THE COLORED JONES POLYNOMIAL OF A CABLE OF THE FIGURE-EIGHT KNOT

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Abstract. We study the asymptotic behavior of the \( N \)-dimensional colored Jones polynomial of a cable of the figure-eight knot, evaluated at \( \exp(\xi/N) \) for a real number \( \xi \). We show that if \( \xi \) is sufficiently large, the colored Jones polynomial grows exponentially when \( N \) goes to the infinity. Moreover the growth rate is related to the Chern–Simons invariant of the knot exterior associated with an \( SL(2; \mathbb{R}) \) representation.

1. Introduction

For a knot \( K \) in the three-sphere \( S^3 \), let \( J_N(K; q) \) be the \( N \)-dimensional colored Jones polynomial [11, 16]. Here we normalize \( J_N(K; q) \) so that \( J_N(U; q) = 1 \) for the unknot \( U \), and that when \( N = 2 \), it satisfies the following skein relation:

\[
qJ_2(\underset{2}{\,}; q) - q^{-1}J_2(\underset{2}{\,}; q) = \left(q^{1/2} - q^{-1/2}\right)q_2(\underset{2}{\,}; q).
\]

Note that it is different from that of Jones’ original paper [11].

R. Kashaev [13] proposed a conjecture stating that his invariant parametrized by an integer \( N \geq 2 \) introduced in [12] grows exponentially with growth rate proportional to the hyperbolic volume of the knot complement for any hyperbolic knot. J. Murakami and the first author [26] proved that Kashaev’s invariant coincides with \( J_N(K; \exp(2\pi\sqrt{-1}/N)) \) and generalized Kashaev’s conjecture for general knots.

Conjecture 1.1 (Volume Conjecture). Let \( K \) be a knot. Then we have

\[
\lim_{N \to \infty} \frac{\log|J_N(K; \exp(2\pi\sqrt{-1}/N))|}{N} = \frac{\text{Vol}(S^3 \setminus K)}{2\pi},
\]

where \( \text{Vol} \) is \( v_3 \) times the simplicial volume. Here \( v_3 \) is the volume of the ideal, hyperbolic, regular tetrahedron.

Note that when \( K \) is hyperbolic, that is, \( S^3 \setminus K \) possesses a unique complete hyperbolic structure with finite volume, then \( \text{Vol} \) is just the hyperbolic volume. So the volume conjecture is a generalization of Kashaev’s conjecture.

It is well known that the knot complement \( S^3 \setminus K \) can be decomposed into hyperbolic pieces and Seifert fibered pieces by a system of tori (Jaco–Shalen–Johannson decomposition [8, 9]) and that \( \text{Vol}(S^3 \setminus K) \) is the sum of the hyperbolic volumes of the hyperbolic pieces. Hence if there are no hyperbolic pieces, then \( \text{Vol}(S^3 \setminus K) = 0 \). In fact, Kashaev and O. Tirkkonen [14] proved that when \( K \) is a torus knot, the limit in (1.1) is zero, proving the volume conjecture.

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For hyperbolic knots, T. Ekmolm showed that the volume conjecture is true for the figure-eight knot $4_1$. The conjecture is also proved for $5_2$ by T. Ohtsuki \cite{29}, and for $6_1, 6_2, 6_3$ by Ohtsuki and Y. Yokota \cite{30}.

For non-hyperbolic knots with non-zero volumes, H. Zheng \cite{33} proved the volume conjecture in the case of Whitehead doubles of the torus knot of type $(2, 2b+1)$, and Lê and the second author proved it for cables of the figure-eight knot \cite{18}.

What happens when we replace $2\pi\sqrt{-1}$ in $J_N\left(K; e^{2\pi\sqrt{-1}/N}\right)$ with another complex number $\eta$?

In the case of the figure-eight knot $E$, the following results about the asymptotic behavior of $J_N\left(E; e^{\eta/N}\right) (N \to \infty)$ are known so far.

(i) $|2\pi\sqrt{-1} - \eta|$ is small enough. If $\eta$ is not purely imaginary, then the limit
\[
\lim_{N \to \infty} \frac{\log J_N(E; e^{\eta/N})}{N}
\]
exists and it determines the volume and the Chern–Simons invariant of the three-manifold obtained by generalized Dehn surgery corresponding to $\eta - 2\pi\sqrt{-1}$ \cite{21} (see also \cite{24}).

(ii) $\eta$ is purely imaginary. If $2\pi/|\eta|$ is irrational, its irrationality measure is finite, and $5\pi/3 < |\eta| < 7\pi/3$, then
\[
\lim_{N \to \infty} \log \left| \frac{J_N\left(E; e^{\eta/N}\right)}{N} \right| = \frac{\text{Vol}(E_{\eta - 2\pi\sqrt{-1}})}{\eta},
\]
where $E_\theta$ is the cone-manifold along $E$ with cone-angle $\theta$ \cite{21} Theorem 1.2 (see \cite{25} for a correction).

(iii) $\eta$ is real. If $\eta > 2\kappa$ with $\kappa := \arccosh(3/2)/2 = \log((3 + \sqrt{5})/2)/2$, then
\[
\lim_{N \to \infty} \log \left| \frac{J_N\left(E; e^{\eta/N}\right)}{N} \right| = \frac{1}{\eta} S(\eta/2),
\]
where
\[
S(\xi) := \text{Li}_2\left(e^{\varphi(\xi) - 2\xi}\right) - \text{Li}_2\left(e^{\varphi(\xi) - 2\xi}\right) + 2\xi \varphi(\xi)
\]
with $\varphi(\xi) := \arccosh(\cosh(2\xi) - 1/2)$ and $\text{Li}_2(z) := -\int_0^z \log(1-x)/x \, dx$. See \cite{21} Theorem 8.1 and \cite{23} Theorem 6.9.

(iv) $\eta = 2\kappa$. In this case $J_N\left(E; e^{\eta/N}\right)$ grows polynomially with respect to $N$. Moreover we have
\[
J_N\left(E; e^{2\kappa/N}\right) \sim \frac{\Gamma(1/3)}{(6\kappa)^{2/3}} N^{2/3},
\]
where $\Gamma(x)$ is the gamma function, and $f(N) \sim g(N)$ means $\lim_{N \to \infty} f(N)/g(N) = 1$ (\cite{42} Theorem 1.1).

(v) $|\eta|$ is small enough. Then $\lim_{N \to \infty} \log |J_N(E; e^{\eta/N})| = 0$. Moreover we have
\[
\lim_{N \to \infty} J_N\left(E; e^{\eta/N}\right) = \frac{1}{\Delta(E; e^{\eta})},
\]
where $\Delta(K; t)$ is the Alexander polynomial of a knot $K$ normalized so that $\Delta(K; 1) = 1$ and $\Delta(K; t^{-1}) = \Delta(K; t)$ \cite{22}. This is also true for any knot \cite{42} Theorem 1.3.

See \cite{21} for more results.

In this paper, we study the colored Jones polynomial of the $(2, 2b+1)$-cable of the figure-eight knot (Figure 11), which we denote by $E_{(2, 2b+1)}$. 
We put \( \kappa := \arccosh(3/2)/2 \) as before. We also put \( \mathcal{F} := S^3 \setminus \text{Int} N(E^{(2,2b+1)}) \), where \( N(E^{(2,2b+1)}) \) is a regular neighborhood of \( E^{(2,2b+1)} \) in \( S^3 \) and Int denotes the interior. We prove

**Theorem (Theorem 5.3).** Let \( \xi \) be a real number with \( \xi > \kappa \), then we have

\[
\xi \lim_{N \to \infty} \frac{\log J_N (E^{(2,2b+1)}; e^{\xi/N})}{N} = S(\xi),
\]

where we put

\[
S(\xi) := \text{Li}_2 \left( e^{-\varphi(\xi)/2} - e^{\varphi(\xi)/2} \right) - 2\xi \varphi(\xi)
\]

with

\[
\varphi(\xi) := \arccosh \left( \cosh(\xi) - \frac{1}{2} \right).
\]

Moreover, \( S(\xi) \) defines the Chern–Simons invariant \( \text{CS}_F([\rho_\xi]; \xi, v(\xi)) \) of the knot exterior \( F \) associated with a representation \( \rho_\xi : \pi_1(F) \to \text{SL}(2; \mathbb{C}) \) and \( (\xi, v(\xi)) \) (see Section 6 for the definition) in the following way:

\[
\text{CS}_F([\rho_\xi]; \xi, v(\xi)) = S(\xi) - \frac{\xi v(\xi)}{4}.
\]

Here \( \rho_\xi \) sends the meridian of \( \partial F \) to \( \begin{pmatrix} e^{\xi/2} & 0 \\ 0 & e^{-\xi/2} \end{pmatrix} \) and the longitude to \( \begin{pmatrix} e^{\varphi(\xi)/2} & 0 \\ 0 & e^{-\varphi(\xi)/2} \end{pmatrix} \) up to conjugation.

