A new relation between quark and lepton mixing matrices

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Abstract

We propose a new relation between quark mixing matrix and lepton mixing matrix. Since the parameters in the quark sector are well determined, we employ them to describe the mixing of leptons. Phenomenologically, we study the neutrino oscillation probabilities for different channels, which can be measured precisely in forthcoming reactor and accelerator experiments. As an example of the applicability of our assumption, CP violation in the lepton sector is also discussed. In the latest T2K experiment, the range of the mixing angle $\theta_{13}$ is measured, and our prediction of $\theta_{13}$ is compatible with their result.

Key words: neutrino oscillation; lepton mixing; quark-lepton complementarity; quark mixing
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The results of many neutrino oscillation experiments in the last decade have demonstrated that there exist physics beyond standard model in neutrino sector. It is commonly accepted that neutrinos are massive and mixing \cite{1}. Neutrino oscillation is governed by two mass square differences $\Delta m^2_{21}$, $\Delta m^2_{31}$ and the lepton mixing matrix proposed by Pontecorvo, Maki, Nakawaga and Sakata (PMNS) \cite{2}

\begin{equation}
U_{\text{PMNS}} = \begin{pmatrix}
U_{e1} & U_{e2} & U_{e3} \\
U_{\mu1} & U_{\mu2} & U_{\mu3} \\
U_{\tau1} & U_{\tau2} & U_{\tau3}
\end{pmatrix}.
\end{equation}

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If neutrinos are of the Majorana type, there should be an additional diagonal matrix with two Majorana phases $P = \text{diag}(e^{i\alpha_1/2}, e^{i\alpha_2/2}, 1)$ multiplied to Eq. (1) from the right. But the two Majorana phases do not affect neutrino oscillations, thus we do not include them in our calculations.

Before more underlying theory of the origin of the mixing is found, parametrizing the mixing matrix properly is helpful in understanding the mixing pattern and analyzing experimental results. A commonly used form is the standard parametrization proposed by Chau and Keung (CK) [3]

$$U_{\text{CK}} = \begin{pmatrix}
    c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
    -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\
    s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} - c_{12}s_{23}s_{13}e^{i\delta} & c_{23}c_{13}
\end{pmatrix}, \quad (2)$$

where $s_{ij} = \sin \theta_{ij}$ and $c_{ij} = \cos \theta_{ij}$ ($i, j = 1, 2, 3$) are the mixing angles, and $\delta$ is the CP-violating phase. The current global fits of the neutrino mixing parameters are given at the 1(3)$\sigma$ level by [4]

$$\theta_{12} = 34.4 \pm 1.0 \ (^{+3.2}_{-2.9})^{\circ}, \quad \theta_{23} = 42.8^{+4.7}_{-2.9} \ (^{+10.7}_{-7.3})^{\circ}, \quad \theta_{13} = 5.6^{+3.0}_{-2.7} \ (\leq 12.5)^{\circ}; \quad (3)$$

$$\Delta m_{21}^2 = 7.59 \pm 0.20 \ (^{+0.61}_{-0.09}) \times 10^{-5} \text{eV}^2, \quad \Delta m_{31}^2 = \begin{cases}
    -2.36 \pm 0.11 \ (\pm 0.37) \times 10^{-3} \text{eV}^2, \\
    +2.46 \pm 0.12 \ (\pm 0.37) \times 10^{-3} \text{eV}^2.
\end{cases} \quad (4)$$

These results for angles are compatible with the tri-bimaximal (TB) matrix [5]

$$U_{\text{TB}} = \begin{pmatrix}
    2/\sqrt{6} & 1/\sqrt{3} & 0 \\
    -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\
    1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2}
\end{pmatrix}. \quad (5)$$

Therefore, it is widely accepted that $U_{\text{TB}}$ is a good approximation to reality [6].

