Abstract

A triangulated category \( \mathcal{T} \) whose suspension functor \( \Sigma \) satisfies \( \Sigma^m \cong \text{Id}_\mathcal{T} \) is called an \( m \)-periodic triangulated category. Such a category does not have a tilting object by the periodicity. In this paper, we introduce the notion of an \( m \)-periodic tilting object in an \( m \)-periodic triangulated category, which is a periodic analogue of a tilting object in a triangulated category, and prove that an \( m \)-periodic triangulated category having an \( m \)-periodic tilting object is triangulated equivalent to the \( m \)-periodic derived category of an algebra under some homological assumptions. As an application, we construct a triangulated equivalence between the stable category of a self-injective algebra and the \( m \)-periodic derived category of a hereditary algebra.

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1 Introduction

1.1 Background

\textit{Tilting theory} describes a way to relate a triangulated category and the derived category of an algebra. It plays an important role in representation theory and gives connections between various areas of mathematics. For example, it appears in a classification of representations of Dynkin quivers, Cohen-Macaulay representations of Gorenstein singularities and commutative/non-commutative algebraic geometry. See [Tilt07] for a broad review of the theory. The aim of this paper is to give a starting point of \textit{tilting theory for periodic triangulated categories}.

Let \( m \) be a positive integer. A triangulated category \( \mathcal{T} \) whose suspension functor \( \Sigma \) satisfies \( \Sigma^m \cong \text{Id}_\mathcal{T} \) is called an \( m \)-\textit{periodic triangulated category}. A basic example of a periodic triangulated category is an
m-periodic derived category, the localization of the category of m-periodic complexes with respect to quasi-isomorphisms. See \( \text{§3} \) for the precise definition. It appears in the categorification of Lie algebras and quantum groups via Ringel-Hall algebras. Peng and Xiao [PX97, PX00] used two-periodic complexes to construct full semisimple Lie algebras. Bridgeland [Br13] investigated the two-periodic derived category of a hereditary abelian category and constructed full quantum groups. Motivated by these studies, several authors analyzed the structure of m-periodic derived categories. Peng and Xiao proved in [PX97] that the m-periodic derived category \( D_m(\Lambda) \) of a hereditary algebra \( \Lambda \) is triangulated equivalent to the orbit category \( D^b(\Lambda)/\Sigma^m \) of the bounded derived category \( D^b(\Lambda) \). In [Zhao14], Zhao generalized this result to an algebra \( \Lambda \) of finite global dimension, and proved that the m-periodic derived category \( D_m(\Lambda) \) is triangulated equivalent to the triangulated hull of the orbit category \( D^b(\Lambda)/\Sigma^m \). In particular, the m-periodic derived category is a derived invariant of algebras. See also [Fu12, Go13, Go18]. We will give a relevant explanation in \( \text{§3.1} \).

Other examples of periodic triangulated categories are given by the stable category of maximal Cohen-Macaulay modules of a hypersurface singularity, and the stable category of a self-injective algebra of finite-representation type. We will treat some of these examples in \( \text{§5.2} \).

In the ordinary tilting theory, we have the tilting theorem, which states that a triangulated category with a tilting object is triangulated equivalent to the perfect derived category of the endomorphism ring of the tilting object. See Fact \ref{fact:tilting} for the precise statement. The motivation of this paper is to give a periodic analogue of this tilting theorem. The main Theorem \ref{thm:main} gives a sufficient condition under which a periodic triangulated category is triangulated equivalent to the periodic derived category of an algebra.

1.2 Main results

Let \( \mathcal{T} \) be a triangulated category with the suspension functor \( \Sigma \) over a field \( k \). We first recall the tilting objects and the tilting theorem in the standard sense. As usual, a thick subcategory of a triangulated category means a full subcategory closed under taking cones, shifts and direct summands.

Definition. An object \( T \) of \( \mathcal{T} \) is called tilting if \( T(T, \Sigma^iT) = 0 \) for any \( i \neq 0 \) and the smallest thick subcategory containing \( T \) is equal to \( \mathcal{T} \).

A triangulated category is called algebraic if it is triangulated equivalent to the stable category of a Frobenius exact category.

Fact 1.1 (The tilting theorem [Kel94]). Let \( \mathcal{T} \) be an idempotent complete algebraic triangulated category. If \( \mathcal{T} \) has a tilting object \( T \), then there exists a triangulated equivalence

\[
\mathcal{T} \overset{\sim}{\to} K^b(\text{proj} \Lambda),
\]

where \( \Lambda := \text{End}_T(T) \) is the endomorphism ring of \( T \), and \( K^b(\text{proj} \Lambda) \) is the homotopy category of bounded complexes of projective \( \Lambda \)-modules.

An m-periodic triangulated category does not have a tilting object by the periodicity, and cannot be triangulated equivalent to the derived category of an algebra. Instead, we will introduce an m-periodic tilting object which is a periodic analogue of a tilting object, and prove a periodic analogue of the tilting theorem.

Definition 1.2. Let \( \mathcal{T} \) be an m-periodic triangulated category. An object \( T \) of \( \mathcal{T} \) is called tilting if \( T(T, \Sigma^iT) = 0 \) for any \( i \notin m\mathbb{Z} \) and the smallest thick subcategory containing \( T \) is equal to \( \mathcal{T} \).

