AN IMPROVED BOUND ON THE NUMBER OF POINT-SURFACE INCIDENCES IN THREE DIMENSIONS

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Abstract. We show that $m$ points and $n$ smooth algebraic surfaces of bounded degree in $\mathbb{R}^3$ satisfying suitable nondegeneracy conditions can have at most $O(m^{k-1} n^{\frac{3k-3}{2}} + m + n)$ incidences, provided that any collection of $k$ points have at most $O(1)$ surfaces passing through all of them, for some $k \geq 3$. In the case where the surfaces are spheres and no three spheres meet in a common circle, this implies there are $O((mn)^{3/4} + m + n)$ point-sphere incidences. This is a slight improvement over the previous bound of $O((mn)^{3/4} \beta(m, n) + m + n)$ for $\beta(m, n)$ an (explicit) very slowly growing function. We obtain this bound by using the discrete polynomial ham sandwich theorem to cut $\mathbb{R}^3$ into open cells adapted to the set of points, and within each cell of the decomposition we apply a Turan-type theorem to obtain crude control on the number of point-surface incidences. We then perform a second polynomial ham sandwich decomposition on the irreducible components of the variety defined by the first decomposition. As an application, we obtain a new bound on the maximum number of unit distances amongst $m$ points in $\mathbb{R}^3$.

1. Introduction

In [6], Clarkson, Edelsbrunner, Guibas, Sharir, and Welzl obtained the following bound on the number of incidences between points and spheres in $\mathbb{R}^3$:

**Theorem 1** (Clarkson et al.). The number of incidences between $m$ points and $n$ spheres in $\mathbb{R}^3$ with no three spheres meeting at a common circle is

$$O((mn)^{3/4} \beta(m, n) + m + n),$$

where $\beta(m, n)$ is a very slowly growing function of $m$ and $n$. In particular, $\beta(m, n) \leq 2^{C \alpha(m^3/n)^2}$, where $\alpha(s)$ is the inverse Ackerman function and $C$ is a large constant.

We obtain the following slight sharpening:

**Theorem 2.** Let $k \geq 3$, and let $\mathcal{P} \subset \mathbb{R}^3$ be a collection of $m$ points and $\mathcal{S}$ a collection of $n$ smooth algebraic surfaces of bounded degree (the degree is allowed to depend on $k$) such that for some constant $C$ we have $|S \cap S' \cap S''| \leq C$ for all $S, S', S'' \in \mathcal{S}$, and for any collection of $k$ points in $\mathbb{R}^3$, there are at most $C$ surfaces that contain all $k$ points. Then the number of incidences between points in $\mathcal{P}$ and surfaces in $\mathcal{S}$ is

$$O(m^{k-1} n^{\frac{3k-3}{2}} + m + n),$$

where the implicit constant depends only on $k$, $C$, and the degree of the algebraic surfaces.

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In particular, the number of incidences between \( m \) points and \( n \) spheres in \( \mathbb{R}^3 \) with no three spheres meeting at a common circle is
\[
O((mn)^{3/4} + m + n).
\]

Remark 3. The requirements that every three surfaces meet in \( C \) points and that every \( k \) points have at most \( C \) surfaces passing through them are analogous to the definition of “curves with \( k \) degrees of freedom” from [18], though in [18] the curves do not need to be algebraic.

Remark 4. The requirement that every three surfaces meet in a complete intersection, or some variant thereof, is necessary to prevent the situation in which all of the surfaces meet in a common curve and all of the points lie on that curve, yielding \( mn \) incidences (i.e. if we don’t place any restrictions on how the surfaces can intersect, then the trivial bound of \( mn \) incidences is sharp).

Remark 5. Theorem 1 can be extended to the more general case of bounded degree algebraic surfaces using the decomposition techniques described in [1, §8.3] to obtain an analogue of (2). Doing so yields a bound of \( O(m^{2k^3}n^{k^3-1} + \beta(m, n) + m + n) \) for \( \beta \) a slowly growing function.

1.1. Previous results. Similar results to Theorem 1 and 2 have been obtained by Laa and Solymosi in [16] and by Iosevich, Jorati, and Laba in [12]. In [16] and [12], however, the authors consider a more general class of surfaces (they need not be algebraic), but they require that the point set be “homogeneous” in a suitable sense.

Our techniques do not work well when \( k = 2 \), i.e. for obtaining bounds on point-hyperplane incidences, but this case has been studied by other authors (see e.g. [7], where the authors obtain sharp bounds on point-hyperplane incidences under a slightly different set of non-degeneracy conditions).

1.2. Update 7/4/2011. The author has recently become aware that concurrently with this paper, Kaplan, Matoušek, Safernová, and Sharir in [13] obtained results similar to the bound (3) using similar methods. Kaplan et. al. are able to avoid some of the technical difficulties present in this paper by using an explicit parameterization of the sphere by rational functions.

1.3. Proof sketch. Clarkson et al. obtain Theorem 1 through their “Canham threshold plus divide and conquer” technique: the arrangement of spheres in \( \mathbb{R}^3 \) is subdivided into smaller collections through a careful partitioning of \( \mathbb{R}^3 \), and the number of incidences between these smaller collections of spheres and points is controlled by a Turan-type bound on the number of edges in a bipartite graph with certain forbidden subgraphs.

In this paper, we employ similar ideas, except instead of dividing the problem into smaller subproblems by partitioning \( \mathbb{R}^3 \) into cells using a decomposition adapted to the collection of spheres (or more general nonsingular algebraic surfaces), we employ a partition adapted to the collection of points. This partition is obtained from the discrete polynomial ham sandwich theorem recently used to great effect by Guth and Katz in [11] and more recently by Solymosi and Tao in [19] and by Kaplan, Matoušek, and Sharir in [14]. Specifically, we find a polynomial \( P \) such that the complement of the zero set of \( P \) consists of open “cells,” none of which contain too many points. We can then apply a Turan-type bound to the points
and surfaces inside each cell. However, some points may lie on the zero set of \( P \), and thus do not lie in any of the cells. To deal with these points, we perform a second polynomial ham sandwich decomposition to find a polynomial \( Q \) whose zero set partitions the zero set of \( P \) into cell-like objects, and we apply the Turan-type bound to each of these “cells.” While it is possible that a point could lie in the zero set of both \( P \) and \( Q \), we can use Bézout-type theorems to control how often this can occur.

1.4. Some difficulties with real algebraic sets. There are several technical difficulties that have to be dealt with while executing the above strategy. In contrast to the situation over \( \mathbb{C} \), there exist polynomials \( P_1, \ldots, P_d \in \mathbb{R}[x_1, \ldots, x_d] \) of degrees \( D_1, \ldots, D_d \) such that \( \{ P_1 = 0 \} \cap \ldots \cap \{ P_d = 0 \} \) contains more than \( D_1 \ldots D_d \) isolated points, i.e. the naive analogue of Bézout’s theorem fails over \( \mathbb{R} \). To deal with this problem, we will sometimes be forced to embed our varieties into \( \mathbb{C} \) and use the (usual) Bézout’s theorem (though we have to be careful that the intersection of the embedded varieties does not contain new, unexpected components of positive dimension).

