PERMUTATION MODULE DECOMPOSITION OF THE SECOND COHOMOLOGY OF A REGULAR SEMISIMPLE HESSENBerg VARIETY

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Abstract. Regular semisimple Hessenberg varieties admit actions of associated Weyl groups on their cohomology spaces of each degree. In this paper, we consider the module structure of the cohomology spaces of regular semisimple Hessenberg varieties of type $A$. We define a subset of the Bialynicki-Birula basis of the cohomology space which becomes a module generator set of the cohomology module of each degree. We use these generators to construct permutation submodules of the degree two cohomology module to form a permutation module decomposition. Our construction is consistent with a known combinatorial result by Chow on chromatic quasisymmetric functions.

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1. Introduction

Since De Mary, Proceci, and Shayman introduced Hessenberg varieties in the 1990s ([9] and [8]), many researchers in various fields have increasingly focused on them. Hessenberg varieties form a family of subvarieties of the full flag varieties and many interesting varieties appear as Hessenberg varieties. For example, full flag varieties and permutohedral varieties

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2020 Mathematics Subject Classification. Primary 14M15; Secondary 05E14, 14L30.

Key words and phrases. Hessenberg varieties, representations of symmetric groups, permutation module decompositions.

Cho was supported by NRF-2020R1A2C1A01011045. Hong was supported by the Institute for Basic Science (IBS-R032-D1). Lee was supported by the Institute for Basic Science (IBS-R003-D1).
are Hessenberg varieties. Flag varieties are central objects in the intersection of algebraic geometry and algebraic combinatorics. Hessenberg varieties share similar features with flag varieties.

A **regular semisimple Hessenberg variety** \( \text{Hess}(S, h) \) is a subvariety of the full flag variety \( F(\mathbb{C}^n) \) which is determined by two data: a weakly increasing function \( h: [n] \to [n] \), called a **Hessenberg function**, and a regular semisimple linear operator \( S \) (see Definition 2.1 for a precise definition). Here, we use \([n]\) to denote the set \( \{1, \ldots, n\} \). Tymoczko defined an action of the symmetric group \( S_n \) on the cohomology space \( H^2k(\text{Hess}(S, h); \mathbb{C}) \) in [16, 17], which is called the dot action. On the other hand, a Hessenberg function \( h \) determines a graph \( G_h \), called the incomparability graph, of the corresponding unit interval order.

Shareshian and Wachs [13] refined a long standing conjecture proposed by Stanley and Stembridge [15, 14] on chromatic symmetric functions as a conjecture on chromatic quasisymmetric functions.

**Conjecture 1.1** ([15, 14], [13]). Let \( h: [n] \to [n] \) be a Hessenberg function and let \( \text{Hess}(S, h) \) be the regular semisimple Hessenberg variety associated with \( h \) and a regular semisimple linear operator \( S \). Let \( X_{G_h}(x, t) \) be the chromatic quasisymmetric function of the graph \( G_h \) associated with \( h \). Then, for \( 0 \leq k \leq N_h := \sum_{i=1}^{n} (h(i) - i) \), the coefficient of \( t^k \) of \( X_{G_h}(x, t) \) is positively expanded as a sum of elementary symmetric functions.

Furthermore, they proposed another conjecture in the same paper [13] that Conjecture 1.1 is equivalent to a conjecture regarding the \( S_n \)-module structure of \( H^* \text{(Hess}(S, h)) \). The latter conjecture has been proved by Brosnan and Chow [4], and by Guay-Paquet [12].

**Theorem 1.2** (Conjectured in [13]; proved in [4, 12]). Let \( h: [n] \to [n] \) be a Hessenberg function and let \( \text{Hess}(S, h) \) be the regular semisimple Hessenberg variety associated with \( h \) and a regular semisimple linear operator \( S \). Let \( X_{G_h}(x, t) \) be the chromatic quasisymmetric function of the graph \( G_h \) associated with \( h \). Then, we have

\[
\sum_{k=0}^{N_h} \text{ch}H^{2k}(\text{Hess}(S, h)) t^k = \omega X_{G_h}(x, t),
\]

where \( \text{ch} \) is the Frobenius characteristic map and \( \omega \) is the involution on the symmetric function algebra sending elementary symmetric functions to complete homogeneous symmetric functions.

To a partition \( \lambda = (\lambda_1, \ldots, \lambda_\ell) \) of \( n \) we associate a module \( M^\lambda \) of \( S_n \), called a permutation module of type \( \lambda \), which is defined by the vector space of formal linear sums of the ordered tuples \( (J_1, \ldots, J_\ell) \) of disjoint subsets of \( \{1, 2, \ldots, n\} \) satisfying \( |J_s| = \lambda_s, \ s = 1, \ldots, \ell \), and

1In what follows, we consider cohomology rings with coefficients in \( \mathbb{C} \) and we will not indicate the coefficient ring that we are working on.
PERMUTATION MODULE DECOMPOSITION OF $H^2(\text{Hess}(S, h))$

$|\bigcup_s J_s| = n$, where the permutations in $\mathfrak{S}_n$ act on $(J_1, \ldots, J_t)$ naturally. Under the Frobenius characteristic map, $M^\lambda$ corresponds to the complete homogeneous symmetric function $h_\lambda$. In Theorem 1.2 and throughout the paper, by a permutation module we mean a permutation module $M^\lambda$ of type $\lambda$ for some partition $\lambda$ of $n$.

Theorem 1.2 inspired efforts to understand combinatorial works on chromatic quasisymmetric functions in the language of Hessenberg varieties, and to prove Conjecture 1.1 using geometric methods. One of them is to construct nice bases $\{\sigma_{w,h}\}_{w \in S_n}$ of $H^2(\text{Hess}(S, h))$ using Bialynicki-Birula cell decompositions of the Hessenberg varieties, called Bialynicki-Birula bases (BB bases) (see Definition 2.4), and to investigate their properties; see [5].

Our work in this paper provides a general method of using the BB bases to construct a permutation module decomposition of $H^2(\text{Hess}(S, h))$. More specifically, we define a subset $G^k_h$ of $S_n$ for a Hessenberg function $h: [n] \to [n]$:

$$G^k_h := \{ w \in S_n \mid \ell_h(w) = k, \ w^{-1}(w(j) + 1) \leq h(j) \text{ for } w(j) \in [n - 1] \}.$$

We then prove that the corresponding BB basis elements form a module generator set of the cohomology of degree $2k$. We use related results in both combinatorics of chromatic quasisymmetric functions and geometry of Hessenberg varieties.

Theorem A (Theorem 3.11). The set $\{\sigma_{w,h} \mid w \in G^k_h\}$ has the cardinality $\dim \mathbb{C} H^{2k}(\text{Hess}(S, h))^{S_n}$ and it generates the $S_n$-module $H^{2k}(\text{Hess}(S, h))$, that is,

$$H^{2k}(\text{Hess}(S, h)) = \sum_{w \in G^k_h} M(\sigma_{w,h}),$$

where $M(\sigma_{w,h})$ denotes the $S_n$-module generated by $\sigma_{w,h}$.

We remark that the sum in Theorem A is not a direct sum, the question is whether we can reduce modules $M(\sigma_{w,h})$ to get a direct sum decomposition of permutation modules as stated in the following conjecture.

Conjecture 1.3. For each $w \in G^k_h$, there is $\tilde{\sigma}_{w,h} \in M(\sigma_{w,h})$ such that the $S_n$-module $M(\tilde{\sigma}_{w,h})$ generated by $\tilde{\sigma}_{w,h}$ is a permutation module and

$$H^{2k}(\text{Hess}(S, h)) = \bigoplus_{w \in G^k_h} M(\tilde{\sigma}_{w,h}).$$

Here, for a precise description of a candidate of $\tilde{\sigma}_{w,h}$, see Definition 7.2.

\footnote{In the literature, a permutation module or a permutation representation of the symmetric group $\mathfrak{S}_n$ is a representation of $\mathfrak{S}_n$ with a basis permuted by the action of $\mathfrak{S}_n$; or more restrictively, a representation of $\mathfrak{S}_n$, in which every point stabilizer is parabolic, in addition. The condition that we impose in this paper is the most restrictive one.}
In [5], Conjecture 1.3 was proved for permutohedral varieties, which, in turn, provided a geometric proof of Conjecture 1.1. In this paper, we use generators $\sigma_{w,h}$, $w \in \mathcal{G}_h^1$, to construct mutually disjoint permutation submodules of $H^2(\text{Hess}(S, h))$ that constitute the entire module $H^2(\text{Hess}(S, h))$; that is, we prove Conjecture 1.3 when $k = 1$.

**Theorem B (Theorem 6.4).** For each $w \in \mathcal{G}_h^1$, there is $\sigma_{w,h} \in M(\sigma_{w,h})$ such that the $S_n$-module $M(\sigma_{w,h})$ generated by $\sigma_{w,h}$ is a permutation module and

$$H^2(\text{Hess}(S, h)) = \bigoplus_{w \in \mathcal{G}_h^1} M(\sigma_{w,h}).$$

This provides a geometric proof of a known result of Conjecture 1.1 on chromatic quasisymmetric functions when $k = 1$, done by Chow in [7]; see Theorem 6.1.

Our paper is structured as follows. In Section 2, we recall the basics on regular semisimple Hessenberg varieties and symmetric group action on their cohomology spaces; especially focusing on the results in [5]. We define a special set $\{\sigma_{w,h} \mid w \in \mathcal{G}_h^k\}$ of module generators of $H^{2k}(\text{Hess}(S, h))$ in Section 3. After restricting ourselves to $k = 1$, we classify the elements in $\mathcal{G}_h^1$ and compute the stabilizer subgroup of the generators $\sigma_{w,h}$ for $w \in \mathcal{G}_h^1$ in Section 4. In Section 5, we consider a partition of $\mathcal{G}_h$ defined by a certain equivalence relation such that there is a bijective correspondence between the set $\mathcal{G}_h^1$ and the equivalence classes. Furthermore, we enumerate the cardinality of each class and the dimension of $H^2(\text{Hess}(S, h))$. Section 6 is devoted to construct a permutation module decomposition of $H^2(\text{Hess}(S, h))$. In the final section, we suggest a more refined conjecture.

2. Preliminaries

2.1. Hessenberg varieties. For a positive integer $n$, we use $S_n$ to denote the symmetric group on $[n]$. Hessenberg varieties are certain subvarieties of the full flag variety $F\ell(\mathbb{C}^n)$:

$$F\ell(\mathbb{C}^n) := \{V_\bullet = (\emptyset \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{n-1} \subseteq \mathbb{C}^n) \mid \dim_{\mathbb{C}} V_i = i \text{ for all } 1 \leq i \leq n\}.$$  

Hessenberg varieties in $F\ell(\mathbb{C}^n)$ are parametrized by $n \times n$ matrices, and Hessenberg functions $h: [n] \to [n]$ satisfying the following two properties

- $h(1) \leq h(2) \leq \cdots \leq h(n)$, and
- $i \leq h(i)$ for all $i = 1, \ldots, n$.

A Hessenberg function is frequently described by writing its values in a parenthesis; $h = (h(1), \ldots, h(n))$.

**Definition 2.1.** Let $h$ be a Hessenberg function and let $S$ be a regular semisimple linear operator with $n$ distinct eigenvalues. The regular semisimple Hessenberg variety $\text{Hess}(S, h)$ is a subvariety of the flag variety defined as

$$\text{Hess}(S, h) = \{V_\bullet \in F\ell(\mathbb{C}^n) \mid SV_i \subset V_{h(i)} \text{ for all } 1 \leq i \leq n\}.$$
We note that if \( h(i) \geq i + 1 \) for all \( 1 \leq i \leq n - 1 \), then Hess\((S, h)\) is irreducible. Because of this, we assume \( h(i) \geq i + 1 \) for all \( 1 \leq i \leq n - 1 \) throughout this paper. The notion of the length \( \ell(w) \) of a permutation \( w \in S_n \), which counts the number of inversions of \( w \), is extended to a quantity respecting a given Hessenberg function \( h \);

\[
\ell_h(w) := |\{(j, i) \mid 1 \leq j < i \leq h(j), w(j) > w(i)\}|.
\]

Note that \( \ell(w) = \ell_h(w) \) for \( h = (n, n, \ldots, n) \).

Some fundamental properties of the Hessenberg varieties are stated in the following proposition. See Section 2 of [5] and the references therein for more details.

**Proposition 2.2.** Let \( S \) be a regular semisimple linear operator and let \( h: [n] \to [n] \) be a Hessenberg function satisfying \( h(i) \geq i + 1 \) \((1 \leq i \leq n - 1)\).

1. Hess\((S, h)\) is a smooth variety of \( \mathbb{C} \)-dimension \( \sum_{i=1}^{n} (h(i) - i) \).
2. The complex torus \( T := (\mathbb{C}^*)^n \) acts on Hess\((S, h)\) by left multiplication, and the fixed points of this action can be identified with the permutations in \( S_n \).
3. (Białynicki-Birula decomposition) Hess\((S, h)\) is decomposed into their plus cells \( X_{w,h}^\circ \) and also into their minus cells \( \Omega_{w,h}^\circ \);

\[
\text{Hess}(S, h) = \bigsqcup_{w \in S_n} X_{w,h}^\circ = \bigsqcup_{w \in S_n} \Omega_{w,h}^\circ .
\]

The plus cell \( X_{w,h}^\circ \) (the minus cell \( \Omega_{w,h}^\circ \), respectively) is the intersection of Hess\((S, h)\) with the Schubert cell \( X_{w}^\circ \) (the opposite Schubert cell \( \Omega_{w} \), respectively) of \( Fl(\mathbb{C}^n) \).

4. The dimension of plus cells and minus cells are given by

\[
\dim_{\mathbb{C}} X_{w,h}^\circ = \ell_h(w) \quad \text{and} \quad \dim_{\mathbb{C}} \Omega_{w,h}^\circ = \dim_{\mathbb{C}}(\text{Hess}(S, h)) - \ell_h(w),
\]

and therefore the Poincaré polynomial \( \text{Poin}(\text{Hess}(S, h), q) \) is given by

\[
\text{Poin}(\text{Hess}(S, h), q) := \sum_{k \geq 0} \dim_{\mathbb{C}} H^k(\text{Hess}(S, h)) q^k = \sum_{w \in S_n} q^{2\ell_h(w)}.
\]

5. For any regular semisimple linear operator \( S' \), \( H^*(\text{Hess}(S, h)) \cong H^*(\text{Hess}(S', h)) \) as graded \( \mathbb{C} \)-algebras.

