On the gravitational energy of the Bonnor spacetime

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Abstract
In this paper we consider the gravitational energy and its flux in the Bonnor spacetime. We construct some non-local expressions from the Einstein canonical energy–momentum pseudotensor of the gravitational field which show that the gravitational energy and its flux in this spacetime are different from zero and do not vanish even outside of the material source (a stationary beam of the null dust) of this spacetime. Thus, the hypothesis which states that the energy and momentum in Bonnor’s spacetime are confined to the regions of non-vanishing energy–momentum tensor of matter cannot be correct.

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1. Introduction

In the paper [1] W B Bonnor gave the solution to the Einstein equations which describes a stationary beam of light (strictly speaking—a stationary beam of null dust). The line element $ds^2$ for this solution reads ($G = c = 1$)

$$ds^2 = (1 + m)\, dt^2 - 2m\, dt\, dz - (1 - m)\, dz^2 - dx^2 - dy^2,$$

where

$$\Delta m := \frac{\partial^2 m}{\partial x^2} + \frac{\partial^2 m}{\partial y^2} = 16\pi \rho$$

and $\rho = \mu (u^0)^2 > 0$.

$\mu > 0$ is the rest density of the null dust and $:= \text{means ‘by definition’}$. In the paper [2] the total energy–momentum ‘densities’, matter and gravity were calculated by using standard energy–momentum complexes of Einstein, Landau–Lifschitz, Papapetrou and Weinberg. All the calculations were performed after transforming the line element (1) to the Kerr–Schild form (see e.g., [2]).
It was shown in [2] that the used four energy–momentum complexes give compatible results in the case and 'localize' total energy–momentum ‘densities’ in the Kerr–Schild coordinates to the domains occupied by sources of the gravitational field only, i.e., to the domains in which energy–momentum tensor of the null dust does not vanish. Vacuum regions of the Bonnor spacetime give no energy–momentum contribution, i.e., the gravitational energy–momentum pseudotensors of Einstein, Landau–Lifschitz, Papapetrou and Weinberg *globally vanish* in the Kerr–Schild coordinates outside of the stationary beam of the considered null dust.

Then, the analogical calculations were repeated in [3] but in Bonnor’s coordinates \( x^0 = t, x^1 = z, x^2 = x \) and \( x^3 = y \). The results were the same as the results obtained earlier in [2]: in the used coordinates \((t, z, x, y)\) the total energy and momentum ‘densities’ were confined to the regions of non-vanishing energy–momentum tensor of the null dust, i.e., again the vacuum regions of the Bonnor spacetime gave no energy–momentum contribution.

Despite the fact that their results are *coordinate-dependent*, the authors of the papers [2, 3] conclude that their results sustain the hypothesis which states that the energy and momentum in general relativity (GR) are confined to the regions of non-vanishing energy–momentum tensor of matter\(^1\). From this hypothesis it follows e.g., that the gravitational waves carry no energy–momentum. Of course, this contradicts basic results of the sixties due to Bondi, Pirani, Sachs, Trautman, Penrose and others.

In the following we will show that the conclusion of such a kind cannot be correct.

The paper is organized as follows. In section 2 we firstly present some information about the Bonnor spacetime, about canonical gravitational superenergy tensor and about averaged gravitational energy–momentum tensor. Then, we consider in this section the gravitational energy and its flux in Bonnor’s spacetime. In section 3 we give a conclusion and remarks concerning energy and momentum of the cosmological models.

2. The gravitational energy and its flux in the Bonnor spacetime outside of the beam

Firstly, let us observe that one can write the line element (1) in the form

\[
\mathrm{d}s^2 = \mathrm{d}t^2 + m(\mathrm{d}t - \mathrm{d}z)^2 - \mathrm{d}z^2 - \mathrm{d}x^2 - \mathrm{d}y^2.
\]  

(3)

Then, after introducing new variables \( u = t - z, v = t + z \) we get

\[
\mathrm{d}s^2 = m \, \mathrm{d}u^2 + \mathrm{d}u \, \mathrm{d}v - \mathrm{d}x^2 - \mathrm{d}y^2.
\]  

(4)

Finally, by introducing \( U = \frac{u}{\sqrt{2}}, V = \frac{v}{\sqrt{2}}, X = x, Y = y \) one gets the line element (1) in the form

\[
\mathrm{d}s^2 = 2m \, \mathrm{d}U^2 + 2 \, \mathrm{d}U \, \mathrm{d}V - \mathrm{d}X^2 - \mathrm{d}Y^2.
\]  

(5)

In vacuum, i.e., outside the stationary beam of the null dust which is the source of the Bonnor spacetime, where \( \rho = 0 \), we obtain from (5)

\[
\mathrm{d}s^2 = 2m(X, Y) \, \mathrm{d}U^2 + 2 \, \mathrm{d}U \, \mathrm{d}V - \mathrm{d}X^2 - \mathrm{d}Y^2,
\]  

(6)

where

\[
\Delta m = \frac{\partial^2 m}{\partial X^2} + \frac{\partial^2 m}{\partial Y^2} = 0.
\]  

(7)

i.e., we obtain special *plane-fronted gravitational wave with parallel rays* (p–p wave, see, e.g., [4]).

