Irreducible Multi-Qutrit Correlations in Greenberger-Horne-Zeilinger Type States

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I. INTRODUCTION

Coherent superposition is the essential distinction between a quantum system and a classical one. The distinction is more significant in composite systems, which appears in the non-classical correlations in the quantum systems. Many concepts have been presented to describe the correlations. For instance, the entanglement \cite{1} depicts the nonseparability of the state of a composite quantum system, and the nonlocality is characterized by violation of a Bell inequality \cite{2}, which means the local measurement outcomes of the state cannot be described by a local hidden variables model.

In the information based viewpoint, the correlation in a quantum system be viewed as the relationship of the whole state and its subsystem. Namely, it measures the degree a quantum state can be described by the reduced states of its subsystems. The total correlation \cite{3} in a multipartite quantum system is defined as the difference between the sum of the von Neumann entropies of all the subsystems and the von Neumann entropy of the whole system. The so called quantum Discord \cite{4–6}, widely studied in very recent years, is considered as the quantum part (opposite the classical one) of the total correlation, in a scheme to classify the correlations. In the current paper, we concerns us in another classification, in which the part (opposite the classical one) of the total correlation, is considered as the quantum system. The so called quantum Discord \cite{4–6}, widely studied in very recent years, is considered as the quantum part (opposite the classical one) of the total correlation, in a scheme to classify the correlations. In the current paper, we concerns us in another classification, in which the part (opposite the classical one) of the total correlation, is considered as the quantum system. The so called quantum Discord \cite{4–6}, widely studied in very recent years, is considered as the quantum part (opposite the classical one) of the total correlation, in a scheme to classify the correlations.

Linden \textit{et al.} \cite{7} proposed the concept of irreducible \(n\)-party correlation in an \(n\)-partite quantum state. This concept which is based on the principle of maximum entropy describes how much more information in the \(n\)-party level than what is contained in the \(n-1\)-partite reduced states. A surprising result was given in the original work of Linden and his collaborators \cite{7, 8} that almost all \(n\)-party pure states are be determined by their reduced density matrices. In \(n\)-qubit case, the only pure states undetermined by their reduced density matrices are proved to be the generalized Greenber-Horne-Zeilinger (GHZ) states \cite{9, 10}. This indicates among \(n\)-qubit pure states, only in the generalized GHZ states the irreducible \(n\)-party correlation has a nonzero value, which is derived explicitly in \cite{11}. For the arbitrary dimensions system, Feng \textit{et al.} \cite{12} introduced the generalized Schmidt decomposition (GSD) states and prove them to be the \(n\)-partite pure states undetermined \textit{among pure states} by their reduced density matrices. It is still an open question whether the GSD states identified in \cite{12} are precisely the pure states undetermined by their reduced density matrices \textit{among arbitrary states} (pure or mixed). In other words, it is under confirmation that, whether the irreducible \(n\)-party correlation in a \(n\)-partite non-GSD states is nonzero or not.

In Zhou’s recent work \cite{11}, he generalized the concept of irreducible \(n\)-party correlation to \(m\)-party (\(2 \leq m \leq n\)) levels, which construct a classification of the total correlation in an \(n\)-partite state. For a given \(n\)-partite quantum state \(\rho^{[n]}\), Zhou introduced a sequence of density matrices. The \(m\)-th one, \(\rho^{[m]}_m\) (\(1 \leq m \leq n\)), has the same \(m\)-partite reduced density matrix as \(\rho^{[n]}\), and maximizes the value of von Neumann entropy, which is considered to contain the \(m\)-party level information of the given state but without the higher level information. The irreducible \(m\)-party correlation is defined as

\[ C^{(m)}(\rho^{[n]}) = S(\tilde{\rho}^{[m-1]}_m) - S(\tilde{\rho}^{[m]}_m), \]

where \(S(\sigma) = -\text{Tr}(\sigma \ln \sigma)\) is the von Neumann entropy. For the states \(\rho^{[n]}\) with maximal \(m\)-partite, \(\rho^{[m]}_m\) is proved to have a standard exponential form

