AUTOMORPHISMS OF PROFINITE MAPPING CLASS GROUPS

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Abstract. For \( S = S_{g,n} \) a closed orientable differentiable surface of genus \( g \) from which \( n \) points have been removed, such that \( \chi(S) = 2 - 2g - n < 0 \), let \( \Gamma(S) \) be the pure mapping class group of \( S \) and \( \Pi\Gamma(S) \) and \( \Pi\Gamma(S) \) be, respectively, its profinite and its congruence completions. The latter can be identified with the image of the natural representation \( \Pi\Gamma(S) \to \text{Out}(\pi_1(S)) \), where \( \pi_1(S) \) is the profinite completion of the fundamental group of the surface \( S \). Let \( \text{Out}^0(\Pi\Gamma(S)) \) and \( \text{Out}^0(\Pi\Gamma(S)) \) be the groups of outer automorphisms which preserve the conjugacy class of a procyclic subgroup generated by a nonseparating Dehn twist and let \( \text{GT} \) be the profinite Grothendieck-Teichmüller group. We then prove that there is a natural faithful representation:

\[ \text{GT} \to \text{Out}^0(\Pi\Gamma(S)) \]

and, letting \( \Sigma_n \) be the symmetric group on the \( n \) punctures of \( S \), for \( \chi(S) < g - 2 \), a natural isomorphism:

\[ \text{Out}^0(\Pi\Gamma(S)) \cong \Sigma_n \times \text{GT}. \]

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1. Introduction

Let \( S = S_{g,n} \) be a closed orientable differentiable surface of genus \( g(S) = g \) from which \( n(S) = n \) points have been removed. We denote by \( \chi(S) = 2 - 2g - n \) the Euler characteristic of \( S \), which we assume to be negative, and by \( d(S) \) the modular dimension of \( S \), that is to say the number of moduli of a Riemann surface diffeomorphic to \( S \). More precisely, to the topological surface \( S \), we associate the moduli stacks \( M(S) \), which parameterizes smooth projective curves whose complex models are diffeomorphic to \( S \), and \( \text{PM}(S) \), which parameterizes the same curves but on which an order of the punctures has been fixed. They are smooth irreducible DM stacks defined over \( \text{Spec}(\mathbb{Z}) \) of dimension \( d(S) = 3g - 3 + n \).

We let \( \Gamma(S) \) be the mapping class group of \( S \), that is to say the group of isotopy classes of orientation preserving diffeomorphisms of \( S \), and \( \Pi\Gamma(S) \) the pure mapping class group of \( S \), that is to say the kernel of the natural representation \( \Gamma(S) \to \Sigma_{n(S)} \), where \( \Sigma_{n(S)} \) denotes the symmetric group on the punctures of \( S \). For an algebraic stack \( X \) over \( \text{Spec}(\mathbb{Z}) \) and a field \( K \), let \( X_K := X \times \text{Spec}(K) \). The groups \( \Gamma(S) \) and \( \Pi\Gamma(S) \) can then be identified, respectively, with the topological fundamental groups of the complex DM stacks \( M(S)_C \) and \( \text{PM}(S)_C \).

A fundamental result of Ivanov states that, for \( g \geq 3 \), the outer automorphism group of \( \Gamma(S) \) is a cyclic group of order 2 (cf. [19]). McCarthy later computed this group also in genus \( \leq 2 \) (cf. [20]). If we let \( \text{Aut}^0(\Gamma(S)) \) be the group of automorphisms of \( \Gamma(S) \) which
preserve the conjugacy class of a cyclic subgroup generated by a nonseparating Dehn twist and \( \text{Out}^\lambda(\Gamma(S)) \) be its quotient by the group of inner automorphisms, all the previous results can be stated in a uniform way by saying that there are, for \( d(S) > 1 \), a natural isomorphism:

\[
\text{Out}^\lambda(\Gamma(S)) \cong \{ \pm 1 \};
\]

and, for \( d(S) > 1 \) and \( S \neq S_{1,2} \), a natural isomorphism:

\[
\text{Out}^\lambda(\Gamma'(S)) \cong \Sigma_n(S) \times \{ \pm 1 \},
\]

where the symmetric group \( \Sigma_n(S) \) identifies with \( \text{Inn}(\Gamma(S))/\text{Inn}(\Gamma'(S)) \) and \( \{ \pm 1 \} \) with its centralizer in \( \text{Out}^\lambda(\Gamma'(S)) \).

Note that, for \( d(S) > 1 \) and \( Z(\Gamma(S)) = \{ 1 \} \), namely for \( (g,n) \neq (0,4), (1,1), (1,2) \) and \( (2,0) \), the \( \mathbb{L}_0 \)-condition is always satisfied. However, for \( Z(\Gamma(S)) \neq \{ 1 \} \), for instance, we have that \( \text{Aut}^\lambda(\Gamma(S)) \subset \text{Aut}(\Gamma(S)) \) (cf. (3) of Theorem 1 in [20]).

The profinite mapping class group \( \hat{\Gamma}(S) \) and the pure profinite mapping class group \( \hat{\Pi}(S) \) are, respectively, the profinite completions of \( \Gamma(S) \) and \( \Pi(S) \). The procongruence mapping class group \( \hat{\Gamma}(S) \) and the pure procongruence mapping class group \( \hat{\Pi}(S) \) are, respectively, the images of \( \hat{\Gamma}(S) \) and \( \hat{\Pi}(S) \) in the profinite group \( \text{Out}(\hat{\Gamma}_1(S)) \).

Let \( \mathcal{C}(S) \to \mathcal{M}(S) \) and \( \mathcal{PC}(S) \to \mathcal{PM}(S) \) be the universal punctured curves. The profinite mapping class groups \( \hat{\Gamma}(S) \) and \( \hat{\Pi}(S) \) then identify, respectively, with the \( \text{\acute{e}tale} \) fundamental groups of \( \mathcal{M}(S)_{\overline{\mathbb{Q}}} \) and \( \mathcal{PM}(S)_{\overline{\mathbb{Q}}} \), while the procongruence mapping class groups \( \hat{\Gamma}(S) \) and \( \hat{\Pi}(S) \) identify, respectively, with the images of the universal monodromy representations associated to the curves \( \mathcal{C}(S)_{\overline{\mathbb{Q}}} \to \mathcal{M}(S)_{\overline{\mathbb{Q}}} \) and \( \mathcal{PC}(S)_{\overline{\mathbb{Q}}} \to \mathcal{PM}(S)_{\overline{\mathbb{Q}}} \).

The congruence subgroup problem for mapping class groups asks whether the natural epimorphism \( \hat{\Gamma}(S) \to \hat{\Gamma}(S) \), or equivalently \( \hat{\Pi}(S) \to \hat{\Pi}(S) \), is an isomorphism. This is known to be true only for \( g(S) \leq 2 \) (cf. [2] and [5], for instance).