\(^1\) This old hypothesis was recently restored e.g., by F Cooperstock.
In an older paper [5] it was shown that in the coordinates \((U, V, X, Y)\) the canonical Einstein’s gravitational energy–momentum pseudotensor \(\varepsilon_t^k\) globally vanishes for the \(p–p\) waves giving zero gravitational ‘energy density’ and no gravitational energy flux, in agreement with the results of the papers [2, 3]. But if one transforms the line element (6), (7) to the coordinates used by MTW [6], then one will obtain a non-vanishing pseudotensor \(\varepsilon_t^k\) and non-zero (negative) ‘energy density’ and its non-zero flux (see e.g., [5]).

So, as it is commonly known, the results obtained by using the energy–momentum pseudotensors of the gravitational field (and complexes) are coordinate-dependent and one must not use them for supporting the hypothesis about ‘localization of the gravitational energy to the regions of the non-vanishing energy–momentum tensor of the matter and all non-gravitational fields’.

The energy–momentum complexes, matter and gravitation, and in consequence the gravitational energy–momentum pseudotensors, can only be reasonably used in a case of very precisely defined asymptotically flat (in null or in spatial infinities) spacetime giving global (or integral) energy and momentum. They can also be considered as giving some kind of quasi-local quantities [7–9]; but they cannot have physical meaning for localization of the gravitational energy and momentum.

In order to do a reasonable and coordinate-independent local analysis of the real gravitational field one should use covariant expressions constructed from the curvature tensor, e.g., one should use the canonical superenergy tensor \(S_a^b (P; v')\) of the gravitational field (and canonical angular supermomentum tensor). The tensor of such a kind was introduced in the series of our papers [10–15].

Let us recall here the constructive, general definition of the canonical superenergy tensor \(S_a^b (P)\) applicable to gravitational field and also to any matter field (for details see e.g., [10–15]).

In normal coordinates \(NC(P)\), we define

\[
S_{(a)}^{(b)} (P) = S_a^b (P) := (-) \lim_{\Omega \to P} \frac{\int_{\Omega} \left[ T_{(a)}^{(b)} (y) - T_{(a)}^{(b)} (P) \right] d\Omega}{1/2 \int_{\Omega} \sigma (P; y) d\Omega},
\]

where

\[
T_{(a)}^{(b)} (y) := T_k^i (y) e^i_{(a)} (y) e_k^{(b)} (y),
\]
\[
T_{(a)}^{(b)} (P) := T_k^i (P) e^i_{(a)} (P) e_k^{(b)} (P) = T^{(b)}_a (P)
\]

are the so-called physical or tetrad components of the tensor (or pseudotensor) field \(T_k^i (y)\) which describes energy–momentum densities. \(y\) denotes the collection of normal coordinates with respect to the origin \(P\), and \(e^i_{(a)} (y), e_k^{(b)} (y)\) mean an orthonormal tetrad such that \(e^i_{(a)} (P) = \delta^i_a\) and its dual \(e_k^{(b)} (P) = \delta_k^b\) parallelly propagated along geodesics through \(P\) (\(P = \text{origin of the normal coordinates } NC(P)\)).

We have

\[
e^{(a)} (y) e^{(R)} (y) = \delta^{(b)}_a.
\]

\(\Omega\) is a sufficiently small ball defined by

\[
y^0^2 + y^1^2 + y^2^2 + y^3^2 \leq R^2
\]

and it can be covariantly described in terms of the auxiliary positive-definite metric

\[
h^{ik} := 2v^i v^k - g^{ik},
\]

where \(v^i\) are the components of the velocity \(\vec{v}\) of a fiducial observer \(O\) being at rest in the beginning \(P\) of the used normal coordinates \(NC(P)\).

\[
\sigma (P; y) = \frac{1}{2} \left( y^0^2 - y^1^2 - y^2^2 - y^3^2 \right)
\]
is the two-point world function introduced by J L Synge \[16\]. \( \Omega \) means that the equation is only valid for special coordinates. The latter can be covariantly defined by the eikonal-like equation

\[
g^{jk} \delta_t \sigma \delta_t \sigma = 2 \sigma \tag{14}
\]

together with \( \sigma(P, P) = 0, \delta_t \sigma(P, P) = 0 \). Then \( \Omega \) is given by the inequality

\[
h^{ik} \delta_t \sigma \delta_t \sigma \leq R^2. \tag{15}
\]

Because at the point \( P \) the tetrad and normal components are equal, we write the components of any quantity attached to the point \( P \) without tetrad brackets, e.g., we write \( S_a^b(P) \) instead of \( (a)(b)(P) \) and so on.

