GENERAL INITIAL DATA FOR A CLASS OF PARABOLIC EQUATIONS INCLUDING THE CURVE SHORTENING PROBLEM

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ABSTRACT. The Cauchy problem for a class of non-uniformly parabolic equations including (4) is studied for initial data with less regularity. When \( m \in (1, 2] \), it is shown that there exists a smooth solution for \( t > 0 \) when the initial data belongs to \( L^p_{\text{loc}}, p > 1 \). When \( m > 2 \), the same results holds when the initial data belongs to \( W^{1,p}_{\text{loc}}, p \geq m - 1 \). An example is displayed to show that a smooth solution may not exist when the initial data is merely in \( L^p_{\text{loc}}, p > 1 \). Solvability of the weak solution is also studied.

1. Introduction. Consider the one dimensional non-uniformly parabolic equation

\[
u_t = (a(u_x))_x, \quad (x, t) \in \mathbb{R} \times (0, T),
\]

where \( a \) is a smooth function defined on \( \mathbb{R} \) satisfying, for all \( p \in \mathbb{R}, \)

\[
a(0) = 0, \quad |a(p)| \leq C_0, \quad C_0 \text{ a constant,}
\]

and

\[
a'(p) > 0.
\]

While (3) is imposed to ensure parabolicity, (2) that requires \( a \) be bounded globally
is the most distinctive feature in this paper. A model case is given by

\[
u_t = \frac{u_{xx}}{(1 + u_x^2)^{m/2}}, \quad m > 1,
\]

that is, \( a'(p) = (1 + p^2)^{-m/2} \) in (1). The special cases \( m = 2 \) and 3 correspond
respectively to the curve shortening flow Gage-Hamilton[17], Grayson[20] and the
one-dimensional case of the mean curvature equation studied in Gerhardt[18] and

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Ecker[11]. In Chou-Kwong[7] it is proved that the Cauchy problem for $(1)$ under (3) and the condition
\[
\int^\infty \frac{a'(s)}{s} \, ds < \infty,
\] (5)
admits a smooth solution for every smooth initial data. A preceding work is Ecker-Huisken[12] where the long time existence of the evolution of an entire graph under the mean curvature flow is established for any smooth initial graph. This property is in sharp contrast with uniformly parabolic equations such as the heat equation where its Cauchy problem is well-posed only for initial data under certain exponential growth.

Here we study to what extent that global solvability remains valid when the regularity of the initial data is substantially reduced. Recall the celebrated theorem of Widder[24] asserting that to every non-negative solution of the heat equation $u$, there associates a unique Radon measure $\mu$ satisfying
\[
\mu([-x,x]) \leq Ce^{\alpha x^2}, \text{ some } \alpha > 0,
\] such that
\[
\lim_{t \to 0^+} \int u(x,t)\varphi(x) \, dx = \int \varphi(x) \, d\mu, \quad \forall \varphi \in C_c(\mathbb{R}),
\] (6)
and, conversely, there exists a unique non-negative smooth solution of the heat equation to each Radon measure $\mu$ satisfying this growth condition so that (6) holds. This theorem was extended to second order uniformly parabolic linear equations in divergence form in arbitrary dimensions in Aronson[3]. The measure $\mu$ is called the initial trace for the solution. Initial traces for the porous medium equation are studied in Aronson-Caffarelli [4], Bénilan et al [5] and Dahlberg-Kenig [9]. For the closely related evolution $p$-Laplacian equation one may consult also Zhao[26] and Wu, et al [25]. As the curve shortening flow is the Euclidean invariant version of the heat equation, it is natural to ask whether there is an initial trace for this or the more general equation (1). Indeed, it is not hard to show that the initial trace really exists. Specifically, for every non-negative solution of (1) under (2) and (3), there exists a Radon measure $\mu$ so that (6) holds. Yet unlike the heat equation, there is no growth restriction on the measure. Furthermore, it can be shown that the initial trace is a continuous measure, that is, $\mu(\{x\}) = 0$ for every $x$. Thus, for instance, the Dirac delta, which is the initial trace of the heat kernel, cannot be the initial trace of any solution to (1) under (2). Can we solve (1) under (2) and (3) for any given initial continuous Radon measure? This basic question has prompted us to a systematic study on the solvability of (1) for very general initial data. According to the Lebesgue decomposition of measures, every continuous Radon measure $\mu$ on the real line can be written as $u_0\mathcal{L}^1 + \lambda$ where $u_0$ is locally integrable with respect to the Lebesgue measure $\mathcal{L}^1$ and $\lambda$ is a continuous singular Radon measure. Disregarding the singular part, our first result in this direction is

**Theorem A.** Consider (4) where $m \in (1, 2]$. For every $u_0 \in L^p_{\text{loc}}(\mathbb{R})$, $p > 1$, there exists a solution $u$ in $C^\infty(\mathbb{R} \times (0, \infty))$ which converges locally to $u_0$ in $L^p$-norm as $t \to 0^+$. 
There exists a unique solution \( \text{Theorem B.} \)

Consider data from \( L \) to retain solvability, we find that it is necessary to fortify the regularity of the initial data. For instance, in Angenent\([1,2]\), a general parabolic flow for closed curves on a surface whose normal velocity is driven by some curvature function is studied for initial curves which are either locally Lipschitz continuous or possess integrable curvature. In the level set approach for the mean curvature flow Evans-Spruck\([13,14,15]\) and Chen-Giga-Goto\([6]\), viscosity solutions are found for initial integrable curvature. In the level set approach for the mean curvature flow Evans-studied for initial curves which are either locally Lipschitz continuous or possess curves on a surface whose normal velocity is driven by some curvature function is initial data. For instance, in Angenent\([1,2]\), a general parabolic flow for closed see\([17]\) and\([8]\) for details. Nevertheless, there are some works for more general solutions, do we need to push up the regularity of the initial data from \( L \) to \( W^{1,p} \)? The answer is affirmative. We will construct an initial data by a sequence of smooth initial data and solve the corresponding problem to obtain a sequence of approximating smooth solutions. Then, by establishing various \textit{a priori} estimates, we can pass to limit to obtain the desired

The curve shortening problem was mainly investigated in the smooth category, see \([17]\) and \([8]\) for details. Nevertheless, there are some works for more general initial data. For instance, in Angenent\([1,2]\), a general parabolic flow for closed curves on a surface whose normal velocity is driven by some curvature function is studied for initial curves which are either locally Lipschitz continuous or possess integrable curvature. In the level set approach for the mean curvature flow Evans-Spruck\([13,14,15]\) and Chen-Giga-Goto\([6]\), viscosity solutions are found for initial integrable curvature. In the level set approach for the mean curvature flow Evans-studied for initial curves which are either locally Lipschitz continuous or possess curves on a surface whose normal velocity is driven by some curvature function is initial data. For instance, in Angenent\([1,2]\), a general parabolic flow for closed see\([17]\) and\([8]\) for details. Nevertheless, there are some works for more general solutions, do we need to push up the regularity of the initial data from \( L \) to \( W^{1,p} \)? The answer is affirmative. We will construct an initial data by a sequence of smooth initial data and solve the corresponding problem to obtain a sequence of approximating smooth solutions. Then, by establishing various \textit{a priori} estimates, we can pass to limit to obtain the desired

\[ \int u(x,t)\zeta(x,t)\,dx - \int u(x,s)\zeta(x,s)\,dx = \int_s^t \int (u\zeta_t - a(u_x)\zeta_x)\,dxdt. \]  

\[ t \to 0^+. \]  

\textbf{Theorem C.} Consider \( a \) a satisfies \( (2) \), \( (3) \) and

\[ a'(p) \leq \frac{C}{(1 + p^2)^{\delta/2}}, \quad \forall p \in \mathbb{R}, \]  

for positive \( \delta \). For every \( u_0 \in W^{1,p}_{\text{loc}}(\mathbb{R}) \), \( p > 1 \), there exists a weak solution in \( \mathbb{R} \times (0,\infty) \) which converges to \( u_0 \) uniformly on every bounded interval as \( t \to 0^+ \).

Our approach to these theorems is quite standard, namely, first we approximate the initial data by a sequence of smooth initial data and solve the corresponding problem to obtain a sequence of approximating smooth solutions. Then, by establishing various \textit{a priori} estimates, we can pass to limit to obtain the desired
solutions. Theorems A, B and C will be proved in the first three sections. In Section 4, we construct a weak solution of (4) in $W^{1,p}(\mathbb{R})$, $1 < p < m/2$, for $m > 2$, whose gradient blows up at the origin for all time. It illustrates the fact that instant regularizing effect does not hold when the initial datum is not sufficiently regular. In the appendix, we discuss the existence of initial trace for the solution of (1) when $a$ is continuous and bounded.

2. Some uniform estimates. In this section we derive some a priori estimates to be used in the proofs of the theorems. In order to obtain smooth solutions, we need uniform bounds on the solution as well as its gradient.

Throughout this paper let $I \subseteq J$ be two bounded, open intervals and $d$ the distance from $I$ to the boundary of $J$. Also let $\varphi$ be a smooth function in $J$ which is positive in the interior of $J$ and vanishes at its boundary.

Our first estimate is concerned with a uniform bound of the solution in $I \times [t_1, T]$ in terms of the $L^p$-norm of the initial data in $J$. Although it is primarily concerned with equation (4), we start by examining the more general equation (1) under (2) and (3). According to these assumptions, we have the structural inequality, for all $p \in \mathbb{R}$,

$$|p| \leq \alpha a(p) + \beta ,$$

(9)

where $\alpha = \max\{a(1)^{-1}, |a(-1)|^{-1}\}$ and $\beta = \max\{a(1), |a(-1)|\}$.

