Entropy numbers and Marcinkiewicz-type discretization theorem *

F. Dai, A. Prymak, A. Shadrin, V. Temlyakov, and S. Tikhonov

Abstract

This paper studies the behavior of the entropy numbers of classes of functions with bounded integral norms from a given finite dimensional linear subspace. Upper bounds of these entropy numbers in the uniform norm are obtained and applied to establish a Marcinkiewicz type discretization theorem for integral norms of functions from a given finite dimensional subspace.

1 Introduction

We start with some necessary notations and definitions. Let $X$ be a Banach space and let $B_X(g, r)$ denote the closed ball $\{f \in X : \|f - g\| \leq r\}$ with center $g \in X$ and radius $r > 0$. For a compact set $A$ in $X$ and a positive number $\varepsilon$, the covering number $N_\varepsilon(A, X)$ is defined as

\[ N_\varepsilon(A, X) := \min \{ n \in \mathbb{N} : \exists g^1, \ldots, g^n \in X, \ A \subset \bigcup_{j=1}^n B_X(g^j, \varepsilon) \}. \]

*The first named author’s research was partially supported by NSERC of Canada Discovery Grant RGPIN 04702-15. The second named author’s research was partially supported by NSERC of Canada Discovery Grant RGPIN 04863-15. The third named author’s research was supported by the Russian Federation Government Grant No. 14.W03.31.0031. The fifth named author’s research was partially supported by MTM 2017-87409-P, 2017 SGR 358, and the CERCA Programme of the Generalitat de Catalunya.
The $\varepsilon$-entropy $H_\varepsilon(A;X)$ of the compact set $A$ in $X$ is defined as $\log_2 N_\varepsilon(A,X)$, and the entropy numbers $\varepsilon_k(A,X)$ of the set $A$ in $X$ are defined as

$$\varepsilon_k(A,X) := \inf\{\varepsilon > 0 : H_\varepsilon(A;X) \leq k\}, \quad k = 1, 2, \ldots$$

Note that in our definition here we do not require $y^j \in A$, whereas in the definitions of $N_\varepsilon(A,X)$ and $\varepsilon_k(A,X)$ in [6], this requirement is imposed. However, it is well known (see [18, p.208]) that these characteristics may differ at most by a factor 2.

Next, let $\Omega$ be a nonempty set equipped with a probability measure $\mu$. For $1 \leq p < \infty$, let $L_p(\Omega)$ denote the real Lebesgue space $L_p$ defined with respect to the measure $\mu$ on $\Omega$, and $\| \cdot \|_p$ the norm of $L_p(\Omega)$. Let $X_N$ be an $N$-dimensional linear subspace of $L_\infty(\Omega)$ and set

$$X_N^p := \{f \in X_N : \|f\|_p \leq 1\}, \quad 1 \leq p < \infty.$$ 

Here and throughout the paper, the index $N$ always stands for the dimension of $X_N$, and we assume that each function $f \in X_N$ is defined everywhere on $\Omega$.

By discretization of the $L_p$ norm we understand a replacement of the measure $\mu$ by a discrete measure $\mu_m$ with support on a set $\xi = \{\xi^\nu\}_{\nu=1}^{m} \subset \Omega$. This means that integration with respect to the measure $\mu$ is replaced by evaluation of an appropriate weighted sum of values of a function $f$ at a finite set of points. This is why we call this way of discretization sampling discretization. Discretization is a very important step in making a continuous problem computationally feasible. An important example of a classical discretization problem is the problem of metric entropy (covering numbers, entropy numbers); see [10, Ch.15], [18, Ch.3], [23, Ch.7], [4], [16] and the recent papers [19], [8]. Another prominent example of discretization is the problem of numerical integration, which requires many fundamental results for constructing optimal (in the sense of order) cubature formulas (see, e.g., [7, Ch.8]).

There are different ways to discretize: use coefficients from an expansion with respect to a basis, or more generally, use linear functionals. We discuss here the way which uses function values at a fixed finite set of points. Our main interest is the problem of discretization of the $L_p$ norms of functions from a given finite dimensional subspace. This problem arises in a very natural way in many applications. Indeed, a typical approach to solving a continuous problem numerically – the Galerkin method – suggests to look
for an approximate solution from a given finite dimensional subspace, while a standard way to measure an error of approximation is an appropriate discretization of an $L_p$ norm, $1 \leq p \leq \infty$. The first results in this direction were obtained by Marcinkiewicz and by Marcinkiewicz-Zygmund (see [24]) for discretization of the $L_p$ norms of the univariate trigonometric polynomials in 1930s. This is why discretization results of this kind are called the Marcinkiewicz-type theorems. We now proceed to the detailed presentation.

**Marcinkiewicz problem.** We say that a linear subspace $X_N$ of $L_p(\Omega)$, $1 \leq p < \infty$, admits the Marcinkiewicz-type discretization theorem with parameters $m \in \mathbb{N}$ and $p$ if there exist a set $\{\xi^\nu \in \Omega : \nu = 1, \ldots, m\}$ and two positive constants $C_j(d, p)$, $j = 1, 2$, such that for any $f \in X_N$ we have

$$C_1(d, p) \|f\|_p^p \leq \frac{1}{m} \sum_{\nu=1}^{m} |f(\xi^\nu)|^p \leq C_2(d, p) \|f\|_p^p. \quad (1.1)$$

In the case $p = \infty$ we ask for

$$C_1(d) \|f\|_\infty \leq \max_{1 \leq \nu \leq m} |f(\xi^\nu)| \leq \|f\|_\infty. \quad (1.2)$$

**Marcinkiewicz problem with weights.** We say that a linear subspace $X_N$ of the $L_p(\Omega)$, $1 \leq p < \infty$, admits the weighted Marcinkiewicz-type discretization theorem with parameters $m \in \mathbb{N}$ and $p$ if there exist a set of knots $\{\xi^\nu \in \Omega\}$, a set of weights $\{\lambda_\nu\}$, $\nu = 1, \ldots, m$, and two positive constants $C_j(d, p)$, $j = 1, 2$, such that for any $f \in X_N$ we have

$$C_1(d, p) \|f\|_p^p \leq \sum_{\nu=1}^{m} \lambda_\nu |f(\xi^\nu)|^q \leq C_2(d, p) \|f\|_p^p. \quad (1.3)$$

The most complete results on sampling discretization are obtained in the case $q = 2$. The problem is basically solved in the case of subspaces of trigonometric polynomials. By $Q$ we denote a finite subset of $\mathbb{Z}^d$, and $|Q|$ stands for the number of elements in $Q$. Let

$$T(Q) := \left\{ f : f = \sum_{k \in Q} c_k e^{i(k, x)}, \ c_k \in \mathbb{C} \right\}.$$

In [21] it was shown how to derive the following result from the recent paper by S. Nitzan, A. Olevskii, and A. Ulanovskii [12], which in turn is based on the paper of A. Marcus, D.A. Spielman, and N. Srivastava [11].
Theorem 1.1. [21] There are three positive absolute constants $C_1$, $C_2$, and $C_3$ with the following properties: For any $d \in \mathbb{N}$ and any $Q \subset \mathbb{Z}^d$ there exists a set of $m \leq C_1 |Q|$ points $\xi^j \in \mathbb{T}^d$, $j = 1, \ldots, m$ such that for any $f \in T(Q)$ we have

$$C_2 \|f\|_2^2 \leq \frac{1}{m} \sum_{j=1}^{m} |f(\xi^j)|^2 \leq C_3 \|f\|_2^2.$$

Some results are obtained under an extra condition on $X_\mathbb{N}$, which we will call Condition E for consistency with prior work (see, e.g., [5]).

Condition E. There exists a constant $K_1 > 0$ such that for all $x \in \Omega$

$$w(x) := \sum_{i=1}^{N} u_i(x)^2 \leq K_1 N.$$

The reader can find the following result, which is a slight generalization of the Rudelson’s [14] celebrated result, in [22].

