Duality of large $N$ Yang-Mills Theory on $T^2 \times R^n$.

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We find aspects of electrically confining large $N$ Yang-Mills theories on $T^2 \times R^{d-2}$ which are consistent with a $GL(2, \mathbb{Z})$ duality. The modular parameter associated with this $GL(2, \mathbb{Z})$ is given by $\frac{m}{N} + i \Lambda^2 A$, where $A$ is the area of the torus, $m$ is the t'Hooft twist on the torus, and $\Lambda^2$ is the string tension. $N$ is taken to infinity keeping $\frac{m}{N}$ and $g^2 N$ fixed. This duality may be interpreted as T-duality of the QCD string if one identifies the magnetic flux with a two-form background in the string theory. Our arguments make no use of supersymmetry. While we are not able to show that this is an exact self duality of conventional QCD, we conjecture that it may be applicable within the universality class of QCD. We discuss the status of the conjecture for the soluble case of pure two dimensional Euclidean QCD on $T^2$, which is almost but not exactly self dual. For higher dimensional theories, we discuss qualitative features consistent with duality. For $m = 0$, such a duality would lead to an equivalence between pure QCD on $R^4$ and QCD on $R^2$ with two adjoint scalars. When $\Lambda^2 A << m^2/N^2$, the proposed duality includes exchanges of rank with twist. This exchange bears some resemblance, but is not equivalent, to Nahm duality. A proposal for an explicit perturbative map which implements duality in this limit is discussed.
1. Introduction

The purpose of this paper is to present evidence suggesting the existence of a $GL(2, Z)$ duality of confining large $N$ gauge theory which resembles T-duality of a string description. We will consider $SU(N)/Z_N$ gauge theory with two spatial directions compactified on a torus. For simplicity we take this torus to be square. This theory has superselection sectors labeled by the t’Hooft magnetic flux $m$ through the torus \([1]\). We shall study the t’Hooft large $N$ limit with $g^2N$ and $m/N$ fixed, and focus almost entirely on the planar limit. Supersymmetry will play no role in our present discussion. Unlike Olive Montonen duality, too much supersymmetry ruins the conjecture, which depends critically on confinement.

Two generators of the conjectured $GL(2, Z)$ are trivially realized even at finite $N$. One such generator takes $m$ to $m + N$. This is a symmetry because the t’Hooft twist $m$ is only defined modulo $N$. The other trivial generator corresponds to parity and takes $m$ to $-m$. If one could also exchange $N$ and $m$, then a $GL(2, Z)$ symmetry would be generated. Such an exchange is often referred to as Nahm duality. Nahm duality \([2][3]\) has been studied in the context of non-confining super Yang-Mills theories with 16 supersymmetries \([4]\). In this case $GL(2, Z)$ cannot be an exact duality of the theory, in part because exchange of $N$ and $m$ does not make sense for $m = 0^*$. This difficulty does not arise in our conjecture, for which the modular parameter is not $\tau = \frac{m}{N}$, but

$$\tau = \frac{m}{N} + i\Lambda^2 A$$

(1.1)

Here $A$ is the area of the torus and $\Lambda^2$ is the string tension. $GL(2, Z)$ is generated by

$$\tau \rightarrow \tau + 1,$$
$$\tau \rightarrow -\bar{\tau},$$
$$\tau \rightarrow -\frac{1}{\tau}$$

(1.2)

Since $m$ and $N$ are integers, the last generator makes sense for non-zero $\Lambda^2 A$ only if $N$ is taken to infinity. In this limit $\frac{m}{N}$ becomes a continuous parameter.

In section 2 we will attempt to motivate the conjecture by discussing qualitative properties of confining Yang-Mills theory which are consistent with T-duality of a string

\* Yang-Mills theory on a non-commutative torus may have an exact duality analogous to Nahm duality \([5][6]\).
description. The $GL(2, Z)$ we propose resembles string T-duality if one identifies $\frac{m}{N}$ with a two form modulus in the string description.

If this duality exists for pure QCD, then large $N$ pure QCD on $R^4$ is dual to a large $N$ QCD on $R^2$ with two adjoint scalars. Some evidence for this is discussed in section 3. At the same time we mention some possible problems which could invalidate $GL(2, Z)$ duality. It may be that T-duality relates theories in the same universality class as QCD, but is not a self duality of conventional QCD.