A second difficulty concerns the failure of the Nullstellensatz for varieties defined over \( \mathbb{R} \). In contrast to the complex case, If \( (P) \) is a principal prime ideal and \( Q \) is a real polynomial, it need not be the case that if \( Q \) vanishes identically on \( \{ x \in \mathbb{R}^d : P = 0 \} \) then \( Q \in (P) \). Luckily, there is a special type of ideal known as a “real ideal” for which an analogue of the Nullstellensatz does hold. Frequently we will be required to replace our polynomials with new polynomials that generate real ideals.

Finally, if \( P \in \mathbb{R}[x_1, \ldots, x_d] \) then the dimension of \( \{ x \in \mathbb{R}^d : P = 0 \} \) may be less than \( d - 1 \), and even if \( P \) is squarefree, \( \nabla P \) may vanish on \( \{ P = 0 \} \). Again, we can remedy this problem by working with (irreducible) polynomials that generate real ideals.

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2. Main Result

2.1. Notation. Throughout the paper, \( c \) and \( C \) will denote sufficiently small and large constants, respectively, which are allowed to vary from line to line. We will write \( A \lesssim B \) to mean \( A < CB \), we will write \( A \sim B \) to mean \( cB < A < CB \), and we say that a quantity is \( O(A) \) if it is \( \lesssim A \).

Let \( \mathcal{S} \) be a collection of smooth (real) surfaces and \( \mathcal{P} \) a collection of points. Then \( I(\mathcal{P}, \mathcal{S}) \) is the number of incidences between the surfaces in \( \mathcal{S} \) and the points in \( \mathcal{P} \). If \( S \in \mathcal{S} \) is a surface, then \( f_S \) is the polynomial whose zero set is \( S \).

All ideals and varieties will be assumed to be affine. Unless otherwise specified, all ideals are subsets of \( \mathbb{R}[x_1, \ldots, x_d] \), and all varieties are defined over \( \mathbb{R} \) and thus are subsets of \( \mathbb{R}^d \), though sometimes we will specialize to the case \( d = 3 \). If \( P \) is a polynomial, \( (P) \subset \mathbb{R}[x_1, \ldots, x_d] \) is the ideal generated by \( P \).
Special emphasis will be placed on “real ideals.” These are described in Definition 21 of Appendix A and they should not be confused with ideals that are merely subsets of $\mathbb{R}[x_1, \ldots, x_d]$. On the other hand, a “real variety” is merely a variety defined over $\mathbb{R}$ (as opposed to $\mathbb{C}$).

If $I$ is an ideal, we use
\[ Z(I) = \{ x \in \mathbb{R}^d : P(x) = 0 \text{ for all } P \in I \} \]
to denote the zero set of $I$. If $P$ is a polynomial we shall abuse notation and use $Z(P)$ to denote $Z((P)) = \{ x \in \mathbb{R}^d : P(x) = 0 \}$. If $Z \subset \mathbb{R}^d$ is a real variety, then we define
\[ I(Z) = \{ P \in \mathbb{R}[x_1, \ldots, x_d] : P(x) = 0 \text{ for all } x \in Z \} \]
to be the ideal of polynomials that vanish on $Z$.

If $Z \subset \mathbb{R}^d$ is a real variety, then $Z^* \subset \mathbb{C}^d$ denotes the smallest complex variety containing $Z$. Conversely, if $Z \subset \mathbb{C}^d$ is a complex variety, then $\mathfrak{R}(Z) \subset \mathbb{R}^d$ is its set of real points.

If $Q \subset \mathbb{R}[x_1, \ldots, x_d]$ is a collection of real polynomials, then we can partition $\mathbb{R}^d \setminus \bigcup_{Q \in Q} Z(Q)$ into a collection of open sets such that on each open set, the polynomials from $Q$ do not change sign. These sets will be called the realizations of realizables strict sign conditions of $Q$. Similarly, if $Z \subset \mathbb{R}^d$ is a variety, then we can consider the restriction of the above open sets to $Z$, and these are called the realizations of realizables strict sign conditions of $Q$ on $Z$. These notions are defined more precisely in Appendix A.

2.2. Preliminaries. Following [4], we shall need the following Turan-type bound:

**Theorem 6** (Kővari, Sós, Turán [15]). Let $s, t$ be fixed, and let $G = G_1 \sqcup G_2$ be a bipartite graph with $|G_1| = m$, $|G_2| = n$ that contains no copy of $K_{s,t}$. Then $G$ has at most $O(nm^{1-1/s} + n)$ edges. Symmetrically, $G$ has at most $O(nm^{1-1/t} + n)$ edges. All implicit constants depend only on $s$ and $t$.

In our case, we have that $|S \cap S' \cap S''| \leq C$ for every three surfaces $S, S', S''$, and any $k$ points have at most $C$ surfaces passing through all of them. Thus we have the bounds

\[ I(P, S) \lesssim |P||S|^{1-1/k} + |S|, \]

\[ I(P, S) \lesssim |P|^{2/3}|S| + |P|. \]

Recall the discrete polynomial partitioning theorem from [11]:

**Theorem 7.** Let $P$ be a collection of points in $\mathbb{R}^d$, and let $D > 0$. Then there exists a non-zero polynomial $P$ of degree at most $D$ such that each connected component of $\mathbb{R}^d \setminus Z(P)$ contains at most $O(|P|/D^2)$ points of $P$.

**Remark 8.** Without loss of generality, we can assume that $P$ is squarefree. Indeed if $P$ is not squarefree then we can replace $P$ by its squarefree part, and the new polynomial still has all of the desired properties.

**Example 9.** Consider the set of $24$ points
\[ P_1 = \{ (0, \pm 1, \pm 1), (0, \pm 2, \pm 2), (\pm 1, \pm 1, \pm 1), (\pm 2, \pm 2, \pm 2) \} \subset \mathbb{R}^3, \]
and let $D = 3$. Then the polynomial $P_1(x_1, x_2, x_3) = x_1 x_2 x_3$ partitions $\mathbb{R}^3$ into $8$ octants, each of which contains $2$ points from $P_1$. 
Remark 10. Note as well that in the above example, the 8 points \( \{0, \pm 1, \pm 1\}, \{0, \pm 2, \pm 2\} \) lie on the set \( \mathbf{Z}(\mathcal{P}_1) \) and thus they do not lie inside any of the open components of \( \mathbb{R}^3 \setminus \mathbf{Z}(\mathcal{P}_1) \). This is not merely a consequence of us choosing \( \mathcal{P}_1 \) poorly; it is an unavoidable phenomena that occurs when performing the discrete polynomial partitioning decomposition. In order to control the number of incidences between points lying on \( \mathbf{Z}(\mathcal{P}_1) \) and surfaces in \( \mathcal{S} \), we shall have to perform a second polynomial partitioning decomposition “on” the surface \( \mathbf{Z}(\mathcal{P}_1) \). For technical reasons, we cannot simply consider the complement of our ham sandwich “cut” as a union of relatively open subsets of \( \mathbf{Z}(\mathcal{P}_1) \). Instead, we need to perform a somewhat more detailed decomposition that partitions \( \mathbf{Z}(\mathcal{P}_1) \) into sets that are realizations of realizable strict sign conditions of a certain family of polynomials. This is made precise in the theorem below. See Appendix A for the definition of a real ideal, a strict sign condition, and the realization of a strict sign condition.