Theorem 1.2. [22] Let $\{u_i\}_{i=1}^{N}$ be a real orthonormal system satisfying condition E. Then for every $\epsilon > 0$ there exists a set $\{\xi^j\}_{j=1}^{m} \subset \Omega$ with

$$m \leq C \frac{K_1}{\epsilon^2} N \log N$$

such that for any $f = \sum_{i=1}^{N} c_i u_i$ we have

$$(1 - \epsilon) \|f\|_2^2 \leq \frac{1}{m} \sum_{j=1}^{m} f(\xi^j)^2 \leq (1 + \epsilon) \|f\|_2^2.$$

For the general case of $1 \leq p < \infty$, it turns out that certain estimates of the entropy numbers $\epsilon_k(X_N^p, L_\infty)$ of the class $X_N^p$ in $L_\infty$-norm play a crucial role in the proof of an Marcinkiewicz-type discretization theorem for the $L_p$ norm of functions from the space $X_N \subset L_\infty(\Omega)$. This can be seen from the following conditional result, proved recently in [22] for $p = 1$, and in [6] for $1 < p < \infty$:

Theorem 1.3. [22, 6] Let $1 \leq p < \infty$. Suppose that $X_N$ is an $N$-dimensional subspace of $L_\infty(\Omega)$ satisfying the condition

$$\epsilon_k(X_N^p, L_\infty) \leq B(N/k)^{1/p}, \quad 1 \leq k \leq N$$

(1.5)
with the constant $B$ satisfying that $B \geq 1$ and $\log_2(2B) \leq C_1(p)N$. Then for any $\epsilon \in (0,1)$, there exist a constant $C(p,\epsilon)$ depending only on $\epsilon$ and $p$ and a set of 

$$m \leq C(p,\epsilon)NB^p(\log_2(2N))^2$$

(1.6)

points $\xi^j \in \Omega$, $j = 1, \ldots, m$, such that for any $f \in X_N$ we have 

$$(1 - \epsilon)\|f\|_p^p \leq \frac{1}{m} \sum_{j=1}^{m} |f(\xi^j)|^p \leq (1 + \epsilon)\|f\|_p^p.$$ 

(1.7)

Theorem [1.3] was proved in [22, 6] for the case $\epsilon = \frac{1}{2}$ only, but the proof there with slight modifications works equally well for $\epsilon \in (0,1)$. For later applications, we also give the following remark here.

**Remark 1.1.** It is worthwhile to point out that the estimate (1.5) for $k = 1$ implies the following Nikol’skii type inequality for $X_N$, 

$$\|f\|_\infty \leq 4BN^{1/p}\|f\|_p$$ 

for any $f \in X_N$,

while the estimate (1.6) for $k = N$ implies

$$\varepsilon_k(X^p_N, L^\infty) \leq 6B2^{-k/N}$$ 

for $k > N$.

The proofs of these two facts can be found in [6].

Note that bounds for the entropy numbers of function classes are important by themselves and also have important connections to other fundamental problems (see, for instance, [18, Ch.3] and [7, Ch.6]). Furthermore, the study of the entropy numbers is a highly nontrivial and intrinsically interesting subject.

The aim of this paper is twofold. Firstly, we conduct a detailed study of the entropy numbers $\varepsilon_k(X^p_N, L^\infty)$ of $X^p_N$ in the $L^\infty$ norm for $1 \leq p \leq 2$. Secondly, we shall apply Theorem [1.3] and the obtained estimates of the entropy numbers to obtain a Marcinkiewicz-type discretization theorem for the $L_p$ norms of functions from the space $X_N$. In this paper, we will focus mainly on the case of $1 \leq p \leq 2$, where rather complete results can be obtained. It turns out that there are significant differences between the cases $1 \leq p \leq 2$ and $p > 2$. The main results of this paper will be summarized in the next section. We present a detailed discussion of these results in Section 6.

Throughout this paper, the letter $C$ denotes a general positive constant depending only on the parameters indicated as arguments or subscripts, and we will use the notation $|A|$ to denote the cardinality of a finite set $A$. 

5
2 Main results

In this section, we shall summarize our main results on the entropy numbers $\varepsilon_k(X^p_N, L_\infty)$ and the Marcinkiewicz discretization of $L_p$ norms of functions from $X_N$. As stated in the introduction, we only deal with the case $1 \leq p \leq 2$ in this paper.

Firstly, we prove the following estimates of the entropy numbers.

**Theorem 2.1.** Assume that $X_N$ is an $N$-dimensional subspace of $L_\infty(\Omega)$ satisfying the following two conditions:

(i) There exists a constant $K_1 > 1$ such that
\[
\|f\|_\infty \leq (K_1N)^{\frac{1}{2}}\|f\|_2, \quad \forall f \in X_N.
\] (2.1)

(ii) There exists a constant $K_2 > 1$ such that
\[
\|f\|_\infty \leq K_2\|f\|_{\log N}, \quad \forall f \in X_N.
\] (2.2)

Then for each $1 \leq p \leq 2$, there exists a constant $C_p > 0$ depending only on $p$ such that
\[
\varepsilon_k(X^p_N, L_\infty) \leq C_p(K_1K_2^2 \log N)^{\frac{1}{p}} \left\{ \begin{array}{ll}
\left(\frac{N}{k}\right)^{\frac{1}{p}}, & \text{if } 1 \leq k \leq N, \\
2^{-k/N}, & \text{if } k > N.
\end{array} \right.
\] (2.3)

The key ingredient in the proof of Theorem 2.1 is the following new inequality on $\varepsilon$-entropy, which appears to be of independent interest (see Lemma 3.3): for $1 \leq p < 2 < q \leq \infty$ and $\theta := (\frac{1}{2} - \frac{1}{q})/(\frac{1}{p} - \frac{1}{q})$, we have
\[
\mathcal{H}_\varepsilon(X^p_N; L_q) \leq \sum_{s=0}^{\infty} \mathcal{H}_{2^{-3a-1}\varepsilon^\theta}(X^{2^s}_N; L_q) + \mathcal{H}_{\varepsilon^\theta}(X^{2^s}_N; L_q), \quad \varepsilon > 0,
\] (2.4)

where $a = a(\theta) = 2^{1-\theta}$. Indeed, using inequality (2.4), and slightly modifying the proof of Theorem 2.1, we can deduce the following more general estimates under the conditions (2.1) and (2.2): for $1 \leq p \leq 2 < q \leq \infty$,
\[
\varepsilon_k(X^p_N; L_q) \leq C_{p,q}(K_1K_2^2 \log N)^{\frac{1}{p} - \frac{1}{q}} \left\{ \begin{array}{ll}
\left(\frac{N}{k}\right)^{\frac{1}{p} - \frac{1}{q}}, & \text{if } 1 \leq k \leq N; \\
2^{-k/N}, & \text{if } k \geq N.
\end{array} \right.
\] (2.5)
The detailed proof of (2.5) will be given in the last section, Section 6.

Secondly, in Section 4 we prove the following Marcinkiewicz discretization theorem for the \( L_p \) norms of functions from \( X_N \).

**Theorem 2.2.** Let \( X_N \) be an \( N \)-dimensional subspace of \( L_\infty(\Omega) \) satisfying the condition (2.1) with \( \log K_1 \leq \alpha \log N \) for some constant \( \alpha > 1 \). If \( 1 \leq p \leq 2 \) then for any \( \varepsilon \in (0, 1) \), there exists a set of \( m \leq C_p(\alpha, \varepsilon) K_1 N \log^3 N \) points \( \xi_1, \ldots, \xi_m \in \Omega \) such that

\[
(1 - \varepsilon) \|f\|_p^p \leq \frac{1}{m} \sum_{j=1}^m |f(\xi_j)|^p \leq (1 + \varepsilon) \|f\|_p^p, \quad \forall f \in X_N, \tag{2.6}
\]

where the constant \( C_p(\alpha, \varepsilon) \) depends only on \( p, \varepsilon \) and \( \alpha \).

Our proof of Theorem 2.2 relies on the estimates of the entropy numbers in Theorem 2.1 and the conditional theorem, Theorem 1.3. However, special efforts are also required as condition (2.2) is assumed in Theorem 2.1 but not assumed in Theorem 2.2.

Finally, we prove the following unconditional weighted discretization result for the \( L_p \) norms of functions from \( X_N \) in Section 5.

**Theorem 2.3.** Given \( 1 \leq p \leq 2 \), an arbitrary \( N \)-dimensional subspace \( X_N \) of \( L_p(\Omega) \) and any \( \varepsilon \in (0, 1) \), there exist \( \xi_1, \ldots, \xi_m \in \Omega \) and \( \lambda_1, \ldots, \lambda_m > 0 \) such that \( m \leq C_p(\varepsilon) N \log^3 N \) and

\[
(1 - \varepsilon) \|f\|_p \leq \left( \sum_{j=1}^m \lambda_j |f(\xi_j)|^p \right)^{\frac{1}{p}} \leq (1 + \varepsilon) \|f\|_p, \quad \forall f \in X_N. \tag{2.7}
\]

The proof of Theorem 2.3 is based on Theorem 2.2 and a change of density argument from functional analysis. While the weights \( \lambda_j \) in (2.7) are in general not equal, Theorem 2.3 is applicable to every \( N \)-dimensional subspace \( X_N \) of \( L_p \) without any additional assumptions.

We conclude this section with a few remarks on our results. Firstly, we point out that condition (2.1) is equivalent to Condition E in the introduction. This can be seen from the following well-known result.
Proposition 2.1. Let $X_N$ be an $N$-dimensional subspace of $L_\infty$. Then for any orthonormal basis $\{u_i\}_{i=1}^N$ of $X_N \subset L_2$ we have that for $x \in \Omega$

$$\sup_{f \in X_N; f \neq 0} |f(x)|/\|f\|_2 = w(x) := \left(\sum_{i=1}^N u_i(x)^2 \right)^{1/2}.$$  \hfill (2.8)

Clearly, (2.8) follows from

$$\sup_{f \in X_N; f \neq 0} |f(x)|/\|f\|_2 = \sup_{(c_1, \ldots, c_N) \in \mathbb{R}^N} \left| \sum_{i=1}^N c_i u_i(x) \right| = w(x)$$

for any $f(x) = \sum_{i=1}^N c_i(f) u_i(x)$, $c_i(f) \in \mathbb{R}$, $1 \leq i \leq N$.