If \( T^i_j(y) \) are the components of the symmetric energy–momentum tensor of any matter field, then we get from (8) (for details see e.g., \[10–15\])

\[
mS^b_a(P) = \delta_{mn} \nabla_m \nabla_n \hat{T}^b_a. \tag{16}
\]

The ‘hat’ over a quantity denotes its value at the point \( P \), i.e., at the origin of the \( \text{NC}(P) \).

Using the four-velocity \( \vec{v} \) of the observer \( O \) being at rest at the origin \( P \) of the normal coordinates \( \text{NC}(P) \) and the local metric \( \hat{\eta}^{ab} \hat{=} \eta_{ab} \) one can write this covariantly as

\[
mS^b_a(P; vl) = (2\vec{v} \cdot \hat{v}^m - \hat{g}^{lm}) \nabla_m \nabla_l \hat{T}^b_a. \tag{17}
\]

The expression (17) gives us the components of the canonical superenergy tensor of a matter field.

For the gravitational field we will obtain from (8), after substituting there \( T^i_j(y) = \hat{E}^i_j(y) \), where \( \hat{E}^i_j(y) \) mean the components of the Einstein canonical energy–momentum pseudotensor of the gravitational field

\[
gS^b_a(P; vl) = (2\vec{v} \cdot \hat{v}^m - \hat{g}^{lm}) \hat{T}^b_a. \tag{18}
\]

Here

\[
\begin{align*}
\hat{T}^b_a \ &= \ \frac{2\alpha}{9} \left[ \hat{B}^{b}_{alm} + \hat{P}^{b}_{alm} - \frac{1}{2} \hat{g}^{b}_{a} \hat{R}^{ijk}_{m} (\hat{R}_{ijkl} + \hat{R}_{ijkl}) \\
& \quad + 2\hat{g}^{b}_{a} \hat{E}_{(l}(g \hat{E}^{k)}_{m)} - 3\hat{\beta}^{2} \hat{E}_{a(l)(i} \hat{E}^{b)}_{m)} + 2\hat{\beta} \hat{R}^{b}_{(ag)(l|m) \hat{E}^{e}_{g)}_{m)} \right], \\
\alpha \ &= \ \frac{c^4}{16\pi G} = \frac{1}{2\hat{\beta}}, \quad E^i_{k} := T^i_{k} - \frac{1}{2} T.
\end{align*}
\]

\[
B^{b}_{alm} := 2R^{b}_{(i}(l|R_{ikm)} - \frac{1}{2} \hat{g}^{b}_{a} R^{ijk}_{i} R_{iklm} \tag{20}
\]

\[
P^{b}_{alm} := 2R^{b}_{(i}(l|R_{akm)} - \frac{1}{2} \hat{g}^{b}_{a} R^{ijk}_{i} R_{ikjm} \tag{21}
\]

are the components of the Bel–Robinson tensor and

\[
\hat{g}S_{ab}(P; vl)^{a} \hat{v}^{b} \quad \text{is positive-definite for unit timelike} \ \hat{v}.
\]

\[
\hat{g}S_{b}^{b}(P; vl) = \frac{8\alpha}{9} (2\vec{v} \cdot \hat{v}^m - \hat{g}^{ab}) \left[ R^{(k(m)}_{a(l|R_{l|m)h)} - \frac{1}{2} \hat{g}^{k}_{l} R^{(l|m)}_{(a|R_{l|m)h)} \right]. \tag{23}
\]
In the papers [11–15] we have generalized the universal idea of the superenergy and canonical superenergy tensors to the angular momentum too and constructed in analogical way as it was done for the energy–momentum, the canonical angular supermomentum tensors, for gravitation and for matter.

Up to now we have analysed, locally and globally, the canonical superenergy and the canonical angular supermomentum tensors to some well-known solutions of the Einstein equations (for details see the papers [10–15]):

(i) plane and plane-fronted gravitational waves (exact and in weak-field limit)
(ii) Friedman universes,
(iii) Schwarzschild and Kerr spacetimes and
(iv) Gödel–like spacetimes.

The obtained local results were very interesting and coordinate-independent. Global results also were interesting, especially the following fact: in the case of a closed system and in the case of Trautman’s radiative spacetimes, the integrals on global superenergetic and angular supermomentum quantities have better convergence than the corresponding integrals on global energy–momentum and on global angular momentum. But, most probably, the standard ADM and Bondi–Sachs energy cannot be recovered in this way\(^2\).

In our older papers [17, 18] the Newtonian limits of the canonical superenergy tensors, matter and gravity were studied. For vacuum gravitational field the canonical gravitational superenergy tensor is, in this limit, a positive-definite quadratic function of the second derivatives of the Newtonian gravitational potential.

We have considered also the canonical superenergy for matter in the framework of special relativity (see e.g., [12–15]).

