Dynamic modeling of geometrically nonlinear electrostatically actuated microbeams (Corotational Finite Element formulation and analysis)

H Borhan and M T Ahmadian

Sharif University of Technology, Center of Excellence for Design, Robotics and Automation, School of Mechanical Engineering, P.O. Box 11365-9567, Tehran, Iran

E-mail: borhan@mehr.sharif.edu

Abstract. In this paper, a complete nonlinear finite element model for coupled-domain MEMS devices with electrostatic actuation and squeeze film effect is developed. For this purpose, a corotational finite element formulation for the dynamic analysis of planar Euler beams is employed. In this method, the internal nodal forces due to deformation and intrinsic residual stresses, the inertial nodal forces, and the damping effect of squeezed air film are systematically derived by consistent linearization of the fully geometrically nonlinear beam theory using d’Alamber and virtual work principles. An incremental-iterative method based on the Newmark direct integration procedure and the Newton-Raphson algorithm is used to solve the nonlinear dynamic equilibrium equations. Numerical examples are presented and compared with experimental findings which indicate properly good agreement.

Nomenclature

\begin{align*}
L & \quad \text{microbeam length} \\
b & \quad \text{microbeam width} \\
h & \quad \text{microbeam thickness} \\
g & \quad \text{air gap} \\
A & \quad \text{cross sectional area} \\
i & \quad \text{moment of inertia} \\
\varepsilon & \quad \text{electrical permittivity} \\
\beta & \quad \text{total rotation of local coordinate system} \\
\beta_0 & \quad \text{initial element slope} \\
\varepsilon_0 & \quad \text{convergence criteria} \\
C_d & \quad \text{coefficient of damping per unit length} \\
\mu & \quad \text{coefficient of air viscosity}
\end{align*}
1. Introduction

Microstructures, actuated by electrostatic force are widely used in various tools and applications such as capacitive switches, resonators, accelerometers, pressure sensors, multi-electrode tunable capacitors, material properties measurements and so on. Therefore dynamic and vibration analysis of micro flexible structures, especially electrostatically actuated micro beams are of paramount importance. Because of low mass and high flexibility of these structures, geometric nonlinear effects due to large deformation cannot be ignored. This nonlinearity causes stiffening of the structure and reducing the deflection response relative to that of a linear system. Linear analysis does not account for these effects and consequently may significantly over-predict the response. Also because of constant flexibility in linear analyses, an unstable response is obtained when the potential exceeds specific voltages which are commonly known as Pull-in voltages. Therefore to design and control such structures and reach to an efficiently performance, an accurate estimation of deflection and dynamic behavior of these micro structures is necessary. Attempts have been made by several authors to derive closed form expressions for evaluating pull-in voltages with higher accuracy [1-4]. However, research on large deformation effect of micro structures with electrostatic actuation is scarce. Besides, the nonlinear dynamics and vibration analysis of these kinds of micro structures has remained relatively unexplored. Almost majority of previous developed dynamic analyses of these micro devices are based on linearization of the nonlinear applied electrostatic load to obtain approximate solutions [5, 6]. Also other attempts have been made to develop a lumped spring-mass model considering the squeeze film effect as a constant dissipative damper [7, 8]. The objective of this research is to develop a complete model of electrostatically actuated microbeams considering geometrical nonlinearity with nonlinear squeezed film effect. For this purpose, a consistent corotational formulation for the dynamic analysis of these micro-actuators or switches is employed and developed [9].

2. Nonlinear Corotational Finite Element (NCFE) Method

2.1. Co-rotational Kinematics

The main idea of CR method is to decompose the motion of the element into a rigid body and a pure deformational part, through the use of a local coordinate system \((x_l, y_l)\) (Figure 1). This local coordinate may undergo large displacements and rotations with the element, and thus the geometrical non-linearity is included by the motion of the local coordinate system. Assuming that the pure deformations of the element are always small relative to the local CR framework, standard small-strain measures can be applied.

\[
\begin{bmatrix}
\theta_1 \\
\theta_2 \\
\theta_3 \\
\theta_4 \\
\theta_5 \\
\theta_6 \\
\end{bmatrix}
= \begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4 \\
u_5 \\
u_6 \\
\end{bmatrix}
\]

