A NEW ALGORITHM FOR THE VOLUME OF A CONVEX POLYTOPE

JEAN B. LASSERRE AND EDUARDO S. ZERON

Abstract. We provide two algorithms for computing the volume of the convex polytope \( \Omega := \{ x \in \mathbb{R}^n_+ \mid Ax \leq b \} \), for \( A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^n \). Both algorithms have a \( O(n^m) \) computational complexity which makes them especially attractive for large \( n \) and relatively small \( m \), when the other methods with \( O(m^n) \) complexity fail. The methodology which differs from previous existing methods uses a Laplace transform technique that is well suited to the half-space representation of \( \Omega \).

1. Introduction

In this paper, we are interested in the exact computation of the volume of the convex polytope \( \Omega := \{ x \in \mathbb{R}^n_+ \mid Ax \leq b \} \), for some given matrix \( A \in \mathbb{R}^{m \times n} \) and vector \( b \in \mathbb{R}^m \).

Computing the volume of a convex polytope \( \Omega \) is difficult. Basically, methods for exact computation of this volume can be classified according to whether one has a half-space representation of \( \Omega \) as above, or a vertex representation, that is, when \( \Omega \) is given by its list of vertices (triangulation methods), or when both descriptions are available. For instance, Lasserre’s algorithm [8] requires a half-space description, whereas Delaunay’s triangulation (see e.g. [3]) or Von Hohenbalken’s simplicial decomposition [11] require the list of vertices. On the other hand, both Lawrence’s formula [9] and Cohen and Hickey’s triangulation method [4] require the double (half-space and vertex) description of the polytope. For an updated review of the above methods and their computational complexity, the interested reader is referred to Büeler et al [3]. In particular, improved versions of some of the above algorithms are also described in [3]. The computational complexity is also discussed in Dyer and Frieze [7]. In a different spirit, Barvinok [2] approximates the volume by computing the integral of \( \exp \langle c, x \rangle \) over \( \Omega \) for a small \( c \). The latter integral reduces to evaluate at each vertex \( v \) of \( \Omega \), the integral of \( \exp \langle c, x \rangle \) over the smallest convex cone \( K_v \) at \( v \), which contains \( \Omega \). Interestingly, the latter integrals are computed via a Fourier transform technique.

In general, when \( \Omega \) has a half-space representation, the methods described in Büeler et al [3] have a computational complexity that is exponential in...
n, the dimension of the underlying affine space. While those methods work well for relatively small n and possibly large m, they become very time-consuming and even fail for large (or even not so large) n. This was the motivation for an alternative method that could work in the “dual” context of possibly large n and relatively small m.

Here we suppose given a half-space representation of Ω. The alternative method that we propose is conceptually very simple (as well as the computations involved) and differs from previous existing methods. The idea is to consider the volume of Ω = \{x ≥ 0; Ax ≤ b\} as a function g : \(\mathbb{R}^m\)→\(\mathbb{R}_+\) of the right-hand side b \(\in\ \mathbb{R}^m\) for which we provide a simple explicit expression of its Laplace transform G : \(\mathbb{C}^m\)→\(\mathbb{C}\) in closed form. It then suffices to apply the inverse Laplace transform to G, which, in the present context, can be done efficiently by repeated applications of Cauchy’s Residue Theorem for the evaluation of one-dimensional complex integrals. We propose and describe two such algorithms.

As already mentioned, the \(O(n^m)\) computational complexity of both algorithms makes the method especially attractive for large n and relatively small m, when the other methods with computational complexity \(O(m^2)\) would fail. This method can also be viewed as “dual” of the latter methods which work in the original space \(\mathbb{R}^n\) with the matrix A, as we instead work in the space \(\mathbb{R}^m\) of “dual” variables associated with the constraints, and the cone \(\{u ≥ 0, A'u ≥ 0\}\) (via the Laplace transform), which explains the computational complexity \(O(n^m)\) (in lieu of \(O(m^2)\)).

2. Main result

Let \(e_i := (1,1,\cdots)\) be the unit vector of \(\mathbb{R}^i\) for \(i \geq 1\). Let \(y \in \mathbb{R}^m\) and \(A \in \mathbb{R}^{m\times n}\) be real-valued matrices such that the convex polyhedron

\[
\Omega(y) := \{x \in \mathbb{R}^n_+ \mid Ax \leq y\}
\]

(2.1)

is compact, that is, \(\Omega(y)\) is a convex polytope. The notation \(\mathbb{R}_+\) stands for the semi-closed interval \([0, \infty) \subset \mathbb{R}\).

Now consider the function \(g : \mathbb{R}^m \to \mathbb{R}\) defined by

\[
y \mapsto g(y) := \int_{\Omega(y)} dx = \text{vol}(\Omega(y)),
\]

(2.2)

and let \(G : \mathbb{C}^m \to \mathbb{C}\) be its m-dimensional two-sided Laplace transform, that is,

\[
\lambda \mapsto G(\lambda) := \int_{\mathbb{R}^m} e^{-\langle \lambda, y \rangle} g(y) \, dy.
\]

(2.3)

We have the following result:

**Theorem 2.1.** Let \(\Omega(y)\) be the convex polytope in (2.1), functions \(g\) and \(G\) are defined as in (2.2) and (2.3) respectively, and assume that \(x = 0\) is the
only solution of the system \( \{ x \geq 0, Ax \leq 0 \} \). Then:

\[
G(\lambda) = \frac{1}{\prod_{i=1}^{m} \lambda_i} \times \frac{1}{\prod_{j=1}^{n} (A'\lambda)_j}, \quad \Re(\lambda) > 0, \quad \Re(A'\lambda) > 0.
\]

Moreover,

\[
g(y) = \frac{1}{(2\pi i)^m} \int_{c_1-i\infty}^{c_1+i\infty} \cdots \int_{c_m-i\infty}^{c_m+i\infty} e^{(\lambda,y)} G(\lambda) d\lambda
\]

where the real constants \( c > 0 \) satisfies \( A'c > 0 \).

