Synchronization of Coupled Stochastic Systems Driven by Non-Gaussian Lévy Noises

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Abstract: We consider the synchronization of the solutions to coupled stochastic systems of N-stochastic ordinary differential equations (SODEs) driven by Non-Gaussian Lévy noises (\(N \in \mathbb{N}\)). We discuss the synchronization between two solutions and among different components of solutions under certain dissipative and integrability conditions. Our results generalize the present work obtained in Liu et al (2010) and Shen et al (2010).

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1 Introduction

The synchronization of coupled systems is a well-known phenomenon in both biology and physics. Description of its diversity of occurrence can be founded in [5], [6], [7], [8], [16], [17], [18]. Synchronization of deterministic coupled systems has been investigated mathematically in [8], [19], [21] for autonomous cases and in [12] for non-autonomous systems. For the stochastic cases, we can refer to the coupled system of Itô SODEs with additive noise [9], [11] and multiplicative noise [10], [15]. Recently, Shen et al. [15] generalized the multiplicative case to \(N\)-Stratonovich SODEs. These dissipative dynamical systems discussed above are focused on the Gaussian noises (in terms of Brownian motion). However, complex systems in engineering and science are often subjected to non-Gaussian fluctuations or uncertainties. The coupled dynamical systems under non-Gaussian Lévy noises are considered in [13], [14] and [23].

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, where \(\Omega = D(\mathbb{R}, \mathbb{R}^d)\) of càdlàg functions with the Skorohod metric as the canonical sample space and denote by \(\mathcal{F} := B(D(\mathbb{R}, \mathbb{R}^d))\) the Borel \(\sigma\)-algebra on \(\Omega\). Let \(\mu_L\) be the (Lévy) probability measure on \(\mathcal{F}\) which is given by the distribution of a two-sided Lévy process with paths in \(\Omega\), i.e. \(\omega(t) = L_t(\omega)\).

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Define $\theta = (\theta_t, t \in \mathbb{R})$ on $\Omega$ the shift by

$$(\theta_t \omega)(s) := \omega(t+s) - \omega(t).$$

Then the mapping $(t, \omega) \rightarrow \theta_t \omega$ is continuous and measurable \cite{1}, and the (Lévy) probability measure is $\theta$-invariant, i.e.

$$\mu_L(\theta_t^{-1}(A)) = \mu_L(A),$$

for all $A \in \mathcal{F}$; see \cite{2} for more details. Consider the following SODEs system driven by non-Gaussian Lévy noises in $\mathbb{R}^{Nd}$,

$$dX_t^{(j)} = f^{(j)}(X_t^{(j)})dt + c_j dL_t^{(j)}, \quad j = 1, \cdots, N,$$

where $c_j \in \mathbb{R}^d$, are constants vectors with no components equal to zero, $L_t^{(j)}$ are independent two-sided scalar Lévy processes on $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying proper conditions which will be specified later, and $f^{(j)}$, $j = 1, \cdots, N$, are regular enough to ensure the existence and uniqueness of solutions and satisfy the one-sided dissipative Lipschitz conditions

$$\langle x_1 - x_2, f^{(j)}(x_1) - f^{(j)}(x_2) \rangle \leq -l\|x_1 - x_2\|^2, \quad j = 1, \cdots, N$$

on $\mathbb{R}^d$ for some $l > 4$. In addition to \cite{12}, we further assume the following integrability condition: There exists $m_0 > 0$ such that for any $m \in (0, m_0]$, and any càdlàg function $X : \mathbb{R} \rightarrow \mathbb{R}^d$ with sub-exponential growth it follows

$$\int_{-\infty}^t e^{ms}\|f^{(j)}(X(s))\|^2 ds < \infty, \quad j = 1, \cdots, N.$$ \hfill (1.3)

Without lose of generality, we also assume the Lipschitz constant $l \leq m_0$.

Set

$$x^{(j)}(t, \omega) = X_t^{(j)} - \bar{X}_t^{(j)}, \quad t \in \mathbb{R}, \omega \in \Omega, j = 1, \cdots, N,$$

where

$$\bar{X}_t^{(j)} = c_j e^{-t} \int_{-\infty}^t e^s dL_s^{(j)}, \quad j = 1, \cdots, N,$$

are the stationary solutions of the Langevin equations

$$dX_t^{(j)} = -X_t^{(j)} dt + c_j dL_t^{(j)}, \quad j = 1, \cdots, N.$$

Then system (1.1) can be translated into the following random ordinary differential equations (RODEs), with right-hand derivative in time

$$\frac{dx_t^{(j)}}{dt} = F^{(j)}(x_t^{(j)}, \bar{X}_t^{(j)})$$

$$:= f^{(j)}(x_t^{(j)} + \bar{X}_t^{(j)}) + x_t^{(j)} + \bar{X}_t^{(j)}, \quad j = 1, \cdots, N.$$ \hfill (1.4)
Now we consider the linear coupled RODEs of (1.4)

\[ \frac{dx^{(j)}}{dt} + \lambda (x^{(j-1)} - 2x^{(j)} + x^{(j+1)}) + \lambda (x^{(j-1)} - 2x^{(j)} + x^{(j+1)}) \]

with the coupled coefficient \( \lambda > 0 \), where \( x^{(0)} = x^{(N)} \) and \( x^{(N+1)} = x^{(1)} \). Hence (1.5) can be written as the following equivalent SODEs

\[ dX^{(j)} = f^{(j)}(X^{(j)}) + \lambda (X^{(j-1)} - 2X^{(j)} + X^{(j+1)}) - \lambda (X^{(j-1)} - 2X^{(j)} + X^{(j+1)}) + c_j dL^{(j)} , \quad j = 1, \ldots, N, \]

where \( X^{(0)} = X^{(N)} \) and \( X^{(N+1)} = X^{(1)} \). For synchronization of solutions to RODEs system (1.5), there are two cases: one for any two solutions and the other for components of solutions. When \( N = 2 \), Liu et al. \cite{13} consider both types of synchronization. Under the one-sided dissipative Lipschitz condition (1.2) and the integrability condition (1.3), they firstly proved that synchronization of any two solutions occurs and the random dynamical system generated by the solution of (1.5) has a singleton sets random attractor, then they obtained that the synchronization between any two components of solutions occurs as the coupled coefficient \( \lambda \) tends to infinity. The synchronization result implies that coupled dynamical system share a dynamical feature in some asymptotic sense. Based on the work of \cite{13} and \cite{15}, we consider the synchronization of solutions of (1.5) in the case of \( N \geq 3 \) and obtain the similar results.

