Three-Loop Reducible Radiative Photon Contributions to Lamb Shift and Hyperfine Splitting

Michael I. Eides *
Department of Physics and Astronomy, University of Kentucky, Lexington, KY 40506, USA
and Petersburg Nuclear Physics Institute, Gatchina, St.Petersburg 188300, Russia

Valery A. Shelyuto †
D. I. Mendeleev Institute of Metrology, St.Petersburg 198005, Russia

Abstract

Corrections of order $\alpha^3(Z\alpha)^5m$ to the Lamb shift and corrections of order $\alpha^3(Z\alpha)E_F$ to hyperfine splitting generated by the insertions of the three-loop one-particle reducible diagrams with radiative photons in the electron line are calculated. The calculations are performed in the Yennie gauge.

I. INTRODUCTION

The theory of Lamb shift in light hydrogenlike atoms is rapidly developing. Calculations of the last corrections which are significantly larger than 1 kHz for the 1S state in hydrogen were completed recently. These are corrections of orders $\alpha^3(Z\alpha)^4m$, $\alpha(Z\alpha)^nm$, and $\alpha^2(Z\alpha)^6m$. The previously unknown correction to the Lamb shift of order $\alpha^3(Z\alpha)^4m$ is connected with the three-loop contribution to the slope of the Dirac form factor, and was obtained in [1]. Corrections of orders $\alpha(Z\alpha)^nm$ for $n = 4, 5, 6$ are already well known perturbatively for some time (see, e.g., review [2]), but relatively large magnitude of the contributions with $n = 6$, and high precision of the experimental data [3,4] required calculation of the corrections of higher order in $Z\alpha$. All such corrections were obtained numerically without expansion in $Z\alpha$ [5,6]. Corrections of order $\alpha(Z\alpha)^6m$ for higher energy levels were also calculated recently with high accuracy [7,8]. From the practical point of view the corrections of order $\alpha(Z\alpha)^nm$ are no more a significant source of theoretical uncertainty for the Lamb shift, for example for the lowest $S$ and $P$ states in hydrogen these contributions are now known with uncertainty about 1 Hz.

*E-mail address: eides@pa.uky.edu, eides@thd.pnpi.spb.ru

†E-mail address: shelyuto@vniim.ru
Another recent success is connected with the corrections of order $\alpha^2(Z\alpha)^6m$. The leading logarithm cubed correction of this order was calculated in the pioneering work [9]. This paper was followed by a heated discussion about the magnitude of the nonleading logarithmic contributions of order $\alpha^2(Z\alpha)^6m$, and even the magnitude of the leading logarithm cubed term was put under suspicion (see, e.g., review [2]). The doubts about the magnitude of the leading logarithm cubed term were put to rest in [10], where all logarithmically enhanced terms of order $\alpha^2(Z\alpha)^6m$ were calculated in the Coulomb gauge. Finally, the dominant part of the nonlogarithmic contribution of order $\alpha^2(Z\alpha)^6m$ was calculated in [11]. At the present stage remaining uncertainty of the contributions of order $\alpha^2(Z\alpha)^6m$ to the Lamb shift is about 0.9 kHz and 0.1 kHz for the $1S$ and $2S$ states in hydrogen, respectively.

The largest still unknown contributions to the Lamb shift in hydrogen are corrections of order $\alpha^3(Z\alpha)^5m$. The magnitude of these corrections can be easily estimated multiplying the corrections of order $\alpha^2(Z\alpha)^5m$ [14–16] by an extra factor $\alpha/\pi$. As a result of this simple exercise we see that the corrections of order $\alpha^3(Z\alpha)^5m$ should be about 1 kHz for the $1S$ state in hydrogen. In [17] we calculated radiative corrections to the Lamb shift of order $\alpha^3(Z\alpha)^5m$ and radiative corrections to hyperfine splitting of order $\alpha^3(Z\alpha)E_F$ generated by the diagrams with insertions of radiative photons and electron polarization loops in the graphs with two external photons.

Below we calculate in the Yennie gauge corrections of order $\alpha^3(Z\alpha)^5m$ to the Lamb shift and corrections of order $\alpha^3(Z\alpha)E_F$ to hyperfine splitting generated by the diagrams in Fig. 1 with insertions of the three-loop one-particle reducible diagrams with radiative photons in the electron line.

