Analytical and numerical assessment of accuracy of the approximated nuclear symmetry energy in the Hartree-Fock theory

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1. Introduction

The nuclear symmetry energy, which is defined as the second derivative of the energy per nucleon with respect to the proton-neutron asymmetry, is an important quantity for prediction of masses of neutron-rich nuclei and structure of neutron stars. Calculation of the symmetry energy, however, is not always easy. Analytical approaches are impossible unless energy of the system is given by a twice-differentiable function with respect to the asymmetry, while a numerical evaluation requires precise calculation of energy of the system. For this reason, the symmetry energy is sometimes approximated as energy difference between the pure neutron matter and symmetric matter. Conversely, this approximation of the symmetry energy corresponds to the quadratic approximation of the neutron-matter energy with respect to the asymmetry, in its estimation from the symmetric-matter energy.

As Bethe originally pointed out [1], this quadratic approximation of energy is applicable to small asymmetry but its validity is not obvious for systems with large asymmetry like the neutron matter. Accuracy of this approximation has been studied in several works. Chen et al. investigated this approximation at the saturation density with a modified Gogny interaction [2]. This approximation was examined in the Brueckner-Hartree-Fock theory with the Paris potential by Bombaci and Lombardo [3], and with the AV18 plus three-nucleon force by Zuo et al. [4]. Drischler et al. also studied it in the chiral effective field theory [5].

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It should be mentioned that Wellenhofer et al. pointed out that the approximation is not valid at certain temperatures [6].

We reinvestigate accuracy of the approximation of the symmetry energy at zero temperature within the Hartree-Fock theory, by using effective interactions which have been tested for structure of finite nuclei. This enables analytical arguments, by which origin of the errors can be examined term by term. Then the errors can be numerically estimated by inserting values of the parameters. In practice we adopt six effective interactions; the Skyrme interactions SkM∗ [7] and SLy4 [8], the Gogny interactions D1S [9] and D1M [10], and the M3Y-type interactions M3Y-P6 and M3Y-P7 [11].

This paper is organized as follows. In Sect. 2, we decompose the nuclear-matter energy, the symmetry energy, and its error. From Sect. 3 to 5, we evaluate the relative errors of individual terms. Section 3 is devoted to the errors whose analytical forms are independent of the effective Hamiltonian. The errors specific to the Skyrme and finite-range (Gogny and M3Y-type) effective interactions are analytically investigated in Sect. 4 and 5, respectively. The numerical results of errors for symmetry energy are displayed and discussed in Sect. 6. Section 7 provides a summary of the paper.

2. Decomposition of energies and errors

In this section, we shall give expressions of the symmetry energy $a_t(\rho)$ and its approximation $\tilde{a}_t(\rho)$, by decomposing them into several terms. Correspondingly, the errors of the symmetry energy can be decomposed as well.

2.1. Effective Hamiltonian

In this paper, we handle the homogeneous nuclear matter with the spin degeneracy in the Hartree-Fock theory.

Because of the homogeneity, the single-particle wave function is written as a plane wave,

$$\varphi_{k\sigma\tau}(r) = \frac{1}{\sqrt{\Omega}} e^{i\mathbf{k} \cdot \mathbf{r}} \chi_\sigma \chi_\tau ,$$

(1)

where $\mathbf{k}$ denotes the momentum and $\Omega$ is the volume of the system, for which we shall take $\Omega \to \infty$ later. $\chi_\sigma$ ($\chi_\tau$) is the spin (isospin) wave function.

The effective Hamiltonian of the system consists of the kinetic energy and the effective interaction,

$$H = K + V; \quad K = \sum \frac{p_i^2}{2M}, \quad V = \sum_{i<j} v_{ij},$$

(2)

where $i$ and $j$ are indices of nucleons. The effective interaction $V$ is comprised of the central term, the LS term and the tensor (TN) term. In the homogeneous matter, the LS term and the TN term can be neglected. We therefore treat only the central term, which may contain usual two-body interaction and density-dependent interaction. For the latter we assume contact form. We then have

$$v_{12} = v_{12}^{(C)} + v_{12}^{(DD)};$$

$$v_{12}^{(C)} = \sum_n \left( t_n^{(SE)} P_{SE} + t_n^{(TE)} P_{TE} + t_n^{(SO)} P_{SO} + t_n^{(TO)} P_{TO} \right) f_n(r_{12}),$$

(3)

$$v_{12}^{(DD)} = t_\rho^{(SE)} P_{SE} [\rho(\mathbf{r}_1)]^\alpha \delta(\mathbf{r}_{12}) + t_\rho^{(TE)} P_{TE} [\rho(\mathbf{r}_1)]^\beta \delta(\mathbf{r}_{12}),$$

(4)
where \( f_n(\tau) \) is an appropriate function of \( r_{12} = |r_{12}| \) with \( r_{12} = r_1 - r_2 \). In this paper, we treat the Skyrme, the Gogny, and the M3Y-type interactions. Correspondingly, \( f_n(\tau) \) is the \( \delta \)-function (with and without momentum-dependence), the Gauss function, or the Yukawa function. The effective interactions have range parameters, and the index \( n \) distinguishes the range. All kinds of \( t \) in the above equation are strength parameters. \( P_{TE}, P_{TO}, P_{SE}, \) and \( P_{SO} \) are projection operators to Singlet-Even, Triplet-Even, Singlet-Odd, and Triplet-Odd two-particle states, and they are related to the spin exchange operator \( P_\sigma \) and the isospin exchange operator \( P_\tau \) as follows,

\[
 P_{SE} = \frac{1 - P_\sigma 1 + P_\tau }{2} , \quad P_{TE} = \frac{1 + P_\sigma 1 - P_\tau }{2} ,
\]

\[
 P_{SO} = \frac{1 - P_\sigma 1 - P_\tau }{2} , \quad P_{TO} = \frac{1 + P_\sigma 1 + P_\tau }{2} . \tag{4}
\]

