Sharp spectral asymptotics for generic 4-dimensional Schrödinger operator with the strong magnetic field

Victor Ivrii
November 1, 2018

Abstract

I consider 4-dimensional Schrödinger operator with the generic non-degenerating magnetic field and for a generic potential I derive spectral asymptotics with the remainder estimate $O(\mu^{-1}h^{-3})$ and the principal part $\asymp h^{-4}$ where $h \ll 1$ is Planck constant and $\mu \gg 1$ is the intensity of the magnetic field. For general potentials remainder estimate $O(\mu^{-1}h^{-3} + \mu^2h^{-2})$ is achieved.

0 Introduction

Sharp spectral asymptotics for multidimensional Magnetic Schrödinger were obtained in [Ivr4, Ivr5] in full- and non-full-rank cases respectively$^1$. The results, as one could expect from the analysis of 2- and 3-dimensional cases, were rather different.

However there are two problems with these papers: first, the rank of the magnetic intensity matrix there was supposed to be constant which is not necessarily the case even if only generic magnetic fields are considered; second, while both the maximal rank$^2$ and microhyperbolicity$^3$ conditions are stable with respect to the small perturbations, they both are not generic in the sense that even in a small but fixed domain a general Magnetic Schrödinger operator is not necessarily approximated by operators satisfying any of these conditions: exactly in the same way as stationary points are not necessarily removable.

However the analysis of [Ivr4] and [Ivr6] leads me to the following conjecture for even-dimensional Magnetic Schrödinger operators:

---

$^1$ I mean the rank of magnetic intensity matrix $(F_{jk})$.

$^2$ I mean that the rank of $(F_{jk})$ is $2 \lfloor d/2 \rfloor$ at each point.

$^3$ Which was assumed in [Ivr4].
**Conjecture 1.** As $\mu \leq \chi h^{-1}$ the main part of asymptotics is given by $E^{MW}$ while

(i) For fixed $(g^{jk})$ and general $(V_j)$, $V$ the remainder estimate is $O(\mu h^{1-d})$;

(ii) For fixed $(g^{jk})$ and generic $(V_j)$ (i.e. $(V_1, ... , V_d) \notin \mathfrak{A}_g$ which is nowhere dense closed set) and general $V$ the remainder estimate is $O(\mu^{-1}h^{1-d} + \mu d/2 h^{-d/2})$ as $(F_{jk})$ has the full rank everywhere and $O(\mu^{-1/2}h^{1-d} + \mu^{d/2} h^{-d/2})$ otherwise;

(iii) For fixed $(g^{jk})$ and generic $(V_j, V)$ (i.e. $(V_1, ... , V_d, V) \notin \mathfrak{A}_g$ which is nowhere dense closed set) the remainder estimate is $O(\mu^{-1}h^{1-d})$ as $(F_{jk})$ has the full rank everywhere and $O(\mu^{-1/2}h^{1-d})$ otherwise.

While (i) is trivial, (ii) and (iii) are rather difficult and my goal is rather limited: to prove them as $d = 4$. For $d = 2$ it was done in [Ivr6, Ivr9]). Further, in the small vicinity of the set $\{x : \text{rank}(F_{jk})(x) \leq 2\}$ it was done in [Ivr8]. So, I will need to investigate the case of $\text{rank}(F_{jk}) = 1$ at every point.

So, operator in question is

\[ A = \frac{1}{2} \left( \sum_{j,k} P_j g^{jk}(x) P_k - V \right), \quad P_j = D_j - \mu V_j \]

with smooth\(^4\) symmetric positive definite matrix $(g^{jk})$ and smooth real-valued potentials $(V_1, ... , V_d; V)$ and $\mu \gg 1$, $h \ll 1$. Assuming that $A$ is self-adjoint, let $E(\tau)$ be the spectral projector of $A$ and $e(x, y, \tau)$ be its Schwartz’ kernel. These assumptions are fulfilled during the whole article.

Magnetic field is characterized by a skew-symmetric matrix $(F_{jk})$, $F_{jk} = \partial_j V_k - \partial_k V_j$ and more precisely by $(F'_{jk}) = (g^{js})(F_{sk})$ and its eigenvalues $\pm i f_j$, $f_j \geq 0$. As $d = 4$ these are $f_1$ and $f_2$.

It is proven [Ma] that

\[ \text{(0.2)} \quad \text{For generic } (V_1, ... , V_4) \ f_1 \text{ and } f_2 \text{ do not vanish simultaneously.} \]

**Remark 0.1.** (i) If one of $f_1, f_2$ vanishes then (for generic $(V_1, ... , V_4)$ locally situation of [Ivr8] occurs and the remainder estimate is $O(\mu^{-1/2}h^{-3})$ for generic $V$ and $O(\mu^{-1/2}h^{-3} + \mu^2 h^{-2})$ for general $V$ (detailed assumptions see in [Ivr8]) and I exclude this case from the analysis, assuming that

\[ \text{(0.3)} \quad f_1 \geq \epsilon_0, \ f_2 \geq \epsilon_0. \]

\(^4\) Smooth means either infinitely smooth or belonging to $C^K$ with large enough $K$. 

(ii) As condition (0.4) is fulfilled and \( \mu h \geq \epsilon \) then the main part of asymptotics is 0 and the remainder estimate is \( O(\mu^{-s}) \) with an arbitrarily large exponent \( s \) (see f.e. [Ivr3]) and therefore in what follows I assume that

(0.4) \hspace{1cm} 1 \leq \mu \leq ch^{-1}.

My goal is to prove two following theorems:

**Theorem 0.2.** Let \( (g^{jk}) \) be fixed and \( (V_1, \ldots, V_4) \) be generic. Further, let conditions (0.3), (0.4) be fulfilled and \( \psi \) be a smooth function. Then

(0.5) \hspace{1cm} | \int (e(x, x, 0) - \mathcal{E}^{MW}(x, 0)) - \mathcal{E}^{MW}_{\text{corr}}(x, 0) \psi(x) \, dx | \leq C \mu^{-1} h^{-3} + C \mu^2 h^{-2}

where

(0.6) \hspace{1cm} \mathcal{E}^{MW}(x, \tau) = (2\pi)^{-2} \mu^2 h^{-2} \sum_{(m,n) \in \mathbb{Z}^+} \theta(2\tau + V - (2m + 1) \mu h f_1 - (2n + 1) \mu h f_2) f_1 f_2 \sqrt{g}

is Magnetic Weyl Expression, \( g = \det(g^{jk})^{-1} \), \( \mathcal{E}^{MW}_{\text{corr}} \) is a correction term, defined by (4.23).

**Theorem 0.3.** Let \( (g^{jk}) \) be fixed and \( (V_1, \ldots, V_4; V) \) be generic. Further, let conditions (0.3), (0.4) be fulfilled and \( \psi \) be a smooth function. Then

(0.7) \hspace{1cm} | \int (e(x, x, 0) - \mathcal{E}^{MW}(x, 0)) \psi(x) \, dx | \leq C \mu^{-1} h^{-3}.

**Remark 0.4.** (i) More precise conditions describing conditions for \( (V_1, \ldots, V_d) \) in Theorem 0.2 and for \( (V_1, \ldots, V_d; V) \) in Theorem 0.3 will be formulated below in section 4.

(ii) I was able to prove that one can skip \( \mathcal{E}^{MW}_{\text{corr}}(x, 0) \) and preserve remainder estimate (0.5) unless \( h^{-1/3+\delta} \leq \mu \leq h^{-1/3-\delta} \); in the latter case the remainder estimate (0.6) should be replaced by \( h^{-8/3-\delta} \); however I suspect that one can always skip \( \mathcal{E}^{MW}_{\text{corr}}(x, 0) \) and preserve remainder estimate (0.5).

(iii) With the exception of section 5 I assume that

(0.8) \hspace{1cm} V \geq \epsilon_0.

In section 5 I will get rid off this condition.
1 GEOMETRY AND PRELIMINARY ANALYSIS

Plan of the paper. Section 1 is devoted to the geometry (discussion of what is the generic case) and the preliminary analysis in the cases when results of [Ivr4] imply theorem 0.3 immediately.

In section 2 I tackle the weak magnetic field case ($\mu \leq h^{-\delta_0}$ with sufficiently small exponent $\delta_0 > 0$) proving the standard Weyl formula with the remainder $O(\mu^{-1}h^{-3})$.

Sections 3, 4 are devoted to the case when $h^{-\delta_0} \leq \mu \leq ch^{-1}$ with arbitrarily small exponent $\tilde{\delta} > 0$ and one can reduce operator to the microlocal canonical form. More precisely, in section 3 I prove asymptotics with the announced remainder canonical estimates but with the implicitly given main part

\[(0.9) \quad h^{-1} \sum_i \int_{-\infty}^0 \left( F_{t \to h^{-1}r} \bar{\chi}_{T_i}(t) \Gamma uQ_{iY}^t \right) d\tau \]

where $u$ is the Schwartz' kernel of the propagator $e^{ih^{-1}tA}$ and $Q_t$ are pseudo-differential partition elements (see details in my previous papers); in this proof estimate of

\[(0.10) \quad |\left( F_{t \to h^{-1}r} \bar{\chi}_{T_i}(t) \Gamma uQ_{iY}^t \right)| \]

plays the crucial role.

In section 4 I replace (0.10) by the standard Magnetic Weyl formula and estimate an error arising from this.

Finally, in section 5 I consider the case when condition (0.8) is violated.

1 Geometry and Preliminary Analysis

1.1 Geometry

Proposition 1.1. Let either $g^{jk}$ be fixed and then $V_j$ be generic, or, alternatively, $V_j$ be fixed and then $g^{jk}$ be generic. Further, let (0.3) be fulfilled. Then

\[(1.1) \quad \Sigma \overset{def}{=} \{ x : f_1 = f_2 \} \text{ is a smooth 2-dimensional manifold: } \Sigma = \{ x : v_1 = v_2 = 0 \} \text{ with appropriate smooth functions } v_1 \text{ and } v_2 \text{ such that } \nabla v_1 \text{ and } \nabla v_2 \text{ linearly independent at any point of } \Sigma \text{ and} \]

\[(1.2) \quad \text{dist}(x, \Sigma) \asymp |f_1 - f_2| = 2(v_1^2 + v_2^2)^{1/2};\]

furthermore, $f_1 + f_2$ and $(f_2 - f_1)^2$ are smooth functions;
Consider symplectic form corresponding to $F_{jk}$

$$\omega = \frac{1}{2} \sum_{jk} F_{jk} dx_j \wedge dx_k;$$

then $\omega|_{\Sigma}$ is either non-degenerate or it is generic degenerate\(^5\) i.e. $\Theta_1 \overset{\text{def}}{=} \{ x \in \Sigma, \{ v_1, v_2 \} = 0 \}$ is a submanifold of dimension 1 and $\nabla_{\Sigma} \{ v_1, v_2 \}$ is disjoint from 0 at $\Theta_1$;

Function $f_1 f_2^{-1}$ has only non-degenerate critical points outside of $\Sigma$ and near $\Sigma | \nabla (f_1 f_2^{-1})|$ is disjoint from 0.

**Proof.** I leave an easy proof to the reader. \(\square\)

### 1.2 Microhyperbolicity Condition

1.2.1 First, let me discuss the case when magnetic intensities $f_1$ and $f_2$ are disjoint. Then they are smooth. It still does not exclude third-order resonances $f_1 = 2 f_2$ and $f_2 = 2 f_1$\(^6\).

Then there are two cyclotron movements and the drift with the velocity $O(\mu^{-1})$ which I want to calculate. Let

$$p_j = \xi_j - \mu V_j.$$

Note that

$$x_j' \overset{\text{def}}{=} x_j - \mu^{-1} \sum_l \phi^{jl} p_l,$$

$$\left( \phi^{**} \right) \overset{\text{def}}{=} \left( F^{**} \right)^{-1}$$

satisfy

$$\{ p_k, x_j' \} = -\mu^{-1} \sum_l \{ p_k, \phi^{jl} \} p_l$$

and therefore

$$\left\{ \frac{1}{2} \sum_{k,m} g^{km} p_k p_m, x_j' \right\} = \frac{1}{2} \sum_{k,m} \{ g^{km}, x_j' \} p_k p_m - \mu^{-1} \sum_{k,m,l} g^{km} \{ p_k, \phi^{jl} \} p_m p_l =$$

$$\mu^{-1} \sum_{k,m,l} \left( \frac{1}{2} \phi^{jl} \{ p_l, g^{km} \} - \{ p_l, \phi^{jk} \} g^{ml} \right) p_k p_m$$

\(^5\) Exactly as $\Sigma$ was in [Ivr6].

\(^6\) Higher-order resonances are not a problem according to [Ivr4], at least under microhyperbolicity condition.
**Proposition 1.2.** Let $f_1 \neq f_2$ in $\Omega$; then one can correct

\[(1.11) \quad x_j' \mapsto x_j'' \defeq x_j' - \frac{1}{2} \mu^{-1} \sum_{k,m} \beta^{km} p_k p_m \]

so that

\[(1.12) \quad \left\{ \frac{1}{2} \sum_{k,m} \epsilon^{km} p_k p_m, x_j'' \right\} = \mu^{-1} \left( \{ f_1, x_j \} F b_1 + \{ f_2, x_j \} F b_2 \right) + O(\mu^{-2}) \]

where in the Birkhoff normal form $a = f_1 b_1 + f_2 b_2 + \ldots$ and

\[(1.13) \quad \{ w_1, w_2 \} F \defeq \sum_{k,l} \phi^{kl} \partial_k w_1 \cdot \partial_l w_2 \]

are Poisson brackets associated with symplectic form $\omega = \omega_F$.

**1.2.2 Microhyperbolicity condition** of [Ivr4] then means for $d = 4$ and $f_1 \neq f_2$ that

\[(1.14) \quad | V - (2p + 1)f_1 \mu h - (2n + 1)f_2 \mu h | \leq \epsilon \quad \implies \quad | \langle \ell, \nabla (V - (2p + 1)f_1 \mu h - (2n + 1)f_2 \mu h) \rangle | \geq \epsilon \]

where $\ell$ (|$\ell$| $\approx 1$) is the microhyperbolicity direction (at the given point $x$) and $(n, p)$ are magnetic indices. However there actually were two microhyperbolicity conditions in [Ivr4]: in the weaker condition $\ell$ could depend not only on $x$ but also on magnetic indices while in the stronger one it was assumed that $\ell$ is the same for all pairs $(n, p)$ connected by the third order resonance. Respectively, the remainder estimates derived in [Ivr4] were $O(h^{1-d})$ under weaker condition and $O(\mu^{-1} h^{1-d})$ under stronger one.

In the relatively simple case of $d = 4$ the resonance means $k f_1 = l f_2$ where $k, l \in \mathbb{Z}^+$ are coprimes, $(k + l)$ is the order of the resonance. So under the stronger microhyperbolicity condition if either $2f_1 - f_2 = 0$ or $f_1 - 2f_2 = 0$ at $x$ then $\ell$ should not depend on $(n, p)$ satisfying the the left-hand inequality in (1.14).

**Remark 1.3.** Case $d = 4$ is relatively simple because there are only two magnetic intensities $f_1$ and $f_2$ and thus it is impossible to have collisions between two second-order resonances (of the type $f_j = f_k$), two third order resonances (of the types $f_j = 2f_k$ or $f_j = f_k + f_l$) or the second and the third order resonances.

Further, as $|kf_1 - lf_2| \leq \epsilon$ there could be other resonances only of order $m(\epsilon)$ or higher; $m(\epsilon) \to +\infty$ as $\epsilon \to +0$. 

So, as \( d = 4 \) and \( f_1 \) is disjoint from \( f_2 \) and \( \mu h \leq \epsilon_1 \) (where \( \epsilon_1 > 0 \) depends on \( \epsilon \) in the microhyperbolicity condition) both microhyperbolicity conditions are equivalent to

\[
|\nabla (\alpha \log f_1 + (1 - \alpha) \log f_2 - \log V)| \geq \epsilon \quad \forall \alpha : 0 \leq \alpha \leq 1.
\]

On the other hand, as \( \mu h \geq \epsilon_1 \) condition (1.15) should be checked only at points where ellipticity condition

\[
|V - (2p + 1)f_1\mu h - (2n + 1)f_2\mu h| \geq \epsilon \quad \forall (p, n) \in \mathbb{Z}^+^2
\]

is violated.

**Remark 1.4.** I remind that under condition (1.16) on \( \text{supp} \psi \) the remainder estimate is \( O(\mu^{-s}h^s) \) with an arbitrarily large exponent \( s \) and the similar result would hold in any dimension provided \( \text{rank}(F_{jk}) = d \) at each point.

So, main theorems 0.3, 0.5, 0.7 of [Iv4] imply

**Proposition 1.5.** Let \( d = 4 \) and conditions (0.8), \( f_1 \neq f_2 \) and (1.15) be fulfilled at \( \text{supp} \psi \). Then the remainder is \( O(\mu^{-1}h^{-3}) \) while the main part of asymptotics is given by the Magnetic Weyl formula.

**1.2.3** This leaves us with bad points where condition (1.15) is violated. So let us consider set \( \Lambda_\alpha \) of critical points of

\[
\phi_\alpha \overset{\text{def}}{=} \alpha \log f_1 + (1 - \alpha) \log f_2 - \log V.
\]

**Proposition 1.6.** Let \( g^{ij}, V_j \) be fixed and let \( V \) be generic. Then

(i) \( \Lambda_\alpha \) is a finite set, continuously depending on \( \alpha \in [0, 1] \);

(ii) There exist \( 0 < \alpha_1 < \alpha_2 < \cdots < \alpha_J < 1 \) such that for \( \alpha \notin \{\alpha_1, \ldots, \alpha_J\} \) function \( \phi_\alpha \) has non-degenerate critical points, while for \( \alpha = \alpha_j \) function \( \phi_\alpha \) has also isolated critical points but in one of them \( \text{rank Hess} \phi_\alpha = 3 \);

(iii) As \( \bar{\alpha} \in \{\alpha_1, \ldots, \alpha_J\} \) and \( \bar{x} \) is a critical point of \( \phi_{\bar{\alpha}} \) with \( \text{rank Hess} \phi_{\bar{\alpha}}(\bar{x}) = 3 \), in an appropriate (smooth) coordinate system in vicinity of \( \bar{x} = 0 \)

\[
\phi_\alpha = \frac{1}{3}x_1^3 - (\alpha - \bar{\alpha})x_1 \pm_2 \frac{1}{2}x_2^2 \pm_3 \frac{1}{2}x_3^2 \pm_4 \frac{1}{2}x_4^2
\]

with independent signs \( \pm_k \).

**Proof.** An easy proof is left to the reader. \( \square \)
Remark 1.7. (i) Assume that at some point \( \bar{x} \)
\[
|\nabla(f_1f_2^{-1})| \geq \epsilon_0.
\]
Then in the vicinity of \( \bar{x} \)
\[
|\nabla \phi_\alpha| \geq \epsilon|\alpha - \bar{\alpha}(x)| \quad \forall \alpha : 0 \leq \alpha \leq 1
\]
with smooth function \( \bar{\alpha}(x) \).

(ii) On the other hand, if \( \bar{x} \) is an isolated critical point of \( f_1f_2^{-1} \) then as \( V \) is generic \( |\nabla \phi_\alpha| \geq \epsilon \) for all \( \alpha \in [0, 1] \) in the vicinity of \( \bar{x} \).

1.3 Analysis near \( \Sigma \): Geometry

1.3.1 Consider now \( \Sigma = \{ x : f_1 = f_2 \} \) assuming that (1.1) holds; then the microhyperbolicity condition at point \( x \in \Sigma \) means exactly that

\[
\frac{1}{2} \ell((f_1 + f_2)V^{-1}) \geq ((\ell v_1)^2 + (\ell v_2)^2)^{1/2}V^{-1} + \epsilon
\]

where \( \ell \) again is the microhyperbolicity direction, or, equivalently,

\[
\frac{1}{2} \nabla ((f_1 + f_2)V^{-1}) - (\beta_1 \nabla v_1 + \beta_2 \nabla v_2)V^{-1} \geq \epsilon \quad \forall \beta = (\beta_1, \beta_2) \in \mathbb{R}^2 : |\beta| \leq 1.
\]

Remark 1.8. Note that the microhyperbolicity condition could be violated only at stationary points of \( f_1 V^{-1} |_{\Sigma} \) and only at those of them where

\[
\frac{1}{2} \nabla ((f_1 + f_2)V^{-1}) = (\beta_1 \nabla v_1 + \beta_2 \nabla v_2)V^{-1}
\]

with \( \beta_1^2 + \beta_2^2 \leq 1 \).

Let \( \Sigma_0 \) be the set of stationary points of \( f_1 V^{-1} |_{\Sigma} \); at each point of \( \Sigma_0 \) decomposition (1.23) holds; let us denote by \( \Sigma_0^+, \Sigma_0^- \) \( \Sigma_0^0 \) the subsets of \( \Sigma_0 \) where \( \beta_1^2 + \beta_2^2 > 1 \), \( \beta_1^2 + \beta_2^2 < 1 \) and \( \beta_1^2 + \beta_2^2 = 1 \) respectively. Then the microhyperbolicity condition holds at \( \Sigma_0^+ \).

