A 4D hyperchaotic Sprott S system with multistability and hidden attractors

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Abstract. This paper derived a new simple hyperchaotic system from the famous Sprott, S system via the linear state feedback control. Compared with the available systems, the new system has eight terms, one constant, two parameters control, and a single quadratic nonlinear term. So this system is considered a simple relying on the number of terms and nonlinearities. The proposed system without equilibrium points and exhibits chaotic hidden attractors. Also, multistability or coexisting attractors are found through experimental simulations using phase portraits and the Lyapunov spectrum. Finally, anti-synchronization is implemented in the new system.

1. Introduction

In 2011, Leonov et al. introduced a new concept in the behavior of dynamic systems based on the attractors of these systems and it is classified into two categories: hidden and self-excited attractors [1]. Most of the famous traditional systems such as Lorenz 1963, Rossler 1976, Chen 1999, and Liu systems 2004 have self-excited attractors [2-6]. Whereas the hidden attractor occur sin three cases; systems with stable equilibrium points [7-9], no-equilibrium points [10-16], many or line equilibria [17]. Hidden attractors are difficult to discover compared with self-excited attractors, since the location of the equilibrium points not necessary as it is in self-excited attractors.

In literature, many works have been concentrated on the discovery of simple chaotic and hyperchaotic systems with no equilibrium points. In 1994, the researcher Sprott presented 19 systems of three-dimensional that were classified into two groups based on the number of terms and nonlinearities, the first group of systems consisting of five terms with two nonlinear (A-E) whereas the second group consisting of six terms with one nonlinear (F-S) [18]. The number of terms and number of quadratic or cubic nonlinearities are important factors within the simplest systems. In 2011, Wei reported 3D chaotic system without equilibrium points containing five terms which include two [11]. In 2012, Wang et al. [15] presented a no-equilibrium 4D hyperchaotic system from on the Sprott A system, this system has a total of nine terms including five nonlinearities. In 2013, Jafari et al. [19] proposed 17 systems without equilibrium points with quadratic nonlinearities. In 2014, Wei et al. introduced a no-equilibrium 4D hyperchaotic system which has a total of eight terms, including two nonlinearities [20]. In 2019, Zhang and Zeng reported a no-equilibrium simple 3D system has a quadratic term and a square nonlinearartly [21]. More recently, Zhang et al. in 2020, present a new simple 3D system without equilibrium points have six terms without any nonlinear term [22].
Recently, many synchronization phenomena have implementation of these systems due to their important applications in such as engineering [23, 24], and encryption [25-28]. Various schemes of synchronization were reported such as complete synchronization (CS) [2, 3, 29, 30], anti-synchronization (AS) [31-33], hybrid synchronization (HS) [34-36], projective synchronization (PS) [32-34].

The following points summarize the main contribution of this work, the proposed system as below:

(i) A 4D simple hyperchaotic Sprott S is constructed from the famous 3D Sprott-S system.
(ii) Without equilibrium point.
(iii) Consists of eight terms containing one constant two parameters control and one nonlinear term. So it can be considered one of the simplest systems.
(iv) Exhibit coexisting of attractors.
(v) Implementing the phenomenon of anti-synchronization of this system.

2. Dynamics of the 4D hyperchaotic Sprott S system

The Sprott S system [18] is depicted as:

\[
\begin{align*}
\dot{x} &= -x - 4y, \\
\dot{y} &= x + z^2, \\
\dot{z} &= 1 + x,
\end{align*}
\]  

(1)

in which \([x, y, z]^T \in \mathbb{R}^3\) is a state vector. Based on the state feedback control [37, 38] by adding a linear controller to the second equation of (1), a new simple 4D chaotic quadratic system is obtained as:

\[
\begin{align*}
\dot{x} &= -x - 4y, \\
\dot{y} &= x + z^2 + aw, \\
\dot{z} &= 1 + x, \\
\dot{w} &= -by,
\end{align*}
\]  

(2)

in which \([x, y, z, w]^T \in \mathbb{R}^4\) is state vector. The parameters \(a\) and \(b\) are two positive constants. This system has eight terms with six linear terms, single quadratic nonlinearity, and one constant. With \(a = 0.01\), \(b = 0.1\) and \(x(0) = (0.1, 0.1, 0.1, 0.1)^T\). This system exhibit chaotic hidden attractors as shown in Figure 1.
2.1 Equilibrium points and Stability

The equilibrium points are established by solving the following equations:

\[
\begin{align*}
-x - 4y &= 0 \\
x + z^2 + au &= 0 \\
1 + x &= 0 \\
-b y &= 0
\end{align*}
\]  

From the third and fourth equations of (3), it is seen that \(x = 1, y = 0\), respectively. This leads to a contradiction with the first equation of (3). Consequently, the system (2) without equilibrium points concludes that this system with hidden attractors.