Below we calculate in the Yennie gauge corrections of order $\alpha^3(Z\alpha)^5m$ to the Lamb shift and corrections of order $\alpha^3(Z\alpha)E_F$ to hyperfine splitting generated by the diagrams in Fig. 1 with insertions of the three-loop one-particle reducible diagrams with radiative photons in the electron line.

![Reducible three-loop diagrams](image)

**FIG. 1.** Reducible three-loop diagrams  

**II. FACTORIZED CONTRIBUTIONS TO THE LAMB SHIFT AND HYPERFINE SPLITTING**

**A. Skeleton Diagram Contributions**

The diagrams for nonrecoil corrections of order $\alpha^3(Z\alpha)^5m$ to the Lamb shift and corrections of order $\alpha^3(Z\alpha)E_F$ to hyperfine splitting in Fig. 1 can be obtained by three-loop radiative insertions in the skeleton diagram in Fig. 2. Respective corrections of lower orders in $\alpha$ generated by one- and two-loop radiative insertions are already well known (see, e.g., review [2]). All corrections of order $\alpha^n(Z\alpha)^5m$ and $\alpha^n(Z\alpha)E_F$ may be calculated in the scattering approximation (see, e.g., [13]).

Calculation of all these contributions starts with the skeleton diagram in Fig. 2. Contribution of each of the diagrams in Fig. 1 to the Lamb shift is described by the integral
\[-\frac{16(Z\alpha)^5}{\pi n^3} \left(\frac{m_r}{m}\right)^3 m \int_0^\infty \frac{d|\mathbf{k}|}{|\mathbf{k}|^4} L(\mathbf{k}) \delta_{10},\]  

(1)

where \( m \) is the electron mass, \( M \) is the proton mass, \( m_r = m/(1 + m/M) \) is the reduced mass, \( \alpha \) is the fine structure constant, \( Z \) is the nucleus charge in terms of the electron charge (\( Z = 1 \) for hydrogen and muonium), and \( |\mathbf{k}| \) is the magnitude of the dimensionless momentum of the external photons measured in the units of the electron mass. The function \( L(\mathbf{k}) \) describes radiative corrections to the skeleton diagram, and should be calculated for each particular diagram in Fig. 1. It is normalized to the contribution of the skeleton numerator, \( L_{skel}(\mathbf{k}) = 1 \). The skeleton contribution to the Lamb shift is infrared divergent. For some diagrams in Fig. 1 this infrared divergence survives in the Feynman gauge even after insertion of the factor \( L(\mathbf{k}) \) which describes radiative insertions. In order to avoid such spurious infrared divergencies which anyway cancel in the gauge-invariant sets of diagrams we use infrared safe Yennie gauge for radiative photons in the calculations below.

Contribution of each of the diagrams in Fig. 1 to hyperfine splitting is described by the integral

\[ \frac{8Z\alpha}{\pi n^3} E_F \int_0^\infty \frac{d|\mathbf{k}|}{|\mathbf{k}|^2} F(\mathbf{k}), \]

(3)

where \( F(\mathbf{k}) \) describes radiative corrections to the skeleton diagram, and, as the function \( L(\mathbf{k}) \) in the case of Lamb shift, should be calculated for each particular diagram in Fig. 1. It is normalized to the contribution of the skeleton numerator, \( F_{skel}(\mathbf{k}) = 1 \).

\[ \text{FIG. 2. Skeleton two-photon diagram} \]

B. Mass Operator and One-Loop Vertex with One On-Mass-Shell Leg in the Yennie Gauge

To calculate contributions of the diagrams in Fig. 1 we use explicit expressions for the electron mass operator and the electron-photon vertex in the Yennie gauge. The one-loop

\[ \text{FIG. 2. Skeleton two-photon diagram} \]

\[ \text{B. Mass Operator and One-Loop Vertex with One On-Mass-Shell Leg in the Yennie Gauge} \]

\[ \text{We define the Fermi energy} \ E_F \text{ as} \]

\[ E_F = \frac{16}{3} Z^4 \alpha^2 \frac{m_r}{M} (1 + a_\mu) \left(\frac{m_r}{m}\right)^3 c h R_\infty, \]

(2)

where \( m \) is the electron mass, \( M \) is the muon mass, \( m_r = m/(1 + m/M) \) is the reduced mass, \( c \) is the velocity of light, \( h \) is the Planck constant, \( R_\infty \) is the Rydberg constant, and \( a_\mu \) is the muon anomalous magnetic moment.
electron self-energy in the Yennie gauge renormalized on the mass-shell is well known, and has the form (see, e.g. [13])

\[ \Sigma(p - k) = -\frac{3\alpha}{4\pi}(\hat{p} - \hat{k} - 1)^2(\hat{p} - \hat{k})M(k), \]  

where

\[ M(k) = \frac{1}{1 - k^2} + \frac{k^2}{(1 - k^2)^2} \ln k^2, \]  

all momenta are dimensionless, and are measured in the units of the electron mass \( m = 1 \), and \( p_\mu = (1, 0), pk = 0, k^\mu = (0, k) \).

