Fundamental thermodynamical equation
of a self-gravitating system

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Abstract

The features of the fundamental thermodynamical relation (expressing entropy as function of state variables) that arise from the self-gravitating character of a system are analyzed. The models studied include not only a spherically symmetric hot matter shell with constant particle number but also a black hole characterized by a general thermal equation of state. These examples illustrate the formal structure of thermodynamics developed by Callen as applied to a gravitational configuration as well as the phenomenological manner in which Einstein equations largely determine the thermodynamical equations of state. We consider in detail the thermodynamics and quasi-static collapse of a self-gravitating shell. This includes a discussion of intrinsic stability for a one-parameter family of thermal equations of state and the interpretation of the Bekenstein bound. The entropy growth associated with a collapsing sequence of equilibrium states of a shell is computed under different boundary conditions in the quasi-static approximation and compared with black hole entropy. Although explicit expressions involve empirical coefficients, these are constrained by physical conditions of thermodynamical origin. The absence of a Gibbs-Duhem relation and the associated scaling laws for self-gravitating matter systems are presented.

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I. INTRODUCTION

Although we expect the line separating “gravitational” from “material” degrees of freedom of a self-gravitating system to disappear completely in an unified treatment of quantum interactions, it remains a subject of study at the phenomenological level. Its exploration may clarify the features of thermodynamics intrinsic to the gravitational field and in turn provide us with useful physical guidance in the search of a quantum description of spacetime. In this respect, we propose to revisit an old problem in this paper and evaluate the fundamental thermodynamical relation for a single component, simple system consisting of a spherically symmetric self-gravitating shell at finite temperature.

This investigation allows us to address two related issues. The first one concerns the properties of the fundamental equation of a system which arise solely from its self-gravitating character. In other words, we desire to restrict as much as possible the physical form of the entropy function by using thermodynamical arguments based only on the phenomenological characteristics of the gravitational field (with only minimal assumptions about the structure of the matter fields making the system). As we show below, the content of Einstein equations enters into the thermodynamical formalism as the phenomenological gravitational piece of the thermodynamical equations of state. The approach followed in this paper to the fundamental equation of a system seems therefore to agree in spirit (although from a different perspective) with the thermodynamical view of Einstein equations suggested in Ref. [1].

The second issue concerns the amount of entropy that might arise from the quasi-static collapse of a matter system. Under certain conditions, a quasi-static sequence of equilibrium states of a shell can be used to simulate a realistic physical collapse resulting into a black hole. Despite the fact that a spherical shell possesses no “gravitational entropy” [2–5], it is natural to ask how much of the entropy of the final black hole can be traced to the (quasi-static) matter entropy of the shell since not every quasi-static process is reversible. It is therefore reasonable to calculate the maximum amount of entropy that might be produced from
quasi-static collapse processes obeying different types of boundary conditions and compare it with black hole entropy. The “remaining” entropy must necessarily have its origin in the irreversible non-equilibrium late stages of collapse, where a quasi-static approach breaks down. This type of analysis, besides having a clear thermodynamical interpretation, does not require a precise description of complex dynamical processes.

Although some of these questions have been suggested and partially addressed before [4–6], we believe that they have not been satisfactorily answered. Davies, Ford, and Page [4] considered a spherically symmetric black hole surrounded concentrically by a cold thin shell of matter and found no inconsistency in taking the gravitational entropy as only that of the black hole. However, they did not take explicit account of the thermodynamics of the matter distribution: the shell had only the passive effect of depressing the temperature of the black hole. Since its state variables were held fixed, the matter entropy of the shell remained a (negligible) constant. The limitations of this approach are further discussed in Section IV. Hiscock [6] considered a dust shell collapsing into a pre-existing black hole and suggested defining its entropy as (one-fourth of) the difference between the surface areas of the apparent and event horizons. Although in the context of a dynamical collapse, this approach did not involve the thermodynamics of the shell itself and remains an indirect proposal. In Ref. [5] it was shown by using a path integral representation of a canonical partition function that the additivity of actions of gravity and matter implies that the total entropy of a black hole and shell system is the simple sum of the black hole entropy and the ordinary matter entropy. Even if the latter was formally identified, its explicit dependence on the state variables could not be determined in that approach.

We wish to employ the present simple model to stress the formal structure of thermodynamics developed by Callen [7] as applied to a gravitational system. Our strategy in this paper is the following: We obtain the fundamental equation by direct integration of the first law of thermodynamics. This requires finding the equations of state for the system. It turns out that a phenomenological consequence of the self-gravitational character of the system is to fix completely its pressure and partially its thermal (and chemical) equations.
of state. The fundamental equation can then be evaluated essentially up to an arbitrary function of the horizon size $r_+$. To specify further the entropy one naturally requires an empirical or microscopic description of the matter fields forming the shell. This is not a failure of the model or of our analysis but a natural consequence of a thermodynamical treatment. Following a procedure common in thermodynamics, we will adopt the simplest physical choice for the undetermined function in the thermal equation of state. This choice involves empirical coefficients characteristic of the matter fields. Since we desire to illustrate how far one can proceed in an entirely phenomenological approach we appeal to physically reasonable macroscopic conditions. These provide constraints on the fundamental equation of our system by restricting the values of its empirical coefficients. Besides assuming the weakest possible restrictions on the matter fields (namely, that stress-energies obey the ordinary phenomenological energy conditions), we select physical equilibrium states as the ones satisfying the following properties: (1) A particular normalization for entropy, (2) intrinsic local stability, (3) Bekenstein bound on entropy and/or validity of the generalized second law of thermodynamics under quasi-static collapse to a black hole, and (4) the third law of thermodynamics for matter fields. Although these conditions do not have either a fundamental character or form a complete set which maximally restricts the fundamental equation, they are physical conditions of a purely thermodynamical origin which provide insight into the structure of physically acceptable fundamental equations.

The thermodynamical results of this paper apply to any gravitational system whose pressure equation of state as a function of its state variables possesses the particular particular simple form stated below. These systems include not only a self-gravitating shell but also a black hole spatially bounded by a spherically symmetric surface characterized by a thermal equation of state which does not necessarily coincide with Hawking’s one. These slightly generalized equations of state might be relevant in studies of quantum corrections to Hawking’s formula beyond the semiclassical approximation.

