SMOOTH AFFINE SHEAR TIGHT FRAMES WITH MRA STRUCTURE

BIN HAN AND XIAOSHENG ZHUANG

Abstract. Finding efficient representations is one of the most challenging and heavily sought problems in mathematics. Representation using shearlets recently receives a lot of attention due to their desirable properties in both theory and applications. Using the framework of frequency-based affine systems as developed in [17], in this paper we introduce and systematically study affine shear tight frames which include all known shearlet tight frames as special cases. Our results in this paper will resolve several key questions on shearlets. We provide a complete characterization for an affine shear tight frame and then use it to obtain smooth affine shear tight frames with all their generators in the Schwarz class. Though multiresolution analysis (MRA) is the foundation and key feature of wavelet analysis for fast numerical implementation of a wavelet transform, all the known shearlets so far do not possess any MRA structure and filter banks. In order to study affine shear tight frames with MRA structure, following the lines developed in [17], we introduce the notion of a sequence of affine shear tight frames and then we provide a complete characterization for it. Based on our characterizations, we present two different approaches, i.e., non-stationary and quasi-stationary, for the construction of sequences of affine shear tight frames with MRA structure such that all their generators are smooth (in the Schwarz class) and they have underlying filter banks. Consequently, their associated transforms can be efficiently implemented using filter banks similarly as a fast wavelet transform does.

1. Introduction and Motivation

In the era of information, everyday and everywhere, huge amount of information is acquired, processed, stored, and transmitted in the form of high-dimensional digital data through Internet, TVs, cell phones, satellites, and various other modern communication technologies. One of the main goals in today’s scientific research is on the efficient representation and extraction of information in high-dimensional data. It is well known that high-dimensional data usually exhibit anisotropic phenomena due to data clustering of various types of structures. For example, cosmological data normally consist of many morphological distinct objects concentrated near lower-dimensional structures such as points (stars), filaments, and sheets (nebulae). The anisotropic features of high-dimensional data thus encode a large portion of significant information. Mathematical representation systems that are capable of capturing such anisotropic features are therefore undoubtedly the key for the efficient representation of high-dimensional data.

During the past decade, directional multiscale representation systems become more and more popular due to their abilities of resolving anisotropic features in high-dimensional data [3, 4, 5, 14, 20, 27, 30]. Our focus of this paper is on investigation and construction of a new type of directional multiscale representation systems: affine shear tight frames. Such a type of directional multiscale representation systems has many desirable properties including directionality, multiresolution analysis, smooth generators, etc. Moreover, the affine shear systems have an underlying filter banks associated with the directional affine (wavelet) systems as considered in [17].

Before proceeding further, let us first introduce some necessary notation and definitions. For a function $f \in L_1(\mathbb{R}^d)$, the Fourier transform of $f$ is defined to be

$$\hat{f}(\xi) = \mathcal{F} f(\xi) := \int_{\mathbb{R}^d} f(x) e^{-i x \cdot \xi} dx, \quad \xi \in \mathbb{R}^d,$$

which can be extended to square-integrable functions in $L_2(\mathbb{R}^d)$ and tempered distributions. Note that the Plancherel identity holds in $L_2(\mathbb{R}^d)$:

$$\langle f, g \rangle = \frac{1}{(2\pi)^d} \langle \hat{f}, \hat{g} \rangle, \quad f, g \in L_2(\mathbb{R}^d),$$

where $\langle f, g \rangle := \int_{\mathbb{R}^d} f(x) g(x) dx$. Let $U$ be a $d \times d$ real-valued invertible matrix. Define $\|f\|_2^2 := \langle f, f \rangle$ and

$$f_{U,k,n}(x) := |\det U|^{1/2} f(Ux - k) \cdot e^{-i n \cdot U x}, \quad k, n, x \in \mathbb{R}^d.$$

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Here \( U, k, \) and \( n \) refer to dilation, translation, and modulation, respectively. We shall adopt the convention that \( f_{U,k}: = f_{U,k,0} \) and \( h_{k,n}: = f_{I,d,k,n} \) with \( I_d \) being the \( d \times d \) identity matrix. It is trivial to check that \( \| f_{U,k,n} \|_2 = \| f \|_2 \) and \( f_{U,k,n+1} = f_{U,k} g_{U,k,n} = (f,g) \).

Following the standard notation, we denote by \( \mathcal{S}(\mathbb{R}^d) \) the linear space of all compactly supported \( C^\infty \) (test) functions with the usual topology, and \( \mathcal{S}'(\mathbb{R}^d) \) the linear space of all distributions; that is, \( \mathcal{S}'(\mathbb{R}^d) \) is the dual space of \( \mathcal{S}(\mathbb{R}^d) \). \( \mathcal{S}(\mathbb{R}^d) \) denotes the Schwarz class of all rapidly decreasing functions on \( \mathbb{R}^d \) and \( \mathcal{S}'(\mathbb{R}^d) \) is its dual space of tempered distributions on \( \mathbb{R}^d \). Note that an element \( f \in \mathcal{S}'(\mathbb{R}^d) \) has the property that its derivatives of any orders belong to \( C^\infty(\mathbb{R}^d) \) and have polynomial decay of arbitrary orders. Moreover, the Fourier transform \( \mathcal{F}: \mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d) \) is well-defined, one-to-one, and onto. By \( L_p^\text{loc}(\mathbb{R}^d), 1 \leq p < \infty \) we denote the linear space of all (Lebesgue) measurable functions \( f \) on \( \mathbb{R}^d \) such that \( \int_X |f(x)|^p dx < \infty \) for every compact set \( X \subset \mathbb{R}^d \).

Though all the discussion and results in this paper can be carried over to any high dimensions \( \mathbb{R}^d \) with \( d \geq 2 \), for simplicity of presentation, we restrict ourselves to the two-dimensional case only, which is the most important case in the area of directional multiscale representations. We shall use the following matrices throughout this paper.

\[
E := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad S^\tau := \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix}, \quad S_T := \begin{bmatrix} 1 & 0 \\ \tau & 1 \end{bmatrix}, \quad A_\lambda := \begin{bmatrix} \lambda^2 & 0 \\ 0 & \lambda \end{bmatrix}, \quad B_\lambda := (A_\lambda)^{-T} = \begin{bmatrix} \lambda^{-2} & 0 \\ 0 & \lambda^{-1} \end{bmatrix}, \quad (1.1)
\]

where \( \tau \in \mathbb{R} \) and \( \lambda > 1 \). \( S_T \) and \( S^\tau \) are the shear operations while \( A_\lambda \) is the dilation matrix. Define \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \) and \( \delta : \mathbb{Z}^d \rightarrow \mathbb{R} \) denotes the Dirac sequence such that \( \delta(0) = 1 \) and \( \delta(k) = 0 \) for all \( k \in \mathbb{Z}^d \setminus \{0\} \). We shall use boldface letters to indicate functions in the frequency domain; that is, \( \mathbf{f} \) usually means \( \mathbf{f} = \mathcal{F} f \) for some function \( f \in L_2(\mathbb{R}^d) \). A frequency-based affine shear system is obtained by applying shear, dilation, and translation (modulation) to generators at different scales. Note that the \( \mathcal{F}(S_{T \cdot}) \) could be highly tilted when \( \tau \) is very large for a compactly supported function \( f \). To balance the shear operation, one usually considers cone-adapted systems \( \mathbf{10} \ [12] \ [15] \ [21] \). A cone-adapted system usually consists of three parts: one subsystem covers the low frequency region, one subsystem covers the horizontal cone \( \{ \xi \in \mathbb{R}^2 : |\xi_2/\xi_1| \leq 1 \} \), and one subsystem covers the vertical cone \( \{ \xi \in \mathbb{R}^2 : |\xi_1/\xi_2| \leq 1 \} \). The vertical-cone subsystem could be constructed to be the ‘flipped’ version of the horizontal-cone subsystem. More precisely, a function \( \mathbf{f} \in L_2^\text{loc}(\mathbb{R}^2) \) will serve as the generator for the low frequency region, a function \( \mathbf{f} \in L_2^\text{loc}(\mathbb{R}^2) \) will generate an affine system covering certain region of the horizontal cone, and \( \{ \psi_j^\lambda, k \in \mathbb{Z}^2, |\xi_j| = r_j + 1, \ldots, s_j \} \) are generators at scale \( j \) that will generate elements along the seamlines (diagonal directions \( \{ \xi \in \mathbb{R}^2 : |\xi_2/\xi_1| = \pm 1 \} \)) to serve the purpose of tightness of the system. Note that \( \psi_j^\lambda, |\xi| = r_j + 1, \ldots, s_j \) may not come from a single generator. Define \( \mathbf{y}_j \) to be

\[
\mathbf{y}_j := \{ \psi_j(S_T^\ell) : \ell = -r_j, \ldots, r_j \} \cup \{ \psi_j^\ell(S_T): |\ell| = r_j + 1, \ldots, s_j \}, \quad (1.2)
\]

where \( r_j \) and \( s_j \) are nonnegative integers. A frequency-based affine shear system is then defined to be

\[
\text{FAS}(\mathbf{f}; \{ \mathbf{y}_j \})_{j=0}^\infty = \{ \mathbf{f}_{0,k} : k \in \mathbb{Z}^2 \} \cup \{ \mathbf{h}_{B_j^0,0,k} : k \in \mathbb{Z}^2, h \in \mathbf{y}_j \}_{j=0}^\infty \cup \{ \mathbf{h}_{B_j^0,0,k} : k \in \mathbb{Z}^2, h \in \mathbf{y}_j \}_{j=0}^\infty. \quad (1.3)
\]

For a function \( f \) on \( \mathbb{R}^2 \), observe that \( f_{E_0}(x,y) = f(x,y) \); that is, \( f_{E_0} \) is the ‘flipped’ version of \( f \) along the line \( y = x \). Note that the system \( \{ h_{B_j^0,0,k} : k \in \mathbb{Z}^2, h \in \mathbf{y}_j \} \) is the high frequency region at scale \( j \) with respect to the horizontal cone, while its ‘flipped’ version \( \{ h_{B_j^0,0,k} : k \in \mathbb{Z}^2, h \in \mathbf{y}_j \} \) is for the high frequency region at scale \( j \) with respect to the vertical cone.

Suppose \( \varphi, \psi, \psi_j^\ell \in \mathcal{S}'(\mathbb{R}^2) \). Then there are tempered distributions \( \varphi, \psi, \psi_j^\ell \in \mathcal{S}'(\mathbb{R}^2) \) such that \( \mathcal{F}\varphi = \varphi, \mathcal{F}\psi = \psi, \) and \( \mathcal{F}\psi_j^\ell = \psi_j^\ell \). Let

\[
\Psi_j := \{ \psi_j(S_T^\ell) : \ell = -r_j, \ldots, r_j \} \cup \{ \psi_j^\ell(S_T^\ell) : |\ell| = r_j + 1, \ldots, s_j \}.
\]

Note that \( f_{U,k,0} = f_{U,-r_0,k} \). Then the system defined as in \( (1.3) \) in the spatial domain is equivalent to

\[
\text{AS}(\varphi; \{ \mathbf{y}_j \})_{j=0}^\infty = \{ \varphi(\cdot - k) : k \in \mathbb{Z}^2 \} \cup \{ h_{B_j^0,k}, h_{B_j^0,k} : k \in \mathbb{Z}^2, h \in \mathbf{y}_j \}_{j=0}^\infty. \quad (1.4)
\]

In other words, \( \text{FAS}(\varphi; \{ \mathbf{y}_j \})_{j=0}^\infty \) is the image of \( \text{AS}(\varphi; \{ \mathbf{y}_j \})_{j=0}^\infty \) under the Fourier transform. Though within the framework of tempered distributions the frequency-based approach and the space-domain approach are equivalent to each other, as argued in \( [17] \) it is often easier to deal with a frequency-based system. Therefore, for simplicity of presentation, in the rest of this paper we shall mainly discuss the frequency-based affine shear system \( \text{FAS}(\varphi; \{ \mathbf{y}_j \})_{j=0}^\infty \). It is trivial to see that a frequency-based affine shear system \( \text{FAS}(\varphi; \{ \mathbf{y}_j \})_{j=0}^\infty \) defined in \( (1.3) \) is a special case of the nonhomogeneous affine (wavelet) systems discussed in \( [17] \) Section 3).

1.1. Related work. In 1D, it is well known that wavelet representation systems provide optimally sparse representation for functions \( f \in L_2(\mathbb{R}) \) that are smooth except for finitely many discontinuity jumps \( [6] \ [23] \). In high dimensions, wavelet representation systems could be obtained by using tensor product of 1D wavelet representation systems. However, tensor product representation systems usually lack directionality since they only favor
certain directions such as horizontal and vertical directions. The limitation of directionality selectivity is one of the main reasons that the tensor product wavelets fail to provide optimally sparse approximation for 2D smooth functions with singularities along a closed smooth curve (anisotropic features) \[8\]. In order to achieve flexible directionality selectivity, additional operation other than dilation and translation is needed.

Directional framelets [11, 16, 17], directly built from the frequency plane, achieve directionality by separating the frequency plane into annulus at different scales and further splitting each annulus into different wedge shapes. More precisely, in the frequency domain, considering the polar coordinate \((r, \theta)\) (i.e., \((x, y) = (r \cos \theta, r \sin \theta)\), one first constructs a pair \(\{\eta(r), \zeta(r)\}\) of 1D scaling and wavelet functions such that \(|\eta|^2 + \sum_{j \in \mathbb{Z}} |\zeta(2^{-j})|^2 = 1\). Then, a 2D scaling function \(\varphi\) in the frequency domain is defined to be \(\varphi(r, \theta) := \eta(r)\), while the 2D radial wavelet function \(\psi\) is defined to be \(\psi(r, \theta) := \zeta(r)\). The function \(\psi(2^{-j}\cdot)\) is supported on an annulus \(\{(r, \theta) : 2^j \epsilon_1 \leq r \leq 2^j \epsilon_2, \theta \in [0, 2\pi)\}\). Obviously, \(\psi\) is an isotropic function. But directionality can be easily achieved by splitting \(\psi\) in the angular direction \(\theta\) with a smooth partition of unity \(\alpha_{j, \ell}(\theta)\) for \([0, 2\pi)\): \(\sum_{\ell=1}^{N} |\alpha_{j, \ell}(\theta)|^2 = 1\), \(\theta \in [0, 2\pi)\). Generators at scale \(j\) is given by \(\psi^{j, \ell}(r, \theta) = \zeta(r) \alpha_{j, \ell}(\theta), \ell = 1, \ldots, s_j\). The directional framelet systems are then obtained by applying isotropic dilation \(N := 2^{-j} L_2\) and translation to the generators, which result in wavelet atoms of the form \(\psi^{j, \ell}_{R_{\theta, \ell}}\) and the whole system is a tight frame for \(L_2(\mathbb{R}^2)\) with all its generators in the Schwarz class.

Although directional framelets can easily achieve directionality, yet they still use the isotropic dilation matrices. The system is thus too ‘dense’ to provide optimally sparse approximation for 2D \(C^2\) functions with singularity along a close \(C^2\) curve. By using parabolic dilation \(\Lambda = \text{diag}(2, \sqrt{2})\) instead of an isotropic dilation, the curvelets introduced in [2] not only can achieve directionality selectivity, but also provide optimally sparse approximation for 2D \(C^2\) functions away from a close \(C^2\) curve; see [2, 12, 24, 25] for more details on the optimally sparse approximation. The curvelet atom is of the form \(\psi^{j, \ell}_{R_{\theta, \ell}}\) with \(R_{\theta, \ell}\) being a rotation operation determined by the angle \(\theta_{j, \ell}\). In other words, each generator \(\psi^{j, \ell}\) is afforded with a dilation matrix \(N_{j, \ell} := \Lambda^{-1} R_{\theta_{j, \ell}}\) that is determined by both scaling and rotation.

The curvelets use parabolic scaling and rotation and can achieve both directionality and optimally sparse approximation. However, the rotation operation \(R_{\theta}\) destroys the preservation of integer lattice since \(R_{\theta}\mathbb{Z}^2\) is not necessarily an integer lattice, yet the integer lattice preservation is a very much desired property in applications. Shearlets, introduced in [9, 10, 12], replace rotation \(R_{\theta}\) by shear \(S_{\theta}\). The shear operator not only preserves the integer lattice \(S_{\theta}\mathbb{Z}^2 = \mathbb{Z}^2\), but also enables a shearlet system with only a few generators; that is, \(\psi^{j, \ell}\) could come from the shear versions of several generators (even one single generator in the case of non-cone-adapted shearlets [11]). Let \(A_{hk} := \text{diag}(4, 2)\) and \(A_h := \text{diag}(2, 4)\). A cone-adapted shearlet system in [10, 12] is generated by three generators \(\varphi, \psi^h, \psi^v := \psi^h(\mathbb{E}_\cdot)\) through shear, parabolic scaling, and translation:

\[
\begin{align*}
\text{CSH}(\varphi; \{\psi^h, \psi^v\}) = & \{(\varphi(\cdot - k) : k \in \mathbb{Z}^2) \\
& \cup \{2^{3j/2} \psi^h(\mathcal{S}^j A_{h}^j \cdot - k) : \ell = -2^j, \ldots, 2^j, k \in \mathbb{Z}^2\}_{j=0}^\infty \\
& \cup \{2^{3j/2} \psi^v(\mathcal{S}^j A_{v}^j \cdot - k) : \ell = -2^j, \ldots, 2^j, k \in \mathbb{Z}^2\}_{j=0}^\infty \}.
\end{align*}
\]

(1.5)

It is obvious that the above shearlet system is indeed a special case of the affine shearlet systems defined as in [13] by noting that \(2^{3j/2} \psi^h(\mathcal{S}^j A_{h}^j \cdot - k) = 2^{3j/2} \psi(\mathcal{S}^j (A_{h}^j \cdot - \mathcal{S}^{-j} k)) = \tilde{\psi}_{A_{h}^j, S^{-j} k}\) with \(\tilde{\psi} := \psi(\mathcal{S}^j \cdot)\). The system defined above in [16] is in general not a tight frame for \(L_2(\mathbb{R}^2)\). In the case of bandlimited generators, such a system can be modified into a tight frame for \(L_2(\mathbb{R}^2)\) by using projection techniques [12], which cut the seamlne elements \(\psi^h(\mathcal{S}^j A_{h}^j \cdot - k), \psi^v(\mathcal{S}^j A_{v}^j \cdot - k)\) with \(\ell = \pm 2^j\) into half pieces and then restrict them strictly in each cone. Such projection techniques will result in non-smooth shearlets along the seamlnes: \(\psi^{h, \pm}(\mathcal{S}^{\pm j} A_{h}^j \cdot - k), \psi^{v, \pm}(\mathcal{S}^{\pm j} A_{v}^j \cdot - k)\).

The non-smoothness of the seamlne elements breaks down the arguments in the proof of the optimally sparse approximation for 2D \(C^2\) functions with singularities along a close \(C^2\) curve in [12], in which at least twice differentiability is needed for the shearlet atoms. Guo and Labate in [15] proposed another type of shearlet-like construction. The idea is still the frequency splitting; but this time for the rectangular strip from the Fourier transform \(\varphi\) of the Meyer 2D tensor product scaling function. The splitting is applied to \(\psi^j := \sqrt{\mathcal{S}^j |\varphi(2^{-2j}-\cdot)|^2 - |\varphi(2^{-2j})|^2}\). A gluing procedure is applied to the two slices along the seamlnes coming from different cones. With appropriate construction, the gluing procedure is smooth and the system in [15] consists of smooth shearlet-like atoms. However, due to the inconsistency of two cones, a different dilation matrix is needed for the glued shearlet-like atom. We shall discuss the connections of such systems to our affine shear systems in more details in Section 4.

Though there are several constructions of various shearlets available in the literature [10, 12, 15, 21], several key problems remain unresolved. In particular, the following three issues:
Q1) Existence of smooth shearlets. The cone-adapted shearlet system is obtained by applying shear, parabolic scaling, and translation to a few generators. To achieve tightness of the system, the shearlet atoms along the seamlines need to be cut into half pieces. One way to achieve smoothness is by using the gluing procedure as in [15]. However, the system no longer has a full shear structure and is not affine-like. Are there shear tight frames using one or a few generators?

Q2) Shearlets with MRA structure. The cone-adapted shearlets achieve directionality by using a parabolic dilation \( A_k \) (in fact it essentially uses two parabolic dilations: \( A_k = \text{diag}(\lambda^2, \lambda) \) for horizontal cone, and \( EA_k E = \text{diag}(\lambda, \lambda^2) \) for the vertical cone) and the shear matrix in (1.3) to keep the generators \( \psi \) at all scales to be the same. In essence, directionality is achieved in a shear system by using infinitely many dilation matrices so that the initial direction of the generator \( \psi \) is dilated and sheared to other directions. This is the main difficulty to build a shear system having a multiresolution structure where only a single dilation matrix is employed. It is shown in [19] that there is no traditional shearlet MRA \( \{ Y_j \}_{j \in \mathbb{Z}} \) with scaling function \( \varphi \) having nice decay property, where \( Y_j = \{ \varphi_{s \lambda A_k^j, \lambda} : k \in \mathbb{Z}^2, \ell \in I_j \} \). In this case, the space \( Y_j \) uses many (possibly infinitely many) dilation matrices. Are there MRA structures in certain setting for a shearlet system?

Q3) Filter bank association. Once we have an MRA for a shear system, it is then natural to ask whether there also exists an associated filter bank system for the shear system. [18, 20] have studied the filter bank system with shear operation directly in the discrete setting and provide characterization for such a shear filter bank system. However, it is still not clear whether a filter bank system exists and can be naturally induced from the constructed shear system.

1.2. Our contributions. In this paper, since shear operation has many nice properties in both theory (optimally sparse approximation, rich group structures, etc., see [22, 24]) and applications (edge detection, inpainting, separation, etc., see [13, 14, 20, 23]), we shall focus on the construction of directional multiscale representation systems with shear operation: affine shear systems. Along the way, we will focus on the above issues as discussed in Q1 – Q3.

For smoothness, we show that by using one inner smooth generator \( \psi \) and only a few smooth boundary generators \( \psi^{j, \ell} \) (8 boundary generators in total for each scale \( j \) and they are actually generated by only 2 generators through shear and ‘flip’ for the non-stationary construction), we can indeed construct smooth affine shear tight frames. In addition, in this paper, we study sequences of affine shear systems. We show that a sequence of affine shear tight frames naturally induces an MRA structure. We would like to point out here that almost all existing approaches [10, 12, 15] study only one shear system, while it is of fundamental importance to investigate sequences of shear systems as promoted in [17].

We propose two approaches for the construction of sequences of smooth affine shear tight frames. One is non-stationary construction and the other is quasi-stationary construction. The function \( \varphi^j \) for the non-stationary construction is different at different scale \( j \), while the quasi-stationary construction is with a fixed scaling function \( \varphi^j \equiv \varphi \). These two approaches actually share the similar idea of frequency splitting as that for the construction of directional frames: at scale \( j \), a smooth 2D wavelet function \( \omega^j = \{ (|\varphi^j| + 1)^2 - |\varphi^j|^2 \}^{1/2} \) is constructed in the frequency domain; then a smooth partition of unity \( \gamma^j_{\ell, s} = 1, \ldots, s_j \), for \( \mathbb{R}^2 \) such that \( \sum_{\ell=1}^{s_j} |\gamma^j_{\ell, s}|=1 \) is created using shear operations for two cones instead of rotation for the case of directional frames or curvelets; eventually, generators \( \psi^{j, \ell} \) at scale \( j \) are obtained by applying \( \gamma^j_{\ell, s} \) to \( \omega^j \).

By carefully designing the function \( \omega^j \), we show that we can indeed generate a smooth affine shear tight frame (or a sequence of affine shear tight frames), which contains a subsystem (or a sequence of subsystems) that is generated by only one generator. In fact, for the non-stationary case, we will see that \( \psi^{j, \ell} = \psi \) for all \( \ell \) except those \( \ell \) with respect to seamline elements (8 in total and they can be generated by only 2 elements). In other words, the shear operations in the non-stationary construction can reach arbitrarily close to the seamlines. For the quasi-stationary construction, we will see that \( \psi^{j, \ell} = \psi \) for a total number of \( \ell \) that is proportional to \( \lambda^2 \). In this case, the shear operators in each scale are restricted inside an area with a fixed opening angle. We shall discuss these two types of constructions in Section 4 with more details.