**Proposition 2.1.** Let $u$ be a smooth solution of (1) under (2) and (3) in $\mathbb{R} \times [0, T]$, $T \in (0, \infty)$. Given $I, J, t_1$ and $p$ where $t_1 \in (0, T]$ and $p > 1$, there exists a constant $C$ depending on $d, t_1, T, p, \parallel u(\cdot, 0) \parallel_{L^p(J)}$ such that

$$\parallel u \parallel_{L^\infty(I \times [t_1, T])} \leq C .$$

(10)

**Proof.** We will establish the following two estimates. First, for each $q \geq p$ and $t_1 \in (0, T]$, there is some constant $C_1$ such that

$$\parallel u(\cdot, t) \parallel_{L^q(I)} \leq C_1 , \quad \forall \ t \in [t_1, T] .$$

(11)

Next, for each $t_1 \in (0, T]$, there is some constant $C_2$ such that

$$\parallel u_x(\cdot, t) \parallel_{L^1(I)} \leq C_2 , \quad \forall \ t \in [t_1, T] .$$

(12)

Clearly (10) follows from combining (11) and (12).

**Step 1.** For $\varepsilon \in (0, 1]$, let $u_\varepsilon = (u^2 + \varepsilon)^{1/2}$. $u_\varepsilon$ is positive, smooth and tends to $|u|$ uniformly in compact sets as $\varepsilon \to 0^+$. For $p > 1$, we have

$$\frac{d}{dt} \int u_\varepsilon^p \varphi^p dx = p \int u_\varepsilon^{p-2} u(a(u_x)) \varphi^p dx .$$

Performing integration by parts,

$$\frac{d}{dt} \int u_\varepsilon^p \varphi^p dx + p \int u_\varepsilon^{p-4}((p-1)u^2 + \varepsilon) u_x a(u_x) \varphi^p dx$$

$$= -p^2 \int u_\varepsilon^{p-2} u(a(u_x)) \varphi^{p-1} \varphi_x dx .$$

(13)
The right hand side of (13) can be estimated by
\[
\left| p^2 \int u_x^{p-2} u a(u_x) \varphi x^{p-1} \varphi x \, dx \right| \leq p^2 C_0 \int u_x^{p-1} \varphi x^{p-1} |\varphi x| \, dx
\leq p^2 C_0 \left( \int u_x^p \varphi x \, dx \right)^{\frac{p-1}{p}} \left( \int |\varphi x|^p \, dx \right)^{\frac{1}{2}}
\leq C_3 \left( \int u_x^p \varphi x \, dx \right)^{\frac{p-1}{p}}, \quad C_3 = \frac{8p^2 C_0}{d^{1-1/p}}. (14)
\]

Using this estimate in (13), after dropping the second integral on the left hand side, we have
\[
\frac{d}{dt} \int u_x^p \varphi x \, dx \leq C_3 \left( \int u_x^p \varphi x \, dx \right)^{\frac{p-1}{p}}.
\]

Integrating this differential inequality from \( s \) to \( t \) yields
\[
\left( \int u_x^p (\cdot, t) \varphi x \, dx \right)^{1/p} \leq \left( \int u_x^p (\cdot, s) \varphi x \, dx \right)^{1/p} + C_4 (t - s) , \quad 0 \leq s < t \leq T , (15)
\]
where \( C_4 \) depends on \( C_3 \) and \( p \).

Now, integrate (13) and use (14) and (15) \((s = 0)\) to get
\[
\int_0^t \int u_x^{p-1} ((p - 1) u^2 + \varepsilon) u_x a(u_x) \varphi x \, dx \, ds
\leq C \left( 1 + \int u_x^p (\cdot, 0) \varphi x \, dx \right).
\]

In view of (9) and \((p - 1) u^2 + \varepsilon \geq \min\{1, p - 1\} u_x^2\), we have
\[
\int_0^t \int u_x^{p-2} |u_x| \varphi x \, dx \, ds \leq C_5 \left( 1 + \int u_x^p (\cdot, 0) \varphi x \, dx \right), \quad (16)
\]
where \( C_5 \) depends further on \( C_4, T, p \) and \( \alpha \).

**Step 2.** Recall the Gagliardo-Nirenberg interpolation inequality [16]: For a smooth function \( w \) on the bounded interval \( K \), there is some constant \( C \) such that
\[
\left( \int_K |w|^q \, dx \right)^{\frac{1}{q}} \leq C \left[ \left( \int_K |w|^p \, dx \right)^{1 - \theta} \left( \int_K |w_x| \right)^{\theta} + \left( \int_K |w|^p \right)^{\frac{1}{p}} \right],
\]
where \( 1/q = (1 - \theta)/p, p > 1, \) and \( \theta \in (0, 1) \). We choose \( q = p + 1 \) so that \( q \theta = 1 \) and raise this inequality to the \( q \)-th power to get
\[
\int_K |w|^{p+1} \, dx \leq C \left[ \left( \int_K |w|^p \, dx \right) \left( \int_K |w_x| \right) + \left( \int_K |w|^p \right)^{\frac{p+1}{p}} \right],
\]
Next we replace \( p \) by \( p/(p - 1) \) in this inequality, fix the interval \( K, I \subset K \subset J \) and let \( w = u_x^{p-1} \) to get
\[
\int_K u_x^{2p-1} \, dx \leq C_6 \left[ \left( \int_K u_x^p \, dx \right) \left( \int_K u_x^{p-3} |u_x|^2 \, dx \right) + \left( \int_K u_x^p \right)^{\frac{2p-1}{p}} \right].
\]
we arrive at
\[ \varepsilon \]
Now, letting \( \varphi \) be chosen so that it is identically equal to 1 on \( K \), we arrive at
\[
\int_0^t \int_K u_\varepsilon^{2p-1} \leq C_6 \left[ \left( \sup_{(0,T)} \int_K u_\varphi^p \right) \left( \int_0^t \int_K u_\varepsilon^{p-2} |u_x| \right) + \int_0^t \left( \int_K u_\varphi^p dx \right)^{\frac{2p-1}{p}} \right].
\] 
By (15) \((s = 0)\) and (16) where \( \varphi \) is chosen so that it is identically equal to 1 on \( K \), we arrive at
\[
\int_0^t \int_K u_\varepsilon^{2p-1} dt dx \leq C_7 \left( 1 + \left( \int_J u_\varphi^p (\cdot, 0) dx \right)^{\frac{2p-1}{p}} \right),
\] 
where \( C_7 \) depends on \( C_4, C_5 \) and \( C_6 \).

On the other hand, we integrate (15) from 0 to \( t \) to get
\[
\int u_\varepsilon^p (\cdot, t) \varphi^p dx \leq C_4 \left( 1 + \frac{1}{t} \int_0^t \int u_\varphi^p \varphi^p dx ds \right),
\] 
where this time we choose \( \varphi \) to be supported on \( K \) and equals to 1 on \( I \). This estimate is valid for all \( p > 1 \). In particular, applying it to \( 2p - 1 \) and combining with (17), we finally obtain
\[
\int_I u_\varepsilon^{2p-1} (\cdot, t) dt \leq C \left( 1 + \frac{1}{t} + \frac{1}{t} \left( \int_J u_\varphi^p (\cdot, 0) dx \right)^{\frac{2p-1}{p}} \right).
\]
Now, letting \( \varepsilon \) tend to 0 to conclude
\[
\int_I |u|^{2p-1} (\cdot, t) dt \leq C, \quad t \in [t_1, T],
\] 
for some constant \( C \) depending on \( C_0, \alpha, d, t_1, T, p \) and \( \|u(\cdot, 0)\|_{L^p(J)} \).

We are ready to prove (11). Clearly it suffices to show it for \( q = p + k(p - 1) \) for all \( k \geq 2 \). WLOG assume \( I = (x_0 - r, x_0 + r) \) and \( J = (x_0 - r - d, x_0 + r + d) \). Let \( I_j = (x_0 - r - jd/k, x_0 + r + jd/k) \) so that \( I_0 = I \) and \( I_k = J \) and let \( s_j = jt_1/k, j = 0, \ldots, k \) so that \( s_0 = 0 \) and \( s_k = t_1 \). Applying (18) to the pair \( I_{j-1} \) and \( I_j \) taking \( s_j \) to be the initial time, we see that
\[
\int_{I_{j-1}} |u|^{p+j(p-1)} dx, \quad t \in [s_j, T],
\] 
can be controlled by
\[
\int_{I_{j-1}} |u|^{p+j(p-1)} dx.
\] 
Therefore, beginning from \( j = 0 \), we can iterate this process and obtain our desired conclusion in \( k \) many steps.

**Step 3.** We prove (12). In view of (13), WLOG we may assume \( p \geq 2 \). Taking \( p = 2 \) in (16) (replace 0, \( t \) by \( t_1/2, T \) respectively) yields
\[
\int_{t_1/2}^T \int_I |u_x| dx dt \leq C,
\] 
for some constant \( C \).