**Proof.** Apply the definition (2.3) of \( G(\lambda) \), to obtain:

\[
G(\lambda) = \int_{\mathbb{R}^m} e^{-\langle \lambda, y \rangle} \left[ \int_{x \geq 0, Ax \leq y} dx \right] dy
\]

\[
= \int_{\mathbb{R}_+^n} \left[ \int_{y \geq Ax} e^{-\langle \lambda, y \rangle} dy \right] dx
\]

\[
= \frac{1}{\prod_{i=1}^{m} \lambda_i} \int_{\mathbb{R}_+^n} e^{-\langle A'\lambda, x \rangle} dx, \quad \Re(\lambda) > 0
\]

\[
= \frac{1}{\prod_{i=1}^{m} \lambda_i} \times \frac{1}{\prod_{j=1}^{n} (A'\lambda)_j}, \quad \text{with } \left\{ \begin{array}{l} \Re(\lambda) > 0 \\ \Re(A'\lambda) > 0 \end{array} \right. .
\]

And (2.5) is obtained by a direct application of the inverse Laplace transform. It remains to show that, indeed, the domain \( \{ \Re(\lambda) > 0, \Re(A'\lambda) > 0 \} \) is nonempty. However, this fact follows from a special version of Farkas’ lemma due to Carver (see e.g. Schrijver [10, (33), p. 95]), which (adapted to the present context) states that \( \{ u > 0, A'u > 0 \} \) has an admissible solution \( u \in \mathbb{R}^m \) if and only if \( (x, y) = 0 \) is the only solution of the system \( \{ Ax + y = 0, x \geq 0, y \geq 0 \} \). In other words, \( x = 0 \) is the only solution of \( \{ x \geq 0, Ax \leq 0 \} \).

**Remark 2.2.** A necessary and sufficient condition for \( \Omega(y) \) to be compact is that there exists some \( u \in \mathbb{R}_+^m \) such that \( A'u \geq e_n \). This is a consequence of the well-known Farkas Lemma.

As mentioned above, computing \( g(y) \) via (2.5) reduces to computing the Laplace inverse of \( G(\lambda) \). In our case, this can be done quite efficiently even for large size problems. We first slightly modify our problem as follows:

Suppose that we want to compute the volume of the convex polytope \( \{ x \geq 0; Ax \leq b \} \) with \( b > 0 \), that is, we want to evaluate \( g(y) \) at the point \( y := b \in \mathbb{R}^m \). We may and shall assume, without loss of generality, that \( y_i = 1 \) for every \( i = 1, \ldots m \). Otherwise, just divide by \( y_i > 0 \).

The problem is then to compute the value \( h(1) \) of the function \( h : \mathbb{R}_+ \rightarrow \mathbb{R} \) given by

\[
h(z) = g(e_m z) = \frac{1}{(2\pi i)^m} \int_{c_1-i\infty}^{c_1+i\infty} \cdots \int_{c_m-i\infty}^{c_m+i\infty} e^{z(\lambda, e_m)} G(\lambda) d\lambda,
\]
where the real vector \( 0 < c \in \mathbb{R}^m \) satisfies \( A'c > 0 \). Computing the complex integral (2.6) can be done in two ways that are explored below. We do it directly in §3 by integrating first with respect to (w.r.t.) \( \lambda_1 \), then w.r.t. \( \lambda_2 \), etc..., or indirectly in §4, by first computing the one-dimensional Laplace transform \( H \) of \( h \) and then computing the Laplace inverse of \( H \).

3. The direct method

To better understand the methodology behind the direct method and for illustration purpose, consider the case of a convex polytope \( \Omega \) with only two \((m = 2)\) nontrivial constraints.

3.1. The \( m = 2 \) non trivial constraints example. Let \( A \in \mathbb{R}^{2 \times n} \) be such that \( x = 0 \) is the only solution of \( \{x \geq 0, Ax \leq 0\} \). Moreover, suppose that \( A' := [a | b] \) with \( a, b \in \mathbb{R}^n \). For ease of exposition, assume that

- \( a_j b_j \neq 0 \) and \( a_j \neq b_j \) for all \( j = 1, 2, \ldots n \).
- \( a_j/b_j \neq a_k/b_k \) for all \( j, k = 1, 2, \ldots n \)

Observe that these assumptions are satisfied with probability one if we add to every coefficient \( a_i, b_i \) a perturbation \( \epsilon \), randomly generated under a uniform distribution on \([0, \bar{\epsilon}]\), with \( \bar{\epsilon} \) very small.

