UNIVERSALITY OF EINSTEIN’S GENERAL RELATIVITY

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Abstract

Among relativistic theories of gravitation the closest ones to general relativity are the scalar-tensor ones and these with Lagrangians being any function \( f(R) \) of the curvature scalar. A complete chart of relationships between these theories and general relativity can be delineated. These theories are mathematically (locally) equivalent to general relativity plus a minimally coupled self-interacting scalar field. Physically they describe a massless spin-2 field (graviton) and a spin-0 component of gravity. It is shown that these theories are either physically equivalent to general relativity plus the scalar or flat space is classically unstable (or at least suspected of being unstable). In this sense general relativity is universal: it is an isolated point in the space of gravity theories since small deviations from it either carry the same physical content as it or give rise to physically untenable theories.

1 Introduction

General relativity is just a point in the "space" of all existing and conceivable theories of gravitational interactions. Nevertheless all the theories other than Einstein’s one, named "alternative theories of gravity", have rather bad reputation among most relativists. General relativity is enough complicated in itself and well confirmed by all known empirical data so that there is no point in considering more intricate theories whose empirical basis is, as a rule, either smaller than that of Einstein’s theory or presently non-existing at all. In fact, the alternative theories of gravity are some generalizations of general relativity, which invariably serves as a reference point for constructing them. These mod-
ifications go in all possible directions [1] making their theoretical investigations and attempts to confront them with experiment so difficult.

On the other hand in the last years there has been considerable revival of interest in some alternative theories. If one seeks for a deeper relationship between gravitational physics and other interactions, particularly in the realm of elementary particles, then one finds signals that some modifications of Einstein’s theory are inevitable or at least desirable. In the low-energy field theory limit of superstring effective action one recovers Einstein-Hilbert Lagrangian plus higher order corrections in the curvature [2]. Superstring theory gives also rise to a scalar field, the dilaton, which in this limit is non-minimally coupled to the string metric [3] and is viewed as a spin-0 partner of metric gravity. Nonrenormalizability of Einstein’s theory and renormalizability of a quadratic Lagrangian suggests that a classical limit (to be defined when quantum theory of gravity will become a fact rather than a fancy) of quantum gravity may be a theory with dynamics more complicated than that of general relativity. Quadratic and higher order in Riemann curvature Lagrangians are possible candidates for a theory avoiding spacetime singularities [4]. Scalar fields are copious in modern particle theories but in these contexts they do not exhibit very specific properties needed in the currently popular models of inflationary evolution of the early universe; usually one puts an inflaton field in the theory just by hand. This is why scalar-tensor theories of gravity, in which the metric field is supplemented by a scalar having arbitrarily prescribed features, seem to be very promising, particularly to particle physicists (and some of them deal with these theories in a rather careless way) for describing the primordial universe. (Hyper)extended inflation [5] and kinetic inflation [6] are just few examples. Finally, and here is a difference to the previous cases, scalar-tensor gravity theories can be used as test theories probing gravitational physics: they agree with general relativity in the stationary weak-field limit (the post-Newtonian approximation) and deviate from it in a strong-field regime or for radiative fields [7].

What I have said above does not mean that these (and possibly other) modifications of general relativity are in a sense superior to it. This is just an argument for treating them more seriously and a motivation for investigation of their physical content and relationship to general relativity. And since GR11 (Stockholm 1986) where a special workshop session was held on alternative theories of gravitation, a significant progress has been made and it is possible now to delineate a complete chart of relationships between scalar-tensor gravity theories, theories with nonlinear Lagrangians and general relativit y.

Accordingly, I will consider here two modifications of general relativity: either the Einstein-Hilbert Lagrangian is replaced by an arbitrary scalar function of the Riemann tensor or there is a spin-0 i.e. scalar-field component of gravitational interaction, which is non-minimally coupled to the metric. These theories are referred to as metric theories of gravity. Similar techniques can be applied to purely affine and metric-affine gravity theories, see a review [8].
these two can also be considered and dealt with in the same way; for simplicity I will omit it here. All other axioms of general relativity hold for these theories. (By Einstein’s theory I mean general relativity in any dimension $d \geq 4$. I assume $d = 4$ because in Kaluza-Klein theory difficulties arise due to the existence of multiple ground states corresponding to various topologies of the extra dimensions while in the case of $d > 4$ uncompactified dimensions the theory is in obvious conflict with experiment. Nonetheless, formally all the constructions presented below work, with slight modifications, for any $d \geq 4$.) In this sense the modified theories form a densely populated neighbourhood of general relativity in the space of gravity theories. This in turn raises a fundamental question: is Einstein’s theory merely a point of this neighbourhood? As a topology of the space is undefined, the problem at this level of reasoning has imprecise, intuitive sense. In other terms: is general relativity distinguished merely by tradition and computational simplicity or does it take a preferred position with respect to theories that surround it? The message of my talk is: these theories are mathematically equivalent to general relativity and there are convincing arguments that they are also physically equivalent to it. In a sense these theories represent Einstein’s theory in disguise. General relativity is not surrounded by theories different from it, its neighbourhood consists of its own versions in distinct variables. General relativity is an isolated point in the space of gravity theories. All these notions will be given a more precise meaning below.

While investigating these theories it is always assumed that they are fundamental i.e. independent theories. This means that even if they arise from other theories (strings, quantum gravity etc.) they are not subject to rules which are not inherent to them.