We show that the random dynamical system (RDS) generated by the solution of the coupled RODEs system (1.5) has a singleton sets random attractor which implies the synchronization of any two solutions of (1.5). Moreover, the singleton set random attractor determines a stationary stochastic solution of the equivalently coupled SODEs system (1.6). We also show that any two solutions of RODEs system (1.5) converge to a solution \( Z(t, \omega) \) of the averaged RODE

\[ \frac{dZ}{dt} = \frac{1}{N} \sum_{j=1}^{N} f^{(j)}(X^{(j)} + Z) + \frac{1}{N} \sum_{j=1}^{N} (X^{(j)} + Z), \]

as the coupling coefficient \( \lambda \to \infty \). It is worth mentioning that the generalization is not trivial because new techniques similar to \cite{15} are needed.

### 2 Auxiliary Lemmas

We will frequently use the following auxiliary results.

**Lemma 2.1.** \cite{13} *(Pathwise boundedness and convergence.)* Let \( L_t \) be a two-sided Lévy motion on \( \mathbb{R}^d \) for which \( \mathbb{E}|L_1| < \infty \) and \( \mathbb{E}|L_1| = \gamma \). Then we have
Lemma 2.2. (Gronwall type inequality.) Suppose that $D(t)$ is a $n \times n$ matrix and $\Phi(t), \Psi(t)$ are $n$-dimensional vectors on $[T_0, T]$ ($T \geq T_0, T, T_0 \in \mathbb{R}$) which are sufficiently regular. If the following inequality holds in the componentwise sense

$$\frac{d}{dt} \Phi(t) \leq D(t) \Phi(t) + \Psi(t), \ t \geq T_0, \quad (2.1)$$

where $\frac{d}{dt} \Phi(t) := \lim_{h \to 0^+} \frac{\Phi(t+h) - \Phi(t)}{h}$ is right-hand derivative of $\Phi(t)$. Then

$$\Phi(t) \leq \exp \left( \int_{T_0}^{t} D(s)ds \right) \Phi(T_0) + \int_{T_0}^{t} \exp(\int_{\tau}^{t} D(s)ds) \Psi(\tau) d\tau, \ t \geq T_0. \quad (2.2)$$

Proof. See Lemma 2.8 in [22] and the proof of Lemma 2.2 in [15].

Lemma 2.3. ([13]) (Random attractor for càdlàg RDS.) Let $(\theta, \phi)$ be an RDS on $\Omega \times \mathbb{R}^d$ and let $\phi$ be continuous in space, but càdlàg in time. If there exists a family $B = \{B(\omega), \omega \in \Omega\}$ of non-empty measurable compact subsets $B(\omega)$ of $\mathbb{R}^d$ and a $\bar{T}_{D,\omega} \geq 0$ such that

$$\phi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)) \subset B(\omega), \ \forall t \geq \bar{T}_{D,\omega},$$

for all families $D = \{D(\omega), \omega \in \Omega\}$ in a given attracting universe, then the RDS $(\theta, \phi)$ has a random attractor $\mathcal{A} = \{\mathcal{A}(\omega), \omega \in \Omega\}$ with the component subsets defined for each $\omega \in \Omega$ by

$$\mathcal{A}(\omega) = \bigcap_{s>0} \bigcup_{t \geq s} \phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)).$$

Furthermore, if the random attractor consist of singleton sets, i.e. $\mathcal{A}(\omega) = \{X^*(\omega)\}$ for some random variable $X^*$, then $X^*_t(\omega) = X^*_t(\theta_{t}\omega)$ is a stationary stochastic process.

3 Synchronization of Two Solutions

Consider the coupled RODEs system ([15])

$$\frac{dx(j)}{dt^+} = F^{(j)}(x(j), X(j)) + \lambda(x(j-1) - 2x(j) + x(j+1)), \ j = 1, \ldots, N, \quad (3.1)$$

with initial data

$$x(j)(0, \omega) = x_{0}^{(j)}(\omega) \in \mathbb{R}^d, \ \omega \in \Omega, \ j = 1, \ldots, N, \quad (3.2)$$
Thus, the differential inequalities can be written as a simple form

$$F^{(j)}(x^{(j)}, X^{(j)}_t) := f^{(j)}(x^{(j)} + X^{(j)}_t) + x^{(j)} + X^{(j)}_t, \quad j = 1, \cdots, N. \quad (3.3)$$

Here $f^{(j)}$ are regular enough to ensure the existence and uniqueness of global solutions on $\mathbb{R}$ and satisfy the one-sided dissipative Lipschitz condition (1.2) and integrability condition (1.3) for $j = 1, \cdots, N$.

First, we have the result of existence of stationary solutions.

**Lemma 3.1.** Suppose the assumptions (1.2) and (1.3) be satisfied. Then the coupled RODEs system (3.1) with initial condition (3.2) has a unique stationary solution.

**Proof.** For any two solutions $(x^{(1)}_1(t), x^{(1)}_2(t), \cdots, x^{(N)}_1(t))^T$ and $(x^{(1)}_2(t), x^{(2)}_2(t), \cdots, x^{(N)}_2(t))^T$ of RODEs system (3.1)-(3.2). By the dissipative Lipschitz condition (1.2), for $j = 1, \cdots, N$, we have

$$\frac{d}{dt_+} \|x^{(j)}_1(t) - x^{(j)}_2(t)\|^2 = 2\langle x^{(j)}_1(t) - x^{(j)}_2(t), \frac{d}{dt_+}x^{(j)}_1(t) - \frac{d}{dt_+}x^{(j)}_2(t) \rangle$$

$$= 2\langle f^{(j)}(x^{(j)}_1 + X^{(j)}_t) - f^{(j)}(x^{(j)}_2 + X^{(j)}_t), x^{(j)}_1(t) - x^{(j)}_2(t) \rangle$$

$$+ (2 - 4\lambda)\|x^{(j)}_1(t) - x^{(j)}_2(t)\|^2$$

$$+ 2\lambda\|x^{(j-1)}_1(t) - x^{(j-1)}_2(t), x^{(j)}_1(t) - x^{(j)}_2(t)\|^2$$

$$+ 2\lambda\|x^{(j+1)}_1(t) - x^{(j+1)}_2(t), x^{(j)}_1(t) - x^{(j)}_2(t)\|^2$$

$$\leq (2 - 2l - 2\lambda)\|x^{(j)}_1(t) - x^{(j)}_2(t)\|^2$$

$$+ \lambda\|x^{(j-1)}_1(t) - x^{(j-1)}_2(t)\|^2$$

$$+ \lambda\|x^{(j+1)}_1(t) - x^{(j+1)}_2(t)\|^2. \quad (3.4)$$

Define for $t \in \mathbb{R}$,

$$x(t) = (\|x^{(1)}_1(t) - x^{(1)}_2(t)\|^2, \|x^{(2)}_1(t) - x^{(2)}_2(t)\|^2, \cdots, \|x^{(N)}_1(t) - x^{(N)}_2(t)\|^2)^T,$$

and

$$D_\lambda = \begin{pmatrix}
2 - 2l - 2\lambda & \lambda & 0 & \cdots & 0 & \lambda \\
\lambda & 2 - 2l - 2\lambda & \lambda & 0 & \cdots & 0 \\
0 & \lambda & 2 - 2l - 2\lambda & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \lambda & 2 - 2l - 2\lambda & \lambda \\
\lambda & 0 & \cdots & 0 & \lambda & 2 - 2l - 2\lambda
\end{pmatrix}_{N \times N}.$$

