Holomorphic Discs and Surgery Exact Triangles

Bijan Sahamie

We show a connection between a surgery exact sequence in knot Floer homology and the sequence derived in [18]. As a consequence of this relationship we see that the exact sequence in [18] also works with coherent orientations and admits refinements with respect to Spin$^c$-structures. As an application of this discussion, we prove that the ranks of the image and kernel of certain cobordism maps between knot Floer homologies can be computed combinatorially by relating them to a count of certain moduli spaces of holomorphic disks.

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1 Introduction

Heegaard Floer homology was introduced by Peter Ozsváth and Zoltan Szabó in [13] (see [19] for a detailed introduction) and has turned out to be a useful tool in the study of low-dimensional topology. They also defined variants of this homology theory which are topological invariants of a pair $(Y, K)$ where $Y$ is a closed, oriented 3-manifold and $K \subset Y$ a null-homologous knot (see [12]). One of the main features of this homology theory is the existence of exact sequences which serve as a main tool in calculations. The exact sequences Ozsváth and Szabó provided either just contained Heegaard Floer homology groups or they just contained knot Floer homology groups.

Theorem 1.1 (Theorem 8.2. of [12], cf. Theorem 2.7. of [11]) Let $Y$ be a closed, oriented 3-manifold with framed knot $C \subset Y$ with framing $n$ and let $K \subset Y\setminus C$ be a null-homologous knot. Denote by $Y_n(C)$ the result of a surgery along $C$ in $Y$ with framing $n$ and denote by $Y_{n+1}(C)$ the result of a $(-1)$-surgery along a meridional curve of $C$ in $Y_n(C)$. Finally, we can interpret $Y$ itself as a result of a surgery along a framed knot in $Y_{n+1}(C)$. Then each surgery can be described by a Heegaard triple diagram which induces a map defined by counting holomorphic triangles. These fit into the following sequence

$$\ldots \xrightarrow{\bar{F}_3} \widehat{HF}(Y, K) \xrightarrow{\bar{F}_1} \widehat{HF}(Y_n(C), K) \xrightarrow{\bar{F}_2} \widehat{HF}(Y_{n+1}(C), K) \xrightarrow{\bar{F}_3} \ldots$$

which is exact.
In [18] we introduced a new exact sequence which, in contrast to the known sequences, contains both Heegaard Floer homology groups and knot Floer homology groups: We proved that the following sequence is exact (cf. Sequence (2–2))

\[(1–1) \quad \ldots \xrightarrow{f} \widehat{HF}(Y, K) \xrightarrow{\Gamma_1} \widehat{HF}(Y_{-1}(K)) \xrightarrow{\Gamma_2} \widehat{HF}(Y_0(K), \mu) \xrightarrow{f^*} \ldots\]

where \(\mu\) is a meridian of \(K\), interpreted as sitting in \(Y_0(K)\). The maps \(\Gamma_1\) and \(\Gamma_2\) are not defined by counting holomorphic triangles and, in fact, the map \(f^*\) is given by counting holomorphic disks in a suitable Heegaard diagram. Comparing both sequences, i.e. Sequence (1–1) and the sequence given in Theorem 1.1, we see that in the Dehn twist sequence, \(K\) serves as the knot and the surgery curve. This is a situation which in general violates the assumption in Theorem 1.1 that the knot is null-homologous in the complement of the surgery curve. We think that it is natural to pose the following question.

**Question 1** Is it possible to relate the Dehn twist sequence with the sequence given in Theorem 1.1 or with one which is defined, similarly?

Providing a discussion and a possible answer to this question will be the main goal of this article. To do that we will start proving the following statement.

**Proposition 1.2** There is an exact sequence

\[(1–2) \quad \ldots \xrightarrow{\partial_\ast} \widehat{HF}(Y, K) \xrightarrow{\tilde{\Gamma}_i} \widehat{HF}(Y_{-1}(K)) \xrightarrow{\tilde{\Gamma}_2} \widehat{HF}(Y_0(K), \mu) \xrightarrow{\partial_\ast} \ldots\]

where the \(\tilde{\Gamma}_i, i = 1, 2\), are maps defined by counting holomorphic triangles in suitable doubly-pointed Heegaard triple diagrams, and \(\partial_\ast\) is a connecting morphism.

This result is set up using different techniques than Ozsváth and Szabó utilized for the surgery exact sequence in knot Floer homology. It is based on bringing the attaching circles of the underlying Heegaard triple diagrams into an opportune position (see Figure 5) and then providing a careful analysis of the underlying doubly-pointed Heegaard triple diagrams (see Lemma 3.1 and Lemma 3.2). Although not essential, it is opportune to work with Heegaard diagrams that are induced by open books. In this particular situation the analysis of the Heegaard diagrams and of the domains of Whitney triangles is easier. Comparing this sequence with Theorem 1.1, notice, that we do not impose any condition on \(K\), i.e. the knot may be homologically essential. Furthermore, notice, that the third map in the sequence, \(\partial_\ast\), is a connecting morphism which is not given by counting holomorphic triangles. As a side-effect of our analysis we will see that the Sequences (1–2) and (1–1) interact in a commutative diagram.
\textbf{Theorem 1.3} Let \( Y \) be a closed, oriented 3-manifold and \( K \subset Y \) a knot. Denote by \( Y_{-1}(K) \) (resp. \( Y_0(K) \)) the result of performing a \((-1)\)-surgery (resp. \(0\)-surgery) along \( K \). We denote by \( \mu \) a meridian of \( K \). Then, all triangles and boxes in the following diagram commute.

\[
\begin{array}{cccccc}
\cdots & \cdots & \partial_* & \widehat{\text{HF}}(Y,K) & \widehat{F}_1 & \widehat{\text{HF}}(Y_{-1}(K)) & \widehat{F}_2 & \widehat{\text{HF}}(Y_0(K),\mu) & \partial_* & \cdots \\
\downarrow{\rho_4} & \downarrow{\rho_5} & \downarrow{\varphi} & \downarrow{\varphi} & \downarrow{\varphi} & \downarrow{\varphi} & \downarrow{\varphi} & \downarrow{\varphi} & \downarrow{\varphi} & \cdots \\
\widehat{\text{HF}}(Y,K) & \widehat{\text{HF}}(Y_{-1}(K)) & \widehat{\text{HF}}(Y_0(K),\mu) & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

Here, the isomorphisms \( \widehat{F}_4 \) and \( \widehat{F}_5 \) are also defined by counting holomorphic triangles in suitable doubly-pointed Heegaard triple diagrams.

The horizontal sequence is, in fact, the Sequence (1–2) and the diagonal sequence is the Dehn twist sequence. Thus, we may interpret this theorem as a possible answer to Question 1. This result, indeed, has some implications we would like to discuss.

\textbf{Implication I.} For a contact manifold \((Y, \xi)\) Ozsváth and Szabó introduced a contact invariant \( \hat{c}(\xi) \), called the contact element, which is an element of \( \widehat{\text{HF}}(\text{−}\text{Y}, s_\xi) \) where \( s_\xi \) is the Spin\(^c\)-structure associated to \( \xi \) (see [15] and cf. [3]). Furthermore, Lisca, Ozsváth, Stipsicz and Szabó introduced in [9] an invariant \( \widehat{L}(L) \) of a Legendrian knot \( L \) which sits in \( \widehat{\text{HF}}(\text{−}\text{Y}, L, s_\xi) \). The contact element has proved to be a powerful obstruction to overtwistedness (see [6, 7, 8]). Being interested in contact geometry, to us, besides Question 1, it was interesting to know if the Dehn twist sequences or the maps involved in these sequences can be defined with coherent orientations (and, hence, with \( \mathbb{Z} \)-coefficients) and if there is a refined version (with respect to Spin\(^c\)-structures).

Our interest in this question originates from [18, Theorem 6.1].

\textbf{Theorem 1.4} (Theorem 6.1 of [18]) Let \((Y, \xi)\) be a contact manifold and \( L \) an oriented Legendrian knot. Let \( W \) be the cobordism induced by \((+1)\)-contact surgery along \( L \) and denote by \((Y_L, \xi_L)\) the contact manifold we obtain from \((Y, \xi)\) by performing this surgery. Then, the cobordism \( −W \) induces a map

\[
\Gamma_{−W} : \widehat{\text{HF}}(−Y, L) \longrightarrow \widehat{\text{HF}}(−Y_{+1}(L))
\]

such that \( \Gamma_{−W}(\widehat{L}(L)) = \hat{c}(\xi_L) \).

There exists a similar naturality property that connects the contact elements before and after the contact surgery (see [5, Theorem 2.3]). In the calculations and applications
given by Lisca and Stipsicz (see [6, 7, 8, 5]) this naturality property was one of the main calculational tools. Additionally, in their work, the understanding of the map providing the naturality was a major ingredient. In light of this, it is natural to ask for refinements and for coherent orientations of $\Gamma_{-W}$.

As a matter of fact, Theorem 1.3 may be applied to introduce coherent orientations and refinements into the Dehn twist sequence: The maps $\hat{F}_i$ all admit refinements (see §2.3) and, as such, we are able to refine the Sequence (1–2). So, we may apply the commutative diagram given in Theorem 1.3 to provide a refined version of the Dehn twist sequence. The same strategy may be applied to bring coherent orientations into the Dehn twist sequence. We outline this at the end of Section 3. We summarize this briefly with the following corollary.

