Finite and torsion cohomology for solvable groups of finite rank

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Abstract
Let $G$ be a solvable group with finite rank that is either finitely generated or virtually torsion-free. Suppose that $A$ is a $\mathbb{Z}G$-module whose underlying additive group has finite rank. If $A$ is torsion-free as an abelian group, then we specify a condition on $A$ that renders $H^n(G, A)$ and $H_n(G, A)$ finite for $n \geq 0$. Moreover, under the assumption that the additive group of $A$ is a Černikov group, we describe a condition on $A$ that ensures that $H^n(G, A)$ is a Černikov group for all $n \geq 0$.

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1 Introduction

The present paper continues the study of the homology and cohomology of solvable groups of finite rank initiated by the author and P. H. Kropholler in [3]. For $G$ a solvable group of finite rank and $A$ a $\mathbb{Z}G$-module that has finite rank as an abelian group, the article [3] identifies a condition on $A$ that ensures that $H^n(G, A)$ and $H_n(G, A)$ are torsion with finite exponent. The condition, which applies when $G$ is virtually torsion-free and $A$ torsion-free as an abelian group, stipulates that $A$ must not have a nonzero submodule whose spectrum is contained in that of $G$. We remind the reader that we use the term spectrum to refer to the set of primes for which the group has a quasicyclic section. Moreover, if $\pi$ is a set of primes, then a solvable group of finite rank is said to be $\pi$-spectral if its spectrum is contained in $\pi$.

Our objective in the present paper is twofold. First, we wish to improve the theorem from [3] described above in two important respects: (1) by showing that $H^n(G, A)$ and $H_n(G, A)$ are actually finite for $n \geq 0$; (2) by establishing that the result also holds for all finitely generated solvable groups of finite rank. In other words, our first aim is to prove Theorem A below.

**Theorem A.** Let $G$ be a solvable group of finite rank with spectrum $\pi$ such that $G$ is either finitely generated or virtually torsion-free. Assume that $A$ is a $\mathbb{Z}G$-module whose underlying abelian group is torsion-free and has finite rank. Suppose further that $A$ does not have any nontrivial $\mathbb{Z}G$-submodules that are $\pi$-spectral as abelian groups. Then $H^n(G, A)$ and $H_n(G, A)$ are finite for all $n \geq 0$. 

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The second purpose of this article is to examine the situation where the underlying additive group of $A$ is torsion; more specifically, we will suppose that $A$ is a Černikov group, meaning that it is a finite extension of a direct sum of finitely many quasicyclic groups. We wish to specify a condition, analogous to the one above for torsion-free modules, that guarantees that $H^n(G,A)$ is a Černikov group for all $n \geq 0$. The precise result that we prove is Theorem B below.

**Theorem B.** Let $G$ be a solvable group of finite rank with spectrum $\pi$ such that $G$ is either finitely generated or virtually torsion-free. Assume that $A$ is a $\mathbb{Z}G$-module whose underlying abelian group is a Černikov $\omega$-group for a set of primes $\omega$. Suppose further that $A$ fails to have an infinite $\mathbb{Z}G$-submodule that is a quotient of a $\mathbb{Z}G$-module that is torsion-free of finite rank and $\pi$-spectral as an abelian group. Then, for all $n \geq 0$, $H^n(G,A)$ is a Černikov $\omega$-group.

Observe that the conclusion that $H^n(G,A)$ is a Černikov group is the best possible outcome with the hypotheses of Theorem B: very easy examples reveal that the cohomology groups may not have finite exponent. Also, there is no mention of homology in the theorem because $H_n(G,A)$ will always be Černikov if $A$ is Černikov (Lemma 2.5(ii)).

The proofs of Theorems A and B draw extensively upon the techniques and findings of [3]. For both theorems, the strategy is to reduce to the case where $A$ is divisible and the Fitting subgroup $N$ of $G$ acts trivially on $A$. It is shown in [3] that these two circumstances give rise to a spectral sequence $\{E^p_{pq}\}$ converging to $H^n(G,A)$ in which $E^2_{pq} = \text{Ext}^p_{\mathbb{Z}G}(H_q N,A)$, where $Q = G/N$. From this spectral sequence, we can obtain the conclusions of Theorems A and B by establishing that $\text{Ext}^p_{\mathbb{Z}G}(H_q N,A)$ is either finite or Černikov, respectively. To accomplish this when $Q$ is a finitely generated abelian group, we identify the following two properties of modules, the first of which is an extension of [3, Proposition A].

**Proposition A.** Let $G$ be a finitely generated abelian group. Assume that $A$ and $B$ are $\mathbb{Z}G$-modules that are torsion-free and have finite rank as abelian groups. If $A$ fails to contain a nonzero submodule that is isomorphic to a submodule of $B$, then $\text{Ext}^p_{\mathbb{Z}G}(A,B)$ and $\text{Tor}^n_{\mathbb{Z}G}(A,B)$ are finite for $n \geq 0$.

**Proposition B.** Let $G$ be a finitely generated abelian group. Assume that $A$ is a $\mathbb{Z}G$-module that has finite rank qua abelian group. Suppose further that $B$ is a $\mathbb{Z}G$-module whose underlying additive group is a divisible Černikov $\omega$-group for a set of primes $\omega$. If $A$ fails to have a $\mathbb{Z}G$-module quotient that is isomorphic to a finite extension of a nonzero quotient of $B$, then, for $n \geq 0$, $\text{Ext}^n_{\mathbb{Z}G}(A,B)$ is a Černikov $\omega$-group.

Propositions A and B are established in §2; the proofs rely on [3, Proposition A], as well as an elementary property of Pontryagin duals of modules (Lemma 2.8). In order to apply these propositions to the $E_2$-page of the spectral sequence described above, we need to investigate the integral homology groups of a nilpotent normal subgroup of a solvable group of finite rank with spectrum $\pi$. We undertake this task in §3, where we demonstrate that, in some cases, these homology groups, regarded as modules over the ambient group, can be constructed from modules that are virtually torsion-free of finite rank and $\pi$-spectral by forming finitely many quotients and extensions. This is shown for torsion-free nilpotent subgroups in [3], but the requirements of the present paper compel us to establish the property for certain groups that possess infinite torsion. In particular, with the aid of the insights of L. Breen in [1] regarding the integral homology of abelian groups, we prove that
the integral homology of the Fitting subgroup of a finitely generated solvable group of finite rank displays this characteristic.

The last section of the article is devoted to the discussion of Theorems A and B. Equipped with Proposition A, we tackle the former result right away, employing an argument very close to that for [3, Theorem A]. The proof of Theorem B, on the other hand, demands some further preparation. First, we must show that the condition on the \(ZG\)-submodules of \(A\) in the theorem also holds for \(ZG_0\)-submodules, where \(G_0\) is any subgroup of finite index (Lemma 4.2). In addition, we state a result of J. C. Lennox and D. J. S. Robinson about the cohomology of nilpotent groups (Theorem 4.4) that is needed to reduce Theorem B to the case where the Fitting subgroup acts trivially on the module. After proving Theorem B and enunciating a corollary, we conclude the paper by presenting two examples that demonstrate the necessity of the hypotheses of the theorem.

Below we describe the notation and terminology that we will employ throughout the paper. In addition, we state a proposition summarizing the fundamental structural properties of solvable groups of finite rank. For the proofs of these properties, we refer the reader to [5].

**Notation and terminology.** If \(p\) is a prime, then \(\hat{\mathbb{Z}}_p\) is the ring of \(p\)-adic integers and \(\mathbb{Z}_p^\infty\) the quasicyclic \(p\)-group.