Now let \( \mathcal{A}(p) \) be the primitive function of \( a \) satisfying \( \mathcal{A}(0) = 0 \). We have
\[
\frac{d}{dt} \int \mathcal{A}(u_x) \varphi^2 dx = - \int a'(u_x)^2 u_{xx}^2 \varphi^2 dx - 2 \int a(u_x)a'(u_x)u_{xx} \varphi^2 dx
\]
Proposition 2.2. Let \( \varphi \) have

\[
\frac{1}{2} \int a'(u_x)^2 u_{xx}^2 \varphi^2 \, dx + 2 \int a^2(u_x) \varphi_x^2 \, dx.
\]

Choosing \( \varphi \) equals to 1 on \( I \) and \( |\varphi_x| \leq 2/d \), we have

\[
\int A(u_x(\cdot, t)) \varphi^2 \, dx + \frac{1}{2} \int_s^t \int a'(u_x) u_{xx} \varphi^2 \, dx dt
\]

\[
\leq \int A(u_x(\cdot, s)) \varphi^2 \, dx + \frac{16C_2^2 T}{d}, \quad 0 < s < t < T.
\]

Integrating this inequality in \( s \) from \( t_1/2 \) to \( t \) and applying (19) to the RHS, we have

\[
(t - \frac{t_1}{2}) \int_I A(u_x(\cdot, t)) \varphi \, dx \leq C(1 + T).
\]

Now (12) follows after observing that for \( |p| \geq 1 \),

\[
A(p) \geq \beta(|p| - 1), \quad \beta = \max\{a(1), |a(-1)|\}.
\]

Next, we derive a uniform interior gradient estimate for the solution.

**Proposition 2.2.** Consider (1) where \( a \) satisfies (2), (3) and the following condition

\[
p a''(p) + (1 + \delta) a'(p) \leq 0, \quad \forall p, \quad |p| \geq p_0,
\]

for some \( \delta \in (0, 1) \) and \( p_0 \). For every smooth solution of (1) in \( J \times [0, T] \),

\[
\|u_x(\cdot, t)\|_{L^\infty(I)} \leq C, \quad \forall t \in [0, T],
\]

where \( C \) depends on \( \delta, d, \|u\|_{L^\infty(J \times [0, T])} \) and \( \|u_x(\cdot, 0)\|_{L^\infty(J)} \). If in addition

\[
a'(p) \geq \frac{\rho}{p^2}, \quad \forall p, \quad |p| \geq p_0,
\]

for some positive \( \rho \), then

\[
\|u_x(\cdot, t)\|_{L^\infty(I)} \leq \frac{C}{t}, \quad t \in (0, T),
\]

holds where the constant \( C \) depends on \( \delta, \rho, d \) and \( \|u(\cdot, 0)\|_{L^\infty(J \times [0, T])} \).

For the equation (4), (20) holds for \( m > 1 \) and (22) holds for \( m \leq 2 \).

**Proof.** We adapt the method in [23]. Consider the function

\[
g(x, t) = \eta(t) \varphi(x)(u + M + 1) \log(1 + u_x^2),
\]

where \( \eta \) is either the constant 1 or \( t \) and \( M = \sup_{J \times [0, T]} |u| \) on \( J \times [0, T] \). This function vanishes along the lateral boundary. Let us assume its maximum is attained at some \( (x_1, t_1) \in J \times (0, T) \) where \( u_x(x_1, t_1) \geq 1 \). (The case where \( u_x(x_1, t_1) \leq -1 \) can be handled similarly.)

At \( (x_1, t_1) \), we have \( (\log g)_x = 0, \ (\log g)_{xx} \leq 0, \) and \( (\log g)_t \geq 0 \), that is,

\[
\frac{\varphi_x}{\varphi} + \frac{u_x}{u + M + 1} + \frac{2u_x u_{xx}}{(1 + u_x^2) \log(1 + u_x^2)} = 0;
\]

\[
\frac{\varphi_{xx}}{\varphi} - \frac{\varphi_x^2}{\varphi^2} + \frac{u_{xx}}{u + M + 1} - \frac{u_x^2}{(u + M + 1)^2} + \frac{2u_{xx}}{(1 + u_x^2) \log(1 + u_x^2)} = 0.
\]
By (1), this inequality simplifies to
\[ u \text{whenever}\]
\[ 2\eta \frac{u t}{u + M + 1} + \frac{2u_x u_{xt}}{(1 + u_x^2) \log(1 + u_x^2)} \geq 0 . \]  
(26)

By differentiating (1), we have
\[ u_{tx} = a'(u_x)u_{xxx} + a''(u_x)u_{xx}^2 . \]  
(27)

Using (25), (27) and (26),
\[ \frac{(1 + u_x^2) \log(1 + u_x^2)}{2u_x} \left[ -\frac{\varphi_{xx}}{\varphi} + \frac{\varphi_x^2}{\varphi^2} - \frac{u_{xx}}{u + M + 1} + \frac{u_x^2}{(u + M + 1)^2} \right] \]
\[ - \frac{u_x^2}{u_x} + \frac{2u_x u_{xx} \log(1 + u_x^2) + 1}{(1 + u_x^2) \log(1 + u_x^2)} \geq u_{xxx} \]
\[ \geq \frac{1}{a'(u_x)} \left[ -\frac{\eta(1 + u_x^2) \log(1 + u_x^2)}{2\eta u_x} - \frac{u_t(1 + u_x^2) \log(1 + u_x^2)}{2(u + M + 1)u_x} - a''(u_x)u_{xx}^2 \right] . \]

By (1), this inequality simplifies to
\[ \frac{a'(u_x)(1 + u_x^2) \log(1 + u_x^2)}{2u_x} \left[ -\frac{\varphi_{xx}}{\varphi} + \frac{\varphi_x^2}{\varphi^2} + \frac{u_x^2}{(u + M + 1)^2} \right] \]
\[ - \frac{a'(u_x)u_{xx}^2}{u_x} + \frac{2a'(u_x)u_{xx}^2 \log(1 + u_x^2) + 1}{(1 + u_x^2) \log(1 + u_x^2)} \geq \frac{\eta(1 + u_x^2) \log(1 + u_x^2)}{2\eta u_x} - a''(u_x)u_{xx}^2 . \]

Moving the terms containing \( u_{xx}^2 \) to the right hand side, the coefficient of \( u_{xx}^2 \) is given by
\[ \frac{a'(u_x)}{u_x} - \frac{2a'(u_x)u_{xx} \log(1 + u_x^2) + 1}{(1 + u_x^2) \log(1 + u_x^2)} - a''(u_x) . \]

In view of (20), we can find some \( C_1 \geq \max\{1,p_0\} \) depending on \( \delta \) such that this coefficient is greater than
\[ \frac{\delta a'(u_x)}{2u_x} \]
whenever \( u_x(x_1,t_1) \geq C_1 \). Assuming this, we have
\[ \frac{\delta a'(u_x)}{2u_x} \]
\[ \leq \frac{a'(u_x)(1 + u_x^2) \log(1 + u_x^2)}{2u_x} \left[ -\frac{\varphi_{xx}}{\varphi} + \frac{\varphi_x^2}{\varphi^2} + \frac{u_x^2}{(u + M + 1)^2} \right] \]
\[ + \frac{\eta(1 + u_x^2) \log(1 + u_x^2)}{2\eta u_x} . \]  
(28)

Now, we take \( \eta \equiv 1 \) to get
\[ \delta u_{xx}^2 \leq \left[ -\frac{\varphi_{xx}}{\varphi} + \frac{\varphi_x^2}{\varphi^2} + \frac{u_x^2}{(u + M + 1)^2} \right] (1 + u_x^2) \log(1 + u_x^2) . \]
Using (24) to eliminate \( u_{xx} \),
\[
\delta \left( \frac{\varphi_x}{\varphi} + \frac{u_x}{u + M + 1} \right)^2 \frac{(1 + u_x^2) \log(1 + u_x^2)}{4u_x^2} \leq - \frac{\varphi_{xx}}{\varphi} + \frac{\varphi_x^2}{\varphi^2} + \frac{u_x^2}{(u + M + 1)^2}.
\]

Using \((a + b)^2 \geq a^2/2 - b^2\) in the LHS of this inequality,
\[
\frac{\delta}{2} \frac{u_x^2}{(u + M + 1)^2} \times \frac{(1 + u_x^2) \log(1 + u_x^2)}{4u_x^2} \leq \frac{\delta \varphi_x^2}{\varphi^2} \times \frac{(1 + u_x^2) \log(1 + u_x^2)}{4u_x^2} - \frac{\varphi_{xx}}{\varphi} + \frac{\varphi_x^2}{\varphi^2} + \frac{u_x^2}{(u + M + 1)^2}.
\]

For large \( u_x \), the last term in the RHS can be absorbed to the LHS. Hence, there is some constant \( C_2 \geq C_1 \) such that whenever \( u_x(x_1, t_1) \geq C_2 \),
\[
\frac{\delta}{4} \left( \frac{u_x^2}{(u + M + 1)^2} \times \frac{(1 + u_x^2) \log(1 + u_x^2)}{4u_x^2} \right) \leq \frac{\delta \varphi_x^2}{\varphi^2} \times \frac{(1 + u_x^2) \log(1 + u_x^2)}{4u_x^2} - \frac{\varphi_{xx}}{\varphi} + \frac{\varphi_x^2}{\varphi^2}.
\]

Hence either
\[
\frac{\delta}{4} \left( \frac{u_x^2}{(u + M + 1)^2} \times \frac{(1 + u_x^2) \log(1 + u_x^2)}{4u_x^2} \right) \leq 2 \frac{\delta \varphi_x^2}{\varphi^2} \times \frac{(1 + u_x^2) \log(1 + u_x^2)}{4u_x^2},
\]
or
\[
\frac{\delta}{4} \left( \frac{u_x^2}{(u + M + 1)^2} \times \frac{(1 + u_x^2) \log(1 + u_x^2)}{4u_x^2} \right) \leq -2 \frac{\varphi_{xx}}{\varphi} + 2 \frac{\varphi_x^2}{\varphi^2}.
\]

In both cases, we have \( g(x_1, t_1) \) in controlled as asserted, and (21) follows.

