Unitary matrix integrals in the framework of Generalized Kontsevich Model

1. Brezin-Gross-Witten Model

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ABSTRACT

We advocate a new approach to the study of unitary matrix models in external fields which emphasizes their relationship to Generalized Kontsevich Models (GKM) with non-polynomial potentials. For example, we show that the partition function of the Brezin-Gross-Witten Model (BGWM), which is defined as an integral over unitary $N \times N$ matrices, $\int [dU] e^{\text{Tr}(JU+J^TU)}$, can also be considered as a GKM with potential $V(X) = \frac{1}{X}$. Moreover, it can be interpreted as the generating functional for correlators in the Penner model. The strong and weak coupling phases of the BGWM are identified with the "character" (weak coupling) and "Kontsevich" (strong coupling) phases of the GKM, respectively. This sort of GKM deserves classification as $p = -2$ one (i.e. $c = 28$ or $c = -2$) when in the Kontsevich phase. This approach allows us to further identify the Harish-Chandra-Itzykson-Zuber (IZ) integral with a peculiar GKM, which arises in the study of $c = 1$ theory and, further, with a conventional 2-matrix model which is rewritten in Miwa coordinates. Inspired by the considered unitary matrix models, some further extensions of the GKM treatment which are inspired by the unitary matrix models which we have considered are also developed. In particular, as a by-product, a new simple method of fixing the Ward identities for matrix models in an external field is presented.

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1 Introduction

The full partition function, i.e. the generating functional of all of the relevant correlators of an eigenvalue matrix model, can be reduced to an integral over matrix eigenvalues only. In this process, the integration over "angular variables" is simultaneously factored out of all of the correlation functions. Such models appear to exhibit a novel integrable structure, which is intimately related to certain topological structures and which has been the subject of much investigation over the last few years. There are essentially two interesting types of eigenvalue models which are in fact closely related: one is the family of Hermitean multi-matrix models and the other is the generalized Kontsevich Model (GKM).

However, not all of the interesting matrix models are of the eigenvalue type. For example, in models which describe lattice gauge theory, the angular degrees of freedom of unitary matrices represent the gauge (Yang-Mills) bosons. Such degrees of freedom are absent in an eigenvalue model. Moreover, in the context of correlators of string theory, it is thought that non-eigenvalue models should be studied in order to overcome the \( d = 2 \) ("\( c = 1 \)"") barrier which separates purely topological theories from those with non-trivial perturbative spectra. A very interesting and important direction for the investigation of these models is to examine in what sense the integrable structure which is important in understanding eigenvalue models is still present, if at all, in non-eigenvalue models. The full answer to this question, which in the future should probably involve a "non-Cartanian" generalization of today’s concept of integrable hierarchies of the (multicomponent) KP and Toda type, is as yet unknown. (See ref.[1] for more detailed presentation of the problem.)

A broad and important class of non-eigenvalue unitary matrix models occurs in the situation where one is interested in correlators of arbitrary matrix elements of the unitary matrices. In that case, an action which depends only on the eigenvalues of the unitary matrices is not sufficient to form a generating functional for all of the correlation functions. This sort of model is essentially of the non-eigenvalue type. Conventional analyses of unitary matrix integrals usually lead to representations in terms of group theoretical quantities (such as characters and their multi-variable analogues) which, although intimately related to integrability theory, are not transparent enough to reveal the relevant structures. Results of this kind are particularly unsuitable for taking large-\( N \) limits, which are often important for relating matrix models to physical reality.

Another important feature of unitary matrix integrals, familiar from studies of lattice Yang-Mills theories, is the occurrence in the infinite \( N \) limit of a third order phase transition separating weak and strong coupling phases. As first emphasized in ref.[2] two analogous “phases” are in fact also present even at finite \( N \) in the GKM. The phase structure of unitary matrix models thus complements, rather than contradicts their relationship with the GKM and integrability.

In this paper we describe what we believe is a natural step toward understanding integrable properties of non-eigenvalue models. We investigate the simplest example: the Brezin-Gross-Witten Model (BGWM) [3, 4] with partition function

\[
Z_{BGWM}(J, J^\dagger) \equiv \frac{1}{V_N} \int_{N \times N} [dU] e^{\text{Tr}(J^\dagger U + J U^\dagger)},
\]  

where the integration is over \( N \times N \) unitary matrices with Haar measure \([dU]\) and \( V_N = \int_{N \times N} [dU] \) is the volume of unitary group. The “external field” \( J \) is an \( N \times N \) complex matrix. This model is \( a \ priori \) of the non-eigenvalue type, since there is no way to reduce the integral in (1.1) to that over eigenvalues of \( U \), and also because derivatives with respect to \( J \) and \( J^\dagger \) generate correlators of any matrix elements of \( U \) and \( U^\dagger \). Still this particular model is simple enough to fit into the traditional integrability framework. Because of the invariance of Haar measure \( Z(J, J^\dagger) \) depends only on \( N \) rather than \( N^2 \) parameters: the eigenvalues of the matrix \( M = JJ^\dagger \). Thus, it is not a complete surprise that it can also be represented as the partition
function of an eigenvalue model, namely of a certain example of a GKM:

\[ Z(J, J^\dagger) = \frac{Z_N(M)}{Z_N(0)}, \quad M = JJ^\dagger, \]

\[ Z_N(M) \equiv \frac{1}{V_N} \int_{N \times N} dX e^{\text{Tr}(MX - N \log X + \frac{1}{X})} \]

(1.2)

The integral now is over \( N \times N \) Hermitean matrices \( X \).

Being a particular example of a GKM, this model satisfies different kinds of Virasoro constraints in the weak and strong-field limits which, in the terminology of ref.[2]), represent the “character” and “Kontsevich” phases, respectively. It is integrable in both limits and, when expressed in terms of the appropriate variables \( (t_k^+ = \frac{1}{k} \text{Tr} M^k \text{ or } t_k^-(2) = \frac{1}{k} \text{Tr} M^{-k/2}) \) \( Z_{BGWM} \) it is the \( \tau \)-function of the Toda chain. In the Kontsevich (strong-field) phase, which corresponds to the most interesting weak-coupling phase of the original BGWM, it is in fact a reduced - MKdV \( \tau \)-function, which was studied earlier in [5], while the relevant Virasoro constraints were first derived in [6].

This model has various representations which display its connection to other theories. It can also be rewritten as

\[ Z_N(M) \equiv \frac{1}{V_N} \int_{N \times N} dX e^{\text{Tr}(J^\dagger X - N \log X + J \dagger)} \]

(1.3)

or, after the change of variables \( X \to Y = 1/X \), as

\[ Z_N(M) = \frac{1}{V_N} \int_{N \times N} dY e^{\text{Tr}(M \dagger - N \log Y + Y)}. \]

(1.4)

Representation (1.3) is in fact nothing but the original integral (1.1) with \( U \) substituted by \( X \) and the Haar measure \([dU]\) - by the left- and right- invariant measure \(^1\)

\[ \langle dX \rangle \equiv \frac{dX}{\text{Det}X^N}. \]

However, this “change of notation” (or of integration contour) reveals an association with the GKM and allows one to put the consideration of the BGWM, - and, consequently, of generic unitary matrix integrals - into a context which is well adjusted for the study of integrable structures.

The representation (1.4) is particularly important from this point of view. In this representation it is possible to “untie” the size of the matrix \( Y \) from the size of the matrices \( J \) and \( J^\dagger \), substituting (1.4) by

\[ Z_N^+(L) = \frac{1}{V_n} \int_{n \times n} dY e^{\text{tr}(L \dagger - N \log Y + Y)}. \]

(1.7)

If it is considered as a function of the \( t_k^+ = \frac{1}{k} \text{tr} L^k \), this quantity is essentially independent of \( n \) (though it depends non-trivially on \( N \)), and thus can be used to reproduce \( Z_N(M) \).

\(^1\)This measure has the properties:

\[ \langle d(GX) \rangle = \langle d(XG) \rangle = \langle dX \rangle \text{ for any } G; \quad \langle d\frac{1}{X} \rangle = \langle dX \rangle, \]

(1.5)

which are characteristic of a Haar measure, thus the only real difference (if at all) between \( X \) and \( U \) integrals can be the reality condition (the choice of integration contour). Furthermore, to avoid any confusion, we remind the reader that in the theory of the GKM the word “Hermitean” does not mean much more than the “flatness” of the measure \( dX \equiv \prod_{i,j} dX_{ij} \). In many cases it is assumed that the integrals go along some appropriate contours in the complex space (of matrix eigenvalues) rather than over real hypersurfaces. The matrices can be complex in this sense, they should be diagonalizable by unitary conjugation but the eigenvalues are allowed to be complex in the way appropriate to making the integral in the GKM well defined.
of the extra parameter $n$ is important for establishing relation both to the GKM and to Penner theory [7, 8].

Though it is defined as a single matrix (one-link) integral, the BGWM is known to have much to do with $2d$ Yang-Mills theory [3]. It can be also relevant to $d > 2$ theories, if considered in the mean-field approximation. An example of more sophisticated unitary matrix model, is provided by the Harish-Chandra-Itzykson-Zuber (IZ) integral. Though still much simpler than generic unitary matrix actions, which are usually defined on the plaquettes, the IZ theory can be used for construction of some oversimplified gauge theories beyond 2 dimensions. Making use of our results for the BGWM, this integral can be rewritten as a GKM with potential of generic shape and, further, as an Hermitean 2-matrix model:

$$Z_{IZ}(\Phi, \bar{\Phi}) \equiv \frac{1}{V_N} \int_{N \times N} [dU] e^{Tr \Phi U \bar{\Phi} U^\dagger} = \left( \frac{\det e^{\phi_i \bar{\phi}_j} \prod_{i<j} (\phi_i - \phi_j) \prod_{i<j} (\bar{\phi}_i - \bar{\phi}_j)}{\prod_{i<j} (\phi_i - \phi_j) \prod_{i<j} (\bar{\phi}_i - \bar{\phi}_j)} \right), \quad \phi_i, \bar{\phi}_j \in \text{spec } \Phi, \bar{\Phi}$$

$$= \frac{1}{V_N} \int_{N \times N} dH e^{Tr \sum_{k \geq 0} T_k^\pm H^\pm k} e^{Tr \Phi H^{-1}} e^{Tr \bar{\Phi} H} = \frac{1}{V_N} \int_{N \times N} d\bar{H} e^{Tr \sum_{k \geq 0} \bar{T}_k^\pm \bar{H}^\pm k} =$$

$$= cN \int \int_{N \times N} dHd\bar{H} e^{Tr \sum_{k \geq 0} (T_k^\pm H^\pm k + \bar{T}_k^\pm \bar{H}^\pm k)}$$

(1.8)

The two sets of times (with “plus” or “minus” superscript), which appear at the r.h.s.

$$T_k^\pm \equiv \frac{1}{k} \text{Tr } \Phi^\pm k, \quad \bar{T}_k^\pm = \frac{1}{k} \text{Tr } \bar{\Phi}^\pm k, \quad k \geq 0,$$

(1.9)

are relevant for description of the two different phases and are in a good sense conjugate to each other.\footnote{Definition of the “zero-times” $T_0^\pm$ deserves a comment: actually the item with $k = 0$ in the sums in the exponent in (1.8) should be ascribed different meanings for different choices of superscripts:}

$$T_0^\pm H^0 - \text{Tr}(I \otimes \log H) = -N \text{Tr} \log H;$$

while

$$T_0^- H^{-0} \rightarrow -\text{Tr}(\Phi \otimes I) = -N \text{Tr} \log \Phi.$$
2 Two “phases” and two versions of the GKM

Let us recall that the main object in the theory of the GKM is the “Kontsevich integral”

$$\mathcal{F}_V(N, L) \equiv \frac{1}{V_n} \int_{n \times n} dX e^{i\text{tr}(LX - N\log X + V(X))}$$ (2.1)

over $n \times n$ Hermitean matrices $X$ with an Hermitean external field $L$ and some “potential” function $V(x)$. This is actually an eigenvalue model, since both matrices $L$ and $X$ can be diagonalized and the angular variables can be integrated away with the help of the IZ formula (first line of (1.8)). We refer the reader to [1] for a review of the basic theory of Kontsevich integrals and a list of the relevant references. For the purposes of this paper some novel considerations are necessary. In particular, it is important to distinguish between four different situations, where the issues of integrability and Ward identities deserve separate analyses.

First of all, it is important to know where the singularities $x_s$ of the potential lie. We shall distinguish two cases: $x_s = \infty$ (the “polynomial” case) and $x_s = 0$ (the “antipolynomial” case). Actually in this paper we mostly deal with monomial potentials, $V(x) = -\frac{x^{p+1}}{p+1}$ where $p$ can be either a positive or a negative integer. Below we use $P$ to denote the absolute value of $p$: $P \equiv |p|$.

Another distinction is between two different limits (“phases”) of the GKM. These can be viewed simply as two different asymptotics of the Kontsevich integral (2.1): for small and large external fields $L$. The term “phases” is not really accurate in the theory of the GKM where the infinite-dimensional phase space is analyzed. This terminology derives from the fact that the weak- and strong-field limits of (certain versions of) the GKM are identified with the strong- and weak-coupling “phases” of lattice Yang-Mills models.

The currently-available results about the four subtopics of the GKM theory are summarized in the following table:

| Character phase | Polynomial | Antipolynomial |
|-----------------|------------|----------------|
| $p$-reduced KP $\tau$-function | $W^{(r)}_k$-const., $k \geq 1 - r$, $r \leq p$ | $W^{(r)}_k$-const., $k \geq 2 - r$, $r \leq |p|$ |
| string eq. $L_{-p}\tau = 0$ | string eq. $W^{(|P|)}_{p(p+2)}\tau = 0$ |

Only one of the four cases has been thoroughly investigated, that of the Kontsevich phase of the polynomial model. The structure of the character phase has so far been addressed only in ref.[2], but knowledge of its detailed structure is thus far incomplete. For comparison with the BGWM we require, for the most part, the antipolynomial model which to the best of our knowledge has never been analyzed in detail. We shall find that, especially in the case of the antipolynomial potential $V(x) = 1/x$ which is most relevant to the BGWM, some results are accessible. The purpose of this section is to describe the content of table 1 in a little more detail. Many things remain obscure and deserve further investigation.

For the BGWM only the piece of a generic GKM theory which deals with monomial potentials is actually relevant, therefore we often restrict consideration in this section to monomials, thus leaving a very interesting piece of the GKM theory beyond the scope of this paper.
2.1 Time-variables

Kontsevich integral $\mathcal{F}_V(L)$, introduced in (2.1), is a symmetric (Weyl-group invariant) function of eigenvalues $l_a$ and has different asymptotics for small and large $l$'s. In this paper we discuss only the two simplest situations: when all the $l$'s are either large or small at once. In the first limit ("character phase") $\mathcal{F}_V(L)$ can be considered as a function of Weyl-invariant quantities $t_k^+ \equiv \frac{1}{k} \text{tr} L^k$, and

$$Z_{\text{GKM}}^+(t^+|n,N,V) \equiv \mathcal{F}_V(N,L). \tag{2.2}$$

In the second limit ("Kontsevich phase") the integral is first rewritten in terms of eigenvalues of $X$ and $L$ and then is evaluated by the saddle-point method. The saddle-point $x_0(l)$ is defined from the equation

$$l + V'(x_0(l)) = 0. \tag{2.3}$$

For large $l$ the eigenvalues $x_0(l)$ are close to singularities of $V(x)$ and much depends on the degree of the singularity. If $V'(x) \sim (x - x_s)^{-p}$, then $x_0 - x_s \sim l^{-1/p}$ and perturbation theory around the saddle point is expandable in terms of $t_k^-(p)$

$$t_{kp+i}^- \equiv \frac{1}{p} \text{tr} L^{-k - \frac{i}{p}}, \quad 0 \leq i \leq |p| - 1. \tag{2.4}$$

In order to define $Z_{\text{GKM}}^-(t^-)$ it remains to factor out the quasi-classical contribution to $\mathcal{F}_V(L)$:

$$Z_{\text{GKM}}^-(t^-|N,V) \bigg|_{t_{kp+i}^- = \frac{1}{k+i/p} \text{tr} L^{-k - \frac{i}{p}}} \equiv (\mathcal{C}_V(N,L))^{-1} \mathcal{F}_V(N,L), \tag{2.5}$$

where

$$\mathcal{C}_V(N,L) \equiv \frac{e^{\text{tr}(LX_0 + V(X_0))}}{\text{Det}X_0^V \text{Det}^{1/2} \left( \frac{\partial}{\partial X_{tr}} \otimes \frac{\partial}{\partial X_{tr}} V(X_0) \right)}. \tag{2.6}$$

2.2 Integrable structure

Integrable structure (of cartanian type) of the GKM is essentially due to existence of determinant representation

$$\mathcal{F}_V(N,L) = \frac{1}{\Delta(l)} \prod_a \int \frac{dx_a}{x_a^N} e^{x_a l_a + V(x_a)} \Delta(x) = \frac{\det a \Psi_a(N(l))}{\Delta(l)}, \tag{2.7}$$

where $\Delta(x) = \prod_{a > b} (x_a - x_b)$ and

$$\Psi_a(l) = \int x_a^{n-1} e^{tx + V(x)} dx. \tag{2.8}$$

This expression should be compared with the standard representation of (1-component) KP $\tau$-function in Miwa coordinates,

$$\tau_{\beta}(t_k) = \frac{\det a \lambda^\beta \Psi_{a-\beta}(\lambda_b)}{\Delta(l)} \bigg|_{t_k = \frac{1}{k} \sum_{a=1}^n \lambda_a^{-k}}. \tag{2.9}$$

1. We reserve a simplified notation $t_k^-(1)$ for $t_k^-(p)$, i.e. for $p = 1$. Everywhere below $k \geq 0$, but the zero-times $t_0$ are defined in a somewhat tricky way: $t_0^+ \sim \frac{1}{k} \text{tr} L; t_0^- \sim \frac{1}{k} \text{tr} L^{-k}$ as $k \to +0$. This means that $t_0^+$ is usually substituted by $t_0 = -\text{tr} \log L$, and do not contribute to the sums $\sum_{k \geq 0} k t_k^+$, while $t_0^-$ on the contrary, does contribute, and $k t_k^+ |_{k=0} = \text{tr} L = n$.

