Two consensus formation methods
for allocation of persons to positions

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Abstract
We consider how to assign candidate persons to suitable positions based on preference data with candidate persons’ preference for each position. There are n voters, m candidates to be assigned for k positions. Each voter ranks candidates up to s ranks according to their preference with respect to each position. Two methods are proposed to build a consensus by aggregating these voting data. Based on the consensus, we assign candidates to positions. Finally we discuss further research problems

Key words: Voting data, Aggregating methods, Consensus formation methods, Allocation of candidates

1. Introduction
For ranking of candidates, one of the most familiar methods is to compare the weighted sum of their votes, after determining suitable weights of each alternative. Borda [1] initially proposed the “Method of Marks” more than two hundred years ago so as to obtain an agreement among different opinions. His method is surely a useful method evaluating consumers’ preferences of commodities in marketing, or in ranking social policies in political sciences, for instance. It is, however, difficult to determine suitable weight of each alternative a priori. In this context, Cook and Kress [2] formulated the measure to automatically decide on the total rank order weight in order to hold the advantage using Data Envelopment Analysis model. Later, Green et al. [3] evolved the measure so as to make it possible to decide on the total rank order of all candidates. We consider how to assign candidate persons to suitable positions based on preference data with respect to candidate persons for each position. Based on these, we propose two consensus formation models to allocate persons to positions.

First we review related work briefly in section 2. After that section 3 formulates one model and proposes an allocation method of persons to positions. Section 4 formulates the other method and proposes another allocation method of persons to positions. Section 5 discusses possibility of other suitable methods of allocation as further research problems.

2. Related Works
Mathematical method to make a consensus is one of group decision method ([7]) and though it is a DEAlike method based on the voting data. Therefore First we review Data envelopment analysis (abbreviated as DEA) is a mathematical evaluation method for measuring the efficiency of decision-making units (DMU) on the basis of the observed data practiced in comparable DMUs, such as public departments (governments, universities, libraries, hospitals, etc), banking, etc. DEA is originated by Charnes et al.[8] and extended by Banker et al. [9].

The Basic DEA models are known as CCR and BCC named after the authors’ initials. For measuring efficiency of a DMU, we use the virtual input and output aggregating inputs and output by weights as follows:

Let an input vector of DMU be \( (x_i, x_{i2}, \ldots, x_{im}) \) and an output \( (y_{ij}, y_{i2j}, \ldots, y_{iks}) \), that is, \( m \) kind of inputs and \( s \) kind of outputs. We aggregate inputs by weights \( v_1, v_2, \ldots, v_m \) and make a virtual input \( v_1x_{ij} + v_2x_{i2j} + \cdots + v_mx_{im} \). Also aggregating outputs by weights \( u_1, u_2, \ldots, u_s \), we make a virtual output \( u_1y_{ij} + u_2y_{i2j} + \cdots + u_sy_{iks} \). Then we define

\[
\text{efficiency} = \frac{\text{virtual output}}{\text{virtual input}}.
\]

These weights are decided by the following linear fractional programming for each DMU \( O_i \), \( O = 1, 2, \ldots, n \).
place be 12, preference. That is, each person has votes and votes \( k \) as the \( k \)-th place. Second, we calculate the weighted sum of captured votes by \( k \) candidates, as follows. That is, using favorable weights for each DMU, weights between rank \( t \) and \( t+1 \) are equal, we lose the information and if \( w_{jk} = 0 \), it is equivalent to vote till rank \( k-1 \). Therefore we replace it by the following condition

\[
 w_{ji} \geq 2w_{j2} \geq 3w_{j3} \geq \cdots \geq k w_{jk}, \quad w_{jk} \geq \frac{1}{(k + \cdots + 1) \times k}
\]

(see [9]) and consider the following linear programming problem.

\[
\begin{align*}
\max & \quad \phi_i = \sum_{t=1}^k w_{ij} y_{jt} \\
\text{subject to} & \quad \phi_i = \sum_{t=1}^k w_{ij} y_{jt} \leq 1, \quad q = 1, 2, \ldots, m, \quad i = 1, 2, \ldots, k, \\
& \quad w_{1j} \geq w_{2j} \geq \cdots \geq w_{kj} \geq 0
\end{align*}
\]

From the optimal values of these linear programming problem, we make the following table.

