Gyrogroups associated with groups

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ABSTRACT
In this paper, we have studied the properties of the associated gyrogroup \( G^1 \) of a given group \( G \) of nilpotency class 3. We have proved that if 3 does not divide the order of the group \( G \), then the nilpotency class of the associated gyrogroup \( G^1 \) is same as that of the group \( G \). We have also studied the problem of abelian inner mapping group in this context.

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1. Introduction
The study of gyrogroups was initiated by Ungar in [6]. Gyrogroups are grouplike structures and non-associative generalization of groups, modeled on Einstein velocity addition law of special relativity [8]. A groupoid \((L, \cdot)\) with identity is called a right loop if the equation \( x \cdot a = b \) has a unique solution in \( L \), for all \( a, b \in L \). Let \((L, \cdot)\) be a right loop and \( y, z \in L \). Then, there is a bijective map \( f(y, z) \) from \( L \) to \( L \) such that

\[
(xy)z = f(y, z)(x)(yz), \quad \text{for all } x \in L.
\]

A right loop \( L \) is called a loop if the equation \( a \cdot x = b \) has a unique solution in \( L \). A right loop \( L \) is called a gyrogroup if \( f(a, b) = f(ab, a)^{-1} \) and \( f(a, b) \) are automorphisms of \( L \) for all \( a, b \in L \). Gyrogroups are loops. Let \( G \) be a group. Define a binary operation \( \circ \) on \( G \) by \( x \circ y = y^{-1}xy \). Foguel and Ungar in [3] proved that \((G, \circ)\) is a gyrogroup if and only if \( G \) is central by a 2-Engel group (see [3, Theorem 3.7]). In particular, if \( G \) is a nilpotent group of class 3, then \((G, \circ)\) is a gyrogroup. It is also shown that the associated right gyrogroup is a group if and only if the group \( G \) is an nilpotent group of class 2 (see [3, Theorem 3.6]). We will denote the associated gyrogroup \((G, \circ)\) by \( G^1 \). Throughout the paper, \( G \) will denote the finite nilpotent group of class 3, otherwise will be stated separately.

Let \( L \) be a loop and \( N \) be a subloop of \( L \). Then \( N \) is called a normal subloop of the loop \( L \) if, for all \( x, y \in L \), we have, (i) \( xN = Nx \), (ii) \( x(yN) = (xy)N \), and (iii) \( (Nx)y = N(xy) \). In a loop \( L \), we have the following important subsets:

(i) The set \( N_x(L) = \{ a \in L | (ax)y = a(xy) \forall x, y \in L \} \) is called the left nucleus.
(ii) The set \( N_y(L) = \{ a \in L | (xa)y = x(ay) \forall x, y \in L \} \) is called the middle nucleus.
(iii) The set \( N_y(L) = \{ a \in L | (xy)a = x(ya) \forall x, y \in L \} \) is called the right nucleus.
(iv) The set \( N(L) = N_x(L) \cap N_y(L) \cap N_z(L) \) is called the nucleus of \( L \).
(v) The set \( C(L) = \{ a \in L | xa = ax \ \forall x \in L \} \) is called the commutant of \( L \).
(vi) The set \( Z(L) = C(L) \cap N(L) \) is called the center of \( L \).

The commutant \( C(L) \) is not necessarily a subloop of \( L \). Except this, all are subloops of \( L \) in the above list. The center \( Z(L) \) is the normal subloop of \( L \). Let \( x, y, z \in L \). Then, the commutator \([x, y]\) and the associator \( A(x, y, z) \) are defined as the unique solutions of the following equations respectively,

\[
xy = [x, y](yx), \\
(\text{and} \ (xy)z = A(x, y, z)(x(yz))).
\]

Let \([L, L]\) and \( \mathcal{A}(L) \) denotes the collection of all the commutators and the associators of the loop \( L \). The commutator of the associated gyrogroups will be denoted by \( ^o[x, y] \). \( [G, G, G] \) denotes the subgroup of the group \( G \) generated by the triple commutators \([x, y, z] = [[x, y], z] \), for all \( x, y, z \in G \).

In Section 2, we have studied the properties of nucleus, commutant and center of the associated gyrogroup. Moreover, we have proved that the commutant \( C(^oG) \), the center \( Z(^oG) \) of the loop \( ^oG \) and the center \( Z(G) \) of the group \( G \) all coincide, if 3 does not divide the order of \( G \). In Section 3, we have studied the nilpotency class of associated gyrogroup. We have proved that, if 3 does not divide the order of the group \( G \), then \( ^oG \) is a nilpotent loop of class 3. In Section 4, we have studied the open problem for abelian inner mapping group for loop of class 3 in the case of associated gyrogroups.

## 2. Nuclei, commutant and center of the associated gyrogroup

In this section, we prove some properties of the nuclei, commutant and the center of the associated gyrogroup. Now, we give the definitions of gyrogroups and gyrogroups associated with groups.

**Definition 2.1** [7, Definition 2.2, p. 888]. Let \((S, \circ)\) be a groupoid with a right identity \( e \) and a right inverse \( a' \) for each element \( a \in S \) such that \( a \circ a' = e \). Then \((S, \circ)\) is called a gyrogroup if the following holds:

(i) For any \( x, y, z \in S \), there exists a unique element \( f(y, z)(x) \in S \) such that

(ii) \( (x \circ y) \circ z = f(y, z)(x) \circ (y \circ z) \),

(iii) The map \( f(y, z) : S \to S \) given by \( x \mapsto f(y, z)(x) \) is an automorphism of \((S, \circ)\).

(iv) For all \( x, y \in S, f(x, x \circ y) = f(x, y) \).

A groupoid \((S, \circ)\) with a right identity \( e \) and a right inverse \( a' \) for each element \( a \in S \) such that \( a \circ a' = e \) is a right gyrogroup if it satisfies the axioms (i) and (ii).

**Definition 2.2** [3, Definition 3.5, p. 6]. The associated right gyrogroup with a group \( G \) is the right gyrogroup \((G, \circ)\) whose right gyrogroup operation \( \circ \) is given by \( x \circ y = y^{-1}xy^2 \) for all \( x, y \in S \).

By [4, Proposition 4.4], the commutant \( C(^oG) \) is the characteristic subgroup of the group \( G \). We prove below that it is a normal subloop of \(^oG\).