2. Note that logarithmic piece is not included into potential $V(x)$ and is treated separately in (2.3) and (2.6). This prescription is important for the nice integrability properties of $Z_{\text{GKM}}^-$. Note that it is self-consistent provided $N/x^\sim l$ for large $l$ (what will usually be the case in our considerations below).
As usual, this formula describes the restriction of \( \tau \)-function on the \( n \)-dimensional hypersurface in the infinite-dimensional space of time-variables, and this is the only source of \( n \)-dependence at the l.h.s. as the shape of the function \( \tau\{t_k\} \) does not depend on any parameter like \( n \). \( \tau \)-function depends on choice of the (\( n \)-independent) functions (called basis vectors) \( \psi \), which are only restricted to have definite asymptotics at large \( \lambda \):

\[
\psi_a(\lambda) \sim \lambda^{a-1}(1 + O(1/\lambda)),
\]

where the r.h.s. should be expandable in negative integer powers of \( \lambda \). Classes of equivalence of such \( \psi \) (modulo analytic at \( \lambda = \infty \) transformations of \( \lambda \)) label the points of infinite Grassmannian (the “universal module space”). Parameter \( \beta \) is referred to as ”zero-time”. It can be considered as a remnant of embedding of KP hierarchy into a more general Toda-lattice hierarchy. The hierarchy is \( P \)-reduced, whenever

\[
\lambda^P \psi_a(\lambda) = \psi_{a+P}(\lambda) + \sum_{b=1}^{a+P-1} Q_{ab} \psi_b(\lambda)
\]

with any \( \lambda \)-independent \( Q_{ab} \). Characteristic feature of the \( P \)-reduced \( \tau \)-function is that

\[
\frac{\partial \log \tau\{t\}}{\partial t_{P_k}} = \text{const}, \text{ for any integer } k, l,
\]

i.e. the \( \tau \)-function is essentially independent of all the time-variables \( t_{P_k} \).

In order to obtain the \( \tau \)-function interpretation of the GKM it is only necessary to adjust the \( \Psi \)-functions (2.8) so that they satisfy conditions (2.10). For this purpose one can connect properly \( l \) and \( \lambda \), change normalization of functions \( \Psi_a \) and also change the labeling (\( a \) indices). Besides, we should do linear combinations of \( \phi_a \)'s not changing the determinant (3.30). Therefore, we can add to the second vector the first one with an arbitrary coefficient, to the third one both the first and the second ones with arbitrary coefficients and so on (so it corresponds to triangle substitution which does not change the determinant). Then we adjust these linear combinations such that new vectors \( \bar{\Psi}_a(l) \) will have the asymptotics \( l^{a-1} \). Let us also normalize them to have the unit coefficient in the first term.

To conclude these introductory comments, let us briefly explain why the determinant representation (3.30) with \( \psi_a(\lambda) \) independent of \( n \) and the asymptotics (2.10) guarantees \( n \)-independence of (3.30) as a function of times. In fact, it means that changing \( n + 1 \to n \) does not change the functional form of (3.30), but only the point in the time space, i.e. times are now parameterized by \( n \) instead of \( n + 1 \) Miwa coordinates. Put differently, this means that one can tend \( \lambda_{n+1} \) to \( \infty \), and the determinant in (3.30) will be of the same form, but of \( n \times n \) matrix. This is actually the case provided by the condition (2.10) and unit coefficient in the leading term. This unit coefficient is also consistent with the normalization of (2.1) which implied to be unit when all \( \lambda \) tends to \( \infty \), i.e. all the times are equal to zero. We postpone more detailed discussion to the conclusion of the paper, when we will have some manifest examples of how it works.

### 2.2.1 Character phase

In the character phase \( \Psi_a(l) \) are expanded in positive integer powers of \( l \), therefore \( \lambda \) should be identified with \( 1/l \). However, \( \Psi_a(l) \) as defined in (2.8) is \( \Psi_a(l) \sim 1 + O(l = 1/\lambda) \) rather than the what is required in (2.10). However, we are still free to perform a linear transformation of the set \( \{\Psi_a\} \) which does not change the determinant (3.30). Namely, we can add to the second vector the first one with an arbitrary coefficient, to the third one both the first and the second ones with arbitrary coefficients and so on (thus the linear transformation is triangular and leave determinant intact). These linear combinations can be adjusted so that the new vectors \( \bar{\phi}_a(l) \)
have asymptotics $t^{a-1}$. Then, we can also normalize them to have unit coefficients in front of the leading terms: this gives rise to an overall $\lambda$-independent constant.

For these new functions we have:

$$\tilde{\Psi}_{a-N}(l) = t^{a-N-1}(1 + O(l)), \quad (2.13)$$

but this is still “inverse” with respect to (2.10): $\lambda^{1-a}$ appears instead of $\lambda^{a-1}$. Now comes the crucial step, which one should always make in the character phase [2]: let us relabel the $\Psi$-functions according to the rule

$$\tilde{\Psi}_a = \tilde{\Psi}_{n-a+1}. \quad (2.14)$$

i.e. just replace $a \to n - a + 1$. This transformation does not change the determinant:

$$\det_{ab} \Psi_a(l_b) = \det_{ab} \tilde{\Psi}_a(l_b) = \det_{ab} \tilde{\Psi}_a(l_b) \quad (2.15)$$

Then

$$\tilde{\Psi}_a = t^{n-a}(1 + O(l)) = \lambda^{1-n}[\lambda^{a-1}(1 + O(\lambda))] \equiv \lambda^{1-n}\tilde{\psi}_a(\lambda) \quad (2.16)$$

and taking into account the change of Van-der-Monde determinant in (2.9) when $l \to \lambda^{-1}$, one finally obtains

$$\tilde{\psi}_a = \lambda^{a-1}(1 + O(\lambda)). \quad (2.17)$$

These functions already possess correct asymptotics behavior (2.10), but instead they can depend on $n$. Indeed, starting from $n$-independent functions we performed the transformation (2.14) which introduced manifest dependence of $n$. In general situation this is a disastrous obstacle for the character phase partition function to be $\tau$-function. Nevertheless, sometimes by special adjustment of the coefficients in the potential in (2.8) it is possible to get rid of this $n$-dependence. An example of such a situation is provided by BGWM and will be considered in section 3.2.3.

### 2.2.2 Kontsevich phase. Polynomial model

In the Kontsevich phase things will be somewhat different in the polynomial and in the antipolynomial situations. Let us begin with the polynomial case which was considered in detail in [9, 10, 11]. The Kontsevich phase corresponds to a large $l$ expansion. Therefore, using the saddle-point method, the integral (2.8) can be expanded in a series of inverse powers of $l$. Indeed, it is straightforward to derive

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5Within the character phase it is also interesting to consider discrete sums (over eigenvalues of $X$) rather than integral in the definition of $F_v(L)$. Such discretized GKM is intimately related to representation theory of compact groups and in this context it can be also reasonable to consider expansion in powers of $t^{+}_k \equiv \frac{1}{L} \text{tr} kl = \sum_{k=0}^{\infty} \frac{e^{-kl}}{k!}$ If in Kontsevich phase the saddle point occurs at the large value of $x = x_0$, then there is no real difference between discretized and continuous GKM. While this condition is satisfied for potentials $\mathcal{V}(x)$ which are polynomials in $x$ (so that $x_0 = \infty$ and $x_0(l) \sim l^{1/p}$, it is no longer true for polynomials in $x^{-1}$ i.e. $x_0 = 0$ and $x_0(l) \sim l^{-1/p}$, which is actually the case in our discussion of the BGWM: see eq.(3.9). In this case, however, the better representation is in terms of $Y = 1/X$, and $y_0(l) \sim l^{1/p}$ is large in Kontsevich phase.

Yet another possibility to represent (2.8) in the form of a $\tau$-function was proposed in [2], where all $n$-dependence of basis vectors is ascribed to $n$ additional Miwa variables parameterizing the negative times of Toda lattice hierarchy. Then, basis vectors are $n$-independent but instead only on negative times. However, putting these new parameters to be unity, one gets the coefficients of basis vectors manifestly depending on $n$. In this case the $n$-dependence is completely due to the special choice of the negative times, and the determinant (3.30) still describes $\tau$-function, but that of Toda lattice hierarchy.
\[
\Psi_{a-N}(l) = \sqrt{V''(\lambda) e^{V'(\lambda)-V(\lambda)} \lambda^{-N} \alpha^{-1}(1+O(1/\lambda))} = \sqrt{V''(\lambda) e^{V'(\lambda)-V(\lambda)} \lambda^{-N} \psi_{a-N}(\lambda)} \equiv s^{-1}(\lambda) [\lambda^N \psi_{a-N}(\lambda)],
\]

where \(\lambda\) is solution of the saddle-point equation:

\[
l + V'(\lambda) = 0.
\]

Then, in accordance with our general rules, we should suit normalization of \(\Psi_a\) to the asymptotics conditions (2.10). This normalization factor in matrix form is just \(C_V\) of (2.6). Simultaneously, we automatically obtain basis vectors which do not depend on \(n\) and the coefficient \(N\) in front of logarithm plays the role of “zero time”.

Consider the case when the potential is a monomial of degree \(p+1\): \(V = -x^{p+1}/(p+1)\). Then, by examining small variations of \(x\) in the integral (2.8), one obtains the recurrence relation

\[
l \Psi_a(\lambda) - \Psi_{a+p}(\lambda) = (a-1)\Psi_{a-1}(\lambda)
\]

and a similar one for the functions \(\psi_a(\lambda)\). Using (2.19), i.e. \(l = \lambda^p\), one immediately gets the reduction condition (2.11):

\[
\lambda^p \psi_a(\lambda) = \psi_{a+p}(\lambda) + (a-1)\psi_{a-1}(\lambda).
\]

Thus, we prove that the GKM in the Kontsevich polynomial phase with monomial potential of the degree \(p\) describes the \(\tau\)-function of \(p\)-reduced KP hierarchy.

We demonstrate now that antipolynomial model is nothing but “analytical continuation” of the polynomial model to negative values of \(p\).

### 2.2.3 Kontsevich phase. Antipolynomial model

Thus, let us consider antipolynomial Kontsevich phase, i.e. potential \(V(x)\) in (2.8) which is polynomial in inverse degrees of \(x\). Again we can apply the saddle-point method to get the same saddle-point equation (2.19) which usually allows one to connect \(l\) and \(\lambda\), and we can read off the correct normalization factor from (2.6).

Nevertheless, there is one subtlety which requires some accuracy. To demonstrate the point, let us consider again the monomial potential \(V(x) = -x^{p+1}/(p+1)\), where \(P = |p| = -p\). Then, \(l = x_0^p = x_0^{-P}\), where \(x_0\) is the saddle point, and we choose \(\lambda = x_0^{-1}\) (as the Kontsevich phase requires \(\lambda\) to increase with \(l\)) and we get for the asymptotics expansion of correctly normalized functions \(\tilde{\Psi}_a(\lambda)\) (these are still not correct \(\psi_a(\lambda)\)):

\[
\tilde{\Psi}_{a-N}(\lambda) \equiv \lambda^{1-a} e^{-P^{p+1}} \psi_{a-N}(\lambda) = \lambda^{N-a}(1+O(1/\lambda)).
\]

For given \(n\) (the size of determinant in (2.9)) this set of functions can be transformed to that with increasing power of \(\lambda\), as required in (2.10), by relabelling. Namely, one can substitute \(a \rightarrow n-a+1\) and get

\[
\tilde{\Psi}_{n-a-N+1}(\lambda) = \lambda^{a-1+N-n}(1+O(1/\lambda)).
\]

These vectors have correct asymptotics behavior, but depend manifestly on \(n\). Therefore, as the last step one should choose \(N = n - \beta\) to get rid of any \(n\) dependence:
\[
\psi_{a-\beta}(\lambda) \equiv \tilde{\psi}_{\beta-a+1}(\lambda) \equiv \tilde{\Psi}_{\beta-a+1}(\lambda) = s(\lambda) \int x^{\beta-a} e^{x\lambda P + \nu(x)} dx.
\] (2.24)

This actually means that, while the integrand (2.1) contains manifest \( n \) dependence, the integral as a function of times does not! Simultaneously we get interpretation of the difference \( n - N \) as the zero time.

Now let us look at the recurrence relation satisfied by the basis vectors \( \psi_a(\lambda) \). It can be derived similarly to (2.20) and looks like

\[
\lambda^P \tilde{\psi}_a(\lambda) - \tilde{\psi}_{a-P}(\lambda) = (a - 1) \tilde{\psi}_{a-1}(\lambda).
\] (2.25)

After relabelling, one obtains the reduction condition (2.11) with \( P = -|p| \):

\[
\lambda^P \psi_a(\lambda) = \psi_{a+P}(\lambda) - a \psi_{a+1}(\lambda).
\] (2.26)

Therefore, we proved that the Kontsevich antipolynomial phase is described by the \( \tau \)-function of \(|p|\)-reduced KP hierarchy. This implies that any GKM in Kontsevich phase can be described as the \( \tau \)-function of \(|p|\)-reduced KP hierarchy, \( p \) being the degree of the potential.

### 2.3 String equation

Besides integrability, another important property which characterizes the GKM is that its partition function satisfies some constraint algebra, which determines it unambiguously. In this sense, one does not need any specific information about integrability. However, it is usually a very difficult problem to find this constraint algebra. Considerably easier is to use integrability properties (which fix the partition function as a \( \tau \)-function) and one more equation, namely, "string equation"\(^6\), which specifies \( \tau \)-function uniquely (i.e. fixes the point of the Grassmannian unambiguously). The string equation is usually the first constraint of the algebra. We will see in the next section that the constraint algebra can be easily obtained in the character phase. However, we think it is too early to recognize the first constraint of this algebra as string equation, since to acquire its full status it should be sufficient for restoration of the whole constraint algebra when supplemented by appropriate integrability property. But these properties are still undiscovered in the character phase. This is why there remain empty places in the table 1.

Unlike this, in the Kontsevich phase we know the integrable properties, but the entire constraint algebra can be found only after some tedious work. Therefore, in this case, the string equation, which can be easily found, is a very efficient tool. Below we derive string equations in both models in Kontsevich phase.

#### 2.3.1 Kontsevich polynomial phase

In this subsection we shall shortly repeat the derivation of string equation in the polynomial case, addressing reader to [9] for more details. We shall demonstrate that this derivation crucially uses only the following information:

1) given asymptotics of \( \psi_a(\mu) \sim \mu^{a-1} \);

2) the manifest form of normalization factor depending on \( M \) in the GKM integral;

3) the manifest form of linear term in \( X \) in exponential in the integrand in (2.1).

