General formula for symmetry factors of Feynman diagrams

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Abstract

General formula for symmetry factors (S-factor) of Feynman diagrams containing fields with high spins is derived. We prove that symmetry factors of Feynman diagrams of well-known theories do not depend on spins of fields. In contributions to S-factors, self-conjugate fields and non self-conjugate fields play the same roles as real scalar fields and complex scalar fields, respectively. Thus, the formula of S-factors for scalar theories — theories include only real and complex scalar fields — works on all well-known theories of fields with high spins. Two interesting consequences deduced from our result are: (i) S-factors of all external connected diagrams consisting of only vertices with three different fields, e.g., spinor QED, are equal to unity; (ii) some diagrams with different topologies can contribute the same factor, leading to the result that the inverse S-factor for the total contribution is the sum of inverse S-factors, i.e., \(1/S = \sum_i(1/S_i)\).

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1 Introduction

In literature, using perturbation theory and Feynman rules, a general Green's function of an arbitrary theory can be written in terms of sum of Feynman diagrams. Each diagram is associated with a factor known as symmetry factor (S-factor). There are some ways to calculate this factor such as given in [1] (for more details, see [2–5]) using functional derivative method. A computer program [6] is also written based on this method to find out S-factors of higher-order diagrams from the lower-order ones. Other independent approaches base directly on computer programs such as [7,8]. S-factors can also be evaluated by using Wick’s theorem, available in many textbooks (see, for example [9–13]). But the disadvantage of these books as well as the methods mentioned above is none of them give out any general formulas. We can see that Ref. [9] has an expression for connected diagrams of real scalar theories, Ref. [13] has some comments about S-factors for scalar electrodynamics, Ref. [10] for real scalar \(\phi^4\) and some particular illustrations in Standard Model. Especially, the very detailed investigation into S-factors, which are very close to weights of Feynman diagrams in \(\phi^4\) theories was presented in [15]. Refs. [16,17] also contain S-factors of some particular diagrams in QCD.

This paper is the development of [14] in which we derived the S-factors for Feynman diagrams of theories with scalar fields. The definition of S-factor can be found in [10]. We can understand this as follows. Using Wick’s theorem for expanding a
Green function one often encounters many terms whose contractions are different but contributions are the same. The S-factor is the number of identical terms which are repeatedly counted. In language of Feynman diagram, this factor turns out to be the product of total number of (symmetry) permutations of all vertices and all internal propagators in the diagram, which create new identical diagrams with factors caused by bubbles. In Ref. [14] we have concentrated on two types of fields, namely real and complex scalar fields, and have noted that the distinction between these fields is very important because they contribute different factors to the formula of S-factor [14]:

$$S = g^2 \beta^2 \prod_n (n!)^{\alpha_n},$$  \hspace{1cm} (1)

where \(g\) is the number of interchanges of vertices leaving the diagram topologically unchanged, \(\beta\) is the number of lines connecting a vertex to itself (\(\beta\) is zero if the field is complex), \(d\) is the number of double bubbles, and \(\alpha_n\) is the number of sets of \(n\)-identical lines connecting the same two vertices.

In this paper, by considering some particular cases, we will indicate precisely that in calculating S-factors, we can classify all well-known fields into two classes. The first class comprises self-conjugate fields for which the particle is the same as the antiparticle, such as the real Higgs scalar \(\sigma\) in the Standard Model, the photon and the \(Z\) boson. We will often refer to this class to be the real scalar-like. The second, all non self-conjugate fields — such as charged particles — will be referred to as complex scalar-like. In analogy with the leptons where \(e^-\), \(\mu^-\) and \(\tau^-\) are called “particles” and \(e^+, \mu^+\) and \(\tau^+\) are called “antiparticles”, hereafter we adopt the convention that the negative electric charged scalar/vector fields (for example \(\pi^-, W^-\)) will be called particles. Keeping these remarks in mind, we then redefine parameters \(g, d, n\) and \(\alpha_n\) of (1) (detailed in the conclusion), then the formula works on all cases.

One more interesting point we would like to mention about this paper is that a simple method of calculating the SF of a particular Feynman diagram emerges directly from the graphical form itself. Especially the \(g\)-factor, the most complicated factor appearing in our formula (1) as well as [9] and many others textbooks, will be naturally made clear through our calculation. It relates strictly with graphical symmetry properties of the diagram.

The outline of our work in this paper is as follows. In the second section, we recall T-product expansions of interaction Lagrangians into N-products and introduce a new definition of vertices and their factors in Feynman diagrams. This helps us simplify our calculation because, for every interaction Lagrangian, we will find factors that really contribute to the SFs and omit other unnecessary factors. The third section is devoted to spinor QED case, the most simple case that contains spinor fields. As mentioned in the abstract, we will show that in the spinor QED, the S-factor of an arbitrary diagram at any order in pertubative expansion is always equal to 1. This is very useful, for example, in calculating high order QED contributions to lepton Anomalous Magnetic Moment \((g - 2)\) [20]. We will also prove that spinor fields behave the same as scalar complex fields. In addition, we will discuss in detail one interesting way of practically determining the \(g\) factor from the geometric symmetries of a particular Feynman diagram. In the next three sections some particular cases are illustrated to point out that when calculating S-factor of a diagram, all well-known fields always
belong to one of two classes mentioned above. In the last section, we will derive the final formula of the S-factor for general cases. An expression for $g$ factor is also presented in order to determine it from connected diagrams of the total diagram. Examples of the S-factors are illustrated in Appendices A and B.