Figure 1. 2D CR beam element undergoing large displacement

The CR formulation is developed based on the parameters presented in figure 2. In the global frame \((x, y)\), the coordinates of the nodes 1 and 2 of the element are presented by \((x_1, y_1)\) and \((x_2, y_2)\) respectively. Global and Local displacement vectors are defined by

\[
\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4 \\
u_5 \\
u_6 \\
\end{bmatrix}
= \begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4 \\
u_5 \\
u_6 \\
\end{bmatrix}
\]
\[ r_i = \begin{bmatrix} 0 & 0 & \bar{\theta}_1 & \bar{\theta}_2 \end{bmatrix}^T \]  

(2)

Where

\[ \bar{\theta}_1 = \theta_1 - \alpha \]  

(3)

\[ \bar{\theta}_2 = \theta_2 - \alpha \]  

(4)

\[ \theta = \theta_1 - \theta_2 \]  

(5)

\[ l_0, l_c \] denote the initial and current lengths of the element and \( \alpha \) denotes the rigid rotation of the local coordinate system. Thus

\[ l_0 = \left[ (x_2 - x_1)^2 + (y_2 - y_1)^2 \right]^{1/2} \]  

(6)

\[ l_c = \left[ (x_2 + u_2 - x_1 - u_1)^2 + (y_2 + w_2 - y_1 - w_1)^2 \right]^{1/2} \]  

(7)

\[ \sin \alpha = c_0 s - s_0 c \]  

(8)

\[ \cos \alpha = c_0 c + s_0 s \]  

(9)

Where

\[ c_0 = \cos \beta_0 = \frac{1}{l_0} (x_2 - x_1) \]  

(10)

\[ s_0 = \sin \beta_0 = \frac{1}{l_0} (y_2 - y_1) \]  

(11)

\[ c = \cos \beta = \frac{1}{l_c} (x_2 + u_2 - x_1 - u_1) \]  

(12)

\[ s = \sin \beta = \frac{1}{l_c} (y_2 + w_2 - y_1 - w_1) \]  

(13)

So considering \[ |\alpha| < \pi \], \( \alpha \) can be computed. Considering the axial strain in the element as \( \bar{\theta} / l_0 \), and residual stresses, the local internal force vector can be evaluated. By equating the internal static virtual work in both local and global coordinate systems and differentiation of the results with respect to the global displacement variables, the components of the element stiffness matrix \( k_T \), can be calculated [11].

3. CR modeling of microbeam

Hermitian functions are employed to interpolate the transverse bending displacements and linear interpolation is used to interpolate the longitudinal displacements. Considering normalized local coordinates to be \( \xi_1 = 1 - \frac{x_j}{l_0} \) and \( \xi_2 = \frac{x_j}{l_0} \), and contribution of the bending to the axial deformation, the local transverse and axial displacements are obtained by

\[ \hat{v} = N^T_{\xi} \xi \]  

(14)

\[ \hat{u} = N^T_{\eta} \xi \]  

(15)

With

\[ N^T_{\xi} = \begin{bmatrix} 0 & \xi_1^2 (3 - 2\xi_1) & l_0 \xi_1^2 \xi_2 & 0 & \xi_2^2 (3 - 2\xi_2) & l_0 \xi_2^2 \xi_1 \end{bmatrix} \]  

(16)

\[ N^T_{\eta} = \begin{bmatrix} \xi_1 \frac{6 y_j \xi_1 (1 - \xi_1)}{l_0} & \eta_j \xi_1 (2 - 3\xi_1) & \xi_2 \frac{6 y_j \xi_2 (1 - \xi_2)}{l_0} & \eta_j \xi_2 (2 - 3\xi_2) \end{bmatrix} \]  

(17)

Then the local strain and stress are defined as
\[
\hat{\sigma} = E\hat{\varepsilon}; \quad \hat{\varepsilon} = \frac{\partial \hat{u}}{\partial x} = \hat{B}\hat{u}
\]  

(18)

Where \(\hat{B}\) is a local strain-displacement matrix. Applying the d'Alembert and virtual work principles with respect to local coordinate systems and ignoring the velocity coupling terms, the mass and dissipative matrices with respect to global coordinate systems can be obtained too [9,10].