| Candi | Date 1 | Date 2 | … | Date n |
|-------|--------|--------|---|--------|
| Candidate 1 | \( \phi_1 \) | \( \phi_2 \) | … | \( \phi_m \) |
| … | \( \phi_1 \) | \( \phi_2 \) | … | \( \phi_m \) |
| … | \( \phi_1 \) | \( \phi_2 \) | … | \( \phi_m \) |
| Candidate n | \( \phi_n \) | \( \phi_n \) | … | \( \phi_n \) |

Table 1. Cross evaluation

Based on Table 1, we calculate the geometric mean \( \tilde{\phi}_i = \sqrt[k]{\phi_1 \cdot \phi_2 \cdots \phi_m} \) for each candidate \( i = 1, 2, \ldots, n \). We rank candidates according to the value of geometric mean, that is, candidate with greatest one as the first rank, the next greatest, the second so on. This ranking method is called cross evaluation (see [3]). For more methods, please see Ogawa [11].

### 3. The First Model

There are \( n \) voters, \( m \) candidates to be assigned for \( k \) positions. Each voter makes ranking candidates till \( s \) ranks according to their preference with respect to each position.

Let denote the ranking data for each position and each voter with

\[ A'(t) = \{ a_{ij}(t) \} \]

where

\[ a_{ij}(t) = \begin{cases} 
1 & \text{if voter } \ell \text{ considers candidate } i \\
0 & \text{as rank } j \text{ with respect to position } t \\
\ell & \text{otherwise}
\end{cases} \]

for \( i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, s \quad \ell = 1, 2, \ldots, n, \quad t = 1, 2, \ldots, k. \)

Note that if voter \( \ell \) considers some candidate \( T \) is not suitable to position \( t \) at all, then
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\[ a^7_{r_1}(t) = a^7_{r_2}(t) = \cdots = a^7_0(t) = 0, \] that is, all components of \( T \) row in the ranking matrix \( A^t(t) \) equal 0.

One method of consensus formation for a suitable allocation of candidates to positions is as follows:

1. We calculate the sum
   \[ \sum_{i=1}^{n} a'_o(t) = v_o(t), \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, s \]

2. Based on the data \( v_o(t) \), we make ranking of candidates with respect to each position \( t = 1, 2, \ldots, k \) utilizing the cross evaluation method. Note that each component is positive and sum of components is 1 corresponding to the eigenvalue \( k \) of the pairwise comparison matrix (4) since \( Fw = \lambda w, \quad w = (w_1, w_2, \ldots, w_k)' \) as is easily checked. This is an ideal case and an actual case, the pairwise comparison matrix includes inconsistency. For example consider the following pairwise comparison matrix:

   \[
   F = \begin{pmatrix}
   1 & 1 & 1 \\
   3 & 1 & 2 \\
   5 & 2 & 1
   \end{pmatrix}
   \]

   Ideally (1,2) component \( F \) should be the product of (1,3) and (3,2) = \( \frac{1}{5} \times 2 = \frac{2}{5} \) but it is \( \frac{1}{3} \). But if the consistency index \( c_i = \frac{\lambda_{\max} - k}{k-1} \lambda_{\max} \) : the greatest eigenvalue of \( F \) and ideally =0) is small (<1), we can use an eigen vector corresponding to eigenvalue \( k \) as a weight vector. For above \( F \), \( \lambda_{\max} \approx 3.004 \), \( k=3 \) and \( c_i = \frac{3.004-3}{2} = 0.002 \approx 0.1 \).

   So we calculate the weight vector corresponding to the eigenvalue 3 and it becomes
   \[ w = (w_1, w_2, w_3)' = \left( \frac{41}{215}, \frac{84}{215}, \frac{18}{43} \right)' = (0.1907, 0.3907, 0.4186)' \]

   As an example, consider 3 positions with the weight as above, 4 candidates and 5 voters with the following preference matrices for each positions.

   (Position 1):

   \[
   A^t(1) = \begin{pmatrix}
   1 & 0 & 0 \\
   0 & 1 & 0 \\
   0 & 0 & 1
   \end{pmatrix},
   A^t(1) = \begin{pmatrix}
   1 & 0 & 0 \\
   0 & 1 & 0 \\
   0 & 0 & 1
   \end{pmatrix},
   A^t(1) = \begin{pmatrix}
   1 & 0 & 0 \\
   0 & 1 & 0 \\
   0 & 0 & 1
   \end{pmatrix}
   \]