So, main theorems 0.3, 0.5, 0.7 of [Ivr4] imply

**Proposition 1.9.** Let \( d = 4 \) and conditions (1.1) and (0.8) be fulfilled. Let \( \psi \) be supported in the small vicinity of \( \bar{x} \in \Sigma_0^+ \). Then the remainder is \( O(\mu^{-1}h^{-3}) \) while the main part of asymptotics is given by magnetic Weyl formula.
1.3.2 One can prove easily two following propositions:

**Proposition 1.10.** Let $d = 4$ and condition (1.1) be fulfilled. Let $V$ be generic satisfying (0.8). Then

(1.24) $\Sigma_0$ consists of the finite number of non-degenerate stationary points of $f_1 V^{-1}|_{\Sigma}$; these points are generic;

(1.25) $0 < \beta_1^2 + \beta_2^2 < 1$ at each point of $\Sigma_0 \setminus \Sigma^+_0$.

**Proposition 1.11.** Let $d = 4$ and condition (1.1) be fulfilled. Let $\bar{x} \in \Sigma_0^-$ with $0 < \beta_1^2 + \beta_2^2 < 1$ in decomposition (1.23). Then in the vicinity of $\bar{x}$

(1.26) $\Lambda \overset{\text{def}}{=} \{ x : f_1 \neq f_2 \text{ and } \exists \alpha \in [0, 1] : \nabla \phi_\alpha = 0 \} \cup \{ \bar{x} \}$

is a smooth curve passing through $\bar{x}$ and transversal to $\Sigma$; moreover

(1.27) $\bar{\alpha}(\bar{x}) = \frac{1}{2}(1 \pm (\beta_1^2 + \beta_2^2)^{1/2})$

2 Weak Magnetic Field Case

In this section I am going to prove the remainder estimate $O(\mu^{-1} h^{-3})$ for general $V$ as

(2.1) $c \leq \mu \leq h^{-\delta_0}$

with small exponent $\delta_0 > 0$.

2.1.1 Assume first that condition (1.19) is fulfilled.

**Proposition 2.1.** Let conditions $f_1 \neq f_2$ and (1.19) be fulfilled at $\text{supp } \psi$. Then under condition (2.1) the remainder is $O(\mu^{-1} h^{-3})$ while the main part of asymptotics is given by Weyl formula.

**Proof.** Part I. (i) Assume first that there are no resonances of order lesser than $M$ on $\text{supp } \psi$:

(2.2) $|kf_1 - lf_2| \geq \epsilon, \quad \forall k, l \in \mathbb{Z}^+: k + l \leq M$.

Then one can reduce operator to the normal form without cubic and “unbalanced” fourth order terms and then along Hamiltonian trajectories

(2.3) $\frac{d}{dt} x_j = \frac{\partial}{\partial \xi_j} a, \quad \frac{d}{dt} \xi_j = -\frac{\partial}{\partial x_j} a$
of

\[ a(x, \xi) \overset{\text{def}}{=} \frac{1}{2} \left( \sum_{j,k} p_j g^{jk}(x) p_k - V \right), \quad p_j = \xi_j - \mu V_j \]

the following relations hold:

\[ \frac{d}{dt} b_j = \{ a, b_j \} = O(\mu^{-2}) \]

where \( b_j = \frac{1}{2}(\xi_j \circ \Psi)(x_j^2 + \xi_j^2) + O(\mu^{-2}) \) are quadratic terms in the Birkhoff normal form; see details in [Ivr4] or below.

Therefore in the "corrected" coordinates \( x'' \) defined by (1.11) along Hamiltonian trajectories

\[ \frac{d}{dt} x''_j = \mu^{-1} (\beta_{j1}(x'') b_1(x, \xi) + \beta_{j2}(x'') b_2(x, \xi)) + O(\mu^{-3}) \]

and

\[ |\frac{d^2}{dt^2} x''| \leq C \mu^{-1} |\frac{d}{dt} x''| + C(\mu^{-4}). \]

Thus as

\[ \mu |\frac{d}{dt} x''| \approx \rho \geq C \mu^{-2} \]

then for time \( T_1 = \epsilon \mu \) this relation (2.8) is retained and also variation of \( dx''/dt \) would be less than \( C \epsilon \rho \mu^{-1} \).

Then there is a fixed direction \( \ell, |\ell| \approx 1 \) such that for \( |t| \leq T_1 \)

\[ \langle \ell, \frac{d}{dt} x'' \rangle \geq \epsilon_0 \mu^{-1} \rho; \]

\( \ell \) is a sort of the microhyperbolicity direction. Without any loss of the generality one can assume that \( \ell = (1, 0, 0, 0) \) and therefore the shift of \( x''_1 \) for time \( T \) is exactly of magnitude \( \rho \mu^{-1} T \).

According to the logarithmic uncertainty principle this shift is microlocally observable as

\[ \rho \mu^{-1} T \times \rho \geq Ch|\log h| \]
Plugging $T = T_0 = \epsilon \mu^{-1}$ one gets $\rho^2 \geq C \mu^2 h|\log h|$ which would hold for $\rho \geq C \mu^{-2}$ as $\mu \leq h^{-\delta}$.

Therefore in this case the contribution of zone (2.6) to the remainder does not exceed

$$C h^{-3} \rho \times T_1^{-1} \lesssim C \mu^{-1} h^{-3} \rho$$

where factor $\rho h^{-3}$ is due to the estimate of

\((2.11)\)

$$|F_{t \to h^{-1} \tau}(\tilde{T}_0(t)\Gamma Qu)| \leq C \rho h^{-3},$$

$Q$ is the cut-off operator in zone (2.8), $u$ is the Schwartz kernel of $e^{-ih^{-1}tA}$ and other notations of [Ivr3, Ivr8] are used.

To prove estimate (2.11) let us make $\rho$-admissible partition in $\xi$; then one can prove easily that the contribution of each such element to the left hand expression does not exceed $C \rho^3 h^{-3}$ because the propagation speed with respect to $x$ is $\simeq 1$ as $|t| \leq T_0$ and therefore one can trade $T_0$ to $\tilde{T} = Ch^{-1}|\log h|$ in the left hand expression of (2.11) with a negligible error and in the estimate even to $\tilde{T} = C \rho^{-1} h$ (one can prove easily by the rescaling). On the other hand, the number of the partition elements for each $x$ (and therefore in its vicinity) so that $hD_t - A$ is not elliptic and (2.8) holds is obviously $O(\rho^{-2})$; so the left-hand expression of (2.11) does not exceed $C \rho^3 h^{-3} \times \rho^{-2}$.

Summation of $O(\rho^{-1} h^{-3})$ with respect to $\rho$ results in $C \mu^{-1} h^{-3}$ and therefore I have proven that

\((2.12)\) In frames of proposition and (2.2) the total contribution of zone

\((2.13)\)

$$\{\mu \left| \frac{d}{dt} x'' \right| \geq \bar{\rho} \overset{\text{def}}{=} C \mu^{-2} \}$$

to the remainder is $O(\mu^{-1} h^{-3})$.

On the other hand, contribution of zone

\((2.14)\)

$$\{\mu \left| \frac{d}{dt} x'' \right| \leq \bar{\rho} = C \mu^{-2} \}$$

to the remainder does not exceed $C \bar{\rho} h^{-3} \times T_0^{-1} = C \mu^{-1} h^{-3}$ where for cut-off operator in this zone estimate (2.11) holds with $\rho = \bar{\rho}$.

So, the remainder does not exceed $C \mu^{-1} h^{-3}$ while the main part of the asymptotics is given by the standard Tauberian formula (0.9) with $T = Ch|\log h|$. One can easily rewrite (0.9) with $T = Ch|\log h|$ as

$$\int E^W(x,0) \psi(x) dx + O(\mu^2 h^{-2}).$$
Furthermore, we can replace here $E^W(x,0)$ by $E^{MW}(x,0)$ with the same error. The proof is standard, easy and left to the reader.

\[\text{2.1.2 Let us allow resonances.}\]

Proof, Part II. (i) Let us consider now the case when $(k_1 - l_2)$ is not disjoint from 0. I will analyze the third order resonance which is worst case scenario leaving the easier case of higher order resonances to the reader.

So, now $(f_1 - 2f_2)$ is not disjoint from 0 (but then one can assume that $(k_1 - l_2)$ is as $l \neq 2k, k \leq M$). Then one can reduce an operator to a (pre)canonical form

\[
\frac{1}{2} \left( f_1(x')Z_1^*Z_1 + f_2(x')Z_2^*Z_2 + \mu^{-1} \Re \beta(x')Z_1^*Z_2^2 \right) + O(\mu^{-2})
\]

with

\[
\begin{align*}
\{Z_j, Z_k\} &\equiv 0, \quad \{Z_j^*, Z_k\} \equiv 2\mu \delta_{jk} \mod O(\mu^{-s}). \\
\{Z_j, x'_k\} &\equiv \{Z_j^*, x'_k\} \equiv 0 \mod O(\mu^{-s}), \\
\{x'_j, x'_k\} &\equiv \mu^{-1} \phi_{jk}(x') \mod O(\mu^{-2})
\end{align*}
\]

where

\[
x' \equiv x \mod O(\mu^{-1}).
\]

Due to (1.19) one can assume without any loss of the generality that

\[
f_1 - 2f_2 = x_1.
\]

(ii) Now let us consider elements of the partition with

\[
|x_1| \asymp \gamma, \quad \gamma \geq C\mu^{-1}.
\]

Then one can get rid off the cubic term in (2.15) by means of the transformation with the generating function $\gamma^{-1}\mu^{-2} \Re \beta Z_1^*Z_2^2$ with $\gamma$-admissible function $\beta$ (all other cubic terms are “regular” and one can get rid of them in the regular way), leading to the error which, as one can easily check, is the sum of terms of the types $\gamma^{-1}\mu^{-2} \Re \beta Z_j^* Z_j Z_2^* Z_2$ and also some smaller terms; one can continue this process getting rid of all “unbalanced” terms up to order $M$. Also one can introduce corrected $x''$ so that

\[
\frac{d}{dt} x''_j = \mu^{-1} (\beta_{j1}(x'') b_1 + \beta_{j2}(x'') b_2) + O(\mu^{-3}\gamma^{-1})
\]
(compare with (2.6); however this would be slightly short of what is needed and one can improve a term $O(\mu^{-3}\gamma^{-1})$ in (2.22) to
\[ \sum_{k+l=2} \beta_{kl}(x'') b_k^* b_l^2 + O(\mu^{-3-\kappa}\gamma^{-1-\kappa}) \]
with $\kappa > 0$. Then the contribution of the zone
\[ \{|x_1| \approx \gamma, \mu|\frac{d}{dt}x''| \leq C\mu^{-2-\kappa}\gamma^{-1-\kappa} \} \]
to the remainder does not exceed $C\mu h^{-3}\gamma \times \mu^{-2-\kappa}\gamma^{-1-\kappa} \approx C\mu^{-1-\kappa} h^{-3}\gamma^{-\kappa}$ where the first factor $\gamma$ is the measure. Proof is similar to one in Part I, with estimate (2.11) replaced by
\[ |F_t \to h^{-1}\tau (\bar{\chi}_T(t)\Gamma Qu)| \leq C\rho \gamma h^{-3}, \]
now $Q$ is the cut-off operator in zone (2.8) intersected with $\{|x_1| \approx \gamma\}$. Obviously summation with respect to $\gamma \geq C\mu^{-1}$ results in $O(\mu^{-1} h^{-3})$.

So one needs to consider the zone where
\[ \mu|\frac{d}{dt}x''| \approx \rho \geq C\mu^{-2-\kappa}\gamma^{-\kappa}, \]
which implies that
\[ |\mu|\frac{d}{dt}(\beta_{j_1}(x'')b_1 + \beta_{j_2}(x'')b_2)| \leq C\rho + C\mu^{-\kappa}\gamma^{-\kappa} \]
and therefore one can take
\[ T_1 = \epsilon \min(\mu\gamma\rho^{-1}, \mu, \rho\mu^{K}\gamma^K) \]
where the restriction $T \leq \epsilon\gamma\rho^{-1}$ preserves the magnitude of $|x_1|$. Therefore the contribution of the zone
\[ \{|x_1| \approx \gamma, \mu|\frac{d}{dt}x''| \approx \rho \} \]
with $\gamma \geq C\mu^{-1}, \rho \geq C\mu^{-2-\kappa}\gamma^{-\kappa}$ to the remainder does not exceed
\[ C\mu h^{-3}\rho\gamma \left(\mu^{-1}\gamma^{-1}\rho + 1 + \mu^{-\kappa}\gamma^{-\kappa}\right) \approx C\mu^{-1} h^{-3} \left(\rho^2 + \rho\gamma + \mu^{1-\kappa}\gamma^{1-\kappa}\right). \]
Obviously summation with respect to $\rho, \gamma$ results in $C\mu^{-1} h^{-3} |\log \mu|$. 
(iii) Now I want to improve this estimate getting rid off \( \log \mu \) factor. Note first that the second term in (2.25) sums to \( O(\mu^{-1}h^{-3}) \), while the first and third terms sum to \( O(\mu^{-1}h^{-3}) \) over zones complemental to \( \{ \rho \geq \max(\gamma, |\log \mu|^{-1/2}) \} \) and \( \{ \gamma \leq \min(\rho, \mu^{-1+\kappa}) \} \) respectively.

However in the first zone one can take \( T_1 = \epsilon \mu \gamma \) since in an appropriate time direction \( |x_1| \gtrsim \gamma \) for this time. Then an extra factor \( \gamma^\kappa \) in the remainder estimate prevents appearance of the logarithmic factor.

Let us introduce \( W(x') = Vf_1^{-1}|x_1=0 \) and a scaling function \( \zeta = \epsilon |\nabla' W| + \gamma \). Then one can upgrade \( T_1 \) to

\[
T_1 = \epsilon \min(\mu \gamma \zeta^{-1}, \mu, \rho^{1-\kappa} \mu K \gamma K)
\]

Really, \( \mu |\frac{d}{dt} x_1| \leq C \zeta \) and one can select direction of time to replace \( \rho \) by \( \rho^{1-\kappa} \). Then the total contribution to the remainder of all partition elements with \( \zeta \leq \gamma^\kappa \) is \( O(\mu^{-1}h^{-3}) \).

On the other hand, as \( \zeta \geq \gamma^\kappa \) and \( \gamma \leq \mu^{-1+\kappa} \) one can obviously take \( T_1 = \epsilon \mu \zeta \) and then the total contribution of such partition elements to the remainder does not exceed \( C \mu^{-1} \gamma^{-1} h^{-3} \gamma \ll \mu^{-1} h^{-3} \).

(iv) Finally, let us consider now zone where

\[
|x_1| \leq \gamma = C \mu^{-1}.
\]

Then I am not getting rid off the cubic terms and should take \( T_1 = \epsilon \rho \) which one can easily upgrade to \( T_1 = \epsilon \rho^{1-\kappa} \) leading to the contribution of this zone to the remainder \( C \mu^{-1} h^{-3} \) where factor \( \mu^{-1} \) is the measure of zone defined by (2.29).

2.1.3 Now let us allow \( f_1 f_2^{-1} \) to have critical points; however for the generic magnetic field these critical points are not resonances.

**Proposition 2.2.** Let conditions \( f_1 \neq f_2, (2.2) \) and

\[
(2.30)_q \quad |\nabla (f_1 f_2^{-1})| \leq \epsilon_0 \implies \text{Hess}(f_1 f_2^{-1}) \text{ has at least } q \text{ eigenvalues with absolute values greater than } \epsilon_0
\]

be fulfilled at \( \text{supp } \psi \) with \( q \geq 3 \). Then under condition (2.1) the remainder is \( O(\mu^{-1} h^{-3}) \) while the main part of asymptotics is given by Weyl formula.

\footnote{One can see easily that in this zone \( \frac{\partial}{\partial x_1} b_j = \{a, b_j\} = O(1) \) and \( (\frac{\partial}{\partial x_1})^2 b_j = \{a, \{a, b_j\}\} = O(1) \); one can prove that the microhyperbolicity is preserved with respect to the same vector \( \ell \rho^{-1} \).}
Proof. Let \( \gamma = \epsilon |\nabla (f_1 f_2^{-1})| + \bar{\gamma}, \bar{\gamma} = C \mu^{-1} \). Let us consider \( \gamma \)-admissible partition with respect to \( x \).

Then similarly to part (i) of the proof of proposition 2.1 one can take \( T_1 = \epsilon \mu^{2-\delta^\gamma} \) with \( \rho \) defined by (2.8). Note that

\[
|\alpha \nabla (f_1 V^{-1}) + (1 - \alpha) \nabla (f_2 V^{-1})| \geq \epsilon |\alpha - \bar{\alpha}(x)| \quad \forall \alpha : 0 \leq \alpha \leq 1
\]

and therefore the measure in \( \xi \)-space gets a factor \( \gamma^{-1} \) but the measure in \( x \)-space gets a factor \( \gamma^q \) due to condition condition (2.30).

Then (2.11) and (2.23) are replaced by the similar estimate with the right hand expression \( C \rho \gamma^{q-1} h^{-3} \).

So, the total contribution \( (\rho, \gamma) \)-elements to the remainder does not exceed

\[
C \mu^{-1} h^{-3} \rho \gamma^{q-1} \times \rho^{\kappa-1} \gamma^{-1} \asymp C \mu^{-1} h^{-3} \gamma^{q-2} \rho^\kappa.
\]

Summation with respect to \( (\rho, \gamma) \) results in \( C \mu^{-1} h^{-3} \) as \( q \geq 3 \) and in \( C \mu^{-1} h^{-3} |\log \mu| \) as \( q = 2 \) and this is the total contribution of zone \( \{ \rho \gamma \geq C \mu^{-2} \} \).

On the other hand, contribution of zone \( \{ \rho \gamma \leq C \mu^{-2} \} \) to the remainder is \( O(\mu h^{-3} \times \mu^{-2}) \) since its measure is \( O(\mu^{-2}) \) under condition (2.30)\( _3 \); under condition (2.30)\( _2 \) an extra logarithmic factor appears as well.

\[ \square \]

Remark 2.3. Probably one can get rid off logarithmic factors as \( q = 2 \) and derive some estimate as \( q = 1 \). I leave it to the curious reader.

### 2.1.4 Now let us consider the vicinity of \( \Sigma = \{ x : f_1 = f_2 \} \).

**Proposition 2.4.** Let \( |f_1 - f_2| \asymp \text{dist}(x, \Sigma) \) where \( \Sigma \) is a 2-dimensional manifold. Let \( \psi \) be supported in the small vicinity of \( \Sigma \). Then under condition (2.1) the remainder is \( O(\mu^{-1} h^{-3}) \) while the main part of asymptotics is given by Weyl formula.

**Proof.** Let us introduce a scaling function

\[
\gamma = \epsilon \text{dist}(x, \Sigma) + \frac{1}{2} \bar{\gamma}, \quad \bar{\ell} = C \mu^{-1}.
\]

Then

\[
|\alpha \nabla (f_1 V^{-1}) + (1 - \alpha) \nabla (f_2 V^{-1})| \geq \epsilon |\alpha - \bar{\alpha}(x)| \quad \forall \alpha : 0 \leq \alpha \leq 1
\]
holds with $\gamma$-admissible $\bar{\alpha}$ and also (2.8),(2.9) hold. Therefore one can take $T_1 = \epsilon \rho^{1-\kappa} \gamma \mu$. Then the contribution of all $(\gamma, \rho)$ elements to the remainder does not exceed (2.32) with $q = 3^8$ and summation over $\{\gamma \geq C \mu^{-1}, \rho \gamma \geq C \mu^{-2}\}$ results in $O(\mu^{-1} h^{-3})$.

Meanwhile the contributions of zones $\{x : \gamma \geq C \mu^{-1}, \rho \gamma \leq C \mu^{-2}\}$, $\{x : \gamma \leq C \mu^{-1}\}$ to the remainder are $O(\mu^{-1} h^{-3} \times \mu^{-2}) = O(\mu^{-1} h^{-3})$ since the measures of these zones are $O(\mu^{-2})$.

2.1.5 Summarizing what is proven one gets

**Proposition 2.5.** Let $(g^{jk})$ be fixed and then $(V_j)$ be generic, more precisely:

(i) Outside of $\Sigma = \{x : f_1 = f_2\}$ critical points of $f_1 f_2^{-1}$ satisfy (2.30)$_3$ and (2.2);

(ii) $\Sigma$ be smooth 2-dimensional manifold and $|f_1 - f_2| \simeq \text{dist}(x, \Sigma)$.

Finally, let $V$ be general but satisfying (0.8) at supp $\psi$. Then under condition (2.1) the remainder is $O(\mu^{-1} h^{-3})$ while the main part of asymptotics is given by Weyl formula.

**Proposition 2.6.** In frames of proposition 2.5 Weyl and Magnetic Weyl expressions differ by (far less than) $O(\mu^{-1} h^{-3})$.

*Proof.* An easy proof is left to the reader. $\square$

**Remark 2.7.** Definitely $M$ in condition (2.2) “critical points of $f_1 f_2^{-1}$ are not resonances of order not exceeding $M$” should not be too large and I leave to the curious reader to investigate it.

# 3 Stronger Magnetic Field Case: Estimates

## 3.1 Canonical form

From now one can assume that

\begin{equation}
(3.1) \quad h^{-\delta_0} \leq \mu \leq ch^{-1}
\end{equation}

with some small fixed exponent $\delta_0 > 0$. Then I can reduce operator to a canonical form (depending on additional assumptions) and also make decomposition with respect to Hermite functions, thus arriving to 2-parametric matrices of 2D $\mu^{-1} h$-PDOs $A_{pn}(x', \mu^{-1} hD')$ where ere and below $x' = (x_1, x_2)$.