2.2 Lyapunov exponents and dissipativity

Wolf Algorithm has received increasing attention over the past decades due to the applications of calculating Lyapunov exponents \(\text{LE}_{\mathcal{S}}\). With parameters \(a = 0.02, b = 0.001\) and the initial condition \((0.5, 0.5, 0.5, 0.5)^T\), step \(\Delta t = 0.25\), observation time \(T = 300\), the \(\text{LE}_{\mathcal{S}}\) of the 4D system (2) are numerically obtained in MATLAB simulation as:

\[
\begin{align*}
\text{LE}_1 &= 0.1658 \\
\text{LE}_2 &= 0 \\
\text{LE}_3 &= -0.0299 \\
\text{LE}_4 &= -1.1353
\end{align*}
\]  

Figure 2. depict the corresponding Lyapunov spectrum of (4) and the Lyapunov dimension (Kaplan-Yorke dimension) [17] of this system is calculated as:

\[
D_{\text{LE}} = j + \frac{1}{|\text{LE}_{j+1}|} \sum_{i=1}^{j} \text{LE}_i = 3 + \frac{\text{LE}_1 + \text{LE}_2 + \text{LE}_3}{|\text{LE}_4|} = 3.1197
\]

Since that the sum of the Lyapunov exponents \(\sum \text{LE}_{\mathcal{S}} = -1\) is negative, which indicate its dissipative system and meanwhile the divergence of the proposed system is equal \(-1\) i.e.

\[
\nabla v = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} + \frac{\partial \dot{u}}{\partial u} = -1
\]

Which is equal or almost to the sum of the Lyapunov exponents. So, that they fulfill the Lyapunov exponents (4) such as \(\sum \text{LE}_{\mathcal{S}} = \nabla v\).
Multistability phenomena which has multistability the same set of parameters and different combinations of coexisting attractors or Multistability means that

3. Sign of

is one of the important requirements for generating hyperchaotic attractors. But we don’t get

With various values of the system’s parameters and various initial, Tables 1-2 depicts more details about calculating $\text{LE}_S$.

**Table 1.** LE$_S$ of system (2) for $a = 0.02$ with varying the $b$ and $(0.5, 0.5, 0.5, 0.5)^T$.

| No. | $b$   | LE$_S$          | Sum of LE$_S$ | Sign of the LE$_S$ |
|-----|-------|-----------------|---------------|---------------------|
| 1.  | 0.001 | $0.1658, -0.0005, -0.03, -1.1353$ | $-1$          | $(+, -, -, -)$      |
| 2.  | 0.004 | $0.1886, 0.0004, -0.0253, -1.1637$ | $-1$          | $(+, -, -, -)$      |
| 3.  | 0.005 | $0.1621, 0.0004, -0.0246, -1.1379$ | $-1$          | $(+, -, -, -)$      |
| 4.  | 0.0002 | $0.1942, 0.0009, -0.0353, -1.1599$ | $-1.0001$       | $(+, -, -, -)$    |
| 5.  | 0.0006 | $0.1903, 0.0005, -0.0316, -1.1592$ | $-1$          | $(+, -, -, -)$      |
| 6.  | 0.0008 | $0.1469, 0.0005, -0.0307, -1.1168$ | $-1.0001$       | $(+, -, -, -)$    |

**Table 2.** LE$_S$ of system (2) for $b = 0.04$ with varying the $a$ and $(0.7, 0.7, 0.7, 0.7)^T$.

| No. | $a$   | LE$_S$          | Sum of LE$_S$ | Sign of the LE$_S$ |
|-----|-------|-----------------|---------------|---------------------|
| 1.  | 0.006 | $0.1964, 0.0009, -0.0165, -1.1808$ | $-1$          | $(+, -, -, -)$      |
| 2.  | 0.0004 | $0.2103, 0.0003, -0.0165, -1.194$ | $-0.9999$       | $(+, -, -, -)$    |
| 3.  | 0.0006 | $0.1849, -0.0001, -0.0165, -1.1682$ | $-0.9999$       | $(+, -, -, -)$    |
| 4.  | 0.0008 | $0.1954, -0.0007, -0.0165, -1.1782$ | $-1$          | $(+, -, -, -)$      |

It is seen from Tables 1-2, although the proposed system has four dimensions, this extended dimension is one of the important requirements for generating hyperchaotic attractors. But we don’t get $(+, +, -, -)$ sign of LE$_S$ which indicates its the system (2) with chaotic hidden attractors. Also, the sum of the LE$_S$ is almost equal to $(-1)$ , and that matches with divergence.