   Voter 1 considers candidate 1 as the first rank, candidate 2 as the second rank, candidate 3 as the third rank and candidate 4 is not suitable (no rank). Voter 2 considers
candidate 1 as rank 1, candidate 2 as rank 3, candidate 3 as rank 2 and candidate 4 no rank. Voter 3 considers candidate 1 as rank 1, candidate 2 as rank 2, candidate 3 no rank and candidate 4 as rank 3. Voter 4 considers candidate 1 as rank 2, candidate 2 as rank 1, candidate 3 as rank 3 and candidate 4 as no rank. Preference of voter 5 is same as voter 1. Then captured vote number $v_{ij}$ are as follows:

\[
\begin{array}{cccccc}
11 & 12 & 13 & 21 & 22 & 23 \\
31 & 32 & 33 & 41 & 42 & 43 \\
\end{array}
\]

Based on $v_{ij}$, we construct the matrix $B(1)$ using cross evaluation.

| Table 2. Result of Cross evaluation for position 1 |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1               | 2               | 3               | 4               | Mean            |
| 1               | 1               | 1               | 1               | 1               |
| 2               | 1/2             | 1/2             | 1/2             | 1/2             |
| 3               | 1/3             | 1/3             | 1/3             | 1/3             |
| 4               | 1/18            | 1/18            | 1/27            | 2/27            |
| $w_1$           | 1/48            | 1/3             | 2/9             | 2/9             |
| $w_2$           | 1/12            | 1/6             | 1/9             | 1/9             |
| $w_3$           | 1/18            | 1/18            | 2/27            | 2/27            |

Second column of Table 2 is be obtained by solving the following linear programming problem.

Maximize $\phi_i = 4w_i + w_{ij}$

subject to $4w_i + w_{ij} \leq 1$, $w_i + 3w_{ij} + w_{ij} \leq 1$, $w_i + 3w_{ij} \leq 1$, $w_i \leq 1$, $w_i \geq 2w_{ij} \geq 3w_{ij}$, $w_{ij} \geq \frac{1}{18}$.

Therefore we rank candidates as follows: Candidate 1 the first rank, candidate 2 the second rank, candidate 3 the third rank and candidate 4 the fourth rank.

That is,

\[
B(1) = \begin{bmatrix} 1000 \\ 0100 \\ 0010 \\ 0001 \end{bmatrix}.
\]

(Applying the cross evaluation, we have the following ranking: Candidate 1 as rank 2, candidate 2 as rank 4, candidate 3 as rank 3 and candidate 4 as rank 1. Therefore

\[
B(3) = \begin{bmatrix} 0100 \\ 0010 \\ 0001 \\ 1000 \end{bmatrix}.
\]

Based on $B(1)$, $B(2)$ and $B(3)$, we have

\[
A(2) = \begin{bmatrix} 100 \\ 010 \\ 000 \\ 001 \end{bmatrix}, A^T(2) = \begin{bmatrix} 100 \\ 001 \end{bmatrix}, A(3) = \begin{bmatrix} 100 \\ 000 \end{bmatrix}, A^T(3) = \begin{bmatrix} 000 \\ 001 \end{bmatrix}, A^T(4) = \begin{bmatrix} 001 \\ 010 \end{bmatrix}.
\]

Therefore

\[
A(3) = \begin{bmatrix} 000 \\ 001 \\ 010 \\ 100 \end{bmatrix}, A^T(3) = \begin{bmatrix} 100 \\ 010 \end{bmatrix}.
\]

Applying the cross evaluation, we have the following ranking: Candidate 1 as rank 2, candidate 2 as rank 4, candidate 3 as rank 3 and candidate 4 as rank 1.

Therefore

\[
B(3) = \begin{bmatrix} 0100 \\ 0010 \\ 0001 \\ 1000 \end{bmatrix}.
\]
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\[
p_{i1} = w_1 = \frac{41}{215}, \quad p_{i2} = 2w_2 = \frac{168}{215}, \quad p_{i3} = 2w_3 = \frac{36}{43},
\]
\[
p_{i1} = 2w_1 = \frac{82}{215}, \quad p_{i2} = 2w_2 = \frac{84}{215}, \quad p_{i3} = 4w_3 = \frac{72}{43},
\]
\[
p_{i1} = 3w_1 = \frac{123}{215}, \quad p_{i2} = 3w_2 = \frac{252}{215} = \frac{54}{43},
\]
\[
p_{i1} = 4w_1 = \frac{164}{215} \quad p_{i2} = 4w_2 = \frac{336}{215}, \quad p_{i3} = w_3 = \frac{18}{43}
\]

