The Super $W_\infty$ Symmetry of the Manin-Radul Super KP Hierarchy

A. Das,\(^1\) E. Sezgin,\(^2\)\(^\dagger\) and S.J. Sin \(^3\)\(^\ddagger\)

\(^1\)Department of Physics and Astronomy, University of Rochester, Rochester, NY 14627, USA
\(^2\)Center for Theoretical Physics, Texas A&M University, College Station, TX 77843-4242, USA
\(^3\)Department of Physics, University of Florida, Gainesville, FL 32611, USA

ABSTRACT

We show that the Manin-Radul super KP hierarchy is invariant under super $W_{\infty}$ transformations. These transformations are characterized by time dependent flows which commute with the usual flows generated by the conserved quantities of the super KP hierarchy.
1. Introduction

There has been a great deal of interest in the study of conformal symmetries in the past several years. In the context of high energy physics, the motivation for such studies came from string theories and has already led to some remarkable results. The conformal symmetries, in turn, have led in a natural way to the study of $W_N$ algebras [1]. While $W_N$ algebras do not define a Lie algebra, $W_\infty$ or $W_{1+\infty}$ algebras [2] are proper Lie algebras and are expected to play a role in the understanding of string theories [3][4].

In a completely parallel development, the integrable models have also attracted a lot of attention in recent years. For example, it is known now that one of the Hamiltonian structures (Poisson brackets) of the KdV equation is nothing other than the Virasoro algebra [5]. Similarly, it has become clear that the $W_N$ algebras arise as Hamiltonian structures of various integrable models in 1+1 dimensions [6]. The integrable models also play an important role in the study of various 2D gravity theories. For example, the gravitational Ward identity for pure 2D gravity in the light-cone gauge turns out to be none other than the KdV hierarchy equation [7] and similarly for other gravity theories.

The KP (Kadomtsev-Petviashvili) equation [8] is a 2+1 dimensional integrable system which leads to a large class of 1+1 dimensional integrable models upon appropriate reduction. In terms of the dynamical variable $u(x, y, t)$, the KP equation reads

$$\frac{\partial}{\partial x} \left( \frac{\partial u}{\partial t} - \frac{1}{4} \frac{\partial^3 u}{\partial x^3} - 3u \frac{\partial u}{\partial x} \right) = 3 \frac{\partial^2 u}{\partial y^2} \tag{1.1}$$

As is obvious, in the absence of y-dependence, Eq. (1.1) reduces to the KdV equation which, we have argued, plays an important role in the study of strings. It is, therefore, natural to expect that the KP equation may provide further understanding of the various string theories. In fact, it has already been pointed out that one of the Hamiltonian structures of the KP equation is isomorphic to the $W_{1+\infty}$ algebra [9] which is expected to play a significant role in the understanding of strings. It has also been argued that the KP hierarchy admits symmetries which satisfy the $W_{1+\infty}$ algebra [10][3].

One can, of course, supersymmetrize various integrable systems. In fact, the KP equation allows for more than one supersymmetric generalization [11][12]. And it is, of course, ultimately the superstring theory which is of physical interest. Therefore, it is the structure of the supersymmetries and the algebra of the symmetries for the supersymmetric systems that will be of direct physical significance. It is with this goal that we have undertaken, in this paper, the study of symmetries for the simplest of the super KP hierarchies, namely, the Manin-Radul hierarchy [11]. In Sec. II, we describe our notation and recapitulate, very briefly, the essential features of the Manin-Radul hierarchy. The symmetry conditions are discussed in detail in Sec. III and explicit symmetry generators as well as the symmetry transformations for the Manin-Radul hierarchy are given in Sec. IV. We would like to emphasize here that the usual conserved quantities associated with an integrable system, of
course, generate symmetries of the system. But what we are interested in are additional symmetries—in general time dependent. We find an infinite set of bosonic and fermionic generators of symmetry for the super KP hierarchy. In Sec. V, we show that the algebra generated by these generators is precisely the super $W_\infty$. Our conclusions are presented in Sec. VI.

