THE TOPOLOGY OF INDEPENDENCE COMPLEXES OF SQUARE GRIDS

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Abstract. The independence complex of a graph $G$ is a simplicial complex whose simplices are the independent sets in $G$. In the last couple of decades, the independence complexes of square grids (with various boundary conditions) have gained much attention because of their connections with the hard square model from statistical physics. In this article, we prove that if $G$ is an $m \times n$ grid with open or cylindrical boundary condition then its independence complex is homotopy equivalent to a wedge of spheres. A part of this result settles a conjecture of Iriye.

1. Introduction

A subset $I$ of the vertex set of a (simple) graph $G$ is called independent, if the induced subgraph of $G$ on $I$ is a collection of isolated vertices. The independence complex of a graph $G$, denoted $\text{Ind}(G)$, is the simplicial complex whose simplices are all the independent sets in $G$.

The study of independence complexes of graphs and their applications to various combinatorial problems has drawn significant attention in the last couple of decades. In Babson and Kozlov’s proof of Lovász’s conjecture [6] regarding odd cycles and graph homomorphism complexes, the independence complexes of cycle graphs played an important role. In 2003, Meshulam [29] established a connection between the domination number of a graph $G$ and homological connectivity of $\text{Ind}(G)$. Properties of independence complexes have been used to study the Tverberg graphs [12] and the independent system of representatives [5]. These complexes also find applications in combinatorial commutative algebra [9, 14, 17] and statistical physics [13, 19, 21]. Furthermore, the independence complexes are closely related to the Vietoris-Rips complexes, which have appeared very often in the area of topological data analysis (see [2, 3, 4, 8]). The independence complexes also coincide with the clique complexes and have direct connections with the matching complexes, which have been studied comprehensively in the last couple of decades (see [22, 25, 26, 31, 33, 34] for instance).

The main purpose of this article is to determine the homotopy type of the independence complexes of planar and cylindrical square grids. For two positive integers $m, n$, the (planar) $m \times n$ square grid, denoted $\Gamma_{m,n}$, is a graph with vertex set $V(\Gamma_{m,n})$ and edge set $E(\Gamma_{m,n})$ defined as follows:

$$V(\Gamma_{m,n}) = \{(i, j) \in \mathbb{N}^2 : i \in [m], j \in [n]\}, \text{ and}$$
$$E(\Gamma_{m,n}) = \{((i, j), (i', j')) : |i - i'| + |j - j'| = 1\}.$$ 

Here, $[m]$ denotes the set $\{1, 2, \ldots, m\}$. 

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For $m > 1$ and $n \geq 1$, the cylindrical square grid (square grid with cylindrical identification) $C_{m,n}$ is defined as follows:

\[ V(C_{m,n}) = \{(i,j) \in \mathbb{N}^2 : i \in [m], j \in [n]\}, \quad \text{and} \]
\[ E(C_{m,n}) = \{((i,j), (i',j')) : |i - i'| + |j - j'| = 1\} \cup \{((1,i), (m,i)) : i \in [n]\}. \]

The topological study of independence complexes of grids in some particular cases has been done by many mathematicians and has been considered quite difficult for general grids. Kozlov [27] computed the homotopy type of independence complex of $\Gamma_{1,n}$. In 2012, Adamaszek [1] determined the homotopy type of $\text{Ind}(\Gamma_{2,n})$ and $\text{Ind}(\Gamma_{3,n})$ for all $n$. Okura [30] determined the reduced Euler characteristic of $\text{Ind}(\Gamma_{4,n})$ and Bousquet-Mélou, Linusson, and Nevo [7] studied the independence complexes of several types of square grid graphs. Very recently, Matsushita and Wakatsuki [28] showed that, for any $n \geq 1$, the complexes $\text{Ind}(\Gamma_{4,n})$ and $\text{Ind}(\Gamma_{5,n})$ are homotopy equivalent to a wedge of spheres. As one of the main results of this article, we prove that the same is true for any $m \times n$ grid. More precisely, we have the following.

**Theorem 1.1.** For any $m, n \geq 1$, the complex $\text{Ind}(\Gamma_{m,n})$ is homotopy equivalent to a wedge of spheres.

In fact, we prove a more general result by showing that the independence complex of any solid grid graph (see Definition 3.1) is homotopy equivalent to a wedge of spheres (cf. Theorem 3.2).