Theorem A (Theorem \ref{thm:main}). Let \( \mathcal{T} \) be an idempotent complete algebraic m-periodic triangulated category having an m-periodic tilting object \( T \), and \( \Lambda := \text{End}_T(T) \) the endomorphism algebra of \( T \) in \( \mathcal{T} \). If

\[
\text{proj.dim}_{\Lambda^e \otimes \Lambda} \Lambda \leq m, \quad (1.1)
\]

then there exists a triangulated equivalence

\[
\mathcal{T} \overset{\sim}{\to} D_m(\Lambda),
\]

where \( D_m(\Lambda) \) is the m-periodic derived category of \( \Lambda \) (\( \text{§3} \)).
At the first look, one may worry about the condition (1.1) of projective dimension as bimodules. In fact, we can replace it to the condition of global dimension of rings under a condition on the base field.

**Theorem B** (Corollary 5.4). Let $\mathcal{T}$ be an idempotent complete algebraic $m$-periodic triangulated category over a perfect field $k$. If $\mathcal{T}$ has an $m$-periodic tilting object $T$ which satisfies the condition

$$\Lambda := \text{End}_\mathcal{T}(T)$$

is a finite dimensional $k$-algebra of global dimension $d \leq m$,

then there exists a triangulated equivalence

$$\mathcal{T} \simeq D_m(\Lambda).$$

A key statement to prove Theorem A is Theorem C, which states that the Laurent polynomial ring is intrinsically formal (Definition 2.2) under some assumptions.

**Theorem C** (Lemma 4.2). Let $\Lambda$ be an algebra which has a finite projective dimension $d$ as a $\Lambda$-bimodule. We regard the Laurent polynomial ring $\Lambda[t, t^{-1}]$ over $\Lambda$ as a graded algebra by setting $\deg(t) := m$. If $d \leq m$, then $\Lambda[t, t^{-1}]$ is intrinsically formal.

The proof of this statement is inspired by [HK19].

**Organization**

This paper is organized as follows. In Section 2, we recall some basic facts on DG categories and Hochschild cohomology which will be used in the proof of Theorem A. In Section 3, we collect the basic definitions and properties of $m$-periodic complexes from the viewpoint of DG categories. In Section 4, we give a proof of Theorem A and Theorem C. In Section 5, we prove Theorem B and give some applications. We also give an example that Theorem C fails for an algebra of infinite global dimension by using a DG category of matrix factorizations.

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**Conventions and notation**

- $\mathbb{Z}_m$ denotes the cyclic group of order $m \in \mathbb{Z}_{>0}$.
- Throughout this paper we work over a fixed field $k$. All categories and functors are $k$-linear.
- A strictly full subcategory means a full subcategory which is closed under isomorphism.
- Let $\mathcal{C}$ be a category.
  - $\text{Ob} \mathcal{C}$ denotes the class of objects of $\mathcal{C}$.
  - We denote by $\mathcal{C}(M, N) = \text{Hom}_\mathcal{C}(M, N)$ the set of morphisms from $M$ to $N$ for $M, N \in \mathcal{C}$.
- We denote by $\Sigma$ the suspension functor of a triangulated category.
- Let $\mathcal{I}$ be a collection of objects in a triangulated category $\mathcal{T}$.
  - $\overline{\mathcal{I}}$ denotes the smallest strictly full subcategory of $\mathcal{T}$ containing $\mathcal{I}$.
  - $\text{add}_\mathcal{T}(\mathcal{I})$ denotes the smallest additive strictly full subcategory of $\mathcal{T}$ which contains $\mathcal{I}$ and is closed under direct summands.
  - $\text{thick}_\mathcal{T}(\mathcal{I})$ denotes the smallest triangulated strictly full subcategory of $\mathcal{T}$ which contains $\mathcal{I}$ and is closed under direct summands.
- By the word “algebra” we mean an associative unital algebra over the base field $k$.
- By the word “module” we mean a right module.
- Let $\Lambda$ be an algebra.
  - $\text{rad} \Lambda$ denotes the Jacobson radical of $\Lambda$.
  - $\text{gl.dim} \Lambda$ denotes the global dimension of $\Lambda$.
  - For a $\Lambda$-module $M$, $\text{proj.dim}_\Lambda M$ denotes the projective dimension of $M$. 
assumption, DG algebras are determined by its cohomology ring $H^0([Kel94,\text{Fact } 2.1]}.

Let $A$ be a graded algebra.

- Gr $A$ denotes the category of graded $A$-modules, i.e., a category whose objects are graded $A$-module and whose morphisms are homogeneous $A$-linear maps.
- For $\ell \in \mathbb{Z}$, $(\ell)$ denotes the $\ell$-degree shift, i.e., $M(\ell)^t = M^{t+\ell}$ for $M \in \text{Gr} A$.

## 2 Preliminaries

### 2.1 DG categories

In this subsection, we recall basic notions on differential graded (DG) categories. See [Kel04] for more details.

Let $\mathcal{A}$ be a (small) DG category. We define categories $Z^0(\mathcal{A})$ and $H^0(\mathcal{A})$. $Z^0(\mathcal{A})$ has the same objects as $\mathcal{A}$ and its morphism space is defined by

$$Z^0(\mathcal{A})(A, B) := Z^0(\mathcal{A}(A, B)),$$

the 0th cocycle of complex $\mathcal{A}(A, B)$ for $A, B \in \mathcal{A}$. Similarly, $H^0(\mathcal{A})$ has the same objects as $\mathcal{A}$ and its morphism space is defined by

$$H^0(\mathcal{A})(A, B) := H^0(\mathcal{A}(A, B)),$$

the 0th cohomology of complex $\mathcal{A}(A, B)$ for $A, B \in \mathcal{A}$. We call $H^0(\mathcal{A})$ the homotopy category of $\mathcal{A}$. Note that $H^0(\mathcal{A})$ is not necessarily a triangulated category.