In section 4 we discuss the status of the conjecture for pure two dimensional QCD on $T^2$, for which the large $N$ partition function is calculable. In this case the partition function is indeed a function of the modular parameter $\tau = \frac{B}{\Lambda^2} + i\frac{A}{2\pi}A$, with $\lambda = g^2 N$. The partition function is almost but not exactly modular invariant. A very simple modification of the partition function removes the anomaly.

There are domains in which the proposed $GL(2, Z)$ takes weak coupling to weak coupling, and should therefore be visible perturbatively. This includes the limit in which $\Lambda^2 A << m^2/N^2$. In section 4 we construct an explicit map which relates theories with the same greatest common divisor of $N$ and $m$ in the limit of vanishing coupling. This map treats $\frac{m}{N}$ as a modular parameter. Under $\tau \rightarrow -\frac{1}{\tau}$ the area is mapped linearly, instead of being inverted. In this respect our proposal differs from Nahm duality, which simultaneously exchanges rank with flux and inverts the area.

2. Duality and confinement

It has been suspected for some time that confining Yang-Mills theories have a string description [7]. Pure two dimensional QCD is known to be a string theory [8] [9] [10] [11], except on a sufficiently small 2-sphere [12]. In higher dimensions the situation is less clear, although some progress has been made [13] [14] [15] [16]. In this section we will assume that a string description exists, and discuss some qualitative features of a confining large $N$ Yang-Mills theory on $T^2 \times R^n$ which suggest that this string theory may have a T-duality. For a review of T-duality in the context of critical strings, the reader is referred to [17].

The T-duality group for compactifications on $T^2$ includes a $GL(2, Z)$ subgroup for which the modular parameter is $\tau = B + i\Lambda^2 A$. $B$ is a flat two form background on $T^2$, $\Lambda$ is the area of $T^2$, and $\Lambda$ is the string tension. It is convenient to use complex coordinates and define $w = w^1 + iw^2$ where $w^i$ are the string winding numbers, and $P = P_1 + iP_2$ where $P_i$ are the integer string momenta. The generator $\tau \rightarrow -\frac{1}{\tau}$ is accompanied by the
exchange $w \leftrightarrow P$. Parity takes $\tau \rightarrow -\bar{\tau}$, $w \rightarrow \bar{w}$ and $P \rightarrow \bar{P}$. The remaining generator, $\tau \rightarrow \tau + 1$, is accompanied by $P \rightarrow P + iw$. This momentum shift arises because the canonical string momentum $P$ has a contribution $iBw$ arising from the two form term in the world sheet action, $2\pi \int \Sigma B$. The $GL(2, \mathbb{Z})$ invariant energy of a multiplet of mass $M$ is given by

$$E = \sqrt{M^2 + \frac{\Lambda^2}{Im(\tau)}|P - i\tau w|^2}. \quad (2.1)$$

Now consider an electrically confining Yang-Mills theory on $T^2 \times \mathbb{R}^n$ with the time direction lying in $\mathbb{R}^n$. The energy of an t’Hooft electric flux $[1]$ on a large (square) torus is given by the confining potential $e\Lambda^2\sqrt{A}$. This energy equals that of a Kaluza Klein mode on a torus with the area inverted, $A' = 1/(\Lambda^4 A)$. This is consistent with T-duality for vanishing $B$ if one identifies the electric flux with the string winding number. A more difficult question is whether the energy of an electric flux on a small torus is equal to the energy of a state with momentum on a large dual torus. This question will be discussed in the next section.

One must also identify the quantity in the Yang-Mills theory corresponding to the two form modulus. This quantity should be continuous and periodic. A natural candidate is $\frac{m}{N}$, where $m$ is the $SU(N)$ t’Hooft magnetic flux, which is defined modulo $N$. $m/N$ becomes a continuous parameter in the $N \rightarrow \infty$ limit. With this identification, the two form contribution to the canonical string momentum $P_i = B_{ij}w^j + \ldots$ has the form one expects when written in terms of Yang-Mills variables; $P_i = \frac{m_{ij}}{N}e^j + \ldots$.