The non-stationary construction and quasi-stationary construction induce two types of MRA structure: non-stationary MRA and stationary MRA. Both of these two types of MRAs are the traditional wavelet MRA in the sense that the space \( Y_j \) is generated by the function (or \( \varphi \) or \( \varphi^j \)) using a fixed dilation matrix \( M = \lambda^2 I_2 \). On the other hand, the space \( Y_j \) is generated by \( \psi \) and \( \psi^{j, \ell} \) using many dilation matrices determined by shears and parabolic scalings. We show that such types of constructions have a very close relation with the directional framelets developed in [16, 17]. By a simple modification, we show that the construction of directional framelets developed in [16, 17] using tensor product on the polar coordinate can be easily adapted to the setting of Cartesian coordinate under the cone-adapted setting. For the directional framelets, it is natural and easy to build a directional tight frame with MRA structure and with an underlying filter bank. We show that certain affine shear tight frames
can be regarded as a subsystem of certain directional framelets. Therefore, such affine shear tight frames have an inherited MRA structure and filter banks from the corresponding directional framelets. This observation implies that the transform of such affine shear tight frames can be implemented through the filter banks of their corresponding directional framelets.

1.3. Contents. The structure of this paper is as follows. In Section 2 we shall provide a characterization of a frequency-based affine shear system to be a tight frame in $L_2(\mathbb{R}^2)$. Based on the characterization, simple characterization conditions can be obtained for frequency-based affine shear systems with nonnegative generators.

Then, we shall present a toy example of frequency-based affine shear tight frames with characteristic function generators. In Section 3 since sequences of frequency-based affine shear systems play a very important role in our study of the MRA structure of affine shear systems, we shall characterize a sequence of frequency-based affine shear systems to be a sequence of affine shear tight frames for $L_2(\mathbb{R}^2)$. Correspondingly, simple characterization conditions on sequences of frequency-based affine shear tight frames with nonnegative generators shall be given.

Based on the characterization results, in Section 4 we provide details for the construction of smooth frequency-based affine shear systems to other existing shear systems shall also be addressed. In Section 5 we shall investigate the relation between our frequency-based affine shear systems and the directional framelets in \cite{16, 17}. By modifying the generators for directional framelets, we shall construct cone-adapted directional framelets, with which a natural filter bank is associated. We shall show that for $\Lambda_\lambda$ with $\lambda > 1$ being an integer, a frequency-based affine shear tight frame is in fact a subsystem of a cone-adapted directional framelet and therefore a frequency-based affine shear system has also an inherited filter bank. Some extension and discussion shall be given in the last section.

2. Frequency-based Affine Shear Tight Frames

Affine systems and their properties have been studied by many researchers, e.g., see \cite{3, 7, 16, 17, 29}. In this section we characterize frequency-based affine shear tight frames. Based on the characterization, we show that very simple characterization conditions could be obtained for frequency-based affine shear tight frames with nonnegative generators. To prepare our study of smooth affine shear tight frames in later sections, we shall present a toy example of frequency-based affine shear tight frame at the end of this section.

For $\text{FAS}(\varphi; \{\Psi_j\}_{j=0}^\infty)$ given as in \cite{1, 3} with $\Psi_j$ being given as in \cite{1, 3}, we define the following functions:

$$T^k_\varphi(\xi) := \overline{\varphi(\xi)} \varphi(\xi + 2\pi k), \quad k \in \mathbb{Z}^2, \quad \xi \in \mathbb{R}^2,$$

$$T^k_\Psi(\xi) := \sum_{\ell = s_j} \psi^{j,\ell}(S_j^*(\xi + 2\pi k)) \psi^{j,\ell}(S_j(\xi + 2\pi k)), \quad k \in \mathbb{Z}^2, \quad \xi \in \mathbb{R}^2; \quad \psi^{j,\ell} := \psi \text{ for } |\ell| \leq r_j,$$

$$T^k_\lambda(\xi) = T^k_\Psi(\xi) = 0, \quad k \in \mathbb{R}^2 \setminus \mathbb{Z}^2, \quad \xi \in \mathbb{R}^2.$$

We say that $\text{FAS}(\varphi; \{\Psi_j\}_{j=0}^\infty)$ is a frequency-based affine shear tight frame for $L_2(\mathbb{R}^2)$ if all generators $\{\varphi\} \cup \{\Psi_j\}_{j=0}^\infty \subseteq L_2(\mathbb{R}^2)$ and

$$(2\pi)^2 \|f\|^2_2 = \sum_{k \in \mathbb{Z}^2} |\langle f, \varphi_0, k \rangle|^2 + \sum_{j=0}^\infty \sum_{h \in \mathcal{J}_j} \sum_{k \in \mathbb{Z}^2} (|\langle f, h_{B_j \xi,0,k} \rangle|^2 + |\langle f, h_{B_j \xi,0,k} \rangle|^2) \quad \forall f \in L_2(\mathbb{R}^2).$$

Using Plancherel identity, \cite{2, 3} is equivalent to saying that $\text{AS}(\varphi; \{\Psi_j\}_{j=0}^\infty)$ is a tight frame for $L_2(\mathbb{R}^2)$:

$$||f||^2_2 = \sum_{k \in \mathbb{Z}^2} |\langle f, \varphi(\cdot-k) \rangle|^2 + \sum_{j=0}^\infty \sum_{h \in \mathcal{J}_j} \sum_{k \in \mathbb{Z}^2} (|\langle f, h_{A_j,\xi,k} \rangle|^2 + |\langle f, h_{A_j,\xi,k} \rangle|^2) \quad \forall f \in L_2(\mathbb{R}^2),$$

where $\mathcal{F}\varphi = \varphi$ and $\{\mathcal{F}h = h : h \in \mathcal{J}_j\} = \Psi_j$ for $j \in \mathbb{N}_0$.

2.1. Characterization of frequency-based affine shear tight frames. We next characterize the system in \cite{1, 3} to be a frequency-based affine shear tight frame. We have the following characterization.

**Theorem 1.** Let $\Lambda_\lambda, B_\lambda, S_\ell, E$ be defined as in \cite{14} with $\lambda > 1$ and let $\text{FAS}(\varphi; \{\Psi_j\}_{j=0}^\infty)$ be defined as in \cite{1, 3} with $\{\varphi\} \cup \{\Psi_j\}_{j=0}^\infty \subseteq L_2^{oc}(\mathbb{R}^2)$. Define $\Lambda := \cup_{j=0}^\infty [A_j \mathbb{Z}^2] \cup [EA_j \mathbb{Z}^2].$ Then $\text{FAS}(\varphi; \{\Psi_j\}_{j=0}^\infty)$ is a frequency-based affine shear tight frame for $L_2(\mathbb{R}^2)$ if and only if

$$T^0_\varphi(\xi) + \sum_{j=0}^\infty [T^0_\Psi(B_j \xi) + T^0_\lambda(B_j \xi E)] = 1, \quad a.e. \xi \in \mathbb{R}^2$$

(4.4)
and
\[ T^k_\varphi(\xi) + \sum_{j=0}^{\infty} |(f, \varphi_{S, B^j_\varphi(\xi)} + T^j_{\varphi}(B^j_\varphi(\xi))] = 0, \quad a.e. \xi \in \mathbb{R}^2, k \in \Lambda \setminus \{0\}, \] (2.5)
where the sum in (2.4) converges absolutely and the infinite sum in (2.5) is finite for almost every \( \xi \in \mathbb{R}^2 \).

**Proof.** The proof essentially follows the lines developed in [17, Theorem 11]. For \( f \in \mathcal{D}(\mathbb{R}^2) \), define
\[ S^j(f) := \sum_{k \in \mathbb{Z}^2} |(f, \varphi_{S, B^j_\varphi(\xi)})|^2 + \sum_{j=0}^{J} \sum_{r=-r_j}^{r_j} \sum_{k \in \mathbb{Z}^2} \{(f, \psi_{S, B^j_{S,B^j_\varphi(\xi)}\{0,0\},k})|^2 + |(f, \psi_{S, B^j_{S,B^j_\varphi(\xi)}\{0,0\},k})|^2 \}
+ \sum_{j=0}^{J} \sum_{|\ell|=r_j+1} \sum_{k \in \mathbb{Z}^2} \{(f, \psi_{S, B^j_{S,B^j_\varphi(\xi)}\{0,0\},k})|^2 + |(f, \psi_{S, B^j_{S,B^j_\varphi(\xi)}\{0,0\},k})|^2 \}. \] (2.6)
Using [17, Lemma 10], we have
\[ S^j(f) = (2\pi)^2 \int_{\mathbb{R}^2} \sum_{k \in \Lambda} f(\xi)^2 \left( T^0_{\varphi}(\xi) + \sum_{j=0}^{J} T^j_{\varphi}(B^j_{\varphi}(\xi)) \right) d\xi. \] (2.7)
Since \( \lambda > 1 \), we have \( B_r(0) \cap \Lambda \) is finite for any ball \( B_r(0) \) with radius \( r > 0 \). Hence, \( \Lambda \) has no accumulation point. Moreover, we have
\[ \{ j \in \mathbb{Z} \mid j \geq 0, B^j_{\varphi} \in \mathbb{Z}^2 \text{ or } B^j_{\varphi} \in \mathbb{Z}^2 \} \]
in fact, let \( k \in \Lambda \setminus \{0\} \). Then there exists \( j_0 \in \mathbb{Z} \) and \( k_0 \in \mathbb{Z}^2 \) such that \( k = A^k_{\Lambda} k_0 \) or \( k = E A^k_{\Lambda} k_0 \). Then, for \( k = A^k_{\Lambda} k_0, B^j_{\varphi} = A^{j-\lambda} k_0 \) or \( B^j_{\varphi} = (B^j_{\varphi} E A^k_{\Lambda} k_0) \). Since \( \Lambda \) is expansive and \( j \geq 0 \), there are only finitely many \( j \) such that \( A^{j-\lambda} k_0 \) or \( (B^j_{\varphi} E A^k_{\Lambda} k_0) \) is in \( \mathbb{Z}^2 \). The same is true for \( k = E A^k_{\Lambda} k_0 \). Consequently, (2.6) holds.

On the one hand, since \( f \) is compactly supported, there exists a constant \( c > 0 \) such that \( f(\xi) = 2\pi k = 0 \)
for all \( \xi \in \mathbb{R}^2 \) and \( |k| \leq c \). On the other hand, \( \Lambda \cap B_r(0) \) is a finite set. Therefore, by (2.6), there exists a positive integer \( J \) such that \( B^j_{\varphi} \notin \mathbb{Z}^2 \) and \( B^j_{\varphi} \in \mathbb{Z}^2 \) for all \( j \geq J \) and \( k \in \Lambda \setminus \{0\} \) with \( |k| < c \). Then for all \( J > J' \), (2.7) becomes
\[ S^j(f) = (2\pi)^2 \int_{\mathbb{R}^2} \sum_{k \in \Lambda \setminus \{0\}} f(\xi)^2 \left( T^0_{\varphi}(\xi) + \sum_{j=0}^{J} T^j_{\varphi}(B^j_{\varphi}(\xi)) \right) d\xi. \] (2.8)
Since (2.5) holds, the above equation (2.7) becomes
\[ S^j(f) = (2\pi)^2 \int_{\mathbb{R}^2} \sum_{k \in \Lambda \setminus \{0\}} f(\xi)^2 \left( T^0_{\varphi}(\xi) + \sum_{j=0}^{J} T^j_{\varphi}(B^j_{\varphi}(\xi)) \right) d\xi, \quad J > J'. \] (2.9)
Note that all \( T^0_{\varphi} \) and \( T^j_{\varphi} \) are nonnegative functions. By (2.4) and the Lebesgue Dominated Convergence Theorem, we have
\[ \lim_{J \to \infty} S^j(f) = (2\pi)^2 \int_{\mathbb{R}^2} \sum_{k \in \Lambda \setminus \{0\}} f(\xi)^2 \left( T^0_{\varphi}(\xi) + \sum_{j=0}^{J} T^j_{\varphi}(B^j_{\varphi}(\xi)) \right) d\xi = (2\pi)^2 \|f\|^2_{L^2}, \quad f \in \mathcal{D}^2(\mathbb{R}^2). \] (2.10)
Therefore, by that \( \mathcal{D}^2(\mathbb{R}^2) \) is dense in \( L_2(\mathbb{R}^2) \), we conclude that \( \{ \varphi \} \cup \{ \psi_j \}_{j=0}^\infty \subseteq L_2(\mathbb{R}^2) \) and (1.3) is a frequency-based affine shear tight frame for \( L_2(\mathbb{R}^2) \).

Conversely, suppose that (1.3) is a frequency-based affine shear tight frame for \( L_2(\mathbb{R}^2) \). Consider
\[ S^j(f, g) := \sum_{k \in \mathbb{Z}^2} \langle f, \varphi_{S, B^j_\varphi(\xi)} \rangle \varphi_{B^j_\varphi(\xi)} \rangle g + \sum_{j=0}^{J} \sum_{k \in \mathbb{F}_j} \langle f, h_{B^j_{S,B^j_\varphi(\xi)}\{0,0\}, k} \rangle \langle h_{B^j_{S,B^j_\varphi(\xi)}\{0,0\}, k}, g \rangle + \langle f, h_{B^j_{S,B^j_\varphi(\xi)}\{0,0\}, k} \rangle \langle h_{B^j_{S,B^j_\varphi(\xi)}\{0,0\}, k}, g \rangle. \] (2.11)
Then by the polarization identity, that (1.3) is tight is equivalent to that
\[ \lim_{J \to \infty} S^j(f, g) = (2\pi)^2 \langle f, g \rangle \quad \forall f, g \in \mathcal{D}^2(\mathbb{R}^2). \]
Similarly, by [17, Lemma 10], we have
\[
S^j(f, g) = (2\pi)^2 \int_{\mathbb{R}^2} f(\xi) \overline{g(\xi)} \left( T^0_\varphi(\xi) + \sum_{j=0}^J \left[ T^0_{\psi_j}(B_\lambda^j \xi) + I^0_{\psi_j}(B_\lambda^j E \xi) \right] \right) d\xi \\
+ (2\pi)^2 \int_{\mathbb{R}^2} \sum_{k \in \Lambda \setminus \{0\}} f(\xi) \overline{g(\xi + 2\pi k)} \left( T^0_\varphi(\xi) + \sum_{j=0}^\infty \left[ T^B_{\psi_j}(B_\lambda^j \xi) + I^B_{\psi_j}(B_\lambda^j E \xi) \right] \right) d\xi.
\]
(12.12)
Note that the set $\Lambda$ is discrete and closed; that is, for any $k \in \Lambda$, $\varepsilon_k := 2\pi \cdot \inf_{y \in \Lambda} \|y - k\|_2/2 > 0$. For any $k \in \Lambda \setminus \{0\}$ and $\zeta_0 \in \mathbb{R}^2$, considering $f, g \in \mathcal{D}(\mathbb{R}^2)$ such that $\text{supp } f \subseteq B_{\zeta_0}(<\zeta_0)$ and $\text{supp } g \subseteq B_{\zeta_0}(<\zeta_0 - 2\pi k)$, then we have
\[
(2\pi)^2 \int_{\mathbb{R}^2} f(\xi) \overline{g(\xi + 2\pi k)} \left( T^0_\varphi(\xi) + \sum_{j=0}^\infty \left[ T^B_{\psi_j}(B_\lambda^j \xi) + I^B_{\psi_j}(B_\lambda^j E \xi) \right] \right) d\xi = \lim_{j \to \infty} S^j(f, g) = (2\pi)^2 \langle f, g \rangle = 0.
\]
(2.13)
Now from the above relation as in (2.13) we deduce that (2.5) holds for almost every $\xi \in B_{\zeta_0}(\zeta_0)$. Since $\zeta_0$ is arbitrary, we see that (2.5) must hold for almost every $\xi \in \mathbb{R}^2$. Consequently, we conclude from (2.3) that
\[
\lim_{j \to \infty} (2\pi)^2 \int_{\mathbb{R}^2} |f(\xi)|^2 \left( T^0_{\varphi_j}(\xi) + \sum_{j=0}^\infty \left[ T^B_{\psi_j}(B_\lambda^j \xi) + I^B_{\psi_j}(B_\lambda^j E \xi) \right] \right) d\xi = \lim_{j \to \infty} S^j(f) = (2\pi)^2 \|f\|_2^2
\]
(2.14)
for all $f \in \mathcal{D}(\mathbb{R}^2)$. Since all $T^0_{\varphi_j}$ and $T^B_{\psi_j}$ are nonnegative functions, by Monotone Convergence Theorem, it follows from (2.14) that
\[
(2\pi)^2 \int_{\mathbb{R}^2} |f(\xi)|^2 \left( T^0_{\varphi_j}(\xi) + \sum_{j=0}^\infty \left[ T^B_{\psi_j}(B_\lambda^j \xi) + I^B_{\psi_j}(B_\lambda^j E \xi) \right] \right) d\xi = (2\pi)^2 \|f\|_2^2
\]
for all $f \in \mathcal{D}(\mathbb{R}^2)$, from which we deduce that (2.5) must hold.

When all generators $\varphi, \psi, \psi^{j,\ell}$ are nonnegative, the characterization in Theorem 1 is simplified as follows.

**Corollary 1.** Let $A_\lambda, B_\lambda, S_\ell, E$ be defined as in (1.1) with $\lambda > 1$ and let $\text{FAS}(\varphi; \{\Psi_j\}_{j=0}^\infty)$ be defined as in (1.3) with $\{\varphi\} \cup \{\Psi_j\}_{j=0}^\infty \subseteq L^2_{\text{loc}}(\mathbb{R}^2)$. Suppose
\[
h(\xi) := 0, \quad \text{a.e. } \xi \in \mathbb{R}^2, \quad \forall \varphi \in \{\varphi\} \cup \{\Psi_j\}_{j=0}^\infty.
\]
(2.15)
Then $\text{FAS}(\varphi; \{\Psi_j\}_{j=0}^\infty)$ is a frequency-based affine shear tight frame for $L^2(\mathbb{R}^2)$ if and only if
\[
|\varphi(\xi)|^2 + \sum_{j=0}^\infty \sum_{\ell=\ell_{r_j}}^{s_j} \left[ |\psi(S_\ell B_\lambda^j \xi)|^2 + |\psi(S_\ell B_\lambda^j E \xi)|^2 \right] + \sum_{r_{j+1} = r_j + 1}^{s_{j+1}} \left[ \psi^{j,\ell}(S_\ell B_\lambda^j \xi)|^2 + |\psi^{j,\ell}(S_\ell B_\lambda^j E \xi)|^2 \right] = 1
\]
(2.16)
for a.e. $\xi \in \mathbb{R}^2$ and
\[
h(\xi) h(\xi + 2\pi k) = 0, \quad \text{a.e. } \xi \in \mathbb{R}^2, \quad \forall k \in \mathbb{Z}^2 \setminus \{0\}, \quad \text{and } \forall \varphi \in \{\varphi\} \cup \{\Psi_j\}_{j=0}^\infty.
\]
(2.17)
**Proof.** Obviously, (2.2) is equivalent to (2.10). When all generators are nonnegative, (2.5) is equivalent to $\varphi(\xi) \psi(\xi + 2\pi k) = 0$, $\varphi(\xi) \psi^{j,\ell}(\xi + 2\pi k) = 0$, and $\psi^{j,\ell}(\xi) \psi^{j,\ell}(\xi + 2\pi k) = 0$ for almost every $\xi \in \mathbb{R}^2$, $|\ell| = r_j + 1, \ldots, s_j$, and $k \in \mathbb{Z}^2 \setminus \{0\}$, which is (2.17).

By Corollary 1, we see that when all generators are nonnegative, condition (2.14) is essentially saying that a partition of unity on the frequency plane is required for the system $\text{FAS}(\varphi; \{\Psi_j\}_{j=0}^\infty)$ to be a tight frame for $L^2(\mathbb{R}^2)$. Condition (2.17) says that each generator should not overlap with its 2π-shifted version. In summary, the characterization in Theorem 1 is simplified to a partition of unity condition and a non-overlapping condition.

### 2.2 A toy example of frequency-based affine shear tight frames

To prepare for our study of smooth affine shear tight frames in later sections, we next give a simple example of frequency-based affine shear tight frames generated by characteristic functions.

Let $\lambda > 1$ and define $\ell_\lambda := |\lambda^2 - 1/2| + 1$. Choose $\rho > 0$ such that either $0 < \rho < 1$ for any $\lambda > 1$, or $2 \rho^2 < \rho^2 < 2$ for $1 < \lambda \leq \sqrt{2}$. Let
\[
E := \{\xi \in \mathbb{R}^2 : -1/2 \leq \xi_j/\xi_1 \leq 1/2, |\xi_1| \in (\lambda^{-2} \rho_0, \rho_0]\},
E_{j,+} := \{\xi \in \mathbb{R}^2 : -1/2 \leq \xi_j/\xi_1 \leq \lambda - \ell_\lambda, |\xi_1| \in (\lambda^{-2} \rho_0, \rho_0]\},
E_{j,-} := \{\xi \in \mathbb{R}^2 : -\lambda^{-2} \ell_\lambda \leq \xi_j/\xi_1 \leq 1/2, |\xi_1| \in (\lambda^{-2} \rho_0, \rho_0]\}.
\]
Define

\[ \varphi := \chi_{[-\lambda^{-2}\rho, \lambda^{-2}\rho]^2}, \quad \psi := \chi_E, \quad \psi^{U}, \psi^{\ell} := \chi_{E_{\ell}}, \quad (2.18) \]

Let

\[ \Psi_{\ell} := (\psi(S_{\ell}) : \ell = -\ell, \ldots, \ell, -1) \cup \{ \psi^{U}(S_{\ell}), \psi^{\ell}(S_{-\ell}) \}. \quad (2.19) \]

Figure 1. Frequency tilings of \( \text{FAS}(\varphi; \{ \Psi_{\ell} \}_{j=0}^{\infty}) \) generated by characteristic functions. Middle rectangle: \( \varphi \). Middle rectangle: \( \psi, \psi^{U}, \psi^{\ell}(S_{\ell}), \psi^{U}(S_{\ell}) \) (color parts) and their flipped versions. Outer rectangle: \( \psi(S_{\ell}B_{2}^{2}), \ell = -1, 0, 1, \psi^{U}(S_{-\ell}B_{2}^{2}), \psi^{U}(S_{2}B_{2}^{2}) \) (color parts) and their flipped versions.

By Corollary 1, we have the following result.

**Corollary 2.** Let \( A_{\lambda}, B_{\lambda}, S_{\ell}, E \) be defined as in (1.1) with \( \lambda > 1 \) and let \( \text{FAS}(\varphi; \{ \Psi_{\ell} \}_{j=0}^{\infty}) \) be defined as in (1.3) with \( \varphi \) and \( \Psi_{\ell} \) being given as in (2.18) and (2.19). Then \( \text{FAS}(\varphi; \{ \Psi_{\ell} \}_{j=0}^{\infty}) \) is a frequency-based affine shear tight frame for \( L_{2}(\mathbb{R}^{2}) \).

**Proof.** Note that for a fixed \( j, \psi(S_{\ell}B_{2}^{2}) = \chi_{E_{\ell}} \) with

\[ E_{\ell} := \{ \xi \in \mathbb{R}^{2} : \lambda^{-j}(\ell - 1/2) \leq \xi_{2}/\xi_{1} \leq \lambda^{-j}(\ell + 1/2), |\xi_{1}| \in (\lambda^{2j-2}\rho, \lambda^{2j}\rho) \} \]

and \( \psi^{U}(S_{\ell}B_{2}^{2}) = \chi_{E_{\ell}} \) with

\[ E_{\ell} := \{ \xi \in \mathbb{R}^{2} : \lambda^{-j}(\ell - 1/2) \leq \xi_{2}/\xi_{1} \leq 1, |\xi_{1}| \in (\lambda^{2j-2}\rho, \lambda^{2j}\rho) \}, \]

Thus, we have,

\[ \mathcal{I}_{\Psi_{\ell}}^{0}(B_{2}^{2}) = \sum_{\ell=1}^{\ell_{j}} \left| \psi(S_{\ell}B_{2}^{2}) \right|^{2} + \left| \psi^{U}(S_{\ell}B_{2}^{2}) \right|^{2} \]

Similarly, we have

\[ \mathcal{I}_{\Psi_{\ell}}^{0}(B_{2}^{2}) = \chi_{\ell} \in \mathbb{R}^{2} : -1 \leq \xi_{2}/\xi_{1} \leq 1, |\xi_{1}| \in (\lambda^{2j-2}\rho, \lambda^{2j}\rho) \}

Consequently, we obtain

\[ \mathcal{I}_{\varphi}(\xi) + \sum_{j=0}^{\infty} \mathcal{I}_{\Psi_{\ell}}^{0}(B_{2}^{2}) \xi_{1} + \mathcal{I}_{\Psi_{\ell}}^{0}(B_{2}^{2}) \xi_{2} = 1, \text{ a.e.} \ \xi \in \mathbb{R}^{2}. \]

Hence, (2.11) holds.