A regular semisimple Hessenberg variety Hess\((S, h)\) is a GKM manifold [16, 17], and the equivariant cohomology ring of Hess\((S, h)\) can be described in terms of its GKM graph \((V, E, \alpha)\) as a subring of the direct sum of polynomial rings \( \bigoplus_{v \in V} \mathbb{C}[t_1, \ldots, t_n] \). We use \( s_{j,i} \) to denote the transposition in \( S_n \) that exchanges \( j \) and \( i \) for \( 1 \leq j < i \leq n \).

**Theorem 2.3 ([11]).** Let Hess\((S, h)\) be a regular semisimple Hessenberg variety. The GKM graph \((V, E, \alpha)\) of Hess\((S, h)\) comprises the set \( E = S_n \) of vertices, the set of (directed)
edges $E = \{(v \rightarrow w) \mid w = vs_{j,i}, \text{ for } j < i \leq h(j)\}$, and the edge labeling $\alpha$ such that $\alpha(v \rightarrow vs_{j,i}) = t_{v(i)} - t_{v(j)} \in \mathbb{C}[t_1, \ldots, t_n]$. Moreover,

$$H^*_T(\text{Hess}(S,h)) \cong \left\{ (p(v)) \in \bigoplus_{v \in \mathcal{S}_n} \mathbb{C}[t_1, \ldots, t_n] \mid \alpha(v \rightarrow w) | (p(v) - p(w)) \text{ for all } (v \rightarrow w) \in E \right\}.$$

We note that for the set $E$ of edges in the statement, we have $(v \rightarrow w) \in E$ if and only if $(w \rightarrow v) \in E$. In particular, $\alpha(v \rightarrow w) = -\alpha(w \rightarrow v)$.

The main concern of this article is on the structure of the cohomology $H^*(\text{Hess}(S,h))$ which can be described from Theorem 2.3 using the following ring isomorphism;

$$H^*(\text{Hess}(S,h)) \cong H^*_T(\text{Hess}(S,h))/(t_1, \ldots, t_n),$$

where we use $t_i$ to indicate the element in $H^*_T(\text{Hess}(S,h))$ whose value at each $v \in \mathcal{S}_n$ is $t_i$.

2.2. Białyńnicki-Birula basis and symmetric group action. The closure $\Omega_{w,h} := \overline{\Omega^c_{w,h}}$ of a minus cell $\Omega^c_{w,h}$ defines a class $[\Omega_{w,h}]$ in the equivariant Chow ring $A^*_T(\text{Hess}(S,h))$ graded by the codimension, and the cycle map from the Chow ring to the cohomology ring is an isomorphism

$$c^T_{\text{Hess}(S,h)}: A^*_T(\text{Hess}(S,h)) \xrightarrow{\cong} H^*_T(\text{Hess}(S,h)).$$

**Definition 2.4** ([5 Definition 2.9]). Let $\text{Hess}(S,h)$ be a regular semisimple Hessenberg variety. For $w \in \mathcal{S}_n$, the Białyńcki-Birula class (BB class) $\sigma^T_{w,h} \in H^*_{T}(\text{Hess}(S,h))$ is the image of the class $[\Omega_{w,h}] \in A^*_{T}(\text{Hess}(S,h))$ under the cycle map $c^T_{\text{Hess}(S,h)}$. We let $\sigma_{w,h} \in H^2_{T}(\text{Hess}(S,h))$ be the corresponding class of $\sigma^T_{w,h} \in H^*_{T}(\text{Hess}(S,h))$.

Because of Proposition 2.2([3], the BB classes $\sigma^T_{w,h}$ ($\sigma_{w,h}$, respectively) form a basis of the equivariant cohomology space (ordinary cohomology space, respectively) of $\text{Hess}(S,h)$.

**Proposition 2.5.** Let $\text{Hess}(S,h)$ be a regular semisimple Hessenberg variety. Then the classes $\sigma^T_{w,h}$, $w \in \mathcal{S}_n$, form a basis called the Białyńcki-Birula basis (BB basis) of $H^*_{T}(\text{Hess}(S,h))$, and the classes $\sigma_{w,h}$, $w \in \mathcal{S}_n$, form a basis called the Białyńcki-Birula basis (BB basis) of $H^*(\text{Hess}(S,h))$.

In [16, 17], Tymoczko defined an action called the dot action of the symmetric group $\mathcal{S}_n$ on the equivariant cohomology $H^*_{T}(\text{Hess}(S,h))$ as follows. For $u \in \mathcal{S}_n$ and $\sigma = (\sigma(v))_{v \in \mathcal{S}_n} \in H^*_{T}(\text{Hess}(S,h)) \subset \bigoplus_{v \in \mathcal{S}_n} \mathbb{C}[t_1, \ldots, t_n]$,

$$(u \cdot \sigma)(v) := (\sigma(u^{-1}v))(t_{u(1)}, \ldots, t_{u(n)}).$$

Since the ideal $(t_1, \ldots, t_n) \subset H^*_{T}(\text{Hess}(S,h))$ is invariant under the dot action on $H^*_{T}(\text{Hess}(S,h))$, it induces an action on $H^*(\text{Hess}(S,h))$ which we call the dot action as well.
We review the description of \( s_i \cdot \sigma_{w,h} \) provided in [5, Section 4]. For two permutations \( v, w \in \mathfrak{S}_n \) such that \( w = vs_{i,j} \) and \( \ell(v) > \ell(w) \), we use \( v \rightarrow w \) to mean that \( (v \rightarrow w) \) is an edge of the GKM graph of Hess\((S, h)\); and \( v \rightarrow w \) to mean that \( (v \rightarrow w) \) is not an edge of the GKM graph of Hess\((S, h)\).

Let \( w \) be a permutation in \( \mathfrak{S}_n \) and let \( s_i = s_{i,i+1} \) be a simple reflection such that \( w \rightarrow s_iw \). The Bialynicki-Birula decomposition of \( \Omega_{s_iw, h} \) is given by

\[
\Omega_{s_iw, h} = \bigcup_{u \in \Omega_{s_iw, h}} (\Omega_u^o \cap \Omega_{s_iw, h}).
\]

Here, for a given \( T \)-invariant variety \( X \), we denote by \( X^T \) the set of \( T \)-fixed points.

Define \( A_{s_i, w} \) by the set of all \( u \in \Omega_{s_iw, h}^T \cap \Omega_{w, h}^T \) such that \( \dim_{\mathbb{C}}(\Omega_u^o \cap \Omega_{s_iw, h}) = \dim_{\mathbb{C}} \Omega_{w, h} \) and \( u \rightarrow s_iu \). For \( u \in A_{s_i, w} \), define \( \tau_u \) and \( \sigma_{s_iu} \) by the closures of \( \Omega_u^o \cap \Omega_{s_iw, h} \) and \( \Omega_{s_iu} \cap \Omega_{s_iw, h} \), and let \( \tau_u \) and \( \tau_{s_iu} \) denote the classes in \( H^*(\text{Hess}(S, h)) \) induced by \( \tau_u \) and \( \tau_{s_iu} \), respectively.

We recall the following proposition.

**Proposition 2.6** ([5, Theorem B]). Let \( w \) be an element in \( \mathfrak{S}_n \) and let \( s_i = s_{i,i+1} \) be a simple reflection.

1. If \( w \rightarrow s_iw \) or \( s_iw \rightarrow w \), then \( s_i \cdot \sigma_{w, h} = \sigma_{s_iw, h} \).
2. If \( s_iw \rightarrow w \), then \( s_i \cdot \sigma_{w, h} = \sigma_{w, h} \).
3. If \( w \rightarrow s_iw \), then

\[
(s_i \cdot \sigma_{w, h} + \sum_{u \in A_{s_i, w}} \tau_{s_iu}) = \sigma_{w, h} + \sum_{u \in A_{s_i, w}} \tau_u.
\]

and the intersection \( A_{s_i, w} \cap s_iA_{s_i, w} \) is empty.

3. **Module generators**

In view of Theorem 1.2, the number of permutation modules whose direct sum is \( H^{2k}(\text{Hess}(S, h)) \) is expected to be the same as \( m := \dim_{\mathbb{C}} \text{Hess}(S, h)^{\mathfrak{S}_n} \). In this section, we will show that there are \( m \) classes \( \sigma_{w, h} \in H^{2k}(\text{Hess}(S, h)) \), generating \( H^{2k}(\text{Hess}(S, h)) \) as an \( \mathfrak{S}_n \)-module.

**Definition 3.1.** For a Hessenberg function \( h : [n] \rightarrow [n] \), let

\[
\mathcal{G}_h := \{w \in \mathfrak{S}_n \mid w^{-1}(w(j) + 1) \leq h(j) \text{ for } w(j) \in [n-1]\}
\]

and

\[
\mathcal{G}_h^k := \{w \in \mathcal{G}_h \mid \ell_h(w) = k\}.
\]

In other words,

\[
\mathcal{G}_h = \mathfrak{S}_n \setminus \{w \in \mathfrak{S}_n \mid w(i) + 1 = w(i) \text{ for some } i > h(j)\} = \{w \in \mathfrak{S}_n \mid w(j) + 1 \neq w(i) \text{ for any } i > h(j)\}.
\]
Proposition 3.2 (cf. [1] Lemma 2.3). For each $k \geq 0$, we have
\[ |G_h^k| = \dim \mathbb{C}(H^{2k}(\text{Hess}(S, h))^\mathfrak{S}_n). \]

Proof. We first notice that Brosnan and Chow [4, Theorem 127] proved
\[ \dim \mathbb{C}(H^{2k}(\text{Hess}(S, h))^\mathfrak{S}_n) = \dim \mathbb{C}(H^{2k}(\text{Hess}(N, h))) \]
for every $k \geq 0$. Here, $N$ is the Jordan canonical form of a regular nilpotent element in $\mathfrak{gl}_n(\mathbb{C})$:
\[
N = \begin{pmatrix}
0 & 1 \\
0 & 1 \\
\ddots & \ddots \\
0 & 1 \\
0 & 0
\end{pmatrix}.
\]

Note that for each $w \in \mathfrak{S}_n$, the intersection of Schubert cell $X^c_w$ and $\text{Hess}(N, h)$ is nonempty if and only if
\[ w^{-1}(w(j) - 1) \leq h(j) \quad \text{for } j \in [n] \] (see [1] Lemma 2.3). Here, we use the convention that $w(0) = 0$. Moreover, for such $w$, the dimension of the intersection is given by
\[ \dim \mathbb{C}(\text{Hess}(N, h) \cap X^c_w) = |\{(j, i) \mid 1 \leq j < i \leq h(j), w(j) > w(i)\}| \] (see [2] Section 2.2, [3] Theorem 35 and the remark after it).

To complete the proof, it is enough to show that there is a bijective correspondence between the following two sets:
\[ \{w \in \mathfrak{S}_n \mid \dim \mathbb{C} \Omega^c_{w,h} = k, w^{-1}(w(j) + 1) \leq h(j) \quad \text{for } w(j) \in [n-1]\}, \]
\[ \{w \in \mathfrak{S}_n \mid \dim \mathbb{C}(\text{Hess}(N, h) \cap X^c_w) = k, w^{-1}(w(j) - 1) \leq h(j) \quad \text{for } j \in [n]\}. \]

Note that by the dimension formula in Proposition 2.2[4], we obtain
\[ \dim \mathbb{C} \Omega^c_{w,h} = |\{(j, i) \mid 1 \leq j < i \leq h(j), w(j) < w(i)\}|. \]

We consider the involution $\iota: \mathfrak{S}_n \rightarrow \mathfrak{S}_n$ given by $(\iota(w))(i) = n - w(i) + 1$. Then, we get
\[ (\iota(w))^{-1}(i) = w^{-1}(n - i + 1) \] and
\[ (\iota(w))^{-1}((\iota(w))(j) + 1) = (\iota(w))^{-1}(n - w(j) + 1 + 1) = (\iota(w))^{-1}(n - w(j) + 2) = (\iota(w))^{-1}(n - (w(j) - 1) + 1) = w^{-1}(w(j) - 1). \]
Moreover, for $1 \leq j < i \leq h(j)$, we have $w(j) > w(i)$ if and only if $\iota(w)(j) = n - w(j) + 1 < n - w(i) + 1 = \iota(w)(i)$. Therefore, the involution gives a desired bijective correspondence between two sets in (3.3) and (3.4). This completes the proof. □

**Example 3.3.** Suppose that $n = 4$ and $h = (2, 4, 4, 4)$. Then the elements $w \in S_4$ satisfying the condition $w^{-1}(w(j) - 1) \leq h(j)$ for $j \in [4]$ and their involutions $\iota(w)$ are given as follows.

| $\dim_{\mathbb{C}}(\text{Hess}(N, h) \cap X^0_w)$ | $w$       | $\iota(w)$ |
|------------------------------------------|---------|----------|
| 4                                        | 4321    | 1234     |
| 3                                        | 4312, 3241, 1432 | 1243, 2314, 4123 |
| 2                                        | 3214, 2143, 1423, 1342 | 2341, 3412, 4132, 4213 |
| 1                                        | 2134, 1324, 1243 | 3421, 4231, 4312 |
| 0                                        | 1234    | 4321     |

**Definition 3.4.** Let $h$ be a Hessenberg function.

(1) The *incomparability graph* $G_h$ of $h$ is the graph with the vertex set $[n]$ and the edge set $\{(j, i) \mid j < i \leq h(j)\}$.

(2) For $w \in S_n$, define a directed graph $G_{w, h}$ with the vertex set $[n]$ such that for each pair of indices $1 \leq j < i \leq n$, there is an edge $j \rightarrow i$ in $G_{w, h}$ if and only if

$$ j < i \leq h(j), \quad w(j) < w(i), $$

and define $\overline{G}_{w, h}$ by adding edges $j \leftarrow i$ to $G_{w, h}$ for any pair $(j, i)$ satisfying $j < i \leq h(j)$ and $w(j) > w(i)$. Then $\overline{G}_{w, h}$ is the incomparability graph of $h$ with an acyclic orientation, denoted by $o_h(w)$.

(3) Denote by $\mathcal{O}_h$ the set of all acyclic orientations of the incomparability graph $G_h$ of $h$.

For each $k$, define $\mathcal{O}_h^k$ by the set of all acyclic orientations of $G_h$ such that the number of edges $i \leftarrow j$ with $i < j$ is $k$. Then $\mathcal{O}_h = \bigsqcup_k \mathcal{O}_h^k$.

(4) Define an equivalence relation $\sim_h$ on $S_n$ by $v \sim_h w$ if $G_{v, h} = G_{w, h}$. In this case, we say that $v$ and $w$ have the same graph type. Denote by $[w]_h$ the equivalence class containing $w$.

