R-MATRIX FORMULATION OF THE QUANTUM INHOMOGENEOUS GROUPS ISO$q,r(N)$ AND IS$p,q,r(N)$.

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Abstract

The quantum commutations $RTT = TTR$ and the orthogonal (symplectic) conditions for the inhomogeneous multiparametric $q$-groups of the $B_n, C_n, D_n$ type are found in terms of the $R$-matrix of $B_{n+1}, C_{n+1}, D_{n+1}$. A consistent Hopf structure on these inhomogeneous $q$-groups is constructed by means of a projection from $B_{n+1}, C_{n+1}, D_{n+1}$. Real forms are discussed: in particular we obtain the $q$-groups $ISO_{q,r}(n+1, n-1)$, including the quantum Poincaré group.
Inhomogeneous groups play an important role in many physical situations, for instance when translations enter the game. One fundamental example is Einstein-Cartan gravity, whose algebraic basis is the Poincaré group. After the discovery of \(q\)-deformed simple Lie groups \([1, 2]\), it was natural to construct the corresponding \(q\)-deformed gauge theories \([3, 4]\). A similar program can be applied to inhomogeneous \(q\)-groups, and indeed in ref. \([5]\) a \(q\)-deformation of Poincaré gravity was found, one of the main motivations being the possibility of \(q\)-regularizing gravity. The gauge program relies on the bicovariant calculus on \(q\)-groups: for a review see for ex. \([6]\) and references quoted therein.

We present in this Letter the \(R\)-matrix formulation of multiparametric inhomogeneous \(q\)-groups, whose homogeneous part are the \(B_n, C_n, D_n\) \(q\)-groups. This extends to the orthogonal and symplectic case the treatment of ref. \([7, 8]\), where the multiparametric (uniparametric in \([7]\)) \(q\)-groups \(IGL_{q,r}(N)\) and their associated differential calculi were constructed via a projection from \(GL_{q,r}(N+1)\). Some of the references on the quantum inhomogeneous groups are collected in \([9]\).

The method used in \([7, 8]\) to obtain \(IGL_{q,r}(N)\), and in this Letter to obtain \(ISO_{q,r}(N)\) and \(ISp_{q,r}(N)\), is based on a consistent projection from the corresponding quantum groups of higher rank \(A_{n+1}, B_{n+1}, C_{n+1}, D_{n+1}\). By consistent we mean that it is compatible (or “commutes”) with the Hopf structure of the \(q\)-groups, as we will see in the sequel. This method was in fact already exploited in ref. \([5]\) to obtain bicovariant differential calculi on \(ISO_{q,r=1}(N)\) and \(ISp_{q,r=1}(N)\).

We give here the explicit structures of \(ISO_{q,r}(N)\) and \(ISp_{q,r}(N)\), and show that they are Hopf algebras by proving that they can be obtained as quotients of \(SO_{q,r}(N+2)\) and \(Sp_{q,r}(N+2)\) with respect to a suitable Hopf ideal. The projection from \(SO_{q,r}(N+2)\) \([Sp_{q,r}(N+2)]\) to the quotient is introduced and found to be an Hopf algebra epimorphism. The (bicovariant) differential calculi on the multiparametric \(ISO_{q,r}(N)\) \([ISp_{q,r}(N)]\), found by means of this projection method, will be presented in a separate publication.

We begin by recalling some basic facts about the \(B_n, C_n, D_n\) multiparametric quantum groups. They are freely generated by the noncommuting matrix elements \(T_{ab}\) (fundamental representation) and the identity \(I\). The noncommutativity is controlled by the \(R\) matrix:

\[
R^{ab}_{\quad ef} T^{e}_{c} T^{f}_{d} = T^{b}_{f} T^{a}_{e} R^{ef}_{\quad cd} \tag{1}
\]

which satisfies the quantum Yang-Baxter equation

\[
R^{a_{1}b_{1}}_{\quad a_{2}b_{2}} R^{a_{2}c_{1}}_{\quad a_{3}c_{2}} R^{b_{2}c_{2}}_{\quad b_{3}c_{3}} = R^{b_{1}c_{1}}_{\quad b_{2}c_{2}} R^{a_{1}c_{1}}_{\quad a_{2}c_{2}} R^{a_{2}b_{2}}_{\quad a_{3}b_{3}}, \tag{2}
\]

a sufficient condition for the consistency of the “\(RTT\)” relations \([1]\). The \(R\)-matrix components \(R^{ab}_{\quad cd}\) depend continuously on a (in general complex) set of parameters \(q_{ab}, r\). For \(q_{ab} = q, r = q\) we recover the uniparametric \(q\)-groups of ref. \([2]\). Then
$q_{ab} \to 1, r \to 1$ is the classical limit for which $R_{cd}^{ab} \to \delta_c^a \delta_d^b$: the matrix entries $T^a_b$ commute and become the usual entries of the fundamental representation. The multiparametric $R$ matrices for the $A, B, C, D$ series can be found in [10] (other ref.s on multiparametric $q$-groups are given in [11, 12]). For the $B, C, D$ case they read:

