Rotationally non-invariant aspects of scattering amplitudes with graviton exchange

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Abstract
We consider the s-channel scattering of massive fermion or vector-boson pairs with equal helicities, mediated by a graviton in the linearized Einstein theory. We show that, although in general both spin-2 and spin-0 components are present in the exchanged graviton, there is a special set of reference frames where the spin-0 graviton component vanishes. This is connected to the dependence of the trace of the graviton propagator on the effective dimension of the space-time spanned by the sources. On the other hand, one finds a non-vanishing (Lorentz-invariant) interference of the graviton amplitude with the scalar-exchange amplitude, that in principle could give measurable effects.

1 Introduction
In a previous work [1], we considered the s-channel scattering amplitude of pairs of on-shell massive fermion/gauge-bosons mediated by a graviton, in the linearized limit of quantum gravity in the Einstein theory. The amplitude is manifestly gauge invariant. We computed its interference with the corresponding amplitude where a scalar (Higgs boson) field replaces the graviton, and found a non-vanishing result. This interference is of course a Lorentz-invariant quantity, that in principle could give measurable effects, and calls for the presence of a spin-0 component in the virtual graviton propagator.

On the other hand, a spin-0 virtual graviton component should correspond to a non-vanishing trace in the effective graviton propagator. By effective we mean the “active”
part of the propagator, that is the one selected by the non-vanishing components of the source tensors.

In this paper, we will show that, for the scattering process considered in Figure 1, the spin content of the effective graviton propagator depends on the choice of the reference frame. In particular, while in a generic frame one can trace back a spin-2 plus a spin-0 component in the effective propagator, there is a special set of reference frames where the spin-0 component vanishes. We will show explicitly this anomalous behavior by decomposing the effective propagator in angular momentum eigenstates. In particular, when projecting the angular momentum along the direction orthogonal to the scattering plane in the c.m. frame, the spin-0 eigenstate is canceled by one of the spin-2 components. The same holds also in any frame obtained from the latter by a Lorentz boost orthogonal to the scattering plane, when projecting the spin along the boost direction.

This cancellation resembles the one occurring in the on-shell limit of the graviton propagator [2, 3], that reduces the spin degrees of freedom of the virtual graviton to the two (±2) helicities allowed for an on-shell (massless) graviton.

The appearance/disappearance of a spin-0 component, depending on the reference frame, shows that rotational invariance is not respected by the graviton-matter vertex, for massive matter fields. This does not contrast with what one expects from the Noether’s theorem, since, as we will show, there is a negative-norm (scalar) state contributing to the graviton propagator. The latter evades the basic hypothesis of the quantum version of the Noether’s theorem, where only positive-norm states are assumed.

The plan of the paper is the following. In Section 2, we study the dependence of the graviton propagator trace on the effective dimensions of the source tensors, and notice the possibility of a critical dimension. In Section 3, we give an explicit example of scattering processes where the source tensors can live in a critical-dimension subspace, giving rise to rotationally non invariant aspects of the corresponding amplitude, as also shown in Section 4. In Section 5, we present some concluding remarks.

For comparison, we discuss in Appendix A the corresponding photon exchange amplitude, where, of course, no negative-norm state propagates and, consequently, rotational invariance holds. In Appendix B, we provide some relevant Lorentz covariant expressions for the matrix elements of the energy-momentum tensor.
The spin-0 component in the virtual graviton can be scrutinized by considering the trace of the graviton propagator tensor, that is naturally connected to the rotationally invariant graviton component. We will see that the effective propagator trace can vanish depending on the effective dimension of the source tensors.

Let us consider the general form of the graviton propagator in the linearized Einstein theory. When contracted with conserved energy-momentum tensors (i.e., for $q_{\mu}T_{\mu\nu} = 0$, with $q_{\mu}$ the momentum flowing in the propagator), terms proportional to the momentum vanish, and the effective graviton propagator becomes

$$G_{\mu\nu\alpha\beta}(q) = i \frac{P_{\mu\nu\alpha\beta}}{q^2 + i\epsilon}; \quad P_{\mu\nu\alpha\beta} = \frac{1}{2} (\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}),$$ \hspace{1cm} (1)

where we use the Minkowski metric $\eta_{\mu\nu} = \text{Diag}(1, -1, -1, -1)$.