The history of these alternative theories of gravity is long, rich and begins with the celebrated Weyl’s theory in 1918 [9]. I do not intend to make a survey of this history. Just to give some idea of how our understanding of relationships of these theories to general relativity developed in time, I mention here some works. If a paper on the subject is omitted, it occurs merely due to incompleteness of the list. Special cases were studied by Higgs 1959 [10], Dicke 1962 [11], Bicknell 1974 [12], Bekenstein 1974 [13], Stelle 1977–78 [14], Whitt 1984 [15], Barrow and Cotsakis 1988 [16], Maeda 1989 [17], Schmidt 1990 [18], Cho 1987–93 [19], Damour, Farèse and Nordtvedt 1992–93 [7] and Wands 1993 [20]. General theory was developed mainly in a series of papers by Magnano, Ferraris and Francaviglia 1987–90 [21-23], Jakubiec and Kijowski 1988–89 [24] and Magnano and Sokolowski 1994 [25]. I will begin with a brief presentation of what is known in the most general case of nonlinear Lagrangians.
2 Structure of a general metric nonlinear gravity theory

A general metric nonlinear gravity (NLG) theory (sometimes named, rather improperly as will be shown below, ”a higher-derivative gravity”) is based on a Lagrangian
\[ L = \sqrt{-g}f(g_{\alpha\beta}, \partial g, \partial^2 g) \]
where \( f \) is an arbitrary scalar function. Due to general covariance of the theory, in vacuum the Lagrangian depends only on the metric and its Riemann tensor,
\[ f = f(g_{\alpha\beta}, R_{\alpha\beta\mu\nu}). \]
Except for Einstein-Hilbert Lagrangian and the Euler-Poincaré topological invariant density (Gauss-Bonnet term), the Lagrangian gives rise to fourth-order field equations.

It is now both common and useful to refer to the set of dynamical variables in a gravitational theory as a **conformal frame** (not to be confused with the notion of a reference frame); the meaning of ”conformal” will become clear later. In the case of a NLG theory it is Jordan conformal frame (JCF) and for vacuum theory it consists of the metric alone, JCF = \{ \( g_{\mu\nu} \) \}.

As it is very difficult to study the physical content of a fourth-order theory one should first lower the order of the field equations. The best method is to use a canonical Hamiltonian formalism. It should be stressed that it represents a covariant field-theory version of the well-known formalism in classical mechanics and it has nothing to do with the ADM formalism; actually it does not apply to general relativity at all. In the case of second-order Lagrangians the Hamiltonian formalism is far from being unique [23]. However in the case of a NLG theory it is natural to write
\[ L = \sqrt{-g}f(g_{\alpha\beta}, R_{\alpha\beta}, C_{\alpha\beta\mu\nu}) \]
and following a purely affine gravity theory to define a momentum tensor canonically conjugated to the Christoffel connection as [24]
\[ h_{\alpha\beta} \equiv (-g)^{-1/2}|\text{det}(\partial f/\partial R_{\alpha\beta})|^{-1/2} \frac{\partial f}{\partial R_{\alpha\beta}}. \]  
(1)

(This is in four dimensions, for \( d \geq 4 \) the exponents are appropriately altered.) The formalism works if the \( 10 \times 10 \) Hessian
\[ \text{det}(\frac{\partial^2 f}{\partial R_{\alpha\beta}\partial R_{\mu\nu}}) \neq 0. \]  
(2)

Then the definition (1) can be inverted and the Ricci tensor is expressed in terms of the momenta and all other fields,
\[ R_{\alpha\beta}(g) = r_{\alpha\beta}(h^{\mu\nu}, g_{\mu\nu}, C). \]  
(3)

The condition (2) means that the Lagrangian is truly nonlinear in the curvature. Furthermore, assuming that the momentum is a nonsingular matrix, i.e. \( \text{det}(\frac{\partial}{\partial R_{\alpha\beta}}) \neq 0 \), one can view \( h_{\alpha\beta} \) as a **new spacetime metric**. This corresponds to a mapping from the Lorentzian manifold \((M, g_{\mu\nu})\) to another one, \((M, h_{\mu\nu})\). The conformal frame for the gravity theory on \((M, h_{\mu\nu})\), in this case
named Einstein conformal frame (ECF), consists of three independent fields, ECF = \{h_{\mu\nu}, g_{\mu\nu}, C_{\alpha\beta\mu\nu}\}. The tensor fields $g_{\mu\nu}$ and $C_{\alpha\beta\mu\nu}$ lose their geometrical meaning they had in the initial spacetime and now can be viewed as a "matter source" for the new metric; actually they represent additional degrees of freedom for gravity.

