PRE-THRESHOLD FRACTIONAL SUSCEPTIBILITY FUNCTION: 
HOLOMORPHY AND RESPONSE FORMULA

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ABSTRACT. For certain smooth unimodal families with negative Schwarzian derivative, we construct a set of Collet-Eckmann and subexponentially recurrent parameters Ω, whose complement set has sufficiently fast decaying density, on which exponential mixing with uniform rates occurs. We use this construction to establish holomorphy of the ‘true’ fractional susceptibility function of the logistic family, in a disk of radius larger than one, for differentiation index $0 \leq \eta < 1/2$, as recently conjectured by Baladi and Smania. We also obtain a fractional response formula.

1. Introduction

This paper is concerned with smooth families of smooth maps $(f_t)_t$, where $f_t : X \to X$ admits a physical measure $\mu_t$ for a positive Lebesgue measure set of parameters $\Omega$. An invariant measure $\mu_t$ of $f_t$ is physical when its ergodic basin, that is the set of $x \in X$ such that the time average of Dirac masses along the orbits, $\frac{1}{n} \sum_{k=0}^{n-1} \delta_{f_t^k(x)}$, converges weakly towards $\mu_t$ when $n \to \infty$, has positive Lebesgue measure. In so many words, a physical measure describes the asymptotic average behavior of a positive (Lebesgue) measure set of initial conditions. For instance, if $f = f_{t_0}$ admits an attractive fixed point $a \in X$, the Dirac measure at the fixed point $\delta_a$ is a physical measure; at the other end, if $f = f_{t_0}$ admits an ergodic, absolutely continuous (w.r.t. Lebesgue) invariant measure, then it is a physical measure: this is a simple consequence of Birkhoff ergodic theorem.

Response theory is concerned with the study of the regularity of the map $t \mapsto \mu_t$. More precisely, one looks at $\mu_t$ as a Radon measure (or a distribution), i.e. one considers the map $R_\phi(t) := \int_X \phi d\mu_t$ for a continuous (or smooth) observable $\phi : X \to \mathbb{R}$. When the map $R_\phi$ is continuous at $t = t_0$, one says that statistical stability holds. When $R_\phi$ is Hölder-continuous at $t = t_0$, one says that fractional response holds, and when it is differentiable, one says that linear response holds.

For instance, when $(f_t)_{t \in (t_0 - \epsilon, t_0 + \epsilon)}$ is a $C^2$ family of $C^3$ uniformly expanding maps of the circle (i.e. $\inf_{x \in \mathbb{S}^1} |f'_t(x)| \geq \lambda > 1$), linear response is known to hold $[3,22]$: in this setting, the physical measure $d\mu_t = \rho_t \cdot dm$ is absolutely continuous w.r.t. Lebesgue measure $m$, with invariant density $\rho_t \in C^2(\mathbb{S}^1)$, and if one has $[\partial_t f_t]_{t = t_0} = X_{t_0} \circ f_{t_0}$ for some $C^1$ $X_{t_0} : \mathbb{S}^1 \to \mathbb{R}$, one may write a linear response formula

\begin{equation}
\frac{dR_\phi}{dt}(t_0) = \sum_{n=0}^{\infty} \int_X \phi \circ f^n_{t_0} \cdot (X_{t_0}\rho_{t_0})' dm.
\end{equation}

This formula lead to introduce the Ruelle susceptibility function, defined for a $C^1$ observable $\phi$ as the formal series

\begin{equation}
\Psi_\phi(z) := \sum_{n=0}^{\infty} z^n \int_X \phi \circ f^n_{t_0} \cdot (X_{t_0}\rho_{t_0})' dm.
\end{equation}
For a smooth family of uniformly expanding maps of the circle, it is easy to show that the previous formal series defines a holomorphic function, in a disk of radius larger than one: in particular, $\Psi(1)$, which is the R.H.S in (1), is well-defined.

1.1. **Logistic family: a quick overview.** The hope was that the susceptibility function would allow to study (linear) response in more general situations, most notably for the logistic family

$$f_t(x) := tx(1-x),$$

$t \in (0,4]$, defined on $X = I = [0,1]$, with a (unique) critical point at $c = 1/2$; we will denote the post-critical orbit $c_{k,t} = f^k_t(c)$, $k \geq 1$. As this family of systems, or closely related ones, are the central focus of the present paper, we recall some essential notions and results:

- If $f_t$ admits an attracting periodic orbit, we say that $t$ is a *regular* parameter. The set $\mathcal{R}$ of regular parameters is open and dense [14].
- If $f_t$ admits an absolutely continuous invariant probability measure, $t$ is then called a *stochastic* parameter.

Let us denote by $\mathcal{S}$ the set of stochastic parameters, on which we will focus from now on. Collet and Eckmann [11] famously introduced the following condition: the map $f_t$ (or the parameter $t$) satisfies the Collet-Eckmann condition if there is $\lambda_c > 1$ and $H_0 > 0$ such that

$$|(f^n_t)'(c_{1,t})| \geq \lambda^n_c \text{ for } n \geq H_0,$$

i.e. one requires the Lyapunov exponent computed along the post-critical orbit to be positive. We will call them Collet-Eckmann, or (CE), parameters. This condition implies the existence of a unique ergodic, absolutely continuous invariant measure $\mu_t$: in particular, parameters $t$ satisfying (4) are in $\mathcal{S}$.

Collet-Eckmann parameters are known to be *abundant*, i.e. to form a positive Lebesgue measure subset: this was shown first by Jakobson [15]. Benedicks and Carleson gave another proof of that result, based on the following exponential recurrence condition: $t$ is said to be exponentially recurrent of order $a$, or to satisfy the Benedicks-Carleson condition, if it is (CE) and there exist $0 < a < \log(\lambda_c)/4$, $H_0 \in \mathbb{N}$ such that

$$|f^n_t(c) - c| \geq Ce^{-an} \text{ for } n \geq H_0.$$

We will say that $t$ is (ER)$_a$ if it satisfies (5). Benedicks and Carleson [9, 10] (see also Rychlik [20]) then proved that there is a set $\Omega_{BC}$ of positive Lebesgue measure, such that any $t \in \Omega_{BC}$ satisfies (CE) and (ER)$_a$ for some $a > 0$.

Closer to the focus of the present paper, Tsuji [25] proved a similar, but stronger result: he showed the existence of a positive Lebesgue measure set of transversal Collet-Eckmann, subexponentially recurrent stochastic parameters. More precisely, [25, Theorem A] establishes that, if $t_0$ is a (CE) parameter satisfying the *transversality* condition\(^1\)

$$\mathcal{J}(t_0) := \sum_{k=0}^{\infty} \left[ \frac{\partial_t f_t(c_{k,t})}{(f^k_t)'(c_{1,t_0})} \right]_{t=t_0} \neq 0,$$

\(^1\)A $t_0$ such that $\mathcal{J}(t_0) = 0$ is called horizontal. For horizontal perturbations of smooth unimodal maps, linear response is known to hold [6].
then there is a positive Lebesgue measure set $\Omega_T = \Omega_T(t_0)$ of parameters $t$ satisfying (CE) and the following Topologically Slow Recurrent (TSR) condition:

$$\lim_{\eta \to 0} \liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \frac{\log(|f'_t(c_{k,t})|)}{|c_{k,t} - c_{k-1,t}|} = 0,$$

and such that for any $t \in \Omega_T$,

$$m(\Omega_T \cap [t - \varepsilon, t + \varepsilon]) = o(\varepsilon^3).$$

In particular, any $t \in \Omega_T$ is a Lebesgue density point, and the (TSR) condition implies (ER)$_a$ for all $a > 0$ (see [6, Prop 5.2]).

Finally, a most important Theorem of Lyubich [18] asserts that a.e. $t$ is either regular or stochastic, so that $f_t$ admits a physical measure Lebesgue almost surely.

1.2. Response in the logistic family: A paradox? The question we want to study in the present paper relates to fractional response for the physical measure $\mu_t$ of the logistic family (3), that is the Hölder regularity of the map $t \mapsto R_\phi(t)$ for suitable observables $\phi$. On the statistical stability front, Thuëmberg [24] showed that, for continuous $\phi$, the map $t \in (0, 4] \mapsto R_\phi(t)$ is discontinuous at every $t \in \Omega_{BC}$. Furthermore, he showed that this same map was discontinuous on any full-measure subset of $(0, 4]$. This suggests to restrict $R_\phi$ domain to suitable subsets (in particular, regularity is to be understood in the sense of Whitney [27]). In this perspective, Freitas [13] and Tsuji [26] (see also [19]) showed that $t \in \Omega_{BC} \mapsto R_\phi(t)$ (resp. $t \in \Omega_T \mapsto R_\phi(t)$) is continuous.

When looking at higher regularity, seemingly contradictory results coexist: on the one hand, Jiang and Ruelle [16] showed that for a $C^1$ observables $\phi$, and a mixing, Misiurewicz-Thurston parameter $t_0$, the susceptibility function $\Psi_\phi$ (2) is meromorphic on $\mathbb{C}$, with no poles on the unit circle: in particular, $\Psi_\phi(1)$ is well-defined, which suggests that, when restricted to a suitable parameter subset $\Omega$, the limit $\lim_{t \to t_0} \frac{R_\phi(t) - R_\phi(t_0)}{t - t_0}$ exists. On the other hand, Baladi, Benedicks and Schnellmann [4] showed that, if $t$ is mixing, (MT) and transversal, there exist a sequence $(t_n)_{n \geq 0}$ of (MT) parameters, converging to $t$, a $C^1$ observable $\phi$ and a constant $C > 0$ such that

$$C^{-1}|t_n - t|^{1/2} \leq |R_\phi(t_n) - R_\phi(t)| \leq C|t_n - t|^{1/2}.$$

In particular, this entails that $R_\phi$ is no better than $C^{1/2}$ at $t$ !

We note that, before [4] result, the hope raised by [16], namely that linear response holds in a suitable sense for the logistic family, was already diminished by the papers [2, 5], which, in the easier setting of piecewise expanding unimodal maps, exhibited smooth families where linear response fails, and highlighted the importance of horizontality of the perturbation for linear response to hold.

1.3. Fractional susceptibility functions: definitions, conjectures and results.

“Aiming to shed some light on this puzzling state of affairs”, Baladi and Smania [7] (see also [1]) introduced so-called fractional susceptibility functions, whose connection to fractional response is similar to the one between classical susceptibility function and linear response (i.e, the value at 1 of the (fractional) susceptibility function, if it is well-defined, is the (fractional) derivative of $R_\phi$ at $t = t_0$):

\textit{t} is a Misiurewicz-Thurston (MT) parameter if the post-critical orbit is preperiodic to a hyperbolic periodic orbit.
The response fractional susceptibility function:\(^3\)

\[
\Psi_{\phi}^{\text{rp}}(\eta, z) := -\sum_{j=0}^{\infty} z^j \int_{\Omega} \phi \circ f_{t_0}^j(x) M^\eta[\rho_{t_0}](x) dx,
\]

where we denoted \(M^\eta[g]\) the two-sided \(\eta\) Marchaud derivative of a\(^4\) bounded, \(\alpha\)-Hölder \((\alpha > \eta)\) \(g : \mathbb{R} \to \mathbb{R}\), defined as

\[
M^\eta[g] := \frac{\eta}{2\Gamma(1-\eta)} \int_{\mathbb{R}} \frac{g(x+y) - g(x)}{\text{sgn}(y)|y|^{1+\eta}} dy
\]

We refer to, e.g. [23, Section 2] for a list of relevant properties of Marchaud derivatives in our setting, to [12] for a quick introduction and to [21] for a full treatise on the subject.

The frozen fractional susceptibility function:

\[
\Psi_{\phi}^{\text{f}}(\eta, z) := \sum_{j=0}^{\infty} z^j \int_{\Omega} \phi \circ f_{t_0}^j(x) M^\eta_{\ell}[\mathcal{L}_t \rho_{t_0}]_{t=t_0} dx,
\]

where we denoted \(M^\eta_{\ell}\) the two-sided Marchaud derivative w.r.t the parameter \(t\).