**Theorem 11** (Discrete polynomial partitioning theorem on a hypersurface). Let \( \mathcal{P} \) be a collection of points in \( \mathbb{R}^d \) lying on the set \( Z = \mathbf{Z}(\mathcal{P}) \) for \( P \) an irreducible polynomial of degree \( D \) such that \( (P) \) is a real ideal. Let \( \rho > 0 \) be a small constant, and let \( E \geq \rho D \). Then there exists a collection of polynomials \( Q \subset \mathbb{R}[x_1, \ldots, x_d] \) with the following properties:

1. \( |Q| \leq \log_3(DE^{d-1}) + O(1) \).
2. \( \sum_{\mathcal{Q}} \deg Q \leq E \).
3. None of the polynomials in \( Q \) vanish identically on \( Z \).
4. The realization of each of the \( O(DE^{d-1}) \) strict sign conditions of \( Q \) on \( Z \) contains at most \( O(\frac{|P|}{DE^{d-1}}) \) points of \( \mathcal{P} \).

All implicit constants depend only on \( \rho \) and the dimension \( d \).

We shall defer the proof of Theorem 11 to Appendix C. In our applications, we will always have \( d = 3 \).

**Example 12.** Let us continue Example 9. The polynomial \( \mathcal{P}_1 \) from Example 9 was not irreducible, but we can factor it into the three irreducible factors \( x_1, x_2, x_3 \). All of the points lying on \( \mathbf{Z}(\mathcal{P}_1) \) actually lie on the irreducible component \( \mathbf{Z}(x_1) \), so we let \( \mathcal{P}_2(x_1, x_2, x_3) = x_1 \). Note that \( (\mathcal{P}_2) = (x_1) \) is a real ideal and let \( D = \deg(\mathcal{P}_2) = 1 \). Select \( E = 2 \) (which is larger than \( D \)). Then the collection of polynomials \( Q = \{x_2, x_3\} \) satisfies the requirements of Theorem 11. The realizations of realizable strict sign conditions of \( Q \) on \( Z \) are the 4 sets of the form

\[
\{(x_1, x_2, x_3) : x_1 = 0, \pm x_2 > 0, \pm x_3 > 0\}.
\]

Note that each of these sets contains 2 points of \( \mathcal{P}_1 \cap \mathbf{Z}(\mathcal{P}_2) \). Two coincidences occur in this example that are not present in general. First, in this example the realizations of the four strict sign conditions of \( Q \) on \( Z \) correspond to the four connected components of \( Z \setminus \bigcup_{Q} \mathbf{Z}(Q) \). In general, each realization of a strict sign condition may be a union of multiple connected components of \( Z \setminus \bigcup_{Q} \mathbf{Z}(Q) \). Second, each of the polynomials in \( Q \) were irreducible factors of \( \mathcal{P}_1 \). In general this does not occur.

We are now ready to prove Theorem 2.

2.3. **Proof of Theorem 2.**
Proof. Let $S$ and $P$ be as in the statement of Theorem 2. From (4) and (5), we have that if $n > cm^k$ or $m > cn^3$ for some fixed small constant $c > 0$ to be specified later, then Theorem 2 immediately holds. Thus we may assume

\begin{align*}
    n < cm^k, \\
m < cn^3.
\end{align*}

(7)

Let $P$ be a squarefree polynomial of degree at most $D$ ($D$ will be determined later, but the impatient reader can jump to (25)) that cuts $\mathbb{R}^3$ into $\sim D^3$ cells with $O(m/D^3)$ points in each cell, and let $Z = Z(P)$. Let $m_i$ be the number of points lying in the $i$–th cell of the above decomposition, and let $n_i$ be the number of surfaces that meet the interior of the $i$–th cell.

Lemma 13.

\[ \sum n_i \lesssim D^2 n. \] (8)

Proof. Let $S$ be a surface that is not contained in $Z$ and is not entirely contained in the closure of one cell. Since there are finitely many cells, we can select a large closed ball $B \subset \mathbb{R}^3$ so that the number of cells that meet $S \cap B$. We can apply a small generic translation to $S$ and doing so can only increase the number of cells that meet $S \cap B$. Select a generic vector $v \in \mathbb{R}^3$ and let $T(x) = v \wedge \nabla f_S(x) \wedge \nabla P(x)$, so if $x \in S \cap Z$ and $\nabla f_S(x)$ and $\nabla P(x)$ are non-zero and non-collinear, then $T(x) = 0$ if the curve $S \cap Z$ is tangent at $x$ to a plane with normal vector $v$.

For every cell $\Omega$ that meets $S$, there is a point $x \in \partial \Omega \cap S$ satisfying the following properties.

1. $x$ is a smooth point of the space curve $Z \cap S$.
2. $x$ is a non-singular intersection point of $Z(T) \cap Z \cap S$.
3. $x$ is a smooth point of $\partial \Omega$.

These three properties follow from the fact that $v$ is generic and we picked a generic translation of $S$. From Item 3 each point $x$ satisfying the above properties can be associated to at most 2 distinct cells $\Omega, \Omega'$. By Item 2 and the real Bézout inequality (see e.g. [3, §4.7]), there can be at most $\deg(P) \deg(T) \deg(f_s) = O(D^2)$ such points, and thus $S$ can enter at most $O(D^2)$ such cells. Since there are $n$ surfaces $S \in S$, the result follows. \hfill \Box

Using Lemma 13 and the bound from (4) we can control the number of incidences between points not lying in $Z$ and surfaces in $S$:

\[ \mathcal{I}(P \setminus Z, S) = \sum_i \mathcal{I}(P \cap \Omega_i, S) \]

\[ \lesssim \sum_i m_i n_i^{1-1/k} + n_i \]

\[ \lesssim \left( \sum_i m_i^k \right)^{1/k} \left( \sum_i n_i \right)^{1-1/k} + D^2 n \]

\[ \lesssim \left( \frac{D^3 m^k}{D^{3k}} \right)^{1/k} (D^2 n)^{1-1/k} + D^2 n \]

\[ \lesssim \frac{mn^{1-1/k}}{D^{1-1/k}} + D^2 n. \] (9)
We must now control $\mathcal{I}(\mathcal{P} \cap Z, \mathcal{S})$. We have
\[
\mathcal{I}(\mathcal{P} \cap Z, \mathcal{S}) = \mathcal{I}(\mathcal{P} \cap Z, \mathcal{S}_1) + \mathcal{I}(\mathcal{P} \cap Z, \mathcal{S}_2),
\]
where $\mathcal{S}_1$ is the set of surfaces contained in $Z$, and $\mathcal{S}_2$ are the remaining surfaces. Since $Z$ has degree $D$, $Z$ can contain at most $D$ surfaces from $\mathcal{S}$, i.e. $|\mathcal{S}_1| \leq D$. By (9),
\[
\mathcal{I}(\mathcal{P} \cap Z, \mathcal{S}_1) \lesssim |\mathcal{S}_1| |\mathcal{P}|^{2/3} + |\mathcal{P}|
\lesssim Dm^{2/3} + m.
\]

Thus it remains to control $\mathcal{I}(\mathcal{P} \cap Z, \mathcal{S}_2)$. Write $P = P_1 \ldots P_t$, where each $P_j$ is irreducible of degree $D_j$, and let $Z_j = Z(P_j)$. Thus we have $D_1 + \ldots + D_t \leq D$, and $Z = \bigcup Z_j$. We would like to use Lemma 11 perform a second discrete polynomial sandwich decomposition on each variety $Z_j$, but if $(P_j)$ is not a real ideal then we cannot apply the lemma. Luckily, the following lemma lets us remedy this situation.