---

$^8$) In comparison with proposition 2.2 there is no factor $\gamma$ in the right hand expression of (2.34) and therefore no factor $\gamma^{-1}$ in the estimate of the measure in $\xi$ space.
3 STRONGER MAGNETIC FIELD CASE: ESTIMATES

More precisely, assuming that there are no resonances of order not exceeding $M$:

(3.2) $|k f_2 - l f_1| \geq \epsilon \quad \forall (k, l) \in \mathbb{Z}^+ : k + l \leq M,$

a canonical form contains diagonal elements

(3.3) $A_{pn} = \frac{1}{2} \left( f_1^\# (2p + 1) \mu h + f_2^\# (2n + 1) \mu h - V^\# + \sum_{l + m + k + j \geq 2} b_{lmkj} (2p + 1) \mu h)^l (2n + 1) \mu h)^m \mu^{3 - 2l - 2m - 2k - j} h^j \right)$

with $f_j^\# = f_j^\# (x', \mu^{-1} h D')$, $V^\# = V^\# (x', \mu^{-1} h D')$, $b_{lmkj} = b_{lmkj} (x', \mu^{-1} h D')$ while all non-diagonal elements are $O(\mu^{2 - M}).$

I will discuss later an alternative form as $M = 2.$

3.2 General estimates at regular points

Assume first that there are no resonances of order not exceeding large $M = M(\delta_0)$. Then under condition (3.1) perturbation $O(\mu^{2 - M})$ is negligible and (3.3) is a true diagonal canonical form (with a negligible perturbation).

3.2.1 In this case an analysis is easy:

**Proposition 3.1.** Let there be no resonances of order not exceeding (large enough) $M$ and condition (1.19) be fulfilled at $\text{supp} \psi$. Then under condition (3.1)

(i) The standard implicit asymptotic formula (0.9) holds with the remainder estimate $O(\mu^{-1} h^{-3} + \mu^2 h^{-2}).$

(ii) In particular, remainder estimate is $O(\mu^{-1} h^{-3})$ as $\mu \leq h^{-1/3}.$

**Proof.** (i) Let us for each pair $(p, n)$ introduce scaling function

(3.4) $\rho_{pn} = \epsilon (|A_{pn}| + |\nabla A_{pn}|^2)^{1/2} + \bar{\rho}, \quad \bar{\rho} = C(\mu^{-1} h |\log h|)^{1/2} + C\mu^{-2}$

and a corresponding partition. Then

(3.5) The contribution of each group $(p, n, \text{element})$ to the main part of asymptotics is $\lesssim \mu^2 h^{-2} \rho^{4/9}.$

---

9) And often enough is is of this amplitude, so summation results in the correct magnitude of the main part.
On the other hand, one can take
\begin{equation}
T_1 = \epsilon \mu
\end{equation}

since the propagation speed is of magnitude $\mu^{-1}$ and also
\begin{equation}
T_0 = Ch\rho^{-2}|\log h|.
\end{equation}

Note that $T_0 \leq \epsilon_0 \mu^{-1}$ provided
\begin{equation}
\rho \geq \bar{\rho}_1 = C(\mu h|\log h|)^{1/2}
\end{equation}

and in this case one can trade $T_0$ to $\bar{T} = Ch|\log h|$ in (0.9) as $\sum Q_i = I$; there will be also the correction term arising from elements failing condition (3.8); see section 4. Moreover, (3.9) In the estimate of expression (0.10) one can replace $T_0$ by $T_0^* = Ch\rho^{-2}$.

So, the contribution of an element to the remainder does not exceed
\begin{equation}
C\mu^2 h^{-2} \rho^4 \times h \rho^{-2} \times \mu^{-1} \times (\rho^2 (\mu h)^{-1} + 1) \times (\rho(\mu h)^{-1} + 1) \times \rho^{-4}
\end{equation}

where $Ch\rho^{-2}|\log h|$ and $Ch\rho^{-2}$ play the roles of $T_0$ and $T_0^*$ in formulae (0.9) and (0.10) respectively. Also the numbers of indices $n$\(^{10}\) and $p$\(^{11}\) are estimated by $C(\rho^2 (\mu h)^{-1} + 1)$ and $C(\rho(\mu h)^{-1} + 1)$ respectively.

One can rewrite expression (3.9) as
\begin{equation}
C\mu^{-1} h^{-3} \rho + Ch^{-2} \rho^{-1} + C\mu h^{-1} \rho^{-2};
\end{equation}

then in the zone
\begin{equation}
\{ \rho \geq \bar{\rho} \overset{\text{def}}{=} C(\mu^{-1} h|\log h|)^{1/2} \}
\end{equation}

the first term sums to $C\mu^{-1} h^{-3}$ while the last two terms sum to their values as $\rho = \bar{\rho}$, which are $O(h^{-5/2} \mu^{1/2}) = O(\mu^{-1} h^{-3} + \mu^2 h^{-2})$ for sure and $O(\mu^2 h^{-2})$ respectively.

(ii) Meanwhile with the same main part the total contribution to the remainder of all groups with $\{ \rho \leq \bar{\rho} \}$ trivially does not exceed
\begin{equation}
C\mu^2 h^{-2} (\rho^2 (\mu h)^{-1} + 1) \times (\bar{\rho}(\mu h)^{-1} + 1) \leq C\mu h^{-3} \bar{\rho} + C\mu^2 h^{-2}.
\end{equation}

\(^{10}\) For which ellipticity is violated for a given $\rho$.

\(^{11}\) Such that $\rho_{pn} \approx \rho$ as $n$ violates ellipticity.
Obviously, this expression is $O(\mu^{-1}h^{-3})$ as $\mu \leq C(h|\log h|)^{-1/3}$ and $O(\mu^2h^{-2})$ as $\mu \geq Ch^{-1/3}|\log h|^{1/3}$.

(iii) To finish the proof I need to reconsider contribution to the remainder of the elements with $\{\rho_{pn} \leq \tilde{\rho}\}$ in the border case

\begin{equation}
\label{3.14}
h^{-1/3}|\log h|^{-1/3} \leq \mu \leq h^{-1/3}|\log h|^{1/3}.
\end{equation}

Let us introduce another scaling function

\begin{equation}
\label{3.15}
\varrho = \epsilon|\nabla^2 A_{pn}| + \frac{1}{2}\tilde{\varrho}, \quad \tilde{\varrho} = |\log h|^{-K},
\end{equation}

calculated with $(p, n)$ delivering minimum to $\rho_{pn}$ and let us introduce the corresponding partition.

Then for any element with $\varrho \geq \tilde{\varrho}$ one can calculate easily that the relative measure of the zone $\{(x', \xi') : \min_{p, n} \rho_{pn} \leq C\tilde{\rho}\}$ is $O(\rho|\log h|^K)$ and then the total contribution of zone $\{(x', \xi') : \rho \leq \tilde{\rho}, \varrho \geq C\tilde{\varrho}\}$ to the remainder is much less than $O(\mu^{-1}h^{-3})$.

On the other hand, let us consider elements with $\varrho \leq C\tilde{\varrho}$. Since the total contribution of subelements with $\rho \leq \rho_1^{\ast} = C(\mu h)^{1/2}$ to the remainder is estimated properly, one needs to consider only subelements with $\rho_1^{\ast} \leq \rho \leq \tilde{\rho}_1$.

But on such subelements $\rho \varrho^{-1}$ is a scaling function as well and using it one can easily decrease $T_0$ to $CH|\log h|\tilde{\varrho}^{-2}$ leaving $T_1 = \epsilon\mu$; this will add an extra factor $|\log h|^{2-K}$ to the estimate of the contribution of this zone to the remainder and this factor leads to the needed estimate $O(\mu^{-1}h^{-3})$.

\[\square\]

3.2.2 Assume now that the is a critical point of $f_1f_2^{-1}$:

**Proposition 3.2.** Let there be no resonances of order not exceeding (large enough) $M$ and condition (2.30) be fulfilled at $\text{supp} \, \psi$. Then under condition (3.1)

(i) The standard implicit asymptotic formula holds with the remainder estimate $O(\mu^{-1}h^{-3} + \mu^2h^{-2})$.

(ii) In particular, the remainder estimate is $O(\mu^{-1}h^{-3})$ as $\mu \leq Ch^{-1/3}$.

**Proof.** (i) The arguments of the proof of proposition 3.1 still work without condition (1.19) with the exception of the estimate $C(\rho(\mu h)^{-1} + 1)$ of the number of the indices \(\rho\).

However, let us introduce another scaling function

\begin{equation}
\label{3.16}
\gamma \overset{\text{def}}{=} \epsilon_1|\nabla(f_1f_2^{-1})|
\end{equation}
and if on some group $\rho_{pn} \leq \gamma$ then the number of indices “$p$” should be estimated by $C(\rho(\gamma \mu h)^{-1} + 1)$; otherwise this number should be estimated by $C((\mu h)^{-1} + 1)$.

Note that if condition $\rho_{pn} \lesssim \gamma$ (or $\rho_{pn} \gtrsim \gamma$) is fulfilled at some point of $\rho$-element, then the same condition (with another implicit constant) is fulfilled at any other point of this element.

Anyway, this modification adds no more than one factor $\gamma^{-1}$ to the estimate, but the factor $\gamma^q$ comes from condition (2.30) which assumes the microhyperbolicity condition holds with any $\ell$ such that $\langle \ell, \nabla (V f)^{-1} \rangle > 0$ and therefore the following statement is generic:

\[(3.20)\] Let $\nabla (f f_2^{-1}) = 0$ and $\nabla (V f_1^{-1}) \neq 0$. Then for any $\psi$ supported in the small vicinity of $\bar{x}$ asymptotics with the magnetic Weyl main part and remainder estimate $O(\mu^{-1} h^{-3})$ holds.

Remark 3.3. Probably one can get rid off logarithmic factors as $q = 1$.

### 3.3 Sharp asymptotics at regular points

The purpose of this and the next odd-numbered subsections is to consider the case

\[(3.18)\] $ch^{-1/3} \leq \mu \leq ch^{-1}$

and derive some under non-degeneracy condition remainder estimate $O(\mu^{-1} h^{-3})$. 

#### 3.3.1 Note first that

\[(3.19)\] In the generic case $\nabla (f f_2^{-1})(\bar{x}) = 0$ implies $\nabla (V f_1^{-1})(\bar{x}) \neq 0$ and then microhyperbolicity condition holds with any $\ell$ such that $\langle \ell, \nabla (V f_1^{-1}) \rangle > 0$.

and therefore the following statement is generic:

\[(3.20)\] Let $\nabla (f f_2^{-1})(\bar{x}) = 0$ and $\nabla (V f_1^{-1})(\bar{x}) \neq 0$. Then for any $\psi$ supported in the small vicinity of $\bar{x}$ asymptotics with the magnetic Weyl main part and remainder estimate $O(\mu^{-1} h^{-3})$ holds.
3.3.2 Therefore in what follows one can assume that $\nabla(f_1 f_2^{-1})$ is disjoint from 0. Further, one should consider only vicinities of points where microhyperbolicity condition (1.15) is violated.

**Proposition 3.4.** Let conditions (0.8) and (3.18) be fulfilled. Moreover, let us assume that on supp $\psi$ there are no resonances of order not exceeding $M$ and also $|\nabla(f_1 f_2^{-1})| \geq \epsilon_0$. Furthermore, let condition (3.21) be fulfilled with $q > 1$.

$$(3.21)_q \quad \nu(\rho) \overset{\text{def}}{=} \operatorname{mes}\{(x, \alpha) : 0 \leq \alpha \leq 1, \quad |\nabla (\alpha \log f_1 + (1 - \alpha) \log f_2 - \log V)| \leq \rho\} = O(\rho^q) \quad \text{as} \quad \rho \to +0$$

be fulfilled with $q > 1$. Then asymptotics with the standard implicit main part (0.9) and the remainder estimate

$$(3.22)_q \quad O(\mu^{-1}h^{-3} + \mu^2 h^{-2}(\mu^{-1}h)^{(q-1)/2} \log h^{(q+1)/2})$$

holds.

**Proof.** As $T = Ch\rho^{-2} \log h$ the contribution to the remainder of $\rho$-elements does not exceed (3.10) multiplied by $\nu(\rho)\rho^{-1}$

$$C\mu h^{-1}\rho^{-2}(\rho^2(\mu h)^{-1} + 1) \times (\rho(\mu h)^{-1} + 1) \times \nu(\rho)\rho^{-1} \approx C\mu h^{-1}(\rho^3(\mu h)^{-2} + \rho(\mu h)^{-1} + 1)\nu(\rho)\rho^{-3} \leq C\mu h^{-3}\rho^q + Ch^{-2}\rho^{q-2} + C\mu h^{-1}\rho^{q-3}$$

where the last factor in the left-hand expression is the total measure of $\rho$-elements.

Here the first term in the right-hand expression always sums to $O(\mu^{-1}h^{-3})$ while the second and the third factor sum to their values as $\rho = 1$ (i.e. $O(\mu^{-1}h^{-3})$ for sure) plus their values as $\rho = \bar{\rho} = C(\mu^{-1}h \log h)^{1/2}$ which do not exceed the second term in $(3.22)_q$. As $q = 2, 3$ the second or the third term respectively acquires an extra logarithmic factor but it does not change the estimate.

Furthermore, contribution of zone $\{\rho \leq \bar{\rho}\}$ to the remainder does not exceed $C\mu h^{-3}\nu(\bar{\rho}) + C\mu^2 h^{-2}\nu(\bar{\rho})\bar{\rho}^{-1}$ which is the second term in $(3.22)_q$. \[\square\]

**Corollary 3.5.** In frames of proposition 3.4 remainder estimate $O(\mu^{-1}h^{-3})$ holds as $q > 3$.

**Remark 3.6.** In the generic case condition (3.21)$_q$ is fulfilled.
3 STRONGER MAGNETIC FIELD CASE: ESTIMATES

3.3.3 Let us consider a special case

\[(3.23)\quad \epsilon h^{-1} \leq \mu \leq ch^{-1}.\]

**Proposition 3.7.** Let conditions \((0.8)\) and \((3.23)\) be fulfilled. Moreover, let us assume that on \(\text{supp} \psi\) there are no resonances of order not exceeding \(M\) and also \(|\nabla(f_1 f_2^{-1})| \geq \epsilon_0\).

Furthermore, let condition \((3.24)\) be fulfilled with \(r \geq 1\). Then asymptotics with the standard implicit main part \((0.9)\) and the remainder estimate \((3.22)\) holds.

**Proof.** Follows easily the proof of proposition 3.4. \(\square\)

**Remark 3.8.** One can get rid off the logarithmic factors in estimate \((3.22)\) but I do not care since I am interested only in the generic cases \(q = 4, r = 4\).

3.4 General asymptotics at resonances

Now assume that there are \((k, l)\) resonances of order \(m = k + l \geq 3\). However considering \(\epsilon_1\)-vicinity of any point one can assume that

\[(3.25)\quad \text{There are no } (k', l')\text{-resonances with } k' + l' \leq M \text{ unless } k'/k = l'/l \in \mathbb{Z}.\]

3.4.1 So, let us consider

\[(3.26)\quad \Xi_{kl} = \{x : f_1 k = f_2 l\}\]

which under condition \((3.4)\) is a smooth surface. One can assume without any loss of the generality that the analogue of \((2.20)\) holds: \(k f_1 - l f_2 = x_1\) while \((x_2, \xi_1, \xi_2)\) are coordinates on \(\Xi_{kl}\). Let us introduce a scaling function

\[(3.27)\quad \gamma = \epsilon |x_1| + \frac{1}{2} \tilde{\gamma}, \quad \tilde{\gamma} = \mu^{-1+\delta}\]

with arbitrarily small exponent \(\delta > 0\).
Proposition 3.9. Let conditions (3.25) with \( k + l = m \geq 3 \) and (1.19) be fulfilled at \( \text{supp} \psi \). Then under condition (3.1) asymptotics with the standard implicit main part (0.9) and the remainder estimate \( O(\mu^{-1}h^{-3} + \mu^2h^{-2}) \) holds.

Proof, Part I. I will give the proof working the worst-case scenario \( m = 3 \); as \( m \geq 4 \) one can simplify the proof. In this part I am going to prove that

\[(3.28) \text{The contribution of zone } \{|x_1| \geq \tilde{\gamma}\} \text{ to the remainder is } O(\mu^{-1}h^{-3} + \mu^2h^{-2}). \]

After reduction to precanonical form with non-diagonal terms corresponding to resonances, in this zone one can get rid off non-diagonal terms (modulo \( O(\mu^{-M}) \)). Let us consider scaling function \( \rho_{pn} \) introduced by (3.4) for a full symbol of \( A_{pn} \).

(i) Consider first subzone

\[(3.29) \\{x : |x_1| \geq \max(\rho, \tilde{\gamma})\} \]

Then one can apply the same arguments as in the proof of proposition 3.1; however there is a problem\(^{12} \): as function of \( x_1 \) \( \rho_{pn} \) remains \( \gamma \)-admissible only as

\[(3.30) \gamma \geq \tilde{\gamma}_1 \overset{\text{def}}{=} \mu^{(4-2m)/3}. \]

So far the arguments as in the proof of proposition 3.1 result in\(^{13} \)

\[(3.31) \text{The contribution of zone } \{|x_1| \geq \max(\rho, \tilde{\gamma}_1)\} \text{ with } \tilde{\gamma}_1 = \mu^{-2/3} \text{ to the remainder is } O(\mu^{-1}h^{-3} + \mu^2h^{-2}). \]

(ii) Consider now subzone

\[(3.32) \{\gamma \geq \tilde{\gamma}, \rho \geq \gamma\}. \]

Note that here the derivatives of \( A_{pn} \) with respect to \( x_1 \) and to \( (x_2, \xi_1, \xi_2) \) have different values: while the derivative with respect to \( x_1 \) measures speed with respect to \( \xi_1 \) and the shift with respect to \( \xi_1 \) is quantum observable as \( |\nabla_{x_1} A_{pn}| \times \gamma \geq C\mu^{-1}h|\log h| \) provided \( \gamma \) is the scale with respect to \( x_1 \), other derivatives measure speed with respect to \( (\xi_2, x_1, x_2) \) and the shift is quantum observable as \( |\nabla' A_{pn}| \times \zeta \geq C\mu^{-1}h|\log h| \), provided \( \zeta \) is the scale with respect to these variable where here and below \( \nabla' \overset{\text{def}}{=} \nabla_{x_2, \xi_1, \xi_2} \).

\(^{12} \) It is not a problem at all as \( m = 5 \) and rather a marginal problem as \( m = 4 \) but for \( d = 3 \) this requires a certain attention.

\(^{13} \) The border case (3.14) (may be with the different powers of logarithm) should be covered only in zone \( \{|\log h|^{-K_1} \leq |x_1| \leq \epsilon\}. \)
So, let us introduce the third scaling function

\begin{equation}
\zeta_{pn} = \epsilon (|A_{pn}| + |\nabla' A_{pn}|^2)^{1/2} + (\rho \gamma)^{1/2}
\end{equation}

where the last term will be actually included later.

Then one can make a $\zeta$-admissible partition. So I have now $(\gamma, \rho, \zeta)$ elements with $\gamma \leq \zeta \leq \rho$ where $\zeta$ is the scale with respect to $(x_2, \xi_1, \xi_2)$ and $\gamma$ is the scale with respect to $x_1$ while $\rho$ at this moment lost its scaling role. However while $\zeta$ is $\gamma$-admissible function with respect to $x_1$, $\rho$ preserves its magnitude in as $x_1$ varies by $O(\gamma)$ only under condition

\begin{equation}
\rho \geq \mu^{-1} \gamma^{-1}.
\end{equation}

I claim that that then

\begin{equation}
T_0 = C h |\log h| (\rho \gamma + \zeta^2)^{-1}, \quad T_0^* = C h (\rho \gamma + \zeta^2)^{-1}
\end{equation}

while

\begin{equation}
T_1 = C \mu \min (\zeta \rho^{-1}, \gamma \zeta^{-1}) \approx C \mu \zeta \gamma (\rho \gamma + \zeta^2)^{-1}
\end{equation}

Really, as $\rho \gamma \approx \zeta^2$ propagation speed with respect to $\xi_1$ is $\approx \rho$ it is dual to $x_1$ of the scale $\gamma$. Meanwhile speeds with respect to all other variables are bounded by $C \zeta$ and for given $T_1$ magnitudes of $\zeta, \gamma, \rho$ are preserved.

On the other hand, as $\rho \gamma \approx \rho_1 \approx \zeta$ and they are dual to $(\xi_1, \xi_2, x_2)$ of the scale $\zeta$ while propagation speed with respect to $\xi_1$ is bounded by $C \rho_1$ and for given $T_1$ magnitudes of $\zeta, \gamma, \rho$ are preserved.

I leave to the reader the standard justification on the quantum level (based on energy estimates approach).

Thus $T_0^* T_1^{-1} \approx \mu^{-1} h \gamma^{-1} \zeta^{-1}$; in virtue of the last term in the definition of $\zeta$ one can skip $\rho \gamma$ in $(\zeta^2 + \rho \gamma)$ here and below.

