Using a nonlinear electrodynamics coupled to teleparallel theory of gravity, three regular charged spherically symmetric solutions are obtained. The nonlinear theory reduces to the Maxwell one in the weak limit and the solutions correspond to charged spacetimes. The third solution contains an arbitrary function from which we can generate the other two solutions. The metric associated with these spacetimes is the same, i.e., a regular charged static spherically symmetric black hole. In calculating the energy content of the third solution using the gravitational energy-momentum given by Møller, within the framework of the teleparallel geometry, we find that the resulting form depends on the arbitrary function. Using the regularized expression of the gravitational energy-momentum we get the value of energy.
1. Introduction

Energy-momentum, angular momentum and electric charge play central roles in modern physics. The conservation of the first two is related to the homogeneity and isotropy of spacetime respectively while charge conservation is related to the invariance of the action integral under internal U(1) transformations. Local quantities such as energy-momentum, angular momentum and charge densities are well defined if gravitational fields are not present in the system. However, in general relativity theory a well-behaved energy-momentum and angular momentum densities have not yet been defined, although total energy-momentum and total angular momentum can be defined for an asymptotically flat spacetime surrounding an isolated finite system. The equality of the gravitational mass and the inertial mass holds within the framework of general relativity [1, 2, 3]. However, such equality is not satisfied for the Schwarzschild metric when it is expressed in a certain coordinate system [4].

At present, teleparallel theory seems to be popular again. There is a trend of analyzing the basic solutions of general relativity with teleparallel theory and comparing the results. It is considered as an essential part of generalized non-Riemannian theories such as the Poincaré gauge theory [5]∼[11] or metric-affine gravity [12]. Physics relevant to geometry may be related to teleparallel description of gravity [13, 14]. Teleparallel approach is used for positive-gravitational-energy proof [15]. A relation between spinor Lagrangian and teleparallel theory is established [16]. Leclerc [17] has shown that the teleparallel equivalent of general relativity (TEGR) is not consistent in presence of minimally coupled spinning matter. Mielke [18] demonstrated the consistency of the coupling of the Dirac fields to the TEGR. However, Obukhov and Pereira [19] have shown that this demonstration is not correct. They also [20] have studied the general teleparallel gravity model within the framework of the metric affine gravity theory.

For a satisfactory description of the total energy of an isolated system it is necessary that the energy-density of the gravitational field is given in terms of first- and/or second-order derivatives of the gravitational field variables. It is well-known that there exists no covariant, nontrivial expression constructed out of the metric tensor. However, covariant expressions that contain a quadratic form of first-order derivatives of the tetrad field are feasible. Thus it is legitimate to conjecture that the difficulties regarding the problem of defining the gravitational energy-momentum are related to the geometrical description of the gravitational field rather than are an intrinsic drawback of the theory [21, 22]. Møller has shown that the problem of energy-momentum complex has no solution in the framework of gravitational field theories based on Riemannian spacetime [23]. In a series of papers, [23]∼[26] he was able to obtain a general expression for a satisfactory energy-momentum complex in the teleparallel spacetime.

In the last years an increasing revival of nonlinear electrodynamics (NLE) theories is observed [27]. A nonlinear electrodynamics was first proposed by Born and Infeld [28] at the 30’s in order to obtain a finite-energy-electron model; they succeeded in determining an electron of finite radius. After these first achievements were carried out, as Plebánski mentioned in 1970 at the introduction of his monograph [29]: If in recent times the interest in NLE cannot be said to be very popular, it is not due to the fact that one could rise some serious objections against this theory. It is simply rather difficult in its mathematical formulation. What causes that it is very unlikely to derive some concrete results in closed form. It is the aim of the present paper to find asymptotically flat solutions with spherical symmetry which is different from Schwarzschild solution in the teleparallel theory of gravity [26].
geometry in nonlinear electrodynamics. Applying this philosophy, we obtain three different 
exact analytic solutions. The singularities of these solutions are studied. We also using the 
superpotential of Mikhail et al. [30] to calculate the energy.

In §2, we briefly review the teleparallel theory of gravitation coupled to nonlinear electrodynamics. The tetrad field with three unknown functions of the radial coordinate in spherical polar coordinates is applied to the field equations and three different exact asymptotically flat solutions are obtained also in §3. The energy of the gravitating source is calculated using the superpotential method in §4. In §5 the energy recalculated using the regularized expression of the gravitational energy-momentum. Discussion and conclusion of the obtained results are given in §6 *.

2. The tetrad theory of gravitation

In a spacetime with absolute parallelism the parallel vector fields $e_i^{\mu*}$ define the nonsymmetric affine connection

$$\Gamma^\lambda_{\mu\nu} \overset{\text{def.}}{=} e_i^\lambda e_i^{\mu, \nu},$$

(1)

where $e_i^{\mu, \nu} = \partial_{\nu} e_i^{\mu}$. The curvature tensor defined by $\Gamma^\lambda_{\mu\nu}$ is identically vanishing, however.

Møller’s constructed a gravitational theory based on this spacetime. In this theory the field variables are the 16 tetrad components $e_i^{\mu*}$, from which the metric tensor is defined by

$$g^{\mu\nu} \overset{\text{def.}}{=} \eta^{ij} e_i^{\mu} e_j^{\nu},$$

(2)

where $\eta^{ij}$ is the Minkowski metric $\eta_{ij} = \text{diag}(+1, -1, -1, -1)$.