Then:

\[
G(\lambda) = \frac{1}{\lambda_1 \lambda_2} \times \frac{1}{\prod_{j=1}^{n} (a_j \lambda_1 + b_j \lambda_2)}, \quad \left\{ \begin{array}{l} \Re(\lambda) > 0 \\ \Re(a \lambda_1 + b \lambda_2) > 0 \end{array} \right.
\]

Next, fix \( c_1 \) and \( c_2 > 0 \) such that \( a_j c_1 + b_j c_2 > 0 \) for every \( j = 1, 2, \ldots n \), and compute the integral (2.6) as follows. We first evaluate the integral

\[
I_1 = \frac{1}{2\pi i} \int_{c_1 - i\infty}^{c_1 + i\infty} e^{z\lambda_1} \frac{1}{\lambda_1 \prod_{j=1}^{n} (a_j \lambda_1 + b_j \lambda_2)} d\lambda_1,
\]

using classical Cauchy’s residue technique. That is, we: (a) close the path of integration adding a semicircle \( \Gamma \) of radius \( R \) large enough, (b) evaluate the closed integral using Cauchy’s Residue Theorem [5, Th. 2.2, p. 112], and (c) show that the integral along \( \Gamma \) converges to zero when \( R \to \infty \).

Now, since we are integrating w.r.t. \( \lambda_1 \) and we want to evaluate \( h(z) \) at \( z = 1 \), we must add the semicircle \( \Gamma \) on the left side of the integration path \( \Re(\lambda_1) = c_1 \) because \( e^{\lambda_1} \) converges to zero when \( \lambda_1 \to -\infty \). Therefore, we must consider only the poles of \( G(\lambda_1, \cdot) \) whose real part is strictly less than \( c_1 \) (with \( \lambda_2 \) being fixed). Then, the evaluation of (3.1) follows easily, and

\[
I_1 = \frac{1}{\lambda_2} \sum_{j=1}^{n} \frac{-e^{-(b_j/a_j)z\lambda_2}}{b_j \lambda_2 \prod_{k \neq j} (-a_k b_j + a_j + b_k)}.
\]

Recall that \( \Re(-\lambda_2 b_j/a_j) = -c_2 b_j/a_j < c_1 \) for each \( j = 1, 2, \ldots n \), and \( G(\lambda_1, \cdot) \) has only poles of first order (with \( \lambda_2 \) being fixed).
Therefore, 
\[
h(z) = \frac{1}{2\pi i} \int_{c_2-i\infty}^{c_2+i\infty} \frac{e^{z\lambda_2}}{\lambda_2} I_1 d\lambda_2 \\
= \frac{1}{2\pi i} \int_{c_2-i\infty}^{c_2+i\infty} \frac{e^{z\lambda_2}}{\lambda_2^{n+1}} \prod_{j=1}^{n} b_j d\lambda_2 - \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{c_2-i\infty}^{c_2+i\infty} \frac{a_j^n e^{(1-b_j/a_j)z\lambda_2}}{\lambda_2^{n+1} a_j b_j \prod_{k\neq j} (b_k a_j - a_k b_j)} d\lambda_2.
\]

These integrals must be evaluated according to whether \((1 - b_j/a_j)y\) is positive or negative. Thus, recalling that \(z > 0\), each integral is equal to
- its residue at the pole \(\lambda_2 = 0 < c_2\) when \(1 - b_j/a_j\) is positive, and
- zero if \(1 - b_j/a_j\) is negative because there is no pole on the right side of \(\Re (\lambda_2) = c_2\).

That is, 
\[
(3.2) \quad h(z) = \frac{z^n}{n!} \left[ \frac{1}{\prod_{j=1}^{n} b_j} - \sum_{a_j/b_j < 1} \frac{(a_j - b_j)^n}{a_j b_j \prod_{k\neq j} (b_k a_j - a_k b_j)} \right],
\]

Observe that the formula is not symmetrical in the parameters \(a, b\). This is because we have chosen to integrate first w.r.t. \(\lambda_1\); and the set \(\{ j \mid a_j/b_j < 1 \}\) is different from \(\{ j \mid a_j/b_j > 1 \}\), which would have been considered had we integrated first w.r.t. \(\lambda_2\). In the latter case, we would have obtained 
\[
(3.3) \quad h(z) = \frac{z^n}{n!} \left[ \frac{1}{\prod_{j=1}^{n} a_j} - \sum_{a_j/b_j < 1} \frac{(b_j - a_j)^n}{a_j b_j \prod_{k\neq j} (a_k b_j - b_k a_j)} \right],
\]

which is (3.2) by interchanging \(a\) and \(b\). Moreover, moving terms around, we get for free the following identity
\[
(3.4) \quad \sum_{j=1}^{n} a_j b_j \prod_{k\neq j} (b_k a_j - a_k b_j) = \frac{1}{\prod_{j=1}^{n} b_j} - \frac{1}{\prod_{j=1}^{n} a_j}.
\]

3.2. The direct method algorithm. The above methodology easily extends to an arbitrary number \(m\) of non trivial constraints. One evaluates the integral of the right-hand side of (2.6) by integrating first w.r.t. \(\lambda_1\), then w.r.t. \(\lambda_2\), and so on. The resulting algorithm can be described with a tree of depth \(m + 1\) (\(m + 1\) “levels”). Let \(0 < c \in \mathbb{R}^m\) be such that \(A' c > 0\).