Secondly, note that inequality (2.1) also implies the following Nikolskii inequalities for $X_N$:

$$\|f\|_q \leq (K_1 N)^{\frac{1}{p} - \frac{1}{q}} \|f\|_p, \quad \forall f \in X_N, \quad 1 \leq p \leq 2, \quad p < q \leq \infty.$$  \hfill (2.9)

Indeed, if $1 \leq p \leq 2$ and $q = \infty$, then using (2.1), we obtain that for any $f \in X_N$,

$$\|f\|_\infty \leq (K_1 N)^{\frac{1}{2}} \|f\|_2 \leq (K_1 N)^{\frac{1}{2}} \|f\|_p \|f\|_\infty^{1 - \frac{1}{p}}$$

which implies

$$\|f\|_\infty \leq (K_1 N)^{\frac{1}{2}} \|f\|_p.$$  \hfill (2.10)

If $1 \leq p \leq 2$, and $q > p$, then using (2.10), we have that

$$\|f\|_q \leq \|f\|_\infty^{1 - \frac{1}{q}} \|f\|_p^{\frac{1}{q}} \leq (K_1 N)^{\frac{1}{p} - \frac{1}{q}} \|f\|_p.$$

Finally, we point out that despite the fact that Theorem 2.2, the Marcin-kiewicz discretization theorem, holds without condition (2.2), the entropy number estimates (2.3) in Theorem 2.1 are no longer true if (2.2) is not assumed. This can be seen from the following example, which was kindly communicated to us by B. Kashin.

Example. For each $k \in \mathbb{N}$, let $r_k : [0, 1] \rightarrow \{1, -1\}$ denote the $k$-th Rademacher function defined by

$$r_k(t) := \text{sign}(\sin(2^{k+1} \pi t)),$$
where \( \text{sign}(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ -1, & \text{if } x < 0. \end{cases} \) Then \( \{r_k\}_{k=1}^\infty \) is an orthonormal system with respect to the Lebesgue measure \( d\mu(t) = dt \) on \( \Omega = [0,1] \). Let

\[
X_N := \text{span}\{r_j : 1 \leq j \leq N\}.
\]

For each \( f_a = \sum_{j=1}^N a_j r_j \in X_N \) with \( a := (a_1, \ldots, a_N) \in \mathbb{R}^N \), we have

\[
\max_{t \in [0,1]} |f_a(t)| = \| \sum_{j=1}^N a_j r_j \|_\infty = \sum_{j=1}^N |a_j|,
\]

(2.11)

where the last step uses the fact that

\[
\mu\{ t \in [0,1] : r_k(t) = \text{sign}(a_k) \text{ for all } k = 1, 2, \ldots, N \} = 2^{-N} > 0.
\]

On the one hand, since the orthonormal system \( \{r_k\}_{k=1}^\infty \) satisfies Condition E, the space \( X_N \) satisfies condition (2.1). On the other hand, however, (2.11) implies that the space \( X_N \) does not satisfy condition (2.2). Let us show that the estimates (2.3) do not hold for any \( 1 \leq p < 2 \). To this end, let \( \tilde{\ell}_q^N \) denote the space \( \mathbb{R}^N \) equipped with the norm

\[
\|x\|_q := \left( \sum_{j=1}^N |x_j|^q \right)^{1/q}, \quad x = (x_1, \ldots, x_N) \in \mathbb{R}^N.
\]

By monotonicity of the \( L_p \) norms, \( X_N^2 \subset X_N^p \) for each \( 1 \leq p \leq 2 \). Thus, using (2.11), we have that

\[
N_\varepsilon(X_N^p, L_\infty) \geq N_\varepsilon(X_N^2, L_\infty) = N_\varepsilon(B_2^N, \tilde{\ell}_1^N), \quad 1 \leq p < 2, \quad \varepsilon > 0,
\]

where \( B_2^N := \{ x \in \mathbb{R}^N : \|x\|_2 \leq 1 \} \). By the standard volume comparison argument, we obtain

\[
N_{2\varepsilon}(B_2^N, \tilde{\ell}_1^N) \geq \frac{\text{Vol}(B_2^N)}{2^N \varepsilon^N \text{Vol}(B_1^N)} = \frac{\pi^{N/2}}{\Gamma(\frac{N}{2})} \frac{N!}{4^N \varepsilon^N} = \frac{\pi^{N-1/2} \Gamma(\frac{N+1}{2})}{2^N \varepsilon^N}.
\]

This together with Stirling’s formula implies that for \( 1 \leq p < 2 \) and \( \varepsilon > 0 \),

\[
\mathcal{H}_\varepsilon(X_N^p; L_\infty) = \log_2 N_\varepsilon(X_N^p, L_\infty) \geq \log_2 \left[ \left( \frac{\sqrt{\pi}}{2\varepsilon} \right)^N \left( \frac{N}{2\varepsilon} \right)^{\frac{N}{2}} \right] - C
\]

\[
\geq N \log_2 \frac{\sqrt{N}}{\varepsilon} - CN,
\]

9
where $C > 0$ is an absolute constant. Thus, there exists an absolute constant $c_0 \in (0, 1)$ such that for any $1 \leq p < 2$ and $0 < \varepsilon < c_0 \sqrt{N}$,

$$\mathcal{H}_\varepsilon(X_N^p; L_\infty) \geq N,$$

which in turn implies that for $1 \leq k < N$,

$$\varepsilon_k(X_N^p, L_\infty) \geq c_0 \sqrt{N}. \quad (2.12)$$

This means that if $N^\alpha \leq k_N \leq N$ for some parameter $1 - \frac{p}{2} < \alpha < 1$, then

$$\liminf_{N \to \infty} \frac{\sqrt{N}}{\left(\frac{N \log N}{k_N}\right)^{\frac{1}{p}}} = \infty.$$

### 3 Proof of Theorem 2.1

This section is devoted to the proof of the estimates (2.3) of the entropy numbers $\varepsilon_k(X_N^p, L_\infty)$ for $1 \leq p \leq 2$. By the definition of the entropy numbers and Remark 1.1, it suffices to show that for $1 \leq p \leq 2$,

$$\mathcal{H}_\varepsilon(X_N^p; L_\infty) \leq C_p K_1 K_2^2 \frac{N \log N}{\varepsilon^p}, \quad \forall \varepsilon > 0, \quad (3.1)$$

where the constant $C$ depends only on $p$. We divide the proof of (3.1) into two different cases: $p = 2$ and $1 \leq p < 2$. The estimate (3.1) for $p = 2$ is essentially known (see, e.g., [2]), but for the sake of completeness, we will summarize its proof in Section 3.1. The proof of (3.1) for the remaining case $1 \leq p < 2$ will be given in Section 3.2.

#### 3.1 Case 1. $p = 2$.

Let $S^{N-1}$ denote the unit sphere of the Euclidean space $\mathbb{R}^N$ equipped with the surface Lebesgue measure $\sigma$ normalized by $\sigma(S^{N-1}) = 1$. Given an $N$-dimensional normed linear space $X = (\mathbb{R}^N, \| \cdot \|_X)$, let $B_X := \{ x \in X : \| x \|_X \leq 1 \}$ and define

$$M_X := \int_{S^{N-1}} \| x \|_X d\sigma(x).$$

We also denote by $X^*$ the dual $(\mathbb{R}^N, \| \cdot \|_{X^*})$ of $X = (\mathbb{R}^N, \| \cdot \|_N)$.

We need the following lemma, which can be found in Lemma 2.4 and Propositions 4.1 and 4.2 of [3].
Lemma 3.1. Let $X$ denote the space $\mathbb{R}^N$ endowed with some norm $\| \cdot \|_X$. Then the following statements hold:

(i) For $0 < \varepsilon \leq 1$,
$$N \log \frac{1}{\varepsilon} \leq H_\varepsilon(B_X, X) \leq N \log(1 + \frac{2}{\varepsilon}). \quad (3.2)$$

(ii) There exists a universal constant $C > 0$ such that
$$H_\varepsilon(B_X, \mathbb{R}^N) \leq CN(\frac{M_X}{\varepsilon})^2 \quad \text{and} \quad H_\varepsilon(B^N_2, X) \leq CN(\frac{M_X}{\varepsilon})^2,$$

where $B^N_2$ denotes the Euclidean unit ball of $\mathbb{R}^N$.

Clearly, for the proof of estimate (3.1) for $p = 2$, it is enough to show the following lemma.