\[ \tilde{\rho}^{[m]}_m = \text{Exp}(Q^{[m]}), \]

where \(Q^{[m]}\) is a sum of \(m\)-partite hermitian operators.
Directly, $\rho_{m}^{[n]} = \rho^{[n]}$ is itself and $\rho_{i}^{[n]} = \otimes_{i=1}^{n} \rho^{(i)}$ is the direct product of all the single partite reduced density matrices. To derive the degrees of the irreducible multiparty correlations in quantum states with nonmaximal ranks, Zhou presented a continue approach based on the fact that a multipartite state without maximal rank can always be regarded as the limit of a series states with maximal rank, such $\rho_{m}^{[n]}(\gamma)_{\gamma \to +\infty} = \rho^{[n]}$. By constructing the sequence of density matrices $\tilde{\rho}_{m}^{[n]}(\gamma)$, one can obtain the irreducible $m$-party correlation of $\rho^{[n]}(\gamma)$

$$C^{(m)}(\rho^{[n]}(\gamma)) = S(\tilde{\rho}_{m-1}^{[n]}(\gamma)) - S(\tilde{\rho}_{m}^{[n]}(\gamma)).$$  \hspace{1cm} (3)

Then $C^{(m)}(\rho^{[n]}) = C^{(m)}(\rho^{[n]}(\gamma))_{\gamma \to +\infty}$ give the correlations of the state $\rho^{[n]}$.

In this approach, Zhou gave the results of the the $n$-qubit stabilizer states and generalized GHZ states. It is worth to note that, only irreducible 2-party and $n$-party correlations are nonzero in the $n$-qubit generalized GHZ states, which are the only pure states with nonzero $n$-qubit correlation [9, 10]. However, a systematic method to construct the standard exponential form density matrices in Eq. (2) for a given state with maximal rank has not been found. Consequently, it is also difficult to analytically obtain the correlation distributions in the states without maximal ranks, of interesting in many-particle physics or quantum information.

To the best of our knowledge, the known correlation distributions, both the analytical [11] and the numerical [13], in multipartite quantum states are restricted in $n$-qubit systems. The main purpose of this paper is to derive the irreducible multiparty correlations of some typical quantum states without the maximal rank in $n$-qutrit system. On one hand, these will offer the instances to analyze the difference between the correlation distributions in qubit systems and the ones in high-dimensional systems. And on the other hand, more analytical results will provide a basis to construct a systematic method to calculate the irreducible multiparty correlations, at least for a family of states.

Our main results are given in the second section. As the first trial, we concerns us in two families of $n$-qutrit GHZ-like states, in which the pure states belongs to the GSD states defined in [12]. The maximal slice (MS) states [14] can be viewed as another generalization of the original GHZ states. We introduce an $n$-qutrit version MS state, which is non-GSD according to the results in [12], and obtain its irreducible correlations. Based on these results, we give a discussion in the last section about the feasibility to solve the open problem about the GSD states mentioned above in the continuity approach.

## II. MULTI-QUTRIT CORRELATIONS

To derive the irreducible multiparty correlations in the three families of $n$-qutrit states studied in this paper, we adopt Zhou’s continuity approach with a little improvement. Namely, the results in the original work of Zhou [11] indicate that, the standard exponential form state $\tilde{\rho}_{m}^{[n]}(\gamma)$ contains the maximal von Neumann entropy among the states with the same $m$-party reduced density matrices, and this property is holden when its parameter $\gamma \to +\infty$. Therefore, for a given state, $\rho^{[n]}$, with nonmaximal rank, we don’t construct the series states $\rho^{[n]}(\gamma)$ and corresponding $\tilde{\rho}_{m}^{[n]}(\gamma)$, but construct a sequence of states $\sigma_{m}^{[n]}(\gamma_{m})$ in the standard exponential form (2) whose limit $\sigma_{m}^{[n]}|_{\gamma_{m} \to +\infty}$ has the same $m$-party reduced density matrices as $\rho^{[n]}$. Here, for different $m$, the states $\sigma_{m}^{[n]}(\gamma_{m})$ are independent, and the same are true of their parameters $\gamma_{m}$. Then the degree of the irreducible $m$-party correlation in the state $\rho^{[n]}$ is given by

$$C^{(m)}(\rho^{[n]}) = S(\sigma_{m-1}^{[n]}|_{\gamma_{m-1} \to +\infty}) - S(\sigma_{m}^{[n]}|_{\gamma_{m} \to +\infty}).$$  \hspace{1cm} (4)

