Introduction to Arithmetic Mirror Symmetry

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Abstract We describe how to find period integrals and Picard-Fuchs differential equations for certain one-parameter families of Calabi-Yau manifolds. These families can be seen as varieties over a finite field, in which case we show in an explicit example that the number of points of a generic element can be given in terms of $p$-adic period integrals. We also discuss several approaches to finding zeta functions of mirror manifolds and their factorizations. These notes are based on lectures given at the Fields Institute during the thematic program on Calabi-Yau Varieties: Arithmetic, Geometry, and Physics.

1 Introduction

The mirror conjecture is an important early result in mirror symmetry which suggests that counting rational curves on a Calabi-Yau threefold, an enumerative problem, can be done in terms of Hodge theory and period integrals on its mirror partner. An arithmetic counterpart to these ideas that was introduced by Candelas, de la Ossa, and Rodriguez-Villegas in can be stated as follows. Let $M$ denote the one-parameter family of quintic threefolds, consisting of hypersurfaces

$$X_\psi: \left\{ 0 = x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5\psi x_1 x_2 x_3 x_4 x_5 \right\} \subseteq \mathbb{P}^4,$$

where we exclude those $\psi$ which give singular fibers. This family is typically defined over $\mathbb{C}$, but if we take $\psi$ to be an element of a finite field $k$, then we can consider $X_\psi$ as a variety defined over $k$. It then turns out that the number of points of $X_\psi$ can be given in terms of a $p$-adic version of certain periods on $X_\psi$. The purpose
of these notes is to describe in detail the techniques and ideas behind this calculation and some related work arising from the intersection of arithmetic and mirror symmetry.

A more detailed overview is in order. A period of $X_{\psi}$ (defined over $\mathbb{C}$) is an integral $\int_\gamma \omega$ of the unique holomorphic top-form $\omega$ on $X_{\psi}$ over some 3-cycle $\gamma$. There are 204 independent periods of $X_{\psi}$ in total, owing to the fact that $\dim H_3(X_{\psi}; \mathbb{C}) = 204$. Four of these 204 period integrals can also be seen as periods of the mirror family $W$. These four period integrals are functions of the parameter $\psi$, and are, in fact, all of the solutions of an ordinary differential equation $L f(\psi) = 0$ called the Picard-Fuchs equation of $W$, where for $\lambda = 1/(5\psi)^3$ and $\vartheta = \lambda \frac{d}{d\lambda}$ we define

$$L := \vartheta^4 - 5\lambda \prod_{i=1}^{4} (5\vartheta + i).$$

This is a hypergeometric differential equation with fundamental solution around $\lambda = 0$ given by

$$a_0 = \sum_{m=0}^{\infty} \frac{(5m)!}{(m!)^5} \lambda^m = \sum_{m=0}^{\infty} \frac{\Gamma(5k+1)}{\Gamma(k+1)^5} \lambda^m.$$  

Consider now each $X_{\psi}$ as a variety over the finite field $k = \mathbb{F}_p$ with $p$ elements and assume that $5 \nmid (p-1)$. The number of points $N(X_{\psi})$ on $X_{\psi}$ with coordinates in $k$ is given by the expression

$$N(X_{\psi}) = 1 + p^4 + \sum_{m=1}^{p-2} \frac{G_{5m}}{G_m} \text{Teich}^m(\lambda),$$

where $\text{Teich}(\lambda)$ is the Teichmüller lifting of $\lambda$ to the $p$-adic numbers $\mathbb{Z}_p$, and $G_m$ is a Gauss sum proportional to the $p$-adic gamma function. This expression for $N(X_{\psi})$ can be seen as a $p$-adic analog of the hypergeometric series (3). A way to illustrate this point is to reduce (4) modulo $p$,

$$N(X_{\psi}) \equiv \sum_{m=0}^{\lfloor p/5 \rfloor} \frac{(5m)!}{(m!)^5} \lambda^m \mod p,$$

which is a truncation of (3). In fact, the number of $\mathbb{F}_p$-rational points on $X_{\psi}$ can be written as a modulo $p^5$ expression (24) involving all of the solutions of (2), as well as an additional term arising from a so-called semi-period.

Arithmetic of varieties appearing in the context of mirror symmetry can also be studied through their zeta functions, defined for a variety $X$ over a finite field $\mathbb{F}_q$ with $q = p^n$ elements by

$$Z(X, T) = \exp \left( \sum_{r=1}^{\infty} N_r(X) \frac{T^r}{r} \right),$$
where \( N_r(X) \) denotes the number of points of \( X \otimes_{\mathbb{F}_q} \mathbb{F}_q \) rational over \( \mathbb{F}_q \). It turns out that the zeta function of \( X \) contains all the terms appearing in the zeta function of its mirror manifold, and the terms not appearing in the mirror zeta function exhibit interesting factorization properties. In the context of mirror symmetry, zeta functions were first considered by Candelas, de la Ossa, and Rodriguez-Villegas in [3]. Due to the explicit nature of their calculation and a large overlap with point counting in terms of period integrals we will focus on their exposition. However, we will also discuss other approaches to calculating zeta functions and their factorizations that are of a more conceptual nature.

The notes are organized as follows. In Section 2.1 we more carefully define period integrals. In Section 2.2 we discuss differentials on hypersurfaces and relations between them. These relations enable us to find Picard-Fuchs equations satisfied by the periods, and by solving them the periods themselves, in Sections 2.3 and 2.4. We then switch gears and talk about counting points and what we mean by \( p \)-adic periods in Section 3.1. Finally, we discuss zeta functions of mirror manifolds and their factorizations in Section 4.

2 Periods and Picard-Fuchs Equations

2.1 Period Integrals

Let \( \pi: \mathcal{X} \to B \) be a proper submersion defining a family of smooth \( n \)-dimensional Kähler manifolds. Ehresmann’s Fibration Theorem [25, Theorem 9.1] then implies that for each \( \psi \in B \) there exists an open set \( U \) containing \( \psi \), and a diffeomorphism \( \varphi \) such that the diagram

\[
\begin{array}{ccc}
\pi^{-1}(U) & \xrightarrow{\varphi} & U \\
\downarrow & \searrow & \swarrow \\
U & \rightarrow & U \times \mathcal{X}_\psi
\end{array}
\]

commutes. In other words, \( \pi \) is a locally trivial fibration. In these notes, the fiber \( \mathcal{X}_\psi := \pi^{-1}(\psi) \) is a nonsingular projective hypersurface for each \( \psi \in B \). Let \( \mathcal{F} \) be a sheaf on \( \mathcal{X} \). Mapping \( \mathcal{F} \) to the direct image sheaf \( \pi_* \mathcal{F} \) on \( B \), determined by \( \pi_* \mathcal{F}(U) := \mathcal{F}(\pi^{-1}(U)) \), defines a covariant functor from sheaves on \( \mathcal{X} \) into sheaves on \( B \). This functor is left exact, but in general not right exact. In fact, \( R^k \pi_* \mathcal{F} \) is the sheafification of the presheaf \( H^k(\pi^{-1}(\cdot), \mathcal{F} \mid_{\cdot}) \). Consider the case \( k = n \) and \( \mathcal{F} = \underline{\mathbb{C}} \), the constant sheaf valued in \( \mathbb{C} \). Stalks are determined on contractible open sets, so for \( U \ni \psi \) such that \( \pi^{-1}(U) \cong U \times \mathcal{X}_\psi \) we have

\[
(R^n \pi_* \underline{\mathbb{C}})_\psi \cong H^n(X_\psi, \mathbb{C}).
\]

The groups on the right are canonically isomorphic for all \( \psi \in U \), which means that \( (R^n \pi_* \underline{\mathbb{C}}) \mid U \) defines a locally constant sheaf on \( B \), i.e., a local system \( H \) of complex vector spaces. Tensoring with the structure sheaf \( \mathcal{O}_B \), we obtain a locally free \( \mathcal{O}_B \)-module.
module $\mathcal{H} = H \otimes \mathcal{O}_B$ which canonically admits the Gauss-Manin connection

$$\nabla: \mathcal{H} \to \mathcal{H} \otimes \Omega_B^1$$

defined by

$$\nabla \left( \sum_i \alpha_i \sigma_i \right) := \sum_i \sigma_i \otimes d\alpha_i,$$

where $\{\sigma_i\}$ is any local basis of $H$, $\Omega_B^1$ is the sheaf of holomorphic 1-forms on $B$, and $\alpha_i \in \mathcal{O}_B$. This connection can be extended to a map $\nabla: \mathcal{H} \otimes \Omega_B^k \to \mathcal{H} \otimes \Omega_B^{k+1}$ by defining $\nabla(\sigma \otimes \omega) = (\nabla \sigma) \wedge \omega$. More details are available in [25], for instance.

For any $\psi \in U$, choose a basis of $n$-cycles $\{\gamma_i\}$ on $X_\psi$ such that the corresponding homology classes generate $H_n(X_\psi; \mathbb{C})$. We can choose this basis to be dual to $\{\sigma_i\}$ and extend it to nearby fibers. For $s(\psi) \in \Gamma(U, \mathcal{H})$ varying holomorphically and $\gamma$ a homology class, we obtain a holomorphic function $\langle s(-), \gamma \rangle: U \to \mathbb{C}$ via the Poincaré pairing,

$$\langle s(\psi), \gamma \rangle = \int_\gamma s(\psi).$$

The sheaf generated by such functions is called the period sheaf. By the de Rham theorem [25, Section 4.3.2] we can think of $s(\psi)$ as a holomorphic family of differential forms.

**Definition 1.** Let $\omega$ be a holomorphic $n$-form on an $n$-dimensional complex manifold $X$. Integrals of the form

$$\int_\gamma \omega \quad \text{for } \gamma \in H_n(X; \mathbb{C})$$

are called period integrals (periods) of $X$ with respect to $\omega$.

Extending the (co)homology basis to all of $B$ can lead to nontrivial monodromy on the fibers of $\mathcal{X}$, which will in turn induce monodromy on the periods. However, in these notes we are only interested in the periods locally. In the case that the parameter space $B$ is one dimensional, the Gauss-Manin connection is locally given by differentiation $\nabla_\psi := \frac{d}{d\psi} \int_\gamma \omega(\psi)$ with respect to the parameter $\psi \in B$, and satisfies

$$\frac{d}{d\psi} \int_\gamma \omega(\psi) = \int_\gamma \frac{d}{d\psi} \omega(\psi).$$

From here on, we are working only with one-parameter families.

**Proposition 1.** The periods with respect to $\omega(\psi)$ satisfy an ordinary differential equation of the form

$$\frac{d^s f}{d\psi^s} + \sum_{j=0}^{s-1} C_j(\psi) \frac{d^j f}{d\psi^j} = 0,$$

where $s$ is a natural number. This equation is called the Picard-Fuchs equation for $\omega(\psi)$.
Proof (given in [22]). Let $\psi$ be the parameter on an open set $U \subseteq B$, and for $j \in \mathbb{Z}$ define
\[
v_j(\psi) = \frac{d^j}{d\psi^j} \left( \begin{array}{c} \int_{\gamma_1} \omega(\psi) \\ \vdots \\ \int_{\gamma_r} \omega(\psi) \end{array} \right) \in \mathbb{C}(\psi)^r.
\]
For $i \in \mathbb{N}^+$ and nearby values of $\psi$, the vector spaces
\[V_i(\psi) := \text{span} \{v_0(\psi), \ldots, v_i(\psi)\}\]
vary together smoothly with respect to $\psi$. Since for a particular value of $\psi$ each $V_i(\psi) \in \mathbb{C}^r$, we also have that $\dim V_i(\psi) \leq r$. Therefore, there is a smallest $s \leq r$ such that $v_s(\psi) \in \text{span}\{v_0(\psi), \ldots, v_{s-1}(\psi)\}$, giving the equation
\[v_s(\psi) = -\sum_{i=0}^{s-1} C_i(\psi) v_j(\psi)\]
satisfied by $\int_{\gamma_i} \omega(\psi)$ for each $\gamma_i$, as claimed. □

Picard-Fuchs equations can in general have non-period solutions, but we will not encounter them in these notes.