$$R_{cd}^{ab} = \frac{\delta_c^a \delta_d^b}{\epsilon_a \epsilon_d} + \frac{r}{r-1} \delta_c^a \delta_d^b \left(1 - \delta^{an^2} + \delta_a^b \delta_c^d \delta_d^a \delta_c^b \right)$$

where $\theta^{ab} = 1$ for $a > b$ and $\theta^{ab} = 0$ for $a \leq b$; we define $n_2 \equiv \frac{N+1}{2}$ and primed indices as $\alpha' \equiv N + 1 - \alpha$. The indices run on $N$ values ($N$-dimension of the fundamental representation $T^a_b$), with $N = 2n + 1$ for $B_n[SO(2n + 1)]$, $N = 2n$ for $C_n[Sp(2n)]$, $D_n[SO(2n)]$. The terms with the index $n_2$ are present only for the $B_n$ series. The $\epsilon_a$ and $\rho_a$ vectors are given by:

$$\epsilon_a = \begin{cases} +1 & \text{for } B_n, D_n; \\ +1 & \text{for } C_n \text{ and } a \leq n; \\ -1 & \text{for } C_n \text{ and } a > n. \end{cases}$$

Moreover the following relations reduce the number of independent $q_{ab}$ parameters [10]:

$$q_{aa} = r, \quad q_{ba} = \frac{r^2}{q_{ab}}$$

$$q_{ab} = \frac{r^2}{q_{ba}} = \frac{r^2}{q_{a'b'}} = q_{a'b'}$$

where (9) also implies $q_{aa'} = r$. Therefore the $q_{ab}$ with $a < b \leq \frac{N}{2}$ give all the $q$’s.

It is useful to list the nonzero complex components of the $R$ matrix (no sum on repeated indices):

$$\begin{align*}
R_{aa}^{aa} &= r, & a \neq n_2 \\
R_{aa'}^{aa'} &= r^{-1}, & a \neq n_2 \\
R_{a}^{n_2n_2} &= 1 \\
R_{ab}^{ab} &= \frac{r}{q_{ab}}, & a \neq b, a' \neq b \\
R_{ab}^{ba} &= r - r^{-1}, & a > b, a' \neq b \\
R_{a'a'}^{a'a'} &= (r - r^{-1})(1 - r^2) \epsilon_a \epsilon_d \rho_{a'} \rho_{a'}, & a > d' \\
R_{a'b'}^{b'a'} &= -(r - r^{-1}) \epsilon_a \epsilon_d \rho_{a'} \rho_{b'}, & a > b, a' \neq b
\end{align*}$$

where $\epsilon = \epsilon_a \epsilon_{a'}$, i.e. $\epsilon = 1$ for $B_n, D_n$ and $\epsilon = -1$ for $C_n$.

Remark 1: The matrix $R$ has the following symmetry:

$$R_{cd}^{ab} = R_{a'b'}^{c'd'}$$

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Remark 2: If we denote by \( q, r \) the set of parameters \( q_{ab}, r \), we have
\[
R_{q,r}^{-1} = R_{q^{-1},r^{-1}}^{-1}
\] (10)
The inverse \( R^{-1} \) is defined by \((R^{-1})^{ab}_{cd} R^{cd}_{ef} = \delta^a_e \delta^b_f = R^{ab}_{cd} (R^{-1})^{cd}_{ef}\). Eq. (10) implies that for \(|q| = |r| = 1\), \( \hat{R} = R^{-1} \).

Remark 3: Let \( R_r \) be the uniparametric \( R \) matrix for the \( B, C, D \) \( q \)-groups. The multiparametric \( R_{q,r} \) matrix is obtained from \( R_r \) via the transformation
\[
R_{q,r} = F^{-1} R_r F^{-1}
\] (11)
where \((F^{-1})^{ab}_{cd}\) is a diagonal matrix in the index couples \( ab, cd \):
\[
F^{-1} \equiv \text{diag}(\sqrt{\frac{r}{q_{11}}}, \sqrt{\frac{r}{q_{12}}}, \ldots, \sqrt{\frac{r}{q_{NN}}})
\] (12)
where \( ab, cd \) are ordered as in the \( R \) matrix. Since \( \sqrt{\frac{r}{q_{ab}}} = (\sqrt{\frac{q_{ba}}{r}})^{-1} \) and \( q_{aa} = q_{bb} \), the non diagonal elements of \( R_{q,r} \) coincide with those of \( R_r \). The matrix \( F \) satisfies \( F_{12} F_{21} = 1 \) i.e. \( F^{ab}_{ef} F^{fe}_{dc} = \delta^a_c \delta^b_d \), the quantum Yang-Baxter equation \( F_{12} F_{13} F_{23} = F_{23} F_{13} F_{12} \) and the relations \((R_r)_{12} F_{13} F_{23} = F_{23} F_{13} (R_r)_{12}\).