Lemma 3.2. If $X_N$ satisfies condition (2.1), then for $2 \leq q \leq \infty$,
$$H_\varepsilon(X^2_N, L_q) \leq CK_1Nq\varepsilon^{-2}, \quad \varepsilon > 0. \quad (3.4)$$

If $X_N$ satisfies both conditions (2.1) and (2.2), then
$$H_\varepsilon(X^2_N, L_\infty) \leq CK_1K_2\varepsilon^{-2}N \log N, \quad \varepsilon > 0. \quad (3.5)$$

Proof. By the rotation invariance of the measure $d\sigma$ on $S^{N-1}$, we have that for any $x \in \mathbb{R}^N$ and $1 \leq q \leq \infty$,
$$\left( \int_{S^{N-1}} |x \cdot y|^q d\sigma(y) \right)^{\frac{1}{q}} = \|x\|_2 \left( \frac{2\Gamma(\frac{N}{2})}{\Gamma(\frac{N-1}{2})} \int_0^1 x^q(1 - x^2)^{\frac{N-3}{2}} dx \right)^{\frac{1}{q}}$$
$$= \left( \frac{\Gamma(\frac{N}{2})\Gamma(\frac{q+1}{2})}{\Gamma(\frac{q}{2})\Gamma(\frac{N+q}{2})} \right)^{\frac{1}{q}} \|x\|_2 \sim \frac{\sqrt{q}}{\sqrt{N+q}} \|x\|_2 \quad (3.6)$$
with absolute constants of equivalence. Using (3.6) and (2.1) (or the equivalent Condition E), we obtain that for an orthonormal basis $\{\varphi_j\}_{j=1}^N$ of $X_N \subset L_2(\Omega)$, and $2 \leq q \leq \infty$,
$$M_{q,X_N} := \int_{S^{N-1}} \left\| \sum_{j=1}^N \xi_j \varphi_j \right\|_{L_q(d\mu)} d\sigma_N(\xi)$$
$$\leq \left( \int_{S^{N-1}} \left\| \sum_{j=1}^N \xi_j \varphi_j \right\|^{q}_{L_q(d\mu)} d\sigma_N(\xi) \right)^{1/q} \leq C \sqrt{K_1q}. \quad (3.7)$$
It then follows by the second inequality in (3.3) that for $1 \leq q < \infty$ and $\varepsilon > 0$,
\[
\mathcal{H}_\varepsilon(X_N^2; L_q) \leq C N \left( \frac{M_{X,q}}{\varepsilon} \right)^2 \leq C K_1 q \varepsilon^{-2}.
\] (3.8)
This proves estimate (3.4). Finally, by (2.2), we have
\[
\mathcal{H}_\varepsilon(X_N^2; L_\infty) \leq \mathcal{H}_\varepsilon(X_N^2; L_{\log N}),
\]
which, using (3.4) with $q = \log N$, leads to the estimate (3.5). \hfill \Box

### 3.2 Case 2. $1 \leq p < 2$.

In this subsection, we shall prove (3.1) for $1 \leq p < 2$. Our proof relies on the following

**Lemma 3.3.** For $1 \leq p < 2 < q \leq \infty$ and $\theta := (\frac{1}{2} - \frac{1}{q})/(\frac{1}{p} - \frac{1}{q})$, we have
\[
\mathcal{H}_\varepsilon(X_N^p; L_q) \leq \sum_{s=0}^{\infty} \mathcal{H}_{2^{-3} a s^{-1} \varepsilon^\theta}(X_N^2; L_q) + \mathcal{H}_{\varepsilon^\theta}(X_N^2; L_q), \quad \varepsilon > 0,
\] (3.9)
where $a = a(\theta) = 2 \frac{\theta}{1 - \theta}$.

For the moment, we take Lemma 3.3 and proceed with the proof of (3.1). Using Lemma 3.3 with $q = \infty$ and $\theta = p/2$, we obtain
\[
\mathcal{H}_\varepsilon(X_N^p; L_\infty) \leq \sum_{s=0}^{\infty} \mathcal{H}_{2^{-3} a s^{-1} \varepsilon^{p/2}}(X_N^2; L_\infty) + \mathcal{H}_{\varepsilon^{p/2}}(X_N^2; L_\infty),
\]
which in light of (3.5) is bounded above by
\[
\leq C K_1 K_2^2 N \log N \varepsilon^{-p} \sum_{s=0}^{\infty} a^{-2s} \leq C \varepsilon^{-p} K_1 K_2^2 N \log N.
\]
This proves (3.1) for $1 \leq p < 2$.

It remains to prove Lemma 3.3.

**Proof of Lemma 3.3.** We use the inequality
\[
\mathcal{H}_\varepsilon(X_N^p; L_q) \leq \mathcal{H}_{\varepsilon^{1-\theta}}(X_N^p; L_2) + \mathcal{H}_{\varepsilon^\theta}(X_N^2; L_q).
\]
Thus, setting $\varepsilon_1 := \varepsilon^{1-\theta}$, we reduce to showing that
\[ H_{\varepsilon_1}(X^p_N; L_2) \leq \sum_{s=0}^{\infty} H_{2^{-3s+1}\varepsilon^\theta}(X^2_N; L_q). \tag{3.10} \]

It will be shown that for $s = 0, 1, \ldots$,
\[ H_{2s\varepsilon_1}(X^p_N; L_2) - H_{2^{s+1}\varepsilon_1}(X^p_N; L_2) \leq H_{2^{-3s+1}\varepsilon^\theta}(X^2_N; L_q), \tag{3.11} \]
from which (3.10) will follow by taking the sum over $s = 0, 1, \ldots$

To show (3.11), for each nonnegative integer $s$, let $F_s \subset X^p_N$ be a maximal $2^s\varepsilon_1$-separated subset of $X^p_N$ in the metric $L_2$; that is, $\|f - g\|_2 \geq 2^s\varepsilon_1$ for any two distinct functions $f, g \in F_s$, and $X^p_N \subset \bigcup_{f \in F_s} B_{L_2}(f, 2^s\varepsilon_1)$. Then
\[ H_{2s\varepsilon_1}(X^p_N; L_2) \leq \log_2 |F_s| \leq H_{2^{s+1}\varepsilon_1}(X^p_N; L_2). \tag{3.12} \]

Let $f_s \in F_{s+2}$ be such that
\[ |B_{L_2}(f_s, 2^{s+2}\varepsilon_1) \cap F_s| = \max_{f \in F_{s+2}} \left| B_{L_2}(f, 2^{s+2}\varepsilon_1) \cap F_s \right|. \]

Since
\[ F_s = \bigcup_{f \in F_{s+2}} \left( B_{L_2}(f, 2^{s+2}\varepsilon_1) \cap F_s \right) \subset X^p_N, \]
it follows that
\[ |F_s| \leq |F_{s+2}| \left| B_{L_2}(f_s, 2^{s+2}\varepsilon_1) \cap F_s \right|. \tag{3.13} \]

Set
\[ \mathcal{A}_s := \left\{ \frac{f - f_s}{2^{s+2}\varepsilon_1} : f \in B_{L_2}(f_s, 2^{s+2}\varepsilon_1) \cap F_s \right\}. \]

Clearly, for any $g \in \mathcal{A}_s$,
\[ \|g\|_2 \leq 1, \quad \|g\|_p \leq (2^{s+1}\varepsilon_1)^{-1}. \tag{3.14} \]

On the one hand, using (3.12) and (3.13) implies that
\[ \log_2 |\mathcal{A}_s| \geq \log_2 |F_s| - \log_2 |F_{s+2}| \geq H_{2^{s+1}\varepsilon_1}(X^p_N; L_2) - H_{2^{s+1}\varepsilon_1}(X^p_N; L_2). \tag{3.15} \]
On the other hand, since \( \frac{1}{2} = \frac{q}{p} + \frac{1-p}{q} \), using (3.14) and the fact that \( \mathcal{F}_s \) is \( 2^s \varepsilon_1 \)-separated in the \( L_2 \)-metric, we have that for any two distinct \( g', g \in \mathcal{A}_s \),

\[
2^{-2} \leq \|g' - g\|_2 \leq \|g' - g\|_p^\theta g^\theta = (2^{s+1} \varepsilon_1)^{1-\theta} \|g - g'\|_q^\theta.
\]

which implies that

\[
\|g' - g\|_q \geq 2^{-2}(2^{s-1} \varepsilon_1)^{1-\theta} = 2^{-2} \alpha^{s-1} \varepsilon^\theta.
\]

This together with (3.14) means that \( \mathcal{A}_s \) is a \( 2^{-2} \alpha^{s-1} \varepsilon^\theta \)-separated subset of \( X^2_N \) in the metric \( L_q \). We obtain

\[
\log_2 |\mathcal{A}_s| \leq \mathcal{H}_{2^{-3} \alpha^{s-1} \varepsilon^\theta}(X^2_N; L_q), \tag{3.16}
\]

Thus, combining (3.16) with (3.15), we prove inequality (3.11). \( \square \)

4 Proof of Theorem 2.2

In this section we prove the Marcinkiewicz discretization theorem for the \( L_p \) norms of functions from \( X_N \) with \( 1 \leq p \leq 2 \). More precisely, for a fixed \( 1 \leq p \leq 2 \) and each \( \varepsilon \in (0, 1) \), we shall show that under the condition (2.1) with \( \log K_1 \leq \alpha \log N \), there exists a set of \( m \leq C_p(\alpha, \varepsilon)K_1 N \log^3 N \) points \( \xi^1, \ldots, \xi^m \in \Omega \) such that

\[
(1 - \varepsilon)\|f\|_p^p \leq \frac{1}{m} \sum_{j=1}^m |f(\xi^j)|^p \leq (1 + \varepsilon)\|f\|_p^p, \quad \forall f \in X_N. \tag{4.1}
\]

Note that this result cannot be deduced straightforwardly from Theorem 2.1 and Theorem 1.3 since we do not assume condition (2.2) here.