We can rewrite \( v_{12}^{(C)} \) as follows,

\[
 v_{12}^{(C)} = \sum_n (t_n^{(W)} + t_n^{(B)} P_\sigma - t_n^{(H)} P_\tau - t_n^{(M)} P_\sigma P_\tau) f_n(\tau) ; \tag{5}
\]

\[
 t_n^{(W)} = (t_n^{(SE)} + t_n^{(TE)} + t_n^{(SO)} + t_n^{(TO)})/4 ,
\]

\[
 t_n^{(B)} = (-t_n^{(SE)} + t_n^{(TE)} + t_n^{(SO)} + t_n^{(TO)})/4 ,
\]

\[
 t_n^{(H)} = (-t_n^{(SE)} + t_n^{(TE)} - t_n^{(SO)} - t_n^{(TO)})/4 ,
\]

\[
 t_n^{(M)} = (t_n^{(SE)} + t_n^{(TE)} - t_n^{(SO)} - t_n^{(TO)})/4 .
\]

\[\]

### 2.2. Decomposition of energy per nucleon

With the wave function of Eq. (1) and the effective Hamiltonian of Eqs. (2), (3), the energy per nucleon \( E \) can be expressed in the following manner, as was derived in Ref. [12]. We consider a function \( W \) as

\[
 W(k_\tau, k_\tau') = \int_{k_\tau \leq k_\tau'} d^3k_1 \int_{k_\tau \leq k_\tau'} d^3k_2 \tilde{f}_n(|k_{12} - k'_{12}|), \tag{6}
\]

where \( \tilde{f}(q) \) signifies the Fourier transform of \( f(r) ; \tilde{f}(q) = \int d^3r f(r) e^{-iq\cdot r} \). \( k_{12} \) and \( k'_{12} \) represent relative momenta of the initial and the final states. Since \( k_{12} = k'_{12} \) in the Hartree (i.e., direct) term and \( k_{12} = -k'_{12} \) in the Fock (i.e., exchange) term, contributions of these terms to the energy is represented by the function \( W \) as follows,

\[
 W_n^H(k_\tau, k_\tau') = \int_{k_\tau \leq k_\tau'} d^3k_1 \int_{k_\tau \leq k_\tau'} d^3k_2 \tilde{f}_n(0) = \frac{16\pi^2}{9} k_\tau^2 k_{12}^2 \tilde{f}_n(0) \quad \text{(Hartree term)},
\]

\[
 W_n^F(k_\tau, k_\tau') = \int_{k_\tau \leq k_\tau'} d^3k_1 \int_{k_\tau \leq k_\tau'} d^3k_2 \tilde{f}_n(2k_{12})
\]

\[
 = 8\pi^2 \left[ \int_0^{(k_\tau - k_{12})/2} dk_{12} \frac{16}{3} k_\tau^2 k_{12}^2 \tilde{f}(2k_{12}) + \int_0^{(k_\tau + k_{12})/2} dk_{12} \left\{ -\frac{1}{2} (k_\tau^2 - k_{12}^2)^2 k_{12} + \frac{8}{3} (k_\tau^2 + k_{12}^2) k_{12}^3 + \frac{8}{3} k_{12}^5 \right\} \tilde{f}(2k_{12}) \right] \quad \text{(Fock term).} \tag{7}
\]
We can express the total energy by using these functions. Under the spin degeneracy, the total energy is represented by

\[
E = \frac{\Omega}{10\pi^2 M} \left(k_5^5 + k_{n_5}^5\right) + \frac{\rho^2 \Omega}{36\pi^4} \left[t^{(SE)}_i \rho^\alpha (k_5^6 + k_{n_5}^6) + (t^{(SE)}_p \rho^\alpha + 3t^{(TE)}_p \rho^3) k_5^3 k_{n_5}^3\right] + \frac{\Omega}{(2\pi)^6} \sum_{n} \left[2t^{(W)}_n + t^{(B)}_n - 2t^{(H)}_n - t^{(M)}_n\right] \left\{W^n_H (k_p, k_p) + W^n_H (k_n, k_n)\right\}
\]

\[
+ 2 \left[2t^{(W)}_n + t^{(B)}_n\right] W^n_H (k_p, k_n) + \left(2t^{(M)}_n + t^{(H)}_n - 2t^{(B)}_n - t^{(W)}_n\right) \left\{W^n_F (k_p, k_p) + W^n_F (k_n, k_n)\right\} + 2 \left(2t^{(M)}_n + t^{(H)}_n\right) W^n_F (k_p, k_n),
\]

where \(k_p\) (\(k_n\)) denotes the Fermi momentum of protons (neutrons), and they are connected with the density \(\rho_p\) (\(\rho_n\)) or with the total density \(\rho\) and the asymmetry \(\eta_t\) as

\[
k_p = (3\pi^2 \rho_p)^{1/3} = \left\{\frac{3\pi^2}{2} \rho (1 - \eta_t)\right\}^{1/3}, \quad k_n = (3\pi^2 \rho_n)^{1/3} = \left\{\frac{3\pi^2}{2} \rho (1 + \eta_t)\right\}^{1/3};
\]

\[
\rho = \rho_p + \rho_n, \quad \eta_t = \frac{\rho_n - \rho_p}{\rho}.
\]