Remark 4: Let \( \hat{R} \) the matrix defined by \( \hat{R}^{ab}_{cd} \equiv R^{ba}_{cd} \). Then the multiparametric \( \hat{R}_{q,r} \) is obtained from \( \hat{R}_r \) via the similarity transformation
\[
\hat{R}_{q,r} = F \hat{R}_r F^{-1}
\] (13)
The characteristic equation and the projector decomposition of \( \hat{R}_{q,r} \) are therefore the same as in the uniparametric case, and we have
\[
(\hat{R} - r I)(\hat{R} + r^{-1} I)(\hat{R} - \epsilon r^{-N} I) = 0
\] (14)
\[
\hat{R} = r P_S - r^{-1} P_A + \epsilon r^{-N} P_0
\] (15)
with
\[
P_S = \frac{1}{r + r^{-1}} [\hat{R} + r^{-1} I - (r^{-1} + \epsilon r^{-N}) P_0]
\]
\[
P_A = \frac{1}{r + r^{-1}} [-\hat{R} + r I - (r - \epsilon r^{-N}) P_0]
\]
\[
P_0 = Q_N(r) K
\]
\[
Q_N(r) \equiv (C^{ab} C^{ab})^{-1} = \frac{1 - r^2}{(1 - r^{N+1}) (1 + r^{N+1} + \epsilon)}
\]
\[
K^{ab}_{cd} = C^{ab} C^{cd}
\]
(16)
I = P_S + P_A + P_0
Orthogonality (and symplecticity) conditions can be imposed on the elements \( T^a_{\ b} \), consistently with the \( RTT \) relations [:]
\[
C^{bc} T^a_{\ b} T^d_{\ c} = C^{ad} I
\]
\[
C_{ac} T^a_{\ b} T^d_{\ c} = C_{bd} I
\] (17)
where the (antidiagonal) metric is:

\[ C_{ab} = \epsilon_ar^{-\rho_a}\delta_{ab'} \]  

(18)

and its inverse \( C^{ab} \) satisfies \( C^{ab}C_{bc} = \delta_a^c = C_{cb}C^{ba} \). We see that for the orthogonal series, the matrix elements of the metric and the inverse metric coincide, while for the symplectic series there is a change of sign: \( C^{ab} = \epsilon C_{ab} \). Notice also the symmetry \( C_{ab} = C_{b'a'} \).

The consistency of (17) with the RTT relations is due to the identities:

\[ C_{ab}R^{bc}_{\phantom{bc}de} = (R^{-1})^{cf}_{\phantom{cf}ad}C^{fe} \]  

(19)

\[ R^{bc}_{\phantom{bc}de}C^{ea} = C^{bf}(R^{-1})^{ca}_{\phantom{ca}fd} \]  

(20)

These identities hold also for \( R \mapsto R^{-1} \) and can be proved using the explicit expression (8) of \( R \).

We note the useful relations, easily deduced from (15):

\[ C_{ab}\hat{R}^{ab}_{\phantom{ab}cd} = \epsilon\epsilon_{r^l}C_{cd}^{l} \]  

(21)

\[ \hat{R}^{ab}_{\phantom{ab}cd}C_{cd}^{\phantom{cd}ef} = C_{ab}^{ef}C_{ab}^{\phantom{ab}ef} \]  

(22)

The metric \( C \) can be used to express the symmetry property (9) in the covariant notation:

\[ R^{ab}_{\phantom{ab}cd} = C^{cp}_{\phantom{cp}Cq}R^{pq}_{\phantom{pq}ef}C^{eq}_{\phantom{eq}Cf}C^{bf}_{\phantom{bf}ae} = C^{ae}_{\phantom{ae}Cf}R^{pq}_{\phantom{pq}ef}C^{qc}_{\phantom{qc}Cp}C^{qd}_{\phantom{qd}Cq} \]  

(23)

The co-structures of the \( B, C, D \) multiparametric quantum groups have the same form as in the uniparametric case: the coproduct \( \Delta \), the counit \( \varepsilon \) and the coinverse \( \kappa \) are given by

\[ \Delta(T^a_{\phantom{a}b}) = T^a_{\phantom{a}b} \otimes T^b_{\phantom{b}c} \]  

(23)

\[ \varepsilon(T^a_{\phantom{a}b}) = \delta^a_b \]  

(24)

\[ \kappa(T^a_{\phantom{a}b}) = C^{ac}T^d_{\phantom{d}c}C_{db} \]  

(25)

A conjugation (i.e. algebra antihomomorphism, coalgebra homomorphism and involution, satisfying \( \kappa((T^*)^*) = T \)) can be defined

- trivially as \( T^* = T \). Compatibility with the RTT relations (1) requires \( R_{q,r} = R_{q^{-1},r^{-1}} \), i.e. \(|q| = |r| = 1\). Then the CTT relations are invariant under \(*\)-conjugation. The corresponding real forms are \( SO_{q,r}(n,n;\mathbb{R}) \), \( SO_{q,r}(n,n+1;\mathbb{R}) \) (for \( N \) even and odd respectively) and \( Sp_{q,r}(n;\mathbb{R}) \).

