Stochastic Lévy Differential Operators and Yang-Mills Equations
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Abstract: The relationship between the Yang-Mills equations and the stochastic analogue of Lévy differential operators is studied. The value of the stochastic Lévy Laplacian is found by means of Césaro averaging of directional derivatives on the stochastic parallel transport. It is shown that the Yang-Mills equations and the Lévy-Laplace equation for such Laplacian are not equivalent as in the deterministic case. An equation equivalent to the Yang-Mills equations is obtained. The equation contains the stochastic Lévy divergence. It is proved that the Yang-Mills action functional can be represented as an infinite-dimensional analogue of the Dirichlet functional of chiral field. This analogue is also derived using Césaro averaging.

keywords: Lévy Laplacian, Lévy divergence, stochastic parallel transport, Yang-Mills equations
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Introduction

The original Lévy Laplacian was defined for functions on the space $L^2(0,1)$ in the twenties of the last century. Paul Lévy suggested different approaches to define this operator. The value of the Lévy Laplacian on a function can be determined as an integral functional generated by the special form of the second-order derivative or as the Césaro mean of second directional derivatives along vectors of some orthonormal basis in $L^2(0,1)$ (see [20, 14, 22]). Both approaches are used to define Laplacians acting on functions on the various functional spaces and these differential operators are also called Lévy Laplacians. One of them has been defined as an integral functional in [4, 3] by L. Accardi, P. Gibilisco, I.V. Volovich and has been used for the description of solutions of the Yang-Mills equations. In these papers it has been proved that the connection in the trivial bundle over a Euclidean space satisfies the Yang-Mills equations if and only if the corresponding parallel transport is a harmonic functional for the Lévy Laplacian (see also [1]). The case of a nontrivial bundle over a compact Riemannian manifold has been considered in [19] by R. Leandre and I. V. Volovich. Also in [19] the stochastic Lévy Laplacian has been defined as a special integral functional. For this operator the similar theorem holds: the stochastic parallel transport satisfies the Lévy-Laplace equation if and only if the corresponding connection satisfies the Yang-Mills equation. The present paper focuses on the second approach to the Lévy Laplacian. We study the stochastic Lévy Laplacian defined as the Césaro mean of the second directional derivatives, associated with this Laplacian divergence and their connection to the Yang-Mills equations. In the deterministic case the second approach to
the definition of the Lévy Laplacian has been used for study the Yang-Mills equations in

\[25, 27, 30\].

Let us recall the general scheme of the definition of homogeneous linear differential operators from [10]. Let \(X, Y, Z\) be real normed vector spaces. Let \(C^n(X, Y)\) be the space of \(n\) times Fréchet differentiable \(Y\)-valued functions on \(X\). Then for every \(x \in X\) it is valid that \(f^{(n)}(x) \in L_n(X, Y)\) \(^1\). Let \(S\) be a linear mapping from \(domS \subset L_n(X, Y)\) to \(Z\). The domain \(domD^{n,S}\) of a differential operator \(D^{n,S}\) of order \(n\) generated by the operator \(S\) consists of all \(f \in C^n(X, Y)\) such that \(f^{(n)}(x) \in domS\) for all \(x \in E\). Then \(D^{n,S}\) is a linear mapping from \(domD^{n,S}\) to space \(\mathcal{F}(X, Z)\) of all \(Z\)-valued functions on \(X\) defined by the formula

\[D^{n,S}f(x) = S(f^{(n)}(x)).\]

If we choose \(X = \mathbb{R}^d, Y = Z = \mathbb{R}\) and \(S = tr\) (trace), then the operator \(D^{2,tr}\) is Lévy Laplacian \(\Delta\). If we choose \(X = Y = \mathbb{R}^d, Z = \mathbb{R}\) and \(S = tr\), then the operator \(D^{1,tr}\) is divergence \(div\).

Let \(E\) be a real normed space. Let \(E\) be continuously embedded in a separable Hilbert space \(H\). Let the image of this embedding be dense in \(H\). Then \(E \subset H \subset E^*\) is a rigged Hilbert space. Let \(\{e_n\}\) be an orthonormal basis in \(H\). We assume that each element of \(\{e_n\}\) belongs to \(E\). The Lévy trace \(tr_L\) defined by

\[tr_L(R) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (R e_k, e_k),\]

where \(dom\ tr_L\) consists of all \(R \in L(E, E^*)\) for which the right side of (11) exists (see [21]). If we choose \(X = E, Y = Z = \mathbb{R}\) and \(S = tr_L\), then \(D^{2,tr_L}\) is the Lévy Laplacian \(\Delta\). If \(X = E, Y = E^*, Z = \mathbb{R}\) and \(S = tr_L\), then \(D^{1,tr_L}\) is the Lévy divergence \(div_L\). An important case is then \(H = L_2([0, 1], \mathbb{R}^d)\), \(E\) is some space of curves (for example, we can choose the Cameron-Martin space of the Wiener measure as \(E\)), and \(\{e_n\}\) is the orthonormal basis in \(L_2([0, 1], \mathbb{R}^d)\) defined by \(e_n(t) = p_{a_n}(t)\sqrt{2} \sin (b_n t)\), where \(\{p_1, p_2, \ldots, p_d\}\) is an orthonormal basis in \(\mathbb{R}^d, a_n = n - 4[\frac{n-1}{4}]\) and \(b_n = \left[\frac{n+3}{4}\right]\) (see [24, 25, 30]). Then the Lévy-Laplacian associated with this basis coincides with the Lévy-Laplacian defined in [11, 12]. The theorem about equivalence of the Yang-Mills equations and the Lévy-Laplace equation holds for this Laplacian.

These definitions of Lévy differential operators can be transferred to the stochastic case. Let \(f\) be an element of the Sobolev space \(W^2_2(P)\) over the Wiener measure \(P\) and \(B\) be a continuous linear mapping from the Cameron-Martin space of \(P\) to \(W^2_2(P)\). Let \(D^2 f\) be the second derivative of \(f\) and \(DB\) be the first derivative of \(B\). We define the value of the stochastic Lévy Laplacian on \(f\) and the Lévy divergence on \(B\) by \(tr_L D^2 f\) and \(tr_L DB\) respectively. We find the value of the stochastic Lévy Laplacian defined as the Césaro mean.

\(^1\)Everywhere below \(L(X, Y)\) is space of linear continuous mappings from \(X\) to \(Y\). The space \(L_n(X, Y)\) is defined by induction: \(L_1(X, Y) = L(X, Y)\) and \(L_n(X, Y) = L(X, L_{n-1}(X, Y))\).

\(^2\)In [19] a functional divergence has been defined as an integral functional. It is possible to show that the restriction of the Lévy divergence associated with basis \(\{e_n\}\) to the domain of the functional divergence coincides with that divergence.

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of directional derivatives on the stochastic parallel transport. We show that, unlike the deterministic case, the equivalence of the Yang-Mills equations and the Lévy-Laplace equation is not valid for such Laplacian. Hence, this Laplacian does not coincide with the operator introduced in [19].