The independence complexes of grids with certain boundary identifications have a beautiful connection with statistical physics, which was first discussed by Jonsson [21] where he proved two conjectures from [13]. This connection was later explored by Bousquet-Mélou, Linusson, and Nevo [7]. After that, the homological study of independence complexes of various grids was done by several authors (see [11, 18, 24]). Our focus in this article would be on the grids with cylindrical identification. Fendley et al. [13] and Jonsson [21] observed that the cylindrical case seemed to have interesting properties. Thus it is natural to try to understand the topology of $\text{Ind}(C_{m,n})$. Kozlov [27] computed the homotopy type of $\text{Ind}(C_{m,1})$ and Thappar [32] determined the homotopy type of $\text{Ind}(C_{m,n})$ for $m = 2, 3, 4$ and 5. In 2012, Iriye [20] showed that, for $m = 6, 7$, the complex $\text{Ind}(C_{m,n})$ is homotopy equivalent to a wedge of spheres and conjectured the following.

**Conjecture 1.2** (cf. [20, Conjecture 1.8]). For any $m > 1$, the complex $\text{Ind}(C_{m,n})$ is homotopy equivalent to a wedge of spheres.

Adamaszek [1] proved various homotopy equivalences among $\text{Ind}(C_{m,n})$ when one of the index ($m$ or $n$) is small. He also provided a method of recursively calculating the Euler characteristic of $\text{Ind}(C_{m,n})$ for even $n$. For odd $n$, the computation of the Euler characteristic of
Ind($C_{m,n}$) was done by Jonsson [23]. In this article we prove that, for each $m, n$, Ind($C_{m,n}$) is homotopy equivalent to a wedge of spheres, thereby showing that the Conjecture 1.2 holds true.

The organization of this article is as follows: In Section 2, we recall some basic definitions and results which are necessary for this article. Section 3 and Section 4 are devoted towards the proofs of Theorem 1.1 and Conjecture 1.2 respectively.

2. Preliminaries

An (abstract) simplicial complex $\Delta$ is a collection of finite sets such that if $\tau \in \Delta$ and $\sigma \subset \tau$, then $\sigma \in \Delta$. The elements of $\Delta$ are called the simplices (or faces) of $\Delta$. If $\sigma \in \Delta$ and $|\sigma| = k + 1$, then $\sigma$ is said to be $k$-dimensional. A 0-dimensional simplex of $\Delta$ is called its vertex and the set of vertices of $\Delta$ is denoted by $V(\Delta)$. A subcomplex of a simplicial complex $\Delta$ is a simplicial complex whose simplices are contained in $\Delta$. In this article, we always assume that empty set is a simplex in any simplicial complex and we consider any simplicial complex as a topological space, namely its geometric realization. For the definition of geometric realization, we refer the reader to Kozlov’s book [26].

For a simplex $\sigma \in \Delta$, define

$$\text{lk}(\sigma, \Delta) := \{ \tau \in \Delta : \sigma \cap \tau = \emptyset, \sigma \cup \tau \in \Delta \},$$

$$\text{del}(\sigma, \Delta) := \{ \tau \in \Delta : \sigma \not\subset \tau \}.$$

The simplicial complexes $\text{lk}(\sigma, \Delta)$ and $\text{del}(\sigma, \Delta)$ are called link of $\sigma$ in $\Delta$ and (face) deletion of $\sigma$ in $\Delta$ respectively. The join of two simplicial complexes $\Delta_1$ and $\Delta_2$, denoted as $\Delta_1 * \Delta_2$, is a simplicial complex whose simplices are disjoint union of simplices of $\Delta_1$ and $\Delta_2$. Let $\Delta^S$ denotes a $(|S| - 1)$-dimensional simplex with vertex set $S$. The cone on $\Delta$ with apex $a$, denoted as $C_a(\Delta)$, is defined as

$$C_a(\Delta) := \Delta * \Delta^{\{a\}}.$$ 

For $a, b \not\in V(\Delta)$, the (reduced) suspension of $\Delta$, denoted as $\Sigma(\Delta)$, is defined as

$$\Sigma(\Delta) := \Delta * \{a\} \cup \Delta * \{b\}.$$ 

Observe that for any vertex $v \in V(\Delta)$, we have

$$\Delta = C_v(\text{lk}(v, \Delta)) \cup \text{del}(v, \Delta)$$

and

$$C_v(\text{lk}(v, \Delta)) \cap \text{del}(v, \Delta) = \text{lk}(v, \Delta).$$

Clearly, $C_v(\text{lk}(v, \Delta))$ is contractible. Therefore, from [16, Example 0.14], we have the following.

