An Improvement of the Lower Bound on the Minimum Number of $\leq k$-Edges

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Abstract: In this paper, we improve the lower bound on the minimum number of $\leq k$-edges in sets of $n$ points in general position in the plane when $k$ is close to $\frac{2}{3}$. As a consequence, we improve the current best lower bound of the rectilinear crossing number of the complete graph $K_n$ for some values of $n$.

Keywords: combinatorial geometry; $\leq k$-edges; rectilinear crossing number; optimization; complete graphs

1. Introduction

The search for lower bounds for the minimum number of $\leq k$-edges in sets of $n$ points of the plane for $n \geq 2 k + 2 (e_{\leq k}(n))$ is an important task in Combinatorial Geometry, due to its relation with the rectilinear crossing number problem. The most well-known case of the rectilinear crossing number problem aims to find the number $\sigma(P)$ of crossings in a complete graph with a set of vertices $P$ consisting of $n$ points in the plane (in general position) and edges represented by segments and the minimum number of crossings over $P$, $\sigma(n)$ (see the definitions below). The idea of determining $\sigma(n)$ for each $n$ was firstly considered by Erdős and Guy (see [1,2]). Determining $\sigma(n)$ is equivalent to finding the minimum number of convex quadrilaterals defined by $n$ points in the plane. These kinds of problems belong to classical combinatorial geometry problems (Erdős-Szekeres problems). The study of $\sigma(n)$ is also interesting from the point of view of Geometric Probability. It is connected with the Sylvester Four-Point Problem, in which Sylvester studies the probability of four random points in the plane forming a convex quadrilateral.

Nowadays, finding the value of $\sigma(n)$ continues to be a challenging open problem. The exact value of $\sigma(n)$ is known for $n \leq 27$ and $n = 30$. The search of lower and upper asymptotic bounds of $\sigma(n)$ constitutes a relevant task due to its connection with the problem of finding the value of the Sylvester Four-Point Constant $q_s$. In order to define properly $q_s$, it is necessary to consider a convex open set $R$ in the plane with finite area. Let $q(R)$ be the probability that four points chosen randomly from $R$ define a convex quadrilateral. Whence, $q_s$ is defined as the infimum of $q(R)$ taken over all open sets $R$.

In particular, the connection between $q_s$ and $\sigma(n)$ is given by the following expression:

$$q_s = \lim_{n \to \infty} \frac{\sigma(n)}{\binom{n}{4}}$$

For more details, see [3].

The rigorous definitions of the above-presented concepts are the following:
Definition 1. Given a finite set of points in general position in the plane P, assume that we join each pair of points of P with a straight line segment. The rectilinear crossing number of P (cr(P)) is the number of intersections out of the vertices of said segments. The rectilinear crossing number of n (cr(n)) is the minimum of cr(P) over all the sets P with n points.

Definition 2. Given a set of points in general position, A = \{p_1, ..., p_n\} and an integer number k such that 0 ≤ k ≤ \frac{n-2}{2}, a k-edge of A is a line R that joins two points of A and leaves exactly k points of A in one of the open half-planes (it is named the k-half plane of R).

Definition 3. Given a set of points in general position, A = \{p_1, ..., p_n\}, a ≤ k-edge of A is an i-edge of A with i ≤ k.

Notation 1. We call \(e_k(P)\) the number of k−edges of the set P and \(e_k(n)\) the maximum number of \(e_k(P)\) over all the sets P with n points.