We note that, associated with any tetrad field $e_i^{\mu*}$ there is a metric field defined uniquely by (2), while a given metric $g^{\mu\nu}$ does not determine the tetrad field completely; for any local Lorentz transformation of the tetrads $e_i^{\mu*}$ leads to a new set of tetrads which also satisfy (2). The Lagrangian $L$ is an invariant constructed from $\gamma_{\mu\nu\rho}$ and $g^{\mu\nu}$, where $\gamma_{\mu\nu\rho}$ is the contorsion tensor given by

$$\gamma_{\mu\nu\rho} \overset{\text{def.}}{=} e_i^{\mu} e_i^{\nu; \rho},$$

(3)

where the semicolon denotes covariant differentiation with respect to Christoffel symbols. The most general Lagrangian density invariant under the parity operation is given by the form [31]

$$\mathcal{L} \overset{\text{def.}}{=} \sqrt{-g} \left( \alpha_1 \Phi^\mu \Phi^\mu + \alpha_2 \gamma_{\mu\nu\rho} \gamma_{\mu\nu\rho} + \alpha_3 \gamma^{\mu\nu\rho} \gamma_{\rho\nu\mu} \right),$$

(4)

where

$$g \overset{\text{def.}}{=} \det(g_{\mu\nu}),$$

(5)

and $\Phi^\mu$ is the basic vector field defined by

$$\Phi^\mu \overset{\text{def.}}{=} \gamma^\rho_{\mu\rho}.$$

*Computer algebra system Maple 6 is used in some calculations.

*In this paper Latin indices ($i, j, ...$) represent the vector number, and Greek indices ($\mu, \nu, ...$) represent the vector components. All indices run from 0 to 3. The spatial part of Latin indices are denoted by ($a, b, ...$),
Here $\alpha_1, \alpha_2,$ and $\alpha_3$ are constants determined by Møller such that the theory coincides with general relativity in the weak fields:

$$
\alpha_1 = -\frac{1}{\kappa}, \quad \alpha_2 = \lambda \frac{1}{\kappa}, \quad \alpha_3 = \frac{1}{\kappa} (1 - 2\lambda),
$$

(7)

where $\kappa$ is the Einstein constant and $\lambda$ is a free dimensionless parameter. The same choice of the parameters was also obtained by Hayashi and Nakano [32].

The NLE Lagrangian has the form [27]

$$
\mathcal{H}_{NLE} \overset{\text{def}}{=} -\frac{1}{4} P_{\mu\nu} P^{\mu\nu},
$$

(8)

with

$$
P_{\mu\nu} = \mathcal{L}(\mathcal{F})_F F_{\mu\nu}, \quad \mathcal{L}(\mathcal{F})_F = \frac{\partial \mathcal{L}(\mathcal{F})}{\partial \mathcal{F}}, \quad \mathcal{L}(\mathcal{F}) = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}
$$

and $F_{\mu\nu}$ being given by

$$
F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}, \text{ where } A_{\mu} \text{ is the vector potential.} \quad (9)
$$

The gravitational and NLE field equations for the system described by $\mathcal{L}_G + \mathcal{H}_{NLE}$ have the following [27, 31]:

$$
G_{\mu\nu} + H_{\mu\nu} = -\kappa T_{\mu\nu},
$$

$$
K_{\mu\nu} = 0,
$$

$$
\partial_{\nu} \left( \sqrt{-g} P^{\mu\nu} \right) = 0,
$$

(10)

where $G_{\mu\nu}$ is the Einstein tensor, $H_{\mu\nu}$ and $K_{\mu\nu}$ are defined by

$$
H_{\mu\nu} \overset{\text{def}}{=} \lambda \left[ \gamma_{\rho\sigma\mu} \gamma^{\rho\sigma\nu} + \gamma_{\rho\sigma\mu} \gamma^{\nu\rho\sigma} + \gamma_{\rho\sigma\nu} \gamma^{\mu\rho\sigma} + g_{\mu\nu} \left( \gamma_{\rho\sigma\tau} \gamma^{\tau\rho\sigma} - \frac{1}{2} \gamma_{\rho\sigma\tau} \gamma^{\rho\sigma\tau} \right) \right],
$$

(11)

and

$$
K_{\mu\nu} \overset{\text{def}}{=} \lambda \left[ \Phi_{\mu,\nu} - \Phi_{\nu,\mu} - \Phi_{\rho} \left( \gamma^{\rho}_{\mu\nu} - \gamma^{\rho}_{\nu\mu} \right) + \gamma_{\mu\nu} \gamma_{,\rho} \right],
$$

(12)

and they are symmetric and skew symmetric tensors, respectively.

Møller assumed that the energy-momentum tensor of matter fields is symmetric. In the Hayashi-Nakano theory, however, the energy-momentum tensor of spin-1/2 fundamental particles has non-vanishing antisymmetric part arising from the effects due to intrinsic spin, and the right-hand side of antisymmetric field equation (10) does not vanish when we take into account the possible effects of intrinsic spin.