We denote by $C_{dg}(k)$ the DG category of complexes of $k$-modules. A (right) DG $A$-module is a DG functor $\mathcal{A}^{\text{op}} \to C_{dg}(k)$. We denote by $C_{dg}(A)$ the DG category of DG $\mathcal{A}$-modules. Each object $X \in \mathcal{A}$ produces a DG $\mathcal{A}$-module

$$X^\wedge := \mathcal{A}(-, X),$$

which is called a representable DG $\mathcal{A}$-module. This gives a fully faithful DG functor $\mathcal{A} \hookrightarrow C_{dg}(\mathcal{A})$, which is called the DG Yoneda embedding. We write $C(\mathcal{A}) := Z^0(C_{dg}(\mathcal{A}))$ and $K(\mathcal{A}) := H^0(C_{dg}(\mathcal{A}))$. $C(\mathcal{A})$ is a Frobenius exact category. The stable category of $C(\mathcal{A})$ coincides with the homotopy category $K(\mathcal{A})$. In particular, $K(\mathcal{A})$ is an algebraic triangulated category. The derived category $D(\mathcal{A})$ of $\mathcal{A}$ is the localization of $K(\mathcal{A})$ with respect to quasi-isomorphisms. Let $F$ be the set of representable DG $\mathcal{A}$-modules. We call

$$\text{per}(\mathcal{A}) := \text{thick}_{D(\mathcal{A})}(F)$$

the full subcategory consisting of perfect DG $\mathcal{A}$-modules.

We can define the shift of a DG $\mathcal{A}$-module, and the cone of a morphism in $C(\mathcal{A})$ in a similar manner to complexes. Note that $\mathcal{A}$ can be viewed as a DG subcategory of $C_{dg}(\mathcal{A})$ by DG Yoneda embedding. $\mathcal{A}$ is said to be pre-triangulated if $\mathcal{A}$ has the zero object and, is closed under taking shifts and cones in $C_{dg}(\mathcal{A})$. In this case, $H^0(\mathcal{A})$ becomes a triangulated category, and a fully faithful functor $H^0(\mathcal{A}) \to K(\mathcal{A})$ induced by the DG Yoneda embedding is a triangulated functor. When a triangulated category $\mathcal{T}$ is triangulated equivalent to $H^0(\mathcal{A})$ for a pre-triangulated DG category $\mathcal{A}$, we say that $\mathcal{A}$ a DG enhancement of $\mathcal{T}$.

A DG category with one object is exactly a DG algebra. Hence the notation above is also used for DG algebras.

We recall some facts which we use in this paper.

**Fact 2.1 ([Kel94] §4.3).** Let $\mathcal{T}$ be an idempotent complete algebraic triangulated category, and $T \in \mathcal{T}$. Then there exists a DG algebra $B$ satisfying the following conditions.

1. There exists a triangulated equivalence $\text{thick}_{\mathcal{T}}(T) \simeq \text{per}(B) \subset D(B)$.
2. $H^*(B) \simeq \oplus_{\ell \in \mathbb{Z}} \text{Hom}_{\mathcal{T}}(T, \Sigma^\ell T)$ as graded algebras.

    Hence the triangulated structure of $\text{thick}_{\mathcal{T}}(T)$ reflects the structure of DG algebra $B$. Under some assumption, DG algebras are determined by its cohomology ring $H^*(B)$.
Definition 2.2. Let $B$ be a DG algebra.

1. $B$ is formal if it is quasi-isomorphic to $H^*(B)$.
2. A graded algebra $A$ is intrinsically formal if any DG algebra $B$ satisfying $H^*(B) \simeq A$ as graded algebras is formal.

2.2 Hochschild cohomology

To describe a sufficient condition of being intrinsically formal, we cite from [RW11] a fact on Hochschild cohomology. See [VW14] §9 for the basics and the detail on Hochschild cohomology.

Let $A$ be a graded algebra. We denote by $A^e := A^q \otimes A$ the enveloping algebra of $A$. Thus $A^e$-modules are nothing but $A$-A-bimodules.

Definition 2.3. The Bar resolution of $A$ is the complex of graded $A^e$-modules

$$B_\bullet : \cdots \rightarrow A^q \rightarrow d_q A \rightarrow \cdots \rightarrow A^0 \rightarrow 0,$$

where $B_q := A^q$ and $d_q$ is given by

$$d_q(a_1 \cdots a_{q+2}) := \sum_{i=1}^{q+1} (-1)^i a_1 \cdots a_i \cdot a_{i+1} \otimes a_{q+2}.$$

Definition 2.4. Let $M$ be a graded $A$-module. The Hochschild cohomology of $A$ valued in $M$ is defined by

$$HH^{p,q}(A, M) := H^p(Hom_{Gr}(B_\bullet, M(q)))$$

for $p, q \in \mathbb{Z}$. To shorten the notation, we write $HH^{p,q}(A)$ instead of $HH^{p,q}(A, A)$.