The relation between the number of ≤k-edges of P and cr(P) is given by the expression:

\[
\text{cr}(P) = \sum_{k=0}^{\frac{n-2}{2}} (n-2k-3) e_{\leq k}(P) - \frac{3}{4} \left( \binom{n}{3} \right) + \left( 1 + (-1)^{n+1} \right) \frac{1}{8} \left( \binom{n}{2} \right),
\]

where \(e_{\leq k}(P)\) is the number of ≤k-edges of the set P with \(|P| = n\) (see [4,5]). This implies that

\[
\text{cr}(n) \geq \sum_{k=0}^{\frac{n-2}{2}} (n-2k-3) e_{\leq k}(n) - \frac{3}{4} \left( \binom{n}{3} \right) + \left( 1 + (-1)^{n+1} \right) \frac{1}{8} \left( \binom{n}{2} \right). \tag{2}
\]

This way, improvements of the lower bound of \(e_{\leq k}(n)\) for \(k \leq \left\lfloor \frac{n-2}{2} \right\rfloor - 2\) yield an improvement of the lower bound of the rectilinear crossing number of n. The exact value of \(e_{\leq k}(n)\) is known for \(k \leq \left\lfloor \frac{4n-11}{9} \right\rfloor\) (see [4,6,7]). For \(k \geq \left\lfloor \frac{4n-11}{9} \right\rfloor\), the current best lower bound of \(e_{\leq k}(n)\) is \(e_{\leq k}(n) \geq u_k\) for the sequence \(u_k\) defined in [6].

Taking into account the asymptotic equivalence of \(u_k\), we have

\[
e_{\leq k}(n) \geq \left( \binom{n}{2} \right) - \frac{1}{9} \sqrt{n - 2k - 2} \left( 5n^2 + 19n + 31 \right). \tag{3}
\]

For \(k\) close to \(\left\lfloor \frac{n-2}{2} \right\rfloor - 2\), namely \(k = \left\lfloor \frac{n-1}{2} \right\rfloor\) for some fixed constant \(t\), the bound (3) gives

\[
e_{\leq k}(n) \geq \left( \binom{n}{2} \right) - O(n^3). \tag{4}
\]

For these values of \(k\), if we define \(P\) as a set for which \(e_{\leq k}(n)\) is attained and \(e_{\leq k}(P)\) as the number of \(s\)-edges of \(P\) (see the definitions below), then we have that the identity:

\[
e_{\leq k}(n) = \left( \binom{n}{2} \right) - \left( e_{k+1}(P) + \ldots + e_{\left\lfloor \frac{n-2}{2} \right\rfloor}(P) \right)
\]

together with the current best upper bound of \(e_s(P)\) (due to Dey, see [8]) yield a lower bound that is asymptotically better than (4). More precisely, in [8] was shown the existence of a constant \(C \leq 6.48\) such that

\[
e_s(P) \leq Cn(s + 1)^{\frac{3}{2}}, \tag{5}
\]

for \(s < \frac{n-2}{2}\) and

\[
e_s(P) \leq Cn \left( \frac{n-1}{2} \right)^{\frac{3}{2}}, \tag{6}
\]

for \(s = \frac{n-2}{2}\).
for \( s = \frac{n-2}{2} \). To do this, Dey in [8] applied the crossing lemma and the following values for \( E(\leq s)(n) \), the maximum number of \( (\leq s) \)-edges due to [9]

\[
E(\leq s)(n) = s(k+1) \text{ for } s < (n-2)/2, E(\leq (n-2)/2)(n) = n(n-1)/2.
\]

The best values for \( C \) are \( C = \left( \frac{31.827}{2} \right)^{1/3} \) for \( s < \frac{n-2}{2} \) and \( C = \left( \frac{31.827}{2} \right)^{1/3} \) for \( s = \frac{n-2}{2} \), for \( n \) an even number, if \( e_k(P) \geq \frac{103n}{6} \), (see [10,11]). Notice that this condition is satisfied for large \( n \) and \( s \) close to \( \frac{n}{2} \) due to the best lower bound of \( e_k(n) \). As an example, for \( s = \frac{n-2}{2} \) we have the upper bound (5) for \( n \geq 327 \) and, for \( s = \frac{n-3}{2} \), we have the upper bound (5) for \( n \geq 329 \).