3.1. Residual stress gradient modeling

Commonly as the result of microfabrication process, undesirable residual stresses created in thin film, deposited on substrates. The gradient of residual stress has caused the cantilever microbeam to curl [12-14]. To model these undesirable effects, a general uniaxial residual stress field may be presented by a polynomial [13]

\[
\sigma_r = \sum_{k=0}^{\infty} \sigma_{r,k}\left(\frac{\bar{y}}{h/2}\right)^k, \quad \bar{y} \in (-h/2, h/2)
\]  

(19)

Where, \(\bar{y}\) is the coordinate across the thickness, \(h\), with an origin at the midplane of microbeam element. In the first approximation, the element's intrinsic moment can be obtained as [12]

\[
T_r = -\int_A \sigma_r \bar{y} \, dA = -\frac{1}{6}Ah\sigma_{r,1}
\]  

(20)

3.2. Electrostatic applied force modeling

For a cantilever microbeam, the electrostatic distributed applied force considering the electrical fringing effect, can be modeled as [1],

\[
q_{ES} = -\frac{d}{dv}\left(\frac{1}{2}C_eV^2\right)
\]  

(21)

Where \(C_e\) is electrical capacity of micro structure with electrostatic actuation. To evaluate the electrostatic element nodal force vector \(f_{e}\), the average force per unit length, \(\bar{q}_{ES}\) is evaluated and it can be obtained that,

\[
f_e = \bar{q}_{ES}l_0 \int_0^1 N^T \, d_{\bar{y}}, \quad \bar{q}_{ES} = q_{ES}(\bar{x}) ; \quad \bar{x} = \frac{x_1 + x_2}{2}
\]  

(22)

3.3. Squeezed-film air damping effect

Based on conventional Reynolds' equation for squeezed-film air damping, the averaged coefficient of damping force considering \(\bar{v}\) as the averaged deflection of the microbeam can be obtained as [16],

\[
\bar{C}_d = \frac{\mu b^3}{(g - \bar{v})^3}
\]  

(23)

4. Solution Technique of NCFE method for dynamical Analysis

The dynamic equilibrium equation for an electrostatically actuated microstructure based on the developed finite element matrices can be presented into a standard assembled form as,

\[
M^{t+\Delta t}\ddot{\mathbf{R}}_g + C_{sq}^{t+\Delta t}\dot{\mathbf{R}}_g + K_{r}^{t+\Delta t}\mathbf{R}_g = ^tF_e - ^tF
\]  

(24)

Where, \(\mathbf{R}_g\), is the incremental global displacement vector; \(^{t+\Delta t}\dot{\mathbf{R}}_g\) and \(^{t+\Delta t}\ddot{\mathbf{R}}_g\) are the velocity and acceleration vectors at time \(t + \Delta t\); \(K_{r}\) is the tangent stiffness matrix; \(M\) is the mass matrix; \(C_{sq}\) is damping matrix; \(^tF_e\) and \(^tF\) are the electrostatic external load and the balanced force vectors, respectively. Accuracy with the incremental solution procedure requires the use of a small time step since the nonlinear damping changes over the time increment (\(\Delta t\)) can be ignored. Improved accuracy
can be achieved by modifying Eq. 13 to calculate the iterative displacement $\delta R_g$. One iterative solution methodology is the modified Newton-Raphson iteration method, i.e.,

$$M^t \Delta \hat{R}_g^{(i)} + C_{sq}^t \Delta \hat{R}_g^{(i)} + K_T^t \delta R_g = F_e^t - F^{(i)}$$  \tag{25}$$

where $i$ is the iteration number. In predictor-corrector form, the proposed time-integration scheme may be presented as [11]:

**Prediction:** $i \leftarrow 0$

$$t + \Delta t \hat{R}_g^{(1)} = t \hat{R}_g$$

$$\delta R_g^{(1)} = 0$$

$$t + \Delta t F^{(1)} = t F$$

**Solution:** $i \leftarrow i + 1$

$$\hat{t} \hat{K} = \frac{1}{\alpha \Delta t^2} M + \frac{\delta}{\alpha \Delta t} (t C_{sq}) + t K_T$$

$$t + \Delta t \hat{F}_e = t + \Delta t M \frac{1}{\alpha \Delta t} \left[ \hat{t} \hat{R}_g - t + \Delta t \hat{R}_g^{(i)} \right] + \frac{1}{\alpha \Delta t} t \hat{R}_g + \left( \frac{1}{2\alpha} - 1 \right) \hat{t} \hat{R}_g$$

$$+ \left( \frac{\delta}{\alpha} - 1 \right) \hat{t} \hat{R}_g + \left( \frac{\delta}{2\alpha} - 1 \right) \hat{t} \hat{R}_g$$

$$t + \Delta t \hat{R}_g^{(i+1)} = t + \Delta t \hat{F}_e - t + \Delta t F^{(i)}$$

