Mesoscopic fluctuations and intermittency in aging dynamics

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Abstract. – Mesoscopic aging systems are characterized by large intermittent noise fluctuations. In a record dynamics scenario [P. Sibani and J. Dall, Europhys. Lett. 64, 2003] these events, quakes, are treated as a Poisson process with average $\alpha \ln(1 + t/t_w)$, where $t$ is the observation time, $t_w$ is the age and $\alpha$ is a parameter. Assuming for simplicity that quakes constitute the only source of de-correlation, we present a model for the probability density function (PDF) of the configuration autocorrelation function. Beside $\alpha$, the model has the average quake size $1/q$ as a parameter. The model autocorrelation PDF has a Gumbel-like shape, which approaches a Gaussian for large $t/t_w$ and becomes sharply peaked in the thermodynamic limit. Its average and variance, which are given analytically, depend on $t/t_w$ as a power-law and a power-law with a logarithmic correction, respectively. Most predictions are in good agreement with data from the literature and with the simulations of the Edwards-Anderson spin glass carried out as a test.

Introduction. – After a rapid quench of an external parameter, e.g. the temperature, many complex materials age, i.e. their properties slowly change with the waiting time, $t_w$, elapsed from the quench. Ever since the initial observations in polymers [1], evidence has accumulated that spin-glasses [2], type II superconductors [3], glasses [4], and soft condensed matter [5], among others, age in similar ways, e.g.: For observation times $t \ll t_w$ physical averages are nearly constant, and autocorrelations and their conjugate linear response functions are connected by an equilibrium-like fluctuation-dissipation theorem (FDT). Conversely, for $t \gg t_w$ they visibly drift and the FDT is violated. As was recently discovered, the drift happens in an intermittent fashion [6, 7], i.e. through rare, large, and spatially heterogeneous re-arrangements, which appear as non-Gaussian tails in the probability density function (PDF) of configurational probes such as colloidal particle displacement [8, 9] and correlation [10] or voltage noise fluctuations in glasses [11].

As aging phenomena are similar for a broad class of interactions, we seek a mesoscopic description, and assume that intermittent events, for short quakes, are the main source of de-correlation in non-equilibrium aging. In the framework of record dynamics [12,13], quakes are irreversible and are triggered by (energy) fluctuations of record magnitude. We show how this leads to a description of the configurational autocorrelation function, more specifically, the dependence of the shape of its PDF on $t$, $t_w$, the temperature $T$ and the system size $N$, which resembles observations for colloidal gels [10] spin-glasses and kinetically constrained models [15,16]. The model PDF is closely approximated by the Gumbel distributions widely
used in the literature [16,17]. The average and variance are given in close form as a function of $t/t_w$, $T$ and $N$. The average and the PDF, standardized to zero mean and unit variance, are in excellent agreement with spin-glass simulations. The agreement is rather poor for the variance itself, mainly because pseudo-equilibrium fluctuations are neglected.

The configuration auto-correlation PDF. – In the model, a set of $N$ binary variables defines the system configuration. Without further loss of generality, and with an eye to the simulations of the E-A spin-glass model [18], we refer to these variables as spins, and to their changes of state as ‘flips’.

Configuration changes are gauged by the number of spins with different orientations at times $t_w$ and $t_w + t$. This Hamming distance, $H$, is simply related to the autocorrelation $C$ by

$$C(t_w, t) = 1 - 2H(t_w, t_w + t)/N.$$  

Initially, we focus on the probability $P_H(h, t_w + t | 0, t_w)$ for $H = h$ at time $t_w + t$, given $H = 0$ at $t_w$, which we write as the average

$$P_H(h, t_w + t | 0, t_w) = \sum_{s=0}^{\infty} P_S(s) P_H(h | s)$$  

over the conditional probability $P_H(h | s)$ for $H = h$ given $s$ flips ($s = 0, 1, \ldots \infty$). The weight function $P_S(s)$ is the probability for exactly $s$ flips during $[t_w, t_w + t)$.

Assuming for simplicity that flips occur at any site with probability $1$/$N$ over the conditional probability $P_H(h | s)$ for $H = h$ given $s$ flips ($s = 0, 1, \ldots \infty$). The weight function $P_S(s)$ is the probability for exactly $s$ flips during $[t_w, t_w + t)$.