Remark 2.5. Since the Bar resolution of $A$ is a projective resolution of $A$ as a graded $A^e$-module, we have $HH^{p,q}(A, M) = Ext^p_{Gr}(A, M(q))$. Hence for any other projective resolution $P_\bullet$ of $A$ as a graded $A^e$-module, We have $HH^{p,q}(A, M) = H^p(Hom_{Gr}(P_\bullet, M(q)))$.

The next fact gives a useful criterion for a graded algebra to be intrinsically formal.

Fact 2.6 (RW11 Corollary 1.9). Let $A$ be a graded algebra. If $HH^{q,2-q}(A) = 0$ for $q \geq 3$, then $A$ is intrinsically formal.

3 $m$-periodic complexes

We gather some definitions and basic properties of $m$-periodic complexes. We refer to [Go13] and [Zhao14] for detailed explanation. See also [Br13] for the periodic derived category of a hereditary algebra.

Hereafter let $m$ be an integer greater than 1.

3.1 $m$-periodic derived categories

Let $\mathcal{C}$ be an additive category. We define the DG category $\mathcal{C}_{m,dg}$ of $m$-periodic complexes over $\mathcal{C}$ as follows.

- The objects are families $M = (M^i, d^i_M)_{i \in \mathbb{Z}_m}$, where $M^i \in \mathcal{C}$ and $d^i_M : M^i \rightarrow M^{i+1}$ is a morphism in $\mathcal{C}$ satisfying $d^{i+1}_M d^i_M = 0$ for all $i \in \mathbb{Z}_m$.
- The Hom-complexes are complexes $(\text{Hom}_{\mathcal{C}}(M, N), d)$ for $M, N \in \text{Ob}(\mathcal{C}_{m,dg})$, where

$$\text{Hom}_{\mathcal{C}}(M, N)^p := \bigoplus_{i \in \mathbb{Z}_m} \text{Hom}_{\mathcal{C}}(M^i, N^{i+p}),$$

$$d(f) := d_M f - (-1)^p fd_M (f \in \text{Hom}_{\mathcal{C}}(M, N)^p).$$

We call $\mathcal{C}_{m} := Z^0(\mathcal{C}_{m,dg})$ the category of $m$-periodic complexes over $\mathcal{C}$. An $m$-periodic complex $M$ is called a stalk complex if there exists $i_0 \in \mathbb{Z}_m$ such that $M^{i_0} \neq 0$ and $M^i = 0$ for all $i \neq i_0$. In this case, we also say that $M$ is concentrated in degree $i_0$. Abusing the notation, we denote by $M \in \mathcal{C}_{m,dg}$ a stalk complex whose $0$th component is $0 \neq M \in \mathcal{C}$. 

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Let $M$ be an $m$-periodic complex and $l \in \mathbb{Z}_m$. The $l$-th shift of $M$, denoted by $\Sigma^l M$, is the $m$-periodic complex with

\[ \Sigma^l M := M^{l+1}, \quad d^i_{\Sigma M} := (-1)^i d^{i+1}_M. \]

Let $f \in \text{Hom}_{C_m(\mathcal{C})}(M, N)$. The cone of $f$ is the $m$-periodic complex $\text{Cone}(f) = (C^i_f, d^i_f)_{i \in \mathbb{Z}_m}$ where

\[ C^i_f := N^i \oplus M^{i+1}, \quad d^i_f := \begin{bmatrix} d^i_N & f^{i+1} \\ 0 & d^{i+1}_M \end{bmatrix}. \]

By this construction, $C_m(\mathcal{C})_{dg}$ is a pre-triangulated DG category. Hence $K_m(\mathcal{C}) := H^0 C_m(\mathcal{C})_{dg}$ is a triangulated category. We call it the homotopy category of $m$-periodic complexes over $\mathcal{C}$.

Let $\mathcal{B}$ be an abelian category, and $\mathcal{P}$ be the full subcategory of $\mathcal{B}$ consisting of projective objects. We denote by $D_m(\mathcal{B})$ the localization of $K_m(\mathcal{B})$ with respect to quasi-isomorphisms. We call it the $m$-periodic derived category of $\mathcal{B}$. It is also a triangulated category.

Note that $C_m(\mathcal{B})$ is also an abelian category. Let $\mathcal{C}_1$ be the full subcategory of $C_m(\mathcal{B})$ consisting of stalk complexes, and for $n \in \mathbb{Z}_{>1}$ let $\mathcal{C}_n$ be the full subcategory of $C_m(\mathcal{B})$ consisting of extensions of an object of $C_{n-1}$ by an object of $\mathcal{C}_1$ in the abelian category $C_m(\mathcal{B})$. Then we denote by $\mathcal{B} := \cup_{n \geq 1} \mathcal{C}_n$ the full subcategory of $C_m(\mathcal{B})$ consisting of objects in $\mathcal{C}_n$'s. The objects of $\mathcal{B}$ are iterated extensions of stalk complexes in $C_m(\mathcal{B})$. Similarly, we denote by $\mathcal{P}$ the full subcategory of $C_m(\mathcal{P})$ consisting of complexes which are iterated extensions of stalk complexes in $C_m(\mathcal{P})$.

The next fact is a summary of [Go13 §9].

**Fact 3.1 (Go13 §9).** Let $\mathcal{B}$ be an enough projective abelian category of finite global dimension, and $\mathcal{P}$ be the full subcategory of $\mathcal{B}$ consisting of projective objects. Then we have the following statements.