**Lemma 2.1.** Let $\Delta$ be a simplicial complex and $v$ be a vertex of $\Delta$. If $\text{lk}(v, \Delta)$ is contractible in $\text{del}(v, \Delta)$ then

$$\Delta \simeq \text{del}(v, \Delta) \vee \Sigma(\text{lk}(v, \Delta)),$$

where $\vee$ denotes the wedge of topological spaces.

A (simple) graph is an ordered pair $G = (V(G), E(G))$, where $V(G)$ is called the set of vertices and $E(G) \subseteq (V(G))^2$, the set of (unordered) edges of $G$. The vertices $v_1, v_2 \in V(G)$ are said to be adjacent, if $(v_1, v_2) \in E(G)$.

The following observation directly follows from the definition of independence complexes of graphs.
Lemma 2.2. Let $G_1 \sqcup G_2$ denotes the disjoint union of two graphs $G_1$ and $G_2$. Then

$$\text{Ind}(G_1 \sqcup G_2) \simeq \text{Ind}(G_1) \ast \text{Ind}(G_2).$$

Let $G$ and $H$ be two graphs. A map $f : V(G) \to V(H)$ is said to be a graph homomorphism if $(f(v), f(w)) \in E(H)$ for all $(v, w) \in E(G)$. A graph homomorphism is called an isomorphism if it is bijective and its inverse map is also a graph homomorphism. Two graphs $G$ and $H$ are said to be isomorphic if there is an isomorphism between them and we denote it by $G \cong H$.

For a subset $A \subseteq V(G)$, the set of neighbours of $A$ is $N_G(A) = \{x \in V(G) : (x, a) \in E(G) \text{ for some } a \in A\}$. The closed neighbourhood set of $A \subseteq V(G)$, is $N_G[A] = N_G(A) \cup A$. If $A = \{v\}$ is a singleton set, then we write $N_G(v)$ (resp. $N_G[v]$) for $N_G(\{v\})$ (resp. $N_G[\{v\}]$).

A graph $H$ with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ is called a subgraph of the graph $G$. For a nonempty subset $U \subseteq V(G)$, the induced subgraph $G[U]$, is the subgraph of $G$ with $V(G[U]) = U$ and $E(G[U]) = \{(a, b) \in E(G) : a, b \in U\}$. The graph $G[V(G) \setminus A]$ is denoted by $G - A$, for $A \subseteq V(G)$. For a subset $B \subseteq E(G)$, we let $G - B$ to be the graph with the vertex set $V(G - B) = V(G)$ and the edge set $E(G - B) = E(G) \setminus B$. For $A \subset V(G)$ and $B \subseteq E(G)$, the notation $G - A - B$ will be used for the induced subgraph $(G - A) - B$.

We now state a few results related to the independence complexes that will be used repeatedly in this article.

Lemma 2.3. [15, Lemma 4] Let $G$ be a graph and $\{a, b\}$ be a 1-simplex in $\text{Ind}(G)$. If $\text{Ind}(G - N_G[\{a, b\}])$ is contractible, then

$$\text{Ind}(G) \simeq \text{Ind}(\tilde{G}),$$

where $V(\tilde{G}) = V(G)$ and $E(\tilde{G}) = E(G) \cup \{(a, b)\}$.

Lemma 2.4. [10, Lemma 2.4] Let $G$ be a graph and $u, u' \in V(G)$, $u \neq u'$ such that $N_G(u) \subseteq N_G(u')$. Then

$$\text{Ind}(G) \simeq \text{Ind}(G - \{u'\}).$$

For $r \geq 1$, the path graph $P_r$ is a graph with $V(P_r) = \{0, 1, \ldots, r\}$ and $E(P_r) = \{(i, i+1) : i \in \mathbb{Z}\}$.

Lemma 2.5. [27, Proposition 4.6] For $r \geq 1$,

$$\text{Ind}(P_r) \simeq \begin{cases} S^{k-1} & \text{if } r = 3k, \\ pt & \text{if } r = 3k + 1, \\ S^k & \text{if } r = 3k + 2. \end{cases}$$

3. The case of square grids with open boundary

In this section we prove that, for any $m, n \geq 1$, the independence complex of $m \times n$ grid is homotopy equivalent to a wedge of spheres. In doing so, the following class of graphs will play a crucial role.