By means of the stochastic Lévy divergence, we study relationship between the Yang-Mills equations and infinite dimensional chiral fields. Let us recall, that the field \( g: \mathbb{R}^d \rightarrow SU(N) \) is a general chiral field (see [13]). Its Dirichlet functional has a form
\[
\frac{1}{2} \int_{\mathbb{R}^d} tr(\partial_\mu g(x) \partial_\mu g^{-1}(x)) dx = -\frac{1}{2} \int_{\mathbb{R}^d} tr(A_\mu(x) A^\mu(x)) dx, \tag{2}
\]
where \( A_\mu = g^{-1}(x) \partial_\mu g(x) \). The equation of motion has a form
\[
\sum_{\mu=1}^d \partial_\mu(g^{-1}(x) \partial_\mu g(x)) = 0
\]
or
\[
\begin{cases}
\text{div} A = 0 \\
\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = 0,
\end{cases}
\]
where \( A = (A_1, \ldots, A_d) \). In this paper it is proved that the connection satisfies the Yang-Mills equations if and only if the corresponding stochastic parallel transport satisfies the analogue of the equation of motion of chiral field, where divergence is replaced with the stochastic Lévy divergence. Also it is proved that the Yang-Mills action functional can be represented as an analogue of [4], where the chiral field is replaced with the stochastic parallel transport, the sum is replaced with the Cesàro mean and the Lebesgue measure is replaced with the product of the Lebesgue measure and the Wiener measure (cf. [24]). There is no analogue of this representation in the deterministic case. On the connection between infinite dimensional analogues of chiral fields and the Yang-Mills equations in the deterministic case see [23, 25]. Note that the Dirichlet form associated with the Lévy Laplacian has been studied in [2]. Another approach to representation of the Yang-Mills action functional by the stochastic parallel transport has been used in [11]. For the recent development in the study of Lévy Laplacians see [31, 30, 26].

For the particular case of Maxwell’s equations the results of the paper have been formulated and proved in [28] and [29].

The paper is organized as follows. In Sec. 1 we find the second derivative of the stochastic parallel transport. In Sec. 2 we prove some technical lemmas. In Sec. 3 we define the stochastic Lévy Laplacian as the Cesàro mean of the second-order directional derivatives and find its value on the stochastic parallel transport. In Sec. 4 we define the stochastic Lévy divergence and obtain the equation with this divergence which is equivalent to the Yang-Mills equations.

\[3\text{In [9], in particular, an analogue of the equation of motion of chiral field for the functional divergence has been considered. It has been shown that the deterministic parallel transport is a solution of this equation if the corresponding connection satisfies the Yang-Mills equations.}\]
In Sec. 5 we show that the Yang-Mills action functional can be represented as an infinitely dimensional analogue of the Dirichlet functional of chiral fields.

1 Stochastic parallel transport and its derivatives

In this section we establish some probabilistic and geometric prerequisites and we find the second derivative of the stochastic parallel transport.

In the paper we use the Einstein summation convention. Let \((\Omega, \mathcal{F}, P)\) be a canonical probability space associated with \(d\)-dimensional space Brownian motion on interval \([0, 1]\). I.e.

\[
\Omega = C_0([0, 1], \mathbb{R}^d) := \{ \gamma \in C([0, 1], \mathbb{R}^d): \gamma(0) = 0 \},
\]

\(\mathcal{F}\) is the completion of the Borel \(\sigma\)-field on \(C_0([0, 1], \mathbb{R}^d)\) with respect to the Wiener measure \(P\). We denote the \(d\)-dimensional Brownian motion by \(b_t = (b_1^t, \ldots, b_d^t)\). We denote by \((\mathcal{F}_t)\) the increasing family of \(\sigma\)-fields generated by \(b_t\). In this paper Itô differentials and Stratonovich differentials are denoted by \(\text{d}b\) and by \(\circ \text{d}b\) respectively.

The space \(W^{2,1}_0([0, 1], \mathbb{R}^d) = \{ \gamma \text{ is absolutely continuous, } \gamma(0) = 0, \dot{\gamma} \in L_2((0, 1), \mathbb{R}^d) \}\) is the Cameron-Martin space of the Wiener measure.

Let \(\mathcal{H}\) be a real or complex Hilbert space. By the symbol \(\| \cdot \|_p\) we denote the norm of \(L^p(P, \mathcal{H})\). The Sobolev norm \(\cdot \|_{p,r}\) on the space \(\mathcal{F}C^\infty(\mathcal{H})\) of \(\mathcal{H}\)-valued \(C^\infty\)-smooth cylindrical function with compact support on \(C_0([0, 1], \mathbb{R}^d)\) is defined by

\[
\| f \|_{p,r} = \sum_{k=1}^r (E( \sum_{i_1, \ldots, i_k = 1}^{\infty} \| \partial_{g_{i_1}} \cdots \partial_{g_{i_k}} f \|_{\mathcal{H}}^2)^{p/2})^{1/p},
\]

where \(\{g_n\}\) is an arbitrary orthonormal basis in \(W^{2,1}_0([0, 1], \mathbb{R}^d)\). The Sobolev space \(W^p_r(P, \mathcal{H})\) is the completion of \(\mathcal{F}C^\infty(\mathcal{H})\) with respect to the norm \(\cdot \|_{p,r}\). Here \(W^\infty_r(P, \mathcal{H})\) is the projective limit \(\lim_{p \to +\infty} W^p_r(P, \mathcal{H})\).

For any \(h \in W^{2,1}_0([0, 1], \mathbb{R}^d)\) and for any \(p \geq 1\) the operator \(\partial_h\) can be closed as an operator from \(W^p_r(P, \mathcal{H})\) to \(L_p(P, \mathcal{H})\). We denote the closure of \(\partial_h\) by the same symbol and by the symbol \(D^h\). If \(F \in W^p_r(P)\), then there exists such \(\mathcal{H}\)-valued stochastic process \(D_tF\) for which the equality \(D^hF = \partial_hF = \int_0^1 \langle D_tF, \dot{h}(t) \rangle_\mathcal{H} \circ \text{d}t\) holds for every \(h \in W^{2,1}_0([0, 1], \mathbb{R}^d)\)\footnote{\(D_tF\) is defined almost everywhere with respect to \(\lambda \times P\), where \(\lambda\) is the Lebesgue measure on \([0, 1]\).} The higher order derivatives are defined similarly (see \([21]\)). (About different ways to define the Sobolev classes over Gaussian measures see for example \([12]\) and the references cited therein.)
A connection in the trivial vector bundle with base \( \mathbb{R}^d \), fiber \( \mathbb{C}^N \) and structure group \( U(N) \) is defined below as \( u(N) \)-valued \( C^\infty \)-smooth 1-form \( A(x) = A_\mu(x)dx^\mu \) on \( \mathbb{R}^d \). If \( \phi \in C^1(\mathbb{R}^d, u(N)) \), then the covariant derivative \( \phi \) in the direction of the vector field \( \frac{\partial}{\partial x^\mu} \) is defined by the formula \( \nabla_\mu \phi = \partial_\mu \phi + [A_\mu, \phi] \). Everywhere below it is supposed that the connection \( A \) and all its derivatives of the first and the second orders are bounded on \( \mathbb{R}^d \). The curvature corresponding to the connection \( A \) is the 2-form \( F(x) = \sum_{\mu<\nu} F_{\mu\nu}(x)dx^\mu \wedge dx^\nu \), where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \). Every connection \( A \) satisfies the Bianchi identities
\[
\nabla_\lambda F_{\mu\nu} + \nabla_\nu F_{\lambda\mu} + \nabla_\mu F_{\nu\lambda} = 0
\]
(3)
The Yang-Mills action functional has a form
\[
- \int_{\mathbb{R}^d} tr(F_{\mu\nu}(x)F^{\mu\nu}(x))dx
\]
(4)
The Yang-Mills equations are the Euler-Lagrange equations for (4) and have a form:
\[
\nabla^\mu F_{\mu\nu} = 0.
\]
(5)
These equations are equations on a connection \( A \).