It can be shown [33] that the tensors, $H_{\mu\nu}$ and $K_{\mu\nu}$, consist of only those terms which are linear or quadratic in the axial-vector part of the torsion tensor, $a_{\mu}$, defined by

$$
a_{\mu} \overset{\text{def}}{=} \frac{1}{3} \epsilon_{\mu\nu\rho\sigma} \gamma^{\nu\rho\sigma}, \quad \text{where} \quad \epsilon_{\mu\nu\rho\sigma} \overset{\text{def}}{=} \sqrt{-g} \delta_{\mu\nu\rho\sigma},
$$

(13)

where $\delta_{\mu\nu\rho\sigma}$ being completely antisymmetric and normalized as $\delta_{0123} = -1$. Therefore, both $H_{\mu\nu}$ and $F_{\mu\nu}$ vanish if the $a_{\mu}$ is vanishing. In other words, when the $a_{\mu}$ is found to vanish from the antisymmetric part of the field equations, (10), the symmetric part of Eq. (10)
coincides with the Einstein field equation in teleparallel equivalent of general relativity. The energy-momentum tensor $T^{\mu\nu}$ is defined by

$$T^{\mu\nu} \overset{\text{def}}{=} 2 \left( \mathcal{H}_P P^\mu \lambda P^\nu \lambda - \delta^\mu\nu \left[ 2 \mathcal{P} \mathcal{H}_P - \mathcal{H} \right] \right), \quad \text{where} \quad P \overset{\text{def}}{=} (1/4) (P_\mu P^\mu). \quad (14)$$

3. Spherically Symmetric Solutions

Let us begin with the tetrad having a spherical symmetry [34]

$$\left( e_i^{\mu} \right) = \begin{pmatrix}
A & D r & 0 & 0 \\
0 & B \sin \theta \cos \phi & \frac{B}{r} \cos \theta \cos \phi & -\frac{B \sin \phi}{r \sin \theta} \\
0 & B \sin \theta \sin \phi & \frac{B}{r} \cos \theta \sin \phi & \frac{B \cos \phi}{r \sin \theta} \\
0 & B \cos \theta & -\frac{B}{r} \sin \theta & 0
\end{pmatrix}, \quad (15)$$

where $A, D, B$, are unknown functions of $r = (x^\alpha x^\alpha)^{1/2}$. Applying (15) to the field equations (10) we obtain a system of non linear differential equations [35].

$$\kappa T_{00} = \frac{1}{r A^2 B^4} \left[ \left( 3 D^2 + 8 B^2 \right) D - 2 \left( 2 D B'' + B' D' \right) B \right] r^3 B^2 D -$$

$$\left\{ 2 \left( D B'' + B' D' \right) B - 5 D B^2 \right\} r^5 D^3 - \left( 2 B B'' - 3 D^2 - 3 B^2 \right) r B^4 +$$

$$2 \left( B D' - 4 D B' \right) r^4 B D^3 + 2 \left( B D' - 6 D B' \right) r^2 B^3 D - 4 B^5 B',$$

$$\kappa T_{01} = \frac{D}{A B^4} \left[ 2 \left( D B'' + B' D' \right) B - 5 D B^2 \right] r^3 D + \left( 2 B B'' - 3 D^2 - 3 B^2 \right) r B^2 -$$

$$2 \left( B D' - 4 D B' \right) r^2 B D + 4 B^3 B',$$

$$\kappa T_{11} = \frac{1}{r A B^4} \left[ \left( 3 D^2 + B^2 \right) A + 2 B A' B' \right] r B^2 - \left\{ 2 \left( D B'' + B' D' \right) B - 5 D B^2 \right\} r^3 A D +$$

$$2 \left( B D' - 4 D B' \right) r^2 A B D - 2 A B^3 B' - 2 B^2 A',$$

$$\kappa T_{22} = \frac{r}{A^2 B^4} \left[ \left( D A'' + 3 A' D' \right) B - 3 D A' B' \right] A B D + \left\{ 2 D B'' + 5 B' D' \right\} B D -$$

$$\left( D D'' + D^2 \right) B^2 - 5 D^2 B^2 \right\} A^2 - 2 B^2 D^2 A^2 \right\} r^3 +$$

$$\left\{ B^2 - 3 D^2 \right\} A^2 - A B^2 A'' - B'' B A^2 + 2 B^2 A^2 \right\} r B^2.$$
\[ 2 \left\{ \left( 3BD' - 4DB' \right) A - 2BDA' \right\} r^2 ABD + A^2 B^3 B' + AB^4 A' \],

\[ T_{33} = \sin^2 \theta T_{22}, \]  

where \( A' = \frac{dA}{dr}, B' = \frac{dB}{dr} \) and \( D' = \frac{dD}{dr} \).

Now we are interested in solving the above differential equations:

**Special solutions:**

A first non-trivial solution can be obtained by taking \( D(r) = 0 \), and solving for \( A(r) \) and \( B(r) \), then we obtain

\[ A = \sqrt{1 - \frac{2m}{R} \left[ 1 - \tanh \left( \frac{q^2}{2mR} \right) \right]}, \quad B = \int \frac{1}{R} \left( 1 - \sqrt{1 - \frac{2m}{R} \left[ 1 - \tanh \left( \frac{q^2}{2mR} \right) \right]} \right) dR, \]  

(17)

where \( R \) is a new radial coordinate defined by \( R = r/B \).