This last point requires some clarification. The division of the momentum into a contribution from $SU(N)$ t’Hooft fluxes and a contribution from everything else is more natural in if one considers instead a $U(N)$ theory, which has vanishing total twist. The $U(N)$ theory is locally $U(1) \times SU(N)/\mathbb{Z}_N$. The requirement of vanishing total twist amounts to the statement that the $U(1)$ magnetic flux is equal, modulo N, to the $SU(N)$ magnetic fluxes $[18]$:

$$m' = m + lN. \quad (2.2)$$

A similar statement,

$$e' = e + kN, \quad (2.3)$$
holds for the electric fluxes, defined as eigenvalues of certain large gauge transformations \[\text{[1]}\]. The Yang-Mills momentum is

\[
P_i^{YM} = \int Tr F_{ij} F_{0j} = p_i + \frac{m'_{ij}}{N} e^i j
\]

(2.4)

where the second term is the zero-mode contribution, which comes from the $U(1)$ electric and magnetic flux quantum numbers. The $U(1)$ fluxes $e', m'$ are well defined, however the $SU(N)$ t’Hooft fluxes $e$ and $m$ are defined modulo $N$. Therefore the momentum defined by

\[
P_i = p_i + \frac{m_{ij}}{N} e^i j
\]

(2.5)

has the same transformation properties under integer shifts of $\frac{m}{N}$ as the canonical string momentum. We are not really interested in the $U(N)$ theory however, which in dimensions greater than two has a decoupled massless photon having nothing to do with the QCD string. In the $SU(N)/Z_N$ case, it is less clear how to define a momentum with the correct transformation properties, since the fluxes are not related to integrals of local operators and do not contribute to the momentum in any direct way. One possibility is to define the quantity $p_i$ as $p_i = P_i^{YM} - \frac{m_{ij}}{N} e^j$, for $-N/2 < m < N/2$. Then the momentum (2.5) has the correct transformation properties under $\tau \rightarrow \tau + 1$ and parity.

Note that there is another context in which momentum and electric flux are exchanged by a T-duality \[\text{[19]}\text{[20]}\]. This arises for compactifications of M theory on $S^1_1 \times T^d$, which is believed to be described by a non-confining $d + 1$ dimensional $U(N)$ Yang-Mills theory with maximal supersymmetry \[\text{[21]}\text{[22]}\text{[19]}\text{[20]}\]. However in this case the M theory spectrum does not correspond to confinement in the Yang-Mills theory. The momentum which gets exchanged with electric flux is an M theory momentum and not a momentum in the world volume of the Yang-Mills theory. Furthermore the Yang-Mills energy corresponds to the M theory light cone Hamiltonian $P^+_M$ rather than $P^0_M$. The energy of a BPS electric flux is equal to that of a classical $U(1)$ electric flux spread evenly over the torus. On the other hand for a confining Yang-Mills theory, one expects the Yang-Mills energy to equal the string theory energy.
3. $D = 4 \leftrightarrow D = 2$ duality

If the conjectured duality exists, there is a remarkable consequence for electrically confining large N Yang-Mills theories on $R^d$. One can obtain $R^d$ from $R^{d-2} \times T^2$ by making the torus very large. Under inversion of the area of the torus (for $m = 0$), one obtains a dimensionally reduced theory. Thus duality would imply that large $N$ pure QCD in four dimensions is equivalent to Large $N$ QCD with two adjoint scalars in two dimensions. In fact such a two dimensional model has been used to approximate the dynamics of pure QCD in 4 dimensions QCD [23] [24] [25]. The adjoint scalars in this model play the role of transversely polarized gluons. In [25] the spectrum of this two dimensional model, computed by discrete light cone quantization, was compared to the glueball spectrum of pure 4-d QCD computed using Monte-Carlo simulation. The degree of numerical accuracy allows only crude comparison, however the spectra have some qualitatively agreement. Perhaps in the $N \to \infty$ limit the agreement is more than just qualitative.