We have proposed to consider the canonical superenergy tensor and the canonical angular supermomentum tensor of the gravitational field as some substitutes of the non-existing in GR gravitational energy–momentum and angular momentum tensors (see our papers [10–15] for details). This is a sensible proposition but it has some defects because the components of these tensors have improper dimensionality: the dimensions of the components of the superenergy tensors (and angular supermomentum tensors also) are equal to the dimensions of the components of an energy–momentum (or angular momentum) tensor (or pseudotensor) multiplied by \(m^{-2}\).

To avoid this defect one can propose another averaging of the energy–momentum differences in \(\text{NC}(\mathbf{P})\) which is very similar to the averaging (8) and which gives the averaged quantities with proper dimensionality of the energy–momentum densities (for details see e.g., [19]), but which needs introduction of a fundamental length \(L\).

Namely, one can propose the following general definition of the averaged tensor (or pseudotensor) \(\langle T^b_a(\mathbf{P}) \rangle\):

\[
\langle T^b_a(\mathbf{P}) \rangle := \lim_{\varepsilon \to 0} \frac{\int_{\Omega} \left[ T^b_a(y) - T^b_a(P) \right] \, d\Omega}{\varepsilon^2 / 2 \int_{\Omega} d\Omega}.
\]

(24)

Here and in the following, the notation is the same as in the formula (8) and in the formulae (9)–(12) with one exception: now we put the radius \(R\) of the ball \(\Omega\) in the form

\[
R = \varepsilon L,
\]

(25)

i.e., we write \(R^2 = \varepsilon^2 L^2\) in the formula (11), where \(\varepsilon\) means a small parameter, \(\varepsilon \in (0; 1)\) and \(L\) is a fundamental length. Of course, the fundamental length \(L\) must be infinitesimally small because its existence violates local Lorentz invariance.

\(^2\) This problem was not studied yet.
From the last averaging formula one can easily get (for details see \[19\]) the following averaged energy–momentum tensor for matter

$$\langle m T_{a}^{b}(P; v') \rangle = m S_{a}^{b}(P; v') \frac{L^2}{6}, \quad (26)$$

where $m S_{a}^{b}(P; v')$ is the canonical superenergy tensor for matter and the following averaged gravitational energy-momentum tensor:

$$\langle \sigma T_{a}^{b}(P; v') \rangle = \sigma S_{a}^{b}(P; v') \frac{L^2}{6}, \quad (27)$$

where the tensor $\sigma S_{a}^{b}(P; v')$ is the canonical superenergy tensor for the gravitational field obtained from the Einstein canonical gravitational energy–momentum pseudotensor.

The averaged energy–momentum tensors $\langle m T_{a}^{b}(P; v') \rangle$ and $\langle \sigma T_{a}^{b}(P; v') \rangle$ can be considered as the averaged tensors of the relative energy–momentum. They can also be interpreted as the fluxes of the appropriate canonical superenergy. It is easily seen from the formulae (24), (26) and (27).

The averaged energy–momentum tensors differ from the canonical superenergy tensors only by the constant scalar multiplicator $\frac{L^2}{6}$, where $L$ means some fundamental length. Thus, from the mathematical point of view these two kinds of tensors are equivalent. Physically they are not because their components have different dimensionality. Moreover, the averaged energy–momentum tensors depend on a fundamental length $L$. Owing to the last fact and owing to the link between the canonical superenergy tensors and averaged energy–momentum tensors given by the formulae (26), (27), it seems that the canonical superenergy tensors are more fundamental than the averaged energy–momentum tensors. This is the main reason why we have used (and still use) superenergy tensors in our papers. But one should emphasize that the averaged energy–momentum tensors have an important superiority over the canonical superenergy tensors: their components have proper dimensions of the energy–momentum densities.

The averaged tensors

$$\langle m T_{a}^{b}(P; v') \rangle, \quad \langle \sigma T_{a}^{b}(P; v') \rangle, \quad (28)$$

depend on the four-velocity $\vec{v}$ of a fiducial observer $\textbf{O}$ which is at rest at the beginning $P$ of the normal coordinates $\text{NC}(P)$ used for averaging and on some fundamental length $L$. After fixing the length $L$ one can determine univocally these tensors along the world line of an observer.

In general one can unambiguously determine these tensors (after fixing $L$) in the whole spacetime or in some domain $\Omega$ if in the spacetime or in the domain $\Omega$ a geometrically distinguished timelike unit vector field $\vec{v}$ is given.

One can try to establish\(^3\) the fundamental length $L$ by using loop quantum gravity (LQG). Namely, one can take as $L$ e.g., the smallest length $l$ over which the classical model of the spacetime is admissible.

Following LQG [20–30] one can say about continuous classical differential geometry, which is already just a few orders of magnitude above the Planck scale, e.g., for distances $l \gtrsim 100L_P = 100\sqrt{\frac{\hbar G}{c^5}} \approx 10^{(-33)} \text{ m}$. So, one can take as the fundamental length $L$ the value $L = 100L_P \approx 10^{(-33)} \text{ m}$.

