MARTINGALE SOLUTION TO A STOCHASTIC CHEMOTAXIS SYSTEM WITH POROUS MEDIUM DIFFUSION

ERIKA HAUSENBLAS, DEBOPRIYA MUKHERJEE, AND ALI ZAKARIA

Abstract. In this paper, we study the classical Keller–Segel system on a two-dimensional domain perturbed by a pair of Wiener processes, where the leading diffusion term is replaced by a porous media term. In particular, we investigate the coupled system

\[
\begin{align*}
\dot{u} &= r_u \Delta |u|^{\gamma-1} u - \chi \text{div}(u \nabla v) + \sigma_u u \circ dW_1, \\
\dot{v} &= r_v \Delta v - \alpha v + \beta u + \sigma_v v \circ dW_2,
\end{align*}
\]

for \( \gamma > 1 \), with initial condition \((u_0, v_0)\) on a filtered probability space \(\mathcal{F}\) and \((W_1, W_2)\) be a pair of time homogeneous spatial Wiener processes over \(\mathcal{F}\). Here \(u\) is the cell density and \(v\) is the concentration of the chemical signal, \(\sigma_u\) and \(\sigma_v\) are positive constants. The positive terms \(r_u\) and \(r_v\) are the diffusivity of the cells and chemotactant, respectively. The positive value \(\chi\) is the chemotactic sensitivity, \(\alpha \geq 0\) is the so-called damping constant and \(\beta \geq 0\) is the production weight corresponding to \(u\).

Since the randomness is intrinsic, interpretation of the stochastic integral in Stratonovich sense is natural. We construct a solution (integral) operator, and establish its continuity and compactness properties in an appropriately chosen Banach space. In this manner, we formulate a stochastic version of the Schauder–Tychonoff Type Fixed Point Theorem which is specific to our problem to obtain a solution. In kind, we achieve the existence of a martingale solution.

Keywords and phrases: Chemotaxis, porous media equation, nonlinear diffusion, the Keller-Segel model, Stochastic Partial Differential Equations, Stochastic Analysis, Mathematical Biology

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1. Introduction

The importance of model organisms with the goal of the uniformity of the mathematical structure and its related biological factors is fundamental. The celebrated Keller–Segel model introduced by Keller and Segel [15], and Patlak [8] illustrate the aggregation of Dictyostelium discoideum in view of pattern formation. For adequate references on the phenomenological analysis of the aforesaid class of models we refer to Horstmann [9, 10], Hillen and Painter [58], Bellomo et al. [49], the works of Biler [50], and of Perthame [6].

Chemotaxis can be defined as the movement or orientation of a population (bacteria, cell or other single or multicellular organisms) induced by a chemical concentration gradient either towards or away from the chemical signals. Chemotaxis is a fundamental and universal phenomenon which includes (but not limited to)

- fertilization and reproduction; see [44, 54] etc.
- bacterial motility; see [53, 65] etc.

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• development of axons in nervous system; see Chapter 8 in [44] and references therein
• molecular mechanisms in amobae; see [45] etc.

In the classical Keller–Segel model for chemotaxis, the chemoattractant is emitted by the cells that react according to biased random walk inducing linear diffusion operators. In view of [69], migration of the cells in porous media and the cell motility is a nonlinear function of the cell density. Hence, there is a genuine need to study the chemotaxis systems with porous medium diffusion. It is well-known that concentration gradients in porous media display a complex topology and are highly variable in terms of magnitude and direction, leading to the occurrence of non-linear effects on bacteria motility.

Finally, Porter et al. [46] developed a multiscale model of chemotaxis in porous media where transport of bacteria is expressed in terms of effective medium parameters. The simplest form of the model is

\[
\begin{align*}
\frac{du(t,x)}{dt} &= \left( r_u \Delta u(t,x) - \chi \text{div}(u(t,x)\nabla v(t,x)) \right) dt, \quad t \geq 0, \ x \in \Omega, \\
\frac{dv(t,x)}{dt} &= \left( r_v \Delta v(t,x) + \beta u(t,x) - \alpha v(t,x) \right) dt, \quad t \geq 0, \ x \in \Omega, \\
(u(0,x), v(0,x)) &= (u_0(x), v_0(x)), \quad x \in \Omega,
\end{align*}
\]

for \( \gamma > 1 \) and \( \Omega \) is a bounded domain in \( \mathbb{R}^d \). Here, \( \xi^{[\gamma]} \) is an abbreviation for \( \xi|\xi|^{\gamma-1} \) for \( \xi \in \mathbb{R} \), and \( \Delta \) denotes the Laplacian with Dirichlet (or Neumann) boundary condition. Furthermore, \( u \) denotes the cell density and \( v \) is the concentration of the chemical signal. The term \( -u \cdot \nabla v \) is the chemotactic flux, which describes the transportation of the amount of the bacteria in the direction of the flow. This gives rise to the evolution of bacterial concentration by diffusion and transport along the flow. This essentially produces \( -\chi \text{div}(u\nabla v) \) illustrating cross diffusive effects into the model, where the positive constant \( \chi \) represents the chemotactic sensitivity. Here, \( \Delta u^{[\gamma]} \) is the migration of the bacteria, for which, the motility depends on the bacterial density and \( \Delta v \) is the diffusion of chemoattractant. The positive terms \( r_u \) and \( r_v \) are the diffusivity of the cells and chemoattractant, respectively. In the signal concentration model, \( \alpha \geq 0 \) is the so–called damping constant and \( \beta \geq 0 \) is the production weight corresponding to \( u \). The leading term is not parabolic at all points, but only degenerate parabolic, a fact that has mathematical consequences, both qualitative and quantitative. For a quick survey, we again refer to [6, 9, 10, 49, 50, 58]. For a brief review on the behaviour of classical Keller–Segel model by a degenerate diffusion of porous medium type in the deterministic framework, we refer the articles of Carrillo et al. [27, 28, 29] and the references cited therein.

The Keller–Segel model is a macroscopic model derived from the limiting behaviour of the microscopic model; see, e.g. [4]. Here, one relies on fundamental balance laws and Fick’s law of diffusion. Consequently, significant aspects of microscopic dynamics such as fluctuations of molecules are disregarded. Hence, in the derivation of the above macroscopic equations, fluctuations around the mean value are neglected. Secondly, in natural systems random disturbances and so-called environmental noise is inevitable and may create together with the nonlinearity a change of the dynamical behaviour. For a more realistic model, it is necessary to consider essential features of the natural environment which are non-reproducible. Hence, the model should include random spatio-temporal forcing.

The randomness leads to a variate of new phenomena and may have highly non–trivial impact on the behaviour of the solution. It should be stressed that adding a stochastic driving term to a partial differential equation can have highly non–trivial impact implication on the behaviour of the solution. The presence of the stochastic term (noise) in the models often leads
to qualitatively new types of behaviour, which is most helpful in understanding the real processes and is also often more realistic. For example, there exist deterministic systems of PDEs, the Navier-Stokes equations, for example, which have non-ergodic invariant measure. However, adding a noise term leads to the existence and uniqueness of an invariant measure, and, hence, an ergodic invariant measure. Besides, researcher from applied science investigate in biological systems disturbed by some noise. In Karig et al. [11], the authors explore whether the stochastic extension leads to a broader range of parameter with Turing patterns by a genetically engineered synthetic bacterial population in which the signalling molecules form a stochastic activator–inhibitor system. Kolinichenko and Ryashko [3], respectively, Bashkirtseva et al. [2] addresses multistability and noise-induced transitions between different states.

In this article, we perturb the density of cells $u$ and concentration of chemoattractant $v$ by time-homogeneous spatial Wiener processes. Due to the non-linearities in the system, one fails to use semigroup approach for the equation perturbing cell density. Thus the standard methods to show existence and uniqueness of solutions cannot be applied here. Here we formulate a stochastic version of Schauder-Tychonoff type Fixed Point Theorem which is specific to our problem to obtain a solution on $[0,T]$ for $T > 0$. In this manner, we achieve only the existence of a martingale solution, but not the uniqueness of the solution.

Solvability and boundedness to the chemotaxis model with porous medium diffusion in deterministic set-up has gained much interest in recent years (see for instance [16, 22, 31], and the references therein). The authors are not aware of any work which treats the stochastic modelling of the coupled Keller–Segel model with porous medium diffusion. Recently, the authors in [12, 13] have treated the simple stochastic Keller–Segel model in one and two dimensions. In these works, the authors rely on the entities controlled by a Lyapunov functional. We refer the very recent work [60], where the deterministic Keller–Segel model is coupled with the stochastic Navier–Stokes equations. The stochastic porous media equation itself is the topic of the recent book of Barbu, Da Prato and Röckner, [62], where we used several results for the stochastic porous media equation. In the article of Dareiotis, Gerencser, and Gess [41] the solvability of the stochastic porous media equation in dimension one with space time white noise is shown. In the article of Dareiotis, Gess, and Tsatsoulis [40] the long time behaviour of the stochastic porous media equation is treated.

**Notation 1.1.** For a Banach space $E$ and $0 \leq a < b < \infty$, let $C_b^\xi([a,b];E)$ denote a set of all continuous and bounded functions $u : [a,b] \to E$ such that

$$
\|u\|_{C_b^\xi([a,b];E)} := \sup_{a \leq t \leq b} |u(t)|_E + \sup_{a \leq s,t \leq b, t \neq s} \frac{|u(t) - u(s)|_E}{|t - s|^{\zeta}}
$$

is finite. The space $C_b^\xi([a,b];E)$ endowed with the norm $\| \cdot \|_{C_b^\xi([a,b];E)}$ is a Banach space.

**Notation 1.2.** Let $1 \leq p < \infty$ and $s \in \mathbb{R}$, then $H^s_p$ denotes the Bessel Potential space or Sobolev space of fractional order defined by

$$
H^s_p(\mathbb{R}^d) := \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : |f|_{H^s_p} := \left| \langle (1 + \xi^2)^{\frac{s}{2}} \hat{f} \rangle \right|_{L^p} < \infty \right\}.
$$

Here, we denote the Fourier transform of a function $f$ by $\hat{f}$ and its inverse by $f^\vee$. Presently, $H^s_p(\mathcal{O})$ is the restriction of $H^s_p(\mathbb{R}^d)$ to $\mathcal{O}$, with

$$
\|f\|_{H^s_p(\mathcal{O})} = \inf_{g|_{\mathcal{O}} = f} \|g\|_{H^s_p(\mathbb{R}^d)}.
$$
Here, \( g|_O \in \mathcal{D}'(O) \) denotes the restriction of \( g \in \mathcal{D}'(\mathbb{R}^d) \) to \( O \) in the sense of the theory of distributions.

2. Problem description and main result

In this section we introduce the definition of martingale solution to the stochastic system and present our main result. Before, we formulate the necessary assumptions on the noise and the initial conditions.

Let \( \mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}) \) be a complete probability space and filtration \( \mathbb{F} = (\mathcal{F}_t)_{t \geq 0} \) satisfying the usual conditions i.e.,

(i) \( \mathbb{P} \) is complete on \( (\Omega, \mathcal{F}) \),
(ii) for each \( t \geq 0 \), \( \mathcal{F}_t \) contains all \( (\mathcal{F}, \mathbb{P}) \)-null sets,
(iii) and the filtration \( \mathbb{F} \) is right-continuous.

Let \( O \subset \mathbb{R}^2 \) be a bounded domain, with smooth boundary (or the rectangle \( O = [0,1]^2 \)). Let \( H_1 \) and \( H_2 \) be two Hilbert spaces, and \( W_j, j = 1,2 \), be two cylindrical Wiener processes defined on \( H_1 \) and \( H_2 \), respectively. In this paper, we consider the following system of equations

\[
\begin{aligned}
&du(t) = \left( r_u \Delta u(t) - \chi \text{div}(u(t)\nabla v(t)) \right) dt + \sigma_u u(t) \circ dW_1(t) \\
&dv(t) = \left( r_v \Delta v(t) + \beta u(t) - \alpha v(t) \right) dt + \sigma_v v(t) \circ dW_2(t), \quad t \in \mathbb{R}_0^+,
\end{aligned}
\]

where \( \sigma_u \) and \( \sigma_v \) are positive constants and the positive terms \( r_u \) and \( r_v \) are the diffusivity of the cells and chemoattractant, respectively. Also, \( \alpha \geq 0 \) is the so-called damping constant and \( \beta \geq 0 \) is the production weight corresponding to \( u \). Here \( A = \Delta \) denotes the Laplace operator with Neuman boundary conditions, i.e. \( D(A) := \{ u \in H^2(O) : \frac{\partial u}{\partial n}(x) = 0 \text{ on } \partial O \} \), where \( n \) denotes the (typically exterior) normal to the boundary \( \partial O \). Since the randomness is intrinsic, interpretation of the stochastic integral in Stratonovich sense is natural. For a detailed explanation of the Stratonovich integral, we refer to the book by Duan and Wang [32] or to the original work of Stratonovich [33]. To show the existence of the solution, the Wiener perturbation have to satisfy regularity assumptions. Both processes \( W_1 \) and \( W_2 \) are cylindrical Wiener processes over real-valued Hilbert spaces \( H_1 \) and \( H_2 \), respectively. Then, due to the spectral representation of Wiener processes, \( W_1 \) and \( W_2 \) can be written formally by the sum (possibly infinite)

\[
W_1(t,x) = \sum_{k \in \mathbb{I}_1} \psi_k^{(1)}(x) \beta_k^{(1)}(t) \quad \text{and} \quad W_2(t,x) = \sum_{k \in \mathbb{I}_2} \psi_k^{(2)}(x) \beta_k^{(2)}(t),
\]

where \( \{ \psi_k^{(1)} : k \in \mathbb{I}_1 \} \) and \( \{ \psi_k^{(2)} : k \in \mathbb{I}_2 \} \) are some orthonormal basis in \( H_1 \) and \( H_2 \), \( \mathbb{I}_1 \) and \( \mathbb{I}_2 \) are the corresponding index sets, \( \{ \beta_k^{(1)} : k \in \mathbb{I}_1 \} \) and \( \{ \beta_k^{(2)} : k \in \mathbb{I}_2 \} \) are two mutually independent standard Brownian motions over \( \mathfrak{A} \). In order to get the existence of a solution, we introduce the following hypothesis.