In contrast to the large mixing in lepton sector, the observed Cabibbo-Kobayashi-Maskawa (CKM) quark mixing matrix $V_{\text{CKM}}$ is close to the unit matrix. Although it seems that the mixing of quarks and leptons are unrelated with each other, there indeed exist phenomenological relations between mixing angles called quark-lepton complementarity (QLC) [8], given by

$$\theta_{12}^Q + \theta_{12} = \pi/4, \quad \theta_{23}^Q + \theta_{23} = \pi/4, \quad \theta_{13}^Q \sim \theta_{13} \sim 0. \quad (6)$$

We pointed out in previous works [9] that the unit matrix pattern for quark mixing is connected with the bimaximal matrix pattern [10] for lepton mixing.
through QLC relations. But present data imply that the tri-bimaximal matrix
$U_{TB}$ is closer to reality than bimaximal matrix. Therefore, a natural idea is
to connect the unit matrix in quark sector with the tri-bimaximal matrix in
lepton sector. Based on this consideration, we propose here a simple relation
between the lepton and quark mixing matrices

$$V_{CKM}^\dagger U_{PMNS}V_{CKM} = U_{TB},$$  \hspace{1cm} (7)

under an Ansatz that $U_{PMNS}$ becomes an exact tri-bimaximal mixing $U_{TB}$
in a limit $V_{CKM} = 1$ \[11\]. Such a relation maybe comes from certain flavor
symmetries, and the corresponding symmetry breaking effects may induce
the deviations of the observed $V_{CKM}$ and $U_{PMNS}$ from the exact unit matrix
and tri-bimaximal matrix. By assuming Eq. (7), we can employ one set of
parameters to describe both quark and lepton mixing matrices, thus Eq. (7)
can be regarded as the quark-lepton complementarity in matrix form.

Currently, the quark mixing matrix is well determined. For the Wolfenstein
parametrization \[12\]

$$V_{CKM} = \begin{pmatrix}
1 - \frac{1}{2} \lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\
-\lambda & 1 - \frac{1}{2} \lambda^2 & A\lambda^2 \\
A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1
\end{pmatrix} + O(\lambda^4),$$ \hspace{1cm} (8)

the up-to-date fit gives \[13\]

$$\lambda = 0.2253 \pm 0.0007, \quad A = 0.808^{+0.022}_{-0.015},$$
$$\rho(1 - \lambda^2/2 + \ldots) = 0.132^{+0.022}_{-0.014}, \quad \eta(1 - \lambda^2/2 + \ldots) = 0.341 \pm 0.013.$$ \hspace{1cm} (9)

By inserting Eq. (8) into Eq. (7), one can easily get the PMNS matrix in terms
of $\lambda$, $A$, $\rho$ and $\eta$, given by \cite{1}

\[
U_{\text{PMNS}} = V_{\text{CKM}} U_{\text{TB}} V_{\text{CKM}}^* = U_{\text{TB}} + \lambda \left( \begin{array}{ccc}
\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{6}} & -\frac{1+\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\
-\frac{1+\sqrt{2}}{\sqrt{3}} & -\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{6}} & 0 \\
-\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & 0
\end{array} \right) + \lambda^2 \left( \frac{A}{\sqrt{6}} + \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}} \left( \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} \right) A + \frac{1+\sqrt{2}}{\sqrt{3}} \left( \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} \right) A - \frac{1}{2\sqrt{2}} \right) + \mathcal{O}(\lambda^3).
\]

\[10\]

Clearly, we describe the deviations of $V_{\text{CKM}}$ from the unit matrix and $U_{\text{PMNS}}$ from the tri-bimaximal matrix with the same set of parameters. In other words, relation Eq. (7) provides a unified way of parametrizing both quark and lepton mixing matrices.

The lepton mixing angles are given by

\[
\sin^2 \theta_{13} = \frac{\lambda^2}{2} - \sqrt{\frac{2}{3}} A \lambda^3 + \mathcal{O}(\lambda^4),
\]
\[
\sin^2 \theta_{23} = \frac{1}{2} - \frac{1}{12} \left( 4(\sqrt{6} - 3)A + 3 \right) \lambda^2 + \frac{A}{2\sqrt{3}} \left( 2\rho + \sqrt{2} - 2 \right) \lambda^3 + \mathcal{O}(\lambda^4),
\]
\[
\sin^2 \theta_{12} = \frac{1}{3} - \frac{2}{3}(\sqrt{2} - 1)\lambda + \frac{1}{6}(3 - 2\sqrt{2})\lambda^2 + \frac{1}{9} \left( 2A(\sqrt{6} - 3\rho) + 9\sqrt{2} - 12 \right) \lambda^3 + \mathcal{O}(\lambda^4).
\]

\[11\]