Thus, the differential inequalities can be written as a simple form

$$\dot{x}(t) \leq D_\lambda x(t), \text{ componentwise}. \quad (3.5)$$
By Lemma 2.2, it yields from (3.5) that
\[
x(t) \leq \exp(\int_0^t D\lambda ds) x(0), \text{ componentwise.} \quad (3.6)
\]

Now, we firstly to estimate the upper bound of eigenvalues of the real symmetric matrix \( \int_0^t D\lambda ds \). The quadratic form satisfies
\[
f(\zeta_1, \zeta_2, \cdots, \zeta_N) = \zeta^T (\int_0^t D\lambda ds) \zeta
\]
\[
= (2 - 2t - 2\lambda)t \sum_{j=1}^N \zeta_j^2 + 2\lambda t \sum_{j=1}^N \zeta_j \zeta_{j-1}
\]
\[
\leq (2 - t) t \sum_{j=1}^N \zeta_j^2 - lt \sum_{j=1}^N \zeta_j^2,
\]
where \( \zeta = (\zeta_1, \zeta_2, \cdots, \zeta_N)^T \in \mathbb{R}^N \) and \( \zeta_0 = \zeta_N \). Due to the Lipschitz constant \( l > 4 \), we have
\[
f(\zeta_1, \zeta_2, \cdots, \zeta_N) \leq -lt \sum_{j=1}^N \zeta_j^2,
\]
which implies that the quadratic form is negative definite and eigenvalues of \( \int_0^t D\lambda ds \) satisfy
\[
\max \{ \mu^{(1)}_\lambda, \mu^{(2)}_\lambda, \cdots, \mu^{(N)}_\lambda \} \leq -lt. \quad (3.7)
\]

Because of the real and symmetric properties of matrix \( \int_0^t D\lambda ds \), for \( j = 1, \cdots, N \), we obtain
\[
\| \exp(\int_0^t D\lambda ds) x(0) \|^2 \leq \| x(0) \|^2 \exp(2 \max \{ \mu^{(1)}_\lambda, \mu^{(2)}_\lambda, \cdots, \mu^{(N)}_\lambda \})
\]
\[
\leq \| x(0) \|^2 \exp(-2lt), \quad (3.8)
\]
which leads to
\[
\lim_{t \to \infty} \| x^{(j)}_1(t) - x^{(j)}_2(t) \| = 0, \ j = 1, \cdots, N,
\]
that is, all solutions of the coupled RODEs system (3.1)-(3.2) converge pathwise to each other as time \( t \) tends to infinity. The proof is finished.

Now, we use the theory of random dynamical systems which generated by SDEs driven by Lévy motion to find what the solutions of (3.1)-(3.2) will converge to. It is easy to see from [13] that the solution
\[
\phi(t, \omega) = (x^{(1)}(t, \omega), x^{(2)}(t, \omega), \cdots, x^{(N)}(t, \omega))^T, \ \omega \in \Omega
\]
of system (3.1)-(3.2) generates a càdlàg RDS over \((\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})\) with state space \( \Omega \times \mathbb{R}^{Nd} \). The RDS \((\theta, \phi)\) is continuous in space but càdlàg in time. Recall that a stationary solution
$X^*$ is a stationary solution of a stochastic differential equation system may be characterized as a stationary orbit of the corresponding RDS $(\theta, \phi)$ generated by the stochastic differential equation system, namely, $\phi(t, \omega)X^*(\omega) = X^*(\theta \omega)$.

Then, we have the result for this RDS.

**Theorem 3.2.** Under the conditions (1.2) and (1.3), the RDS $\phi(t, \omega), t \in \mathbb{R}, \omega \in \Omega$, has a singleton sets random attractor given by

$$A_{\lambda}(\omega) = \{(x^{(1)}_{\lambda}(\omega), \bar{x}^{(2)}_{\lambda}(\omega), \ldots, \bar{x}^{(N)}_{\lambda}(\omega))^T\},$$

which implies the synchronization of any two solutions of system (3.1)-(3.2). Furthermore,

$$(\bar{x}^{(1)}_{\lambda}(\theta \omega) + \bar{X}^{(1)}_t, \bar{x}^{(2)}_{\lambda}(\theta \omega) + \bar{X}^{(2)}_t, \ldots, \bar{x}^{(N)}_{\lambda}(\theta \omega) + \bar{X}^{(N)}_t)^T$$

is the stationary stochastic solution of the equivalent coupled SODEs (3.1).

**Proof.** For $j = 1, \ldots, N$, we have

$$\frac{d}{dt+}\|x^{(j)}(t)\|^2 = 2\langle x^{(j)}(t), \frac{d}{dt} x^{(j)}(t) \rangle$$

$$= 2\langle f^{(j)}(x^{(j)}(t) + \bar{X}^{(j)}_t), x^{(j)}(t) \rangle + 2\langle x^{(j)}(t) + \bar{X}^{(j)}_t, x^{(j)}(t) \rangle$$

$$- 2\lambda\|x^{(j)}(t)\|^2 + 2\lambda\|x^{(j)}(t), x^{(j-1)}(t)\| + 2\lambda\|x^{(j)}(t), x^{(j+1)}(t)\|$$

$$\leq 2\langle f^{(j)}(x^{(j)}(t) + \bar{X}^{(j)}_t) - f^{(j)}(\bar{X}^{(j)}_t), x^{(j)}(t) \rangle + 2\langle f^{(j)}(\bar{X}^{(j)}_t), x^{(j)}(t) \rangle$$

$$+ (2 - 4\lambda)\|x^{(j)}(t)\|^2 + 2\langle \bar{X}^{(j)}_t, x^{(j)}(t) \rangle$$