To define a fractional susceptibility function in the spirit of [1], more care is needed. Indeed, as the invariant density \(\rho_t\) is not defined for every parameter \(t\), one has to consider integrals over some suitable positive measure subset \(\Omega \subset S\). We set

\[
\Psi_{\phi}^{\text{f}}(\eta, z) := \frac{\eta}{2\Gamma(1-\eta)} \sum_{j=0}^{\infty} z^j \int_{\Omega} \int_{\Omega-\ell_{-t_0}} \phi \circ f_{t_0}^j \frac{(\mathcal{L}_{t_0+t} - \mathcal{L}_{t_0}) \rho_{t_0}}{|t|^{1+\eta}} \text{sgn}(t) dt dx.
\]

We will call this the true fractional susceptibility function. As an intermediary object between the frozen and true fractional susceptibility, one may introduce the semifreddo susceptibility function (see [7, Section 7.2]), defined by

\[
\Psi_{\phi}^{\text{sf}}(\eta, z) := \frac{\eta}{2\Gamma(1-\eta)} \sum_{j=0}^{\infty} z^j \int_{\Omega} \int_{\Omega-\ell_{-t_0}} \phi \circ f_{t_0}^j \frac{(\mathcal{L}_{t_0+t} - \mathcal{L}_{t_0}) \rho_{t_0}}{|t|^{1+\eta}} \text{sgn}(t) dt dx.
\]

We also recall the notion of Whitney-Marchaud fractional derivative [7, §7.2], defined for \(0 \leq \eta < 1\), \(\Omega \subset \mathbb{R}\) of positive measure, \(x \in \Omega\) a Lebesgue density point, and \(g \in C^\alpha(\Omega, \mathbb{R})\), \(\alpha > \eta\) by

\[
M^\eta_{\Omega}[g](x) := \frac{\eta}{2\Gamma(1-\eta)} \int_{\Omega-x} \frac{g(x+y) - g(x)}{\text{sgn}(y)|y|^{1+\eta}} dy.
\]

For the quadratic family, Baladi and Smania [7, Conjecture A] formulated a set of conjectures for those fractional susceptibility functions associated with compactly supported \(C^1\) observables \(\phi\):

- For (fixed) \(0 < \eta < 1/2\), holomorphy in a disk \(D_\eta\) of radius greater than one and a fractional response formula.
- At \(\eta = 1/2\), decomposition as a sum of meromorphic functions, related to the presence of poles on the unit circle.
- For \(1/2 < \eta < 1\), holomorphy in a disk of radius smaller than one.

\(^{3}\text{In [7, Def 2.3], another definition of the response fractional susceptibility is given. We note that the two definitions coincide for } 0 \leq \eta < 1/2 \text{ and } \phi \in C^1 \text{ compactly supported (see also [7, Lemma 5.2])}\)

\(^{4}\text{The (Whitney-)Marchaud derivative may be defined for larger classes of functions, see [21, §5.4]}\)
We refer to the discussion in [7, §1.1] to see how those conjectures help solving the paradox. In particular, [7, Thm. C] establishes the second item, existence of the decomposition at the threshold value $\eta = \frac{1}{2}$, for the response and frozen fractional susceptibilities.

In the easier setting of piecewise expanding unimodal maps, the invariant density $\rho_t$ is defined for every $t$ in some small interval around $t_0$, so that one may take $\Omega = (t_0 - \varepsilon, t_0 + \varepsilon)$ in (12). Replacing $1/2$ by $1$, [1] established holomorphy of the true fractional susceptibility function, as well as a fractional response formula.

1.4. Contributions of the present paper. In [23, Theorem 13] we established, for mixing Misiurewicz parameters, holomorphy of (9), (11) and (13) in a disk of radius greater than one, for $0 < \eta < \frac{1}{2}$. In the present paper, we extend those results to the true fractional susceptibility function: in Theorem 1, we construct a positive measure set of Collet-Eckmann and subexponentially recurrent parameters $\Omega \subset S$, and an $\varepsilon > 0$, such that for any $t_0 \in \Omega$, setting $\Omega(t_0) = \Omega \cap (t_0 - \varepsilon, t_0 + \varepsilon)$ the fractional susceptibility function $\Psi^{(t_0)}(\eta, \cdot)$ is holomorphic on a disk $D_\eta$ of radius larger than one, for $0 < \eta < \frac{1}{2}$. Furthermore, a fractional response formula (20) holds: this is the content of Theorem 5.

To relate the value at 1 of the fractional susceptibility function, and regularity properties of the response function $R_\phi$ (see Corollary 6), it is of paramount importance that $\Omega$ satisfies more than just having positive Lebesgue measure: some form of quantitative estimate on Lebesgue density (see (8)) seems necessary. This is why the construction of our parameter set $\Omega$ rely on previous work by Tsuchii [25].

The proof of Theorem 5 relies on Theorem 4, that gives uniform (exponential) mixing rates and renormalization periods for parameters $t \in \Omega(t_0)$, a result of independent interest. At the heart of the proof of Theorem 4 lies a delicate tower construction, that we essentially borrowed from [6, 4] (see also [8] for a first occurrence). Most of the work consists in checking that, in our transversal setting, one can replace the polynomially decaying towers and associated estimates from [4] with the exponentially decaying ones from [6]: this requires a special attention to uniformity of the constants w.r.t. certain dynamical quantities (called goodness, see Definition 3) appearing in the various estimates (see e.g. Lemma 7 or Lemma 14). Nevertheless, we improve on the previous constructions, e.g. by remarking (see Remark 2) that a uniform choice of $\varepsilon$ in Theorem 1 entails that the renormalization period is constant over our parameter set $\Omega(t_0)$, or that the decomposition (57) in Corollary 18, very close to [6, Prop. 2.1], implies a new regularity result on the invariant density for (TSR) parameters (namely, that it is in Sobolev spaces $H^s_p$ for $0 \leq s < 1/2$, $1 < p < \frac{1}{1/2+s}$).

We note that the results we obtained are not exactly those conjectures in [7, Conjecture A]: indeed, we do not construct a set of good parameters $\Omega(t)$ for any mixing and (TSR) $t$. Instead, we first construct the set $\Omega$ of parameters with the required properties, and then take parameters $t \in \Omega$. This forces us to consider parameters with non trivial renormalization period, i.e. non-mixing.

2. Setting and statement of main results

We consider a $C^2$ family $(f_t)_{t \in \Delta}$, where $\Delta \subset \mathbb{R}$ is a closed interval, each $f_t : I \cap$ is a $C^4$ map on the interval $I$, having negative Schwarzian derivative and a non-degenerate critical point of quadratic type $c \in \hat{I}$ (i.e. $f'_t(c) = 0$ and $f''_t(c) < 0$), satisfying, at every
These are constants hence (up to reducing it), one may choose $\varepsilon$.  

Remark (18) Assume that there is a transversal Collet-Eckmann parameter $t_1 \in \Delta$. There are constants $\lambda_c > 1$, $H_0 \geq 1$, and a set $\Omega = \Omega(\lambda_c, H_0) \subset \Delta$ such that for any $a > 0$, any $t \in \Omega$, $t$ is $(\lambda_c, H_0)$ Collet-Eckmann and exponentially recurrent of order $a > 0$.  

Furthermore, for any $t \in \Omega$, (8) hold for some $1 < \beta < 2$ (in particular, any $t \in \Omega$ is a Lebesgue density point).  

Finally, for any $\delta > 0$, any $t_0 \in \Omega$, there is $\varepsilon = \varepsilon(\delta, t_0) > 0$, s.t. there are constants $C > 0$, $0 < \rho < 1$, s.t for any $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$  

\begin{align}
(17) & \quad |(f_1^n)'(x)| \geq C_0 \rho^n \forall x \text{ s.t } f_1^j(x) \notin (-\delta, \delta), \forall 0 \leq j < n \\
(18) & \quad |(f_1^n)'(x)| \geq C_0 \rho^n \forall x \text{ s.t } |f_1^j(x)| \geq \delta, \forall 0 \leq j < n, |f_1^n(x)| \leq \delta.
\end{align}

The next Remark will be important later on:  

Remark 2. The last part of the Theorem is in fact [25, Lemma 5.1]. We remark that the $\varepsilon(\delta)$ given there depends a priori also on the parameter $t_0$. Going over the proof of [25, Lemma 5.1], one sees that this $\varepsilon(t_0, \delta)$ is constructed as the size of the maximal interval $[t_0, t_0 + \varepsilon(t_0, \delta)]$ for which certain properties $(A_i)$ and $(B_i)$ hold. Those properties hold on open sets by continuity of a certain application, which is in fact uniformly continuous: it follows that the size of the interval on which those properties hold is independent of $t_0$, hence (up to reducing it), one may choose $\varepsilon(t_0, \delta) = \varepsilon(\delta)$ to be independent of $t_0$.

We show Theorem 1 in Section 4. This result leads us to introduce the following notion:  

Definition 3. We say that the map $f_1$ (or the parameter $t$) is good w.r.t. $\lambda_c > 1$, $H_0 \geq 1$, $a \geq 0$, $\rho > 1$ and $C_0 > 0$ if it satisfied the $(\lambda_c, H_0)$ CE condition (4), if it is exponentially recurrent of order $a$ (5), and if the expansions (17), (18) are satisfied for $\rho$ and $C_0$.

Let $t_0 \in \Omega$, good for $(\lambda_c, H_0, a, C_0, \rho)$ and set $\Omega(t_0) := \Omega \cap (t_0 - \varepsilon, t_0 + \varepsilon)$, where $\varepsilon > 0$ is given by the second part of Theorem 1. Then $\Omega(t_0)$ is a set of good parameters for $(\lambda_c, H_0, a, C_0, \rho)$, that satisfies (8).
We can now state our second main result: uniformity of renormalization periods and rates of mixing over $\Omega(t_0)$.

**Theorem 4.** Let $t_0 \in \Omega$, good for $(\lambda, h_0, s, C_0, \rho)$, and consider $\Omega(t_0)$. Then all $t \in \Omega(t_0)$ have constant renormalization period $P_t \equiv P_0$, where $P_0$ is the renormalization period of $t_0$. Furthermore, there exists $\kappa > 1$ and $C > 0$ such that for any $t \in \Omega(t_0)$, any $L^\infty$ observable $\phi$ and $H^s_p$ ($s > 0$, $p > 1$) observable $\psi$, one has:

\[
\left| \int \phi L_t^k \psi \, dm - \int \phi_0 \psi \, dm \right| \leq C\kappa^{-k} \| \phi \|_{L^\infty} \| \psi \|_{H^s_p}
\]

Theorem 4 is a simple consequence of Lemma 21 and the mollification trick from [7, Lemma 5.3] or [23, Lemma 14] to go from $W^{1,1}$ to $H^s_p$ observables.

We now state our last main Theorem: holomorphy of the fractional susceptibility function on a certain disk of radius larger than one, as well as a fractional response formula.

**Theorem 5.** Let $t_0 \in \Omega$, and consider $\Omega(t_0)$. Let $\phi : I \to \mathbb{R}$ be a $C^1$ function.

- For any $0 \leq \eta < 1/2$, the fractional susceptibility function $\Psi^\Omega(t_0)(\eta, \cdot)$ is holomorphic on a disk $D_\eta$ of radius greater than one.
- For any $0 \leq \eta < 1/2$, the fractional response formula holds:

\[
M^\eta(\Omega(t_0))(R_{\phi})(t_0) = \Psi^\Omega(t_0)(\eta, 1),
\]

where $M^\eta(\Omega(t_0))$ is the Whitney-marchaud derivative (14).