Let \(G\) be a group. If \(g, x \in G\), then \(x^g = g^{-1}xg\). Moreover, for \(H < G\) and \(g \in G\), \(H^g = g^{-1}Hg\).

The join of all the nilpotent normal subgroups of a group \(G\), known as the Fitting subgroup, is denoted \(\text{Fitt}(G)\).

We designate the join of all the torsion normal subgroups of a group \(G\) by \(\tau(G)\).

A solvable group is said to have finite rank if its elementary abelian sections are all finite.

If \(G\) is a solvable group such that \(G/\tau(G)\) has finite rank, then \(h(G)\) denotes the Hirsch rank of \(G\), namely, the number of infinite cyclic factors in any series of finite length whose factors are all either infinite cyclic or torsion. By the Schreier refinement theorem, this number is independent of the particular series selected.

A solvable group is minimax if it has a series of finite length in which each factor is either cyclic or quasicyclic. If \(\pi\) is a set of primes, then a \(\pi\)-minimax group is a solvable minimax group that is \(\pi\)-spectral.

If \(G\) is a Černikov group, then the number of quasicyclic summands in the divisible subgroup of finite index is called the Černikov rank of \(G\), written \(c(G)\).

A section of a group or module is a quotient of one its subgroups or submodules, respectively.

Let \(R\) be a ring. An \(R\)-module \(A\) is said to be bounded if there exists \(r \in R\) such that \(rA = 0\).

Suppose that \(R\) is a principal ideal domain and \(A\) an \(R\)-module. Let \(F\) be the field of fractions of \(R\). The torsion-free rank of \(A\), denoted \(r_0(A)\), is defined by

\[
r_0(A) = \dim_F(A \otimes_R F).
\]
Moreover, for each prime $p$ in $R$, the $p$-rank, $r_p(A)$, is defined by

$$r_p(A) = \dim_{R/pR}(\text{Hom}(R/pR, A)).$$

Finally, the total rank of $A$ is the sum of $r_0(A)$ and the $p$-ranks for all primes $p$.

Let $G$ be a group and $A$ a $\mathbb{Z}G$-module. If $A \otimes \mathbb{Q}$ is a simple $\mathbb{Q}G$-module, then we will refer to $A$ as rationally irreducible. In other words, $A$ is rationally irreducible if and only if the additive group of $A$ is not torsion and, for every submodule $B$ of $A$, either $B$ or $A/B$ is torsion as an abelian group.

Assume that $G$ is a group. Let $A$ be a $\mathbb{Z}G$-module and $B$ a subgroup of the underlying additive group of $A$. We define

$$C_G(A) = \{g \in G : ga = a \ \forall a \in A\}; \quad N_G(B) = \{g \in G : gb \in B \ \forall b \in B\}.$$

**Proposition 1.1.** Let $G$ be a solvable group of finite rank.

(i) $G$ is virtually torsion-free if and only if $\tau(G)$ is finite.

(ii) If $\tau(G)$ is a Černikov group, then $\text{Fitt}(G)$ is nilpotent and $G/\text{Fitt}(G)$ virtually abelian.

(iii) If $\tau(G)$ is finite, then $G/\text{Fitt}(G)$ is finitely generated.

(iv) If $G$ is finitely generated, then $G$ is minimax.

## 2 Propositions A and B

Propositions A and B are based on the following result from [3]. There it is stated for the case $R = \mathbb{Z}$, but the proof applies to any principal ideal domain satisfying the finiteness condition given.

**Proposition 2.1.** (Kropholler, Lorensen, and Robinson) Let $G$ be an abelian group and $R$ a principal ideal domain such that $R/Ra$ is finite for every nonzero element $a$ of $R$. Assume that $A$ and $B$ are $RG$-modules that are $R$-torsion-free and have finite $R$-rank. Suppose further that $A$ fails to contain a nontrivial $RG$-submodule that is isomorphic to a submodule of $B$. Then $\text{Ext}^n_{RG}(A, B)$ and $\text{Tor}^n_{RG}(A, B)$ are bounded $R$-modules for all $n \geq 0$. □

In proving Proposition 2.1, one requires the observation below from [3] concerning near splittings of modules. We will invoke this property in proving Propositions A and B, too.

**Proposition 2.2.** Let $R$ be a ring. Suppose that $A$ and $B$ are $R$-modules such that, for every positive integer $m$, both the kernel and the cokernel of the map $b \mapsto mb$ from $B$ to $B$ are finite. Let $0 \to B \to E \to A \to 0$ be an $R$-module extension and $\xi$ the element of $\text{Ext}^1_R(A, B)$ that corresponds to this extension. Then $\xi$ has finite order if and only if $E$ has a submodule $X$ such that $B \cap X$ and $E/B + X$ are finite. □

Two other results from [3] that we require are the following proposition and its corollary.

**Proposition 2.3.** Let $G$ be a group and $R$ a commutative ring. Suppose that $A$ and $B$ are $RG$-modules and regard $\text{Ext}^n_R(A, B)$ and $\text{Tor}^n_R(A, B)$ as $\mathbb{Z}G$-modules via the diagonal action for $n \geq 0$. Then the following two statements hold.
(i) There is a cohomology spectral sequence whose $E_2$-page is given by

$$E_{pq}^2 = H_p(G, \text{Ext}_R^q(A, B)),$$

and that converges to $\text{Ext}_{RG}^n(A, B)$.

(ii) There is a homology spectral sequence whose $E_2$-page is given by

$$E_{pq}^2 = H_p(G, \text{Tor}_R^q(A, B)),$$

and that converges to $\text{Tor}_{RG}^n(A, B)$.

\[\blacksquare\]

Corollary 2.4. Let $G$ be a group and $R$ a commutative ring. Suppose that $A$ and $B$ are $RG$-modules and view both $\text{Hom}_R(A, B)$ and $A \otimes_R B$ as $ZG$-modules under the diagonal actions.

(i) If either $A$ is projective or $B$ is injective as an $R$-module, then, for $n \geq 0$,

$$\text{Ext}_{RG}^n(A, B) \cong H^n(G, \text{Hom}_R(A, B)).$$

(ii) If $B$ is flat as an $R$-module, then, for $n \geq 0$,

$$\text{Tor}_{RG}^n(A, B) \cong H_n(G, A \otimes_R B).$$

\[\blacksquare\]

Of considerable relevance to our discussion are the two elementary homological properties of solvable groups of finite rank described in the following lemma. Both of these assertions can be established very easily by inducting on the length of a series in $G$ whose factors are all either infinite cyclic or torsion (see [6, Lemma 4.3]).

Lemma 2.5. Let $G$ be a solvable group of finite rank and $A$ a $ZG$-module. Suppose further that $\omega$ is a set of primes.

(i) If the additive group of $A$ is a finite $\omega$-group, then $H^n(G, A)$ and $H_n(G, A)$ are finite $\omega$-groups for $n \geq 0$.

(ii) If the additive group of $A$ is a \v{C}ernikov $\omega$-group, then $H_n(G, A)$ is a \v{C}ernikov $\omega$-group for $n \geq 0$.

\[\blacksquare\]

Before establishing Propositions A and B, we discuss a key tool employed in both proofs, namely, the Pontryagin dual of a module, defined for the convenience of the reader below.

