Affine Toda field theory from tree unitarity

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Abstract

Elasticity property (i.e. no-particle creation) is used in the tree level scattering of scalar particles in 1+1 dimensions to construct affine Toda field theory (ATFT) associated with the root systems of groups $a_2^{(2)}$ and $c_2^{(1)}$. A general prescription is given for constructing ATFT (associated with rank two root systems) with two self conjugate scalar fields. It is conjectured that the same method could be used to obtain the other ATFT associated with higher rank root systems.

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1 Introduction:

The present note is motivated by the opening section of the paper “Exact S-matrices” by Patrick Dorey [1], which in turn was inspired by a remark in an article by Goebel on the sine-Gordon S-matrix [2].

The aim of this paper is construction of affine Toda field theory (ATFT) from well known scalar field theory by demanding elasticity property (i.e. no particle production) in the scattering of particles at tree level. We would show that the tree-level calculation...
would suffice for this purpose. Once the coupling ratios are determined the higher order elasticity follows. We will see that the three-point couplings (a “fusing rule” for which was proposed in Ref. [3]) play an important role in this.

In the following we give a very brief description of affine Toda field theory. Affine Toda field theory\[4] is a massive scalar field theory with exponential interactions in $1+1$ dimensions described by the Lagrangian,

$$\mathcal{L} = \frac{1}{2} \partial \phi \cdot \partial \phi - \frac{m^2}{\beta^2} \sum_{i=0}^{r} n_i e^{\beta \alpha_i \cdot \phi}.$$  

(1.1)

The field $\phi$ is an $r$-component scalar field, $r$ is the rank of a compact semi-simple Lie algebra $G$. $\alpha_i; i = 1, \ldots, r$ are simple roots and $\alpha_0$ is the affine root of $G$. The roots are normalized so that long roots have length $\sqrt{2}$, i.e. $\alpha_L^2 = 2$. The Kac-Coxeter labels $n_i$ are such that $\sum_{i=0}^{r} n_i \alpha_i = 0$, with the convention $n_0 = 1$. The quantity, $\sum_{i=0}^{r} n_i$, is denoted by ‘$h$’ and known as the Coxeter number. ‘$m$’ is a real parameter setting the mass scale of the theory and $\beta$ is a real coupling constant, which is relevant only in the quantum theory.

ATFT is the best theoretical laboratory for understanding quantum field theory ‘beyond perturbation’. ATFT with real coupling is one of the best understood field theories at classical and quantum levels. ATFT is integrable at the classical level [5, 6] due to the presence of an infinite number of conserved quantities. Based on the assumption that the infinite set of conserved quantities be preserved after quantization, only the elastic processes are allowed and the multi-particle $S$-matrices are factorized into a product of two particle elastic $S$-matrices [7]. In ATFT, it is well-known that these conserved quantities are related with the Cartan matrix of the associated finite Lie algebra. Higher-spin quantum conserved currents are discussed in Ref. [8]. Exact quantum $S$-matrices for all simply laced ATFT were evaluated in Refs. [9-14]. Most of the non-simply laced ATFT exact $S$-matrices were calculated in Ref. [15] with the beautiful idea of floating masses. These $S$-matrices respect crossing symmetry and bootstrap principle [7]. The exact quantum $S$-matrices for the remaining non-simply laced theory were constructed in Ref. [16] where generalized bootstrap principle was introduced and more insight to the mechanism was provided. The singularity structure of the $S$-matrices of simply laced theories, which in some cases contain poles up to 12-th order [11], is beautifully explained in terms of the singularities of the corresponding Feynman diagrams [17], so called Landau sin-

\[1\]For an excellent review see Ref. [4]
gularities. Finally Affine Toda field theory is one place where one can see explicitly the recently popular strong-weak coupling duality. It is known that exact Toda $S$-matrices for simply laced systems are invariant under this duality.

Next section presents the results obtained in the opening section of Ref. [1]. Section 3, solves the exercise suggested at end of the opening section of the Ref. [1] to obtain the $a_n^{(2)}$ ATFT or the Bullough-Dodd theory. Section 4, works with two scalar fields of different masses (one has a mass $\sqrt{2}$ times the other) and an interaction between them. This problem leads to an ATFT associated with $c_2^{(1)}$ root system, which is non-simply laced. Section 5 will deal with a general approach towards theories with two scalar particles which are self conjugate. In this section the allowed values of mass ratio and 3-point couplings would be obtained and the final theory would come out to be an ATFT associated with a rank two root system. Section 6 is reserved for conclusions and a conjecture.

2 sinh-Gordon or $a_1^{(1)}$ theory

This section is shamelessly lifted from the “Introduction” of the paper [1]. Starting from scalar $\phi^4$ theory in 1+1 dimensions the simplest possible ATFT i.e. sinh-Gordon or $a_1^{(1)}$ is obtained.

We begin with 1+1 D scalar $\phi^4$ Lagrangian,

$$\mathcal{L} = \frac{1}{2}(\partial \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4.$$  \hspace{1cm} (2.1)

The Feynman rules are:

$$\frac{i}{p^2 - m^2 + i\epsilon} = -i\lambda$$

where $p$ is momentum and $m$ is the mass of the particles. We use light-cone coordinates,

$$(p, \bar{p}) = (p^0 + p^1, p^0 - p^1).$$

Using the mass-shell condition $p\bar{p} = m^2$, in and out momenta are written as

$$(p_a, \bar{p}_a) = (ma, ma^{-1}), \quad (p_b, \bar{p}_b) = (mb, mb^{-1})$$
and so on, with \( a, b, \ldots \) real numbers, positive for particle traveling forward in time. One now calculates the connected \( 2\phi \rightarrow 4\phi \) production amplitude at tree level. For this one looks at the diagrams of \( 3\phi \rightarrow 3\phi \) processes, with implicit understanding that one of the in momenta will be crossed to out at the end. The in particles are labeled as \( a, b, c \) and the out particles as \( d, e, f \). In terms of these variables crossing from \( 3\phi \rightarrow 3\phi \) to \( 2\phi \rightarrow 4\phi \) amounts to a continuation from \( c \) to \(-c\). For the \( 3\phi \rightarrow 3\phi \) amplitude at tree level there are just the following two classes of diagrams (Fig. 1) as shown in the Ref. [1],

As one of the in momenta is actually going out, thus the propagator is not on the mass-shell so removal of \( i\epsilon \) terms is allowed. Thus the internal momentum \( p = m(a + b - d, a^{-1} + b^{-1} - d^{-1}) \), and the contribution to the propagator from above Fig. 1 (a) is

\[
\frac{i}{p^2 - m^2} = \frac{i}{m^2[(a + b - d)(a^{-1} + b^{-1} - d^{-1}) - 1]} = \frac{-iabd}{m^2(a + b)(a - d)(b - d)}. \tag{2.2}
\]