**Assumption 2.1.** Let us assume that the embeddings \( H_1 \hookrightarrow L^\infty(O) \) and \( H_2 \hookrightarrow L^3(O) \) are \( \gamma \)-radonifying.

**Assumption 2.2.** Let \( H_1 \) and \( H_2 \) be isomorphic to Bessel potential spaces. For example, \( H_1 = H^{\delta_1}(O) \) for \( \delta_1 > 1 \), and \( H_2 = H^{\delta_2}(O) \) for \( \delta_2 > 1 \). Let us assume that there exist \( \delta_i > 1; i = 1,2 \) such that the embeddings \( H_2^{\delta_i}(O) \) in \( H_1^{\delta_i} \) is a Hilbert-Schmidt for \( i = 1,2 \).
Remark 2.3. Let \( \{ \phi_k : k \in \mathbb{N} \} \) be the eigenfunctions of \((A, D(A))\) with the corresponding eigenvalues \( \{ \nu_k : k \in \mathbb{N} \} \). Then, \( \{ \phi_{\delta,k} : k \in \mathbb{N} \} \) with \( \phi_{\delta,k} := \nu_k^{-1/2} \phi_k \) is an orthonormal basis in \( H^3_2(O) \) for \( i = 1, 2 \). Then, we can write for \( W_1 \) and \( W_2 \) as the following sum

\[
W_1(t,x) = \sum_{k \in \mathbb{N}} \psi_k^{(1)}(x) \beta_k^{(1)}(t) \quad \text{and} \quad W_2(t,x) = \sum_{k \in \mathbb{N}} \psi_k^{(2)}(x) \beta_k^{(2)}(t).
\]

Example 2.4. In the case of a single dimension, a complete orthonormal system of the underlying Lebesgue space \( L^2([0,2\pi]) \) is given by sine and cosine functions

\[
\theta_m(x) = \begin{cases} 
\sqrt{2} \sin(mx) & \text{if } m \geq 1, \\
\sqrt{2} & \text{if } m = 0, \\
\sqrt{2} \cos(mx) & \text{if } m \leq -1.
\end{cases}
\]

The extension to \([0,1]^2\) relies on tensor products, i.e., for a multiindex \( m = (m_1, m_2) \in \mathbb{Z}^2 \) we have

\[
\Theta_m(x) = \theta_{m_1}(x_1) \theta_{m_2}(x_2), \quad x = (x_1, x_2) \in O.
\]

The corresponding eigenvalues are given by

\[
\lambda_m = -4\pi^2(m_1^2 + m_2^2), \quad m = (m_1, m_2) \in \mathbb{Z}^2.
\]

Example 2.5. Let \( O = \mathcal{D} := \{ x \in \mathbb{R}^d : |x| \leq 1 \} \) be the unit disk, then for a multiindex \( m = (m_1, m_2) \in \mathbb{N} \times \mathbb{Z} \), we have as eigenfunction

\[
\psi(x_1, x_2) = J_{m_1}(\kappa_{m_1,m_2} r) \theta_{m_2}(\theta),
\]

where \( r = \sqrt{x_1^2 + x_2^2} \), \( \theta = \tan(x_1/x_2) \), \( J_{m_1} \) is the \( m_1 \)th Bessel function, \( \kappa_{m_1,m_2} \) is the \( m_2 \)th root of \( J_{m_1} \). The eigenvalues \( \{ \lambda_m : m \in \mathbb{N} \times \mathbb{Z} \} \) are given by \( \lambda_m = \kappa_{m_1,m_2}^2, m \in \mathbb{N} \times \mathbb{Z} \). Note, the \( n \)th maxima of the \( j \)th Bessel function is for \( n,j \to \infty \) approximately at \( n \), hence \( \lambda_m \sim m_2^2 \).

Since the cell density, \( u \), and the concentration of the chemical signal, \( v \), of the chemotaxis system have to be non-negative, the initial conditions \( u_0 \) and \( v_0 \) have to be non-negative as well. Besides, we have to impose some regularity assumptions to get the existence of a solution.

Assumption 2.6. Let \( u_0 \in H^{-1}_2(O) \) and \( v_0 \in L^2(O) \) be two random variables over \( \mathfrak{A} \) such that

a.) \( u_0 \geq 0 \) and \( v_0 \geq 0 \);

b.) \( (u_0, v_0) \) is \( \mathcal{F}_0 \)-measurable;

c.) \( \mathbb{E}|u_0|_{L^{r+1}}^{r+1} \) and \( \mathbb{E}v_0|_{L^4}^4 < \infty \).

As mentioned before, in the proof for the existence of the solution, we are using compactness arguments, which causes the loss of the original probability space. This essentially means the solution will only be a weak solution in the probabilistic sense. Thereupon, we have to construct another probability space and obtain a martingale solution.

Definition 2.7. A martingale solution to the problem (2.1) is a system

\[
(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, (W_1, W_2), (u, v))
\]

such that

- \( \mathfrak{A} := (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}) \) is a complete filtered probability space with a filtration \( \mathbb{F} = \{ \mathcal{F}_t : t \in [0, T] \} \) satisfying the usual conditions,

- \( W_1 \) and \( W_2 \) are \( H^1 \)-valued, respectively \( H^2 \)-valued Wiener processes over the probability space \( \mathfrak{A} \) with covariance \( Q_1 \) and \( Q_2 \).
Theorem 2.8. Let Assumption 2.1 be satisfied. Then, for all initial conditions \((u_0, v_0) \in H_2^{-1}(\Omega) \times L^4(\Omega)\) satisfying Assumption 2.6 and for all \(T > 0\), there exists a martingale solution \((\mathfrak{A}, (W_1, W_2), (u, v))\) to the system (2.1) satisfying the following properties:

(i) \(\mathbb{P} \otimes \text{Leb}-\text{a.s.}\) \(u(x, t) \geq 0\) and \(v(x, t) \geq 0\);

(ii) there exists a positive constant \(C_1 = C_1(T, u_0, v_0)\) such that

\[
\mathbb{E}\left\{ \sup_{0 \leq s \leq T} |u(s)|_{H^{-1}}^{\gamma + 1} + \frac{\gamma}{2} (\gamma + 1) \int_0^T |u(t)\nabla u(t)|_{L^2}^2 \, ds \right\} \leq C_1;
\]

(iii) there exists a constant \(C_2 = C_2(T, u_0, v_0) > 0\) such that

\[
\mathbb{E}\left[ \sup_{0 \leq s \leq T} |u(s)|_{H^{-1}}^{\gamma + 1} + \int_0^T |u(t)|_{L^{\gamma+1}}^{\gamma+1} \, ds \right] \leq C_2;
\]

(iv) there exist constants \(C_3 = C_3(T, u_0, v_0)\) and \(C_4 = C_4(T, u_0, v_0) > 0\) such that

\[
\mathbb{E}\|v\|_{L^{\gamma+1}(0,T;H_2^1)}^4 \leq \mathbb{E}\|v_0\|_{L^4}^4 + C_3 \quad \text{and} \quad \mathbb{E}\|v\|_{C([0,T];L^4)}^4 \leq \mathbb{E}\|v_0\|_{L^4}^4 + C_4.
\]

In Section 3, we will show the existence of a martingale solution to the system (2.1). In Section 4 we present several auxiliary propositions which are essential for the existence theory.

3. Existence of a martingale solution to the system (2.1)

As mentioned in the introduction, we will show first the existence of a martingale solution, which is given below. Before starting with the actual proof, we first recall the drawback of the Stratonovich stochastic integral that this integral is not a martingale and the Burkholder–Davis–Gundy inequality does not hold here. Hence, it is convenient to work the equation in Itô form, which is presented below.

\[
\begin{align*}
(3.1) \quad du(t) &= \left( r_u \Delta u^{[\beta]}(t) - \chi \text{div}(u(t) \nabla v(t) - \mu_1 u(t)) \right) dt + \sigma_u u(t) dW_1(t) \\
& \quad + dv(t) = \left( r_v \Delta v(t) + \beta u(t) - (\alpha + \mu_2) v(t) \right) dt + \sigma_v v(t) dW_2(t), \quad t \in \mathbb{R}_+^+.
\end{align*}
\]

Adding the correction term \(\mu_1\) and \(\mu_2\), we see that system (3.1) is equivalent to (2.1), where the stochastic integral is interpreted as the Stratonovich integral. The advantage of the Itô stochastic integral is, that the Itô integral is a local martingale and we can apply the Burkholder–Davis–Gundy inequality. For detailed discussion about the correction term and the conversion between the Itô and Stratonovich forms, we refer the readers to Section 2.1 of [12]. For simplicity we renamed \(\alpha\); in particular, \(\alpha\) corresponds to \(\alpha + \mu_2\) and \(\mu = \mu_1\). In this way we end up with the following system

\[
\begin{align*}
(3.2) \quad du(t) &= \left( r_u \Delta u^{[\beta]}(t) - \chi \text{div}(u(t) \nabla v(t) + \mu u(t)) \right) dt + \sigma_u u(t) dW_1(t) \\
& \quad + dv(t) = \left( r_v \Delta v(t) + \beta u(t) - \alpha v(t) \right) dt + \sigma_v v(t) dW_2(t), \quad t \in \mathbb{R}_0^+.
\end{align*}
\]

Hence, due to the drawbacks of the Stratonovich integral, in the proof, we will show the existence of a solution to system (3.2). However, by the correction term, system (3.2) is equivalent to system (2.1).
Proof of Theorem 2.8. In this proof, we will show the existence of a solution (in the sense of Definition 2.7) to the system (3.2). The proof is split into three main steps. First, we construct an integral operator on an appropriate space and show its compactness. Then, we formulate a stochastic version of the Schauder–Tychonoff Fixed Point Theorem to obtain a solution. In this manner, we achieve only the existence of a martingale solution.

Step (I) Definitions of the underlying spaces: Let us consider
\[ X_\mathcal{A} = \left\{ \xi : [0, T] \times \Omega \to H_2^{-1}(\mathcal{O}) \text{ be progressively measurable over } \mathcal{A} \right\}, \]
(3.3)
equipped with the norm
\[ \|\xi\|_{X_\mathcal{A}} = \left\{ \mathbb{E} \sup_{0 \leq s \leq T} |\xi(s)|_{H_2^{-1}}^2 < \infty \right\}^{\frac{1}{2}}. \]

For \( \sigma \geq 1 \) and for a Banach space \( E \), let us define the collection of processes
\[ M_\sigma^E(0, T; E) = \left\{ \xi : [0, T] \times \Omega \to E \text{ be a progressively measurable process over } \mathcal{A}, \right. \]
(3.4)
\[ \text{such that } \mathbb{E} \left[ \int_0^T |\xi(s)|_E^\sigma \, ds \right] < \infty \}
equipped with the norm
\[ \|\xi\|_{M_\sigma^E(0, T; E)} := \left( \mathbb{E} \left[ \int_0^T |\xi(s)|_E^\sigma \, ds \right] \right)^{\frac{1}{\sigma}}. \]

Finally, for \( R_1, R_2 > 0 \) and fixed \( \gamma \), let us define the following subspace
\[ X_\mathcal{A}(R_1, R_2) := \left\{ \xi \in X_\mathcal{A} : \xi \geq 0 \text{ } \mathbb{P}-\text{Leb-a.s.} \right\}, \]
\[ \mathbb{E} \left( \sup_{0 \leq s \leq T} |\xi(s)|_{L^{\gamma+1}} + \frac{r_u}{2} (\gamma + 1) \gamma \int_0^T |\xi(s)|_{L^2} |\nabla \xi(s)|_{L^2} \, ds \right) \leq R_1, \]
\[ \mathbb{E} \left( \sup_{0 \leq s \leq T} |\xi(s)|_{H_2^{-1}}^2 + \int_0^T |\xi(s)|_{L^{\gamma+1}} \, ds \right) \leq R_2 \} \right\}. \]

Step (II) Definition of the operator:
Let us define the operator \( \mathcal{T} \) acting on \( X_\mathcal{A} \) as follows. For \( \eta \in X_\mathcal{A} \), let \( \mathcal{T}\eta := u \), where \((u, v)\) solves the following system
\[ du(t) = \left( r_u \Delta u^{[\gamma]}(t) - \chi \text{div}(\eta(t) \nabla v(t)) + \mu u(t) \right) \, dt + \sigma_u u(t) \, dW_1(t), \]
(3.5)
and
\[ dv(t) = \left( r_v \Delta v(t) + \beta \eta(t) - \alpha v(t) \right) \, dt + \sigma_v v(t) \, dW_2(t), \quad t \in [0, T]. \]
(3.6)
First, note that due to Proposition 4.2, the operator is well define on \( X_\mathcal{A}(R_1, R_2) \subset X_\mathcal{A} \).
In particular, for any \( \eta \in X_\mathbb{A}(R_1, R_2) \), there exist processes \( u \) and \( v \) solving system (3.3)–(3.6) such that \( u \in X_\mathbb{A} \) and \( \mathbb{P}\) a.s. \( v \in L^4(0, T; H^2_\mathcal{O}) \cap C([0, T]; L^4(\mathcal{O})) \). Due to Proposition 4.4, there exist \( R_1 > 0 \) and \( R_2 > 0 \) such that \( T \) maps \( X_\mathbb{A}(R_1, R_2) \) into itself. Proposition 4.5 gives the continuity of the operator \( T \) from \( X_\mathbb{A}(R_1, R_2) \) into \( X_\mathbb{A} \) and by Proposition 4.6 we know that the operator \( T \) maps \( X_\mathbb{A}(R_1, R_2) \) to a precompact set.