2. The Manin-Radul Super KP Hierarchy

The supersymmetric extension of the KP hierarchy introduced by Manin and Radul [11] is a system of nonlinear equations for an infinite set of even and odd functions, depending on a pair of odd and even space variables $(\xi, x)$ and the odd-even times $(\tau_1, t_2, \tau_3, t_4, \ldots)$ \(^\dagger\). The manifold on which the solutions are defined, in this case, is a graded manifold. On this manifold, we can, of course, define the usual supercovariant derivative

$$\theta = \frac{\partial}{\partial \xi} + \xi \frac{\partial}{\partial x}$$

which satisfies

$$[\theta, \theta] = 2 \frac{\partial}{\partial x}$$

where the graded commutator is defined by $[A, B] = AB - (\mathbf{1})^{ab}BA$ with $a, b$ denoting the gradings of $A$ and $B$ and taking values 0 and 1 depending on whether the variable is bosonic or fermionic. A formal inverse of $\theta$ is defined to be

$$\theta^{-1} = \xi + \frac{\partial}{\partial \xi} \left( \frac{\partial}{\partial x} \right)^{-1}$$

On this manifold, we can also define the even and odd time derivatives as

$$\theta_{2i} = \frac{\partial}{\partial t_{2i}}, \quad \theta_{2i-1} = \frac{\partial}{\partial \tau_{2i-1}} + \sum_{j=1}^{\infty} \tau_{2j-1} \frac{\partial}{\partial t_{2i+2j-2}} \quad i, j = 1, 2, \ldots$$

These time derivatives satisfy the algebra

$$\begin{align*}
[\theta_{2i}, \theta_{2j}] &= 0 \\
[\theta_{2i}, \theta_{2j-1}] &= 0 \\
[\theta_{2i-1}, \theta_{2j-1}] &= 2\theta_{2i+2j-2}
\end{align*}$$

\(^\dagger\) $t^2$ and $t^4$ are to be identified with $y$ and $t$, respectively. In the bosonic case, the KP equation (1.1) then arises as the lowest member of a hierarchy of equations.
Furthermore, it is easy to see that both the even and the odd time derivatives have vanishing commutator with $\theta$, namely,

$$\left[ \theta, \theta_{2i} \right] = 0 = \left[ \theta, \theta_{2i-1} \right]$$

(2.6)

In other words, these derivatives are covariant with respect to supersymmetry transformations.

With these preliminaries, we can define the Lax operator for the Manin-Radul super KP hierarchy as a pseudo-differential operator on this graded manifold with the form

$$L = \theta + \sum_{i=1}^{\infty} U_i \theta^{-i}$$

(2.7)

where the $U_i$’s are functions of all the even and odd variables with the grading $(i+1)$. This is completely parallel to the bosonic KP hierarchy where the Lax operator is defined as a pseudo-differential operator involving $\partial / \partial x$. The generalized Leibnitz rule for the supercovariant derivatives is given by

$$\theta^i U = \sum_{j=0}^{\infty} (-1)^{u(i-j)} \begin{pmatrix} i \\ j \end{pmatrix} (\theta^j U) \theta^{i-j}$$

(2.8)

where the super-binomial coefficients are defined for $i \geq 0$ by

$$\begin{pmatrix} i \\ j \end{pmatrix} = \begin{cases} 0 & \text{for } j < 0 \text{ or } j > i \text{ or } (i, j) = (0, 1) \text{ mod } 2 \\ \begin{pmatrix} \frac{i}{2} \\ \frac{j}{2} \end{pmatrix} & \text{for } 0 \leq j \leq i \text{ and } (i, j) \neq (0, 1) \text{ mod } 2 \end{cases}$$

(2.9)

For $i < 0$, we define

$$\begin{pmatrix} i \\ j \end{pmatrix} = (-1)^{[\frac{i}{2}]} \begin{pmatrix} i + j - 1 \\ j \end{pmatrix}$$

(2.10)

