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Algebraic approach to Rump’s results on relations between braces and pre-Lie algebras

Agata Smoktunowicz

Abstract

In 2014, Wolfgang Rump showed that there exists a correspondence between left nilpotent right \( R \)-braces and pre-Lie algebras. This correspondence, established using a geometric approach related to flat affine manifolds and affine torsors, works locally. In this paper, we explain Rump’s correspondence using only algebraic formulae. An algebraic interpretation of the correspondence works for fields of sufficiently large prime characteristic as well as for fields of characteristic zero.

1 Introduction

In [19], Wolfgang Rump showed that there exists a correspondence between left nilpotent right \( R \)-braces and pre-Lie algebras. Braces were introduced by Rump in 2005 to describe all involutive non-degenerate set-theoretic solutions of the Yang-Baxter equation. This approach subsequently found applications in several other research areas.

The algebraic formula for Rump’s connection from strongly nilpotent braces to pre-Lie algebras is particularly simple and enables rapid production of examples of pre-Lie algebras from braces.

Question 1. Is there a formula for a passage from finite pre-Lie algebras to finite \( F \)-braces, where \( F \) is a finite field?

The algebraic formula for a passage from pre-Lie algebras to \( F \)-braces in Rump’s correspondence is not easy to use in examples, which raises the question:

Question 2. Is there an easy way to obtain braces from pre-Lie algebras?

In this paper all considered braces are left braces. Recall that if we change the multiplication in a left brace to the opposite multiplication we will obtain a right brace, and vice-versa. In Rump’s papers [18], [19], a brace always means a right brace. In [19] Rump presented a correspondence between right \( R \)-braces and RSA - the right symmetric algebras. By taking the opposite multiplication in both the brace and the RSA it gives the correspondence between left braces and LSA, left symmetric algebras, also called pre-Lie algebras.

Notice that it is known that pre-Lie algebras are in correspondence with the \( \acute{e} \)tale affine representations of nilpotent Lie algebras [6], and with Lie algebras with 1-cocycles [19],[4], [5], they also appear in noncommutative geometry. Braces have also been linked to other research areas, for example, in [11], Gateva-Ivanova showed that there is a correspondence between braces and braided groups with an involutive braiding operator, whereas in [2],
Bachiller observed that there is a connection between braces and Hopf-Galois extensions of abelian type (see also the appendix to [22] for some further results). In [2, 7, 11, 19, 22] braces and skew braces have been shown to be equivalent to several concepts in group theory (1-cocycles, regular subgroups, matched pairs of groups). Moreover, two-sided braces are exactly the Jacobson radical rings [8, 18]. In [9], applications of braces in quantum integrable systems were investigated, and in [20] R-matrices constructed from braces were studied. Solutions of the reflection equation related to braces have also been investigated by several authors [10, 14, 23].

In this paper, we use purely algebraic methods to look closely at Rump’s correspondence between braces and pre-Lie algebras and give algebraic formulas for this correspondence.

2 Background information

A pre-Lie algebra $A$ is a vector space with a binary operation $(x, y) \to xy$ satisfying

$$(xy)z - x(yz) = (yx)z - y(xz),$$

for every $x, y, z \in A$. We say that a pre-Lie algebra $A$ is nilpotent if, for some $n \in \mathbb{N}$, all products of $n$ elements in $A$ are zero.

Recall that a set $A$ with binary operations $+$ and $*$ is a left brace if $(A, +)$ is an abelian group and the following version of distributivity combined with associativity holds:

$$(a + b + a * b) * c = a * c + b * c + a * (b * c),$$

and

$$a * (b + c) = a * b + a * c,$$

for all $a, b, c \in A$, moreover $(A, \circ)$ is a group, where we define $a \circ b = a + b + a * b$.

See [18] for the original definition. For a shorter equivalent definition using group theory see [8]. In what follows we will use the definition in terms of operation ‘$\circ$’, presented in [8]: a set $A$ with binary operations of addition $+$, and multiplication $\circ$ is a brace if $(A, +)$ is an abelian group, $(A, \circ)$ is a group and for every $a, b, c \in A$

$$a \circ (b + c) + a = a \circ b + a \circ c.$$

We now recall Definition 2 from [19] which we state for left braces, as it was originally stated for right braces.

**Definition 1.** Let $F$ be a field. We say that a left brace $A$ is an $F$-brace if its additive group is an $F$-vector space such that $a * (\alpha b) = \alpha (a * b)$ for all $a, b \in A$, $\alpha \in F$. Here $a * b = a \circ b - a - b$.

$F$-braces were considered by Catino and Rizzo [7], who called them circle algebras.