The Legendre transformation (1) is followed in the standard way by replacing the Lagrangian by a Hamiltonian defined as

$$H(h, g, C) \equiv \sqrt{-h} h^{\mu\nu} r_{\mu\nu}(h, g, C) - \sqrt{-g} f(g, r, C).$$

In classical mechanics the canonical Hamilton equations arise as the variational Euler-Lagrange equations (see section 4) from the Helmholtz Lagrangian (I use this name following Poincaré and Levi-Civita [26]). In the present case it reads

$$L_H(h, g, C) \equiv \sqrt{-h} h^{\mu\nu} R_{\mu\nu}(g) - H(h, g, C)$$

and using an identity relating Ricci tensors for any two distinct metrics [21], it can be re-expressed, after some manipulations and discarding a full divergence, in the following form:

$$L_H(h, g, C) = \sqrt{-h}[R(h) + K(h, g) - h^{\mu\nu} r_{\mu\nu}(h, g, C)] + \sqrt{-g} f(g, r_{\alpha\beta}, C),$$

where $R(h)$ denotes the curvature scalar for $h_{\mu\nu}$. This Helmholtz Lagrangian has remarkable properties. Firstly, the free Lagrangian for the metric is exactly the Einstein-Hilbert one. Secondly, the kinetic Lagrangian for the "matter" field $g_{\mu\nu}$, $K(h, g) = \frac{1}{2} h^{\mu\nu} g^{\alpha\sigma} g^{\beta\gamma} [\nabla_\alpha g_{\beta\gamma} (\nabla_\mu g_{\sigma\nu} - \frac{1}{2} \nabla_\sigma g_{\mu\nu}) + \nabla_\alpha g_{\tau\rho} (\nabla_\mu g_{\sigma\rho} - \nabla_\sigma g_{\mu\rho}) - \frac{1}{2} \nabla_\mu g_{\sigma\rho} \nabla_\nu g_{\alpha\tau}]$ is universal i.e. it bears no trace of $f$ [24, 21] ($\nabla_\mu$ denotes the covariant derivative with respect to $h_{\alpha\beta}$). Thirdly, the whole information on the original nonlinear Lagrangian $L$ is encoded in the potential terms depending also on the $C_{\alpha\beta\mu\nu}$ field. The variation with respect to $h^{\mu\nu}$ yields Einstein field equations,

$$G_{\mu\nu}(h) = T_{\mu\nu}(h, g, g, g, C),$$

where all the terms in (6) except for $R(h)$ contribute to the "matter" energy-momentum tensor. It is clear from the form (5) of $L_H$ that these equations are equivalent to the inverse Legendre transformation, $R_{\mu\nu}(g) = r_{\mu\nu}(h, g, C)$. Independent variations $\delta g^{\mu\nu}$ and $\delta C_{\alpha\beta\mu\nu}$ provide second-order equations of motion for the fields. Unfortunately, the equations for $g^{\mu\nu}$ are intractably complex even in the simplest case of a quadratic $L$ (actually it turns out that the full complexity of the NLG theory arises already on this level and for other Lagrangians,
polynomial or non-polynomial in the curvature, the complexity does not substantially increase). Using the relation $R_{\mu\nu} = \tau_{\mu\nu}$ one gets $L_H(h, g, C) = L(g)$, therefore the equations of motion are equivalent to the fourth-order field equations $\frac{\delta L}{\delta g_{\mu\nu}} = 0$ of the original theory (see [24] for the detailed proof).

Thus there is a dynamical equivalence of any vacuum NLG theory to general relativity with Lagrangian $L_H(h, g, C)$ for ”matter” fields $g_{\mu\nu}$ and $C_{\alpha\beta\mu\nu}$. This means that although the action integrals for $L$ and $L_H$ are different in general, their stationary points (i.e. classical equations of motion) are the same, what is equivalent to the statement that the spaces of classical solutions for both theories are isomorphic. In other terms, general relativity with $L_H$ is a universal Hamiltonian counterpart of any NLG theory.

As an example let’s take the most frequently studied quadratic case without Weyl tensor, $L = \sqrt{-g}(R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu})$. Here JCF = \{g_{\mu\nu}\} while ECF = \{h_{\mu\nu}, \psi_{\mu\nu}, \psi\}. Using field propagators in linear approximation [14, 27] one gets a direct particle interpretation of the fields:

- $h_{\mu\nu}$ is the massless graviton (spin-2) with 2 degrees of freedom (d.o.f.),
- $\psi_{\mu\nu}$ is a massive spin-2 field carrying 5 d.o.f.,
- $\psi$ is a massive spin-0 particle with 1 d.o.f.

The coefficients should satisfy the no-tachyon conditions $m^2_0 = (6\alpha + 2\beta)^{-1} > 0$ and $m^2_2 = \frac{-1}{\beta} > 0$. The original metric $g_{\mu\nu}$ is a unifying field carrying together 8 d.o.f. with different spins.

From this example one sees that the mathematically equivalent theories have different interpretation. While in the NLG theory one views the massive fields $\psi_{\mu\nu}$ and $\psi$ as finite-range components of the gravitational interaction, in Einstein’s theory one interpretes them as particular species of elementary particles and describes gravitation in terms of the metric $h_{\mu\nu}$ alone. Despite the difference, the particle content of both theories (revealed in ECF) is the same.

