Thermally isolated Luttinger liquids with noisy Hamiltonians

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We study the dynamics of a quantum-coherent thermally isolated Luttinger liquid with noisy Luttinger parameter. To characterize the fluctuations of the absorbed energy in generic noise-driven systems, we first identify two types of energy moments, which can help tease apart the effects of classical (sample-to-sample) and quantum sources of fluctuations. One type of moment captures the total fluctuations due to both sources, while the other one captures the effect of the classical source only. We then demonstrate that in the Luttinger liquid case, the two types of moments agree in the thermodynamic limit, indicating that the classical source dominates. In contrast to equilibrium thermodynamics, in this driven system the relative fluctuations of energy do not decay with the system size. Additionally, we study the deviations of equal-time correlation functions from their ground-state value, and find a simple scaling behavior.

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I. INTRODUCTION

Recent experimental developments with ultracold atoms have motivated numerous studies of the nonequilibrium dynamics of thermally isolated many-body quantum systems. Most of these studies focus on deterministic quantum evolution, generated, e.g., by a sudden quench or gradual ramping of the Hamiltonian (see Ref. [1] and the references therein). However, stochastic driving of thermally isolated systems with noisy Hamiltonians[2–4] is much less studied, and, in particular, the role of quantum coherence remains largely unexplored.

Understanding the unitary dynamics generated by stochastic Hamiltonians is of interest from both experimental and fundamental viewpoints. On the experimental side, the preparation of strongly correlated ground-state wave functions by controlled unitary evolution (for the purpose of quantum simulations, for example) is an important goal in cold-atom physics. However, in any real experiment, the implementation of the desired (time-independent) Hamiltonian is not perfect and noisy fluctuations are unavoidable. Additionally, prescribed time-dependent protocols (such as, e.g., complex optimal control protocols[2,4]) have inaccuracies, which may be in the form of noisy fluctuations. In such cases, the stochasticity has a detrimental effect, which one needs to minimize. An in-depth understanding of the effect of noisy Hamiltonian evolution is thus crucial for, on one hand, correctly predicting the results of experiments and, on the other hand, designing sophisticated experimental setups, which are robust against the effect of the noise[4].

On the fundamental side, evolution with noisy Hamiltonian is a natural extension of the physics of disorder to the time domain. Disorder (in real space) has been widely studied for time-independent Hamiltonians starting from the pioneering work of Anderson[9] and has turned into a rich area of research. The advances in nonequilibrium dynamics motivate the study of the effect of disorder in time for time-dependent Hamiltonians. To address this question, it is necessary to analyze an ensemble of unitary evolutions characterized by different realization of noise and develop methods for computing ensemble-averaged quantities.

Another new question of fundamental interest is understanding the role of different sources of fluctuations. In stochastically driven systems, the fluctuations of physical observables stem from two distinct sources: (i) the classical (sample-to-sample) stochastic nature of the driving (different realizations of noise result in different wave functions) and (ii) inherent quantum fluctuations (each wave function can be a coherent superposition of eigenstates). Our objective in this paper is to (i) characterize the effect of the two sources above on energy fluctuations, and (ii) understand the effects of such stochastic driving on correlation functions.

In this paper, we consider the following general setup: a quantum system with a local Hamiltonian $H(g)$, which depends on some parameter $g$ (such as for example a coupling constant). We assume the system is initially in the ground state of $H_0 \equiv H(g_0)$. For $t > 0$, the parameter $g$ fluctuates in time: $g(t) = g_0 + \delta g(t)$, and, by assumption, $|\delta g(t)| \ll |g_0|$. For each realization $\delta g(t)$ of noise, the system is then described by a pure-state wave function, which evolves unitarily with Hamiltonian $H(g(t))$ (generically $H(g(t))$ does not commute with $H_0$). Thus, the quantum evolution creates excitations with respect to the ground state of $H_0$, increasing the system’s energy. The driving is assumed external, i.e. there is no feedback action from the quantum system on $\delta g(t)$. Therefore, in isolation from a thermal environment (without a mechanism for dissipation), the system can absorb energy ad infinitum. Such noisy systems can, for example, be realized by cold atoms in optical potentials, with $\delta g(t)$ generated by small fluctuations in the optical potential. Note that this setup can be readily generalized to the case of several fluctuating parameters.

The absorbed energy of such noise-driven thermally isolated systems is a time-dependent random variable $\epsilon$ (as opposed to the steady states which emerge in noise-driven systems coupled to a heat bath[15]). What is the precise meaning...
Hamiltonian with a fluctuating LL parameter, i.e., interaction quenches) has been a subject of intense studies. Luttinger liquid (LL) theory. The nonequilibrium dynamics then perform explicit analytical calculations of these motions. (b) For a Luttinger liquid in the thermodynamic limit, the two distribution functions are close (classical fluctuations dominate).

of $\epsilon$ for systems described by coherent superpositions of energy eigenstates? How can we characterize the average $E(\epsilon)$ and the variance $\text{Var}(\epsilon)$ of this random variable as a function of time? How are the correlation functions affected by this stochastic driving?

To answer the above questions, we first identify two types of quantum- and noise-averaged moments of energy, which make the notion of a random absorbed energy precise, and, help tease apart the effects of the classical and quantum sources of fluctuations discussed above (see Fig. 1). We then perform explicit analytical calculations of these moments, as well as different noise-averaged correlation functions, for generic one-dimensional systems described by the Luttinger (LL) theory. The non-equilibrium dynamics of Luttinger liquids (due to deterministic protocols such as interaction quenches) has been a subject of intense studies with various methods. In this work, we consider a LL Hamiltonian with a fluctuating LL parameter, i.e., $H(K(t))$, and use a method based on the (first-quantized) evolution of momentum-mode wave functions, which proves extremely powerful in the analysis of stochastic driving.

Our main results are as follows. We find an exact analytical expression for the noise- and quantum-averaged absorbed energy:

$$\overline{\langle \epsilon \rangle} = \frac{L}{8\pi K_0^2 W^2 t} \left( e^{2K_0^2 W^2 t} - 2\pi^2 K_0^2 W^2 t - 1 \right),$$  

(1)

where $L$ is the system size, $t$ is the time since the beginning of the evolution, and $W^2$ is a constant with dimension of time which characterizes the strength of noise in the Luttinger parameter: $1/K(t) = 1/K_0 + \delta\alpha(t)$ with $\delta\alpha(t_1)\delta\alpha(t_2) = W^2(\delta(t_1) - \delta(t_2))$. Here $1/K_0$ and $\delta\alpha(t)$ correspond respectively to $g_0$ and $\delta\alpha(t)$ introduced earlier. We find that, in the thermodynamic limit, the dominant contribution to the energy fluctuations stems from the classical source. In the thermodynamic limit, and in the regime of validity of the LL description, we find a general relationship (independent of the strength of noise and the Luttinger parameter) between the average and the variance of energy:

$$\text{Var}(\epsilon) = \mathcal{F}(\pi t) [E(\epsilon)]^2,$$  

(2)

where $\mathcal{F}(x)$ is a dimensionless function of $x = \pi tu/a$ (with velocity $u$ and lattice spacing $a$ set to unity), which decays as $1/x$ for large $x$. In contrast to equilibrium thermodynamics, where the relative fluctuations $\sqrt{\text{Var}(\epsilon)/E(\epsilon)}$ scale as $1/\sqrt{t}$, in this case, the relative fluctuations are independent of the systems size and, instead, decay off as $1/\sqrt{t}$ for large $t$.