The density estimate (8) for our parameter set $\Omega$ allows us to apply [7, Prop. F], which gives the following corollary (see the discussion in [7, §1.1]):

**Corollary 6.** In the setting of Theorem 5, we have

\[
\lim_{\zeta \to \eta} \frac{\Psi^\Omega(t_0)(\zeta, 1)}{\Gamma(\eta - \zeta)} = \frac{1}{\Gamma(1 - \eta)} \lim_{t_0 \to t_0} \frac{R_{\phi}(t + t_0) - R_{\phi}(t_0)}{\text{sgn}(t)|t|^\eta}.
\]

In particular, both limits are well-defined.

### 3. Proof of Theorem 5

Let $t_0 \in \Omega$, and consider $t \in \Delta$ such that $t + t_0 \in \Omega(t_0)$. As the renormalization period is constant over $\Omega(t_0)$, we may always assume that it is equal to one\(^5\), i.e. that $t_0$ is mixing. Let $\phi \in L^\infty$, and consider the following (a priori improper) integral

\[
\int_I \phi L_t^{j+t_0} \left( \frac{(L_{t+t_0} - L_{t_0}) \rho_0}{|t|^{1+\eta}} \right) \, dm.
\]

By Corollary 18, $\rho_0 \in H^s_p(I)$ for $0 \leq s < 1/2$ and $1 < p < \frac{1}{1/2+s}$. Arguing as in [23, Proof of Theorem 13], we see that $(L_{t+t_0} - L_{t_0}) \rho_0 \in H^s_p(I)$ for the same range of $s$, $p$, and that

\[
\| (L_{t+t_0} - L_{t_0}) \rho_0 \|_{H^s_p} \leq C|t|^{s-\delta} \| \rho_0 \|_{H^s_p}
\]

\(^5\)In particular, if one $t \in \Omega(t_0)$ is mixing then all $t \in \Omega(t_0)$ are.

\(^6\)If the renormalization period $P_0 \neq 1$ on $\Omega(t_0)$, write $j = \ell P_0 + r$ for $\ell \in \mathbb{N}$ and apply the argument to $\phi \circ f_{t+t_0}^\ell$ and $L_{t+t_0}^{\ell}$. 

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for $0 \leq \tilde{s} < s$. Let us choose $0 \leq \eta < s - \tilde{s}$. Applying (19) and (22) we get
\[
\left| \int_I \phi L^i_{t+t_0} \left( \frac{(L_{t+t_0} - L_{t_0}) \rho_{t_0}}{|t|^{1+\eta}} \right) dm \right| \leq C \kappa^{-j} \| \phi \|_{L^\infty} \left\| \frac{(L_{t+t_0} - L_{t_0}) \rho_{t_0}}{|t|^{1+\eta}} \right\|_{H^\delta_p} \\
\leq C \kappa^{-j} \| \phi \|_{L^\infty} |t|^{s-\tilde{s}-1-\eta} \| \rho_{t_0} \|_{H^\delta_p}.
\]

As this last function of $t$ is integrable at 0, we get
\[
\left| \int_{\Omega(t_0)-t_0} \int_I \phi L^i_{t+t_0} \left( \frac{(L_{t+t_0} - L_{t_0}) \rho_{t_0}}{|t|^{1+\eta}} \right) dm \operatorname{sgn}(t) dt \right| \\
\leq C \kappa^{-j} \| \phi \|_{L^\infty} \| \rho_{t_0} \|_{H^\delta_p} \int_{(-\varepsilon,\varepsilon)} |t|^{s-\tilde{s}-1-\eta} dt \\
\leq C \kappa^{-j} \| \phi \|_{L^\infty} \| \rho_{t_0} \|_{H^\delta_p}.
\]

From this bound, we obtain in particular that the integral on the LHS is well-defined. Since this integral is also the coefficient of $z^j$ in the formal series (12), it follows that, for any $0 \leq \eta < 1/2$, $\Psi^{\Omega(t_0)}_{\phi,t_0}(\eta, \cdot)$ is a holomorphic function on a disk $D_\eta$ of radius larger than $\kappa > 1$: this is the first part of Theorem 5.

To prove the second part, we start by remarking that, by the proof of the first part, $\Psi^{\Omega(t_0)}_{\phi,t_0}(\eta, 1)$ is well-defined, and that we may use Fubini to permute the integrals and sums appearing in its definition however we please. Hence, denoting $C_\eta = \frac{n}{2(1-\eta)}$, we may write,
\[
\Psi^{\Omega(t_0)}_{\phi,t_0}(\eta, 1) = C_\eta \sum_{j=0}^\infty \int_{\Omega(t_0)-t_0} \int_I \phi L^i_{t+t_0} \left( \frac{(L_{t+t_0} - L_{t_0}) \rho_{t_0}}{|t|^{1+\eta}} \right) dm \operatorname{sgn}(t) dt \\
= C_\eta \int_{\Omega(t_0)-t_0} \int_I \phi (I - L_{t+t_0})^{-1} \left( \frac{(I - L_{t_0})(\rho_{t+t_0} - \rho_{t_0})}{|t|^{1+\eta}} \right) dm \operatorname{sgn}(t) dt \\
= C_\eta \int_{\Omega(t_0)-t_0} \left( \int_I \phi (\rho_{t+t_0} - \rho_{t_0}) dm \right) \frac{\operatorname{sgn}(t)}{|t|^{1+\eta}} dt \\
= C_\eta \int_{\Omega(t_0)-t_0} (R_{\phi}(t + t_0) - R_{\phi}(t_0)) \frac{\operatorname{sgn}(t)}{|t|^{1+\eta}} dt \\
= [M^{\eta,\Omega(t_0)} R_{\phi}](t_0),
\]
where we used the identity $(I - L_{t+t_0})(\rho_{t+t_0} - \rho_{t_0}) = (L_{t+t_0} - L_{t_0}) \rho_{t_0}$. This proves (20), and concludes the proof of Theorem 5.

### 4. Building the parameter set: proof of Theorem 1

Let us start by recalling Tsuji’s main result in [25] in our setting: starting from a transversal Collet-Eckmann parameter $t_1$, Tsuji constructs a set $\Omega_T \subset \Delta$ of positive Lebesgue measure such that:

- For any $t \in \Omega_T$, the density estimate (8) holds.
- For $t \in \Omega_T$, $f_t$ is (CE) and (TSR).

Since the (TSR) condition implies Benedicks-Carleson exponential recurrence of any order [6, Prop. 5.2], Tsuji’s result almost gives us what we are after: what is lacking is the uniform control over the constants appearing in (CE), (ER), (17) and (18).
Proof of Theorem 1. Consider the set $\Omega_T$ given by [25, Theorem A]. Any $t \in \Omega_T$ satisfies CE for some $(\lambda_c, H_0)$ and ER for any $\alpha > 0$ and is a Lebesgue density point of $\Omega$. For $j, H, \ell \geq 1$, let us set

$$\Omega_{j, H, \ell} := \{ t \in \Omega, |c_{k,t} - c| \geq e^{- j^{-1} k}, |(f_{t}^{k})'(c_{1,t})| \geq e^{k/\ell}, \forall k \geq H \}.$$ 

Up to keeping only the Lebesgue density points in $\Omega_{j, H, \ell}$, it is easy to see that, modulo a zero-measure set, $\Omega = \bigcap_{j \geq 1} \bigcup_{H \geq 1} \bigcup_{\ell \geq 1} \Omega_{j, H, \ell}$. In particular, the Lebesgue dominated convergence Theorem entails that for any $j \geq 1$, any large enough $H$ and $\ell$, $\Omega_{j, H, \ell}$ satisfies (8) for some $\beta > 1$. Let us fix some $H_0, \ell_0$ for which this last property is satisfied, and set

$$\Omega := \bigcap_{j \geq 1} \Omega_{j, H_0, \ell_0}.$$ 

By construction, any $t \in \Omega$ is good w.r.t. $1/\ell_0, H_0$ and $1/j$ for any $j \geq 1$.

The previous argument gives us uniformity of the Collet-Eckmann and exponential recurrence constants over $\Omega$. To establish uniformity of all $t \in \Omega$ w.r.t. (the same) $\rho > 1$, $C_0 > 0$, we may argue exactly as in [4, Proof of Prop 2.1], applying [25, Lemma 5.1]. □

5. Building the tower extension

Recall the set $\Omega$ constructed in Theorem 1. Our goal in this section is to build, for $t \in \Omega$, a tower extension $\hat{f}_t : \hat{I}_t \circ$ over $f_t$ such that the associated transfer operator $\hat{L}_t$ has uniform in $t$ spectral properties on a Banach space $B_t$. We will deduce from this construction Theorem 4, i.e. uniform mixing rates and renormalization periods for $f_t$, $t \in \Omega$.

5.1. Definitions and preliminary estimates. We now give the construction of the tower extension of $f_t$, inspired by [4, 6]. Consider $t$ good for $(\lambda_c, H_0, \alpha, \rho, C_0)$, and fix two constants $b, L$ such that

$$\frac{3}{2} a < b < 2a \quad \text{and} \quad L > 1.$$ 

Define the tower $\hat{I}_t$ as $\hat{I}_t = \bigcup_{k \geq 0} E_{k,t}$, where $E_{k,t} = B_{k,t} \times \{k\}$, and satisfies:

- $B_{0,t} = B_0 = I$.
- For $k \geq 1$, the interval $B_{k,t}$ is centered at $c_{k,t}$ and such that

$$[c_{k,t} - e^{-bk}/L^3, c_{k,t} + e^{-bk}/L^3] \subset B_{k,t} \subset [c_{k,t} - e^{-bk}/L, c_{k,t} + e^{-bk}/L].$$

Observe that for $k \geq H_0$, $c \notin B_{k,t}$ given our choice of $b$. Given $\lambda_c > 1$ and $H_0 \geq 1$, we will assume that $\delta > 0$ is so small that we may find a $C_0 > 0$ such that, for any $|x - c| \leq \delta$, any $(\lambda_c, H_0)$ Collet-Eckmann map $f_t$,

$$\inf_{1 \leq j \leq H_0} |f_t^j(x) - c| \geq C_0$$

For $(x, k) \in E_{k,t}$, we define the tower map $\hat{f}_t$ by

$$\hat{f}_t(x, k) := \begin{cases} (f_t(x), k + 1) \text{ if } k \geq 1 \text{ and } f_t(x) \in B_{k+1,t} \\ (f_t(x), k + 1) \text{ if } k = 0 \text{ and } x \in [c - \alpha, c + \alpha] \\ (f_t(x), 0) \text{ otherwise.} \end{cases}$$

If $\pi : \hat{I}_t \rightarrow I, \pi(x, k) = x$, then $\pi \circ \hat{f}_t = f_t \circ \pi$. Define $H(\delta) := \min \{k \geq 1, \exists x \in [c - \delta, c + \delta] \text{ such that } f_t^{k+1}(x, 0) \in E_0\}$. 

i.e. $H(\delta)$ represents the minimal number of levels the orbit of a point $x \in [c - \delta, c + \delta]$ has to climb. We will assume\(^7\) that $\delta > 0$ is so small that $H(\delta) \geq \max(2, H_0)$.

Let us decompose $[c - \delta, c + \delta] \setminus \{c\}$ as $\bigcup_{j \geq H(\delta)} I_{j,t}$, where

$$I_{j,t} = I_{j,t}^+ \cup I_{j,t}^-$$

(27) $I_{j,t}^\pm := \{ |x| < \delta, \pm x > 0, \hat{f}_t^j(x,0) \in E_{\ell,t} \forall 0 \leq \ell < j \text{ and } \hat{f}_t^j(x,0) \in E_0 \}$,

i.e. $I_{j,t}$ represents the set of points that climb to level $j - 1$ and fall back to level 0 at the $j$-th step.

