Decay properties of the Hardy-Littlewood-Sobolev systems of the Lane-Emden type

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Abstract
In this paper, we study the asymptotic behavior of positive solutions of the nonlinear differential systems of Lane-Emden type \(2k\)-order equations
\[
\begin{aligned}
(-\Delta)^k u &= v^q, \ u > 0 \quad \text{in } \mathbb{R}^n, \\
(-\Delta)^k v &= u^p, \ v > 0 \quad \text{in } \mathbb{R}^n,
\end{aligned}
\]
and the Hardy-Littlewood-Sobolev (HLS) type system of nonlinear equations
\[
\begin{aligned}
u(x) &= \frac{\int_{\mathbb{R}^n} v^q(y)dy}{|x-y|^{n-\alpha}}, \ u > 0 \quad \text{in } \mathbb{R}^n, \\
v(x) &= \frac{\int_{\mathbb{R}^n} u^p(y)dy}{|x-y|^{n-\alpha}}, \ v > 0 \quad \text{in } \mathbb{R}^n.
\end{aligned}
\]
Such an integral system is related to the study the extremal functions of the HLS inequality. We point out that the bounded solutions \(u, v\) converge to zero either with the fast decay rates or with the slow decay rates when \(|x| \to \infty\) under some assumptions. In addition, we also find a criterion to distinguish the fast and the slow decay rates: if \(u, v\) are the integrable solutions (i.e. \((u, v) \in L^{r_0}(\mathbb{R}^n) \times L^{s_0}(\mathbb{R}^n))\), then they decay fast; if the bounded solutions \(u, v\) are not the integrable solutions (i.e. \((u, v) \notin L^{r_0}(\mathbb{R}^n) \times L^{s_0}(\mathbb{R}^n))\), then they decay almost slowly. Here, for the HLS type system, \(r_0 = \frac{n(pq-1)}{\alpha(q+1)}, s_0 = \frac{n(pq-1)}{\alpha(p+1)},\) and for the Lane-Emden type system, \(r_0, s_0\) are still the forms above where \(\alpha\) is replaced by \(2k\).

Keywords: Lane-Emden equations, Hardy-Littlewood-Sobolev type integral equations, decay rates, finite energy solution, bounded decaying solution

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1 Introduction
Let \(n \geq 3\) and \(p > 1\). In this paper, we are concerned with the asymptotic behavior of positive solutions of the Hardy-Littlewood-Sobolev (HLS) type system
of nonlinear equations
\[
\begin{aligned}
  u(x) &= \int_{\mathbb{R}^n} \frac{v(y)dy}{|x-y|^{n-\alpha}}, \quad u > 0 \quad \text{in } \mathbb{R}^n, \\
v(x) &= \int_{\mathbb{R}^n} \frac{u(y)dy}{|x-y|^{n-\alpha}}, \quad v > 0 \quad \text{in } \mathbb{R}^n.
\end{aligned}
\] (1.1)

Under the assumption of the non-subcritical condition
\[
\frac{1}{p+1} + \frac{1}{q+1} \leq \frac{n-\alpha}{n},
\] (1.2)
we obtain that if \( u, v \) are the integrable solutions, then they converge to 0 with the fast decay rates when \( |x| \to \infty \). Moreover, we also point out that the equivalence relation of the integrable solutions, the finite energy solutions, and the bounded solutions with fast decay rates. On the other hand, we prove that the bounded solutions decay almost slowly if the solutions are not integrable solutions. Those decay rates are helpful to understand the existence of positive solutions: in the supercritical case, the energy of positive solutions is infinite and hence the variational methods cannot use to investigate the existence. We can search for positive solutions in the functions class whose elements decay with the slow rates.

Recall the asymptotic behavior of the positive solutions of the Lane-Emden equation
\[
-\Delta u = u^p, \quad u > 0 \quad \text{in } \mathbb{R}^n.
\] (1.3)

(R1): When \( |x| \to \infty \), then \( u(x) \) converges to zero either fast by \( u(x) \simeq |x|^{2-n} \) or slowly by \( u(x) \simeq |x|^{-\frac{2-n}{k}} \) (cf. [20]).

Here \( f(x) \simeq g(x) \) means there exists \( C > 0 \) such that \( \frac{g(x)}{C} \leq f(x) \leq Cg(x) \) when \( |x| \to \infty \). Similar results are also found in [14] and [25].

We expect to generalize this result (R1) to the positive solutions of the higher-order Lane-Emden type systems
\[
\begin{aligned}
  (-\Delta)^ku &= v^q, \quad u > 0 \quad \text{in } \mathbb{R}^n, \\
(-\Delta)^kv &= u^p, \quad v > 0 \quad \text{in } \mathbb{R}^n.
\end{aligned}
\] (1.4)
Here \( k \in [1, n/2) \) is an integer, \( p, q > 0 \) and \( pq > 1 \). The classification of the solutions of (1.4) has provided an important ingredient in the study of the prescribing scalar curvature problem. The positive solutions of (1.4) and its corresponding single equation were studied rather extensively (cf. [1], [3], [4], [11], [18], [23], [30] and the references therein).

The decay rates of the positive solutions play an important role in study the properties of the Lane-Emden type PDEs (cf. [10], [15] and [31]). Recently, Chen and Li [7] proved the equivalence between (1.4) and the system involving the Riesz potentials
\[
\begin{aligned}
u(x) &= \int_{\mathbb{R}^n} \frac{v^q(y)dy}{|x-y|^{n-2k}}, \quad u > 0 \quad \text{in } \mathbb{R}^n, \\
v(x) &= \int_{\mathbb{R}^n} \frac{u^p(y)dy}{|x-y|^{n-2k}}, \quad v > 0 \quad \text{in } \mathbb{R}^n.
\end{aligned}
\]
Thus, we investigate the more general Hardy-Littlewood-Sobolev (HLS) type integral system \((1.1)\) where \(\alpha \in (0, n)\) and \(p, q > 0, \ pq > 1\). The positive solutions \(u, v\) of \((1.1)\) are called the integrable solutions if \((u, v) \in L^r(R^n) \times L^s(R^n)\). Here

\[
\begin{align*}
 r_0 &= \frac{n(pq - 1)}{\alpha(q + 1)}, \\
 s_0 &= \frac{n(pq - 1)}{\alpha(p + 1)}.
\end{align*}
\]

Moreover, if the critical condition

\[
\frac{1}{p + 1} + \frac{1}{q + 1} = \frac{n - \alpha}{n} \tag{1.5}
\]

holds, then \(r_0 = p + 1\) and \(s_0 = q + 1\). The positive solutions \((u, v) \in L^{p+1}(R^n) \times L^{q+1}(R^n)\) are called the finite energy solutions.

The system \((1.1)\) is related to the Euler-Lagrange system of the extremal functions of the HLS inequality (cf. [9], [21], [22]). For the finite energy solutions, Chen, Li and Ou [8] proved the radial symmetry. Jin and Li [13] obtained the optimal integrability intervals. Hang [12] proved the smoothness. The fast decay rates was obtained in [17]. For the integrable solutions, Chen and Li [6] proved the radial symmetry. In this paper, we will establish the integrability and the estimate the decay rates.

Recall some existence results. An important conjecture is that the HLS type systems \((1.1)\) has no any positive solution under the subcritical condition:

\[
\frac{1}{p + 1} + \frac{1}{q + 1} > \frac{n - \alpha}{n} . \tag{1.6}
\]

When \(\alpha = 2\), it is the well known Lane-Emden conjecture. It is still open except for \(n \leq 4\) (cf. [26], [27], [28]). Chen and Li [6] proved the nonexistence of the integrable solutions of \((1.1)\). The nonexistence of the radial solution can be seen in [2] and [24].

In the critical case, we have the following result.

**Proposition 1.1.** (Theorem 1.2 in [15]) The system \((1.1)\) has the finite energy solutions if and only if the critical condition \((1.5)\) holds.

So the existence was proved by Lieb who pointed out that the extremal functions of the HLS inequality solve \((1.1)\) (cf. [22]).

Proposition 1.1 implies that the energy of the solutions is infinite in the supercritical case. Therefore, it seems difficult to prove the existence of positive solutions by the variational methods.