2 Feynman diagrams and symmetry factors

Let us start using Wick’s theorem to expand $T$-products of interaction Lagrangians into sums of $N$-products \[^{1,10}\]:

1. Real scalar $\phi^3$ theory:

\[
\mathcal{L}_{\text{int}}^r(x) = \frac{\lambda}{3!} \phi^3(x),
\]

\[
\frac{1}{3!} \phi^3(x) \sim \frac{1}{3!} T \left[ \phi^3(x) \right] = N \left[ \frac{1}{3!} \phi^3(x) \right] + \frac{3}{3!} \phi(x) \dot{\Delta}(x) \tag{2}
\]

where each $\dot{\Delta}(x) \equiv \phi(x) \phi(x)$ corresponds to a bubble located at $x$-coordinate in some Feynman diagram.

2. Real scalar $\phi^4$ theory:

\[
\mathcal{L}_{\text{int}}^r(x) = \frac{\lambda}{4!} \phi^4(x),
\]

\[
\frac{1}{4!} \phi^4(x) \sim \frac{1}{4!} T \left[ \phi^4(x) \right] = N \left[ \frac{1}{4!} \phi^4(x) \right] + \frac{6}{4!} N \left[ \phi^2(x) \right] \dot{\Delta}(x) + \frac{3}{4!} \dot{\Delta}(x) \dot{\Delta}(x). \tag{3}
\]

3. Complex scalar $\varphi^4$ theory:

\[
\mathcal{L}_{\text{int}}^c(x) = \frac{\rho}{4} [\varphi(x) \varphi^*(x)]^2,
\]

\[
\frac{1}{4} [\varphi(x) \varphi^*(x)]^2 \sim \frac{1}{4} T [\varphi(x) \varphi^*(x)]^2 = N \left[ \frac{1}{4} [\varphi(x) \varphi^*(x)]^2 \right] + \frac{4}{4} N \left[ \varphi(x) \varphi^*(x) \right] \dot{\Delta}(x) + \frac{2}{4} \dot{\Delta}(x) \dot{\Delta}(x). \tag{4}
\]

Each term in right hand sides (RHS) of (2), (3) and (4) changes into one particular kind of vertex in the language of Feynman diagram. They are illustrated in Fig. 1 where propagators of real fields are represented as dash lines without directions (arrows), while complex cases are represented as dash lines with directions. Vertices are different from each others in numbers of lines and kinds of line they have. This is because terms in RHSs of (2,4) are different in fields and contractions. Now, for a given interaction Lagrangian, we can show exactly all kinds of vertex in the theory. This is very important for us to find out not only $g$ factor relating with vertices but also contributory factors of different kinds of vertex to S-factors. Vertices themselves have well-known factors
as *vertex factors*, which can be ignored due to the fact that S-factors are independent on them.

Vertex factors, in scalar theories, are simply $i\lambda$ or $i\rho$, while in the others such as in scalar electrodynamics or in quantum chromodynamics, where there exist interactions containing derivatives, are more complicated. For our method, the S-factor determined from (1) depends on values of factors, for example: $1/(3!)$ in $\phi^3$, of interacting terms in Lagrangian. These factors are obtained by taking partial derivatives of respective interacting terms. Furthermore, we need to write down each interacting term of the Lagrangian in form of [vertex factor \times $T$-product], then omit this vertex factor in our calculation. The symmetry factor now depends only on $T$-product.

A general Feynman diagram derived from the expansion of a general Green’s function consists of many connected pieces (subdiagrams) disconnected with each others. We will call pieces connected vacuum diagrams if they have not any external legs and connected external diagrams if they have at least one external leg [for example, see Fig. 4(a.10)]. Every connected subdiagram has its private S-factor. In this work, we concentrate on only aspect of S-factor calculation. Other symmetries, such as the charge conjugation under which diagrams in the QED with odd number of external photon legs give vanishing contributions, are outside the scope.

Now we turn to theories of fields with high spins.

### 3 Symmetry factors in spinor QED

In spinor Quantum Electrodynamics (QED), the interaction Lagrangian of one fermion field $\psi$ is given by

$$L^{QED}_{int}(x) = eq\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x),$$

where $e$ is the electromagnetic coupling constant, $q$ is the electric charge of fermion $\psi$ in units of positron charge and $A_\mu(x)$ is the electromagnetic field. For the sake of brevity, from now on we will write $L(x)$ instead of $L_{int}(x)$.

The above Lagrangian has only one interacting term with a vertex factor $[ieq\gamma^\mu]$. $T$-product expansion gives:

$$T \left[ \bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x) \right] = N \left[ \bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x) \right] + \bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x)$$

$$= N \left[ \bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x) \right] + iS_\beta^\alpha(x)(\gamma^\mu)^\beta_\alpha A_\mu(x),$$

(6)
where $S(x)$ is a fermion bubble.

The last expression in (6) has two terms corresponding to two kinds of vertices: The first has one photon leg, one incoming and one outgoing electron leg. The second has one photon leg and one fermion bubble. These vertices are illustrated in Fig. 2.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2}
\caption{Vertices of QED}
\end{figure}

For simplicity in calculating, let us denote two terms in the LHS of (6) as follows:

\begin{align}
  a_1 &= N \left[ \bar{\psi}(x) \gamma^\mu \psi(x) A_\mu(x) \right], \\
  a_2 &= i S^\alpha_\beta(x) (\gamma^\mu)_\alpha A_\mu(x) \equiv i S(x) \gamma^\mu A_\mu(x)
\end{align}

(7)

Thus, (6) is rewritten as:

\begin{align}
  T \left[ \bar{\psi}(x) \gamma^\mu \psi(x) A_\mu(x) \right] &= a_1(x) + a_2(x) \\
  \text{(8)}
\end{align}

The $n$-point Green’s function $G^{QED}(x_1, x_2, ..., x_n)$ of QED is defined as

\begin{align}
  G^{QED}(x_1, x_2, ..., x_n) &= \sum_{p=0}^{\infty} \frac{(-i)^p}{p!} \int dy_1 dy_2 ... dy_p \langle 0 | T [\phi(x_1) \phi(x_2) ... \phi(x_n) \\
  \times \mathcal{L}(y_1) \mathcal{L}(y_2) ... \mathcal{L}(y_p)] | 0 \rangle,
\end{align}