**Example 3.5.** Let $h = (2, 4, 4, 4)$. There are twelve different acyclic orientations on the incomparability graph $G_h$ of $h$, each of which is described by $\overline{G}_{w, h}$ for $w \in G_h$. We describe
Proposition 3.6. Let $h : [n] \to [n]$ be a Hessenberg function. Then, we have

$$|G^k_h| = |\mathcal{O}^k_h|.$$  

Proof. For each $\lambda \vdash n$, let $c^h_\lambda(t)$ be the coefficient of the elementary symmetric function $e_\lambda$ in the $e$-basis expansion of the chromatic quasisymmetric function $X_{G_h}(x, t)$ of the incomparability graph $G_h$ of $h$. By Theorem 5.3 of [13],

$$\sum_{\lambda \in \text{Par}(n, j)} c^h_\lambda(t) = \sum_{o \in \mathcal{O}(G_h, j)} t^{\text{asc}(o)},$$  

where $\text{Par}(n, j)$ is the set of partitions of $n$ of length $j$, $\mathcal{O}(G_h, j)$ is the set of acyclic orientations of $G_h$ with $j$ sinks, and $\text{asc}(o)$ is the number of directed edges $(a, b)$ of $o$ for which $a < b$.

On the other hand, by Theorem 1.2, we have $\sum_{\lambda \vdash n} c^h_\lambda(t)M^\lambda = \sum_k t^k \text{dim} \mathbb{C} H^2_k(\text{Hess}(S, h))$ in the ring of $\mathfrak{S}_n$-representations, from which it follows that

$$\sum_{\lambda \vdash n} c^h_\lambda(t) = \sum_k t^k \text{dim} \mathbb{C} H^2_k(\text{Hess}(S, h)) \mathfrak{S}_n$$  

because $\text{dim} \mathbb{C}(M^\lambda) \mathfrak{S}_n = 1$ for any permutation module $M^\lambda$. By Proposition 3.2, the right-hand side is equal to $\sum_k t^k |\mathcal{G}^k_h|$. Therefore, $|\mathcal{O}^k_h| = |\mathcal{G}^k_h|$. 

We recall the following lemma related to the graph type.

Lemma 3.7 ([5, Lemma 4.4]). Let $v$ be a permutation in $\mathfrak{S}_n$ and let $s_i$ be a simple reflection. If $v \xrightarrow{s_i} v$, then $G_{v, h} = G_{s_i v, h}$.

For two permutations $v, w \in \mathfrak{S}_n$ such that $w = s_i v$ and $v \xrightarrow{s_i} w$, we denote by $v \xrightarrow{s_i} w$. 


d this correspondence.

\begin{align*}
1 &\xrightarrow{s_1} 2 \xrightarrow{s_2} 3 \xrightarrow{s_3} 4 \\
G_{1234, h} &\xrightarrow{s_1} G_{4123, h} \xrightarrow{s_2} G_{3142, h} \xrightarrow{s_3} G_{2314, h} \xrightarrow{s_4} G_{1234, h} \\
1 &\xrightarrow{s_2} 2 \xrightarrow{s_1} 3 \xrightarrow{s_3} 4 \\
G_{4132, h} &\xrightarrow{s_2} G_{1342, h} \xrightarrow{s_1} G_{3412, h} \xrightarrow{s_3} G_{2341, h} \xrightarrow{s_4} G_{1324, h} \\
1 &\xrightarrow{s_3} 2 \xrightarrow{s_4} 3 \xrightarrow{s_1} 4 \\
G_{4321, h} &\xrightarrow{s_3} G_{2341, h} \xrightarrow{s_4} G_{3421, h} \xrightarrow{s_1} G_{4312, h} \xrightarrow{s_2} G_{4231, h} \\
1 &\xrightarrow{s_4} 2 \xrightarrow{s_3} 3 \xrightarrow{s_2} 4 \\
G_{4231, h} &\xrightarrow{s_4} G_{2314, h} \xrightarrow{s_3} G_{3142, h} \xrightarrow{s_2} G_{1423, h} \xrightarrow{s_1} G_{1234, h}
\end{align*}
Proposition 3.8. For any $u \in \mathfrak{S}_n$, there exists a unique $w \in \mathcal{G}_h$ such that $G_{u,h} = G_{w,h}$. In this case, we have

$$w \xrightarrow{s_{i_1}} \cdots \xrightarrow{s_{i_r}} u$$

and $\sigma_{u,h} = s_{i_r} \cdots s_{i_1} \cdot \sigma_{w,h}$ for some simple reflections $s_{i_1}, \ldots, s_{i_r}$.

Proof. We claim that a maximal element with respect to the Bruhat order in each equivalence class $[\ ]_h$ is an element of $\mathcal{G}_h$. Let $v \in \mathfrak{S}_n$ which is not an element of $\mathcal{G}_h$. Then there is $j < k$ with $v(j) = i$ and $v(k) = i + 1$ and the vertex $j$ is not connected by an edge to the vertex $k$ in $G_{v,h}$. Thus we have $s_i v \rightarrow v$ and $G_{s_i v, h} = G_{v, h}$ by Lemma 3.7. Therefore, $v$ is not a maximal element with respect to Bruhat order in its equivalence class. This completes the proof of the claim. By the claim, the map $\mathcal{G}_h \rightarrow \mathfrak{S}_n/\sim_h$ assigning $w \in \mathcal{G}_h$ to its equivalent class $[w]_h$ is surjective.

On the other hand, an equivalence class $[w]_h$ induces an element $o_h(w)$ in $\mathcal{O}_h$ (see Definition 3.4(2)). By a similar argument as in the proof of Proposition 4.1 of [6], the map

$$[w]_h \in \mathfrak{S}_n/\sim_h \mapsto o_h(w) \in \mathcal{O}_h$$

is surjective, whereas provide a proof for the convenience of the reader.

Given an acyclic orientation $o$ on $G_h$, we define a directed graph $\Gamma_\ell$ with $n - \ell + 1$ vertices and a vertex $s_\ell$ of $\Gamma_\ell$ for $\ell \in [n]$ inductively, as follows. For $\ell = 1$, let $\Gamma_1$ be the graph $G_h$ with the acyclic orientation $o$ and define $s_1$ by the maximal source of $o$. Assume that we have defined $\Gamma_{\ell-1}$ and $s_{\ell-1}$. Define $\Gamma_\ell$ by the directed graph obtained from $\Gamma_{\ell-1}$ by deleting $s_{\ell-1}$ and every edge from $s_{\ell-1}$, and define $s_\ell$ by the maximal source of $\Gamma_\ell$. Let $w \in \mathfrak{S}_n$ be the permutation defined by $w(s_\ell) = \ell$ for $\ell \in [n]$.

We now claim that $o_h(w) = o$. It is enough to show that $o_h(w)$ and $o$ have the same set of directed edges $j \rightarrow i$ with $j < i$ to prove the claim. To see this, let $s_\ell$ and $s_k$ ($\ell < k$) be two vertices which are connected by an edge in $G_h$. Then this edge is directed as $s_\ell \rightarrow s_k$ in $o$ because $s_\ell$ is a source in the directed graph $\Gamma_\ell$ which has $s_k$ as a vertex. Since we have $w(s_\ell) = \ell < w(s_k) = k$, the edge $\{s_\ell, s_k\}$ is directed from $s_\ell$ to $s_k$ in $G_{w,h}$. Therefore, $o_h(w)$ and $o$ have the same set of directed edges $j \rightarrow i$ with $j < i$. Consequently, we get $o_h(w) = o$.

Therefore, the composition $\mathcal{G}_h \rightarrow \mathfrak{S}_n/\sim_h \rightarrow \mathcal{O}_h$ defines a surjective map $\mathcal{G}_h \rightarrow \mathcal{O}_h$. By Proposition 3.6, we have $|\mathcal{G}_h| = |\mathcal{O}_h|$. Consequently, the map $\mathcal{G}_h \rightarrow \mathfrak{S}_n/\sim_h$ is injective. It follows that each equivalence class $[\ ]_h$ has a unique maximal element.

Furthermore, the proof of the claim implies that for any element $u$ in the same equivalence class as $w \in \mathcal{G}_h$, there exist simple reflections $s_{i_1}, \ldots, s_{i_r}$ such that $w \xrightarrow{s_{i_1}} \cdots \xrightarrow{s_{i_r}} u$. Then, by Proposition 2.6(1), we get $\sigma_{u,h} = s_{i_r} \cdots s_{i_1} \cdot \sigma_{w,h}$.

Remark 3.9. In the proof of Proposition 3.8 for a given acyclic orientation $o$ on $G_h$, we assign a permutation $w$ with $o_h(w) = o$. In fact, such a permutation $w$ is an element of $\mathcal{G}_h$.

Suppose that $s_{\ell-1} < s_\ell$ and there is no edge connecting $s_{\ell-1}$ and $s_\ell$ in $G_h$. Then $s_\ell$ is a source
in $\Gamma_{\ell - 1}$ because $s_\ell$ is a source in $\Gamma_\ell$ which is obtained from $\Gamma_{\ell - 1}$ by removing $s_{\ell - 1}$ and all edges from $s_{\ell - 1}$, contradicting to the condition that $s_{\ell - 1}$ is the maximal source of the directed graph $\Gamma_{\ell - 1}$. Therefore, if $s_{\ell - 1} < s_\ell$, then there is an edge connecting $s_{\ell - 1}$ and $s_\ell$ in $G_h$. In this case, the orientation is from $s_{\ell - 1}$ to $s_\ell$ because $w(s_{\ell - 1}) = \ell - 1 < w(s_\ell) = \ell$.

For $w \in G^k_h$, define a subset $P_{w,h}$ of $\mathfrak{S}_n$ by

$$P_{w,h} := \{u \in \mathfrak{S}_n \mid G_u, h = G_w, h\} = [w]_h$$

and denote by $M(\sigma, w, h)$ the $\mathfrak{S}_n$-module generated by $\sigma, w, h \in H^{2k}(\text{Hess}(S, h))$. Then Proposition 3.8 can be rephrased as follows.

**Proposition 3.10.** For each $w \in G^k_h$, any element in $\{\sigma, u, h \mid u \in P_{w,h}\}$ is contained in $M(\sigma, w, h)$, and $\{P_{w,h} \mid w \in G^k_h\}$ defines a partition on the set $\{\sigma, u, h \mid u \in \mathfrak{S}_n, \ell(u) = k\}$, that is,

$$\{\sigma, u, h \mid u \in \mathfrak{S}_n, \ell(u) = k\} = \bigcup_{u \in G^k_h} \{\sigma, u, h \mid u \in P_{w,h}\}.$$

**Theorem 3.11.** The set $\{\sigma, w, h \mid w \in G^k_h\}$ has the cardinality $\dim_{\mathbb{C}} H^{2k}(\text{Hess}(S, h))^\mathfrak{S}_n$, and it generates the $\mathfrak{S}_n$-module $H^{2k}(\text{Hess}(S, h))$, that is,

$$H^{2k}(\text{Hess}(S, h)) = \sum_{w \in G^k_h} M(\sigma, w, h),$$

where $M(\sigma, w, h)$ denotes the $\mathfrak{S}_n$-module generated by $\sigma, w, h$.

**Proof.** The first statement follows from Proposition 3.2. By Proposition 3.10, for any $u \in \mathfrak{S}_n$, there is $w \in G_h$ with $\sigma, u, h \in M(\sigma, w, h)$. Since $\{\sigma, u, h \mid u \in \mathfrak{S}_n\}$ is a $\mathbb{C}$-basis of $H^*(\text{Hess}(S, h))$, the set $\{\sigma, w, h \mid w \in G_h\}$ generates the $\mathfrak{S}_n$-module $H^*(\text{Hess}(S, h))$. \qed

We recall the definition of permutation modules of the symmetric group $\mathfrak{S}_n$. A composition $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ of $n$ is a sequence of positive integers such that $\sum_{i=1}^\ell \alpha_i = n$. For a composition $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ of $n$, we let $\mathfrak{S}_\alpha$ be the **Young subgroup** of $\mathfrak{S}_n$ defined as

$$\mathfrak{S}_\alpha = \mathfrak{S}_{\{1, \ldots, \alpha_1\}} \times \mathfrak{S}_{\{\alpha_1 + 1, \ldots, \alpha_1 + \alpha_2\}} \times \cdots \times \mathfrak{S}_{\{n - \alpha_\ell + 1, \ldots, n\}},$$

and let $M^\alpha$ be the **permutation module** of $\mathfrak{S}_n$ associated to $\alpha$ defined as the induced module $1_{\mathfrak{S}_\alpha}^{\mathfrak{S}_n}$. Indeed, $M^\alpha$ is isomorphic to $M^\lambda$ for a partition $\lambda$ obtained by rearranging the parts of $\alpha$ in nonincreasing order.

We remark that the sum $\sum_{w \in G^k_h} M(\sigma, w, h)$ is generally not a direct sum, and the question is whether we can reduce modules $M(\sigma, w, h)$ so that we get a direct sum decomposition into permutation modules, as in Conjecture 1.3. A natural question related to this conjecture is the following.

**Question 3.12.** Let $w \in G_h$.

1. Is the stabilizer $\text{Stab}_{\mathfrak{S}_n}(\sigma, w, h)$ of $\sigma, w, h$ in $\mathfrak{S}_n$ a Young subgroup?
(2) Is the $\mathfrak{S}_n$-module $M(\sigma_{w,h})$ generated by $\sigma_{w,h}$ a permutation module?

For $w \in \mathfrak{S}_n$, let $J_{w,h}$ be the subset of $[n - 1]$ consisting of $i \in [n - 1]$ such that $w \mapsto s_i w$, $s_i w \mapsto w$, or $w \mapsto s_i w$ and \{ $u \in \Omega_{s_i w,h}^T \cap \Omega_w^T$ | $u \mapsto s_i u$ and $\dim \mathbb{C}(\Omega_u^T \cap \Omega_{s_i w,h}) = \dim \mathbb{C} \Omega_{w,h}$ \} $\neq \emptyset$.

Then the Young subgroup

$$\mathfrak{S}_{\alpha_w} = \langle s_i | i \notin J_{w,h} \rangle$$

stabilizes $\sigma_{w,h}$ by Proposition \ref{prop-stabilizer}. We expect that $\mathfrak{S}_{\alpha_w}$ is indeed the stabilizer $\text{Stab}_{\mathfrak{S}_n}(\sigma_{w,h})$, proving that $\text{Stab}_{\mathfrak{S}_n}(\sigma_{w,h})$ is a Young subgroup.