Similarly for the Fig. 1 (b) the contribution to the propagator is:

\[
\frac{i}{p^2 - m^2} = \frac{iabc}{m^2(a + b)(a + c)(b + c)}. \tag{2.3}
\]

Taking all the terms in accounts the amplitude of \( \text{in} \rightarrow \text{out} \) is:

\[
\langle \text{out} | \text{in} \rangle_{\text{tree}} = -\frac{i\lambda^2}{m^2}A_{\text{legs}}H(a, b, c, d, e, f), \tag{2.4}
\]

where \( A_{\text{legs}} \) contains all the common factors on external legs, and

\[
H(a, b, c, d, e, f) = \left[ \sum_{\text{cyc}(a,b,c)} \left( \frac{-abd}{(a + b)(a - d)(b - d)} \right) + \frac{abc}{(a + b)(b + c)(c + a)} \right]. \tag{2.5}
\]

Using \( a + b + c = d + e + f \) and \( a^{-1} + b^{-1} + c^{-1} = d^{-1} + e^{-1} + f^{-1} \). i.e. the conservation of left- and right-light-cone momenta respectively, one finds \( H(a, b, c, d, e, f) = -1 \). As above argument does not contain the sign of any momenta, it holds for \(-c\) also.

Figure 1: (a) and (b), \( 3\phi \rightarrow 3\phi \) process

- a, d
- b, e
- c, f

- a, d
- b, e
- c, f
So we find in \(1+1\) D \(\lambda \phi^4\) theory, the amplitude of \(2\phi \rightarrow 4\phi\) is constant at tree level. By adding a term \(-\frac{\lambda^2}{6!m^2}\phi^6\) to the original Lagrangian (2.1) one can make the \(2\phi \rightarrow 4\phi\) amplitude to vanish. Defining \(\beta^2 = \lambda/m^2\), the new Lagrangian up to \(\phi^6\) order is,

\[
\mathcal{L} = \frac{1}{2}(\partial \phi)^2 - \frac{m^2}{\beta^2} \left[ \frac{1}{2!} \beta^2 \phi^2 + \frac{1}{4!} \beta^4 \phi^4 + \frac{1}{6!} \beta^6 \phi^6 \right].
\]

(2.6)

Now one calculates \(2\phi \rightarrow 6\phi\) tree level amplitude with this \(-\frac{\lambda^2}{6!m^2}\phi^6\) term added to the Lagrangian (2.1) and finds it to be a constant, which can be canceled by a judiciously chosen \(\phi^8\) term and so on. At each stage a residual constant piece can be removed by a (uniquely determined) higher-order interaction. After adding infinitely many terms in this way assuring no particle production at tree level one finds,

\[
\mathcal{L} = \frac{1}{2}(\partial \phi)^2 - \frac{m^2}{\beta^2} \left[ \cosh(\beta \phi) - 1 \right] = \mathcal{L}_{\alpha_1^{(1)}}.
\]

(2.7)

The above Lagrangian, the simplest ATFT, is sinh-Gordon or \(\alpha_1^{(1)}\) Lagrangian and is well studied in the literature. Araf’eva and Korepin showed in Ref. [18] that the elasticity is maintained at one loop level for the above Lagrangian. The well known sine-Gordon Lagrangian could be obtained by sending the coupling \(\beta\) to imaginary.

### 3 Bullough-Dodd model or \(\alpha_2^{(2)}\) theory

What would happen if we played the same game with a \(\phi^3\) theory? This was suggested as an exercise in [1]. The solution follows here in detail. The Lagrangian we begin with has the following form,

\[
\mathcal{L} = \frac{1}{2}(\partial \phi)^2 - \frac{1}{2}m^2 \phi^2 - \frac{\eta}{3!} \phi^3.
\]

(3.1)

Again the Feynman rules are:

\[
\resizebox{0.5\textwidth}{!}{\begin{align*}
\frac{i}{p^2 - m^2 + i\epsilon} &= -i\eta
\end{align*}}
\]
First we consider a $2\phi \to 3\phi$ process for which we have the following tree level diagram (Fig. 2). For making our calculations easier we take both in momenta ($(a, a^{-1})$ and $(b, b^{-1})$) equal to $(1,1)$ and one of the out momenta, $(e, e^{-1})$, equal to $(1 + \delta, (1 + \delta)^{-1})$, $\delta$ need not be small.

![Figure 2: 2φ → 3φ process with three φ³ vertices](image)

Now the conservation of left- and right-light-cone momenta would give,

$$c + d = 1 - \delta, \quad c^{-1} + d^{-1} = \frac{1 + 2\delta}{1 + \delta}, \quad cd = \frac{1 - \delta^2}{1 + 2\delta}, \quad cd^{-1} + c^{-1}d = -\frac{1 + \delta + 2\delta^2}{1 + \delta}, \quad (3.2)$$

where $(c, c^{-1})$ and $(d, d^{-1})$ are momenta of other two outgoing particles. There are altogether fifteen diagrams of the above type (details of their individual contributions are given in appendix A1). Summing all the diagrams using above relations, (3.2), we obtain,

$$\frac{i\eta^3}{m^4} \left[ -\frac{3}{2} \frac{(1 + \delta)}{\delta^2} - 1 + \frac{3}{2} \frac{(4 + \delta)}{(1 + \delta + \delta^2)} + \frac{9}{2} \frac{\delta}{(1 + \delta + \delta^2)^2} \right]. \quad (3.3)$$

For stopping the particle production at tree level we add a counter term $-\frac{\lambda}{4!}\phi^4$ to the Lagrangian (3.1). This would produce a new Feynman rule,

$$\times = -i\lambda,$$

giving the following class of new diagrams (Fig. 3) for the tree level $2\phi \to 3\phi$ process.

![Figure 3: 2φ → 3φ process with a φ³ vertex and a φ⁴ vertex](image)
There are total of ten diagrams of the above type and we sum them again using the relations (3.2). The total contribution is (for individual details and contributions of the diagrams see appendix A2),

\[ \frac{i\eta\lambda}{m^2} \left[ \frac{1}{2} \frac{(1 + \delta)}{\delta^2} + 2 - \frac{1}{2} \frac{(4 + \delta)}{(1 + \delta + \delta^2)} - \frac{3}{2} \frac{\delta}{(1 + \delta + \delta^2)^2} \right]. \tag{3.4} \]

Adding (3.4) and (3.3) we obtain total tree level contribution of the \(2\phi \rightarrow 3\phi\) process with \(\phi^3\) and \(\phi^4\) terms in the Lagrangian as,

\[ (\text{Tree})_{abcde} = \left( \frac{i2\eta\lambda}{m^2} - \frac{i\eta^3}{m^4} \right) + \left( \frac{\eta\lambda}{m^2} - \frac{i\eta^3}{m^4} \right) \left[ \frac{1}{2} \frac{(1 + \delta)}{\delta^2} - \frac{1}{2} \frac{(4 + \delta)}{(1 + \delta + \delta^2)} - \frac{3}{2} \frac{\delta}{(1 + \delta + \delta^2)^2} \right]. \tag{3.5} \]