(9)

where $\phi(x)$ implies a spinor $\bar{\psi}(x), \psi(x)$ or an $A_\mu(x)$. The $p$th-order term in this expression is:

\begin{align}
  G^{QED(p)}(x_1, x_2, ..., x_n) &= \frac{(-i)^p}{p!} \int dy_1 dy_2 ... dy_p \langle 0 | T [\phi(x_1) \phi(x_2) ... \phi(x_n) \\
  \times \mathcal{L}(y_1) \mathcal{L}(y_2) ... \mathcal{L}(y_p)] | 0 \rangle
\end{align}

(10)

Note that QED has some features different from scalar cases. The Lagrangian of QED contains nonvanishing-spin fields, namely half-integer spin fields and spin-1 photon. Spinor fields follow anti-communication relations so when positions of these fields are changed in a product, a minus sign will appear. However, it does not affect the S-factors. Furthermore, every interacting term always has even number of spinor fields so the value of total product in (10) is unchanged regardless positions of these terms. This conclusion is correct for any theories. Then the method used in [14] can again be used as we will discuss next.

In a resulting product $[\mathcal{L}(y_1) \mathcal{L}(y_2) ... \mathcal{L}(y_p)]$ of (10), all terms consisting of the same numbers of $a_i$, have the same contributions. Then the sum of these terms is
presented as a product of single symbolic term multiplied by a factor deduced from the multinomial formula:

\[(x_1 + x_2 + \cdots + x_r)^p = \sum_{p_1, p_2, \ldots, p_r} \frac{p!}{p_1!p_2! \cdots p_r!} x_1^{p_1} \cdots x_r^{p_r},\]  
with \[p_1 + p_2 + p_3 + \cdots + p_r = p.\]

In case of QED, sum of all terms in (10) which have \[p_1\] factors of \(a_1\) and \[p_2\] factors of \(a_2\) are all presented as product of a single term \(a_1^{p_1}a_2^{p_2}\) \((p_1 + p_2 = p)\) multiplied by a factor:

\[\frac{p!}{p_1!p_2!}\]  
(12)

In order to do contractions between internal fields of \(a_1^{p_1}a_2^{p_2}\) and external fields \(\phi(x_1), \phi(x_2), \ldots, \phi(x_n)\), we write the T-product as follows:

\[\langle 0 | T[\phi(x_1)\phi(x_2)\ldots\phi(x_n)a_1^{p_1}a_2^{p_2}] | 0 \rangle = \langle 0 | T[\phi(x_1)\phi(x_2)\ldots\phi(x_n)N[\psi(y_1)\gamma^{\mu_1}\psi(y_1)A_{\mu_1}(y_1)]\ldots\nonumber\]
\[\times N[\bar{\psi}(y_r)\gamma^{\mu_r}\psi(y_r)A_{\mu_r}(y_r)](-i)S(y_{r+1})\gamma^{\mu_{r+1}}A_{\mu_{r+1}}(x)\ldots(-i)S(y_{r+s})\gamma^{\mu_{r+s}}A_{\mu_{r+s}}(x)] | 0 \rangle\]  
(13)

Performing contractions between fields, we rewrite (13) as a sum of terms containing only contractions. The terms with similar contractions lead to the fact that a total contribution of these terms can be presented as a product of a term (now corresponding to a Feynman diagram) multiplied by a new additional factor. This new factor contributes to our S-factor and will be calculated by performing permutations of propagators and vertices.

Let us concentrate on symmetries of (13) because this will help us count the number of different terms having the same contribution. There are factors created by two kinds of permutation: (i) permutation of propagators (lines) in each vertex and (ii) permutation of vertices in one diagram. For propagator permutations, first, there is no permutation in the vertex of type \((a_2)\) (figure 2b) having only one leg. Second, according to Wick’s theorem, each vertex of type \((a_1)\) has three different fields with possibility of contraction with internal fields of other vertices or external fields. These three contractions present as three different lines (see fig 2a). Again, there are no permutations of these lines. Therefore the factor caused by propagator permutations in any vertex is \(f_1 = 1\).

Next, consider symmetries (or equivalences) between vertices. Vertices in the same kind, which play the same roles in doing contractions, will create distinguishable terms with identical contributions. For QED, there are two types of vertex, number of these terms is given by a factor:

\[f_2 = \frac{p_1!p_2!}{g}\]  
(14)

Let us explain how to obtain this factor. \((p_1!p_2!)\) is the permutation number of vertices \((p_1\) of \(a_1\) and \(p_2\) of \(a_2\)) to get new terms, including repeatedly permutations. Factor
$g$ cancels the repeat, i.e., permutations are repeatedly counted. It will be more clear when we discuss directly on Feynman diagrams.