Question 3.12(2) is trickier than Question 3.12(1) as the following example shows.

Example 3.13. Let $h = (2, 4, 4, 4)$ and $w = 3412$. Then $J_{w,h}$ is \{ 2 \} and the stabilizer $\text{Stab}_{\mathfrak{S}_4}(\sigma_{w,h})$ is the Young subgroup $\mathfrak{S}_{(2, 2)}$ but the $\mathfrak{S}_n$-module $M(\sigma_{w,h})$ generated by $\sigma_{w,h}$ is the permutation module $M^{(3, 1)}$ of type $(3, 1)$. More precisely, the set \{ $u \cdot \sigma_{w,h} | u \in \mathfrak{S}_4$ \} is generated by $\sigma_{3412, h} \rightarrow \sigma_{2413, h}$ $s_2 \rightarrow \sigma_{1423, h}$ $s_1 \rightarrow \sigma_{1243, h}$ $s_3 \rightarrow \sigma_{2413, h}$ $s_1 \rightarrow \sigma_{1423, h}$ $s_3 \rightarrow \sigma_{1423, h}$ $+(\sigma_{3412, h} - \sigma_{3214, h})$ $+(\sigma_{3412, h} - \sigma_{3214, h})$ $+(\sigma_{3412, h} - \sigma_{3214, h})$ $+(\sigma_{3412, h} - \sigma_{3214, h})$ $+(\sigma_{3412, h} - \sigma_{3214, h})$

From this, we see that six elements

$\sigma_{w,h}$,  $s_2 \cdot \sigma_{w,h}$,  $s_1 s_2 \cdot \sigma_{w,h}$,  $s_3 s_2 \cdot \sigma_{w,h}$,  $s_3 s_1 s_2 \cdot \sigma_{w,h}$,  $s_2 s_3 s_1 s_2 \cdot \sigma_{w,h}$

are all distinct but they span a 4-dimensional vector space in $H^4(\text{Hess}(S, h))$. Indeed, there are two linear relations:

$s_3 s_1 s_2 \cdot \sigma_{w,h} = s_3 s_2 \cdot \sigma_{w,h} - s_2 \cdot \sigma_{w,h} + s_1 s_2 \cdot \sigma_{w,h}$,  $s_2 s_3 s_1 s_2 \cdot \sigma_{w,h} = -\sigma_{w,h} + s_3 s_2 \cdot \sigma_{w,h} + s_1 s_2 \cdot \sigma_{w,h}$.

We will give an affirmative answer to Question 3.12 when $w \in \mathfrak{G}^1_h$ in Propositions \ref{prop-linearrelations} and Theorem \ref{thm-linearrelations}. Also, see Remark \ref{rem-linearrelations}.

4. $\mathfrak{S}_n$-action on generators

Recall from Definition \ref{def-generators} and Theorem \ref{thm-generators} that the set $\mathfrak{G}^1_h$ of permutations whose corresponding classes form a generator set of $H^2(\text{Hess}(S, h))$ is given as follows

$$\mathfrak{G}^1_h = \{ w \in \mathfrak{S}_n | \ell_h(w) = 1 \text{ and } w^{-1}(w(j) + 1) \leq h(j) \text{ for } w(j) \in [n - 1] \}.$$
Definition 4.1. For a Hessenberg function $h: [n] \rightarrow [n]$, we define

$$T = T_h := \{i \in [n-1] \mid h(i-1) > (i-1) + 1 = i\} \subset [n-1].$$

Here, we set $h(0) = 2$.

We note that $1 \in T$ always holds because $h(0) = 2 > 1$.

Remark 4.2. For each element $i$ in $T$, we construct a trivial representation (see Theorems 6.1 and 6.4). The notation $T$ stands for the word ‘trivial’.

For a permutation $w \in \mathfrak{S}_n$, we say that $i \in [n-1]$ is a descent of $w$ if $w(i) > w(i+1)$. Let $\text{Des}(w)$ be the set of descents of $w$, and let $\text{des}(w) = |\text{Des}(w)|$ be the number of descents of $w$.

Definition 4.3. Let $h: [n] \rightarrow [n]$ be a Hessenberg function. We define $w[i] \in \mathfrak{S}_n$ for $i = 1, 2, \ldots, n-1$ according to the values of $h$ as follows. We place $|$ between the $i$th and $(i+1)$st values of $w[i]$, and provide the graph $G_{w[i], h}$ below the definition of $w[i]$ for each case. Let $T = T_h$.

1. Suppose that $i \in T$.
   (a) If $i + 1 \in T \cup \{n\}$, then we let
   $$w[i] := 1 2 \cdots i - 1 i + 1 | i i + 2 \cdots n = s_i.$$

2. Suppose that $i \notin T$.
   (a) If $i + 1 \notin T \cup \{n\}$, then we let
   $$w[i] := 2 \cdots i i + 1 | 1 i + 2 \cdots n.$$
(b) If $i + 1 \notin T \cup \{n\}$, then we let 

$$w^{[i]} := n - i + 1 \cdots n - 1 \; n \; 1 \cdots n - i.$$ 

Here, for each vertex $j$ in the graph $\overline{G}_{w^{[i]}, h}$, we write the value $w^{[i]}(j)$ in blue below $j$.

**Lemma 4.4.** The number of acyclic orientations that have exactly one directed edge from a vertex $j$ to a vertex $i$ with $i < j$ is at most $n - 1$.

**Proof.** If $j \to i$ with $i + 1 < j$ is an edge in an orientation of $G_h$, then the directed edges 

$$i \to i + 1, \ldots, j - 1 \to j, j \to i$$

form a cycle. Therefore, a unique edge from $j$ to $i$ where $i < j$ must occur only when $j = i + 1$. \qed

It is easy to check that the permutations $w^{[i]}$, $i = 1, \ldots, n - 1$, are all distinct and contained in $G^1_h$. We have $|G^1_h| = |\mathcal{O}^1_h|$ by Proposition 3.6 and combining this with the previous lemma, we obtain the following proposition.

**Proposition 4.5.** For a Hessenberg function $h: [n] \to [n]$, we have

$$G^1_h = \{w^{[i]} \mid i \in [n - 1]\}.$$ 

We now consider the $\mathfrak{S}_n$-action on the generators $\sigma_{w^{[i]}, h}$ of $H^2(\text{Hess}(S, h))$ to analyze their stabilizer subgroup. Recall that we can use Proposition 2.6 to compute the action of simple reflections, while we need to understand the set $A_{s_i, w}$ when $w = s_i$ for the explicit computation.

We denote by $\mathcal{A}_i$ the set $A_{s_i, s_i}$ of permutations that is used to describe the class $s_i \cdot \sigma_{s_i, h}$ in Proposition 2.6. By the definition of $\mathcal{T}_u$, we have

$$\mathcal{T}_u = \Omega_u \cap \Omega_{s_i, s_i, h} = \Omega_u \cap \Omega_{e, h} = \Omega_u \cap \text{Hess}(S, h) = \Omega_{u, h}. $$

Here, $e$ is the identity element in $\mathfrak{S}_n$. Accordingly, we get

$$\left( s_i \cdot \sigma_{s_i, h} + \sum_{u \in \mathcal{A}_i} \sigma_{s_i, u, h} \right) = \sigma_{s_i, h} + \sum_{u \in \mathcal{A}_i} \sigma_{u, h}. $$
Furthermore, the intersection \( A_i \cap s_i A_i \) is empty by Proposition 2.6. As a direct consequence of (4.2), we obtain the following lemma.

**Lemma 4.6.** For \( 1 \leq i \leq n - 1 \), the set \( A_i \) is empty if and only if \( s_i \cdot \sigma_{s_i,h} = \sigma_{s_i,h} \), that is, the \( s_i \)-action stabilizes the class \( \sigma_{s_i,h} \).

**Proof.** By (4.2), if the set \( A_i \) is empty, then we obtain \( s_i \cdot \sigma_{s_i,h} = \sigma_{s_i,h} \). Suppose that the set \( A_i \) is nonempty. Since the intersection \( A_i \cap s_i A_i \) is empty and the classes \( \{ \sigma_{u,h} \} \) form a basis by Proposition 2.5, the sum

\[
\sum_{u \in A_i} \sigma_{s_i,u,h} - \sum_{u \in A_i} \sigma_{u,h} = \sum_{v \in s_i A_i} \sigma_{v,h} - \sum_{u \in A_i} \sigma_{u,h}
\]

is nonzero. Therefore, we get \( s_i \cdot \sigma_{s_i,h} \neq \sigma_{s_i,h} \). \( \square \)

**Proposition 4.7.** For \( i \in [n-1] \), we have

\[ A_i = \{ u \in \mathcal{S}_n \mid u^{-1}(i+1) \leq i, \ u^{-1}(i) \geq i, \ dim \mathcal{T}_u = dim \mathcal{C} \mathcal{O}_{s_i} \}, \]

for all \( 1 \leq i \leq n \), where \( \mathcal{T}_u \) is the set of \( \mathcal{C} \mathcal{O}_{s_i} \) for \( u \) and \( \mathcal{C} \mathcal{O}_{s_i} \) is the set of \( \mathcal{C} \mathcal{O}_{s_i} \) for \( u \).

**Proof.** Recall from Proposition 2.6 that we have
\[ A_i = \{ u \in \mathcal{O}_{s_i} \mid \mathcal{C} \mathcal{O}_{s_i} \Omega_{s_i} = \mathcal{C} \mathcal{O}_{s_i} \Omega_{s_i} = \{ u \mid u \geq s_i \} \}. \] By Theorem 4.1, we get \( \mathcal{C} \mathcal{O}_{s_i} \Omega_{s_i} = \mathcal{C} \mathcal{O}_{s_i} \Omega_{s_i} \). Therefore, \( \mathcal{C} \mathcal{O}_{s_i} \Omega_{s_i} = \mathcal{C} \mathcal{O}_{s_i} \Omega_{s_i} \).

Now the description of \( A_i \) in (4.3) becomes
\[ A_i = \{ u \in \mathcal{S}_n \mid u \geq s_i, \ \ell_h(u) = 1, \ u \rightarrow s_i u \}. \]

Recall from [3, §3.2] a property of the Bruhat order. For a set \{\( a_1, \ldots, a_k \)\} of distinct integers, \{\( a_1, \ldots, a_k \)\} denotes the ordered \( k \)-tuple obtained from \{\( a_1, \ldots, a_k \)\} by arranging its elements in ascending order. Moreover, we use the order to compare two ordered \( k \)-tuples defined as follows: \( a_1, \ldots, a_k \geq b_1, \ldots, b_k \) if and only if \( a_i \geq b_i \) for all \( 1 \leq i \leq k \). For a permutation \( w \in \mathcal{S}_n \), we denote by \( w^{(k)} \) for the ordered \( k \)-tuple \{\( w(1), \ldots, w(k) \)\}. For \( w_1, w_2 \in \mathcal{S}_n \), we have \( w_1 \geq w_2 \) in Bruhat order if and only if
\[ w_1^{(k)} \geq w_2^{(k)} \text{ for all } 1 \leq k \leq n. \]

A permutation \( u \) satisfies \( u \geq s_i \) if and only if \( u^{(k)} \geq (s_i)^{(k)} \) for all \( k \). Note that we have
\[ (s_i)^{(k)} = \begin{cases} (1, 2, \ldots, k) & \text{if } k \neq i, \\ (1, 2, \ldots, i-1, i+1) & \text{if } k = i. \end{cases} \]

Since \( (1, 2, \ldots, k) \) is the minimum among the ordered \( k \)-tuples of integers, for a permutation \( u \),
\[ u \geq s_i \iff u^{(i)} \geq (1, 2, \ldots, i-1, i+1). \]
We set
\[ B = \{ u \in \mathfrak{S}_n \mid u^{-1}(i+1) \leq i, \ u^{-1}(i) > i, \ h(u^{-1}(i+1)) < u^{-1}(i), \ \ell_h(u) = 1 \} \]
in the statement of the proposition. We first claim that \( B \subset \mathcal{A}_i \). Take \( u \in B \). Since \( u^{-1}(i+1) \leq i \), the number \( i+1 \) appear in the first \( i \) letters so we have \( u^{(i)} \geq (1, 2, \ldots, i-1, i+1) \). This implies that \( u \geq s_i \) by (4.6). Moreover, the condition \( u \rightarrow s_i \) is equivalent to saying that in one-line notation of \( u \), the number \( i+1 \) appears ahead of \( i \) and the locations of \( i \) and \( i+1 \) in the one-line notation of \( u \) are far from each other. More precisely, \( u^{-1}(i+1) < u^{-1}(i) \) and \( h(u^{-1}(i+1)) < u^{-1}(i) \). Accordingly, \( u \) is an element of \( \mathcal{A}_i \).

We claim that \( \mathcal{A}_i \subset B \). Take \( u \in \mathcal{A}_i \). Because of the condition \( u \rightarrow s_i \), we have \( u^{-1}(i+1) < u^{-1}(i) \) and \( h(u^{-1}(i+1)) < u^{-1}(i) \). It is enough to check that \( u^{-1}(i+1) \leq i \) and \( u^{-1}(i) > i \). Since \( h \) satisfies \( h(j) \geq j+1 \) for all \( j \), we have
\[ \ell_h(u) \geq \text{des}(u). \]
Accordingly, because of the assumption \( \ell_h(u) = 1 \), there exists only one descent in \( u \). Now assume on the contrary that \( u^{-1}(i+1) > i \). Then there exists \( k > i+1 \) such that \( u^{-1}(k) \leq i \) since \( u \geq s_i \) and (4.6). This implies that the numbers \( k > i+1 \geq i \) satisfy \( u^{-1}(k) < u^{-1}(i+1) < u^{-1}(i) \). This produces at least two descents in \( u \). Therefore, we have \( u^{-1}(i+1) \leq i \).

Now consider the condition \( u^{-1}(i) > i \). Assume on the contrary that \( u^{-1}(i) \leq i \) so we get \( u^{-1}(i+1) < u^{-1}(i) \) \( i \). Therefore, there exists \( k < i \) such that \( u^{-1}(k) > i \). This implies that the numbers \( i+1 > i > k \) satisfy \( u^{-1}(i+1) < u^{-1}(i) < u^{-1}(k) \). This produces at least two descents in \( u \). Therefore, we have \( u^{-1}(i) > i \). This proves \( \mathcal{A}_i \subset B \) so we are done.

**Example 4.8.** Let \( h = (2, 3, 5, 6, 6, 6) \). Using the description of \( \mathcal{A}_i \) in Proposition 4.7 we have the following computations.