We also evaluate certain noise-averaged equal-time correlation functions, and find that in the limit of small absorbed energies, correlation functions scaling as $x^{-2}$ in the ground state, deviate from their ground-state value by amounts proportional to $W^2 t^2 / x^{2+2}$. Interestingly, we are able to find an exact analytical result for the current-current correlation function,

$$\overline{C(x)} = \frac{K_0}{2\pi} \int_0^\pi dq \cos(q x) \exp \left( 2K_0^2 W^2 q^2 t \right),$$  

(3)

which can be written in closed form in terms of the error functions.

II. TWO TYPES OF ENERGY MOMENTS

Let us now discuss the two aforementioned moments of energy. As the evolution with different noise realizations can lead to different pure-state wave functions $\psi$ at time $t$, it is helpful to introduce a wave-function probability distribution $f(\psi, t)$, which encodes this stochastic effect. The inherent quantum fluctuations, on the other hand, stem from the internal structure of the wave functions, which are, generically, coherent superpositions of all energy eigenstates: $|\psi\rangle = \sum_n c_n |n\rangle$, where $H_0|n\rangle = \epsilon_n |n\rangle$. At $t = 0$, the system is in the ground state of $H_0$ so $f(\psi, 0)$ is a delta function (there are no classical fluctuations). Additionally, since the ground state is an eigenstate of the energy, there are no quantum fluctuations either at $t = 0$. For $t > 0$ the evolution both broadens the distribution $f(\psi, t)$ (classical source), and makes the wave function $|\psi\rangle$ a coherent superposition of multiple eigenstates (quantum source), giving rise to (total) energy fluctuations which originate from both sources above.

Experimentally, the outcomes of an energy measurement at time $t$ (with respect $H_0^{\Delta E}$) is described by the distribution function $f(\epsilon, t)$, which mixes both classical and quantum contributions (see Fig. 1). We can characterize $f(\epsilon, t)$ through the moments $\langle \epsilon^m \rangle = \int d\epsilon f(\epsilon) \epsilon^m$, where the brackets (overline) indicate a quantum (noise) average. Moreover, these moments can be written as $\langle \epsilon^m \rangle = \sum_n \epsilon_n^m \mathcal{P}(\epsilon_n)$, where

$$\mathcal{P}(\epsilon) = \int d\psi f(\psi, t) |\psi\rangle |\psi\rangle \langle \psi| \langle \psi| \langle \psi|$$

is the probability of measuring $\epsilon_n$, which yields

$$\langle \epsilon^m \rangle = \int d\psi f(\psi, t) \langle \psi| H_0^m |\psi\rangle = \text{tr} \left[ H_0^m \rho(t) \right],$$  

(4)

where $\rho(t) = \int d\psi f(\psi, t) |\psi\rangle \langle \psi|$ is the density matrix at time $t$.

To separate the contributions of the two sources above, we can consider the fluctuations of the expectation value of energy, $\langle \epsilon \rangle \equiv \langle \psi| H_0 |\psi\rangle$, over different realizations of noise, which are characterized by the following moments:

$$\langle \epsilon^m \rangle = \int d\epsilon f(\epsilon, t) \langle \epsilon \rangle^m = \int d\psi f(\psi, t) \langle \langle \psi| H_0 |\psi\rangle \rangle^m.$$  

(5)
As \langle e \rangle is a quantum-averaged quantity, its fluctuations stem solely from the classical stochastic driving. Thus, the difference between these two types of moments [Eqs. 4 and 5] can serve as a theoretical diagnostic for the relative importance of the two sources of fluctuations (see Fig. 1). Note that the moments 5 can not be written in terms of the density matrix for \( m > 1 \). Fluctuations of quantities other than energy can be similarly studied by expanding the wave function in the corresponding eigenbasis.

### III. WAVE-FUNCTION METHOD

Let us now turn to the specific model studied in this paper, namely, a Luttinger liquid, which is a universal low-energy description of interacting fermions and bosons in one dimension. Since a noise-driven dissipationless system can keep absorbing energy, this low-energy description will eventually break down for most experimentally realistic scenarios. In this paper, we focus on dynamics over a finite time scale where the LL description remains valid. Notice that such time scales can be extended by decreasing the strength of noise. There are numerous proposals for realizing the LL physics (with negligible coupling to the environment) with both bosonic and fermionic atoms. Luttinger liquids have already been realized with bosonic atoms in the Tonks-Girardeau gas and in elongated quasicondensates.

In terms of the Luttinger parameter \( K \) and velocity \( u \), the LL Hamiltonian is given by

\[
H(K) = u \sum_{q > 0} \left( K \Pi_q \Pi_{-q} + \frac{1}{K} q^2 \Phi_q \Phi_{-q} \right),
\]

where \( \Phi_q \) are bosonic fields and \( \Pi_q \) their conjugate momenta. The Hamiltonian above can be written as \( \sum_{q > 0} \left( H_q^{(1)} + H_q^{(3)} \right) \), where \( H_q^{(1,3)} \) is the Hamiltonian of a single harmonic oscillator involving only the real (imaginary) part of \( \Phi_q \). It is convenient to shift the Hamiltonian by a constant, i.e., \( H_q^{(1,3)} \rightarrow H_q^{(1,3)} - uq^2/2 \), so that the energies are measured with respect to the ground state. We consider a system initially in the ground state of \( H_0 = H(K_0) \), which evolves with \( H(K(t)) = H(K_0 + \delta K(t)) \) for \( t > 0 \), where \( \delta K(t) \ll K_0 \) represents the noise. As the fluctuations of velocity \( u \) correspond to a trivial rescaling of the Hamiltonian, we set \( u = 1 \) throughout this paper.

Expanding the Hamiltonian in \( \delta K \) results in quadratic (in the bosonic fields) noise terms. Thus, integrating out the noise at the outset (as in Refs. 3–5) leads to an interacting (quartic) effective action, which is difficult to treat exactly. In our case it is convenient to use an alternative approach, i.e., the wave-function approach, which consists of the following steps. (i) We parametrize the many-body time-dependent wave function of the system with complex numbers \( z_q \). (ii) We transform the time-dependent Schrödinger equation to an equation of motion for the parameters above, and then expand these equations in the noise terms to obtain a stochastic Langevin equation. (iii) We express the observables of interest in terms of the parameters above, and study their stochastic dynamics with the Langevin equation. This method allows us to reduce the quantum dynamics in Hilbert space to a set of equations of motion (in our case Langevin equations) for the parameters. Generically, the number of such parameters grows exponentially with system size, but for exactly solvable models such as Luttinger liquids a much smaller number of parameters may be necessary.

