EXISTENCE AND MULTIPlicity OF NONTRIVIAL SOLUTIONS FOR A SEMILINEAR BIHARMONIC EQUATION WITH WEIGHT FUNCTIONS

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ABSTRACT. We consider the existence and multiplicity of nontrivial solutions for a semilinear biharmonic equation with the concave-convex nonlinearities \( f(x) |u|^{q-1} u \) and \( h(x) |u|^{p-1} u \) under certain conditions on \( f(x) \), \( h(x) \), \( p \) and \( q \).

Applying the Nehari manifold method along with the fibering maps and the minimization method, we study the effect of \( f(x) \) and \( h(x) \) on the existence and multiplicity of nontrivial solutions for the semilinear biharmonic equation.

When \( h(x)^+ \neq 0 \), we prove that the equation has at least one nontrivial solution if \( f(x)^+ = 0 \) and that the equation has at least two nontrivial solutions if \( \int_\Omega |f^+|^r \, dx \in (0, \Lambda^r) \), where \( r \) and \( \Lambda \) are explicit numbers. These results are novel, which improve and extend the existing results in the literature.

1. Introduction. In this paper, we are concerned with the following semilinear biharmonic elliptic problem

\[
\begin{cases}
\Delta^2 u = f(x)|u|^{q-1} u + h(x)|u|^{p-1} u, & \text{in } \Omega, \\
u = \Delta u = 0, & \text{on } \partial \Omega,
\end{cases}
\]

where \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^N \) \((N > 4)\), \( 0 < q < 1 < p < \frac{N+4}{N-4} \), and \( f, h \in C(\bar{\Omega}) \) are weight functions with \( h(x)^+ := \max\{h(x), 0\} \neq 0 \).

In recent years, there has been continuous attention on the semilinear second-order elliptic problem

\[
\begin{cases}
-\Delta u = g_\lambda(x, u), & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\]

2020 Mathematics Subject Classification. 35J35, 35J40, 35J65.

Key words and phrases. Biharmonic equation, Nehari manifold, concave-convex nonlinearity, nontrivial solution, weight function.

This work is supported National Science Foundation of China No. 11871250.
where $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$ ($N \geq 3$), $g_\lambda : \Omega \times \mathbb{R} \to \mathbb{R}$ and parameter $\lambda \in \mathbb{R}_0^+$ ($\mathbb{R}_0^+ = (0, +\infty)$), see [2–4, 6, 13, 16, 18, 24] and the references therein. As $g_\lambda$ is sublinear, for example, $g_\lambda = \lambda u^q$, $0 < q < 1$, the monotone iteration scheme or the upper and lower solution method are effective, see [12]. As $g_\lambda$ is superlinear and subcritical, for instance, $g_\lambda = \lambda u + |u|^{p-1}u$, $1 < p < \frac{N+2}{N-2}$, variational methods can be applied, see [23]. In contrast to the sublinear case and the superlinear case, Ambrosetti-Brezis-Cerami [3] considered the problem (2) in the case that $\lambda \in \mathbb{R}_0^+$ small enough. Sun-Li [16] considered a similar problem

$$
\begin{cases}
-\Delta u = u^q + \lambda u^p, & \text{in } \Omega,
0 \leq u \in H_0^1(\Omega),
\end{cases}
$$

where $0 < q < 1 < p \leq \frac{N+2}{N-2}$, and showed that the problem (3) admits at least two positive solutions for $\lambda \in \mathbb{R}_0^+$ small enough. Sun-Li [16] considered a similar problem

$$
\begin{cases}
-\Delta u = u^q + \lambda u^p, & \text{in } \Omega,
0 \leq u \in H_0^1(\Omega),
\end{cases}
$$

with $0 < q < 1 < p = \frac{N+2}{N-2}$. They investigated the value of $\Lambda$ the supremum of the set $\lambda$ associated with the existence and multiplicity of positive solutions and established the uniform lower bounds for $\Lambda$. Wu [24] considered the subcritical case of the problem (3) with $\lambda u^q$ being replaced by $\lambda f(x)u^q$, where $f(x) \in C(\Omega)$ is a sign-changing function, and proved that the problem (3) has at least two positive solutions as $\lambda \in \mathbb{R}_0^+$ is sufficiently small.

Some interesting extensions of (3) have been carried out in the framework of quasi-linear elliptic equations or systems, semilinear second-order elliptic systems and semilinear fourth-order elliptic equations, see [9, 11, 15, 20, 25, 26]. For instance, Yang-Wang [25] showed that there exists at least two positive solutions when $\lambda \in \mathbb{R}_0^+$ is small enough for the semilinear biharmonic elliptic problem

$$
\begin{cases}
\Delta^2 u = \lambda f(x)|u|^{q-2}u + h(x)|u|^{p-2}u, & \text{in } \Omega,
\Delta u = 0, & \text{on } \partial\Omega,
\end{cases}
$$

where $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$ ($N > 4$), $0 < q < 1 < p < \frac{N+4}{N-4}$, $\lambda \in \mathbb{R}_0^+$, and $f(x) \in C(\overline{\Omega}, \mathbb{R})$ and $h(x) \in C(\overline{\Omega}, \mathbb{R}_0^+)$ are weight functions.

In the present paper, our goal is to study the existence and multiplicity of nontrivial solutions for the problem (1). For simplicity and convenience, we denote the norm of $u$ in $L^s(\Omega)$ by $|u|_s = (\int_\Omega |u(x)|^s)^{1/s}$ and the norm of $u$ in $C(\overline{\Omega})$ by $|u|_\infty = \max_{x \in \overline{\Omega}} |u(x)|$. We also denote $H_0^1(\Omega) \cap H^2(\Omega)$ by $H(\Omega)$, endowed with the norm $\|u\| = |\Delta u|_2$. Let $S$ be the best Sobolev constant for the embedding of $H(\Omega)$ in $L^{p+1}(\Omega)$, precisely, $|u|_{p+1} \leq S\|u\|$ for all $u \in H(\Omega)$ (see [1]).

We define

$$
J(u) = \frac{1}{2}\|u\|^2 - \frac{1}{q+1} \int_\Omega f(x)|u|^{q+1} \, dx - \frac{1}{p+1} \int_\Omega h(x)|u|^{p+1} \, dx, \quad u \in H(\Omega).
$$

It is well-known that the nontrivial weak solutions of the problem (1) are the nonzero critical points of the energy functional $J$ (see [5, 14]).

In order to find the nonzero critical points of the functional $J$, we focus on the Nehari minimization problem:

$$
\alpha = \inf\{J(u) : u \in \mathcal{N}\},
$$
where $N = \{ u \in H(\Omega) \setminus \{0\} \mid \langle J'(u), u \rangle = 0 \}$. To this end, we define
\[
\psi(u) := \langle J'(u), u \rangle = ||u||^2 - \int_{\Omega} f(x)|u|^{q+1} \, dx - \int_{\Omega} h(x)|u|^{p+1} \, dx.
\]
For $u \in N$, there holds
\[
\langle \psi'(u), u \rangle = 2||u||^2 - (q+1) \int_{\Omega} f(x)|u|^{q+1} \, dx - (p+1) \int_{\Omega} h(x)|u|^{p+1} \, dx
\]
\[
= (1-p)||u||^2 - (q-p) \int_{\Omega} f(x)|u|^{q+1} \, dx
\]
\[
= (1-q)||u||^2 - (q-p) \int_{\Omega} h(x)|u|^{p+1} \, dx
\]
\[
= (1-q) \int_{\Omega} f(x)|u|^{q+1} \, dx - (p-1) \int_{\Omega} h(x)|u|^{p+1} \, dx.
\]
We split $N$ into three parts
\[
N^+ = \{ u \in N \mid \langle \psi'(u), u \rangle > 0 \},
\]
\[
N^0 = \{ u \in N \mid \langle \psi'(u), u \rangle = 0 \},
\]
\[
N^- = \{ u \in N \mid \langle \psi'(u), u \rangle < 0 \}.
\]

Note that all nonzero solutions of (1) are obviously in the Nehari Manifold-$N$. Hence, our approach to solve the problem (1) is to equivalent to analyzing the structure of $N$, and the minimization problem for $J$ on $N^+$ and $N^-$ by applying the Nehari manifold method along with the fibering maps and the minimization method.