Combining two factors of $(12)$, $(14)$ and the expanding factor $(1/p!)$ in $(10)$ we get a total factor of a particular Feynman diagram appearing in $(10)$:

$$\frac{1}{S} = \frac{1}{p!} \times \frac{p!}{p_1!p_2!} \times \frac{p_1!p_2!}{g}$$

$$\frac{1}{g}$$

Hence, in QED the symmetry factor is only $g$, the factor belongs to $f_2$ factor. It is more easy to understand $g$ in language of Feynman diagram. We will see that $g$ is the number of repeatedly vertex permutations, i.e., the number of vertex permutations that creates identical diagrams. Determining $g$ is rather complicated because we have to make clear relations not only between vertices themselves but also vertices and propagators. Fortunately, we can exploit relations between permutation symmetries and geometrical symmetries of a diagram to solve this problem. Further, $g$ factor of a general diagram can be established from $gs$ of connected pieces. Therefore, to determine $g$ we just consider particular connected subdiagrams based on geometric symmetries of itself. Let us illustrate this by some examples in Fig.3

$$\begin{align*}
(a) & \quad S = g = 1 \\
(b) & \quad S = g = 2 \\
(c) & \quad S = g = 4 \\
(d) & \quad S = g = 6
\end{align*}$$

Figure 3: Examples of symmetry factors in QED

Let us look at figures 3(b), (c) and (d). Figure (c) has only rotational symmetries of a square, and (d)-a regular hexagon, because fermion lines have directions. With these three diagrams we can rotate (b) an angle $180^0$, (c) three angles $90^0, 180^0, 270^0$ and (c)-$k \times 60^0, k = 2,..., 5$ to get the same diagrams as the origins. Clearly, the number of rotative symmetries (including trivial rotation) is exactly equal to $g$ factor found in [2]. Let us go to other examples in figure 4. Diagram in figure 4a has three identical fermion loops, lying on three vertices of a regular triangle, and three photon propagators (no direction). Then $g$ of the diagram is $3! = 6$, is also equal to six symmetries of a regular triangle (two rotation symmetries, three axial and the identical). In Fig 4b, there is no symmetry because of two fixed external propagators. Fig4: has three connected pieces: two external connected pieces do not contribute any factor while the third (connected vacuum piece) causes a factor $g = 2$ by a $180^0$ rotation. From the above discussion, all S-factors of diagrams in the QED given in Ref. [2], can be derived. It is worth noting that in the case of spinor QED, all connected pieces relating with external legs, have $S = 1$ because external legs cancel their geometrical symmetries.
with those in $p$ only on field having momentum $\phi$. This term should be considered in momentum-space where $A = \text{closed fermion line}$. For the photon $A_\mu$, it has properties of a real scalar field as we will prove in the next section. Thus $S$ in (15) is a special case of formula (11). In appendix A S-factors of the QED up to fourth order are presented. Ours results are consistent with those in [2].

4 Symmetry factors in scalar Quantum Electrodynamics

In scalar Quantum Electrodynamics (sQED), the interaction Lagrangian consists of both $A_\mu$ and a complex scalar field:

$$\mathcal{L}^\text{sQED}(x) = ieq A^\mu(x)[\varphi^*(x)\partial_\mu\varphi(x) - (\partial_\mu\varphi^*(x))\varphi(x)] + e^2 q^2 A_\mu(x)A^\mu(x)\varphi^*(x)\varphi(x), \quad (16)$$

where $q$ is the electric charge of the complex scalar field $\varphi$. First, we pay attention to the term with derivative. This term should be considered in momentum-space where $\varphi(x)$, $\varphi^*(x)$ and $A_\mu(x)$ have respective momenta $p$, $p'$ and $k$. If $\partial_\mu^p$ denotes that $\partial_\mu$ acts only on field having momentum $p$ then we can rewrite:

$$[\varphi^*(x)\partial_\mu\varphi(x) - (\partial_\mu\varphi^*(x))\varphi(x)] \equiv (\partial_\mu^p - \partial_\mu^{p'})[\varphi^*(x)\varphi(x)] \quad (17)$$

This definition helps us easily write down the vertex factor of derivative term in momentum space as $[ieq(p + p')^\mu]$.

$T$-product expansion of the Lagrangian into sum of $N$-products gives:

$$T \{ ieq A^\mu(x)[\partial^p - \partial^{p'}][\varphi^*(x)\varphi(x)] + e^2 q^2 A_\mu(x)A^\mu(x)\varphi^*(x)\varphi(x) \}$$

$$= ieq(\partial^p - \partial^{p'})[\partial_\mu A_\mu(x)N[\varphi^*(x)\varphi(x)] + ieq(\partial^p - \partial^{p'})[\partial_\mu A_\mu(x)\hat{\Delta}(x)$$

$$+ e^2 g^{\mu\nu}q^2 N[A_\mu(x)A_\nu(x)\varphi^*(x)\varphi(x)] + (2e^2 q^2 g^{\mu\nu})\frac{1}{2}N[A_\mu(x)A_\nu(x)]\hat{\Delta}(x)$$

$$+ (2e^2 q^2 g^{\mu\nu})\frac{1}{2}\hat{\Delta}_{\mu\nu}(x)N[\varphi^*(x)\varphi(x)] + (2e^2 q^2 g^{\mu\nu})\frac{1}{2}\hat{\Delta}_{\mu\nu}(x)\hat{\Delta}(x)$$

$$= [ieq(\partial^p - \partial^{p'})[ a_1 + [ieq(\partial^p - \partial^{p'})[ a_2 + (2e^2 q^2 g^{\mu\nu})\frac{1}{2}a_3$$

$$+ (2e^2 q^2 g^{\mu\nu})\frac{1}{2}a_4 + (2e^2 q^2 g^{\mu\nu})\frac{1}{2}a_5 + (2e^2 q^2 g^{\mu\nu})\frac{1}{2}a_6, \quad (18)$$

Figure 4: Examples in calculation of symmetry factors

One more new important conclusion for this section is: in our calculation, fermion fields behave exactly the same as complex scalar fields, except a minus sign for each closed fermion line.
where

\[
\begin{align*}
  a_1 &= A_\mu(x)N[\varphi^*(x)]\varphi(x) \\
  a_2 &= A_\nu(x)\Delta(x) \\
  a_3 &= N[A_\nu(x)A_\mu(x)\varphi^*(x)\varphi(x)] \\
  a_4 &= N[A_\nu(x)A_\mu(x)]\Delta(x) \\
  a_5 &= \tilde{\Delta}_{\mu\nu}(x)N[\varphi^*(x)\varphi(x)] \\
  a_6 &= \tilde{\Delta}_{\mu\nu}(x)\Delta(x).
\end{align*}
\]