Definition. Assume that $p$ is a prime. If $A$ is an abelian $p$-group with finite rank, then the Pontryagin dual $A^*$ of $A$ is the group $\text{Hom}_Z(A, \mathbb{Z}_{p^\infty})$. The dual $A^*$ is a finitely generated $\mathbb{Z}_p$-module. If $A$ happens to be endowed with a $ZG$-module structure for a group $G$, then we equip $A^*$ with a $G$-action by letting $(g\phi)(x) = \phi(g^{-1}x)$ for every $g \in G$, $\phi \in A^*$, and $x \in A$. In this way, we may view $A^*$ as a $\mathbb{Z}_pG$-module.

For any finitely generated $\mathbb{Z}_p$-module $M$, the Pontryagin dual $M^*$ of $M$ is the group $\text{Hom}_Z(M, \mathbb{Z}_{p^\infty})$. In this case, the dual $M^*$ is an abelian $p$-group with finite rank. If $M$ is a $ZG$-module for a group $G$, then we regard $M^*$ as a $ZG$-module using the same $G$-action as above.

Some elementary properties of Pontryagin duals are described in the following two lemmas. For proofs of these, we refer the reader to [7].
Lemma 2.6. Let $p$ be a prime.

(i) If $A$ and $B$ are abelian $p$-groups of finite rank, then
\[ \text{Hom}_{\mathbb{Z}}(A, B) \cong \text{Hom}_{\hat{\mathbb{Z}}_p}(B^*, A^*). \]

(ii) If $M$ and $N$ are finitely generated $\hat{\mathbb{Z}}_p$-modules, then
\[ \text{Hom}_{\mathbb{Z}}(M, N) \cong \text{Hom}_{\hat{\mathbb{Z}}_p}(M^*, N^*). \]

\[ \square \]

Lemma 2.7. Let $p$ be a prime and $G$ a group.

(i) If $A$ is a $\mathbb{Z}G$-module whose additive group is a $p$-group of finite rank, then $A$ is isomorphic to its double dual $A^{**}$.

(ii) If $M$ is a $\hat{\mathbb{Z}}_pG$-module that is finitely generated as a $\hat{\mathbb{Z}}_p$-module, then $M$ is isomorphic to its double dual $M^{**}$.

The crucial property of Pontryagin duals for our purposes is described in Lemma 2.8 below, which appeared in a preliminary version of [3]. This fact is undoubtedly well known; however, as we are unaware of any reference to it in the published literature, we include a proof.

Lemma 2.8. Assume that $p$ is a prime and $G$ a group. Let $A$ and $B$ be $\mathbb{Z}G$-modules whose additive groups are $p$-groups of finite rank. Suppose further that, as an abelian group, $B$ is divisible. Then
\[ \text{Ext}^n_{\mathbb{Z}G}(A, B) \cong \text{Ext}^n_{\hat{\mathbb{Z}}_pG}(B^*, A^*) \]
for $n \geq 0$.

Proof. We have the following chain of isomorphisms.
\[ \text{Ext}^n_{\mathbb{Z}G}(A, B) \cong H^n(G, \text{Hom}_{\mathbb{Z}}(A, B)) \cong H^n(G, \text{Hom}_{\hat{\mathbb{Z}}_p}(B^*, A^*)) \cong \text{Ext}^n_{\hat{\mathbb{Z}}_pG}(B^*, A^*). \]

The first and last isomorphisms in this chain are consequences of Corollary 2.4(i). The first arises from the divisibility of the underlying abelian group of $B$ and the last from the resulting projectivity of $B^*$ as a $\hat{\mathbb{Z}}_p$-module.

We are now prepared to prove Propositions A and B.

Proof of Proposition A. It suffices to consider the case where $A$ and $B$ are both rationally irreducible. Once that case has been established, the general result will follow by the argument adduced in the last two paragraphs of the proof of [3, Proposition A]. First we investigate $\text{Ext}^n_{\mathbb{Z}G}(A, B)$. Since $B \otimes \mathbb{Q}$ fails to contain any nonzero $\mathbb{Z}G$-submodule that is isomorphic to a submodule of $A$, it follows from Proposition 2.1 that $\text{Ext}^n_{\mathbb{Z}G}(A, B \otimes \mathbb{Q})$ is a bounded $\mathbb{Z}$-module for each $n \geq 0$. But $\text{Ext}^n_{\mathbb{Z}G}(A, B \otimes \mathbb{Q})$ is a vector space over $\mathbb{Q}$, so that $\text{Ext}^n_{\mathbb{Z}G}(A, B \otimes \mathbb{Q}) = 0$ for $n \geq 0$. Thus the conclusion of the proposition will follow from the long exact Ext-sequence if we can show that $\text{Ext}^n_{\mathbb{Z}G}(A, (B \otimes \mathbb{Q})/B)$ is finite for $n \geq 0$. To
accomplish this, we let \( A_0 \) be a nontrivial finitely generated submodule of \( A \). Then \( A/A_0 \) is torsion as an abelian group. Since \( ZG \) is Noetherian, \( A_0 \) is of type \( FP_\infty \), implying that \( \text{Ext}_{\hat{Z}G}^n(A_0, (B \otimes \mathbb{Q})/B) \) has finite rank for \( n \geq 0 \). However, by Proposition 2.1 and the long exact Ext-sequence, \( \text{Ext}_{\hat{Z}G}^n(A_0, (B \otimes \mathbb{Q})/B) \) is a bounded \( \mathbb{Z} \)-module for \( n \geq 0 \). It follows, then, that \( \text{Ext}_{\hat{Z}G}^n(A_0, (B \otimes \mathbb{Q})/B) \) must be finite for \( n \geq 0 \).

For each prime \( p \), let \( T_p \) be the \( p \)-torsion subgroup of \( A/A_0 \) and \( S_p \) the \( p \)-torsion subgroup of \( (B \otimes \mathbb{Q})/B \). Then, for \( n \geq 0 \), we have

\[
\text{Ext}_{\hat{Z}G}^n(A/A_0, (B \otimes \mathbb{Q})/B) \cong \prod_p \text{Ext}_{\hat{Z}G}^n(T_p, (B \otimes \mathbb{Q})/B) \cong \prod_p \text{Ext}_{\hat{Z}G}^n(T_p, S_p).
\]

Our plan is to establish that \( \text{Ext}_{\hat{Z}G}^n(T_p, S_p) \) is a finite \( p \)-group for each prime \( p \) and \( n \geq 0 \). For this purpose, we employ the Pontryagin duals \( S_p^* \) and \( T_p^* \). Since \( \hat{Z}_pG \) is Noetherian and \( S_p^* \) is a finitely generated \( \hat{Z}_pG \)-module, \( S_p^* \) has type \( FP_\infty \) as a \( \hat{Z}_pG \)-module. This implies that \( \text{Ext}_{\hat{Z}G}^n(S_p^*, T_p^*) \) has finite total \( \hat{Z}_p \)-rank. However, \( \text{Ext}_{\hat{Z}G}^n(S_p^*, T_p^*) \cong \text{Ext}_{\hat{Z}G}^n(T_p, S_p) \); moreover, Proposition 2.1 and the long exact Ext-sequence yield that \( \text{Ext}_{\hat{Z}G}^n(T_p, S_p) \) is bounded as a \( \mathbb{Z} \)-module. Hence \( \text{Ext}_{\hat{Z}G}^n(S_p^*, T_p^*) \), and therefore \( \text{Ext}_{\hat{Z}G}^n(T_p, S_p) \), is a finite \( p \)-group. But \( \text{Ext}_{\hat{Z}G}^n(A/A_0, (B \otimes \mathbb{Q})/B) \), too, is bounded. Consequently, as a direct product of finite groups whose orders are pairwise relatively prime, \( \text{Ext}_{\hat{Z}G}^n(A/A_0, (B \otimes \mathbb{Q})/B) \) must be finite for \( n \geq 0 \). Thus the same is true for \( \text{Ext}_{\hat{Z}G}^n(A, B) \).