This gives:

\[
e_{\leq k}(n) \geq \left( \frac{n}{2} \right) - Cn \sum_{i=k+1}^{\left\lfloor \frac{n-2}{2} \right\rfloor} (i+1)^{1/3}, \tag{7}
\]

for \( n \) an odd number and

\[
e_{\leq k}(n) \geq \left( \frac{n}{2} \right) - Cn \sum_{i=k+1}^{\left\lfloor \frac{n-4}{2} \right\rfloor} (i+1)^{1/3} + Cn\left( \frac{n-1}{2} \right)^{4/3}, \tag{8}
\]

for \( n \) an even number. In this paper we improve in, at most, \( \left\lfloor \frac{n}{2} \right\rfloor \) the bounds (7) and (8) for \( k = \left\lfloor \frac{n-2}{2} \right\rfloor \) and some big values of \( n \). In this way, we achieve the best lower bound of \( e_{\leq k}(n) \) for these values of \( k \) and \( n \). As a consequence, we improve the lower bound of the rectilinear crossing number of \( K_n \).

The outline of the rest of the paper is as follows: In Section 2 we give the improvement of the lower bound of \( e_{\leq k}(n), k = \left\lfloor \frac{n-2}{2} \right\rfloor \), for the cases \( t = 7 \) (\( n \) is an odd number) and \( t = 8 \) (\( n \) is an even number). In Section 3, we generalize the achieved results in Section 2, and in Section 4 we give some concluding remarks.

2. The Improvement of the Lower Bound

In order to get the improvement of the lower bound of \( e_{\leq k}(n) \), we need the following lemma:

**Lemma 1.** Let \( k \) and \( n \) be positive integers, and let \( P \) be a set of \( n \) points in general position in the plane. If \( k < \left\lfloor \frac{n-2}{2} \right\rfloor \), then

\[
e_k(n-1) \geq \frac{n-k-2}{n}e_k(P) + \frac{k+1}{n}e_{k+1}(P). \tag{9}
\]

**Proof.** Each \((k+1)\)-edge of \( P \) leaves \( k + 1 \) points of \( P \) in its \((k+1)\)-half plane, and each \( k \)-edge of \( P \) leaves \( n - k - 2 \) points of \( P \) in one of its half-planes. Therefore, the total number of points of \( P \) in these planes, allowing repetitions, is

\[
(n - k - 2)e_k(P) + (k + 1)e_{k+1}(P), \tag{10}
\]

and then there is a point of \( P \), say \( p_n \), that belongs to \( s \) half-planes with

\[
s \geq \frac{n-k-2}{n}e_k(P) + \frac{k+1}{n}e_{k+1}(P). \tag{11}
\]

If we remove \( p_n \), then we obtain a set \( Q = \{p_1, \ldots, p_{n-1}\} \) such that the \((k+1)\)-edges of \( P \) corresponding to the \( s \) half-planes are now \( k \)-edges of \( Q \), because they have \((k+1) - 1 = k \) points of \( Q \) in one of the open half-planes.
Moreover, the $k$-edges of $P$ corresponding to the $s$ half-planes are now $k$-edges of $Q$ because they still have $k$ points of $Q$ in one of the open half-planes. Therefore, we have that

$$e_k(n - 1) \geq e_k(Q) \geq s \geq \frac{n - k - 2}{n} e_k(P) + \frac{k + 1}{n} e_{k+1}(P)$$

(12)

as desired. \qed

**Corollary 1.** Let $k$ and $n$ be positive integers, and let $P$ be a set of $n$ points in general position in the plane. If $k < \left\lfloor \frac{n^2}{2} \right\rfloor$, then

$$\min\{e_k(P), e_{k+1}(P)\} \leq \left\lfloor \frac{n}{n - 1} e_k(n - 1) \right\rfloor.$$  

(13)

**Proof.** Applying Lemma 1, we obtain

$$e_k(n - 1) \geq \frac{n - k - 2}{n} e_k(P) + \frac{k + 1}{n} e_{k+1}(P) \geq \frac{n - 1}{n} \min\{e_k(P), e_{k+1}(P)\}.$$  