As before, here we have denoted \(\tilde{\Delta}_{\mu\nu}(x) \equiv A_\mu(x)A_\nu(x)\). Let us explain a reason for the factor \(\frac{1}{2}\) associated with \(a_i, i = 3, 4, 5, 6\). As mentioned in section [2] we have to write: \([e^2q^2A_\mu(x)A^\nu(x)\varphi^*(x)\varphi(x)] = [(2e^2g_{\mu\nu})] \times [\frac{1}{2}A_\mu(x)A_\nu(x)\varphi^*(x)\varphi(x)]\) because \([(2e^2g_{\mu\nu})]\) is well-known vertex factor in literature so the factor \(1/2\) (where \(2\) is derived by taking derivatives of \([A_\mu(x)A^\nu(x)\varphi^*(x)\varphi(x)]\) with respect to all fields) is needed for our method. We note that the photon \(A_\mu\) is real field. Factors \(ieq (\partial^\mu - \partial^\nu)\) and \((2e^2g^\mu\nu)\) are vertex factors, we again ignore them. The vertices in (19) are illustrated in Fig.5. Applying (11), each term of the total Green’s function of scalar QED is product of \([a_1^{p_1}a_2^{p_2}a_3^{p_3}a_4^{p_4}a_5^{p_5}a_6^{p_6}\] and a factor \(f_1\):

\[
f_1 = \frac{p!}{p_1!p_2!p_3!p_4!p_5!p_6!}a_1^{p_1}a_2^{p_2}\left[\frac{a_3}{2}\right]^{p_3}\left[\frac{a_4}{2}\right]^{p_4}\left[\frac{a_5}{2}\right]^{p_5}\left[\frac{a_6}{2}\right]^{p_6} = f_1a_1^{p_1}a_2^{p_2}a_3^{p_3}a_4^{p_4}a_5^{p_5}a_6^{p_6},
\]

Now, as an example, we investigate contractions of a vertex of type \(a_3\) with other vertices. This vertex has two identical lines, so that in some case we can change roles of these two lines to create new terms.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{vertices}
\caption{Vertices of scalar QED}
\end{figure}

Each vertex of kind \(a_1\) or \(a_6\) has different fields, \(a_5\) has no relation with other vertices. Vertices \(a_3\) and \(a_4\), each has two identical lines. All different contractions of \(a_{1,2,3,4,5,6}\) create a new factor:

\[
f_2 = \frac{2^{p_3}2^{p_4}}{\Pi_n(n!)^{\alpha_n}}.
\]

where \(n\) and \(\alpha_n\) were mentioned in (11).
Next, similar to QED case, a factor caused from making contractions of vertices is given by:

$$p_1^!p_2^!p_3^!p_4^!p_6^! \not\!
\frac{g'}{g'} = f_3,$$

where $g'$ is number of vertex permutations of types $a_{1,2,3,4}$ and $a_6$ creating identical diagrams. Then, the total factor is:

$$f = \frac{1}{S} = \frac{1}{p!f_1f_2f_3}
= \frac{1}{p!} \frac{p!}{2p_3^!+p_4^!+p_6^!} \frac{2p_3^!2p_4^!}{p_1^!p_2^!p_3^!p_4^!p_6^!}
\frac{\prod_n(n!)^{\alpha_n}}{g'}
= \frac{1}{g^2p_5^!p_6^!\prod_n(n!)^{\alpha_n}},$$

where $g = g'p_5^!$ now is different from $g'$-permutation number of all vertices. We note that $g_5^!$ related with $a_5$ now is included in $g$. It is easy to realize that the quantity $\beta$ that appears in Eq. (21) is given by $p_5^!+p_6^!$ for this case.

From (23) we come to following conclusions:

1. $a_2$ and $a_4$ do not contribute to $\beta$. Remember that this important property of complex scalar fields leads to the discrimination against real ones. Also, non self-conjugate bubble in $a_5$ does not create any new factors. $a_5$ in $\beta$ therefore comes from bubbles of $A_\mu$s. As a consequence, we conclude that $A_\mu$s play equivalent roles to real scalar fields.

2. Although this theory contains vertex $a_5$ with two different bubbles, the factor $2^d$ does not appear. Clearly, $d$ is only the number of vertices with double identical bubbles. This new result is very important for theories with many different fields such as $[\phi^2\phi^2]$. .

Our formula can be verified by results of Ref. 3.

5 Symmetry factors in QCD

The Lagrangian in the QCD is given by

$$\mathcal{L}^{QCD} = \sum_{i=1}^{3} \bar{\psi}(iD_\mu\gamma^\mu - m)\psi - \frac{1}{4} F^{a}_{\mu\nu} F^{a}_{\mu\nu},$$

where $D_\mu = \partial_\mu - ig_s t_a A^a_\mu$, $t_a$s are representation matrices of $SU(3)_C$, and $t_a = \frac{\lambda_a}{2}$ for the basic representation, $F^{a}_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc} A^b_\mu A^c_\nu$. The indices $a, b, c = 1, 2, ..., 8$, $\psi$ has three color components: $\psi = (\psi^R, \psi^G, \psi^B)^T$, $A^a_\mu$s are gauge gluon fields.