That \( \text{Tor}_{\hat{Z}G}^n(A, B) \) is finite follows very quickly from its boundedness. To see this, notice that \( A \otimes \mathbb{Z} B \) has finite rank. Therefore, since \( G \) has type \( FP_\infty \), Corollary 2.4(ii) implies that \( \text{Tor}_{\hat{Z}G}^n(A, B) \) has finite rank. Thus, being bounded, \( \text{Tor}_{\hat{Z}G}^n(A, B) \) must be finite.

**Proof of Proposition B.** Without incurring any significant loss of generality, we can assume that \( \omega \) contains just a single prime \( p \). First we treat the case where either \( A \) is rationally irreducible or the additive group of \( A \) is torsion. Under this assumption, we have a finitely generated submodule \( A_0 \) of \( A \) such that the additive group of \( A/A_0 \) is torsion. Moreover, since \( ZG \) is Noetherian, \( A_0 \) is of type \( FP_\infty \), implying that \( \text{Ext}_{\hat{Z}G}^n(A_0, B) \) is a Černikov \( p \)-group for \( n \geq 0 \). Let \( P \) be the divisible part of the \( p \)-torsion subgroup of \( A/A_0 \). We claim that \( \text{Ext}_{\hat{Z}G}^n(P, B) \) is a finite \( p \)-group for \( n \geq 0 \), which will then yield the conclusion of the proposition. To establish this claim, we use the fact that \( \text{Ext}_{\hat{Z}G}^n(P, B) \) and \( \text{Ext}_{\hat{Z}G}^n(B^*, P^*) \) are isomorphic for all \( n \geq 0 \). Since \( \hat{Z}_pG \) is Noetherian and \( B^* \) is a finitely generated \( \hat{Z}_pG \)-module, \( B^* \) has type \( FP_\infty \) as a \( \hat{Z}_pG \)-module. As a consequence, \( \text{Ext}_{\hat{Z}G}^n(B^*, P^*) \) has finite total \( \hat{Z}_p \)-rank. Suppose now that \( \text{Ext}_{\hat{Z}G}^n(B^*, P^*) \) is not a finite \( p \)-group for some \( n \geq 0 \). Then \( \text{Ext}_{\hat{Z}G}^n(B^*, P^*) \) is not bounded as a \( \hat{Z}_p \)-module. However, by Proposition 2.1, this means that \( P^* \) and \( B^* \) have isomorphic nontrivial \( \hat{Z}_pG \)-submodules. Dualizing a second time, we deduce that \( P \) and \( B \) have isomorphic infinite \( ZG \)-module quotients, thereby contradicting our hypothesis. Therefore, for each \( n \geq 0 \), \( \text{Ext}_{\hat{Z}G}^n(B^*, P^*) \), and hence \( \text{Ext}_{\hat{Z}G}^n(P, B) \), must be a finite \( p \)-group. This concludes the argument for the case where \( A \) is either rationally irreducible as a module or torsion qua abelian group.

We handle the general case by inducting on \( h(A) \), the case \( h(A) = 0 \) having already been established above. Suppose \( h(A) \geq 1 \). Let \( U \) be a submodule of \( A \) with \( h(U) \) as large as possible while still remaining less than \( h(A) \). It follows from the case proved in the first
paragraph that \( \Ext^n_{\mathbb{Z}G}(A/U, B) \) is a Černikov \( p \)-group for \( n \geq 0 \). Consequently, the conclusion of the theorem will follow from the inductive hypothesis if we succeed in showing that \( U \) does not have a quotient that is isomorphic to a finite extension of a nontrivial quotient of \( B \). To accomplish this, suppose that \( U \) possesses such a quotient and take \( V \) to be a submodule of \( U \) such that \( U/V \cong W \), where \( W \) is a finite extension of a nontrivial quotient of \( B \). By the case of the proposition established above, \( \Ext^1_{\mathbb{Z}G}(A/U, U/V) \) is torsion. According to Proposition 2.2, this means that \( A \) has a quotient that is a finite extension of a nontrivial quotient of \( W \). But the presence of such a quotient contradicts our hypothesis. Therefore, \( U \) cannot possess a quotient that is isomorphic to a finite extension of a nontrivial quotient of \( B \). This completes the proof of Proposition B.

\[ \Box \]

3 The Class \( \mathcal{C}(G, \pi) \)

To prove Theorems A and B, we need to understand the following class of modules. This class is a slight extension of the class with the same designation discussed in [3].

**Definition.** Assume that \( G \) is a group and \( \pi \) a set of primes. Let \( \mathcal{C}(G, \pi) \) be the smallest class of \( \mathbb{Z}G \)-modules with the following two properties.

(i) The class \( \mathcal{C}(G, \pi) \) contains every finite \( \mathbb{Z}G \)-module as well as every \( \mathbb{Z}G \)-module whose underlying abelian group is torsion-free of finite rank and \( \pi \)-spectral.

(ii) The class \( \mathcal{C}(G, \pi) \) is closed under forming \( \mathbb{Z}G \)-module quotients and extensions.

The class \( \mathcal{C}(G, \pi) \) has the following two additional closure properties, which can be established by arguments in the same vein as those for [3, Lemmas 2.4, 2.5]. The basic strategy is to induct on the number of closure operations from (ii) that are necessary to construct the module from ones that are either finite or torsion-free of finite rank and \( \pi \)-spectral. We leave the small task of supplying complete proofs to the reader.

**Lemma 3.1.** For any group \( G \) and set of primes \( \pi \), the class \( \mathcal{C}(G, \pi) \) is closed under forming \( \mathbb{Z}G \)-module sections. \( \Box \)

**Lemma 3.2.** Assume that \( G \) is a group and \( \pi \) a set of primes. Let \( A \) and \( B \) be \( \mathbb{Z}G \)-modules and, for each \( n \geq 0 \), regard \( \Tor^n_{\mathbb{Z}G}(A, B) \) as a \( \mathbb{Z}G \)-module under the diagonal action. If \( A \) and \( B \) belong to \( \mathcal{C}(G, \pi) \), then \( \Tor^n_{\mathbb{Z}G}(A, B) \) lies in \( \mathcal{C}(G, \pi) \) for \( n \geq 0 \). \( \Box \)

As a consequence of Proposition A, we have the following lemma. This can be proved in the same manner as [3, Lemma 2.6], though in place of [3, Proposition A] we must invoke Proposition A.

**Lemma 3.3.** Let \( \pi \) be a set of primes and \( G \) a finitely generated abelian group. Assume that \( B \) is a \( \mathbb{Z}G \)-module whose additive group is torsion-free with finite rank. Suppose further that there are no nontrivial \( \mathbb{Z}G \)-submodules of \( B \) that are \( \pi \)-spectral as abelian groups. If \( A \) is a \( \mathbb{Z}G \)-module in \( \mathcal{C}(G, \pi) \) and \( \bar{B} \) a \( \mathbb{Z}G \)-module quotient of \( B \), then \( \Ext^n_{\mathbb{Z}G}(A, \bar{B}) \) and \( \Tor^n_{\mathbb{Z}G}(A, B) \) are finite for \( n \geq 0 \).