### A. First GHZ-type states

We introduce the $n$-qutrit states in the subspace $\{{|0^{[n]}\rangle, |1^{[n]}\rangle, |2^{[n]}\rangle}\}$ as the first family of GHZ-type states, where $|i^{[n]}\rangle = |i_{1}...i_{n}\rangle$ with $i = 0, 1, 2$, denotes the direct product of the basis $|i\rangle$ for $n$ qutrits. They can be expressed as

$$\mathcal{G} = \sum_{i=0, j=0}^{2, 2} c_{ij} |i^{[n]}\rangle \langle j^{[n]}|,$$  \hspace{1cm} (5)

with $c_{ij} = c_{ij}^{\ast}$ and the positive real numbers $c_{ij}$ satisfying $\sum_{i,j} c_{ij} = 1$. One can write the diagonal elements in spherical coordinate as $(c_{00}, c_{11}, c_{22}) = (\sin^{2} \theta \cos^{2} \phi, \cos^{2} \theta, \sin^{2} \theta \sin^{2} \phi)$ with $\theta, \phi \in [0, \pi/4]$.

**Theorem 1.** The irreducible multiparty correlations of the $n$-qutrit GHZ-type state $\mathcal{G}$ in Eq. (5) are given by

$$C^{(2)} = (n-1) H_{3}(\theta, \phi),$$

$$C^{(n)} = H_{3}(\theta, \phi) - S(\mathcal{G}),$$  \hspace{1cm} (6)

and $C^{(m)} = 0$ for $m = 3, 4, ..., n-1$.

*Proof.\textit{ Let $Z_{j} = |0\rangle \langle 0| - |2\rangle \langle 2|$ be the spin-1 operator in $z$-axis of the $j$-th qutrit, the 2-partite operators defined as

$$Q_{ij} = \frac{2}{3}[1 + \cos \frac{2\pi}{3} (Z_{i} - Z_{j})]$$  \hspace{1cm} (7)

satisfies $Q_{ij}^{2} = Q_{ij}$ and $\text{Tr}_{i,j} Q_{ij} = 3$. We construct an*
the result returns to the \( \theta \) for the first \( m \) and \( g_{ij} \) under local unitary transformations, which belongs to the \( \tilde{G}_0 \) state \( |\tilde{G}_0\rangle \). The state \( \tilde{G}_0 \) has the same \( (n-1) \)-partite reduced matrices as the states \( G \), and there exists only irreducible 2-party correlation in \( \sigma_g(\gamma) \). Therefore, one can take \( \tilde{G}_m = \tilde{D}_g \) for \( m = 2, 3, \ldots n - 1 \) and obtain the results in Eq. (6).

The pure states in this family are always equivalent to the generalized GHZ states for \( n \)-qutrit system

\[
|G^n\rangle = \cos \theta |0^{[n]}\rangle + \sin \theta \cos \phi |1^{[n]}\rangle + \sin \theta \sin \phi |2^{[n]}\rangle
\]

under local unitary transformations, which belongs to the GSD states in [12] apparently. There are \( n \) \( H_3(\theta, \phi) \) correlations in \( |G^n\rangle \), \( H_3(\theta, \phi) \) of which is irreducible \( n \)-party correlation and the others are irreducible 2-party correlation. This distribution is the same as the generalized \( n \)-qubit GHZ states in [11]. When \( \phi = 0 \), \( H_3(\theta, 0) = H_2(\theta) \), this result returns to the \( n \)-qubit case.