Now if one chooses \(\lambda = \frac{3\eta^2}{m^4}\), one gets rid of all the terms involving parameter \(\delta\) and the total contribution becomes a constant equal to \(i5\eta^3/m^4\). This constant contribution could be killed if another counter term, \(-\frac{5\eta^3}{5!m^4}\phi^5\), is added to the Lagrangian (3.1). After adding \(\phi^4\) and \(\phi^5\) terms, the new Lagrangian looks like,

\[ L = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\eta}{3!}\phi^3 - \frac{3\eta^2}{4!m^2}\phi^4 - \frac{5\eta^3}{5!m^4}\phi^5 \cdots \tag{3.6} \]

The above Lagrangian, (3.6) would produce vanishing result for the tree level \(2\phi \rightarrow 3\phi\) process. To fix the dots in the above Lagrangian one should look for \(2\phi \rightarrow 4\phi\) tree level process. Next one demands vanishing contribution for this \(2\phi \rightarrow 4\phi\) tree level process to fix the \(\phi^6\) term and one can proceed so on order by order.

Setting \(\beta = \frac{\eta}{m^2}\) we observe that the Lagrangian, (3.6), contains first four terms of the following Lagrangian after a power series expansion,

\[ L_{a_2}^{(2)} = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{6\beta^2} \left[ e^{2\beta\phi} + 2e^{-\beta\phi} - 3 \right]. \tag{3.7} \]

The above, (3.7), is another well studied Lagrangian known as the Bullough-Dodd model or the \(a_2^{(2)}\) ATFT in the literature. We mention in passing that if one starts with a theory with both \(\phi^3\) and \(\phi^4\) terms absent in the Lagrangian then no particle production at tree level would lead to a free theory that is having all higher couplings vanishing.
4 \( c_2^{(1)} \) theory

In this section we consider two interacting self conjugate scalar fields. The starting Lagrangian in this case is chosen to be

\[ L = \frac{1}{2} (\partial \phi_1)^2 + \frac{1}{2} (\partial \phi_2)^2 - m^2 \phi_1^2 - \frac{1}{2} m^2 \phi_2^2 + \frac{\xi}{2!} \phi_1 \phi_2^2. \]  

(4.1)

Notice that the particle \( \phi_1 \) is \( \sqrt{2} \) times heavier than particle \( \phi_2 \) and there is only one interaction term, viz. \( \phi_1 \phi_2^2 \).

The Feynman rules are the following,

\[
\begin{align*}
\phi_1 \text{ propagator} : & \quad \frac{i}{p^2 - 2m^2 + i\epsilon} \\
\phi_2 \text{ propagator} : & \quad \frac{i}{p^2 - m^2 + i\epsilon} \\
& = i\xi
\end{align*}
\]

First we consider the process \( 2\phi_2 \to 2\phi_1 \). At tree level, there are following two diagrams (Fig. 4),

![Diagram](image_url)

Figure 4: \( 2\phi_2 \to 2\phi_1 \) process

The in particles have momenta \((a, a^{-1})\) and \((b, b^{-1})\) whereas the out ones have momenta \((\sqrt{2} c, \sqrt{2} c^{-1})\) and \((\sqrt{2} d, \sqrt{2} d^{-1})\) without loss of generality since the out particles are \( \sqrt{2} \) times heavy as the in ones. The conservation of left-and right-light-cone momenta gives the following relations,

\[ c + d = \frac{a + b}{\sqrt{2}}, \quad c^{-1} + d^{-1} = \frac{a^{-1} + b^{-1}}{\sqrt{2}}, \quad cd = ab, \quad cd^{-1} + c^{-1}d + 1 = \frac{1}{2}(ab^{-1} + a^{-1}b). \]

(4.2)

Individual contributions of the above two diagrams are,

\[ \frac{-i\xi^2}{m^2[2 - \sqrt{2}(ac^{-1} + a^{-1}c)] + i\epsilon} + \frac{-i\xi^2}{m^2[2 - \sqrt{2}(ad^{-1} + a^{-1}d)] + i\epsilon}. \]

(4.3)
Summing the above two expressions using relations (4.2) and then taking the limit $\epsilon \to 0$ we obtain a constant equal to $\frac{i\xi^2}{m^2}$. This constant contribution will be killed if we add a term, $-\frac{1}{2!2!} \frac{\xi^2}{m^2} \phi_1^2 \phi_2^2$, to the Lagrangian (4.1). This new term gives an additional Feynman rule,

$$\begin{array}{c}
\end{array} = -\frac{i\xi^2}{m^2}.$$

Next we look at the process $2\phi_2 \to 2\phi_2 + \phi_1$. Again for simplifying our calculations we take in momenta as (1,1) and outgoing particle $\phi_2$ momentum as $\sqrt{2e}$, $\sqrt{2e^{-1}}$. We have the following two types of diagrams (Fig. 5), one having three $\phi_1 \phi_2^2$ vertices and the other containing one $\phi_1 \phi_2^2$ vertex and one $\phi_1^2 \phi_2^2$ vertex.

Figure 5: $2\phi_2 \to 2\phi_2 + \phi_1$ process (a) with 3 $\phi_1 \phi_2^2$ vertices and (b) with one $\phi_1 \phi_2^2$ vertex and one $\phi_1^2 \phi_2^2$ vertex.

We have 12 diagrams of the former type and 6 diagrams of the latter type (details of which are presented in the appendix B1). Summing all 18 diagrams one obtains (using momentum conservation relations of course),

$$-\frac{i\xi^3 e (8e - 3\sqrt{2} e^2 - 3\sqrt{2})}{m^4 (2e - \sqrt{2} e^2 - \sqrt{2})^2} \tag{4.4}$$

Now if one adds a counter term $-\frac{\zeta}{4!} \phi_2^4$ to the Lagrangian, (4.1) one will have a new vertex of the following type,

$$\begin{array}{c}
\end{array} = -i\zeta$$

This new vertex in turn would add the following 4 more diagrams (Fig. 6) to the above process $2\phi_2 \to 2\phi_2 + \phi_1$. Contribution of these four diagrams when added is equal to,
Adding (4.4) and (4.5) we have,

\[ \text{Tree} = -\frac{i\xi\zeta}{m^2} + \frac{i\xi\zeta}{m^2} \frac{e(8e - 3\sqrt{2}e^2 - 3\sqrt{2})}{(2e - \sqrt{2}e^2 - \sqrt{2})^2}. \]

