Pauli energy spectrum for twist-deformed space-time

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Abstract

In this article, we define the Pauli Hamiltonian function for twist-deformed N-enlarged Newton-Hooke space-time provided in article [12]. Further, we derive its energy spectrum, i.e., we find the corresponding eigenvalues as well as the proper eigenfunctions.
The suggestion to use noncommutative coordinates goes back to Heisenberg and was firstly formalized by Snyder in [1]. Recently, there were also found formal arguments based mainly on Quantum Gravity [2], [3] and String Theory models [4], [5], indicating that space-time at the Planck scale should be noncommutative, i.e., it should have a quantum nature. Consequently, there appeared a lot of papers dealing with noncommutative classical and quantum mechanics (see e.g. [6], [7]) as well as with field theoretical models (see e.g. [8], [9]), in which the quantum space-time is employed.

In accordance with the Hopf-algebraic classification of all deformations of relativistic [10] and nonrelativistic [11] symmetries, one can distinguish three basic types of space-time noncommutativity (see also [12] for details):

1) Canonical ($\theta^{\mu\nu}$-deformed) type of quantum space [13]-[15]

\[
[x_{\mu}, x_{\nu}] = i\theta^{\mu\nu} ,
\]

2) Lie-algebraic modification of classical space-time [15]-[18]

\[
[x_{\mu}, x_{\nu}] = i\theta^{\rho\mu\nu} x_{\rho} ,
\]

and

3) Quadratic deformation of Minkowski and Galilei spaces [15], [18]-[20]

\[
[x_{\mu}, x_{\nu}] = i\theta^{\rho\tau\mu\nu} x_{\rho} x_{\tau} ,
\]

with coefficients $\theta^{\mu\nu}$, $\theta^{\rho\mu\nu}$ and $\theta^{\rho\tau\mu\nu}$ being constants.

Moreover, it has been demonstrated in [12], that in the case of the so-called N-enlarged Newton-Hooke Hopf algebras $U_0^{(N)}(NH_{\pm})$ the twist deformation provides the new space-time noncommutativity of the form

\[
4) \quad [t, x_i] = 0 \quad , \quad [x_i, x_j] = if_{\pm}\left(\frac{t}{\tau}\right)\theta_{ij}(x) ,
\]

with time-dependent functions

\[
f_+\left(\frac{t}{\tau}\right) = f\left(\sinh\left(\frac{t}{\tau}\right) , \cosh\left(\frac{t}{\tau}\right)\right) \quad , \quad f_-\left(\frac{t}{\tau}\right) = f\left(\sin\left(\frac{t}{\tau}\right) , \cos\left(\frac{t}{\tau}\right)\right) ,
\]

$\theta_{ij}(x) \sim \theta_{ij} = \text{const}$ or $\theta_{ij}(x) \sim \theta^{k}_{ij} x_k$ and $\tau$ denoting the time scale parameter - the cosmological constant. Besides, it should be noted, that the above mentioned quantum

\[1x_0 = ct.\]

\[2\] The discussed space-times have been defined as the quantum representation spaces, so-called Hopf modules (see e.g. [13], [14]), for the quantum N-enlarged Newton-Hooke Hopf algebras.
spaces 1), 2) and 3) can be obtained by the proper contraction limit of the commutation relations 4).

In this article we investigate the impact of the twisted N-enlarged Newton-Hooke space-time \[12\] on the Pauli energy spectrum \[21\]. Firstly, however, we remained the basic facts concerning the Pauli model defined on classical space. In this aim we start with the following canonical commutation relations for momentum and position operators \((x_i, p_i)\)

\[
[x_i, x_j] = 0 = [p_i, p_j] , \quad [x_i, p_j] = i\hbar\delta_{ij} ,
\]