Compare this with the following, which was proved in [21, Theorem 8.1] and [23, Theorem 6.9] (see (1.2)). We put \( E = S^3 \setminus \text{Int} N(E) \).

**Theorem (Theorem 5.4).** Let \( \eta \) be a real number with \( \eta > 2\kappa \), then we have

\[
\eta \lim_{N \to \infty} \frac{J_N (E^{(2,2b+1)}; e^{\eta/N})}{N} = S(\eta/2).
\]

Moreover we have

\[
\text{CS}_E([\sigma_\eta]; \eta, v_E(\eta)) = S(\eta/2) - \frac{\eta v_E(\eta)}{4},
\]

where \( \sigma_\eta \) sends the meridian of \( \partial E \) to \( \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix} \) and the longitude to \( \begin{pmatrix} e^{\varphi(\eta)/2} & 0 \\ 0 & e^{-\varphi(\eta)/2} \end{pmatrix} \) up to conjugation.
2. Preliminaries

In this section, we use linear skein theory (see for example [19, Chapters 13, 14]) to calculate the colored Jones polynomial. For another way to calculate it, see [18].

We denote by \(\begin{array}{c}
\vline \\
\begin{array}{c}
E \\

(1,1)
\end{array}
\end{array}\) the \((1,1)\)-tangle and by \(\begin{array}{c}
\vline \\
\begin{array}{c}
E \\

(2,2)
\end{array}
\end{array}\) the \((2,2)\)-tangle that is the two-parallel of \(\begin{array}{c}
\vline \\
\begin{array}{c}
E \\

(1,1)
\end{array}
\end{array}\).

The \(N\)-dimensional colored Jones polynomial of \(E^{(2,2b+1)}\) is given by

\[
\frac{1}{\Delta_{N-1}} \left( (-1)^{N-1} A^{(N-1)^2 + 2(N-1)} \right)^{-(2b+1)} \, \langle \begin{array}{c}
\vline \\
\begin{array}{c}
E \\

(2,2b+1)
\end{array}
\end{array} \rangle, 
\]

where the number \(N - 1\) beside a line indicates that we put the \((N - 1)\)-th Jones–Wenzl indempotent \([10, 32]\) along the line, \(\langle D \rangle\) is the Kauffman bracket \([15]\) of a link diagram \(D\), and \(\Delta_k := (-1)^k \frac{A^{2k+1} - A^{-2k+1}}{A - A^{-1}}\). Note that the writhe of \(\begin{array}{c}
\vline \\
\begin{array}{c}
E \\

(2b+1)\text{ half-twists}
\end{array}
\end{array}\) is \((2b + 1)\) and so we need to multiply by

\[
\left( (-1)^{N-1} A^{(N-1)^2 + 2(N-1)} \right)^{-(2b+1)}
\]

to obtain a unframed knot invariant (see [19 Lemma 14.1]). We use the notation described in [19 Chapters 13, 14].
By using identities in [19 Chapter 14], we have

\[ (2.1) \]

\[
\begin{array}{c}
\sum_{c: \text{ even}, 0 \leq c \leq 2(N-1)} (-1)^{N-1-c/2} A^{(2b+1)(-2N+2+c-(N-1)^2+c^2/2)} \frac{\Delta_c}{\theta(N-1, N-1, c)} \langle \begin{array}{c}
E^2 \bigg\{ \begin{array}{c}
\bullet \\
\times \\
\bullet \\
\bullet \\
\times \\
\bullet \\
\bullet \\
N-1 \\
2b+1 \text{ half-twists}
\end{array}
\end{array} \rangle
\end{array}
\]

\[ = \]

\[ (\text{Reidemeister moves II and III}) \]

\[
\sum_{c} (-1)^{N-1-c/2} A^{(2b+1)(-N^2+1+c+c^2/2)} \frac{\Delta_c}{\theta(N-1, N-1, c)} \langle \begin{array}{c}
E^2 \\
N-1 \\
\{ \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet
\end{array}
\end{array} \rangle
\]

Now \( \langle E \rangle = \left( J_{c+1}(E; t) \bigg|_{t=A^4} \right) \Delta_{c+1} \). So we have the following proposition.
Proposition 2.1. We have

\[
J_N \left( E_{(2,2b+1)}; t \right) = \frac{(-1)^{N-1} t^{N/2}}{t_1/2 - t^{-1/2}} \sum_{d=0}^{N-1} \left( -1 \right)^d \frac{t^{(2d+1)/2} - t^{-(2d+1)/2}}{t^{1/2} - t^{-1/2}} \sum_{k=0}^{2d} \left( \frac{t^{(2d+1+k)/2} - t^{-(2d+1+k)/2}}{(2d+1+k)/2 - t^{-(2d+1+k)/2}} \right). 
\]

Proof. From (2.1), we have

\[
(\xi) \left( t^{(2d+1)/2} - t^{-(2d+1)/2} \right) \left( t^{(2d+1)/2} - t^{-(2d+1)/2} \right) = (-1)^{N-1} A(N-1)^2 + 2(N-1) - (2b+1) \]

\[
\times \sum_{0 \leq c \leq 2(N-1)} (-1)^{N-1-c/2} A^{(2b+1)}(-N^{2}+1+c+\xi^{2}/2) \left( J_{c+1}(E; t) \bigg|_{t=A^{c}} \right) \Delta_{c+1} \bigg|_{A:=t^{1/4}} = \sum_{0 \leq c \leq 2(N-1)} (-1)^{N-1-c/2} A^{(2b+1)}(-N^{2}+1+c+\xi^{2}/2) \left( J_{c+1}(E; t) \bigg|_{t=A^{c}} \right) \Delta_{c+1} \bigg|_{A:=t^{1/4}}
\]

\[
(\xi) = 2d \sum_{d=0}^{N-1} \frac{t^{(2d+1)/2} - t^{-(2d+1)/2}}{t^{1/2} - t^{-1/2}} \sum_{k=0}^{2d} \left( \frac{t^{(2d+1+k)/2} - t^{-(2d+1+k)/2}}{(2d+1+k)/2 - t^{-(2d+1+k)/2}} \right) = (-1)^{N-1} A(N-1)^2 + 2(N-1) - (2b+1) \]

Now using the following formula of the \( m \)-dimensional colored Jones polynomial of the figure-eight knot by K. Habiro and T. Lê \[6] [20]

\[
J_m(E; t) = \sum_{l=0}^{m-1} \prod_{k=1}^{l} \left( t^{(m+k)/2} - t^{-(m+k)/2} \right) \left( t^{(m-k)/2} - t^{-(m-k)/2} \right),
\]

we obtain the required formula. \qed

3. Limit

Fix a positive real number \( \xi \). Then from proposition 2.1 we have

\[
J_N \left( E_{(2,2b+1)}; e^{\xi/N} \right) = \frac{(-1)^{N-1} \exp \left( -\frac{(2b+1)(N-1)\xi}{2N} \right)}{2 \sinh(\xi/2)} \sum_{d=0}^{N-1} \sum_{l=0}^{2d} (-1)^d f_{d,l}
\]

with

\[
f_{d,l} = \exp \left( \frac{(2b+1)(d^2 + d)\xi}{2N} \right) \times 2 \sinh \left( \frac{(2d+1)\xi}{2N} \right) \times \prod_{k=1}^{l} 4 \sinh \left( \frac{(2d+1 + k)\xi}{2N} \right) \sinh \left( \frac{(2d+1 - k)\xi}{2N} \right).
\]