For the scalar equation of \((1.4)\) with \(k = 1\)

\[-\Delta u = u^p,\]

paper [10] shows the existence of positive solutions with slow decay rate in the supercritical case. Recently, Li [19] proved the existence of positive solutions of \((1.4)\) under the supercritical condition

\[
\frac{1}{p + 1} + \frac{1}{q + 1} < \frac{n - 2k}{n}
\]

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by means of the shooting method. Therefore, we always assume in this paper the non-subcritical condition (1.2) holds.

Next, we list the decay results stated by four theorems.

**Theorem 1.1.** Let $p \leq q$, and $u, v$ be positive solutions of (1.1). Then there exists $c > 0$ such that as $|x| \to \infty$,

$$u(x) \geq \frac{c}{|x|^{n-\alpha}}; \quad v(x) \geq \frac{c}{|x|^{\min\{n-\alpha, pn-(p+1)\alpha\}}}.$$  

Moreover, if $u, v$ are bounded decaying solutions, and there exists some $\epsilon_0 > 0$ such that for $|y| \leq |x|$, $u(y) \geq \epsilon_0 u(x)$ or $v(y) \geq \epsilon_0 v(x)$, then there exists $C > 0$ such that as $|x| \to \infty$,

$$u(x) \leq C|x|^{-\frac{a(q+1)}{pq-1}}; \quad v(x) \leq C|x|^{-\frac{a(p+1)}{pq-1}}.$$  

**Remark 1.1.** When $\alpha > 2$, $u, v$ are monotonicity decreasing and hence satisfy the condition in Theorem 1.1 (2), as long as $u, v$ are radially symmetric or bounded. On the contrary, if the radial solutions $u, v$ are not bounded, then (1.1) has the singular solutions $(u, v) = (C|x|^{-\frac{a(q+1)}{pq-1}}, C|x|^{-\frac{a(p+1)}{pq-1}})$ with some $C > 0$.

Let $p \leq q$. According to Theorem 1.5 (2) in [15], we know that $pq > 1$ and $\frac{a(q+1)}{pq-1} < n - \alpha$. This implies $\frac{a(q+1)}{pq-1} < \min\{n - \alpha, pn - (p+1)\alpha\}$. When $|x| \to \infty$, the exponents $n - \alpha$ and $\min\{n - \alpha, pn - (p+1)\alpha\}$ of $|x|^{-1}$ are called the fast decay rates of $u$ and $v$ respectively. The exponents $\frac{a(q+1)}{pq-1}$ and $\frac{a(p+1)}{pq-1}$ are called the slow ones of $u$ and $v$.

Theorem 1.1 shows that the decay rates of $u, v$ cannot be larger than the fast rates. Moreover, if $u$ or $v$ has some monotonicity, then their decay exponents must be between the fast and the slow rates.

The following result shows that if $u, v$ are the integrable solutions, then they decay fast.

**Theorem 1.2.** Let $p \leq q$, and $(u, v)$ be a pair of positive solutions of (1.1) with the non-subcritical condition (1.2). The following three items are equivalent:

1. $(u, v) \in L^{r_0}(\mathbb{R}^n) \times L^{q_0}(\mathbb{R}^n)$, i.e. $u, v$ are the integrable solutions.
2. $u, v$ are bounded, and decay fast when $|x| \to \infty$:

   $$u(x) \simeq |x|^n;$$
   $$v(x) \simeq |x|^{n-n} \quad \text{if } p(n-\alpha) > n;$$
   $$v(x) \simeq |x|^{n-n} \ln |x| \quad \text{if } p(n-\alpha) = n;$$
   $$v(x) \simeq |x|^{n-(n-p+1)n} \quad \text{if } p(n-\alpha) < n.$$  

3. $(u, v) \in L^{p+1}(\mathbb{R}^n) \times L^{q+1}(\mathbb{R}^n)$, i.e. $u, v$ are the finite energy solutions.
Remark 1.2. It should be pointed out that the condition \( p \leq q \) in Theorems 1.1 and 1.2 is not essential. If \( q \leq p \), then the conclusions also hold as long as the positions of \( u \) and \( v \) are exchanged.

According to Proposition 1.1 if the supercritical condition

\[
\frac{1}{p+1} + \frac{1}{q+1} < \frac{n-\alpha}{n} \tag{1.7}
\]

holds, then the positive solutions are not finite energy solutions. Theorem 1.2 shows that these solutions are not integrable solutions and do not decay with the fast rates. The following result shows that these solutions decay ‘almost slowly’.

**Theorem 1.3.** Let \( u, v \) be positive bounded solutions of (1.1). Then

1. there does not exist \( C > 0 \) such that either
   \[
   \begin{cases}
   u(x) \geq C(1 + |x|)^{-\theta_1}, & \text{or} \\
   v(x) \geq C(1 + |x|)^{-\theta_2},
   \end{cases}
   \]
   where \( \theta_1 < \frac{\alpha(q+1)}{pq-1} \), \( \theta_2 < \frac{\alpha(p+1)}{pq-1} \).

2. Moreover, if \( (u, v) \notin L^{p_0}(R^n) \times L^{q_0}(R^n) \) (i.e., they are not integrable solutions, particularly in the supercritical case), then \( u, v \) decay with rates not larger than the slow rates. Namely, there does not exist \( C > 0 \) such that either
   \[
   \begin{cases}
   u(x) \leq C(1 + |x|)^{-\theta_3}, & \text{or} \\
   v(x) \leq C(1 + |x|)^{-\theta_4},
   \end{cases}
   \]
   where \( \theta_3 > \frac{\alpha(q+1)}{pq-1} \), \( \theta_4 > \frac{\alpha(p+1)}{pq-1} \).

**Remark 1.3.**

1. The reason why we consider the bounded solutions is there exists singular solutions \( (u, v) = (C|x|^\frac{-\alpha(q+1)}{pq-1}, C|x|^\frac{-\alpha(p+1)}{pq-1}) \) with some \( C > 0 \).

2. According to Theorems 1.1, 1.3, we see that the solutions obtained by the shooting method in [19] must decay with the slow rates.

So far, we only obtain the ‘almost slow’ decay result as Theorem 1.3. It is still open whether the exactly slow decay result holds. If \( u, v \) are monotony like the condition in Theorem 1.1 (2), then they decay slowly. In addition, assume the solutions are polynomially decaying

\[
\begin{align*}
  u(x) &\approx (1 + |x|)^{-\theta_1}, \\
v(x) &\approx (1 + |x|)^{-\theta_2},
\end{align*}
\tag{1.8}
\]

then the following theorem shows that \( u, v \) must decay slowly as long as the supercritical condition (1.7) holds.
Theorem 1.4. Let $u, v$ be bounded positive solutions of (1.1). If there exist $\theta_1, \theta_2 > 0$ such that $u, v$ satisfy (1.8) as $|x| \to \infty$, then (1.2) must hold, and

$$
\theta_1 \geq \frac{\alpha(q+1)}{pq-1}, \quad \theta_2 \geq \frac{\alpha(p+1)}{pq-1}.
$$

Furthermore,

1. if one strict inequality holds, then (1.5) must be true, and $u, v$ are the finite energy solutions decaying fast like Theorem 1.2.

2. If the supercritical condition (1.7) holds, then the decay rates must be the slow ones:

$$
\theta_1 = \frac{\alpha(q+1)}{pq-1}, \quad \theta_2 = \frac{\alpha(p+1)}{pq-1}.
$$

Remark 1.4.

1. The fast decay rates of the finite energy solutions were obtained in [17] and [29], which is coincident with Theorem 1.2.

2. By virtue of $pq > 1$, (1.4) is equivalent to (1.1) with $\alpha = 2k$ (cf. [7]). Therefore, Theorems 1.1-1.4 with $\alpha = 2k$ are still true for (1.4).

3. If $p = q$ and $u \equiv v$, the system (1.1) is reduced to the single equation. Therefore, Theorems 1.1-1.4 with $p = q$ and $u \equiv v$ are still true. In particular, when $\alpha = 2k$, Theorems 1.1-1.4 still hold for the single $2k$-order PDE, which is coincident with (R1).