For other Lagrangians, however, hard problems arise. Assume simple purely quadratic Lagrangian, $L = \sqrt{-g}R_{\mu\nu}R^{\mu\nu}$. Then $h^{\mu\nu}$ is proportional to $R^{\mu\nu}$ and in the subspace of solutions for which $\det(R_{\mu\nu}) \neq 0$ the tensor $h^{\mu\nu}$ in general does not have Lorentz signature and thus cannot describe the physical spacetime metric. Furthermore, the non-geometric components of gravity, $g_{\mu\nu}$ and $C_{\alpha\beta\mu\nu}$, are formally interpreted in ECF as sources of metric gravity, i.e. as matter fields. What can then be said about their energy? In the light of the well-known inconsistency of minimal coupling of the linear spin-2 field to geometry [28] one expects that a physical theory of these fields, although free of inconsistencies (since they are absent in JCF they cannot arise in any frame dynamically equivalent to it), will not be easy to formulate (e.g. it is known that the tensor field is a ghost). These are open questions and at the present level of art it is prudent to say that for a generic NLG theory its mathematical equivalence to general relativity (including the ”matter” fields) needs not to imply their physical equivalence.
In what follows I will confine myself to the restricted NLG theories, where these problems do not arise. It is convenient to consider first the scalar-tensor theories of gravity.

3 Scalar-tensor gravity

These are theories in which gravitational interactions are described by a doublet consisting of a spacetime metric and a scalar field. It is here that the notion of Jordan conformal frame was introduced [29]. Thus JCF = \{g_{\mu\nu}, \varphi\}, we consider first a vacuum theory. It was first recognized by Pauli in early fifties (quoted in Sect. 28 of ref. [30]) that in such a system one can always make a field redefinition (a change of variables) to another conformal frame via a conformal mapping \( g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = \Omega^2(\varphi)g_{\mu\nu} \) and \( \varphi \rightarrow \tilde{\varphi} = \tilde{\varphi}(\varphi) \) with arbitrary \( \Omega \) and \( \tilde{\varphi} \). Thus the theory can be expressed in terms of infinite number of conformally related frames. A general Lagrangian for a scalar-tensor gravity (STG) theory,

\[
L = \sqrt{-g} \left[ \varphi R - \frac{\omega(\varphi)}{\varphi} g^{\mu\nu} \varphi,\mu \varphi,\nu + 2 \varphi V(\varphi) \right],
\]

can then take, among others, the equivalent forms

\[
L = \sqrt{-\tilde{g}} \left[ f(\tilde{\varphi}) \tilde{R} + \tilde{V}(\tilde{\varphi}) \right] = \sqrt{-\bar{g}} \left[ \bar{R}(\bar{g}) - \bar{g}^{\mu\nu} \phi,\mu \phi,\nu - \bar{V}(\phi) \right].
\]

The first of these contains no kinetic term for \( \tilde{\varphi} \) and a propagation equation for the scalar arises due to the nonminimal coupling to the curvature. The other form represents just a self-interacting scalar \( \phi \) minimally coupled to gravity and is designated as Einstein conformal frame, ECF = \{\bar{g}_{\mu\nu}, \phi\}, with \( \bar{g}_{\mu\nu} = \phi g_{\mu\nu} \) and \( d\phi \equiv [\omega(\varphi) + \frac{3}{2}]^{1/2} d\varphi \) for \( \omega > -\frac{3}{2} \). It should be stressed that these two forms are not special cases of the general Lagrangian, but are equivalent to it for any \( \omega(\varphi) \) [25, 27].

Thus apparently different STG theories can be mapped onto each other by conformal mappings. Consider for simplicity the case without self-interaction, \( V(\varphi) = 0 \). Then all the theories are divided in two classes: those with \( \omega = 0 \) (this class actually contains only one member) and with \( \omega \neq 0 \). The conformal transformations map theories within the classes and from one class to the other, the potential is always zero. Each STG theory can be transformed into general relativity plus conformally invariant scalar field \( \chi \) [25]. The latter is usually interpreted as different from a STG theory: the conformally invariant scalar is commonly viewed as a special kind of matter rather than being a spin-0 component of gravity; in early seventies it was believed that the field would exhibit more interesting features in quantum theory than the ordinary scalar. Despite the traditional interpretation, the scalar \( \chi \) fits the general framework of STG theories. It had been discovered by Bekenstein [13] and remained unnoticed for many years that the conformally invariant field is equivalent under a conformal...
map to the massless linear scalar minimally coupled to Einstein gravity. Now it is known that this is merely a special case of a generic feature: each STG theory is equivalent to general relativity plus the massless linear scalar field [25]. In other terms this ordinary scalar may be represented in disguise in infinite number of ways as the spin-0 gravity component in any STG theory or as the conformally invariant field.

What about interactions with matter? Among all conformally related frames one distinguishes two frames: JCF and ECF. As scalar fields have not yet been observed in nature, one can in principle assume any form of their interaction with ordinary matter (here collectively denoted by \( \psi \)). Actually only two possibilities seem to be physically interesting and reasonable.

1. Matter minimally couples to the metric in JCF,

\[
L(g, \varphi, \psi) = \frac{1}{16\pi} (\varphi R - \frac{\omega}{\varphi} g^{\mu\nu} \varphi_{,\mu,\nu}) + L_{\text{mat}}(g, \psi),
\]

"to the only physically meaningful frame whenever one has to deal with a Jordan-Brans-Dicke-type theory" (S. Matarrese).

2. Minimal coupling in ECF,

\[
L = \frac{1}{16\pi} [\bar{R}(\bar{g}) - \bar{g}^{\mu\nu} \phi_{,\mu,\nu}] + L_{\text{mat}}(\bar{g}, \psi).
\]

One first makes the conformal transformation to ECF and then couples matter to metric gravity in it. Clearly no trace of the original STG theory survives in this frame and most advocates of these theories reject this form of coupling.