(14)

This implies the desired result. \qed

**Corollary 2.** Let $k$ and $n$ be positive integers, and let $P$ be a set of $n$ points in general position in the plane. If $k < \left\lfloor \frac{n^2}{2} \right\rfloor$, then

$$\min\{e_k(P), e_{k+1}(P)\} \leq \left\lfloor \frac{n}{n - 1} \left( \frac{31,827}{2^{10}} \right)^{\frac{1}{3}} (n - 1) (k + 1)^{\frac{1}{3}} \right\rfloor.$$  

(15)

**Proof.** The result follows from Corollary 1 and inequality (5). \qed

**Remark 1.** For fixed $k$ and some values of $n$, the bound in Corollary 2 may improve by one the following upper bound of $\min\{e_k(P), e_{k+1}(P)\}$ derived from (5)

$$\min\{e_k(P), e_{k+1}(P)\} \leq \left\lfloor \left( \frac{31,827}{2^{10}} \right)^{\frac{1}{3}} n(k + 1)^{\frac{1}{3}} \right\rfloor.$$  

(16)

We will apply this improvement to shift the lower bound on the number of $\leq k$-edges for sets with $n$ points in the cases $k = \frac{n - 7}{2}$ and $k = \frac{n - 8}{2}$ for some values of $n$.

**Corollary 3.** Let $n \geq 7$ be an odd integer, and let $k := (n - 7)/2$. Then

$$e_{\leq k}(n) \geq \frac{n^2 - n}{2} - \left\lfloor \frac{n}{n - 1} \left( \frac{31,827}{2^{10}} \right)^{\frac{1}{3}} (n - 1) (n - 3)^{\frac{1}{3}} \right\rfloor.$$  

(17)

**Proof.** Let $P$ be a set of $n$ points in general position attaining $e_{\leq k}(n)$. From (7), it follows that

$$e_{\leq k}(n) = \frac{n^2 - n}{2} - e_{\leq k}(P) - e_{\leq k}(P) = \frac{n^2 - n}{2} - \min\left\{e_{\leq k}(P), e_{\leq k}(P)\right\}$$

$$- \max\left\{e_{\leq k}(P), e_{\leq k}(P)\right\}.$$  

(18)

Thus, we obtain the desired result by applying Corollary 2 to $k = \frac{n - 5}{2}$ and the following upper bound of $\max\left\{e_{\leq k}(P), e_{\leq k}(P)\right\}$ derived from (5)
Let $n$ be an odd integer with $n \geq 33,627$. For these values of $n$, the lower bound $(5)$ derived from (5) is better than the lower bound for $e_{\leq n-\frac{7}{2}}(n)$ of [6]. As an example, we get the improvement for the following odd values of $n$:

33,627, 33,629, 33,637, 33,639, 33,641, 33,647, 33,649, 33,651, 33,653, 33,661, 33,663, 33,665, 33,667, 33,677, 33,679, 33,681, 33,683, 33,685, 33,687, 33,713, 33,715, 33,717, 33,719, 33,721, 33,723.

Remark 3. Plugging (17) in (2), we obtain an improvement of 4 for the lower bound of $\overline{c}(n)$ for the aforementioned odd values of $n$ in the range $[33623, 33723]$ because the coefficient of $e_{\leq n-\frac{7}{2}}(n)$ in (2) is 4.