**Corollary 1.5** The Dehn twist Sequences (2–2) can be defined with coherent orientations. Furthermore, these sequences refine with respect to $\text{Spin}^c$-structures. □

The element $\hat{c}(\xi_L)$ is the element in homology, induced by a special generator $eh_L$ of $\hat{C}(\partial(Y_-,L))$ one can specify (see [3]). Theorem 1.4 is proved by identifying a generator $eh_L$ (in fact, the element for which $[eh_L] = \hat{c}(L)$) of $\hat{C}(\partial(Y,L))$ which is mapped onto $eh_L$ under $\Gamma_{-W}$. The fact that the element $[eh_L] = \hat{c}(\xi_L)$ is invariant under all choices made in its definition (see [3]) together with the invariance properties we proved in [19, §3] for $\Gamma_{-W}$ may be assembled to an alternative proof of the fact that $[eh_L]$ does not depend on the choices made in its definition. So, providing coherent orientations additionally gives us evidence that the invariant $\hat{c}$ can also be defined in $\mathbb{Z}$-coefficients and for Legendrian knots which are homologically essential. Of course, what was done in [9] can be slightly altered to provide these generalizations, as well. Because of that and since we do not write down this alternative approach we do not state this as a result, here.

**Implication II.** We prove the following statement.

**Theorem 1.6** Given a closed, oriented 3-manifold and a knot $L \subset Y$ with framing $n$, then denote by $K$ a push-off of $L$ which corresponds to the $(n+1)$-framing of $L$. Let $Y'$ be the result of a $n$-surgery along $L$ and let $K'$ be the knot $K$ represents in $Y'$. Furthermore, let

$$\hat{F} : \hat{HFK}(Y,K) \rightarrow \hat{HFK}(Y',K')$$

be the map defined by counting holomorphic triangles in a suitable Heegaard triple diagram associated to the surgery. Then, (as part of a Dehn twist sequence) there is a map

$$f_* : \hat{HFK}(Y,K) \rightarrow \hat{HFK}(Y',K')$$
which is defined by counting holomorphic disks in a suitable Heegaard diagram, such that \( \ker(\tilde{F}) = \ker(f_\ast) \) and \( \text{im}(\tilde{F}) = \text{im}(f_\ast) \).

This result tells us that the rank of the image and the kernel of the map \( \tilde{F} \) can be computed using the map \( f_\ast \). However, since \( f_\ast \) appears as part of a Dehn twist sequence, by its definition, it is part of a Heegaard Floer differential (cf. Proposition 2.3). The Sarkar-Wang algorithm (see [21]) presents a way to combinatorially compute Heegaard Floer differentials. We will prove that this algorithm may as well be applied in this particular situation. Thus, we get the following result.

**Proposition 1.7** Given a closed, oriented 3-manifold \( Y \) and a framed knot \( L \subset Y \) with framing \( n \), then denote by \( K \) a push-off of \( L \) which corresponds to the \((n + 1)\)-framing of \( L \). Let \( Y' \) be the manifold obtained by performing a surgery along \( L \) and let \( K' \) be the knot \( K \) represents in \( Y' \). In this situation we may define a map

\[
\tilde{F}: \hat{HF}(Y, K) \rightarrow \hat{HF}(Y', K')
\]

by counting holomorphic triangles in a suitable Heegaard triple diagram associated to the surgery. The rank of the kernel and image of \( \tilde{F} \) can be computed combinatorially.

Here, again it is opportune to work with open books: Plamenevskaya showed in [17] that Heegaard diagrams induced by open books can be made nice using the Sarkar-Wang algorithm by just using deformations that are induced by isotopies of the monodromy. This leads to a simplification of our discussion (see proof of Proposition 1.7).

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## 2 Preliminaries

### 2.1 Heegaard Floer homologies

The Heegaard Floer homology group \( \hat{HF}(Y) \) of a 3-manifold \( Y \) was introduced in [13]. The definition was extended for the case where \( Y \) is equipped with a knot \( K \subset Y \) to the variant \( \hat{HF}(Y, K) \) in [12] (cf. [18]).
A 3-manifold $Y$ can be described by a Heegaard diagram, which is a triple $(\Sigma, \alpha, \beta)$, where $\Sigma$ is an oriented genus-$g$ surface and $\alpha = \{\alpha_1, \ldots, \alpha_g\}$, $\beta = \{\beta_1, \ldots, \beta_g\}$ are two sets of pairwise disjoint simple closed curves in $\Sigma$ called attaching circles. Each set of curves $\alpha$ and $\beta$ is required to consist of linearly independent curves in $H_1(\Sigma, \mathbb{Z})$. In the following we will talk about the curves in the set $\alpha$ (resp. $\beta$) as $\alpha$-curves (resp. $\beta$-curves). Without loss of generality we may assume that the $\alpha$-curves and $\beta$-curves intersect transversely. To a Heegaard diagram we may associate the triple $(\text{Sym}^g(\Sigma), \mathbb{T}_\alpha, \mathbb{T}_\beta)$ consisting of the $g$-fold symmetric power of $\Sigma$,

$$\text{Sym}^g(\Sigma) = \Sigma^g / S_g,$$

and the submanifolds $\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_g$ and $\mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_g$. We define $\hat{\text{CF}}(\Sigma, \alpha, \beta)$ as the free $\mathbb{Z}_2$-module generated by the set $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. In the following we will just write $\hat{\text{CF}}$. For two intersection points $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ define $\pi_2(x, y)$ to be the set of homology classes of holomorphic Whitney discs $\phi: D \to \text{Sym}^g(\Sigma)$ ($D \subset \mathbb{C}$) that connect $x$ with $y$. The map $\phi$ is called Whitney if $\phi(D \cap \{Re < 0\}) \subset \mathbb{T}_\alpha$ and $\phi(D \cap \{Re > 0\}) \subset \mathbb{T}_\beta$. We call $D \cap \{Re < 0\}$ the $\alpha$-boundary of $\phi$ and $D \cap \{Re > 0\}$ the $\beta$-boundary of $\phi$. Such a Whitney disc connects $x$ with $y$ if $\phi(i) = x$ and $\phi(-i) = y$. Note that $\pi_2(x, y)$ can be interpreted as the subgroup of elements in $H_2(\text{Sym}^g(\Sigma), \mathbb{T}_\alpha \cup \mathbb{T}_\beta)$ represented by discs with appropriate boundary conditions. We endow $\text{Sym}^g(\Sigma)$ with a symplectic structure $\omega$. By choosing a path of almost complex structures $J_s$ on $\text{Sym}^g(\Sigma)$ suitably (cf. [13]) all moduli spaces of holomorphic Whitney discs are Gromov-compact manifolds. Denote by $\mathcal{M}_\phi$ the set of holomorphic Whitney discs in the equivalence class $\phi$, and $\mu(\phi)$ the formal dimension of $\mathcal{M}_\phi$. Denote by $\hat{\mathcal{M}}_\phi = \mathcal{M}_\phi / \mathbb{R}$ the quotient under the translation action of $\mathbb{R}$ (cf. [13]). Define $H(x, y, k)$ to be the subset of classes in $\pi_2(x, y)$ that admit moduli spaces of dimension $k$. Fix a point $z \in \Sigma \setminus (\alpha \cup \beta)$ and define the map

$$n_z: \pi_2(x, y) \to \mathbb{Z}, \phi \mapsto \#(\phi, \{z\} \times \text{Sym}^{g-1}(\Sigma)).$$

A boundary operator $\hat{\partial}: \hat{\text{CF}} \to \hat{\text{CF}}$ is given by defining it on the generators $x$ of $\hat{\text{CF}}$ by

$$\hat{\partial}x = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\phi \in H(x, y, 1)} \#(\hat{\mathcal{M}}_\phi \cdot U_{\phi}(\phi), y).$$

These homology groups are topological invariants of the manifold $Y$. We would like to note that not all Heegaard diagrams are suitable for defining Heegaard Floer homology; there is an additional condition that has to be imposed called admissibility. This is a technical condition in the compactification of the moduli spaces of holomorphic Whitney discs. A detailed knowledge of this condition is not important in the remainder of the present article since all constructions are done nicely so that there will never be a problem. We advise the interested reader to [13].
2.2 Knot Floer homology

Given a knot $K \subset Y$, we can specify a certain subclass of Heegaard diagrams.

**Definition 2.1** A Heegaard diagram $(\Sigma, \alpha, \beta)$ is said to be adapted to the knot $K$ if $K$ is isotopic to a knot lying in $\Sigma$ and $K$ intersects $\beta_1$ once transversely and is disjoint from the other $\beta$-circles.

Since $K$ intersects $\beta_1$ once and is disjoint from the other $\beta$-curves we know that $K$ intersects the core disc of the 2-handle represented by $\beta_1$ once and is disjoint from the others (after possibly isotoping the knot $K$). Every pair $(Y, K)$ admits a Heegaard diagram adapted to $K$. Having fixed such a Heegaard diagram $(\Sigma, \alpha, \beta)$ we can encode the knot $K$ in a pair of points. After isotoping $K$ onto $\Sigma$, we fix a small interval $I$ in $K$ containing the intersection point $K \cap \beta_1$. This interval should be chosen small enough such that $I$ does not contain any other intersections of $K$ with other attaching curves. The boundary $\partial I$ of $I$ determines two points in $\Sigma$ that lie in the complement of the attaching circles, i.e. $\partial I = z - w$, where the orientation of $I$ is given by the knot orientation. This leads to a doubly-pointed Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$. Conversely, a doubly-pointed Heegaard diagram uniquely determines a topological knot class: Connect $w$ with $z$ in the complement of the attaching circles $\alpha$ and $\beta \setminus \beta_1$ with an arc $\delta$ that crosses $\beta_1$, once. Connect $z$ with $w$ in the complement of $\beta$ using an arc $\gamma$. The union $\delta \cup \gamma$ is represents the knot class $K$ represents. The orientation on $K$ is given by orienting $\delta$ such that $\partial \delta = z - w$.

