of $\mathbb{R}$ on the two-torus $T^2 = S^1 \times S^1$, i.e. $\mathcal{O} := \{ (e^{it\omega_1}, e^{it\omega_2}) : t \in \mathbb{R} \} \subset T^2$ with irrational quotient $\omega_1/\omega_2$, covers the torus densely, hence it is not a submanifold of $T^2$. As it is shown in a Kirillov’s example [2] (cited and reproduced in [1, 14.1.(f), p.449]), such a pathologically looking case is possible also in the cases of finite-dimensional coadjoint orbits. The more one could expect such a phenomenon in the case of infinite-dimensional orbits of Banach Lie groups.

Let $\mathcal{O}_\rho(\mathcal{U}) = \mathcal{U}/\mathcal{U}_\rho$ be the homogeneous space of the unitary group $\mathcal{U} := \mathcal{U}(\mathcal{H})$ of the infinite-dimensional Hilbert space $\mathcal{H}$ corresponding to an orbit of the action $u \mapsto upu^*$ on the space $\mathfrak{T}_s(\rho)$ of symmetric trace class operators in $\mathcal{L}(\mathcal{H})$. The space $\mathfrak{T}_s$ is naturally identified with the predual $\mathcal{L}(\mathcal{H})_s^*$ of the Lie algebra Lie $\mathcal{U}(\mathcal{H}) := i\mathcal{L}(\mathcal{H})_s \sim \mathcal{L}(\mathcal{H})_s (\mathcal{U}_\rho$ is the stability subgroup of $\mathcal{U}$ at $\rho$, namely $\mathcal{U}_\rho = \{ \rho \}' \cap \mathcal{U}$, $\rho^* = \rho \in \mathfrak{T}_s$, with $\{ A \}'$ being the commutant in $\mathcal{L}(\mathcal{H})$ of $A$).

In the paper [3], the topology of the orbits $\mathcal{O}_\rho(\mathcal{U})$, as well as the topology of their natural injection into the dual B-space (containing the predual $\mathfrak{T}_s$) were investigated.\footnote{Let us note that a far reaching generalization of some of structures developed and investigated in [3] is contained in very elegant paper [5]; that paper was also stimulating for the here reported research.} It was proved there (cf. [3, Proposition 2.1.5]), that orbits trough symmetric trace-class operators are injectively immersed into $\mathfrak{T}_s$ iff they are going trough operators with finite rank. There was not completed, however, the proof of an assertion on regularity of this embedding (in the terminology of [9]) of such “finite-rank” orbits, which claim was contained in the text of the Proposition 2.1.5.\footnote{The claim of “regularity of the embedding” was, however, superfluous (nevertheless correct, as it could be seen from what we shall prove here) with respect to the validity of the Proposition 2.1.5 (without the requirement of regularity of the embedding in its item (iv)) , as well as with respect to its actual applications in all the paper [3].} One of the aims of this paper is to fill this gap.

Let us note, that the posed question of whether the orbit is also a submanifold of the “ambient” space in which the group acts is easily and positively answered in the case of finite-dimensional Hilbert space $\mathcal{H}$: In that case the group $\mathcal{U}(\mathcal{H})$ is compact, so that the orbits are also compact and a continuous bijection of any compact space into a Hausdorff space is a closed mapping, hence a homeomorphism. For an infinite-dimensional
\( \mathcal{H} \), however, the orbits \( \mathcal{O}_\rho(\mathfrak{U}) \) are noncompact.

The proof of the main theorem is contained in the next Section 2. The presented proof is based on a simple idea, and it does not contain any “sophisticated mathematics”; it needed just some linear algebra and elementary topology to be presented in details. In the last Section 3 some additional facts (including a proof of the fact that the orbits consisting of finite-rank density matrices are closed subsets of the “ambient” space) are presented. Also an independent proof of regularity of the embedding (we use here the definitions adopted from [9] differing from those introduced in [1], also to keep continuity with [3]) of the projective Hilbert space is presented: It indicates also an alternative way for proving the main Theorem 2.6 for the general case.

## 2 A proof of regularity of the embedding

We shall accept here some conventions and results from [3], mainly from the proof of Proposition 2.1.5 and Theorem 2.1.19. The proof of the following Theorem 2.6 completes the missing part of the proof of Proposition 2.1.5 in [3] concerned the regularity of the embedding of \( \mathcal{O}_\rho(\mathfrak{U}) \subset \mathfrak{I}_s \). Some of the constructions built and used in the run of the proof might be, perhaps, also of independent interest.

Let us describe first in more details a formulation of the problem, and our strategy to approach it. It is known, [6, Proposition 37, Chap. III. §3], that the unitary group \( \mathfrak{U} := \mathfrak{U}(\mathcal{H}) \) of the \( C^* \)-algebra \( \mathcal{L}(\mathcal{H}) \) of all bounded operators on a complex Hilbert space \( \mathcal{H} \) is a Banach Lie group, and its Lie algebra \( \text{Lie}(\mathfrak{U}) \) consists of all antisymmetric bounded linear operators \( i\mathcal{L}(\mathcal{H}) \), which is B-space isomorphic to \( \mathcal{L}(\mathcal{H})_s \). The adjoint representation of \( \mathfrak{U} \) in the B-space \( \mathcal{L}(\mathcal{H})_s \) is \( \text{Ad} : \mathfrak{U} \to \mathcal{L} (\mathcal{L}(\mathcal{H})_s) \), \( u \mapsto \text{Ad}(u) \), with \( \text{Ad}(u)B := uBu^*, \forall B \in \mathcal{L}(\mathcal{H})_s \). The representation we are mostly interested in here is the coadjoint representation consisting of the transposed mappings \( \text{Ad}^*(u) := \text{Ad}(u^{-1})^* \) to \( \text{Ad}(u^{-1}) \)'s, hence acting on continuous linear functionals \( \nu \in \mathcal{L}(\mathcal{H})_s^* \), \( \nu : \mathcal{L}(\mathcal{H})_s \to \mathbb{C} \), \( B \mapsto \langle \nu; B \rangle \); the mapping \( \text{Ad}^*(u) : \mathcal{L}(\mathcal{H})_s^* \to \mathcal{L}(\mathcal{H})_s^* \) is determined by \( \langle \text{Ad}^*(u)\nu; B \rangle := \langle \nu; \text{Ad}(u^{-1})B \rangle \). The subset of symmetric normal linear functionals can be identified with the B-space \( \mathfrak{I}_s \subset \mathcal{L}(\mathcal{H})_s^* \) of symmetric trace-class operators : \( \nu \in \mathfrak{I}_s \) : \( B \mapsto \langle \nu; B \rangle := \text{Tr}(\nu B) \); the space of normal (i.e. continuous in the topology on \( \mathcal{L}(\mathcal{H}) \) given by the seminorms \( p_\nu : B \mapsto p_\nu (B) := |\text{Tr}(\nu B)| \), \( \nu \in \mathfrak{I}_s \) symmetric functionals is a Banach space \( \mathfrak{I}_s \) with the trace-norm \( \|\nu\|_1 := |\text{Tr}| \nu | \), with the absolute value of the operator \( \nu \) defined as the operator \( |\nu| := \sqrt{\nu^* \nu} \in \mathcal{L}(\mathcal{H}) \).

