THE RELAXED MAXIMUM PRINCIPLE FOR G-STOCHASTIC CONTROL SYSTEMS WITH CONTROLLED JUMPS

H. Ben Gherbal¹, A. Redjil, and O. Kebiri

ABSTRACT. This paper is concerned with optimal control of systems driven by G-stochastic differential equations (G-SDEs), with controlled jump term. We study the relaxed problem, in which admissible controls are measure-valued processes and the state variable is governed by a G-SDE driven by a counting measure valued process called relaxed Poisson measure such that the compensator is a product measure. Under some conditions on the coefficients, using the G-chattering lemma, we show that the strict and the relaxed control problems have the same value function. Additionally, we derive a maximum principle for this control problem.

1. INTRODUCTION

We consider a stochastic control problem where the state variable is a solution of a SDE driven by a $G$-Brownian motion with jumps, the control enters both the drift and the jump term. More precisely the system evolves according to the SDE

¹corresponding author

2020 Mathematics Subject Classification. 93E20, 60H07, 60H10, 60H30.
Key words and phrases. Relaxed optimal control, G-Brownian motion, sublinear expectation, stochastic control, relaxed maximum principle, jump process.
Submitted: 07.12.2022; Accepted: 22.12.2022; Published: 24.12.2022.
\[ dx_t = b(t, x_t, u_t)dt + \sigma(t, x_t)dB_t + \gamma(t, x_t, u_t)d\langle B \rangle_t + \int_0^t f(t, x_{t-}, \theta, u_t)\tilde{N}(dt, d\theta), \]

\[ x_0 = x, \]

on some space of sublinear expectation \((\Omega, H, \mathbb{E}, F_P)\), where \(F_P\) is the universal filtration, with \(P\) is a tight family of possibly mutually singular probability measures, and \(b, \sigma, \gamma, f\) are given deterministic functions, \(u\) is the control process. We consider here an independent Poisson random measure \(N\); whose compensator is given by \(v(d\theta)dt\).

The expected cost to be minimized over the class of admissible controls is defined by:

\[ J(x; u) = \sup_{P \in P} \mathbb{E}^P \left[ g(x_T) + \int_0^T h(t, x_t, u_t)dt \right] = \mathbb{E} \left[ g(x_T) + \int_0^T h(t, x_t, u_t)dt \right], \]

where \(x\) is the initial condition of the process \((x_t)_{t \in [0, T]}\).

We defined then the value function \(V\) by:

\[ V(x) := \inf_{u \in U} J(x; u), \]

where \(U\) is the set of admissible controls, a control process that verify \((1.3)\) is called optimal.

In the recent years the framework of G-expectation has found increasing application in the domain of finance and economics, e.g., Epstein and Ji [16,17] study the asset pricing with ambiguity preferences, Beissner [5] who studies the equilibrium theory with ambiguous volatility, and many others see e.g [6,48,49], also see [25–27] for numerical methods. The motivation is that many systems are subject to model uncertainty or ambiguity due to incomplete information, or vague concepts and principles. Aspects of model ambiguity such as volatility uncertainty have been studied by Peng (2007, 2008, 2010, [39–41]) who introduced a sublinear expectation with a process called \(G\)-Brownian motion, also by Denis and Martini [13] who suggested a structure based on quasi-sure analysis from abstract potential theory to construct a similar structure using a tight family \(P\) of possibly mutually singular probability measures.
The strict control problem may fail to have an optimal solution, if we don’t impose some kind of convexity assumption. In this case, we must embed the space of strict controls into a larger space that has nice properties of compactness and convexity. This space is that of probability measures on $A$, where $A$ is the set of values taken by the strict control. These measure valued processes are called relaxed controls. In the classical framework, the first existence result of an optimal relaxed control is proved by Fleming [18], for the SDEs with uncontrolled diffusion coefficient and no jump term. For such systems of SDEs a maximum principle has been established in [2,3,34]. The case where the control variable appears in the diffusion coefficient has been solved in [14]. The existence of an optimal relaxed control of SDEs, where the control variable enters in the jump term was derived by Kushner [31], also recently the work given by H. Ben Gherbal and B. Mezerdi [7] of relaxed stochastic maximum principle in optimal control of diffusions with controlled jumps, for which the state variable is governed by a SDE driven by a counting measure valued process called relaxed Poisson measure, where the existence of an optimal relaxed control and a Pontryagin maximum principle were proved.

In the $G$-framework, the existence of an optimal relaxed control is established in 2018, by Redjil and Choutri [42], where a stochastic differential equation is considered without jump term and an uncontrolled diffusion coefficient, the celebrated Chattering lemma was generalized in the $G$-framework and the existence of relaxed optimal control was proved. The same result in the case with jump term is recently proved by A. Redjil, H. Ben Gherbal and O. Kebiri [43].

In this paper, we establish a Pontryagin maximum principle for the relaxed control problem given by (2.1) and (2.2). More precisely we derive necessary conditions for optimality satisfied by an optimal control. The proof is based on the results obtained in [43], Pontryagin’s maximum principle for nearly optimal strict controls and some stability results of trajectories and adjoint processes with respect to the control variable. In our case the diffusion coefficient is uncontrolled because our control set is not convex, by this the controlled case require more work and two-adjoint equation. We let this case as a future work.

The motivation of our work came from applications in finance when a jump process models the stock price where we can’t estimate exactly the coefficients
The noise coefficient will produce a G-SDE with jump, if we want to control this dynamic, this leads to a controlled G-SDE with jump.

The rest of the paper is organized as follows: in the next section, we formulate the control problem, and introduce the assumptions of the model. Section 3 is devoted to the proof of the approximation and stability results. In the last section, we state and prove a maximum principle for our relaxed control problem, which is the main result of this paper.

2. Formulation of the Problem

2.1. G-Strict control problem. We consider a control problem of systems governed by stochastic differential equations on some sublinear expectation space \((\Omega, H, \mathbb{E}, F^P)\), such that \(F^P\) the universal filtration defined by \(F^P = \{ \mathcal{F}^P_t \}_{t \geq 0}\), where \(\mathcal{F}^P_t := \bigcap_{P \in \mathbb{P}^c} (\mathcal{F}^P_t \vee \mathcal{N}_P)\) for \(t \geq 0\), such that \(\mathcal{F}^P_t := \mathcal{F}^P_t \vee \mathcal{N}^P(\mathcal{F}^P_t)\) is the right continuous \(\mathbb{P}\)-completed filtrations generated by a \(G\)-Brownian motion \(B\) and an independent Poisson measure \(N\), with compensator \(\nu(d\theta)dt\), and \(\mathcal{N}^P := \bigcap_{P \in \mathbb{P}} \mathcal{N}^P(\mathcal{F}_\infty)\) the family of polar sets, where \(\mathcal{N}^P(\mathcal{F}_\infty)\) is the \(\mathbb{P}\)-negligible sets. The jumps are confined to a compact set \(\Gamma\), and set

\[
\tilde{N}(dt, d\theta) = N(dt, d\theta) - \nu(d\theta)dt.
\]

Consider a compact set \(A\) in \(\mathbb{R}^k\) and let \(\mathcal{U}\) the class of measurable, adapted processes \(u : [0; T] \times \Omega \rightarrow A\), such that \(u \in M^2_G(0, T)\). For any \(u \in \mathcal{U}\), we consider the following stochastic differential equation (SDE)

\[
\begin{cases}
    dx_t = b(t, x_t, u_t)dt + \sigma(t, x_t)dB_t + \gamma(t, x_t, u_t)\tilde{N}(dt, d\theta) \\
    x_0 = x,
\end{cases}
\]

where

\[
\begin{align*}
    b : [0; T] \times \mathbb{R}^n &\times A \rightarrow \mathbb{R}^n \\
    \sigma : [0; T] \times \mathbb{R}^n &\rightarrow M_{n \times d}(\mathbb{R})
\end{align*}
\]
\[ \gamma: [0; T] \times \mathbb{R}^n \times A \rightarrow \mathbb{R}^{n \times n} \]

\[ f: [0; T] \times \mathbb{R}^n \times \Gamma \times A \rightarrow \mathbb{R}^n \]

are bounded, measurable and continuous functions.

The expected cost is given by

\[ J(u) = \sup_{\mathcal{P} \in \mathcal{P}} \mathbb{E}^\mathcal{P} \left[ g(x_T) + \int_0^T h(t, x_t, u_t) dt \right] = \mathbb{E}^\mathcal{E} \left[ g(x_T) + \int_0^T h(t, x_t, u_t) dt \right], \]

where,

\[ g: \mathbb{R}^n \rightarrow \mathbb{R} \]

\[ h: [0; T] \times \mathbb{R}^n \times A \rightarrow \mathbb{R}, \]

be bounded and continuous functions.

The problem is to minimize the functional \( J(.) \) over \( \mathcal{U} \). A control that solves this problem is called optimal.

We note that from the result of [35], equation (2.1) has a unique solution, under these assumptions (A):

(A1) Let be \( b, \sigma, \gamma \) and \( f \) bounded and lipschitz continuous with respect to the state variable \( x \) uniformly in \((t, u)\), also we suppose that \( \gamma(t, x, .) \) is a symmetric \( d \times d \) matrix with each element.

(A2) For all \((t, x, \theta) \in [0, T] \times \mathbb{R}^n \times \Gamma\) the functions \( b(t, x, .), f(t, x, \theta, .) \) and \( \gamma(t, x, .) \) are continuous in \( u \in \mathcal{U} \).

(A3) \( b(., x, .), \gamma(., x, .) \) and \( \sigma(., x) \) taking value in \( M^2_G(0, T) \) and \( f(., x, ., .) \) takes value in \( \hat{H}^2_G(0, T) \).

(A4) The functions \( g \) and \( h(., x, .) \) are taking value in \( M^2_G(0, T) \) and bounded. Moreover we suppose that \( g \) is lipschitz continuous, and \( h \) is lipschitz continuous with respect to the state variable \( x \) uniformly in time and control \((t, u)\).