**Lemma 14.** Let $\mathcal{A} \subset \mathbb{R}[x_1, \ldots, x_d]$ be a collection of irreducible polynomials. Then we can find a new collection $\mathcal{A}'$ of irreducible polynomials such that:

1. $\bigcup_{P \in \mathcal{A}} \mathbb{Z}(P) \subset \bigcup_{P \in \mathcal{A}'} \mathbb{Z}(P)$.
2. $\sum_{P \in \mathcal{A}} \deg P \leq \sum_{P \in \mathcal{A}'} \deg P$.
3. $(P)$ is a real ideal for each $P \in \mathcal{A}'$.

**Proof.** We shall proceed by induction on $\sum_{P \in \mathcal{A}} \deg P$. If the sum is 1 then the result is trivial since in that case $\mathcal{A}$ consists of a single linear polynomial, so we can let $\mathcal{A}' = \mathcal{A}$. Suppose the lemma has been established for all families $\mathcal{A}$ with $\sum_{P \in \mathcal{A}} \deg P < w$, and let $\sum_{P \in \mathcal{A}} \deg P = w$. If $(P)$ is a real ideal for every $P \in \mathcal{A}$ then the result is immediate. If not, select $P \in \mathcal{A}$ such that $(P)$ is not a real ideal. By Proposition 22 in Appendix B, $\nabla P$ vanishes on $\mathbb{Z}(P)$. Let $v \in \mathbb{R}^d$ be a generic unit vector. Then $\mathbb{Z}(P) \subset \mathbb{Z}(\nabla_v P)$ and $\deg(\nabla_v P) < \deg P$. Write $\nabla_v P = Q_1 \ldots Q_a$ as a product of irreducible components, and let $\tilde{\mathcal{A}} = \mathcal{A} \cup \{Q_1, \ldots, Q_a\} \setminus \{P\}$. We have $\sum_{P \in \tilde{\mathcal{A}}} \deg P < \sum_{P \in \mathcal{A}} \deg P = w$, and $\bigcup_{P \in \tilde{\mathcal{A}}} \mathbb{Z}(P) \subset \bigcup_{P \in \mathcal{A}} \mathbb{Z}(P)$. Apply the induction hypothesis to $\tilde{\mathcal{A}}$ to obtain a family $\tilde{\mathcal{A}}'$ satisfying Properties 14 with $\tilde{\mathcal{A}}'$ in place of $\mathcal{A}$. We can verify that $\tilde{\mathcal{A}}'$ has the desired properties. \qed

After applying Lemma 14 we can assume that each irreducible polynomial $P_j$ in the decomposition of $P$ generates a real ideal. Write $\mathcal{P} \cap Z = \bigcup \mathcal{P}_j$, where $\mathcal{P}_j$ consists of those points lying in $Z_j$. If a point lies on two or more such varieties, place it into only one of the sets. We need to distinguish between several cases. Let
\[
\mathcal{A}_1 = \{j : |\mathcal{P}_j|^k < D_j^k n\},
\mathcal{A}_2 = \{j : D_j^k n \leq |\mathcal{P}_j|^k < cD_j^{3k-1} n\},
\mathcal{A}_3 = \{j : |\mathcal{P}_j|^k \geq cD_j^{3k-1} n\},
\]
where $c$ is a small constant to be determined later. For each $j \in \mathcal{A}_1$ we have
\[
\mathcal{I}(\mathcal{P} \cap Z_j, \mathcal{S}_2) \lesssim |\mathcal{P}_j| n^{1-1/k} + n
\lesssim D_j n,
\]

\[
(12)
\]
where the second inequality uses the assumption $|P_j| < D_j n^{1/k}$. Summing (12) over all $j \in A_1$, we obtain

$$I(P \cap \bigcup_{A_1} Z_j, S_2) \lesssim \sum_{A_1} D_j n^{1/k} \leq Dn. \tag{13}$$

Now we must control the incidences between surfaces and points lying on varieties $Z_j$, $j \in A_2$ or $A_3$. If $j \in A_2$, use Theorem 7 to select a squarefree polynomial $Q_j$ of degree at most $E_j$,

$$E_j = \left(\frac{|P_j|^k}{n D_j^k}\right)^{1/(2k-1)}, \tag{14}$$

that cuts $\mathbb{R}^3$ into $E_j^3$ cells, each of which contains $\lesssim |P_j|/E_j^3$ points of $P_j$. Recall that $P_j$ is irreducible, $(P_j)$ is real, and $j \in A_2$ implies $\deg(Q_j) \leq E_j < \deg(P_j)$. Thus $Q_j$ does not vanish identically on $Z_j$. Let $Q_j = \{Q_j\}$ and let $W_j = Z(Q_j)$.

If $j \in A_3$, let $E_j$ be as in (14) and use Theorem 11 (with $E = E_j$) to find a family $Q_j$ of polynomials satisfying properties 1–4 of the theorem. In particular, the realizations of the realizable strict sign conditions of $Q_j$ on $Z_j$ partition $Z_j$ into $\sim D_j E_j^2$ (not necessarily connected) sets, each of which contains $\lesssim |P_j|/D_j E_j^2$ points, plus the “boundary” $Z_j \cap \bigcup_{Q_j} Z(Q)$. Define $W_j = \bigcup_{Q_j} Z(Q)$ (thus the definition of $W_j$ depends on whether $j \in A_2$ or $j \in A_3$).

Regardless of whether $j \in A_2$ or $A_3$, have

$$I(P_j, S_2) = I(P_j \setminus W_j, S_2) + I(P_j \cap W_j, S_2). \tag{15}$$

We shall begin by bounding the first term of (15). If $j \in A_2$, then through the same computation performed in (9) we have

$$I(P_j \setminus W_j, S_2) \lesssim \frac{|P_j|^n}{E_j^{1/k}} + n E_j^2 \tag{16}$$

$$\lesssim \frac{|P_j|^n}{E_j^{1-k}} + n D_j E_j.$$ 

If $j \in A_3$, then let $\Omega_{ij}$ be the realization of the $i$–th realizable strict sign condition of $Q_j$ on $Z_j$. Let $m_{ij} = |P_j \cap \Omega_{ij}|$, and let $n_{ij}$ be the number of surfaces in $S_2$ that intersect $\Omega_{ij}$.

**Lemma 15.**

$$\sum_i n_{ij} \lesssim n D_j E_j. \tag{17}$$

**Proof.** If a surface $S \in S_2$ lies in $W_j$ then it does not contribute to the above sum, so we need only consider those surfaces $S$ that do not lie in $Z_j$ or $W_j$. First, we can replace each $Q \in Q$ by the polynomial $Q + \epsilon$ for $\epsilon > 0$ a sufficiently small constant. If $S \cap \{x \in \mathbb{R}^3 : Q(x) > 0\} \cap Z_j \neq \emptyset$, then there must be a point on $S \cap Z_j$ where $Q$ is positive, so $S \cap \{x \in \mathbb{R}^3 : Q(x) + \epsilon > 0\} \cap Z_j \neq \emptyset$ for $\epsilon$ sufficiently small, and similarly for $S \cap \{x \in \mathbb{R}^3 : Q(x) < 0\} \cap Z_j$. Thus replacing each $Q \in Q$ by $Q + \epsilon$ does not increase the number of realizations of realizable strict sign conditions that meet $S$. We shall select a small generic (with respect to $S$ and $Z_j$) choice of $\epsilon$.