After fixing the fundamental length $L$ we have the averaged energy–momentum tensors established with the same precise as the canonical superenergy tensors.

\(^3\) But this is not necessary. One can effectively use the averaged energy–momentum tensors without fixing $L$. 

The averaged energy–momentum tensors (with $L$ fixed or no)
\[ \langle g_{\alpha\beta}(P; v') \rangle, \quad \langle m_{\alpha\beta}(P; v') \rangle \] (29)
give us a good tool to a local (and also to global) analysis of the gravitational and matter fields; as good as the canonical superenergy tensors
\[ g_{\alpha\beta}(P; v') m_{\alpha\beta}(P; v') \] (30)
give. For example, one can apply the averaged energy–momentum tensors to analyse all the problems which have been studied in the papers [10–15].

Of course, one can also consider in GR the averaged angular momentum tensors for matter and for gravitation. The constructive definition of these tensors is analogical to the definition of the averaged energy–momentum tensors (for details see [19]).

In this paper we will apply the averaged gravitational energy–momentum tensor in order to justify our expressions on pure gravitational energy density and its flux in the Bonnor spacetime which are given by the formulae (39) and (40).

The superenergy tensor (18)–(22) was obtained as a result of a special averaging of the differences $E_{ik}(y) - E_{ik}(P)$ in a Riemann normal coordinates $NC(P)$ introduced in a sufficiently small vicinity of an arbitrary point $P$ (for details see e.g., [10–15]). So, it is some kind of a non-local construction obtained from the Einstein canonical energy–momentum pseudotensor $E_{ik}$.

The fiducial observer $O$ is at rest in this $NC(P)$. In Bonnor’s coordinates $(t, z, x, y)$ the four-velocity $\vec{v}$ of this observer has the following components
\[ v^i = \frac{\delta^i_0}{\sqrt{g_{00}}} = \frac{\delta^i_0}{\sqrt{1 + m}}, \quad \Rightarrow v_i = \frac{g_{00}}{\sqrt{1 + m}}. \] (31)

In the paper [5] the components $g_{Sik}(P; v')$ of the canonical superenergy tensor were calculated (18)–(22) for a gravitational p–p wave in the 1-form basis
\[ \delta^0 = m \, dU + dV, \quad \delta^1 = dU, \quad \delta^2 = dX, \quad \delta^3 = dY \] (32)
determined by the line element (6), (7). There was obtained the only one component
\[ g_{S0}^0 = \frac{16\alpha}{9} (v^1)^2 [(m_{xx})^2 + 2(m_{xy})^2 + (m_{yy})^2] \] (33)
of the $g_{Sik}(P; v')$ that is different from zero in the basis (32).

Here and in the following $m_{xx} := \frac{\partial^2 m}{\partial x^2}, m_{xy} := \frac{\partial^2 m}{\partial x \partial y}, m_{yy} := \frac{\partial^2 m}{\partial y^2}$.

By using (33), one can easily calculate the canonical superenergy density $g_{\epsilon_1}$ of the p–p wave (6), (7) in the 1-form basis
\[ g_{\epsilon_1} := g_{S0}^0 (v')^2 = g_{S1}^0 (v_j)^2 = \frac{4\alpha}{9(m + 1)^2} [(m_{xx})^2 + 2(m_{xy})^2 + (m_{yy})^2] > 0. \] (34)

The expression (34), like the expression (33), is positive-definite and it does not vanish in any coordinates or frames.

Also the spatial Poynting’s supervector
\[ g_{P^i} := (\delta^i_k - v^i v_k) g_{S0}^0 v^j \] (35)

\[ 4 \] At every point $P$ we have chosen the unit timelike vector of the $NC(P)$ to be proportional to the timelike vector of the holonomic frame at the point determined by Bonnor’s coordinates.

\[ 5 \] For simplicity we still use in the formulae (34), (36) and (37) the four-velocity of the observer $O$ which is at rest at the beginning of the $NC(P)$, but this is not necessary and one can use in these formulae a four-velocity of another observer.
does not vanish in the case and it has the two non-vanishing components \( P^0, P^1 \) in the basis (32)

\[
P^0 \pm \frac{8\alpha}{9\sqrt{2(m+1)^3}}[(m_{xx})^2 + 2(m_{xy})^2 + 2(m_{yy})^2],
\]

(36)

\[
P^1 \pm \frac{-8\alpha}{9(m+1)\sqrt{2(m+1)^3}}[(m_{xx})^2 + 2(m_{xy})^2 + 2(m_{yy})^2] = -\frac{P^0}{(m+1)}.
\]

(37)

So, we have for Bonnor spacetime (1), (2) the non-vanishing gravitational superenergy density and non-vanishing gravitational superenergy flux even outside of the sources of this spacetime, i.e., outside of the domains in which \( T^k_l \neq 0 \).