In [18] Rump introduced left nilpotent and right nilpotent braces and radical chains $A^{i+1} = A * A^i$ and $A^{(i+1)} = A^{(i)} * A$, for a left brace $A$, where $A = A^1 = A^{(1)}$ (the original construction of Rump is for right braces, but we give the natural translation of it to left braces here). Recall that a left brace $A$ is left nilpotent if there is a number $n$ such that $A^n = 0$, where inductively $A^i$ consists of sums of elements $a * b$ with $a \in A, b \in A^{i-1}$. A left brace $A$ is right nilpotent if there is a number $n$ such that $A^{(n)} = 0$, where $A^{(i)}$
consists of sums of elements \( a \ast b \) with \( a \in A^{(i-1)}, b \in A \). Strongly nilpotent braces and the chain of ideals \( A^{[i]} \) of a brace \( A \) were defined in [21]. Define \( A^{[i]} = A \) and \( A^{[i+1]} = \sum_{j=1}^{i} A^{[j]} \ast A^{[i+1-j]} \). A left brace \( A \) is strongly nilpotent if there is a number \( n \) such that \( A^{[n]} = 0 \), where \( A^{[i]} \) consists of sums of elements \( a \ast b \) with \( a \in A^{[j]}, b \in A^{[i-j]} \) for all \( 0 < j < i \). A brace is strongly nilpotent if and only if it is both left nilpotent and right nilpotent [21].

Various other radicals in braces were subsequently introduced, in analogy with ring theory and group theory. Recall that solvable braces were introduced in [3], and in [16] the connection between prime radical and solvable braces was investigated. See also [15] for some further results on solvable braces. In [13], the radical of a brace was introduced as the intersection of all maximal ideals in a given brace. This radical enjoys good properties, and is very useful for describing the structure of a given brace.

3 Passage from pre-Lie algebras to braces

In [1], Agrachev and Gamkrelidze introduced the formal group of flows constructed from a pre-Lie algebra. Notice that this group of flows combined with the same addition is a brace, and it is the same brace as obtained in Rump’s correspondence. As mentioned by Rump in a private correspondence the addition in the pre-Lie algebra and in the corresponding brace is always the same. In his survey [17], Manchon mentions this group of flows along with explanations of their structure. To summarise from [1], [17], let \( A \) with operations \( \cdot, + \) be a pre-Lie algebra over \( F \), so

\[
a \cdot (b \cdot c) - (a \cdot b) \cdot c = b \cdot (a \cdot c) - (b \cdot a) \cdot c,
\]

for all \( a, b, c \in A \). Following Rump’s correspondence [19], define the \( F \)-brace \((A, +, \circ) \) with the same addition as in pre-Lie algebra \( A \) and with the multiplication \( \circ \) defined as in the group of flows as follows. The following is based on [1], [17], [19]. We additionally assume that \( A \) is a nilpotent pre-Lie algebra, and that \( F \) is a field of characteristic zero (or of characteristic larger than the nilpotency index of \( A \)). We use notation from [17].

1. Let \( a \in A \), and let \( L_a : A \to A \) denote the left multiplication by \( a \), so \( L_a(b) = a \cdot b \). Define \( L_c \cdot L_b(a) = L_c(L_b(a)) = c \cdot (b \cdot a) \). Define

\[
e^{L_a}(b) = b + a \cdot b + \frac{1}{2!} a \cdot (a \cdot b) + \frac{1}{3!} a \cdot (a \cdot (a \cdot b)) + \cdots
\]

2. We can formally consider element 1 such that \( 1 \cdot a = a \cdot 1 = a \) in our pre-Lie algebra (as in [17]) and define

\[
W(a) = e^{L_a}(1) - 1 = a + \frac{1}{2!} a \cdot a + \frac{1}{3!} a \cdot (a \cdot a) + \cdots
\]

Notice that \( W : A \to A \) is a bijective function, provided that \( A \) is a nilpotent pre-Lie algebra.

3. Let \( \Omega : A \to A \) be the inverse function to the function \( W(a) \), so \( \Omega(W(a)) = W(\Omega(a)) = a \). Following [17] the first terms of \( \Omega \) are

\[
\Omega(a) = a - \frac{1}{2} a \cdot a + \frac{1}{4} (a \cdot a) \cdot a + \frac{1}{12} a \cdot (a \cdot a) + \cdots
\]
4. Define

\[ a \circ b = a + e^{L_\Omega(a)}(b). \]

Here, the addition is the same as in the pre-Lie algebra \( A \). It was shown in [1] that \((A, \circ)\) is a group. It is immediate to see that \((A, +, \circ)\) is a left brace because

\[
a \circ (b + c) + a = a + e^{L_\Omega(a)}(b + c) + a = (a + e^{L_\Omega(a)}(b)) + (a + e^{L_\Omega(a)}(c)) = a \circ b + a \circ c.
\]

Notice that the above correspondence works globally provided that \( A \) is a nilpotent pre-Lie algebra, so \( A^n = 0 \) for some \( n \).

