Network cluster detecting in associated bi-graph view

Zhe He, Yi-Ming Huang, Rui-Jie Xu, Bing-Hong Wang and Zhong-Can Ou-Yang

We find there is relationship between the associated bigraph and the cluster (or community) detecting on network. By imbedding the associated bigraph of some network (suppose it has cluster structures) into some space, we can identify the clusters on this network, which is a new method for network cluster detecting. And this method, of which the physical meaning is clear and the time complexity is acceptable, may provide us a new point to understand the structure and character of networks. In this paper, We test the methods on several computer-generated networks and real networks. A computer-generated network with 128 vertices and the Zachary Network, which presents the structure of a karate club, can be partitioned correctly by these methods. And the Dolphin network, which presents the relationship between 62 dolphins on the coast of New Zealand, is partitioned reasonably.

Keywords: cluster, community, networks, associated bigraph

INTRODUCTION

As an essential problem in network science, network cluster detecting is significant for computer science, biology, communication and social networks, and marketing strategy. And it gains lots of attention from researchers in related fields. Especially in recent years, understanding networks deeper and deeper, people get rich harvests in the study of network cluster detecting. Many detecting algorithms and evaluation criteria are proposed. Some of the algorithms are based on some operating processes on network structures, and some are based on spectrum analysis algorithms and Benchmark graphs. From the angle of category, there are overlapping clustering and non-overlapping clustering.

The new network cluster detecting method we put forward is based on a measure on associated bigraph (AG). In this paper, discrete equidistant imbedding (DEI) and continuous imbedding (CI) separately provide two different measures and two different methods.

AG AND THE TWO METHODS

In this section, we give a definition to Associated Bi-graph (AG) and from it we propose two methods, DEI method and CI method. And to test the methods, we give partitions to a computer-generated network, Zachary network and Dolphin network. At last, we compare the results by our methods with those by modularity method in Gephi.

Definition of AG and DEI

Suppose graph $G = (V, E)$, $V$ refers to its vertex set, $E$ refers to its edge set, $|V| = N$. Then, the associated bigraph of $G$ is $G_A = (V_1 \cup V_2, E_A)$. If $V = \{v_1, v_2, \ldots, v_N\}$, then $V_1 = \{v_{11}, v_{12}, \ldots, v_{1N}\}, V_2 = \{v_{21}, v_{22}, \ldots, v_{2N}\}$, for any $i$, $v_i$ corresponds to $v_{i1}$ and $v_{i2}$. If and only if $(v_i, v_j) \in E$, $(v_{i1}, v_{j2}) \in E_A$. It is easy to know, $G_A$ is a bigraph. $V_1$ and $V_2$ are two parts of it, $|E_A| = |E|$ (suppose undirected edges be bidirectional edges). If we merge the corresponding vertexes in $V_1$ and $V_2$, $G_A$ is equal to $G$. See FIG.1. We place the vertexes of AG as in FIG.1. With equal interval, place the vertexes from sets $V_1$, $V_2$ on two parallel lines ($L_1$ & $L_2$) and let those with corresponding labels be at corresponding positions. We call the placing pattern described above discrete equidistant imbedding (DEI). Of a given graph, the AG has $N!$ kinds of different DEIs. (If the graph has some symmetry, the number will decrease.) Without loss of generality, we let the allowed coordinate of vertexes in DEI successively be $1, 2, 3 \ldots N$. Thus, the distance between adjacent vertexes is 1.

![Graph and its AG are equidistantly imbedded on the two lines.](image_url)
DEI method

Now, we consider the simple graphs (undirected, non-weighted, acyclic, non-multiple edges). If there are clusters structures in such graphs, among different DEIs of an AG, at least there is one that the vertexes are arrayed in the sequence of cluster. That is to say, vertexes of a same cluster will be placed together. In detail, different clusters will be placed nearer if they have closer relations. In the interior of a cluster, vertexes with closer relations are placed nearer. This arrangement is called optimal DEI.