**Step (III) Application of the Schauder–Tychonoff type Theorem:** This Step coincide literally with Step III in our earlier paper [12]. Each process \( \eta \in X_\mathbb{A}(R_1, R_2) \) is assigned a value \( \mathcal{T}\eta := u \) and another value \( \mathcal{R}\eta := v \) where \( (u, v) \) is the unique solution to system (3.2). In this way, we formulate Schauder-Tychonoff-Type Fixed Point Theorem and show that there exists an \( u^* \in X_\mathbb{A}(R_1, R_2) \) and a corresponding \( v^* \) solving equation (3.6) with \( \eta = u^* \), and the pair \( (u^*, v^*) \) is a solution to system (3.2).

\[ \Box \]

## 4. Auxiliary Propositions specifying the properties of system (3.5)–(3.6)

We shortly give the exact form of the Burkholder–Davis–Gundy inequality which will be useful during the course of analysis. Given a Wiener process \( W \) being cylindrical on \( \mathcal{H} \) over \( \mathbb{A} \), and a progressively measurable process \( \xi \in M^2_\mathbb{A}(0, T; \mathcal{L}(\mathcal{H}, E)) \), let us define \( \{Y(t) : t \in [0, T]\} \) by

\[ Y(t) := \int_0^t \xi(s) \, dW(s), \quad t \in [0, T]. \]

Here, \( \xi = m(u) \), where \( u \) is function-valued stochastic process and for each \( t \in [0, T] \) \( \xi(t) = m(u(t)) \) is interpreted as a multiplication operator acting on the elements of \( \mathcal{H} \), namely, \( m(u): \mathcal{H} \ni \psi \mapsto u\psi \in \mathcal{S}'(\mathbb{R}) \). Taking the representation of \( W \) by its sum

\[ W(t) = \sum_{k \in \mathbb{I}} \psi_k \beta_k(t) \]

then, in case \( E \) is a Hilbert space, the Hilbert-Schmidt norm of \( m(u) \) is given by

\[ |m(u)|_{LHS(\mathcal{H}, E)} := \left( \sum_{k \in \mathbb{I}} |u \psi_k|^2_E \right)^{\frac{1}{2}}. \]

Consequently, for any \( p \geq 1 \), we get for any progressively measurable process \( \xi \in M^2_\mathbb{A}(0, T; L_{HS}(\mathcal{H}, E)) \),

\[ \mathbb{E} \left[ \sup_{t \in [0, T]} |Y(t)|^p_E \right] \leq C_p \mathbb{E} \left[ \int_0^T |\xi(t)|^2_{L_{HS}(\mathcal{H}, E)} \, dt \right]^{\frac{p}{2}}. \tag{4.1} \]

Let \( E = H_{-1}^2(\mathcal{O}) \), then we have for all \( \delta > 0 \) (see [59, Theorem 3, p. 179])

\[ |m(u)|^2_{L_{HS}(\mathcal{H}, E)} \leq \sum_{k \in \mathbb{I}} |u \psi_k|^2_{H_{-1}^2} \leq |u|^2_{H_2^2} \sum_{k \in \mathbb{I}} |\psi_k|^2_{L^2}. \tag{4.2} \]

By the interpolation and the Young inequalities we infer that for all \( \varepsilon > 0 \), there exists a constant \( C = C(\varepsilon) > 0 \) such that

\[ |m(u)|^2_{L_{HS}(\mathcal{H}, E)} \leq \varepsilon |u|^2_{H_2^2} + C(\varepsilon)|u|^2_{H_{-1}^2} \sum_{k \in \mathbb{I}} |\psi_k|^2_{H_2^4}. \tag{4.3} \]
For $E = L^2(\mathcal{O})$ we know for any $\delta > 2$, $q$ with $\frac{1}{q} + \frac{1}{\delta} \geq \frac{1}{2}$,
\[
|m(u)|^2_{L^{H\delta}(\mathcal{H},E)} \leq \sum_{k \in I} |u\psi_k|^2_{L^2} \leq \sum_{k \in I} |u|^2_{L^q} |\psi_k|^2_{L^\delta}.
\]
Again, by the interpolation and the Young inequalities we declare that for all $\varepsilon > 0$, there exists a constant $C = C(\varepsilon) > 0$ such that
\[
(4.4) \quad |m(u)|^2_{L^{H\delta}(\mathcal{H},E)} \leq \varepsilon |u|^2_{H^1_2} + C(\varepsilon) |u|^2_{H^{+1}_4} \sum_{k \in I} |\psi_k|^2_{L^\delta}.
\]
In case, $E$ is a Banach space, the Banach space $\gamma(\mathcal{H}, E)$ is defined as the completion of $\mathcal{H} \otimes E$ with respect to the norm
\[
(4.5) \quad \left( \left| \sum_{k=1}^{N} \tilde{\psi}_k \otimes x_k \right|^2 \right)^{\frac{1}{2}} := \left( \left| \mathbb{E} \sum_{k=1}^{N} r_k x_k \right|^2 \right)^{\frac{1}{2}},
\]
for all finite sequences $x_1, \ldots, x_N \in E$ and $\{\tilde{\psi}_k : k = 1, \ldots, N\}$ are assumed to be orthonormal in $\mathcal{H}$. Here $(r_k)_{k \geq 1}$ is a Rademacher sequence; for more resources we cite [38]. Throughout the paper, let us denote the norm introduced in (4.5) by $\gamma(\mathcal{H}, E)$. Similar to (4.4), for any $p \geq 1$ we get for any progressively measurable process $\xi \in M^2_p(0, T; \gamma(\mathcal{H}, E))$,
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} |Y(t)|^p_E \right] \leq C_p \mathbb{E} \left[ \int_0^T |\xi(t)|^2_{\gamma(\mathcal{H}, E)} dt \right]^\frac{p}{2}.
\]
For further details we refer to the survey on $\gamma$–radonifying operator [38]. Let $E = L^4(\mathcal{O})$ and $v \in H_{-1}^4(\mathcal{O}) \cap H_{1}^4(\mathcal{O})$ then we get by the Hölder inequality and the Sobolev embedding
\[
|u|^2_{H_{-1}^4} \sum_{k \in I} |\psi_k|^2_{L^5} \leq \varepsilon |u|^2_{H_{1}^4} + C |u|^2_{H_{-1}^4} \sum_{k \in I} |\psi_k|^2_{L^5}.
\]
Let $E = L^{\gamma+1}(\mathcal{O})$. Then we get by similar calculations as above for any $0 < \delta < \frac{1}{\gamma}$
\[
|u|^2_{H_{-1}^4} \sum_{k \in I} |\psi_k|^2_{L^5} \leq \varepsilon |u|^2_{H_{1}^4} + C |u|^2_{H_{-1}^4} \sum_{k \in I} |\psi_k|^2_{L^5}.
\]
Let $E = H_{\gamma+1}^{2+3}(\mathcal{O})$ and $u \in H_{\gamma+1}^{2+3}(\mathcal{O}) \cap H_{\gamma+1}^{2-1}(\mathcal{O})$ then we get
\[
|u|^2_{H_{\gamma+1}^{2+3}} \sum_{k \in I} |\psi_k|^2_{H_{\gamma+1}^{2-1}} \leq \varepsilon |u|^2_{H_{\gamma+1}^{2+3}} + C |u|^2_{H_{\gamma+1}^{2-1}} \sum_{k \in I} |\psi_k|^2_{H_{\gamma+1}^{2-1}}.
\]
and finally for $E = L^{\gamma+1}(\mathcal{O})$ we achieve in an alternate fashion
\[
|u|^2_{L^{\gamma+1}(\mathcal{O})} \leq \sum_{k \in I} |\psi_k|^2_{E} \leq C |u|^2_{L^{\gamma+1}} \sum_{k \in I} |\psi_k|^2_{H_{\gamma+1}^{2-1}} \leq C |u|^2_{L^{\gamma+1}} \sum_{k \in I} |\psi_k|^2_{H_{\gamma+1}^{2-1}}.
\]
At once, we compare Lemma 3.5 and Lemma 3.6 in [42] and state the following technical Proposition in our settings.
Technical Proposition 4.1. Let $\gamma > 1$. Then, for any $\theta$ with $0 < \theta < \frac{1}{\gamma}$, there exists a constant $C > 0$ such that for all $w$ with $|w|^\gamma \in H_2^2(\mathcal{O})$,

$$C|w|^{\gamma}_{H_2^2} \leq ||w|^\gamma|_{H_2^1}.$$  

Furthermore, for any $\eta$ with $\int_0^T |\eta^{\gamma-1}(s)\nabla \eta(s)|^2_{L^2} ds < \infty$, there holds

$$\|\eta\|_{L^2(0,T; H_2^2)}^{2\gamma} \leq \int_0^T |\eta^{\gamma-1}(s)\nabla \eta(s)|^2_{L^2} ds.$$  

Proof. We know by Runst and Sickel, [59, p. 365] that for any $p \in (1, \infty)$, $s \in (0,1)$, $\varepsilon \in (0, \infty)$, and $\mu \in (0,1)$

$$||w|^\mu|^{s-\varepsilon}_{H_p^{\mu-s} \leq C||w|^\mu|_{F_p^{\mu-s}} \leq C|w|^\mu|_{F_p^{\mu-s}} = C|w|^\mu|_{H_p^{\mu}}, \ w \in H_2^2(\mathcal{O}).$$

Reformulating, we have that for any $p \in (1, \infty)$, $\mu, s \in (0,1)$ and $\varepsilon \in (0, \infty)$ there exists a constant $C > 0$ such that

(4.9) $$||w|^\mu|^{s-\varepsilon}_{H_p^{\mu-s}} \leq C||w|^\mu|_{H_p^{\mu}}, \ w \in H_2^2(\mathcal{O}).$$

From (4.9) we know that for any $\gamma > 1$, $\theta \in (0, \frac{1}{\gamma})$, $\varepsilon > 0$, and $p > \gamma$, there exists a constant $C > 0$ such that

$$|w|^\gamma|_{H_p^\theta} \leq C||w|^\gamma|_{H_2^1}.$$  

In particular, for any $\theta < \frac{1}{\gamma}$ and $p = 2\gamma$, there exists a constant $C > 0$ such that for all $w$ with $|w|^\gamma \in H_2^1(\mathcal{O})$,

$$C|w|^{\gamma}_{H_2^2} \leq ||w|^\gamma|_{H_2^1}.$$  

This proves the first part of the proposition. Again, since we know that

$$\int_0^T |\eta^{\gamma-1}(s)\nabla \eta(s)|^2_{L^2} ds = \int_0^T |\nabla \eta^{\gamma}(s)|^2_{L^2} ds \sim \int_0^T |\eta^{\gamma}(s)|^2_{H_2^1} ds \leq R_2 < \infty,$$

hence, for any $0 < \theta < \frac{1}{\gamma}$ and $p = 2\gamma$

$$\int_0^T |\eta^{\gamma}(s)|^2_{H_2^\theta} ds \leq C \int_0^T |\eta^{\gamma-1}(s)\nabla \eta(s)|^2_{L^2} ds.$$  

This finishes the proof of the proposition. \qed

With these estimates at hand, we proceed to the next proposition to declare that the mapping $\mathcal{T}$ on $\mathbb{X}_2$ is well defined.