Next, we take \( \eta(t) \equiv t \) so that \( g \) vanishes also at \( t = 0 \). Now (28) becomes an inequality of the form \( A \leq B + C \) where
\[
C = \frac{(1 + u_x^2) \log(1 + u_x^2)}{2tu_x}.
\]

Then either \( A \leq 2B \) or \( A \leq 2C \) must hold. In the first case we control \( g(x_1, t_1) \) as before. In the second case, by (24) and (22),
\[
\delta \rho \left( \frac{\varphi_x}{\varphi} + \frac{u_x}{u + M + 1} \right)^2 \frac{(1 + u_x^2) \log(1 + u_x^2)}{4u_x^2} \leq \frac{2}{\ell}.
\]

Using \((a + b)^2 \geq a^2/2 - b^2\),
\[
\frac{\delta \rho u_x^2 (1 + u_x^2) \log^2(1 + u_x^2)}{2(u + M + 1)^2} \leq \frac{\delta \rho \varphi_x^2 (1 + u_x^2) \log^2(1 + u_x^2)}{\varphi^2} + \frac{2}{\ell}.
\]

Again this inequality is in the form \( A \leq B + C \). Both cases \( A \leq 2B \) and \( A \leq 2C \) lead to (23).

3. Existence of smooth solutions. In this section we prove Theorems A and B.

**Proposition 3.1.** Let \( \{ u_n \} \) be a sequence of smooth solutions of (1) in \( \mathbb{R} \times [0, T] \) satisfying (2) and (3) such that, for every \( t \in (0, T] \), \( u_n(\cdot, t) \) converges to some \( u(\cdot, t) \) in local \( L^p \)-norm for some \( p > 1 \). Suppose that \( \{ u_n(\cdot, 0) \} \) converges locally to some \( u_0 \) in \( L^p \)-norm. Then \( u(\cdot, t) \) converges locally to \( u_0 \) in \( L^p \)-norm as \( t \to 0^+ \).
Proof. Setting \( v = u_m \) and \( w = u_n \) for simplicity, we have, for \( \varphi \) as given in Section 1,
\[
\frac{d}{dt} \int ((w - v)^2 + \varepsilon)^{\frac{p}{2}} \varphi^p dx
\]
\[
= -p \int [(w - v)^2 + \varepsilon]^{\frac{p-2}{2}} [(p-1)(w - v)^2 + \varepsilon] (v_x - w_x) (a(v_x) - a(w_x)) \varphi^p dx
\]
\[
- kp \int [(w - v)^2 + \varepsilon]^{\frac{p-2}{2}} (w - v) (a(v_x) - a(w_x)) \varphi^{p-1} \varphi_x dx
\]
\[
\leq C \int [(w - v)^2 + \varepsilon]^{\frac{p-2}{2}} |v - w| \varphi^{p-1} \varphi_x dx
\]
\[
\leq C \left( \int [(w - v)^2 + \varepsilon]^{\frac{p}{2}} \varphi^p dx \right)^{\frac{p-1}{p}} \left( \int |\varphi|^{2p} dx \right)^{\frac{1}{p}}
\]
By integrating this differential inequality and then letting \( \varepsilon \to 0^+ \) and \( m \to \infty \), we obtain
\[
\left( \int |u_n(x,t) - u(x,t)|^p \varphi^k dx \right)^{1/p} \leq \left( \int |u_n(x,0) - u_0|^p \varphi^k dx \right)^{1/p} + C_1 t \frac{1}{p}. \tag{29}
\]
Now, given \( \varepsilon > 0 \), there is some \( n_0 \) such that \( \|u_n(\cdot,0) - u_0\|_{L^p(I)} < \varepsilon \) for all \( n \geq n_0 \). In view of (29), there is some \( t_1 \) such that \( \|u_n(\cdot,t) - u(\cdot,t)\|_{L^p(I)} < 2\varepsilon \) for all \( n \geq n_0 \) and \( t \in (0,t_1) \). By assumption, there is some \( t_2 \) such that \( \|u_{n_0}(\cdot,t) - u_{n_0}(\cdot,0)\|_{L^p(I)} < \varepsilon \) for all \( t \in (0,t_2) \). Therefore, for \( t, 0 < t \leq \min\{t_1, t_2\} \), we have
\[
\|u(\cdot,t) - u_0\|_{L^p(I)}
\]
\[
\leq \|u(\cdot,t) - u_{n_0}(\cdot,t)\|_{L^p(I)} + \|u_{n_0}(\cdot,t) - u_{n_0}(\cdot,0)\|_{L^p(I)} + \|u_{n_0}(\cdot,0) - u_0\|_{L^p(I)}
\]
\[
< 4\varepsilon.
\]

Proof of Theorem A. For \( u_0 \in L^p_{\text{loc}}(\mathbb{R}) \), fix a sequence of smooth functions \( \{u_{n0}\} \) which converges to \( u_0 \) in \( L^p_{\text{loc}} \)-norm locally as \( n \to \infty \). Let \( u_n \) be the corresponding smooth solution of (4) taking \( u_{n0} \) as its initial datum. The existence of this solution is guaranteed by theorem 2.2 in [7] (note that the \( a \) in [7] corresponds to \( a' \) here). Using Proposition 1.1 and Proposition 1.2 (where (20) and (22) hold when \( m \in (1,2) \)), \( \{u_{nx}\} \) is uniformly bounded in every compact subset of \( \mathbb{R} \times (0,\infty) \). Observing that \( w = u_{nx} \) satisfies the equation \( w_t = (a'(w)w_x)_x \), by standard parabolic theory, we obtain uniform estimates of all order on \( \{u_n\} \) over every compact set in \( \mathbb{R} \times (0,\infty) \). By taking a diagonal subsequence, there is a subsequence \( \{u_n\} \) converging smoothly in every compact subset of \( \mathbb{R} \times (0,\infty) \) to a smooth function \( u \) in \( \mathbb{R} \times (0,\infty) \). Clearly, \( u \) solves (4). By Proposition 2.1, \( u(\cdot,t) \) converges to \( u_0 \) in \( L^p \)-norm on every bounded interval as \( t \to 0^+ \), Theorem A holds.

Remark 1. According to proposition 2.3 in [7], the solution constructed in Theorem A is unique when the initial datum belongs to \( L^p_{\text{loc}}(\mathbb{R}) \) when \( p \geq 2 \). It is also unique
when \( u_0 \in L^p(\mathbb{R}), p > 1 \). To see this, let \( u_1 \) and \( u_2 \) be two solutions with the same initial function. First, by (1.3), they are uniformly bounded in \( L^p \)-norm. Next, take \( v = u_2 \) and \( w = u_1 \) in the proof of Proposition 2.1 where we let \( \varepsilon \to 0^+ \) to get

\[
\frac{d}{dt} \iint |u_2 - u_1|^p \varphi^2 dx \leq C \iint |u_2 - u_1|^p |\varphi \varphi_x| dx.
\]

An integration yields

\[
\iint |u_2 - u_1|^p (x,t) \varphi^2 dx \leq C \int_0^t \iint |u_2 - u_1|^p |\varphi \varphi_x| dx dt.
\]

We now choose \( \varphi \) such that \( \varphi(x) = 1, \ x \in (-R, R) \) and vanishes outside \((-R - 1, R + 1)\) with \( |\varphi_x| \leq 2 \). As both \( u_1 \) and \( u_2 \) are in \( L^p(\mathbb{R}) \), the integral on the right hand side tends to 0 as \( R \) goes to infinity, so \( u_2 \) and \( u_1 \) are equal.

**Remark 2.** By examining the proof of Theorem A, we see that this theorem holds also for the general equation (1) satisfying (2), (3), (20) and (22). Note that (5), which ensures the existence of smooth solutions for smooth initial data, holds under (20).

Next, we prove Theorem B.