The stochastic parallel transport \( U^x(b, t) (x \in \mathbb{R}^d) \) associated with the connection \( A \) is a solution to the differential equation (in the sense of Stratonovich):
\[
U^x(b, t) = I_N - \int_0^t A_\mu(x + b_s)U^x(b, s) \circ db^\mu_s
\]
(6)
Since the connection \( A \) and its first-order derivatives are bounded, this equation has a unique strong solution.

We consider stochastic processes
\[
L^x_{\mu\nu}(b, t) = U^x(b, t)^{-1}F_{\mu\nu}(x + b_t)U^x(b, t)
\]
and
\[
J^x_{\lambda\mu\nu}(b, t) = U^x(b, t)^{-1}\nabla_\nu F_{\mu\nu}(x + b_t)U^x(b, t).
\]

**Proposition 1.** The following equalities hold:
\[
\int_0^t L^x_{\mu\nu}(b, s)u^\mu(s) \circ db^\nu_s = \int_0^t L^x_{\nu\mu}(b, s)u^\nu(s)db^\mu_s + \frac{1}{2} \int_0^t J^x_{\nu\mu} \nu(b, s)u^\mu(s)ds,
\]
\[
\int_0^t J^x_{\mu\nu} \nu(b, s) \circ db^\nu_s = \int_0^t J^x_{\nu\mu} \nu(b, s)db^\mu_s.
\]
(7)
**Proof.** Indeed, since \( dU^x(b, s)^{-1} = U^x(b, s)A_\mu(x + b_s) \circ db^\mu_s \) and \( dU^x(b, s) = -A_\mu(x + b_s)U^x(b, s) \circ db^\mu_s \), we have
\[
dL^x_{\mu\nu}(b, s) = J^x_{\mu\nu}(b, s) \circ db^\lambda_s.
\]
Let us show that

\[
\int_0^t L^x_{\mu \nu}(b, s) u^\mu(s) \circ db^\nu_s = \int_0^t L^x_{\mu \nu}(b, s) u^\mu(s) db^\nu_s + \frac{1}{2} \int_0^t u^\mu(s)(dL^x_{\mu \nu}(b, s)) \cdot (db^\nu_s) = \]

\[
= \int_0^t L^x_{\mu \nu}(b, s) u^\mu(s) db^\nu_s + \frac{1}{2} \int_0^t u^\mu(s)(dL^x_{\mu \nu}(b, s)) \cdot (db^\nu_s) = \]

\[
= \int_0^t L^x_{\mu \nu}(b, s) u^\mu(s) db^\nu_s + \frac{1}{2} \int_0^t \nabla \nu \frac{\partial}{\partial \nu} U^x(b, s) u^\mu(s) ds
\]

Similarly, we obtain

\[
\int_0^t J^x_{\nu \mu}(b, s) \circ db^\nu_s = \int_0^t J^x_{\nu \mu}(b, s) db^\nu_s + \frac{1}{2} \int_0^t (dJ^x_{\nu \mu}(b, s)) \cdot (db^\nu_s) = \]

\[
= \int_0^t J^x_{\nu \mu}(b, s) db^\nu_s + \frac{1}{2} \int_0^t U^x(b, s)^{-1} \nabla \nu \frac{\partial}{\partial \nu} F^{\mu \nu}(x + b_s)U^x(b, s) ds
\]

but it is valid that \( \nabla \nu \frac{\partial}{\partial \nu} F^{\mu \nu} = 0 \). So, we obtain 7.

**Proposition 2.** For any \( t \in [0, 1] \) it is hold that \( U^x(b, t) \in W^2_{\infty}(P, M_N(\mathbb{C})) \). If \( u \in W^{2,1}_0([0, 1], \mathbb{R}^d) \), then

\[
\frac{\partial}{\partial u} U^x(b, t) = -U^x(b, t) \int_0^t L^x_{\mu \nu}(b, s) u^\mu(s) \circ db^\nu_s - A_\mu(x + b_t) u^\mu(t) U^x(b, t) = \]

\[
= -U^x(b, t) \int_0^t L^x_{\mu \nu}(b, s) u^\mu(s) db^\nu_s - \frac{1}{2} U^x(b, t) \int_0^t J^x_{\nu \mu}(b, s) u^\mu(s) dt - A_\mu(x + b_t) u^\mu(t) U^x(b, t) \tag{8}
\]

**Proof.** We consider two-parameter stochastic process \( Z^x(b, t, s) \) defined by the formula

\[
Z^x_\nu(b, t, s) = \begin{cases} 
-U^x(b, t)U^x(b, s)^{-1} A_\nu(x + b_s)U^x(b, s) - \\
-U^x(b, t) \int_0^s U^x(b, r)^{-1} \partial_\nu A_\mu(x + b_r)U^x(b, r) \circ db^\mu_r, \text{ if } s \leq t \\
0, \text{ otherwise.}
\end{cases}
\tag{9}
\]

Let us show that \( D^x_\nu U^x(b, t) = Z^x_\nu(b, t, s) \).

Theorem 2.2.1 from 21 implies that

\[
U^x(b, t) \in W^1_{\infty}(P, M_N(\mathbb{C}))
\]

and \( D^x_\nu U^x(b, t) \) is a solution to the equation

\[
D^x_\nu U^x(b, t) = -A_\nu(x + b_t) U^x(b, s) Ind_{s \leq t}(s, t) - \\
\int_0^t (D^x_\nu A_\mu(x + b_r) U^x(b, r) + A_\mu(x + b_r) D^x_\nu U^x(b, r)) \circ db^\mu_r, \tag{10}
\]

\[\text{If } y_1 \text{ and } y_2 \text{ are semi-martingales } dy_1 \cdot dy_2 \text{ denotes the differential of the quadratic co-}
\]

\[\text{variation of } y_1 \text{ and } y_2.\]

\[\text{The symbol } Ind_{s \leq t} \text{ denotes the indicator of the set } \{(s, t): s \leq t\}\]
where
\[ D^\nu_s A_\mu(x + b_t) = \partial_\nu A_\mu(x + b_t) I_{nd_s \leq t}(s, t) \]

We verify whether \( Z^\nu(x, t, s) \) is the solution to equation (10). If \( s > t \), then \( Z^\nu(x, t, s) \) is equal to zero and obviously satisfies (10). For \( s \leq t \) we have

\[-A_\nu(x + b_s) U^\nu(x, t, s) - \int_0^t \left( D^\nu_s A_\mu(x + b_r) U^\nu(b, r) + A_\mu(x + b_r) Z^\nu(b, s) \right) \circ db^\mu_r = \]
\[= -A_\nu(x + b_s) U^\nu(x, t, s) - \int_s^t \left( \partial_\nu A_\mu(x + b_r) U^\nu(b, r) - A_\mu(x + b_r) U^\nu(b, s) - A_\mu(x + b_r) \int_s^r U^\nu(b, p) - \partial_\nu A_\lambda(x + b_p) U^\nu(b, p) \circ db^\lambda_p \right) \circ db^\nu_r = \]
\[= (-A_\nu(x + b_s) U^\nu(x, t, s) + \int_s^t A_\mu(x + b_r) U^\nu(b, r) U^\nu(b, s) - A_\mu(x + b_s) U^\nu(b, s) \circ db^\nu_r) - \]
\[ - \int_s^t \left( \partial_\nu A_\mu(x + b_r) U^\nu(b, r) - A_\mu(x + b_r) \int_s^r U^\nu(b, p) - \partial_\nu A_\lambda(x + b_p) U^\nu(b, p) \circ db^\lambda_p \right) \circ db^\nu_r. \quad (11)\]