- via the metric as \( T^* = (\kappa(T))^t \). The condition on \( R \) is \( R_{q,r}^{ab}_{\phantom{ab}cd} = R_{q,r}^{dc}_{\phantom{dc}ba} \), which happens for \( q_{ab}q_{ab} = r^2, r \in \mathbb{R} \). Again the CTT relations are \(*\)-invariant. The metric on a “real” basis has compact signature \(+,+,+,...+\) so that the real form is \( SO_{q,r}(N;\mathbb{R}) \).

- as \( (T^a_{\phantom{a}b})^* = T^{a'}_{\phantom{a'}b'} \). This conjugation, as far as we know, has never been discussed in the literature. The conditions on \( R \) are \( R_{q,r}^{ab}_{\phantom{ab}cd} = R_{q,r}^{b'a'}_{\phantom{b'a'}d'e'} \), and due to (2)
they turn out to be the same as for the preceding conjugation. The compatibility with the CTT relations follows from \( \tilde{C}_{ab} = C_{ba} \) (when \( r \in \mathbb{R} \)).

- there is also a fourth way \( [3] \) to define a conjugation on the orthogonal quantum groups \( SO_{q,r}(2n, \mathbb{C}) \), which extends to the multiparametric case the one proposed by the authors of ref. \([13]\) for \( SO_{q}(2n, \mathbb{C}) \). The conjugation is defined by:

\[
(T^a)_{b}^* = \mathcal{D}_c^a T^c_d \mathcal{D}_b^d
\]

\( \mathcal{D} \) being the matrix that exchanges the index \( n \) with the index \( n+1 \). This conjugation is compatible with the coproduct: \( \Delta(T^*) = (\Delta T)^* \); for \( |r| = 1 \) it is also compatible with the orthogonality relations \([17]\) (due to \( \bar{C} = C^T \) and also \( \mathcal{D}_{CT} \mathcal{D} = C \)) and with the antipode: \( \kappa(\kappa(T^*)) = T \). Compatibility with the RTT relations is easily seen to require

\[
(\bar{R})_{n+n+1} = R^{-1},
\]

which implies

i) \( |q_{ab}| = |r| = 1 \) for and \( b \) both different from \( n \) or \( n+1 \);

ii) \( q_{ab}/r \in \mathbb{R} \) when at least one of the indices \( a, b \) is equal to \( n \) or \( n+1 \).

This last conjugation leads to the real form \( SO_{q,r}(n+1, n-1; \mathbb{R}) \), and is in fact the one needed to obtain \( ISO_{q,r}(3,1; \mathbb{R}) \), as discussed in ref. \([4]\) and later in this Letter.

Finally, we consider the \( R \) matrix for the \( SO_{q,r}(N+2) \) and \( Sp_{q,r}(N+2) \) quantum groups. Using formula \([3]\) or \([5]\), we find that it can be decomposed in terms of \( SO_{q,r}(N) \) and \( Sp_{q,r}(N) \) quantities as follows (splitting the index \( A \) as \( A=(o, a, \bullet) \), with \( a = 1, \ldots N \)):

\[
R^{AB}_{\quad CD} = \begin{pmatrix}
\circ o & \bullet \circ & \bullet \circ & \circ d & \bullet d & \circ c & \bullet c & cd \\
\circ o & r & 0 & 0 & 0 & 0 & 0 & 0 \\
\bullet o & 0 & r^{-1} & 0 & 0 & 0 & 0 & 0 \\
\bullet o & 0 & f(r) & r^{-1} & 0 & 0 & 0 & 0 \\
\bullet o & 0 & 0 & r & 0 & 0 & 0 & 0 \\
\bullet o & 0 & 0 & 0 & r & 0 & 0 & 0 \\
\circ b & 0 & 0 & 0 & 0 & \frac{r q_{ob}}{q^b_d} \delta^b_d & 0 & 0 & 0 \\
\bullet b & 0 & 0 & 0 & 0 & 0 & \frac{r q_{ob}}{q^b_d} \delta^b_d & 0 & \lambda^b_d \\
\circ a & 0 & 0 & 0 & 0 & \lambda^a_d & 0 & \frac{r q_{oa}}{q^a_c} \delta^a_c & 0 \\
\bullet a & 0 & 0 & 0 & 0 & 0 & \frac{r q_{oa}}{q^a_c} \delta^a_c & 0 & 0 \\
ab & 0 & -C^{ba} \lambda r^{-\rho} & 0 & 0 & 0 & 0 & 0 & R^{ab}_{\quad cd}
\end{pmatrix}
\]

where \( R^{ab}_{\quad cd} \) is the \( R \) matrix for \( SO_{q,r}(N) \) or \( Sp_{q,r}(N) \), \( C_{ab} \) is the corresponding metric, \( \lambda \equiv r - r^{-1} \), \( \rho = \frac{N+1}{2} \) (\( r^\rho = C_{\bullet} \)) and \( f(r) \equiv \lambda(1 - e^{-2 \rho}) \). The sign \( \epsilon \) has been defined after eq. s \([8]\).