By our choice of \( \rho \), we have \( \rho \leq \lambda^{2} \). Then \( -\lambda^{-2}\rho + 2\pi \geq \lambda^{-2}\rho \), which implies

\[ (-\lambda^{-2}\rho + 2\pi, \lambda^{-2}\rho + 2\pi) \cap (-\lambda^{-2}\rho, \lambda^{-2}\rho) = \emptyset. \]

Hence, \( \varphi(\xi) \varphi(\xi + 2\pi) = 0, \text{ a.e.} \ \xi \in \mathbb{R}^{2} \) and \( k \in \mathbb{Z}^{2} \setminus \{0\} \). Similarly, the condition for \( \psi \) that

\[ (-\rho + 2\pi, -\lambda^{-2}\rho + 2\pi) \cup (\lambda^{-2}\rho + 2\pi, \rho + 2\pi) \cap (-\rho, -\lambda^{-2}\rho) \cup (\lambda^{-2}\rho, \rho) = \emptyset \]

Hence, \( \varphi(\xi) \varphi(\xi + 2\pi) = 0, \text{ a.e.} \ \xi \in \mathbb{R}^{2} \) and \( k \in \mathbb{Z}^{2} \setminus \{0\} \). Similarly, the condition for \( \psi \) that
and
\[
\left(-\frac{1}{2}\rho^2 + 2\pi, \frac{1}{2}\rho^2 + 2\pi\right) \cap \left(-\frac{1}{2}\rho^2, \frac{1}{2}\rho^2\right) = \emptyset,
\]
is equivalent to
\[
\rho \leq \frac{2\lambda^2}{\lambda^2 - 1} \quad \text{and} \quad \rho \leq 1 \quad \text{or} \quad \rho \leq \frac{2\lambda^2}{\lambda^2 - 1} \quad \text{and} \quad \rho \geq \lambda^2 \quad \text{and} \quad \rho \leq 2,
\]
which is equivalent to our choice of \( \rho \) and implies \( \psi(\xi)\psi(\xi + 2k\pi) = 0 \), a.e. \( \xi \in \mathbb{R}^2 \) and \( k \in \mathbb{Z}^2 \setminus \{0\} \). The case that \( \psi^J \in \mathcal{F} \) (\( \psi^J \in \mathcal{F} + 2k\pi \)) = 0, a.e. \( \xi \in \mathbb{R}^2 \) and \( k \in \mathbb{Z}^2 \setminus \{0\} \) can be argued in the same way. Hence, (2.17) holds.

Note that all generators are nonnegative. Therefore, by Corollary 1 FAS\((\psi; \{\Psi_j\}_{j=0}^\infty)\) with \( \psi \) and \( \Psi_j \) being given as in (2.18) and (2.19) is a frequency-based affine shear tight frame for \( L_2(\mathbb{R}^2) \).

For an illustration of FAS\((\psi; \{\Psi_j\}_{j=0}^\infty)\) with \( \lambda = 2 \), see Figure 1. One of the main goals of this paper is to construct smooth frequency-based affine shear tight frames that in certain sense can be regarded as the smoothed version of FAS\((\psi; \{\Psi_j\}_{j=0}^\infty)\) as discussed in Corollary 2.

### 3. Sequences of Frequency-Based Affine Shear Tight Frames

Most current papers in the literature have investigated only one single affine system. However, to have MRA structure, as argued in [17], it is of fundamental importance to study a sequence of affine systems. In order to study the MRA structure of frequency-based affine shear systems, we next study sequences of frequency-based affine shear systems. We first characterize a sequence of frequency-based affine shear systems to be a sequence of frequency-based affine shear tight frames for \( L_2(\mathbb{R}^2) \). Then, corresponding to Corollary 1, a simple characterization shall be given for a sequence of frequency-based affine shear tight frames with nonnegative generators.

We need an additional dilation matrix \( M_\lambda := \lambda^2 I_2 \) and we define \( N_\lambda := M_\lambda^{-1} = \lambda^{-2} I_2 \). Let \( J \) be an integer and \( \varphi, \psi, \psi^J, |\ell| = r_j + 1, \ldots, s_j, j \geq J_0 \) be functions in \( L_2^0(\mathbb{R}^2) \). Let \( \Psi_j \) be defined as in (1.2) and \( \Lambda, B_\lambda, S_\ell, E \) be defined as before. A frequency-based affine shear system FAS\(_J(\psi^J; \{\Psi_j\}_{j=J}^\infty)\) is then defined to be

\[
\text{FAS}_J(\psi^J; \{\Psi_j\}_{j=J}^\infty) := \{\varphi_{\Psi_j}^J : k \in \mathbb{Z}^2\} \cup \{h_{B_\lambda^J k}^j : r_j \in \mathbb{Z}^2, \ell \in \mathbb{Z}^2, j \geq J_0\}. 
\]

We have the following characterization for a sequence of affine systems FAS\(_J(\psi^J; \{\Psi_j\}_{j=J}^\infty)\), \( J \geq J_0 \) to be a sequence of affine shear tight frames for \( L_2(\mathbb{R}^2) \).

**Theorem 2.** Let \( M_\lambda, N_\lambda, A_\lambda, B_\lambda, S_\ell, E \) be defined as before and \( J_0 \) be an integer. Let FAS\(_J(\psi^J; \{\Psi_j\}_{j=J}^\infty)\) be defined as in (3.1) with all generators \( \{\varphi^J\} \cup \{\Psi_j\}_{j=J}^\infty \subseteq L_2^0(\mathbb{R}^2) \) for all \( J \geq J_0 \). Then the following statements are equivalent to each other.

1. FAS\(_J(\psi^J; \{\Psi_j\}_{j=J}^\infty)\) is a frequency-based affine shear tight frame for \( L_2(\mathbb{R}^2) \), i.e., all generators are from \( L_2(\mathbb{R}^2) \) and for all \( f \in L_2(\mathbb{R}^2) \),

\[
(2\pi)^2 ||f||_2^2 = \sum_{k \in \mathbb{Z}^2} ||f, \varphi_{\Psi_j}^J_{k,0}||^2 + \sum_{j=J}^{\infty} \sum_{h \in \mathbb{Z}^2} \sum_{k \in \mathbb{Z}^2} (||f, h_{B_\lambda^J k}^j||^2 + ||f, h_{B_\lambda^J k}^j||^2)^2
\]

for every integer \( J \geq J_0 \).

2. The following identities hold:

\[
\lim_{j \to \infty} \sum_{k \in \mathbb{Z}^2} ||f, \varphi_{\Psi_j}^J_{k,0}||^2 = (2\pi)^2 ||f||_2^2 \quad \forall f \in \mathcal{D}(\mathbb{R}^2)
\]

and for all \( j \geq J_0 \),

\[
\sum_{k \in \mathbb{Z}^2} ||f, \varphi_{\Psi_j}^{J+1}_{k+1,0}||^2 = \sum_{k \in \mathbb{Z}^2} ||f, \varphi_{\Psi_j}^{J}_{k,0}||^2 + \sum_{\ell=-r_j}^{s_j} \sum_{k \in \mathbb{Z}^2} (||f, \psi_{\Psi_j}^{\ell J}_{k,0}||^2 + ||f, \psi_{\Psi_j}^{\ell J}_{k,0}||^2)^2
\]

\[
+ \sum_{\ell=r_j+1}^{s_j} \sum_{k \in \mathbb{Z}^2} (||f, \psi_{\Psi_j}^{\ell J}_{k,0}||^2 + ||f, \psi_{\Psi_j}^{\ell J}_{k,0}||^2)^2 \quad \forall f \in \mathcal{D}(\mathbb{R}^2).
\]

(3) The following identities hold:

\[
\lim_{j \to \infty} (||\varphi(\Psi_j^J)||^2, h) = (1, h) \quad \forall h \in \mathcal{D}(\mathbb{R}^2)
\]

and for all integers \( j \geq J_0 \),

\[
\mathcal{T}_{\varphi_j}^{N_\lambda^J k}(N_\lambda^J) + \left(T_{\Psi_j}^{B_\lambda^J k}(B_\lambda^J) + \mathcal{T}_{\Psi_j}^{B_\lambda^J k}(B_\lambda^J E k)\right) = \mathcal{T}_{\varphi_j}^{N_\lambda^J+1 k}(N_\lambda^J+1)\]

(3.6)
for a.e. $\xi \in \mathbb{R}^2$, $k \in ([M^j_{\lambda}]Z^2) \cup [M^j_{\lambda}+1Z^2] \cup [A^j_{\lambda}Z^2] \cup [EA^j_{\lambda}Z^2])$, where $T^j_{\varphi'}$, $T^j_{\psi}$, are similarly defined as in (2).

**Proof.** (1)$\Rightarrow$(2). Considering $\|\cdot\|_j$ with two consecutive $J$ and $J+1$ with $J \geq J_0$, the difference gives (5.4). Now by (5.4), it is easy to deduce that,

$$
\sum_{k \in \mathbb{Z}^2} |(f, \varphi'_{N^{j}_{\lambda},0,k})|^2 = \sum_{k \in \mathbb{Z}^2} |(f, \varphi'_{N^{j}_{\lambda},0,k})|^2 + \sum_{j=J}^{J-1} \left( \sum_{l=-\infty}^{\infty} \sum_{k \in \mathbb{Z}^2} \left( |(f, \psi_{S^lB^j_{\lambda},0,k})|^2 + |(f, \psi_{S^lB^{j+1}_{\lambda},0,k})|^2 \right) \right) + \sum_{|\ell|=r_j+1} \sum_{k \in \mathbb{Z}^2} \left( |(f, \psi'_{S^\ell B^j_{\lambda},0,k})|^2 + |(f, \psi'_{S^\ell B^{j+1}_{\lambda},0,k})|^2 \right) \quad \forall J' \geq J.
$$

(3.7)

Therefore, by letting $J' \to \infty$, we see that (5.3) holds.

(2)$\Rightarrow$(3). By (5.4), we deduce that (3.7) holds. By letting $J' \to \infty$ and in view of (3.3), we conclude that (5.2) holds.

(2)$\Rightarrow$(3). By [17, Lemma 10], we can show that (5.3) is equivalent to

$$
\int_{\mathbb{R}^2} \sum_{k \in \Lambda_j} f(\xi) f(\xi + 2\pi k) \left( \mathcal{I}_{\varphi'}^{N^j_{\lambda}}(N^j_{\lambda}) + \mathcal{I}_{\psi}^{B^j_{\lambda}}(B^j_{\lambda}) + \mathcal{I}_{\psi}^{E_{\lambda}}(B^j_{\lambda}) \right) - \mathcal{I}_{\varphi'}^{N^j_{\lambda}}(N^j_{\lambda}) d\xi = 0,
$$

(3.8)

where $\Lambda_j = [M^j_{\lambda}]Z^2 \cup [M^j_{\lambda}+1Z^2] \cup [A^j_{\lambda}Z^2] \cup [EA^j_{\lambda}Z^2]$. Since $M_{\lambda} = \lambda^2 I_2$ and $A_{\lambda} = \text{diag}(\lambda^2, \lambda)$ with $\lambda > 1$, we see that the lattice $\Lambda_j$ is discrete. By the same argument as in the proof of Theorem 1, we see that (3.8) is equivalent to (3.6).

By [17, Lemma 10], we see that (3.3) is equivalent to

$$
\lim_{j \to \infty} \int_{\mathbb{R}^2} \sum_{k \in [M^j_{\lambda}]Z^2} f(\xi) f(\xi + 2\pi k) \mathcal{I}_{\varphi'}^{N^j_{\lambda}}(N^j_{\lambda}) = (2\pi)^2 \|f\|^2_2, \quad \forall f \in \mathcal{D}(\mathbb{R}^2).
$$

(3.9)

Since $f \in \mathcal{D}(\mathbb{R}^2)$ is compactly supported, there exists $c > 0$ such that $f(\xi) f(\xi + 2\pi k) = 0$ for all $\xi \in \mathbb{R}^2$ and $|k| > c$. Also, by our assumption on $M_{\lambda}$ and $A_{\lambda}$, it is easy to show that

$$
\{j \in \mathbb{Z}^2 : j \geq J_0, [N^j_{\lambda}B_{\ell}(0)] \cap \mathbb{Z}^2 = \{0\}\}
$$

is a finite set for every $c \in [1, \infty)$. Hence, there exists $J'' \geq J_0$ such that $f(\xi) f(\xi + 2\pi k) \mathcal{I}_{\varphi'}^{N^j_{\lambda}}(N^j_{\lambda}) = 0$ for all $\xi \in \mathbb{R}^2$, $k \in [M^j_{\lambda}]Z^2 \setminus \{0\}$, and $j \geq J''$. Consequently, for $j \geq J''$, (3.9) becomes

$$
\lim_{j \to \infty} \int_{\mathbb{R}^2} |f(\xi)|^2 \mathcal{I}_{\varphi'}^{j}(N^j_{\lambda}(\xi)) = (2\pi)^2 \|f\|^2_2, \quad \forall f \in \mathcal{D}(\mathbb{R}^2),
$$

(3.11)

which is equivalent to (5.3).

If all elements $\varphi', \psi, \psi^j, \ell$ are nonnegative, we have the following simple characterization; also see [17, Cor. 18].

**Corollary 3.** Let $M_{\lambda}, N_{\lambda}, A_{\lambda}, B_{\lambda}, S_{\ell}, E$ be defined as before and $J_0$ be an integer. Let $\text{FAS}_J(\varphi'^j; \{\Psi_j\}_{i=J}^\infty)$ be defined as in (3.1) with all generators $\{\varphi'^j\} \cup \{\Psi_j\}_{i=J}^\infty \subseteq L^2_{\text{loc}}(\mathbb{R}^2)$ for all $J \geq J_0$. Suppose that

$$
h \geq 0 \quad \forall h \in \{\varphi'^j, \psi, \psi^j, \ell : j \geq J_0, |\ell| = r_j + 1, \ldots, s_j\}.
$$

(3.12)

Then, for all integers $J \geq J_0$, $\text{FAS}_J(\varphi'^j; \{\Psi_j\}_{i=J}^\infty)$ is a frequency-based affine shear tight frame for $L_2(\mathbb{R}^2)$ if and only if

$$
h(\xi) h(\xi + 2\pi k) = 0, \quad \text{a.e.} \xi \in \mathbb{R}^2, k \in \mathbb{Z}^2 \setminus \{0\}, \text{ and } h \in \{\varphi'^j, \psi, \psi^j, \ell : j \geq J_0, |\ell| = r_j + 1, \ldots, s_j\}.
$$

(3.13)

$$
|\varphi'^{j+1}(N^j_{\lambda}(\xi))|^2 = |\varphi'^j(N^j_{\lambda}(\xi))|^2 + \sum_{|\ell|=r_j} |(\psi(S_{\ell}B^j_{\lambda}(\xi))|^2 + |\psi(S_{\ell}B^{j+1}_{\lambda}(\xi))|^2)
$$

(3.14)

$$
+ \sum_{|\ell|=r_j+1} |(\psi^j(S_{\ell}B^j_{\lambda}(\xi))|^2 + |\psi^j(S_{\ell}B^{j+1}_{\lambda}(\xi))|^2), \quad \text{a.e.} \xi \in \mathbb{R}^2, j \geq J_0,
$$

and (3.3) holds.

**Proof.** When (3.12) holds, by item (3) of Theorem 2 for $k \in \mathbb{Z}^2 \setminus \{0\}$, (5.3) is equivalent to (3.13). For $k = 0$, (3.6) is equivalent to (3.14). Together with the condition (3.3) and by item (3) of Theorem 2, the claim follows from the equivalence between item (1) and item (3) of Theorem 2. \qed
The condition in (3.15) can be further simplified as in the following lemma.

**Lemma 1.** Suppose that there exist two positive numbers \(c\) and \(C\) such that
\[
|\varphi^j(\xi)| < C, \text{ a.e. } \xi \in [-c, c]^2 \text{ and } \forall j \geq j_0.
\] (3.15)
Assume that \(g(\xi) := \lim_{j \to \infty} |\varphi^j(N^2_{j}\xi)|^2\) exists for almost every \(\xi \in \mathbb{R}^2\). Then (3.15) holds if and only if \(g(\xi) = 1\), a.e. \(\xi \in \mathbb{R}^2\).

**Proof.** Since \(h \in \mathcal{D}(\mathbb{R}^2)\) has compact support and \(N_{j}^{-1} = M_{\lambda}\) is expansive, there exists \(J \in \mathbb{N}\) such that
\[
|\varphi^j(N^2_{j}\xi)|^2 |h(\xi)| \leq C^2 |h(\xi)| \quad \forall j \geq J, \text{ a.e. } \xi \in \mathbb{R}^2.
\]
Since \(h \in L_1(\mathbb{R}^2)\), by Lebesgue Dominated Convergence Theorem, we have
\[
\lim_{j \to \infty} |\varphi^j(N^2_{j}\xi)|^2 h = \lim_{j \to \infty} |\varphi^j(N^2_{j}\xi)|^2 = \langle g, h \rangle.
\]
Now it is trivial to see that (3.15) holds if and only if \(\langle g, h \rangle = \langle 1, h \rangle\) for all \(h \in \mathcal{D}(\mathbb{R}^2)\), which is equivalent to \(g(\xi) = 1\) for almost every \(\xi \in \mathbb{R}^2\). \(\square\)

Consider the toy example in Corollary 2. Now define \(\varphi^j := \varphi\) and \(\Psi_j := \{\psi(S^j_{\ell}) : \ell = -\ell_{\lambda} + 1, \ldots, \ell_{\lambda} - 1\} \cup \{\psi^{\pm,\ell_{\lambda},\nu}\}^{\pm,\ell_{\lambda},\nu}\) being constructed as in Corollary 2. Then condition (3.15) holds by Lemma 1 since \(\varphi^j\) satisfies (3.15) and \(g(\xi) = \lim_{j \to \infty} |\varphi^j(N^2_{j}\xi)|^2 = 1\) a.e., \(\xi \in \mathbb{R}^2\). Condition (3.15) directly follows from the proof of Corollary 2. Condition (3.14) holds by our construction. Therefore, by Corollary 2, FAS \(f(\varphi^j; \{\Psi_j\}_{j=1}^{\infty})\) is a frequency-based affine shear tight frame for \(L_2(\mathbb{R}^2)\) for any integer \(J \geq 0\).

A sequence of frequency-based affine shear tight frames naturally induces an MRA structure \(\{\mathcal{V}_j\}_{j=0}^{\infty}\) with \(\mathcal{V}_j := \text{span}(\varphi^j(M^{-1}_{\lambda}k) : k \in \mathbb{Z}^2)\), where \(\varphi = \mathcal{D}^{-1}\varphi^j\). But so far, the toy example and its induced sequence of systems are not smooth since all their generators are discontinuous. In the next section, we shall focus on the construction of smooth frequency-based affine shear tight frames for \(L_2(\mathbb{R}^2)\) that have many desirable properties. We shall show that not only our systems can achieve smoothness of generators, but also they have shear structure and more importantly, an MRA structure could be deduced from such type of systems.

### 4. Construction of Smooth Frequency-based Affine Shear Tight Frames

In this section we shall provide two types of constructions of smooth frequency-based affine shear tight frames: one is non-stationary construction and the other is quasi-stationary construction. Both these two types of constructions use the idea of normalization. In essence, we first construct a smooth frequency-based affine shear frame for \(L_2(\mathbb{R}^2)\) and then a normalization procedure is applied to such a frame. The non-stationary construction uses different functions \(\varphi^j\) for different scales \(j\), while the quasi-stationary construction employs a single function \(\varphi\) for every scale. We first need some auxiliary results and then provide details on the two types of constructions.

**4.1. Auxiliary results.** We shall use a function \(\nu \in C^\infty(\mathbb{R})\) such that \(\nu(x) = 0\) for \(x \leq -1\), \(\nu(x) = 1\) for \(x \geq 1\), and \([\nu(x)]^2 + [\nu(-x)]^2 = 1\) for all \(x \in \mathbb{R}\). There are many choices of such functions. For example, define \(f(x) := e^{-1/x^2}\) for \(x > 0\) and \(f(x) := 0\) for \(x \leq 0\), and let \(g(x) := f(1 + t)f(1 - t)dt\). Define
\[
\nu(x) := \frac{g(x)}{\sqrt{g(x)^2 + g(-x)^2}}, \quad x \in \mathbb{R}.
\] (4.1)
Then \(\nu \in C^\infty(\mathbb{R})\) is a desired function. Using such a function \(\nu\), we now construct our building block \(f_{[a,b];\varepsilon_1,\varepsilon_2}\) with \(a, b \in \mathbb{R}, \varepsilon_1, \varepsilon_2 > 0\) and \(\varepsilon_1 + \varepsilon_2 \leq b - a\) as follows.
\[
f_{[a,b];\varepsilon_1,\varepsilon_2}(x) = \begin{cases} \nu\left(\frac{x - a}{\varepsilon_1}\right) & x < a + \varepsilon_1 \\ 1 & a + \varepsilon_1 \leq x \leq b - \varepsilon_2 \\ \nu\left(\frac{x - b}{\varepsilon_2}\right) & x > b - \varepsilon_2. \end{cases}
\] (4.2)
Then \(f_{[a,b];\varepsilon_1,\varepsilon_2} \in C^\infty(\mathbb{R})\) and \(\text{supp} f_{[a,b];\varepsilon_1,\varepsilon_2} = [a - \varepsilon_1, b + \varepsilon_2]\).

Define \(\alpha_{\lambda,t,\rho}, \beta_{\lambda,t,\rho}\) with \(\lambda > 1\), \(0 < t \leq 1\), and \(0 < \rho \leq \lambda^2\) as follows:
\[
\alpha_{\lambda,t,\rho}(\xi) := f_{[a,b];\varepsilon_1,\varepsilon_2}(\xi) \quad \text{and} \quad \beta_{\lambda,t,\rho}(\xi) := \left(\frac{\alpha_{\lambda,t,\rho}(\lambda^{-2}\xi)}{\alpha_{\lambda,t,\rho}(\xi)}\right)^{1/2}, \quad \xi \in \mathbb{R},
\] (4.3)
where \([a, b] = [-\lambda^{-2}(1-t)/2, \lambda^{-2}(1-t)/2], \varepsilon_1 = \varepsilon_2 = \lambda^{-2}t\rho/2\) and we have \(\alpha_{\lambda,t,\rho} \in \mathcal{D}(\mathbb{R}^2)\). We have \(\text{supp} \alpha_{\lambda,t,\rho} = [-\lambda^{-2}\rho, \lambda^{-2}\rho]\) and \(\text{supp} \beta_{\lambda,t,\rho} = [-\rho, -\lambda^{-2}(1-t)/2] \cup [\lambda^{-2}(1-t)/2, \rho/2]\). Furthermore, define a \(2\pi\)-periodic function \(\mu_{\lambda,t,\rho}\)
and \( v_{\lambda,t,\rho} \) as follows:

\[
\begin{align*}
\mu_{\lambda,t,\rho}(\xi) & := \begin{cases} 
\frac{\alpha_{\lambda,t,\rho}(\lambda^2 \xi)}{\alpha_{\lambda,t,\rho}(\xi)} & |\xi| \leq \lambda^{-2} \rho \pi, \\
0 & \lambda^{-2} \rho \pi < |\xi| \leq \pi,
\end{cases} \\
v_{\lambda,t,\rho}(\xi) & := \begin{cases} 
\frac{\beta_{\lambda,t,\rho}(\lambda^2 \xi)}{\alpha_{\lambda,t,\rho}(\xi)} & \lambda^{-4}(1 - t) \rho \pi \leq |\xi| \leq \lambda^{-2} \rho \pi, \\
g_{\lambda,t,\rho}(\xi) & \xi \in [-\pi, \pi] \setminus \text{supp} \beta_{\lambda,t,\rho}(\lambda^2 \xi),
\end{cases}
\end{align*}
\]

where \( g_{\lambda,t,\rho} \) is a function in \( C^\infty(T) \) such that \( \frac{m}{\lambda} g_{\lambda,t,\rho}(\xi) \big|_{\xi = \pm \lambda^{-2} \rho \pi} = \delta(n) \) for all \( n \in \mathbb{N}_0 \). The purpose of \( g_{\lambda,t,\rho} \) is to make the function \( v_{\lambda,t,\rho} \) smooth. Such a \( g_{\lambda,t,\rho} \) exists. In fact, noting that \( \frac{\alpha_{\lambda,t,\rho}(\lambda^2 \xi)}{\alpha_{\lambda,t,\rho}(\xi)} \equiv 1 \) for \( |\xi| \geq \lambda^{-4} \rho \pi \) and \( \frac{\beta_{\lambda,t,\rho}(\lambda^2 \xi)}{\alpha_{\lambda,t,\rho}(\xi)} \equiv 0 \) for \( |\xi| \leq \lambda^{-4}(1 - t) \rho \pi \), we can simply define \( g_{\lambda,t,\rho} \) to be \( g_{\lambda,t,\rho}(\xi) := 1 \) for \( \lambda^{-4} \rho \pi \leq |\xi| \leq \pi \) and \( g_{\lambda,t,\rho}(\xi) \equiv 0 \) for \( |\xi| \leq \lambda^{-4}(1 - t) \rho \pi \). In this case, \( g_{\lambda,t,\rho} \) extends periodically as a constant 1 near the boundary of \( T \). If \( \lambda^{-2} \rho < 1 \), then another way to make \( v_{\lambda,t,\rho}(\xi) \) smooth is by defining \( g_{\lambda,t,\rho} \) to be \( g_{\lambda,t,\rho}(\xi) \equiv 1 \) for \( \lambda^{-4} \rho \pi \leq |\xi| \leq \lambda^{-2} \rho \pi \), and \( g_{\lambda,t,\rho}(\xi) \equiv 0 \) for \( |\xi| \leq \lambda^{-4}(1 - t) \rho \pi \) or \( \lambda^{-2} \rho \pi \leq |\xi| \leq \pi \) with \( \rho \pi \) being a positive constant such that \( \lambda^{-2} \rho < \lambda^{-2} \rho \pi < 1 \), which can be achieved by using smoothing kernel. We have the following result.