Then, the Hamiltonian (Pauli) function for nonrelativistic electron with spin, moving in the external electric field \(\vec{E} = -\text{grad}\phi\) and in the magnetic field \(\vec{B} = \text{rot}\vec{A}\), is defined as a sum of two mutually commuting terms

\[
H = H_1(\vec{p}, \vec{x}) + H_2(\vec{x}, \vec{\sigma}) ,
\]

such that

\[
H_1(\vec{p}, \vec{x}) = \frac{1}{2m} \left( \vec{p} + \frac{e}{c} \vec{A}(\vec{x}) \right)^2 - e\phi(\vec{x}) ,
\]

and

\[
H_2(\vec{x}, \vec{\sigma}) = \frac{1}{2m} \vec{\sigma} \cdot \vec{B}(\vec{x}) ,
\]

with Pauli matrices vector \(\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)\). Besides, it is easy to see, that in so-called symmetric gauge framework

\[
\vec{A}(\vec{x}) = \left[ -\frac{B}{2}x_2, \frac{B}{2}x_1, 0 \right] ,
\]

and for the following choice of the electric field \(\phi(\vec{x})\)

\[
\phi(\vec{x}) = -Ex_1 ,
\]

the above operators take the form

\[
H_1(\vec{p}, \vec{x}) = \frac{1}{2m} \left[ p_1 - \frac{eB}{2c}x_2 \right]^2 + \left[ p_2 + \frac{eB}{2c}x_1 \right]^2 + eEx_1 ,
\]

\[
H_2(\vec{x}, \vec{\sigma}) = H_2(\sigma_3) = \frac{1}{2m} \sigma_3 B = \frac{B}{4m} \left[ 1 \quad 0 \\
0 \quad -1 \right] .
\]

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3Such a result indicates that the twisted N-enlarged Newton-Hooke Hopf algebra plays a role of the most general type of quantum group deformation at nonrelativistic level.

4In the formula (5) \(\kappa_a\) denotes the deformation parameter such that \(\lim_{\kappa \to 0} f_{\kappa}(t) = 0\).

5It means, that from physical point of view, we investigate the impact of high-energy (transplanckian) regime on the one of the most important quantum system, such as the nonrelativistic electron with spin \(s = \frac{1}{2}\), moving in constant electric and magnetic fields simultaneously.

6The symbols \(m, e, c\) denote mass, electric charge and speed of light respectively.

7From now, for simplicity, we consider electron moving in \((x_1, x_2)\)-plane.
The energy spectrum of first term (12) has been found with use of creation/annihilation operator technique in paper [22]. It looks as follows

$$H_1(\bar{p}, \bar{x})\psi_{(n,\alpha)}(\bar{x}) = E_{(n,\alpha)}\psi_{(n,\alpha)}(\bar{x}) \, ,$$

where

$$\psi_{(n,\alpha)}(\bar{x}) = \exp i \left[ \alpha x_2 + \frac{m\omega}{2\hbar} x_1 x_2 \right] \cdot \frac{1}{\sqrt{(2m\hbar\omega)^n n!}} (a^\dagger)^n |0\rangle \, ,$$

$$E_{(n,\alpha)} = \frac{\hbar\omega}{2} (2n + 1) - \frac{1}{m} \left[ \hbar\lambda\alpha - \frac{\lambda^2}{2} \right] \, ,$$

with $n = 0, 1, 2, \ldots$, real parameter $\alpha$, cyclotron frequency $\omega = \frac{eB}{mc}$, parameter $\lambda = \frac{mcE}{B}$ and with $(a, a^\dagger)$-objects given by

$$a^\dagger = -2ip^* + \frac{eB}{2c} x + \lambda \, , \quad a = 2ip + \frac{eB}{2c} x^* + \lambda \, ,$$

$$x = x_1 + ix_2 \, , \quad p = \frac{1}{2}(p_1 - ip_2) \, ,$$

$$(\alpha + i\beta)^* = \alpha - i\beta \, , \quad a|0\rangle = 0 \, .$$

The eigenvectors and eigenvalues of the second term (13) can be find as well. They take the form

$$H_2(\sigma_3)\psi_{\pm}(\bar{x}) = E_{\pm}\psi_{\pm}(\bar{x}) \, ,$$

where

$$\psi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \, , \quad \psi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \, , \quad E_{\pm} = \pm B \frac{2m}{2m}$$

Consequently, the energy spectrum of the whole Hamiltonian operator (7) looks as follows

$$H(\bar{p}, \bar{x}) = \psi_{\pm(n,\alpha)}(\bar{x}) = E_{\pm(n,\alpha)}\psi_{\pm(n,\alpha)}(\bar{x}) \, ,$$

with

$$\psi_{\pm(n,\alpha)}(\bar{x}) = \psi_{(n,\alpha)}(\bar{x}) \otimes \psi_{\pm} \, ,$$

and

$$E_{\pm(n,\alpha)} = E_{(n,\alpha)} + E_{\pm} = \frac{\hbar\omega}{2} (2n + 1) - \frac{1}{m} \left[ \hbar\lambda\alpha - \frac{\lambda^2}{2} \pm \frac{B}{2} \right] \, ,$$

respectively.