The anstaz of the anti-symmetric field \( P_{\mu \nu} \), the nonlinear electrodynamics source used to derive this solution and the energy-momentum tensor have the form

\[ P = \frac{q}{R^2} dt \wedge dR, \quad \mathcal{H} = -\frac{q^2}{2R^4} \text{sech}^2 \left( \frac{q^2}{2mR} \right), \quad T_{0}^{0} = T_{1}^{1} = \frac{q^2 e^{(q^2/mR)}}{2\pi R^4 (1 + e^{(q^2/mR)})^2}, \]

\[ T_{2}^{2} = T_{3}^{3} = \frac{q^2 e^{(q^2/mR)}}{4\pi m R^5 (1 + e^{(q^2/mR)})^3} \left[ q^2 \left( e^{(q^2/mR)} - 1 \right) - 2mR(1 + e^{(q^2/mR)}) \right]. \]  

(18)

Using (17), the tetrad (15) takes the form

\[
(e_i^\mu) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
\sqrt{1 - \frac{2m}{R} \left[ 1 - \tanh \left( \frac{q^2}{2mR} \right) \right]} & 0 & \sin \theta \cos \phi & \cos \theta \cos \phi \\
0 & 1 - \frac{2m}{R} \left[ 1 - \tanh \left( \frac{q^2}{2mR} \right) \right] & \cos \theta \cos \phi & -\sin \phi \cos \phi \\
0 & 0 & \sin \theta \sin \phi & 1 - \frac{2m}{R} \left[ 1 - \tanh \left( \frac{q^2}{2mR} \right) \right] \\
0 & 0 & \cos \theta \sin \phi & 1 - \frac{2m}{R} \left[ 1 - \tanh \left( \frac{q^2}{2mR} \right) \right] \\
0 & 0 & \cos \phi & -\frac{\sin \theta}{R} \\
0 & \frac{\sin \theta}{R} & 0 & 0
\end{pmatrix},
\]

(19)

with the associated Riemannian metric

\[ ds^2 = -\eta_1 dt^2 + \frac{dR^2}{\eta_1} + R^2 d\Omega^2, \quad \text{where} \quad \eta_1 = 1 - \frac{2m}{R} \left[ 1 - \tanh \left( \frac{q^2}{2mR} \right) \right] \simeq 1 - \frac{2m}{R} + \frac{q^2}{R^2}, \]

(20)

where \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \). Eq. (20) represents a static spherically symmetric regular black hole solution [27]

A second non-trivial solution can be obtained by taking \( A(r) = 1, B(r) = 1, D(r) \neq 0 \).
directly to give

\[ D(r) = \sqrt{\frac{2m \left[ 1 - \tanh \left( \frac{q^2}{2mr} \right) \right]}{r^3}}, \quad (21) \]

Substituting for the value of \( D(r) \) as given by (21) into (15), we get

\[
(e_i^\mu) = \begin{pmatrix}
1 & \sqrt{\frac{2m \left[ 1 - \tanh \left( \frac{q^2}{2mr} \right) \right]}{r}} & 0 & 0 \\
0 & \sin \theta \cos \phi & \frac{\cos \theta \cos \phi}{r} & -\frac{\sin \phi}{r \sin \theta} \\
0 & \sin \theta \sin \phi & \frac{\cos \theta \sin \phi}{r} & \frac{\cos \phi}{r \sin \theta} \\
0 & \cos \theta & -\frac{\sin \theta}{r} & 0
\end{pmatrix}, \quad (22)
\]

with the associated metric

\[
ds^2 = -\left[ 1 - \frac{2m \left[ 1 - \tanh \left( \frac{q^2}{2mr} \right) \right]}{r} \right] dt^2 - 2\sqrt{\frac{2m \left[ 1 - \tanh \left( \frac{q^2}{2mr} \right) \right]}{r}} dr dt + dr^2 + r^2 d\Omega^2, \quad (23)
\]

it is to be noted that \( m \) & \( q \) appear in the metric (20) and (23) are constant of integration that will play the role of mass and charge producing the field in the calculations of energy. Also the anstaz of the anti-symmetric field \( P_{\mu \nu} \), the nonlinear electrodynamics source used to derive this solution and the energy-momentum tensor have the form (18) with \( R = r \).

Using the coordinate transformation

\[
dT = dt + \frac{Dr}{1 - D^2r^2} dr, \quad (24)
\]

we can eliminate the cross term of (23) to obtain

\[
ds^2 = -\eta_1 dT^2 + \frac{dr^2}{\eta_1} + r^2 d\Omega^2, \quad (25)
\]

where \( \eta_1 \) is defined by (20) with \( R = r \).

It is our purpose to find a general solution for the tetrad (15) when the stress-energy momentum tensor is not vanishing and has the form given by [36]

\[
T^0_0 = T^1_1, \\
T^2_2 = T^3_3,
\]

where all the other mixed spatial components equal to zero [36]. Then the left hand side of the second equation of (16) is equal zero and we can find a solution of the unknown function \( D \) in terms of the unknown function \( B \) in the form

\[
D = \frac{1}{(1 - rB')^3} \sqrt{\frac{k_1 B^3}{r^3} \left[ 1 - \tanh \left( \frac{q^2}{2mr} \right) \right] + \frac{BB'}{r} \left( \frac{rB'}{B} - 2 \right)}. \quad (27)
\]
where \( k_1 \) is a constant of integration. From the first and third equations of (16) using (26) and (27), we get the unknown function \( A \) in the form

\[
A = \frac{k_2}{1 - \frac{rB'}{B}}. \tag{28}
\]

with \( k_2 \) being another constant of integration. The general solution (27) and (28) satisfy the differential equations (16).