**Proposition 1.** Let \( \lambda > 1, 0 < t \leq 1, \) and \( 0 < \rho \leq 2 \). Let \( \alpha_{\lambda,t,\rho}, \beta_{\lambda,t,\rho}, \) and \( \mu_{\lambda,t,\rho}, v_{\lambda,t,\rho} \) be defined as in (4.3) and (4.4), respectively. Then \( \alpha_{\lambda,t,\rho}, \beta_{\lambda,t,\rho}, v_{\lambda,t,\rho} \in C^\infty(R) \). Moreover,

\[
|\alpha_{\lambda,t,\rho}(\xi)|^2 + |\beta_{\lambda,t,\rho}(\xi)|^2 = |\alpha_{\lambda,t,\rho}(\lambda^{-2} \xi)|^2, \quad \xi \in \mathbb{R},
\]

and

\[
\alpha_{\lambda,t,\rho}(\lambda^2 \xi) = \mu_{\lambda,t,\rho}(\alpha_{\lambda,t,\rho}(\xi)), \quad \beta_{\lambda,t,\rho}(\lambda^2 \xi) = v_{\lambda,t,\rho}(\alpha_{\lambda,t,\rho}(\xi)), \quad \xi \in \mathbb{R}.
\]

**Proof.** Explicitly, we have

\[
\alpha_{\lambda,t,\rho}(\xi) = \begin{cases} 
1 & \text{if } |\xi| \leq \lambda^{-2}(1 - t) \rho \pi, \\
\nu(\frac{\lambda^2 |\xi| + (1 - t) \rho \pi}{t \rho \pi}) & \text{if } \lambda^{-2}(1 - t) \rho \pi < |\xi| \leq \lambda^{-2} \rho \pi, \\
0 & \text{otherwise}.
\end{cases}
\]

(4.5)

Hence, by the smoothness of \( \nu \), we have \( \alpha_{\lambda,t,\rho} \in C^\infty(R) \).

If \( 1 - t \geq \lambda^{-2} \), by the definition, \( \beta_{\lambda,t,\rho} \) can be written as \( \beta_{\lambda,t,\rho}(\xi) = f_{\lambda,t,\rho}(\xi) + \tilde{f}_{\lambda,t,\rho}(\xi) = f_{\lambda,t,\rho}(\xi) \) with \( [a,b] = [\lambda^{-2}(2 - t) \rho \pi/2, (2 - t) \rho \pi/2] \) and \( \epsilon_1 = \lambda^{-2} \rho \pi/2, \epsilon_2 = \rho \pi/2 \); that is,

\[
\beta_{\lambda,t,\rho}(\xi) = \begin{cases} 
\nu(\frac{\lambda^2 |\xi| - (2 - t) \rho \pi}{t \rho \pi}) & \text{if } \lambda^{-2}(1 - t) \rho \pi \leq |\xi| < \lambda^{-2} \rho \pi, \\
1 & \text{if } \lambda^{-2}(1 - t) \rho \pi \leq |\xi| < (1 - t) \rho \pi, \\
\nu(\frac{-2 |\xi| + (2 - t) \rho \pi}{t \rho \pi}) & \text{if } (1 - t) \rho \pi \leq |\xi| \leq \rho \pi, \\
0 & \text{otherwise}.
\end{cases}
\]

(4.6)

Again, by the smoothness of \( \nu \), we have \( \beta_{\lambda,t,\rho} \in C^\infty(R) \).

If \( 0 \leq 1 - t < \lambda^{-2} \), then \( \beta_{\lambda,t,\rho} \) is given by

\[
\beta_{\lambda,t,\rho}(\xi) = \begin{cases} 
(\nu(\frac{\lambda^2 |\xi| - (2 - t) \rho \pi}{t \rho \pi}) - \nu(\frac{2 |\xi| + (2 - t) \rho \pi}{t \rho \pi}))^2/2 & \text{if } \lambda^{-2}(1 - t) \rho \pi \leq |\xi| \leq \rho \pi, \\
0 & \text{otherwise}.
\end{cases}
\]

(4.7)

Note that \( \tilde{\nu}(\xi) := (\nu(\frac{\lambda^2 |\xi| - (2 - t) \rho \pi}{t \rho \pi}) - \nu(\frac{2 |\xi| + (2 - t) \rho \pi}{t \rho \pi}))^2/2 > 0 \) for all \( \xi \) such that \( |\xi| \in (\lambda^{-2}(1 - t) \rho \pi, \rho \pi) \). Hence, \( \beta_{\lambda,t,\rho}(\xi) = \sqrt{\tilde{\nu}(\xi)} \) is infinitely differentiable for all \( \xi \in (\lambda^{-2}(1 - t) \rho \pi, \rho \pi) \). For all other \( \xi \) such that \( |\xi| \notin (\lambda^{-2}(1 - t) \rho \pi, \rho \pi) \), all the derivatives of \( \tilde{\nu}(\xi) \) vanish. Then, using the Taylor expansion for \( \beta_{\lambda,t,\rho} = \sqrt{\tilde{\nu}} \), we see that all the derivatives of \( \beta_{\lambda,t,\rho} \) vanish for all \( \xi \notin (\lambda^{-2}(1 - t) \rho \pi, \rho \pi) \). Hence, \( \beta_{\lambda,t,\rho} \in C^\infty(R) \).

Therefore, \( \alpha_{\lambda,t,\rho}, \beta_{\lambda,t,\rho} \in C^\infty(R) \). By the definition of \( \beta_{\lambda,t,\rho} \), we have \( |\alpha_{\lambda,t,\rho}(\xi)|^2 + |\beta_{\lambda,t,\rho}(\xi)|^2 = |\alpha_{\lambda,t,\rho}(\lambda^{-2} \xi)|^2 \) for all \( \xi \in \mathbb{R} \).

Similar to the cases of \( \beta_{\lambda,t,\rho} \), if \( 1 - t \geq \lambda^{-2} \), then we have

\[
\mu_{\lambda,t,\rho}(\xi) = \begin{cases} 
\alpha_{\lambda,t,\rho}(\lambda^2 \xi) & |\xi| \leq \lambda^{-4} \rho \pi, \\
0 & \lambda^{-4} \rho \pi < |\xi| \leq \pi,
\end{cases}
\]

\[
v_{\lambda,t,\rho}(\xi) = \begin{cases} 
\nu(\frac{\lambda^2 |\xi| - (2 - t) \rho \pi}{t \rho \pi}) & \lambda^{-4}(1 - t) \rho \pi \leq |\xi| \leq \lambda^{-2} \rho \pi, \\
g_{\lambda,t,\rho}(\xi) & \xi \in [-\pi, \pi] \setminus \text{supp} \beta_{\lambda,t,\rho}(\lambda^2 \xi).
\end{cases}
\]

(4.8)
In this case, obviously, \( \mu_{\lambda,t,\rho} \in C^\infty(\mathbb{T}) \). Note that \( \left[ \frac{d^m}{d\xi^n} \nu\left(\frac{-2\lambda^2(|\xi|+(2-t)\rho\pi)}{t\rho\pi}\right) \right]_{\xi = \pm \lambda^{-2} \rho\pi} = \delta(n) \). By our choice of \( g_{\lambda,t,\rho} \), we see that \( v_{\lambda,t,\rho} \in C^\infty(\mathbb{T}) \).

If \( 0 \leq t < \lambda^{-2} \), then we have

\[
\mu_{\lambda,t,\rho}(\xi) = \begin{cases} \frac{\nu\left(\frac{-2\lambda^2(|\xi|+(2-t)\rho\pi)}{t\rho\pi}\right)}{\nu\left(\frac{-2\lambda^2(|\xi|+(2-t)\rho\pi)}{t\rho\pi}\right)} & |\xi| \leq \lambda^{-4} \rho\pi \\ 0 & \lambda^{-4} \rho\pi < |\xi| \leq \rho\pi \end{cases}
\]

and

\[
v_{\lambda,t,\rho}(\xi) = \begin{cases} \left(\frac{\nu\left(\frac{-2\lambda^2(|\xi|+(2-t)\rho\pi)}{t\rho\pi}\right)}{\nu\left(\frac{-2\lambda^2(|\xi|+(2-t)\rho\pi)}{t\rho\pi}\right)}\right)^{1/2} & |\xi| \leq \lambda^{-4} \rho\pi \\ g_{\lambda,t,\rho}(\xi) & \lambda^{-4} \rho\pi < |\xi| \leq \rho\pi \end{cases}
\]

Note that the function \( \nu\left(\frac{-2\lambda^2|\xi|+(2-t)\rho\pi}{t\rho\pi}\right) \) is strictly positive for all \( \xi \) such that \( |\xi| \leq \lambda^{-4} \rho\pi \). Hence, \( \mu_{\lambda,t,\rho}(\xi) \) is infinitely differentiable for all \( \xi \) such that \( |\xi| \leq \lambda^{-4} \rho\pi \). Since \( \nu \) is \( C^\infty \), the product of \( \nu\left(\frac{-2\lambda^2|\xi|+(2-t)\rho\pi}{t\rho\pi}\right) \) and \( \frac{1}{\nu\left(\frac{-2\lambda^2|\xi|+(2-t)\rho\pi}{t\rho\pi}\right)} \) is infinitely differentiable for all \( \xi \) such that \( |\xi| \leq \lambda^{-4} \rho\pi \). For \( \xi \) such that \( \lambda^{-4} \rho\pi \leq |\xi| \leq \rho\pi \), all the derivatives of \( \mu_{\lambda,t,\rho}(\xi) \) vanish. Consequently, \( \mu_{\lambda,t,\rho} \in C^\infty(\mathbb{T}) \). Observe that \( \left[\nu\left(\frac{-2\lambda^2|\xi|+(2-t)\rho\pi}{t\rho\pi}\right)\right]^2 = \left(\nu\left(\frac{-2\lambda^2(|\xi|+(2-t)\rho\pi)}{t\rho\pi}\right)\right)^2 \) for \( \lambda^{-2}(1-t)\rho\pi \leq |\xi| \leq \lambda^{-2} \rho\pi \). By similar arguments, we conclude that \( v_{\lambda,t,\rho} \in C^\infty(\mathbb{T}) \).

Therefore, \( \mu_{\lambda,t,\rho}, v_{\lambda,t,\rho} \in C^\infty(\mathbb{T}) \). By their constructions, it is easy to check that \( \alpha_{\lambda,t,\rho}(\lambda^2 \xi) = \mu_{\lambda,t,\rho}(\xi) \alpha_{\lambda,t,\rho}(\xi) \) and \( \beta_{\lambda,t,\rho}(\lambda^2 \xi) = v_{\lambda,t,\rho}(\xi) \alpha_{\lambda,t,\rho}(\xi) \) for \( \xi \in \mathbb{R} \).

The functions \( \alpha_{\lambda,t,\rho} \) and \( \beta_{\lambda,t,\rho} \) shall be used for the horizontal direction. We next define \( \gamma_\varepsilon \) for splitting pieces along the vertical direction. In what follows, \( \varepsilon \) shall be fixed as a constant such that \( 0 < \varepsilon \leq 1/2 \). Define a function \( \gamma_\varepsilon \) to be \( \gamma_\varepsilon := \mathbb{1}_{[-1/2,1/2],\varepsilon} \); that is,

\[
\gamma_\varepsilon(x) = \begin{cases} \mathbb{1}_{[-|x|+1/2,\varepsilon]} & \text{if } |x| \leq 1/2 - \varepsilon; \\ \mathbb{1}_{[1/2 - \varepsilon, |x|+1/2]} & \text{if } 1/2 - \varepsilon \leq |x| \leq 1/2 + \varepsilon; \\ 0 & \text{otherwise.} \end{cases}
\]

Then it is easy to check that \( \gamma_\varepsilon \in C^\infty(\mathbb{R}) \) and \( \sum_{\ell \in \mathbb{Z}} |\gamma_\varepsilon(\cdot + \ell)|^2 \equiv 1 \).

For \( \lambda \in \mathbb{R} \), define \( \ell_\lambda := [\lambda - (1/2 + \varepsilon)] + 1 = [\lambda + (1/2 - \varepsilon)] \). Define the corner pieces \( \gamma_{\lambda,\varepsilon,\ell_\varepsilon} \) by

\[
\begin{align*}
\gamma_{\lambda,\varepsilon,\ell_\varepsilon}(x - \ell_\lambda) := & \begin{cases} 
\mathbb{1}_{[\lambda - (1/2 + \varepsilon), \lambda - 1/2 - \varepsilon]} & \text{if } x \leq \lambda^{-1}(\ell_\lambda - 1/2 + \varepsilon) \\
\mathbb{1}_{[\ell_\lambda - 1/2 + \varepsilon, \lambda^{-1}(\ell_\lambda - 1/2) + \varepsilon]} & \text{if } x \leq \lambda^{-1}(\ell_\lambda - 1/2 + \varepsilon) \\
0 & \text{otherwise.}
\end{cases} \\
\gamma_{\lambda,\varepsilon,\ell_\varepsilon}(x + \ell_\lambda) := & \begin{cases} 
\mathbb{1}_{[-1/2 + \varepsilon, \lambda^{-1}(\ell_\lambda + 1/2 - \varepsilon) + \varepsilon]} & \text{if } x \leq \lambda^{-1}(\ell_\lambda + 1/2 - \varepsilon) \\
\mathbb{1}_{[\lambda^{-1}(\ell_\lambda + 1/2) + \varepsilon, \lambda^{-1}(\ell_\lambda + 1/2 + \varepsilon) + \varepsilon]} & \text{if } x \leq \lambda^{-1}(\ell_\lambda + 1/2 + \varepsilon) \\
0 & \text{otherwise.}
\end{cases}
\end{align*}
\]

That is,

\[
\begin{align*}
\gamma_{\lambda,\varepsilon,\ell_\varepsilon}(x) = & \begin{cases} 
\mathbb{1}_{[\lambda - (1/2 + \varepsilon), \lambda - 1/2 - \varepsilon]} & \text{if } x \leq -1/2 + \varepsilon \\
\mathbb{1}_{[-1/2 + \varepsilon, \lambda^{-1}(\lambda + 1/2 - \varepsilon)]} & \text{if } x \leq \lambda^{-1}(\lambda + 1/2 - \varepsilon) \\
\mathbb{1}_{[\lambda^{-1}(\lambda + 1/2) + \varepsilon, \lambda^{-1}(\lambda + 1/2 + \varepsilon)]} & \text{if } x \leq \lambda^{-1}(\lambda + 1/2 + \varepsilon) \\
0 & \text{otherwise.}
\end{cases} \\
\gamma_{\lambda,\varepsilon,\ell_\varepsilon}(x) = & \begin{cases} 
\mathbb{1}_{[\lambda^{-1}(\lambda + 1/2) + \varepsilon, \lambda^{-1}(\lambda + 1/2 + \varepsilon)]} & \text{if } x \leq -1/2 - \varepsilon \\
\mathbb{1}_{[-1/2 - \varepsilon, \lambda^{-1}(\lambda + 1/2)]} & \text{if } x \leq \lambda^{-1}(\lambda + 1/2) \\
\mathbb{1}_{[\lambda^{-1}(\lambda + 1/2 + \varepsilon), \lambda^{-1}(\lambda + 1/2 + \varepsilon)]} & \text{if } x \leq \lambda^{-1}(\lambda + 1/2 + \varepsilon) \\
0 & \text{otherwise.}
\end{cases}
\end{align*}
\]

Note that \( \gamma_{\lambda,\varepsilon,\ell_\varepsilon} \) are \( C^\infty \) functions. Then, for \( \lambda \geq 1 \),

\[
\sum_{\ell_\varepsilon = -\ell_\lambda + 1}^{\ell_\varepsilon - 1} |\gamma_\varepsilon(\lambda x + \ell_\varepsilon)|^2 + |\gamma_{\lambda,\varepsilon,\ell_\varepsilon}(\lambda x \mp \ell_\lambda)|^2 = 1 \quad \forall |x| \leq 1
\]

and

\[
\sum_{\ell_\varepsilon = -\ell_\lambda}^{\ell_\varepsilon} |\gamma_\varepsilon(\lambda x + \ell_\varepsilon)|^2 = 1 \quad \forall |x| \leq \frac{\ell_\lambda + 1/2 - \varepsilon}{\lambda}.
\]

Equations (4.13) and (4.14) will be used to construct two types of smooth frequency-based affine shear tight frames. One is non-stationary construction with \( \phi^\varepsilon \) changing at different scales and the other is quasi-stationary construction, in which case the function \( \phi \) is fixed. We next discuss the details of these two types of constructions.
4.2. Non-stationary construction. We first discuss the non-stationary construction. For such a type of construction, the shear operations could reach arbitrarily close to the seamlines when \( j \) goes to infinity. The idea of constructing such a smooth frequency-based affine shear tight frame in the non-stationary setting is simple. We first construct a frequency-based affine shear frame from only a few generators and then apply normalization to such a frame to obtain a tight frame.

More precisely, let \( \lambda > 1, 0 < t, \rho \leq 1 \), and \( 0 < \varepsilon \leq 1/2 \). Let \( A_\lambda, B_\lambda, M_\lambda, N_\lambda, \alpha_{\lambda,t,\rho}, \beta_{\lambda,t,\rho}, \) and \( \gamma_{\varepsilon}, \gamma_{\lambda,t,\varepsilon,\alpha}, \, \ell_\lambda \) be defined as before. Define

\[
\begin{align*}
\eta(\xi_1, \xi_2) &:= \alpha_{\lambda,t,\rho}(\xi_1) \gamma_{\varepsilon}(\xi_2/\xi_1), \quad (\xi_1, \xi_2) \in \mathbb{R}^2, \\
\xi(\xi_1, \xi_2) &:= \beta_{\lambda,t,\rho}(\xi_1) \gamma_{\varepsilon}(\xi_2/\xi_1), \quad (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\},
\end{align*}
\]

as well as the corner pieces

\[
\begin{align*}
\eta^{j,\ell}(\xi_1, \xi_2) &:= \alpha_{\lambda,t,\rho}(\xi_1) \gamma_{\varepsilon}(\xi_2/\xi_1), \quad (\xi_1, \xi_2) \in \mathbb{R}^2, \\
\xi^{j,\ell}(\xi_1, \xi_2) &:= \beta_{\lambda,t,\rho}(\xi_1) \gamma_{\varepsilon}(\xi_2/\xi_1), \quad (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}.
\end{align*}
\]

For \( \xi = 0 \), \( \zeta(0) = 0 \) and \( \zeta^{j,\ell}(0) = 0 \). Since the support of \( \beta_{\lambda,t,\rho} \) is away from the origin, we have \( \zeta, \zeta^{j,\ell} \in C^\infty(\mathbb{R}^2) \). Let \( \varphi(\xi) := \alpha_{\lambda,t,\rho}(\xi_1) \alpha_{\lambda,t,\rho}(\xi_2), \xi \in \mathbb{R}^2 \). Then, \( \varphi \) is also a function in \( C^\infty(\mathbb{R}^2) \). For a nonnegative integer \( J_0 \), define

\[
\Theta^{\mathcal{J}_0}(\xi) := |\varphi(N_\lambda^{J_0}\xi)|^2 + \sum_{j=J_0}^{J_{\lambda J_0}} \sum_{\ell = -\ell_{\lambda J_0}}^{\ell_{\lambda J_0}} (|\xi^{j,\ell}(S_\ell B_\lambda^{J_0}\xi)|^2 + |\xi^{j,\ell}(S_\ell B_\lambda^{J_0}\varepsilon\xi)|^2)
\]

for \( \xi \in \mathbb{R}^2 \), where \( \eta^{j,\ell} := \eta \) and \( \xi^{j,\ell} := \xi \) for \( |\ell| < \ell_{\lambda J_0} \). We have the following result concerning the function \( \Theta^{\mathcal{J}_0} \).

**Proposition 2.** Let \( \lambda > 1, 0 < \varepsilon \leq 1/2, 0 < t, \rho \leq 1 \). Let \( J_0 \) be a nonnegative integer and \( \Theta^{\mathcal{J}_0} \) be defined as in (4.13). Choose \( \varepsilon_0 > 0 \) such that \( \varepsilon_0 < 1/2 \lambda^{J_0-1} \). Then \( \Theta^{\mathcal{J}_0} \) has the following properties.

(i) \( \Theta^{\mathcal{J}_0} \in C^\infty(\mathbb{R}^2) \), \( \Theta^{\mathcal{J}_0} = \Theta^{\mathcal{J}_0}(E) \), and \( 0 < \Theta^{\mathcal{J}_0} \leq 2 \).

(ii) \( \Theta^{\mathcal{J}_0}(\xi) = \Theta^{\mathcal{J}_0}(\varepsilon\xi) \quad \forall \xi \in \cup_{j=J_0+1}^{J_{\lambda J_0}} \cup_{|\ell| = \ell_{\lambda J_0}+2}^{\ell_{\lambda J_0}} \text{supp } \xi^{j,\ell} \cdot S_\ell B_\lambda^{J_0} E \).

**Proof.** Since the generators \( \varphi, \zeta^{j,\ell} \) are compactly supported functions in \( C^\infty(\mathbb{R}^2) \) and for any bounded open set \( E \subseteq \mathbb{R}^2 \), \( \Theta^{\mathcal{J}_0}(\xi) \) is the summation of finitely many terms from \( \varphi, \zeta^{j,\ell} \) for all \( \xi \in E \), the function \( \Theta^{\mathcal{J}_0} \) is thus also a function in \( C^\infty(\mathbb{R}^2) \). By its definition, it is obvious that \( \Theta^{\mathcal{J}_0}(E) = \Theta^{\mathcal{J}_0}. \)

For simplicity of presentation, we denote \( \chi(\xi ||\xi_1|, |\xi_2| < 1/2) \) and \( \chi(\xi ||\xi_1|, |\xi_2| < 1/2) \) and similar notation applies for others.