**Proposition 3.2.** Let \( u \) be a smooth solution of (4) for \( m > 2 \) in \( \mathbb{R} \times [0,T] \) for \( T \in (0, \infty) \). Given \( I, J \) and \( t_1 \in (0,T] \), there exists a constant \( C \) depending on \( d, t_1, T \) and \( \|u(\cdot, 0)\|_{W^{1,m-1}}(1) \) such that

\[
\|u_x(\cdot, t)\|_{L^\infty(I)} \leq C, \quad t \in [t_1,T]. \tag{30}
\]

**Proof.** We have

\[
\frac{d}{dt} \int (u_x^2 + 1)^\frac{p}{2} \varphi^2 dx
\]

\[
= -p \int (u_x^2 + 1)^\frac{p-2-m}{2} (1 + (p-1)u_x^2) u_{xx} \varphi^2 dx
\]

\[
-2p \int (u_x^2 + 1)^\frac{p-2-m}{2} u_x u_{xx} \varphi \varphi_x dx.
\]

Using

\[
2p \int (u_x^2 + 1)^\frac{p-2-m}{2} |u_x||u_{xx}||\varphi \varphi_x| dx
\]

\[
\leq \varepsilon \int (u_x^2 + 1)^\frac{p-2-m}{2} (1 + (p-1)u_x^2) u_{xx} \varphi^2 dx + C \int (u_x^2 + 1)^\frac{p-2-m}{2} \varphi_x^2 dx,
\]

By choosing a suitably small \( \varepsilon \), we have

\[
\frac{d}{dt} \int (u_x^2 + 1)^\frac{p}{2} \varphi^2 dx \leq -\rho \int (u_x^2 + 1)^\frac{p-2-m}{2} u_{xx} \varphi^2 dx + C \int (u_x^2 + 1)^\frac{p-2-m}{2} \varphi_x^2 dx,
\]

for some \( \rho > 0 \). Now, take \( p = m - 1 \) and integrate from 0 to \( t \) to get

\[
\int (u_x^2(x,t) + 1)^\frac{m-1}{2} \varphi^2 dx + \rho \int_0^t \int \frac{u_{xx}^2 \varphi^2}{(u_x^2 + 1)^\frac{p}{2}} dx dt \\
\leq \int (u_x^2(x,0) + 1)^\frac{m-1}{2} \varphi^2 dx + CT. \tag{31}
\]
Applying the mean value theorem to the second integral on the LHS in \((0, t_1/2)\), there is some \(\tau \in (0, t_1/2)\) such that

\[
\int \frac{u_x^2(\cdot, \tau) \varphi^2}{(u_x^2(\cdot, \tau) + 1)^{\frac{3}{2}}} \, dx \leq \frac{2C}{p^2 t_1}.
\]  

(32)

For \(\varphi = 1\) on an interval \(K\) satisfying \(I \subseteq K \subseteq J\). Using (31) and (32), we conclude

\[
\|u_x(\cdot, \tau)\|_{L^\infty(K)} \leq C,
\]

where \(C\) depends on \(t_1\) and \(\|u_x(\cdot, 0)\|_{L^{m-1}(J)}\). Finally, we apply Proposition 1.2 on \(K \times [\tau, T]\) to deduce (30) from (21).

Proof of Theorem B  

With the help of Proposition 2.2, the proof of the existence of a smooth solution in \(\mathbb{R} \times (0, \infty)\) follows along the same lines as in the proof of Theorem A. Moreover, according to Proposition 2.1, the solution \(u(\cdot, 0)\) converges to \(u_0\) in \(L^p\) on every bounded interval as \(t \to 0^+\). Using the fact that the local \(W^{1, m-1}\)-norm of \(u\) is uniformly bounded (see (2.3)), by interpolation,

\[
\|u(\cdot, t) - u_0\|_{L^\infty(I)} \leq C\left(\|u(\cdot, t) - u_0\|_{W^{1,p}(I)}^{\frac{1}{p}} \|u(\cdot, t) - u_0\|_{L^{m-1}(I)}^{\frac{m-2}{p}} + \|u(\cdot, t) - u_0\|_{L^{m-1}(I)}\right).
\]

Hence \(u\) converges to \(u_0\) uniformly on every bounded interval. Uniqueness of the solution follows from proposition 2.3 in [7]. The proof of Theorem B is completed.

4. Proof of Theorem C  

In this section we prove Theorem C. Instead of the uniform estimates in Section 1, we need some integral estimates for this purpose. First, we have

**Proposition 4.1.** Let \(u\) be a smooth solution of (1) under (2) and (3) in \(\mathbb{R} \times [0, T]\) and \(p > 1\). For each pair of \(I\) and \(J\), there is a constant \(C_1\) depending on \(C_0, d\) and \(p\) such that

\[
\left(\int_I |u(x, t)|^p \, dx\right)^{1/p} \leq \left(\int_J |u(x, 0)|^p \, dx\right)^{1/p} + C_1 t, \quad t \in [0, T].
\]  

(33)

If in addition (8) holds, there is a constant \(C_2\) further depending on \(\delta\) (appearing in (8)) such that

\[
\left(\int_I |u_x(x, t)|^p \, dx\right)^{\delta/p} \leq \left(\int_J |u_x(x, 0)|^p \, dx\right)^{\delta/p} + C_2 t, \quad t \in [0, T].
\]  

(34)
Proof. The first estimate can be obtained by letting $\varepsilon \to 0^+$ in (15). For the second estimate, letting $w_\varepsilon = (u_x^2 + \varepsilon)^{1/2}$, we compute
\[
\frac{d}{dt} \int w_\varepsilon^p \varphi^s \, dx = -p \int w_\varepsilon^{p-4} (p-1) u_x^2 \varphi^s \, dx + ps \int w_\varepsilon^{p-2} u_x a'(u_x) \varphi^{s-1} \varphi_x \, dx \leq \frac{p}{2} \int w_\varepsilon^{p-4} (p-1) u_x^2 \varphi^s \, dx + C \int w_\varepsilon^{p-2} u_x^2 \varphi^{s-2} \varphi_x^2 \varphi'(u_x) \, dx.
\]
WLOG we may assume $\delta \in (0, 1)$ in (8). Using it, we have
\[
\int w_\varepsilon^{p-2} u_x^2 \varphi^{s-2} \varphi_x^2 \varphi'(u_x) \, dx \leq C \int w_\varepsilon^{p-\delta} \varphi^{s-2} \varphi_x^2 \, dx \leq C \left( \int w_\varepsilon^p \varphi^s \, dx \right)^{\frac{p-\delta}{p}},
\]
after choosing $s = 2p/\delta$. Plugging this into (35), we have
\[
\frac{d}{dt} \int w_\varepsilon^p \varphi^s \, dx + \frac{p}{2} \int w_\varepsilon^{p-2} u_x^2 \varphi'(u_x) \varphi^s \, dx \leq C \left( \int w_\varepsilon^p \varphi^s \, dx \right)^{\frac{p-\delta}{p}}. \tag{36}
\]
Ignoring the second integral on the left of (36) and integrating the inequality, we have
\[
\left( \int w_\varepsilon^p (\cdot, t) \varphi^s \, dx \right)^{\frac{\delta}{p}} \leq \left( \int w_\varepsilon^p (\cdot, 0) \varphi^s \, dx \right)^{\frac{\delta}{p}} + \frac{\delta C}{p} t, \quad t \in [0, T],
\]
and (34) follows by letting $\varepsilon \to 0^+$ in this inequality. \hfill \square

**Proposition 4.2.** Let $u$ be a smooth solution of (1) under (2) and (3) in $\mathbb{R} \times [0, T]$. For each pair of $I$ and $J$, there exists a constant $C$ depending on $d, T$ and $\|u(\cdot, 0 \|_{L^2(J)}$ such that
\[
\|u_\cdot(t)\|_{L^2(I)} \leq \frac{C}{t}, \quad \forall \ t \in (0, T]. \tag{38}
\]
Proof. We multiply (1) by $t^{1/2} u_x \varphi^2$ and integrate to get,
\[
\int_0^t \int s^2 u_x^2 \varphi^2 \, dx ds = -2 \int_0^t \int s^2 a(u_x) a(u_x) \varphi \varphi_x \, dx ds - \int_0^t \int s^2 A(u_x) \varphi^2 \, dx ds.
\]
On one hand,
\[
\left| \int_0^t \int s^2 a(u_x) \phi \phi_x dx ds \right| = \int_0^t \int s^2 a(u_x)^2 |(\phi \phi_x)_x| dx ds \\
\leq Ct^{\frac{3}{2}}.
\]

On the other hand, using \( \mathcal{A}(p) \leq pa(p) \),
\[
\left| \int_0^t \int s^2 a(u_x) \phi^2 dx ds \right| \leq \frac{3}{2} \int_0^t \int s^2 u_x a(u_x) \phi^2 dx ds \\
+ \int t^{\frac{3}{2}} u_x(x, t)a(u_x(x, t)) \phi^2 dx.
\]

Moreover,
\[
\int_0^t \int s^2 u_x a(u_x) \phi^2 dx ds \\
= -2 \int_0^t \int s^2 u_x (u_x) \phi \phi_x dx ds - \frac{1}{2} \int_0^t \int s^2 (u^2)_t \phi^2 dx ds \\
\leq 2C0 \int_0^t \int s^2 |u| \phi \phi_x dx ds + \frac{1}{4} \int_0^t \int s^{-\frac{3}{2}} u^2 \phi^2 dx ds - \frac{1}{2} \int t^{\frac{3}{2}} u^2(x, t) \phi^2 dx.
\]

From (33), the \( L^2 \)-norm of \( u(\cdot, t) \) can be controlled by the \( L^2 \)-norm of \( u(\cdot, 0) \) locally. Therefore, the integral above is bounded by some constant multiple of \( t^{1/2} \). Putting things together, we have
\[
\int_0^t \int s^2 u_t^2 \phi^2 dx ds \leq Ct^{1/2}, \quad t \in (0, T].
\]

Choosing \( \phi \) to be equal to 1 on \( K \), an interval satisfying \( I \subset K \subset J \), we have
\[
\int_0^t \int_K s^2 u_t^2 dx ds \leq Ct^{\frac{1}{2}}, \quad (39)
\]

