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Universal deformation rings and fusion

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UNIVERSAL DEFORMATION RINGS AND FUSION

by

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A thesis submitted in partial fulfillment of the requirements for the Doctor of Philosophy degree in Mathematics in the Graduate College of The University of Iowa

August 2015

Thesis Supervisor: Professor Frauke Bleher
CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Mathematics at the August 2015 graduation.

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ABSTRACT

This thesis is on the representation theory of finite groups. Specifically, it is about finding connections between fusion and universal deformation rings.

Two elements of a subgroup $N$ of a finite group $\Gamma$ are said to be fused if they are conjugate in $\Gamma$, but not in $N$. The study of fusion arises in trying to relate the local structure of $\Gamma$ (for example, its subgroups and their embeddings) to the global structure of $\Gamma$ (for example, its normal subgroups, quotient groups, conjugacy classes). Fusion is also important to understand the representation theory of $\Gamma$ (for example, through the formula for the induction of a character from $N$ to $\Gamma$).

Universal deformation rings of irreducible mod $p$ representations of $\Gamma$ can be viewed as providing a universal generalization of the Brauer character theory of these mod $p$ representations of $\Gamma$.

It is the aim of this thesis to connect fusion to this universal generalization by considering the case when $\Gamma$ is an extension of a finite group $G$ of order prime to $p$ by an elementary abelian $p$-group $N$ of rank 2. We obtain a complete answer in the case when $G$ is a dihedral group, and we also consider the case when $G$ is abelian. On the way, we compute for many absolutely irreducible $\mathbb{F}_p\Gamma$-modules $V$, the cohomology groups $H^2(\Gamma, \text{Hom}_{\mathbb{F}_p}(V, V))$ for $i = 1, 2$, and also the universal deformation rings $R(\Gamma, V)$. 
This thesis discusses to what extent the universal deformation rings of representations can be used to detect fusion in group theory. Groups, rings, and representations are three important areas of study in Abstract Algebra. This thesis connects group-theoretic phenomena to representations of certain infinite families of groups by associating a ring, the universal deformation ring, to special representations of each group. In the cases considered, the universal deformation rings of these representations will typically be the same. However, the knowledge of when the universal deformation rings are different can be used to detect the fusion of a certain normal subgroup in each larger group under consideration. In this sense, universal deformation rings can “see” fusion. This thesis develops the machinery necessary to investigate this connection in general. It then analyzes the connection completely for some infinite classes of groups.
# TABLE OF CONTENTS

## CHAPTER

1  INTRODUCTION AND BACKGROUND ............................................. 1

1.1 Introduction ................................................................. 1
1.2 General Homological Algebra ............................................. 3
1.3 Spectral Sequences .......................................................... 10
1.4 Group Representations ..................................................... 22
1.5 Group Cohomology ........................................................... 26
1.6 Universal Deformation Rings ............................................. 35

2  COHOMOLOGY ................................................................. 38

3  MAIN RESULTS ..................................................................... 52

4  COHOMOLOGY FOR DIHEDRAL GROUPS .................................... 56

5  UNIVERSAL DEFORMATION RINGS FOR DIHEDRAL GROUPS ........... 60

6  FUSION FOR A DIHEDRAL GROUP .......................................... 62

7  PROOF OF THE MAIN RESULTS ........................................... 67

8  ABELIAN GROUPS ............................................................. 69

APPENDIX ............................................................................. 73

REFERENCES ......................................................................... 79
CHAPTER 1
INTRODUCTION AND BACKGROUND

1.1 Introduction

Let $p$ be an odd prime, and let $\Gamma$ be a finite group. In this thesis, we look at the case when $\Gamma$ is an extension of a group $G$ whose order is relatively prime to $p$ by an elementary abelian $p$-group $N$ of rank 2. Note that, in this case, every absolutely irreducible $\mathbb{F}_p\Gamma$-module $V$ is inflated from an absolutely irreducible $\mathbb{F}_pG$-module, which we also denote by $V$. We consider the question to what extent the universal deformation ring $R(\Gamma, V)$ for various absolutely irreducible $\mathbb{F}_pG$-modules $V$ recognizes the fusion of $N$ in $\Gamma$. (Recall that two elements of $N$ are said to be fused in $\Gamma$ if they are conjugate in $\Gamma$ but not in $N$). Since, in general, $R(\Gamma, V)$ cannot always be readily determined, we also study a variation of this question where we replace $R(\Gamma, V)$ by the $\mathbb{F}_p$-dimension $d_{iV}$ of the cohomology groups $H^i(\Gamma, \text{Hom}_{\mathbb{F}_p}(V, V))$ for $i = 1, 2$. This variation on the question is natural because the structure of $R(\Gamma, V)$ is intimately connected to the cohomology groups $H^i(\Gamma, \text{Hom}_{\mathbb{F}_p}(V, V))$ for $i = 1, 2$. We call an absolutely irreducible $\mathbb{F}_pG$-module $V_0$ cohomologically maximal for $\Gamma$ if $d_{iV_0}^2$ is maximal among all $d_{iV}^2$.

We answer the above two questions completely in the case when $n \geq 3, p \equiv 1 \pmod{n}$, $G$ is a dihedral group of order $2n$, and the action of $G$ on $N$ is given by a two-dimensional irreducible representation $\phi$. More precisely, we determine a set $\Omega$ of two-dimensional irreducible representations of $G$ over $\mathbb{F}_p$ such that for all
\( \phi \in \Omega \), the fusion of \( N \) in \( \Gamma \) is uniquely determined by the set of the kernels of the action of \( G \) on the absolutely irreducible \( \mathbb{F}_pG \)-modules \( V \) that are cohomologically maximal for \( \Gamma \). We further show that the absolutely irreducible \( \mathbb{F}_pG \)-modules \( V \) that are cohomologically maximal for \( \Gamma \) coincide with those for which \( R(\Gamma, V) \not\cong \mathbb{Z}_p \). The latter uses a modified argument from [3]. On the other hand, if \( \phi \) is irreducible but not in \( \Omega \), then the fusion of \( N \) in \( \Gamma \) cannot typically be detected by \( R(\Gamma, V) \), and also not by \( d_i^V \) for \( i = 1, 2 \), for all irreducible \( \mathbb{F}_pG \)-modules \( V \).

The question arises if we can expect a similar result in the “simpler” case when \( G \) is abelian. We show that this is not possible. The main reason is that all absolutely irreducible \( \mathbb{F}_pG \)-modules \( V \) are 1-dimensional when \( G \) is abelian. This implies that \( R(\Gamma, V) \) is the same ring for all these \( V \), and hence too coarse to detect the full fusion of \( N \) in \( \Gamma \).

This dissertation is organized as follows. In this chapter, we recall the definitions of deformations and deformation rings, and include some basic results from representation theory and homological algebra. In Chapter 2, we concentrate on the case when \( \Gamma \) is an extension of a finite group \( G \) by an elementary abelian \( p \)-group of rank \( \ell \geq 2 \). We give an explicit formula for the cohomology groups \( H^i(\Gamma, \text{Hom}_{\mathbb{F}_p}(V, V)) \) for \( i = 1, 2 \) for all projective \( \mathbb{F}_pG \)-modules \( V \) which are viewed as \( \mathbb{F}_p\Gamma \)-modules by inflation. In Chapter 3, we state our main results, Theorems 3.3 and 3.4, on the connection between fusion and universal deformation rings, respectively cohomology groups, in the case when \( G \) is a dihedral group. In Chapter 4, we compute \( H^i(\Gamma, \text{Hom}_{\mathbb{F}_p}(V, V)) \) for \( i = 1, 2 \) for \( G \) dihedral. In Chapter 5, again in the case of a
dihedral $G$, we explicitly compute the universal deformation rings for all absolutely irreducible 2-dimensional $\mathbb{F}_p \Gamma$-modules $V$. In Chapter 6 we determine the fusion for a dihedral group. In Chapter 7 we prove our main results stated in Chapter 3. Then, in Chapter 8 we briefly discuss the case when $G$ is abelian and compare this case to the dihedral one. We also include an appendix giving an alternate proof of a result used in Chapter 2.

Some of the material in this dissertation was submitted for publication, see [16]. I would like to thank my advisor Professor Frauke Bleher, for all of her advice and guidance.

### 1.2 General Homological Algebra

In this section we give a brief introduction to the general homological algebra required for this thesis. For additional background, see [19], [18], and [22]. Unless otherwise stated, all rings $R$ have a multiplicative identity $1$, and all modules are finitely generated, unital, left $R$-modules.

**Definition 1.1.** An $R$-module $A$ is simple if $A \neq 0$, and $A$ has no proper submodules. We say $A$ is semisimple if it is isomorphic to a direct sum of simple $R$-modules. We say that $R$ is semisimple, if it is semisimple as a module over itself.

**Theorem 1.2.** Let $R$ be a ring. Then, the following are equivalent:

1. $R$ is semisimple;

2. every $R$-module is semisimple;
3. every $R$-module is injective;

4. every short exact sequence of $R$-modules splits;

5. every $R$-module is projective.

Proof. See [18, Theorem 4.13].

Definition 1.3. Let $R$ be a ring. A complex (or chain complex) $D$ of $R$-modules is a sequence of $R$-modules and $R$-module homomorphisms

$$D = \ldots \rightarrow D_{n+1} \xrightarrow{d_{n+1}} D_n \xrightarrow{d_n} D_{n-1} \rightarrow \ldots$$

with $d_n \circ d_{n+1} = 0$, for all $n \in \mathbb{Z}$. The maps $d_n$ are called differentials. We may write $(D, d)$ for $D$ when we need to emphasize the differentials.

If the indices are increasing instead of decreasing, we call $B$ a cocomplex. Note that the condition $d_n \circ d_{n+1} = 0$ is equivalent to

$$\text{im}(d_{n+1}) \subseteq \text{ker}(d_n), \text{ for all } n.$$ 

Thus, we may define the homology of a complex (or alternatively the cohomology of a cocomplex).

Definition 1.4. Let $(D, d)$ be a complex of $R$-modules. Then the $n$-th homology of $D$ is defined to be the $R$-module

$$H_n(D) = \text{ker}(d_n)/\text{im}(d_{n+1}).$$
We call the elements of $\text{ker}(d_n)$, $n$-cycles and the elements of $\text{im}(d_{n+1})$, $n$-boundaries. We will write

\[
\text{ker}(d_n) = Z_n(D) = Z_n
\]

\[
\text{im}(d_{n+1}) = B_n(D) = B_n
\]

and therefore,

\[
H_n(D) = Z_n(D)/B_n(D).
\]

We may similarly define the $n$-th cohomology $H^n(B)$ of a cocomplex $B$. We say a complex (resp. cocomplex) is exact if its $n$-th homology (resp. $n$-th cohomology) is zero for all $n \in \mathbb{Z}$. We will now define the notion of morphisms between complexes.

**Definition 1.5.** Let $D, D'$ be complexes of $R$-modules. A chain map $f : D \to D'$ is a sequence of $R$-module homomorphisms $f_n : D_n \to D'_n$ for all $n \in \mathbb{Z}$, such that the following diagram commutes:

\[
\begin{array}{ccccccccccc}
... & \rightarrow & D_{n+1} & \xrightarrow{d_{n+1}} & D_n & \xrightarrow{d_n} & D_{n-1} & \xrightarrow{d_{n-1}} & \rightarrow & ...
\end{array}
\]

\[
\begin{array}{ccccccccccc}
... & \rightarrow & D'_{n+1} & \xrightarrow{d'_{n+1}} & D'_{n} & \xrightarrow{d'_n} & D'_{n-1} & \xrightarrow{d'_{n-1}} & \rightarrow & ...
\end{array}
\]

\[
\begin{array}{ccccccccccc}
... & \xrightarrow{f_{n+1}} & D_n & \xrightarrow{f_n} & D_{n-1} & \xrightarrow{f_{n-1}} & \rightarrow & ...
\end{array}
\]

\[
\begin{array}{ccccccccccc}
... & \xrightarrow{d'_{n+1}} & D'_{n} & \xrightarrow{d'_n} & D'_{n-1} & \xrightarrow{d'_{n-1}} & \rightarrow & ...
\end{array}
\]

Note that for all $n$, $H_n$ is a functor from the category of complexes of $R$-modules to the category of $R$-modules. We now make explicit what $H_n$ does at the level of morphisms.

**Definition 1.6.** Let $f : D \to D'$ be a chain map between complexes of $R$-modules.
Define $H_n(f) : H_n(D) \to H_n(D')$ by the formula

$$z_n + B_n(D) \xrightarrow{H_n(f)} f_n(z_n) + B_n(D'),$$
for all $z_n \in Z_n(D)$.

It is easy to check that $H_n(f)$ is a well-defined $R$-module homomorphism. We abbreviate $H_n(f)$ by $f_*$. 

**Definition 1.7.** Let $(A, d)$ be a complex of $R$-modules. Then, $(A', d')$ is a subcomplex of $(A, d)$ if $A'_n$ is a submodule of $A_n$ for all $n$, and $d'_n$ is the restriction of $d_n$ to $A'$. In this case, we can define a quotient complex

$$A/A' = \ldots \to A_n/A'_n \xrightarrow{d'_n} A_{n-1}/A'_{n-1} \to \ldots$$

where

$$a_n + A'_n \xrightarrow{d'_n} d_n(a_n) + A'_{n-1}$$

for all $a_n \in A_n$.

If $f : D \to D'$ is a chain map between complexes of $R$-modules, one may define the complexes $\ker(f)$, $\im(f)$, and $\coker(f)$ in the obvious way. For example,

$$\ker(f) = \ldots \to \ker(f_{n+1}) \xrightarrow{d'_{n+1}} \ker(f_n) \xrightarrow{d'_n} \ker(f_{n-1}) \to \ldots$$

where $d'_n$ is the restriction of the differential $D_n \xrightarrow{d_n} D_{n-1}$ to $\ker(f_n)$. Given a sequence of chain maps between complexes of $R$-modules

$$A' \xrightarrow{f} A \xrightarrow{g} A'',$$
we define this to be exact at $A$ if $\ker(g) = \im(f)$ as complexes.
**Theorem 1.8.** Let \( 0 \to A' \xrightarrow{i} A \xrightarrow{p} A'' \to 0 \) be an exact sequence of complexes of \( R \)-modules. For each \( n \), there is a so-called connecting homomorphism

\[
\delta_n : H_n(A'') \to H_{n-1}(A').
\]

Using these connecting homomorphisms we obtain a long exact sequence of \( R \)-modules

\[
\ldots \to H_n(A') \xrightarrow{i^*} H_n(A) \xrightarrow{p_*} H_n(A'') \xrightarrow{\delta} H_{n-1}(A') \xrightarrow{i^*} H_{n-1}(A) \to \ldots.
\]

**Proof.** See [18, Theorems 6.2 and 6.3]. \( \square \)

**Definition 1.9.** Let \( M \) be an \( R \)-module. Let \( X \) be a complex of \( R \)-modules of the form

\[
X = \ldots \to X_1 \to X_0 \to M \to 0
\]

then, the complex obtained by suppressing \( M \),

\[
X_M = \ldots \to X_1 \to X_0 \to 0
\]

is called the deleted complex of \( X \).

Similarly, we define the deleted complex \( Y_M \) of the complex

\[
Y = 0 \to M \to Y_0 \to Y_1 \to \ldots
\]

to be the complex

\[
Y_M = 0 \to Y_0 \to Y_1 \to \ldots.
\]

**Definition 1.10.** Let \( M \) be an \( R \)-module. An exact complex \( P \) is called a projective resolution of \( M \) if it is of the form

\[
P = \ldots \to P_1 \to P_0 \to M \to 0,
\]
where each $P_i$ is projective. Similarly, an exact complex $I$ is called an injective resolution of $M$ if it is of the form

$$I = 0 \rightarrow M \rightarrow I_0 \rightarrow I_1 \rightarrow \ldots$$

where each $I_i$ is injective.

Every $R$-module has both injective and projective resolutions. Shortly, we will use injective and projective resolutions to define left and right derived functors.

**Theorem 1.11. (Comparison Theorem)** Consider the diagram

$$
\begin{array}{c}
\cdots \rightarrow X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \xrightarrow{\epsilon} A \xrightarrow{f} 0 \\
\cdots \rightarrow X_2' \xrightarrow{d_2'} X_1' \xrightarrow{d_1'} X_0' \xrightarrow{\epsilon'} A' \xrightarrow{f'} 0
\end{array}
$$

where the rows are complexes of $R$-modules such that each $X_n$ is projective, and the bottom row is exact. Then, there is a chain map $\hat{f}$ between the deleted complexes $X_A$ and $X_{A'}$, making the following diagram commute:

$$
\begin{array}{c}
\cdots \rightarrow X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \xrightarrow{\epsilon} A \xrightarrow{f} 0 \\
\downarrow{f_1} \downarrow{f_0} \downarrow{f} \\
\cdots \rightarrow X_2' \xrightarrow{d_2'} X_1' \xrightarrow{d_1'} X_0' \xrightarrow{\epsilon'} A' \xrightarrow{f'} 0
\end{array}
$$

Moreover, any two such chain maps induce the same map at the level of homology.

**Proof.** See [18, Theorems 6.8 and 6.9].

Now, suppose $R$ and $S$ are rings. Given an additive covariant functor $T$ from the category of $R$-modules to the category of $S$-modules, we now describe the
left derived functors $L_nT$. More precisely, for each $n \geq 0$, $L_nT$ will again be an
additive covariant functor from the category of $R$-modules to the category of $S$-
modes. For an $R$-module $A$, we choose and fix a projective resolution of $A$, and we
let $P_A$ denote the corresponding deleted complex. We then form the complex $T(P_A)$
and take homology.

**Definition 1.12.** Using the notation in the preceding paragraph, define

$$(L_nT)A = H_n(T(P_A)) = \ker(Td_n)/\text{im}(Td_{n+1})$$

For each $R$-module homomorphism $f : A \to B$, we obtain $\hat{f} : P_A \to P_B$ as in the
statement of Theorem 1.11. Then, define

$$(L_nT)f : (L_nT)A \to (L_nT)B$$

by

$$(L_nT)f = H_n(T\hat{f}) = (T\hat{f})_*$$

Then, $L_n$ is the $n$-th left derived functor for $T$.

**Proposition 1.13.** For each $n \geq 0$, $L_nT$ is a well-defined additive functor. Moreover,
up to natural isomorphism, $L_nT$ is independent of the choices of projective resolutions.

**Proof.** See [18, Theorems 6.10 and 6.11].

We define the right derived functors $R^\alpha T$ by using injective resolutions and
taking cohomology. If $T$ is contravariant, we obtain the left derived functors using
injective resolutions, and the right derived functors using projective resolutions.
**Definition 1.14.** An $R$-module $A$ is said to be left $T$-acyclic if $(L_nT)A = 0$ for all $n \geq 1$. Similarly, $A$ is said to be right $T$-acyclic if $(R^nT)A = 0$ for all $n \geq 1$.