If such an equivalence exists, duality must map the QCD scale nontrivially. The mass gap is only proportional to the QCD scale, defined in terms of the running coupling, if the torus is very large. In general the mass gap, or string tension $\Lambda^2$, depends on both the QCD scale $\Lambda_{QCD4}$ and the area of the torus. Let us fix the mass gap and take the size of the torus to infinity. $\Lambda_{QCD4}$ on the small dual torus with area $A$ can be found by matching the coupling of the two dimensional reduced theory to the 4 dimensional running coupling at the Kaluza Klein scale:

$$\sqrt{A}g_{2D} = \sqrt{A}\Lambda = g_{4D}(\frac{1}{\sqrt{A}}, \Lambda_{QCD4}).$$

(3.1)

The notion that large $N$ can generate extra dimensions is not novel. The $d = 2 \leftrightarrow d = 4$ equivalence we suggest here is similar to Eguchi-Kawai reduction [26], although our discussion pertains to the continuum theory. Assuming the proposed duality exists, the momentum in the two hidden dimensions of the reduced theory must correspond to the electric flux on the vanishingly small torus in the unreduced theory. More precisely, the hidden momentum should be related to the length of a string wrapped around the small torus, $p_i = e_i\sqrt{A}\Lambda^2$, which is held fixed as $A \to 0$. For $SU(N)/Z_N$, the flux $e_i$ is defined [1] by considering certain large gauge transformations $\hat{T}_i$ which leave the boundary conditions on the torus invariant. Such gauge transformations correspond to elements of $SU(N)$ satisfying

$$g_i(x + 2\pi R_s) = g_i(x)e^{2\pi i \frac{e_i}{N}}.$$  

(3.2)
where $R_s$ is the radius of a cycle of the small (square) torus. $\hat{T}_i^N$ is a “small” gauge transformation leaving physical states invariant. The electric fluxes are defined by the eigenvalues of $\hat{T}_i$, $\exp(2\pi i e_i N)$. To see what $e_i$ becomes in the reduced theory, consider a Wilson loop around the $i$'th cycle of $T^2$. Under a large gauge transformation

$$Pe^{i\oint A} \to e^{\frac{2\pi}{N}} Pe^{i\oint A},$$

(3.3)

In the reduced theory, this transformation becomes a global symmetry:

$$e^{iR_X} \to e^{\frac{2\pi}{N}} e^{iR_X}.$$  

(3.4)

Here $X^i$ are the adjoint scalars of the reduced theory. If hidden momenta exist in the $d = 2$ theory, they should be the generators of this transformation. Since $N$ such transformations give the identity map, we wish ultimately to interpret them as translations around a large discretized hidden torus which becomes continuous as $N \to \infty$.

A problem arises because the naive reduction of pure $SU(N)$ gauge theory,

$$S = \frac{N}{\lambda_{2d}} \int d^2x Tr(F_{\mu\nu}^2 + D_{\mu}X^i D^{\mu}X^i + [X^2, X^3]^2),$$

(3.5)

is not a finite theory and requires a mass counterterm. It also does not have a symmetry corresponding to (3.4). It is tempting to consider a $U(N)$ theory instead, in which case the naive reduced theory has a continuous $U(1) \times U(1)$ symmetry. However in this case there is a free photon and no mass gap. States charged under the $U(1) \times U(1)$ have energies inconsistent with (2.1). Therefore we will only consider the $SU(N)$ theory. For $R_s$ finite but small compared to $\frac{1}{\Lambda}$, an effective action which is $Z_N \times Z_N$ symmetric may be written in terms of the $SU(N)$ Wilson loops $h_i = \exp(iR_X X)$;

$$S_{SU(N)} = \frac{N}{\lambda_{2d}} \int d^2x Tr \left( F_{\mu\nu}^2 + \frac{1}{R_s^2}(h_i D_{\mu} h_i^{\dagger})^2 + \frac{1}{R_s^3}[h_2, h_3][h_2^{\dagger}, h_3^{\dagger}] \right).$$

(3.6)

The naive reduced action is recovered by writing $h_i = 1 + \exp(iR_X X) + \ldots$ and taking $R_s \to 0$ with $X_i$ fixed. The metric of this sigma model is proportional to $1/R_s^2$, which corresponds to the area of the hidden torus on which $Z_N \times Z_N$ acts as a translation. Note that a mass counterterm proportional to $\frac{1}{R_s^2} \sum_i Tr h_i$ is prohibited by the $Z_N \times Z_N$ symmetry.