The $s$-channel $p_1 \bar{p}_1 \rightarrow p_2 \bar{p}_2$ scattering amplitude mediated by a graviton is then given by

$$A = \frac{-i}{M_P^2} \left( \hat{T}_{\nu}^i \hat{T}_{\mu}^i G_{\mu\nu\alpha\beta}(q) \hat{T}_{\alpha}^i \right),$$ \hspace{1cm} (2)

where $\hat{T}_{\mu\nu}^{i(f)}$ are the matrix elements of the conserved energy-momentum tensors between the vacuum and the initial (final) state, and $M_P$ is the Planck mass.

Apart from propagating spin-2 states, the $G_{\mu\nu\alpha\beta}(k)$ tensor can propagate a scalar field associated to the trace $P_{\mu\nu\alpha\beta}$.

Let us now consider a given reference frame where the source tensor $\hat{T}_{\mu\nu}$ has non-vanishing components only in a $d$-dimensional subspace (with $d \leq 4$). Then, $\hat{T}_{\mu\nu}$ can be decomposed as

$$\hat{T}_{\mu\nu} = \hat{R}_{\mu\nu} + \hat{T} \tilde{\eta}_{d}^d_{\mu\nu},$$ \hspace{1cm} (3)

where $\hat{R}_{\mu\nu} = 0$, $\hat{T} = \hat{T}_{\mu\nu} / d$, and $\tilde{\eta}_{d}^d_{\mu\nu}$ is defined as follows: $\tilde{\eta}_{d}^d_{\mu\nu} = \eta_{\mu\nu}$ for $(\mu, \nu)$ belonging to the $d$-dimensional subspace, and $\tilde{\eta}_{d}^d_{\mu\nu} = 0$ for all the other components. Note that $\tilde{\eta}_{d}^d_{\mu\nu}$ is not a tensor. Its definition depends on the reference frame.
Let $d'$ (with $1 \leq d' \leq 4$) be the smallest of the dimensions relative to the two subspaces spanned by $\hat{T}^i_{\mu\nu}$ and $\hat{T}^f_{\mu\nu}$, respectively. Then, let $d$ (with $d' \leq d \leq 4$) be the dimension of the minimal subspace that contains all the non-vanishing components of both the matrix elements. Then, the $G_{\mu\nu\alpha\beta}(k)$ tensor will propagate a scalar field associated to the trace

$$\tilde{\eta}^{d'}_{\mu\nu} P_{\mu\nu\alpha\beta} \tilde{\eta}^d_{\alpha\beta} = -\frac{1}{2} d' (d - 2).$$

(4)

The latter contributes to the scattering amplitude in Eq. (2) via

$$A^{J=0} = -\frac{(d - 2)}{2 d M_P^2} \left( \hat{T}^{f\dagger}_{\mu\mu} \hat{T}^i_{\nu\nu} \right),$$

(5)

whose coefficient depends on the reference frame, through the definition of $d$. The minus sign in Eq. (5) is due to the negative norm of the associated propagating (ghost) scalar field.

Let us now assume that there exists some reference frame where the external sources live on the same plane, so that only their components in a $d = 2$ subspace can differ from zero (we will give an explicit example of this case in Section 3). Then, the scalar amplitude in Eq. (5) vanishes. In this case, no spin-0 graviton component will propagate in the process.

We observe that the absence of this spin-0 component does not occur in any reference frame. For instance, let us rotate the above frame around an axis not orthogonal to the plane corresponding to the $d = 2$ subspace, where the source-tensor components can differ from zero. Then, the source tensors will acquire in general non-vanishing components along a third dimension. Correspondingly, the trace of the effective propagator, according to Eq. (4), will be different from zero, giving rise to the propagation of a spin-0 component of the graviton in Eq. (5).

In Section 3, we will show that there are particular scattering processes, where the source tensors can live in a two-dimensional subspace, giving rise to the rotationally non-invariant aspects discussed above. We will show that the partial-wave decomposition of the amplitude includes a spin-0 component, and, at the same time, in a particular frame, the effective tensor $P_{\mu\nu\alpha\beta}$ propagating in the scattering process is two dimensional and traceless, according to Eq. (4).