Note that every projective $R$-module is left $T$-acyclic if $T$ is covariant, and right $T$-acyclic if $T$ is contravariant. Similarly, every injective $R$-module is left $T$-acyclic if $T$ is contravariant and right $T$-acyclic if $T$ is covariant.

**Definition 1.15.** Let $A$, $B$ be $R$-modules. We define $\text{Ext}^n_R(A, B)$ to be the $n$-th right derived functor of the contravariant functor $T = \text{Hom}_R(\cdot, B)$.

**Definition 1.16.** Let $A$ be a right $R$-module, and let $B$ be a left $R$-module. We define $\text{Tor}^n_R(A, B)$ to be the $n$-th left derived functor of the covariant functor $A \otimes_R \cdot$.

### 1.3 Spectral Sequences

Spectral sequences can be used as a method for computing the homology of a total complex $\text{Tot}(M)$ of a bicomplex $M = (M_{p,q})$. In this section we give a brief introduction to spectral sequences, focusing ultimately on Grothendieck spectral sequences. For additional background, see [15], [19], [18], and [22]. First, we need some preliminary definitions. We make all definitions using complexes, though analogous results hold for cocomplexes.

Let $R$ be a ring. We first define graded $R$-modules and graded maps.

**Definition 1.17.** A graded $R$-module is an indexed family $M = (M_p)_{p \in \mathbb{Z}}$ of $R$-modules. We will often denote graded modules by $M_\bullet$.

**Definition 1.18.** Let $M$, $N$ be graded $R$-modules, and let $a \in \mathbb{Z}$. A graded map of degree $a$ from $M$ to $N$ is a family of $R$-module homomorphisms $f = (f_p : M_p \to N_{p+a})_{p \in \mathbb{Z}}$. 
The integer $a$ is said to be the degree of $f$, written $\text{deg}(f) = a$.

**Remark 1.19.** If $(D, d)$ is a complex of $R$-modules, then $(D_n)_{n \in \mathbb{Z}}$ is a graded $R$-module and $(d_n : D_n \to D_{n-1})_{n \in \mathbb{Z}}$ is a graded map of degree $-1$. If $f : D \to D'$ is a chain map, then $(f_n : D_n \to D'_n)_{n \in \mathbb{Z}}$ is a graded map of degree $0$. \hfill $\square$

Note that graded $R$-modules together with graded maps form a category (see [19, Section 10.1]). In particular, if $f : M \to N$ is a graded map of degree $a$ and $g : N \to P$ is a graded map of degree $b$, then the composition $g \circ f$ is a graded map of degree $a + b$ (see [19, Proposition 10.3]).

Following [19, Section 10.1] we work towards the definition of exactness in this category. Let $M' = (M'_p)_{p \in \mathbb{Z}}$ and $M = (M_p)_{p \in \mathbb{Z}}$ be two graded $R$-modules. We say $M'$ is a submodule of $M$ if $M'_p \subseteq M_p$ for all $p$. Note that these inclusions correspond to a graded map of degree $0$. If $M'_p \subseteq M_p$ for all $p$, then we define the quotient $M/M' = (M_p/M'_p)_{p \in \mathbb{Z}}$. The natural projection $M \to M/M'$ is a graded map of degree $0$.

**Definition 1.20.** Let $M, N$ be two graded $R$-modules and let $f : M \to N$ be a graded map of degree $a$. Then $\text{ker}(f) = (\text{ker}(f_p))_{p \in \mathbb{Z}}$ is a submodule of $M$. Analogously, $\text{im}(f) = (\text{im}(f_{p-a}))_{p \in \mathbb{Z}}$ is a submodule of $N$. We say a sequence of graded modules $A \xrightarrow{f} B \xrightarrow{g} C$ is exact, if $\text{im}(f) = \text{ker}(g)$. In other words, $\text{im}(f_{p-a}) = \text{ker}(g_p)$ for all $p$, where $a = \text{deg}(f)$. 

Note that given a short exact sequence of complexes of $R$-modules

$$0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$$

the long exact sequence of homology modules can be represented as a so-called exact triangle:

$$\begin{array}{ccc}
H_\bullet(X) & \xrightarrow{\alpha_*} & H_\bullet(Y) \\
\downarrow{\delta} & & \downarrow{\beta_*} \\
H_\bullet(Z) & & \\
\end{array}$$

Regarding each vertex as a graded module, the arrows are graded maps with $\alpha_*$ and $\beta_*$ having degree 0, while the connecting homomorphism $\delta$ has degree $-1$. More generally, given an exact triangle of graded $R$-modules and graded maps,

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{h} & & \downarrow{g} \\
C & & \\
\end{array}$$

with $f, g, h$ having degree $a, b, c$, respectively, we can reconstruct corresponding long exact sequences as follows. For any $p \in \mathbb{Z}$, we have an exact sequence of $R$-modules:

$$\ldots \to B_{p-b-c} \xrightarrow{g} C_{p-c} \xrightarrow{h} A_p \xrightarrow{f} B_{p+a} \xrightarrow{g} C_{p+a+b} \xrightarrow{h} A_{p+a+b+c} \to \ldots$$

We now discuss bigraded $R$-modules and bigraded maps.

**Definition 1.21.** A bigraded $R$-module is a doubly indexed family

$$M = (M_{p,q})_{(p,q) \in \mathbb{Z} \times \mathbb{Z}}$$
of $R$-modules. Bigraded modules are often denoted by $M_{\bullet \bullet}$.

One may picture bigraded $R$-modules in the $pq$-plane with an $R$-module occupying each lattice point.

**Definition 1.22.** Let $M, N$ be two bigraded $R$-modules, and let $(a, b) \in \mathbb{Z} \times \mathbb{Z}$. A bigraded map of bidegree $(a, b)$, denoted by $f : M \rightarrow N$, is a family of homomorphisms

$$f = (f_{p,q} : M_{p,q} \rightarrow N_{p+a,q+b})_{(p,q) \in \mathbb{Z} \times \mathbb{Z}}.$$

The bidegree of $f$ is $(a, b)$, written $\text{deg}(f) = (a, b)$.

As with graded $R$-modules, bigraded $R$-modules together with bigraded maps form a category (see [19, Section 10.1]), and, when composing bigraded maps, bidegrees add. As in Definition 1.20, we define kernels, images, and the notion of exactness for this category. More precisely, let $M' = (M'_{p,q})_{(p,q) \in \mathbb{Z} \times \mathbb{Z}}$ and $M = (M_{p,q})_{(p,q) \in \mathbb{Z} \times \mathbb{Z}}$ be two bigraded $R$-modules. We say $M'$ is a submodule of $M$ if $M'_{p,q} \subseteq M_{p,q}$ for all $(p, q)$. Note that these inclusions correspond to a bigraded map of bidegree $(0, 0)$. If $M'_{p,q} \subseteq M_{p,q}$ for all $(p, q)$ then we define the quotient $M/M' = (M_{p,q}/M'_{p,q})_{(p,q) \in \mathbb{Z} \times \mathbb{Z}}$. Again, the natural projection $M \rightarrow M/M'$ is a bigraded map of bidegree $(0, 0)$.

**Definition 1.23.** Let $M, N$ be two bigraded $R$-modules and let $f : M \rightarrow N$ be a bigraded map of bidegree $(a, b)$. Then $\text{ker}(f) = (\text{ker}(f_{p,q}))_{(p,q) \in \mathbb{Z} \times \mathbb{Z}}$ is a submodule of $M$. Also, $\text{im}(f) = (\text{im}(f_{p-a,q-b}))_{p \in \mathbb{Z}}$ is a submodule of $N$. We say a sequence of bigraded modules

$$A \xrightarrow{f} B \xrightarrow{g} C$$
is exact, if \( \text{im}(f) = \ker(g) \). In other words, \( \text{im}(f_{p-a,q-b}) = \ker(g_{p,q}) \) for all \((p, q)\), where \((a, b) = \text{deg}(f)\).

As with graded \(R\)-modules and graded maps, we may consider either an exact triangle \((A, B, C, f, g, h)\) of bigraded \(R\)-modules and bigraded maps,

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{h} & & \downarrow{g} \\
C & \xleftarrow{g} & A
\end{array}
\]

or for each \((p, q) \in \mathbb{Z} \times \mathbb{Z}\), a long exact sequence of \(R\)-modules

\[
\cdots \to C_{p-c,q-c'} \xrightarrow{h} A_{p,q} \xrightarrow{f} B_{p+a,q+a'} \xrightarrow{g} C_{p+a+b,q+a'+b'} \to \cdots
\]

where \(f, g, h\) have bidegrees \((a, a'), (b, b'), (c, c')\), respectively.

**Definition 1.24.** A bicomplex (or double complex) of \(R\)-modules is a triple \((M, d', d'')\), where \(M = (M_{p,q})\) is a bigraded module and \(d', d'' : M \to M\) are differentials of bidegree \((-1, 0)\) and \((0, -1)\), respectively, such that for all \(p, q\)

\[
d'_{p,q-1}d''_{p,q} + d''_{p-1,q}d'_{p,q} = 0.
\]

As with a bimodule, a bicomplex \(M\) may be pictured as a family of \(R\)-modules in the plane, with \(M_{p,q}\) sitting at the lattice point \((p, q)\). The differential \(d'_{p,q}\) points to the left, while the differential \(d''_{p,q}\) points down. The identity \(d'_{p,q-1}d''_{p,q} + d''_{p-1,q}d'_{p,q} = 0\) says that each square anticommutes.

We are now ready to define the total complex of a bicomplex.
Definition 1.25. Let \((M, d', d'')\) be a bicomplex of \(R\)-modules. Then, its total complex \(Tot(M)\) is the complex of \(R\)-modules, whose \(n\)-th term is

\[
Tot(M)_n = \bigoplus_{p+q=n} M_{p,q}
\]

with differential \(D_n : Tot(M)_n \to Tot(M)_{n-1}\) given by

\[
D_n = \sum_{p+q=n} (d'_{p,q} + d''_{p,q}).
\]

Note that by [19, Lemma 10.5], the total complex \(Tot(M)\) is indeed a complex of \(R\)-modules.

Definition 1.26. A first quadrant bicomplex is a bicomplex \((M_{p,q})\) for which \(M_{p,q} = 0\), whenever \(p\) or \(q\) is negative. Similarly, if \((M_{p,q})\) is concentrated in the third quadrant we call it a third quadrant bicomplex.

Definition 1.27. Let \(M\) be an \(R\)-module. A filtration of \(M\) is a family \((M_p)_{p \in \mathbb{Z}}\) of submodules of \(M\) such that

\[
\ldots \subseteq M_{p-1} \subseteq M_p \subseteq M_{p+1} \subseteq \ldots.
\]

The factor modules of this filtration form the graded module \((M_p/M_{p-1})_{p \in \mathbb{Z}}\). Of course, one can define filtrations in any abelian category. In particular, a filtration of a complex \(C\) of \(R\)-modules is a family of subcomplexes \((F^pC)_{p \in \mathbb{Z}}\) with

\[
\ldots \subseteq F^{p-1}C \subseteq F^pC \subseteq F^{p+1}C \subseteq \ldots.
\]

In other words, a filtration of \(C\) is a commutative diagram of \(R\)-modules such that for each \(n\), the \(n\)-th column is a filtration of \(C_n\). This can be pictured as follows:
Here are two particularly important filtrations of the total complex. Let $(M, d', d'')$ be a bicomplex of $R$-modules.

**Definition 1.28.** The first filtration of $\text{Tot}(M)$ is $^{I}F^{p}\text{Tot}(M)$, which is given by

$$(^{I}F^{p}\text{Tot}(M))_{n} = \bigoplus_{i \leq p} M_{i, n-i}$$

$$= \cdots \oplus M_{p-2, q+2} \oplus M_{p-1, q+1} \oplus M_{p, q}$$

where $p + q = n$. By [19, Section 10.2], $^{I}F^{p}\text{Tot}(M)$ is a subcomplex of $\text{Tot}(M)$.

**Definition 1.29.** The second filtration of $\text{Tot}(M)$ is $^{II}F^{p}\text{Tot}(M)$, which is given by

$$(^{II}F^{p}\text{Tot}(M))_{n} = \bigoplus_{j \leq p} M_{n-j, j}$$

$$= \cdots \oplus M_{q+2, p-2} \oplus M_{q+1, p-1} \oplus M_{q, p}$$

where $p + q = n$. By [19, Section 10.2], $^{II}F^{p}\text{Tot}(M)$ is also a subcomplex of $\text{Tot}(M)$.

We now introduce exact couples.
**Definition 1.30.** An exact couple over $R$ is a 5-tuple $(D, E, \alpha, \beta, \gamma)$, where $D$ and $E$ are bigraded $R$-modules, $\alpha, \beta, \gamma$ are bigraded maps, and there is an exact triangle of bigraded $R$-modules as follows:

```
\[
\begin{array}{ccc}
D & \xrightarrow{\alpha} & D \\
\downarrow{\gamma} & & \downarrow{\beta} \\
E & & E
\end{array}
\]
```

**Proposition 1.31.** Every filtration $(F^p C)_{p \in \mathbb{Z}}$ of a complex $C$ of $R$-modules determines an exact couple over $R$ whose bigraded maps have the displayed bidegrees.

**Proof.** See [19, Proposition 10.8].

**Definition 1.32.** A differential bigraded $R$-module is a tuple $(M, d)$, where $M$ is a bigraded $R$-module and $d : M \to M$ is a differential bigraded map, i.e. $dd = 0$. If $(M, d)$ is a differential bigraded $R$-module and $d$ has bidegree $(a, b)$, then its homology $H(M, d)$ is the bigraded $R$-module whose $(p, q)$ term is

\[
H(M, d)_{p, q} = \ker(d_{p,q}) / \text{im}(d_{p-a,q-b}).
\]
Note that a bicomplex \((M, d', d'')\) of \(R\)-modules gives rise to two differential bigraded \(R\)-modules \((M, d')\) and \((M, d'')\). Since \(d' + d'': M \to M\) is not a bigraded map, \((M, d' + d'')\) is not a differential bigraded \(R\)-module.

**Proposition 1.33.** If \((D, E, \alpha, \beta, \gamma)\) is an exact couple over \(R\), then \(d^1 = \beta \gamma\) is a differential on \(E\). Moreover, there is an exact couple \((D^2, E^2, \alpha^2, \beta^2, \gamma^2)\) over \(R\), called the derived couple, where \(E^2 = H(E, d^1)\):

\[
\begin{array}{ccc}
D^2 & \alpha^2 & D^2 \\
\beta^2 & \downarrow & \beta^2 \\
E^2 & \gamma^2 & \downarrow \\
\end{array}
\]

**Proof.** See [19, Proposition 10.9].

**Definition 1.34.** For \(r \geq 1\), we inductively define the \((r + 1)\)-st derived couple of \((D, E, \alpha, \beta, \gamma)\) to be the derived couple of the \(r\)-th derived couple \((D^r, E^r, \alpha^r, \beta^r, \gamma^r)\).

We now come to the definition of a spectral sequence.

**Definition 1.35.** A spectral sequence over \(R\) is a sequence \(\{(E^r, d^r)\}_{r \geq 1}\) of differential bigraded \(R\)-modules such that \(E^{r+1} = H(E^r, d^r)\) for all \(r\).

Thus, if we relabel \((D, E, \alpha, \beta, \gamma)\) as \((D^1, E^1, \alpha^1, \beta^1, \gamma^1)\), we see that each exact couple yields a spectral sequence by Proposition 1.33. Moreover, by Proposition 1.31, each filtration of a complex determines a spectral sequence. If \(\{(E^r, d^r)\}\) is a spectral sequence, then \(E^r\) is a subquotient of \(E^{r-1}\). Say \(E^r = Z^r/B^r\), where \(Z^r\) represents
the cycles and the $B^r$ represents the boundaries. Then, there is a chain of differential bigraded $R$-modules

$$B^2 \subseteq \ldots \subseteq B^r \subseteq Z^r \subseteq \ldots \subseteq Z^2 \subseteq E^1$$

by [19, Section 10.3].

**Definition 1.36.** Given a spectral sequence $\{(E^r, d^r)\}$ over $R$, define $Z^\infty = \bigcap_r Z^r$ and $B^\infty = \bigcup_r B^r$. Then $B^\infty \subseteq Z^\infty$, and the limit term of the spectral sequence is the bigraded module $E^\infty$ defined by

$$E^\infty_{p,q} = Z^\infty_{p,q}/B^\infty_{p,q}.$$

**Definition 1.37.** If $(F^pC)_p$ is a filtration of a complex $C$ of $R$-modules and $\iota^p : F^pC \to C$ are the inclusions, define

$$\Phi^p H_n(C) = \text{im}(\iota^p).$$

We call $(\Phi^p H_n(C))_p$ the induced filtration of $H_n(C)$.

**Definition 1.38.** A filtration $(F^pM)$ of a graded $R$-module $M = (M_n)$ is bounded if for each $n$, there exist integers $s = s(n)$ and $t = t(n)$ such that

$$F^s M_n = 0 \text{ and } F^t M_n = M_n.$$

If $(F^pC)$ is a bounded filtration of a complex $C$ of $R$-modules, then the induced filtration on homology is also bounded.

**Definition 1.39.** We say that a spectral sequence $\{(E^r, d^r)\}_{r \geq 1}$ over $R$ converges to a graded $R$-module $H$ if there is some bounded filtration $(\Phi^p H_n)$ of $H$ with

$$E^\infty_{p,q} \cong \Phi^p H_n / \Phi^{p-1} H_n.$$
for all \(n\), where \(p + q = n\). If this is the case we write

\[
E^2_{p,q} \xrightarrow{p} H_n
\]

In the above definition, note the significance of the superscript \(p\) in the notation \(E^2_{p,q} \xrightarrow{p} H_n\). Indeed, the isomorphism involves the limit term \(E^\infty_{p,q}\) which need not be equal to \(E^\infty_{q,p}\).

**Lemma 1.40.** Let \(M = (M_{p,q})\) be a bicomplex of \(R\)-modules. Then the first and second filtrations \(^1F^p\text{Tot}(M)\) and \(^1H^p\text{Tot}(M)\) of \(\text{Tot}(M)\) are bounded if and only if for each \(n\), there are only finitely many non-zero \(M_{p,q}\) on the line \(p + q = n\). In particular, if \(M\) is a first or third quadrant bicomplex, then both filtrations of \(\text{Tot}(M)\) are bounded.