\(^6\)In the literature, it is used to call string equation the derivative of the appropriate constraint over the first time variable. For the sake of brevity, we usually call string equation the constraint itself.
The main idea is to consider the following derivative of the GKM partition function \( \tau = \frac{\det \psi_a(\lambda_b)}{\Delta(\lambda)} \equiv \frac{\det s(\lambda_b)\Psi_a(\lambda_b)}{\Delta(\lambda)} \) (see (2.9), (2.18)):

\[
\text{Tr} \left\{ \frac{1}{V'(\Lambda)} \frac{\partial}{\partial \Lambda} \log \tau \right\}
\]  
(2.27)

where, as usual, \( L \equiv V'(\Lambda) \), and rewrite it as

\[
\sum_{p>0} \text{Tr} \frac{1}{V'(\Lambda)} \frac{\partial}{\partial t_p} \partial \log \tau = - \sum_{p>0} \text{Tr} \frac{1}{V'(\Lambda)} \frac{1}{\Delta^{p+1}} \frac{\partial}{\partial t_p} \log \tau.
\]  
(2.28)

On the other hand, the derivative (2.27) is equal to two pieces: the first one originates from the derivative of factors \( s(\lambda) \) and \( \Delta(\lambda)^{-1} \) (here we use the information of point 2) and is equal to:

\[
\frac{1}{2} \sum_{a,b} \frac{V''(\lambda_a)V''(\lambda_b)}{\lambda_a - \lambda_b} V''(\lambda_a) - V''(\lambda_b),
\]  
(2.29)

where potential \( V(\lambda) \) generally contains logarithmic term. The remaining second piece can be transformed to derivative over \( t_1 \) essentially using the correct asymptotics of \( \psi_a(\lambda) \) (point 1):

\[
\text{Tr} \left\{ \frac{1}{V'(\Lambda)} \frac{\partial}{\partial \Lambda} \log \det \Psi_a(\lambda_b) \right\} = \frac{\partial}{\partial t_1} \log \tau.
\]  
(2.30)

Thus, we finally obtain the string equation in the form:

\[
A_\tau = 0, \quad A^{(V)} \equiv \sum_{p>0} T^{(V^+)}_{p+1} \frac{1}{\partial t_p} + \frac{\partial}{\partial t_1} + \frac{1}{2} \sum_{a,b} \frac{V''(\lambda_a)V''(\lambda_b)}{\lambda_a - \lambda_b} V''(\lambda_a) - V''(\lambda_b)
\]  
(2.31)

with

\[
T^{(V)}_p \equiv \text{Tr} \frac{1}{V'(\Lambda)\Delta^p}.
\]  
(2.32)

Now let us describe some different approach to the string equation proposed in [12]. For the sake of simplicity, we consider only monomial potential.

The idea of approach is to use the integrability of the system, i.e. the fact that its partition function is a \( \tau \)-function. This means that, instead of considering operators acting in the space of functions of times, one can immediately operate in the infinite-dimensional Grassmannian. Our problem now is to find the analog of the string equation in these terms to fix unambiguously the point of the Grassmannian. We already specified the \( \tau \)-function of polynomial model to be a \( p \)-reduced \( \tau \)-function. This means that the element of the Grassmannian we are looking for lies in the subspace \( V \) satisfying the condition:

\[
\lambda^p V = lV \subset V.
\]  
(2.33)

Now we are searching for one more condition on \( V \). For this purpose, let us note that all functions \( \Psi_a(\lambda) \) are connected by the transformation

\[
\Psi_{a+1}(l) = \frac{\partial}{\partial l} \Psi_a(l),
\]  
(2.34)

or, equivalently,
\[
\psi_{a+1}(l) = \left[ s(l) \frac{\partial}{\partial l} s^{-1}(l) \right] \psi_a(l) \equiv A \psi_a(l). \quad (2.35)
\]

The set of basis vectors \( \psi_a(\lambda) \) gives an element of the Grassmannian. On the other hand, the operator \( A \) maps the set of basis vectors onto itself. Thus, we get the following new condition on the subspace of the Grassmannian:

\[
AV \subset V. \quad (2.36)
\]

This condition is equivalent to the string equation (see also [12]) and, along with (2.33), specifies the point of the Grassmannian, i.e. the GKM partition function. In particular, one can see that this is actually generating the constraint algebra. Indeed, conditions (2.33) and (2.36) imply that any commutator of products of \( l \) and \( A \) also leaves point of \( V \) in \( V \). On the other hand, the commutator \([A,l] = 1\), which means that all possible products \( l^a A^b \), \( a, b \leq 0 \) generate a subalgebra of \( W^{(\infty)} \)-algebra. However, this algebra does not annihilate the \( \tau \)-function, as this is the algebra in the space of the parameter \( l \). Since the true spectral parameter is \( \lambda \) (i.e. all times are made of integer degrees of \( \lambda \)), one should make a transformation \( \lambda^p \to \lambda \). It turns out to be extremely non-trivial problem [13]. We demonstrate in section 2.5 how to find the constraint algebra without explicitly making this transformation.

To conclude, let us remark that the operator \( \lambda^a \frac{\partial}{\partial \lambda} \) (in the space of the spectral parameter) corresponds trivially to the operator \( W^{(b+1)} \). This means that the operator \( A \) corresponds to the \( \mathcal{L}_{-p} \)-constraint, which is, in fact, the string equation operator \( A \) (2.31). Besides, it implies that the subalgebra of \( W^{(\infty)} \)-algebra, annihilating the GKM in polynomial case, contains only modes which are zero by modulo \( p \), and, moreover, of \( W^{(p)}_{pk} \)-generators, only those with \( k \geq 1 - r \) are presented. We return to this issue in section 2.5.

### 2.3.2 Kontsevich antipolynomial phase

Now let us repeat the derivation of the string equation for the antipolynomial model in Kontsevich phase. It is difficult to reproduce the derivation of the first part of the previous subsection for the general case of antipolynomial model. For illustrative purpose, we demonstrate this derivation in the case of \( p = -2 \) in section 3.3.4. But now we are going to get the string equation in terms of the Grassmannian.

In the antipolynomial case we have again the reduction condition

\[
\lambda^p V = lV \subset V. \quad (2.37)
\]

Now we need to construct the operator \( A \). To do this, let us note that the derivative \( \frac{\partial}{\partial l} \) shifts the index of the basis vector (2.24) in wrong direction: \( \left[ s(\lambda) \frac{\partial}{\partial l} s^{-1}(\lambda) \right] \psi_a \to \psi_{a-1} \), i.e. \( \left[ s(\lambda) \frac{\partial}{\partial l} s^{-1}(\lambda) \right] V \not\subset V \). However, we can use the reduction condition (2.37) to shift it back. Thus, the operator \( l \left[ s(\lambda) \frac{\partial}{\partial l} s^{-1}(\lambda) \right] \psi_a \to \psi_{a+p-1} + ... \), where dots stand for terms which can be removed by the linear low-triangle transformation of basis vectors. This operator is a good candidate for \( A \)-operator and it satisfies (2.36). Along with the reduction operator, it generates the subalgebra of \( W^{(\infty)} \)-algebra with leading terms \( l^a \frac{\partial}{\partial \lambda} \) with \( a \geq b \). But this is not the maximal possible algebra. Indeed, let us note that we can take as many as \( P - 1 \) derivatives of basis vector and after this use the reduction condition. This procedure induced by operator \( A = \left[ ls(\lambda) \frac{\partial^{P-1}}{\partial \lambda^{P-1}} s^{-1}(\lambda) \right] \) transforms the basis vectors \( \psi_a \to \psi_{a+1} \). This is just the ”minimal” operator we need. Surely, it still satisfies the equation (2.36).

It mean, finally, that operator \( A \) acting on the Grassmannian has the form
\[ A \equiv \left[ l_s(\lambda) \frac{\partial P^{-1}}{\partial P^{-1}} s^{-1}(\lambda) \right] \sim \lambda - P^2 + 3P^{-1} \frac{\partial P^{-1}}{\partial \lambda} + \ldots, \quad (2.38) \]

and we obtain the string equation (2.36). In accordance with the general rules above this operator corresponds to \( W^{(P)}_{-(P-2)P} \)-constraint. It proves the corresponding statement in the table 1.

Let us note that one can again construct the subalgebra of \( W^{(\infty)} \)-algebra from products of the operators \( l = \lambda P \) and \( A \): \( l^a A^b \). It again contains only zero by modulo \( p \) modes, but different restrictions will be produced: in the \( W^{(r)}_{pk} \)-algebra only \( k \geq 2 - r \) modes are presented for \( r \leq p \) and \( k \geq 2 - p \) for all other \( r \). Let us note that we can consider a \( W \)-algebra defined at the vicinity of \( \infty \), instead of 0. This will result into the change of sign of all modes. We will use in future just this convention (and it is used in the table 1) to have common approach to both polynomial and antipolynomial models.

### 2.4 Ward identities

All the Ward identities for the GKM follow from the matrix-valued equation of motion, \( \langle L - \frac{N}{X} + V'(X) \rangle = 0 \), which can be rewritten as

\[
\left\{ L - N \left( \frac{\partial}{\partial L_{tr}} \right)^{-1} + V' \left( \frac{\partial}{\partial L_{tr}} \right) \right\} F_V(N, L) = 0. \quad (2.39)
\]

If \( N \neq 0 \) and/or \( V'(x) \) contains some negative powers of \( x \), the reasonable identity arises as some \( L \)-derivative of this relation, so that it becomes differential rather than integro-differential equation (eq.(3.8) is a particular example).

#### 2.4.1 \( \tilde{W} \)-operators

Ward identities for the GKM in the character and Kontsevich phases arise when \( Z_{GKM}^\pm(t^\pm) \) from (2.2) and (2.5) are substituted for \( F \) into (2.39). The resulting equations are expressed in terms of the differential \( \tilde{W} \)-operators. These are defined by any of the following three relations:

\[
\left( \frac{\partial}{\partial L_{tr}} \right)^{m+1} f(t^\pm) = \sum_{s \geq 1} L^{s-1} \tilde{W}^{(\pm, m+1)}_{s \pm m}(t^\pm) f(t^\pm) \bigg|_{t^\pm_k = t^{\pm}_{k+1}}, \quad (2.40)
\]

or

\[
\tilde{W}^{(\pm, m+1)}_{s \pm m}(t) e \sum_{k \geq 0} t^\pm_k L^{\pm k} = \left\{ \text{tr} \left( \frac{\partial}{\partial L_{tr}} \right)^m L^{\pm s} \right\} e \sum_{k \geq 0} t^\pm_k L^{\pm k}, \quad (2.41)
\]

or

\[
\tilde{W}^{(\pm, m+1)}_{s \pm m}(t) = \sum_{k \geq 0} k t^\pm_k \tilde{W}^{(\pm, m)}_{s+k \pm m}(t) + \sum_{k=1}^{s-1} \frac{\partial}{\partial t^\pm_k} \tilde{W}^{(\pm, m)}_{s-k \pm m}(t). \quad (2.42)
\]

The last recurrence relation should be supplemented by “initial condition”

\[
\tilde{W}^{(\pm, 1)}_s = \frac{\partial}{\partial t_s}, \quad s \geq 1 \quad (2.43)
\]

or even

\[
\tilde{W}^{(\pm, 0)}_s = \delta_{s,0}. \quad (2.44)
\]

\(^7\text{It is in eqs.}(2.41)\text{ and } (2.42)\text{ that the convention } k t^\pm_k \big|_{k=0} = \text{tr} I = n, \text{ introduced in the footnote 3 is essential.}\)
These relations define operators $\tilde{W}_{s\pm m}^{(\pm,m+1)}(t)$ for $s \geq 1/2 \pm 1/2$, no reasonable definition of harmonics with $s < 1/2 \pm 1/2$ is known. This and the recurrence relation (2.42) are their most striking differences from the conventional Zamolodchikov’s operators $W(t)$, defined by the standard bosonization procedure.

2.4.2 Character phase

Relation (2.40) can be used directly to derive the Ward identities in the character phase of the polynomial GKM with $N = 0$. If $V(x) = -\frac{\partial^{p+1}}{\partial x^{p+1}}$, $p > 0$, eq.(2.39) turns into:

$$\sum_{s \geq 1} L^{s-1} \left\{ \left( \tilde{W}_{s+p-1}^{(+p)}(t^+) - \delta_{s,2} \right) Z_{\text{GKM}}^{+}(t^+) \right\} = 0,$$

(2.45)

from which we conclude, that

$$\tilde{W}_{s+p-1}^{(+p)}(t^+) Z_{\text{GKM}}^{+}(t^+) = \delta_{s,2} Z_{\text{GKM}}^{+}(t^+), \quad s \geq 1.$$

(2.46)

If potential is not a monomial, a sum over $p$ arises at the l.h.s.

If $N \neq 0$, one should consider an $L$-derivative of (2.39) in order to get rid of the integral operator $(\partial/\partial L_{tr})^{-1}$:

$$\left\{ \frac{\partial}{\partial L_{tr}} L - NI + \frac{\partial}{\partial L_{tr}} V' \left( \frac{\partial}{\partial L_{tr}} \right) \right\} F_V(N, L) = 0.$$

(2.47)

This can be again rewritten in terms of the $\tilde{W}$-operators, but there are two essential differences from the case of $N = 0$. First, operator $\tilde{W}_{s+p}^{(+p+1)}$ will appear instead of the $\tilde{W}_{s}^{(+p)}$. Second, explicit dependence on $n$, i.e. on the size of the matrix $L$ will arise, because when $\frac{\partial}{\partial L_{tr}}$ acts on $L$, it produces $nI$. (It is not just a commutator because the contraction of matrix indices remain intact.) Keeping these two remarks in mind, we obtain:

$$\sum_{s \geq 1} L^{s-1} \left\{ \left( \tilde{W}_{s+p}^{(+p+1)}(t^+) - (N - n) \delta_{s,1} + \tilde{W}_{s+1}^{(+1)}(1 - \delta_{s,1}) \right) Z_{\text{GKM}}^{+}(t|N) \right\} = 0,$$

(2.48)

or

$$\left( \tilde{W}_{s+p}^{(+p+1)}(t^+) + \frac{\partial}{\partial L_{tr}}(1 - \delta_{s,1}) \right) Z_{\text{GKM}}^{+}(t|N) = (N - n) \delta_{s,1} Z_{\text{GKM}}^{+}(t|N),$$

(2.49)

$s \geq 1$.

If $N = 0$ this relation is of course a corollary of (2.46) and (2.42).

Now it is clear, what should be done in the anti-polynomial case. For $p < 0$ and $V(x) = -\frac{\partial^{p+1}}{\partial x^{p+1}} = \frac{\partial^{1-p}}{\partial x^{1-p}}$, $P = |p| = -p$, one should take as many as $P$ derivatives of (2.39):

$$\left\{ \left( \frac{\partial}{\partial L_{tr}} \right)^{P} L - N \left( \frac{\partial}{\partial L_{tr}} \right)^{P-1} + \left( \frac{\partial}{\partial L_{tr}} \right)^{P} V' \left( \frac{\partial}{\partial L_{tr}} \right) \right\} F_V(N, L) = 0.$$

(2.50)

Then the last item at the l.h.s. is just $-I$, while the first one can be rewritten as

$$\sum_{a+b=p} \left\{ \left( \frac{\partial}{\partial L_{tr}} \right)^a \text{Tr} \left( \frac{\partial}{\partial L_{tr}} \right)^b + n \left( \frac{\partial}{\partial L_{tr}} \right)^{P-1} + \left[ L_{tr} \left( \frac{\partial}{\partial L_{tr}} \right)^P \right]_{tr} \right\} F_V(N, L) = 0.$$

(2.51)
From (2.40) we deduce that

$$\sum_{a+b-p-1 \geq 1} \left( \frac{\partial}{\partial L_{tr}} \right)^a \text{Tr} \left( \frac{\partial}{\partial L_{tr}} \right)^b = \sum_{a+b-p-1 \geq 1} L^{a-1} \tilde{W}_{s+a-1}^{(+,a)} \left( (q-1)t_{q-1} \tilde{W}_{q+b-1}^{(+,b)} \right);$$

and, putting everything together and using again (2.40), we obtain:

$$\sum_{s \geq 1} L^{s-1} \left\{ \sum_{a+b-p-1 \geq 1} \tilde{W}_{s+a-1}^{(+,a)} \left( (q-1)t_{q-1} \tilde{W}_{q+b-1}^{(+,b)} \right) + (n-\mathcal{N}) \tilde{W}_{s+p-2}^{(+,P-1)} + \tilde{W}_{s+p-2}^{(+,P)} (1 - \delta_{s,1} - \delta_{s,1}) \right\} Z_{GKM}^+(t^+|\mathcal{N}) = 0,$$

or

$$\sum_{a+b-p-1 \geq 1} \tilde{W}_{s+a-1}^{(+,a)} \left( (q-1)t_{q-1} \tilde{W}_{q+b-1}^{(+,b)} \right) + (n-\mathcal{N}) \tilde{W}_{s+p-2}^{(+,P-1)} + \tilde{W}_{s+p-2}^{(+,P)} (1 - \delta_{s,1}) \right\} Z_{GKM}^+(t^+|\mathcal{N}) = \delta_{s,1} Z_{GKM}^+(t^+|\mathcal{N}).$$

Of all the $\tilde{W}^{(+)}$-operators in this paper we need only

$$L^{(d)}_s(n,t) \equiv \tilde{W}_s^{(+,2)} = n \frac{\partial}{\partial t_s} + \sum_{k=1}^s k t_k \frac{\partial}{\partial t_{k+s}} + \sum_{k=1}^{s-1} \frac{\partial^2}{\partial t_k \partial t_{s-k}}.$$

We emphasize, that this operator depends explicitly on the parameter $n$ (the size of $L$ matrix), which is hidden in the previous formulas in the $k = 0$ item in the sums over $k$ (according to our convention, $kt_0|_{k=0} = n$).