**Theorem 1.1.** The problem (1) has at least one nontrivial solution if $f^+ = \max\{f, 0\} = 0$.

**Theorem 1.2.** The problem (1) has at least two nontrivial solutions for any $f \in \{f \mid f^+, f^- \in (0, \Lambda)\}$, where $r = \frac{p+1}{p-q}$ and $\Lambda = \frac{p-1}{p-q} \cdot \left[ \frac{1-q}{\|h\|_{\infty}} \right]^{\frac{1}{p-q}} S^{\frac{2(p-q)}{p+q}}$.

Applying Theorems 1.1-1.2 to the problem (4), we can obtain the following results immediately.

**Corollary 1.3.** The problem (4) has at least one nontrivial solution for $\lambda \geq 0$ (resp. $\lambda \leq 0$) if $f^+ = 0$ (resp. $(-f)^+ = 0$).

**Corollary 1.4.** The problem (4) has at least two nontrivial solutions for any $\lambda \in \mathcal{S}_{\lambda, f} := \{ \lambda \in \mathbb{R} \mid (\lambda f)^+ \in (0, \Lambda) \}$.

**Remark 1.** Since $\lambda \cdot f = (-\lambda) \cdot (-f)$, it is easy to see that condition $(\lambda f)^+ \neq 0$ covers the following cases
(i) $\lambda > 0$ and $f^+ \neq 0$; (ii) $\lambda < 0$ and $(-f)^+ \neq 0$; (iii) $\lambda \neq 0$ and $(\pm f)^+ \neq 0$. Obviously, we have $|h^+|_{\infty} \leq |h|_{\infty}$ and $|f^+|^r \leq |f|^r$. It implies that if conditions (i) or (iii) holds, then $(0, \lambda_*) \subset \mathcal{S}_{\lambda, f}$, where $\lambda_* = \frac{p-1}{p-q} \cdot \left[ \frac{1-q}{\|h\|_{\infty}} \right]^{\frac{1}{p-q}} S^{\frac{2(p-q)}{p+q}} |f|^r$. In particular, compared with the existing results in the literature, for example, see [25, Theorem 1.1], the larger subset $\mathcal{S}_{\lambda, f}$ is determined such that the problem (4) has at least two nontrivial solutions for any $\lambda \in \mathcal{S}_{\lambda, f}$.

**Remark 2.** As pointed out in [8], we do not claim in the statements of Theorems 1.1 and 1.2 that the solutions are nonnegative. The reason is that we can not use,
for a local minimizer \( u \in H(\Omega) \), the relation \( J(u) = J(|u|) \) to deduce nonnegativity of solution, since it can happen that \( |u| \notin H(\Omega) \) even if \( u \in H(\Omega) \).

The paper is organized as follows. In Section 2, we present some technical lemmas. In Section 3, we prove Theorems 1.1 and 1.2, respectively.

2. Preliminaries. In this section, we present several lemmas, which will be used in the proofs of our main results.

Lemma 2.1. For any \( f \in \{ f \mid |f^+|_r \in [0, A] \} \) (where \( r \) and \( A \) given in Theorem 1.2), there holds \( N^0 = \emptyset \).

Proof. Assume that \( N^0 \neq \emptyset \). If \( u \in N^0 \), then we have
\[
\|u\|^2 = \int_{\Omega} f(x)|u|^{p+1} \, dx + \int_{\Omega} h(x)|u|^{p+1} \, dx \tag{7}
\]
and
\[
2\|u\|^2 = (q+1) \int_{\Omega} f(x)|u|^{p+1} \, dx + (p+1) \int_{\Omega} h(x)|u|^{p+1} \, dx. \tag{8}
\]
By (7)-(8), it follows from Sobolev’s inequality and Hölder’s inequality that
\[
\|u\|^2 = \frac{p-q}{1-q} \int_{\Omega} h(x)|u|^{p+1} \, dx \leq \frac{p-q}{1-q} \cdot |h^+|_\infty S^{p+1} \|u\|^{p+1} \tag{9}
\]
and
\[
\|u\|^2 = \frac{p-q}{p-1} \int_{\Omega} f(x)|u|^{p+1} \, dx \leq \frac{p-q}{p-1} \cdot |f^+|_r S^{q+1} \|u\|^{q+1}. \tag{10}
\]
Using (9) and (10) yields
\[
|f^+|_r \geq \frac{p-1}{p-q} \cdot S^{-q-1} \left(\frac{1-q}{p-q} |h^+|_\infty S^{-1} \right)^\frac{1-q}{p-1} \tag{11}
\]
\[
= \frac{p-1}{p-q} \cdot \left(\frac{1-q}{(p-q)|h^+|_\infty} \right)^\frac{1-q}{p-1} S^{\frac{2(p-q)}{p-1}} = A.
\]
Hence, by (11) we arrive at the desired result. \( \square \)

By Lemma 2.1, for \( f \in \{ f \mid |f^+|_r \in [0, A] \} \) we write \( N = N^+ \cup N^- \) and define
\[
\alpha^+ = \inf_{u \in N^+} J(u), \quad \alpha^- = \inf_{u \in N^-} J(u).
\]
The following lemma shows that the local minimizers on \( N \) are the critical points for \( J \).

Lemma 2.2. For \( f \in \{ f \mid |f^+|_r \in [0, A] \} \), if \( u_0 \) is a local minimizer of \( J \) on \( N \), then \( J'(u_0) = 0 \) in \( [H(\Omega)]^* \), where \( [H(\Omega)]^* \) is the dual space of \( H(\Omega) \).

Proof. If \( u_0 \) is a local minimizer of \( J \) on \( N \), then \( u_0 \) is a solution of the optimization problem
minimize \( J(u) \) subject to \( \psi(u) = 0 \).

By using the Lagrange multipliers, there exists \( \theta \in \mathbb{R} \) such that
\[
J'(u_0) = \theta \psi'(u_0) \text{ in } [H(\Omega)]^*. \tag{12}
\]
Thus, we get
\[
\langle J'(u_0), u_0 \rangle = \theta \langle \psi'(u_0), u_0 \rangle. \tag{13}
\]
From \( u_0 \in N \) and Lemma 2.1, we have \( \langle J'(u_0), u_0 \rangle = 0 \) and \( \langle \psi'(u_0), u_0 \rangle \neq 0 \). Thus \( \theta = 0 \) due to (13), which together with (12) implies that \( J'(u_0) = 0 \) in \( [H(\Omega)]^* \). \( \square \)
Lemma 2.3. The following two statements are true.
(i) If \( u \in \mathbb{N}^+ \), then \( \int_{\Omega} f(x)|u|^{q+1} \, dx > 0 \).
(ii) If \( u \in \mathbb{N}^- \), then \( \int_{\Omega} h(x)|u|^{p+1} \, dx > 0 \).