**Proof.** See [18, Lemma 11.16].

**Theorem 1.41.** Let \(M = (M_{p,q})\) be a first or third quadrant bicomplex of \(R\)-modules, and let \(\{(^1E^r, ^1d^r)\} \) and \(\{(^1H^r, ^1Hd^r)\}\) be the spectral sequences determined by the first and second filtrations of \(\text{Tot}(M)\), respectively. Then

1. For each \((p, q)\), we have \(^1E^r_{p,q} = \text{\(E^\infty_{p,q}\)} and \(^1H^r_{p,q} = \text{\(H^\infty_{p,q}\)} for \(r\) sufficiently large.

2. \(^1E^2_{p,q} \xrightarrow{p} H_n(\text{Tot}(M))\) and \(^1H^2_{p,q} \xrightarrow{p} H_n(\text{Tot}(M))\).

**Proof.** See [18, Theorem 11.17].

**Definition 1.42.** Let \(M = (M_{p,q})\) be a first or third quadrant bicomplex of \(R\)-modules, and let \(\{E^r\}\) denote either \(\{(^1E^r, ^1d^r)\}\) or \(\{(^1H^r, ^1Hd^r)\}\). We say that \(\{E^r\}\) collapses if \(E^2_{p,q} = 0\) for all \(q \neq 0\).
So collapsing says that the bigraded module $E^2$ may have nonzero terms only on the $p$-axis.

**Proposition 1.43.** Let $M = (M_{p,q})$ be a first or third quadrant bicomplex of $R$-modules, and let $\{E^r\}$ denote either $\{(I^r, I^r d^r)\}$ or $\{(II^r, II^r d^r)\}$. Suppose that $\{E^r\}$ collapses. Then

$$E_p^\infty = E_p^2, \text{ for all } (p, q) \text{ and } H_n(Tot(M)) \cong E^2_{n,0}.$$

**Proof.** See [18, Lemma 11.20].

We are now ready to define Grothendieck spectral sequences. The following theorem is due to Grothendieck. Let $A, B$ and $C$ be module categories.

**Theorem 1.44.** (Grothendieck) Let $G : A \to B$ and $F : B \to C$ be functors where $F$ is left exact and $G$ sends every injective module in $A$ to a right $F$-acyclic module in $B$. Then, for each module $A$ in $A$ there exists a third quadrant spectral sequence with

$$E_p^\infty = R^p F(R^q G(A)) \Rightarrow R^n (F \circ G)(A).$$

where $n = p + q$, as before.

**Proof.** See [18, Theorem 11.38].

A spectral sequence that arises in this way is called a Grothendieck spectral sequence. Thus a Grothendieck spectral sequence relates the right derived functors of a composition of two functors, to the right derived functors of the constituent functors.

We are interested in a particular application of Grothendieck spectral sequences to
group cohomology. Recall that for a group $G$ and a left $G$-module $A$, $H^n(G, A) = \text{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, A)$ for all $n \geq 0$, where $\mathbb{Z}$ denotes the trivial $G$-module. We will give a brief introduction to group cohomology in Section 1.5.

**Theorem 1.45.** (Lyndon-Hochschild-Serre) Let $\Gamma$ be a group with normal subgroup $N$. Then, for each $\Gamma$-module $A$, there is a third quadrant spectral sequence with

$$E_2^{p,q} = H^p(\Gamma/N, H^q(N, A)) \Longrightarrow H^n(\Gamma, A).$$

**Proof.** See [18, Theorem 11.45]. □

### 1.4 Group Representations

In this section, we briefly introduce group representations of finite groups. For additional background, see [1], [21], [6], and [7]. Let $G$ be a finite group, and let $K$ be a field. For background on groups, see [17], [12], [2], and [11].

**Definition 1.46.** A $K$-representation of the group $G$ is a homomorphism

$$\rho : G \to \text{Aut}_K(V),$$

where $V$ is some finite dimensional $K$-vector space. Thus a $K$-representation of $G$ can be represented as a pair $(\rho, V)$, where $V$ is a finite dimensional $K$-vector space and $\rho$ is a homomorphism from $G$ into $\text{Aut}_K(V)$.

**Definition 1.47.** Let $(\rho, V), (\tau, W)$ be two $K$-representations of $G$. We say that $L : V \to W$ is an intertwining map if $L$ is a $K$-linear map from $V$ to $W$, and for all $g \in G$ and all $v \in V$, $L(\rho(g)(v)) = \tau(g)(L(v))$. 
In other words, for every $g \in G$, the following diagram commutes:

\[
\begin{array}{ccc}
V & \xrightarrow{L} & W \\
\downarrow{\rho(g)} & & \downarrow{\tau(g)} \\
V & \xrightarrow{L} & W.
\end{array}
\]

Note that the $K$-representations of $G$ together with intertwining maps form a category and that there is a categorical equivalence between $K$-representations of $G$ and finitely generated $KG$-modules (see [9, Section 18.1]). We briefly summarize the correspondence at the level of objects. Let $V$ be a finite dimensional $K$-vector space, let $\rho : G \to Aut_K(V)$ be a $K$-representation of $G$, and for $g \in G$, let $a_g \in K$. Then the definition

\[
(\sum_{g \in G} a_g g) \cdot v = \sum_{g \in G} a_g (\rho(g)(v)), \text{ for all } v \in V
\]

provides a $KG$-module structure on $V$ [9, p. 843]. Moreover, if $V$ is a finitely generated $KG$-module, then $V$ is a finite dimensional $K$-vector space and $\rho : G \to Aut_K(V)$ defined by $\rho(g)(v) = g \cdot v$ for all $g \in G$ and $v \in V$, is a group homomorphism. It is easy to see that these assignments are inverses of each other [9, p. 843]. Thus, one may take a representation-theoretic or module-theoretic viewpoint wherever convenient. In particular, by using this equivalence of categories, we have implicitly defined direct sums of representations, irreducible representations, indecomposable representations, subrepresentations, and quotient representations, as those representations in correspondence with the appropriate modules. For the remainder of the section, we adopt the module-theoretic perspective. If not stated otherwise, all our modules are assumed to be finitely generated left modules.
Definition 1.48. Let $V, W$ be $KG$-modules. Then $V \otimes_K W$ is a $KG$-module with so-called diagonal $G$-action. More precisely, if $v \otimes w$ is a simple tensor, then we define $g \cdot (v \otimes w) = g \cdot v \otimes g \cdot w$ for all $g \in G$.

Note that unless stated otherwise, all tensor products will be taken over $K$.

Definition 1.49. Let $V, W$ be $KG$-modules. Then $\text{Hom}_K(V, W)$ is a $KG$-module where the $G$-action is given by $(g \cdot f)(v) = g \cdot f(g^{-1} \cdot v)$ for all $g \in G$, $f \in \text{Hom}_K(V, W)$ and all $v \in V$.

A special case of Definition 1.49 is when $W$ is taken to be the trivial simple $KG$-module, i.e. $W = K$ with trivial $G$-action.

Definition 1.50. Let $K$ denote the trivial simple $KG$-module. Then the dual module of $V$, $V^*$, is defined to be the $KG$-module $\text{Hom}_K(V, K)$.

We now define induction and restriction, which are two very important concepts in the representation theory of finite groups.

Definition 1.51. Let $H$ be a subgroup of $G$ and let $W$ be a $KH$-module. Then, the induced module $\text{Ind}_H^G(W)$ is the $KG$-module $KG \otimes_{KH} W$, where $KG$ is viewed as a $(KG, KH)$-bimodule in the natural way.

Definition 1.52. Let $H$ be a subgroup of $G$ and let $V$ be a $KG$-module. Then, the restricted module $\text{Res}_H^G(V)$ is the $KH$-module obtained by restricting the action of $G$ on $V$ to $H$.

Proposition 1.53. Let $H$ be a subgroup of a group $G$. Let $V, V_1, V_2$ be $KH$-modules and let $U, U_1, U_2$ be $KG$-modules. Then,
1. \( \text{Hom}_K(U_1, U_2) \cong U_1^* \otimes U_2 \) as \( KG \)-modules.

2. \( (U^*)^* \cong U \) as \( KG \)-modules.

3. If \( V \) is a free (resp. projective) \( KH \)-module then \( \text{Ind}^G_H(V) \) is a free (resp. projective) \( KG \)-module.

4. \( \text{Ind}^G_H(V^*) \cong (\text{Ind}^G_H(V))^* \) as \( KG \)-modules.

5. \( \text{Ind}^G_H(V_1 \oplus V_2) \cong \text{Ind}^G_H(V_1) \oplus \text{Ind}^G_H(V_2) \) as \( KG \)-modules.

Proof. See [6, Section 10].

While the representation type of \( KG \) may be infinite tame or wild [10], Maschke’s Theorem and its converse provide necessary and sufficient conditions for \( KG \) to be semisimple.

**Proposition 1.54.** The algebra \( KG \) is semisimple if and only if the characteristic of \( K \) does not divide the order of \( G \).

Proof. See [1, Theorem 3.1].

In particular, if \( K \) has characteristic 0, \( KG \) is always semisimple. Moreover, if \( K = \mathbb{C} \), we can use character theory to distinguish \( KG \)-modules.

**Definition 1.55.** Let \( V \) be a \( KG \)-module with corresponding \( K \)-representation \( \rho : G \to \text{Aut}_K(V) \). For \( g \in G \), define

\[
\chi(g) = \text{Tr}(\rho(g))
\]

where \( \text{Tr} \) denotes the trace of a matrix. We say that \( \chi \) is the character of \( V \) or of \( \rho \).
Proposition 1.56. Let $V, \rho, \chi$ be as in Definition 1.55. Suppose $K = \mathbb{C}$. Then

1. $\chi$ is a well-defined function on $G$ that is constant on conjugacy classes.

2. Two $KG$-modules have the same character if and only if they are isomorphic.

Proof. See [21, Sections 2.1 and 2.3].

Definition 1.57. Let $K$ be a field, and let $m$ be the least common multiple of the orders of the elements of $G$. We say that $K$ is sufficiently large for $G$, if $K$ contains all the $m$-th roots of unity.

Definition 1.58. We say that a field $K$ is a splitting field for $G$ if every irreducible representation of $G$ over any extension field of $K$ is equivalent to a $K$-representation.

Theorem 1.59. Let $G$ be a group and let $K$ be a sufficiently large field for $G$. Then, $K$ is a splitting field for $G$ and all of its subgroups.

Proof. See [6, Theorem 17.1].

Therefore, if $KG$ is semisimple and $K$ is sufficiently large for $G$, then the $K$-representation theory of $G$ looks like that of the complex numbers.

1.5 Group Cohomology

In this section, we give a brief introduction to group cohomology. For additional background, see [19], [18], [5] and [20]. We confine our attention entirely to extensions of finite groups by abelian groups. For a discussion on non-abelian group
cohomology, see [20]. Before discussing group cohomology in general, we briefly dis-
cuss the second cohomology group and the interpretation of its elements as group
extensions.

Let $G$ be a finite group, and let $A$ be an abelian group, where we write $G$
multiplicatively and $A$ additively. If $G$ acts on $A$, then $A$ is a $\mathbb{Z}G$-module. We fix a
particular action of $G$ on $A$ and use the symbol $\cdot$ to denote this action. We consider
a short exact sequence of groups

$$0 \to A \xrightarrow{\iota} \Gamma \xrightarrow{\pi} G \to 1$$

such that $\iota(A)$ is normal in $\Gamma$, $\iota(A) = \ker(\pi)$, and the natural quotient map $\pi$ is an
isomorphism from $\Gamma/\iota(A)$ to $G$. In the following, we sometimes write $a$ instead of
$\iota(a)$ for $a \in A$. By picking a section $\lambda$ of $\pi$ such that $\lambda(1_G) = 1_\Gamma$, we define a new
action of $G$ on $A$, by

$$g \odot a = \lambda(g)a(\lambda(g))^{-1} \text{ for all } g \in G \text{ and } a \in A,$$

where the multiplication takes place inside $G$. (Note we do not require that the
section $\lambda$ is a group homomorphism.) Also, since $A$ is normal in $G$, $g \odot a \in A$. In
addition, this action is independent of the choice of section. Namely if $\tilde{\lambda}$ is another
section, then for all $g \in G$, $\tilde{\lambda}(g) = \lambda(g)a_g$, for some $a_g \in A$ in $A$. For all $a \in A$ and
\(g \in G\), we have,

\[
\tilde{\lambda}(g)a(\tilde{\lambda}(g))^{-1} = \lambda(g)a_g a(a_g \lambda(g))^{-1} \\
= \lambda(g)(a_g a(a_g)^{-1})(\lambda(g))^{-1} \\
= \lambda(g)a(\lambda(g))^{-1} \\
= g \odot a,
\]

since \(A\) is abelian.

**Definition 1.60.** With an action \(\cdot\) of \(G\) on \(A\) fixed as above, we say \(0 \to A \to \Gamma \to G \to 1\) realizes the operators if for all \(g \in G\) and for all \(a \in A\), we have \(g \cdot a = g \odot a\) (see [19, Section 9.1]).

**Proposition 1.61.** Suppose the action of \(G\) on \(A\) is trivial. Then a sequence

\[
0 \to A \to \Gamma \to G \to 1,
\]

realizes the operators if and only if \(A \subset Z(\Gamma)\).

**Proof.** See [19, Proposition 9.2]. \(\square\)

**Definition 1.62.** Given a short exact sequence \(0 \to A \to \Gamma \to G \to 1\) and a section \(\lambda : G \to \Gamma\) as above, note that for \(x, y \in G\), \(\lambda(xy)\) and \(\lambda(x)\lambda(y)\) both represent the same right coset in \(A \Gamma\). Thus, \(\lambda(x)\lambda(y)\) must be equal to \(f(x, y)\lambda(xy)\), for some function \(f : G \times G \to A\). The function \(f\) is called the factor set corresponding to the section \(\lambda\).

**Proposition 1.63.** Let \(0 \to A \to \Gamma \to G \to 1\) be an extension realizing the operators, and suppose \(\lambda\) is a section with corresponding factor set \(f\). Then, for all \(x, y\) in \(G\),
\( f(1, y) = 0 = f(x, 1) \), where 0 denotes the identity in \( A \) and 1 is the identity in \( G \). In addition, the so-called two-cocycle identity holds, that is, for all \( x, y, z \) in \( G \), we have

\[ f(x, y) + f(xy, z) = x \cdot f(y, z) + f(x, yz). \]

**Proof.** See [19, Proposition 9.7]. \( \square \)

Moreover, given an action of \( G \) on \( A \), and a function \( f : G \times G \to A \) which satisfies the two-cocycle condition and \( f(1, y) = 0 = f(x, 1) \) for all \( x, y \in G \), we may construct a group \( \Gamma \) and a sequence as above. More precisely, we have the following result.

**Proposition 1.64.** Suppose \( G, A, f \) are as in the previous paragraph. Let \( \Gamma = A \times G \) as a set. Define \( \lambda : G \to \Gamma \), by \( \lambda(g) = (0, g) \). Then, \( \Gamma \) is a group, with operation given by

\[ (a_1, g_1) \cdot (a_2, g_2) = (a_1 + g_1 \cdot a_2 + f(g_1, g_2), g_1 g_2). \]

Additionally, we have an extension \( 0 \to A \xrightarrow{\iota} \Gamma \xrightarrow{\pi} G \to 1 \) realizing the operators, and \( \lambda \) is a section with corresponding factor set \( f \).

**Proof.** See [19, Theorem 9.8]. \( \square \)

We denote the extension \( \Gamma \) defined in Proposition 1.64 by \( \text{Gr}(G, A, f) \). It is clear that such an extension \( \text{Gr}(G, A, f) \) generalizes semi-direct products \( A \rtimes G \), since the semi-direct products correspond to the choice of the trivial factor set \( f \). Moreover, this exhausts all such extensions of \( G \) by \( A \).
Theorem 1.65. Let $G$ be a group, let $A$ be a $\mathbb{Z}G$-module, and let $0 \to A \xrightarrow{\lambda} \Gamma \xrightarrow{\pi} G \to 1$ be an extension realizing the operators. Then, there exists a factor set $f$ such that $\Gamma \cong Gr(G, A, f)$.

Proof. See [19, Theorem 9.9].

Proposition 1.66. Given a group $G$, a $\mathbb{Z}G$-module $A$, and an extension $0 \to A \xrightarrow{\lambda} \Gamma \xrightarrow{\pi} G \to 1$ realizing the operators. If $\lambda_1, \lambda_2$ are two sections corresponding to factor sets $f_1, f_2$, respectively, then there exists a function $h : G \to A$ such that for all $x, y$ in $G$, $f_1(x, y) - f_2(x, y) = x \cdot h(y) - h(xy) + h(x)$. Moreover, one can take $h(1_G) = 0$ in $A$.

Proof. See [19, Lemma 9.10].

Definition 1.67. A function $g : G \times G \to A$ is called a coboundary if there exists a function $h : G \to A$, with $h(1) = 0$, such that $g(x, y) = x \cdot h(y) - h(xy) + h(x)$ for all $x, y \in G$.

Definition 1.68. Given $G, A$ as above with an action of $G$ on $A$ fixed. Define

$Z^2(G, A) = \{ f : G \times G \to A : f \text{ is a factor set} \}$ and

$B^2(G, A) = \{ g : G \times G \to A : g \text{ is a coboundary} \}$.

Proposition 1.69. With operation given by point-wise addition, $B^2(G, A)$ is an abelian subgroup of $Z^2(G, A)$.

Proof. See [19, Proposition 9.11].

Definition 1.70. The second cohomology group $H^2(G, A)$ is defined to be the abelian group $Z^2(G, A)/B^2(G, A)$. 
Definition 1.71. Two factor sets are said to be equivalent if their difference is a coboundary. Two extensions realizing the operators are said to be equivalent if they have equivalent factor sets.

Thus, factor sets are equivalent exactly when they define the same coset in $H^2(G, A)$. The significance of the abelian group $H^2(G, A)$ is illustrated by the next proposition.

Proposition 1.72. The extensions

$$0 \to A \xrightarrow{\iota_1} \Gamma \xrightarrow{\pi_1} G \to 1$$

and

$$0 \to A \xrightarrow{\iota_2} \Gamma \xrightarrow{\pi_2} G \to 1$$

realizing the operators are equivalent if and only if there exists a group homomorphism $\gamma$ yielding the commutative diagram:

\[
\begin{array}{c}
0 \to A \xrightarrow{\iota_1} \Gamma \xrightarrow{\pi_1} G \to 1 \\
\downarrow \text{Id} \downarrow \gamma \downarrow \text{Id} \\
0 \to A \xrightarrow{\iota_2} \Gamma \xrightarrow{\pi_2} G \to 1
\end{array}
\]

Proof. See [19, Proposition 9.12].

Note that if such a $\gamma$ exists, then the Short Five Lemma will ensure that $\gamma$ is an isomorphism. Therefore, equivalence in the above sense defines an equivalence relation on the collection of extensions of $G$ by $A$ that realize the operators. Thus,
we may view the abelian group $H^2(G, A)$ as being in one-to-one correspondence with
equivalence classes of extensions, where two extensions are equivalent if and only if
their factor sets define the same element in $H^2(G, A)$. We will now shift our focus
away from $H^2(G, A)$ specifically, and turn our attention to general group cohomology.
As before, $G$ is a finite group.