As we have used only tensorial expressions in our analysis of the Bonnor spacetime, our results are valid in any coordinates or frames. Thus, one can conclude that in the case of the Bonnor spacetime one has a positive-definite gravitational superenergy density and non-vanishing gravitational superenergy flux in vacuum, i.e., outside of the stationary beam of the null dust which is the source of this spacetime\(^6\).

But this means that the real free gravitational field in the Bonnor spacetime for which \( R_{\text{kin}} \neq 0 \) also possesses its own gravitational (relative) energy density and the non-zero of this (relative) gravitational energy flux. It is most easily seen from the gravitational averaged energy–momentum tensor and it is also visible from the following considerations which use only the canonical gravitational superenergy tensor.

Let us consider an observer \( O \) which is studying gravitational field. His world-line is \( x^a = x^a(s) \) and \( \vec{v} : v^a = \frac{dx^a}{ds} \) represents his four-velocity. At any point \( P \) of the world-line one can define an instantaneous, local 3-space of the observer \( O \) orthogonal to \( \vec{v} \). This instantaneous 3-space has the following interior proper Riemannian metric

\[
\gamma_{ab} = v_av_b - g_{ab} \quad \dot{=} \quad \gamma_{ab} \quad \dot{=} \quad (-)g_{a\beta},
\]

(38)

where the Greek indices run over the values 1, 2, 3 (see e.g., [31]).

Then, by using the gravitational superenergy density \( \varepsilon_{\gamma} \) and its flux \( P^i \), one can easily construct in such instantaneous local 3-space of the observer \( O \) e.g., the following non-local expressions which have proper dimensions of the energy density and its flux

\[
\varepsilon_{\gamma} := \oint_{S_2} \varepsilon_{\gamma}(P) \ d^2S \approx \varepsilon_{\gamma}(P) \oint_{S_2} \ d^2S = 4\pi R^2 \varepsilon_{\gamma}(P),
\]

(39)

\[
P^i := \oint_{S_2} P^i(P) \ d^2S \approx \oint_{S_2} P^i(P) \ d^2S = 4\pi R^2 \ v^i(P).
\]

(40)

Here \( S_2 \) means an infinitesimal sphere \( \gamma_{\alpha\beta} x^\alpha x^\beta = R^2 \) in the instantaneous local 3-space of the observer \( O \).

\[
d^2S = \sqrt{\gamma_{\alpha\beta} x^\alpha x^\beta},
\]

(41)

\(^6\) Of course, the analogical conclusion is also true inside of the beam. However, in this domain the suitable expression on \( g_{\text{kin}}(P; \vec{v}) \) is slightly modified by additional terms which depend on the matter tensor \( T^k_l \). On the other hand, the canonical superenergy tensor of matter \( m_{\gamma k}(P; \vec{v}) \) given by (17) is confined to the domains occupied by matter, i.e., it is confined to the beam only.
where
\[ \sigma^\alpha = \gamma^{\alpha\beta} \sigma_\beta. \] (42)

Here \( \gamma^{\alpha\beta} \) means the inverse to the interior metric \( \gamma_{\alpha\beta} \), i.e., \( \gamma^{\alpha\beta} \gamma_{\beta\gamma} = \delta^\alpha_\gamma \), and
\[ \sigma_\beta = \frac{1}{2} \epsilon_{\beta\gamma\delta} \, dx^\gamma \wedge dx^\delta \] (43)
is a covector density–valued 2-form which defines two-dimensional integration element over \( S_2 \).

\( \sigma_\beta \) is usually treated as covariant components of a vector density normal to the sphere \( S_2 \) which length is equal to \( d^2 \).

\( \epsilon_{\alpha\beta\gamma} \) is the three-dimensional Levi–Civita pseudotensor, i.e., the components of the 3-form
\[ \epsilon = \frac{1}{3!} \epsilon_{\alpha\beta\gamma} \, dx^\alpha \wedge dx^\beta \wedge dx^\gamma, \] (44)
established by the condition
\[ \epsilon_{123} = \sqrt{\gamma}, \] (45)
where \( \gamma := \det(\gamma_{\alpha\beta}) \).

The expressions (39), (40) give us the relative gravitational energy density\(^7\) and its flux for an observer \( O \) in his instantaneous 3-space orthogonal to \( \vec{v} \) and they are in full agreement with the suitable expressions which one can easily obtain by using the averaged gravitational energy–momentum tensor\(^8\)
\[ \langle g_{ab}(P; \vec{v}) \rangle = g_{ab}(P; \vec{v}) \frac{L^2}{d^2}. \] This last fact gives a justification of the definitions (39), (40).

The values of the quantities \( \epsilon_\alpha \) and \( P^i \) depend on radius \( R \) of the sphere \( S_2 \). In order to get unique \( \epsilon_\alpha \) and \( P^i \) one should take as the radius \( R \) of this sphere \( S_2 \) a fundamental length \( L \). Putting as in the foregoing \( R = L = 100L_p \approx 10^{13} \) m one can evaluate the integrals (39), (40) in the way as we have already done.

