Seismic Analysis of Simply Supported Damped Rayleigh Beams on Elastic Foundation

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Abstract

In this paper, the flexural analysis of a simply supported damped Rayleigh beam subjected to distributed loads and with damping due to resistance to the transverse displacement resting on elastic foundation is obtained. The characteristics of the beam are assumed uniform over the beam length while the foundation is considered of Winkler type. In order to evaluate the vibration characteristics of the dynamical system, the Fourier sine integral transformation in conjunction with the asymptotic method of Struble is used to solve the governing equations for the transversal vibrations in the beam structure induced by moving load. The effect of prestress and other structural parameters were considered. Numerical results show that the structural parameters have significant influence on the behaviour of the dynamical system.

Keywords: Flexural analysis; simply supported; rayleigh beam; dynamical system; damped beam.

1 Introduction

The study of characteristics of elastic structure such as beams under various form of moving loads is of great significant importance in the field of Engineering and Mathematical Physics. As such, many practical engineering applications as railroad track, highway pavement, buried pipelines and foundation beams are modeled as beams resting on elastic foundation [1-3]. The beam is a simple model of one-dimensional continuous system. This simple structure is important from theoretical and engineering points of view, and

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He did not incorporate the damping effect of the system. In the present work, the model is extended to supported by an elasti
loads. He obtained the close form solutions of the one exponentially varying thickness resting on Vlasov foundation under variable harmonic load moving with this end, Jimoh [15]
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deflection beam theories agree well with each other. Ogunbamike and Oni [14] carried out the dynamic
response to partially distributed moving masses of non-prismatic Rayleigh beam with classical boundary
conditions and resting on Vlasov elastic foundation moving with variable velocities. From the above
literature survey, it is noted that the effect of damping on the dynamical system, which is one of the most
important as well difficult and complicated problems in vibration theory of mechanical system has been
neglected. It is important in vibration-isolation system to have damping to attenuate excessive vibration near
resonance. In order to properly understand the control and dynamic response of vibrating structures to
moving loads, it is imperative to carry out objective analyses of the effect of damping on such structures. To

Celep et al. [10] considered the static and dynamic responses of a completely free elastic beam resting on a
two-parameter tensionless Pasternak foundation. Governing equations of the problem are solved by paying
attention on the boundary conditions including the concentrated edge foundation reaction in the case of complete contact and lift-off condition of the beam. Later, Civalek and Kaircioğlu [11] considered the free vibration analysis of Timoshenko beam using discrete singular convolution method for numerical solutions of equation of motion. Clamped, pinned and sliding boundary conditions and their combinations are taken into account. Typical results are presented for different parameters and boundary conditions. Numerical results show that the results obtained are in good agreement with that available in literature. The buckling analysis of material sandwich plate based on a two-parameter elastic foundation under various boundary conditions on the basis of a new theory of refined trigonometric shear deformation was tackled by Chikr et al. [12]. The principle of virtual displacements is used to obtain the governing equations of the boundary conditions. Galerkin’s approach is utilized to solve the buckling problem for the six boundary conditions of the sandwich plates. They carried out a detailed numerical study to examine the influence of the plate aspect ratio, elastic foundation ratio, side-to-thickness ratio and boundary conditions on the buckling response of FG sandwich plate. Much later, the microstructures-dependent buckling behaviour of single walled carbon nanotubes surrounded by a two-parameter elastic foundation is investigated by Akgoz and Civalek [13]. The equation of motion and the corresponding boundary conditions are achieved by implementing minimum total potential energy principle via modified strain gradient theory and several beam theories. In their work, they employed the Navier solution procedure for simply supported boundary conditions to solve the resulting equation analytically. Numerical results reveal that the classical buckling loads evaluated by all shear deformation beam theories agree well with each other. Ogunbamike and Oni [14] carried out the dynamic response to partially distributed moving masses of non-prismatic Rayleigh beam with classical boundary conditions and resting on Vlasov elastic foundation moving with variable velocities. From the above
include the effect of foundation stiffness, axial force and damping due to resistance to the transverse displacement on the vibration of the flexural beams. The foundation reaction is modeled as Winkler type.