The knot chain complex $\widehat{CFK}(Y, K)$ is the free $\mathbb{Z}_2$-module generated by the intersections $T_\alpha \cap T_\beta$. The boundary operator $\widehat{\partial}^\mu$, for $x \in T_\alpha \cap T_\beta$, is defined by

$$\widehat{\partial}^\mu(x) = \sum_{y \in \pi_2(x, y) \cap T_\beta} \sum_{\phi \in H(x, y, 1)} \#M_{\phi} \cdot y,$$

where $H(x, y, 1) \subset \pi_2(x, y)$ are the homotopy classes with $\mu = 1$ and $n_z = n_w = 0$. We denote by $\widehat{HFK}(Y, K)$ the associated homology theory $H_*(\widehat{CFK}(Y, K), \widehat{\partial}^\mu)$.

2.3 Maps Induced by Cobordisms

Here, we briefly give the construction of cobordism maps between knot Floer homologies. We restrict ourselves to the case that the cobordism is given by a single 2-handle attachment. We point the interested reader to [20].
Given a closed, oriented 3-manifold $Y$ with knot $K \subset Y$ and a cobordism $W$ by performing a surgery along a second knot $L \subset Y$ which is disjoint from $K$. The surgered manifold will be denoted by $Y'$. These data determine a cobordism

$$W = [0, 1] \times Y \cup_b h^{4,2}$$

together with a canonical embedding $[0, 1] \times S^1 \hookrightarrow W$ such that

$$\{0\} \times S^1 \hookrightarrow K \subset Y$$
$$\{1\} \times S^1 \hookrightarrow K' \subset Y'. $$

A cobordism $W$ together with such an embedding will be called a cobordism between $(Y, K)$ and $(Y', K')$. Choose a doubly-pointed Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$ which is adapted to $K$ and $L$ and whose pair of base points $(w, z)$ encode the knot $K$. Performing the surgery along $L$, by the constructions given by Ozsváth and Szabó, determines a third set of attaching circles $\gamma = \{\gamma_1, \ldots, \gamma_g\}$. As we know from work of Ozsváth and Szabó the cobordism associated to the triple diagram $(\Sigma, \alpha, \beta, \gamma)$ is diffeomorphic to $W$ (see [16, Proposition 4.3]). The information of the knot $K$ is encoded in the pair of base points $(w, z)$ we include into our triple diagram. The doubly-pointed Heegaard triple $(\Sigma, \alpha, \beta, \gamma, w, z)$ not only determines the cobordism $W$ but also determines the knots $K$ and $K'$: By definition, the Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$ determines $(Y, K)$, the diagram $(\Sigma, \alpha, \gamma, w, z)$ determines $(Y', K')$ and, observe, that the doubly-pointed Heegaard diagram $(\Sigma, \beta, \gamma, w, z)$ determines the pair

$$(S^3 \#_{g-1}(S^2 \times S^1), U)$$

with $U$ the unknot. The associated knot Floer homology of the third diagram is isomorphic to

$$\widehat{HF}(S^3 \#_{g-1}(S^2 \times S^1))$$

which admits a top-dimensional generator $\widehat{\Theta}_{\beta\gamma}^+$ (see [16, §2.4]). This generator is uniquely represented by an intersection point $z_{\beta\gamma}^+$ in the diagram $(\Sigma, \beta, \gamma)$. For intersections $x \in T_\alpha \cap T_\beta$, $y \in T_\alpha \cap T_\gamma$ we define a Whitney triangle $\phi: \Delta \to \text{Sym}^g(\Sigma)$ connecting $x$ with $y$ and $z_{\beta\gamma}^+$ as a continuous map from the unit disc in $\mathbb{C}$ into $\text{Sym}^g(\Sigma)$ with boundary conditions like indicated in Figure 1. We denote by $M^{\alpha\beta\gamma}(x, z_{\beta\gamma}^+, y)$ the Maslov index-1 moduli space of holomorphic Whitney triangles connecting the indicated intersection points with $n_w = n_z = 0$. We define

$$\widehat{F}_{\alpha, \beta\gamma}: \widehat{CFK}(\Sigma, \alpha, \beta, w, z) \to \widehat{CFK}(\Sigma, \alpha, \gamma, w, z)$$

by sending an intersection point $x \in T_\alpha \cap T_\beta$ to

$$\widehat{F}_{\alpha, \beta\gamma}(x) = \sum_{y \in T_\alpha \cap T_\gamma} #M^{\alpha\beta\gamma}(x, z_{\beta\gamma}^+, y) \cdot y.$$
This map descends to a map
\[ \tilde{F}_L: \widehat{HFK}(Y, K) \to \widehat{HFK}(Y', K') \]
between the associated knot Floer homology groups and just depends on the framed knot \( L \) in an analogue way as this was proved for the maps coming from 2-handle attachments in Heegaard Floer homologies (see [20]). The triple diagram \((\Sigma, \alpha, \beta, \gamma)\) used to define the cobordism map \( \tilde{F}_{\alpha,\beta\gamma} \) also defines a cobordism \( X_{\alpha,\beta,\gamma} \) (see [16, §2.2]) with boundary components \(-Y, Y'\) and \(-S^3\#^{g-1}(S^2 \times S^1)\) (see [16, §2.2]). Closing up the third boundary component with \#^{g-1}(B^3 \times S^1)\), the cobordism is diffeomorphic to the cobordism \( W \) which is determined by the 2-handle attachment along \( L \) (see [16, Proposition 4.3]). The same way as it was done in [16], it is possible to define refinements \( \tilde{F}_{L,s} \) of \( \tilde{F}_L \) for \( s \in \text{Spin}^c(W) \).

Cobordism maps between knot Floer homologies provide a surgery exact triangle like in the Heegaard Floer case (see [12, Theorem 8.2] and [11, Theorem 2.7]). The proof from the Heegaard Floer case carries over verbatim to the knot Floer homology case.

### 2.3.1 Open Books and Heegaard Diagrams

We start by recalling some facts about open book decompositions of 3-manifolds. For details we point the reader to [2].

An open book is a pair \((P, \phi)\) consisting of an oriented genus-\( g \) surface \( P \) with boundary and a homeomorphism \( \phi: P \to P \) that is the identity near the boundary of \( P \). The surface \( P \) is called page and \( \phi \) the monodromy. Recall that an open book \((P, \phi)\) gives rise to a 3-manifold by the following construction: Let \( c_1, \ldots, c_k \) denote the boundary components of \( P \). Observe that

\[
(P \times [0, 1])/(p, 1) \sim (\phi(p), 0)
\]
is a 3-manifold with boundary given by the tori
\[(c_i \times [0, 1])/(p, 1) \sim (p, 0) \cong c_i \times S^1.\]

Fill in each of the holes with a full torus $D^2 \times S^1$: we glue a meridional disc $D^2 \times \{\star\}$ onto $\{\star\} \times S^1 \subset c_i \times S^1$. In this way we define a closed, oriented 3-manifold $Y(P, \phi)$. Denote by $B$ the union of the cores of the tori $D^2 \times S^1$. The set $B$ is called binding. Observe that the definition of $Y(P, \phi)$ defines a fibration
\[P \hookrightarrow Y(P, \phi) \backslash B \rightarrow S^1.\]

Consequently, an open book gives rise to a Heegaard decomposition of $Y(P, \phi)$ and, thus, induces a Heegaard diagram of $Y(P, \phi)$. To see this we have to identify a splitting surface of $Y(P, \phi)$, i.e. a surface $\Sigma$ that splits the manifold into two components. Observe that the boundary of each fiber lies on the binding $B$. Thus gluing together two fibers yields a closed surface $\Sigma$ of genus $2g$. The surface $\Sigma$ obviously splits $Y(P, \phi)$ into two components and can therefore be used to define a Heegaard decomposition of $Y(P, \phi)$ (cf. [3]). Let $a = \{a_1, \ldots, a_n\}$ be a cut system of $P$, i.e. a set of disjoint properly embedded arcs such that $P \backslash \{a_1, \ldots, a_n\}$ is a disc. One can easily show that being a cut system implies that $n = 2g$. Choose the splitting surface
\[\Sigma := P \times \{1/2\} \cup_{\partial} (-P) \times \{1\}\]
and let $\overline{a}_i$ be the curve $a_i \subset P \times \{1/2\}$ with opposite orientation, interpreted as a curve in $(-P) \times \{0\}$. Then define $\alpha_i := a_i \cup \overline{a}_i$. The curves $b_i$ are isotopic push-offs of the $a_i$. We choose them like indicated in Figure 2: We push the $b_i$ f the $a_i$ by following with $\partial b_i$ the positive boundary orientation of $\partial P$. Finally set $\beta_i := b_i \cup \phi(b_i)$. The data $(\Sigma, \alpha, \beta)$ define a Heegaard diagram of $Y(P, \phi)$ (cf. [3]).