Proof. (i) In view of (6) and \( u \in \mathbb{N}^+ \), we have
\[
(1 - p)|u|^2 - (q - p) \int_{\Omega} f(x)|u|^{q+1} \, dx = \langle \psi'(u), u \rangle > 0,
\]
which implies
\[
\int_{\Omega} f(x)|u|^{q+1} \, dx > \frac{p-1}{p-q} \|u\|^2 > 0.
\]
(ii) In view of (6) and \( u \in \mathbb{N}^- \), we get
\[
(1 - q)|u|^2 - (p - q) \int_{\Omega} h(x)|u|^{p+1} \, dx = \langle \psi'(u), u \rangle < 0,
\]
and then
\[
\int_{\Omega} h(x)|u|^{p+1} \, dx > \frac{q-1}{p-q} \|u\|^2 > 0.
\]

For each \( u \in H(\Omega) \setminus \{0\} \), we write
\[
t_{\max} = t_{\max}(u) = \begin{cases} 
\left(\frac{1-q}{p-q} \int_{\Omega} h(x)|u|^{p+1} \, dx\right)^{\frac{1}{p+1}}, & \int_{\Omega} h(x)|u|^{p+1} \, dx > 0, \\
+\infty, & \int_{\Omega} h(x)|u|^{p+1} \, dx \leq 0.
\end{cases}
\]

Then, we have the following lemma.

Lemma 2.4. For \( u \in H(\Omega) \setminus \{0\} \) and \( f \in \{f | |f^+|, \in [0, A]\} \), the following two statements are true.
(i) If \( \int_{\Omega} h(x)|u|^{p+1} \, dx > 0 \) and \( \int_{\Omega} f(x)|u|^{q+1} \, dx \leq 0 \), then there exists a unique \( t^- > t_{\max} \) such that \( t^- u \in \mathbb{N}^- \) and
\[
J(t^- u) = \max_{t \geq 0} J(tu).
\]
(ii) If \( \int_{\Omega} f(x)|u|^{q+1} \, dx > 0 \), then there exist unique \( t^+ \) and \( t^- \) such that \( 0 < t^- < t_{\max} < t^-, t^+ u \in \mathbb{N}^+, t^- u \in \mathbb{N}^- \) and
\[
\begin{cases}
J(t^+ u) = \min_{0 \leq t \leq t^-} J(tu), & J(t^- u) = \max_{t \geq t^+} J(tu), \quad \int_{\Omega} h(x)|u|^{p+1} \, dx > 0; \\
J(t^+ u) = \min_{t \geq 0} J(tu), & \int_{\Omega} h(x)|u|^{p+1} \, dx \leq 0.
\end{cases}
\]

Proof. For \( u \in H(\Omega) \setminus \{0\} \), we set
\[
s(t) = t^{1-q}\|u\|^2 - t^{p-q} \int_{\Omega} h(x)|u|^{p+1} \, dx, \quad t \in \mathbb{R}^+ := [0, +\infty).
\]
It is easy to see that
\[
\begin{cases}
s(t) \text{ increases in } [0, t_{\max}], \text{ and decreases in } [t_{\max}, +\infty), \quad \text{as } \int_{\Omega} h(x)|u|^{p+1} \, dx > 0; \\
s(t) \text{ increases in } [0, +\infty), \quad \text{as } \int_{\Omega} h(x)|u|^{p+1} \, dx \leq 0.
\end{cases}
\]
Moreover, if \( \int_{\Omega} h(x) \lvert u \rvert^{p+1} \, dx > 0 \), then
\[
s(t_{\text{max}}) = t_{\text{max}}^{\frac{1-q}{p}} \lVert u \lVert^2 - t_{\text{max}}^{\frac{p-q}{p}} \int_{\Omega} h(x) \lvert u \rvert^{p+1} \, dx
\]
\[
= \lVert u \lVert^2 \left[ \frac{1-q}{p-q} \left( \frac{1}{\int_{\Omega} h(x) \lvert u \rvert^{p+1} \, dx} \right) - \frac{p-q}{p-q} \right]
\]
\[
\geq \lVert u \lVert^2 \left[ \frac{1-q}{p-q} \left( \frac{1}{\int_{\Omega} h(x) \lvert u \rvert^{p+1} \, dx} \right) - \frac{p-q}{p-q} \right]
\]  
\[\tag{15}\]

(i) If \( \int_{\Omega} h(x) \lvert u \rvert^{p+1} \, dx > 0 \) and \( \int_{\Omega} f(x) \lvert u \rvert^{q+1} \, dx \leq 0 \), in view of \( s(t_{\text{max}}) > 0 \) and \( s(t) \to -\infty \) as \( t \to +\infty \), there is a unique \( t^- > t_{\text{max}} \) such that \( s(t^-) = \int_{\Omega} f(x) \lvert u \rvert^{q+1} \, dx \) and \( s'(t^-) < 0 \). Then
\[
\langle J'(t^-)u, t^-u \rangle = \lVert t^-u \lVert^2 - \int_{\Omega} f(x) \lvert t^-u \rvert^{q+1} \, dx - \int_{\Omega} h(x) \lvert t^-u \rvert^{p+1} \, dx
\]
\[
= (t^-)^{q+1} \left[ s(t^-) - \int_{\Omega} f(x) \lvert u \rvert^{q+1} \, dx \right] = 0,
\]
and
\[
\langle \psi'(t^-)u, t^-u \rangle = (1-q)\lVert t^-u \lVert^2 - (p-q) \int_{\Omega} h(x) \lvert t^-u \rvert^{p+1} \, dx
\]
\[
= (t^-)^{2+q} \left[ (1-q)(t^-)^{-q} \lVert u \lVert^2 - (p-q)(t^-)^{p-q} \int_{\Omega} h(x) \lvert u \rvert^{p+1} \, dx \right]
\]
\[
= (t^-)^{2+q} s'(t^-) < 0.
\]
This implies that \( t^-u \in \mathcal{N}^- \). By a direct calculation, we have
\[
\frac{dJ(tu)}{dt} = t \lVert u \lVert^2 - t^q \int_{\Omega} f(x) \lvert u \rvert^{q+1} \, dx - t^p \int_{\Omega} h(x) \lvert u \rvert^{p+1} \, dx
\]
\[
= t^q \left[ s(t) - \int_{\Omega} f(x) \lvert u \rvert^{q+1} \, dx \right].
\]
Clearly, \( \frac{dJ(tu)}{dt} > 0 \) for \( t \in (0, t^-) \) and \( \frac{dJ(tu)}{dt} < 0 \) for \( t \in (t^-, +\infty) \). Hence, \( J(tu) \) attains its maximum at \( t^- \), that is, \( J(t^-u) = \max_{t_{\geq 0}} J(tu) \).

(ii) If \( \int_{\Omega} h(x) \lvert u \rvert^{p+1} \, dx > 0 \) and \( \int_{\Omega} f(x) \lvert u \rvert^{q+1} \, dx > 0 \), from (15) and
\[
s(0) = 0 < \int_{\Omega} f(x) \lvert u \rvert^{q+1} \, dx \leq \lvert f \rvert_{\lVert S \rVert^{q+1}} \lVert u \lVert^{q+1}
\]
\[
< \lVert u \lVert^{q+1} \left[ \frac{p-1}{p-q} \left( \frac{1-q}{p-q} \right) \left( \frac{1}{\lVert h \rVert_{\infty} \lVert S \rVert^{p+1}} \right) \right],
\]
there exist unique \( t^+ \) and \( t^- \) such that \( 0 < t^+ < t_{\text{max}} < t^- \), and
\[
s(t^+) = \int_{\Omega} f(x) \lvert u \rvert^{q+1} \, dx = s(t^-)
\]
and
\[s'(t^+) > 0 > s'(t^-).
\]
Similar to (i), we have \( t^+u \in \mathcal{N}^+, t^-u \in \mathcal{N}^- \), and
\[
J(t^+u) = \min_{0 \leq t \leq t^-} J(tu), \quad J(t^-u) = \max_{t \geq t^+} J(tu).
\]
If \( \int_{\Omega} h(x) |u|^{p+1} \, dx \leq 0 \) and \( \int_{\Omega} f(x) |u|^{q+1} \, dx > 0 \), by the monotonous increase of \( s(t) \) with \( s(0) = 0 \) and \( \lim_{t \to +\infty} s(t) = +\infty \), there is a unique \( t^+ \in (0, +\infty) \) such that 
\[
 s(t^+) = \int_{\Omega} f(x) |u|^{q+1} \, dx \quad \text{with} \quad s'(t^+) > 0,
\]
which leads to \( J(t^+ u) = \min_{t \geq 0} J(t u) \) with \( t^+ u \in N^+ \).