For the Luttinger liquid above, the Luttinger parameter \( K(t) \) is assumed spatially uniform. Therefore, momentum is a good quantum number throughout the noisy evolution. Moreover, the ground state of \( H_0 \) is a direct product of Gaussian wave functions for different momentum modes \( q \). Since each mode evolves with a quadratic Hamiltonian, the time-dependent wave function of the system, i.e., the solution of the time-dependent Schrödinger equation, retains the form

\[
Ψ(\Phi_q(t), t) = \prod_{q > 0} \left( \frac{2q |ℜz_q(t)|}{\pi} \right)^{\frac{1}{2}} \exp \left[ -q z_q(t) |Φ_q|^2 \right],
\]

where the parameter \( z_q(t) \) now satisfies the Riccati equation

\[
iδz_q = \frac{q}{K_0^2} (K_0^2 z_q^2 - 1) - q (K_0^2 z_q^2 + 1) δα,
\]

where \( δα(t) = −δK(t)/K_0^2 \). Note that there is no dissipative term in the above equation. Also notice that although different modes are decoupled in the Langevin equations above, different \( z_q(t) \) evolve with the same \( δα(t) \). Therefore the many-mode quantities must be computed by taking the noise-induced correlations into account.

We now assume Gaussian noise with zero average and second moments characterized by strength \( W \) and correlation time \( τ \) as in the Ornstein-Uhlenbeck process:

\[
\frac{dα(t_1)δα(t_2)}{2τ} = \frac{W^2}{2τ} e^{−|t_1−t_2|/τ}.
\]

In the limit \( τ → 0^+ \), the right-hand side of Eq. 9 reduced to \( W^2 δ(t_1 − t_2) \) describing Gaussian white noise. We leave a discussion of the effects of colored noise (finite \( τ \)) to Appendix B and only consider the \( τ → 0^+ \) limit here. As the white noise is thought of as a limit of a continuous process, we use the Stratonovich interpretation for Eq. 8. The stochastic differential equation described by Eqs. 8 and 9 is one of the key equations of this paper: it governs the stochastic evolution of \( z_q(t) \), which in turn determines the many-body wave function, and, consequently, all the observables of the system.

Several observables can be simply written in terms of \( z_q \). For instance, the first moment of the absorbed energy and the
equal-time correlation function of the bosonic fields are respectively given by (see Appendix [B]):

\[ \langle H_q(K_0) \rangle = \frac{q}{2} \left[ \frac{1}{2K_0 \mathcal{R}_{zq}} \left( 1 + K_0^2 |q|^2 \right)^2 - 1 \right], \]  

(10)

\[ \langle \Phi(x) \Phi(x') \rangle = \frac{1}{L} \sum_{q \neq 0} \cos \left[ q(x - x') \right] \frac{q \mathcal{R}_{zq}}{q^2 \mathcal{R}_{zq}}. \]  

(11)

IV. ENERGY FLUCTUATIONS

We now describe how to compute different moments of the energy moments using the wave-function approach introduced above. First, we present, in Sec. [A] a perturbative treatment of the Langevin equation [8], valid for small deviation from the initial values, i.e. \( z_q(t) \approx K_0^{-1} \). In Sec. [B] we then present a treatment valid for large deviation from the initial value based on the Fokker-Planck equation.

A. Perturbative treatment

As we are interested in the low-energy limit of \( |\delta z_q(t)| \ll K_0^{-1} \), where \( z_q(t) = K_0^{-1} + \delta z_q(t) \), we can linearize Eq. (9) in \( \delta z_q \) and obtain a simple linear equation

\[ i \delta z_q = 2q \left( \delta z_q - \delta \alpha \right), \]  

(12)

which admits the explicit solution \( \delta z_q(t) = 2i q \int_0^t dt' e^{2i q |t-t'|} \delta \alpha(t') \). Our strategy for computing the leading contribution in \( \delta z_q \) of a generic noise-averaged function of \( z_q \) is as follows. We expand this function to leading order in \( \delta z_q \), and insert the explicit solution above (in terms of \( \delta \alpha \)) into the resulting expression. The average over noise can then be done by using the Wick’s theorem, which relates \( \delta \alpha(t_1) \delta \alpha(t_2) \ldots \delta \alpha(t_n) \) to two-point functions [9]. As the \( n \)-point function above vanishes for odd \( n \) this approach is possible only when the leading contribution is even in \( \delta \alpha \). Luckily, this is the case for all energy moments of interest. Note that the expansion to the next-to-leading order in \( \delta z \) is not consistent with the linearization approximation made in Eq. (12).} Finally, we perform the required integrations over the time arguments to find the noise average.

Let us now compute the two types of energy moments [3] and [5]. Here, we only consider the first and the second moments (see Appendix [C] for a discussion of higher moments). As stated above, all observables, including \( \langle H_q^m(K_0) \rangle \), can be written in terms of \( z_q \). The expressions for \( \langle H_q^m(K_0) \rangle \) become more and more involved as \( m \) increases, but if we expand these expressions to leading order in \( \delta z_q \), we find the simple relationship \( \langle H_q^m(K_0) \rangle = q^m 2^{m-3} K_0^2 |\delta z_q|^2 + O(|\delta z_q|^3) \).

Using the strategy outlined above, we then obtain \( \langle H_q^2 \rangle \approx 2q^2 K_0^2 W^2 t^2 \) and \( \langle H_q^4 \rangle \approx q^4 K_0^2 W^4 \left[ 2t^2 + \sin^2(2qt)/4 \right] \) (see Appendix [D] for details). Clearly, for a single mode, the energy fluctuations are significantly affected by the quantum source at least in the limit of small excess energy, i.e. \( \langle H_q^2 \rangle \neq \langle H_q^4 \rangle \).