The line-element squared of (15) takes the form

\[
ds^2 = -\frac{(B^2 - D^2r^2)}{A^2B^2}dt^2 - \frac{2Dr}{AB^2}drdt + \frac{1}{B^2}(dr^2 + r^2d\Omega^2). \tag{29}
\]

We assume \( B(r) \) to be nonvanishing so that the surface area of the sphere of a constant \( r \) be finite. We also assume that \( A(r) \) and \( B(r) \) satisfy the asymptotic condition, \( \lim_{r \to \infty} A(r) = \lim_{r \to \infty} B(r) = 1 \) and \( \lim_{r \to \infty} rB' = 0 \). Then, we can show from (27), (28) and (29) that

1. \( k_2 = 1 \),
2. \( B(r) > 0 \),
3. \( \lim_{r \to \infty} rD(r) = 0 \), and
4. if \( B - rB' \) vanishes at some point, then \( 1 - \frac{B(r)k_1}{r} \left[ 1 - \tanh \left( \frac{q^2}{2mR} \right) \right] < 0 \) at that point.

Using the coordinate transformation

\[
\frac{dT}{dt} = dt + \frac{ADr}{B^2 - D^2r^2}dr, \tag{30}
\]

we can eliminate the cross term of (29) to obtain

\[
ds^2 = -\eta_2dT^2 + \frac{1}{\eta_2 \frac{A^2B^2}{B^2}} \frac{dr^2}{r^2} + \frac{r^2}{B^2}d\Omega^2 \tag{31}
\]

with \( \eta_2 = \frac{(B^2 - D^2r^2)}{A^2B^2} \). Taking the new radial coordinate \( R = r/B \), we finally get

\[
ds^2 = -\eta_2dT^2 + \frac{dR^2}{\eta_2} + R^2d\Omega^2, \tag{32}
\]

where

\[
\eta_2(R) = \left( 1 - \frac{k_1}{R} \left[ 1 - \tanh \left( \frac{q^2}{2mR} \right) \right] \right) \approx 1 - \frac{k_1}{R} + \frac{q^2}{R^2}. \tag{33}
\]

Then, (32) coincides with the charged regular black hole solution given before [28] with the mass, \( m = k_1/2 \), and hence the general solution in the case of the spherically symmetric tetrad when the stress-energy momentum tensor is nonvanishing gives no more than the regular charged black hole solution when \( 1 - rB'/B \) has no zero and \( R \) is monotonically increasing function of \( r \). If \( 1 - rB'/B \) has zeroes, the line-element (29) is singular at these zeroes which lie inside the event horizon as is seen from the property (4) mentioned above. We shall study in the future whether this singularity at zero-points of \( 1 - rB'/B \) is physically acceptable or not.
After using the above transformations, the tetrad (15) can be put in the form

\[
(e_i^\mu) = \begin{pmatrix}
\frac{A}{1 - D^2 R^2} & DR(1 - RB') & 0 & 0 \\
\frac{ADR \sin \theta \cos \phi}{1 - D^2 R^2} & (1 - RB') \sin \theta \cos \phi & \frac{\cos \theta \cos \phi}{R} & -\frac{\sin \phi}{R \sin \theta} \\
\frac{ADR \sin \theta \sin \phi}{1 - D^2 R^2} & (1 - RB') \sin \theta \sin \phi & \frac{\cos \theta \sin \phi}{R} & \frac{\cos \phi}{R \sin \theta} \\
\frac{ADR \cos \theta}{1 - D^2 R^2} & (1 - RB') \cos \theta & -\frac{\sin \theta}{R} & 0
\end{pmatrix}.
\] (34)

Here \(A\) and \(D\) are given in terms of the unknown function \(B(R)\) as

\[
A(R) = \frac{1}{1 - RB'}, \quad D(R) = \frac{1}{1 - RB'} \sqrt{\frac{2m [1 - \tanh \left( \frac{q^2}{2mr} \right)]}{R^3} + \frac{B'}{R} (RB' - 2)},
\] (35)

where \(B' = \frac{dB(R)}{dR}\). It is of interest to note that the general solution (35) satisfies the field equations (10) when the anstaz of the anti-symmetric field \(P_{\mu\nu}\), the nonlinear electrodynamics source and the energy-momentum tensor have the form (18).

The previously obtained solutions can be verified as special cases of the general solution (35). The choice

\[
B(R) = 1,
\] (36)

reproduces solution (21). On the other hand, the choice

\[
B(R) = \int \frac{1}{R} \left( 1 - \sqrt{1 - \frac{2m(1 - \tanh \left( \frac{q^2}{2mr} \right))}{R^3}} \right) dR,
\] (37)

reproduces solution (17). It is of interest to note that if \(q = 0\) then the general solution (35) reduces to that obtained before by Mikhail et al. [37] and the two choices (37) and (38) will give the Schwarzschild solution in its standard form.

### 4. Energy associated with the third solution

To make the consequence of calculations more clear, let us calculate the energy associated with solution (35).