By Corollary 26 in Appendix B we can assume that each irreducible component of each polynomial in $Q_j$ generates a real ideal.
Lemma 17. For \( j \in \mathcal{A}_2 \cup \mathcal{A}_3 \), let \( Z_j, W_j, P_j \), and \( S_2 \) be as above. Then
\[
\mathcal{I}(P_j \cap W_j, S_2) \lesssim nD_jE_j + |P_j|.
\]
Proof. We shall write
\[ \mathcal{I}(P_j \cap W_j, S_2) = \mathcal{I}_1(P_j \cap W_j, S_2) + \mathcal{I}_2(P_j \cap W_j, S_2), \]
where \( \mathcal{I}_1 \) counts those incidences between points \( p \in P_j \cap W_j \) and surfaces \( S \in S_2 \) such that \( p^* \) lies on a 1 (complex) dimensional component of \( S^* \cap Z^*_j \cap W^*_j \), and \( \mathcal{I}_2 \) counts the remaining incidences. To control \( \mathcal{I}_2 \), note that by Bézout’s inequality (over \( \mathbb{C} \)), for each \( S \in S_2 \), \( S^* \cap Z^*_j \cap W^*_j \) contains \( O(D_j E_j) \) isolated points. Since \( |S_2| \leq n \) we obtain
\[ \mathcal{I}_2(P_j \cap W_j, S_2) \leq n D_j E_j. \]
Thus it remains to control \( \mathcal{I}_1 \). First, we shall replace \( Q_j \) with a new family of polynomials \( \check{Q}_j \) with the following properties:

1. \( Z_j \cap W_j \subset Z_j \cap \bigcup_{Q \in \check{Q}_j} Z(Q) \).
2. \( \sum_{Q \in \check{Q}_j} \deg Q \leq E_j \).
3. Each \( Q \in \check{Q}_j \) is irreducible.
4. For each \( Q \in \check{Q}_j \), every irreducible component of \( Z^*_j \cap Z(Q)^* \) that contains a real point has (complex) dimension 1.

The procedure will be similar to that in the proof of Lemma 14. For each \( Q \in Q_j \), write \( Q = Q_1, \ldots, Q_n \) as a product of irreducible factors. Discard those factors \( Q_b \) with \( Z(Q_b) \cap Z_j = \emptyset \). Of the remaining factors, place each irreducible factor that generates a real ideal in \( \check{Q}_j \). If \( Q_b \) is a factor that does not generate a real ideal then consider \( \nabla_{v} Q_b \) for \( v \) a generic vector. By assumption, \( Q_b \) does not vanish identically on \( Z_j \), but it does vanish at least one point on \( Z_j \). Thus \( Q_b \) is not constant on \( Z_j \), so \( \nabla Q_j \) does not vanish identically on \( Z_j \) and hence if \( v \) is a generic vector then \( \nabla_{v} Q_b \) does not vanish identically on \( Z_j \). Thus we can repeat the above procedure with \( \nabla_{v} Q_b \) in place of \( Q \). This process will eventually terminate, and the resulting collection of polynomials \( \check{Q}_j \) has the desired properties; Properties 1, 3 are immediate. To obtain Property 4, suppose that \( Z^*_j \cap Z(Q)^* \) fails to be a complete intersection for some \( Q \in \check{Q}_j \). Then there exists some variety \( Y \) that is an irreducible component of both \( Z^*_j \) and \( Z(Q)^* \). By Proposition 27 in Appendix B, \( R(Y) \) is an irreducible component of \( Z_j \) and \( Z(Q) \), and thus either \( R(Y) = \emptyset \) or \( R(Y) = Z_j = Z(Q) \). The latter is impossible since \( Z_j \) and \( Z(Q) \) have dimension 2, while \( Z_j \cap Z(Q) \) has dimension at most 1.

Let \( \check{W}_j = \bigcup_{Q \in \check{Q}_j} Z(Q) \).
We can write
\[ Z^*_j \cap \check{W}_j^* = \bigcup_{j=1}^{n} Y_j \]
as a union of irreducible (complex) varieties. By Property 4 above, we need only consider those components with (complex) dimension 1. We shall discard all components that have dimension 2. Let
\[ \hat{P}_j = \{ p \in P_j : \text{there exists a (Euclidean) neighborhood } U \subset \mathbb{C}^3 \text{ of } p^* \text{ such that } Z^*_j \cap \check{W}_j^* \cap U \text{ is a (topological) 1–complex-dimensional curve} \}. \]
We shall establish several claims.

1. \( Z^*_j \cap \check{W}_j^* \) is a union of \( O(D_j E_j) \) irreducible varieties.
(2) If \( p \in \tilde{P}_j \) then \( p^* \) lies on at most one of the irreducible component from (22).
(3) Let \( Y \) be a variety from the above decomposition. If there exist three surfaces \( S_1, S_2, S_3 \in S_2 \) such that \( Y \subset S_i^*, i = 1, 2, 3, \) then \( |P_j \cap \mathcal{R}(Y)| \leq C. \)
(4) If \( S \in S_2 \), then there are \( O(D_j E_j) \) points \( p \notin \tilde{P}_j \) such that \( p^* \) is contained in a 1–dimensional component of \( S^* \cap Z_j^* \cap W_j^* \).

For Item 1 see e.g. [10]. Item 2 follows from the assumption that every variety in the decomposition (22) has dimension 1. Item 3 follows from the requirement that any three surfaces intersect in at most \( C \) points. To obtain Item 4, suppose that \( D_j \leq E_j \) (if not, we can interchange the roles of \( Z_j \) and \( W_j \)). Note if \( p \) satisfies the requirements of Item 4, then \( p^* \) is a point of \( S^* \cap Z_j^* \) at which \( S^* \cap Z_j^* \) fails to be (locally) a 1–dimensional (complex) curve. Thus after a generic rotation of the coordinate axis, the image of \( p^* \) under the projection \( (x_1, x_2, x_3) \mapsto (x_1, x_2) \) is a singular point of the (complex) plane curve \( \mathbf{Z}(\text{res}_{x_3}(f_S, P_j))^* \), where \( \text{res}_{x_3} \) is the bivariate polynomial obtained by taking the resultant of \( f_S \) and \( P_j \) in the \( x_3 \) variable. This curve has degree \( O(D_j) \) and thus has \( O(D_j^2) = O(D_j E_j) \) singular points.

