Numerical method in riemann invariant form for a submerged bar breakwater model

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Abstract. Recently, Geotextiles are widely used for coastal protection in Indonesia. Geotextile is one of the implementations of a submerged bar as a breakwater. Here, we will study the damping mechanism of the Geotextile to reduce the incoming wave amplitudes through a mathematical model. The model that we use based on the Shallow Water Equations. Analytical solutions for wave transmission coefficient are derived using the Separation of Variables Method. Numerically, we construct a scheme using the Riemann Invariant Method to approximate the analytical model. For validation, the numerical results are compared against the analytical wave transmission coefficient, which resulted in a very good agreement between the two findings. Further, we investigate the effect of the changes in the dimension of a submerged bar to the reduction of the transmitted wave amplitude.

Keyword: Submerged bar breakwater, Shallow Water Equations, Riemann Invariant Method, Transmission coefficient.

1. Introduction
Geotextiles are permeable textile materials used in various civil engineering, coastal engineering, and other geotechnical applications. In the coastal protection and off-shore engineering area, geotextiles are mainly used for flood and water control. However, they also perform more basic functions such as shoreline protection, erosion control, and sediment control [1-3]. Wave's interaction with geotextiles appears to be somewhat identical to the behavior of waves with breakwaters, both emerged and submerged. Such interaction induces wave shoaling and wave breaking phenomena, which then generate transmitted and reflected waves. Therefore, geotextiles are often considered to be one of the applications of breakwaters and are widely used to protect coastal regions. In Indonesia, a number of studies have been undertaken on the implementation of geotextiles as submerged breakwaters on certain beaches, such as Tanjung Kait Beach, Tangerang, West Java, Indonesia [4] and Sigandu Beach, Central Java, Indonesia [5].

In addition, numerous studies have been performed on submerged breakwater itself for years, beginning with an analytical approach to investigating the propagation of waves over the breakwater in [6-9]. The above findings are extended in [10-13], where wave propagation experiments are conducted over various types of breakwater. To resolve the disadvantages of an experimental approach involving
excessive costs, many researchers investigated the interaction of waves with breakwater using a numerical method, such as [14-18]. However, many of these studies used complicated equations, such as the Boussinesq-type model, the Navier-Stokes Equations, or the RANS model, which are challenging to solve analytically and numerically. As a response, a few other researchers, such as [19-22], were using Shallow Water Equations as the governing equations, as it is easier to solve but still as accurate as other wave models. The numerical method used in these studies is the Staggered Grid Finite Volume Method. While the Finite Volume Method produces a numerical scheme that is easier and faster than other methods, it is difficult to compute the transmitted amplitude and the reflected amplitude separately. Consequently, we have not been able to properly analyze both amplitudes individually and there will always be an error caused by the problem. Meanwhile, the Riemann Invariant Method provides a very beneficial property where the transmitted and reflected wave can be calculated independently. Moreover, when we use the Riemann Invariant Method, the boundary conditions of the corresponding case can be treated more conveniently.

In this paper, we investigate the propagation of waves over a submerged breakwater using a mathematical model based on the Shallow Water Equations. The model formulation is presented in Section 2. Then, in Section 3, the equations are solved analytically using the Separation of Variables Method to determine the wave transmission coefficient. The transmission coefficient will be used as an indicator of the efficiency of the breakwater. Further, the model will be solved numerically using the Riemann Invariant Method. The explanation of the method as well as the numerical scheme can be found in Section 4. The results and discussion in Section 5 present the numerical simulation results as well as the comparison between the analytical solution and the numerical simulation. The comparison is provided to validate our numerical model and to analyze how precise our numerical scheme for simulating wave propagation over submerged breakwater is. Finally, the conclusion of this paper is given in Section 6.

2. Mathematical Model
In this section, we will formulate a mathematical model of wave propagation over a submerged breakwater to estimate the reduction of wave height caused by that breakwater. The governing equations we use here is the Shallow Water Equations (SWEs), which consist of two equations: mass conservation and momentum balance equation. In this case, we only use the Linear SWEs which are written as

\[ \eta_t + (h u)_x = 0, \]  
\[ u_t + g \eta_x = 0, \]  

where \( \eta \) is the surface elevation, \( u \) denotes the horizontal wave velocity, and the gravity acceleration is defined by \( g \). In the Linear SWEs, we define \( h(x) \) as the total water depth calculated from the undisturbed water condition.