2 Integrable solution and finite energy solution

Theorem 2.1. Let $(u, v) \in L^r(\mathbb{R}^n) \times L^s(\mathbb{R}^n)$ be a pair of positive solutions of (1.1). If $p \leq q$, then $(u, v) \in L^r(\mathbb{R}^n) \times L^s(\mathbb{R}^n)$ for all

$$
\frac{1}{r} \in (0, \frac{n-\alpha}{n}), \quad \frac{1}{s} \in (0, \min\{\frac{n-\alpha}{n}, \frac{pn-\alpha(p+1)}{n}\}).
$$

Proof. Let $\frac{1}{r} \in (\frac{\alpha(q-p)}{n(pq-1)}, \frac{n-\alpha}{n})$ and $\frac{1}{s} \in (0, 1 - \frac{\alpha(q-p)(p+1)}{n(pq-1)})$ satisfy

$$
\frac{1}{r} - \frac{1}{s} = \frac{1}{r_0} - \frac{1}{s_0}.
$$

By (2.1) and the values of $r_0$ and $s_0$, we have

$$
\frac{1}{r} + \frac{\alpha}{n} = \frac{q-1}{s_0} + \frac{1}{s}, \quad \frac{1}{s} + \frac{\alpha}{n} = \frac{q-1}{s_0} + \frac{1}{s}.
$$

For $A > 0$, set $u_A = u$ when $u > A$ or $|x| > A$; $u_A = 0$ when $u \leq A$ and $|x| \leq A$. Similarly, $v_A$ is the same definition.
For \( g \in L^r(\mathbb{R}^n) \) and \( f \in L^s(\mathbb{R}^n) \), define
\[
(T_1g)(x) = \int_{\mathbb{R}^n} \frac{v_A^{q-1}(y)g(y)dy}{|x-y|^{n-\alpha}}, \quad (T_2f)(x) = \int_{\mathbb{R}^n} \frac{u_A^{p-1}(y)f(y)dy}{|x-y|^{n-\alpha}}.
\]
Noting (2.2), we can use the HLS inequality and the Hölder inequality to obtain
\[
\|T_1g\|_r \leq C\|v_A^{q-1}g\|_s^{\frac{r}{n+\alpha}} \leq C\|v_A\|_{q_0}^{q-1}\|g\|_s,
\]
\[
\|T_2f\|_s \leq C\|u_A^{p-1}f\|_r^{\frac{s}{n+\alpha}} \leq C\|u_A\|_{p_0}^{p-1}\|f\|_r.
\]
Choosing \( A \) sufficiently large such that
\[
C\|v_A\|_{q_0}^{q-1} \leq \frac{1}{4}, \quad C\|u_A\|_{p_0}^{p-1} \leq \frac{1}{4},
\]
we see that \( T = (T_1, T_2) \) is a contraction map from \( L^r(\mathbb{R}^n) \times L^s(\mathbb{R}^n) \) to itself for all \((\frac{1}{r}, \frac{1}{s}) \in I_1\), where
\[
I_1 := \left(\frac{\alpha(q-p)}{n(pq-1)}, \frac{n-\alpha}{n} \right) \times (0, 1 - \frac{\alpha(q-1)(p+1)}{n(pq-1)}),
\]
and the norm
\[
\|T(g, f)\|_{L^r(\mathbb{R}^n) \times L^s(\mathbb{R}^n)} = \|T_1g\|_r + \|T_2f\|_s.
\]
In view of \((\frac{1}{r_0}, \frac{1}{s_0}) \in I_1\), \( T \) is also a contraction map from \( L^{r_0}(\mathbb{R}^n) \times L^{s_0}(\mathbb{R}^n) \) to itself.
Define
\[
G = \int_{\mathbb{R}^n} \frac{(v-v_A)^q(y)dy}{|x-y|^{n-\alpha}}, \quad F = \int_{\mathbb{R}^n} \frac{(u-u_A)^p(y)dy}{|x-y|^{n-\alpha}}.
\]
Then the HLS inequality leads to \((G, F) \in L^r(\mathbb{R}^n) \times L^s(\mathbb{R}^n)\).
Since \((u, v)\) solves
\[
(g, f) = T(g, f) + (G, F),
\]
we can use the lifting lemma (Lemma 2.1 in [13]) to obtain
\[
(u, v) \in L^r(\mathbb{R}^n) \times L^s(\mathbb{R}^n), \quad \forall \left(\frac{1}{r}, \frac{1}{s}\right) \in I_1.
\]
Next we extend the integrability domain from \(I_1\) to
\[
(0, \frac{n-\alpha}{n}) \times (0, \min\{\frac{n-\alpha}{n}, \frac{pn-(p+1)\alpha}{n}\}).
\]
First we claim
\[
q > \frac{q\alpha(p+1)(q-1)}{pq-1} + \frac{\alpha}{n}.
\]
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In fact, by the non-subcritical condition (1.2) and $p \leq q$, we have $\frac{1}{q+1} < \frac{2n-\alpha}{2n}$. This leads to

$$1 - \frac{\alpha}{qn} > \frac{q-1}{q+1}.$$  

Using again the non-subcritical condition (1.2), we get

$$1 - \left( \frac{1}{p+1} + \frac{1}{q+1} \right) > \frac{\alpha}{n} \frac{q-1}{n} + \frac{1}{q} \left( 1 - \left( \frac{1}{p+1} + \frac{1}{q+1} \right) \right).$$

Multiplying by $(p+1)(q+1)$ yields

$$pq - 1 > \frac{n}{\alpha} \left[ (p+1)(q-1) + \frac{pq - 1}{q} \right].$$

Multiplying by $\frac{q}{pq - 1}$ again, we see (2.4).

Second we claim

$$q \frac{\alpha}{n} (p+1)(q-1) - \frac{\alpha}{n} > \alpha \frac{(q-p)}{n(pq - 1)}.$$ (2.5)

In fact, by the non-subcritical condition (1.2), we have

$$\frac{q}{q-1} \left[ 1 - \left( \frac{1}{p+1} + \frac{1}{q+1} \right) \right] > \frac{\alpha}{n}.$$  

This leads to

$$q > \frac{\alpha}{n} \frac{q-1}{1 - \left( \frac{1}{p+1} + \frac{1}{q+1} \right)} = \frac{\alpha}{n} \frac{\frac{(p+1)(q-1)}{pq - 1}}.$$  

This is (2.5).

Using the HLS inequality, we have

$$\|u\|_r \leq C\|v\|^{\frac{n}{q_n\bar{c}}} \leq C\|v\|^{\frac{n}{n+\alpha}}.$$  

Noting $v \in L^s(R^n)$ for all $\frac{1}{s} \in (0, 1 - \frac{\alpha}{n} \frac{(p+1)(q-1)}{pq - 1})$ implied by (2.3), we get $u \in L^r(R^n)$ for all

$$\frac{1}{r} \in (0, q - \frac{\alpha}{n} \frac{(p+1)(q-1)}{pq - 1} - \frac{\alpha}{n}).$$

Eq. (2.4) means this interval makes sense. Combining this with (2.3), from (2.5) we deduce

$$u \in L^r(R^n), \quad \forall \frac{1}{r} \in (0, \frac{n-\alpha}{n}).$$ (2.6)

Similarly, using the HLS inequality, we get

$$\|v\|_s \leq C\|u\|^{\frac{p_{\infty}}{n+\alpha}}.$$
By means of (2.6) we get
\[ v \in L^s(R^n), \quad \forall \frac{1}{s} \in (0, \min\left\{\frac{n - \alpha}{n}, \frac{pm - (p + 1)\alpha}{n}\right\}). \]

\[ \Box \]

**Theorem 2.2.** If \( u, v \) are integrable solutions of (1.1) with (1.2), then \( u, v \) are the finite energy solutions.

**Proof.** First, (1.2) implies
\[ \frac{1}{p + 1}, \frac{1}{q + 1} \in (0, \frac{n - \alpha}{n}), \quad (2.7) \]
and
\[ \frac{1}{p + 1} + \frac{1}{q + 1} < \frac{p}{(p + 1)(q + 1)} + \frac{n - \alpha}{n}. \]
Thus,
\[ \frac{\alpha}{n} < \frac{p}{(p + 1)(q + 1)} + 1 - \left(\frac{1}{p + 1} + \frac{1}{q + 1}\right) = \frac{p + pq - 1}{(p + 1)(q + 1)}. \]
This result leads to
\[ \frac{1}{q + 1} < p - (p + 1)\frac{\alpha}{n} = \frac{pm - (p + 1)\alpha}{n}. \]
Combining this with (2.7), and using Theorem 2.1 we obtain
\[ (u, v) \in L^{p+1}(R^n) \times L^{q+1}(R^n). \]
Namely, \( u, v \) are the finite energy solutions.