Once matter has been coupled to gravity in a frame one has freedom to make conformal transformations to any other frame. E.g., if matter is minimally coupled in JCF (version 1) then the theory is described in ECF by

\[
L' = \sqrt{-\bar{g}} \left[ \bar{R}(\bar{g}) - \bar{g}^{\mu\nu} \phi_{,\mu,\nu} + \left( \frac{16\pi}{\varphi(\bar{\phi})} \right)^2 L_{\text{mat}}(\varphi(\bar{\phi}), \bar{g}, \psi) \right];
\]

in this frame the scalar directly interacts with matter. Similar interactions arise when one transforms the theory with matter minimally coupled in ECF (version 2) back to JCF. The two versions of coupling matter to gravity generate two physically different gravity theories. Their difference lies in distinct effects they predict for matter while it should be emphasized that each theory is internally consistent in any frame; e.g. conservation laws hold for them in all frames [25].

## 4 Restricted nonlinear gravity theories

These are metric gravity theories in which the Lagrangian depends on the Riemann tensor solely via the curvature scalar. In vacuum \( L = \sqrt{-g} f(R) \) and JCF = \( \{g_{\mu\nu}\} \), where \( f \) is any smooth function (later I shall assume that \( f \) is
analytic around \( R = 0 \). Except for \( f = R \) the field equations are of fourth order. To lower their order one cannot use exactly the same canonical formalism as in the general case since the Legendre map is degenerate,

\[
\det \left( \frac{\partial^2 f}{\partial R_{\alpha\beta} \partial R_{\mu\nu}} \right) = 0.
\]

One introduces instead a scalar momentum canonically conjugated to a linear combination of connection,

\[
p \equiv \frac{1}{\sqrt{-g}} \frac{\partial L}{\partial R} = f'(R).
\]

The regularity conditions are then \( f'(R) > 0 \) and \( f''(R) \neq 0 \). Then the definition (9) can be inverted to yield \( R(g) = r(p) \), e.g. for \( f = R + aR^2 \) there is \( r(p) = \frac{1}{2a}(p - 1) \). The field Hamiltonian is then

\[
H(p, g) \equiv \sqrt{-g}[pr(p) - f(r(p))].
\]

The Hamilton equations for a mechanical system arise as stationary points of the action for the Helmholtz Lagrangian defined as a function on the tangent bundle to the phase space:

\[
L_H(q, p, \dot{q}, \dot{p}) \equiv p\dot{q} - H(q, p).
\]

In fact, the independent variations \( \delta q \) and \( \delta p \) of the action yield correspondingly

\[
\dot{p} = -\frac{\partial H}{\partial q} \quad \text{and} \quad \dot{q} = \frac{\partial H}{\partial p}.
\]

The Helmholtz Lagrangian for a NLG theory reads

\[
L_H(p, g) \equiv \sqrt{-g} pR(g) - H(p, g).
\]

Since \( p \) is an independent degree of freedom we are now working in Helmholtz-Jordan conformal frame (HJCF) consisting of \( g_{\mu\nu} \) and \( p \). The Hamilton equations following from \( L_H \) are of second order,

\[
\frac{\delta L_H}{\delta \dot{p}} = 0 \quad \Rightarrow \quad R(g) = r(p),
\]

\[
\frac{\delta L_H}{\delta g^{\alpha\beta}} = 0 \quad \Rightarrow \quad G_{\alpha\beta} = \frac{1}{p}(\nabla_{\alpha} \nabla_{\beta} p - g_{\alpha\beta} \nabla_{\mu} \nabla^{\mu} p) \quad -\frac{1}{2}[r(p) - \frac{1}{2}f(r(p))]g_{\alpha\beta} \equiv \theta_{\alpha\beta}(g, p).
\]

By introducing the scalar momentum one not only reduces the fourth-order field equations to second-order ones but moreover these are precisely Einstein
equations with the momentum generating a "matter source". I stress that this has been achieved without altering the spacetime metric [23, 25] and the theory is not inherently higher-derivative one.

Investigation of the Cauchy problem for $L = \sqrt{-g} f(R)$ shows that the scalar $p$ represents an independent dynamical degree of freedom [31].

The NLG theory with the Lagrangian $L$ in JCF is dynamically equivalent to the theory with Helmholtz Lagrangian $L_H$ expressed in HJCF. The latter describes 3 d.o.f., hence the metric $g_{\mu\nu}$ in JCF unifies spin 2 and spin 0. This is a special case of the general rule: the number of spin d.o.f. carried by a field depends on its tensorial character and the equations of motion it satisfies. If the metric Lagrangian does not contain explicitly the Ricci tensor, the number of d.o.f. decreases from 8 to 3.