Lemma 3.1. Let \( l \) and \( d \) be integers with \( 1 \leq d \leq N - 1 \) and \( 0 \leq l \leq 2d - 2 \), we have \( f_{d,l} > f_{d-1,l} \).
Proof. We first note that $f_{d,l}$ is positive. We have

$$ \frac{f_{d,l}}{f_{d-1,l}} = \exp\left(\frac{(2b+1)d\xi}{N}\right) \frac{\sinh\left(\frac{(2d+1)\xi}{2N}\right)}{\sinh\left(\frac{(2d-1)\xi}{2N}\right)} \prod_{k=1}^{l} \frac{\sinh\left(\frac{(2d+1-k)\xi}{2N}\right)}{\sinh\left(\frac{(2d-1-k)\xi}{2N}\right)}. $$

Since $\xi > 0$, we have

$$ \exp\left(\frac{(2b+1)d\xi}{N}\right) > 1, $$
$$ \frac{\sinh\left(\frac{(2d+1)\xi}{2N}\right)}{\sinh\left(\frac{(2d-1)\xi}{2N}\right)} > 1, $$
$$ \frac{\sinh\left(\frac{(2d+1+k)\xi}{2N}\right)}{\sinh\left(\frac{(2d-1+k)\xi}{2N}\right)} > 1, $$
$$ \frac{\sinh\left(\frac{(2d+1-k)\xi}{2N}\right)}{\sinh\left(\frac{(2d-1-k)\xi}{2N}\right)} > 1. $$

Therefore we have the required inequality.

□

Corollary 3.2. For any $l$ and $d$ with $0 \leq d \leq N - 2$ and $0 \leq l \leq 2d$, we have $f_{d,l} < f_{N-1,l}$. 

Define

$$ S := (-1)^{N-1} \sum_{d=0}^{N-1} \sum_{l=0}^{2d} (-1)^d f_{d,l} $$

so that

$$ J_N \left( E^{(2,2b+1)} ; e^{\xi/N} \right) = \frac{\exp\left(\frac{(2b+1)(N^2-1)\xi}{2N}\right)}{2 \sinh(\xi/2)} S. $$

Then we have the following lemma.

Lemma 3.3. The following inequality holds.

$$ S > (1 - e^{-\xi/2}) \sum_{l=0}^{2N-2} f_{N-1,l}. $$

Proof. From (3.1) and the subsequent inequalities we have

$$ \frac{f_{N-1,l}}{f_{N-2,l}} > e^{(2b+1)(N-1)\xi/N} \geq e^{(N-1)\xi/N} \geq e^{\xi/2} $$

since $b \geq 0$ and $N \geq 2$.

Suppose that $N$ is even. We have

$$ S = \sum_{d=0}^{N-1} \sum_{l=0}^{2d} (-1)^d f_{d,l} $$
$$ = \sum_{k=0}^{(N-2)/2} \left( -\sum_{l=0}^{4k} f_{2k,l} + \sum_{l=0}^{4k+2} f_{2k+1,l} \right) $$
$$ + \sum_{k=0}^{(N-2)/2} \left( \sum_{l=0}^{4k} (f_{2k+1,l} - f_{2k,l}) + f_{2k+1,4k+1} + f_{2k+1,4k+2} \right) $$
Since \( f_{2k+1,l} - f_{2k,l} > 0, f_{2k+1,4k+1} > 0, \) and \( f_{2k+1,4k+2} > 0, \) we have

\[
S > \sum_{l=0}^{2N-4} (f_{N-1,l} - f_{N-2,l}) + f_{N-1,2N-3} + f_{N-1,2N-2} > (1 - e^{-\xi/2}) \sum_{l=0}^{2N-2} f_{N-1,l}
\]

from (52).

Suppose that \( N \) is odd. We have

\[
S = \sum_{d=0}^{N-1} \sum_{l=0}^{2d} (-1)^d f_{d,l}
\]

\[
= f_{0,0} + \sum_{k=0}^{(N-3)/2} \left( - \sum_{l=0}^{4k+2} f_{2k+1,l} + \sum_{l=0}^{4k+4} f_{2k+2,l} \right)
\]

\[
= f_{0,0} + \sum_{k=0}^{(N-3)/2} \left( \sum_{l=0}^{4k+2} (f_{2k+2,l} - f_{2k+1,l}) + f_{2k+2,4k+3} + f_{2k+2,4k+4} \right).
\]

As in the case where \( N \) is even, we have

\[
S > \sum_{l=0}^{2N-4} (f_{N-1,l} - f_{N-2,l}) + f_{N-1,2N-3} + f_{N-1,2N-2} > (1 - e^{-\xi/2}) \sum_{l=0}^{2N-2} f_{N-1,l}.
\]

The proof is complete. \( \square \)

Now we look for the maximum of \( \{ f_{N-1,l} \mid l = 0, 1, 2, \ldots, 2N-2 \}. \) We have

**Lemma 3.4.** Assume that \( 1 \leq l \leq 2N-2 \) and let \( \delta \) be a positive real number.

(i) If \( \cosh \left( \frac{\xi}{N} \right) \geq \cosh(2\xi) - \frac{1}{2} \), then \( f_{N-1,l-1} < f_{N-1,l}. \)

(ii) If \( \cosh \left( \frac{\xi}{N} \right) < \cosh(2\xi) - \frac{1}{2} - \delta \), then there exists \( N_0 \) such that \( f_{N-1,l-1} < f_{N-1,l} \) for \( N > N_0. \)

**Proof.** First of all, we have

\[
f_{N-1,l-1} = 4 \sinh \left( \frac{(2N-1 + l)\xi}{2N} \right) \sinh \left( \frac{(2N-1-l)\xi}{2N} \right)
\]

\[
= 2 \cosh \left( \frac{2\xi}{N} \right) - 2 \cosh \left( \frac{l \xi}{N} \right).
\]

(i) If \( \cosh \left( \frac{\xi}{N} \right) \geq \cosh(2\xi) - \frac{1}{2} \), it follows that

\[
\frac{f_{N-1,l-1}}{f_{N-1,l-1}} \leq 2 \cosh \left( \frac{2\xi}{N} \right) - 2 \cosh(2\xi) + 1 < 1.
\]

(ii) If \( \cosh \left( \frac{\xi}{N} \right) < \cosh(2\xi) - \frac{1}{2} - \delta \), then it follows that

\[
\frac{f_{N-1,l-1}}{f_{N-1,l-1}} > 2 \cosh \left( \frac{2\xi}{N} \right) - 2 \cosh(2\xi) + 1 + 2\delta.
\]

Therefore if we choose \( N_0 \) so that

\[
\cosh(2\xi) - \cosh \left( \frac{2\xi}{N_0} \right) < \delta,
\]

the inequality \( f_{N-1,l-1} < f_{N-1,l} \) holds for \( N > N_0. \)

The proof is complete. \( \square \)
Now we define
\[ \kappa := \frac{1}{2} \arccosh \left( \frac{3}{2} \right) = \frac{1}{2} \log \left( \frac{3 + \sqrt{5}}{2} \right), \]
\[ \varphi(\xi) := \arccosh \left( \cosh(2\xi) - \frac{1}{2} \right) \]
\[ = \log \left( \frac{1}{2} \left( 2 \cosh(2\xi) - 1 + \sqrt{(2 \cosh(2\xi) + 1)(2 \cosh(2\xi) - 3)} \right) \right). \]

Remark 3.5. If \( \xi > \kappa \), then \( \varphi(\xi) \) is real with \( 0 < \varphi(\xi) < 2\xi \) because \( \cosh(\varphi(\xi)) = \cosh(2\xi) - \frac{1}{2} \), which is between 1 and \( \cosh(2\xi) \).