Note that (6) and the Itô formula together imply

\[ \int_s^t \partial_\nu A_\mu(x + b_r) U^\nu(b, r) - A_\mu(x + b_r) U^\nu(b, r) \int_s^r \left( U^\nu(b, p) - \partial_\nu A_\lambda(x + b_p) U^\nu(b, p) \circ db^\lambda_p \right) \circ db^\nu_r = \]
\[= \int_s^t U^\nu(b, r) \left( U^\nu(b, r) - \partial_\nu A_\lambda(x + b_r) U^\nu(b, r) \right) -\]
\[ - A_\mu(x + b_r) U^\nu(b, r) \left( \int_s^r U^\nu(b, p) - \partial_\nu A_\lambda(x + b_p) U^\nu(b, p) \circ db^\lambda_p \right) \circ db^\nu_r = \]
\[= U^\nu(b, t) \int_s^t \left( U^\nu(b, r) - \partial_\nu A_\lambda(x + b_r) U^\nu(b, r) \right) \circ db^\nu_r. \quad (12)\]

From (6) it follows that \( U^\nu(b, t) = U^\nu(b, s) - \int_s^t A_\mu(x + b_r) U^\nu(b, r) \circ db^\nu_r. \) This equality implies

\[ A_\nu(x + b_s) U^\nu(b, s) - \int_s^t A_\mu(x + b_r) U^\nu(b, r) U^\nu(b, s) - A_\mu(x + b_s) U^\nu(b, s) \circ db^\nu_r = \]
\[= (U^\nu(b, s) - \int_s^t A_\mu(x + b_r) U^\nu(b, r) \circ db^\nu_r) U^\nu(b, s) - A_\mu(x + b_s) U^\nu(b, s) = \]
\[= U^\nu(b, t) U^\nu(b, s) - A_\mu(x + b_s) U^\nu(b, s). \quad (13)\]
The equalities (12) and (13) together imply that (11) is equal to

\[ - A_\nu(x + b_s)U^\nu(b, s) - U^\nu(b, t) \int_t^1 U^\nu(b, r) \partial_\nu A_\mu(x + b_r)U^\mu(b, r) \od dB^\nu_r = Z^\nu(b, t, s). \]

So \( Z^\nu(b, t, s) \) coincides with \( D^\nu_s U^\nu(b, t) \).

Using the Itô formula, we finally have

\[
\partial_u U^\nu(b, t) = D^\nu_s U^\nu(b, t) = \int_0^1 Z^\nu_s(b, t, s)\dot{u}^\nu(s)ds = \\
= - \int_0^1 (U^\nu(b, t)U^\nu(b, s))^{-1}A_\mu(x + b_s)U^\mu(b, s) + \\
+ U^\nu(b, t) \int_s^1 U^\nu(b, r) \partial_\mu A_\nu(x + b_r)U^\mu(b, r) \od dB^\mu_r \dot{u}^\nu(s)ds = \\
= -U^\nu(b, t) \int_0^1 L^\nu_{\mu\nu}(b, s)u^\mu(s) \od dB^\nu_s + A_\mu(x + b_s)u^\mu(t)U^\nu(b, t) = \\
= -U^\nu(b, t) \int_0^1 L^\nu_{\mu\nu}(b, s)u^\mu(s) + db^\nu_s - A_\mu(x + b_s)u^\mu(t)U^\nu(b, t)
\]

Since the connection \( A \) and all its derivatives of the first and the second orders are bounded, the theorem 2.2.2 from [21] implies \( U^\nu(b, t) \in W^2_\infty(P, M_N(\mathbb{C})) \).

\[ \square \]

**Proposition 3.** For any \( t \in [0, 1] \) it is valid that \( U^\nu(b, t)^{-1} \in W^2_\infty(P, M_N(\mathbb{C})) \). If \( u \in W^{2,1}_0([0, 1], \mathbb{R}^d) \), then

\[
\partial_u U^\nu(b, t)^{-1} = \\
= (\int_0^1 L^\nu_{\mu\nu}(b, s)u^\mu(s) \od dB^\nu_s)U^\nu(b, t)^{-1} + U^\nu(b, t)^{-1}A_\mu(x + b_s)u^\mu(t) \quad (14)
\]

**Proof.** The proof is similar to the previous one. \[ \square \]

**Proposition 4.** If \( u, v \in W^{2,1}_0([0, 1], \mathbb{R}^d) \) and \( u(1) = v(1) = 0 \), then

\[
\partial_v \partial_u U^\nu(b, 1) = \\
= U^\nu(b, 1) \int_0^1 L^\nu_{\mu\lambda}(b, t)u^\mu(t) \int_0^t L^\nu_{\nu\omega}(b, s)v^\nu(s) \od dB^\nu_s + \\
+ U^\nu(b, 1) \int_0^1 L^\nu_{\nu\lambda}(b, t)v^\nu(t) \int_0^t L^\nu_{\mu\lambda}(b, s)u^\mu(s) \od dB^\lambda_s - \\
- U^\nu(b, 1) (\frac{1}{2} \int_0^1 (J^\nu_{\nu\lambda}(b, t) + J^\nu_{\mu\lambda}(b, t))u^\mu(t)v^\nu(t) \od dB^\lambda_s + \\
+ \frac{1}{2} \int_0^1 L^\nu_{\nu\lambda}(b, t)(\dot{u}^\nu(t)v^\nu(t) + \dot{v}^\nu(t)u^\mu(t))dt). \quad (15)
\]
Proof. Lemma 1.3.4 from \[21\] implies

\[
D^\nu \partial_u U^x(b, 1) = -Z^x(b, 1, s) \int_0^1 L^x_{\mu \lambda}(b, t) u^\mu(t) \circ db^\lambda_t -
\]

\[
- U^x(b, 1) \int_0^1 L^x_{\mu \lambda}(x + b_t) u^\mu(t) U^x(b, t)^{-1} Z^x_v(b, t, s) \circ db^\lambda_t -
\]

\[
+ U^x(b, 1) \int_0^1 U^x(b, t)^{-1} Z^x_v(b, t, s) L^x_{\mu \lambda}(b, t) u^\mu(t) \circ db^\lambda_t -
\]

\[
- U^x(b, 1) \int_s^1 U^x(b, t)^{-1} \partial_v F_{\mu \lambda}(x + b_t) u^\mu(t) U^x(b, t) \circ db^\lambda_t -
\]

\[- U^x(b, 1) L^x_{\mu \nu}(b, s) u^\mu(s) .
\]

Since

\[
\partial_v \partial_u U^x(b, 1) = \sum_{\nu=1}^d \int_0^1 \dot{v}^\nu(s) D^\nu \partial_u U^x(b, 1) ds,
\]

using the Itô formula, we obtain

\[
\partial_v \partial_u U^x(b, 1) =
\]

\[
= U^x(b, 1) \int_0^1 L^x_{\mu \lambda}(b, t) u^\mu(t) \left( \int_0^t L^x_{\nu \lambda}(b, s) v^\nu(s) \circ db^\lambda_s \right) \circ db^\lambda_t +
\]

\[
+ U^x(b, 1) \int_0^1 L^x_{\nu \lambda}(b, t) v^\nu(t) \left( \int_0^t L^x_{\mu \lambda}(b, s) u^\mu(s) \circ db^\lambda_s \right) \circ db^\lambda_t -
\]

\[
- U^x(b, 1) \int_0^1 J^x_{\nu \lambda}(b, t) u^\mu(t) v^\nu(t) \circ db^\lambda_t +
\]

\[
+ \int_0^1 L^x_{\mu \nu}(b, t) \dot{v}^\nu(t) \circ dt . \quad (16)
\]