$$+ 2\lambda\|x^{(j)}(t), x^{(j-1)}(t)\| + 2\lambda\|x^{(j)}(t), x^{(j+1)}(t)\|$$

$$\leq \|\bar{X}^{(j)}_t\|^2 + \|f^{(j)}(\bar{X}^{(j)}_t)\|^2 + (4 - 2\lambda)\|x^{(j)}(t)\|^2$$

$$+ \lambda\|x^{(j-1)}(t)\|^2 + \lambda\|x^{(j+1)}(t)\|^2.$$
and
\[
\tilde{D}_\lambda = \begin{pmatrix}
4 - 2l - 2\lambda & \lambda & 0 & \cdots & 0 & \lambda \\
\lambda & 4 - 2l - 2\lambda & \lambda & 0 & \cdots & 0 \\
0 & \lambda & 4 - 2l - 2\lambda & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \lambda & 4 - 2l - 2\lambda & \lambda \\
\lambda & 0 & \cdots & 0 & \lambda & 4 - 2l - 2\lambda 
\end{pmatrix}_{N \times N}.
\]

Then by Lemma 2.2,
\[
y(t) \leq \exp(\int_{t_0}^{t} \tilde{D}_\lambda ds)y(t_0) + \int_{t_0}^{t} \exp(\int_{\tau}^{t} \tilde{D}_\lambda ds)g(\tau)d\tau, \quad t \geq t_0.
\]

Similar to Lemma 3.1 we have
\[
\|\exp(\int_{t_0}^{t} \tilde{D}_\lambda ds)y(t_0)\| \leq \|y(t_0)\|\exp(-l(t-t_0)), \quad t \geq t_0.
\]

Define
\[
\rho_\lambda(\omega) := \int_{-\infty}^{0} \exp(\int_{\tau}^{0} \tilde{D}_\lambda ds)g(\tau)d\tau, \quad (3.9)
\]
and
\[
R^2_\lambda(\omega) = 1 + \|\rho_\lambda(\omega)\|^2, \quad (3.10)
\]
and let \(B_\lambda\) be a random ball in \(\mathbb{R}^{Nd}\) centered at the origin with radius \(R_\lambda(\omega)\). Obviously, the infinite integral on the right-hand side of (3.9) is well-defined by Lemma 2.1 and the integrability condition (1.3). Hence by Lemma 2.3, the coupled system has a random attractor \(A_\lambda = \{A_\lambda(\omega), \omega \in \Omega\}\) with \(A_\lambda(\omega) \subseteq B_\lambda\). By Lemma 3.1 all solutions of (3.1)-(3.2) converge pathwise to each other, therefore, \(A_\lambda(\omega)\) consists of singleton sets, that is
\[
A_\lambda(\omega) = \{(\bar{x}_\lambda^{(1)}(\omega), \bar{x}_\lambda^{(2)}(\omega), \ldots, \bar{x}_\lambda^{(N)}(\omega))\}^T.
\]

We transform the coupled RODEs (3.1) back to the coupled SODEs (1.6), the corresponding pathwise singleton sets attractor is then equal to
\[
(\bar{x}_\lambda^{(1)}(\theta_1\omega) + X_t^{(1)}(\theta_1\omega), \bar{x}_\lambda^{(2)}(\theta_1\omega) + X_t^{(2)}(\theta_1\omega), \cdots, \bar{x}_\lambda^{(N)}(\theta_1\omega) + X_t^{(N)}(\theta_1\omega))^T,
\]
which is exactly a stationary stochastic solution of the coupled SODEs (1.6) because the Ornstein-Uhlenbeck process is stationary. \(\square\)
4 Synchronization of Components of Solutions

It is known in Section 3 that all solutions of the coupled RODEs system (3.1)-(3.2) converge pathwise to each other in the future for a fixed positive coupling coefficient \( \lambda \). Here, we would like to discuss what will happen to solutions of the coupled RODEs system (3.1)-(3.2) as \( \lambda \to \infty \).

First, we will give some lemmas which play an important role in this section.

We need the following estimations. Suppose that \( (x^{(1)}_\lambda(t), x^{(2)}_\lambda(t), \ldots, x^{(N)}_\lambda(t))^T \) is a solution of the coupled RODEs system (3.1)-(3.2). For any two different components \( x^{(j)}_\lambda(t), x^{(k)}_\lambda(t) \) of the solution for \( \forall j, k \in \{1, 2, \ldots, N\} \),

\[
        d^{k,j}_\lambda(t) = 2(x^{(j)}_\lambda(t) - x^{(k)}_\lambda(t), F^{(j)}(x^{(j)}_\lambda, \bar{X}^{(j)}_t) - F^{(k)}(x^{(k)}_\lambda, \bar{X}^{(k)}_t)) \\
        = 2(x^{(j)}_\lambda(t) - x^{(k)}_\lambda(t), f^{(j)}(x^{(j)}_\lambda + \bar{X}^{(j)}_t) - f^{(k)}(x^{(k)}_\lambda + \bar{X}^{(k)}_t)) \\
        \quad + 2\|x^{(j)}_\lambda(t) - x^{(k)}_\lambda(t)\|^2 + 2(x^{(j)}_\lambda(t) - x^{(k)}_\lambda(t), \bar{X}^{(j)}_t - \bar{X}^{(k)}_t) \\
        \leq -2(\|x^{(j)}_\lambda(t)\|^2 - \|x^{(k)}_\lambda(t)\|^2) + 2\|x^{(j)}_\lambda(t) - x^{(k)}_\lambda(t)\|^2 \\
        \quad + 2(f^{(j)}(\bar{X}^{(j)}_t) - f^{(k)}(\bar{X}^{(k)}_t), x^{(j)}_\lambda(t) - x^{(k)}_\lambda(t)) \\
        \quad + 2(x^{(j)}_\lambda(t) - x^{(k)}_\lambda(t), \bar{X}^{(j)}_t - \bar{X}^{(k)}_t) \\
        \leq 2\|x^{(j)}_\lambda(t) - x^{(k)}_\lambda(t)\|\|f^{(j)}(\bar{X}^{(j)}_t)\| + \|\bar{X}^{(j)}_t\|) \\
        \quad + 2\|x^{(j)}_\lambda(t) - x^{(k)}_\lambda(t)\|\|f^{(k)}(\bar{X}^{(k)}_t)\| + \|\bar{X}^{(k)}_t\|),
\]

thus, for fixed \( \alpha > 0 \), we have

\[
        -\alpha\lambda\|x^{(j)}_\lambda(t) - x^{(k)}_\lambda(t)\|^2 + d^{k,j}_\lambda(t) \\
        \leq \frac{1}{\lambda} \left[ \frac{4}{\alpha} \|f^{(j)}(\bar{X}^{(j)}_t)\|^2 + \frac{4}{\alpha} |\bar{X}^{(j)}_t|^2 \right] \\
        + \frac{1}{\lambda} \left[ \frac{4}{\alpha} \|f^{(k)}(\bar{X}^{(k)}_t)\|^2 + \frac{4}{\alpha} |\bar{X}^{(k)}_t|^2 \right].
\]