The interest in these groups comes from the observation, made by Grothendieck in the Esquisse d’un Programme [11], that, by Belyi theorem, their outer automorphism groups contain a copy of the absolute Galois group of the rationals \( G_\mathbb{Q} \). Grothendieck suggested that the \( \text{\acute{e}tale} \) fundamental groups of the moduli stacks \( \mathcal{PM}(S) \) should be assembled together in what he called the Teichmüller tower and that the automorphism group of this tower should be already determined by its truncation at modular dimension 2. He did not give many details on how to proceed in what was and remained the sketch of a research plan. It was not even clear what should be done with this automorphism group but a reasonable guess is that he hoped that it would provide a combinatorial description of the absolute Galois group \( G_\mathbb{Q} \) (cf. Conjecture 2.10).

In the paper [8], Drinfeld initiated the study of the genus 0 stage of the Grothendieck-Teichmüller tower and, to this end, he introduced the Grothendieck-Teichmüller group \( \hat{G}_\mathbb{T} \). From the point of view which is relevant here, this can be described as the group of automorphisms of the first two levels (modular dimensions 1 and 2) of the genus 0 stage of the Teichmüller tower which respect suitable inertia conditions (cf. [12]).

Drinfeld then speculated, following Grothendieck, that \( \hat{G}_\mathbb{T} \) should be the automorphism group of the Teichmüller tower sketched by Grothendieck but he did not give many details...
on how the tower should, in a precise way, be defined and such statement proved. In this paper, we will adopt the definition of the Teichmüller tower (cf. Section 2.8) proposed by Hatcher, Lochak and Schneps in [13].

The study of the genus 0 case was essentially completed by Harbater and Schneps (cf. Main Theorem in [12]). More recently, Hoshi, Minamide and Mochizuki have improved this result by showing that the so called ”inertia conditions” are in fact automatically satisfied (cf. Corollary C in [15]). Putting everything together, the conclusion is that, for \( g(S) = 0 \), there is a natural isomorphism:

\[
\text{Out}(\hat{\Gamma}(S)) \cong \Sigma_{n(S)} \times \hat{\Gamma},
\]

where the symmetric group \( \Sigma_{n(S)} \) identifies with the subgroup \( \text{Inn}(\hat{\Gamma}(S))/\text{Inn}(\hat{\Gamma}(S)) \) of \( \text{Out}(\hat{\Gamma}(S)) \) and \( \hat{\Gamma} \) with the centralizer of this subgroup.

In this paper, we generalize these results to higher genus in the procongruence setting. A basic advantage in dealing with procongruence mapping class groups is that they have a well developed combinatorial theory (cf. [3], [4] and [5]).

By a classical result of Grossman the natural homomorphism \( \Gamma(S) \to \hat{\Gamma}(S) \) is injective and so, in particular, we also have an embedding \( \Gamma(S) \hookrightarrow \hat{\Gamma}(S) \). We then identify \( \Gamma(S) \) with its image in \( \hat{\Gamma}(S) \) (resp. \( \hat{\Gamma}(S) \)). Hence, we associate to a simple closed curve \( \gamma \) on \( S \) the Dehn twist \( \tau_{\gamma} \in \Gamma(S) \subset \hat{\Gamma}(S) \) (resp. \( \subset \hat{\Gamma}(S) \)). The set of \textit{profinite Dehn twists} in \( \hat{\Gamma}(S) \) (resp. \( \hat{\Gamma}(S) \)) is the closure of the set of Dehn twist. For the procongruence mapping class group \( \hat{\Gamma}(S) \), there is a more intrinsic description of profinite Dehn twists in terms of \textit{profinite simple closed curves} on \( S \) (cf. Section 2.6 and Section 4 in [3]).

Let \( \text{Aut}^{lo}(\hat{\Gamma}(S)) \) (resp. \( \text{Aut}^{lo}(\hat{\Gamma}(S)) \)) be the group of automorphisms of \( \hat{\Gamma}(S) \) (resp. \( \hat{\Gamma}(S) \)) which preserve the conjugacy class of a procyclic subgroup generated by a nonseparating Dehn twist and let \( \text{Out}^{lo}(\hat{\Gamma}(S)) \), \( \text{Out}^{lo}(\hat{\Gamma}(S)) \) be the corresponding outer automorphism groups. In a simplified form, one of the main results of the paper can then be formulated as follows (Theorem 6.5 is in fact substantially stronger):

\textbf{Theorem A.}  
(i) For \( d(S) > 1 \), there is a natural isomorphism:

\[
\text{Out}^{lo}(\hat{\Gamma}(S)) \cong \hat{\Gamma} \times \hat{\Gamma},
\]

(ii) For \( d(S) > 1 \) and \( S \neq S_{1,2} \), there is a natural isomorphism:

\[
\text{Out}^{lo}(\hat{\Gamma}(S)) \cong \Sigma_{n(S)} \times \hat{\Gamma},
\]

where the symmetric group \( \Sigma_{n(S)} \) identifies with \( \text{Inn}(\hat{\Gamma}(S))/\text{Inn}(\hat{\Gamma}(S)) \) and \( \hat{\Gamma} \) with the centralizer of this subgroup.

The proof of Theorem A essentially consists of two steps. In the first one, for \( S \) and \( S' \) hyperbolic surfaces such that \( g(S) \geq g(S') \) and \( \chi(S) \leq \chi(S') < 0 \), we construct a natural homomorphism

\[
\mu_{S,S'} : \text{Out}^{lo}(\hat{\Gamma}(S)) \to \text{Out}^{lo}(\hat{\Gamma}(S'))
\]

and we show that, essentially as a consequence of Theorem 8.1 in [7] (cf. Lemma 4.4), \( \mu_{S,S'} \) is injective.
In particular, by the genus 0 case of the theorem (cf. the isomorphism (1) and Proposition 3.8), we get a natural monomorphism:

$$\mu_S : \text{Out}^0(\hat{\Gamma}(S)) \hookrightarrow \hat{\Gamma}.$$ 

The second step then consists in showing that there is a natural (e.g. compatible with the homomorphism \(\mu_S\)) homomorphism \(\Psi_S : \hat{\Gamma} \to \text{Out}^0(\hat{\Gamma}(S)).\)

In order to construct \(\Psi_S\), we first prove what is, in its own right, one of the main results of the paper (cf. Theorem 6.11 for a stronger statement):

**Theorem B.** For a hyperbolic surface \(S\), there is a natural faithful representation:

$$\hat{\rho}_{\hat{\Gamma}} : \hat{\Gamma} \hookrightarrow \text{Out}^0(\hat{\Gamma}(S)),$$

where \(\text{Out}^0(\hat{\Gamma}(S))\) is the group of outer automorphisms of \(\hat{\Gamma}(S)\) which preserve the conjugacy class of the pro-cyclic subgroup generated by a nonseparating Dehn twist.