So far we have considered a single mode \( q \) with Hamiltonian \( H_q^3 \) or \( H_q^2 \). We now turn to the (many-mode) LL with Hamiltonian \( H = 2 \sum_q h_q \) where the factor of 2 accounts for the contributions of \( H_q^3 \) and \( H_q^2 \). The average energy can be simply written as \( \langle \epsilon \rangle = 2 \sum_q \langle H_q \rangle \), and using \( \sum_{q \neq 0} = \frac{1}{2} \sum_0^n \), we then obtain

\[ \langle \epsilon \rangle \approx L \pi^2 K_0^2 W^2 t^4/4. \]  

(13)

As for the second moment, we have \( \langle \epsilon^2 \rangle \equiv 4 \sum_{q, q' > 0} \langle H_q H_q \rangle \) and \( \langle \epsilon^2 \rangle \equiv 4 \sum_{q, q' > 0} \langle H_q H_q \rangle \langle H_{q'} H_{q'} \rangle \). Noting that the many-mode wave function is a direct product of wave functions for different modes \( q \) (in the \( R \) and \( F \) sectors), we can then write \( \langle \epsilon^2 \rangle = \langle \epsilon \rangle^2 + 4 \sum_q \langle (H_q^2 - \langle H_q \rangle^2) \rangle \). In the previous expression, both \( \langle \epsilon \rangle^2 \) and \( \langle \epsilon^2 \rangle \) scale as \( L^2 \), while the sum over \( q \) scales as \( L \). Therefore, if we take the thermodynamic limit before any other limit, the two types of moments will be, to leading order, identical [see Fig. (1b)]. As we will see below, this is also the case for the two types of cumulants obtained by subtracting \( \langle \epsilon \rangle^2 \) from the moments above. Such subtraction does not change the scaling with \( L^2 \) as an explicit calculation gives (see Appendix [D]):

\[ \langle \epsilon^2 \rangle - \langle \epsilon \rangle^2 = \frac{1}{16} \pi^6 K_0^6 W^4 t^2 L^2 F(\pi t), \]  

(14)

where the function \( F(x) \) has the following asymptotic behavior: \( F(x) \approx 2 \left( 1 - \frac{x^2}{3} \right) \) for \( x \ll 1 \), and \( F(x) \approx \frac{4 \log 4}{x} \) for \( x \gg 1 \). This indicates that the cumulant above crosses over from quadratic growth in \( t \) for short times to linear growth in \( t \) at longer times. Combining Eqs. (13) and (14) leads to the important relationship [2]. The regime of validity for these results can be extended by decreasing the strength of the noise \( W \). A comment is in order on the scaling of Eq. (14) with \( L^2 \). In equilibrium and for short-range interactions, all cumulants of energy are expected to scale linearly with the system size. The scaling of Eq. (14) with \( L^2 \) is a nonequilibrium feature, which indicates that the fluctuation of the absorbed energy from one noise realization to the other is extensive.

B. Fokker-Planck approach

Our results so far have been obtained by doing perturbative calculations in the limit of small \( \delta z_q \). This limit coincides with the regime of validity of the LL description and is of great experimental interest. From a theoretical perspective, however, it is interesting to study the effects of nonlinearities in Eq. (8). An alternative approach, which allows us to go beyond the limit of small \( \delta z_q \), is through the Fokker-Planck (FP) equation for the wave-function probability distribution. Using such FP equation, we obtain below an exact nonperturbative expression for the average energy of the system at time \( t \). Note that the FP approach is valid in the white-noise limit.

Let us briefly review the general formalism for a vector \( \vec{d} \) of stochastic variables satisfying the Langevin equation

\[ \partial_t d_i = h_i(d) + g_i(d)\gamma(t), \]  

where \( h_i \) and \( g_i \) are arbitrary functions of \( d \) and \( \gamma(t)\gamma(t') = 2\delta(t-t') \), the probability distribution
f(\vec{a}, t) evolves according to the FP equation \( \partial_t f = Df \), with the differential operator \( D = -\frac{\partial}{\partial a_i} h_i - \frac{\partial}{\partial a_i} g_j \frac{\partial}{\partial a_i} g_j + \frac{\partial}{\partial a_i} \frac{\partial}{\partial a_i} g_j \); (summation over repeated indices is implied). The noise-averaged expectation value of an arbitrary function \( G(\vec{a}) \) of the stochastic variables \( \vec{a} \) can then be computed at time \( t \) as an integral over \( \prod_i da_i \), weighted by the formal solution of the FP equation, \( f(\vec{a}, t) = e^{\Delta t} f(\vec{a}, 0) \).

In analogy with the Heisenberg picture of quantum dynamics, we can evolve the observable \( G(\vec{a}) \) instead of the distribution function \( f(\vec{a}) \) (using repeated integration by parts), and write

\[
\overline{G(\vec{a})} = \int \prod_i da_i \, f(\vec{a}, 0) \, e^{D't} G(\vec{a}),
\]

(15)

where \( D' \equiv \left( h_i + \frac{\partial}{\partial a_i} g_j \right) \frac{\partial}{\partial a_i} + g_j \frac{\partial}{\partial a_i} g_j \). By expanding the exponential operator above as \( e^{D't} = \sum_n \frac{D'^t}{n!} \), we can then compute the noise average of \( G(\vec{a}) \) as a Taylor expansion in \( t \). Since we are interested in finite time scales, such expansion is indeed very useful even when truncated at a finite order.

For the average energy \( \langle e \rangle \) and the current-current correlation function [see Eq. (17)], it turns out that the Taylor series can be expanded up to first order in time, even when truncated at a finite order. Note that the integration over the initial distribution \( \delta a \) is trivial due to a \( \delta \)-function initial distribution \( f(\vec{a}, 0) = \delta(a_1 - K^{-1}_0 a_2) \) (see also Ref. 52, Eq. (10)). We then obtain

\[
\langle e \rangle = \frac{L}{8\pi K_0} \frac{W^2 t}{2} \left( e^{2K_0^2 W^2 t^2} - 2\pi^2 K_0^2 W^2 t - 1 \right).
\]

(16)

As expected, the above expression, up to first order in time, agrees with Eq. (13).

Using the Wigner-function approach (see also Ref. 52), we can additionally show that for a thermal initial state (which is subsequently decoupled from the thermal environment during the evolution), the absorbed single-mode energy above is simply multiplied by a prefactor \( \coth(q/2k_BT_0) \), where \( T_0 \) is the initial temperature, and \( k_B \) the Boltzmann constant (see Appendix F). The many-mode energy can then be similarly computed by integration over \( q \). The FP approach can be used to compute other quantities such as the second moment of energy. However, in this case, one needs to construct a FP equation for four stochastic variables, \( \vec{a} = (Rz_q, Jz_q, R\bar{z}_q, J\bar{z}_q) \), and so on and so forth for higher moments.