With the central values in Eq. \cite{9}, we get numerically

\[\theta_{12} \cong 31.80^\circ, \quad \theta_{23} \cong 44.66^\circ, \quad \theta_{13} \cong 7.67^\circ,\]

\[12\]

which are compatible with the fit results in Eq. \cite{3}. Comparing with the exact tri-bimaximal mixing angles

\[\theta_{12} = \arcsin \frac{1}{\sqrt{3}}, \quad \theta_{23} = \pi/4, \quad \theta_{13} = 0,\]

\[13\]

our result for $\theta_{12}$ is more closer to the prediction of QLC since the $1\sigma$ range for the corresponding quark mixing angle reads \cite{11} $\theta_{12}^Q = 13.03(\pm 0.05)^\circ$.

\footnote{For simplicity, we present expressions to the second order of $\lambda$ here, but all the results below result from $U_{\text{PMNS}}$ to $\mathcal{O}(\lambda^3)$.}
Denoting $\epsilon_{ij}$ the deviations from the exact tri-bimaximal angles, we have

$$
\epsilon_{12} \equiv \theta_{12} - \arcsin \frac{1}{\sqrt{3}} \approx -3.46^\circ, \quad \epsilon_{23} \equiv \theta_{23} - \frac{\pi}{4} \approx -0.34^\circ, \quad \epsilon_{13} \equiv \theta_{13} \approx 7.67^\circ.
$$

(14)

As it shows, a relatively large $\theta_{13}$ is predicted from our assumption, and such a prediction can be tested precisely in future reactor and accelerator experiments.

An important property of the relation in Eq. (7) is that different phase conventions of $V_{\text{CKM}}$ give different predictions on lepton mixing angles. We give a brief argument here. If $V_{\text{CKM}}$ is rephased by taking

$$
V'_{\text{CKM}} = \Psi_1 V_{\text{CKM}} \Psi_2^\dagger,
$$

(15)

in which $\Psi_1 \equiv \text{diag}(e^{iu}, e^{ic}, e^{it})$ and $\Psi_2 \equiv \text{diag}(e^{id}, e^{is}, e^{ib})$ consist of arbitrary phases and can be absorbed into the redefinition of quark phases, Eq. (7) turns into

$$
U = V'_{\text{CKM}} U_{\text{TB}} V'_{\text{CKM}}^\dagger = \Psi_1 V_{\text{CKM}} \Psi_2^\dagger U_{\text{TB}} \Psi_2 V'_{\text{CKM}} \Psi_1^\dagger.
$$

(16)

Because $V_{\text{CKM}}$ ($V'_{\text{CKM}}$) does not commute with $\Psi_2^\dagger$ ($\Psi_2$), it is generally impossible to absorb the phases in $\Psi_2$ ($\Psi_2^\dagger$) into the redefinition of lepton phases. As a result, the magnitudes of elements in $U_{\text{PMNS}}$ and consequently lepton mixing angles depend on the rephasing matrix $\Psi_2$ ($\Psi_2^\dagger$), which does not bring any difference in $V_{\text{CKM}}$. Therefore, predictions on lepton mixing resulting from Eq. (7) will be changed if we change the parametrization of quark mixing matrix. Similar arguments can be applied for assumptions discussed in Ref. [11]. For generality, detailed discussion concerning the behavior of mixing angles on phases in $\Psi_2^\dagger$ ($\Psi_2$) is given in the Appendix of this letter. As an instance, another parametrization [15] for $V_{\text{CKM}}$ is also employed to determine leptonic mixing angles in the Appendix.