Then, in Section 4, we will analyze, in different frames, the rotational-group representations contributing to the graviton polarization states that are effectively propagating in the process.
3 Scattering of equal-helicity matter fields

Let us consider, in its c.m. frame, the $s$-channel scattering, mediated by a graviton, of massive fermion and vector-boson pairs (Figure 1)

$$p_1(k, \lambda_1) + \bar{p}_1(k', \lambda_1) \rightarrow p_2(p, \lambda_2) + \bar{p}_2(p', \lambda_2) ,$$  \(6\)

where initial(final) particles have equal and non-vanishing helicity $\lambda_1(\lambda_2)$. The 4-momenta of the initial and final particles, $k/k'$ and $p/p'$, respectively, can be expressed in terms of their spatial 3-momenta $k/k'$ and $p/p'$:

$$k = (E, k), \quad k' = (E, -k),$$

$$p = (E, p), \quad p' = (E, -p),$$  \(7\)

with $E^2 = |k|^2 + m_1^2$, $E^2 = |p|^2 + m_2^2$, and $E = \sqrt{s}/2$.

We then define the on-shell matrix element of the energy-momentum tensor on an initial $p\bar{p}$ state, $\hat{T}_{\mu\nu}[p, \bar{p}]$, as follows

$$\hat{T}_{\mu\nu}[p, \bar{p}] \equiv <0|T_{\mu\nu}|p, \bar{p}>.$$  \(8\)

Due to the conservation of the on-shell matrix elements of the energy-momentum tensor, in the c.m. frame one has $\hat{T}_{0\nu}[p, \bar{p}] = 0$, and only the spatial components $\hat{T}_{ij}$ ($i, j = 1, 2, 3$) can differ from zero.

For equal-helicity fermions and transverse vector bosons, the on-shell matrix elements of the energy-momentum tensor can be expressed as [see Appendix B]

$$\hat{T}_{\mu\nu}[p, \bar{p}] \sim \frac{1}{\sqrt{s}} \frac{4\lambda m_f}{\beta_f \sqrt{s}} |k_i k_j| ,$$  \(9\)

where $\beta_f = \sqrt{1 - m_f^2/E^2}$, and $m_f$ is the fermion mass.

For equal-helicity transverse real vector bosons, we obtain from Eq. (33) in the Appendix B

$$\hat{T}_{0\nu}[V_\pm, \bar{V}_\pm] = 0 , \quad \hat{T}_{ij}[V_\pm, \bar{V}_\pm] = -\frac{8 m_V^2}{\beta_V^2 s} |k_i k_j| ,$$  \(10\)

*Only equal helicities for initial/final particles can give rise to $J = 0$ components in the scattering amplitudes.*
where $\beta_V = \sqrt{1 - m_V^2 / E^2}$, and $m_V$ is the vector boson mass.

Note that both the matrix elements $\hat{T}_{\mu\nu}[f, \bar{f}]$ and $\hat{T}_{\mu\nu}[V\pm, V\pm]$:

- vanish for massless states;
- are proportional to $k_i k_j$, in the spatial sector.

The amplitude for the gravitational scattering $p_1 \bar{p}_1 \rightarrow p_2 \bar{p}_2$ is given by [cf. Eq. (2)]

$$A[p_1 \bar{p}_1 \rightarrow p_2 \bar{p}_2] = -\frac{i}{M_P^2} \hat{T}_{\mu\nu}^\dagger[p_2 \bar{p}_2] G_{\mu\nu\alpha\beta}(q) \hat{T}_{\alpha\beta}[p_1 \bar{p}_1],$$

where $G_{\mu\nu\alpha\beta}(q)$ is the graviton propagator of momentum $q = k + k' = p + p'$. By inserting Eqs. (11) and (11) into Eq. (12), we obtain the following expressions for the polarized scattering amplitudes in the c.m. frame

$$A[f, \bar{f} \rightarrow f', \bar{f}'] = \frac{s}{6} \lambda' \beta_f \beta_f' \sqrt{r_f r_f'} \left[4 d_{0,0}^2(\theta) - d_{0,0}^0(\theta)\right],$$

$$A[f, \bar{f} \rightarrow V\pm, V\pm] = -\frac{s}{3} \lambda \beta_f \sqrt{r_f} r_V \left[4 d_{0,0}^2(\theta) - d_{0,0}^0(\theta)\right],$$