With (39) at our disposal, we compute
\[
\frac{d}{dt} \int u_t^2 \phi^2 dx = 2 \int a'(u_x) u_x u_t \phi \phi_x dx - 2 \int a'(u_x) u_{xx}^2 \phi^2 dx \\
\leq \frac{1}{2} \int a'(u_x) u_{xx}^2 \phi^2 dx,
\]

where now \( \phi \) is chosen to be equal to 1 on \( I \) and vanishes outside \( K \). Integrating from \( s \) to \( t \) yields
\[
\int u_t^2(x, t) \phi^2 dx \leq \int u_t^2(x, s) \phi^2 dx + C \int_s^t \int_K u_t^2(\cdot, \tau) dx d\tau.
\]

Multiply both sides of this inequality by \( s^{3/2} \) and then use (39) to get
\[
s^\frac{3}{2} \int u_t^2(x, t) \phi^2 dx \leq \int s^\frac{3}{2} u_t^2(x, s) \phi^2 dx + C \int_s^t \int_K u_t^2(\cdot, \tau) dx d\tau \\
\leq C \int_K s^\frac{3}{2} u_t^2(x, s) dx + Ct^{1/2}.
\]

Then a further integration over \([0, t]\) gives (38).
Proof of Theorem C  Let \( \{ u_n \} \) be a sequence consisting of smooth solutions to (1) in \( \mathbb{R} \times (0, \infty) \) whose smooth initial data \( \{ u_{n0} \} \) converges to \( u_0 \) in local \( W^{1,p} \)-norm as \( n \to \infty \). The existence of these smooth solutions is due to theorem 2.2 in [7]. By (33), (34) and (38), for each bounded interval \( T < \infty \), \( \{ u_n \} \) is uniformly bounded in both \( C^{\alpha}(I \times [t_1, T]) \) and \( L^\infty(0, T; W^{1,p}(I)) \). Therefore, by Ascoli’s theorem and Rellich-Kondrachov compactness theorem, by passing to a subsequence if necessary, we may assume there is some \( u \) belongs to both \( C^{\alpha}_{loc}(\mathbb{R} \times (0, \infty)) \) and \( L^\infty(0, \infty; W^{1,p}_{loc}(\mathbb{R})) \) such that

(a) \( \{ u_n \} \) converges uniformly to \( u \) in every compact subset of \( \mathbb{R} \times (0, \infty) \); and

(b) For each bounded \( I \) and \( T > 0 \),

\[
\int_0^T \int_I |u_n - u| \, dx \, dt \to 0, \quad \text{as} \quad n \to \infty.
\]

Note that (b) follows from (a) and (33).

Each approximate solution \( u_n \) satisfies the defining equation for the weak solution, namely, for \( \zeta \in C^1(\mathbb{R} \times (0, T)) \), vanishing outside \( I \times (0, T) \) where \( I \) is a finite interval, and \( 0 < s < t \),

\[
\int u_n(x,t)\zeta(x,t) \, dx - \int u_n(x,s)\zeta(x,s) \, dx = \int_s^t \int (a u_n) \zeta \, dx \, dt. \quad (40)
\]

In the lemma below, we will show that \( \{ u_{nx} \} \) converges to \( u_x \) in \( L^1(0, T; L^1_{loc}(\mathbb{R})) \) for every \( T \). It follows that we can find a subsequence converging pointwisely to \( u \). Since \( a \) is bounded, applying Lebesgue’s dominated convergence theorem and passing limit in (40), we conclude that \( u \) satisfies (7), that is, it is a weak solution to (1).

Finally, we claim that \( u(\cdot, t) \) converges uniformly to \( u_0 \) on every bounded interval \( I \) as \( t \to 0^+ \). Since \( u(\cdot, t) \) is uniformly bounded in \( W^{1,p}(I) \) for \( t \in (0, 1] \), by interpolation it suffices to show its convergence in \( L^2(I) \). We have

\[
\frac{d}{dt} \int |u_n(x,t) - u_{n0}(x)|^2 \varphi^2 \, dx
\]

\[
= -2 \int (u_n(x,t) - u_{n0}(x)) a(u_n) \varphi \, dx - 4 \int (u_n - u_{n0}) a(u_n) \varphi \varphi_x \, dx.
\]

Therefore, taking \( \varphi \equiv 1 \) on \( I \), in view of (33) and (34),

\[
\int |u_n(x,t) - u_{n0}(x)|^2 \, dx \leq C \int_0^t \int (|ux| + |u_{nx}| + |u| + |u_0|) \, dx \, dt.
\]

By letting \( n \to \infty \), we conclude

\[
\int |u(x,t) - u_0(x)|^2 \, dx \leq C \int_0^t \int (|ux| + |u_{0x}| + |u| + |u_0|) \, dx \, dt \to 0,
\]

as \( t \to 0^+ \). The proof of Theorems C is completed except for the following lemma.

**Lemma 4.3.** Let \( u_0 \in W^{1,p}_{loc}(\mathbb{R}) \) for some \( p > 1 \). Let \( \{ u_n \} \) be a sequence of smooth solutions of (1) under (2), (3) and (8) in \( \mathbb{R} \times [0, \infty) \) with initial data \( \{ u_{n0} \} \) converging to \( u_0 \) in \( W^{1,p}_{loc}(\mathbb{R}) \). Suppose that \( \{ u_n \} \) converges to some \( u \) in \( L^1_{loc}(\mathbb{R} \times [0, \infty)) \). Then \( \{ u_{nx} \} \) converges to \( u_x \) in \( L^1_{loc}(\mathbb{R} \times [0, \infty)) \).
Proof. Let \( v = u_n \) and \( w = u_m \) for simplicity. By differentiating the integral of \((v - w)^2 \varphi \) over \( \mathbb{R} \) and using (1), we obtain the estimate
\[
\int_0^T \int_I (a(v_x) - a(w_x))(v_x - w_x) \, dx \, dt \\
\leq \frac{1}{2} \int_0^T |v_0 - w_0|^2 \, dx + C \int_0^T \int_J |v - w| \, dx \, dt .
\] (41)

where \( C \) is independent of \( v \) and \( w \). By (b), the RHS of this inequality tends to 0 as \( n, m \to \infty \). For \( M > 1 \) to be specified later, we express \( J \times [0, T] \) as the union of the following four subsets:
\[
A = \{ |v_x|, |w_x| \leq M \} , \\
B = \{ |v_x| \in (M, M+1), |w_x| \leq M \} \cup \{ |w_x| \in (M, M+1), |v_x| \leq M \} , \\
C = \{ |v_x| \geq M+1, |w_x| \leq M \} \cup \{ |w_x| \geq M+1, |v_x| \leq M \} , \\
D = \{ |v_x|, |w_x| \geq M \} .
\]

First, by
\[
C(M) \int_0^T \int_A |v_x - w_x|^2 \, dx \, dt \\
\leq \int_0^T \int_A (a(v_x) - a(w_x))(v_x - w_x) \, dx \, dt ,
\]
where \( C(M) = \inf_{|p| \leq M} a'(p) \),
\[
\int_0^T \int_A |v_x - w_x| \, dx \, dt \\
\leq \sqrt{\frac{|I| T}{C(M)}} \sqrt{\int_0^T \int_J (a(v_x) - a(w_x))(v_x - w_x) \, dx \, dt}.
\]

Similarly, observing that \( C(M+1) > 0 \) on \( B \), we have
\[
\int_0^T \int_B |v_x - w_x| \, dx \, dt \\
\leq \sqrt{\frac{|I| T}{C(M+1)}} \sqrt{\int_0^T \int_J (a(v_x) - a(w_x))(v_x - w_x) \, dx \, dt}.
\]

Next, since there exists a positive \( \tilde{C}(M) \) so that
\[
(p - q)(a(p) - a(q)) \geq \tilde{C}(M)|p - q|,
\]
for \( p, q, |q| \leq M, \ |p| \geq M+1 \), or \( |p| \leq M, \ |q| \geq M+1 \), we have
\[
\int_0^T \int_C |v_x - w_x| \, dx \, dt \\
\leq \frac{1}{C(M)} \int_0^T \int_J (a(v_x) - a(w_x))(v_x - w_x) \, dx \, dt.
\]
Finally, by (34),
\[
\int_0^T \int_D |v_x - w_x| \, dx \, dt \\
\leq \frac{1}{M^{p-1}} \int_0^T \int_D (|v_x|^p + |w_x|^p) \, dx \, dt \\
\leq \frac{C_1}{M^{p-1}}.
\]
These estimates together with (41) implies that
\[
\int_0^T \int_I |v_x - w_x| \, dx \, dt \leq \frac{C_1}{M^{p-1}} + C_2(L + L^{1/2}) ,
\]
where \( L \) is given by the RHS of (41). Letting \( m \to \infty \), we obtain
\[
\int_0^T \int_I |u_{nx} - u_x| \, dx \, dt \leq \frac{C_1}{M^{p-1}} + C_2(L_1 + L_1^{1/2}) ,
\]
where now
\[
L_1 = \frac{1}{2} \int_I |u_{00} - u_0| \, dx \, dt + C \int_0^T \int_I |u_n - u| \, dx \, dt .
\]
We know that \( L_1 \) tends to 0 as \( n \to \infty \). Given \( \varepsilon > 0 \), we fix \( M \) so that \( C_1/M^{p-1} < \varepsilon \).
Then there exists some \( n_0 \) such that \( C_2(L_1 + L_1^{1/2}) < \varepsilon \). Hence, for all \( n \geq n_0 \), we conclude that
\[
\int_0^T \int_I |u_{nx} - u_x| \, dx \, dt < 2\varepsilon .
\]

5. A genuine weak solution. To show the necessity of studying weak solutions, we are going to construct a non-trivial weak solution of (4). In this section we take \( m > 2 \) and \( p \in (1, m/2) \), and show that for certain initial data, existence of classical solution is impossible. We note that interior blow-up of the gradient of solutions for equations similar to (4) is studied in Giga[18].