### B. Second GHZ-type States

A generalization of the family of \( n \)-qutrit states \( G \) is the one in the subspace \( \{|0^{[m]}\rangle, |1^{[m]}\rangle, |2^{[m]}\rangle\} \) with \( m \) being a positive integer less than \( n \), and \( |0^{[m]}|2^{[n-m]}\rangle \) denoting the direct product of basis \( |0\rangle \) for the first \( m \) qutrits and \( |2\rangle \) for the others. Denoting the basis \( \{|0\rangle, |1\rangle, |2\rangle\} = \{|0^{[m]}\rangle, |1^{[m]}\rangle, |2^{[m]}\rangle\} \), the states in this family can be written as

\[
G_2 = \sum_{i=0, j=0}^{2.2} c_{ij} |i\rangle \langle j|,
\]

with the same constraint on \( c_{ij} \) as Eq. (5), and the diagonal elements \( c_{ii} \) also being expressed in the spherical coordinate \( \theta \) and \( \phi \).

**Theorem 2.** The irreducible multiparty correlations of the second family of GHZ-type state \( G_2 \) in Eq. (11) are given by

\[
\begin{align*}
C^{(2)} &= m H_2(\theta) + (n - m - 1) H_3(\theta, \phi), \\
C^{(m)} &= H_3(\theta, \phi) - H_3(\theta, \phi), \\
C^{(n)} &= H_3(\theta, \phi) - S(G_2),
\end{align*}
\]

and \( C^{(k)} = 0 \) for the other integer numbers \( 2 \leq k \leq n \), where the value of \( \phi \) is given by \( \cos^2 2\phi = \cos^2 2\phi + 4|c_{02}|^2 / \sin^4 \theta \).

**Proof.** The quantum states \( D_2 = \sum c_{ij} |i\rangle \langle i| \) has the same \( k \)-partite reduced matrices as \( G_2 \) for \( k < n - m \). It can be proved to be the limit of a state in the form (2).

Let us construct a 2-partite operator \( P_{ij} = (2|Z_i^2 - 1\rangle \langle j| \) by using the spin operator \( Z_i \) and the third Gell-Mann matrix of the \( j \)-th qutrit, \( \lambda_3^{(j)} = |0\rangle \langle 0| - |1\rangle \langle 1| \). It satisfies \( P_{ij}^{2} = \lambda_3^{(j)} \) and \( P_{ij}^3 = P_{ij} \). Then, the basis \( |0\rangle, |1\rangle, |2\rangle \) are the three eigenvectors of \( \Omega = \sum_{j=1}^{m} P_{m+1,j} + \sum_{l=m+2}^{n} Q_{l+1,l} \) corresponding to the maximal eigenvalue \( \omega_{max} = n - 1 \). Choosing the values of \( \gamma_1 \) and \( \gamma_2 \) the same as the ones in Eq. (8), and \( \exp \gamma = (2 \cos \gamma + 1)^{-n} \exp \gamma + 2^{-n-m+1}(\exp \gamma + 2 \cos \gamma \gamma)^{-1} \), the quantum state with only irreducible 2-party correlation

\[
\sigma_2(\gamma) = \exp(\eta + \gamma \Omega - \gamma^2 Z_{m+1} + \gamma Z_{m+1})
\]

has the limit \( \sigma_2 |\gamma \rightarrow +\infty = D_2 \). Accordingly, we can choose \( G_{2,k} = D_2 \) for \( k = 2, 3, \ldots n - m - 1 \).