**Correction:**

$$t + \Delta t \hat{R}_g^{(i+1)} = t + \Delta t \hat{R}_g^{(i)} + \delta R_g^{(i+1)}$$

$$t + \Delta t \hat{R}_g^{(i+1)} = \frac{1}{\alpha \Delta t^2} \left[ t + \Delta t \hat{R}_g^{(i+1)} - t \hat{R}_g \right] + \frac{1}{\alpha \Delta t} \hat{t} \hat{R}_g^{(i+1)} + \left( \frac{1}{2\alpha} - 1 \right) \hat{t} \hat{R}_g^{(i+1)}$$

$$t + \Delta t \hat{R}_g^{(i+1)} = \frac{\delta}{\alpha \Delta t} \left[ t + \Delta t \hat{R}_g^{(i+1)} - t \hat{R}_g \right] + \frac{1}{\alpha \Delta t} \hat{t} \hat{R}_g^{(i+1)} + \Delta t \left( \frac{\delta}{2\alpha} - 1 \right) \hat{t} \hat{R}_g^{(i+1)}$$

5. Model validation and discussion

To validate the developed NCFE model of electrostatic micro-actuators, a number of numerical examples are presented and compared with experimental findings. The main focus is on investigating the geometrical nonlinear effects on microbeam deflections and dynamic behavior under various applied voltages.

5.1. Numerical results of static analysis

An isotropic cantilever beam of length 500 (µm) with an initial air gap of 2.5 (µm) and residual strain gradient of $1.5 \times 10^{-5}$ / µm under various difference voltages is considered. Using NCFE Method, a programming code is developed base on MATLAB software. The number of elements in the beam is assumed 250 and with displacement convergence criteria of 0.001. The static beam deflection versus beam length is presented in figure 2. Increasing electrostatic force leads to a continuous increase in the microstructure deflection. This behavior continues until a physical contact is made with the stationary substrate. In contrast by linear or lumped modeling, as applied voltage exceeds a specific value called pull-in voltage, the response becomes unstable and it has been assumed that the microbeam contacts substrate suddenly. So by considering geometrical nonlinearity and its nonlinear stiffening effect, there is no sudden instability in the microbeam deflection.
5.2. Numerical results of dynamic analysis

In this section, the transient response of cantilever type electrostatic micro-actuators under various applied voltages is investigated. For this purpose, the transient response of a polysilicon microbeam with 800 (µm) length, 40 (µm) width and 2.25 (µm) thickness with nominal gap height of 2 (µm) under various step-input electrostatic voltages have been simulated. To numerically simulate the dynamic response of these electrostatic micro-actuators using developed NCFE model, 250 beam elements with time step of 0.00001 (s) are employed. Figure 3 presents the time history plots of microbeam tip displacement. The results show that as the deflection of microbeam is increased via applying higher voltages, the distributed damping effect of squeeze film and dynamic rise time are also increased. In contrast, as presented in figure 4, by increasing the initial nominal gap and under equivalent deflections, the squeeze-film air damping and consequently dynamic rise time have been decreased and the response is more oscillatory. This phenomenon is in good agreement with conventional Reynolds' model for squeeze-film air damping of microplates [16] and also experimental modal analysis of micro structures [17]. Similar results can be observed as the width of the microbeam increases (figure 3).
5.3. Comparing pull-in voltage with NCFE Contact Voltage

In linear modeling of electrostatically actuated microbeams, the nonlinear stiffening effect of large deformation is neglected. Consequently at a specific gap, the stability of the equilibrium is broken and pull-in occurs. For this purpose, various closed form formulations have been developed by several authors to determine the pull-in voltage with higher accuracy. They have tried to include nonlinear effects such as fringing, charge-redistribution, axial residual stresses and anchor compliance [1-4]. However in the developed NCFE model considering geometric nonlinearity effect, as the applied voltage increases, microbeam deflection also increases until a certain voltage called 'contact voltage', when the beam touches the substrate. Figure 5 compares contact voltages of various cantilever microbeams based on different methods of analysis including closed form formulas, linear models and NCFE method. It is clear that NCFE method is in very good agreement with experimental findings especially when large deformation effect is dominated [18].