The energy per nucleon \(\mathcal{E} = E/A\) is acquired by dividing \(E\) by the nucleon number \(A = \rho \Omega\). We here decompose \(\mathcal{E}\) into the kinetic term \(\mathcal{E}_K\), the density-dependent term \(\mathcal{E}_{DD}\), the Hartree term between like nucleons \(\mathcal{E}_{HO}\) and between unlike nucleons \(\mathcal{E}_{HX}\), the Fock term between like nucleons \(\mathcal{E}_{FO}\) and between unlike nucleons \(\mathcal{E}_{FX}\),

\[
\mathcal{E}(\rho, \eta_t) = \mathcal{E}_K + \mathcal{E}_{DD} + \sum_n \left(\mathcal{E}_{HO_n} + \mathcal{E}_{HX_n} + \mathcal{E}_{FO_n} + \mathcal{E}_{FX_n}\right);
\]

\[
\mathcal{E}_K = \frac{1}{10\pi^2 M \rho} (k_5^5 + k_{n_5}^5),
\]

\[
\mathcal{E}_{DD} = \frac{1}{36\pi^4 \rho} \left[t^{(SE)}_i \rho^{\alpha - 1} (k_5^6 + k_{n_5}^6 + k_5^3 k_{n_5}^3) + 3t^{(TE)}_p \rho^3 k_5^3 k_{n_5}^3\right],
\]

\[
\mathcal{E}_{HO_n} = \frac{1}{(2\pi)^6 \rho} \cdot \left(2t^{(W)}_n + t^{(B)}_n - 2t^{(H)}_n - t^{(M)}_n\right) \left\{W^n_H (k_p, k_p) + W^n_H (k_n, k_n)\right\},
\]

\[
\mathcal{E}_{HX_n} = \frac{1}{(2\pi)^6 \rho} \cdot \left(2t^{(M)}_n + t^{(H)}_n - 2t^{(B)}_n - t^{(W)}_n\right) \left\{W^n_F (k_p, k_p) + W^n_F (k_n, k_n)\right\},
\]

\[
\mathcal{E}_{FO_n} = \frac{1}{(2\pi)^6 \rho} \cdot \left(2t^{(M)}_n + t^{(H)}_n - 2t^{(B)}_n - t^{(W)}_n\right) \left\{W^n_F (k_p, k_p) + W^n_F (k_n, k_n)\right\},
\]

\[
\mathcal{E}_{FX_n} = \frac{1}{(2\pi)^6 \rho} \cdot \left(2t^{(M)}_n + t^{(H)}_n\right) \left\{W^n_F (k_p, k_p)\right\}.
\]

2.3. Decomposition of symmetry energy

The symmetry energy \(a_t (\rho)\) is defined by the second-order derivative of \(\mathcal{E}\) with respect to the asymmetry \(\eta_t\),

\[
a_t (\rho) := \frac{1}{2} \frac{\partial^2 \mathcal{E}}{\partial \eta_t^2} \bigg|_{\eta_t = 0}.
\]

Let us denote the \(\nu\)-th order partial derivative of a function \(\mathcal{F}\) with respect to \(\eta_t\) by \(\mathcal{F}^{(\nu)} = \left(\partial^{(\nu)} / \partial \eta_t^{(\nu)}\right) \mathcal{F}\). Corresponding to the decomposition of \(\mathcal{E}\), we also decompose the symmetry
energy as follows,
\[
\begin{align*}
    a_t(\rho) &= \frac{1}{2} \mathcal{E}^{(2)} \bigg|_{\eta_t=0} = a_K + a_{DD} + \sum_n \left( a_{HO_n} + a_{HX_n} + a_{FO_n} + a_{FX_n} \right); \\
    a_i &= \frac{1}{2} \mathcal{E}_i^{(2)} \bigg|_{\eta_t=0} (i = K, DD, HO_n, HX_n, FO_n, FX_n).
\end{align*}
\]
(12)

Formulas for calculating each \( a_i \) are given in Appendix B. Note that \( W_H^H(k_p, k_p)^{(\nu)} \big|_{k_p=k_F} \neq W_H^H(k_p, k_n)^{(\nu)} \big|_{k_p=k_n=k_F} \).

2.4. Approximation of symmetry energy and its error

The symmetry energy \( a_t(\rho) \) is often approximated by the difference of \( \mathcal{E} \) between \( \eta_t = 1 \) and \( \eta_t = 0 \),
\[
\tilde{a}_t(\rho) := \mathcal{E}(\rho, \eta_t = 1) - \mathcal{E}(\rho, \eta_t = 0)
= a_t(\rho) + \sum_{\nu=2}^{\infty} \frac{\mathcal{E}(\rho, \eta_t)^{(2\nu)}}{(2\nu)!} \bigg|_{\eta=0}.
\]
(13)

This coincides with the quadratic approximation of \( \mathcal{E} \) with respect to \( \eta_t \),
\[
\mathcal{E}(\rho, \eta_t) \approx \mathcal{E}(\rho, \eta_t = 0) + a_t(\rho) \eta_t^2.
\]
(14)

Notice that \( \mathcal{E} \) is an even function of \( \eta_t \) under the charge symmetry.

We are now ready to consider accuracy of the approximation of Eq. (13). As measures of the approximation, we will estimate the absolute error \( \delta_t = \tilde{a}_t - a_t \) and the relative error \( \Delta_t = (\tilde{a}_t - a_t)/a_t \). In correspondence to each term of Eq. (12) \( (a_i(\rho); i = K, DD, HO_n, HX_n, FO_n, FX_n) \), we define its approximated value \( \tilde{a}_i(\rho) \) by
\[
\tilde{a}_i(\rho) := \mathcal{E}_i(\rho, \eta_t = 1) - \mathcal{E}_i(\rho, \eta_t = 0).
\]
(15)

Then the relative error of each term \( \Delta_i = (\tilde{a}_i - a_i)/a_i \) can be estimated analytically, as shown below. The full errors \( \delta_t \) and \( \Delta_t \) are expressed by using \( \Delta_i \),
\[
\begin{align*}
    \delta_t &= \tilde{a}_t - a_t = \sum_i \Delta_i a_i, \\
    \Delta_t &= \frac{\tilde{a}_t - a_t}{a_t} = \sum_i \Delta_i \frac{a_i}{a_t}.
\end{align*}
\]
(16)

Thus, through \( \Delta_i \), the error of the symmetry energy can be examined term by term. We shall estimate \( \Delta_i \) in the following three sections.

3. Terms with constant relative errors

The relative errors \( (\Delta_i) \) for the kinetic term, the density-dependent term, and the Hartree term of central force are constant, or even vanish, independent of the density. In this section we discuss \( \Delta_i \) for these terms.