To obtain \( G_{2,k} \) for the other values of \( k \), we introduce three \( (n - m) \)-partite operators \( \Sigma_0 = \prod_{l=1}^{m} \lambda_{ij}^{(l)} \), \( \Sigma_2 = \lambda_{m+1}^{(l)} \prod_{l=m+2}^{n} \lambda_{ij}^{(l)} \) \( \Sigma_3 = Z_{m+1} \prod_{l=m+2}^{n} Z_{ij}^{(l)} \), with \( \lambda_{ij}^{(l)} = |0\rangle \langle 0| + |2\rangle \langle 2| \) and \( \lambda_{ij}^{(l)} = -i|0\rangle \langle 2| + i|2\rangle \langle 0| \) being the forth and fifth Gell-Mann matrices of the \( j \)-th qutrit. They satisfy the relations \( \Sigma_0 \Sigma_\beta = \xi_{\alpha \beta} \Sigma_\alpha \), \( \Sigma_0^2 = \prod_{l=m+1}^{n} Z_{ij}^{(l)} \) and \( \Sigma_0, \Omega = 0 \), where \( \alpha, \beta, \gamma = 1, 2, 3 \), and can be viewed as the Pauli matrices in the subspace \( \{|0\rangle, |2\rangle\} \). Let \( \tan \gamma = 2 \cos \phi \) and \( \exp \gamma = 2 \cos \gamma \gamma \cos^2 \theta \), we can define an \( n \)-qutrit density matrix

\[
\tau_2(\gamma) = \exp(\eta + \gamma \Omega - \gamma_1^2 + \gamma_2^2 \Sigma),
\]

where \( \gamma_\tau = \cos \gamma \Sigma_3 + \sin \gamma \cos \Sigma_1 \Sigma_3 + \sin \gamma \sin \Sigma_2 \Sigma_3 \) with the parameters \( \xi = \frac{1}{2} \arg(c_{02}^2/c_{02}) \) and \( \xi = \arccos(2\phi/(c_{20}^2/c_{02}^2)) \), and \( \eta \) is determined by the normalization condition \( \text{Tr} \tau_2(\gamma) = 1 \). It has no irreducible \( (n - m + 1) \)-party or higher level correlation, and approaches the quantum state \( B_2 = D_2 + c_{02}|0\rangle \langle 2| + c_{20}|2\rangle \langle 0| \) when the parameter \( \gamma \rightarrow +\infty \). The \( n \)-qutrit state \( G_2 \) has the same \( (n - 1) \)-partite reduced density matrices as \( B_2 \). Therefore, we can take \( G_{2,k} = B_2 \) for \( k = n - m, n - m + 1, \ldots n - 1 \). One can yield the results in Eq. (12) via a direct calculation.

For the pure state in this family

\[
|G_2^n\rangle = \sin \theta \cos \phi |0\rangle + \cos \theta |1\rangle + \sin \theta \sin \phi |2\rangle,
\]
the variable \( \varphi = 0 \), the total correlation with the value \( C^n = nH_2(\varphi) + (n - m)nH_3(\varphi) \) is divided into three nonzero irreducible multi-qubit correlations as \( C^{(2)} = nH_2(\varphi) + (n - m - 1)nH_3(\varphi) \), \( C^{(n)} = \sin^2 \theta H_3(\varphi) \) and \( C^{(m)} = H_2(\varphi) \). This result is different with the \( n \)-qubit case, which indicates there could exist nonzero irreducible multiparty correlations between the highest \( n \)-party level and the lowest 2-party level in an \( n \)-qubit pure states.

C. MS states

The set of MS states [14] for qubit case is an important example to investigate the fundamental concepts in multiparite system [15]. We generalize the definition of MS states to \( n \)-qubit system

\[
|S\rangle = \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle),
\]

where \(|i\rangle = [\cos \alpha + \sin \alpha (X_1 + X_1^T)/\sqrt{2}]|i^{(n)}\rangle \) with the operator of \( j \)-th qutrit \( X_j = |0\rangle\langle 1| + |1\rangle\langle 2| + |2\rangle|1\rangle \) and \( \alpha \in (0, \arctan \sqrt{2}) \). The operators \( X_j \) satisfy \( X_j^2 = X_j \), \( X_j^T = X_j \) and \( X_jX_j^T = X_j^TX_j = 1 \).