From Lemma [3.4] we have

**Proposition 3.6.** If \( \xi \leq \kappa \), then the maximum of \( \{ f_{N-1,l} \mid l = 0, 1, \ldots, 2N - 2 \} \) is \( f_{N-1,0} \). Moreover from Corollary 3.3, \( f_{N-1,0} \) is indeed the maximum of \( \{ f_{d,l} \mid 0 \leq l \leq 2d, 0 \leq d \leq N - 1 \} \).

If \( \xi > \kappa \), the maximum of \( \{ f_{N-1,l} \mid l = 0, 1, \ldots, 2N - 2 \} \) is \( f_{N-1,\lceil \varphi(\xi)N/\xi \rceil - 1} \) for sufficiently large \( N \), where \( \lfloor x \rfloor \) is the greatest integer less than or equal to \( x \). Moreover from Corollary 3.3, \( f_{N-1,\lceil \varphi(\xi)N/\xi \rceil - 1} \) is indeed the maximum of \( \{ f_{d,l} \mid 0 \leq l \leq 2d, 0 \leq d \leq N - 1 \} \).

**Proof.** If \( \xi \leq \kappa \), that is, \( \cosh(2\xi) - \frac{1}{2} \leq 1 \), then any \( l \) satisfies the assumption of (i) in Lemma [3.3]. So \( f_{N-1,l} \) is decreasing with respect to \( l \) (\( 0 \leq l \leq 2N - 2 \)) and the maximum is \( f_{N-1,0} \).

If \( \xi > \kappa \), that is, \( \cosh(2\xi) - \frac{1}{2} > 1 \), then we choose \( \delta > 0 \) such that \( \cosh(2\xi) - \frac{1}{2} - \delta > 1 \). Now if \( N \) is sufficiently large, there exists \( l \) such that \( \cosh(lN/\xi) < \cosh(2\xi) - \frac{1}{2} - \delta \). Since \( \cosh(\varphi(\xi)) = \cosh \left( \frac{\varphi(\xi)N}{\xi} \times \frac{\xi}{N} \right) = \cosh(2\xi) - \frac{1}{2} \), we see that
\[ \cosh \left( \left\lfloor \frac{\varphi(\xi)N}{\xi} \right\rfloor \times \frac{\xi}{N} \right) \leq \cosh(2\xi) - \frac{1}{2} < \cosh \left( \left( \left\lfloor \frac{\varphi(\xi)N}{\xi} \right\rfloor + 1 \right) \times \frac{\xi}{N} \right). \]

This means that \( \left\lfloor \frac{\varphi(\xi)N}{\xi} \right\rfloor \) is the maximum of all integers \( l \) satisfying \( \cosh(lN/\xi) \leq \cosh(2\xi) - \frac{1}{2} \). So if \( l < \left\lfloor \frac{\varphi(\xi)N}{\xi} \right\rfloor \), then we can choose \( \delta' > 0 \) such that \( \cosh \left( \frac{\varphi(\xi)N}{\xi} \right) < \cosh(2\xi) - \frac{1}{2} - \delta' \) and so \( f_{N-1,l+1} < f_{N-1,l} \) from (ii) in Lemma [3.3]. If \( l \geq \left\lfloor \frac{\varphi(\xi)N}{\xi} \right\rfloor \), then \( \cosh \left( \frac{\varphi(\xi)N}{\xi} \right) \geq \cosh(2\xi) - \frac{1}{2} \) and so \( f_{N-1,l} > f_{N-1,l-1} \) from (i) in Lemma [3.3]. Therefore the maximum is \( f_{N-1,\lceil \varphi(\xi)N/\xi \rceil - 1} \). \( \Box \)

From Lemma [3.3] and Proposition 3.6, we have the following inequalities when \( \xi > \kappa \):
\[ (1 - e^{-\xi/2})f_{N-1,\lceil \varphi(\xi)N/\xi \rceil - 1} < S < N^2 \times f_{N-1,\lceil \varphi(\xi)N/\xi \rceil - 1}. \]

Here the second inequality holds since there are \( \sum_{d=0}^{N-1}(2d + 1) = N^2 \) terms in \( S \).

Taking log and dividing by \( N \) we have
\[ \frac{\log(1 - e^{-\xi/2})}{N} + \frac{\log \left( f_{N-1,\lceil \varphi(\xi)N/\xi \rceil - 1} \right)}{N} < \frac{\log S}{N} < \frac{2 \log N}{N} + \frac{\log \left( f_{N-1,\lceil \varphi(\xi)N/\xi \rceil - 1} \right)}{N}. \]
Since the limits \( \lim_{N \to \infty} \frac{\log(1 - e^{-\xi/2})}{N} \) and \( \lim_{N \to \infty} \frac{2\log N}{N} \) vanish, we have from the squeeze theorem

\[
\lim_{N \to \infty} \frac{\log S}{N} = \lim_{N \to \infty} \frac{\log \left( J_{N-1,1} \right)}{N} = \lim_{N \to \infty} \frac{1}{N} \log \left( 2 \sinh \left( \frac{(2N-1)\xi}{2N} \right) \right) + \lim_{N \to \infty} \frac{\log \left( 2 \sinh \left( \frac{(k-1)\xi}{2N} \right) \right)}{N}
\]

\[
+ \lim_{N \to \infty} \frac{1}{N} \log \left( 2 \sinh \left( \frac{(k+1)\xi}{2N} \right) \right) + \lim_{N \to \infty} \frac{\log \left( 2 \sinh \left( \frac{(k+1)\xi}{2N} \right) \right)}{N} = \frac{(2b+1)\xi}{2} + \frac{2}{\xi} \int_{\xi}^{\xi+\varphi(\xi)/2} \log(2 \sinh x) \, dx + \frac{2}{\xi} \int_{\xi-\varphi(\xi)/2}^{\xi} \log(2 \sinh x) \, dx
\]

\[
= \frac{(2b+1)\xi}{2} + \frac{2}{\xi} \int_{\xi}^{\xi+\varphi(\xi)/2} \log(2 \sinh x) \, dx + \frac{2}{\xi} \int_{\xi}^{\xi+\varphi(\xi)/2} \log(1 - e^{-2x}) + x \, dx
\]

(putting \( y := e^{-2x} \))

\[
= \frac{(2b+1)\xi}{2} - \frac{2}{\xi} \int_{e^{-\varphi(\xi)-2\xi}}^{e^{-\varphi(\xi)-2\xi}} \frac{\log(1 - y)}{2y} \, dy + 2\varphi(\xi)
\]

\[
= \frac{(2b+1)\xi}{2} + \frac{1}{\xi} \text{Li}_2(e^{-\varphi(\xi)-2\xi}) - \frac{1}{\xi} \text{Li}_2(e^{\varphi(\xi)-2\xi}) + 2\varphi(\xi).
\]

Since \( J_N \left( E^{(2,2b+1)}; \frac{\xi}{N} \right) = \frac{e^{-(2b+1)(N^2-1)\xi/(2N)}}{2 \sinh(\xi/2)} S \), we finally have

\[
\lim_{N \to \infty} \frac{\log J_N \left( E^{(2,2b+1)}; \frac{\xi}{N} \right)}{N} = \frac{1}{\xi} \text{Li}_2(e^{-\varphi(\xi)-2\xi}) - \frac{1}{\xi} \text{Li}_2(e^{\varphi(\xi)-2\xi}) + 2\varphi(\xi).
\]

4. Representation

In this section, we introduce a representation \( \rho_n \) from \( \pi_1(S^3 \setminus E^{(2,2b+1)}) \) to \( \text{SL}(2; \mathbb{C}) \). In fact, it turns out to be a representation to \( \text{SL}(2; \mathbb{R}) \).