Adding (4.4) and (4.5) we have,

\[ \text{Tree} = -\frac{i\xi\zeta}{m^2} \left( \frac{e(8e - 3\sqrt{2}e^2 - 3\sqrt{2})}{(2e - \sqrt{2}e^2 - \sqrt{2})^2} - \frac{i\xi}{m^4} \frac{e(8e - 3\sqrt{2}e^2 - 3\sqrt{2})}{(2e - \sqrt{2}e^2 - \sqrt{2})^2} \right) \]

(4.6)

The above expression, (4.6), clearly shows if we choose \( \zeta = \frac{\xi^2}{m^2} \) then the above expression becomes independent of the parameter \( e \) and the total sum becomes \( -\frac{i\xi^3}{m^4} \). This constant contribution will be killed if one adds a new counter term \( \frac{1}{4!} \frac{\xi^3}{m^4} \phi_1 \phi_2 \) to the Lagrangian (4.1). At this stage our Lagrangian reads,

\[ \mathcal{L} = \frac{1}{2} (\partial \phi_1)^2 + \frac{1}{2} (\partial \phi_2)^2 - m^2 \phi_1^2 - m^2 \phi_2^2 - \frac{1}{2} \phi_1 \phi_2 - \frac{1}{2!} \frac{\xi}{m^2} \phi_1 \phi_2^2 - \frac{1}{2!} \frac{\xi^2}{m^2} \phi_1^2 \phi_2^2 - \frac{1}{4!} \frac{\xi^3}{m^4} \phi_1 \phi_2^3 \phi_1 \phi_2. \]

(4.7)

To fix the remaining quartic and quintic interactions we concentrate on \( 2\phi_1 \rightarrow 2\phi_2 + \phi_1 \) process which possesses following two classes of diagrams (Fig. 7), viz. one with three \( \phi_1 \phi_2 \) vertices and the other containing one \( \phi_1 \phi_2^2 \) vertex and one \( \phi_1^3 \phi_2^2 \) vertex like before.

\[ \begin{align*}
\text{Tree} &= -\frac{i\xi\zeta}{m^2} + \frac{i\xi\zeta}{m^2} \frac{e(8e - 3\sqrt{2}e^2 - 3\sqrt{2})}{(2e - \sqrt{2}e^2 - \sqrt{2})^2},
\end{align*} \]

Figure 7: a) and b) \( 2\phi_1 \rightarrow 2\phi_2 + \phi_1 \) process
This case we choose in momenta for both the particles as \((\sqrt{2}, \sqrt{2})\) and outgoing particle \(\phi_1\) momentum is designated by \((\sqrt{2} e, \sqrt{2} e^{-1})\). There are 6 diagrams of each kind (see appendix B2 for details). Summing all 12 diagrams we obtain,

\[
-\frac{i\xi^3}{m^4} \frac{e}{2m^4 (e - 1)^2}
\]  

(4.8)

From the above expression \(4.8\) it is easy to fix the quartic \(\phi_1^4\) counter term so that the parameter \(e\) dependent term is killed. For this we add a term \(-\frac{\gamma}{4!}\phi_1^4\) to the Lagrangian \(4.7\) to obtain the following new diagram for the above process.

\[
-\frac{i\xi^3 \gamma}{4m^2 (e - 1)^2}
\]  

(4.9)

Now it is very clear from the above expressions \(4.8\) and \(4.9\), if we choose \(\gamma = \frac{2\xi^2}{m^2}\) all \(e\) dependent terms would cancel from the tree level \(2\phi_1 \rightarrow 2\phi_2 + \phi_1\) process and the result would be a constant equal to \(-\frac{i\xi^3}{m^4}\). This constant contribution is canceled by a vertex of the following type,

\[
\frac{i\xi^3}{m^4}
\]  

(4.10)

The above vertex corresponds to adding a \(\frac{1}{3!2!} \frac{\xi^3}{m^4} \phi_1 \phi_2^2\) term to the Lagrangian \(4.7\). The final Lagrangian with all 5-point interaction vertices becomes,

\[
\mathcal{L} = \frac{1}{2} (\partial\phi_1)^2 + \frac{1}{2} (\partial\phi_2)^2 - m^2 \phi_1^2 - \frac{1}{2} m^2 \phi_2^2 + \frac{\xi}{2!} \phi_1 \phi_2^2 - \frac{2}{4!} m^2 \phi_1^4 - \frac{1}{2!} \frac{\xi^2}{m^2} \phi_1 \phi_2^2 - \frac{1}{4!} \frac{\xi^2}{m^4} \phi_1 \phi_2^4 + \frac{1}{4!} \frac{\xi^3}{m^4} \phi_1 \phi_2^4 + \frac{1}{3!2!} \frac{\xi^3}{m^4} \phi_1 \phi_2^2.
\]  

(4.11)

The above Lagrangian, \(4.11\) contains the first eight terms (after expansion) of the following \(c_2^{(1)}\) ATFT Lagrangian,

\[
\mathcal{L}_{c_2^{(1)}} = \frac{1}{2} \partial \phi \cdot \partial \phi - \frac{m^2}{2\beta^2} \left[ e^{\beta \alpha_0 \cdot \phi} + 2 e^{\beta \alpha_1 \cdot \phi} + e^{\beta \alpha_2 \cdot \phi} - 4 \right],
\]  

(4.12)
where field $\phi$ has two components i.e. $\phi_1$ and $\phi_2$. $\alpha_0$ is affine and $\alpha_1$ and $\alpha_2$ are simple roots of algebra $c_2^{(1)}$ ($\alpha_1 = (1, 0)$, $\alpha_2 = (-1, 1)$, $\alpha_0 = (-1, -1)$) and $\beta = \frac{\xi}{m^2}$. This is another integrable model which is well studied. Again all the higher $n$-point couplings ($n > 5$) could be fixed by studying the various other tree level processes.

5 Other theories with two self conjugate scalar fields

In this section we give a general method for constructing various other integrable theories associated with rank two root systems. We have seen in the previous section that the sole three point interaction decides the fate of other terms if one maintains elasticity property order by order at tree level. One can verify that elasticity is maintained if one goes to loop diagrams. Here we start with the most general Lagrangian with two self conjugate scalar fields with all possible three-point interactions,

$$L = \frac{1}{2} (\partial \phi_1)^2 + \frac{1}{2} (\partial \phi_2)^2 - \frac{q}{2} m^2 \phi_1^2 - \frac{1}{2} m^2 \phi_2^2 + \frac{\xi}{2!} \phi_1 \phi_2^2 - \frac{r \xi}{3!} \phi_3^3 - \frac{s \xi}{3!} \phi_2^3 - \frac{t \xi}{2!} \phi_1^2 \phi_2^2. \quad (5.1)$$

Note that we have fixed the strength of one mass term and only one of the three-point interactions. The other mass and couplings have strengths relative to these. Our objective is to determine these relative strengths (i.e. $q$, $r$, $s$ and $t$) for an integrable theory. Feynman rules are given by,