2.2 Differentials on Hypersurfaces

Let $\pi: \mathcal{X} \to B$ be a one-parameter family of hypersurfaces $\{X_{\psi}\}_{\psi \in B}$ in projective space, and let $\omega = \omega(\psi)$ be a top-form on the family. A basic strategy for finding Picard-Fuchs equations is to express the forms $\frac{d^i}{d\psi^i} \omega$ in terms of a particular basis of forms on $X_{\psi}$, and exploit this description to find relations between them. We will now show how to find a basis of forms on $X_{\psi}$ in the first place, by relating them via residue maps to rational forms on projective space which we can write down explicitly.

2.2.1 The Adjunction Formula and Poincaré Residues

We begin by defining the residue map for differentials with a simple pole in projective space. Throughout this section, $Y$ denotes an $n$-dimensional compact complex manifold and $X$ a hypersurface on $Y$. An example to keep in mind is $Y = \mathbb{P}^4$ and $X \subset \mathbb{P}^4$ a generic element of $[1]$. A reference for this section is [13]. Recall that the normal bundle on $X$ is given by the quotient $N_X = T_Y|_X / T_X$ of tangent bundles, that its dual $N_X^\ast$ is called the conormal bundle, and that the canonical bundle of (any manifold) $X$ is defined as
\[K_X := \bigwedge^n \Omega_X^1 = \Omega_X^n,
\]
where $\Omega^1_X = T^*_X$. The sections of the canonical bundle are given by holomorphic $n$-forms on $X$, which locally look like $\omega = f(z)dz$, where $z$ is a local coordinate and $f(z)$ is holomorphic. In the discussion that follows, $[X]$ is the line bundle associated with the divisor $X$.

**Proposition 2 (The Adjunction Formula).** For $Y$ and $X$ defined as above we have the isomorphism

$$K_Y \cong K_X |_X \otimes N_X.$$ 

**Proof.** From the conormal exact sequence for $X$,

$$0 \rightarrow N^*_X \rightarrow \Omega^1_Y |_X \rightarrow \Omega^1_X \rightarrow 0,$$

we have that

$$K_Y |_X \cong \bigwedge^n (\Omega^1_Y |_X) \cong N^*_X \otimes \bigwedge^{n-1} \Omega^1_X \cong N^*_X \otimes K_X.$$ 

Tensoring with $N_X$ gives the result. $\square$

The map on sections corresponding to the adjunction formula is called the Poincaré residue map. To describe it, we need to set up some notation and observe a few basic facts. Denote by $\Omega^n_Y(X)$ the sheaf of meromorphic differentials on $Y$ with a pole of order one along $X$. Tensoring by a section of $[X]$ provides the isomorphism

$$\Omega^n_Y(X) \cong \Omega^n_Y \otimes [X],$$

where we are abusing notation and writing $\Omega^n_Y \otimes [X]$ for $\theta(\Omega^n_Y \otimes [X])$. The line bundle $[X]$ is given by transition functions $g_{ij} = f_i/f_j$, where $f_i$ and $f_j$ are local functions of $X$ on open sets $U_i$ and $U_j$ with nontrivial intersection. Using the product rule then shows that a section $df_i$ of the conormal bundle on $X$ can be written as $g_{ij} df_j$, which means that $[X] \otimes N^*_X$ has a nonzero global section $\{df_i\}$ and is consequently trivial. Dualizing, we obtain $N_X = [X] |_X$. The above isomorphism and Proposition 2 then imply that sections of $\Omega^n_Y(X)$ correspond to sections of $\Omega^{n-1}_Y$. The former are locally given by meromorphic $n$-forms with a single pole along $X$ and holomorphic elsewhere,

$$\omega = \frac{g(z)}{f(z)}dz_1 \wedge \ldots \wedge dz_n,$$

where $z = (z_1, z_2, \ldots, z_n)$ are the local coordinates on $Y$, and $X$ is locally given by $f(z)$. If we write $df = \sum_{i=1}^n \frac{\partial f}{\partial z_i} dz_i$, it follows that for any $i$ such that $\frac{\partial f}{\partial z_i} \neq 0$, the form $\omega'$ on $X$ defined by

$$\omega' = (-1)^i \frac{g(z)dz_1 \wedge \ldots \wedge \hat{dz}_i \wedge \ldots \wedge dz_n}{\partial f/\partial z_i}$$

(5)

satisfies

$$\omega = \frac{df}{f} \wedge \omega'.$$
The Poincaré residue map $\text{Res}: \Omega^n(X) \rightarrow \Omega^{n-1}_X$ can then locally be given by $\omega \mapsto \omega' |_{f=0}$.

**Example 1.** Let $\mathbb{P}^2$ have coordinates $[x_1 : x_2 : x_3]$. The Fermat family of elliptic curves is the one-parameter family of hypersurfaces $Z_\psi \subset \mathbb{P}^2$ given by

$$Z_\psi: \{ F_\psi := x_1^3 + x_2^3 + x_3^3 - 3\psi x_1 x_2 x_3 = 0 \}$$

for $\psi \in \mathbb{C} \setminus \{ \xi_1, \xi_2, \xi_3 \}$, where we exclude $\xi_n = (e^{2\pi i/3})^n$ as values of the parameter $\psi$ since they yield singular fibers. Let

$$\omega = \frac{1}{F_\psi} (x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2)$$

be a section of $\Omega^2_{\mathbb{P}^2}(Z_\psi)$. We will directly compute $\omega'$ on $U_3 = \{ [x_1 : x_2 : x_3] | x_3 \neq 0 \}$ with coordinates $z_1 = x_1 / x_3$ and $z_2 = x_2 / x_3$. Since

$$d z_1 = \frac{\partial z_1}{\partial x_1} dx_1 + \frac{\partial z_2}{\partial x_2} dx_2 = \frac{x_3 dx_1 - x_1 dx_3}{x_3^2} \quad \text{and} \quad d z_2 = \frac{x_3 dx_2 - x_2 dx_3}{x_3^2},$$

we have

$$d z_1 \wedge d z_2 = \frac{1}{x_3^2} (x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2).$$

If we let $f := F_\psi |_{U_3}$, then it follows that

$$\omega = \frac{x_3}{F_\psi} d z_1 \wedge d z_2 = \frac{x_3^3}{x_3^3 (\frac{x_1^3}{x_3^3} + \frac{x_2^3}{x_3^3} + 1 - 3\psi \frac{x_1 x_2 x_3}{x_3^3})} d z_1 \wedge d z_2$$

$$= \frac{1}{f} d z_1 \wedge d z_2.$$

We wish to solve for $A(z_1, z_2)$ and $B(z_1, z_2)$ in $\omega' = A d z_1 + B d z_2$ satisfying

$$\omega' \wedge \frac{df}{f} = \frac{1}{f} d z_1 \wedge d z_2.$$

Taking $B = 0$ and evaluating

$$A d z_1 \wedge \frac{1}{f} \left( \frac{df}{dz_1} d z_1 + \frac{df}{dz_2} d z_2 \right)$$

yields the relation

$$\frac{\partial f}{\partial z_2} A d z_1 \wedge d z_2 = \frac{1}{f} d z_1 \wedge d z_2,$$
which implies $A = \frac{1}{\pi^2}$ and consequently $\omega' = \frac{\partial f}{\partial z}$.

Remark 1. Another way to realize the Poincaré residue map is as integration over a tube $\tau(X)$ along the hypersurface $X$. The map $\text{Res}: H^n(\mathbb{P}^n \setminus X) \mapsto H^{n-1}(X)$ is given by

$$\omega \mapsto \omega' = \frac{1}{2\pi i} \int_{\tau(X)} \omega.$$ 

### 2.2.2 Higher Order Poles and Reduction of Pole Order

In this section we will generalize the residue map to rational forms with higher order poles in order to later more easily find Picard-Fuchs equations. Let $\mathbb{P}^n$ have coordinates $[x_0: \ldots : x_n]$ and $J \in \mathcal{J} = \{(j_1, \ldots, j_k): j_1 < j_2 < \ldots < j_k\}$. Consider a rational $k$-form on $\mathbb{C}^{n+1}$ given by

$$\phi = \frac{1}{B(x)} \sum_{J \in \mathcal{J}} A_J(x) dx_J,$$

where $x = (x_0, \ldots, x_n)$, $dx_J = dx_{j_1} \wedge dx_{j_2} \wedge \ldots \wedge dx_{j_k}$, and $A_J, B$ are homogeneous polynomials. By [14], this $k$-form comes from a $k$-form on $\mathbb{P}^n$ if and only if $\deg B(x) = \deg A_J(x) + k$ and $\theta(\phi) = 0$, where $\theta := \sum_{i=0}^n x_i \frac{\partial}{\partial x_i}$ is the Euler vector field. This fact allows us to express rational forms on $\mathbb{P}^n$ in a way suitable for later calculations.

Lemma 1. Rational $(n+1-l)$-forms on $\mathbb{P}^n$ may all be written as

$$\omega = \frac{1}{B(x)} \sum_{J \in \mathcal{J}} \left[ (-1)^{\sum_{j=1}^l} \left( \sum_{i=1}^l (-1)^i x_i A_{j_1, \ldots, j_i} (x) \right) \right] dx_J,$$

where $\deg B = \deg A_{j_1, \ldots, j_i} + (n + 2 - l)$, and $dx_J$ denotes the $(n+1-l)$-form with $dx_j$ omitted if $j \in J$.

Proof. This is [14] Theorem 2.9. \qed
where $P$ and $Q$ are relatively prime and $\deg P = k \deg Q - (n + 1)$. In this case we say that $\omega$ has a pole of order $k \geq 1$ along $X$. Let

$$\mathcal{R} := \left\{ \frac{P}{Q} \mid \deg P = k \deg Q - (n + 1) \right\}$$

be the set of all rational $n$-forms with a pole along $X$. By [15], there is an isomorphism between $\mathcal{R}$ modulo exact forms and $H_n(\mathbb{P}^n \setminus X; \mathbb{C})$.

**Definition 2.** Let $Q(x) \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial. The *Jacobian ideal* of $Q$ is given by

$$J(Q) = \langle \frac{\partial Q}{\partial x_0}, \ldots, \frac{\partial Q}{\partial x_n} \rangle.$$

**Proposition 3.** We can reduce the order of the pole of $\omega = \frac{P(x)}{Q(x)^k} \Omega$ from $k \geq 2$ to $k - 1$ by adding an exact form if and only if $P$ is in the Jacobian ideal $J(Q)$.