(1) \( \mathcal{A}_1 = \{ 23 \mid 1456, 24 \mid 1356, 25 \mid 1346, 26 \mid 1345, 234 \mid 156 \}, \)
(2) \( \mathcal{A}_2 = \{ 3 \mid 12456, 34 \mid 1256, 35 \mid 1264, 36 \mid 1245, 345 \mid 1256 \}, \)
(3) \( \mathcal{A}_3 = \{ 4 \mid 12356, 14 \mid 2356, 24 \mid 1356, 45 \mid 1236, 46 \mid 1235 \}, \)
(4) \( \mathcal{A}_4 = \{ 5 \mid 12346, 15 \mid 2346, 25 \mid 1346, 35 \mid 1246, 56 \mid 1234 \}, \)
(5) \( \mathcal{A}_5 = \{ 6 \mid 12346, 16 \mid 2345, 26 \mid 1345, 36 \mid 1245, 46 \mid 1235 \}. \)

We decorate the places where descents appear.

Proposition 4.7 provides the following corollary.

**Corollary 4.9.** For \( i \in [n-1] \), let \( \mathcal{A}_i \) be the set used in the description 4.2. Then, for \( u \in \mathcal{A}_i \), we have \( \sigma_{s_i, u} = s_i \cdot \sigma_{u, h} \). In particular, we get
\[ s_i \cdot \sigma_{s_i, h} = \sigma_{s_i, h} + \sum_{u \in \mathcal{A}_i} \sigma_{u, h} - \sum_{u \in \mathcal{A}_i} s_i \cdot \sigma_{u, h}. \]
Proof. As we have seen in (4.4) in the proof of Proposition 4.7, $u \rightarrow s_i u$ for any $u \in A_i$. Accordingly, we obtain

$$s_i \cdot \sigma_{u,h} = \sigma_{s_i u,h}$$

by Proposition 2.6(1). Using (4.2), the result follows.

□

Lemma 4.10. If $T = [n - 1]$, that is, $h(i) > i + 1$ for all $i = 1, 2, \ldots, n - 2$, then $A_k = \emptyset$ for all $k = 1, 2, \ldots, n - 1$.

Proof. Assume on the contrary that there exists $w \in A_k$. Let $i_{k+1} = w^{-1}(k+1)$ and $i_k = w^{-1}(k)$. Then, by Proposition 4.7 and the assumption,

$$i_{k+1} \leq k < i_k \quad \text{and} \quad i_{k+1} + 1 < h(i_{k+1}) < i_k$$

must hold and we can see that $i_{k+1} + 2 \leq i_k$ and the unique descent, say $d$, must be between $i_{k+1}$ and $i_k$, that is, $i_{k+1} \leq d < i_k$.

If $d = i_{k+1}$, then we have

$$w(d) = k + 1 > w(d + 1) \quad \text{and} \quad w(d + 2) \leq w(i_k) = k.$$

Hence, both $(d, d + 1)$ and $(d, d + 2)$ are counted in $\ell_h(w)$ contradicting to $w \in A_k$.

If $d > i_{k+1}$, then we obtain

$$w(i_{k+1}) = k + 1 \leq w(d - 1) < w(d) \quad \text{and} \quad w(d + 1) \leq w(i_k) = k.$$

Hence, both $(d, d + 1)$ and $(d - 1, d + 1)$ are counted in $\ell_h(w)$ contradicting to $w \in A_k$.

This shows that there is no $w$ in $A_k$. □

With explicit descriptions of $A_i$ for $i = 1, 2, \ldots, n - 1$, we completely determine the stabilizer subgroup of $\sigma_{w[i], h}$ in most cases: See Propositions 4.12 and 4.13. The following lemma can be derived using known results given at the beginning of Section 8.5 and Theorem 3.3A in [10]. We provide a proof for the readers’ convenience.

Lemma 4.11. Let $H$ be a subgroup of $\mathfrak{S}_n$, $n > 2$, that contains $\mathfrak{S}_{i, n-i}$ but does not contain the transposition $s_i$ for some $i$.

1. If $i \neq n - i$, then $H = \mathfrak{S}_{i, n-i}$.

2. If $i = n - i$ and there is no $\alpha \in H$ such that $\alpha(\{1, 2, \ldots, i\}) = \{i + 1, i + 2, \ldots, n\}$, then $H = \mathfrak{S}_{i, n-i}$.

Proof. Assume on the contrary that there is $\beta \in H \setminus \mathfrak{S}_{i, n-i}$. If $i \neq n - i$, we may assume that $i > n - i$ and there must be elements $k, l \in \{1, \ldots, i\}$ such that

$$\beta(k) \in \{1, \ldots, i\} \quad \text{and} \quad \beta(l) \notin \{1, \ldots, i\}. \quad (4.7)$$

On the other hand, if $i = n - i$ and there is no $\alpha \in H$ such that $\alpha(\{1, 2, \ldots, i\}) = \{i + 1, i + 2, \ldots, n\}$, then there also exist elements $k, l \in \{1, \ldots, i\}$ satisfying (4.7).
The element $\beta$ produces $\beta(k) \beta^{-1} = (\beta(k) \beta(l)) \in H$ and we have
\[(\beta(l) i + 1)(\beta(k) i)(\beta(k) \beta(l))(\beta(k) i)(\beta(l) i + 1) = (i i + 1) = s_i \in H,
\]
which is a contradiction. Here, we denote by $(k l)$ the transposition $s_{k,l}$. We conclude that $H = \mathfrak{S}_{(i,n-i)}$. \(\square\)

Let $\text{Stab}(\sigma_{w,h})$ denote the stabilizer subgroup of $\sigma_{w,h}$ in $\mathfrak{S}_n$. We use Proposition 2.6 to compute $\text{Stab}(\sigma_{w[i],h})$.

**Proposition 4.12.** Let $h: [n] \to [n]$ be a Hessenberg function and $i \in T = T_h$.

(1) Suppose that $i \in T$ and $i + 1 \in T \cup \{n\}$, then
(a) $\text{Stab}(\sigma_{w[i],h}) = \mathfrak{S}_{(i,n-i)}$ if $n - i \neq i$ and $T \neq [n - 1]$.
(b) $\mathfrak{S}_{(i,n-i)} \leq \text{Stab}(\sigma_{w[i],h}) < \mathfrak{S}_n$ if $n - i = i$ and $T \neq [n - 1]$.
(c) $\text{Stab}(\sigma_{w[i],h}) = \mathfrak{S}_n$ otherwise, that is, $T = [n - 1]$.

(2) If $i + 1 \notin T \cup \{n\}$, then $\text{Stab}(\sigma_{w[i],h}) = \mathfrak{S}_{(n-1,1)}$.

**Proof.** (1) Suppose that $i \in T$ and $i + 1 \in T \cup \{n\}$, that is, $h(i - 1) > i$ and $h(i) > i + 1$; or $i + 1 = n$, then $w[i] = s_i$ by Definition 1.3. For $k \neq i$, $s_kw[i] \to w[i]$ and $s_k \cdot \sigma_{w[i],h} = \sigma_{w[i],h}$. Accordingly, we have
\[\mathfrak{S}_{(i,n-i)} \leq \text{Stab}(\sigma_{w[i],h}) < \mathfrak{S}_n.\]

On the other hand, we have $w[i] = s_i \to s_iw[i] = c$, so we apply Lemma 4.10 to get that $A_i$ is empty if and only if $s_i \in \text{Stab}(\sigma_{w[i],h})$.

where $A_i$ is described in Proposition 4.7.

Suppose that $T \neq [n - 1]$. Choose $j \in [n - 1] \setminus T$. Then $h(j - 1) = j$. If such $j$ satisfies $i < j$, then
\[u = 12 \cdots i - 1 i + 1 \cdots j + 1 i j + 2 \cdots n\]
is a permutation in $A_i$ which has the unique descent at $j$. If any $j \in [n - 1] \setminus T$ satisfies $j < i$, then take
\[u = i - j + 1 i - j + 2 \cdots i - 1 i + 1 1 2 \cdots i - j i i + 2 \cdots n.\]

We notice that $u$ is a permutation in $A_i$ which has the unique descent at $j$. We thus have $A_i \neq \emptyset$ and $s_i \notin \text{Stab}(\sigma_{w[i],h})$ if $T \neq [n - 1]$.

Lemma 4.11 and 4.8 now imply that $\text{Stab}(\sigma_{w[i],h}) = \mathfrak{S}_{(i,n-i)}$ if $n - i \neq i$, which proves 1-(a). The statement 1-(b) follows from 1-(a).

If $T = [n - 1]$, then by Lemma 4.10, $A_i$ is empty and $s_i \in \text{Stab}(\sigma_{w[i],h})$. Therefore, $\text{Stab}(\sigma_{w[i],h}) = \mathfrak{S}_n$, which proves 1-(c).

(2) Suppose that $i \in T$ and $i + 1 \notin T \cup \{n\}$, that is, $h(i) = i + 1$. Then $s_kw[i] \to w[i]$ and $s_k \cdot \sigma_{w[i],h} = \sigma_{w[i],h}$ for $k = 1, \ldots, n - 2$. Moreover, $w[i] \to s_{n-1}w[i]$ and we have $s_{n-1} \cdot \sigma_{w[i],h} = \sigma_{s_{n-1}w[i],h} \neq \sigma_{w[i],h}$. This, due to Lemma 4.11, proves the claim. \(\square\)
We also obtain the following proposition using similar arguments used in the proof of Proposition 4.12.

**Proposition 4.13.** Let $h : [n] \rightarrow [n]$ be a Hessenberg function and $i \notin T = T_h$.

1. If $i + 1 \in T \cup \{n\}$, then $\text{Stab}(\sigma_{w[i],h}) = \mathfrak{S}_{(1,n-1)}$.
2. If $i + 1 \notin T \cup \{n\}$, then
   
   a. $\text{Stab}(\sigma_{w[i],h}) = \mathfrak{S}_{(n-1,i)}$ if $n - i \neq i$.
   b. $\mathfrak{S}_{(n-i,i)} \subseteq \text{Stab}(\sigma_{w[i],h}) < \mathfrak{S}_n$ if $n - i = i$.

**Proof.**

1. Suppose that $i \notin T$ and $i + 1 \in T \cup \{n\}$. Then we have $w[i] \rightarrow s_1w[i]$ and $s_kw[i] \rightarrow w[i]$ for $k = 2, \ldots, n - 1$. Thus we have $s_1 \notin \text{Stab}(\sigma_{w[i],h})$ and $s_2, \ldots, s_{n-1} \in \text{Stab}(\sigma_{w[i],h})$, proving the first part of the proposition due to Lemma 4.11.

2. Suppose that $i \notin T$ and $i + 1 \notin T \cup \{n\}$. Then $w[i] \rightarrow s_{n-i}w[i]$ and $s_kw[i] \rightarrow w[i]$ for $k \neq n - i$. Thus we have $s_{n-i} \notin \text{Stab}(\sigma_{w[i],h})$ and $s_k \in \text{Stab}(\sigma_{w[i],h})$ for $k \neq n - i$, proving the second part of the proposition due to Lemma 4.11.

**Remark 4.14.** We will see in Corollary 6.9 that for $i$ satisfying $i \notin T$ and $i + 1 \notin T \cup \{n\}$, we have $\text{Stab}(\sigma_{w[i],h}) = \mathfrak{S}_{(n-i,i)}$ while we have an inclusion in Proposition 4.13.

5. A PARTITION OF $\mathfrak{S}_n$

In this section, we provide a partition of $\mathfrak{S}_n$ each class of which consists of permutations having the same graph type in Proposition 5.3. Using this description, we also enumerate the dimension of $H^2(\text{Hess}(S,h))$.

**Lemma 5.1.** Let $T = T_h$. For $1 \leq i \leq n - 1$, we have

$$\left( \bigcup_{u \in A_i} \text{Des}(u) \right) \cap \{ j \in T \mid j \geq i \} = \emptyset.$$

**Proof.** Assume on the contrary that there exists $u \in A_i$ such that the element in the descent set $\text{Des}(u) = \{ j \}$ satisfies $j \in T$ and $j \geq i$. By the definition of $T$, we have $h(j - 1) > j$, so a part of the graph $G_{u,h}$ can be depicted as follows:

```
  1 ----> ⋮ ----> i ----> ⋮ ----> j - 1 ----> j ----> j + 1 ----> ⋮ ----> n
```

Indeed, the permutation $u$ satisfies

$$(5.1) \quad u(1) < \cdots < u(i) < \cdots < u(j - 1) \quad < \quad u(j + 1) < u(j + 2) < \cdots < u(n)$$

$$\quad < \quad u(j)$$
Accordingly, \( u(j) \geq j \) and in the one-line notation of \( u \), the numbers \( \{n\} \setminus \{u(j)\} \) are displayed in ascending order. Therefore, \( u(k) = k \) for all \( 1 \leq k < j \), which contradicts the assumption \( u \in \mathcal{A}_i \) because any elements \( u \) in \( \mathcal{A}_i \) should satisfy \( u^{-1}(i + 1) \leq i \) and \( h(u^{-1}(i + 1)) < u^{-1}(i) \) by Proposition 4.7. Here, if \( j = i \) and \( u(j) = u(i) = i + 1 \), then we should have \( u(i + 1) = i \) and such a permutation \( u \) cannot satisfy the condition \( h(u^{-1}(i + 1)) < u^{-1}(i) \). Hence, the result follows.

**Example 5.2.** We demonstrate Lemma 5.1 in this example. Suppose that \( h = (2, 3, 5, 6, 6, 6) \). We obtain \( T = T_h = \{1, 4, 5\} \). Considering the computations of \( \mathcal{A}_i \) in Example 4.8, we obtain

\[
\bigcup_{u \in \mathcal{A}_1} \text{Des}(u) \cap \{j \in T \mid j \geq 1\} = \{2, 3\} \cap \{1, 4, 5\} = \emptyset,
\]

\[
\bigcup_{u \in \mathcal{A}_2} \text{Des}(u) \cap \{j \in T \mid j \geq 2\} = \{1, 2, 3\} \cap \{4, 5\} = \emptyset,
\]

\[
\bigcup_{u \in \mathcal{A}_3} \text{Des}(u) \cap \{j \in T \mid j \geq 3\} = \{1, 2\} \cap \{4, 5\} = \emptyset,
\]

\[
\bigcup_{u \in \mathcal{A}_4} \text{Des}(u) \cap \{j \in T \mid j \geq 4\} = \{1, 2\} \cap \{4, 5\} = \emptyset,
\]

\[
\bigcup_{u \in \mathcal{A}_5} \text{Des}(u) \cap \{j \in T \mid j \geq 5\} = \{1, 2\} \cap \{5\} = \emptyset.
\]

In the remaining part of this section, we provide a geometric construction of the permutation module decomposition of \( H^2(\text{Hess}(S, h)) \) exhibited in Theorem 6.1. We separate the set \( [n - 1] \) of indices into the following two families: \( i \in T \) or \( i \notin T \).