**Proposition 2.3.** The commutant \( C(^oG) \) is a subloop of the loop \(^oG\).

**Proof.** Let \( a \in G \) be any element. Then, from the proof of the Proposition [4, Proposition 4.4, p. 1450058-10], we have,
\[ a \in C(\theta G) \iff (ax)^3 = a^3x^3 \forall x \in G. \quad (1) \]

First, we prove that \( C(\theta G) \) is closed under the binary operation \( \circ \). Let \( a, b \in C(\theta G) \). Then for all \( x \in \theta G \), we have

\[
\begin{align*}
(a \circ b)^3x^3 &= (b^{-1}ab^2)^3x^3 \\
&= ((b^{-1}ab)b)^3x^3 \\
&= ((b^{-1}ab)b^3)x^3, \text{ (using (1))} \\
&= (b^{-1}ab)^3(bx)^3, \text{ (using (1))} \\
&= (b^{-1}ab)(bx)^3, \text{ (using (1))} \\
&= (b^{-1}ab^2x)^3 \\
&= ((a \circ b)x)^3.
\end{align*}
\]

Therefore, \( a \circ b \in C(\theta G) \).

Let \( a, b \in C(\theta G) \). Note that, \( aba^{-2} \in C(\theta G) \) and \( aba^{-2} \circ a = a^{-1}aba^{-2}a^2 = b \). Thus, for any two elements \( a, b \in C(\theta G) \), the equation \( X \circ a = b \) has a unique solution in \( C(\theta G) \). Therefore, \( C(\theta G) \) is a subloop of \( \theta G \). \( \square \)

**Proposition 2.4.** The commutant \( C(\theta G) \) is a normal subloop in \( \theta G \).

**Proof.**

(i) Clearly, \( x \circ C(\theta G) = C(\theta G) \circ x \) for all \( x \in \theta G \).

(ii) For all \( a \in C(\theta G) \) and \( x, y \in \theta G \),

\[
a \circ (x \circ y) = a \circ (y^{-1}xy^2) \\
= y^{-2}x^{-1}yay^{-1}xy^2y^{-1}xy^2 \\
= y^{-1}(y^{-1}x^{-1}yay^{-1}xy)xy^2 \\
= y^{-1}x^{-1}(xy^{-1}x^{-1})a(y^{-1}xy^{-1})x^2y^2 \\
= y^{-1}x^{-1}[x, y^{-1}]a[y^{-1}, x]x^2y^2 \\
= y^{-1}(x^{-1}bx^2)y^2 \\
= (b \circ x) \circ y.
\]

Thus, \( a \circ (x \circ y) = (b \circ x) \circ y \), where \( b = [x, y^{-1}]a[y^{-1}, x] \). Since \( C(\theta G) \) is a normal subgroup of the group \( G \), \( b \in C(\theta G) \). Therefore, \( C(\theta G) \circ (x \circ y) = (C(\theta G) \circ x) \circ y \), for all \( x, y \in \theta G \).

(iii) Using (i) and (ii), we have \( x \circ (y \circ C(\theta G)) = (x \circ y) \circ C(\theta G) \iff x \circ (C(\theta G) \circ y) = C(\theta G) \circ (x \circ y) \iff x \circ (C(\theta G) \circ y) = (C(\theta G) \circ x) \circ y \iff x \circ (C(\theta G) \circ y) = (x \circ C(\theta G)) \circ y \). Note that, for all \( a \in C(\theta G) \) and \( x \in \theta G \), \( (ax)^3 = a^3x^3 \iff (ax)^2 = x^2a^2 \). Now, for all \( a \in C(\theta G) \) and \( x, y \in \theta G \), we have

\[
x \circ (a \circ y) = x \circ (y^{-1}ay^2) \\
= y^{-2}a^{-1}yxy^{-1}ay^2 \\
= y^{-2}a^{-1}yxy^{-1}(ay)^2y \\
= y^{-2}a^{-1}yxy^{-1}y^2a^2y \\
= y^{-2}(a^{-1}yx^2a) \\
= y^{-2}(yx \circ a) \circ y
\]
\[ y^{-2}(a \circ xy)y = y^{-2}y^{-1}xy^{-1}ayxyxyy = y^{-3}x^{-1}y^{-1}axyx^2y^2 \\
= y^{-1}x^{-1}(xy^{-2}x^{-1})^{-1}a(xy^2x^{-1})x^2y^2 = y^{-1}x^{-1}(xy^2x^{-1})^{-1}a(xy^2x^{-1})x^2y^2 \\
= y^{-1}(x^{-1}bx^2)y^2 = (b \circ x) \circ y. \]

Thus, \( x \circ (a \circ y) = (x \circ b) \circ y \), where \( b = (xy^2x^{-1})^{-1}a(xy^2x^{-1}). \) Since \( C^G(G) \) is a normal subgroup of the group \( G \), \( b \in C^G(G). \) Therefore, \( x \circ (y \circ C^G(G)) = (x \circ y) \circ C^G(G) \) for all \( x, y \in G. \)

Hence, \( C^G(G) \) is a normal subloop of the loop \( ^G G. \)

Lemma 2.5. Let \( G \) be a group. Then for all \( x, y, z \in G \) following holds,

\begin{enumerate}[(i)]
\item \( [xy, z] = [x, [y, z]][y, z][x, z], \)
\item \( [x, yz] = [x, y][y, z][x, z]. \)
\end{enumerate}

Now, we will prove that the left, the middle and the right nuclei of \( ^G G \) are characteristic subgroups of the group \( G. \) First, note that, the center \( Z(G) \) of the group \( G \) is contained in the left, the middle and the right nuclei of \( ^G G. \)

Proposition 2.6. The left nucleus \( N_l(^G G) \) is a subgroup of the group \( G. \)

Proof. Let \( x, y \in ^G G. \) Then \( a \in N_l(^G G) \)

\[ [ab, [x, y^{-1}]] = [a, [b, [x, y^{-1}]][b, [x, y^{-1}]] [a, [x, y^{-1}]] = 1 \text{ (for } [G, G, G] \subseteq Z(G)). \]

Thus \( ab \in N_l(^G G) \). Also, for \( a \in N_l(^G G) \) and \( x, y \in ^G G, 1 = [1, [x, y^{-1}]] = [aa^{-1}, [x, y^{-1}]] = [a, [a^{-1}, [x, y^{-1}]][a^{-1}, [x, y^{-1}]] [a, [x, y^{-1}]] = [a^{-1}, [x, y^{-1}]]. \) Therefore, \( a^{-1} \in N_l(^G G). \) Hence, \( N_l(^G G) \) is a subgroup of the group \( G. \)

Proposition 2.7. The middle nucleus \( N_m(^G G) \) is a subgroup of \( G. \)

Proof. Let \( x, y \in ^G G. \) Then, by the similar argument as in the proof of Proposition 2.6,

\[ a \in N_m(^G G) \iff [x, [a, y^{-1}]] = 1. \]

Now, for all \( a, b \in N_m(^G G), \) by the similar argument as in the proof of Proposition 2.6, using Lemma 2.5(i), we have \( [x, [ab, y^{-1}]] = 1 \) and \( [x, [a^{-1}, y^{-1}]] = 1. \) Thus, \( a^{-1}, ab \in N_m(^G G). \) Hence, \( N_m(^G G) \) is a subgroup of \( G. \)
Proposition 2.8. The right nucleus $N_\rho(G)$ is a subgroup of $G$.