**Proof.** A rational $(n - 1)$-form $\varphi$ with a pole of order $k - 1$ along $X$ can by Lemma[1] be written as

$$\varphi = \frac{1}{Q(x)^k-1} \sum_{i<j} (-1)^{i+j} (x_i A_j(x) - x_j A_i(x)) \, dx_{i,j}.$$

A brief calculation shows that

$$d\varphi = \left[ (k - 1) \sum_{i=0}^n \left( A_j(x) \frac{\partial Q(x)}{\partial x_i} \right) \right] \, \frac{Q(x)^{k-1}}{Q(x)^k} \Omega,$$

which after rearranging the terms is equivalent to

$$\frac{R(x)}{Q(x)^{k-1}} \Omega = \omega + d\varphi$$

for some polynomial $R(x)$, proving the result. $\square$

Let

$$\mathcal{J} := \left\{ \frac{P \Omega}{Q^k} \mid P \in J(Q) \right\} \subset \mathcal{R},$$

be the forms whose pole order can be reduced by an exact form. Then there is a natural filtration of $H^n(\mathbb{P}^n \setminus X; \mathbb{C})$ by pole order,

$$B_1 \to B_2 \to \ldots \to B_i \to B_{i+1} \to \ldots$$

where
$B_i = \left( \mathcal{B} \right)_i := \left\{ \left[ \frac{P\Omega}{Q} \right] \mid \deg(P) = i \deg(Q) - (n+1) \right\}$.

Furthermore, by a theorem due to Macaulay [14, Theorem 4.11], for any $P\Omega/Q^m$ such that $Q$ is nonsingular and $\deg P \geq (n+1)(\deg Q - 2)$, we have that $P \in J(Q)$. This means that the filtration stabilizes, since this inequality is satisfied for $m > n$.

In other words, we can find a vector space basis $\mathcal{B}_1 \cup \ldots \cup \mathcal{B}_n$ of $H^n(\mathbb{P}^n \setminus X; \mathbb{C})$, where $\mathcal{B}_i$ is a basis of $B_i$ consisting of forms with a pole of order $i$.

**Remark 2.** If $Q$ and some coefficient of $P$ depend on a parameter $\psi$, and we denote $d/d\psi = f'$ for any polynomial $f$, then

\[
\frac{d}{d\psi} \left( \frac{P\Omega}{Q'} \right) = \frac{(QP' - lPQ')\Omega}{Q^{'2}}.
\]

In other words, pole order increases by one when differentiating with respect to $\psi$.

It is shown in [15] that $B_l$ can be identified with the Hodge filtration $F_{n-l}^{n-1}H^n(X)$, in which case the above equation is a manifestation of Griffiths transversality. For more details, see [25, Section 10.2.2].

**Definition 3.** Fix an $(n-1)$-cycle $\gamma$ on $X$. The generalized residue map

\[
\text{Res} : H^n(\mathbb{P}^n \setminus X; \mathbb{C}) \rightarrow PH^{n-1}(X)
\]

is determined by the relation

\[
\frac{1}{2\pi i} \int_{\tau(\gamma)} \frac{P}{Q'} \Omega = \int_{\tau} \text{Res} \left( \frac{P}{Q'} \Omega \right),
\]

where $\tau(\gamma)$ is a tube around $\gamma$, and $PH^{n-1}(X)$ is the primitive cohomology of $X$. If $H$ represents a hyperplane class, primitive cohomology is defined as

\[
PH^{n-1}(X) = \{ \eta \in H^{n-1}(X; \mathbb{C}) \mid \eta \cdot H = 0 \}.
\]

The residue map is surjective in general, and in the case that $n-1$ is odd, primitive cohomology captures all of the cohomology of $X$ (for a proof, see [14]). Therefore, if we are working with an odd-dimensional one-parameter family of hypersurfaces $X_\psi : \{ Q = 0 \}$, as is the case in equation (1), then $H^{n-1}(X_\psi; \mathbb{C})$ has a basis of residues. Moreover, we have that

\[
\frac{d^k}{d\psi^k} \int_{\gamma} \text{Res} \left( \frac{P\Omega}{Q'} \right) = \frac{d^k}{d\psi^k} \left( \frac{1}{2\pi i} \int_{\tau(\gamma)} \frac{P\Omega}{Q'} \right) = \frac{1}{2\pi i} \int_{\tau(\gamma)} \frac{d^k}{d\psi^k} \left( \frac{P\Omega}{Q'} \right).
\]

So, in order to find relations amongst $\frac{d^k}{d\psi^k} \text{Res} \left( \frac{P\Omega}{Q'} \right)$, we can work with meromorphic forms on $\mathbb{P}^n$.

**Remark 3.** If $n = 1$, the residue map is the familiar contour integral. For instance, for a one form $\frac{p(z)}{q(z)}dz$ with $p, q \in \mathbb{C}[z]$ we have
\[ \int_{\Gamma} \frac{p(z)}{q(z)} \, dz = \frac{1}{2\pi i} \left[ \sum_{P_j} \text{Res}_{P_j} \left( \frac{p(z)}{q(z)} \right) \right], \]

where \( \Gamma \) encircles all the poles \( P_j \) of \( \frac{p(z)}{q(z)} \).

### 2.3 Determining Picard-Fuchs Equations

By this point we have established sufficient background material to determine Picard-Fuchs equations for one-parameter families of hypersurfaces in several ways.

#### 2.3.1 The Griffiths-Dwork Method

Let \( X_\psi \subset \mathbb{P}^n \) be an element of a one-parameter family of hypersurfaces parameterized by \( \psi \). Suppose that \( X_\psi \) is given by \( \{ Q = 0 \} \) and choose a form \( P \Omega / Q \) whose residue is a holomorphic \((n-1)\)-form \( \omega(\psi) \) on \( X_\psi \). Finding the Picard-Fuchs equation satisfied by the period \( \int_{\Gamma} \omega(\psi) \) amounts to finding a relation between \( \omega(\psi) \) and its derivatives. The description of forms on \( X_\psi \) as residues of meromorphic forms on \( \mathbb{P}^n \) gives rise to the following algorithm for finding Picard-Fuchs equations called the Griffiths-Dwork method, also described in [4, 7].

1. Find a basis \( B \) of meromorphic differentials for \( H^n(\mathbb{P}^n \setminus X_\psi; \mathbb{C}) \). This amounts to finding a basis for the ring \( \mathbb{C}(\psi)[x_1, \ldots, x_n]/J(Q) \), where \( \mathbb{C}(\psi) \) emphasizes that coefficients are rational functions in \( \psi \).
2. Starting with a form \( P \Omega / Q \) as above, calculate \( |B| \) of its derivatives with respect to \( \psi \) and express them in terms of forms in the basis and forms with numerators in \( J(Q) \). Pole order increases with differentiation due to Remark 2, so use Proposition 3 to reduce the pole order.
3. The \( |B| + 1 \) forms obtained from \( \omega \) and its derivatives must have a relation between them. This is the Picard-Fuchs equation satisfied by \( \omega(\psi) \).

**Example 2.** We follow [4] to illustrate the Griffiths-Dwork method on the mirror \( \mathcal{W} \) of the one-parameter family \( \mathcal{M} \) of quintic threefolds whose elements are given by

\[ X_\psi: \quad \{ Q := \sum_{i=1}^5 x_i^5 - 5\psi x_1 x_2 x_3 x_4 x_5 = 0 \}. \]

Let us roughly describe the mirror construction. Let \( \eta_i \) be a fifth root of unity, and define \( \mathcal{G} \) be the group of diagonal automorphisms

\[ g: (x_1, x_2, x_3, x_4, x_5) \mapsto (\eta_1 x_1, \eta_2 x_2, \eta_3 x_3, \eta_4 x_4, \eta_5 x_5) \]

which preserve the holomorphic 3-form on \( X_\psi \), modulo those that come from the scaling action of projective space. The mirror family \( \mathcal{W} \) is then given by the resolu-
tion of singularities of the quotient $\mathcal{M}/\mathcal{G}$. For a generic pair $M \in \mathcal{M}$ and $W \in \mathcal{W}$, Hodge numbers are exchanged according to $h^{p,q}(M) = h^{3-p,3-q}(W)$ by mirror symmetry. So, since $b^{1,1}(W) = 1$, we have that $h_3(W) = 4$. Moreover, cohomology of $\mathcal{W}$ contains the $\mathcal{G}$-invariant cohomology of $\mathcal{M}$.

We therefore choose residues of four meromorphic 4-forms $\omega_1, \ldots, \omega_4$ that are invariant under $\mathcal{G}$. This will give a basis for the cohomology of the mirror family. Specifically, for any $l \geq 1$ we define $P_l = (-1)^{l-1}(l-1)!\psi^l(\prod_{i=1}^5 x_i)\psi^{l-1}$ and $\omega_l = P_l \Omega/Q$. Our goal is to find the Picard-Fuchs equation of $\text{Res}(\omega_1)$, i.e., the relation between derivatives of $\omega_1$ with respect to $\psi$. It is convenient to define $w = \psi^{-5}$ and differentiate using the operator $\partial_w := w \frac{d}{dw} = -\frac{1}{5} \psi \frac{d}{d\psi}$. We have that

$$\partial_w \omega_l = -\frac{l}{5} \omega_l + \omega_{l+1},$$

which after repeated application to $\omega_1$ yields

$$\begin{pmatrix}
\omega_1 \\
\partial_w \omega_1 \\
\partial_w^2 \omega_1 \\
\partial_w^3 \omega_1 \\
\partial_w^4 \omega_1
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 & 0 \\
-\frac{5}{6} & 1 & 0 & 0 \\
\frac{3}{125} & -\frac{6}{5} & 1 & 0 \\
-\frac{3}{125} & \frac{22}{5} & -\frac{6}{5} & 1 \\
-\frac{1}{125} & \frac{22}{5} & -\frac{6}{5} & 1
\end{pmatrix}
\begin{pmatrix}
\omega_1 \\
\omega_2 \\
\omega_3 \\
\omega_4 \\
\omega_5
\end{pmatrix}. \quad (8)$$

We need to differentiate one more time in order to get a non-trivial relation. Since forms can be written in terms of the basis, and $\partial_w^5 \omega_1$ has a pole of order 5 by Remark 2, it follows that

$$\partial_w^4 \omega_1 = c_1(\psi) \omega_1 + \ldots + c_4(\psi) \omega_4 + \frac{(\sum A_i B_j) \Omega}{Q^5}$$

for some $c_j(\psi) \in \mathbb{C}(\psi)$, $A_i(x) \in \mathbb{C}[x_1, \ldots, x_5]$, where $\{B_j\}$ constitutes a Gröbner basis for $J(Q)$. We can reduce the pole order of the last term using Proposition 3 and again express the lower order form in terms of the basis $\{\omega_i\}$. These calculations can be done using a computer (see [7] for details and source code), and the result is the Picard-Fuchs equation (3).