Then the total contribution of all $(\rho, \zeta, \gamma)$ elements to the remainder does not exceed

\begin{equation}
C \mu^2 h^{-2} \times \gamma \times T_0^* T_1^{-1} \times ((\rho \gamma + \zeta^2)(\mu h)^{-1} + 1) \times (\rho (\mu h)^{-1} + 1)
\end{equation}

where $C ((\rho \gamma + \zeta^2)(\mu h)^{-1} + 1) \approx C ((\zeta^2 (\mu h)^{-1} + 1)$ is an upper bound for a number of indices $n$ violating ellipticity for a given index $\rho$.

If one picks up only “1” from both factors with the parentheses in (3.37) and replaces $T_0 T_1^{-1}$ by $1$, then summation with respect to partitions results in $C \mu^2 h^{-2}$; on the other hand, since $\zeta^2 \leq \rho$, one can rewrite the above expression (3.37) as

\begin{equation}
C \mu h^{-1} \times \zeta^{-1}((\zeta^2 (\mu h)^{-1} + 1) \times (\rho (\mu h)^{-1} \approx C h^{-2}((\zeta (\mu h)^{-1} + \zeta^{-1}) \rho.
\end{equation}
3 STRONGER MAGNETIC FIELD CASE: ESTIMATES

Then summation with respect to $\zeta$ from $(\rho \gamma)^{1/2}$ to $\rho$ results in

\[(3.39) \quad C h^{-2} \rho (\rho(\mu h)^{-1} + \rho^{-1/2} \gamma^{-1/2}) \simeq C \mu^{-1} h^{-3} \rho^2 + Ch^{-2} \rho^{1/2} \gamma^{-1/2}.\]

The second term in the right hand expression sums with respect to $(\rho, \gamma)$ to $C \mu^{-1} h^{-3} |\log h|$ and this logarithmic factor appears due to summation with respect to $\gamma$.

To get rid off this factor let us notice that only case $\mu \leq h^{-1/3} \log h^{1/3}$ needs to be addressed and only zone $\{\zeta \geq \gamma^{\kappa}\}$ should be reconsidered (with an arbitrarily small exponent $\kappa > 0$); in this case $\zeta > (\rho \gamma)^{1/2}$. Moreover, only term $C h^{-4} \rho^{2} \gamma \times T_{0}^{*} T_{1}^{-1}$ in (3.37) should be reexamined.

However then one does not need to use the canonical form but rather a weak magnetic field approach and take $T_{0}^{*} = h^{-2}$ and $T_{1} = \mu \zeta$ and the contribution of this zone to the term in question does not exceed $C \mu^{-1} h^{-3} \int \zeta^{-1} d\gamma \leq C \mu^{-1} h^{-3}$. Therefore

\[(3.40) \quad \text{The contribution of zone } \{\rho \geq \max(\gamma, \mu^{-2} \gamma^{-2}), \gamma \geq \bar{\gamma}\} \text{ to the remainder is } O(\mu^{-1} h^{-3} + \mu^{2} h^{-2}).\]

(iii) Now let us consider the remaining part of the zone $\{\epsilon \geq \gamma \geq \bar{\gamma}\}$. In this zone let us introduce the scaling function

\[(3.41) \quad \eta = \mu^{2} \gamma^{3} \rho.\]

Then $\rho$ as a function of $x_{1}$ is $\eta$-admissible. Let us modify definition (3.33) etc replacing $\gamma$ by $\eta$:

\[(3.33)^{*} \quad \zeta_{p n} = \epsilon (|A_{p n}| + |\nabla' A_{p n}|^{2})^{1/2} + (\rho \gamma)^{1/2}\]

and then (3.35), (3.36) are also modified in the same way (anyway, terms $\rho \gamma$ originally and $\rho \eta$ now are not important):

\[(3.35)^{*} \quad T_{1} = \epsilon \mu \min(\zeta \rho^{-1}, \eta \zeta^{-1}) \simeq \epsilon \mu \eta \zeta^{-1}\]

and $T_{0}^{*} T_{1}^{-1} \simeq \mu^{-1} \eta h^{-1} \zeta^{-1}$.

Then modified (3.37) and (3.38) expressions

\[(3.37)^{*} \quad C \mu h^{-1} \times \zeta^{-1} (\zeta^{2}(\mu h)^{-1} + 1) \times \rho(\mu h)^{-1} \gamma^{-1} \simeq C h^{-2} \times (\zeta(\mu h)^{-1} + \zeta^{-1}) \rho \gamma \eta^{-1}\]

estimate contribution of all $(\rho, \gamma, \eta, \zeta)$ elements. Here an (unpleasant) factor $\gamma \eta^{-1}$ appears since the measure of the strip remains $\gamma$ and $\gamma^{-1}$ is replaced by $\eta^{-1}$ in $T_{0}^{*} T_{1}^{-1}$.  

Plugging $\eta$ into (3.37)* one gets

$$C\mu^{-3}h^{-3}\nabla^2 \zeta + C\mu^{-2}h^{-2}\nabla^2 \zeta^{-1}$$

and the first term sums to $O(\mu^{-1}h^{-3})$.

The second term sums with respect to $\zeta$ to its value at the smallest $\zeta$ satisfying conditions $T_1 = \mu \eta \zeta^{-1} \geq T_0 = h\zeta^{-2}$ and $\zeta \geq (\rho \eta)^{1/2}$:

$$\zeta = \max(\mu^{-1}h\eta^{-1}, (\rho \eta)^{1/2}) = \max(\mu^{-3}h\rho^{-1}\zeta^{-3}, \rho \mu \gamma^3/2)$$

so one gets

$$C\mu^{-2}h^{-2}\nabla^2 \zeta^{-2} \min(h^{-1}\rho^3 \gamma^{-3}, \rho^{-1} \mu^{-1}\zeta^{-3/2})$$

which sums with respect to $\rho$ to $C\mu^{-1}h^{-5/2}\nabla^2 \gamma^{-5/4}$ and then with respect to $\gamma$ to $O(\mu^{1/4}h^{-5/2}) \ll (\mu^{-1}h^{-3} + \mu^2 h^{-2})$.

Meanwhile if $T_1 \leq T_0$ then one just replaces $T_1$ by $T_0$ and gets $Ch^{-4}\nabla^2 \rho \gamma + C\mu h^{-3} \rho \gamma$ with $\zeta = (\rho \eta)^{1/2}$ i.e.

$$Ch^{-4}(\rho \eta)^{1/2} \rho \gamma + C\mu h^{-3} \rho \gamma \approx Ch^{-4}\mu^2 \gamma^{5/2} + C\mu h^{-3} \rho \gamma$$

and only if $(\rho \eta)^{1/2} \leq \mu^{-1}h\eta^{-1}$ or equivalently $\rho \leq \bar{\rho} \equiv \mu^{-2}h^{1/2}\gamma^{-9/4}$. Plugging $\bar{\rho}$ in (3.44) I get $C(h^{-3}\mu^{-3}\nabla^2 \gamma^2 + \mu^{-1}h^{-5/2}\nabla^2 \gamma^{-5/4})$; summation with respect to $\gamma$ results in $O(\mu^{-1}h^{-3} + \mu^{1/4}h^{-5/2})$. Therefore

Statement (3.28) is proven.

Proof of Proposition 3.9, Part II. I am left with zone $\{|x_1| \leq \tilde{\gamma}\}$; its contribution to the remainder does not exceed $C\mu h^{-3} \tilde{\gamma} = C\mu^4 h^{-3}$ which is $O(\mu^2 h^{-2})$ as $\mu \geq h^{-1/2-\delta'}$ and therefore only case $\mu \leq h^{-1/2-\delta'}$ needs to be addressed. In this zone I use precanonical form with non-diagonal matrix elements but without singular terms.

3 STRONGER MAGNETIC FIELD CASE: ESTIMATES

(3.45) Let $A^0_{pn}$ denote diagonal matrix elements,

One can see easily that

$$|\nabla' A^0_{pn}| \equiv |\nabla' \log(f_j V^{-1})| \mod O(\gamma) \quad \text{as } |x_1| \leq \gamma$$

(as $A_{pn}$ is non-elliptic) does not actually depend on $p$; let us define

$$\zeta = |\nabla' \log(f_j V^{-1})|_{x_1=0} + (\rho \gamma)^{1/2}. $$
(3.48) Let us define $\rho_{pn}$ as before but with $A_{pn}^0$ instead of $A_{pn}$.

in contrast to $\zeta \rho_{pn}$ strongly depends on $\rho$.

Then, as before $\mu^{-1}\rho$ controls the propagation speed with respect to $x_1$ and thus bounds the propagation speed with respect to $\zeta$ while $\mu^{-1}\zeta$ controls the propagation speed with respect to $(x_1, x_2, \xi_2)$ as long as $\zeta \geq C\gamma$, $\rho \geq C\gamma$. I remind that $\gamma \geq C\mu^{-1}$.

However the propagation speed with respect to $\rho$ is a different matter. Considering commutator of

$$A = f_1 V^{-1}Z_1^*Z_1 + f_2 Z_2^*Z_2 + 2\mu^{-1} \text{Re}(\beta Z_1^*Z_2^*) + ...$$

(assuming that resonance is $2f_2 = f_1$) with

$$(3.49) \quad \partial_{x_i}A^0 = \kappa Z_1^*Z_1 + \omega A^0 + ..., \quad \kappa = f_1(\partial_{x_i}(\log(f_1f_2^{-1})))$$

(with $\cdots = O(\gamma)$) one can see easily that

$$(3.50) \quad \kappa^{-1} [A, \partial_{x_i}A^0] \equiv [\text{Re} \beta Z_1^*Z_2^*, Z_1^*Z_1] = 2\kappa \text{Re} \beta Z_1^*Z_2^* \mod O(\gamma)$$

which is bounded by 1. Therefore

(3.51) The propagation speed with respect to $\rho$ does not exceed 1. Furthermore, $\rho$ is properly defined as $\rho \geq C\mu h|\log h|$ (the logarithmic uncertainty principle).

Without nondegeneracy condition there is not much use of $\zeta$; let us consider elements with $\rho \geq C\gamma$. Due to (3.51) I can pick up $T_1 = \epsilon\rho$. Then the contribution to the remainder of all such elements does not exceed

$$(3.52) \quad C\mu^2 h^{-2}\gamma \times h\zeta^{-2}\rho^{-1}(\zeta^2(\mu h)^{-1} + 1) \times (\rho(\mu h)^{-1} + 1)$$

and as long I include $O(\mu^2 h^{-2})$ into final remainder estimate I can skip “+1” in the last factor (due to the same arguments as before) resulting in

$$(3.53) \quad C\mu h^{-2}\gamma \zeta^{-2}(\zeta^2(\mu h)^{-1} + 1) \approx Ch^{-3}\gamma + C\mu h^{-2}\zeta^{-2}\gamma \leq Ch^{-3}\gamma + C\mu h^{-2}\rho^{-1};$$

as $\gamma = \tilde{\gamma}$ this expression does not exceed $Ch^{-3}\tilde{\gamma} + C\mu h^{-2}\rho^{-1}$ which sums with respect to $\rho$ to $Ch^{-3}\tilde{\gamma}|\log h| + C\mu h^{-2}\rho^{-1}$ and the last term is $O(\mu^2 h^{-2})$.

On the other hand, in the zone $\{\rho \leq \tilde{\rho}\}$ I pick up $T_0^* T_1^{-1} = 1$ and its contribution to the remainder does not exceed $C\rho^2 \gamma^2 h^{-\delta} + C\mu h^{-3}\tilde{\rho}\tilde{\gamma} + C\mu^2 h^{-2}$.

Therefore, as $h^{-1/3-\delta'} \leq \mu \leq h^{-1/2-\delta'}$, the contribution to the remainder of the zone $\{|x_1| \leq \tilde{\gamma} = \mu^{-1+\delta}\}$ with sufficiently small $\delta = \delta(\delta') > 0$ does not exceed $C\mu^2 h^{-2}$. In this case proposition 3.9 is also proven. \qed
I will need the following

**Proposition 3.10.** Let conditions (0.8) and (3.1) and \(|f_1 - f_2| \geq \epsilon\) be fulfilled. Let us consider the precanonical form. Finally, let \(Q\) be a \(\rho\)-admissible partition element element in \((q_1, q_2)\), then quantized as \(Q((\hbar^2 D_3^2 + \mu^2 x_3^2), (\hbar^2 D_4^2 + \mu^2 x_4^2))\) with

\[
(3.54) \quad \rho \geq C_\mu h |\log h| + C_{\gamma_0}^k \gamma^{1-k}
\]

and \(\psi\) be \(\gamma\)-admissible with respect to \(x_1\), either supported in \(|x_1| \approx \gamma\) as \(\gamma > \gamma_0 \equiv C^{-1}\mu^{-1}\) or supported in \(|x_1| \lesssim \gamma\) as \(\gamma = \gamma_0\). Then

\[
(3.55) \quad |F_{t \mapsto h^{-1/2}} T(t) \Gamma(\psi \nu Q^\dagger)| \leq Ch^k,
\]

and

\[
(3.56) \quad |F_{t \mapsto h^{-1/2}} T(t) \Gamma(\psi \nu Q^\dagger)| \leq C \rho \gamma h^{-3}
\]

as \(|\tau| \leq \epsilon\)

\[
(3.57) \quad Ch |\log h| \rho^{-1} \leq T \leq \epsilon \mu^{-1}.
\]

**Proof.** Consider the propagation with respect to either \((x_3, \mu^{-1}hD_3)\) or \((x_4, \mu^{-1}hD_4)\). Due to (0.8) on energy levels close to 0 at least one of \(\mu^2 x_j^2 + h^2 D_j^2\) is of magnitude 1 \((j = 3, 4)\). The propagation speed with respect to \((x_3, x_4, \mu^{-1}hD_3, \mu^{-1}hD_4)\) is \(\simeq 1\). Therefore one can trade \(T \leq \epsilon \mu^{-1}\) to \(T = Ch |\log h|\) and in the estimate of Fourier transform to \(T^* = Ch\). The remaining part of the proof is easy and left to the reader. \(\square\)

**Proof of Proposition 3.9, Part III.** Therefore only case \(h^{-\delta_0} \leq \mu \leq h^{-1/3-\delta'}\) remains to be addressed where \(\delta_0 > 0\) is small and fixed and \(\delta' > 0\) is arbitrarily small. It follows from Parts I,II that exponents \(\delta > 0\) in the definition of \(\gamma\) and \(\delta' > 0\) are **independently** small.

(i) Let us consider elements with \(|\nabla' V^{-1} f_j| \approx \varsigma \geq C\gamma\). Then as \(T_0 = Ch |\log h| (\varsigma^2 + \rho\gamma)^{-1} \leq \epsilon \rho\), i.e. as

\[
(3.58) \quad \rho \geq \epsilon \equiv C \min(h |\log h| \varsigma^{-2}, (\gamma^{-1} h |\log h|)^{1/2}) + C\gamma,
\]

\(\rho\) is preserved on the time interval \(T_0\) which can be traded to \(T_1 = \epsilon \mu \varsigma\). Therefore the contribution to the remainder of all such elements does not exceed

\[
(3.59) \quad C \rho \gamma (\varsigma^2 + \mu h) h^{-4} T_0^* T_1^{-1} \approx C \mu^{-1} h^{-3} \rho \gamma \varsigma^{-1} + Ch^{-2} \rho \gamma \varsigma^{-3};
\]
summation with respect to $\varsigma, \rho, \gamma$ trivially results in $O(\mu^{-1}h^{-3} + \mu^2h^{-2})$.

On the other hand, one can see easily that if (3.58) is violated, then $C\varrho$ remains an upper bound for $\rho$ at time $|t| \leq T_0 = C|\log h|\varsigma^{-1}$; therefore contribution of such elements to the remainder does not exceed

\begin{equation}
(C\rho\varsigma^2 + \rho\gamma + \mu h)h^{-4}T_0^*T_1^{-1} \approx C\mu^{-1}h^{-3}\gamma\varsigma^{-1} + C\mu^{-1}h^{-3}\rho^2\gamma^2\varsigma^{-3} + Ch^{-2}\rho\gamma\varsigma^{-3}
\end{equation}

which does not exceed the same expression with $\varsigma = \gamma$ and corresponding $\varrho$; one can see easily that $\varrho \leq \gamma h^{-5\delta}$ and (3.60) is $o(\mu^{-1}h^{-3})$.

Therefore only elements with $|\nabla'V^{-1}f_j| \leq C\gamma$ remain to be treated, where either $|x_1| \leq \gamma = C\mu^{-1}$ or $C\mu^{-1} \leq |x_1| \asymp \gamma \leq \mu^{-1+\delta}$.

(ii) Let us consider the propagation speed with respect to $\rho$ more precisely. Note that as $|x_1| \asymp \gamma \geq C\mu^{-1}$ one can translate non-diagonal term $\mu^{-1}\text{Re}(\omega Z^*_1Z^*_2)$ into

$$
\mu^{-2}x_1^{-1}|\omega|^2(Z^*_1Z_1 - 4z^*_2Z_2)Z^*_2Z_2 + ...
$$

with

$$
\rho = -\mu^{-2}x_2^{-1}|\omega|^2(Z^*_1Z_1 - 4z^*_2Z_2)Z^*_2Z_2 + ...
$$

Then $[a, \rho] = O(\mu^{-s}\gamma^{-s} + \mu^{-1})$ and furthermore along trajectories $[a, \rho](t) = [a, \rho](0) + O((\mu^{-s}\gamma^{-s}|t|)$ (where $s$ is an arbitrarily large exponent and $\delta = \delta(s) > 0$ is small enough) and therefore

\begin{equation}
T_1 \overset{\text{def}}{=} \epsilon\rho^{1/2} \mu^{s/2} |\gamma|^{s/2} \quad \text{as } \gamma_0 = C\mu^{-1} \leq \gamma \leq \bar{\gamma}, \quad \rho \geq C\mu^{-s/2}\gamma^{-1-s/2}
\end{equation}

(where $\rho = \rho(0)$). Therefore the contribution of the corresponding strip to the remainder does not exceed

$$
G(\rho, \gamma) \overset{\text{def}}{=} C\mu^2h^{-2}\gamma \times h\rho^{-2} \times ((\rho^2 + \gamma^2)(\mu h)^{-1} + 1) \times \rho(\mu h)^{-1} \times \rho^{-1/2}\mu^{-s/2}\gamma^{s/2}.
$$

Thus after summation over $\rho \geq C\mu^{-s/6}\gamma^{1-s/6}$ I arrive to $G(1, \gamma) + G(\mu^{-s/6}\gamma^{1-s/2})$ and for large enough $s$ summation with respect to $\gamma$ results in $G(1, \mu^{-1}) + G(\mu^{-1}\mu^{-1})$ which is $O(\mu^{-1}h^{-3})$.

Meanwhile the contribution of the zone $\{|x_1| \asymp \gamma, \rho \leq \rho \overset{\text{def}}{=} C\mu^{-s/6}\gamma^{1-s/6}\}$ to the remainder due to proposition 3.10 does not exceed $C\rho\gamma h^{-3} \approx C\mu^{-s/6}\gamma^{1-s/6}h^{-3}$ and summation with respect to $\gamma$ results in $C\mu^{-1}h^{-3}$. Therefore

\begin{equation}
\text{Contribution of zone } \{\gamma_0 \leq |x_1| \leq \bar{\gamma}\} \text{ to the remainder is } O(\mu^{-1}h^{-3}).
\end{equation}
(iii) Similar arguments work for zones \( \{|x_1| \leq \bar{\gamma}_0, \rho \geq \varrho \} \) with \( T_1 = C \rho^{1/2} \) and for \( \{|x_1| \leq \bar{\gamma}_0, \rho \leq \varrho \} \) and therefore Contribution of zone \( \{|x_1| \leq \bar{\gamma}_0 \} \) to the remainder is \( O(\mu^{-1}h^{-3}) \).

This concludes Part III and the whole proof.

3.5 Sharp asymptotics at resonances

In this subsection I am going to prove sharp remainder estimate under generic assumptions to \( V \). I know from proposition 1.6 that in the generic situation critical points of \( \phi_\alpha = \alpha \log f_1 + (1 - \alpha) \log f_2 - \log V \) are non-degenerate except of discrete values of \( \alpha = \alpha_j \). One can prove easily that

(3.64) In the generic case degenerate critical points \( \alpha_j \) of \( \phi_\alpha \) are not resonances. Then as \( \alpha \neq \alpha_j \) the set of critical points is a smooth 1-dimensional curve parametrized by \( \alpha \) and resonance surface is 3D surface.

(3.65) In the generic case these curve and resonance surface \( \Xi_{kl} \) meet at isolated points and are transversal in them.

Then

\[
(3.66)_{q,r} \quad \tilde{v}(\rho, \zeta, \gamma) \overset{\text{def}}{=} \text{mes}\left\{ (x, \alpha) : |kf_1 - lf_2| < \gamma, \right. \\
|\nabla' (\alpha \log f_1 + (1 - \alpha) \log f_2 - \log V)| \leq \zeta, \\
\left. |\nabla (\alpha \log f_1 + (1 - \alpha) \log f_2 - \log V)| \leq \rho \right\} \leq C \rho^{q-r} \zeta^r \gamma
\]

as \( \rho \geq \gamma, \rho \geq \zeta \geq \left(\rho \gamma\right)^{1/2} \)

with \( r = 3, q = 4 \).