We now turn to the application of the assumption Eq. (7) to neutrino oscillations. Let us denote $P_{\alpha\beta} = P(\nu_{\alpha} \to \nu_{\beta})$ the probability of transition from a neutrino flavor $\alpha$ to a neutrino flavor $\beta$. Similar to the discussion in Ref. [16], the probability can be found as $P_{\alpha\beta} = |S_{\beta\alpha}(t, t_0)|^2$, in which $S(t, t_0)$ is the evolution matrix such that

$$
|\nu(t)\rangle = S(t, t_0)|\nu(t_0)\rangle, \quad S(t_0, t_0) = 1.
$$

(17)

For simplicity, we neglect the effects due to interactions between neutrinos and matters in which the neutrino beam propagates and only deal with oscillation
probabilities in vacuum, the evolution matrix can be given by

\[
S_{\beta\alpha}(t, t_0) = \sum_{i=1}^{3} (U_{\alpha i})^* U_{\beta i} e^{-iE_i L}, \quad \alpha, \beta = e, \mu, \tau,
\]  

(18)

where \( L = t - t_0 \) is the length of the baseline in neutrino experiment and \( E_i \) are the eigenvalue of the effective hamiltonian

\[
H \simeq \frac{1}{2E} U \text{diag}(0, \Delta m_{21}^2, \Delta m_{31}^2) U^\dagger,
\]

(19)

in which \( E \) is the average energy of the neutrino beam.

Inspecting the values of the mass square difference in Eq. (4), one can find \( \Delta m_{21}^2 \) is much less than \( \Delta m_{31}^2 \), thus can be neglected to a good precision. The calculation of \( P_{\alpha\beta} \) is now straightforward, such that by combining Eq. (7), Eq. (8) and Eq. (18), we get the evolution matrix \( S_{\beta\alpha} \) and consequently the oscillation probability \( P_{\alpha\beta} \). Expanded to \( \lambda^3 \), oscillation probabilities can be expressed in a matrix form as

\[
P = \begin{pmatrix}
1 - 2\lambda^2 \Delta + 4\sqrt{2} A \lambda^3 \Delta & \lambda^2 \Delta - 2\sqrt{2} A \lambda^3 \Delta & \lambda^2 \Delta - 2\sqrt{2} A \lambda^3 \Delta \\
\lambda^2 \Delta - 2\sqrt{2} A \lambda^3 \Delta & 1 - \Delta & \Delta - \lambda^2 \Delta + 2\sqrt{2} A \lambda^3 \Delta \\
\lambda^2 \Delta - 2\sqrt{2} A \lambda^3 \Delta & \Delta - \lambda^2 \Delta + 2\sqrt{2} A \lambda^3 \Delta & 1 - \Delta
\end{pmatrix} + \mathcal{O}(\lambda^4),
\]

(20)

where we have defined \( \Delta \equiv \sin^2 \left( \frac{L \Delta m_{31}^2}{4E} \right) \).

Let us have a first look at the oscillation probability matrix in Eq. (20). Apparently, it exhibits a hierarchical structure among different channels of neutrino oscillation. The diagonal elements \( P_{ee}, P_{\mu\mu} \) and \( P_{\tau\tau} \) which measure the disappearance probabilities for \( \nu_e, \nu_\mu \) and \( \nu_\tau \) beams are of \( \mathcal{O}(1) \). It is also not difficult to understand that the \( \nu_\mu \leftrightarrow \nu_\tau \) probabilities \( P_{\mu\tau} \) and \( P_{\tau\mu} \) are of \( \mathcal{O}(1) \), since \( \nu_\mu \) and \( \nu_\tau \) are maximally mixing implied by data. Other terms measuring probabilities of \( \nu_e \leftrightarrow \nu_\mu \) and \( \nu_e \leftrightarrow \nu_\tau \) are of \( \mathcal{O}(\lambda^2) \). Interestingly, there are no terms proportional to \( \lambda \), that is because such terms are suppressed by \( \Delta m_{21}^2 \), which we neglect here.