2 Governing Equations

The dynamic transverse displacement \( W(x, y) \) of a Rayleigh beam when it is resting on a Winkler elastic foundation and traversed by several moving distributed masses is governed by the fifth order partial differential equation given by

\[
\frac{\partial^2}{\partial x^2} \left[ EI \frac{\partial^2 W(x, t)}{\partial x^2} + C^0 I \frac{\partial^3 W(x, t)}{\partial x^3 \partial t} \right] + \mu \frac{\partial^2 W(x, t)}{\partial t^2} - N \frac{\partial^2 W(x, t)}{\partial x^2} - \mu R^0 \frac{\partial^4 W(x, t)}{\partial x^4 \partial t^2} + KW(x, t) = MgH \left[ x - vt \right] \left[ 1 - \frac{\Delta^*}{g} W(x, t) \right]
\]

(1)

where \( E \) is the Young’s Modulus, \( I \) is the Moment of inertia, \( C^0 \) is the damping due to resistance to the transverse displacement, \( \mu \) is the mass per unit length of the beam, \( N \) is the axial force, \( K \) is the elastic foundation constant, \( W(x, t) \) is the transverse displacement, \( M \) is the mass of the distributed load, \( x \) is the spatial coordinate, the time \( t \) is assumed to be limited to that interval of time within which the mass on the beam, that is

\[
0 \leq vt \leq L
\]

(2)

\( g \) is the acceleration due to gravity and \( \Delta^* \) is the convective acceleration operator defined in Fryba [1]

\[
\Delta^* = \frac{\partial^2}{\partial t^2} + 2v \frac{\partial^2}{\partial x \partial t} + v^2 \frac{\partial^2}{\partial x^2}
\]

(3)

where \( v \) is the initial velocity and \( H[x - vt] \) is the Heaviside function defined by

\[
H[x - vt] = \begin{cases} 
0 & \text{if } x < vt \\
1 & \text{if } x > vt \end{cases}
\]

(4)

The boundary conditions of the structure under consideration are

\[
W(0, t) = W(L, t) = 0; \quad \frac{\partial^2 W(0, t)}{\partial x^2} = \frac{\partial^2 W(L, t)}{\partial x^2} = 0
\]

(5)

The initial conditions, without any loss of generality is taken as is taken to be

\[
W(x, 0) = 0 = \frac{\partial W(x, 0)}{\partial t}
\]

(6)

Substituting equation (3) into equation (1) after some simplifications and rearrangements, one obtains
\[
\frac{\partial^2}{\partial x^2} \left[ EI \frac{\partial^2 W(x,t)}{\partial x^2} + C^0 I \frac{\partial W(x,t)}{\partial t^2} \right] + \mu \frac{\partial^2 W(x,t)}{\partial t^2} - N \frac{\partial^2 W(x,t)}{\partial x^2\partial t} - \mu R^0 \frac{\partial^4 W(x,t)}{\partial x^2\partial t^2} \\
+ KW(x,t) + MH \left[ x - vt \right] \left( \frac{\partial^2}{\partial t^2} + 2v \frac{\partial^2}{\partial x\partial t} + v^2 \frac{\partial^2}{\partial x^2} \right) W(x,t) = MgH [x - vt] \tag{7}
\]

3 Analytical Approximate Solution

In this section, use is made of the Fourier sine integral transform. The integral transformation technique is defined by

\[
W(m,t) = \int_0^L W(x,t) \sin \frac{m\pi x}{L} \, dx \tag{8}
\]

with inverse

\[
\overline{W}(x,t) = \frac{2}{L} \sum_{m=1}^{\infty} W(m,t) \sin \frac{m\pi x}{L} \tag{9}
\]

Applying the Finite Fourier Sine Integral Transform (8), equation (7) becomes

\[
EIF_x W(m,t) + C^0 IF_x W_x (m,t) + \mu W_x (m,t) - NF_x W(m,t) + KW(m,t) \\
+ \mu R^0 F_x W_x (m,t) + F_C (t) + 2vF_D (t) + v^2 F_E (t) = Mg \int_0^L H(x - vt) \sin \frac{m\pi x}{L} \, dx \tag{10}
\]

Where

\[
F_A = \left( \frac{m\pi}{L} \right)^4 \int_0^{\frac{L}{m\pi}} \sin \frac{m\pi x}{L} \, dx \tag{11}
\]

\[
F_B (t) = \left( \frac{m\pi}{L} \right)^2 \int_0^{\frac{L}{m\pi}} \sin \frac{m\pi x}{L} \, dx \tag{12}
\]

\[
F_C (t) = M \int_0^L H(x - vt) \frac{\partial^2 W}{\partial t^2} \sin \frac{m\pi x}{L} \, dx \tag{13}
\]

\[
F_D (t) = M \int_0^L H(x - vt) \frac{\partial^2 W}{\partial x\partial t} \sin \frac{m\pi x}{L} \, dx \tag{14}
\]

\[
F_E (t) = M \int_0^L H(x - vt) \frac{\partial^2 W}{\partial x^2} \sin \frac{m\pi x}{L} \, dx \tag{15}
\]
Using the property of the Heaviside function defined as

\[ H(x - vt) = \frac{1}{4} + \frac{1}{\pi} \sum_{n=0}^{\infty} \sin(2n+1)\pi(x - vt) \frac{2n+1}{2n+1} \]  