For the case of water channel with a submerged breakwater, we separate the domain into two kind of sub-domains, which are the free domain where there is no breakwater and the domain with a breakwater. Thus, as described in Figure (1), we can divide the domain into three sub-domains: the free domain in front of the breakwater (\( \Omega_1 \)), the domain on top of the breakwater (\( \Omega_2 \)), and the free domain behind the breakwater (\( \Omega_3 \)). Mathematically, the water depth of the whole domain can be written as

\[ h(x) = \begin{cases} 
  h_0 & \text{if } x \leq 0 \\
  h_1 & \text{if } 0 \leq x \leq L \\
  h_0 & \text{if } x \geq L 
\end{cases} \]  

with \( h_0 \) is the water depth on the free domains, while \( h_1 \) is the water depth on top of the breakwater.
3. Analytical Solution

Now, we will solve Equation (1) and (2) analytically using Separation of Variables Method to obtain the transmission coefficient. Later, this transmission coefficient, which is the ratio between the transmitted wave amplitude and the initial wave amplitude, will be used to determine the effectiveness of the breakwater on reducing wave amplitude. To solve the equations, first, we assume that the incoming wave is a monochromatic wave with \( \eta(x, t) \) and \( u(x, t) \) follow the functions below.

\[
\eta(x, t) = F(x)e^{-i\omega t},
\]

(4)

\[
u(x, t) = G(x)e^{-i\omega t}.
\]

(5)

Substituting Equation (4) and (5) into Equation (1) and (2) will yield

\[
F_{xx} + \frac{\omega^2}{gd} F = 0,
\]

(6)

\[
G(x) = \frac{g}{i\omega} F_x.
\]

(7)

Using the Characteristic Method, we get the solution of Equation (6) as shown by the following function:

\[
F(x) = A_i e^{ik_0 x} + A_r e^{-ik_0 x},
\]

(8)

where \( A_i \) is the initial wave amplitude and \( A_r \) is the reflected amplitude. The wave number \( k_0 \) follows this dispersion relation \( k_0 = \sqrt{\frac{\omega^2}{gh_0}} \). Next, we differentiate Equation (8) with respect to \( x \) then substitute the result to Equation (7), we obtain

\[
G(x) = \left( \frac{gk_0}{\omega} \right) \left( A_i e^{ik_0 x} - A_r e^{-ik_0 x} \right).
\]

(9)

For domain \( \Omega_2 \), we use the same method and step to find the solution for \( F(x) \) and \( G(x) \), read as follow:

\[
F(x) = C_1 e^{ik_1 x} + C_2 e^{-ik_1 x},
\]

(10)

\[
G(x) = \left( \frac{gk_1}{\omega} \right) \left( C_1 e^{ik_1 x} - C_2 e^{-ik_1 x} \right),
\]

(11)

with \( k_1 = \sqrt{\frac{\omega^2}{gh_1}} \) as the dispersion relation in domain \( \Omega_2 \), while \( C_1 \) and \( C_2 \) are amplitudes of the wave propagates to the right and left side, respectively. Lastly, for domain \( \Omega_3 \), we have the identical solution
as in domain $\Omega_1$ for the same depth and bottom structure, but in domain $\Omega_3$ we assume that the water channel has an absorbing boundary on the right side, means that there is no wave that propagates to the left side in this domain (reflected wave) and there is only the transmitted wave. So, the solutions for domain $\Omega_3$ are

$$F(x) = A_t e^{ik_0x},$$

$$G(x) = \left(\frac{gk_0}{\omega}\right)(A_t e^{ik_0x}),$$

with $A_t$ is the transmitted wave amplitude. Now, we substitute all the solutions of each sub-domain to Equation (4) and (5) to get the full solution of the whole domain, written as

$$\eta(x, t) = \begin{cases} 
 A_t e^{i(k_0x-\omega t)} + A_r e^{-i(k_0x+\omega t)} & , \text{if } x \leq 0 \\
 C_1 e^{i(k_1x-\omega t)} + C_2 e^{-i(k_1x+\omega t)} & , \text{if } 0 \leq x \leq L \\
 A_t e^{i(k_0x-\omega t)} & , \text{if } x \geq L 
\end{cases}$$