- $\phi_1$ propagator: $\quad \quad \quad \quad \quad \quad = \quad \frac{i}{p^2 - q m^2 + i\epsilon}$

- $\phi_2$ propagator: $\quad \quad \quad \quad \quad \quad = \quad \frac{i}{p^2 - m^2 + i\epsilon}$

\[\quad \quad \quad = -i r \xi, \quad \quad \quad = i \xi, \quad \quad \quad = -i t \xi, \quad \quad \quad = -i s \xi.\]

We start with the process $2\phi_2 \rightarrow 2\phi_1$, as done in the previous section, calculate all possible tree level diagrams with the above Lagrangian. For some particular combinations of $q$, $r$, $s$ and $t$ only the contribution of the $2\phi_2 \rightarrow 2\phi_1$ process comes out to be a constant i.e. independent of $in$ momenta, and in that case this constant contribution can be killed by adding a judiciously chosen $\phi_1^2 \phi_2^2$ term to the above Lagrangian (5.1). This way one decides all possible three-point functions for a particular theory to be constructed. Next
one proceeds in manner explained in the previous section, viz. studying the other tree level processes and fixing the higher order interaction terms. Each of these combination of three-point functions (i.e. combination of $q, r, s$ and $t$) gives an integrable model associated with a rank 2 root system. In this section we would only fix the 3-point couplings by studying $2\phi_2 \rightarrow 2\phi_1$ process in detail. Following are the six diagrams (Figs. 8, 9, 10) contributing to the process $2\phi_2 \rightarrow 2\phi_1$.

\[ \frac{i\xi^2}{m^2(2 - q + x)} , \quad \frac{-i\xi^2}{m^2(1 + x)} \]

Figure 8: a) and b) $2\phi_2 \rightarrow 2\phi_1$ process

where $x \equiv ab^{-1} + a^{-1}b$. Using conservation of left- and right-light-cone momenta,

\[ \sqrt{q}(c + d) = a + b \quad \text{and} \quad \sqrt{q}(c^{-1} + d^{-1}) = a^{-1} + b^{-1} \]  

respectively one obtains $x = q(cd^{-1} + c^{-1}d) + 2(q - 1)$.

\[ \frac{1}{m^2(q - \sqrt{q}(ac^{-1} + a^{-1}c))} , \quad \frac{1}{m^2(q - \sqrt{q}(ad^{-1} + a^{-1}d))} \]

Figure 9: a) and b) $2\phi_2 \rightarrow 2\phi_1$ process with two $\phi_1\phi_2^2$ vertices

\[ \frac{-i\xi^2}{m^2(1 - \sqrt{q}(ac^{-1} + a^{-1}c))} , \quad \frac{-i\xi^2}{m^2(1 - \sqrt{q}(ad^{-1} + a^{-1}d))} \]

Figure 10: a) and b) $2\phi_2 \rightarrow 2\phi_1$ process with two $\phi_2^2\phi_2$ vertices

Summing both the diagrams of Fig. 9 we get

\[ \frac{i\xi^2}{m^2(q^2 - 4q + 2 + x)} \]

using (5.2). Adding both the diagrams of Fig. 10 we obtain

\[ \frac{t^2 x}{m^2(1 - 2q + qx)} \]

using (5.2) again. Total
contribution of all the six diagrams then becomes,

\[
\frac{i\xi^2}{m^2} \left[ \frac{r}{(2 - q + x)} - \frac{st}{(1 + x)} + \frac{(2 - 2q + x)}{(q^2 - 4q + 2 + x)} + \frac{t^2x}{(1 - 2q + qx)} \right]
\]

(5.3)

Now one looks for cases for which expression (5.3) is a constant, i.e. independent of \(x\) (or incoming momenta), so that it could be killed by adding a \(\phi_1^2\phi_2^2\) term to the Lagrangian (5.1) with a suitably chosen coefficient.

Case I: \(t = 0^2\);

This gives a contribution (from the right hand side of (5.3)),

\[
\frac{i\xi^2}{m^2} \left[ \frac{r}{(2 - q + x)} + \frac{(2 - 2q + x)}{(q^2 - 4q + 2 + x)} \right],
\]

(5.4)

To have constant contribution one must now equate the coefficients of various powers of \(x\) in numerator and denominator within the square bracket of expression (5.4). In this case we get two equations. Equating coefficients of \(x\) we get,

\[
q^2 - 2q = r,
\]

(5.5)

and equating the constants we have,

\[
q^3 + q^2(r - 4) + 4q(1 - r) + 2r = 0.
\]

(5.6)

Using (5.5) in (5.6), we get three solutions, viz. \((q = 0, r = 0)\); \((q = 2, r = 0)\) and \((q = 3, r = 3)\).

a) First of these, viz. \((q = 0, r = 0)\) is not acceptable as it sets mass of particle \(\phi_1\) as zero.

b) The second solution \((q = 2, r = 0)\) is already discussed in detail in the last section and leads to \(c_2^{(1)}\) ATFT. Moreover one has to choose \(s = 0\) for that. It is clear that \(s\)

---

\(^2\)One need not fix \(t = 0\) (or \(r = s = 0\) as done in the case II discussed later) a priori. One could study the entire expression (5.3) as done in the case III later. The constancy constraint on (5.3) would produce cases I and II as solutions.
cannot be determined from the above as it vanishes from the expression once one chooses $t = 0$ in (5.3). Of course it can be fixed demanding zero contribution from other tree level processes.

c) The third solution ($q = 3, r = 3$) will lead to two different theories depending on the values chosen for $s$. Allowed values of $s$ can be again fixed by studying other tree level processes.

i) If $s = 0$, then the theory is $a_3^{(2)}$ with the Lagrangian,

$$L_{a_3^{(2)}} = \frac{1}{2} \partial \phi \cdot \partial \phi - \frac{m^2}{\beta^2} \left[ \sum_{i=0}^{2} e^{\beta \alpha_i \cdot \phi} - 3 \right],$$

with simple and affine roots $\alpha_1 = (\sqrt{2}, 0), \alpha_2 = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), \alpha_0 = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ and $\beta = \sqrt{2} \xi / m^2$.

ii) If $s = -\frac{2}{\sqrt{3}}$, then the theory is $g_2^{(1)}$ and the corresponding Lagrangian becomes,

$$L_{g_2^{(1)}} = \frac{1}{2} \partial \phi \cdot \partial \phi - \frac{m^2}{2 \beta^2} \left[ e^{\beta \alpha_0 \cdot \phi} + 3 e^{\beta \alpha_1 \cdot \phi} + 2 e^{\beta \alpha_2 \cdot \phi} - 6 \right],$$

with simple and affine roots $\alpha_1 = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}), \alpha_2 = (\sqrt{2}, 0), \alpha_0 = (-\frac{1}{\sqrt{2}}, -\frac{3}{\sqrt{2}})$ and $\beta = \sqrt{2} \xi / m^2$.