**Definition 1.73.** For $ZG$-modules $A$ and $B$, $\text{Hom}_{ZG}(A, B)$ can be made into a $ZG$-
module with the so-called adjoint action (compare to Definition 1.49)

$$(g \cdot f)(a) = gf(g^{-1} \cdot a),$$
for all $g \in G$, $f \in \text{Hom}_{ZG}(A, B)$, and $a \in A$.

**Definition 1.74.** For a $ZG$-module $C$, the $G$ fixed points of $C$ are defined as

$$C^G = \{ c \in C : \forall g \in G, g \cdot c = c \}.$$

**Definition 1.75.** We define a functor $\text{Fix}^G : ZG\text{-Mod} \rightarrow Z\text{-Mod}$, called the fixed-
point functor, by

$\text{Fix}^G(C) = C^G$, and $\text{Fix}^G(\phi) = \phi|_{C^G}$, for all $ZG$-modules $C$ and all $ZG$-module
homomorphisms $\phi$.

It is clear that $\text{Fix}^G$ is a functor. For example, if $\phi$ is in $\text{Hom}_{ZG}(A, B)$, then

$\phi(A^G) \subset B^G$, since for $a \in A^G, g \in G$, we have $g \cdot \phi(a) = \phi(g \cdot a) = \phi(a)$.

Using this notation, we obtain

$$\text{Fix}^G(\text{Hom}_Z(A, B)) = \text{Hom}_Z(A, B)^G = \text{Hom}_{ZG}(A, B).$$

This follows since for all $g \in G$ and $f \in \text{Hom}_Z(A, B)$, $g \cdot f = f$ if and only if

$$(g \cdot f)(x) = f(x)$$
for all $x \in A$. This, on the other hand, happens if and only if for all
Proposition 1.76. Let $\mathbb{Z}$ be the trivial $\mathbb{Z}G$-module. Then

$$\text{Fix}^G \cong \text{Hom}_{\mathbb{Z}G}(\mathbb{Z}, -).$$

In particular, $\text{Fix}^G$ is left exact.

Proof. See [19, Proposition 9.22].

Definition 1.77. Let $G$ be a group, and let $A$ be a $\mathbb{Z}G$-module. The $n$-th cohomology group of $G$ with coefficients in $A$ is defined as

$$H^n(G, A) = \text{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, A),$$

where $\mathbb{Z}$ is the trivial $\mathbb{Z}G$ module.

Thus, by Proposition 1.76, we view group cohomology as the right derived functors for $\text{Fix}^G$. At this point, $H^2(G, A)$ has two definitions. We will now introduce the so-called bar resolution to show that there is no ambiguity.

Definition 1.78. Let $B_0$ be the free $\mathbb{Z}G$-module of rank 1 with basis the symbol $[\ ]$, so $B_0 \cong \mathbb{Z}G$. For $n \geq 1$, let $B_n$ be the free $\mathbb{Z}G$-module with basis the cartesian product $G^n = \prod_{i=1}^{n} G$. Following the standard convention in [19, Section 9.3], we write $[x_1]|...|x_n]$ instead of $(x_1, ..., x_n)$ to denote elements of $G^n$. 

$$f(g \cdot a) = (g \cdot f)(g \cdot a)$$
$$= gf(g^{-1} \cdot (g \cdot a))$$
$$= gf(a).$$
Proposition 1.79. With the above notation, the sequence of $G$-modules

$$B_3 \xrightarrow{d_3} B_2 \xrightarrow{d_2} B_1 \xrightarrow{d_1} B_0 \xrightarrow{\epsilon} \mathbb{Z} \rightarrow 0$$

is exact, and can be taken to be the beginning of a free $\mathbb{Z}G$-module resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$, where for $x_1, x_2, x_3 \in G$,

$$d_3[x_1|x_2|x_3] = x_1[x_2|x_3] - [x_1x_2|x_3] + [x_1|x_2x_3] - [x_1|x_2]$$

$$d_2[x_1|x_2] = x_1[x_2] - [x_1x_2] + [x_1]$$

$$d_1[x] = x[ ] - [ ]$$

$$\epsilon[ ] = 1$$

Proof. See [19, Lemma 9.34].

Thus, $H^2(G, A) = \text{Ext}^2_{\mathbb{Z}G}(\mathbb{Z}, A) = \text{ker}(d_3)/\text{im}(d_2)$. Note that this is consistent with the definition of $H^2(G, A)$ in Definition 1.70.

We now state and prove a technical lemma which is well-known to experts.

Lemma 1.80. If $N$ is an elementary abelian $p$-group of rank $\ell \geq 2$, then $H^a(N, \mathbb{C}^*)$ is an elementary abelian $p$-group for all $a \geq 1$, when $\mathbb{C}^*$ has trivial $N$-action.

Proof. We prove this by induction on $\ell$, using the Künneth formula [5, Proposition I.0.8 together with V.5.4]. By the Künneth formula we have for finite groups $G_1, G_2$ and $a \geq 0$:

$$H^a(G_1 \times G_2, \mathbb{C}^*) \cong \bigoplus_{u+v=a} H^u(G_1, \mathbb{C}^*) \otimes \mathbb{Z} H^v(G_2, \mathbb{C}^*)$$
For \( \ell = 2 \), we set \( G_1 = G_2 = \mathbb{Z}/p\mathbb{Z} \) to obtain

\[
H^a(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}, \mathbb{C}^*) \cong \left( \bigoplus_{u+v=a} H^u(\mathbb{Z}/p\mathbb{Z}, \mathbb{C}^*) \otimes_{\mathbb{Z}} H^v(\mathbb{Z}/p\mathbb{Z}, \mathbb{C}^*) \right) \oplus \left( \bigoplus_{u+v=a+1} \text{Tor}_1^{\mathbb{Z}}(H^u(G_1, \mathbb{C}^*), H^v(G_2, \mathbb{C}^*)) \right).
\]

Since for any finite group \( G \), \( |G| \) annihilates \( H^i(G, \mathbb{C}^*) \) for \( i \geq 1 \) [18, Theorem 10.26], \( H^i(\mathbb{Z}/p\mathbb{Z}, \mathbb{C}^*) \) is an elementary abelian \( p \)-group for all \( i \geq 1 \). Note that the tensor product over \( \mathbb{Z} \) of an elementary abelian \( p \)-group with \( \mathbb{C}^* \) is trivial, and \( \text{Tor}_1^{\mathbb{Z}} \) of an elementary abelian \( p \)-group with \( \mathbb{C}^* \) is elementary abelian. Thus, the result follows for \( \ell = 2 \). For \( \ell \geq 2 \), the result follows by induction.

\[ \square \]

### 1.6 Universal Deformation Rings

In this section, we give a brief introduction to universal deformation rings and deformations. For more background material, we refer the reader to [14] and [8].

Let \( \Gamma \) be a finite group, let \( k \) be a perfect field of characteristic \( p > 0 \), and let \( V \) be a finitely generated \( k\Gamma \)-module. Let \( \hat{\mathcal{C}} \) be the category of all complete local commutative Noetherian rings with residue field \( k \) with specified projection to \( k \).

More precisely, the objects in \( \hat{\mathcal{C}} \) are pairs \((R, \tau)\) where \( R \) is a ring in \( \hat{\mathcal{C}} \), and \( \tau \) is a particular surjection from \( R \) to the residue field \( k \). The morphisms in \( \hat{\mathcal{C}} \) are local homomorphisms of local rings which induce the identity on the residue field \( k \).

**Definition 1.81.** A lift of \( V \) over \( R \) is a pair \((M, \phi)\) where \( M \) is a finitely generated \( R\Gamma \)-module \( M \) which is free over \( R \) and \( \phi \) is a \( k\Gamma \)-module isomorphism \( \phi : k \otimes M \to V \).
Two lifts \((M, \phi)\) and \((M', \phi')\) of \(V\) over \(R\) are said to be isomorphic if there exists an \(R\Gamma\)-module isomorphism \(f : M \to M'\) such that \(\phi' \circ (id_k \otimes f) = \phi\). A deformation of \(V\) over \(R\) is an isomorphism class \([M, \phi]\) of a lift \((M, \phi)\) of \(V\) over \(R\). The set of such deformations is denoted by \(\text{Def}_R(V, R)\).

**Definition 1.82.** The deformation functor

\[
\hat{F}_V : \hat{C} \to \text{Sets}
\]

sends an object \(R\) in \(\hat{C}\) to \(\text{Def}_R(V, R)\) and a morphism \(f : R \to R'\) in \(\hat{C}\) to the map \(\text{Def}_R(V, R) \to \text{Def}_R(V, R')\) defined by \([M, \phi] \mapsto [R' \otimes_{R, f} M, \phi']\), where \(\phi' = \phi\) after identifying \(k \otimes_{R'} (R' \otimes_{R, f} M)\) with \(k \otimes_R M\).

**Definition 1.83.** If there exists an object \(R(\Gamma, V)\) in \(\hat{C}\) and a deformation \([U(\Gamma, V), \phi_U]\) of \(V\) over \(R(\Gamma, V)\) such that for each \(R\) in \(\hat{C}\) and for each lift \((M, \phi)\) of \(V\) over \(R\) there is a unique morphism \(\alpha : R(\Gamma, V) \to R\) in \(\hat{C}\) such that \(\hat{F}_V(\alpha)([U(\Gamma, V), \phi_U]) = [M, \phi]\) and the assignment is natural, then we call \(R(\Gamma, V)\) the universal deformation ring of \(V\) and \([U(\Gamma, V), \phi_U]\) the universal deformation of \(V\).

In other words, if \(R(\Gamma, V)\) exists, \(R(\Gamma, V)\) represents the functor \(\hat{F}_V\) in the sense that \(\hat{F}_V\) is naturally isomorphic to \(\text{Hom}_c(R(\Gamma, V), -)\). In the case when the morphism \(\alpha : R(\Gamma, V) \to R\) relative to any lift \((M, \phi)\) of \(V\) over \(R\) is only known to be unique if \(R\) is the ring of dual numbers \(k[\epsilon]\) but may not be unique for other \(R\), \(R(\Gamma, V)\) is called the versal deformation ring of \(V\) and \([U(\Gamma, V), \phi_U]\) is called the versal deformation of \(V\).
By [14], every finitely generated \( k\Gamma \)-module \( V \) has a versal deformation ring \( R(\Gamma, V) \). Moreover, if \( V \) is an absolutely irreducible \( k\Gamma \)-module, then \( R(\Gamma, V) \) is universal. The following result shows the connection between \( R(\Gamma, V) \) and certain first and second cohomology groups of \( \Gamma \) that are related to \( V \).

**Theorem 1.84.** Suppose \( V \) is an absolutely irreducible \( k\Gamma \)-module, let \( W \) be the ring of infinite Witt vectors over \( k \), and let \( d^1_V = \dim_k \mathcal{H}^1(\Gamma, \text{Hom}_k(V, V)) \) for \( i = 1, 2 \). Then \( R(\Gamma, V) \) is isomorphic to a quotient algebra \( W[[t_1, \ldots, t_r]]/J \) where \( r = d^1_V \), and \( r \) is minimal with this property. Moreover, \( d^2_V \) is an upper bound on the minimal number of generators of \( J \).

**Proof.** See [14, Section 1.6] and [4, Theorem 2.4].

For more background on Witt vectors see [20, Section II.6]. Note that when \( k = \mathbb{F}_p \), then \( W = \mathbb{Z}_p \) (see [20, p. 43]).
CHAPTER 2
COHOMOLOGY

In this chapter we will develop a computational technique we wish to apply in subsequent chapters. Let \( p \geq 3 \) be a prime, and consider a short exact sequence of groups

\[
0 \to N \to \Gamma \to G \cong \Gamma/N \to 1
\]

where \( N \) is an elementary abelian \( p \)-group of rank \( \ell \geq 2 \) and \( G \) is a finite group. We identify \( G \) with \( \Gamma/N \) in the following. Note that since \( N \) is elementary abelian, the action of \( G = \Gamma/N \) on \( N \) corresponds to an \( \mathbb{F}_p \)-representation of \( G \) denoted by \( \phi \). Let \( V \) be a projective \( \mathbb{F}_p \)\( G \)-module, and view \( V \) as an \( \mathbb{F}_p \)\( \Gamma \)-module by inflation. Let \( \tilde{\phi} \) be the contragredient of \( \phi \) (i.e. \( \tilde{\phi} \) is the representation such that \( \tilde{\phi}(g) \) is the transpose of \( \phi(g^{-1}) \) for all \( g \in G \)). Let \( V_{\tilde{\phi}} \) (resp. \( V_{\tilde{\phi} \wedge \tilde{\phi}} \)) denote the \( \mathbb{F}_p \)\( \Gamma \)-module associated to \( \tilde{\phi} \) (resp. \( \tilde{\phi} \wedge \tilde{\phi} \)). If \( X \) is a \( G \)-module, let \( X^G \) denote the fixed points of the action of \( G \).

Let \( \otimes \) stand for the tensor product over \( \mathbb{F}_p \). We prove the following result.

**Theorem 2.1.** Using the above notation,

\[
H^2(\Gamma, \text{Hom}_{\mathbb{F}_p}(V, V)) \cong [(V_{\tilde{\phi}} \otimes V^* \otimes V) \oplus (V_{\tilde{\phi} \wedge \tilde{\phi}} \otimes V^* \otimes V)]^G.
\]

If \( N \) is elementary abelian of rank two, the representation \( \tilde{\phi} \wedge \tilde{\phi} \) is the one-dimensional representation \( \text{det} \circ (\tilde{\phi}) \).

In the case when \( \mathbb{F}_p G \) is semisimple, this result will provide a way of using character theory to compute the first and second cohomology groups of \( \Gamma \) with coefficients
in Hom_{\mathbb{F}_p}(V, V). To prove Theorem 2.1 we need the following result.

**Proposition 2.2.** Let $A$ be a projective $\mathbb{F}_pG$-module. Then for all $i \geq 1$,

$$A \otimes H^i(N, \mathbb{F}_p) \cong H^i(N, A)$$

as $\mathbb{F}_pG$-modules, and

$$H^i(\Gamma, A) \cong H^0(\Gamma/N, H^i(N, A)) \cong [H^i(N, A)]^G.$$ 

**Proof.** Let $i \geq 1$. We first show that $H^i(N, A) \cong A \otimes H^i(N, \mathbb{F}_p)$ as $\mathbb{F}_pG$-modules for $i \geq 1$, where we identify $G$ with $\Gamma/N$ as before. Let $Z^i(N, A)$ denote the space of $i$-cocycles of $N$ with coefficients in $A$, and let $B^i(N, A)$ denote the space of $i$-coboundaries of $N$ with coefficients in $A$. Let $\{e_j\}$ be an $\mathbb{F}_p$-basis for $A$. Since $A$ is an $\mathbb{F}_pG$-module, $N$ acts trivially on $A$.

Consider the maps

$\Phi : A \otimes Z^i(N, \mathbb{F}_p) \to Z^i(N, A), \quad a \otimes c \mapsto \Delta_{c,a},$ for all $(a, c)$ in $A \times Z^i(N, \mathbb{F}_p)$

$\Psi : Z^i(N, A) \to A \otimes Z^i(N, \mathbb{F}_p), \quad d \mapsto \sum_j e_j \otimes (e_j^* \circ d),$ for all $d$ in $Z^i(N, A)$

where $\Delta_{c,a}(n_1, n_2, ..., n_i) = c(n_1, n_2, ..., n_i)a$ and $\{e_j^*\}$ is the dual basis to $\{e_j\}$. We will show that $\Psi$ and $\Phi$ are $\mathbb{F}_pG$-module isomorphisms. First, note that if $c$ is an element of $Z^i(N, \mathbb{F}_p)$, then $\Delta_{c,a}$ is a cocycle with coefficients in $A$ for every $a \in A$. Similiarly, for every $j$, $e_j^* \circ d$ is a cocycle with coefficients in $\mathbb{F}_p$, for every cocycle $d$ with coefficients in $A$. Now, let $g \in G$, and let $a \otimes c$ be a simple tensor in $A \otimes Z^i(N, \mathbb{F}_p)$. 
Then,

\[(g \cdot \Phi(a \otimes c))(n_1, n_2, ..., n_i) = (g \cdot \Delta_{c,a})(n_1, n_2, ..., n_i)\]
\[= g \Delta_{c,a}(g^{-1} \cdot n_1, g^{-1} \cdot n_2, ..., g^{-1} \cdot n_i)\]
\[= gc(g^{-1} \cdot n_1, g^{-1} \cdot n_2, ..., g^{-1} \cdot n_i)a\]
\[= c(g^{-1} \cdot n_1, g^{-1} \cdot n_2, ..., g^{-1} \cdot n_i)ga.\]

Also,

\[\Phi(g \cdot (a \otimes c))(n_1, n_2, ..., n_i) = \Delta_{g \cdot c, g \cdot a}(n_1, n_2, ..., n_i)\]
\[= (g \cdot c)(n_1, n_2, ..., n_i)(g \cdot a)\]
\[= c(g^{-1} \cdot n_1, g^{-1} \cdot n_2, ..., g^{-1} \cdot n_i)(ga).\]

Thus, \(\Phi\) is a \(G\)-module map.

Now, \(\Psi(\Phi(a \otimes c)) = \sum_j e_j \otimes (e_j^* \circ \Delta_{c,a}).\) But if \(a = \sum_i a_i e_i,\) then \(e_k^* \circ \Delta_{c,a} = e_k^* \circ c \sum_i a_i e_i = ca_k.\) Therefore,

\[\Psi(\Phi(a \otimes c)) = \sum_j e_j \otimes ca_j\]
\[= \sum_j a_j e_j \otimes c\]
\[= a \otimes c.\]
Also,

\[
\Phi(\Psi(d)) = \Phi(\sum_j e_j \otimes (e_j^* \circ d)) \\
= \sum_j \Phi(e_j \otimes (e_j^* \circ d)) \\
= \sum_j \Delta_{e_j^* \circ d, e_j} \\
= d.
\]

Thus, \( \Phi \) and \( \Psi \) are inverses of each other, and \( \Psi \) is a \( G \)-module map. Next, note that \( \Psi \) and \( \Phi \) restrict to isomorphisms between \( B^i(N, A) \), and \( A \otimes B^i(N, \mathbb{F}_p) \). Thus,

\[
H^i(N, A) \cong \frac{A \otimes Z^i(N, \mathbb{F}_p)}{A \otimes B^i(N, \mathbb{F}_p)}
\]

as \( \mathbb{F}_pG \)-modules.