$$A[V\pm, V\pm \rightarrow V'\pm, V'\pm] = \frac{2s}{3} r_V r_{V'} \left[4 d_{0,0}^2(\theta) - d_{0,0}^0(\theta)\right],$$

where

$$d_{0,0}^2(\theta) = \frac{1}{2} \left(3 \cos^2 \theta - 1\right) \quad \text{and} \quad d_{0,0}^0(\theta) = 1$$

are the Wigner functions (as defined in [5]) corresponding to the $J = 2$ and $J = 0$ eigenstates (with $J$ the total angular momentum), and $\theta$ is the scattering angle between the final and initial momenta, $p$ and $k$. Furthermore, we defined $s = s/M_P^2$, $r_f = m_f^2 / s$, and $r_{V(\bar{V})} = m_{V(\bar{V})}^2 / s$. Eq. (13) has been previously derived in [1] for any fermion-helicity combination (see, in particular, Eq. (29) in [1]).

4 Graviton polarization analysis in different frames

In this Section, we will show that, in case the matrix elements of the energy-momentum tensor in the amplitude in Eq. (12) match the expression in Eq. (9), the total angular-momentum decomposition of the propagating graviton depends on the choice of the reference frame. Note that rotational invariance requires that the eigenstates of the $J^2$ Casimir operator belonging to different $SO(3)$ representations do not mix under rotation.

In the c.m. frame, the scattering amplitude in Eq. (12) is proportional to

$$A \sim \sum_{i,j,\ell,m} \hat{T}_{ij}^\dagger(p) \ P_{ij\ell m} \ \hat{T}_{\ell m}(k),$$

where
with
\[ \hat{T}^\dagger_{ij}(p) \sim p_i p_j, \quad \hat{T}_{\ell m}(k) \sim k_\ell k_m, \]
where the indices \((i, j, \ell, m)\) run from 1 to 3, \(k\) and \(p\) are defined as in Eq. (7), and the projector \(P_{ij\ell m}\) is given by
\[ P_{ij\ell m} = \frac{1}{2}(\delta_{im}\delta_{j\ell} + \delta_{i\ell}\delta_{jm} - \delta_{ij}\delta_{\ell m}). \] (17)

The fact that the spatial \(T_{ij}\) matrix elements are proportional to the tensorial product of the corresponding external 3-momenta, \(k_i k_j\) and \(p_i p_j\), makes the effective geometry of the problem planar.

In the following, the relevant angular-momentum representations for the graviton polarization tensor in the 3-dimensional space, \(\epsilon_{ij}^{(J,J_3)}\), are given by the \(J = 2\) and \(J = 0\) representations [4, 6]
\[ \epsilon^{(2,+2)} = \frac{1}{2} \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \epsilon^{(2,-2)} = \frac{1}{2} \begin{pmatrix} 1 & -i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \] (18)
\[ \epsilon^{(2,0)} = -\sqrt{\frac{2}{3}} \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \epsilon^{(0,0)} = -\frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \] (19)
where \(J_3\) is the angular momentum projection along the 3rd axis.

If one chooses a frame where both \(k\) and \(p\) lie in the \((1, 2)\) plane, the only non-vanishing components of the energy momentum tensors are \(\hat{T}_{11}, \hat{T}_{22},\) and \(\hat{T}_{12}\). We can then write explicitly the sum in Eq. (16) as
\[ A \sim \frac{1}{2} \left[ \hat{T}_{11}^\dagger(p) - \hat{T}_{22}^\dagger(p) \right] \left[ \hat{T}_{11}(k) - \hat{T}_{22}(k) \right] + 2 \hat{T}_{12}^\dagger(p) \hat{T}_{12}(k). \] (20)

Hence, terms proportional to the trace \(\left[ \hat{T}_{11}^\dagger(p) + \hat{T}_{22}^\dagger(p) \right] \left[ \hat{T}_{11}(k) + \hat{T}_{22}(k) \right]\) do not contribute. On the other hand, Eq. (20) can be checked to be equivalent to the sum over the virtual graviton polarizations
\[ A \sim \sum_{i,j,\ell,m} \left\{ \hat{T}_{ij}^\dagger(p) \left[ \epsilon_{ij}^{(2,+2)*} \epsilon_{\ell m}^{(2,+2)} + \epsilon_{ij}^{(2,-2)*} \epsilon_{\ell m}^{(2,-2)} \right] \hat{T}_{\ell m}(k) \right\}, \] (21)
where only the \(J = 2, J_3 = \pm 2\) representations of \(SO(3)\) contribute in this case.