Our proof relies on two known lemmas.

Lemma 4.1. [3] Lemma 2.1] Let \( \{g_j\}_{j=1}^m \) be independent random variables with mean \( 0 \) on some probability space \( (\Omega_0, \mu) \), which satisfy

\[
\max_{1 \leq j \leq m} \|g_j\|_{L_1(d\mu)} \leq M_1, \quad \max_{1 \leq j \leq m} \|g_j\|_{L_\infty(d\mu)} \leq M_\infty.
\]
for some constants $M_1$ and $M_\infty$. Then for any $0 < \varepsilon < 1$, the inequality
\[
\left| \frac{1}{m} \sum_{j=1}^{m} g_j \right| \geq \varepsilon
\]
holds with probability $\leq 2e^{-\frac{m\varepsilon^2}{4M_1 M_\infty}}$.

**Lemma 4.2.** [Lemma 2.5] Let $T : X \to Y$ be a bounded linear map from a normed linear space $(X, \|\cdot\|_X)$ into another normed linear space $(Y, \|\cdot\|_Y)$. Let $\varepsilon \in (0, 1)$ and let $F$ be an $\varepsilon$-net of the unit ball $B_X := \{x \in X : \|x\|_X \leq 1\}$. Assume that there exist constants $C_1, C_2 > 0$ such that
\[
C_1 \|x\|_X \leq \|Tx\|_Y \leq C_2 \|x\|_X, \quad \forall x \in F.
\]
Then
\[
C_1(\varepsilon) \|z\|_X \leq \|Tz\|_Y \leq C_2(\varepsilon) \|z\|_X, \quad \forall z \in X,
\]
where
\[
C_1(\varepsilon) := C_1(1 - \varepsilon) - C_2 \varepsilon \frac{1 + \varepsilon}{1 - \varepsilon}, \quad C_2(\varepsilon) := C_2 \frac{1 + \varepsilon}{1 - \varepsilon}.
\]

To prove Theorem 2.2, we start with the following weaker result.

**Lemma 4.3.** Let $1 \leq p < \infty$ be a fixed number. Assume that $X_N$ is an $N$-dimensional subspace of $L_\infty(\Omega)$ satisfying the following condition for some parameter $\beta > 0$ and constant $K \geq 2$:
\[
\|f\|_\infty \leq (KN)^{\beta} \|f\|_p, \quad \forall f \in X_N.
\]  
(4.2)

Let $\{\xi_j\}_{j=1}^{\infty}$ be a sequence of independent random points selected uniformly from the probability space $(\Omega, \mu)$. Then there exists a positive constants $C_\beta$ depending only on $\beta$ such that for any $0 < \varepsilon \leq \frac{1}{2}$ and
\[
m \geq C_\beta K^\beta \varepsilon^{-2} (\log \frac{2}{\varepsilon}) N^{\beta + 1} \log N,
\]  
(4.3)

the inequality
\[
(1 - \varepsilon) \|f\|_p \leq \frac{1}{m} \sum_{j=1}^{m} |f(\xi_j)|^p \leq (1 + \varepsilon) \|f\|_p,
\]  
(4.4)

holds with probability $\geq 1 - m^{-N/\log K}$.
Proof. The proof is quite standard, and we only sketch the main steps. By Lemma 3.1 (i), given \( \varepsilon \in (0, 1) \), there exists an \( \varepsilon \)-net \( \mathcal{F} \subset X_N^p \) of \( X_N^p \) in the space \( L^p \) such that

\[
|\mathcal{F}| \leq \left( 1 + \frac{2}{\varepsilon} \right)^N \leq 2^{-1} \exp\left( \frac{m\varepsilon^2}{16N_1^\beta} \right)
\]  

whenever

\[
m \geq C_0 K^\beta \varepsilon^{-2} (\log \frac{2}{\varepsilon}) N_1^{\beta+1}.
\]  

Next, using Lemma 4.1, we have that the inequalities

\[
\left| \frac{1}{m} \sum_{j=1}^m |f(\xi_j)|^p - \int_\Omega |f|^p \, d\mu \right| \leq \varepsilon \|f\|_p^p, \quad \forall f \in \mathcal{F}
\]  

hold under condition (4.6) with probability

\[
1 - 2|\mathcal{F}| \exp\left( -\frac{m\varepsilon^2}{8N_1^\beta} \right) \geq 1 - \exp\left( -\frac{m\varepsilon^2}{16N_1^\beta} \right).
\]

To complete the proof, we just need to observe that the function \( \frac{x}{\log x} \) is increasing on \((e, \infty)\), and hence, by a straightforward calculation, the condition (4.3) with a sufficiently large constant \( C_\beta \) implies both (4.6) and

\[
1 - \exp\left( -\frac{m\varepsilon^2}{16N_1^\beta} \right) \geq 1 - m^{-N/\log K}.
\]

\[ \square \]

Proof of Theorem 2.2. Given each positive integer \( m \) and each \( z = (z_1, \ldots, z_m) \in \Omega^m \), we define the operator \( T_{m,z} : X_N \to \mathbb{R}^m \) by \( T_{m,z} f = (f(z_1), \ldots, f(z_m)) \).

We denote by \( \ell_p^m \) the space \( \mathbb{R}^m \) equipped with the norm

\[
\|x\|_{\ell_p^m} := \begin{cases} 
\left( \frac{1}{m} \sum_{j=1}^m |x_j|^p \right)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty, \\
\max_{1 \leq j \leq m} |x_j|, & \text{if } p = \infty,
\end{cases} \quad x = (x_1, \ldots, x_m) \in \mathbb{R}^m.
\]

By (2.9) and (2.1), we have

\[
\|f\|_\infty \leq (K_1 N)^\frac{1}{p} \|f\|_p, \quad \forall f \in X_N, \quad 1 \leq p \leq 2.
\]
Thus, for each fixed $1 \leq p \leq 2$, and each $\varepsilon \in (0, 1)$, by Lemma 4.3 applied to $\alpha = 1$, there exists a vector $z = (z_1, \ldots, z_{m_1}) \in \Omega^{m_1}$ and a constant $C(\varepsilon) > 1$ such that

\[ C^{-1}(\varepsilon)K_1N^2 \log N \leq m_1 \leq C(\varepsilon)K_1N^2 \log N, \]

and the inequalities

\[ \left| \|f\|_q - \|T_{m, \mathbf{z}} f\|_{\ell_q^{m_1}} \right| \leq \frac{\varepsilon}{4} \|f\|_q, \quad \forall f \in X^q_N, \quad q = 2, p \quad (4.8) \]

hold simultaneously. Consider the $N$-dimensional subspace $\tilde{X}_N := T_{m, \mathbf{z}}(X_N)$ of $\ell_p^{m_1}$. Using (2.1) and (4.8), we have that for any $f \in X_N$,

\[ \|T_{m, \mathbf{z}} f\|_{\ell_2^{m_1}} \leq \|f\|_{\infty} \leq (K_1)^{\frac{1}{2}} \|f\|_2 \leq (1 + \varepsilon)(K_1)^{\frac{1}{2}} \|T_{m, \mathbf{z}} f\|_{\ell_2^{m_1}}. \quad (4.9) \]

On the other hand, since $\log m_1 \leq C_\varepsilon(\log K_1 + \log N)$ and

\[ \|x\|_{\ell_2^{m_1}} \leq m_1^{\frac{1}{q_1} - \frac{1}{q_2}} \|x\|_{\ell_1^{m_1}}, \quad \forall x \in \mathbb{R}^{m_1}, \quad 0 < q_1 < q_2 \leq \infty, \]

in light of $\log K_1 \leq \alpha \log N$, it follows that

\[ \|x\|_{\ell_2^{m_1}} \leq e^{\alpha \varepsilon} \|x\|_{\ell_{q_N}^{m_1}}, \quad q_N = \log N, \quad \forall x \in \mathbb{R}^{m_1}. \]