3.1. Kinetic term

We first discuss the kinetic energy term in the exact and the approximated symmetry energy, \( a_K \) and \( \tilde{a}_K \). The \( 2\nu \)-th order coefficient of kinetic energy per nucleon \( (1 \leq \nu) \) can be derived
This is because for the density-dependent contact term, we have

\[
\Delta K = \frac{\tilde{a}_K - a_K}{a_K} = \sum_{\nu=2}^{\infty} \left. \left( \frac{\mathcal{E}_K^{(2\nu)}}{\mathcal{E}_K^{(2)}} \right) \right|_{\eta=0}^{\nu=0} = \sum_{\nu=2}^{\infty} \prod_{i=1}^{\nu-1} \frac{(6i - 5)(6i - 2)}{(6i + 3)(6i + 6)} = 0.057. \tag{18}
\]

In Eq. (6) of Ref.[6], the result equivalent to Eq. (18) is presented, \((\tilde{a}_K - a_K)/\tilde{a}_K = \Delta_K/(1 + \Delta_K) = 0.054.

### 3.2. Density-dependent term

For the density-dependent contact term, we have

\[
\tilde{a}_{DD} = a_{DD}, \quad \Delta_{DD} = 0. \tag{19}
\]

This is because \(\mathcal{E}_{DD}\) can be written up to the quadratic term with respect to \(\eta_t\),

\[
\mathcal{E}_{DD}(\rho, \eta_t) = \frac{1}{8} \left[ \alpha(TE) \rho^{\alpha+1}(3 + \eta_t^2) + 3\beta(TE) \rho^{\beta+1}(1 - \eta_t^2) \right]. \tag{20}
\]

### 3.3. Hartree-term

The Hartree term of the central force in the energy per nucleon also depends quadratically on \(\eta_t\). In practice, Eq. (7) yields

\[
\begin{align*}
\mathcal{W}_n^H(k, k) &= 4\pi^6 \rho^2 (1 - \eta_t^2)^2 \tilde{f}_n(0), \\
\mathcal{W}_n^H(k, k) &= 4\pi^6 \rho^2 (1 + \eta_t^2) \tilde{f}_n(0), \\
\mathcal{W}_n^H(k, k) &= 4\pi^6 \rho^2 (1 - \eta_t^2) \tilde{f}_n(0),
\end{align*}
\tag{21}
\]

leading to \(\Delta_{HOn} = \Delta_{HXn} = 0.

### 4. Skyrme interaction

We next discuss the terms depending on the function types of the interaction, through which full expression of errors of the symmetry energy is obtained.

The Skyrme interaction has momentum-dependent terms, additional to Eq. (3), while some of the terms in Eq. (5) become equivalent to exchange terms of others. The interaction is expressed, instead of Eq. (3),

\[
v_{12} = t_0 (1 + x_0 P_\sigma) \delta(r_{12})
\]

\[
+ \frac{1}{2} t_1 (1 + x_1 P_\sigma) \left[ P_{12}^2 \delta(r_{12}) + \delta(r_{12}) P_{12}^2 \right] + t_2 (1 + x_2 P_\sigma) P_{12}' \cdot \delta(r_{12}) P_{12}
\]

\[
+ \frac{1}{2} t_3 (1 + x_3 P_\sigma) \left[ P_{12}^3 \delta(r_{12}) + \delta(r_{12}) P_{12}^3 \right] + t_4 (1 + x_4 P_\sigma) P_{12}' \cdot \delta(r_{12}) P_{12}
\]

Here \(P_{12} = (\nabla_1 - \nabla_2)/(2i)\), and \(P_{12}'\) is the hermitian conjugate of \(P_{12}\) acting on the left. For the \(t_0\) term that has \(f(r_{12}) = \delta(r_{12})\), we have

\[
\mathcal{W}_0^H(k, k) = \mathcal{W}_0^E(k, k) = \frac{16\pi^2}{9} k_0^3 k_r^3.
\tag{23}
\]
For the $t_1$ and $t_2$ terms, $\frac{1}{2}(p_{12}^2 \delta (r_{12}) + \delta (r_{12}) p_{12}^2)$ and $p_{12} \cdot \delta (r_{12}) p_{12}$ yield [12]

$$\mathcal{W}_1^H(k_r, k_r') = \mathcal{W}_1^F(k_r, k_r') = \frac{4 \pi^2}{15} k_r^3 k_r'(k_r^2 + k_r'^2) \quad (t_1 \text{ term}),$$

$$\mathcal{W}_2^H(k_r, k_r') = -\mathcal{W}_2^F(k_r, k_r') = \frac{4 \pi^2}{15} k_r^3 k_r'(k_r^2 + k_r'^2) \quad (t_2 \text{ term}),$$

respectively.

Let us define

$$\mathcal{W}_c^o = \frac{16 \pi^2}{9} (k_p^6 + k_n^6), \quad \mathcal{W}_c^x = \frac{16 \pi^2}{9} k_p^3 k_n^3,$$

$$\mathcal{W}_p^o = \frac{8 \pi^2}{15} (k_p^8 + k_n^8), \quad \mathcal{W}_p^x = \frac{8 \pi^2}{15} k_p^3 k_n^3 (k_p^2 + k_n^2),$$

where the superscript $o$ ($x$) indicates like- (unlike-) nucleon contribution and the subscript $c$ ($p$) corresponds to the $t_0$ ($t_1$ or $t_2$) term. Then the energy per nucleon $\mathcal{E}$ is decomposed as follows.

$$\mathcal{E} = \mathcal{E}_K + \mathcal{E}_c + \mathcal{E}_{pO} + \mathcal{E}_{pX} + \mathcal{E}_{DD};$$

$$\mathcal{E}_c = \frac{1}{(2\pi)^6 \rho} \left( t_c^o \mathcal{W}_c^o + t_c^x \mathcal{W}_c^x \right),$$

$$\mathcal{E}_{pO} = \frac{1}{(2\pi)^6 \rho} t_p^o \mathcal{W}_p^o, \quad \mathcal{E}_{pX} = \frac{1}{(2\pi)^6 \rho} t_p^x \mathcal{W}_p^x,$$

where

$$t_c^o = t_0(1 - x_0), \quad t_c^x = t_0(2 + x_0),$$

$$t_p^o = t_1(1 - x_1) + 3t_2(1 + x_2), \quad t_p^x = t_1(2 + x_1) + t_2(2 + x_2).$$