On the contrary, if (1.5) is true, we obtain \( p + 1 = r_0 \) and \( q + 1 = s_0 \). So we also have the following result which shows that the finite energy solutions are also the integrable solutions.

**Theorem 2.3.** If \( (u, v) \in L^{p+1}(R^n) \times L^{q+1}(R^n) \) solves (1.1), then \( (u, v) \in L^{r_0}(R^n) \times L^{s_0}(R^n) \).

**Proof.** First, Proposition 1.1 implies that (1.5) holds when \( u, v \) are the finite energy solutions. According to Theorem 1.1 in [13], the finite energy solutions \( (u, v) \) of (1.1) with (1.5) have the following integrability: \( (u, v) \in L^r(R^n) \times L^s(R^n) \) for all \( r, s \) satisfying
\[ \frac{1}{r} \in (0, \frac{n - \alpha}{n}), \quad \frac{1}{s} \in (0, \min\left\{\frac{n - \alpha}{n}, \frac{pm - (p + 1)\alpha}{n}\right\}). \]
So we only need to prove that \( \frac{1}{r_0} \) and \( \frac{1}{s_0} \) belong to the corresponding intervals.
First (1.5) implies
\[ \frac{n}{p+1} < n - \alpha. \]
Eq. (1.5) also leads to
\[ \frac{n}{p+1} = \frac{\alpha}{p+1} (1 - \frac{1}{p+1} - \frac{1}{q+1})^{-1}. \]
Combining these results yields
\[ \frac{\alpha(q+1)}{pq-1} < n - \alpha, \tag{2.8} \]
which means \( \frac{1}{s_0} \in (0, \frac{a-n}{n}) \).

Noting \( p \leq q \), we also have \( \frac{1}{s_0} \in (0, \frac{a-n}{n}) \) by the same argument above. In addition, (2.8) leads to \( \frac{\alpha(p+1)}{pq-1} < n \). Thus, \( \frac{\alpha(p+1)}{pq-1} + (p+1)\alpha < pn \), which implies \( \frac{1}{s_0} < \frac{pn-(p+1)\alpha}{n} \). Therefore, \( \frac{1}{s_0} \) belongs to the integrability interval. \( \square \)

Theorems 2.2 and 2.3 show that (1) and (3) in Theorem 1.2 are equivalent.

3 Integrable solutions are bounded

Theorem 3.1. If \((u, v) \in L^{q_0}(R^n) \times L^{s_0}(R^n)\) is a pair of positive solutions of (1.1), then \( u, v \) are bounded and converge to zero when \(|x| \to \infty\).

Proof. (1) Both the solutions \( u \) and \( v \) of (1.1) are bounded.

By exchanging the order of the integral variables, we have
\[ u(x) \leq C \left( \int_0^1 \frac{\int_{B_t(x)} v^q(y)dy dt}{t^{n-\alpha}} \right)^{1/l} + \int_1^\infty \frac{\int_{B_t(x)} v^q(y)dy dt}{t^{n-\alpha}} \frac{dt}{t} \]
\[ := C(H_1 + H_2). \]

By Hölder’s inequality, for any \( l > 1 \),
\[ \int_{B_t(x)} v^q(y)dy \leq C \|v^q\|_l |B_t(x)|^{1-1/l}. \]

Take \( l \) sufficiently large such that \( \frac{1}{l} = \varepsilon \) is sufficiently small. According to Theorem 2.1, \( \|w^q\|_l < \infty \). Therefore,
\[ H_1 \leq C \int_0^1 \frac{|B_t(x)|^{1-q\varepsilon} dt}{t^{n-\alpha}} \leq C \int_0^1 t^{\alpha-\varepsilon} dt \leq C. \]
If $z \in B_\delta(x)$, then $B_t(x) \subset B_{t+\delta}(z)$. For $\delta \in (0, 1)$ and $z \in B_\delta(x)$, it follows

$$H_2 = \int_1^\infty \int_{B_t(x)} v^q(y) dy \frac{dt}{t^{n-\alpha}} \leq \int_1^\infty \int_{B_{t+\delta}(z)} v^q(y) dy \frac{\alpha t^{n-\alpha}}{t + \delta} \frac{dt}{t + \delta} \leq (1 + \delta)^{n-\alpha+1} \int_{1+\delta}^\infty \int_{B_t(x)} v^q(y) dy \frac{dt}{t^{n-\alpha}} \leq Cu(z).$$  \hspace{1cm} (3.1)

Combining the estimates of $H_1$ and $H_2$, we have

$$u(x) \leq C + Cu(z), \; \text{for} \; z \in B_\delta(x),$$

where $\delta \in (0, 1)$. Integrating on $B_\delta(x)$, we get

$$u(x) \leq C + C \int_{B_\delta(x)} u(z) dz \leq C + C\|u\| \alpha |B_\delta(x)|^{1-\frac{\alpha}{n}} \leq C.$$  

This shows $u$ is bounded in $R^n$. Similarly, $v$ is also bounded.

(2) We claim the solutions $u, v$ of (1) satisfy

$$\lim_{|x| \to \infty} u(x) = 0, \; \lim_{|x| \to \infty} v(x) = 0.$$  \hspace{1cm} (3.2)

Take $x_0 \in R^n$. By (1), $\|v\| < \infty$. Thus, $\forall \varepsilon > 0$, there exists $\delta \in (0, 1)$ such that

$$\int_0^\delta \int_{B_t(x_0)} v^q(z) dz \frac{dt}{t^{n-\alpha}} \leq C\|v\| \delta \int_0^\delta \frac{dt}{t} < \varepsilon.$$  

On the other hand, similarly to the derivation of (3.1), as $|x - x_0| < \delta$,

$$\int_0^\infty \int_{B_t(x_0)} v^q(z) dz \frac{dt}{t^{n-\alpha}} \leq \int_0^\infty \int_{B_{t+\delta}(z)} v^q(y) dy \frac{\alpha t^{n-\alpha}}{t + \delta} \frac{dt}{t + \delta} \leq C \int_0^\infty \int_{B_t(x)} v^q(z) dz \frac{dt}{t^{n-\alpha}} \leq Cu(x).$$

Combining these estimates, we get

$$u(x_0) < \varepsilon + Cu(x), \; \text{for} \; |x - x_0| < \delta.$$  

Since $u \in L^\infty(R^n)$, there holds $\lim_{|x| \to \infty} \int_{B_\delta(x_0)} u^\rho(x) dx = 0$. Thus, we have

$$u^\rho(x_0) = |B_\delta(x_0)|^{-1} \int_{B_\delta(x_0)} u^\rho(x_0) dx \leq C\varepsilon^\rho + C|B_\delta(x_0)|^{-1} \int_{B_\delta(x_0)} u^\rho(x) dx \to 0$$

when $|x_0| \to \infty$ and $\varepsilon \to 0$. Similarly, $v$ has the same result. Thus, (3.2) holds.
4 Fast decay of integrable solutions

In this section, we always assume that \((u, v)\) is a pair of positive solutions of the system (1.1) with (1.2).

First we verify the integrable solutions decay fast. This argument includes five propositions.

**Proposition 4.1.** \(B_0 := \int_{\mathbb{R}^n} v^q(y) dy < \infty\).

**Proof.** By (1.2) and \(q \geq p\), we have

\[
\frac{1}{q} \leq \frac{n - \alpha}{n + \alpha} < \frac{n - \alpha}{n}.
\]

On the other hand, in view of \(\frac{1}{p+1} < 1 - \frac{1}{q+1} = \frac{q}{q+1}\), it follows from (1.2) that

\[
\frac{q + 1}{q(p + 1)} = \frac{1}{p + 1} + \frac{1}{q(p + 1)} < \frac{1}{p + 1} + \frac{1}{q + 1} < \frac{n - \alpha}{n}. \tag{4.1}
\]

Therefore, \(1 + \frac{1}{q} < \frac{n - \alpha}{n}\). This implies

\[
\frac{1}{q} < \frac{p n - (p + 1) \alpha}{n}.
\]

According to Theorem 2.1, \(v \in L^q(\mathbb{R}^n)\).

