RELATIVE DERIVED DIMENSIONS FOR COTILTING MODULES

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Abstract. For a Noetherian ring $R$ and a cotilting $R$-module $T$ of injective dimension at least 1, we prove that the derived dimension of $R$ with respect to the category $\mathcal{X}_T$ is precisely the injective dimension of $T$ by applying Auslander-Buchweitz theory and Ghost Lemma. In particular, when $R$ is a commutative Noetherian local ring with a canonical module $\omega_R$ and $\dim R \geq 1$, the derived dimension of $R$ with respect to the category of maximal Cohen-Macaulay modules is precisely $\dim R$.

1. Introduction

The notion of dimension of a triangulated category was introduced by Rouquier [7] based on work of Bondal and Van den Bergh [4] on Brown representability. A relative version of this notion was introduced in [3], which counts how many extensions are needed to build the triangulated category out of a given subcategory. The aim of this paper is to give an explicit value of the relative dimension when the subcategory is associated with a cotilting module.

In this paper, we denote by $R$ a Noetherian ring. All $R$-modules are finitely generated right $R$-modules. We denote by mod $R$ the abelian category of $R$-modules and by $D^b (\text{mod } R)$ the derived category of mod $R$.

Then our main result is the following, which completes a main result Theorem 5.3 in [1].

Theorem 1.1. Let $R$ be a Noetherian ring and $T$ a cotilting $R$-module with $\text{inj.dim } T \geq 1$. Then we have an equality

$$\mathcal{X}_T - \text{tri. dim } D^b (\text{mod } R) = \text{inj.dim } T.$$

The inequality $\leq$ was shown in [1, Theorem 5.3]. In this paper, we will prove the converse inequality by applying Auslander-Buchweitz theory and Ghost Lemma.

We apply Theorem 1.1 to the following settings. For a commutative Noetherian local ring $R$ with a canonical module $\omega_R$, we denote by $\text{CM } R$ the category of maximal Cohen-Macaulay modules. We call an $R$-algebra $\Lambda$ an $R$-order if $\Lambda \in \text{CM } R$. We denote by $\text{CMA}$ the category of maximal Cohen-Macaulay $\Lambda$-modules (i.e. $\Lambda$-modules $X$ satisfying $X \in \text{CM } R$). As a special case of Theorem 1.1, we obtain the following results, which completes the inequalities (1.2.1) and (4.2.1) in [1].

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Corollary 1.2. Let $R$ be a commutative Noetherian local ring with a canonical module $\omega_R$ and $\dim R \geq 1$. Then

1. We have an equality
   $$(\text{CMR}) - \text{tri.dim } \mathcal{D}^b(\text{mod } R) = \dim R.$$

2. More generally, for an $R$-order $\Lambda$, we have an equality
   $$(\text{CMA}) - \text{tri.dim } \mathcal{D}^b(\text{mod } \Lambda) = \dim R.$$

Proof. Since $\omega_R$ (respectively, $\omega_\Lambda := \text{Hom}_R(\Lambda, \omega_R)$) is a cotilting module with injective dimension $\dim R$, the assertion follows from Theorem 1.1. \qed

Remark 1.3. [1, Remark 5.4] If $\text{inj.dim } \omega_\Lambda = \dim R = 0$, then the equality in Corollary 1.2 (2) does not necessarily hold in general. Namely, let $\Lambda$ be a finite dimensional non-semisimple self-injective algebra over a field. Then the right $\Lambda$-module $\Lambda$ is a cotilting module with $\text{inj.dim } \Lambda = 0$ and $\mathcal{X}_\Lambda = \text{mod } \Lambda$. However, $\langle \text{mod } \Lambda \rangle$ is different from $\mathcal{D}^b(\text{mod } \Lambda)$.

2. Preliminaries

In this section, we will introduce several concepts.

Definition 2.1. (Aihara-Araya-Iyama-Takahashi-Y [1])

Let $\mathcal{T}$ be a triangulated category with shift $[1]$ and $\mathcal{X}, \mathcal{Y}$ full subcategories of $\mathcal{T}$.