We are interested in comparison of two topologies introduced on the orbits \( \mathcal{O}_\rho(\mathfrak{U}) := \)
\{u \rho u^*: u^{-1} = u^* \in \mathcal{U} \subset \mathcal{L}(\mathcal{H})\} of the coadjoint representation. Let us denote \(\mathcal{U}_\rho : \{u \in \mathcal{U} : u \rho = \rho u\}\) \((\rho \in \mathfrak{T}_s)\). Then \(\mathcal{U}_\rho\) is a Lie subgroup of \(\mathcal{U}\), [3, Lemma 2.1.2], and the factor-space \(\mathcal{U}/\mathcal{U}_\rho\) (which can be canonically identified, as a set, with \(\mathcal{O}_\rho(\mathcal{U})\)) endowed with the factor-topology of the analytic Banach Lie group \(\mathcal{U}\) is an analytic Banach manifold, [6, III.1.6, Proposition 11]. On the other side, the orbit \(\mathcal{O}_\rho(\mathcal{U})\) is naturally a subset of the Banach space \(\mathfrak{T}_s\) endowed with the norm-topology given by the trace-norm \(\|\cdot\|_1\). The topology induced on \(\mathcal{O}_\rho(\mathcal{U})\) from this B-space topology on \(\mathfrak{T}_s\) need not coincide with the analytic manifold topology of \(\mathcal{U}/\mathcal{U}_\rho\). It is known that this coincidence is not the case for any \(\rho\) with infinite-dimensional range, cf. [3, Proposition 2.1.5]. The coincidence of these two topologies means that the immersed subset \(\iota(\mathcal{U}/\mathcal{U}_\rho) = \mathcal{O}_\rho(\mathcal{U})\) of \(\mathfrak{T}_s\) endowed with the topology of \(\mathcal{U}/\mathcal{U}_\rho\) is a submanifold of \(\mathfrak{T}_s\), or equivalently, that the inclusion mapping \(\iota : \mathcal{U}/\mathcal{U}_\rho \to \mathfrak{T}_s\) (provided that \(\iota\) is immersion) is a homeomorphism of \(\mathcal{U}/\mathcal{U}_\rho\) onto the topological subspace \(\mathcal{O}_\rho(\mathcal{U}) \subset \mathfrak{T}_s\), [7, 5.8.3].

We intend to prove that, for any \(\rho = \rho^* \in \mathfrak{F}\) (:= the linear space of finite-rank operators in a complex Hilbert space \(\mathcal{H}\)), the topology induced on the subset \(\mathcal{O}_\rho(\mathcal{U}) := \{u \rho u^* : u^{-1} = u^* \in \mathcal{U} \subset \mathcal{L}(\mathcal{H})\}\) from the overlying (resp. “ambient”) Banach space of symmetric trace-class operators \(\mathfrak{T}_s\) is equivalent to the topology of the set \(\mathcal{O}_\rho(\mathcal{U})\) considered as the factor-space \(\mathcal{U}/\mathcal{U}_\rho\). If the inclusion \(\iota : \mathcal{U}/\mathcal{U}_\rho \to \mathcal{O}_\rho(\mathcal{U}) \subset \mathfrak{T}_s\), \([u]_\rho \mapsto \iota([u]_\rho) := u \rho u^*\), where \([u]_\rho := \{v \in \mathcal{U} : v \rho v^* = u \rho u^*\}\) is an (injective) immersion, and if it were also homeomorphism of \(\mathcal{U}/\mathcal{U}_\rho\) onto \(\iota(\mathcal{U}/\mathcal{U}_\rho) = \mathcal{O}_\rho(\mathcal{U})\), then \(\mathcal{O}_\rho(\mathcal{U})\) would be a submanifold of \(\mathfrak{T}_s\), cf. [7, 5.8.3].

Let us sketch our “strategy” of proving this claim here. It was proved in [3, Proposition 2.1.5], that \(\mathcal{O}_\rho(\mathcal{U})\) is an immersed submanifold (i.e. the inclusion \(\iota : \mathcal{U}/\mathcal{U}_\rho \to \mathcal{O}_\rho(\mathcal{U}) \subset \mathfrak{T}_s\) is an immersion, [7, 5.7.1]) of \(\mathfrak{T}_s\) for \(\dim(\rho) := \text{rank}(\rho) < \infty\). We are going to prove that the inverse mapping \(\iota^{-1} : \mathcal{O}_\rho(\mathcal{U}) \to \mathcal{U}/\mathcal{U}_\rho\) is also continuous. It will be useful to our technique to use the metric-space description of continuity of mappings, i.e. the “\(\epsilon \leftrightarrow \delta\) language”.

It is useful to realize for this that the considered orbits \(\mathcal{O}_\rho(\mathcal{U})\) are all (for \(\dim(\rho) < \infty\)) Riemann manifolds endowed with strong riemannian metrics, [3, Theorem 2.1.19]. Then the manifold topology is given by the corresponding distance function, [8, Proposition 4.64], hence all the considered topologies are metric ones, i.e. the topology on \(\mathcal{U}\) given by the operator norm \(\|u - v\|\), the riemannian topology on \(\mathcal{O}_\rho(\mathcal{U})\) represented by a distance function \(d_\rho(\rho', u \rho' u^*)\),\(^3\) and also the topology of the space \(\mathfrak{T}_s\) into which is \(\mathcal{O}_\rho(\mathcal{U})\) embedded is given by the norm \(\|\rho' - u \rho' u^*\|_1\).

We have to prove that, for any \(\rho' \in \mathcal{O}_\rho(\mathcal{U})\), and for an arbitrary (small) \(\epsilon' > 0\) there

\(^3\)The distance \(d_\rho\) will not be explicitly calculated here.
is a $\delta' > 0$ such that if there is an element $\rho'' \in \mathcal{O}_\rho(\mathcal{U})$ with $\|\rho'' - \rho'\|_1 < \delta'$, then it is also $d_\rho(\rho', \rho'') < \epsilon'$. The projection $\Pi_\rho : \mathcal{U} \to \mathcal{O}_\rho(\mathcal{U})$, $u \mapsto [u]_\rho \approx u\rho^*$ is continuous (here $[u]_\rho \approx u\rho^*$ means the canonical identification of the left cosets $[u]_\rho \subset \mathcal{U}$ with their realization as the points $u\rho^*$ of $\mathcal{O}_\rho(\mathcal{U})$). We can use this continuity to avoid necessity of (possibly complicated) calculation of explicit forms of $d_\rho$, cf., however, Proposition 3.2: Since $\Pi_\rho$ is uniformly continuous (due to obvious invariance of both metrics), to any $\epsilon > 0$ there is an $\epsilon' > 0$ such that if $\|u - v\| < \epsilon$, then also $d_\rho(u\rho^*, v\rho^*) < \epsilon'$. So, if we could find to any $\epsilon > 0$ and a $\rho' \in \mathcal{O}_\rho(\mathcal{U})$ such a $\delta' > 0$ that for any $\rho'' := u\rho'/u^* : \rho' - \rho'' \|_1 < \delta'$ it is possible to find an unitary $v$ such that also $\rho'' = v\rho'v^*$, and such that also $\|I_{\mathcal{H}} - v\| < \epsilon$, then the continuity will be proved. We shall proceed essentially in this way, but to avoid explicit calculation of dependence $\epsilon \mapsto \delta'(\epsilon)$, we shall use also another known continuity, namely the continuous dependence of the spectral projections $F_j(\rho'')$ of $\rho'' := \sum_j \lambda_j F_j \in \mathcal{O}_\rho(\mathcal{U})$ onto the $\rho''$ itself. Also the homogeneity of the orbit and of its “ambient” space $\mathfrak{T}_s$ will lead to a simplification.

The following lemma provides a reader with a ‘freedom’ in dealing with various topologies induced on the considered orbits.

2.1 Lemma. The topologies coming from the trace class B-space $L^1(\mathcal{H}) := \mathfrak{T}(\mathcal{H}) \supset \mathfrak{T}_s \supset \mathfrak{F}_N$, from the Hilbert-Schmidt B-space $L^2(\mathcal{H}) := \mathfrak{H} \supset \mathfrak{H}_s \supset \mathfrak{T}_s \supset \mathfrak{F}_N$, as well as from the C*-algebra of all bounded operators $L^\infty(\mathcal{H}) := \mathcal{L}(\mathcal{H}) \supset \mathcal{L}(\mathcal{H})_s \supset \mathfrak{H}_s \supset \mathfrak{T}_s \supset \mathfrak{F}_N$, induced on the subset of symmetric finite rank operators $\mathfrak{F}_N$ with a fixed maximal dimension $N$ of their ranges are all equivalent. ♣

Proof. These topologies are equivalent in finite dimensional linear spaces. Explicitly, in our case: Let $N$ be maximal dimension of ranges of the considered operators $A,B \in \mathfrak{F}_N$, $A = A^*$, $B = B^*$, hence the ranges of the operators $A - B$ are of maximal dimension $2N$. The considered topologies are all metric topologies induced on $\mathfrak{F}_N$ by the corresponding norms from the “above lying” spaces. The distances between $A$ and $B$ are correspondingly given by $\|A - B\|_1 := Tr|A - B|$, $\|A - B\|_2 := \sqrt{Tr|A - B|^2}$, and $\|A - B\|_\infty := \|A - B\|_\infty = \text{the maximal eigenvalue of } |A - B|$, where $|A - B|$ denotes the absolute value of the operator $A - B$, $|A - B| := \sqrt{(A - B)^*(A - B)}$. Generally it is $\|C\|_\infty \equiv \|C\| \leq \|C\|_2 \leq \|C\|_1$ for any trace-class operator $C$. Conversely, also due to the mentioned inequalities, one clearly has $\|A - B\|_2 \leq \|A - B\|_1 \leq 2N\|A - B\|_\infty \leq 2N\|A - B\|_2$ for $A,B \in \mathfrak{F}_N$. This shows that all the three metric topologies are on $\mathfrak{F}_N$ mutually equivalent. □

We shall need a rather indirect, but a quite “faithful” expression for “proximity” of finite-rank operators on the same orbit considered as a subset of the B-space $\mathfrak{T}_s$, which
would be more difficult to express directly with the help the usual norms of their differences. To this end we shall need the following lemma.