The proof of Theorem B can also be split in two steps. In the first one, we use the theory of profinite hyperelliptic mapping class groups to reduce the cases of genus \(\leq 2\) to those of genus 0 (cf. Section 6.6 to Section 6.12). In the second step, thanks to a profinite version (cf. Corollary 6.19) of a presentation given by Gervais in [10] for pure mapping class groups, we are able to extend the \(\hat{\Gamma}\)-action on profinite pure mapping class groups from genus \(\leq 2\) to higher genus. The representation \(\Psi_S\) is obtained observing that the representation \(\hat{\rho}_{\hat{\Gamma}}\) preserves the congruence kernel and then composing with the natural homomorphism \(\text{Out}^0(\hat{\Gamma}(S)) \to \text{Out}^0(\hat{\Gamma}(S))\) obtained in Theorem 4.3.

Some partial results in the direction of Theorem B had been obtained by Lochak, Nakamura and Schneps (cf. [13] and [23]) who constructed a faithful representation to \(\text{Out}^0(\hat{\Gamma}(S))\) from a subgroup of \(\hat{\Gamma}\), the so called ”new” Grothendieck-Teichmüller group, obtained adding one more equation to those which define \(\hat{\Gamma}\).

We then apply the previous results to the study of the Grothendieck-Teichmüller tower. For the precise definition of the profinite and the pro-congruence Grothendieck-Teichmüller towers \(\hat{T}\) and \(\hat{T}^{\text{out}}\), we refer the reader to Section 2.8. Here it suffices to say that, in order to get enough morphisms between objects, it is necessary to consider also relative mapping class groups of surfaces with boundary. An almost immediate consequence of the general version of Theorem A (cf. Theorem 6.5) is then the pro-congruence version of the Drinfeld-Grothendieck conjecture:

**Theorem C.** There is a natural isomorphism \(\hat{\Gamma} \cong \text{Aut}(\hat{T}^{\text{out}})\).

From the stronger version of Theorem B which we prove (cf. Theorem 6.11), it also follows that there is a natural faithful representation (cf. Theorem 6.4):

$$\hat{\rho}_{\hat{\Gamma}} : \hat{\Gamma} \hookrightarrow \text{Aut}(\hat{T}^{\text{out}}).$$

This last result shows, in particular, that, even if the congruence subgroup problem had a negative answer, there are no extra restrictions on Galois actions coming from the profinite rather than the pro-congruence Grothendieck-Teichmüller tower.
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2. Notations, definitions and preliminary results

2.1. Surfaces. In this paper, a hyperbolic surface is a connected orientable differentiable surface of negative Euler characteristic. We denote by $S = S_{g,n}^k$ a closed oriented surface of genus $g(S) = g$ from which $n(S) = n$ points and $k(S) = k$ open discs have been removed. The modular dimension of $S$ is defined to be $d(S) := 3g - 3 + n + k$.

For $k = 0$ (resp. $n = 0$), we let $S_{g,n} := S_{g,n}^0$ (resp. $S_g^k := S_g^k$). Also, we let $S_g := S_{g,0}$. The boundary of $S$ is denoted by $\partial S$ and we let $\hat{S} := S \setminus \partial S$, so that $\hat{S} \cong S_{g,n+k}$.

Let then $\mathcal{S}$ be the category with objects hyperbolic surfaces and maps embeddings of surfaces $S \hookrightarrow S'$ such that a 1-punctured disc of $S$ is mapped either to another 1-punctured disc or to a closed disc.

2.2. Profinite and procongruence mapping class groups. For $S = S_{g,n}^k$ a hyperbolic surface, we let $\Gamma(S)$ and $\Pi(S)$ be respectively the mapping class group and the pure mapping class group associated to $S$. Note that the inclusion $\hat{\mathcal{S}} \hookrightarrow \mathcal{S}$ induces, by restriction, a natural homomorphism $\Gamma(S) \hookrightarrow \Gamma(\hat{\mathcal{S}})$ whose image is the stabilizer of the partition of the punctures of $\hat{\mathcal{S}}$ into those which have a boundary in $S$ and those which do not.

For $S = S_{g,n}$, sometimes, to simplify notation, we will denote the corresponding mapping and pure mapping class groups by $\Gamma_{g,\{n\}}$ and $\Gamma_{g,n}$, respectively. These groups are then related by the short exact sequence $1 \to \Pi(\hat{\mathcal{S}}) \to \Gamma(\hat{\mathcal{S}}) \to \Sigma_n \to 1$, where $\Sigma_n$ is the symmetric group on $n$ letters.

The profinite and the pure profinite mapping class groups $\hat{\Gamma}(S)$ and $\hat{\Pi}(S)$ are defined to be the profinite completions of the groups $\Gamma(S)$ and $\Pi(S)$, respectively.

Let $\Pi$ be the fundamental group of the surface $S$ with respect to some base point and let $\hat{\Pi}$ be its profinite completion. Since $\Pi$ is conjugacy separable and class automorphisms of $\Pi$ are inner, the group $\text{Out}(\Pi)$ naturally embeds in $\text{Out}(\hat{\Pi})$. Moreover, since (topologically) finitely generated profinite groups are strongly complete, $\text{Out}(\hat{\Pi})$ is a profinite group.

Therefore, both groups $\Gamma(S)$ and $\Pi(S)$ naturally embed in the profinite group $\text{Out}(\hat{\Pi})$ and inherit a profinite topology which is called the congruence topology. The completions of $\Gamma(S)$ and $\Pi(S)$ with respect to these topologies are denoted by $\hat{\Gamma}(S)$ and $\hat{\Pi}(S)$ and called the procongruence and the pure procongruence mapping class group, respectively.

For surfaces with empty boundary, these groups are studied in detail in [3] and [5]. In what follows, we will recall the basic properties we need and prove a few more for surfaces with boundary.

2.3. The congruence subgroup problem. The congruence subgroup problem asks if there is an isomorphism $\hat{\Gamma}(S) \cong \Gamma(S)$ (equiv. $\hat{\Pi}(S) \cong \Pi(S)$). This is known to be true only for $g(S) \leq 2$ (cf. [1], [2] and [5]). In the sequel, we will identify the profinite and the procongruence mapping class groups for $g(S) \leq 2$. 
2.4. **Profinite Dehn twists and profinite braid twists.** Let \( \mathcal{L}(S) \) be the set of isotopy classes of simple closed curves on \( S \). We then denote by \( \mathcal{L}(S)_0 \) the subset of \( \mathcal{L}(S) \) consisting of nonperipheral curves. The set \( \mathcal{L}(S)_0 \) parameterizes the set of \textit{Dehn twists} in \( \Gamma(S) \) (cf. Section 3.1.1 in [9] for the definition). For \( \gamma \in \mathcal{L}(S)_0 \), we denote by \( \tau_\gamma \in \text{PI}(S) \subseteq \Gamma(S) \) the associated Dehn twist.