When the underlying pre-Lie algebra is a ring the obtained brace is with the familiar multiplication \( a \circ b = a + b + ab \) (see [17]).

**Remark about connections with the BCH formula.** Notice that the above formula can also be written using the Baker-Campbell-Hausdorff formula and Lazard’s correspondence, see [1], [17] for details. In particular, the following formula for the multiplication \( \circ \) holds in the brace obtained above:

\[
W(a) \circ W(b) = W(C(a, b)),
\]

where \( C(a, b) \) is obtained using the Baker-Campbell-Hausdorff series in the Lie algebra \( L(A) \). Recall that the Lie algebra \( L(A) \) is obtained from a pre-Lie algebra \( A \) by taking \([a, b] = a \cdot b - b \cdot a\), and has the same addition as \( A \). By the Baker-Campbell-Hausdorff formula the element \( C(a, b) \) can be represented in the form of a series in variables \( a, b \), multiplication by scalars and commutation in the Lie algebra \( L(A) \), and

\[
C(a, b) = a + b + \frac{1}{2}[a, b] + \frac{1}{12}([a, [a, b]] + [b, [b, a]]) + \cdots.
\]

For more details see [1], page 1658, [17], page 91.

**Example.** Let \((A, +, \cdot)\) be a pre-Lie algebra such that \( A^4 = 0 \) (that means that any product of any 4 elements in this pre-Lie algebra is zero). We calculate the formula for the multiplication in the corresponding brace \((A, +, \circ)\).

We know that \( \Omega(a) = a - \frac{1}{2}a \cdot a + c \) for some \( c \in A^3 \) following the formula from [17]. We obtain:

\[
e^{L_\Omega(a)}(b) = b + \Omega(a) \cdot b + \frac{1}{2} \Omega(a) \cdot (\Omega(a) \cdot b) + c',
\]

where \( c' \in A^4 \), so \( c' = 0 \). Therefore

\[ a \circ b = a + b + a \cdot b - \frac{1}{2}(a \cdot a) \cdot b + \frac{1}{2} a \cdot (a \cdot b), \]

hence

\[ a * b = a \cdot b - \frac{1}{2}(a \cdot a) \cdot b + \frac{1}{2} a \cdot (a \cdot b). \]

Observe that the following result follows from the above construction.
Theorem 2. Let \((A, +, \circ)\) be a nilpotent pre-Lie algebra over a field \(\mathbb{F}\) of characteristic zero, and let \((A, +, \circ)\) be the brace obtained as above. Denote \(a \circ b = a \circ b - a - b\). Then
\[
a \ast b = a \cdot b + \sum_{x \in B} \alpha_x x
\]
where \(\alpha_x \in \mathbb{F}\) and \(B\) is the set of all products of elements \(a\) and \(b\) from \((A, \cdot)\) with \(b\) appearing only at the end, and an appearing at least two times in each product.

\textbf{Proof.} This follows immediately from the construction of \(\Omega(a)\), which is a sum of \(a\) and a linear combination of all possible products of more than one element \(a\) with any distribution of brackets, which can be proved by induction. \qed

Question 3. What braces are obtained from the known types of pre-Lie algebras?

Question 4. What braces are obtained from Novikov algebras?

4 Some supporting lemmas

We recall Lemma 4.1 from [21]:

\textbf{Lemma 3.} Let \(s\) be a natural number and let \((A, +, \circ)\) be a left brace such that \(A^s = 0\). Let \(a, b \in A\), and as usual define \(a \ast b = a \circ b - a - b\). Define inductively elements \(d_i = d_i(a, b), d'_i = d'_i(a, b)\) as follows: \(d_0 = a, d'_0 = b\), and for \(i \geq 1\) define \(d_{i+1} = d_i + d'_i\) and \(d'_{i+1} = d_i \ast d'_i\) (* defined as usual, so \(x \ast y = x + y + x \ast y\)). Then for every \(c \in A\) we have
\[
(a + b) \ast c = a \ast c + b \ast c + \sum_{i=0}^{2s} (-1)^{i+1} ((d_i \ast d'_i) \ast c - d_i \ast (d'_i \ast c)) \tag{1}
\]

\textbf{Notation 1.} Let \(A\) be a strongly nilpotent brace with operations \(+, \circ\). Let \(x, y \in A\). Consider elements \(x \ast y, x \ast (x \ast y), \ldots\) which are all products (with any distribution of brackets) of some non-zero number of elements \(x\) and one element \(y\) at the end. The set of all such elements will be denoted as \(E_{x,y}\). Notice that this set is finite, because \(A\) is a strongly nilpotent brace. We can list elements from the set \(E_{x,y}\) in a such way that shorter products always appear before longer products, and then we can make it into a vector, which we will denote as \(V_{x,y}\).