Let us now turn to the main aim of our investigations - to the derivation of Pauli energy levels for quantum space-times [5]. For this purpose, we extend the deformed spaces to the whole algebra of momentum and position operators as follows

$$[\hat{x}_1, \hat{x}_2] = i\hbar \kappa_a(t) \, , \quad [\hat{p}_i, \hat{p}_j] = 0 \ , \quad [\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij} \, .$$
Next, by analogy to the commutative case, we define the twist-deformed Hamiltonian operator as (see \(^{(12)},\, (13)\))
\[
\hat{H}(t) = \hat{H}(\hat{p}, \hat{x}) = \frac{1}{2m} \left[ \left( \hat{p}_1 - \frac{e}{2c} \hat{B}(f_{\kappa_a}(t))\hat{x}_2 \right)^2 + \left( \hat{p}_2 + \frac{e}{2c} \hat{B}(f_{\kappa_a}(t))\hat{x}_1 \right)^2 \right] + e\hat{E}(f_{\kappa_a}(t))\hat{x}_1 + \frac{1}{2m}\sigma_3\hat{B}(f_{\kappa_a}(t)) ,
\]
where
\[
\hat{B}(f_{\kappa_a}(t)) = \frac{2}{f_{\kappa_a}(t)} \left[ (1 - f_{\kappa_a}(t)B)^{-\frac{1}{2}} - 1 \right] , \tag{25}
\]
\[
\hat{B}(f_{\kappa_a}(t)) = \frac{B}{1 + f_{\kappa_a}(t)B} , \tag{26}
\]
\[
\hat{E}(f_{\kappa_a}(t)) = \frac{E}{1 + f_{\kappa_a}(t)E} . \tag{27}
\]

It should be noted, that the form of the above magnetic inductions (\(\hat{B}(f_{\kappa_a}(t)), \hat{B}(f_{\kappa_a}(t))\)) as well as the electric field \(\hat{E}(f_{\kappa_a}(t))\) is dictated by the results of papers \(^{[5]}\) and \(^{[24]}\), which concern the classical electrodynamics defined for the canonical noncommutativity \(^{[1]}\). Then, as we shall see for a moment, for the most simple case \(f_{\kappa_a}(t) = \kappa_a = \theta = \text{const.},\) we reproduce from the obtained in present article energy levels the correct Pauli spectrum provided in \(^{[24]}\). Besides, one can check, that in terms of commutative variables \((x_i, p_i)\) the operator (24) takes the form
\[
\hat{H}(t) = \hat{H}_1(t) + \hat{H}_2(t) , \tag{28}
\]
with
\[
\hat{H}_1(t) = \frac{1}{2m} \left[ \left( (1 - \alpha_{\kappa_a}(t))p_1 - \frac{e}{2c} \hat{B}(f_{\kappa_a}(t))x_2 \right)^2 + \left( (1 - \alpha_{\kappa_a}(t))p_2 + \frac{e}{2c} \hat{B}(f_{\kappa_a}(t))x_1 \right)^2 \right] + e\hat{E}(f_{\kappa_a}(t)) \left[ \frac{x_1 - f_{\kappa_a}(t)}{2\hbar} p_2 \right] = \hat{H}(\hat{p}, \hat{x}, t) , \tag{29}
\]
\[
\hat{H}_2(t) = \frac{1}{4m} \sigma_3\hat{B}(f_{\kappa_a}(t)) = \hat{H}_2(\sigma_3, t) , \quad \alpha_{\kappa_a}(t) = \frac{e}{4\hbar c} f_{\kappa_a}(t) \hat{B}(f_{\kappa_a}(t)) , \tag{30}
\]
and
\[
[ \hat{H}_1(t), \hat{H}_2(t) ] = 0 . \tag{31}
\]