The superpotential is given by [30]

\[
U_{\mu\nu} = \frac{(-g)^{1/2}}{2\kappa} P_{\chi^\rho} \tau^\nu \lambda \left[ \Phi^\rho g^\sigma g_{\mu\tau} - \lambda g_{\tau\mu} \gamma^\lambda \chi^\rho \sigma - (1 - 2\lambda) g_{\tau\mu} \gamma^\sigma \chi^\rho \right],
\] (38)

where \(P_{\chi^\rho} \tau^\nu \lambda\) is
with $g_{\rho\sigma}^{\nu\lambda}$ being a tensor defined by

$$g_{\rho\sigma}^{\nu\lambda} \overset{\text{def.}}{=} \delta^\nu_\rho \delta^\lambda_\sigma - \delta^\nu_\sigma \delta^\lambda_\rho. \quad (40)$$

The energy is expressed by the surface integral [38]

$$E = \lim_{r \to \infty} \int_{r=\text{constant}} U_0^{0\alpha} n_\alpha dS, \quad (41)$$

where $n_\alpha$ is the unit 3-vector normal to the surface element $dS$.

Now we are in a position to calculate the energy associated with solution (35) using the superpotential (39). It is clear from (41) that, the only components which contribute to the energy is $U_0^{0\alpha}$. Thus substituting from solution (35) into (39) we obtain the following non-vanishing value

$$U_0^{0\alpha} = \frac{2X^\alpha}{\kappa R^3} \left( 2m - 2m \tanh \left( \frac{q^2}{2mR} \right) - R^2 B'(R)' \right). \quad (42)$$

Substituting from (42) into (41) we get

$$E(R) = 2m \left[ 1 - \tanh \left( \frac{q^2}{2mR} \right) \right] - R^2 B'(R)' \quad (43)$$

which is depends on the arbitrary function $B(R)$.

5. Regularized expression for the gravitational energy-momentum

An important property of the tetrad fields that satisfy the condition

$$\epsilon_{ij} \cong \eta_{ij} + 1/2 \eta_{ij}(1/r), \quad (44)$$

is that in the flat space-time limit $\epsilon^i_\mu(t, x, y, z) = \delta^i_\mu$, and therefore the torsion tensor defined by

$$T^\lambda_\mu\nu \overset{\text{def.}}{=} \epsilon_a^\lambda T^a_\mu\nu = \Gamma^\lambda_\mu\nu - \Gamma^\lambda_\nu\mu, \quad (45)$$

is vanishing, i.e., $T^\lambda_\mu\nu = 0$. Hence for the flat space-time it is normally to consider a set of tetrad fields such that $T^\lambda_\mu\nu = 0$ in any coordinate system. However, in general an arbitrary set of tetrad fields that yields the metric tensor for the asymptotically flat space-time does not satisfy the asymptotic condition given by (44). Moreover for such tetrad fields the torsion $T^\lambda_\mu\nu \neq 0$ for the flat space-time [39]~[42]. It might be argued, therefore, that the expression for the energy given by (41) is restricted to particular class of tetrad fields, namely, to the class of frames such that $T^\lambda_\mu\nu = 0$ if $\epsilon^i_\mu$ represents the flat space-time tetrad field [40]. To
explain this, let us calculate the flat space-time tetrad field of (34) using (35) which is given by

\[
(E_i^\mu) = \begin{pmatrix}
(1 - RB') & \sqrt{R^2B'^2 - 2RB'} & 0 & 0 \\
\sqrt{R^2B'^2 - 2RB'} \sin \theta \cos \phi & (1 - RB') \sin \theta \cos \phi & \frac{\cos \theta \cos \phi}{R} & -\frac{\sin \phi}{R \sin \theta} \\
\sqrt{R^2B'^2 - 2RB'} \sin \theta \sin \phi & (1 - RB') \sin \theta \sin \phi & \frac{\cos \theta \sin \phi}{R} & \frac{\cos \phi}{R \sin \theta} \\
\sqrt{R^2B'^2 - 2RB'} \cos \theta & (1 - RB') \cos \theta & \frac{\sin \theta}{R} & 0
\end{pmatrix}.
\] (46)

Expression (46) yields the following non-vanishing torsion components:

\[
T_{001} = B', \quad T_{112} = -r \cos(\theta) \cos \phi B', \quad T_{113} = \sin(\theta) \sin \phi B', \quad T_{114} = -\frac{\sin(\theta) \cos \phi \sqrt{B'} (1 - B')}{\sqrt{R^2B'^2 - 2R}},
\]

\[
T_{124} = \cos \theta \cos \phi \sqrt{R^2B'^2 - 2RB'}, \quad T_{134} = -\sin(\theta) \sin \phi \sqrt{R^2B'^2 - 2RB'}, \quad T_{212} = -R \cos(\theta) \sin \phi B',
\]

\[
T_{213} = -R \sin(\theta) \cos \phi B', \quad T_{214} = -\frac{\sin \theta \sin \phi \sqrt{B'} (1 - B')}{\sqrt{R^2B'^2 - 2R}}, \quad T_{224} = \cos(\theta) \sin \phi \sqrt{R^2B'^2 - 2RB'},
\]

\[
T_{234} = \sin(\theta) \cos \phi \sqrt{R^2B'^2 - 2RB'}, \quad T_{312} = R \sin(\theta) B', \quad T_{314} = -\frac{\cos(\theta) \sqrt{B'} (1 - B')}{\sqrt{R^2B'^2 - 2R}},
\]

\[
T_{324} = -\sin(\theta) \sqrt{R^2B'^2 - 2RB'}.
\] (47)