If, instead, one puts the initial particle 3-momenta along the 3rd axis, Eq. (16) becomes
\[ A \sim -\frac{1}{2} \hat{T}_{33}^\dagger(p) \left[ \hat{T}_{11}(k) - \hat{T}_{33}(k) \right]. \] (22)
The decomposition of Eq. (22) in terms of the graviton polarization tensors is then

\[ A \sim \sum_{i,j,\ell,m} \left\{ \hat{T}^\dagger_{ij}(p) \left[ \epsilon^{(2,0)*}_{ij} \epsilon^{(2,0)}_{\ell m} - \frac{1}{2} \epsilon^{(0,0)*}_{ij} \epsilon^{(0,0)}_{\ell m} \right] \hat{T}_{\ell m}(k) \right\} , \quad (23) \]

Note that, in Eq. (20), one is projecting \( J \) along the direction orthogonal to the scattering plane, while, in Eq. (22), the \( J_3 \) direction in the one of the incoming particle momenta.

One can see that the presence of the \( J = 0 \) representation in the effective graviton propagator depends on the choice of the reference frame. This, of course, breaks the rotational invariance, but is not in contrast with the Noether’s theorem due to the presence of a negative-norm state. This phenomenon is connected in a straightforward way to the vanishing of the effective propagator trace in two dimensions, discussed in Section 2. Indeed, the second term in Eq. (23) has a negative sign, arising from a ghost field.

In Appendix A, for comparison, we discuss the corresponding \( s \)-channel amplitude angular-momentum decomposition for the photon propagator. In the latter case, no ghost field is present, and rotational invariance holds.

5 Concluding remarks

We have explicitly shown that, in the \( s \)-channel amplitudes of massive fermion or vector-boson pairs with equal helicities mediated by a graviton, when projecting the graviton angular momentum along the direction orthogonal to the scattering plane in the c.m. frame, only the graviton polarizations corresponding to the \( J_3 = \pm 2 \) projections are exchanged [cf. Eq. (21)]. This angular momentum decomposition also holds in any frame that is obtained from the latter by a Lorentz boost orthogonal to the scattering plane. This decomposition definitely contrasts with the presence of a \( J = 0 \) component both in Eq. (23) and in the partial-wave decomposition of the amplitudes in Eqs. (13)–(15).

On the other hand, in the Wigner functions entering Eqs. (13)–(15), the angular momentum is projected along the collision axis. Then, the presence of a \( J = 0 \) component in the partial-wave decomposition seems to match the Eq. (23) content. Nevertheless, in the partial-wave decomposition in Eqs. (13)–(15), the presence of a \( J = 0 \) representation can not be rotated away to recover Eq. (21). Indeed, the non-vanishing interference of the graviton amplitude with the scalar exchange amplitude computed in [1] [that is directly connected to the projection of the graviton amplitudes in Eqs. (13)–(15) on the \( J = 0 \) representation] is a Lorentz-invariant quantity. The
latter in principle could give measurable effects in the Standard Model, where the scalar field is given by the Higgs boson \cite{1}.

Note that the decomposition in Eq. (21) can be viewed as the result of the cancellation in the “active” two-dimensional (1,2) space between the $J = 0$ (ghost) component $\epsilon^{(0,0)}$ [arising in the graviton propagator in Eq. (5)] and the $J = 2, J_3 = 0$ polarization state corresponding to the $SO(3)$ representation $\epsilon^{(2,0)}$ [cf. Eq. (19)].

In conclusion, the Noether’s theorem and the $SO(3,1)$ invariance of the minimal coupling of gravity in perturbation theory would imply the angular-momentum conservation in the off-shell graviton coupling to matter fields \cite{8}, in case off-shell (gauge-invariant) ghost fields were absent. We found instead that the virtual graviton exchanged in the $s$-channel scattering of on-shell massive matter fields does not respect the rotational invariance. This effect is due to the presence, in the graviton propagator, of an off-shell (gauge-independent) scalar state with negative norm, that evades the assumptions required by the Noether’s theorem.