**Proposition 3.11.** Let us assume that \( |f_1 - f_2| \geq \epsilon_0 \) and \( |\nabla (f_1 f_2^{-1})| \geq \epsilon_0 \) on \( \text{supp} \psi \). Furthermore, let conditions (0.8), (3.21)\(_q\) with \( q > 3 \) and (3.66)\(_{q,r}\) be fulfilled with \( r > 2, q > 3 \). Then under condition (3.18) asymptotics with the standard implicit main part (0.9) and the remainder estimate \( O(\mu^{-1}h^{-3}) \) holds.

**Proof, Part I.** I will follow the proof of proposition 3.7 and use the same numbering of its parts.

(i) Repeating arguments leading to the proof of statements of proposition 3.4 one can prove easily the following analogue of (3.40):
(3.67) Under condition $\text{(3.21)}_q$ the contribution of zone $\{x : |x_1| \geq \max(\rho, \mu^{-2/3})\}$ to the remainder does not exceed $(3.22)_q$.

(ii) Consider zone: $\{\rho \geq \max(\gamma, \mu^{-2}\gamma^{-2}, \gamma \geq \bar{\gamma}\}$. Then the total contribution of all $(\rho, \zeta, \gamma)$ elements does not exceed expression $(3.38)$ multiplied by $\bar{\nu}(\rho, \zeta, \gamma)\rho^{-1}$:

$$C\mu h^{-1}(\zeta(\mu h)^{-1} + \zeta^{-1}) \times (\rho(\mu h)^{-1} + 1) \times \rho^{q-r-1}\zeta'$$

where I used $(3.66)_{q,r}$ to estimate $\bar{\nu}$. Then as $r > 1$ summation with respect to $\zeta$ results in the same expression as $\zeta = \rho$; further as $q > 3$ summation with respect to $\rho$ results in the same expression as $\rho = 1$ which is $C\mu^{-1}h^{-3}$ and summation with respect to $\gamma$ results in $C\mu^{-1}h^{-3} |\log h|$. One can get rid of the logarithmic factor using the same arguments as in the Part I (ii) of proof of proposition 3.4.

(iii) Consider the remaining part of zone $\{\epsilon \geq \gamma \geq \bar{\gamma}\}$ and introduce scaling function $\eta$ by $(3.41)$. Then one gets $(3.37)^*$ modified in the same way as in the Part I (iii) of proof of proposition 3.4 and multiplied by $\eta^{-1}\bar{\nu}(\rho\gamma)^{-1}$:

$$C\mu h^{-1}(\zeta(\mu h)^{-1} + \zeta^{-1}) \times (\rho(\mu h)^{-1} + 1) \times \eta^{-1} \times \rho^{q-r-s-1}\zeta'.$$

Then as in (ii) summation with respect to $\zeta$ results in its value as $\zeta = \rho$:

$$C\mu h^{-1}(\rho(\mu h)^{-1} + \rho^{-1}) \times (\rho(\mu h)^{-1} + 1) \times \mu^2 \mu^{-2}\gamma^{-2} \rho^{-1} \times \rho^{q-s-1}\gamma^{-s-1}$$

and summation with respect to $\rho$ returns the above expression at its largest value which is $\mu^{-2}\gamma^{-2}\mu^2$; the result does not exceed $C\mu^{-1-\delta'}h^{-3}$ and summation with respect to $\gamma$ returns $O(\mu^{-1}h^{-3}).$ \hfill $\Box$

**Remark 3.12.** (i) Again as the order of resonance $m \geq 4$, analysis of (iii) is not needed; (ii) Furthermore, the rough remainder estimate of zone $\{|x_1| \leq \bar{\gamma}\}$ returns $O(\mu^{-1}h^{-3})$ as $m \geq 5$ and $\mu \leq h^{\delta'-1}$ and $O(\mu^{-1+\delta}h^{-3})$ as either $m = 4$ or $m = 5$, $h^{\delta'-1} \leq \mu \leq ch^{-1}$.

**Proof, Part II.** (i) Analysis in zone $\{|x_1| \leq \bar{\gamma}\}$ is now simpler. Note first that the contribution of zone $\{\zeta \leq \bar{\gamma}\}$ to the remainder does not exceed $C\mu h^{-3}\bar{\gamma}^{r+1} = O(\mu^{-1}h^{-3})$ as $r > 1$ and $\delta < \delta(r)$.

(ii) Consider zone $\{\zeta \geq C\bar{\gamma}\}$. Then defining $T_0, T_0^*$ by $(3.35)$ and $T_1$ by $(3.36)$ one estimates contribution of $(\rho, \bar{\gamma}, \zeta)$ elements by $(3.52)$ multiplied by $\bar{\nu}(\rho\bar{\gamma})^{-1}$:

$$C\mu^2 h^{-2}\bar{\gamma} \times h\zeta^{-2} \times (\zeta^2(\mu h)^{-1} + 1) \times \rho(\mu h)^{-1} \times (\mu^{-1}\bar{\gamma}^{-1/2} + \rho^{-1}) \times \rho^{q-r-2}\zeta'.$$
and summation with respect to $\zeta$ results in its value as $\zeta = \rho$ (now I need $r > 2$)

$$C\mu h^{-2\gamma} \times (\rho^2(\mu h)^{-1} + 1) \times (\mu^{-1}h^{-1/2} + \rho^{-1}) \times \rho^{q-3}$$

and summation with respect to $\rho$ results in $Ch^{-3\gamma}$ which is marginally worse than $O(\mu^{-1}h^{-3})$. To improve it one can sum to $\zeta \leq \mu - \kappa$ and in zone $\{\zeta \geq \mu^{-\kappa}\}$ one can take $T_1 = \epsilon \mu \zeta$. \hfill $\Box$

### 3.6 General estimates near $\Sigma$

Now let us consider the vicinity of $\Sigma = \{f_1 = f_2\} = \{v_1 = v_2 = 0\}$ where

$$(3.70) \quad v_1|_\Sigma = v_2|_\Sigma = 0, \quad (\nabla v_1)|_\Sigma \text{ and } (\nabla v_2)|_\Sigma \text{ are linearly independent}$$

and near $\Sigma$

$$(3.71) \quad f_{1,2} = f \pm (v_1^2 + v_2^2)^{1/2}, \quad f > 0.$$

This analysis is simpler than near third order resonances because codimension of $\Sigma$ is 2 and everywhere factor $\gamma^1$ reflecting measure should be replaced by $\gamma^2$.

**Proposition 3.13.** Let condition (1.1) – (1.2) be fulfilled and let $\psi$ be supported in the small vicinity of $\Sigma$.

The standard implicit asymptotic formula (0.9) holds with the remainder estimate $O(\mu^{-1}h^{-3} + \mu^2h^{-2})$.

**Proof.** (i) Note first that

$$(3.72) \quad \text{The contribution of zone } \{\text{dist}(x, \Sigma) \leq \gamma\} \text{ to the remainder does not exceed } C\mu h^{-3}\gamma^2$$

and as

$$(3.73) \quad \gamma \overset{\text{def}}{=} \text{dist}(x, \Sigma) \overset{\sim}{\leq} |f_1 - f_2| \leq \bar{\gamma}_1 \overset{\text{def}}{=} c\mu^{-1} + c(\mu h)^{1/2}$$

this contribution is $O(\mu^{-1}h^{-3} + \mu^2h^{-2})$.

On the other hand,

$$(3.74) \quad \text{One can reduce operator to the canonical form without non-diagonal terms as long as}$$

$$(3.75) \quad \gamma \geq \bar{\gamma} \overset{\text{def}}{=} \mu^{-1/2}h^{1/2-\delta'} + \mu^{-2}h^{-\delta'}.$$
The second term in (3.75) appears because one needs to get rid of terms $Z_i^1 Z_j^1 Z_k^1 Z_l^1$ with $i + k = j + l$ but $(i, j) \neq (k, l)$ and these terms are of magnitude $O(\mu^{-2})$ unless $i + j + k + l = 2$ in which case one just diagonalizes the quadratic form and it is where the first term in in (3.75) comes from.

Important is that $\bar{\gamma} \leq \bar{\gamma}_1$. In the quest for remainder estimate $O(\mu^{-1} h^{-3})$ one would need to take $\bar{\gamma} = c\mu^{-1}$ and $\bar{\gamma} \leq \bar{\gamma}_1$ would hold as $\mu \leq h^{g-1}$.

(ii) Making $\epsilon\gamma$-admissible partition and $\rho$-admissible subpartition with

\begin{equation}
(3.76) \quad \rho = \epsilon |\nabla \phi_\alpha| \gamma + \frac{1}{2} \bar{\rho}, \quad \bar{\rho} = (C \mu^{-1} h |\log h|)^{1/2}
\end{equation}

one can take

\begin{equation}
(3.77) \quad T'_0 = C h \rho^{-2} \gamma, \quad T_1 = \epsilon \mu \gamma
\end{equation}

and the total contribution to the remainder of $(\gamma, \rho)$ subelements with $\rho \geq \rho$ to the remainder does not exceed

\begin{equation}
(3.78) \quad C \mu^2 h^{-2} \times \mu^{-1} h \rho^{-2} \times (\rho^2 (\mu h \gamma)^{-1} + 1) \times (\rho(\mu h \gamma)^{-1} + 1) \times \gamma^2 \cong \nonumber
\end{equation}

\begin{equation}
\nonumber C \mu^{-3} h^{-3} \rho + C \rho^{-1} h^{-2} \gamma + C \mu h^{-1} \rho^{-2} \gamma^2
\end{equation}

where $\gamma^2$ is their total measure. The right-hand expression sums with respect to $\rho \in (\bar{\rho}, \gamma)$ to $G(\gamma) \overset{\text{def}}{=} C \mu^{-1} h^{-3} \gamma + C \rho^{-2} \bar{\rho}^{-1} \gamma + C \mu h^{-1} \rho^{-2} \gamma^2$. Then with respect to $\gamma$ it sums to $G(1) = O(\mu^{-1} h^{-3} + \mu^{1/2} h^{-5/2} + \mu^2 h^{-2})$ where the middle term is less than the sum of two others.

(iii) Meanwhile the total contribution to the remainder of $(\bar{\rho}, \gamma)$ subelements does not exceed

\begin{equation}
(3.79) \quad C \mu^2 h^{-2} \times (\bar{\rho}^2 (\mu h \gamma)^{-1} + 1) \times (\bar{\rho}(\mu h \gamma)^{-1} + 1) \times \gamma^2 \cong \nonumber
\end{equation}

\begin{equation}
\nonumber C h^{-4} \bar{\rho}^3 + C \mu h^{-3} \gamma \bar{\rho} + C \mu^2 h^{-2} \gamma^2.
\end{equation}

This expression sums with respect to $\gamma$ to its value as $\gamma = 1$ resulting in

\begin{equation}
C \mu^{-3/2} h^{-5/2} |\log h|^{3/2} + C \mu^{1/2} h^{-5/2} |\log h|^{1/2} + C \mu^2 h^{-2}.
\end{equation}

Note that the first and the third term are properly estimated and the second terms is properly estimated save border case $h^{-1/3} |\log h|^{-K} \leq \mu \leq h^{-1/3} |\log h|^{K}$ which is treated as in the part (iii) of the proof of proposition 3.1.

(iv) Finally, as $\gamma \leq \bar{\gamma}$ one does not need a subpartition; the contribution to the remainder does not exceed $C \mu h^{-3} \gamma^2$ due to an analogue of proposition 3.10 below:

**Proposition 3.14.** Proposition 3.10 remains true near $\Sigma$ (i.e. without condition $|f_1 - f_2| \geq \epsilon$ provided at point $\bar{x}$ main part of precanonical form is $f_1 (\mu^2 x^2 + h^2 D^2_1) + f_2 (\mu^2 x^2 + h^2 D^2_2)$.)

**Proof.** Proof basically repeats the one of proposition 3.10. \qed
3.7 Sharp asymptotics near $\Sigma$

3.7.1 Now let us improve the results of the previous subsection. Let us note that

(3.80) Microhyperbolicity condition holds at $\bar{x} \in \Sigma$ iff in frames of $\nabla(fV^{-1})$ is not a linear combination of $\nabla(v_1V^{-1})$, $\nabla(v_2V^{-1})$ with coefficients $(\beta_1, \beta_2) \in \mathbb{R}^2 \cap B(0, 1)$ \(^{(14)}\).

In virtue of [Ivr4] I have already

**Proposition 3.15.** If (1.22) is fulfilled on $\text{supp} \psi$ then the standard formula holds with $\bar{T} = Ch|\log h|$.

Now let us analyze the meaning of (1.22). First of all, it is fulfilled as $\nabla_\Sigma(fV^{-1}) \neq 0$. So one needs to consider only set $\Sigma_0$ of the critical points of $fV^{-1}|_\Sigma$:

(3.81) $\Sigma_0 = \{x \in \Sigma, \nabla_\Sigma(fV^{-1}) = 0\}$.

Then

**Proposition 3.16.** For generic $V$

(3.82) $\Sigma_0$ consists of separate non-degenerate points;

(3.83) Magnetic form $\omega_F$ restricted to $\Sigma$ is the generic closed form on $\Sigma$ and thus degenerates on the smooth curve $\{\{v_1, v_2\} = 0\}$;

(3.84) $\omega_M$ does not degenerate on $\Sigma_0$ (which is equivalent to $\{v_1, v_2\} \neq 0$ on $\Sigma_0$).

One can write down many generic properties, but they are overkill.

3.7.2 First let us improve the remainder under condition (3.85) below (I remind that $f = \frac{1}{2}(f_1 + f_2)$):

**Proposition 3.17.** Let at some point $\bar{x} \in \Sigma$

(3.85) $|\nabla(fV^{-1})| \geq \epsilon_0$.

Then as $\psi$ is supported in the small enough vicinity of $\bar{x}$ the standard implicit formula holds with remainder $O(\mu^{-1}h^{-3} + \mu^{3/2}h^{-3/2}|\log h|)$.

\(^{(14)}\) In the uniform sense, i.e. (1.22)
\[ \nabla (fV^{-1}) = \beta_1 \nabla (v_1 V^{-1}) + \beta_2 \nabla (v_2 V^{-1}) \quad \text{at } \bar{x}, \quad \beta = (\beta_1, \beta_2) \in \mathbb{R}^2 : 1 \geq |\beta| \geq \epsilon_0 \]

where \(|\beta| \geq \epsilon_0\) due to (3.85).

Let us consider \( \phi_\alpha = \alpha(w_1 V^{-1}) + (1 - \alpha)(w_2 V^{-1}) \); note that one can extend \( \beta_j \) to the vicinity of \( \bar{x} \) so that
\[ fV^{-1} = \tilde{f}(w) + \beta_1(w) v_1 V^{-1} + \beta_2(w) v_2 V^{-1} + O(|v|^2) \]
where \( w = (w_1, w_2) \) are coordinates on \( \Sigma \).

Without any loss of the generality one can assume that \( \beta_2(w) = 0 \); then under conditions (3.85), (3.86)
\[ |\nabla \phi_\alpha| \geq \epsilon_1 \left( |(v_2 - \omega v_2^2)(v_1^2 + v_2^2)^{-1/2}| + |\alpha - \tilde{\alpha}| \right) \]
where \( \omega = \omega(w, v_1) \) is a smooth function.

(ii) Let us follow the proof of proposition 3.13. In part (ii) estimate (3.78) (for contribution of all elements with \( \rho \geq \rho \)) gains a factor \( \rho \gamma^{-1} \) and becomes
\[ C \mu^{-1} h^{-3} \rho^2 \gamma^{-1} + Ch^{-2} + C \mu h^{-1} \rho^{-1} \gamma. \]
This expression sums with respect to \( \rho \) ranging from \( \bar{\rho} \) to \( \gamma \) to
\[ C \mu^{-1} h^{-3} \gamma + Ch^{-2} |\log h| + C \mu h^{-1} \bar{\rho}^{-1} \gamma. \]
Then summation with respect to \( \gamma \) results in
\[ C \mu^{-1} h^{-3} + Ch^{-2} |\log h|^2 + C \mu^{3/2} h^{-3/2} \]
which is less than the announced estimate.

(iii) In part (ii) estimate (3.79) (for contribution of all elements with \( \rho \leq \bar{\rho} \)) gains a factor \( \bar{\rho} \gamma^{-1} \) and becomes
\[ Ch^{-4} \bar{\rho}^4 \gamma^{-1} + C \mu h^{-3} \bar{\rho}^2 + C \mu^2 h^{-2} \bar{\rho} \gamma. \]
This expression sums with respect to \( \gamma \) to
\[ C \mu^{-2} h^{-2} |\log h|^2 \gamma^{-1} + Ch^{-2} |\log h|^2 + C \mu^{3/2} h^{-3/2} |\log h|^{1/2} \]
which is also below than announced remainder estimate.

(iv) Finally, as \( \gamma \leq \bar{\gamma} \) one does not need a subpartition; the contribution to the remainder does not exceed \( C \mu h^{-3} \bar{\gamma}^2 = C \mu^{-1} h^{-3} + C \mu^{-1/2} h^{8-5/2} \) and the second term here is obviously much less than the announced estimate.

So, condition (3.85) is a kind of non-degeneracy condition, improving remainder estimate.
3.7.3 Let us now use the remaining non-degeneracy conditions.

**Proposition 3.18.** Let conditions (3.18) and

\[(3.88), \quad |(2p + 2)\mu hfV^{-1} - 1| + |\nabla_{\Sigma}((2p + 2)\mu hfV^{-1} - 1)| \leq \epsilon_0 \implies \text{Hess}_{\Sigma}((2p + 2)\mu hfV^{-1} - 1) \text{ has at least } r \text{ eigenvalues with absolute values greater than } \epsilon_0\]

be fulfilled. Then

(i) The total remainder is given by (3.21)$_{r+1}$ while the main part is given by the standard implicit formula (0.9).

(ii) Under condition (3.85) the total remainder is given by (3.21)$_{r+2}$ as $r = 1$ while the main part is given by the standard implicit formula (0.9);

(iii) Under conditions (3.85) and (3.84) the total remainder is $O(\mu^{-1}h^{-3})$ as $r = 2$ while the main part is given by the standard implicit formula (0.9).

**Proof.** (a) Under condition (3.85) the total contribution to the remainder of all $(\gamma, \rho)$-elements does not exceed (3.78)* i.e.

\[(3.89)_k \quad C\mu^{-1}h^{-3}\rho^{k+1}\gamma^{-k} + C\rho^{k-1}\gamma^{1-k}h^{-2} + C\mu h^{-1}\gamma^{2-k}\rho^{k-2}\]

with $k = 1$ while without it it does not exceed the same expression with $k = 0$.

Further, under extra condition (3.88), this expression acquires factor $C\rho\gamma^{-r}$. Therefore again (3.89)$_q$ with $q = k + r$ gives a proper estimate for the total contribution of all $(\gamma, \rho)$-elements to the remainder.

Consider summation with respect to $\bar{\rho} \leq \rho \leq \gamma \leq 1$. The first term sums to $C\mu^{-1}h^{-3}$ independently on $q$: the second term sums to $Ch^{-2}|\log h|^2$ as $q = 1$ and to $Ch^{-2}|\log h|$ as $q \geq 2$; the third term sums to $C\mu h^{-1}\bar{\rho}^{-1}$ as $q = 1$, to $C\mu h^{-1}|\log h|^2$ as $q = 2$ and to $C\mu|\log h|$ as $q = 3$.

Therefore in all cases but one the remainder estimate (3.21)$_q$ with $q = k + r$ is proven; this exceptional case is $q = 3, \mu \geq h^{-1}|\log h|^{-1}$ when the estimate $O(h^{-2}|\log h|)$ is recovered; I remind that only zone $\{\rho \geq \bar{\rho}, \gamma \geq \bar{\gamma}\}$ is covered so far.

(b) To cover the remaining case let us introduce a scaling function $\zeta = \epsilon|\nabla_{\Sigma}fV^{-1}| + \bar{\rho}$ on $\Sigma$; one can extend this function to

\[(3.90) \quad \zeta = \epsilon|\nabla_{\Sigma}fV^{-1}| + \gamma.\]
Then in arguments above one can replace factor $\rho^2 \gamma^{-2}$ by $\zeta^2$ and then contribution of zone $\{\zeta \leq \gamma^{\delta'}\}$ would be $O(\mu^{-1}h^{-3})$ as $q = 3$; so only subzone $\{\zeta \geq \gamma^{\delta}\}$ remains to be treated. In this subzone one can take

$$(3.91) \quad T_0^* = Ch\zeta^{-2}, \quad T_1 = \epsilon \mu \gamma$$

where one can take $T_0^* = Ch\zeta^{-2}$ rather than $T_0^* = Ch\zeta^{-1}\gamma^{-1}$ due to (3.84). Then the total contribution of such $(\gamma, \zeta)$ elements to the remainder does not exceed

$$C\mu^2 h^{-2} \zeta^2 \gamma^2 \times h\zeta^{-2} \times \mu^{-1} \zeta^{-1} \times ((\zeta^2 + \gamma)(\mu h)^{-1} + 1) \times (\mu h)^{-1}$$

where $\zeta^2 \gamma^2$ is the measure and $(\mu h)^{-1}$ and $((\zeta^2 + \gamma)(\mu h)^{-1} + 1)$ are estimates of the numbers of indices “$p$” and corresponding indices “$n$”; noting that $\zeta^2 \geq \gamma$ one can rewrite this expression as

$$C\mu^{-1} h^{-3} \gamma^2 \zeta + Ch^{-2} \gamma^2 \zeta^{-1}$$

and the summation with respect to $\zeta \geq \gamma^{\delta'}$ and $\gamma$ results in $O(\mu^{-1}h^{-3})$.