V. CORRELATION FUNCTIONS

In this section, we compute some noise-averaged correlation functions of the system. We start by the current-current correlation function (see Appendix B):

\[
C(x - x') \equiv \langle \partial_x \Phi(x) \partial_x \Phi(x') \rangle = \frac{K_0}{L} \sum_{q=0} q \cos[q(x - x')] \frac{W q}{2} [1 - K_0(Re\delta z_q) + K_0^2(Re\delta z_q)^2 + \cdots].
\]

(17)

Using the perturbative method of Sec. IV A, we can expand the above expression in \( \delta z_q = z_q - K^{-1}_0 \) as

\[
C(x) = \frac{K_0}{L} \sum_{q=0} q \cos(qx) \left[ 1 - K_0(Re\delta z_q) + K_0^2(Re\delta z_q)^2 + \cdots \right].
\]

(18)

If we now use the solution of the linearized Langevin equation (12), we find that the first-order term in \( \delta z \) vanishes upon noise averaging, and the second-order term gives a leading contribution of order \( W^2 \):

\[
(Re\delta z_q)^2 = \frac{W^2 q}{2} [4qt - \sin(4qt)].
\]

(19)

Due to the presence of a linear term in Eq. (18), however, we need to also expand the Langevin equation (8) to second order in \( \delta z \):

\[
i \delta z_q = 2q \left( \delta z_q - \delta \alpha \right) + K_0 q \delta z_q^2 - 2K_0 q \delta z_q \delta \alpha,
\]

(20)

which may lead to a contribution of order \( W^2 \) to \( Re\delta z_q \). We then proceed by computing the next correction to \( \delta z \) iteratively: we write \( \delta z = \delta z^{(1)} + \delta z^{(2)} \), where \( \delta z^{(1)} \) is first-order in \( \delta \) and satisfies the linear equation (12) (it is given by the explicit integral expression below Eq. (12)). We then insert the above \( \delta z \) into Eq. (20) to obtain an equation for the evolution of \( \delta z^{(2)} \):

\[
i \delta z^{(2)}_q = 2q \left[ \delta z^{(2)}_q - K_0(\delta z^{(1)} \delta \alpha + K_0(\delta z^{(1)} \delta \alpha)) / 2 \right],
\]

(21)

where we have kept only the second-order terms in \( \delta \alpha^2 \). The above linear equation for \( \delta z^{(2)} \) yields an explicit expression in terms of \( \delta z^{(1)} \) and \( \delta \alpha \). Replacing \( \delta z^{(1)} \) by the explicit solution of Eq. (12) leads to an integral expression for \( \delta z^{(2)}_q \) in terms of \( \delta \). After some algebra, we can then write the leading contribution to \( Re\delta z_q \) as

\[
Re\delta z_q(t) \approx -\frac{K_0 W^2}{2} q \sin(4qt),
\]

(22)

which is of the same order in \( W \) as Eq. (19). Inserting Eqs. (22) and (19) into Eq. (13), and performing an integral over \( q \) gives

\[
C(x) \approx -\frac{K_0}{2\pi} \left( \frac{1}{x^2} + 12K_0^2 W^2 \frac{1}{x^4} \right),
\]

(23)
where we have neglected fast oscillatory terms for asymptotically large $x$.

We now turn to the correlation functions of vertex operators:

$$\mathcal{V}(x - x') \equiv \langle e^{i\Phi(x)} e^{-i\Phi(x')} \rangle = \exp \left[ -\frac{\nu^2}{2} (\Phi(x)\Phi(x')) \right],$$

(24)

where $\nu$ is a generic constant (which depends on $K_0$ for realistic models), and the correlator in the argument of the exponential is given by Eq. (11). As in Eq. (15), we expand the above expression up to second order in $\delta z_q$ as $\mathcal{V}(x) \approx e^{-\frac{\nu^2}{2} \sum_{q>0} K_0 \cos(q_{x}) (1 - K_0 (\mathcal{R}\delta z_q))^2}$, which gives

$$\mathcal{V}(x) \approx e^{-\frac{\nu^2}{2} \sum_{q>0} K_0 \cos(q_{x})} \left[ 1 + \frac{\nu^2}{2L} K_0^2 \sum_{q>0} \cos(q_{x}) \left( (\mathcal{R}\delta z_q) - K_0 (\mathcal{R}\delta z_q)^2 \right) + \frac{\nu^4}{8L^2} K_0^4 \sum_{q_1,q_2>0} \frac{\cos(q_{1,x}) \cos(q_{2,x})}{q_1 q_2} (\mathcal{R}\delta z_{q_1})(\mathcal{R}\delta z_{q_2}) \right].$$

(25)

To perform the averaging over noise we need Eqs. (19) and (22) as well as $(\mathcal{R}\delta z_{q_1})(\mathcal{R}\delta z_{q_2})$, which can be computed from the solution of the linearized Langevin equation and gives:

$$\mathcal{C}(x) = q_{1,q_2} W^2 \left[ \frac{\sin(2(q_{1} - q_{2})t)}{q_{1} - q_{2}} - \frac{\sin(2(q_{1} + q_{2})t)}{q_{1} + q_{2}} \right].$$

(26)

Note that the above expression reduces to Eq. (19) in the limit of $q_1 \rightarrow q_2$.

If we now take the asymptotic limit of large $x$ in Eq. (25), we find that the main contributions to the integrals over momenta (we are using $\sum_{q>0} \rightarrow \frac{1}{2\pi} \int_{0}^{\infty}$) come from small $q$. For fixed time $t \ll x$, we can then expand $(\mathcal{R}\delta z_{q_1})(\mathcal{R}\delta z_{q_2}) \approx \frac{16}{\nu^4} W^4 t^2 q_{1}^2 q_{2}^2$. Upon integration over momenta (and neglecting the fast oscillations) the term proportional to $\nu^2$ in (25) gives a contribution scaling as $t/x^2$, while the term proportional to $\nu^4$ gives a subleading contribution scaling as $t^3/x^4$. We then find that the leading correction (in the limit $t \ll x$) to the ground-state correlation function is $\mathcal{V}(x) \approx \mathcal{V}(0)(x)(1 + \mathcal{V}(x))$ with $\mathcal{V}(x) \approx \nu^2 K_0^2 W^2 t x^{-2}$.

Interestingly, for the current-current correlation function (17), the FP approach can provide an exact solution. Since this correlation function is the sum of contributions for each momentum mode, we can use the FP approach for each modes separately. In this case the variable of interest is $(\mathcal{R}z_q)^{-1}$ [see Eq. (17)]. By repeating the procedure used to compute the excess energy (see Appendix F), we obtain a Taylor series that can be resummed to give the exact expression

$$\mathcal{C}(x) = \frac{K_0}{2\pi} \int_{0}^{\pi} dq q \cos(q_{x}) \exp(2K_0^2 W^2 q_{x}^2),$$

(28)

which agrees with Eq. (23) in the limit of small absorbed energy.

VI. CONCLUSION

In summary, we studied a thermally isolated LL, randomly driven with a noisy Luttinger parameter, and undergoing coherent quantum evolution for each realization of noise. We computed noise-averaged correlation functions, and studied the energy fluctuations. We characterized these fluctuations by two types of energy moments: one that mixes the classical and quantum sources of fluctuations and one that is only affected by the classical source. We found that while for a single mode, the two types of moments lead to very different results, for the many-mode problem, the difference disappears in the thermodynamic limit. This indicates that many-body properties of such noise-driven coherent systems likely exhibit effective decoherence. Our approach to the dynamics of noisy LLs is based on a mapping of the quantum problem to one of nonequilibrium classical statistical mechanics, which provides powerful tools, such as the FP equation, for performing exact calculations.