For \( 1 \leq i \leq n - 1 \), define a subset \( P_i \) of \( \mathfrak{S}_n \) by

\[
P_i := P_{w[i]} = \{u \in \mathfrak{S}_n \mid G_{u,h} = G_{w[i],h}\}.
\]

By Propositions 3.10 and 4.5 we have \( \{u \in \mathfrak{S}_n \mid \ell_h(u) = 1\} = P_1 \sqcup \cdots \sqcup P_{n-1} \), and moreover,

\[
P_i = \{u \in \mathfrak{S}_n \mid \ell_h(u) = 1\} \cap \{u \in \mathfrak{S}_n \mid \text{Des}(u) = \{i\}\}.
\]

**Proposition 5.3.** For \( 1 \leq i \leq n - 1 \), we describe \( P_i \) as follows.

(1) If \( i \in T \), then

\[
P_i = \begin{cases} 
\{w[i]\} & \text{if } i + 1 \in T \cup \{n\}, \\
\{s_{j+1}s_{j+2}\cdots s_{i-1}w[i] \mid i \leq j < n\} & \text{if } i + 1 \notin T \cup \{n\}.
\end{cases}
\]

(2) If \( i \notin T \), then

\[
P_i = \begin{cases} 
\{s_{j}w[i] \mid 0 \leq j < i\} & \text{if } i + 1 \in T \cup \{n\}, \\
\{u \in \mathfrak{S}_n \mid \text{Des}(u) = \{i\}\} & \text{if } i + 1 \notin T \cup \{n\}.
\end{cases}
\]
Proof. Before providing case-by-case analysis, we recall from Proposition 3.8 that for any element \( u \in P_1 \), there exists a sequence of simple reflections \( s_{i_1}, \ldots, s_{i_r} \) such that \( w[i] \xrightarrow{s_{i_1}} \cdots \xrightarrow{s_{i_r}} u \). Accordingly, by Lemma 3.7 to obtain all elements in \( P_1 \), it is enough to consider simple reflections \( s_j \) satisfying that \( u \xrightarrow{s_j} s_j u \) repeatedly for \( u \in P_1 \). We notice that we may assume that \( n > 2 \) because when \( n = 2 \), there is only one Hessenberg function satisfying \( h(i) \geq i + 1 \), which is \( h = (2, 2) \). In this case, the Hessenberg variety is the full flag variety \( \mathcal{F}_l(\mathbb{C}^2) \) and we have \( T = \{ 1 \} \) and \( P_1 = \{ w[i] \} = \{ 21 \} \). This proves the statement when \( n = 2 \).

We provide a proof when \( n > 2 \) using case-by-case analysis and recall the descriptions of \( w[i] \) from Definition 4.3 for each case.

Case 1-1. Suppose that \( i \in T \) and \( i + 1 \in T \cup \{ n \} \). Then, we have

\[
w[i] = 12 \cdots i - 1 i + 1 i i + 2 \cdots n \in \mathcal{G}_n^1.
\]

Indeed, \( w[i] = s_i \). In this case, there is no simple reflection \( s_j \) satisfying \( s_i \xrightarrow{s_j} s_j s_i \). Therefore, \( P_1 = \{ w[i] \} = \{ s_i \} \), proving the claim.

Case 1-2. Suppose that \( i \in T \) and \( i + 1 \notin T \cup \{ n \} \). Then we have

\[
w[i] = 12 \cdots i - 1 n i i + 1 \cdots n - 1 \in \mathcal{G}_n^1.
\]

Since the stabilizer subgroup of \( \sigma_{w[i], h} \) is \( \mathcal{S}_{(n-1,1)} \) by Proposition 4.12, it is enough to consider the action of \( s_{n-1} \) on \( w[i] \).

\[
s_{n-1}w[i] = 12 \cdots i - 1 n i \cdots n - 2 n.
\]

Since \( w[i] \xrightarrow{s_{n-1}} s_{n-1}w[i] \), we obtain \( s_{n-1}w[i] \in P_1 \) by Lemma 3.7.

On the other hand, the class \( \sigma_{s_{n-1}w[i], h} = s_{n-1} \cdot \sigma_{w[i], h} \) has the stabilizer \( s_{n-1}\mathcal{S}_{(n-1,1)}s_{n-1} = \mathcal{S}_{(1,2,\ldots,n-2,n)} \times \mathcal{S}_{(n-1)} \). Therefore, \( s_{n-2} \) is the only element we have to care about and it produces an element \( s_{n-2}s_{n-1}w[i] \in P_1 \) because \( \ell_h(s_{n-2}s_{n-1}w[i]) = 1 \) and \( \text{Des}(s_{n-2}s_{n-1}w[i]) = \{ 1 \} \). Continuing a similar process, we obtain

\[
s_j s_{j+1} \cdots s_{n-1}w[i] \in P_1 \quad \text{for } i < j \leq n.
\]

For the case when \( j = i + 1 \), we obtain

\[
s_{i+1} \cdots s_{n-1}w[i] = 12 \cdots i - 1 i + 1 i \cdots n - 1 n = s_i.
\]

Since there is no simple reflection \( s_j \) satisfying \( s_i \xrightarrow{s_j} s_j s_i \), we obtain

\[
P_1 = \{ s_{j+1} \cdots s_{n-1}w[i] \mid i \leq j < n \},
\]

which proves the claim.

Case 2-1. Suppose that \( i \notin T \) and \( i + 1 \in T \cup \{ n \} \). Then we have

\[
w[i] = 2 \cdots i i + 1 i i + 2 \cdots n \in \mathcal{G}_n^1.
\]
In this case, the stabilizer subgroup of \( \sigma_{w[i], h} \) is \( \mathfrak{S}_{(1,n-1)} \) by Proposition 4.13. Hence, it is enough to consider the action of \( s_1 \):

\[
s_1 w[i] = 1\, 3\, \cdots \, i \, i + 1 \mid 2\, i + 2 \cdots \, n \in P_i.
\]

Continuing a similar process as in Case 1-2, we obtain

\[
s_j s_{j-1} \cdots s_1 w[i] \in P_i \quad \text{for } 0 \leq j < i.
\]

For the case when \( j = i - 1 \), we obtain

\[
s_{i-1} s_{i-2} \cdots s_1 w[i] = 1\, 2\, \cdots \, i - 1 \, i + 1 \mid i \, i + 2 \cdots \, n = s_i.
\]

Since there is no simple reflection \( s_j \) satisfying \( s_i \overset{s_j}{\rightarrow} s_j s_i \), we obtain

\[
P_i = \{ s_j s_{j-1} \cdots s_1 w[i] \mid 0 \leq j < i \},
\]

proving the claim.

Case 2-2. Suppose that \( i \notin T \) and \( i + 1 \notin T \cup \{ n \} \). Because of [5.3], it is enough to show that any element having one descent at \( i \) satisfies \( \ell_h(u) = 1 \). Since the Hessenberg function \( h \) is weakly increasing and \( h(i - 1) = i \), we have \( h(k) \leq i \) for \( k \leq i - 1 \). Accordingly, any \( u \in \mathfrak{S}_n \) such that \( \text{Des}(u) = \{ i \} \) satisfies \( \ell_h(u) = 1 \), proving the claim.

Example 5.4. Let \( h = (2, 3, 5, 6, 6, 6) \). By Definition 4.3 we have

\[
w[1] = 6|12345, \quad w[2] = 56|1234, \quad w[3] = 234|156, \quad w[4] = 1235|46, \quad w[5] = 12346|5.
\]

By Proposition 5.3 we obtain

\[
P_1 = \{ s_{j+1} s_{j+2} \cdots s_5 w[1] \mid 1 \leq j < 6 \} = \{ 6|12345, 5|12346, 4|12356, 3|12456, 2|13456 \},
\]

\[
P_2 = \{ u \in \mathfrak{S}_6 \mid \text{Des}(u) = \{ 2 \} \}
\]

\[
= \{ 13|2456, 14|2356, 15|2346, 16|2345, 23|1456, 24|1356, 25|1346, 26|1345, 34|1256, 35|1246, 36|1245, 45|1236, 46|1235, 56|1234 \},
\]

\[
P_3 = \{ s_j s_{j-1} \cdots s_1 w[3] \mid 0 \leq j < 3 \} = \{ 234|156, 134|256, 124|356 \},
\]

\[
P_4 = \{ w[4] \} = \{ 1235|46 \},
\]

\[
P_5 = \{ w[5] \} = \{ 12346|5 \}.
\]

Proposition 5.3 leads us to compute the cardinality of the set \( \{ u \in \mathfrak{S}_n \mid \ell_h(u) = 1 \} \), which is the same as the dimension of \( H^2(\text{Hess}(S, h)) \). For \( i \in [n - 1] \), we define \( d_i \) to be

\[
d_i = \begin{cases} 
1 & \text{if } i \in T, \\
n & \text{if } i \notin T \text{ and } i + 1 \in T \cup \{ n \}, \\
\binom{n}{i} & \text{if } i \notin T \text{ and } i + 1 \notin T \cup \{ n \}.
\end{cases}
\]
Proposition 5.5. The cardinality of the set \( \{ u \in \mathfrak{S}_n \mid \ell_h(u) = 1 \} \) is \( \sum_{i=1}^{n-1} d_i \), which is the same as the dimension of \( H^2(\text{Hess}(S, h)) \).

Proof. By Proposition 5.3, we know that the cardinality of each \( P_i \) is given by

\[
|P_i| = \begin{cases} 
1 & \text{if } i \in T \text{ and } i + 1 \in T \cup \{n\}, \\
-n+i & \text{if } i \in T \text{ and } i + 1 \notin T \cup \{n\}, \\
i & \text{if } i \notin T \text{ and } i + 1 \in T \cup \{n\}, \\
\binom{n}{i} - 1 & \text{if } i \notin T \text{ and } i + 1 \notin T \cup \{n\}.
\end{cases}
\]  

(5.5)

We consider indices \( i \in T \). We denote by \( T = \{ t_1 < t_2 < \cdots < t_s \} \). Here, \( s = |T| \). Using the elements in \( T \), we consider a partition \( (B_1, \ldots, B_s) \) of the set \( \{ n - 1 \} \), where

\[ B_a = [t_a, t_{a+1}) \quad \text{for } 1 \leq a \leq s. \]

Here, we set \( t_{s+1} = n \). To prove the proposition, it is enough to prove that

\[ \sum_{i \in B_a} d_i = \sum_{i \in B_a} |P_i| \quad \text{for } 1 \leq a \leq s. \]

(5.6)

We first consider the case when \( |B_a| = 1 \), i.e., \( B_a = \{ t_a \} \). In this case, \( t_a \in T \) and \( t_a + 1 = t_{a+1} \) is again contained in \( T \cup \{n\} \). Accordingly, \( d_{t_a} = 1 \) by (5.4) and \( |P_{t_a}| = 1 \) by (5.5), providing (5.6).

Now we suppose that \( |B_a| = x > 1 \), i.e., \( B_a = \{ t_a, t_a+1, \ldots, t_a+x-1 \} \). In this case, \( t_a \in T \) and \( t_a + x = t_{a+1} \in T \cup \{n\} \), whereas \( t_a+1, \ldots, t_a+x-1 \notin T \cup \{n\} \). Considering \( |P_i| \) for \( i \in B_a \), by (5.5), we obtain

\[ |P_i| = \begin{cases} 
n - t_a & \text{if } i = t_a, \\
\binom{n}{i} - 1 & \text{if } t_a < i < t_a + x - 1, \\
t_a + x - 1 & \text{if } i = t_a + x - 1.
\end{cases} \]

This provides

\[
\sum_{i \in B_a} |P_i| = n - t_a + \sum_{t_a < i < t_a + x - 1} \left( \binom{n}{i} - 1 \right) + t_a + x - 1
\]

\[ = n + x - 1 + \sum_{t_a < i < t_a + x - 1} \left( \binom{n}{i} - 1 \right) \]

\[ = 1 + \sum_{t_a < i < t_a + x - 1} \left( \binom{n}{i} \right) + n \]

\[ = \sum_{i \in B_a} d_i, \]

proving (5.6). Hence, we are done. □
6. Geometric construction

In this section, we consider a geometric construction of the permutation module representation of $H^2(\text{Hess}(S, h))$ using the BB basis. Before going on, we recall the result by Chow [7]. Chow showed that $H^2(\text{Hess}(S, h))$ is decomposed as a sum of permutation modules by computing the $e$-expansion of the chromatic quasisymmetric function corresponding to the Hessenberg function $h$, which is equivalent to the following theorem.

**Theorem 6.1 ([7]).** Let $h: [n] \to [n]$ be a Hessenberg function such that $h(i) \geq i + 1$ for all $i < n$ and let $T = T_h$. Then, as $\mathfrak{S}_n$-modules,  

$$H^2(\text{Hess}(S, h)) \cong \bigoplus_{i=1}^{n-1} M^{\alpha^i},$$

where for $1 \leq i \leq n - 1$,

$$\alpha^i = \begin{cases} (n) & \text{if } i \in T, \\ (1, n - 1) & \text{if } i \notin T \text{ and } i + 1 \in T \cup \{n\}, \\ (i, n - i) & \text{if } i \notin T \text{ and } i + 1 \notin T \cup \{n\}. \end{cases}$$

**Example 6.2.** Let $n = 8$ and $h = (2, 3, 6, 6, 7, 8, 8)$. Then, $T = T_h = \{1, 4, 5\}$ and we obtain

$$w^{[1]} = 8|1234567, \quad w^{[2]} = 78|123456, \quad w^{[3]} = 234|15678, \quad w^{[4]} = 1235|4678,$$

$$w^{[5]} = 1234|567, \quad w^{[6]} = 34567|12, \quad w^{[7]} = 234567|1.$$  
Moreover, we have

$$\alpha^1 = (8), \quad \alpha^2 = (2, 6), \quad \alpha^3 = (1, 7), \quad \alpha^4 = (8), \quad \alpha^5 = (8), \quad \alpha^6 = (6, 2), \quad \alpha^7 = (1, 7).$$

We therefore have

$$H^2(\text{Hess}(S, h)) \cong M^{(8)} \oplus M^{(2,6)} \oplus M^{(1,7)} \oplus M^{(8)} \oplus M^{(8)} \oplus M^{(6,2)} \oplus M^{(1,7)}$$

by Theorem 6.1.