For \( \xi \in \{ \xi \in \mathbb{R}^2 : \max\{|\xi_1|, |\xi_2| < 1/2\} \} \), by the property of \( \gamma_{\varepsilon} \) as in (4.14), we have

\[
\Theta^{\mathcal{J}_0}(\xi) \geq |\varphi(N_\lambda^{J_0}\xi)|^2 + \sum_{j=J_0}^{J_{\lambda J_0}} \sum_{\ell = -\ell_{\lambda J_0}}^{\ell_{\lambda J_0}} (|\xi^{j,\ell}(S_\ell B_\lambda^{J_0}\varepsilon\xi)|^2 + |\xi^{j,\ell}(S_\ell B_\lambda^{J_0}\varepsilon\xi)|^2)
\]

\[
\geq |\alpha_{\lambda,\rho}(\lambda^{-2J_0 \xi_1}) \alpha_{\lambda,\rho}(\lambda^{-2J_0 \xi_2})|^2 + |\beta_{\lambda,\rho}(\lambda^{-2J_0 \xi_1})|^2 \chi(\xi ||\xi_1|, |\xi_2| < 1/2) \]

\[
+ |\beta_{\lambda,\rho}(\lambda^{-2J_0 \xi_2})|^2 \chi(\xi ||\xi_1|, |\xi_2| < 1/2) > 0,
\]

and for \( \xi \in \{ \xi \in \mathbb{R}^2 : \max\{|\xi_1|, |\xi_2| > 1/2\} \} \), we have

\[
\Theta^{\mathcal{J}_0}(\xi) \geq \sum_{j=J_0}^{J_{\lambda J_0}} \sum_{\ell = -\ell_{\lambda J_0}}^{\ell_{\lambda J_0}} (|\xi^{j,\ell}(S_\ell B_\lambda^{J_0}\varepsilon\xi)|^2 + |\xi^{j,\ell}(S_\ell B_\lambda^{J_0}\varepsilon\xi)|^2)
\]

\[
\geq \sum_{j=J_0}^{J_{\lambda J_0}} \left[ |\beta_{\lambda,\rho}(\lambda^{-2j \xi_1}) \chi(\xi ||\xi_1|, |\xi_2| < 1/2) \chi(\xi ||\xi_1|, |\xi_2| > 1/2) \right] > 0.
\]

Consequently, \( \Theta^{\mathcal{J}_0} > 0 \).

We next show that \( \Theta^{\mathcal{J}_0} \leq 2 \). By the property of \( \gamma_{\varepsilon} \) as in (4.14), we have

\[
\sum_{\ell = -\ell_{\lambda J_0}}^{\ell_{\lambda J_0}} |\eta(S_\ell B_\lambda^{J_0}\varepsilon\xi)|^2 \chi(\xi ||\xi_1|, |\xi_2| < 1/2) \chi(\xi ||\xi_1|, |\xi_2| > 1/2) = 0, \quad \xi \neq 0
\]
and similarly,
\[ \sum_{\ell = -\ell_j}^{\ell_j} |\chi(S\ell_j^o B^j_{\lambda})^2| = |\beta_{\lambda, t, \rho} (\lambda^{-2j} \xi)|^2 \chi([\xi_2/\xi_1 \leq 1]) (\xi), \quad \xi \neq 0. \]

Hence,
\[
\sum_{\ell = -\ell_j}^{\ell_j} (|\eta(S\ell_j^o B^j_{\lambda})^2| + |\chi(S\ell_j^o B^j_{\lambda})^2|) \chi([\xi_2/\xi_1 \leq 1]) (\xi) \\
= (|\alpha_{\lambda, t, \rho} (\lambda^{-2j} \xi)|^2 + |\beta_{\lambda, t, \rho} (\lambda^{-2j} \xi)|^2) \chi([\xi_2/\xi_1 \leq 1]) (\xi) \\
= \sum_{\ell = -\ell_j+1}^{\ell_j+1} |\eta^{j+1, \ell}(S\ell_j^o B^j_{\lambda})^2| \chi([\xi_2/\xi_1 \leq 1]) (\xi), \quad \xi \neq 0.
\]

Therefore, we have
\[
\lim_{J \to \infty} \left( \sum_{\ell = -\ell_J}^{\ell_J} |\eta^{j, \ell}(S\ell_J^o B^j_{\lambda})^2| + \sum_{J = J_0}^{J-1} \sum_{\ell = -\ell_J}^{\ell_J} |\chi^{j, \ell}(S\ell_J^o B^j_{\lambda})^2| \chi([\xi_2/\xi_1 \leq 1]) (\xi) \right) \\
= \lim_{J \to \infty} |\alpha_{\lambda, t, \rho} (\lambda^{-2j} \xi)|^2 \chi([\xi_2/\xi_1 \leq 1]) (\xi) = \chi([\xi_2/\xi_1 \leq 1]) (\xi), \quad \xi \neq 0.
\]

Now, we define
\[
\Theta^{j_0}(\xi) := \sum_{\ell = -\ell_J}^{\ell_J} (|\eta^{j_0, \ell}(S\ell_J^o B^j_{\lambda})^2| + |\eta^{j_0, \ell}(S\ell_J^o B^j_{\lambda} E \xi)|^2) \\
+ \sum_{J = J_0}^{\infty} \sum_{\ell = -\ell_J}^{\ell_J} (|\chi^{j, \ell}(S\ell_J^o B^j_{\lambda})^2| + |\chi^{j, \ell}(S\ell_J^o B^j_{\lambda} E \xi)|^2), \quad \xi \neq 0.
\]

Then,
\[
\Theta^{j_0}(\xi) = \sum_{\ell = -\ell_J}^{\ell_J} (|\eta^{j_0, \ell}(S\ell_J^o B^j_{\lambda})^2| + |\eta^{j_0, \ell}(S\ell_J^o B^j_{\lambda} E \xi)|^2)(\chi([\xi_2/\xi_1 \leq 1]) (\xi) + \chi([\xi_2/\xi_1 > 1]) (\xi)) \\
+ \sum_{J = J_0}^{\infty} \sum_{\ell = -\ell_J}^{\ell_J} (|\chi^{j, \ell}(S\ell_J^o B^j_{\lambda})^2| + |\chi^{j, \ell}(S\ell_J^o B^j_{\lambda} E \xi)|^2)(\chi([\xi_2/\xi_1 \leq 1]) (\xi) + \chi([\xi_2/\xi_1 > 1]) (\xi)) \\
= 1 + (|\eta^{j_0, \ell_0}(S\ell_0^o B^j_{\lambda})^2| + \sum_{J = J_0}^{\infty} |\chi^{j, \ell_0}(S\ell_0^o B^j_{\lambda})^2| \chi([\xi_2/\xi_1 > 1]) (\xi)) \\
+ \sum_{J = J_0}^{\infty} |\eta^{j_0, \ell_0}(S\ell_0^o B^j_{\lambda} E \xi)|^2 + \sum_{J = J_0}^{\infty} |\chi^{j, \ell_0}(S\ell_0^o B^j_{\lambda} E \xi)|^2 \chi([\xi_2/\xi_1 \leq 1]) (\xi)) \\
= 1 + I(\xi) + I(E \xi) - \sqrt{I(\xi) I(E \xi)}, \quad \xi \neq 0,
\]

where
\[
I(\xi) := (|\eta^{j_0, \ell_0}(S\ell_0^o B^j_{\lambda})^2| + \sum_{J = J_0}^{\infty} |\chi^{j, \ell_0}(S\ell_0^o B^j_{\lambda})^2| \chi([\xi_2/\xi_1 > 1]) (\xi), \quad \xi \neq 0.
\]

By the construction of \(\alpha_{\lambda, t, \rho}\) and \(\beta_{\lambda, t, \rho}\), we have \(I \leq 1\). Therefore, \(1 \leq \Theta^{j_0} \leq 2\). Observe that \(\Theta^{j_0}(\xi) \leq \Theta^{j_0}(\xi) \leq 2\) for \(\xi \neq 0\) and \(\Theta^{j_0}(0) = 1\), we conclude that item (i) holds.

We next show that item (ii) holds. We have
\[
\Theta^{j_0}(\xi) = \Theta^{j_0}(\xi) + |\varphi(N^{j_0}_\lambda)|^2 - \sum_{\ell = -\ell_J}^{\ell_J} (|\eta^{j_0, \ell}(S\ell_J^o B^j_{\lambda})^2|).
\]

Note that \(\varphi(N^{j_0}_\lambda)\) is inside the support of \(\sum_{\ell = -\ell_J}^{\ell_J} (|\eta^{j_0, \ell}(S\ell_J^o B^j_{\lambda})^2|). \) Hence, for \(\xi\) outside the support of \(\sum_{\ell = -\ell_J}^{\ell_J} (|\eta^{j_0, \ell}(S\ell_J^o B^j_{\lambda})^2|), \) we have \(\Theta^{j_0} = \Theta^{j_0} \). By that \(\Theta^{j_0} = 1 + I(E \xi) - \sqrt{I(\xi) I(E \xi)}, \) we hence only need to check the overlapping coming from \(I\) and \(I(E \xi)\). In fact, at scale \(j\), the seamline element on the horizontal cone...
with respect to $\ell = -\ell_{j_0}$ has part of the piece overlapping with the other cone. By the support of $\gamma_{\lambda,\xi,\epsilon,\rho}$, for this seamline element, we have its support satisfying $\xi_1/\xi_2 \leq 1 + \frac{2\epsilon}{\lambda^2 j_0}$. Moreover, by the support of $\beta_{\lambda,\ell,\rho}$, this seamline element can only affect other elements in the vertical cone with respect to scales $j_0 = j - 1, j, j + 1$. Now, the support of the vertical cone element corresponding to scale $j_0 = j - 1$ and $\ell = -\ell_{j_0} + s$ with $s$ being a nonnegative integer satisfying
\[
\lambda^{j_0} \xi_1/\xi_2 - \ell_{j_0} + s \leq \frac{1}{2} + \epsilon,
\]
which implies $\xi_1/\xi_2 \leq \frac{1}{2} + \frac{\lambda^{j_0} \xi_1/\xi_2 - \ell_{j_0} + s}{\lambda^2 j_0}$. Consequently, the seamline element on the horizontal cone affecting the elements in the vertical cone at scale $j_0$ means
\[
1 + \frac{2\epsilon_0}{\lambda^{j_0}} \geq \frac{\lambda^{j_0}}{1/2 + \epsilon + \ell_{j_0} - s},
\]
which implies
\[
s \leq \frac{1}{2} + \epsilon + \ell_{j_0} - \frac{\lambda^{2j_0}}{\lambda^{2j_0} + 2\epsilon_0} \leq \frac{1}{2} + \epsilon + \frac{\lambda^{2j_0} + 2\epsilon_0}{\lambda^{2j_0} + 2\epsilon_0} \leq \frac{1}{\lambda^{2j_0} - 1} \leq \frac{2\epsilon_0}{2\epsilon_0}.
\]

Hence, by that $s$ is a nonnegative integer, we deduce that $s$ is either 0 or 1. By symmetry, same result holds for seamline elements on vertical cone affecting the horizontal cone. Therefore, we have $\Theta^{j_0}(\xi) \equiv \Theta^{j_0}(\xi) \equiv 1$ for $\xi$ in the support of those $C^\ell(S_\ell B_\lambda^\ell)$ with $|\ell| < \ell_{j_0} - 1$ and $j \geq J_0 + 1$. That is, item (ii) holds.

The function $\Theta^{j_0}$ is for the normalization of the frame generated by $\xi^{j_0}$. We next define a function $\Gamma^j$, which shall be used for frequency splitting. The function $\Gamma^j$ is defined to be
\[
\Gamma^j(\xi) := \sum_{j = -\ell_{j_0} + 1}^{\ell_{j_0} - 1} \left[ |\gamma_c(\lambda^j \xi_2/\xi_1 + \ell)|^2 + |\gamma_c(\lambda^j \xi_1/\xi_2 + \ell)|^2 \right] + |\gamma^{\xi}_{\lambda,\xi,\epsilon,\rho}(\lambda^j \xi_2/\xi_1 + \ell_\lambda)|^2 + |\gamma^{\xi}_{\lambda,\xi,\epsilon,\rho}(\lambda^j \xi_1/\xi_2 + \ell_\lambda)|^2, \quad \xi \neq 0.
\]

For $\Gamma^j$, we have the following result.

**Proposition 3.** Let $\Gamma^j$ be defined as in (4.19). Then $\Gamma^j$ is a function in $C^\infty(\mathbb{R}^2 \setminus \{0\})$ and has the following properties.

(i) $1 \leq \Gamma^j(\xi) \leq 2$, $\Gamma^j(t\xi) = \Gamma^j(\xi)$, and $\Gamma^j(t\xi) = \Gamma^j(\xi)$ for all $t \neq 0$ and $\xi \neq 0$.

(ii) $\Gamma^j$ satisfies
\[
\Gamma^j(\xi) \equiv 1, \quad \xi \in \left\{ \xi \in \mathbb{R}^2 \setminus \{0\} : \max\{|\xi_2/\xi_1|, |\xi_1/\xi_2|\} \leq \frac{\lambda^2 j}{\lambda^{2j} + 2\epsilon_0} \right\}.
\]

**Proof.** Since $\gamma_c$ is a compactly supported function in $C^\infty(\mathbb{R})$ and the function $f(\xi) := \xi_2/\xi_1$ or $\xi_1/\xi_2$ is infinitely differentiable for all $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ such that both $\xi_1 \neq 0$ and $\xi_2 \neq 0$, we see that by Taylor expansion $\Gamma^j$ is infinitely differentiable for $\xi \in \mathbb{R}^2$ such that both $\xi_1 \neq 0$ and $\xi_2 \neq 0$. For a fixed $\xi_1 \neq 0$, we have
\[
\Gamma^j(\xi) = \sum_{j = -\ell_{j_0} + 1}^{\ell_{j_0} - 1} |\gamma_c(\lambda^j \xi_2/\xi_1 + \ell)|^2 + |\gamma^{\xi}_{\lambda,\xi,\epsilon,\rho}(\lambda^j \xi_2/\xi_1 + \ell_\lambda)|^2
\]
for $|\xi_2|$ small enough in view of the supports of $\gamma_c$ and $\gamma^{\xi}_{\lambda,\xi,\epsilon,\rho}$, which implies that $\Gamma^j(\xi)$ is infinitely differentiable at $(\xi_1, 0)$ with $\xi_1 \neq 0$. Similarly, we have $\Gamma^j(\xi)$ is infinitely differentiable at $(0, \xi_2)$ with $\xi_2 \neq 0$. Hence, we have $\Gamma^j \in C^\infty(\mathbb{R}^2 \setminus \{0\})$. By its definition as in (4.19), $\Gamma^j(t\xi) = \Gamma^j(\xi)$ and $\Gamma^j(t\xi) = \Gamma^j(\xi)$ for $t \neq 0$ and $\xi \neq 0$.

By the property of $\gamma_c$, $\gamma^{\xi}_{\lambda,\xi,\epsilon,\rho}$ as in (4.14), it is easily seen that $1 \leq \Gamma^j(\xi) \leq 2$ for $\xi \neq 0$. Now to see that (4.20) holds, we notice that the seamline element on the horizontal cone with respect to $\ell = -\ell_{j_0}$ has part of the piece overlapping with the other cone. By the support of $\gamma^{\xi}_{\lambda,\xi,\epsilon,\rho}$, for this seamline element, we have $\xi_2/\xi_1 \leq 1 + \frac{2\epsilon_0}{\lambda^{2j} + 2\epsilon_0}$. Hence, those elements on the vertical cone with support satisfying $|\xi_1/\xi_2| \leq \frac{\lambda^2 j}{\lambda^{2j} + 2\epsilon_0}$ is not affected by that seamline elements. By symmetry, same result holds for seamline elements on vertical cone affecting the horizontal cone. Therefore, (4.20) holds.
Define \( \varphi^{j+1} \) to be
\[
\varphi^{j+1}(N^j_\lambda \xi) := \left( |\varphi^j(N^j_\lambda \xi)|^2 + |\omega^{j}_{\lambda,t,\rho}(N^j_\lambda \xi)|^2 \right)^{1/2}.
\]
(4.22)

Now, we split the function \( \omega^{j}_{\lambda,t,\rho} \) as follows. Let \( D_\lambda := \text{diag}(1, \lambda) \). For \( \xi \neq 0 \), define
\[
\psi^{j,\ell}(\xi) := \omega^{j}_J(D^{-j}_\lambda \mathcal{S}_{-\ell,\xi}) \gamma^\nu_\lambda,\ell,\rho_\alpha(\xi^\nu_\lambda) = -\ell_\lambda + 1, \ell_\lambda - 1,
\]
and
\[
\psi^{j,+,\ell,\lambda}(\xi) := \omega^{j}_J(D^{-j}_\lambda \mathcal{S}_{-\ell,\xi}) \gamma^\nu_\lambda,\ell,\rho_\alpha(\xi^\nu_\lambda).
\]
(4.24)

For \( \xi = 0 \), we define \( \psi^{j,0}(0) := 0 \). Since the support of \( \omega^{j}_J \) is away from the origin and in view of the properties of \( \mathbf{D} \), we deduce that \( \psi^{j,\ell} \) are functions in \( C^\infty(\mathbb{R}^2) \). Let
\[
\Psi_j := \{ \psi^{j,\ell}(S_j \cdot \xi) : \ell = -\ell_\lambda, \ldots, \ell_\lambda \}
\]
(4.25)

with \( \psi^{j,\ell} \) being given as in (4.23) and (4.24). The frequency-based affine shear system \( \text{FAS}_J(\varphi^j; \{ \Psi_j \}_{j=0}^\infty) \) is then defined as follows:
\[
\text{FAS}_J(\varphi^j; \{ \Psi_j \}_{j=0}^\infty) := \bigcup_{k \in \mathbb{Z}^2} \{ h_{\mathcal{B}_{j}(0,k)} : k \in \mathbb{Z}^2, h \in \Psi_j \}_{j=0}^\infty.
\]
(4.26)

Note that, explicitly, we have,
\[
\text{FAS}_J(\varphi^j; \{ \Psi_j \}_{j=0}^\infty) = \bigcup_{k \in \mathbb{Z}^2} \{ \psi^{j,\ell}(S_j \cdot \xi) : \ell = -\ell_\lambda, \ldots, \ell_\lambda \}_{j=0}^\infty.
\]

With the property of \( \Theta^J_0 \) as in item (ii) of Proposition 2 we can show that the system defined as in (4.26) can have shear structure for elements inside each cone. Moreover, with the scale \( j \) going to infinity, the shear operation could reach the seamline arbitrarily close. Indeed, we have the following result.

**Theorem 3.** Let \( \lambda > 1, 0 < \varepsilon \leq 1/2, 0 < t, \rho \leq 1 \) such that \( 1/\rho - 1/2 - \varepsilon > 0 \). Let \( J_0 \) be a nonnegative integer. Choose \( \varepsilon_0 > 0 \) such that
\[
\varepsilon_0 < \min \left\{ \frac{\lambda^2 J_0 - 1}{2}, \lambda^2 J_0, (1/\rho - 1/2 - \varepsilon) J_0 \right\}.
\]
Then the system \( \text{FAS}_J(\varphi^j; \{ \Psi_j \}_{j=0}^\infty) \) defined as in (4.26) with \( \varphi^j \) and \( \Psi_j \) being given as in (4.22) and (4.25), respectively, is a frequency-based affine shear tight frame for \( \mathcal{H}_2(\mathbb{R}^2) \) for all \( J \geq J_0 \). All elements in \( \text{FAS}_J(\varphi^j; \{ \Psi_j \}_{j=0}^\infty) \) are compactly supported functions in \( C^\infty(\mathbb{R}^2) \). Moreover,
\[
\left\{ \xi(S_j \cdot \xi) : |\ell| < \varepsilon_0, j \geq J_0 + 1, \right\} \subseteq \Psi_j, \quad j \geq J_0 + 1,
\]
and
\[
\left\{ \xi(S_j \cdot \xi) : |\ell| < \varepsilon_0, j \geq J_0 + 1, \right\} \subseteq \text{FAS}_J(\varphi^j; \{ \Psi_j \}_{j=0}^\infty), \quad J \geq J_0 + 1.
\]

**Proof.** By the property of \( \Theta^J_0 \) as in Proposition 2 we have \( \omega^{j}_J(D^{-j}_\lambda \mathcal{S}_{-\ell,\xi}) = \beta_{\lambda,t,\rho}(\lambda^{-2} J_0,1) \) for \( \xi \in \text{supp} \xi^{j,\ell}(S_j \mathcal{B}_{\lambda}^0) \) with \( |\ell| < \ell_\lambda - 1, j \geq J_0 + 1 \), and \( \omega^{j}_J(\xi) = \beta_{\lambda,t,\rho}(\lambda^{-2} J_0,1) \) for \( \xi \in \text{supp} \xi^{j,\ell}(S_j \mathcal{B}_{\lambda}^0) \) with \( |\ell| < \ell_\lambda - 1 \) and \( j \geq J_0 + 1 \). Hence, it is easily seen that for \( j \geq J_0 + 1 \),
\[
\psi^{j,\ell}(\xi) = \beta_{\lambda,t,\rho}(\xi,\ell_\lambda J_0) = \xi(\ell_\lambda J_0) = |\ell| < \varepsilon_0.
\]

Define \( \tilde{\Psi}_j := \{ \xi(S_j \cdot \xi) : |\ell| < \ell_\lambda + 1, \ell \neq \ell_\lambda, \ell_\lambda - 2 \} \cup \{ \psi^{j,\ell}(S_j \cdot \xi) : |\ell| = \ell_\lambda - 1, \ell \neq \ell_\lambda \} \). Then for \( j \geq J_0 + 1 \), \( \{ \xi(S_j \cdot \xi) : |\ell| < \ell_\lambda - 1 \} \subseteq \tilde{\Psi}_j \), and \( \text{FAS}_J(\varphi^j; \{ \Psi_j \}_{j=0}^\infty) = \text{FAS}_J(\varphi^j; \{ \tilde{\Psi}_j \}_{j=0}^\infty) \).

By the construction, (2.18) and (3.10) hold. Moreover, all generators are nonnegative. Noting that \( \text{supp} \varphi^j \subseteq [-\varepsilon, \varepsilon] \), \( \text{supp} \varphi^j \subseteq [-\varepsilon, \varepsilon] \), and \( \text{supp} \varphi^j \subseteq [-\varepsilon, \varepsilon] \), \( \text{supp} \varphi^j \subseteq [-\varepsilon, \varepsilon] \), together with \( \rho \leq 1 \) and \( 0 < \varepsilon < 1/2 \), we see that \( \text{supp} \psi^{j,\ell} \subseteq [-\varepsilon, \varepsilon] \) for \( |\ell| < \ell_\lambda - 1 \). Hence, we have
\[
\text{supp} \psi^{j,\ell}(\xi) \psi^{j,\ell}(\xi + 2\pi k) = 0, \quad \text{a.e., } \xi \in \mathbb{R}^2 \text{ and } k \in \mathbb{Z}^2 \setminus \{0\} \text{ for } |\ell| < \ell_\lambda - 1.
\]

For \( j \geq J_0 + 1 \), we have
\[
\text{supp} \psi^{j,\ell}(\xi) \subseteq \{ \xi \in \mathbb{R}^2 : \xi \in [-\varepsilon, \varepsilon], -1/2 - \varepsilon < \xi_2/\xi_1 < \lambda^2(1 + 2\varepsilon J_0/\lambda^2) - \ell_\lambda \}.
\]

Since \( 0 < \Theta^J_0 \leq 2 \), we can take the square root of \( \Theta^J_0 \), which is still a smooth function. Moreover, \( 1/\sqrt{\Theta^J_0} \) is also a smooth function. Define \( \varphi^j := \varphi^{j,\ell} \) and
\[
\omega^{j}_J(x) := \left( \sum_{\ell \neq -\ell_\lambda} (|\varphi^j(S_j B^0_\lambda \xi)|^2 + |\omega^j_{\lambda,t,\rho}(S_j B^0_\lambda \xi)|^2) \right)^{1/2} \sqrt{\Theta^J_0(\xi)}.
\]
(4.21)

Define \( \varphi^{j+1} \) to be
\[
\varphi^{j+1}(N^j_\lambda \xi) := \left( |\varphi^j(N^j_\lambda \xi)|^2 + |\omega^j_{\lambda,t,\rho}(N^j_\lambda \xi)|^2 \right)^{1/2}.
\]
(4.22)
Since \( \varepsilon_0 \leq \lambda^0 (2/\rho - 1 - 2 \varepsilon) \), we have,
\[
(\lambda^1(1 + 2 \varepsilon_0/\lambda^3) - \lambda^1(1 + 1/2 + \varepsilon) \leq \lambda^1(1 + 2 \varepsilon_0/\lambda^3) - (\lambda^1 + 1/2 + \varepsilon) + 1 + 1/2 + \varepsilon) \leq \frac{2 \varepsilon}{\lambda^3} + 1 + 2 \varepsilon \leq 2/\rho.
\]
Hence, we conclude that \( \psi^{j, \pm \ell_{\lambda^1}}(\xi, \psi^{j, \pm \ell_{\lambda^1}}(\xi + 2 \pi k) = 0 \), a.e., \( \xi \in \mathbb{R}^2 \) and \( k \in \mathbb{Z}^2 \setminus \{0\} \).