We define the distance in DEI as follow: the distance between \( v_i \) and \( v_j \) is \(|x_i - x_j|\), where \( x_i \) and \( x_j \) are the coordinates of \( v_i \) and \( v_j \). If edge \((v_{i1}, v_{j2})\) exists, we define the length of \((v_{i1}, v_{j2})\) as \(|x_i - x_j|\). Let

\[
Z = \sum_{ij} a_{ij} |x_i - x_j|, \tag{1}
\]

We treat \( Z \) as an objective function, and minimize it under the condition of DEI, the solution of which is the optimal DEI.

If an edge \( a \) connects \( v_{i1} \) and \( v_{j2} \) (suppose \( i < j \)), we can find it in the interval of \( k \) and \( k + 1 \) (FIG.1), where \( i \preceq k < k + 1 \preceq j \). The number of edges found in the interval of \( k \) and \( k + 1 \) is defined as the cross of \( k \). Let the number of crosses be \( \{m_1, m_2, \cdots, m_{N-1}\} \), it is easy to see that \( Z = \sum_i m_i \), which means the optimal DEI corresponds to ‘the minimum sum sum of crosses’.

This optimization equals to the follow operation in the adjacency matrix \( A \): \(|x_i - x_j|\) suggests the absolute difference of the element \( a_{ij} \)’s column number and row number, which can measure the distance of the element and main diagonal. In order to minimize \( Z \), by swapping vertexes we move the non-zero elements to the main diagonal as near as possible.

Actually, this definition of \( Z \) may give ‘greater rights’ to vertexes with larger degree. In order to minimize \( Z \), some vertexes with large degree may draw connected vertexes close to themselves. This may drown the structures of other vertexes. Or say, edges from some vertexes is so large a proportion of total edges that these vertexes affect the arrangement too strong and the effect of other vertexes are unimportant. In order to avoid this, we have to correct \( Z \). A natural correction is to average the weight of each vertex.

\[
Z = \sum_{ij} \frac{a_{ij}|x_i - x_j|}{k_i}. \tag{2}
\]

Since it is a undirected graph, Eq.2 is equivalent to

\[
Z = \frac{1}{2} \sum_{ij} a_{ij} \left( \frac{1}{k_i} + \frac{1}{k_j} \right) |x_i - x_j|. \tag{3}
\]

Although we correct \( Z \) by \( 1/k \), people may have different opinions on whether this correction is reasonable or not. Some people may believe that vertex with large degree should have a greater effect. While, our suggestion is: the definition of clustering do not need to be unique. Different definitions should be allowed in different case. It is more important that a good definition should match the practical problem. However, the simulation results suggest that \( Z \) corrected by \( 1/k \) has a higher resolution power. (FIG.3)

![FIG. 2: are graphs of R matrix of computer-generating network without correction and with 1/k correction.](image)

The network is generated as follow: First generate four ER networks of 32 vertexes with \( p_1 = 0.6 \). Then randomly construct edges among different networks with \( p_2 = 0.2 \). In this picture, vertexes are arranged in the order of four networks. We can see that our methods can uncover the four clusters correctly.

We use Simulated Annealing Algorithm to find the solution of minimum \( Z \).

A consequent question is even though we have found the optimal DEI, how can we know which vertexes belong to the same cluster? A intuitive idea is to count the crosses between adjacent vertexes. The sum of crosses between adjacent vertexes belonging to the same cluster will be larger than that between adjacent vertexes belonging to different clusters. (considering that a large degree vertex may drown the information of other vertexes, we can give each edge a weight \( 1/k \)). However, after the simulation, we find that this method is not so effective and cannot show the ‘bond’ of real cluster well. Thus, we adopt the following movement correlation method.

Suppose we move some vertex pair \((v_{i1}, v_{j2})\) on \( L_1 \) and \( L_2 \), some other vertexes have to move in order to keep \( Z \) as small as possible. If a vertex always follows another, it is that there is movement correlation between them. We use the strength of the movement correlation measured by Pearson correlation coefficient matrix \( R \) to partition clusters. Vertexes in the same cluster are with strong correlation. where

\[
R_{ij} = \frac{\text{Cov}(x_i, x_j)}{\sqrt{\text{Cov}(x_i, x_i) \text{Cov}(x_j, x_j)}}.
\]

In the simulation, we randomly fix a small part of vertexes (e.g. 5%) and minimize \( Z \). Repeat this operation...
several times, we can get many optimal DEIs in this case and calculate the coordinate correlation of different vertices. Then we get matrix $R$.