Definition 3.1. An induced subgraph of a square grid is called a solid grid graph (SGG) if it can be embedded in the plane such that the vertices lie on integer coordinates, the edges have unit length and all interior faces have unit area.

For example, in Figure 2, the graph $G_1$ is a SGG whereas the graphs $G_2$ and $G_3$ are not SGGs.
Another way to visualize SGGs is that these graphs can be obtained from square grids by recursively removing one vertex at a time from the boundary. Clearly, all square grids are SGGs. Therefore, Theorem 1.1 is an immediate consequence of the following result.

**Theorem 3.2.** The independence complex of any SGG is homotopy equivalent to a wedge of spheres.

**Proof.** We prove this by induction on the number of vertices (say $n$) in the graph $G$. If $n \leq 2$, then the result follows easily. Let the result be true for any graph satisfying the hypothesis and having at most $|V(G)| - 1$ vertices. If $G$ is disconnected then the result follows from induction and Lemma 2.2. Thus, let $G$ is connected.

Consider that $G$ has a leaf vertex $\ell$. Let $N_G(\ell) = \{w\}$ and $N_G(w) = \{\ell, v_1, \ldots, v_t\}$ for some $t \leq 3$. Since $N_G(\ell) \subseteq N_G(v_i)$ for all $i \in \{1, \ldots, t\}$, from Lemma 2.4, we get that $\text{Ind}(G) \cong \text{Ind}(G - \{v_1, \ldots, v_t\})$. Moreover, it is easy to see that $G - \{v_1, \ldots, v_t\}$ is isomorphic to $G - N_G[w] \cup P_2$ and $G - N_G[w]$ is a SGG on at most $n - 1$ vertices. Therefore, from induction, Lemma 2.2 and Lemma 2.5, we have that $\text{Ind}(G) \cong \sum (\text{Ind}(G - N_G[w]))$ implying that $\text{Ind}(G)$ is homotopy equivalent to a wedge of spheres.

Now consider that $G$ does not have any leaf vertex and let

\[
\begin{align*}
  r &= \max\{i \in \mathbb{Z} : (i, j) \in V(G) \text{ for some } j\}, \\
  s &= \min\{j \in \mathbb{Z} : (r, j) \in V(G)\}.
\end{align*}
\]

We split the remaining proof in two cases.

**Case 1:** $((r, s), (r, s + 1)) \in E(G)$ but $((r, s + 1), (r, s + 2)) \notin E(G)$.

**Case 2:** $((r, s), (r, s + 1)) \notin E(G)$ but $((r, s + 1), (r, s + 2)) \in E(G)$.

**Figure 2.** Examples of solid grid graphs

**Figure 3**
Observe that \( N_G((r, s)) \subseteq N_G((r - 1, s + 1)) \) and \( N_G((r, s + 1)) \subseteq N_G((r - 1, s)) \). Therefore, using Lemma 2.4 and Lemma 2.2, we get \( \text{Ind}(G) \simeq \text{Ind}(G - \{(r - 1, s), (r - 1, s + 1)\}) \simeq \text{Ind}(H \cup P_2) \simeq \sum \text{Ind}(H) \), where \( H \) is the induced subgraph of \( G \) on vertex set \( V(G) \setminus \{(r, s), (r, s + 1), (r - 1, s), (r - 1, s + 1)\} \) (cf. Figure 3). Clearly, \( H \) is a SGG on at most \( n - 2 \) vertices. Thus, induction hypothesis implies that \( \text{Ind}(H) \) is homotopy equivalent to a wedge of spheres and hence so is \( \text{Ind}(G) \simeq \sum \text{Ind}(H) \).