**Theorem 1:** the quantum inhomogeneous groups \( ISO_{q,r}(N) \) and \( ISp_{q,r}(N) \) are freely generated by the non-commuting elements

\[
T^a_{\quad b}, x^a, v, u \equiv v^{-1} \text{ and the identity } I \quad (a = 1, \ldots N)
\]

(29)
modulo the relations:

\[
R^a_{\, \, ef} T^e_c T^f_d = T^b_f T^a_e R^c_{\, \, fd} \quad (30)
\]
\[
T^a_c C^{bc} T^d_c = C^{ad} I \quad (31)
\]
\[
T^a_c C^{ac} T^c_d = C^{bd} I \quad (32)
\]
\[
T^b_d x^a = \frac{r}{q_d} R^a_{\, \, ef} T^e_c T^f_d \quad (33)
\]
\[
P^a_{\, \, cd} x^c x^d = 0 \quad (34)
\]
\[
T^b_d v = \frac{q_b v}{q_d} T^b_d \quad (35)
\]
\[
x^b v = q_b v x^b \quad (36)
\]
\[
u v = vu = I \quad (37)
\]
\[
u x^b = q_b x^b u \quad (38)
\]
\[
u T^b_d = \frac{q_b}{q_d} T^b_d u \quad (39)
\]

where \( q_{\bullet} \) are \( N \) free complex parameters. The matrix \( P_A \) in eq. (34) is the \( q \)-antisymmetrizer for the \( B, C, D \) \( q \)-groups given by (cf. (10)):

\[
P^a_{\, \, cd} = -\frac{1}{r + r^{-1}} (\hat{R}^a_{\, \, cd} - r \delta^a_c \delta^b_d + \frac{r - r^{-1}}{\epsilon r N - 1 - \epsilon + 1} C^{ab} C_{cd}) \quad (40)
\]

The co-structures are given by:

\[
\Delta(T^a_b) = T^a_c T^c_b \quad (41)
\]
\[
\Delta(x^a) = T^a_c x^c + x^a \otimes v \quad (42)
\]
\[
\Delta(v) = v \otimes v \quad (43)
\]
\[
\Delta(u) = u \otimes u \quad (44)
\]
\[
\kappa(T^a_b) = C^{ac} T^d_c C_{db} = \epsilon_a \epsilon_b r^{-\rho_a + \rho_b} T^{b'}_{\, \, a'} \quad (45)
\]
\[
\kappa(x^a) = -\kappa(T^a_c) x^c u \quad (46)
\]
\[
\kappa(v) = \epsilon u \quad (47)
\]
\[
\kappa(u) = \epsilon v \quad (48)
\]
\[
\epsilon(T^a_b) = \delta^a_b \quad ; \quad \epsilon(x^a) = 0 \quad ; \quad \epsilon(u) = \epsilon(v) = \epsilon(I) = 1 \quad (49)
\]

In the commutative limit \( q \to 1, r \to 1 \) we recover the algebra of functions on \( ISO(N) \) and \( ISp(N) \) (plus the dilatation \( v \) that can be set to the identity).

**Proof**: our strategy will be to prove that the quantum groups \( ISO_{q,r}(N) \) and \( ISp_{q,r}(N) \) can be derived as the quotients

\[
\frac{SO_{q,r}(N + 2)}{H}, \quad \frac{Sp_{q,r}(N + 2)}{H} \quad (50)
\]
where $H$ is a suitable Hopf ideal in $SO_{q,r}(N + 2)$ or $Sp_{q,r}(N + 2)$. Then the Hopf structure of the groups in the numerators of (50) is naturally inherited by the quotient groups $[14]$. We indicate by $T^A_B$ the basic elements of $SO_{q,r}(N + 2)$ or $Sp_{q,r}(N + 2)$, with the index convention $A=(_\circ, a, _\bullet)$, $a = 1, ..., N$, induced by the $R$ matrix of (28).

The space $H$ is defined as the space of all sums of monomials containing at least an element of the kind $T^\circ_a T^\bullet_b T^\bullet_o$.

We introduce the following convenient notations: $T$ stands for $T^\circ_a T^\bullet_b$, $T^\bullet_o$ or $T^\bullet_b T^\circ_o$, $S_{q,r}(N + 2)$ stands for either $SO_{q,r}(N + 2)$ or $Sp_{q,r}(N + 2)$, and we indicate by $\Delta_{N+2}$, $\varepsilon_{N+2}$ and $\kappa_{N+2}$ the corresponding co-structures.