Proposition 4.2. For any $R_1, R_2 > 0$ and $\eta \in \mathbb{X}_2(R_1, R_2)$, there exists a unique pair $(u, v)$ of solutions to the systems (3.5) - (3.6) such that the following holds:

$$\mathbb{E}\left[\sup_{0 \leq s \leq T} |u(s)|_{H_2^2}^2 + \int_0^T |u(s)|_{L^2}^\gamma ds\right] < \infty.$$  

There exist two constants $C_1, C_2 > 0$ such that

$$\mathbb{E}[v|_{L^4(0,T; H_4^1)}^4 \leq C\mathbb{E}[|\eta||_{L^2(0,T; H_2^1)}^2].$$
and
\[ E\|v\|_{C([0,T];H^4_{\gamma+1})}^4 \leq E|v_0|^4_{H^4_{\gamma+1}} + C E\|\eta\|_{L^2(0,T;H^4_{\gamma+1})}^2. \]

If, in addition, \( E|v_0|_{H^4_{\gamma+1}}^{\gamma+1} < \infty \), then there exist two constants \( C_1, C_2 > 0 \) such that
\[ (4.10) \quad E\left[ \sup_{0\leq s \leq T} \|v(s)\|_{H^4_{\gamma+1}}^{\gamma+1} \right] \leq E|v_0|_{H^4_{\gamma+1}}^{\gamma+1} + C_1 E\|\eta\|_{L^2(0,T;H^4_{\gamma+1})}^{\gamma+1}, \]
and
\[ (4.11) \quad E\|v\|_{L^2(0,T;H^4_{\gamma+1})}^{\gamma+1} \leq E|v_0|_{H^4_{\gamma+1}}^{\gamma+1} + C_2 E\|\eta\|_{L^2(0,T;H^4_{\gamma+1})}^{\gamma+1}. \]

Remark 4.3. If \( \gamma \geq 3 \), then \( \frac{\gamma+1}{\gamma-1} \leq 2 \) and we obtain
\[ E\|v\|_{L^2(0,T;H^4_{\gamma+1})}^{\gamma+1} \leq C \left( E\|v\|_{L^2(0,T;H^4_{\gamma+1})}^2 \right)^{\frac{\gamma+1}{\gamma-1}} \leq C \left( E\|v\|_{L^2(0,T;H^4_{\gamma+1})}^{\gamma+1} \right)^{\frac{1}{\gamma-1}}. \]

Proof. The proof consists of two steps. First in Claim 4.1 we will investigate equation (3.6) and show that there exists a unique solution \( v \) to (3.6) and specify the integrability and regularity properties of \( v \). Then as a second step in Claim 4.2 we will give the existence of the unique solution \( u \) to (3.5) and show that \( u \in M^2_{\gamma+1}(0,T;L^2(T;\mathcal{O})). \)

Claim 4.1. Let us consider the equation
\[ dv(t) = \left( r_v \Delta v(t) + \beta u(t) - \alpha v(t) \right) dt + \sigma_v v(t)dW_2(t), \quad t \in \mathbb{R}_0^+, \]
with initial condition \( v(0) = v_0 \), where \( v_0 \) is a \( \mathcal{F}_0 \)-measurable random data.

a) If \( E|v_0|_{L^4}^2 < \infty \) and \( \eta \in M^2_{\gamma}(0,T;H^{-1}_{4}(\mathcal{O})) \), then a unique solution \( v \) to (4.12) exists and there exists a constant \( C > 0 \) such that
\[ E\|v\|_{L^2(0,T;H^4_{\gamma})} \leq E|v_0|_{L^2}^2 + C E\|\eta\|_{L^2(0,T;H^{-1}_{4})}^2, \]
and
\[ E\|v\|_{C([0,T];L^2)}^2 \leq E|v_0|_{L^2}^2 + C E\|\eta\|_{L^2(0,T;H^{-1}_{4})}^2. \]

b) If \( E|v_0|_{L^4}^2 < \infty \) and \( \eta \in M^2_{\gamma}(0,T;H^{-1}_{4}(\mathcal{O})) \), then a unique solution \( v \) to (4.12) exists and there exists a constant \( C > 0 \) such that
\[ E\|v\|_{L^4(0,T;H^4_{\gamma})} \leq E|v_0|_{L^4}^2 + C E\|\eta\|_{L^2(0,T;H^{-1}_{4})}^2, \]
and
\[ E\|v\|_{C([0,T];L^4)}^2 \leq E|v_0|_{L^4}^2 + C E\|\eta\|_{L^2(0,T;H^{-1}_{4})}^2. \]

c) If \( E|v_0|_{H^4_{\gamma+1}}^{\gamma+1} < \infty \) and \( \eta \in M^2_{\gamma}(0,T;H^{-1}_{\gamma+1}(\mathcal{O})) \), then a unique solution \( v \) to (4.12) exists and there exist constants \( C_1, C_2 > 0 \) such that
\[ E\left[ \sup_{0\leq s \leq T} |v(s)|_{H^4_{\gamma+1}}^{\gamma+1} \right] \leq E|v_0|_{H^4_{\gamma+1}}^{\gamma+1} + C_1 E\|\eta\|_{L^2(0,T;H^{-1}_{\gamma+1})}^{\gamma+1}. \]
and
\[ \mathbb{E}\|v\|^{\gamma+1}_{L^2(0,T;H^{2}_{\gamma+1})} \leq \mathbb{E}\|v_0\|^{\gamma+1}_{H^{2}_{\gamma+1}} + C_2 \mathbb{E}\|\eta\|^{\gamma+1}_{L^2(0,T;H^{2}_{\gamma+1})}. \]

**Proof of Claim 4.1.** By Example 3.2 (1)-(4) [35] we know that the Laplace operator with Neumann boundary condition has a bounded \( H^\infty \)-calculus on \( H^{-1}_2(\mathcal{O}), L^2(\mathcal{O}) \) and \( L^{\gamma+1}(\mathcal{O}) \). The item (a) to (d) in this claim is an application of Theorem 4.5 in [35]. To show item (a), let us put in Theorem 4.5 \( p = 2 \), \( X_0 = H^{-1}_2(\mathcal{O}), X_1 = H^1_2(\mathcal{O}), B(v)[\psi] := v \psi, \psi \in \mathcal{H}, F(v) = \alpha v \), (and incorporating \( f(t) := \eta(t) \)). Note, that the Hilbert-Schmidt norm of \( B \) is evaluated in \([14] \) and due to Assumption 2.1 we know \( B \) satisfies Hypothesis (HB) (see \([35 \text{ p. } 1384] \)). It follows that there exists a constant \( C > 0 \) such that
\[ \mathbb{E}\|v\|_{L^2(0,T;H^2)} \leq \mathbb{E}\|v_0\|_{L^2} + C \mathbb{E}\|\eta\|_{L^2(0,T;H^{-1}_2)}. \]

To show (b), let us substitute in Theorem 4.5 \( p = 2 \), \( X_0 = H^{-1}_2(\mathcal{O}), X_1 = H^1_2(\mathcal{O}), B(v)[\psi] := v \psi, \psi \in \mathcal{H}, F(v) = \alpha v \), (and incorporating \( f(t) := \eta(t) \)). Note, that \( |B(u)|_{\gamma(\mathcal{H},L^4)} \) is evaluated in \([4.6] \) and due to Assumption 2.1 we infer that \( B \) satisfies Hypothesis (HB) (see \([35 \text{ p. } 1384] \)). It is elementary to see that there exists a constant \( C > 0 \) such that
\[ \mathbb{E}\|v\|_{L^2(0,T;H^4)} \leq \mathbb{E}\|v_0\|_{L^2} + C \mathbb{E}\|\eta\|_{L^2(0,T;H^{-1}_4)}, \]
and
\[ \mathbb{E}\|v\|_{C([0,T];L^4)} \leq \mathbb{E}\|v_0\|_{L^4} + C \mathbb{E}\|\eta\|_{L^2(0,T;H^{-1}_4)}. \]

To show item (c), let us continue using Theorem 4.5 with \( p = \gamma + 1 \),
\[ X_0 = H^{-1/\gamma+1}(\mathcal{O}), \quad X_1 = H^{2/\gamma+1}(\mathcal{O}), \quad X_{1/2} = H^{2/\gamma+1}(\mathcal{O}), \quad X_{1-1/\gamma} = H^{1}(\mathcal{O}). \]
Furthermore, let us put \( B(v)[\psi] := v \psi, \psi \in \mathcal{H}, F(v) = \alpha v \), (and incorporating \( f(t) := \eta(t) \)). Note, that \( |B(u)|_{\gamma(\mathcal{H},X_{1/2})} \) is evaluated in \([4.7] \) and owing to Assumption 2.1 we know \( B \) satisfies Hypothesis (HB) (see \([35 \text{ p. } 1384] \)). Consequently, there exists a constant \( C > 0 \) such that
\[ \mathbb{E}\|v\|_{L^2(0,T;H^{2/\gamma+1})} \leq \mathbb{E}\|v_0\|_{L^2} + C \mathbb{E}\|\eta\|_{L^{\gamma+1}(0,T;X_0)}, \]
and
\[ \mathbb{E}\|v\|_{L^{\gamma+1}(0,T;H^{2/\gamma+1})} \leq \mathbb{E}\|v_0\|_{L^{\gamma+1}} + C \mathbb{E}\|\eta\|_{L^{\gamma+1}(0,T;X_0)}. \]

Finally, to show item (d), let us continue Theorem 4.5 with \( p = \gamma + 1 \),
\[ X_0 = L^{\gamma+1}(\mathcal{O}), \quad X_1 = H^1_{\gamma+1}(\mathcal{O}), \quad X_{1/2} = H^1_{\gamma+1}(\mathcal{O}), \quad X_{1-1/\gamma} = H^{1/\gamma+1}(\mathcal{O}). \]
Furthermore, we again incorporate \( B(v)\psi := v \psi, \psi \in \mathcal{H}, F(v) = \alpha v, \) (and take \( f(t) := \eta(t) \)). Once again we note that \( |B(u)|_{N} \) (see [35, p. 1384]). It follows that there exists a constant \( C > 0 \) such that (d) holds.

\[ \text{Claim 4.2. For } \gamma > 1, \text{ any } R_{1}, R_{2} > 0 \text{ and } \eta \in \mathcal{X}_{R}(R_{1}, R_{2}), \text{ there exists a unique pair } (u, v) \text{ of solutions to the systems (3.5)–(3.6) such that} \]

\[ \mathbb{E} \left[ \sup_{0 \leq s \leq T} |u(s)|_{H_{2}^{-1}}^{2} + \int_{0}^{T} |u(s)|_{L^{\gamma+1}}^{\gamma+1} ds \right] < \infty. \]  

\[ \text{Proof of Claim 4.2.} \] Let us fix \( R_{1}, R_{2} > 0 \) and \( \eta \in \mathcal{X}_{R}(R_{1}, R_{2}). \) We know by Claim 4.1 that there exists unique \( v \in C([0, T]; H_{2}^{\gamma+1}(\Omega)) \) \( \mathbb{P}-a.s \) to (3.6) and \( v \) satisfies the estimate (4.12). We now show that there exists a solution \( u \) to (3.5) by verifying the assumptions of Theorem 5.1.3 in [63] for \( \gamma > 1 \). Let us consider the Gelfand triple

\[ V \subset \mathcal{H} \cong \mathcal{H}^* \subset V^*, \]

with \( \mathcal{H} := H_{2}^{-1}(\Omega), \) the dual space \( \mathcal{H}^* \) of \( H_{2}^{1}(\Omega) \) (corresponding to Neumann boundary conditions) and set \( V := L^{p}(\Omega) \) for some fix \( p = \gamma + 1 \) and its conjugate \( p' = \frac{\gamma+1}{\gamma}. \) Let \( V^* = \{ u \in \mathcal{D}(\Omega) : u = -\Delta v, v \in L^{p'} \} \). The duality \( \mathcal{V} \cdot (\cdot, \cdot)_{V} \) is defined as

\[ \mathcal{V} \cdot (u, w)_{V} = \int_{\Omega} (-\Delta)^{-1}u(x) w(x) dx. \]

We set

\[ \mathcal{A}(t, u, \omega) := \mathcal{A}_{\upsilon}(u) := ru_{\upsilon} \Delta u_{\upsilon} - \chi \text{div} (\eta(t, \omega) \nabla v(t, \omega)). \]

Firstly, let us investigate the process

\[ \xi : [0, T] \times \Omega \ni (t, \omega) \mapsto \text{div} (\eta(t, \omega) \nabla v(t, \omega)), \]

where \( \upsilon \) solves (3.6). We note that

\[ \mathcal{V} \cdot (\xi(t, \omega), w)_{V} = \int_{\Omega} (-\Delta)^{-1} (\text{div} (\eta(t, \omega) \nabla v(t, \omega))) w(x) dx. \]

Now using the fact that \( L^{1}(\Omega) \hookrightarrow H_{\frac{1}{\gamma+1}}(\Omega) \) for \( d = 1, 2 \) and the Young inequality, we see that for any \( \varepsilon_{1}, \varepsilon_{2} > 0, \) there exists a constant \( C = C(\varepsilon_{1}, \varepsilon_{2}) > 0 \) such that

\[ |\mathcal{V} \cdot (\xi(t, \omega), w)_{V} | \leq |\eta(t, \omega) \nabla v(t, \omega)|_{L^{-\frac{1}{\gamma+1}}} |w|_{L^{1}} + |\eta(t, \omega) \nabla v(t, \omega)|_{L^{\gamma+1}} |w|_{L^{\gamma+1}} \]

\[ \leq \varepsilon_{1} \eta(t, \omega)|_{L^{\gamma+1}} |\nabla v(t, \omega)|_{L^{\frac{1}{\gamma+1}}} + C |\nabla v(t, \omega)|_{L^{\frac{\gamma+1}{\gamma+1}}} + \varepsilon_{2} |w|_{L^{\gamma+1}}. \]