Renormalized vertex operator with one on-mass-shell leg in the Yennie gauge has the form [18] (we omit terms proportional to the momentum \( k_\mu \) because they do not contribute to the Lamb shift and hyperfine splitting)

\[ \Lambda_\mu = \frac{\alpha}{2\pi} \{ A(k)k^2\gamma_\mu + B(k)\gamma_\mu(\hat{p} - \hat{k} - 1) + C(k)p_\mu(\hat{p} - \hat{k} - 1) + E(k)\sigma_{\mu\nu}k^\nu \}, \]

where

\[ A(k) = -\left( \frac{2}{|k|^3} + \frac{1}{2|k|} \right) \Phi(k) + \frac{2}{k^2}S(k) - \frac{3}{2}M(k) - 2 \frac{\ln k^2}{k^2} - \frac{3}{2} \frac{\ln k^2}{1 - k^2}, \]  

\[ B(k) = -\left( \frac{1}{|k|} + \frac{|k|}{8} \right) \Phi(k) + \frac{1}{2}S(k) - \frac{5}{4}M(k) + \frac{1}{4} - \frac{1}{8} \ln k^2 - \frac{7}{8} \frac{\ln k^2}{1 - k^2}, \]

\[ C(k) = \frac{1}{|k|}\Phi(k) - S(k) - \frac{1}{2}M(k) + \frac{1}{2} \ln k^2 - \frac{3}{2} \frac{\ln k^2}{1 - k^2}, \]

\[ E(k) = -\frac{|k|}{8} \Phi(k) - \frac{1}{2}S(k) - \frac{1}{4}M(k) + \frac{1}{4} + \frac{3}{8} \ln k^2 - \frac{3}{8} \frac{\ln k^2}{1 - k^2}, \]

and

\[ \Phi(k) = |k| \int_0^1 \frac{dz}{1 - k^2z^2} \ln \frac{1 + k^2z(1 - z)}{k^2z}, \]

\[ = \text{Li}(1 - |k|) - \text{Li}(1 + |k|) + 2 \left[ \text{Li} \left( 1 + \frac{\sqrt{k^2 + 4} + |k|}{2} \right) - \text{Li} \left( 1 - \frac{\sqrt{k^2 + 4} + |k|}{2} \right) - \frac{\pi^2}{4} \right], \]

\[ S(k) = \frac{\sqrt{k^2 + 4}}{2|k|} \ln \frac{\sqrt{k^2 + 4} + |k|}{\sqrt{k^2 + 4} - |k|}. \]

Euler dilogarithm \( \text{Li} \) is defined here as in [13], and the function \( \Phi(k) \) usually arises in calculations of the diagrams with factorized radiative insertions in the electron line, see, e.g., [19].


C. Factorized Corrections of Order $\alpha^3(Z\alpha)^5m$ to the Lamb Shift

We use explicit expression for the self-energy operator in eq.(4), and calculating the spinor projection on the Lamb shift obtain the contribution of the graph $a$ in Fig. 1 to the Lamb shift:

$$\Delta E^a_L = \frac{\alpha^3(Z\alpha)^5}{\pi^2 n^3} \left( \frac{m_r}{m} \right)^3 m \left( -\frac{27}{8\pi^2} \right) \int_0^\infty dk M^3(k) (1 - k^2)$$

$$= \left( -\frac{135}{1024}\pi^2 + \frac{81}{64} \right) \frac{\alpha^3(Z\alpha)^5}{\pi^2 n^3} \left( \frac{m_r}{m} \right)^3 m.$$  

Performing similar calculations for the graph $b$ in Fig. 1 we obtain

$$2\Delta E^b_L = \frac{\alpha^3(Z\alpha)^5}{\pi^2 n^3} \left( \frac{m_r}{m} \right)^3 m \left( -\frac{9}{2\pi^2} \right) \int_0^\infty dk M^2(k) (1 - k^2) (B(k) + C(k) - E(k))$$

$$= \left( -\frac{153}{512}\pi^2 + \frac{63}{32} \right) \frac{\alpha^3(Z\alpha)^5}{\pi^2 n^3} \left( \frac{m_r}{m} \right)^3 m.$$  