Next, we tensor the short exact sequence of \( \mathbb{F}_pG \)-modules

\[
0 \to B^i(N, \mathbb{F}_p) \to Z^i(N, \mathbb{F}_p) \to H^i(N, \mathbb{F}_p) \to 0
\]

with \( A \) over \( \mathbb{F}_p \). Since \( A \) is flat, we obtain \( A \otimes H^i(N, \mathbb{F}_p) \cong \frac{A \otimes Z^i(N, \mathbb{F}_p)}{A \otimes B^i(N, \mathbb{F}_p)} \) as \( \mathbb{F}_pG \)-modules. Therefore, \( A \otimes H^i(N, \mathbb{F}_p) \cong H^i(N, A) \) as \( \mathbb{F}_pG \)-modules. This implies that, in particular, \( H^i(N, A) \) is a projective \( \mathbb{F}_pG \)-module, since the tensor product of a projective \( \mathbb{F}_pG \)-module with an arbitrary \( \mathbb{F}_pG \)-module is projective.

Next, consider the Lyndon-Hochschild-Serre spectral sequence (see Theorem 1.45)

\[
H^{p_0}(\Gamma/N, H^{q_0}(N, A)) \Rightarrow H^{p_0+q_0}(\Gamma, A).
\]

By the above argument, \( H^{q_0}(N, A) \) is a projective \( \mathbb{F}_pG \)-module for all \( q_0 \geq 1 \). Since \( H^0(N, A) \cong A^N \cong A \) which is also projective, the terms corresponding to \( (p_0, q_0) = \)
$(1, i - 1), (2, i - 2), \ldots, (i, 0)$ vanish for $i = p_0 + q_0 \geq 1$. Therefore, the spectral sequence degenerates, and $H^i(\Gamma, A) \cong H^0(\Gamma/N, H^i(N, A))$. \hfill \square

We are now ready to show the main result of the section.

Proof of Theorem 2.1. Recall that $V$ is assumed to be a projective $\mathbb{F}_pG$-module, where we identify $G$ with $\Gamma/N$. By Proposition 2.2,

$$H^2(N, \text{Hom}_{\mathbb{F}_p}(V, V)) \cong \text{Hom}_{\mathbb{F}_p}(V, V) \otimes H^2(N, \mathbb{F}_p)$$

as $\mathbb{F}_pG$-modules.

Consider the Kummer sequence $1 \to \mu_p \xrightarrow{i} \mathbb{C}^* \xrightarrow{p} \mathbb{C}^* \to 1$, where $\mu_p$ is the group of $p$-th roots of unity in $\mathbb{C}^*$ and $\mathbb{C}^* \xrightarrow{p} \mathbb{C}^*$ denotes the map given by $z \mapsto z^p$. Since $\mathbb{C}^*$ is divisible, this sequence is exact in $\mathbb{Z}$-$\text{Mod}$. We view this as a sequence in $\mathbb{Z}N$-$\text{Mod}$, where each module is endowed with trivial $N$-action.

Applying the functor $\text{Hom}_{\mathbb{Z}N}(\mathbb{Z}, -)$ we obtain the long exact sequence of $\mathbb{F}_pG$-modules

$$\ldots \xrightarrow{\delta} H^1(N, \mu_p) \xrightarrow{i} H^1(N, \mathbb{C}^*) \xrightarrow{p} H^1(N, \mathbb{C}^*) \xrightarrow{\delta} H^2(N, \mu_p) \xrightarrow{i} \ldots$$

$$H^2(N, \mathbb{C}^*) \xrightarrow{p} H^2(N, \mathbb{C}^*) \xrightarrow{\delta} H^3(N, \mu_p) \xrightarrow{i} \ldots$$

where $\delta$ denotes the appropriate connecting morphism. Since $N$ is an elementary abelian $p$-group, $H^i(N, \mathbb{C}^*) \xrightarrow{p} H^i(N, \mathbb{C}^*)$ is trivial for $i \geq 1$ (see Lemma 1.80). Identifying $\mathbb{F}_p = \mu_p$, we get a short exact sequence of $\mathbb{F}_pG$-modules

$$0 \to H^1(N, \mathbb{C}^*) \xrightarrow{\delta} H^2(N, \mathbb{F}_p) \xrightarrow{i} H^2(N, \mathbb{C}^*) \to 0.$$
Hom\(_{\mathbb{F}_p}(V, V)\), we obtain
\[ H^2(\Gamma, Hom_{\mathbb{F}_p}(V, V)) \cong [(H^1(N, \mathbb{C}^*) \otimes Hom_{\mathbb{F}_p}(V, V))]^G \oplus [H^2(N, \mathbb{C}^*) \otimes Hom_{\mathbb{F}_p}(V, V)]^G. \]

Therefore, Theorem 2.1 follows once we have shown that
\[ H^1(N, \mathbb{C}^*) \cong V_{\hat{\phi}} \]
and
\[ H^2(N, \mathbb{C}^*) \cong V_{\hat{\phi} \wedge \hat{\phi}} \]
as \(\mathbb{F}_pG\)-modules.

Since \(N\) is an elementary abelian \(p\)-group which acts trivially on \(\mathbb{C}^*\),
\[ H^1(N, \mathbb{C}^*) = Hom(N, \mathbb{C}^*) \cong Hom_{\mathbb{F}_p}(N, \mathbb{F}_p) \]
as \(\mathbb{F}_pG\)-modules, which implies \(H^1(N, \mathbb{C}^*) \cong V_{\hat{\phi}}\). Explicitly, for \(H^1(N, \mathbb{C}^*)\), fix a primitive \(p\)-th root of unity \(\eta\) in \(\mathbb{C}^*\). Let \(\{e_1, e_2\}\) generate \(N = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}\).

Define \(\theta : H^1(N, \mathbb{C}^*) \rightarrow V_{\hat{\phi}}\) by \(\theta(f) = \begin{pmatrix} i \\ j \end{pmatrix}\), where \(f(e_1) = \eta^i, f(e_2) = \eta^j\). Then, \(\theta\) is an isomorphism of \(\mathbb{F}_p\)-vector spaces. We now prove that \(\theta\) is an \(\mathbb{F}_pG\)-module isomorphism between \(H^1(N, \mathbb{C}^*)\) and \(V_{\hat{\phi}}\). Let \(f, \eta, \theta\) be as above. Let \(g \in G\) with \(g^{-1}\) corresponding to the matrix \(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\) in coordinates relative to \(\{e_1, e_2\}\). Then,
\[ \theta(g \cdot f)(e_1) = f(g^{-1} \cdot e_1) = f(ae_1 + ce_2) = f(e_1)^a f(e_2)^c = \eta^{ai} \eta^{jc} = \eta^{ai+cj}. \]
Also,
\[ \theta(g \cdot f)(e_2) = f(g^{-1} \cdot e_2) = f(be_1 + de_2) = f(e_1)^b f(e_2)^d = \eta^{ib} \eta^{jd} = \eta^{bi+dj}. \]

Therefore,
\[ \theta(g \cdot f) = \begin{pmatrix} ai + cj \\ bi + dj \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T \begin{pmatrix} i \\ j \end{pmatrix}. \]
which implies that $\theta$ is an $\mathbb{F}_p$-module isomorphism.

It remains to determine the $G$-module structure of $H^2(N, \mathbb{C}^*)$. (See Appendix A for an independent argument in the case when $N$ is elementary abelian of rank two, $\mathbb{F}_p G$ is semisimple and $\mathbb{F}_p$ is a sufficiently large field for $G$. Note that this more special result is sufficient for the subsequent results.) We use the general result (see [17, 11.4.15 and 11.4.16] and [12, Hauptsatz V.23.5]) that for any finite group $H$,

$$H^2(H, \mathbb{C}^*) \cong (H/[H,H]) \wedge (H/[H,H]),$$

where the wedge product is taken over $\mathbb{Z}$. Since $N$ is abelian,

$$H^2(N, \mathbb{C}^*) \cong N \wedge N.$$ 

As $N$ is an elementary abelian $p$-group, the wedge product may be taken over $\mathbb{F}_p$. Therefore, $\{e_i \wedge e_j\}_{1 \leq i < j \leq \ell}$ defines an $\mathbb{F}_p$-basis for $H^2(N, \mathbb{C}^*)$, for any $\mathbb{F}_p$-basis $\{e_1, e_2, ..., e_\ell\}$ of $N$. We give an explicit $\mathbb{F}_p$-basis for $H^2(N, \mathbb{C}^*)$ when $\ell = 3$. As above, $N \wedge N$ is an elementary abelian $p$-group of rank 3 with $\mathbb{F}_p$-basis $\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$.

We define three 2-cocycles $c_{1,2}, c_{1,3}, c_{2,3} : N \times N \to \mathbb{C}^*$ such that $\{c_{1,2}, c_{1,3}, c_{2,3}\}$ is an $\mathbb{F}_p$-basis for $H^2(N, \mathbb{C}^*)$.

Define $c_{1,2} : N \times N \to \mathbb{C}^*$, by

$$c_{1,2} : \left( \begin{array}{c} n_1 \\ n_2 \\ n_3 \end{array} \right) \left( \begin{array}{c} m_1 \\ m_2 \\ m_3 \end{array} \right) \rightarrow c_{1,2} \eta^{n_1 m_2 - m_1 n_2},$$

where $\eta$ is a fixed $p$-th root of unity in $\mathbb{C}^*$. We will show that $c_{1,2}$ satisfies the 2-
cocyce condition. Let $\bar{h} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}$, $\bar{k} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$, $\bar{\ell} = \begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{pmatrix}$ be three elements of $N$. Then,

$$c_{1,2}(\bar{h}, \bar{k}) \cdot c_{1,2}(\bar{h} + \bar{k}, \bar{\ell}) = \eta^{h_1 k_2 - h_2 k_1} \cdot \eta^{(h_1 + k_1) \ell_2 - (h_2 + k_2) \ell_1}$$

$$= \eta^{h_1 k_2 - h_2 k_1 + h_1 \ell_2 + k_1 \ell_2 - h_2 \ell_1 - k_2 \ell_1}$$

$$= \eta^{k_1 \ell_2 - k_2 \ell_1} \cdot \eta^{h_1(k_2 + \ell_2) - h_2(k_1 + \ell_1)}$$

$$= c_{1,2}(\bar{k}, \bar{\ell}) \cdot c_{1,2}(\bar{h}, \bar{k} + \bar{\ell})$$

$$= (\bar{h} \cdot c_{1,2}(\bar{k}, \bar{\ell})) \cdot c_{1,2}(\bar{h}, \bar{k} + \bar{\ell})$$

as required, where we write the 2-cocycle condition multiplicatively. Note that the last equality holds since $N$ acts trivially on $C^*$. Also, $c_{1,2}$ is not a coboundary. If, for a contradiction, $f : N \to C^*$ was any function such that $\delta f = c_{1,2}$, then for all $\bar{n}, \bar{m} \in N$,

$$c_{1,2}(\bar{n}, \bar{m}) = (\delta f)(\bar{n}, \bar{m})$$

$$= (n \cdot f(\bar{m})) \cdot (f(\bar{n} + \bar{m}))^{-1} \cdot f(\bar{n})$$

$$= f(\bar{n}) \cdot f(\bar{m}) \cdot (f(\bar{n} + \bar{m}))^{-1},$$

where we again write the 2-coboundary condition multiplicatively. Setting $\bar{n} = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$
and $\tilde{m} = \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}$ and solving for $f(\tilde{n} + \tilde{m})$, we see that $f$ sends

$$\begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \mapsto f(\tilde{n}) \cdot f(\tilde{m}) \cdot \eta^{-ab}.$$

Similarly, by switching the roles of $\tilde{n}$ and $\tilde{m}$ and solving, we see that $f$ must send

$$\begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \mapsto f(\tilde{n}) \cdot f(\tilde{m}) \cdot \eta^{ab}.$$

Setting $a = b = 1$, we obtain $\eta = \eta^{-1}$. Since $\eta$ is a primitive $p$-th root of unity, and $p \geq 3$, this contradiction shows that $c_{1,2}$ is not a coboundary. We denote the corresponding non-trivial element in $H^2(N, \mathbb{C}^*)$ also by $c_{1,2}$. Analogously, define $c_{1,3}, c_{2,3} : N \times N \to \mathbb{C}^*$, by

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \mapsto c_{1,3} \eta^{n_1m_3 - m_1n_3},$$

and

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \mapsto c_{2,3} \eta^{n_2m_3 - m_2n_3}.$$
By the previous argument, \( c_{1,3}, c_{2,3} \) also define non-trivial elements of \( H^2(N, \mathbb{C}^*) \) which we again denote by \( c_{1,3}, c_{2,3} \). We now show that \( \{c_{1,2}, c_{1,3}, c_{2,3}\} \) is a linearly independent set over \( \mathbb{F}_p \). Say

\[
c = c_{1,2}^\alpha \cdot c_{1,3}^\beta \cdot c_{2,3}^\gamma
\]

is trivial, where we write multiplicatively, and where \( \alpha, \beta, \gamma \in \mathbb{F}_p \). In other words, \( c \) is a coboundary, say \( c = \delta f \) for some \( f : N \to \mathbb{C}^* \). As above, this means that for all \( \bar{n}, \bar{m} \in N \),

\[
c(\bar{n}, \bar{m}) = (\delta f)(\bar{n}, \bar{m}) = f(\bar{n}) \cdot f(\bar{m}) \cdot (f(\bar{n} + \bar{m}))^{-1}.
\]

Arguing as above, we see that this implies that \( f \) must send

\[
\begin{pmatrix}
a \\
b \\
0
\end{pmatrix}
\]

\( \mapsto f(\bar{n}) \cdot f(\bar{m}) \cdot \eta^{-\alpha ab} \)

when \( \bar{n} = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \) and \( \bar{m} = \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix} \). On the other hand, switching the roles of \( \bar{n}, \bar{m} \), we see that \( f \) must send

\[
\begin{pmatrix}
a \\
b \\
0
\end{pmatrix}
\]

\( \mapsto f(\bar{n}) \cdot f(\bar{m}) \cdot \eta^{\alpha ab} \).

Again, since \( p \) is odd, this is a contradiction unless \( \alpha = 0 \). Similarly, \( \beta = \gamma = 0 \).

Thus, since \( H^2(N, \mathbb{C}^*) \) has \( \mathbb{F}_p \)-dimension 3, \( \{c_{1,2}, c_{1,3}, c_{2,3}\} \) is an \( \mathbb{F}_p \)-basis for \( H^2(N, \mathbb{C}^*) \).
Now, define

$$\Phi : H^2(N, C^*) \rightarrow N \wedge N,$$

by

$$c_{1,2}^\alpha \cdot c_{1,3}^\beta \cdot c_{2,3}^\gamma \xrightarrow{\Phi} \alpha \cdot e_1 \wedge e_2 + \beta \cdot e_1 \wedge e_3 + \gamma \cdot e_2 \wedge e_3.$$ 

We will show $\Phi$ is an $\mathbb{F}_p \Gamma$-module isomorphism, when $N \wedge N$ is endowed with the $G$-action given by $\tilde{\phi} \wedge \tilde{\phi}$. Let $\mu \in G$, and using the basis $\{e_1, e_2, e_3\}$, suppose $\mu^{-1}$ corresponds to the matrix

$$\begin{pmatrix}
  a & b & c \\
  d & e & f \\
  g & h & i
\end{pmatrix}$$

under $\phi$. Then, by direct computation,

$$\Phi(\mu \cdot (c_{1,2}^\alpha \cdot c_{1,3}^\beta \cdot c_{2,3}^\gamma)) = \Phi((c_{1,2}^{\alpha (ae-db) + \beta (ah-gb) + \gamma (dh-ge)})
\cdot (c_{1,3}^{\alpha (af-dc) + \beta (ai-cg) + \gamma (di-fg)})
\cdot (c_{2,3}^{\alpha (bf-ce) + \beta (bi-hc) + \gamma (ei-hf)}))$$

$$= [\alpha (ae - db) + \beta (ah - gb) + \gamma (dh - ge)] e_1 \wedge e_2$$
$$+ [\alpha (af - dc) + \beta (ai - cg) + \gamma (di - fg)] e_1 \wedge e_3$$
$$+ [\alpha (bf - ce) + \beta (bi - hc) + \gamma (ei - hf)] e_2 \wedge e_3$$

$$= \alpha [(ae - db)e_1 \wedge e_2 + (af - dc)e_1 \wedge e_3 + (bf - ce)e_2 \wedge e_3]$$
$$+ \beta [(ah - gb)e_1 \wedge e_2 + (ai - cg)e_1 \wedge e_3 + (bi - hc)e_2 \wedge e_3]$$
$$+ \gamma [(dh - ge)e_1 \wedge e_2 + (di - fg)e_1 \wedge e_3 + (ei - hf)e_2 \wedge e_3]$$

$$= \alpha \phi(\mu^{-1})^T e_1 \wedge \phi(\mu^{-1})^T e_2$$
$$+ \beta \phi(\mu^{-1})^T e_1 \wedge \phi(\mu^{-1})^T e_3$$
$$+ \gamma \phi(\mu^{-1})^T e_2 \wedge \phi(\mu^{-1})^T e_3$$

$$= \mu \cdot \Phi(c_{1,2}^\alpha \cdot c_{1,3}^\beta \cdot c_{2,3}^\gamma).$$
A similar, though notationally cumbersome, proof works for arbitrary \( \ell \geq 2 \). Thus, 
\[
H^2(N, \mathbb{C}^*) \cong V_{\phi^\wedge_\phi} \text{ as } \mathbb{F}_p G\text{-modules.}
\]
This completes the proof of Theorem 2.1.

As a consequence of the proof of Theorem 2.1 we obtain the following result.

**Corollary 2.3.** *Under the general hypothesis of Theorem 2.1, we obtain:*

a. \( H^1(\Gamma, \text{Hom}_{\mathbb{F}_p}(V, V)) \cong (V_\phi \otimes V^* \otimes V)^G \).

b. \( H^1(\Gamma, \text{Hom}_{\mathbb{F}_p}(V, V)) \) is a summand of \( H^2(\Gamma, \text{Hom}_{\mathbb{F}_p}(V, V)) \).

c. \( \dim_{\mathbb{F}_p}(H^1(\Gamma, \text{Hom}_{\mathbb{F}_p}(V, V))) \leq \dim_{\mathbb{F}_p}(H^2(\Gamma, \text{Hom}_{\mathbb{F}_p}(V, V))) \).