Proof. Let $x,y \in G$. Then, by the similar argument as in the proof of Proposition 2.6,

$$a \in N_\rho(G) \iff [x, [y, a^{-1}]] = 1.$$  

By the similar argument as in the proof of Proposition 2.6, one can show that $N_\rho(G)$ is a subgroup of $G$. \hfill \Box

Proposition 2.9. $N_i(G)$, where $i \in \{\lambda, \mu, \rho\}$ are characteristic subgroups of the group $G$ and are of nilpotency class atmost 2.

Proof. Let $\psi \in Aut(G, \cdot)$ and $a \in N_i(G)$. Then for all $x,y \in G$, we have

$$(\psi(a) \circ x) \circ y = (\psi(a) \circ \psi(u)) \circ \psi(v), \text{ where } x = \psi(u), \text{ and } y = \psi(v)$$

$$= \psi(v)^{-1}(\psi(u)^{-1}\psi(a)\psi(u)^2)^2$$

$$= \psi(v^{-1}(u^{-1}au)^2v^2)$$

$$= \psi((a \circ u) \circ v)$$

$$= \psi(a \circ (u \circ v))$$

$$= \psi(a \circ (v^{-1}uv^2))$$

$$= \psi((v^{-1}uv^2)^{-1}a(v^{-1}uv^2)^2)$$

$$= \psi(v^{-1}uv^2)^{-1}\psi(a)^{-1}(v^{-1}uv^2)^2$$

$$= (\psi(v)^{-1}\psi(u)\psi(v)^2)^{-1}\psi(a)(\psi(v)^{-1}\psi(u)\psi(v)^2)^2$$

$$= (\psi(u) \circ \psi(v))^{-1}\psi(a)(\psi(u) \circ \psi(v))^2$$

$$= \psi(a) \circ (\psi(u) \circ \psi(v))$$

$$= \psi(a) \circ (x \circ y).$$

Thus, $\psi(a) \in N_i(G)$. Hence, $N_i(G)$ is a characteristic subgroup of $G$. Also, by the proof of Proposition 2.6, one notes that, $[N_\lambda(G), N_\mu(G), N_\rho(G)] = \{1\}$. Hence, $N_i(G)$ is a nilpotent group of class atmost 2. By the similar argument, $N_\rho(G)$ and $N_\mu(G)$ are characteristic subgroups of the group $G$ of nilpotency class atmost 2. \hfill \Box

Corollary 2.10. $(N_i(G), \circ)$, where $i \in \{\lambda, \mu, \rho\}$ are groups with the induced binary operation $\circ$.

Proof. Follows immediately by Propositions 2.9 and Theorem [3, Theorem 3.6]. \hfill \Box

Proposition 2.11. The following relations between the nuclei holds

(i) $N_\rho(G) = N_\rho(G)$,

(ii) $N_\rho(G) \subseteq N_\rho(G)$.

Proof.

(i) Let $a \in N_\rho(G)$. Then, for all $x,y \in G$,

$$[x, [a, y^{-1}]] = [x, [y^{-1}, a^{-1}]]$$

$$= [x, [y^{-1}, a]]^{-1}, \text{ as } [G, G, G] \subseteq Z(G)$$

$$= 1, \text{ as } a \in N_\rho(G).$$
Thus, \([x, [a, y^{-1}]] = 1\), for all \(x, y \in G\). Therefore, \(a \in N_\mu(G)\) and \(N_\mu(G) \subseteq N_\mu(G)\).

Conversely, let \(a \in N_\mu(G)\). Then, for all \(x, y \in G\),

\[
[x, [y, a^{-1}]] = [x, [a^{-1}, y^{-1}]^{-1}] = [x, [a^{-1}, y]^{-1}]^{-1} = [x, [a^{-1}, y]]^{-1} = [1, a^{-1}, y^{-1}] \subseteq Z(G).
\]

Therefore, \(a \in N_\rho(G)\) and \(N_\mu(G) \subseteq N_\rho(G)\). Hence, \(N_\mu(G) = N_\rho(G)\).

(ii) Let \(a \in N_\mu(G)\). Then, for all \(x, y \in G\),

\[
[a, [x, y]] = a[x, y]a^{-1}[x, y]^{-1} = a[y, a^{-1}]^{-1}([y, a^{-1}] [x, y] a^{-1} y^{-1} x^{-1}) y^{-1} x^{-1} = a[y, a^{-1}] [x, y] [y, a^{-1}] a^{-1} y^{-1} x^{-1} \text{ (for } [G, G] \text{ is abelian)}
\]

\[
= ay^{-1}(xy^{-1} y^{-1} a^{-1} x) y^{-1} x^{-1} = ay^{-1} a^{-1} x(x^{-1} a y x^{-1} a^{-1} x) y^{-1} x^{-1} = ay^{-1} a^{-1} x^2(x^{-2} a y x^{-1} a^{-1} x^{-1} a^{-1} x^2) x^{-2} a y^{-1} x^{-1} = ay^{-1} a^{-1} x^2 ((x^{-1} a^{-1} x^2) a^{-1} y^{-1} (x^{-1} a^{-1} x^2)) x^{-2} a y^{-1} x^{-1} = ay^{-1} a^{-1} x^2 (y \circ (a^{-1} \circ x)) x^{-2} a y^{-1} x^{-1} = ay^{-1} a^{-1} x^2 ((y \circ a^{-1}) \circ x) x^{-2} a y^{-1} x^{-1} = ay^{-1} a^{-1} x^2 (x^{-1} a y a^{-2} a^{-2} x^2) x^{-2} a y^{-1} x^{-1} = ay^{-1} a^{-1} x y a^{-1} y^{-1} x^{-1} = [y, a] x[a, y] x^{-1} = [[y, a], x] = [a, y]^{-1}, x]
\]

\[
= [[a, y], x]^{-1}, \text{ as } [G, G, G] \subseteq Z(G)
\]

\[
= [x, [a, y]] = [x, [a^{-1}, y^{-1}]] = 1, \text{ as } a \in N_\mu(G).
\]

Hence, \(a \in N_\mu(G)\). Therefore, \(N_\mu(G) \subseteq N_\mu(G)\).

**Proposition 2.12.** \(N_\mu(G)\) is a normal subloop of the loop \(G\).

**Proof.**

(i) Clearly, \((N_\mu(G) \circ x) \circ y = N_\mu(G) \circ (x \circ y)\) for all \(x, y \in G\).