The Helmholtz Lagrangian (12) describes a STG theory in a frame where the kinetic part of the Lagrangian for the scalar $p$ is absent, $\omega = 0$. Thus the restricted NLG theory with $L = \sqrt{-g} f(R)$ is equivalent to an STG theory with a self-interacting scalar field. The latter theory can, as we have seen, be transformed into general relativity with a self-interacting minimally coupled scalar field $\phi$. The conclusion is that any restricted vacuum NLG theory is dynamically equivalent to the nonlinear scalar $\phi$ in Einstein's gravity theory. The equivalence is attained in two steps. First one maps a given NLG theory via the Legendre map into a STG theory and then the latter is transformed with the aid of a conformal change of variables, $\bar{g}_{\alpha\beta} \equiv p g_{\alpha\beta}$, $\phi \equiv \sqrt{\frac{2}{3}} \ln p$, into general relativity plus $\phi$ with

$$L_H(\bar{g}, \phi) = \sqrt{-\bar{g}}[\bar{R}(\bar{g}) - \bar{g}^{\alpha\beta} \phi,_{\alpha} \phi,_{\beta} - 2 \bar{V}(\phi)],$$

(15)

where the potential is

$$\bar{V}(\phi) = \frac{1}{2} \left( \frac{r(p)}{p} - \frac{f(r)}{p^2} \right),$$

(16)

and ECF = $\{\bar{g}_{\mu\nu}, \phi\}$.

The equivalence means that the conformal transformation and Legendre map can (at least in principle) be inverted. In fact, for any self-interaction potential $\bar{V}(\phi) \neq 0$ in ECF the inverse problem of nonlinear gravity has a solution [25]: there exists a vacuum JCF = $\{g_{\mu\nu}\}$ and a Lagrangian $L = \sqrt{-g} f(R)$ in it such that $L$ is equivalent to $L_H$ given in (15). In most cases, however, $L$ cannot be expressed in terms of elementary functions. One of few exceptions is provided by the Liouville field theory with $\bar{V}(\phi) = A \exp(\sqrt{\frac{2}{3}} \phi)$, then $f(R) = 4A(6AR)^{3/2}$. The linear massless ($\bar{V} = 0$) field $\phi$ is not equivalent to a restricted NLG theory but to a STG theory. It is worth to emphasize the difference between the two steps. STG theories are equivalent to Einstein's theory plus the scalar in the sense of equality of the action integrals for the theories, thus the equivalence holds not only for the solutions of the classical field equations. Yet for NLG theories on one hand and for STG theories and general relativity on the other,
the action integrals are different in general and these are the spaces of classical solutions that are isomorphic. (Actually the isomorphism is usually local since the Legendre map is only locally invertible, I will not discuss here this difficult problem.)

When ordinary matter is taken into account the same problem appears as in the case of STG theories: to which metric should it be minimally coupled? If it is minimally coupled in JCF,

\[ L = \sqrt{-g} [f(R) + 2L_{\text{mat}}(g, \psi)], \]  

(17)

then after making the Legendre map and conformally transforming it takes on the form in ECF:

\[ L_H(\bar{g}, \phi, \psi) = \sqrt{-\bar{g}} [\bar{R}(\bar{g}) - \bar{g}^{\mu\nu}\phi,_{\mu}\phi,_{\nu} - 2\bar{V}(\phi) + 2e^{-2\sqrt{2/3}\phi}L_{\text{mat}}(e^{-\sqrt{2/3}\phi}\bar{g}, \psi)]. \]  

(18)

On the other hand, if matter is minimally coupled in ECF,

\[ L_H = \sqrt{-g} [\bar{R}(\bar{g}) - \bar{g}^{\mu\nu}\phi,_{\mu}\phi,_{\nu} + 2L_{\text{mat}}(\bar{g}, \psi)], \]  

(19)

one can make a Legendre transformation to absorb the scalar into the metric field and obtain again a NLG theory, this time in the presence of matter. Contrary to a naive view the Legendre map needed to this aim is not the inverse of the map from the original JCF to ECF. In other words one has an open chain of frame mappings: \{g_{\mu\nu}\} \rightarrow \{\bar{g}_{\mu\nu}, \phi\} \rightarrow \{\bar{g}_{\mu\nu}, \phi, \psi\} \rightarrow \{\hat{g}_{\mu\nu}, \psi\} with \( g_{\mu\nu} \neq \hat{g}_{\mu\nu} \); interaction with matter results in an appropriate change of the Jordan frame metric [25]. In matter Jordan conformal frame \{\hat{g}_{\mu\nu}, \psi\} the metric and matter variables are inextricably intertwined in the resulting nonlinear Lagrangian.

The two ways of coupling with matter give rise to two physically distinct gravity theories. Contrary to some claims (there was a debate in Phys. Rev. D, 1995) both theories are consistent in any frame [25]. The fundamental problem is then: which frame (if any) contains the spacetime metric of the physical world? Is the problem meaningful at all?

5 Energy and the choice of a physical frame

Physical laws are not conformally invariant and properties of elementary particles are altered under a conformal map. The most general argument regarding particle masses is provided by quantum mechanics. Under a conformal rescaling of the spacetime metric, \( g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu} \), the particle’s wave function transforms as \( \psi \rightarrow \Omega^{-3/2}\psi \). The covariant momentum operator \( p_{\mu} = -i\frac{\partial}{\partial x^\mu} \) is independent of the metric. The momentum eigenfunctions of a free particle should satisfy the equation \( p_{\mu} p^{\mu} \psi = m^2 \psi^2 \) both in the original metric and in the rescaled one. Assuming that the conformal factor \( \Omega \) varies slowly on the distances of the order of the particle’s Compton wavelength, one finds that the particle masses
scale uniformly under the conformal mapping: $m \mapsto \Omega^{-1}m$. (It is amusing to notice that some authors prove this relation in the flat cosmological Friedmann model and derive from it far-reaching conclusions.)