### 2.4.3 Kontsevich phase

We have demonstrated how to generate Ward identities in the character phase and how to transform them to $\tilde{W}$-operators. This is, however, somewhat less straightforward in the Kontsevich phase, because one should take into account the $\mathcal{C}_V$ factor and the difference between the proper variables $t^{(-,p)}$ and $t^- = t^{(-,1)}$, which appear in (2.40). Indeed, what one should do is to take into account correctly the terms which are constituted Ward identities derivatives of the factor $\mathcal{C}_V$. This adds to the standard GKM Ward identities (2.39) some new pieces. This pieces are of importance as (2.39) failed to be expanded properly into traces of negative powers of $L$ such that the result depends only on times $t_{pk+i}$ (2.4). But it can be done after taking into account all the contributions from $\mathcal{C}_V$.

Simultaneously it generates the system of $W$-constraints imposed on the partition function. The drawback of such a calculation is that this is very long and tedious and was presented only in the cases of $p = 1$ [15, 10], $p = 2$ [16, 17, 6], $p = 3$ [18] and $p = -2$ [6]. (We will return to the last case in section 3.3.) Thus, we need some other way to determine the constraint algebra. One more way was proposed in the paper [13], who manifestly demonstrated the tranformation $l \rightarrow \lambda$ at the level of $W$-algebra (see discussion in section 2.3). This calculation is also rather tedious and was not completed in full. Instead, in the next section we propose a very simple, a bit heuristic way to determine the constraint algebra which is imposed on the partition function. It allows us, in particular, to determine this in the antipolynomial phase.
2.5 Ward identities as recursive relations. A new way to deal with Ward identities in matrix models

Now we are going to explain new and to our knowledge the most effective method to deal with constraint algebras in matrix models though it is not very rigid, but rather heuristic.

It is based on a specifics of matrix models that the Ward identities are essentially the same as equations of motion and thus define the partition function unambiguously, at least in the form of a formal series,

$$Z = 1 + \sum_s a_s t_s + \sum_{s_1, s_2} a_{s_1, s_2} t_{s_1} t_{s_2} + \ldots \quad (2.56)$$

We shall now explain how this actually works in different models and phases and how it can be used to fix constraint algebra.

All kinds of $\mathcal{W}$-operators (including Virasoro, $\hat{\mathcal{W}}^{(\pm)}$, Zamolodchikov’s $W$ etc) possess the following property: operator with the subscript $r$ is a linear combination of terms like

$$k_1 \hat{t}_{k_1} \ldots k_a \hat{t}_{k_a} \frac{\partial^b}{\partial t_{l_1} \ldots \partial t_{l_b}}$$

with

$$l_1 + \ldots + l_b - k_1 - \ldots - k_a = r \quad \text{and all} \quad l_1, \ldots, l_b \geq 1,$$

where $\hat{t}_{k}$’s are the times $t_{k}$’s, may be shifted by a constant (see below). A property of the systems of constraints, arising in the study of matrix models, which is responsible for the uniqueness of their solutions, is that every constrain has an item $\frac{\partial}{\partial t_r}$, i.e. with linear derivative and $t$-independent coefficient, and every integer $r \geq 1$ appears in one and exactly one constraint.

The origin of such terms is somewhat different in different situations.

2.5.1 Character phase

In the character phase $\hat{t}_k = t_k$, $\hat{s} = s$ and such terms arise from the contributions with $k t_k^r |_{k=0} = n$ to $\hat{\mathcal{W}}$-operators. $\frac{\partial}{\partial t_r}$ appears for this reason in the operator $\hat{\mathcal{W}}^{(\pm, p)}_r$ and preserve the gradation. In order to have all the integer $r \geq 1$ represented exactly once in the systems of constraints (2.48) or (2.53) the labels $r$ in these systems are restricted to be $r \geq 1$.

Whenever such system of constraints is given, it unambiguously fixes the perturbative expansion of the partition function. To prove this, one should just act iteratively starting from the first gradation level. It is possible as the coefficients $\{a_{s_1, \ldots, s_k}\}$ has the determined gradation level $s_1 + s_2 + \ldots + s_k$ and so do constraint algebras (2.48) and (2.53). We illustrate the procedure for the simplest example of $P = 2$ antipolynomial phase. Then we get from (2.54) and (2.55)

$$\left( N \frac{\partial}{\partial t_s} + \sum_{k>0} k t_k \frac{\partial}{\partial t_{k+s}} + \sum_{k=1}^{s-1} \frac{\partial^2}{\partial t_k \partial t_{s-k}} \right) Z = \delta_{s,1} Z. \quad (2.58)$$

As the first step, we calculate $a_1$ using $\mathcal{L}_1$-constraint at all $t_k = 0$:

$$N a_1 = 1, \quad a_1 = \frac{1}{N}. \quad (2.59)$$

Now we have two coefficients at the second level (in the gradation) - $a_2$ and $a_{1,1}$. They can be fixed by two equations obtained from $\mathcal{L}_2$-constraint and by differentiating $\mathcal{L}_1$-constraint in the first time, both taken at all $t_k = 0$:
\[ N_{a_2} + 2a_{1,1} = 0, \]
\[ 2Na_{1,1} + a_2 = a_1, \]
\[ \text{i.e. } a_{1,1} = \frac{1}{2(N^2 - 1)}, \quad a_2 = -\frac{1}{N(N^2 - 1)}. \]

This procedure can be evidently continued to build all the expansion (2.56). In particular, at the \( k \)-th level there are \( P(k) \) unknown coefficients \( a_k, a_{k-1,1}, a_{k-2,1,1}, \ldots, P(k) \) being the number of partitions of \( k \) into integers, which can be determined from \( P(k) \) equations: constraints \( L_k, \frac{\partial}{\partial t_1}, L_{k-1}, \frac{\partial}{\partial t_1}L_{k-2}, \ldots \), all taken at \( t_k = 0 \). It is clear that the number of unknown coefficients coincides with the number of equations.

Besides, it is clear that the same procedure is applicable for any constraint systems of (2.48) or (2.53) type (note that no algebras of several different spins simultaneously presented here - in constrast to Kontsevich phase below). Indeed, we could guess these constraints even without concrete calculations. This is not of great importance in the present trivial case, but gives new and very efficient method in the complicated case of Kontsevich phase.

### 2.5.2 Kontsevich phase

In Kontsevich phase there is a “shift of times” \( kt_k = kt_k - p\delta_{k,p+1} \), or, better,
\[ \hat{t}_{pk+i}^{-(p)} = t_{pk+i}^{-(p)} - \frac{p}{p + 1}\delta_{k,1}\delta_{i,1} \] (2.61)
both for positive and negative \( p \). \(^9\) Such shift breaks the gradation rule (2.57). Nevertheless, the reasoning above is still applicable, as this shift goes in “correct” direction, which means that it preserves hierarchical structure of equations for the coefficients \( \{a_{s_1,\ldots,s_k}\} \) in (2.56) and they still can be found iteratively.

Now let us consider the concrete structure of Kontsevich phase with the deformed gradation rules given by the shift (2.61). We should take into account that there should appear only \( W_{pk}^{(r)} \)-operators, as we need the operator algebra which respects \( p \)-gradation, i.e. does not depend on \( t_{pk} \) and have natural \( p \)-gradation. Then, the \( \frac{\partial}{\partial t_s} \) term arises from the term \( k_1^1 \hat{t}_{k_1} \ldots k_r^1 \hat{t}_{k_r} \frac{\partial}{\partial s_{r+1}} \) in the \( W_{s-r(p+1)}^{(r+1)} \)-operator. Actually \( s - r(p + 1) = pk \), thus \( s = r \mod p \). This is the main complication as compared to the character phase which involves separate consideration of the constraints of different spins. Indeed, we need the constraint system with each \( \frac{\partial}{\partial t_s} \) term appearing in exactly one equation (for all positive, or for all negative \( s \) depending on the direction of gradation).

As the first example, let us consider polynomial case. Then, \( \frac{\partial}{\partial t_s} \) term can be obtained from \( L_r \)-constraint, \( \frac{\partial}{\partial t_{p+1}} \) term – from \( L_0 \)-constraint, \( \ldots \) , and, generally, \( \frac{\partial}{\partial t_{kp+1}} \) term – from \( L_{(k-1)p} \)-constraint. Similarly we can get all other constraints and obtain finally that the constraint algebra fixing our partition function unambiguously is the system of \( W_{pk}^{(r)} \)-operators with \( r = 1, \ldots, p \) and \( k \geq 1 - r \). Let us note that the terms with \( \frac{\partial}{\partial t_{pk}} \) arise from the \( W_{pk}^{(1)} \) constraints, expressing the independence of the GKM partition function in Kontsevich phase of the \( t_{pk} \)-variables (this is a little more than just \( p \)-reduction – see (2.12)).

Now let us consider the antipolynomial case. This time \( \frac{\partial}{\partial t_{p+1}} \) term again arises from \( L_0 \)-constraint, but we do not need \( \frac{\partial}{\partial t_1} \) term, i.e. now our \( L_{pk} \)-constraints will be limited to \( k \geq 0 \).

\(^9\)We discuss here the implications of this shift. As to its origin and relation to topological models and quasi-classical hierarchies, we refer to the detailed discussion in ref.[19].
Analogously, the term $\frac{\partial}{\partial \epsilon_{p+2}}$ appears in $W^{(3)}$-constraint and etc. up to $W^{(|p|)}_{p_k}$-constraint which is limited this time by $k \geq 2 - |p|$. It means that this time we have the system of $W^{(r)}_{p_k}$-operators with $r = 1,...,|p|$ and $k \geq 2 - r$. It coincides with the statement of the table 1.

Thus, we obtained that the modes of $W$-constraint are positive or negative depending on where is the singularity of the potential. Certainly, it is rather natural and could be guessed from the very beginning.

Let us say some words to justify the procedure we have proposed. We already stated in section 2.3 that string equation operator $A$ (2.36) along with reduction condition imply the presence of constraint algebra which fixes unambiguously (as perturbative series) the partition function. We obtained in this section all constraints which are necessary for this purpose.

3 The BGWM versus the GKM

After the presentation of generic theory of the GKM, we are now equipped to discussion of the BGWM as a particular example antipolynomial GKM. However, to begin with we still need to give a little more comments on the reasons, why the BGWM can indeed be identified as a GKM. In making this identification we shall be naturally lead to the introduction of a concept of “universal” BGWM, which in a certain sense is unifying such models for all the unitary groups.

3.1 Ward identities for the BGWM and their GKM-like solution

Integrability of matrix models is usually a corollary of huge covariance of the (matrix) integral, which is used to define the full partition function. Since the action is of generic type, arbitrary change of integration variables results into some transformation of coupling constants, and invariance of the integral under such change implies restrictive constraints on the functional dependence of partition function on the coupling constants (external fields). Some of constraints are explicitly resolvable, but others form less trivial closed algebras. When these are isomorphic to (subalgebras of) some natural cartanian-type algebras (like Virasoro, $W$- or $\tilde{W}$-), solutions to the constraints are naturally $\tau$-functions of conventional (cartanian) integrable hierarchies (i.e. of (multicomponent) KP or Toda type). The set of constraints can be considered as invariant description of partition function, of which the original matrix model is nothing but particular integral representation. Other representations can differ by choices of integration contours and provide a kind of analytic continuation of the original function. See [1] for more details.

According to this description, one should begin analysis of integrability structure, if any, of a given matrix model from identification of the adequate changes of integration variables and derivation of the corresponding constraints (Ward identities). The next step should be the choice of the coupling constants which brings these Ward identities to some standard form. The last step - identification of integrable structure - is yet not always possible, because the theory of non-cartanian hierarchies (i.e. those, not associated with the level $k = 1$ simply-laced Kac-Moody algebras) is not worked out in any detail. This, however, will not be an obstacle in our first example of the BGWM, which appears to suit into the standard Toda-lattice pattern.

The BGWM partition function is defined by the integral (1.1):

$$Z_{\text{BGWM}}(J, J^\dagger) \equiv \frac{1}{V_N} \int_{N \times N} [dU] e^{\text{Tr}(J^\dagger U + JU^\dagger)}.$$  (3.1)

The “coupling constants” of the model are represented by the $N \times N$ matrix $J$ (external matrix field). Among the admissible changes of integration variable $U$ are left multiplications $U \rightarrow VU$ by any unitary $V$, which leave the Haar measure $[dU]$ invariant. The associated Ward identities
read just

\[ Z_{\text{BGWM}}(J, J^\dagger) = Z_{\text{BGWM}}(JV, V^\dagger J^\dagger), \] (3.2)

and together with their analogues, reflecting the right-multiplication invariance of \([dU]\), they imply that \(Z_N\) is in fact a symmetric function of only \(N\) variables: eigenvalues \(m_i, \ i = 1 \ldots N\) of the matrix \(M \equiv JJ^\dagger\): in particular,

\[ Z_{\text{BGWM}}(J, J^\dagger) = Z_N(M). \] (3.3)

Dependence on these remaining variables is defined by the more involved Ward identities, which can not be resolved in such a simple way. It is most convenient to write them in the form of the matrix-valued identity \([3, 4]\):

\[ \frac{\partial}{\partial J_{\text{tr}}^\dagger} \cdot \frac{\partial}{\partial J_{\text{tr}}} Z_{\text{BGWM}}(J, J^\dagger) = I \cdot Z_{\text{BGWM}}(J, J^\dagger). \] (3.4)

This relation holds just because the derivatives at the l.h.s. produce a product \(U \cdot U^\dagger = I\) under the integral. If \(Z_N(M)\) is now substituted instead of \(Z_{\text{BGWM}}(J, J^\dagger)\) into (3.4), we get (see Appendix):

\[ \frac{\partial}{\partial M_{\text{tr}}} M \frac{\partial}{\partial M_{\text{tr}}} Z_N(M) = I \cdot Z_N(M). \] (3.5)

This is the equation that can look somewhat familiar from the theory of the GKM. Kontsevich integral \((2.1)\),

\[ \mathcal{F}_V(N, L) \equiv \frac{1}{V_N} \int_{n \times n} dX e^{\text{tr}(LX - N\log X + V(X))} \] (3.6)

is defined with the “flat” Hermitian Haar measure \(dX = \prod_{a,b} dX_{ab}\). This measure is invariant under conjugation \(X \to VXV^\dagger\) with any unitary matrix \(V\), thus

\[ \mathcal{F}_V(L) = \mathcal{F}_V(VLV^\dagger), \] (3.7)

what implies that \(\mathcal{F}_V\) is a symmetric function of eigenvalues \(l_a, a = 1 \ldots n\) of \(L\) only.