**Proof.** For a., we note that since \( \mathbb{C}^* \xrightarrow{\alpha} \mathbb{C}^* \) is surjective, and \( H^1(N, \mathbb{C}^*) \xrightarrow{\alpha^*} H^1(N, \mathbb{C}^*) \) is trivial, \( H^1(N, \mu_p) \cong H^1(N, \mathbb{C}^*) \). Thus, by Proposition 2.2,

\[
H^1(\Gamma, \text{Hom}_{\mathbb{F}_p}(V, V)) \cong [H^1(N, \text{Hom}_{\mathbb{F}_p}(V, V))]^G \\
\cong [H^1(N, \mathbb{F}_p) \otimes \text{Hom}_{\mathbb{F}_p}(V, V)]^G \\
\cong [H^1(N, \mathbb{C}^*) \otimes V^* \otimes V]^G \\
\cong (V_\phi \otimes V^* \otimes V)^G.
\]

Thus, b. follows from Theorem 2.1, and c. follows trivially.

In the following section, we apply the above result to the special case

\[
0 \to N \to \Gamma \to G = \Gamma/N \to 1
\]

where \( \mathbb{F}_p G \) is semisimple, \( \mathbb{F}_p \) is a sufficiently large field for \( G \), and \( V \) is an irreducible \( \mathbb{F}_p G\)-module. As above, let \( \phi \) denote the action of \( G \) on \( N \). Our goal is to relate the
cohomology groups $H^i(\Gamma, \text{Hom}_{\mathbb{F}_p}(V, V))$ for $i = 1, 2$ to the fusion of $N$ in $\Gamma$.

For what follows, we will need the following definition.

**Definition 2.4.** Let $N, \Gamma, G, \phi$ be as above.

(a) For every irreducible $\mathbb{F}_pG$-module $V$, let $d^i_V = \dim_{\mathbb{F}_p}(H^i(\Gamma, \text{Hom}_{\mathbb{F}_p}(V, V))$ for $i = 1, 2$. Note that this number depends on $\phi$. We say an irreducible $\mathbb{F}_pG$-module $V_0$ is cohomologically maximal for $\phi$ if $d^2_{V_0}$ is maximal among all $d^2_V$.

We say an irreducible representation $\rho$ of $G$ over $\mathbb{F}_p$ is cohomologically maximal for $\phi$ if $\rho$ corresponds to an $\mathbb{F}_pG$-module with this property.

(b) We call the orbits of the action $\phi$ of $G$ on $N$ the fusion orbits of $\phi$. For all $m \geq 1$, let $F_{\phi, m}$ be the number of fusion orbits of $\phi$ with cardinality $m$. We call the sequence $\{F_{\phi, m}\}_{m \geq 1}$ the fusion numbers of $\phi$. We call the fusion orbits of $\phi$ the fusion of $\phi$.

Note that if $n_1, n_2 \in N$ are fused in $\Gamma$, then there exists an element $(n, g) \in \Gamma$ (i.e. $n \in N, g \in G$) such that

$$(n_2, 1) = (n, g)(n_1, 1)(g^{-1} \cdot (-n), g^{-1})$$

$$= (n + g \cdot n_1, g)(g^{-1} \cdot (-n), g^{-1})$$

$$= (n + g \cdot n_1 - n, 1)$$

$$= (g \cdot n_1, 1).$$

Hence two elements in $N$ are conjugate in $\Gamma$ if and only if they lie in the same fusion orbit of $\phi$. Thus, the fusion of $N$ in $G$ is uniquely determined by the fusion orbits of
\( \phi \), and hence by the fusion of \( \phi \). In the next chapter we state our main results.
CHAPTER 3
MAIN RESULTS

In this chapter, we consider the case when $\ell = 2, n \geq 3$ and $\Gamma/N = G$ is the dihedral group $D_{2n}$ of order $2n$. That is, we have a short exact sequence of groups

$$0 \to N \to \Gamma \to G = \Gamma/N \to 1$$

where $G$ is dihedral and $N$ is an elementary abelian $p$-group of rank two. As before, $p$ is an odd prime. Moreover, we assume $\mathbb{F}_pG$ is semisimple and $\mathbb{F}_p$ is a sufficiently large field for $G$. These conditions together guarantee that the representation theory of $G$ over $\mathbb{F}_p$ can be identified with the representation theory of $G$ over $\mathbb{C}$. Again, we let $\phi$ denote the action of $G$ on $N$, and we now assume $\phi$ is irreducible (and, hence, absolutely irreducible). Our main results, Theorems 3.3 and 3.4, show how the first and second cohomology groups, respectively the universal deformation rings, associated to certain $\mathbb{F}_pG$-modules $V$, can detect the fusion of $N$ in $\Gamma$, i.e. the fusion of $\phi$.

**Definition 3.1.** Let $\text{Rep}_2(G)$ be a complete set of representatives of isomorphism classes of 2-dimensional representations of $G$ over $\mathbb{F}_p$. Let $\text{Irr}_2(G) \subset \text{Rep}_2(G)$ be the subset of isomorphism classes of irreducible 2-dimensional representations. For $\rho$ in $\text{Irr}_2(G)$, let $V_\rho$ be an irreducible $\mathbb{F}_pG$-module with representation $\rho$. Note that $\text{Irr}_2(G) = \text{Irr}_2(\Gamma)$ since all irreducible $\mathbb{F}_p\Gamma$-modules are inflated from irreducible $\mathbb{F}_pG$-modules.

We let $n \geq 3$, and consider the standard presentation for $G = D_{2n}$, given by
Moreover, we assume \( p \equiv 1 \pmod{n} \). This requirement ensures that \( \mathbb{F}_p G \) is semisimple and \( \mathbb{F}_p \) is a sufficiently large field for \( G \). Recall that all 2-dimensional irreducible representations of \( G \) over \( \mathbb{F}_p \) are of the form \( \theta_i \) with

\[
\begin{pmatrix}
\omega^i & 0 \\
0 & \omega^{-i}
\end{pmatrix}
\]

for \( 1 \leq i < \frac{n}{2} \), and \( \omega \) a primitive \( n \)-th root of unity in \( \mathbb{F}_p^\times \). Note that \( \theta_i = \text{Ind}_{(r)}^G(\chi_i) \), where \( \chi_i \) is the one-dimensional representation of \( \langle r \rangle \) with \( \chi_i(r) = \omega^i \). Let \( \{e_1, e_2\} \) denote the \( \mathbb{F}_p \)-basis corresponding to the matrices above.

**Definition 3.2.** Define the set map \( T : \text{Irr}_2(G) \to \text{Rep}_2(G) \) by

\[
T(\theta_i) = T(\text{Ind}_{(r)}^G(\chi_i)) = \text{Ind}_{(r)}^G(\chi_i^2).
\]

If \( n \) is odd, let \( \Omega = \text{Irr}_2(G) = T(\text{Irr}_2(G)) \). If \( n \) is even, let \( \Omega = \text{Irr}_2(G) \cap T(\text{Irr}_2(G)) \).

Note that in the latter case, for all \( \psi \) in \( \Omega \), \( |T^{-1}(\psi)| = 2 \).

**Theorem 3.3.** If \( \phi \in \Omega \), then the fusion of \( \phi \) is uniquely determined by the set

\[
\{ \ker(\psi) : \psi \in \text{Irr}_2(G) \text{ is cohomologically maximal for } \phi \} =
\]

\[
\{ \ker(\psi) : \psi \in \text{Irr}_2(G) \text{ with } R(\Gamma, V_\psi) \not\cong \mathbb{Z}_p \} =
\]

\[
\{ \ker(\psi) : \psi \in \text{Irr}_2(G) \text{ with } R(\Gamma, V_\psi) \cong \mathbb{Z}_p[[t]]/(t^2, pt) \}.
\]

**Theorem 3.4.** Let \( G = D_{2n} \). Let \( T \) and \( \Omega \) be as above.

a. Let \( n \) be arbitrary and let \( \phi \) be in \( \Omega \). Then, for any \( \psi \) in \( \text{Irr}_2(G) \), \( \psi \) is cohomologically maximal for \( \phi \) if and only if \( T(\psi) = \phi \).
b. Let \( n \) be odd, and let \( \phi_1, \phi_2 \in \text{Irr}_2(G) = \Omega \). Then \( \phi_1 \) and \( \phi_2 \) have the same fusion if and only if \( T^{-1}(\phi_1) \) and \( T^{-1}(\phi_2) \) have the same kernel.

c. Let \( n \) be even, and let \( \phi_1, \phi_2 \in \Omega \). Then \( \phi_1 \) and \( \phi_2 \) have the same fusion if and only if \( \{ \text{kernel of } \psi : \psi \in T^{-1}(\phi_1) \} = \{ \text{kernel of } \psi : \psi \in T^{-1}(\phi_2) \} \).

Theorems 3.3 and 3.4 say that for \( \phi \) in \( \Omega \), the fusion of \( N \) in \( \Gamma \) can be detected by the cohomology groups, respectively the universal deformation rings, in the following sense. Given \( \phi \) in \( \Omega \), we may determine the irreducible representations \( \psi \) such that \( \psi \) is cohomologically maximal for \( \phi \). Additionally, this assignment is reversible. That is, given a collection of irreducible representations that are cohomologically maximal for some \( \phi \) in \( \Omega \), we may determine \( \phi \). Moreover, given only the fusion of \( \phi \) in \( \Omega \) we can determine the kernels of the representations that are cohomologically maximal for \( \phi \). Analogously, this assignment is again reversible. In addition, since \( \psi \) is cohomologically maximal for \( \phi \) in \( \Omega \) if and only if \( R(\Gamma, V_\psi) \not\cong \mathbb{Z}_p \) if and only if \( R(\Gamma, V_\psi) \cong \mathbb{Z}_p[[t]]/(t^2, pt) \), the fusion of \( N \) in \( \Gamma \) can also be determined by the knowledge of the universal deformation rings.

Thus, for \( \phi \) in \( \Omega \) we have the following one-to-one correspondences:

\[
\begin{align*}
\phi & \leftrightarrow \{ \psi \in \text{Irr}_2(G) : \psi \text{ is cohomologically maximal for } \phi \} \\
\phi & \leftrightarrow \{ \psi \in \text{Irr}_2(G) : R(\Gamma, V_\psi) \not\cong \mathbb{Z}_p \} \\
\phi & \leftrightarrow \{ \psi \in \text{Irr}_2(G) : R(\Gamma, V_\psi) \cong \mathbb{Z}_p[[t]]/(t^2, pt) \}.
\end{align*}
\]
And also,

Fusion of $\phi \rightsquigarrow \{\ker(\psi) : \psi \in \text{Irr}_2(G) \text{ is cohomologically maximal for } \phi\}$

Fusion of $\phi \rightsquigarrow \{\ker(\psi) : \psi \in \text{Irr}_2(G) \text{ and } R(\Gamma, V_\psi) \not\cong \mathbb{Z}_p\}$

Fusion of $\phi \rightsquigarrow \{\ker(\psi) : \psi \in \text{Irr}_2(G) \text{ and } R(\Gamma, V_\psi) \cong \mathbb{Z}_p[[t]]/(t^2, pt)\}$

In Chapters 4-7, we prove our main results. In Chapter 8, we briefly discuss the case when $G = \Gamma/N$ is an abelian group and compare this case to the dihedral case.
In this chapter we determine $H^2(\Gamma, \text{Hom}_{F_p}(V_\psi, V_\psi))$ for $\phi$ in $\Omega$ and $\psi$ in $\text{Irr}_2(G)$. We make the same assumptions as in Chapter 3. In particular, $n \geq 3$, $G = D_{2n}$, and $p \equiv 1(\text{mod } n)$, which means that $F_pG$ is semisimple and $F_p$ is a sufficiently large field for $G$. Recall, we have that $T : \text{Irr}_2(G) \to \text{Rep}_2(G)$ is given by $T(\theta_i) = T(\text{Ind}^G_{(r)}(\chi_i)) = \text{Ind}^G_{(r)}(\chi_i^2)$. Recall also that for $n$ odd, $T$ is a bijection from $\text{Irr}_2(G)$ to $\text{Irr}_2(G) = \Omega$. For $n$ even, $\Omega = \text{Irr}_2(G) \cap T(\text{Irr}_2(G))$ and $T$ is a two-to-one set map.

**Proposition 4.1.** Let $G = D_{2n}$. Let $\Omega$ and $T$ be as above.

a. Let $n$ be odd, and let $\phi$ be an element of $\text{Irr}_2(G) = \Omega$. Then, there exists a unique $\psi = T^{-1}(\phi)$ in $\text{Irr}_2(G)$ with $d^2_V = 2$. For all other $V$, $d^2_V = 1$. So $V_\psi$ is cohomologically maximal for $\phi$.

b. Let $n$ be even, and let $\phi$ be an element of $\Omega$. Then, there exist exactly two $\psi$ in $\text{Irr}_2(G)$ with $d^2_V = 2$. For all other $V, d^2_V = 1$. Thus, there are precisely two $\psi$ that are cohomologically maximal for $\phi$. These representations are exactly the elements of $T^{-1}(\{\phi\})$.

The proposition follows from the following two lemmas.

**Lemma 4.2.** Let $G = D_{2n}$, let $1 \leq i < \frac{n}{2}$, let $V = V_{\theta_i}$, and let $\phi = T(\theta_i)$. Then,

$$V^* \otimes V \cong F_p \oplus V_{\chi_1} \oplus V_\phi,$$
as \( \mathbb{F}_p \)-modules, where \( \mathbb{F}_p \) is the trivial simple \( \mathbb{F}_p G \)-module and \( \chi_1 \) is the sign representation.

More precisely, identifying \( V^* \otimes V = \text{Hom}_{\mathbb{F}_p}(V, V) = M_2(\mathbb{F}_p) \) with the adjoint action of \( \theta \), and using the \( \mathbb{F}_p \)-basis \( \{e_1, e_2\} \) for \( V \), we obtain:

a. The \( \mathbb{F}_p \)-span of
\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]
is isomorphic to the trivial simple \( \mathbb{F}_p G \)-module \( \mathbb{F}_p \).

b. The \( \mathbb{F}_p \)-span of
\[
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\]
is isomorphic to \( V_{\chi_1} \).

c. The \( \mathbb{F}_p \)-span of
\[
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix}
\]
is isomorphic to \( V_\phi \) which is isomorphic to \( V_{\hat{\phi}} \).

Proof. As before, let \( G = D_{2n} = \langle r, s \rangle \). For a., let \( f \) be the identity in \( \text{Hom}_{\mathbb{F}_p}(V, V) \), then
\[
(r \cdot f)(e_1) = r(f(r^{-1}e_1)) = r(f(\omega^{-i}e_1)) = r(\omega^{-i}e_1) = e_1 = f(e_1).
\]
Also,
\[
(r \cdot f)(e_2) = e_2 = f(e_2),
\]
thus \( r \) acts trivially on the subspace generated by \( f \). Similarly, \( (s \cdot f)(e_1) = s(f(s^{-1}e_1)) = s(f(e_2)) = se_2 = e_1 = f(e_1) \), and by the same argument \( (s \cdot f)(e_2) = e_2 = f(e_2) \). Thus, \( s \) acts trivially on \( f \), and the \( \mathbb{F}_p \) subspace generated by \( f \) has trivial \( G \) action.

For b., let \( f \) correspond to the matrix in b. Then,
\[
(r \cdot f)(e_2) = r(f(r^{-1}e_2)) = r(f(\omega e_2)) = r(-\omega e_2) = -\omega \omega^{-i}e_2 = -e_2 = f(e_2).
\]
Also,
\[
(r \cdot f)(e_1) = r(f(r^{-1}e_1)) = r(f(\omega^{-i}e_1)) = \omega^i\omega^{-i}e_1 = e_1 = f(e_1),
\]
so \( r \) acts trivially on \( f \). Also,
\[
(s \cdot f)(e_2) = sf(s^{-1}e_2) = sf(e_1) = s(e_1) = e_2 = -f(e_2).
\]
A similar arguments show that \( s \cdot f = -f \), thus the \( \mathbb{F}_p \) subspace generated by \( f \) is isomorphic to the sign representation.
Statement $c.$ follows from the identities $r \cdot u = \omega^{2i} u$, $r \cdot v = \omega^{-2i} v$, $s \cdot u = v$, and $s \cdot v = u$. 

**Lemma 4.3.** Let $G = D_{2n}$, $1 \leq i, j < \frac{n}{2}$, let $V = V_{\theta_i}$, and $\phi = \theta_j$. Then, \( d^2_V = d^1_V + 1 \) and 

\[
\begin{align*}
    d^1_V &= \begin{cases} 
        0, & \text{if } \theta_j \neq T(\theta_i) \\
        1, & \text{if } \theta_j = T(\theta_i).
    \end{cases}
\end{align*}
\]

**Proof.** Define $T(V) = V_{T(\theta_i)}$. By Lemma 4.2, we have $V^* \otimes V \cong \mathbb{F}_p \oplus V_{\chi_1} \oplus T(V)$ as $\mathbb{F}_p G$-modules. Note that for $\phi = \theta_j$, we see that 

\[
\begin{align*}
    (\det \circ (\tilde{\phi}))(r) &= \det(\phi(r^{-1})^T) \\
    &= \det(\phi(r^{-1})) \\
    &= \det \left( \begin{pmatrix} \omega^{-j} & 0 \\ 0 & \omega^j \end{pmatrix} \right) \\
    &= 1
\end{align*}
\]

and 

\[
\begin{align*}
    (\det \circ (\tilde{\phi}))(s) &= \det(\phi(s^{-1})^T) \\
    &= \det(\phi(s^{-1})) \\
    &= \det \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \\
    &= -1.
\end{align*}
\]

Therefore, $\det \circ (\tilde{\phi}) = \chi_1$. Since we assume $\mathbb{F}_p G$ is semisimple, the $\mathbb{F}_p$-dimension of the $G$-fixed points of any $\mathbb{F}_p G$-module is the multiplicity of the trivial simple $\mathbb{F}_p G$-module.
as a summand. Recall that we identify $G = \Gamma/N$. By Theorem 2.1 and Corollary 2.3, we have that $H^2(\Gamma, \text{Hom}_{\mathbb{F}_p}(V, V)) \cong [(V_{\tilde{\phi}} \otimes V^* \otimes V) \oplus (\text{det}(\tilde{\phi}) \otimes V^* \otimes V)]^G$, and $H^1(\Gamma, \text{Hom}_{\mathbb{F}_p}(V, V)) = (V_{\tilde{\phi}} \otimes V^* \otimes V)^G$. Hence,

$$H^2(\Gamma, \text{Hom}_{\mathbb{F}_p}(V, V)) \cong [(V_{\tilde{\phi}} \otimes (\mathbb{F}_p \oplus V_{\chi_1} \oplus T(V)))^G \oplus [(V_{\chi_1} \otimes (\mathbb{F}_p \oplus V_{\chi_1} \oplus T(V)))^G$$

$$\cong [(V_{\tilde{\phi}} \otimes (\mathbb{F}_p \oplus V_{\chi_1} \oplus T(V)))^G \oplus [V_{\chi_1} \oplus \mathbb{F}_p \oplus T(V)]^G$$

$$\cong [V_{\tilde{\phi}} \oplus V_{\tilde{\phi}} \oplus (V_{\tilde{\phi}} \otimes T(V))]^G \oplus [V_{\chi_1} \oplus \mathbb{F}_p \oplus T(V)]^G.$$

It is clear that the trivial simple $\mathbb{F}_pG$-module appears as a summand of the second term with multiplicity 1. Additionally, the trivial simple $\mathbb{F}_pG$-module $\mathbb{F}_p$ is a summand of the first term if and only if $\mathbb{F}_p$ is a summand of $(V_{\tilde{\phi}} \otimes T(V))^G \cong \text{Hom}_{\mathbb{F}_pG}(V_{\tilde{\phi}}, T(V))$ if and only if $V_{\tilde{\phi}} \cong T(V)$, i.e. $\phi = \theta_j = T(\theta_i)$. \hfill \qed

Observe that we have shown that for all $\phi$ not in $\Omega$, $H^2(\Gamma, \text{Hom}_{\mathbb{F}_p}(V, V))$ is 1-dimensional for every 2-dimensional irreducible $\mathbb{F}_pG$-module $V$. Hence, in this case, every $V$ in $\text{Irr}_2(G)$ is cohomologically maximal. Thus, in this case, we cannot typically detect the fusion of $N$ in $\Gamma$ by letting $V$ range. For certain choices of $n$, however, the situation is actually better. More precisely, if $n$ is either a power of 2, or $n = 2q$ for some odd prime $q$, then the fusion of $N$ in $\Gamma$ can always be determined by the knowledge of all $R(\Gamma, V)$. 