**Case 2:** \( \{(r, s), (r, s + 1), (r + 1, s), (r + 1, s + 2)\} \in E(G) \).

Since \( G \) does not have any leaf vertex, definitions of \( r \) and \( s \) imply that \( \{(r - 1, s), (r, s)\} \in E(G) \). In this case we analyze the link and deletion of vertex \( (r - 1, s) \) in \( \text{Ind}(G) \). Clearly both the graphs \( G - \{(r - 1, s)\} \) and \( G - N_G[(r - 1, s)] \) are SGGs on at most \( n - 1 \) vertices. Therefore, from induction hypothesis both the complexes \( \text{del}((r - 1, s), \text{Ind}(G)) = \text{Ind}(G - \{(r - 1, s)\}) \) and \( \text{lk}((r - 1, s), \text{Ind}(G)) = \text{Ind}(G - N_G[(r - 1, s)]) \) are homotopy equivalent to a wedge of spheres. Hence, to prove our result using Lemma 2.1, it is enough to show that \( \text{lk}((r - 1, s), \text{Ind}(G)) \) is contractible in \( \text{del}((r - 1, s), \text{Ind}(G)) \).

![Diagram](image_url)

**Figure 4**

For simplicity of notations, denote the graph \( G - N_G[(r - 1, s)] \) by \( G' \) (cf. Figure 4b). Since \( G \) is an induced subgraph of a square grid, either \( (r - 1, s + 1) \in N_G[(r - 1, s)] \) or \( (r - 1, s + 1) \notin V(G) \). Therefore, vertex \((r, s + 1)\) is a leaf vertex in \( G' \). Moreover, since \( G \) does
not have a leaf vertex, \(((r-1, s+2), (r, s+2)), ((r, s+3), (r, s+2))\) \(\cap E(G') \neq \emptyset\). We assume that \(((r-1, s+2), (r, s+2)) \in E(G')\). The other case, i.e., \(((r, s+3), (r, s+2)) \in E(G')\) is exactly similar to this case. Define

\[
G'' = \begin{cases} 
G' - \{(r, s + 3)\} & \text{if } ((r, s + 3), (r, s + 2)) \in E(G'), \\
G' & \text{if } ((r, s + 3), (r, s + 2)) \notin E(G').
\end{cases}
\]

If \(((r-1, s+2), (r, s+2)), ((r, s+3), (r, s+2))\) \(\subseteq E(G')\), then using Lemma 2.4, we get that \(\text{Ind}(G') \simeq \text{Ind}(G' - \{ (r, s + 3) \})\) (since \(N_{G'}((r, s + 1)) = \{ (r, s + 2) \} \subseteq N_{G'}((r, s + 3))\)). Hence \(\text{Ind}(G') \simeq \text{Ind}(G'')\).

Let \(G''\) be the graph \(G'' - \{ ((r-2, s+2), (r-1, s+2)), ((r-1, s+3), (r-1, s+2)) \}\) (cf. Figure 4d). Observe that, if \(((r-2, s+2), (r-1, s+2)) \in E(G'')\) then \(G'' - N_{G''}\{ ((r-2, s+2), (r-1, s+2)) \}\) has \((r, s+1)\) as an isolated vertex and hence \(\text{Ind}(G'' - N_{G''}\{ ((r-2, s+2), (r-1, s+2)) \})\) is contractible. Therefore, from Lemma 2.3, \(\text{Ind}(G'') \simeq \text{Ind}(G' - \{ ((r-2, s+2), (r-1, s+2), (r-1, s+2)) \})\). Similar argument shows that, if \(((r-1, s+3), (r-1, s+2)) \in E(G'')\) then \(\text{Ind}(G'') \simeq \text{Ind}(G' - \{ ((r-2, s+2), (r-1, s+2), (r-1, s+2)) \})\).

Since \(N_{G''}\{ (r-1, s+2) \} = N_{G''}\{ (r, s+1) \}\), using Lemma 2.4, we get \(\text{Ind}(G'') \simeq \text{Ind}(G' - \{ (r, s+1) \})\). Moreover, \(V(G'') = \{ (r, s+1) \}\) \(\cap N_{G' - \{ (r-1), s \}}((r, s)) = \emptyset\) implies that \(\text{Ind}(G'' - \{ (r, s+1) \}) \ast \{ (r, s) \} \subseteq \text{Ind}(G - \{ (r-1), s \})\) and therefore the inclusion map \(\text{Ind}(G'' - \{ (r, s+1) \}) \hookrightarrow \text{Ind}(G - \{ (r-1), s \})\) is nullhomotopic.