2.2. \textbf{The G-relaxed control problem.} Let \((A, d)\) be a separable metric space and \( \mathcal{P}(A) \) be the space of probability measures on the set \( A \) endowed with its Borel \( \sigma \)-algebra \( \mathcal{B}(A) \). The class \( M([0, T] \times A) \) of relaxed controls we consider in this paper is a subset of the set \( M([0, T] \times A) \) of Radon measures \( \nu(dt, da) \) on \([0, T] \times A\) equipped with the topology of stable convergence of measures, whose projections
on \([0, T]\) coincide with the Lebesgue measure \(dt\), and whose projection on \(A\) coincide with some probability measure \(\mu_t(da) \in \mathcal{P}(A)\) i.e. \(\nu(da, dt) := \mu_t(da)dt\). The topology of stable convergence of measures is the coarsest topology which makes the mapping
\[
q \mapsto \int_0^T \int_A \varphi(t, a)q(dt, da)
\]
continuous, for all bounded measurable functions \(\varphi(t, a)\) such that for fixed \(t\), \(\varphi(t, \cdot)\) is continuous. Equipped with this topology, \(M := M([0, T] \times A)\) is a separable metrizable space. Moreover, it is compact whenever \(A\) is compact. The topology of stable convergence of measures implies the topology of its weak convergence. For further details see \([14,15]\).

Now we present the following definitions:

**Definition 2.1.** \(\text{Lip}(\Omega)\) is the set of random variables of the form
\[
\xi := \varphi(B_{t_1}, B_{t_2}, \ldots, B_{t_n})
\]
for some bounded Lipschitz continuous function \(\varphi\) on \(\mathbb{R}^{d \times n}\) and \(0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq T\). The coordinate process \((B_t, t \geq 0)\) is called \(G\)-Brownian motion whenever \(B_1\) is \(G\)-normally distributed under \(\mathbb{E}\) and for each \(s, t \geq 0\) and \(t_1, t_2, \ldots, t_n \in [0, t]\) we have
\[
\mathbb{E}[\varphi(B_{t_1}, \ldots, B_{t_n}, B_{t+s} - B_t)] = \mathbb{E}[\psi(x_1, \ldots, x_n, \sqrt{s}B_1)],
\]
where \(\psi(x_1, \ldots, x_n) = \mathbb{E}[\varphi(x_1, \ldots, x_n, \sqrt{s}B_1)]\). This property implies that the increments of the \(G\)-Brownian motion are independent and that \(B_{t+s} - B_t\) and \(B_s\) are \(N(0, s\Sigma)\)-distributed.

Next, we introduce the class of relaxed stochastic controls on \((\Omega_T, \mathcal{H}, \mathbb{E})\), where \(\mathcal{H}\) is a vector lattice of real functions on \(\Omega\) such that \(\text{Lip}(\Omega_T) \subset \mathcal{H}\).

**Definition 2.2.** A relaxed stochastic control on \((\Omega_T, \text{Lip}(\Omega_T), \mathbb{E})\) is a random measure \(q(\omega, dt, da) = \mu_t(\omega, da)dt\) such that for each subset \(C \in \mathcal{B}(A)\), the process \((\mu_t(C))_{t \in [0, T]}\) is \(F^p\)-progressively measurable i.e. for every \(t \in [0, T]\), the mapping \([0, t] \times \Omega \rightarrow [0, 1]\) defined by \((s, \omega) \mapsto \mu_s(\omega, A)\) is \(B([0, t]) \otimes \mathcal{F}^t\)-measurable. In particular, the process \((\mu_t(C))_{t \in [0, T]}\) is adapted to the universal filtration \(F^p\). We denote by \(\mathcal{R}\) the class of relaxed stochastic controls.

The set \(\mathcal{U}([0, T])\) of strict controls constituted of \(F^p\)-adapted processes \(u\) taking values in the set \(A\), embeds into the set \(\mathcal{R}\) of relaxed controls through the
mapping:
\[ \Phi : \mathcal{U}([0, T]) \ni u \mapsto \Phi(u)(dt, da) = \delta_{u_t}(da)dt \in \mathcal{R}. \]

**Definition 2.3.** Let \( \mu \) a relaxed representation of an admissible control \( u \), for each \( \Gamma_0 \subset \Gamma \), \( \Gamma_0 \) is a Borel set \( (\Gamma_0 \in \mathcal{B}(\Gamma)) \) and \( A_0 \subset A \), \( (A_0 \in \mathcal{B}(A)) \), we define:
\[
N^\mu([0, t] \times A_0, \Gamma_0) := \int_0^t \int_{\Gamma_0} 1_{A_0}(u_s)N(ds, d\theta).
\]

\( N^\mu \) is the number of jumps of \( R_{t_0}R_{\Gamma_0}N_{t_0}(ds, d\theta) \) on \([0, t]\) with values in \( \Gamma_0 \) and where \( u_s \in A_0 \) at the jump times \( s \).

Since \( 1_{A_0}(u_s) = \mu_s(A_0) \); then the compensator of the counting measure valued process \( N^\mu \) is
\[
v(d\theta)\mu_t(da)dt = \mu_t \otimes v(da, d\theta).
\]

**Definition 2.4.** A relaxed Poisson measure \( N^\mu \) is a counting measure valued process such that its compensator is the product measure of the relaxed control \( \mu \) with the compensator \( v \) of \( N \) such that for each \( \Gamma_0 \subset \Gamma \), \( \Gamma_0 \) is a Borel set \( (\Gamma_0 \in \mathcal{B}(\Gamma)) \) and \( A_0 \subset A \), \( (A_0 \in \mathcal{B}(A)) \), the processes
\[
Z^\mu = \tilde{N}^\mu(t, A_0, \Gamma_0) = N^\mu(t, A_0, \Gamma_0) - \mu(t, A_0)\nu(\Gamma_0)
\]
are \( \mathcal{F}_t^P \)-martingales and orthogonal for disjoint \( \Gamma_0 \times A_0 \), because according to [7], the processes \( Z^\mu \) are \( \mathcal{F}_t^P \)-martingales for each \( \mathbb{P} \in \mathcal{P} \), and so is an \( \mathcal{F}_t^P \)-martingale, also are orthogonal for disjoint \( \Gamma_0 \times A_0 \).

**Proposition 2.1.** For any bounded measurable function \( \varphi \) with real values, the process \( Y \) given by:
\[
Y_t = \int_0^t \int_{\Gamma} \int_A \varphi(s, x_{s-}, \theta, a)N^\mu(dt, d\theta, da) - \int_0^t \int_{\Gamma} \int_A \varphi(s, x_{s-}, \theta, a)v(d\theta)\mu_s(da)ds
\]
is an \( \mathcal{F}_t^P \)-martingale.

**Proof.** By the definition of \( G \)-martingale [41] \( Y \) is an \( \mathcal{F}_t^P \)-martingale means that \( Y \) is an \( \mathcal{F}_t^P \)-supermartingale for each probability sufficiently. And this is verified in our case because from H. Ben Gherbal and B. Mezerdi [7] the process \( Y \) is an \( \mathcal{F}_t^P \)-martingale for each \( \mathbb{P} \in \mathcal{P} \).

**Proposition 2.2.** Consider a sequence of \( \mu^\mu_n \) converging weakly to \( \mu_s \otimes \nu \) on \( \Omega \times [0, T] \times A \times \Gamma \), there exists a sequence of orthogonal martingale measures \( \tilde{N}^\mu_n \)
defined on $\Omega \times [0, T] \times A \times \Gamma$, such that for each bounded function $\varphi$:

$$
\int_0^t \int_A \int_\Gamma \varphi(s, X^\mu_x, \theta, a) \tilde{N}^\mu(ds, d\theta, da)
\rightarrow_{n \to \infty} \int_0^t \int_A \int_\Gamma \varphi(s, X^\mu_x, \theta, a) \tilde{N}^\mu(ds, d\theta, da) \text{ quasi-surely.}
$$

**Proof.** Given a fixed probability measure, we have from [7]

$$
P \in \mathcal{P} \Rightarrow \int_0^t \int_A \int_\Gamma \varphi(s, x^\mu_x, \theta, a) \tilde{N}^\mu(ds, d\theta, da)
\rightarrow \int_0^t \int_A \int_\Gamma \varphi(s, x^\mu_x, \theta, a) \tilde{N}^\mu(ds, d\theta, da) \mathbb{P}\text{-surely,}
$$

this means that we have the convergence outside a polar set, which means that we have quasi surely convergence. \qed

Now we present our relaxed controlled system, the $G$-SDE with controlled jumps in terms of relaxed Poisson measure is given by:

$$
\begin{align*}
\left\{ \begin{array}{l}
\frac{dx_t^\mu}{dt} = \int_A b(t, x_t^\mu, a) \mu_t(da)dt + \sigma(t, x_t^\mu) dB_t + \int_A \gamma(t, x_t^\mu, a) \mu_t(da) d\langle B \rangle_t \\
+ \int_A \int_\Gamma f(t, x_t^\mu, \theta, a) \tilde{N}^\mu(dt, d\theta, da) \\
x_0^\mu = 0.
\end{array} \right.
\end{align*}
$$

(2.3)

The cost functional is given by:

$$
J(\mu) = \mathbb{E} \left[ \int_0^T \int_A h(t, x_t^\mu, a) \mu_t(da) dt + g(x_T^\mu) \right].
$$

3. APPROXIMATION OF TRAJECTORIES AND STABILITY RESULTS

The next lemma, which called G-chattering lemma gives the approximation of a relaxed control by a sequence of strict controls order for the relaxed control problem. This result is considered essential in showing that the relaxed control problem is a truly an extension of the strict one. We refer to [43] to more detail of this subsection.