In this chapter we determine the universal deformation ring $R(\Gamma, V)$ for every 2-dimensional irreducible $\mathbb{F}_p G$-module $V$, which we view as an $\mathbb{F}_p \Gamma$-module by inflation. We make the same assumptions as in Chapter 3. In particular, we continue to assume that $\mathbb{F}_p G$ is semisimple and $\mathbb{F}_p$ is a sufficiently large field for $G$. We use a result from [3] to show that if $H^2(\Gamma, \text{Hom}_{\mathbb{F}_p}(V, V))$ is two-dimensional, then $R(\Gamma, V) \cong \mathbb{Z}_p[[t]]/(t^2, pt)$. Recall, we have shown in Lemma 4.3 that for $G = D_{2n}$, $d^2_V = \dim_{\mathbb{F}_p}(H^2(\Gamma, \text{Hom}_{\mathbb{F}_p}(V, V)))$ is two-dimensional if and only if $d^1_V = 1$. Otherwise $d^2_V = 1$ and $d^1_V = 0$. In the latter case, $R(\Gamma, V)$ is a quotient of $\mathbb{Z}_p$. Since any $V$ has a lift to $\mathbb{Z}_p$, it follows that in this case the universal deformation ring is $\mathbb{Z}_p$.

**Proposition 5.1.** Let $n \geq 3$, $G = D_{2n}$, let $\phi$ be in $\Omega$, and let $V$ be a 2-dimensional irreducible $\mathbb{F}_p G$-module. Then,

$$R(\Gamma, V) = \begin{cases} \mathbb{Z}_p & \text{if } V \text{ is not cohomologically maximal for } \phi, \\ \mathbb{Z}_p[[t]]/(t^2, pt) & \text{if } V \text{ is cohomologically maximal for } \phi. \end{cases}$$

Additionally, for any $\phi$ in $\text{Irr}_2(G)$, $R(\Gamma, V) \cong \mathbb{Z}_p[[t]]/(t^2, pt)$ if and only if $d^2_V$ is equal to two. Thus, for $\phi$ not in $\Omega$, $R(\Gamma, V) \cong \mathbb{Z}_p$.

**Proof.** By our comments before the statement of the proposition, we only need to consider the case when $d^2_V = 2$. Following the proof of [3, Theorem 3.1], let $W = \mathbb{Z}_p$ and $R = W[[t]]/(t^2, pt)$. Since $d^2_V = 2$, it follows from Lemmas 4.2 and 4.3 that $V_\phi$
is a summand of $V^* \otimes V$. Identifying $N = \mathbb{F}_p \times \mathbb{F}_p$ and using Lemma 4.2 part c., we obtain an injective group homomorphism $\iota : N \to M_2(\mathbb{F}_p) \cong M_2(W/pW)$ given by $\iota((n_1, n_2)) = n_1u + n_2v = \begin{pmatrix} 0 & n_1 \\ n_2 & 0 \end{pmatrix}$. Hence, we have a commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & N & \longrightarrow & \Gamma & \longrightarrow & G & \longrightarrow & 1 \\
\downarrow{\iota} & & \downarrow{\rho_R} & & \downarrow{\rho_W} & & \\
0 & \longrightarrow & M_2(W/pW) & \longrightarrow & GL_2(R) & \longrightarrow & GL_2(W) & \longrightarrow & 1
\end{array}
\]

where $d(X) = 1 + tX$ as in diagram (3.1) in the proof of [3, Theorem 3.1]. We notice that all the arguments in the proof of [3, Theorem 3.1] go through once we have proved that the image under $\iota$ of the group $N$ contains two elements which do not commute with each other under multiplication in $M_2(W/pW)$. Using the notation in Lemma 4.2 part c., we see that $u \cdot v \neq v \cdot u$. Thus, $R(\Gamma, V) \cong \mathbb{Z}_p[t]/(t^2, pt)$. \qed

Thus we have shown that if there exists an absolutely irreducible 2-dimensional $\mathbb{F}_p\Gamma$-module $V$ with $d^2_V = 2$, then $R(\Gamma, V) \cong \mathbb{Z}_p[t]/(t^2, pt)$, and $V$ is cohomologically maximal. If, on the other hand, no such $V$ exists, then, for all $V$, $d^2_V = 1$, and $R(\Gamma, V) \cong \mathbb{Z}_p$. 

In this chapter, we make the same assumptions as in Chapter 3. We determine the fusion of $\phi \in \text{Irr}_2(G)$. This uniquely determines the fusion of $N$ in $\Gamma$, in the case when the action of $G = \Gamma/N$ on $N$ is given by $\phi$ (see the remark after Definition 2.4).

**Proposition 6.1.** Let $n \geq 3, G = D_{2n}, 1 \leq i_0 < \frac{n}{2}$ and $\phi = \theta_{i_0}$. Let $(i_0, n)$ denote the greatest common divisor of $i_0$ and $n$. Then, for $\begin{pmatrix} x \\ y \end{pmatrix} \in N = F_p \times F_p$, let $\text{Orb} \begin{pmatrix} x \\ y \end{pmatrix}$ denote the orbit of $\begin{pmatrix} x \\ y \end{pmatrix}$ under the action of $G$ on $N$. We have

$$\text{Orb} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} 1, & \text{if } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \frac{n}{(i_0, n)}, & \text{if } \begin{pmatrix} x \\ y \end{pmatrix} \in F_p^* \times F_p^*, \frac{y}{x} \in \langle \omega^{i_0} \rangle \\ \frac{2n}{(i_0, n)}, & \text{otherwise.} \end{cases}$$

**Proposition 6.2.** Use the same notation as in Proposition 6.1, and let $k = \frac{n}{(i_0, n)}$.

The fusion orbits of $\phi$, i.e. the orbits of the elements in $N$ under the action $\phi$ of $G$ on $N$ are as follows:
1. \[ \text{Orb} \left( \begin{array}{c} 0 \\ 0 \end{array} \right) = \left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\} \]

2. For \( \left\{ \begin{array}{c} x \\ y \end{array} \right\} \in \mathbb{F}_p^* \times \mathbb{F}_p^* \), \( y/x \notin \langle \omega^j \rangle \) we have

\[ \text{Orb} \left( \begin{array}{c} x \\ y \end{array} \right) = \left\{ \begin{array}{c} x \\ y \\ \vdots \\ x \end{array} \right\} \begin{pmatrix} \omega^j x \\ \omega^{-j} y \\ \vdots \\ \omega^{-(k-1)j} y \end{pmatrix}, \begin{pmatrix} \omega^{(k-1)j} x \\ \omega^{-(k-1)j} y \end{pmatrix} \]

3. For \( \left\{ \begin{array}{c} x \\ y \end{array} \right\} \in \mathbb{F}_p^* \times \mathbb{F}_p^* \), \( y/x \in \langle \omega^j \rangle \) we have

\[ \text{Orb} \left( \begin{array}{c} x \\ y \end{array} \right) = \left\{ \begin{array}{c} x \\ y \\ \vdots \\ x \end{array} \right\} \begin{pmatrix} \omega^j x \\ \omega^{-j} y \\ \vdots \\ \omega^{-(k-1)j} y \end{pmatrix}, \begin{pmatrix} \omega^{(k-1)j} x \\ \omega^{-(k-1)j} y \end{pmatrix} \]

4. For \( \left\{ \begin{array}{c} x \\ y \end{array} \right\} \neq \left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\} \notin \mathbb{F}_p^* \times \mathbb{F}_p^* \),

\[ \text{Orb} \left( \begin{array}{c} x \\ y \end{array} \right) = \left\{ \begin{array}{c} x \\ y \\ \vdots \\ x \end{array} \right\} \begin{pmatrix} \omega^j x \\ \omega^{-j} y \\ \vdots \\ \omega^{-(k-1)j} y \end{pmatrix}, \begin{pmatrix} \omega^{(k-1)j} x \\ \omega^{-(k-1)j} y \end{pmatrix} \]

Proof of Propositions 6.1 and 6.2. As before, \( G = \langle r, s \mid r^n, s^2, sr^{-1}s^{-1} \rangle \). First,
clearly \( \text{Orb} \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \). Next, note that \( r^j \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \) if and only if \( \omega x \cdot x = x \) and \( \omega^{-i} \cdot y = y \). Thus, for \( \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{F}_p^* \times \mathbb{F}_p^* \), the intersection of the stabilizer of \( \begin{pmatrix} x \\ y \end{pmatrix} \) with \( \langle r \rangle \) is \( \langle r^{n/(i_0)} \rangle = \langle r^k \rangle \), which is precisely the kernel of \( \theta_{i_0} \).

Also \( sr^j \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \) if and only if \( x = \omega^{-i} \cdot y \) and \( y = \omega^i \cdot x \) which implies \( y/x \in \langle \omega^i \rangle \) (if \( x, y \neq 0 \)). Hence, for \( \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{F}_p^* \times \mathbb{F}_p^* \) with \( y/x \notin \langle \omega^i \rangle \), the stabilizer of \( \begin{pmatrix} x \\ y \end{pmatrix} \) is \( \langle r^{n/(i_0)} \rangle = \langle r^k \rangle \) which implies \( \text{Orb} \left( \begin{pmatrix} x \\ y \end{pmatrix} \right) = 2n/(i_0, n) \). Choosing \( \{1, r, r^2, \ldots r^{k-1}, s, sr, \ldots sr^{k-1}\} \) as a collection of coset representatives for \( G/\langle r^k \rangle \), we see that the orbit of \( \begin{pmatrix} x \\ y \end{pmatrix} \) is as in part 2. of Proposition 6.2.

On the other hand, if \( y/x = \omega^{i_0 j_0} \in \langle \omega^i \rangle \), then \( sr^j \) is in the stabilizer if and only if \( \omega^i = \omega^{i_0 j_0} \), if and only if \( j \equiv j_0 \pmod{n/(i_0, n)} \). Therefore, for these \( \begin{pmatrix} x \\ y \end{pmatrix} \),

\[
|\text{Stab} \left( \begin{pmatrix} x \\ y \end{pmatrix} \right)| = 2(i_0, n), \text{ which means } |\text{Orb} \left( \begin{pmatrix} x \\ y \end{pmatrix} \right)| = n/(i_0, n). \]

Specifically, the stabilizer of \( \begin{pmatrix} x \\ y \end{pmatrix} \) is \( \langle r^{n/(i_0), sr^j_0} \rangle = \langle r^k, sr^j_0 \rangle \). Choosing \( \{1, r, r^2, \ldots r^{k-1}\} \) as a col-
lection of coset representatives for $G$ modulo the stabilizer of \begin{pmatrix} x \\ y \end{pmatrix}, we see that the orbit of \begin{pmatrix} x \\ y \end{pmatrix} is as in part 3. of Proposition 6.2.

Lastly, if exactly one of $x, y$ lies in $\mathbb{F}_p^*$, then $Stab \begin{pmatrix} x \\ y \end{pmatrix} = \langle r^{n/(i_0,n)} \rangle = \langle r^k \rangle$, so $|Orb \begin{pmatrix} x \\ y \end{pmatrix}| = 2n/(i_0, n)$. Picking the same collection of coset representatives as for \begin{pmatrix} x \\ y \end{pmatrix} $\in \mathbb{F}_p^* \times \mathbb{F}_p^*$ with $y/x \notin \langle \omega^{i_0} \rangle$, we obtain that the orbit of \begin{pmatrix} x \\ y \end{pmatrix} is as in part 4. of Proposition 6.2. This completes the proof of Propositions 6.1 and 6.2.

**Corollary 6.3.** With the same notation as in Propositions 6.1 and 6.2, the fusion of $\phi$ is uniquely determined by the greatest common divisor $(i_0, n)$. Moreover, the fusion numbers of $\phi$ uniquely determine the fusion of $\phi$.

**Proof.** We only need to show the statement about the fusion numbers. Recall from Definition 2.4 that the fusion numbers of $\phi$ are given by the sequence \{ $F_{\phi,m} \}_{m \geq 1}$ where $F_{\phi,m}$ is the number of fusion orbits of $\phi$ of cardinality $m$. From Proposition
6.1, we obtain (letting $k = n/(i_0, n)$ as before)

\[
    F_{\phi,1} = 1
\]

\[
    F_{\phi,k} = p - 1
\]

\[
    F_{\phi,2k} = \frac{(p - 1)(p + 1 - k)}{2k}
\]

\[
    F_{\phi,m} = 0 \text{ for all other } m \geq 1.
\]

(Note by the assumptions in Chapter 3, \( p \equiv 1(\text{mod} \ n) \), say \( p - 1 = \ell n \). Thus, \( (p - 1)(p + 1 - k)/2k = \frac{\ell n}{k} \cdot \frac{p+1}{2} - \frac{p-1}{2} \) is an integer.) This proves Corollary 6.3. \qed

In particular, two representations \( \theta_i, \theta_i \in \text{Irr}_2(G) \) have the same fusion if and only if \( (i, n) = (i_0, n) \).
CHAPTER 7
PROOF OF THE MAIN RESULTS

In this chapter we complete the proof of our main results Theorems 3.3 and 3.4. Again, we make the same assumptions as in Chapter 3. Note that the equality of the sets in Theorem 3.3 follows from Propositions 4.1 and 5.1. Theorem 3.4 part a. also follows from Proposition 4.1. Therefore, to complete the proof of Theorems 3.3 and 3.4, it remains only to prove the one-to-one correspondence for \( \phi \in \Omega \):

\[
\text{Fusion of } \phi \iff \{ \ker(\psi) : \psi \in \text{Irr}_2(G) \text{ is cohomologically maximal for } \phi \}.
\]

It follows from the definition of the 2-dimensional irreducible representations of \( G = D_{2n} \) that for any \( 1 \leq i_0 < \frac{n}{2} \), the kernel of \( \theta_{i_0} \) is \( \langle r^{n/(i_0,n)} \rangle \). Recall that in Chapter 6 we showed that the fusion of \( \theta_{i_0} \) is also completely determined by \( (i_0, n) \) (see Corollary 6.3). Moreover, for \( 1 \leq i < \frac{n}{2} \), we have \( T(\theta_i) = \begin{cases} 
\theta_{2i} & \text{if } 2i < \frac{n}{2} \\
\theta_{n-2i} & \text{otherwise.}
\end{cases} \)

Therefore, for \( n \) odd, the result follows from Proposition 4.1 since \( (i, n) = (2i, n) = (n - 2i, n) = (i_0, n) \) when \( T(\theta_i) = \theta_{i_0} \). In the case when \( n \) is even, let \( \theta_{i_0} \in \Omega \), i.e. \( 1 \leq i_0 \leq \frac{n}{2} - 1 \) and \( i_0 = 2d_0 \) for some \( d_0 \). Moreover, \( T^{-1}(\theta_{i_0}) = \{ \theta_{d_0}, \theta_{k-d_0} \} \) for \( k = \frac{n}{2} \). But \( \psi \in \text{Irr}_2(G) \) is cohomologically maximal for \( \phi = \theta_{i_0} \) if and only if \( \psi \) is an element of \( T^{-1}(\theta_{i_0}) \), by Proposition 4.1. Therefore, for \( n \) even, the result follows from the following lemma, since the fusion of \( \theta_{i_0} \) is completely determined by \( (i_0, n) \), and Lemma 7.1 shows that the set \( \{(d_0, n), (k-d_0, n)\} \) determines \( (i_0, n) \).
Lemma 7.1. Let \( n \) be even, \( k = \frac{n}{2} \), and write \( n = 2^\lambda \cdot m \), for some odd \( m \) and some \( \lambda \geq 1 \). Let \( \theta_0 \in \Omega \) and write \( i_0 = 2d_0 \). Define \( a_0 = (d_0, k) \). Then \( \{(d_0, n), (k - d_0, n)\} = \{(a_0, n), (k - a_0, n)\} \). Moreover, \( (i_0, n) = 2a_0, (a_0, n) = a_0 \), and \( (k - a_0, n) \in \{a_0, 2a_0\} \).