Thus estimate (3.21)$_q$ for contribution of zone $\{\rho \geq \bar{\rho}, \gamma \geq \bar{\gamma}\}$ is established.

(c) Further, contribution of $(\gamma, \rho)$ elements with $\rho \leq \bar{\rho}$, $\gamma \geq \bar{\gamma}$ to the remainder does not exceed (3.78) as $k = 0$ or (3.78)* as $k = 1$ multiplied by $\bar{\rho}^{r-1} \gamma^{-1}$ i.e.

$$(3.92)_q \quad Ch^{-4} \bar{\rho}^{q+3} \gamma^{-q} + C\mu h^{-3} \bar{\rho}^{q+1} \gamma^{1-q} + C\mu^2 h^{-2} \bar{\rho}^q \gamma^{3-q}$$

and one can check easily that summation with respect to $\gamma \geq \bar{\gamma}$ results in expression not exceeding (3.19)$_q$.

(d) Finally, contribution of zone $\{\gamma \leq \bar{\gamma}\}$ to the remainder does not exceed $C\mu h^{-3} \bar{\gamma}^2$ which is $O(\mu^{-1}h^{-3})$ unless $\mu \geq h^{24-1}$ and even in this case it is less than (3.19)$_1$.

Therefore as $q \geq 2$ I need some better arguments in this zone. Again, one needs to consider only part of it with $\{\zeta \geq \mu^{-\delta}\}$ as contribution of zone $\{\zeta \leq \zeta\}$ does not exceed $C\mu h^{-3} \bar{\gamma}^2 \zeta^{-2}$.

As using precanonical form the speed would be $O(\mu^{-1})$ and since only a magnitude of $\zeta$ is important for us and also inequality $\zeta \geq \gamma^{\delta'}$, one can take

$$(3.93) \quad T_0^* = Ch^{-1} \bar{\gamma}^{-1}, \quad T_1 = \epsilon \mu \bar{\gamma}^{1-\delta'}$$

where an appropriate time direction for this $T_1$ is taken. In these arguments I do not assume (3.84) and and thus $T_0$ is not as it was in (b) (surely, some improvements are possible but not needed).

Then the total contribution to the remainder of $\zeta$-elements does not exceed

$$Ch^{-4} \zeta^r \bar{\gamma}^2 \times h\zeta^{-1} \bar{\gamma}^{-1} \times \mu^{-1} \zeta^{\delta''-1} \times C\mu^{-1} h^{-3} \zeta^{r-1} \bar{\gamma}^{\delta''}$$

which sums with respect to $\zeta$ to $C\mu^{-1} h^{-3} \log h \mid \bar{\gamma}^{\delta''} = o(\mu^{-1} h^{-3})$ even as $r = 1$. \qed
Remark 3.19. (i) Since in the generic case in the critical points of \( fV^{-1} \) are non-degenerate (i.e. \( (3.82) \) is fulfilled) and also \( (3.83), (3.84) \) hold, the remainder estimate is \( O(\mu^{-1}h^{-3}) \).

(ii) I think one can get rid off logarithmic factors in the above estimates but I do not care.

3.8 Summary

Proposition 3.20. (I) Let \( (g^{jk}) \) be fixed and then \( (V_j) \) be generic, more precisely:

(i) Outside of \( \Sigma = \{ x : f_1 = f_2 \} \) critical points of \( f_1f_2^{-1} \) satisfy \( (2.30)_3 \) and \( (2.2) \);

(ii) \( \Sigma \) be smooth 2-dimensional manifold and \( |f_1 - f_2| \approx \text{dist}(x, \Sigma) \).

Let \( V \) be general but satisfying \( (0.8) \) at \( \text{supp} \psi \). Then under condition \( (3.1) \) the remainder is \( O(\mu^{-1}h^{-3} + \mu^{2}h^{-2}) \) while the main part of asymptotics is given by implicit formula \( (0.9) \).

(II) Furthermore, let \( V \) be generic i.e.

(iii) Outside of \( \Sigma \) and resonances condition \( (3.21)_4 \) be fulfilled;

(iv) Near resonances condition \( (3.66)_{4,3} \) be fulfilled;

(v) At \( \Sigma \) conditions \( (3.82) - (3.85) \) be fulfilled\(^{15}\).

Let \( V \) satisfy \( (0.8) \) at \( \text{supp} \psi \). Then under condition \( (3.18) \) the remainder is \( O(\mu^{-1}h^{-3}) \) while the main part of asymptotics is given by implicit formula \( (0.9) \).

4 Calculations

The purpose of this section is to pass from the implicit formula \( (0.9) \) to more explicit one, namely either \( (0.7) \) or \( (0.5) \) with more or less explicit expression for \( E^{\text{MW corr}}_\epsilon \). It will be done by different methods depending on the magnitude of \( \mu \) and also non-degeneracy conditions and the methods applied will be used in the classification. My main concern will be either no non-degeneracy condition or the generic non-degeneracy condition.

4.1 Temperate magnetic field

In this subsection I will use formula \( (0.9) \) with \( T_0 \leq \epsilon \mu^{-1} \) and derive a remainder estimate. More precisely, if in section 3 \( T \) was given by \( (0.9) \) with \( T_0 \geq \epsilon \mu^{-1} \) I replace it by \( T_0 = \epsilon \mu^{-1} \) and estimate an error.

\(^{15}\) Actually these conditions should be fulfilled at \( \Sigma^+_0 \cup \Sigma^0_0 \) only.
This error actually is the contribution of the affected domain to the remainder with some $T_0 \ll \mu^{-1}$ (usually $T_0 = Ch|\log h|$ under condition (0.8)) and $T_1 = \epsilon \mu^{-1}$.

Further, under condition (0.8) I can trade $T \leq \epsilon \mu^{-1}$ to $T = Ch|\log h|$ and then I can apply the standard approach; I will assume here by default that

\[(4.1)\quad h^{-\delta_0} \leq \mu \leq h^{-1+\delta_0}\]

leaving case $\leq h^{-1+\delta_0} \leq \mu \leq ch^{-1}$ for a separate consideration.

4.1.1 Let us consider regular points first.

**Proposition 4.1.** Assume that there are no resonances of order not exceeding $M$ and also $|\nabla f_1 f_2^{-1})| \geq \epsilon_0$ on $\text{supp } \psi$. Further, assume that condition (3.21) with $q \geq 1$ is fulfilled. Then under condition (4.1) the remainder estimate is given by

\[(4.2)_q \quad O\left(\mu^{-1}h^{-3} + (\mu h)^{(q+2)/2}h^{-4} |\log h|^{q/2}\right)\]

while the main term of asymptotics given by (0.9) with any $T \geq \epsilon \mu^{-1}$.

**Proof.** First of all, as $q = 1$ this remainder estimate is $O(\mu^{-1}h^{-3} + (\mu h)^3h^{-4} |\log h|^{1/2})$ which is no smaller than $O(\mu^{-1}h^{-3} + \mu^2h^{-2})$ given by proposition 3.1 and as $q > 1$ remainder estimate $(4.2)_q$ is no smaller than $(3.22)_q$ given by proposition 3.4; so both of these propositions could be applied and one needs to estimate a substitution $T_0 \mapsto \epsilon \mu^{-1}$ error.

According to the proofs of these propositions $T_0 = Ch\rho^{-2} |\log h|$ which is less than $\epsilon \mu^{-1}$ as

\[(4.3) \quad \rho \geq \tilde{\rho}_1 \overset{\text{def}}{=} (C \mu h |\log h|)^{1/2};\]

So one can take there $T_0 = \epsilon \mu^{-1}$.

On the other hand, contribution of zone $\{\rho \leq \tilde{\rho}_1\}$ to the remainder with $T = \epsilon \mu^{-1}$ does no exceed $C \mu h^{-3}v(\tilde{\rho}_1) \leq C \mu h^{-3}\tilde{\rho}_1^q$ which is exactly the second term in $(4.2)_q$. \qed

**Corollary 4.2.** Let conditions of proposition 4.1 be fulfilled. Then under assumption (0.8)

(i) The remainder estimate is given by $(4.2)_q$ while the main term of asymptotics given by

\[(4.4) \quad \int C^{MW}(x,0)\psi(x) \, dx.\]

(ii) In particular the remainder estimate is $O(\mu^{-1}h^{-3})$ as $\mu \leq \bar{\mu}_q \overset{\text{def}}{=} C(h |\log h|)^{-q/(q+4)}$ where in the general case $\bar{\mu}_1 = C(h |\log h|)^{-1/5}$ and in the generic case $\bar{\mu}_4 = C(h |\log h|)^{-1/2}$. 

4 CALCULATIONS

Proof. According to proposition 4.1 with the remainder estimate (4.2)\(_q\) the main part of asymptotics is given by (0.9) with \(T = \epsilon \mu^{-1}\). However then under condition (0.8) one can trade \(T = \epsilon \mu^{-1}\) to any \(T \geq \overline{T} \equiv C \hbar |\log \hbar|\). The rest is proven by the standard successive approximation method, applied to the original operator (rather than to the canonical form); one can take as an unperturbed operator the same operator with the coefficients frozen at \(y\) which leads to the Weyl expression perturbed by \(\sum_{m \geq 0, n \geq 1} \Gamma_{mn} \hbar^{-4+2m+2n} \mu^{2n}\) where terms with \(n \geq 2\) or \(m \geq 1\) do not exceed the announced remainder estimate. Alternatively one can take as an unperturbed operator the same operator with \(g^{jk}, V\) frozen at \(y\) and with \(V_j\) replaced by \(V_j(y) + \sum_k (\partial_k V_j)(y)(x_k - y_k)\) which leads directly to Magnetic Weyl expression. Details see in (1.11) of [Ivr3]. \(\square\)

4.1.2 Now I want to improve the result in the general case and allow non-degenerate critical points of \(f_1 f_2^{-1}\).

Proposition 4.3. Assume that \(f_1 \neq f_2\) and there are no resonances of order not exceeding \(M\) and also critical points of \(f_1 f_2^{-1}\) are non-degenerate on \(\text{supp} \, \psi\). Then under condition (4.1) the remainder estimate is given by (4.2)\(_q\) with any \(q < 2\) arbitrarily close to 2 while the main term of asymptotics given by the standard implicit formula with any \(T \geq \epsilon \mu^{-1}\).

Proof. Again, this remainder estimate is no smaller than \(O(\mu^{-1} h^{-3} + \mu^2 h^{-2})\) given by proposition 3.2, so again one needs to estimate an error arising from the substitution \(T_0 \mapsto \epsilon \mu^{-1}\).

Proof follows ideas of the proof of proposition 4.5 of [Ivr8]. Let us introduce functions \(\ell_k, k = 1, \ldots, K\) in the same proof but with \(\gamma = 1\) and again let \(\overline{\ell}_k = (\mu \hbar |\log \hbar|)^{\delta/(k+1)}\).

(i) Assume first that \(|\nabla (f_1 f_2^{-1})| \geq \epsilon_0\). Then the arguments of the mentioned proof survive with this simplification.

(ii) Assume now that \(f_1 f_2^{-1}\) has one non-degenerate critical point \(\bar{x}\). Consider first zone where \(\ell_K(x) \leq \gamma(x) \equiv \frac{1}{2}|x - \bar{x}|\).

Then the number of indices \(\ll q\) does not exceed \(C(\ell_{K+1}^K(\mu h \gamma)^{-1} + 1)\) (where the second term is the smallest one) while the number of indices \(\ll n\) for each \(p\) does not exceed \(C(\ell_{K+1}^K(\mu h)^{-1} + 1)\) and the total contribution to the asymptotics of all such elements with \(\ell_K \ll \overline{\ell}_K\) and fixed magnitude of \(\gamma(x) \approx \gamma\) for some \(k\) does not exceed

\[
C \mu^2 h^{-2} h^{-2} \times (\ell_{K+1}^K(\mu h)^{-1} + 1) \times \ell_{K}^K(\mu h \gamma)^{-1} \times \gamma^4
\]

where \(\gamma^4\) is the total measure of zone \(\{\gamma(x) \approx \gamma\}\). This expression does not exceed \(C \mu^2 h^{-2} \overline{\ell}_K^{-1} \gamma^3 |\log \hbar|^J\) and summation with respect to \(\gamma\) results in \(C \mu^2 h^{-2} \overline{\ell}_K^{-1} |\log \hbar|^J \lesssim C(\mu h)^{2-\delta} h^{-4}\) as \(\delta > 1/K\).
4 CALCULATIONS

On the other hand, repeating arguments of the proof of proposition 4.5 of [Ivr8] one can see easily that the total contribution to the substitution error (when one replaces $T_0 \geq \epsilon \mu^{-1}$ by $T_0 = \epsilon \mu^{-1}$) of all elements with $\ell_k \geq C_0 \ell_k$ does not exceed $C \mu^{-1} h^{-3} + C \mu^2 h^{-2} \ell_k^{-1}$ which results in the same estimate.

(iii) Alternatively, assume that $\ell_j \leq \gamma \leq \ell_{j+1}$ for some $j \leq K - 1$. Again one needs to consider elements with $\ell_k \approx \tilde{\ell}_k \leq \gamma$ for some $k \leq j$. Then in virtue of the same arguments of the proof of proposition 4.5 of [Ivr8] the contribution to the asymptotics does not exceed

\[ C \mu^2 h^{-2} \ell_k^{-1} \gamma^{-1} |\log h|^J \times \gamma^4 \times \ell_k \gamma^{-1} \approx C \mu^2 h^{-2} |\log h|^J \gamma^2 \]

where the last factor $\ell_k \gamma^{-1}$ is the upper bound of the relative measure of $\ell_k$ elements to $\gamma$.

Again summation with respect to $\gamma$ results in $C \mu^2 h^{-2} |\log h|^J$.

(iv) Finally, consider elements with $\ell_1 \geq \gamma$. Then I redefine $\gamma = \ell_1$ and only $\ell \approx \tilde{\ell}_1$ should be considered since otherwise $T_0 \leq \epsilon \mu^{-1}$.

Contribution of such elements to the remainder with $T_0 = \epsilon \mu^{-1}$ does not exceed $C \mu h^{-3} \tilde{\ell}_1^4 = O(\mu^3 h^{-1} |\log h|^J)$.

4.1.3 Now I want to attack resonances.

Proposition 4.4. Assume that that $|f_1 - f_2| \geq \varepsilon_0$ and $|\nabla(f_1 f_2^{-1})| \geq \varepsilon_0$ on $\text{supp} \psi$. Let consider asymptotics with the main part given by the standard implicit formula with any $T \geq \epsilon \mu^{-1}$. Then under condition (4.1) the remainder does not exceed (4.2)' $16$ with $q < 2$ arbitrarily close to 2.

Proof, Part I. Let us consider first zone $\{|x_1| \geq \bar{\gamma}\}$ and apply to the canonical form the same method as in the proof of proposition 4.3. Again I assume that there is just one resonance surface $\Xi = \{x_1 = 0\}$.

However, definition of $\ell_k$ involves derivatives of order $k$ and they are unbounded (with respect to $x_1$) as $\gamma^{k+1} \leq \mu^{4-2m}$. To avoid this problem I rescale first $B(y, \gamma(y))$ into $B(0, 1)$ and then apply this method. However, such rescaling would replace $\rho$ by $\rho \gamma$ and $\ell$ by $\ell \gamma$ and leave $\rho \ell \geq C \mu h |\log h|$ intact. However then the main part of remainder estimate gets factor $\gamma^{-1}$ since the number of indices “$p$” will be $\rho (\mu h \gamma)^{-1}$ because the derivative with respect to $x_1$ for “$p$” and “$(p + 1)$” would differ by $\mu h \gamma$.

So, the contribution of zone $\{x : \gamma(x) \approx \gamma\}$ to the error estimate in question does not exceed $C(\mu h)^q h^{-4}$ with $q < 2$ arbitrarily close to 2; factor $\gamma^{-1}$ discussed above is compensated by the same factor coming from the measure. Summation with respect to $\gamma$ results in the similar answer (extra factor $|\log h|$ is covered by miniscule decrease of $q$).

\[16\] Where ’ denotes that the power of $|\log h|$ could be larger.
This works as long as \(|x_1| \geq \bar{\gamma}\) with
\[
\bar{\gamma} = \mu^{2-m+\delta'} + \mu^{-2}
\]  
(4.5)
where the last term dominates as \(m \geq 5\) only and guarantees that \(\gamma \geq C\mu^{-1}h^{1-\delta}\).

Therefore,
\[
(4.6) \text{ Contribution of zone } \{\bar{\gamma} \leq |x_1| \leq \epsilon\} \text{ to the remainder does not exceed } C\mu^{-1}h^{-3} + C(\mu h)^q h^{-4}. 
\]

In particular, as \(m = 5\) proposition is proven since the contribution of zone \(|x_1| \leq \bar{\gamma}\) to the remainder does not exceed \(C\mu h^{-3}\bar{\gamma}\). \(\square\)

**Proof of Proposition 4.4, Part II.** On the other hand, in zone \(|x_1| \leq \bar{\gamma}\) one can use pre-canonical form and contribution of subzone \(\{\rho \geq \bar{\rho} \overset{\text{def}}{=} C \max(\bar{\gamma}, \mu h |\log h|^{-1})\}\) to the error is negligible while contribution of subzone \(\{\rho \leq \bar{\rho}\}\) to the remainder does not exceed \(C\mu h^{-3}\bar{\gamma}\bar{\rho} = C\mu h^{-3}\bar{\gamma}\bar{\rho}^2 + C\mu^2 h^{-2}|\log h|\); picking \(\bar{\gamma}\) as \(m = 4\) results in the proper estimate then.

However as \(m = 3\) one recovers only remainder estimate \(C\mu^{-1+2\delta'}h^{-3} + C(\mu h)^q h^{-4}\) which is the required estimate as \(\mu \geq h^{-1/3+\delta'}\) and only marginally worse otherwise. To recover proper estimate one needs to reexamine zone \(\{|x_1| \leq \bar{\gamma}, \rho \leq \bar{\gamma}\}\) \(\{|x_1| \leq \bar{\gamma}_0, \rho \leq \bar{\gamma}_0\}\) since the contribution to the remainder of the latter does not exceed \(C\mu^{-1}h^{-3}\). Without any loss of the generality one can assume that non-diagonal term does not depend on \(x_1\) since one can remove term divisible by \(x_1\) by the same method as for \(|x_1| \geq \bar{\gamma}\) was removed the whole term.

In subzone \(\{\max(\rho, \bar{\gamma}_0) \leq |x_1| \leq \bar{\gamma}\}\) one can pick up \(T_0 = Ch |\log h| \rho^{-2}\) which is less than \(\epsilon\mu^{-1}\) unless \(\rho \leq \bar{\rho} \overset{\text{def}}{=} C(\mu h |\log h|)^{1/2}\) (which is less than \(\mu^{-1}\)) and the contribution to the remainder of \(\{|x_1| \leq \bar{\gamma}, \rho \leq \bar{\rho}\}\) does not exceed
\[
C\mu h^{-4}\bar{\gamma}\bar{\rho} = C\mu^{-1}h^{-3} \times \mu^3(\mu^3|\log h|)^{1/2} \leq C\mu^{-1}h^{-3}
\]
as \(\mu \leq h^{-1/3+\delta'}\) and \(\delta = \delta'\).

In subzones \(\{|x_1| \sim \gamma \leq \rho\}\) with \(\bar{\gamma}_0 \leq \gamma \leq \bar{\gamma}_1\) and \(\{|x_1| \leq \gamma = \bar{\gamma}_0 \leq \rho\}\) one can take \(T_0 = Ch |\log h| (\rho \gamma)^{-1}\) which is less than \(\epsilon\mu^{-1}\) here for sure. \(\square\)

**Remark 4.5.** It can happen that \(T_1\) described in section 3 is less than \(\epsilon\mu^{-1}\). Then one can take \(T_0 \leq \epsilon\mu^{-1}\) anyway.
4.1.4 Let us derive sharp remainder estimates in the resonance case:

**Proposition 4.6.** Assume that that $|f_1 - f_2| \geq \epsilon_0$ and also $|\nabla(f_1 f_2^{-1})| \geq \epsilon_0$ on supp $\psi$. Further, assume that conditions (4.1) and (3.21) are fulfilled with $q > 2$ and that on (any) resonance surface $\Xi$

$$\tag{4.7}, \quad \text{Hess}_\Xi(\mu hfV^{-1} - 1) \text{ has at least } r \text{ eigenvalues}$$

with absolute values greater than $\epsilon_0$

Then the remainder estimate is given by (4.2) while the main term of asymptotics given by the standard implicit formula with any $T \geq \epsilon \mu^{-1}$.

**Proof.** Due to proposition 4.3 one should cover only case $\mu \geq h^{-1/3+\delta}$. Assumptions of proposition imply that condition (3.66)$_{r+1}$ holds. While one can apply proposition 3.10 directly only as $r > 2$, the proof of it yields that under condition (4.7) with $r = 1, 2$ the remainder estimate $O(\mu^{-1} h^{-3} + \mu^{2-r/2} h^{2+r/2} |\log h|^2)$ holds, which is not worse than (4.2)$_{r+1}$.

Let us apply the same partition to zones as in propositions 3.9, 3.10.