1. $C_m(\mathcal{B}) = \mathcal{B}$.
2. $C_m(\mathcal{P}) = \mathcal{P}$.
3. For any $P \in C_m(\mathcal{P})$ and $M \in C_m(\mathcal{B})$, we have $\text{Hom}_{K_m(\mathcal{B})}(P, M) = \text{Hom}_{D_m(\mathcal{B})}(P, M)$.
4. For any $M, N \in C_m(\mathcal{B})$, we have $\text{Ext}^1_{C_m(\mathcal{B})}(M, N) = \text{Hom}_{K_m(\mathcal{B})}(M, \Sigma^p N)$ for all $p \geq 1$.
5. For any $M \in C_m(\mathcal{B})$, there exists a surjective quasi-isomorphism $P \rightarrow M$ with $P \in C_m(\mathcal{P})$.

Hence $K_m(\mathcal{P})$ is triangulated equivalent to $D_m(\mathcal{B})$ by the natural inclusion functor.

Let $\Lambda$ be an algebra of finite global dimension. To simplify the notation, we write $D_m(\Lambda)$ instead of $D_m(\text{mod } \Lambda)$. Since $K_m(\text{proj } \Lambda)$ is a full subcategory of $K_m(\text{Proj } \Lambda)$, we can regard $D_m(\Lambda)$ as a full subcategory of $D_m(\text{mod } \Lambda)$.

**Fact 3.2 (Zhao14).** $D_m(\Lambda)$ is the triangulated hull of the orbit category $D^b(\Lambda) / \Sigma^m$ of the bounded derived category $D^b(\Lambda)$ by $\Sigma^m$.

In particular, $D_m(\Lambda)$ is idempotent complete by the construction of the triangulated hull of an orbit category [Ko05 §5].

**Proposition 3.3.** We have triangulated equivalences $\text{thick}_{D_m(\text{mod } \Lambda)}(\Lambda) = K_m(\text{proj } \Lambda) \simeq D_m(\Lambda)$.

**Proof.** Since $K_m(\text{proj } \Lambda)$ is triangulated equivalent to $D_m(\Lambda)$, $K_m(\text{proj } \Lambda)$ is a thick subcategory of $D_m(\text{mod } \Lambda)$. Hence $\text{thick}_{D_m(\text{mod } \Lambda)}(\Lambda) \subset K_m(\text{proj } \Lambda)$ as $\Lambda \in K_m(\text{proj } \Lambda)$. By Fact 3.3[2] we have $K_m(\text{proj } \Lambda) = \text{thick}_{K_m(\text{proj } \Lambda)}(\Lambda)$. Thus we have $K_m(\text{proj } \Lambda) \subset \text{thick}_{D_m(\text{mod } \Lambda)}(\Lambda)$.

3.2 $m$-periodic complexes as DG modules

Let $\Lambda$ be an algebra of finite global dimension. We consider the Laurent polynomial ring $\Lambda[t, t^{-1}]$ over $\Lambda$ as a graded algebra by setting $\text{deg}(t) := m$. Then DG $\Lambda[t, t^{-1}]$-modules are the same as $m$-periodic complexes over $\text{mod } \Lambda$. Indeed, for $(M, d) \in C_{dg}(\Lambda[t, t^{-1}])$, the right multiplication $r_t : M_i \rightarrow M_{i+m}$ by $t$ is a $\Lambda$-isomorphism. Then we obtain an $m$-periodic complex $\tilde{M}$ by setting $\tilde{M}^i := M^i$ and

\[ d^i_{\tilde{M}} := \begin{cases} d^i & i \neq m - 1 \\ r_{t^{-1}} \circ d^{m-1} & i = m - 1. \end{cases} \]

This correspondence induces a DG equivalence between $C_{dg}(\Lambda[t, t^{-1}])$ and $C_m(\text{mod } \Lambda)_{dg}$. Hence we have a triangulated equivalence $D_m(\text{mod } \Lambda) \simeq D(\Lambda[t, t^{-1}]), \Lambda \mapsto \Lambda[t, t^{-1}]$. 

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Lemma 3.4. The triangulated equivalence $D_m(\text{Mod } \Lambda) \sim D(\Lambda[t, t^{-1}])$ induces a triangulated equivalence $D_m(\Lambda) \sim \text{per}(\Lambda[t, t^{-1}])$.

Proof. This follows from the discussion above and Lemma 3.3.

4 A proof of the main theorem

We prepare some lemmas for the proof of the main theorem. The arguments in this section are inspired by [HK19].

Lemma 4.1. Let $\Lambda$ be an algebra which has a finite projective dimension $d$ as a $\Lambda$-bimodule. We regard the Laurent polynomial ring $\Lambda[t, t^{-1}]$ over $\Lambda$ as a graded algebra by setting $\deg(t) := m$. Let $p, q \in \mathbb{Z}$ and consider the Hochschild cohomology $HH^{p,q}(\Lambda[t, t^{-1}])$. If $p \geq d + 2$ or $q \neq 0 \mod m$, then we have

$$HH^{p,q}(\Lambda[t, t^{-1}]) = 0.$$

Proof. Let $A := \Lambda[t, t^{-1}]$. Since $\text{proj.dim}_A \Lambda = d$, we have a projective resolution $F_* : 0 \to F_d \to \cdots \to F_0 \to \Lambda \to 0$ of $A$ of length $d$ as a $\Lambda^e$-module. Let $k[t, t^{-1}]$ be the Laurent polynomial ring over $k$ with $\deg(t) = m$. Then we have $S := k[t, t^{-1}]^e \cong k[x, x^{-1}, y, y^{-1}]$ as graded algebras with $\deg(x) = \deg(y) = m$. As a graded $S$-module, $k[t, t^{-1}]$ is a free resolution of $A$ by using the projective resolution $H_*$, we have

$$HH^{p,q}(A) = \text{Ext}^p_{\text{Gr}}(A, A(q)) = H^p(\text{Hom}_{\text{Gr}}(H_*, A(q))).$$