Similarly, from Proposition B we deduce
Lemma 3.4. Let \( \pi \) be a set of primes and \( G \) a finitely generated abelian group. Assume that \( B \) is a \( \mathbb{Z}G \)-module whose additive group is a divisible Černikov \( \omega \)-group for some set of primes \( \omega \). Suppose further that there is no nontrivial \( \mathbb{Z}G \)-module quotient of \( B \) that is also a quotient of a \( \mathbb{Z}G \)-module that is torsion-free of finite rank and \( \pi \)-spectral as an abelian group. If \( A \) is a \( \mathbb{Z}G \)-module in \( \mathcal{C}(G, \pi) \), then, for each \( n \geq 0 \), \( \text{Ext}^n_{\mathbb{Z}G}(A, B) \) is a Černikov \( \omega \)-group. \( \Box \)

Our principal goal in this section is to venture beyond what was shown in [3] about \( \mathcal{C}(G, \pi) \) by showing, first, that the class is closed under the integral homology functor. This will then permit us to identify a large class of nilpotent normal subgroups of solvable groups of finite rank whose integral homology lies within \( \mathcal{C}(G, \pi) \). In order to establish the former property, we will avail ourselves of the results of Breen in [1] concerning the integral homology of abelian groups. We begin by defining the \( m \)-fold torsion product of abelian groups, following Breen’s notation and terminology.

**Definition.** Assume that \( A_1, \ldots, A_m \) are abelian groups. For each \( i = 1, \ldots, m \), let \( F^i \) be a \( \mathbb{Z} \)-flat resolution of \( A_i \). Then, for \( n \geq 0 \), we define

\[
\text{Tor}^\mathbb{Z}(A_1, \ldots, A_m) = H_n(F^1 \otimes_\mathbb{Z} \cdots \otimes_\mathbb{Z} F^m).
\]

If \( m = 2 \), this group is isomorphic to \( \text{Tor}^\mathbb{Z}(A_1, A_2) \) as it is usually defined.

In [1, p. 209], Breen cites a result of A. Grothendieck asserting that there is a spectral sequence involving lower order torsion products that converges to the \( m \)-fold tensor product.

**Proposition 3.5.** (Grothendieck [2, 6.8.3]) Let \( A_1, \ldots, A_m \) be abelian groups. Then, for each \( i = 1, \ldots, m-1 \), there is a natural spectral sequence \( \{ E^r_{pq} \} \) that converges to \( \text{Tor}^\mathbb{Z}_n(A_1, \ldots, A_m) \) such that

\[
E^2_{pq} = \bigoplus_{q_1+q_2=q} \text{Tor}^\mathbb{Z}_{q_1}(A_1, \ldots, A_i), \text{Tor}^\mathbb{Z}_{q_2}(A_{i+1}, \ldots, A_m)).
\]

\( \Box \)

The above spectral sequence, in conjunction with Lemma 3.2, immediately gives rise to the following closure property for \( \mathcal{C}(G, \pi) \).

**Lemma 3.6.** Assume that \( G \) is a group and \( \pi \) a set of primes. Let \( A_1, \ldots, A_m \) be \( \mathbb{Z}G \)-modules and, for each \( n \geq 0 \), regard \( \text{Tor}^\mathbb{Z}_n(A_1, \ldots, A_m) \) as a \( \mathbb{Z}G \)-module under the diagonal action. If \( A_1, \ldots, A_m \) belong to \( \mathcal{C}(G, \pi) \), then \( \text{Tor}^\mathbb{Z}_n(A_1, \ldots, A_m) \) lies in \( \mathcal{C}(G, \pi) \) for \( n \geq 0 \). \( \Box \)

Equipped with the preceding lemma, we can apply two of the results in [1] to obtain our most important closure property for \( \mathcal{C}(G, \pi) \).

**Lemma 3.7.** Assume that \( G \) is a group and \( \pi \) a set of primes. If \( A \) is a \( \mathbb{Z}G \)-module that belongs to \( \mathcal{C}(G, \pi) \), then \( H_nA \) lies in \( \mathcal{C}(G, \pi) \) for \( n \geq 0 \).

**Proof.** According to [1, Proposition 1.1], there is a natural spectral sequence \( \{ E^r_{pq} \} \) converging to \( H_nA \) such that \( E^2_{pq} = L_pA^qA \) for \( p, q \geq 0 \). Here \( L_pA^qA \) represents the \( q \)-th derived functor of the \( q \)-th exterior power of \( A \). Moreover, by [1, Theorem 4.7], there is a \( \mathbb{Z}G \)-module monomorphism from \( L_pA^qA \) to \( \text{Tor}^\mathbb{Z}_{q}(A_1, \ldots, A) \) for \( q > 0 \). It follows, then, from Lemmas 3.6
and 3.1 that $E^2_{pq}$ lies in $\mathfrak{C}(G, \pi)$ for $q > 0$. In addition, $E^2_{pq}$, being trivial, belongs to $\mathfrak{C}(G, \pi)$. Since the differentials in Breen’s spectral sequence are $\mathbb{Z}G$-module homomorphisms, $E^\infty_{pq}$ can be regarded as a $\mathbb{Z}G$-module section of $E^2_{pq}$. Hence $E^\infty_{pq}$ must be a member of $\mathfrak{C}(G, \pi)$ for $p, q \geq 0$. Furthermore, $H_nA$ has a $\mathbb{Z}G$-module series whose factors are the modules $E^\infty_{pq}$ for $p + q = n$. Therefore, $H_nA$ must belong to $\mathfrak{C}(G, \pi)$ for $n \geq 0$.

Now we prove the desired result concerning the integral homology of a nilpotent normal subgroup. This proposition will serve as a crucial ingredient in the proofs of Theorems A and B.

**Proposition 3.8.** Assume that $G$ is a group with a nilpotent normal subgroup $N$ of finite rank. Let $\pi = \text{spec}(N)$ and $Q = G/N$. Suppose further that $N$ has a $G$-invariant central series in which each infinite factor can be expressed as a quotient of a $\mathbb{Z}Q$-module whose underlying abelian group is torsion-free of finite rank and $\pi$-spectral. Then $H_nN$ belongs to $\mathfrak{C}(Q, \pi)$ for $n \geq 0$.

**Proof.** Let $1 = N_0 < N_1 < \cdots < N_r = N$ be a central series with the properties described. We prove the proposition by induction on $r$, the case $r = 0$ being trivial. Assume $r > 0$. By the inductive hypothesis, $H_n(N/N_1)$ is a member of $\mathfrak{C}(Q, \pi)$ for $n \geq 0$. Moreover, according to Lemma 3.7, $H_nN_1$ lies in $\mathfrak{C}(Q, \pi)$ for $n \geq 0$. We now consider the Lyndon-Hochschild-Serre (LHS) spectral sequence associated to the central extension $1 \to N_1 \to N \to N/N_1 \to 1$. In this spectral sequence, $E^0_{pq} = H_p(N/N_1, H_qN_1)$, and so there is an exact sequence

$$0 \to H_p(N/N_1) \otimes_{\mathbb{Z}} H_qN_1 \to E^2_{pq} \to \text{Tor}^\mathbb{Z}_1(H_{p-1}(N/N_1), H_qN_1) \to 0.$$ 

An appeal to Lemma 3.2 allows us to deduce that $E^2_{pq}$ lies in $\mathfrak{C}(Q, \pi)$. Since the differentials in the spectral sequence are $\mathbb{Z}Q$-module homomorphisms, it follows from Lemma 3.1 that $E^\infty_{pq}$ belongs to $\mathfrak{C}(Q, \pi)$. As a result, $H_nN$ is a member of $\mathfrak{C}(Q, \pi)$ for $n \geq 0$.