The equation has the formal solution

$$P_H(h | s) = T^s P_H(h | 0),$$  

where $T$ is the (bi-diagonal) stochastic matrix implicitly given by Eq. 3 and where the vector $P_H$ has elements $P_H(0 | s), P_H(1 | s), \ldots P_H(N | s)$. The $s$ dependence of the conditional average and variance of $H$, $\mu_H(s)$ and $\sigma_H^2(s)$ can be gleaned from the moment generating function $\sum_{h=0}^{N} P_H(h | s) z^h$, $|z| \leq 1$. Omitting the details, one finds

$$\mu_H(s) = N/2(1 - (1 - 2/N)^s)$$  

and

$$\sigma_H^2(s) = N/4(1 - (1 - 4/N)^s) + N^2/4((1 - 4/N)^s - (1 - 2/N)^{2s}).$$  

Since $\sigma_H(s) \ll \mu_H(s)$ for large $N$, the r.h.s of Eq. 2 is dominated in this limit by the term with index $s(h)$ implicitly given by $h = \mu_H(s)$. As a consequence, $P_H$ and $P_S$ acquire very similar shapes when standardized to zero average and unit variance.

To calculate $P_S(s)$ we need the probability that $i$ quakes occur between $t_w$ and $t_w + t$ and the distribution of the number of flips, for short ‘size’, of each quake. According to refs. [12,13], $i$ has a Poisson distribution with average

$$n_i(t_w, t) = \alpha(N) \ln(1 + t/t_w).$$  

(8)
Fig. 1 - (a): Three PDF’s of the model auto-correlation function $C$ are shown in the main panel, for different values of $n_I$. The PDF’s are all shifted to zero average. The inserts show the $n_I$ dependence of the average and the variance, the latter multiplied by $N$. The symbols are from numerical evaluations, and the full lines are according to Eqs. 15 and 16. (b) The model PDF for $n_I = 16.22$ is shown together with its best Gumbel approximation, with both curves shifted to zero mean and rescaled to unit variance. The left insert shows the linear relation between the model parameter $n_I$ and the $g$ value for the best Gumbel fit. In the right panel the approximation error, (multiplied by 1000), is plotted versus $n_I$.

The property $\alpha(N) \propto N$, which removes the $N$ dependence of the exponent $\lambda$, (see Eq. [14]), arises when the intermittent signal results from independent intermittent processes, stemming e.g. from locally thermalized clusters [13]. The $T$ independence of $\alpha$ reflects the noise insensitivity of record dynamics [12], and holds within the low temperature range for which the description applies.

For the quake size, simulations of vortex dynamics [19] yield a near exponential distribution. The same form is consistent with the (asymptotically) exponential distribution of the energy released [13, 20] by intermittent events. Hence, glossing over the integer nature of the sizes, we treat them as independent stochastic variables $X_k$, $k = 1, 2, \ldots, i$, with the PDF

$$P_{X_k}(x) = q(T) \exp(-q(T)x).$$

(9)

A temperature dependence of the reciprocal average quake size, $q(T)$, is allowed (but not required) by the theory, and is directly observable through the exponent $\lambda$, see Fig. 2. For typographical clarity this dependence is left understood, together with the dependence of $n_I$ on $\ln(1 + t/t_w)$.

Considering first the conditional probability for $S_i$ flips for a given number $i$ of quakes, we note that $S_i = \sum_{k=1}^{i} X_k$ has the gamma density

$$P_{S_i}(x) = \frac{(qx)^{i-1}}{(i-1)!} \exp(-qx). \quad i > 0.$$  

(10)

Averaging the above expression over the Poisson distribution of $i$, and taking into account
that \( \delta(s) \exp(-n_I) \) is the probability of no flips \( (i = 0) \), one finds

\[
P_S(s) = \sum_{i=1}^{\infty} P_S_i(s) \frac{1}{i!} \exp(-n_I) + \delta(s) \exp(-n_I).
\]

With the variable \( z = (4qsn_I)^{1/2} \), this is rewritten as

\[
P_S(s) = 2n_Ia \exp(-qs - n_I)I_1(z)/z + \delta(s) \exp(-n_I),
\]

where \( I_1 \) is the modified Bessel function of order one (See e.g. Abramowitz & Stegun, 9.6.10).