Some advocates of STG and restricted NLG theories claim that Jordan frame is physical since the physical frame is the one in which "atomic masses are constant". In fact, if matter is minimally coupled in JCF as is done in most papers on the subject, then in ECF particle masses are spacetime-dependent. It is clear however from what I have said above that this argument is based on the \textit{petitio principii} error. Assuming that in JCF the Lagrangian has the form (17) amounts to assuming that this frame is physical in the common sense of the term and the argument is a trivial check of internal consistency of the assumption. One can equally well assume that in ECF the correct form of the Lagrangian is given by (19) and then the particle masses are physical constants in this frame while in matter Jordan conf. frame the highly nonminimal coupling makes the masses variable. The problem of which frame is physical cannot be solved in this way.

The problem should be somehow addressed by anyone who deals with STG and NLG theories. There are possible four answers to the problem and all are found in the current literature. The first is that JCF is physical while ECF is merely a useful computational tool; this view is shared by most authors applying these theories to various problems in cosmology and high energy physics. The second is that ECF is physical, the motivation is less obvious and various authors in this group use different arguments to support the view: (i) ADM Hamiltonian formalism works in this frame and ADM and Bondi-Sachs masses for an isolated system are well defined; (ii) quantization of scalar field fluctuations should be done in ECF while in JCF the procedure is at least suspect ("it appears as if the quantization and a conformal transformation are two mutually noncommutable procedures"); (iii) the massless spin-2 graviton is described by the difference $\bar{g}_{\mu\nu} - \eta_{\mu\nu}$ while in JCF the fluctuations of $g_{\mu\nu}$ about the flat space represent a mixture of the tensor and scalar fields; (iv) ECF is singled out by dimensional reduction of higher-dimensional Einstein's gravity. Authors in the third group claim that classically JCF and ECF are physically equivalent: since mass ratios of elementary particles are unchanged by conformal transformations then "physics cannot distinguish between conformal frames". They seem to overlook that other physical quantities, notably energy, are sensitive to these transformations. The fourth answer, implicitly contained in some works is just to avoid addressing the problem (see [25] for references). All authors admit that ECF is \textit{always} (except for very special cases) computationally advantageous.

The problem should be ultimately solved by experiment but it is clear that we are very far from it. On purely theoretical grounds one can provide convincing arguments in favour of Einstein frame.

Consider a nonrelativistic charged particle in an external electromagnetic field and perform in its phase space the canonical transformation renaming the positions and momenta: $P_i = -x_i$ and $Q_i = p_i$ ($x_i$ being Cartesian coordinates).
Then its Hamiltonian reads

\[ H = \frac{1}{2m} \sum_i \left[ Q_i - \frac{e}{c} A_i(-P_k, t) \right]^2 + eA_0(-P_k, t) \]

and the rule of minimal coupling to electromagnetic field, symbolically expressed as \( p \rightarrow p - eA \), clearly does not hold. The rule applies to the momenta canonically conjugated to the variables on which the field depends. These variables—the physical coordinates—can be determined in the absence of the field by requiring that the Hamiltonian of the free particle be independent of them. Due to homogeneity of the space and time the Hamiltonian should depend only on the particle’s momentum.

In classical general relativity all observable quantities can be determined by taking into account the physical nature and gravitational interactions of material bodies forming the reference frame (see e.g. [32]). This seems to undermine our conjecture that the correct physical metric can be determined for a vacuum theory. However this is not so since the role of a material reference frame is here played by the scalar field which inevitably arises in the gravity theories we consider. In a sense the scalar field is analogous to the electromagnetic field in the example and the spacetime homogeneity is replaced by total energy (the scalar cannot be, however, "switched off"). In fact, energy plays a distinguished role in gravitational physics being effectively a conserved (nonnegative) charge. In this respect the scalar should not substantially differ from known (classical) matter.

Accordingly, the crucial conjecture is that the physical metric should be singled out for vacuum theory and this should be done by considering the interaction of a given metric with the scalar field. Matter is minimally coupled to the physical metric. Then the form of the coupling in any conformal frame determined by the relation of the metric in this frame to the physical metric. The physical metric need not exist for any gravity theory; actually it exists only for physically viable theories. A theory of gravity is physical if there exists a classically stable maximally symmetric ground state solution for it. (A theory may have several ground states and some of them can be semiclassically unstable.) A physical theory can be expressed in terms of various conformal frames of which only few are physical. A conformal frame is physical if the dynamical variables constituting it are (at least in principle) measurable and their fluctuations around the ground state solution have positive energy. In other terms the ground state solution represents the minimum of energy expressed in physical variables. We shall employ the well known relation between stability and positivity of energy. (In practice one does not compute the total energy of the system; a given solution is stable if any perturbations have positive energy density. In a theory of gravity expressed in an unphysical frame the energy density for a fluctuation is indefinite in general and the relationship to stability is broken.)
Let
\[ L = \sqrt{-g} f(R) = \sqrt{-\bar{g}} (R + aR^2 + bR^3 + \cdots), \quad a \neq 0 \] (20)
and \( \Lambda = 0 \). Then \( g_{\mu\nu} = \eta_{\mu\nu} = \bar{g}_{\mu\nu} \) is a candidate ground state solution. Near Minkowski space i.e. for \( R \approx 0 \) one finds \( p = f'(R) = 1 + 2aR + \cdots > 0 \) and \( f'' = 2a + \cdots \neq 0 \), thus the Legendre map is invertible and the metric \( \bar{g}_{\mu\nu} \) has the correct signature. This shows the local (in the vicinity of flat space) equivalence of Jordan and Einstein frames. Furthermore let the spacetime \((M, g_{\mu\nu})\) be asymptotically flat,

\[ g_{\mu\nu} = \eta_{\mu\nu} + O(r^{-1/2-\epsilon}), \quad g_{\mu\nu,\alpha} = O(r^{-3/2-\epsilon}), \quad \epsilon > 0. \] (21)