(ii) Let \(x \in G\) and \(a \in N_\mu(G)\). Then

\[
x \circ a = a^{-1} x a^2 = a^{-1} [x, a] a x a = [x, a] x a, \text{ as } [a^{-1}, [x, a]] = 1
\]

\[
x a x^{-1} a^{-1} x a = x^{-1} (x^2 a^{-1} a^{-1} x a x^{-2}) x^2 = b \circ x, \text{ where }
\]
where \( b = x^2ax^{-1}a^{-1}xax^{-2} \). Since \( N_\mu(\mathbb{G}) \) is a normal subgroup of the group \( G \), \( b \in N_\mu(\mathbb{G}) \).
Thus, \( x \circ N_\mu(\mathbb{G}) = N_\mu(\mathbb{G}) \circ x \), for all \( x \in \mathbb{G} \).

(iii) Let \( x, y \in \mathbb{G} \) and \( a \in N_\mu(\mathbb{G}) \). Then

\[
x \circ (y \circ a) = x \circ (a^{-1}ya^2)
= a^{-2}ya^{-1}axa^{-1}ya^2
= a^{-1}y^{-1}(ya^{-1}axa^{-1}y)a^2
= a^{-1}y^{-1}([y, a^{-1}]x[a^{-1}, y]x^{-1})xy^2a^2
= a^{-1}y^{-1}([y, a^{-1}], x)xy^2a^2
= b^{-1}y^{-1}xy^2b^2\text{, where } b = a[[y, a^{-1}], x]
= (x \circ y) \circ b.
\]

Since \( [y, a^{-1}, x] \in Z(\mathbb{G}) \subseteq N_\mu(\mathbb{G}) \), \( b \in N_\mu(\mathbb{G}) \). Thus, \( x \circ (y \circ N_\mu(\mathbb{G})) = (x \circ y) \circ N_\mu(\mathbb{G}) \). Hence, 
\( N_\mu(\mathbb{G}) \) is a normal subloop.

**Proposition 2.13.** \( N_\mu(\mathbb{G}) \) is a normal subloop of the loop \( \mathbb{G} \).

**Proof.** Clearly, by Proposition 2.11(i) and (ii), \( (N_\mu(\mathbb{G}) \circ x) \circ y = N_\mu(\mathbb{G}) \circ (x \circ y) \) and \( x \circ (y \circ N_\mu(\mathbb{G})) = (x \circ y) \circ N_\mu(\mathbb{G}) \) for all \( x, y \in \mathbb{G} \). Now, let \( x \in \mathbb{G} \) and \( a \in N_\mu(\mathbb{G}) \). Then

\[
x \circ a = a^{-1}xa^2
= [a^{-1}, x]xa
= x[a^{-1}, x]a, \text{ because } [x, [a^{-1}, x]] = 1
= x^{-1}(x^2a^{-1}xax^{-1}a^{-2}x^2)
= b \circ x,
\]

where \( b = x^2a^{-1}xax^{-1}a^{-2} \). Since \( N_\mu(\mathbb{G}) \) is a normal subgroup of the group \( G \), \( b \in N_\mu(\mathbb{G}) \).
Thus, \( x \circ N_\mu(\mathbb{G}) = N_\mu(\mathbb{G}) \circ x \) for all \( x \in \mathbb{G} \). Hence, \( N_\mu(\mathbb{G}) \) is a normal subloop of the loop \( \mathbb{G} \).

**Corollary 2.14.** The nucleus \( N(\mathbb{G}) \) is a normal subloop of the loop \( \mathbb{G} \). Moreover, \( (N(\mathbb{G}), \circ) \) is group with the induced binary operation \( \circ \).

**Proof.** Since \( N(\mathbb{G}) = N_x(\mathbb{G}) \cap N_\mu(\mathbb{G}) \cap N_\rho(\mathbb{G}) \), \( N(\mathbb{G}) = N_\mu(\mathbb{G}) = N_\rho(\mathbb{G}) \) using Proposition 2.11. Hence, the corollary follows by Proposition 2.13 and Corollary 2.10.

**Lemma 2.15.** View \( C(\mathbb{G}) \) as a subgroup of the group \( G \). For \( a \in C(\mathbb{G}) \) and \( x \in \mathbb{G} \), we have

(i) \( [a, x, x] = 1 \) and \( [a, x, a] = 1 \),
(ii) \( [x^{-1}, a^{-1}] = [a, x^{-1}] = [x, a] \),
(iii) \( [x^2, a] = [x, a^2] = [x, a^{-1}] = [x^{-1}, a] \),
(iv) \( [a, x^3] = [a, x]^3 = [a^3, x] = 1 \).

**Proof.** Let \( a \in C(\mathbb{G}) \) and \( x \in G \). Then, we have

(i) \( [a, x, x] = [a, x, a] = axa^{-1}xax^{-1}a^{-1}x = ax(a^{-1}xa^2)(a^{-1}x^{-1})^2 = ax(x^{-1}ax^{-1})(a^{-1}x^{-1})^2 = a^2x^2x^{-2}a^{-2} = 1 \). By similar argument, one can obtain \( [a, x, a] = 1 \).
(ii) \[ [x^{-1}, a^{-1}] = x^{-1}a^{-1}xa = (x^{-1}a^{-1}x^2)x^{-1}a = axa^{-2}x^{-1}a = ax(a^{-1}xa^2)^{-1} = axx^{-2}a^{-1}x = [a, x^{-1}], \]
Similarly, \([a, x^{-1}] = [x, a].\)

\[
[x^2, a] = [x, [x, a]][x, a]^2, \text{ using the Lemma 2.5(i)}
\]
\[
= [x, [a, x]]^{-1}[x, a]^2
\]
\[
= [x, [a, x]]^{-1}[x, a]^2, \text{ as } [G, G, G] \subseteq Z(G)
\]
\[
= [(a, x), x][x, a]^2
\]
\[
= [x, a]^2, \text{ using part (i)}
\]

By the similar argument, \([x, a^2] = [x, a]^2.\) Now, \([x^2, a] = x^2ax^{-2}a^{-1} = x(a \circ x^{-1})a^{-1} = x(x^{-1} \circ a)a^{-1} = xax^{-1}a^2a^{-1} = xax^{-1} = [a, x^{-1}].\) By the similar argument, \([x, a^2] = [x^{-1}, a].\)

\[
[a, x^3] = [a, x^2][x^2, [a, x]][x, a], \text{ using the Lemma 2.5(ii)}
\]
\[
= [a, x^2][(a, x), x^2]^{-1}[a, x]
\]
\[
= [a, x^2][a, x, x^2]^{-2}[a, x], \text{ as } [G, G, G] \subseteq Z(G)
\]
\[
= [a, x]^3, \text{ using parts (i) and (iii).}
\]