Then zone

$$\{ \gamma \geq \max(\rho, \tilde{\gamma}_2 \text{def } C \max(\mu^{-2/3}, (\mu h |\log h|)^{1/2})) \}$$

is covered by the arguments used in the proof of proposition 4.1; its contribution to the error does not exceed (4.2)$_q$. Meanwhile in the subzone $\{ \rho \geq \gamma \geq \tilde{\gamma}_2 \}$ one can take $T_0 = Ch |\log h|^{1/2}$ which is less than $\epsilon \mu^{-1}$.

Further, in subzone $\{ \gamma \leq |x_1| > \tilde{\gamma}_2 \}$ one can take $T_0 = Ch |\log h|^{1/2}$ which is less than $\epsilon \mu^{-1}$ unless $\xi \leq C(\mu h |\log h|)^{1/2}$. Similarly, in the subzone $\{ |x_1| \leq \tilde{\gamma}_2 \}$ one can take $T_0 = Ch |\log h|^{1/2}$ as well. Note that the contribution of the zone

$$\{ |x_1| \leq \tilde{\gamma}_2, \xi \leq \tilde{\xi} \text{def } C(\mu h |\log h|)^{1/2} \}$$

to the remainder does not exceed $C \mu h^{-3} \tilde{\gamma}_2 \tilde{\xi}^r$ which does not exceed the second term in (4.2)$_{r+1}$ as $\tilde{\xi} \approx \tilde{\gamma}_2$ i.e. $\mu^{-2/3} \leq C(\mu h |\log h|)^{1/2}$ i.e. as $\mu \geq C(h |\log h|)^{-3/7}$.

As $\mu \leq C(h |\log h|)^{-3/7}$ let us consider zone $\{ |x_1| \leq \tilde{\gamma}_2 \}$ and use the precanonical form here. Then contribution of the subzone $\{ |x_1| \leq \tilde{\gamma}_2, \rho \leq \tilde{\rho}_2, \xi \leq \tilde{\xi} \}$ to the remainder does not exceed $C \mu h^{-3} \tilde{\gamma}_2 \tilde{\rho}_2 \tilde{\xi}^r$ which does not exceed the second term in (4.2)$_{r+1}$ as $\tilde{\xi} \approx \tilde{\gamma}_2 \tilde{\rho}_2$. Therefore one can pick up $\tilde{\rho}_2 = \tilde{\xi} \tilde{\gamma}_2^{-1}$; one can see easily that $\tilde{\rho}_2 \gg \mu^{-2/3}$ and $\tilde{\rho}_2 \tilde{\gamma}_2 \gg C(\mu h |\log h|)$ and then $T_0 \leq Ch h^{1/2} h^{1/2-1/2} \ll \epsilon \mu^{-1}$ and the contribution of the zones $\{ |x_1| \leq \tilde{\gamma}_2, \rho \geq \tilde{\rho}_2 \}$ and $\{ |x_1| \leq \tilde{\gamma}_2, \xi \leq \tilde{\xi} \}$ to the error is negligible. \qed
4.1.5 Finally, let us consider the vicinity of $\Sigma$.

**Proposition 4.7.** Let condition (1.1)–(1.2), (4.1) be fulfilled and let $\psi$ be supported in the small vicinity of $\Sigma$.

Then the remainder estimate is given by $(4.2)'_q$ with any $q < 2$ arbitrarily close to 2 while the main term of asymptotics given by the standard implicit formula with any $T \geq \epsilon \mu^{-1}$.

**Proof.** Again this remainder estimate is no smaller than $O(\mu^{-1} h^{-3} + \mu^2 h^{-2})$ delivered by proposition 3.13.

Applying the same approach as in the Part I of the proof of proposition 4.4 (which is applicable as $\gamma \geq \tilde{\gamma}_2 \stackrel{\text{def}}{=} \max((\mu h|\log h|)^{1/2}, \mu^{-1})$ now) I conclude after summation with respect to $\gamma$ that the contribution of the zone $\{x : \gamma(x) > \gamma\}$ (where $\gamma(x) = \text{dist}(x, \Sigma)$) to the error does not exceed $C(\mu h)^{qh^{-4}} \gamma$ with $q < 2$ arbitrarily close to 1. Here an extra factor $\gamma$ appears because the measure of $\{x : \gamma(x) > \gamma\}$ is $\approx \gamma^2$ rather than $\approx \gamma$ as it was before. Then after summation I conclude that

$$\gamma \geq \gamma_2 \stackrel{\text{def}}{=} \max((\mu h|\log h|)^{1/2}, \mu^{-1})$$

The contribution of the zone $\{x : \gamma(x) \leq \gamma_2\}$ to the error is $O((\mu h)^{qh^{-4}} \gamma)$. In particular, it is $O((\mu h)^{qh^{-4}})$ as $\gamma = \epsilon$.

Meanwhile, the contribution of the zone $\{x : \gamma(x) \leq \gamma\}$ to the error does not exceed $C\mu h^{-3} \gamma_2 \approx (\mu^{-1} h^{-3} + \mu^2 h^{-2})$. $\square$

4.1.6 Let us improve the above estimate under generic conditions.

**Proposition 4.8.** Let conditions (1.1)–(1.2), (4.1) and (3.88), with $r = 1, 2$ be fulfilled. Let $\psi$ be supported in the small enough vicinity of $\Sigma$. Then

(i) The total remainder is given by $(4.2)'_{r+1}$ while the main part is given by the standard implicit formula with any $T \geq \epsilon \mu^{-1}$.

(ii) Under condition (3.85) the total remainder is given by $(4.2)'_{r+2}$ as $r = 0, 1$ while the main part is given by the standard implicit formula with any $T \geq \epsilon \mu^{-1}$;

(iii) Under conditions (3.85) and (3.84) the total remainder is given by $(4.2)'_{r+2}$ as $r = 2$ while the main part is given by the standard implicit formula with any $T \geq \epsilon \mu^{-1}$.

**Proof.** Again let us note that the remainder estimate given by proposition 3.17 as $r = 0$ and 3.18 as $r = 1, 2$ is no worse than one announced here.

In zone $\{\gamma \geq \max((\mu h|\log h|)^{1/2}, \mu^{-1})\}$ the same arguments as in the proof of proposition 4.3 are applied, and conditions (3.88), and (3.85) add factors $(\mu h|\log h|)^{r/2}$ and $(\mu h|\log h|)^{1/2} \gamma^{-1}$ respectively to the measure of zone $\{\rho \leq (\mu h|\log h|)^{1/2}\}$. 


Then after summation with respect to \( \gamma \) one arrives to the estimate \((4.2)_{r+k+1}'\) of the contribution of this zone to the error estimate and \( k=1 \) under condition \((3.85)\) and \( k=0 \) otherwise.

Meanwhile \( T_0 \leq \epsilon \mu^{-1} \) in zone \( \{ \rho \geq \gamma \geq C \max((\mu h|\log h|)^{1/2}, \mu^{-1}) \} \).

So, one needs to consider zone \( \{ \gamma \leq C\bar{\gamma}_2 \} \). In this zone condition \((3.75)\) has no value. Contribution of this zone to the remainder estimate obviously does not exceed \((4.2)_2'\). Therefore case \( r+k \leq 1 \) is covered. Let us introduce \( \zeta \) as before. As either \( \zeta \leq \gamma \) or condition \((3.84)\) is fulfilled, one can take \( T_0 = Ch\zeta^{-2} \) and then only subzone \( \{ \zeta \leq C(\mu h|\log h|)^{1/2} \} \) should be considered; then condition \((3.88)\), adds an extra factor \((\mu h|\log h|)^{1/2} \) to the measure and to the estimate which becomes \((4.2)'_{r+2}\).

That leaves us with the analysis of zone \( \{ \zeta \geq \gamma \} \cap \{ \gamma \leq \bar{\gamma}_2 \} \) and only without condition \((3.84)\), in which case only estimate \((4.2)'_4\) should be proven under condition \((3.88)_1\). However then one can take \( T_0 = C\mu h|\log h|\gamma^{-1}\zeta^{-1} \) and then only subzone \( \zeta \leq C\mu h|\log h|\gamma^{-1} \) remains to be considered. Its contribution to the remainder does not exceed \( C\mu h^{-3} \times \mu h|\log h|\gamma^{-1}\zeta^{-1} \) which sums with respect to \( \gamma \) ranging from \( \bar{\gamma}_1 \) to \( C\mu h^{-3} \times \mu h|\log h|\gamma^{-1}\zeta^{-1} \) which is properly estimated. Meanwhile contribution of \( \{ x : \gamma(x) \leq \bar{\gamma}_1 \} \) to the remainder does not exceed \( C\mu h^{-3}\bar{\gamma}_2 \) and is properly estimated as well.

\[ \square \]

### 4.1.7 Conclusion

**Corollary 4.9.** Let conditions of one of propositions 4.3 – 4.8 be fulfilled. Further, let condition \((0.8)\) be fulfilled. Then

(i) The remainder estimate is \((4.2)'_q\) where

(a) \( q < 2 \) is arbitrarily close to 2 in the general case (propositions 4.3, 4.4, 4.7) and

(b) \( q \) is described in the corresponding case (propositions 4.1, 4.6, 4.8),

while the main part of asymptotics is given by \((4.4)\).

(ii) In particular the remainder estimate is \( O(\mu^{-1}h^{-3}) \) as \( \mu \leq \bar{\mu}_q \) defined \( Ch^{-q/(q+4)}|\log h|^{-K} \). In particular

(a) in the general case \( \bar{\mu}_1 = Ch^{1/3} \)

(b) in the generic case \( \bar{\mu}_4 = h^{1/2}|\log h|^{-K} \).

**Proof.** Proof coincides with the proof of corollary 4.2.

Then I get immediately

**Corollary 4.10.** (i) Theorem 0.2 is proven with \( \mathcal{E}_{corr}^{MW} = 0 \) as \( \mu \leq h^{4/13} \);

(ii) Theorem 0.3 is proven as \( \mu \leq Ch^{-1/2}|\log h|^{-K} \).
4.2 Strong magnetic field

In this subsection I assume that

\[ h^{-1/3} \leq \mu \leq ch^{-1} \]  

sometimes making separate considerations for the case of the superstrong magnetic field

\[ h^{-1} |\log h|^{-K} \leq \mu \leq ch^{-1} \]

as needed. Also in the general case I consider a special range

\[ h^{-1/3+\delta} \leq \mu \leq h^{-1/3-\delta} \]

4.2.1 I start from the regular points.

**Proposition 4.11.** Assume that there are no resonances of order not exceeding \( M \) and also \( |\nabla (f_1 f_2^{-1})| \geq \epsilon_0 \) on \( \text{supp} \psi \). Further, assume that condition (3.21) with \( q \geq 1 \) is fulfilled and

\[ \mu^{-1} h |\log h| \leq \epsilon \leq \mu h \]

with \( \epsilon = \mu^{-2} \) here.

Then the remainder does not exceed

\[ C \mu^{-3} h + Ch^{-4} (\mu h |\log h|)^{q/2} + \begin{cases} 0 & q \geq 3, \\ \mu^{-2} h^{-2} \epsilon^{(q-1)/2} & 2 \leq q < 3, \\ (\mu^{-2} h^{-2} \epsilon^{(q-1)/2} + \mu h^{-3} \epsilon^{q/2}) & 1 \leq q < 2 \end{cases} \]

while the main term of asymptotics given by (4.4).

**Proof.** The proof follows the sequence of the proofs of propositions 4.15, 4.17 with \( \epsilon = \mu^{-2} \), 4.20(i) from [Ivr8]:

(i) First, using the method of successive approximations, I rewrite the implicit formula as expression (4.4) plus a (temporary) correction term

\[ \sum_i \int \left( \hat{E}^{MW}_Q(x, 0) - E^{MW}_Q(x, 0) \right) dx \]

with an integrand defined by (4.33)-(4.34), [Ivr8]:

\[ \hat{E}^{MW}_Q(x, 0) \overset{\text{def}}{=} \text{const} \sum_{n,p} \left( \theta(A_{pn}) \hat{Q} \right) \left( x'' \xi'' = \Psi^{-1}(x) \right) \times f_1(x) f_2(x) \mu^2 h^{-2} \]
and

\[(4.15) \quad \mathcal{E}^\text{MW}_Q(x, 0) \overset{\text{def}}{=} \text{const} \sum_{n,p} \theta \left( V(x) - (2n + 1)\mu h f_2(x) - (2p + 1)\mu h f_1(x) \right) f_1(x) f_2(x) \bar{Q} \mu^2 h^{-2} \]

where \( Q_i \) cover zone with \( T_0 \geq \epsilon \mu^{-1} \) (i.e. \( \rho \leq C(\mu h | \log h |)^{1/2} \)); here \text{const} meant the same constant as in the definition of \( \mathcal{E}^\text{MW} \).

(ii) Then I need to estimate expression \((4.13)\). Now, however, one needs only analysis which was used in \([\text{Ivr8}]\) in the strictly outer zone. The crucial moment is the estimate of the correction term as in \((4.17)\) of \([\text{Ivr8}]\) which was

\[(4.16) \quad C \mu^2 h^{-2} \times h \rho^{-2} \times \epsilon h^{-1} \times (\rho^2(\mu h)^{-1} + 1) \times (\rho(\mu h)^{-1} + 1) \rho^{q-1} \times C \mu^2 h^{-2} \epsilon (\rho^2(\mu h)^{-1} + 1) \times (\rho(\mu h)^{-1} + 1) \rho^{q-3} \]

and one needs to sum this expression with respect to \( \rho \) ranging from \( \bar{\rho} \) defined as \( C(\mu h | \log h |)^{1/2} \).

Then

(a) As \( q > 3 \) expression \((4.16)\) sums to its value as \( \rho = \bar{\rho} \), which is exactly

\[ C \epsilon(\mu h | \log h |)^{q/2} h^{-4} \]

(b) As \( 1 < q \leq 3 \) an extra term due to summation of \( C \mu^2 h^{-2} \epsilon \rho^{q-3} \) appears in the estimate. As \( q < 3 \) this results in the value of this term as \( \rho = \epsilon^{1/2} \) which is \( C \mu^2 h^{-2} \epsilon (\rho^{-1})^{1/2} \) while as \( q = 3 \) it results in \( C \mu^2 h^{-2} \epsilon \log(\mu h | \log h |) \epsilon^{-1} \).

(c) Further, as \( 1 \leq q \leq 2 \) one should take in account also \( C \mu h^{-3} \epsilon \rho^{q-2} \) which sums to \( C \mu h^{-3} \epsilon^{q/2} \) as \( 1 < q < 2 \) and \( C \mu h^{-3} \epsilon | \log h | \epsilon^{q} \) as \( q = 2 \).

(iii) Finally in zone \( \{ \rho \leq \epsilon^{1/2} \} \) one simply considers its contribution to the remainder with \( T = \epsilon \mu^{-1} \) rather than to the correction:

\[(4.17) \quad C \mu^2 h^{-2} \times (\rho^2(\mu h)^{-1} + 1) \times (\rho(\mu h)^{-1} + 1) \rho^{q-1} \bigg|_{\rho = \epsilon^{1/2}} \approx C \mu h^{-3} \epsilon^{q/2} + C \mu^2 h^{-2} \epsilon^{(q-1)/2} \]

\( \square \)
To cover properly the case of the *superstrong* magnetic field (4.10) let us rewrite the implicit formula as

\[(4.18) \quad \int \hat{E}^\text{MW}_{i}(x,0) \, dx\]

and then as \(q > 3\) rewrite it as the three term decomposition with respect to \(\epsilon\); then the third (remainder) term is \(o(\mu^{-1}h^{-3})\) while the second term is given by a two-dimensional Riemannian sum proportional \(\mu^{-2}\) with the steps \(2f_1\mu h\) and \(2f_2\mu h\). Replacing this Riemannian sum by an integral one can see easily that again with an error \(o(\mu^{-1}h^{-3})\) the second term is \(\propto \mu^{-2}h^{-4}\) which disagrees with expression as \(\mu \leq h^{-1}|\log h|^{-K}\) unless \(\epsilon = 0\). Combining with the previous proposition I arrived to

**Proposition 4.12.** In frames of proposition 4.11 with \(q > 3\) asymptotics with the main part (4.4) and \(O(\mu^{-1}h^{-3})\) remainder holds.

Combining propositions 4.11, 4.12, corollary 4.2 and results of section 2 I conclude that

\[(4.19) \text{in frames of proposition 4.11 in the generic case } q > 3 \text{ estimate } O(\mu^{-1}h^{-3}) \text{ is proven for the complete range of } \mu \quad (1 \leq \mu \leq ch^{-1}).\]

Meanwhile in the general case \(q = 1\) estimate \(O(\mu^2h^{-2})\) is proven as \(h^{-1/2}|\log h|^K \leq \mu \leq ch^{-1}\).

### 4.2.2

Now I want to improve result in the general case.

**Proposition 4.13.** Assume that \(f_1 \neq f_2\) and there are no resonances of order not exceeding \(M\) and also critical points of \(f_1f_2^{-1}\) are non-degenerate on \(\text{supp}\, \psi\). Then as \(\mu \geq h^{-1/3}\) the remainder estimate is given by (4.12) with any \(q < 2\) arbitrarily close to 2 while the main term of asymptotics given by (4.4).

**Proof.** Proof follows ideas of the proof of proposition 4.22 of [Ivr8]. Let us introduce functions \(\ell_k, k = 1, \ldots, K\) in the same proof but with \(\gamma = 1\) and again let \(\tilde{\ell}_k = (\mu h|\log h|^J)^{1/(k+1)}\).

(i) Assume first that \(|\nabla(f_1f_2^{-1})| \geq \epsilon_0\). Then the arguments of the mentioned proof survive with this simplification.

(ii) Assume now that \(f_1f_2^{-1}\) has one non-degenerate stationary point \(\bar{x}\). Assume first that \(\ell_K \leq \gamma\) where \(\gamma = \frac{1}{2}|x - \bar{x}|\). Then the number of indices “\(p\)” does not exceed \(C\ell_k^k/(\mu h\gamma)\) and the contribution of all such elements with \(\ell_k \sim \tilde{\ell}_k\) for some \(k = 1, \ldots, (K - 1)\) to the correction term does not exceed \(C(\mu h)^{1-\delta} \cdot \mu^{-2}h^{-4}\gamma^{-1} \times \gamma^4|\log h|^J\) where \(\gamma^4\) is the total measure of such elements, \(\delta = 1/K\) and \(J\) is large enough. Summation with respect to \(\gamma\) results in the announced estimate.
On the other hand, the total contribution to the correction of all elements with \( \ell_k \geq C_0 \bar{\ell}_k \) does not exceed \( C\mu^{-1}h^{-3} + C\mu h^{-3} \cdot \mu^{-2}\bar{\ell}_k^{-1} \).

(iii) Alternatively, assume that \( \ell_j \leq \gamma \leq \ell_{j+1} \). Again one needs to consider elements with \( \ell_k \approx \bar{\ell}_k \leq \gamma \) for some \( k \leq j \). Then the contribution to the correction term does not exceed \( C\mu h^{-3} \times \mu^{-2}\gamma^{-1} \times \gamma^4 \times \gamma^{-1} \log h \) where the second factor \( \gamma^{-1} \) appears since the relative measure of \( \ell_k \) elements to \( \gamma \) elements does not exceed \( \ell_k \gamma^{-1} \). Again summation with respect to \( \gamma \) results in \( C\mu^{-1}h^{-3} \log h \).

(iv) Finally, consider elements with \( \ell_1 \geq \gamma \). Then I redefine \( \gamma = \ell_1 \) and only \( \ell_1 \approx \bar{\ell}_1 \) should be considered. Contribution of such elements to the correction does not exceed \( Ch^{-4} \times \mu^{-2}\bar{\ell}_1^4 = O(\mu^{-1}h^{-3}) \).

Therefore

\[
\text{(4.20)} \quad \text{In the general case asymptotics with the main part (4.4) and the remainder estimate } O(\mu^{-1}h^{-3} + \mu^2h^{-2}) \text{ holds unless }
\]

\[
\text{(4.21)} \quad h^{-1/3+\delta} \leq \mu \leq h^{-1/3-\delta}
\]

in which case the remainder estimate is \( O(\mu^{-1}h^{-3-\delta}) \) containing an extra factor \( h^{-\delta} \); \( \delta > 0 \) is an arbitrarily small exponent.

4.2.3 Let us recover remainder estimate \( O(\mu^{-1}h^{-3} + \mu^2h^{-2}) \) in the latter case (4.21) (introducing some correction term). Using the same arguments as above I can purge from \( A_{\rho n} \) in (4.14) all terms which are even marginally less than \( \mu^{-2} \); it includes higher order terms and also the difference between \( \mu^{-2}B_{\rho n} \) and \( \mu^{-2}B_{\bar{\rho},\bar{n}} \) where \( \bar{\rho} = \alpha/(2f_1\mu h) \) and \( \bar{n} = (1 - \alpha)/(2f_2\mu h) \), \( \alpha = \alpha(x) \) is the minimizer of \( |\nabla \phi_\alpha|^2 \).