The \( \delta(s) \) term in Eq. 12 will be neglected, since \( n_I \) is large except for \( t \ll t_w \). With the term discarded, the standardized \( P_S(s) \) has no \( T \) dependence. This is seen, in brief, as follows: Using \( \mu_S = n_I/q \) and \( \sigma_S^2 = 2n_I/q^2 \) for the average and variance of \( S \), the standardized PDF, \( \sigma_SP_S((s - \mu_S)/\sigma_S) \), has no \( q \) dependence. However, as \( q \) carries the model full \( T \) dependence, the latter disappears as well. Furthermore, due to its similarity with \( P_S \), the standardized \( P_H \) is also independent of \( T \), as confirmed by Fig. 3.

Averaging Eqs. 6 and 7 over \( P_S(s) \), reintroducing the time dependence of \( n_I \) and making the approximation \( -\ln(1 - 2/N) \approx 2/N \), one finds

\[
\mu_H(t_w, t) = N \left( 1 - (1 + t/t_w)^{\lambda(T)} \right),
\]

where

\[
\lambda(T) = -2\alpha(N) \frac{1}{N} \frac{1}{q(T)}.
\]

The average (macroscopic) form of the correlation function \( C \), is obtained from Eqs. 13 and 11 as

\[
\mu_C(t_w, t) = (1 + t/t_w)^{\lambda(T)}.
\]

Similar steps lead from Eq. 7 to

\[
N\sigma_C^2(t_w, t) = 1 - (1 + t/t_w)^{2\lambda(T)} (1 - 2\lambda(T) \ln(1 + t/t_w)).
\]

Panel (a) of Fig. 4 shows, for three different values of \( n_I = \alpha(N) \ln(1 + t/t_w) \), the model PDF given by Eqs. 12, 6 and 7. The two inserts show the \( n_I \) dependence of \( \mu_C \) and \( N\sigma_C^2 \), from a numerical evaluation of the model equations (circles) and from Eqs. 13 and 16 with the \( n_I(t, t_w) \) dependence reintroduced (lines).

In standard form (see e.g. ref [16]), the one-parameter family of Gumbel densities is given by \( \Phi_g(y) = \frac{\exp(b(y - y_0) - e^{b(y - y_0)})}{1 + \exp(-b(y - y_0))} \), with \( b = \sqrt{(\Psi'(g))} \) and \( y_0 = \ln(g) - \Psi(g) \), where \( \Psi \) denotes the digamma function, \( \Psi' \) its derivative and \( g \) is a real number. Gumbel densities empirically describe fluctuations in complex systems [16, 17]. Fig. 4 (b) shows that, except for \( n_I < 1 \), our model PDF is closely approximated by the Gumbel PDF whose \( g \) value minimizes the \( L_1 \) distance between the two. The left insert of the figure shows that this optimal \( g \) value is linearly related to \( n_I \) as \( g = 0.300n_I - 0.185 \).

Comparison with simulation data. – The (average) autocorrelation function \( \mu_C \) for spin-glasses is well investigated [14, 16, 21, 22]. For \( t > t_w \), \( \mu_C \) is nearly a function of \( t/t_w \), and can be fitted by a power-law with a temperature dependent exponent. E.g. Picco et al. [21] found an excellent scaling using the variable \( \ln(t/t + t) - \ln(t_w) \). The autocorrelation PDF for the E-A spin glass model nearly follows \( t/t_w \) scaling according to Castillo et al. [15] Chamon et al. [16] also consider a kinetically constrained model with trivial statics. In both cases, the
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Fig. 2 – (a): The mean of the autocorrelation function in an E-A spin-glass of size \( N = 16^3 \) for different values of \( t_w \) and \( t \) is shown by symbols versus \( \ln(1 + \frac{t}{t_w}) \), for the indicated temperatures. The full lines are power-law fits according to Eq. 15. The insert shows the temperature dependence of the exponent \( \lambda(T) \). (b) For the same temperatures, the estimated variances of the autocorrelation function multiplied by \( \langle \rangle \) versus \( \ln(1 + \frac{t}{t_w}) \) together with 1\( \sigma \) error-bars. The full lines are fitted to a shifted and rescaled form of Eq. 16, as detailed in the text.

autocorrelation PDF, shifted to zero mean and rescaled to unit variance, is empirically fitted to time evolving Gumbel distributions, which are numerically equivalent to our model results (see Fig. 1).