By the similar argument, \([a^3, x] = [a, x]^3.\) Now, using parts (ii) and (iii), we have \([x, a]^3 = [x, a]^2[x, a] = [x^{-1}, a][a, x^{-1}] = 1.\) Thus, \([a, x^3] = 1.\) Similarly, \([a^2, x] = 1.\)

**Corollary 2.16.** \(C(\circ)^3 \subseteq Z(G).\)

**Proof.** Follows directly from Lemma 2.15(iv).

**Proposition 2.17.** \(Z(\circ) = C(\circ) \cap N_i(\circ),\) where \(i \in \{\lambda, \mu, \rho\}.\)

**Proof.** Let \(x, y \in \circ G.\) Then \(a \in C(\circ) \cap N_i(\circ) G\)

\[
\Leftrightarrow (a \circ x) \circ y = a \circ (x \circ y)
\]
\[
\Leftrightarrow (x \circ a) \circ y = (x \circ y) \circ a
\]
\[
\Leftrightarrow y^{-1}a^{-1}xa^2y^2 = a^{-1}y^{-1}xy^2a^2
\]
\[
\Leftrightarrow (yay^{-1}a^{-1})x = x(y^2a^2y^{-2}a^{-2})
\]
\[
\Leftrightarrow [y, a][x = x(y^2, a^2)]
\]
\[
\Leftrightarrow [y, a][x = x(y, a^4], \text{ using the Lemma 2.15(iii)}
\]
\[
\Leftrightarrow [y, a][x = x(y, a^4], \text{ using the Lemma 2.15(iv)}
\]
\[
\Leftrightarrow [a^{-1}, y][x = x(a^{-1}, y], \text{ using the Lemma 2.15(ii)}
\]
\[
\Leftrightarrow a^{-1}yay^{-1}x = xa^{-1}yay^{-1}
\]
\[
\Leftrightarrow a^{-1}xy = a^{-1}y^{-1}axa^{-1}ya
\]
\[
\Leftrightarrow a^{-1}y^{-1}xy^2a^2 = a^{-2}y^{-1}axa^{-1}yaya^2
\]
\[
\Leftrightarrow (x \circ y) \circ a = (a^{-1}ya^2)^{-1}x(a^{-1}ya^2)^2
\]
\[
\Leftrightarrow (x \circ y) \circ a = x \circ (y \circ a)
\]
\[
\Leftrightarrow a \in C(\circ) \cap N_i(\circ).\]

Thus, \(C(\circ) \cap N_i(\circ) = C(\circ) \cap N_i(\circ).\) Using Proposition 2.11(i), \(C(\circ) \cap N_i(\circ) = C(\circ) \cap N_i(\circ) = C(\circ) \cap N_i(\circ).\) Hence, \(Z(\circ) = C(\circ) \cap N(\circ) = C(\circ) \cap N_i(\circ),\) for any \(i \in \{\lambda, \mu, \rho\}.\)
In [4, Proposition 4.4, p. 1450058-10], it is proved that if 3 does not divide the order of G, then \( C(\circ G)^2 \subseteq Z(G) \). Below, we prove that the commutant of \( \circ G \) and the center of the group G are equal in this case.

**Theorem 2.18.** Let 3 does not divide the order of G. Then \( C(\circ G) = Z(G) \).

**Proof.** Let \( a \in C(\circ G) \) and \( x \in G \). Then \((xa)^2 = a^2x^2\). Since \( C(\circ G)^2 \subseteq Z(G) \), \((xa)^2 = a^2x^2 = x^2a^2\). This implies that \( ax = xa \) for all \( x \in G \). Hence, \( a \in Z(G) \). Therefore, \( C(\circ G) \subseteq Z(G) \). Since \( Z(G) \subseteq C(\circ G) \), \( C(\circ G) = Z(G) \). \( \square \)

**Corollary 2.19.** Let 3 does not divide the order of G. Then \( C(\circ G) = Z(G) = Z(\circ G) \).

**Proposition 2.20.** \( \circ G/C(\circ G) \) is a group.

**Proof.** For all \( x, y, z \in \circ G/C(\circ G) \), we have
\[
\bar{x} \circ (\bar{y} \circ \bar{z}) = x \circ C(\circ G) \circ ((y \circ C(\circ G)) \circ (z \circ C(\circ G)))
\]
\[
= (x \circ (y \circ z)) \circ C(\circ G)
\]
\[
= (x \circ z^{-1}yz^2) \circ C(\circ G)
\]
\[
= z^{-2}y^{-1}xzx^{-1}yz^2z^{-1}yz^2 \circ C(\circ G)
\]
\[
= z^{-1}(y^{-1}y)z^{-1}y^{-1}xzx^{-1}yzy^{-1}y)yz^2 \circ C(\circ G)
\]
\[
= z^{-1}y^{-1}(yz^{-1}xzx^{-1}yzy^{-1})yz^2 \circ C(\circ G)
\]
\[
= z^{-1}y^{-1}[y, z^{-1}]x[z^{-1}, y]y^2z^2 \circ C(\circ G)
\]
\[
= z^{-1}y^{-1}[y, z^{-1}], x]xy^2z^2 \circ C(\circ G)
\]
\[
= z^{-1}(y^{-1}xy^2)^2 \circ C(\circ G), \text{as } [y, z^{-1}, x] \in Z(G) \subseteq C(\circ G)
\]
\[
= ((x \circ y) \circ z) \circ C(\circ G)
\]
\[
= (\bar{x} \circ \bar{y}) \circ \bar{z}
\]
Hence, \( \circ G/C(\circ G) \) is a group. \( \square \)

**Theorem 2.21.** \( (\circ G/C(\circ G), \circ) = (\circ (G/C(\circ G)), \circ) \).

**Proof.** Since \( G \) is a nilpotent group of class 3, \( G/Z(G) \) is a nilpotent group of class 2. Now, define a group homomorphism \( \phi : G/Z(G) \to G/C(\circ G) \) by \( Z(G)x \to C(\circ G)x \). Then, one can easily observe that \( \phi \) is a surjective homomorphism. Hence, \( G/C(\circ G) \) being the homomorphic image of the group \( G/Z(G) \) is a nilpotent group of class at most 2. Therefore, by [3, Theorem 3.6], the associated right gyrogroup \( \circ (G/C(\circ G)) \) is a group.