2.2 Lemma. Let us consider a subset $\mathfrak{F}_\sigma$ of bounded symmetric operators $\rho \in \mathcal{L}(\mathcal{H})$ with a given purely discrete finite spectrum $\sigma := \{\lambda_0, \lambda_1, \ldots, \lambda_n\} \subset \mathbb{C}$. Their spectral projections $F_j \equiv F_j(\rho), (j = 0, 1, 2, \ldots, n)$ are continuous functions of $\rho \in \mathfrak{F}_\sigma$:

$$\rho := \sum_{j=0}^{n} \lambda_j \cdot F_j,$$

in the operator norm topology of $\mathcal{L}(\mathcal{H})$.

Proof. The spectral projections of any symmetric operator $\rho$ are uniquely determined by that operator, hence for a given spectrum (e.g. $\rho \in \mathfrak{F}_\sigma$) the projections corresponding to fixed spectral values are uniquely determined functions of the operators $\rho \in \mathfrak{F}_\sigma$. By a use of a spectral functional calculus one can choose some functions $p_j : \mathbb{R} \to \mathbb{R}$ such, that $p_j(\lambda_k) \equiv \delta_{jk}$. Then $p_j(\rho) = F_j := F_j(\rho), \forall j$. Let us choose for the functions $p_j$ polynomials; we define for any complex $z \in \mathbb{C}$

$$p_j(z) := \prod_{k(\neq j)=0}^{n} \frac{z - \lambda_k}{\lambda_j - \lambda_k}, \quad (2.1)$$

what gives $p_j(\rho) = F_j(\rho)$, and the continuity of $\rho \mapsto F_j(\rho)$ on (any subset of) $\mathfrak{F}_\sigma$ is explicitly seen.

This two Lemmas lead immediately to

2.3 Corollary. The spectral projections $F_j$ of finite rank operators $\rho \in \mathfrak{F}_\sigma \cap \mathfrak{F}_N$ are (on the set $\mathfrak{F}_\sigma \cap \mathfrak{F}_N$) continuous functions $\rho \mapsto F_j(\rho)$ of these operators in any of the considered (i.e. trace, Hilbert-Schmidt, and $\mathcal{L}(\mathcal{H})$) topologies (taken independently on the domain-, or range-sides).

We shall use in the following text also the Dirac notation for vectors and operators in a complex Hilbert space: $|x\rangle := x \in \mathcal{H}$ will denote a vector, $\langle x|y \rangle$ is the scalar product of such vectors (linear in the second factor), and $|x\rangle\langle y|$ is the operator of one-dimensional range such, that $|x\rangle\langle y| : \sum_j c_j |z_j\rangle \mapsto |x\rangle\langle y| : \sum_j c_j |z_j\rangle := \left(\sum_j c_j \langle y|z_j \rangle\right) |x\rangle$.

The constructions needed in the proof of the main theorem use also a more detailed description of consequences of “proximity” of two projections described in the following

2.4 Lemma. Let $E$, $F$ be two orthogonal projections of finite-dimensional ranges of equal dimensions $N := \dim E = \dim F := Tr(E)$ in an infinite-dimensional Hilbert space $\mathcal{H}$. 
Assume that $E \land F = 0$, i.e. the subspaces $\mathcal{E} := E \mathcal{H}$ and $\mathcal{F} := F \mathcal{H}$ have no nonzero common vectors. Let us also denote $\mathcal{E} \lor \mathcal{F} := \mathcal{E} + \mathcal{F} = (E \lor F) \mathcal{H}$ the $2N$-dimensional linear hull in $\mathcal{H}$ of $\mathcal{E} \cup \mathcal{F}$. Let

$$\text{Tr}[(E - F)^2] \equiv \|E - F\|^2 < 2. \quad (2.2)$$

Then:

(i) For any one-dimensional projections given by normalized vectors $e \in \mathcal{E}, f \in \mathcal{F}$: $|e\rangle\langle e| =: P_e \leq E$ (i.e. $P_e \cdot E = P_e$), and $|f\rangle\langle f| =: P_f \leq F$, it is: $P_e \cdot F \neq 0$, and $P_f \cdot E \neq 0$.

(ii) There exists an orthonormal basis $\{e_j : j = 1, 2, \ldots, N := \text{dim} E\} \subset \mathcal{H}$ in $\mathcal{E}$, i.e. $\sum_j P_{e_j} = E$, such that one can find to it an orthonormal basis of $\mathcal{F}$: $\{f_j : j = 1, 2, \ldots, N\} \subset \mathcal{H}$ (i.e. $\sum_j P_{f_j} = F$), satisfying the relations

$$P_{f_j}(E - P_{e_j}) = 0, \quad P_{e_j}(F - P_{f_j}) = 0, \quad \forall j. \quad (2.3)$$

(iii) This means that these orthonormal systems $\{e_j : j = 1, 2, \ldots, N\}$, and $\{f_j : j = 1, 2, \ldots, N\}$, decomposing $E$ and $F$, are in a certain strong sense mutually “affiliated”:

$$F|e_j\rangle = |f_j\rangle\langle f_j|e_j\rangle, \quad \forall j = 1, 2, \ldots N, \ 0 \neq \langle f_j|e_j\rangle \in \mathbb{C}, \ \langle e_j|e_j\rangle \equiv 1 \equiv \langle f_j|f_j\rangle, \quad (2.4)$$

i.e. from a specific orthonormal ‘decomposition’ $\{e_j : j = 1, 2, \ldots, N\}$ of $\mathcal{E}$ the orthonormal system $\{f_j : j = 1, 2, \ldots, N\}$ ‘decomposing’ $\mathcal{F}$ and satisfying (2.3) is obtained, uniquely up to a nonzero numerical factor, simply by element-wise orthogonal projections of $e_j$’s onto $\mathcal{F} := F\mathcal{H}$.

(iv) The above mentioned specific orthonormal basis $\{e_j : j = 1, 2, \ldots, N\}$ determines also (up to ‘phase factors’) an orthonormal basis $\{e_j^+ : j = 1, 2, \ldots, N\}$ of $\mathcal{E}^\perp := (E \lor F) - E] \mathcal{H} = \mathcal{E} \lor \mathcal{F} \ominus \mathcal{E}$, such that $f_j = \alpha_j e_j + \beta_j e_j^+, \ \alpha_j \cdot \beta_j \neq 0, \ (\forall j)$. ♦

Proof.

(i): Let there be a projection $P_e \leq E$ such that $P_e F = 0$. Let $e_1 := e$, and let $\{e_j : j = 1, 2, \ldots N\}$ be an orthonormal system decomposing $E$, $E = \sum_{j=1}^N P_{e_j}$. Then

$$\text{Tr}(EF) = \text{Tr}[(E - P_e)F] = \sum_{j=2}^N \text{Tr}(P_{e_j} F) \leq N - 1, \quad (2.5)$$
since always it is $Tr(P_2 F) \leq 1, \forall x \in \mathcal{H}$. The estimate (2.5) would be then in contradiction with the assumption (2.2), since $Tr[(E - F)^2] = 2(N - Tr(EF))$. Due to the symmetry of the assumed conditions with respect to the exchange $E \leftrightarrow F$, one obtains also $P_IF \neq 0$. This implies validity of (i).

