RAINBOW HAMILTON CYCLE IN HYPERGRAPH SYSTEMS

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ABSTRACT. Rödl, Ruciński and Szemerédi proved that every n-vertex k-graph $H$, $k \geq 3$, $\gamma > 0$ and $n$ is sufficiently large, with $\delta_{k-1}(H) \geq (1/2 + \gamma)n$ contains a tight Hamilton cycle, which can be seen as a generalization of Dirac’s theorem in hypergraphs. In this paper, we extend this result to the rainbow setting as follows. A k-graph system $H = \{H_i\}_{i \in [n]}$ is a family of not necessarily distinct k-graphs on the same n-vertex set $V$, a k-graph $G$ on $V$ is rainbow if $E(G) \subseteq \bigcup_{i \in [n]} E(H_i)$ and $|E(G) \cap E(H_i)| \leq 1$ for $i \in [n]$. Then we show that given $k \geq 3$, $\gamma > 0$, sufficiently large $n$ and an n-vertex k-graph system $H = \{H_i\}_{i \in [n]}$, if $\delta_{k-1}(H_i) \geq (1/2 + \gamma)n$ for $i \in [n]$, then there exists a rainbow tight Hamilton cycle.

1. Introduction

1.1. Dirac-type problems. The problem of finding Hamilton cycles in graphs and hypergraphs is a central area with a profound history. One of the classical theorems in graph theory is the result of Dirac [17] which states that every n-vertex graph with minimum degree at least $n/2$, $n \geq 3$, contains a Hamilton cycle.

A k-uniform hypergraph (k-graph) $H = (V, E)$ consists of a vertex set $V$ and an edge set $E \subseteq V^k$. Berge [8] defined a Hamilton cycle in an n-vertex hypergraph $H$ as a cyclic ordering $v_1 \cdots v_n$ such that for $i \in [n]$, there exist distinct edges $e_i$ of $E$ with $\{v_i, v_{i+1}\} \subseteq e_i$. The degree of a vertex $v$ in the hypergraph, is the number of edges containing $v$. Bermond et al. [9] studied the existence of Berge Hamilton cycles under the degree condition. In many applications, the notion of Berge Hamilton cycles appears to be not strong enough. Katona and Kierstead [26] defined another type of cycles in hypergraphs, which has been studied extensively.

A k-uniform tight Hamilton cycle $H = (V, E)$ can be seen as a cyclic ordering $v_1 \cdots v_n$ of $V$ in such a way that $\{v_i, \ldots, v_{i+k-1}\} \in E$ for $i \in [n]$. In order to give the Dirac-type results, the definition of degree must be extended. For any $S \subseteq V(H)$, the degree of $S$ in $H$, denoted by $\deg_H(S)$, is the number of edges containing $S$. For any integer $\ell \geq 0$, let $\delta_{\ell}(H) := \min\{\deg_H(S) : S \in \binom{V(H)}{\ell}\}$.

Throughout the rest of this paper, we refer to k-uniform tight Hamilton cycles as Hamilton cycles. Katona and Kierstead [26] gave a sufficient condition for finding a Hamilton cycle in a k-graph with minimum $(k-1)$-degree: every n-vertex k-graph $H$ with $\delta_{k-1}(H) > (1 - 1/2k)n + 4 - k - 5/2k$ admits a Hamilton cycle. They conjectured that the bound on the minimum $(k-1)$-degree can be reduced to roughly $n/2$, which was confirmed asymptotically by Rödl, Ruciński and Szemerédi in [43, 44]. The same authors gave the exact version for $k = 3$ in [45].
Theorem 1.1 (\cite{44, 45}). Let \( k \geq 3, \gamma > 0 \) and \( H \) be an \( n \)-vertex \( k \)-graph, where \( n \) is sufficiently large. If \( \delta_{k-1}(H) \geq (1/2 + \gamma)n \) edges, then \( H \) contains a Hamilton cycle. Furthermore, when \( k = 3 \) it is enough to have \( \delta_2(H) \geq \lfloor n/2 \rfloor \).

For more problems and results on Dirac-type problems, we refer the readers to \cite{5, 6, 7, 12, 16, 21, 22, 23, 27, 29, 30, 32, 34, 41} and the recent surveys of Rödl and Ruciński \cite{42}, Simonovits and Szemerédi \cite{46} and Zhao \cite{47}.

1.2. A rainbow setting. The study of rainbow structures in graph systems has not received much attention until recently. Aharoni et al. \cite{1} conjectured a rainbow version of the Dirac’s theorem in graph systems: for \( |V| = n \geq 3 \) and \( G = \{G_i\}_{i \in [n]} \) on \( V \), if \( \delta(G_i) \geq n/2 \) for each \( i \in [n] \), then there exists a rainbow Hamilton cycle: a cycle with edge set \( \{e_1, \ldots, e_n\} \) such that \( e_i \in E(G_i) \) for \( i \in [n] \). This was recently verified asymptotically by Cheng, Wang and Zhao \cite{14}, and completely by Joos and Kim \cite{25}. In \cite{11}, Bradshaw, Halasz, and Stacho strengthened the Joos-Kim result by showing that given an \( n \)-vertex graph system \( G = \{G_i\}_{i \in [n]} \) and each \( G_i \) has minimum degree at least \( n/2 \), \( G \) has exponentially many rainbow Hamilton cycles. Similarly, a degree condition of Moon and Moser \cite{40} for Hamiltonicity in bipartite graphs has been generalized to the rainbow setting by Bradshaw in \cite{10}.

Other recent results on rainbow structures include works on matchings \cite{2, 19, 24, 28, 31, 35, 36, 37, 38}, factors \cite{13, 15, 39} and so on.

For convenience, we use \( [i, j] \), \( i, j \in \mathbb{Z} \), to denote the set \( \{i, i+1, \ldots, j\} \). The set \( [1, n] \) is denoted by \([n]\) in short.

Definition 1.2. Let \( m \) be an integer and \( H = \{H_i\}_{i \in [m]} \) be a \( k \)-graph system, where each \( H_i \) can be seen as the collection of edges with color \( i \). Then a \( k \)-graph \( G \) on \( V \) is rainbow if \( E(G) \subseteq \bigcup_{i \in [m]} E(H_i) \) and \( |E(G) \cap E(H_i)| \leq 1 \) for \( i \in [m] \).

The main goal of this paper is to give a sufficient condition forcing a rainbow Hamilton cycle in a \( k \)-graph system. For every \( k \geq 3, \gamma > 0 \), we say that an \( n \)-vertex \( k \)-graph system \( H = \{H_i\}_{i \in [n]} \) is a \((k, n, \gamma)\)-graph system if \( \delta_{k-1}(H_i) \geq (1/2 + \gamma)n \) for \( i \in [n] \).

Theorem 1.3. For every \( k \geq 3, \gamma > 0 \), there exists \( n_0 \) such that the following holds for \( n \geq n_0 \). A \((k, n, \gamma)\)-graph system \( H = \{H_i\}_{i \in [n]} \) admits a rainbow Hamilton cycle.

This result can be regarded as a generalization of Theorem 1.1 to the rainbow setting.

2. Notation and Proof Strategy

2.1. Notation. A tight-path \( P = v_1 v_2 \cdots v_t \) is a \( k \)-graph whose vertices can be ordered in such a way that each edge consists of \( k \) consecutive vertices and two consecutive edges intersect in exactly \( k-1 \) vertices. We say that \( P \) connects \((v_1, \ldots, v_{k-1})\) and \((v_t, \ldots, v_{t+k-2})\). The ordered \((k-1)\)-sets \((v_1, \ldots, v_{k-1})\) and \((v_t, \ldots, v_{t+k-2})\) are called the ends of \( P \). In this paper, tight paths are referred as paths for convenience.
Given a $k$-graph $G$ and a $k$-graph system $H = \{H_i\}_{i \in [n]}$, we define $\{i : E(H_i) \cap E(G) \neq \emptyset\}$ as the color set $C(G)$ of $G$. We call $P = x_1 \cdots x_{2k-2}$ a path with color pattern $(c_1,\ldots,c_{k-1})$ if $\{x_i,\ldots,x_{i+k-1}\} \in E(H_{c_i})$ for $i \in [k-1]$. Recall that $P$ is rainbow if $c_i \neq c_j$ for all $i,j \in [k-1]$.

**Definition 2.1.** Let $P = \{P_1,\ldots,P_m\}$ be a family of vertex-disjoint paths. If each path is rainbow and $C(P_i) \cap C(P_j) = \emptyset$ for distinct $i,j \in [m]$, then we call this family a rainbow family of paths. Denote $\bigcup_{i \in [m]} V(P_i)$ by $V(P)$.

We use by now a common hierarchical notation for constants, writing $x \ll y$ to mean that there is a fixed positive nondecreasing function on $(0,1]$ such that the subsequent statements hold for $x \leq f(y)$. While multiple constants appear in a hierarchy, they are chosen from right to left.

In this paper, we use the following concentration inequalities.

**Proposition 2.2 ([4]).** Suppose that $X$ has the binomial distribution and $0 < a < 3/2$. Then $\Pr(|X - \mathbb{E}X| \geq a\mathbb{E}X) \leq 2e^{-a^2\mathbb{E}X/3}$.