By the definition of \( \varphi^j \) and that \( \varepsilon_0 \leq \lambda^2/((\lambda^2 - 1)/2) \), we have
\[
supp \varphi^j \subseteq [-\lambda^{-2} p(1 + 2 \varepsilon_0/\lambda^3) \pi, \lambda^{-2} p(1 + 2 \varepsilon_0/\lambda^3) \pi] \subseteq [-\pi, \pi]^2.
\]
Hence, we conclude that \( \varphi^j(\xi, \frac{\varphi^j(\xi + 2 \pi k)}{\lambda^2} = 0 \) for all \( k \in \mathbb{Z}^2 \setminus \{0\} \) and for almost every \( \xi \in \mathbb{R}^2 \). Therefore, (3.13) holds. By the result of Corollary 3, \( \mathbf{FAS}_j(\varphi^j, \{ \Psi_j \}_{j \geq 0}) \) is a frequency-based affine shear tight frame for \( L_2(\mathbb{R}^2) \) for all \( J \geq J_0 \). Since all generators are compactly supported functions in \( C^\infty(\mathbb{R}^2) \), all elements in \( \mathbf{FAS}_j(\varphi^j, \{ \Psi_j \}_{j \geq 0}) \) are compactly supported functions in \( C^\infty(\mathbb{R}^2) \).

From Theorem 3 we see that
\[
\zeta(S_{t, \varepsilon_{\lambda^1}} \mathcal{B}_j(\xi) = \beta_{\lambda, t, \rho}(\lambda^{-2} \xi_1) \gamma_{\lambda}(\lambda^2 \xi_2 / \xi_1 - \ell_{\lambda^1} + 2), \quad \xi \in \mathbb{R}^2
\]
has support satisfying \( \xi_2 / \xi_1 \leq \xi_1 - \lambda^0 \xi_1 / \lambda^3 \) \( \to 1 \) as \( j \to \infty \). In other words, the shear operation reaches arbitrarily close to the seamlines \( \{ \xi | \xi \in \mathbb{R}^2 : |\xi_2 / \xi_1 | = \pm 1 \} \).

4.3. Quasi-stationary construction. Let us next discuss the quasi-stationary construction. The idea is to use the tensor product of functions in 1D to obtain rectangular bands for different scales, and then a frequency splitting using \( \gamma_{\lambda} \) is applied to produce generators with respect to different shears. More precisely, let \( \lambda > 1 \) and \( 0 \leq t, \rho \leq 1 \). Consider \( \varphi(\xi) := \alpha_{\lambda, t, \rho}(\xi) \alpha_{\lambda, t, \rho}(\xi), \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \) and define
\[
\omega_{\lambda, t, \rho}(\xi) := \sqrt{\varphi(\lambda^{-2} \xi)^2 - \varphi(\xi)^2}, \quad \xi \in \mathbb{R}^2.
\]
(4.27)
Then \( \omega_{\lambda, t, \rho} \in C^\infty(\mathbb{R}^2) \). In fact, it is easy to show that if \( \varphi(\xi_0) = 0 \) or 1, then all the derivatives of \( \varphi \) vanish at \( \xi_0 \). Now if \( \omega_{\lambda, t, \rho}(\xi) := |\varphi(\lambda^{-2} \xi)^2 - \varphi(\xi)^2| \) is not vanishing for \( \xi = \xi_0 \), then it is trivial to see that \( \omega_{\lambda, t, \rho} \equiv \sqrt{\omega_{\lambda, t, \rho}} \) is infinitely differentiable at \( \xi = \xi_0 \). If \( \omega_{\lambda, t, \rho}(\xi) = 0 \) at \( \xi = \xi_0 \), then we must have \( \varphi(\xi_0) = \varphi(\lambda^{-2} \xi_0) = 0 \) or \( \varphi(\xi_0) = \varphi(\lambda^{-2} \xi_0) = 1 \). Then, all the derivatives of \( \omega_{\lambda, t, \rho} \) vanish at \( \xi_0 \). By the Taylor expansion, we see that \( \omega_{\lambda, t, \rho} \equiv \sqrt{\omega_{\lambda, t, \rho}} \) must be infinitely differentiable at \( \xi_0 \) with all its derivatives at \( \xi_0 \) being zero. Therefore, \( \omega_{\lambda, t, \rho} \in C^\infty(\mathbb{R}^2) \).

In view of the construction of \( \varphi \), the refinable structure is clear. We have \( \varphi(\lambda^2 \xi) = a(\xi) \varphi(\xi), \xi \in \mathbb{R}^2 \) with \( a = \mu_{\lambda, t, \rho} \otimes \mu_{\lambda, t, \rho} \) being the tensor product of the 1D mask \( \mu_{\lambda, t, \rho} \) given as in (4.4). Moreover, we have \( \omega(\lambda^2 \xi) = b(\xi) \varphi(\xi) \) with \( b \in C^\infty(\mathbb{T}^2) \) being given by \( b(\xi) = (g(\xi) - |a(\xi)|^2)^{1/2} \) for any smooth function \( g \in C^\infty(\mathbb{T}^2) \) such that \( g \equiv 1 \) on the support of \( \varphi \).

Proposition 4. Define
\[
\Gamma_j(\xi) := \sum_{\ell = -\lambda_{\lambda^1}}^{\lambda_{\lambda^1}} |g_{\lambda}(\lambda^1 \xi_2 / \xi_1 + \ell)|^2 + |g_{\lambda}(\lambda^1 \xi_2 / \xi_1 + \ell)|^2, \quad j \in \mathbb{N}_0.
\]
(4.28)
Then \( \Gamma_j \in C^\infty(\mathbb{R}^2 \setminus \{0\}) \) having the following properties.

(i) \( 0 < \Gamma_j(\xi) \leq 2 \), \( \Gamma_j(\xi) = \Gamma_j(\xi), \) and \( \Gamma_j(t, \xi) = \Gamma_j(\xi) \) for all \( t \neq 0 \) and \( \xi \neq 0 \).

(ii) \( \Gamma_j \) satisfies
\[
\Gamma_j(\xi) \equiv 1, \quad \xi \in \left\{ \xi \in \mathbb{R}^2 : \max\{|\xi_2 / \xi_1|, |\xi_1 / \xi_2|\} \leq \frac{\lambda_j}{|\xi_{\lambda^1} + 1/2 + \varepsilon|} \right\}.
\]
(4.29)

Proof. The proof for \( \Gamma_j \in C^\infty(\mathbb{R}^2 \setminus \{0\}) \) is similar to that for \( \Gamma^j \) in Proposition 3. By its definition in (4.28),
\[
\Gamma_j(E) \equiv \Gamma_j, \quad \Gamma_j(t, \xi) = \Gamma_j(\xi) \text{ for } t \neq 0.
\]

By the property of \( \gamma_{\lambda} \) as in (4.13), it is easily seen that \( 0 < \Gamma_j \leq 2 \). Now to see that (4.29) holds, we notice that the seamline element on the horizontal cone with respect to \( \xi = -\xi_{\lambda^1} \) has part of the piece overlapping with the other cone. By the support of \( \gamma_{\lambda^1} \), for this seamline element, we have
\[
|\lambda^1 \xi_2 / \xi_1 - \xi_{\lambda^1} | \leq \frac{1}{2 + \varepsilon},
\]
which implies \( \xi_2 / \xi_1 \leq \frac{1/2 + \varepsilon}{\lambda^1} \). Hence, it only affects elements in other cone with support satisfying \( |\xi_1 / \xi_2| > 1/2 + \varepsilon \). By symmetry, the same result holds for seamline elements on vertical cone affecting the horizontal cone. Therefore, (4.29) holds.

\[ \square \]
Since $0 < \Gamma_j \leq 2$ and $\Gamma_j$ is in $C^\infty(\mathbb{R}^2 \setminus \{0\})$, we have that $\sqrt{\Gamma_j}$ is infinitely differentiable for all $\xi \in \mathbb{R}^2 \setminus \{0\}$. Let $A_\lambda, B_\lambda, M_\lambda, N_\lambda, D_\lambda$ with $\lambda > 1$ be defined as before. Let $\Psi_j := \{\psi_{j,\ell}(S_\ell) : \ell = -\ell_\lambda, \ldots, \ell_\lambda\}$ with

$$
\psi_{j,\ell}(\xi) := \omega_{\lambda,t,\rho}(D_\lambda^{-j} S_\ell \xi) \gamma_c(\xi_2/\xi_1) \frac{\gamma_c(\xi_2/\xi_1)}{\sqrt{\Gamma_j((S_\ell B_\lambda^t)^{-1})}}, \quad \xi \in \mathbb{R}^2 \setminus \{0\},
$$

(43.30)

and $\psi_{j,\ell}(0) := 0$, which gives

$$
\psi_{j,\ell}(S_\ell B_\lambda^t \xi) = \omega_{\lambda,t,\rho}(N_\lambda^t \xi) \gamma_c(\lambda^j \xi_2/\xi_1 + \ell) \frac{\gamma_c(\lambda^j \xi_2/\xi_1 + \ell)}{\sqrt{\Gamma_j(\xi)}}.
$$

By the properties of $\Gamma_j$ and that the support of $\omega_{\lambda,t,\rho}$ is away from the origin, we see that $\psi_{j,\ell}$ are functions in $C^\infty(\mathbb{R}^2)$. We can define the following system:

$$
\text{FAS}_j(\varphi; \{\Psi_j\}_{j=2}^\infty) := \{\varphi_{N_\lambda^t(0,k): k \in \mathbb{Z}^2} \cup \{h_{B_\lambda^t(0,k_0),h_{B_\lambda^t(0,k)}: k \in \mathbb{Z}^2, h \in \Psi_j\}_{j=2}^\infty

= \{\varphi_{N_\lambda^t(0,k): k \in \mathbb{Z}^2} \cup \{\psi_{j,\ell}(S_\ell B_\lambda^t(0,k) : k \in \mathbb{Z}^2, \ell = -\ell_\lambda, \ldots, \ell_\lambda, j=2\}.

(43.31)

At first glance, such a system does not have shear structure at all due to that the function $\omega_{\lambda,t,\rho}$ is not shear-invariant. However, we next shall show that such a system do have certain affine and shear structure in the sense that a sub-system of this system is from shear dilation and shear of single generator.

**Theorem 4.** Let $\lambda > 1$ and $0 < t, \rho < 1$. Let $\text{FAS}_j(\varphi; \{\Psi_j\}_{j=2}^\infty)$ be defined as in (43.31) with $\varphi = \alpha_{\lambda,t,\rho} \otimes \alpha_{\lambda,t,\rho}$ and $\psi_{j,\ell}$ being given by (43.30). Then $\text{FAS}_j(\varphi; \{\Psi_j\}_{j=2}^\infty)$ is a frequency-based affine shear tight frame for $L^2(\mathbb{R}^2)$ for all $J \geq 0$. All elements in $\text{FAS}_J(\varphi; \{\Psi_j\}_{j=2}^\infty)$ are compactly supported functions in $C^\infty(\mathbb{R}^2)$. Moreover, we have

$$\{\zeta(S_\ell) : \ell = -r_j, \ldots, r_j\} \subseteq \Psi_j, \quad j \geq J,$n

where $r_j := \lfloor \lambda^{j-1}(1-t)\rho - (1/2 + \varepsilon) \rfloor$ and $\zeta(\xi) := \beta_{\lambda,t,\rho}(\xi_1) \gamma_c(\xi_2/\xi_1), \xi \in \mathbb{R}^2$. In other words,

$$\{\zeta_{S_\ell B_\lambda^t(0,k),k : k \in \mathbb{Z}^2, \ell = -r_j, \ldots, r_j\}_{j=2}^\infty \subseteq \text{FAS}_J(\varphi; \{\Psi_j\}_{j=2}^\infty).

**Proof.** By our construction, we have

$$\| \varphi(N_\lambda^t \xi) \|^2 + \sum_{\ell = -r_j}^{r_j} \left| \psi_{j,\ell}(S_\ell B_\lambda^t \xi) \right|^2 + \left| \psi_{j,\ell}(S_\ell B_\lambda^t E \xi) \right|^2

= \| \varphi(N_\lambda^t \xi) \|^2 + \frac{\| \omega_{\lambda,t,\rho}(N_\lambda^t \xi) \|^2}{\gamma_c(\xi)} \sum_{\ell = -r_j}^{r_j} \left| \gamma_c(\lambda^j \xi_2/\xi_1 + \ell) \right|^2 + \left| \gamma_c(\lambda^j \xi_2/\xi_1 + \ell) \right|^2 \right|

= | \varphi(N_\lambda^t \xi) \|^2 + \left| \omega_{\lambda,t,\rho}(N_\lambda^t \xi) \right|^2 = | \varphi(N_{\lambda+1}^t \xi) \|^2, \quad \xi \in \mathbb{R}^2.

Hence, (43.31) holds. By the definition of $\varphi$, (43.32) also holds. Note that all generators $\psi_{j,\ell}$ are nonnegative and are defined in $[-\rho, \rho\pi]^2$ with $\rho < 1$. Hence, (43.33) is true. Now, by Corollary 3, we conclude that $\text{FAS}_j(\varphi; \{\Psi_j\}_{j=2}^\infty)$ is a frequency-based affine shear tight frame for $L^2(\mathbb{R}^2)$ for all $J \geq 0$. Since all generators $\varphi, \psi_{j,\ell}$ are compactly supported functions in $C^\infty(\mathbb{R}^2)$, all elements in $\text{FAS}_J(\varphi; \{\Psi_j\}_{j=2}^\infty)$ are compactly supported functions in $C^\infty(\mathbb{R}^2)$.

By the definition of $\omega_{\lambda,t,\rho}$, it is easy to see that

$$| \omega_{\lambda,t,\rho}(\xi,\xi_2) |^2 = | \alpha_{\lambda,t,\rho}(\xi_1) \beta_{\lambda,t,\rho}(\xi_2) |^2 + | \beta_{\lambda,t,\rho}(\xi_1) \alpha_{\lambda,t,\rho}(\xi_2) |^2 + | \beta_{\lambda,t,\rho}(\xi_1) \beta_{\lambda,t,\rho}(\xi_2) |^2.

And for $| \xi_2 | \leq \lambda^{j-1}(1-t)\rho$, we have

$$\omega_{\lambda,t,\rho}(\xi_1,\xi_2) = \beta_{\lambda,t,\rho}(\xi_1) \alpha_{\lambda,t,\rho}(\xi_2) = \beta_{\lambda,t,\rho}(\xi_1).

Consequently, if the support $\{\xi = (\xi_1,\xi_2) : \xi \in \text{supp } \psi_{j,\ell}(S_\ell B_\lambda^t(0,k))\}$ satisfies $| \xi_2 | \leq \lambda^{j-1}(1-t)\rho$, then we have

$$\psi_{j,\ell}(S_\ell B_\lambda^t(0,k)) = \lambda^{-3j/2} \omega_{\lambda,t,\rho}(\lambda^{-2j} \xi_1) \gamma_c(\lambda^j \xi_2/\xi_1 + \ell) e^{-ik \cdot S_\ell B_\lambda^t(0,k)}

= \lambda^{-3j/2} \beta_{\lambda,t,\rho}(\lambda^{-2j} \xi_1) \gamma_c(\lambda^j \xi_2/\xi_1 + \ell) e^{-ik \cdot S_\ell B_\lambda^t(0,k)}

\zeta_{S_\ell B_\lambda^t(0,k)}(\xi) = \lambda^{-3j/2} \beta_{\lambda,t,\rho}(\lambda^{-2j} \xi_1) \gamma_c(\lambda^j \xi_2/\xi_1 + \ell) e^{-ik \cdot S_\ell B_\lambda^t(0,k)}

with $\zeta(\xi) := \beta_{\lambda,t,\rho}(\xi_1) \gamma_c(\xi_2/\xi_1), \xi \in \mathbb{R}^2$. Now let us find the range of $\ell$ such that the support constrain holds.
At scale $j$, we have
\[ \text{supp} \omega_{\lambda,t,\rho}(\lambda^{-2j}) \subseteq [-\lambda^{2j}\rho, \lambda^{2j}\rho]^2 \cup [-\lambda^{2j-2}(1-t)\rho, \lambda^{2j-2}(1-t)\rho]^2. \]

Then, the support constrain means that at scale $j$, one needs $|\xi_2/\xi_1| \leq \lambda^{-2}(1-t)\rho$. Hence, the support of $\gamma_\ell(\lambda^j\xi_2/\xi_1 + \ell)$ must satisfy
\[ -\lambda^{-2}(1-t)\rho \leq -\lambda^{-2}(1/2 + \varepsilon + \ell) \leq \xi_2/\xi_1 \leq \lambda^{-2}(1/2 + \varepsilon - \ell) \leq \lambda^{-2}(1-t)\rho. \]

Consequently, we obtain
\[ -\lambda^{-2}(1-t)\rho + (1/2 + \varepsilon) \leq \ell \leq \lambda^{-2}(1-t)\rho - (1/2 + \varepsilon). \]
That is, $|\ell| \leq \lambda^{-2}(1-t)\rho - (1/2 + \varepsilon)$. In summary, letting $r_j := [\lambda^{-2}(1-t)\rho - (1/2 + \varepsilon)]$, we have
\[ \{\zeta(S_\ell) : \ell = -r_j, \ldots, r_j\} \subseteq \Psi_j, \quad j \geq J, \]
and
\[ \{\zeta_{S_{\beta_\varepsilon},\alpha,k}, \zeta_{S_{\beta_\varepsilon},\alpha,k} : j \geq J, k \in \mathbb{Z}^2, \ell = -r_j, \ldots, r_j\} \subseteq \text{FAS}_J(\varphi; \{\Psi_j\}_{j=0}^\infty). \]

Note that when $\ell = -r_j$, the support $\zeta(S_{\beta_\varepsilon}(\xi)) = \beta_{\lambda,t,\rho}(\lambda^{-2j}\xi_1)\gamma_\ell(\lambda^j\xi_2/\xi_1 - r_j)$ satisfies
\[ \xi_2/\xi_1 \leq \lambda^{-2}(r_j + 1/2 + \varepsilon) \leq \lambda^{-2}((1-t)\rho - 1/2 - \varepsilon)] + 1/2 + \varepsilon \leq \lambda^{-2}(1-t)\rho. \]

Hence, by the symmetry property of $\Gamma_j$, we see that the shear operation generates a subsystem of $\text{FAS}_J(\varphi; \{\Psi_j\}_{j=0}^\infty)$ inside the cone area $\{\xi \in \mathbb{R}^2 : \max(|\xi_2/\xi_1|, |\xi_1/\xi_2|) \leq \lambda^{-2}(1-t)\rho\}$.

### 4.4. Connections to other directional multiscale representation systems
In this subsection, we shall discuss the connections of our affine shear tight frames to those shearlet systems in [10] or shearlet-like systems in [15].

Define corner pieces
\[ \gamma^+(\lambda_\ell, \xi) := \begin{cases} 
\gamma_\ell(x) & \text{if } -1/2 - \varepsilon \leq x \leq -1/2 + \varepsilon; \\
1 & \text{if } -1/2 + \varepsilon \leq x \leq \lambda - \ell_\lambda; \\
0 & \text{otherwise,}
\end{cases} \tag{4.32} \]
\[ \gamma^-(\lambda_\ell, \xi) := \begin{cases} 
\gamma_\ell(x) & \text{if } 1/2 - \varepsilon \leq x \leq 1/2 + \varepsilon; \\
1 & \text{if } -\lambda + \ell_\lambda \leq x \leq 1/2 - \varepsilon; \\
0 & \text{otherwise.}
\end{cases} \]
These are the corner pieces that shall be used to achieve tightness of the system or for gluing two seamline elements together smoothly. Let $\{\alpha_{\lambda,t,\rho}, \beta_{\lambda,t,\rho}, \gamma_\ell, \gamma^\pm_\lambda\}$ be defined as before. Similar to the half pieces for the system generated by the characteristic functions as in [2.15], we define $\psi, \psi^{\pm,\pm}_\lambda$ by
\[ \psi(\xi) = \beta_{\lambda,t,\rho}(\xi_1)\gamma_\ell(\xi_2/\xi_1), \quad \psi^{\pm,\pm}_\lambda(\xi) = \beta_{\lambda,t,\rho}(\xi_1)\gamma^\pm_\lambda(\xi_2/\xi_1), \quad \xi \neq 0 \]
and $\psi(0) := 0$. The scaling function $\varphi$ is defined to be
\[ \varphi := \varphi^h + \varphi^v \tag{4.33} \]
with $\varphi^h(\xi) = \alpha_{\lambda,t,\rho}(\xi_1)\chi_{\xi \in \mathbb{R}^2 : |\xi_2/\xi_1| \leq 1}(\xi), \xi \in \mathbb{R}^2$ and $\varphi^v = \varphi^h(E) = \alpha_{\lambda,t,\rho}(\xi_2)\chi_{\xi \in \mathbb{R}^2 : |\xi_2/\xi_1| \leq 1}(\xi), \xi \in \mathbb{R}^2$.

Now define
\[ \Psi_j := \{\psi(S_{\ell}) : \ell = -\ell_\lambda + 1, \ldots, \ell_\lambda - 1\} \cup \{\psi^{\pm,\pm}(S_{\ell}) : \ell = \pm \ell_\lambda\}. \tag{4.34} \]

Note that $\psi$ is smooth while the corner pieces $\psi^{\pm,\pm}_\lambda$ are not smooth.

We have the following result.

**Corollary 4.** Let $A_\lambda, B_\lambda, M_\lambda, N_\lambda, S_{\ell}, E$ be defined as before with $\lambda > 1$. Let $0 < t, \rho < 1$ and $0 < \varepsilon < \frac{1}{2}$. Then the system $\text{FAS}_J(\varphi; \{\Psi_j\}_{j=0}^\infty)$ defined as in [3.1] with $\varphi, \Psi_j$ being given by [4.33], [4.34], respectively, is a frequency-based affine shear tight frame for $L_2(\mathbb{R}^2)$ for all $J \geq 0$. 
Proof. By the definition of $\gamma_\ell$ and $\gamma_\lambda^\perp$, for a fixed $j \geq 0$, it is easy to show that
\[
\sum_{\ell = -\ell_j + 1}^{\ell_j - 1} |\gamma_\ell(\lambda^j \xi_2 / \xi_1 + \ell)|^2 + |\gamma_\lambda^\perp(\lambda^j \xi_2 / \xi_1 + \ell_\lambda)|^2 = \chi_{\{\xi_2 / \xi_1 \leq 1\}}(\xi), \quad \xi \neq 0.
\]
Hence, we have
\[
|\varphi^h(N_j^1 \xi)|^2 + \sum_{h \in \Psi_j} |h(B_j^1 \xi)|^2 = (|\alpha_{\lambda,t,\rho}(\lambda^{-2j} \xi_1)|^2 + |\beta_{\lambda,t,\rho}(\lambda^{-2j} \xi_1)|^2) \chi_{\{\xi_2 / \xi_1 \leq 1\}}(\xi)
= |\varphi^h(N_j^1 \xi)|^2, \quad \xi \in \mathbb{R}^2.
\]
Similarly, we have
\[
|\varphi^n(N_j^1 \xi)|^2 + \sum_{h \in \Psi_j} |h(B_j^1 E \xi)|^2 = (|\alpha_{\lambda,t,\rho}(\lambda^{-2j} \xi_2)|^2 + |\beta_{\lambda,t,\rho}(\lambda^{-2j} \xi_2)|^2) \chi_{\{\xi_2 / \xi_1 \leq 1\}}(\xi)
= |\varphi^n(N_j^1 \xi)|^2, \quad \xi \in \mathbb{R}^2.
\]
Consequently, we have
\[
|\varphi(N_j^1 \xi)|^2 + \sum_{h \in \Psi_j} (|h(B_j^1 \xi)|^2 + |h(B_j^1 E \xi)|^2) = |\varphi(N_j^1 \xi)|^2, \quad a.e. \xi \in \mathbb{R}^2.
\]
Hence (4.14) holds.

Moreover, we have $h(\xi)h(\xi + 2k\pi) = 0$ for all $h \in \{\varphi \cup \{\Psi_j\}_{j=0}^\infty$ and $k \in \mathbb{Z}^2 \setminus \{0\}$. In fact, if $k = (k_1, k_2) \in \mathbb{Z}^2$ with $k_1 \neq 0$, then $h(\xi)h(\xi + 2k\pi) = 0$ due to that $\alpha_{\lambda,t,\rho}, \beta_{\lambda,t,\rho}$ are supported on $[-\rho \pi, \rho \pi)$ with $\rho \leq 1$. If $k_1 = 0$ but $k_2 \neq 0$, then by that $\gamma_\ell((\xi_2 + 2k_2\pi)/\xi_1) \gamma_\ell(\xi_2/\xi_1) = \gamma_\ell((\xi_2 + 2k_2\pi)/\xi_1) \gamma_\ell(\xi_2/\xi_1) = 0$ for $\xi_1 \in [-\rho \pi, \rho \pi]$, we have $h(\xi)h(\xi + 2k\pi) = 0$ as well. Hence, (4.13) is satisfied. Obviously, (4.13) is true by our construction of $\varphi$.