**Example 3.** The quintic threefold family (1) can be seen as a deformation of the Fermat quintic $x_1^5 + \ldots + x_5^5$. This polynomial belongs to a larger class of invertible polynomials. A quasi-homogeneous polynomial

$$G(x) = \sum_{i=1}^n c_i \prod_{j=1}^n x_i^{a_{ij}}$$

with (reduced) weights $(q_1, \ldots, q_n)$ is invertible if the exponent matrix $(a_{ij})$ is invertible and the ring $\mathbb{C}[x]/J(G)$ has a finite basis. In general, the zero set of an invertible polynomial defines a variety in a weighted projective space $\mathbb{P}(q_1, \ldots, q_n)$, which can be realized as the quotient of the usual projective space $\mathbb{P}^{n-1}$ by an abelian group action (for more details about varieties in weighted projective space, see [3]). We can obtain a one-parameter family from polynomials such as $G(x)$ via
\[ F(x) = G(x) + \psi \prod_{i=1}^{n} x_i. \]

Elements of this family define Calabi-Yau hypersurfaces if \( \sum g_i = \text{deg } G \) by [5], so such deformations provide a large class examples of one-parameter Calabi-Yau families. A combinatorial method for calculating the Picard-Fuchs equations of such families based on the Griffiths-Dwork method is presented in [7]. For instance, the family of K3 surfaces

\[ \{ x_1^8 + x_2^4 + x_1 x_3^3 + x_4^3 \prod_{i=1}^{n} x_i^3 = 0 \} \]

in \( \mathbb{P}(3, 6, 7, 8) \) has Picard-Fuchs equation \( \mathcal{L} f = 0 \), where for \( \vartheta_\psi = \psi \frac{\partial}{\partial \psi} \) we have

\[ \mathcal{L} : = \psi^{12} \vartheta_\psi^3 (\vartheta_\psi + 3)(\vartheta_\psi + 6)(\vartheta_\psi + 9) \]
\[ - 2^8 3^9 (\vartheta_\psi - 1)(\vartheta_\psi - 2)(\vartheta_\psi - 5)(\vartheta_\psi - 7)(\vartheta_\psi - 10)(\vartheta_\psi - 11). \]

### 2.4 Finding the Periods

In this section we describe how to find series solutions to the Picard-Fuchs equation (2). The solutions correspond to periods of the mirror, and thus by solving the equation we obtain a series description of the periods.

#### 2.4.1 Hypergeometric Series

Let \( Q \) be the defining polynomial of the quintic threefold \( X_\psi \) in equation (1). It turns out that it is possible to directly calculate the period on \( X_\psi \) with respect to

\[ \frac{d x_2}{\frac{d Q}{\partial x_2}}. \]

**Example 4.** We will show here the analogous calculation on the Fermat family of elliptic curves \( Z_\psi : \{ F_\psi = 0 \} \) defined in Example [1] which can be applied, mutatis mutandis, to \( X_\psi \). The latter appears in [1]. Denote by \( \gamma \) the cycle on \( \{ Q = 0 \} \) determined by \( |x_1| = \delta \) for some small \( \delta \), and consider

\[ \pi_0(\psi) = 3 \psi \frac{1}{2 \pi i} \int_{\gamma} \frac{x_3 dx_1}{\frac{d Q}{\partial x_2}}. \]

Since \( 1 = \frac{1}{2 \pi i} \int_{\gamma} \frac{dx_3}{x_3} \) and
\[
\text{Res} \left( \frac{dx_2}{f(x)} \right) = \frac{1}{2\pi i} \int_{\gamma_2} \frac{dx_2}{F_\psi} = -\frac{1}{\partial F_\psi / \partial x_2}
\]

we have

\[
\pi_0(\psi) = -3\psi \frac{1}{(2\pi i)^3} \int_{\gamma_1 \times \gamma_2 \times \gamma_3} \frac{dx_1 dx_2 dx_3}{F_\psi(x)}
\]

\[
= -3\psi \frac{1}{(2\pi i)^3} \int_{\gamma_1 \times \gamma_2 \times \gamma_3} \frac{dx_1 dx_2 dx_3}{x_1^3 + x_2^3 + x_3^3 - 3\psi x_1 x_2 x_3}
\]

\[
= 3\psi \frac{1}{(2\pi i)^3} \int_{\gamma_1 \times \gamma_2 \times \gamma_3} \frac{dx_1 dx_2 dx_3}{3\psi x_1 x_2 x_3} \left( 1 - \frac{x_1^3 + x_2^3 + x_3^3}{3\psi x_1 x_2 x_3} \right)
\]

\[
= -\frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{\gamma_1 \times \gamma_2 \times \gamma_3} \frac{dx_1 dx_2 dx_3}{x_1 x_2 x_3} \frac{(x_1^3 + x_2^3 + x_3^3)^n}{(3\psi)^n (x_1 x_2 x_3)^n},
\]

with the expansion performed for large enough \(\psi\). We now wish to evaluate the integral using residues. The integral for each \(n\) is a rational function in the variables \(x_i\) and therefore vanishes for all powers of \(x_i\) except \(-1\). This happens when the term \((x_1 x_2 x_3)^n\) occurs in the expansion of \((x_1^3 + x_2^3 + x_3^3)^n\). To see when this is the case, consider

\[
(x_1^3 + x_2^3 + x_3^3)^n = \sum_{k_1 + k_2 + k_3 = n} \binom{n}{k_1, k_2, k_3} x_1^{3k_1} x_2^{3k_2} x_3^{3k_3},
\]

and note that we want \(k_1 = k_2 = k_3 = k\) so that \(n = 3k\). The coefficient we need is then \(\binom{3k}{k, k, k} = \frac{(3k)!}{(k!)^3}\) and the expression for \(\pi_0(\psi)\) becomes

\[
\pi_0(\psi) = \sum_{k=0}^{\infty} \frac{(3k)!}{(k!)^3} \frac{1}{(3\psi)^{3k}}
\]

(10)

for \(\psi\) large enough.

Similarly, the integral \(\sigma_0(\psi)\) of (9) on \(X_\psi\) is given by equation (3), i.e.,

\[
\sigma_0(\psi) = \sum_{m=0}^{\infty} \frac{(5m)!}{(m!)^5} \lambda^m,
\]

where \(\lambda = \frac{1}{(5\psi)^5}\). This gives one period of \(X_\psi\) and, as we will see in a moment, a solution of \(\mathcal{L} f = 0\) for \(\mathcal{L}\) defined in (7). What about the other solutions? Recall that

\[
f(z) = \sum_k C(k) z^k
\]

is a (generalized) hypergeometric series if the ratio of consecutive terms is a rational function of \(k,

\[
\frac{C(k+1)}{C(k)} = c \frac{(k+a_1) \ldots (k+a_p)}{(k+b_1) \ldots (k+b_q)(k+1)}
\]

(12)
where \( c \) is a constant \([5]\). In the case of the Fermat family of elliptic curves, we indeed have

\[
\frac{(3(k+1))^i/((k+1)i)^3}{(3k)!/(k!)^3} = \frac{3(k + \frac{2}{3})(k + \frac{4}{3})}{(k+1)(k+1)}.
\]

(13)

The standard notation for a hypergeometric function given by \([11]\) is

\[ f(z) = \, _pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z), \]

in which case \( f(z) \) satisfies the hypergeometric differential equation

\[
\left[ \vartheta_z \frac{\partial}{\partial z} \prod_{i=1}^{q} \left( \vartheta_z + b_i - 1 \right) - z \prod_{i=1}^{p} \left( \vartheta_z + a_i \right) \right] f(z) = 0,
\]

(14)

where \( \vartheta_z := \frac{z}{\psi} \). The Picard-Fuchs equation \([2]\) is hypergeometric, where one solution is given by \( \omega_0 = _4F_3(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1, 1, 1; \psi^{-5}) \). We will next explain how to find the remaining solutions.

### 2.4.2 Frobenius Method

Hypergeometric differential equations can be solved using the Frobenius method. We will illustrate the basic technique for the Picard-Fuchs equation \([2]\). We already have the series description \([3]\) of one solution around \( \lambda = 0 \)

\[ \omega_0 = \sum_{m=0}^{\infty} \frac{\Gamma(5k+1)}{r(k+1)^3} \lambda^m, \]

where \( \lambda = \frac{1}{(5\psi)^3} \). So our goal is to obtain the remaining three solutions of the differential equation \( \mathcal{L} f(z) = 0 \), where \( \mathcal{L} \) is defined in equation \([2]\). Before we do so, we remark that there is a more systematic way of finding the first solution of a hypergeometric differential equation than the direct calculation of the integral \( \omega_0 \). Since it is not critical to what follows, we illustrate with a quick example.

**Example 5.** The differential equation satisfied by the period \( \pi_0 \) of Example \([1]\) is by equation \([13]\) given by

\[
\mathcal{L} f(z) = \left[ \vartheta_z^2 - z(\vartheta_z + \frac{1}{3})(\vartheta_z + \frac{2}{3}) \right] f(z) = 0
\]

(15)

in terms of \( z = 1/(3\psi)^3 \). We now make the ansatz

\[ f(z) = z^c + \sum_{k=1}^{m} a_k z^{k+c} \]

(16)

for the solution around the regular singular point \( z = 0 \). Applying the differential equation and setting coefficients to zero, we obtain
\[ c^2 = 0 \quad \text{and} \quad (k + c)^2 a_k - \left( (k - 1 + c)^2 + (k - 1 + c) \left( \frac{2}{9} \right) \right) a_{k-1} = 0 \]

from the \( z^c \) term and the \( z^{k+c} \) terms for \( k \geq 1 \), respectively. The first equation is called the \textit{indicial equation}, and implies that \( c = 0 \). Using this in the second equation gives for \( k \geq 1 \)

\[ a_k = \frac{(\frac{1}{3} + k - 1)(\frac{2}{3} + k - 1)}{k^2} a_{k-1}. \]

We can then iterate to obtain the solution

\[ f_1(z) = \sum_{k=0}^{\infty} \frac{(\frac{1}{3})_k (\frac{2}{3})_k}{(1)_k} z^k, \]

where \( (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)\ldots(a+k-1) \) is the Pochhammer symbol. It is easily checked that this solution is equivalent to (10). There is, of course, a second solution of (15). The indicial equation is of degree two (which is implied by the fact that \( z = c \) is a \textit{regular} singular point), but has a repeated root and cannot be used again as above. The idea is to show that \( \frac{\partial f}{\partial c} \bigg|_{c=0} \) is a solution, the analog of which we will tackle for the quintic directly.

We now return to the case of the quintic threefold family (1). Define

\[ \sigma(\lambda, s) = \sum_{k=0}^{\infty} \frac{\Gamma(5(k + s) + 1)}{\Gamma(k + s + 1)} \lambda^{k+s} \]

and note that \( \sigma_0(\lambda) = \sigma(\lambda, 0) \). A direct calculation shows that \( \mathcal{L} \sigma(\lambda, s) = s^4 \lambda^s + \mathcal{O}(s^5) \), from which it follows that for \( 0 \leq i \leq 3 \) we have

\[ \frac{\partial^i}{\partial s^i} \mathcal{L} \sigma(\lambda, s) \bigg|_{s=0} = \mathcal{L} \frac{\partial^i}{\partial s^i} \sigma(\lambda, s) \bigg|_{s=0} = 0. \]

This gives us a set of solutions,

\[ \sigma_i(\lambda) = \frac{\partial^i}{\partial s^i} \sigma(\lambda, s) \bigg|_{s=0}. \]

To describe them more explicitly and see that they are linearly independent, let

\[ a_k(s) := \frac{\Gamma(5(k + s) + 1)}{\Gamma(k + s + 1)} \quad \text{and} \quad g_i(z) = \sum_{k=0}^{\infty} \frac{\partial^i a_k(s)}{\partial s^i} \bigg|_{s=0} \lambda^k. \]

We calculate
Iterating, we obtain a full set of solutions. Namely, for $0 \leq i \leq 3$ the solutions are

$$\sigma_i(\lambda) = \sum_{j=0}^{i} \left( \begin{array}{c} i \\ j \end{array} \right) g_j(\lambda)(\log \lambda)^{i-j}. \quad (18)$$

In the case of the family given in (2), these solutions correspond to the periods by $[\Pi]$.

### 3 Point Counting

In this chapter we will show how to obtain the expression (4) for the number of points on a quintic threefold (1) defined over a finite field $k = \mathbb{F}_p$ of characteristic $p$. The basic idea is to use $p$-adic character formulas to mimic the behavior of period integrals via $p$-adic analysis techniques. We will also explain how to calculate the zeta functions of several varieties over finite fields, and discuss the relationship of zeta functions and mirror symmetry.