Corresponding to the above decomposition, the symmetry energy and its error are decomposed into $a_i$ and $\Delta_i$ ($i = K, c, pO, pX, DD$). $\delta_i$ and $\Delta_i$ are expressed by

$$\delta_i = \tilde{a}_i - a_i = \Delta_K a_K + \Delta_p a_p + \Delta_p a_pX ,$$

$$\Delta_i = \tilde{a}_i - a_i = \Delta_K \frac{a_K}{a_t} + \Delta_p \frac{a_pO}{a_t} + \Delta_p \frac{a_pX}{a_t}.$$

We have shown, in the previous section, $\Delta_K = 0.057$ and $\Delta_{DD} = 0$. Moreover, we obviously have $\Delta_c = 0$. For the momentum-dependent terms, we consider the ratios $\tilde{a}_{pO}/a_{pO} = 1 + \Delta_{pO}$ and $\tilde{a}_{pX}/a_{pX} = 1 + \Delta_{pX}$, where $a_{pO} = \frac{1}{2} E_{pO}(2)_{\eta_t=0}$ and $a_{pX} = \frac{1}{2} E_{pX}(2)_{\eta_t=0}$. Since

$$\mathcal{W}_{pO}^{(2\nu)}(\rho, \eta_t) \bigg|_{\eta_t=0} = \frac{16 \pi^2}{15} k_F^{2\nu} \prod_{i=1}^{2\nu} \frac{11 - 3i}{3},$$

$$\mathcal{W}_{pX}^{(2\nu)}(\rho, \eta_t) \bigg|_{\eta_t=0} = \frac{16 \pi^2}{15} k_F^{2\nu} (3\nu - 2) \prod_{i=1}^{2\nu} \frac{11 - 3i}{3},$$

the ratios are

$$\tilde{a}_{pO}/a_{pO} = \frac{\sum_{\nu=1}^{\infty} E_{pO}^{(2\nu)}/(2\nu)!}{E_{pO}^{(2)} / 2} \bigg|_{\eta_t=0} = \frac{9}{20} \sum_{\nu=1}^{\infty} \left[ \frac{1}{3^{2\nu}(2\nu)!} \prod_{i=1}^{2\nu} (11 - 3i) \right] = 0.979,$$

$$\tilde{a}_{pX}/a_{pX} = \frac{\sum_{\nu=1}^{\infty} E_{pX}^{(2\nu)}/(2\nu)!}{E_{pX}^{(2)} / 2} \bigg|_{\eta_t=0} = \frac{9}{20} \sum_{\nu=1}^{\infty} \left[ \frac{3\nu - 2}{3^{2\nu}(2\nu)!} \prod_{i=1}^{2\nu} (11 - 3i) \right] = 0.900.$$
Therefore $\delta_t$ and $\Delta_t$ for the Skyrme interaction is calculated by the following formulas,

$$\delta_t = \tilde{a}_t - a_t = 0.057a_K - 0.021a_{pO} - 0.100a_{pX},$$

$$\Delta_t = \frac{\tilde{a}_t - a_t}{a_t} = 0.057a_K - 0.021a_{pO} - 0.100a_{pX}. \quad (31)$$

5. Finite-range interactions

In this section, errors of each term in the symmetry energy are analytically investigated for finite-range effective interactions. The Gauss function $f_n(r_{12}) = e^{-(\mu_n r_{12})^2}$ and the Yukawa function $f_n(r_{12}) = e^{-\mu_n r_{12}}/\mu_n r_{12}$ are handled in practice, which are used in the central channels of the Gogny and M3Y-type interactions.

As we showed in Sect. 3, the density-dependent term and the Hartree term give no errors. Then the errors of the approximated symmetry energy can be expressed as follows,

$$\delta_t = \tilde{a}_t - a_t = \Delta_K a_K + \sum_n \Delta_{FOn} a_{FOn} + \sum_n \Delta_{FXn} a_{FXn},$$

$$\Delta_t = \frac{\tilde{a}_t - a_t}{a_t} = \Delta_K a_K + \sum_n \Delta_{FOn} a_{FOn} + \sum_n \Delta_{FXn} a_{FXn} \quad (32)$$

It is recalled that $\Delta_K$ is constant. Unlike the Skyrme interaction, the relative errors of Fock terms in the symmetry energy depend on the density. However, for a given function form, each of $\Delta_{FOn}$ and $\Delta_{FXn}$ depends only on $k_F/\mu_n$, where $1/\mu_n$ is the range parameter. Therefore, we calculate the relative errors $\Delta_{FOn}$ and $\Delta_{FXn}$ as functions of $k_F/\mu_n$. We depict the results in Fig. 1 for the Gauss and the Yukawa functions, whose analytic expressions are given in Appendix C. It is emphasized that these results are determined only by the function type of the effective interaction, independent of parameters. As $k_F/\mu_n$ increases, $|\Delta_{FOn}|$ and $|\Delta_{FXn}|$ tend to deviate from zero. The longest range of the nucleon-nucleon interaction should be given by the Compton wavelength of the pion (1.414 fm). Corresponding to the range $0 \leq 1/\mu_n \leq 1.414$ fm and the density $0 \leq \rho \leq 0.64$ fm$^{-3}$ (i.e., $0 \leq k_F \leq 2.11$ fm$^{-1}$), whose upper bound is about four times of the normal density, $\Delta_{FOn}$ and $\Delta_{FXn}$ are displayed for $0 \leq k_F/\mu_n \leq 3$.

The dashed vertical line in Fig. 1(a) means $k_F/\mu_n$ for $\rho = 0.32$ fm$^{-3}$ and $1/\mu_n = 1.2$ fm, which is the longer range in D1S. We find that $|\Delta_{FXn}|$ may reach about 0.25 at $\rho = 0.32$ fm$^{-3}$, while $|\Delta_{FOn}|$ is kept within 0.05 in the full range in Fig. 1(a).