Figure 5. Contact and pull-in voltage comparison with experimental measurements

6. Conclusion

To develop a complete model for dynamic analysis of electrostatically actuated microstructures, a CR formulation of geometrically nonlinear Euler-Bernoulli beam element has been developed and employed. The nonlinear finite element model accounts the effects of the fringing filed, air squeeze film, electrostatic actuation and residual intrinsic stresses. The dynamics response of some electrostatic micro-actuators has been simulated and investigated. The results indicate that the transient dynamic response behaviors such as oscillatory motion and rising time are very sensitive to the nonlinear squeeze film damping effect. Applying developed NCFE model considering geometrical nonlinearity induced stiffening effect, causing stable response of the microbeam. The obtained contact voltages are in good agreement with experimental findings.
References
[1] S. Pamidighantam, R. Puers, K. Baert and H. A C Tilmans Pull-in voltage analysis of electrostatically actuated beam structures with fixed–fixed and fixed–free end conditions, J. Micromech. Microeng. 12 458-464
[2] S Chowdhury, M Ahmadi and W C Miller 2005 A closed-form model for the pull-in voltage of electrostatically actuated cantilever beams J. Micromech. Microeng. 15
[3] C O'Mahony1, M Hill2, R Duane1 and A Mathewson1 2003 Analysis of electromechanical boundary effects on the pull-in of micro machined fixed–fixed beams, J. Micromech. Microeng. 13 S75-S80
[4] M. Lishchynska, N, Corderom O. Slattery 2005 Modeling electrostatic behavior of microcantileves incorporating residual stress and non-ideal anchors J. Micromech. Microeng. 15 S10-S14
[5] Hu Y C, Chang C M and Huang S C 2004 Some design considerations on the electrostatically actuated microstructures Sensors Actuators A 112 155–61
[6] W. Zhang and G. Meng 2003 Nonlinear dynamical system of micro-cantilever under combined parametric and forcing excitations in MEMS Journal Sensor and Actuators A: Physical, Vol. /Iss. 119
[7] Nayfeh, A. and Younis, M. 2004 A new approach to the modeling and simulation of flexible microstructures under the effect of squeeze-film damping, J. Micromech. Microeng. Vo. 14, pp. 170-18
[8] Ahn Y, Guckel H and Zook JD 2001 Capacitive microbeam. resonator design J. Micromech. Microeng. 11 70–80.
[9] K. M. Hsiao and J. Y. Yang 1990 Nonlinear dynamic analysis of elastic frames, Comput. Struct., 33, 1057-1063
[10] K. M. Hsiao, R. T. Yang and A. C. Lee 1994 A consistent finite element formulation for nonlinear dynamic analysis of planar beam, International J. of Numerical Methods in Eng., Vol. 37, 75-89
[11] M.A. Crisfield 1997 Nonlinear Finite Element Analysis of Solids and Structures, Advanced Topics, vol. 2, Wiley, Chichester
[12] Senturia S D 2001 Microsystems Design (Boston, MA: Kluwer) pp 222-8
[13] W Fang and J A Wickert Determining mean and gradient residual stresses in thin films using micromachined cantilevers Journal of Micromechanics and Microengineering Vol. 6 Issue 3 Article 002
[14] Chun-Hsien Lee and Hsin-Hua Hu the buckling behavior of micromachined beams J. Micromech. and Microeng. Vol. 9 Issue 3 Article 304
[15] Hughes P J, Forde M, O’Neill B, Hill M, Berneck D, O’Mahony C and Lane W A 2000 MEMswitch: CMOS compatible surface micromachined switches and relays Proc. Micromechanics Europe (MME2000)
[16] M. Bao, H. Yang, Y. Sun and Y. Wang 2003 Squeeze-film air damping of thick hole plate Sensors and Actuators A: Physical, Vol. 108, pp. 212-217,
[17] Ozdoganlar, O. B., Hansche, B. D., and Carne, T. G. 2003 Experimental modal analysis for Microsystems Proc. of 21st International Modal Analysis Conference (IMAC)
[18] R. K. Gupta, P. M. Ostberg and S.D. Senturia 1996 Material property measurement of micromechanical polysilicon beam SPIE 1996 conference