**Proposition 2.3 ([20]).** Let $\binom{[N]}{r}$ be the set of $r$-subsets of $\{1,\ldots,N\}$ and let $h : \binom{[N]}{r} \to \mathbb{R}$ be given. Let $C$ be a uniformly random element of $\binom{[N]}{r}$. Suppose that there exists $\alpha \geq 0$ such that

$$|h(A) - h(A')| \leq \alpha$$

for any $A, A' \in \binom{[N]}{r}$ with $|A \cap A'| = r - 1$. Then

$$\mathbb{E}e^{h(C)} = \exp(\mathbb{E}h(C) + a),$$

where $a$ is a real constant such that $0 \leq a \leq \frac{1}{8} \min\{r, N - r\} \alpha^2$. Furthermore, for any real $t > 0$,

$$\Pr(|h(C) - \mathbb{E}h(C)| \geq t) \leq 2 \exp\left(-\frac{2t^2}{\min\{r, N - r\} \alpha^2}\right).$$

### 2.2. Proof strategy.

Let us briefly comment on the proofs. Our proof follows the popular absorption approach in this line of research, in which the job splits into several tasks (see Lemmas 2.4–2.6). Our contribution is on showing that these subtasks can be adapted into the rainbow setting. Similar as in [13], when dealing with structures with a finite number of colors, it is relatively straightforward to extend the arguments for single host $k$-graph to the rainbow setting, which is the case for Lemmas 2.4 and 2.5. In Lemma 2.4 we also have an innovation point where we use a new gadget for connecting a family of paths to a cycle, where we append the connectors given by Lemma 2.5 to the absorbing cycle.

On the other hand, Lemma 2.6 is proved by using the regularity method, which converts the problem to finding matching type structures in the reduced $k$-graphs. Note that a rainbow path cover uses almost $n$ colors (so in contrast to the constructions of absorbers and connectors), which needs a different strategy. We use an auxiliary hypergraph first used in [38] where we add $n$ new vertices for the set of colors.

We apply the weak regularity lemma [21] on this auxiliary hypergraph. Then it is proved that the reduced hypergraph contains an almost perfect matching, where we apply the results on matchings in [3, 13]. Finally the rainbow tight path cover is obtained by constructing long rainbow paths in the regular tuples, via a variant of hypergraph paths.
We state the following lemmas, whose proofs will be given later.

**Lemma 2.4 (Absorbing lemma).** For every \( k \geq 3, \gamma > 0 \) and \( \kappa := (3k - 3)^{-6}2^{1-2k}\gamma^{4k-4} \), there exists \( n_0 \) such that the following holds for \( n \geq n_0 \). Let \( H = \{H_i\}_{i \in [n]} \) be a \((k, n, \gamma)\)-graph system on \( V \). There exists a rainbow cycle \( A \) with at most \( n_0 \) vertices such that for any rainbow family of paths \( \mathcal{P} \) and any vertex set \( U \) in \( V \setminus V(A) \) with \(|\mathcal{P}|, |U| \leq \kappa n \), there exists a rainbow cycle \( A' \) with vertex set \( V(A) \cup U \cup V(\mathcal{P}) \) and \( C(A) \subseteq C(A') \).

In other words, Lemma 2.4 gives us a short rainbow cycle \( A \) such that every small subset of vertices and small rainbow family of paths can be absorbed into a long rainbow cycle. The following lemma ensures the existence of a rainbow path with constant length between any two disjoint ends.

**Lemma 2.5 (Connecting lemma).** For every \( k \geq 3, \gamma > 0 \) and \( c := [2k\gamma^{-2}] - (k - 1) \), there exists \( n_0 \) such that the following holds for \( n \geq n_0 \). Let \( H = \{H_i\}_{i \in [c]} \) be a \((k, n, \gamma)\)-graph system and \( u, v \) be two disjoint \((k - 1)\)-tuples of vertices. There exists a rainbow path \( P \) from \( u \) to \( v \) with at most \( c + k - 1 \) vertices.

Next, our next result constructs a rainbow family of paths, which almost covers all vertices of \( V \setminus V(A) \).

**Lemma 2.6 (Path cover lemma).** For every \( k \geq 3, \gamma, \delta > 0 \), there exist \( n_0 \) and \( L \) such that every \((k, n, \gamma)\)-graph system \( H = \{H_i\}_{i \in [n]} \) on \( V \), \( n \geq n_0 \), contains a rainbow family \( \mathcal{P} \) of at most \( L \) paths, covering at least \((1 - \delta)n \) vertices of \( V \).

### 2.3. Proof of Theorem 1.3.

For any \( k \geq 3 \) and \( \gamma > 0 \), let \( 1/n \ll 1/L \ll \gamma \), \( \kappa := (3k - 3)^{-6}2^{1-2k}\gamma^{4k-4} \) and \( H \) be a \((k, n, \gamma)\)-graph system on \( V \).

**Step 1.** By Lemma 2.4, we obtain a rainbow absorbing cycle \( A \) with at most \( \gamma n/2 \) vertices such that the following property holds.

(Q) For any rainbow family of paths \( \mathcal{P} \) and any vertex set \( U \) in \( V \setminus V(A) \) with \(|\mathcal{P}|, |U| \leq \kappa n \), there exists a rainbow cycle \( A' \) with vertex set \( V(A) \cup U \cup V(\mathcal{P}) \) and \( C(A) \subseteq C(A') \).

**Step 2.** Set \( H' = \{H'_i\}_{i \in C} \) where \( C = [n]\setminus C(A), H'_i = H_i[V \setminus V(A)] \) for \( i \in C \) and \( n' = n - |V(A)| \). Note that \( H' \) is a \((k, n', \gamma/2)\)-graph system where \( n' > (1 - \gamma/2)n \). Applying Lemma 2.6 to \( H' \) with \( \delta = \kappa \), we obtain a rainbow family of paths \( \mathcal{P} = \{P_1, \ldots, P_p\} \), where \( p \leq L \leq \kappa n \), which covers all but at most \( \kappa n' \) vertices of \( V \setminus V(A) \). Denote the set of uncovered vertices by \( T \). Thus, \(|T| \leq \kappa n' \leq \kappa n \).

**Step 3.** Using property (Q), we obtain a long rainbow cycle with vertex set \( V(A) \cup T \cup V(\mathcal{P}) \), which is a rainbow Hamilton cycle in \( H \).

### 3. Rainbow Absorption Method

Given a vertex \( x \) and a color \( c \), we say that a path \( P \) is a rainbow absorber for \((x, c)\) in an \( n \)-vertex \( k \)-graph system if the following holds:

- \( x \notin V(P) \);
- \( P = x_1 \cdots x_{2k-2} \) is a rainbow path of length \( k - 1 \), whose color pattern is denoted by \((c_1, \ldots, c_{k-1})\);
\* $x_1 \cdots x_{k-1} x_k \cdots x_{2k-2}$ is a rainbow path of length $k$, whose color pattern is denoted by $(c, c_1, \ldots, c_{k-1})$.

**Figure 1.** Absorber for $(x, c)$ when $k = 3$

Given two disjoint $(k-1)$-tuples of vertices $u = (u_1, \ldots, u_{k-1})$, $v = (v_1, \ldots, v_{k-1})$ and a $(k-1)$-tuple $(o_1, \ldots, o_{k-1})$ of colors, we say that a path $P$ is a *rainbow absorber* for $(u, v; o_1, \ldots, o_{k-1})$ in an $n$-vertex $k$-graph system if the following holds:

\* $V(P) \cap \{u_1, \ldots, u_{k-1}\} = \emptyset, V(P) \cap \{v_1, \ldots, v_{k-1}\} = \emptyset$;
\* $P = x_1 \cdots x_{2k-2}$ is a rainbow path of length $k-1$, whose color pattern is denoted by $(c_1, \ldots, c_{k-1})$;
\* $x_1 \cdots x_{k-1} u_1 \cdots u_{k-1}$ and $v_1 \cdots v_{k-1} x_k \cdots x_{2k-2}$ are rainbow paths of length $k-1$, whose color patterns are denoted by $(c_1, \ldots, c_{k-1})$, $(o_1, \ldots, o_{k-1})$ respectively.

**Figure 2.** Absorber for $((u_1, u_2), (v_1, v_2); o_1, o_2)$ when $k = 3$

Given a vertex $x \in V$ and a color $c \in \{1, 2, \ldots, n\}$, let $\mathcal{L}(x; c)$ be the set of rainbow absorbers for $(x, c)$. Similarly, given two disjoint $(k-1)$-tuples $u$ and $v$ of $V$ and a $(k-1)$-tuple $(o_1, \ldots, o_{k-1})$ of $\{1, 2, \ldots, n\}$, let $\mathcal{L}(u, v; o_1, \ldots, o_{k-1})$ be the set of rainbow absorbers for $(u, v; o_1, \ldots, o_{k-1})$. We need the following simple result.

**Fact 3.1.** Let $S$ be a $(k-1)$-subset of $V$ and $V_0 \subseteq V \setminus S$. For any $i \in \{1, 2, \ldots, n\}$, we have

$$|N_{\mathcal{H}_i}(S) \cap V_0| \geq |V_0| - \frac{1}{2} n + \gamma n + k - 1.$$
In particular, for two \((k - 1)\)-subsets of vertices \(S_1\) and \(S_2\), we obtain that for any \(i, j \in [n]\),

\[|N_{H_i}(S_1) \cap N_{H_j}(S_2)| \geq 2\gamma n + |S_1 \cap S_2|\]

**Proof.** We have \(|N_{H_i}(S) \cup V_0| \leq n - k + 1\) and thus

\[|N_{H_i}(S) \cap V_0| \geq |V_0| + |N_{H_i}(S)| - (n + k - 1) \geq |V_0| - \frac{1}{2}n + \gamma n + k - 1.\]

For the second statement, we apply the first one with \(S = S_1\) and \(V_0 = N_{H_i}(S_2) \setminus S_1\) and note that \(|V_0| \geq (\frac{1}{2} + \gamma)n - (k - 1 - |S_1 \cap S_2|)\). \(\Box\)

**Proposition 3.2.** For every \(k \geq 3, \gamma > 0\), there exists \(n_0\) such that the following holds for \(n \geq n_0\). Suppose \(H = \{H_1, \ldots, H_n\}\) is a \((k, n, \gamma)\)-graph system on \(V\), then \(|\mathcal{L}(x; c)| \geq 2^{2^k - 2}n^{3k-3}\) for every vertex \(x \in V\) and color \(c \in [n]\), \(|\mathcal{L}(u, v, o_1, \ldots, o_{k-1})| \geq 2^{1-k}\gamma 2^{k-2}n^{3k-3}\) for every two disjoint \((k - 1)\)-tuples \(u\) and \(v\) of \(V\) and a \((k - 1)\)-tuple \((o_1, \ldots, o_{k-1})\) of \([n]\).