To get the anti-neutrino oscillation probabilities \( P_{\bar{\alpha}\bar{\beta}} \), one needs to take the replacement \( U_{\text{PMNS}} \rightarrow U_{\text{PMNS}}^* \), which, in our case, is equivalent to reverse the sign of \( \eta \), i.e.

\[
P_{\bar{\alpha}\bar{\beta}} = P_{\alpha\beta}(\eta \rightarrow -\eta).
\]

(21)

However, the CP-violating parameter \( \eta \) is missing in Eq. (20), meaning that
to this approximation, we have

\[ P_{\bar{\alpha}\bar{\beta}} = P_{\alpha\beta}. \]  

(22)

Therefore, CP symmetry is preserved, and resulting from the CPT theorem, time reversal symmetry is also preserved, i.e., \( P_{\alpha\beta} = P_{\beta\alpha} \). Consistently, Eq. (20) is a symmetric matrix.

Defining the asymmetries in neutrino oscillations as

\[ A_{\alpha\beta}^{\text{CP}} = P_{\alpha\beta} - P_{\bar{\alpha}\bar{\beta}}, \]  

(23)

the probabilities in Eq. (20) imply \( A_{\alpha\beta}^{\text{CP}} = 0 \). The reason for the vanishing of \( A_{\alpha\beta}^{\text{CP}} \) is that we neglect the contribution of the smaller mass square difference \( \Delta m^2_{21} \). Now taking this into account, we get nonzero values for CP asymmetries as

\[
A^{\text{CP}} = \begin{pmatrix}
0 & a & -a \\
-a & 0 & a \\
a & -a & 0
\end{pmatrix},
\]  

(24)

in which

\[
a = \frac{2}{9}(2\sqrt{3} - 3)A\eta\lambda^3 \left( \sin \left( \frac{L\Delta m^2_{31}}{2E} \right) - \sin \left( \frac{L(\Delta m^2_{31} - \Delta m^2_{21})}{2E} \right) - \sin \left( \frac{L\Delta m^2_{21}}{2E} \right) \right) + O(\lambda^4).
\]  

(25)

The structure of asymmetries in Eq. (24) comes from the unitarity of PMNS matrix and the conservation of probability. One also has [17]

\[
A_{\mu e}^{\text{CP}} = -A_{\tau e}^{\text{CP}} = A_{\tau\mu}^{\text{CP}} = 4J_{\text{CP}} \left( \sin \left( \frac{L\Delta m^2_{13}}{2E} \right) + \sin \left( \frac{L\Delta m^2_{23}}{2E} \right) + \sin \left( \frac{L\Delta m^2_{21}}{2E} \right) \right),
\]  

(26)

in which

\[
J_{\text{CP}} \equiv \text{Im}(U_{\mu 3}U_{e 2}U_{e 3}^*U_{\mu 2}^*),
\]  

(27)

is the rephasing invariant [18]. Combining Eq. (24), Eq. (25) and Eq. (26), we get

\[
J_{\text{CP}} = -\frac{1}{18}(2\sqrt{3} - 3)A\eta\lambda^3 + O(\lambda^4).
\]  

(28)

Compared with the results in Ref. [11], where they arrive at \( J_{\text{CP}} \sim O(\lambda) \) by assuming \( V_{\text{CKM}}U_{\text{PMNS}} = U_{\text{TB}} \) or \( V_{\text{CKM}}^\dagger U_{\text{PMNS}} = U_{\text{TB}} \), prediction of \( J_{\text{CP}} \) here
is quite smaller. This is because that the exact $U_{TB}$ implies $J_{CP} = 0$, thus $J_{CP}$ only depends on the deviation of $U_{PMNS}$ from $U_{TB}$ (which we denote by $D$ below). Since $V_{CKM}$ is close to the unit matrix, it can be regarded as the measurement of $D$. Then it is not difficult to understand the smallness of $J_{CP}$ in Eq. (28) as one has

$$D \sim \mathcal{O}(V_{CKM}) \quad \text{and} \quad J_{CP} \sim \mathcal{O}(\lambda) \quad (29)$$

in Ref. [11], while

$$D \sim \mathcal{O}(V_{CKM}^2) \quad \text{and} \quad J_{CP} \sim \mathcal{O}(\lambda^3) \quad (30)$$

in this letter.