Expanding the above Lagrangian, we have an interaction lagrangian:

$$\mathcal{L}^{QCD}_{int} = g_s \bar{\psi}\gamma^\mu t^a \psi A^a_\mu - g_s f^{abc}(\partial_\mu A^a_\nu)A^{b\mu}A^{c\nu} - \frac{1}{4} g_s^2 (f^{eab} A^a_\mu A^b_\nu)(f^{ecd} A^c_\mu A^d_\nu)$$
We emphasize that QCD is different from QED, gluon gauge fields of QCD $A^a_\mu$s are labeled by a color quantum number $a$ and belong to adjoint representation of $SU(3)_C$. But all of $A^a_\mu$s are real fields, or self-conjugate fields. Since quarks carry colors so gluons must carry them too and physical gauge fields are combinations of $A^a_\mu$s. However, due to the assumption that all observed particles only are color singlet, we can work with just $A^a_\mu$s [17]. It is easy to see that the first interacting term identifies with T-product. Hence, we can write this T-product in terms of sum of $N$-products [17]:

$$T\{\bar{\psi}\gamma^\mu t^a A^a_\mu \psi\} = \frac{N}{4} \bar{\psi} \gamma^\mu t^a A^a_\mu \psi + \frac{N}{4} \bar{\psi}(x)\gamma^\mu t^a A^a_\mu \psi(x)$$

where $\alpha, \beta$ are Dirac indices, and $i, k$ are $SU(3)_C$ ones. For the third term of [25], firstly we rewrite it in new form [17]:

$$\frac{1}{4}(f^{eabc} A^a_\mu A^b_\nu)(f^{eabcd} A^{\mu c} A^{\nu d}) = \left[f^{eabc} f^{eabcd} (g^{\mu \alpha} g^{\nu \beta} - g^{\mu \beta} g^{\nu \alpha}) + f^{eabc} f^{eabd} (g^{\mu \alpha} g^{\nu \beta} - g^{\mu \beta} g^{\nu \alpha}) + f^{eabc} f^{eabcd} (g^{\mu \alpha} g^{\beta \nu} - g^{\beta \alpha} g^{\mu \nu})\right] \frac{1}{4!} A^a_\mu A^b_\nu A^c_\alpha A^d_\beta$$

Next we choose T-product of this term as $\frac{1}{4!} T\left[A^a_\mu A^b_\nu A^c_\alpha A^d_\beta\right]$. The rest is vertex factor. Then we will get:

$$\frac{1}{4!} T(A^a_\mu A^b_\nu A^c_\alpha A^d_\beta) = \frac{1}{4!} N [A^a_\mu A^b_\nu A^c_\alpha A^d_\beta] + \frac{6}{4!} \underbrace{A^a_\mu A^b_\nu} N [A^c_\alpha A^d_\beta]$$

$$+ \underbrace{A^a_\mu A^b_\nu} N [A^c_\alpha A^d_\beta].$$

We see that (28) is a part of (27) which brings out S-factors. This part is almost identical with the expansion of real scalar theory [3], except indices $a$ and $\mu$ of gluon fields. However these indices are quiet. Four field components $A^a_\mu, A^b_\nu, A^c_\alpha$ and $A^d_\beta$ are the same after doing contractions to form internal lines without directions. Hence, we can consider them as four identical real scalar fields. Thus, the S-factor formula of this case is also given by the formula (11).

The second term in (25) contains derivatives, so it is easier to work in momentum-space. In this space, if we denote momentum of $A^a_\mu, A^b_\nu$ and $A^c_\sigma$ correspond to $p, k$ and $q$ then we can write $\partial^a A^c_\sigma \equiv (\partial^c_\mu) A^a_\mu$, etc..., namely:

$$f^{abc}(\partial^a A^c_\mu) A^a_\mu A^c_\nu \equiv f^{abc}(\partial^c_\mu)(A^a_\nu A^b_\mu A^c_\sigma) g^{\sigma \nu},$$

in sense that $\partial^a_\mu$ does not operate on any fields except those having momentum $p$.

Now (29) can be rewritten in the form [17]:

$$f^{abc}(\partial^a A^c_\mu) A^a_\mu A^c_\nu = f^{abc}[g^{\mu \nu}(\partial_p - \partial_k)^a + g^{\mu \sigma}(\partial_k - \partial_q)^\nu + g^{\sigma \nu}(\partial_q - \partial_p)^\mu] \times \frac{1}{6} A^a_\mu A^b_\nu A^c_\sigma$$

Again, the first factor in (30) is the three-gluon interaction vertex factor in momentum-space and the second, the T-product, is the same as interacting term of $\varphi^3$ real scalar theory.
In conclusion for calculating S-factor of QCD, all fermion fields can be considered as scalar complex fields, and gluon fields play the roles of real scalar fields. Then, the formula (11) is applicable to the QCD. Some examples are given in Fig. 6. These results also agree with those given in Ref. [10] and [16].

\[ a. \alpha_2 = 1, g = 1 \]
\[ S = 2! = 2 \]
\[ b. \alpha_3 = 1, g = 1 \]
\[ S = 3! = 6 \]

Figure 6: Examples of S-factors in QCD

6 An example of Standard Model

Now we turn to the Standard Model. Let us consider a particular coupling between W with charged currents:

\[ \mathcal{L}_{CC}^{ij} = \frac{g}{\sqrt{2}} \left[ W_{\mu}^{+} J_{\mu}^{i} + W_{\mu}^{-} J_{\mu}^{i} \right] \]
\[ = \sum_{i=1}^{3} \frac{g}{2\sqrt{2}} \left\{ W_{\mu}^{+} \bar{\nu}_i \gamma_{\mu}(1-\gamma_5) e_i + \bar{u}_i \gamma_{\mu}(1-\gamma_5) d_i \right\} + W_{\mu}^{-} \bar{e}_i \gamma_{\mu}(1-\gamma_5) \nu_i + \bar{d}_i \gamma_{\mu}(1-\gamma_5) u_i \right\} \]

This Lagrangian has twelve terms in the same form as \( g/(2\sqrt{2}) \bar{\psi}_i \gamma_{\mu}(1-\gamma_5) \psi' W_{\mu} \) in which all terms have the same vertex factor \( g/(2\sqrt{2}) \gamma_{\mu}(1-\gamma_5) \). By our choice, T-products have form \( \bar{\psi}_i \psi' W_{\mu} \). It is very simple to calculate because T-product is equal to N-product. In similarity with the case of QED, we easily prove that W field behaves the same way as a complex scalar field.