Proof. Suppose first that \( 2^{\lambda} \nmid d_0 \). Then \((d_0, n) | k \), and hence \((d_0, n) = (d_0, k) \) which is equal to \( a_0 \) by definition. If \( 2^{\lambda-1} \nmid d_0 \), then \((k - d_0, n) = (k - d_0, k) \) and \( k - d_0 \) and \( k - a_0 \) are even, and so \((k - d_0, n) = 2(k - d_0, k) = 2a_0 = 2(k - a_0, k) = (k - a_0, n) \). On the other hand, if \( 2^\lambda \mid d_0 \), then \( 2^\lambda \mid (k - d_0) \) but \( 2^{\lambda-1} \mid (k - d_0) \) and \( 2^{\lambda-1} \mid (k - a_0) \). Hence we can use the above argument to obtain \((d_0, n) = (k - (k - d_0), n) = 2(k - (k - d_0), k) = 2(d_0, k) = 2a_0 = 2(k - a_0, k) = (k - a_0, n) \). \( \square \)

Thus, we have shown that for \( \phi \in \Omega \), one can determine the fusion of \( N \) in \( \Gamma \) from knowledge of \( R(\Gamma, V) \) for all absolutely irreducible 2-dimensional \( \mathbb{F}_p \Gamma \)-modules \( V \). As stated previously, when \( n \) is odd, every irreducible \( \phi \) is in \( \Omega \). For a typical even \( n \), one cannot do better. If, however, \( n \) is either a power of 2 or equal to \( 2q \) for some odd prime \( q \), then \( \phi \notin \Omega \) if and only if \( \phi \) is faithful. Thus, if one knows that \( R(\Gamma, V) \cong \mathbb{Z}_p \), for all absolutely irreducible 2-dimensional \( \mathbb{F}_p \Gamma \)-modules \( V \), then it must be the case that the fusion of \( N \) in \( \Gamma \) corresponds to \((1, n)\) in the sense of Corollary 6.3. On the other hand, if \( n \) is even, but not as above, then there must exist some odd prime \( v \) such that \( \theta_v \notin \Omega \). But then \( \theta_1 \) and \( \theta_v \) have different fusion, but in both cases \( R(\Gamma, V) \cong \mathbb{Z}_p \), for all absolutely irreducible 2-dimensional \( \mathbb{F}_p \Gamma \)-modules \( V \).
CHAPTER 8
ABELIAN GROUPS

In this chapter, we briefly discuss the case when $\Gamma/N = G$ is an abelian group, and compare this case to the dihedral case discussed in the previous chapters. In other words, we consider the short exact sequence of groups

$$0 \to N \to \Gamma \to G = \Gamma/N \to 1$$

where $G$ is abelian and finite and $N$ is an elementary abelian $p$-group of rank two. As before, we assume $\mathbb{F}_p G$ is semisimple and $\mathbb{F}_p$ is a sufficiently large field for $G$.

Let $V$ be an irreducible $\mathbb{F}_p G$-module viewed as an $\mathbb{F}_p \Gamma$-module via inflation. Let $\phi$ denote the action of $G$ on $N$. Since $G$ is abelian, $V$ is one-dimensional, and $\phi$ splits into a direct sum of two one-dimensional representations. Let $\phi = (\theta_1, \theta_2)$, where $\theta_i : G \to \mathbb{F}_p^*$. We again analyze the extent to which the universal deformation ring $R(\Gamma, V)$ can see the fusion of $N$ in $\Gamma$. In contrast to the dihedral case, if $G$ is abelian, then $R(\Gamma, V)$ will only be able to detect some information about fusion.

**Proposition 8.1.** Let $G$ be abelian, $V$ and $\phi$ be as above. Let $\{F_{\phi,m}\}_{m \geq 1}$ be the fusion numbers of $\phi$.

a. The universal deformation ring $R(\Gamma, V) \cong \mathbb{Z}_p$ if and only if $F_{\phi,1} = 1$ if and only if both $\theta_1$ and $\theta_2$ are not trivial if and only if $d^1_V = 0$.

b. The universal deformation ring $R(\Gamma, V) \cong \mathbb{Z}_p[\mathbb{Z}/p\mathbb{Z}]$ if and only if $F_{\phi,1} = p$ if and only if exactly one of $\theta_1, \theta_2$ is trivial if and only if $d^1_V = 1$. 
c. The universal deformation ring \( R(\Gamma, V) \cong \mathbb{Z}_p[\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}] \) if and only if 
\( F_{\phi,1} = p^2 \) if and only if both \( \theta_1, \theta_2 \) are trivial if and only if \( d^1_V = 2 \).

In the statement of the proposition, we have added brackets to the group rings for clarity. The above proposition illustrates the extent to which fusion can be detected by universal deformation rings in the case when \( G \) is abelian. In contrast to the dihedral case, we get no information by letting \( V \) range over all absolutely irreducible \( \mathbb{F}_p \Gamma \)-modules. This is because both \( R(\Gamma, V) \) and \( d^i_V \) for \( i = 1, 2 \), are constant with respect to \( V \) (that is, depend only on \( \phi \)). In the abelian case, while some information about the fusion of \( N \) in \( \Gamma \) may be detected by the universal deformation ring (and indeed by the cohomology), it is simply too coarse to completely determine fusion (compare with Theorems 3.3 and 3.4). Instead, the universal deformation ring only sees the number of orbits of size 1. This does not correspond to the full fusion of \( N \) in \( \Gamma \), but only to the number of \( \theta_i \) which are trivial.

Proof of Proposition 8.1. Let \( G \) be abelian, and let \( V \) and \( \phi = (\theta_1, \theta_2) \) be as above. We first determine the size of the fusion orbits of \( \phi \). Let \( n \in N \) and write \( n = \begin{pmatrix} x \\ y \end{pmatrix} \) in coordinates corresponding to \( \theta_1, \theta_2 \). By computing stabilizers we get

\[
\text{Orb}
\begin{pmatrix}
  x \\
  y
\end{pmatrix}
= \begin{cases}
  |G/\ker(\theta_2)|, & \text{if } x = 0, y \neq 0 \\
  |G/\ker(\theta_1)|, & \text{if } y = 0, x \neq 0 \\
  |G/\ker(\theta_1) \cap \ker(\theta_2)|, & \text{if } x \neq 0, y \neq 0 \\
  1, & \text{if } x = 0 = y.
\end{cases}
\]
Note that it follows that the fusion $F_{\theta,1} = p^j$, where $j$ is the number of $\theta_i, i = 1, 2$ which are trivial. This also shows that the fusion of $N$ in $\Gamma$ depends on more than just $F_{\phi,1}$, at least when $G$ does not act trivially on $N$.

Next, we determine $d_V^i, i = 1, 2$. By Theorem 2.1 and Corollary 2.3, we calculate $(V_{\phi} \otimes V^* \otimes V)^G$ and $(V_{\det(\phi)} \otimes V^* \otimes V)^G$. Since $V$ is one-dimensional, $V^* \otimes V$ is trivial, thus $d_V^i$ is independent of $V$ for $i = 1, 2$. Since $\phi = (\theta_1, \theta_2)$, $d_V^1$ counts the multiplicity of the trivial simple $F_{\theta_2}G$-module in $V_{\theta_1}^* \oplus V_{\theta_2}^*$. Thus, $d_V^1$ is the number of $j$ such that $\theta_j$ is trivial simple. Also, $d_V^2 - d_V^1$ is 1 if $\theta_2 = \theta_1^{-1}$, and it is 0 otherwise, since $\det \circ (\phi)$ is just the one-dimensional representation $\theta_1^{-1} \otimes \theta_2^{-1}$. Therefore, we obtain the following cases:

i. $d_V^1 = 0, d_V^2 = 0$ if and only if both $\theta_1, \theta_2$ are not trivial, and $\theta_2 \neq \theta_1^{-1}$,

ii. $d_V^1 = 0, d_V^2 = 1$ if and only if both $\theta_1, \theta_2$ are not trivial, and $\theta_2 = \theta_1^{-1}$,

iii. $d_V^1 = 1, d_V^2 = 1$ if and only if exactly one of $\theta_1, \theta_2$ are trivial,

iv. $d_V^1 = 2, d_V^2 = 3$ if and only if both of $\theta_1, \theta_2$ are trivial.

Finally, we determine $R(\Gamma, V)$. Since $G$ is abelian, it follows by [14, Section 1.4] that $R(\Gamma, V) = \mathbb{Z}_p[\Gamma^{ab,p}]$, where $\Gamma^{ab,p}$ denotes the maximal abelian $p$-quotient of $\Gamma$. Since the order of $G$ is relatively prime to $p$, $\Gamma^{ab,p}$ can be either the trivial group, or $\mathbb{Z}/p\mathbb{Z}$, or $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Since $d := d_V^1$ is minimal such that $R(\Gamma, V)$ is a quotient of $\mathbb{Z}_p[[t_1, t_2, ..., t_d]]$, it follows that:

a. $d_V^1 = 0$ if and only if $R(\Gamma, V) \cong \mathbb{Z}_p$. 

b. \( d^1_V = 1 \) if and only if \( R(\Gamma, V) \cong \mathbb{Z}_p[\mathbb{Z}/p\mathbb{Z}] \),

c. \( d^1_V = 2 \) if and only if \( R(\Gamma, V) \cong \mathbb{Z}_p[\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}] \).

This completes the proof of Proposition 8.1.
In this appendix, we give an alternate proof to the one given in Chapter 2 in the proof of Theorem 2.1 of the following (more special) result.

**Lemma 2.** Let $G$ be a finite group such that $\mathbb{F}_pG$ is semisimple and $\mathbb{F}_p$ is a sufficiently large field for $G$. Let $N$ be an elementary abelian $p$-group of rank $\ell = 2$ on which $G$ acts by the $\mathbb{F}_p$-representation $\phi$. Then $H^2(N, \mathbb{C}^*) \cong \mathbb{F}_p$ with $G$-action given by $\det o(\phi)$.

**Proof.** Since the rank of $N$ is two, it is well known that the Schur multiplier of $N$ is $\mathbb{Z}/p\mathbb{Z}$ as an abelian group [12, Satz V.25.10]. Alternatively, we may use Hopf’s formula [12, Hauptsatz V.23.5]. For any group $H$, let $\hat{H}$ be the set of linear ordinary characters of $H$, and let $H' = [H, H]$. Using this notation, if

- $\mathfrak{F}$ is a finitely generated free group, and
- $\mathfrak{K}$ is a normal subgroup generated by relations with $\mathfrak{F}/\mathfrak{K} \cong N$,

then there is a $\mathbb{Z}$-module isomorphism

$$\tau : (\mathfrak{K} \cap \mathfrak{F})/[\mathfrak{K}, \mathfrak{F}] \to H^2(N, \mathbb{C}^*).$$

Let $\mathfrak{F} = \langle x, y \rangle$, and let $\mathfrak{K} = \langle x^p, y^p, [x, y] \rangle_{\text{normal}}$. We now use the construction of $\tau$ to show that $\mathbb{Z}/p\mathbb{Z} \cong \langle [x, y][\mathfrak{K}, \mathfrak{F}] \rangle \to H^2(N, \mathbb{C}^*)$ is an isomorphism of abelian groups. Following [12, Proof of Hauptsatz V.23.5], define

$$\epsilon : \hat{L} \to H^2(N, \mathbb{C}^*) \text{ by } \epsilon(\mu) = \mu \circ J, \text{ where}$$
• \( L = \mathcal{R}/[\mathcal{R}, \mathfrak{F}] \),

• \( \lambda \) is a section of the natural projection \( \mathfrak{F}/[\mathcal{R}, \mathfrak{F}] \to N \) and

• \( J : N \times N \to L \) is a function satisfying \( \lambda(g_1g_2) = J(g_1, g_2)\lambda(g_1)\lambda(g_2) \).

As demonstrated in [12, Proof of Hauptsatz V.23.5], \( \varepsilon \) will satisfy the two-cocycle condition, and \( \bar{L}/\ker(\varepsilon) \to H^2(N, \mathbb{C}^*) \) is an isomorphism. Moreover, the elements of \( \ker(\varepsilon) \) are exactly the characters in \( \bar{L} \) which are trivial on \( K = (\mathcal{R} \cap [\mathfrak{F}, \mathfrak{F}])/[\mathcal{R}, \mathfrak{F}] \). Since \( \mathbb{C}^* \) is divisible as a \( \mathbb{Z} \)-module, every character on \( K \) can be extended to \( L \), so \( \bar{L}/\ker(\varepsilon) \cong \hat{K} \). In other words, for \( \mathcal{R}, \mathfrak{F} \) as above, \( \tau \) is the composition \( \hat{K} \twoheadrightarrow \bar{L}/\ker(\varepsilon) \overset{\bar{\varepsilon}}{\to} H^2(N, \mathbb{C}^*) \).

With \( \mathcal{R}, \mathfrak{F} \) as above, note that in \( L \),

\[
[x, y][x, y] = xyx^{-1}y^{-1}xyx^{-1}y^{-1} = (xyx^{-1}y^{-1})x^{-1}(xyx^{-1}y^{-1})x = xyx^{-1}y^{-1}yx^{-1}y^{-1}x = xyx^{-2}y^{-1}x = x^2yx^{-2}y = [x^2, y].
\]

By the same argument, \( [x, y][x, y] = [x, y^2] \). Analogously, \( [x^a, y] = [x, y]^a = [x, y^a] \), and \( [x^a, y^b] = [x, y]^{ab} \). Since \( [x, y]^p = [x, y^p] = 1_L \), this argument shows that \( K = (\mathcal{R} \cap \mathfrak{F}'/[\mathcal{R}, \mathfrak{F}] = \mathfrak{F}'/[\mathcal{R}, \mathfrak{F}] = [\langle x, y|\mathcal{R}, \mathfrak{F}\rangle] \cong \mathbb{Z}/p\mathbb{Z} \).
Thus, we have bijective maps
\[
\mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow \langle [x, y][\mathfrak{A}, \mathfrak{B}] \rangle = \hat{K} \rightarrow \hat{L}/\ker(\epsilon) \xrightarrow{\tilde{\epsilon}} H^2(N, \mathbb{C}^*)
\]
(1)
the composition of the last two maps being $\tau$.

Let $\eta$ be a primitive $p$-th root of unity in $\mathbb{C}$. Explicitly, at the level of elements, the composition of maps in (A.1) does the following to $\bar{1} \in \mathbb{Z}/p\mathbb{Z}$:
\[
\bar{1} \rightarrow (\bar{1} \mapsto \eta) \rightarrow ([x, y] \mapsto \eta) \rightarrow \mu_1 + \ker(\epsilon) \xrightarrow{\tilde{\epsilon}} \mu_1 \circ \epsilon
\]
where $\mu_1$ denotes the extension to $\hat{L}$ of the character $[x, y] \mapsto \eta$ of $K$.

We now choose a section $\lambda$ and compute $J$.

- Let $\{e_1, e_2\}$ generate $N = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ and write $\begin{pmatrix} i \\ j \end{pmatrix}$ for $ie_1 + je_2$.

- Define the section $\lambda : N \rightarrow \mathfrak{F}/[\mathfrak{A}, \mathfrak{B}]$, by $\begin{pmatrix} i \\ j \end{pmatrix} \xrightarrow{\lambda} x^iy^j[\mathfrak{A}, \mathfrak{B}]$.

We next compute the function $J : N \times N \rightarrow L$.

Let $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ be two elements of $N$. 


We set

\[
\lambda \left( \begin{pmatrix} a + c \\ b + d \end{pmatrix} \right) = x^{a+c} y^{b+d} [\mathfrak{R}, \mathfrak{F}]
\]

\[
= J' \lambda \left( \begin{pmatrix} a \\ b \end{pmatrix} \right) \lambda \left( \begin{pmatrix} c \\ d \end{pmatrix} \right)
\]

\[
= J' x^a y^b x^c y^d [\mathfrak{R}, \mathfrak{F}],
\]

where \( J' \) denotes the image under \( J \) of \( \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \).

Therefore, we need to find \( J' \) such that \( y^{-b-d} x^{-a-c} J' x^a y^b x^c y^d \in [\mathfrak{R}, \mathfrak{F}] \). We note that if we let \( J'' = x^{-a} J' x^a \), then

\[
y^{-b-d} x^{-a-c} J' x^a y^b x^c y^d \in [\mathfrak{R}, \mathfrak{F}] \iff y^{-b-d} x^{-c} J'' y^b x^c y^d \in [\mathfrak{R}, \mathfrak{F}] \iff y^d y^{-b-d} x^{-c} J'' y^b x^c y^d y^{-d} \in [\mathfrak{R}, \mathfrak{F}] \iff y^{-b} x^{-c} J'' y^b x^c \in [\mathfrak{R}, \mathfrak{F}].
\]

Defining \( J'' = x^c y^b x^{-c} y^{-b} \), we may define \( J' = x^a x^c y^b x^{-c} y^{-b} x^{-a} \). Since \( [\mathfrak{R}, \mathfrak{F}] \leq \mathfrak{F} \), we may conjugate by \( x^a \) again, so that we can define \( J' = [x^c, y^b] [\mathfrak{R}, \mathfrak{F}] = [x^c, y^b] \).
Now,
\[ \tilde{\epsilon}(\mu_1 + \ker(\epsilon)) \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = (\mu_1 \circ J) \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \mu_1([x^c, y^b]) = \mu_1([x, y]^{cb}) = (\mu_1([x, y])^{cb} = \eta^{cb}. \]

Identify \( \hat{K} = \langle [x, y]\mathbb{R}, \mathbb{F} \rangle \) with \( \hat{L}/\ker(\epsilon) \) and let \( \gamma, \kappa \) be characters in \( \hat{K} \) with \( \gamma([x, y]) = \eta^i \) and \( \kappa([x, y]) = \eta^j \). Then,
\[ \tilde{\epsilon}(\gamma \cdot \kappa) \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = (\gamma \cdot \kappa) \circ J \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = (\gamma \cdot \kappa)([x^c, y^b]) = (\gamma \cdot \kappa)([x, y]^{cb}) = \gamma([x, y]^{cb}) \kappa([x, y]^{cb}) = \eta^{icb} \eta^{jcb} = \eta^{(i+j)cb}
\]
\[ = (\tilde{\epsilon}(\gamma) \cdot \tilde{\epsilon}(\kappa)) \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}. \]

Thus, our composition defined in (A.1) is an explicit isomorphism of abelian groups.

We now show that the \( \mathbb{F}_pG \)-module structure on \( \mathbb{H}^2(N, \mathbb{C}^*) \) is given by \( \det \circ (\tilde{\varphi}) \).

Let \( g \in G \). Then since \( \mathbb{F}_pG \) is semisimple and \( \mathbb{F}_p \) is a sufficiently large field for \( G \),
there is an \( \mathbb{F}_p \)-basis for \( N \), \( \{e_1, e_2\} \), such that the matrix corresponding to \( g^{-1} \) is diagonalizable with respect to \( \{e_1, e_2\} \). Say in \( \{e_1, e_2\} \) coordinates, for \( x, y \in \mathbb{F}_p^* \), \( g^{-1} \) corresponds to \( \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \).

By (A.1), every element in \( H^2(N, \mathbb{C}^*) \) is of the form \( \bar{\epsilon}(\mu_i) \) for a unique \( i \in \{0, 1, ..., p-1\} \), where \( \mu_i([x, y]) = \eta^i \). We obtain, (again in \( \{e_1, e_2\} \) coordinates),

\[
(g \cdot \bar{\epsilon}(\mu_i)) \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} = \bar{\epsilon}(\mu_i) \begin{pmatrix} g^{-1} \\ \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} = \bar{\epsilon}(\mu_i) \begin{pmatrix} u & c \\ v & d \end{pmatrix} = (\mu_i \circ J) \begin{pmatrix} u & c \\ v & d \end{pmatrix} = \mu_i([x, y])^{u_{acb}} = \eta^{u_{acb}} = \eta^{i_{acb}} = \eta^{i \det(g^{-1}) cb} = \eta^{i \det(\tilde{\phi}(g)) cb} = \bar{\epsilon}(\mu_{\det(\tilde{\phi}(g))}) \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}
\]

as required. \( \square \)
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