**Theorem 3.** In the \( n \)-qutrit MS state (16), only irreducible 2-party and \((n - 1)\)-party correlation exist as

\[
C^{(2)} = H_3(\chi; \pi/4) + (n - 2)\ln 3,
\]

\[
C^{(n-1)} = \ln 3,
\]

where \( \chi \) is determined by the eigenvalue of the reduced density matrix \( \rho^{(3)}_1 \) for the first qutrit, \( \cos^2 \chi = \frac{1}{3}(3 - \cos 2\alpha + 2\sqrt{2}\sin 2\alpha) \).

**Proof.** The operator \( Q = \sum_{j=1}^{n} Q_{2j} \) has 9 eigenvalues as \(|i\rangle \otimes |j^{(n-1)}\rangle \) with \( i, j = 0, 1, 2 \), corresponding to its maximal eigenvalues \( q_{\text{max}} = n - 2 \). We construct an operator \( X = \cos \beta Q_{12} + \frac{1}{\sqrt{2}} \sin \beta (X_1 + X_1^T) + \sum_{j=1}^{n-2} X_j + \sum_{j=2}^{n-1} X_j^T \) commuting with \( Q \). Thus, the eigenvector of \( Q + X \) with the maximal eigenvalue is the one of \( X \) in the subspace \(|i\rangle \otimes |j^{(n-1)}\rangle \). Choosing the value of \( \beta \) satisfying \( \cot \beta = \sqrt{2}(\cot \alpha - \tan \alpha) + 1 \), one can check that the MS state (16) is the unique eigenvector corresponding to the maximal eigenvalues (UEME) of \( Q + X \). Therefore, the MS state can be viewed as the limit of a state without irreducible \( n \)-party correlation

\[
\rho_n = |S\rangle\langle S| = \lim_{\gamma \to +\infty} \text{Exp}(\eta + \gamma Q + \gamma X),
\]

where \( \eta = -\ln \text{Tr}[|\text{Exp}(\gamma Q + \gamma X)|] \), which leads to \( \rho_{n-1} = \rho_n \).

The two-party operator \( R_{12} = [\cos \alpha + \sin \alpha (X_1 + X_1^T)/\sqrt{2}]Q_{12}[\cos \alpha + \sin \alpha (X_1 + X_1^T)/\sqrt{2}] \) shares the similar properties with \( Q_{12} \), \( R_{12}^2 = R_{12} \) and \( \text{Tr}_{12}R_{12} = 3 \). The maximal eigenvalue of \( Q + R_{12} \) is triple degenerated with the eigenvectors \(|i\rangle \). Consequently, the quantum state with only irreducible 2-party correlation

\[
\sigma_s(\gamma) = \text{Exp}(\eta + \gamma Q + \gamma R_{12})
\]

has the limit \( \sigma_s|_{\gamma \to +\infty} = D_s = \sum_{i=0}^{2} \frac{1}{2} |i\rangle \langle i| \), where the value of \( \eta \) is determined by the normalization condition \( \text{Tr}\sigma_s(\gamma) = 1 \). Thus, we can take \( \rho_{s, m} = D_s \) for \( m = 2, 3, \ldots, n - 2 \) and obtain the results in Eq. (17).

According to the results by Feng et al. [12], it is easy to identify the \( n \)-qubit MS state \(|S\rangle\rangle \) can be determined by its \((n - 1)\)-partite reduced density matrices among \( n \)-qubit pure states. Our results show that there is no irreducible \( n \)-qubit correlation in the state \(|S\rangle\rangle \), which indicates it can be determined by its \((n - 1)\)-partite reduced density matrices among arbitrary \( n \)-qubit states (pure or mixed).