**Proof.** Given \(k, \gamma\), we choose \(n\) such that \(1/n \ll \gamma/k\). Fixing vertex \(x \in V\) and color \(c \in [n]\), we construct a rainbow absorber \(P = x_1 \cdots x_{2^k-2}\) for \((x, c)\). We choose \((c_1, \ldots, c_{k-1})\) arbitrarily, so there are \((n - 1) \cdots (n - k + 1) \geq 2^{2^k - k}\n^{k-1}\) choices. Furthermore, \(x_1, \ldots, x_{k-2}\) can be chosen arbitrarily in \((n - 1) \cdots (n - k + 2) \geq 2^{2^k - k}\n^{k-2}\) ways. For \(x_{k-1}\), there are at least \((\frac{1}{2} + \gamma)n\) choices such that \(\{x_1, \ldots, x_{k-1}, x\} \in E(H_c)\). By Fact 3.1, there are at least \(2\gamma n + k - 2\) choices for \(x_j, j \in [k, 2k - 2]\), such that \(\{x_j \cdots x_{j+k-2}, x_j, x\} \in E(H_{c_j})\). For \(j \in [k+1, 2k-2]\), \(x_j\) should be different from \(x_1, \ldots, x_{j-k}\). Thus, the number of choices for each \(x_j\) is at least \(2\gamma n + k - 2 - (j - k) \geq 2\gamma n, j \in [k, 2k - 2]\), yielding together at least \(2^{1-k}\n^{k-1}2^{2-k}\n^{k-2}(\frac{1}{2} + \gamma)n(2\gamma n)^{k-1} \geq 2^{2^{k-1}-k}\n^{k-3}\) rainbow absorbers for \((x, c)\).

Given \(u = (u_1, \ldots, u_{k-1}), v = (v_1, \ldots, v_{k-1})\) and \((o_1, \ldots, o_{k-1})\), we construct a rainbow absorber \(P = x_1 \cdots x_{2^k-2}\) for \((u, v; o_1, \ldots, o_{k-1})\). There are \((n - k + 1) \cdots (n - 2k + 2) \geq 2^{1-k}\n^{k-1}\) choices for \((c_1, \ldots, c_{k-1})\). There are at least \((\frac{1}{2} + \gamma)n - (k - 1) \geq \gamma n\) choices for \(x_{k-1}\) such that \(\{u_1, \ldots, u_{k-1}, x_{k-1}\} \in E(H_{c_{k-1}})\) and \(x_{k-1}\) should be different from \(v_1, \ldots, v_{k-1}\). For \(x_i, i \in [k-2]\), there at least \((\frac{1}{2} + \gamma)n - (2k - 3) \geq \gamma n\) choices such that \(\{u_{k-i}, \ldots, u_{k-1}, x_{k-1}, \ldots, x_{i+1}, x_i\} \in E(H_{c_i})\), and it should be different from \(v_1, \ldots, v_{k-1}, u_1, \ldots, u_{k-1-i}\).

By Fact 3.1, there are at least \(2\gamma n\) choices for \(x_k\) such that \(\{x_1, \ldots, x_{k-1}, x_k\} \in E(H_{c_k})\) and it is different from \(u_1, \ldots, u_{k-1}\). For \(x_i, i \in [k + 1, 2k - 2]\), the number of choices is at least \(2\gamma n + k - 2 - (k - 1 + 2(i - k)) \geq \gamma n\), such that \(\{x_{i-(k-1)}, \ldots, x_i\} \in E(H_{c_{i-(k-1)}})\), \(\{v_{i-(k-1)}, \ldots, v_{k-1}, x_{i-(k-1)}, \ldots, x_i\} \in E(H_{o_{i-(k-1)}})\) and it should be different from \(u_1, \ldots, u_{k-1}, x_1, \ldots, x_{i-k}, v_1, v_{i-1}, v_{i-k}\). The inequality holds since \(1/n \ll \gamma/k\), thus, there are at least \(2^{1-k}\n^{k-1}(\gamma n)^{k-1} \geq 2^{2^{k-1}-k-2}\n^{k-3}\) rainbow absorbers for \((u, v; o_1, \ldots, o_{k-1})\). \(\Box\)

**Lemma 3.3.** For every \(k \geq 3, \gamma > 0, 0 < \zeta < 1\), there exists \(n_0\) such that the following holds for \(n \geq n_0\). Let \(H = \{H_1, \ldots, H_n\}\) be an \(n\)-vertex \(k\)-graph system on \(V\) and \(u, v\) be distinct \((k - 1)\)-tuples of \(V\). If \(|\mathcal{L}(x; c)| \geq \zeta n^{3k-3}\) for every vertex \(x \in V, c \in [n]\) and \(|\mathcal{L}(u, v; o_1, \ldots, o_{k-1})| \geq \zeta n^{3(k-3)}\) for all disjoint \((k - 1)\)-tuples \(u\) and \(v\) of \(V\) and \((k - 1)\)-tuple \((o_1, \ldots, o_{k-1})\) of \([n]\), then there exists a rainbow family of paths \(\mathcal{F}'\), where each path is of length \(k - 1\), satisfying the following properties.

\[|\mathcal{F}'| \leq (3k - 3)^{-3}\zeta n, \quad |\mathcal{F}' \cap \mathcal{L}(x; c)| \geq (3k - 3)^{-6}\zeta n,\]
\[ |F' \cap L(u, v; o_1, \ldots, o_{k-1})| \geq (3k - 3)^{-6} \zeta^2 n, \]

for every vertex \( x \in V \), \( c \in [n] \), two disjoint \((k - 1)\)-tuples \( u \) and \( v \) of \( V \) and \( (o_1, \ldots, o_{k-1}) \) of \([n]\).

**Proof.** Each rainbow path \( x_1 x_2 \cdots x_{2k-2} \) with color pattern \((c_1, \ldots, c_{k-1})\) can be considered as a \((3k - 3)\)-tuple \((x_1, x_2, \ldots, x_{2k-2}, c_1, \ldots, c_{k-1})\). Choose a family \( F \) of \((3k - 3)\)-tuples from \( \binom{V}{n} \times \binom{[n]}{k-1} \) by including each of the \( n^{n-1} \) possible \((3k - 3)\)-tuples independently at random with probability

\[
p = (3k - 3)^{-4} \zeta \left( \frac{n - (2k - 2)!}{n - (k - 1)!} \cdot \frac{(n - (k - 1))!}{n!} \right) \geq (3k - 3)^{-4} \zeta n^{-(3k-4)}.
\]

Note that \(|F|, |L(x, c) \cap F|, |L(u, v; o_1, \ldots, o_{k-1}) \cap F|\) are binomial random variables,

\[
\mathbb{E}|F| = p \frac{n! \cdot n!}{(n - (2k - 2))! \cdot (n - (k - 1))!} = (3k - 3)^{-4} \zeta n,
\]

\[
\mathbb{E}|L(x, c) \cap F| = p |L(x, c)| \geq (3k - 3)^{-4} \zeta^2 n,
\]

\[
\mathbb{E}|L(u, v; o_1, \ldots, o_{k-1}) \cap F| = p |L(u, v; o_1, \ldots, o_{k-1})| \geq (3k - 3)^{-4} \zeta^2 n,
\]

for every vertex \( x \in V \), \( c \in [n] \), two disjoint \((k - 1)\)-tuples \( u \) and \( v \) of \( V \) and \( (o_1, \ldots, o_{k-1}) \) of \([n]\). By Proposition 3.2, with probability \( 1 - o(1) \), the family \( F \) satisfies the following properties

\[
|F| \leq 2 \mathbb{E}|F| = 2(3k - 3)^{-4} \zeta n \leq (3k - 3)^{-3} \zeta n,
\]

\[
|L(x, c) \cap F| \geq 2^{-1} \mathbb{E}|L(x, c) \cap F| \geq 2^{-1}(3k - 3)^{-4} \zeta^2 n,
\]

\[
|L(u, v; o_1, \ldots, o_{k-1}) \cap F| \geq 2^{-1} \mathbb{E}|L(u, v; o_1, \ldots, o_{k-1})| \geq 2^{-1}(3k - 3)^{-4} \zeta^2 n,
\]

for every vertex \( x \in V \), \( c \in [n] \), two disjoint \((k - 1)\)-tuples \( u \) and \( v \) of \( V \) and \( (o_1, \ldots, o_{k-1}) \) of \([n]\). We say that two \((3k - 3)\)-tuples \((x_1, x_2, \ldots, x_{2k-2}, c_1, \ldots, c_{k-1})\) and \((y_1, y_2, \ldots, y_{2k-2}, f_1, \ldots, f_{k-1})\) are intersecting if \( x_i = y_j \) for some \( i, j \in [2k - 2] \) or \( c_m = f_\ell \) for some \( m, \ell \in [k - 1] \). We can bound the expected number of pairs of \((3k - 3)\)-tuples in \( F \) that are intersecting from above by

\[
\frac{n! \cdot n!}{(n - (2k - 2))! \cdot (n - (k - 1))!} \left( \frac{3k - 3}{n - (2k - 2)!} \cdot \frac{(n - (k - 1))!}{n!} \right)^2 \cdot \frac{(n - 1)! \cdot n!}{(n - (2k - 2))! \cdot (n - (k - 1))!} \geq (3k - 3)^{-6} \zeta^2 n.
\]