Put \( \mathcal{E} := S^3 \setminus \text{Int} N(E) \) and \( \mathcal{F} := S^3 \setminus \text{Int} N(E^{(2,2b+1)}) \). Let \( L \) be a knot in a solid torus \( D := D^2 \times S^1 \) depicted in Figure 2. Put \( C := D \setminus \text{Int} N(L) \).

**Figure 2. A knot \( L \) in a solid torus**

Then \( \mathcal{F} \) is obtained from \( \mathcal{C} \) and \( \mathcal{E} \) by identifying \( \partial D^2 \times S^1 \subset \partial D \) with \( \partial \mathcal{E} \) so that \( \{ \text{point} \} \times S^1 \) is identified with \( \lambda_{E\mu E} \) and \( \partial D^2 \times \{ \text{point} \} \) is identified with...
\(\mu_E\), where \(\lambda_E \subset \partial E\) and \(\mu_E \subset \partial E\) are the preferred longitude and the meridian of \(E\) (Figure 3).

\[\begin{align*}
\mu_E &\subset \partial E \quad \text{and} \quad \lambda_E \subset \partial E, \\
\end{align*}\]

**Figure 3.** \(\mathcal{F}\) is obtained from \(E\) (left) and a knot \(L\) in the solid torus (right).

We calculate the fundamental group of \(S^3 \setminus E^{(2,2b+1)}\). From now on, we identify \(\pi_1(S^3 \setminus E)\) with \(\pi_1(\mathcal{E})\), \(\pi_1(S^3 \setminus E^{(2,2b+1)})\) with \(\pi_1(\mathcal{F})\), and \(\pi_1(D \setminus L)\) with \(\pi_1(\mathcal{C})\), where we choose basepoints in \(\partial E\) that is identified with \(\partial D\). If we choose Wirtinger generators as in Figure 4, then the fundamental group \(\pi_1(\mathcal{E})\) is given as

\[\langle x, y \mid xy^{-1}x^{-1}yx = yxy^{-1}x^{-1}y \rangle.\]

**Figure 4.** Figure-eight knot \(E\)

Since \(\mathcal{C}\) is homeomorphic to the exterior of the \((2,4)\)-torus link (Figure 5), the fundamental group \(\pi_1(\mathcal{C})\) is given as

\[\pi_1(\mathcal{C}) = \langle p, r \mid prpr = rprp \rangle,\]

if we choose generators as in Figure 5.

Observe that the meridian of the solid torus \(D\) is given as \(prpr^{-1}\) (as you can read off by going along the inner circle of the \((2,4)\)-torus knot in Figure 5) and that the longitude \(\lambda_E\) of the figure-eight knot is given as \(x y^{-1}x y^{-2}y x y^{-1}x^{-1}\). By van Kampen’s theorem, the fundamental group of \(S^3 \setminus E^{(2,2b+1)}\) is given as follows:

\[\pi_1(\mathcal{F}) = \langle x, y \mid xy^{-1}x^{-1}yx = yxy^{-1}x^{-1}y \rangle \bigcup_{r=\lambda_E, \mu_E} \langle p, r \mid prpr = rprp \rangle\]

\[= \langle x, y, p, r \mid x y^{-1}x^{-1}yx = yxy^{-1}x^{-1}y, prpr = rprp, x = prpr^{-1}, \lambda_E = rx^{-b} \rangle,\]

where \(\lambda_E \subset \partial E\) and \(\mu_E \subset \partial E\) are the preferred longitude and the meridian of \(E\) (Figure 3).
Figure 5. $(2,4)$-torus knot (left) and $C$ (right)

where $\lambda_E := xy^{-1}xyx^{-2}yxy^{-1}x^{-1}$ and we choose $x$ as the meridian $\mu_E$ of the figure-eight knot. Note that we do not need $r$ as a generator.

For a complex number $u$, we define $\rho_u : \pi_1(F) \to \text{SL}(2; \mathbb{C})$ as follows. First define $\rho_u \big|_{\pi_1(E)}$ as

$$
\rho_u(x) := \begin{pmatrix} e^u & 1 \\ 0 & e^{-u} \end{pmatrix}, \\
\rho_u(y) := \begin{pmatrix} e^u & 0 \\ -\delta(u) & e^{-u} \end{pmatrix},
$$

where

$$\delta(u) := \frac{1}{2} \left( e^{2u} + e^{-2u} - 3 + \sqrt{(e^{2u} + e^{-2u} + 1)(e^{2u} + e^{-2u} - 3)} \right).$$

Note that from [31], any non-Abelian representation $\pi_1(E) \to \text{SL}(2; \mathbb{C})$ is given as above. The longitude $\lambda_E$ of $E$ is sent to

$$
\rho_u(\lambda_E) = \begin{pmatrix} \ell(u) & (e^u + e^{-u})\sqrt{(e^{2u} + e^{-2u} + 1)(e^{2u} + e^{-2u} - 3)} \\ 0 & (e^{2u} + e^{-2u} + 1)(e^{2u} + e^{-2u} - 3) \end{pmatrix},
$$

where

$$\ell(u) := \frac{1}{2} \left( e^{4u} - e^{2u} - 2 - e^{-2u} + e^{-4u} + \frac{e^{2u} - e^{-2u}}{2} \sqrt{(e^{2u} + e^{-2u} + 1)(e^{2u} + e^{-2u} - 3)} \right).$$

We extend it to $\pi_1(F)$ so that its restriction to $\pi_1(C)$ is Abelian. We put

$$\rho_u(p) = \begin{pmatrix} e^{u/2} & \frac{1}{2 \cosh(u/2)} \\ 0 & e^{-u/2} \end{pmatrix},$$

so that $\rho_u(p)^2 = \rho_u(x)$. Since

$$\rho_u(x)^b = \begin{pmatrix} e^{bu} & e^{bu-e^{-bu}} \\ 0 & e^{-bu} \end{pmatrix},$$

we have

$$\rho_u(r) = \rho_u(\lambda_E)\rho_u(x)^b$$

$$= \begin{pmatrix} \ell(u)e^{bu} & e^{bu}(e^u + e^{-u})\sqrt{(e^{2u} + e^{-2u} + 1)(e^{2u} + e^{-2u} - 3)} + \ell(u)^{-1}e^{-bu} \\ 0 & \ell(u)^{-1}e^{-bu} \end{pmatrix}.\]

We can see that $\rho_u(p)$ and $\rho_u(r)$ commute and so $\rho_u$ is well-defined.
Remark 4.1. Note that $\delta(\kappa) = 0$ since $e^{2\kappa} = \frac{4+\sqrt{7}}{2}$, which is a zero of $t - 3 + t^{-1}$. Note also that $e^\kappa$ is a zero of the Alexander polynomial of $E^{(2,2b+1)}$ that is given as

$$
\Delta(E^{(2,2b+1)}; t) = (-t^2 + 3 - t^{-2}) \times \frac{t^{(2b+1)/2} + t^{-(2b+1)/2}}{t^{1/2} + t^{-1/2}}.
$$

The preferred longitude $\lambda$ of $E^{(2,2b+1)}$ goes twice along $E$. Since $\rho_u \big|_{\pi_1(G)}$ is Abelian, it is sent to

$$\rho_u(\lambda) = \rho_u(\lambda_E)^2 = \left(\begin{array}{cc} \ell(u)^2 & 0 \\ 0 & \ell(u)^{-2} \end{array}\right).$$

Remark 4.2. The function $\ell(u)$ satisfies the following equation:

$$\ell(u) - (e^{4u} - e^{2u} - 2 - e^{-2u} + e^{-4u}) + \ell(u)^{-1} = 0,$$

and so $\ell(u)^2$ satisfies

$$\ell(u)^2 + \ell(u)^{-2} = (e^{4u} - e^{2u} - 2 - e^{-2u} + e^{-4u})^2 - 2.$$

Compare them with the A-polynomial of the figure-eight knot ([11 Appendix]):

$$1 - (m^4 - m^2 - 2 - m^{-2} + m^{-4}) + \Gamma^{-1}$$

and the A-polynomial of $E^{(2,2b+1)}$ ([23 Example 2.11]) given by

$$\left(1 + m^{2(2b+1)}\right) \left(1 - (m^8 - m^4 - 2 - m^{-4} + m^{-8})^2 - 2 + \Gamma^{-1}\right).$$

Note that the meridian of $E$ ($E^{(2,2b+1)}$, respectively) is given by $x = p^2$ ($p$, respectively).