Let us denote by \( \mathcal{L}^b(S) \) the subset of \( \mathcal{L}(S) \) consisting of simple closed curves bounding a 2-punctured disc on \( S \). For \( \gamma \in \mathcal{L}^b(S) \), let \( D \) be the disc bounded by \( \gamma \). Then, the \textit{braid twist} \( b_\gamma \) about \( \gamma \) is the isotopy class of a self-homeomorphism of \( S \) which is the identity outside of \( D \), swaps the punctures of \( B \) and satisfies the identity \( b_\gamma^2 = \tau_\gamma \).

The mapping class groups \( \text{PI}(S) \) and \( \Gamma(S) \) are generated, respectively, by Dehn twists and by Dehn twists and braid twists (cf. Corollary 4.15 in [9]).

A \textit{profinite Dehn twist} (resp. \textit{profinite braid twist}) of \( \hat{\Gamma}(S) \) is an element which lies in the closure of the set of Dehn twists (resp. braid twists) inside the profinite group \( \hat{\Gamma}(S) \) (where we identify \( \Gamma(S) \) with a subgroup of the latter group).

It is a remarkable and nontrivial fact that the profinite Dehn twists of \( \hat{\Gamma}(S) \) are parameterized by the profinite set of \textit{nonperipheral profinite simple closed curves} \( \hat{\mathcal{L}}(S)_0 \) (cf. Theorem 5.1 in [3]). This set is realized, roughly speaking, as the closure of \( \mathcal{L}(S)_0 \) inside the profinite set of \textit{"unoriented"} conjugacy classes in \( \hat{\Pi} \) (cf. Section 4 in [3] and Section 3.2 in [5] for the precise definition). As above, we then denote by \( \tau_\gamma \in \text{PI}(S) \), for \( \gamma \in \hat{\mathcal{L}}(S)_0 \), the associated profinite Dehn twist.

Let \( \hat{\mathcal{L}}^b(S) \) be the subset of \( \hat{\mathcal{L}}(S) \) consisting of profinite simple closed curves whose topological type (cf. Definition 4.7 in [7]) is that of a simple closed curve bounding a 2-punctured disc. Then, the the profinite braid twists of the procongruence mapping class group \( \hat{\Gamma}(S) \) are parameterized by this profinite set and are denoted by \( b_\gamma \), for \( \gamma \in \hat{\mathcal{L}}^b(S) \).

2.5. **Multicurves and their stabilizers.** A \textit{multicurve} \( \sigma \) on \( S \) is a set of isotopy classes of disjoint simple closed curves on \( S \) such that \( S \setminus \sigma \) does not contain a disc or a one-punctured disc. The \textit{topological type} of \( \sigma \) is the topological type of the surface \( S \setminus \sigma \). Multicurves parameterize sets of commuting Dehn twists or braid twists in \( \Gamma(S) \). The complex of curves \( C(S) \) is the abstract simplicial complex whose simplices are multicurves on \( S \). Its combinatorial dimension is \( d(S) - 1 \).

**Definition 2.1.** For \( \sigma \in C(S) \), we denote by \( I_\sigma \) the abelian subgroup of \( \Gamma(S) \) generated by the Dehn twists \( \tau_\gamma \), for \( \gamma \in \sigma \). For \( H \) a subgroup of \( \Gamma(S) \), we then let \( I_\sigma(H) := I_\sigma \cap H \). The \textit{topological type} of \( I_\sigma \) is the topological type of \( \sigma \).

There is a natural simplicial action of \( \Gamma(S) \) on \( C(S) \). Let us describe the stabilizer of a simplex \( \sigma \in C(S) \). Let \( S \setminus \sigma = S_1 \coprod \ldots \coprod S_k \), let \( B_i \) be the set of punctures of \( S_i \) bounded by a curve in \( \sigma \), for \( i = 1, \ldots, k \), and let \( \Sigma_{\sigma^\pm} \) be the symmetric group on the set of oriented circles \( \sigma^\pm := \sigma^+ \cup \sigma^- \). Let us also denote by \( \Gamma(S_i)_{B_i} \) the pointwise stabilizer of the subset of punctures \( B_i \) for the action of the mapping class group \( \Gamma(S_i) \), for \( i = 1, \ldots, k \). Then,
the stabilizer $\Gamma(S)_\sigma$ is described by the two exact sequences:

$$1 \to \Gamma(S)_{\bar{\sigma}} \to \Gamma(S)_\sigma \to \Sigma_{\sigma^\pm},$$

$$1 \to I_{\sigma} \to \Gamma(S)_{\bar{\sigma}} \to \Gamma(S_1)_{B_1} \times \ldots \times \Gamma(S_k)_{B_k} \to 1.$$

A similar description but simpler applies to the stabilizer $P\Gamma(S)_\sigma$ for the action of the pure mapping class group $P\Gamma(S)$ on $C(S)$:

$$1 \to P\Gamma(S)_{\bar{\sigma}} \to P\Gamma(S)_\sigma \to \Sigma_{\sigma^\pm},$$

$$1 \to I_{\sigma} \to P\Gamma(S)_{\bar{\sigma}} \to P\Gamma(S_1) \times \ldots \times P\Gamma(S_k) \to 1.$$

2.6. The procongruence curve complex. A central object for the study of procongruence mapping class groups is a profinite version of the complex of curves. For every $k \geq 0$, there is a natural embedding $C(S)_k \hookrightarrow \mathcal{P}_{k+1}(\hat{C}(S))$, where $\mathcal{P}_{k+1}(\hat{C}(S))$ is the profinite set of unordered subsets of $k + 1$ distinct elements in $\hat{C}(S)$. We then define:

**Definition 2.2.** The complex of profinite curves $\hat{C}(S)$ (cf. Section 4 in [3], for more details) is the abstract simplicial profinite complex whose set $\hat{C}(S)_k$ of $k$-simplices is the closure of $C(S)_k$ inside $\mathcal{P}_{k+1}(\hat{C}(S))$. In particular, $\hat{C}(S)_0 = \hat{C}(S)_0$. A simplex $\sigma$ of $\hat{C}(S)$ is also called a profinite multicurve. Its topological type is the topological type of a simplex in the intersection of the orbit $\hat{\Gamma}(S) \cdot \sigma$ with $C(S) \subset \hat{C}(S)$.

There are continuous natural actions of the procongruence mapping class groups $\hat{\Gamma}(S)$ and $P\hat{\Gamma}(S)$ on $\hat{C}(S)$. Theorem 4.5 in [3] describes the stabilizers for the action of $P\hat{\Gamma}(S)$. This result implies a similar description for the action of $\hat{\Gamma}(S)$ (cf. Theorem 4.10 in [7]):

**Theorem 2.3.** For $\sigma \in C(S) \subset \hat{C}(S)$, let $S \triangleleft \sigma = S_1 \coprod \ldots \coprod S_k$ and $B_i$ be the set of punctures of $S_i$ bounded by a curve in $\sigma$, for $i = 1, \ldots, k$. Then, we have:

(i) The stabilizer $\hat{\Gamma}(S)_\sigma$ is described by the two exact sequences:

$$1 \to \hat{\Gamma}(S)_{\bar{\sigma}} \to \hat{\Gamma}(S)_\sigma \to \Sigma_{\sigma^\pm},$$

$$1 \to \hat{I}_{\sigma} \to \hat{\Gamma}(S)_{\bar{\sigma}} \to \hat{\Gamma}(S_1)_{B_1} \times \ldots \times \hat{\Gamma}(S_k)_{B_k} \to 1,$$

where $\hat{I}_{\sigma}$ is the profinite completion of the free abelian group $I_{\sigma}$ and $\hat{\Gamma}(S_i)_{B_i}$ is the pointwise stabilizer of the subset of punctures $B_i$ in $\hat{\Gamma}(S_i)$, for $i = 1, \ldots, k$.