**Case II: $r = s = 0$;**

In this case again we demand contribution (from the right hand side of (5.3)),

$$\text{Tree} = \frac{i \xi^2}{m^2} \left[ \frac{(2 - 2q + x)}{(q^2 - 4q + 2 + x)} + \frac{t^2 x}{(1 - 2q + qx)} \right],$$

$$= \frac{i \xi^2}{m^2} \left[ \frac{(1 + \frac{t^2}{q}) x^2 + (t^2 q - 4t^2 - 2q + \frac{1}{q} + \frac{2q^2}{q}) x + (2 - 2q)(\frac{t}{q} - 2)}{x^2 + (q^2 - 4q + \frac{1}{q}) x + (q^2 - 4q + 2)(\frac{1}{q} - 2)} \right],$$

for the process $2\phi_2 \rightarrow 2\phi_1$ to be a constant. Proceeding exactly like the previous way (i.e. matching the coefficients of various powers $x$ in numerator and denominator) we have the following two distinct solutions, viz. ($q = \frac{3 + \sqrt{5}}{2}, t = \frac{1 + \sqrt{5}}{2}$) and ($q = 1, t^2 = -1$).

a) First solution, ($q = \frac{3 + \sqrt{5}}{2}, t = \frac{1 + \sqrt{5}}{2}$), leads to the $a_4^{(2)}$ ATFT. The Lagrangian for which is given by,

$$L_{a_4^{(2)}} = \frac{1}{2} \partial \phi \cdot \partial \phi - \frac{2m^2}{5\beta^2} \frac{1}{(1 - \sin 2\theta)} \left[ e^{\beta \alpha_0 \cdot \phi} + 2 e^{\beta \alpha_1 \cdot \phi} + 2 e^{\beta \alpha_2 \cdot \phi} - 5 \right],$$

(5.10)
where affine and simple roots are $\alpha_0 = (-\sqrt{2} \sin(\pi/4 + \theta), \sqrt{2} \cos(\pi/4 + \theta))$, $\alpha_1 = (\cos \theta, \sin \theta)$, $\alpha_2 = (-\frac{1}{\sqrt{2}} \sin(\pi/4 - \theta), -\frac{1}{\sqrt{2}} \cos(\pi/4 - \theta))$, with $2 \tan 2\theta = 1$ and $\beta = \frac{(1 - \csc 2\theta)}{(\sin \theta - \cos \theta) m^2}$.

b) It is clear from expression (5.9) that the second solution ($q = 1, t^2 = -1$) will result in vanishing contribution for the above process. This also asks for an imaginary coupling $t$. This, we believe, should lead to $a_2^{(1)}$ ATFT which is another rank 2 ATFT available with mass ratio of two fields as unity (i.e. $q = 1$). One must note that in $a_2^{(1)}$ theory the two fields are not self conjugate but mutually conjugate.

**Case III:** $r \neq 0, s \neq 0, t \neq 0$;

In this case we obtain four equations (equating various powers of $x$ in the numerator and the denominator of the expression (5.3), as done earlier) by demanding the contribution of the same $2\phi_2 \rightarrow 2\phi_1$ process be a constant. After some cumbersome algebra we reach the following solution, ($q = 2 + \sqrt{3}, r = -3 - 2\sqrt{3}, t = 2 + \sqrt{3}, s = -\sqrt{3}$). This leads to the last remaining rank 2 ATFT, viz. theory associated with $d_4^{(3)}$ root system. Lagrangian for which is,

$$\mathcal{L}_{d_4^{(3)}} = \frac{1}{2} \partial \phi \cdot \partial \phi - \frac{\sqrt{3}m^2}{2(\sqrt{3} - 1)\beta^2} \left[ e^{\beta \alpha_0 \cdot \phi} + e^{\beta \alpha_1 \cdot \phi} + 2e^{\beta \alpha_2 \cdot \phi} - 4 \right],$$

(5.11)

where simple and affine roots are $\alpha_1 = (\frac{1+\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}), \alpha_2 = (-\frac{1+\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2})$, $\alpha_0 = (\frac{\sqrt{3}+1}{2\sqrt{3}}, \frac{\sqrt{3}+1}{2\sqrt{3}})$ and $\beta = -2\sqrt{3} \xi/m^2$.

This completes our list of distinct solutions. There are other solutions like ($q = \frac{3 + \sqrt{5}}{2}, t = -\frac{1 + \sqrt{5}}{2}$), ($q = \frac{3 - \sqrt{5}}{2}, t = \frac{1 - \sqrt{5}}{2}$) and ($q = 2 - \sqrt{3}, r = -3 + 2\sqrt{3}, t = 2 - \sqrt{3}, s = \sqrt{3}$) etc. which would keep expression (5.3) constant but these are not distinct in a sense that they would not produce any new ATFT. The first one could be obtained from the case II a) by changing the field $\phi_2$ to $-\phi_2$. The second of the above is again same as the solution II a). In this case the roles of the fields $\phi_1$ and $\phi_2$ are interchanged. Both of these would lead to the same $d_4^{(2)}$ theory. The last solution viz.$(q = 2 - \sqrt{3}, r = -3 + 2\sqrt{3}, t = 2 - \sqrt{3}, s = \sqrt{3})$ is again same as case III with $\phi_1 \leftrightarrow \phi_2$ and leads to $d_4^{(3)}$ ATFT.

The above cases exhaust all possible solutions or acceptable values of $q, r, s$ and $t$ which will respect elasticity and also exhaust all possible ATFT associated with rank two root systems, viz. $a_2^{(1)}, a_4^{(2)}, c_2^{(1)}, d_3^{(2)}, d_4^{(3)}$ and $g_2^{(1)}$. S-matrices and other details about these models can be found in Ref.[11-16].
Our Lagrangians for various ATFT may look little different from the ones existing in the literature. This is due to the fact in the expressions of these Lagrangians we have chosen the simple and affine roots such a way that the mass matrix becomes diagonal.

6 Summary and Results

Here we summarize the results. In a pedagogical way we have introduced the way of constructing integrable models in $1+1$ dimensions. Starting with simple scalar field theories we have exploited the elasticity property (no particle production) at tree level in the scattering of scalar particles for constructing affine Toda field theory associated with rank one and rank two root systems. It has been shown that the relative masses and three-point couplings could be fixed by vanishing amplitude of 4-point function ($2\phi_2 \rightarrow 2\phi_1$) in case of two scalar fields. We summarize the findings of the section 5 in the following table.