The conclusion of the above discussion is that the following composition of maps is nullhomotopic:

\[
\text{lk}((r-1, s), \text{Ind}(G)) \xrightarrow{\simeq} \text{Ind}(G'') \xrightarrow{\simeq} \text{Ind}(G'' - \{ (r, s+1) \}) \rightarrow \text{del}((r-1, s), \text{Ind}(G)).
\]

This completes the proof of Theorem 3.2.

In [7], the authors considered various subgraphs of square grids with open boundary conditions and computed the homotopy type of their independence complexes by using discrete Morse theory. Observe that all those subgraphs are solid grid graphs and hence Theorem 3.2 implies their results. However, we are not giving the number of spheres in the homotopy type.

4. The case of square grids with cylindrical identification

As in the case of square grids here also we make use of some induced subgraphs of \(C_{m,n}\) (defined below) in order to determine the homotopy type of \(\text{Ind}(C_{m,n})\).

**Definition 4.1.** A graph \(G\) is called a solid cylindrical grid graph (SCGG) if \(G\) can be obtained from \(C_{m,n}\) (for some \(m, n\)) by recursively removing the boundary vertices.

![Figure 5. Examples of SCGGs](image-url)
For example, in Figure 5, the graphs $G_1$ and $G_2$ are SCGGs whereas the graph $G_3$ is not a SCGG. The graph $G_1$ is obtained from $C_{4,3}$ by removing vertex $(4, 3)$, and $G_2$ is obtained from $C_{4,3}$ by removing vertex $(2, 3)$ first and then removing vertex $(3, 2)$.

We now state the main result of this section.

**Theorem 4.2.** For any $m > 1$ and $n \geq 1$, the independence complex of $C_{m,n}$ is homotopy equivalent to a wedge of spheres.

**Proof.** The proof is by induction on $n$. If $m \leq 7$ or $n \leq 3$, the result follows from [27], [32] and [20]. Therefore, assume that $m \geq 8$ and $n \geq 4$. Let the result be true for any $t < n$, i.e., $\text{Ind}(C_{m,t})$ is homotopy equivalent to a wedge of spheres for any $t < n$. Our aim now is to prove that the result is true for $t = n$.

For simplicity of notations, denote the graphs $C_{m,n} - \{(2, n-1)\}$ and $C_{m,n} - N_{C_{m,n}}[\{(2, n-1)\}]$ by $G_{m,n}$ and $H_{m,n}$ respectively (see Figure 6). Clearly, $\text{del}((2, n-1), \text{Ind}(C_{m,n})) = \text{Ind}(G_{m,n})$ and $\text{lk}((2, n-1), \text{Ind}(C_{m,n})) = \text{Ind}(H_{m,n})$. Thus, to prove our result using Lemma 2.1, it is enough to show that $\text{Ind}(G_{m,n})$ and $\text{Ind}(H_{m,n})$ are homotopy equivalent to a wedge of spheres, and $\text{Ind}(H_{m,n})$ is contractible in $\text{Ind}(G_{m,n})$.

![Diagram](image)

(a) $G_{m,n} = C_{m,n} - \{(2, n-1)\}$

(b) $H_{m,n} = C_{m,n} - N_{C_{m,n}}[\{(2, n-1)\}]$

**Claim 4.3.** (1) Let $G$ be a SCGG. If $G$ is a proper subgraph of $C_{m,n}$, then $\text{Ind}(G)$ is homotopy equivalent to a wedge of spheres.

(2) The inclusion map $\text{Ind}(H_{m,n}) \hookrightarrow \text{Ind}(G_{m,n})$ is nullhomotopic.

**Proof of Claim 4.3.** (1) The proof is by induction on the number of vertices of $G$. If $G$ has at most 2 vertices then the result follows easily. Let the result be true for any graph satisfying the hypothesis and having at most $|V(G)| - 1$ vertices.

Observe that, if every vertex of $G$ has degree at least 3 then $G$ is isomorphic to $C_{m,i}$ for some $i < n$ and thus the result follows from the induction hypothesis. Now assume that $G$ has a vertex of degree less than 3. In this case, the proof is similar to the proof of Theorem 3.2. To explain in short, if $G$ has a leaf vertex $\ell$ with $N_G(\ell) = \{w\} \text{ and } N_G(w) = \{\ell, v_1, \ldots, v_i\}$ then Lemma 2.4 implies that $\text{Ind}(G) \simeq \text{Ind}(G - \{v_1, \ldots, v_i\}) \cong \text{Ind}(H \sqcup P_3)$ where $H = G - \{\ell, w, v_1, \ldots, v_i\}$ satisfies the hypothesis of Claim 4.3(1). Hence the result follows from induction and Lemma 2.2.