Since length of $H_*$ is $d + 1$, $HH^{p,q}(A) = 0$ for $p \geq d + 2$. Note that $H_i$ is concentrated in degrees $0$ and $m$ for all $i$, and $\Lambda[t, t^{-1}]$ is concentrated in degrees of multiples of $m$. For $q \neq 0 \mod m$, $\text{Hom}_{\text{Gr}}(H_*, A(q)) = 0$ and hence $HH^{p,q}(A) = 0$.

Lemma 4.2. Under the assumption of Lemma 4.1, $\Lambda[t, t^{-1}]$ is intrinsically formal if $d \leq m$.

Proof. By Lemma 4.1, $HH^{q,2-q}(\Lambda[t, t^{-1}]) = 0$ if $q \geq d + 2$ or $q \neq 0 \mod m$. Since $m + 2 \geq d + 2$ by the assumption, $HH^{q,2-q}(\Lambda[t, t^{-1}]) = 0$ for $q \geq 3$. Hence $\Lambda[t, t^{-1}]$ is intrinsically formal by Fact 2.6.

Theorem 4.3. Let $\mathcal{T}$ be an idempotent complete algebraic $m$-periodic triangulated category having an $m$-periodic tilting object $T$, and $\Lambda$ the endomorphism algebra of $T$ in $\mathcal{T}$. If $\text{proj.dim}_\Lambda \Lambda \leq m$, then there exists a triangulated equivalence

$$\mathcal{T} \simeq D_m(\Lambda).$$

Proof. It follows from Fact 2.1, Lemma 4.2 and Lemma 3.3.

5 Some remarks and applications

5.1 A remark on Theorem 4.3

In this subsection we give a restatement of Theorem 4.3 in a convenient form. We first recall homologically smoothness of algebras. We refer to [RR20, Section 3] for a detailed explanation.

Definition 5.1. Let $\Lambda$ be an algebra.

1. $\Lambda$ is homologically smooth of dimension $d$ if $\Lambda$ has finite projective dimension equal to $d$ as an $\Lambda^e$-module.

2. $\Lambda$ is separable if $\Lambda$ is homologically smooth of dimension $0$. 7
For finite dimensional algebras, we have a useful characterization of homologically smoothness.

**Fact 5.2** ([RR20, Corollary 3.19, 3.22]). A finite dimensional algebra $\Lambda$ is homologically smooth of dimension $d$ if and only if $\Lambda / \text{rad} \Lambda$ is separable and $\text{gl.dim} \Lambda = d$.

Since semisimple algebras over a perfect field are separable (c.f. [RR 20, Lemma 3.3]), we have next corollary.

**Corollary 5.3.** A finite dimensional algebra $\Lambda$ over a perfect field $k$ is homologically smooth of dimension $d$ if and only if $\text{gl.dim} \Lambda = d$.

We restate Theorem 4.3 in a convenient form.

**Corollary 5.4.** Let $\mathcal{T}$ be an idempotent complete algebraic $m$-periodic triangulated category over a perfect field $k$. If $\mathcal{T}$ has an $m$-periodic tilting object $T$ whose endomorphism algebra $\Lambda := \text{End}_\mathcal{T}(T)$ is a finite dimensional algebra of global dimension $d \leq m$, then there exists a triangulated equivalence $\mathcal{T} \simeq D_m(\Lambda)$.

5.2 Examples

We give examples of $m$-periodic tilting objects and an application of Corollary 5.4.

The first example is a trivial one but indicates the possibility that we can drop the assumption $\text{proj.dim}_\Lambda \Lambda \leq m$ in Theorem 4.3.

**Example 5.5.** Let $\Lambda$ be an algebra of finite global dimension. Then $\Lambda$ itself is an $m$-periodic tilting object of $D_m(\Lambda)$ for an arbitrary positive integer $m$ by Fact 3.1.

Next we construct an equivalence between the stable category of a self-injective algebra and the periodic derived category of a hereditary algebra by Corollary 5.4.

**Example 5.6.** Let $k$ be a perfect field, and $Q = (Q_0, Q_1)$ be the cyclic quiver in Figure 5.1.

![Figure 5.1: The cyclic quiver Q.](image)

We identify the numbers assigned to vertices of $Q$ with the elements of $\mathbb{Z}_n$ as in Figure 5.1. Then the Auslander-Reiten (AR) quiver of $\Lambda := kQ / (\text{rad} kQ)^n$ is Figure 5.2 where $M(a, l)$ is the indecomposable $\Lambda$-module of length $l$ whose top is the simple $\Lambda$-module associated to the vertex $a$.