(ii) The stabilizer $P\hat{\Gamma}(S)_\sigma$ is described by the two exact sequences:

$$1 \to P\hat{\Gamma}(S)_{\bar{\sigma}} \to P\hat{\Gamma}(S)_\sigma \to \Sigma_{\sigma^\pm},$$

$$1 \to \hat{I}_{\sigma} \to P\hat{\Gamma}(S)_{\bar{\sigma}} \to P\hat{\Gamma}(S_1) \times \ldots \times P\hat{\Gamma}(S_k) \to 1.$$
2.7. **Profinite and procongruence relative mapping class groups.** For a hyperbolic surface \( S = S_{g,n}^k \), we let \( \Gamma(S, \partial S) \) be the relative mapping class group of the surface with boundary \((S, \partial S)\), that is to say the group of relative, with respect to the boundary \( \partial S \), isotopy classes of orientation preserving diffeomorphisms of \( S \). Let \( \delta_1, \ldots, \delta_k \) be the connected components of the boundary \( \partial S \). Then, the group \( \Gamma(S, \partial S) \) is described by the natural short exact sequence:

\[
1 \to \prod_{i=1}^k \tau_{\delta_i} \to \Gamma(S, \partial S) \to \Gamma(S) \to 1.
\]

The relative pure mapping class group \( P\Gamma(S, \partial S) \) is the subgroup of \( \Gamma(S, \partial S) \) consisting of those elements which admit representatives fixing pointwise the punctures and the boundary of \( S \). It is described by the short exact sequence:

\[
1 \to P\Gamma(S, \partial S) \to \Gamma(S, \partial S) \to \Sigma_n \times \Sigma_k \to 1.
\]

**Remark 2.4.** Let \( \tilde{S} = S_{g,n+2k} \), with \( d(\tilde{S}) > 1 \) and \( \sigma = \{ \delta_1, \ldots, \delta_k \} \in C(\tilde{S}) \), where \( \delta_i \) is a simple closed curve bounding an open disc \( D_i \) containing the punctures indexed by \( n + 2i - 1 \) and \( n + 2i \), for \( i = 1, \ldots, k \). Let \( S := \tilde{S} \setminus \cup_{i=1}^k D_i \cong S_{g,n}^k \).

Then, there is an isomorphism \( \Gamma(\tilde{S})_\sigma \cong (\Sigma_2)^k \times \Gamma(S, \partial S) \), where \( \Gamma(S, \partial S) \) acts on the product \((\Sigma_2)^k\) by permuting its factors through the natural representation \( \Gamma(S, \partial S) \to \Sigma_k \), and a natural isomorphism \( P\Gamma(S, \partial S) \cong P\Gamma(\tilde{S})_\sigma \).

The profinite (resp. profinite pure) relative mapping class group of the surface with boundary \((S, \partial S)\) is defined to be the profinite completion \( \hat{\Gamma}(S, \partial S) \) (resp. \( \hat{P}\Gamma(S, \partial S) \)) of \( \Gamma(S, \partial S) \) (resp. of \( P\Gamma(S, \partial S) \)). From (ii) of Theorem 2.3 and Remark 2.4, it follows that these profinite groups are described by the short exact sequences:

\[
1 \to \prod_{i=1}^k \tau_{\delta_i} \to \hat{\Gamma}(S, \partial S) \to \hat{\Gamma}(S) \to 1
\]

and

\[
1 \to \hat{P}\Gamma(S, \partial S) \to \hat{\Gamma}(S, \partial S) \to \Sigma_n \times \Sigma_k \to 1.
\]

**Definition 2.5.** For a surface \( S = S_{g,n}^k \) with boundary \( \partial S = \cup_{i=1}^k \delta_i \), we let \( \tilde{S} \cong S_{g,n+2k} \) be the surface obtained from \( S \) glueing a 2-punctured disc over each boundary component \( \delta_i \), for \( i = 1, \ldots, k \).

By Remark 2.4, the embedding \( S \hookrightarrow \tilde{S} \) induces a monomorphism of mapping class groups \( \Gamma(S, \partial S) \hookrightarrow \Gamma(\tilde{S}) \), which identifies \( \Gamma(S, \partial S) \) with a subgroup of the stabilizer \( \Gamma(\tilde{S})_\sigma \).

The congruence topology on \( \Gamma(\tilde{S}) \) then induces a profinite topology on \( \Gamma(S, \partial S) \) which we call the congruence topology on \( \Gamma(S, \partial S) \). This clearly also defines a profinite topology on \( P\Gamma(S, \partial S) \subseteq \Gamma(S, \partial S) \), which we call the congruence topology on \( P\Gamma(S, \partial S) \).

The completions of \( \Gamma(S, \partial S) \) and \( P\Gamma(S, \partial S) \) with respect to these congruence topologies are then denoted by \( \hat{\Gamma}(S, \partial S) \) and \( \hat{P}\Gamma(S, \partial S) \), respectively, and are called the procongruence (resp. pure) relative mapping class group of the surface with boundary \((S, \partial S)\).
By Theorem 2.3, they fit in the short exact sequences:

\[ 1 \to \prod_{i=1}^{k} \tau_{\delta_i}^2 \to \hat{\Gamma}(S, \partial S) \to \hat{\Gamma}(S) \to 1 \]

and

\[ 1 \to \text{PF}(S, \partial S) \to \hat{\Gamma}(S, \partial S) \to \Sigma_n \times \Sigma_k \to 1. \]  

An immediate consequence is then:

**Proposition 2.6.**

(i) For \( g(S) \leq 2 \), we have \( \hat{\Gamma}(S, \partial S) = \hat{\Gamma}(S) \) and \( \text{PF}(S, \partial S) = \text{PF}(S, \partial S) \).

(ii) For \( S \neq S_2, S_{1,2} \) and \( S_1^2 \), we have \( Z(\hat{\Gamma}(S, \partial S)) = (\prod_{i=1}^{k} \tau_{\delta_i})^2 \). Otherwise, the center \( Z(\hat{\Gamma}(S, \partial S)) \) is generated by \( (\prod_{i=1}^{k} \tau_{\delta_i})^2 \) and the hyperelliptic involution.