Our analysis leads to a general principle: interactions such as given in (31), Yukawa couplings, etc, are similar to interactions in the spinor QED. Hence we conclude that: S-factors of all external connected (sub-)diagrams containing only vertices with three different fields, are equal to unity. Illustrations can be found in appendix B. We must remember that W boson is complex scalar-like, in S-factor calculation, although in diagrams we do not draw its propagator direction.

It is emphasized that Majorana neutrinos belong to real scalar-like. For more details, interested readers can find in Refs. [18][19].

7 The vacuum diagrams factorization

Every Feynman diagram consists of two kinds of well-known connected pieces, namely external connected and vacuum connected subdiagrams. One diagram may
include many identical vacuum connected pieces. Conversely, all external connected pieces are different from each others because they connect to different external legs. Each piece has its private S-factor which is independent on the others.

It is interesting that the S-factor of a total diagram can be presented as a product of private S-factors of connected pieces, that is well known vacuum diagrams factorization. These private S-factors can be clearly evaluated from our analysis. If there are \(i\) different kinds of connected piece (easily classified by geometric properties) then we can label an index \(i\) for any thing related with a piece of the \(i\)th kind, such as \(g_i, \beta_i, d_i, n_i, \alpha_{n_i}\) without losing original meanings of \(g, \beta, d, \alpha\). Then, each connected piece of kind \(i\) contributes a factor \(S_i\) to the total S-factor:

\[
S_i = g_i \beta_i^2 d_i n_i! \alpha_{n_i} \prod n_i, \quad (32)
\]

which is the same as \(\mathbb{I}\) except an extra index \(i\). Each set consisting of all \(k_i\) indistinguishable pieces (pieces in kind \(i\)) causes a factor \(\Pi k_i! (n_i!)^{\alpha_{n_i}}\). The total S-factor now is presented as a new expression:

\[
S = \prod_i k_i! (S_i)^{k_i} = \prod_i k_i! (g_i)^{\beta_i} \times 2 \sum_i k_i \beta_i \times 2 \sum_i k_i d_i \times \Pi n_i! (n_i!)^{\alpha_{n_i}} \quad (33)
\]

Comparing with \(\mathbb{I}\) we have \(\beta = \sum_i k_i \beta_i, d = \sum_i k_i d_i\) and the factor \(\Pi n_i! (n_i!)^{\alpha_{n_i}}\) can be replaced by \(\Pi n!^{\alpha_n}\), where \(\alpha_n = \sum_i k_i \alpha_{n_i}\) because \((n, n_1, n_2, ... = 1, 2, 3, ...\) are running indices. Especially, the relation between \(g\) and \(g_i\)s:

\[
g = \prod_i k_i! (g_i)^{k_i}, \quad (34)
\]

can help us practically calculate \(g\) from \(g_i\)s. The most convenient property of \(g_i\) is that it is equal to the number of graphical symmetry transformations of a connected piece in the \(i\)th kind.

In practice, \(\alpha_n, d\) and \(\beta\) can directly be deduced from the particular graphical properties of diagram itself. Taking into account of \(34\), we determine \(g\) from \(g_i\).

To see the vacuum factorization, from \(33\), we group all factors related with connected vacuum pieces (private S-factors, \(S_i\)s, and factors \(k_i!\)-rising from \(k_i\) identical pieces) in a single factor called vacuum symmetry factor \(S_v\) and the remaining-external connected ones in another factor \(S_c\), then the S-factor of the diagram is divided into two factors \(S = S_v \times S_c\). If we sum all of total diagrams in all orders with their S-factors we will receive results mentioned in Refs. \(9, 10\).

\[
\begin{align*}
\text{(a) } g = 1; \alpha_2 = 2 & \quad \text{(b) } S = 1 \quad \text{(c) } g = 1; \alpha_2 = 2 \\
S = 4 & \quad S = 4
\end{align*}
\]

Figure 7: S-factors for diagrams with different propagator directions.
One more remark we point out here: normally, when drawing a diagram, we just pay attention to directions of momenta while omitting directions of propagators (for example, W boson). This makes us confused in counting β and we may lose some diagrams because there are diagrams differing only in directions of propagators. For example, with self-interacting term of W boson we have a diagram without directions of propagators in figure 7a which has the same S-factor as the one in figure 7c - the figure including charged transition directions of W (not direction of momentum). But in case of this directional field, there is another different diagram in figure 7b which does distinguish from the one in figure 7c by only their directions of lines. Both of them have same contributions but different S-factor values. The S-factor now is related with not only figure 7a or 7c but also with both of 7b and 7c. For simplicity, we can define a diagram without directions in lines, and call it equivalent diagram. This kind of diagram stands for all diagrams which have the same geometrical shape and contribution but are different in directions of lines. Then the S-factor of an equivalent diagram is different from usual: it is S-factor for the total contribution and the inverse of this factor is the sum of inverse ones of directional diagrams.

To determine S-factor of an equivalent diagram (for example, see, Fig.7a), we have to find out S-factors of all possible directional diagrams coming from this non-directional diagram, and denote S-factors $S_1, S_2, \ldots$, respectively. Then the S-factor of the equivalent diagram is obtained as follows:

$$\frac{1}{S} = \sum_n \frac{1}{S_n}.$$ \hspace{1cm} (35)

One of our new results that has never been mentioned before: All well-known formulas for S-factors, including our formula in this paper, only work on directional diagrams where all directions of complex scalar-like fields are pointed out. For example, in (35) our calculation is only used for $S_n$, not for S. From now on, our formula implies S-factor of $S_n$.