(16)

Equations (13), (14) and (15) can then be simplify as

\[
F_E(t) = \frac{M}{\eta_k} \sum_{k=1}^{N} \left[ \frac{1}{4} \int_{0}^{L} \sin \left( \frac{k\pi x}{L} \right) \sin \left( \frac{m\pi x}{L} \right) dx + \frac{1}{\pi} \sum_{n=0}^{\infty} \sin(2n+1)\pi vt \right].
\]

(17)

\[
F_F(t) = \frac{M}{\eta_k} \sum_{k=1}^{N} \left[ \frac{1}{4} \int_{0}^{L} \cos \left( \frac{k\pi x}{L} \right) \sin \left( \frac{m\pi x}{L} \right) dx + \frac{1}{\pi} \sum_{n=0}^{\infty} \cos(2n+1)\pi vt \right].
\]

(18)

\[
F_P(t) = \frac{M}{\eta_k} \sum_{k=1}^{N} \left[ \frac{1}{4} \int_{0}^{L} \sin \left( \frac{k\pi x}{L} \right) \cos \left( \frac{m\pi x}{L} \right) dx - \frac{1}{\pi} \sum_{n=0}^{\infty} \sin(2n+1)\pi vt \right].
\]

(19)

Using equations (17), (18) and (19) in equation (10), one obtains

\[
\left[ 1 + R^i \left( \frac{m\pi}{L} \right)^2 \right] W_n(m,t) + \frac{C_{ij}^i}{\mu} \left( \frac{m\pi}{L} \right)^4 W_i(m,t) + \left[ \frac{E\theta}{\mu} \left( \frac{m\pi}{L} \right)^4 + \frac{N}{\mu} \left( \frac{m\pi}{L} \right)^2 + \frac{K}{\mu} \right] W(m,t)
\]

+ \varepsilon_0 L \sum_{k=1}^{N} \left[ \frac{1}{4} H_0 + \frac{1}{\pi} \sum_{n=0}^{\infty} \cos(2n+1)\pi vt \frac{2n+1}{2n+1} H_2 - \frac{1}{\pi} \sum_{n=0}^{\infty} \sin(2n+1)\pi vt \frac{2n+1}{2n+1} H_3 \right] W_n(k,t)

+ \varepsilon L \left[ \frac{1}{2} H_4 + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{2n+1}{2n+1} H_5 - \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{2n+1}{2n+1} H_6 \right] W_i(k,t)

+ \varepsilon \left[ \frac{1}{4} H_7 + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{2n+1}{2n+1} H_8 - \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{2n+1}{2n+1} H_9 \right] W(k,t)

= \frac{MgL}{m\pi\mu} \left[ - \cos m\pi + \cos \frac{mvt}{L} \right]

(20)
Where

\[ e_0 = \frac{M}{\mu L}, \]  

\[ H_1 = \frac{1}{\eta_k} \int_0^L \sin \frac{k\pi x}{L} \sin \frac{m\pi x}{L} \, dx, \quad H_2 = \frac{1}{\eta_k} \int_0^L \sin(2n+1)\pi x \sin \frac{k\pi x}{L} \sin \frac{m\pi x}{L} \, dx \]

\[ H_3 = \frac{1}{\eta_k} \int_0^L \cos(2n+1)\pi x \sin \frac{k\pi x}{L} \sin \frac{m\pi x}{L} \, dx, \quad H_4 = \frac{k\pi}{\eta_k L} \int_0^L \sin \frac{k\pi x}{L} \sin \frac{m\pi x}{L} \, dx \]

\[ H_5 = \frac{k_2\pi^2}{\eta_k L^2} \int_0^L \sin \frac{k\pi x}{L} \sin \frac{m\pi x}{L} \, dx, \quad H_6 = \frac{k_2\pi^2}{\eta_k L^2} \int_0^L \cos(2n+1)\pi x \sin \frac{k\pi x}{L} \sin \frac{m\pi x}{L} \, dx \]

\[ H_7 = \frac{k_2\pi^2}{\eta_k L^2} \int_0^L \sin \frac{k\pi x}{L} \sin \frac{m\pi x}{L} \, dx, \quad H_8 = \frac{k_2\pi^2}{\eta_k L^2} \int_0^L \cos(2n+1)\pi x \sin \frac{k\pi x}{L} \sin \frac{m\pi x}{L} \, dx \]