The tetrad field (46) when written in the Cartesian coordinate will have the form

\[
(E_i^\mu(t, x, y, z)) = \begin{pmatrix}
1 - RB' & n^a \sqrt{R^2B'^2 - 2RB'} \\
n^\alpha \sqrt{R^2B'^2 - 2RB'} & \delta^\alpha _\alpha - n_\alpha n^\alpha RB'
\end{pmatrix}.
\] (48)

In view of the geometric structure of (48), we see that, Equation (34) using (35) do not display the asymptotic behavior required by (44). Moreover, in general the tetrad field (48) is adapted to accelerated observers [39]~[42]. To explain this, let us consider a boost in the x-direction of Eq. (48). We find

\[
(E_i^\mu(t, x, y, z)) = \begin{pmatrix}
\frac{vx \sqrt{R^2B'^2 - 2B'} + v \sqrt{R^2(1 - RB')}}{\gamma^{3/2} \sqrt{R}} & \frac{x \sqrt{R^2B'^2 - 2B'} + v \sqrt{R^2(1 - RB')}}{\gamma^{3/2} \sqrt{R}} & \frac{y \sqrt{R^2B'^2 - 2B'}}{R} & \frac{z \sqrt{R^2B'^2 - 2B'}}{R} \\
\frac{x \sqrt{R^2B'^2 - 2B'} + v \sqrt{R^2(1 - RB')}}{\gamma^{3/2} \sqrt{R}} & \frac{x \sqrt{R^2B'^2 - 2B'} + v \sqrt{R^2(1 - RB')}}{\gamma^{3/2} \sqrt{R}} & -\frac{xyB'}{R} & -\frac{xzB'}{R} \\
\frac{y \sqrt{R^2B'^2 - 2B'} - x \sqrt{RB'}}{(\gamma R)^{3/2}} & \frac{y \sqrt{R^2B'^2 - 2B'} - x \sqrt{RB'}}{(\gamma R)^{3/2}} & -\frac{y^2B'}{R} & -\frac{yzB'}{R} \\
\frac{z \sqrt{R^2B'^2 - 2B'} - x \sqrt{RB'}}{(\gamma R)^{3/2}} & \frac{z \sqrt{R^2B'^2 - 2B'} - x \sqrt{RB'}}{(\gamma R)^{3/2}} & -\frac{z^2B'}{R} & 1 - \frac{x^2B'}{R}
\end{pmatrix},
\] (49)

where \( v \) is the speed of the observer and \( \gamma = \sqrt{1 - v^2} \). It can be shown that along an observer’s trajectory whose velocity is determined by

\[
u^\mu = E_0^\mu = \begin{pmatrix}
\frac{vx \sqrt{R^2B'^2 - 2B'} + v \sqrt{R^2(1 - RB')}}{\gamma^{3/2} \sqrt{R}} \\
\frac{x \sqrt{R^2B'^2 - 2B'} + v \sqrt{R^2(1 - RB')}}{\gamma^{3/2} \sqrt{R}} \\
\frac{y \sqrt{R^2B'^2 - 2B'}}{R} \\
\frac{z \sqrt{R^2B'^2 - 2B'}}{R}
\end{pmatrix},
\]

the quantities \( \phi ^b = \nu^b \cdot (E^{\beta}_b, E^{\gamma}_b) \)
constructed out from (50) are non vanishing. This fact indicates that along the observer’s path the spatial axis $E_i^\mu$ rotate [39]~[42]. In spite of the above problems discussed for the tetrad field of Eq. (34) using (35) yield a satisfactory value for the total gravitational energy-momentum, as we will discussed.

Maluf et al. [39]~[42] discussed the above problems in the framework of TEGR and constructed a regularized expression for the gravitational energy-momentum in this frame. They checked this expression for a tetrad field that suffer from the above problems and obtain a very satisfactory results [40]. In this section we will follow the same procedure to derive a regularized expression for the gravitational energy-momentum defined by Eq. (41). It can be shown that the gravitational energy-momentum contained within an arbitrary volume $V$ of the three-dimensional spacelike hypersurface has the form [30, 31]

$$P_\mu = \int_V d^3x \partial_\alpha U_\mu^{0\alpha},$$

(51)

where $U_\mu^{\nu\lambda}$ is given by Eq. (38). Expression (51) bears no relationship to the ADM energy-momentum [41]. $P_\mu$ transforms as a vector under the global SO(3,1) group. It describes the gravitational energy-momentum with respect to observers adapted to $e_i^\mu$. These observers are characterized by the velocity field $u^\mu = e_0^\mu$ and by the acceleration $f^\mu$ given by [40, 41]

$$f^\mu = \frac{Du'^\mu}{ds} = \frac{De_0^\mu}{ds} = u^a \nabla_a e_0^\mu.$$

(52)

Our assumption is that the space-time be asymptotically flat. In this case the total gravitational energy-momentum is given by

$$P_\mu = \oint_{S \to \infty} dS_\alpha U_\mu^{0\alpha}.$$

(53)

The field quantities are evaluated on a surface $S$ in the limit $r \to \infty$.