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Appendix A: EIGENSTATE EXPANSION OF THE SINGLE-MODE WAVE FUNCTION

Here we compute the overlaps of the single-mode wave function, Eq. (7), with the eigenstates of the single-mode Hamiltonian $H_q$. The Hamiltonian $H_q$ is shorthand for either $H_q^R$ or $H_q^S$:

$$H_q(K(t)) = u \left( -\frac{K(t)}{4} \frac{\partial^2}{\partial \phi^2} + \frac{1}{K(t)} q^2 \phi^2 \right). \quad (A1)$$

In the absence of noise, the Hamiltonian is given by $H_0 = H_q(K_0)$. For $z_q = 1/K_0$, the wave function (7) is the ground state of Hamiltonian (A1). For arbitrary $z_q$, however, it is a superposition $\psi_q(\phi) = \sum_{n=0}^{\infty} c_n(z_q) \psi_n(\phi)$, where $\psi_n(\phi)$ is an eigenfunction of $H_0$ with energy $(n + \frac{1}{2}) u q$, which can be written explicitly in terms of the Hermite polynomials. Through direct integration, we can compute the amplitudes $c_n(z_q) = \int dq \psi_n(\phi) \psi_n(\phi)$, which vanish for odd $n$, and are given by the following expression for even $n$:

$$|c_{2n}(z_q)|^2 = \frac{(2m)! \frac{1}{2^{m+1/2}}}{2^{2m} n! m!} \left[ (x_q - 1)^2 + y_q^2 \right]^{m+1/2} \left[ (x_q + 1)^2 + y_q^2 \right]^{m-1/2},$$

where $x_q = K_0 (\Re z_q(t))$ and $y_q = K_0 (\Im z_q(t))$. Thus, the overlaps $|c_n(z_q)|$ are identically zero for odd $n$, and decay exponentially for even $n$.

Appendix B: EXPRESSION FOR OBSERVABLES IN TERMS OF $z_q$

In this appendix, we express some of the observables of the system in terms of $z_q(t)$. Measuring the energies with respect to the ground state of $H_q(K_0)$, and using the explicit form of the wave function (7), we can obtain Eq. (10) by direct integration over the bosonic fields:

$$\langle H_q(K_0) \rangle \equiv \int_{-\infty}^{\infty} dq \psi_q(\phi, t)^* H_q(K_0) \psi_q(\phi, t)$$

$$= \frac{q^2}{2} \left[ \frac{1}{2 K_0 \Re z_q} \left( 1 + K_0^2 |z_q|^2 \right) - 1 \right]. \quad (B1)$$

where $u$ is set to unity. Similarly, we can compute the second moment of energy:

$$\langle H_q^2(K_0) \rangle - \langle H_q(K_0) \rangle^2 \approx \frac{q^2}{8 K_0^2 \Re z_q^2} \left[ 1 - 2 K_0^2 \Re z_q - 3 z_q^2 + K_0^2 |z_q|^4 \right]. \quad (B2)$$

Note that both expressions above vanish for the ground state ($z_q = K_0^{-1}$). This simple approach can be used to compute higher moments of energy, but the resulting expressions become more cumbersome. Using MATHEMATICA®, we have calculated the moments $\langle H_q^m(K_0) \rangle$ for $m \leq 20$, and checked that to leading order in $\delta z_q = z_q - K_0^{-1}$, they can be written as

$$\langle e_q^m \rangle = \langle H_q^m(K_0) \rangle = q^m 2^{m-3} K_0^2 |\delta z_q|^2 + O(\delta z_q^3). \quad (B3)$$

Equal-time correlation functions are also simple to compute in terms of $z_q(t)$. We can expand $\Phi(x)$ in Fourier modes, $\Phi(x) = \sum_q e^{i \epsilon_q t \delta_q} \Phi_q = \frac{1}{\sqrt{2 \pi}} \sum_q [\cos(qx) \Re \Phi_q - \sin(qx) \Im \Phi_q]$, and make use of the relation $\langle \Re \Phi_q \Re \Phi_{q'} \rangle = \langle \Re \Phi_q \Re \Phi_{q'} \rangle = \delta_{q, q'} / 4q \Re z_q$ and $\langle \Re \Phi_q \Im \Phi_{q'} \rangle = \langle \Re \Phi_q \Im \Phi_{q'} \rangle = 0$ to obtain Eq. (11):

$$\langle \Phi(x) \Phi(x') \rangle = \frac{1}{L} \sum_{q \neq 0} \frac{\cos[q(x - x')]}{q \Re z_q}.$$ \quad (B4)

Other correlation functions for the current and the vertex operator can be obtained by simple differentiation and exponentiation.

Appendix C: HIGHER MOMENTS OF ENERGY

Focusing on a single mode for simplicity, we calculate the noise-averaged moments $\langle e_q^m \rangle$. To leading order in $\delta z_q$, we can write $\langle e_q^m \rangle = \left( \frac{K_0 \delta z_q}{|z_q|^2} \right)^m$ [see Eq. (B3)]. After substituting the explicit solution of the linear Langevin equation (12) we need to compute

$$\langle e_q^m \rangle \approx q^m (K_0^2 W_2^2 t)^m \sum_{p=0}^{[m/2]} S_{m,p} \frac{\left( \sin 2qt \right)^{2p}}{2qt},$$

(C2)

where $[x]$ indicates the floor value of $x$, and $S_{m,p}$ represents the number of ways for having $p$ contractions of type (i), $p$ contractions of type (ii), and, consequently, $m - 2p$ contractions of type (iii). The simple combinatorial argument below gives

$$S_{m,p} = \left[ \left( \frac{m}{2p} \right) \cdot (2p - 1)!! \right] (m - 2p)!!.$$ \quad (C3)
Appendix D: DERIVATION OF EQ. (4)

To compute $F(x)$ in Eq. (4), we start from the expression $\langle \epsilon \rangle = 2 \sum_q \langle H_q \rangle$ (the factor of 2 accounts for the contribution of $H^2_q$ and $H'^2_q$), and write $\langle \epsilon^2 \rangle = 4 \sum_{q_1,q_2} \langle H_{q_1} \rangle \langle H_{q_2} \rangle$. After inserting Eq. (B3) for $m = 1$, we obtain

$$\langle \epsilon^2 \rangle \approx \frac{K^2_0}{4} \sum_{q_1,q_2} \frac{q_1^2 q_2^2}{\langle \delta z_{q_1} \rangle^2 \langle \delta z_{q_2} \rangle^2},$$

which, upon substituting the explicit solution of Eq. (12), yields

$$\langle \epsilon^2 \rangle \approx 4 K^2_0 \sum_{q_1,q_2} \frac{q_1^2 q_2^2}{\langle \delta z_{q_1} \rangle^2 \langle \delta z_{q_2} \rangle^2} \int_0^\infty dt_1 dt_2 dt'_1 dt'_2 \ e^{2q_1(t-t')/\xi} e^{2q_2(t-t')/\xi} \times \delta(\xi) \delta(\xi') \delta(\alpha(t)) \delta(\alpha(t')).$$