Let us consider the existence of the least energy (or ground state) solutions for the semilinear biharmonic problem
\[
\begin{aligned}
\begin{cases}
\Delta^2 u = f(x)|u|^{q-1} u, & \text{in } \Omega, \\
u = \Delta u = 0, & \text{on } \partial \Omega,
\end{cases}
\end{aligned}
\tag{16}
\]
where \( \Omega \subset \mathbb{R}^N \) (\( N > 4 \)) is an open bounded domain with the smooth boundary, the weight function \( f \in C(\Omega) \) and \( 0 < q < 1 \). The energy functional is given by
\[
E_f(u) = \frac{1}{2} \|u\|^2 - \frac{1}{q+1} \int_{\Omega} f(x)|u|^{q+1} \, dx, \quad u \in H(\Omega),
\]
and the corresponding Nehari minimization problem is
\[
\gamma_f = \inf\{E_f(u) \mid u \in \mathcal{M}_f\},
\]
where \( \mathcal{M}_f = \{u \in H(\Omega) \setminus \{0\} \mid \langle E'_f(u), u \rangle = 0\} \). Then, we have the following result.

**Lemma 2.5.** If \( f^+ \neq 0 \), then the problem (16) has a nonzero solution \( w_f \in \mathcal{M}_f \) such that
\[
E_f(w_f) = \gamma_f < 0.
\]

**Proof.** We separate the proof into the following steps.

**Step 1.** We claim that \( \mathcal{M}_f \neq \emptyset \).

Let
\[
\phi(x) = \chi_{(r^+) \setminus 0}(x) := \begin{cases} 1, & f^+(x) > 0, \\
0, & f^+(x) = 0.
\end{cases}
\]

Then \( \phi \in L^\infty(\Omega) \), and there exists a sequence of the smooth functions \( \{\phi_n\}_{n=1}^{\infty} \subset C^\infty(\Omega) \) such that \( \phi_n \to \phi \) (as \( n \to \infty \)) in \( L^{p+1}(\Omega) \). It follows from Minkowski’s and Hölder’s inequalities that
\[
\left| \left( \int_{\Omega} f^\pm |\phi_n|^{q+1} \, dx \right)^{\frac{1}{q+1}} - \left( \int_{\Omega} f^\pm |\phi|^{q+1} \, dx \right)^{\frac{1}{q+1}} \right|
\leq \left( \int_{\Omega} f^\pm |\phi_n - \phi|^{q+1} \, dx \right)^{\frac{1}{q+1}}
\leq \left( \int_{\Omega} |f^\pm| \, dx \right)^{\frac{1}{q+1}} \left( \int_{\Omega} |\phi_n - \phi|^{p+1} \, dx \right)^{\frac{1}{p+1}},
\]
where \( f^\pm = \max\{\pm f, 0\} \), which implies that
\[
\lim_{n \to \infty} \int_{\Omega} f|\phi_n|^{q+1} \, dx = \lim_{n \to \infty} \int_{\Omega} (f^+ - f^-)|\phi_n|^{q+1} \, dx
\leq \int_{\Omega} (f^+ - f^-)|\phi|^{q+1} \, dx = \int_{\Omega} f^+|\phi|^{q+1} \, dx.
\]
Note that $f^+ \neq 0$ and
\[
0 \leq \int_{\{f^+ > 0\}} f^+ \, dx = \int_{\Omega} f^+ |\phi|^{q+1} \, dx \leq \left( \int_{\Omega} |f^+|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |\phi|^{q+1} \, dx \right)^{\frac{q+1}{2}} < +\infty.
\]

So, $\int_{\Omega} f^+ |\phi|^{q+1} \, dx \in \mathbb{R}^+_0$. We now choose $n \in \mathbb{N}$ large enough such that
\[
\int_{\Omega} f|\phi_n|^{q+1} \, dx > \frac{1}{2} \int_{\Omega} f^+ |\phi|^{q+1} \, dx > 0.
\]

We consider
\[
L(t) := \langle E'_f(t\phi_n), \phi_n \rangle = \frac{d}{dt} E_f(t\phi_n) = t||\phi_n||^2 - t^q \int_{\Omega} f|\phi_n|^{q+1} \, dx.
\]

It is clear that there exists a unique $t_n \in \mathbb{R}^+_0$ such that $L(t_n) = 0$, that is, $\langle E'_f(t_n\phi_n), t_n\phi_n \rangle = 0$, i.e. $t_n\phi_n \in M_f$. Hence, $M_f \neq \emptyset$.

**Step 2.** We show that $E_f$ is bounded below on $H(\Omega)$ and $\gamma_f < 0$.

By using Hölder’s, Sobolev’s and Young’s inequalities, we have
\[
E_f(w) = \frac{1}{2}||w||^2 - \frac{1}{q+1} \int_{\Omega} f(x)|w|^{q+1} \, dx \\
\geq \frac{1}{2}||w||^2 - \frac{1}{q+1} |f^+|_r S^{q+1}||w||^{q+1} \\
\geq \frac{1}{2}||w||^2 - \left[ 2^{\frac{q+1}{q}} \left( |f^+|_r S^{q+1} \right)^{\frac{2}{q+1}} + \frac{1}{2} ||w||^2 \right] \\
\geq - \left( 2|f^+|_r S^{q+1} \right)^{\frac{2}{q+1}}, \quad w \in H(\Omega).
\]

Thus, $E_f$ is bounded below on $H(\Omega)$ and $E_f$ is bounded below on $M_f$. Hence, $\gamma_f = \inf\{E_f(w) \mid w \in M_f\}$ is finite. Moreover, fixing a $w^* \in M_f$, we get
\[
\gamma_f \leq E_f(w^*) = \frac{1}{2}||w^*||^2 - \frac{1}{q+1} \int_{\Omega} f(x)|w^*|^{q+1} \, dx = \left( \frac{1}{2} - \frac{1}{q+1} \right)||w^*||^2 < 0.
\]

**Step 3.** We prove the existence of the minimizers for $E_f$ on $M_f$.

For $w \in M_f$, it follows from Hölder’s and Sobolev’s inequalities that
\[
||w||^2 = \int_{\Omega} f(x)|w|^{q+1} \, dx \leq |f^+|_r S^{q+1}||w||^{q+1},
\]
which implies
\[
||w|| \leq \left( |f^+|_r S^{q+1} \right)^{\frac{q}{q+1}}, \quad w \in M_f. \tag{17}
\]