Solving integrals (22) and substituting into equation (20) one obtains

\[
\begin{align*}
& \left[ 1 + R^0 \left( \frac{m\pi}{L} \right)^2 \right] W_n(m,t) + \frac{C_0 L}{\mu L} \left( \frac{m\pi}{L} \right)^4 W_n(m,t) + \frac{E I}{\mu} \left( \frac{m\pi}{L} \right)^4 + \frac{N}{\mu} \left( \frac{m\pi}{L} \right)^2 + \frac{K}{\mu} \right] W(m,t) \\
& + e_0 L \left[ \frac{1}{4} + \frac{L}{\pi^2} \sum_{n=0}^\infty \frac{(-1)^n \cos(2n+1)\pi L}{(2n+1)\pi L^2 - 4m^2} \left( \frac{\pi}{2} \right) \cos(2n+1)\pi \nu t - \frac{\nu^2 m^2 \pi^2}{L^2} \right] W_n(m,t) \\
& - 2m \sum_{n=0}^\infty \frac{(-1)^n \cos(2n+1)\pi L}{(2n+1)\pi L^2 - 4m^2} \left( \frac{\pi}{2} \right) \cos(2n+1)\pi \nu t - \frac{\nu^2 m^2 \pi^2}{L^2} \right] W(m,t) \\
& + \frac{L^2}{2\pi^2} \sum_{n=0}^\infty \frac{(-1)^n \cos(2n+1)\pi L}{(2n+1)\pi L^2 - 4m^2} \left( \frac{\pi}{2} \right) \cos(2n+1)\pi \nu t \\
& + \frac{2m}{2\pi^2} \sum_{n=0}^\infty \frac{(-1)^n \cos(2n+1)\pi L}{(2n+1)\pi L^2 - 4m^2} \left( \frac{\pi}{2} \right) \cos(2n+1)\pi \nu t \\
& - \frac{m}{k^2 - m^2} + \frac{1}{\pi} \sum_{n=0}^\infty \frac{k(m) (-1)^k \cos(2n+1)\pi L}{(2n+1)\pi L^2 - (k-m)^2} - \frac{m}{k^2 - m^2} + \frac{1}{\pi} \sum_{n=0}^\infty \frac{k(m) (-1)^k \cos(2n+1)\pi L}{(2n+1)\pi L^2 - (k-m)^2} \right] W_n(k,t) + v k \left[ - \frac{v^2 k^2 \pi^2}{4} \sum_{n=0}^\infty \frac{(-1)^n \cos(2n+1)\pi L}{(2n+1)\pi L^2 - (k-m)^2} \left( \frac{\pi}{2} \right) \cos(2n+1)\pi \nu t \right] W(k,t) \\
& = \frac{MgL}{m \pi \mu} \left[ - \cos m \pi + \cos \frac{m \nu t}{L} \right]
\end{align*}
\]
Equation (23) is the transformed equation governing the problem when the Rayleigh beam resting constant elastic foundation has simple support at all its edges. In what follows, two cases of the equation (23) are considered.

**Case I: Simply supported beam traversed by a moving force**

When \( \varepsilon_0 = 0 \) in equation (23), an approximate model of the system when the inertia effect of the moving mass \( M \) is neglected, we have the moving force problem associated with the system. Thus the differential equation (23) reduces to

\[
\left[ 1 + R_0^0 \left( \frac{m\pi}{L} \right)^2 \right] W_\varepsilon(m,t) + \frac{C_0^0 I \left( \frac{m\pi}{L} \right)^4}{\mu} W_t(m,t) + \left[ \frac{EI \left( \frac{m\pi}{L} \right)^4}{\mu} + \frac{N \left( \frac{m\pi}{L} \right)^2}{\mu} + \frac{K}{\mu} \right] W(m,t)
\]

which when rearranged gives

\[
W_\varepsilon(m,t) + T_{\varepsilon} W_t(m,t) + \Omega_{mf}^2 W(m,t) = P_{mf} + \sigma_{mf} \cos \Phi t
\]  
(25)