It is worth mentioning that overlap is allowed in this method (FIG 3).

Although we get $R$, we do not have a clear-cut criterion for clustering. For example, in FIG 3 if we set different resolutions, we can get different results of clustering. Maybe two clusters, maybe three, maybe four. What partition is reasonable is worth of discussion. Further more, we have done an elementary analysis. We can consider this partition criterion from two aspects. 1. amplitude criterion, 2. step criterion. The first criterion means that we can set a value and when a element of matrix $R$ is less than this value, we set it to zero. At last a non-zero diagonal block is a cluster(allow overlapping). The second criterion means that we can consider the step (difference) of adjacent elements. We can confirm the ‘bond’ of a cluster by finding a position with big step. We can achieve this by high-pass filtering. Criterion 2 is affected seriously by the sequence of vertexes in figure of matrix $R$. In fact, ‘how many the clusters is there’ is just a question has more than one answer, for when the criterion is ‘loose’, there may be two clusters, and when the criterion is ‘strict’, there may be four clusters.
Actually, $Z$ in Eq. (1) and Eq. (2) is regarded as $L^1$ norm. We can generally define the measure based on $L^p$ norm. Here we write two possible definitions of $L^2$ norm. Uncorrected:

$$Z = \sum_{ij} a_{ij} (x_i - x_j)^2.$$ 

Corrected:

$$Z = \sum_{ij} \frac{a_{ij} (x_i - x_j)^2}{k_i} \text{ or } Z = \sum_{ij} \frac{a_{ij} (x_i - x_j)^2}{k_i^2}.$$ 

On the previous two real networks, similar partitions can be made by $L^2$ norm and by $L^1$ norm. But the resolution of $L^2$ norm method is lower than that of $L^1$ norm.

**CI method**

We can change DEI to continuous imbedding (CI). CI means that a vertex can be placed at any point on the line and the vertex pair with same label still should has the same coordinate. Comparing CI and DEI, the objective function $Z$ does not change, but the feasible region changes from all arrangements of $\{1, 2, \ldots, N\}$ to $R^N$. Corresponding to the discrete imbedding, here we set the constraints of continuous imbedding:

For $L^1$ norm: $1. \sum x_i = 0, 2. \sum |x_i| = 1$.

For $L^2$ norm: $1. \sum x_i = 0, 2. \sum x_i^2 = 1$.

There is a special relation between $L^2$ norm CI method and spectral method [12, 13]. Next, we only discuss $L^1$ case with $1/k$ correction.

For $L^1$ case with $1/k$ correction, in simulations, the vertexes with minimum $Z$ are always scattered in two groups (FIG 5). Thus, we can put forward a CI method measured by $L^1$ norm with $1/k$ correction. The algorithm for a given network $G$ is as follow:

**Step A:** minimize $Z$ of $G$ and get the optimal solution $X$, of which the positive components the induced subgraph affiliating with is called $G_1$, and the induced subgraph of remaining vertexes is called $G_2$.

**Step B:** respectively redo Step A on $G_1$ and $G_2$ until each vertex is a induced subgraph.

This process generates a binary tree called cluster tree.

How can we take advantage of the cluster tree to uncover the clusters?

Criteria are needed. Here, we adopt $Q$ function. In detail, we partition the leaves of the cluster tree with maximized $Q$ in all possible ways and each part is a cluster. For our method is based on numerical optimization, among solutions of each computing, usually, there is a little difference always observed in the overlapping vertexes or between clusters with close relation.

**DISCUSSION**

Theoretically, we can expand the method to directed graph and weighted graph, for ‘undirected’ or ‘unweighted’ is not the necessary condition. We just need to replace adjacent matrix with weighted adjacent matrix for weighted graph.