**Lemma 3.1** (see [42]). Let $(A, d)$ be a separable metric space and assume that $A$ is a compact set. Let $(\mu_t)_t$ be an $\mathcal{F}^P$-progressively measurable process with values in $P(A)$. Then there exists a sequence $(u^\mu_n)_n \geq 0$ of $\mathcal{F}^P$-progressively measurable processes
with values in $A$ such that the sequence of random measures $\delta_{\mu^n_t}(da)dt$ converges in the sense of stable convergence (thus, weakly) to $\mu_t(da)dt$ quasi-surely:

$$\mu^n_t(da)dt = \delta_{\mu^n_t}(da)dt \rightarrow \mu_t(da)dt \quad \text{quasi-surely.}$$

**Lemma 3.2** (See [43]). Under our assumption (A), for every $P \in \mathcal{P}$, it holds that

1. $$\lim_{n \to \infty} \mathbb{E}^P \left[ \sup_{0 \leq t \leq T} |x^n_t - x^\mu_t|^2 \right] = 0,$$
   and

2. $$\lim_{n \to \infty} J^P(u^n) = J^P(\mu).$$

(2) Moreover,

3. $$\inf_{u \in U} J^P(u) = \inf_{\mu \in R} J^P(\mu),$$
   and there exists a relaxed control $\hat{\mu}_P \in R$ such that $J^P(\hat{\mu}_P) = \inf_{\mu \in R} J^P(\mu)$.

The next theorem gives the stability of the stochastic differential equations with respect to the control variable, and that the two problems has the same infimum of the expected costs.

**Theorem 3.1** (See [43].). Under our assumption (A) we have:

1. Let $\mu$ be a relaxed control and let $x^\mu$ the corresponding trajectory. Then there exists a sequence $(u^n)$ of strict controls such that:
   $$\lim_{n \to \infty} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |x^n_t - x^\mu_t|^2 \right] = 0,$$
   where $x^n_t$ denotes the trajectory associated to $u^n$.

2. Let $J(u^n)$ and $J(\mu)$ be the cost functional corresponding respectively to $u^n$ and $\mu$ (where $dt\delta_{u^n_t}(da)$ converges weakly to $dt\mu_t(da)$ quasi-surely). Then, there exists a subsequence $(u^{n_k})$ of $(u^n)$ such that
   $$\lim_{k \to \infty} J(u^{n_k}) = J(\mu).$$
4. Maximum Principle for Relaxed Control Problems

Our main goal in this section is to establish optimality necessary conditions for relaxed control problem, where the system is described by a G-SDE driven by a relaxed Poisson measure. The proof is based on the G-chattering lemma, we derive necessary conditions of near optimality satisfied by a sequence of strict controls. By using stability properties of the state equations and adjoint processes, we obtain the maximum principle for our relaxed problem.

4.1. The maximum principle for strict control. Under the above hypothesis, (2.1) has a unique strong solution and the cost functional (2.2) is well defined from $U$ into $\mathbb{R}$, for more details see [43]. The purpose of this subsection is to derive optimality necessary conditions satisfied by an optimal strict control. The proof is based on the strong perturbation of the optimal control $u^*$, which defined by:

$$u^h_t = \begin{cases} \nu_t & \text{if } t \in [t_0; t_0 + h] \\ u^*_t & \text{otherwise,} \end{cases}$$

where $0 \leq t_0 < T$ is fixed, $h$ is sufficiently small, and $\nu$ is an arbitrary $A$–valued $\mathcal{F}_{t_0}^\mathbb{P}$–measurable random such that under every $\mathbb{P} \in \mathcal{P}$, $\mathbb{E}^\mathbb{P} |\nu|^2 < \infty$. Let $x^h_t$ denotes the trajectory associated with $u^h_t$, then

$$\begin{cases} x^h_t = x^*_t ; t \leq t_0 \\
x^h_t = b(t, x^h_t, \nu_t)dt + \sigma(t, x^h_t)dB_t + \gamma(t, x^h_t, \nu_t)d\langle B \rangle_t \\
\int f(t, x^h_t, \theta, \nu_t)\tilde{N}(dt, d\theta) ; t_0 < t < t_0 + h \\
\int f(t, x^h_t, \theta, u^*_t)\tilde{N}(dt, d\theta) ; t_0 + h < t < T. \end{cases}$$

We first have

Lemma 4.1. Under assumptions (A1)-(A2), we have for every $\mathbb{P} \in \mathcal{P}$, it holds that

$$\lim_{h \to 0} \mathbb{E}^\mathbb{P} \left[ \sup_{0 \leq t \leq T} \left| x^h_t - x^*_t \right|^2 \right] = 0,$$

(4.1)
and,

\[(4.2) \quad \lim_{h \to 0} \mathbb{E}^P \left[ \sup_{t \in [t_0; T]} \left| x_t^h - x_t^* \right|^2 \right] = 0.\]

**Proof.** Under every $\mathbb{P} \in \mathcal{P}$, the G-SDEs (2.1) and (2.3) becomes standard SDEs driven by a standard Brownian motion $B$ and a Poisson measure $\tilde{N}$, the proof of (4.1) follows from H. Ben Gherbal and B. Mezerdi [7], we sketch it here. Using the fact that under $\mathbb{P} \in \mathcal{P}$, $\tilde{N}$ is a martingale and $B$ is a continuous martingale whose quadratic variation process $\langle B \rangle_t$ is such that $\pi_t = \frac{d\langle B \rangle_t}{dt}$ is bounded by a deterministic $d \times d$ symmetric positive definite matrix $\sigma$, and $x^h$ satisfy

\[dx_t^h = b(t, x_t^h, \nu_t)dt + \sigma(t, x_t^h)dB_t + \pi_t \gamma(t, x_t^h, \nu_t)dt + \int_{\Gamma} f(t, x_t^h, \theta, \nu_t)\tilde{N}(dt, d\theta),\]

then the result gives by a standard arguments from stochastic calculus, for more detail see [7].

For the second limit, set

\[\varsigma_h = \sup_{t \in [t_0; T]} \left| x_t^h - x_t^* \right|^2,\]

if there is a $\theta > 0$ such that $\mathbb{E}\left[\varsigma_h\right] \geq \theta$, we can find a probability $\mathbb{P} \in \mathcal{P}$ such that $\mathbb{E}\left[\varsigma_h\right] \geq \theta - \varepsilon; \varepsilon \to 0$. Since $\mathcal{P}$ is weakly compact, there exists a subsequence $(\mathbb{P}_{nk})_{k \geq 1}$ that converges weakly to some $\mathbb{P} \in \mathcal{P}$, hence

\[\lim_{h \to 0} \mathbb{E}\left[\varsigma_h\right] = \lim_{h \to 0} \lim_{k \to \infty} \mathbb{E}^{\mathbb{P}_{nk}}[\varsigma_h] \geq \lim_{k \to \infty} \inf_{k \to \infty} \mathbb{E}^{\mathbb{P}_{nk}}[\varsigma_h] \geq \theta.\]

This contradicts (4.1). This complete the proof. \hfill $\Box$

Since $u^*$ is optimal, then

\[J(u^*) \leq J(u^h) = J(u^*) + h \frac{dJ(u^h)}{dh} \bigg|_{h=0} + o(h).\]

Thus, a necessary condition for optimality is that

\[\frac{dJ(u^h)}{dh} \bigg|_{h=0} \geq 0.\]
Note that under every $\mathbb{P} \in \mathcal{P}$, the following properties hold, because $b(t, x, u)$, $h(t, x, u)$, $\gamma(t, x, u)$ and $f(t, x_{t-}, \theta, u)$ are sufficiently integrable.

\begin{align}
(4.3) & \quad \frac{1}{h} \int_{t}^{t+h} \mathbb{E} \left[ |k(s, x_s, u_s) - k(t, x_t, u_t)|^2 \right] \, h \to 0 \, dt - a.e \\
(4.4) & \quad \frac{1}{h} \int_{t}^{t+h} \mathbb{E} \left[ |\gamma(s, x_s, u_s) - \gamma(t, x_t, u_t)|^2 \right] \, h \to 0 \, d\langle B \rangle_t - a.e \\
(4.5) & \quad \frac{1}{h} \int_{t}^{t+h} \mathbb{E} \left[ |f(s, x_{s-}, \theta, u_s) - f(t, x_{t-}, \theta, u_t)|^2 \right] \, v(d\theta) \, h \to 0 \, dt - a.e,
\end{align}

where $k$ stands for $b$ or $h$.

**Lemma 4.2.** Under assumptions (A$_1$)-(A$_3$), it holds that

$$
\lim_{h \to 0} \mathbb{E} \left[ \left\| \frac{x^h_t - x^*_t}{h} - z_t \right\|^2 \right] = 0.
$$

**Proof.** We proceed as in H. Ben Gherbal and B. Mezerdi [7], let

$$
y^h_t = \frac{x^h_t - x^*_t}{h} - z_t.
$$

Then, we have for $t \in [t_0; t_0 + h]$

\begin{align}
\left\{ 
\begin{array}{l}
\frac{d y^h_t}{dt} = \frac{1}{h} \left[ b(t, x^*_t + h(y^h_t + z_t), \nu_t) - b(t, x^*_t, u^*_t) - h b_x(t, x^*_t, u^*_t) z_t \right] dt \\
+ \frac{1}{h} \left[ \sigma(t, x^*_t + h(y^h_t + z_t), \nu_t) - \sigma(t, x^*_t, \nu_t) - h \sigma_x(t, x^*_t, u^*_t) z_t \right] dB_t \\
+ \frac{1}{h} \left[ \gamma(t, x^*_t + h(y^h_t + z_t), \nu_t) - \gamma(t, x^*_t, u^*_t) - h \gamma_x(t, x^*_t, u^*_t) z_t \right] d\langle B \rangle_t \\
+ \frac{1}{h} \int_{t}^{t+h} \left[ f(t, x^*_{t-} + h(y^h_{t-} + z_{t-}), \nu_t) - f(t, x^*_{t-}, u^*_{t-}) - h f_x(t, x^*_{t-}, u^*_{t-}) z_{t-} \right] \tilde{N}(dt, d\theta) \\
y^h_{t_0} = - \left[ b(t_0, x^*_{t_0}, \nu_{t_0}) - b(t_0, x^*_{t_0}, u^*_{t_0}) \right].
\end{array}
\right.
\end{align}