Therefore, by Corollary 3 of SMFAS ($\varphi; \{\Psi_j\}_{j=0}^\infty$) defined as in (1.3) with $\varphi, \Psi_j$ being given by (4.33), (4.34), respectively, is a frequency-based affine shear tight frame for $L_2(\mathbb{R}^2)$ for all $J \geq 0$.

Now, it is easy to show that the cone-adapted shearlet system constructed in [12] is indeed the initial system of a sequence of frequency-based affine shear tight frames. In fact, let $\lambda = 2$, and $A_h := A_\lambda, A_s := EA_s$. Let $\varphi, \psi, \psi^{j,\pm \ell_j}$ be the inverse Fourier transforms of $\varphi, \psi^h, \psi^{j,\pm \ell_j}$; that is, $\varphi = \mathcal{F}\varphi, \psi = \mathcal{F}\psi^h,$ and $\psi^{j,\pm \ell_j} = \mathcal{F}\psi^{j,\pm \ell_j}$. Let $\psi_n := \psi^h(E_\ell).$ It is easy to show that
\[
\mathcal{F}(2^{3j/2} \psi^h(S^{\ell_j} A_h - k)) = 2^{-3j/2} \psi(S_{-\ell_j} B_j^1 e^{-i\ell_j k} B_j^1) = \psi_{S_{-\ell_j} B_j^1(0,0,k)}.
\]
Similarly, we have $\mathcal{F}(2^{3j/2} \psi^h(S^{\ell_j} A_h - k)) = \psi_{S_{-\ell_j} B_j^1(0,0,k)}$. Noting that $\mathcal{F}(2^{3j/2} \psi^h(S^{\ell_j} A_h - k)) = \psi_{S_{-\ell_j} B_j^1(0,0,k)}$, and the symmetry of the range of $\ell$ for each scale $j$, we see that the cone-adapted shearlet system in (1.3) with modified seamline elements is the frequency-based affine shear tight frame $\mathcal{F}S_j(\varphi; \{\Psi_j\}_{j=0}^\infty)$ defined as in (1.3) with $\varphi, \Psi_j$ being given by (4.33), (4.34), and $\lambda = 2$. Moreover, it is the initial system of the sequence of frequency-based affine shear tight frames $\mathcal{F}S_j(\varphi; \{\Psi_j\}_{j=0}^\infty), \ J \in \mathbb{N}^0$ defined as in (1.11) with $\varphi, \Psi_j$ being given by (4.33), (4.34), respectively.

For the smooth shearlet-like systems constructed in [13], it is also a special case of the follow system. Note that $\gamma_\lambda, \gamma_\lambda^\perp$ satisfy
\[
\left. \frac{d^n}{dx^n} \gamma_\lambda^\perp(\lambda x + \ell_\lambda) \right|_{x = \pm 1} = \delta(n) \quad \forall n \in \mathbb{N}_0,
\]
which guarantees the smooth gluing of two corner pieces.

Let $\Psi_j := \{\psi^{j,\ell}(S_{-\ell_j} : \ell = -\ell_j, \ldots, \ell_j) \}$ with elements strictly inside the cone, i.e., $\ell = -\ell_j + 1, \ldots, \ell_j - 1$ being given by
\[
\psi^{j,\ell}(\xi) := \omega_{\lambda,t,\rho}(D_{-\ell_j} S_{-\ell_j} \xi) \gamma_\ell(\xi_2 / \xi_1), \quad \omega_{\lambda,t,\rho}(\xi_1, \lambda^{-j} (-\xi_1 \ell + \xi_2)) \gamma_\ell(\xi_2 / \xi_1), \quad \xi \in \mathbb{R}^2,
\]
which gives
\[
\psi^{j,\ell}(S_{-\ell_j} B_j^1 \xi) = \omega_{\lambda,t,\rho}(\lambda^{-2\ell} \xi_2) \gamma_\ell(\lambda^j \xi_2 / \xi_1 + \ell);
\]
and those elements on the seamlines, i.e., for $\ell = \pm \ell_j$ and $j \geq 1$, being given by gluing two pieces along the seamline on two cones together,
\[
\psi^{j,\pm \ell_j}(S_{\pm \ell_j} B_j^1 / 2 \xi) := \begin{cases} \omega_{\lambda,t,\rho}(\lambda^{-2\ell} \xi_2) \gamma_\ell^\perp(\lambda^j \xi_2 / \xi_1 \pm \ell_\lambda) & |\xi_2 / \xi_1| \leq 1 \\ \omega_{\lambda,t,\rho}(\lambda^{-2\ell} \xi_2) \gamma_\ell^\perp(\lambda^j \xi_2 / \xi_1 \pm \ell_\lambda) & |\xi_2 / \xi_1| \geq 1. \end{cases}
\]
For $j = 0$,
\[
\psi^{\pm 1}(S_{\pm 1}\xi) := \begin{cases}
\omega_{\lambda, t, \rho}(\xi)\gamma_{\ell}(\xi_2/\xi_1 \pm 1) & |\xi_2/\xi_1| \leq 1 \\
\omega_{\lambda, t, \rho}(\xi)\gamma_{\ell}(\xi_2/\xi_1 \pm 2) & |\xi_2/\xi_1| \geq 1.
\end{cases}
\]

Let $B^j_{\lambda} := B^j_\lambda$ for $j \geq 1$ and $\ell < \ell_{\lambda'}$, $B^{j,\ell}_{\lambda} := B^j_\lambda/2$ for $j \geq 1$, and for $j = 0$, $B^j_{\lambda} := I_2$. Then, we can define the following system
\[
FAS(\varphi; \{\Psi_j\}_{j=0}^\infty) = \{\varphi_{0,k} : k \in \mathbb{Z}^2\} \cup \\
\cup \{h_{B^j_{\lambda},0,k} : k \in \mathbb{Z}^2, h \in \Psi_j \setminus \{\psi^{j,\ell}(S\cdot) : \ell = \pm \ell_{\lambda'}\}\}_{j=0}^\infty
\]
(4.37)

**Corollary 5.** $FAS(\varphi; \{\Psi_j\}_{j=0}^\infty)$ defined as in (4.37) is a frequency-based affine shear tight frame for $L_2(\mathbb{R}^2)$ and all elements in $FAS(\varphi; \{\Psi_j\}_{j=0}^\infty)$ are compactly supported functions in $C^\infty(\mathbb{R}^2)$.

**Proof.** By our construction, we have
\[
|\varphi|^2 + \sum_{j=0}^\infty \sum_{\ell = -\ell_{\lambda'}}^{\ell_{\lambda'}-1} (|\psi^{j,\ell}(S\cdot B^j_\lambda)|^2 + |\psi^{j,\ell}(S\cdot B^j_\lambda E \cdot)|^2) + \\
\sum_{j=0}^\infty \sum_{\ell = -\ell_{\lambda'}}^{\ell_{\lambda'}+1} |\psi^{j,\ell}(S\cdot B^j_\lambda/2)|^2 + |\psi^{j,\ell}(S\cdot B^j_\lambda E \cdot/2)|^2 = 1, \quad a.e. \xi \in \mathbb{R}^2.
\]

Moreover, all generators $\psi^{j,\ell}$ are nonnegative and defined in $[-\pi, \pi]^2$. Note that dilation matrices of the seamline generators $\psi^{j,\ell_{\lambda'}}$ are $B^j_{\lambda}/2$. A simple adaptation of the proof of Theorem 4 gives that $FAS(\varphi; \{\Psi_j\}_{j=0}^\infty)$ is a frequency-based tight frame. By the definition of $\gamma, \gamma^\pm$, $\psi^{j,\ell}$ are smooth. Consequently, all elements in $FAS(\varphi; \{\Psi_j\}_{j=0}^\infty)$ are smooth.

When $\lambda = 2, t = 1 - \lambda^{-2}$ and $\rho = 1$, $FAS(\varphi; \{\Psi_j\}_{j=0}^\infty)$ defined as in (4.37) is essentially the system defined as in [15]. For their construction, the shear operations can reach only up to slope (in absolute value) $\lambda^{-4} = 1/16$. Here, our construction is more general and in our construction, the shear can reach up to slope $\lambda^{-2}(1-t)\rho$ with any $0 < t, \rho \leq 1$. Comparing to our quasi-stationary construction, the gluing procedure is somewhat unnatural since one can see that a different dilation matrix $B^j_{\lambda}/2$ needs to be applied to the gluing elements at this scale $j$ while all other generators use the dilation matrix $B^j_{\lambda}$. On the other hand, all atoms of our affine shear systems, either under quasi-stationary construction or non-stationary construction, obey the parabolic rule and more importantly, at all scales $j$, for each cone, the dilation matrix is fixed as $B^j_{\lambda}$ for all generators.

5. MRA STRUCTURES AND FILTER BANKS

In this section we shall study the MRA structure of sequences of frequency-based affine shear tight frames constructed in Section 4 and investigate their underlying filter banks.

The sequence of systems $FAS_j(\varphi^j; \{\Psi_j\}_{j=0}^\infty)$, $J \geq J_0$ defined as in (3.1) has two different dilation matrices $A_\lambda$ and $EA_\lambda E$ for two cones. On the one hand, the functions $\varphi^j$, $J \geq J_0$ with the dilation matrix $M_\lambda$ induce an MRA $\{\mathcal{V}_j\}_{j \geq J_0}$ with
\[
\mathcal{V}_j := \mathcal{S}pan\{\varphi^j(M_\lambda \cdot -k) : k \in \mathbb{Z}^2\},
\]
where $\varphi^j := \mathcal{F}^{-1}\varphi$. The function space $\mathcal{V}_j$ is shift-invariant on the lattice $N_\lambda^j \mathbb{Z}^2$. On the other hand, at scale $j$, let $\psi := \mathcal{F}^{-1}\psi$ and $\psi^j, \ell := \mathcal{F}^{-1}\psi^{j,\ell}, |\ell| = r_j + 1, \ldots, s_j$. For convention, $\psi^j, 0 := \psi$ for $|\ell| \leq r_j$. Then the wavelet subspaces $\mathcal{W}_j$ are given by
\[
\mathcal{W}_j := \mathcal{S}pan\{\psi^j(S\cdot A_\lambda \cdot -k), \psi^j(S\cdot A_\lambda E \cdot -k) : \ell = -s_j, \ldots, s_j, k \in \mathbb{Z}^2\}, \quad j \geq J_0,
\]
and we have $\mathcal{V}_j \subseteq \mathcal{V}_{j+1}$, $\mathcal{W}_j \subseteq \mathcal{W}_{j+1}$, and $\mathcal{V}_j + \mathcal{W}_j = \mathcal{V}_{j+1}$ provided $FAS_j(\varphi^j; \{\Psi_j\}_{j=0}^\infty)$ is a tight frame for $L_2(\mathbb{R}^2)$ for all $J \geq J_0$. While $\mathcal{V}_j$ is shift-invariant on the lattice $N_\lambda^j \mathbb{Z}^2$, $\mathcal{W}_j$ is not shift-invariant on the lattice $N_\lambda^j \mathbb{Z}^2$. We shall next discuss other types of sequences of frequency-based affine systems that can be regarded as a ‘shift-invariant’ extension of $FAS_j(\varphi^j; \{\Psi_j\}_{j=0}^\infty)$. 
5.1. Non-stationary MRA and filter banks. Let us first discuss the MRA and filter bank structure for the non-stationary construction in Section 4.

Let \( \eta, \zeta \) be defined as in (4.10). Fixed \( J_0 \geq 0 \), let \( \Theta^{J_0}, \varphi^J, \psi_J \) be defined as in (4.11), (4.22), (4.23), respectively. Let \( \text{FAS}_J(\varphi^J; \{ \Psi^J \}) \) be defined with these \( \varphi^J, \psi_J \). By modifying elements from original \( \psi^{J,f} \) in (4.23), we define \( \tilde{\Psi}^J \) as follows:

\[
\tilde{\Psi}^J := \{ \psi^{J,f} : \ell = -\ell_M, \ldots, \ell_M \}
\]

with

\[
\tilde{\psi}^{J,f}(\xi) := \omega_{\lambda, t, \rho, \epsilon, \epsilon}(\xi) \frac{\gamma_\epsilon(\lambda(\ell + \xi_1 + \ell) \Gamma(\xi)}{\Gamma(\xi)}, \quad \xi \neq 0, \quad |\ell| \leq \ell_M - 1,
\]

\[
\tilde{\psi}^{J,0}(\xi) := \omega_{\lambda, t, \rho, \epsilon, \epsilon}(\xi) \frac{\gamma_{\epsilon}^{\lambda}(\lambda(\ell + \xi_1 + \ell) \Gamma(\xi)}{\Gamma(\xi)}, \quad \xi \neq 0,
\]

and \( \tilde{\psi}^{J,0}(0) = 0 \). Again, it is trivial that \( \tilde{\psi}^{J,f}, \ell = -\ell_M, \ldots, \ell_M \) are compactly supported functions in \( C^\infty(\mathbb{R}^2) \). We then use the dilation matrix \( M_\lambda \) for all generators \( \varphi^J \) and \( \tilde{\psi}^{J,f} \). The frequency-based affine system is then defined to be

\[
\text{FAS}_J(\varphi^J; \{ \tilde{\Psi}^J \}) := \{ \varphi^{N_0}_{J',0,k} : k \in \mathbb{Z}^2 \} \cup \{ h_{N_0,0,k} : k \in \mathbb{Z}^2, h \in \tilde{\Psi}^J \}
\]

\[
= \{ \varphi^{N_0}_{J',0,k} : k \in \mathbb{Z}^2 \} \cup \{ \tilde{\psi}^{J,0}_{N_0,0,k} : k \in \mathbb{Z}^2, \ell = -\ell_M, \ldots, \ell_M \}.
\]

We now discuss connection of (5.3) with the directional tight framelets as discussed in (17).

**Theorem 5.** Retaining all the conditions for \( \lambda, t, \rho, \epsilon, \epsilon, J_0 \) as in Theorem 3. Let \( \text{FAS}_J(\varphi^J; \{ \tilde{\Psi}^J \}) \) be defined as in (5.4) with \( \varphi^J \) and \( \tilde{\Psi}^J \) being given by (4.22) and (5.1), respectively. Then \( \text{FAS}_J(\varphi^J; \{ \tilde{\Psi}^J \}) \) is a frequency-based affine tight frame for \( L_2(\mathbb{R}^2) \) for all \( J \geq J_0 \); that is, \( \{ \varphi^J \} \cup \{ \tilde{\Psi}^J \} \subseteq L_2(\mathbb{R}^2) \) and

\[
(2\pi^2\|f\|^2 \leq \sum_{k \in \mathbb{Z}^2} |(f, \varphi^{N_0}_{J',0,k})|^2 + \sum_{j \geq 0} \sum_{k \in \mathbb{Z}^2} |(f, h_{N_0,0,k})|^2 + |(f, h_{N_0,0,k})|^2 \quad \forall f \in L_2(\mathbb{R}^2).
\]

All elements of \( \text{FAS}_J(\varphi^J; \{ \tilde{\Psi}^J \}) \) are compactly supported functions in \( C^\infty(\mathbb{R}^2) \). Moreover, let \( \text{FAS}_J(\varphi^J; \{ \tilde{\Psi}^J \}) \) be the frequency-based affine shear tight frame with \( \Psi^J = \{ \psi^{J,f} : \ell = -\ell_M, \ldots, \ell_M \} \) being given as in (4.23). Then \( \text{FAS}_J(\varphi^J; \{ \tilde{\Psi}^J \}) \) and \( \text{FAS}_J(\varphi^J; \{ \tilde{\Psi}^J \}) \) are connected to each other by the following relations:

\[
\psi^{J,f}_{Sb^{j,k}} = \lambda^{j/2} \tilde{\psi}^{J,f}_{N_0,0,0, D^{j,k}} \quad \text{and} \quad \psi^{J,f}_{Sb^{j,k}} = \lambda^{j/2} \tilde{\psi}^{J,f}_{N_0,0,0, D^{j,k}}
\]

**Proof.** It is obvious that

\[
\varphi^J(\xi) = \sum_{h \in \tilde{\Psi}^J} (h(\xi) + |h(E\xi)|^2 = |\varphi^{J+1}(N_0)\|^2, \quad \forall \xi \in \mathbb{R}^2.
\]

By our construction, (5.13) and (5.5) hold. Now, the conclusion follows from (17) Corollary 18 that \( \text{FAS}_J(\varphi^J; \{ \tilde{\Psi}^J \}) \) is a frequency-based affine tight frame for \( L_2(\mathbb{R}^2) \) for all \( J \geq J_0 \). Obviously, all elements of \( \text{FAS}_J(\varphi^J; \{ \tilde{\Psi}^J \}) \) are compactly supported functions in \( C^\infty(\mathbb{R}^2) \).

Since

\[
\psi^{J,f}_{Sb^{j,k}}(\xi) = \lambda^{j/2} \psi^{J,f}(Sb^{j,k} \xi) e^{-ik\xi} Sb^{j,k} \xi = \lambda^{-j/2} \omega_{\lambda, t, \rho, \epsilon, \epsilon}(N_0) \frac{\gamma_x(\lambda(\ell + \xi_1 + \ell) \Gamma(\xi)}{\Gamma(\xi)} e^{-ik\xi} Sb^{j,k} \xi
\]

and

\[
\tilde{\psi}^{J,f}_{N_0,0,0, D^{j,k}}(\xi) = \lambda^{-j/2} \tilde{\psi}^{J,f}(N_0) \xi e^{-ikN_0} = \lambda^{j/2} \omega_{\lambda, t, \rho, \epsilon, \epsilon}(N_0) \frac{\gamma_x(\lambda(\ell + \xi_1 + \ell) \Gamma(\xi)}{\Gamma(\xi)} e^{-ik\xi} Sb^{j,k} \xi
\]

by noting that \( Sb^{j,k} = Sb^{j,k} \). (5.6) is obviously true. \( \square \)

When \( \lambda \) is an integer, we have \( D^{j}_{\lambda,0,0,2^j} \subseteq \mathbb{Z}^2 \). Equation (5.6) shows that when \( \lambda \) is an integer, the frequency-based affine shear tight frame \( \text{FAS}_J(\varphi^J; \{ \tilde{\Psi}^J \}) \) is indeed a subsystem of the frequency-based affine tight frame \( \text{FAS}_J(\varphi^J; \{ \tilde{\Psi}^J \}) \) through subsampling. Since both of these two systems share the same refinable functions \( \varphi^J, \tilde{\Psi}^J \) of the MRA for these two systems are the same. However, at scale \( j \), the wavelet subspace \( \tilde{W}_j \) of \( \text{FAS}_J(\varphi^J; \{ \tilde{\Psi}^J \}) \) generated by \( \tilde{\psi}^{J,f} := 2^{-j} \tilde{\psi}^{J,f} \) are

\[
\tilde{W}_j = \text{span}\{ \tilde{\psi}^{J,f}(M^{J,f}_\lambda \cdot -k) : \ell = -\ell_M, \ldots, \ell_M, k \in \mathbb{Z}^2 \}, \quad j \geq J_0.
\]

It is trivial to see that \( \tilde{W}_j \) is shift-invariant on the lattice \( N_0 \mathbb{Z}^2 \) and moreover, we have \( \tilde{W}_j \subseteq \tilde{W}_j \).
Let us next study the filter bank structure of the frequency-based affine tight frame \( \text{FAS}_j(\varphi^j; \{ \tilde{\Psi}_j \}_{j \in J}) \). By our construction and requiring \( 0 < \rho < 1 \), we can choose \( \varepsilon_0 < \lambda^2 (\rho_0 / \rho - 1) / 2 \) so that \( \text{supp} \varphi^j(M_{\lambda}) \subseteq \text{supp} \varphi^{j+1} \subseteq [-\rho_0 \pi, \rho_0 \pi]^2 \) and \( \text{supp} \tilde{\psi}^{j, \ell} (M_{\lambda}) \subseteq \text{supp} \varphi^{j+1} \subseteq [-\rho_0 \pi, \rho_0 \pi]^2 \) for some \( 0 < \rho < \rho_0 < 1 \). Let \( 2 \pi \mathbb{Z}^2 \)-periodic functions \( a^j, b^{j, \ell}, j \geq J_0 \) be defined as follows.

\[
a^j(\xi) := \begin{cases} \frac{\varphi^j(M_{\lambda})}{\varphi^{j+1}(\xi)} & \xi \in \text{supp} \varphi^j(M_{\lambda}), \\ 0 & \xi \in [-\pi, \pi]^2 \setminus \text{supp} \varphi^j(M_{\lambda}), \end{cases}
\]

\[
b^{j, \ell}(\xi) := b^j(\xi) \frac{\gamma_{\varphi^{j+1}}(\lambda^j \xi_2 / \xi_1 + \ell)}{\Gamma_j(\xi)}, \quad |\ell| < \ell_{j,\lambda} - 1,
\]

\[
b^{j, \pm \ell_{j,\lambda}}(\xi) := b^j(\xi) \frac{\gamma_{\varphi^{j+1}}(\lambda^j \xi_2 / \xi_1 \pm \ell_{j,\lambda})}{\Gamma_j(\xi)},
\]

where \( b^j(\xi) = \sqrt{g^j(\xi) - |a^j(\xi)|^2} \) for some function \( g^j \) defined on \( \mathbb{T}^2 \) satisfying \( g^j \equiv 1 \) on the support of \( \varphi^{j+1} \). By [17, Corollary 18 and Theorem 17], we have the following result.

**Corollary 6.** Retaining all the conditions for \( \lambda, \ell, \rho, \varepsilon, \varepsilon_0, J_0 \) as in Theorem 18 with \( \lambda \) being an integer, \( 0 < \rho < 1 \), and \( \varepsilon_0 < \lambda^2 (\rho_0 / \rho - 1) \) such that \( \text{supp} \varphi^j \) and \( \text{supp} \psi^j \) are both inside \( [-\rho_0 \pi, \rho_0 \pi]^2 \) for some \( 0 < \rho < \rho_0 < 1 \). Let \( \text{FAS}_j(\varphi^j; \{ \tilde{\Psi}_j \}_{j \in J}) \), \( J \geq J_0 \) be defined as in [5.3] with \( \tilde{\Psi}_j \) being given as in [5.1] and let \( a^j, b^{j, \ell} \) be defined as in [5.7]. Then there exist \( g^j \in C^\infty(\mathbb{T}^2), j \geq J_0 \) such that \( a^j, b^{j, \ell} \in C^\infty(\mathbb{T}^2) \) for all \( j \geq J_0, \ell = -\ell_{j,\lambda}, \ldots, \ell_{j,\lambda} \), and we have

\[
\varphi^j(M_{\lambda} \xi) = a^j(\xi) \varphi^{j+1}(\xi) \quad \text{and} \quad \tilde{\psi}^{j, \ell} (M_{\lambda} \xi) = b^{j, \ell}(\xi) \varphi^{j+1}(\xi), \quad j \geq J_0
\]

for \( \text{a.e.} \ \xi \in \mathbb{R}^2 \). Moreover, \( \{a^j, b^{j, \ell}, \ell : \ell = -\ell_{j,\lambda}, \ldots, \ell_{j,\lambda}\} \) is a filter bank having the perfect reconstruction property, i.e.,

\[
|a^j(\xi)|^2 + \sum_{|\ell| = -\ell_{j,\lambda}}^{\ell_{j,\lambda}} (|b^{j, \ell}(\xi)|^2 + |b^{j, \ell}(\xi)|^2) = 1, \quad \text{a.e.} \ \xi \in \sigma_{j+1},
\]

and

\[
\overline{\sigma}^j(\xi) a^j(\xi + 2 \pi \omega) + \sum_{|\ell| = -\ell_{j,\lambda}}^{\ell_{j,\lambda}} \left[ b^{j, \ell}(\xi) b^{j, \ell}(\xi + 2 \pi \omega) + \overline{b^{j, \ell}(\xi)} b^{j, \ell}(\xi + 2 \pi \omega) \right] = 0
\]

for \( \text{a.e.} \ \xi \in \sigma_{j+1} \cap \sigma_{j+1}, \) and for \( \omega \in \Omega_{M_{\lambda}} \setminus \{0\} \) with \( \Omega_{M_{\lambda}} = [M_{\lambda}^{-1} \mathbb{Z}^2] \cap [0, 1)^2 \) and \( \sigma_{j+1} := \{ \xi \in \mathbb{R}^2 : \sum_{k \in \mathbb{Z}^2} |\varphi^j(\xi + 2 \pi k)|^2 \neq 0 \} \).