and

$$u(x, t) = \begin{cases} 
 \sqrt{g} \left( A_t e^{i(k_0x-\omega t)} - A_r e^{-i(k_0x+\omega t)} \right) & , \text{if } x \leq 0 \\
 \sqrt{g} \left( C_1 e^{i(k_1x-\omega t)} - C_2 e^{-i(k_1x+\omega t)} \right) & , \text{if } 0 \leq x \leq L \\
 \sqrt{g} \left( A_t e^{i(k_0x-\omega t)} \right) & , \text{if } x \geq L 
\end{cases}$$

To obtain the wave transmission coefficient, we apply the matching conditions along the discontinuity points at $x = 0$ and $x = L$. The matching conditions are based on the continuity of free surface elevation ($\eta$) and horizontal flux ($hu$) at the discontinue points which can be written mathematically as

$$\eta|_{x=0^-} = \eta|_{x=0^+} \quad \text{and} \quad hu|_{x=0^-} = hu|_{x=0^+}$$

$$\eta|_{x=L^-} = \eta|_{x=L^+} \quad \text{and} \quad hu|_{x=L^-} = hu|_{x=L^+}.$$

From the matching conditions above, we get equations below.

$$\begin{align*}
A_l + A_r &= C_1 + C_2, \\
\sqrt{h_0}(A_l - A_r) &= \sqrt{h_1}(C_1 - C_2), \\
(C_1 e^{ik_1L} + C_2 e^{-ik_1L}) &= A_t e^{ik_0L}, \\
\sqrt{h_1}(C_1 e^{ik_1L} - C_2 e^{-ik_1L}) &= \sqrt{h_0}A_t e^{ik_0L}. 
\end{align*}$$

Solving Equation (16) and (17) using Elimination Method, we get

$$C_1 = \frac{H_+ A_l - H_- A_r}{2},$$
\[ C_2 = \frac{-H_- A_i + H_+ A_r}{2}, \]

where \( H_+ = \left( \frac{h_0}{\sqrt{h_1}} + 1 \right) \) and \( H_- = \left( \frac{h_0}{\sqrt{h_1}} - 1 \right) \). Substituting \( C_1 \) and \( C_2 \) to Equation (18) and (19), then solve them using Elimination Method, we obtain the analytical transmission coefficient \( (C_t) \) written as

\[
\begin{vmatrix}
A_t \\
A_i
\end{vmatrix} = \begin{vmatrix}
\left( \alpha_- \beta_+ + \alpha_+ \beta_- \right) & \frac{h_0}{\sqrt{h_1}} \\
\beta_+ + \beta_- & \frac{h_0}{2e^{i k_0 L}} \end{vmatrix},
\]

(20)

with

\[
\alpha_- = H_+ e^{ik_1 L} - H_- e^{-ik_1 L},
\]
\[
\alpha_+ = H_+ e^{ik_1 L} + H_- e^{-ik_1 L},
\]
\[
\beta_- = H_+ e^{-ik_1 L} - H_- e^{ik_1 L},
\]
\[
\beta_+ = H_+ e^{-ik_1 L} + H_- e^{ik_1 L}.
\]

As we can see in Equation (20), the value of \( C_t \) depends on \( \omega \) and the dimension of the submerged breakwater. We have the ratio between the water depth at the top of breakwater and the free domain water depth as the value of \( \sqrt{\frac{h_0}{h_1}} \) as well as the length of the breakwater as the value of \( L \).

4. Numerical Method in Riemann Formulation

In this section, we will formulate a numerical scheme in Riemann Invariant Form for each domain based on the SWEs (1) and (2). For domain \( \Omega_1 \) with water depth \( h(x) = h_0 \), the equation can be written in the form of matrix

\[
\begin{pmatrix}
\eta_t \\
u_t
\end{pmatrix} = \begin{pmatrix}
0 & -h_0 \\
-g & 0
\end{pmatrix} \begin{pmatrix}
\eta_x \\
u_x
\end{pmatrix}.
\]