This and (4.9) mean that the $N$-dimensional subspace $\tilde{X}_N$ of $\ell_{q_N}^{m_1}$ satisfies both the conditions (2.1) and (2.2). It then follows by Theorem 2.1 that

\[ \varepsilon_k(\tilde{X}_N, \ell_{q_N}^{m_1}) \leq C_p(\varepsilon)K_1^\frac{1}{2} e^{\frac{2\alpha \varepsilon}{N_0}} (\log N)^\frac{1}{p} \begin{cases} \left( \frac{N}{2} \right)^\frac{1}{p}, & \text{if } 1 \leq k \leq N, \\ 2^{-k/N}, & \text{if } k > N, \end{cases} \]

where $\tilde{X}_N^p := \{ x \in \tilde{X}_N : \|x\|_{\ell_{q_N}^{m_1}} \leq 1 \}$. Thus, applying Theorem 4.3 to the subspace $\tilde{X}_N$ in $\ell_{q_N}^{m_1}$, we can find a subset $\Lambda \subset \{1, 2, \ldots, m_1\}$ with $|\Lambda| \leq C_p(\alpha, \varepsilon)K_1N \log^3 N$ such that

\[ \left| \frac{1}{|\Lambda|} \sum_{j \in \Lambda} |T_{m, \mathbf{z}} f(j)|^p - \|T_{m, \mathbf{z}} f\|_{\ell_{p_N}^{m_1}}^p \right| \leq \varepsilon \left\| T_{m, \mathbf{z}} f \right\|_{\ell_{p_N}^{m_1}}^p. \]

Now using (4.8) with $q = p$ and a sufficiently small parameter $\varepsilon \in (0, 1)$, we obtain

\[ (1 - \varepsilon)\|f\|_p^p \leq \frac{1}{|\Lambda|} \sum_{j \in \Lambda} |f(z_j)|^p \leq (1 + \varepsilon)\|f\|_p^p, \quad \forall f \in X_N. \]

\[ \square \]
5 Proof of Theorem 2.3

Here we derive the unconditional weighted Marcinkiewicz discretization theorem for $L_p$ norms of functions from a general $N$-dimensional subspace $X_N \subset L_p$. We need the following result.

**Lemma 5.1.** [3 Lemma 7.1] Let $X_N$ be an $N$-dimensional subspace of $L_p(\Omega, d\mu)$ with $1 \leq p < \infty$. Then there is a basis $\{\varphi_j\}_{j=1}^N$ of $X_N$ so that the function $F = \left(\sum_{j=1}^N \varphi_j^2\right)^{\frac{1}{2}}$ satisfies that $\|F\|_p = 1$ and for all scalars $\{\lambda_j\}_{j=1}^N \subset \mathbb{R}$,

$$\int_{\Omega} \left| \sum_{j=1}^N \lambda_j \varphi_j(x) \right|^2 F(x)^{p-2} d\mu(x) = N^{-1} \sum_{j=1}^N \lambda_j^2. \quad \text{(5.1)}$$

**Proof of Theorem 2.3.** Let $1 \leq p \leq 2$, and let $\{\varphi_j\}_{j=1}^N$ be a basis of $X_N$ for which the function $F = \left(\sum_{j=1}^N \varphi_j^2\right)^{\frac{1}{2}}$ has the properties in Lemma 5.1. Then $d\nu := F^p d\mu$ is a probability measure on $\Omega$. Define the mapping $U : L_p(d\mu) \to L_p(d\nu)$ by

$$Uf(x) = \begin{cases} \frac{f(x)}{F(x)}, & \text{if } F(x) \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Note that if $x \in \Omega$ and $F(x) = 0$, then $f(x) = 0$ for all $f \in X_N$. It follows that

$$\|Uf\|_{L_p(d\nu)} = \|f\|_{L_p(d\mu)}, \quad \forall f \in X_N. \quad \text{(5.2)}$$

Next, let $\psi_j = \sqrt{N}U\varphi_j$ for $1 \leq j \leq N$. Then (5.1) implies that $\{\psi_j\}_{j=1}^N$ is an orthonormal basis of the space $\tilde{X}_N := UX_N$ equipped with the norm of $L_2(\Omega, d\nu)$. Moreover, for any $x \in \Omega$ with $F(x) \neq 0$,

$$\frac{1}{N} \sum_{j=1}^N \psi_j(x)^2 = \frac{1}{F(x)^2} \sum_{j=1}^N \varphi_j(x)^2 = 1. \quad \text{(5.3)}$$

Note that (5.3) with $=1$ replaced by $\leq 1$ holds trivially if $F(x) = 0$, in which case $\psi_j(x) = 0$ for all $1 \leq j \leq N$. This implies that

$$\|g\|_{L_\infty(\Omega)} \leq N^{\frac{1}{p}}\|g\|_{L_2(d\nu)}, \quad \forall g \in \tilde{X}_N.$$
Finally, applying Theorem 2.2 to the space $\tilde{X}_N \subset L_p(\Omega, d\nu)$ with $K_1 = 1$, for any $\epsilon \in (0, 1)$, we can find a set of $m$ points $x_1, \ldots, x_m \in \Omega$ with $m \leq C_p(\epsilon)N \log^3 N$ such that $F(x_j) > 0$ for all $j = 1, \ldots, m$ and

$$(1 - \epsilon)\|Uf\|_{L_p(d\nu)}^p \leq \frac{1}{m} \sum_{j=1}^m |Uf(x_j)|^p \leq (1 + \epsilon)\|Uf\|_{L_p(d\nu)}^p, \quad \forall f \in X_N.$$ 

To complete the proof, we just need to observe that $Uf(x_j) = \frac{f(x_j)}{F(x_j)}$, $j = 1, \ldots, m$ for all $f \in X_N$ and recall (5.2).

\[\square\]

6 Discussion

In [6] we have recently proved a conditional result, Theorem 1.3. This theorem guarantees the existence of good Marcinkiewicz-type discretization results for a given $N$-dimensional subspace $X_N$ under condition on the behavior of the entropy numbers of the unit $L_p$-ball of $X_N$ in the uniform norm $L_\infty$.

In this paper we concentrate on establishing good upper bounds for the corresponding entropy numbers $\varepsilon_k(X^p_N, L_\infty)$. The problem of estimating the entropy numbers of different compacts, including function classes, is a deep fundamental problem of analysis (see, e.g., [18], [7], and [23]). In this section we discuss some known techniques and results and compare them with our new results. We concentrate on the case of entropy numbers in the $L_\infty$ norm. The first step of our technique is the following well-known result (see [13]).

**Theorem 6.1.** Let $X$ be $\mathbb{R}^n$ equipped with $\| \cdot \|$ and

$$M_X = \int_{S^{n-1}} \|x\| d\sigma(x).$$

Then we have

$$\varepsilon_k(B^n_2, X) \leq CM_X \begin{cases} (n/k)^{1/2} & \text{if } k \leq n, \\ 2^{-k/n} & \text{if } k \geq n. \end{cases}$$

Theorem 6.1 is a dual version of the corresponding result from [17].

Another technique to estimate the entropy numbers in the $L_\infty$ norm (see [20], [22], [23]) is based on the connection between the entropy numbers and
the best \( m \)-term approximations with respect to a dictionary and it does not use Theorem 6.1. In estimating the best \( m \)-term approximations greedy-type algorithms are used. In a certain sense this technique provides a constructive way of building good point nets. This technique was applied in [22] to prove Marcinkiewicz-type discretization results in the case of \( L_1 \) norm.

Theorem 6.1 is used for estimation of \( \varepsilon_k(X_N^2, L_\infty) \) (see Subsection 3.1 above). To obtain bounds for \( \varepsilon_k(X_N^p, L_\infty), 1 \leq p < 2 \), in contrast with [2], we use in Subsection 3.2 a new technique, which allows us to derive bounds of \( \varepsilon_k(X_N^p, L_\infty), 1 \leq p < 2 \), from the bounds for \( \varepsilon_k(X_N^2, L_\infty) \).

In [6] we discussed some results on the entropy numbers of the \( L_q \)-balls of subspaces \( \mathcal{T}(Q_n) \) of trigonometric polynomials with frequencies from the hyperbolic cross \( Q_n \). We continue this discussion with one more example here. We remind some notation. For \( s \in \mathbb{Z}_d^+ \), define

\[
\rho(s) := \{ k \in \mathbb{Z}^d : [2^{s_j-1}] \leq |k_j| < 2^{s_j}, \quad j = 1, \ldots, d \},
\]

where \([x]\) denotes the integer part of \( x \). We define the step hyperbolic cross \( Q_n \) as follows

\[
Q_n := \bigcup_{|s| \leq \rho(s)}\mathbb{Z}^d,
\]

and the corresponding set of the hyperbolic cross polynomials as

\[
\mathcal{T}(Q_n) := \{ f : f = \sum_{k \in Q_n} c_k e^{i(k,x)} , \; c_k \in \mathbb{R} \},
\]

\[
\mathcal{T}(Q_n)^q := \{ f : f \in \mathcal{T}(Q_n), \| f \|_q \leq 1 \}.
\]

It is well known (see, for instance, [6]) that the bound

\[
\varepsilon_N(X_N^p, L_q) \leq B
\]

implies the bound

\[
\varepsilon_k(X_N^p, L_q) \leq 6B 2^{-k/N} \quad k > N.
\]

Thus, we only compare bounds for \( 1 \leq k \leq N \). The following bound is known (see [23 p.350]) for all dimensions \( d \): with \( N := |Q_n| \) and \( \beta := 1/p - 1/q \),

\[
\varepsilon_k(\mathcal{T}(Q_n)^p, L_q) \leq C(q, p, d)(N/k)^\beta(\log(N/k))^\beta, \quad 1 < p \leq 2 \leq q < \infty.
\]

(6.1)
Moreover, for \( d = 2 \) one has (see [23, p.361])

\[
\varepsilon_k(\mathcal{T}(Q_n)^1, L_q) \leq C(q, p, d)(N/k)^\beta(\log(N/k))^\beta, \quad 2 \leq q < \infty.
\] (6.2)

To demonstrate the strength of our technique, we derive Theorem 6.2, which applies to more general subspaces than the \( \mathcal{T}(Q_n) \) and, moreover, covers the cases \( p = 1 \) and \( q = \infty \). However, we point out that in some cases bounds (6.1) and (6.2) are better for proving the upper bounds for the classes of functions with mixed smoothness. Typically, the extra factor \((\log(N/k))^\beta\) in (6.1) and (6.2) does not contribute in the final upper bound for a class, while the extra factor \((\log N)^\beta\) in Theorem 6.2 will increase the power of the corresponding log factor by \( \beta \).