Hence,

$$
y^h_{t_0 + h} = \frac{1}{h} \int_{t_0}^{t_0+h} \left[ b(t, x^*_t + h(y^h_t + z_t), \nu_t) - b(t, x^*_t, \nu_t) \right] dt.
$$
\begin{align*}
&\frac{1}{h} \int_{t_0}^{t_0+h} \left[ b(t, x_t^*, \nu_t) - b(t, x_{t_0}^*, \nu_t) \right] dt \\
&+ \frac{1}{h} \int_{t_0}^{t_0+h} \left[ b(t, x_{t_0}^*, \nu_t) - b(t_0, x_{t_0}^*, \nu_t) \right] dt + \frac{1}{h} \int_{t_0}^{t_0+h} \left[ b(t_0, x_{t_0}^*, u_{t_0}^*) - b(t, x_t^*, u_t^*) \right] dt \\
&+ \frac{1}{h} \int_{t_0}^{t_0+h} \left[ \sigma(t, x_t^* + h(y_t^h + z_t)) - \sigma(t, x_t^*) \right] dB_t \\
&+ \frac{1}{h} \int_{t_0}^{t_0+h} \left[ \gamma(t, x_t^* + h(y_t^h + z_t), \nu_t) - \gamma(t, x_t^*, \nu_t) \right] dB_t \\
&+ \frac{1}{h} \int_{t_0}^{t_0+h} \left[ \gamma(t, x_t^*, \nu_t) - \gamma(t_0, x_{t_0}^*, \nu_t) \right] dB_t \\
&+ \frac{1}{h} \int_{t_0}^{t_0+h} \left[ \gamma(t_0, x_{t_0}^*, u_{t_0}^*) - \gamma(t, x_t^*, u_t^*) \right] dB_t \\
&+ \frac{1}{h} \int_{t_0}^{t_0+h} \int_{\Gamma} \left[ f(t, x_t^* + h(y_t^h + z_t), \theta, \nu_t) - f(t, x_{t_0}^*, \theta, \nu_t) \right] \tilde{N}(dt, d\theta) \\
&+ \frac{1}{h} \int_{t_0}^{t_0+h} \int_{\Gamma} \left[ f(t, x_{t_0}^*, \theta, \nu_t) - f(t, x_{t_0}^*, \theta, \nu_t) \right] \tilde{N}(dt, d\theta) \\
&+ \frac{1}{h} \int_{t_0}^{t_0+h} \int_{\Gamma} \left[ f(t_0, x_{t_0}^*, \theta, \nu_t) - f(t_0, x_{t_0}^*, \theta, u_{t_0}^*) \right] \tilde{N}(dt, d\theta)
\end{align*}
Then, under every $\mathbb{P} \in \mathcal{P}$, we have

\begin{align*}
\mathbb{E}^\mathbb{P} \left| y_{t_0+h}^h \right|^2 & \leq C \left[ \mathbb{E}^\mathbb{P} \sup_{t_0 \leq t \leq t_0+h} \left| x_t^h - x_t^\ast \right|^2 + \sup_{t_0 \leq t \leq t_0+h} \mathbb{E}^\mathbb{P} \left| b(t, x_{t_0}^h, \nu_{t_0}) - b(t_0, x_{t_0}^\ast, \nu_{t_0}) \right|^2 dt \\
+ \frac{1}{h} \mathbb{E}^\mathbb{P} \int_{t_0}^{t_0+h} \left| \gamma(t, x_{t_0}^h, \nu_{t_0}) - \gamma(t_0, x_{t_0}^\ast, \nu_{t_0}) \right|^2 d\langle B \rangle_t \right] \\
& \quad + \frac{1}{h} \mathbb{E}^\mathbb{P} \int_{t_0}^{t_0+h} \left| \sigma_x(t, x_t^h)z_t \right|^2 dB_t - \int_{t_0}^{t_0+h} \int \left| f(t, x_{t_0}^h, \theta, u_{t_0}^h) - f(t, x_{t_0}^\ast, \theta, u_{t_0}^\ast) \right|^2 \nu(d\theta) dt \\
& \quad + \frac{1}{h} \mathbb{E}^\mathbb{P} \int_{t_0}^{t_0+h} \left| \sigma_x(t, x_t^h)z_t \right|^2 dB_t - \int_{t_0}^{t_0+h} \int \left| f(t, x_{t_0}^h, \theta, u_{t_0}^h) - f(t, x_{t_0}^\ast, \theta, u_{t_0}^\ast) \right|^2 \nu(d\theta) dt \\
& \quad - \frac{1}{h} \mathbb{E}^\mathbb{P} \int_{t_0}^{t_0+h} \left[ f(t, x_{t_0}^h, \theta, u_{t_0}^h) - f(t, x_{t_0}^\ast, \theta, u_{t_0}^\ast) \right] dB_t - \int_{t_0}^{t_0+h} \int \left| f(t, x_{t_0}^h, \theta, u_{t_0}^h) - f(t, x_{t_0}^\ast, \theta, u_{t_0}^\ast) \right|^2 \nu(d\theta) dt.
\end{align*}

(4.6)

By lemma (4.1), and the properties (4.3), (4.4) and (4.5), it is easy to see that for each $\mathbb{P} \in \mathcal{P}$, $\mathbb{E}^\mathbb{P} \left| y_{t_0+h}^h \right|^2$ tends to 0 as $h \to 0$.

Finally, we deduce that $\mathbb{E} \left| y_{t_0+h}^h \right|^2$ tends to 0 as $h \to 0$ by the same way as in the proof of lemma (4.1). For $t \in [t_0 + h; T]$, we denote $x_{t_0}^{h, \lambda} = x_t^\ast + \lambda h(y_t^h + z_t)$, then
**y^h_t** satisfies the following SDE:

\[
dy^h_t = \frac{1}{h} \left[ b(t, x^*_t + h(y^h_t + z_t), u^*_t) - b(t, x^*_t, u^*_t) \right] dt \\
+ \frac{1}{h} \left[ \sigma(t, x^*_t + h(y^h_t + z_t)) - \sigma(t, x^*_t) \right] dB_t \\
+ \frac{1}{h} \left[ \gamma(t, x^*_t + h(y^h_t + z_t), u^*_t) - \gamma(t, x^*_t, u^*_t) \right] d\langle B \rangle_t \\
+ \frac{1}{h} \int \left[ f(t, x^*_t + h(y^h_t + z_t), u^*_t) - f(t, x^*_t, u^*_t) \right] \tilde{N}(dt, d\theta) \\
- b_x(t, x^*_t, u^*_t) z_t dt - \sigma_x(t, x^*_t) z_t dB_t - \int f_x(t, x^*_t, \theta, u^*_t) z_t \tilde{N}(dt, d\theta),
\]

then,

\[
y^h_t = y^h_{t_0+h} + \int_{t_0+h}^{t} b_x(s, x^*_s, u^*_s) y^h_s d\lambda ds + \int_{t_0+h}^{t} \sigma_x(s, x^*_s) y^h_s d\lambda dB_s \\
+ \int_{t_0+h}^{t} \gamma_x(s, x^*_s, u^*_s) y^h_s d\lambda d\langle B \rangle_s + \int_{t_0+h}^{t} \int_{\Gamma} f_x(s, x^*_s, \theta, u^*_s) y^h_s d\lambda d\tilde{N}(ds, d\theta) + \rho^h_t,
\]

where

\[
\rho^h_t = \int_{t_0+h}^{t} b_x(s, x^*_s, u^*_s) z_s d\lambda ds + \int_{t_0+h}^{t} \sigma_x(s, x^*_s) z_s d\lambda dB_s \\
+ \int_{t_0+h}^{t} \gamma_x(s, x^*_s, u^*_s) z_s d\lambda d\langle B \rangle_s + \int_{t_0+h}^{t} \int_{\Gamma} f_x(s, x^*_s, \theta, u^*_s) z_s d\lambda d\tilde{N}(ds, d\theta) \\
- \int_{t_0+h}^{t} b_x(s, x^*_s, u^*_s) z_s ds - \int_{t_0+h}^{t} \gamma_x(s, x^*_s, u^*_s) z_s d\langle B \rangle_s - \int_{t_0+h}^{t} \sigma_x(s, x^*_s) z_s dB_s \\
- \int_{t_0+h}^{t} \int_{\Gamma} f_x(s, x^*_s, \theta, u^*_s) z_s d\tilde{N}(ds, d\theta).
\]
Hence, under every $\mathbb{P} \in \mathcal{P}$, we have

$$
\mathbb{E}^\mathbb{P} |y_t^h|^2 \leq \mathbb{E}^\mathbb{P} |y_{t_0+h}^h|^2 + K \mathbb{E}^\mathbb{P} \int_{t_0+h}^t \left| \int b_x(s, x_s^{h, \lambda}, u_s^h)y_s^h d\lambda \right|^2 ds \\
+ K \mathbb{E}^\mathbb{P} \int_{t_0+h}^t \left| \int \sigma_x(s, x_s^{h, \lambda})y_s^h d\lambda \right|^2 ds + K \mathbb{E}^\mathbb{P} \int_{t_0+h}^t \left| \int \gamma_x(s, x_s^{h, \lambda}, u_s^h)y_s^h d\lambda \right|^2 d(B)_s \\
+ K \mathbb{E}^\mathbb{P} \int_{t_0+h}^t \int \left| \int f_x(s, x_s^{h, \lambda}, \theta, u_s^h)y_s^h d\lambda \right|^2 v(d\theta) ds + K \mathbb{E}^\mathbb{P} |\rho_t^h|^2.
$$

Since $b_x, \sigma_x, \gamma_x$ and $f_x$ are bounded, then

$$
\mathbb{E}^\mathbb{P} |y_t^h|^2 \leq \mathbb{E}^\mathbb{P} |y_{t_0+h}^h|^2 + C \mathbb{E}^\mathbb{P} \int_0^t |y_s^h|^2 ds + K \mathbb{E}^\mathbb{P} |\rho_t^h|^2.
$$

We conclude by the continuity of $b_x, \sigma_x, \gamma_x$ and $f_x$, and the dominated convergence that $\lim_{h \to 0} \rho_t^h = 0$. Hence by the Gronwall lemma, and (4.6) we get

$$
\lim_{h \to 0} \sup_{t_0 \leq t \leq T} \mathbb{E}^\mathbb{P} |y_t^h|^2 = 0.
$$

Finally, we deduce that $\hat{\mathbb{E}} |y_t^h|^2$ tends to 0 as $h \to 0$ by the same way as in the proof of lemma (4.1).