Let $\{w_n\}$ be a minimizing sequence for $E_f$ on $M_f$. By (17) and the compact imbedding theorem, there exist a subsequence still denoted by $\{w_n\}$ and $w_f$ in $H(\Omega)$ such that
\[
w_n \rightharpoonup w_f \text{ weakly in } H(\Omega) \tag{18}
\]
and
\[
w_n \rightarrow w_f \text{ strongly in } L^{p+1}(\Omega). \tag{19}
\]
We claim that \( \int_{\Omega} f(x)|w_j|^{q+1} \, dx \neq 0 \). Otherwise, by Minkowski’s and Hölder’s inequalities with \( f \in L^r(\Omega) \) and (19), there holds
\[
\left( \int_{\Omega} f^\pm(x)|w_n|^{q+1} \, dx \right)^{1 \over q+1} \rightarrow \left( \int_{\Omega} f^\pm(x)|w_j|^{q+1} \, dx \right)^{1 \over q+1}, \quad \text{as } n \rightarrow \infty,
\]
which implies
\[
\int_{\Omega} f(x)|w_n|^{q+1} \, dx \rightarrow \int_{\Omega} f(x)|w_j|^{q+1} \, dx = 0, \quad \text{as } n \rightarrow \infty.
\]
Then
\[
E_f(w_n) = {1 \over 2} \|w_n\|^2 - {1 \over q+1} \int_{\Omega} f(x)|w_n|^{q+1} \, dx
\]
\[
= \left( {1 \over 2} - {1 \over q+1} \right) \int_{\Omega} f(x)|w_n|^{q+1} \, dx \rightarrow 0, \quad \text{as } n \rightarrow \infty.
\]
This yields a contradiction with \( E_f(w_n) \rightarrow \gamma_j < 0 \), as \( n \rightarrow \infty \). Thus, \( \int_{\Omega} f(x)|w_j|^{q+1} \, dx \neq 0 \). In particular, \( w_j \neq 0 \).

We now show that \( w_n \rightarrow w_j \) strongly in \( H(\Omega) \). Otherwise, \( \|w_j\| < \liminf_{n \rightarrow \infty} \|w_n\| \).

That is,
\[
\|w_j\|^2 - \int_{\Omega} f(x)|w_j|^{q+1} \, dx < \liminf_{n \rightarrow \infty} \left( \|w_n\|^2 - \int_{\Omega} f(x)|w_n|^{q+1} \, dx \right) = 0,
\]
which implies \( \int_{\Omega} f(x)|w_j|^{q+1} \, dx > \|w_j\|^2 > 0 \). There exists a unique \( t_0 \in (1, +\infty) \) such that
\[
\begin{align*}
\left\{ \begin{array}{l}
\langle E_f(t_0w_j), t_0w_j \rangle = t_0^2 \left( \|w_j\|^2 - t_0^{-1} \int_{\Omega} f(x)|w_j|^{q+1} \, dx \right) = 0, \\
{d \over dt} E_f(tw_j) = t \left( \|w_j\|^2 - t^{q-1} \int_{\Omega} f(x)|w_j|^{q+1} \, dx \right) < 0, \quad t \in (0, t_0).
\end{array} \right.
\end{align*}
\]
Thus, \( t_0w_j \in \mathcal{M}_f \) and
\[
\gamma_j \leq E_f(t_0w_j) < E_f(w_j) < \lim_{n \rightarrow \infty} E_f(w_n) = \gamma_j.
\]
This is a contradiction. Hence, \( w_n \rightarrow w_j \) strongly in \( H(\Omega) \). This implies \( w_j \in \mathcal{M}_f \) and
\[
E_f(w_j) = \lim_{n \rightarrow \infty} E_f(w_n) = \gamma_j.
\]
That is, \( w_j \) is a minimizer of \( E_f \) on \( \mathcal{M}_f \).

**Step 4.** We show the existence of nonzero weak solutions of the problem (16).

If \( w_0 \) is a local minimizer of \( E_f \) on \( \mathcal{M}_f \), then \( w_0 \) is a solution of the optimization problem
\[
\text{minimize } E_f(w) \text{ subject to } \Psi_f(w) = 0,
\]
where \( \Psi_f(w) = \langle E'_f(w), w \rangle, w \in H(\Omega) \). Thus, by the theory of Lagrange multipliers, there exists \( \ell \in \mathbb{R} \) such that
\[
E'_f(w_0) = \ell \Psi'_f(w_0) \text{ in } [H(\Omega)]^*.
\]
Then
\[
\ell \langle \Psi'_f(w_0), w_0 \rangle = \langle E'_f(w_0), w_0 \rangle = \Psi_f(w_0) = 0.
\]
Since \( \langle \Psi'_f(w_0), w_0 \rangle = (1 - q)\|w_0\|^2 > 0 \), we have \( \ell = 0 \). Hence, \( E'_f(w_0) = 0 \). In other words, a minimizer of \( E_f \) on \( \mathcal{M}_f \) is a nonzero critical point of \( E_f \).
Proof. It follows from Lemma 2.5 that

\[ \Gamma \]

Thus, by (21) and (23)-(26), we obtain \( \alpha \leq \alpha^+ < \Gamma_f < 0 \).

Lemma 2.6. \( J \) is coercive and bounded below on \( N \).

Proof. If \( u \in N \), by Hölder’s and Young’s inequalities, it follows from (5)-(6) that

\[
J(u) = \left( \frac{1}{2} - \frac{1}{p+1} \right) \| u \|^2 - \frac{1}{q+1} \| u \|^2 - \frac{1}{p+1} \int f(x) \| u \|^{q+1} \, dx
\]

Thus, \( J \) is coercive and bounded below on \( N \).

Lemma 2.7. Assume that \( f \in \{ f \mid |f^+|_r \in (0,A) \} \). Then there exists a constant \( \Gamma_f < 0 \) such that \( \alpha \leq \alpha^+ < \Gamma_f < 0 \).

Proof. It follows from Lemma 2.5 that

\[
\int f(x) w_f^{q+1} \, dx > 0
\]

and

\[
J(w_f) = E_f(w_f) - \frac{1}{p+1} \int h(x) w_f^{p+1} \, dx
\]

\[
= \frac{1}{2} \| w_f \|^2 - \frac{1}{q+1} \int f(x) w_f^{q+1} \, dx - \frac{1}{p+1} \int h(x) w_f^{p+1} \, dx.
\]

By Lemma 2.4 (ii), if \( \int h(x) w_f^{p+1} \, dx > 0 \), then there exist \( t^+ = t^+(w_f) \in (0,t_{\max}) \) and \( t^- = t^-(w_f) \in (t_{\max},+\infty) \) with \( t_{\max} = t_{\max}(w_f) \) such that

\[
t^+ w_f \in N^+, \quad t^- w_f \in N^-,
\]

and

\[
J(t^+ w_f) = \min_{0 < t \leq t^-} J(t w_f).
\]

By Lemma 2.5, (21) and (22), it is easy to see that \( t^- \in (1,+\infty) \), and

\[
J(t^+ w_f) \leq J(w_f) < \gamma_f < 0.
\]

By Lemma 2.4 (ii), if \( \int h(x) w_f^{p+1} \, dx \leq 0 \), then there exist \( t^+ = t^+(w_f) \in (0,+\infty) \) and sufficiently small \( t_0 \in (0,1) \) such that

\[
t^+ w_f \in N^+
\]

and

\[
J(t^+ w_f) = \min_{t \geq 0} J(t w_f) \leq J(t_0 w_f) < -\frac{t_0^{q+1}}{2} \cdot \| w_f \|^2 < 0.
\]

Let

\[
\Gamma_f = \begin{cases} 
\gamma_f, & \text{if } \int h(x) w_f^{p+1} \, dx > 0, \\
-\frac{t_0^{q+1}}{2} \cdot \| w_f \|^2, & \text{if } \int h(x) w_f^{p+1} \, dx \leq 0.
\end{cases}
\]

Thus, by (21) and (23)-(26), we obtain \( \alpha \leq \alpha^+ < \Gamma_f < 0 \).
As a conclusion in this section, we give the similar results to [25, Lemmas 2.8-2.9] whose proofs can be processed based on [19] in a straightforward way, so we omit them.