The important question is whether this symmetry is spontaneously broken below a critical $R_s$. The two dimensional theory can not generate extra dimensions as $N \to \infty$. 


if $R_s < R_s^c$. If there is a T-duality, the dual torus should have an unbroken translation symmetry. However, from experience with finite temperature deconfinement transitions, one might conclude that $Z_N \times Z_N$ should be broken for sufficiently small $R_s$. Note that in our case, symmetry breaking would not be interpreted as deconfinement, since the order parameters are spacelike Wilson loops, rather than a timelike Wilson loop*.

The question of whether symmetry breaking occurs at finite $R_s$ is closely tied to the question of whether the limit we wish to take exists. This limit is $N \to \infty$ followed by $R_s \to 0$ while tuning $\Lambda_{QCD}$ to keep the mass gap fixed. If $Z_N$ can be thought of as a continuous $U(1)$ in the $N \to \infty$ limit, then the broken phase would have a goldstone boson. Thus there can not be any symmetry breaking if the reduced theory has a gap in the large $N$ limit. Furthermore if a string description remains valid for small $R_s$, one would not expect symmetry breaking for $N \to \infty$. If there were symmetry breaking, states with different $Z_N \times Z_N$ charges, or “hidden momenta,” would become degenerate and condense. Recall that the hidden momenta are given by $e_i R_s \Lambda^2$. $e_i R_s$ is the minimum length of a QCD string wrapping $e_i$ times around a cycle of the small torus. Since $e_i$ is defined modulo $N$, the minimum length can be at most $(N - 1)R_s$. If $N$ is infinite, then the minimum length is unbounded and can be held fixed as $R_s \to 0$ by scaling $e_i$ like $1/R_s$. For fixed string tension $\Lambda^2$ one would expect wrapped strings with arbitrarily large minimum lengths to be heavy, in which case they could not condense.

If there is no symmetry breaking at infinite $N$, then the vacuum has zero electric flux. Introducing even the minimum amount of flux, or a singly wrapped string, would lead to an energy in excess of the mass gap. A possible explanation for such unusual behavior is that the string spreads in the non-compact direction as $R_s \to 0$.

If it can be shown that the $Z_N \times Z_N$ symmetry is unbroken in the large $N$ limit, one must then show that the spectrum of the $R_s \to 0$ theory has a hidden four dimensional Lorentz invariance. This might be testable numerically using the two dimensional description. Having found candidates for momentum in the hidden directions, it might also be possible to see if a four dimensional Lorentz algebra exists. However we leave these tests for the future.

It is entirely possible that the $Z_N \times Z_N$ symmetry is spontaneously broken in conventional pure QCD on a sufficiently small torus. Nonetheless, $GL(2, \mathbb{Z})$ might still relate

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* In the finite temperature theory, the Wilson loop wrapped around the Euclidean time direction is an order parameter of the deconfining phase transition [27]. In this case, the $Z_N$ symmetry is unbroken in the confining phase.
theories in the same universality class as QCD. Also if the reduced theory (3.6), has a symmetry breaking transition at finite $R_s$, it could be interesting to study this theory for $R_s > R_c$, since in this domain the large $N$ limit may generate extra dimensions. This action is also interesting since it appears to be a more suitable action to describe dimensionally reduced pure QCD, due to the absence of a mass renormalization.

4. Modular invariance in two dimensions

While the QCD string may fail to have a T-duality in four dimensions, it comes very close to having a T-duality in two dimensions. In this section we study pure Euclidean $SU(N)$ Yang-Mills theory on $T^2$ in the t’Hooft large $N$ limit. Although this theory has no dynamics, we can use it to test the conjecture that the t’Hooft twist and the area combine into a modular parameter (1.1). The partition function of this theory on a surface of arbitrary genus is known in terms of a sum over representations of $SU(N)$ [28][29]. On a two torus, the partition function in the absence of twisted boundary conditions is given by

$$Z = \sum_R e^{g^2 AC_2(R)},$$

where $C_2(R)$ is the quadratic casimir in the representation $R$. In a sector with t’Hooft twist $m$, the partition function may be computed by continuum methods [30], and is also easily computed using the heat kernel action on a $T^2$ lattice with twisted boundary conditions. The result is

$$Z = \sum_R e^{g^2 AC_2(R)} \frac{X_R(D_m)}{d_R}$$

where $X_R(D_m)$ is the trace of the element $D_m$ in the center of $SU(N)$ corresponding to the t’Hooft twist. In a representation whose Young Tableaux has $n_R$ boxes,