We will now prove a supporting lemma which will be useful in Section 5. In what follows, by \(2^n\) we denote the sum of \(2^n\) copies of element \(c \in A\).

\textbf{Lemma 4.} Let \((A, +, \circ)\) be a strongly nilpotent brace over the field of rational numbers. Then for every \(a, b \in A\) the limit \(\lim_{n \to \infty} 2^n (\frac{1}{2^n} a) \ast b\) exists.

Moreover, there are square matrices \(P, T\), not depending on \(a, b\), such that
\[
2^n (\frac{1}{2^n} a) \ast b = E_1 P T^n P^{-1} V_{a,b},
\]
where \(T\) is a matrix in the Jordan block form with the first Jordan block of dimension 1 equal to 1 and all other Jordan blocks with eigenvalues smaller than 1. Moreover \(E_1 = [1, 0, 0, \ldots, 0]\) and entries of \(P, T\) are rational numbers and
\[
2^n V_{\frac{1}{2^n}a,b} = P T^n P^{-1} V_{a,b}.
\]
Proof. Let notation for $E_{x,y}$ and $V_{x,y}$ be as in Notation 1 above. By Lemma 3 (applied several times) every element from the set $E_{2x,y}$ can be written as a linear combination of elements from $E_{x,y}$, with coefficients which do not depend on $x$ and $y$. We can then organise these coefficients in a matrix, which we will call $M = \{m_{ij}\}$, so that we obtain

$$MV_{x,y} = V_{2x,y}.$$  

Notice that elements from $E_{x,y}$ (and from $E_{2x,y}$) which are shorter appear before elements which are longer in our vectors $V_{x,y}$ and $V_{2x,y}$. Therefore by Lemma 3 it follows that $M$ is an upper triangular matrix. Observe that the first element in the vector $E_{2x,y}$ is $(2x) \ast y$ and that this element can be written as $2(x \ast y)$ plus elements of degree larger than 2 (by Lemma 3). It follows that the first diagonal entry in $M$ equals 2, so $m_{1,1} = 2$. Observe that the following diagonal entries will be equal to 4 or more, because for example $(2x) \ast ((2x) \ast y)$ can be written using Lemma 3 as $4x \ast (x \ast y)$ plus elements of degree larger than 3.

Therefore $M$ has exactly one eigenvalue equal to 2 with exactly one corresponding eigenvector, and all other eigenvalues equal $2^i$ for some $i > 1$ (because diagonal entries of $M$ are its eigenvalues). Notice that $M$ does not depend on $x$ and $y$, as we only used relations from Lemma 3 to construct it. We can write $M = PJP^{-1}$ where $J$ consists of Jordan blocks, and the first block is the $1 \times 1$ diagonal entry $J_{1,1} = 2$. Notice that since the eigenvalues are real it is possible to find such matrices $P, J$ with entries from the field of rational numbers.

It follows that for every $n$, $M^nV_{x,y} = V_{2^n x,y}$, therefore

$$2^nV_{2^{-n}x,y} = (2M^{-1})^nV_{x,y}.$$  

Notice that $2M^{-1} = PJ'P^{-1}$ where $J' = 2J^{-1}$ is the matrix with Jordan blocks, and the first block is $1 \times 1$ block with eigenvalue 1. The remaining blocks have eigenvalues $2^{-i}$ for $i > 0$. It follows that we can define the limit

$$\lim_{n \to \infty} 2^nV_{2^{-n}x,y} = \lim_{n \to \infty} PJ'^nP^{-1}V_{x,y} = PE_{1,1}P^{-1}V_{x,y},$$

where $E_{1,1}$ is the matrix with the first entry equal to 1 and all other entries equal to zero. The first entry of vector $V_{2^{-n}x,y}$ is $(2^{-n}x) \ast y$, therefore $\lim_{n \to \infty} 2^n(2^{-n}x) \ast y = p_{1,1}R_1V_{x,y}$, where $p_{1,1}$ is the first diagonal entry of matrix $P$, and $R_1$ is the first row of matrix $P^{-1}$. \hfill \Box

**Notation 2.** Let $A$ be a brace with operations $+, \circ, \ast$ defined as usual so $x \circ y = x + y + x \ast y$. For $x, y, z \in A$ and let $E_{x,y,z} \subseteq A$ denote the set consisting of any product of elements $x$ and $y$ and one element $z$ at the end of each product under the operation $\ast$, in any order, with any distribution of brackets, each product consisting of at least 2 elements from the set $\{x, y\}$, each product having $x$ and $y$ appear at least once, and having element $z$ at the end. Notice that $E_{x,y,z}$ is finite provided that $A$ is a strongly nilpotent brace. Let $V_{x,y,z}$ be a vector obtained from products of elements $x, y, z$ arranged in such a way that shorter products of elements are situated before longer products.