Lemma 2.8. [25] For any \( u \in \mathbb{N}^\pm \), there exist \( \epsilon > 0 \) and a differentiable functional \( \xi : B(0; \epsilon) \subset H(\Omega) \rightarrow \mathbb{R}^+ \) such that \( \xi(0) = 1 \), the function \( \xi(0)(u + v) \in \mathbb{N}^\pm \) and

\[
\langle \xi'(0), v \rangle = \frac{2 \int_\Omega \Delta u \Delta v - (q + 1) \int_\Omega f|u|^{q-1}uv - (p + 1) \int_\Omega h|u|^{p-1}uv}{(1 - q)||u||^2 - (p - q) \int_\Omega h(x)|u|^{p+1} dx}
\]

for all \( v \in H(\Omega) \).

Lemma 2.9. [25] For any \( u \in \mathbb{N}^- \), there exist \( \epsilon > 0 \) and a differentiable functional \( \xi^- : B(0; \epsilon) \subset H(\Omega) \rightarrow \mathbb{R}^+ \) such that \( \xi^-(0)(u + v) \in \mathbb{N}^- \) and

\[
\langle (\xi^-)'(0), v \rangle = \frac{2 \int_\Omega \Delta u \Delta v - (q + 1) \int_\Omega f|u|^{q-1}uv - (p + 1) \int_\Omega h|u|^{p-1}uv}{(1 - q)||u||^2 - (p - q) \int_\Omega h(x)|u|^{p+1} dx}
\]

for all \( v \in H(\Omega) \).

3. Proof of Theorems 1.1-1.2. We start with the existence of the minimizing sequences for \( J \) on \( \mathbb{N} \) and \( \mathbb{N}^- \) as \( \|f^+\|_r \) is sufficiently small.

Proposition 3.1. (i) If \( f \in \{ f \mid \|f^+\|_r \in (0, A) \} \), then there exists a minimizing sequence \( \{u_n\} \subset \mathbb{N} \) such that

\[
J(u_n) = \alpha + o(1), \quad \text{and} \quad J'(u_n) = o(1) \text{ in } [H(\Omega)]^*;
\]

(ii) If \( f \in \{ f \mid \|f^+\|_r \in [0, A) \} \), then there exists a minimizing sequence \( \{u_n\} \subset \mathbb{N}^- \) such that

\[
J(u_n) = \alpha^- + o(1), \quad \text{and} \quad J'(u_n) = o(1) \text{ in } [H(\Omega)]^*.
\]

Proof. (i) By Lemma 2.6 and the Ekeland variational principle [10], there exists a minimizing sequence \( \{u_n\} \subset \mathbb{N} \) such that

\[
J(u_n) < \alpha + \frac{1}{n}
\]

and

\[
J(u_n) < J(w) + \frac{1}{n} \|w - u_n\| \text{ for each } w \in \mathbb{N}.
\]

By Lemma 2.6 and (27), \( \{\|u_n\|\} \) is bounded. Taking \( n \) large, from Lemma 2.7 and (27), we have

\[
J(u_n) = \left( \frac{1}{2} - \frac{1}{p + 1} \right) \|u_n\|^2 - \left( \frac{1}{q + 1} - \frac{1}{p + 1} \right) \int_\Omega f(x)|u_n|^{q+1} dx
\]

\[
< \alpha + \frac{1}{n} < \Gamma_f.
\]

This implies

\[
\|f^+\|_r S^{q+1} \|u_n\|^{q+1} \geq \int_\Omega f(x)|u_n|^{q+1} dx > \frac{(p + 1)(q + 1)\Gamma_f}{q - p} > 0.
\]

That is,

\[
\|u_n\| > \left[ \frac{(p + 1)(q + 1)\Gamma_f}{(q - p)\|f^+\|_r S^{q+1}} \right]^{\frac{1}{q+1}} =: C_1.
\]

To show that

\[
\langle J'(u_n), \varphi \rangle \rightarrow 0 \text{ as } n \rightarrow \infty, \quad \varphi \in H(\Omega),
\]
In view of Lemma 2.8, we use the suitable functionals \( \xi_n(v) > 0 \) to \( u_n \) and get
\[
\xi_n(v)(u_n + v) \in \mathbb{N}, \quad v \in H(\Omega), \quad \|v\| < \epsilon_n. \tag{32}
\]
Hence, if \( \varphi \in H(\Omega) \) and \( s > 0 \) small, we take \( v = s\varphi \). It follows from (32) and (28) that
\[
\frac{1}{n}[\|\xi_n(s\varphi) - 1\| \cdot \|u_n\| + \xi_n(s\varphi)\|s\varphi\|]
\geq J(u_n) - J(\xi_n(s\varphi)(u_n + s\varphi))
= \frac{1}{2}\|u_n\|^2 - \frac{1}{q + 1} \int_{\Omega} f(x)|u_n|^{q+1} \, dx - \frac{1}{p + 1} \int_{\Omega} h(x)|u_n|^{p+1} \, dx
- \frac{1}{2}\xi_n^2(s\varphi)\|u_n + s\varphi\|^2 + \frac{1}{q + 1}\xi_n^{q+1}(s\varphi) \int_{\Omega} f(x)|u_n + s\varphi|^{q+1} \, dx
+ \frac{1}{p + 1}\xi_n^{p+1}(s\varphi) \int_{\Omega} h(x)|u_n + s\varphi|^{p+1} \, dx
= -\frac{\xi_n^2(s\varphi) - 1}{2}\|u_n + s\varphi\|^2 - \frac{1}{2}(\|u_n + s\varphi\|^2 - \|u_n\|^2)
+ \frac{\xi_n^{q+1}(s\varphi) - 1}{q + 1} \int_{\Omega} f(x)|u_n + s\varphi|^{q+1} \, dx + \frac{1}{q + 1} \int_{\Omega} f(x)(|u_n + s\varphi|^{q+1} - |u_n|^{q+1}) \, dx
+ \frac{\xi_n^{p+1}(s\varphi) - 1}{p + 1} \int_{\Omega} h(x)|u_n + s\varphi|^{p+1} \, dx + \frac{1}{p + 1} \int_{\Omega} h(x)(|u_n + s\varphi|^{p+1} - |u_n|^{p+1}) \, dx.
\]
Divided by \( s > 0 \) and by passing to the limit as \( s \to 0 \), there holds
\[
\frac{1}{n}[\|\xi_n'(0)\varphi\||u_n|| + ||\varphi||]
\geq -\frac{[\xi_n'(0)\varphi][\|u_n\|^2 - \int_{\Omega} f(x)|u_n|^{q+1} \, dx - \int_{\Omega} h(x)|u_n|^{p+1} \, dx]
- \int_{\Omega} \Delta u_n \Delta \varphi \, dx + \int_{\Omega} f(x)|u_n|^{q-1} u_n \varphi \, dx + \int_{\Omega} h(x)|u_n|^{p-1} u_n \varphi \, dx
= -\int_{\Omega} \Delta u_n \Delta \varphi \, dx + \int_{\Omega} f(x)|u_n|^{q-1} u_n \varphi \, dx + \int_{\Omega} h(x)|u_n|^{p-1} u_n \varphi \, dx
= (J'(u_n), -\varphi).
\]
Since
\[
\xi_n'(0)\varphi = \frac{2 \int_{\Omega} \Delta u_n \Delta \varphi - (q + 1) \int_{\Omega} f|u_n|^{q-1} u_n \varphi - (p + 1) \int_{\Omega} h|u_n|^{p-1} u_n \varphi}{(1-q)\|u_n\|^2 - (p-q) \int_{\Omega} h(x)|u_n|^{p+1} \, dx},
\]
in view of the boundedness of \( u_n \) we get
\[
\|\xi_n'(0)\| \leq \frac{d_0}{(1-q)\|u_n\|^2 - (p-q) \int_{\Omega} h(x)|u_n|^{p+1} \, dx} \tag{34}
\]
for a suitable positive constant \( d_0 \).
Now, it suffices to show that
\[
\left| (1-q)\|u_n\|^2 - (p-q) \int_{\Omega} h(x)|u_n|^{p+1} \, dx \right| > d_1 \tag{35}
\]
for some \( d_1 > 0 \) and the large \( n \). Assume by contradiction that there exists a subsequence still denoted by \( \{u_n\} \) such that
\[
(1-q)\|u_n\|^2 - (p-q) \int_{\Omega} h(x)|u_n|^{p+1} \, dx = o(1),
\]
which implies
\[ \|u_n\|^2 = \frac{p - q}{1 - q} \int \Omega h(x)|u_n|^{p+1} \, dx + o(1). \]  
(36)