As a hyperbolic problem, matrix coefficient of the Linearized Shallow Water Equations can be diagonalized, so that the matrix coefficient can be written as

\[
\begin{pmatrix}
0 & -h_0 \\
-g & 0
\end{pmatrix} = \Lambda \begin{pmatrix}
-c_0 & 0 \\
0 & c_0
\end{pmatrix} \Lambda^{-1}
\]

with \( c_0 = \sqrt{g h_0} \) are its eigen values. \( \Lambda \) is a matrix that contains both eigen vectors, while \( \Lambda^{-1} \) is its inverse. Those matrices, respectively, can be written as

\[
\Lambda = \frac{1}{2} \begin{pmatrix}
1 & 1 \\
\frac{g}{h_0} & \frac{g}{h_0}
\end{pmatrix} \quad \text{and} \quad \Lambda^{-1} = \begin{pmatrix}
\frac{h_0}{g} & 0 \\
1 & -\frac{h_0}{g}
\end{pmatrix}.
\]

Now our model can be rewritten in matrix form as

\[
\begin{pmatrix}
\eta_t \\
u_t
\end{pmatrix} = \Lambda \begin{pmatrix}
-c_0 & 0 \\
0 & c_0
\end{pmatrix} \Lambda^{-1} \begin{pmatrix}
\eta_x \\
u_x
\end{pmatrix}.
\]
If we multiply both sides with $\Lambda^{-1}$, then the equation becomes

$$\Lambda^{-1}(\eta_t) = \Lambda^{-1}A\begin{pmatrix}-c_0 & 0 \\ 0 & c_0\end{pmatrix}\Lambda^{-1}(u_x).$$ \quad (21)

Define $\begin{pmatrix}R_+ \\ R_-\end{pmatrix} = \Lambda^{-1}(\eta)$, then Equation (21) can be rewritten as

$$\begin{pmatrix}R_+ \\ R_-\end{pmatrix}_t = \begin{pmatrix}-c_0 & 0 \\ 0 & c_0\end{pmatrix}\begin{pmatrix}R_+ \\ R_-\end{pmatrix}_x.$$

Using the same steps, Riemann formulation of our model in domain $\Omega_2$, where $h(x) = h_1$, will becomes

$$\begin{pmatrix}R_+ \\ R_-\end{pmatrix}_t = \begin{pmatrix}-c_1 & 0 \\ 0 & c_1\end{pmatrix}\begin{pmatrix}R_+ \\ R_-\end{pmatrix}_x,$$

with $c_1 = \sqrt{gh_1}$. Hence, for each sub-domain, our model can be written in Riemann formulation as

$$\begin{align*}
d\var_{t}R_+ &= -c_i \var_{x}R_+, \\
\var_{t}R_- &= c_i \var_{x}R_-,
\end{align*}$$ \quad (22)

where $i = 0$ for domain $\Omega_1$ and $\Omega_3$, while $i = 1$ for domain $\Omega_2$. Now, we can formulate our numerical scheme based on our new equation system (22).

Consider a numerical domain $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$, where $\Omega_1 = [-X, 0)$, $\Omega_2 = [0, L]$, and $\Omega_3 = (L, X]$. We discretized the domain into $-X = x_0, x_1, x_2, ..., x_{nx} = X$, with $nx = \left\lceil\frac{2X}{\Delta x}\right\rceil + 1$ and $\Delta x$ is the width of each partition of the domain. Using finite difference method, we write Equation (22) in its discrete form as

$$\begin{align*}
\frac{R_{+j}^{n+1} - R_{+j}^{n}}{\Delta t} &= -c_i \frac{R_{+j}^{n} - R_{+j-1}^{n}}{\Delta x}, \\
\frac{R_{-j}^{n+1} - R_{-j}^{n}}{\Delta t} &= -c_i \frac{R_{-j+1}^{n} - R_{-j}^{n}}{\Delta x},
\end{align*}$$ \quad (23)

with $j = 0, 1, 2, ..., nx$ and $n = 0, 1, 2, ..., Nt$, where $Nt = \left\lceil\frac{T}{\Delta t}\right\rceil + 1$ and $\Delta t$ is the time step for each iteration. The equation contains $R_+$ is computed using Forward Time and Backward Space approximation, while the other one that contains $R_-$ is computed using Forward Time and Forward Space approximation. Numerical scheme (23) will stable if $\frac{\Delta t}{\Delta x}\sqrt{gh_0} \leq 1$ is satisfied.