### 2.4 The Dehn Twist Sequence

We will briefly recall the results given in [18]. Especially we will focus on the derived surgery exact sequence we will call the Dehn twist sequence. Given an abstract open book $(P, \phi)$ and let $\delta \subset P$ be a homologically essential simple closed curve. We try to determine how the groups $\widehat{HF}$-change if we compose the monodromy $\phi$ with a Dehn twist along $\delta$, here we stick to positive Dehn twists. Let $(\Sigma, \alpha, \beta)$ be a Heegaard diagram induced by the pair $(P, \phi)$ such that $\delta$ intersects $\beta_1$ once transversely and is disjoint from the other $\beta$-circles. We denote by $\beta_1'$ the curve $D_\delta^+(\beta_1)$. Note that the set of attaching circles $\beta'$ which is given by
\[\{\beta_1', \beta_2, \ldots, \beta_g\}\]
and determines the manifold after the surgery. A third set of attaching circles

\[ \delta = \{ \delta, \beta_2, \ldots, \beta_g \} \]

is formed. In the left portion of Figure 3 we see a neighborhood of \( \delta \cap \beta_1 \) in the Heegaard surface (cf. [18]).

Denote by \( Y^\delta \) the manifold determined by the Heegaard diagram \((\Sigma, \alpha, \beta')\). In [18] we have shown that in this particular situation the homology groups \( \hat{HF}(Y^\delta) \) can be interpreted as a mapping cone of the complexes \((\hat{CF}(\alpha, \beta), \partial_{w, \alpha, \beta})\) and \((\hat{CF}(\alpha, \delta), \partial_{w, \alpha, \delta})\) with the chain map \( f \) given in following definition (cf. Proposition 2.3).

**Definition 2.2** Define a map

\[
f : \hat{CFK}(\Sigma, \alpha, \delta, z, w) \longrightarrow \hat{CFK}(\Sigma, \alpha, \beta, z, w)
\]

by sending an element \( x \in T_\alpha \cap T_\delta \) to

\[
f(x) = \sum_{\phi \in H(x, y, 1)} \hat{M}_\phi \cdot y
\]

where \( H(x, y, 1) \) are classes in \( \pi_{2, \alpha, \beta'}(x, y) \) with \( \mu = 1 \) and whose pair \((n_*(\phi), n_{**}(\phi))\) does not equal \((0, 0)\). Here, \( n_*(\phi) \) and \( n_{**}(\phi) \) denote the multiplicities of \( \phi \) at the domains \( D_* \) and \( D_{**} \) (cf. right portion of Figure 3).

We would like to explain the main idea of the proof: The chain complex \( \hat{CF}(\alpha, \beta') \) is generated by the intersection points \( T_\alpha \cap T_{\beta'} \). It is easy to observe that this generating set can be canonically identified with the disjoint union

\[
T_\alpha \cap T_\beta \sqcup T_\alpha \cap T_\delta.
\]

We will call those intersections in \( T_\alpha \cap T_{\beta'} \) corresponding to the intersections \( T_\alpha \cap T_\beta \) as \( \alpha \beta \)-intersections and call the others \( \alpha \delta \)-intersections. Due to the positioning
of the point \( z \) we observe that there is no holomorphic disc connecting an \( \alpha \beta \)-intersection with and \( \alpha \delta \)-intersection. Furthermore, we can identify moduli-spaces of holomorphic discs connecting an \( \alpha \beta \)-intersection with an \( \alpha \beta \)-intersection with the moduli-spaces of holomorphic discs appearing in the differential \( \hat{\partial}_w \). Moreover, we can identify the moduli-spaces of holomorphic discs connecting an \( \alpha \delta \)-intersection with an \( \alpha \delta \)-intersection with the moduli-spaces of holomorphic discs appearing in the differential \( \hat{\partial}_w \). Finally, there might be discs connecting an \( \alpha \delta \)-intersection with an \( \alpha \beta \)-intersection. We can explicitly characterize whose homotopy classes of Whitney discs belong to this class of discs. So, using this characterization we are able to define with them a chain map \( f \) as it is done in Definition 2.2. By construction the associated mapping cone of this map \( f \) is isomorphic to the Heegaard Floer homology of the chain complex \((\hat{\text{CF}}(\alpha, \beta'), \hat{\partial}_{\alpha \beta'})\).

**Proposition 2.3** Let \((\Sigma, \alpha, \beta)\) be a \( \delta \)-adapted Heegaard diagram of \( Y \) and denote by \( Y^\delta \) the manifold obtained from \( Y \) by composing the gluing map given by the attaching curves \( \alpha, \beta \) with a positive Dehn twist along \( \delta \) as indicated in Figure 3. Then the following holds:

\[
\hat{\text{HF}}(Y^\delta) \cong H_s(\hat{\text{CF}}(\alpha, \beta) \oplus \hat{\text{CF}}(\alpha, \delta), \partial^f),
\]

where \( \partial^f \) is of the form

\[
\begin{pmatrix}
\hat{\partial}_w^\alpha \beta & f \\
0 & \hat{\partial}_w^\alpha \delta
\end{pmatrix}
\]

with \( f \) the chain map between \((\hat{\text{CF}}(\alpha, \delta), \hat{\partial}_w^\alpha \delta)\) and \((\hat{\text{CF}}(\alpha, \beta), \hat{\partial}_w^\alpha \beta)\) given in Definition 2.2.
As a consequence of this fact we deduce the existence of two exact sequences which we call the Dehn twist sequences

\[(2-2) \ldots \xrightarrow{f_*} \widehat{\text{HF}}(Y, K) \stackrel{\Gamma_1}{\longrightarrow} \widehat{\text{HF}}(Y_{-1}(K)) \stackrel{\Gamma_2}{\longrightarrow} \widehat{\text{HF}}(Y_{0}(K), \mu) \xrightarrow{f_*} \ldots\]

\[(2-3) \ldots \xrightarrow{f_*} \widehat{\text{HF}}(Y_{0}(K), \mu) \stackrel{\Gamma_2}{\longrightarrow} \widehat{\text{HF}}(Y_{+1}(K)) \stackrel{\Gamma_1}{\longrightarrow} \widehat{\text{HF}}(Y, K) \xrightarrow{f_*} \ldots\]

where \(\mu\) is a meridian of \(K\) in \(Y\), interpreted as sitting in \(Y_{0}(K)\). The Dehn twist sequences admit some invariance properties which are similar to those of the surgery exact sequences in Heegaard Floer theory. For details we point the reader to [18].

### 3 Surgery Exact Triangle and Dehn Twist Sequence

The purpose of this section is to study the Dehn twist sequences and their relationship to maps induced by cobordisms. Recall that the maps involved in the definition of the Dehn twist sequence are not the usual cobordism maps. Suppose we are given a closed, oriented 3-manifold \(Y\), a knot \(K \subset Y\) and a framed knot \(L\) disjoint from \(K\).

We can define a map induced by a surgery along \(L\) in the following way (cf. §2.3): We choose a Heegaard diagram \((\Sigma, \alpha, \beta)\) of \(Y\) which is adapted to the link \(K \sqcup L\). The link \(K \sqcup L\) is isotopic to a two-component link on the Heegaard surface, each of its components being a longitude of a torus component of \(\Sigma\). The knot \(K\) induces a pair of points \((w, z)\) on \(\Sigma\) such that the doubly-pointed Heegaard diagram \((\Sigma, \alpha, \beta, w, z)\) encodes the pair \((Y, K)\). Performing a surgery along \(L\) induces a third set of attaching circles \(\gamma = \{\gamma_1, \ldots, \gamma_g\}\), \(g\) being the genus of \(\Sigma\). The doubly-pointed Heegaard triple diagram \((\Sigma, \alpha, \beta, \gamma, w, z)\) induces a map

\[\widehat{F}_{\alpha, \beta, \gamma}: \widehat{\text{CFK}}(\Sigma, \alpha, \beta, w, z) \longrightarrow \widehat{\text{CFK}}(\Sigma, \alpha, \gamma, w, z)\]

by counting holomorphic triangles with \(n_{z} = n_{w} = 0\) like introduced in §2.3. This map descends to a map between the associated homology theories.