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Appendix A

As a comparative example, in this appendix we provide the angular momentum decomposition of the $s$-channel scattering amplitude $A^{J=1}$ mediated by a virtual photon. The corresponding Feynman diagram is the same as in Figure 1, where the intermediate graviton propagator is now replaced by the photon one. The corresponding amplitude is given by

$$A^{J=1} = -i e^2 \hat{J}_\mu^f D^\mu\nu(q) \hat{J}_\nu^i ,$$  \hspace{1cm} (24)

where $e$ is the electromagnetic coupling, $q_\mu$ is the momentum flowing in the photon propagator $D_{\mu\nu}(q)$. The vectors $\hat{J}_\mu^f$ and $\hat{J}_\mu^i$ stand for the on-shell matrix elements of the electromagnetic current $J_\mu$ for the final and initial states, respectively. The expression for $D_{\mu\nu}(q)$ in the $\xi$ covariant gauge is given by

$$D_{\mu\nu}(q) = \frac{i}{q^2 + i\varepsilon} \left( -g_{\mu\nu} + (1 - \xi) \frac{q_\mu q_\nu}{q^2} \right).$$  \hspace{1cm} (25)

Due to the conservation of the electromagnetic current ($q^\mu \hat{J}_\mu^f = q^\mu \hat{J}_\mu^i = 0$), the amplitude in Eq.(24) is gauge invariant, and independent of the $\xi$ gauge-fixing term. Then, it can be simplified as follows

$$A^{J=1} = e^2 q^2 \left( \hat{J}_0^f \hat{J}_0^i + \hat{J}_1^f \hat{J}_1^i + \hat{J}_2^f \hat{J}_2^i + \hat{J}_3^f \hat{J}_3^i \right) .$$  \hspace{1cm} (26)

Let us now consider a virtual photon with momentum $q_\mu$ along the third axis, $q_\mu = (E, 0, 0, k)$. By using the current conservation, implying $E \hat{J}_0^{(f,i)} = k \hat{J}_3^{(f,i)}$, one can write Eq.(26) as follows

$$A^{J=1} = \frac{e^2}{q^2} \left( \hat{J}_1^f \hat{J}_1^i + \hat{J}_2^f \hat{J}_2^i + \frac{q^2}{E^2} \hat{J}_3^f \hat{J}_3^i \right) .$$  \hspace{1cm} (27)

The virtual photon contains both transverse and longitudinal polarizations. The corresponding polarization vectors are given by

$$\varepsilon^\mu_{\lambda=+1} = \frac{1}{\sqrt{2}} (0, -1, -i, 0)$$

$$\varepsilon^\mu_{\lambda=-1} = \frac{1}{\sqrt{2}} (0, 1, -i, 0)$$

$$\varepsilon^\mu_{\lambda=0} = \frac{1}{\sqrt{q^2}} (k, 0, 0, E) ,$$  \hspace{1cm} (28)

where $\lambda$ stands for the photon helicity, or the projection of the angular momentum $J$ along the third axis, $J_3$. Then, by making use of the current conservation, the
expression in Eq. (27) can be rewritten as

\[ A^{J=1} = \frac{e^2}{q^2} \hat{j}_\mu^f \left( \sum_{\lambda=-1}^{+1} \varepsilon^\mu_\lambda \varepsilon^\nu_\lambda \right) \hat{j}_\nu^i, \]  

(29)

where the sum runs over the three \( J = 1 \) polarization vectors. Notice that for a real photon one has \( q^2 = 0 \), and only the \( \hat{j}_1^f \hat{j}_1^i + \hat{j}_2^f \hat{j}_2^i \) contribution survives in Eq. (27), corresponding to the two transverse physical polarizations of the photon. This result shows that only the components of the \( J = 1 \) multiplet contribute in the photon propagator. There is no \( J = 0 \) component.