![Figure 5.2: The AR quiver of $\Lambda$ for $n = 3$.](image)
The indecomposable projective modules are \( M(a, n) (a \in \mathbb{Q}_0) \), and they are also injective modules. Thus \( \Lambda \) is a self-injective algebra and the stable category \( \text{mod} \Lambda \) is a triangulated category. Since the sequence \( 0 \to M(a+l, n-l) \to M(a+l, n) \to M(a, l) \to 0 \) is exact, we have \( \Sigma M(a, l) = M(a+l, n-l) \). Thus \( \text{mod} \Lambda \) is 2-periodic. Then for a vertex \( a \in \mathbb{Q}_0 \), the object \( T(a) := \bigoplus_{l=1}^{n-1} M(a, l) \) is a 2-periodic tilting object of \( \text{mod} \Lambda \). Indeed, \( T(a) \) is a rigid thick generator of \( \text{mod} \Lambda \) by Figure 5.2.

Figure 5.2 also says that the endomorphism algebra of \( T(a) \) is isomorphic to a hereditary algebra \( kA_{n-1} \), where \( A_{n-1} \) is the quiver \( 1 \leftarrow 2 \leftarrow \cdots \leftarrow n-1 \). By Corollary 5.4 there exists a triangulated equivalence \( \text{mod} \Lambda \cong D_2(\Lambda) \).

Example 5.7. Let \( S := \mathbb{C}[[x, y]] \) be the formal power ring of two variables over the complex number field \( \mathbb{C} \). Consider the one dimensional simple singularity \( R_n := S/(x^2 - y^{n+1}) \) of type \( A_{n-1} \). Then the stable category \( \text{CM}(R_n) \) of maximal Cohen-Macaulay modules over \( R_n \) is 2-periodic, and the AR translation \( \tau \) coincides with the syzygy functor \( \Omega \). (In general, stable categories of maximal Cohen-Macaulay modules over hypersurface singularities are 2-periodic triangulated categories [Yo90, Chapter 9].)

When \( n = 2\ell \) is even, the AR quiver of \( \text{CM}(R_{2\ell}) \) is given as in Figure 5.3. In this case there exists no rigid maximal Cohen-Macaulay modules, and hence there exists no 2-periodic tilting objects.

When \( n = 2\ell - 1 \) is odd, the AR quiver of \( \text{CM}(R_{2\ell-1}) \) is given as in Figure 5.4. In this case the maximal Cohen-Macaulay \( R_{2\ell-1} \)-modules \( N_{\pm} := \mathbb{C}[[x, y]]/(x \pm y^\ell) \) are 2-periodic tilting objects of \( \text{CM}(R_{2\ell-1}) \), but the endomorphism algebras of \( N_{\pm} \) are both isomorphic to \( k[x]/(x^\ell) \), which is of infinite global dimension [BIKR08, Proposition 2.4]. Thus we cannot apply Theorem 4.3.

We will continue the study of Example 5.7 in §5.3 Example 5.11.

5.3 Remarks on Lemma 4.2

In this subsection, we give an example of an algebra of infinite global dimension for which Lemma 4.2 fails. Using Example 5.7, we will construct a DG algebra whose cohomology ring is the Laurent polynomial ring over \( k[x]/(x^\ell) \), but which is not formal. First we introduce the DG category of matrix factorizations, following [BRTV18, §2.2]. Let \( S \) be a commutative ring, and \( 0 \neq f \in S \). The DG category \( \text{MF}_S(f)_{dg} \) of matrix factorizations is defined as follows.

- The objects are pairs \((P, \delta)\) consisting of
  - a \( \mathbb{Z}_2 \)-graded \( S \)-module \( P = P_0 \oplus P_1 \) with \( P_0, P_1 \in \text{proj} S \), and
  - a graded \( S \)-linear map \( \delta : P \to P \) of degree 1 satisfying \( \delta^2 = f \cdot \text{id}_P \).
• The morphism complex \( MF_S(f)_{dg}((P, \delta), (Q, \epsilon)) \) is defined as follows.
  - \( MF_S(f)_{dg}((P, \delta), (Q, \epsilon))^n \) is the set of graded \( S \)-linear maps of degree \( n \) from \( P \) to \( Q \).
  - The differential is defined by \( d(t) := e - (-1)^n t \delta \) for all \( t \in MF_S(f)_{dg}((P, \delta), (Q, \epsilon))^n \).

Then \( MF_S(f)_{dg}((P, \delta), (Q, \epsilon)) \) is a complex. Indeed, we have
\[
d^2(t) = d(et - (-1)^n t \delta) = e(e - (-1)^n t \delta) - (-1)^{n+1} e et - (-1)^{n+1} t \epsilon t \delta = e(t - (-1)^n t \delta) - (-1)^{n+1} e t \delta - f \cdot t = 0.
\]

We regard an object \((P, \delta) \in MF_S(f)_{dg} \) as the 4-tuple \((P_0, P_1, \delta_0 : P_0 \to P_1, \delta_1 : P_1 \to P_0)\), and denote it by \((\delta_0, \delta_1)\) to simplify the notation. Let \( MF_S(f) := \mathcal{Z}^0(MF_S(f)_{dg}) \), and \( MF_S(f) \) be the quotient category of \( MF_S(f) \) by the ideal generated by the morphisms factoring through the trivial matrix factorizations \((f, 1) = (S \oplus S, [f^1]) \) and \((1, f) = (S \oplus S, [1 f^1]) \).

The following fact is fundamental for Cohen-Macaulay representations.

**Fact 5.8** (Yo90 Chapter 7). Let \( S \) be a complete regular local commutative ring with the maximal ideal \( m \) and \( 0 \neq f \in m \). Then there exists an equivalence
\[
MF_S(f) \sim \mathcal{CM}(S/(f)), \quad (\delta_0, \delta_1) \mapsto \text{Coker} \delta_0,
\]
where \( \mathcal{CM}(S/(f)) \) is the stable category of maximal Cohen-Macaulay modules over the Cohen-Macaulay ring \( S/(f) \).