Corollary 4. Let $n \geq 8$ be an even integer, and let $k := (n - 8)/2$. Then

\[
e_{\leq k}(n) \geq \frac{n^2 - n}{2} - \left[ \frac{n}{n-1} \left( 2^{11} \right)^{\frac{1}{3}} (n-4)(n-3)^{\frac{1}{3}} \right] - \left[ \left( 2^{11} \right)^{\frac{1}{3}} (n-1)^{\frac{1}{3}} \right].
\]

Proof. Let $P$ be a set of $n$ points in general position attaining $e_{\leq k}(n)$. From (8), it follows that

\[
e_{\leq k}(n) = \frac{n^2 - n}{2} - \min \left\{ e_{\leq k-6}(P), e_{\leq k-4}(P) \right\} - \max \left\{ e_{\leq k-6}(P), e_{\leq k-4}(P) \right\} - e_{\leq 2}(P).
\]

Then we obtain the desired result by applying Corollary 2 to $k = \frac{n-6}{2}$, (6) and the following upper bound of $\max \left\{ e_{\leq k-6}(P), e_{\leq k-4}(P) \right\}$ derived from (5):

\[
\max \left\{ e_{\leq k-6}(P), e_{\leq k-4}(P) \right\} \leq \max \left\{ \left( 2^{11} \right)^{\frac{1}{3}} n(n-4)^{\frac{1}{3}}, \left( 2^{11} \right)^{\frac{1}{3}} n(n-2)^{\frac{1}{3}} \right\} = \left( 2^{11} \right)^{\frac{1}{3}} n(n-2)^{\frac{1}{3}}.
\]
Remark 4. Comparing with the upper bound of $u_{\frac{n-t}{2}}$ included in Lemma 1 of [6], we obtain that for $n \geq 63,370$, the lower bound
\[
e_{\frac{n-t}{2}}(n) \geq \frac{n^2 - n}{2} - \sum_{s=0}^{\frac{n-t}{2}} \left( \left\lfloor \frac{n}{n-1} \right\rfloor \left( \frac{31,827}{211} \right)^{\frac{1}{3}} (n-1)(n-(4s+3))^{\frac{1}{3}} \right) - \left( \left\lfloor \frac{31,827}{211} \right\rfloor n(n-1)^{\frac{1}{3}} \right)
\]
is better than the lower bound for $e_{\frac{n-t}{2}}(n)$ of [6]. For these values of $n$, the lower bound included in Corollary 4 sometimes improves (24) by one, and then it is the best current lower bound of $e_{\frac{n-t}{2}}(n)$. As an example, we get the improvement for the following values of $n$:
63,374, 63,380, 63,386, 63,392, 63,398, 63,404, 63,408, 63,410, 63,414, 63,416, 63,420, 63,426, 63,430, 63,436, 63,440, 63,444, 63,450, 63,454, 63,456, 63,460, 63,464, 63,468.

Remark 5. Plugging the lower bound included in Corollary 4 in (2), we obtain an improvement of 5 for the lower bound of $e\tau(n)$ for the aforementioned values of $n$ in the range $[63,370,63,470]$ because the coefficient of $e_{\frac{n-t}{2}}(n)$ in (2) is 5.

3. Generalization

We can apply Corollary 2 to improve the lower bound of $e_{\frac{n-t}{2}}(n)$ in at most $\left\lceil \frac{n}{4} \right\rceil$ for fixed $t$, $n > t$, $n$ and $t$ with the same parity, by a generalization of the Corollaries 3 and 4.

Proposition 1. It is satisfied that
\[
e_{\frac{n-t}{2}}(n) \geq \frac{n^2 - n}{2} - \sum_{s=0}^{n-t} \left( \left\lfloor \frac{n}{n-1} \right\rfloor \left( \frac{31,827}{211} \right)^{\frac{1}{3}} (n-1)(n-(4s+3))^{\frac{1}{3}} \right) + \left( \left\lfloor \frac{31,827}{211} \right\rfloor n(n+2-(4s+3))^{\frac{1}{3}} \right)
\]

for odd $n$, $t \equiv 3(4)$, $t \geq 7$,
\[
e_{\frac{n-t}{2}}(n) \geq \frac{n^2 - n}{2} - \sum_{s=0}^{n-t} \left( \left\lfloor \frac{n}{n-1} \right\rfloor \left( \frac{31,827}{211} \right)^{\frac{1}{3}} (n-1)(n-(4s+1))^{\frac{1}{3}} \right) + \left( \left\lfloor \frac{31,827}{211} \right\rfloor n(n+2-(4s+1))^{\frac{1}{3}} \right) - \left( \left\lfloor \frac{31,827}{211} \right\rfloor n(n-1)^{\frac{1}{3}} \right)
\]