The paper is organized as follows. We review briefly in Section II the entropy representation in ordinary thermodynamics and compute the gravitational contribution to the entropic
fundamental equation. The resulting general expression is applied to the particular case of a power law thermal equation of state. Physical requirements imposed on the fundamental equation are presented and discussed in detail in Section III. Quasi-static processes involving equilibrium states of the shell are studied in Section IV for different types of boundary conditions. For a closed system, the quasi-static motion of the shell is reversible. Examples of irreversible processes are presented. These results generalize the results of Ref. [4] to configurations that include explicitly the entropy content of matter. Although the amount of entropy in a quasi-static shell collapse depends on the precise values of the empirical coefficients in the thermal equation of state, we calculate the maximum values of the entropy for a one-parameter family of equations of state and compare them with the entropy of the resulting black hole. Finally, we illustrate in Section V the scaling laws for self-gravitating matter systems and the associated absence of a Gibbs-Duhem relation. These laws are in clear contrast to the ones familiar in ordinary flat-space thermodynamics. Concluding remarks are presented in Section VI. Henceforth we adopt units for which \( c = k_{\text{Boltzmann}} = 1 \), but explicitly display \( G \) and \( \hbar \).

II. FUNDAMENTAL RELATION

The fundamental relation of a thermodynamical system in the so-called “entropy representation” expresses the entropy as the function \( S = S(M, A, N) \).

\[
S = S(M, A, N) .
\] (2.1)

The entropic state variables of the system are its proper local energy (denoted here by \( M \)), its size (denoted by \( A \)), and an arbitrary number of conserved quantities (denoted generically by \( N \)). Once known, the fundamental equation (2.1) contains all the thermodynamical information about the system [7]. In this paper we focus our attention on the entropy representation of equilibrium states (as opposed to the alternative energy representation) because we are interested in finding the entropy as a function of energy and size.
There exist several methods to find (2.1) in ordinary thermodynamics where self-gravitational effects are considered negligible. One possible way is by direct integration of the first law of thermodynamics

\[ T \, dS = dM + p \, dA - \mu \, dN \]  

if one knows the three equations of state

\[ \beta = \beta(M, A, N) \]
\[ p = p(M, A, N) \]
\[ \mu = \mu(M, A, N) \]  

where \( \beta = 1/T \) denotes the temperature function, \( p \) the pressure conjugate to \( A \), and \( \mu \) the chemical potential conjugate to \( N \). In fact, two equations of state are sufficient to determine the fundamental relation up to an undetermined integration constant \[7\]. Clearly, any single equation of state contains less information than the fundamental equation. A second alternative method in standard thermodynamics consists in substituting the three equations of state (2.3) in the so-called Euler relation \[7\]. However, as we will show in Section V, the ordinary form of the Euler relation is not the correct one for a self-gravitating matter system. We will use therefore in this paper the first approach to the fundamental equation and point thereafter the appropriate form of the Euler relation for the system. This approach is technically simple and, most importantly, conceptually transparent. Alternative methods which involve calculations of partition functions or density of states in terms of functional integrals \[5,8,9\] will not be explored here.

What are the thermodynamical state variables and equations of state (2.3) for a self-gravitating shell in both thermal and mechanical equilibrium with itself? To answer this question, consider Israel’s massive thin-shell formalism \[10,11\]. As it is well known, an exterior Schwarzschild solution and an interior flat solution are joined together across an infinitely thin, spherically symmetric matter shell. The position of the shell is denoted by its circumferential radius \( r(\tau) \) which is a function of the shell proper time \( \tau \). We consider in this
paper only equilibrium configurations, namely, static (or effectively static) configurations in which the shell remains at rest for proper time periods much longer than the thermalization period of the material on the shell. The position of the equilibrium configuration is denoted by \( r = R \), and the surface area of the shell by \( A \equiv 4\pi R^2 \).

The junction conditions at the shell require the induced metric to be continuous and the discontinuity in the extrinsic curvature to be proportional to the stress-energy tensor in this hypersurface \([10]\). The latter can be decomposed in terms of a surface energy density \( \sigma \) and a surface pressure \( p \). The proper, locally defined mass \( M \) of the shell is related to the surface energy density \( \sigma \) by \( M \equiv 4\pi R^2 \sigma \). The junction conditions imply that the ADM energy \( m \) is given in terms of the energy \( M \) and position \( R \) as \([10, 4]\)

\[
m(M, R) = M - \frac{GM^2}{2R}.
\]

The ADM energy is the sum of the proper energy \( M \) and the gravitational binding energy associated with building the shell \([12, 4, 5]\). This equation can be rewritten in the familiar form

\[
M = \frac{R}{G}(1 - k),
\]

where it is useful to introduce the notation \( k \equiv (1 - r_+/R)^{1/2} \). The quantity \( r_+(M, R) \equiv 2Gm(M, R) \) denotes the Schwarzschild radius of the shell. The junction conditions also determine the value of the equilibrium pressure. For a shell to be effectively static its tangential pressure must have the precise form

\[
p(M, R) = \frac{GM^2}{16\pi R^3} \left(1 - \frac{GM}{R}\right)^{-1} = \frac{1}{16\pi GRk}(1 - k)^2.
\]

The thermodynamical state variables for the system in the entropy representation are the local energy \( M \), the surface area \( A \equiv 4\pi R^2 \), and the conserved number \( N \). (Because of spherical symmetry, we use \( R \) and \( A \) interchangeably in what follows.) We will assume throughout the equilibrium surface energy density \( \sigma \) and pressure \( p \) to be non-negative. The state variables \( (M, R) \) are therefore non-negative. We will also assume that \( R \geq r_+ \geq \ldots \)
$l_p$, where $l_p = (G\hbar)^{1/2}$ denotes the Planck length. In particular, this implies that the thermodynamic state space is such that $0 \leq k \leq 1$.

Since both $\sigma$ and $p$ are non-negative, the shell matter automatically satisfies the weak energy and time-like convergence conditions [13,14]. It is well known that the dominant energy condition $p \leq \sigma$ further constraints the position of the shell to obey $R \geq 25/24 r_+$, or equivalently, $k \geq 1/5$.

We wish to evaluate the entropy using a minimal set of assumptions about the matter fields making the shell. Consequently, we focus attention in the case when the number of particles $N$ is constant and ignore the form of the chemical potential. We therefore require only two equations of state to determine the fundamental relation up to an additive constant [7]. Observe that Eq. (2.6) does indeed provide the desired pressure equation of state for the shell. We emphasize that this equation is a unique consequence of the gravitational equations across the shell hypersurface and is independent of the nature of the matter fields making the shell.