Finally, let

$$J_{k,t} := \{ c \} \cup \bigcup_{j \geq k+1} I_{j,t}$$

be the set of $x \in [c - \delta, c + \delta]$ that climb the tower at least $k$ levels. Clearly, $J_{k,t} = [c - \delta, c + \delta]$ if $0 \leq k \leq H(\delta) - 1$.

We can now state our distortion lemmas:

**Lemma 7.** [Lemma 3.3 \cite{6}, Lemma 2.3 \cite{4}] Let $t$ be good for $(\lambda_c, H_0, a, \rho, C_0)$, and assume that (25) holds. There is $C > 1$, depending only on the goodness constants and $b$, such that for every $j \geq 1$, every $k \leq j$, every $x, y \in f_t(J_{j,t})$

$$C^{-1} \leq \left( \frac{|f_t^j(x)|}{|f_t^j(y)|} \right) \leq C. \tag{29}$$

*Proof. For any $1 \leq \ell \leq k \leq j$, take $x_\ell, y_\ell \in f_t^\ell(J_{j,t}) \subset B_{\ell,t}$. Observe that there is a constant $C > 0$ such that $|f_t^\ell(y)| \geq |y - c|/C$ for any $y \in I$ and $t \in \Omega$. One may then write

$$\prod_{\ell=1}^k \left| \frac{f_t^\ell(x_\ell)}{f_t^\ell(y_\ell)} \right| \leq \prod_{\ell=1}^k \left( 1 + \sup_{y_\ell} \left| \frac{f_t^{\ell+1}}{f_t^{\ell}} \right| |x_\ell - y_\ell| \right)$$

$$\leq \prod_{\ell=1}^k \left( 1 + C \sup_{y_\ell} \left| \frac{f_t^{\ell+1}}{f_t^{\ell}} \right| |x_\ell - y_\ell| \right)$$

$$\leq \prod_{\ell=1}^\infty \left( 1 + C e^{-b\ell} \right) < +\infty,$$

where we used $|x_\ell - y_\ell| \leq 2L^{-1} e^{-b\ell}$ and $|y_\ell - c| \geq e^{-a\ell} - e^{-b\ell} \geq C e^{-b\ell}$ for $\ell > H_0$. If $1 \leq \ell \leq H_0$, we use (25). Taking $x_\ell = f_t^\ell(x)$ and $y_\ell = f_t^\ell(y)$ gives the upper bound, reversing the choice gives the lower bound in (29). \qed \]

**Lemma 8.** Let $t$ be good w.r.t. $(\lambda_c, H_0, a, \rho, C_0)$. Then there is a constant $C$, depending only on the goodness parameters and $b$, such that:

- For all $j \geq 1$, $x \in J_{j-1,t}$, one has

$$|x - c| \leq C |B_{j-1,t}|^{1/2} |(f_t^{j-2})'(c_{1,t})|^{-1/2}. \tag{30}$$

- For all $x \in I_{j,t}$, $j \geq H(\delta)$, one has

$$|f_t^j(x)| \geq C^{-1} |(f_t^{j-1})'(c_{1,t})|^{-1/2} |f_t^j(x) - c_{j,t}|^{1/2} \geq C^{-1} |(f_t^{j-1})'(c_{1,t})|^{-1/2} L^{-3/2} e^{-jb/2} \tag{31}$$

$$|(f_t^j)'(x)| \geq C^{-1} |(f_t^{j-1})'(c_{1,t})| |f_t^j(x) - c_{j,t}|^{1/2} \geq C^{-1} |(f_t^{j-1})'(c_{1,t})|^{3/2} L^{-3/2} e^{-jb/2} \tag{32}$$

\(^7\)See the discussion in \cite[p.12]{4} for why its possible to assume so.
Proof. Let \( x \in J_{j-1,t}, j \geq 1 \). The mean value theorem applied to \( f_i^{j-2} \) entails that there exists a \( f_i(y) \in [f_i(x), f_i(c)] \) such that
\[
|(f_i^{j-2})'(f_i(y))| |f_i(x) - f_i(c)| = |f_i^{j-1}(x) - f_i^{j-1}(c)| \leq |B_{j-1,t}|
\]
where we used that \( f_i^{j-1}(J_{j-1,t}) \subset B_{j-1,t} \). Using Lemma 7, and the fact that \( |f_i(x) - f_i(c)| \geq C^{-1}|x - c|^2 \) for all \( x \in I \) and \( t \in \Omega \), we get (30).

Next, we let \( x \in I_{j,t} \) and apply the mean value theorem to \( f_i^{j-1} \) on the interval \([f_i(x), f_i(c)]\) to get a \( f_i(y) \) such that
\[
|(f_i^{j-1})'(f_i(y))| |f_i(x) - f_i(c)| = |f_i^{j-1}(x) - c_{j,t}| \geq 2L^{-3}e^{-jb},
\]
since \( f_i^{j}(x) \notin B_{j,t} \). Together with \( |f_i(x) - f_i(c)| \leq C|x - c|^2 \) and Lemma 7, we get
\[
|x - c| \geq C^{-1} \frac{|f_i^{j}(x) - c_{j,t}|^{1/2}}{|(f_i^{j-1})'(f_i(y))|^{1/2}} \geq C^{-1}|(f_i^{j-1})'(c_{1,t})|^{-1/2}L^{-3/2}e^{-jb/2},
\]
using again that \( f_i^{j}(x) \notin B_{j,t} \). This yields the first bound in (31).

For the second bound, we decompose \( |(f_i^{j})'(x)| = |(f_i^{j-1})'(f_i(x))||f_i^{j}(x)| \) and apply Lemma 7 to get
\[
|(f_i^{j-1})'(f_i(x))| \geq C^{-1}|(f_i^{j-1})'(c_{1,t})|,
\]
and we conclude by the first bound in (31). \( \square \)

Remark 9. We can reverse the inequalities in the previous proof to obtain
\[
|(f_i^{j})'(x)| \leq C|(f_i^{j-1})'(c_{1,t})(f_i^{j}(x) - c_{j,t})|^{1/2},
\]
with \( C \) depending only on the goodness.

Notice also that both the first inequalities in (31), (32) and the previous inequality hold for \( x \in J_{j,t} \).

As an easy corollary to (30), we obtain the exponential decay of the size of the interval \( I_{k,t} \) and \( J_{k,t} \).

Corollary 10. There exists a \( C > 0 \), depending only on the goodness constants and on \( L \), such that for any \( j \geq H_0 \)
\[
|I_{k+1,t}| \leq |J_{k,t}| \leq Ce^{-kb/2}|(f_i^{k-1})'(c_{1,t})|^{-1/2}
\]

The following lemma is also an important result: as it is very close to [6, Prop. 3.7] (see also [4, Prop. 2.4]), we omit its proof.

Lemma 11. Let \( t \) be good for \((\lambda_c, H_0, a, \rho, C_0)\). There exists \( C > 0 \), depending only on the goodness constants, such that for every \( j \geq 0 \)
\[
\sum_{k \geq j+1} \frac{1}{|(f_i^{k-1})'(f_i^{k}(c_{1,t}))|} \leq Ce^{aj}.
\]

Let us introduce, for \( k \geq 1 \), the interval \( U_{k,t,+} \) (resp. \( U_{k,t,-} \)) which is the monotonicity interval of \( f_i^k \) containing \( c \) and located to its right (resp. to the left of \( c \)). We then set
\[
f_{1,t,\pm}^k := (f_i^k |_{U_{k,t,\pm}})^{-1}.
\]
We have the following lemma, drawn from [6, Lemma 4.1] or [4, Lemma 2.5].
Lemma 12. Let \( t \) be good. There is a constant \( C \), depending only on the goodness constants and \( L \), such that for all \( j \geq H(\delta) \), all \( x \in f_t^j(I_{j,t}) \) and \( \sigma = \pm \)

\[
\left| \frac{\partial_x}{(f_t^j)'(f_{-\sigma}^j(x))} \right| \leq C \frac{e^{5aj}}{|(f_t^j)'(f_{-\sigma}^{-j})(c_{1,t})|^{1/2}}
\]

Proof. We only consider the case \( \sigma = + \), the other case being essentially identical. First, we remark that, as noted in the proof of Lemma 7, for \( y \in J_{k,t} \)

\[
|f_t'(f_t^j(y))| \geq C^{-1}e^{-nj}, \quad \forall 1 \leq j \leq k,
\]

where \( C \) depends on the goodness and \( b \).

Next, by Lemma 7 and 11, one gets a \( C > 0 \) such that, for all \( y \in J_{k,t} \), all \( 1 \leq j \leq k - 1 \)

\[
\sup_{y \in J_{k,t}} \frac{1}{|(f_t^j)'(y)|} \leq C \sum_{\ell=1}^{\infty} \frac{1}{|(f_t^{\ell})'(f_t^{\ell-1})(c_{1,t})|} \leq Ce^{3aj}.
\]

Combining the last two inequalities gives

\[
\sup_{y \in J_{k,t}} \frac{1}{|(f_t^j)'(y)|} \leq Ce^{2ak}
\]

By (31),

\[
\sup_{y \in J_{k,t}} \frac{1}{|(f_t^j)'(y)|} \leq Ce^{k/b}|(f_t^{k-1})(c_{1,t})|^{-1/2}.
\]

Finally, if \( x \in f_t^k(I_{k,t}) \), setting \( y = f_t^{-k}(x) \), one gets

\[
\partial_x \frac{1}{|(f_t^j)'(f_t^{-k}(x))|} = \frac{1}{|(f_t^j)'(f_t^{-k}(x))|} \cdot \partial_y \frac{1}{|(f_t^j)'(y)|}
\]

The first bound in (37) now follows from the two previous inequalities and the definition of \( b \). \[ \square \]

Remark 13. Reasoning as in the proof of [6, Lemma 4.1] or [4, Lemma 2.5], we can also show

\[
\left| \frac{\partial^2_x}{(f_t^j)'(f_{-\sigma}^j(x))} \right| \leq C \frac{e^{5aj}}{|(f_t^j)'(f_{-\sigma}^{-j})(c_{1,t})|^{1/2}},
\]

with \( C \) depending only on the goodness constants and \( L \), and for all \( j \geq H(\delta) \), all \( x \in f_t^j(I_{j,t}) \) and \( \sigma = \pm \).

The next lemma, essentially taken from [4, Lemma 2.6] gives us uniformity of goodness and distortion constants for maps \( f_s \) close to a given good map \( f_{t_0} \). For simplicity we assume \( t_0 = 0 \).