Thus, using Markov’s inequality, we derive that with probability at least \( 1/2 \), \( F \) contains at most \( 2(3k - 3)^{-6} \zeta^2 n \) intersecting pairs of \((3k - 3)\)-tuples. Remove one \((3k - 3)\)-tuple from each intersecting pair in such a family \( F \) and remove the \((3k - 3)\)-tuples that are not absorbing paths for any \( x \in V \), \( c \in [n] \) or \((k - 1)\)-tuples \( u \) and \( v \) of \( V \) and \( (o_1, \ldots, o_{k-1}) \) of \([n]\). We get a subfamily \( F' \) consisting of pairwise disjoint \((3k - 3)\)-tuples, which satisfies

\[
|L(x, c) \cap F'| \geq 2^{-1}(3k - 3)^{-4} \zeta^2 n - 2(3k - 3)^{-6} \zeta^2 n \geq (3k - 3)^{-6} \zeta^2 n,
\]

for any \( x \in V \), \( c \in [n] \), and a similar statement holds for \(|L(u, v; o_1, \ldots, o_{k-1}) \cap F'|\) for any two disjoint \((k - 1)\)-tuples \( u \) and \( v \) of \( V \) and a \((k - 1)\)-tuple \( (o_1, \ldots, o_{k-1}) \) of \([n]\). Since each \((3k - 3)\)-tuple in \( F' \) is a rainbow absorber, \( F' \) is a rainbow family of paths, where each path is of length \( k - 1 \).

**3.1. Proof of Lemma 2.4.** Set \( \zeta := 2^{1-k} \gamma^{2k-2} \) and \( \kappa := 2^{-1}(3k - 3)^{-6} \zeta^2 \). Let \( H = \{ H_i \}_{i \in [n]} \) be a \((k, n, \gamma)\)-graph system on \( V \). By Proposition 3.2, we obtain \(|L(x, c)| \geq \zeta n^{3k-3} \) for every vertex \( x \in V \) and \( c \in [n] \), \(|L(u, v; o_1, \ldots, o_{k-1})| \geq \zeta n^{3k-3} \) for every two disjoint \((k - 1)\)-tuples \( u \) and \( v \)
of $V$ and a $(k-1)$-tuple $(o_1, \ldots, o_{k-1})$ of $[n]$. By Lemma 3.3, there is a rainbow family of paths $\mathcal{F}' = \{P_1, \ldots, P_q\}$, where $q \leq (3k-3)^3\zeta n$ and $|V(P_i)| = 2k-2$ for $i \in [q]$, $|\mathcal{F}' \cap \mathcal{L}(x; e)| \geq 2\kappa n$ for every vertex $x \in V$, $c \in [n]$, $|\mathcal{F}' \cap \mathcal{L}(u, v; o_1, \ldots, o_{k-1})| \geq 2\kappa n$ for every two disjoint $(k-1)$-tuples $u$ and $v$ of $V$ and $(o_1, \ldots, o_{k-1})$ of $[n]$.

Next, we describe our connecting process. Suppose we have connected $P_1, \ldots, P_j$ into one path $P$, using each time at most $(1 - \gamma/k^2) + 1$ vertices from outside $V(\mathcal{F}')$, the next path from $\mathcal{F}'$ to connect is $P_{j+1}$. Let $e = (u_1, \ldots, u_{k-1})$ and $f = (v_1, \ldots, v_{k-1})$ be one end of $P$ and one end of $P_{j+1}$ respectively. Let $H'_i$ be the induced subgraph of $H_i$ obtained by removing the vertices of $V(\mathcal{F}') \cup V(P)$ except $e$ and $f$. The number of vertices removed is at most

$$|V(\mathcal{F}') \cup V(P)| \leq q(2k-2) + (q-1) \left(\left\lceil \frac{8k}{\gamma^2} \right\rceil - (2k-2)\right) < q \left\lceil \frac{8k}{\gamma^2} \right\rceil < \frac{\gamma n}{2},$$

where the last inequality holds since $q \leq (3k-3)^3\zeta n$ and $k \geq 3$.

We get a $(k, n', \gamma/2)$-graph system $H' = \{H'_i\}_{i \in C}$ where $C = [n] \setminus (C(P) \cup C(\mathcal{F}'))$ and $n' = |V(H')|$. Taking a $([8k\gamma^{-2}] - (k-1))$-subset $C'$ of $C$, we apply Lemma 2.5 to $\{H'_i\}_{i \in C'}$, $e' = (u_{k-1}, \ldots, u_1)$ and $f' = (v_1, \ldots, v_1)$, obtaining a rainbow path $P'$ connecting $e'$ and $f'$ such that $|V(P')| \leq [8k\gamma^{-2}]$. Thus, $P \cup P' \cup P_{j+1}$ forms a rainbow path in $H$.

After connecting all rainbow paths in $\mathcal{F}'$, we obtain a rainbow $(k-1)$-cycle $A$ with at most

$$q(2k-2) + q \left(\left\lceil \frac{8k}{\gamma^2} \right\rceil - (2k-2)\right) \leq \frac{\gamma n}{2}$$

vertices. Furthermore, by the property of rainbow absorbers, $A$ can absorb a vertex set and a rainbow family of paths with sizes at most $\kappa n$.

4. Rainbow Almost Path Cover

4.1. Proof Sketch. The main goal of this section is to find a rainbow family of paths in a $(k, n, \gamma)$-graph system $H = \{H_i\}_{i \in [n]}$ on $V$. We transform the initial problem in $H$ into a new problem in an auxiliary hypergraph. To construct an auxiliary hypergraph, we use the following definition.

**Definition 4.1.** We call a hypergraph $H$ a $(1, k)$-graph ($(1, k)$-partite), if $V(H)$ can be partitioned into $V_1$ and $V_2$ such that every edge contains exactly one vertex of $V_1$ and $k$ vertices of $V_2$.

Given a partition of $V(H) = V_1 \cup V_2$, a $(1, k-1)$-subset $S$ of $V(H)$ contains one vertex in $V_1$ and $k-1$ vertices in $V_2$. Let $\delta_{1,k-1}(H) := \min\{\deg_H(S) : S \text{ is a (1, k-1)-subset of } V(H)\}$.

**Construct a (1, k)-graph.** We construct an auxiliary hypergraph $H^*$ with vertex set $[n] \cup V$ and edge set $E(H^*) = \{\{i\} \cup e : e \in E(H_i), i \in [n]\}$. Obviously, $H^*$ is a $(1, k)$-graph and we have $\delta_{1,k-1}(H^*) \geq (1/2 + \gamma)n$.

**Obtain a cluster hypergraph $K$.** With an initial partition $[n] \cup V$, by a weak regularity lemma, we partition the $(1, k)$-graph $H^*$ and obtain a cluster hypergraph $K$ where $V(K) = \mathcal{I} \cup \mathcal{W}$, $|\mathcal{I}| = |\mathcal{W}| = t$, $\mathcal{I} = \{I_1, \ldots, I_t\}$ is an equitable partition of $[n]$ and $\mathcal{W}$ is an equitable partition of $V$. Note that $K$ is a $(1, k)$-graph. We will prove in Section 4.3 that $K$ almost “inherits” the $(1, k-1)$-degree condition of $H^*$.

**Obtain many matchings in $K$.** We equally split $\mathcal{I}$ into $k$ parts $\mathcal{I}_i = \{I_{(i-1)t/k+1}, \ldots, I_{it/k}\}$ for
$i \in [k]$. Considering each $(1,k)$-graph $F_i$ of $K$ induced on $\mathcal{T}_i \cup \mathcal{W}$, we take a random partition of $\mathcal{T}_i \cup \mathcal{W}$. Most induced subgraphs of each $F_i$ on these parts “inherit” the $(1,k-1)$-degree condition of $H^*$, which can be proved in Section 4.3. For each subgraph of $F_i$, which “inherits” the $(1,k-1)$-degree condition, $i \in [k]$, we use a combination of results in ([13] Theorem 1.7) and ([3] Theorem 1.2), obtaining many matchings in $K$.

**Lemma 4.2 ([3, 13]).** For every $\gamma > 0$, there exists $n_0 \in \mathbb{N}$, such that the following holds for $n \geq n_0$ and $k \mid n$. Every $n$-vertex $k$-graph system $H = \{H_i\}_{i \in [n/k]}$ with $\delta_{k-1}(H_i) \geq (1/2 + \gamma)n$ for each $i$ admits a rainbow perfect matching.

That is, for every $\gamma > 0$, there exists $n_0 \in \mathbb{N}$, such that the following holds for $n \geq n_0$ and $k \mid n$. Every $(1,k)$-graph $H$ on $[n/k] \cup V$ with $\delta_{1,k-1}(H) \geq (1/2 + \gamma)n$ admits a perfect matching, where $|V| = n$.

**Embed the rainbow paths.** Each matching in $K$ can be blown up into a rainbow family of paths in $H$. We obtain a rainbow family of paths almost covering all vertices in Section 4.4.