The spectrum of the above \(\hat{H}_1\)-term can be derived with use of ladder operator scheme in the same manner as in paper \(^{[23]}\). It is given by
\[
\hat{H}_1(t)\psi_{(n,\alpha,\kappa_a)} = E_{(n,\alpha,\kappa_a)}(t)\psi_{(n,\alpha,\kappa_a)}(t) , \tag{32}
\]
We use the standard link between both types of operators defined by: \(\hat{x}_i = x_i - \epsilon_{ij} \frac{f_{\kappa_a}(t)}{2\hbar} p_j\) and \(\hat{p}_i = p_i\) (see e.g. \(^{[25]}\)); the calculations are performed in the same way as in article \(^{[23]}\).
where \( n = 0, 1, 2, \ldots, \alpha \in \mathbb{R} \) and

\[
\psi_{(n,\alpha,\kappa)}(t) = \exp\left[ i \left( \alpha x_2 + \frac{m\omega(t)}{2\hbar\beta^2(t)} x_1 x_2 \right) \cdot ((2m\hbar\omega(t))^n n!)^{-1/2} \right] \\
\cdot \left( a^\dagger(t) \right)^n |0> \nonumber ,
\]

\[
E_{(n,\alpha,\kappa)}(t) = \frac{\hbar\omega(t)}{2} (2n + 1) - \frac{1}{m} \hbar \beta(t) \lambda_+(t) \alpha - \frac{m}{2} \lambda_-^2(t) ,
\]

\[
a^\dagger(t) = -2i\tilde{p}^\dagger(t) + \frac{e}{2c} \tilde{B}(f_{\kappa}(t)) x + \lambda_- (t) ,
\]

\[
a(t) = 2i\tilde{p}(t) + \frac{e}{2c} \tilde{B}(f_{\kappa}(t)) x^* + \lambda_- (t) ,
\]

\[
\beta(t) = (1 - \alpha_{\kappa}(t)) , \quad \lambda_{\pm}(t) = \lambda_{\pm} \frac{em}{4 \beta(t) \hbar} \hat{E}(f_{\kappa}(t)) f_{\kappa}(t) ,
\]

\[
\omega(t) = \beta(t) \omega , \quad \bar{p}(t) = \beta(t) p(t) .
\]

The eigenvalues and eigenvectors of the second term \( \hat{H}_2(t) \) are defined by the equation \([19]\). Consequently, due to the commutation relations \([31]\), the twist-deformed (Pauli) energy levels look as follows

\[
\psi_{\pm(n,\alpha,\kappa)}(t) = \exp\left[ i \left( \alpha x_2 + \frac{m\omega(t)}{2\hbar\beta^2(t)} x_1 x_2 \right) \cdot ((2m\hbar\omega(t))^n n!)^{-1/2} \right] \\
\cdot \left( a^\dagger(t) \right)^n \otimes \psi_{\pm} ,
\]

\[
E_{\pm(n,\alpha,\kappa)}(t) = E_{(n,\alpha,\kappa)}(t) + E_{\pm} = \frac{\hbar\omega(t)}{2} (2n + 1) - \frac{\hbar}{m} \beta(t) \lambda_+(t) \alpha + \nonumber \\
- \frac{m}{2} \lambda_-^2(t) \pm \frac{1}{2m} \hat{B}(f_{\kappa}(t)) .
\]

It is easy to see, that in obvious way they depend on time, and such a property probably follows from nonstationary character of the background space-time noncommutativity \([7]\). In other words, from physical point of view, the above mentioned attribute can be interpreted as an direct impact of quantum space dynamics just on the total energy of moving particle. However, unfortunately, the deeper understanding of such a behavior of the discussed system seems to be at this stage rather difficult, and it is postponed for the future investigations. Apart of that, one should also observe, that the spectrum \([34]\) and \([35]\) structurally appears very similar to its commutative counterpart, given by the formulas \([21]\) and \([22]\), i.e., it still remains linear for example in frequency parameter \( \omega(t) \), and it stays quadratic in function \( \lambda_-(t) \). Probably such a feature follows from the specific, central charge-like form of the commutation relations \([5]\), and in the case of another, nontrivially deformed quantum spaces, it should be rather lost. Of course, for \( f_{\kappa}(t) = 0 \) the above energy levels become classical, while for the most simple (canonical) space-time noncommutativity \([1]\) - given by \( f_{\kappa}(t) = \theta \) - they reproduce for \( E = 0 \) the results of article \([24]\).
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