The Manin-Radul super KP hierarchy can, then, be described in terms of the Lax equations [11]

$$\theta_i L = \left[ (L^i)_{\pm}, L \right] \quad i = 1, 2, \ldots$$

(2.11)

where by $A_+, A_-$, we will understand the parts of the pseudo differential operator $A$ containing nonnegative and only negative powers of $\theta$ respectively. The structure of Eq. (2.11) is quite analogous to the Lax equation for the bosonic KP hierarchy. Let us note here that the equations in Eq. (2.11) can also be written in an equivalent alternate form as

$$\begin{align*}
\theta_{2i} &= -\left[ (L^{2i})_+, L \right] \\
\theta_{2i-1} &= -\left[ (L^{2i-1})_+, L \right] + 2L^{2i}
\end{align*}$$

(2.12)
Furthermore, we can introduce a dressing operator, as in the bosonic case, by

\[ L = K \theta K^{-1} \]  \tag{2.13}

with

\[ K = 1 + \sum_{i=1}^{\infty} K_i \theta^{-i}. \]  \tag{2.14}

Consistency would then require that \( K_1 + \frac{1}{2} K_0 = 0 \). The Lax equation (2.11) can now be written as

\[ \theta_i K = (L^i)_- K \]  \tag{2.15}

With a little bit of algebra, one can easily show that the Lax equation (2.11) or equivalent Eq.(2.15) are consistent with the algebraic structure in Eq.(2.5). For the even time derivatives, then, we obtain the zero curvature condition

\[ \theta_{2i} (L^{2j})_- - \theta_{2j} (L^{2i})_- - \left[ (L^{2i})_-, (L^{2j})_- \right] = 0. \]  \tag{2.16}

3. The Symmetries of the Manin-Radul Super KP Hierarchy

The symmetries of the super KP hierarchy can be discussed equivalently at various levels. For example, one can study the symmetries of the linear equation or the symmetries of the evolution of the dressing operator or the symmetries of the Lax equation. From our point of view, it is most useful to study the evolution of the dressing operator. We know that (see Eq. (2.15))

\[ \theta_i K = (L^i)_- K \]  \tag{3.1}

Consider an infinitesimal deformation of the dressing operator \( K \), such that

\[ \delta K = \epsilon Q_- K \]  \tag{3.2}

where \( \epsilon \) is an infinitesimal constant parameter of deformation. Note that we have restricted the operator \( Q \) to its negative part. As we shall see, this restriction turns out to simplify considerably the symmetry condition on \( Q \) which we derive below. One can think of the deformation as being generated through a flow equation

\[ \phi K = Q_- K \]  \tag{3.3}

This deformation will be a symmetry of the super KP equation (namely, of Eq. (3.1) if \( \delta K \) satisfies

\[ \theta_i \delta K = (\delta L^i)_- K + (L^i)_- \delta K \]  \tag{3.4}
One can show quite easily from the relations in Eqs. (2.13) and (3.2) that
\[ \delta (L^i)_- = \epsilon [Q_-, L^i]_- \] (3.5)

The symmetry condition, Eq. (3.4), can now be equivalently written as
\[ \theta_i Q_- = - [L^i, Q_-] + [(L^i)_-, Q_-] \] (3.6)

This can be easily seen to take the simple form
\[ \theta_i Q_- = - [(L^i)_+, Q_-] \] (3.7)

In obtaining this simple form, it was essential that the symmetry operator Q is restricted to its negative part in (3.2). It is often convenient to express the symmetry generator as
\[ Q = KV K^{-1} \] (3.8)

The symmetry condition, Eq. (3.7), can then be shown to be equivalent to
\[ \theta_i V = - [\theta^i, V] \] (3.9)

The construction of symmetries of the super-KP hierarchy is, then, equivalent to determining \( V \)'s which satisfy Eq. (3.9). As a familiar example, let us note that
\[ V_n = \theta^{2n} \] (3.10)

automatically satisfies the above relation. These are nothing other than the familiar flows generated by the bosonic conserved quantities of the theory.