We will construct a basis $\{\hat{\sigma}_{w,h} \mid w \in G_h^1\}$ of $H^2(\text{Hess}(S, h))$ generating permutation modules in the right hand side of the formula in Theorem 6.1 by modifying the basis element $\sigma_{w^{[1]},h}$. We are going to modify $\sigma_{w^{[1]},h}$ in such a way that its stabilizer subgroup agrees with the Young subgroup $\mathfrak{S}_{\alpha^1}$, where $\alpha^1$ is the composition defined in Theorem 6.1.

**Definition 6.3.** Let $h: [n] \to [n]$ be a Hessenberg function such that $h(i) \geq i + 1$ for all $i < n$. For $1 \leq i \leq n - 1$, we define $\hat{\sigma}_{i,h}$ as follows.

$$\hat{\sigma}_{i,h} := \begin{cases} \sum_{w \in \mathfrak{S}_n} v \cdot \sigma_{w^{[1]},h} & \text{if } i \in T, \\ \sigma_{w^{[1]},h} & \text{otherwise.} \end{cases}$$
Now we state the main theorem of this section which provides a geometric construction of the permutation module decomposition of $H^2(\text{Hess}(S, h))$ exhibited in Theorem 6.1.

**Theorem 6.4.** For $1 \leq i \leq n - 1$, let $M_{i, h} = \mathbb{C}\mathcal{S}_n(\hat{\sigma}_{i, h})$ be the $\mathcal{S}_n$-module generated by $\hat{\sigma}_{i, h}$. Then we have the following.

1. $M_{i, h} \cong M^a$;
2. $\text{Stab}(\hat{\sigma}_{i, h}) = \mathcal{S}_{n-1}$; and
3. $H^2(\text{Hess}(S, h)) = \bigoplus_{i=1}^{n-1} M_{i, h}$.

Before providing a proof of Theorem 6.4, we prepare terminologies and three lemmas. We consider the sum

$$M := \sum_{i=1}^{n-1} M_{i, h}$$

of the modules which is contained in $H^2(\text{Hess}(S, h))$ by the construction.

**Definition 6.5.** For two elements $\tau_1$ and $\tau_2$ in $H^2(\text{Hess}(S, h))$, we say that they are the same modulo $M$, denoted by

$$\tau_1 \equiv \tau_2 \mod M,$$

if their difference $\tau_1 - \tau_2$ is contained in $M$.

Since $M$ is invariant under the $\mathcal{S}_n$-action, for any $u \in \mathcal{S}_n$, if $\tau_1 \equiv \tau_2 \mod M$, then $u \cdot \tau_1 \equiv u \cdot \tau_2 \mod M$. By Proposition 3.10, we obtain the following lemma.

**Lemma 6.6.** For $i \notin T$ and $u \in P_i$, we have

$$\sigma_{u, h} \in M_{i, h} \subset M.$$  

**Proof.** As is defined in Definition 6.3, the class $\hat{\sigma}_{i, h}$ is $\sigma_{w[i], h}$. On the other hand, by Proposition 3.10, any class $\sigma_{u, h}$ for $u \in P_{w[i]} = P_i$ is contained in the $\mathcal{S}_n$-module $M(\sigma_{i, h}) = \mathbb{C}\mathcal{S}_n(\sigma_{w[i], h})$. This proves the lemma. \hfill $\square$

**Lemma 6.7.** For $i \in [n - 1]$, the class $s_i \cdot \sigma_{s_i, h}$ can be expressed as follows:

$$s_i \cdot \sigma_{s_i, h} = \sigma_{s_i, h} + \sum_{u \in P_{i} \setminus \ell \in T, \ell < i} a_u \sigma_{u, h} + \sum_{u \in P_{i} \setminus \ell \in T} b_u \sigma_{u, h} \quad \text{for some } a_u, b_u \in \mathbb{C}. \quad (6.1)$$

**Proof.** By (4.2), we have

$$s_i \cdot \sigma_{s_i, h} = \sigma_{s_i, h} + \sum_{v \in \mathcal{A}_i} \sigma_{v, h} - \sum_{v \in \mathcal{A}_i} \sigma_{s_i v, h}. \quad (6.2)$$

The classes determined by $\mathcal{A}_i$ and $s_i \mathcal{A}_i$ are all of degree 1, that is,

$$\mathcal{A}_i \sqcup s_i \mathcal{A}_i \subset P_1 \sqcup P_2 \sqcup \cdots \sqcup P_{n-1}.$$
Moreover, by Corollary 4.19, for any \( v \in A_i \), we have \( \text{Des}(v) = \text{Des}(s_i v) \) and hence \( v \) and \( s_i v \) are in the same \( P_\ell \). Accordingly, the second and third sum in the equation (6.2) can be expressed as follows:

\[
\sum_{v \in A_i} \sigma_{v,h} - \sum_{v \in A_i} \sigma_{s_i v,h} = \sum_{v \in A_i} (\sigma_{v,h} - \sigma_{s_i v,h}) \\
= \sum_{\ell=1}^{n-1} \sum_{v \in A_i \cap P_\ell} (\sigma_{v,h} - \sigma_{s_i v,h}).
\]

By applying Lemma 5.1 and (5.3), we get \( A_i \cap P_\ell = \emptyset \) if \( \ell \in T \) and \( \ell \geq i \), so the result follows.

We let \( \tau_i \) be the sum of the second and third terms in (6.1) for later use. To provide the last lemma, we prepare terminologies. Suppose that \( i \in T = \{ t_1 < t_2 < \cdots < t_k \} \) and \( i + 1 \notin T \cup \{ n \} \). In this case, \( P_i = \{ s_{j+1}s_{j+2} \cdots s_{n-1}w[i] \mid i \leq j < n \} \) by Proposition 5.3(1). Here, we notice that the one-line notation of an element in \( P_i \) is given as follows:

\[
s_{j+1}s_{j+2}\cdots s_{n-1}w[i] = 12 \cdots i - 1 \ j + 1 \ i \ i + 1 \ \cdots \ j \ j + 2 \cdots n \ \text{ for } i \leq j < n.
\]

We denote by

\[
\rho_j := \sigma_{s_{j+1}s_{j+2}\cdots s_{n-1}w[i],h} \ \text{ for } i \leq j < n.
\]

By the proof of Proposition 5.3, the classes \( \rho_j \) satisfy the relations

\[
s_j \cdot \rho_j = \rho_{j-1} \ \text{ for } i < j < n,
\]

\[
s_k \cdot \rho_j = \rho_j \ \text{ for } i \leq j < n \text{ and } k \neq j, j + 1.
\]

**Lemma 6.8.** Suppose that \( i \in T \) and \( i + 1 \notin T \cup \{ n \} \). For \( j \) satisfying \( i = t_a < j < t_{a+1} \), we get

\[
s_j \cdot \sigma_{s_j,h} = \sigma_{s_j,h} + \sum_{\substack{v \in P_i \ \\ \ell \notin T}} a'_v \sigma_{v,h} + (\rho_j - \rho_{j-1}) + \sum_{\substack{v \in P_i \ \\ \ell \in T, \ell < i}} b'_v \sigma_{v,h} \ \text{ for some } a'_v, b'_v \in \mathbb{C}.
\]

**Proof.** Using the formula (6.1) in Lemma 6.7 we have

\[
s_j \cdot \sigma_{s_j,h} = \sigma_{s_j,h} + \sum_{\substack{v \in P_i \ \\ \ell \notin T}} a'_v \sigma_{v,h} + \sum_{\substack{v \in P_i \ \\ \ell \in T, \ell < j}} b'_v \sigma_{v,h}
\]

\[
= \sigma_{s_j,h} + \sum_{\substack{v \in P_i \ \\ \ell \notin T}} a'_v \sigma_{v,h} + \sum_{\substack{v \in P_i \ \\ \ell \in T, \ell < j}} b'_v \sigma_{v,h} + \sum_{\substack{v \in P_i \ \\ \ell \in T, \ell \leq i}} b'_v \sigma_{v,h}
\]

\[
= \sigma_{s_j,h} + \sum_{\substack{v \in P_i \ \\ \ell \notin T}} a'_v \sigma_{v,h} + (\rho_j - \rho_{j-1}) + \sum_{\substack{v \in P_i \ \\ \ell \in T, \ell < i}} b'_v \sigma_{v,h}
\]
for some $a'_i, b'_i \in \mathbb{C}$. Here, the third equality comes from the following: For $i = t_a < j < t_{a+1}$, we have

\begin{equation}
A_j \cap P_i = \{s_{j+1}s_{j+2}\ldots s_{n-1}w[i]\}
\end{equation}

because of Proposition 6.7 (6.3), and moreover, $s_j \cdot \rho_j = \rho_{j-1}$ (see (6.5)). This proves the lemma.

\[ \tag{6.7} \]

Proof of Theorem 6.4. Let $M = \sum_{i=1}^{n-1} M_{i,h}$ be the sum of the modules which is contained in $H^2(\text{Hess}(S,h))$ by the construction. We first notice that for each $1 \leq i \leq n - 1$, we have

\[ \text{dim}_\mathbb{C} M_{i,h} \leq \frac{|\mathcal{S}_n|}{|\text{Stab}(\tilde{\sigma}_{i,h})|} \leq \text{dim}_\mathbb{C} M_{\alpha^i} \]

by the orbit-stabilizer theorem and Proposition 4.13. Hence, we obtain

\[ \text{dim}_\mathbb{C} M \leq \sum_{i=1}^{n-1} \text{dim}_\mathbb{C} M_{i,h} \leq \sum_{i=1}^{n-1} \text{dim}_\mathbb{C} M_{\alpha^i} = \text{dim}_\mathbb{C} H^2(\text{Hess}(S,h)). \]

Here, the last equality comes from Proposition 5.5. Accordingly, if we prove $M = H^2(\text{Hess}(S,h))$, then we get

1. $M_{i,h} \cong M_{\alpha^i}$;
2. $\text{Stab}(\tilde{\sigma}_{i,h}) = \mathcal{S}_{\alpha^i}$; and
3. $M$ is a direct sum of $M_{i,h}$, that is, $M = \bigoplus_{i=1}^{n-1} M_{i,h} = H^2(\text{Hess}(S,h))$,

proving the theorem.

Note that

\[ H^2(\text{Hess}(S,h)) = \text{span}_\mathbb{C}\{\sigma_{u,h} \mid \ell_h(u) = 1\} = \text{span}_\mathbb{C}\{\sigma_{u,h} \mid u \in P_1 \sqcup P_2 \sqcup \cdots \sqcup P_{n-1}\}. \]

Claim: For any $i \in [n - 1]$ and $u \in P_i$, we have $\sigma_{u,h} \in M$.

If the claim holds, then we get the desired result $H^2(\text{Hess}(S,h)) \subset M$. We provide a proof using case-by-case analysis.

Case 1. First of all, if $i \notin T$, then

\[ \sigma_{u,h} \in M_{i,h} \subset M \quad \text{for } u \in P_i \]

by Lemma 6.6. This proves the claim for $i \notin T$.

We note that since $\sigma_{s_i,h} \in M_{i,h}$, we have

\begin{equation}
\tag{6.8}
s_i \cdot \sigma_{s_i,h} \in M_{i,h} \subset M \quad \text{for } i \notin T.
\end{equation}

Case 2. We now consider indices $i \in T$. We denote by $T = \{t_1 < t_2 < \cdots < t_s\}$. Here, $s = |T|$. We will prove the claim using an induction argument on $1 \leq a \leq s$.

We first consider the case when $a = 1$, that is, $i = t_1 = 1$ and consider two cases separately:

\[ 2 \in T \cup \{n\}; \] or \[ 2 \notin T \cup \{n\}. \]
Case 2-1. Suppose that $i = 1 \in T$ and $i+1 \in T \cup \{n\}$. In this case, $P_i = \{w[i]\}$, and moreover, $w[i] = s_i$ by Proposition 5.3(1). The class $\hat{\sigma}_{i,h}$ is defined by

$$\hat{\sigma}_{i,h} = \sum_{v \in \mathfrak{S}_n} v \cdot \sigma_{s_i,h} \in M.$$  

By Proposition 4.12, the stabilizer subgroup of $\sigma_{s_i,h}$ is $\mathfrak{S}_n$; or it is a proper subgroup containing $\mathfrak{S}_{(i,n-i)}$. If the stabilizer subgroup is $\mathfrak{S}_n$, then $\hat{\sigma}_{i,h} = n! \sigma_{s_i,h} \in M$, proving that the class $\sigma_{s_i,h}$, the only element in $P_i$, is contained in $M$.

On the other hand, suppose that the stabilizer subgroup of $\sigma_{s_i,h}$ is not the whole group $\mathfrak{S}_n$ but contains $\mathfrak{S}_{(i,n-i)}$. In this case, the third term on the right-hand side of (6.11) does not exist. Moreover, because of Lemma 6.6, the second term on the right-hand side of (6.1) is contained in $M$, so $\tau_i \in M$. This proves

$$s_i \cdot \sigma_{s_i,h} \equiv \sigma_{s_i,h} \mod M.$$  

Moreover, because the stabilizer group of $\sigma_{s_i,h}$ contains $\mathfrak{S}_{(i,n-i)}$, we have

$$s_j \cdot \sigma_{s_i,h} = \sigma_{s_i,h} \quad \text{for } j \neq i.$$  

Considering equations (6.9) and (6.10), for any $v \in \mathfrak{S}_n$, we have $v \cdot \sigma_{s_i,h} \equiv \sigma_{s_i,h} \mod M$. Accordingly, we obtain

$$\hat{\sigma}_{i,h} = \sum_{v \in \mathfrak{S}_n} v \cdot \sigma_{s_i,h} \equiv \sum_{v \in \mathfrak{S}_n} \sigma_{s_i,h} = n! \sigma_{s_i,h} \mod M.$$  

Since $\hat{\sigma}_{i,h} \in M$, the class $\sigma_{s_i,h}$, the only element in $P_i$, is contained in $M$. This proves the claim when $i = 1 \in T$ and $i+1 \in T \cup \{n\}$.