#### 3.1 Character Formulas

Let us first establish some basics about characters of finite groups. Let $K = \mathbb{C}$ or $\mathbb{C}_p$, let $G$ be a nontrivial finite abelian group, and take a non-trivial character $\chi : G \to K$. We then have that

$$\sum_{x \in G} \chi(x) = 0$$

since for $y \in G$ such that $\chi(y) \neq 1$ we have $\chi(y) \sum_{x \in G} \chi(x) = \sum_{x \in G} \chi(x)$, and so $(\chi(y) - 1) \sum_{x \in G} \chi(x) = 0$. It is also easy to see that

$$\sum_{\chi \in \hat{G}} \chi(x) = \begin{cases} 0, & \text{if } x \neq 1; \\ |G|, & \text{if } x = 1, \end{cases} \quad (19)$$

and

$$\chi(x^{-1}) = \frac{1}{\chi(x)} = \frac{1}{\chi(x)},$$

where if the character $\chi$ maps into $\mathbb{C}_p$, we define $\overline{\chi(x)} = \chi^{-1}(x)$. Our goal is to count points on hypersurfaces defined over $k = \mathbb{F}_q$, where $q = p^n$ with coordinates in some $k_r =$degree $r$ extension of $k$. Denote the trivial character by $\varepsilon$. If $\chi : k_r \to K$
is a multiplicative character we can define
\[ \chi(0) = \begin{cases} 
0, & \chi \neq \varepsilon; \\
1, & \chi = \varepsilon,
\end{cases} \]
so that we can consider it as a homomorphism
\[ \chi : k_r \to K. \]

Fixing a non-trivial additive character \( \psi : k_r \to (K, \times) \), we define the Gauss sum
\[ g(\chi) = \sum_{x \in k_r} \chi(x) \psi(x) \]
which for non-trivial \( \chi \) equals \( g_0(\chi) := \sum_{x \in k_r^*} \chi(x) \psi(x) \) and otherwise
\[ g(\varepsilon) = \sum_{x \in k_r} \psi(x) = 0 \quad \text{and} \quad g_0(\varepsilon) = -1, \]
since \( \psi \) is non-trivial.

Gauss sums \( g_0 \) are proportional to Fourier transforms: consider \( \psi : k_r^* \to K \) as a \( K \)-valued function on \( k_r^* \), and let the Fourier transform of \( f \) to be the \( K \)-valued function on the group \( \hat{k}_r^* \) of multiplicative characters \( \chi : k_r^* \to K \) given by
\[ \hat{f}(\chi) = \frac{1}{q^r - 1} \sum_{x \in k_r^*} \psi(x) \chi(x). \]
We also get Fourier inversion, i.e., we can express \( f(x) \) in terms of characters. Consider the sum over all multiplicative characters \( \chi : k_r^* \to K \) for any \( x \neq 0 \),
\[ \sum_{\chi} g_0(\overline{\chi}) \chi(x) = \sum_{\chi} \left( \sum_{y \in k_r^*} \overline{\chi}(y) \psi(y) \right) \chi(x) 
= \sum_{\chi} \sum_{y \in k_r^*} \chi(y^{-1} x) \psi(y) 
= \sum_{y \in k_r^*} \psi(y) \sum_{\chi} \chi(y^{-1} x) 
= (q^r - 1) \psi(x). \]
Therefore, for all \( x \neq 0 \) we have
\[ \psi(x) = \frac{1}{q^r - 1} \sum_{\chi} g_0(\overline{\chi}) \chi(x) = \frac{1}{q^r - 1} \sum_{\chi} g_0(\chi) \overline{\chi}(x). \quad (20) \]
This is the Fourier inversion formula for \( f = \psi \) (up to an unconventional choice for what we are conjugating):
\[ f(x) = \sum_{\chi \in \hat{G}} \hat{f}(\chi) \hat{\varphi}(x). \]

**Remark 4.** If \( x = 0 \), then \( \psi(0) = 1 \) and since \( \varphi(0) = 0 \) unless \( \chi = \varepsilon \) we have the right hand side equal to \( \frac{1}{q'-1}g_0(\varepsilon) = \frac{1}{q'-1}(-1) \neq 1 \). So the formula does not hold for \( x = 0 \). This will prove to be a minor annoyance when counting points.

**Remark 5.** Since \( g(\varepsilon)\varphi(x) = g_0(\varepsilon)\varepsilon(x) + \varepsilon(0)\psi(0)\varphi(x) = g_0(\varepsilon)\varphi(x) + 1 \) and for \( \chi \neq \varepsilon \) \( g(\chi) = g_0(\chi) \) we have
\[
\psi(x) = \frac{1}{q'-1} \left( \sum_{\chi} g(\chi) \varphi(x) - 1 \right),
\]
which is harder to work with, even though \( g(\varepsilon) = 0 \) and \( g_0(\varepsilon) \neq 0 \). Therefore, we will be using the Gauss sums \( g_0(\chi) \) as opposed to \( g(\chi) \).

### 3.1.1 \( p \)-adic Characters

We will now construct a concrete multiplicative and additive character into the \( p \)-adic numbers \( \mathbb{C}_p \) to use with the formulas above. For now, we will use characters from \( k = \mathbb{F}_p \), leaving finer fields for later. Given \( n = mp^v \in \mathbb{Z} \), where \( (p, m) = 1 \) define the \( p \)-adic norm \( |n|_p = \frac{1}{p^v} \). The completion of \( \mathbb{Z} \) with respect to this norm gives the \( p \)-adic integers \( \mathbb{Z}_p \), which can be written as sequences
\[
\mathbb{Z}_p := \lim_{\rightarrow\leftarrow} \mathbb{Z}/p^n\mathbb{Z} = \{a_0 + a_1p + a_2p^2 + \ldots | a_i \in [0, p-1]\}
\]
with the last expression being thought of as giving increasingly better approximations of the corresponding \( p \)-adic integer as \( n \to \infty \). Taking the field of fractions gives \( \mathbb{Q}_p = \text{Frac}(\mathbb{Z}_p) \) whose algebraic closure \( \overline{\mathbb{Q}}_p \) is not complete. The completion of \( \overline{\mathbb{Q}}_p \) is \( \mathbb{C}_p \), and is also algebraically closed.

**Lemma 2 (Hensel’s Lemma).** Suppose that \( \overline{f} \in \mathbb{F}_p[x] \) and let \( f \in \mathbb{Z}_p[x] \) be any lift (so that \( f \equiv \overline{f} \pmod{p} \)). If \( \overline{\alpha} \in \mathbb{F}_p \) is a simple root of \( \overline{f} \), then there exists a unique \( \alpha \in \mathbb{Z}_p \) such that
\[
F(\alpha) = 0 \quad \text{and} \quad a \equiv \overline{\alpha} \pmod{p}.
\]

**Proof.** See [20]. \( \square \)

**Proposition 4.** For each \( x \in \mathbb{F}_p^\times \), there is a unique \((p-1)\)-st root of unity in \( \mathbb{Z}_p^\times \), denoted \( \text{Teich}(x) \) or \( T(x) \) such that \( T(x) \equiv x \pmod{p} \). The map \( T : \mathbb{F}_p^\times \to \mathbb{Z}_p^\times \) given by \( x \mapsto T(x) \) gives a multiplicative character called the Teichmüller character.
Proof. The elements of $\mathbb{F}_p^*$ are the roots of $\overline{f}(X) = X^{p-1} - 1$ since each $x \in \mathbb{F}_p^*$ satisfies the equation $\overline{f}(X) = 0$. Let $f(X) = X^{p-1} - 1 \in \mathbb{Z}_p[X]$ be a lift of $\overline{f}(X)$. By Hensel’s Lemma, each $x \in \mathbb{F}_p^*$ lifts to a unique $(p-1)$-st root of unity $T(x) \in \mathbb{Z}_p^*$ such that $T(x) \equiv x \mod p$. Since a product of roots of unity is still a root of unity, for $x, y \in \mathbb{F}_p^*$ we have that

$$(T(x)T(y))^{p-1} = 1$$

in $\mathbb{Z}_p$. Since we also have that

$$T(x)T(y) \equiv xy \mod p$$

it must be the case that

$$T(x)T(y) = T(xy)$$

by uniqueness in Hensel’s Lemma. \qed

We will use an explicit description of $T(x)$ as in [2]. Let $x$ denote an integer representative of $x \in \mathbb{F}_p^*$. We have that

$$x^{p-1} = 1 + O(p) \text{ in } \mathbb{Z},$$

and consequently that

$$(x^{p-1})^p = 1 + \binom{p}{1} O(p) + O(p^2) = 1 + O(p^3) \text{ in } \mathbb{Z}.$$  

By raising both sides of this equation to the $p$-th power repeatedly, it follows that

$$x^{p^k(p-1)} = 1 + O(p^{k+1}),$$

which is equivalent to

$$x^{p^k+1} = x + O(p^{k+1}).$$

Define

$$S(x) := \lim_{n \to \infty} x^{p^n}. \tag{21}$$

The character $T$ is uniquely determined by the conditions $T(x)^{p-1} = 1$ and $T(x) \equiv x \mod p$ for all $x \in \mathbb{F}_p^*$, or equivalently the conditions $T(x)^p = T(x)$ and $T(x) \equiv x \mod p$ for all $x \in \mathbb{F}_p^*$. Since the expression in equation (21) satisfies both of these conditions, we conclude that $S(x) = T(x)$ for all $x \in \mathbb{F}_p^*$. In fact, defining $T^i : \mathbb{F}_p^* \to \mathbb{Z}_p^*$ by $T^i(x) = T(x)^i$ for $i \in \{0, \ldots, p-2\}$ gives a full set of characters from $\mathbb{F}_p^*$ to $\mathbb{Z}_p^*$. Note that when applied to $\chi = T^i$, equation (19) takes the form

$$\sum_{x \in \mathbb{F}_p^*} T^i(x) = \begin{cases} 0, & \text{if } i \not\equiv 0 \mod p-1; \\ p-1, & \text{if } i \equiv 0 \mod p-1. \end{cases}$$
We now turn to constructing an additive character $\theta : \mathbb{F}_p \to \mathbb{C}^\times_p$. Let $\zeta_p$ be a $p$-th root of unity in the $p$-adic numbers and define

$$\theta(x) = \zeta_p^{T(x)}.$$ 

Since for $Z \in \mathbb{Z}_p$ we have

$$T(x + y) = T(x) + T(y) + pZ$$

it follows that

$$\theta(x + y) = \theta(x) + \theta(y),$$

so $\theta$ is indeed an additive character. For the root of unity we can take $\Theta(x) = \exp(\pi(x - x^p))$ and set $\zeta_p := \Theta(1)$, as shown in (20).

Recalling Remark 5, we consider the Gauss sum associated to these characters,

$$G_n = \sum_{x \in \mathbb{F}_p^\times} \theta(x) T^n(x),$$

where $n \in \mathbb{Z}$.

**Remark 6.** This expression can be thought of as a $p$-adic analog of the classical Gamma function,

$$\Gamma(s) = \int_0^\infty \frac{dt}{t} t^{s-1},$$

where we think of $T(x)$ as the analog of $t \mapsto t^s$, of $\theta(x)$ as the analog of $t \mapsto e^{-t}$, and of summation over $\mathbb{F}_p^\times$ as the analog of integration with respect to the Haar measure $\frac{dt}{t}$. In fact, relations can be proven in terms of these characters for $\Gamma(s)$ can also be given for $G_n$.