We conclude this section by showing that the Fitting subgroup of a finitely generated soluble group of finite rank satisfies the hypotheses of Proposition 3.8. Hence the integral homology of such a Fitting subgroup lies in the eponymous class, with respect to the ambient group.

**Proposition 3.9.** Assume that $G$ is a finitely generated minimax group with spectrum $\pi$ and Fitting subgroup $N$. Then $N$ has a $G$-invariant central series in which each infinite factor can be expressed as a quotient of a $\mathbb{Z}Q$-module whose underlying abelian group is torsion-free of finite rank and $\pi$-spectral.

**Proof.** Let $Q = G/N$. Let $i$ be a positive integer and consider the canonical $\mathbb{Z}Q$-module epimorphism $\theta_i : N_{ab} \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} N_{ab} \to \gamma_iN/\gamma_{i+1}N$. Since $G$ is finitely generated as a group and $Q$ polycyclic, $N_{ab}$ is finitely generated as a $\mathbb{Z}Q$-module. As a result, $N_{ab}$ is a Noetherian $\mathbb{Z}Q$-module and hence virtually torsion-free as an abelian group. Therefore, each factor in the lower central series of $N$ is a homomorphic image of a $\mathbb{Z}Q$-module whose underlying additive group is virtually torsion-free of finite rank and $\pi$-spectral. This, then, implies the conclusion of the lemma.
4 Theorems A and B

This section is devoted to the proofs of the main results of the paper, Theorems A and B. We begin with the former, as it presents the least difficulty.

**Proof of Theorem A.** It will be convenient for us to establish the conclusion under the more general hypotheses that $G$ has a nilpotent normal subgroup $N$ such that $Q = G/N$ is finitely generated and virtually abelian, and that $N$ has a $G$-invariant central series in which each infinite factor can be realized as a $\mathbb{Z}Q$-module quotient of a $\mathbb{Z}Q$-module whose underlying abelian group is torsion-free of finite rank and $\pi$-spectral. Of essential importance to our reasoning is the observation that, if $G$ and $N$ satisfy the hypotheses and $M$ is a normal subgroup of $G$ such that $G/M$ is virtually abelian, then $G$ and $M \cap N$ also fulfill the conditions.

First we treat the case where $A$ is rationally irreducible. Under this assumption, $G/C_G(A)$ is virtually abelian (see [5, 3.1.6]). Hence, by replacing $N$ with $N \cap C_G(A)$, we can assume that $N < C_G(A)$. Let $G_0$ be a normal subgroup of $G$ with finite index such that $Q_0 = G_0/N$ is abelian. Since $H_n^1(N)$ lies in $G(Q_0, \pi)$ by Proposition 3.8, we can argue exactly as in the second and third paragraphs of the proof of [3, Theorem A], except that now we invoke Lemma 3.3 rather than [3, Lemma 2.6]. This reasoning allows us to deduce that $H^n(G_0, A)$ and $H_n(G_0, A)$ are finite for $n \geq 0$. Thus the same holds for $H^n(G, A)$ and $H_n(G, A)$.

We handle the general case by inducting on $h(A)$, the case $h(A) = 0$ being trivial. Assume $h(A) \geq 1$. Choose $B$ to be a submodule of $A$ with $h(B)$ as large as possible subject to the conditions that $h(B) < h(A)$ and the additive group of $A/B$ is torsion-free. By the inductive hypothesis, $H^n(G, B)$ and $H_n(G, B)$ are finite for $n \geq 0$. Since $A/B$ is rationally irreducible, the conclusion will then follow from the case proved above if we can show that $A/B$ fails to contain a nonzero $\mathbb{Z}G$-submodule that is torsion-free and $\pi$-spectral as an abelian group. Suppose that $A/B$ contains such a submodule, and let $C$ be a submodule of $A$ properly containing $B$ such that $C/B$ is torsion-free and $\pi$-spectral qua abelian group. Because $G/C_G(C/B)$ is virtually abelian, we may assume $N < C_G(C/B)$. Now, as in the proof of [3, Theorem A], we put $\Gamma = C \rtimes G$. Also, let $\bar{N} = C \rtimes N$. Then $\Gamma/B$ and $\bar{N}/B$ satisfy the hypotheses of the statement we have opted to prove, with respect to the module $B$. As a result, $H^2(\Gamma/B, B)$ is finite. This allows us to derive a contradiction by adding the argument employed at the end of the proof of [3, Theorem A]. Therefore, we are compelled to conclude that $A/B$ fails to possess a nonzero $\mathbb{Z}G$-submodule that is torsion-free and $\pi$-spectral as an abelian group. This completes the proof of the theorem.

To lay the groundwork for Theorem B, we prove two lemmas concerning modules that can be expressed as quotients of modules that are torsion-free of finite rank.

**Lemma 4.1.** Suppose that $\pi$ is a set of primes. Let $G$ be a group and $G_0 < G$ with $[G : G_0] < \infty$. Assume that $A$ is a $\mathbb{Z}G$-module that can be expressed as a $\mathbb{Z}G_0$-module quotient of a $\mathbb{Z}G_0$-module whose underlying abelian group is torsion-free, has finite rank, and is $\pi$-spectral. Then $A$ can also be realized as a $\mathbb{Z}G$-module quotient of a $\mathbb{Z}G_0$-module whose additive group possesses the same three properties.

**Proof.** Assume that $M_0$ is a $\mathbb{Z}G_0$-module that is torsion-free of finite rank and $\pi$-spectral as an abelian group, and that there is a $\mathbb{Z}G_0$-module epimorphism $\phi_0 : M_0 \to A$. Set $\Gamma = A \rtimes G$, $\Gamma_0 = A \rtimes G_0$, and $\Omega_0 = M_0 \rtimes G_0$. Moreover, let $\phi_0 : \Omega_0 \to \Gamma_0$ be the epimorphism induced by $\phi_0$. In addition, suppose that $\theta : F \to \Gamma$ is a group epimorphism such that $F$ is a free
Lemma 4.2. Suppose that \( \pi \) is a set of primes. Let \( G \) be a solvable group of finite rank and \( G_0 < G \) with \( [G : G_0] < \infty \). Assume that \( A \) is a \( \mathbb{Z}G \)-module and \( B \) a \( \mathbb{Z}G_0 \)-submodule of \( A \). Suppose further that \( B \) can be expressed as a quotient of a \( \mathbb{Z}G_0 \)-module whose underlying abelian group is torsion-free, has finite rank, and is \( \pi \)-spectral. Then the \( \mathbb{Z}G \)-submodule of \( A \) generated by \( B \) can be realized as a \( \mathbb{Z}G \)-module quotient of a \( \mathbb{Z}G \)-module whose additive group enjoys the same three properties.