Using the Wick’s theorem to break the four-point function $\delta(\xi) \delta(\xi') \delta(\alpha(t)) \delta(\alpha(t'))$ into a sum of products of two-point functions, we get three different types of contractions. Upon inserting these delta functions and performing the integrals over time, we then find

$$\langle \epsilon^2 \rangle = 4 K^2_0 W^4 \sum_{q_1,q_2} \frac{q_1^2 q_2^2}{\langle \delta z_{q_1} \rangle^2 \langle \delta z_{q_2} \rangle^2} \int_0^\infty dt_1 dt_2 dt'_1 dt'_2 \ e^{2q_1(t-t')/\xi} e^{2q_2(t-t')/\xi} \times \delta(\xi) \delta(\xi') \delta(\alpha(t)) \delta(\alpha(t')) + t^2.$$

The term proportional to $t^2$ is exactly equal to $\langle \epsilon^2 \rangle^2$. To obtain the asymptotic value of $F(x)$, we convert the sums to integrals, and define the new variables $q_i = \pi \xi_i$ and $x = \pi t$, which leads to

$$F(x) = 16 \int_0^1 dx_1 \int_0^1 dx_2 \ e^{x_1^2 x_2^2} \times \left\{ \frac{\sin^2 [(q_1 + q_2)x]}{(q_1 + q_2)^2} + \frac{\sin^2 [(q_1 - q_2)x]}{(q_1 - q_2)^2} \right\}.$$

For $x \leq 1$, the asymptotic behavior is obtained by a simple Taylor expansion in $x$, which gives $F(x) \sim 2 (1 - 4x^2)$. For $x \gg 1$, the main contribution comes from $\xi_1 = \xi_2$, then we can approximate $F(x) \sim 16 \int_0^1 dq \int_0^1 dq' \ e^{q_1^2 q_2^2} \times \frac{\sin(q_1x)}{q_1^2 q_2^2} \left\{ \frac{\sin((q_1 - q_2)x)}{(q_1 - q_2)^2} \right\}$ where we have defined $Q = \xi_1 + \xi_2$ and $q = \xi_1 - \xi_2$, and the factor 1/2 comes from the Jacobian of the transformation. Since $Q \gg q$ the dominant contribution comes from the term proportional to $Q^6$. The final change of variable $k = qx$ leads to $F(x) \sim 16 \int_0^1 dq \int_0^\infty \frac{dt}{2t} \ e^{q_1^2 q_2^2} = 16 \tau.$

Appendix E: EFFECTS OF COLORED NOISE

In this appendix, we discuss the effects of finite correlation time $\tau$ in the Ornstein-Uhlenbeck process:

$$\frac{\delta(\xi)}{\delta(\xi')} = W^2 \frac{2t}{2\tau} e^{-(h-t)/\tau}.$$

For the single-mode energy moments $\langle \epsilon^m \rangle \approx q^m 2^{m-3} K^2_0 \langle \delta z_{q} \rangle^2$, we need to evaluate

$$\langle \delta z_{q} \rangle^2 = (2q)^2 \int_0^\infty dt_1 \int_0^\infty dt'_1 \ e^{2q(t-t')/\xi} \delta(\xi) \delta(\alpha(t)) \delta(\alpha(t')).$$

Substituting Eq. (E1), and performing the integral yields

$$\langle \delta z_{q} \rangle^2 = 4q^2 W^2 t \left[ \frac{1}{1 + (2q^2 \tau)^2} + \frac{\tau}{t} \left( e^{-\frac{1}{1+2q^2 \tau}} - 1 \right) \right].$$

Note that, for $t \gg \tau$, the above expression simplifies to $\langle \delta z_{q} \rangle^2 = 4q^2 W^2 \tau$, while for white noise, $\tau \to 0^+$, we obtain $\langle \delta z_{q} \rangle^2 = 4q^2 W^2 t$. We then conclude that, for $t \gg \tau$, the single-mode energy can be obtained from the corresponding expression for white noise by the simple rescaling $W \to W \tau^{-1/2}$. The average energy can then be calculated by integration over momenta:

$$\langle \epsilon \rangle = 2 \sum_q \langle \epsilon_q \rangle = 2 \sum_q \left( \frac{K^2_0 W^2 L}{\pi} \int_0^\infty dq \frac{q^3}{1 + (2q^2 \tau)^2} \right).$$

For white noise, $\tau \to 0^+$, reduces to $\langle \epsilon \rangle = L N^3 K^2_0 W^2 / 4 t$. Let us now consider the higher moments of the single-mode energy, which are given by $\langle \epsilon_q \rangle^m \approx \left( \frac{K^2_0 W^2}{4 \langle \delta z_{q} \rangle^2} \right)^m$. Using the method of Appendix C and taking the limit of $t \gg \tau$, we can write the analog of Eq. (C2) as

$$\langle \epsilon_q \rangle^m \approx \left( \frac{K^2_0 W^2}{1 + (2q^2 \tau)^2} \right)^m \times \sum_{p=0}^{\lfloor m/2 \rfloor} S_{m,p} \frac{(\sin(2qt) - 2q \tau \cos(2qt))^2}{2q^2 \tau}.$$

where $S_{m,p}$ is defined in Eq. (C3). We thus observe that finite correlation time does not qualitatively change the white-noise behavior. Finally using the same method as in Appendix D we can generalize Eq. (D4) to

$$\langle \epsilon^2 \rangle = \langle \epsilon \rangle^2 + \frac{L^2 K^4_0 W^4 t^2}{16} F(\pi t, \pi t),$$

where, for $t \gg \tau$, we have
In writing the above expression, we have defined $q_i = \pi \xi_i, \ \ x = \pi t, \ \ y = \pi \tau$. Note that this expression is valid only for $t \gg \tau$, which implies $x \gg y$. The asymptotic behavior at long times, $t \gg 1$, is given by $F(x, y) \approx \frac{16g}{\sqrt{\pi}} \left(1 - \frac{36g^2}{9} y^2 + 0(y^4)\right)$. This implies that the finite correlation time ($y > 0$) simply lowers, by an overall prefactor, the white-noise ($y = 0$) result.

The short-time behavior can be seen in Fig. 2 for several values of $y = \pi \tau$. In the physically relevant region, $x \gg y$, the curves show a rapid decay followed by a slower one. This behavior is qualitatively similar to the white-noise case, which is also plotted in Fig. 2 for comparison.