In this setting, formula (20) is

$$\theta(x) = \frac{1}{p-1} \sum_{m=0}^{p-2} G_{-m} T^m(x), \quad (22)$$

and if $p - 1 \nmid n$, we also obtain the relation

$$G_n G_{-n} = (-1)^n p. \quad (23)$$

### 3.1.2 Relationship with the Periods

To actually count points, we note that for any polynomial $P(x) \in \mathbb{F}_p[x]$, we have that

$$\sum_{y \in \mathbb{F}_p} \theta(y P(x)) = \begin{cases} p, & \text{if } P(x) = 0; \\ 0, & \text{if } P(x) \neq 0, \end{cases}$$

so that
\[ \sum_{y \in \mathbb{F}_p} \sum_{x \in \mathbb{F}_p} \theta(yP(x)) = p \cdot N^*(X), \]

where \( N^*(X) \) is the number of points on \( X : \{ P(x) = 0 \} \) with coordinates in \( \mathbb{F}_p^* \). We will illustrate the use of this formula using the Fermat family of elliptic curves.

**Example 6.** Let \( F_\psi = \sum_{i=1}^{3} x_i^3 - 3 \psi x_1 x_2 x_3 \in \mathbb{F}_p[x] \), and let \( N^*(Z_\psi) \) denote the number of nonzero \( \mathbb{F}_p \)-rational points on \( Z_\psi : \{ F_\psi = 0 \} \). We have that
\[
pN^*(Z_\psi) - (p - 1)^3 = \sum_{y, x_i \in \mathbb{F}_p} \theta(yF_\psi(x)),
\]
and by (22) that
\[
\theta(yF_\psi(x)) = \left( \prod_{i=1}^{3} \theta(yx_i^3) \right) \theta(-3 \psi x_1 x_2 x_3)
\]
\[
= \frac{1}{p-1} \sum_{m=0}^{p-2} G_{-m}T^m(-3\psi) \prod_{i=1}^{3} \theta(yx_i^3)T^m(x_i).
\]

Therefore,
\[
pN^*(Z_\psi) - (p - 1)^3 = \sum_{x_i \in (\mathbb{F}_p^*)^3} \frac{1}{p-1} \sum_{m=0}^{p-2} G_{-m}T^m(-3\psi) \prod_{i=1}^{3} \theta(yx_i^3)T^m(x_i)
\]
\[
= \frac{1}{p-1} \sum_{m=0}^{p-2} G_{-m}T^m(-3\psi) \sum_{y \in \mathbb{F}_p^*} \left( \sum_{w \in \mathbb{F}_p^*} \theta(yw^3)T^m(w) \right)^3,
\]

where we have used the fact that \( \sum_{x \in (\mathbb{F}_p^*)^3} \theta(0F(x)) = (p - 1)^3 \) in the first step, and renamed the variables \( x_i \) to \( w \), since for each \( x_i \) the sum is identical. Suppose that \( 3 \nmid (p - 1) \). Since 3 and \( (p - 1) \) are relatively prime, there exist \( a, b \in \mathbb{Z} \) such that \( 3a + b(p - 1) = 1 \). In particular, we have that
\[
3a \equiv 1 \pmod{p - 1}.
\]

Since \( T^{l(p-1)} \) is the identity character for any \( l \in \mathbb{Z} \), this implies that
\[
T^m = T^{3am} = (T^am)^3
\]
and so
\[ pN^*(Z_\psi) - (p - 1)^3 = \frac{1}{p - 1} \sum_{m=0}^{p-2} G_{-m}T^m(-3\psi) \left( \sum_{y \in \mathbb{F}_p^*} \theta(y^3)T^m(w)T^{ma}(y) \right)^3 \]

\[ = \frac{1}{p - 1} \sum_{m=0}^{p-2} G_{-m}T^m(-3\psi) \left( \sum_{y \in \mathbb{F}_p^*} \theta(y^3)T^{ma}(w^3)T^{ma}(y) \right)^3 \]

\[ = \frac{1}{p - 1} \sum_{m=0}^{p-2} G_{-m}T^m(-3\psi)((p - 1)G_{ma}^3) \]

\[ = p \sum_{m=0}^{p-2} \frac{G_{ma}^3}{G_m}T^m(-3\psi)(-1)^m. \]

To simplify this expression, note that the \( m \)-th Gauss sum \( G_m = G_m + l(p - 1) \) depends only on the class of \( m \) modulo \( p - 1 \), since \( m \) only appears in the exponent of \( T \) within the sum. Define the map \( \phi \in \text{End}(\mathbb{Z}/(p - 1)\mathbb{Z}) \) by \( m \mapsto am \). Its inverse is given by \( \phi^{-1} : k \mapsto 3k \), which allows us to rewrite the expression above as

\[ N^*(Z_\psi) = \sum_{k=0}^{p-2} \frac{G^3_k}{G_{3k}} T^{3k}(-3\psi)(-1)^{3k} + (p - 1)^3 \]

\[ = 1 + \sum_{k=1}^{p-2} \frac{G^3_k}{G_{3k}} T^{3k}(3\psi) + (p - 1)^3, \]

where we have used the fact that \( T(-1) = -1 \) and that

\[ G_0 = \sum_{x \in \mathbb{F}_p^*} \theta(x) = -1. \]

The analogous calculation for the quintic threefold is performed in [2]. The result after accounting for points for which \( x_i = 0 \) for some coordinate in the case \( 5 \nmid (p - 1) \) given by the expression

\[ N(X_\psi) = 1 + p^4 + \sum_{m=1}^{p-2} \frac{G^5_m}{G_{5m}}\text{Teich}^{-m}(\lambda), \]

where \( \lambda = 1/(5\psi)^5 \). Using (23) and \( -m \mapsto (p - 1) - m \), which does not change the expression, we obtain exactly equation (4). If we keep Remark 6 in mind, this allows us to interpret the number of points \( N(X_\psi) \) as the \( p \)-adic analog to the period (3). We can relate the number of points to the periods further. Let \( g(z) \) be defined as in equation (17) and \( \vartheta = \lambda \frac{d}{dz} \). Then it can be shown [2] equation (6.1) that
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\[ N(X_\psi) = (p-1)g_0(\lambda^p) + \left( \frac{p}{1-p} \right)^{(p-1)}(\partial g_1)(\lambda^p) \]
\[ + \frac{1}{2!} \left( \frac{p}{1-p} \right)^{(p-1)}(\partial^2 g_2)(\lambda^p) \]
\[ + \frac{1}{3!} \left( \frac{p}{1-p} \right)^{(p-1)}(\partial^3 g_3)(\lambda^p) \]
\[ + \frac{1}{4!} \left( \frac{p}{1-p} \right)^{(p-1)}(\partial^4 g_4)(\lambda^p) \mod p^5. \]  

(24)

The last term makes an appearance in a period-like integral called a semi-period in [2]. In particular, analogously to equation (18), let

\[ \varpi_4(z) = \sum_{j=0}^{4} \binom{4}{j} g_j(z) (\log z)^{4-j}. \]

This expression can also be given as \( \gamma' \), but where \( \partial \gamma' \neq 0 \). While this calculation directly demonstrates that periods calculate the number of points, it does not explain the reason for this phenomenon. More conceptual approaches are considered in Section 4.

An alternative point of view can be given by realizing period and semi-period integrals in a different form. Let \( Q_\psi := \sum_{i=1}^{5} x_i^5 - 5 \psi \prod_{j=1}^{5} x_j \). Using the calculation in Example 1 we can write the fundamental period as

\[ \varpi_0 = \frac{5 \psi}{(2\pi i)^4} \int_{\gamma_2 \times \ldots \times \gamma_5} x_1 dx_2 \ldots dx_5 \frac{1}{Q_\psi(x)}, \]

where \( x_1 \) is kept constant and \( \gamma_i \) is a circle around the origin as in the example. Using the integral

\[ \frac{1}{Q_\psi} = \int_0^\infty e^{-tQ_\psi} dt \]

we can rewrite this period as

\[ \varpi_0 = \frac{5 \psi}{(2\pi i)^4} \int_{\gamma_2 \times \ldots \times \gamma_5} x_1 e^{-tQ_\psi(x)} dx_2 \ldots dx_5 dt \]
\[ = \frac{5 \psi}{(2\pi i)^4} \int_{\tilde{\gamma}_2 \times \ldots \times \tilde{\gamma}_5} e^{-tQ_\psi(x)} dx_1 \ldots dx_5, \]

where the last step follows from re-absorbing the parameter \( t \) into the variables via \( x_j \mapsto s^{-\frac{1}{5}} x_j \), and where \( \tilde{\gamma}_i \) is the corresponding change in the domain of integration. This point of view gives a more algebraic description of the periods which we now describe in rough terms. We can consider periods (and semi-periods) of some hypersurface \( \{ Q(x_1, \ldots, x_n) = 0 \} \) as integrals of the form
\[ \int_{\Gamma} x^v e^{-Q} \, dx_1 \ldots dx_n, \]  

(25)

where \( x^v \) is a monomial in \( \mathbb{C}[x_1, \ldots, x_n] \), and \( \Gamma \) is a cycle on \( \mathbb{C}^n \) such that \( e^{-Q} \) goes to zero sufficiently quickly. Such integrals have a cohomological analog. In particular, we can think of the integral (25) as the element \( x^v \) in the module \( M = \mathbb{C}[x] e^{-Q} \) of the Weyl algebra \( A_n \) generated by \( x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \) modulo the relations

\[ [\partial_i, \partial_j] = 0, \quad [x_i, x_j] = 0, \quad \text{and} \quad [\partial_i, x_j] = \delta_{i,j}, \]

where the action of \( \partial_i \) is taken to be \( \frac{\partial}{\partial x_i} - \frac{\partial Q}{\partial x_i} \) (capturing the chain rule). Assuming that \( Q \) is non-singular, it can be shown that the algebraic de Rham cohomology of the module \( M \) is given by

\[ H(M) = \frac{M}{\partial_1 M + \ldots + \partial_n M} \, dx_1 \ldots dx_n, \]

and that integrals of the form (25) are independent of the choice of representative of the cohomology class of \( x^v \). The module \( M \) can be similarly defined over \( \mathbb{C}_p \), in which case there exists an action of Frobenius on cohomology that can be used to study arithmetic of the periods. For a more formal explanation and further details, we refer the reader to [23, 24].

**Remark 7.** While the results for the quintic were given in the case \( 5 \nmid (p - 1) \), in [2] the case \( 5 \mid (p - 1) \) in which additional technical difficulties arise is also covered.

**Remark 8.** Since the calculation for the number of points is determined by character formulas which are valid for any finite group, we can calculate the number of points on (1) with coordinates in field extensions \( k_r = \mathbb{F}_{p^r} \) of \( k = \mathbb{F}_p \) by producing multiplicative and additive characters from these fields. If \( q = p^r \), then the Teichmüller character \( T: k_r \to \mathbb{Q}_p \) is given by the expression

\[ T(x) = \lim_{n \to \infty} x^{q^n}, \]

while an additive character \( \theta: k_r \to \mathbb{Q}_p \) is given by composing \( \theta \) with the (additive) trace map \( \text{Tr}: k_r \to k \) given by \( \text{Tr}(x) = x + x^p + \ldots + x^{p^{r-1}} \).