Now, for each \( S \in S_2 \), at most \( O(D_j E_j) \) points \( p \in \mathcal{P}_j \setminus \tilde{P}_j \) can contribute to \( \mathcal{I}_1(\mathcal{P}_j \cap W_j, S_2) \), so the total contribution from all surfaces in \( S_2 \) is \( O(n D_j E_j) \). To control the remaining incidences, use Item 3 to write \( \{Y_j\} = \{Y_j'\} \cup \{Y_j''\} \), where the first set consists of varieties that are contained in at most 2 surfaces \( S \in S_2 \), and the second consists of varieties that contain at most \( C \) points. Each point \( p \in \tilde{P}_j \) with \( p^* \in \bigcup Y_j' \) can be incident to at most two surfaces, so the total contribution from such points is \( O(|\mathcal{P}_j|) \). On the other hand, by Item 1 at most \( O(D_j E_j) \) points can be contained in \( \mathcal{R}(\bigcup Y_j'') \), so these points can contribute at most \( O(n D_j E_j) \) incidences.

Combining (16), (18), and (19) and optimizing in \( E_j \), we see that our choice of \( E_j \) from (14) yields the bound

\[
\mathcal{I}(\mathcal{P}_j, S_2) \lesssim |\mathcal{P}_j|^{\frac{k}{2k-1}} n^{\frac{2k-2}{2k-1}} D_j^{\frac{2k-1}{k-1}} + m_j.
\]  

(23)

Summing (23) over all \( j \in A_2 \cup A_3 \) and noting that \( \frac{2k-1}{k} \) and \( \frac{2k-1}{k-1} \) are conjugate exponents, we obtain

\[
\mathcal{I}(\mathcal{P} \cap \bigcup_{A_2 \cup A_3} Z_j, S_2) \lesssim \sum_{A_2 \cup A_3} |\mathcal{P}_j|^{\frac{k}{2k-1}} n^{\frac{2k-2}{2k-1}} D_j^{\frac{2k-1}{k-1}} + |\mathcal{P}_j| \\
\lesssim n^{\frac{2k-2}{2k-1}} \left( \sum_j |\mathcal{P}_j| \right)^{\frac{k}{2k-1}} \left( \sum_j D_j \right)^{\frac{2k-1}{k-1}} + m 
\]  

(24)

Finally, selecting

\[
D = m^{\frac{k}{2k-1}} n^{\frac{1}{2k-1}},
\]  

(25)
which by (7) satisfies $D > C$, and combining (7), (9), (11), (13), and (24), we obtain

$$
\mathcal{I}(\mathcal{P}, \mathcal{S}) \lesssim D^2n + m + \frac{mn^{1-1/k}}{D^{1-1/k}} + Dm^{2/3} + nD + m + m^{\frac{1}{3k-1}}n^{\frac{3k-2}{3k-1}}D^{\frac{1}{3k-1}} \\
\lesssim m^{\frac{1}{3k-1}}n^{\frac{3k-3}{3k-1}} + m^{\frac{1}{3k-1}}n^{\frac{1}{3k-1}} + m + n \\
\lesssim m^{\frac{1}{3k-1}}n^{\frac{3k-3}{3k-1}} + m + n.
$$

(26)

3. Applications

In [8, 9], Erdős asked how many unit distances there could be amongst $m$ points in the plane or in $\mathbb{R}^3$. Theorem 2 yields new bounds for the $\mathbb{R}^3$ version of this question. Let $\mathcal{P}$ be a collection of $m$ points in $\mathbb{R}^3$, and let $\mathcal{S}$ be a collection of unit spheres centered about the points in $\mathcal{P}$. We can immediately verify that any three spheres have at most $O(1)$ points in common, so Theorem 2 tells us that there are $O(m^{3/2})$ point-sphere incidences, i.e.

**Theorem 18.** The maximum number of unit-distance pairs in a set of $m$ points in $\mathbb{R}^3$ is $O(m^{3/2})$.

This is a slight improvement over the previous bound of $O(m^{3/2}\beta(m))$ from [9], where $\beta$ is a very slowly growing function.

As observed in [9], theorem 2 combined with the method outlined in [5] can be used to establish bounds on the number of incidences between points and spheres in $\mathbb{R}^d$. Specifically, we have the following theorem:

**Theorem 19.** The maximum number of incidences between $m$ points and $n$ spheres in $\mathbb{R}^d$ is

$$
O(m^{d/(d+1)}n^{d/(d+1)} + m + n),
$$

(27)

provided no $d$ of the spheres intersect in a common circle.

Again, this is a slight improvement (by a $\beta(m, n)$ factor) from the analogous bounds established in [9]. See [9] \S 6.5 for additional applications of Theorem 2. In each case, we are able to slightly sharpen the bound from [9] by removing the $\beta(m)$ factor.

4. Generalizations to higher dimension

It is reasonable to ask whether Theorem 2 can be generalized to incidences between points and hypersurfaces in higher dimensions. This task appears to be quite involved, as the necessary algebraic geometry becomes more difficult. In particular, it appears that in order to generalize the proof of Theorem 2 to (say) spheres in $\mathbb{R}^d$, we need to perform $d-1$ polynomial ham sandwich decompositions, with each successive decomposition performed on the variety defined by the previous decompositions. As $d$ increases, the number of cases to be considered increases dramatically, and certain difficulties such as the failure of the connected components of a complete intersection to themselves be a complete intersection, the failure of an arbitrary complete intersection to be a nonsingular complete intersection, etc. become increasingly problematic.
One could also consider dimension 2 surfaces in $\mathbb{R}^d$, $d > 3$, and this appears to be more promising. However, the analogues of (11) and Lemma 17 become more difficult: an algebraic variety of dimension $d - 1$ can contain many 2-dimensional surfaces without obvious constraints being imposed on its structure, and in higher dimensions there are more (and more complicated) ways in which varieties can fail to intersect completely. Nevertheless, this is certainly a promising area for future work.

**APPENDIX A. DEFINITIONS**

**Definition 20.** Let $Q \subset \mathbb{R}[x_1, \ldots, x_d]$ be a collection of non-zero real polynomials. A *strict sign condition* on $Q$ is a map $\sigma: Q \to \{\pm 1\}$. If $Q \in Q$, we will denote the evaluation of $\sigma$ at $Q$ either by $\sigma Q$ or $\sigma(Q)$, depending on context. If $\sigma$ is a strict sign condition on $Q$ we define its *realization* by

$$\text{Real}(\sigma, Q) = \{x \in \mathbb{R}^d: Q(x)\sigma Q > 0 \text{ for all } Q \in Q\}. \quad (28)$$

If $\text{Real}(\sigma, Q) \neq \emptyset$ then we say that $\sigma$ is realizable. We define

$$\Sigma_Q = \{\sigma: \text{Real}(\sigma, Q) \neq \emptyset\}, \quad (29)$$

and

$$\text{Real}(Q) = \{\text{Real}(\sigma, Q): \sigma \in \Sigma_Q\}. \quad (30)$$

We call $\text{Real}(Q)$ the collection of “realizations of realizable strict sign conditions of $Q$.”

If $Z \subset \mathbb{R}^d$ is a variety, and $\sigma$ is a strict sign condition on $Q$, then we can define the *realization of $\sigma$ on $Z$* by

$$\text{Real}(\sigma, Q, Z) = \{x \in Z: Q(x)\sigma Q > 0 \text{ for all } Q \in Q\}, \quad (31)$$

and we can define analogous sets

$$\Sigma_{Q,Z} = \{\sigma: \text{Real}(\sigma, Q, Z) \neq \emptyset\}, \quad (32)$$

and

$$\text{Real}(Q, Z) = \{\text{Real}(\sigma, Q, Z): \sigma \in \Sigma_{Q,Z}\}. \quad (33)$$

We call $\text{Real}(Q, Z)$ the collection of “realizations of realizable strict sign conditions of $Q$ on $Z$.” Note that if some $Q \in Q$ vanishes identically on $Z$ then $\Sigma_{Q,Z} = \emptyset$ and thus $\text{Real}(Q, Z) = \emptyset$.