Consider now the first law of thermodynamics for a hot shell whenever the total number of particles $N$ is constant. By virtue of the pressure equation of state (2.6) and the differential form of the local energy expression (2.5) the first law can be suggestively written as a total differential of the form:

$$T dS = dM + p dA = \frac{1}{2Gk} dr_+ ,$$

(2.7)

where $T \equiv 1/\beta$ denotes the local temperature of the shell at the equilibrium position $r = R$. The result (2.7) is non-trivial despite its familiar form. The shell possesses a local mass $M \neq m$, and a non-zero pressure $p$ which keeps it static at an equilibrium position. The identification of $-p dA$ as mechanical work and $T dS$ as heat transfer refer to quasi-static processes of the shell itself. The particular form (2.7) of the first law is a consequence of the pressure equation of state (2.6), and therefore, of the gravitational junction conditions.

The local temperature $T$ appears in the first law as an integrating factor. The integra-
bility condition for the entropy $S$ provides an equation for $\beta(M,R)$ of the form

\[
\left( \frac{\partial \beta}{\partial A} \right)_M = \left( \frac{\partial \beta p}{\partial M} \right)_A ,
\] (2.8)

with the pressure given by Eq. (2.6). Under a change of variables from $(M, R)$ to $(r_+, R)$ the integrability equation becomes

\[
\left( \frac{\partial \beta}{\partial R} \right)_{r_+} = \frac{\beta}{2 R k^2} (1 - k^2) ,
\] (2.9)

whose general solution is

\[
\beta(M,R) = b(r_+) k ,
\] (2.10)

where $b(r_+)$ is an arbitrary function of the quantity $r_+(M,R)$. The function $b(r_+)$ can therefore be interpreted as the inverse temperature the shell would possess if located at spatial infinity. Equation (2.10) is a consequence of the integrability conditions for entropy and naturally represents the equivalence principle [15] as applied to a self-gravitating system at non-zero temperature. While the integrability condition forces the function $b$ to be dependent on the state variables through the quantity $r_+(M,R)$, it does not determine its precise dependence. This is physically reasonable and expected from other grounds: in a path integral description of the partition function for the system, the flat spacetime geometry in the region inside the shell can be periodically identified with any proper period [5].

Substitution of the inverse temperature (2.10) into the first law (2.7) implies

\[
dS = \frac{1}{2G} b(r_+) dr_+ .
\] (2.11)

The fundamental equation for the system is therefore

\[
S(M,R) = \frac{1}{2G} \int b(r_+) dr_+ + S_0 ,
\] (2.12)

where $S_0$ is an integration constant. Notice that the entropy is a function of the state variables $(M,R)$ only through the quantity $r_+(M,R)$. In general, the quantity $S_0$ is only a function of the number of particles $N$. Since the latter is constant in our model, the quantity $S_0$ is a number.
The entropy expression (2.12) is a consequence of the self-gravitating character of the model and constitutes one of the main results of this section. It follows directly from the gravitational junction conditions (2.4) and (2.6) and the equivalence principle (2.10). The former determine the pressure equation of state whereas the latter determines the redshift factor in the temperature equation of state. (As is expected the equivalence principle also determines the redshift factor in the chemical potential equation of state [5].) Equations (2.10) and (2.12) apply to every self-gravitating shell with $N = \text{const.}$ independently of its matter composition. A concrete form of the function $b(r_+)$ in the fundamental equation has to originate in an explicit model of the matter fields.

The calculation of the entropy (2.12) is clearly reminiscent of the calculation of black hole entropy. This is so because the quasilocal energy $E$ and pressure $s$ characteristic of a Schwarzschild geometry of ADM mass $\tilde{m} = \tilde{r}_+/2G$ enclosed inside a boundary surface of radius $r_0$ are respectively \[ E = \frac{r_0}{G} (1 - \tilde{k}) , \] \[ s = \frac{1}{16\pi G r_0 k} (1 - \tilde{k})^2 , \] where $\tilde{k} \equiv (1 - \tilde{r}_+/r_0)^{1/2}$. Both quasilocal energy (2.13) and pressure (2.14) associated to a black hole are defined in terms of the two-dimensional surface that contains the system, and possess the same functional form as expressions (2.5) and (2.6) for a shell. However, the thermal equation of state for a black hole in thermal equilibrium with a heat bath is given uniquely in the semiclassical approximation by Hawking’s temperature formula \[ \beta_H(r_0) = \frac{4\pi G \tilde{r}_+}{\nu^2} \tilde{k} . \] \[ (2.15) \]

The integration left undone in (2.12) can be carried out explicitly for a black hole, yielding the well-known Bekenstein-Hawking formula \[ S_{BH} = \pi \left( \frac{\tilde{r}_+}{\nu} \right)^2 , \] \[ (2.16) \] where the entropy is normalized to zero for a zero mass black hole.
A. Power law equation of state

Consider now the simplest possible choice for the function \( b(r_+) \) in the thermal equation of state (2.10). This is clearly a power law expression of the form

\[
b(r_+; \eta, a) = \frac{2 G \eta}{l_p^{1+a}} r_+^a ,
\]

where \( \eta \) and \( a \) are two empirical coefficients that characterize the matter fields in the shell. We treat \( a \) as a real parameter. For simplicity and with no loss of generality, we consider \( \eta \) as a dimensionless number and display the units explicitly in the previous equation. In later sections we will pay special attention to the case when \( \eta \) is of order one. This condition simplifies the model and allows to study the contribution of the terms involving \( r_+ \) in order of magnitude estimates of the shell entropy.

By substituting (2.17) in (2.10) we obtain the simplest one-parameter family of thermal equations of state. Positivity and finiteness of temperature imply that \( \eta \) is non-negative. With this choice for the thermal equation of state the entropy becomes

\[
S(M, R; \eta, a) = \frac{\eta}{a + 1} \left( \frac{r_+}{l_p} \right)^{(a+1)} + S_0
\]

for parameter values \( a \neq -1 \), and

\[
S(M, R; \eta) = \eta \ln(r_+) + S_0
\]

in the case \( a = -1 \). In our model \( S_0 \) is itself a numerical constant. Observe that for \( a = 1 \) the thermal equation of state is linear in \( r_+ \). In this case the entropy has the same functional dependence on \( r_+ \) as the Bekenstein-Hawking black hole entropy (2.16). In contrast, the temperature equation of state is independent of the extensive variables \( (M, R) \) in the case \( a = 0 \).