### 4 Zeta Functions and Mirror Symmetry

Let \( k = \mathbb{F}_q \) be a finite field with \( q = p^k \) elements, \( k_r \) an extension of degree \( r \), and \( X \) a smooth variety set of dimension \( d \) over \( k \). We will usually take \( k = \mathbb{F}_p \). If we let \( N_r(X) \) denote the number of points of \( \bar{X} := X \times \bar{k} \) rational over \( k_r \), then the generating function

\[ Z(X,T) = \exp \left( \sum_{r=1}^{\infty} N_r(X) \frac{T^r}{r} \right) \]
is called the zeta function of X. From the Weil Conjecture (since proved; see [17] for a broader overview) we know that

\[ Z(X, T) = \frac{P_0(T)P_2(T)\ldots P_{2d-1}(T)}{P_0(T)P_2(T)\ldots P_{2d}(T)}, \]

where \( P_0(T) = 1 - T, P_{2d}(T) = 1 - q^d T \) and \( P_i(T) \in 1 + T\mathbb{Z}[T] \) for each \( 1 \leq i \leq 2d - 1 \). Furthermore, the degree of \( P_i \) equals the \( i \)-th Betti number \( b_i \) of \( X \) and

\[ P_i(T) = \det (I - T\text{Frob}^* | H^i(X)) = \prod_{j=1}^{b_i} (1 - \alpha_{ij} T), \]

where \( \alpha_{ij} \) are algebraic integers such that \( |\alpha_{ij}| = q^{i/2} \), \( H^i(X) \) is a suitable cohomology theory, for instance Étale cohomology, and \( \text{Frob}^* \) is the map on cohomology induced from the Frobenius morphism \( \text{Frob} : X \to \overline{X} \) given by \( (x_i) \mapsto (x_i^q) \).

What can be said about the relationship between zeta functions of a pair of mirror quintic threefolds \( M \) and \( W \) belonging, respectively, to the quintic family \( \mathcal{M} \) defined by (1), and its mirror \( \mathcal{W} \) outlined in Example 2? The Weil Conjecture and the Hodge diamond of \( M \) imply that if \( Z(M, T) = N(T)/D(T) \), then we have \( \deg N(T) = 2h^{2,1}(M) + 2 = 204 \) and \( \deg D(T) = 2h^{1,1}(M) + 2 = 4 \). Since \( h^{2,1} \) and \( h^{1,1} \) are exchanged under mirror symmetry, we might hope that there is some kind of zeta function, which Candelas et al. call the “quantum” zeta function in [3], that satisfies \( Z(M, T) = Z(W, T)^{-1} \). This zeta function cannot be the usual zeta function, since that would imply the impossible relation \( N_r(M) = -N_r(W) \). However, numerical calculations by Candelas, de la Ossa, and Rodriguez Villegas in [3] show that if \( |p - 1| \), then there is a relation

\[ Z(M, T) \equiv \frac{1}{Z(W, T)} \equiv (1 - pT)^{100}(1 - p^2T)^{100} \mod 5^2. \]

This congruence can be seen as coming from the fact that the zeta functions of \( M \) and \( W \) share certain terms. In particular,

\[ Z(M, T) = \frac{R_e(T, \psi) \prod_j R_j(T, \psi)}{(1 - T)(1 - pT)(1 - p^2T)(1 - p^3T)(1 - p^4T)}, \]

and

\[ Z(W, T) = \frac{R_e(T, \psi)}{(1 - T)(1 - pT)^{101}(1 - p^2T)^{101}(1 - p^3T)^{101}(1 - p^4T)}, \]

where \( R_e(T, \psi) \) is of degree 4, and each \( R_j(T, \psi) \) comes from a period of \( M_\psi \) as described below. The relationship between the zeta function of a family of manifolds and the solutions of the Picard-Fuchs equation was first observed in greater generality by Katz in [13]. However, because of its computational nature, we will first illustrate the numerical calculation of [3], and then proceed to outline more conceptual explanations due to Kadir and Yui [27], Kloosterman [19], and Goutet [21].
4.1 Computational Observations

We have seen in Section 2.2 that periods of a hypersurface $X \subset \mathbb{P}^n$ given by $\{Q(x) = 0\}$ are determined by monomials $x^v = x_1^{v_1} \cdots x_n^{v_n}$ modulo those in the Jacobian ideal $J(Q)$ of $Q$, since we can write every period in terms of $\varpi_v$:

$$\varpi_v := \int_{\Gamma} \frac{x^v \Omega}{Q^{k(v)+1}},$$

(28)

where $\Gamma$ is a cycle on $\mathbb{P} \setminus V$ described earlier, and $k(v)$ is determined by $k(v) \deg Q = (v_1 + \cdots + v_5) + (n + 1)$. Using this description and Griffiths’s formula (6) we can find relations amongst the periods, and in fact also Picard-Fuchs equations, in a diagrammatic way. In the case of the quintic $X_\psi$ given in (1), choose some $i \in \{1, \ldots, n\}$ and set $A_i = \frac{1}{5} x^5 \psi$ as well as $A_j = 0$ for $j \neq i$, where $e_i$ is the standard basis vector of $\mathbb{Z}^n$ with 1 in the $i$-th slot and zeros elsewhere. If $\epsilon = (1, \ldots, 1)$, then Griffiths’s formula gives

$$x^v (x^5 \epsilon_i - \psi x^5) \Omega = \frac{1}{5k(v)} (v_1 + 1) x^v \Omega$$

up to an exact form. If we use the shorthand $v = (v_1, v_2, v_3, v_4, v_5)$ for the period (28) determined by the monomial $x^v$, then integrating this expression is a relation between the three periods $v$, $v + 5e_i$ and $v + \epsilon$ which we can encode in the diagram

$$v \rightarrow v + \epsilon \quad \downarrow D_i \quad v + 5e_i$$

where $D_i = \frac{\partial}{\partial x_i} \circ x_i$ denotes the operator which gave rise the this relation. To get a differential equation with respect to $\psi$ out of such relations we can use

$$\frac{d}{d\psi} \frac{x^v \Omega}{Q^{k(v)+1}} = \frac{-5k(v) x^{v+\epsilon} \Omega}{Q^{k(v)+1}},$$

(29)

which allows us to exchange $v + \epsilon$ for a derivative of $v$.

Example 7. Simply because the diagrams are more manageable, we will illustrate this method on the Fermat family of elliptic curves. Following [2], we will also change the form of the period integral encoded by the vector $v$ to

$$v = \frac{1}{(2\pi i)^3} \int F^k(v+1) \, dx_1 \, dx_2 \, dx_3,$$

where $Z_\psi$: $\{F_\psi = 0\}$ defines an element of the family, and $\Gamma$ is now a product of tubes around the loci $\partial Q/\partial x_i = 0$. Define $E := x^E = x_1 x_2 x_3$ and for $n \geq 1$ let

$$E_n := \frac{E_{n-1}}{Q^n} \quad \text{and} \quad I_n := \frac{1}{(2\pi i)^3} \int_{\Gamma} E_n \, dx_1 \, dx_2 \, dx_3.$$
Applying the procedure above to \((0,0,0), (1,1,1), \) and \((2,2,2)\) we have the diagram

\[
\begin{align*}
(2, 2, -1) & \rightarrow (3, 3, 0) \\
\downarrow D_3 & \\
(0, 0, 0) & \rightarrow (1, 1, 1) \rightarrow (2, 2, 2) \\
\downarrow D_1 & \\
(3, 0, 0) & \rightarrow (4, 1, 1) \\
\downarrow D_2 & \\
(3, 3, 0) &
\end{align*}
\]

in which the upper right dependence corresponds to

\[
D_3 \left( \frac{x_1^2 x_2^2}{x_3 Q^2} \right) = \frac{\partial}{\partial x_3} \left( \frac{x_1^2 x_2^2}{Q^2} \right) = \frac{6\psi x_1^3 x_2^3}{Q^3} - \frac{6E^2}{Q^3}. \tag{30}
\]

The relations coming from differentiation are

\[
I_{n+1} = \frac{1}{3n} \frac{d}{d\psi} I_n \quad \text{and} \quad I_{n+1} = \frac{1}{3n!} \frac{d^n}{d\psi^n} I_1, \tag{31}
\]

which we use in the dependence diagram above. We start computing the actual relations starting from the bottom of the diagram, replacing terms until we have a relation between only the periods corresponding to \((0,0,0), (1,1,1), \) and \((2,2,2)\).

We get rid of the \((3,3,0)\) period because it “loops around” the diagram by equation (30). The end result is

\[
E_1 + 3\psi E_2 + 6 \left( \frac{\psi^2 - 1}{\psi} \right) E_3 = \frac{\partial}{\partial x_1} \left( \frac{x_1^2 x_2 x_3}{Q^2} + \frac{x_1}{Q} \right) + \frac{\partial}{\partial x_2} \left( \frac{x_2^2 x_3}{Q^2} \right) + \frac{\partial}{\partial x_3} \left( \frac{x_1^2 x_2^2}{\psi Q^2} \right),
\]

which results in the Picard-Fuchs equation

\[
\left[ 3 + 3\psi \frac{\partial}{\partial \psi} + \left( \frac{\psi^2 - 1}{\psi} \right) \frac{\partial^2}{\partial \psi^2} \right] f(\psi) = 0
\]

satisfied by the period

\[
I_1 = \int f \frac{dx_1 dx_2 dx_3}{F_{\psi}}.
\]

Candelas et al. use such diagrams to find the Picard-Fuchs equations for all 204 periods of the quintic family \((1)\). Note that \(x^\epsilon\) is invariant under the diagonal symmetry group \(G\) of \(X_\psi\) defined in Example 2. By equation (29) this means that the periods \(\varpi_\psi\) and \(\varpi_{\psi+\epsilon}\) correspond to the same representation of the group \(G\). Moreover, the periods can be classified according to the transformation of \(x^\epsilon\) under the group into the sets
{1, x^6, x^{2e}, x^{3c}}, \{x_2^4, x_3^4, x_5^4\},
\{x_2^7, x_2^2, x_3^2, x_5^2\},
\{x_2 x_3 x_5, x_2^2 x_3, x_5^2, x_5^2\}.

Given up to permutation of the variables. In [2], the diagrammatic method is applied to each group of monomials. Choose a representative v of each one, and denote the corresponding Picard-Fuchs equation, which turns out to be hypergeometric in each case, by L^v. In the p-adic setting the hypergeometric expressions for these allow, by comparison of coefficients, to rewrite the number of points on the quintic (1) in terms of all the periods as

$$N(X_\psi) = p^d + \sum_v \chi_v \sum_{m=0}^{p-2} \beta_{v,m} \text{Teich}^m(\lambda),$$

where the outer sum is over representative monomials in the sets above, \(\chi_v\) accounts for the number of permutations in each group, and \(\beta_{v,m}\) is a ratio of Gauss sums

$$\beta_{v,m} = p^d \frac{G_{5m}}{\prod_{i=1}^{5} G_{m+kv_i}}.$$  

A consequence is an expression for the number of points with coordinates in \(k\) that decomposes as

$$N_r(X_\psi) = N_{e,r}(X_\psi) + \sum_v N_{v,r}(X_\psi),$$

so that \(R_v(T, \psi)\) arises as \(\sum_{r>0} N_{e,r}(X_\psi)^r\). At the \(\psi = 0\) (or Fermat) point of the moduli space, equation (32) can equivalently be given in terms of Fermat motives. This is a consequence of the Kadir-Yui monomial-motive correspondence, which is a one-to-one correspondence between the monomial classes given above and explicitly realized Fermat motives. For more details and applications to mirror symmetry, see [27]. The zeta function (27) can also be found by considering monomial classes, by understanding the mirror \(W\) torically and using Cox variables instead of \(x^v\). For more details, we refer the reader to [3].