For a more detailed comparison, we simulated the E-A spin-glass [18] on a cubic lattice with \( N = 16^3 \) using an event driven simulation technique [23], whose ‘intrinsic’ time unit corresponds, for large systems, to one Monte Carlo sweep. The data are sampled at 20 time points, which are separated by a multiplicative factor of 1.5, with start at \( t = 100 \) and end at \( t \approx 2.2 \times 10^5 \). Among these points, any ordered pair can be chosen for \( t_w \) and \( t_w + t \). For each set of physical parameters, 1000 runs are performed with independent noise and couplings realizations, producing e.g. 1000 data points for \( t/t_w = 2216 \), and 20000 points for \( t/t_w = 1.5 \).

For a range of temperatures, the average spin-glass autocorrelation is plotted versus \( t/t_w \) (symbols). Deviations from \( t/t_w \) scaling appear for approx. \( t/t_w < 4 \) and \( T > 0.5 \), as seen in the left panel of Fig. 2 from the poor data collapse. Away from this parameter region, the data are well described by Eq.15 (full line). The \( T \) dependence of the exponent is shown in the insert (circles), where the fit \( \lambda(T) = -0.25T/T_g \) is also shown (full line). \( T_g = 0.97 \) is the critical temperature of the model [24]. The same linear form, and with a similar slope coefficient, is found numerically in Kisker et al. [14]. They, however, introduce a small \( t_w \) dependence of \( \lambda \), which is beyond the present model.

Summarizing, for low \( T \) and not too small \( t/t_w \), the model is able to describe the time dependences of the average autocorrelation with no free parameters. Furthermore, Eq. 14 links \( \lambda(T) \) to a linear temperature increase of the average quake size.

To improve the statistics of the variance and PDF data in the \( t/t_w \) scaling region where a comparison with the model is most interesting, we estimate the variance and its error-bar as the mean value and standard deviation over the set of variances for all pairs \( t_w, t \) having the same ratio \( t/t_w \). Similarly, the empirical frequencies of the \( C \) values are calculated based on...
Fig. 3 – From the left to the right panel, the \( n_I \) values indicated correspond to \( t/t_w = 2.3, 7.6 \) and 25.6. Within each panel, the scaled and shifted autocorrelation PDF is shown for the model (full line) together with four sets of simulation data for temperatures \( T = 0.15, 0.25, 0.35 \) and 0.5.

all data with the same \( t/t_w \).

The value of \( N \) stands in the model for an (unknown) number of thermally active spins, and appears in the autocorrelation variance, which vanishes linearly with \( 1/N \). This leads to an undetermined \( T \) dependent scale factor, \( f_1 \), between model and simulation variance. Secondly, and more importantly, the data cannot be fitted without a second, ad hoc, offset parameter \( f_2 \), likely because the de-correlating effect of the pseudo-equilibrium fluctuations is altogether neglected. Hence, with respect to the variance, the theory only provides a qualitative description, which is captured by the empirical formula

\[
N\sigma^2_{\text{emp}}(t_w,t) = f_1(N\sigma^2_C(t_w,t) + f_2).
\]

For completeness, the latter is plotted (lines) in the right panel of Fig. 4 together with the simulation data with error bars. The parameter \( f_1 \) increases with \( T \) within the range \( 1 \to 20 \), and \( f_2 \) remains close to \( 1/10 \).

By contrast, an excellent agreement between predictions and data for the standardized PDF is achieved by a simple adjustment of the vertical scale of the latter. This scale, which is undetermined from the outset, is fitted to the properly normalized model PDF. The centering and rescaling are done using the data average and standard deviation. The results are plotted with 1\( \sigma \) error-bars in Fig. 5. The three panels of the figure correspond, from left to right, to \( t/t_w = 2.3, 7.6 \) and 25.6. In each panel, the data shown are for \( T = 0.15, 0.25, 0.35 \) and 0.5. Their collapse confirms the anticipated \( T \) independence of the standardized autocorrelation PDF. The model predictions (full lines) contain one parameter, \( \alpha(N) \), whence it is possible to determine one value of \( n_I \) by data fitting. This was done for \( (t/t_w = 25.6) \)—in the rightmost panel—yielding \( n_I = 78 \). Eq. 8 then gives \( \alpha = 24 \), whence \( n_I = 20 \) and 49 for \( t_w = 2.3 \) and 7.6 respectively.
Conclusion. – Based on the record dynamics description of intermittency [12, 13] the model develops the aging properties of the configuration autocorrelation after a deep quench. Its predictions for the average autocorrelation and the standardized PDF are accurate at low temperatures and for $t > t_w$. Together with allied efforts [19, 25–27], the present results support the view that record-sized fluctuations are important for aging in metastable glassy systems.

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