- Level 0 is the root of the tree.
- Level 1 is the integration w.r.t. \(\lambda_1\) and consists of at most \(n + 1\) nodes associated with the poles \(\lambda_1 := \rho^1_j, j = 1, \ldots n + 1\, of the rational function \(\prod_i \lambda_i^{-1} \prod_j (A\lambda_j)^{-1}\), seen as a function of \(\lambda_1\) only. By the assumption on \(c\), there is no pole \(\rho^1_j\) on the line \(\Re(\lambda_1) = c_1\). By Cauchy’s Residue Theorem, only the poles at the left side of the integration path \(\Re(\lambda_1) = c_1\), say \(\rho^1_j, j \in I_1\), are selected.
Level 2 is the integration w.r.t. $\lambda_2$. After integration w.r.t. $\lambda_1$, each of the poles $\rho^1_j$, $j \in I_1$, generates a rational function of $\lambda_2, \lambda_3, \ldots, \lambda_m$, which, seen as a function of $\lambda_2$ only, has at most $n+1$ poles $\rho^2_i(j)$, $i = 1, \ldots, n+1$, $j \in I_1$. Thus level 2 has at most $(n+1)^2$ nodes associated with the poles $\rho^2_j$. Assuming no pole $\rho^2_j$ on the line $\Re(\lambda_2) = c_2$, by Cauchy’s Residue Theorem, only the poles $\rho^2_j(j), (j, i) \in I_2$, located on the correct side of the integration path $\Re(\lambda_2) = c_2$ are selected.

Level $k$, $k \leq m$, consists of at most $(n+1)^k$ nodes associated with the poles $\rho^k(i_1, i_2, \ldots, i_{k-1}), (i_1, i_2, \ldots, i_{k-1}) \in I_{k-1}$, $s = 1, \ldots, n+1$, of some rational functions of $\lambda_k, \ldots, \lambda_m$, seen as functions of $\lambda_k$ only. Assuming no pole on the line $\Re(\lambda_k) = c_k$, only the poles $\rho^k(i_1, i_2, \ldots, i_{k-1}), (i_1, i_2, \ldots, i_k) \in I_k$, located on the correct side of the integration path $\Re(\lambda_k) = c_k$, are selected. And so on.

The last level $m$ consists of at most $(n+1)^m$ nodes and the integration w.r.t. $\lambda_m$ is trivial as it amounts to evaluate integrals of the form

$$(2\pi i)^{-1} \int_{c_m-i\infty}^{c_m+i\infty} A\lambda_m^{-(n+1)} e^{\alpha z\lambda_m} d\lambda_m,$$

for some coefficients $A, \alpha$, which yields $A(\alpha z)^n/n!$ for those $\alpha > 0$. Summing up over all the nodes provides the desired value.

Only simple elementary arithmetic operations are needed to compute the nodes at each level, as in Gauss elimination for solving linear systems. Therefore, the computational complexity is easily seen to be $O(n^m)$.

However, some care must be taken with the choice of the integration paths as we assume that at each level $k$ there is no pole on the integration path $\Re(\lambda_k) = c_k$. This issue is discussed in §3.3. The algorithm is illustrated on the following simple example with $n = 2, m = 3$.

Example: Let $\Omega(z, e_2) \subset \mathbb{R}^2$ be the polytope

$$\Omega(z, e_2) := \{ x \in \mathbb{R}_+^2 \mid x_1 + x_2 \leq z; -2x_1 + 2x_2 \leq z; 2x_1 - x_2 \leq z \},$$

whose area is $17z^2/48$.

Choose $c_1 = 3$, $c_2 = 2$ and $c_3 = 1$, so that $c_1 > 2c_2 - 2c_3$ and $c_1 > c_3 - 2c_2$.

$$h(z) = \frac{1}{(2\pi i)^3} \int_{c_1-i\infty}^{c_1+i\infty} \cdots \int_{c_3-i\infty}^{c_3+i\infty} e^{(\lambda_1 + \lambda_2 + \lambda_3)z} G(\lambda) \, d\lambda,$$

with

$$G(\lambda) = \frac{1}{\lambda_1 \lambda_2 \lambda_3 (\lambda_1 - 2\lambda_2 + 2\lambda_3)(\lambda_1 + 2\lambda_2 - \lambda_3)}.$$

Integrate first w.r.t. $\lambda_1$; that is, evaluate the residues at the poles $\lambda_1 = 0$, $\lambda_1 = 2\lambda_2 - 2\lambda_3$ and $\lambda_1 = \lambda_3 - 2\lambda_2$ because $0 < z$, $0 < c_1$, $\Re(2\lambda_2 - 2\lambda_3) < c_1$ and $\Re(\lambda_3 - 2\lambda_2) < c_1$. We obtain

$$h(z) = \frac{1}{(2\pi i)^2} \int_{c_2-i\infty}^{c_2+i\infty} \int_{c_3-i\infty}^{c_3+i\infty} I_2 + I_3 + I_4 \, d\lambda_2 d\lambda_3,$$
where
\[ I_2 = \frac{-e^{(\lambda_2 + \lambda_3)z}}{2\lambda_2 \lambda_3 (\lambda_3 - \lambda_2)(\lambda_3 - 2\lambda_2)}, \]
\[ I_3 = \frac{e^{(3\lambda_2 - \lambda_3)z}}{6\lambda_2 \lambda_3 (\lambda_3 - \lambda_2)(\lambda_3 - 4\lambda_2/3)}, \]
\[ I_4 = \frac{e^{(2\lambda_3 - \lambda_2)z}}{3\lambda_2 \lambda_3 (\lambda_3 - 2\lambda_2)(\lambda_3 - 4\lambda_2/3)}. \]

Next, integrate \( I_2 \) w.r.t. \( \lambda_3 \). We must consider the poles on the left side of \( \Re(\lambda_3) = 1 \), that is, the pole \( \lambda_3 = 0 \) because \( \Re(\lambda_2) = 2 \). Thus, we get
\[ -\frac{e^{z\lambda_2}/4\lambda_3^2}{8}, \]
and the next integration w.r.t. \( \lambda_2 \) yields \( -z^2/8 \).