Let

\[
        C^{j,k,\alpha}_{T_1, T_2}(\lambda, \omega) = \frac{4}{\alpha} \sup_{t \in [T_1, T_2] [\|f^{(j)}(\bar{X}^{(j)}_t)\|^2 + |\bar{X}^{(j)}_t|^2 + \|f^{(k)}(\bar{X}^{(k)}_t)\|^2 + |\bar{X}^{(k)}_t|^2]}
\]

in any bounded interval \([T_1, T_2]\). Note that \( \rho_\lambda(\omega) \) in (3.9) satisfies

\[
        \frac{d}{d\lambda} \|\rho_\lambda(\omega)\|^2 = 2\langle \rho_\lambda(\omega), \frac{d}{d\lambda} \rho_\lambda(\omega) \rangle \leq 0,
\]

and consequently, \( \rho_\lambda(\omega) \leq \rho_1(\omega) \) for \( \lambda \geq 1 \). Hence, \( C^{j,k,\alpha}_{T_1, T_2}(\lambda, \omega) \) is uniformly bounded in \( \lambda \) and

\[
        -\alpha\lambda\|x^{(j)}_\lambda(t) - x^{(k)}_\lambda(t)\|^2 + d^{k,j}_\lambda(t) \leq \frac{1}{\lambda} C^{j,k,\alpha}_{T_1, T_2}(\lambda, \omega)
\]

uniformly for \( t \in [T_1, T_2] \) with

\[
        C^{j,k,\alpha}_{T_1, T_2}(\omega) = \sup_{\lambda \geq 1} C^{j,k,\alpha}_{T_1, T_2}(\lambda, \omega).
\]

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Now let us estimate the difference between any two components of a solution of the coupled RODEs system (3.1)-(3.2) as $\lambda \to \infty$.

**Lemma 4.1.** Provided conditions (1.2) and (1.3) are satisfied, then any two components of a solution $(x_{\lambda}^{(1)}(t), x_{\lambda}^{(2)}(t), \ldots, x_{\lambda}^{(N)}(t))^T$ of the coupled RODEs system (3.1)-(3.2) uniformly vanish in any bounded time interval when the coupling coefficient $\lambda \to \infty$, that is, for any bounded interval $[T_1, T_2]$ and $\forall t \in [T_1, T_2]$, it yields

$$
\lim_{\lambda \to \infty} \|x_{\lambda}^{(j)}(t) - x_{\lambda}^{(k)}(t)\| = 0, \ \forall j, k \in \{1, 2, \ldots, N\}.
$$

**Proof.** To prove the result, we can equivalently estimate the difference between any two adjacent components only because the first and the last components of the solution are considered to be adjacent. We will notice that only one new term appears in each step which continuous the process, except the last step that ends the process.

For the difference of the first part of the solution $(x_{\lambda}^{(1)}(t), x_{\lambda}^{(2)}(t), \ldots, x_{\lambda}^{(N)}(t))^T$,

$$
\frac{d}{dt} \|x_{\lambda}^{(1)}(t) - x_{\lambda}^{(2)}(t)\|^2 = 2\langle x_{\lambda}^{(1)}(t) - x_{\lambda}^{(2)}(t), F^{(1)}(x_{\lambda}^{(1)}, \bar{X}_{\lambda}^{(1)}) - F^{(2)}(x_{\lambda}^{(2)}, \bar{X}_{\lambda}^{(2)}) \rangle
$$

$$
-6\lambda \|x_{\lambda}^{(1)}(t) - x_{\lambda}^{(2)}(t)\|^2
+ 2\lambda \|x_{\lambda}^{(1)}(t) - x_{\lambda}^{(2)}(t), x_{\lambda}^{(N)}(t) - x_{\lambda}^{(3)}(t)\|
\leq -5\|x_{\lambda}^{(1)}(t) - x_{\lambda}^{(2)}(t)\|^2 + \lambda \|x_{\lambda}^{(N)}(t) - x_{\lambda}^{(3)}(t)\|^2 + d_{\lambda}^{1,2}(t)
\leq -\beta \lambda \|x_{\lambda}^{(1)}(t) - x_{\lambda}^{(2)}(t)\|^2 + \lambda \|x_{\lambda}^{(N)}(t) - x_{\lambda}^{(3)}(t)\|^2
+ \frac{1}{\lambda} C_{1,2,5}^{1,2,5,\beta}(\omega)
$$

uniformly for $t \in [T_1, T_2]$ by (4.1). Here, we can take

$$
\beta = \begin{cases} 
1 - \cos \frac{N\pi}{N+2}, & \text{N is even,} \\
1 - \cos \frac{(N-1)\pi}{N+1}, & \text{N is odd.}
\end{cases}
$$

In fact, from Lemma 4.1 in [15], we can take any $\beta \in (-2\cos \frac{N\pi}{N+2}, 2)$ when $N$ is even and any $\beta \in (-2\cos \frac{(N-1)\pi}{N+1}, 2)$ when $N$ is odd.

We have seen that the estimations in (1.2) generate $x_{\lambda}^{(3)}(t) - x_{\lambda}^{(N)}(t)$. Now, we have

$$
\frac{d}{dt} \|x_{\lambda}^{(3)}(t) - x_{\lambda}^{(N)}(t)\|^2 = 2\langle x_{\lambda}^{(3)}(t) - x_{\lambda}^{(N)}(t), F^{(3)}(x_{\lambda}^{(3)}, \bar{X}_{\lambda}^{(3)}) - F^{(N)}(x_{\lambda}^{(N)}, \bar{X}_{\lambda}^{(N)}) \rangle
$$

$$
-4\lambda \|x_{\lambda}^{(3)}(t) - x_{\lambda}^{(N)}(t)\|^2
+ 2\lambda \|x_{\lambda}^{(3)}(t) - x_{\lambda}^{(N)}(t), x_{\lambda}^{(2)}(t) - x_{\lambda}^{(1)}(t)\|
+ 2\lambda \|x_{\lambda}^{(3)}(t) - x_{\lambda}^{(N)}(t), x_{\lambda}^{(4)}(t) - x_{\lambda}^{(N-1)}(t)\|
\leq -\beta \lambda \|x_{\lambda}^{(3)}(t) - x_{\lambda}^{(N)}(t)\|^2 + \lambda \|x_{\lambda}^{(1)}(t) - x_{\lambda}^{(2)}(t)\|^2
+ \lambda \|x_{\lambda}^{(4)}(t) - x_{\lambda}^{(N-1)}(t)\|^2
+ \frac{1}{\lambda} C_{3,2,5}^{1,2,5,\beta}(\omega)
$$
uniformly for \( t \in [T_1, T_2] \).