In the following, suppose that the knots \(L\) and \(K\) are isotopic, more precisely, \(K\) is a push-off of \(L\) representing its framing. In this particular situation, we can choose a Heegaard diagram which is adapted to both \(K\) and \(L\) such that \(K\) and \(L\) sit in the same torus component of \(\Sigma\). This situation is somewhat special and can be realized in this particular situation, only. Without loss of generality, we write \(\Sigma\) as \(T^2 \# \Sigma'\) and we may assume that both \(K\) and \(L\) sit in \(T^2\). Hence, \(\beta_1\) is a meridian of \(K\) (and hence of \(L\)). There is a longitude \(\lambda\) of \(T^2\) which represents the framing of \(L\). In the following, the framing of \(L\) will be our reference framing. The points \(w\) and \(z\) will lie one on each side of the curve \(\beta_1\) (see left part of Figure 4). We perform a \((-1)\)-surgery along \(L\).
which changes the $\beta_1$-curve to $\gamma_1 = \beta_1 + \lambda$. By choosing $\gamma_i$, $i \geq 2$ as small isotopic translates of the $\beta_i$, the set $\gamma = \{\gamma_1, \ldots, \gamma_g\}$ is a Heegaard diagram of the surgered manifold. The effect of the surgery to the $\beta$-circles is, that we applied to $\beta_1$ a Dehn twist about $\lambda$. For our purposes, it is necessary to move the point $w$ over $\lambda$, once (see right part of Figure 4). This movement corresponds to an isotopy of $K$ in $Y$ which makes the knot $K$ cross $L$, once. The new knot will correspond to the $(-1)$-framing of $L$. Denote by $W$ the cobordism induced by the surgery. We see that after the surgery the knot induced by the pair of base points at the right of Figure 4 will be the unknot, i.e. we have a map

$$\tilde{F}_1 : \hat{\text{HFK}}(Y, K) \longrightarrow \hat{\text{HFK}}(Y_{-1}(L), U) = \hat{\text{HF}}(Y_{-1}(K)).$$

We would like to show that this map sits in the exact triangle (see Proposition 1.2)

$$\ldots \longrightarrow \partial_* \hat{\text{HFK}}(Y, K) \longrightarrow \tilde{F}_1 \hat{\text{HFK}}(Y_{-1}(K)) \longrightarrow \tilde{F}_2 \hat{\text{HFK}}(Y_{0}(K), \mu) \longrightarrow \partial_* \ldots$$

where the maps $\tilde{F}_i$, $i = 1, 2$ are defined by counting holomorphic triangles in suitable Heegaard triple diagrams. We will try to relate this sequence to the Dehn twist sequences introduced in [18]. To do that we will carefully analyze the involved Heegaard diagrams. Although not essential, it is opportune to work with Heegaard diagrams that are induced by open books. In this particular situation the analysis of the Heegaard diagrams and of the domains of Whitney triangles become easier. So, suppose we are given an abstract open book $(P, \phi)$. This abstract open book induces a Heegaard diagram $(\Sigma, \alpha, \beta, z)$ by the algorithm given in §2.3.1. Without loss of generality we may think this Heegaard diagram to be adapted to the knot $K$ (and hence adapted to $L$). Indeed, we may think $L$ to be isotopic to a homologically essential, simple closed curve $\delta$ on the page $P$ of the abstract open book which intersects $\beta_1$ once, transversely and is disjoint from the other $\beta$-circles (see [2, Corollary 4.23] and [18, Lemma 4.2]). We continue by defining the following sets of attaching circles

$$\beta' = \{\beta'_1, \ldots, \beta'_g\}$$

$$\tilde{\delta} = \{\delta, \beta''_2, \ldots, \beta''_g\},$$

where $\beta'_1 = D^+_\delta(\beta_1)$ and $D^+_\delta$ denotes a positive Dehn twist about $\delta$. The $\beta'_i$, $i \geq 2$, are isotopic push-offs of the $\beta_i$ such that $\beta_i$ and $\beta'_i$ intersect in a cancelling pair of intersection points. Furthermore, let $\beta''_i$, $i \geq 2$, be push-offs of the $\beta'_i$. As above, the push-offs are chosen such that the $\beta''_i$ and $\beta'_i$ intersect in a cancelling pair of intersection points. The curve $\tilde{\delta}$ is given as a perturbation of the curve $\delta$, like indicated in Figure 5. Since the Heegaard diagram is induced by an open book, the $\alpha_1$-curve in Figure 5 is surrounded by the domain of the base point $z$. Figure 5 additionally shows
Holomorphic Discs and Surgery Exact Triangles

Figure 4: Movement of $w$ representing an isotopy of $K$ that makes $K$ cross $L$, once.

Figure 5: The relevant attaching circles.

the surface orientation. We will have a close look at the following two cobordism maps:

$$
\tilde{F}_{\alpha, \beta' \delta} : \hat{CFK}(\alpha, \beta, z, w) \longrightarrow \hat{CF}(\alpha, \beta', z).
$$

Since the Heegaard surface $\Sigma$ remains fixed throughout our discussion we suppressed it from the notation. These two cobordism maps correspond to the maps $\tilde{F}_1$ and $\tilde{F}_2$ of the Sequence (1–2). Denote by $\delta$ the set of attaching circles $\{\delta, \beta'_2, \ldots, \beta'_g\}$. By considerations from [18] (cf. §2.4) we see that we have a short exact sequence of chain
complexes.

\[(3-1) \quad 0 \xrightarrow{\Gamma_1} \CFK(\alpha, \beta, z, w) \xrightarrow{\Gamma_2} \CF(\alpha, \beta', z) \xrightarrow{\Gamma_3} \CFK(\alpha, \delta, z, w) \xrightarrow{\Gamma_4} 0.\]

We see that the sequences given in (1–2) and (3–1) coincide at the middle term, namely at \(\CF(\alpha, \beta', z)\).

**Lemma 3.1** The maps \(\tilde{F}_{\alpha, \beta'}\) and \(\tilde{F}_{\alpha' \beta} \circ \tilde{F}_{\alpha, \beta'}\) respect the splitting of \(\CF(\alpha, \beta', z)\), given in Proposition 2.3, i.e. given by the Sequence (3–1).

**Proof** We show that the claim is true for the map \(\tilde{F}_{\alpha, \beta'}\). We look at Figure 6 and try to show that there is no holomorphic triangle from an \(\alpha \beta\)-intersection to an \(\alpha \delta\)-intersection (cf. §2.4) that contributes to \(\tilde{F}_{\alpha, \beta'}\): Let \(\phi\) be a triangle that connects a point \(x \in T_\alpha \cap T_\beta\) with a point \(y \in T_\alpha \cap T_\delta \subset T_\alpha \cap T_\beta'\). The triangle \(\phi\) connects \(y\) with \(\hat{\Theta}\) along its \(\beta'\)-boundary. In Figure 7, we illustrate the two possible ways to do that. In both cases the \(\beta'\)-boundary of \(\phi\) follows the black arrow pictured there. We either cause a non-negative intersection number \(n_w\) (cf. left of Figure 7) or a non-negative intersection number \(n_z\) (cf. right part of Figure 7). Thus, \(n_w(\phi) \neq 0\) or \(n_z(\phi) \neq 0\), which shows that \(\phi\) does not contribute to \(\tilde{F}_{\alpha, \beta'}\). A similar line of arguments can be used to prove the claim for \(\tilde{F}_{\alpha' \beta}\). \(\square\)

As a consequence of the last lemma we see that

\[\tilde{F}_{\alpha' \beta} \circ \tilde{F}_{\alpha, \beta'} = 0.\]

Indeed, we can prove the following result.

**Lemma 3.2** The diagram

\[
\begin{array}{ccc}
\CFK(\alpha, \beta, z, w) & \xrightarrow{\tilde{F}_{\alpha, \beta'}} & \CF(\alpha, \beta', z) \\
\CFK(\alpha, \beta, z, w) & \xleftarrow{\iota} & \CF(\alpha, \beta', z)
\end{array}
\]

commutes where \(\iota\) denotes the inclusion induced by a natural identification of generators.

For simplicity let us denote by \(h\) the map \(\tilde{F}_{\alpha, \beta'}\) and by \(g\) the map \(\tilde{F}_{\alpha' \beta}\). There is a canonical inclusion

\[\iota : \CFK(\alpha, \beta, z, w) \rightarrow \CF(\alpha, \beta', z, w)\]
induced by an identification of intersection points. Namely, observe that
\[
T_{\alpha} \cap T_{\beta'} = T_{\alpha} \cap T_{\beta} \cup T_{\alpha} \cap T_{\delta}
\]
\[
= T_{\alpha} \cap T_{\beta} \cup T_{\alpha} \cap T_{\delta}
\]
in case \( \bar{\beta} \) is a suitable perturbation of \( \beta \) we will define in a moment. We define \( \bar{\beta}_i = \beta_i \), for all \( i \geq 2 \), and \( \bar{\beta}_1 \) as indicated in Figure 8 (see also Figure 5). We would like to show that \( h = \iota \circ g \).

**Definition 3.3** Let \((\Sigma, \alpha, \beta, z)\) be a Heegaard diagram and denote by \( D_1, \ldots, D_k \) the components of \( \Sigma \setminus \{ \alpha \cup \beta \} \). We say that a Whitney disc \( \phi \) does not use a domain \( D_i \), \( i \in \{ 1, \ldots, k \} \), if the domain \( D_i \) does not appear in \( D(\phi) \), i.e. writing \( D(\phi) \) as

\[
D(\phi) = \sum_{j=1}^{k} d_j \cdot D_j,
\]

the coefficient \( d_i \) vanishes. We also say that the domain \( D(\phi) \) does not use \( D_i \).

The main idea is to first prove that given intersections \( x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \), all positive domains \( D \), i.e. all coefficients in \( D \) are greater than or equal to 0, connecting \( x \) and \( y \), with \( n_w(D) = n_z(D) = 0 \), do not use certain components of \( \Sigma \setminus \{ \alpha \cup \beta \} \) or \( \Sigma \setminus \{ \alpha \cup \beta' \} \). Which domains are expected not to be used is indicated in Figure 8, the left part illustrating the situation for \( h \), the right part illustrating the situation for \( g \). With this information, we compare the boundary conditions of holomorphic triangles for \( h \) and \( g \). The conclusion will be that, with its \( \beta' \)-boundary, the holomorphic triangles counted by \( h \) always stay inside \( \mathbb{T}_{\beta'} \cap \mathbb{T}_{\bar{\beta}} \), and, with their \( \bar{\beta} \)-boundary, holomorphic triangles counted by \( g \) stay inside \( \mathbb{T}_{\bar{\beta}} \cap \mathbb{T}_{\beta'} \). Thus, we are able to identify the moduli spaces of holomorphic triangles contributing to \( h \) and \( g \) with arguments similar to those used in the proof of Proposition 2.3.