1. The full subcategory $\mathcal{X} \ast \mathcal{Y}$ of $\mathcal{T}$ is defined as follows:
   $$\mathcal{X} \ast \mathcal{Y} := \{ M \in \mathcal{T} \mid \exists \text{ a triangle } : X \to M \to Y \to X[1] \text{ with } X \in \mathcal{X}, Y \in \mathcal{Y} \}.$$

   Note that $(\mathcal{X} \ast \mathcal{Y}) \ast \mathcal{Z} = \mathcal{X} \ast (\mathcal{Y} \ast \mathcal{Z})$ holds by the octahedral axiom.

2. Set $\langle \mathcal{X} \rangle := \text{add} \{ X[i] \mid X \in \mathcal{X}, i \in \mathbb{Z} \}$. And, for any positive integer $n$,
   $$\langle \mathcal{X} \rangle_n := \text{add} \left( \langle \mathcal{X} \rangle \ast \langle \mathcal{X} \rangle \ast \cdots \ast \langle \mathcal{X} \rangle \right).$$

   Clearly, $\langle \mathcal{X} \rangle_n$ is closed under shifts.

3. The dimension of $\mathcal{T}$ with respect to a subcategory $\mathcal{X}$ is defined as follows:
   $$\mathcal{X} - \text{tri.dim } \mathcal{T} := \inf \{ n \geq 0 \mid \mathcal{T} = \langle \mathcal{X} \rangle_{n+1} \}.$$

   When $\mathcal{X} = \text{add } M$ for some object $M \in \mathcal{T}$, one can recover the dimension of triangulated category in the sense of Rouquier [7].

The relative (derived) dimensions realize the several invariants for rings. For instance, we have the following fact, which was proved by Krause and Kussin.

Example 2.2. (Krause-Kussin [5, Lemma 2.4 and 2.5])

$$(\text{proj } R) - \text{tri.dim } \mathcal{D}^b(\text{mod } R) = \text{gl.dim } R,$$

where $\text{proj } R$ is the subcategory of $\text{mod } R$ consisting of projective modules and $\text{gl.dim } R$ is the global dimension of $R$. This is a special case of our main result.
For an $R$-module $T$, we define the full subcategory $\perp T$ of $\text{mod} \ R$ as follows:

$$\perp T := \{ X \in \text{mod} \ R \mid \text{Ext}_R^i(X, T) = 0 \text{ for any } i > 0 \}.$$ 

Then we will introduce the concept of a cotilting module.

**Definition 2.3.** An $R$-module $T$ is called **cotilting** if it satisfies the following three conditions:

1. The injective dimension $\text{inj.dim} \ T$ of $T$ is finite.
2. $T \in \perp T$.
3. For any $X \in \perp T$, there exists a short exact sequence

$$0 \to X \to T' \to X' \to 0$$

with $T' \in \text{add} \ T$, $X' \in \perp T$.

For a cotilting module $T$, we will write $\mathcal{X}_T$ instead of $\perp T$. Moreover, we set

$$\mathcal{Y}_T := (\mathcal{X}_T)^{\perp} = \{ Y \in \text{mod} \ R \mid \text{Ext}_R^i(X, Y) = 0 \text{ for any } i > 0 \text{ and any } X \in \mathcal{X}_T \}.$$ 

Then by Auslander-Buchweitz approximation theory \cite{3}, we have the following fact.

**Proposition 2.4.** Let $T$ be a cotilting $R$-module with injective dimension $d$ and $M \in \text{mod} \ R$. Then

1. there exists an exact sequence $0 \to Y \to X \to M \to 0$ with $X \in \mathcal{X}_T$ and $Y \in \mathcal{Y}_T$.
2. $M$ belongs to $\mathcal{Y}_T$ if and only if there exists an exact sequence

$$0 \to T_d \to \cdots \to T_1 \to T_0 \to M \to 0$$

with $T_i \in \text{add} \ T$.

A typical example of a cotilting module is the following.

**Example 2.5.** (1) The canonical module over a commutative Noetherian local ring $R$ is a cotilting $R$-module.
(2) For a finite dimensional algebra $R$ over a field $k$ and a tilting $R^{\text{op}}$-module $T$ in the sense of \cite{4}, the $k$-dual of $T$ is a cotilting $R$-module.