When integrating \( I_3 \) w.r.t. \( \lambda_3 \), we have to consider the poles \( \lambda_3 = \lambda_2 \) and \( \lambda_3 = 4\lambda_2/3 \), on the right side of \( \Re(\lambda_3) = 1 \); and we get
\[ -\frac{1}{\lambda_3^3} \left[ \frac{e^{2z\lambda_2}}{2} + \frac{3e^{z\lambda_25/3}}{8} \right]. \]

Recall that the path of integration has a negative orientation, so we have to consider the negative value of residues. The next integration w.r.t. \( \lambda_2 \) yields \( z^2(1 - 25/48) \).

Finally, when integrating \( I_4 \) w.r.t. \( \lambda_3 \), we must consider only the pole \( \lambda_3 = 0 \), and we get \( e^{-z\lambda_2}/8\lambda_3^3 \); the next integration w.r.t. \( \lambda_2 \) yields zero. Hence, adding up the above three partial results, yields
\[ h(z) = z^2 \left[ -\frac{1}{8} + 1 - \frac{25}{48} \right] = \frac{17 z^2}{48}, \]
which is the desired result.

3.3. Paths of integration. In choosing the integration paths \( \Re(\lambda_k) = c_k \), \( k = 1, \ldots, m \), we must determine a vector \( 0 < c \in \mathbb{R}^m \) such that \( A'c > 0 \). However, this may not be enough when we want to evaluate the integral (2.6) by repeated applications of Cauchy’s Residue Theorem. Indeed, we have seen in the tree description of the algorithm (cf. [1,2]), that at each level \( k > 1 \) of the tree (integration w.r.t. \( \lambda_k \)), one assumes that there is no pole on the integration path \( \Re(\lambda_k) = c_k \).

For instance, had we set \( c_1 = c_2 = c_3 = 1 \) (instead of \( c_1 = 3, c_2 = 2 \) and \( c_1 = 1 \)) in the above example, we could not use Cauchy’s Residue Theorem to integrate \( I_2 \) or \( I_3 \) because we would have the pole \( \lambda_2 = \lambda_3 \) exactly on the path of integration (recall that \( \Re(\lambda_2) = \Re(\lambda_3) = 1 \)); fortunately, this case is pathological as it happens with probability zero in a set of problems with randomly generated data \( A \in \mathbb{R}^{m \times n} \) and, therefore, this issue could be ignored in practice. However, for the sake of mathematical rigor, in addition to the constraints \( c > 0 \) and \( A'c > 0 \), the vector \( c \in \mathbb{R}^m \) must satisfy additional constraints to avoid the above mentioned pathological problem. We next describe one way to proceed to ensure that \( c \) satisfies these additional constraints.
In §3.2 we have described the algorithm as a tree of depth $m$ (level $i$ being the integration w.r.t. $\lambda_i$) where each node has at most $n+1$ descendants (one descendant for each pole on the correct side of the integration path $\Re(\lambda_i) = c_i$). The volume is then the summation of all partial results obtained at each leaf of the tree (that is, each node of level $m$). We next describe how to “perturbate” on-line the initial vector $c \in \mathbb{R}^m$ if at some level $k$ of the algorithm there is a pole on the corresponding integration path $\Re(\lambda_k) = c_k$.

- **Step 1. Integration w.r.t. $\lambda_1$.** Choose a real vector $c := (c_1, \ldots , c_m) > 0$ such that $A'c > 0$ and integrate $\text{(2.4)}$ along the line $\Re(\lambda_1) = c_1^1$. From Cauchy’s Residue Theorem, this is done by selecting the (at most $(\Re)\lambda_1$ located at the left or the right of the line $\Re(\lambda)$, and then integrate w.r.t. $\Re(\lambda)$.

Each pole $\rho_1^j, j = 1, \ldots , n + 1$ (with $\rho_1^1 := 0$) is a linear combination $\beta_{j2}(1)\lambda_2 + \ldots + \beta_{jm}(1)\lambda_m$ with real coefficients $\{\beta_{jk}^{(1)}\}$, because $A$ is a real-valued matrix. Observe that by the initial choice of $c$,

$$\delta_1 := \min_{j=1, \ldots , n+1} |c_1^1 - \sum_{k=2}^m \beta_{jk}^{(1)} c_k^1| > 0.$$ 

- **Step 2. Integration w.r.t. $\lambda_2$.** For each of the poles $\rho_1^j, j \in I_1$, selected at step 1, and after integration w.r.t. $\lambda_1$, we now have to consider a rational function of $\lambda_2$ with at most $n+1$ poles $\lambda_2 := \rho_1^2(j) := \sum_{k=3}^m \beta_{ik}^{(2)}(j)\lambda_k$, $i = 1, \ldots , n + 1$. If

$$\delta_2 := \min_{j \in I_1} \min_{i=1, \ldots , n+1} |c_2^1 - \sum_{k=3}^m \beta_{ik}^{(2)}(j)c_k^1| > 0,$$

then integrate w.r.t. $\lambda_2$ on the line $\Re(\lambda_2) = c_2^1$. Otherwise, if $\delta_2 = 0$ we set $c_2^3 := c_2^1 + \epsilon_2$ and $c_k^2 := c_k^1$ for all $k \neq 2$, by choosing $\epsilon_2 > 0$ small enough to ensure that