**Step 1.** The construction of sub-solution and super-solution. We set
\[
v(x, t) = \eta(t)x^{m-2}, \quad m > 2, \quad x \in (0, \infty), \quad t \geq 0.
\]
Then,
\[
v_t = \eta x^{m-2}, \quad v_x = \left(\frac{m-2}{m}\right) \eta x^{\frac{m-2}{m}}, \quad v_{xx} = -\frac{2(m-2)}{m^2} \eta x^{\frac{2-m}{m}}.
\]
Substituting this into the equation
\[
v_t - \frac{v_{xx}}{v_x} = 0, \quad x \in (0, \infty),
\]
we see that \( v \) is a solution to this equation if we choose
\[
\eta(t) = C(m)(T-t)^{1/m}, \quad C(m) = \frac{2(m-2)^{1-m}}{m^2-m}.
\]
With this choice of \( \eta \), \( v \) is positive and concave on \((0, \infty)\) with derivative blowing up at origin. It satisfies
\begin{align*}
\begin{cases}
v_t - \frac{v_{xx}}{(1 + v_2^2)^{m/2}} \leq 0, & m > 2, \quad \text{for } (x,t) \in (0, \infty) \times (0,T), \\
v(0,t) = 0, & t \in (0,T), \\
v(x,0) = \eta(0)x^{m/2}, & x \in (0, \infty).
\end{cases}
\end{align*}

Consequently, \( v \) is a subsolution of (4) in \((0, \infty) \times (0,T)\). We extend \( v(t) \) as an odd function to \((-\infty, 0]\). Then it satisfies
\[
v_t - \frac{v_{xx}}{(1 + v_2^2)^{m/2}} \geq 0 \quad \text{on } (-\infty, 0] \times (0,T),
\]
and is therefore a supersolution on the negative side.

**Step 2.** Consider the Cauchy problem
\begin{align*}
\begin{cases}
u_t - \frac{u_{xx}}{(u_2^2 + 1)^{m/2}} = 0, & m > 2, \quad (x,t) \in (-\infty, \infty) \times (0,T), \\
u(x,0) = v(x,0), & x \in (-\infty, \infty).
\end{cases}
\end{align*}

We note that \( v(x,0) \in W^{1,p}_{\text{loc}}(\mathbb{R}) \) with \( p \in (1, m/2) \). By Theorem C, (43) admits a weak solution \( u \).

We will see that \( u \) is not a classical solution for \( t > 0 \). Indeed, assuming the contrary, suppose \( u \) is classical, then \( u(\cdot, t) \) must be odd due to our uniqueness result in [7] (proposition 2.3).

**Step 3.** Finally, by the following proposition, since the derivative of \( v \) blows up at the origin, so does the derivative of \( u \). This contradiction with our smoothness hypothesis on \( u \) shows that \( u \) cannot be a classical solution.

**Proposition 5.1.** Let \( u \) be the smooth solution of (43). Then \( u(x, t) \geq v(x, t) \) for \( t > 0 \) and \( x \in (0, \infty) \), \( u(x, t) \leq v(x, t) \) for \( t > 0 \) and \( x \in (-\infty, 0) \).

**Proof.** **Step 1.** For \( \rho > 0 \) and \( \sigma \in (0, 1) \), we fix a non-negative cut-off function \( \zeta \) such that \( \zeta \equiv 1 \) on \([0, \rho]\), supported on \([0, (1 + \sigma)\rho]\) and satisfying \( |\zeta_x| \leq 1/\sigma \rho \).

Setting
\[ a(s) = \frac{1}{(1 + s^2)^{m/2}}, \]
and \( A(s) \) its primitive function satisfying \( A(0) = 0 \). Recalling that \( v \) is a subsolution of (4) on \((0, \infty)\), we have
\[
(v - u)_t - (A(v_x) - A(u_x))_x \leq 0.
\]

We multiply the equation in (43) by the test function \((v - u)_+ \zeta^2\) and integrate to get
\[
\int_{B_{(1+\sigma)\rho}} (v(T) - u(T))^2 \zeta^2 \, dx \, dt \\
+ 2 \int_0^T \int_{B_{(1+\sigma)\rho \cap \{v > u\}}} |\zeta| |v_x - u_x| |v_x - u_x| \zeta^2 \, dx \, dt \\
\leq \frac{4}{\sigma \rho} \int_0^T \int_{B_{(1+\sigma)\rho \cap \{v > u\}}} |A(v_x) - A(u_x)| |v - u| \zeta \, dx \, dt,
\]
where \( B_\rho \equiv (0, \rho) \), and we have taken into account the oddness of \( u \) and \( v \) so that \( u(0, t) = v(0, t) = 0 \).
Step 2. Next we note that,
\[ |A(v_x) - A(u_x)| = \left| \int_{u_x}^{v_x} a(s) \, ds \right| \leq \|a\|_{L^p(0,\infty)} |v_x - u_x|^{1-\frac{2}{p}}. \]
Setting
\[ E = \left\{ (x, t) \in B_{(1+\sigma)p} \times (0, T) : v(x, t) > u(x, t), \quad (v - u) < \frac{\sigma p}{4} |v_x - u_x| \right\}, \]
and
\[ F = \left\{ (x, t) \in B_{(1+\sigma)p} \times (0, T) : v(x, t) > u(x, t), \quad (v - u) > \frac{\sigma p}{4} |v_x - u_x| \right\}. \]

The right hand side of (44)
\[ \leq \int_E |A(v_x) - A(u_x)||v_x - u_x| \zeta^2 + \frac{4\|a\|_{L^p(\mathbb{R}^+)}}{\sigma p} \int_F |v_x - u_x|^{1-\frac{2}{p}} (v - u)_+ \zeta \]
\[ \leq \int_E |A(v_x) - A(u_x)||v_x - u_x| \zeta^2 + \frac{4\|a\|_{L^p(\mathbb{R}^+)}}{\sigma p} \int_F \frac{1}{(\sigma p \zeta)^{1-\frac{2}{p}}} (v - u)_+^{2-\frac{2}{p}} \zeta \]
\[ = \int_E |A(v_x) - A(u_x)||v_x - u_x| \zeta^2 + \frac{C(p)\|a\|_{L^p(\mathbb{R}^+)}}{(\sigma p)^{2-\frac{2}{p}}} \int_F (v - u)_+^{2-\frac{2}{p}} \zeta \]
Therefore,
\[ \int_{B_p} (v(T) - u(T))^2 \leq \frac{C(p)\|a\|_{L^p(\mathbb{R}^+)}}{(\sigma p)^{2-\frac{2}{p}}} \int_{B_{(1+\sigma)p}} (v - u)_+^{2-\frac{2}{p}} \]
where \(2 - 1/p \in (1, 2)\). This estimate remains valid when the variable \(T\) on the LHS is replaced by \(t\). Integrating both sides with respect to \(t\) yields
\[ \int_0^T \int_{B_p} (v - u)_+^{2} \, dx \, dt \leq \frac{C(p)\|a\|_{L^p(\mathbb{R}^+)}}{(\sigma p)^{2-\frac{2}{p}}} \int_0^T \int_{B_{(1+\sigma)p}} (v - u)_+^{2-\frac{2}{p}} \, dx \, dt \]
Invoking Hölder’s Inequality, we obtain the key estimate
\[ \int_0^T \int_{B_p} (v - u)_+^{2} \, dx \, dt \leq \frac{C(p, T)\|a\|_{L^p(\mathbb{R}^+)}}{(\sigma p)^{2-\frac{2}{p}}} \left[ \int_0^T \int_{B_{(1+\sigma)p}} (v - u)_+^{2} \right]^{1-\frac{2}{p}} \, dx \, dt. \quad (45) \]
Step 3. Now that for \(n \geq 0, \) set
\[ \rho_n = \sum_{j=0}^n \frac{\rho}{2^j}, \quad \sigma_n = \frac{1}{2^{n+1}}, \]
and
\[ Y_n = \int_0^T \int_{B_{\rho_n}} (v(t) - u(t))^2. \]
We can express (45) in the form
\[ Y_n \leq \frac{C(p, T)\|a\|_{L^p(\mathbb{R}^+)}}{\rho^2-\frac{2}{p}} Y_n^{1-\frac{2}{p}} Y_{n+1}^{\frac{2}{p}}. \quad (46) \]
Note that
\[ Y_n \leq \int_0^T \int_{B_2} (v(t)) - u(t))^2 \, \rho, \]
so \( \{Y_n\} \) is uniformly bounded.

Noting that (46) is an iterative inequality of the form
\[ Y_n \leq Cb^n Y_1^{1-\alpha} - \alpha^{n+1}, \]
where
\[ C = \frac{C(p, T)}{\rho^{2-\frac{1}{p}}}, \quad b = 2^{2-\frac{1}{p}} > 1, \]
and \( \alpha = 1/2p \in (0, 1) \). Then standard results, see for instance, [21], imply
\[ Y_0 \leq \left( \frac{2C}{b^{1-\frac{1}{p}}} \right)^{\frac{1}{\alpha}}. \]
Thus,
\[ \int_0^T \int_{B_2} (v(t)) - u(t))^2 \leq \frac{C(p, T, \|a\|_{L^p(\mathbb{R})})}{\rho^{4p-3}}. \]
We obtain the desired conclusion after letting \( \rho \) tend to infinity. Similarly, one can show that \( u \leq v \) on \((\infty, 0) \times (0, T)\).