Lemma 14. Let \( f = f_{t_0} \) be good for \( (\lambda_c,H_0,a,\rho,C_0) \), and assume that (25) holds. There exist constant \( C \geq 1 \) and \( \varepsilon > 0 \), depending only on the goodness constants and \( b \) such that, for any pair \( (s,M) \), \( M \geq 1 \), \( s \in (-\varepsilon,\varepsilon) \) satisfying

\[
|(f_t^{k-1})'(c_1)| \cdot s \leq e^{-kb}, \quad \forall 1 \leq k \leq M,
\]

the following hold:

- We have, for any \( x \in f(J_{k-1}) \), any \( 1 < k \leq M \),

\[
C^{-1} \leq \left| \frac{(f_s^{k-1})'(x)}{(f_t^{k-1})'(x)} \right| \leq C.
\]
• Furthermore, for any $x \in I_k$, any $1 \leq k \leq M$,

$$| (f^k_s)'(x) | \geq C^{-1} L \cdot 3/2 \cdot e^{-kb/2} |(f^{k-1})'(c_1)|^{1/2} .$$  \hspace{1cm} (44) $$

• For $x \in J_{k-1}$, $1 \leq k \leq M$,

$$| \partial_s f^k_s(x) | \leq C |(f^{k-1})'(c_1)|,$$

and, if the transversality condition $J(0) \neq 0$ (where $J(0)$ is given by (6)) holds, then for sufficiently small $\delta > 0$ we have, for all $x \in J_{k-1}$, $H(\delta) \leq k \leq M$,

$$| \partial_s f^k_s(x) | \geq C^{-1} |J_0 \cdot (f^{k-1})'(c_1)|$$  \hspace{1cm} (46) $$

• Finally, for any $1 \leq k < l \leq M - 1$, we have

$$\left| \frac{(f^k_s)'(c_{1,s})}{(f^l_s)'(c_{1,s})} \right| \leq C e^{ak},$$  \hspace{1cm} (47) $$

Proof. First note that (42) implies $|s| \leq e^{-Mb}$. Let us introduce, for $x \in J_{M-1}$ and $s$ satisfying (42)

$$D_k = D_k(x) := \sup_{t \in [0,s]} |f^k(x) - f^k_t(x)|.$$  

We claim that if $b_0$ satisfies $3a/2 < b_0 < b$,

$$D_k \leq e^{-kb_0}, \forall 1 \leq k \leq M - 1.$$  \hspace{1cm} (48) $$

We give a proof by induction over $k$. Fix $k_0 \leq M - 1$. Up to shrinking $\varepsilon > 0$, we may assume (48) hold for $k \leq k_0$. Assume now that $1 \leq k \leq M - 1$, and that (48) hold for all $i \leq k - 1$. Note that for any $(y,t)$

$$\partial_s f^k_t(y) = \sum_{j=1}^{k} (f^{k-j}_t)(f^j_t(y)) \cdot \partial_s f^j_t(f^{j-1}_t(y)),$$  \hspace{1cm} (49) $$

and that there exists $\tilde{C} > 1$ such that $\max_{y,t} (| \partial_s f^k_t(y) |, | f^j_t(y) |, | \partial_s f^j_t(y) |) \leq \tilde{C}$. It then follows from the mean value theorem that, for any $t \in [0,s]$

$$| f^j_t(f^k_t(x)) - f^j_t(f^k(x)) | \leq \tilde{C} | f^j_t(f^k(x)) - f^j_t(f^k_t(x)) | + | f^j_t(f^k(x)) - f^j_t(f^k_t(x)) |$$

$$\leq \tilde{C} (D_k + |t|).$$

Thus $| f^j_t(f^k_t(x)) | \leq | f^j_t(f^k(x)) | + \tilde{C} (D_k + |t|)$, which gives, together with (49)

$$\left| \frac{\partial_s f^k_t(x)}{(f^{k-1})'(f(x))} \right| \leq \tilde{C} \sum_{j=1}^{k} \frac{1}{(f^{j-1})'(f(x))} \prod_{i=j}^{k-1} \left(1 + \frac{\tilde{C}(|t| + D_i)}{|f^j_t(f^i_t(x))|} \right).$$  \hspace{1cm} (50) $$

As in the proof of Lemma 7, we remark that $f^{i+1}(J_{M-1}) \subset B_{i+1}$, if $i \leq M - 2$, implies that $| f^i(x) - c | \geq e^{-ia} - e^{-ib} \geq C^{-1} e^{-ia}$, for all $i \leq M - 2$. Since the critical point is nondegenerate, this entails $| f^i(f^i(x)) | \geq C^{-1} | f^i(x) - c | \geq C^{-2} e^{-ia}$. Together with the assumptions on $s$ and the induction hypothesis on $D_i$, this yields

$$\frac{\tilde{C}(|t| + D_i)}{|f^j_t(f^i_t(x))|} \leq \tilde{C} e^{ia} (\lambda + D_i) \leq \tilde{C} e^{ia} (e^{-Mb} + e^{-ib_0}) \leq 2\tilde{C} e^{i(a-b_0)}.$$  

By summability of this last term, the product in the RHS of (50) is bounded by a constant $C'$. Thus, via the mean value theorem and Lemma 7 (for $t = 0$), we conclude that

$$D_k \leq \tilde{C} C' C |s| |(f^{k-1})'(c_1)| (\lambda_c - 1)^{-1} \leq \tilde{C} C' C (\lambda_c - 1)^{-1} e^{-kb} \leq e^{-kb_0},$$
up to enlarging $k_0$ (and thus shrinking $\epsilon$ further).

We can now turn to the wanted estimates: Proceeding as in (50), we have

$$
\left| \frac{(f_{s}^{k-1})'(x)}{(f_{s}^{k})'(x)} \right| \leq \prod_{i=0}^{k-2} \left( 1 + \frac{\tilde{C}(D_{i} + |s|)}{|f'(f_{i}(x))|} \right), \quad \forall x \in f(J_{k-1}),
$$

and we obtain as in (50) that this product is uniformly bounded by a $C'$, which gives the upper-bound in (43). We get the lower bound by exchanging the role of $(f_{s}^{k-1})'(x)$ and $(f_{s}^{k})'(x)$.

To obtain the next bound, we start from the decomposition

$$
|(f_{s}^{k})'(x)| = |(f_{s}^{k-1})'(f_{s}(x))| \cdot |f_{s}'(x)| \geq C^{-1}|x - c| \cdot |(f_{s}^{k-1})'(f_{s}(x))|
$$

and we conclude with (33), (43) and Lemma 7.

A consequence of (50) and the subsequent argument is that its RHS is uniformly bounded, i.e. there is a $C' > 0$ such that, for any $x \in J_{k-1}$,

$$
|\partial_{s}f_{s}^{k}(x)| \leq C'|(f_{s}^{k-1})'(f(x))|,
$$

which, together with Lemma 7, easily gives the third bound.

The proof of the next estimate is as in [4, Lemma 2.6]. For the last inequality, we observe that by Lemma 11 (using also (17) and (18)), there is a constant $C > 0$, depending only on the goodness parameters such that, for any $l > k \geq 1$

$$
\left| \frac{(f_{s}^{k})'(c_{1,s})}{(f_{s}^{l})'(c_{1,s})} \right| \leq C e^{ak}
$$

□

The previous result leads us to introduce the notion of admissible pairs: we assume that $t_0 = 0$.

**Definition 15.** [Admissible pairs] Let $C > 1, b, \epsilon > 0$ be given constants. We say that the pair $(M, t)$ with $M \geq 1$ and $t \in (-\epsilon, \epsilon)$ is admissible if

$$
|(f^{k-1})'(c_{1})| \cdot |t| \leq C e^{-kb}
$$

for any $1 \leq k \leq M$.

By Lemma 14, this admissibility condition gives us uniform control over the distortion constants appearing in some estimates. This will play a key role in the next section, allowing us to construct tower extensions such that the associated transfer operator can be compared for different parameters values.

### 5.2. Transfer operators and Banach spaces.

We now turn to the definition of the (tower) transfer operator and the Banach spaces on which it will act.

Start by fixing

$$
1 < \lambda < \min(\epsilon^{3a/4}, \lambda_{c}^{1/2}, \rho^{1/2}).
$$

Recall that, for an interval $I$, and an integer $r \geq 1$, $W^{r,1}(I)$ is the set of $\phi \in L^{1}(I)$ such that $\phi', \phi'', \ldots, \phi^{(r)} \in L^{1}(I)$, endowed with the norm

$$
\|\phi\|_{W^{r,1}} := \max_{k=0,\ldots,r} \|\phi^{(k)}\|_{L^{1}}.
$$
Definition 16. Let $\Psi = (\psi_k)_{k \geq 0}, \psi_k : I \to \mathbb{R}$, be a sequence of $W^{1,1}$ functions, such that: $\text{supp}(\psi_0) \subset \tilde{I}$ and for $k \geq 1$, $\text{supp}(\psi_k) \subset J_{k,t}$.

We set $\|\cdot\|_{B_t}$ to be

$$\|\Psi\|_{B_t} := \sum_{k \geq 0} \|\psi_k\|_{W^{1,1}},$$

and $\|\cdot\|_{B^t_1}$ to be

$$\|\Psi\|_{B^t_1} := \sum_{k \geq 0} \lambda^k \|\psi_k\|_{L^1}$$

for sequences $\Psi = (\psi_k)_{k \geq 0}$ with $\psi_k \in L^1(I)$. We then set $B_t := \{\Psi, \|\Psi\|_{B_t} < +\infty\}$ and $B^t_1 := \{\Psi, \|\Psi\|_{B^t_1} < +\infty\}$.

It is easy to see that $B_t \subset B^t_1$ continuously, and even compactly by Rellich-Kondrachov, uniformly in $t$, i.e.

$$\|\cdot\|_{B^t_1} \leq C\|\cdot\|_{B_t},$$

with a constant $C > 0$ depending only on the goodness of $t$.

Let us also define the projection $\Pi_t(\Psi)$ for $\Psi \in B_t$ by

$$\Pi_t(\Psi) := \sum_{k \geq 0, \sigma = \pm} \lambda^k \frac{|(f_k^t)'(f_{-k}^t(x))|}{\psi_k(f_{-k}^t(x))}$$

We notice that $\Pi_t$ defines a bounded operator from $B_t$ to $L^1(I)$. Indeed, since $\psi_k \circ f_{-k}^t$ is supported in $f_k^t(J_{k,t}) \subset B_{k,t}$, using (38), we get that

$$\int_{f_k^t(J_{k,t})} \lambda^k \frac{|(f_k^t)'(f_{-k}^t(x))|}{\psi_k(f_{-k}^t(x))} dx \leq C e^{-3a/4} \lambda^k \psi_k$$

which gives that $\Pi_t(\Psi) \in L^1(I)$ if $\Psi \in B_t$, since $\lambda < \sqrt{\lambda_c}$.

Furthermore, $\|\Pi_t(\Psi)\|_{L^1} \leq C\|\Psi\|_{B_t}$, where $C$ only depends on the goodness constants of $t$.

Before introducing the transfer operator on our tower, we introduce $C^\infty$ cutoff functions $\xi_{k,t} : I \to [0, 1]$, that will be needed to insure a bounded action of our transfer operator on $B_t$: we choose a $\xi_0$ such that $\text{supp}(\xi_0) = [c-\delta, c+\delta]$ and $\xi_0|_{[c-\delta, c+\delta/2]} \equiv 1$. For $k \geq 1$, we set $\xi_{k,t} \equiv 1$ if $I_{k+2,t} = \emptyset$, and choose $\xi_{k,t} \sigma$ such that

- $\xi_{k,t} \sigma = \psi_k^{-(k+1)}(x)$ on $I_{k+2,t} \cap ([c_{k+1,t} - e^{-b(k+1)/2L^3}, c_{k+1,t} + e^{-b(k+1)/2L^3}]$)
- $\text{supp}(\xi_{k,t}) = J_{k+1,t}$
- For any $j \in \{1, 2, 3\}$, $\|\partial_j^j \xi_{k,t}\|_{\infty} \leq C |J_{k+1,t}|^{-1}$ for some constant $C$ independent on $t$.

We also remark that, when $\xi_{k-1,t} \neq 1$, $I_{k+1,t} \neq \emptyset$ so that $f_{k+1}^t(J_{k+1,t}) \geq e^{-(k+1)b}/2L^3$ (it is both adjacent to the boundary of $B_{k+1,t}$ and contains $c_{k+1,t}$). Using (33), one gets

$$|J_{k+1,t}| \geq C^{-1} L^{-3/2} e^{-(k+1)b/2}|(f_k^t)'(c_{k,t})|^{-1/2}$$

Together with the remark that we may choose our $\xi_{k,t}$ so that $\{x \in \text{supp}(\xi_{k,t}), \xi_{k,t} < 1\} \subset I_{k+2,t}$ (see the beginning of the proof of Lemma 20), we can deduce from the previous estimate, and the definition of $\xi_{k,t}$ that for $j \in \{1, 2, 3\}$

$$\|\xi_{k,t} \circ f_{-k}^t\|_{C^j} \leq C e^{k,j}$$

for some constant $C$ that depends on the goodness, $L$ and $\sup_t |f_t|$ (see also [4, Eq.(63)] and [6, Eq (75)]).
We can now introduce the (tower extended) transfer operator \( \hat{\mathcal{L}}_t \), defined for \( \Psi \in \mathcal{B}_t \) by

\[
(\hat{\mathcal{L}}_t \Psi)_k(x) := \begin{cases} \frac{\xi_{k-1,t}(x)}{\lambda} \psi_{k-1}(x) & \text{if } k \geq 1 \\ \sum_{j \geq 0, \sigma = \pm} \lambda^j (1 - \xi_{j,t}(f_{t,t}^{-j-1}(x))) \psi_j(f_{t,t}^{-j-1}(x)) & \text{if } k = 0 \end{cases}
\]

We have, as in \([6, \text{p.896}]\), the commutation relation \( \Pi_t \circ \hat{\mathcal{L}}_t = \mathcal{L}_t \circ \Pi_t \) on \( \mathcal{B}_t \), where \( \mathcal{L}_t \) is the classical transfer operator associated with \( f_t \).