**Lemma 5.** Let $(A, +, \circ)$ be a strongly nilpotent brace over the field of rational numbers $\mathbb{Q}$. Let $a, b \in A$. Denote

$$a \cdot b = \lim_{n \to \infty} 2^n\left(\frac{1}{2^n}a\right) \ast b.$$
Then, for \( \alpha, \gamma \in \mathbb{Q} \) and \( a, b, c \in A \) we have
\[
(\alpha a + \gamma b) \cdot c = \alpha (a \cdot c) + \gamma (b \cdot c)
\]
and
\[
a \cdot (\alpha b + \gamma c) = \alpha (a \cdot b) + \gamma (a \cdot c).
\]

**Proof.** By the definition of a left \( \mathbb{Q} \)-brace we immediately get that
\[
(\alpha a + \gamma b) \cdot c = \alpha (a \cdot c) + \gamma (b \cdot c).
\]

Assume that \((a + b) \cdot c = a \cdot c + b \cdot c\) for \(a, b, c \in A\), then \((na) \cdot b = (a + \cdots + a) \cdot b = a \cdot b + \cdots + a \cdot b = n(a \cdot b)\) where \(n\) is a natural number. Let \(d = na\), then
\[
d \cdot b = (na) \cdot b = n(a \cdot b) = n(\frac{1}{n}d) \cdot b.
\]
Thus, for \(p, q \in \mathbb{Z}, q \neq 0\) we have
\[
(\frac{p}{q}a) \cdot b = p((\frac{1}{q})a) \cdot b = \frac{p}{q}(a \cdot b).
\]

Therefore, to prove our lemma we only need to show that for \(a, b, c \in A\) we have
\[
(a + b) \cdot c = a \cdot c + b \cdot c.
\]

Observe that Lemma 3 yields
\[
(a + b) \cdot c = \lim_{n \to \infty} 2^n \left( \frac{1}{2^n}a + \frac{1}{2^n}b \right) \cdot c = \lim_{n \to \infty} 2^n \left( \frac{1}{2^n}a \right) \cdot c + 2^n \left( \frac{1}{2^n}b \right) \cdot c + 2^n C(n),
\]
where \(C(n)\) is a sum of some products of elements \(\frac{1}{2^n}a\) and \(\frac{1}{2^n}b\) and an element \(c\) at the end (because \(A\) is a strongly nilpotent brace). Moreover, each product has at last one occurrence of element \(\frac{1}{2^n}a\) and also at last one occurrence of element \(\frac{1}{2^n}b\) and an element \(c\) at the end.

To show that \((a + b) \cdot c = a \cdot c + b \cdot c\) it suffices to prove that \(\lim_{n \to \infty} 2^n C(n) = 0\). We may consider a vector \(V_{\frac{1}{2^n}a, \frac{1}{2^n}b, c}\) obtained as in Notation 2 from products of elements \(\frac{1}{2^n}a, \frac{1}{2^n}b, c\).

Using similar methods as in the proof of Lemma 4 we can show that for an appropriate upper triangular matrix \(Q\) with diagonal entries smaller than \(\frac{1}{2}\) (equal to \(2^{-i}\) for some \(i > 1\)) we have
\[
V_{\frac{1}{2^n}a, \frac{1}{2^n}b, c} = Q^n V_{a, b, c},
\]
hence the limit \(2^n V_{2^{-n}a, 2^{-n}b, c}\) exists and is equal to zero, which implies that \(\lim_{n \to \infty} 2^n C(n) = 0\). This concludes the proof.

We will now explain in more detail how to obtain the matrix \(Q\): By Lemma 3 (applied several times) every element from the set \(E_{x, y, z}\) can be written as a linear combination of elements from \(E_{x, y, z}\), with coefficients which do not depend on \(x, y\) and \(z\). We can then organise these coefficients in a matrix, which we will call \(M = \{m_{i,j}\}\), so that we obtain
\[
MV_{x, y, z} = V_{2x, 2y, z}.
\]

Notice that elements from \(E_{x, y, z}\) (and from \(E_{2x, 2y, z}\)) which are shorter appear before elements which are longer in our vectors \(V_{x, y, z}\) and \(V_{2x, 2y, z}\). Therefore by Lemma 3 it follows that \(M\) is an upper triangular matrix.