Proof. Since \( G_0 \) contains a \( G \)-invariant subgroup of finite index, it suffices to consider the case where \( G_0 \triangleleft G \). Let \( g_1, \ldots, g_r \) be a complete list of coset representatives of \( G_0 \) in \( G \). Then \( B = g_1B + \cdots + g_rB \) is the \( \mathbb{Z}G \)-submodule of \( A \) generated by \( B \). According to our hypotheses, there is a \( \mathbb{Z}G_0 \)-module \( M \) and a \( \mathbb{Z}G_0 \)-module epimorphism \( \phi : M \to B \). Now, for each \( i = 1, \ldots, r \), we construct a new \( \mathbb{Z}G_0 \)-module \( M_i \) with the same underlying abelian group as \( M \). The new action, denoted \( \circ_i \) for each \( i \), is defined in terms of the old action as follows: \( g_0 \circ_i x = g_0^{g_i}x \) for all \( x \in M_i \) and \( g_0 \in G_0 \). Next set \( \tilde{M} = M_1 \oplus \cdots \oplus M_r \), and define the map \( \tilde{\phi} : \tilde{M} \to B \) by

\[
\tilde{\phi}(x_1, \ldots, x_r) = g_1\phi(x_1) + \cdots + g_r\phi(x_r)
\]

for all \((x_1, \ldots, x_r) \in \tilde{M}\). It is easily checked that \( \tilde{\phi} \) is an \( \mathbb{Z}G_0 \)-module epimorphism. Therefore, the conclusion of the lemma follows by applying Lemma 4.1.

Another elementary property required for the proof of Theorem B is described in the following lemma.

Lemma 4.3. Assume that \( \pi \) is a set of primes. Let \( G \) be a group and \( A \) a \( \mathbb{Z}G \)-module that is divisible as an abelian group. Suppose that \( A \) has a finite submodule \( B \), and that there is an epimorphism \( \theta : U \to A/B \), where \( U \) is a \( \mathbb{Z}G \)-module whose additive group is torsion-free of finite rank and \( \pi \)-spectral. Then the module \( A \), too, can be realized as a quotient of a \( \mathbb{Z}G \)-module whose additive group is torsion-free of finite rank and \( \pi \)-spectral.
Proof. From the map \( \theta : U \to A/B \), we can construct a \( \mathbb{Z}G \)-module extension \( 0 \to B \to M \to U \to 0 \) that fits into a commutative diagram of the form

\[
\begin{array}{ccccccc}
0 & \longrightarrow & B & \longrightarrow & M & \longrightarrow & U & \longrightarrow & 0 \\
& & \| & & \downarrow & & \downarrow \theta & \\
0 & \longrightarrow & B & \longrightarrow & A & \longrightarrow & A/B & \longrightarrow & 0.
\end{array}
\]

Since \( M \) is virtually torsion-free, we can select a torsion-free submodule \( M_0 \) of \( M \) with finite index such that \( M_0 \cap B = 0 \). The submodule \( M_0 \), then, is mapped surjectively to \( A \), thereby witnessing the property desired. \( \square \)

The proof of Theorem B depends upon three homological properties. The first two are contained in Lemma 3.4 and Proposition 3.8; the third is articulated in the following special case of a theorem due to Lennox and Robinson (see [4] and [5, 10.3.6]).

**Theorem 4.4.** (Lennox and Robinson) Assume that \( \omega \) is a set of primes. Let \( N \) be a nilpotent group with finite rank and \( A \) a \( \mathbb{Z}N \)-module whose underlying additive group is a Cernikov \( \omega \)-group. If \( AN \) is finite, then \( H^n(N, A) \) is a finite \( \omega \)-group for \( n \geq 0 \). \( \square \)

In the original statement of Lennox and Robinson’s theorem, \( N \) is not assumed to have finite rank, which leads to the weaker conclusion that \( H^n(N, A) \) is merely bounded as a \( \mathbb{Z} \)-module. A close examination of their argument, however, reveals that, under our more restrictive hypothesis, one can deduce the stronger conclusion that \( H^n(N, A) \) is a finite \( \omega \)-group.

Armed with the results above, we proceed with the proof of our second theorem.

**Proof of Theorem B.** As in the proof of Theorem A, we establish the conclusion under the more general hypotheses that \( G \) has a normal nilpotent subgroup \( N \) such that \( Q = G/N \) is finitely generated and virtually abelian, and that \( N \) has a \( G \)-invariant central series in which each infinite factor can be realized as a \( \mathbb{Z}Q \)-module quotient of a \( \mathbb{Z}Q \)-module whose underlying abelian group is torsion-free of finite rank and \( \pi \)-spectral. In the course of the proof, we will use the fact that, if \( G, N, \) and \( A \) satisfy the hypotheses, then, for any finite-index subgroup \( G_0 \) of \( G \), the groups \( G_0 \) and \( G_0 \cap N \) also fulfill these conditions with respect to any \( \mathbb{Z}G_0 \)-submodule of \( A \). Notice that this observation depends on Lemma 4.2. Furthermore, if \( V \) is a \( \mathbb{Z}G \)-module such that \( N < C_G(V) \) and \( V \) can be expressed as a quotient of a \( \mathbb{Z}Q \)-module whose underlying abelian group is torsion-free of finite rank and \( \pi \)-spectral, then any extension \( G \) of \( V \) by \( G \), together with the inverse image of \( N \) in \( G \), satisfies the hypotheses with respect to the module \( A \)

Without any real loss of generality, we may assume that \( A \) is divisible \( qua \) abelian group.

In this paragraph, we treat the case where, for every subgroup \( H \) of \( G \) of finite index, each proper \( \mathbb{Z}H \)-submodule of \( A \) is finite. Consider first the situation where \( AN \neq A \). Under this assumption, Lennox and Robinson's result yields that \( H^n(N, A) \) is a finite \( \omega \)-group for \( n \geq 0 \). It follows, then, from the LHS spectral sequence that \( H^n(G, A) \) is a finite \( \omega \)-group for \( n \geq 0 \). Next we suppose \( N < C_G(A) \). Let \( G_0 \) be a normal subgroup of \( G \) with finite index such that \( N < G_0 \) and \( Q_0 = G_0/N \) is a finitely generated abelian group. We will employ the LHS spectral sequence for the group extension \( 1 \to N \to G_0 \to Q_0 \to 1 \) to investigate \( H^n(G_0, A) \). The universal coefficient theorem and Corollary 2.4 give rise to the following chain of isomorphisms for \( p, q \geq 0 \).

\[
H^p(Q_0, H^q(N, A)) \cong H^p(Q_0, \text{Hom}_\mathbb{Z}(HqN, A)) \cong \text{Ext}^p_{\mathbb{Z}Q_0}(H_qN, A).
\]
According to Lemma 4.3, A fails to have a quotient that is also a quotient of a $\mathbb{Z}Q_0$-module that is torsion-free of finite rank and $\pi$-spectral as an abelian group. Moreover, Proposition 3.8 implies that $H_qN$ belongs to $C(Q_0, \pi)$ for $q \geq 0$. Therefore, by Lemma 3.4, $\text{Ext}^p_{\mathbb{Z}Q_0}(H_qN, A)$ is a Černikov $\omega$-group for $p, q \geq 0$. It follows, then, that $H^n(G_0, A)$ is a Černikov $\omega$-group for $n \geq 0$. Finally, appeals to the LHS spectral sequence for the extension $1 \to G_0 \to G \to G/G_0 \to 1$ and [6, Lemma 4.3] permit the deduction that $H^n(G, A)$ is a Černikov $\omega$-group for $n \geq 0$.