Remark 4.3. If we use the hyperbolic functions, we have the following expressions for $\ell(u)$.

$$\ell(u) = \cosh(4u) - \cosh(2u) - 1 + \sinh(2u)\sqrt{(2\cosh(2u) + 1)(2\cosh(2u) - 3)}$$

$$= 2\cosh^2(2u) - \cosh(2u) - 2 + \sinh(2u)\sqrt{(2\cosh(2u) + 1)(2\cosh(2u) - 3)}$$

$$= \cosh(4u) - \cosh(2u) - 1 + 2\sinh(2u)\sinh(\varphi(u)).$$

Here the last equality follows since $\cosh(\varphi(u)) = \cosh(2u) - \frac{1}{2}$.

Since $\cosh(2\kappa) = \frac{4}{7}$ by definition, we have the following equality:

$$\ell(\kappa) = 1.$$

For $u$ greater than $\kappa$ we have the following lemma.

**Lemma 4.4.** If $u > \kappa$, then $\ell(u)$ is a real number with $\ell(u) > 1$.

**Proof.** Since $\cosh(2u) > \frac{4}{7}$ when $u > \kappa$, we see that $\ell(u)$ is real. We also see that

$$\ell(u) > 2\cosh^2(2u) - \cosh(2u) - 2 > 1.$$

\[\square\]

Remark 4.5. If $u$ is real and $u > \kappa$, then the representation $\rho_u$ is to $\text{SL}(2;\mathbb{R})$. 
5. Chern–Simons invariant

We will calculate the \( \text{PSL}(2; \mathbb{C}) \) Chern–Simons invariant of \( \mathcal{F} \) associated with the representation \( \rho_u \).

A practical definition of the Chern–Simons invariant is as follows. See [17] for the precise definition. Let \( M \) be a three-manifold with boundary \( \partial M \) a torus.

The \( \text{PSL}(2; \mathbb{C}) \) character variety \( X(\partial M) \) of \( \pi_1(\partial M) \) is the quotient set of \( \text{Hom}(\pi_1(\partial), \text{PSL}(2; \mathbb{C})) \), where two representations are equivalent if they share the same trace. It can be described as follows. Let \( \{\mu, \lambda\} \) be a generator set of \( \pi_1(\partial M) \sim \mathbb{Z}^2 \) that are presented by oriented simple closed curves on \( \partial M \). For a \( \text{PSL}(2; \mathbb{C}) \) representation \( \gamma \), we can assume

\[
\gamma(\mu) = \pm \left( e^{2\pi \sqrt{-1} g(\gamma)} \begin{pmatrix} 1 & 0 \\ 0 & e^{-2\pi \sqrt{-1} h(\gamma)} \end{pmatrix} \right),
\gamma(\lambda) = \pm \left( e^{2\pi \sqrt{-1} h(\gamma)} \begin{pmatrix} 1 & 0 \\ 0 & e^{-2\pi \sqrt{-1} h(\gamma)} \end{pmatrix} \right)
\]

after suitable conjugation. Then the map \( \gamma \mapsto (g(\gamma), h(\gamma)) \) gives a one-to-one correspondence between \( X(\partial M) \) and the quotient space \( \mathbb{C}^2/H \), where the group \( H \) is given as

\[
H := \langle X, Y, b \mid [X, Y] = XbXb = YbYb = b^2 = 1 \rangle,
\]

and it acts on \( \mathbb{C}^2 \) by

\[
X \cdot (x, y) := \left( x + \frac{1}{2}, y \right),
Y \cdot (x, y) := \left( x, y + \frac{1}{2} \right),
\]

\[
b \cdot (x, y) = (-x, -y).
\]

We denote by \( [x, y; z] \in E(\partial M) \) the equivalence class of \((x, y, z)\), that is,

\[
\begin{align*}
[x, y; z] & = \left[ x + \frac{1}{2}, y ; z \times e^{-4\pi \sqrt{-1} } \right], \\
[x, y; z] & = \left[ x, y + \frac{1}{2} ; z \times e^{4\pi \sqrt{-1} } \right], \\
[x, y; z] & = \left[ -x, -y ; z \right].
\end{align*}
\]

Then we can see that \( q: E(\partial M) \to X(\partial M) \) \((q: [x, y; z] \mapsto [x, y])\) is a \( \mathbb{C}^* \)-bundle.

Now the Chern–Simons function \( c_{3,M} \) is a map from the \( \text{PSL}(2, \mathbb{C}) \) character variety \( X(M) \) to \( E(\partial M) \) such that the following diagram commutes:

\[
\begin{array}{ccc}
\varphi_{3,M} & X(M) & \to X(\partial M) \\
\downarrow & & \downarrow \\
\varphi_{3,M} & X(M) & \to X(\partial M)
\end{array}
\]

Here \( r: X(M) \to X(\partial M) \) is the restriction map.
Definition 5.1. If \( cs_M([\gamma]) = \left[ \frac{u}{4\pi\sqrt{-1}}, \frac{v}{4\pi\sqrt{-1}}; z \right] \), we put \( CS_M([\gamma]; u, v) := \frac{1}{2\pi} \log z \mod \pi^2 \mathbb{Z} \) and call it the Chern–Simons invariant of \( M \) associated with \([\gamma]\) and the representative \( \left( \frac{u}{4\pi\sqrt{-1}}, \frac{v}{4\pi\sqrt{-1}} \right) \) of \( X(\partial M) \), where we denote by \([\gamma]\) \( \in X(M) \) the equivalence class of the representation \( \gamma \).

To calculate the Chern–Simons invariant, we use the following theorem of Kirk and Klassen [17].

Theorem 5.2 (Kirk–Klassen). Let \( M \) a three-manifold with boundary \( \partial M \) a torus. Let \( \gamma_t: \pi_1(M) \to \text{PSL}(2; \mathbb{C}) \) be a smooth path of representations such that

\[
\begin{align*}
\gamma_t(\mu_M) &= \left( e^{u(t)/2}^* \begin{array}{cc} 0 & 1 \\ 0 & e^{-u(t)/2} \end{array} \right), \\
\gamma_t(\lambda_M) &= \left( e^{v(t)/2}^* \begin{array}{cc} 0 & 1 \\ 0 & e^{-v(t)/2} \end{array} \right)
\end{align*}
\]

up to conjugation, where \( \mu_M, \lambda_M \in \pi_1(\partial M) \) are the meridian and the longitude of \( \partial M \).

If \( cs_M([\rho_t]) = \left[ \frac{u(t)}{4\pi\sqrt{-1}}, \frac{v(t)}{4\pi\sqrt{-1}}; z(t) \right] \), then we have

\[
\frac{z(1)}{z(0)} = \exp \left( \frac{-1}{2\pi} \int_0^1 \frac{d}{dt} u(t) - v(t) \right).
\]

We calculate the Chern–Simons invariant of \( F \) by using Theorem 5.2. Putting \( u_t := (u - \kappa)t + \kappa \), we define two paths of representations \( \alpha_t \) and \( \beta_t \) \( (0 \leq t \leq 1) \) as follows.

\[
\begin{align*}
\alpha_t: p &\mapsto \left( \begin{array}{cc} e^{tk/2} & 0 \\ 0 & e^{-tk/2} \end{array} \right), & x &\mapsto \left( \begin{array}{cc} e^{tk} & 0 \\ 0 & e^{-tk} \end{array} \right), & y &\mapsto \left( \begin{array}{cc} 0 & 1 \\ 0 & e^{-tk} \end{array} \right), \\
\beta_t: p &\mapsto \left( \begin{array}{cc} e^{ut/2} & 0 \\ 0 & e^{-ut/2} \end{array} \right), & x &\mapsto \left( \begin{array}{cc} e^{ut} & 1 \\ 0 & e^{-ut} \end{array} \right), & y &\mapsto \left( \begin{array}{cc} e^{ut} & 0 \\ 0 & e^{-ut} \end{array} \right).
\end{align*}
\]

Note that \( \alpha_1 \) and \( \beta_0 \) share the same trace and that \( \beta_0 \) is upper-triangular.