The following proposition seems to be well-known, but since we cannot find a proof in literatures, let us write down it.

**Proposition 5.9.** There exists an equivalence \( H^0(MF_S(f)_{dg}) \simeq MF_S(f) \).

**Proof.** First we show that the natural functor \( MF_S(f) \to H^0(MF_S(f)_{dg}) \) induces a functor \( MF_S(f) \to H^0(MF_S(f)_{dg}) \). It is enough to prove that the trivial matrix factorizations \((1, f)\) and \((f, 1)\) are zero objects in \( H^0(MF_S(f)_{dg}) \). The diagram
\[
\begin{array}{ccc}
S & \xrightarrow{f} & S \\
\downarrow & & \downarrow \\
S & \xrightarrow{0} & S
\end{array}
\]
shows \( \text{id}_{(1, f)} \in B^0(MF_S(f)_{dg})((1, f), (1, f)) \). Hence \((1, f)\) is a zero object in \( H^0(MF_S(f)_{dg}) \). Similarly we can show that \((f, 1)\) is a zero object in \( H^0(MF_S(f)_{dg}) \).

Next we show that the natural functor \( MF_S(f) \to MF_S(f) \) induces a functor \( H^0(MF_S(f)_{dg}) \to MF_S(f) \). Let \( \phi \in B^0(MF_S(f)_{dg})((P, \delta), (Q, \epsilon)) \). It is sufficient to prove that \( \phi \) factors through a direct sum of \((1, f)\) and \((f, 1)\). Let \( \alpha : P \to Q \) be a graded \( S \)-linear map of degree 1 satisfying \( e \alpha + \alpha \delta = \phi \). Then the diagram
\[
\begin{array}{ccc}
P_1 & \xrightarrow{\delta_1} & P_0 \\
\downarrow & & \downarrow \\
Q_0 \oplus Q_1 & \xrightarrow{\epsilon_1} & Q_0 \oplus Q_1
\end{array}
\]
is commutative, where \( A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \) and \( B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \). Since \( AB = f \cdot \text{id} \) and \( BA = f \cdot \text{id} \), we have \((A, B) \in MF_S(f) \). \( A \) and \( B \) can be transformed into \([1 f^1] \) and \([f 1] \) respectively by elementary operations of matrix. Hence \( \phi \) factors through a direct sum of \((1, f)\) and \((f, 1)\).

It is easy to see that these functors are quasi-inverse of each others.

**Corollary 5.10.** Let \( S \) be a complete regular local commutative ring with the maximal ideal \( m \) and \( 0 \neq f \in m \). Then \( MF_S(f)_{dg} \) is a DG enhancement of \( \mathcal{CM}(S/(f)) \).
Example 5.11. We continue Example 5.7. In the odd case $n = 2\ell - 1, N_+ \in \mathbf{CM}(R)$ corresponds to the matrix factorization $(x + y^\ell, x - y^\ell) = (S^{\otimes 2}, \left[ \begin{array}{c} x + y^\ell \\ x - y^\ell \end{array} \right])$. Consider the DG endomorphism algebra $A := MF_S(f)_{dg}((x + y^\ell, x - y^\ell), (x + y^\ell, x - y^\ell))$. $A$ is 2-periodic as a complex, and we denote it by
\[ (d^0 : A^0 \to A^1, d^1 : A^1 \to A^0). \]

By the definition of morphism complexes, we can regard $A^i$ as an $S$-submodule of the matrix ring $M_2(S)$ over $S$ for $i \in \mathbb{Z}_2$. In this situation,
\[ A^0 = e_{11}S + e_{22}S, \quad A^1 = e_{21}S + e_{12}S, \]
where $e_{ij}$ (1 \leq i,j \leq 2) is the standard free $S$-basis for $M_2(S)$. Then
\[
\begin{bmatrix}
0 & -2x \\
0 & 2y^m
\end{bmatrix}, \quad
\begin{bmatrix}
2y^m & 2x \\
0 & 0
\end{bmatrix}
\]
are the representation matrices of $d^0$ and $d^1$ with respect to bases $e_{11} + e_{22}$, $e_{11} - e_{22}$ and $e_{21} + e_{12}, e_{21} - e_{12}$, respectively. Hence we have $Z^0(A) = (e_{11} + e_{22})S \cong S$. On the other hand, the cohomology ring $H^0(A)$ of $A$ is isomorphic to the Laurent polynomial ring $(k[y]/(y^\ell))[t, t^{-1}]$ over $k[y]/(y^\ell)$ with deg$(t) = 2$.

Since $Z^0(A) \cong S$ is an integral domain and $(k[y]/(y^\ell))[t, t^{-1}] = k[y]/(y^\ell)$ is not reduced, there is no quasi-isomorphism $(k[y]/(y^\ell))[t, t^{-1}] \to A$ of DG algebras. Because $(k[y]/(y^\ell))[t, t^{-1}]$ is a complex concentrated in even degrees, it is $K$-projective as a complex. Thus $(k[y]/(y^\ell))[t, t^{-1}]$ is not quasi-isomorphic to $A$. Hence $A$ is not formal and $(k[y]/(y^\ell))[t, t^{-1}]$ is not intrinsically formal.

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