Observe that the first four elements in the vector \(V_{2x, 2y, z}\) are \((2x) * ((2y) * z), ((2x) * (2y)) * z, (2y) * ((2x) * z)\) and \(((2y) * (2x)) * z\) (arranged in some order). We can assume that \((2x) * (2y) * z\) is the first entry in the vector \(V_{2x, 2y, z}\) (so \(x * (y * z)\) is the first entry in the vector \(V_{x, y, z}\)). Observe that \((2x) * (2y) * z\) can be written as \(4(x * (y * z))\) plus elements of degree larger than 3 (by Lemma 3). It follows that the first diagonal entry in \(M\) equals 4, so \(m_{1,1} = 4\). Observe that the following diagonal entries will be equal to 4 or more, because for example \((2x) * ((2x) * ((2x) * y))\) can be written using Lemma 3 as \(8(x * (x * (x * y)))\) plus elements of degree larger than 4.
Therefore $M$ is an upper triangular matrix with all diagonal entries larger or equal to 4. Therefore, $M$ has all eigenvalues larger than 2 (because diagonal entries of $M$ are its eigenvalues). Notice that $M$ does not depend on $x$ and $y$, as we only used relations from Lemma 3 to construct it. It follows that for every $n$, $M^nV_{x,y,z} = V_{2^n x, 2^n y, z}$, therefore
\[
2^n V_{2^{-n} x, 2^{-n} y, z} = (2M^{-1})^n V_{x,y,z}.
\]
Notice that $2M^{-1} = PJP^{-1}$ where $J$ is a matrix in the Jordan block form, with all eigenvalues smaller than 1 and larger than 0. It follows that we can define the limit
\[
limit_{n \to \infty} 2^n V_{2^{-n} x, 2^{-n} y, z} = \lim_{n \to \infty} PJ^n P^{-1} V_{x,y,z} = 0.
\]
We can now take $Q = 2M^{-1}$.

5 Passage from braces to pre-Lie algebras

We now explain how to obtain a pre-Lie algebra from a brace. It is the same pre-Lie algebra as in Rump’s correspondence, but we developed an algebraic method to obtain this algebra instead of geometric methods used by Rump. In a brace $(A, +, \circ)$ we will denote as usual $a \ast b = a \circ b - a - b$.

**Theorem 6.** Let $(A, +, \circ)$ be a strongly nilpotent brace over the field of rational numbers. For $a, b \in A$ define
\[
a \cdot b = \lim_{n \to \infty} 2^n (\frac{1}{2n} a) \ast b.
\]
Then $(A, +, \cdot)$ is a pre-Lie algebra.

**Proof.** Observe first that that $\lim_{n \to \infty} 2^n (\frac{1}{2n} a) \ast b$ exists by Lemma 4.

We will now show that for every $a, b, c \in A$ we have
\[
a \cdot (b \cdot c) - (a \cdot b) \cdot c = b \cdot (a \cdot c) - (b \cdot a) \cdot c.
\]

By Lemma 3 we get
\[
(x + y) \ast z = x \ast z + y \ast z + x \ast (y \ast z) - (x \ast y) \ast z + d(x, y, z),
\]
\[
(y + x) \ast z = x \ast z + y \ast z + y \ast (x \ast z) - (y \ast z) \ast z + d(y, x, z),
\]
where $d(x, y, z) = E^T V_{x,y,z}$ for some vector $E$ which does not depend on $x, y, z$, and where $V_{x,y,z}$ is as in Notation 2 (moreover $d(x, y, z)$ is a combination of elements with at least 3 occurrences of elements from the set \{x, y\}). It follows that
\[
x \ast (y \ast z) - (x \ast y) \ast z - y \ast (x \ast z) + (y \ast x) \ast z = d(y, x, z) - d(x, y, z).
\]

Let $a, b, c \in A$ and let $m, n$ be natural numbers. Applying it to $x = \frac{1}{2^n} a$, $y = \frac{1}{2^m} b$, $z = c$ we get
\[
(\frac{1}{2^n} a) \ast ((\frac{1}{2^m} b) \ast c) - ((\frac{1}{2^n} a) \ast (\frac{1}{2^m} b)) \ast c + d(\frac{1}{2^n} a, \frac{1}{2^m} b, c) = \]
This, together with Lemma 5, implies that

\[ \lim_{m \to \infty} \lim_{n \to \infty} 2^{m+n} \left( \frac{1}{2^m} b \right) \cdot \left( \frac{1}{2^n} a \right) - \left( \frac{1}{2^n} b \right) \cdot \left( \frac{1}{2^m} a \right) \cdot c = a \cdot (b \cdot c) - (a \cdot b) \cdot c \]

and

\[ \lim_{m \to \infty} \lim_{n \to \infty} 2^{m+n} \left( \frac{1}{2^m} b \right) \cdot \left( \frac{1}{2^n} a \right) - \left( \frac{1}{2^n} b \right) \cdot \left( \frac{1}{2^m} a \right) \cdot c = b \cdot (a \cdot c) - (b \cdot a) \cdot c. \]