### 4.2. Weak Regularity Lemma for Hypergraphs

A $k$-graph $H$ is $k$-partite if there is a partition $V(H) = V_1 \cup \cdots \cup V_k$ such that every edge of $H$ intersects each set $V_i$ in precisely one vertex for $i \in [k]$. Given a $k$-partite $k$-graph $H$ on $V_1 \cup \cdots \cup V_k$ and subsets $A_i \subseteq V_i$, $i \in [k]$, we define $e_H(A_1, \ldots, A_k)$ to be the number of edges in $H$ with one vertex in each $A_i$ and the density of $H$ with respect to $(A_1, \ldots, A_k)$ as

$$d_H(A_1, \ldots, A_k) = \frac{e_H(A_1, \ldots, A_k)}{|A_1| \cdots |A_k|}.$$

We say that a $k$-partite $k$-graph $H$ is $\varepsilon$-regular if for all $A_i \subseteq V_i$ with $|A_i| \geq \varepsilon|V_i|$, $i \in [k]$, we have

$$d_H(A_1, \ldots, A_k) - d_H(V_1, \ldots, V_k) \leq \varepsilon.$$

We give a straightforward generalization of the graph regularity lemma.

**Lemma 4.3** (Weak regularity lemma for hypergraphs [21]). For all $k \geq 2$, every $\varepsilon > 0$ and every integer $t_0$, there exist $T_0$ and $n_0$ such that the following holds. For every $k$-graph $H$ on $n \geq n_0$ vertices there is, for some $t \in \mathbb{N}$ with $t_0 \leq t \leq T_0$, a partition $V(H) = V_0 \cup V_1 \cup \cdots \cup V_t$ such that $|V_0| \leq \varepsilon n$, $|V_1| = |V_2| = \cdots = |V_t|$ and for all but at most $\varepsilon t^k$ sets $\{i_1, \ldots, i_k\} \in \binom{[t]}{k}$, the induced $k$-partite $k$-graph $H[V_{i_1}, \ldots, V_{i_k}]$ is $\varepsilon$-regular.

The partition in Lemma 4.3 is called an $\varepsilon$-regular partition of $H$. For an $\varepsilon$-regular partition of $H$ and $d \geq 0$, we refer to the sets $V_i$, $i \in [t]$ as clusters and define thecluster hypergraph $K = K(\varepsilon, d)$ with vertex set $[t]$ and $\{i_1, \ldots, i_k\} \in \binom{[t]}{k}$ being an edge if and only if $(V_{i_1}, \ldots, V_{i_k})$ is $\varepsilon$-regular and $d(V_{i_1}, \ldots, V_{i_k}) \geq d$.

Let $H^*$ be a $(1,k)$-graph with $\delta_{1,k-1}(H^*) \geq (1/2+\gamma)n$. With an initial partition $[n] \cup V$ of $V(H^*)$, we apply Lemma 4.3 with $\varepsilon$, $t_0$, and obtain a partition $V(H^*) = V_0^* \cup I_1 \cup \cdots \cup I_{t_1} \cup W_1 \cup \cdots \cup W_{t_2}$ where $|I_i| = |W_j| = m$ for $i \in [t_1]$ and $j \in [t_2]$, $|V_0^*| \leq 2\varepsilon n$. By throwing at most $2\varepsilon n/m$ clusters into $V_0^*$, we rename it as $V_0$ if necessary. We have $V(H^*) = V_0 \cup I_1 \cup \cdots \cup I_t \cup W_1 \cup \cdots \cup W_t$, where each cluster keep the size, $|V_0| \leq 4\varepsilon n$, $I_i \subseteq [n]$ and $W_j \subseteq V$ for $i, j \in [t]$. Then the corresponding cluster hypergraph $K$ is still a $(1,k)$-graph with partite sets $\{I_1, I_2, \ldots, I_t\}$ and $\{W_1, W_2, \ldots, W_t\}$. The
following proposition shows that the cluster hypergraph almost “inherits” the minimum degree property of the original hypergraph.

**Proposition 4.4.** For $0 < \varepsilon \leq \gamma^2/16$ and $t_0 \geq 3k/\gamma$, given a $(1,k)$-graph $H$ with $\delta_{1,k-1}(H) \geq (1/2+\gamma)n$ and an $\varepsilon$-regular partition $V(H) = V_0 \cup V_1 \cup \ldots \cup V_t \cup \ldots \cup V_r$, let $K := K(\varepsilon, \gamma/6)$ be the cluster hypergraph. The number of $(1,k-1)$-subsets $S = \{I_{i_0}, W_{i_1}, \ldots, W_{i_k-1}\}$ of $V(K)$ violating $\deg_K(S) \geq (1/2 + \gamma/4)t$ is at most $k\sqrt{\varepsilon} t^k$ where $I_{i_0} \in \{I_1, \ldots, I_t\}$ and $W_{i_j} \in \{W_1, \ldots, W_t\}$ for $j \in [k-1]$.

**Proof.** Note that the cluster hypergraph $K(\varepsilon, \gamma/6)$ can be written as the intersection of two hypergraphs $D := D(\gamma/6)$ and $R := R(\varepsilon)$ both defined on the vertex set $\{I_1, \ldots, I_t, W_1, \ldots, W_t\}$ and

- $D$ consists of all sets $\{I_{i_0}, W_{i_1}, \ldots, W_{i_k}\}$ such that $d(I_{i_0}, W_{i_1}, \ldots, W_{i_k}) \geq \gamma/6$,
- $R$ consists of all sets $\{I_{i_0}, W_{i_1}, \ldots, W_{i_k}\}$ such that $(I_{i_0}, W_{i_1}, \ldots, W_{i_k})$ is $\varepsilon$-regular.

For any $(1,k-1)$-set $S$, we first show that

\[
\deg_D(S) \geq \left(\frac{1}{2} + \frac{\gamma}{2}\right)t.
\]

Note that $n/t \geq m := |W_i| = |I_j|$ for $i, j \in [t]$. We now consider the number $z$ of edges in $H$ which intersect each of $I_{i_0}, W_{i_1}, \ldots, W_{i_k}$ in exactly one vertex. From the condition on $\delta_{1,k-1}(H)$, we have

\[
z \geq m^k \left(\frac{1}{2} + \gamma\right) n - (k-1)m \geq tm^{k+1}\left(\frac{1}{2} + \frac{2\gamma}{3}\right),
\]

since $t \geq t_0 \geq 3k/\gamma$.

On the other hand, if (3) does not hold, then

\[
z < \left(\frac{1}{2} + \frac{\gamma}{2}\right) tm^{k+1} + t\frac{\gamma}{6} m^{k+1},
\]

a contradiction with (4).

Note that there are at most $\varepsilon t^{k+1}$ edges not belonging to $R$. Denote the set of such edges by $\overline{R}$. Let $S$ be the family of all $(1,k-1)$-element subsets $S$ for which $\deg_{\overline{R}}(S) > \sqrt{\varepsilon} t$. We have $|S| \leq k\sqrt{\varepsilon} t^k$. That is, all but at most $k\sqrt{\varepsilon} t^k$ sets $S$ satisfy $\deg_{\overline{R}}(S) \geq (1 - \sqrt{\varepsilon})t$. This, together with (3) and $\varepsilon \leq \gamma^2/16$, implies the property. \qed

4.3. **Random Partition.** The following lemma shows that random subgraphs of a hypergraph typically “inherit” minimum degree conditions. It first appeared in [18].

**Lemma 4.5** (Partition Lemma). Suppose that $k \geq 3$ and $\eta \ll 1/Q \ll \lambda, \gamma$, the following holds for $t \in QN$. If $H$ is a $(1,k)$-graph on $[\frac{t}{k}] \cup V$ with $|V| = t$ where all but at most $\frac{\eta}{k}$ of the $(1,k-1)$-subsets of $V(H)$ have degree at least $(1/2 + \gamma)(t-k+1)$, then there is a partition $V(H) = S_1 \cup \cdots \cup S_{t/Q}$ such that all but at most $\lambda t/Q$ classes satisfy $\delta_{1,k-1}(H[S_i]) \geq (1/2 + \gamma/2)(Q-k+1)$ where each $S_i$ consists of a $Q/k$-subset $I_i$ of $\lceil t/k \rceil$ and a $Q$-subset $V_i$ of $V$.

**Proof.** Partition $[t/k]$ into $t/Q$ sets $I_1, \ldots, I_{t/Q}$ such that $|I_i| = Q/k$ for $i \in [t/Q]$ uniformly at random. We randomly order $V$ as $v_1, \ldots, v_t$ and then partition $V$ into $t/Q$ classes $V_1, \ldots, V_{t/Q}$
such that $V_i = \{v_{(i-1)Q+1}, \ldots, v_{iQ}\}$ for $i \in [t/Q]$. Let $S_i = I_i \cup V_i$ for $i \in [t/Q]$. Note that each $V_i$ is a random subset of $V$. Let $M^*$ be the collection of $(1, k - 1)$-subsets with degree less than $(1/2 + \gamma)(t - k + 1)$ in $H$. We will prove that for $i \in [t/Q]$ and every $(1, k - 1)$-subset $S$ of $S_i$, 

$$
\Pr \left[ \deg_{H[S_i]}(S) < \left( \frac{1}{2} + \frac{\gamma}{2} \right) (Q - k + 1) \right] \leq \eta + e^{-\Omega(\gamma^2 Q)}.
$$

First note that the probability of the event $S \in M^*$, is at most $\eta$. Now let $A_S$ be the event that $S$ is not in $M^*$. The set $V_i \setminus S$ is a uniformly random set of $V$ other than $S$. Let $A$ denote the event that a vertex $v$ in $V_i \setminus S$ is the neighbor of $S$. Note that

$$
\Pr[A|A_S] \geq \frac{(1/2 + \gamma)(t - k + 1)(t-k)}{(Q - (k - 1))(t - (k - 1))} = \frac{1}{2} + \gamma,
$$

then we have

$$
\mathbb{E} \left[ \deg_{H[S_i]}(S) | A_S \right] \geq \left( \frac{1}{2} + \gamma \right) (Q - k + 1).
$$