We emphasize that in the very recent T2K experiment [19], observations of $\nu_\mu \rightarrow \nu_\ell$ oscillation indicate that at 90% C.L., the data are consistent with

$$0.03(0.04) < \sin^2 2\theta_{13} < 0.28(0.34) \quad (31)$$

for $\delta = 0$ and normal (inverted) hierarchy. Such a result implies an apparent deviation from $U_{TB}$ and is important in the establishment of lepton mixing pattern, in which new symmetries among leptons may hide. If $\theta_{13}$ is really large, the test of CP violation in the lepton sector is possible for future neutrino experiments since $\delta$ is always multiplied by $\theta_{13}$ in $U_{PMNS}$. With Eq. (31), straightforward calculations give

$$4.99^\circ(5.77^\circ) < \theta_{13} < 15.97^\circ(17.83^\circ) \quad (32)$$

at 90% C.L., for $\delta = 0$ and normal (inverted) hierarchy. We find that our prediction, i.e., $\theta_{13} \approx 7.67$ is compatible with the T2K result. Thus the relation Eq. (7) may serve as a good description of the deviation of $U_{PMNS}$ from $U_{TB}$.

In summary, we propose a new relation between quark mixing matrix $V_{CKM}$ and lepton mixing matrix $U_{PMNS}$ given by Eq. (7) as the quark-lepton complementarity in matrix form. Based on this relation, we parametrize $U_{PMNS}$ with the quark mixing parameters in Eq. (10), and determine the deviations of mixing angles from the exact tri-bimaximal angles. Especially, our prediction of the mixing angle $\theta_{13}$ agrees with the latest T2K result. For neutrino oscillations, we derive oscillation probabilities for different channels given by Eq. (20). As we can see, to a good precision, the expressions for each probability are quite simple. Furthermore, the CP violation in neutrino flavor transitions are discussed.

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Appendix : Dependence of lepton mixing angles on the phase convention of CKM matrix

With discussions concerning Eq. (16), we have pointed out that the predicted lepton mixing angles are dependent on the phase convention of \( V_{CKM} \), i.e., \( \Psi_2 (\Psi_2^\dagger) \) in our notation. By substituting the Wolfenstein matrix Eq. (8) into Eq. (16), we arrive at the general expression of \( U_{PMNS} \), which includes explicitly the phases in \( \Psi_1 \) and \( \Psi_2 \). Since phases in \( \Psi_1 (\Psi_1^\dagger) \) can be absorbed into the redefinition of lepton fields, the physical mixing angles are thus dependent only on phase parameters \( d, s \) and \( b \). To a good accuracy, mixing angles are determined by

\[
\sin^2 \theta_{12} = \frac{1}{3} - \frac{2}{3} \lambda \left( \sqrt{2} \cos \alpha - \cos \alpha \right) + \frac{1}{6} \lambda^2 \left( 2 \sqrt{2} \cos 2\alpha - 4 \sqrt{2} + 3 \right) + \mathcal{O}(\lambda^3),
\]

\[
\sin^2 \theta_{23} = \frac{1}{2} + \lambda^2 \left( - \sqrt{\frac{2}{3}} A \cos \beta + A \cos \beta - \frac{1}{4} \right) + \mathcal{O}(\lambda^3),
\]

\[
\sin^2 \theta_{13} = \frac{\lambda^2}{2} - \sqrt{\frac{2}{3}} A \lambda^3 \cos \gamma + \mathcal{O}(\lambda^4),
\]

where we define \( \alpha \equiv d - s, \beta \equiv b - s \) and \( \gamma \equiv b + d - 2s \), and the last equation is to \( \mathcal{O}(\lambda^3) \) because of the smallness of \( \theta_{13} \). It is easy to see that the sensitivities to phases of \( \sin^2 \theta_{12}, \sin^2 \theta_{23} \) and \( \sin^2 \theta_{13} \) are of \( \mathcal{O}(\lambda), \mathcal{O}(\lambda^2) \) and \( \mathcal{O}(\lambda^3) \) respectively. With the best fit values in Eq. (9), the dependence of each mixing angle is illustrated in Fig. 1 and Fig. 2 which show that, the prediction of \( \theta_{12} \) is strongly dependent on the phase, such that some areas of \( \alpha \) are excluded by data and, the result in Eq. (12) can be improved by choose a particular value for \( \alpha \). However, the dependence of the predicted \( \theta_{23} \) and \( \theta_{13} \) on phases is negligibly small compared with current data.