We also introduce the truncation operator: for any \( M \geq 1, \Psi \in \mathcal{B}_t \), set

\[
T_M(\Psi) := \begin{cases} \psi_k & k \geq M \\ 0 & k > M \end{cases}
\]

It is easy to see that \( \|T_M \Psi\|_{\mathcal{B}_t} \leq \|\Psi\|_{\mathcal{B}_t} \), so that \( T_M \) defines a bounded operator on \( \mathcal{B}_t \).

We may now define the truncated transfer operator, \( \hat{\mathcal{L}}_{t,M} \), as

\[\hat{\mathcal{L}}_{t,M} := T_M \hat{\mathcal{L}}_t T_M.\]

Finally, let us set, for \( x \in I \) and \( k \geq 0 \), \( w(x,k) = \lambda^k \), and define \( \nu \) the measure on \( \bigcup_{k \geq 0} I \times \{k\} \) whose density w.r.t. Lebesgue is \( w \). The next two results give the spectral picture for \( \hat{\mathcal{L}}_t \) and \( \hat{\mathcal{L}}_{t,M} \) on \( \mathcal{B}_t \).

**Proposition 17.** Let \( t \) be good for \( (\lambda_c, H_0, a, C_0, \rho) \). Fix \( \delta > 0 \), \( L > 1 \), \( b > 3a/2 \) and \( \lambda \) as previously. Then:

- \( \hat{\mathcal{L}}_t \) is bounded on \( \mathcal{B}_t \).
- The spectral radius of \( \hat{\mathcal{L}}_t \) on \( \mathcal{B}_t \) is 1, and its essential spectral radius is bounded by some \( \Theta_0^{-1} < 1 \).
- On the unit circle, there is only a finite number \( P_t \) (where \( P_t \) is the renormalization period of \( f_t \)) of simple eigenvalues, \( 1, e^{2\pi i/P_t}, \ldots, e^{2\pi i(P_t - 1)/P_t} \).
- If \( \hat{\mathcal{L}}_t \hat{\rho}_t = \rho_t \) is the unique fixed point of \( \hat{\mathcal{L}}_t \) normalized by \( \nu(\hat{\rho}_t) = 1 \), then \( \hat{\rho}_{t,0} \in W^{2,1} \), with \( \|\hat{\rho}_{t,0}\|_{W^{2,1}} \leq \hat{C} \), with \( \hat{C} \) only depending on the goodness.

As a corollary, we derive a decomposition of the invariant density \( \rho_t \), which is very close to the one obtained in \([6, \text{Prop 2.7}]\).

**Corollary 18.** Let \( \rho_t = \Pi_t(\hat{\rho}_t) \). Then \( \rho_t \) is the invariant density of \( f_t \), and it satisfies

\[
\rho_t = \hat{\rho}_{t,0} + \sum_{j \geq 1, \sigma = \pm} \sum_{\ell \leq 0} \frac{\lambda^j}{\|(f_{t,t}^j)'(f_{t,t}^{-\ell})\|} \hat{\rho}_{t,\ell} \circ f_{t,t}^{-j}.
\]

In particular, \( \rho_t \in H^s_p(I) \) for \( 0 \leq s < 1/2 \) and \( 1 < p < \frac{1}{1/2 + s} \).

**Proof.** Let \( \hat{\rho}_t \) be the fixed point of \( \hat{\mathcal{L}}_t \) given by Proposition 17. We have \( \hat{\rho}_{t,0} \in W^{2,1} \subset C^1 \) by Sobolev embedding, so that

\[
\rho_t = \Pi_t(\hat{\rho}_t) = \hat{\rho}_{t,0} + \sum_{j \geq 1, \sigma = \pm} \frac{\lambda^j}{\|(f_{t,t}^j)'(f_{t,t}^{-\ell})\|} \hat{\rho}_{t,\ell} \circ f_{t,t}^{-j}.
\]

Finally:

\[
\rho_t = \hat{\rho}_{t,0} + \sum_{j \geq 1, \sigma = \pm} \frac{\lambda^j}{\|(f_{t,t}^j)'(f_{t,t}^{-\ell})\|} \hat{\rho}_{t,\ell} \circ f_{t,t}^{-j}.
\]
by using \( \hat{\rho}_{j,t} = (\hat{L}_t \hat{\rho})_j = \frac{1}{\xi_j} \xi_{j-1} \hat{\rho}_{j-1,t} \).

To obtain the announced regularity on \( \hat{\rho}_t \), we show that the sum in the RHS of (57) converges exponentially fast in the Sobolev space \( H^s_p \), for \( 0 \leq s < 1/2 \) and \( 1 < p < \frac{1}{1/2 + s} \).

We start with the observation that \( \frac{1}{|f_{t,s}^k|} \in H^s_p \), for \( \sigma \in \{\pm\} \). We start by noting that (32) and Remark 9 imply, for any \( x \in f_{k,t}^k(J_{k,t}) \),

\[
C^{-1}|(f_{t}^{k-1})'(c_{1,t})(x - c_{k,t})|^{-1/2} \leq \frac{1}{|(f_{t}^{k-1})'(f_{t}^{-x})|} \leq C|(f_{t}^{k-1})'(c_{1,t})(x - c_{k,t})|^{-1/2},
\]

for a \( C \) depending only on the goodness and \( L \). If \( x, y \in f_{k,t}^k(J_{k,t}) \), \( \text{sgn}((f_{t}^{k})(f_{t}^{-x}(y))) = \text{sgn}((f_{t}^{k})(f_{t}^{-x}(y))) \), as they are on the same side of \( c_{k,t} \), so we deduce from the previous estimate

\[
\frac{C^{-1}}{|(f_{t}^{k-1})'(c_{1,t})|^{1/2}} M^+_{\epsilon,c_{k,t}} \left( \frac{1}{\sqrt{|x - c_{k,t}|}} \right) \leq M^+_{\epsilon,c_{k,t}} \left( \frac{1}{|(f_{t}^{k-1})'(f_{t}^{-x})|} \right) \leq \frac{C}{|(f_{t}^{k-1})'(c_{1,t})|^{1/2}} M^+_{\epsilon,c_{k,t}} \left( \frac{1}{\sqrt{|x - c_{k,t}|}} \right),
\]

where

\[
M^+_{\epsilon,c_{k,t}} \phi(x) := \int_{c_{k,t}}^{x - \epsilon} \phi(y) - \phi(y) \frac{dy}{y - x} \geq \frac{1}{\sqrt{|x - c_{k,t}|}},
\]

is the (truncated) local Marchaud derivative. As established in [23, Lemma 11], \( \frac{1}{\sqrt{|x - c_{k,t}|}} \) is a \( H^s_p(I) \) function for \( 0 \leq s < 1/2 \), \( 1 < p < \frac{1}{1/2 + s} \), and its \( H^s_p \) norm is uniformly bounded in \( k, t \). In particular, it follows that the previous truncated local Marchaud derivative converges in \( L^p \) when \( \epsilon \to 0 \), and that \( \sup_{\epsilon > 0} \left\| M^+_{\epsilon,c_{k,t}} \left( \frac{1}{(f_{t}^{k-1})'(f_{t}^{-x})} \right) \right\|_{L^p} < \infty \). By [21, Theorem 6.2], this entails that \( \frac{1}{(f_{t}^{k-1})'(f_{t}^{-x})} \in I^s(L^p(I)) = H^s_p(I) \) by [21, Cor. to Theorem 18.2]. Furthermore, by (59) and the second part of [23, Lemma 12], there is a constant \( \hat{C} \), depending only on the goodness and \( L \), such that

\[
\left\| \frac{1}{(f_{t}^{k-1})'(f_{t}^{-x})} \right\|_{H^s_p} \leq \hat{C}|(f_{t}^{k-1})'(c_{1,t})|^{-1/2}
\]

Next, we show that \( \| \hat{\rho}_{t,0} \circ f_{t}^{-k} \|_{C^1} \) is bounded uniformly in \( L \). First, we notice that since \( \hat{\rho}_{t,0} \in W^{2,1} \) by Prop. 17, with a \( W^{2,1} \) norm bounded uniformly in the goodness and \( L \), we get \( \| \hat{\rho}_{t,0} \circ f_{t}^{-k} \|_{\infty} \leq \hat{C} \), with \( \hat{C} \) depending only on the goodness and \( L \).

Second, by virtue of Proposition 17 and (58), we compute for \( x, y \in f_{k,t}^k(J_{k,t}) \),

\[
|\hat{\rho}_{t,0}(f_{t}^{-k}(x)) - \hat{\rho}_{t,0}(f_{t}^{-k}(y))| \leq \| \hat{\rho}_{t,0} \|_{C^1} |f_{t}^{-k}(x) - f_{t}^{-k}(y)|
\]

\[
= \| \hat{\rho}_{t,0} \|_{C^1} \left| \int_{[x,y]} |(f_{t}^{k})(f_{t}^{-z})| \right|\left| dz \right|
\]

\[
\leq \| \hat{\rho}_{t,0} \|_{C^1} \frac{C}{|(f_{t}^{k})(c_{1,t})|^{1/2}} \int_{[x,y]} \frac{dz}{\sqrt{|z - c_{k,t}|}}
\]

\[
\leq C \sqrt{|x - y|},
\]

with \( C \) depending only on the goodness and \( L \). This is the announced bound. To conclude, we use (54), the Benedicks-Carleson and Collet-Eckmann conditions to get that the series
in the RHS of (57) converges (exponentially fast) in $H^s_p$, for $0 \leq s < 1/2$ and $1 < p < 1/2 + s$.

**Proof of Proposition 17.** It is easy to see that

$$\|\hat{\mathcal{L}}_t\|_{\mathcal{B}} \leq 1.$$  

Note that [6, Eq.(80),(81)] (see also [4, Eq. (136),137]) hold in our setting: hence we have, for any $k \geq 1$, any $1 \leq j \leq k$,

$$\|((\hat{\mathcal{L}}_t^j)^\prime)_{k_{j}}\|_{L^1} \leq \frac{Cj}{\lambda^j} \|\psi'_{j-k}\|_{L^1}.$$  

Recalling (31), (39) and (41), we get

$$\|((\hat{\mathcal{L}}_t\Psi)'_{k})\|_{L^1} \leq \frac{C}{c(\delta)}(\|\psi'_{0}\|_{L^1} + \|\psi_{0}\|_{L^1} + \sup_{\psi_{0}})$$

$$+ \sum_{k \geq B_{\delta}(\delta)} \frac{\lambda^k e^{2ak}}{|(f'_{k+1})(c,\ell)|^{1/2}}(\|\psi'_{k}\|_{L^1} + \|\psi_{k}\|_{L^1} + \sup_{\psi_{k}})$$

Taking into account (52) and the Sobolev embedding [6, Eq.(80)], we get that $\hat{\mathcal{L}}_t$ is bounded on $\mathcal{B}$.