for odd $n$, $t \equiv 1(4)$, $t \geq 5$,
\[
e_{\frac{n-t}{2}}(n) \geq \frac{n^2 - n}{2} - \sum_{s=0}^{n-t} \left( \left\lfloor \frac{n}{n-1} \right\rfloor \left( \frac{31,827}{211} \right)^{\frac{1}{3}} (n-1)(n-4s)^{\frac{1}{3}} \right) + \left( \left\lfloor \frac{31,827}{211} \right\rfloor n(n+2-4s)^{\frac{1}{3}} \right) - \left( \left\lfloor \frac{31,827}{211} \right\rfloor n(n-1)^{\frac{1}{3}} \right)
\]

for even $n$, $t \equiv 0(4)$, $t \geq 4$ and
\[ \epsilon_{\leq \frac{n}{2}}(n) \geq \frac{n^2 - n}{2} - \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \left( \left\lfloor \frac{n}{n-1} \left( \frac{31,827}{2^{11}} \right)^{\frac{1}{3}} (n-1)(n-(4s+2))^{\frac{1}{3}} \right\rfloor \right) + \left\lfloor \frac{31,827}{2^{11}} \frac{1}{3} n(n+2-(4s+2))^{\frac{1}{3}} \right\rfloor - \left\lfloor \frac{31,827}{2^{11}} \frac{1}{3} n(n-2)^{\frac{1}{3}} \right\rfloor - \left\lfloor \frac{31,827}{2^{11}} \frac{1}{3} (n-1)^{\frac{1}{3}} \right\rfloor \tag{28} \]

for even \( n, t \equiv 2(4), t \geq 6. \)

**Proof.** Assume that \( P \) is a set in which \( \epsilon_{\leq \frac{n}{2}}(n) \) is attained.

For odd \( n, t \equiv 3(4), t \geq 7 \) we have that:

\[
e_{\leq \frac{n}{2}}(n) = \frac{n^2 - n}{2} - \sum_{s=0}^{\frac{n-3}{2}} \left( e_{\frac{n-5}{2}}(P) + e_{\frac{n-7}{2}}(P) \right) = \]

\[
n^2 - n - \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \left( \min \left\{ e_{\frac{n-5}{2}}(P), e_{\frac{n-7}{2}}(P) \right\} + \max \left\{ e_{\frac{n-5}{2}}(P), e_{\frac{n-7}{2}}(P) \right\} \right) - \epsilon_{\frac{n}{2}}(P). \tag{29} \]

For odd \( n, t \equiv 1(4), t \geq 5 \) we have that:

\[
e_{\leq \frac{n}{2}}(n) = \frac{n^2 - n}{2} - \sum_{s=1}^{\frac{n-3}{2}} \left( e_{\frac{n-5}{2}}(P) + e_{\frac{n-7}{2}}(P) \right) - \epsilon_{\frac{n}{2}}(P) = \]

\[
n^2 - n - \sum_{s=1}^{\lfloor \frac{n}{2} \rfloor} \left( \min \left\{ e_{\frac{n-5}{2}}(P), e_{\frac{n-7}{2}}(P) \right\} + \max \left\{ e_{\frac{n-5}{2}}(P), e_{\frac{n-7}{2}}(P) \right\} \right) - \epsilon_{\frac{n}{2}}(P). \tag{30} \]