Case 2-2. Suppose that $i = 1 \in T$ and $i+1 \notin T \cup \{n\}$. By Proposition 4.12(2), we have

$$\hat{\sigma}_{i,h} = \sum_{v \in \mathfrak{S}_n} v \cdot \sigma_{w[i],h}$$

$$= (n-1)! (\sigma_{w[i],h} + s_{n-1} \cdot \sigma_{w[i],h} + (s_{n-2}s_{n-1}) \cdot \sigma_{w[i],h} + \cdots + (s_1 \cdots s_{n-1}) \cdot \sigma_{w[i],h})$$

$$= (n-1)! (\rho_{n-1} + \rho_{n-2} + \cdots + \rho_i + s_i \cdot \rho_i + (s_i s_i) \cdot \rho_i + \cdots + (s_1 \cdots s_i) \cdot \rho_i) \in M.$$  

Here, $\rho_j$’s are the classes defined in (6.3), and the last equality comes from (6.5). Accordingly, since $\hat{\sigma}_{i,h} \in M$, the following class $\hat{\sigma}_{i,h}'$ is contained in $M$.

$$\hat{\sigma}_{i,h}' := \rho_{n-1} + \rho_{n-2} + \cdots + \rho_{i+1} + \rho_i + s_i \cdot \rho_i + (s_i s_i) \cdot \rho_i + \cdots + (s_1 \cdots s_i) \cdot \rho_i \in M.$$  

Since $i = 1$, the third term on the right-hand side of (6.11) does not exist. Moreover, because of Lemma 6.6, the second term on the right-hand side of (6.1) is contained in $M$, so $\tau_i \in M$. This provides the modular equality (6.9), that is, $s_i \cdot \sigma_{s_i,h} \equiv \sigma_{s_i,h} \mod M$, and

$$(s_k s_{k+1} \cdots s_{i-1} s_i) \cdot \rho_i = (s_k s_{k+1} \cdots s_{i-1} s_i) \cdot \sigma_{s_i,h}$$

$$\equiv (s_k s_{k+1} \cdots s_{i-1}) \cdot \sigma_{s_i,h} \mod M.$$
for 1 \leq k \leq i - 1. Accordingly, we obtain
\begin{equation}
\hat{s}_{i,h} = \rho_{n-1} + \rho_{n-2} + \cdots + \rho_{i+1} + (i+1) \rho_i \mod M,
\end{equation}
and moreover, by (6.11), we get
\begin{equation}
\rho_{n-1} + \rho_{n-2} + \cdots + \rho_{i+1} + (i+1) \rho_i \in M.
\end{equation}
On the other hand, for 1 = t_a < j < t_2 - 1, by Lemma [6.8] we get
\begin{equation}
s_j \cdot \sigma_{s_j,h} = \sigma_{s_j,h} + \sum_{v \in P_j, \ell \notin T} a'_v \sigma_{v,h} + (\rho_j - \rho_{j-1}) + \sum_{v \in P_j, \ell \in T, \ell < i} b'_v \sigma_{v,h} \quad \text{for some } a'_v, b'_v \in \mathbb{C}
\end{equation}
\begin{equation}
\equiv \rho_j - \rho_{j-1} \mod M.
\end{equation}
Here, the modular equality holds because the fourth term does not exist (since i = 1) and the second term and \sigma_{s_j,h} are contained in M because of Lemma [6.6]. Since j \notin T and s_j \cdot \sigma_{s_j,h} \in M as seen in (6.8), we obtain
\begin{equation}
\rho_j - \rho_{j-1} \in M \quad \text{for } t_a < j < t_{a+1} - 1.
\end{equation}
For the index j = t_{a+1} - 1, the module M_{j,h} is spanned by
\begin{equation}
\left\{\sigma_{u,h} \mid u \in P_j\right\} \cup \left\{s_j \cdot \sigma_{s_j,h}, (s_{j-1}s_j) \cdot \sigma_{s_{j-1},h}, \ldots, (s_1 \cdot s_j) \cdot \sigma_{s_1,h}\right\}.
\end{equation}
A similar computation in (6.14) leads to
\begin{equation}
s_j \cdot \sigma_{s_j,h} \equiv \rho_j - \rho_{j-1} \mod M,
\end{equation}
and, by (6.5), we have the following.

\begin{align*}
(s_k s_{k+1} \cdots s_{j+2}s_j s_{j+1}s_j) \cdot \sigma_{s_j,h} &= (s_k s_{k+1} \cdots s_{j+2}s_j s_{j+1}) \cdot (s_j \cdot \sigma_{s_j,h}) \\
&\equiv (s_k s_{k+1} \cdots s_{j+2}s_j s_{j+1}) \cdot (\rho_j - \rho_{j-1}) \mod M \\
&\equiv (s_k s_{k+1} \cdots s_{j+2}) \cdot (\rho_{j+1} - \rho_{j-1}) \mod M \\
&\equiv \rho_k - \rho_{j-1} \mod M
\end{align*}
fors \leq k \leq n - 1. Accordingly, we obtain
\begin{equation}
\rho_k - \rho_{t_{a+1}-2} \in M \quad \text{for } t_{a+1} - 1 \leq k \leq n - 1.
\end{equation}
Combining (6.13), (6.15), and (6.16), we obtain \rho_j \in M for i \leq j < n, which proves the desired claim when i = 1 \in T and i + 1 \notin T \cup \{n\}.

**Case 2-1** and **Case 2-2** cover the initial case a = 1 (that is, i = t_1 = 1 \in T) of the induction argument. Now we suppose that 1 \neq i = t_a \in T, and moreover, the claim holds for the smaller indices than a. Under this setting, we prove the claim for such a, i.e., when i = t_a \neq 1.
For such an index \( i = t_a \in T \), because of Lemma \([6.6]\) and the induction hypothesis, the second and third terms on the right-hand side of (6.1) are already contained in \( M \). This provides \( \tau_i \in M \) and the modular equality \([6.9]\).

If \( i + 1 \in T \cup \{n\} \), then by the same argument we used in Case 2-1, the claim holds. Otherwise, by applying the same argument we used in Case 2-2, the claim holds.

This completes the proof of the theorem. \( \square \)

For \( i \notin T \), we have \( \hat{\sigma}_{t,h} = \sigma_{w[i],h} \). Accordingly, for this case, we can restate Theorem \([6.4]2\) as follows.

**Corollary 6.9.** Let \( h : [n] \to [n] \) be a Hessenberg function. Suppose that \( i \notin T = T_h \) and \( i + 1 \notin T \cup \{n\} \). Then, \( \text{Stab}(\sigma_{w[i],h}) = \mathfrak{S}_{(n-i,i)} \).

**Remark 6.10.** By Corollary \([6.9]\) also by Propositions \([4.12]\) and \([4.13]\) we explicitly compute the stabilizer \( \text{Stab}(\sigma_{w[i],h}) \) except one case \( i \in T \), \( i + 1 \in T \cup \{n\} \), \( 2i = n \), and \( T \subset [n-1] \).

Moreover, from Definition \([6.3]\) the class \( \hat{\sigma}_{i,h} = \sigma_{w[i],h} \) if \( i \notin T \). Therefore, by Theorem \([6.4]1\), the module \( M(\sigma_{w[i],h}) = M_{i,h} \) becomes a permutation module. This provides an affirmative answer to Question \([3.12]\) when \( w \in \mathcal{S}_h^1 \).

**Example 6.11.** Let \( h = (2, 3, 5, 6, 6, 6) \). Following the notation in Theorem \([6.1]\) we obtain

\[
\alpha^1 = (6), \quad \alpha^2 = (2, 4), \quad \alpha^3 = (1, 5), \quad \alpha^4 = (6), \quad \alpha^5 = (6).
\]

Since \( T = \{1, 4, 5\} \), we have \( \hat{\sigma}_{i,h} = \sigma_{w[i],h} \) for \( i = 2 \) or \( 3 \), and

\[
M_{i,h} = \text{span}_C(\{\sigma_{u,h} \mid u \in P_1\} \cup \{s_i \cdot \sigma_{s_{i,h}}\}).
\]

Accordingly, \( \sigma_{u,h} \in M = \sum_{i=1}^5 M_{i,h} \) for any \( u \in P_2 \cup P_3 \) and \( s_i \cdot \sigma_{s_{i,h}} \in M \) for \( i = 2, 3 \).

We demonstrate that for \( u \in P_1 \), the class \( \sigma_{u,h} \) is contained in \( M \) in this example. The element \( \hat{\sigma}_{1,h} \) becomes

\[
\hat{\sigma}_{1,h} = \sigma_{612345,h} + \sigma_{512346,h} + \sigma_{412356,h} + \sigma_{312456,h} + \sigma_{213456,h} + s_1 \cdot \sigma_{s_{1,h}}.
\]

Moreover, using Example \([4.8]\) and Corollary \([4.9]\) we obtain

\[
s_1 \cdot \sigma_{s_{1,h}} = \sigma_{s_{1,h}} + \sigma_{23|1456,h} + \sigma_{24|1356,h} + \sigma_{25|1346,h} + \sigma_{26|1345,h} + \sigma_{324|156,h} - (\sigma_{13|2456,h} + \sigma_{14|2356,h} + \sigma_{15|2346,h} + \sigma_{16|2345,h} + \sigma_{134|256,h})
\]

\[
\equiv \sigma_{s_{1,h}} \mod M.
\]

Here, we draw a box for presenting elements in \( M_{2,h} \cup M_{3,h} \subset M \). Hence, we get

\[
(6.17) \quad M \ni \hat{\sigma}_{1,h} = \sigma_{612345,h} + \sigma_{512346,h} + \sigma_{412356,h} + \sigma_{312456,h} + 2\sigma_{213456,h} \mod M.
\]

On the other hand, we get

\[
(6.18) \quad M \ni s_2 \cdot \sigma_{s_{2,h}} = \sigma_{3|12456,h} - \sigma_{2|13456,h} \mod M,
\]
and

\[ M \equiv s_3 \cdot \sigma_{s_3, h} = \sigma_{4, 12356, h} - \sigma_{3, 12456, h} \mod M, \]

\( (6.19) \)

\[ M \equiv (s_4 s_3) \cdot \sigma_{s_3, h} = \sigma_{5, 12346, h} - \sigma_{3, 12456, h} \mod M, \]

\[ M \equiv (s_5 s_4 s_3) \cdot \sigma_{s_3, h} = \sigma_{6, 12345, h} - \sigma_{3, 12456, h} \mod M. \]

Accordingly, by equations (6.18) and (6.19), we obtain

\[ \sigma_{3, 12456, h} - \sigma_{2, 12456, h} \in M, \]

\[ \sigma_{4, 12356, h} - \sigma_{3, 12456, h} \in M, \]

\[ \sigma_{5, 12346, h} - \sigma_{3, 12456, h} \in M, \]

\[ \sigma_{6, 12345, h} - \sigma_{3, 12456, h} \in M. \]

Because of (6.17), for any \( u \in P_1 \), we have \( \sigma_u, h \in M \).

7. Concluding remarks

Let \( h: [n] \to [n] \) be a Hessenberg function. Recall that we assign an acyclic orientation \( o_h(w) \) of \( G_h \) to each \( w \in G_h \) in Definition 3.4. In fact, the orientation \( o_h(w) \) we assign to \( w \in G_h \) is opposite to the usual one. More precisely, to be compatible with the previous work in the case when \( \text{Hess}(S, h) \) is a permutohedral variety or when the cohomology has degree two, we will consider the source set instead of the sink set.

Lemma 7.1. For \( w \in G_h \), write the source set of \( o_h(w) \) as \( \{s_1, \ldots, s_{\ell+1}\} \), where \( s_1 > \cdots > s_{\ell+1} \). Then \( w(s_1) < \cdots < w(s_{\ell+1}) \).

Proof. Let \( w \in G_h \). Suppose that \( s \) and \( s' \) are sources of \( o_h(w) \) with \( w(s) < w(s') \). We define a sequence \( j_1, \ldots, j_r \) of indices by

\[ j_x = w^{-1}(w(s) + x - 1) \quad \text{for} \quad 1 \leq x \leq w(s') - w(s) + 1 =: r. \]

In fact, we obtain \( w(j_1) = w(s), w(j_2) = w(s) + 1, \ldots, w(j_r) = w(s) + (w(s') - w(s) + 1) - 1 = w(s') \). By the definition of \( G_h \),

\[ j_x \to j_{x+1} \text{ is an edge in } G_{w, h} \quad \text{if} \quad j_{x+1} > j_x. \]

Using this property, we will show that \( j_x > s' \) for any \( 1 \leq x \leq r - 1 \) varying \( x \) from \( r - 1 \) to \( 1 \) inductively.

For \( x = r - 1 \), since \( s' = j_r \) is a source and \( w(j_{r-1}) < w(j_r) \), there is no edge in \( G_h \) connecting \( j_{r-1} \) and \( s' \). In particular, by (7.1), we have \( j_{r-1} > s' \). If \( j_{x+1} > j_x \) and \( j_x < s' \) for some \( 1 \leq x < r - 1 \), then by (7.1), \( j_x \) and \( j_{x+1} \) is connected by an edge in \( G_h \), and so is \( j_x \) and \( s' \) because of \( j_x < s' < j_{x+1} \), a contradiction. Therefore, \( j_x > s' \) for any \( 1 \leq x \leq r - 1 \).

Consequently, we have \( s > s' \).

Definition 7.2. For \( w \in G_h \), we define an element \( \tilde{\sigma}_{w, h} \) in \( M(\sigma_{w, h}) \) as follows.
PERMUTATION MODULE DECOMPOSITION OF $H^2(\text{Hess}(S, h))$

Let $\{s_1 > \cdots > s_{\ell+1}\}$ be the source set of the acyclic orientation $o_h(w)$ associated with $w$. Define $K_i := \{w(s_i), w(s_i) + 1, \ldots, w(s_{i+1}) - 1\}$ for $1 \leq i \leq \ell$ and $K_{\ell+1} := \{w(s_{\ell+1}), \ldots, n\}$. Denote by $\mathfrak{S}_{K_{\ell+1}}$ the Young subgroup $\mathfrak{S}_{K_1} \times \cdots \times \mathfrak{S}_{K_{\ell+1}}$.

(2) Define an element $\tilde{\sigma}_{w,h}$ in $M(\sigma_{w,h})$ by

$$\tilde{\sigma}_{w,h} := \sum_{u \in \mathfrak{S}_{K_{w,h}}} u \cdot \sigma_{w,h}.$$ 

Conjecture 7.3. For any $w \in G_h$, the $\mathfrak{S}_n$-module $M(\tilde{\sigma}_{w,h})$ generated by $\tilde{\sigma}_{w,h}$ is a permutation module and

$$H^2k(\text{Hess}(S, h)) = \bigoplus_{w \in G^k_h} M(\tilde{\sigma}_{w,h}).$$

We remark that $\tilde{\sigma}_{w,h}$ in Definition 7.2 is a constant multiple of the class we considered when $\text{Hess}(S, h)$ is a permutohedral variety (see [5]) or when $k = 1$ (see Definition 6.3).

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