The Itô formula implies:

\[
- \frac{1}{2} U^x(b, 1) \int_0^1 L^x_{\mu \nu}(b, s) u^\mu(s) v^\nu(s) ds =
\]

\[
= \frac{1}{2} U^x(b, 1) \int_0^1 J^x_{\mu \nu}(b, s) u^\mu(s) v^\nu(s) \circ db^\lambda_s +
\]

\[
+ \frac{1}{2} U^x(b, 1) \int_0^1 L^x_{\mu \nu}(b, s) \dot{u}^\mu(s) v^\nu(s) ds =
\]

\[
= - \frac{1}{2} U^x(b, 1) \int_0^1 J^x_{\mu \nu}(b, s) u^\mu(s) v^\nu(s) \circ db^\lambda_s -
\]

\[
- \frac{1}{2} U^x(b, 1) \int_0^1 J^x_{\nu \lambda}(b, s) u^\mu(s) v^\nu(s) \circ db^\lambda_s +
\]

\[
+ \frac{1}{2} U^x(b, 1) \int_0^1 L^x_{\mu \nu}(b, s) \dot{u}^\mu(s) v^\nu(s) ds . \quad (17)
\]
The last equality holds due to the Bianchi identities (3). Then equalities (16) and (17) together imply (15).

2 Technical lemmas

There are certain technical results obtained in this section that are referred to in proofs of the theorems in the following sections.

Let $h_n(t) = \sqrt{2} \sin n\pi t$ and $l_n(s, t) = \frac{1}{n} \sum_{k=1}^{n} h_k(s) h_k(t)$. Note that

$$\lim_{n \to \infty} l_n(s, t) = \begin{cases} 1, & \text{if } t = s \text{ and } (t, s) \neq (0, 0), (1, 1) \\ 0, & \text{otherwise} \end{cases}$$

(18)

and for all $n \in \mathbb{N}$ and $(t, s) \in [0, 1] \times [0, 1]$ the following inequality holds:

$$|l_n(s, t)| \leq 2.$$  

(19)

Below $\| \cdot \|$ denotes the Frobenius norm on the space $M_N(\mathbb{C})$.

Lemma 1. Let $H_\mu, M_\mu$ be adapted with $(\mathcal{F}_t)$ bounded $M_N(\mathbb{C})$-valued stochastic processes. Then in $L^2(P, M_N(\mathbb{C}))$ the following equalities hold

$$\lim_{n \to \infty} \int_0^1 \left( \int_0^t H_\mu(b, s)l_n(s, t)db_\nu^\mu(db)^\nu_t \right) M_\nu(b, t)db_\nu^\nu = 0,$$

(20)

$$\lim_{n \to \infty} \int_0^1 M_\nu(b, t) \left( \int_0^t H_\mu(b, s)l_n(s, t)db_\nu^\mu(db)^\nu_t \right) db_\nu^\nu = 0.$$  

(21)

Proof. In the proof of the lemma we denote

$$\int_0^1 \left( \int_0^t H_\mu(b, s)l_n(s, t)db_\nu^\mu(db)^\nu_t \right) M_\nu(b, t)db_\nu^\nu$$

by $R_n(b)$. Due to the Fubini–Tonelli theorem we have

$$\|R_n\|_2^2 = E(\text{tr}(\int_0^1 (\int_0^t H_\mu(b, s)l_n(s, t)db_\nu^\mu)M_\nu(b, t)db_\nu^\nu) \times \int_0^1 (\int_0^t H_\mu(b, s)l_n(s, t)db_\nu^\nu)M_\nu(b, t)db_\nu^\nu)^*) =$$

$$= \int_0^1 E(\text{tr}(\int_0^t H_\mu^*(b, s)l_n(s, t)db_\nu^\mu)(\int_0^t H_\mu(b, s)l_n(s, t)db_\nu^\nu)M_\nu(b, t)M_\nu^*(b, t)))dt$$

Since the processes $M_\mu$ are bounded, there exists such a constant $C > 0$ that

$$\|R_n\|_2^2 \leq C \int_0^1 E(\|\int_0^t H_\mu(b, s)l_n(s, t)db_\nu^\mu\|^2)dt =$$

$$= C \int_0^1 \int_0^t E(\text{tr}(H_\mu^*(b, s)H_\mu(b, s)))l_n^2(s, t)dsdt.$$  

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Then equality (20) follows from (18), (19), boundedness of processes $H_\mu$ and Lebesgue’s dominated convergence theorem. Equality (21) can be proved similarly.

**Lemma 2.** Let $M_\mu, K$ be adapted with $(\mathcal{F}_t)$ bounded $M_N(\mathbb{C})$-valued processes. Then in $L^2(P, M_N(\mathbb{C}))$ the following equalities hold

$$\lim_{n \to \infty} \int_0^1 \left( \int_0^t K(b, s)l_n(s, t)ds \right) M_\nu(b, t)db_\nu^t = 0,$$

(22)

$$\lim_{n \to \infty} \int_0^1 M_\nu(b, t)\left( \int_0^t K(b, s)l_n(s, t)ds \right)db_\nu^t = 0.$$  

(23)

**Proof.** In the proof of the lemma we denote $\int_0^1 \int_0^t K(b, s)l_n(s, t)ds M_\nu(b, t)db_\nu^t$ by $R_n(b)$. Due to the Fubini–Tonelli theorem we have

$$\|R_n\|_2^2 = E(tr(\int_0^1 (\int_0^t K(b, s)l_n(s, t)ds)M_\nu(b, t)db_\nu^t) \times$$

$$\times (\int_0^1 (\int_0^t K(b, s)l_n(s, t)ds)M_\nu(b, t)db_\nu^t)^*)) =$$

$$= E(tr(\int_0^1 K^*(b, s)l_n(s, t)ds)(\int_0^t K(b, s)l_n(s, t)ds)M^\nu(b, t)M^*_\nu(b, t)dt)) =$$

$$= \int_0^1 \int_0^t E(tr(K^*(b, s_1)K(b, s_2)M^\nu(b, t)M^*_\nu(b, t)))l_n(s_1, t)l_n(s_2, t)ds_1ds_2dt$$

Since the processes $M_\mu, K$ are bounded, there exists such a constant $C > 0$ that

$$\|R_n\|_2^2 \leq C \int_0^1 \int_0^t l_n(s_1, t)l_n(s_2, t)ds_1ds_2dt.$$

Then equality (22) follows from (18) and (19) and Lebesgue’s dominated convergence theorem. Equality (23) can be proved similarly.

**Lemma 3.** Let $H_\mu, K$ be adapted with $(\mathcal{F}_t)$ bounded $M_N(\mathbb{C})$-valued processes. Then in $L^2(P, M_N(\mathbb{C}))$ the following equalities hold

$$\lim_{n \to \infty} \int_0^1 \left( \int_0^t H_\mu(b, s)l_n(s, t)db_\nu^t \right)K(b, t)dt = 0,$$

(24)

$$\lim_{n \to \infty} \int_0^1 K(b, t)\left( \int_0^t H_\mu(b, s)l_n(s, t)db_\nu^t \right)dt = 0.$$  

(25)
Proof. In the proof of the lemma we denote \( \int_0^1 (\int_0^t K(b, s) l_n(s, t) ds) M_\nu(b, t) db'_\nu \) by \( R_n(b) \). Due to the Fubini–Tonelli theorem we have

\[
\|R_n\|_2^2 = \int_0^1 \int_0^1 E(\text{tr}((\int_0^{t_1} H^\mu(b, s_1) l_n(s_1, t_1) db^\mu_{s_1}) \times 
\times (\int_0^{t_2} H^\mu(b, s_2) l_n(s_2, t_2) db^\mu_{s_2}) K(b, t_2) K^*(b, t_1)))) dt_1 dt_2
\]