We can write

\[
\begin{align*}
CS_F([\alpha_t]) := \left[ \frac{tk}{4\pi\sqrt{-1}}; \frac{0}{4\pi\sqrt{-1}}; z(t) \right], \\
CS_F([\beta_t]) := \left[ \frac{ut}{4\pi\sqrt{-1}}; \frac{4\log\ell(u_t)}{4\pi\sqrt{-1}}; w(t) \right],
\end{align*}
\]

since

\[
\begin{align*}
\alpha_t(\lambda) &= \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), \\
\beta_t(\lambda) &= \left( \begin{array}{cc} \ell(u_t)^2 & * \\ 0 & \ell(u_t)^{-2} \end{array} \right).
\end{align*}
\]
Note that since $\ell(u) > 1$ (Lemma 4.3), $\log \ell(u) > 0$. Therefore we have $z(1) = z(0) = 1$ and
\[
\frac{w(1)}{w(0)} = \exp \left( \frac{\sqrt{-1} \pi}{2} \int_0^1 \left( \frac{4d}{dt} \log \ell(u) - 4 \log \ell(u) \right) dt \right)
\]
\[
= \exp \left( \frac{2\sqrt{-1} \pi}{\kappa} \left( \left[ \log \ell(u) \right]_0^1 - 2(\kappa - \kappa) \int_0^1 \log \ell(u) dt \right) \right)
\]
= \exp \left( \frac{2\sqrt{-1} \pi}{\kappa} \left( u \log \ell(u) - 2 \int_0^u \log \ell(s) ds \right) \right).
\]
Because $\alpha_1$ and $\beta_0$ share the same trace, $cs_F([\alpha_1]) = cs_F([\beta_0])$. Since
\[
cs_F([\alpha_1]) = \left[ \frac{\kappa}{4\pi \sqrt{-1}}, 0; 1 \right],
\]
and
\[
\frac{4\log \ell(u)}{4\pi \sqrt{-1}}, 0; w(0) \right],
\]
we have
\[
\left[ 0; 1 \right] = \left[ \frac{\kappa}{4\pi \sqrt{-1}}, 0; 1 \right] = \left[ \frac{\kappa}{4\pi \sqrt{-1}}, 0; w(0) \right]
\]
since $\ell(\kappa) = 1$. So we have $w(0) = 1$.

Therefore we have
\[
w(1) = \exp \left( \frac{2\sqrt{-1} \pi}{\kappa} \left( u \log \ell(u) - 2 \int_0^u \log \ell(s) ds \right) \right)
\]
and so
\[
CS_F ([\rho_u]; u, v(u)) = 2 \int_0^u \log \ell(s) ds - u \log \ell(u),
\]
if we put $v(u) := 4 \log \ell(u)$.

Now we put
\[
S(u) := \text{Li}_2(e^{-\varphi(u) - 2u}) - \text{Li}_2(e^{\varphi(u) - 2u}) + 2u \varphi(u).
\]
We will show that $CS_F ([\rho_u]; u, v(u)) = S(u) - u \log \ell(u)$, that is,
\[
(5.2) \quad S(u) = 2 \int_0^u \log \ell(s) ds.
\]
Since $e^{\varphi(u)} + e^{-\varphi(u)} = e^{2u} + e^{-2u} - 1 = 2 \cosh(2u) - 1$, we have
\[
\ell(u) = \frac{1}{2} \left( (e^{2u} + e^{-2u})^2 - (e^{2u} + e^{-2u}) - 4 \right) + \sinh(2u) \sqrt{(2 \cosh(2u) + 1)(2 \cosh(2u) - 3)}
\]
\[
= \cosh(4u) - \cosh(2u) - 1 + \sin(2u) \sqrt{(2 \cosh(2u) + 1)(2 \cosh(2u) - 3)}.
\]
Therefore we have
\[
\exp \left( \frac{d}{du} \left( 2 \int_0^u \log \ell(s) ds \right) \right)
\]
\[
= \left( \cosh(4u) - \cosh(2u) - 1 + \sin(2u) \sqrt{(2 \cosh(2u) + 1)(2 \cosh(2u) - 3)} \right)^2.
\]
Since
Therefore we conclude
Here we use the following equalities:
we have
Now we calculate
On the other hand, we have
Since
we have
Now we calculate
Expanding and simplifying, we obtain:
Here we use the following equalities:
Therefore we conclude
where we put $x$ to $\log J_N \left( E^{(2,2b+1)}; e^{\xi/N} \right)$, where 

\[
\lim_{N \to \infty} \frac{\log J_N \left( E^{(2,2b+1)}; e^{\xi/N} \right)}{N} = S(\xi),
\]

Thus, we obtain the following theorem.

**Theorem 5.3.** Let $\xi$ be a positive real number with $\xi > \frac{1}{4} \arccosh \left( \frac{2}{3} \right)$, then we have

\[
\xi \lim_{N \to \infty} \frac{\log J_N \left( E^{(2,2b+1)}; e^{\xi/N} \right)}{N} = S(\xi),
\]

where we put

\[
S(\xi) := \text{Li}_2 \left( e^{-\varphi(\xi) - 2\xi} \right) - \text{Li}_2 \left( e^{\varphi(\xi) - 2\xi} \right) + 2\xi \varphi(\xi)
\]

with

\[
\varphi(\xi) := \arccosh \left( \frac{\cosh(2\xi) - 1}{2} \right).
\]

We have also shown that $S(\xi)$ defines the Chern–Simons invariant $CS_F([\rho_\xi]; \xi, v(\xi))$ of the knot exterior $F = S^3 \setminus \text{Int} \left( E(2,2b+1) \right)$ associated with $\rho_\xi$ and $(\xi, v(\xi))$ defined in Section 4 in the following way:

\[
CS_F([\rho_\xi]; \xi, v(\xi)) = S(\xi) - \frac{\xi v(\xi)}{4}.
\]

Here the meridian $p \in \pi_1(F)$ is sent to \( e^{\xi/2} \begin{pmatrix} 1 & 0 \\ 0 & e^{-\xi/2} \end{pmatrix} \) and the longitude $\lambda \in \pi_1(F)$ is sent to \( e^{-\varphi(\xi)/2} \begin{pmatrix} -e^{-\varphi(\xi)/2} & 0 \\ 0 & -e^{-\varphi(\xi)/2} \end{pmatrix} \).

Compare this with the result about the figure-eight knot. In [21, Theorem 8.1] and [23, Theorem 6.9], the first author proved the following theorem.

**Theorem 5.4 (21).** Let $\eta$ be a real number with $\eta > 2\kappa$, then we have

\[
\eta \lim_{N \to \infty} \frac{J_N \left( E; e^{\eta/N} \right)}{N} = S(\eta/2).
\]

We can express the Chern–Simons invariant of $\mathcal{E}$ in terms of $S(\eta/2)$ as follows.