**Proof.** By our choice of \( \varepsilon, \rho, \varepsilon_0, \lambda, J_0, \) \( \text{FAS}_j(\varphi^j; \{ \tilde{\Psi}_j \}_{j \in J}) \) defined as in [5.3] with \( \tilde{\Psi}_j \) being given in [5.1] is a frequency-based affine tight frame for \( L_2(\mathbb{R}^2) \) for all \( J \geq J_0 \). By the construction of \( a^j, b^{j, \ell} \), it is easily seen that [5.5] holds. Now by [17, Theorem 17], [5.9] and [5.10] hold.

Since \( \text{supp} \varphi^j(M_{\lambda}) \) is strictly inside \( \text{supp} \varphi^{j+1} \), by the smoothness of \( \varphi^j \) and \( \varphi^{j+1} \), it is trivial that \( a^j \in C^\infty(\mathbb{T}^2) \). We next show that there exist \( g^j \in C^\infty(\mathbb{T}^2) \) such that \( b^{j, \ell} \in C^\infty(\mathbb{T}^2) \). Since \( \varphi^{j+1} \) and \( \psi^{j, \ell} \) is inside \( [-\rho_0 \pi, \rho_0 \pi]^2 \). Then one can construct a function \( g^j \in C^\infty(\mathbb{T}^2) \) such that \( g^j(\xi) \equiv 1 \) for \( \xi \in [-\rho_0 \pi, \rho_0 \pi]^2 \) and \( g^j(\xi) \equiv 0 \) for \( \xi \in \mathbb{T}^2 \setminus [-\rho_1 \pi, \rho_1 \pi]^2 \) and for some \( \rho_1 \) such that \( 0 < \rho_0 < \rho_1 < 1 \). Since \( \omega_{j,t,\rho}(M_{\lambda}) \subseteq \text{supp} \varphi^{j+1} \), we have

\[
\omega_{j,t,\rho}^{j+1}(\xi) = (|\varphi^j(\xi)|^2 - |\varphi^j(M_{\lambda})|^2)^{1/2} = (|\varphi^{j+1}(\xi)|^2 - |a^j(\xi)|^2 \varphi^{j+1}(\xi))^{1/2} = (1 - |a^j(\xi)|^2)^{1/2} \varphi^{j+1}(\xi).
\]

Obviously, \( (g^j(\xi) - |a^j(\xi)|^2)^{1/2} \in C^\infty(\mathbb{T}^2) \). Then,

\[
\tilde{\psi}^{j, \ell} (M_{\lambda} \xi) = \omega_{j,t,\rho}^{j+1}(M_{\lambda} \xi) \frac{\gamma_{\varphi^{j+1}}(\lambda^j \xi_2 / \xi_1 + \ell)}{\Gamma_j(\xi)} = b^{j, \ell}(\xi) \varphi^{j+1}(\xi)
\]

with \( b^{j, \ell}(\xi) = b^j(\xi) \frac{\gamma_{\varphi^{j+1}}(\lambda^j \xi_2 / \xi_1 + \ell)}{\Gamma_j(\xi)} \) being a function in \( C^\infty(\mathbb{T}^2) \). Similarly,

\[
\tilde{\psi}^{j, \pm \ell_{j,\lambda}} (M_{\lambda} \xi) = \omega_{j,t,\rho}^{j+1}(M_{\lambda} \xi) \frac{\gamma_{\varphi^{j+1}}(\lambda^j \xi_2 / \xi_1 \pm \ell_{j,\lambda})}{\Gamma_j(\xi)} = b^{j, \pm \ell_{j,\lambda}}(\xi) \varphi^{j+1}(\xi).
\]

We are done. \( \square \)
5.2. Quasi-stationary MRA and filter banks. Next, let us discuss the MRA and filter bank structure for the quasi-stationary construction. For the quasi-stationary case, the function \( \varphi' \) for the system \( \text{FAS}_J(\varphi'; \{ \Psi_j \}_{j=0}^{\infty}) \) is fixed as \( \varphi' \equiv \varphi \). Consider the system \( \text{FAS}_J(\varphi; \{ \Psi_j \}_{j=0}^{\infty}) \) as defined in (4.31). It is also from a ‘denser’ directional framelet system. Recall that for this system \( \varphi_0(\xi) = \alpha_{\lambda_1, \ldots, \lambda_j}(\xi_1) \alpha_{\lambda_1, \ldots, \lambda_j}(\xi_2), \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \) and \( \Psi_j := \{ \psi_j^{\ell} \lambda_{\lambda_1, \ldots, \lambda_j}(\xi) \} \), with \( \psi_j^{\ell}(\xi) = \omega_{\lambda_1, \ldots, \lambda_j}(\xi_1 + \ell) \eta_{\lambda_1, \ldots, \lambda_j}(\xi_2) \), which gives

\[
\psi_j^{\ell}(\xi) = \omega_{\lambda_1, \ldots, \lambda_j}(\xi_1 + \ell) \eta_{\lambda_1, \ldots, \lambda_j}(\xi_2) \in \mathbb{R}^2,
\]

Now define a new set

\[
\tilde{\psi}_j := \{ \tilde{\psi}_j^{\ell}, \ell = -\ell_{\lambda_1}, \ldots, \ell_{\lambda_j} \}
\]

with \( \tilde{\psi}_j^{\ell}(\xi) = \omega_{\lambda_1, \ldots, \lambda_j}(\xi_1 + \ell) \eta_{\lambda_1, \ldots, \lambda_j}(\xi_2) \), which gives

\[
\tilde{\psi}_j^{\ell}(\xi) = \omega_{\lambda_1, \ldots, \lambda_j}(\xi_1 + \ell) \eta_{\lambda_1, \ldots, \lambda_j}(\xi_2) e^{-ikN_j \ell},
\]

We then use a fixed dilation matrix \( M_\lambda \) for all generators \( \varphi \) and \( \tilde{\psi}_j^{\ell} \). The frequency-based affine system is then defined to be

\[
\tilde{\psi}_j^{\ell}(\xi) = \omega_{\lambda_1, \ldots, \lambda_j}(\xi_1 + \ell) \eta_{\lambda_1, \ldots, \lambda_j}(\xi_2) e^{-ikN_j \ell}.
\]

Now similar to the result in Theorem 5 for the non-stationary construction, we have the following result for the quasi-stationary construction.

**Theorem 6.** Let \( \lambda > 1 \) and \( 0 < \varepsilon \leq 1/2, 0 < t, \rho \leq 1 \). Let \( \text{FAS}_J(\varphi; \{ \tilde{\psi}_j \}_{j=0}^{\infty}) \) be defined as in (5.12) with \( \varphi \) and \( \tilde{\psi}_j \) being given by (4.9) and (5.11), respectively. Then \( \text{FAS}_J(\varphi; \{ \tilde{\psi}_j \}_{j=0}^{\infty}) \) is a frequency-based affine tight frame for \( L_2(\mathbb{R}^2) \) for all \( J > 0 \); that is, \( \{ \varphi \} \cup \{ \tilde{\psi}_j \}_{j=0}^{\infty} \subseteq L_2(\mathbb{R}^2) \) and

\[
(2\pi)^2\|f\|^2 = \sum_{j=0}^{\infty} \sum_{h=\tilde{\psi}_j, k \in \mathbb{Z}^2} \left( \sum_{k \in \mathbb{Z}^2} |(f, \tilde{\psi}_j^{\ell})|^2 + \sum_{k \in \mathbb{Z}^2} |(f, \tilde{\psi}_j^{\ell})|^2 \right) + \sum_{k \in \mathbb{Z}^2} \left( \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^2} |(f, \tilde{\psi}_j^{\ell})|^2 + |(f, h_{N_j, 0, k})|^2 \right) \quad \forall f \in L_2(\mathbb{R}^2).
\]

All elements of \( \text{FAS}_J(\varphi; \{ \tilde{\psi}_j \}_{j=0}^{\infty}) \) are compactly supported functions in \( C^\infty(\mathbb{R}^2) \). Moreover, let \( \text{FAS}_J(\varphi; \{ \tilde{\psi}_j \}_{j=0}^{\infty}) \) be the frequency-based affine tight frame with \( \tilde{\psi}_j = \{ \tilde{\psi}_j^{\ell} : \ell = -\ell_{\lambda_1}, \ldots, \ell_{\lambda_j} \} \) being given in (5.31). Then \( \text{FAS}_J(\varphi; \{ \tilde{\psi}_j \}_{j=0}^{\infty}) \) and \( \text{FAS}_J(\varphi; \{ \psi_j \}_{j=0}^{\infty}) \) are connected to each other by the following relations:

\[
\tilde{\psi}_j^{\ell}(\xi) = \psi_j^{\ell}(\xi) = \lambda^{\ell/2} \tilde{\psi}_j^{\ell}(N_\lambda \xi)
\]

**Proof.** The proof is essentially the same as the proof of Theorem 5.

Similarly, if \( \lambda \) is an integer, then \( D_{\ell}^{\ell} S^{\ell} : \mathbb{Z}^2 \subseteq \mathbb{Z}^2 \), and we can define a sequence of filter banks. In this case, the low-pass filter \( a \) of \( 2\pi \mathbb{Z}^2 \)-periodic function for \( \varphi \) is fixed as follows

\[
a(\xi) = \mu_{\lambda_1, \ldots, \lambda_j}(1), \quad \xi \in \mathbb{T}^2.
\]

with \( \mu_{\lambda_1, \ldots, \lambda_j} \) being given by (4.3). Note that \( \text{supp} \tilde{\psi}_j^{\ell}(M_\lambda \xi) \subseteq \text{supp} \varphi \). Define \( 2\pi \mathbb{Z}^2 \)-periodic functions \( b^{\ell} \) for \( \tilde{\psi}_j^{\ell}, j \geq 0 \) as follows.

\[
b^{\ell}(\xi) := b(\xi) \frac{\eta_{\lambda_1, \ldots, \lambda_j}(\xi_1 + \ell)}{\Gamma_j(\xi_1)}, \quad |\ell| \leq \ell_{\lambda_1}.
\]

with \( b(\xi) := \sqrt{g(\xi)} - |a(\xi)|^2 \) for some function \( g \) defined on \( \mathbb{T}^2 \) satisfying \( g \equiv 1 \) on the support of \( \varphi \). Then, we have the following result.

**Corollary 7.** Let \( \lambda > 1 \) be an integer. Choose \( 0 < \varepsilon \leq 1/2, 0 < t, \rho < 1 \). Let \( \text{FAS}_J(\varphi; \{ \tilde{\psi}_j \}_{j=0}^{\infty}) \), \( J > 0 \) be defined as in (5.12) with \( \tilde{\psi}_j \) being given in (5.11) and let \( a, b^{\ell} \) be defined as in (5.15) and (5.16), respectively. Then there exists \( g \in C^\infty(\mathbb{T}^2) \) such that \( a, b^{\ell} \in C^\infty(\mathbb{T}^2) \) for all \( j \geq 0, \ell = -\ell_{\lambda_1}, \ldots, \ell_{\lambda_j} \) and we have

\[
\varphi(M_\lambda \xi) = a(\xi) \varphi(\xi) \quad \text{and} \quad \tilde{\psi}_j^{\ell}(M_\lambda \xi) = b^{\ell}(\xi) \varphi(\xi)
\]
for a.e. $\xi \in \mathbb{R}^2$. Moreover, $\{a; b^{j,\ell}(E): \ell = -\ell_j, \ldots, \ell_j\}$ is a filter bank having the perfect reconstruction property, i.e.,

$$|a(\xi)|^2 + \sum_{\ell=-\ell_j}^{\ell_j} \left( |b^{j,\ell}(\xi)|^2 + |b^{j,\ell}(E \xi)|^2 \right) = 1,$$

and

$$\overline{a(\xi)} a(\xi + 2\pi \omega) + \sum_{\ell=-\ell_j}^{\ell_j} \left[ \overline{b^{j,\ell}(\xi)} b^{j,\ell}(\xi + 2\pi \omega) + \overline{b^{j,\ell}(E \xi)} b^{j,\ell}(E \xi + 2\pi \omega) \right] = 0$$

for a.e. $\xi \in \sigma_{\varphi}$ and $(\omega - 2\pi \omega) \in \Omega_{M_{\lambda}} \setminus \{0\}$ with $\Omega_{M_{\lambda}} = [M_{\lambda}^{-1}\mathbb{Z}^2] \cap [0,1)^2$ and $\sigma_{\varphi} := \{\xi \in \mathbb{R}^2 : \sum_{k \in \mathbb{Z}^2} |\varphi(\xi + 2\pi k)|^2 \neq 0\}$.

**Proof.** By our choice of $\varepsilon, \rho, \lambda, t$, FAS$_{j}(\varphi; \{\tilde{\Psi}_j\}_{j=0}^{\infty})$ defined as in (5.12) with $\tilde{\Psi}_j$ being given in (5.11) is a frequency-based affine tight frame for $L_2(\mathbb{R}^2)$ for all $J \geq 0$. By the construction of $a, b^{j,\ell}$, it is easily seen that (5.17) holds. Now by [17, Theorem 17], (5.18) and (5.19) hold.

Since $\text{supp} \varphi(M_{\lambda})$ is strictly inside $\text{supp} \varphi$, by the smoothness of $\varphi$, it is trivial that $a \in C^\infty(\mathbb{T}^2)$. We next show that there exists $g \in C^\infty(\mathbb{T}^2)$ such that $b^{j,\ell} \in C^\infty(\mathbb{T}^2)$. Since $\rho < 1$, we have $|\varphi| = [\lambda^2 - \rho^2, \lambda^2 - \rho^2] \subseteq [-\rho^2, \rho^2]$ and $\text{supp} \varphi \subseteq [-\rho^2, \rho^2]$. Then one can construct a function $g \in C^\infty(\mathbb{T}^2)$ such that $g(\xi) \equiv 1$ for $\xi \in [-\rho^2, \rho^2]$ and $g(\xi) \equiv 0$ for $\xi$ outside $[-\rho^2, \rho^2]$. Note that $\text{supp} \varphi \subseteq [-\rho^2, \rho^2]$ and we have

$$\omega_{\lambda, t, \rho}(\lambda^2 \xi^2) = (|\varphi(\xi)|^2 - |\varphi(\rho^2 \xi^2)|^2)^{1/2} = (|\varphi(\xi)|^2 - |a(\xi)|^2)^{1/2} \varphi(\xi) = (|\varphi(\xi)|^2 - |a(\xi)|^2)^{1/2} \varphi(\xi).$$

Obviously, $(g(\xi) - |a(\xi)|^2)^{1/2} \in C^\infty(\mathbb{T}^2)$. Then,

$$\tilde{\psi}^{j,\ell}(M_{\lambda} \xi) = \omega_{\lambda, t, \rho}(\lambda^2 \xi^2) g_{j}(\xi) \varphi(\xi)$$

with $b^{j,\ell}(\xi) := \frac{g_{j}(\xi) \varphi(\xi)}{\omega_{\lambda, t, \rho}(\lambda^2 \xi^2)}$ being a function in $C^\infty(\mathbb{T}^2)$.

**6. Discussion and Extension**

In this paper, we mainly investigate affine shear systems in $L_2(\mathbb{R}^2)$ for the purpose of simplicity of presentation. Our characterization and construction can be easily extended to higher dimensions. In $\mathbb{R}^d$ with $d \geq 2$, the shear operator $S^\tau$ with $\tau = (\tau_2, \ldots, \tau_d) \in \mathbb{R}^{d-1}$ and $A_\lambda$ are of the form:

$$S^\tau = \begin{bmatrix} 1 & \tau_2 & \cdots & \tau_d \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \quad \text{and} \quad A_\lambda = \begin{bmatrix} \lambda^2 & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda \end{bmatrix}.$$

Define $S_\tau := (S^\tau)^T$ and denote $E_n$ to be the elementary matrix corresponding to the coordinate exchange between the first axis and the nth one. For example, $E_1 = I_d$ and $E_2 = \text{diag}(E, I_{d-2})$. For $d = 2$, we have $E_2 = E$. Let $\Psi_j$ be given by

$$\Psi_j := \{\psi(S^\ell \xi) : \ell = -r_j, \ldots, r_j, n = 2, \ldots, d\} \cup \{\psi(S^\ell E \xi) : |\ell| = r_j + 1, \ldots, s_j, n = 2, \ldots, d\}$$

with $\ell = (\ell_2, \ldots, \ell_d) \in \mathbb{Z}^{d-1}$ and $\psi$ and $\psi^{j,\ell}$ being functions in $L^2_{\text{loc}}(\mathbb{R}^d)$. For the low frequency part, it corresponds to a function $\varphi_j \in L^2_{\text{loc}}(\mathbb{R}^d)$. Then a frequency-base affine shear system in $\mathbb{R}^d$ is defined to be

$$\text{FAS}_{j}^{\varphi_j; \{\tilde{\Psi}_j\}_{j=0}^{\infty}} := \{\varphi_{N_{j}^{\ell} \in \mathbb{Z}^d} : k \in \mathbb{Z}^d\} \cup \{h_{B}(E_{n} \in \mathbb{Z}^d) : k \in \mathbb{Z}^d, n = 1, \ldots, d, \ h \in \Psi_j \}_{j=0}^{\infty},$$

where $N_{j}^{\ell} := \lambda^{-2} I_d$ and $B_\lambda := (A_\lambda)^{-1}$.

All the characterizations for frequency-based affine shear tight frames and sequences of frequency-based affine shear frames can be carried over to the $d$-dimensional case for the system defined as in (5.1). Since the essential idea of our smooth non-stationary construction and smooth quasi-stationary construction is frequency splitting, our 2D construction thus can be easily extended to any high dimensions once an $\omega^1$ is constructed in a way satisfying $\omega^1 = (|\varphi|^2 + |N_{j}^{\ell} |^2 - |\varphi|^2)^{1/2}$ for both the non-stationary and quasi-stationary construction. Filter banks associated with high-dimensional frequency-based affine systems can be obtained as well as their connection to cone-adapted high-dimensional directional framelets.

Several problems remain open in our study of affine shear tight frames. For example, the existence and construction of affine shear tight frames with compactly supported generators in the spatial domain. If we
drop the tightness requirement, there are indeed compactly supported shearlet frames, e.g., see [21]. In view of the connection between affine shear tight frames and cone-adapted directional framelets, one might want to consider the existence and construction of cone-adapted directional framelets with compactly supported generators first. Another problem is the existence of affine shear tight frames with only one smooth generator; that is, $\Psi_j := \{ \varphi(S_j^r) : r \in I_j \}$ is from one generator $\psi$. Considering that the shear operator along the seamlines is not consistent for both cones, our conjecture is that there is even no affine shear tight frame with one single generator that is continuous. In other words, additional seamline generators seem to be unavoidable when considering cone-adapted construction. When $\lambda > 1$ is an integer, we know that an affine shear tight frame can be regarded as a subsystem of a directional framelet through sub-sampling, from which an underlying filter bank exists for the affine shear tight frame. However, when $\lambda > 1$ is not an integer, though an affine shear tight frame is still related to a cone-adapted directional framelet via $\{5.14\}$ or $\{5.11\}$, the lattice $D^1_\lambda S^2 \mathbb{Z}^2$ is no longer an integer lattice, the sub-sampling procedure thus fails and we do not know whether there is still an underlying filter bank for such an affine shear tight frame.

**References**

[1] J.-P. Antoine, R. Murenzi, and P. Vanderheyndt, Directional wavelets revisited: Cauchy wavelets and symmetry detection in pattern, *Appl. Comput. Harmon. Anal.* 6 (3) (1999) 314 – 345.

[2] E. J. Candès and D. L. Donoho, New tight frames of curvelets and optimal representations of objects with piecewise $C^2$ singularities, *Comm. Pure Appl. Math.* 57 (2) (2004) 219–266.

[3] C. K. Chui, W. He, and J. Stöckler, Compactly supported tight and sibling frames with maximum vanishing moments, *Appl. Comput. Harmon. Anal.* 13 (3) (2002) 224 – 262.

[4] M. N. Do and M. Vetterli, Contourlets, in G. V. Welland, editor, Beyond Wavelets, Academic Press, 2008.

[5] M. Elad, J.-L. Starck, P. Querre, D. L. Donoho, Simultaneous cartoon and texture image inpainting using morphological component analysis (MCA), *Appl. Comput. Harmon. Anal.* 19 (3) (2005) 340–358.

[6] I. Daubechies, Ten Lectures on Wavelets, CBMS-NSF Regional Conference Series in Applied Mathematics, 61, SIAM, Philadelphia, PA, 1992.

[7] I. Daubechies, B. Han, A. Ron, and Z. Shen, Framelets: MRA-based constructions of wavelet frames, *Appl. Comput. Harmon. Anal.* 14 (1) (2003) 1–46.

[8] D. L. Donoho, Sparse components of images and optimal atomic decomposition, *Constr. Approx.* 17 (2001) 353–382.

[9] K. Guo, D. Labate, W. Lim, G. Weiss, and E. Wilson, Wavelets with composite dilations, *Electr. Res. Ann. AMS* 10 (2004) 78–87.

[10] K. Guo, G. Kutyniok, and D. Labate, Sparse multidimensional representations using anisotropic dilation and shear operators, Wavelets and Splines (Athens, GA, 2005), Nashboro Press, Nashville, TN (2006) 189-201.

[11] K. Guo, D. Labate, W. Lim, G. Weiss, and E. Wilson, Wavelets with composite dilations and their MRA properties, *Appl. Comput. Harmon. Anal.* 20 (2006) 231–249.

[12] K. Guo and D. Labate, Optimal sparse multidimensional representation using shearlets, *SIAM J. Math. Anal.* 9 (2007) 298–318.

[13] K. Guo, and D. Labate, Characterization and analysis of edges using the continuous shearlet transform, *SIAM J. Imag. Sci.* 2 (2009) 959–986.

[14] K. Guo, and D. Labate, Analysis and detection of surface discontinuities using the 3D continuous shearlet transform, *Appl. Comput. Harmon. Anal.* 30 (2011) 231–242.

[15] K. Guo, D. Labate, The construction of smooth Parseval frames of shearlets, *Math. Model. Nat. Phenom.*., to appear.

[16] B. Han, On dual wavelet tight frames, *Appl. Comput. Harmon. Anal.* 4 (1997) 380–413.

[17] B. Han, Nonhomogeneous wavelet systems in high dimensions, *Appl. Comput. Harmon. Anal.* 32 (2012) 169–196.

[18] B. Han, G. Kutyniok, and Z. Shen, Adaptive multiresolution analysis structures and shearlet systems, *SIAM J. Num. Anal.* 49 (2011) 1921–1946.

[19] R. Houska, The nonexistence of shearlet scaling functions, *Appl. Comput. Harmon. Anal.* 32 (1) (2012) 28–44.

[20] E. J. King, G. Kutyniok, and X. Zhuang, Analysis of data separation and recovery problems using clustered sparsity, J. Math. Imaging Vis., to appear.

[21] P. Kittipoom, G. Kutyniok, and W.-Q. Lim, Construction of compactly supported shearlet frames, *Constr. Appr.* 35 (1) (2012) 21–72.

[22] G. Kutyniok and D. Labate, Resolution of the wavefront set using continuous shearlets. *Trans. Amer. Math. Soc.* 361 (2009) 2719–2754.

[23] G. Kutyniok and D. Labate, Shearlets: Multiscale Analysis for Multivariate Data, Birkhäuser, 2012.

[24] G. Kutyniok, J. Lemvig, and W.-Q Lim, Adaptively sparse multiresolution analysis structures and shearlet systems, *SIAM J. Num. Anal.* 49 (2011) 1921–1946.

[25] G. Kutyniok and W.-Q. Lim, Compactly supported shearlets are optimally sparse, *J. Approx. Theory* 163 (11) (2011) 1564–1589.

[26] G. Kutyniok and T. Sauer. Adaptive directional subdivision schemes and shearlet multiresolution analysis, *SIAM J. Math. Anal.* 41 (2009) 1436–1471.

[27] G. Kutyniok, M. Shahram, and X. Zhuang, ShearLab: A rational design of a digital parabolic scaling algorithm, *SIAM J. Imaging Sci.* 5 (4) (2012) 1201–1332.

[28] S. Mallat, A Wavelet Tour of Signal Processing, Academic Press, 2008.

[29] A. Ron and Z. Shen, Affine systems in $L^2(\mathbb{R}^d)$: The analysis of the analysis operator, *J. Funct. Anal.* 148 (2) (1997) 408–447.

[30] I. W. Selesnick, R. G. Baraniuk, and N. G. Kingsbury, The dual-tree complex wavelet transform, *IEEE Signal Process. Mag.* 22 (6) (2005) 123–151.