Let \( x, y \in \circ G \). Since \( C(\circ G) \) is a normal subgroup of \( G \), \( C(\circ G) \circ x = x^{-1}C(\circ G)x = (x^{-1}C(\circ G)x)x = C(\circ G)x \). Therefore, \( (C(\circ G) \circ x) \circ (C(\circ G) \circ y) = C(\circ G) \circ (x \circ y) = C(\circ G)(x \circ y) = (C(\circ G)x) \circ (C(\circ G)y) \). Thus, \( \circ (G/C(\circ G), \circ) = \circ (G/C(\circ G), \circ) \). \( \square \)

Let 3 does not divide the order of G. Then, by **Corollary 2.19** and **Theorem 2.21**, the exact sequence
\[
0 \to Z(G) \xrightarrow{i} G \xrightarrow{\pi} G/Z(G) \to 1
\]
(2)
of the groups induces the exact sequence
\[
0 \to Z(\circ G) \xrightarrow{i} \circ G \xrightarrow{\pi} \circ (G/Z(\circ G)) \to 1
\]
(3)
of the loops. Note that, \( Z(\circ G) \) and \( G/Z(\circ G) \) are groups. Now, \( G \) can be identified with
\{(a, \tilde{x}) | a \in Z(G), \tilde{x} \in G/Z(G)\},

with the binary operation defined for all \((a, \tilde{x}), (b, \tilde{y}) \in G\) as,

\[(a, \tilde{x}) \cdot (b, \tilde{y}) = (abf(\tilde{x}, \tilde{y}), \tilde{x}\tilde{y}),\]

where \(f : G/Z(G) \times G/Z(G) \to Z(G)\) is a normalized function given by the equation

\[(\tilde{x} \cdot \tilde{y}) \cdot \tilde{z} = \tilde{x}f(\tilde{y}, \tilde{z}) \cdot (\tilde{y} \cdot \tilde{z}), \forall \tilde{x}, \tilde{y}, \tilde{z} \in G/Z(G).\]

Let \(Z^2(G/Z(G), G)\) and \(B^2(G/Z(G), Z(G))\) denotes the group of all the 2-cocycles and 2-coboundaries associated to the group extension \(G\) of the group \(Z(G)\) by the group \(G/Z(G)\). Then, for all \((a, \tilde{x}), (b, \tilde{y}) \in \text{G}\),

\[
(a, \tilde{x}) \circ (b, \tilde{y}) = (b^{-1}f(\tilde{y}, \tilde{y}^{-1})^{-1}, b^2f(\tilde{y}, \tilde{y})),
\]

By [2, Section 2], if

\[
0 \to Z(L) \xrightarrow{i} L \xrightarrow{\pi} Q \to 1
\]

is the central extension of the loop \(L\) by the loop \(Q\), then \(L\) is identified with \(Z(L) \times Q\) with the binary operation

\[
(a, \tilde{x}) \cdot (b, \tilde{y}) = (ab\phi(\tilde{x}, \tilde{y}), \tilde{x}\tilde{y}),
\]

where \(\phi : Q \times Q \to Z(L)\) is a 2-cocycle.

Thus, we define

\[
(a, \tilde{x}) \circ (b, \tilde{y}) = (a \circ b \circ f(\tilde{x}, \tilde{y}), \tilde{x} \circ \tilde{y}),
\]

where \(a \circ b = b^{-1}ab^2, \tilde{x} \circ \tilde{y} = \tilde{y}^{-1}\tilde{x}\tilde{y}^2\), and \(f(\tilde{x}, \tilde{y}) = f(\tilde{y}, \tilde{y}^{-1})^{-1}f(\tilde{y}^{-1}, \tilde{x})f(\tilde{y}, \tilde{y})f(\tilde{y}^{-1}, \tilde{x}^2), \) for all \(a, b \in Z(G)\) and \(\tilde{x}, \tilde{y} \in \text{O}(G/Z(G))\). Since, \(G\) is a nilpotent group of class 3, the associated right gyrogroup \(\text{O}(G, o)\) is actually a gyrogroup. Let \(f, g \in Z^2(G/Z(G), Z(G))\) be two 2-cocycles associated to the group extension \(G\). Then there exists a map \(\tau : G/Z(G) \to Z(G)\) such that

\[
g(\tilde{x}, \tilde{y}) = \tau(\tilde{x}) \cdot \tau(\tilde{y}) \cdot f(\tilde{x}, \tilde{y}) \cdot \tau(\tilde{x} \cdot \tilde{y})^{-1}
\]

for all \(\tilde{x}, \tilde{y} \in G/Z(G)\). Now, let \(\text{O} f, \text{O} g\) be 2-cocycles associated to the loop extension \(\text{O} G\) of the group \(Z(G)\) by the group \(\text{O} (G/Z(G))\). Then,

\[
\text{O} f(\tilde{x}, \tilde{y}) = f(\tilde{y}, \tilde{y}^{-1})^{-1}f(\tilde{y}^{-1}, \tilde{x})f(\tilde{y}, \tilde{y})f(\tilde{y}^{-1}, \tilde{x}^2)
\]

and \(\text{O} g(\tilde{x}, \tilde{y}) = g(\tilde{y}, \tilde{y}^{-1})^{-1}g(\tilde{y}^{-1}, \tilde{x})g(\tilde{y}, \tilde{y})g(\tilde{y}^{-1}, \tilde{x}^2)\).

Since, \(Z(G)\) is the center of the group \(G\), using Equation (9), we get

\[
\text{O} g(\tilde{x}, \tilde{y}) = \tau(\tilde{x}) \cdot \tau(\tilde{y}) \cdot f(\tilde{x}, \tilde{y}) \cdot \tau(\tilde{x} \cdot \tilde{y})^{-1}
\]

Thus, we define a map \(\text{O} \tau : \text{O} (G/Z(G)) \to Z(G)\) by \(\text{O} \tau(\tilde{x}) = \tau(\tilde{x})\) for all \(\tilde{x} \in \text{O} (G/Z(G))\). We will denote the map \(\text{O} \tau\) by the map \(\text{O} \tau\).

Let \(\text{O} Z^2(\text{O} (G/Z(G)), Z(G))\) (we will write \(\text{O} Z^2\) in short) be the collection of all associated 2-cocycles \(\text{O} f\) associated to the loop extensions \(\text{O} G\) of the normal subloop \(Z(G)\) by the group \(\text{O} (G/Z(G))\). Then we define a relation \(\sim\) on the set \(\text{O} Z^2\) as, for any two 2-cocycles \(\text{O} f, \text{O} g \in \text{O} Z^2\), we say that \(\text{O} f \sim \text{O} g\) if there exists a normalized map \(\tau : \text{O} (G/Z(G)) \to Z(G)\) satisfying Equation (10). One can easily observe that \(\sim\) is an equivalence relation on the set \(\text{O} Z^2\). Let
\[ \mathcal{H}^2((G/Z(G)), Z(G)) = \{ [f] \mid f \in \mathcal{Z} \} \] be the set of all the equivalence classes \([f]\) of the elements in \(\mathcal{Z}\). One can observe that, \(\mathcal{H}^2 \subseteq \mathcal{H}^2\).