However, we are interested in additional time dependent symmetries of the theory. Thus, we make an ansatz for V of the following form
\[ V = \alpha \xi + x(\beta \theta + \gamma \xi \partial) + \sum_{n \geq 1} \left( \alpha_n^1 \theta + \alpha_n^2 \xi \partial \right) t_{2n-1} \partial^{2n-1} \]
\[ + \sum_{n \geq 1} \left( \alpha_n^3 + \alpha_n^4 \xi \theta \right) t_{2n-1} \partial^{2n-1} \]
\[ + \sum_{n,k \geq 1} \left( \alpha_{nk}^5 \theta + \alpha_{nk}^6 \xi \partial \right) t_{2n-1} t_{2k-1} \partial^{2n+k-2} \] (3.11)

where \( \alpha, \beta, \gamma, \alpha_n^1, \ldots, \alpha_{nk}^6 \) are constant coefficients to be determined. Requiring the symmetry condition to hold for even flows, we obtain
\[ \alpha_n^1 = -\beta n \]
\[ \alpha_n^2 = -\gamma n \] (3.12)
Similarly, requiring the symmetry condition to hold for the odd flows we determine all but two constants. Thus, a symmetry generator satisfying Eq. (3.9) can be written as

\[ V = \alpha \xi + \beta x \bar{\theta} - \beta \sum_{n \geq 1} nt_{2n} \partial^{n-1} \theta + \sum_{n \geq 1} (-\alpha + (n-1)\beta) \tau_{2n-1} \partial^{n-1} \]

\[ - \beta \sum_{n \geq 1} (2n-1) \tau_{2n-1} \partial^{n-1} \xi \theta + \beta \sum_{n,k \geq 1} n \tau_{2n-1} \tau_{2k-1} \partial^{n+k-2} \theta \]

where \( \alpha, \beta \) are the two arbitrary constants and

\[ \bar{\theta} = \frac{\partial}{\partial \xi} - \xi \frac{\partial}{\partial x} \]  

which satisfies

\[ [\theta, \bar{\theta}] = 0 \]

\[ \bar{\theta}, \bar{\theta} = -2\partial \]  

(3.15)

4. Algebraic Structure of the Symmetry Transformations

In this section, we would like to construct a canonical basis of the symmetry operators which would lead to a natural (graded) Lie-algebraic structure. From experience, we realize that in the coordinate basis, it is most convenient to isolate the symmetry operators into the form \( x + \ldots, \xi + \ldots, \partial + \ldots, \frac{\partial}{\partial \xi} + \ldots \). It is trivially seen from the symmetry condition in Eq. (3.9) that the operator \( \partial = \frac{\partial}{\partial x} \) is a symmetry operator. Furthermore, by choosing special values of the parameters \( \alpha \) and \( \beta \) in Eq. (3.13) we obtain

\[ \alpha = 1, \beta = 0 : \quad V_1 = \xi - \sum_{n \geq 1} \tau_{2n-1} \partial^{n-1} \equiv T \]  

(4.1)

\[ \alpha = 0, \beta = 1 : \quad V_2 = X \bar{\theta} \]  

(4.2)

where

\[ X = x - \sum_{n \geq 1} nt_{2n} \partial^{n-1} - \frac{1}{2} \sum_{n \geq 1} \tau_{2n-1} \partial^{n-2} ((2n-1)\theta - \bar{\theta}) \]

\[ + \sum_{n,k \geq 1} n \tau_{2n-1} \tau_{2k-1} \partial^{n+k-2} \]  

(4.3)

While \( T \) is already in the desired form, we need to reduce \( V_2 \) to the canonical form. To this end, we note that the commutator of two symmetry operators is, itself, a symmetry operator. Therefore, since

\[ [\partial, V_2] = \bar{\theta} \]  

(4.4)

we conclude that \( \bar{\theta} \) is a symmetry operator. (In fact, it generates space supersymmetry.) Furthermore, from the structure of \( V_2 \), which is a symmetry operator, it is then obvious that
$X$ must also be a symmetry operator. To find a symmetry operator with the coordinate form $\frac{\partial}{\partial \xi} + \ldots$, we note that

$$S \equiv \overline{\theta} + T \partial = \frac{\partial}{\partial \xi} - \sum \tau_{2n-1} \partial^n$$

is also a symmetry operator.