**Definition 21.** An ideal $I \subset \mathbb{R}[x_1, \ldots, x_d]$ is real if for every sequence $a_1, \ldots, a_\ell \in \mathbb{R}[x_1, \ldots, x_d]$, $a_1^2 + \ldots + a_\ell^2 \in I$ implies $a_j \in I$ for each $j = 1, \ldots, \ell$.

**APPENDIX B. PROPERTIES OF REAL VARIETIES**

The following proposition shows that real principal prime ideals and their corresponding real varieties have some of the “nice” properties of ideals and varieties defined over $\mathbb{C}$.

**Proposition 22** (see [4, §4.5]). Let $(P) \subset \mathbb{R}[x_1, \ldots, x_d]$ be a principal prime ideal. Then the following are equivalent:

1. $(P)$ is real.
2. $(P) = I(Z(P))$.
3. $\dim(Z(P)) = d - 1$.
4. $\nabla P$ does not vanish identically on $Z(P)$.
5. The sign of $P$ changes somewhere on $\mathbb{R}^d$. 

Definition 23. If $P \subset \mathbb{R}[x_1, \ldots, x_d]$ is a polynomial and $P = P_1, \ldots, P_\ell$ is its factorization, we define $\bar{P}$ to be the polynomial obtained by removing those irreducible components that generate ideals that aren’t real. If every irreducible component of $P$ generates an ideal that is not real, then we define $\bar{P} = 1$.

Example 24. Let $P = (x_1^2 + x_2^2 + x_3^2 - 1)(x_4^2 + x_2^2 - 1)$. Then $\bar{P} = x_1^2 + x_2^2 + x_3^2 - 1$. Geometrically, if $\bar{P} \neq 0$, then $\mathbf{Z}(P)$ is a $(d-1)$-dimensional (real) variety, but some of the components of $\mathbf{Z}(P)$ may have dimension less than $d-1$. $\bar{P}$ keeps those factors that generate components that have dimension $d-1$.

The existence of polynomials that do not generate real ideals complicates our analysis, but since the zero sets of such polynomials have codimension at least 2, we can ignore them when we are computing the number of times a surface meets the realization of a realizable strict sign condition of a family of polynomials. The following theorem helps make this statement precise.

Theorem 25. Let $Q \subset \mathbb{R}[x_1, \ldots, x_d]$, $d \geq 3$ be a collection of real polynomials and let $\hat{Q} = \{Q : Q \in Q\}\{0\}$. Then there exists a bijection

$$\tau : \text{Reali}(Q) \to \text{Reali}(\hat{Q})$$

such that

$$X \subset \tau(X) \text{ for every } X \in \text{Reali}(Q). \tag{34}$$

Similarly, if $Z = \mathbf{Z}(P)$ where $P \in \mathbb{R}[x_1, \ldots, x_d]$ generates a real ideal and no polynomial $Q \in Q$ vanishes identically on $Z$, then there exists a bijection

$$\tau : \text{Reali}(Q, Z) \to \text{Reali}(\hat{Q}, Z)$$

such that (34) holds with $\text{Reali}(Q, Z)$ in place of $\text{Reali}(Q)$.

Proof. First, by Item 5 of Proposition 22 for each $Q \in Q$ we have that $Q/\hat{Q} \geq 0$ or $Q/\hat{Q} \leq 0$ on all of $\mathbb{R}^d$. Choose $\varepsilon_Q \in \{\pm 1\}$ so that $\varepsilon_Q Q/\hat{Q} \geq 0$. Now, note that if there exist $Q_1, Q_2 \in Q$ with $\hat{Q}_1 \neq \hat{Q}_2$ and if $\sigma$ is a strict sign condition on $Q$, then either $\varepsilon_Q\sigma(Q_1) = \varepsilon_Q\sigma(Q_2)$ or $\text{Reali}(\sigma, Q) = \emptyset$. Thus if $\sigma$ is a realizable strict sign condition on $Q$, then we can define $\hat{\sigma} : \hat{Q} \to \{\pm 1\}$ by $\hat{\sigma}(T) = \varepsilon_Q\sigma(Q)$, where $Q \in Q$ satisfies $T = \hat{Q}$, and $\hat{\sigma}$ is well-defined.

We shall show that the map $\Sigma_Q \to \Sigma_{\hat{Q}}$, $\sigma \mapsto \hat{\sigma}$ is a bijection. To prove injectivity, note that if distinct $\sigma_1, \sigma_2$ both map to the same element $\hat{\sigma}$, then $\varepsilon_Q\sigma_1(Q) = \varepsilon_Q\sigma_2(Q)$ for all $Q \in Q$, so clearly $\sigma_1 = \sigma_2$. To establish surjectivity, note that each $\sigma_1 \in \Sigma_{\hat{Q}}$ has a pre-image under the map $\sigma \mapsto \hat{\sigma}$. Thus every element of $\Sigma_{\hat{Q}}$ may be written as $\hat{\sigma}$ for some strict sign condition $\sigma$ on $Q$. All that we must establish is that $\sigma$ is realizable. For each $Q \in Q$, we have

$$\dim \left(\{Q > 0\}\{\varepsilon_Q Q > 0\}\right) \leq d - 2, \tag{35}$$

(see i.e. 4 for the dimension of a semi-algebraic set). On the other hand, the realization of each realizable strict sign condition of $Q$ has dimension $d$. Thus if $\text{Reali}(\sigma, Q) \neq \emptyset$ then $\text{Reali}(\sigma, Q)$ can be written as a (non-empty) dimension $d$ semi-algebraic set minus a dimension $d - 2$ semi-algebraic set, and in particular, $\text{Reali}(\sigma, Q) \neq \emptyset$.

Thus the map $\text{Reali}(Q) \to \text{Reali}(\hat{Q})$ which takes $X = \text{Reali}(\sigma, Q) \to \text{Reali}(\hat{\sigma}, \hat{Q})$ is well-defined and is a bijection. Now, note that by Items 3 and 5 of Proposition
\[ \{ \varepsilon Q > 0 \} \subset \{ \hat{Q} > 0 \}, \text{ and similarly with } ">" \text{ replaced by } "<". \] Thus
\[ \text{Reali}(\sigma, Q) \subset \text{Reali}(\hat{\sigma}, \hat{Q}), \] (36)
so (34) holds.

The same arguments establish the second part of the theorem. The only new thing that must be verified is that the map \( \Sigma_{Q, Z} \to \Sigma_{Q, Z}, \sigma \mapsto \hat{\sigma} \) is onto. However, this is established by (35) plus the fact that the realization of each realizable strict sign condition of \( Q \) on \( Z \) has dimension \( d - 1 \).