III. PHYSICAL CONSTRAINTS

Thermodynamics alone cannot fix uniquely either the empirical coefficients \( \eta \) and \( S_0 \) or the empirical parameter \( a \). Their precise values must necessarily arise from a description of
the micro-physics of the physical shell. However, the next step in our approach is to investigate conditions which physical equilibrium states of the system must necessarily satisfy. We focus attention on the restrictions imposed phenomenologically by these conditions on the range of values of the empirical coefficients appearing in the fundamental equation (2.18).

A. Normalization of entropy

The entropy can be defined up to an absolute constant. However, it seems physically reasonable to assume that a zero mass shell must possess zero entropy. By Eq. (2.12), this condition restricts the area under the function $b(r_+)$ in the limit of zero mass, namely

$$\int b(r_+) dr_+ + S_0 \to 0 \quad as \quad M \to 0.$$  (3.1)

Consider the power law thermal equation of state (2.17) and the entropy (2.18). Since the quantity $r_+$ also vanishes if the proper mass $M$ vanishes, the above condition is satisfied for $a > -1$ whenever $S_0 = 0$. In contrast, the entropy diverges for $a \leq -1$ as $M$ tends to zero for any finite value of $S_0$. This is clearly not physical. The normalization condition therefore constrains the coefficients to be $a > -1$ and $S_0 = 0$.

B. Intrinsic Stability

We can study the intrinsic stability of the thermodynamic equilibrium states by direct inspection of the fundamental relation. Global stability in the entropy representation requires that the entropy hypersurface lies everywhere below its tangent two-dimensional planes. We focus our attention here on the local intrinsic stability conditions which, although weaker than the concavity of the entropy stated above, insure that the entropy function does not increase due to inhomogeneities of the state variables. (Stronger stability criteria for a shell which may or may not overlap with the one adopted here are briefly discussed in the concluding section.) Since $N$ is assumed constant, we deal only with a three-dimensional
thermodynamic space defined by the variables \((S, M, R)\). In terms of the fundamental equation, local intrinsic stability is guaranteed if the following three inequalities are satisfied simultaneously, namely

\[
\left( \frac{\partial^2 S}{\partial M^2} \right)_A \leq 0, \quad (3.2)
\]

\[
\left( \frac{\partial^2 S}{\partial A^2} \right)_M \leq 0, \quad (3.3)
\]

\[
\left( \frac{\partial^2 S}{\partial M^2} \right) \left( \frac{\partial^2 S}{\partial A^2} \right) - \left( \frac{\partial^2 S}{\partial M \partial A} \right)^2 \geq 0. \quad (3.4)
\]

Conditions (3.2) and (3.3) insure that the intersection of the entropy surface with planes of constant \(M\) or \(A\) have negative curvature, whereas the “fluting” condition (3.4) insures the equivalent under coupled inhomogeneities of \(M\) and \(A\).

Although the criteria (3.2)-(3.4) can be expressed as a set of differential inequalities for the function \(b\), we do not write these explicitly. Instead, consider them as applied to the fundamental equation (2.18)-(2.19). Our goal is to find the stability regions in the thermodynamical state space \((M, R)\) of the shell as a function of the parameter \(a\). Consider first the case \(a \neq -1\). It is easy to see that the condition (3.2) is automatically satisfied by any physical \(k\) (that is, in the range \(0 \leq k \leq 1\)) if \(a \leq 0\), whereas it is satisfied for \(a > 0\) provided

\[
k \leq \sqrt{\frac{1}{2a + 1}}, \quad (3.5)
\]

or equivalently, if \(R \leq (1 + 1/2a)r_+\). Notice that the dominant energy condition requires \(k \geq 1/5\). Therefore, the stability condition (3.2) and the dominant energy condition jointly restrict the value of the parameter \(a\) to be \(a \leq 12\).

Condition (3.3) is satisfied automatically by any \(0 \leq k \leq 1\) if \(a \leq 3\), whereas for \(a \geq 3\) it requires

\[
k \geq 1 - \frac{6}{(a + 3)}, \quad (3.6)
\]

or equivalently, that \(R \geq (a + 3)^2/(12a) r_+\). Observe that for parameter values \(a > 3 + \sqrt{6} \approx 5.45\), the two conditions (3.2) and (3.3) cannot be satisfied simultaneously.
Consider now the “fluting” condition (3.4). It is not difficult to show that it implies the inequality

\[(3a + 1)k^2 + 2ak + (a - 1) \leq 0.\]  

This relation can not be satisfied for parameter values \(a \geq 1\) if \(k\) is in the range \(0 \leq k \leq 1\). (For \(a = 1\) the inequality is marginally satisfied at \(k = 0\), or equivalently, at \(R = r_+\).) The left-hand side of (3.7) is always smaller than \(6a\). This implies that the inequality is automatically satisfied for \(a \leq 0\). The non-trivial range to analyze is therefore \(1 > a > 0\). For values of \(a\) slightly larger than zero, Eq. (3.7) restricts the values of \(k\) to be slightly smaller than one, whereas for values of \(a\) slightly smaller than one, the inequality restricts \(k\) to be slightly larger than zero. For example, if \(a = 1/3\), \(k \leq 0.44\) \((R \leq 1.24 r_+)\), whereas if \(a = 2/3\), \(k \leq 0.175\) \((R \leq 1.03 r_+)\). Interestingly, the inequality (3.7) implies \(k = 1/5\) for the particular parameter value

\[a = \frac{12}{19}.\]  

For values of \(a > (12/19)\), \(k\) is restricted to take values smaller than \(1/5\). This clearly contradicts the dominant energy condition. Therefore a shell may satisfy simultaneously the stability condition (3.4) and the dominant energy condition provided \(a \leq 12/19\). This restriction on the parameter \(a\) is more stringent than the ones implied by conditions (3.2) and (3.3). In fact, condition (3.5) is superseded by (3.7): it can be shown that (3.7) is automatically satisfied if (3.5) is satisfied, whereas the inverse is not true.

For completeness, consider the case \(a = -1\). The stability criteria are automatically satisfied by the entropy formula (2.19) for all values of the state variables satisfying \(R \geq r_+\). In particular, the condition (3.4) implies \(k^2 + k + 1 \geq 0\), which is valid for any physical \(k\).