Appendix. Initial trace. The following theorem asserts that the initial trace of a weak solution of (1) is the distributional derivative of a function in \( L^\infty_{\text{loc}}(\mathbb{R}) \), and in fact a continuous Radon measure when it is non-negative.

**Theorem A.1.** Consider (1) where \( a \) is bounded and continuous in \( \mathbb{R} \). For every weak solution \( u \) of (1), there exists an \( h \) in \( L^\infty_{\text{loc}}(\mathbb{R}) \) such that
\[ \lim_{t \to 0^+} \int u(x, t) \varphi(x) \, dx = - \int h \varphi \, dx, \quad \forall \varphi \in C^1_c(\mathbb{R}). \] (47)
When \( u \) is non-negative almost everywhere, \( h \) is increasing, continuous and there is a continuous Radon measure \( \mu \) such that
\[ - \int h \varphi \, dx = \int \varphi \, d\mu, \quad \forall \varphi \in C^1_c(\mathbb{R}). \] (48)

**Proof.** Let \([a, b]\) be any finite interval and \( \delta > 0 \) a small number. We pick \( \zeta = \zeta(x) \) to be the test function which is equal to one on \([a + \delta, b - \delta] \), vanishes at \( a \) and \( b \), and linear on \([a, a + \delta]\) and \([b - \delta, b]\) respectively. Using \( \zeta \) in (7) and then letting \( \delta \) go to 0, we obtain
\[ \left| \int_a^b u(x, t) \, dx - \int_a^b u(x, s) \, dx \right| \leq C|t - s|, \quad 0 < s < t < 1, \]
where the constant \( C \) depends only on \( a(p) \). Since \( a \) and \( b \) are arbitrary, the function
\[ U(x, t) = \int_0^x u(y, t) \, dy \]
is uniformly bounded in \( t \in (0, 1) \) on every finite interval. Moreover, the limit \( \lim_{t \to 0^+} U(x, t) \) exists at each \( x \).
Let \( \varphi(x) = \int_a^x \psi dx \) where \( \psi \in C_c((a,b)) \) is of zero mean. Then \( \varphi \in C^1_c((a,b)) \) and we can plug it into (7) and perform integration by parts:

\[
- \int U(x,t) \psi(x) dx + \int U(x,s) \psi(x) dx = \int_s^t \int a(U_{xx}) \psi(x) dx.
\]

Taking \( t = 1 \),

\[
\left| \int U(x,s) \psi(x) dx \right| \leq \left| \int U(x,t) \psi(x) dx + \int_0^1 \int a(U_{xx}) \psi(x) dx \right|,
\]

which implies that

\[
\lim_{s \to 0^+} \int U(x,s) \psi(x) dx
\]

defines a linear functional \( T \) on the subspace of the space \( C_c((a,b)) \) consisting of functions of zero mean satisfying

\[
|T\psi| = \lim_{s \to 0^+} \int U(x,s) \psi(x) dx \leq C\|\psi\|_{L^1}.
\]

By Hahn-Banach theorem, \( T \) can be extended to become a bounded linear functional on the space \( L^1((a,b)) \). Furthermore, by \( L^1-L^\infty \) duality, there exists some \( h \in L^\infty((a,b)) \) such that

\[
\lim_{t \to 0^+} \int U(x,t) \psi(x) dx = \int \psi h dx, \quad \forall \psi \in C_c((a,b)), \quad \int \psi dx = 0. \tag{49}
\]

As \( U(\cdot, t) \) is uniformly bounded in \( t \in (0,1) \), by an approximation argument (49) holds for all \( \psi \in C([a,b]) \) with zero mean. For any \( \psi \in C([a,b]) \), \( \psi - \bar{\psi} \) has zero mean (\( \bar{\psi} \) is the mean of \( \psi \)). Plugging \( \psi - \bar{\psi} \) into (49), we have

\[
\lim_{t \to 0^+} \int U(x,t) \psi(x) dx - \int \psi h dx = c \bar{\psi},
\]

where

\[
c = \lim_{t \to 0^+} \int U(x,t) dx - \int h(x) dx.
\]

Setting \( V = U - c/(b-a) \), we have

\[
\int \psi h dx = \lim_{t \to 0^+} \int V(x,t) \psi(x) dx, \quad \forall \psi \in C([a,b]).
\]

Now (47) follows by taking \( \psi \) to be some \( \varphi_x \), \( \varphi \in C^1_c((a,b)) \), in this formula and then integrating by parts.

Let \( h_n \) be the function \( h \) corresponding to the interval \( (-n, n) \), \( n \geq 1 \). From (47), it is clear that \( h_n \) and \( h_{n+1} \) differ by a constant on \( (-n, n) \). By adding a suitable constant to each \( h_n \), \( n \geq 2 \), we obtain an \( h \in L^\infty_{loc}(\mathbb{R}) \) for which (47) holds. Its uniqueness up to a constant is evident.

The function \( h \) is increasing when \( u \) is non-negative. To see this, denote the mollified \( \varphi \) by \( \varphi^\varphi \). By (49), we have

\[
\int h_x^\varphi(x) \varphi(x) dx = - \int h(x) \varphi_x^\varphi(x) dx = \lim_{t \to 0^+} \int u(x,t) \varphi^\varphi(x) dx \geq 0,
\]
for every non-negative \( \varphi \). It follows that \( h^{-\varepsilon} \) is increasing. As \( h^{-\varepsilon} \) converges to \( h \) in \( L^1 \), say, we can extract a subsequence \( \{h^{\varepsilon_j}\} \) converging to \( h \) almost everywhere. By modifying the values of \( h \) on a set of measure zero, we may assume that \( h \) is increasing everywhere.

Given an interval \( (a, b) \), define

\[
\mu((a, b)) = h_+(b) - h_+(a),
\]

where \( h_+(x) \) is right limit of \( h \) at \( x \). It is known \cite{22} that \( \mu \) extends to a unique Radon measure, still denoted by \( \mu \), which satisfies (48). Moreover, \( \mu(\{x\}) = h_+(x) - h_-(x) \) \( (h_-(x) \) is the left limit of \( h \) at \( x ) \) holds. Thus, to show that \( h \) is continuous, it suffices to prove that \( \mu \) is a continuous measure. Assuming that \( \mu(\{x_0\}) = \alpha > 0 \) at some \( x_0 \), we will draw a contradiction. Indeed, by letting \( s \to 0^+ \) in (7), we have

\[
- \int u(x, t) \varphi(x)dx + \int \varphi d\mu = - \int_0^t \int a(u_x) \varphi_x(x)dxdt. \tag{50}
\]

Taking \( \varphi \) to be the test function which is equal to 1 at \( x_0 \), 0 at \( x_0 \pm \delta \) and linear on \( (x_0 - \delta, x_0) \) and \( (x_0, x_0 + \delta) \) respectively, we have

\[
\left| \int_0^t \int a(u_x) \varphi_x(x)dx \right| \leq C_1 t,
\]

for some \( C_1 \) independent of \( \delta \). We fix some \( t_1 \) such that \( C_1 t_1 \leq \alpha/2 \). Then by letting \( \delta \) tend to 0 in (50) at \( t_1 \), the left hand side of (50) tends to \( \alpha \) while the absolute value of the right hand side is less than \( \alpha/2 \), contradiction holds. Hence \( \mu \) must be a continuous measure.

We remark that it is possible to obtain the Radon measure \( \mu \) without appealing to the primitive function \( U \) when the solution is nonnegative. First of all, for \( \varphi \in C^1_c(\mathbb{R}) \), (7) gives

\[
\left| \int u(x, t) \varphi(x)dx - \int u(x, s) \varphi(x)dx \right| \leq C|t - s|. \tag{51}
\]

On the other hand, by taking \( \varphi \) to be a cut-off function in (7), we have

\[
\int_a^b u(x, t)dx \leq \int_{a-1}^{b+1} u(x, 1)dx + C(1 - t), \quad t \in (0, 1), \quad a < b. \tag{52}
\]

Now, for a given \( \varphi \in C_c((a, b)) \) and \( \varepsilon > 0 \), we pick some \( \psi \in C^1_c((a, b)) \) such that \( \|\psi - \varphi\|_C \leq \varepsilon \). By (51) and (52),

\[
\left| \int u(x, t) \varphi dx - \int u(x, s) \varphi dx \right| \leq \left| \int u(x, t)(\varphi - \psi)dx \right| + \left| \int u(x, s)(\psi - \varphi)dx \right|
\]

\[
+ \left| \int u(x, t) \psi dx - \int u(x, s) \psi dx \right|
\]

\[
\leq C(|t - s| + 2\varepsilon),
\]

which shows that the limit \( \lim_{t \to 0^+} \int u(x, t) \varphi dx \) exists. From (52) we know that

\[
\left| \lim_{t \to 0^+} \int u(x, t) \varphi dx \right| \leq C\|\varphi\|_{L^\infty},
\]
so the existence of a Radon measure $\mu$ satisfying

$$\int \varphi d\mu = \lim_{t \to 0^+} \int u(x,t)\varphi(x)dx$$

follows from the Riesz representation theorem.

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