Now we tackle the general case, proceeding by induction on $c(A)$. As the case $c(A) = 0$ is trivial, we suppose $c(A) > 0$. Select an additive subgroup $B$ of $A$ with $c(B)$ as large as possible subject to the conditions that $c(B) < c(A)$ and $N_G(B)$ has finite index in $G$. Let $G_1$ be a normal subgroup of $G$ contained in $N_G(B)$. Our choice of $B$ guarantees that, for any subgroup $H$ of $G_1$ with finite index, every proper $ZH$-submodule of $A/B$ is finite. Notice that, for $n \geq 0$, $H^n(G_1, B)$ is a Černikov $\omega$-group, in view of the inductive hypothesis. Hence the conclusion of the theorem will follow if we manage to establish that $H^n(G_1, A/B)$ is a Černikov $\omega$-group for $n \geq 0$. Set $N_1 = G_1 \cap N$. If $N_1$ fails to act trivially on $A/B$, then we deduce that $H^n(G_1, A/B)$ is a finite $\omega$-group from Lennox and Robinson's theorem. Suppose that $N_1$ acts trivially on $A/B$. Our plan is to show that, in this case, $A/B$ cannot be a $ZG_1$-module quotient of a $ZG_1$-module whose additive group is torsion-free of finite rank and $\pi$-spectral; then it will follow from the case proved above that $H^n(G_1, A/B)$ is a Černikov $\omega$-group for $n \geq 0$, thus yielding the desired conclusion.

Suppose that $A/B$ is a quotient of a $ZG_1$-module whose additive group is torsion-free of finite rank and $\pi$-spectral. Let $\Gamma = A \rtimes G_1$ and $\tilde{N}_1 = A \rtimes N_1$. Then, with respect to the module $B$, the group $\Gamma/B$ and its nilpotent subgroup $\tilde{N}_1/B$ satisfy the hypotheses of the statement we have elected to prove. Consequently, by the inductive hypothesis, $H^n(\Gamma/B, B)$ is torsion for $n \geq 0$. That this holds for $n = 2$ implies the existence of a subgroup $X$ of $\Gamma$ such that $B \cap X$ is finite and $[\Gamma : BX] < \infty$ (see [5, 10.1.15]). Putting $G_2 = BX$, we have that $A \cap X$ and $B \cap X$ are $ZG_2$-submodules of $A$. Moreover, $A \cap X/B \cap X$ and $A/B$ are isomorphic as $ZG_1$-modules. Thus Lemma 4.3 yields that $A \cap X$ is a $ZG_2$-module quotient of a $ZG_1$-module whose underlying additive group is torsion-free of finite rank and $\pi$-spectral. In view of Lemma 4.2, this contradicts our hypothesis concerning $A$. Therefore, $A/B$ cannot be a $ZG_1$-module quotient of a $ZG_1$-module whose additive group is torsion-free of finite rank and $\pi$-spectral. This concludes the proof of the theorem.

In Corollary 4.5, we identify a common situation in which Theorem B applies, namely, where the action of the group on the module is transcendental.

**Corollary 4.5.** Let $G$ be a solvable group of finite rank that is either finitely generated or virtually torsion-free. Assume that $A$ is a $ZG$-module whose additive group is a Černikov $\omega$-group for a set of primes $\omega$, and whose proper submodules are all finite. Suppose further that $\phi : G \to \text{Aut}(A)$ is the homomorphism arising from the action of $G$ on $A$. If, for some $g \in G$, $\phi(g)$ fails to satisfy any polynomial over $\mathbb{Z}$, then, for $n \geq 0$, $H^n(G, A)$ must be a Černikov $\omega$-group.

**Proof.** If $A$ could be realized as a quotient of a $ZG$-module whose additive group were torsion-free of finite rank, then $\phi(g)$ would plainly satisfy a polynomial over $\mathbb{Z}$ for every $g \in G$. Hence it must be impossible to express $A$ as such a quotient. Theorem A, therefore, yields that $H^n(G, A)$ is a Černikov $\omega$-group for $n \geq 0$. \qed
We conclude the paper with two examples illustrating that, in Theorem B, we cannot dispense with either the restriction on $G$ or the condition on the submodules of $A$.

**Example 4.6.** Let $p$ and $q$ be distinct primes. Take $Q$ to be an infinite cyclic group, and let $A$ be the $\mathbb{Z}Q$-module obtained by letting the generator of $Q$ act on $\mathbb{Z}_{p^\infty}$ by multiplication by $q$. Set $G = A \times Q$. We may also regard $A$ as a $\mathbb{Z}G$-module via the epimorphism $G \to Q$. First we claim that $A$ fails to be a quotient of a torsion-free $\mathbb{Z}G$-module that has finite rank and is $\{p\}$-spectral. To show this, we let $B$ be the underlying additive group of the ring $\mathbb{Z}[1/q]$ and endow $B$ with a $\mathbb{Z}Q$-module structure by letting the generator of $Q$ act again by multiplication by $q$. By Proposition 2.3(i),

$$\text{Ext}^1_{\mathbb{Z}Q}(A, B) \cong \text{Ext}^1_{\mathbb{Z}}(A, B)^{Q}.$$ 

Moreover, a simple calculation reveals $\text{Ext}^1_{\mathbb{Z}}(A, B) \cong \hat{\mathbb{Z}}_p$. Since the diagonal action of $Q$ on $\text{Ext}^1_{\mathbb{Z}}(A, B)$ is trivial, this means $\text{Ext}^1_{\mathbb{Z}Q}(A, B) \cong \hat{\mathbb{Z}}_p$. However, according to Lemma 2.3, if $A$ were a quotient of a torsion-free $\{p\}$-minimax $\mathbb{Z}G$-module, then $\text{Ext}^2_{\mathbb{Z}Q}(A, B)$ would have to be finite. Hence $A$ cannot be a quotient of such a module.

Now consider the cohomology group $H^1(G, A)$, which is isomorphic to $\text{Hom}(A, A)^Q$. With the diagonal action being trivial, this yields $H^1(G, A) \cong \hat{\mathbb{Z}}_p$. Therefore, we have described an example where all the hypotheses of Theorem A are satisfied save the stipulation that $G$ is either virtually torsion-free or finitely generated, and yet the cohomology fails to be torsion.

**Example 4.7.** Let $Q = \langle t \rangle$ be an infinite cyclic group. Form a $\mathbb{Z}Q$-module $B_1$ by letting $t$ act on $\mathbb{Z}[1/p]$ by multiplication by $p$. Moreover, let $B_2$ be the $\mathbb{Z}Q$-module with the same underlying additive group but where $t$ acts instead by multiplication by $1/p$. Next set $G = (B_1 \oplus B_2) \times Q$. Then $G$ is a finitely generated torsion-free solvable minimax group. Take $A$ to be the trivial $\mathbb{Z}G$-module with underlying abelian group $\mathbb{Z}_{p^\infty}$. We will use the LHS spectral sequence $\{E^p_\ast\}$ associated to the extension $1 \to B_1 \oplus B_2 \to G \to Q \to 1$ to compute $H^2(G, A)$. For this purpose, we let $V = \text{Hom}_G(\mathbb{Z}[1/p], A)$, so that $V$ is a vector space over $Q$ with dimension equal to the cardinality of the continuum. We thus have $H^1(B_1 \oplus B_2, A) \cong V \oplus V$, where $Q$ acts on the first copy of $V$ via multiplication by $p$ and on the second by multiplication by $1/p$. Hence every derivation from $Q$ to $V \oplus V$ is inner, rendering $E_2^{01} = 0$. Also, the universal coefficient theorem and Künneth formula yield $H^2(B_1 \oplus B_2, A) \cong V$, where the action of $Q$ on $V$ is trivial. This means $E_2^{02} \cong V$. Consequently, $H^2(G, A) \cong V$. This example, then, demonstrates the necessity of the condition in Theorem A on the submodules of $A$.

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