(ii): We have to prove existence of the bases $\{e_j\} := \{e_j : j = 1, 2, \ldots N := \dim E\}$, and $\{f_j : j = 1, 2, \ldots N = \dim F\}$ of $\mathcal{E}$, resp. $\mathcal{F}$ satisfying (2.3).

This means to find an orthonormal basis $\{e_j : j = 1, 2, \ldots N\}$ of $\mathcal{E}$ such, that its element-wise projections are proportional to $f_j$’s, cf. (2.4). This also means that for such a basis $\{e_j\} \subset \mathcal{E}$ the projections $F|e_j\rangle \in \mathcal{F}$ are nonzero and mutually orthogonal.

The statement (i) ensures that all the projections $F|e\rangle$ of all nonzero vectors $e \in \mathcal{E}$ are nonzero, i.e. that the restriction $EFE \in L(\mathcal{E})$ of the projector $F$ to the subspace $\mathcal{E} \subset \mathcal{H}$ has trivial kernel: Ker$_\mathcal{E}(EFE) = 0$. This implies that the bounded operator $EFE = (FE)^*FE$ on $\mathcal{E}$ is strictly positive and there is an orthonormal basis $\{e_j\}$ of $\mathcal{E}$ in which the matrix $\langle e_j|EFE|e_k\rangle = \langle e_j|F|e_k\rangle$ is diagonal, with strictly positive diagonal elements $\|F e_j\|^2$.

Let us define then, e.g., $f_j := \|F e_j\|^{-1} \cdot F e_j, \ j = 1, 2, \ldots, N$; these elements form the wanted decomposition of $\mathcal{F}$, resp. of the projector $F$ satisfying together with the just found basis $\{e_j\}$ the relations (2.3). This proves (ii).

(iii): That statement is just a rephrasing of (ii); the uniqueness also is seen from (2.3).

(iv): Since each $f_j \in \mathcal{F}$ constructed as above is orthogonal to all the $e_k (k \neq j)$, and $\langle f_j|e_j\rangle \neq 0$, but it is also $E^\perp f_j \neq 0$, with $E^\perp := E \vee F - E$, $f_j$ is expressible in the form

$$f_j := \alpha_j e_j + \beta_j e_j^\perp, \ \forall j,$$

where $e_j^\perp \in \mathcal{E}^\perp := E^\perp \mathcal{H}$ is some normalized vector determined by $f_j$ up to a ‘phase factor’, e.g.: $e_j^\perp := \|E^\perp f_j\|^{-1} E^\perp f_j$.

We also see that all $\alpha_j \cdot \beta_j \neq 0$, since all $f_j \notin \mathcal{E}$, but also $f_j \notin \mathcal{E}^\perp$.

The orthogonality between the vectors $f_j$’s: $\langle f_j|f_k\rangle \equiv \delta_{jk}$ implies also the orthogonality relations for $e_j^\perp$’s: $\langle e_j^\perp|e_k^\perp\rangle = \delta_{jk}$. \hfill \Box

The following lemma is an illustration of one of the main tools used in the proof of the forthcoming theorem:

**2.5 Lemma.** Let $E, F$ be two orthogonal projections in an infinite-dimensional complex Hilbert space $\mathcal{H}$ of the same finite dimension $N = Tr(E) = Tr(F)$. Let us choose $0 < \epsilon < 2$, and assume that $N - Tr(EF) < \epsilon^2/4 \ (< 1)$. Then there is a unitary operator $u \in \mathcal{U}$ such that $\|u - I_{\mathcal{H}}\| < \epsilon$, and that $F = u E u^*$. \hfill \ blacksquare
Proof. Let us denote \( Q := E \wedge F, \) \( E' := E - Q, \) \( F' := F - Q, \) \( N' := \text{Tr}(E') = \text{Tr}(F') = N - \dim Q, \) \( E'^\perp := (E' \vee F') - E', \) \( F'^\perp := (E' \vee F') - F'. \) Then \( E' \wedge F' = E'^\perp \wedge F'^\perp = 0. \)

We have now also \( \text{Tr}(E'^\perp) = \text{Tr}(F'^\perp) = N' =: N'^\perp. \) Moreover,

\[
1 > e^2/4 > N - \text{Tr}(EF) = \frac{1}{2} \text{Tr}((E - F)^2) = \frac{1}{2} \text{Tr}((E' - F')^2) = N' - \text{Tr}(E'F') = \frac{1}{2} \text{Tr}((E'^\perp - F'^\perp)^2) = N'^\perp - \text{Tr}(E'^\perp F'^\perp). \tag{2.7}
\]

We can now apply Lemma 2.4 to the both couples, i.e. to \((E'; F'),\) as well as to \((E'^\perp; F'^\perp),\) of projections. Let \( \{e_j : j = 1, 2, \ldots N'\}, \) resp. \( \{e_j^\perp : j = 1, 2, \ldots N'\} \) be the orthonormal decompositions of \( \mathcal{E}' := E'\mathcal{H}, \) resp. \( \mathcal{E}'^\perp := (E' \vee F' - E')\mathcal{H}, \) with the corresponding orthonormal decompositions \( \{f_j : j = 1, 2, \ldots N'\} \) of \( \mathcal{F}' := F'\mathcal{H}, \) resp. \( \{f_j^\perp : j = 1, 2, \ldots N'\} \) of \( \mathcal{F}'^\perp := F'^\perp\mathcal{H} \) constructed according to Lemma 2.4. Because the normalized vectors \( f_j, f_j^\perp \) remained specified, according to Lemma 2.4, up to arbitrary phase factors, we shall choose them so that the scalar products \( \langle f_j | e_j \rangle > 0, \) \( \langle f_j^\perp | e_j^\perp \rangle > 0. \) Remember also that for \( j \neq k : \langle f_j | e_k \rangle = 0, \) \( \langle f_j^\perp | e_k^\perp \rangle = 0. \) We have constructed two orthonormal decompositions of the space \( \mathcal{E}' \vee \mathcal{F}' := (E' \vee F')\mathcal{H}, \) i.e. \( \{e_j, e_j^\perp : j = 1, 2, \ldots N'\}, \) as well as \( \{f_j, f_j^\perp : j = 1, 2, \ldots N'\}. \) We could also define formally \( \{e_j = f_j : j = N' + 1, \ldots N\} \) as an arbitrary orthonormal decomposition of \( Q = E - E' = F - F', \) but it will not be used now.

Let us define now the wanted unitary operator \( u \in \mathcal{U} \) by:

\[
ux := x \quad \text{for } x \in \mathcal{H} \ominus (\mathcal{E}' \vee \mathcal{F}'); \quad ue_j := f_j, \quad ue_j^\perp := f_j^\perp \quad (j = 1, 2, \ldots N'), \tag{2.8}
\]

and this prescription is completed by linearity to a unique unitary operator \( u \) on \( \mathcal{H}. \) Let us prove that this operator has the wanted property. It is clear that \( uE^*u^* = F: \) \( uE^*u^* = uQu^* + uE^*u^* = Q + \sum_{j=1}^{N'} u|e_j\rangle \langle e_j|u^* = Q + \sum_{j=1}^{N'} |f_j\rangle \langle f_j| = Q + F' = F. \) Since on the complement of the finite-dimensional subspace \( \mathcal{E}' \vee \mathcal{F}' \) of \( \mathcal{H} \) the operator \( u \) coincides with \( I_\mathcal{H}, \) their difference \( u - I_\mathcal{H} \) can be nonzero just on the finite dimensional subspace \( \mathcal{E}' \vee \mathcal{F}' \). Hence the norm \( \|u - I_\mathcal{H}\| \) can be calculated as the norm of the restriction to the subspace \( \mathcal{E}' \vee \mathcal{F}', \) and we can deal with this operator \( u - I_\mathcal{H} \) as with a finite-dimensional matrix. Or, the operator \( u - I_\mathcal{H} \) is of finite rank in \( \mathcal{H}. \) Let us denote \( Tr'(C) \) the trace of the restriction of \( C \in \mathcal{L}(\mathcal{H}) \) to the \( 2N' \)-dimensional subspace \( \mathcal{E}' \vee \mathcal{F}' : \) \( Tr'(C) := Tr[(E' \vee F')C]. \) We have
∥u − I_{\mathcal{H}}∥^2 \leq ∥u − I_{\mathcal{H}}∥^2 = Tr'[\{u^* − I_{\mathcal{H}}\}(u − I_{\mathcal{H}})] = Tr'[2I_{\mathcal{H}} − u − u^*] =

\begin{align*}
4N' &− \sum_{j=1}^{N'} [\langle e_j | f_j \rangle + \langle e_j^\perp | f_j^\perp \rangle + \langle f_j | e_j \rangle + \langle f_j^\perp | e_j^\perp \rangle] = \\
4N' &− 2\sum_{j=1}^{N'} [\|\langle e_j | f_j \rangle\| + \|\langle e_j^\perp | f_j^\perp \rangle\|],
\end{align*}