(a) $$A'\epsilon^2 > 0$$

(b) $$\delta_2 := \min_{j \in I_1} \min_{i=1, \ldots , n+1} |c_2^1 - \sum_{k=3}^m \beta_{ik}^{(2)}(j)c_k^1| > 0$$

(c) $$\max_{j=1, \ldots , n+1} |\beta_{j2}^{(1)}(\epsilon_2)| < \delta_1$$

The condition (a) is basic whereas (b) ensures that there is no pole on the integration path $\Re(\lambda_2) = c_2^3$. Moreover, what has been done in step 1 remains valid because from (c), $c_1^1 - \sum_{k=2}^m \beta_{jk}^{(1)} c_k^1$ has the same sign as $c_1^1 - \sum_{k=2}^m \beta_{jk}^{(1)} c_k^1$, and, therefore, none of the poles $\rho_1^j$ has crossed the integration path $\Re(\lambda_1) = c_1^1 = c_2^1$, that is, the set $I_1$ is unchanged.

Then integrate w.r.t. $\lambda_2$ on the line $\Re(\lambda_2) = c_2^2$, which is done via Cauchy’s Residue Theorem by selecting the (at most $(n+1)^2$) poles $\rho_2^2(j), (j, i) \in I_2$, located at the left or the right of the line $\Re(\lambda_2) = c_2^2$, depending on the sign of the coefficient of the argument in the exponential.
- Step 3. Integration w.r.t. $\lambda_3$. Likewise, for each of the poles $\rho^2_i(j)$, $(j,i) \in I_2$, selected at step 2, we now have to consider a rational function of $\lambda_3$ with at most $n+1$ poles $\rho^3_s(j,i) := \sum_{k=1}^m \beta^{(3)}_{sk}(j,i) \lambda_k$, $s = 1, \ldots, n+1$. If

$$\delta_3 := \min_{(j,i) \in I_2} \min_{s=1,\ldots,n+1} |c_3^2 - \frac{m}{k=4} \beta^{(3)}_{sk}(j,i) c_k^2| > 0,$$

then integrate w.r.t. $\lambda_3$ on the line $\Re(\lambda_3) = c_3^2$. Otherwise, if $\delta_3 = 0$, set $c_3^2 := c_3^2 + \epsilon_3$ and $c_k^2 := c_k^2$ for all $k \neq 3$, by choosing $\epsilon_3 > 0$ small enough to ensure that

(a) $A' c_3 > 0$

(b) $\delta_3 := \min_{(j,i) \in I_2} \min_{s=1,\ldots,n+1} |c_3^2 - \sum_{k=4}^m \beta^{(3)}_{sk}(j,i) c_k^2| > 0$

(c) $\max_{j \in I_1} \max_{i=1,\ldots,n+1} |\beta^{(2)}_{i3}(j) \epsilon_3| < \delta_2$

(d) $\max_{j=1,\ldots,n+1} |\beta^{(1)}_{j2} \epsilon_2 + \beta^{(1)}_{j3} \epsilon_3| < \delta_1$

As in previous steps, condition (a) is basic. The condition (b) ensures that there is no pole on the integration path $\Re(\lambda_3) = c_3^2$. Condition (c) (resp. (d)) ensures that none of the poles $\rho^2_i(j)$ considered at step 2 (resp. none of the poles $\rho^3_i$ considered at step 1) has crossed the line $\Re(\lambda_2) = c_2^2 = c_2^2$ (resp. the line $\Re(\lambda_1) = c_1^2 = c_1^2$). That is, both sets $I_1$ and $I_2$ are unchanged.

Then integrate w.r.t. $\lambda_3$ on the line $\Re(\lambda_3) = c_3^2$, which is done by selecting the (at most $(n+1)^3$) poles $\rho_s(j,i), (j,i,s) \in I_3$, located at the left or right of the line $\Re(\lambda_3) = c_3^2$, depending on the sign of the argument in the exponential.

And so on. It is important to notice that the $\epsilon_k$’s and $c_k^2$’s play no (numerical) role in the integration itself. They are only used to (i) ensure the absence of a pole on the integration path $\Re(\lambda_k) = c_k^2$, and (ii) to locate the poles on the left or the right of the integration path. Their numerical value (which can be very small) has no influence on the computation.

4. THE ASSOCIATED TRANSFORM ALGORITHM

An alternative to the direct method permits to avoid evaluating integrals of exponential functions in (2.4) by making the following simple change of variable. Let $\lambda_1 = p - \sum_{j=2}^m \lambda_j$ and $d = \sum_{j=1}^m c_j$ in (2.4), so that

$$h(z) = \frac{1}{(2\pi i)^m} \int_{c_m+i\infty}^{c_m-i\infty} \cdots \int_{c_2+i\infty}^{c_2-i\infty} \left[ \int_{d-i\infty}^{d+i\infty} e^{zp} \widehat{G}(p) \, dp \right] d\lambda_2 \ldots d\lambda_m,$$

where

$$(4.1) \quad \widehat{G} = G(p - \sum_{j=2}^m \lambda_j, \lambda_2, \ldots, \lambda_m).$$
We can rewrite \( h(z) \) as

\[
(4.2) \quad h(z) = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} e^{zp} H(p) dp, \quad \text{with}
\]

\[
(4.3) \quad H(p) := \frac{1}{(2\pi i)^{m-1}} \int_{c_2-i\infty}^{c_2+i\infty} \cdots \int_{c_m-i\infty}^{c_m+i\infty} \hat{G} d\lambda_2 \cdots d\lambda_m.
\]

