THE GROTHENDIECK PROPERTY FROM AN ORDERED POINT OF VIEW

OMID ZABETI

Abstract. In this note, we consider several notions related to the Grothendieck property. Among them, we introduce the notion “unbounded Grothendieck property” in a Banach lattice as an unbounded version of the known Grothedieck property in the Banach space theory. Beside other results, surprisingly, we show that spaces with the unbounded Grothendieck property are exactly the reflexive Banach lattices.

1. Motivation and Preliminaries

Let us start with some motivation. There are several known and important concepts in the category of all Banach spaces such as the Schur property, the Banach-Saks property, the Grothendieck property and so on. When we are dealing with a Banach lattice, as a special case of Banach spaces, the order structure comes to the mind as a fruitful tool. This structure enables us to consider different types of notions related to the property using both topological and order structures. Recall that a Banach space $X$ has the Grothendieck property if for every sequence $(x_n') \subseteq X'$, we have $x_n' \overset{w^*}{\to} 0$ implies that $x_n' \overset{w}{\to} 0$. It is known that every reflexive space has the Grothendieck property. Moreover, in the setting of the Banach lattice theory, every $\sigma$-order complete $AM$-space with unit has this property, as well (see [1, Theorem 4.44]). In this paper, we consider several notions related to the Grothendieck property using order and topological structures. We investigate some relations between them. In particular, we characterize reflexive Banach lattices in terms of them. Before anything, let us recall some preliminaries. Suppose $E$ is a Banach lattice. A sequence $(x_n') \subseteq E'$ is said to be unbounded weak* convergent to $x' \in E'$ (in notation, $x_n' \overset{uw^*}{\to} x'$) provided that $|x_n' - x'| \wedge u' \overset{w^*}{\to} 0$ for each $u' \in E'_+$. Two elements $x, y \in E$ are called disjoint if $|x| \wedge |y| = 0$. For more information about unbounded convergences as well as a comprehensive context regarding Banach lattices, see [1, 5].

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2. MAIN RESULTS

First, we consider the following definition.

**Definition 1.** Suppose $E$ is a Banach lattice. $E$ is said to have

(i) The Grothendieck property (GP, for short) if for every sequence $(x_n') \subseteq E'$, $x_n' \xrightarrow{\text{w}} 0$ implies that $x_n' \xrightarrow{\text{w}^*} 0$.

(ii) The positive Grothendieck property (PGP, for short) if for every sequence $(x_n') \subseteq E'_+$, $x_n' \xrightarrow{\text{w}^*} 0$ implies that $x_n' \xrightarrow{\text{w}} 0$.

(iii) The weak Grothendieck property (WGP, for short) if for every disjoint sequence $(x_n') \subseteq E'$, $x_n' \xrightarrow{\text{w}} 0$ implies that $x_n' \xrightarrow{\text{w}^*} 0$.

(iv) The disjoint Grothendieck property (DGP, for short) if for every norm bounded disjoint sequence $(x_n') \subseteq E'$, we have $x_n' \xrightarrow{\text{w}} 0$.

(v) The unbounded Grothendieck property (UGP, for short) if for every norm bounded sequence $(x_n') \subseteq E'$, $x_n' \xrightarrow{\text{uaw}^*} 0$ implies that $x_n' \xrightarrow{\text{w}} 0$.

The first part is the original definition of the Grothendieck property. The second part of the definition was initially defined in [4]. The third part was proposed in [2] at first. Parts (iv) and (v) are new definitions using unbounded convergence and disjointness. Observe that in parts (i), (ii), (iii), norm boundedness of the proposed sequence is guaranteed by the weak*-convergence but in parts (iv), (v), we need assume that the proposed sequence is norm bounded.

First, we shall consider some elementary implications between them.

**Lemma 2.** Suppose $E$ is a Banach lattice. Then, consider the following statements.

(i) $E$ possesses the UGP.

(ii) $E$ possesses the DGP.

(iii) $E$ possesses the WGP.

Then, we have

$$(i) \Rightarrow (ii) \Rightarrow (iii)$$

**Proof.** (i) $\Rightarrow$ (ii). Suppose $(x_n') \subseteq E'$ is a bounded disjoint sequence. By [5] Lemma 2], $x_n' \xrightarrow{\text{uaw}} 0$ so that $x_n' \xrightarrow{\text{uaw}^*} 0$. By the assumption, $x_n' \xrightarrow{\text{w}} 0$, as desired.

(ii) $\Rightarrow$ (iii). It is trivial. $\square$

Observe that the opposite implications proposed in Lemma 2 do not hold, in general. Consider $\ell_\infty$; it possesses the WGP but it fails to have DGP; put $u_n = (0, \ldots, 0, \frac{1}{2n}, \frac{1}{2n+1}, 0, \ldots)$. It is easy to see that the sequence $(u_n) \subseteq \ell_\infty'$ is disjoint.
Nevertheless, it is not weakly null. Furthermore, assume that \( E = \ell_1 \); it possesses the DGP; suppose \((x_n') \subseteq \ell_\infty\) is a disjoint sequence so that it is weakly null by \([5, \text{Lemma 2, Theorem 7}]\). But it can not have the UGP as the \(uaw^*\)-null sequence \((v_n)\) defined via \(v_n = (0, \ldots, 0, 1, 1, \ldots)\) is not weakly null.

By \([1, \text{Theorem 4.44}]\), \(\ell_\infty\) possesses the GP; we shall show that it fails to have the UGP.