### 3. Nilpotency class of the associated gyrogroup \(\mathcal{G}\)

In this section, we will find the nilpotency class of the associated gyrogroup \(\mathcal{G}\). It was proved in [4, Proposition 4.5, 1450058-11] that a group \(G\) is a nilpotent group of class 2 if and only if \(\mathcal{G}\) is a nilpotent group of class at most 2. It should be noted that ‘at most’ is missed in this proposition and this was confirmed with the author of the paper through personal communication.

**Theorem 3.1.** Let \(G\) be a group such that 3 does not divide the order of \(G\). Then \(G\) is a nilpotent group of class exactly 2 if and only if \(\mathcal{G}\) is a nilpotent group of class exactly 2.

**Proof.** Let \(G\) be a nilpotent group of class exactly 2 such that 3 does not divide the order of \(G\). Then the associated right gyrogroup \(\mathcal{G}\) is a group of class at most 2. If class of \(\mathcal{G}\) is 1, then \(Z(\mathcal{G}) = \mathcal{G}\). By Corollary 2.19, \(Z(G) = Z(\mathcal{G})\). This means that \(G = Z(G)\). Therefore, \(G\) is of class 1. Conversely, let \(\mathcal{G}\) be a nilpotent group of class 2. Then, by [3, Theorem 3.6], \(G\) is a nilpotent group of class 2. \(\square\)

**Theorem 3.2.** Let \(G\) be a group such that 3 does not divide the order of \(G\). Then \(G\) is a nilpotent group of class 3 if and only if \(\mathcal{G}\) is a nilpotent loop of class 3.

**Proof.** Let \(G\) be a nilpotent group of class 3 such that 3 does not divide the order of \(G\). Then \(G/Z(G)\) is a nilpotent group of class 2. By Theorem 3.1, the associated right gyrogroup \(\mathcal{G}(G/Z(G))\) is a nilpotent group of class 2. Also, by Theorem 2.21 and Corollary 2.19, \(\mathcal{G}(Z(G))\) is a group of class 2. Hence, \(\mathcal{G}\) is a nilpotent loop of class 3. Conversely, let \(\mathcal{G}\) be a nilpotent loop of class 3. Then, \(\mathcal{G}/Z(\mathcal{G}) = \mathcal{G}(G/Z(G))\) is a nilpotent group of class 2. Hence, by [3, Theorem 3.6], \(G/Z(G)\) is a nilpotent group of class 2. Therefore, \(G\) is a nilpotent group of class 3. \(\square\)

**Theorem 3.3.** If \(G\) is a nilpotent group of class 3, then \(\mathcal{G}\) is nilpotent loop of class 2 if and only if \([x, y]^3 \in Z(\mathcal{G})\) for all \(x, y \in G\).

**Proof.** Let \(x, y \in \mathcal{G}\). Since \(x \circ y = [x, y] \circ (y \circ x)\),

\[\begin{align*}
\mathcal{G}[x, y] &= (x \circ y) \circ (y \circ x)^{-1} \\
&= y^{-1}xy^2 \circ (x^{-1}yx^2)^{-1} \\
&= x^{-1}yx^2y^{-1}xy^2(x^{-2}y^{-1}x)^2 \\
&= x^{-1}yx^2y^{-1}x^2y^2x^{-2}y^{-1}x^{-1}y^{-1}x \\
&= (x^{-1}yx)(xy^{-1})(xy^{-1}x)(x^{-1}y^{-1}x) \\
&= (x^{-1}yx)(xy^{-1})(xy^{-1}x)(x^2y^{-1}x)(xy^{-1}x) \\
&= [x^{-1}, y][y, x][x^2, y^2][y, x^{-1}] \\
&= [y, x][x^2, y^2], \text{ as } [G, G] \text{ is abelian} \\
&= [y, x][x^2, y^2][y, [x, y]], \text{ using Lemma 2.5(ii), and } [G, G, G] \subseteq Z(G) \\
&= [y, x][x, [x, y]][x, y]^2[y, [x, y]][x, y]^2, \text{ using Lemma 2.5(i),} \\
&= [y, x][x, [x, y]^2][y, [x, y]][x, y]^2[y, [x, y]][x, y]^2 \text{ and Lemma 2.5(i),} \\
&= [y, x][x, [x, y]^2][y, [x, y]]^4[y, [x, y]][x, y]^2. 
\end{align*}\]
Thus,
\[ ^0[x, y] = [x, y]^3[x, [x, y]]^2[y, [x, y]]^2. \tag{11} \]

Now, \(^oG\) is nilpotent of class 2 \iff \(^oG/Z(\^oG)\) is abelian \iff \(^o[x, y] \in Z(\^oG)\) for all \(x, y \in \^oG\). Since \(G\) is a nilpotent group of class 3, \([G, G, G] \subseteq Z(G)\) and \(Z(G) \subseteq Z(\^oG) \subseteq C(\^oG)\), \(^oG\) is nilpotent loop of class 2 \iff \([x, y]^3 \in Z(\^oG)\).

**Corollary 3.4.** If \(G\) is a nilpotent group of class 3, then \(^oG\) is nilpotent loop of class 2 if and only if \([x, y]^3 \in C(\^oG)\) for all \(x, y \in G\).

**Proof.** Since, for all \(u, v \in \^oG, [u, [x, y]^3, v] = 1\), \([x, y]^3 \in N_p(\^oG)\). Also, by Proposition 2.11, \(N_p(\^oG) = N(\^oG)\). Therefore, if \([x, y]^3 \in C(\^oG)\), then \([x, y]^3 \in Z(\^oG)\). Hence, using Theorem 3.3, \(^oG\) is nilpotent loop of class 2 \iff \([x, y]^3 \in C(\^oG)\).

**Corollary 3.5.** Let \(G\) be a 2-Engel group. Then, \(^oG\) is a nilpotent loop of class 2.

**Proof.** Let \(G\) be a 2-Engel group. Then, \(G\) is of class at most 3 and \([x, y, z]^3 = 1\) for all \(x, y, z \in G\). Therefore, \([[[x, y]^3], z] = 1\) which implies that \([x, y]^3 \in Z(G)\). Thus, \([x, y]^3 \in C(\^oG)\) and the corollary follows from Theorem 3.3.

**Corollary 3.6.** Let \(G\) be a group of exponent 3, then \(^oG\) is a nilpotent loop of class 2.

**Proof.** Follows immediately from Corollary 3.5 and the fact that every group of exponent 3 is a 2-Engel group.

### 4. Problem of abelian inner mapping groups

It was an open problem whether there exists a loop of nilpotency class 3 with abelian inner mapping group. Csorgo in [1] gave its answer in affirmative by giving a loop of order \(2^7\). This problem is still open for odd case, that is whether there exists an odd order loop of nilpotency class 3 whose inner mapping group is abelian. In this section, we investigate when one hopes to find the answer of this problem for the loop of order \(3^n\) for some positive integer \(n\).