Exchanging any element with an element outside $V_i \setminus S$ affects $\deg_{H[S_i]}(S)$ by at most 1. Fixing $i$, we use Proposition 2.3 with $S \notin M^*$, the probability that $S$ has degree less than $(1/2 + \gamma/2)(Q - k + 1)$ in $H[S_i]$ is at most

$$
2 \exp \left( -2 \left( \frac{1}{2} \left( \frac{2}{Q} (Q - k + 1) \right)^2 \right) \right) = e^{-\Omega(\gamma^2 Q)}.
$$

We say that $S_i$ is poor if some $(1, k - 1)$-tuple in the induced graph $H[S_i]$ has degree less than $(1/2 + \gamma/2)(Q - k + 1)$. Thus, $\Pr[S_i$ is poor$] \leq \frac{Q}{k} \left( \frac{Q}{k - 1} \right) (\eta + e^{-\Omega(\gamma^2 Q)})$ for $i \in [t/Q]$. Let $X$ be the number of poor classes and by Markov’s inequality, we obtain

$$
\Pr \left[ X \geq \frac{\lambda t}{Q} \right] \leq \frac{Q}{\lambda k} \left( \frac{Q}{k - 1} \right) (\eta + e^{-\Omega(\gamma^2 Q)})
$$

now we choose $Q, \eta$ such that the following holds

$$
\frac{Q}{k} \left( \frac{Q}{k - 1} \right) (\eta + e^{-\Omega(\gamma^2 Q)}) < \lambda,
$$

thus, we have $\Pr[X \geq \frac{\lambda t}{Q}] < 1$. With positive probability, we get a partition $V(H^*) = S_1 \cup \cdots \cup S_{t/Q}$, where $S_i = I_i \cup V_i$, such that at least $(1 - \lambda)t/Q$ classes of them satisfy $\delta_{1,k-1}(H^*[S_i]) \geq (1/2 + \gamma/2)(Q - k + 1)$. 

4.4. Path Embeddings. A $(0, k - 1)$-path $P$ of length $t$ in $H$ is a $(k + 1)$-graph with vertex set $\{c_1, \ldots, c_t\} \cup \{v_1, \ldots, v_{t+k-1}\}$ where $\{c_1, \ldots, c_t\} \subseteq V_0$, $\{v_1, \ldots, v_{t+k-1}\} \subseteq V_1 \cup \cdots \cup V_k$ and edge set $\{e_1, \ldots, e_t\}$ such that $e_i = \{c_i, v_i, \ldots, v_{i+k-1}\}$.

**Figure 3.** A $(0, k - 1)$-path for $k = 3$ (the vertices with the same color are from the same part.)
Note that a rainbow path in a $k$-graph system is a $(0, k - 1)$-path in the auxiliary $(1, k)$-graph $H^*$. We call that a $(k - 1)$-subset $S$ of $V(H)$ is legal if $|S \cap V_i| \leq 1$ for $i \in [k]$ and $|S \cap V_0| = 0$.

**Lemma 4.6.** Given $c, m > 0$ and $k \geq 2$, every $(k + 1)$-partite $(k + 1)$-graph $H$ on $V_0 \cup V_1 \cup \cdots \cup V_k$ with at most $m$ vertices in each part and with at least $cm^{k+1}$ edges contains a $(0, k - 1)$-path of at least $cm/k$ vertices.

**Proof.** There are at most $k \cdot m^{k-1}$ legal $(k - 1)$-subsets of $V(H)$. We proceed by iteratively deleting the edges as follows. If there is a legal $(k - 1)$-subset $S$, which is contained in less than $cm^2/k$ edges in the current hypergraph, then all the edges containing $S$ will be deleted. The process terminates at a nonempty hypergraph $H_0$ since less than $km^{k-1}(cm^2/k) = cm^{k+1}$ edges have been deleted in total. In $H_0$, every legal $(k - 1)$-subset has degree either zero or at least $cm^2/k$.

Let $P$ be a longest $(0, k - 1)$-path in $H_0$ with vertex set $\{c_1, \ldots, c_t\} \cup \{v_1, \ldots, v_{t+k-1}\}$ for some integer $t$. We have $|V(P) \cap V_0| = t$ and $|V(P) \cap V_i| \leq t$ since each edge contains at most one vertex of $V_i$ for $i \in [k]$. Consider $S_t = \{v_{t+1}, \ldots, v_{t+k-1}\}$, which is a legal $(k - 1)$-subset of $V(H)$. Furthermore, $\deg_{H_0}(S_t) \geq cm^2/k$ since $S_t$ has positive degree. All the edges containing $S_t$ must intersect $(V(P) \cap V_0) \cup (V(P) \cap V_j)$ by the maximality of $P$, where the index $j$ is determined such that $S_t \cap V_j = \emptyset$. Thus, we have

$$
\frac{cm^2}{k} \leq |V(P) \cap V_0| \cdot |V_i| + |V(P) \cap V_i| \cdot |V_0| \leq 2tm,
$$

which implies $t \geq cm/(2k)$. Note that $|V(P)| = t + t + k - 1$ and thus $|V(P)| \geq cm/k$. □

The next result enables us to find a collection of vertex-disjoint long $(0, k - 1)$-paths which covers almost all vertices in $V_0$ in an $\varepsilon$-regular $(k + 1)$-partite $(k + 1)$-graph.

**Lemma 4.7.** Let $0 < \varepsilon < \alpha < 1$. Given an $\varepsilon$-regular $(k + 1)$-partite $(k + 1)$-graph $H$ with density at least $\alpha$ and $V(H) = V_0 \cup \cdots \cup V_k$ where $|V_0| = m$ and $m/k \leq |V_i| \leq m$ for $i \in [k]$, $m$ is sufficiently large, we obtain that $H$ contains a family $\mathcal{P}$ of vertex-disjoint $(0, k - 1)$-paths such that for each $P \in \mathcal{P}$, $|V(P)| \geq \varepsilon(\alpha - \varepsilon)m/k$ and $\sum_{P \in \mathcal{P}} |V(P) \cap V_0| \geq (1 - 2k\varepsilon)m$.

**Proof.** We call a $(0, k - 1)$-path $P$ good if $|V(P)| \geq \varepsilon(\alpha - \varepsilon)m/k$. Let $\mathcal{P} = \{P_1, \ldots, P_p\}$ be a largest family of good, vertex-disjoint $(0, k - 1)$-paths and $|V(P_i) \cap V_0| = t_i$ for $i \in [p]$. Note that $|V(P_i) \cap V_j| = \lceil \frac{t_i + k - 1}{k} \rceil$ or $\lceil \frac{t_i + k - 1}{k} \rceil$ for $i \in [p]$ and $j \in [k]$. Suppose to the contrary that $\mathcal{P}$ covers less than $(1 - 2k\varepsilon)m$ vertices of $V_0$ and let $W = V(H) - \bigcup_{P \in \mathcal{P}} V(P)$ be the set of vertices uncovered by $\mathcal{P}$. Then we have $|W \cap V_0| \geq 2k\varepsilon m$. Hence, by the observation that $|V(P_i) \cap V_j| \leq \lceil \frac{t_i + k - 1}{k} \rceil \leq \frac{t_i}{k} + 2$ for each $i \in [p], j \in [k]$ and the fact that $p = |\mathcal{P}| \leq (k + 1)m/\varepsilon(\alpha - \varepsilon)m/k = k(k + 1)(\varepsilon(\alpha - \varepsilon))^{-1}$, and $m$ is sufficiently large, we have that

$$
|W \cap V_i| = |V_i| - |V_i \cap V(P)| \geq \frac{m}{k} - \sum_{i \in [p]} \left(\frac{t_i}{k} + 2\right) \geq \frac{m}{k} - \frac{(1 - 2k\varepsilon)m}{k} - 2p \geq \varepsilon m + 1.
$$

Let $W_i \subseteq W \cap V_i, i \in \{0, 1, \ldots, k\}$ be such that

$$
|W_0| = |W_1| = \cdots = |W_k| = \varepsilon m \geq \varepsilon |V_i|.
$$
Finally, let $\hat{H}$ be the subhypergraph of $H$ induced on the vertex set $W_0 \cup W_1 \cup \cdots \cup W_k$. By the maximality of $\mathcal{P}$, $\hat{H}$ is a $(k+1)$-partite $(k+1)$-graph containing no $(0,k-1)$-path of order at least $\varepsilon(\alpha - \varepsilon)m/k$.

On the other hand, since $H$ is $\varepsilon$-regular, we have

$$d_H(W_0, W_1, \ldots, W_k) \geq d_H(V_0, V_1, \ldots, V_k) - \varepsilon \geq \alpha - \varepsilon,$$

or equivalently,

$$|E(\hat{H})| \geq (\alpha - \varepsilon)(\varepsilon m)^{k+1},$$

and then Lemma 4.6 implies that there is a $(0,k-1)$-path in $\hat{H}$ on at least $\varepsilon(\alpha - \varepsilon)m/k$ vertices, contrary to the maximality of $\mathcal{P}$.

\[\square\]

4.5. Proof of Lemma 2.6.

Proof. We choose the following parameters

$$1/n \ll 1/T_0 \ll \varepsilon, 1/t_0 \ll 1/Q \ll \lambda \ll \delta, \gamma.$$  

Given a $(k, n, \gamma)$-graph system $H = \{H_i\}_{i \in [n]}$ on $V$, we construct a $(1, k)$-graph $H^*$ with vertex set $[n] \cup V$ and edge set $\{i \cup e : e \in H_i, i \in [n]\}$. With an initial partition $[n] \cup V$ of $V(H^*)$, we apply Lemma 4.3 with $\varepsilon$, $t_0$ and obtain a partition $V(H^*) = V_0^* \cup I_1 \cup \cdots \cup I_t \cup W_1 \cup \cdots \cup W_t$ where $t_0 \leq t_1$, $t_2 \leq T_0$, $|I_i| = |W_j| = m$ for $i \in [t_1]$ and $j \in [t_2]$, $|V_0^*| \leq 2\varepsilon n$. By throwing at most $2\varepsilon n/m$ clusters into $V_0^*$, rename it as $V_0$ if necessary, we have $V(H^*) = V_0 \cup I_1 \cup \cdots \cup I_t \cup W_1 \cup \cdots \cup W_t$, where each cluster keeps the size, $|V_0| \leq 4\varepsilon n$. Let $U \subseteq [n]$ and $W_0 \subseteq V$ for $i, j \in [t]$. Let $L := [3kT_0/\varepsilon(\gamma/6 - \varepsilon)]$.