For even \( n, t \equiv 0(4), t \geq 4 \) we have that:

\[
e_{\leq \frac{n}{2}}(n) = \frac{n^2 - n}{2} - \sum_{s=1}^{\frac{n-4}{2}} \left( e_{\frac{n-6}{2}}(P) + e_{\frac{n-8}{2}}(P) \right) - \epsilon_{\frac{n}{2}}(P) = \]

\[
n^2 - n - \sum_{s=1}^{\lfloor \frac{n}{2} \rfloor} \left( \min \left\{ e_{\frac{n-6}{2}}(P), e_{\frac{n-8}{2}}(P) \right\} + \max \left\{ e_{\frac{n-6}{2}}(P), e_{\frac{n-8}{2}}(P) \right\} \right) - \epsilon_{\frac{n}{2}}(P). \tag{31} \]

For even \( n, t \equiv 2(4), t \geq 6 \) we have that:

\[
e_{\leq \frac{n}{2}}(n) = \frac{n^2 - n}{2} - \sum_{s=1}^{\frac{n-4}{2}} \left( e_{\frac{n-6}{2}}(P) + e_{\frac{n-8}{2}}(P) \right) - \epsilon_{\frac{n}{2}}(P) - \epsilon_{\frac{n}{2}}(P) = \]

\[
n^2 - n - \sum_{s=1}^{\lfloor \frac{n}{2} \rfloor} \left( \min \left\{ e_{\frac{n-6}{2}}(P), e_{\frac{n-8}{2}}(P) \right\} + \max \left\{ e_{\frac{n-6}{2}}(P), e_{\frac{n-8}{2}}(P) \right\} \right) \]

\[ - \epsilon_{\frac{n}{2}}(P) - \epsilon_{\frac{n}{2}}(P). \tag{32} \]

Then we have the desired results by applying the bound of Corollary 2, (5), and (6). \( \square \)
Remark 6. As an example, for \( t = 11 \equiv 3(4) \) and \( n \) an odd number, we obtain that for \( n \geq 122,487 \), the lower bound

\[
e_{\frac{31,827}{2^{11}}} (n) \geq \frac{n^2 - n}{2} - \left( \frac{31,827}{2^{11}} \right)^\frac{1}{3} n(n - 3)^\frac{1}{3} - \left( \frac{31,827}{2^{11}} \right)^\frac{1}{3} n(n - 7)^\frac{1}{3} - \left( \frac{31,827}{2^{11}} \right)^\frac{1}{3} n(n - 5)^\frac{1}{3}
\]  

(33)

is better than the lower bound for \( e_{\frac{31,827}{2^{11}}} (n) \) of [6]. For these values of \( n \), the lower bound included in Proposition 1 sometimes improves (33) by two, and then it is the best current lower bound of \( e_{\frac{31,827}{2^{11}}} (n) \). As a matter of fact, we get the improvement for every odd value of \( n \) in the range \([122,487,122,587]\) except for the following values: 122,533, 122,547, 122,577, 122,583.

4. Conclusions

We have improved the current lower bound on the maximum number of \( \leq k \)-edges for planar sets of \( n \) points when \( k \) is close to \( \frac{n}{2} \) for some values of \( n \). To do this, we have applied an upper bound of \( \min\{e_k(P), e_{k-1}(P)\} \) that is a function of \( e_k(n - 1) \), where \( e_k(P) \) is the number of \( s \)-edges of a set \( P \) of \( n \) points, and \( e_k(n - 1) \) is the maximum number of \( k \)-edges over all the sets \( Q \) with \( n - 1 \) points. This sometimes improves by one the upper bound of \( \min\{e_k(P), e_{k-1}(P)\} \) due to Dey (see [8]).

As a consequence, we have shifted the lower bound of the rectilinear crossing number of \( n \) points in the plane for some large values of \( n \). This reduces the gap with the current best upper bound for these values of \( n \), closing in the exact value of \( \pi(n) \).

An open problem is to determine whether these improvements are attained for infinite values of \( n \). In order to do this, it is enough to prove that, for \( k \) close to \( \frac{n}{2} \) and, for infinite values of \( n \), the bound of expression (15) improves by one unit the bound of (16).

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