**Appendix F: FOKKER-PLANCK APPROACH FOR THE EXCESS ENERGY**

In this section we explicitly construct the differential operator $D^\dagger$ for the single mode FP equation and we use it to compute exactly the noise averaged absorbed energy.

Defining $\mathcal{H}_i \equiv \mathbb{R} \mathbb{E}_{q_i}$, and $\mathcal{J}_i \equiv \mathbb{J} \mathbb{E}_{q_i}$, we can write the non-linear Langevin equation (8) as

\[
\mathcal{H}_i = 2K_0 q_i \mathcal{J}_i \mathcal{J}_i - 2K_0 q_i \mathcal{H}_i \mathcal{J}_i \alpha, \\
\mathcal{J}_i = K_0 q_i \left( \mathcal{J}_i^2 - \mathcal{H}_i^2 + K_0^{-2} \right) - K_0 q_i \left( \mathcal{J}_i^2 - \mathcal{H}_i^2 - K_0^{-2} \right) \alpha.
\]  

(F1)

As the total energy is a sum of single-mode energies, we only need a single-mode Fokker-Planck equation, and we can drop the subscript $i$. Rescaling $\alpha \rightarrow \frac{\alpha}{\sqrt{2}} \gamma(t)$, with $\langle \gamma(t) \gamma(t') \rangle = 2\delta(t - t')$, we can read off $f_i$ and $g_i$ for $i = 1, 2$ [see discussion after Eq. (15)]:

\[
f_1 = 2K_0 q_1 a_2, \\
g_1 = -\sqrt{2}W K_0^2 q_1 a_2, \\
f_2 = K_0 g \left( a_2^2 - a_1^2 + K_0^{-2} \right), \\
g_2 = -\frac{W}{\sqrt{2}} K_0^2 g \left( a_2^2 - a_1^2 - K_0^{-2} \right),
\]

where $\vec{a} \equiv (a_1, a_2) = (\mathcal{H}, \mathcal{J})$. In terms of the above $f_i$ and $g_i$, we can write

\[
D^\dagger = D_1^{(1)} \partial_{a_1} + D_2^{(1)} \partial_{a_2} + D_{1,2}^{(2)} \partial_{a_1}^2 + D_{2,2}^{(2)} \partial_{a_2}^2 + 2D_{1,2}^{(2)} \partial_{a_1} \partial_{a_2}, \quad (F2)
\]

no sum is implied. By applying Eq. (12) repeatedly on the expression for the mean energy (see Eq. (10)):

\[
\langle H_q \rangle = \frac{q}{2} \left[ \frac{1}{2K_0 a_1} (1 + K_0^{-1} (a_1^2 + a_2^2)) - 1 \right] \quad (F3)
\]

and evaluating the terms at $a_1 = K_0^{-1}$ and $a_2 = 0$ we obtain a Taylor expansion which can be resummed to give the Equation $\langle H_q \rangle = \frac{q}{2} \left[ \exp \left( 2q^2 W^2 K_0^2 \right) - 1 \right]$. The total noise averaged excess energy is obtained by summing the previous expression over the momentum.

Note that if we are interested in two-mode quantities, we need to repeat the procedure above with $\vec{a} = (\mathcal{H}, \mathcal{J}, \mathcal{H}_2, \mathcal{J}_2)$, identify $f_i, g_i$ for $i = 1, 2, 3, 4$, and build the four-dimensional vector $D^{(1)}$ and the $4 \times 4$ matrix $D^{(2)}$. In general, if we need the correlations of $m$ momenta, $D^{(1)}$ is $2m$-dimensional and $D^{(2)}$ is a $2m \times 2m$ matrix.

**Appendix G: WIGNER-FUNCTION APPROACH**

An alternative approach to the dynamics of the problem is through the Wigner-function representation. This approach allows us to treat, on the same footing, initial conditions given by a finite-temperature thermal density matrix as well as the
zero-temperature ground state considered so far. Note that the system is still thermally isolated during the evolution, i.e., it is decoupled from a finite-temperature heat bath at time \( t = 0 \).

Let us briefly review the formalism. In terms of phase-space variable \( x \) and \( p \) (which can be vectors for multidimensional problems), the Weyl symbol \( \Omega_w(x, p) \) for a quantum operator \( \hat{\Omega} \) is defined as

\[
\Omega_w(x, p) = \int ds \left( x - \frac{s}{2} \right) \left[ \hat{\Omega} \right] \left| x + \frac{s}{2} \right| \exp \left[ \frac{i}{\hbar} p \cdot s \right]. \tag{G1}
\]

The Wigner function \( W(x, p) \) is, by definition, the Weyl symbol for the density matrix \( \hat{\rho} \), written as

\[
W(x, p) \equiv \frac{1}{(2\pi\hbar)^2} \tr \left[ \exp\left(-\frac{i}{\hbar} \hat{p} \hat{x} \right) \hat{\rho} \right].
\]

Let us now consider the single-mode Hamiltonian \( \hat{H}(K(t)) = -\frac{\hbar}{2} \dot{\hat{p}}^2 + \frac{1}{K(t)} \hat{q}^2 \) [see Eq. (F2)], where we have set \( u = 1 \) and suppressed the \( \Re \) or \( \Im \) superscripts. Here, \( \hat{\chi} \) and \( \hat{\rho} \) respectively represent the real (or imaginary) part of operators \( \hat{\Phi}_q \) and \( \hat{\Pi}_q \), and we have used the hat notation to distinguish quantum operators from phase-space variables. For a system evolving from an initial thermal density matrix, we have

\[
\hat{\rho}_0 = \exp \left( -\hat{H}(K_0)/k_b T_0 \right) / \tr \left[ \exp \left( -\hat{H}(K_0)/k_b T_0 \right) \right], \tag{G4}
\]

with the Hamiltonian above (for arbitrary \( K(t) \)), the Wigner function retains the following form:

\[
W(x, p) = \mathcal{N} \exp \left[ -\frac{A}{2} x^2 - \frac{B}{2} p^2 + C x p \right], \tag{G5}
\]

where \( A, B, C, \) and \( \mathcal{N} \) are potentially time-dependent functions with the following initial conditions:

\[
A(t = 0) = \frac{4q}{K_0 R}, \quad B(t = 0) = \frac{K_0}{q R^2}, \quad C(t = 0) = 0, \tag{G6}
\]

\[
\mathcal{N}(t = 0) = \frac{2}{R}, \quad R = \coth \left( \frac{q}{2k_b T_0} \right). \tag{G7}
\]

Note that the normalization condition \( \int_{-\infty}^{+\infty} \frac{dx dp}{2\pi} W(x, p) = 1 \) sets

\[
\mathcal{N} = \sqrt{AB - C^2}. \tag{G7}
\]

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