**Proposition 4.2.**

\[
\lim_{|x| \to \infty} u(x)|x|^{n - \alpha} = B_0.
\]

**Proof.** For fixed \(R > 0\), write

\[
L_1 := \int_{B_R} v^q(y)\left(\frac{|x|^{n - \alpha}}{|x - y|^{n - \alpha}} - 1\right) dy.
\]

When \(y \in B_R\) and \(|x| \to \infty\),

\[
v^q(y)\frac{|x|^{n - \alpha}}{|x - y|^{n - \alpha}} - 1 \leq 3v^q(y) \in L^1(\mathbb{R}^n)
\]

by virtue of Proposition 4.1. Using Lebesgue’s dominated convergence theorem yields

\[
|L_1| \to 0, \quad \text{as} \quad |x| \to \infty.
\]

This result leads to

\[
\lim_{R \to \infty} \lim_{|x| \to \infty} \int_{B_R} v^q(y)\frac{|x|^{n - \alpha}}{|x - y|^{n - \alpha}} dy = B_0.
\]

Next, we write

\[
L_2 := \int_{(\mathbb{R}^n \setminus B_R) \setminus B(x, |x|/2)} v^q(y)\frac{|x|^{n - \alpha}}{|x - y|^{n - \alpha}} dy.
\]
Clearly, \(|x - y| \geq |x|/2\) when \(y \in (R^n \setminus B_R) \setminus B(x, |x|/2)\). Therefore, when \(R \to \infty\),

\[ L_2 = C \int_{R^n \setminus B_R} v^q(y) dy \to 0. \]

We write

\[ L_3 := \int_{B(x, |x|/2)} v^q(y) \frac{|x|^{n-\alpha}}{|x-y|^{n-\alpha}} dy \]

and

\[ L_4 := \frac{L_3}{|x|^{\alpha}} = \int_{B(x, |x|/2)} \frac{v^q(y) dy}{|x-y|^{n-\alpha}}. \]

According to Theorem 3.1 in [6], \(v\) is radially symmetric and decreasing about \(x_0 \in R^n\). Without loss of generality, we can view \(x_0\) as the origin when \(|x|\) is sufficiently large. Therefore,

\[ L_4 \leq v^q(x/2) \int_{B(x, |x|/2)} \frac{dy}{|x-y|^{n-\alpha}} \leq \frac{C v^q(x/2)}{|x|^{-\alpha}}. \]

Write \(r = |x|\), and define \(\tilde{v}(r) = v(x)\). Thus,

\[ L_4 \leq C \tilde{v}^q(|x|/2)|x|^\alpha \quad (4.2) \]

On the other hand, Theorem 2.1 shows that \(v \in L^s(R^n)\), \(\frac{1}{s} = \frac{1}{n} \min\{n - \alpha, p(n - \alpha) - \alpha\} - \varepsilon\). Here \(\varepsilon > 0\) is sufficiently small. This integrability result, together with the decreasing property of \(v\), implies

\[ \tilde{v}^s(|x|/2)|x|^n \leq C \int_{B(0, \frac{|x|}{2})} v^s(y) dy \leq C. \quad (4.3) \]

We claim

\[ |x|^{n-\alpha} L_4 = o(1), \quad \text{as} \quad |x| \to \infty. \quad (4.4) \]

We will prove (4.4) in two cases.

Case 1. When \(n - \alpha \leq p(n - \alpha) - \alpha\), (4.3) means

\[ \tilde{v}^s(|x|/2)|x|^{q(n-\alpha-\varepsilon)} \leq C. \]

Combining this consequence with (4.2) yields

\[ |x|^{q(n-\alpha-\varepsilon)-\alpha} L_4 \leq C \tilde{v}^q(|x|/2)|x|^{q(n-\alpha-\varepsilon)} \leq C. \]

In view of \(q(n - \alpha) > n\), we have \(q(n - \alpha) - \alpha > n - \alpha\). Choosing \(\varepsilon\) properly small and letting \(|x| \to \infty\) in the result above, we can deduce (4.4).

Case 2. When \(n - \alpha > p(n - \alpha) - \alpha\), (4.3) implies

\[ \tilde{v}^s(|x|/2)|x|^{q(p(n-\alpha)-\alpha-\varepsilon)} \leq C. \]
Combining with (4.2) yields
\[ |x|^q(p(n-\alpha)-\alpha-\varepsilon) L_4 \leq C. \]

By (4.1), it follows \( q(p+1)(n-\alpha) > n + qn \). This is equivalent to \( q[p(n-\alpha)-\alpha] > n \). Choosing \( \varepsilon \) properly small and letting \( |x| \to \infty \) in the result above, we can also obtain (4.4).

Inserting (4.4) into \( L_3 \), we derive that
\[ L_3 \to 0, \quad \text{as} \quad |x| \to \infty. \]

Combining all the estimates of \( L_1, L_2 \) and \( L_3 \), we complete the proof.

According to Proposition 4.2, there exists a properly large constant \( R > 0 \) such that
\[ u(x) = \frac{B_0 + o(1)}{|x|^{n-\alpha}}, \quad \text{for} \quad x \in \mathbb{R}^n \setminus B_R. \]

Hereafter, we will use (4.5) to investigate the decay rate of \( v \).

**Proposition 4.3.** If \( p(n-\alpha) > n \), then \( B_1 = \int_{\mathbb{R}^n} u^p(y)dy < \infty \). In addition,
\[ \lim_{|x| \to \infty} v(x)|x|^{n-\alpha} = B_1. \]

**Proof.** According to Theorem 2.1, by \( p(n-\alpha) > n \) we can also prove \( B_1 < \infty \).

When \( y \in B_R \) for large \( R > 0 \), \( \lim_{|x| \to \infty} \frac{|x|}{|x-y|} = 1 \). By \( B_1 < \infty \) and the Lebesgue dominated convergence theorem, it follows that
\[ \lim_{R \to \infty} \lim_{|x| \to \infty} \int_{B_R} \frac{|x|^{n-\alpha}u^p(y)dy}{|x-y|^{n-\alpha}} = B_1. \]

In view of \( p(n-\alpha) > n \), it is not difficult to deduce that
\[ \int_{\mathbb{R}^n \setminus B_R} \frac{|x|^{n-\alpha}dy}{|x-y|^{n-\alpha}|y|^{p(n-\alpha)}} = o(1), \quad \text{as} \quad |x| \to \infty, R \to \infty. \]

By virtue of (4.6), we have
\[ |x|^{n-\alpha}v(x) = \int_{B_R} \frac{|x|^{n-\alpha}u^p(y)dy}{|x-y|^{n-\alpha}} + \int_{\mathbb{R}^n \setminus B_R} \frac{(B_0 + o(1))|x|^{n-\alpha}dy}{|x-y|^{n-\alpha}|y|^{p(n-\alpha)}}. \]

Inserting (4.6) and (4.7) into the identity above, we get
\[ |x|^{n-\alpha}v(x) \to B_1, \quad \text{as} \quad |x| \to \infty. \]

Proposition 4.3 is proved.
Proposition 4.4. If $p(n-\alpha) = n$, then
\[
\lim_{|x| \to \infty} |x|^{n-\alpha} (\ln |x|)^{-1} v(x) = B_0^p |S^{n-1}|.
\]

Proof. By virtue of (4.5), for large $R > 0$, we have
\[
\frac{|x|^{n-\alpha}}{\ln |x|} v(x) = \frac{1}{\ln |x|} \int_{B_R} \frac{|x|^{n-\alpha} u^p(y) dy}{|x-y|^{n-\alpha}} + \frac{(B_0 + o(1))^p}{\ln |x|} \int_{R^n \setminus B_R} \frac{|x|^{n-\alpha} dy}{|x-y|^{n-\alpha}|y|^n}.
\]

Eq. (4.10) implies that as $|x| \to \infty$,
\[
\frac{1}{\ln |x|} \int_{B_R} \frac{|x|^{n-\alpha} u^p(y) dy}{|x-y|^{n-\alpha}} = o(1).
\]