(iii) For \( S \neq S_2, S_{1,2}, S_1^2 \) and \( U \) an open subgroup of \( \text{PF}(S, \partial S) \), we have \( Z_{\hat{\Gamma}(S, \partial S)}(U) = \prod_{i=1}^{k} \tau_{\delta_i}^2 \). Otherwise, \( Z_{\hat{\Gamma}(S, \partial S)}(U) \) is generated by \( \prod_{i=1}^{k} \tau_{\delta_i}^2 \) and the hyperelliptic involution.

**Proof.** The first item follows from the congruence subgroup property in genus \( \leq 2 \) for surfaces without boundary and the short exact sequences (2) and (3). The second and the third item from the same exact sequences, Corollary 6.2 in [3] and Theorem 4.14 in [7]. \( \square \)

The procongruence (resp. pure) relative mapping class group \( \hat{\Gamma}(S, \partial S) \) (resp. \( \text{PF}(S, \partial S) \)) acts on the procongruence curve complex \( \check{C}(S) \) through the natural homomorphism to \( \hat{\Gamma}(S) \) (resp. \( \text{PF}(S) \)). For \( \sigma \in C(S) \), let \( I_{\sigma} \) be the subgroup of \( \Gamma(S, \partial S) \) generated by the Dehn twists \( \tau_{\gamma} \), for \( \gamma \in \sigma \) (cf. Definition 2.1). Then, from Theorem 2.3, it easily follows:

**Theorem 2.7.** For \( \sigma \in C(S) \subset \check{C}(S) \), let \( S \setminus \sigma = S_1 \bigsqcup \ldots \bigsqcup S_k \) and \( B_i \) be the set of punctures of \( S_i \) bounded by a curve in \( \sigma \), for \( i = 1, \ldots, k \). Then, we have:

(i) The stabilizer \( \hat{\Gamma}(S, \partial S)_{\sigma} \) is described by the two exact sequences:

\[ 1 \to \hat{\Gamma}(S, \partial S)_{\sigma} \to \hat{\Gamma}(S, \partial S)_{\sigma} \to \Sigma_{\sigma} \]

\[ 1 \to \hat{\Gamma}(S, \partial S)_{\sigma} \to \hat{\Gamma}(S, \partial S)_{B_1} \times \ldots \times \hat{\Gamma}(S, \partial S)_{B_k} \to 1, \]

where \( \hat{\Gamma}(S, \partial S)_{\sigma} \) is the profinite completion of the free abelian group \( I_{\sigma} \) and \( \hat{\Gamma}(S, \partial S)_{B_i} \) is the pointwise stabilizer of the subset of punctures \( B_i \) in \( \hat{\Gamma}(S, \partial S)_{\sigma} \), for \( i = 1, \ldots, k \).

(ii) The stabilizer \( \text{PF}(S, \partial S)_{\sigma} \) is described by the two exact sequences:

\[ 1 \to \text{PF}(S, \partial S)_{\sigma} \to \text{PF}(S, \partial S)_{\sigma} \to \Sigma_{\sigma} \]

\[ 1 \to \text{PF}(S, \partial S)_{\sigma} \to \text{PF}(S, \partial S)_{B_1} \times \ldots \times \text{PF}(S, \partial S)_{B_k} \to 1. \]

From the definition of the congruence topology and Corollary 4.12 in [7], it then follows a description of the centralizers of profinite multitwists in procongruence relative mapping class groups:
Theorem 2.8. For \( \sigma = \{ \gamma_0, \ldots, \gamma_k \} \) a profinite multicurve on \( S \) and \( m \in \mathbb{Z} \setminus \{ 0 \} \), we have:

\[
Z_{\Gamma(S, \partial S)}(\tau_{\gamma_0}^m \cdots \tau_{\gamma_k}^m) = N_{\Gamma(S, \partial S)}((\tau_{\gamma_0}^m \cdots \tau_{\gamma_k}^m)^{\hat{\sigma}}) = N_{\Gamma(S, \partial S)}((\tau_{\gamma_0}^m, \ldots, \tau_{\gamma_k}^m)) = \hat{\Gamma}(S, \partial S)_{\sigma},
\]

where \( \hat{\Gamma}(S, \partial S)_{\sigma} \) is the stabilizer of \( \sigma \) described in (i) of Theorem 2.7. A similar result holds for the procongruence pure relative mapping class group \( \hat{\Gamma}(S, \partial S) \).

2.8. The Grothendieck-Teichmüller tower. Let \( \eta : S \hookrightarrow S' \) be a map in the category of surfaces \( S \) defined in Section 2.1. Then, \( \eta \) induces a homomorphism of pure relative mapping class groups \( \eta_* : \Pi\Gamma(S, \partial S) \to \Pi\Gamma(S', \partial S') \) and hence of profinite pure relative mapping class groups \( \hat{\eta}_* : \hat{\Pi}\Gamma(S, \partial S) \to \hat{\Pi}\Gamma(S', \partial S') \). From Remark 2.4 and (ii) of Theorem 2.3, it follows that \( \hat{\eta}_* \) also induces a homomorphism of procongruence pure relative mapping class groups \( \hat{\eta}_* : \hat{\Pi}\Gamma(S, \partial S) \to \hat{\Pi}\Gamma(S', \partial S') \).

Let \( \mathcal{G} \) be the category of profinite groups and continuous homomorphisms. We have then defined two functors \( \hat{\hat{\mathcal{F}}} \) and \( \hat{\mathcal{F}} \) from \( \mathcal{S} \) to \( \mathcal{G} \). It is not difficult to see that the functor \( \hat{\mathcal{F}} \) factors through the functor \( \hat{\hat{\mathcal{F}}} \).

There is a natural way to realize the above profinite mapping class groups as geometric étale fundamental groups of suitable geometric objects. We will associate to a hyperbolic surface \( S \) a logarithmic formal DM stack \( P_{\hat{\mathcal{M}}}(S) \) (cf. [27] for a similar construction).

For a surface \( S \) with empty boundary, let \( P_{\mathcal{M}}(S) \) be the DM stack of smooth curves whose complex model is diffeomorphic to \( S \) with an order on the set of punctures and let \( P_{\mathcal{M}}(S) \) be the DM compactification of \( P_{\mathcal{M}}(S) \). Let then \( P_{\mathcal{M}}(S)_{\log} \) be the log DM stack with the logarithmic structure associated to the DM boundary. There is a natural fully faithful functor from the category of DM stacks to the category of formal DM stacks. We then denote by \( P_{\mathcal{M}}(S, \emptyset)_{\log} \) the formal log DM stack associated to the log DM stack \( P_{\mathcal{M}}(S)_{\log} \) and assign this to the surface \( S \). There is a series of isomorphisms:

\[
\pi_1^{\text{et}}(P_{\mathcal{M}}(S, \emptyset)_{\log}) \cong \pi_1^{\text{et}}(P_{\mathcal{M}}(S)_{\log}) \cong \pi_1^{\text{et}}(P_{\mathcal{M}}(S)_{\log}) \cong P_{\hat{\mathcal{F}}}(S).
\]

For a surface \( S \) with nonempty boundary, let \( \hat{S} \cong S_{g,n+2k} \) be as in Definition 2.5. The simplex \( \sigma = \{ \delta_1, \ldots, \delta_k \} \in C(\hat{S}) \) determines a closed stratum \( \delta_\sigma \) in the boundary of \( P_{\mathcal{M}}(\hat{S}) \) and we let \( P_{\mathcal{M}}(S, \partial S) \) be the formal neighborhood of \( \delta_\sigma \) in \( P_{\mathcal{M}}(\hat{S}) \).