As mentioned before, one recovers the Bekenstein-Hawking expression for entropy in the limit \(a = 1\) and \(\eta = 2\pi\) (“black hole case”). The previous analysis shows that a system characterized by this fundamental equation is not intrinsically stable. This has been noted in the context of statistical ensembles in Refs. [18].
Table 1 illustrates the stability regions for different values of the parameter $a$. Summarizing, the stability criteria (3.2)-(3.4) do restrict the range of the exponent $a$ to be $-\infty < a < 1$. In the parameter range $-\infty < a \leq 0$ the shell is intrinsically stable for every position $R \geq 25/24 r_+$. In the range $0 < a < 1$ stability restricts the possible values of its position according to Eq. (3.7). Intrinsic local stability together with the normalization (3.1) of the entropy necessarily imply $-1 < a < 1$. Finally, the dominant energy condition further restricts the range to

$$-1 < a \leq \frac{12}{19}. \tag{3.9}$$

Only for these values of the parameter $a$ can an intrinsically stable shell be located at a position $(25/24) r_+ \leq R < \infty$. The smaller the value of $a$, the larger the range of possible radii $R$ for the shell.

We close this subsection with a final remark. The result (3.9) implies that the exponent of $r_+$ in the entropy formula (2.18) for an intrinsically stable shell obeying energy conditions has an upper limit $(a + 1) \leq (31/19) \approx 1.63 < 2$.

C. Second Law and Bekenstein bound

A quasi-static sequence of equilibrium states describing a collapsing shell is described in detail in the following section. Assuming that a black hole forms as the end-point of this process, the generalized second law of thermodynamics would require the entropy $S_{BH}$ of the black hole to be larger or equal than the entropy $S$ of the shell as it crosses its own horizon. We will not discuss in this paper whether the second law is satisfied because of buoyancy forces of the type discussed in Refs. [19] or because of a fundamental bound in entropy of the type introduced in Refs. [20,21]. In the spirit of thermodynamics, we wish only to emphasize the restrictions it imposes on thermodynamical parameters and discuss the interpretation of the thermodynamic quantities involved in the Bekenstein bound for the system. For a shell obeying the fundamental equation (2.18) the second law will be satisfied provided
\[ S = \frac{\eta}{(a+1)} \left( \frac{r_+}{l_p} \right)^{(a+1)} \leq S_{BH} = \pi \left( \frac{r_+}{l_p} \right)^2, \]  
where we have assumed \( S_0 = 0 \) and \( a > -1 \). This in turn restricts the numerical value of the coefficient \( \eta \) to be

\[ \eta \leq \pi (a + 1) \left( \frac{r_+}{l_p} \right)^{(1-a)}. \]  

The second law can be satisfied with values for \( \eta \) not necessarily of order one. However, in our general model (2.17) the coefficient \( \eta \) was assumed to be a number and not a function dependent on the quantity \( r_+ \). For simplicity, its value must remain unchanged for every choice of \( r_+ \), either in the case of \( r_+ \) being constant (closed system) or in the case when \( r_+ \) varies in an open quasi-static collapse. (These processes are discussed in Section IV.) In particular, we desire to satisfy (3.11) for all \( r_+ \geq l_p \) with a single value of \( \eta \). A sufficient (but not necessary) condition that guarantees the second law and the above requirements is obtained by taking the infimum of the right-hand side of (3.11). Since \( (1-a) \geq 0 \), this implies

\[ \eta \leq \pi (a + 1). \]  

This inequality is perhaps too restrictive in general since for given \( r_+ \) the second law can be satisfied for larger values of \( \eta \). However, as already mentioned, it allows us to illustrate in the next section the order of magnitude of the contribution to the entropy (2.18) arising from powers of \( r_+ \).

Notice that Eq. (3.11) restricts the values of temperature. Substitution into (2.17) yields

\[ T_\infty \geq T_c \equiv \frac{2}{(a+1)} T_H, \]  
where \( T_\infty \) is the temperature the shell would have if located at spatial infinity, and \( T_H \equiv h/(4\pi r_+) \) denotes Hawking’s asymptotic temperature. It appears therefore that for asymptotic values of the temperature of the shell smaller than the critical value \( T_c \) the generalized second law would be violated. Apparent paradoxes of this type have appeared
in other systems (see for example Ref. [22]). They indicate that in this regime statistical fluctuations do become dominant and a thermodynamical description is inappropriate. Nevertheless, in our model we assume Eq. (3.12) as the defining value of $\eta$. This implies that the local temperature is such that

$$T_\infty \geq \frac{2}{(a + 1)} \left(\frac{r_+}{l_p}\right)^{(1-a)} T_H.$$  \hspace{1cm} (3.14)

Since $a < 1$, the temperature remains much larger than $T_c$ for macroscopic shell configurations for which $r_+ \gg l_p$.

Consider now the entropy bound proposed by Bekenstein [20,21] in the particular case of a self-gravitating shell. For an object of maximal radius $R$ and total energy $\mathcal{E}$ the proposed bound reads

$$S \leq \frac{2\pi R \mathcal{E}}{\hbar}.$$  \hspace{1cm} (3.15)

In our model the bound restricts the area under the function $b(r_+)$:

$$\int b(r_+)dr_+ \leq \frac{2\pi R \mathcal{E}}{\hbar} - S_0.$$  \hspace{1cm} (3.16)

How are the quantities $R$ and $\mathcal{E}$ to be interpreted in this case? It seems natural to assume $R = \mathcal{R} = (A/4\pi)^{1/2}$, since the latter is a unique measure of the size of the shell. There are however at least two possibilities for interpreting $\mathcal{E}$: either as the local mass $M$ or as the ADM mass $m$. If $\mathcal{E}$ is interpreted as $M$, the bound implies

$$S \leq 2\pi \left(\frac{R^2}{l_p^2}\right) (1 - k).$$  \hspace{1cm} (3.17)

As the shell crosses its own horizon ($R \rightarrow r_+$), the right hand side of (3.17) tends to $2S_{BH}$. On the other hand, if $\mathcal{E}$ is interpreted as $m$, the bound implies

$$S \leq \pi \left(\frac{R r_+}{l_p^2}\right).$$  \hspace{1cm} (3.18)

As the shell crosses its horizon, the right hand side of (3.18) tends to $S_{BH}$. Hence, if a black hole forms in the limit $R = r_+$ of a quasi-static collapse, the Bekenstein bound (3.15)
guarantees the validity of the second law if the quantity $\mathcal{E}$ is interpreted not as the local proper energy $M$ but as the ADM energy $m$. This is simply because, as $R \to r_+$, $M \to 2m$ and not to $m$, as can be seen from Eq. (2.5). The bound that guarantees the second law can be written in terms of the local state variables $(M, R)$ as

$$S \leq \frac{2\pi R M}{\hbar} \left(1 - \frac{M}{2R}\right). \quad (3.19)$$

D. Third law and Summary

In its simplest form, the third law of standard thermodynamics requires the entropy to vanish in the state of zero temperature. From Eq. (2.10), the latter occurs for our system whenever the function $b(r_+)$ diverges. If this function has the power law dependence (2.17) it will diverge for $a > 0$ if $r_+$ diverges, though in this state the entropy (2.18) diverges (assuming finite $\eta$). For $a \leq -1$, $b$ diverges as $r_+$ tends to zero, but in this state the entropy diverges as well. In contrast, for $-1 < a < 0$ the function $b$ diverges and the entropy vanishes as $r_+$ tends to zero. Notice that the temperature cannot be zero for the case $a = 0$. If the above mentioned states could be reached by a quasi-static sequence of thermodynamic equilibrium states, the third law in the above mentioned form would further restrict the parameter $a$ to the range $-1 < a \leq 0$.