As in Fig. 1(a), the dashed line in Fig. 1(b) means $k_F/\mu_n$ for $\rho = 0.32$ fm$^{-3}$ and the longest range of the M3Y-type interaction 1.414 fm$^{-1}$. In contrast to the Gauss function, $|\Delta_{FOn}|$ and $|\Delta_{FXn}|$ stay within 0.1 in $0 \leq \rho \leq 0.32$ fm$^{-3}$ for the Yukawa interaction. Even in the higher density region up to $k_F/\mu_n = 3$, they are 0.13 at largest.

6. Numerical results for several interactions

In Fig. 2, we show the numerical results of $\Delta_t$ and $\delta_t$ as a function of $\rho$, which are calculated by inputting the values of the parameters given in Appendix A. Figure 2(a) reveals interaction-dependence of $\Delta_t$. In $0 < \rho \leq 0.32$ fm$^{-3}$, we find that the relative error is $|\Delta_t| < 0.1$ for all of the effective interactions. To be more precise, it is within $|\Delta_t| < 0.05$ except SkM*. In the higher density region of $0.32 \leq \rho < 0.64$ fm$^{-3}$, $|\Delta_t| < 0.05$ holds for SLy4 and the M3Y-type interactions. In contrast, $|\Delta_t|$ exceeds 0.20 for SkM* and the Gogny interactions. This
Fig. 1: $k_F/\mu_n$ dependence of $\Delta_{\text{FO}_n}$ and $\Delta_{\text{FX}_n}$, for (a) the Gauss interaction and (b) the Yukawa interaction. The dashed lines are explained in the text.

Fig. 2: Errors for the symmetry energy (a) $\Delta_t$ and (b) $\delta_t$ for several effective interactions.

tendency is also seen for $\delta_t$ in Fig. 2(b). $|\delta_t|$ of M3Y-P6, M3Y-P7 and SLy4 are significantly smaller than that of D1M, D1S and SkM$^*$. Interestingly, although SkM$^*$ and SLy4 have the same function type, $\delta_t$ behaves quite differently between them. Origin of this difference may be accounted for on the basis of the decomposition in Sect. 4, especially by Eq. (31). In Fig. 3, $a_K(\rho)$, $a_{\text{pO}}(\rho)$ and $a_{\text{pX}}(\rho)$ for SkM$^*$ and SLy4 are presented. As recognized from Eq. (31) and Fig. 3, $\Delta_{\text{pX}}a_{\text{pX}} = -0.1a_{\text{pX}}$ contributes to $\delta_t$ positively. The contribution ($-a_{\text{pX}}$) is larger for SkM$^*$ than for SLy4. Moreover, $\Delta_{\text{pO}}a_{\text{pO}} = -0.021a_{\text{pO}}$ contributes negatively and it tends to cancel the positive contribution of $a_{\text{pX}}$ for SLy4, while $a_{\text{pO}}$ almost vanishes for SkM$^*$.
Figure 1 accounts for the difference of $\delta_t$ between the Gogny and M3Y-type interactions. The large $|\Delta_{Fx}|$ for the Gauss function at high $k_F/\mu_n$ makes $|\delta_t|$ large. Note that $\Delta_t$ in D1S diverges at $\rho = 0.55\text{fm}^{-3}$ as shown in Fig. 2(a), because $a_t$ reaches 0.

7. Summary

We have investigated accuracy of approximation of the symmetry energy in the nuclear matter within the Hartree-Fock theory. As measures of the accuracy of $\tilde{a}_t(\rho)$ (difference of the neutron-matter energy from the symmetric-matter energy) relative to $a_t(\rho)$ (second derivative of the energy with respect to the asymmetry $\eta_t$), we have estimated the absolute error $\delta_t$ and the relative error $\Delta_t$. The errors are decomposed into several terms, in association with the decomposition of the nuclear-matter energy in Eq. (10). With this decomposition, $\delta_t$ can be expressed in terms of $a_i$ and $\Delta_i$, where $i$ represents individual components. We derive analytical expressions for each of $\Delta_i$, which makes origin of $\delta_t$ and $\Delta_t$ transparent. On this basis, $\delta_t$ and $\Delta_t$ are estimated for the six effective interactions by inserting values of the parameters. We find $-2\text{MeV} \lesssim \delta_t \lesssim 4\text{MeV}$, which means $|\Delta_t| < 0.1$, up to twice of the normal density for all of the effective interactions. At higher densities, they depend much on the effective interactions. The errors is kept small for the two M3Y-type interactions and SLy4, while the accuracy is worse for the two Gogny interactions and SkM*. We have explained this tendency from $\Delta_i$, i.e., the relative errors of individual terms in the symmetry energy.

It is recalled that SLy4, D1M, M3Y-P6 and M3Y-P7 are the parameter-sets which are fitted to the microscopic calculation of the equation-of-state of the neutron matter. With an exception of D1M, this constraint might help keeping the parameter-sets to have small $\delta_t$ and $\Delta_t$ even at high density. For the Skyrme interactions, all of $\Delta_i$ are constant, and $\delta_t$ can be written by only three terms, the kinetic term $a_K$, the momentum-dependent term between like nucleons $a_{pO}$, and that between unlike nucleons $a_{pX}$. Then, we have found that the parameters of the momentum-dependent terms $t^c_p$ and $t^x_p$ make the different behavior of $\delta_t$ between SkM* and SLy4. For the finite-range interactions, $\delta_t$ can be expressed by
the kinetic term $a_K$, the Fock terms between like nucleons $a_{\text{FO}n}$ and that between unlike nucleons $a_{\text{FX}n}$. For given function types, the coefficients of the Fock terms $\Delta_{\text{FO}n}$ and $\Delta_{\text{FX}n}$ are single-variable functions of $k_F/\mu_n$, which are independent of the parameters. As shown in Fig. 1, $\Delta_{\text{FX}n}$ produce significant difference of $\delta_t$ and $\Delta_t$ between the Gauss and the Yukawa functions at high density. This suggests that the function type plays a certain role in the accuracy of approximation.