On the other hand, for the large constant $R > 0$ and the small constant $\delta \in (0, 1/2)$,
\[
\frac{1}{\ln |x|} \int_{R^n \setminus B_R} \frac{|x|^{n-\alpha} dy}{|x-y|^{n-\alpha}|y|^n} = \frac{1}{\ln |x|} \int_{B^\delta} \int_{S^{n-1}} \frac{dr}{|y|^{n-\alpha}} + \frac{1}{\ln |x|} \int_{R^n \setminus B_R} \frac{dz}{|z|^{n-\alpha}}.
\]

Indeed, the second term of the right hand side is finite since $n-\alpha < n$ near $e$; and $n-\alpha + n > n$ near infinity. Moreover, the upper bound only depends on $\delta$. Letting $|x| \to \infty$, we have
\[
\frac{1}{\ln |x|} \int_{B^\delta} \int_{S^{n-1}} \frac{dr}{|y|^{n-\alpha}} = o(1).
\]

When $r \in (0, \delta)$, $1-\delta \leq |e-rw| \leq 1+\delta$. There exists $\theta \in (-1, 1)$ such that $|e-rw| = 1+\theta \delta$. Thus, the first term of (4.10)
\[
\frac{1}{\ln |x|} \int_{B^\delta} \int_{S^{n-1}} \frac{dr}{|y|^{n-\alpha}} = \frac{|S^{n-1}|}{(1+\theta \delta)^{n-\alpha} \ln |x|} (\ln \delta - \ln R + \ln |x|) \to |S^{n-1}| (|x| \to \infty)
\]
\[
\to |S^{n-1}| (\delta \to 0).
\]

Substituting this result and (4.11) into (4.10), we have
\[
\frac{1}{\ln |x|} \int_{R^n \setminus B_R} \frac{|x|^{n-\alpha} dy}{|x-y|^{n-\alpha}|y|^n} \to |S^{n-1}|, \text{ as } |x| \to \infty.
\]
Combining with (4.8) and (4.9) we can complete Proposition 4.4.
**Proposition 4.5.** If \( p(n - \alpha) < n \), then

\[
B_3 := B_0^p \int_{\mathbb{R}^n} |z|^{-(n-\alpha)p} |e - z|^{-n+\alpha} \, dz < \infty.
\]

In addition,

\[
\lim_{|x| \to \infty} |x|^{pn-\alpha(p+1)} v(x) = B_3.
\]

**Proof.** It is easy to see \( B_3 < \infty \), since we observe that \( \frac{1}{p+1} < \frac{n-\alpha}{n} \) means that the integral decays at the rate \((n-\alpha)(p+1) > n\) near infinite, \( n - \alpha < n \) near \( e \), and \( p(n-\alpha) < n \) near the origin.

For large \( R > 0 \), using (4.5) we have

\[
|x|^{p(n-\alpha)-n} \int_{B_R} \frac{|x|^{n-\alpha} u^p(y) \, dy}{|x - y|^{n-\alpha}} + (B_0 + o(1))^p \int_{R^n \setminus B_R} \frac{|x|^{(p+1)(n-\alpha)-n} \, dy}{|x - y|^{n-\alpha}|y|^{p(n-\alpha)}}.
\]

When \( y \in B_R \), and \( |x| \to \infty \),

\[
|x|^{p(n-\alpha)-n} \int_{B_R} \frac{|x|^{n-\alpha} u^p(y) \, dy}{|x - y|^{n-\alpha}} \leq C|x|^{p(n-\alpha)-n} \to 0,
\]

since \( p(n-\alpha) < n \).

On the other hand, when \( |x| \to \infty \),

\[
\int_{R^n \setminus B_R} \frac{|x|^{(p+1)(n-\alpha)-n} \, dy}{|x - y|^{n-\alpha}|y|^{p(n-\alpha)}} = \int_{R^n \setminus B_R / |x|} \frac{dz}{|z|^{(p-1)(n-\alpha)}} \to B_3.
\]

Inserting this result and (4.13) into (4.12), we complete the proof of Proposition.

\[\Box\]

Next, we verify that the bounded solutions with fast decay rates must be the integrable solutions.

**Proposition 4.6.** Let \( u, v \) solve (1.1) with (1.2). If they are bounded and decay fast, then \( (u, v) \in L^r_\infty(R^n) \times L^s_\infty(R^n) \).

**Proof.** From (1.2), we can see easily that \( p > \frac{\alpha}{n-\alpha} \). This results together with (1.2) lead to

\[
\frac{\alpha}{(n-\alpha)(p+1)} < 1 - \left( \frac{1}{p+1} + \frac{1}{q+1} \right).
\]

Multiplying by \( (p+1)(q+1) \) yields \( \frac{\alpha(q+1)}{n-\alpha} < pq - 1 \). This implies \( n < (n-\alpha) r_0 \).

Since \( u \) is bounded and decay fast, there holds

\[
\int_{R^n} u^{r_0}(x) \, dx \leq C + \int_{R} \infty r^{-(n-\alpha) r_0} \frac{dr}{r} < \infty.
\]
Namely, \( u \in L^{r_0}(R^n) \).

Noting \( p \le q \), we also have \( n - (n - \alpha)s_0 < 0 \). If \( v \) is bounded and decaying with the rate \( |x|^{\alpha-n} \), we also deduce \( v \in L^{s_0} \) by the same argument above.

If \( v \) is bounded and decaying with the rate \( |x|^{\alpha-n} \ln |x| \), then there exists a suitably large \( R > 0 \) such that \( (\ln |x|)^{s_0} \leq |x|^\epsilon \) for \( |x| > R \), where \( \epsilon > 0 \) is sufficiently small. Then, by \( n - (n - \alpha)s_0 < 0 \), we also get

\[
\int_{R^n} v^{s_0}(x)dx \leq C + C \int_{R}^{\infty} r^{n-(n-\alpha)s_0+\epsilon} \frac{dr}{r} < \infty.
\]

Let \( v \) be bounded and decaying with the rate \( |x|^{(\alpha-n)(p+1)+n} \). From (1.2) we have \( \frac{1}{p+1} < \frac{n-\alpha}{n} \). This and (1.2) lead to \( \frac{\alpha(p+1)}{pq} < n - \alpha \), which implies \( \frac{pn(pq-1)}{\alpha(p+1)} > pq \). From this we deduce that \( n - [pn - \alpha(p+1)]s_0 < 0 \), and hence

\[
\int_{R^n} v^{s_0}dx \leq C + \int_{R}^{\infty} r^{n-[(n-\alpha)(p+1)+n]} \frac{dr}{r} < \infty.
\]

This means \( v \in L^{s_0}(R^n) \).

The argument in Sections 3 and 4 shows that (1) and (2) in Theorem 1.2 are equivalent. Combining with the argument in Section 2, we complete the proof of Theorem 1.2.

In addition, we can also prove directly the following result.

**Proposition 4.7.** Items (2) and (3) in Theorem 1.2 are also equivalent.

**Proof.** (3)\(\Rightarrow\)(2): First, according to Proposition 1.1 we see the critical condition (1.5) holds. By Theorems 1.1 and 1.3 in [13], we also obtain the optimal integrability of the finite energy solutions \( u, v \). Based on this result, [5] proved the boundedness of \( u \) and \( v \). In addition, Theorem 2 in [17] shows the fast decay rates of \( u, v \) as Theorem 1.2 (see also Corollary 1.3 (2) in [29]).

(2)\(\Rightarrow\)(3): Eq. (1.2) leads to \( n < (p+1)(n - \alpha) \). Hence, from the boundedness and the fast decay rate of \( u \), we have

\[
\int_{R^n} u^{p+1}(x)dx \leq C + C \int_{R}^{\infty} r^{n-(p+1)(n-\alpha)} \frac{dr}{r} < \infty.
\]

Similarly, (1.2) also leads to \( n < (q+1)(n - \alpha) \). We also deduce that \( v \in L^{q+1}(R^n) \) when \( p(n-\alpha) \geq n \).