In Eqs. (51) and (53) it is implicitly assumed that the reference space is determined by a set of tetrad fields $e_i^\mu$ for flat space-time such that the condition $T_\lambda^{\mu\nu} = 0$ is satisfied. However, in general there exist flat space-time tetrad fields for which $T_\mu^{a\mu} \neq 0$. In this case Eq. (51) may be generalized [40, 41] by adding a suitable reference space subtraction term, exactly like in the Brown-York formalism [43, 44].

We will denote $T_\mu^{a\mu}(E) = \partial_\mu E_\nu^{a\nu} - \partial_\nu E_\mu^{a\mu}$ and $U_\mu^{0\alpha}(E)$ as the expression of $U_\mu^{0\alpha}$ constructed out of the flat tetrad $E_i^\mu$. The regularized form of the gravitational energy-momentum $P_\mu$ is defined by

$$P_\mu = \int_V d^3x \partial_\alpha \left[ U_\mu^{0\alpha}(e) - U_\mu^{0\alpha}(E) \right].$$

(54)

This condition guarantees that the energy-momentum of the flat space-time always vanishes. The reference space-time is determined by tetrad fields $E_i^\mu$, obtained from $e_i^\mu$ by requiring the vanishing of the physical parameters like mass, angular momentum, etc. Assuming that the space-time is asymptotically flat then Eq. (54) can have the form

$$P_\mu = \oint_{S \to \infty} dS_\alpha \left[ U_\mu^{0\alpha}(e) - U_\mu^{0\alpha}(E) \right].$$

(55)
where the surface $S$ is established at spacelike infinity. Eq. (55) transforms as a vector under the global SO(3,1) group [31]. Now we are in a position to proof that the tetrad field (34) sing (35) yield a satisfactory value for the total gravitational energy-momentum.

We will integrate Eq. (55) over a surface of constant radius $x^1 = R$ and require $R \to \infty$. Therefore, the index $\alpha$ in (55) takes the value $\alpha = 1$. We need to calculate the quantity $U_0^{01}$ which has the form

$$U_0^{01}(e) \approx -\frac{1}{4\pi} R \sin(\theta) \left[ \frac{2m}{R} \left( 1 - \tanh \left( \frac{q^2}{2mR} \right) \right) - RB' + R^2 B'^2 \right], \quad (56)$$

and the expression of $U_0^{01}(E)$ is obtained by just making $m = 0$ in Eq.(56). It is given by

$$U_0^{01}(E) \approx -\frac{1}{4\pi} R \sin(\theta) (R^2 B'^2 - RB'). \quad (57)$$

Thus the gravitational energy contained within a sphere of radius $R_1$ is given by

$$P_0 \approx \int_{R \to R_1} d\theta d\phi \frac{1}{4\pi} \sin(\theta) \left\{ -R \left( \frac{2m}{R} \left[ 1 - \tanh \left( \frac{q^2}{2mR} \right) \right] - RB' + R^2 B'^2 \right) \right\} = 2m \left[ 1 - \tanh \left( \frac{q^2}{2mR_1} \right) \right]. \quad (58)$$

### 6. Main results and Discussion

We have studied the charged solutions in the tetrad theory of gravitation. The axial vector part of the torsion, $a^\mu$ for these solutions is identically vanishing and the theory reduces to teleparallel equivalent of general relativity coupled to nonlinear electrodynamics.

Three different exact analytic solutions of the field equations are obtained for the case of spherical symmetry. These solutions give rise to the same Riemannian metric spacetime, i.e., ”regular charged spherically symmetric spacetime”. The exact solutions (17), (21) and (35) represent a regular charged which contain the Reissner-Nordström black hole. Solution (35) is a general solution that we can generate from it the other solutions (17) and (21) by giving the arbitrary function $B(r)$ the values (36) & (37).

It was shown by Møller [24] that a tetrad description of a gravitational field equation allows a more satisfactory treatment of the energy-momentum complex than does general relativity. Therefore, we have applied the superpotential method given by Mikhail et al.[30] to calculate the energy of the central gravitating body. Calculating the energy associated with solution (35) we obtain the formula (44) which depends on the arbitrary function $B(R)$. This formula is not accepted in physics because it makes the energy depends on the arbitrary function.

Maluf et al. [40]-[42] have derived a simple expression for the energy-momentum flux of the gravitational field. This expression is obtained on the assumption that Eq.(36) represent the energy-momentum of the gravitational field on a volume $V$ of the three-dimensional spacelike hypersurface. They [40, 41] gave this definition for the gravitational energy-momentum.
space-time. They extended this definition to the case where the flat space-time tetrad fields $E^a_\mu$, yield $T^\lambda_{\mu\nu}(E) \neq 0$. They show that [41] in the context of the regularized gravitational energy-momentum definition it is not strictly necessary to stipulate asymptotic boundary conditions for tetrad fields that describe asymptotically flat space-times.

Following the same procedure given by Maluf et al. [39, 40, 41] to derive a regularized expression for the gravitational energy-momentum in the framework of TTEGR, we derive a similar expression of the regularized energy-momentum in the framework of tetrad theory of gravitation. Then we use definition (45) of the torsion tensor and apply it to the tetrad field (34) using (35) showing that the flat space-time associated with this tetrad field has a non-vanishing torsion components Eq. (47) and it is adapted to an accelerated observer (50). However, using Eq. (55) and calculate all the necessary components we finally get Eq. (58) which shows that the total gravitational mass does not depend on the arbitrary function.

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