III. CONCLUSION AND DISCUSSION

Following the idea of Zhou’s continuity approach, for an \( n \)-qubit quantum state \( \rho^{(n)} \) without maximal rank, we construct \( \rho^{(n)}_m \) as the limit of a series \( n \)-qubit states in the standard exponential form (2). In this way, we obtain the irreducible multiparty correlations in three families \( n \)-qubit states, which can be viewed as three generalizations of the original GHZ state. The distribution of the total correlations in the generalized GHZ state (10) is the same as the \( n \)-qubit case. Whereas, in the \( n \)-qubit pure state (15), there exist three nonzero irreducible correlations which are \( C^{(2)}, C^{(n)} \) and \( C^{(m)}(2 < m < n) \). By contrast, only \( C^{(2)} \) and \( C^{(n)} \) are nonzero in the \( n \)-qubit generalize GHZ state which is the only pure \( n \)-qubit state with irreducible \( n \)-party correlation. This indicates the classification of the total correlations for a multipartite pure state in high dimensional system would be more complicated. It is also raised an interesting question that, which of the irreducible multiparty correlations can simultaneously be nonzero in a pure states.

To prove the absence of irreducible \( n \)-qubit correlation in the MS state (16), we construct an operator \( Q + X \) which is a sum of \((n - 1)\)-partite operators and with \(|S\rangle\rangle \) being its UEMS. The conclusion is proved by Eq. (18) in the continuity approach. This indicates the open problem about the GSD states is equivalent to the following one in the sense of limit: Whether the GSD state precisely the pure states which can’t be viewed as the UEMS of an operator \( Q^{(n-1)} \), a sum of \((n - 1)\)-partite operators. It is straightforward to prove the sufficiency part that, there exists no \( Q^{(n-1)} \) whose UEMS being a GSD state. In the results of [12], for a \( n \)-partite GSD state \(|\psi\rangle\rangle \), there exist two projectors \( P_1 = \bigotimes_{j=1}^{n} P^{(1)}_j \) and \( P_2 = \bigotimes_{j=1}^{n} P^{(2)}_j \), such that \(|\psi\rangle = P_1|\psi\rangle + P_2|\psi\rangle \). The projectors \( P^{(1)}_j \) and \( P^{(2)}_j \) for the \( j \)-th partite satisfy \( P^{(1)}_j|\psi\rangle \neq 0, P^{(2)}_j|\psi\rangle \neq 0 \) and \( P^{(1)}_j \perp P^{(2)}_j \) for
any $Q^{[n-1]}$, $\langle \psi | Q^{[n-1]} | \psi \rangle = \langle \psi' | Q^{[n-1]} | \psi' \rangle$, where $| \psi' \rangle = P_1 | \psi \rangle - P_2 | \psi \rangle$. Then, the necessity part of this question is left as, how to construct a sum of $(n - 1)$-partite operators, $Q^{[n-1]}$, whose UEMS is the given non-GSD state. We hope to find an universal generalization of the construction in Theorem 3 to an arbitrary non-GSD state in our subsequent investigation.

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[1] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. 47, 777 (1935).
[2] J. S. Bell, Physics 1, 195 (1964).
[3] S. Wantanabe, IBM Journal of research and development 4, 66 (1960).
[4] H. Ollivier and W. H. Zurek, Phys. Rev. Lett. 88, 017901 (2001).
[5] L. Henderson and V. Vedral, Journal of Physics A: Mathematical and General 34, 6899 (2001).
[6] A. Datta, A. Shaji, and C. Caves, Physical review letters 100, 50502 (2008).
[7] N. Linden, S. Popescu, and W. Wootters, Phys. Rev. Lett. 89, 207901 (2002).
[8] N. Linden and W. Wootters, Phys. Rev. Lett. 89, 277906 (2002).
[9] S. N. Walck and D. W. Lyons, Phys. Rev. Lett. 100, 050501 (2008).
[10] S. N. Walck and D. W. Lyons, Phys. Rev. A 79, 032326 (2009).
[11] D. L. Zhou, Phys. Rev. Lett. 101, 180505 (2008).
[12] Y. Feng, R. Duan, and M. Ying, Quantum Information & Computation 9, 0997 (2009).
[13] D. Zhou, preprint arXiv:0909.3700 (2009).
[14] H. A. Carteret and A. Sudbery, J. Phys. A 33, 4981 (2000).
[15] S. Ghose, N. Sinclair, S. Debnath, P. Rungta, and R. Stock, Phys. Rev. Lett. 102, 250404 (2009).