For example, \( \lim_{m \to \infty} \lim_{n \to \infty} 2^{m+n} \left( \frac{1}{2^m} b \right) \cdot \left( \frac{1}{2^n} a \right) \cdot c = \lim_{m \to \infty} 2^m \left( \frac{1}{2^m} b \right) \cdot a \cdot c = \lim_{m \to \infty} (\frac{1}{2^m} b) \cdot a \cdot c \), by Lemma 5. We know that \( \lim_{m \to \infty} 2^m \left( \frac{1}{2^m} b \right) \cdot a \cdot c = a \cdot b \), hence \( 2^m \left( \frac{1}{2^m} b \right) \cdot a \cdot c = a \cdot b + \sum_{w \in E_{a,b,c}} t \cdot a \cdot c \) for some \( t \in Q \) such that \( \lim_{m \to \infty} t = 0 \). This, together with Lemma 5, implies that \( \lim_{m \to \infty} 2^m \left( \frac{1}{2^m} b \right) \cdot a \cdot c = (a \cdot b) \cdot c \).

Consequently

\[ a \cdot (b \cdot c) - (a \cdot b) \cdot c = b \cdot (a \cdot c) - (b \cdot a) \cdot c \]

provided that

\[ \lim_{m \to \infty} \lim_{n \to \infty} 2^{m+n} \left( \frac{1}{2^m} a \right) \cdot \left( \frac{1}{2^n} b \right) = \lim_{m \to \infty} \lim_{n \to \infty} 2^{m+n} \left( \frac{1}{2^m} b \right) \cdot \left( \frac{1}{2^n} a \right) = 0. \]

It suffices to show that \( \lim_{m \to \infty} \lim_{n \to \infty} 2^{m+n} \left( \frac{1}{2^m} a \right) \cdot \left( \frac{1}{2^n} b \right) = 0 \), as we can denote \( d(a,b,c) \) and use the same proof to show that \( \lim_{m \to \infty} \lim_{n \to \infty} d(a,b,c) = 0 \). Let \( V_{a,b,c} \) be the vector which has the same entries as the vector \( V_{a,b,c} \) from Notation 2 but in a different order, namely we first form vector \( P_{a,b,c} \) of these entries from \( V_{a,b,c} \) which have \( a \) appearing only once (arranged in the same order as in \( V_{a,b,c} \)). After that we form vector \( S_{a,b,c} \) of these entries from \( V_{a,b,c} \) which have \( b \) appearing only once. After that we form vector \( Q_{a,b,c} \) of all the remaining entries from \( V_{a,b,c} \) in the same order as in \( V_{a,b,c} \). We then form \( V'_{a,b,c} \) by listing first vector \( P_{a,b,c} \) and after that listing vector \( S_{a,b,c} \) and after that listing \( Q_{a,b,c} \). Notice that each entry of \( V_{a,b,c} \) belongs to \( E_{a,b,c} \) and therefore it will have at least three occurrences of elements from the set \( \{ a, b \} \). By using Lemma 3 several times (similarly as in the proof of Lemma 5) there exists a matrix \( M \) such that \( V_{a,b,c} = 2M V'_{a,b,c} \) for every \( a, b, c \in A \). Observe that \( M \) will be a block matrix consisting of 9 blocks. The first block in \( M \) is the identity matrix \( I \) of the same dimension as vector \( P_{a,b,c} \) the second block will be zero, and the last block in the row one will be some matrix \( C \). In the next row the first block will be zero, and the second block will be some upper triangular matrix \( D \) whose eigenvalues are at \( 2^i \) for some \( i \geq 1 \) (because \( a \) appears at least twice in entries of \( S_{a,b,c} \), and the third block will be an upper triangular matrix whose eigenvalues are \( 2^i \) for some \( i \geq 1 \). Notice that the limit \( \lim_{m \to \infty} M_{a,b,c} \) exists and equals the matrix whose first block is the identity matrix, the second block is zero, and the last block in row one is some matrix \( A' \), moreover all the remaining blocks \( 4 \) - \( 9 \) are zero. We will denote this matrix as \( Z \).