Based on DEI and movement correlation, we can put forward a different method as follow:

1. Derive optimal DEI

2. Extend the feasible region of objective function $Z$ to $R^N$. The solution vector is a list of coordinates of all vertexes. Randomly select a small part of vertexes and with certain probability, add a small displacement to the optimal DEI coordinates of these vertexes.
3. Repeat step 2. Each time we can get a $N$ dimensional column vector, and they together form a matrix. Calculate all correlation coefficients between any two rows of the matrix, then we can get the correlation matrix $R$.

4. Rearrange the vertexes which corresponds to the elements of $R$ in the order of optimal DEI.

This method will lesson the time complexity greatly comparing with the method mentioned above. The validity should be tested in future work.

In this paper, we imbed AG in 1-D ‘line’ space. While we can imbed that in high dimensional spaces or some spaces with very different topological structures (e.g. 1-D ‘ring’ space [24]). What is optimum structure? This is a question. Considering the high dimensional ‘line’ space, maybe there is some $M$, in any ‘line’ space whose dimension is higher than $M$, the configurations of the optimal imbedding are the same, which is called faithful imbedding. $M_0$, the infimum of $M$, is able to reflect the complexity of clustering structure (E.g., we can define the quantity $M_0/(N - 1)$). Otherwise, we can coarse grain the configurations of faithful imbedding, each grain is a cluster and coarse grained topologies shows the network’s skeleton. There are a lot of related questions worth considering and studying.

---

**FIG. 6:** (a) is the cluster tree of Zachary Network by $L^1$ norm CI method with $1/k$ correction. The partition by the dashed has the maximum $Q$. □ and ○ refers to vertexes of the two divided clubs in history. Different colors marks different clusters of Zachary Network. (b) shows Dolphin Network clustering by $L^1$ norm CI method with $1/k$ correction. Different colors marks different clusters. We use modularity tool of Gephi to detect clusters on Dolphin Network. We set mode random and resolution 1.
[7] C. Homburg, O. Jensen, and H. Krohmer, Journal of Marketing 72, 133 (2008).
[8] S. Fortunato, Physics Reports 486, 75 (2010).
[9] F. Radicchi, C. Castellano, F. Cecconi, V. Loreto, and D. Parisi, Proceedings of the National Academy of Sciences of the United States of America 101, 2658 (2004).
[10] A. Clauset, M. E. Newman, and C. Moore, Physical Review E 70, 066111 (2004).
[11] A. Clauset, M. E. Newman, and C. Moore, Physical Review E 70, 066111 (2004).
[12] A. Capocci, V. D. Servedio, G. Caldarelli, and F. Colaiori, Physica A: Statistical Mechanics and its Applications 352, 669 (2005).
[13] L. Donetti and M. A. Munoz, Journal of Statistical Mechanics: Theory and Experiment 2004, P10012 (2004).
[14] H. Zhou, Physical Review E 67, 061901 (2003).
[15] M. E. Newman and M. Girvan, Physical Review E 69, 026113 (2004).
[16] Y.-Y. Ahn, J. P. Bagrow, and S. Lehmann, Nature 466, 761 (2010).
[17] A. Lancichinetti, S. Fortunato, and F. Radicchi, Physical Review E 78, 046110 (2008).
[18] G. Palla, I. Derényi, I. Farkas, and T. Vicsek, Nature 435, 814 (2005).
[19] W. W. Zachary, Journal of Anthropological Research, 452 (1977).
[20] D. Lusseau, Evolutionary Ecology 21, 357 (2007).
[21] V. D. Blondel, J.-L. Guillaume, R. Lambiotte, and E. Lefebvre, Journal of Statistical Mechanics: Theory and Experiment 2008, P10008 (2008).
[22] B. Bollobás, Random graphs, Graduate Texts in Mathematics, Vol. 184 (Springer, 1998).
[23] S. Fortunato and M. Barthelemy, Proceedings of the National Academy of Sciences 104, 36 (2007).
[24] D. Holten, Visualization and Computer Graphics, IEEE Transactions on 12, 741 (2006).