Where

\[
T_{\varepsilon} = \frac{C_0^0 I \left( \frac{m\pi}{L} \right)^4}{1 + R_0^0 \left( \frac{m\pi}{L} \right)^2}
\]  
(26)

\[
\Omega_{mf}^2 = \frac{\left[ \frac{EI \left( \frac{m\pi}{L} \right)^4}{\mu} + \frac{N \left( \frac{m\pi}{L} \right)^2}{\mu} + \frac{K}{\mu} \right]}{1 + R_0^0 \left( \frac{m\pi}{L} \right)^2}
\]  
(27)

\[
P_{mf} = \frac{-MgL(-1)^n}{m\pi\mu \left[ 1 + R_0^0 \left( \frac{m\pi}{L} \right)^2 \right]}
\]  
(28)

\[
\sigma_{mf} = \frac{MgL}{m\pi\mu \left[ 1 + R_0^0 \left( \frac{m\pi}{L} \right)^2 \right]}
\]  
(29)

\[
\Phi = \frac{m\pi\nu}{L}
\]  
(30)
Equation (25) is solved using Laplace transforms and convolution theory in conjunction with the initial conditions (6), one obtains

\[ W(m,t) = \frac{1}{\xi_2 - \xi_1} \left\{ \frac{P_{mf}}{\xi_2} \left( e^{\xi_2 t} - 1 \right) - \frac{P_{mf}}{\xi_2} \left( e^{\xi_2 t} - 1 \right) + \frac{\sigma_{mf}}{\xi_2 + \Phi_m^2} \left[ \Phi_m \sin \Phi_m t - \xi_1 \cos \Phi_m + \xi_1 e^{\xi_1 t} \right] \right\} \]

(31)

Where

\[ \xi_1 = -\frac{T_A + \sqrt{Y_A^2 - 4\Omega_{mf}^2}}{2}, \quad \text{and} \quad \xi_2 = -\frac{T_A - \sqrt{Y_A^2 - 4\Omega_{mf}^2}}{2} \]

(32)

In view of equation (9), one obtains the displacement of the simply supported damped beam as

\[ W(x,t) = 2 \sum_{L=1}^{\infty} \frac{1}{\xi_2 - \xi_1} \left\{ \frac{P_{mf}}{\xi_2} \left( e^{\xi_2 t} - 1 \right) - \frac{P_{mf}}{\xi_2} \left( e^{\xi_2 t} - 1 \right) + \frac{\sigma_{mf}}{\xi_2 + \Phi_m^2} \left[ \Phi_m \sin \Phi_m t - \xi_1 \cos \Phi_m + \xi_1 e^{\xi_1 t} \right] \right\} \sin \frac{m\pi x}{L} \]

(33)

**Case II: Simply supported beam traversed by a moving mass**

In this section we seek the solution to the entire equation (23) when no term of the equation is neglected. This problem is solved by the use of the modified asymptotic method of Struble’s already alluded to [17]. To this end we rearrange equation (23) to take the form

\[ W_\alpha(m,t) + \Omega_{mf}^2 W(m,t) + \frac{\epsilon_0 L}{1 + R_0 \left( \frac{m\pi}{L} \right)^2} \left[ \frac{1}{4} + \frac{1}{L} \sum_{n=0}^{\infty} \frac{(2n+1)}{(2n+1)^2} \right] \left( \frac{(-1)^{2m}\cos(2n+1)\pi t - 1}{[(2n+1)\pi]^2 - 4m^2} \right) \]

\[ - \frac{\cos(2n+1)\pi t}{2(n+1)} \frac{\cos(2n+1)\pi t}{2n+1} \left[ \sum_{n=0}^{\infty} \frac{(2n+1)}{(2n+1)^2} \right] \]

\[ \sin(2n+1)\pi t \left( \frac{\epsilon_0 I}{\mu} \left( \frac{m\pi}{L} \right)^2 \right) ^4 \left[ \sum_{n=0}^{\infty} \frac{(2n+1)}{(2n+1)^2} \right] \]

\[ \frac{\cos(2n+1)\pi t}{(2n+1)^2 - 4m^2} - \frac{\cos(2n+1)\pi t}{2n+1} \frac{\cos(2n+1)\pi t}{2n+1} \]

\[ W(m,t) \]