The second estimate is proved in a similar way.

Choose $t_0$ such that (4.3), (4.4) and (4.5) holds, then we have

**Corollary 4.1.** Under assumptions (A_1)-(A_3), one has

$$
0 \leq \frac{dJ(u_t^h)}{dh}_{h=0} \leq \hat{\mathbb{E}} \left[ g(x_t^*) z_T + \int_0^T h_x(t, x_t^*, u_t^*) z_t dt \right],
$$

where the process $z$ is the solution of the linear SDE

$$
z_t = b_x(t, x_t^*, u_t^*) z_t dt + \sigma_x(t, x_t^*) z_t dB_t + \gamma_x(t, x_t^*, u_t^*) z_t d(B)_t \\
+ \int_\Gamma f_x(t, x_t^*, \theta, u_t^*) z_t \tilde{N}(dt, d\theta); \ t_0 \leq t \leq T \\
z_{t_0} = [b(t_0, x_{t_0}, u_{t_0}) - b(t_0, x_{t_0}^*, u_{t_0}^*)].
$$
We use the same notations as in the proof of lemma (4.2), to prove this corollary.

**proof.** We have by the definition of $J$ that

$$
\frac{1}{h} [J(u^h) - J(u^*)] \leq \frac{1}{h} \mathbb{E} \left[ \int_0^T h(t, x_t^h, u_t^h)dt + g(x_T^h) \right] \quad - \mathbb{E} \left[ \int_0^T h(t, x_t^*, u_t^*)dt + g(x_T^*) \right],
$$

then,

$$
0 \leq \frac{1}{h} [J(u^h) - J(u^*)] \leq \frac{1}{h} \mathbb{E} \left[ \int_0^T g_x(x_t^{h,\lambda})z_T d\lambda + \frac{1}{h} \int \int_0^T h_x(t, x_t^{h,\lambda}, u_t^h)z_t^h d\lambda dt \right. \\
+ \int \int_0^T h_u(t, x_t^*, u_t^{h,\lambda})u_t d\lambda dt \right].
$$

From lemma (4.2), we obtain (4.7) by letting $h$ tend to 0. □

Let us introduce the adjoint process, which is a $G$-backward stochastic differential equation (G-BSDE in short). We proceed as in [8], [47] and [7].

By the integration by parts formula, we can see that the solution of $dz_t$ is given by $z_t = \varphi_t \eta_t$ where

$$
\begin{cases}
  d\varphi(t, \tau) = b_x(t, x_t^*, u_t^*)\varphi(t, \tau)dt + \sigma_x(t, x_t^*)\varphi(t, \tau)dB_t \\
  + \int \gamma_x(t, x_t^*, u_t^*)\varphi(t, \tau)N(dt, d\theta) + \gamma_x(t, x_t^*, u_t^*)d\langle B \rangle_t \\ 0 \leq \tau \leq t \leq T, \\
  \varphi(\tau, \tau) = I_d,
\end{cases}
$$

and

$$
\begin{cases}
  d\eta_t = \psi_t \left\{ b_u(t, x_t^*, u_t^*)u_t - \int_{\Gamma} f_u(t, x_{t-}^*, \theta, u_t^*)u_t v(d\theta) \right\} dt \\
  - \psi_t - \int_{\Gamma} (f_x(t, x_{t-}^*, \theta, u_t^*) + I_d)^{-1} f_u(t, x_{t-}^*, \theta, u_t^*)u_t N(dt, d\theta) \\
  + \psi \gamma_u(t, x_t^*, u_t^*)u_t d\langle B \rangle_t \\
  \eta_0 = 0,
\end{cases}
$$

with $\psi_t$ is the inverse of $\varphi$ satisfying suitable integrability conditions, and it is the solution of the following equation.
\[
\left\{ \begin{array}{l}
d\psi(t, \tau) = \{ \sigma_x(t, x^*_t) \psi(t, \tau) - b_x(t, x^*_t, u^*_t) \psi(t, \tau) \\
- \int_{t}^{T} f_x(t, x^*_t, \theta, u^*_t) \psi(t^-, \tau) \nu(d\theta) \} \, dt \\
- \int_{t}^{T} \sigma_x(t, x^*_t) \psi(t, \tau) dB_t - \gamma_x(t, x^*_t, u^*_t) d\langle B \rangle_t \\
- \psi(t^-, \tau) \left( f_x(t, x^*_t, \theta, u^*_t) + I_d \right)^{-1} f_x(t, x^*_t, \theta, u^*_t) N(dt, d\theta) \\
0 \leq \tau \leq t \leq T,
\end{array} \right.
\]
\[\psi(\tau, \tau) = I_d.\]

Remark 4.1.

(1) From Ito's formula, we can easily check that
\[d(\varphi(t, \tau) \psi(t, \tau)) = 0, \text{ and } \varphi(\tau, \tau) \psi(\tau, \tau) = I_d.\]

(2) If \( \tau = 0 \), we simply write \( \varphi(t, 0) = \varphi_t \) and \( \psi(t, 0) = \psi_t \).

Then the equality (4.7) will become

\begin{equation}
\left. \frac{dJ(u^h)}{dt} \right|_{h=0} = E \left[ \int_{0}^{T} \left\{ h_x(t, x^*_t, u^*_t) \varphi_t \eta_t + h_u(t, x^*_t, u^*_t) u_t \right\} dt \\
+ g_x(x^*_T) \varphi_T \eta_T \right].
\end{equation}

Set
\[X = \int_{0}^{T} h_x(t, x^*_t, u^*_t) \varphi^*_t dt + g_x(x^*_T) \varphi^*_T,
\]
\[y_t = E \left[ X \left| \mathcal{F}_t^P \right. \right] - \int_{0}^{t} h_x(s, x^*_s, u^*_s) \varphi^*_s ds + \int_{0}^{t} dk_s,
\]

then, we have
\begin{equation}
y_T = E \left[ X \left| \mathcal{F}_T^P \right. \right] - \int_{0}^{T} h_x(s, x^*_s, u^*_s) \varphi^*_s ds + \int_{0}^{T} dk_t
\end{equation}

\[= X - \int_{0}^{T} h_x(s, x^*_s, u^*_s) \varphi^*_s ds = g_x(x^*_T) \varphi^*_T + \int_{0}^{T} dk_t.
\]
Replacing (4.10) in (4.9), we obtain

\[ \frac{dJ(u^h)}{dh} \bigg|_{h=0} = \mathbb{E} \left[ \int_0^T \{ h_x(t, x_t^*, u_t^*) \varphi_t \eta_t + h_u(t, x_t^*, u_t^*) u_t \} \, dt + y_T \eta_T \right]. \]

By the Ito representation theorem of a $G$-martingale (see [41]), there exist processes $Q \in M_G^2(0, T)$, $S \in S(d)$ and $R \in L_G^2(0, T)$ satisfying

\[
\mathbb{E} \left[ X | \mathcal{F}_t \right] = \mathbb{E} \left[ X \right] + \int_0^t Q_s \, dB_s + \int_0^t \varphi_s^* S_s d\langle B \rangle_s - 2 \int_0^t \varphi_s^* G(S_s) \, ds + \int_0^t R_s(\theta) \tilde{N}(ds, d\theta),
\]

where $G$ the generator $G : S(d) \to \mathbb{R}$ satisfying the uniformly elliptic condition, i.e., there exists a $\beta > 0$ such that, for each $A, \overline{A} \in S(d)$ with $A \geq \overline{A}$,

\[ G(A) - G(\overline{A}) \geq \beta tr[A - \overline{A}]. \]

Hence,

\[ y_t = \mathbb{E} \left[ X \right] - \int_0^t \{ h_x(s, x_s^*, u_s^*) \varphi_s + 2 \varphi_s^* G(S_s) \} \, ds + \int_0^t Q_s \, dB_s + \int_0^t R_s(\theta) \tilde{N}(ds, d\theta) \]

Now, let us calculate $\mathbb{E} \left[ y_T \eta_T \right]$, we have

\[ dy_t = - \{ h_x(s, x_s^*, u_s^*) \varphi_s + 2 \varphi_s^* G(S_s) \} \, dt + Q_t \, dB_t + \int_\Gamma R_t(\theta) \tilde{N}(dt, d\theta) + d\eta_t + \varphi_s^* S_t d\langle B \rangle_t, \]

by the integration by parts formula we get

\[
\int_\Gamma (f_x + Id)^{-1} f_u(t, x_t^*, \theta, u_t^*) u_t \nu(d\theta) \, dt
\]

\[ + \eta_t Q_t \, dB_t + \int_\Gamma \eta_t R_t(\theta) \tilde{N}(dt, d\theta) \]

\[ + \{ y_t \psi_t \gamma_u(t, x_t^*, u_t^*) u_t + q_t \sigma_x \psi_t \varphi_t^* + \nu_t \varphi_t^* S_t \} \, d\langle B \rangle_t \]
If we define the adjoint process by: 
\[ p_t = y_t \psi_t, \]
then
\[ d(y_t \eta_t) = p_t b_u u_t dt - p_t \int_\Gamma f_u u_t v(d\theta) dt - p_t \int_\Gamma (f_x + Id)^{-1} f_u u_t \tilde{N}(dt, d\theta) \]
\[ + \int_\Gamma R_t(\theta) \psi_t (f_x + Id)^{-1} f_u u_t v(d\theta) dt + \eta_t \phi_t^* dk_t. \]