In view of the previous estimates, we can reproduce the arguments of [6, Prop 4.10-4.11 and Appendix B] (see also [4, Appendix B]) to obtain the following Lasota-Yorke inequality: for any $n \geq 1$

$$(60) \quad \|\hat{\mathcal{L}}_t^n\|_{\mathcal{B}} \leq C\Theta_0^n \|\Psi\|_{\mathcal{B}} + \|\Psi\|_{\mathcal{B}},$$

for some $\Theta_0 > 1$, with $C$ depending only on the goodness, $\delta$ and $L$. Together with the remark that $\hat{\mathcal{L}}_t^\nu = \nu$, the first two points of the Proposition follow easily.

The next point is a consequence of the first two items, as well as classical results on nonnegative operators (see [4, Appendix B, p.52] for the full argument). To see that $\hat{\mathcal{L}}_t^\nu \in W^{2,1}$, take a $\Psi \in \mathcal{B}$ such that $\psi_0 \in C^\infty$ with $\int \psi_0 dm = 1$, and $\psi_k = 0$ for all $k \geq 1$. From the previous quasi-compactness argument, we know that

$$\hat{\mathcal{L}}_t = \frac{1}{n} \lim_{n \to \infty} \sum_{k=0}^{n-1} \hat{\mathcal{L}}_t^k(\Psi),$$

in the $\mathcal{B}$ norm. Arguing as in [6, Appendix B] or [4, Appendix B], we get that $\|\hat{\mathcal{L}}_t^n\|_{\mathcal{B}} \leq C$, for some $C$ depending only on the goodness, $L$ and $\delta$: this gives us the wanted conclusion.

**Proposition 19.** Let $t$ be good for $(\lambda, H_0, a, C_0, \rho)$. Fix $\delta > 0$, $L > 1$, $b > 3a/2$ and $\lambda$ as previously. Then:

- The spectral radius of $\hat{\mathcal{L}}_{t,M}|_{\mathcal{B}}$ is bounded by 1, and its essential spectral radius by $\Theta_0^{-1} < 1$, where $\Theta_0$ is the bound on the essential spectral radius of $\hat{\mathcal{L}}_t|_{\mathcal{B}}$.
- There exists some $M_0 > 1$, depending only on the goodness, such that for all $M \geq M_0$, $\hat{\mathcal{L}}_{t,M}$ admits a simple eigenvalue $\Theta_0^{-1} < \kappa_{t,M} \leq 1$, with an associated real, non-negative eigenfunction $\hat{\rho}_{t,M}$, normalized by $\nu(\hat{\rho}_{t,M}) = 1$, and an eigenform $\nu_{t,M}$, normalized by $\nu_{t,M}(\hat{\rho}_{t}) = 1$, and they satisfy: there exists a constant $C_1$, depending only on the goodness, such that

$$(61) \quad \sup_{M} \|\hat{\rho}_{t,M}\|_{\mathcal{B}} \leq C_1 \quad \text{and} \quad \sup_{M} \|\nu_{t,M}\|_{(\mathcal{B}^1)^*} \leq C_1.$$
Furthermore, fixing $v < 1$ yields: there exists a constant $C_t > 1$ such that
\begin{equation}
0 \leq |1 - \kappa_{t,M}| \leq C_t \tau_{t,M}^v \quad \|\hat{p}_{t,M} - \hat{\rho}\|_{B_t^1} \leq C_t \tau_{t,M}^v \quad \|\nu - \nu_{t,M}\|_{B_t^1} \leq C_t \tau_{t,M}^v,
\end{equation}
where
\[\tau_{t,M} := e^{(M-1)\alpha/2} |(f_t^{M-1})'(c_{1,t})|^{-1/2} < 1.\]

**Proof.** To alleviate notation, we will denote $\hat{L}_{t,M} = \hat{L}_M$ and generally remove the subscript $t$.

Arguing as in [4, Prop 3.7] or [6, Lemma 4.12], one sees that $\hat{L}_M$ satisfies the same Lasota-Yorke inequalities as $\hat{\lambda}$, i.e. for any $M \geq 1$, any $n \geq 1$,
\begin{align}
\|\hat{\lambda}_M^n \Psi\|_{B_t^1} &\leq \|\Psi\|_{B_t^1} \\
\|\hat{\lambda}_M^n \Psi\|_{B_t^1} &\leq C\Theta_0^{-n} \|\Psi\|_{B_t^1} + C \|\Psi\|_{B_t^1},
\end{align}
where $\Theta_0$ can be chosen as in Proposition 17 and $C$ depends only on the goodness and our choice of $\delta$ and $L$: this gives the first part of the Proposition. Together with positivity of $\hat{L}_M$, we get the existence of a simple eigenvalue $\Theta_0^{-1} < \kappa_{t,M} \leq 1$ and associated eigenfunction $\hat{\rho}_M$ and eigenform $\nu_M$.

To obtain the announced bounds, we will apply the Keller-Liverani framework [17, Theorem 1] to $\hat{\lambda}$ and $\hat{L}_M$ on $B_t$ and $B_t^1$. We already have the uniform Lasota-Yorke inequalities (63), and it remains to establish closeness in triple-norm. We show that
\begin{equation}
\|(\hat{\lambda} - \hat{L}_M)\Psi\|_{B_t^1} \leq C \tau_M \|\Psi\|_{B_t^1}.
\end{equation}

Indeed, it is easy to see that
\[
\|(\hat{\lambda} - \hat{L}_M)\Psi\|_{B_t^1} \leq \sum_{k \geq M+1} \lambda^{k-1} \|\psi_{k-1}\|_{L^1} = \|(Id - T_M)\Psi\|_{B_t^1},
\]
so that the wanted inequality simply results from the following computation:
\[
\|(Id - T_M)\Psi\|_{B_t^1} = \sum_{k \geq M} \lambda^k \|\psi_k\|_{L^1} \leq \sum_{k \geq M} \lambda^k |J_{k,t}| \|\psi_k\|_{L^\infty}
\leq C \sum_{k \geq M} \lambda^k e^{-kb/2} |(f_t^{k-1})'(c_{1,t})|^{-1/2} \|\psi_k\|_{W^{1,1}}
\leq C \|\Psi\|_{B_t^1} \sum_{k \geq M} \frac{\lambda^{k-M-1} e^{-(a-b)/2}}{|(f_t^{k-M-1})'(c_{1,t})|^{1/2}}
\leq C \|\Psi\|_{B_t^1} \sum_{k \geq M} \frac{1}{|(f_t^{k-M-1})'(c_{1,t})|^{1/2}}
\leq C \|\Psi\|_{B_t^1} \sum_{k \geq M} \frac{\lambda^{k-M-1} e^{(a-b)/2}}{|(f_t^{k-M-1})'(c_{1,t})|^{1/2}}
\leq C \tau_M \|\Psi\|_{B_t^1},
\]
where we used (35), (52), as well as, for the second to last inequality, a reasoning similar to the proof of Proposition 11. We note that the constants appearing in the previous chain of inequalities only depend on the goodness parameters.

We can then apply [17, Thm 1, Cor 1]: fixing $v < 1$, and denoting $P\Psi := \nu(\Psi)\hat{\rho}$ (resp. $P_M\Psi := \nu_M(\Psi)\hat{\rho}_M$) $\hat{\lambda}$ (resp. $\hat{L}_M$) spectral projector onto the eigenvalue 1 (resp. $\kappa_M$), we get
\[
\|(P - P_M)\Psi\|_{B_t^1} \leq C_t \tau_M^v \|\Psi\|_{B_t^1},
\]
which gives (recall our normalization choices for \( \nu \) and \( \nu_M \), for \( \Psi = \hat{\rho}_M \), the first inequality in (62), and for \( \Psi = \hat{\rho} \), the second one.

The same result, now applied to the dual operators \( \hat{\mathcal{L}}^* \) and \( \hat{\mathcal{L}}_M^* \) and associated projectors, gives the third inequality in (62). The bounds (61) follow from our choice of normalizations for \( \nu^*_M, \rho^*_M \) and the fact that \( \rho_M = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \kappa^k \hat{\mathcal{L}}^*_M \hat{\rho} \), which is an easy consequence of the spectral decomposition. \( \square \)

The next result is central to be able to compare truncated transfer operators for different good parameters: it states that for admissible pairs \((M,t)\), one can choose the first \( M \) levels in \( \hat{I}_t \) and \( \hat{I}_0 \) to be essentially the same.

**Lemma 20.** Assume that \( f = f_0 \) is good for parameters \((\lambda_c, H_0, a, C_0, \rho)\). One can take the constant \( L > 1 \) in the definition of levels \( B_k \) and cutoff functions \( \xi_k \) such that: there is a tower \( \hat{I}_t = \hat{I}_0 \), a map \( \hat{f} \) and cutoff functions \( \xi_k \), an \( \epsilon > 0 \), such that, for any \( M > 1 \), any \( |t| < \epsilon \) good for the same parameters, satisfying (42), we may build a tower \( \hat{f}_t : \hat{I}_t \cap \xi_k \) with

\[
J_{k,t} = J_k \quad \text{and} \quad \xi_{k-1,t} = \xi_{k-1} \quad \forall 1 \leq k \leq M.
\]

**Proof.** We start by fixing a \( L > \max(2C^3,4) \), where \( C \) is the maximum of the constants appearing in Lemma 7 and 43 (note that \( C \) does not depend on \( L \)). Let \( \epsilon \) be given by Lemma 14.

We construct the intervals \( B_k \) as follows: for \( k \geq 1 \), set

\[
B_k = \begin{cases} 
  \left[ c_k - \frac{e^{-kb}k}{2L^2}, c_k + \frac{e^{-kb}k}{2L^2} \right], & \text{if } \frac{e^{-kb}k}{L^2} \leq |f^k(J_{k-1})| \leq 2\frac{e^{-kb}k}{L^2} \\
  \left[ c_k - \frac{e^{-kb}k}{L^2}, c_k + \frac{e^{-kb}k}{L^2} \right], & \text{otherwise.}
\end{cases}
\]

This entails the following: if \( I_k \neq \emptyset \), then \( |f^k(J_k)| \leq \frac{1}{2}|f^k(J_{k-1})| \), i.e. \( \frac{|f^k(J_k)|}{|f^k(J_{k-1})|} \geq \frac{1}{2} \). Using that \( C^{-1}|x-c| \leq |f'(x)| \leq C|x-c| \) and Lemma 7 we get:

\[
\frac{|I_k|}{|J_{k-1}|} \geq C^{-2} \left( \frac{|f(I_k)|}{|f(J_{k-1})|} \right)^{1/2} \geq \frac{1}{C^3\sqrt{2}^2}.
\]

In turns, this implies that we may choose our cutoff function \( \xi_{k-2}, k \geq 2 \), such that \( \{x \in J_{k-1}, 0 < \xi_{k-2} < 1 \} \subset I_k \) and \( \|\partial_x^j \xi_{k-2}\|_\infty \leq C'|J_{k-1}|^{-j}, j \in \{1,2,3\} \), the rest of the requirements on \( \xi_{k-2} \) being satisfied as well.

Since no points fall from the tower before level \( H(\delta) \), for any \( j \leq H(\delta) \), \( J_{j,t} = I_j = \emptyset \), \( J_{j-1,t} = J_{j-1} = [c-\delta, c+\delta] \) and (66) is satisfied for any \( k \geq H(\delta) - 2 \). Consider now \( H(\delta) - 1 \leq k \leq M - 1 \), and assume that (66) holds for any \( 0 \leq j \leq k - 1 \).