**Proof** Figure 8 shows the part of the Heegaard triple diagrams where the boundary conditions for the holomorphic triangles involved in the definition of \( h \) and \( g \) differ. The picture illustrates which regions are not used by holomorphic triangles that contribute to \( h \) and \( g \). This has to be shown in the following: We start our discussion with the map \( h \) and look at Figure 9. Each part of Figure 9 covers one of the cases which we will discuss in the following. The different parts of Figure 9 show parts of the Heegaard diagram pictured in the left of Figure 8. We focused on those parts important to our arguments. Denote by \( \phi \) a holomorphic triangle that contributes to \( h \). The domains, which we want to show not to be used by \( \phi \), will be denoted by \( D_{x_i}, i = 1, 2, 3 \). In each of these regions we fix a point \( x_i, i = 1, 2, 3 \). If \( \phi \) uses one of the domains \( D_{x_i} \), the associated intersection number \( n_w \) is non-zero. Suppose the domain \( D(\phi) \) has non-trivial intersection number \( n_{w_i} \) (cf. left part of Figure 9). This means we generate a \( \beta' \)-boundary pointing inside \( D_w \), as indicated by the black arrow in the left part of Figure 9. Consequently, \( n_w \) has to be non-zero. Supposing the domain \( D(\phi) \) would
have non-trivial intersection number $n_x$, $i = 1, 2, 3$ have to be trivial. We continue arguing that holomorphic triangles contributing to $g$, cannot use the domains indicated in the right part of Figure 8: Let $\phi$ be a holomorphic triangle contributing to $g$. Analogous to the discussion done for $h$, we denote the regions not expected to be used by $\phi$ with $D_{x_i}$, $i = 1, 2$. In each of the domains we fix a point $x_i$. We want to show that a non-zero $n_{x_i}$ for $i \in \{1, 2\}$ implies that $n_w \neq 0$ or $n_z \neq 0$. The different parts of Figure 10 show parts of the Heegaard diagram pictured at the right of Figure 8. Suppose the domain $D(\phi)$ has non-trivial intersection number $n_{x_i}$ (cf. left part of Figure 10). Since $n_w = 0$, we generate a $\beta$-boundary pointing inside $D_x$, as it is indicated in the left part of Figure 10 (the boundary follows the black arrow). We see that $n_z \neq 0$. If the domain $D(\phi)$ would have non-trivial intersection number $n_{x_2}$ (cf. right part of Figure 10), then, since $n_z = 0$, we would generate a $\beta$-boundary pointing inside $D_w$ (cf. right part of Figure 10). But would imply that $n_w$ is non-zero. Thus, using arguments that are similar to those applied in the proof of Proposition 2.3, we can identify the moduli spaces of holomorphic triangles that contribute to $h$ and $g$. 

Figure 9: Here we see why $n_{x_i}$, $i = 1, 2, 3$ have to be trivial.
Lemma 3.4  The diagram

\[
\begin{align*}
\widehat{\text{CF}}K(\alpha, \delta, z, w) \\
\Rightarrow & \quad \widehat{\text{CF}}(\alpha, \beta', z) \\
\pi & \Rightarrow \widehat{\text{CF}}K(\alpha, \tilde{\delta}, z, w)
\end{align*}
\]

commutes where \( \pi \) is the projection induced by a natural identification of generators.

Proof  The proof is analogous to the proof of Lemma 3.2. Analogous to \( \iota \) we define
the projection \( \pi \) by identifying

\[
T_{\alpha} \cap T_{\beta'} = T_{\alpha} \cap T_{\beta} \cup T_{\alpha} \cap T_{\delta}
\]

i.e. by identifying \( T_{\alpha} \cap T_{\delta} \) with \( T_{\alpha} \cap T_{\tilde{\delta}} \). This induces a projection \( \pi \) between the respective chain modules. In the following we will denote by \( h \) the map \( \widehat{F}_{\alpha, \beta} \) and by \( g \) the map \( \widehat{F}_{\alpha, \beta} \). This time, we would like to show that \( h = g \circ \pi \). Figure 11 indicates which domains are not used by holomorphic triangles (in the sense of Definition 3.3) that contribute to \( g \) and \( h \). This has to be shown in the following discussion: Observe, that each part of Figure 12 shows a part of the Heegaard diagrams pictured in Figure 11. Each of these portions will be relevant in one of the cases we will have to investigate. There are two domains not to be used by holomorphic triangles contributing to \( g \) (cf. left part of Figure 11). In each of these domains we fix a point \( x_i \) and denote the associated domain by \( D_{x_i}, i = 1, 2 \) (cf. left and middle part of Figure 12). There is one domain not to be used by triangles contributing to \( h \) (cf. right part of Figure 11). We fix a point \( x_3 \) in this domain and denote the associated domain by \( D_{x_3} \) (cf. right of Figure 12). Let \( \phi \) be a holomorphic triangle that contributes to \( g \). Suppose the domain
\( \mathcal{D}(\phi) \) has non-trivial intersection number \( n_{x_1} \) (cf. left part of Figure 12). This generates a \( \beta' \)-boundary like indicated by the black arrow in the left portion of Figure 12. This boundary cannot be killed, i.e. cannot be interpreted as sitting in the interior of \( \mathcal{D}(\phi) \), since \( n_w = 0 \). This \( \beta' \)-boundary, thus, has to emanate from \( \hat{\Theta} \), forcing it to follow the black arrow like indicated. Thus, \( n_z \) is non-zero. Supposing the domain \( \mathcal{D}(\phi) \) would have non-trivial intersection number \( n_{x_2} \) (cf. middle part of Figure 12), this would create a \( \beta' \)-boundary like indicated by the black arrow in the middle portion of Figure 12. This boundary points towards \( \hat{\Theta} \). But recall that the \( \beta' \)-boundary of \( \phi \) has to emanate from \( \hat{\Theta} \), as can be seen by looking at the triangle pictured at the top of the left and middle part of Figure 12. Thus, we would have to generate a \( \beta' \)-boundary going along \( \beta' \) once, completely. But this would imply \( n_w \) to be non-zero. Now, let \( \phi \) be a holomorphic triangle that contributes to \( g \). Assuming the domain \( \mathcal{D}(\phi) \) would have non-trivial intersection number \( n_{x_3} \) (cf. right part of Figure 12), we would generate \( \delta \)-boundary like indicated by the black arrow in the right portion of Figure 12. This boundary cannot be killed, since \( n_z = 0 \). This boundary has to emanate from \( \hat{\Theta} \) as can be seen by looking at the triangle pictured at the top of the right part of Figure 12. But this is impossible, since \( n_w = 0 \). We have seen that holomorphic triangles, that contribute to \( h \) or \( g \), do not use the domains indicated in Figure 11. Again, using arguments that are similar to those applied in the proof of Proposition 2.3, we are able to identify the moduli spaces of holomorphic triangles that contribute to \( h \) and \( g \). This shows that \( h = g \circ \pi \).

**Figure 11:** Comparing the boundary conditions of \( \hat{F}_{\alpha, \beta' \delta} \) and \( \hat{F}_{\alpha, \delta \delta} \).
Figure 12: Here, we see why \( n_i, i = 1, 2 \) have to be trivial for \( f \) and why \( n_3 \) has to be trivial for \( g \).

**Proof of Proposition 1.2** From Lemma 3.2 and Lemma 3.4 we conclude that

\[
0 \xrightarrow{} \hat{\text{CFK}}(\alpha, \beta, z, w) \xrightarrow{\hat{F}_{\alpha, \beta'}} \hat{\text{CF}}(\alpha, \beta', z) \xrightarrow{\hat{\text{P}}_{\alpha, \beta'}} \hat{\text{CFK}}(\alpha, \hat{\delta}, z, w) \xrightarrow{\hat{\text{CFK}}(\alpha, \delta, z, w) \xrightarrow{0}}
\]

is a short exact sequence of chain complexes, since \((3–1)\) is a short exact sequence of chain complexes. Thus, by standard algebraic topology we obtain the following long exact sequence between the respective homologies.

\[
(3–3) \quad \ldots \xrightarrow{\partial_*} \hat{\text{HFK}}(Y, K) \xrightarrow{\hat{F}_1} \hat{\text{HF}}(Y_{-1}(K)) \xrightarrow{\hat{F}_2} \hat{\text{HFK}}(Y_0(K), \mu) \xrightarrow{\partial} \ldots
\]

Observe, that \( \partial_* \) is the connecting homomorphism of the short exact sequence. \( \square \)

We are now ready to prove the main result by combining both Lemma 3.2 and Lemma 3.4. This result provides an answer to Question 1.