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\[ + 2\varepsilon_0 \left[ \frac{L^2}{2\pi} \sum_{n=0}^\infty (2n+1) \left( \frac{(-1)^{k+m} \cos(2n+1)\pi L - 1}{\left[(2n+1)L^2 - [(k+m)]^2\right]} - \frac{(-1)^{k-m} \cos(2n+1)\pi L - 1}{\left[(2n+1)L^2 - [(k-m)]^2\right]} \right) \right] \]

\[ \cos\left(\frac{2n+1}{2} \pi vt\right) W_n(k,t) + \sum_{n=0}^\infty \left[ \frac{-m}{k^2 - m^2} - \frac{1}{\pi} \sum_{n=0}^\infty \left( \frac{(k+m)(-1)^{k+m} \cos(2n+1)\pi L - 1}{\left[(2n+1)L^2 - [(k+m)]^2\right]} - \frac{(k-m)(-1)^{k-m} \cos(2n+1)\pi L - 1}{\left[(2n+1)L^2 - [(k-m)]^2\right]} \right)\right] W_i(k,t) \]

\[ = \frac{MgL}{m \pi \mu_L^2 \left[1 + R^0 \left(m \pi \frac{m \pi L}{L}\right)^2\right]} \left[ (-1)^{k+m} \cos\frac{m \pi vt}{L} \right] \]

By means of the Struble’s technique, one seeks the modified frequency corresponding to the frequency of the free system due to the presence of the moving mass. An equivalent free system operator defined by the modified frequency that replaces equation (34). Thus, the right-hand side of equation (34) is set to zero, and a parameter \( \Lambda_0 < 1 \) is considered for any arbitrary mass ratio \( \varepsilon_0 \), defined as

\[ \Lambda_0 = \frac{\varepsilon_0}{1 + \varepsilon_0} \]  

(35)

Evidently

\[ \varepsilon_0 = \Lambda_0 + O(\Lambda_0^2) \]  

(36)

when \( \Lambda_0 = 0 \), a case corresponding to the situation when the axial force effect is neglected is obtained and the solution of (29) becomes

\[ \overline{W}(m,t) = \chi_m \cos[\Omega_{mf} t - \theta_m] \]  

(37)

where \( \chi_m \) and \( \theta_m \) are constants.

Since \( \Lambda_0 << 1 \), Struble’s technique requires that the asymptotic solution of the homogeneous part of equation (34) be given in [13] by Nayfeh as

\[ \overline{W}(m,t) = \chi(m,t) \cos[\Omega_{mf} t - \theta(m,t)] + \Lambda_0 \overline{W}(m,t) + O(\Lambda_0^2) \]  

(38)

where \( \chi(m,t) \) and \( \theta(m,t) \) are slowly varying functions of time.

To obtain the modified frequency, equation (38) and its derivatives are substituted into equation (34), taking into account the following trigonometric identities
The variational equations are obtained by equating the coefficients of terms on both sides of equation (43) to zero. Thus one obtains

\[
\begin{align*}
\frac{\cos(2n + 1)\pi vt}{2n + 1} \cos[\Omega_{m^f} t - \theta(m,t)] &= \frac{1}{2} \sin \left[ \frac{(2n + 1)\pi vt}{2n + 1} + \Omega_{m^f} t - \theta(m,t) \right] \\
&\quad - \frac{1}{2} \sin \left[ \frac{(2n + 1)\pi vt}{2n + 1} + \Omega_{m^f} t - \theta(m,t) \right] \\
&\quad - \frac{1}{2} \cos \left[ \frac{(2n + 1)\pi vt}{2n + 1} + \Omega_{m^f} t - \theta(m,t) \right] \\
&\quad + \frac{1}{2} \cos \left[ \frac{(2n + 1)\pi vt}{2n + 1} + \Omega_{m^f} t - \theta(m,t) \right] \\
\frac{\sin(2n + 1)\pi vt}{2n + 1} \sin[\Omega_{m^f} t - \theta(m,t)] &= \frac{1}{2} \cos \left[ \frac{(2n + 1)\pi vt}{2n + 1} + \Omega_{m^f} t - \theta(m,t) \right] \\
&\quad - \frac{1}{2} \cos \left[ \frac{(2n + 1)\pi vt}{2n + 1} + \Omega_{m^f} t - \theta(m,t) \right] \\
&\quad - \frac{1}{2} \sin \left[ \frac{(2n + 1)\pi vt}{2n + 1} + \Omega_{m^f} t - \theta(m,t) \right] \\
&\quad + \frac{1}{2} \sin \left[ \frac{(2n + 1)\pi vt}{2n + 1} + \Omega_{m^f} t - \theta(m,t) \right]
\end{align*}
\]  

(39)  

(40)  

(41)  

(42)

\[
2\chi(m,t)\dot{\theta}(m,t)\Omega_{m^f} \cos[\Omega_{m^f} t - \theta(m,t)] - 2\chi(m,t)\Omega_{m^f} \sin[\Omega_{m^f} t - \theta(m,t)] \\
- \frac{\Lambda_0 L}{4\Psi_A} \chi(m,t)\Omega_{m^f}^2 \cos[\Omega_{m^f} t - \theta(m,t)] + \Lambda_0 LT_A \chi(m,t)\Omega_{m^f} \sin[\Omega_{m^f} t - \theta(m,t)] \\
- \frac{v^2 m^2 \pi^2 \Lambda_0}{8\Psi_A} \chi(m,t) \cos[\Omega_{m^f} t - \theta(m,t)] = 0
\]  

(43)

terms to \(O(\Lambda_0^3)\).