As we could have seen from this paper the two quantities \( \epsilon_\alpha \) and \( P^i \) do not vanish in the Bonnor spacetime for every \( O \); especially, they do not vanish outside of the beam which is the source of this spacetime. In consequence, one can conclude that it is easy to attribute to the real gravitational field in Bonnor spacetime the positive-definite (although relative) energy density and its non-zero flux by using e.g., our canonical superenergy tensor of this field\(^9\).

When we only use gravitational pseudotensors which depend on the Levi–Civita connection\(^10\), then this important fact is camouflaged in some coordinates, such as Kerr–Schild coordinates or Bonnor coordinates, by energy and momentum of the inertial forces field (with \( R_{iklm} = 0 \)) which simply cancel out (outside of the beam) with energy and momentum of the real gravitational field (with \( R_{iklm} \neq 0 \)).

Thus, the conclusions of the authors of the papers \([2, 3]\) are incorrect as they resulted from coordinate-dependent pseudotensorial expressions which describe energy–momentum of a mixture of the real gravitational field (\( R_{iklm} \neq 0 \)) and an inertial forces field (\( R_{iklm} = 0 \)). One can call this mixture the total gravitational field.

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\(^7\) The fact that (39), (40) give us some kind of relative quantities (with respect to \( P \)) is seen from the construction of the canonical superenergy tensors.

\(^8\) The suitable expressions which can be obtained by using the averaged gravitational energy–momentum tensor differ from the expressions (39), (40) by a constant coefficient and this difference can be included into the fundamental length \( L \).

\(^9\) The use of the averaged gravitational energy–momentum tensor \( \langle g_{ab}(P; \vec{v}) \rangle \) is simpler and gives, of course, the fully equivalent results.

\(^10\) This connection is a ‘mixture’ of the real gravitational field which has \( R_{iklm} \neq 0 \) and the inertial forces field which has \( R_{iklm} = 0 \). It is a consequence of the Einstein equivalence principle.
In order to get the coordinate-independent information about energy and momentum (and also about angular momentum, see e.g., [14, 15, 19]) of the real gravitational field for which one has \( R_{iklm} \neq 0 \), one must use covariant expressions which depend only on the curvature tensor, such as our canonical gravitational superenergy tensor \( g_{Sab}(P; v') \) or our averaged gravitational energy–momentum tensor \( \langle g_{tab}(P; v') \rangle \) (and our gravitational angular supermomentum tensor or averaged gravitational angular momentum tensor, see e.g., [14, 15, 19]). These tensors extract the covariant and coordinate-independent information about energy and momentum of the real gravitational field \( (R_{iklm} \neq 0) \) which is hidden in the canonical gravitational energy–momentum pseudotensor (and in canonical gravitational angular momentum pseudotensor).

3. Conclusions

We have formulated and justified in the paper an obvious statement which reads: in order to get the coordinate-independent local information about energy–momentum (and also about angular momentum, see e.g., [14, 15, 19]) of the real gravitational field \( (R_{iklm} \neq 0) \) one should use the covariant expressions which depend only on the curvature tensor. The most natural expressions of such a kind are given by our canonical gravitational superenergy tensor \( g_{Sab}(P; v') \) or, very closely related to it, the averaged canonical gravitational energy–momentum tensor \( \langle g_{tab}(P; v') \rangle = g_{Sab}(P; v') \frac{L^2}{2} \). One can obtain these tensors at a point \( P \) by using special averagings of the gravitational energy–momentum differences in the normal coordinates \( NC(P) \).

As an example, we have considered in this paper the gravitational energy of the real gravitational field and its flux in the Bonnor spacetime. We have shown that in this case, the covariantly defined energy of the real gravitational field and its flux do not vanish everywhere in this spacetime; also in vacuum. Thus, our results contradict some earlier investigations in which the coordinate-dependent energy–momentum complexes and gravitational energy–momentum pseudotensors were used and led to the (incorrect) conclusion that the pure gravitational energy–momentum do not exist in this spacetime. The pure gravitational energy, of course, does exist in this spacetime but, if calculated by using energy–momentum pseudotensors, it simply cancels out outside the beam and in the considered coordinates with the energy of the inertial force field.

Of course, our statement is very general; especially it concerns the very popular recent analysis of the energetic content of the cosmological models.

In a series of the papers [32–39], by using the coordinate-dependent energy–momentum complexes, some authors have found that for special cosmological models and in special coordinates that the total energy–momentum ‘densities’, matter and gravitation, or that the integral energy–momentum, matter and gravitation are equal to zero.