Combining (31) with (36), we can find a suitable constant \( d_2 > 0 \) such that
\[ \int \Omega h(x)|u_n|^{p+1} \, dx \geq d_2 \]  
for the sufficiently large \( n \).

(37)

In view of \( \{u_n\} \subset N \) and (36), we derive
\[ \int \Omega f(x)|u_n|^{q+1} \, dx = \|u_n\|^2 - \int \Omega h(x)|u_n|^{p+1} \, dx = \frac{p - 1}{p - q} \int \Omega h(x)|u_n|^{p+1} \, dx + o(1), \]
which implies
\[ I(u_n) := \frac{p - 1}{p - q} \left( \frac{1 - q}{p - q} \right)^{\frac{1}{p+1}} \left( \frac{\|u_n\|^{2p}}{h(x)|u_n|^{p+1} \, dx} \right)^{\frac{1}{p+1}} - \int \Omega f(x)|u_n|^{q+1} \, dx 
\]
\[ = \frac{p - 1}{p - q} \left( \frac{1 - q}{p - q} \right)^{\frac{1}{p+1}} \left( \frac{p - q}{1 - q} \right) \int \Omega h(x)|u_n|^{p+1} \, dx 
\]
\[ - \frac{p - 1}{1 - q} \int \Omega h(x)|u_n|^{p+1} \, dx + o(1) 
\]
\[ = o(1), \]  
(38)

and
\[ \|u_n\|^2 = \frac{p - q}{p - 1} \int \Omega f(x)|u_n|^{q+1} \, dx + o(1). \]

From Hölder’s and Sobolev’s inequalities, it follows that
\[ \|u_n\| \leq \frac{\|f^+\|^{\frac{1}{r}}}{r} \left[ \frac{(p - q)S^{q+1}}{p - 1} \right] \]  
(39)

Combining (31), (37) and (39) yields
\[ I(u_n) = \frac{p - 1}{p - q} \left( \frac{1 - q}{p - q} \right)^{\frac{1}{p+1}} \left( \frac{\|u_n\|^{2p}}{h(x)|u_n|^{p+1} \, dx} \right)^{\frac{1}{p+1}} - \int \Omega f(x)|u_n|^{q+1} \, dx 
\]
\[ \geq \frac{p - 1}{p - q} \left( \frac{1 - q}{p - q} \right)^{\frac{1}{p+1}} \left( \frac{1}{|h^+|_\infty S^{q+1}} \right) \|u_n\| - |f^+||rS^{q+1}||u_n||^{q+1} 
\]
\[ = \|u_n||^{q+1} \left[ \frac{p - 1}{p - q} \left( \frac{1 - q}{p - q} \right) \frac{1}{|h^+|_\infty S^{q+1}} \right] \frac{1}{p+1} \|u_n\|^{-q} - |f^+||rS^{q+1}||u_n||^{q+1} 
\]
\[ > C_1^{q+1} S^{q+1} |f^+|^\frac{2(q+1)}{r} \]  
\[ \left\{ \left( \frac{p - 1}{p - q} \right)^{\frac{1}{p+1}} \left[ \frac{1 - q}{(p - q)|h^+|_\infty} \right] ^{\frac{1}{p+1}} S^{\frac{2(p-q)}{r(2(p-q))}} - |f^+|^\frac{1}{r} + o(1) \right\} 
\]
\[ = C_1^{q+1} S^{q+1} |f^+|^\frac{2}{r} \left[ A^{\frac{1}{r}} - |f^+|^\frac{1}{r} + o(1) \right] > 0. \]

This yields a contradiction to (38). Hence, we have
\[ \left\langle J'(u_n), \frac{u}{\|u\|} \right\rangle \leq C \frac{u}{n}, \]
which implies \( J'(u_n) = o(1) \) in \( [H(\Omega)]^r \).
(ii) According to Lemma 2.6 and the Ekeland variational principle, there is a minimizing sequence \( \{ u_n \} \subset \mathcal{N}^- \) such that
\[
J(u_n) < \alpha + \frac{1}{n}
\] (40)
and
\[
J(u_n) \leq J(w) + \frac{1}{n} \| w - u_n \|, \quad w \in \mathcal{N}^-.
\] (41)
In view of \( \{ u_n \} \subset \mathcal{N}^- \), we have
\[
\| u_n \| > \left[ \frac{1 - q}{(p - q)\| h \|_{\infty} S^{p+1}} \right]^{\frac{1}{p+1}}.
\] (42)
Using Hölder’s, Sobolev's and Young’s inequalities, it follows from (40) that there is \( C_2 = C_2(p,q,S,\alpha) > 0 \) such that
\[
\| u_n \| < C_2.
\] (43)
From (41)-(43), processing an analogous argument as we did in the proof of Proposition 3.1 (i), we can obtain the desired result of \( J'(u_n) = o(1) \) in \( [H(\Omega)]^* \) by virtue of Lemma 2.9.

Now, we establish the existence of a local minimum for \( J \) on \( \mathcal{N}^+ \).

**Theorem 3.2.** If \( f \in \{ f \mid |f^+|_r \in (0, \Lambda) \} \), then the functional \( J \) has a minimizer \( u_0^+ \in \mathcal{N}^+ \) and it satisfies
(i) \( J(u_0^+) = \alpha = \alpha^+ < \Gamma_f < 0 \);
(ii) \( u_0^+ \) is a nontrivial solution of equation (1); and
(iii) \( J(u_0^+) \to 0 \) as \( |f^+|_r \to 0 \).