We start the proof of Theorem 1 by proving first the important Lemma:

**Lemma:** the space $H$ is a Hopf ideal in $S_{q,r}(N + 2)$, that is, if
i) $H$ is a two-sided ideal in $S_{q,r}(N + 2)$,
ii) $H$ is a co-ideal, i.e.
$$\Delta_{N+2}(H) \subseteq H \otimes S_{q,r}(N + 2) + S_{q,r}(N + 2) \otimes H; \quad \varepsilon_{N+2}(H) = 0$$ (51)
iii) $H$ is compatible with $\kappa_{N+2}$:
$$\kappa_{N+2}(H) \subseteq H$$ (52)

**Proof:**

i) $H$ is trivially a subalgebra of $S_{q,r}(N + 2)$. It is a right and left ideal since $\forall h \in H, \forall a \in S_{q,r}(N + 2)$, $ha \in H$ and $ah \in H$. This follows immediately from the definition of $H$ as sums of monomials containing at least a factor $T$. $H$ is the ideal in $S_{q,r}(N + 2)$ generated by the elements $T$.

ii) First notice that $\Delta_{N+2}(T) \in H \otimes S_{q,r}(N + 2) + S_{q,r}(N + 2) \otimes H$. Now by definition of $H$ we have
$$\forall h \in H, \quad h = bTc, \quad b, c \in S_{q,r}(N + 2).$$ (53)
where $bTc$ represents a sum of monomials. Then we find
$$\Delta_{N+2}(h) = \Delta_{N+2}(b)\Delta_{N+2}(T)\Delta_{N+2}(c) \in H \otimes S_{q,r}(N + 2) + S_{q,r}(N + 2) \otimes H.$$. (54)
Moreover, since $\varepsilon_{N+2}$ vanishes on $T$ we have:
$$\varepsilon_{N+2}(h) = 0, \quad \forall h \in H.$$ (55)
These relations ensure that (51) holds.

iii)
$$\kappa_{N+2}(T^\circ_a) = C^{aa'}T^\bullet_{a'}, C^\bullet_{_o}$$ (56)
$$\kappa_{N+2}(T^\bullet_b) = C^\bullet_{_o}T^\circ_{b'} C^\bullet_{_o'}$$ (57)
$$\kappa_{N+2}(T^\bullet_o) = C^\bullet_{_o}T^\circ_{_o} C^\bullet_{_o}$$ (58)
so that $\kappa_{N+2}(T) \propto T$ and therefore

$$\kappa_{N+2}(h) = \kappa_{N+2}(bTc) = \kappa_{N+2}(c)\kappa_{N+2}(T)\kappa_{N+2}(b) \in H$$  \hspace{1cm} (59)

and the Lemma is proved. □

Consider now the quotient

$$\frac{S_{q,r}(N + 2)}{H}, \hspace{1cm} (60)$$

and the canonical projection

$$P : S_{q,r}(N + 2) \rightarrow S_{q,r}(N + 2)/H \hspace{1cm} (61)$$

Any element of $S_{q,r}(N + 2)/H$ is of the form $P(a)$. Also, $P(H) = 0$, i.e. $H = \text{Ker}(P)$.

Since $H$ is a two-sided ideal, $S_{q,r}(N + 2)/H$ is an algebra with the following sum and products:

$$P(a) + P(b) \equiv P(a + b) ; \quad P(a)P(b) \equiv P(ab) ; \quad \mu P(a) \equiv P(\mu a), \quad \mu \in \mathbb{C} \hspace{1cm} (62)$$

We will use the following notation:

$$u \equiv P(T^0_a), \quad v \equiv P(T^*_a), \quad z \equiv P(T^c_a) \hspace{1cm} (63)$$

and with abuse of symbols:

$$T^a_b \equiv P(T^a_b) ; \quad I \equiv P(I) ; \quad 0 \equiv P(0) \hspace{1cm} (65)$$

Using (65) it is easy to show that $T^a_b, x^a, y_b, u, v, z$ and $I$ generate the algebra $S_{q,r}(N + 2)/H$. Moreover from the $RTT$ relations (65) $R_{12}T_1T_2 = T_2T_1R_{12}$ and the $CTT = C$ relations (66) in $S_{q,r}(N + 2)$ we find the "$P(RTT)$" and "$P(CTT)$" relations in $S_{q,r}(N + 2)/H$:

$$P(R_{12}T_1T_2) = P(T_2T_1R_{12}) \quad \text{i.e.} \quad R_{12}P(T_1)P(T_2) = P(T_2)P(T_1)R_{12} \hspace{1cm} (66)$$

$$P(CTT) = C \quad \text{i.e.} \quad CP(T)P(T) = C \hspace{1cm} (67)$$

**Proposition**: The projected relations (66) and (67) are equivalent to the relations (30)-(39), supplemented by the two constraints:

$$y_b = -r^\rho T^a_b C_{ac} x^c u \hspace{1cm} (68)$$

$$z = -\frac{1}{(r^\rho - r^\rho - 2)} x^b C_{ba} x^a u \hspace{1cm} (69)$$

**Proof**: Consider the $R$ matrix decomposition (28). The three kinds of indices $\circ, a, \bullet$ yield 81 $RTT$ relations. Out of these only 41 are independent and give all
the $q$-commutations between the $T_{AB}$ elements: they contain all the information of the $RTT$ relations. We then project these 41 relations to obtain the $P(RTT)$ relations. We proceed in a similar way with the 9 $CTT$ relations to obtain the $P(CTT)$ relations: in particular one finds $uv = vu = I$.