Despite coordinate-dependency of their calculations (and lack of the asymptotic flateness) these authors conclude that the cosmological models which they have considered are energy–momentum-free locally and globally (or only globally). But one can doubt the correctness of such a conclusion because in other coordinates one gets the non-zero results. Rather, one should conclude from such coordinate-dependent calculations that the energy and momentum of the considered cosmological models are not determined either locally (non-tensorial quantities) or globally (lack of asymptotic flateness).

We would like to remark that the coordinate-independent superenergetic analysis or, equivalently, coordinate-independent analysis which uses the averaged energy–momentum tensors, gives non-zero and positive-definite results already in the case of the most simple,
spatially homogeneous and isotropic, Friedman cosmological models, closed, flat and open (see e.g., [10, 12]).

References

[1] Bonnor W B 2000 Gen. Relativ. Gravit. 32 1627
[2] Bringley T 2002 Energy and momentum of a stationary beam of light Preprint gr-qc/0204006
[3] Gad R M 2005 Astrophys. Space Sci. 295 451
[4] Ehlers J and Kundt W 1962 An Article in Gravitation: An Introduction to Current Problems ed L Witten (New York: Wiley)
[5] Garecki J 1997 Rep. Math. Phys. 40 485
[6] Misner C W, Thorne K S and Wheeler J A 1973 Gravitation (San Francisco: Freeman)
[7] Chang C C, Nester J M and Chen C M 1999 Phys. Rev. Lett. 83 1897 (Preprint gr-qc/9809040)
[8] Chang C C, Nester J M and Chen C M 2000 Energy–momentum (quasi)-localization for gravitating systems Gravitation and Astrophysics, The Proc. 4th Int. Workshop on Gravitation and Astrophysics (Beijing) ed L Liu, J Luo, X-Z Li and J-P Hsu (Singapore: World Scientific) pp 163–73 (Preprint gr-qc/9912058)
[9] Szabados L B 2004 Quasi-Local energy–momentum and angular momentum in GR: a review article Living Rev. Relativ. 7 4
[10] Garecki J 1993 Rep. Math. Phys. 33 57
[11] Garecki J 1999 J. Math. Phys. 40 4035
[12] Garecki J 1999 Rep. Math. Phys. 43 397
[13] Garecki J 1999 Rep. Math. Phys. 44 95
[14] Garecki J 2002 Ann. Phys. Lpz 11 441
[15] Dabrowski M P and Garecki J 2002 Class. Quantum Grav. 19 1
[16] Synge J L 1960 Relativity: The General Theory (Amsterdam: North-Holland)
[17] Garecki J 1979 Acta Phys. Pol. B 10 883
[18] Garecki J 1981 The superenergy in general relativity Proc. Einstein Centenary Symposium vol 2 (Nagpur, Maharashtra, India: Duhita Publishers)
[19] Garecki J 2005 The averaged tensors of the relative energy–momentum and angular momentum in general relativity Preprint gr-qc/0505151
[20] Ashtekar A 1999 Quantum mechanics of geometry Preprint gr-qc/9901023
[21] Ashtekar A 2001 Quantum geometry and gravity: recent advances Preprint gr-qc/0112038
[22] Pullin J 2002 Canonical quantization of general relativity: the last 18 years in a nutshell Preprint gr-qc/0209008
[23] Rovelli C 2003 A dialog on quantum gravity Preprint hep-th/0310077
[24] Markopoulou F 2002 Planck-scale models of the Universe Preprint gr-qc/0210086
[25] Ashtekar A 2002 Quantum geometry in action: big-bang and black holes Preprint math-ph/0202008
[26] Bojowald M 2004 Loop quantum cosmology: recent progress Preprint gr-qc/0402053
[27] Ashtekar A and Lewandowski J 2004 Class. Quantum Grav. 21 R53
[28] Smolin L 2003 How far we are from the quantum theory of gravity? Preprint hep-th/0303185
[29] Ashtekar A 2004 Gravity and the quantum Preprint gr-qc/0410054
[30] Perez A 2004 Introduction to loop quantum gravity and spin foams Preprint gr-qc/0409061
[31] Landau L D and Lifschitz E M 1975 The Classical Theory of Fields (Oxford: Pergamon)
[32] Rosen N 1994 Gen. Relativ. Gravit. 26 319
[33] John V B et al 1995 Gen. Relativ. Gravit. 27 323
[34] Banerjee N and Sen S 1997 Pramana J. Phys. 49 609
[35] Xulu S S 2003 The energy–momentum problem in general relativity Preprint hep-th/030870
[36] Mustafa S and Ali H 2005 Energy–momentum in viscous Kasner-type universe in Bergmann–Thomson formulation Preprint gr-qc/0502060
[37] Mustafa S et al 2005 Energy of the universe in Bianchi-type I models in Møller's tetrads theory of gravity Preprint gr-qc/0505079
[38] Mustafa S 2005 Different approaches for Møller’s energy in the Kasner-type spacetime Preprint gr-qc/0505078
[39] Mustafa S 2005 Energy and momentum associated with Kasner-type universes Preprint gr-qc/0506061