Observe that there is a vector \( W \) with entries in \( Q \) such that for every \( a, b, c \in A \)

\[ d(a,b,c) = W^T V'_{a,b,c}. \]
It follows that for a given \( m \), \( \lim_{n \to \infty} 2^n d(\frac{1}{2^n} a, \frac{1}{2^n} b, c) = \lim_{n \to \infty} W^T M^{-n} V'_{a, \frac{1}{2^n} b, c} = W^T Z V'_{a, \frac{1}{2^n} b, c} \). It follows that \( \lim_{n \to \infty} 2^n d(\frac{1}{2^n} a, \frac{1}{2^n} b, c) \) exists and equals a sum of some entries from vectors \( P_{a, \frac{1}{2^n} b, c} \) and \( Q_{a, \frac{1}{2^n} b, c} \). We will denote this limit as \( w(a, \frac{1}{2^n} b, c) \). Let \( W_{a, \frac{1}{2^n} b, c} \) be the vector obtained by listing vector \( P_{a, b, c} \), and after that listing vector \( Q_{a, b, c} \). Observe that then \( w(a, \frac{1}{2^n} b, c) = L^T V_{a, \frac{1}{2^n} b, c} \), where \( L \) is a vector with entries from \( Q_{a, \frac{1}{2^n} b, c} \).

6 The correspondence is one-to-one

In this chapter we show that the correspondence between strongly nilpotent \( F \)-braces and pre-Lie algebras over \( F \) is one-to-one for \( F = \mathbb{Q} \). Recall that \( \mathbb{N} \) denotes the set of natural numbers. We start with the following.

**Proposition 7.** Let \((A, +, \cdot)\) be a nilpotent pre-Lie algebra over a field \( F \) of characteristic zero, and let \((A, +, \circ)\) be the brace obtained as in Section 3, so \((A, \circ)\) is the formal group of flows of the pre-Lie algebra \( A \). Suppose that \( F = \mathbb{R} \) the field of real numbers (or the field of rational numbers). Then for every \( a \in A \) there exists limit

\[
\lim_{n \to \infty} 2^n (\frac{1}{2^n} a) * b.
\]

Moreover

\[
a \cdot b = \lim_{n \to \infty} 2^n (\frac{1}{2^n} a) * b,
\]

where \( n \in \mathbb{N} \).

**Proof.** It follows immediately from the fact that the multiplication in a pre-Lie algebra is bilinear, and from the fact that \( a * b \) can be expressed as in Theorem 2. \( \square \)

We now obtain the ‘reverse’ theorem to Theorem 2:

**Theorem 8.** Let \((A, +, \circ)\) be a strongly nilpotent brace and let \((A, +, \cdot)\) be a nilpotent pre-Lie algebra over the field \( \mathbb{Q} \) obtained from this brace using Theorem 6, so

\[
a \cdot b = \lim_{n \to \infty} 2^n (\frac{1}{2^n} a) * b.
\]

Then \((A, \circ)\) is the group of flows of the pre-Lie algebra \( A \), and \((A, +, \circ)\) can be obtained as in Section 3 from pre-Lie algebra \((A, +, \cdot)\).

**Proof.** Let \( E_{a,b} \) be as in Notation 1. Observe that by Lemma 4 applied several times

\[
a \cdot b = a * b + \sum_{w \in E_{a,b}} \alpha_w w.
\]
where $\alpha_w \in \mathbb{Q}$ does not depend on $a, b$, but only on their arrangement in word $w$ as an element of $E_{a,b}$ (and each $w$ is a product of at least 3 elements from the set $\{a, b\}$). Observe that coefficients $\alpha_w$ do not depend on the brace $A$, as they were constructed using the formula from Lemma 3 which holds in every strongly nilpotent brace (as we can consider $V_{a,b}$ to be an infinite vector with almost all entries zero in $A$). Therefore $a \cdot b = a \cdot b - \sum_{w \in E_{a,b}} \alpha_w w$, and now we can use this formula several times to write every element from $E_{a,b}$ as a product of elements $a$ and $b$ under the operation $\cdot$. In this way we can recover the brace $(A, +, \cdot)$ from the pre-Lie algebra $(A, \cdot, +)$.

Notice that because we know that pre-Lie algebra $(A, +, \cdot)$ can be obtained as in Theorem 6 from the brace which is it’s group of flows (by Theorem 7) we can use the same reasoning and ‘recover’ the group of flows using the same formula.

Therefore $(A, \circ)$ is the group of flows of pre-Lie algebra $A$.

By combining results from Theorems 7 and 8 we get the following corollary.

**Corollary 9.** There is one-to-one correspondence between the set of strongly nilpotent $\mathbb{Q}$-braces and the set of nilpotent pre-Lie algebras over $\mathbb{Q}$.

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