From the structures of the symmetry operators $\partial$, $X$, $S$, $T$ – two of which are bosonic and the other two fermionic – it is easy to verify that

$$[\partial, X] = 1 = [S, T]$$

with all other graded commutators vanishing. One can, therefore, make the following correspondence now, namely,

$$X \leftrightarrow z \quad \partial \leftrightarrow \frac{\partial}{\partial z}$$

$$T \leftrightarrow \kappa \quad S \leftrightarrow \frac{\partial}{\partial \kappa}$$

where $z$ is a bosonic variable while $\kappa$ is a Grassmann variable. With this identification, we can now show that the algebra of the symmetry operators is nothing other than the super $W^\pm_\infty$ algebra. We do this in the next section.

5. The Super $W_\infty$ Structure

The super $W_\infty$ algebra was constructed in Ref. [13]. The bosonic subalgebra is $W_\infty \oplus W_{1+\infty}$ generated by $V^i_m$ and $\bar{V}^i_m$, respectively, where $i + 2$ labels the (quasi) conformal spin of the generator, and $-\infty \leq m \leq \infty$. For $V^i_m$, $i = 0, 1, \ldots$ while for $\bar{V}^i_m$, $i = -1, 0, \ldots$. The fermionic generators are $G^{i\pm}_{m}$, where the (quasi) conformal spin of the generator is now $i + \frac{3}{2}$, $i = 0, 1, \ldots$ A realisation of this algebra in terms of super differential operators was given in Ref. [14]. These operators are as follows

\[
V^i_m = \sum_{\ell=0}^{i+1} \frac{1}{i+1} a^i_{m\ell} \left[ \ell + (i + 1 - \ell) \kappa \frac{\partial}{\partial \kappa} \right] z^{k-m} \left( \frac{\partial}{\partial z} \right)^\ell,
\]

\[
\bar{V}^i_m = \sum_{\ell=0}^{i+1} \frac{1}{2i+1} a^{i\ell}_{m\ell} \left[ -\ell + (i + \ell + 2) \kappa \frac{\partial}{\partial \kappa} \right] z^{k-m} \left( \frac{\partial}{\partial z} \right)^\ell,
\]

\[
G^{i\pm}_{m-\frac{1}{2}} = \sum_{\ell=0}^{i} z^{\ell+1-m} \left( \frac{\partial}{\partial z} \right)^\ell \left[ \frac{2\ell}{i+\ell+1} a^{i\ell}_{m,\ell+1} \kappa \frac{\partial}{\partial \kappa} \pm \frac{i+\ell+1}{2i+1} a^{i-1}_{m-1,\ell+1} \frac{\partial}{\partial \kappa} \right].
\]

\* In Ref. [14], one parameter (denoted by $\lambda$) family of super $W_\infty$ algebras are given, which correspond to different choices of basis. Here we have made the choice $\lambda = 0$, which produces the super $W_\infty$ algebra constructed explicitly in Ref. [13].
where
\[ a^i_{m\ell} = \binom{i + 1}{\ell} \frac{(m - i - 1)_{i+1-\ell}(-i - 1)_{i+1-\ell}}{(i + \ell + 2)_{i+1-\ell}}, \]  
(5.2)

where we have used the definition
\[ (a)_n \equiv \frac{\Gamma(a + n)}{\Gamma(a)} = a(a + 1)(a + 2) \cdots (a + n - 1), \quad \text{with} \quad (a)_0 = 1. \]  
(5.3)

The reason for complicated choices for coefficients is to ensure that the resulting generator has a definite transformation property under the $SL(2, R)$ subalgebra of the Virasoro algebra generated by $V^0_m$. This $SL(2, R)$ covariance property allows one to assign (quasi) conformal spin to each generator, thus facilitating the use of many conformal field theoretic techniques in dealing with this algebra.