Recall that \( G(\lambda) \) is well defined on the domain \( \Re(\lambda) > 0 \) and \( \Re(A'\lambda) > 0 \); moreover, the real vector \( c \) is taken in this domain. Hence, the domain of definition of \( H(p) \) is given by the condition

\[
(\Re(p) - \sum_{j=2}^{m} c_j, c_2, \ldots, c_m) \in \{ y \in \mathbb{R}^m \mid y > 0, A'y > 0 \}.
\]

On the other hand, recall that the system \( \{ x \geq 0, Ax \leq 0 \} \) has only one solution \( x = 0 \) (see the hypotheses of Theorem 2.1). Hence, the function \( h(z) \) is identically equal to zero when \( z \leq 0 \) (see (2.2) and (2.6)). Therefore, \( H(p) \) is the one-sided Laplace transform of \( h(z) \). Moreover, it is also easy to see that there exists a real constant \( C \) such that \( h(z) = z^n C/n! \) when \( z \geq 0 \). Therefore,

\[
H(p) = C/p^{n+1}
\]

and the main problem completely reduces to evaluating the constant \( C = h(1)n! \) by integrating \( \hat{G} \) in (4.3).

Notice that we only need to evaluate \( m - 1 \) integrals. The function \( H(p) \) is called the associated transform of \( G(\lambda) \).

Again, the integral (4.3) can be computed via repeated applications of Cauchy’s Residue Theorem (and as in the direct method algorithm of §3, some care is needed with the domain of integration and the location of the poles of \( \hat{G} \)). The method is illustrated on the same example of two non trivial constraints \( (m=2) \) already considered at the beginning of §3.

4.1. The \( m=2 \) non trivial constraints example. Let \( A \in \mathbb{R}^{2 \times n} \) such that \( x=0 \) is the only solution of \( \{ x \geq 0, Ax \leq 0 \} \). Write \( A' := [a \mid b] \) with \( a, b \in \mathbb{R}^n \). To compare with the direct method, and as in the beginning of §3, assume that \( a_j b_j \neq 0 \) for all \( j = 1, \ldots n \) and \( a_j/b_j \neq a_k/b_k \) for all \( j \neq k \).

Then :

\[
G(\lambda) = \frac{1}{\lambda_1 \lambda_2} \times \prod_{j=1}^{n} \left( \frac{1}{a_j \lambda_1 + b_j \lambda_2} \right), \quad \Re(\lambda) > 0, \quad \Re(A'\lambda) > 0.
\]

Fix \( \lambda_2 = p - \lambda_1 \) and choose a real constant \( c_1 > 0 \) such that the system of inequalities \( \Re(p) > c_1 \) and \( (a_j - b_j)c_1 + b_j \Re(p) > 0 \) for all \( j=1, \ldots n \) has a solution. We already know that there is at least one vector \( u = (c_1, \Re(p) - c_1) \) such that \( u > 0 \) and \( A'u > 0 \). We obtain \( H(p) \) by integrating \( G(\lambda_1, p - \lambda_1) \) w.r.t. \( \lambda_1 \), which yields
Next, we need to determine which poles of \( G(\lambda_1, p - \lambda_1) \) are on the left (right) side of the integration path \( \Re(\lambda_1) = c_1 \) in order to apply Cauchy’s Residue theorem. Let \( J_+ = \{ j | a_j > b_j \} \), \( J_0 = \{ j | a_j = b_j \} \) and \( J_- = \{ j | a_j < b_j \} \). Then, the poles on the left side of \( \Re(\lambda_1) = c_1 \) are \( \lambda_1 = 0 \) and \( \lambda_1 = -b_jp/(a_j - b_j) \) for all \( j \in J_+ \) because \(-b_j\Re(p)/(a_j - b_j) < c_j \). Besides, the poles on the right side of \( \Re(\lambda_1) = c_1 \) are \( \lambda_1 = p \) and \( \lambda_1 = -b_jp/(a_j - b_j) \) for all \( j \in J_- \). Finally, notice that \( G(\lambda_1, p - \lambda_1) \) has only poles or first order.

Hence, computing the residues of poles on the left side of \( \Re(\lambda_1) = c_1 \), yields

\[
H(p) = \frac{1}{p^{n+1}} \left[ \frac{1}{\prod_{j=1}^{n} b_j} - \sum_{b_j/a_j > 1} \frac{(b_j - a_j)^n}{a_j b_j \prod_{k \neq j} (b_j a_k - a_j b_k)} \right],
\]

and one retrieves (3.2) when we take \( J_0 \) to be an empty set, in other words, when its cardinality \( |J_0| = 0 \). Now, computing the negative value of residues of poles on the right side of \( \Re(\lambda_1) = c_1 \) (we need to take the negative value because the path of integration has a negative orientation), yields

\[
H(p) = \frac{1}{p^{n+1}} \left[ \frac{1}{\prod_{j=1}^{n} b_j} - \sum_{b_j/a_j > 1} \frac{(b_j - a_j)^n}{a_j b_j \prod_{k \neq j} (b_j a_k - a_j b_k)} \right],
\]

and we also retrieve (3.2).

4.2. The associated transform algorithm. As for the direct method algorithm, the above methodology easily extends to an arbitrary number \( m \) of nontrivial constraints. The algorithm also consists of \( m \) (one-dimensional integration) steps. At each step, the several one-dimensional complex integrals are evaluated by application of Cauchy’s Residue Theorem [3, Theor. 2.2, p. 112]. For same reasons as in the direct method, the computational complexity is easily seen to be \( O(n^m) \).