Note that \( x^{(1)}_\lambda(t) - x^{(2)}_\lambda(t) \) has been fixed and \( x^{(4)}_\lambda(t) - x^{(N-1)}_\lambda(t) \) is generated. Similarly, it yields

\[
\frac{d}{dt} \| x^{(4)}_\lambda(t) - x^{(N-1)}_\lambda(t) \|^2 \leq -\beta \lambda \| x^{(4)}_\lambda(t) - x^{(N-1)}_\lambda(t) \|^2 + \lambda \| x^{(3)}_\lambda(t) - x^{(N)}_\lambda(t) \|^2 \\
+ \lambda \| x^{(5)}_\lambda(t) - x^{(N-2)}_\lambda(t) \|^2 + \frac{1}{\lambda} C_{T_1,T_2}^{4,N-1,2-\beta}(\omega)
\]

uniformly for \( t \in [T_1, T_2] \).

Continue such estimations, for \( j = 2, 3, \ldots \), we get

\[
\frac{d}{dt} \| x^{(j+3)}_\lambda(t) - x^{(N-j)}_\lambda(t) \|^2 \leq -\beta \lambda \| x^{(j+3)}_\lambda(t) - x^{(N-j)}_\lambda(t) \|^2 \\
+ \lambda \| x^{(j+2)}_\lambda(t) - x^{(N-j+1)}_\lambda(t) \|^2 \\
+ \lambda \| x^{(j+4)}_\lambda(t) - x^{(N-j-1)}_\lambda(t) \|^2 + \frac{1}{\lambda} C_{T_1,T_2}^{j+3,N-j,2-\beta}(\omega)
\]

uniformly for \( t \in [T_1, T_2] \).

We can divide the situation into two cases: \( N \) is even and \( N \) is odd, which just as same as \cite{15} did. When \( N \) is even, we can rewrite the inequalities in the matrix form

\[
\dot{u}(t) \leq H_\lambda u(t) + \frac{1}{\lambda} C,
\]

which uniformly for \( t \in [T_1, T_2] \), where for \( t \in \mathbb{R} \),

\[
u(t) = (\| x^{(1)}_\lambda(t) - x^{(2)}_\lambda(t) \|^2, \| x^{(3)}_\lambda(t) - x^{(N)}_\lambda(t) \|^2, \ldots, \| x^{(N+1)}_\lambda(t) - x^{(N+2)}_\lambda(t) \|^2)^T,
\]

\[
C = (C_{T_1,T_2}^{1,2,5-\beta}(\omega), C_{T_1,T_2}^{3,2,2-\beta}(\omega), \ldots, C_{T_1,T_2}^{N,N+3,2-\beta}(\omega), C_{T_1,T_2}^{N+1,N+1,2+5-\beta}(\omega))^T,
\]

are \( \frac{N}{2} \)-dimensional vectors, and

\[
H_\lambda = \begin{pmatrix}
-\beta \lambda & \lambda & 0 & \cdots & 0 \\
\lambda & -\beta \lambda & \lambda & \ddots & \vdots \\
0 & \lambda & \ddots & \ddots & 0 \\
& \ddots & \ddots & \ddots & -\beta \lambda \\
0 & \cdots & 0 & \lambda & -\beta \lambda
\end{pmatrix}_{\frac{N}{2} \times \frac{N}{2}}
\]

By Lemma 2.2 it follows from (4.3) that

\[
u(t) \leq e^{(t-t_0)H_\lambda} \nu(t_0) + \frac{1}{\lambda} \int_{t_0}^{t} e^{(t-s)H_\lambda} C ds.
\]

By Lemma 4.1 in \cite{15} again, \( \frac{1}{\lambda} H_\lambda \) is negative definite, then we have

\[
\| e^{(t-t_0)H_\lambda} \nu(t_0) \| \leq e^{(t-t_0)\mu_{\max}} \| \nu(t_0) \|,
\]

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where $\mu_{\text{max}} = -\beta - 2\cos\frac{N\pi}{N+2} < 0$ is the maximal eigenvalue of $\frac{1}{\lambda}H$. Thus (4.4) implies that 

$$u(t) \to 0 \quad \text{as} \quad \lambda \to \infty,$$

and

$$\|x_{\lambda}^{(1)}(t) - x_{\lambda}^{(2)}(t)\|^2 \to 0 \quad \text{and} \quad \|x_{\lambda}^{(N+1)}(t) - x_{\lambda}^{(N+2)}(t)\|^2 \to 0,$$

uniformly for $t \in [T_1, T_2]$ as $\lambda \to \infty$.

Similarly, when $N$ is odd, we can rewrite the inequalities in the matrix form

$$\dot{v}(t) \leq \tilde{H}\lambda v(t) + \frac{1}{\lambda}\tilde{C},$$

which uniformly for $t \in [T_1, T_2]$, where for $t \in \mathbb{R}$,

$$v(t) = (\|x_{\lambda}^{(1)}(t) - x_{\lambda}^{(2)}(t)\|^2, \|x_{\lambda}^{(3)}(t) - x_{\lambda}^{(N)}(t)\|^2, \ldots, \|x_{\lambda}^{(N+1)}(t) - x_{\lambda}^{(N+2)}(t)\|^2)^T,$$

$$\tilde{C} = (C_{T_1,T_2}^{1,2,5-\beta} (\omega), C_{T_1,T_2}^{3,N,2-\beta} (\omega), \ldots, C_{T_1,T_2}^{N+1, N+1+3,2-\beta} (\omega), C_{T_1,T_2}^{N+1, N+1+2,5-\beta} (\omega))^T,$$

are $\frac{N-1}{2}$-dimensional vectors, and

$$\tilde{H}_\lambda = \begin{pmatrix}
-\beta\lambda & \lambda & 0 & \cdots & 0 \\
\lambda & -\beta\lambda & \lambda & \ddots & \vdots \\
0 & \lambda & \ddots & \ddots & 0 \\
& \ddots & \ddots & -\beta\lambda & \lambda \\
0 & \cdots & 0 & \lambda & -\beta\lambda \\
\end{pmatrix}_{\frac{N-1}{2} \times \frac{N-1}{2}}.$$ 