It is clear now how to identify the super $W_\infty$ structure of the symmetry transformations of the previous section. Since any power of the basic symmetry operators $\partial, X, S$ and $T$ is also a symmetry operator, using the correspondence (4.7), we have the following symmetry flows with manifest super $W_\infty$ structure:
\[ \frac{\partial}{\partial \varepsilon^i_{m}} K = (K V^i_m K^{-1})_{-} K, \]
\[ \frac{\partial}{\partial \varepsilon^i_{m}} K = (K \bar{V}^i_m K^{-1})_{-} K, \]
\[ \frac{\partial}{\partial \varepsilon^{i \pm}_{m - 1/2}} K = (K G^{i \pm}_{m - 1/2} K^{-1})_{-} K, \]  
(5.4)

These are time dependent, non-isospectral flows which commute with the isospectral super KP flows (2.11). We emphasize that in (5.4), the replacements $z \to X, \frac{\partial}{\partial z} \to \theta^2, \kappa \to T, \frac{\partial}{\partial \kappa} \to S$ are to be made. For example, some of the low lying generators are given explicitly as follows
\[ V^0_m = X^{1-m} \theta^2 - \frac{1}{2}(m - 1)X^{-m}TS, \]
\[ V^1_m = X^{2-m} \theta^4 - \frac{1}{2}(m - 2)(1 + TS)X^{1-m} \theta^2 + \frac{1}{3}(m - 1)(m - 2)X^{-m}TS, \]
\[ \bar{V}^{-1}_m = -X^{-m}TS, \]
\[ \bar{V}^{-1}_m = (-1 + 3TS)X^{1-m} \theta^2 - (m - 1)X^{-m}TS, \]
\[ G^{0\pm}_{m - 1/2} = X^{1-m}(T \theta^2 \pm S), \]
\[ G^{1\pm}_{m - 1/2} = X^{2-m} \theta^2(T \theta^2 \pm S) - \frac{1}{3}(m - 2)X^{1-m}(2T \theta^2 \pm S). \]  
(5.5)

It is straightforward to show that $V^0_m$ obey the Virasoro algebra. Other interesting subalgebras of the full super $W_\infty$ are discussed in Ref. [13], and contraction down to the classical
version super $w_\infty$ which is equivalent to super symplectic diffeomorphism of a suitable supermanifold are discussed in Refs. [13] and [15]. In particular, there exists the subalgebra super $W_\infty^{\nu}$ where $m \leq i + 1$ in $V^i_m$, in which case no negative powers of $z$ occur in (5.1).

6. Conclusions

We have constructed flows (5.4) which commute with the Manin-Radul super KP flows (2.11). These, therefore, represent the symmetries of the Manin-Radul super KP hierarchy. We would like to emphasize here that the usual conserved quantities associated with an integrable system also generate symmetries of the system, but what we have obtained here are additional symmetries which are in general time dependent.

The action of the basic symmetry operators $\partial$, and $T, X, S$ defined in (4.1), (4.3), (4.5), respectively, on the vacuum solution (2.20) can not all be represented in terms of differential operators involving the spectral parameters $\lambda$ and $\eta$ alone. In the case of of the bosonic KP hierarchy this is possible. It may as well be possible, and it would be useful, to construct a similar representation of the symmetry operators in terms of the spectral parameters in the case of super KP hierarchy as well, by taking suitable combinations of the symmetry operators given in this paper.

We conclude by pointing out some further interesting problems which deserve study. In Ref. [16] it has been shown that the KP hierarchy has additional symmetries which obey a Kac-Moody-Virasoro based on a subalgebra of $Vir \oplus \hat{\mathfrak{sl}}(5,R)$. It is not clear whether this means that the symmetry ring of the Manin-Radul system admits a Kac-Moody extended version of $W_\infty^+$ which contains the symmetry algebra of Ref. [16] as a subalgebra. To understand this, it will be useful first to realize such an extended algebra in terms of differential operators in a manner similar to Eq. (5.1).

The KP hierarchy has other supersymmetric extensions as well [12]. It would be interesting to explore the symmetry properties of these extensions in a systematic manner and to compare with one another. What would be very interesting in this context is to find a theory of a relativistic membrane or even higher extended objects whose equations of motion would belong to a KP hierarchy, in which case the large symmetry of the hierarchy, such as those found in this paper, could be utilized to generate infinitely many solutions, and infinitely many relations between scattering matrix elements, thus rendering the theory, if not soluble, at least manageable.

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