Thus, we obtain
\[ y_T \eta_T = \int_0^T p_t b_u u_t dt - \int_0^T p_t f_u u_t v(d\theta) dt - \int_0^T p_t (f_x + Id)^{-1} f_u u_t \tilde{N}(dt, d\theta) \]
\[ - \int_0^T \int_\Gamma p_t (f_x + Id)^{-1} f_u u_t v(d\theta) dt - \int_0^T (\eta_t \phi_t^* h_x + 2\eta_t \phi_t^* G(S_t)) dt + \int_0^T \eta_t Q_t dB_t \]
\[ + \int_0^T \{ p_t \gamma_u(t, x_t^*, u_t^*) u_t + q_t \sigma_x \eta_t \phi_t^* + \eta_t S_t \} d\langle B \rangle_t + \int_0^T \eta_t \phi_t^* dk_t \]
\[ + \int_0^T \int_\Gamma \eta_t R_t(\theta) \tilde{N}(dt, d\theta) + \int_0^T \int_\Gamma R_t(\theta) \psi_t (f_x + Id)^{-1} f_u u_t v(d\theta) dt, \]

and
\[ \tilde{\mathbb{E}} [y_T \eta_T] = \tilde{\mathbb{E}} \left[ \int_0^T p_t b_u u_t dt + \int_0^T \int_\Gamma R_t(\theta) \psi_t (f_x + Id)^{-1} \right. \]
\[ - p_t ((f_x + Id)^{-1} + Id) f_u u_t v(d\theta) dt + \int_0^T \{ p_t \gamma_u(t, x_t^*, u_t^*) u_t + q_t \sigma_x \eta_t \phi_t^* \]
\[ + \eta_t \phi_t^* S_t \} d\langle B \rangle_t + \int_0^T \eta_t \phi_t^* dk_t - \int_0^T (\eta_t \phi_t^* h_x + 2\eta_t \phi_t^* G(S_t)) dt \right]. \]
We define the adjoint process $r$ by

$$r_t(\theta) = R_t(\theta)\psi_t (f_x + Id)^{-1} - p_t ((f_x + Id)^{-1} + Id),$$

hence

$$\widehat{E} [y_T \eta_T] = \widehat{E} \left[ \int_0^T \left\{ p_t b_u u_t + \int_G r_t(\theta) f_u u_t v(d\theta) \right\} dt 
- \int_0^T (\eta_t \varphi_t^* h_x + 2\eta_t \varphi_t^* G(S_t)) dt
+ \int_0^T \left\{ \gamma_u(t, x_t^*, u_t^*) u_t + q_t \sigma_x \eta_t \varphi_t^* + \eta_t \varphi_t^* S_t \right\} d\langle B \rangle_t + \int_0^T \eta_t \varphi_t^* d\langle k \rangle_t \right].$$

By the replacing in (4.11), we get

$$\frac{dJ(u^h)}{dh} \bigg|_{h=0} = \widehat{E} \left[ \int_0^T \left\{ h_u(s, x_s^*, u_s^*) + p_s b_u(s, x_s^*, u_s^*)
+ \int_G r_s(\theta) f_u(s, x_s^*, \theta, u_s^*) v(d\theta) \right\} u_s^* ds
+ \int_0^T \left\{ p_t \gamma_u(t, x_t^*, u_t^*) u_t + q_t \sigma_x \eta_t \varphi_t^* + \eta_t \varphi_t^* S_t \right\} d\langle B \rangle_t
- \int_0^T 2\eta_t \varphi_t^* G(S_t) dt + \int_0^T \eta_t \varphi_t^* d\langle k \rangle_t \right] \geq 0. \tag{4.12}$$

Finally, based on the remark 5.2 in [47] if we assume that in equation (4.12) $k = 0 \ q.s$, and we define the Hamiltonian $H$ from $[0; T] \times \mathbb{R}^n \times A \times \mathbb{R}^n \times \mathbb{R}^{n \times m} \times L_m^2$ into $\mathbb{R}$ by

$$H(t, x, u, p, q, r(.)) = h(t, x_t, u_t) + pb(t, x_t, u_t) + q\sigma(t, x_t) + \int_G r_t(\theta) f(s, x_t, \theta, u_t) v(d\theta), \tag{4.13}$$

and
\[ F(t, x, u, p, q, r(.)) = \int_{0}^{T} \{ p_t \gamma_u(t, x_t^*, u_t^*) u_t + q_t \sigma_x \eta_{t} \varphi_t^* + \eta_{t} \varphi_t^* S_t \} \, d\langle B \rangle_t \]

\[ - \int_{0}^{T} 2 \eta_{t} \varphi_t^* G(S_t) \, dt. \]

We get from (4.12) the next theorem.

**Theorem 4.1** (maximum principle for strict control). Let \( u^* \) be the optimal strict control minimizing the cost \( J(.) \) over \( \mathcal{U} \), and denote by \( x^* \) the corresponding optimal trajectory. Then there exists a unique triple of square integrable adapted processes \( (p, q, r) \) which is the unique solution of the backward G-SDE

\[
\begin{align*}
dp_t &= - \left\{ h_x(t, x_t^*, u_t^*) + p_t b_x(t, x_t^*, u_t^*) \
&\quad + \int_{\Gamma} r_t(\theta) f(t, x_t^*, \theta, u_t^*) \nu(d\theta) \right\} dt \\
&\quad - \{ \gamma_x(t, x_t^*, u_t^*) p_t + q_t \sigma_x(t, x_t^*) \} \, d\langle B \rangle_t \\
&+ q_t dB_t + \int_{\Gamma} r_t(\theta) \tilde{N}(dt, d\theta) + dk_t \\
p_T &= g_x(x_T^*), \quad k_0 = 0,
\end{align*}
\]

such that, if we assume that \( b = 0 \) and \( h = 0 \), then for all \( \nu \in \mathcal{U} \) the following inequality holds

\[
\mathbb{E} [H(t, x_t^*, \nu_t, p_t) - H(t, x_t^*, u_t^*, p_t) + G(t, x_t^*, u_t^*, p, q, r(.))] \geq 0 \, dt \quad a.e.
\]

Where the Hamiltonian \( H \) is defined by (4.13).

**4.2. The maximum principle for near optimal controls.** In this subsection, we establish necessary conditions of near optimality satisfied by a sequence of nearly optimal strict controls. This result is based on Ekeland’s variational principle, which is given by the following lemma.

**Lemma 4.3.** [Ekeland’s variational principle] Let \( (E, d) \) be a complete metric space and \( f : E \to \mathbb{R} \) be lower semicontinuous and bounded from below. Given \( \varepsilon > 0, \)
suppose \( u^\varepsilon \in E \) satisfies \( f(u^\varepsilon) \leq \inf(f) + \varepsilon \). Then for any \( \lambda > 0 \), there exists \( \nu \in E \) such that

- \( f(\nu) \leq f(u^\varepsilon) \)
- \( d(u^\varepsilon, \nu) \leq \lambda \)
- \( f(\nu) \leq f(\omega) + \frac{\lambda}{2} d(\omega, \nu) \) for all \( \omega \neq \nu \).

To apply Ekeland’s variational principle, we have to endow the set \( \mathcal{U} \) of strict controls with an appropriate metric. For any \( u, \nu \in \mathcal{U} \), we set

\[
d(u, \nu) = \mathbb{P} \otimes dt \{ (\omega, t) \in \Omega \times [0; T] ; u(t, \omega) \neq \nu(t, \omega) \},
\]

where \( \mathbb{P} \otimes dt \) is the product measure of \( \mathbb{P} \) with the Lebesgue measure \( dt \).

**Remark 4.2.** It is easy to see that \( (\mathcal{U}, d) \) is a complete metric space, and it well known that the cost functional \( J \) is continuous from \( \mathcal{U} \) into \( \mathbb{R} \). For more detail see [32].

Now, let \( \mu^* \in \mathcal{R} \) be an optimal relaxed control and denote by \( x^{\mu^*} \) the trajectory of the system controlled by \( \mu^* \). From lemma (3.1), there exists a sequence \( (u^n) \) of strict controls such that

\[
\mu^n_t(da)dt = \delta_{u^n_t}(da)dt \rightarrow \mu^*_t(da)dt \quad \text{quasi-surely},
\]

and for every \( \mathbb{P} \in \mathcal{P} \)

\[
\lim_{n \to \infty} \mathbb{E}^\mathbb{P} \left[ \left| x^n_t - x^{\mu^*}_t \right|^2 \right] = 0,
\]

where \( x^n \) is the solution of (2.3) corresponding to \( \mu^n \).

According to the optimality of \( \mu^* \) and lemma (4.3), there exists a sequence \( (\varepsilon_n) \) of positive numbers with \( \lim_{n \to \infty} \varepsilon_n = 0 \) such that

\[
J(u^n) = J(\mu^n) \leq J(\mu^*) + \varepsilon_n = \inf_{u \in \mathcal{U}} J(u) + \varepsilon_n.
\]

A suitable version of lemma (4.3) implies that, given any \( \varepsilon_n > 0 \), there exists \( u^n \in \mathcal{U} \) such that

\[
J(u^n) \leq J(u) + \varepsilon_n d(u^n, u), \forall u \in \mathcal{U}.
\]

Let us define the perturbation

\[
u_t^{n,h} = \begin{cases} \nu_t & \text{if} \ t \in [t_0; t_0 + h] \\ u^n_t & \text{otherwise.} \end{cases}
\]
From (4.15) we have

$$0 \leq J(u^{n+h}) - J(u^n) + \varepsilon_n d(u^{n+h}, u^n).$$

Using the definition of $d$ it holds that

(4.16) $$0 \leq J(u^{n+h}) - J(u^n) + \varepsilon_n C h,$$

where $C$ is a positive constant.

Now, we can introduce the next theorem which is the main result of this section.