**Corollary 26.** Let \( S \subset \mathbb{R}^3 \) be a smooth surface, let \( Q \) be a collection of polynomials, and let \( \hat{Q} \) be as in Theorem 25. Then
\[ |\{ x \in \text{Reali}(Q) : X \cap S \neq \emptyset \}| \leq \left| \{ x \in \text{Reali}(\hat{Q}) : X \cap S \neq \emptyset \} \right|. \] (37)

Similarly, let \( S \subset \mathbb{R}^3 \) be a smooth surface, let \( Z = \mathbb{Z}(P) \) where \( P \in \mathbb{R}[x_1, x_2, x_3] \) generates a real ideal, let \( Q \) be a collection of polynomials, none of which vanish identically on \( Z \), and let \( \hat{Q} \) be as in Theorem 25. Then
\[ |\{ x \in \text{Reali}(Q, Z) : X \cap S \neq \emptyset \}| \leq \left| \{ x \in \text{Reali}(\hat{Q}, Z) : X \cap S \neq \emptyset \} \right|. \] (38)

As noted in Section 1.4, the number of intersection points of a collection of real polynomials may exceed the product of their degrees, even if those polynomials intersect completely. Over \( \mathbb{C} \) things are much better behaved, so there will be times when we will wish to embed everything into \( \mathbb{C} \). The following proposition relates the properties of a variety defined over \( \mathbb{R} \) and the corresponding variety defined over \( \mathbb{C} \):

**Proposition 27** (see [21 §10]). Let \( Z \subset \mathbb{R}^d \) be a real variety and let \( Z^*_1, \ldots, Z^*_e \) be the irreducible components of \( Z^* \). Then \( \mathcal{R}(Z^*_1), \ldots, \mathcal{R}(Z^*_e) \) are the irreducible components of \( Z \).

**Appendix C. Proof of Theorem 11**

Our proof of Theorem 11 will be similar to the original proof of the discrete polynomial ham sandwich theorem in [11 §4], which can be stated as follows:

**Proposition 28** (Discrete polynomial ham sandwich theorem). Let \( V \subset \mathbb{R}[x_1, \ldots, x_d] \) be a vector space of dimension \( \ell \), and let \( F_1, \ldots, F_\ell \subset \mathbb{R}^d \) be finite families of points. Then there exists a polynomial \( P \in V \) such that
\[ |F_j \cap \{ x \in \mathbb{R}^d : P(x) > 0 \}| \leq |F_j|/2, \text{ and} \]
\[ |F_j \cap \{ x \in \mathbb{R}^d : P(x) < 0 \}| \leq |F_j|/2, \quad j = 1, \ldots, \ell. \]

Proposition 28 is proved in [11] only in the special case where \( V \) is the vector space of all polynomials of degree at most \( e \) \((e \) chosen large enough to ensure that \( V \) has the required dimension). However, the proof carries over verbatim to the general case where \( V \) is arbitrary. To prove Theorem 11, we will iterate the following lemma:

**Lemma 29.** Let \( Z = \mathbb{Z}(P) \subset \mathbb{R}^d \) for \( P \) an irreducible polynomial of degree \( D \) such that \( (P) \) is a real ideal. Let \( E > 0 \), and let \( F_1, \ldots, F_\ell \), \( \ell = c \min(E^d, DE^{d-1}) \) be finite families of points in \( \mathbb{R}^d \), with \( F_j \subset Z \) for each \( j \). Then provided \( c \) is
sufficiently small (depending only on $d$), there exists a polynomial $Q$ of degree at most $E$ that does not vanish identically on $\mathbb{Z}(P)$ such that
\[
|F_j \cap \{ x \in \mathbb{R}^d : Q(x) > 0 \}| \leq |F_j|/2, \text{ and }
\]
\[
|F_j \cap \{ x \in \mathbb{R}^d : Q(x) < 0 \}| \leq |F_j|/2, \quad j = 1, \ldots, \ell.
\] (39)

Proof. Let $\mathbb{R}[x_1, \ldots, x_d]_{\leq E}$ be the vector space of all polynomials in $d$ variables of degree at most $E$, and let $(P)_{\leq E}$ be the vector space of all polynomials in the ideal $(P)$ that have degree at most $E$ (of course, if $E < \deg P$ then $(P)_{\leq E} = \emptyset$). We have (41)
\[
\dim(\mathbb{R}[x_1, \ldots, x_d]_{\leq E}) - \dim((P)_{\leq E}) > c \min(E^d, DE^{d-1})
\]
for some (explicit) constant $c$ depending only on the dimension $d$. Thus, we can find a vector space $V \subset \mathbb{R}[x_1, \ldots, x_d]_{\leq E}$ with $\dim(V) > c \min(E^d, DE^{d-1})$ such that $V \cap (P)_{\leq E} = \emptyset$. By Proposition 28 we can find a polynomial $Q \in V$ satisfying (39). Since $Q \in \mathbb{R}[x_1, \ldots, x_d]_{\leq E}$ but $Q \notin (P)_{\leq E}$, we have $Q \notin (P)$. Since $P$ is irreducible and generates a real ideal, by Item 2 of Proposition 28 $Q$ does not vanish identically on $\mathbb{Z}(P)$. \hfill \qed

Proof of Theorem 11 Use Lemma 29 to find a polynomial $Q_1$ of degree $O(1)$ such that
\[
|\{ x \in \mathbb{R}^d : Q_1(x) > 0 \} \cap P | \leq |P|/2, \\
|\{ x \in \mathbb{R}^d : Q_1(x) < 0 \} \cap P | \leq |P|/2.
\] (40)
Let $Q_1 = \{Q_1\}$. For each $i = 2, \ldots, t$, with
\[
t = \lceil \log_2(DE^{d-1}) \rceil,
\] use Lemma 29 to find a polynomial $Q_i$ with
\[
\deg(Q_i) \lesssim \max((2^i/D)^{1/(d-1)}, 2^{i/d})
\]
such that for each $\sigma \in \Sigma_{Q_{i-1}}$, we have
\[
\left| \{ x \in \mathbb{R}^d : Q_i(x) > 0 \} \cap (P \cap \Reali(\sigma, Q_{i-1})) \right| \leq \frac{1}{2} |P \cap \Reali(\sigma, Q_{i-1})|,
\]
\[
\left| \{ x \in \mathbb{R}^d : Q_i(x) < 0 \} \cap (P \cap \Reali(\sigma, Q_{i-1})) \right| \leq \frac{1}{2} |P \cap \Reali(\sigma, Q_{i-1})|.
\] (41)
Some of the above sets may be empty, but this does not pose a problem. Let $Q_i = Q_{i-1} \cup \{Q_i\}$.

None of the polynomials in $Q = Q_i$ vanish on $P$, so Item 3 of the theorem is satisfied. Since $E > \rho D$ we have
\[
\sum_{Q_i} \deg Q \lesssim \sum_{i=1}^t (2^i/D)^{1/(d-1)} + \sum_{i} 2^{i/d}
\]
\[
\lesssim (DE^{d-1}/D)^{1/(d-1)} + (DE^{d-1})^{1/d}
\]
\[
\lesssim E,
\]
which satisfies Item 2. By (41), for each $\sigma \in \Sigma_Q$,
\[
|P \cap \Reali(\sigma, Q)| \lesssim 2^{-t} |P|
\]
\[
\lesssim \frac{|P|}{DE^{d-1}},
\] (42)
which satisfies Item 4. Finally, Item 1 follows from (40). \hfill \qed
AN IMPROVED BOUND ON POINT-SURFACE INCIDENCES IN $\mathbb{R}^3$

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