5. Results and Discussions
Here, we will simulate the wave propagation over a submerged breakwater using the numerical scheme formulated in the previous section. First, we will evaluate how well the Riemann formulation estimates the wave profile when and after it passes over the breakwater by performing a simple simulation of specific wave and breakwater specifications. In that case, consider a numerical domain of $[0, 40]$ m where we placed a breakwater with length of $L = 10$ m at $[15, 25]$ m. The water depth at the top of the breakwater is set to be $h_1 = 1$ m, while the depth of the free water domain is $h_0 = 2$ m. For the incoming wave, we define the wave angular frequency as $\omega = 4 \, s^{-1}$ with amplitude of $A_i = 0.2258$ m. The simulation is presented in Figure (2).
Figure 2. Simulation results of wave propagation over a submerged bar breakwater at every time step (a) and at the end of the observation (b).

Figure (2.a) shows how the wave propagates over the breakwater gradually. Notice that the wave is slowing down at $x = 15$ to $x = 25$ where the breakwater is placed. However, as the velocity decreases in that region, the waves keep coming in with the original velocity, forcing the wave in the breakwater zone to shorten its length, which leads to a higher amplitude. This is called as the wave shoaling effect, where wave amplitude tends to increase as it enters the shallower zone. This shoaling phenomena becomes even more clear in the next picture (Figure (2.b)) where the wave amplitude rises and the wavelength is shortened when the wave propagates over the breakwater. However, we also notice that as the wave travels to the deeper region behind the breakwater, the amplitude is reduced, becomes even smaller than the initial wave amplitude. This statement is confirmed by the value of the transmission coefficient for this particular simulation, which is $C_t = 0.8872$. Since $C_t$ is the ratio between the transmitted wave amplitude and the initial amplitude, this means that the wave transmission amplitude is smaller than the initial amplitude. This ensures that the breakwater has effectively lowered the wave amplitude and that our computational scheme has accurately reproduced the wave shoaling effect as well as the reduction of the wave amplitude induced by the submerged breakwater.

Next, to investigate how accurate our numerical model is on approximating the analytical model, we will compare the numerical results against the analytical solutions. At the same time, we will study how the water depth ratio $\sqrt{h_0/h_1}$ affects the transmission coefficient. All the parameters we use here are the same as the previous simulation, except that we vary the value of $h_1$ in the range of $0.5 \leq h_1 \leq 2$ m. The result of the comparison between analytical solutions and numerical results is shown in Figure (3).

In Figure (3), we can see that the term $\sqrt{h_0/h_1}$ is inversely proportional to the transmission coefficient. The higher the value of $\sqrt{h_0/h_1}$, the lower $C_t$. It means, with the fixed value of $L$ and $h_0$, we will get smaller $C_t$ if we set the value of $h_1$ smaller or if we make the breakwater higher. This result makes sense, because as the term $\sqrt{h_0/h_1}$ becomes bigger or when the breakwater becomes higher, more part of the wave will be reflected and only small part of it will be transmitted. Hence, the transmitted wave amplitude will also becomes smaller. In addition, notice that our numerical result closely approximates the analytical solution. It can be proved by the average relative error between the two solutions, which is 0.03 %. This error is small enough to say that our numerical model has estimated the analytical solutions very well.
Figure 3. Comparison of analytical and numerical $C_t$.

6. Conclusions
This paper provides an investigation of wave propagation over a rectangular submerged breakwater using a mathematical model based on the Shallow Water Equations. The model is solved analytically and numerically using Separation of Variables Method and Riemann Invariant Method, respectively. From the analytical derivation, a formula for transmission coefficient is obtained. It is found that the transmission coefficient is significantly affected by the depth ratio and the length of the breakwater. Meanwhile, a numerical scheme is developed using the Riemann Invariant Method. This scheme is then implemented to simulate wave propagation over a submerged breakwater and compare the results to the analytical solutions. The comparison resulted in an agreement between numerical and analytical model, with error of 0.03 %. This proves that our numerical scheme is accurate enough to approximate the analytical model and to estimate the wave transmission coefficient.

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