The variational equations are obtained by equating the coefficients of \(\sin[\Omega_{m^f} t - \theta(m,t)]\) and \(\cos[\Omega_{m^f} t - \theta(m,t)]\) terms on both sides of equation (43) to zero. Thus

\[
- 2\chi(m,t)\Omega_{m^f} + \Lambda_0 LT_A \chi(m,t)\Omega_{m^f} = 0
\]  

(44)

And

\[
2\chi(m,t)\dot{\theta}(m,t)\Omega_{m^f} - \frac{\Lambda_0 L}{4\Psi_A} \chi(m,t)\Omega_{m^f}^2 - \frac{v^2 m^2 \pi^2 \Lambda_0}{8\Psi_A} \chi(m,t) = 0
\]  

(45)
Solving equations (44) and (45), we have

\[ \chi(m, t) = A_0 e^{-\frac{t}{\xi}} \]  

(46)

And

\[ \theta(m, t) = \frac{\Lambda_0}{16\Omega_{mf}} \left( \frac{2L\Omega^2_{mf} + v^2m^2\pi^2}{\Psi_A} \right) t + \xi_m \]  

(47)

Where

\[ \alpha = -\Lambda_0 LT_A \text{ and } \Psi_A = \left[ 1 + R^0 \left( \frac{m\pi}{L} \right)^2 \right] \]  

(48)

\( \eta_m \) and \( \xi_m \) are constants.

Therefore when the effect of the mass of the particle is considered, the first approximation to the homogeneous system is

\[ W(m, t) = A_0 e^{-\frac{t}{\xi}} \cos [Y_{mn} t - \xi_m] \]  

(49)

Where

\[ Y_{mn} = \Omega_{mf} \left[ 1 - \frac{\Lambda_0}{16} \left( \frac{2L\Omega^2_{mf} + v^2m^2\pi^2}{\Omega_{mf}\Psi_A} \right) \right] \]  

(50)

is called the modified natural frequency representing the frequency of the free system due to the presence of the moving mass.

The homogeneous part of the equation (34) can be written as

\[ W_a(m, t) + Y_{mn}^2 W(m, t) = 0 \]  

(51)

while the entire equation (34) takes the form

\[ W_a(m, t) + Y_{mn}^2 W(m, t) = \left[ -(-1)^m + \cos \frac{m\pi t}{L} \right] \]  

(52)

Using Laplace transformation technique and the convolution theory, expression \( W(m, t) \) is obtained and in view of equation (9), one obtains

\[ \bar{W}(x, t) = \frac{2}{L^2} \sum_{n=1}^{\infty} \frac{1}{\Phi_n^2 - Y_{mn}^2} \left\{ \cos Y_{mn} t - \cos \Phi_n t - \frac{(-1)^m(\Phi_m^2 - Y_{mn}^2)\cos Y_{mn}}{Y_{mn}} \right\} \sin \frac{m\pi x}{L} \]  

(53)

this represents the response to a moving mass of a simply supported Bernoulli-Euler beam on a Winkler foundation.
4 Analysis of the Solution

We shall examine the phenomenon of resonance in this section. Equation (33) reveals clearly that the Rayleigh beam on a Winkler elastic foundation and traversed by a moving force reaches a state of resonance whenever

\[ m_{\frac{\varphi}{1}} = \Phi_m \quad \text{and} \quad m_{\frac{\varphi}{2}} = \Phi_m \]  

while equation (53) shows that the same beam under the action of moving mass reaches the state of resonance effect whenever

\[ Y_{mm} = \Phi_m \]  

Where

\[ Y_{mm} = \Omega_{mf} \left[ 1 - \frac{\Lambda_0}{16} \left( \frac{2L \Omega_{mf} + v^2 m^2 \pi^2}{\Omega_{mf} \Psi_A} \right) \right] \]  

Thus

\[ Y_{mm} = \Omega_{mf} \left[ 1 - \frac{\Lambda_0}{16} \left( \frac{2L \Omega_{mf} + v^2 m^2 \pi^2}{\Omega_{mf} \Psi_A} \right) \right] = \Phi_m \]  

Consequently, show that for the same natural frequency, the critical speed and the natural frequency for the same system of a uniform Rayleigh beam traversed by a moving mass is smaller than that of the same system traversed by a moving force. Thus, resonance is reached earlier in the moving mass system than in the moving force system.