We start by remarking that the basic fact \( C^{-1}|x-c| \leq |f'_i(x)| \leq C|x-c| \), together\(^8\) with (43) and Lemma 7 gives the inequality

\[
C^{-3}|f^i(J_i)| \leq |f'_i(J_i)| \leq C^3|f^i(J_i)|.
\]

Let us now consider two cases:

\^8In fact, we use here a slightly stronger version of (43): for \( x \in f(J_k), 0 \leq j \leq k \leq M - 1 \)

\[
C^{-1} \leq \frac{|(f_j)'(x)|}{|f_j'(x)|} \leq C.
\]
\(I_{k+1} = \emptyset\). Then \(J_{k+1} = J_k\), so that \(f^{k+1}(J_k) \subset B_{k+1}\). In particular, we get from the previous upper bound and our choice of \(L\) that \(|f_t^{k+1}(J_k)| < L^{-1}e^{-(k+1)b}\); setting \(B_{k+1,t} = [c_{k+1,t} - L^{-1}e^{-(k+1)b}, c_{k+1,t} + L^{-1}e^{-(k+1)b}]\), we see that both (24) and \(I_{k+1,t} = \emptyset\) hold, since \(f_t^{k+1}(J_{k+1})\) is not adjacent to the boundary of \(B_{k+1,t}\). By definition, \(\xi_{k-1,t} = \xi_{k-1} \equiv 1\).

- \(I_{k+1} \neq \emptyset\). Then, as previously remarked, \(f^{k+1}(J_{k+1})\) is adjacent to the boundary of \(B_{k+1}\). This suggests to set \(B_{k+1,t} = [c_{k+1,t} - b_{k,t}, c_{k+1,t} + b_{k,t}]\), where \(b_{k,t} := |f_t^{k+1}(J_{k+1})|\). From (67), our choice of \(B_{k+1}\) and \(L\), we see that

\[
e^{-(k+1)b}/2L^3 \leq b_{k,t} \leq e^{-(k+1)b}/2L,
\]

from which \(B_{k+1,t}\) satisfies (24), \(J_{k+1,t} = J_{k+1}\) and thus \(I_{k+1,t} = I_{k+1}\). The conclusion follows.

\[\square\]

From now on, we will use the tower given by Lemma 20. We are now in a position to establish the following key result: uniformity in \(t \in \Omega\) of the renormalization period \(P_t\) and mixing rate \(\theta_t\), by relating those quantities to the spectral data of the transfer operator \(\hat{L}_{t,M}\) for admissible pairs \((M,t)\) (recall (42)). Our uniform mixing result, Theorem 4 follows immediately.

**Lemma 21.** Let \(\theta_t := \sup\{|z|, \ z \in \sigma(\hat{L}_t), \ z \neq 1\}\), and \(P_t\) be the renormalization period of \(f_t\). We have:

- There is a integer \(P_0\), such that for any \(t \in \Omega\), \(P_t \equiv P_0\).
- There is a \(\Theta_1 > 1\), such that \(\theta_t < \Theta_1^{-1}\) for \(t \in \Omega\).

**Proof.** Let \(t_0 \in \Omega\). As per Proposition 19, there is \(\varepsilon > 0\) and \(C \geq 1\) such that for all \(t \in \Omega\), \(|t - t_0| < \varepsilon\), (63) and (65) hold. Let \((M,t)\) be an admissible pair, and set

\[
Q_{t,M}(z) := z - \hat{L}_{t,M}.
\]

We show that there exists a small circle \(\gamma\) centered at 1 and a \(C > 1\) (uniform in the goodness) such that for large enough \(M\) and small enough \(|t - t_0| \leq \varepsilon\), the disk \(D_{\gamma}\) encircled by \(\gamma\) intersects the spectrum of \(\hat{L}_{t,M}\) only at \(\kappa_{t,M}\) (the eigenvalue give by Prop. 19), and such that

\[
(68) \quad \sup_{z \in D_{\gamma}} \|Q_{t,M}^{-1}\|_{B_t} \leq C.
\]

For this, we may apply [17, Theorem 1, Cor. 1] to \(\hat{L}\) and \(\hat{L}_M\), and then to \(\hat{L}_{t,M}\) and \(\hat{L}_M\) arguing exactly as in [4, Proof of Lemma 4.6].

Set \(\Omega(t_0) := \Omega \cap (t_0 - \varepsilon, t_0 + \varepsilon)\). From (68), we get first that for if \(t \in \Omega(t_0)\), \(P_t \leq P_0\), where \(P_0\) is the renormalization period of \(f_{t_0}\).\footnote{Indeed, the number of eigenvalues on the unit circle could drop by perturbing the operators, as remarked in [4, Lemma 4.6].} But by construction of the set \(\Omega\), and since \(\varepsilon\) from the last claim of Theorem 1 can be chosen independently of \(t_0\) (see Remark 2), we may exchange the role of \(t\) and \(t_0\) in the previous arguments, i.e. if \(t \in \Omega(t_0)\), then \(t_0 \in \Omega(t)\). This gives us the reverse inequality \(P_0 \leq P_t\), and thus the announced equality \(P_t \equiv P_0\) for any \(t \in \Omega(t_0)\).

In order to apply [17, Theorem 1, Cor. 1] to \(\hat{L}_{t,M}\) and \(\hat{L}_M\), the only thing that remain
to do is to establish the following triple norm estimate: there is $C > 1$ and $0 < \eta < 1/2$ such that for $\Psi \in \mathcal{B}_t$

\[(69) \quad \| (\hat{\mathcal{L}}_{t,M} - \mathcal{L}_M) \Psi \|_{\mathcal{B}_t^1} \leq C |t|^{\eta} \| \Psi \|_{\mathcal{B}_t}.\]

Note that, since $(M, t)$ is admissible, the tower given by Lemma 20 allows us to compare $\hat{\mathcal{L}}_{t,M} \Psi$ and $\mathcal{L}_M \Psi$.

We start with the remark that, for any $C$ with a constant $M, t$

\[
\| \frac{1}{\lambda} \xi_{k-1} \psi_{k-1} = \frac{1}{\lambda} \xi_{k-1} \psi_{k-1} = [\hat{\mathcal{L}}_{t,M} \Psi]_k.
\]

Hence we only need to focus on the ‘ground floor’ term $k = 0$. Denoting $\tilde{\psi}_j = (1 - \xi_j) \psi_j$, and using the fundamental theorem of calculus, we may write

\[
[\hat{\mathcal{L}}_{t,M} \Psi - \mathcal{L}_M \Psi]_0 = \sum_{j=0}^{M-1} \lambda^j \left( \frac{\tilde{\psi}_j \circ f_{I_\sigma}^{(j+1)}(1)}{(f_{I_\sigma}^{(j+1)})'} \circ f_{I_\sigma}^{(j+1)}(1) \right) - \sum_{j=0}^{M-1} \lambda^j \int_0^t \partial_s \left[ \frac{\tilde{\psi}_j \circ f_{s,\sigma}^{(j+1)}(1)}{(f_{s,\sigma}^{(j+1)})'} \circ f_{s,\sigma}^{(j+1)}(1) \right] ds.
\]

Thus, when estimating $\| [\hat{\mathcal{L}}_{t,M} \Psi - \mathcal{L}_M \Psi]_0 \|_{L^1}$, one needs to bound $\| \partial_s \left[ \frac{\tilde{\psi}_j \circ f_{s,\sigma}^{(j+1)}(1)}{(f_{s,\sigma}^{(j+1)})'} \circ f_{s,\sigma}^{(j+1)}(1) \right] \|_{L^1}$
for $s \in [0, t]$ and $0 \leq j \leq M - 1$. For this, we start by writing

\[
\partial_s \left[ \frac{\tilde{\psi}_j \circ f_{s,\sigma}^{(j+1)}(1)}{(f_{s,\sigma}^{(j+1)})'} \circ f_{s,\sigma}^{(j+1)}(1) \right] = \frac{\partial_s f_{s,\sigma}^{(j+1)}}{(f_{s,\sigma}^{(j+1)})'} \partial_x \left[ \frac{\tilde{\psi}_j \circ f_{s,\sigma}^{(j+1)}(1)}{(f_{s,\sigma}^{(j+1)})'} \circ f_{s,\sigma}^{(j+1)}(1) \right] \tilde{\psi}_j \circ f_{s,\sigma}^{(j+1)}(1).
\]

We now notice that, by Sobolev embedding and (24)

\[(70) \quad \int_{\mathcal{I}} |\tilde{\psi}_j \circ f_{s,\sigma}^{(j+1)}(1)| dm \leq C |\text{supp}(\tilde{\psi}_j \circ f_{s,\sigma}^{(j+1)}(1)||\psi_j||_{\infty} \leq C e^{-(j+1)b}||\psi_j||_{W^{1,1}},\]

with a constant $C$ independent on $s \in [0, t]$ and $j$. We also have, using the change of variables $y = f_{s,\sigma}^{(j+1)}(x)$

\[(71) \quad \int_{\mathcal{I}} |\partial_x (\tilde{\psi}_j \circ f_{s,\sigma}^{(j+1)}(1)| dm \leq \int |\xi_j' \cdot \psi_j| dm + \int |(1 - \xi_j) \psi_j| dm \leq C ||\psi_j||_{W^{1,1}},\]

where we also used [4, Eq (137)-(138)].

Using estimates (44) and (45) from Lemma 14, we get, for $y \in \text{supp}(\tilde{\psi}_j)$

\[(72) \quad \left| \frac{\partial_s f_{s,\sigma}^{(j+1)}(1)}{(f_{s,\sigma}^{(j+1)})'}(y) \right| \leq C^2 L_{3/2} e^{((j+1)b/2)}((f_{s,\sigma}^{(j+1)}(1))^{1/2}.\]

Recall $X_s$, defined such that $X_t \circ f_{s} = [\partial_s f_{s}]_s$. By (49), we may write

\[
\frac{\partial_s f_{s,\sigma}^{(j+1)}(1)}{(f_{s,\sigma}^{(j+1)})'}(y) = \sum_{k=1}^{j+1} X_s(f_{s}^{k}(y)).
\]
so that
\[
\frac{\partial_y \partial_s f_s^{j+1}(y)}{(f_s^{j+1})'(y)} = \sum_{k=1}^{j+1} \left( X_s'(f_s^k(y)) - X_s(f_s^k(y)) \cdot \sum_{\ell=0}^{k-1} \frac{f_s''(f_s^\ell(y))}{(f_s^\ell(y))(f_s^{\ell+1}(y))} \right)
\]
To bound the previous sum, we remark that, for \( y \in \text{supp}(\tilde{\psi}_j) \), its dominant terms appear for \( k = 1, \ell = 0 \). Using (43) (44), we have that
\[
\frac{1}{|(f_s^m)'(y)|} \leq C^3 e^{(j+1)b/2} \frac{|(f_j)'(c_1)|^{1/2}}{|(f_m)'(c_1)|}
\]
for \( 1 \leq m \leq j + 1 \).

By virtue of this last estimate, and taking (29), (43) (44) and (47) into account, we may argue as in the proof of [4, Lemma 4.5, Eq (107)] to obtain
\[
\frac{1}{|(f_s^{j+1})'(y)|} \leq C e^{(j+1)b/2} |(f_j)'(c_1)|
\]
Putting (70), (71), (72), (73) for \( m = j + 1 \) and (74) together, we get:
\[
\left\| \frac{\partial_s}{(f_s^{j+1})'(y)} \right\|_{L^1} \leq C e^{(j+1)b/2} |(f_j)'(c_1)|^{1/2} \left\| \tilde{\psi}_j \right\|_{W^{1,1}},
\]
so that, using the admissibility condition (42)
\[
\left\| (\hat{L}_{t,M} - \hat{L}_M) \Psi \right\|_{B^1_t} = \left\| \left[ (\hat{L}_{t,M} - \hat{L}_M) \Psi \right]_0 \right\|_{L^1}
\leq \sum_{j=0}^{M-1} C \lambda^j |t| e^{(j+1)b/2} |(f_j)'(c_1)|^{1/2} \left\| \tilde{\psi}_j \right\|_{W^{1,1}}
\leq C M \lambda^M |t|^{1/2} \left\| \Psi \right\|_{B_t}
\leq C \lambda^{|t|} \left\| \Psi \right\|_{B_t},
\]
by (52), for some \( 0 < t < 1/2 \), which gives (65) and concludes the proof of Lemma 21.

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