(2.9)
due to the chosen positivity of the scalar products \(\langle e_j | f_j \rangle\), \(\langle e_j^\perp | f_j^\perp \rangle\). According to (2.7), and also from the orthogonality properties of the sets of chosen vectors \(\{e_j, e_j^\perp, f_j, f_j^\perp : j = 1,2,\ldots N'\}\), and because it is \(|\langle e | f \rangle|^2 \leq |\langle e | f \rangle| \leq 1\) for scalar product of any two normalized vectors \(e, f\) in \(\mathcal{H}\), one has

\begin{align*}
2N' &− \sum_{j=1}^{N'} [\|\langle e_j | f_j \rangle\| + \|\langle e_j^\perp | f_j^\perp \rangle\|] \leq 2N' − \sum_{j=1}^{N'} [\|\langle e_j | f_j \rangle\|^2 + \|\langle e_j^\perp | f_j^\perp \rangle\|^2] = \\
(N' − Tr(E' F')) + (N'^\perp − Tr(E'^\perp F'^\perp)) < \frac{\epsilon^2}{2}.
\end{align*}

(2.10)

We have obtained, according to (2.9), the wanted estimate ∥u − I_{\mathcal{H}}∥^2 < \epsilon^2.

We are prepared now to prove the regularity of embeddings into \(\mathfrak{T}_s\) of unitary orbits through finite-rank symmetric operators.

2.6 Theorem. Let \(0 \neq \rho = \rho^* \in \mathfrak{F}\) (:=the set of all finite-rank operators on \(\mathcal{H}\)), \(\mathcal{O}_\rho(\mathfrak{U}) := \{u \rho u^* : u \in \mathfrak{U}\} \subset \mathfrak{T}_s\). The unitary orbit \(\mathcal{O}_\rho(\mathfrak{U})\) is a regularly embedded [9, p. 550] submanifold of the Banach space \(\mathfrak{T}_s\) of symmetric trace-class operators endowed with its trace norm. ♦

Proof. The mapping \(\Pi_\rho : \mathfrak{U} \to \mathcal{O}_\rho(\mathfrak{U}), u \mapsto u \rho u^*\) is an analytic submersion [6, III.§1.6, Prop.11], and the inclusion \(\iota_\rho : \mathcal{O}_\rho(\mathfrak{U}) \to \mathfrak{T}_s\) is an injective immersion (cf. [3, Proposition 2.1.5]), hence the composition \(\iota_\rho \circ \Pi_\rho, \mathfrak{U} \to \mathfrak{T}_s\) is continuous. We want to prove, that the inverse (identity) mapping \(\iota_\rho^{-1} : \mathcal{O}_\rho(\mathfrak{U}) \subset \mathfrak{T}_s \to \mathcal{O}_\rho(\mathfrak{U}) := \{\mathfrak{U}/\mathfrak{U}_\rho\}\) is also continuous, if the “domain copy” \(\mathcal{O}_\rho(\mathfrak{U})\) of \(\iota_\rho^{-1}\) is taken in the relative topology of the corresponding “ambient” space \(\mathfrak{T}_s \subset L^1(\mathcal{H})\). Because of the invariance of all the relevant metrics with respect to the unitary group action (including their invariance in the “ambient” normed spaces), and also because of the continuity of the projection \(\Pi_\rho\), it suffices to prove the wanted continuity in an arbitrary point \(\rho\) of the orbit by showing the following: To any...
positive $\epsilon > 0$ one can find a $\delta' > 0$ such, that if there is some element $\rho' = u\rho u^* \in \mathcal{O}_\rho(\mathcal{U})$ in the $\delta'$—neighbourhood of $\rho$ in the space $\mathcal{T}_s$ : $\|\rho - u\rho u^*\|_1 < \delta'$, then it is possible to find a unitary $v \in \mathcal{U}$ : $\|v - I_\mathcal{H}\| < \epsilon$, such that $v\rho v^* = u\rho u^*$.

Now we can use, for the sake of simplicity of our expression, that the orbit $\mathcal{O}_\rho(\mathcal{U})$ is also a strong riemannian manifold [3, Thm. 2.1.19] with a distance-function $d_\rho(\rho', \rho'')$ generating the topology of $\mathcal{U}/\mathcal{U}_\rho$ (cf. [8, Proposition 4.64]). Now (due to the continuity of $\Pi_\rho$), to any $\epsilon' > 0$ there is an $\epsilon > 0$ such that if $\|v - I_\mathcal{H}\| < \epsilon$, then $d_\rho(\rho, v\rho v^*) < \epsilon'$. We have to prove that, to this $\epsilon$, there exists the corresponding $\delta' > 0$ such that $\|\rho - u\rho u^*\|_1 < \delta' \Rightarrow d_\rho(\rho, u\rho u^*) < \epsilon'$, what means the desired continuity. The proof will be direct: A construction of a unitary $v$ : $\|v - I\| < \epsilon$ for any given $\rho' = u\rho u^*$ lying “sufficiently close” to $\rho$ in $\mathcal{T}_s$, such that it is also $\rho' = v\rho v^*$.

Let us write $\rho = \sum_{j=1}^n \lambda_j E_j$, $0 < n < \infty$, where $\lambda_j \neq \lambda_k$ for $j \neq k$, $E_j$ are the orthogonal projections of the spectral measure of $\rho = \rho^*$, $0 < \dim E_j := Tr(E_j) := N_j < \infty$ ($\forall j \neq 0$), $E_0 := I_\mathcal{H} - \sum_{j=1}^n E_j =: I_\mathcal{H} - E$, $\lambda_0 := 0$, $\sum_{j=1}^n N_j =: N$. Let us denote $F_j := uE_ju^*$ ($\forall j$), hence $\rho' := u\rho u^* = \sum_j \lambda_j F_j$, and also $F := \sum_{j=1}^n F_j$.

It is clear that the nonnegative numbers $N_j - Tr(E_j F_j(\rho'))$ and $N - Tr(E F(\rho'))$ are all continuous functions of $\rho'$, and for $\rho' = \rho$ they are all zero. This can be seen, e.g. by representing the projection operators $F_j \equiv F_j(\rho')$ by polynomials $p_j$ of the operators $\rho'$, as it was done in Lemma 2.2.

These considerations imply that, for all sufficiently small $\delta' > 0$, and for all such $\rho' = u\rho u^*$ that $\|\rho - u\rho u^*\|_1 < \delta'$, one obtains

$$0 \leq N_j - Tr(E_j F_j(\rho')) =: \delta_j < 1, \ j = 1, 2, \ldots n;$$
$$0 \leq N - Tr(E F(\rho')) =: \delta < 1,$$  \hspace{1cm} (2.11)

where $\delta$, $\delta_j$ ($j = 1, 2, \ldots n$) can be chosen arbitrarily small positive numbers (i.e. they can be bounded from above by arbitrarily small positive upper bounds determining the choice of the mentioned $\delta' > 0$, what is possible due to the continuous dependence on $\rho'$ of the expressions entering into (2.11)).

Let us choose now $0 < \epsilon < 1$, and assume that the above mentioned $\delta'$ is such\(^4\) that

$$\delta \leq \sum_{j=1}^n \delta_j < \epsilon^2/4,$$  \hspace{1cm} (2.12)

\(^4\)We need not here any explicit expression for the dependence $\epsilon \mapsto \delta' \equiv \delta'(\epsilon)$; it could be ‘in principle’ obtained, however, from explicit formulas for the functions $\rho' \mapsto F_j(\rho')$, e.g. from those given in the proof of Lemma 2.2.
where the first inequality is a consequence of the definitions (2.11).

We shall now construct, for any \( \rho' = u \rho u^* \) with \( \| \rho' - \rho \|_1 < \delta' \), such a unitary \( v \in \mathfrak{U} \), that \( v \rho v^* = u \rho u^* \), and simultaneously \( \| v - I_\mathcal{H} \| < \epsilon \).