Let $K := K(\varepsilon, \gamma/6)$ be the $(1, k)$-partite cluster hypergraph on $U \cup W$ where $U = \{I_1, \ldots, I_t\}$ and $W = \{W_1, \ldots, W_t\}$. We get a family of $(1, k)$-graphs $F = \{F_1, \ldots, F_k\}$ where $F_i = K([\{I_i \cup \{j\}\}_{j \in [k]}])$.

For each $i \in [k]$, applying Proposition 4.4 and Lemma 4.5 on $F_i$ with $\eta := k\sqrt{\varepsilon}$, we obtain a partition $V(F_i) = S_{i,1} \cup \cdots \cup S_{i,t/Q}$ such that all but at most $\lambda/Q$ classes satisfy $\delta_{i,k-1}(F_i[S_{ij}]) \geq (1/2 + \gamma/2)(Q - k + 1)$. We say that $S_{ij}$ is good if every $(1, k-1)$-tuple in the induced graph $F_i[S_{ij}]$ has degree at least $(1/2 + \gamma/2)(Q - k + 1)$. Denote by $S_i$ the set of indices $j \in [t]$ such that $S_{ij}$ is good. Applying Lemma 4.2 on each $F_i[S_{ij}]$ for $i \in [k]$, $\ell \in S_i$, we obtain perfect matching $M_{i,\ell}$ and let $M_i = \bigcup_{\ell \in S_i} M_{i,\ell}$, $M = \bigcup_{i \in [k]} M_i$. Note that each $M_i$ is a matching in $V_i$. For each $W_j \in W$, let $p_j$ be the number of edges in $M$ that contain $W_j$, $j \in [t]$. Next, we do the following process.

Path Embedding Process:

Given $H^*, U = \{I_1, \ldots, I_t\}$, $W = \{W_1, \ldots, W_t\}$, $M_1, \ldots, M_k$, $W_j := W_j$ for $j \in [t]$ and $i := 1$.

Step 1. For each $e \in M_i$, let $H_e$ be the subgraph of $H^*$ induced on the corresponding clusters constituting the edge $e$, which can be denoted by $I_{e,1}, W_{j_1(e)}, \ldots, W_{j_t(e)}$ where $I_e \in U$.

Step 2. Applying Lemma 4.7 on each $H_e$, $e \in M_i$, we obtain a family $P_e$ of vertex-disjoint $(0,k-1)$-paths that covers all but at most $2k\varepsilon m$ vertices in $I_e$.

Step 3. Let $P_i = \bigcup_{j \leq i} \bigcup_{e \in M_j} P_e$.

Step 4. Update $W_j$ by deleting the vertices used in $P_i$ for $j \in [t]$.

Step 5. Update $i := i + 1$ and do the same from Step 1 to Step 4. When $i = k + 1$, the process terminates.
After the process, we obtain $\mathcal{P} := \mathcal{P}_k$. It follows from the definition of $p_j$ that the size of uncovered vertices of each $W_j$ is

$$|W_j^*| = m - \sum_{W_j \in e, e \in M} |\mathcal{P}_e \cap W_j| \leq m - p_j \left(\frac{(1 - 2k\varepsilon)m + k - 1}{k}\right) \leq m - p_j \left(\frac{1 - 2k\varepsilon}{k}\right).$$

Note that $\sum_{j \in [t]} p_j \geq Q\left(\frac{1}{Q} - \frac{\lambda}{Q}\right)k$. We obtain that $\mathcal{P}$ covers all but

$$|V_0| + \sum_{j \in [t]} |W_j^*| \leq 4\varepsilon n + \sum_{j \in [t]} \left(m - p_j \left(\frac{1 - 2k\varepsilon}{k}\right)\right) \leq (4k + 4)\varepsilon + \lambda n \leq \delta n$$

vertices of $V$. Moreover, since each path in $\mathcal{P}$ has length at least $\varepsilon(\frac{n}{k} - (n - 1))$ and $t \leq T_0$, we have $|\mathcal{P}| < 2n/(\varepsilon(\frac{n}{k} - (n - 1))) < L$. Thus $\mathcal{P}$ is as desired.

5. Concluding Remarks

The threshold for the minimum $(k - 1)$-degree condition in Theorem 1.3 asymptotically equals to the single host hypergraph version in Theorem 1.1. Inspired by a series of very recent successes on rainbow matchings [35, 36, 37, 38] and graph Hamiltonicity, we suspect an even closer relation of this two thresholds.
Conjecture 5.1. For any $\gamma > 0, k \geq 3$, there exists $n_0 \in \mathbb{N}$ such that the following holds. Suppose $H = \{H_i\}_{i \in [n]}$ is an $n$-vertex $k$-graph system on $V$ such that $\delta_{k-1}(H_i) \geq \lfloor (n - k + 3)/2 \rfloor$, then $H$ admits a rainbow Hamilton cycle.

On the other hand, the problem of giving the sufficient condition for the rainbow Hamilton $\ell$-cycle, $\ell \in [k - 2]$, is still open.

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APPENDIX A. THE POSTPONED PROOFS

The idea of the proof is to grow tree-like structures (called cascades) from both designated ends $e_1$ and $e_2$ until they meet, forming the desired rainbow path. This method can be seen in [33, 44]. Before we describe the cascades, it is convenient to introduce the following notation. For two sequences of vertices

$$\omega_1 = (v_1, \ldots, v_r, w_1, \ldots, w_s) \text{ and } \omega_2 = (w_1, \ldots, w_s, u_1, \ldots, u_t)$$

where $r, t \geq 1, s \geq 0$ and all vertices are distinct, we define their concatenation as

$$\omega_1\omega_2 = (v_1, \ldots, v_r, w_1, \ldots, w_s, u_1, \ldots, u_t).$$

This operation can be iterated. For instance, if $\omega_1 = (w_1, \ldots, w_{k-2}), \omega_2 = (w_2, \ldots, w_{k-1})$ and $\omega_3 = (w_3, \ldots, w_k)$ where all $w_i$ are distinct, then $\omega_1\omega_2\omega_3 = (w_1, \ldots, w_k)$. We could write $\omega_1\omega_2w_k$ instead of $\omega_1\omega_2\omega_3$. In this paper, an $r$-element sequence of distinct vertices of $V$ will be referred as $r$-tuple. Let $e_0 = (v_1, \ldots, v_{k-1})$ be a given $(k-1)$-tuple of vertices. We will define the rainbow $e_0$-cascade as an auxiliary sequence of bipartite graphs $G_j, j = 1, 2, \ldots$ with bipartitions $(A_{j-1}, A_j)$, whose vertices are $(k-2)$-tuples of the vertices of $H$ and the edges correspond to some $(k-1)$-tuples of the vertices of $H$. Each node $f \in A_j$ belongs to two graphs $G_j$ and $G_{j+1}$. Its neighbors in $G_j$ belongs to $A_{j-1}$, while its neighbors in $G_{j+1}$ belongs to $A_{j+1}$. For a node $f = (v_1, \ldots, v_{k-2})$ of the rainbow cascade, the vertex $v_1$ is called the prefix, while $v_{k-2}$ is called the suffix of $f$.

We define the rainbow cascade recursively as follows. Let $f_0 = (v_2, \ldots, v_{k-1})$ and let $A_0 = \{f_0\}$. For every vertex $v \notin e_0$, we include the node $g = (v_3, \ldots, v_{k-1}, v)$ in the set $A_1$ if and only if $v_1f_0g = e_0v \in H_{c_1}$ for $c_1 \in [c]$. The graph $G_1$ is the star with center $f_0$ and the arms leading to all the nodes $g \in A_1$.

Further, let $A_2$ be the set of all $(k - 2)$-tuples $h$ such that for some node $g \in A_1$ we have $f_0gh \in H_{c_2}$ where $c_2 \neq c_1$ and $c_2 \in [c]$. Note that each $h \in A_2$ is obtained from a node $g \in A_1$ by dropping the prefix of $g$ and adding a new suffix $u$, we denote such node by $g_u$. The graph $G_2$ consists of all edges $\{g, h\}$ where $g \in A_1, h \in A_2$ and $f_0gh \in H_{c_2}$, it is equal to say $G_2$ consists of all edges $\{g, g_u\}$ where $f_0gu \in H_{c_2}$.

For $j = 3, \ldots, k - 2$, we similarly define

$$A_j = \{h : \exists f \in A_{j-2}, g \in A_{j-1} \text{ such that } \{f, g\} \in G_{j-1}, fgh \in H_{c_j} \text{ where } c_j \neq c_\ell \text{ for } \ell \in [j - 1]\}$$

and $G_j$ as the bipartite graph with bipartition $(A_{j-1}, A_j)$ and the edge set

$$\{\{g, h\} : \exists f \in A_{j-2} \text{ such that } \{f, g\} \in G_{j-1} \text{ and } fgh \in H_{c_j} \text{, where } c_j \neq c_\ell \text{ for } \ell \in [j - 1]\}.$$

In other words, $A_j$ and $G_j$ correspond to the sets of $(k - 2)$-tuples and $(k - 1)$-tuples of the vertices of $V$ which can be reached from $e_0$ in $j$ steps by a rainbow path.