Let us then assign to \( S \) the formal logarithmic DM stack \( P_{\mathcal{M}}(S, \partial S)_{\log} \) obtained by pulling back to the formal DM stack \( P_{\mathcal{M}}(S, \partial S) \) the logarithmic structure of \( P_{\mathcal{M}}(\hat{S}, \emptyset)_{\log} \), via the natural morphism of formal DM stacks \( P_{\mathcal{M}}(S, \partial S) \to P_{\mathcal{M}}(\hat{S}, \emptyset) \). There is an isomorphism:

\[
\pi_1^{\text{et}}(P_{\mathcal{M}}(S, \partial S)_{\log}) \cong P_{\hat{\mathcal{F}}}(S, \partial S).
\]

From the canonical short exact sequence

\[
1 \to \pi_1^{\text{et}}(P_{\mathcal{M}}(S, \partial S)_{\log}) \to \pi_1^{\text{et}}(P_{\mathcal{M}}(S, \partial S)_{\log}) \to G_{\mathbb{Q}} \to 1,
\]

we get a natural outer representation \( \hat{\rho}_S : G_{\mathbb{Q}} \to \text{Out}(P_{\hat{\mathcal{F}}}(S, \partial S)) \). A (nontrivial) consequence of Belyi theorem is that this representation is faithful.
The Grothendieck-Teichmüller functor $\hat{\mathcal{T}}$ factors as the composition of the functor $P\overline{M}_{\mathbb{Q}}^{\log}$, which assigns to the surface $S$ the log formal DM stack $P\overline{M}(S, \partial S)^{\log}_{\mathbb{Q}}$, and the functor which assigns to a log formal DM stack $X^{\log}$ over the rationals its geometric étale fundamental group $\pi_1^{et}(X^{\log}_{\mathbb{Q}})$.

Note that the identification of topological fundamental groups with mapping class groups takes care of base points. However, in order to make keeping track of base points totally irrelevant, it is convenient to modify our definition of the functors $\hat{\mathcal{T}}$ and $\check{\mathcal{T}}$. Let $G_{out}$ be the category with objects profinite groups and maps outer continuous homomorphisms. There is a natural functor $G_{\mathbb{Q}} \to G_{out}$ and we define $\hat{\mathcal{T}}_{out}$ and $\check{\mathcal{T}}_{out}$ to be, respectively, the composition of $\hat{\mathcal{T}}$ and $\check{\mathcal{T}}$ with this functor.

**Definition 2.9.** The full images in $G_{out}$ of the functors $\hat{\mathcal{T}}_{out}$ and $\check{\mathcal{T}}_{out}$ are called the (profinite) Grothendieck-Teichmüller tower and the procongruence Grothendieck-Teichmüller tower, respectively.

From the geometric description of the functor $\hat{\mathcal{T}}_{out}$, it follows that there is a natural faithful representation:

$$\rho_{\hat{\mathcal{T}}}: G_{\mathbb{Q}} \hookrightarrow \text{Aut}(\hat{\mathcal{T}}_{out}),$$

defined by the assignment $\alpha \mapsto \{\hat{\rho}_S(\alpha)\}_{S \in S}$. At the root of Grothendieck-Teichmüller theory, is the following apocryphal conjecture:

**Conjecture 2.10 (Grothendieck?).** The representation $\rho_{\hat{\mathcal{T}}}$ induces a natural isomorphism:

$$G_{\mathbb{Q}} \cong \text{Aut}(\hat{\mathcal{T}}_{out}).$$

Somewhat in this spirit is the I/OM conjecture, which has been proved by Pop (cf. [24] for details).

### 2.9. The Galois action on the procongruence Grothendieck-Teichmüller tower.

For $S$ a hyperbolic surface with empty boundary, let $P\overline{M}(S)^{\log} \to P\overline{M}(S)^{\log}_{\mathbb{Q}}$ be the universal log stable curve (this is the universal stable curve endowed with the logarithmic structure associated to its DM boundary). For $\xi \in P\overline{M}(S)^{\log}_{\mathbb{Q}}$ a $\mathbb{Q}$-point, there is an associated universal monodromy representation:

$$\rho_{\xi}: \pi_1^{et}(P\overline{M}(S)^{\log}_{\mathbb{Q}}) \to \text{Out}(P\overline{M}(S)^{\log}_{\mathbb{Q}}),$$

where $P\overline{M}(S)^{\log}_{\xi}$ is the fiber of the universal curve over $\xi$ endowed with the logarithmic structure induced by $P\overline{M}(S)^{\log}$. The isomorphism $\pi_1^{et}(P\overline{M}(S)^{\log}_{\mathbb{Q}}) \cong P\hat{\Gamma}(S)$ then induces an isomorphism $\text{Im} \rho_{\xi} \cong \hat{\Gamma}(S)$.

For $\psi: S \hookrightarrow \tilde{S}$ the embedding considered in Remark 2.4, there is an induced homomorphism:

$$\hat{\mathcal{T}}(\psi): \pi_1^{et}(P\overline{M}(S, \partial S)^{\log}_{\mathbb{Q}}) \cong P\hat{\Gamma}(S, \partial S) \to \pi_1^{et}(P\overline{M}(\tilde{S}, \emptyset)^{\log}_{\mathbb{Q}}) \cong \pi_1^{et}(P\overline{M}(\tilde{S})^{\log}_{\mathbb{Q}}).$$

We then have $\rho_{\xi} \circ \hat{\mathcal{T}}(\psi)(\pi_1^{et}(P\overline{M}(S, \partial S)^{\log}_{\mathbb{Q}})) \cong P\hat{\Gamma}(S, \partial S).$
The above procedure defines natural transformations of functors:

$$\rho: \tilde{\mathcal{F}} \rightarrow \tilde{\mathcal{F}}$$

and

$$\rho^\text{out}: \tilde{\mathcal{F}}^\text{out} \rightarrow \tilde{\mathcal{F}}^\text{out}.$$  

Since the universal monodromy representation $\rho_{\xi}$ is compatible with the Galois actions, the natural transformation $\rho^\text{out}$ is $G_{\mathbb{Q}}$-equivariant and there is a natural representation (again, Belyi theorem implies that this is a faithful one):

$$\rho_{\xi}: G_{\mathbb{Q}} \hookrightarrow \text{Aut}(\tilde{\mathcal{F}}^\text{out}).$$

2.10. **Levels and level structures.** Let $S$ be a hyperbolic surface with empty boundary. A finite index subgroup $\Gamma^\lambda$ of $\Gamma(S)$ is called a *level*. A *congruence* level is a level which is open for the congruence topology on $\Gamma(S)$. The étale covering $\mathcal{M}^\lambda(S)$ of $\mathcal{M}(S)$ associated to the level $\Gamma^\lambda$ is called a *level structure* over $\mathcal{M}(S)$. The level is *representable* if the associated DM stack $\mathcal{M}^\lambda(S)$ is representable.