Let us denote \( Q_j := E_j \cap F_j, \quad E'_j := E_j - Q_j, \quad F'_j := F_j - Q_j, \quad Q := E \cap F, \quad E' := E - Q, \quad F' := F - Q, \quad E'_{\perp} := (E' \cap F') - E' = E \cap F' - E, \quad F'_{\perp} := (E' \cap F') - F' = E \cap F' - F, \quad N'_j := \dim E_j - \dim Q_j = \dim E'_j = \dim F'_j, \quad N' := \dim E - \dim Q = \dim E' = \dim F' = \dim E'_{\perp} = \dim F'_{\perp}. \) Observe that \( (E - F)^2 = [(E \cap F) - (E \cap F')]^2 = (E'_{\perp} - F'_{\perp})^2. \) Also it is \( Tr(EF) = Tr(E'F' + Q) = Tr(E'F') + N - N', \) and \( \dim(E \cap F) = N + N'. \) So that we obtain

\[
Tr[(E - F)^2] = 2[N - Tr(EF)] = Tr[(E'_{\perp} - F'_{\perp})^2] = 2[N' - Tr(E'_{\perp}F'_{\perp})]. \quad (2.13)
\]

Now we can apply Lemma 2.4 separately to each of the couples of projections

\[
(E'_j; F'_j), \quad j = 1, 2, \ldots n; \quad (E'_{\perp}; F'_{\perp}), \quad (2.14)
\]

and construct the orthonormal systems \( \{e^{(j)}_k : k = 1, 2, \ldots N'_j\} \) forming the convenient bases of every \( \mathcal{E}'_j := E'_j \mathcal{H} \) \( (j = 1, 2, \ldots n) \), and also the basis \( \{e^\perp_k : k = 1, 2, \ldots N'\} \) of \( \mathcal{E}'_{\perp} := E'_{\perp} \mathcal{H} \), such that their respective orthogonal projections onto the spaces \( \mathcal{F}'_j := F'_j \mathcal{H} \) \( (j = 1, 2, \ldots n) \), and \( \mathcal{F}'_{\perp} := F'_{\perp} \mathcal{H} \), corresponding to the second projection in the considered pair of (2.14), are the orthogonal (and afterwards normalized) bases \( \{f^{(j)}_k : k = 1, 2, \ldots N'_j\} \) of \( \mathcal{F}'_j \) \( (j = 1, 2, \ldots n) \), and the orthonormal basis \( \{f^\perp_k : k = 1, 2, \ldots N'\} \) of \( \mathcal{F}'_{\perp} \). Let us choose any orthonormal bases \( \{e^{(j)}_k \equiv f^{(j)}_k : k = N'_j + 1, \ldots N_j\} \) of all the subspaces \( Q_j := Q_j \mathcal{H}, \quad j = 1, 2, \ldots n. \) We have obtained in this way two orthonormal systems \( \{e^{(j)}_k, e^\perp_i : k = 1, 2, \ldots N_j, \quad j = 1, 2, \ldots n, \quad i = 1, 2, \ldots N'\}, \) and \( \{f^{(j)}_k, f^\perp_i : k = 1, 2, \ldots N_j, \quad j = 1, 2, \ldots n, \quad i = 1, 2, \ldots N'\}, \) each forming a basis of the subspace \( \mathcal{E} \cap \mathcal{F} := (E \cap F) \mathcal{H}. \) Remember also the “cross-orthogonality” of the mutually “affiliated” orthonormal systems:

\[
\langle f^{(j)}_k | e^\perp_i \rangle = 0 \quad (j = 1, 2, \ldots n), \quad \langle f^\perp_i | e^{(j)}_l \rangle = 0, \quad \text{for} \quad l \neq k \quad (\forall k, l). \quad (2.15)
\]

Let also the arbitrary phase factors at the all \( f's \) entering into the orthonormal sets be chosen so that for all possible values of the indices it is

\[
\langle f^\perp_i | e^\perp_i \rangle > 0, \quad \langle f^{(j)}_k | e^{(j)}_k \rangle > 0. \quad (2.16)
\]
Now we shall define the wanted unitary \( v \): Let the restriction of \( v \) to \( \mathcal{H} \ominus (\mathcal{E} \vee \mathcal{F}) := (\mathcal{E} \vee \mathcal{F})^\perp \) be the identity (i.e. \( v\rvert_{\mathcal{H} \ominus (\mathcal{E} \vee \mathcal{F})} := I_{\mathcal{H} \ominus (\mathcal{E} \vee \mathcal{F})} \)), and its restriction to \( \mathcal{E} \vee \mathcal{F} \) is defined as the linear transformation between the constructed orthonormal systems forming two bases in \( \mathcal{E} \vee \mathcal{F} \) specified by:

\[
ve^{(j)}_k := f^{(j)}_k, \quad ve^\perp_i := f^\perp_i; \quad \forall i, j, k. \tag{2.17}
\]

It is clear from this definition of \( v \), esp. from (2.17), that \( \sum_{j=1}^n \lambda_j F_j = v(\sum_{j=1}^n \lambda_j E_j)v^* \), i.e. \( \rho' = v\rho v^* \). Let us show next, that \( \|v - I_{\mathcal{H}}\| < \epsilon \). Since \( (v - I_{\mathcal{H}})\rvert_{\mathcal{H} \ominus (\mathcal{E} \vee \mathcal{F})} = 0 \), we shall estimate the Hilbert-Schmidt norm of \( (v - I_{\mathcal{H}}) \) in the subspace \( \mathcal{E} \vee \mathcal{F} \). Let \( Tr' (C) \) will be the trace of the restriction of \( C \in \mathcal{L}(\mathcal{H}) \) to \( \mathcal{E} \vee \mathcal{F} \). We obtain with a help of (2.16):

\[
\|v - I_{\mathcal{H}}\|_2^2 = Tr' (2I_{\mathcal{H}} - v - v^*) = 2(N + N') - 2 \sum_{j=1}^n \sum_{k=1}^{N_j} \langle f^{(j)}_k | e^{(j)}_k \rangle^* - 2 \sum_{j=1}^{N'} \langle f^\perp_j | e^\perp_j \rangle^* \Rightarrow 2 \sum_{j=1}^n \sum_{k=1}^{N_j} [N_j - \sum_{k=1}^{N_j} \langle f^{(j)}_k | e^{(j)}_k \rangle^*] + 2 \sum_{j=1}^{N'} [N' - \sum_{j=1}^{N'} \langle f^\perp_j | e^\perp_j \rangle^*] \leq 2 \sum_{j=1}^n [N_j - Tr(E_j F_j)] + 2 \sum_{j=1}^{N'} [N' - Tr(E'^\perp F'^\perp)] \leq 2 \sum_{j=1}^n [N_j - Tr(E_j F_j)] + 2 \sum_{j=1}^{N'} [N' - Tr(E F)], \tag{2.18}
\]

where we have used again the orthogonality properties (2.15) of the vectors inside each “block” corresponding to \( E_j, j = 1, 2, \ldots n \), as well as to \( E'^\perp : \sum_{j=1}^n E_j + E'^\perp = E \vee F \); the fact that \( |\langle f | e \rangle|^2 \leq \langle f | e \rangle \) for any normalized vectors \( e, f \in \mathcal{H} \), and also the relation (2.13).

Now we shall use the definitions (2.11), and the assumption (2.12). We obtain:

\[
\|v - I_{\mathcal{H}}\|_2^2 \leq \|v - I_{\mathcal{H}}\|_2^2 \leq 2 \sum_{j=1}^n \delta_j + 2\delta \leq 4 \sum_{j=1}^n \delta_j < \epsilon^2, \tag{2.19}
\]

what is the desired result.
Hence, each orbit of the coadjoint action of $\mathfrak{u}$ going through density matrices with only finite number of different eigenvalues is a submanifold of $\mathfrak{t}_s$: There is an open neighbourhood of any point $\nu$ of $\mathcal{O}_\rho(\mathfrak{u}) = \mathfrak{u}/\mathfrak{u}_\rho$ which coincides with intersection of the embedded $\mathcal{O}_\rho(\mathfrak{u})$ into $\mathfrak{t}_s$ with an open neighbourhood of the point $\nu$ in $\mathfrak{t}_s$.