However, we do not consider this restriction fundamental. Firstly, it is not a basic postulate in ordinary thermodynamics. It is not unusual to encounter reasonable fundamental equations in thermodynamics which do not satisfy the third law (an example is the ideal van der Waals fundamental equation [7]). Secondly, it is not clear if the formulation of the third law used above is the correct one for self-gravitating matter. In particular, it does not apply to black holes, where alternative versions exist [23]. Finally, the state of zero temperature may not be reachable by a finite number of quasi-static equilibrium states of a macroscopic shell. Since the thermodynamical treatment breaks down in the limit $r_+ \approx l_p$ due to quantum gravitational fluctuations becoming non-negligible, states for which $r_+ = 0$
cannot be reached by a finite number of steps within the present approximation. Therefore, a violation of the third law implies at most that the fundamental equation is not a very good approximation at very low temperatures.

The results of this section can be summarized as follows. For a power law thermal equation of state, the fundamental equation for an intrinsically stable shell is

\[ S(M, R; a) = \pi \left( \frac{r_+}{r_p} \right)^{(a+1)}, \]  

(3.20)

where \( r_+(M, R) \) is given by Eq. (2.4) and \(-1 < a \leq 12/19\). We have adopted in (3.20) a value for the coefficient \( \eta \) of order one which respects the generalized second law. If enforced, the third law in its standard form may further restrict \( a \) to be smaller or equal than zero.

IV. QUASI-STATIC COLLAPSE AND MAXIMUM ENTROPY

A quasi-static process consists of a dense succession of equilibrium states \([7]\). There are no obstacles in principle in constructing a quasi-static sequence of equilibrium states of a shell that simulates at least part of its dynamical collapse. One can imagine infinitesimal differences between the pressure of the shell and the gravitational pull which will force the shell to collapse gradually. A quasi-static sequence can be expected to be a good approximation to the true dynamical process only if the time of thermalization of the shell with itself is relatively small compared to the characteristic times of collapse. The thermalization time depends on the material in the shell. However, it is physically natural to expect a good agreement for shell radii large compared with the horizon radius. It is not unreasonable to assume that this approximation breaks down for radii in the neighborhood (if not larger) of the minimal radius \( R = 25/24 \) at which the dominant energy condition is marginally satisfied. In this section we will not be interested in the precise distances at which the approximation becomes unphysical but assume the maximal possible range of positions, namely \( 25/24 \leq R < \infty \).

Quasi-static processes can be used to simulate a real dynamical process in a closed system only if the total entropy is a non-decreasing function along the process \([7]\). Thus, the
processes can be either reversible (if the quasi-static increase of entropy $dS$ along the process is zero) or irreversible (if the increase of entropy $dS$ along the process is positive). In fact, although every reversible process coincides with a quasi-static process, not every quasi-static sequence is reversible. It is natural to try to find the maximum amount of entropy whose origin can be ascribed to a quasi-static collapse processes and compare it with the entropy of the latter. As mentioned in the Introduction, the “remaining” entropy must have its origin in the irreversible non-equilibrium stages of collapse.

Consider the fundamental equation (2.18). It implies that a quasi-static sequence of shell configurations for which $r_+ = \text{const.}$ is reversible, whereas a quasi-static sequence for which $r_+$ increases is irreversible. (Throughout this paper we refer only to quasi-static processes for a simple, single shell as opposed to processes for composite systems which could include a shell in interaction with a heat bath.) The reversible process is constructed with equilibrium states whose extensive variables $(M, R)$ obey Eq. (2.4) with fixed constant $m$. It is not difficult to prove that the condition $r_+ = \text{const.}$ defines a closed system: If the shell is imagined located in a finite region bounded by a surface whose radius is larger than $R$, the above condition is a consequence of fixing both the quasilocal energy contained inside the surface that bounds the system and its size [5]. Therefore, if the system is closed, the motion of a shell is reversible.

We emphasize that the above results are not trivial and are not included in the results presented in Ref. [4]. In our case the state variables of the shell are allowed to vary in a quasi-static manner. For variations respecting the constraint $r_+ = \text{const.}$, the quasi-static motion turns out to reversible. In Ref. [4] the state variables of the shell were kept “frozen” during a quasi-static variation of the internal black hole parameters. Naturally the matter entropy remained a negligible constant. This led to the conclusion that the motion of a shell is always reversible. This was only so because the matter entropy was not included in the analysis. To say it differently, a quasi-static sequence is a series of equilibrium states. The entropy for each equilibrium state is naturally a constant. However, for each equilibrium state the constant value of the entropy is different. This is what accounts for an irreversible
growth of entropy even in quasi-static processes. Since in the analysis of Ref. 4 the entropy of the shell was not considered, the growth of entropy in a quasi-static sequence of shell positions could not be addressed.