$$D_m = e^{2\pi i \frac{m}{N} n_R}$$

To compute the large $N$ expansion of the partition function, we repeat the calculations of [3] for the case of nonvanishing twist. This expansion is obtained by considering composite representations $\bar{S}R$ obtained by gluing the Young Tableaux of a representation $R$ with a finite number of boxes onto the right of the complex conjugate of a representation $S$ with a finite number of boxes. The quadratic Casimir of such a representation in the $\frac{1}{N}$ expansion is

$$C_2(\bar{S}R) = n_R N + n_S N + \ldots$$
Furthermore,

\[ D_m = e^{2\pi i \frac{m}{N} (n_R - n_S)} \]  \hspace{1cm} (4.5)

Therefore the free energy at leading order is

\[ F = \ln \left| \sum_{n} \rho(n) e^{g^2 N A n + 2\pi i \frac{m}{N} n} \right|^2 \]  \hspace{1cm} (4.6)

where \( \rho(n) \) counts the number of representations with \( n \) boxes. This sum is computed just as in [8], giving

\[ F = \ln \left| \frac{e^{2\pi i \frac{m}{N} \tau}}{\eta(\tau)} \right|^2 \]  \hspace{1cm} (4.7)

where \( \eta \) is a Dedekind eta function, and

\[ \tau = \frac{m}{N} - \frac{\lambda A}{2\pi i}, \]  \hspace{1cm} (4.8)

with \( g^2 N = \lambda \). Thus the complexification of the area generated by modular transformations corresponds to a non-zero t’Hooft twist. Upon completing this paper we became aware that M. Douglas has also made this observation using a Jevicki-Sakita boson description of 2-d QCD on the torus [31]. As noted in [32] [31] the free energy is almost, but not exactly invariant under inversion of the area. The eta function has the modular properties

\[ \eta(\tau + 1) = \eta(\tau) \]
\[ \eta(-\frac{1}{\tau}) = \sqrt{i\tau} \eta(\tau). \]  \hspace{1cm} (4.9)

A simple modification of the partition function,

\[ Z = Z \frac{e^{\frac{\Delta A}{2\pi i}}}{\sqrt{\lambda A}} \]  \hspace{1cm} (4.10)

is modular invariant. The extra factor exponential in the area is a local term, \( \sim \int \sqrt{\det g} \), corresponding to a modified ground state energy. On the other hand, the non-local \( \sqrt{\lambda A} \) factor poses a problem for modular invariance. Nonetheless the deviation from modular invariance is very simple and we feel deserves better understanding. The two dimensional QCD string has at least an “approximate” T-duality.
5. Exchange of rank and twist

The \( GL(2, \mathbb{Z}) \) we have proposed treats \( \frac{m}{N} \) as a modular parameter as long as one only acts with \( \tau \to -1/\tau \) in the region \( \Lambda^2 AN^2/m^2 \ll 1 \). In this domain the area is mapped linearly and is never inverted. If this duality exists, it should be possible to see it perturbatively in this region. In this section we discuss an attempt to construct a classical map under which \( N \) and \( m \) transform as a doublet of \( GL(2, \mathbb{Z}) \). We are only able to show the validity of the map we construct in the limit of vanishing coupling. However if it does extend to finite coupling, it has the correct qualitative property that the area of the torus is mapped linearly rather than being inverted. While the results of this section are very far from quantitative rigor, we feel they are at least suggestive. The map we propose will take an \( SU(N)/\mathbb{Z}_N \) theory with flux \( m \), or an \( (N, m) \) theory, into a \( (p, 0) \) theory where \( p \) is the greatest common divisor of \( N \) and \( m \). \( GL(2, \mathbb{Z}) \) is then generated by inverting this map to get other theories with the same greatest common divisor.