Next, we will show that for fixed \( t \in [0, T] \) and fixed \( \omega \in \Omega, \) the operator \( \mathcal{A}_{\upsilon} : V \rightarrow V^* \) is indeed a bounded operator. In fact, using (4.13) we obtain,

\[ |\mathcal{V} \cdot (\mathcal{A}_{\upsilon}(u), w)_{V} | = \left| - \int_{\Omega} [ru_{\upsilon} \nabla v(t, \omega) - \chi (-\Delta)^{-1} \text{div} (\eta(t, \omega) \nabla v(t, \omega)) w(x) dx \right| \]

\[ \leq ru_{\upsilon} |w|_{L^{\gamma+1}} + \chi \eta(t, \omega)|_{L^{\gamma+1}} |\nabla v(t, \omega)|_{L^{\frac{1}{\gamma+1}}} |w|_{L^{\gamma+1}} \]

\[ \leq ru_{\upsilon} |w|_{L^{\gamma+1}} + \chi \eta(t, \omega)|_{L^{\gamma+1}} |\nabla v(t, \omega)|_{L^{\frac{1}{\gamma+1}}} |w|_{L^{\gamma+1}}. \]
Verification of (H1) (Hemicontinuity). Let \( \lambda \in \mathbb{R} \). We need to show that
\[
\lambda \mapsto v^* \langle A_v(u_1 + \lambda u_2), w \rangle
\]
is continuous on \( \mathbb{R} \), for any \( u_1, u_2, w \in V, t \in [0, T], \omega \in \Omega \). As the map \( \lambda \mapsto (u_1 + \lambda u_2)^{[\gamma]} \)
is continuous, by the Dominated Convergence Theorem \( \lambda \mapsto \int_0^T (u_1 + \lambda u_2)^{[\gamma]} \, w \, dx \) is continuous. Hence, \( \lambda \mapsto v^* \langle A_v(u_1 + \lambda u_2), w \rangle \) is continuous.

Verification of (H2') (Local monotonicity). Let \( u, w \in L^{\gamma+1}(\mathcal{O}), \ t \in [0, T], \omega \in \Omega \). Then, we get
\[
\begin{align*}
&v^* \langle A_v(u) - A_v(w), u - w \rangle_V + |\sigma_1(u) - \sigma_1(w)|^2_{L^2(H_1, H)} \\
&\leq - \int_\Omega \left[ r_u (w^{[\gamma]} - w^{[\gamma]}) (u - w) \right] dx + C|u - w|^2_{H^1} \\
&\leq Cr_u |u - w|^2_{L^{\gamma+1}} \left( |w|_{L^{\gamma+1}}^{\gamma-1} + |w|_{L^{\gamma+1}}^{\gamma-1} \right) + C|u - w|^2_{H^1} \\
&\leq \left( Cr_u |u|_{L^{\gamma+1}}^{\gamma-1} + |w|_{L^{\gamma+1}}^{\gamma-1} + C \right) |u - w|^2_{H^1}.
\end{align*}
\]
(4.16)
Hence, (H2') of [63, Theorem 5.1.3] holds.

Verification of (H3) (Coercivity). Let \( u, w \in L^{\gamma+1}(\mathcal{O}), \ t \in [0, T], \omega \in \Omega \). Then, by substituting \( \xi \) and taking \( \varepsilon_1 = \frac{1}{4} \) and \( \varepsilon_2 = \frac{1}{4} \) in (1.14) we achieve
\[
\begin{align*}
&v^* \langle A_v(u), u \rangle_V + |\sigma_1(u)|^2_{L^2(H_1, H)} \\
&\quad = -r_u \int_\Omega w^{[\gamma]} u - \chi \int_\Omega ((-\Delta)^{-1} \xi) (x) u(x) dx + |\sigma_1(u)|^2_{L^2(H_1, H)} \\
&\quad \leq - \frac{3}{4} r_u |u|_{L^{\gamma+1}}^{\gamma+1} + C \chi |\nabla v(t, \omega)|_{L^{\gamma+1}}^{\gamma+1} + \frac{1}{4} |\eta(t, \omega)|_{L^{\gamma+1}}^{\gamma+1} + C|u|^2_{H^1}.
\end{align*}
\]
Owing to Remark [4.4] and the assumptions on \( \eta \) it follows that a non-negative adapted process
\[
f(t) := C \chi |\nabla v(t, \omega)|_{L^{\gamma+1}}^{\gamma+1} + \frac{1}{4} |\eta(t, \omega)|_{L^{\gamma+1}}^{\gamma+1}
\]
belongs to \( \mathcal{M}^1_{\mathbb{R}}(0, T; \mathbb{R}) \). This proves (H3) of [63, Theorem 5.1.3].

Verification of (H4') (Growth). Let \( u, \in L^{\gamma+1}(\mathcal{O}), \ t \in [0, T], \omega \in \Omega \). Due to assumption \( \eta \in \mathcal{M}^1_{\mathbb{R}}(0, T; L^{\gamma+1}) \) and Remark [4.3] a non-negative adapted process \( f \) with
\[
f(t) := \left( \varepsilon_1 |\eta(t, \omega)|_{L^{\gamma+1}}^{\gamma+1} + C |\nabla v(t, \omega)|_{L^{\gamma+1}}^{\gamma+1} \right) \in \mathcal{M}^1_{\mathbb{R}}(0, T; \mathbb{R}).
\]
Again using (4.15) we have
\[
|A_v(u)|_{L^{\gamma+1}}^{\gamma+1} \leq r_u |u|_{L^{\gamma+1}}^{\gamma+1} + \chi \left( |\eta(t, \omega)|_{L^{\gamma+1}}^{\gamma+1} |\nabla v(t, \omega)|_{L^{\gamma+1}}^{\gamma+1} \right)^{\gamma+1}.
\]
By the Young inequality for \( p = \gamma \) and \( p' = \frac{\gamma}{\gamma-1} \), we get
\[
|A_v(u)|_{L^{\gamma+1}}^{\gamma+1} \leq r_u |u|_{L^{\gamma+1}}^{\gamma+1} + \frac{1}{4} |\eta(t, \omega)|_{L^{\gamma+1}}^{\gamma+1} + \chi C |\nabla v(t, \omega)|_{L^{\gamma+1}}^{\gamma+1}.
\]
Then, (H4') of [63, Theorem 5.1.3] holds (with \( \alpha = \gamma + 1, \beta = 0 \)).
In this way we have shown that the Hypothesis (H1), (H2'), (H3), and (H4') of [63] Theorem 5.1.3 are satisfied. Therefore, by an application of [63] Theorem 5.1.3, the existence of \( u \) is guaranteed satisfying

\[
\mathbb{E} \left[ \sup_{0 \leq s \leq T} |u(t)|^2_{H^{-1}_2} \right] + \mathbb{E} \left[ \int_0^T |u(t)|^\gamma_{L^{\gamma+1}} \, dt \right] < \infty.
\]

\[\square\]

From Claim 4.1 we have shown that, if \( \gamma > 1 \), for given \( \eta \), there exists an unique solution \( v \) to (3.6) such that

\[
\mathbb{E} \left[ \int_0^T |\nabla v(s)|^{\frac{\gamma+1}{\gamma}} \, \frac{dt}{L^{\gamma+1}} \right] < \infty,
\]

and by Claim 4.2 we have shown that if \( v \) satisfies (4.17), then there exists a unique solution \( u \) to (3.5). Hence, a unique pair \((u, v)\) of solutions to (3.5)--(3.6) exists such that \( u \in \mathcal{X}_{\mathbb{R}} \) and

\[
\mathbb{E} \left[ \sup_{0 \leq s \leq T} |u(s)|^2_{H^{-1}_2} + \int_0^T |u(s)|^\gamma_{L^{\gamma+1}} \, ds \right] < \infty,
\]

proving (4.10), (4.11) and (4.13). This completes the proof of Proposition 4.2 \( \square \)

**Proposition 4.4.** For \( \gamma > 3 \) there exist numbers \( R_1 > 0 \) and \( R_2 > 0 \) such that \( T \) maps \( \mathcal{X}_{\mathbb{R}}(R_1, R_2) \) into itself.

**Proof.** Let \( \eta \in \mathcal{X}_{\mathbb{R}}(R_1, R_2) \). The proposition is shown in three steps. First, in the Step (i), we investigate the norm of \( u \) depending on \( v \) and \( \eta \). In this way, we get an estimate on \( \|u\|_{\mathcal{X}} \) in terms of \( R_2 \), from which we get a condition for \( R_2 \). Now, to get an estimate on \( \mathbb{E} \left[ \sup_{0 \leq s \leq T} |u(s)|^\gamma_{L^{\gamma+1}} \right] \), we first establish in Step (ii) an estimate on \( v \) in a stronger norm, depending on \( R_2 \). From this estimate we get in Step (iii) the estimate \( \mathbb{E} \left[ \sup_{0 \leq s \leq T} |u(s)|^\gamma_{L^{\gamma+1}} \right] \) depending on \( R_2 \), which gives us a condition for \( R_1 \). To start, let \( \eta \in \mathcal{X}_{\mathbb{R}}(R_1, R_2) \) and \((u, v)\) be solution to (3.5)--(3.6).

**Step (i):** Applying the Itô formula to the function \( \Phi(u) := |u|^2_{H^{-1}_2} \) we get

\[
\begin{align*}
|u(t)|^2_{H^{-1}_2} - |u_0|^2_{H^{-1}_2} &= \int_0^t \left[ \langle u(s), r_u \Delta u^{[\gamma]}(s) \rangle_{H^{-1}_2} - \chi \langle u(s), \text{div}(\eta(s) \nabla v(s)) \rangle_{H^{-1}_2} \right] ds \\
&\quad + \sum_{k \in \mathbb{N}} \int_0^t \left[ \langle u(s), \sigma_u u(s) \psi_k^{(1)} \rangle_{H^{-1}_2}, d\beta_k^{(1)}(r) \right] + \frac{1}{2} \sum_{k \in \mathbb{N}} \int_0^t \sigma_u^2 |u(s)\psi_k^{(1)}|^2_{H^{-1}_2} ds.
\end{align*}
\]

Taking supremum over \([0, T]\) and then expectation, it is easy to see that

\[
\begin{align*}
\frac{1}{2} \mathbb{E} \left[ \sup_{0 \leq s \leq T} |u(s)|^2_{H^{-1}_2} \right] - \frac{1}{2} \mathbb{E} |u_0|^2_{H^{-1}_2} &\leq \mathbb{E} \left[ \sup_{0 \leq s \leq T} \sum_{k \in \mathbb{N}} \left| \int_0^t \langle u(r), \sigma_u u(r) \psi_k^{(1)} \rangle_{H^{-1}_2}, d\beta_k^{(1)}(r) \right| \right] + C \frac{\sigma_u^2}{2} \mathbb{E} \left[ \int_0^T |u(s)|^2_{H^{-1}_2} ds \right] \\
&\quad - \mathbb{E} \left[ \sup_{0 \leq s \leq T} \int_0^s \left[ r_u u^{[\gamma]}(r, x) + \chi (-\nabla)^{-1}(\eta(r, x) \nabla v(r, x)) u(r, x) \right] dx \, dr \right].
\end{align*}
\]
The Burkholder-Davis-Gundy inequality (see [14]) and the Young inequality for product term gives for a constant $C > 0$

$$
\mathbb{E} \left[ \sup_{0 \leq s \leq t} \sum_{k \in I_1} \left| \int_0^s \langle u(r), \sigma_u(r) \psi_k^{(1)} \rangle_H^{2-1} \, d\beta_k^{(1)}(r) \right| \right] \leq C \mathbb{E} \left[ \left( \int_0^t \sigma_u^2 |u(s)|_H^{2-1} \, ds \right)^{\frac{3}{2}} \right]
$$

(4.19)

$$
\leq \frac{1}{4} \mathbb{E} \left[ \sup_{0 \leq s \leq t} |u(s)|_H^{2-1} \right] + C \sigma_u^2 \mathbb{E} \left[ \int_0^T |u(s)|_H^{2-1} \, ds \right].
$$

To estimate the nonlinear term we apply first the generalized Hölder inequality with $p = \gamma + 1$ and its conjugate $p' = \frac{2+1}{\gamma}$. Then we apply the Sobolev embedding $L^1(\mathcal{O}) \hookrightarrow H^{-1,\gamma}(\mathcal{O})$, and, finally again the Hölder inequality. In this way we get

$$
\left| \int_\mathcal{O} (-\nabla)^{-1} (\eta(s, x)\nabla v(s, x)) u(s, x) \, dx \right| \leq |u(s)|_{L^{\gamma+1}} |(-\nabla)^{-1} (\eta(s)\nabla v(s))|\left[ L^{\gamma+1} \right]
$$

$$
\leq |u(s)|_{L^{\gamma+1}} |\eta(s)\nabla v(s)|_{L^{\gamma+1}} \leq |u(s)|_{L^{\gamma+1}} |\eta(s)|_{L^{\gamma+1}} |\nabla v(s)|_{L^{\gamma+1}}.
$$