Table 1: Relative strengths of the mass terms and the three-point couplings for rank 2 ATFT.

| Case | mass terms | Three-point interaction terms | Theory |
|------|------------|-------------------------------|--------|
|      | $\phi_1^2$ | $\phi_2^2$ | $\phi_1^2$ | $\phi_2^2$ | $\phi_1 \phi_2$ | $\phi_1 \phi_2$ |        |
|      | $q$ | $r$ | $s$ | $t$ | $-$ |        |        |
| Ib   | 2  | 1  | 0  | 0  | 0  | $-1$  |        | $c_2^{(1)}$ |
| Ic i | 3  | 1  | 3  | 0  | 0  | $-1$  |        | $d_3^{(2)}$ |
| Ic ii| 3  | 1  | 3  | $-2\sqrt{3}$ | 0  | $-1$  |        | $g_2^{(1)}$ |
| II a | $3+\sqrt{5}$ | 1  | 0  | 0  | $1+\sqrt{5}$ | $-1$  |        | $a_4^{(2)}$ |
| II b | $2$  | 1  | 0  | 0  | $\frac{2}{\pm i}$ | $-1$  |        | $a_2^{(1)}$ |
| III  | $2+\sqrt{3}$ | 1  | $-3-2\sqrt{3}$ | $-\sqrt{3}$ | $2+\sqrt{3}$ | $-1$  |        | $d_4^{(3)}$ |

Further it was shown that once the three-point coupling are fixed, the higher order couplings are determined uniquely by demanding vanishing of various other scattering processes at tree level\(^3\). We have calculated 5-point functions and verified explicitly no particle production (i.e. vanishing amplitudes for $2$-particle $\rightarrow$ $3$-particle processes) in $a_2^{(2)}$ and $c_2^{(1)}$ ATFT for the very first time, we believe. Each combination of allowed three-

\(^3\)One can calculate higher order couplings from three point couplings, see Ref. [17]
point couplings produces an ATFT associated with a particular rank two root system. We strongly believe that the same procedure could also be used for constructing ATFT associated with root systems having rank greater than two. It would be nice if one develops a way which works for affine Toda field theories in general. Our effort is just a modest beginning in this direction.

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Appendix A1:

1) \[ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a \\
b \\
c \\
d \\
e \\
\end{array}
\end{array}
\end{array} = -\frac{i\eta^3}{6m^4} \frac{1}{(2 - c - c^{-1})} \]

2) \[ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a \\
b \\
c \\
d \\
e \\
\end{array}
\end{array}
\end{array} = \frac{i\eta^3}{6m^4} \frac{1}{\delta^2} (1 + \delta) \]

3) \[ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a \\
b \\
c \\
d \\
e \\
\end{array}
\end{array}
\end{array} = -\frac{i\eta^3}{2m^4} \frac{(1 + \delta)^2}{\delta^2(1 + \delta + \delta^2)} \]

4) \[ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a \\
b \\
c \\
d \\
e \\
\end{array}
\end{array}
\end{array} = -\frac{i\eta^3}{2m^4} \frac{1}{(1 - c - c^{-1})(2 - c - c^{-1})} \]

5) \[ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a \\
b \\
c \\
d \\
e \\
\end{array}
\end{array}
\end{array} = -\frac{i\eta^3}{m^4} \frac{1}{(1 - c - c^{-1})(1 - d - d^{-1})} \]

6) \[ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 a \\
b \\
c \\
d \\
e \\
\end{array}
\end{array}
\end{array} = \frac{i\eta^3}{m^4} \frac{1}{(1 - c - c^{-1})(1 + \delta + \delta^2)} \]

7) \[ 4) \] (c \leftrightarrow d) = \frac{i\eta^3}{2m^4} \frac{1}{(1 - d - d^{-1})(2 - d - d^{-1})} \]
8) \( (c \leftrightarrow d) = 5) \) \( \frac{i \eta^3}{m^2} \frac{1}{(1 - c^{-1})(1 - d^{-1})} \) (A1.8)

9) \( (c \leftrightarrow d) = 6) \) \( \frac{i \eta^3}{m^2} \frac{1}{(1 - \delta^2)} \) (A1.9)

10) \( (c \leftrightarrow d) = 1) \) \( \frac{i \eta^3}{6m^4} \frac{1}{(2 - d^{-1})} \) (A1.10)

11) \( (a \leftrightarrow b) = 6) \) \( \frac{i \eta^3}{m^4} \frac{1}{(1 - c^{-1})(1 + \delta)} \) (A1.11)

12) \( (a \leftrightarrow b) = 9) \) \( \frac{i \eta^3}{m^4} \frac{1}{(1 + \delta)} \) (A1.12)

13) \( (a \leftrightarrow b) = 3) \) \( \frac{i \eta^3}{2m^4} \frac{(1 + \delta)^2}{\delta^2(1 + \delta + \delta^2)} \) (A1.13)

14) \( (a \leftrightarrow b) = 4) \) \( \frac{i \eta^3}{2m^4} \frac{1}{(1 - c^{-1})(2 - c^{-1})} \) (A1.14)

15) \( (c \leftrightarrow d) = 14) \) \( \frac{i \eta^3}{2m^4} \frac{1}{(1 - d^{-1})(2 - d^{-1})} \) (A1.15)

**Appendix A2:**

1) \( a \rightarrow c \rightarrow d \rightarrow b \) = \( -\frac{i \eta \lambda}{3m^2} \) (A2.1)

2) \( a \rightarrow c \rightarrow d \rightarrow e \) = \( -\frac{i \eta \lambda}{m^2} \frac{1}{(1 - c^{-1})} \) (A2.2)

3) \( a \rightarrow c \rightarrow d \rightarrow e \) = \( \frac{i \eta \lambda}{m^2} \frac{(1 + \delta)}{1 + \delta + \delta^2} \) (A2.3)

4) \( a \rightarrow c \rightarrow d \rightarrow e \) = \( \frac{i \eta \lambda}{2m^2} \frac{(1 + \delta)}{\delta^2} \) (A2.4)

5) \( a \rightarrow c \rightarrow d \rightarrow e \) = \( -\frac{i \eta \lambda}{2m^2} \frac{1}{(2 - d^{-1})} \) (A2.5)

6) \( (a \leftrightarrow b) = 2) \) \( -\frac{i \eta \lambda}{m^2} \frac{1}{(1 - c^{-1})} \) (A2.6)

7) \( (c \leftrightarrow d) = 2) \) \( -\frac{i \eta \lambda}{m^2} \frac{1}{(1 - d^{-1})} \) (A2.7)
\[8) \quad (a \leftrightarrow b) = \frac{i\eta \lambda}{m^2 (1 - d - d^{-1})} \quad \text{(A2.8)}\]

\[9) \quad (a \leftrightarrow b) = \frac{i\eta \lambda}{m^2 (1 + \delta)} \quad \text{(A2.9)}\]

\[10) \quad (c \leftrightarrow d) = -\frac{i\eta \lambda}{2m^2 (2 - c - c^{-1})} \quad \text{(A2.10)}\]