8 Conclusion

Based on cases illustrated above, we conclude that our calculation does not depend on the spins of fields. It only depends on whether fields presenting a particle and its anti-particle are identical or not. In other words, the class of fields is very important in our calculation of S-factors. We have two classes of field, real scalar-like and complex scalar-like, as mentioned in the second section. For practical calculation, in Table 1 we list some known fields.

Now, as our main result, we introduce a general formula of symmetry factor for Feynman diagrams of theories containing many different fields with any spin values. Although it has the same form as the formula for scalar theories (11)

$$S = g^2 \beta^2 \prod_n (n!)^{\alpha_n},$$ \hspace{1cm} (36)

definitions of $\alpha_n$, $d$ and $\beta$ are generalized. They are redefined as:
Table 1: Classification of fields

| Real scalar-like | Complex scalar-like |
|------------------|---------------------|
| Real scalar      | complex scalar      |
| Photon $A_\mu$   | spinor Dirac field  |
| $Z$ boson        | $W$ boson           |
| Gluon            | Ghost               |
| Majarona fields  |                     |

- $\alpha_n$ is the number of sets of $n$ identical lines connecting the same pairs of vertices (there may be more than such one sets in one vertex pair).
- $d$ is the number of vertices with two identical bubbles.
- $\beta$ is sum of all self-conjugate bubbles coming from self-conjugate fields ($\beta$ vanishes if all fields in the theory belong to non-conjugate fields).
- $g$ is the number of vertex permutations keeping the diagram topologically unchanged.

We must emphasize that the most important goal of our work is to find out the general definitions of these parameters in common case. They are more general than [9,14] and others.

The most important thing: formula (36) is applicable to diagrams where all directions of propagators are showed (although they may not be drawn in diagrams).

We remind one interesting property of our result: The diagrams with different topologies can contribute the same, and the inverse symmetry factor for the total contribution is therefore the sum of the inverse symmetry ones, i.e., $1/S = \sum_i (1/S_i)$.

We have showed that the S-factors of all external connected diagrams containing vertices with three different fields such as interactions in spinor QED, Yukawa couplings, etc, *are equal to unity* ($S = 1$). This conclusion is also correct for all diagrams consisting of only vertices with different legs.

We recall that determining the symmetry factor is important because it not only is an important component of modern quantum field theory, but also is used to calculate effective potentials in higher-dimensional theories and cosmological models. Our formula works on all of these.

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A. Examples of Feynman diagrams in QED up to fourth order.

(a1) $S = g = 1$

(b1) $S = g = 1$

(c1) $S = g = 2$

(a2) $S = g = 1$

(b2) $S = g = 2$

(c2) $S = g = 2$

(d2) $S = g = 1$

(a3) $S = g = 1$

(b3) $S = g = 1$

(c3) $S = g = 1/2$

(a4) $S = g = 1$

(b4) $S = g = 2$

(c4) $g_1 = 2$; $S = 2!(g_1)^2 = 8$

(a5) $S = g = 1$

(b5) $S = g = 1$

(c5) $S = g = 1$
(a.6) \( g_1 = 2, k_1 = 2 \)
\[ S = g = 2! \cdot (2)^2 = 8 \]

(b.6) \( S = g = 2 \)

(c.6) \( S = g = 1 \)

(a.7) \( S = g = 1 \)

(b.7) \( S = g = 1 \)

(c.7) \( S = g = 1 \)

(a.8) \( S = g = 1 \)

(b.8) \( S = g = 2 \)

(c.8) \( S = g = 1 \)

(a.9) \( S = g = 2 \)

(b.9) \( g_1 = g_2 = 2; k_1 = k_2 = 1 \)
\[ S = g = g_1 g_2 = 4 \]

(c.9) \( S = g = 2 \)
(a.10) $S = g = 2$

(b.10) $S = g = 1$

(c.10) $S = g = 4$

(a.11) $S = g = 2$

(b.11) $S = g = 2$

(c.11) $g_1 = 2, k_1 = 2$

$S = g = 2!2^2 = 8$

(a.12) $S = g = 1$

(b.12) $S = g = 1$

(c.12) $S = g = 1$

(a.13) $S = g = 2$

(b.13) $S = g = 8$

(c.13) $S = g = 1$

(d.13) $S = g = 1$
B Examples of Feynman diagrams in SM up to tenth order: $\mu^- \rightarrow \nu_\mu + e^- + \bar{\nu}_e$.

This case we must remember that all W-boson lines are directional, though we don’t draw.
\[(a.19) \quad g_1 = 1, k_1 = 2 \quad S = 2\]

\[(b.19) \quad S = 1\]

\[(c.19) \quad S = 1\]

\[(a.20) \quad g_1 = g_2 = 1 \quad k_1 = k_2 = 1 \quad s = 1 \quad S = 3! = 6\]

\[(b.20) \quad g_1 = 1, k_1 = 3 \quad S = 6\]

\[(c.20) \quad g_1 = 1, k_1 = 3\]

\[(a.21) \quad g_1 = g_2 = 1 \quad k_1 = 2, k_2 = 1 \quad S = 2\]

\[(b.21) \quad S = 1\]

\[(c.21) \quad S = 1\]

\[(a.22) \quad S = 2\]

\[(b.22) \quad S = 1\]

\[(c.22) \quad S = 1\]

\[(a.23) \quad g_1 = g_2 = 1 \quad k_1 = k_2 = 2 \quad S = 4\]

\[(b.23) \quad g_1 = g_2 = 1 \quad k_1 = 3, k_2 = 1 \quad S = 6\]

\[(c.23) \quad g_1 = 1, k_1 = 4 \quad S = 24\]