**Proof.** (i) By Proposition 3.1 (i), there is a minimizing sequence \( \{ u_n \} \) for \( J \) on \( \mathcal{N} \) such that
\[
J(u_n) = \alpha + o(1), \quad \text{and} \quad J'(u_n) = o(1) \text{ in } [H(\Omega)]^*.
\] (44)
Then by Lemma 2.6 and the compact imbedding theorem, there exist a subsequence \( \{ u_n \} \) and \( u_0^+ \in H(\Omega) \) such that
\[
u_n \rightharpoonup u_0^+ \text{ weakly in } H(\Omega), \quad u_n \to u_0^+ \text{ strongly in } L^{p+1}(\Omega).
\] (45)
and
\[
u_n \to u_0^+ \text{ strongly in } L^{q+1}(\Omega).
\] (47)
We make a claim that
\[
\int_{\Omega} f(x)|u_0^+|^{q+1} \, dx > 0.
\]
If not, by (47) there holds
\[
\lim_{n \to \infty} \int_{\Omega} f(x)|u_n|^{q+1} \, dx = \int_{\Omega} f(x)|u_0^+|^{q+1} \, dx \leq 0.
\] (48)
Combining \( \{ u_n \} \subset \mathcal{N} \) with (46) and (48) leads to
\[
\int_{\Omega} h(x)|u_0^+|^{p+1} \, dx = \lim_{n \to \infty} \int_{\Omega} h(x)|u_n|^{p+1} \, dx = \lim_{n \to \infty} \| u_n \|^2 - \int_{\Omega} f(x)|u_n|^{q+1} \, dx \geq 0.
\]
Then

\[
\lim_{n \to \infty} J(u_n) = \lim_{n \to \infty} \left[ \frac{1}{2} \|u_n\|^2 - \frac{1}{q+1} \int_{\Omega} f(x)|u_n|^{q+1} \, dx - \frac{1}{p+1} \int_{\Omega} h(x)|u_n|^{p+1} \, dx \right]
\]

\[
= \lim_{n \to \infty} \left[ \left( \frac{1}{2} - \frac{1}{q+1} \right) \int_{\Omega} f(x)|u_n|^q \, dx + \left( \frac{1}{2} - \frac{1}{p+1} \right) \int_{\Omega} h(x)|u_n|^{p+1} \, dx \right]
\]

\[
= \left( \frac{1}{2} - \frac{1}{q+1} \right) \int_\Omega f(x)|u_0^+|^q \, dx + \left( \frac{1}{2} - \frac{1}{p+1} \right) \int_\Omega h(x)|u_0^+|^{p+1} \, dx \geq 0.
\]

This contradicts \( \lim_{n \to \infty} J(u_n) = \alpha < 0 \).

In combination with (44)-(47), it is easy to verify that \( u_0^+ \in \mathbb{N} \) is a nontrivial weak solution of the problem (1).

Now we prove that \( u_n \to u_0^+ \) strongly in \( H(\Omega) \). Suppose the contrary. Then \( \|u_n^+\| < \liminf_{n \to \infty} \|u_n\| \) and

\[
\|u_n^+\|^2 - \int_\Omega f(x)|u_n^+|^{q+1} \, dx - \int_\Omega h(x)|u_n^+|^{p+1} \, dx < \liminf_{n \to \infty} \left( \|u_n\|^2 - \int_\Omega f(x)|u_n|^q \, dx - \int_\Omega h(x)|u_n|^{p+1} \, dx \right) = 0.
\]

This contradicts \( u_0^+ \in \mathbb{N} \). Hence, \( u_n \to u_0^+ \) strongly in \( H(\Omega) \). This implies

\[
J(u_n) \to J(u_0^+) = \alpha \quad \text{as} \quad n \to \infty.
\]

Moreover, we have \( u_0^+ \in \mathbb{N}^+ \). In fact, if \( u_0^- \in \mathbb{N}^- \), by Lemma 2.4 (ii), there exist the unique \( t_0^+ \) and \( t_0^- \) such that \( t_0^+ u_0^- \in \mathbb{N}^+ \), \( t_0^- u_0^+ \in \mathbb{N}^- \) \((0 < t_0^+ < t_0^- = 1)\) and \( \frac{\partial}{\partial t} J(t u_0^+) > 0 , \, t \in (t_0^+, t_0^-) \). So, there holds

\[
\alpha \leq J(t_0^+ u_0^+) < J(t_0^- u_0^-) = J(u_0^+) = \alpha.
\]

This is obviously a contradiction.

(ii) By Lemma 2.2 and Theorem 3.2 (i), \( u_0^+ \) is a nontrivial weak solution to the problem (1).

(iii) By virtue of Theorem 3.2 (i) and the proof of Lemma 2.6, we have

\[
0 > J(u_0^+) \geq - \left[ \frac{4(p+1)S^2}{(p-1)} \right]^{\frac{q+1}{q-1}} |f^+|^\frac{2}{q-1},
\]

which implies that \( J(u_0^+) \to 0 \) as \( |f^+|^r \to 0 \).

Next, we establish the existence of a local minimum for \( J \) on \( \mathbb{N}^- \).

**Theorem 3.3.** If \( f \in \{ f \mid |f^+|^r \in [0, A] \} \), then the functional \( J \) has a minimizer \( u_0^- \in \mathbb{N}^- \) and it satisfies

(i) \( J(u_0^-) = \alpha^- \); and

(ii) \( u_0^- \) is a nontrivial solution of the problem (1).

**Proof.** (i) By Proposition 3.1 (ii), there is a minimizing sequence \( \{ u_n \} \) for \( J \) on \( \mathbb{N}^- \) such that

\[
J(u_n) = \alpha^- + o(1), \quad \text{and} \quad J'(u_n) = o(1) \text{ in } [H(\Omega)]^*.
\]
Then by Lemma 2.6 and the compact imbedding theorem, there exist a subsequence \( \{u_n\} \) and \( u^-_0 \in H(\Omega) \) such that

\[
\begin{align*}
u_n &\rightarrow u^-_0 \quad \text{weakly in } H(\Omega), \\
u_n &\rightarrow u^-_0 \quad \text{strongly in } L^{p+1}(\Omega)
\end{align*}
\]

and

\[
u_n \rightarrow u^-_0 \quad \text{strongly in } L^{q+1}(\Omega).
\]

Due to the fact that \( \langle J'(u^-_0), \varphi \rangle = \lim_{n \rightarrow \infty} \langle J'(u_n), \varphi \rangle = 0 \) for all \( \varphi \in H(\Omega) \), we can see that \( u^-_0 \in N \) is a nontrivial weak solution of the problem (1).

To prove that \( u_n \rightarrow u^-_0 \) strongly in \( H(\Omega) \), we suppose the contrary that \( \|u^-_0\| < \liminf_{n \rightarrow \infty} \|u_n\| \) and then

\[
\begin{align*}
\|u^-_0\|^2 - \int_{\Omega} f(x)|u^-_0|^{q+1} \, dx - \int_{\Omega} h(x)|u^-_0|^{p+1} \, dx 
&< \liminf_{n \rightarrow \infty} \left( \|u_n\|^2 - \int_{\Omega} f(x)|u_n|^{q+1} \, dx - \int_{\Omega} h(x)|u_n|^{p+1} \, dx \right) = 0.
\end{align*}
\]

This contradicts the fact \( u^-_0 \in N \). Hence, \( u_n \rightarrow u^-_0 \) strongly in \( H(\Omega) \). This implies \( J(u_n) \rightarrow J(u^-_0) = \alpha^- \) as \( n \rightarrow \infty \).

In view of \( N^0 = \emptyset \) (see Lemma 2.1) and \( \{u_n\} \subset N \), we obtain \( u^-_0 \in N^- \).

(ii) It follows from Lemma 2.2 and Theorem 3.3 (i) that \( u^-_0 \) is a nontrivial weak solution to the problem (1). \( \square \)

**Proof of Theorem 1.1.** By virtue of Theorem 3.3, the problem (1) has at least one nontrivial solution \( u^-_0 \in N^- \). \( \square \)

**Proof of Theorem 1.2.** By virtue of Theorems 3.2 and 3.3, for any \( f \in \{f| |f^+|_r \in (0, A)\} \) the problem (1) has at least two nontrivial solutions \( u^+_0 \in N^+ \) and \( u^-_0 \in N^- \). Since \( N^+ \cap N^- = \emptyset \), this indicates that \( u^+_0 \) and \( u^-_0 \) are distinct. \( \square \)

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Received January 2020; revised September 2020.

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