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**A. Parameters of interactions**

The parameters of the effective interactions used in this paper, SkM* [7], SLy4 [8], D1S [9], D1M [10], M3Y-P6 and M3Y-P7 [11], are tabulated in Tables A1, A2 and A3.

Table A1: Parameters of SLy4 & SkM*.

| parameters | SLy4  | SkM*  |
|------------|-------|-------|
| $t_0$      | (MeV fm$^3$) | -2488.91 | -2645.0 |
| $t_1$      | (MeV fm$^5$) | 483.13  | 410.0   |
| $t_2$      | (MeV fm$^5$) | -549.40 | -135.0  |
| $x_0$      |          | 0.778  | 0.090   |
| $x_1$      |          | 0.328  | 0.000   |
| $x_2$      |          | -1.000 | 0.000   |
| $t_0^c$    | (MeV fm$^3$) | -2406.95 | -382.291 |
| $t_1^c$    | (MeV fm$^3$) | -5528.05 | -7064.94 |
| $t_2^c$    | (MeV fm$^5$) | 324.663 | 5.000   |
| $t_0^p$    | (MeV fm$^5$) | 575.327 | 550.00  |
| $t_1^p$    | (MeV fm$^5$) | -608.49 | 2599.17 |
| $t_2^p$    | (MeV fm$^5$) | 5166.12 | 2599.17 |
| $\alpha$   |          | 1/6    | 1/6     |
| $\beta$    |          | 1/6    | 1/6     |

**B. Formulas for symmetry energy**

We give some formulas for $a_i$, which is defined in Eq. (12), in this Appendix.

The second-order derivatives with respect to the asymmetry of the kinetic-energy term and the density-dependent term are,

$$
\xi_{\text{K}}^{(2)} = \frac{\partial^2 \xi_{\text{K}}}{\partial \eta_t^2} = \frac{\pi^2 \rho}{4M} (k_p^{-1} + k_n^{-1}),
$$

$$
\xi_{\text{DD}}^{(2)} = \frac{\partial^2 \xi_{\text{DD}}}{\partial \eta_t^2} = \frac{1}{8} \left( t_{\rho}^{\text{(SE)}} \rho^{\alpha+1} - 3 t_{\rho}^{\text{(TE)}} \rho^{\beta+1} \right). \tag{B1}
$$
Table A2: Parameter sets of D1S & D1M.

| parameters     | D1S    | D1M    |
|----------------|--------|--------|
| $\mu_1^{-1}$   | (fm)   | 0.7    | 0.5    |
| $\mu_2^{-1}$   | (fm)   | 1.2    | 1.0    |
| $t_1^{(W)}$    | (MeV)  | −1720.3| −12797.57|
| $t_1^{(B)}$    | (MeV)  | 1300.0 | 14048.85 |
| $t_1^{(H)}$    | (MeV)  | −1813.53| −15144.43|
| $t_1^{(M)}$    | (MeV)  | 1397.6 | 11963.89 |
| $t_2^{(W)}$    | (MeV)  | 103.64 | 490.95 |
| $t_2^{(B)}$    | (MeV)  | −163.48| −752.27 |
| $t_2^{(H)}$    | (MeV)  | 162.81 | 675.12 |
| $t_2^{(M)}$    | (MeV)  | −223.93| −693.57 |
| $t_1^{(SE)}$   | (MeV fm$^{3(1+\alpha)}$) | 0    | 0    |
| $t_1^{(TE)}$   | (MeV fm$^{3(1+\beta)}$) | 2781.2 | 3124.44 |
| $\alpha$       | —      | —      | 1/3    | 1/3    |
| $\beta$        | 1/3    | 1/3    | 1/3    | 1/3    |

Table A3: Parameter sets of M3Y-P6 & M3Y-P7.

| Parameters     | M3Y-P6 | M3Y-P7 |
|----------------|--------|--------|
| $\mu_1^{-1}$   | (fm)   | 0.25   | 0.25   |
| $\mu_2^{-1}$   | (fm)   | 0.40   | 0.40   |
| $\mu_3^{-1}$   | (fm)   | 1.414  | 1.414  |
| $t_1^{(SE)}$   | (MeV)  | 10766  | 10655  |
| $t_1^{(TE)}$   | (MeV)  | 8474   | 9592   |
| $t_1^{(SO)}$   | (MeV)  | −728   | 11510  |
| $t_1^{(TO)}$   | (MeV)  | 12453  | 13507  |
| $t_2^{(SE)}$   | (MeV)  | −3520  | −3556  |
| $t_2^{(TE)}$   | (MeV)  | −4594  | −4594  |
| $t_2^{(SO)}$   | (MeV)  | 1386   | 1283   |
| $t_2^{(TO)}$   | (MeV)  | −1588  | −1812  |
| $t_3^{(SE)}$   | (MeV)  | −10.463| −10.463|
| $t_3^{(TE)}$   | (MeV)  | −10.463| −10.463|
| $t_3^{(SO)}$   | (MeV)  | 31.389 | 31.389 |
| $t_3^{(TO)}$   | (MeV)  | 3.488  | 3.488  |
| $t_1^{(SE)}$   | (MeV fm$^{3(1+\alpha)}$) | 384 | 830 |
| $t_1^{(TE)}$   | (MeV fm$^{3(1+\beta)}$) | 1930 | 1478 |
| $\alpha$       | —      | —      | 1      |
| $\beta$        | 1/3    | 1/3    | 1/3    |
The second-order derivatives of the $W$ functions are represented as follows,

\[
W_n^H(k_p, k_p)^{(2)} + W_n^H(k_n, k_n)^{(2)} = 16\pi^6 \rho^2 \tilde{f}_n(0),
\]

\[
W_n^H(k_p, k_n)^{(2)} = -8\pi^6 \rho^2 \tilde{f}_n(0),
\]

\[
W_n^F(k_p, k_p)^{(2)} + W_n^F(k_n, k_n)^{(2)} = 32\pi^6 \rho^2 \left[ k_p^{-4} \int_0^{k_p} dk k^3 \tilde{f}_n(2k) + k_n^{-4} \int_0^{k_n} dk k^3 \tilde{f}_n(2k) \right],
\]

\[
W_n^F(k_p, k_n)^{(2)} = 4\pi^6 \rho^2 \int_{(k_p-k_n)/2}^{(k_p+k_n)/2} dk \\
\times \left[ -\{(k_p^{-4} + k_n^{-4})(k_p^2 + k_n^2) + 4k_p^{-1}k_n^{-1}\}k + 4(k_p^{-4} - k_n^{-4})k^3 \right] \tilde{f}_n(2k).
\]