The explicit amount of entropy (which remains constant for the quasi-static reversible process mentioned above or increases for the quasi-static irreversible processes discussed later) depends on the model description of the matter in the shell. This would determine the function $b(r_+)$ in (2.12) and the empirical coefficients $a$ and $\eta$ in (2.18). In previous sections we saw how physical considerations of a general character severely restrict the range of the coefficient $a$. It is not so easy to constrain the range of $\eta$ without a precise description of the matter fields. In particular, it is not difficult to see that one could account for most of the black hole entropy if the value of $\eta$ is of order $(r_+/l_p)^{(1-a)}$. If we assume that $\eta$ is of order one, we obtain the fundamental equation (3.20) for a stable shell. For fixed $r_+$, the entropy remains constant as one lowers the shell, and the maximum possible constant value of the entropy is attained for $a = 12/19$. Therefore for a stable shell at constant $r_+$

$$S \leq S_{\text{max}} = \pi \left(\frac{r_+}{l_p}\right)^{(31/19)} = \left(\frac{l_p}{r_+}\right)^{(7/19)} S_{\text{BH}},$$

where $S_{\text{BH}}$ refers to the entropy of the black hole whose ADM mass equals the ADM mass $m$ of the shell. In other words, the entropy of a black hole exceeds the entropy of a stable shell of the same ADM mass by (at least) a factor of order $(r_+/l_p)^{(7/19)}$ whenever the thermal equation of state for the latter takes the form (2.17) and $\eta$ is of order one (namely $\eta = \pi(a + 1)$). If the third law were to be enforced in the manner discussed in Section III, the maximum entropy would occur for $a = 0$. In this case, the entropy of a black hole of radius $r_+$ would exceed the entropy of a stable shell of the same ADM mass by a factor of order $(r_+/l_p)$. In any case, in this simple example the entropy of a shell of given ADM mass would equal the entropy of a black hole of the same mass only if the size of the horizon is of the order of the Planck length. We stress that these estimates are based on the particular form (2.17) of the thermal equation of state whose overall multiplicative coefficient $\eta$ is assumed to be of order one.
Irreversible quasi-static processes characterized by different types of boundary data can be easily constructed. For given boundary conditions defining the process, the sequence of equilibrium states \((M, R)\) is dictated by Eq. (2.5) and the associated entropy at each state by (2.18). As an example, consider a quasi-static process obtained by constraining the position of the shell to be \(R = R_0 = \text{const}\). In thermodynamical language, this is equivalent to assuming internal constraints in the system that induce the shell to be restrictive with respect to size. The incoming energy \(dm \geq 0\) to the system is fully spent in increasing the energy \(M\) (according to Eq. (2.5) with \(R = R_0\)) and none transforms into mechanical work. A possible realization of this kind of quasi-static process is the following: start with an idealized shell whose mass is small. As energy flows into the system, the local mass \(M\) grows quasi-statically until \(r_+ \to R_0\), at which limit we assume that a black hole forms. Its entropy at each equilibrium stage is given by (2.18). However, as we mentioned before, we expect that this expression becomes inaccurate for values \(r_+ \geq (24/25)R_0\). Therefore, for fixed values of \(\eta\) and \(a\) (and \(S_0 = 0\)) the entropy attainable by this quasi-static process is

\[
S \leq S_{\text{max}} = \frac{\eta}{(a + 1)} \left(\frac{24}{25} \frac{R_0}{l_p}\right)^{(a+1)}.
\]  

(4.2)

If one assumes \(\eta = \pi(a + 1)\), the maximum entropy for a stable shell can be obtained from (3.20) with \(a = 12/19\) and equals

\[
S_{\text{max}} = \left(\frac{24}{25}\right)^{(31/19)} \left(\frac{l_p}{R_0}\right)^{(7/19)} S_{BH} \approx 0.94 \left(\frac{l_p}{R_0}\right)^{(7/19)} S_{BH},
\]

(4.3)

where \(S_{BH} = \pi(R_0/l_p)^2\) is the entropy of the black hole formed as \(r_+ \to R_0\).

The previous analysis can be easily generalized to a wide variety of processes involving quasi-static interchange of energy and work between the shell and a reservoir. We do not claim all of these processes to be physical: thermodynamics does not guarantee dynamics. They are simply not forbidden by thermodynamical arguments and provide interesting examples which illustrate the quasi-static behavior of entropy.
V. SCALING AND GIBBS-DUHEM

The integrated form of the first law for our system is

\[ M = (a + 1) TS - 2p A - (a + 1) T S_0 . \] (5.1)

This is easy to verify using Eqs. (2.6), (2.10) and (2.17) for the intensive parameters (β, p) and Eq. (2.18) for the entropy S. This “Euler relation” implies that the entropy \( S(M, A, N) \) is a homogeneous function of degree \((a+1)\) in \( M \) and of degree \((a+1)/2\) in \( A \). (Alternatively, the energy \( M(S, A, N) \) is a homogeneous function of degree \(1/(a+1)\) in \( S \) and of degree one-half in \( A \).) The scaling laws for the self-gravitating system are therefore: \( M \to \lambda M \) \((r_+ \to \lambda r_+)\), \( A \to \lambda^2 A \) \((R \to \lambda R)\), and \( S \to \lambda^{(a+1)} S \). This behavior is due to the fact that the intensive variables have to be rescaled according to \( \beta \to \lambda^a \beta \), and \( p \to \lambda^{-1} p \).

The Euler relation (5.1) illustrates that the scaling laws characteristic of self-gravitating matter systems are different from the ones of ordinary flat-spacetime thermodynamics. We stress that this is so even if no black hole is present in the system. Observe that in the limit \( a = 0 \) one does not recover the ordinary scaling: the work term in (5.1) does not reduce to the familiar form \(-pV\). This is partly a consequence of the role played by area as the variable measuring the “size” of a system. The difference between the scaling laws of standard thermodynamics and the ones characteristic of a black hole has been recognized in Ref. [12]. The expressions for a black hole can be recovered from the above ones by taking the limit \( a = 1 \) and \( \eta = 2\pi \). As pointed out in Ref. [12] in the context of black holes, these scaling properties must play an important role in the description of phase transitions involving self-gravitating systems.

In standard thermodynamics there exists a relationship (the Gibbs-Duhem relation) among the various intensive parameters which is a consequence of the homogeneous first order degree of the fundamental relation [7]. It can be obtained by combining the Euler relation with the first law of thermodynamics and states that the sum of products of the extensive parameters and the differentials of the corresponding intensive parameters vanishes,
namely \( SdT - VdP + Nd\mu = 0 \) in the energy representation. The quantity \( V \) denotes the size of the system and \( \mu \) the chemical potential conjugate to \( N \). Because of the homogeneous properties of Eq. (5.1) discussed above, this form of the Gibbs-Duhem relation is not valid for a self-gravitating system. By differentiating Eq. (5.1) and combining the result with the first law (2.7) one obtains instead

\[
(a + 1) SdT - 2A dp + (a + 1) N d\mu + a dM + (a - 1) p dA = 0 .
\] (5.2)

In the limit \( a = 1 \) this relationship reduces to the corresponding one for a black hole, namely

\[
2SdT - 2A dp + 2N d\mu + dM = 0 .
\] (5.3)

VI. CONCLUDING REMARKS

We have attempted to clarify in this paper the features of the fundamental thermodynamical equation of a matter system which arise solely from its self-gravitating character and are therefore independent of the microscopic structure of the matter fields. The fundamental equation (2.12) is a consequence of the pressure equation of state (2.6), the integrability of the first law of thermodynamics, and the assumption of constant number \( N \). The pressure equation is in turn a direct consequence of the gravitational junction conditions (and therefore of Einstein equations) at the position of a two-dimensional surface in thermal and mechanical equilibrium with itself. To specify the fundamental equation completely one needs to add a phenomenological or quantum description of the matter fields. Different models provide different functions \( b(r_+) \). Our purpose has been not to classify the latter (since this does not concern gravity) but illustrate the method with a simple physical choice.