5 Numerical Results and Discussions of Results

In this section, the effects of the foundation stiffness, axial force, damping due to resistance to the transverse displacement and the rotatory inertia are analyzed. Simply supported boundary conditions are considered and the basic parameters of the system used for the numerical calculations are as follows: length \( L = 12.192 \text{m} \), velocity \( v = 8.123 \text{m/s} \), \( EI = 6.068 \times 10^6 \text{m}^3/\text{s}^2 \) and \( M/\mu L = 0.25 \).

Figs. 1 and 5 show the variations of the deflection of the simply supported beam for both moving distributed force and moving distributed mass of the foundation stiffness \( K \) for various values of \( K \). The figures illustrate that the amplitudes of deflection decreases as the corresponding foundation stiffness get larger. In Fig. 2, it is clear that as the value of axial force \( N \) increases, the displacement response of the simply supported beam to moving distributed force get smaller. Similar results are obtained when the beam is subjected to moving distributed mass as in Fig. 6.

Figs. 3 and 7 show the displacement response of the simply supported beam to moving distributed force and moving distributed mass for various values of damping due to resistance to the transverse displacement \( C \). From the figures, it is found that as the value of \( C \) increases, the response amplitudes of the vibrating system decreases. Figs. 3 and 7 clearly show that the effect of vibration of the dynamical system becomes less pronounced when the damping due to resistance to the transverse displacement \( C \) increases. In Fig. 4, the deflection profile of simply supported beam under the action of moving distributed force for various values
of rotatory inertia $R_0$ is displayed. The figure shows that as $R_0$ increases, the response amplitude of the beam decreases. Similar results are obtained when the beam is subjected to moving distributed mass as shown in Fig. 8. The comparison of the displacement response of the moving distributed force and moving distributed mass of the beam is depicted in Fig. 9. Clearly, the response amplitude of moving distributed force is higher than that of moving distributed mass.

Fig. 1. Response amplitude of moving force for simply supported beam on elastic foundation for various values of foundation stiffness $K$ and fixed values of $N(20000)$, $C(20)$ and $R_0(15)$

Fig. 2. Deflection profile of moving force for simply supported beam on elastic foundation for various values of axial force $N$ and fixed values of foundation $K(40000)$, $C(20)$ and $R_0(15)$

Fig. 3. Response amplitude of moving force for simply supported beam on elastic foundation for various values of damping due to the resistance of the displacement $C$ and fixed values of $K(40000)$, $N(20000)$ and $R_0(15)$
Fig. 4. Transverse displacement of moving force for simply supported beam on elastic foundation for various values of rotatory inertia $R_0$ and fixed values of $K(40000)$, $N(20000)$ and $C(20)$

Fig. 5. Response amplitude of moving mass for simply supported beam on elastic foundation for various values of foundation stiffness $K$ and fixed values of $N(20000)$, $C(20)$ and $R_0(15)$

Fig. 6. Deflection profile of moving mass for simply supported beam on elastic foundation for various values of axial force $N$ and fixed values of $N(20000)$, $C(20)$ and $R_0(15)$
Fig. 7. Response amplitude of moving mass for simply supported beam on elastic foundation for various values of damping due to the resistance of the displacement $C$ and fixed values of $K(40000)$, $N(20000)$ and $R(15)$

Fig. 8. Transverse displacement of moving mass for simply supported beam on elastic foundation for various values of rotatory inertia $R_0$ and fixed values of $K(40000)$, $N(20000)$ and $C(20)$

Fig. 9. Variation of the response of moving force and moving mass for fixed values of $K = 40000$, $N = 20000$, $C = 7.5$, and $R_0 = 10$
6 Conclusion

This study presents an analytical solution to the vibration analysis of a simply supported damped Rayleigh beam under travelling distributed loads. The closed form solutions are analyzed and results in plotted curves are presented. Numerical and analytical results show that increase in the damping due to resistance of the transverse displacement decrease the deflection of the beam. It is also found that, as the foundation constant $K$ and axial force $N$ increased the response amplitudes of the vibrating system decrease. Thus, higher values of the structural parameters minimize the effect of vibration of the dynamical system. Hence the risk factor of resonance in a vibrating system is reduced.

Competing Interests

Author has declared that no competing interests exist.

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