Another possibility of proving this theorem is indicated in the next Section, where such a proof for the specific case of $\mathcal{O}_\rho(\mathfrak{u}) := P(\mathcal{H})$ is given.

3 Some other related results

To give here a proof of the promised closeness of the unitary coadjoint orbit going through any symmetric trace-class operator of finite rank, we shall use an encoding of the spectral invariants (i.e. the spectra, and their multiplicities) of these operators into finite positive measures on $\mathbb{R}$:

3.1 Proposition. The unitary orbits $\mathcal{O}_\rho(\mathfrak{u})$ for finite-rank $\rho \in \mathfrak{t}_s$ are closed subsets of $\mathfrak{t}_s$.

Proof. Let us take now the smooth (although differentiability will not be exploited here) numerical functions $\rho \mapsto a_n(\rho) := Tr(\rho^{n+2})$ determined for all symmetric trace-class operators $\rho \in \mathfrak{t}_s$. It is claimed that fixing the infinite sequence $\{a_n(\rho), n = 0, 1, 2, \ldots\}$ of real numbers one can determine the unitary orbit $\mathcal{O}_\rho(\mathfrak{u}) \subset \mathfrak{t}_s$ (on which the numbers $a_n$ are constant: $a_n(\rho u^* u) \equiv a_n(\rho)$, $\forall u \in \mathfrak{u}, \rho \in \mathfrak{t}_s$) uniquely. This can be seen as follows: The orbit $\mathcal{O}_\rho(\mathfrak{u})$ for a finite-rank $\rho$ is determined by the spectral invariants of any $\nu \in \mathcal{O}_\rho(\mathfrak{u})$, i.e. by its nonzero eigenvalues and their multiplicities. These might be, however, determined by a measure $\mu_\rho$ on $\mathbb{R}$, namely the (not normalized) measure given by the characteristic function $t(\in \mathbb{R}) \mapsto Tr(\rho^2 e^{it\rho})$, the moments of which are exactly the numbers $a_n(\rho)$.

That measure expressed by the nonzero eigenvalues $\lambda_j$ of $\rho$, and their multiplicities $m_j$, has the form

$$\mu_\rho = \sum_j \lambda_j^2 \cdot m_j \cdot \delta_{\lambda_j}, \quad (3.1)$$

where $\delta_{\lambda}$ is the Dirac probabilistic measure concentrated in the point $\lambda$. It is clear that this measure $\mu_\rho$ determines the orbit uniquely. The uniqueness of the solution of the Hamburger problem of moments (see [10, Theorem X.4, and Example 4 in Chap. X.6]) for the moments given by the sequence $\{a_n(\rho), n = 0, 1, 2, \ldots\}$ proves, that the measure $\mu_\rho$ is in turn determined by the sequence $\{a_n(\rho)\}$ uniquely.

5Compare, however, also Proposition 2.1.5 in [3].
Since the functions $\rho \mapsto a_n(\rho)$ are continuous in the trace (and even Hilbert-Schmidt, and on bounded balls in $\mathfrak{T}_s$ also in the operator $\mathcal{L}(\mathcal{H})$- topology, the intersection of the (closed) inverse images $a_n^{-1}[a_n(\rho)]$ ($n \in \mathbb{Z}_+$):

$$O_\rho(\Sigma) = \bigcap_{n=0}^{\infty} \{ \nu \in \mathfrak{T}_s : a_n(\nu) = a_n(\rho) \}$$

(3.2)

is a closed subset of $\mathfrak{T}_s$ in these (induced) topologies.

Next will be given an independent way of proving the above Theorem 2.6, but only for a specific case of the orbit $O_\rho(\Sigma)$ with $\rho = P_x$, i.e. for the projective Hilbert space $P(\mathcal{H})$. A use of that method for other orbits $O_\rho(\Sigma)$ would need calculation of the distance functions $d_\rho(u_\rho u^*, v_\rho v^*)$ on the riemannian manifolds $O_\rho(\Sigma)$ for a general $\rho$ of finite range.

3.2 Proposition. The unitary orbit $O_\rho(\Sigma)$ going through a one dimensional projection $\rho := P_x$ ($0 \neq x \in \mathcal{H}$) is a submanifold of (i.e. it is regularly embedded into) the space $\mathfrak{T}_s$ of symmetric trace-class operators. *\

Proof. It is known, that the riemannian distance function on $P(\mathcal{H})$ is (cf., e.g., the formula (3.2.11) in [3]):

$$d(P_x, P_y) = \sqrt{2 \arccos \sqrt{\text{Tr}(P_x P_y)}}.$$  (3.3)

On the other hand, the distance between the same projections in the “ambient space” $\mathfrak{T}_s$ is

$$\text{Tr}|P_x - P_y| = 2[1 - \text{Tr}(P_x P_y)]^{1/2},$$  (3.4)

what is easily obtained as the sum of absolute values $|\lambda_1| + |\lambda_2|$ of the two nonzero real eigenvalues (if $P_x \neq P_y$: choose $\lambda_1 \geq \lambda_2$) of $P_x - P_y$: Since $\text{Tr}(P_x - P_y) = \lambda_1 + \lambda_2 = 0$, one has $\lambda_1 = -\lambda_2 =: \lambda > 0$. Because $2\lambda^2 = \text{Tr}[(P_x - P_y)^2] = 2[1 - \text{Tr}(P_x P_y)]$, one obtains $\lambda = \sqrt{1 - \text{Tr}(P_x P_y)}$, hence the result (3.4). We see that these two metrics are mutually equivalent.

This implies that the convergence of some sequence $\{P_{y_n} : n \in \mathbb{Z}_+\}$ of points of this orbit to a chosen point $P_x \in O_\rho(\Sigma)$ in the space $\mathfrak{T}_s$ means also its convergence on the orbit $O_\rho(\Sigma)$, what gives the wanted continuity of the inverse $\iota^{-1}$ of the injective immersion (it was proved earlier in [3] that $\iota$ is an immersion) $\iota : \Sigma/O_\rho = O_{P_x}(\Sigma) \to P(\mathcal{H}) \subset \mathfrak{T}_s$ (the set $P(\mathcal{H})$ is taken here in the relative topology of $\mathfrak{T}_s$). This means, that the injection $\iota$ is a homeomorphism, hence $P(\mathcal{H})$ is a submanifold (cf. [7]) of $\mathfrak{T}_s$.  

\textsuperscript{6}Remember that (cf. [3]) for $\rho \in \mathfrak{T}_s$ with infinite range the claim of Theorem 2.6 is false!
It might be useful to formulate an easy generalization of Lemma 2.4. One can see that the condition (2.2) of “proximity” of the two projections $E, F$ was used in the proof of that lemma for proving the item (i) only. Assuming the conclusion (i), one can formulate a generalization of Lemma 2.4 valid also for infinite-dimensional projections, and without any restriction to their mutual “proximity”:

### 3.3 Proposition

Let $E, F$ be two orthogonal projections in a separable (real, or complex) Hilbert space $\mathcal{H}$ with mutually isomorphic ranges: $E\mathcal{H} \sim F\mathcal{H}$. Assume that $E \wedge F = 0$, and that for any one-dimensional projections $P_e \leq E$, and $P_f \leq F$ it is

$$P_e \cdot F \neq 0, \quad P_f \cdot E \neq 0. \quad (3.5)$$

Let also the spectrum of $EFE$ be pure-point (i.e. the eigenvectors form a basis of $\mathcal{H}$).

Then there is an orthonormal decomposition of $E$ to one-dimensional projections $E = \sum_j P_{e_j}$ (the sum is strongly convergent), to which there is a unique orthogonal one-dimensional decomposition of $F : \sum_j P_{f_j} = F$ such that

$$P_{f_j} P_{e_k} = 0 \quad \text{for } j \neq k, \quad P_{f_j} P_{e_j} \neq 0, \quad (3.6)$$

for all values of the indices. ♣

**Proof.** The validity of the proposition in the case of $\dim E = \dim F < \infty$ is seen from the proof of Lemma 2.4. In our case, the proof of the Lemma 2.4 can be essentially used as a first step for proving our claims also for infinite dimensions. Let $\dim E = \dim F = \infty$. The operator $EFE$ restricted to $\mathcal{E} := E\mathcal{H}$ has trivial kernel: $\ker_{\mathcal{E}}(EFE) = 0$, due to the assumption (3.5). Let an orthonormal basis in the subspace $\mathcal{E} := E\mathcal{H}$ consisting of the eigenvectors of $EFE$ be $\{e_j \in \mathcal{H} : j \in \mathbb{N}\}$. It exists because $EFE$ has pure point spectrum. The basis $\{e_j\}$ also determines an orthonormal decomposition $\{P_{e_j}\}$ of $E$.