Now let $G$ does not have any leaf vertex but it has a vertex of degree 2. In this case we can choose $(r, s) \in V(G)$ such that $(r, s)$ is of degree 2 and either $(i, j) \notin V(G)$ for any $i > r$ and $s \in [n]$ or $(i, j) \notin V(G)$ for any $i < r$ and $s \in [n]$. 


Without loss of generality, assume that \((i,j) \notin V(G)\) for any \(i > r\) and \(s \in [n]\) (cf. Figure 7a). Here we analyze the complexes \(lk((r, s - 1), \text{Ind}(G)) = \text{Ind}(G - N_G([r, s - 1]))\) and \(\text{del}(r, s - 1), \text{Ind}(G) = \text{Ind}(G - \{(r, s - 1)\})\). Using similar arguments as in the proof of Theorem 3.2, we get that the inclusion map \(lk((r, s - 1), \text{Ind}(G)) \hookrightarrow \text{del}(r, s - 1), \text{Ind}(G)\) is nullhomotopic. Moreover, observe that both the graphs \(G - N_G([r, s - 1])\) and \(G - \{(r, s - 1)\}\) satisfy the hypothesis of Claim 4.3(1). Therefore induction hypothesis implies that both the complexes \(lk((r, s - 1), \text{Ind}(G))\) and \(\text{del}(r, s - 1), \text{Ind}(G)\) are homotopy equivalent to a wedge of spheres. Hence the result follows from Lemma 2.1.

![Graph](image)

Figure 7

(2) Since \(N_{H_{m,n}}((1, n)) = \{(m, n)\} \subseteq N_{H_{m,n}}((m, n - 1))\) and \(N_{H_{m,n}}((3, n)) = \{(4, n)\} \subseteq N_{H_{m,n}}((4, n - 1))\), using Lemma 2.4, we get that \(\text{Ind}(H_{m,n}) \simeq \text{Ind}(H_{m,n} - \{(4, n - 1), (m, n - 1)\})\) (see Figure 7b). For simplicity of notations, denote the graph \(H_{m,n} - \{(4, n - 1), (m, n - 1)\}\) by \(H'\). It is easy to see that both the graphs \(H' - N_{H'}\{(5, n), (6, n)\}\) and \(H' - N_{H'}\{(5, n), (5, n - 1)\}\) have an isolated vertex \(3n\) and therefore their independence complexes are contractible. Hence, from Lemma 2.3, we get that \(\text{Ind}(H') \simeq \text{Ind}(H' - \{(5, n), (6, n), (5, n), (6, n)\})\). Using similar arguments, we get that \(\text{Ind}(H' - \{(5, n), (6, n), (5, n), (6, n)\}) \simeq \text{Ind}(H'' - \{(5, n), (6, n), (5, n), (6, n), (m - 1, n), (m - 2, n), (m - 1, n), (m - 1, n - 1)\})\). Denote the graph \(H'' - \{(5, n), (6, n), (5, n), (6, n), (m - 1, n), (m - 2, n), (m - 1, n), (m - 1, n - 1)\}\) by \(H''\).

Observe that \(N_{H''}(5, n) \subseteq N_{H''}(3, n)\) and \(N_{H''}(n - 1, n) \subseteq N_{H''}(1, n)\). Thus, using Lemma 2.4, we get \(\text{Ind}(H'') \simeq \text{Ind}(H'' - \{(1, n), (3, n)\})\). Moreover, \(V(H'' - \{(1, n), (3, n)\}) \cap N_{G_{m,n}}(2, n) = \emptyset\) implies that \(\text{Ind}(H'' - \{(1, n), (3, n)\}) \simeq \text{Ind}(G_{m,n})\). Hence the inclusion map \(\text{Ind}(H'' - \{(1, n), (3, n)\}) \hookrightarrow \text{Ind}(G_{m,n})\) is nullhomotopic.

From the above discussion we have that the following composition of maps is nullhomotopic:

\[
\text{Ind}(H_{m,n}) \xrightarrow{\simeq} \text{Ind}(H') \xrightarrow{\simeq} \text{Ind}(H'') \xrightarrow{\simeq} \text{Ind}(H'' - \{(1, n), (3, n)\}) \hookrightarrow \text{Ind}(G_{m,n}).
\]

This completes the proof of Claim 4.3.