Remaining less trivial Ward identities are associated with more general changes of integration variable. Of interest for us will be implication of particular transformation: \(X \to X + X\epsilon X\) with some infinitesimal (\(X\)-independent) matrix \(\epsilon\). Invariance of the integral for any \(\epsilon\) implies the following matrix-valued equation:

\[ \left[ \frac{\partial}{\partial L_{\text{tr}}} L \frac{\partial}{\partial L_{\text{tr}}} + (n - N) \frac{\partial}{\partial L_{\text{tr}}} \right] + \left( \frac{\partial}{\partial L_{\text{tr}}} \right)^2 \mathcal{F}_V(N, L) = 0. \] (3.8)

This equation becomes identical to (3.5), provided \(M = L\), thus \(n = N\), and

\[ N = n, \quad V(x) = \frac{1}{x}. \] (3.9)

Eqs.(3.5) and (3.8) define \(Z_N(M)\) and \(\mathcal{F}_V(L)\) unambiguously, thus equivalence of the equations implies the identity (1.2),

\[ Z_{\text{BGWM}}(J, J^\dagger) = Z_N(M)/Z_N(0), \quad M = JJ^\dagger, \]

\[ Z_N(M) = \mathcal{F}_1/X(N, L = M) = \frac{1}{V_N} \int_{N \times N} dX e^{\text{tr}(MX - N\log X + \frac{1}{x})}. \] (3.10)

\[ This\ conclusion\ would\ not\ be\ true\ for\ SU(N)\ integral.\ In\ that\ case\ \(V \in SU(N)\),\ thus\ \text{Det}V\ = 1\ and\ \(Z_N\)\ is\ a\ function\ of\ \(M = JJ^\dagger,\)\ and\ also\ of\ \text{Det}J,\ \text{Det}J^\dagger.\]

\[ 11\ Subscript\ "tr"\ here\ and\ below\ denotes\ transposed\ matrices,\ \text{I}\ stands\ for\ the\ unit\ matrix.\]
The last relation can be also rewritten in terms of $Y = 1/X$:

$$Z_N(M) = \frac{1}{V_N} \int_{N \times N} dY e^{\text{Tr}(M \frac{1}{Y} - N \log Y + Y)} \tag{3.11}$$

where we used the transformation law for the measure,

$$d\frac{1}{Y} = \frac{dY}{(\det Y)^{2N}} \tag{3.12}$$

and thus $^{12}$

$$\langle dX \rangle \equiv \frac{dX}{(\det X)^N} \bigg|_{X=1/Y} \frac{dY}{(\det Y)^N} = \langle dY \rangle = \langle d\frac{1}{X} \rangle. \tag{3.13}$$

This property in fact implies the symmetry of the integral (3.10) under the change of integration variable $X \leftrightarrow 1/X$. Together with left- and right- invariance of $\langle dX \rangle$,

$$\langle d(GX) \rangle = \langle d(XG) \rangle = \langle dX \rangle \quad \text{for any } G, \tag{3.14}$$

it can be used to restore the symmetry between $J$ and $J^\dagger$ (which is not really explicit at the r.h.s. of (3.10)):

$$Z_{\text{BGWM}}(J, J^\dagger) = \frac{1}{V_N} \int_{N \times N} dX e^{\text{Tr}(JX - N \log X + J^\dagger \frac{1}{X})} \tag{3.15}$$

Invariance under $X \leftrightarrow 1/X$ now implies the required identity

$$Z_{\text{BGWM}}(J, J^\dagger) = Z_{\text{BGWM}}(J^\dagger, J). \tag{3.16}$$

### 3.2 Virasoro constraints and integrability in the character phase of the BGWM

We are now in prepared to continue discussion of the BGWM, considering it as a particular example of the GKM with potential $V(x) = 1/x$.

#### 3.2.1 BGWM in the character phase

In this limit partition function,

$$Z_{\text{BGWM}}(J, J^\dagger) = \frac{Z_N(M = JJ^\dagger)}{Z_N(M = 0)}, \tag{3.17}$$

is expandable in a series in positive powers of $J, J^\dagger$-fields. It can be also considered as a generating functional of the symmetric unitary matrix integral with Haar measure:

$$Z_N^+(t^+) = 1 + \sum_{M \geq 1} \left( \sum_{1 \leq k_1 \leq \ldots \leq k_M} c_N{k_a} k_1^+ k_1 \ldots k_M^+ k_M \right) \tag{3.18}$$

where the coefficients $c_N{k_a}$ are defined as:

$^{12}$ To emphasize the relation of $\langle dX \rangle$ to Haar measure, note that for $X = \frac{1-iH}{1+iH}$, we get the standard expressions for Haar measure [20], $\frac{dX}{(\det X)^N} = \frac{dH}{(\det N(1+H)^2)}$. 

\[
\frac{1}{V_N} \int [dU] U_{ij} U^\dagger_{ji} = \delta_{ii} \delta_{jj} c_N(1);
\]
\[
\frac{1}{V_N} \int [dU] U_{ij} U^\dagger_{ji} U^\dagger_{ik} U_{kj} = \left( \delta_{ii} \delta_{jj} \delta_{kk} \delta_{ll} + \delta_{ik} \delta_{j,l} \delta_{i,j} \right) c_N(1,1) + \left( \delta_{ii} \delta_{j,l} \delta_{k,l} \delta_{i,j} + \delta_{ik} \delta_{j,l} \delta_{i,k} \delta_{l,t} \right) c_N(2);
\]
and so on. \hspace{1cm} (3.19)

The series (3.18) is naturally graduated by the number of \( U - U^\dagger \) pairs in the correlator, which is equal to \( K\{k_a\} = \sum k_a \). Coefficients \( c_N\{k_a\} \) are defined through the recurrence relations, which are nothing but implications of the Ward identities for the BGWM. In order to derive them in explicit form we need to substitute \( Z^+_N(t^+) \) into eq. (3.5):
\[
\left. \frac{\partial}{\partial M_{tr}} M \frac{\partial}{\partial M_{tr}} Z^+_N(t^+) = I \cdot Z^+_N(t^+) \right|_{t^+_k = \frac{1}{k} Tr M^k} \tag{3.20}
\]
and use (2.40) to rewrite it in terms of \( t \)-variables (see [1] for comments on this type of derivations). The result reads:
\[
\sum_{s \geq 0} M^s \left( \tilde{W}^{(+,2)}_{s+1}(t) - \delta_{s,0} \right) Z^+_N(t) = 0. \tag{3.21}
\]

Since \( W^{(+,2)}_s(t) = L^{(d)}_s(N,t) \) are just “discrete-Virasoro” operators,
\[
L^{(d)}_s(N,t) = N \frac{\partial}{\partial t_s} + \sum_{k>0} k t_k \frac{\partial}{\partial t_{k+s}} + \sum_{k=1}^{s-1} \frac{\partial^2}{\partial t_k \partial t_{s-k}}, \quad s \geq +1, \tag{3.22}
\]
we obtain a set of discrete-Virasoro constraints for \( Z^+_N(t^+) \):
\[
L^{(d)}_s(N,t) Z^+_N(t) = \delta_{s,1} Z^+_N(t), \quad s \geq +1. \tag{3.23}
\]

These provide the required set of recursive relations for \( c_N\{k_a\} \), which can be used to derive the somewhat non-trivial explicit examples. If
\[
c_N\{k_a\} \equiv \hat{c}_N\{k_a\} \prod_{l=0}^{K\{k_a\}-1} (N^2 - l^2)^{-1}, \quad K\{k_a\} = \sum k_a, \tag{3.24}
\]
then \( \hat{c}_N(k_1 \ldots k_M) \) are polynomials of degree \( M \) in \( N \), e.g. [21] (see also (2.59)-(2.60)):
\[
\hat{c}_N(1) = N; \quad \hat{c}_N(2) = -N, \quad \hat{c}_N(1,1) = N^2; \quad \hat{c}_N(3) = 4N, \quad \hat{c}_N(1,2) = -3N^2, \quad \hat{c}_N(1,1,1) = N(N^2 - 2); \tag{3.25}
\]
\[
\hat{c}_N(4) = -30N, \quad \hat{c}_N(1,3) = +8(2N^2 - 3), \quad \hat{c}_N(2,2) = +3(N^2 + 6), \quad \hat{c}_N(1,1,2) = -6N(N^2 - 4), \quad \hat{c}_N(1,1,1,1) = N^4 - 8N^2 + 6; \quad \ldots
\]

Alternatively, the same quantities can be directly derived from the GKM-representation of the BGWM.

The fact that poles occur in \( c_N\{k_a\} \) at all \( N < K\{k_a\} \) is referred to as the De Wit-t’Hooft anomaly [22]. These describe the singularities of \( Z^+_N(t) \) at generic values of \( t \)-variables. This
function is a kind of a universal object, describing all the BGWM models (for all unitary groups) at once. Reduction to particular group implies that time-variables are substituted by the $N$-dependent quantities $t^+_k = \frac{1}{k} \text{Tr} (J J^t)^k$, so that only $N$ of them remain independent. Restriction to this hypersurface in the infinite-dimensional space of time-variables spoils some nice properties of $Z^+_N(t)$, but instead on this hypersurface the De Witt-t’Hooft poles are canceled between different terms with the same $K \{ k_a \}$, in accordance with the finiteness of the unitary group integral (3.1).

### 3.2.2 Universal BGWM in character phase

In order to establish connection with the pertinent GKM it is necessary to find a representation of the BGWM partition function in terms of an integral, where the size of the matrix is a new parameter, independent of $N$. The answer to this question depends, of course, on the way how the dependence of $N$ is separated from that of all other variables. We choose eq.(3.18) as the \textit{definition} of the universal BGWM in the character phase (i.e. $t^+$’s and $N$ are considered to be independent variables). Invariant definition of this quantity is provided by the Virasoro constraints (3.23), and our goal now is to find a solution to this equation in the form of an $n \times n$ Hermitian matrix integral with \textit{any} $n$. This is a simple exercise if we return back to eq.(3.8) and make the substitution $V(x) = 1/x$, but do not identify $N$ with $n$ and $N$. Then we have:

$$
\left( \frac{\partial}{\partial \text{tr} L} L \frac{\partial}{\partial \text{tr} L} + (n - N) \frac{\partial}{\partial \text{tr} L} - I \right) F_{1/X}(N, L) = 0
$$

(note that according to eq.(2.1) $F$ is defined as an $n \times n$ integral). Let us now consider $F_{1/X}(N, L) = \hat{Z}(t^+|N)$ as a function of $t^+_k = \frac{1}{k} \text{tr} L^k$, $k \geq 1$. There is no explicit $n$-dependence in $\hat{Z}$! It appears, however, in eq.(3.8). Indeed, the first operator at the l.h.s., when acting on $\hat{Z}$, turns into $\sum_{s \geq 0} L^s \mathcal{L}^{(d)}_{s+1}(n, t^+)$, where $n$ appears explicitly. Moreover, the contribution from the second operator turns into $(n - N) \sum_{s \geq 0} L^s \frac{\partial}{\partial t^+} \mathcal{L}^{(d)}(n, t^+)$ and does not cancel this $n$-dependence, instead it changes $\mathcal{L}^{(d)}(n, t)$ into $\mathcal{L}^{(d)}(2n - N, t)$ and the resulting constraints are

$$
\mathcal{L}^{(d)}_s(2n - N, t) \hat{Z}^+(t^+|N) = \delta_{s,1} \hat{Z}^+(t^+|N). \quad s \geq +1.
$$

We see now that (3.23) can be reproduced, if $N = 2n - N$, i.e. $N = 2n - N$ and

$$
Z^+_N(t) \bigg|_{t_k = \frac{1}{k} \text{tr} L^k} = F_{1/X}(2n - N, L) = \frac{1}{V_n} \int_{n \times n} \text{d}X e^{\text{tr}(LX + (N-2n) \log X + \frac{1}{X})}.
$$

Of course, for $n = N$ we obtain the old formula. Eq.(3.28) can be also rewritten in terms of $Y = 1/X$, thus getting a GKM with non-standard coupling of the $L$-field:

$$
Z^+_N(t) \bigg|_{t_k = \frac{1}{k} \text{tr} L^k} = \frac{1}{V_n} \int_{n \times n} \text{d}Y e^{\text{tr}(LY^{-1} - N \log Y + Y)}.
$$

In this form it can be interpreted as a generating function for correlators in the Penner model [7]. This expression also reminds the theory, discussed in ref.[8], though there are some important differences in our interpretation of this integral.

Since $Z^+_N(t)$ satisfies $n$-independent Virasoro constraints (3.23), it does not actually depend on $n$.\footnote{This is despite explicit $n$-dependence of the action in (3.28)! One should first express everything in terms of $t$-variables and then observe the elimination of $n$-dependence. The issue of $n$-independence is somewhat obscure in ref.[8]. It is not very clear whether it can be preserved for generic potential $V(Y) \neq Y$.} This $n$-independence is crucial for interpretation of $Z^+_N(t)$ as a KP $\tau$-function.
3.2.3 $Z_N^+$ as a $\tau$-function

We proceed now to determinant representation of the integrals (3.28), (3.29). We derive explicit representation in terms of Bessel functions and demonstrate the $n$-independence explicitly.

Application of the usual GKM technique to the case of (3.28) gives:

$$F_V(N, L) \sim \int_{n \times n} dX e^{tr(LX + (N - 2n)\log X + \lambda N)} = \int \prod_i dx_i \frac{\Delta(x)}{\Delta(l)} e^{l_i x_i - N \log x_i + 1/x_i} =$$

$$= \frac{1}{\Delta(l)} \det_{ab} \frac{\partial^{a-1}}{\partial x_{b}^{a-1}} \int \prod_i dx_i e^{l_i x_i - N \log x_i + 1/x_i} = \frac{\det \Psi_a(l_b)}{\Delta(l)},$$

where the would be “basis vectors” are:

$$\Psi_a(l) = \int dx x^{a-1} e^{l x - N \log x + 1/x} = \int dy y^{-1-a} e^{l y + N \log y + 1/y} = \sum_{k=0}^{l} \frac{l^k}{k!} \int dy y^{-1-a-N-k} e^y =$$

$$= \sum_{k=0}^{l} \frac{2\pi i}{\Gamma(a - N + k + 1)} \frac{l^k}{k!} = 2\pi i (2\sqrt{l})^{N-a} I_{N-a}(2\sqrt{l}),$$

(3.31)

where normalization of $\Psi_a(l)$ can be left unspecified at the moment, $I_a(z)$ are modified Bessel functions and (3.31) is correct for all real values of $N$ (in particular, there is an evident symmetry $I_a = I_{-a}$).

To discuss the dependence of $n$ we make explicitly the steps which were explained in section 2.2.1 for the general case.

The basis vectors correspond to character phase, therefore, their asymptotics (at small $l_i$) are $\Psi_a(l) \sim 1 + O(l)$. This means that one should do the transformation $l \rightarrow \lambda^{-1}$ and consider linear combinations of $\phi_a$’s not changing the determinant (3.30) as it explained in section 2.2.1 to adjust these linear combinations such that new vectors $\tilde{\Psi}_a(l)$ will have the asymptotics $l^{a-1}$. Let us also normalize them to have the unit coefficient in the first term.

So far the problem of the normalization was left beyond our consideration. In fact the Ward identities discussed in the previous subsection knew nothing about normalization. Therefore, we are free to fix it arbitrary. In accordance with (1.1) we prefer to use the requirement that $Z = 1$ at all $l_i = 0$ (or all $t_k = 0$). This is achieved by adjusting the coefficients in front of the leading terms in the small-$l$ expansion of $\tilde{\Psi}_a(l)$ to unity.

With these requirements in mind one can derive the following formula for $\tilde{\Psi}_a$:

$$\tilde{\Psi}_a(l) = \sum_{k=a-1}^{l} \frac{\Gamma(2a - N)}{\Gamma(a - N + k + 1) \Gamma(k - a + 2)} \frac{l^k}{k!}.$$

(3.32)

In order to allow comparison with the $\tau$-function theory let us substitute $l \rightarrow 1/\lambda$. It remains to rearrange the indexes, $a \rightarrow n - a + 1$, and obtain:

$$F_V(N, L) \bigg|_{Tr L^k = \sum_i \lambda_i^{-k}} = \frac{\det \psi_a(\lambda_b)}{\Delta(\lambda)},$$

(3.33)

where

$$\psi_a(\lambda) = \tilde{\Psi}_a(\lambda) = \tilde{\Psi}_{n-a+1}(l = 1/\lambda) = \lambda^a \sum_{k=1}^{l} \frac{\Gamma(2n - N - 2a + 2)}{\Gamma((2n - N) - 2a + k + 1) (k - 1)!} \lambda^{-k} \equiv \lambda^a \sum_{k=1}^{l} p_{ak} \lambda^{-k}$$

(3.34)
and matrix $p_{ak}$ can be continued to the non-positive values of $k$ by definition $p_{ak} = 0$, whenever $k \leq 0$ (this definition easily follows from the manifest form of $p_{ak}$ (3.34) due to the poles of $\Gamma$-functions in the denominator).

Thus, we can see that to eliminate any $n$-dependence of the partition function (3.33) we can just choose $N$ to be $2n - N$, where $N$ is a free parameter. Indeed, this parameter $N$ is nothing but the size of matrix in the BGWM and, simultaneously, it appears to be the zero time (times 2), as can be understood from (2.9).

Now let us rewrite the sum (3.34) in terms of modified Bessel functions and in integral form like (3.31):

\[
\hat{\psi}_a(N, \lambda) = \left[ \frac{\Gamma(N - 2a + 2)}{2\pi i} \right] \lambda^{a-1} \int dy y^{2a-N-2} e^{1/(\lambda y) + y} = \frac{[\Gamma(N - 2a + 2)]}{2^{-a} \pi i} \left( \frac{\sqrt{\lambda}}{2} \right)^{N-1} I_{N/2-a}(\frac{2}{\sqrt{\lambda}}).
\] (3.35)

One can display manifestly in the expression (3.34) the dependence of zero time, in accordance with (2.9):

\[
Z_N^+(t) = \tau_{N/2}(t) \left| \frac{1}{k} \sum \lambda_i^{-k} \frac{\det_{ab} \lambda^{N/2} \hat{\psi}_{a-N/2}(0, \lambda)}{\Delta(\lambda)} \right| \equiv \frac{\det_{ab} \lambda^{N/2} \psi_{a-N/2}(\lambda)}{\Delta(\lambda)}.
\] (3.36)

Therefore, we obtained that in the character phase the partition function of the BGWM is the $\tau$-function of KP hierarchy, corresponding to the element of the Grassmannian given by the basis vectors (3.34) with $N/2$ being zero time. These basis vectors give rise to the reduction of the following type. Let us consider infinitesimal additive variation of variable $y$ in (3.35). It induces the recurrence relation for the integrals of the type of those in (3.35). It connects integrals with even and odd degrees of $y$ in the integrand. Applying the procedure successively three times we finally obtain the relation:

\[
\psi_a = \lambda \psi_{a-1} + \frac{1}{2(a-1)(a-2)} \psi_{a-1} - \frac{1}{(2a-3)(2a-4)(2a-5)} \psi_{a-2}.
\] (3.37)

This is a counterpart of the reduction condition (2.11) in the character phase which is, in fact, a sort of Toda chain-like reduction (see sect.2 of the paper [10]).