In this section, we show that the following extension of Theorem 2.1 can be proved.

**Theorem 6.2.** If \( X = X_N \) is an \( N \)-dimensional subspace of \( L_\infty \) satisfying the conditions (2.1) and (2.2), then for \( 1 \leq p \leq 2 < q \leq \infty \) and \( \beta = 1/p - 1/q \),

\[
\varepsilon_k(X^p; L_q) \leq C_{p,q}(K_1 K_2^2 \log N)^\beta \left\{ \begin{array}{ll}
(N/k)^\beta, & \text{if } 1 \leq k \leq N; \\
2^{-k/N}, & \text{if } k \geq N.
\end{array} \right.
\] (6.3)

**Proof.** Since the case \( q = \infty \) is contained in Theorem 2.1, we assume \( q < \infty \). First, we observe that

\[
\mathcal{H}_\varepsilon(X^2_N; L_q) \leq C_q K_1 K_2^2 \varepsilon^{-\frac{2q}{q-2}} N \log N, \quad 2 < q < \infty, \quad \varepsilon > 0.
\] (6.4)

Indeed, setting \( \theta = \frac{2}{q} \), we have that for any \( f, g \in X^2_N \),

\[
\|f - g\|_q \leq \|f - g\|_2^{\theta} \|f - g\|_\infty^{1-\theta} \leq 2^\theta \|f - g\|_\infty^{1-\theta}.
\]

Thus, using (3.5), we obtain

\[
\mathcal{H}_\varepsilon(X^2_N; L_q) \leq \mathcal{H}_{(2^{-\varepsilon})^{1/\theta}}(X^2_N; L_\infty) \leq \frac{C_q K_1 K_2^2 N \log N}{\varepsilon^{2q/2}}.
\]

Next, we apply inequality (3.9) in Lemma 3.3 to obtain that

\[
\mathcal{H}_\varepsilon(X^p_N; L_q) \leq \sum_{s=0}^\infty \mathcal{H}_{2^{-s}\varepsilon^{p-1}s}(X^2_N; L_q) + \mathcal{H}_\varepsilon^\theta(X^2_N; L_q), \quad \varepsilon > 0,
\]

21
where \( \theta := \left( \frac{1}{2} - \frac{1}{q} \right) / \left( \frac{1}{p} - \frac{1}{q} \right) \) and \( a = a(\theta) = 2^{\frac{\theta}{p}} \). Thus, we derive from (6.4) that
\[
\mathcal{H}_\varepsilon(X^p; L_q) \leq C_q K_1 K_2^2 N \log N \varepsilon^{-1/(\frac{1}{p} - \frac{1}{q})},
\]
which implies (6.3).

The primary goal of this paper is to obtain the Marcinkiewicz-type discretization results with equal weights (see (1.1)) for a wide class of finite dimensional subspaces \( X_N \). In Theorem 2.2 above, we only impose one restriction on a subspace \( X_N \), namely, the Nikol’skii inequality (2.1): for any \( f \in X_N \),
\[
\|f\|_\infty \leq \left( K_1 N \right)^{1/2} \|f\|_2.
\] (6.5)
Under that assumption and a minor assumption on \( K_1 \), Theorem 2.2 guarantees that \( X_N \) admits the Marcinkiewicz-type discretization theorem with \( m \) of order \( N(\log N)^3 \). It is clear that our result is optimal with respect to power scale. However, it would be interesting to know if it is possible to replace in Theorem 2.2 the bound \( m \leq C_p(\alpha) K_1 N(\log N)^3 \) by the bound \( m \leq C_p(\alpha) K_1 N \).

Another question is to what extent we can weaken the Nikol’skii inequality (6.5) and still have Theorem 2.2 with the bound \( m \leq C_p(\alpha) K_1 N(\log N)^3 \)? Here we stress that Theorem 2.3 above shows that we can drop the assumption on the Nikol’skii inequality if we allow arbitrary weights instead of equal weights in the discretization theorem. We now make some comments on Theorem 2.3.

We consider two cases: \( p = 2 \) and \( 1 \leq p < 2 \). In the case \( p = 2 \), the recent results from [1] basically solve the discretization problem with weights; see [22]. We present an explicit formulation of this important result in our notations.

**Theorem.** Let \( \Omega_M = \{x^j\}_{j=1}^M \) be a discrete set with the probability measure \( \mu(x^j) = 1/M, \ j = 1, \ldots, M \) and let \( X_N \) be an \( N \)-dimensional subspace of real functions defined on \( \Omega_M \). Then for any number \( b > 1 \) there exist a set of weights \( \lambda_j \geq 0 \) such that \(|\{j : \lambda_j \neq 0\}| \leq bN \) so that for any \( f \in X_N \) we have
\[
\|f\|_2^2 \leq \sum_{j=1}^M \lambda_j f(x^j)^2 \leq \frac{b + 1 + 2\sqrt{b}}{b + 1 - 2\sqrt{b}} \|f\|_2^2.
\] (6.6)
As was observed in [5, Theorem 2.13], this last theorem with a general probability space \((\Omega, \mu)\) in place of the discrete space \((\Omega_M, \mu)\) remains true if \(X_N \subset L^4(\Omega)\). We further remark here that the additional assumption \(X_N \subset L^4(\Omega)\) can be dropped as well; namely, we have

**Theorem 6.3.** If \(X_N\) is an \(N\)-dimensional subspace of \(L^2(\Omega)\), then for any \(b \in (1, 2]\), there exist a set of \(m \leq bN\) points \(x^1, \ldots, x^m \in \Omega\) and a set of nonnegative weights \(\lambda_j, j = 1, \ldots, m\) such that

\[
\|f\|_2^2 \leq \sum_{j=1}^{m} \lambda_j |f(x^j)|^2 \leq \frac{C}{(b-1)^2} \|f\|_2^2, \quad \forall f \in X_N,
\]

where \(C > 1\) is an absolute constant.

**Proof.** Let \(\{\varphi_j\}_{j=1}^{N}\) be an orthonormal basis of \(X_N\), and let

\[
F(x) := \left(\frac{1}{N} \sum_{j=1}^{N} \varphi_j(x)^2\right)^{\frac{1}{2}}, \quad x \in \Omega.
\]

Consider the probability measure \(d\nu := F^2 d\mu\) on the set

\[
\Omega_0 = \{x \in \Omega : F(x) > 0\}.
\]

Define the mapping \(U : L^2(\Omega, d\mu) \to L^2(\Omega_0, d\nu)\) by

\[
Uf(x) = \frac{f(x)}{F(x)}, \quad x \in \Omega_0,
\]

and let \(\tilde{X}_N := UX_N\) be the subspace of \(L^2(\Omega_0, d\nu)\). Then

\[
\|Uf\|_{L^2(\Omega_0, d\nu)} = \|f\|_{L^2(\Omega, d\mu)}, \quad \forall f \in X_N,
\]

and by the Cauchy-Schwarz inequality,

\[
\|g\|_{L^\infty(\Omega_0, d\nu)} \leq N^\frac{1}{2} \|g\|_{L^2(\Omega_0, d\nu)}, \quad \forall g \in \tilde{X}_N.
\]

Now applying Theorem 12 to the space \(\tilde{X}_N \subset L^2(\Omega_0, d\nu)\), we obtain a finite subset \(\Lambda \subset \Omega_0\) such that \(|\Lambda| \leq CN \log N\) and

\[
\frac{1}{2} \|Uf\|_{L^2(\Omega_0, d\nu)}^2 \leq \frac{1}{|\Lambda|} \sum_{\omega \in \Lambda} |Uf(\omega)|^2 \leq \frac{3}{2} \|Uf\|_{L^2(\Omega_0, d\nu)}^2, \quad \forall f \in X_N.
\]
It then follows by (6.7) that for any \( f \in X_N \),
\[
\frac{1}{2} \| f \|_{L^2(\Omega, d\mu)}^2 \leq \frac{1}{|\Lambda|} \sum_{\omega \in \Lambda} |f(\omega)|^2 (F(\omega))^{-2} \leq \frac{3}{2} \| f \|_{L^2(\Omega, d\mu)}^2.
\]  
\[ (6.8) \]