The projected relations obtained in this way are not independent. In fact choosing the lower indices in the $P(CTT)$ relations as $\bullet$ we find the constraint (68). All the projected relations that contain the elements $y$ are a consequence of the remaining projected relations and of (68). Therefore (68) is a consistent constraint. The contraction of the $(^a\bullet^b\bullet) P(RTT)$ relations with the metric $C_{ab}$ gives (69) \[\text{use (21) and } C_{ab}C^{ab} \text{ in (13)}\] . All the other projected relations containing the element $z$ are a consequence of the remaining ones and of (68). Finally all the projected relations containing the element $u$ are a consequence of $u = v^{-1}$.

We thus arrive at the minimal set of $P(RTT)$ and $P(CTT)$ relations given by (30)-(39), (68) and (69). The Proposition is then proved. □

This implies that we can choose as independent generators the set $T_{ab}^a$, $x^a$, $v$, $u \equiv v^{-1}$, and $I$.

Let us indicate by $\Delta_{N+2}, \varepsilon_{N+2}$ and $\kappa_{N+2}$ the costructures of $S_{q,r}(N+2)$, defined by:

\[
\Delta_{N+2}(T_{AB}^a) = T_{AC}^a \otimes T_{CB}^b \quad (70)
\]

\[
\kappa_{N+2}(T_{AB}^a) = C^{AC}T_D^D C_{DB} \quad (71)
\]

\[
\varepsilon_{N+2}(T_{AB}^a) = \delta_{AB} \quad (72)
\]

Since $H$ is a Hopf ideal then $S_{q,r}(N+2)/H$ is also a Hopf algebra with costructures:

$\Delta(P(a)) \equiv (P \otimes P)\Delta_{N+2}(a) \quad \varepsilon(P(a)) \equiv \varepsilon_{N+2}(a) \quad \kappa(P(a)) \equiv P(\kappa_{N+2}(a)) \quad (73)$

Indeed (71) and (72) ensure that $\Delta$, $\varepsilon$, and $\kappa$ are well defined. For example

\[
(P \otimes P)\Delta_{N+2}(a) = (P \otimes P)\Delta_{N+2}(b) \quad \text{if} \quad P(a) = P(b) \quad (74)
\]

In order to prove the Hopf algebra axioms of the Appendix for $\Delta$, $\varepsilon$, $\kappa$ we just have to project those for $\Delta_{N+2}$, $\varepsilon_{N+2}$, $\kappa_{N+2}$ . For example, the first axiom is proved by applying $P \otimes P \otimes P$ to $(\Delta_{N+2} \otimes id)\Delta_{N+2}(a) = (id \otimes \Delta_{N+2})\Delta_{N+2}(a)$. The other axioms are proved in a similar way.

In conclusion, the elements $T_{ab}^a$, $x^a$, $v$, $u \equiv v^{-1}$ and $I$ generate the Hopf algebra $S_{q,r}(N+2)/H$ and satisfy the $P(RTT)$ and $P(CTT)$ relations (30)-(39). The costructures defined in (73) act on them exactly as the co-structures defined in (41)-(49). Therefore the explicit structure of the Hopf algebra $S_{q,r}(N+2)/H$ is the one described in Theorem 1. We have

\[
\mathcal{I}S_{q,r}(N) = \frac{S_{q,r}(N+2)}{H} \quad , \quad (75)
\]
and Theorem 1 is proved. □

The canonical projection $P : S_{q,r}(N+2) \to IS_{q,r}(N)$ is an epimorphism between these two Hopf algebras.

**Note 1:** the consistency of the $P(RTT)$ and $P(CTT)$ relations with the co-structures $\Delta, \varepsilon$ and $\kappa$ is easily proved. For example,

$$\Delta(P(R_{12}T_1T_2) - P(T_2T_1R_{12})) = 0$$

is a particular case of eq. (74). Similarly for $\varepsilon$ and $\kappa$, and for the $P(CTT)$ relations.

We are now able to give a $R$ matrix formulation of the inhomogeneous $ISO_{q,r}(N)$ and $ISp_{q,r}(N)$ groups. Indeed recall that $S_{q,r}(N+2)$ is the Hopf algebra freely generated by the non-commuting matrix elements $T^A_B$ modulo the ideal generated by the $RTT$ and $CTT$ relations [$R$ matrix and metric $C$ of $S_{q,r}(N+2)$]. This can be expressed as:

$$S_{q,r}(N+2) \equiv \frac{< T^A_B >}{[RTT, CTT]}$$

Therefore we have (recall that $H \equiv [T^a_o, T^b\cdot, T^\cdot o] \equiv [T]$):

$$IS_{q,r}(N) = \frac{S_{q,r}(N+2)}{[T]} = \frac{< T^A_B >}{[RTT, CTT]} = \frac{< T^A_B >}{[RTT, CTT, T]}$$