From Claim 4.3, we get that \(\text{Ind}(H_{m,n})\) is homotopy equivalent to a wedge of spheres (since it satisfies the hypothesis of Claim 4.3(1) and the inclusion map \(\text{Ind}(H_{m,n}) \hookrightarrow \text{Ind}(G_{m,n})\)

is nullhomotopic. Therefore to prove Theorem 4.2 using Lemma 2.1, it is enough to prove the following result.

**Lemma 4.4.** The complex $\text{Ind}(G_{m,n})$ is homotopy equivalent to a wedge of spheres.

*Proof.* Here we analyze the link and deletion of vertex $(3, n - 1)$ in $\text{Ind}(G_{m,n})$. Denote the graphs $G_{m,n} - \{(3, n - 1)\}$ and $G_{m,n} - N_{G_{m,n}}[(3, n - 1)]$ by $A_{m,n}$ and $B_{m,n}$ respectively (see Figure 8).

![Figure 8](image)

Clearly the graph $B_{m,n}$ satisfies the hypothesis of Claim 4.3(1) and therefore $\text{Ind}(B_{m,n}) = \text{lk}((3, n - 1), \text{Ind}(G_{m,n}))$ is homotopy equivalent to a wedge of spheres. Moreover, using similar arguments as in the proof of Claim 4.3(2) we get that the inclusion map $\text{lk}((3, n - 1), \text{Ind}(G_{m,n})) \hookrightarrow \text{del}((3, n-1), \text{Ind}(G_{m,n}))$ is nullhomotopic. More precisely, the following composition of maps in nullhomotopic:

$$\text{Ind}(B_{m,n}) \xrightarrow{\sim} \text{Ind}(B') \xrightarrow{\sim} \text{Ind}(B'') \xrightarrow{\sim} \text{Ind}(B'' - \{(2, n), (4, n)\}) \hookrightarrow \text{Ind}(A_{m,n}),$$

where $B' = B_{m,n} - \{(1, n - 1), (5, n - 1)\}$, $B'' = B' - \{(m, n), (m, n - 1)\}, ((m - 1, n), (m, n)), ((6, n), (6, n - 1)), ((6, n), (7, n))\}$. Hence, to prove Lemma 4.4, it is enough to prove the following.

**Claim 4.5.** $\text{Ind}(A_{m,n})$ is homotopy equivalent to a wedge of spheres.

*Proof of Claim 4.5.* This is again done by analyzing the complexes $\text{lk}((4, n), \text{Ind}(A_{m,n})) = \text{Ind}(A_{m,n} - N_{A_{m,n}}[(4, n)])$ and $\text{del}((4, n), \text{Ind}(A_{m,n})) = \text{Ind}(A_{m,n} - \{(4, n)\})$. This time, we have the advantage that both the graphs $A_{m,n} - N_{A_{m,n}}[(4, n)]$ and $A_{m,n} - \{(4, n)\}$ satisfy the hypothesis of Claim 4.3(1), and hence their independence complexes are homotopy equivalent to a wedge of spheres. Moreover, using similar arguments as in previous cases, we get that the inclusion map $\text{lk}((4, n), \text{Ind}(A_{m,n})) \hookrightarrow \text{del}((4, n), \text{Ind}(A_{m,n}))$ is nullhomotopic. More precisely, the following composition of maps in nullhomotopic:

$$\text{Ind}(A_{m,n} - N_{A_{m,n}}[(4, n)]) \xrightarrow{\sim} \text{Ind}(A') \xrightarrow{\sim} \text{Ind}(A'') \xrightarrow{\sim} \text{Ind}(A'' - \{(2, n)\}) \hookrightarrow \text{Ind}(A_{m,n} - \{(4, n)\}),$$

where $A' = A_{m,n} - N_{A_{m,n}}[(4, n)] - \{(1, n - 1)\}$, $B'' = B' - \{(m, n), (m, n - 1)\}, ((m - 1, n), (m, n))\}$. This completes the proof of Claim 4.5.

Thus, Lemma 4.4 follows from the above discussion and Claim 4.5.

Hence, by combining Claim 4.5, Lemma 4.4 and Claim 4.3, we get the proof of Theorem 4.2.
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