Each term of Eq. (12) is given by

\[
a_K = \frac{\pi^2 \rho}{4M k_F^{-1}},
\]

\[
a_{DD} = \frac{\rho}{16} (t_p^{(SE)} \rho^{a+1} - 3t_p^{(TE)} \rho^{b+1}),
\]

\[
a_{HOn} = \left. \frac{1}{2(2\pi)^6} (2t_n^{(W)} + t_n^{(B)} - 2t_n^{(H)} - t_n^{(M)}) \left\{ W_n^H(k_p, k_p)^{(2)} + W_n^H(k_n, k_n)^{(2)} \right\} \right|_{\eta=0}
\]

\[
= \frac{\rho}{8} (2t_n^{(W)} + t_n^{(B)} - 2t_n^{(H)} - t_n^{(M)}) \tilde{f}_n(0),
\]

\[
a_{HXn} = \left. \frac{1}{(2\pi)^6} (2t_n^{(M)} + t_n^{(H)} - 2t_n^{(B)} - t_n^{(W)}) \left\{ W_n^H(k_p, k_p)^{(2)} + W_n^H(k_n, k_n)^{(2)} \right\} \right|_{\eta=0}
\]

\[
= -\frac{\rho}{8} (2t_n^{(M)} + t_n^{(H)} - 2t_n^{(B)} - t_n^{(W)}) \tilde{f}_n(0),
\]

\[
a_{FOn} = \left. \frac{1}{2(2\pi)^6} (2t_n^{(M)} + t_n^{(H)} - 2t_n^{(B)} - t_n^{(W)}) \left\{ W_n^F(k_p, k_p)^{(2)} + W_n^F(k_n, k_n)^{(2)} \right\} \right|_{\eta=0}
\]

\[
= \frac{\rho}{2} (2t_n^{(M)} + t_n^{(H)} - 2t_n^{(B)} - t_n^{(W)}) k_F^{-4} \int_0^{k_p} dk k^3 \tilde{f}_n(2k)
\]

\[
a_{FXn} = \left. \frac{1}{(2\pi)^6} (2t_n^{(M)} + t_n^{(H)}) \left\{ W_n^F(k_p, k_p)^{(2)} \right\} \right|_{\eta=0}
\]

\[
= \frac{\rho}{2} (2t_n^{(M)} + t_n^{(H)}) k_F^{-4} \int_0^{k_p} dk \left( -k_F^2 k + k^3 \right) \tilde{f}_n(2k).
\]

C. Explicit expressions of $\Delta_{FOn}$ and $\Delta_{FXn}$

If the function form is specified, $\Delta_{FOn}$ and $\Delta_{FXn}$ can be expressed in more explicit manner. We here denote $k_F/\mu_n$ by $x$. Then $\Delta_{FOn}$ and $\Delta_{FXn}$ become functions only of $x$ for a given function form.

For the Gauss function $f(x) = e^{-(\mu \nu)^2}$, whose Fourier transform is $\tilde{f}(2k) = (\sqrt{\pi}/\mu)^3 e^{-k(\mu \nu)^2}$, we obtain

\[
\Delta_{FOn} = \frac{3}{x^2} \cdot \frac{1}{1 - e^{-x^2} - x^2 e^{-x^2}} \left[-1 + 3(1 - 2^{-1/3})x^2 + (2 - x^2) e^{-x^2} - (1 - 2^{-1/3}) e^{-2x^2} + \sqrt{\pi} x^3 \left( \text{erf}(2^{1/3} x) - \text{erf}(x) \right) \right] - 1,
\]

\[
\Delta_{FXn} = \frac{3}{x^2} \cdot \frac{2 - 3x^2 - 2e^{-x^2} + x^2 e^{-x^2} + \sqrt{\pi} x^3 \text{erf}(x)}{-1 + x^2 + e^{-x^2}} - 1,
\]

where $\text{erf}(x) = (2/\sqrt{\pi}) \int_0^x e^{-t^2} dt$. 

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For the Yukawa function \( f(r) = e^{-\mu r}/\mu r \), whose Fourier transform is \( \hat{f}(2k) = 4\pi/\mu(\mu^2 + 4k^2) \), \( \Delta_{\text{FOn}} \) and \( \Delta_{\text{FXn}} \) are

\[
\Delta_{\text{FOn}} = \frac{3}{2x^2} \cdot \frac{1}{4x^2 - \log (1 + 4x^2)} \left[ 2(1 - 2^{-1/3})x^2 + 12(2^{1/3} - 1)x^4 
+ \frac{1}{4} \log \frac{1 + 4 \cdot 2^{2/3}x^2}{(1 + 4x^2)^2} + 3 \cdot 2^{2/3}x^2 \log(1 + 4 \cdot 2^{2/3}x^2) - 6x^2 \log(1 + 4x^2) 
- 16x^3 \{ \arctan(2^{4/3}x) - \arctan(2x) \} \right] - 1,
\]

\[
\Delta_{\text{FXn}} = \frac{3}{4x^2} \cdot \frac{4x^2(1 - 6x^2) + 32x^3 \arctan(2x) - (1 + 12x^2) \log(1 + 4x^2)}{4x^2 - (1 + 4x^2) \log(1 + 4x^2)} - 1.
\]

(C2)

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