We begin by reviewing the definition of magnetic flux for \( SU(N)/\mathbb{Z}_N \) Yang-Mills on a torus. Following t’Hooft [1], translations around a cycle of the torus are equivalent to gauge transformations:

\[
A_i(x + a_j) = U_j(x)A_i(x)U_j^\dagger(x) + U_j(x)i\partial_i U_j^\dagger(x)
\]

(5.1)

where \( a_i \) are the periodicities of the torus. For adjoint matter, one has the following constraint:

\[
U_i(x)U_j(x + a_i)U_i^\dagger(x + a_j)U_j^\dagger(x) = e^{2\pi i \frac{m_{ij}}{N}}
\]

(5.2)

where the twist \( m_{ij} \) is an integer, and is taken as the definition of nonabelian magnetic flux. We consider the case of Yang-Mills on \( T^2 \times \mathbb{R}^n \), and drop the indices \( ij \). Note that \( m \to m + jN \) is a manifest symmetry. We shall assume the timelike direction lies in \( \mathbb{R}^n \). In \( A_0 = 0 \) gauge we choose the following twisted boundary conditions

\[
U_1(x) = Q,
\]
\[
U_2(x) = P^m,
\]

(5.3)

where

\[
Q = e^{i\theta}\begin{pmatrix}
1 & e^{\frac{2\pi i}{N}} & \cdots \\
e^{\frac{2\pi i}{N}} & 1 & \cdots \\
\cdots & \cdots & \cdots
\end{pmatrix},
\]
\[
P = e^{i\theta'}\begin{pmatrix}
1 & 1 & \cdots \\
1 & 1 & \cdots \\
\cdots & \cdots & \cdots
\end{pmatrix}
\]

(5.4)
The phases $\theta$ and $\theta'$ are chosen so that $Q$ and $P$ have determinant 1. $Q$ and $P$ satisfy
\[ PQ = QPe^{\frac{2\pi i}{N}}. \] (5.5)

Different choices of twisted boundary conditions with the same magnetic flux are related by large gauge transformations, but in $A_0 = 0$ gauge no paths connect different twisted boundary conditions at different times. Different twisted boundary conditions with the same magnetic flux define equivalent superselection sectors and there is nothing special about our choice. There are other large gauge transformations which leave the boundary conditions invariant. The eigenvalues of these gauge transformation determine the t’Hooft electric flux.

The constraints imposed by twisted boundary conditions can be solved to find the independent perturbative degrees of freedom. To this end we shall work in momentum space. Since $Q^N = 1$, the gauge potentials are periodic in $x^1$ on the interval $[0, Na^1]$. To find the periodicity in $x^2$, one looks for the minimal power to which one must raise $P^m$ to get 1. Writing the pair $(N, m)$ as $(p\alpha, p\beta)$ where $\alpha$ and $\beta$ are relatively prime, one finds that this power is $\alpha$. Therefore the gauge potentials are periodic in $x^2$ on the interval $[0, \alpha a_2]$. The Fourier modes of the gauge field strength $F_{n_1, n_2}$ satisfy the twisted boundary conditions,
\[ e^{2\pi i \frac{n_1}{N}} F_{n_1, n_2} = QF_{n_1, n_2}Q^\dagger \] (5.6)
and
\[ e^{2\pi i \frac{n_2}{\alpha}} F_{n_1, n_2} = P^m F_{n_1, n_2}P^m\dagger. \] (5.7)

Making use of the algebra (5.5), a general solution of (5.6) is
\[ F_{n_1, n_2} = M_{n_1, n_2} P^{n_1}, \] (5.8)
where $M_{n_1, n_2}$ is a diagonal $N \times N$ matrix which is traceless when $n_1 = 0 \text{ mod } N$. Note that if $m$ vanished there would be $N - 1$ degrees of freedom for each Fourier mode with $n_1 = 0 \text{ mod } N$ and $N$ degrees of freedom for all the others, rather than $N^2 - 1$ degrees of freedom for every Fourier mode. This is because the (non-gauge invariant) momentum in the $x^1$ direction of the torus is fractional in units of $\frac{1}{N}$. Heuristically, in going to a more conventional gauge with $U_1 = U_2 = I$, the fractional modes become integral and fill out the Lie algebra.
Now consider arbitrary $m$. The second constraint (5.4) gives

$$P^m M_{n_1,n_2} P^{-m} = M_{n_1,n_2} e^{2\pi i \frac{n_2}{N}} ,$$

(5.9)

Conjugating $M$ by $P^m$ shifts the elements of $M$ cyclically by an amount $m$. Thus we find that the number of independent elements of $M_{n_1,n_2}$ is $p$, the greatest common divisor of $N$ and $m$;

$$M_n = \begin{pmatrix} M'_n & M'_n e^{2\pi i \frac{n_2}{p}} \\ M'_n e^{2\pi i \frac{n_1}{p}} & \ldots \end{pmatrix}$$