3. **Proof of our result**

We need the following to prove Theorem \ref{thm:main}, which is called the **Ghost Lemma**.

**Proposition 3.1.** \cite[Lemma 4.11]{7} Let $H_1, H_2, \cdots, H_{n+1}$ be cohomological functors on a triangulated category $\mathcal{T}$ and $f_i : H_i \to H_{i+1}$ morphisms between them. Let $\mathcal{X}_i$ be subcategories of $\mathcal{T}$ such that $f_i$ vanishes on $\mathcal{X}_i$ and $\mathcal{X}_i = \langle \mathcal{X}_i \rangle$. Then the composite $f_n \circ \cdots \circ f_1$ vanishes on $\text{add}(\mathcal{X}_1 \ast \cdots \ast \mathcal{X}_n)$.

Now we will prove Theorem \ref{thm:main}. 

Proof of Theorem 4.7. In the rest, we show the converse inequality.

Set $\text{inj.dim } T =: d \geq 1$. Then we have $\mathbb{D}^b(\text{mod } R)(M, T[d]) \cong \text{Ext}^d_R(M, T) \neq 0$ for some $M \in \text{mod } R$. We will identify $\mathbb{D}^b(\text{mod } R)(-, -[i])$ with $\text{Ext}^i_R(-, -)$ for any integer $i$ under the natural isomorphism. By Proposition 2.4, we have an exact sequence

$$[\xi_M: 0 \to T_d \xrightarrow{\phi_d} \cdots \xrightarrow{\phi_2} T_1 \xrightarrow{\phi_1} T_0 \xrightarrow{\phi_0} M \to 0] \in \text{Ext}^d_R(M, T),$$

where $T_0 \in \mathcal{X}_T$ and $T_i \in \text{add } T = \mathcal{X}_T \cap \mathcal{Y}_T$ for all $i = 1, \cdots, d$. Note that $\xi_M \neq 0$ since $\text{Ext}^d_R(M, T) \neq 0$. Put $K_i := \text{Im } \phi_i$ for each $i = 0, \cdots, d$. Then for any $i = 1, \cdots, d$, $K_i \in \mathcal{Y}_T$ and we have a short exact sequence

$$\xi_i: 0 \to K_i \to T_{i-1} \to K_{i-1} \to 0.$$ 

Regarding $\xi_M$ (respectively, $\xi_i$) as a morphism from $K_0 = M$ to $K_d[d] = T_d[d]$ (respectively, from $K_{i-1}[i-1]$ to $K_i[i]$) in $\mathbb{D}^b(\text{mod } R)$, we have an equality

$$\xi_M = \xi_d \circ \cdots \circ \xi_1.$$ 

For any $i = 1, \cdots, d$, the morphism $\xi_i$ induces a morphism

$$f_i: \mathbb{D}^b(\text{mod } R)(-, K_{i-1}[i-1]) \to \mathbb{D}^b(\text{mod } R)(-, K_i[i]).$$

Clearly $\mathbb{D}^b(\text{mod } R)(\mathcal{X}_T, K_i[j]) = 0$ holds if $(i, j)$ belongs to $\{0, \cdots, d-1\} \times \mathbb{Z}_{<0}$ or $\{1, \cdots, d\} \times \mathbb{Z}_{>0}$. Therefore $\mathbb{D}^b(\text{mod } R)(\mathcal{X}_T, \xi_i[j]) = 0$ holds for any $i = 1, \cdots, d$ and $j \in \mathbb{Z}$. Namely, $f_i$ vanishes on $\langle \mathcal{X}_T \rangle$. By Proposition 3.1, the composite $f_d \circ \cdots \circ f_1$ vanishes on $\langle \mathcal{X}_T \rangle_d$. But the composite $f_d \circ \cdots \circ f_1$ sends the identity morphism of $K_0 = M$ to $\xi_M = \xi_d \circ \cdots \circ \xi_1 \neq 0$ and hence $M \not\in \langle \mathcal{X}_T \rangle_d$. Namely, we have the inequality

$$\mathcal{X}_T\text{-tri.dim } \mathbb{D}^b(\text{mod } R) \geq \text{inj.dim } T,$$

as desired. \qed

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