By Lemma 2.2, it follows from (4.5) that

$$v(t) \leq e^{(t-t_0)\tilde{H}_\lambda}v(t_0) + \frac{1}{\lambda}\int_{t_0}^{t} e^{(t-s)\tilde{H}_\lambda}\tilde{C}ds.$$  

(4.6)

Just like the even case, for uniform $t \in [T_1, T_2]$, we have

$$\|x_{\lambda}^{(1)}(t) - x_{\lambda}^{(2)}(t)\|^2 \to 0, \quad \text{as} \quad \lambda \to \infty.$$ 

For other adjacent components, the process above can be repeated. Hence, we can draw a conclusion that the difference between any adjacent components of a solution of the coupled RODEs system (3.1)-(3.2) tends to zero uniformly for $t \in [T_1, T_2]$ as the coupling coefficient goes to infinity which completes the proof.

We know that all components of a solution of system (3.1)-(3.2) have the same limit uniformly for $t \in [T_1, T_2]$ as $\lambda \to \infty$. Now, we are in the position to find what they converge to.
Lemma 4.2. If the assumptions (1.2) and (1.3) hold, then the random dynamical system \( \phi(t, \omega) \) generated by the solution of the averaged RODE system
\[
\frac{dZ}{dt} = \frac{1}{N} \sum_{j=1}^{N} f^{(j)}(X^{(j)}_t + Z) + \frac{1}{N} \sum_{j=1}^{N} (X^{(j)}_t + Z)
\]
(4.7)
has a singleton sets random attractor denoted by \( \{ \bar{Z}(\omega) \} \). Furthermore,
\[
\bar{Z}(\theta_t \omega) + \frac{1}{N} \sum_{j=1}^{N} \bar{X}^{(j)}_t
\]
is the stationary stochastic solution of the equivalently averaged SODE system
\[
dz = \frac{1}{N} \sum_{j=1}^{N} f^{(j)}(z)dt + \frac{1}{N} \sum_{j=1}^{N} c_j dL^{(j)}_t.
\]
(4.8)
Proof. Assume that \( Z_1(t) \) and \( Z_2(t) \) are two solutions of (4.7), we have
\[
\frac{d}{dt} \| Z_1(t) - Z_2(t) \| \leq (2-2l) \| Z_1(t) - Z_2(t) \|^2.
\]
It follows from Gronwall’s lemma that
\[
\| Z_1(t) - Z_2(t) \|^2 \leq e^{(2-2l)t} \| Z_1(0) - Z_2(0) \|^2,
\]
which implies
\[
\lim_{t \to \infty} \| Z_1(t) - Z_2(t) \|^2 = 0,
\]
because of the Lipschitz coefficient \( l > 4 \). Then all solutions of (4.7) converge pathwise to each other.

Now, we have to give what they converge to based on the theory of càdlàg random dynamical systems. Let \( Z(t) \) be a solution of (4.7), we get
\[
\frac{d}{dt} \| Z(t) \|^2 \leq (4-2l) \| Z(t) \|^2 + \frac{1}{N} \sum_{j=1}^{N} \| f^{(j)}(\bar{X}^{(j)}_t) \|^2 + \frac{1}{N} \sum_{j=1}^{N} | \bar{X}^{(j)}_t |^2.
\]
From Gronwall’s lemma, it yields for \( t > t_0 \),
\[
\| Z(t) \|^2 \leq e^{(4-2l)(t-t_0)} \| Z(t_0) \|^2 + \frac{1}{N} \sum_{j=1}^{N} \int_{t_0}^{t} e^{(4-2l)(t-\tau)} (\| f^{(j)}(\bar{X}^{(j)}_\tau) \|^2 + | \bar{X}^{(j)}_\tau |^2) d\tau.
\]
By pathwise pullback convergence with \( t_0 \to -\infty \), the random closed ball centered as the origin with random radius \( \bar{R}(\omega) \) is a pullback absorbing set of \( \phi(t, \omega) \), where
\[
\bar{R}^2(\omega) = 1 + \frac{1}{N} \sum_{j=1}^{N} \int_{-\infty}^{0} e^{(2l-4)\tau} (\| f^{(j)}(\bar{X}^{(j)}_\tau) \|^2 + | \bar{X}^{(j)}_\tau |^2) d\tau.
\]
Obviously, by Lemma 2.1 and condition (1.3), the integral defined in the right-hand side is well-defined.

By Lemma 2.3 there exists a random attractor \( \{ \bar{Z}(\omega) \} \) for \( \phi(t, \omega) \). Since all solutions of (4.7) converge pathwise to each other, the random attractor \( \{ \bar{Z}(\omega) \} \) are composed of singleton sets.

Note that the averaged RODE (4.7) is transformed from the averaged SODE (4.8) by the transformation

\[
Z(t, \omega) = z - \frac{1}{N} \sum_{j=1}^{N} \overline{X}^{(j)}_t,
\]

so the pathwise singleton sets attractor \( \bar{Z}(\theta_t \omega) + \frac{1}{N} \sum_{j=1}^{N} \overline{X}^{(j)}_t \) is a stationary solution of the averaged SODE (4.8) since the Ornstein-Uhlenbeck process is stationary.

Now, we will present another main result of this work.

Theorem 4.3. (Synchronization under non-Gaussian Lévy noise.) Let

\[
(\bar{x}^{(1)}_{\lambda_n}(t, \omega), \bar{x}^{(2)}_{\lambda_n}(t, \omega), \cdots, \bar{x}^{(N)}_{\lambda_n}(t, \omega))^T = (\bar{x}^{(1)}_{\lambda_n}(\theta_t \omega), \bar{x}^{(2)}_{\lambda_n}(\theta_t \omega), \cdots, \bar{x}^{(N)}_{\lambda_n}(\theta_t \omega))^T
\]

be the singleton sets random attractor of the càdlàg random dynamical system \( \phi(t, \omega) \) generated by the solution of RODEs system (3.1)-(3.2), then

\[
((\bar{x}^{(1)}_{\lambda_n}(t, \omega), \bar{x}^{(2)}_{\lambda_n}(t, \omega), \cdots, \bar{x}^{(N)}_{\lambda_n}(t, \omega))^T) \rightarrow (\bar{Z}(t, \omega), \bar{Z}(t, \omega), \cdots, \bar{Z}(t, \omega))^T
\]

in Skorohod metric pathwise uniformly for \( t \) belongs to any bounded time-interval \([T_1, T_2]\) for any sequence \( \lambda_n \to \infty \), where \( \bar{Z}(t, \omega) = \bar{Z}(\theta_1 \omega) \) is the solution of the averaged RODE (4.7) and \( \bar{Z}(\omega) \) is the singleton sets random attractor of the càdlàg random dynamical system \( \phi(t, \omega) \) which generated by the solution of averaged RODE (3.7).