From the expressions (3.33)–(3.34) one can see that, indeed, this partition function is normalized to be unity when all times are equal to zero, or all $\lambda_i \to \infty$.

### 3.2.4 Character representation

Let us now demonstrate how it is possible to use these explicit expressions to obtain the coefficients $c_N\{k_a\}$. The expansion coefficients $C_N\{m_a\}$ of a generic GKM in terms of the unitary group characters [2] can be written down for the particular case of (3.28) and comparison of the formula

\[
Z_N^+(t) = \sum_{m_1 \geq m_2 \geq \ldots \geq m_n \geq 0} C_N\{m_a\} \chi_{m_1,m_2,\ldots}(t).
\] (3.38)

with (3.18) provides an expression for $c_N\{m\}$ in terms of $C_N\{m\}$.

The first step is to work out (3.38) in explicit form. Using the standard formulas from [2] (see subsection 5.4 of that paper), one obtains:
Thus, we obtain that the final answer for the expansion (3.39) does not depend on $n$. The only restriction to $n$ is that it should be larger than the number of non-zero $m_a$’s, i.e. there should exist such $p \leq n$ that the representation with these $m_a$’s can be embedded into the group $U(p)$.

For integer $N$ formula (3.39) can be transformed into expression

$$Z = \sum_{m_1 \geq m_2 \geq \ldots \geq m_r \geq 0}^{\infty} \det_{a,b} \frac{\det_{ab} p_{a,m_b} a - b + 1}{\Delta(\mu)} \frac{\det_{ab} \mu_{a}^{-m_b + b - 1}}{\Delta(\mu)} =$$

$$= \sum_{m_1 \geq m_2 \geq \ldots \geq m_r \geq 0}^{\infty} \det_{ab} \frac{\Gamma(N - 2a + 2)}{\Gamma(m_b + a - b + 1) \Gamma(N + m_b - a - b + 2)} \chi_{m_1, m_2, \ldots, m_r}(t),$$

where

$$\chi = \frac{\det_{ab} \lambda_a^{-m_b + b - 1}}{\Delta(\lambda)}.$$  (3.40)

This is the first Weyl formula for (primitive) characters, $\lambda_a$ are the inverse eigenvalues (in contrast to the notation of the paper [2]) of a given unitary matrix, and $\{m_a\}$ are the lengths of the rows of corresponding Young table (therefore, they should be ordered as in formula (3.39)).

Now, one can see immediately from (3.39) that, if some $m_{r+1} = 0$ (and, therefore, so all $m_q$, $q > r$), then $P\{m_i\}$ will be determinant of block matrix:

$$P\{m_a\} = \det \begin{vmatrix} \vdots & \vdots & \vdots \\ A & \ddots & 0 \\ \vdots & \vdots & \vdots \\ B & \ddots & \ddots \\ & \ddots & \vdots \\ & \ddots & C \end{vmatrix} = \det_{r \times r} A = \det_{ab} \frac{\det_{ab} p_{a,m_b} a - b + 1}{\Delta(\mu)}_{a,b \leq r}.  \quad (3.41)$$

Thus, we obtain that the final answer for the expansion (3.39) does not depend on $n$. The only restriction to $n$ is that it should be larger than the number of non-zero $m_a$’s, i.e. there should exist such $p \leq n$ that the representation with these $m_a$’s can be embedded into the group $U(p)$.

For integer $N$ formula (3.39) can be transformed into expression

$$Z = \sum_{m_1 \geq m_2 \geq \ldots \geq m_r \geq 0}^{\infty} \left[ \frac{1}{(m_1!m_2! \ldots m_r)!} \prod_{1 \leq a < b} \left( 1 - \frac{m_b}{m_a + b - a} \right)^2 \right] \times$$

$$\times \left[ \frac{1}{\prod_{1 \leq a < b \leq N} \left( 1 + \frac{m_a - m_b}{b - a} \right)} \right] \chi_{m_1, m_2, \ldots, m_r}(t) = \sum_{m_1 \geq m_2 \geq \ldots \geq m_r \geq 0}^{\infty} C_N\{m_a\} \chi_{m_1, m_2, \ldots, m_r}(t)$$  (3.42)

which coincides with that in the paper [21]. The only difference is that author of [21] considered $N$ equal to $n$, and $N$, therefore, was not just parameter, but the size of matrix. It explains why his answer has no De Witt-Hooft poles, while our universal function, which can be determined from the coefficients of expansion in the characters (see below) does have. Indeed, the term in the first brackets does not depend on $N$ at all, while the term in the second brackets is nothing but inverse of dimension $d_N(m_1, m_2, \ldots, m_r)$ of the representation of the group $U(N)$, which is given by the numbers $m_1, m_2, \ldots, m_r$. The simplest representations have dimensions:
to the hypersurface in the time space corresponding to finite $N$
where again only the first product depends on $N$
the coefficients $C$
convenience let us rewrite the coefficients $N$
This expression, and, therefore, the second factor in the formula (3.42) can be trivially continued
to non-integer $N$ replacing factorials by proper $\Gamma$-functions. Then, one can trivially see the presence of poles at all integer values of $N=0,1,\ldots,r-1$.

Thus, in the universal function, when considering $N$ as a free parameter with no the reduction to the hypersurface in the time space corresponding to finite $n=N$, there are De Wit-t’Hooft poles.

Now let us say some words on how one can effectively obtain from the expression (3.39) the coefficients $C_N\{k_a\}$ of the time-expansion (3.18) of the universal function $Z^+_N$. For further convenience let us rewrite the coefficients $C_N\{m_a\}$ in (3.42), using (3.43), in the form

$$C_N\{m_a\} = \prod_{a=1}^r \frac{\Gamma(N-a+1)}{\Gamma(N+m_a-a+1)(m_a+r-a)!} \times \prod_{a>b} (m_a-m_b+a-b),$$

where again only the first product depends on $N$. The first coefficients are equal to

$$C_N(1) = \frac{1}{N}; \quad C_N(2) = \frac{1}{2N(N+1)}, \quad C_N(1,1) = \frac{1}{2N(N-1)};$$

$$C_N(3) = \frac{1}{6N(N+1)(N+2)}, \quad C_N(2,1) = \frac{1}{3N(N+1)(N-1)}, \quad C_N(1,1,1) = \frac{1}{6N(N-1)(N-2)};$$

$$\begin{cases}
C_N(4) = \frac{1}{24N(N+1)(N+2)(N+3)}; & C_N(3,1) = \frac{1}{2N(N+1)(N+2)(N-1)}, \\
C_N(2,2) = \frac{1}{12N(N+1)(N-1)}; & C_N(2,1,1) = \frac{1}{8N(N+1)(N-1)(N-2)}, \\
C_N(1,1,1,1) = \frac{1}{24N(N-1)(N-2)(N-3)(N-4)}.
\end{cases} \quad (3.45)$$

The next step is to rewrite the characters in terms of times. To do this, let us introduce Schur polynomials $P_k(t)$ defined by the expansion:

$$\exp\left\{\sum_{k=0}^\infty t_k x^k\right\} \equiv \sum_{k=0}^\infty P_k(t)x^k, \quad k \leq 0,$$

the polynomials with negative indices being put zero by definition. Some first polynomials are

$$P_0(t) = 1,$$
$$P_1(t) = t_1,$$
$$P_2(t) = t_2 + \frac{t_2^2}{2},$$
$$P_3(t) = t_3 + t_1 t_2 + \frac{t^3}{6},$$
$$P_4(t) = t_4 + t_3 t_1 + \frac{t_2^2}{2} + \frac{t_2 t_2}{2} + \frac{t_1^4}{24},$$

$$\ldots$$
Now we can use the second Weyl formula for (primitive) characters

\[ \chi_{m_1,m_2,...,m_p}(t) = \det_{ab} P_{m_a - a + b}(t) \]  

(3.48)
to rewrite characters through time variables. Now we can calculate the coefficients \( c_N \{ k_a \} \) immediately using the formulas (3.38), (3.44), (3.46) and (3.48). Say, using some first characters

\[
\begin{align*}
\chi_1(t) &= t_1; \\
\chi_2(t) &= t_2 + \frac{t_1^2}{2}, \quad \chi_{11} = \frac{t_2^2}{2} - t_2; \\
\chi_3(t) &= t_3 + t_1 t_2 + \frac{t_1^3}{6}, \quad \chi_{21}(t) = \frac{t_1^3}{3} - t_3, \quad \chi_{111} = t_3 - t_1 t_2 + \frac{t_1^3}{6}; \\
\chi_4(t) &= t_4 + t_1 t_3 + \frac{t_2^2}{2} + \frac{t_1 t_2}{2} + \frac{t_1^3}{24}, \quad \chi_{31}(t) = \frac{t_1^3 t_2}{2} - t_4 - \frac{t_1^3}{2} + \frac{t_1^3}{8}; \\
\chi_{22}(t) &= \frac{t_2^3}{2} - t_1 t_3 + \frac{t_1^3}{12}, \quad \chi_{211}(t) = t_4 - \frac{t_2^3}{2} - \frac{t_1^3 t_2}{2} + \frac{t_1^3}{8}, \\
\chi_{1111}(t) &= t_1 t_3 - t_4 - \frac{t_1^3 t_2}{2} + \frac{t_1^3}{2}, \\
\ldots
\end{align*}
\]

one trivially reproduces the expressions (3.25).

### 3.3 Virasoro constraints and integrability in the Kontsevich phase of the BGWM

#### 3.3.1 The BGWM in the Kontsevich phase

We need now to consider eqs.(2.3-2.6) with \( L = M \) and \( n = N \) for the particular case of \( \mathcal{V}(X) = \frac{1}{k} \) and \( \mathcal{N} = N \). In such situation \( X_0 = M^{-1/2} \) and \( p = -2 \). The \( C \)-factor, appearing in the definition of partition function \( Z^{-\left(t^{-\left(-2\right)}\right)} \),

\[
Z_{BGWM}(J,J^\dagger) = \frac{Z_N(M = JJ^\dagger)}{Z_N(M = 0)},
\]

(3.50)

\[
Z_N(M) = \mathcal{F}_{1/X}(N,M) \equiv \mathcal{C}_{1/X}(N,M) Z^{-\left(t^{-\left(-2\right)}\right)}\bigg|_{t^{-\left(-2\right)} = -\frac{1}{2k-1}\text{Tr}M^{-k+1/2}}
\]

is given by (2.6):

\[
\mathcal{C}_{1/X} = \frac{e^{2\text{Tr}M^{1/2}}}{(\text{Det}M)^{-N/2}\text{Det}^{1/2} (M^{1/2} \otimes M + M \otimes M^{1/2})},
\]

(3.51)

so that

\[
Z^{-\left(t^{-\left(-2\right)}\right)}\bigg|_{t^{-\left(-2\right)} = -\frac{1}{2k-1}\text{Tr}M^{-k+1/2}} = \prod_{a,b}^N \sqrt{m_a^{1/2} + m_b^{1/2}} e^{-2 \sum a^{m_a^{1/2}}} Z_N(M).
\]

(3.52)

Ward identities for \( Z^{-} \) result from the substitution of \( Z_N(M) \) into (2.39) \(^{14}\). Though general procedure was already written in section 2.4.3, this is the only case which was worked out for antipolynomial model [6]. The result is the set of Virasoro constraints imposed on the partition function:

\[
\mathcal{L}_a Z^{-\left(t^{-\left(-2\right)}\right)} = 0,
\]

(3.53)

\(^{14}\)This substitution was already performed in ref.[6]. Since the relation between the GKM and the BGWM was not discussed in that paper, expression (3.52) for the \( C \)-factor was just guessed and postulated, without reference to generic prescription of the GKM.
where

\[
\mathcal{L}_0 = \frac{1}{2} \sum_{\text{odd } k \atop k < 0} kt_k \frac{\partial}{\partial t_k} + \frac{1}{16} + \frac{\partial}{\partial t_{-1}},
\]

\[
\mathcal{L}_{-2q} = \frac{1}{2} \sum_{\text{odd } k \atop k < 0} kt_k \frac{\partial}{\partial t_{-q}} + \frac{\partial}{\partial t_{-1-q}}.
\]  

(3.54)

Let us point out that, in accordance with the general rule (2.61), the shifted time is \( t_{-1} \), and the first constraint is \( \mathcal{L}_0 \). As it was already explained in section 2.3.2, this constraint is the string equation in \( P = -p = 2 \) case. Along with the statement that the partition function is a \( \tau \)-function of KdV hierarchy it is sufficient to fix the partition function, as well as to reproduce all the tower of Virasoro constraints (3.53).

### 3.3.2 The Universal BGWM in the Kontsevich phase

Now let us emphasize that the Virasoro algebra (3.54) does not contain \( N \) dependence at all. It is, certainly, the trivial consequence of the fact discussed in section 2.2.3, where we have demonstrated that the partition function of the BGWM is a \( (\tau) \)-function of times, which does not depend on \( N \). It means that we do need no additional parameter to define properly universal function in the Kontsevich phase. Indeed, in this case the function \( Z^-(t^{-(2)}) \bigg|_{t_k(2)} \) is just needed universal function which does not depend on the size of matrix. This is why we omitted the subscription \( N \) in the formula (3.52).

Still some additional parameter can be introduced into the partition function, namely, zero time, how it was explained in section 2.2.3. This parameter is completely analogous to the parameter \( N \) in the character phase, but has nothing to do with the BGWM, as the BGWM corresponds to zero value of this parameter (see (3.52)).

It stresses the difference between structure of the universal functions in the different phases. Generally, in the Kontsevich phase the partition function has more regular and universal behavior which reduces the number of essential parameters, unlike the character phase when each change in the potential requires a special treatment, therefore, having many parameters which govern the behavior of the universal function. It explains why the integrable treatment which gives, in a sense, a universal description, is hardly applicable to the character case.

### 3.3.3 \( Z^- \) as a \( \tau \)-function

As we already discussed in subsection 2.2.3 the partition function in general antipolynomial Kontsevich case is the \( \tau \)-function of KP hierarchy specified by the reduction condition (2.26) and string equation. It can be described as well by its explicit determinant representation (2.9) with the basis vectors determined in (2.24).

In the concrete case of BGWM, one should put \( p = -2 \) and get KdV \( \tau \)-function with basis vectors having the manifest expression (see (2.24)-(3.31)):

\[
\psi_{a-N}(\lambda) = 2\sqrt{\pi} e^{-2\lambda} \sqrt{\lambda} \lambda^{a-N-1} I_{a-N-1}(2\lambda) \sim \lambda^{a-N-1} \sum_{k=0}^{\infty} \left(\frac{-1}{4\lambda}\right)^k \frac{\Gamma(a-N+k-1/2)}{k! \Gamma(a-N-k-1/2)} + ... \]

(3.55)

\[15\]In the reference [6] the signs of indices of all Virasoro generators and times were chosed opposite, and the numeration of times is a bit different.
Parameter $N$ here is just zero time. The dots in the asymptotics expansion stands instead of exponentially small at large $\lambda$ terms.