![Fig. 1. Behavior of \( \sin^2 \theta_{12} \) versus \( \alpha \equiv d - s \) (left) and \( \sin^2 \theta_{23} \) versus \( \beta \equiv b - s \) (right), both from \(-\pi\) to \(\pi\). The solid horizontal lines denote the 3\(\sigma\) ranges calculated from Eq. (3), while the dashed lines denote the best fit values.](image)

In order to demonstrate that our predictions of \( U_{PMNS} \) and mixing angles depend on the parametrization of \( V_{CKM} \), we here employ another Wolfenstein-
Fig. 2. Behavior of $\sin^2 \theta_{13}$ versus $\gamma \equiv b + d - 2s$ from $-\pi$ to $\pi$. The solid (dashed) horizontal lines denote 90% C.L. ranges of the newest T2K result with normal (inverted) hierarchy.

like parametrization [15], given by

$$V_{\text{CKM}} = \begin{pmatrix} 1 - \frac{\lambda^2}{2} & \lambda & e^{-i\delta} h \lambda^3 \\ -\lambda & 1 - \frac{\lambda^2}{2} & (f + e^{-i\delta} h) \lambda^2 \\ f \lambda^3 - (f + e^{-i\delta} h) \lambda^2 & 1 \end{pmatrix} + O(\lambda),$$

to deduce the form of $U_{\text{PMNS}}$. The ranges for the parameters are

$$\lambda = 0.2253 \pm 0.0007, \quad h = 0.303^{+0.014}_{-0.010}, \quad f = 0.754^{+0.022}_{-0.018}, \quad \delta^Q = 90.97^{+2.77}_{-4.44}.$$

Further detailed analysis and discussions concerning the relationship of this form of parametrization with others are given in Ref. [20]. By substituting this new Wolfenstein-like parametrization into Eq. (7), we get the expression for $U_{\text{PMNS}}$ to $O(\lambda^2)$ as

$$U_{\text{PMNS}} = V_{\text{CKM}} U_{\text{TB}} V_{\text{CKM}}^\dagger = U_{\text{TB}} + \lambda \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & 0 \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & 0 \end{pmatrix} + \lambda^2 \begin{pmatrix} \frac{f}{\sqrt{6}} + \frac{e^{-i\delta} h}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ -\frac{f}{\sqrt{3}} & \frac{1}{\sqrt{6}} & 0 \\ -\frac{f}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & 0 \end{pmatrix}$$

Comparing with Eq. (10), one can easily find that differences begin to appear in terms of $O(\lambda^2)$. To $O(\lambda^3)$, mixing angles are given by

$$\sin^2 \theta_{13} = \frac{\lambda^2}{2} - \sqrt{\frac{2}{3}} (f + \cos \delta) \lambda^3,$$

$$\sin^2 \theta_{23} = \frac{1}{2} - \frac{1}{12} \left(4(\sqrt{6} - 3)f + 4(\sqrt{6} - 3h) \cos \delta + 3\right) \lambda^2 + \frac{1}{2\sqrt{3}} \left((\sqrt{2} - 2)f + \sqrt{2} \cos \delta\right) \lambda^3,$$

$$\sin^2 \theta_{12} = \frac{1}{3} - \frac{2}{3}(\sqrt{2} - 1) \lambda + \frac{1}{6}(3 - 2\sqrt{2}) \lambda^2 + \frac{1}{9} \left(2\sqrt{6}f + (3(\sqrt{6} - 2)h - \sqrt{6}) \cos \delta + 9\sqrt{2} - 12\right) \lambda^3.$$
in which $\sin^2 \theta_{13}$, $\sin^2 \theta_{12}$ differ from Eq. (11) only in terms proportional to $\lambda^3$ and the difference of $\sin^2 \theta_{23}$ is of $O(\lambda^2)$. Therefore, the numerical results for mixing angles are very close to Eq. (12), as straightforward calculation gives

$$\theta_{12} \cong 31.84^\circ, \quad \theta_{23} \cong 44.61^\circ, \quad \theta_{13} \cong 7.82^\circ.$$ 

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