On the other hand, one can also look at the partial wave decomposition of the photon exchange amplitude, and check that it does not contain any \( J = 0 \) component. As in the graviton case analyzed in this paper, let us consider the case of initial and final fermion pairs with equal helicities \( \lambda (\lambda') \) in the center of mass frame. This helicity combination is the only one that can have a non-vanishing interference with a scalar-exchange amplitude. Then, one has [1]

\[ A^{J=1}\left[ f, \bar{f}_\lambda \rightarrow f', \bar{f}'_{\lambda'} \right] = \lambda \lambda' e^2 \sqrt{r_f r'_{f'}} \beta_f \beta_{f'} d^4_{0,0}(\theta), \]  

(30)

where \( r_{(f,f')} = m^2_{(f,f')}/q^2 \), \( \beta_{(f,f')} = \sqrt{1 - 4m^2_{(f,f')}/q^2} \), \( \lambda(\lambda') = \pm 1 \), and \( d^4_{0,0}(\theta) = \cos \theta \) is the \( J = 1 \) Wigner function as defined in [5]. Hence, a scalar component is present neither in Eq. (29) nor, coherently, in Eq. (30).

There is no problem with rotational invariance in the photon exchange amplitude. This is because, contrary to the graviton case, in QED no negative-norm state is effectively propagating in gauge-invariant amplitudes, and therefore the requirements of the Noether’s theorem regarding the angular momentum conservation are satisfied.
Appendix B

In this Appendix, we provide the Lorentz covariant expressions for the matrix elements of the energy-momentum tensor $T_{\mu\nu}$ between the vacuum and a particle pair, as defined in Eq. (3). We consider two different cases for the external particles: fermions and vector-bosons, with non-vanishing masses [7].

- **Fermions**

\[
\hat{T}_{\mu\nu} [f_\lambda(k), \bar{f}_{\lambda'}(k')] = \frac{1}{4} \left\{ \hat{J}^{\lambda\lambda'}_{\mu} (k - k')_\nu + \hat{J}^{\lambda\lambda'}_{\nu} (k - k')_\mu \right\}
\]

where $\hat{J}^{\lambda\lambda'}_{\mu}$ stands for the matrix element of a $U(1)$ current $\bar{f}_{\lambda'}(k')\gamma_{\mu}f_\lambda(k)$ between the vacuum and a pair of fermion $(f_\lambda)$ and anti-fermion $(\bar{f}_{\lambda'})$ of mass $m_f$ in the initial state, with momenta $k, k'$ (entering the vertex), and helicities $\lambda, \lambda'$, respectively. In the c.m. frame, the vector $\hat{J}^{\lambda\lambda'}_{0} = 0$ due to the current conservation, and, up to a phase, the spatial components are given by

\[
\hat{J}^{\lambda\lambda'}_{1} = \delta_{\lambda,-\lambda'} \sqrt{s} \left( \cos \theta \cos \varphi - i \lambda \sin \varphi \right) - 2\lambda \delta_{\lambda,\lambda'} m_f \sin \theta \cos \varphi
\]

\[
\hat{J}^{\lambda\lambda'}_{2} = \delta_{\lambda,-\lambda'} \sqrt{s} \left( \cos \theta \sin \varphi + i \lambda \cos \varphi \right) - 2\lambda \delta_{\lambda,\lambda'} m_f \sin \theta \sin \varphi
\]

\[
\hat{J}^{\lambda\lambda'}_{3} = \delta_{\lambda,-\lambda'} \sqrt{s} \left( -\sin \theta \sin \varphi \right) - 2\lambda \delta_{\lambda,\lambda'} m_f \cos \theta
\]

where the 3-vector $k = -k'$ is given by $k = \frac{\sqrt{2} \beta_f}{s} \left( \sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta \right)$ and $\beta_f = \sqrt{1 - 4m_f^2/s}$.

- **Real Vector Bosons**

\[
\hat{T}_{\mu\nu} [V_\epsilon(k), V_{\epsilon'}(k')] = \left\{ \left( k \cdot k' + m_V^2 \right) \left( \epsilon_\mu \epsilon'_\nu - \frac{1}{2} \eta_{\mu\nu} (\epsilon \cdot \epsilon') \right) + \frac{1}{2} \eta_{\mu\nu} (k \cdot \epsilon')(k' \cdot \epsilon) - \epsilon_\mu k'_\nu (k' \cdot \epsilon') + \epsilon_\mu k'_\nu (k \cdot \epsilon') \right\} + \{ \mu \leftrightarrow \nu \}
\]

where $\epsilon, \epsilon'$ stand for vector-boson polarization states, and $m_V$ is the vector-boson mass. The momenta $k, k'$ are both entering the vertex.
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