Then \( B_{\rho n} \) becomes \( B_{\bar{\rho},\bar{n}} = \omega(x) \). However, let us include this modified term \( \mu^{-2}B_{\rho n} \) for all \( \rho, n \) and not only for those for which \( T_0 \geq \epsilon \mu^{-1} \). Then one needs to correct this alternation by the term \( \kappa \mu^{-2}h^{-4} \) with \( \kappa \) selected so it would result in the correction term 0, if one replaces the Riemann sum by the corresponding integral because it would provide the result for \( \mu \geq h^{-1/3-\delta} \) and it should agree with the results of the previous subsubsection. I leave the easy details to the reader. Then I arrive to

\[\text{Proposition 4.14. Assume that } f_1 \neq f_2 \text{ and there are no resonances of order not exceeding } M \text{ and also the critical points of } f_1f_2^{-1} \text{ are non-degenerate on } \text{supp } \psi. \text{ Then for } \mu satisfying}\]

\[17^) \text{ Which is the smooth function outside of the critical points of } f_1f_2^{-1}.\]
(4.21) the remainder estimate is $O(\mu^{-1}h^{-3} + \mu^2h^{-2})$ while the main part of the asymptotics is given by

$$\int \left( E^\text{MW}(x, 0) + E^\text{MW}_\text{corr}(x, 0) \right) \psi(x) \, dx. \tag{4.22}$$

with

$$E^\text{MW}_\text{corr}(x, \tau) = \sum_{(p,n) \in \mathbb{Z}^+} \left( \theta(2\tau + V - (2p + 1)\mu h f_1 - (2n + 1)\mu h f_2 - \omega(x)\mu^{-2}) - \right) \left( \theta(2\tau + V - (2p + 1)\mu h f_1 - (2n + 1)\mu h f_2) f_1 f_2 \sqrt{g} + (4\pi)^{-2} \mu^{-2} h^{-4} (2\tau + V) \omega \sqrt{g} \right). \tag{4.23}$$

Remark 4.15. (i) Here

$$\int E^\text{MW}_\text{corr}(x, \tau) \psi(x) \, dx = O(\mu^{-1}h^{-3-\kappa})$$

with an arbitrarily small exponent $\kappa > 0$; (ii) Furthermore under nondegeneracy condition

$$\sum_{|\beta| \leq K} |\nabla^\beta \phi_\alpha| \geq \epsilon_0 \quad \forall x \forall \alpha \in [0, 1] \tag{4.24}$$

with arbitrarily large $K$ one can skip a correction term without deteriorating remainder estimate unless $h^{-1/3} \log h^{-1/3} \leq \mu \leq h^{-1/3} \log h^{1/3}$ and with the remainder estimate $O(h^{-8/3} \log h^4)$ in this border case. I suspect that one can get rid off logarithmic factors and to prove an estimate $O(\mu^{-1}h^{-3} + \mu^2h^{-2})$ even in the border case.

(iii) However I could not find any example demonstrating that this correction term is not superficial and without (4.24) one cannot skip it without penalty. Clarification of this would be interesting.

Correction term in [Ivr8] was not superficial for sure.

---

18) It follows from proposition 4.3
19) Notice that under condition (4.24) $\ell_K \asymp 1$ in the proof of propositions 4.3, 4.13.
4.2.4 Now I want to attack resonances. I start from the generic case. First, after rescaling again in the same manner as before one can see easily that the contribution of zone \(|x_1 \sim \gamma|\) to an error does not exceed \(C \mu h^{1-d} \varepsilon^{1-\kappa}\) where factor \(\gamma^{-1}\) appearing from the calculation of the number of indices "\(p"\) is compensated by factor \(\gamma\) appearing from the measure. Then this contribution does not exceed \(C \mu h^{-3}\(\mu^4-2m\gamma^{-1}\)(1-\kappa)\) if only terms originated from non-diagonal terms are removed and summation with respect to \(\gamma \geq \bar{\gamma}\) results in the value of this expression as \(\gamma = \bar{\gamma}\) and as \(m \geq 4\) it is \(O(\mu^{-1}h^{-3} + \mu^2 h^{-2})\). I remind that \(\bar{\gamma} = \mu^{\delta-2}\) as \(m = 4\) and \(\bar{\gamma} = C\mu^{-2}\) as \(m \geq 5\).

On the other hand in the zone \(|x_1| \leq \bar{\gamma}|\) one can apply proposition 3.10 and get \(C(\mu h)^{3/2}h^{-4}\bar{\gamma} |\log h|^J\) which is \(o(\mu^{-1}h^{-3} + \mu^2 h^{-2})\) as well (as \(m \geq 4\)).

Now after singular terms from operator are removed it can be treated as if there was no resonance resulting in two following statements:

**Proposition 4.16.** Assume that \(f_1 \neq f_2\) and \(|\nabla(f_1f_2^{-1})| \geq \epsilon_0\) on \(\text{supp} \psi\). Furthermore, assume that there are no third-order resonances on \(\text{supp} \psi\).

Then

(i) As \(\mu \geq h^{-1/3-\delta}\) the remainder estimate is \(O(\mu^2 h^{-2})\) while the main term of asymptotics given by (4.4);

(ii) As \(h^{-1/3+\delta} \leq \mu \leq h^{-1/3-\delta}\) statement (i) of remark 4.15 holds; further under condition (4.24) one can skip correction term with \(O(\mu^2 h^{-2} |\log h|^J)\) penalty\(^{20}\).

4.2.5 Consider resonances of order 3 now. As \(\mu \geq h^{-1/3-\delta}\) in the virtue of the above arguments, the contribution of zone

\[
\{ |x_1| \geq \bar{\gamma}_3 \text{ def } = \mu^{-3} h^{-1-\delta} \}
\]

(4.25)

to the correction does not exceed \(C \mu^2 h^{-2}\).

Moreover, contribution of zone

\[
\{ |x_1| \leq \bar{\gamma}_2 \text{ def } = (\mu h)^{1/2} |\log h|^{-1/2} \}
\]

(4.26)

to the correction also does not exceed \(C \mu^2 h^{-2}\). However, zone

\[
\{ \bar{\gamma}_2 \leq |x_1| \leq \bar{\gamma}_3 \}
\]

(4.27)

needs to be reexamined; here \(\bar{\gamma}_2 \geq \bar{\gamma}\) as \(\mu \geq h^{-1/3-\delta}\) and as \(\mu \geq h^{-3/7-\delta'}\) zone (4.27) disappears. In virtue of arguments of subsection 4.2 the contribution of this zone does not exceed \(C \mu^2 h^{-2-\kappa}\) with arbitrarily small \(\kappa > 0\) and now I want to improve it marginally.

I claim that

\(^{20}\) Which is probably superficial.
Proposition 4.17. For any $\delta > 0$ there exists $K = K(\delta)$ such that if $\mu^{-1+\delta} \leq \gamma \leq \mu^{-\delta}$ and $B(y, \gamma(y))$ with $\gamma(y) \asymp \gamma$ is rescaled to $B(0,1)$ then either

\begin{equation}
\sum_{|\alpha| \leq K} |\partial^\alpha_\gamma A_{pn}| \asymp \varsigma \quad \forall z \in B(0,1)
\end{equation}

with some $\varsigma$ or

\begin{equation}
\sum_{|\alpha| \leq K} |\partial^\alpha_\gamma A_{pn}| \leq \varsigma \quad \forall z \in B(0,1)
\end{equation}

with $\varsigma = \mu h^{1-\kappa_1} \gamma$ where (continuous) parameters $p$ and $n$ are selected so

\begin{equation}
A_{pn}(y) = \partial_{x_1} A_{pn}(y) = 0.
\end{equation}

Proof. Note that before rescaling $A_{pn} = A_{pn}^0 + x_1^{-1} \mu^{-2} B_{pn} + ...$ with smooth symbols. Then decomposing all smooth symbols into Taylor series at $y$ one can prove proposition easily since high powers of there contain high powers of $\gamma$ and thus are small.

Proposition 4.18. In frames of proposition 4.17 contribution of zone $\{\mu^{-1+\delta} \leq |x_1| \leq \mu^{-\delta}\}$ to the error estimate estimate is $O(\mu^{-1} h^{-3} + \mu^2 h^{-2} |\log h|)$.

Proof. Proof repeats those as I had before. One needs to construct $\ell_k$ corresponding to $\varsigma A_{pn}$ and then $(\mu h |\log h|)$ as minimal value for $\ell_k^{k+1}$ should be replaced by $\varsigma^{-1}(\mu h |\log h|)$ and there will be no final division by $\ell_K$ which would be $\asymp 1$. That will give $C \mu^2 h^{-2} \times |\log h| \times |\log h| \gamma^{-1} \times \gamma$ where the factors $\gamma^{-1}$ and $\gamma$ appear from division by $(\mu h \gamma)$ and the measure.

Then summation with respect to $\gamma$ results in an extra $|\log h|$ factor.

Remark 4.19. (i) Under condition

\begin{equation}
\sum_{|\beta| \leq K} |\nabla^\beta Vx^{-1}| \geq \epsilon_0 \quad \forall x
\end{equation}

with arbitrarily large $K$ one can skip a correction term without deteriorating remainder estimate unless $h^{-1/3} |\log h|^{-J} \leq \mu \leq h^{-1/3} |\log h|^J$ and with the remainder estimate $O(h^{-8/3} |\log h|^J)$ in this border case. I suspect that one can get rid off logarithmic factors and to prove an estimate $O(\mu^{-1} h^{-3} + \mu^2 h^{-2})$ even in the border case.

(ii) One can easily construct a correction term in the case of third-order resonances but an expression seems rather too complicated. So I leave it to the curious reader.
4.2.6 Let us consider the generic case now:

**Proposition 4.20.** Assume that that $|f_1 - f_2| \geq \epsilon_0$ and also $|\nabla(f_1 f_2^{-1})| \geq \epsilon_0$ on $\text{supp} \psi$. Further let us assume that conditions (3.21)$_4$ and (4.7)$_3$ are fulfilled.

Then the remainder is $O(\mu^{-1}h^{-3})$ while the main term of asymptotics given by (4.4).

*Proof.* Again it is sufficient to consider the case of the single resonance surface $\Xi = \{x_1 = 0\}$.

(i) Combining in zone $\{|x| \propto \gamma \geq \tilde{\gamma} \overset{\text{def}}{=} \mu^{\delta - 1}\}$ arguments of the proofs of proposition 4.6 and 4.11 one can estimate contribution of it to the correction terms by $C(\mu h |\log h|)^{9/2} h^{-4} \gamma^{2} \times \mu^{-2} \gamma^{-1}$ which after summation with respect to $\gamma$ results in $C(\mu h |\log h|)^{9/2} \mu^{-2} h^{-4} |\log h|$, which in turn is $O(\mu^{-1}h^{-3})$ as long as $\mu \leq h^{-1} |\log h|^{-J}$.

(ii) In zone $\{|x| \leq \tilde{\gamma}\}$ one can use arguments of the proofs of proposition 4.7 and estimate the contribution of this zone to the correction term arising as $T \geq T_0$ is replaced by $T = \epsilon \mu^{-1}$ by $C(\mu h |\log h|)^{5/2} h^{-4} \gamma^{2}$ which is $O(\mu^{-1}h^{-3})$ as $\mu \leq h^{-3/5+\delta'}$.

(iii) Let $\mu \geq h^{-5/3+\delta'}$. In zone $\{|x| \leq \tilde{\gamma}\}$ one can consider precanonical form and then the contribution of this zone to the correction term does not exceed $C(\mu h |\log h|)^{3/2} \mu^{-1} h^{-4} \gamma^{2}$ which is $O(\mu^{-1}h^{-3})$ as $\mu \leq h^{-1+\delta}$.

The direct calculation shows that the first approximation term actually vanishes as it comes from the “main perturbation term” and is odd with respect to $x_1$, and its estimate contains an extra factor $\gamma$ otherwise.

So the correction terms associated with zone $\{|x| \leq \tilde{\gamma}\}$ do not exceed

$$C(\mu h)^{3/2} \mu^{-1} h^{-4} \gamma^{2} + C(\mu h)^{1/2} \mu^{-2} h^{-4} \gamma^{2} = o(\mu^{-1}h^{-3})$$

(iv) Finally, arguments of (i) should be slightly improved as $\mu \geq h^{-1} |\log h|^{-J}$. Namely the source of term containing logarithmic factor is the only perturbation of the type $C \mu^{-2} x_1^{-1} B(x', \mu^{-1} h D)$ but then if $\psi$ is even with respect to $x_1$ the results of calculation will be 0 and if $\psi$ contains factor $x_1$ it would compensate $x_1^{-1}$ and no logarithm would appear. \hfill \Box

4.2.7 Now I need to consider the vicinity of $\Sigma$. Let us start from the general case first.

Then scaling $x \to x/\gamma$, $h \to h/\gamma$, $\mu \to \mu \gamma$ and applying the same arguments as in the resonance case I estimate the contribution of zone $\{\gamma(x) \propto \gamma\}$ to the correction by

$$C \mu h^{-3} \gamma^{-1} \times (\mu^{-2} \gamma^{-1})^{1-\kappa} \times \gamma^{2}$$

which in comparison to resonance case gains an extra factor $\gamma$ and thus sums to its values as $\gamma = 1$, which is $C \mu^{-1+\kappa} h^{-3}$ which in turn is $O(\mu^2 h^{-2})$ as $\mu \geq h^{-1/3-\delta'}$. I ignore again term which sums to $C \mu^2 h^{-2}$ in the end of the day.
Meanwhile contribution of zone \( \{ \gamma(x) \leq \tilde{\gamma} \} \) does not exceed \( C \mu h^{-3} \tilde{\gamma}^2 = O(\mu^2 h^{-2}) \).

Thus I arrive to the following statement:

**Proposition 4.21.** Statement of proposition 4.3 remain true in the vicinity of \( \Sigma \).

**Remark 4.22.** (i) Under condition

\[
\sum_{|\beta| \leq K} |\nabla_x^\beta \Phi^{-1}| \geq \epsilon_0 \quad \forall x
\]

with arbitrarily large \( K \) one can skip a correction term without deteriorating remainder estimate unless \( h^{-1/3} |\log h|^{-J} \leq \mu \leq h^{-1/3} |\log h|^{J} \) and with the remainder estimate \( O(h^{-8/3} |\log h|^4) \) in this border case. I suspect that one can get rid off logarithmic factors and to prove an estimate \( O(\mu^{-1} h^{-3} + \mu^2 h^{-2}) \) even in the border case.

(ii) One can easily construct a correction term in the case \( \Sigma \neq \emptyset \) resonances but an expression seems rather too complicated. So I leave it to the curious reader.

4.2.8 Finally let us consider the generic case near \( \Sigma \). Using the same arguments as before I arrive to

**Proposition 4.23.** Under non-degeneracy conditions (3.84) and (3.85) for \( \psi \) supported in the vicinity of \( \Sigma \), the remainder estimate is \( O(\mu^{-1} h^{-3}) \) while the main part is given by the magnetic Weyl formula.

5 Vanishing \( V \) Case

5.1 Generic Case

In the generic case, as \( V = 0, |\nabla V| \geq \epsilon \) and then the microhyperbolicity condition of [Ivr3] holds and then

**Proposition 5.1.** Assume that

\[
|V| + |\nabla V| \geq \epsilon.
\]

Then in any dimension without no condition to \( F_{jk} \) other than \( |\det(F_{jk})| \geq \epsilon \) for \( \psi \) supported in the small vicinity of

\[
\Delta \overset{\text{def}}{=} \{ x : V(x) = 0 \}
\]

the asymptotics with the main part given by Magnetic Weyl expression and the remainder estimate \( O(\mu^{-1} h^{1-d}) \) holds.

This covers the generic case completely.
5.2 General Case

The general case however is more complicated. Let us introduce a scaling function

\[ \ell = \epsilon(|V| + |\nabla V|^2)^{1/2} + \frac{1}{2}\bar{\ell}, \quad \bar{\ell} = \epsilon_0 \max((\mu h)^{1/2}, \mu^{-1}) \]

with dominating the second and the first terms in \( \bar{\ell} \) as \( \mu \leq h^{-1/3} \) and \( \mu \geq h^{-1/3} \) respectively.

Let us apply scaling \( x \mapsto x\ell, \quad h \mapsto h\ell^{-2}, \quad \mu \mapsto \mu, \quad V \mapsto V\ell^{-2} \):

- Let \( \ell \leq \epsilon_0(\mu h)^{1/2} \). Then I am in the classically forbidden zone and the contributions of \( \ell \)-element to the principal part and the remainder estimate are 0 and negligible respectively.

- Let \( \ell \geq \bar{\ell} \) and \( |\nabla V| \approx \ell \). Then after rescaling I am in frames of subsection 5.1 and the contribution of \( \ell \)-element to the remainder is \( O(\mu^{-1}h^{-3}\bar{\ell}^6) \) and the total contribution of all \( \ell \)-elements to the remainder is \( O(\mu^{-1}h^{-3}\bar{\ell}^2) \) and I am done here.

- Also, the contribution of all elements with \( \ell \approx \mu^{-1} \) to the remainder is \( O(\mu h^{-3}\bar{\ell}^2) \) which is \( O(\mu^{-1}h^{-3}) \). So case \( \ell \approx \bar{\ell} \) is completely covered.

- Let us consider \( V \approx \ell^2, \quad \ell \geq \bar{\ell} \); then condition \( V \approx 1 \) is recovered after rescaling but now conditions to \( F_{jk} \) could fail; actually these condition do not fail completely, but are replaced by somewhat weaker condition with extra factors \( \ell \) or \( \ell^2 \) in the estimates from below. However in the weak magnetic field case this weakened condition is enough; I leave the details to the reader:

**Proposition 5.2.** As \( F_{jk} \) is generic and \( \mu \leq h^{-\delta} \) the remainder estimate \( O(\mu^{-1}h^{-3}) \) holds.

Now one can assume that \( h^{-\delta} \leq \mu \leq ch^{-1} \). Then since in section 3 no condition \( "|V| \geq \epsilon" \) was required, the remainder estimate is \( O(\mu^{-1}h^{-3} + \mu^2h^{-2}) \) but the principal part is given by the implicit formula (0.9) rather than by the magnetic Weyl expression (4.4) and now I need to modify arguments of section 4 to pass from (0.9) to (4.4).

Note that in this implicit formula one always can take \( T_0 = \epsilon\mu^{-1} \) with an arbitrarily small constant \( \epsilon > 0 \). Then in the virtue of arguments of the proof of propositions 4.3, 4.4 and 4.8 the remainder estimate \( O(\mu^{-1}h^{-3} + \mu^2h^{-2-\delta}) \) with an arbitrarily small exponent \( \delta > 0 \). However in contrast to the analysis in subsection 4.1, without condition \( "|V| \geq \epsilon" \) it does not translates into \( T_0 = Ch|\log h| \) but rather into \( T_0 = Ch\ell^{-2}|\log h| \) after the above partition is applied. However this last implicit formula (0.9) translates (with the same error) into Weyl or Magnetic Weyl formula. Therefore **Theorem 0.2 with correction term 0 is proven for \( \mu \leq h^{-1/3-\delta} \) with an arbitrarily small exponent \( \delta > 0 \).**
The similar arguments work as $\mu \geq h^{-1/3-\delta'}$ since one again refers to any $T_0 \leq \epsilon \mu^{-1}$. So,

**Theorem 5.3.** Both theorem 0.2 and theorem 0.3

**References**

[BrIvr] M. Bronstein, V. Ivrii. *Sharp Spectral Asymptotics for Operators with Irregular Coefficients. Pushing the Limits*, Comm. Partial Differential Equations, **28** (2003) 1&2, 99–123.

[Dim] M. Dimassi. *Développements asymptotiques de l’oprateur de Schrödinger avec champ magnétique fort*, Comm. Partial Differential Equations, **26** (2001) 3&4, 595–627.

[Ivr1] V. Ivrii. *Microlocal Analysis and Precise Spectral Asymptotics*, Springer-Verlag, SMM, 1998, xv+731.

[Ivr2] V. Ivrii. *Sharp Spectral Asymptotics for operators with irregular coefficients. II. Boundary and Degenerations*, Comm. Partial Differential Equations, **28** (2003) 1&2, 125–156.

[Ivr3] V. Ivrii. *Sharp spectral asymptotics for operators with irregular coefficients. III. Schrödinger operator with a strong magnetic field*, (to appear).

[Ivr4] V. Ivrii. *Sharp spectral asymptotics for operators with irregular coefficients. IV. Multidimensional Schrödinger operator with a strong magnetic field. Full-rank case*, (to appear).

[Ivr5] V. Ivrii. *Sharp spectral asymptotics for operators with irregular coefficients. V. Multidimensional Schrödinger operator with a strong magnetic field. Non-full-rank case*, (to appear).

[Ivr6] V. Ivrii. *Sharp spectral asymptotics for two-dimensional Schrödinger operator with a strong degenerating magnetic field*, (to appear).

[Ivr7] V. Ivrii. *Sharp spectral asymptotics for two-dimensional Schrödinger operator with a strong degenerating magnetic field. II*, (to appear).

[Ivr8] V. Ivrii. *Sharp spectral asymptotics for four-dimensional Schrödinger operator with a strong degenerating magnetic field*, (to appear).
REFERENCES

[Ivr9] V. Ivrii. *Sharp spectral asymptotics for 2-dimensional Schrödinger operator with a strong magnetic field. note about forgotten generic case.*, (to appear).

[Ma] J. Martinet, Sur les singularites des formes differentielles, *Ann. Inst. Fourier*, 20 (1970), 1, 95-178.

[Rou] R. Roussarie. *Modèles locaux de champs et de forms* Astérisque, 30 (1975) 3–179.

Department of Mathematics,
University of Toronto,
40, St.George Str.,
Toronto, Ontario M5S 2E4
Canada
ivrii@math.toronto.edu
Fax: (416)978-4107