Substituting this estimate above we get for $\varepsilon > 0$

$$
\frac{1}{4} \mathbb{E} \left[ \sup_{0 \leq s \leq t} |u(s)|_H^{2-1} \right] - \frac{1}{2} \mathbb{E}|u_0|_H^{2-1} \leq -r_u \mathbb{E} \left[ \int_0^T |u(s)|_{L^{\gamma+1}}^{2+1} \, ds \right] + C \mathbb{E} \left[ \int_0^T |u(s)|_H^{2-1} \, ds \right]
$$

$$
+ \chi C \mathbb{E} \left[ \int_0^T \left\{ |\nabla v(s)|_{L^{\gamma+1}} |\eta(s)|_{L^{\gamma+1}} |u(s)|_{L^{\gamma+1}} \right\} \, ds \right]
$$

$$
\leq -r_u \mathbb{E} \left[ \int_0^T |u(s)|_{L^{\gamma+1}}^{2+1} \, ds \right] + \varepsilon \mathbb{E} \left[ \int_0^T |\eta(s)|_{L^{\gamma+1}}^{2+1} \, ds \right] + C(\varepsilon, r_u, \gamma) \mathbb{E} \left[ \int_0^T |\nabla v(s)|_{L^{\gamma+1}}^{2+1} \, ds \right]
$$

$$
+ \frac{r_u}{2} \mathbb{E} \left[ \int_0^T |u(s)|_{L^{\gamma+1}}^{2+1} \, ds \right] + C \mathbb{E} \left[ \int_0^T |u(s)|_H^{2-1} \, ds \right].
$$

Rearranging gives

$$
\frac{1}{4} \mathbb{E} \left[ \sup_{0 \leq s \leq t} |u(s)|_H^{2-1} \right] - \frac{1}{2} \mathbb{E}|u_0|_H^{2-1} + \frac{r_u}{2} \mathbb{E} \left[ \int_0^T |u(s)|_{L^{\gamma+1}}^{2+1} \, ds \right]
$$

$$
\leq \varepsilon \mathbb{E} \left[ \int_0^T |\eta(s)|_{L^{\gamma+1}}^{2+1} \, ds \right] + C(\varepsilon, r_u, \gamma) \mathbb{E} \left[ \int_0^T |\nabla v(s)|_{L^{\gamma+1}}^{(\gamma+1)/(\gamma-1)} \, ds \right] + C \mathbb{E} \left[ \int_0^T |u(s)|_H^{2-1} \, ds \right].
$$

To estimate the second term on the right hand side, we take into account that for $\gamma > 3$ we know $(\gamma+1)/(\gamma-1) \leq 2$. In addition from Claim 4.11(a) we know that there exists a constant $C > 0$ such that

$$
\mathbb{E}\|v\|_{L^2(0,T;H_2^1)}^2 \leq \mathbb{E}|v_0|_{L^2}^2 + C \mathbb{E}\|\eta\|_{L^2(0,T;H_2^1)}^2.
$$

(4.20)

In this way we obtain

$$
\mathbb{E}\|v\|_{L^{\gamma+1}(0,T;H_2^{\gamma+1})}^{\frac{\gamma+1}{\gamma+1-1}} \leq C \left( \mathbb{E}\|v\|_{L^2(0,T;H_2^1)}^2 \right)^{\frac{\gamma+1}{2(\gamma+1-1)}} \leq C \left( \mathbb{E}|v_0|_{L^2}^2 + \mathbb{E}\|\eta\|_{L^2(0,T;H_2^1)}^2 \right)^{\frac{\gamma+1}{2(\gamma+1-1)}}.
$$

(4.21)

Since $\eta \in \mathcal{A}_3(R_1, R_2)$, we can write

$$
\mathbb{E}\|v\|_{L^{\gamma+1}(0,T;H_2^{\gamma+1})}^{\frac{\gamma+1}{\gamma+1-1}} \leq C \left[ \mathbb{E}|v_0|_{L^2}^2 + R_2 \right]^{\frac{\gamma+1}{2(\gamma+1-1)}}.
$$
Using the Gronwall Lemma we know that for any \( \varepsilon > 0 \) there exist constants \( C_1, C_2 > 0 \) such that
\[
\mathbb{E}\left[ \sup_{0 \leq t \leq T} |u(t)|^2_{H^2} \right] + \frac{r_u}{4} \mathbb{E}\left[ \int_0^T |u(s)|^{\gamma + 1}_{L^{\gamma + 1}} \, ds \right]
\]
\[
\leq \frac{1}{2} \mathbb{E}|u_0|^2_{H^2} + \varepsilon R_2 + C_1 R_2^{\frac{\gamma + 1}{\gamma - 1}} + C_2 \left( \mathbb{E}|v_0|^2_{L^2} \right)^{\frac{\gamma + 1}{\gamma - 1}}.
\]
(4.22)

Taking \( \varepsilon = \frac{\delta}{8} \) and \( R_2 \) so large that
\[
R_2 \geq \frac{8}{r_u} \left\{ C_1 R_2^{\frac{\gamma + 1}{\gamma - 1}} + \frac{1}{2} \mathbb{E}|u_0|^2_{H^2} + C_2 \left( \mathbb{E}|v_0|^2_{L^2} \right)^{\frac{\gamma + 1}{\gamma - 1}} \right\}.
\]
Consequently,
\[
\mathbb{E}\left[ \sup_{0 \leq t \leq T} |\eta(t)|^2_{H^2} \right] \leq R_2,
\]
which essentially implies
\[
\mathbb{E}\left[ \sup_{0 \leq t \leq T} |u(t)|^2_{H^2} \right] + \frac{r_u}{4} \mathbb{E}\left[ \int_0^T |u(s)|^{\gamma + 1}_{L^{\gamma + 1}} \, ds \right] \leq R_2.
\]

Step (ii): Next, we derive a lower estimate for \( R_1 \). Applying the Itô formula to the function \( \Phi(u) := |u|^{\gamma + 1}_{L^{\gamma + 1}} \) and using standard calculation we obtain
\[
d|u(t)|^{\gamma + 1}_{L^{\gamma + 1}} = (\gamma + 1) \left( \int_{\Omega} u^\gamma(s, x) \Delta u^\gamma(t, x) \, dx \right) dt - (\gamma + 1) \left( \int_{\Omega} u^\gamma(t, x) \text{div} (\eta(t, x) \nabla v(t, x)) \, dx \right) dt
\]
\[
+ \frac{(\gamma + 1)}{2} \sigma^2 u \sum_{k \in \mathbb{N}} \int_{\Omega} u^{\gamma + 1}(t, x) \psi_k^{(1)}(x) \psi_k^{(1)}(x) \, dx \, dt
\]
\[
+ (\gamma + 1) \sigma_u \sum_{k \in \mathbb{N}} \int_{\Omega} u^{\gamma + 1}(t, x) \psi_k^{(1)}(x) \, d\beta_k^{(1)}(t).
\]

This implies
\[
d|u(t)|^{\gamma + 1}_{L^{\gamma + 1}} + (\gamma + 1) \gamma^2 \left( \int_{\Omega} u^{2\gamma - 2} |\nabla u(t, x)|^2 \, dx \right) dt
\]
\[
= (\gamma + 1) \gamma \left( \int_{\Omega} u^{\gamma - 1}(t, x) \eta(t, x) \nabla u(t, x) \cdot \nabla v(t, x) \, dx \right) dt + C(\Gamma^{(1)}_{\infty}, \sigma_u, \gamma)|u(t)|^{\gamma + 1}_{L^{\gamma + 1}} dt
\]
\[
+ (\gamma + 1) \sigma_u \sum_{k \in \mathbb{N}} \int_{\Omega} u^{\gamma + 1}(t, x) \psi_k^{(1)}(x) \, d\beta_k^{(1)}(t).
\]

Integrating over \([0, T]\) we obtain
\[
|u(t)|^{\gamma + 1}_{L^{\gamma + 1}} - |u_0|^{\gamma + 1}_{L^{\gamma + 1}} + (\gamma + 1) \gamma^2 \int_0^t \int_{\Omega} u^{2\gamma - 2} |\nabla u(t, x)|^2 \, dx \, ds
\]
Using the H"older inequality with $p = \frac{\gamma + 1}{2}$ and $p' = \frac{\gamma + 1}{\gamma - 1}$, we can write

\begin{align*}
& (\gamma + 1) \gamma \int_{\mathcal{O}} u^{-1}(s, x) \eta(s, x) \nabla u(s, x) \cdot \nabla v(s, x) dx ds \\
& + C_\gamma \int_{\mathcal{O}} \eta^2(s, x) |\nabla v(s, x)|^2 dx \\
& \leq (\gamma + 1) \gamma \int_{\mathcal{O}} u^{-1}(s, x) \eta(s, x) \nabla u(s, x) \cdot \nabla v(s, x) dx ds \\
& + C_\gamma \int_{\mathcal{O}} \eta^2(s, x) |\nabla v(s, x)|^2 dx + \varepsilon_1 |\eta(s)|_{L^{\gamma+1}} + C(\varepsilon_1, \gamma) |\nabla v(s)| \frac{2(\gamma+1)}{2(\gamma+1)}.
\end{align*}

Using (4.24), we obtain from (4.23) that

\begin{align*}
|u(t)|_{L^{\gamma+1}} + (\gamma + 1) \gamma^2 \int_{\mathcal{O}} u^{2\gamma-2} |\nabla u(t, x)|^2 dx ds \\
& \leq (\gamma + 1) \gamma \int_{\mathcal{O}} u^{-1}(s, x) \eta(s, x) \nabla u(s, x) \cdot \nabla v(s, x) dx ds \\
& + C(\varepsilon_1, \gamma) \int_{\mathcal{O}} |\nabla v(s)| \frac{2(\gamma+1)}{2(\gamma+1)} + \varepsilon_1 \int \mathcal{O} |\eta(s, x)|^{\gamma+1} dx ds \\
& + C_\gamma \int_{0}^{t} |u(s)|_{L^{\gamma+1}} dx ds + (\gamma + 1) \sigma_u \sum_{k \in \mathbb{N}} \int_{\mathcal{O}} u^{\gamma+1}(t, x) \psi_k^{(1)}(x) d\beta_k^{(1)}(t).
\end{align*}

Note, that

\begin{align*}
\int_{0}^{T} |\nabla v(s)| \frac{2(\gamma+1)}{2(\gamma+1)} ds & \leq C \left( \int_{0}^{T} |v(s)|_{H_1}^4 ds \right)^{\frac{\gamma+1}{2(\gamma+1)}}.
\end{align*}

Claim 4.1 yields

\begin{align*}
\mathbb{E} \left[ \int_{0}^{T} |\nabla v(s)| \frac{2(\gamma+1)}{2(\gamma+1)} ds \right] & \leq \left\{ \mathbb{E} |v_0|_{L^1}^4 + \mathbb{E} \left[ \int_{0}^{T} |\eta(s)|_{H_1}^4 ds \right] \right\}^{\frac{\gamma+1}{2(\gamma+1)}}.
\end{align*}

Taking supremum over $[0, T]$ and then expectation, using the Burkholder–Davis–Gundy inequality, i.e. (4.8), and substituting (4.26) in (4.25), we get

\begin{align*}
& \mathbb{E} \left[ \sup_{0 \leq s \leq T} |u(s)|_{L^{\gamma+1}}^{\gamma+1} \right] - \mathbb{E} |u_0|_{L^{\gamma+1}}^{\gamma+1} + \frac{(\gamma + 1) \gamma^2}{2} \mathbb{E} \left[ \int_{0}^{T} (u(s, x))^{2(\gamma+1)} |\nabla u(s, x)|^2 dx ds \right] \\
& \leq C(\varepsilon_1, \gamma) \mathbb{E} \left[ \int_{0}^{T} |\nabla v(s)| \frac{2(\gamma+1)}{2(\gamma+1)} ds \right] + \varepsilon_1 \mathbb{E} \left[ \int_{0}^{T} |\eta(s)|_{L^{\gamma+1}}^{\gamma+1} ds \right] + C \mathbb{E} \left[ \int_{0}^{T} |u(s)|_{L^{\gamma+1}}^{\gamma+1} ds \right] \\
& + \varepsilon \mathbb{E} \left[ \int_{0}^{T} |u(s)|_{H_1}^{\gamma+1} ds \right].
\end{align*}
There exists some Proposition 4.5.