**First refinement.** Having defined $A_j$ and $G_j$ for $j \leq k$, beginning with $j = k - 1$ we change the recursive mechanism by getting rid of the nodes in $A_j$ with too small degree in $G_j$. We define auxiliary

$$A'_{k-1} = \{h : \exists f \in A_{k-3}, g \in A_{k-2} \text{ such that } \{f, g\} \in G_{k-2}, fgh \in H_{c_{k-1}} \text{ where } c_{k-1} \neq c_\ell \text{ for } \ell \in [k-2]\}$$
and $G'_{k-1}$ as the bipartite graph with bipartition \((A_{k-2}, A'_{k-1})\) and the edge set

\[
\{\{g, h\} : \exists f \in A_{k-3} \text{ such that } \{f, g\} \in G_{k-2} \text{ and } fgh \in H_{c_{k-1}} \text{ where } c_{k-1} \neq c_\ell \text{ for } \ell \in [k-2]\}.
\]

Then let $A_{k-1}$ be the subset of $A'_{k-1}$ consisting of all nodes $h$ with $\deg_{G'_{k-1}}(h) \geq \sqrt{n}$ and set $G_{k-1} = G'_{k-1}[A_{k-2} \cup A_{k-1}]$. For convenience, we set $A_j^j = A_j$ and $G_j' = G_j$ for all $j \leq k-2$.

**Second refinement.** For $j \geq k$, to form an edge $\{g, h\}$ of $G_j$ we will now require not one but many nodes $f \in A_{j-2}$ to fulfill the above definition.

Set $m = \lceil n^{1/4} \rceil$. Having defined $G_{j-1}$, let $A_j' = \{h : \exists f_1, \ldots, f_m \in A_{j-2}, g \in A_{j-1} \text{ such that for all } i \in [m], \{f_i, g\} \in G_{j-1} \text{ and } f_ig \in H_{c_j} \text{ where } c_j \neq c_\ell \text{ for } \ell \in [j-1]\}$ and let $G_j'$ be the bipartite graph with bipartition $\langle A_{j-1}, A_j' \rangle$ and the edge set $\{\{g, h\} : \exists f_1, \ldots, f_m \in A_{j-2} \text{ such that for all } i \in [m], \{f_i, g\} \in G_{j-1} \text{ and } f_ig \in H_{c_j} \text{ where } c_j \neq c_\ell \text{ for } \ell \in [j-1]\}$.

Finally, let $A_j$ be the subset of $A_j'$ consisting of all nodes $h$ with $\deg_{G_j'}(h) \geq \sqrt{n}$ and let $G_j = G_j'[A_{j-1} \cup A_j]$. The sequence $(G_j, j = 1, 2, \ldots)$, will be called the rainbow $e_0$-cascade.

### A.1. Properties of the cascade.

**Claim A.1 ([44]).** For every $j \geq k-1$ and every edge $\{g, h\}$ of $G_j$ where $g = (w_1, \ldots, w_{k-2}) \in A_{j-1}, h = (w_2, \ldots, w_{k-1}) \in A_j$ and $(g \cup h) \cap e_0 = \emptyset$ and for every set of vertices $W \subset V \setminus (g \cup h \cup e_0)$ such that $j + |W| \leq n^{1/4}$, there is a rainbow $(j+k-1)$-path $P$ in $H$ which connects $(w_{k-1}, \ldots, w_1)$ with $e_0 = (v_1, \ldots, v_{k-1})$ and $V(P) \cap W = \emptyset$.

**Degrees.** Recall that $G_j' = G_j$ for $j \leq k-2$. For a node $g \in A_j$, we set

\[
d^+(g) = \deg_{G'_{j+1}}(g) \text{ and } d^-(g) = \deg_{G_j}(g)
\]

for the forward and backward degree of $g$ in the cascade. Note that in the definition of $d^+(g)$ we consider the forward degree before some small degree vertices of $A'_{j+1}$ are removed. The reason is that we have no control over the effects of the removal on individual forward degrees. On the other hand, for all $f \in A_j$, $\deg_{G_j}(f) = \deg_{G_j'}(f)$, so the backward degree is unaffected unless the node is removed. It is trivial that $d^-(g), d^+(g) \leq n - k + 2$. Observe that $G_1 \cup \cdots \cup G_{k-2}$ is a tree, thus, $d^-(g) = 1$ for all $g \in A_j, j = 1, \ldots, k-2$. Recall that for $j \geq k-1$ the graph $G_j$ is obtained from $G_j'$ by removing nodes $g$ with $\deg_{G_j'}(g) < \sqrt{n}$. Hence our construction guarantees that for all $g \in A_j, j \geq k-1$, we have $d^-(g) \geq \sqrt{n}$.

For all $j \leq k-2$ and all $g \in A_j$,

\[
d^+(g) \geq \left(\frac{1}{2} + \gamma\right)n,
\]

since there are at least $(\frac{1}{2} + \gamma)n$ vertices $u$ such that $fgu \in H_{c_{j+1}}$ where $f$ is the neighbor of $g$ in $A_{j-1}$. Each such vertex $u$ corresponds to a neighbor $gu$ of $g$ in $A_{j+1}$.

For $j \geq k$, the second refinement affects and no lower bound on $d^+(g)$ is obvious. However, the lower bound $d^-(g) \geq \sqrt{n}$ introduced by the first refinement maintains.

**Growth.** By inequality (7), for each $j \in [k-2]$, we have

\[
|G_j| = |A_j| \geq \left(\frac{1}{2} + \gamma\right)^j n^j,
\]
Call a node $f \in A_j$ small if $d^{-}(f) < \frac{1}{2}n$ and denote by $S_j$ the subset of $A_j$ consisting of the small nodes. Assume for simplicity that $1/e^2$ is an integer.

**Claim A.2** ([44]). There exists an index $j_0$, $k - 1 \leq j_0 \leq k - 1 + (k - 1)/\gamma^2$ such that for all $j = j_0, \ldots, j_0 + k - 2$ we have $|S_j| \leq 2\gamma n^{k-2}$.

**Claim A.3** ([44]). Let

$$k\gamma^{2-k} < 2^{-k}$$

and let $j_0$ be as in Claim A.2. Then $|A_{j_0+k-2} \setminus S_{j_0+k-2}| \geq (n - k + 2 - 2^{2-k}n)k^{-2}$.

**A.2. The completion of the proof of Lemma 2.5.** Let $\gamma_0$ satisfy the condition in Claim A.3, i.e. $\gamma_0 := \gamma^{2-k}$ and $k\gamma_0 < 2^{-k}$. Given two disjoint $(k - 1)$-tuples of vertices $e_1$ and $e_2$, we build the rainbow $e_1$-cascade and the rainbow $e_2$-cascade, with the sets of nodes denoted by $A_j$ and $B_j$.

Let $j_1 = j_0 + k - 2$, where $j_0$ is the index guaranteed by Claim A.2 for the rainbow $e_1$-cascade. Then by Claim A.3, with sufficiently large $n$, by Bernoulli inequality, we have

$$|A_{j_1} \setminus S_{j_1}| \geq (n - 2\gamma_0 n)^{k-2} > (1 - 2k\gamma_0)n^{k-2}.$$ 

On the other hand by inequality (8) for $j = k - 2$, we have $|B_{k-2}| > 2^{2-k}n^{k-2}$,

$$|B_{k-2} \cap (A_{j_1} \setminus S_{j_1})| \geq (2^{2-k} - 2k\gamma_0)n^{k-2} \geq \left(\frac{n}{2}\right)^{k-2}.$$ 

Hence, there is a not small node $g = (u_1, \ldots, u_{k-2}) \in A_{j_1}$ such that $g \cap (e_1 \cup e_2) = \emptyset$ and $g' = (u_{k-2}, \ldots, u_1) \in B_{k-2}$.

Let $e_2 = (w_1, \ldots, w_{k-1})$, $S = \{u_1, \ldots, u_{k-2}, w_{k-1}\}$ and $V_0$ be the set of prefixes $v$ of the neighbors $f \in A_{j_1-1}$ of $g$. By Fact 3.1, we have $|N_{H_{j_1}} \cap V_0| > \gamma n$, and thus, there is at least one vertex $v_0 \notin e_2$ such that $\{v_0, u_1, \ldots, u_{k-2}, w_{k-1}\} \in H_{c_{j_1}}$. Besides, $g' = (u_{k-2}, \ldots, u_1) \in B_{k-2}$, which guarantees that there is rainbow path $u_{k-2} \cdots u_1 w_{k-1} \cdots w_1$.

Let $P_1$ be a rainbow $(j_1 + k - 1)$-path from $e_1$ to $(u_{k-2}, \ldots, u_1, v_0)$ which avoids the vertices of $e_2$. The existence of $P_1$ follows from Claim A.1 with $W = e_2$. The path $P$ obtained from $P_1$ by adding the segment $(w_{k-1}, \ldots, w_1)$ and the “hook-up” edge $\{v_0, u_1, \ldots, u_{k-2}, w_{k-1}\}$, is the desired rainbow path connecting $e_1$ and $e_2$.

By the bound on $j_0$ established in Claim A.2 and since $\gamma \leq 1/2$,

$$|V(P)| = j_1 + 2(k - 1) = j_0 + 3k - 4 \leq \frac{k - 1}{\gamma^2} + 4k - 5 \leq \frac{2k}{\gamma^2}.$$
Figure 5. A rainbow path connects two \((k - 1)\)-tuples \(e_1\) and \(e_2\)