**Lemma 3.** \(\ell_\infty\) does not have the UGP.

**Proof.** It is known that \(\ell_1 \subseteq (\ell_\infty)'\) so that consider the standard basis sequence \((e_n) \subseteq (\ell_\infty)'\). It is \(uaw\)-null by \([5, \text{Lemma 2}]\) so that \(uaw^*\)-null. But it is not weakly null in \((\ell_\infty)'\); in this case, it should be weakly null in \(\ell_1\) which is not possible. \(\square\)

Note that the UGP implies the PGP. Now, we have the following surprising fact.

**Theorem 4.** A Banach lattice \(E\) is reflexive if and only if it possesses the UGP.

**Proof.** The direct implication is trivial since \(E\) is reflexive if and only if so is \(E'\). For the other side, suppose \(E\) has the UGP but it fails to be reflexive. By \([1, \text{Theorem 4.71}]\), \(E\) has a lattice copy of either \(c_0\) or \(\ell_1\). Observe that neither \(\ell_1\) nor \(c_0\) has the UGP; for both cases, note that \(\ell_1' = c_0' = \ell_\infty\). Consider the sequence \((u_n) \subseteq \ell_\infty\) defined via \(u_n = (0, \ldots, 0, 1, \ldots)\), in which zero is appeared \(n\)-times. It is easy to see that by the \(w^*\)-topology inherited by either \(\ell_1\) or \(c_0\), the sequence \((u_n)\) is \(uaw^*\)-null. Nevertheless, by the Dini’s theorem (\([1, \text{Theorem 3.52}]\)), it is not weakly null in \(\ell_\infty\), certainly. This shows that \(\ell_1\) or \(c_0\) does not have the UGP. Now, Suppose \(E\) has the UGP. Note that by \([3, \text{Theorem 2.4.12}]\), there is a positive projection \(P\) from \(E\) onto \(c_0\). Moreover, by \([3, \text{Theorem 5.3.13}]\), \(E\) can not have the PGP. This also show that \(E\) fails to have the UGP, as well. Furthermore, by \([3, \text{Proposition 2.3.11}]\), there exists a positive projection \(P_1\) from \(E\) onto \(\ell_1\). Now, consider the positive operator \(T = \iota P_1\), where \(\iota\) denotes the inclusion operator from \(\ell_1\) into \(c_0\). Again, using \([3, \text{Theorem 5.3.13}]\) convinces us that \(E\) fails to have the PGP so that it does not have the UGP. Both statements contradict our assumption. So, \(E\) is reflexive. \(\square\)

Furthermore, we have the following useful observation.

**Theorem 5.** Suppose \(E\) is a \(\sigma\)-order complete Banach lattice whose dual space is order continuous. Then the PGP in \(E\) implies the UGP if and only if \(E\) is order continuous.
Proof. First, assume that $E$ is order continuous and possesses the PGP. Suppose $(x_n') \subseteq E'$ is a positive $uaw^*$-null sequence. By [1, Theorem 4.18], for each arbitrary $\varepsilon > 0$, there exists a $u' \in E_+$ such that for each $x \in E_+$, $x_n'(x) - (x_n' \wedge u')(x) < \varepsilon$. By the assumption, $(x_n' \wedge u')(x) \to 0$ so that $x_n'(x) \to 0$. Therefore, $x_n' \wrightarrow 0$. By the assumption, again, $x_n' \wrightarrow 0$. For the converse, suppose not. By [1, Theorem 4.51], $E$ possesses a lattice copy of $\ell_\infty$. Note that although $\ell_\infty$ has the PGP but it fails to have the UGP by Lemma 3. This implies that $E$ does not have the UGP, as well. \qed

Corollary 6. Suppose $E$ is a $\sigma$-order complete Banach lattice whose dual space is order continuous and possesses the PGP. Then, it is reflexive if and only if it is order continuous.

Note that the GP always implies the PGP. So, we have the following result.

Corollary 7. Suppose $E$ is a $\sigma$-order complete Banach lattice whose dual space is order continuous. Then the GP in $E$ implies the UGP if and only if $E$ is order continuous.

By [2, Proposition 4.9], when $E'$ is order continuous, the WGP implies the PGP. Thus, the following result comes to the mind, readily.

Corollary 8. Suppose $E$ is a $\sigma$-order complete Banach lattice whose dual space is order continuous. Then the WGP in $E$ implies the UGP if and only if $E$ is order continuous.

Remark 9. Note that order continuity of $E'$ is essential in Corollary 8 and can not be removed; $\ell_1$ possesses the WGP but it fails to have UGP although it is order continuous. Furthermore, suppose a Banach lattice $E$ possesses the UGP. By Theorem 4, it is reflexive so that $E$ has the GP, WGP, and PGP.

Now, we consider the following statement; a relation between the remarkable Grothendieck property and the disjoint Grothendieck one.

Theorem 10. A $\sigma$-order complete Banach lattice $E$ is order continuous if and only if the GP in $E$ implies the DGP.

Proof. Suppose $E$ is order continuous and possesses the GP. Assume that $(x_n')$ is a bounded disjoint sequence in $E'$. By [3, Lemma 2], $x_n' \wrightarrow 0$ so that $x_n' \warrow 0$. By [3, Proposition 5], $x_n' \wrightarrow 0$. By the GP, $x_n' \wrightarrow 0$. For the converse, assume that $E$ is not order continuous. By [1, Theorem 4.56], $\ell_\infty$ is lattice embeddable in $E$. By [1, Theorem 4.44], $\ell_\infty$ has the GP; nevertheless, it does not possess the DGP; consider
the standard basis sequence \((e_n) \subseteq \ell_\infty^*\). This implies that \(E\) also can not have the \(\text{DGP}\), as well.

\[ \square \]

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(O. Zabeti) Department of Mathematics, Faculty of Mathematics, Statistics, and Computer science, University of Sistan and Baluchestan, Zahedan, P.O. Box 98135-674, Iran

Email address: o.zabeti@gmail.com