Proof of Proposition 4.5. Using the Gronwall inequality we get

$$\varepsilon_1 E \left[ \int_0^T \sup_{0 \leq s \leq T} |u(s)|_{L_{\gamma+1}^{\gamma+1}} \right] + \varepsilon_1 E \left[ \int_0^T |\eta(s)|_{H_{\gamma+1}^{\gamma+1}} \right]$$

$$\leq C(\varepsilon_1, \gamma) \left( E v_0^4 \right)^{\gamma+1} + \varepsilon_1 R_2 + C(\varepsilon_1, \gamma) R_2^{2(\gamma-1)} + C(\gamma, \varepsilon_1) E \left[ \int_0^T |u(s)|_{L_{\gamma+1}^{\gamma+1}} \right]$$

(4.27)

Note, that by the Technical Proposition 4.1 we cancel the term $\varepsilon E \left[ \int_0^T |u(s)|_{H_{\gamma+1}^{\gamma+1}} \right]$ by

$$E \left[ \int_0^T \int_\Omega (u(s, x))^2(\gamma-1)|\nabla u(s, x)|^2 dx ds \right].$$

Using the Gronwall inequality we get

$$\varepsilon_1 C R_2 + C(\varepsilon_1, \gamma) R_2^{2(\gamma-1)} + C E v_0^4 \left( E v_0^4 \right)^{\gamma+1} \leq R_1,$$

we know by the calculations above that

(4.29) $E \left[ \sup_{0 \leq s \leq T} |u(s)|_{L_{\gamma+1}^{\gamma+1}} \right] + \frac{(\gamma+1)\gamma^2}{2} E \left[ \sup_{0 \leq t \leq T} \int_0^t \int_\Omega (u(s, x))^2(\gamma-1)|\nabla u(s, x)|^2 dx ds \right] \leq R_1.$

Summarising, we have shown that there exists $R_1 > 0$ and $R_2 > 0$ such that $\mathcal{T}$ maps $\mathcal{X}_3(R_1, R_2)$ in itself, which finishes the proof of Proposition 4.4.

In the next Proposition we show the continuity of the solution operator.

**Proposition 4.5.** There exists some $\delta > 0$ and a constant $C > 0$ such that for all $\eta_1, \eta_2 \in \mathcal{X}_3(R_1, R_2)$ we have

$$\|\mathcal{T}[\eta_1] - \mathcal{T}[\eta_2]\|_X \leq C\|\eta_1 - \eta_2\|_X.$$

**Proof of Proposition 4.5.** Let $(u_1, v_1)$ and $(u_2, v_2)$ be solutions of (4.5) and (4.6) corresponding to $\eta_1$ and $\eta_2$ respectively, i.e.,

(4.30) $du_i(t) = \left( r_u u_i^r(t) - \chi \text{div}(\eta_i(t) \nabla v_i(t)) \right) dt + \sigma_u u_i(t) dW_i(t)$

(4.31) $dv_i(t) = \left( r_v \Delta v_i(t) + \beta u_i(t) - \alpha v_i(t) \right) dt + \sigma_v v_i(t) dW_2(t) \quad t \in [0, T], \ i = 1, 2.$
First, let us investigate the difference of \( e = v_1 - v_2 \). Since the system of \( v \) is linear, we know that \( e \) solves
\[
de(t) = (r_u \Delta e(t) + \beta e(t) - \alpha e(t)) \, dt + \sigma v(t) \, dW_2(t) \quad t \in [0, T], \quad i = 1, 2.
\]
Secondly, by Example 3.2-(4) \[35\] we know that the Laplace operator with Neumann boundary condition has a bounded \( H^\infty \)-calculus on \( H^{-1}_2(O) \). Hence, we know by Theorem 4.5-(iii) with \( B(e)[\psi] := e \psi, \psi \in H, F(e) = \alpha e, e_0 = 0 \) (and incorporating \( f(t) := \eta_1(t) - \eta_2(t) \)) that there exists a constant \( C > 0 \) such that
\[
\mathbb{E}\|v_1 - v_2\|_{L^2(0, T; H^1_2)}^2 \leq C \mathbb{E}\left[ \sup_{0 \leq s \leq t} \|\eta_1(s) - \eta_2(s)\|_{H^{-1}_2}^2 \right].
\]
Next, we investigate the difference of \( u_1 - u_2 \). Using Itô formula to \( |u_1(t) - u_2(t)|_{H^{-1}_2}^2 \) and by canonical calculations we obtain
\[
\frac{1}{2}|u_1(t) - u_2(t)|_{H^{-1}_2}^2 \leq \left[ \int_0^t \left( (u_1(s) - u_2(s)), r_u \Delta (u_1^2(s) - u_2^2(s)) \right)_{H^{-1}_2} ds \right]
- \chi \left[ \int_0^t \left( (u_1(s) - u_2(s)), \text{div}(\eta_1(s) \nabla v_1(s) - \eta_2(s) \nabla v_2(s)) \right)_{H^{-1}_2} ds \right]
+ \sum_{k \in \mathbb{N}_1} \int_0^t \left( (\sigma_u u_1(s) - \sigma_u u_2(s)), (u_1(s) - u_2(s)) \psi^{(1)}_k \right)_{H^{-1}_2} d\beta_k^{(1)}(s)
+ \frac{1}{2} \sum_{k \in \mathbb{N}_1} \int_0^t \sigma_u^2 |(u_1(s) - u_2(s)) \psi^{(1)}_k |_{H^{-1}_2}^2 ds
\leq - \int_0^t \int_\Omega \left[ r_u (u_1^2(s, x) - u_2^2(s, x))(u_1(s, x) - u_2(s, x)) \right] dx ds
- \chi \left[ \int_0^t \int_\Omega (-\nabla)^{-1} \left( \eta_1(s, x) \nabla v_1(s, x) - \eta_2(s, x) \nabla v_2(s, x) \right) (u_1(s, x) - u_2(s, x)) \right] dx ds
+ \sum_{k \in \mathbb{N}_1} \int_0^t \left( (\sigma_u u_1(s) - \sigma_u u_2(s)), (u_1(s) - u_2(s)) \psi^{(1)}_k \right)_{H^{-1}_2} d\beta_k^{(1)}(s)
+ \frac{1}{2} \sum_{k \in \mathbb{N}_1} \int_0^t \sigma_u^2 |(u_1(s) - u_2(s)) \psi^{(1)}_k |_{H^{-1}_2}^2 ds
\]
\[
\text{(4.33)} \quad := J_1(t) + J_2(t) + J_3(t) + J_4(t).
\]
Exploiting the fact that \( (u_1^2 - u_2^2)(u_1 - u_2) \geq |u_1 - u_2|^{\gamma+1}, \) we obtain
\[
- r_u \int_0^t \int_\Omega \left[ (u_1^2(s, x) - u_2^2(s, x))(u_1(s, x) - u_2(s, x)) \right] dx ds
\leq - r_u \int_0^1 \int_\Omega |u_1(s, x) - u_2(s, x)|^{\gamma+1} dx ds = - r_u \left[ \int_0^T |u_1(s) - u_2(s)|_{L^{\gamma+1}} ds \right].
\]
Using \(4.34\) and taking supremum over \([0, T]\) and expectation we obtain
\[
\mathbb{E}\left[ \sup_{0 \leq t \leq T} |u_1(t) - u_2(t)|_{H^{-1}_2}^2 \right] + r_u \mathbb{E}\left[ \int_0^T |u_1(s) - u_2(s)|_{L^{\gamma+1}} ds \right]
\]
Now we consider the term $J_2(t)$. First, we split the term into the following sum

$$
\mathbb{E}\left[\sup_{0 \leq t \leq T} \int_0^t \int_0^1 (\nabla)^{-1} \left( \eta_1(s,x) \nabla v_1(s,x) - \eta_2(s,x) \nabla v_2(s,x) \right) (u_1(s,x) - u_2(s,x)) dx ds \right]
$$

$$
= I_1(t)
$$

$$
+ \mathbb{E}\left[\sup_{0 \leq t \leq T} \int_0^t \int_0^1 (\nabla)^{-1} \left( \eta_1(s,x) - \eta_2(s,x) \right) \nabla v_1(s,x) (u_1(s,x) - u_2(s,x)) dx ds \right]
$$

$$
= I_2(t)
$$

(4.35)

Next, we estimate $\mathbb{E}\left[\sup_{0 \leq t \leq T} I_1(t)\right]$. Using the H"older and the Young inequality we know that for any $\varepsilon > 0$ there exists a constant $C > 0$ with

$$
\mathbb{E}\left[\sup_{0 \leq t \leq T} I_1(t)\right] \leq C(\varepsilon) \mathbb{E}\left[ \int_0^T \left| (\eta_1(s) - \eta_2(s)) \nabla v_1(s) \right|_{H^{\gamma+1}_2}^\gamma |u_1(s) - u_2(s)|_{L^{\gamma+1}}^\gamma ds \right]
$$

$$
\leq C(\varepsilon) \mathbb{E}\left[ \int_0^T |(\eta_1(s) - \eta_2(s)) \nabla v_1(s)|_{H^{\gamma+1}_2}^{\gamma+1} ds \right] + \varepsilon \mathbb{E}\left[ \int_0^T |u_1(s) - u_2(s)|_{L^{\gamma+1}}^{\gamma+1} ds \right].
$$

Applying the Sobolev embedding $L^1(\mathcal{O}) \hookrightarrow H^\gamma_2(\mathcal{O})$ and the H"older inequality we get

$$
\mathbb{E}\left[\sup_{0 \leq t \leq T} I_1(t)\right] \leq C(\varepsilon) \mathbb{E}\left[ \int_0^T |\eta_1(s) - \eta_2(s)|_{H^{\gamma+1}_2}^{\gamma+1} |\nabla v_1(s)|_{L^2}^{\gamma+1} ds \right] + \varepsilon \mathbb{E}\left[ \int_0^T |u_1(s) - u_2(s)|_{L^{\gamma+1}}^{\gamma+1} ds \right].
$$

Next, applying complex interpolation we get for some $\rho \in (0, \frac{1}{\gamma})$

$$
(4.36)
$$

$$
\left[ H^{\gamma}_2(\mathcal{O}), H^{-1}_2(\mathcal{O}) \right] = L^2(\mathcal{O})
$$

with $\rho(1 - \theta) + \theta(-1) = 0$ we know

$$
(4.37)
$$

$$
|\eta_1(s) - \eta_2(s)|_{L^2} \leq |\eta_1(s) - \eta_2(s)|_{H^{-1}_2}^{\theta} |\eta_1(s) - \eta_2(s)|_{H^\gamma_2}^{1-\theta}
$$

This gives for $\theta = \frac{\rho}{1+\rho} < \frac{1}{1+\gamma}$ and $1 - \theta > \frac{\gamma}{1+\gamma}$

$$
\mathbb{E}\left[\sup_{0 \leq t \leq T} |I_1(t)|\right] \leq C \mathbb{E}\left[ \int_0^T |\eta_1(s) - \eta_2(s)|_{H^{-1}_2}^{\theta} |\eta_1(s) - \eta_2(s)|_{H^\gamma_2}^{1-\theta} |\nabla v_1(s)|_{L^2}^{\gamma+1} ds \right]
$$
We choose (4.40) This finally gives

\begin{equation}
\mathbb{E}\left[\int_0^T |u_1(s) - u_2(s)|_{L^{\gamma+1}}^2 \, ds\right] 
\leq \mathbb{E}\left[\sup_{0 \leq s \leq T} |\eta_1(s) - \eta_2(s)|_{H^{2\gamma+1}}^2\right] \mathbb{E}\left[\sup_{0 \leq s \leq T} |\nabla v_1(s)|_{L^{2\gamma+1}}^{\gamma+1}\right] + \mathbb{E}\left[\int_0^T |u_1(s) - u_2(s)|_{L^{\gamma+1}}^2 \, ds\right].
\end{equation}

(4.38)

\begin{equation}
\mathbb{E}\left[\int_0^T \left(|\eta_1(s)|_{H^{2\gamma}}^\gamma + |\eta_2(s)|_{H^{2\gamma}}^\gamma\right) \, ds\right] \leq C \left(\mathbb{E}|v_0|_{H^{2\gamma+1}}^\gamma + R_2\right).
\end{equation}

Owing to Technical Proposition 4.1 one achieve

\begin{equation}
\mathbb{E}\left[\int_0^t \left(|\eta_1(s)|_{H^{2\gamma}}^\gamma + |\eta_2(s)|_{H^{2\gamma}}^\gamma\right) \, ds\right] \leq R_1.
\end{equation}

This finally gives

\begin{equation}
\mathbb{E}\left[\int_0^T |u_1(s) - u_2(s)|_{L^{\gamma+1}}^2 \, ds\right] \leq R_1.
\end{equation}

Next, we estimate \(\mathbb{E}\left[\sup_{0 \leq t \leq T} I_2(t)\right]\). Again, using the Hölder and the Young inequality and applying the Sobolev embedding \(L^1(O) \hookrightarrow H^{-1}(O)\), we know that for any \(\varepsilon > 0\) there exists a constant \(C \geq 0\) with

\begin{equation}
\mathbb{E}\left[\sup_{0 \leq t \leq T} I_2(t)\right] \leq C(\varepsilon) \mathbb{E}\left[\int_0^T |\eta_2(s)|_{L^{2\gamma+1}} |\nabla (v_1(s) - v_2(s))|_{L^{2\gamma+1}}^{2\gamma+1} \, ds\right] + \mathbb{E}\left[\int_0^T |u_1(s) - u_2(s)|_{L^{\gamma+1}}^{2\gamma+1} \, ds\right]
\end{equation}

\begin{equation}
\leq C(\varepsilon) \mathbb{E}\left[\left(\sup_{0 \leq s \leq T} |\eta_2(s)|_{L^{2\gamma+1}}^{2\gamma+1}\right) \left(\int_0^T |\nabla (v_1(s) - v_2(s))|_{L^{2\gamma+1}}^{2\gamma+1} \, ds\right)\right]
\end{equation}

\begin{equation}
+ \varepsilon \mathbb{E}\left[\int_0^T |u_1(s) - u_2(s)|_{L^{\gamma+1}}^{\gamma+1} \, ds\right].
\end{equation}