Since the process \( K \) is bounded, there exists such a constant \( C > 0 \) that

\[
\|R_n\|_2^2 \leq C \int_0^1 \int_0^1 E(\| (\int_0^{t_1} H^\mu(b, s_1) l_n(s_1, t_1) db^\mu_{s_1}) \times \n\times (\int_0^{t_2} H^\mu(b, s_2) l_n(s_2, t_2) db^\mu_{s_2}) dt_1 dt_2 ) \leq
\]

Then

\[
\|R_n\|_2^2 \leq \frac{1}{2} C E((\int_0^1 \int_0^1 E(\| (\int_0^{t_1} H^\mu(b, s_1) l_n(s_1, t_1) db^\mu_{s_1}) \times \n\times (\int_0^{t_2} H^\mu(b, s_2) l_n(s_2, t_2) db^\mu_{s_2}) dt_1 dt_2 )^2 + \n+ (\int_0^1 \int_0^1 E(\| (\int_0^{t_1} H^\mu(b, s_1) l_n(s_1, t_1) db^\mu_{s_1})^2 dt_1 dt_2 
\n= C \int_0^1 \int_0^t E(\| (\int_0^{t_1} H^\mu(b, s) l_n(s, t) db^\mu_{s_2})^2 dt)
\n= C \int_0^1 \int_0^t E(\text{tr}(H^\mu(b, s) H^\mu_*(b, s)) l_n^2(s, t) ds dt)
\]

Then equality (24) follows from (18) and (19) and Lebesgue’s dominated convergence theorem. Equality (25) can be proved similarly. \( \square \)

Lemma 4. Let \( R, K \) be adapted with \( (F_t) \) bounded \( M_N(\mathbb{C}) \)-valued processes. Then in \( L_2(P, M_N(\mathbb{C})) \) the following equalities hold

\[
\lim_{n \to \infty} \int_0^t \left( \int_0^1 R(b, s) l_n(s, t) ds \right) K(b, t) dt = 0, \quad (26)
\]

\[
\lim_{n \to \infty} \int_0^1 K(b, t) \left( \int_0^t R(b, s) l_n(s, t) ds \right) dt = 0. \quad (27)
\]

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Proof. Denote \( \int_0^1 (f^t_0 R(b,s)l_n(s,t)ds)K(b,t)dt \) by \( R_n(b) \). Due to the Fubini–Tonelli theorem we have

\[
\|R_n\|_2^2 = \int_0^1 \int_0^1 \int_0^{t_1} \int_0^{t_2} l_n(s_1,t_1)l_n(s_2,t_2) \times \times E(tr(R^*(b_1)R(b_2)K(b_1,t_1)K^*(b_2,t_2)))ds_1ds_2dt_1dt_2
\]

Then equality (26) follows from (18), boundedness of processes \( R, K \) and Lebesgue’s dominated convergence theorem. Equality (27) can be proved similarly. \( \square \)

3 Stochastic Lévy Laplacian

Definition 1. The stochastic Lévy Laplacian \( \Delta_L \) is a linear mapping from \( \text{dom}\Delta_L \) to \( L^2(P, M, \mathbb{R}^d) \) defined as

\[
\Delta_L f(b) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n \sum_{\mu=1}^d \partial_{p_kh_\mu} \partial_{p_kh_\mu} f(b), \quad (28)
\]

where the sequence converges in \( L^2(P, M, \mathbb{R}^d) \) and \( \text{dom}\Delta_L \) consists of all \( f \in W^2_2(P, M, \mathbb{R}^d) \) for which the right-hand side of (28) exists.

Remark 1. If \( f \in \text{dom}\Delta_L \), then \( \Delta_L f(b) = d \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n \partial_{p_kh_\mu} \partial_{p_kh_\mu} f(b) \), where, as it has been mentioned in the introduction, \( \{e_n\} \) is the orthonormal basis in \( L^2([0,1], \mathbb{R}^d) \) defined by \( e_n(t) = p_{a_n}(t)h_{b_n}(t) \), where \( a_n = n - 4\lfloor \frac{n}{4} \rfloor \) and \( b_n = \lfloor \frac{n+3}{4} \rfloor \). This formula precises the definition to \( \text{tr}_L D^2 f \). The stochastic analogue of the Lévy d’Alambertian (see [4, 3, 30, 28]) can be defined as

\[
\square_L f(b) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n \partial_{p_kh_\mu} \partial_{p_kh_\mu} f(b) - \sum_{\mu=2}^d \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n \partial_{p_kh_\mu} \partial_{p_kh_\mu} f(b). \quad (29)
\]

Theorem 1. The following equality holds

\[
\Delta_L U^{\tau}(b,1) = U^{\tau}(b,1)\left( \int_0^1 L^{\tau}_{\mu\nu}(b,t)L^{\tau\mu\nu}(b,t)dt - \int_0^1 J^{\tau\mu\nu}_{\mu\nu}(b,t) \circ dB_t^{\tau} \right)
\]

Proof. Introduce the notations

\[
R_n^{\tau}(b) = \frac{1}{n} \sum_{k=1}^n \sum_{\mu=1}^d \partial_{p_kh_\mu} \partial_{p_kh_\mu} U^{\tau}(b,1)
\]

and

\[
R^{\tau}(b) = U^{\tau}(b,1)\left( \int_0^1 L^{\tau}_{\mu\nu}(b,t)L^{\tau\mu\nu}(b,t)dt - \int_0^1 J^{\tau\mu\nu}_{\mu\nu}(b,t) \circ dB_t^{\tau} \right)
\]
Proposition 4 implies

\[ \| R^x - R^x_n \|_2 = -E(tr( \int_0^1 L^x_{\mu\nu}(b, t)L^x_{\mu\nu}(b, t)(1 - l_n(t, t)) dt + \int_0^1 ( \int_0^t L^x_{\mu\lambda}(b, s)l_n(s, t) \odot db^\lambda_s) L^x_{\nu\rho}(b, t) db^\rho_s + \int_0^1 ( \int_0^t L^x_{\mu\lambda}(b, s)l_n(s, t) \odot db^\lambda_s) J^x_{\mu\nu}(b, t) dt + \int_0^1 J^{x\mu}_{\mu\nu}(b, t)(1 - l_n(t, t)) \odot db^\nu_t)^2 ). \]

It follows from lemmas 1 and 2 that

\[ - \lim_{n \to \infty} E(tr( \int_0^1 ( \int_0^t L^x_{\mu\lambda}(b, s)l_n(s, t) \odot db^\lambda_s) L^x_{\nu\rho}(b, t) db^\rho_s )^2 ) = 0 \]  

(30)

It follows from lemmas 3 and 4 that

\[ - \lim_{n \to \infty} E(tr( \int_0^1 ( \int_0^t L^x_{\mu\lambda}(b, s)l_n(s, t) \odot db^\lambda_s) J^x_{\mu\nu}(b, t) dt )^2 ) = 0 \]  

(31)

Lebesgue’s dominated convergence theorem implies

\[ \lim_{n \to \infty} E(tr( \int_0^1 L^x_{\mu\nu}(b, t)L^x_{\mu\nu}(b, t)(1 - l_n(t, t)) dt )^2 ) = 0 \]  

(32)

and

\[ \lim_{n \to \infty} E(tr( \int_0^1 J^{x\mu}_{\mu\nu}(b, t)(1 - l_n(t, t)) \odot db^\nu_t )^2 ) = 0. \]  

(33)

Then Minkowski inequality and (30), (31), (32), (33) together imply that

\[ \lim_{n \to \infty} \| R^x - R^x_n \|_2 = 0. \]

Remark 2. If the connection A satisfies the Yang-Mills equations (5), it is valid that \( \Delta_L U^x(b, 1) = U^x(b, 1) \int_0^1 L^x_{\mu\nu}(b, t)L^x_{\rho\lambda}(b, t) dt \). It would be interesting to study relationship between the Laplacians introduced in the present paper and in [13].