**Theorem 4.2.** For each $\varepsilon_n > 0$, there exists $(u^n) \in \mathcal{U}$ such that there exists a unique triple of square integrable adapted processes $(p^n, q^n, r^n)$ which is the solution of the backward SDE

$$
\begin{cases}
  dp^n_t = - \{h_x(t, x^n_{t}, u^n_{t}) + p^n_{t} b_x(t, x^n_{t}, u^n_{t}) \\
  + \int_{\Gamma} r^n_{t}(\theta) f(t, x^n_{t-}, \theta, u^n_{t}) \nu(d\theta) \} \, dt \\
  - \{\gamma_x(t, x^n_{t}, u^n_{t}) p^n_{t} + q^n_{t} \sigma_x(t, x^n_{t}) \} \, dB_{t} \\
  + q^n_{t} d B_t + \int_{\Gamma} r^n_{t}(\theta) \tilde{N}(dt, d\theta) + dk^n_{t}
\end{cases}
$$

(4.17) $$p^n_T = g_x(x^n_T), \quad k^n_0 = 0,$$

such that, if we assume that in equation (4.17)

$$\forall n, h_x(t, x^n_{t}, u^n_{t}) = 0, b_x(t, x^n_{t}, u^n_{t}) = 0,$$

then for all $\nu \in \mathcal{U}$

$$\hat{E} [H(t, x^n_{t}, \nu_{t}, p^n_{t}) - H(t, x^n_{t}, u^n_{t}, p^n_{t})]
+ G^n(t, x^n_{t}, u^n_{t}, p, q, r(.)) + C \varepsilon_n \geq 0 \quad dt - a.e..$$

(4.18)

Here $C$ is a positive constant.

**Proof.** From the inequality (4.16), we use the same method as in the previous subsection, we obtain (4.18). $\square$

4.3. **The relaxed stochastic maximum principle.** Now, we can introduce the next theorem, which is the main result of this section.
Theorem 4.3. [The relaxed stochastic maximum principle] Let \( \mu^* \) be an optimal relaxed control minimizing the functional \( J \) over \( \mathcal{R} \), and let \( x^\mu_\tau \) be the corresponding optimal trajectory. Then there exists a unique triple of square integrable and adapted processes \( (p^\mu, q^\mu, r^\mu) \) which is the solution of the backward SDE

\[
\begin{align*}
            dp^\mu_t &= - \left\{ \int_{\mathcal{A}} h_x(t, x^\mu_t, a) \mu^*_t(da) + \int_{\mathcal{A}} p^\mu_t b_x(t, x^\mu_t, a) \mu^*_t(da) \\
                 &\quad + \int_{\mathcal{A}} \Gamma_t r^\mu_t(\theta) f(t, x^\mu_t, \theta, a) \mu^*_t(da) \right\} dt \\
        &\quad - \left\{ \gamma_x(t, x^\mu_t, a)p^\mu_t \mu^*_t(da) + q^\mu_t \sigma_x(t, x^\mu_t) \right\} dB_t + q^\mu_t dB_t \\
        &\quad + \int_{\mathcal{A}} \Gamma_t r^\mu_t(\theta) \tilde{N}^\mu_t(dt, d\theta, da) + dk^\mu_t \\
        p^\mu_T &= g_x(x^\mu_T), \quad k^\mu_0 = 0,
\end{align*}
\]

such that if we assume that \( b = 0 \) and \( h = 0 \), then for all \( \nu \in \mathcal{U} \)

\[
0 \leq \mathbb{E} \left[ H(t, x^\mu_t, u_t, p^\mu_t, q^\mu_t, r^\mu_t(.)) - \int_{\mathcal{A}} H(t, x^\mu_t, a, p^\mu_t, q^\mu_t, r^\mu_t(.)) \mu^*_t(da) \\
+ G^\mu(t, x^\mu_t, u_t^*, p, q, r(.)) \right] dt - \text{a.e.}
\]

The proof of this theorem is based on the following stability result of G-BSDEs with jumps. Note that this theorem is proved in the classical problems by Hu and Peng [23], and by H. Ben Gherbal and B. Mezerdi [7] in the case with jump.

4.3.1. Stability theorem for G-BSDEs with jump. Let us denote by \( M^k_G(0, T) \) the subset of \( M_G^2(0, T) \) consisting of \( \mathcal{F}_t \)—progressively measurable processes. consider the following G-BSDE with jump depending on a parameter \( n \). Using the fact that under \( P \in \mathcal{P} \), \( \tilde{N} \) is a martingale and \( B \) is a continuous martingale whose quadratic variation process \( \langle B \rangle \) is such that \( \pi_t = \frac{d\langle B \rangle_t}{dt} \) is bounded by a deterministic \( d \times d \) symmetric positive definite matrix \( \sigma \), and \( p^\mu_t \) satisfy
\[ dp^n_t = -\left\{ h_t(t, x^n_t, u^n_t) + p^n_t \left( b_t(t, x^n_t, u^n_t) - \pi_t \gamma_t(t, x^n_t, u^n_t) \right) \right. \]
\[ \left. - \pi_t q^n_t \sigma_t(t, x^n_t) + \int_\Gamma r^n_t(\theta)f(t, x^n_t, \theta, u^n_t)\nu(d\theta) \right\} dt \]
\[ + q^n_t dB_t + \int_\Gamma r^n_t(\theta)\tilde{N}(dt, d\theta) + dk^n_t \]
\[ p^n_T = g(x^n_T); \quad k^n_0 = 0. \]

Then we have
\[ p^n_t = p^n_T + \int T \int F^n(s, p^n_s, q^n_s, r^n_s)ds - \int T q^n_s dB_s - \int T r^n_s(\theta)N^n(ds, d\theta) - K^n_t + K^n_t \quad t \in [0; T], \]

with
\[ F^n(s, p^n_s, q^n_s, r^n_s) = -h_t(t, x^n_t, u^n_t) + p^n_t \left( b_t(t, x^n_t, u^n_t) - \pi_t \gamma_t(t, x^n_t, u^n_t) \right) \]
\[ - \pi_t q^n_t \sigma_t(t, x^n_t) + \int_\Gamma r^n_t(\theta)f(t, x^n_t, \theta, u^n_t)\nu(d\theta). \]

Using the linearity of the adjoint equation, it is not difficult to check that the following assumptions are verified:

(1) For any \( n, (p, q, r) \in \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \mathbb{R}, \) \( F^n(., p, q, r) \in M^2_G(0, T) \) and \( p^n_T \in L^2_G(0, T). \)

(2) There exists a constant \( C_0 > 0 \) such that
\[ |F^n(s, p_1, q_1, r_1) - F^n(s, p_2, q_2, r_2)| \leq C_0 \left( |p_1 - p_2| + |q_2 - q_2| + \int_\Gamma |r_1 - r_2|\nu(d\theta) \right) \quad P. a.s \quad a.e \quad t \in [0; T], \]

(3) \( E \left( |p^n_T - p^*_T|^2 \right) \xrightarrow{n \to \infty} 0, \)

(4) \( \forall t \in [0; T], \)
\[ \lim_{n \to \infty} E^P \left[ \left( \int_t^T (F^n(s, p^n_s, q^n_s, r^n_s) - F^*(s, p^n_s, q^n_s, r^n_s)) ds \right)^2 \right] = 0. \]
Theorem 4.4 (Stability theorem for G-BSDEs with jumps). Let \((p^n, q^n, r^n)\) and \((p^*, q^*, r^*)\), be the solutions of (4.17) and (4.19), respectively. We have

\[
\lim_{n \to \infty} \mathbb{E} \left[ |p^n - p^*|^2 + \int_0^T |q^n - q^*|^2 \, ds + \int_0^T |r^n - r^*|^2 \, \nu(d\theta) \, ds + |k^n - k^*|^2 \right] = 0.
\]

Proof. Under every \(P \in 
\mathcal{P}\), we have

\[
\mathbb{E}^P |p^n_t - p^*_t|^2 + \int_0^T |q^n_s - q^*_s|^2 \, ds + \int_0^T |r^n_s - r^*_s|^2 \, \nu(d\theta) \, ds
\leq 2\mathbb{E}^P |\alpha^n_t|^2 + 2\mathbb{E}^P \left( \int_0^T |F^n(s, p^n_s, q^n_s, r^n_s) - F^n(s, p^*_s, q^*_s, r^*_s)| \, ds \right)^2
\leq 2\mathbb{E}^P |\alpha^n_t|^2 + 2(T - t) \mathbb{E}^P \left( \int_t^T |F^n(s, p^n_s, q^n_s, r^n_s) - F^n(s, p^*_s, q^*_s, r^*_s)|^2 \, ds \right)
\]

with

\[
\alpha^n_t = p^n_T - p^*_T + \int_t^T [F^n(s, p^*_s, q^*_s, r^*_s) - F^*(s, p^*_s, q^*_s, r^*_s)] \, ds + (k^*_T - k^n_T)
\]

\[
\quad + (k^*_T - k^n_T).
\]

Because of the assumption 2, we get

\[
\mathbb{E}^P |p^n_t - p^*_t|^2 \leq \frac{2}{3} \mathbb{E}^P |\alpha^n_t|^2 + \frac{1}{6} \int_0^T \mathbb{E}^P |p^n_s - p^*_s|^2 \, ds
\]

\[
\mathbb{E}^P \int_t^T |q^n_s - q^*_s|^2 \, ds \leq \frac{4}{3} \mathbb{E}^P |\alpha^n_t|^2 + \frac{2}{3} \int_t^T \mathbb{E}^P |p^n_s - p^*_s|^2 \, ds
\]

\[
\mathbb{E}^P \int_0^T |r^n_s - r^*_s|^2 \, \nu(d\theta) \, ds \leq \frac{4}{3} \mathbb{E}^P |\alpha^n_t|^2 + \frac{2}{3} \int_0^T \mathbb{E}^P |p^n_s - p^*_s|^2 \, ds.
\]

By the assumptions 3, 4 and the stability theorem of G-BSDE without jump, see [21], we deduce that \(\lim_{n \to \infty} \mathbb{E}^P |\alpha^n_t|^2 = 0\), then \(\lim_{n \to \infty} \mathbb{E}^P |p^n_t - p^*_t|^2 = 0\) and
\[
\lim_{n \to \infty} \mathbb{E}^{P} \int_{t}^{T} |q_{s}^{n} - q_{s}^{*}|^{2} \, ds = 0.
\]

Hence, by (4.24) we get

\[
\lim_{n \to \infty} \mathbb{E}^{P} \int_{t}^{T} \int_{\Gamma} |r_{s}^{n} - r_{s}^{*}|^{2} \, v(d\theta) \, ds = 0.
\]

Finally, by the aggregation property we conclude the desired result. □

**Proof of Theorem (4.3).** By passing to the limit in inequality (4.18), and using lemma (4.3), we get easily the inequality (4.20). □

**References**

[1] D. Aldous: *Stopping Times and Tightness, II.* Ann. Prob. 17 (1989), 586–595.