The methods adopted in this paper to explore the fundamental equation of a self-gravitating system are closely related to methods of finding fundamental equations in ordinary thermodynamics. Our phenomenological description of the gravitational field through Einstein equations provided us with an “empirical” pressure equation of state, in much the
same spirit as one obtains, for example, the van der Walls pressure equation of state in fluid mechanics [4]. A thermal equation of state has then to be found by adopting the simplest expression that is physically reasonable and guarantees integrability for the entropy. In both phenomenological descriptions of simple fluids and self-gravitating bodies, empirical coefficients have to be assumed. The phenomenological approach can be pushed forward by requiring the entropy to satisfy several physical conditions which have thermodynamical origin. Ultimately, their precise values as well as the range of applicability of the resulting fundamental equation have to be determined by experiment or by a fully statistical description of the interactions. It would be very interesting to construct simple but realistic models (involving, for example, scalar fields) of the structure of the shell. They may provide explicit values for the coefficients as well as define the regions in which the quasi-static processes discussed in Section IV are a good approximation to the full dynamical collapse. In any event, the fundamental thermodynamical equations arrived at are very rich and illustrate many of the results and strengths of a thermodynamical approach to quantum-statistical gravitational systems.

A black hole contained inside a spherical boundary in thermal equilibrium with a heat bath and obeying the semiclassical Hawking’s equation (2.15) is not intrinsically stable. Therefore, it is of interest to investigate slightly more general thermal equations of state which would allow black hole stability without altering the pressure equation of state (2.14). This approach may in turn provide some guidance into thermal equations of state which might be considered as ‘corrections’ to Hawking’s equation and which might effectively incorporate back-reaction and higher order effects. The results of this paper apply to a black hole located inside a spherical cavity whose thermal equation of state is generally given by Eqs. (2.10) and (2.17). (Hawking’s formula is recovered formally by taking the limits $a = 1$ and $\eta = 2\pi$.) The stability analysis of Section III can be easily adapted to this type of black holes. It implies that values of $a$ smaller than unity in the thermal equation of state do guarantee stability for (finite) ranges of the boundary surface radii larger than the horizon radius. For example, if $a = 0.9$ the black hole could be stable provided the
boundary radius satisfies $r_0 \leq 1.003 r_+$. The range of values for $r_0$ that guarantees stability increases as $a$ decreases. Of course, in a phenomenological description like the one presented here there is no reason to select a particular value of this coefficient. In any case, this would imply a black hole entropy given by Eq. (3.20) where the exponent of $r_+$ is smaller than two. It would be of interest to see whether alternative expressions for the entropy of a black hole, obtained by including either higher-loop terms or different types of “hair” in the gravitational action (see for example Ref. [24] and references therein) could be expressed (at least partially) in the above mentioned form. We will return to this issue elsewhere.

We have focused our attention on the thermodynamics of equilibrium states and therefore only considered quasi-static (and not dynamical) processes. Whereas the local intrinsic stability conditions studied here are the weakest stability restrictions one can impose to a system based solely on thermodynamics, dynamical stability may impose further restrictions. Mechanical stability under dynamical perturbations of a shell surrounding a black hole has been studied recently in Ref. [25] by examining the equations of motion in the neighborhood of equilibrium configurations. This analysis did not include a thermal behavior for a shell but found that, in the particular case when no black hole is present, the largest region for mechanical stability for a “stiff” shell (for which the speed of sound equals the speed of light) occurs when $k \leq 0.395$, or equivalently, for a radius $R$ larger than approximately $1.185 r_+$. The critical value for stability might indeed be larger for smaller values of the speed of sound [25]. It is not difficult to see that this mechanical stability condition, if applied literally to our equilibrium states with a thermal equation of state given by (2.17) would further restrict the value of the parameter $a$ to be smaller or equal than approximately 0.37. It would be of interest to generalize the dynamical analysis of Ref. [25] to incorporate the thermal behavior of the shell discussed in this paper.
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TABLES

TABLE I. Intrinsic stability conditions for the fundamental equation (2.18). The table shows the range of $k$ in which the conditions are satisfied for different values of the parameter $a$. The symbol “√” indicates that a criterion is satisfied in the full physical range $0 \leq k \leq 1$, whereas the symbol “×” indicates that the criterion is not satisfied in this range.

| RANGE OF $a$ | $(\frac{\partial^2 S}{\partial M^2}) \leq 0$ | $(\frac{\partial^2 S}{\partial A^2}) \leq 0$ | $(\frac{\partial^2 S}{\partial M^2})(\frac{\partial^2 S}{\partial A^2}) - (\frac{\partial^2 S}{\partial A \partial M})^2 \geq 0$ |
|--------------|-----------------------------------|-----------------------------------|-----------------------------------------------|
| $a \geq 3$   | $k \leq \sqrt{\frac{1}{2a+1}}$   | $k \geq 1 - \frac{6}{a+3}$       | ×                                             |
| $3 \geq a \geq 1$ | $k \leq \sqrt{\frac{1}{2a+1}}$ | √                                 | ×, but (a)                                    |
| $1 < a < \frac{12}{19}$ | $k \leq \sqrt{\frac{1}{2a+1}}$ | √                                 | (b)                                           |
| $\frac{12}{19} \geq a > 0$ | $k \leq \sqrt{\frac{1}{2a+1}}$ | √                                 | (c)                                           |
| $0 \geq a$   | √                                 | √                                 | √                                             |

(a) For $a=1$, this condition is satisfied if $k = 0$.
(b) $k$ is a solution of Eq. (3.7), but remains smaller than $1/5$.
(c) $k$ is a solution of Eq. (3.7) always larger or equal (for $a = 12/19$) than $1/5$. See main
text.