This manifest expression for the basis vectors allow us, among other, to prove explicitely the independence of the $\tau$-function of even times, which was noted in [6]. Indeed, the reduction condition to KdV (2.26) already implies that the $\tau$-function can include any even-time dependence only as an exponential, linear in these times (see (2.12)). Then, we can use the trick proposed in [9]. That is, we can choose only two non-vanishing Miwa variables $\lambda_1 = \lambda$ and $\lambda_2 = -\lambda$. Then all the odd times to be zero, and independence of the full $\tau$-function of even times is equivalent to the fact that $\tau = 1$ in this particular case. It is trivial to check that the $\tau$-function is equal to

$$-2\pi \lambda \left[ \{\sqrt{I_N(\lambda)}\} \{\sqrt{-\lambda I_N-1(-\lambda)}\} + \{\sqrt{-\lambda I_N(-\lambda)}\} \{\sqrt{\lambda I_N-1(\lambda)}\} \right]$$

in the point under consideration. The most subtle point is what we should understand by the function $I_c(-z)$ which is ambiguously continuable to the negative values of argument. Indeed, how it is usually understood in GKM approach, we consider this object as continuation of formal power series. This means that we consider instead of $I_c(-z)$ Macdonalds function $\frac{1}{16} K_c(z)$, where the normalization is chosen to have the same asymptotics as in (3.55). The only difference of these two functions as formal series is in exponentially small terms which are just governed by Stock’s phenomenon and are not taken into account in any integrable treatment. Now we can use the identity between functions $I_c(z)$ and $K_c(z)$ [23]

$$I_{c+1}(z)K_c(z) + I_c(z)K_{c+1}(z) = \frac{1}{z}$$

(3.57)

to get finally that (3.56) is equal to unity. Thus, we have proved that the $\tau$-function of $p = -2$ model does not depend on even times at all. The proof for the general case can be done mostly like that proposed in the paper [9].

Now we would like to make some comments on the literature concerning this $\tau$-function. To begin with, let us note that any product of two KdV $\tau$-functions taken at sucessive values of zero time (say, 0 and 1), is an MKdV $\tau$-function (see, for example, [24, 25, 5, 26]. The MKdV $\tau$-function obtained as a product of two $p = -2$ GKM $\tau$-functions can be specified by constraint algebra (or, sufficiently, by its lowest constraint, i.e. by the string equation) (3.53)-(3.54). As the constraint algebra does not contains any $N$-dependence, we can write

$$\mathcal{L}_q \tau_s = 0, \quad q > 0,$$

$$\mathcal{L}_0 \tau_s = 0,$$

(3.58)

where $s = 0, 1$ labels $\tau$-functions with two sucesssive values of zero time.

In the papers [5, 26] it was demonstrated that the general partition function of unitary matrix model [27] (not in the external field, but with arbitrary potential) in the double scaling limit is MKdV $\tau$-function which is the product of two KdV $\tau$-functions satisfying the string equation

$$\mathcal{L}_q \tau_s = 0, \quad q > 0,$$

$$\mathcal{L}_0 \tau_s = \mu \tau_s,$$

(3.59)

16 These terms can effect, in particular, the possibility to present the determinant of basis vectors as a function times.

17 One of them can be chosen to be BGWM partition function, but the other one can not be, as there is no way to introduce zero time to unitary matrix integral. This point caused a problem in ref.[6], where an attempt to construct the BGWM unitary matrix integral for MKdV $\tau$-function was made.

18 Their operator $\mathcal{L}_0$ does not contain the constant 1/16, therefore, our definition of the parameter $\mu$ is a bit different.
all other constraints being just Virasoro algebra (3.54). $\mu$ here is just a free parameter which is not specified at all. Comparing this expression with (3.58), one can see that $p = -2$ GKM gives an explicit example of the double scaled partition function of the unitary matrix model with $\mu = 0$. To construct the general case, one just needs the KdV $\tau$-function which satisfied the deformed string equation (3.59).

### 3.3.4 String equation

In section 2.3 we derived the string equation in some non-manifest way. The different approach to this is to use the direct derivation described in the very beginning of subsection 2.3.1. This derivation is less useful and a bit tedious but it can also be done. We demonstrate this for the simplest case of $p = -2$.

Indeed, let us consider the derivative of the BGWM partition function (cf. with (2.27)):

$$\text{Tr} \left[ \Lambda \frac{\partial}{\partial \Lambda_{\text{tr}}} \right] \log Z^-(t^{-(2)}).$$

(3.60)

It can be written as

$$\sum_{p<0} \text{Tr} \left[ \Lambda \frac{\partial t_{2p+1}}{\partial \Lambda_{\text{tr}}} \right] \frac{\partial}{\partial t_{2p+1}} \log Z^-(t^{-(2)}) = - \sum_{p<0} \text{Tr} \left[ \frac{1}{\Lambda^{-2p-1}} \right] \frac{\partial}{\partial t_{2p+1}} \log Z^-(t^{-(2)}) =$$

$$= - \sum_{p<0} (2p + 1)t_{2p+1} \frac{\partial}{\partial t_{2p+1}} \log Z^-(t^{-(2)}).$$

(3.61)

On the other hand, the derivative in (3.60) can be manifestly calculated. To do this, let us represent the BGWM partition function in the determinant form (2.9), (3.55)

$$Z^-(t^{-(2)}) \mid_{t_k = -\frac{1}{2k+1}} = \prod_{k<0} \text{det} \frac{\psi_a(\lambda_b)}{\Delta(\lambda)}$$

(3.62)

with

$$\psi_a(\lambda) = e^{-2\lambda} \sqrt{\lambda} \int dx x^{-a} e^{\lambda^2 x^2 + 1/\lambda} \equiv e^{-2\lambda} \tilde{F}_a(\lambda).$$

(3.63)

Now we need two facts which are true in the general antipolynomial case, i.e. our proof can be immediately generalized.

Let us note that the action of the operator $\lambda^2 \frac{\partial}{\partial \lambda^2}$ transforms $\tilde{F}_a(\lambda)$ into $\tilde{F}_{a+1}(\lambda) + \sum_{1 \leq b \leq a} \tilde{F}_b(\lambda)$ (we used the reduction condition (2.26)), i.e. there are linear combinations $F_a(\lambda)$ of $\tilde{F}_a(\lambda)$ such that $F_{a+1}(\lambda) = \lambda^2 \frac{\partial}{\partial \lambda^2} F_a(\lambda) = \left( \lambda^2 \frac{\partial}{\partial \lambda^2} \right)^a F_1(\lambda)$. The second important property which is crucial for our derivation is the asymptotics behaviour of $F_a(\lambda)$ at large $\lambda$ (see (3.55)):

$$F_1(\lambda) = e^{2\lambda} \left( 1 - \frac{1}{16} \frac{1}{\lambda} + \ldots \right),$$

$$\frac{\left( \lambda^2 \frac{\partial}{\partial \lambda^2} \right)^a F_1(\lambda)}{F_1(\lambda)} = \lambda^a \left( 1 - \frac{N(N-1)}{4} + o(1/\lambda) \right) \text{ for any } a \geq 1.$$
\[
\text{Tr} \left[ \lambda \frac{\partial}{\partial \lambda_{tr}} \right] \log Z^{-\left(t^{-(-2)}\right)} = -2 \sum_a \lambda_a - \sum_a \lambda_a \frac{\partial}{\partial \lambda_a} \log \Delta(\lambda) + \sum_a \lambda_a \frac{\partial}{\partial \lambda_a} \log \det F_a(\lambda) = \\
= -2 \sum_a \lambda_a + \frac{N(N-1)}{2} + 2 \sum_a \lambda_a^2 \frac{\partial}{\partial \lambda_a^2} \log \det F_a(\lambda).
\]

To calculate the last derivative we use the trick from the paper [9], where it was demonstrated that (the derivative with respect to the first time in [9] is to be replaced by that with respect to \(t \rightarrow t-1\) here)

\[
-\frac{\partial}{\partial t^{-1}} \log Z^{-\left(t^{-(-2)}\right)} = \text{res}_\lambda \left[ \frac{\psi_1(\lambda)}{\prod_{j=1}^N \left(\lambda - \lambda_j\right)} \lambda^N \left( \left\{ 1 - \frac{N(N-1)}{4} \frac{1}{\lambda} + o(1/\lambda) \right\} - \frac{1}{\lambda} \sum_a \lambda_a^2 \frac{\partial}{\partial \lambda_a^2} \log \det F_a(\lambda)[1 + O(1/\lambda)] \right) \right],
\]

if the conditions (3.64) are satisfied along with the determinant representation (2.9) with the entries \((\lambda^2 \frac{\partial}{\partial \lambda^2})^a F_1(\lambda)\). Therefore, using (3.64) we obtain

\[
-\frac{\partial}{\partial t^{-1}} \log Z^{-\left(t^{-(-2)}\right)} = \sum_a \lambda_a + \frac{1}{16} - \frac{N(N-1)}{4} - \sum_a \lambda_a^2 \frac{\partial}{\partial \lambda_a^2} \log \det F_a(\lambda).
\]

Now, collecting together (3.61), (3.65) and (3.67) one can finally obtain the string equation:

\[
\left( \sum_{p<0} (2p+1) t_{2p+1} \frac{\partial}{\partial t_{2p+1}} + 2 \frac{\partial}{\partial t^{-1}} + \frac{1}{8} \right) Z^{-\left(t^{-(-2)}\right)} = 0.
\]

This is evidently \(L_0\)-constraint which coincides with the formula (3.54).

It is interesting to note that in the course of the derivation there arose the manifest dependence of \(n\), but it disappeared from the final result. This reflects the rather complicated \(n\)-structure of antipolynomial Kontsevich phase.

### 4. Itzykson-Zuber integral as a 2-matrix model

#### 4.1. From the BGWM to IZ integral

Since partition function \(Z_N(J, J^\dagger) = \hat{Z}_N(M)\) is a generating functional for any correlators of \(U\)-matrices with Haar measure, it can be used in particular to represent the IZ integral in terms of Hermitian matrix model:

\[
IZ(\Phi, \bar{\Phi} | J, J^\dagger) \equiv \frac{1}{V_N} \int_{J_{N \times N}} [dU] e^{Tr(J^\dagger U + J U^\dagger)} e^{Tr \Phi U \Phi^\dagger} = Z_{BGWM}(J, \Phi, \frac{\partial}{\partial J_{tr}} \Phi + J^\dagger) = Z_N(J, J^\dagger + J \bar{\Phi} \frac{\partial}{\partial J_{tr}} \Phi) = \frac{1}{V_N} \int_{J_{N \times N}} \frac{e^{Tr \Phi \Phi^\dagger} dX}{(\text{Det} X)^N} e^{Tr \left( J \bar{\Phi} \frac{\partial}{\partial J_{tr}} + J^\dagger \right) X}.
\]

"Normal ordering" sign at the r.h.s. implies that all the derivatives \(\partial/\partial J\) should be placed to the left of all \(J\)’s. Pulling back all these derivatives to the right provides the following relation:

\[
Z = \frac{1}{V_N} \int_{J_{N \times N}} \frac{\frac{e^{Tr \Phi \Phi^\dagger} dX}{(\text{Det} X)^N}}{\text{det}^{k} (\Phi X)^k} \left( 1 + \sum_{l \geq 0} \frac{\text{Tr} J^l \Phi^l J^\dagger X (\Phi X)^l}{l!} + O(J^2, (J^\dagger)^2) \right)
\]
If we now put $J = J^\dagger = 0$ the answer arises for the IZ integral \textit{per se}: \footnote{Another version of this formula (for the derivation, see Appendix):}

$$Z_{IZ}(\Phi, \bar{\Phi}) \equiv \frac{1}{V_N} \int_{N \times N} [dU] e^{Tr \Phi U \Phi U^\dagger} = \int_{N \times N} \frac{e^{Tr \Phi}}{(\text{Det}X)^N} e^{\sum_{k>0} \frac{N}{k} \text{Tr}^k \Phi H^k}$$ \hspace{1cm} (4.3)

We can now make a change of matrix variables

$$X \to \bar{H} \equiv \frac{1}{\Phi X}.$$ \hspace{1cm} (4.4)

Invariance of the measure implies that

$$\frac{e^{Tr \frac{1}{X} dX}}{(\text{Det}X)^N} \to \frac{e^{Tr \bar{H} \Phi}}{(\text{Det} \bar{H})^N}.$$ \hspace{1cm} (4.5)

It remains to denote

$$T_k^+ = \frac{1}{k} \text{Tr} \Phi^k, \hspace{1cm} k > 0,$$

$$kT_k^+|_{k=0} = \text{Tr} I = N$$ \hspace{1cm} (4.6)

(compare with our usual definition of the “+”-time variables) in order to get:

$$Z_{IZ}(\Phi, \bar{\Phi}) = \frac{1}{V_N} \int_{N \times N} d\bar{H} e^{\sum_{k \geq 0} \frac{N}{k} \text{Tr} \bar{H}^k} e^{\text{Tr} \bar{H} \Phi / \bar{H}}.$$ \hspace{1cm} (4.7)

This represents IZ integral in the form of the GKM with generic potential $V(x) = \sum_k \frac{N}{k} x^{-k}$. The first representation is identical to expression from \cite{8} for the (non-full) partition function of the $c = 1$ model.

One can also work out similar representations at large, rather than small eigenvalues of $\bar{\Phi}$. It is enough to note, that

$$e^{\sum_{k \geq 0} \frac{N}{k} \text{Tr} \bar{H}^k} = \frac{1}{\text{Det}(\bar{\Phi} \otimes I - I \otimes \bar{H})} = e^{\sum_{k \geq 0} \frac{N}{k} \text{Tr} \bar{H}^k},$$ \hspace{1cm} (4.8)

where

$$\bar{T}_k^- = \frac{1}{k} \text{Tr} \Phi^{-k}, \hspace{1cm} k > 0,$$

$$T_k^-|_{k=0} = -N \text{Tr} \log \Phi$$ \hspace{1cm} (4.9)

Thus (4.7) can be also represented as

$$Z_{IZ}(\Phi, \bar{\Phi}) = \frac{1}{V_N} \int_{N \times N} d\bar{H} e^{\sum_{k \geq 0} \frac{N}{k} \text{Tr} \bar{H}^k} e^{\text{Tr} \bar{H} \Phi / \bar{H}}.$$ \hspace{1cm} (4.10)

These expressions are, however, asymmetric in $\Phi$ and $\bar{\Phi}$. Symmetry is restored in another representation of IZ integral - in terms of conventional 2-matrix model.
4.2 Relation to 2-matrix model: direct proof

Now we are going to state the connection (1.8) between IZ integral and two-matrix model by direct calculation. We need only the formula for the Cauchy determinant

\[
\det \frac{1}{x_i - y_j} = \frac{\Delta(x) \Delta(y)}{\prod_{i,j} (x_i - y_j)}.
\]

and the integration is chosen such that

\[
\int \int f(h) \bar{f}(\bar{h}) dh d\bar{h} = \int f(h) dh \int \bar{f}(\bar{h}) d\bar{h}.
\]

Now one can write the chain of identities:

\[
\int \int \sum_{k \geq 0} (T_k^+ H^+ \bar{T}_k^+ \bar{H}^+) \frac{\Delta(h) \Delta(\bar{h})}{\prod_{i,j} (h_i - \phi_j) \prod_{i,j} (\bar{h}_i - \bar{\phi}_j)} = \frac{1}{N!} \int \int \prod_i dh_i d\bar{h}_i e^{h_i \bar{h}_i} \frac{\Delta(h) \Delta(\bar{h})}{\prod_{i,j} (h_i - \phi_j) \prod_{i,j} (\bar{h}_i - \bar{\phi}_j)}
\]

and we used at the last stage the convention (4.12) which, after manifest writing the determinant as sum over permutations, leads to \(N!\) equal terms of each type (of each possible permutation) and transforms to \(N!\) equal determinants.

Now let us say some words on the integration contour in these expressions. In fact, it is different in the cases of potentials, which are polynomial and antipolynomial in \(H\). Indeed, in the first case we use the expansion of logarithm in the domain where \(h_i/\phi_j < 1\) for any \(i\) and \(j\) (see the first equality in the chain (4.13)). It means that in the course of integration (last equality in (4.13)) all \(\phi_i's\) lie out of the integration contour, or, equivalently, the integration goes around \(\infty\). Unlike this case, antipolynomial potential implies \(\phi_i/h_j < 1\) in the first equality in (4.13), i.e. integration contour is closed around zero, all \(\phi_i's\) being inside it. Under these rules, the integration is to be determined as usual one complex variable integration over closed contour:

\[
\int e^{h \bar{h}} dh = e^{\phi \bar{h}}
\]

and so for the integration over \(\bar{h}\). It completes the proof.

In the next subsection we discuss some properties of the IZ integral which are consequences of this its representation in terms of two-matrix model.