In a talk of the first conference on Artificial Intelligence and Theorem Proving in the year 2016, M. Kinyon along with B. Veroff gives the following theorem (see [link to slides]).

**Theorem 4.1.** (Unpublished). Let \(Q\) be a loop. Then

(i) \(Q/N(Q)\) is an abelian group, \(Q/Z(Q)\) is a group, and \(K(\cdot, \cdot)\) is associative, then \(Inn(Q)\) is abelian.

(ii) If \(Inn(Q)\) is abelian, then the loop commutator is associative.

Therefore, we would like to get the conditions of Theorem 4.1 satisfied to find the answer in affirmative.

**Proposition 4.2.** \(A(\^oG) \subseteq Z(\^oG)\).
Proof. Let \( x, y, z \in G \). Then, for \( A(x, y, z) \in A(G) \) we have,

\[
A(x, y, z) = \left( (x \circ y) \circ z \right) \circ \left( (x \circ (y \circ z)) \right)^{-1}
\]

\[
= (z^{-1}y^{-1}xy^2z^2) \circ (z^{-2}y^{-1}xz^{-1}yz^{-2})^{-1}
\]

\[
= (z^{-2}y^{-1}xz^{-1}yz^{-2}(z^{-1}y^{-1}xy^2z^2)(z^{-2}y^{-1}xz^{-1}yz^{-2})^{-2}
\]

\[
= z^{-2}y^{-1}xz^{-1}yz^{-2}y^{-1}xz^{-1}y^{-1}xz^{-1}z^{-1}y^2
\]

\[
= z^{-2}y^{-1}xz^{-1}yz^{-2}z^{-2}y^{-1}xz^{-1}y^2
\]

\[
= \left[ [z^{-1}, y], x \right] z^{-2}y^{-1}xz^{-1}y^2
\]

\[
= \left[ [z^{-1}, y], x \right], \text{for} \left[ [z^{-1}, y], x \right] \in Z(G).
\]

Thus, \( A(x, y, z) = \left[ [z^{-1}, y], x \right] \in Z(G) \), for all \( x, y, z \in G \). Hence, \( A(G) \subseteq Z(G) \).

\[
\text{Proposition 4.3.} \quad (G^G/N(G), \circ) \text{ is an abelian group.}
\]

Proof. By Proposition 4.2, \( A(G) \subseteq Z(G) \) and the fact that \( Z(G) \subseteq N(G), (G^G/N(G), \circ) \) is a group. Since \( G \) is of class 3, for all \( u, v, x, y \in G \),

\[
[x, [[u, v], y]] = 1.
\]

Therefore, \( [u, v] \in N_u(G) = N(G) \) for all \( u, v \in G \). Hence, by Equation (11), \( ^0[u, v] \in N(G) \). Thus, \( (G^G/N(G), \circ) \) is an abelian group.

\[
\text{Proposition 4.4.} \quad (G^G/Z(G), \circ) \text{ is a group.}
\]

Proof. Follows directly from the fact that \( Z(G) \subseteq Z(G) \) and Proposition 4.2.

\[
\text{Proposition 4.5.} \quad \text{The commutator operation} \quad ^0[\cdot, \cdot] : G^G \times G \to G \text{ is associative if and only if} \quad [[x, y], z]^9 = [x, [y, z]]^9 \quad \text{for all} \quad x, y, z \in G.
\]

Proof. Let \( x, y, z \in G \). Then by Equation (11), we have

\[
^0[\circ, [x, y], z] = \left[ \circ [x, y], z \right]^3 \left[ \circ [x, y], \circ [x, y], z \right]^2 \left[ z, \circ [x, y], z \right]^2.
\]

Now, using the fact that \( G \) is nilpotent group of class 3 and Lemma 2.5(i), \( \circ [x, y], z] = [x, y]^3, z \]. Since, \( [x, y, z] \in Z(G), [[x, y, z], z] = [[x, y], z]^3 \). Therefore, we get

\[
^0[\circ, [x, y], z] = [x, [y, z]]^9.
\]

By the similar argument, we have

\[
^0[\circ, [x, y], z] = [x, [y, z]]^9.
\]

Thus, the commutator operation \( [\cdot, \cdot] \) is associative

\[
\iff^0[\circ, [x, y], z] = [x, [y, z]]^9 \iff [[x, y], z]^9 = [x, [y, z]]^9,
\]

for all \( x, y, z \in G \).

\[
\text{Proposition 4.6.} \quad \text{Let} \quad G \quad \text{be a group such that} \quad 3 \quad \text{does not divide the order of the group} \quad G. \quad \text{Then the commutator operation} \quad ^0[\cdot, \cdot] : G^G \times G \to G \text{ is not associative.}
\]

Proof. On contrary, suppose that the commutator operation \( ^0[\cdot, \cdot] \) is associative. Therefore, by Proposition 4.5, \( [[x, y], z]^9 = [x, [y, z]]^9 \) for all \( x, y, z \in G \). Since, 3 does not divide the order of the
group $G$, 9 does not divide the order of the group $G$. Therefore, there is an isomorphism from $G$ to $G$ given by $x \rightarrow x^9$ which gives $[[x, y], z] = [x, [y, z]]$. Thus, the commutator operation $[\cdot, \cdot] : G \times G \rightarrow G$ is associative. This is a contradiction, by the Levi’s theorem [5].

**Theorem 4.7.** Let $G$ be a group such that 3 does not divide the order of the group $G$. Then $\text{Inn}^G$ is not abelian.

**Proof.** On contrary, suppose that $\text{Inn}^G$ is abelian. Then by Theorem 4.1(ii), the commutator operation $^\circ[\cdot, \cdot]$ is associative. This is a contradiction, by Proposition 4.6.

Note that, if $G_1$ and $G_2$ are two groups, then $^\circ(G_1 \times G_2) = ^\circ G_1 \times ^\circ G_2$. Since $G$ is nilpotent, it is sufficient to discuss about the associated gyrogroup $^\circ G$ for 3-groups $G$ of nilpotency class 3. According to Theorem 3.3, $^\circ G$ is nilpotent loop of class 2 if and only if $[x, y]^3 \in C(^\circ G)$ for all $x, y \in G$. If $[G, G]$ is of exponent 3, then by Theorem 3.3, $^\circ G$ is nilpotent loop of class 2. Therefore, if there is a 3-group for which $[x, y]^3 \not\in C(^\circ G)$, exponent of $[G, G]$ is not 3 and $[[x, y], z]^9 = [x, [y, z]]^9$ for all $x, y, z \in G$, then one can hope to get a loop of class 3 with abelian inner mapping group.

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