Eq. (1.2) implies

\[
\alpha < (1 - \frac{1}{p+1} - \frac{1}{q+1})n + \frac{pn}{(p+1)(q+1)}.
\]

Multiplying by \( (p+1)(q+1) \) yields

\[
\frac{1}{q+1} < \frac{pn - (p+1)\alpha}{n}.
\]
Thus,
\[ \int_{\mathbb{R}^n} v^{q+1} \, dx \leq C + C \int_{\mathbb{R}} r^{n-(q+1)(p_n-(p+1)\alpha)} \frac{dr}{r} < \infty. \]
So, \( v \in L^{q+1}(\mathbb{R}^n) \) when \( p(n-\alpha) < n \).

5 Slow decay of bounded solutions

**Proposition 5.1.** Let \( u, v \) be positive bounded solutions. Then there exists \( c > 0 \) such that as \( |x| \to \infty \),
\[
    u(x) \geq \frac{c}{(1 + |x|)^{n-\alpha}}; \quad (5.1)
\]
\[
    v(x) \geq \frac{c}{(1 + |x|)^{\min\{n-\alpha, pn-(p+1)\alpha\}}}. \quad (5.2)
\]

**Proof.** First, we can find \( c > 0 \) such that \( u(y), v(y) \geq c > 0 \) for \( y \in B_1(0) \).
Therefore,
\[
    u(x) \geq c \int_{B_1(0)} \frac{dy}{|x-y|^{n-\alpha}} \geq c(1 + |x|)^{\alpha-n}. \]
This is (5.1). Similarly, we also have
\[
    v(x) \geq \frac{c}{(1 + |x|)^{n-\alpha}}. \quad (5.3)
\]
Substituting (5.1) into \( v(x) \geq \int_{B(x,|x|/2)} u^p(y)|x-y|^\alpha \, dy \) yields
\[
    v(x) \geq c(1 + |x|)^{p(\alpha-n)} \int_0^{x/2} r^{\alpha} \frac{dr}{r} = c(1 + |x|)^{(p+1)\alpha-pn}. \]
Combining with (5.3), we obtain (5.2). \( \square \)

Theorem 1.2 shows that \( u, v \) decay by the fast rates as long as they are the integrable solutions. If \( u, v \) are not integrable, we conjecture that they decay slowly.

The following result shows that the decay rates of \( u, v \) are not faster than the slow rates \( \frac{\alpha(q+1)}{pq-1} \) and \( \frac{\alpha(p+1)}{pq-1} \) respectively, if \( u, v \) are not integrable.

**Proposition 5.2.** Let \( u, v \) be positive bounded solutions, and \( \theta_3 > \frac{\alpha(q+1)}{pq-1}, \theta_4 > \frac{\alpha(p+1)}{pq-1} \). If \( u, v \) are not the integrable solutions, then there does not exist \( C > 0 \) such that either
\[
    u(x) \leq C(1 + |x|)^{-\theta_3}, \quad \text{or} \quad v(x) \leq C(1 + |x|)^{-\theta_4}. \]

**Proof.** Suppose there exists \( C > 0 \) such that as \( |x| \to \infty \),
\[
    u(x) \leq C(1 + |x|)^{-\theta_3},
\]

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where $\theta_3 > \frac{\alpha(q+1)}{pq-1}$, then
\[
\int_{R^n} u^\alpha(x) dx = \int_{B_R(0)} u^\alpha(x) dx + \int_{R^n \setminus B_R(0)} u^\alpha(x) dx 
\leq C + C \int_{R} \frac{r^{n-\alpha \theta_3}}{r} dr < \infty.
\]

Similarly, if $v(x) \leq C(1 + |x|)^{-\theta_4}$ with $\theta_4 > \frac{\alpha(p+1)}{pq-1}$, then it also belongs to $L^\infty(R^n)$.

Thus, $u$ (or $v$) is integrable solution, which contradicts with the assumption of our proposition.

The following result shows that the decay rates of $u, v$ are not slower than the slow rates $\frac{\alpha(q+1)}{pq-1}$ and $\frac{\alpha(p+1)}{pq-1}$, respectively.

**Proposition 5.3.** Let $u, v$ be positive bounded solutions of (1.1), and $\theta_1 < \frac{\alpha(q+1)}{pq-1}$, $\theta_2 < \frac{\alpha(p+1)}{pq-1}$. Then there does not exist $C > 0$ such that either
\[
u(x) \geq C(1 + |x|)^{-\theta_1}, \quad \text{or} \quad u(x) \geq C(1 + |x|)^{-\theta_2}.
\]

**Proof.** If there exists $C > 0$ such that for large $|x|$, $u(x) \geq C(1 + |x|)^{-\theta_1}$, $\theta_1 < \frac{\alpha(q+1)}{pq-1}$.

By an iteration we can deduce the contradiction.

Denoting $\theta_1$ by $b_0$, we have
\[
v(x) \geq \int_{B(|x|/2)} \frac{u^p(y)dy}{|x-y|^{n-\alpha}} \geq c(1 + |x|)^{-a_1}, \quad a_1 = pb_0 - \alpha.
\]

Using this result, we have
\[
u(x) \geq \int_{B(|x|/2)} \frac{v^q(y)dy}{|x-y|^{n-\alpha}} \geq c(1 + |x|)^{-b_1}, \quad b_1 = qa_1 - \alpha.
\]

By induction, we have two sequences
\[
b_0 = \theta_1, \quad a_j = pb_{j-1} - \alpha, \quad b_j = qa_j - \alpha, \quad j = 1, 2, \ldots.
\]

We claim that there must be $j_0$ such that $b_{j_0} < 0$, which leads to $u(x) = \infty$ for large $|x|$. In fact,
\[
b_j = pqb_{j-1} - \alpha(q+1) = \cdots = (pq)^j b_0 - \alpha(q+1)(1 + pq + \cdots + (pq)^{j-1}).
\]

In view of $pq > 1$, we have
\[
b_j = (pq)^j (b_0 - \frac{\alpha(q+1)}{pq-1}) + \frac{\alpha(q+1)}{pq-1}.
\]
Noting $b_0 - \frac{\alpha(q+1)}{pq-1} < 0$, we can find a large $j_0$ such that $b_{j_0} < 0$. It is impossible since the solution $u$ blows up.

Similarly, if there exists $C > 0$ such that for large $|x|$, $$v(x) \geq C(1 + |x|)^{-\theta_2}, \quad \theta_2 < \frac{\alpha(p+1)}{pq-1}.$$ By an analogous iteration argument above, we can also deduce a contradiction. \(\square\)

Combining Propositions 5.2 and 5.3, we complete the proof of Theorem 1.3.

Remark 5.1. Proposition 5.3 shows that if there exists $C > 0$ such that for large $|x|$, $$u(x) \leq C(1 + |x|)^{-\theta_1}, \quad v(x) \leq C(1 + |x|)^{-\theta_2},$$ then $\theta_1 \geq \frac{\alpha(q+1)}{pq-1}$, $\theta_2 \geq \frac{\alpha(p+1)}{pq-1}$. However, there may be $C_j \to \infty$ such that as some $|x_j| \to \infty$, $$u(x_j) \leq C_j(1 + |x_j|)^{-\theta_1}, \quad v(x_j) \leq C_j(1 + |x_j|)^{-\theta_2}.$$ If $u, v$ have some monotonicity, then the result above does not happen.

Proposition 5.4. Let $u, v$ be positive bounded decaying solutions. If there exists some $\epsilon_0 > 0$ such that for $|y| \leq |x|$, $$u(y) \geq \epsilon_0 u(x), \quad v(y) \geq \epsilon_0 v(x),$$ then there exists $C > 0$ such that $$u(x) \leq C(1 + |x|)^{-\frac{\alpha(q+1)}{pq-1}}, \quad v(x) \leq C(1 + |x|)^{-\frac{\alpha(p+1)}{pq-1}}.$$ Proof. Clearly, $$u(x) \geq cv^q(x) \int_R^{|x|} r^{\alpha} \frac{dr}{r} \geq cv^q(x)|x|^\alpha, \quad \text{for large } |x|.$$ By the monotonicity of $u$ we also deduce the monotonicity of $v$. Thus, we also have $$v(x) \geq cv^p(x)|x|^\alpha.$$ Inserting this result into (5.4), we get $$v(x) \geq cv^{pq}(x)|x|^{(p+1)\alpha},$$ which implies the estimate of $v$. Similarly, $u$ also has the corresponding estimate. \(\square\)

Combining Propositions 5.1 and 5.4, we complete the proof of Theorem 1.4.