**Proof of Theorem 1.3** We put together Lemma 3.2 and Lemma 3.4 to get two short exact sequences of chain complexes that are related like claimed, i.e. we have

\[
0 \xrightarrow{} \hat{\text{CFK}}(\alpha, \delta, z, w) \xrightarrow{0}
\]

To identify the diagonal sequence, i.e. the sequence

\[
0 \xrightarrow{} \hat{\text{CFK}}(\alpha, \hat{\beta}, z, w) \xrightarrow{\hat{\text{CF}}} \hat{\text{CF}}(\alpha, \beta', z) \xrightarrow{\pi} \hat{\text{CFK}}(\alpha, \delta, z, w) \xrightarrow{0}
\]
with the Dehn twist Sequence $(2–2)$, we have to isotope $\tilde{\beta}_1$ a bit. Observe, that $\tilde{\beta}_1$ does not match with the situation presented in §2.4. The isotopy, however, is supported within $D_z \cup D_w$. Furthermore, recall that an isotopy not generating/cancelling intersection points, acts on the Heegaard Floer homology as a perturbation $J_s$ of the path of almost complex structures $J_{s,0}$ (see [13, §6 and §7.3] or cf. [19, §4.2 and §4.3]) used in the definition of the Heegaard Floer homologies. We have to see that the induced map $\hat{\Phi}_{J_s,t}$ (cf. [13, §6]) is the identity on the chain level: In the definition $\hat{\Phi}_{J_s,t}$, we count 0-dimensional components of holomorphic discs with $n_z = n_w = 0$. The family $J_{s,t}$ coincides with $J_{s,0}$ outside of a set, which is contained in $(D_z \cup D_w) \times \text{Sym}^{g–1}(\Sigma)$, since the isotopy perturbing $\tilde{\beta}_1$ is supported in $D_z \cup D_w$. Thus, for $x, y \in T_\alpha \cap T_\beta$, we have an identification

$$\left( M_{J_{s,t}}(x,y) \right)_{\mu=0, n_z=n_w=0} = \left( M_{J_{s,0}}(x,y) \right)_{n_z=n_w=0},$$

where the notation should indicate that we are interested in moduli spaces with Maslov index 0 and whose elements satisfy $n_z = n_w = 0$. The moduli space on the right of Equation (3–4), in the following denoted by $M$, is empty unless $x = y$: Suppose there is a holomorphic Whitney disc $\phi$ connecting $x$ with $y$. Assuming that $x$ and $y$ are not equal, the disc $\phi$ is non-constant. So, because of the translation action (cf. §2.1) the disc $\phi$ comes in a 1-dimensional family. Thus, $\phi$ cannot be an element of $M$. If $x$ and $y$ are the same point, the moduli space $M$ contains the constant holomorphic disc. But it does not contain non-constant holomorphic discs by the same reasoning done for $x \neq y$.

Consequently, the map $\hat{\Phi}_{J_{s,t}}$ is the identity on the chain level. We know from [13, §6] that the map $\hat{\Phi}_{J_{s,t}}$ is a chain map, i.e. we have

$$0 = \partial^w_{J_{s,1}} \circ \hat{\Phi}_{J_{s,t}} - \hat{\Phi}_{J_{s,t}} \circ \partial^w_{J_{s,0}} = \partial^{w}_{J_{s,1}} - \partial^{w}_{J_{s,0}}.$$

Thus, the signed count of holomorphic discs with Maslov index 1 in both $\text{CFK}(\alpha, \beta, z, w)$ and $\text{CFK}(\alpha, \tilde{\beta}, z, w)$ equals for each homotopy class admitting holomorphic representatives. Thus, we may replace the map $\iota$ with $\Gamma_1$. The map $\pi$ already equals $\Gamma_2$.

As a consequence of these results, it is possible to refine the maps $\Gamma_1$ and $\Gamma_2$. We would like to indicate how this is done: Note, that the triple diagram $(\Sigma, \alpha, \beta, \beta')$ comes from a 2-handle attachment (or, more precisely, from a $(-1)$-surgery along $K$). Denote by $W$ the associated cobordism. The map $\tilde{F}_{\alpha, \beta \beta'}$ refines with respect to $\text{Spin}^c$-structures of the cobordism $W$ (see §2.3 and cf. [16, §4.1]). To be precise, for
\( s \in \text{Spin}^c(W) \) denote by \( s_1 \) and \( s_2 \) its restriction onto \( Y \) and \( Y_{-1}(K) \). Then the map \( \hat{F}_1 \) refines to a map

\[
\hat{F}_{1,s} : \hat{HFK}(Y, K; s_1) \longrightarrow \hat{HFK}(Y_{-1}(K); s_2).
\]

The map \( \hat{F}_4 \) is given as the map induced in homology by \( \hat{F}_{\alpha, \beta, \overline{\beta}} \). The underlying triple diagram \((\Sigma, \alpha, \beta, \overline{\beta})\) represents the trivial cobordism, since we obtain \( \overline{\beta} \) from \( \beta \) by isotopies (see Figure 5) which do not create/change any intersection points with the \( \alpha \)-circles. The \( \text{Spin}^c \)-structures of the trivial cobordism \([0, 1] \times Y\) coincide with the \( \text{Spin}^c \)-structures of \( Y \). Thus, \( \hat{F}_4 \) refines to

\[
\hat{F}_{4,s_1} : \hat{HFK}(Y, K; s_1) \longrightarrow \hat{HFK}(Y, K; s_1)
\]

for \( s_1 \in \text{Spin}^c([0, 1] \times Y) = \text{Spin}^c(Y) \). According to Theorem 1.3, if we define \( \Gamma_{1,s} \) to be the restriction of \( \Gamma_1 \) onto \( \hat{HFK}(Y, K; s_1) \), then we get a map

\[
\Gamma_{1,s} : \hat{HFK}(Y, K; s_1) \longrightarrow \hat{HFK}(Y_{-1}(K); s_2).
\]

In a similar fashion, it is possible to define refinements \( \Gamma_{2,s} \) for the map \( \Gamma_2 \) and refinements \( f_{s_1} \) for \( f_s \). Given a \( \text{Spin}^c \)-structure \( t \) of \( Y \setminus \nu K \), denote by \( t(Y) \), \( t(Y_{-1}(K)) \) and \( t(Y_0(K)) \) the set of extensions of \( t \) to \( Y, Y_{-1}(K) \) and \( Y_0(K) \), respectively. There is a refined version of the Dehn twist sequence we may derive, namely

\[
\begin{array}{ccc}
\hat{HFK}(Y, K; t(Y)) & \overset{\Gamma}{\longrightarrow} & \hat{HFK}(Y_{-1}(K); t(Y_{-1}(K))) \\
\downarrow f & & \downarrow \hat{\Gamma} \\
\hat{HFK}(Y_0(K), \mu; t(Y_0(K)))
\end{array}
\]

(3–5)

where \( \Gamma \) is the sum of \( \Gamma_{1,s} \) for all \( \text{Spin}^c \)-structures \( s \in \text{Spin}^c(W) \) extending elements \( t(Y) \) and \( t(Y_{-1}(K)) \) and, analogously, the maps \( \Gamma \) and \( \hat{\Gamma} \) are defined. The exactness of (3–5) follows from Theorem 1.3 and the fact, that we may refine the horizontal sequence in (1–3) in exactly the same way (cf. [10, Theorem 14.3.2])

Similarly, applying [13, Lemma 8.7] we see that for an orientation system \( o \) on \( \hat{HFK}(Y, K) \) there is an orientation system \( o' \) for \( \hat{HFK}(Y_{-1}(K)) \) and an orientation system for \( \hat{F}_1 \) which are compatible (in the sense of [13, Lemma 8.7]), i.e. the map \( \hat{F}_1 \) can be defined with \( \mathbb{Z} \)-coefficients. The same we do with \( \hat{F}_4 \), however, as above, using that the associated cobordism is \([0, 1] \times Y\). We obtain that \( \hat{F}_4 \) can be defined with \( \mathbb{Z} \)-coefficients for the orientation system \( o \) on the source and target \( \hat{HFK}(Y, K) \). Thus, using the commutativity of the diagram in Theorem 1.3, or using the chain level version derived in the proof of Theorem 1.3, we may give a version of \( \Gamma_1 \) with \( \mathbb{Z} \)-coefficients.
4 Proofs of Theorem 1.6 and Proposition 1.7

Suppose we are given a closed, oriented 3-manifold \( Y \) and a framed knot \( L \subset Y \) with framing we denote by \( n \). Denote by \( K \) a push-off of \( L \) corresponding to the \((n + 1)\)-framing of \( L \). The goal of the following discussion will be to associate to a \( n \)-surgery along \( L \) a connecting morphism \( f_* \) (see Definition 2.2 and cf. Proposition 2.3) from a suitable Dehn twist sequence. Denote by \( K' \) a push-off of \( L \) corresponding to the \((n + 1)\)-framing of \( L \). The goal of the following discussion will be to associate to a \( n \)-surgery along \( L \) a connecting morphism \( f_* \) (see Definition 2.2 and cf. Proposition 2.3) from a suitable Dehn twist sequence. Denote by \( (P, \phi) \) an open book decomposition of \( Y' = Y_n(L) \) which is adapted to \( K' \sqcup L' \) (in the sense of §3). Here, \( L' \) denotes the knot \( K' \) represents in the surgered manifold \( Y_n(L) \) with framing given by \( n + 2 \) and \( K' \) is a push-off of \( L' \) representing its \( n + 1 \)-framing. Denote by \( (\Sigma, \alpha, \beta, w, z) \) the doubly-pointed Heegaard diagram induced by \( (P, \phi) \) like constructed in §3. Using the notation from that section, we would like to show that the diagram \( (\Sigma, \alpha, \beta, w, z) \) is a Heegaard diagram of \( Y \) which is adapted to the knot \( K \). If we are able to show this, then the Dehn twist sequence will read

\[
\begin{array}{ccc}
\hat{\text{HFK}}(\Sigma, \alpha, \delta, z, w) & \xrightarrow{f_*} & \hat{\text{HFK}}(\Sigma, \alpha, \beta, z, w) \\
\hat{\text{HF}}(\Sigma, \alpha, \beta', z) & \xleftarrow{\Gamma_1} & \hat{\text{HF}}(\Sigma, \alpha, \beta', z)
\end{array}
\]