We choose \(\varepsilon\) small enough so that the first term in the right hand side of (4.40) can be cancelled using the second term in the left hand side of (4.35). The term \(\mathbb{E}\left[\sup_{0 \leq s \leq T} |\eta_2(s)|_{L^{2\gamma+1}}^{\gamma+1}\right]\) can be estimated by \(R_1\). It remains to estimate

\begin{equation}
\mathbb{E}\|\nabla (v_1(s) - v_2(s))\|_{L^{\gamma+1}(0,T;L^2)}^{\gamma+1}.
\end{equation}
First, note that for $\gamma \geq 3$ we have
\[
\mathbb{E}\|\nabla (v_1(s) - v_2(s))\|_{L^{\gamma+1}(0,T;L^2)}^{\gamma+1} \leq \left\{ \mathbb{E}\|\nabla (v_1(s) - v_2(s))\|_{L^2(0,T;L^2)}^2 \right\}^{\frac{\gamma+1}{2}}.
\]
Using estimate (4.32) this term can be handled. It remains to handle $J_3(t)$ and $J_4(t)$. Using the Burkhholder-Davis-Gundy inequality we obtain
\[
\mathbb{E}\left[ \sup_{0 \leq s \leq t} |J_3(s)| \right] = \mathbb{E}\left[ \sup_{0 \leq s \leq t} \int_0^t \left( \langle \sigma_u u_1(s) - \sigma_u u_2(s), (u_1(s) - u_2(s)) \psi_k^{(1)} \rangle_{H^{-1}_2} d\beta_k^{(1)} \right) \right]
\leq C \left( \mathbb{E}\left[ \int_0^t \sigma_u^2 |u_1(s) - u_2(s)|^2_{H^{-1}_2} ds \right] \right)^{\frac{1}{2}}
\leq \frac{1}{4} \mathbb{E}\left[ \sup_{0 \leq s \leq t} |u_1(s) - u_2(s)|^2_{H^{-1}_2} \right] + C \mathbb{E}\left[ \int_0^T |u_1(s) - u_2(s)|^2_{H^{-1}_2} ds \right].
\] (4.41)
Finally, evaluating the trace we obtain
\[
\mathbb{E}\left[ \sup_{0 \leq s \leq t} |J_4(s)| \right] \leq \frac{1}{2} \mathbb{E}\left[ \int_0^T |u_1(s) - u_2(s)|^2_{H^{-1}_2} ds \right].
\] (4.42)
Collecting altogether and substituting the estimates above into (4.35) and using the Gronwall inequality, one can infer that there exists a constant $C(R_1, R_2) > 0$ and a number $\delta > 0$ such that
\[
\mathbb{E}\left[ \sup_{0 \leq s \leq t} |u_1(t) - u_2(t)|^2_{H^{-1}_2} \right] \leq C(R_1, R_2) \left( \left\{ \mathbb{E}\left[ \sup_{0 \leq s \leq T} |\eta_1(s) - \eta_2(s)|^2_{H^{-1}_2} \right] \right\}^\delta + \mathbb{E}\left[ \sup_{0 \leq s \leq T} |\eta_1(s) - \eta_2(s)|^2_{H^{-1}_2} \right] \right).
\] (4.43)
This completes the proof of Proposition 4.5.

In the next proposition we will show that $\mathcal{T}$ maps $X_{\mathfrak{A}}(R_1, R_2)$ to a precompact set.

**Proposition 4.6.** For any initial condition $(u_0, v_0)$ satisfying Assumption 2.6 and all $R_1 > 0$ and $R_2 > 0$ we know that

(a) there exists $r = r(T, \gamma) > 0$ such that for any $\eta \in X_{\mathfrak{A}}(R_1, R_2)$, we have
\[
\sup_{0 \leq t \leq T} \mathbb{E}\|\mathcal{T}\eta(t)\|_{L^{\gamma+1}}^{\gamma+1} \leq C \mathbb{E}\|\eta\|_{L^{\gamma+1}(0,T;L^{\gamma+1})}^{\gamma+1}.
\] (4.44)
(b) there exists a number $\delta = \delta(T, \gamma) > 0$ and $C = C(\delta, T, \gamma, R_1, R_2) > 0$ such that for any $0 < t_1 < t_2 \leq T$ and $\eta \in X_{\mathfrak{A}}(R_1, R_2)$ we have
\[
\mathbb{E}\|\mathcal{T}\eta(t_1) - \mathcal{T}\eta(t_2)\|_{H^{-1}_2}^2 \leq C|t_1 - t_2|^\delta.
\] (4.45)

**Proof.** Part 4.4(a) is clear, due to the definition of $X_{\mathfrak{A}}(R_1, R_2)$. Let us start with Part 4.4(b). Applying the Itô formula to the function $\Phi(u) := |u - u_0|_{H^{-1}_2}$ and using standard calculations we obtain
\[
d|u(t) - u_0|_{H^{-1}_2}^2 = - (\gamma + 1) \int_0^t \langle (u(s) - u_0), u(s) \rangle_{H^{-1}_2} \mathbb{E}\|\nabla^{-1}(u(s) - u_0), \eta(s)\nabla\eta(s)\|_{H^{-1}_2} \right) ds
+ (\gamma + 1) \int_0^t \langle \nabla^{-1}(u(s) - u_0), \eta(s)\nabla\eta(s)\rangle ds.
\]
By the Young inequality we know that for all
\[
\epsilon > 0
\]
the H"older inequality. In this way we get
\[
\sum_{k \in \mathbb{N}} \int_0^t \langle \nabla^{-1}(u(s) - u_0), \nabla^{-1}(u(s)\psi_k) \rangle d\beta_k^{(1)}(t) + \frac{1}{2} \sum_{k \in \mathbb{N}} \int_0^t |u(s)\psi_k|_{H^{-1}_{2,1}}^2 dt.
\]
This immediately gives
\[
d|u(t) - u_0|_{H^{-1}_{2,1}}^2 + (\gamma + 1) \int_0^t \langle (u(s) - u_0), u(s)[\gamma] - u_0[\gamma] \rangle ds
\]
\[
+ \sum_{k \in \mathbb{N}} \int_0^t \langle \nabla^{-1}(u(s) - u_0), \nabla^{-1}(u(s)\psi_k) \rangle d\beta_k^{(1)}(t) + \frac{1}{2} \sum_{k \in \mathbb{N}} \int_0^t |u(s)\psi_k|_{H^{-1}_{2,1}}^2 dt
\]
\[
= (\gamma + 1) \int_0^t \langle (u(s) - u_0), u_0[\gamma] \rangle ds + (\gamma + 1) \int_0^t \langle \nabla^{-1}(u(s) - u_0), \eta(s)\nabla v(s) \rangle ds
\]
\[
+ \sum_{k \in \mathbb{N}} \int_0^t \langle \nabla^{-1}(u(s) - u_0), \nabla^{-1}(u(s)\psi_k) \rangle d\beta_k^{(1)}(t) + \frac{1}{2} \sum_{k \in \mathbb{N}} \int_0^t |u(s)\psi_k|_{H^{-1}_{2,1}}^2 dt
\]
\[
:= K_1(t) + K_2(t) + K_3(t) + K_4(t).
\]
Applying the Hölder inequality gives for the first term
\[
\mathbb{E}\left[ \sup_{0 \leq s \leq t} K_1(s) \right] \leq (\gamma + 1) \mathbb{E}\left[ \left\{ \int_0^t |u(s) - u_0|_{L^{\gamma+1}_{\gamma+1}}^{\gamma+1} ds \right\}^{\frac{1}{\gamma+1}} \left\{ \int_0^t |u_0[\gamma]|_{L^{\gamma+1}_{\gamma+1}}^{\gamma+1} ds \right\}^{\frac{1}{\gamma+1}} \right].
\]
The Cauchy Schwarz inequality implies
\[
\mathbb{E}\left[ \sup_{0 \leq s \leq t} K_1(s) \right] \leq (\gamma + 1) t^{\frac{1}{\gamma+1}} \mathbb{E}\left[ \int_0^t |u(s) - u_0|_{L^{\gamma+1}_{\gamma+1}}^{\gamma+1} ds \right]^{\frac{1}{\gamma+1}} \left\{ \mathbb{E}\left[ |u_0[\gamma]|_{L^{\gamma+1}_{\gamma+1}}^{\gamma+1} ds \right] \right\}^{\frac{1}{\gamma+1}}.
\]
Applying the Young inequality, we can cancel the \( \mathbb{E}\left[ \int_0^t |u(s) - u_0|_{L^{\gamma+1}_{\gamma+1}}^{\gamma+1} ds \right] \) with the second term on the left hand side. In order to calculate the next term, we apply integration by parts and the Hölder’s inequality. In this way we get
\[
\mathbb{E}\left[ \sup_{0 \leq s \leq t} K_2(s) \right] \leq (\gamma + 1) \int_0^t \langle \nabla^{-1}(u(s) - u_0), \eta(s)\nabla v(s) \rangle ds
\]
\[
\leq (\gamma + 1) \int_0^t |u(s) - u_0|_{H^{\gamma+1}_{\gamma+1}} |\eta(s)\nabla v(s)|_{H^{-1}_{\gamma+1}} ds.
\]
By the Young inequality we know that for all \( \varepsilon > 0 \) there exists a constant \( C(\varepsilon) > 0 \) such that
\[
\mathbb{E}\left[ \sup_{0 \leq s \leq t} K_2(s) \right]
\]
\[
\leq \varepsilon \mathbb{E}\left[ \int_0^t |u(s) - u_0|_{L^{\gamma+1}_{\gamma+1}}^{\gamma+1} ds \right] + \mathbb{E}\left[ \int_0^t |\eta(s)\nabla v(s)|_{H^{-1}_{\gamma+1}}^{\gamma+1} ds \right].
\]
Using the embedding \( L^1(\mathcal{O}) \hookrightarrow H^{-1}_{\gamma+1}(\mathcal{O}) \), and applying the Hölder inequality we get
\[
\mathbb{E}\left[ \sup_{0 \leq s \leq t} K_2(s) \right] \leq \varepsilon \mathbb{E}\left[ \int_0^t |u(s) - u_0|_{L^{\gamma+1}_{\gamma+1}}^{\gamma+1} ds \right] + t \mathbb{E}\left[ \sup_{0 \leq s \leq t} |\eta(s)|_{L^{\gamma+1}_{\gamma+1}}^{\gamma+1} \sup_{0 \leq s \leq t} |\nabla v(s)|_{L^{\gamma+1}_{\gamma+1}}^{\gamma+1} \right].
\]
Applying the Hölder’s inequality again we get
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} K_2(s) \right] \leq \varepsilon \mathbb{E} \left[ \int_0^t |u(s) - u_0|^{\gamma+1}_{L_{\gamma+1}} \, ds \right] + t \left\{ \mathbb{E} \left[ \sup_{0 \leq s \leq t} |\eta(s)|^{\gamma+1}_{L_{\gamma+1}} \right]^\gamma \left\{ \mathbb{E} \left[ \sup_{0 \leq s \leq t} |\nabla v(s)|^{\gamma+1}_{L_{\gamma+1}} \right]^\gamma \right\}. \]

Using Claim 4 and
\[
|\nabla v|^{\gamma+1}_{L_{\gamma+1}} \leq C |v|^{\frac{\gamma}{\gamma+1}}_{H^{\frac{\gamma+1}{\gamma+1}}},
\]
we obtain
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} K_2(s) \right] \leq \varepsilon \mathbb{E} \left[ \int_0^t |u(s) - u_0|^{\gamma+1}_{L_{\gamma+1}} \, ds \right] + t \left\{ \mathbb{E} \left[ \sup_{0 \leq s \leq t} |\eta(s)|^{\gamma+1}_{L_{\gamma+1}} \right]^\gamma \left\{ \mathbb{E} \left[ \sup_{0 \leq s \leq t} |\nabla v(s)|^{\gamma+1}_{L_{\gamma+1}} \right]^\gamma \right\}^\gamma. \]

Observe, that \( w^\gamma - w^\gamma \leq (u - w)^\gamma \). Collecting all together we know that for any \( \varepsilon_1, \varepsilon_2 > 0 \) there exist constants \( C_1, C_2 > 0 \) such that
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} |u(s) - u_0|^{\gamma+1}_{L_{\gamma+1}} \right] \leq C_1(\gamma + 1) t^{\frac{\gamma}{\gamma+1}} \left\{ \mathbb{E} \left[ |u_0|^{\gamma+1}_{L_{\gamma+1}} \right] \right\} \]
\[
+ \varepsilon_2 \mathbb{E} \left[ \int_0^t |u(s) - u_0|^{\gamma+1}_{L_{\gamma+1}} \, ds \right] + t \left\{ \mathbb{E} \left[ \sup_{0 \leq s \leq t} |\eta(s)|^{\gamma+1}_{L_{\gamma+1}} \right]^\gamma \left\{ \mathbb{E} \left[ \sup_{0 \leq s \leq t} |\nabla v(s)|^{\gamma+1}_{L_{\gamma+1}} \right]^\gamma \right\}^\gamma. \]

Taking \( \varepsilon_1 \) and \( \varepsilon_2 \) small enough gives the assertion. This finishes the proof of the proposition. \( \square \)

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