So that we have shown the following

**Theorem 2:** the quantum inhomogeneous groups $ISO_{q,r}(N)$ and $ISp_{q,r}(N)$ are freely generated by the non-commuting matrix elements $T^A_B$ [A=(o, a, \cdot), with $a = 1, \ldots N$] and the identity $I$, modulo the relations:

$$T^a_o = T^\cdot b = T^\cdot o = 0,$$

the $RTT$ relations

$$R^{AE}_{EF} T^E_C T^F_D = T^B_F T^A_E R^{EF}_{CD},$$

and the orthogonality (symplecticity) relations

$$C^{BC} T^A_B T^D_C = C^{AD}$$

$$C_{AC} T^A_B T^C_D = C_{BD}$$

The co-structures of $ISO_{q,r}(N)$ and $ISp_{q,r}(N)$ are simply given by:

$$\Delta(T^A_B) = T^A_C \otimes T^C_B$$

$$\kappa(T^A_B) = C^{AC} T^D_C C_{DB}$$

$$\varepsilon(T^A_B) = \delta^A_B$$

**Note 2:** the $T^A_B$ matrix elements in eq. (80) are really a redundant set: indeed not all of them are independent, see the constraints (68) and (69). This is necessary
if we want to express the $q$-commutations of the $ISO_{q,r}(N)$ and $ISp_{q,r}(N)$ basic group elements as $RTT = TTR$ (i.e. if we want an $R$-matrix formulation). Remark that, in the $R$-matrix formulation for $IGL_{q,r}(N)$, all the $T^A_B$ are independent.

**Note 3:** From the commutations (31) - (39) we see that one can set $u = I$ only when $q_a \odot = 1$ for all $a$. From $q_a \odot = r^2/q_a \bullet$, cf. eq. (7), this implies also $r = 1$, in agreement with the results of ref. [3], where a differential calculus for $ISO_{q,r}(N)$ without dilatations was found only for $r = 1$.

**Note 4:** eq.s (34) are the multiparametric (orthogonal or symplectic) quantum plane commutations. They follow from the $(a \odot b) P(RT) T$ components and (69).

Finally, it is not difficult to see how the real forms of $S_{q,r}(N+2)$ are inherited by $IS_{q,r}(N)$. In fact, only the first and the fourth real forms of $S_{q,r}(N+2)$, discussed after (25), are compatible with the coset structure of $IS_{q,r}(N)$. More precisely, $H$ is a $\ast$-Hopf ideal, i.e. $(H)^\ast \subseteq H$, only for $T^\ast = T$ or $(T^a_b)^\ast = D^a_c T^c_d D^d_b$. Then we can define a $\ast$-structure on $IS_{q,r}(N)$ as $[P(a)]^\ast \equiv P(a^\ast)$, $\forall a \in S_{q,r}(N+2)$. The conditions on the parameters are respectively:

- $|q_{ab}| = |q_a\bullet| = |r| = 1$ for $ISO_{q,r}(n, n; \mathbb{R})$, $ISO_{q,r}(n, n+1; \mathbb{R})$ and $ISp_{q,r}(n; \mathbb{R})$.

- For $ISO_{q,r}(n+1, n-1; \mathbb{R})$, $|r| = 1$; $|q_{ab}| = 1$ for $a$ and $b$ both different from $n$ or $n+1$; $q_{ab}/r \in \mathbb{R}$ when at least one of the indices $a, b$ is equal to $n$ or $n+1$; $q_a\bullet/r \in \mathbb{R}$ for $a = n$ or $a = n+1$.

In particular, the quantum Poincaré group $ISO_{q,r}(3, 1; \mathbb{R})$ is obtained by setting $|q_1\bullet| = |q_2\bullet| = |r| = 1$, $q_{12}/r \in \mathbb{R}$.

**APPENDIX : the Hopf algebra axioms**

A Hopf algebra over the field $K$ is a unital algebra over $K$ endowed with the linear maps:

$$\Delta : A \to A \otimes A$$
$$\varepsilon : A \to K$$
$$\kappa : A \to A$$

satisfying the following properties $\forall a, b \in A$:

$$(\Delta \otimes \text{id})\Delta(a) = (\text{id} \otimes \Delta)\Delta(a)$$

$$(\varepsilon \otimes \text{id})\Delta(a) = (\text{id} \otimes \varepsilon)\Delta(a) = a$$

$$m(\kappa \otimes \text{id})\Delta(a) = m(\text{id} \otimes \kappa)\Delta(a) = \varepsilon(a)I$$

$$\Delta(ab) = \Delta(a)\Delta(b) ; \Delta(I) = I \otimes I$$
\[ \varepsilon(ab) = \varepsilon(a)\varepsilon(b) \ ; \ \varepsilon(I) = 1 \]  

(92)

where \( m \) is the multiplication map \( m(a \otimes b) = ab \). From these axioms we deduce:

\[ \kappa(ab) = \kappa(b)\kappa(a) \ ; \ \Delta[\kappa(a)] = \tau(\kappa \otimes \kappa)\Delta(a) \ ; \ \varepsilon[\kappa(a)] = \varepsilon(a) \ ; \ \kappa(I) = I \]  

(93)

where \( \tau(a \otimes b) = b \otimes a \) is the twist map.

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