Remark 3. There exists the canonical unitary isomorphism between \( L^2(P) \) and \( \Gamma(L_2([0, 1], \mathbb{R}^d)) \) (the boson Fock space over the Hilbert space \( L_2([0, 1], \mathbb{R}^d) \)). In the white noise theory a rigged Hilbert space \( E \subset \Gamma(L_2([0, 1], \mathbb{R}^d)) \subset E^* \) is considered, there \( E \) is the space of white noise test functionals and \( E^* \) is the space of white noise generalized functionals. The Lévy Laplacian on the space of white noise generalized functionals has been defined by T. Hida in [15] (see also [16, 18, 17]). This operator is equal to zero on \( \Gamma(L_2([0, 1], \mathbb{R}^d)) \) (see [18]).
Hence, it does not coincide with the Lévy Laplacian introduced in the present paper. But in the next paper we will show that the last operator coincides with an element from the chain of nonclassical Lévy Laplacians acting on the space of white noise generalized functionals (see \cite{8, 26}). Moreover, we assume that this operator can be represented as \( \int_{|s-t|<\varepsilon} a_{\mu}(s)\dot{a}^\mu(t)dsdt \), where \( \dot{a}^\mu(t) \) is the derivative of the annihilation process.

4 Stochastic Lévy Divergence

**Definition 2.** The stochastic Lévy divergence \( \text{div}_L \) is a linear mapping from \( \text{dom} \text{div}_L \) to \( L_2(P, M_N(\mathbb{C})) \) defined by the formula

\[
\text{div}_LB(b) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{\mu=1}^{d} \partial_{p_{\mu}h_k}B(b)(p_{\mu}h_k),
\]

where the sequence converges in \( L_2(P, M_N(\mathbb{C})) \) and \( \text{dom} \text{div}_L \) consists of all \( B \in L(W_{0,1}^{2,1}([0, 1], \mathbb{R}^d), W_{2}^1(P, M_N(\mathbb{C}))) \) for which the right-hand side of (34) exists.

**Remark 4.** We use the notations of remark 1. If \( B \in \text{dom} \text{div}_L \), then

\[
\text{div}_LB(b) = d \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \partial_{e_n}B(b)e_n.
\]

This formula precises the definition to \( \text{tr}_L DB \).

**Example 1.** Let \( d = 3 \). Let \( \epsilon_{\mu\nu\lambda} \) be totally antisymmetric unit tensor. Let

\[
S^x(b)h = \int_0^1 \epsilon_{\mu\nu\lambda} L^x_{\nu\lambda}(b,t)h^\mu(t)dt
\]

The Bianchi identities imply that

\[
\text{div}_LS^x(b) = \int_0^1 \epsilon_{\mu\nu\lambda} T^x_{\mu\nu\lambda}(b,t)dt = 0.
\]

From the fact that \( U^x(b,1), U^x(b,1)^{-1} \in W_\infty^2(P, M_N(\mathbb{C})) \) it follows that

\[
B^{A,x} \in L(W_{0,1}^{2,1}([0, 1], \mathbb{R}^d), W_{2}^1(P, M_N(\mathbb{C}))),
\]

where \( B^{A,x} \) is defined by the formula

\[
B^{A,x}(b)u = U^x(b,1)^{-1}\partial_{u}U^x(b,1) =
\]

\[
= -\int_0^1 U^x(b,t)^{-1} F_{\mu\nu}(x + b_1)u^\mu(t)U^x(b,t) \circ db^\nu - U^x(b,1)^{-1} A_\mu(x + b_1)u^\mu(1)U^x(b,1).
\]

Note that in fact \( B^{A,x} \in L(W_{0,1}^{2,1}([0, 1], \mathbb{R}^d), W_{2}^1(P, u(N))) \).
Proposition 5. If \( u, v \in W^{2,1}_0([0, 1], \mathbb{R}^d) \) and \( u(1) = v(1) = 0 \), then
\[
\partial_u B^{A,x}(b)v = \int_0^1 \left[ L^x_{\nu\kappa}(b, t)\nu(t), \int_0^t L^x_{\mu\lambda}(b, s)u^\mu(s) \circ db^\lambda_s \right] \circ db^\nu_t - \frac{1}{2} \int_0^1 \left( J^x_{\nu\mu\lambda}(b, t) + J^x_{\mu\nu\lambda}(b, t) \right) u^\mu(t) \nu(t) \circ db^\lambda_t - \frac{1}{2} \int_0^1 L^x_{\mu\nu}(b, t) (\dot{u}^\nu(t) u^\mu(t) + \dot{\nu}^\nu(t) u^\mu(t))dt. \tag{35}
\]

Proof. It is valid that
\[
\partial_u B^{A,x}(b, v) = \partial_v U^{x}(b, 1) - \frac{1}{2} \partial_u U^{x}(b, 1) + U^{x}(b, 1) - \frac{1}{2} \partial_v \partial_u U^{x}(b, 1). \tag{36}
\]
This equality and Propositions 2, 3 and 4 together imply (35).

Theorem 2. The following equality holds
\[
div_L B^x(b) = -\int_0^1 J^{x\mu}_{\mu\nu}(b, t) \circ db^\nu_s = -\int_0^1 U^x(b, s)^{-1} \nabla^\mu F_{\mu\nu}(x + b, s)U^x(b, s)db^\nu_s. \tag{37}
\]

Proof. The proof is similar to the proof of Theorem 1. Due to Proposition 5 we have
\[
\sum_{\mu=1}^d \partial_p^\mu h_n B^{A,x}(b)(p, h_n) = \int_0^1 \left[ L^x_{\mu\lambda}(b, t)h_n(t), \int_0^t L^x_{\mu\kappa}(b, s)h_n(s) \circ db^\kappa_s \circ db^\lambda_t \right] - \int_0^1 J^{x\mu}_{\mu\nu}(b, t)h_n^2(t) \circ db^\nu_s =
\]
\[
= \int_0^1 \left[ L^x_{\mu\lambda}(b, t), \int_0^t L^x_{\mu\kappa}(b, s)l_n(s, t) \circ db^\kappa_s \circ db^\lambda_t \right] + \frac{1}{2} \int_0^1 J^{x\mu}_{\lambda\mu}(b, t), \int_0^t L^x_{\mu\kappa}(b, s)l_n(s, t) \circ db^\kappa_s \circ db^\lambda_t dt - \int_0^1 J^{x\mu}_{\nu\mu}(b, t)h_n^2(t) \circ db^\nu_s. \tag{38}
\]

Then Lemmas 1, 3, 2, 4, Lebesgue’s dominated convergence theorem and the Minkowski inequality imply (37).