The general case is better illustrated on the same example as in (3.2). Again, to avoid the case of poles on the integration path in pathological examples, some care is needed when one specifies the integration path at each step of the algorithm.

Let \( \Omega(ze_2) \subset \mathbb{R}^2 \) be the polytope

\[
\Omega(ze_2) := \{ x \in \mathbb{R}^2 \mid x_1 + x_2 \leq z; -2x_1 + 2x_2 \leq z; 2x_1 - x_2 \leq z \},
\]
whose area is $17z^2/48$.

We can choose $\lambda_3 = p - \lambda_2 - \lambda_1$ and $c_1 = c_2 = 1$ such that $\Re(p) > 2$, $2\Re(p) > 5$ and $\Re(p) < 5$; and so

$$H(p) = \frac{1}{(2\pi i)^2} \int_{1-i\infty}^{1+i\infty} \int_{1-i\infty}^{1+i\infty} M(\lambda, p) \, d\lambda_1 \, d\lambda_2,$$

with

$$M(\lambda, p) = \frac{1}{\lambda_1\lambda_2 (p - \lambda_1 - \lambda_2)(2p - \lambda_1 - 4\lambda_2)(2\lambda_1 + 3\lambda_2 - p)}.$$

We first integrate w.r.t. $\lambda_1$. Only the real parts of the poles $\lambda_1 = 0$ and $\lambda_1 = (p - 3\lambda_2)/2$ are less than 1. Therefore, the residue of the 0-pole yields:

(4.4) $$\frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{1}{\lambda_2 (p - \lambda_2)(2p - 4\lambda_2)(3\lambda_2 - p)} \, d\lambda_2,$$

whereas the residue of the $(p - 3\lambda_2)/2)$-pole yields

(4.5) $$\frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{4}{\lambda_2 (p - 3\lambda_2)(p + \lambda_2)(3p - 5\lambda_2)} \, d\lambda_2.$$

At this point, we have to be careful; observe that $5/2 < \Re(p) < 5$. However, we cannot put $\Re(p) = 3$ because otherwise we will have a pole in the path of integration of (4.4) and (4.5). We thus fix $3 < \Re(p) < 5$. Applying again Cauchy’s Residue Theorem to (4.4) at the pole $\lambda_2 = 0$ (the only one whose real part is less than one), yields $-1/2p^3$.

Similarly, applying Cauchy’s Residue Theorem to (4.5) at the poles $\lambda_2 = 0$ and $\lambda_2 = -p$ (the only ones whose real part is less than one), yields $4/3p^3 - 1/8p^3$.

We finally have that $H(p) = (4/3 - 1/8 - 1/2)/p^3 = 17/24p^3$, and so $h(z) = 17z^2/48$, the desired result.

Concerning the pathological case of some poles on the integration paths at some step of the algorithm, the same remarks and similar remedies as for the direct method are valid (cf. §3.3).

5. Conclusion

We have presented two algorithms for the exact computation of the volume of a convex polytope given by its half-space representation. The methodology behind both algorithms is conceptually simple as it reduces to invert the Laplace transform of the volume (considered as a function of the right-hand-side). Both algorithms are relatively easy to implement (with special care for the choice of the integration paths of the repeated one-dimensional integrals). Their $O(n^m)$ computational complexity can make them especially attractive for large $n$ and small $m$, when the other methods (with half-space representation of $\Omega$) fail because of their $O(m^n)$ computational complexity.
References

[1] E.L. Allgower, P.M. Schmidt. Computing volumes of polyhedra, Math. Comp. **46** (1986), 171–174.

[2] A.I. Barvinok. Computing the volume, counting integral points and exponentials sums, Discr. Comp. Geom. **10** (1993), 123–141.

[3] B. Büeler, A. Enge, K. Fukuda. Exact volume computation for polytopes: A practical study. In: Polytopes - Combinatorics and Computation, G. Kalai, G. M. Ziegler, Eds., Birkhäuser Verlag, Basel, 2000.

[4] J. Cohen, T. Hickey. Two algorithms for determining volumes of convex polyhedra, J. ACM **26** (1979), 401–414.

[5] J.B. Conway. Functions of a complex variable I, 2nd ed., Springer, New York, (1978).

[6] M.E. Dyer. The complexity of vertex enumeration methods. Math. Oper. Res. **8** (1983), 381–402.

[7] M.E. Dyer, A.M. Frieze. The complexity of computing the volume of a polyhedron. SIAM J. Comput. **17** (1988), 967–974.

[8] J.B. Lasserre. An analytical expression and an algorithm for the volume of a convex polyhedron in $\mathbb{R}^n$. J. Optim. Theor. Appl. **39** (1983), 363–377.

[9] J. Lawrence. Polytope volume computation, Math. Comp. **57** (1991), 259–271.

[10] A. Schrijver. Theory of Linear and Integer Programming, John Wiley & Sons, Chichester, 1986.

[11] B. Von Hohenbalken. Finding simplicial subdivisions of polytopes, Math. Prog. **21** (1981), 233–234.

LAAS-CNRS, 7 Avenue du Colonel Roche, 31077 Toulouse Cédex 4, France.
E-mail address: lasserre@laas.fr

DEPTO. Matemáticas, CIVESTAV-IPN, Apdo. Postal 14740, Mexico D.F. 07000, México.
E-mail address: eszeron@math.cinvestav.mx