Next, we prove Theorem 1.5. It is the corollary of the following proposition.
Proposition 5.5. Suppose that the positive bounded solutions $u, v$ satisfy
$$u(x) \simeq (1 + |x|)^{-\theta_1}, \quad v(x) \simeq (1 + |x|)^{-\theta_2}, \quad \text{when } |x| \to \infty. \quad (5.5)$$
Then (1.2) must hold, and
$$
\theta_1 \geq \frac{\alpha (q + 1)}{pq - 1}, \quad \theta_2 \geq \frac{\alpha (p + 1)}{pq - 1}. \quad (5.6)
$$
Furthermore,
(1) if one strict inequality of (5.6) holds, then (1.5) must be true and $u, v$ are the finite energy solutions decaying fast like Theorem 1.2.
(2) If $u, v$ are not the integrable solutions, then
$$
\theta_1 = \frac{\alpha (q + 1)}{pq - 1}, \quad \theta_2 = \frac{\alpha (p + 1)}{pq - 1}.
$$
Proof. Step 1. We first claim
$$
\theta_1 \geq \frac{\alpha (q + 1)}{pq - 1}, \quad \theta_2 \geq \frac{\alpha (p + 1)}{pq - 1}.
$$
In fact, $|x|/2 \leq |y| \leq 3|x|/2$ when $y \in B_{|x|/2}(x)$. Thus, for large $|x|$, from (5.5) it follows that
$$
C(1 + |x|)^{-\theta_1} \geq u(x)
$$
$$
\geq \int_{B(x,|x|/2)} \frac{v^q(y)dy}{|x - y|^{n-\alpha}}
$$
$$
\geq c(1 + |x|)^{-\theta_2} \int_0^{|x|/2} r^{\alpha - q} dr
$$
$$
\geq c(1 + |x|)^{\alpha - q\theta_2}.
$$
This result implies
$$
\theta_1 \leq q\theta_2 - \alpha
$$
since $|x|$ is sufficiently large. Similarly,
$$
\theta_2 \leq p\theta_1 - \alpha.
$$
These two inequalities above show our claim.

Step 2. We claim that the subcritical condition (1.0) is not true. Otherwise,
$$
\theta_2 \geq \frac{\alpha (p + 1)}{pq - 1} = \frac{\alpha}{q + 1} (1 - \frac{1}{p + 1} - \frac{1}{q + 1})^{-1} \geq \frac{n}{q + 1},
$$
which implies that $v \in L^{n+1}(\mathbb{R}^n)$ is a finite energy solution. This contradicts with Proposition 1.1.

Step 3. We prove (1) and (2).
(1) Without loss of generality, we assume $\theta_1 > \frac{\alpha (q + 1)}{pq - 1}$. Then using Proposition 5.2, we know $u \in L^{q\alpha}(\mathbb{R}^n)$ and hence $u$ is the integrable solution. By the HLS inequality, $v$ is also the integrable solution. According to Theorem 1.2 (1.3) is true, and $u, v$ are the finite energy solutions.
(2) Using Proposition 5.2 from (5.5) and (5.6) we can see our conclusion. \( \square \)
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References

[1] L. Caffarelli, B. Gidas, J. Spruck, Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth, Comm. Pure Appl. Math. 42 (1989), 271–297.

[2] G. Caristi, L. D’Ambrosio, E. Mitidieri, Representation formulae for solutions to some classes of higher order systems and related Liouville theorems, Milan J. Math., 76 (2008), 27–67.

[3] A. Chang, P. Yang, On uniqueness of an n-th order differential equation in conformal geometry, Math. Res. Lett., 4 (1997), 91–102.

[4] W. Chen, C. Li, A priori estimates for prescribing scalar curvature equations, Ann. of Math., 145 (1997), 547–564.

[5] W. Chen, C. Li, Regularity of Solutions for a system of Integral Equations, Commun. Pure Appl. Anal., 4 (2005), 1–8.

[6] W. Chen, C. Li, An integral system and the Lane-Emden conjecture, Discrete Contin. Dyn. Syst., 24 (2009), 1167–1184.

[7] W. Chen, C. Li, Super polyharmonic property of solutions for PDE systems and its applications, Commun. Pure Appl. Anal., in press. arXiv:1110.2539.

[8] W. Chen, C. Li, B. Ou, Classification of solutions for a system of integral equations, Comm. Partial Differential Equations, 30 (2005), 59–65.

[9] W. Chen, C. Li, B. Ou, Classification of solutions for an integral equation, Comm. Pure Appl. Math., 59 (2006), 330–343.

[10] J. Davila, M. del Pino, M. Musso, J. Wei, Fast and slow decay solutions for supercritical elliptic problems in exterior domains, Calc. Var. Partial Differential Equations, 32 (2008), 453–480.

[11] B. Gidas, W.-M. Ni, L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$ (collected in the book Mathematical Analysis and Applications, which is vol. 7a of the book series Advances in Mathematics. Supplementary Studies, Academic Press, New York, 1981.)

[12] F. Hang, On the integral systems related to Hardy-Littlewood-sobolev inequality, Math. Res. Lett., 14 (2007), 373–383.

[13] C. Jin, C. Li, Qualitative analysis of some systems of integral equations, Calc. Var. Partial Differential Equations, 26 (2006), 447–457.
[14] Y. Lei, Asymptotic properties of positive solutions of the Hardy-Sobolev type equations, J. Differential Equations, 254 (2013) 1774–1799.

[15] Y. Lei, C. Li, Sharp criteria of Liouville type for some nonlinear systems, arXiv:1301.6235, 2013.

[16] Y. Lei, C. Li, C. Ma, Decay estimation for positive solutions of a γ-Laplace equation, Discrete Contin. Dyn. Syst., 30 (2011), 547–558.

[17] Y. Lei, C. Li, C. Ma, Asymptotic radial symmetry and growth estimates of positive solutions to weighted Hardy-Littlewood-Sobolev system, Calc. Var. Partial Differential Equations., 45 (2012), 43–61.

[18] C. Li, Local asymptotic symmetry of singular solutions to nonlinear elliptic equations, Invent. Math., 123 (1996), 221–231.

[19] C. Li, A degree theory approach for the shooting method, arXiv:1301.6232, 2013.

[20] Y. Li, Asymptotic behavior of positive solutions of equation $\Delta u + K(x)u^p = 0$ in $\mathbb{R}^n$, J. Differential Equations, 95 (1992), 304–330.

[21] Y. Li, Remark on some conformally invariant integral equations: the method of moving spheres, J. Eur. Math. Soc., 6 (2004), 153–180.

[22] E. Lieb, Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities, Ann. of Math., 118 (1983), 349–374.

[23] C. Lin, A classification of solutions of a conformally invariant fourth order equation in $\mathbb{R}^n$, Comm. Math. Helv., 73 (1998), 206–231.

[24] J. Liu, Y. Guo, Y. Zhang, Liouville-type theorems for polyharmonic systems in $\mathbb{R}^n$, J. Differential Equations, 225 (2006), 685–709.

[25] Y. Liu, Y. Li, Y. Deng, Separation property of solutions for a semilinear elliptic equation, J. Differential Equations, 163 (2000), 381-406.

[26] E. Mitidieri, Nonexistence of positive solutions of semilinear elliptic systems in $\mathbb{R}^n$, Differential Integral Equations, 9 (1996), 465–479.

[27] J. Serrin, H. Zou, Non-existence of positive solution of Lane-Emden systems, Differential Integral Equations, 9 (1996), 635–653.

[28] P. Souplet, The proof of the Lane-Emden conjecture in 4 space dimensions, Adv. Math., 221 (2009), 1409–1427.

[29] S. Sun, Y. Lei, Fast decay estimates for integrable solutions of the Lane-Emden type integral systems involving the Wolff potentials, J. Funct. Anal., 263 (2012), 3857–3882.

[30] J. Wei and X. Xu, Classification of solutions of higher order conformally invariant equations, Math. Ann., 313 (1999), 207–228.
[31] H. Zou, *Symmetry of ground states for a semilinear elliptic system*, Trans. Amer. Math. Soc., 352 (2000), 1217–1245.

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