For the diagram $c$ in Fig. 1 we obtain

$$\Delta E^c_L = \frac{\alpha^3(Z\alpha)^5}{\pi^2 n^3} \left( \frac{m_r}{m} \right)^3 m \left( -\frac{3}{2\pi^2} \right) \int_0^\infty dk M(k) \left\{ k^2A(k) \left[ A(k) - 2(B(k) + C(k) - E(k)) \right] \right. $$

$$+ \left. [B(k) + C(k) - E(k)]^2 \right\} = -4.30582 \left( 1 \right) \frac{\alpha^3(Z\alpha)^5}{\pi^2 n^3} \left( \frac{m_r}{m} \right)^3 m.$$  

D. Factorized Corrections of Order $\alpha^3(Z\alpha)E_f$ to Hyperfine Splitting

Like in the case of the Lamb shift we use explicit expression for the self-energy operator in eq.(4), and calculating the spinor projection on hyperfine splitting obtain the contribution of the graph $a$ in Fig. 1 to hyperfine splitting:

$$\Delta E^a_{HFS} = \frac{\alpha^3(Z\alpha)}{\pi^2 n^3} E_f \frac{27}{8\pi^2} \int_0^\infty dk (1 - k^2) (2 - k^2) M^3(k)$$

$$= \left( \frac{1215}{1024}\pi^2 - \frac{81}{64} \right) \frac{\alpha^3(Z\alpha)}{\pi^2 n^3} E_f.$$  

\(^2\)From now on $|k| = k$.  

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Performing similar calculations for the graph $b$ in Fig. 1 we obtain
\[
2\Delta E_{HFS}^b = \frac{\alpha^3(Z\alpha)}{\pi^2 n^3} E_F \frac{9}{2\pi^2} \int_0^\infty dk M^2(k) (1 - k^2) \left[ - k^2 A(k) + 2B(k) + C(k) - E(k) \right] \tag{16}
\]
\[
= -22.064414 \left( \frac{\alpha^3(Z\alpha)}{\pi^2 n^3} E_F \right).
\]

For the graph $c$ in Fig. 1 we obtain
\[
\Delta E_{HFS}^c = \frac{\alpha^3(Z\alpha)}{\pi^2 n^3} E_F \frac{3}{2\pi^2} \int_0^\infty dk M(k) \left\{ k^2 A(k) [k^2 A(k) - 2B(k) - C(k)] \right\} \tag{17}
\]
\[
+ (2 - k^2) B(k) [B(k) + C(k) - E(k)] + k^2 E(k) [B(k) + C(k) - E(k)] \right\}
\]
\[
= 11.723748 \left( \frac{\alpha^3(Z\alpha)}{\pi^2 n^3} E_F \right).
\]

### III. SUMMARY

In this paper we calculated factorized corrections of order $\alpha^3(Z\alpha)^5 m$ to the Lamb shift, and factorized corrections of order $\alpha^3(Z\alpha)E_F$ to hyperfine splitting generated by the diagrams in Fig. 1. Collecting contributions to the Lamb shift in eq.(12), eq.(13), and eq.(14), we obtain
\[
\Delta E_{L}^{tot} = -5.32193 \left( \frac{\alpha^3(Z\alpha)^5}{\pi^2 n^3} \right) m, \tag{18}
\]
or
\[
\Delta E_{L}^{tot} = -0.535 \text{ kHz} \tag{19}
\]
for the $1S$ level in hydrogen.

Collecting all contributions to hyperfine splitting in eq.(15), eq.(16), and eq.(17) we obtain
\[
\Delta E_{HFS}^{tot} = 0.10423 \left( \frac{\alpha^3(Z\alpha)}{\pi^2} \right) E_F, \tag{20}
\]
or
\[
\delta E_{HFS}^{tot} = 0.00013 \text{ kHz} \tag{21}
\]
for the ground state in muonium.

The result in eq.(19) has just the scale we expected on the basis of the general considerations explained in the Introduction, and corrections of this magnitude are phenomenologically relevant at the current level of the experimental and theoretical accuracy (see, e.g. [2]). Work on calculation of nonfactorizable contributions is now in progress, and we postpone discussion of the phenomenological implications of the results in eq.(19) and eq.(20) until its completion.
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