Then the vectors

$$f_j := \|Fe_j\|^{-1}Fe_j, \quad \forall j \in \mathbb{N} \quad (3.7)$$

form an orthonormal system in $\mathcal{F} := F\mathcal{H}$: $\langle f_j | f_k \rangle \propto \langle Fe_j | Fe_k \rangle = \langle e_j, EFEe_k \rangle$ ($\forall j, k$). Let $P_{f_j}$ be the one-dimensional orthogonal projections onto subspaces of $\mathcal{H}$ spanned by $f_j$’s, and define $F_n := \sum_{j=1}^n P_{f_j}$ ($\leq F$). Let also $E_n := \sum_{j=1}^n P_{e_j}$ ($\leq E$).

The projections $E_n$ and $F_n$ are both (finite) $n$-dimensional and fulfill the assumptions of the proposition (by obvious interchange $E \leftrightarrow E_n$, $F \leftrightarrow F_n$). Also it is $Fe_j = F_ne_j$, $j = 1, 2, \ldots, n$, so that the presently defined $P_{f_j}$’s coincide with those obtained according to
the proof of Lemma 2.4. It is clear that also the orthogonality relations (3.6) are fulfilled. It remains to show that

\[ F = s - \lim_{n \to \infty} \sum_{j=1}^{n} P_{f_j} := \bigvee_{n=1}^{\infty} F_n. \] (3.8)

Obviously, it is \( \bigvee_{n=1}^{\infty} F_n \leq F \). We have to prove equality in this relation. Assume that there is a one-dimensional projection \( P_f \leq F \) orthogonal to all \( F_n : P_f \cdot F_n \equiv 0 \). Since, according to (3.5), \( P_f E \not\equiv 0 \), there is at least one \( e_k \) contained in the given orthonormal system \( \{ e_j \in \mathcal{H} : j \in \mathbb{N} \} \) such that \( P_f e_k \not\equiv 0 \). But \( FP_f = P_f F = P_f \), and any vector \( f_k \not\equiv 0 \) corresponding to \( P_f f_k \) is \( f_k \propto F e_k = F_n e_k \) (\( \forall n \geq k \)). Consequently, for all \( n \geq k \) it is \( P_f F_n e_k = P_f F e_k = P_f e_k \not\equiv 0 \), what implies \( P_f F_n \not\equiv 0 \) (\( n \geq k \)). So that any assumed \( P_f \) orthogonal to all \( F_n \)'s does not exist, and the equality in (3.8) holds.

The uniqueness of \( \{ P_{f_j} : j \in \mathbb{N} \} \) corresponding to the decomposition \( \{ P_{e_j} : j \in \mathbb{N} \} \) of \( E \) and determined by eigenvectors \( \{ e_j \} \) of \( EFE \) in \( \mathcal{E} \), with the stated properties follows from the orthogonality relations (3.6): It is obtained by orthogonal projecting of the \( e_j \)'s onto \( \mathcal{F} : f_j \propto F e_j, \forall j \). This proves the proposition. \( \square \)

To see a rather weak connection of the derived properties of considered projections \( E, F \) with their previously discussed mutual “proximity”, we shall consider an explicit representation of these projections. It will show also in which way the point-spectrum of the restriction of \( EFE \) to \( \mathcal{E} := E \mathcal{H} \) can be made an arbitrary countable subset of the open interval \((0, 1) \subset \mathbb{R} \).

3.4 Example. Let \( E \) be an orthogonal projection in a complex Hilbert space \( \mathcal{H} \) and let \( \{ e_j : j \in J \} \) (with an index set \( J \) of cardinality \( \leq \aleph_0 \)) be an orthonormal basis in \( \mathcal{E} := E \mathcal{H} \). Let \( E^\perp \) be another orthogonal projection in \( \mathcal{H} \) with the same “dimension \( J \)” of \( \mathcal{E}^\perp := E^\perp \mathcal{H} \) and orthogonal to \( E : E \cdot E^\perp = 0 \). Let \( \{ e_j^\perp : j \in J \} \) be an orthonormal basis of \( \mathcal{E}^\perp \). Let us choose an arbitrary set of complex numbers \( \{ \alpha_j, \beta_j : j \in J \} \) such that \( \alpha_j \cdot \beta_j \not\equiv 0, |\alpha_j|^2 + |\beta_j|^2 = 1, \forall j \). Let us define in \( \mathcal{H} \) vectors \( f_j := \alpha_j e_j + \beta_j e_j^\perp, \forall j \in J \). The vectors \( \{ f_j : j \in J \} \) form an orthonormal basis in a subspace \( \mathcal{F} \subset \mathcal{H} \) with the orthogonal projection \( F : F \mathcal{H} = \mathcal{F} \). It is clear that the couple of projections \( (E; F) \) satisfies the assumptions of the Proposition 3.3, and that the specified sets of vectors \( \{ e_j : j \in J \} \) and \( \{ f_j : j \in J \} \) determine decompositions of \( E \), and \( F \), respectively, appearing in the assertions of Proposition 3.3.

Now we see that the spectrum of our positive bounded operator \( EFE \) is pure-point and contains the eigenvalues \( \{ |\alpha_j|^2 : j \in J \} \), with the eigenvectors \( e_j \) (\( j \in J \)). But we could choose the \( \alpha_j \)'s arbitrarily with the only restriction \( 0 < |\alpha_j| < 1 \). Hence, the
pure-point spectrum of $EFE$ with $\dim(E) = \aleph_0$ can be made, in this way, an arbitrary countable subset of the real interval $(0, 1)$.

To investigate the question of mutual “proximity” of projectors $E$ and $F$, let us calculate first the distance $\|E - F\|_2^2 = 2(N - Tr(EF))$ in the case of $|J| = N < \infty$. Due to the orthogonality relations (3.6), resp. (2.15), we have (in the Dirac notation) $Tr(EF) = \sum_{j \in J} |\langle e_j | f_j \rangle|^2 = \sum_{j \in J} |\alpha_j|^2$. So that, it is:

$$0 < \|E - F\|_2^2 = 2(N - \sum_{j \in J} |\alpha_j|^2) < 2N,$$

where every value of the open interval $(0, 2N)$ can be reached without violating our general specification of $(E; F)$. General projections of the dimension $N$ could reach all values in the closed interval: $0 \leq \|E - F\|_2^2 \leq 2N$.

With a help of their chosen representation, we can calculate also the “proximity” of the projections $(E; F)$ in the operator norm, i.e. $\|E - F\|$, what can be used also if $|J| = \aleph_0$. This can be easily done, because the two-dimensional subspaces spanned by the couples of vectors $\{e_j; f_j\}$, $j \in J$, are all mutually orthogonal. Then the spectrum of $|E - F|$ can be easily calculated: $|E - F| = \sum_{j \in J} |P_{e_j} - P_{f_j}|$, the spectrum is (cf. proof of the Proposition 3.2) $\sigma(|E - F|) = \{\sqrt{1 - Tr(P_{e_j}P_{f_j})} : j \in J\}$, and the norm is

$$\|E - F\| = \sup_{j \in J} \|P_{e_j} - P_{f_j}\| = \sup_{j \in J} \sqrt{1 - Tr(P_{e_j}P_{f_j})} = \sup_{j \in J} \sqrt{1 - |\alpha_j|^2}.$$

In this case, it is possible to reach all the values $0 < \|E - F\| \leq 1$ for our projections (the equality can be reached for $\dim E = \infty$ only). ☐

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Let us note that, according to a theorem by Naimark (cf. [11, Chap. 9, Theorem 3.2]), any positive operator $A : 0 \leq A \leq I_E$ defined on a Hilbert space $E$ can be extended into the form $A = EFE\mid_E$, where $E, F$ are some orthogonal projections in a Hilbert space $H$, and $E = EH$. 

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