(4–1)

and we see, that the connecting morphism \( f_* \) in this Dehn twist sequence is a morphism between the knot Floer homologies of the pair \((Y, K)\) and of the pair \((Y_n(L), K')\): To prove the claim, we will give a surgical description of the manifold represented by \((\Sigma, \alpha, \delta, w, z)\). Using the notation from §3 we know that the Heegaard diagram \((\Sigma, \alpha, \delta, w, z)\) represents the pair \((Y_{n+2}(L'), \mu)\), where \( \mu \) is a meridian of \( L' \) in \( Y' \), interpreted as sitting in \( Y_{n+2}(L') \). Just note, that in §3 we measured the surgery coefficients with respect to the page framing of \( L' \) induced by \((P, \phi)\). Thus, the 0-framing from §3 corresponds to the specified surgery framing of \( L' \), i.e. the \((n + 2)\)-framing. We obtain the surgery description of \((Y_{n+2}(L'), \mu)\) given in the left of Figure 13. We slide the knot \( \mu \) over \( L' \), once, and obtain the middle portion of Figure 13. Finally, we perform an inverse handle slide of \( L' \) over \( L \) to obtain the right portion of Figure 13. With a slam dunk, we can slide \( L' \) off \( L \) which makes both surgery curves disappear in the surgery diagram. The resulting manifold is \( Y \) (as the surgeries we did now disappeared) and, since \( \mu \) is a parallel of \( L \) (see right of Figure 13), the knot \( \mu \) equals the knot \( K \).

Thus, the connecting morphism which can be defined as in Definition 2.2 induces a map

\[
f_* : \hat{\text{HFK}}(Y, K) \longrightarrow \hat{\text{HFK}}(Y_n(L), K').
\]
Before we continue with the proofs of the main results of this section, we would like to make the following observation.

**Lemma 4.1** In Theorem 1.1 we can drop the condition that \( K \subset Y \setminus C \) is null-homologous.

**Sketch of Proof** We go through the mapping cone proof of the surgery exact triangle (see [14, Theorem 4.5], cf. also the end of the proof of Proposition 2.1 in [14] and cf. [19, §7]) and see that we do not need the condition.

**Proof of Theorem 1.6** Suppose we are given a manifold \( Y' \) with knot \( K' \) in it with framing \( n \). Denote by \( Y \) the result of an \( n \)-surgery along \( K' \) and denote by \( K' \) the knot induced from the push-off of \( K' \) corresponding to the \((n - 1)\)-framing of \( K' \). From the considerations given in this section we know that to this situation we may associate a map \( f_\ast \) (defined as in Definition 2.2) which is part of a Dehn twist sequence, i.e. we may set up a Dehn twist sequence which reads

\[
\widehat{\text{HFK}}(Y', K') \xrightarrow{f_\ast} \widehat{\text{HFK}}(Y, K) \\
\downarrow \quad \downarrow \\
\text{HF}(Y'')
\]

for some manifold \( Y'' \). In this situation, \( Y'' \) will correspond to \( Y_{-1}(K) \) and the pair \((Y', K')\) will equal \((Y_0(K), \mu)\).

On the other hand, we may apply Proposition 1.2 to the pair \((Y, K)\) to get the following exact sequence

\[
(4–2)
\]

Furthermore, by Theorem 1.1 (and the observation formulated in Lemma 4.1) we see that the pair \((Y, K)\) induces a surgery exact triangle which reads

\[
(4–3)
\]
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Since the maps $\tilde{F}_i$, $i = 1, 2$, appear in both the Sequences (4–2) and (4–3), by exactness we conclude that

\begin{equation}
\begin{split}
\text{ker}(\tilde{F}_3) &= \text{ker}(\partial_*) \\
\text{im}(\tilde{F}_3) &= \text{im}(\partial_*).
\end{split}
\end{equation}

Now, we continue going back to the situation of Theorem 1.3, especially considering the notations used there. We consider the commutative diagram

\begin{equation}
\begin{array}{ccc}
\widehat{\text{HFK}}(Y, K) & \xrightarrow{f_*} & \widehat{\text{HFK}}(Y_0(K), \mu) \\
\tilde{F}_4 & \downarrow & \tilde{F}_5 \\
\widehat{\text{HFK}}(Y, K) & \xrightarrow{\partial_*} & \widehat{\text{HFK}}(Y_0(K), \mu)
\end{array}
\end{equation}

where $\tilde{F}_4$ is the map in homology induced by $\tilde{F}_{\alpha,\beta\tilde{\beta}}$ and $\tilde{F}_5$ is the map in homology induced by $\tilde{F}_{\alpha,\delta\tilde{\beta}}$. This diagram is the square from Sequence (1–3) and which commutes according to Theorem 1.3. As we have observed in (4–4), the kernel and the image of $\tilde{F}_3$ coincides with the kernel and image of $\partial_*$. Thus, we have that

\begin{align*}
\text{im}(f_*) &= \text{im}(\tilde{F}_5 \circ \partial_* \circ \tilde{F}_4) = \text{im}(\tilde{F}_5 \circ \tilde{F}_3 \circ \tilde{F}_4) \\
\text{ker}(f_*) &= \text{ker}(\tilde{F}_5 \circ \partial_* \circ \tilde{F}_4) = \ker(\tilde{F}_5 \circ \tilde{F}_3 \circ \tilde{F}_4)
\end{align*}

Observe, that the map $\tilde{F}_3$ is given as the map in homology induced by the Heegaard triple diagram $(\Sigma, \alpha, \tilde{\delta}, \beta)$. The composition $\tilde{F}_5 \circ \tilde{F}_3 \circ \tilde{F}_4$ equals

\begin{equation}
\tilde{F}_{\alpha,\beta\tilde{\beta}} \circ \tilde{F}_{\alpha,\beta\tilde{\beta}} \circ \tilde{F}_{\alpha,\delta\tilde{\beta}} = \tilde{F}_{\alpha,\beta\tilde{\beta}} \circ \tilde{F}_{\alpha,\delta\beta} = \tilde{F}_{\alpha,\delta\beta}.
\end{equation}
where we used the composition law for cobordism maps and the fact that $\tilde{F}_3$ is the map, which is induced by $\tilde{F}_{\alpha,\delta\beta}$ in homology. Hence, the image and kernel of the map $f_*$ coincides with the image and kernel of the map induced by $\tilde{F}_{\alpha,\delta\beta}$ in homology. However, the maps $\tilde{F}_3$ and $\tilde{F}_{\alpha,\delta\beta}$ are induced by the same 2-handle attachment, which can be seen by comparing the Heegaard triple diagrams $(\Sigma, \alpha, \delta, \beta)$ and $(\Sigma, \alpha, \delta, \beta)$.

**Proof of Proposition 1.7** Let $\tilde{F}$ be the map given in the statement of the proposition. By the discussion at the beginning of this section, we know that to the described surgery we may associate a map $f_*$ (as defined in Definition 2.2) which is part of a Dehn twist sequence. Furthermore, by Theorem 1.6 we know that the rank of the image and the kernel of $\tilde{F}$ and $f_*$ agree. Thus, to prove the statement, we have to give a reasoning why the kernel and image of $f_*$ can be computed, combinatorially:

Using the notation from the beginning of this section, we have to prove that it is possible to find an open book decomposition $(P, \phi)$ adapted to $L'$ and $K'$ such that the induced Heegaard diagram $(\Sigma, \alpha, \beta)$ is nice (see [21]). In [17], Plamenevskaya shows that the Sarkar-Wang algorithm (see [21]) can be modified to apply for open books by just using isotopies of the monodromy: To be more precise, it is possible to modify the monodromy $\phi$ with a suitable isotopy $\varphi_1$ such that the open book $(P, \phi')$, where $\phi' = \varphi_1 \circ \phi$, induces a nice Heegaard diagram. We apply her modified version of the Sarkar-Wang algorithm to obtain such an isotopy $\varphi_1$. The knot $K'$ (and respectively $L'$) can be modified with the isotopy $\varphi_1$, as well. The knot $K'$ is isotopic to a curve $\delta$ on the page $P$ of the open book (since it is an adapted open book by definition). The isotopy deforms $\delta$ into $\varphi_1(\delta)$. The open book $(P, \phi')$ is an open book decomposition adapted to $\varphi_1(\delta)$. To complete the proof, we have to see that $(\Sigma, \alpha, \beta')$ is nice, as well: To see this, recall (for instance from [18], proof of Lemma 4.2)) that $\delta$ (on the Heegaard surface) is parallel to $\beta_2$ outside of the region pictured in Figure 3 (cf. Figure 5).

Hence, the domains of the diagram $(\Sigma, \alpha, \beta')$ are, besides $D_z$, all obtained from the domains of $(\Sigma, \alpha, \beta)$ by splitting off a rectangle. The domain of the point $z$ in the new diagram is obtained by joining together $D_z$ and $D_w$ of the old diagram (see Figure 3). Thus, niceness is preserved. As $f_*$, by definition, is a part of the differential $\partial_{\alpha,\beta'}$ (see Proposition 2.3), its image and kernel can be now computed combinatorially.

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*Mathematisches Institut der LMU München, Theresienstrasse 39, 80333 München*

sahamie@math.lmu.de

http://www.math.lmu.de/~sahamie