Finally, applying (6.6) to the subspace \( Y_N := \left\{ \frac{f}{|\Lambda|} : f \in X_N \right\} \) of \( \mathbb{R}^{|\Lambda|} \), we conclude that for any \( b \in (1, 2] \), there exist a set of \( m \) points \( x^1, \ldots, x^m \in \Lambda \subset \Omega_0 \) and a set of weights \( w_j \geq 0, 1 \leq j \leq m \) such that 
\[
\frac{1}{|\Lambda|} \sum_{\omega \in \Lambda} |f(\omega)|^2 (F(\omega))^{-2} \leq \sum_{j=1}^{m} w_j |f(x^j)|^2 F(x^j)^{-2}
\leq C(b-1)^{-2} \frac{1}{|\Lambda|} \sum_{\omega \in \Lambda} |f(\omega)|^2 (F(\omega))^{-2}, \quad \forall f \in X_N.
\]

This together with (6.8) implies that
\[
\| f \|_{L^2(\Omega, d\mu)}^2 \leq 2 \sum_{j=1}^{m} w_j |f(x^j)|^2 F(x^j)^{-2} \leq 3C(b-1)^{-2} \| f \|_{L^2(\Omega, d\mu)}^2.
\]

This completes the proof of the theorem with \( \lambda_j = w_j/F(x^j)^2 \).

\[ \square \]

Finally, let us discuss the case \( 1 \leq p < 2 \). Our proof of Theorem 2.3 demonstrates how deeps results from functional analysis (see Lemma 5.1) can be used in obtaining good discretization theorems with weights. That kind of technique was developed in the following important problem from functional analysis (see, for instance, [3], [15], [9]): Given an \( N \) dimensional subspace \( X_N \) of \( L_q(0,1) \) and \( \varepsilon > 0 \), what is the smallest \( L(X_N, q, \varepsilon) \) such that there is a subspace \( Y_N \) of \( \ell_q^N(X_N, q, \varepsilon) \) with \( d(X_N, Y_N) \leq 1 + \varepsilon \)? Here \( d(X, Y) \) stands for the Banach–Mazur distance between two finite dimensional spaces \( X \) and \( Y \) of the same dimension, that is,
\[
d(X, Y) := \inf \{ \| T \| \| T^{-1} \| : T \text{ is a linear isomorphism from } X \text{ to } Y \}.
\]

This question is related to the following discretization problem. **Marcinkiewicz problem with \( \varepsilon \).** We write \( X_N \in \mathcal{M}(m, q, \varepsilon) \) if (1.1) holds with \( C_1(d, q) = 1 - \varepsilon \) and \( C_2(d, q) = 1 + \varepsilon \).

Certainly, if \( X_N \in \mathcal{M}(m, q, \varepsilon) \) then \( L(X_N, q, C(q)\varepsilon) \leq m \). On the other hand, results on the behavior of \( L(X_N, q, \varepsilon) \) do not imply bounds on \( m \) for
$X_N \in \mathcal{M}(m, q, \varepsilon)$. It is obvious in the case $q = 2$. However, it turns out that the technique developed in this area for studying behavior of $L(X_N, 1, \varepsilon)$ can be used for the Marcinkiewicz-type discretization (see [3], [15], [9]).

Acknowledgement

The authors are extremely grateful to the Isaac Newton Institute (Cambridge, UK): their collaboration has started while they participated in the research program “Approximation, sampling, and compression in high dimensional problems” in 2019. The work was supported by the Russian Federation Government Grant №14.W03.31.0031. The paper contains results obtained in frames of the program “Center for the storage and analysis of big data”, supported by the Ministry of Science and High Education of Russian Federation (contract 11.12.2018№13/1251/2018 between the Lomonosov Moscow State University and the Fond of support of the National technological initiative projects).

References

[1] J. Batson, D.A. Spielman, and N. Srivastava, Twice-Ramanujan Spar-sifiers, *SIAM J. Comput.*, **41** (2012), 1704–1721.

[2] E.S. Belinsky, Interpolation and integral norms of hyperbolic polyno-mials, *Matem. Zametki*, **66** (1999), 20–29.

[3] J. Bourgain, J. Lindenstrauss and V. Milman, Approximation of zonoids by zonotopes, *Acta Math.*, **162** (1989), 73–141.

[4] B. Carl, Entropy numbers, s-numbers, and eigenvalue problem, *J. Func. Analysis*, **41** (1981), 290–306.

[5] F. Dai, A. Prymak, V.N. Temlyakov, and S. Tikhonov, Integral norm discretization and related problems, *Russian Math. Surveys* **74** (2019), 579–630. Translation from *Uspekhi Mat. Nauk* **74** (2019), 3–58; arXiv:1807.01353v1.

[6] F. Dai, A. Prymak, A. Shadrin, V.N. Temlyakov, and S. Tikhonov, Sampling discretization of integral norms, arXiv:2001.09320.
[7] Ding Dung, V.N. Temlyakov, and T. Ullrich, Hyperbolic Cross Approximation, Advanced Courses in Mathematics CRM Barcelona, Birkhäuser, 2018; arXiv:1601.03978v2.

[8] A. Hinrichs, J. Prochno, and J. Vybiral, Entropy numbers of embeddings of Schatten classes, *J. Functional Analysis*, 273 (2017), 3241–3261; arXiv:1612.08105v1.

[9] W.B. Johnson and G. Schechtman, Finite dimensional subspaces of $L_p$, Handbook of the geometry of Banach spaces, Vol. 1 (2001), 837–870, North-Holland, Amsterdam.

[10] G. Lorentz, M. von Golitschek, Y. Makovoz, Constructive Approximation: Advanced Problems. Volume 304 of Grundlehren der Mathematischen Wissenschaften, Springer, Berlin(1996).

[11] A. Marcus, D.A. Spielman, and N. Srivastava, Interlacing families II: Mixed characteristic polynomials and the Kadison-Singer problem, *Annals of Math.*, 182 (2015), 327–350.

[12] S. Nitzan, A. Olevskii, and A. Ulanovskii, Exponential frames on unbounded sets, *Proc. Amer. Math. Soc.*, 144 (2016), 109–118.

[13] A. Pajor and N. Tomczak-Yaegermann (1986), Subspaces of small co-dimension of finite-dimensional Banach spaces, *Proc. Amer. Math. Soc.*, 97 (1986), 637–642.

[14] M. Rudelson, Almost orthogonal submatrices of an orthogonal matrix, *Israel J. Math.*, 111 (1999), 143–155.

[15] G. Schechtman, More on embeddings subspaces of $L_p$ in $\ell^n_p$, *Compositio Math.*, 61 (1987), 159–170.

[16] C. Schütt, Entropy numbers of diagonal operators between symmetric Banach spaces, *J. Approx. Theory*, 40 (1984), 121–128.

[17] V.N. Sudakov, Gaussian random processes and measures of solid angles in Hilbert spaces, *Sov. Math. Dokl.*, 12, (1971), 412–415.

[18] V.N. Temlyakov, Greedy Approximation, Cambridge University Press, 2011.
[19] V.N. Temlyakov, An inequality for the entropy numbers and its application, *J. Approx. Theory*, 173 (2013), 110–121.

[20] V.N. Temlyakov, On the entropy numbers of the mixed smoothness function classes, *J. Approx. Theory*, 207 (2017), 26–56; arXiv:1602.08712v1.

[21] V.N. Temlyakov, The Marcinkiewicz-type discretization theorems for the hyperbolic cross polynomials, *Jaen Journal on Approximation*, 9 (2017), No. 1, 37–63; arXiv: 1702.01617v2.

[22] V.N. Temlyakov, The Marcinkiewicz-type discretization theorems, *Constr. Approx.* 48 (2018), 337–369; arXiv: 1703.03743v1.

[23] V. Temlyakov, Multivariate Approximation, Cambridge University Press, 2018.

[24] A. Zygmund, Trigonometric Series, Cambridge University Press, 1959.

F. Dai, Department of Mathematical and Statistical Sciences
University of Alberta
Edmonton, Alberta T6G 2G1, Canada
E-mail: fdai@ualberta.ca

A. Prymak, Department of Mathematics
University of Manitoba
Winnipeg, MB, R3T 2N2, Canada
E-mail: Andriy.Prymak@umanitoba.ca

A. Shadrin, Department of Mathematics and Theoretical Physics
University of Cambridge
Wilberforce Road, Cambridge CB3 0WA, UK
E-mail: a.shadrin@damtp.cam.ac.uk

V.N. Temlyakov, University of South Carolina,
Steklov Institute of Mathematics,
and Lomonosov Moscow State University
E-mail: temlyak@math.sc.edu

S. Tikhonov, Centre de Recerca Matemàtica
Campus de Bellaterra, Edifici C 08193 Bellaterra (Barcelona), Spain;
ICREA, Pg. Lluís Companys 23, 08010 Barcelona, Spain,
and Universitat Autònoma de Barcelona
E-mail: stikhonov@crm.cat