4.3 IZ integral as a \(\tau\)-function

The real point is that we know many different facts about two-matrix model, and, among others, that its partition function is a \(\tau\)-function of lattice Toda hierarchy. Therefore, we can assert that the IZ integral is a \(\tau\)-function in Miwa parameterization (1.9). It means that we can fix the point of the Grassmannian which corresponds to this \(\tau\)-function and obtain some useful representation just using the theory of integrable hierarchies.
The point of the Grassmannian can be described in two different ways. The first one is to express it through exponential of a bilinear of fermions. It was done in [2]. Another possibility is less manifest, but is directly connected with the formalism of two-matrix model. Namely, using the formalism of orthogonal polynomials, one can rewrite (1.8) in the determinant form [28]:

\[ \Delta(h) \Delta(\bar{h}) = \det_{N \times N} H_{ij}(T, \bar{T}) \]  

with

\[ H_{ij}(T, \bar{T}) = \int dhd\bar{h}e^{-h_i\bar{h}_j + \sum_{k \geq 0} (T_k^+ h_i^\pm k + T_k^- \bar{h}_j^\mp k)} \]

and properties

\[ \frac{\partial}{\partial T_k} H_{ij} = H_{i+k,j} = \left( \frac{\partial}{\partial T_k} \right)^k H_{ij}; \]

\[ \frac{\partial}{\partial \bar{T}_k} H_{ij} = H_{i,j+k} = \left( \frac{\partial}{\partial \bar{T}_k} \right)^k H_{ij}. \]  

In fact, the determinant representation (4.15) along with the properties (4.17) give rise to a generic \( \tau \)-function of Toda lattice hierarchy, and the formula (4.16) specifies it.

There are different possible applications of the integrable structure of the IZ integral. Say, having the formula (4.15) one can trivially find the fermionic representation for the element of the Grassmannian [10] and, then, by the Miwa transformation of times restore the IZ integral (instead of the manifest calculation of the previous subsection). Certainly, it can be also done reversely. But now, as an example, we are interested in only one possible implication of the integrable properties. Namely, we can find out the expansion of the IZ integral at small times like it was done in subsection 3.2.4 in the BGWM. Moreover, it can be done equally for both the polynomial and antipolynomial potentials in (1.8). This means that one can equally construct large- and small- \((\phi, \bar{\phi})\) expansion.

For the definiteness, let us consider the antipolynomial case. Then, the integration rules of the previous subsection implies that

\[ \int x^\alpha y^\beta e^{xy} dx dy = \frac{1}{\Gamma(-\alpha)} \]  

Then, using (4.15), one can obtain the determinant representation in terms of (Schur polynomials of) times

\[ \det_{i,j} e^{\phi_i \bar{\phi}_j} = \det_{m,n} \frac{1}{\Gamma(N + i - n + 1)} P_i(T) P_m(T) \]  

where the sum over \( i \) can be extended to all integer, the real range of summation being determined by poles of the \( \Gamma \)-function and by vanishing Schur polynomials with negative subscriptions. For example, the first term of expansion of antipolynomial case looks is

\[ \text{Certainly, one can trivially write down an analogous expansion also in the polynomial case.} \]

\[ \text{Note that in generic case (i.e. when formula (4.16)) is replaced by a general solution } \]

\[ \tau_N(T, \bar{T}) = \det_{i,j} \sum_{l,m} P_{l+1}(T) P_{m-i-n}(\bar{T}), \]  

which describes the adjoint action of the element of the Grassmannian on fermionic modes.
\[ \det_{\mu \alpha} \sum_{i}^{1} \frac{1}{\Gamma(N + i - n + 1)} P_i(T) P_{\mu+i-n}(\bar{T}) = \frac{1}{\prod_{k=0}^{N-1}(N - k)!} (1 + \ldots). \quad (4.20) \]

Generally speaking, the calculation of the expansion is more complicated in comparison with the BGWM. The underlying reason is that now the number of Miwa variables \( N \) can not be made independent of the size of matrices in IZ integral.

## 5 Conclusion

This paper defines a framework in which both the phase- and integrable structures of matrix models can be discussed simultaneously. The crucial role is played by the concept of universal partition functions (UPF). Each UPF describes a set of matrix integrals, differing by the size \( n \) of the matrices which are integrated over and serve as external fields. UPF is defined to be an \( n \)-independent functional, depending on the form of the action (i.e. on the choice of the matrix model), but not on the matrix size.

This definition of UPF depends on the choice of the variables ("coupling constants"), which are considered to be properly reducible under the certain embedding of the smaller \( n_1 \times n_1 \) matrices into the bigger \( n_2 \times n_2 \) ones. For the simplest embedding, when the smaller matrix is supposed to stand at the left upper corner of the bigger one (which is implicitly accepted in this paper), the example of the properly reducible variables is provided by traces of (powers of) matrices.\(^{22}\) For eigenvalue matrix models this is also a complete set of variables. This means that whenever partition function of a matrix model is expressed in terms of the infinitely many "time-variables" \( t_k^{\pm(p)} = \frac{1}{k} \text{tr}_{n \times n} M_k^{\pm k/p} \), which are treated as independent variables (despite only \( n \) of these are actually independent), and there is no explicit \( n \)-dependence in the shape of this function \( Z(t) \), it is properly reducible and can be considered as UPF. For numerous interesting matrix models such UPF can be further identified with the \( \tau \)-functions of integrable hierarchies (if matrix model is of eigenvalue type, these are usually multicomponent Toda, i.e. "Cartanian" - hierarchies).

The best way to specify UPF is through the system of equations, of which the Virasoro- and \( \hat{W} \)-constraints are the simplest examples. Original matrix models are then particular solutions of these equations, represented in the form of \( n \times n \) matrix integrals, which describe the restriction of the UPF on peculiar \( n \)-dimensional hypersurface in the \( t \)-space.

Construction of the set of the \( n \times n \) integrals, possessing the same UPF is a matter of art. Moreover, given \( N \times N \) integral can be associated with different UPF - depending on the particular choice of the time-variables (even if the abovementioned embedding of matrices is fixed). We illustrated this phenomenon in the main body of the paper by consideration of the same Kontsevich integrals in two different asymptotics (character and Kontsevich "phases"), where the adequate time variables are either \( t_k^+ \) or \( t_k^{-(p)} \). The corresponding universal partition functions are not the same! (For example, \( Z_N^+(t^+) \) from section 3.2 depends on \( N \), while \( Z^-(t^{(2)}) \) from section 3.3 does not.) All these notions and features are most transparent in the study of Generalized Kontsevich model, which is the basic one in the entire field of the eigenvalue models and was in the center of discussion in this paper.

Above scheme can help to reveal the relation of various physically relevant \( U(N) \) lattice Yang-Mills models to integrability theory. Partition functions of these theories are often strongly dependent on the parameter \( N \), and this dependence is physically relevant and, if present, does not need to be eliminated. The corresponding UPF can thus be also \( N \)-dependent (like \( Z_N^+(t^+) \), though sometimes this \( N \)-dependence occasionally disappears, like for \( Z^-(t^{(2)}) \)). It is a partic-

\(^{22}\)Indeed, if \( n_1 \times n_1 \) matrix \( L_1 \) is identified in this way with the \( n_2 \times n_2 \) matrix \( L_2 \) (i.e. \( L_2 \) has all the entries vanishing, except for those in the left upper \( n_1 \times n_1 \) block), we have: \( \text{tr}_{n_2 \times n_2} L_2^k = \text{tr}_{n_1 \times n_1} L_1^k \) for any \( L_1 \) and \( k \).
ular integral representation (of the restriction of this UPF on the $n$-dimensional hypersurface), arising when $n = N$, that reproduces the original unitary matrix model per se.

Usually in the unitary matrix lattice models of YM theories it is the weak-coupling (perturbative) phase is complicated. Our observation in this paper is that the strong-coupling limit in (at least some of) these models can be identified as a weak-field limit of certain versions of the GKM. Both these limits (strong coupled unitary matrices and weak field in the GKM) are reasonably simple. The outcome is that the interesting weak coupling limit in lattice YM-related models can be now identified with the strong-field limit of the GKM, which is rather well understood.

We illustrated all these ideas with the simplest examples of the BGWM and IZ models, though the applications can appear to be much broader. Specifics of these models is that they are single-link theories and thus arise in the study of Yang-Mills theory only under restrictive assumptions (trivial topology in $d = 2$ or the mean-field approximation in higher dimensions). Another specifics - at least of the BGWM and even among the single-link models - is that it is essentially of the eigenvalue type. Going beyond the BGWM towards generic lattice Yang-Mills theories and $c > 1$ models requires broad generalization of Cartanian integrability.\(^{23}\)

The last thing to be mentioned is that the language of universal partition functions is the most adequate for consideration of “phase transitions”. Phase transition is nothing but a singularity of the UPF at some point (or finite-dimensional domain) in the infinite-dimensional space of parameters (= coupling-constants = time-variables). In a given physical system almost all the parameters (the form of Lagrangian) are fixed. Then if the line, associated with non-fixed parameter passes through the singularity, the free energy is actually singular at some point: the phase transition occurs. Singularity can be, however, easily avoided if some other parameters are allowed to change.\(^{24}\) In this sense there are actually no different phases in the framework of UPF: there are rather just different asymptotics of a single analytical function. These become really separated phases only on the low-dimensional hypersurfaces, i.e. when most of external fields are switched off in a given physical system.

These trivial remarks are important to keep in mind in application to particular models, which we discussed in the main text. The BGWM is known to have a phase transition in the $N = \infty$ limit. Actually this result was obtained in [3], when the space of parameters was severely restricted ($J = \sigma I$) and the phase transition can look differently (or disappear) on other lines in the $t$-space. Moreover, unitary matrix integrals are, of course, usually finite for $N < \infty$. This, however, does not imply that, say, $Z_N^+(t^+) \mathrm{has\ no\ singularities\ at\ finite\ values\ of\ }N\mathrm{;\ the\ statement\ is\ only\ that\ the\ }n = N\ \text{hypersurface }t_k = \frac{1}{t} \mathrm{Tr}((J J^T)^k)\ \text{in the }t\text{-space avoids all these singularities (if any). Moreover, }Z_N^+(t^+)\ \text{certainly has singularities as a function of }N,\ \text{which are again avoided by above hypersurfaces. For actual investigation of the phase transitions in the BGWM one can examine the large-}N\ \text{limit of }Z_N^+(t^+)\ \text{by the method of ref.}[10] (\text{see section 4 of that paper or section 5.3 of [1]}).\ \text{Another thing to be taken into account is the switch between different UPF: from }Z_N^+(t^+)\ \text{in the strong-coupling “phase” to }Z^-((t^2)\ \text{in the weak-coupling one. More detailed discussion of phase structures and singularities of }\tau\text{-functions is beyond the scope of the present paper.}}$

\(^{23}\) Cartanian $\tau$-functions (i.e. those of multicomponent Toda systems, including their KP, KdV etc reductions) are defined as correlators of the free $2d$ fermions and thus can be interpreted as determinants of $\not \partial$ operators and described in terms of Grassmannian (Sato-Segal-Wilson theory). The necessary generalizations, first, substitute free fermions (fermionization of $U(q)$, Kac-Moody algebra) by generic $2d$ free-field theory (Wess-Zumino-Witten model), thus going from Segal-Wilson construction to generic description of determinant bundles over the universal module space. This should allow to go from “Cartanian” and thus essentially Abelian hierarchies to generic non-Abelian one (i.e. from commuting Hamiltonian flows to those which from closed non-trivial algebra). Second, to go further beyond single-link theories (e.g. to those involving plaquettes), one should probably abandon 2 dimensions and consider correlators in the free-field theories in $d = 3$. See [1] for more discussion of these issues.

\(^{24}\) The most familiar example is of course elimination of the second-order phase transition by magnetic field. It is only at the line $H = 0$ that the free energy is really singular.
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**Appendix**

This appendix contains proofs of several important formulas, which were omitted from the main body of the paper, but can still be useful for illustrative purposes.

In the first part of the Appendix we consider some subtle points connected with the transition from the external matrix variables $J^\dagger$, $J$ to the radial matrix variable $M \equiv JJ^\dagger$. This is essential for the derivation of the equation (3.5).

Let us consider the following second derivatives

\[
\frac{\partial}{\partial J^\dagger_{ji}} \frac{\partial}{\partial J_{kj}} = \frac{\partial}{\partial J^\dagger_{ji}} \frac{\partial}{\partial J_{kj}} = \frac{\partial}{\partial J^\dagger_{ji}} \frac{\partial}{\partial M_{qs}} \frac{\partial}{\partial M_{ks}} = \frac{\partial}{\partial J^\dagger_{ji}} \frac{\partial}{\partial M_{qs}} \frac{\partial}{\partial M_{ks}} = \delta_{is} \frac{\partial}{\partial M_{ks}} + J^\dagger_{js} \frac{\partial}{\partial J^\dagger_{ji}} \frac{\partial}{\partial M_{ks}} =
\]

\[
= \frac{\partial}{\partial M_{ik}} + J^\dagger_{js} \frac{\partial M_{mn}}{\partial J^\dagger_{ji}} \frac{\partial}{\partial M_{mn}} \frac{\partial}{\partial M_{sk}} = \frac{\partial}{\partial M_{ik}} + J^\dagger_{js} J^\dagger_{mij} \frac{\partial}{\partial M_{im}} \frac{\partial}{\partial M_{sk}} =
\]

\[
(A.1)
\]

for the derivative (A.1) and we used the definition of matrix derivative which is the derivative with respect to matrix elements of transposed matrix. Absolutely analogously one can do change of the variables in the derivative (A.1)

\[
\frac{\partial}{\partial J_{ji}} \frac{\partial}{\partial J^\dagger_{kj}} = \frac{\partial}{\partial J_{ji}} \frac{\partial}{\partial J^\dagger_{kj}} = \frac{\partial}{\partial J_{ji}} \frac{\partial}{\partial M_{qs}} \frac{\partial}{\partial M_{qj}} = \frac{\partial}{\partial J_{ji}} \frac{\partial}{\partial M_{qs}} \frac{\partial}{\partial M_{qj}} = \delta_{ik} \frac{\partial}{\partial M_{qj}} + J_{qk} \frac{\partial}{\partial J_{ji}} \frac{\partial}{\partial M_{qj}} =
\]

\[
= \delta_{ik} \frac{\partial}{\partial M_{jj}} + J_{qk} \frac{\partial M_{mn}}{\partial J_{ji}} \frac{\partial}{\partial M_{mn}} \frac{\partial}{\partial M_{qj}} = \delta_{ik} \frac{\partial}{\partial M_{jj}} + J_{qk} J^\dagger_{in} \frac{\partial}{\partial M_{in}} \frac{\partial}{\partial M_{qj}}.
\]

Unlike the previous case this result cannot be expressed in terms of radial matrix $M$. Nevertheless, it was just this expression which was used to derive Ward identity in the paper [6]. This is resolved by noting that the authors of [6] considered only this expression after changing variables to the eigenvalues $\lambda$ of matrix $M$. In this case, (A.3) can be really rewritten in terms of $\lambda$ and coincides with the formula (A.2) expressed in terms of eigenvalues:
\[
\left[ \frac{\partial}{\partial J_{ij}}, \frac{\partial}{\partial J_{kj}} \right] \rightarrow \sum_b \frac{\partial}{\partial \lambda_b} + \lambda_a \frac{\partial^2}{\partial \lambda_a^2} + \lambda_a \sum_{b \neq a} \frac{\partial \lambda_a}{\partial \lambda_b} - \frac{\partial}{\partial \lambda_b}.
\] (A.4)

This expresses the fact of completeness of these Ward identities which unambiguously define unitary matrix integral (1.1) (this integral satisfies Ward identities with both possible derivatives (A.1) and (A.1)).

In the second part of the Appendix we are going to discuss the leading term at the r.h.s. of (4.2). It can be evaluated as follows. Let

\[ K \equiv \text{tr} \frac{\partial}{\partial J_{tr}} \Phi X J. \] (A.5)

The quantity to evaluate is:

\[ \sum_{n=0}^{\infty} \frac{1}{n!} : K^n : \] (A.6)

Then

\[ k_n \equiv \frac{1}{n!} : K^n := \delta_{n,0} + \frac{1}{n!} \sum_{m=0}^{n-1} \frac{K^m}{m!} \text{tr} \Phi^m \text{tr}(\Phi X)^{n-m} = \delta_{n,0} + \frac{1}{n!} \sum_{m=0}^{n-1} k_m s_{n-m}. \] (A.7)

Introduce

\[ k(t) \equiv \sum_{n=0}^{\infty} t^n k_n; \]
\[ s(t) \equiv \sum_{n=1}^{\infty} s_n t^{n-1} = \sum_{n=1}^{\infty} t^{n-1} \text{tr} \Phi^n \text{tr}(\Phi X)^n = \text{tr} \otimes \text{tr} \frac{\Phi X \otimes \Phi}{I \otimes I - t\Phi X \otimes \Phi}. \] (A.8)

Then

\[ k(t) = 1 + \int_0^t k(t)s(t)dt, \] (A.9)

or

\[ \frac{dk(t)}{dt} = k(t)s(t), \]
\[ k(t) = \exp \int_0^t s(t)dt = \exp \int_0^t \sum_{m=1}^{\infty} \frac{t^m}{m!} \text{tr} \Phi^m \text{tr}(\Phi X)^m = \text{Det}^{-1}(I \otimes I - t\Phi X \otimes \Phi). \] (A.10)

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