and so we consider the following transportation problem

\[
P: \text{Minimize} \quad \frac{41}{215} x_{11} + \frac{168}{215} x_{12} + \frac{36}{43} x_{13} + \frac{82}{215} x_{21} + \frac{84}{215} x_{22} + \frac{72}{43} x_{23} + \frac{123}{215} x_{31} + \frac{252}{215} x_{32} + \frac{54}{43} x_{33} + \frac{164}{215} x_{41} + \frac{336}{215} x_{42} + \frac{18}{43} x_{43}
\]

subject to \(x_{11} + x_{12} + x_{13} \leq 1, \quad x_{21} + x_{22} + x_{23} \leq 1, \quad x_{31} + x_{32} + x_{33} \leq 1, \quad x_{41} + x_{42} + x_{43} \leq 1, \quad x_{11} + x_{21} + x_{31} + x_{41} = 1, \quad x_{12} + x_{22} + x_{32} + x_{42} = 1, \quad x_{13} + x_{23} + x_{33} + x_{43} = 1, \quad x_j = 0 \text{ or } 1, \quad i = 1, 2, 3, 4, \quad j = 1, 2, 3.

Optimal solution of this problem is:

\[
x_{11} = 1, \quad x_{12} = 1, \quad x_{13} = 1, \quad \text{other } x_j = 0.
\]

Therefore we should allocate candidate 1 to position 1, candidate 2 to position 2 and candidate 4 to position 3.

4. The Second Model

We assume that the ranking data for each position and each voter

\[A'(t) = (a'_{ij}(t))\]

same as Section 2 is given. Further importance of each position \(i\) is calculated as \(w_i\) by the pairwise comparison matrix in Section 2. Then the second method is given as follows:

1. Sort the positions according to non-increasing order of importance and let result be \(w_{(1)} \geq w_{(2)} \geq \cdots \geq w_{(4)}\).

2. Assign the candidates to position from the order \(t(1), t(2), \ldots, t(k)\) one by one as follows:

First for \(A' \{t(1)\} = (a'_{ij}(t(1)))\), we calculate

\[v_j(t(1)) = \sum_{i=1}^{n} a'_{ij}(t(1)), \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, s\]

that is, captured vote number by candidate \(i\) as \(j\)-th rank for position \(k\). Then using cross evaluation of ranking method based on DEA, we rank the candidates according to geometric means with respect to scores of cross evaluation for \(t(1)\). Then assign the candidate \((C(1))\) with the greatest mean to the position \(t(1)\). Let \(AC = \{C(1)\}\).

3. For \(t(u), \ u = 1, 2, \ldots, k\), execute the following operation iteratively.

Calculate

\[v_j(t(u)) = \sum_{i=1}^{w} a'_{ij}(t(u)), \quad i \in \{1, 2, \ldots, m\} - AC, \quad j = 1, 2, \ldots, s\]

that is, captured vote number by candidate \(i\) as \(j\)-th rank for position \(t(u)\). Then using cross evaluation of ranking method based on DEA, we rank the candidates according to the geometric means with respect to scores of cross evaluation for \(t(u)\). Let the candidate with greatest mean among \(i \in \{1, 2, \ldots, m\} - AC\) be \(C(u)\) and \(AC = AC \cup \{C(u)\}\).

4. Finally we obtain an allocation of candidate person \(C(u)\) to position \(t(u), \ u = 1, 2, \ldots, k\).

Using the same data for example of the first method in section 3, we illustrate the second method.

5. Since weight vector correspondence to importance, most important position is position 3, second one position 2 and third one position 1. That is, \(t(1) = 3, t(2) = 2, t(3) = 3\).

6. First we consider the position 3 and from the result of cross evaluation shown in the example of section 3, assign candidate 4(\(C(1)\)) to position 3. \(AC = \{4\}\).

7. For position 2

\[v_{i1} = 2, v_{i2} = 1, v_{i3} = 2, v_{i4} = 3, v_{i5} = 1, v_{i6} = 0, v_{j2} = 2, v_{j3} = 1\]

and \(C(2) = 2\) by cross evaluation. This is same as the first method. \(AC = \{4, 2\}\). For position one,

\[v_{i1} = 2, v_{i2} = 1, v_{i3} = 0, v_{i4} = 0, v_{j2} = 1, v_{j3} = 3\]

8. Finally optimal allocation is same as the first method in section 3.

5. Conclusion

We have proposed two methods for allocating candidates to positions. Though there does not exist the best method, depending the situation and data type, we should endeavor to find a better method to aggregate preferences of group, i.e., consensus formation. One important factor is to assure the fairness and the other is the suitability of allocation such as effectiveness.

But in this paper we do not consider the ties nor future change of preferences. For that purpose, we need fuzziness of preference and scenario analysis. These are future research problems.
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