If we define $\sigma_u := \rho_{u/2} \mid_{\pi_1(\mathcal{E})}$, then

\[
\sigma_u(x) = \begin{pmatrix} e^{u/2} & 0 \\ 0 & e^{-u/2} \end{pmatrix}, \quad \sigma_u(y) = \begin{pmatrix} e^{u/2} & 0 \\ -d(u/2) & e^{-u/2} \end{pmatrix},
\]

where $x$ and $y$ are generators of $\pi_1(\mathcal{E})$ given in [14]. The longitude $\lambda_E$ of $\mathcal{E}$ is sent to

\[
\sigma_u(\lambda_E) = \begin{pmatrix} \ell(u/2) & 0 \\ 0 & \ell(u/2)^{-1} \end{pmatrix}.
\]

By a calculation similar to $E^{(2,2b+1)}$, the Chern–Simons function $c_{\mathcal{E}}$ is given as

\[
c_{\mathcal{E}}(\sigma_u) = \left[ \frac{u}{4\pi \sqrt{1}} - \frac{2 \log \ell(u/2)}{4\pi \sqrt{1}}, \exp \left( \frac{2}{\pi \sqrt{1}} \text{CS}[\sigma_u; u, 2 \log \ell(u/2)] \right) \right]
\]

with

\[
\text{CS}[\sigma_u; u, v_E(u)] = \text{Li}_2(e^{-\varphi(u/2) - u}) - \text{Li}_2(e^{\varphi(u/2) - u}) + w \varphi(u/2) - \frac{1}{4} w v_E(u)
\]

\[
= S(u/2) - \frac{w v_E(u)}{4},
\]
where \( v_E(u) := 2 \log \ell(u/2) \). So we have

\[
CS E([\sigma_\eta]; \eta, v_E(\eta)) = 5(\eta/2) - \frac{\eta v_E(\eta)}{4}.
\]

Since \( v_E(u) = \frac{1}{2} v(u/2) \), we have

\[
CS E([\rho_\xi]; \xi, v(\xi)) = CS E \left( \left[ \rho_\xi \mid_{\pi_1(E)} \right]; 2\xi, 2v_E(2\xi) \right).
\]

References

[1] D. Cooper, M. Culler, C. Gillet, D. D. Long, and P. B. Shalen, Plane curves associated to character varieties of 3-manifolds, Invent. Math. 118 (1994), no. 1, 47–84. MR 1288467 (95g:57029)

[2] T. Dimofte and S. Gukov, Quantum field theory and the volume conjecture, Chern-Simons gauge theory: 20 years after, AMS/IP Stud. Adv. Math., vol. 50, Amer. Math. Soc., Providence, RI, 2011, pp. 19–42. MR 2809445

[3] T. Dimofte, S. Gukov, J. Lenells, and D. Zagier, Exact results for perturbative Chern-Simons theory with complex gauge group, Commun. Number Theory Phys. 3 (2009), no. 2, 363–443. MR 2551896 (2010k:58038)

[4] S. Garoufalidis, T. Kohno, and H. Murakami, Colored Jones polynomials with polynomial growth, Math. Ann. 340 (2008), no. 1, 261–277. MR 2454330

[5] K. Habiro, On the colored Jones polynomials of some simple links, Sūrikaisekikenkyūsho Kōkyūroku (2000), no. 1172, 34–43. MR 1805727

[6] P. Kirk and E. Klassen, On the volume conjecture for cables of knots, Math. Proc. Cambridge Philos. Soc. 121 (1997), no. 3, 269–275. MR 1434238

[7] W. B. R. Lickorish, An introduction to knot theory, Graduate Texts in Mathematics, vol. 175, Springer-Verlag, New York, 1997. MR 1472978

[8] G. Masbaum, Skein-theoretical derivation of some formulas of Habiro, Algebr. Geom. Topol. 3 (2003), 537–556 (electronic). MR 1997328 (2004f:57013)

[9] K. Johannson, Seifert fibered spaces in \( \mathbb{R}^3 \) with complex gauge group, J. Knot Theory Ramifications 19 (2010), no. 12, 1673–1691. MR 2755490

[10] A. N. Kirillov and N. Yu. Reshetikhin, Representations of the algebra \( \mathfrak{sl}(2) \) and the hyperbolic volume of knots, Lett. Math. Phys. 153 (1997), no. 3, 269–275. MR 1434238

[11] T. T. Q. Lê and A. T. Tran, On the volume conjecture for cables of knots, J. Knot Theory Ramifications 19 (2010), no. 12, 1673–1691. MR 2755490

[12] W. B. R. Lickorish, An introduction to knot theory, Graduate Texts in Mathematics, vol. 175, Springer-Verlag, New York, 1997. MR 1472978

[13] G. Masbaum, Skein-theoretical derivation of some formulas of Habiro, Algebr. Geom. Topol. 3 (2003), 537–556 (electronic). MR 1997328 (2004f:57013)

[14] H. Murakami, Some limits of the colored Jones polynomials of the figure-eight knot, Kyungpook Math. J. 44 (2004), no. 3, 369–383. MR 2095421

[15] K. Harada, Chern-Simons invariants of 3-manifolds decomposed along tori and the circle bundle over the representation space of \( \mathbb{T}^2 \), Comm. Math. Phys. 153 (1993), no. 3, 521–557. MR 1218931

[16] T. T. Q. Lê and A. T. Tran, On the volume conjecture for cables of knots, J. Knot Theory Ramifications 19 (2010), no. 12, 1673–1691. MR 2755490

[17] W. B. R. Lickorish, An introduction to knot theory, Graduate Texts in Mathematics, vol. 175, Springer-Verlag, New York, 1997. MR 1472978

[18] G. Masbaum, Skein-theoretical derivation of some formulas of Habiro, Algebr. Geom. Topol. 3 (2003), 537–556 (electronic). MR 1997328 (2004f:57013)

[19] H. Murakami, Some limits of the colored Jones polynomials of the figure-eight knot, J. Knot Theory Ramifications 15 (2006), no. 1, 39–64. MR 2229755

[20] T. T. Q. Lê and A. T. Tran, On the volume conjecture for cables of knots, J. Knot Theory Ramifications 19 (2010), no. 12, 1673–1691. MR 2755490

[21] W. B. R. Lickorish, An introduction to knot theory, Graduate Texts in Mathematics, vol. 175, Springer-Verlag, New York, 1997. MR 1472978

[22] G. Masbaum, Skein-theoretical derivation of some formulas of Habiro, Algebr. Geom. Topol. 3 (2003), 537–556 (electronic). MR 1997328 (2004f:57013)
Various generalizations of the volume conjecture. The interaction of analysis and geometry, Contemp. Math., vol. 424, Amer. Math. Soc., Providence, RI, 2007, pp. 165–186. MR MR2316336

The coloured Jones polynomial, the Chern-Simons invariant, and the Reidemeister torsion of the figure-eight knot, J. Topol. 6 (2013), no. 1, 193–216. MR 3029425

Erratum to ‘Some limits of the colored Jones polynomials of the figure-eight knot’, Kyungpook Math. J. 56 (2016), no. 2, 639–645.

H. Murakami and J. Murakami, The colored Jones polynomials and the simplicial volume of a knot, Acta Math. 186 (2001), no. 1, 85–104. MR 1828373

H. Murakami and Y. Yokota, The colored Jones polynomials of the figure-eight knot and its Dehn surgery spaces, J. Reine Angew. Math. 607 (2007), 47–68. MR 2338120

Y. Ni and X. Zhang, Detection of knots and a cabling formula for $A$-polynomials, Algebr. Geom. Topol. 17 (2017), no. 1, 65–109. MR 3604373

T. Ohtsuki, On the asymptotic expansion of the Kashaev invariant of the $5_2$ knot, Quantum Topol. 7 (2016), no. 4, 669–735. MR 3593566

T. Ohtsuki and Y. Yokota, On the asymptotic expansions of the Kashaev invariant of the knots with 6 crossings, Math. Proc. Cambridge Philos. Soc. 165 (2018), no. 2, 287–339. MR 3834003

R. Riley, Nonabelian representations of 2-bridge knot groups, Quart. J. Math. Oxford Ser. (2) 35 (1984), no. 138, 191–208. MR MR745421 (85i:20043)

H. Wenzl, On sequences of projections, C. R. Math. Rep. Acad. Sci. Canada 9 (1987), no. 1, 5–9. MR 88k:46070

H. Zheng, Proof of the volume conjecture for Whitehead doubles of a family of torus knots, Chin. Ann. Math. Ser. B 28 (2007), no. 4, 375–388. MR MR2348452

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