3. Multistability behavior of the proposed system

Coexisting attractors or Multistability means that the dissipative system has different trajectories under the same set of parameters and different combinations of and initial values [14]. The proposed system has multistability behavior or coexistence of chaotic hidden attractors. All of Figures 3. and 4., exhibit multistability phenomena which include four different hidden attractors with various initial values. With the parameters are set as $a = 8$, $b = 80$ and initial values $[-2.7, 2.7, 2.7, 2.7]^T$ (red) and $[2.7, 2.7, 2.7, 2.7]$ (blue) as shown in Figure 3.a whereas in Figure 3.b with initial values $[-10.8, 2.7, 2.7, 2.7]^T$ (red) and $[-0.1, 2.7, 2.7, 2.7]$ (blue). Under the parameters $a = 5$, $b = 20$, with
initial values with initial values \([-0.5, -0.5, -0.5, -5]^T\) (red) and \([-0.5, -0.5, -0.5, -2]\) (blue) as shown in Figure 4.a whereas the initial values are \([-0.5, -0.5, -0.5, -5]^T\) (red) and \([-0.5, -0.5, -0.5, -2]\) as shown in Figure 4.b.

![Figure 3. Coexistsences of two different attractors for system (2) with \(a = 8, b = 80\) and initial conditions \(y(0) = z(0) = w(0) = 2.7\). (a) Coexistsences attractors with \(x(0) = -2.7\) (red) and \((0) = 2.7\) (blue), (b) Coexistsences attractors with \(x(0) = -10.8\) (red) and \((0) = -0.1\).](image)

![Figure 4. Coexistsences of two different attractors for system (2) with \(a = 5, b = 20\) and initial conditions conditions \(x(0) = y(0) = z(0) = -0.5\). (a) Coexistsences attractors with \(w(0) = -0.5\) (red) and \(w(0) = 0.5\) (blue), (b) Coexistsences attractors with \(w(0) = -5\) (red) and \(w(0) = -2\) (blue).](image)

4. Anti-synchronization between identical new systems

In this section, anti-synchronization is realized between two identical system (2) drive and response systems which described as

\[
\begin{align*}
\dot{x} &= Ax + f(x) \\
\dot{y} &= By + g(y) + U
\end{align*}
\]

In which \(x, y \in \mathbb{R}^4\) are the state variables, \(A, B \in \mathbb{R}^{4 \times 4}\) are matrices of system’s parameters, \(f, g: \mathbb{R}^n \to \mathbb{R}^n\) are nonlinear functions, \(U \in \mathbb{R}^4\) is nonlinear controller. The system (2) can be modeled as
Construct Lyapunov function

\[ V(e_i) = e_i^T P e_i \quad \Rightarrow \quad V(e_i) = \begin{bmatrix} e_1 & e_2 & e_3 & e_4 \end{bmatrix} \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ * & 0.5 & 0 & a \\ * & * & 0.5 & 0 \\ * & * & * & 0.05 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} \]  

The symbol (*) which indicates symmetry in the matrix. The time derivative of the above function \( V(e_i) \) gives

\[ \dot{V}(e_i) = e_1 \dot{e}_1 + e_2 \dot{e}_2 + e_3 \dot{e}_3 + 0.2 e_4 \dot{e}_4 \]

\[ \dot{V}(e) = e_1(-e_1 - 4e_2 - e_3) + e_2(4e_1 - e_2 + ae_4 + e_3^2) + e_3(e_1 - e_3 - e_2e_3) + \frac{1}{10} e_4(-be_2 - e_4) \]
\[ \dot{V}(e_i) = -e_1^2 - e_2^2 - e_3^2 - \frac{1}{10} e_4^2 = -[e_1 \quad e_2 \quad e_3 \quad e_4]^T \left[ \begin{array}{cccc} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & \frac{1}{10} \end{array} \right] \begin{array}{c} e_1 \\ e_2 \\ e_3 \\ e_4 \end{array} = -e_i^T Q e_i \quad (14) \]

where \( Q = \text{diag}(1,1,1,\frac{1}{10}) \), leads to \( Q > 0 \). Consequently, \( \dot{V}(e_i) < 0 \) on \( R^4 \). The controller (11) realize the anti-synchronization.

Now, the initial values as \((2,7,10,-5)\) and \((10,1,3,8)\) to illustrate the anti-synchronization that happened between (7) and (8) numerically. Figure 5. validate our theoretical results.

5. Conclusions
A simple 4D Sprott S system without equilibrium point and coexistence of attractors constructed from 3D Sprott-S system via a state feedback controller. This system consists of eight terms, including one constant, two parameters control, and a single quadratic nonlinear term. Besides, chaotic hidden behaviors have been discovered. Finally, anti-synchronization (AS) between two new identical systems realized via nonlinear control by using linearization and Lyapunov stability tools.

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