Theorem 3. The following two assertions are equivalent:

1. A connection \( A \) satisfies the Yang-Mills equations: \( \nabla_{\mu} F_{\mu\nu} = 0 \),
2. \( div_L B^{A,x} = 0 \) for some \( x \in \mathbb{R}^d \).
Proof. Let it is valid for some \( x \in \mathbb{R}^d \) that \( \text{div}_L B^{A,x} = 0 \). Then

\[
0 = -E(\text{tr}(\text{div}_L B^{A,x})^2) = -E \int_0^1 \text{tr}(\nabla_{\mu} F^\mu_{\nu}(x + b_t) \nabla_{\lambda} F^{\lambda\nu}(x + b_t)) dt = \\
= -\int_0^1 \int_{\mathbb{R}^d} \text{tr}(\nabla_{\mu} F^\mu_{\nu}(x + y) \nabla_{\lambda} F^{\lambda\nu}(x + y))(2\pi t)^{-\frac{d}{2}} e^{-\frac{(y,y)^2}{2t}} dy dt
\]

Then we obtain \( \nabla_{\mu} F^\mu_{\nu} = 0 \). The other side of the proof of the theorem is trivial. \( \Box \)

5 Action functional

In this section it is proved that the Yang-Mills action functional (4) can be represented as an infinite-dimensional analogue of the Dirichlet functional of chiral field.

Proposition 6. The following equality holds

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n E(\text{tr}(\partial_{p_n, h_k} U^x(b, 1)^{-1} \partial_{p_n, h_k} U^x(b, 1))) = \\
= -E(\int_0^1 \text{tr}(F_{\mu\nu}(x + b_t) F^{\mu\nu}(x + b_t)) dt).
\]

Proof. Note that Propositions 2 and 3 imply

\[
\text{tr}(\partial_{p_n, h_k} U^x(b, 1)^{-1} \partial_{p_n, h_k} U^x(b, 1)) = -\text{tr}(\int_0^1 L_{\mu\nu}(b, t) h_n(t) \circ d\omega_t)\cdot h_n(t)^2.
\]

Using the Itô formula we have

\[
(\int_0^1 L_{\mu\nu}(b, t) h_n(t) \circ d\omega_t)\cdot h_n(t)^2 = \int_0^1 L_{\mu\nu}(b, t) L_{\mu\nu}(b, t) h_n^2(t) dt + \\
+ 2 \int_0^1 (\int_0^t L_{\mu\nu}(b, s) h_n(s) \circ d\omega_t) J_{\lambda\mu}^\nu(b, t) h_n(t) dt + \\
+ 4 \int_0^1 (\int_0^t L_{\mu\nu}(x + b_s) h_n(s) \circ d\omega_t) J_{\lambda\mu}^\nu(b, t) h_n(t) dt. \quad (39)
\]

It is valid that

\[
E(\int_0^1 (\int_0^t L_{\mu\nu}(b, s) h_n(s) \circ d\omega_t) L_{\mu\lambda}^\nu(b, t) h_n(t) dt) = 0.
\]

Lemmas 3 and 4 imply

\[
\lim_{n \to \infty} E(\int_0^1 (\int_0^t L_{\mu\nu}(b, s) l_n(s, t) \circ d\omega_t) J_{\lambda\mu}^\nu(b, t) dt) = 0.
\]
We obtain

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} E(\operatorname{tr}(\partial_{p_{h_{k}}} U^{x}(b, 1)^{-1} \partial_{p_{h_{k}}} U^{x}(b, 1))) = \\
= - \lim_{n \to \infty} E(\operatorname{tr} \int_{0}^{1} L_{\mu \nu}(b, t)L_{\mu \nu}(b, t)l_{n}(t, t)dt) = \\
= -E(\int_{0}^{1} \operatorname{tr}(F_{\mu \nu}(x + b_{t})F_{\mu \nu}'(x + b_{t}))dt). \tag{40}
\]

The last equality follows from Lebesgue’s dominated convergence theorem. □

**Theorem 4.** If

\[- \int_{\mathbb{R}^{d}} \operatorname{tr}(F_{\mu \nu}(x)F^{\mu \nu}(x))dx < \infty \]

and

\[- \int_{\mathbb{R}^{d}} \operatorname{tr}(\nabla_{\mu} F_{\nu \lambda}(x)\nabla_{\nu} F^{\mu \lambda}(x))dx < \infty, \]

then

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{\mu=1}^{d} E(\operatorname{tr}(\partial_{p_{h_{k}}} U^{x}(b, 1)^{-1} \partial_{p_{h_{k}}} U^{x}(b, 1)))dx) = \\
= - \int_{\mathbb{R}^{d}} \operatorname{tr}(F_{\mu \nu}(x)F^{\mu \nu}(x))dx \tag{41}
\]

**Proof.** Introduce the notations

\[G_{1}(x) = -E(\int_{0}^{1} \operatorname{tr}(F_{\mu \nu}(x + b_{t})F^{\mu \nu}(x + b_{t}))dt), \]

\[G_{2}(x) = -E(\int_{0}^{1} \operatorname{tr}(\nabla_{\mu} F^{\mu \lambda}(x + b_{t})\nabla_{\nu} F^{\mu \lambda}(x + b_{t}))dt). \]

The Fubini–Tonelli theorem implies

\[
\int_{\mathbb{R}^{d}} G_{1}(x)dx = -E(\int_{0}^{1} \int_{\mathbb{R}^{d}} \operatorname{tr}(F_{\mu \nu}(x + b_{t})F^{\mu \nu}(x + b_{t}))dxdt) = \\
= - \int_{\mathbb{R}^{d}} \operatorname{tr}(F_{\mu \nu}(x)F^{\mu \nu}(x))dx < \infty,
\]

\[
\int_{\mathbb{R}^{d}} G_{2}(x)dx = -E(\int_{0}^{1} \int_{\mathbb{R}^{d}} \operatorname{tr}(\nabla_{\mu} F^{\mu \lambda}(x)\nabla_{\nu} F^{\mu \lambda}(x))dxdt) = \\
= - \int_{\mathbb{R}^{d}} \operatorname{tr}(\nabla_{\mu} F^{\mu \lambda}(x)\nabla_{\nu} F^{\mu \lambda}(x))dx < \infty.
\]
Note that

\[-E(\text{tr}(\int_0^1 L^x_{\mu\nu}(b, t)h_k(t) \circ \text{d}b^\nu_t) \, \text{d}t) \leq -2E(\text{tr}(\int_0^1 L^x_{\mu\nu}(x + b_t)h_k(t) \circ \text{d}b^\nu_t) \, \text{d}t) - \frac{1}{2}E(\text{tr}(\int_0^1 J^x_{\nu\mu}(b, t)h_k(t) \, \text{d}t)^2)\]

Since (19) holds, for all \( n \in \mathbb{N} \) we have

\[-\frac{1}{n} \sum_{k=1}^{n} \sum_{\mu=1}^{d} E(\text{tr}(\int_0^1 L^x_{\mu\nu}(x + b_t)h_k(t) \circ \text{d}b^\nu_t) \, \text{d}t)^2) \leq 2G_1(x)\]

and

\[-\frac{1}{n} \sum_{k=1}^{n} \sum_{\mu=1}^{d} E(\text{tr}(\int_0^1 J^x_{\nu\mu}(b, t)h_k(t) \, \text{d}t)^2) =\]

\[-\frac{1}{n} \sum_{k=1}^{n} E(\text{tr}(\int_0^1 \int_0^1 J^x_{\nu\mu}(b, t)J^x_{\lambda\lambda}(b, t)h_k(t)h_k(s) \, \text{d}t \, \text{d}s) \leq \]

\[-\frac{1}{n} \sum_{k=1}^{n} \int_0^1 \int_0^1 E(\text{tr}(J^x_{\nu\mu}(b, t)J^x_{\lambda\lambda}(b, t))h_k^2(t) + \]

\[+ \text{tr}(J^x_{\nu\mu}(b, s)J^x_{\lambda\lambda}(b, s))h_k^2(s) \, \text{d}t \, \text{d}s \leq 2G_2(x)\]

Then equality (41) follows from Proposition 6 and Lebesgue’s dominated convergence theorem.

\(\Box\)

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