\textbf{Appendix B1:}

1) \[a \rightarrow b \rightarrow c \rightarrow d \rightarrow e = \frac{i\xi^3}{4m^4 (2 - d - d^{-1})} \quad \text{(B1.1)}\]

2) \[a \rightarrow b \rightarrow c \rightarrow d \rightarrow e = \frac{i\xi^3}{4m^4 (1 - \sqrt{2} e + e^2)^2} \quad \text{(B1.2)}\]

3) \[\quad = 1) \quad (c \leftrightarrow d) = \frac{i\xi^3}{4m^4 (2 - c - c^{-1})} \quad \text{(B1.3)}\]

4) \[\quad = 2) \quad (a \leftrightarrow b) = \frac{i\xi^3}{4m^4 (1 - \sqrt{2} e + e^2)^2} \quad \text{(B1.4)}\]

5) \[a \rightarrow b \rightarrow c \rightarrow e \rightarrow c = -\frac{i\xi^3}{2m^4 (c + c^{-1})(2 - c - c^{-1})} \quad \text{(B1.5)}\]

6) \[\quad = \frac{i\xi^3}{\sqrt{2} m^4 (d + d^{-1})(e + e^{-1} - \sqrt{2})} \quad \text{(B1.6)}\]

7) \[\quad = 5) \quad (a \leftrightarrow b) = -\frac{i\xi^3}{2m^4 (c + c^{-1})(2 - c - c^{-1})} \quad \text{(B1.7)}\]

8) \[\quad = 5) \quad (c \leftrightarrow d) = -\frac{i\xi^3}{2m^4 (d + d^{-1})(2 - d - d^{-1})} \quad \text{(B1.8)}\]
9) \[ (c \leftrightarrow d) = -\frac{i\xi^3}{2m^4(d + d^{-1})(2 - d - d^{-1})} \] (B1.9)

10) \[ (a \leftrightarrow b) = \frac{i\xi^3}{\sqrt{2} m^4 (d + d^{-1})(e + e^{-1} - \sqrt{2})} \] (B1.10)

11) \[ (c \leftrightarrow d) = \frac{i\xi^3}{\sqrt{2} m^4 (c + c^{-1})(e + e^{-1} - \sqrt{2})} \] (B1.11)

12) \[ (c \leftrightarrow d) = \frac{i\xi^3}{\sqrt{2} m^4 (c + c^{-1})(e + e^{-1} - \sqrt{2})} \] (B1.12)

13) \[ \begin{array}{c}
\begin{array}{c}
 a \\
b
\end{array}
\begin{array}{c}
 \cdots \\
 e \\
d
\end{array}
\begin{array}{c}
 c
\end{array}
\end{array} = \frac{i\xi^3}{2m^4} \] (B1.13)

14) \[ \begin{array}{c}
\begin{array}{c}
 a \\
b
\end{array}
\begin{array}{c}
 \cdots \\
 e \\
d
\end{array}
\begin{array}{c}
 c
\end{array}
\end{array} = -\frac{i\xi^3}{2\sqrt{2}m^4 (e + e^{-1} - \sqrt{2})} \] (B1.14)

15) \[ \begin{array}{c}
\begin{array}{c}
 a \\
b
\end{array}
\begin{array}{c}
 \cdots \\
 e \\
d
\end{array}
\begin{array}{c}
 c
\end{array}
\end{array} = -\frac{i\xi^3}{m^4 (d + d^{-1})} \] (B1.15)

16) \[ (a \leftrightarrow b) = -\frac{i\xi^3}{m^4(d + d^{-1})} \] (B1.16)

17) \[ (c \leftrightarrow d) = -\frac{i\xi^3}{m^4(c + c^{-1})} \] (B1.17)

18) \[ (c \leftrightarrow d) = -\frac{i\xi^3}{m^4(c + c^{-1})} \] (B1.18)

**Appendix B2:**

1) \[ \begin{array}{c}
\begin{array}{c}
 a \\
b
\end{array}
\begin{array}{c}
 \cdots \\
 e \\
d
\end{array}
\begin{array}{c}
 c
\end{array}
\end{array} = \frac{i\xi^3}{m^4(2 - \sqrt{2}(c + c^{-1})) (2 - \sqrt{2}(d + d^{-1}))} \] (B2.1)
\[ \begin{align*}
2) \quad & \quad \frac{a}{b} \quad \frac{c}{e} = \frac{i\xi^3}{4m^4} \frac{1}{(\sqrt{2} - d - d^{-1})} \frac{1}{(2\sqrt{2} - d - d^{-1})} \\
3) \quad & \quad (c \leftrightarrow d) = \frac{i\xi^3}{m^4} \frac{1}{(2 - \sqrt{2}(c + c^{-1}))} \frac{1}{(2 - \sqrt{2}(d + d^{-1}))} \\
4) \quad & \quad (a \leftrightarrow b) = \frac{i\xi^3}{4m^4} \frac{1}{(\sqrt{2} - d - d^{-1})} \frac{1}{(2\sqrt{2} - d - d^{-1})} \\
5) \quad & \quad (c \leftrightarrow d) = \frac{i\xi^3}{4m^4} \frac{1}{(\sqrt{2} - c - c^{-1})} \frac{1}{(2\sqrt{2} - c - c^{-1})} \\
6) \quad & \quad (a \leftrightarrow b) = \frac{i\xi^3}{4m^4} \frac{1}{(\sqrt{2} - c - c^{-1})} \frac{1}{(2\sqrt{2} - c - c^{-1})} \\
7) \quad & \quad \frac{a}{b} \quad \frac{c}{e} \quad \frac{d}{e} = \frac{i\xi^3}{2\sqrt{2} m^4} \frac{1}{(2\sqrt{2} - d - d^{-1})} \\
8) \quad & \quad \frac{a}{b} \quad \frac{c}{e} \quad \frac{d}{e} = \frac{i\xi^3}{\sqrt{2} m^4} \frac{1}{(\sqrt{2} - d - d^{-1})} \\
9) \quad & \quad (c \leftrightarrow d) = \frac{i\xi^3}{2\sqrt{2} m^4} \frac{1}{(2\sqrt{2} - c - c^{-1})} \\
10) \quad & \quad (c \leftrightarrow d) = \frac{i\xi^3}{\sqrt{2} m^4} \frac{1}{(\sqrt{2} - c - c^{-1})} \\
11) \quad & \quad (a \leftrightarrow b) = \frac{i\xi^3}{\sqrt{2} m^4} \frac{1}{(\sqrt{2} - d - d^{-1})} \\
12) \quad & \quad (c \leftrightarrow d) = \frac{i\xi^3}{\sqrt{2} m^4} \frac{1}{(\sqrt{2} - c - c^{-1})} \\
\end{align*} \]

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