Then the positive energy theorem for NLG theories holds \[25\]. Let \( \Sigma \) be an asymptotically flat, nonsingular spacelike hypersurface in a spacetime \((M, g_{\mu\nu})\) topologically equivalent to \( \mathbb{R}^4 \). If (i) the Lagrangian is given by (20), (ii) \( p > 0 \) and \( f''(R) \neq 0 \) everywhere on \( \Sigma \), (iii) a solution \((g_{\mu\nu}, p)\) in HJCF to the field equations (13)–(14) satisfies the condition (21) and (iv) the coefficient \( a > 0 \), then a) the potential \( \bar{V}(\phi) \) given by (16) is non-negative on \( \Sigma \) and

b) the ADM energy formally defined in JCF in the same way as in general relativity, is non-negative,

\[ E_{\text{ADM}}[g] = \frac{1}{2} \int_{S^2} dS_i (g_{ij,j} - g_{jj,i}) \geq 0 \] (22)

and vanishes only in flat space.

One sees that under the assumptions of the theorem Minkowski space is a classically stable ground state solution in both Jordan and Einstein frames. The proof is an extension and modification of that given by Strominger \[33\] for \( f = R + a^2 R^2 \). As in the case of the classical Positive Energy Theorem in general relativity, the proof is based on the dominant energy condition for the source, in this case for the scalar. Therefore the proof goes only in ECF where \( \phi \) is minimally coupled to \( \bar{g}_{\mu\nu} \) and its potential \( \bar{V}(\phi) \geq 0 \). It should be stressed that although the positive energy theorem does hold in JCF (more precisely, in HJCF), it cannot be proven in this frame, the existence of ECF is essential. Energy is well defined for systems described by second-order equations of motion and these are achieved in HJCF. In this frame \( E_{\text{ADM}} \) is equal to energy in ECF. Equality of the total energy in these two frames reflects the fact that (being a charge) it is evaluated at spatial infinity (where \( p \to 1 \)) and is rather loosely related to the interior of the system. The only detailed information about the interior that is needed is whether all local energy flows are timelike or null (dominant energy condition). This connection is lost in JCF. The energy-momentum tensor \( \theta_{\alpha\beta}(g, p) \) for \( p \), defined by (14), is indefinite and \( a \text{ priori} \) negative energy density and superluminal energy flows may occur. These flaws do not reflect the genuine properties of spin-0 gravity and do not imply that the scalar particles are tachyons; these are merely due to an improper choice.
of field variables. $\theta_{\alpha\beta}$ is not the physical energy and momentum density for $p$. Any field redefinition of the scalar will not help and only with the aid of the conformal rescaling of the metric one finds the correct expression for these quantities.

Our conclusions regarding the features of a physical restricted NLG theory are following.

1. Its Lagrangian must contain the linear term $R$. It ensures that the Legendre map to ECF exists near Minkowski space (i.e., for $R \approx 0$) which is supposed to be a ground state solution.

2. $L$ should contain the quadratic term $aR^2$. It ensures regularity (invertibility) of the Legendre map at flat space.

3. The coefficient $a$ determines stability of flat space. For $a > 0$ Minkowski space is a stable ground state solution. For $a < 0$ the potential $\bar{V}(\phi)$ attains maximum for flat space and renders it classically unstable. Existence of another solution with minimum of energy is unclear.

4. If a theory is physical then Einstein frame meets all general requirements of relativistic field theories and is regarded as physical. The metric $\bar{g}_{\mu\nu}$ of this frame determines the spacetime intervals in the physical world. It is related to the original metric variable of JCF by $\bar{g}_{\mu\nu} = f'(R)g_{\mu\nu}$. On the contrary, Jordan frame is never physical: its variables do not provide a meaningful and tractable relationship to the total energy of the system.

5. The transformation from JCF to ECF is physically interpreted as a transition to dynamical variables describing fields with definite spins and for which the local energy flows are causal implying the positive energy theorem. There is a geometrical analogy: ECF = $\{\bar{g}_{\mu\nu}, \phi\}$ is like Cartesian coordinates for dynamical variables while JCF and other frames are a kind of “curvilinear coordinates” which can generate fictitious “coordinate singularities”.

Whenever Einstein frame does not exist in the vicinity of flat space, the theory is either unphysical, e.g., for $f = R^2$ [12], or is suspected of being such and showing that it is a viable one is difficult.

Final conclusion is that a physical restricted NLG theory is nothing but general relativity plus the scalar field in a disguise. Such a theory cannot provide new physical effects different from those existing in Einstein’s theory. If there is a deeper motivation for considering NLG theories (e.g., as being a field-theory limit of string theory) then the function $f(R)$ generates the potential $\bar{V}(\phi)$ for the scalar field.

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