[2] S. Bahlali, B. Djehiche and B. Mezerdi: *The relaxed stochastic maximum principle in singular optimal control of diffusions,* SIAM J. Control Optim. 46(2) (2007), 427–444.

[3] S. Bahlali, B. Djehiche and B. Mezerdi: *Approximation and optimality necessary conditions in relaxed stochastic control problems,* International Journal of Stochastic Analysis 72762 (2006), 1–23.

[4] X. Bai, Y. Lin: *On the existence and uniqueness of solutions to stochastic differential equations driven by G-Brownian motion with integral-Lipschitz coefficients,* Can. J. Math. 30(3) (2014), 589–610.

[5] P. Beissner, *Radner equilibria under ambiguous volatility.* working paper. Institute of Mathematical Economics Working. Paper (2013), No. 493.

P. Beissner: *Radner equilibria under ambiguous volatility,* working paper. Institute of Mathematical Economics Working. (2013), 493.

[6] P. Beissner, L. Denis: *Duality and general equilibrium theory under Knightian uncertainty,* SIAM Journal on Financial Mathematics. 9(1) (2018), 381–400.

[7] H. Ben Gherbal, B. Mezerdi: *The relaxed stochastic maximum principle in optimal control of diffusions with controlled jumps,* Afrika Statistika. 12(2) (2017), 1287–1312.

[8] A. Bensoussan: *Lectures on stochastic control.* In Nonlinear Filtering and Stochastic Control, Springer Berlin Heidelberg, 1983.
[9] Z. Chen, R. Kulperger, L. Jiang: Jensen inequality for G-expectation: part 1, Comptes Rendus Mathematiques. 337(11) (2003), 725–730.
[10] Z. Chen, S. Peng: A general downcrossing inequality for G-martingales, Statistics and Probability Letters. 46(2) (2000), 169–175.
[11] F. Coquet, Y.J. Hu, S. Peng: Iteration-consistent nonlinear expectations and related g-expectations, Probability Theory and Related Fields. 123(1) (2002), 1–27.
[12] L. Denis, M. Hu, S. Peng: Function spaces and capacity related to a sublinear expectation: application to G-Brownian motion paths, Potential Analysis. 34(2) (2011), 139–161.
[13] L. Denis, C. Martini: A theoretical framework for the pricing of contingent claims in the presence of model uncertainty, The Annals of Applied Probability. 16(2) (2006), 827–852.
[14] N. El Karoui, N. Du Hu, M. Jeanblanc: Compactification methods in the control of degenerate diffusions: existence of an optimal control, Stochastics. 20(3) (1987), 169–219.
[15] N. El Karoui, D. Nguyen, M. Jeanblanc: Existence of an optimal Markovian filter for the control under partial observations, SIAM journal on control and optimization. 26(5) (1987), 1025–1061.
[16] L. G. Epstein, S. Ji: Ambiguous volatility, possibility and utility in continuous time, Journal of Mathematical Economics. 50 (2014), 269–282.
[17] L. G. Epstein, J. Shaolin: Ambiguous volatility and asset pricing in continuous time, The Review of Financial Studies. 26(7) (2013), 1740–1786.
[18] W. H. Fleming: Generalized solutions in optimal stochastic control, Differential Games and Control theory II, Proceedings of 2nd Conference, Univ. of Rhode Island, Kingston, RI, 1976, Lect. Notes in Pure and Appl. Math. 30 (1977), 147–165.
[19] W. H. Fleming, M. Nisio: On stochastic relaxed control for partially observed diffusions, Nagoya Mathematical Journal. 93 (1984), 71–108.
[20] F. Gao: Pathwise properties and homeomorphic flows for stochastic differential equations driven by G-Brownian motion, Stochastic Processes and their Applications. 119(10) (2009), 3356–3382.
[21] Y. Hu, Y. Lin, A. S. Hima: Quadratic backward stochastic differential equations driven by G-Brownian motion: Discrete solutions and approximation, Stochastic Processes and their Applications. 128(11) (2018), 3724–3750.
[22] M. Hu, S. Ji, S. Yang: A stochastic recursive optimal control problem under the G-expectation framework, Applied Mathematics Optimization. 70(2) (2014), 253–278.
[23] Y. Hu, S. Peng: A stability theorem of backward stochastic differential equations and its application, Comptes Rendus de l'Académie des Sciences-Series I-Mathematics. 324(9) (1997), 1059–1064.
[24] M. Hu, S. Peng: G-Lévy processes under sublinear expectations, Probability, Uncertainty and Quantitative Risk. 6(1) (2021), 1.
[25] O. KEBIRI, L. NEUREITHER, C. HARTMANN: Singularly perturbed forward-backward stochastic differential equations: application to the optimal control of bilinear systems, Computation. 6(3) (2018), 41.

[26] O. KEBIRI, L. NEUREITHER, C. HARTMANN: Adaptive importance sampling with forward backward stochastic differential equations, the Proceedings of the IHP Trimester: Stochastic Dynamics Out of Equilibrium, Institute Henri Poincaré. (2019), 265–281.

[27] C. HARTMANN, O. KEBIRI, L. NEUREITHER, L. RICHTER: Variational approach to rare event simulation using least-squares regression, Chaos: An Interdisciplinary Journal of Nonlinear Science. 29(6) (2019), 063107.

[28] M. HU, S. JI, S. PENG, Y. SONG: Backward stochastic differential equations driven by G-Brownian motion, Stochastic Processes and their Applications. 124(1) (2014), 759–784.

[29] N. IKEDA, S. WATANABE: Stochastic differential equations and diffusion processes, North Holland, 2014.

[30] J. JACOD, A.N. SHIRYAEV: Limit theorems for stochastic processes, Springer Berlin, Heidelberg, New York, 1987.

[31] H.J. KUSHNER: Jump-diffusions with controlled jumps: Existence and numerical methods, J. Math. Anal. Appl. 249(1) (2000), 179–198.

[32] B. MEZERDI: Necessary conditions for optimality for a diffusion with a non-smooth drift, Stochastics. 24(4) (1988), 305–326.

[33] B. MEZERDI, S. BAHLALI, S. BAHLALI: Approximation in optimal control of diffusion processes, Random Oper. Stochastic Equations. 8(4) (2000), 365–372.

[34] B. MEZERDI, S. BAHLALI: Necessary conditions for optimality in relaxed stochastic control problems, Stochastics and Stoch. Reports. 73(4) (2002), 201–218.

[35] K. PACZKA: Itô calculus and jump diffusions for G-Lévy processes, arXiv preprint arXiv:1211.2973v3. (2012), 196.

[36] S. PENG N. EL KAROUI, L. MAZLIKA: BSDE and related g-expectation, Backward Stochastic Differential Equation, Pitman Res. Notes Math. Ser. (364) (1997), 141–159.

[37] S. PENG: Monotonic limit theorem of BSDE and nonlinear decomposition theorem of doob meyers type, Probability Theory and Related Field. 113(4) (1999), 473–499.

[38] S. PENG: Filtration consistent nonlinear expectations and evaluations of contingent claims, Acta Mathematicae Applicatae Sinica, English Series. 20(2) (2004), 191–214.

[39] S. PENG: G-Brownian motion and dynamic risk measure under volatility uncertainty, ArXiv e-prints arXiv:0711.2834v1. (2007).

[40] S. PENG: Multi-dimensional G-Brownian motion and related stochastic calculus under G-expectation, Stochastic Processes and their Applications. 118(12) (2008), 2223–2253.

[41] S. PENG: Nonlinear expectations and stochastic calculus under uncertainty - with robust CLT and G-Brownian motion, Springer-Verlag GmbH Germany, 2019.
[42] A. Redjil, S. E. Choutri: On relaxed stochastic optimal control for stochastic differential equations driven by G-Brownian motion, ALEA, Lat. Am. J. Probab. Math. Stat. 15(09) (2018), 201-212.

[43] A. Redjil, H. Ben Gherbal, O. Kebiri: Existence of relaxed stochastic optimal control for G-SDEs with controlled jumps, Stochastic Analysis and Applications (2021), 1–19.

[44] A.V. Skorokhod: Studies in the theory of random processes, translated from the Russian Scripta Technica, Inc, Addison-Wesley Publishing Co., Inc., Reading, Mass., originally published in Kiev, 1961.

[45] H. M. Soner, N. Touzi, J. Zhang: Martingale representation theorem for the G-expectation, Stochastic Processes and their Applications. 121(2) (2011), 265–287.

[46] M. Soner, N. Touzi, J. Zhang: Quasi-sure stochastic analysis through aggregation, Electronic Journal of Probability. (16) (2011), 1844–1879.

[47] Z. Sun, X. Zhang, J. Guo: A stochastic maximum principle for processes driven by G-Brownian motion and applications to finance, Optimal Control Applications and Methods. 38(6) (2017), 934–948.

[48] Y. Xu: Multidimensional dynamic risk measure via conditional g-expectation, Mathematical Finance. 26(3) (2016), 638–673.

[49] Y.H. Xu: Robust valuation, arbitrage ambiguity and profit and loss analysis, Journal of the Operations Research Society of China. 6(1) (2018), 59–83.