How can we interpret \(\prod_v R_v(T, \psi)\) appearing in (26)? Candelas et al. numerically observed for small primes, and conjectured for all primes, that this product can be written as

$$R_A(qT, \psi)^{10} R_B(qT, \psi)^{15},$$

where \(R_A(T, \psi)\) and \(R_B(T, \psi)\) arise as numerators of the zeta functions of affine curves

$$A: y^5 = x^2(1-x)^3(x-\psi^5)^2 \quad \text{and} \quad B: y^5 = x^2(1-x)^4(x-\psi^5),$$

respectively. This claim was proven by Goutet in [10] via Gauss sum techniques. In fact, he has proven similar results more generally. An immediate generalization of (1) is the Dwork family of hypersurfaces

$$X_\psi: \{x_1^n + \ldots + x_n^n - n\psi x_1 \ldots x_n = 0\} \subseteq \mathbb{P}^{n-1}_k,$$  

(33)
where \( \psi \in k \) and we only consider nonsingular \( X_\psi \). Arithmetic of this family and its mirror was considered by Wan in \[26\] and Haessig in \[16\]. The mirror family is constructed in two stages, analogously to the quintic case. First we form the quotient

\[ Y_\psi := X_\psi / G \]

where \( G = \{ (\xi_1, \ldots, \xi_n) | \xi_i \in k, \xi_1^n = 1, \xi_1 \ldots \xi_n = 1 \} \) is the group of diagonal symmetries of \( X_\psi \). Wan calls \( Y_\psi \) the singular mirror of \( X_\psi \).

It can be explicitly realized as the projective closure of the affine hypersurface

\[ g(x_1, \ldots, x_{n-1}) = x_1 + \ldots + x_{n-1} + \frac{1}{x_1 \ldots x_{n-1}} - n \psi = 0, \]

in the torus \((k^*)^{n-1}\), which enables the use of Gauss sums to count points. The mirror family \( \{ W_\psi \} \) is obtained by resolving the singularities of \( \{ Y_\psi \} \). Picking a manifold from each family will produce a mirror pair, and if the two parameter values \( \psi \) are equal, then \( \{ X_\psi, W_\psi \} \) is called a strong mirror pair.

Now, reciprocal zeros \( \beta_i \) and poles \( \gamma_j \) of the zeta function

\[ Z(X, T) = \prod_{i} (1 - \beta_i T) / \prod_{j} (1 - \gamma_j T) \]

of some smooth variety \( X \) determine the number of points over various extensions of \( k \), since we have that

\[ \sum_{r=1}^{\infty} N_r(X) T^r = t \frac{d \log(\zeta T)}{dT} = \sum_j \frac{\gamma_j T}{1 - \gamma_j T} - \sum_i \frac{\beta_i T}{1 - \beta_i T}, \]

which implies

\[ N_r(X) = \sum_j \gamma_j^r - \sum_i \beta_i^r. \]

Furthermore, if we define the slope of \( \alpha \in \mathbb{Q} \) as

\[ s(\alpha) = \text{ord}_p(\alpha) \]

where \( \text{ord}_p \) denotes the \( p \)-adic order of \( \alpha \), then \( \beta_i, \gamma_j \) as defined above satisfy

\[ 0 \leq s(\beta_i), s(\gamma_j) \leq 2d \]

and are rational numbers in the range \([0, \text{dim} X]\). We now select a part of the zeta function of \( X \)

\[ Z_{(0,1)}(X, t) = \prod_{\alpha_i \in (\beta_i, \gamma_j), 0 \leq s(\alpha_i) < 1} (1 - \alpha_t \pm 1). \]

A character formula calculation gives the following theorem, which is the main result of \[26\].

**Theorem 1.** For a strong mirror pair \( (X_\psi, W_\psi) \) and \( r \in \mathbb{Z}_{>0} \) we have

\[ N_r(X_\psi) \equiv N_r(Y_\psi) \equiv N_r(W_\psi) \mod q^r, \]

or equivalently

\[ Z_{(0,1)}(X_\psi, T) = Z_{(0,1)}(Y_\psi, T) = Z_{(0,1)}(W_\psi, T). \]
In fact, if \( q \equiv 1 \mod n \) or if \( n \) is prime, it is shown in [26] and [16], respectively, that
\[
Z(X_\psi, T) = \frac{(Q(T, \psi)R(q^s T^s))^{(-1)^{n-1}}}{(1-T)(1-qT)\ldots(1-q^{n-2}T)}
\]
where \( s \) is the order of \( q \) in \((\mathbb{Z}/n\mathbb{Z})^n\), as well as
\[
Z(Y_\psi, T) = \frac{Q(T, \psi)^{(-1)^{n-1}}}{(1-T)(1-qT)\ldots(1-q^{n-2}T)}.
\]

A natural question to ask is whether, analogously to the quintic case, the polynomial \( R(T, \psi) \) can be shown to contain terms appearing in zeta functions of other varieties. As remarked by Wan in [26], in addition to the \( n=5 \) case, this question was answered affirmatively in the cases \( n=3 \) and \( n=4 \) by Dwork. Relying on a result of Haessig [16] and Gauss sum calculations, Goutet [9] has found explicit varieties whose zeta functions have terms appearing in \( R(T, \psi) \). Specifically, if we define \( N_R(q^r) \) by
\[
R(T, \psi) = \exp \left( \sum_{r > 0} N_R(q^r) T^r \right)
\]
Goutet proves the following.

**Theorem 2.** Let \( n \geq 5 \) be a prime congruent to 1 modulo \( n \). Then,
\[
N_R(q^r) = q^{\frac{n-5}{2}} N_1(q^r) + q^{\frac{n-3}{2}} N_3(q^r) + \ldots + N_{n-4}(q^r),
\]
where each \( N_i(q^r) \) is equal to the sum of counts of points of certain varieties of hypergeometric type.

### 4.2 Cohomological Interpretation

While Gauss sum techniques allow us to test and prove conjectures about zeta functions of mirror manifolds, they do not provide a conceptual understanding of what is happening. We have already mentioned the Kadir-Yui monomial-motive correspondence [27] which begins to provide a theoretical explanation. Kloosterman [19] extends this result to a neighborhood of the Fermat fiber, but also considers more general families. In particular, let \( k = \mathbb{F}_q \) be a finite field and consider the family consisting of hypersurfaces
\[
X_\lambda : \left\{ F_\lambda = \sum_{i=0}^n x_i^{d_i} + \lambda \prod_i x_i^{a_i} = 0 \right\}
\]
in weighted projective space \( \mathbb{P}(w) := \mathbb{P}_k(w_0, \ldots, w_n) \), where \( w_i a_i = d_i \), \( a_i \geq 0 \), \( \gcd(q,d) = 1 \), and \( \sum w_i a_i = d \). We will also only work with nonsingular fibers in what follows. If \( U_\lambda = \mathbb{P}(w) \setminus X_\lambda \) denotes the complement of a generic member of this family, then
\[
Z(X_\lambda, T)Z(U_\lambda, T) = Z(\mathbb{P}(w), T),
\]
and we can work with $U_{\lambda}$ instead of $X_{\lambda}$ for the purposes of determining the zeta function of $X_{\lambda}$. One reason for doing so is that there is a $p$-adic cohomology theory resembling de Rham cohomology called Monsky-Washnitzer cohomology that is well understood on hypersurface complements. To work with Monsky-Washnitzer cohomology, we need to lift $X_{\lambda}$ to a $p$-adic context. Let $\lambda$ be the Teichmüller lift of $\overline{\lambda}$ to the fraction field $\mathbb{Q}_q$ of the ring of Witt vectors over $k = \mathbb{F}_q$ (which equals $\mathbb{Z}_p$ if $q = p$). We can then consider $F\lambda$ to have coefficients in $\mathbb{Q}_q$, and work with $X_{\lambda}$ and $U_{\lambda}$ defined in the obvious way over $\mathbb{Q}_q$. Cohomology classes of Monsky-Washnitzer cohomology $H^{\text{MW}}_{\text{et}}(U_{\lambda}, \mathbb{Q}_q)$ are given by differential forms with $\mathbb{Q}_q$ coefficients, and these groups possess an action of Frobenius. It turns out that cohomology is zero except in degree $n$ and degree 0, where it is one-dimensional with trivial action of Frobenius. From this it can be shown that

$$Z(U_{\lambda}, T) = \frac{(\det (I - q^n(Frob_{\lambda})^{-1}T | H^{\text{MW}}_{\text{et}}(U_{\lambda}, \mathbb{Q}_q)))^{(-1)^{n+1}}}{(1 - q^nT)}.$$ 

By a result of Katz [18], $(Frob_{\lambda})^{-1}$ can be given by $A(\lambda)^{-1}Frob_{\lambda, 0}A(\lambda)^{q}$ extended via $p$-adic analytic continuation to a small disc around $\lambda = 0$, where $Frob_{\lambda, 0}$ is the action of Frobenius on the $\lambda = 0$ fiber, and $A(\lambda)$ is a solution of the Picard-Fuchs equation associated with the family $X_{\lambda}$. Therefore, to determine the zeta function of $X_{\lambda}$, we need to understand the action of Frobenius on the Fermat fiber, and to compute the Picard-Fuchs equation of the deformed family. Finding the latter and showing it is hypergeometric is one of the main results of [19]. Additionally, Kloosterman shows that there is a factorization of the zeta function along the lines of the Kadir-Yui monomial-motive correspondence [27]. These ideas were also exploited to calculate zeta functions of certain K3 surfaces in [8].

An alternative theoretical approach, in terms of Étale cohomology, is given by Goutet in [11]. For a nonsingular element $\bar{X}_{\psi} = X_{\psi} \times \bar{k}$ of this family considered over $\bar{k}$, it can be shown that $H^{\text{et}}_{\text{et}}(\bar{X}_{\psi}, \mathbb{Q}_\ell)$ is zero for $i > 2n - 4$ and $i < 0$, as well as for odd $i \neq n - 2$. For the remaining even $i \neq n - 2$ these groups are 1-dimensional. The most interesting part of cohomology is thus the primitive part of $H^{n-2}_{\text{et}}(\bar{X}_{\psi}, \mathbb{Q}_\ell)$, since it can be shown that the action of Frobenius is multiplication by $q^{(n-2)/2}$ on the non-primitive part of $H^{n-2}_{\text{et}}(\bar{X}_{\psi}, \mathbb{Q}_\ell)$, and multiplication by $q^n$ on each $H^{n}_{\text{et}}(\bar{X}_{\psi}, \mathbb{Q}_\ell)$. It follows that

$$Z(X_{\psi}, T) = \frac{(\det (I - T Frob^* | H^{n-2}_{\text{et}}(\bar{X}_{\psi}, \mathbb{Q}_\ell)_{\text{prim}}))^{(-1)^{n+1}}}{(1 - T)(1 - qT) \ldots (1 - q^{n-2}T)}.$$ 

Goutet shows that $H^{n}_{\text{et}}(\bar{X}_{\psi}, \mathbb{Q}_\ell)_{\text{prim}}$ decomposes into a direct sum of linear subspaces which correspond to equivalence classes of irreducible representations of the group of automorphisms of $X_{\psi}$ acting on cohomology. Frobenius stabilizes each of these subspaces, and the zeta function inherits a factor from each summand. The resulting factorization is finer than the one given in [19], and Goutet relates this factorization to the one resulting from Theorem 2 in a recent preprint [12]. An interesting question...
is whether these factors can be explained geometrically in the context of mirror symmetry.

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