Proof. Define

\[
\bar{Z}_\lambda(\omega) = \frac{1}{N} \sum_{j=1}^{N} \bar{x}^{(j)}_{\lambda_n}(\omega),
\]

where \( \{ \bar{x}^{(1)}_{\lambda}(\omega), \bar{x}^{(2)}_{\lambda}(\omega), \cdots, \bar{x}^{(N)}_{\lambda}(\omega) \} \) is the singleton sets random attractor of the càdlàg RDS generated by RODEs system (3.1)-(3.2). Thus, \( \bar{Z}_\lambda(t, \omega) = \bar{Z}_\lambda(\theta_t \omega) \) satisfies

\[
\frac{d\bar{Z}_\lambda(t, \omega)}{dt^+} = \frac{1}{N} \sum_{j=1}^{N} f^{(j)}(\overline{X}^{(j)}_t + \bar{x}^{(j)}_{\lambda_n}(t, \omega)) + \frac{1}{N} \sum_{j=1}^{N} (\bar{X}^{(j)}_t + \bar{x}^{(j)}_{\lambda_n}(t, \omega)),
\]

Then, we get

\[
\| \frac{d\bar{Z}_\lambda(t, \omega)}{dt^+} \|^2 \leq \frac{2}{N} \sum_{j=1}^{N} (\| f^{(j)}(\overline{X}^{(j)}_t + \bar{x}^{(j)}_{\lambda_n}(t, \omega)) \|^2 + \| \overline{X}^{(j)}_t + \bar{x}^{(j)}_{\lambda_n}(t, \omega) \|^2),
\]

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by the càdlàg property of the solutions in [2] and the fact that these solutions belong to the compact ball \( \mathbb{B}_1(\omega) \), it follows that

\[
\sup_{t \in [T_1, T_2]} \| \frac{d\bar{Z}_\lambda(t, \omega)}{dt} \| \leq \left( \frac{2}{N} \sum_{j=1}^N \alpha \frac{C_{\lambda, T_1, T_2}}{T^\alpha} \right)^{\frac{1}{\alpha}} < \infty.
\]

By the Ascoli-Arzelà theorem in \( D([T_1, T_2], \mathbb{R}^d) \) in [3], there exists a subsequence \( \lambda_{nk} \to \infty \) such that \( Z_{\lambda_{nk}}(t, \omega) \) converges to \( \bar{Z}(t, \omega) \) in Skorohod metric as \( n_k \to \infty \).

Since difference between any two components of a solution of the coupled RODEs system (3.1)-(3.2) tends to zero uniformly for \( t \in [T_1, T_2] \) as \( \lambda \to \infty \), from (3.9), we have

\[
\bar{x}_{\lambda_{nk}}^{(j)}(t, \omega) = \bar{Z}_{\lambda_{nk}}(t, \omega) + \frac{1}{N} \sum_{j' \neq j} \sum_{j''} (\bar{x}_{\lambda_{nk}}^{(j'')} - \bar{x}_{\lambda_{nk}}^{(j')})(t, \omega) \to \bar{Z}(t, \omega)
\]

uniformly for \( t \in [T_1, T_2] \) as \( \lambda_{nk} \to \infty \) for \( j = 1, \ldots, N \). Furthermore, it follows from (4.10) that for \( t \geq T_1 \),

\[
\bar{Z}_\lambda(t, \omega) = \bar{Z}(T_1, \omega) + \frac{1}{N} \sum_{j=1}^N \int_{T_1}^t \left( (f(j)(\bar{x}_s^{(j)} + \bar{z}_\lambda^{(j)}(s, \omega)) + (\bar{x}_s^{(j)} + \bar{z}_\lambda^{(j)}(s, \omega))) \right) ds.
\]

Thus,

\[
\bar{Z}(t, \omega) = \bar{Z}(T_1, \omega) + \frac{1}{N} \sum_{j=1}^N \int_{T_1}^t (f(j)(\bar{x}_s^{(j)} + \bar{z}(s, \omega)) + (\bar{x}_s^{(j)} + \bar{z}(s, \omega))) ds,
\]

uniformly for \( t \in [T_1, T_2] \) as \( \lambda_{nk} \to \infty \), which implies that \( \bar{Z}_\lambda(s, \omega) \) solves RODE (4.7). Then, we note that all possible sequences of \( \bar{Z}_{\lambda_{nk}}(t, \omega) \) converges to the same limit \( \bar{Z}(t, \omega) \) uniformly for \( t \in [T_1, T_2] \) as \( \lambda_n \to \infty \). Since the RDS generated by the solutions of RODE (4.7) has a singleton sets random attractor \( \{\bar{Z}(\omega)\} \), the stationary stochastic process \( \bar{Z} (\theta_\omega) \) must be equal to \( \bar{Z}(t, \omega) \), i.e. \( \bar{Z}(t, \omega) = \bar{Z}(\theta_\omega), \) which completes the proof.

As a obvious result of Theorem 3.2, we get

**Corollary 4.4.**

\[
((\bar{x}_\lambda^{(1)}(t, \omega), \bar{x}_\lambda^{(2)}(t, \omega), \ldots, \bar{x}_\lambda^{(N)}(t, \omega)))^T) \to (\bar{Z}(t, \omega), \bar{Z}(t, \omega), \ldots, \bar{Z}(t, \omega))^T
\]

in Skorohod metric pathwise uniformly for \( t \in [T_1, T_2] \) as \( \lambda \to \infty \).

**Remark 4.5.** The results in this paper hold just in almost everywhere sense. In the equation (1.1) we should replace the \( X_t^{(j)} \) with \( X_t^{(j)} \) because we must take the left limit to make sure that càdlàg solution process \( X_t^{(j)} \) is predictable and unique [21]. For the typographical convenience,
however, we will use $X_t^{(j)}$ instead of $X_{t-}^{(j)}$ for the rest of the paper. Moreover, in the case of additive noise, the distinction for left limit or not is not necessary because if we have to consider the integral form of equation (1.1), $f^{(j)}(X_t^{(j)})$ has only countable discontinuous points and is still Riemann and Lebesgue integrable, where $j = 1, \cdots, N$.

5 Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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