(5.10)

where $M'_n$ is a diagonal $p \times p$ matrix. It is now easy to construct a candidate for the gauge field of the dual theory with rank $p$ and vanishing magnetic flux, or a $(p,0)$ theory. We define the dual field strength as

$$F'_n = e^{i\phi(n)} M'_n P'^{n_1}$$

(5.11)

where $P'$ is the $p \times p$ shift matrix. The diagonal phase factor $\phi$ is chosen so that the dual field strength is real, $F'_{-n} = F'^\dagger_n$. This is general solution of the constraint

$$e^{2\pi i \frac{n_1}{p}} F'_n = Q' F'_{n_1,n_2} Q'^\dagger ,$$

(5.12)

where $Q'$ is the $p \times p$ matrix

$$Q' = \begin{pmatrix} 1 \\ e^{\frac{2\pi i}{p}} \\ \vdots \\ e^{2\pi i (p-1)} \end{pmatrix} .$$

(5.13)

Therefore $F'$ is a candidate for a field strength on a dual torus with the twisted boundary conditions given by $U'_1 = Q'$ and $U'_2 = I$, which corresponds to vanishing magnetic flux.

The action of the $(N,m)$ theory is

$$S = \int_{R^n} \sum_n Tr_{N \times N} F_{\mu\nu,n} F'^{\dagger}_{\mu\nu,n}$$

(5.14)

Written in terms of the proposed dual field strength this becomes

$$S = \int_{R^n} \alpha Tr_{p \times p} F'_{\mu\nu,n} F'^{\dagger}_{\mu\nu,n} .$$

(5.15)
However, for duality to hold, we must be able to define a dual gauge potential which solves the Jacobi identity and gives the correct measure in the path integral. Let us define the dual gauge potential the same way we defined the dual field strength. Then the map between the \((N,m)\) gauge potential and the dual \((p,0)\) gauge potential is linear, so one might expect that the measure maps correctly. We will scale the fields such that the coupling constant appears only in the interaction terms. If one neglects the the \([A_\mu, F^{\alpha\beta}]\) terms, it is easy to check that the Jacobi identity is preserved by the map, provided that the periodicities of fields on the original torus are the same as those on the dual torus. In other words \(Na_1 = N'a'1 = pa'1\) and \(\alpha a^2 = \alpha'a'^2 = a'^2\). Thus the area of the torus is mapped linearly. Working in a Hamiltonian formulation, one can easily check that the Hamiltonian, commutation relations, and Gauss law constraint of the \((N,m)\) theory at zero coupling map to those of the \((p,0)\) theory at zero coupling.

At finite coupling however, the Jacobi identity is no longer satisfied. It is violated by terms involving the difference between the \(N \times N\) shift matrix \(P\) and \(\alpha\) copies of the \(p \times p\) shift matrix,

\[
P - \begin{pmatrix} P' & & \\ & P' & \\ & & \ldots \end{pmatrix}.
\]

These matrices are related by moving a finite number of ones and zeros in the limit that \(p \to \infty\). Thus it is very tempting to neglect the difference. However when these matrices are raised to a power of order \(p\), the difference is not always negligible. This occurs when \(n_1\) is of order \(p\). We can not discard such modes from the action, since they may correspond to a finite gauge invariant physical momenta. It may be possible that this discrepancy is negligible at leading order in some weak coupling expansion, however we will not attempt to prove this.

6. Conclusion

We have found properties of large \(N\) confining Yang-Mills theories which are suggestive of a duality resembling T-duality of a string description. Whether such a duality really exists for pure QCD or some theories in the same universality class remains to be seen. If such a duality existed it would be quite useful, since it would relate theories on \(R^4\) to more numerically tractable theories on \(R^2\). Among the the principle obstacles to such a duality is the possibility of \(Z_N \times Z_N\) symmetry breaking, and the absence of a string description on a sufficiently small torus. If the \(Z_N \times Z_N\) symmetry is broken at finite radius, there may
still be interesting phenomena at large $N$ for $R > R_c$ due to the existence of wrapped string states. Also, the two dimensional gauged sigma model given by (3.6) may have interesting properties for $R_s > R_s^c$, such as the generation of extra dimensions in the large $N$ limit.

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