In particular, to the pure mapping class group $\text{PM}(S)$ is associated the level structure $\mathcal{P}_{\mathcal{M}}(S)$. This is the étale Galois covering of $\mathcal{M}(S)$, with Galois group the symmetric group $\Sigma_n$, which parameterizes smooth projective curves with an order on the set of punctures. For $S = S_{g,n}$, in order to simplify notations, the stacks $\mathcal{M}(S)$ and $\mathcal{P}_{\mathcal{M}}(S)$ are sometimes simply denoted by $\mathcal{M}_{g,n}$ and $\mathcal{M}_{g,n}$, respectively.

Let $\overline{\mathcal{M}}(S)$ be the DM compactification of the moduli stack $\mathcal{M}(S)$. This is a DM stack defined over $\mathbb{Z}$ which parameterizes stable $n$-punctured, genus $g$ curves. A *level structure over $\overline{\mathcal{M}}(S)$* is the compactification $\overline{\mathcal{M}}^\lambda(S)$ of a level structure $\mathcal{M}^\lambda(S)$ over $\mathcal{M}(S)$ obtained taking the normalization of the DM stack $\overline{\mathcal{M}}(S)$ in the function field of $\mathcal{M}^\lambda(S)$. This is also a DM stack and, if the level $\Gamma^\lambda$ is contained in an abelian level of order $\geq 3$, then $\overline{\mathcal{M}}^\lambda(S)$ is representable.

2.11. **The procongruence pants graph.** The *Fulton curve* $\mathcal{F}(S)$ is the 1-dimensional closed substack of the moduli stack $\overline{\mathcal{M}}(S)$ which parameterizes curves with at least $d - 1$ singular points. More generally, for a level structure $\overline{\mathcal{M}}^\lambda(S)$ over $\overline{\mathcal{M}}(S)$, the *Fulton curve* $\mathcal{F}^\lambda(S)$ is the inverse image of $\mathcal{F}(S)$ via the natural morphism $\overline{\mathcal{M}}^\lambda(S) \rightarrow \overline{\mathcal{M}}(S)$.

Let $\Gamma^\lambda$ be a representable level of $\Gamma(S)$ contained in an abelian level of order $m \geq 2$. Then, each irreducible component of the Fulton curve $\mathcal{F}^\lambda(S)_{\mathbb{Q}}$ is a covering of $\mathbb{P}^1_{\mathbb{Q}}$ ramified at most over the set $\{0, 1, \infty\}$ and the ramification points are contained in the boundary of $\mathcal{F}^\lambda(S)_{\mathbb{Q}}$ (that is to say the points parameterizing maximally degenerated stable curves). It follows that the analytic complex variety $\mathcal{F}(S)^\text{an}$ admits a natural triangulation with vertices the points parameterizing maximally degenerated stable curves and edges geodesic segments of length 1 joining these boundary points. We denote by $C^\lambda_p(S)$ the 1-skeleton of this triangulation which we regard as a 1-dimensional abstract simplicial complex.

**Definition 2.11.** The *procongruence pants graph* $\hat{C}_p(S)$ is the 1-dimensional abstract simplicial profinite complex obtained as the inverse limit of the finite abstract simplicial complexes $C^\lambda_p(S)$, for $\Gamma^\lambda$ varying over the set of all congruence levels of $\Gamma(S)$.
For a more comprehensive treatment of $\hat{C}_p(S)$, we refer to [7]. Here we only recall the basic properties which we will need later. The vertices of $\hat{C}_p(S)$ are in natural bijective correspondence with the profinite set $\hat{C}(S)_{d(S)-1}$ of facets of the procongruence curve complex and its edges with the profinite set $\hat{C}(S)_{d(S)-2}$ of $(d(S) - 2)$-simplices. To the decomposition in irreducible components of the Fulton curves $F^\lambda(S)$, for all levels $\Gamma^\lambda$, corresponds a decomposition of the procongruence pants graph $\hat{C}_p(S)$ in closed subgraphs called profinite Farey subgraphs. They are parameterized by the profinite set $\hat{C}(S)_{d(S)-2}$. The profinite Farey subgraph associated to $\sigma \in \hat{C}(S)_{d(S)-2}$ is denoted by $\hat{F}_\sigma(S)$ and its set of vertices consists of the profinite set of facets of $\hat{C}_p(S)$ containing $\sigma$.

3. The $I$-condition and the $D$-condition on automorphisms

3.1. Decomposition and inertia groups. Let $S$ be a hyperbolic surface. For an open subgroup $U$ of $\hat{\Gamma}(S)$, we call the stabilizer $U_\sigma = U \cap \hat{\Gamma}(S)_\sigma$ of a simplex $\sigma \in \hat{C}(S)$ for the action of $U$ on $\hat{C}(S)$ the decomposition group of $U$ associated to $\sigma$.

**Definition 3.1.** The group of $D$-automorphisms $\text{Aut}^D(U)$ is the closed subgroup of $\text{Aut}(U)$ consisting of automorphisms preserving the set of decomposition groups $\{U_\sigma\}_{\gamma \in \hat{\Gamma}(S)_0}$.

**Remark 3.2.** In [7], $\text{Aut}^D(U)$ was denoted by $\text{Aut}^I(U)$. According to Proposition 7.2 in [7], the group $\text{Aut}^D(U)$ preserve the set of all decomposition groups $\{U_\sigma\}_{\gamma \in \hat{C}(S)}$ of $U$.

Let us introduce a slightly more restrictive condition in a more general context which we will need later:

**Definition 3.3.**

(i) For $\sigma \in \hat{C}(S)$, we denote by $\hat{I}_\sigma$ the closed abelian subgroup of $\hat{\Gamma}(S)$ topologically generated by the Dehn twists $\tau_\gamma$, for $\gamma \in \sigma$. For $H$ a closed subgroup of $\hat{\Gamma}(S)$, we then let $\hat{I}_\sigma(H) := \hat{I}_\sigma \cap H$ (the inertia group of $H$ associated to $\sigma$). The topological type of $\hat{I}_\sigma$ is the topological type of $\sigma$.

(ii) The group of $I$-automorphisms $\text{Aut}^I(H)$ is the closed subgroup of $\text{Aut}(H)$ consisting of those elements which preserve the set of inertia groups $\{\hat{I}_\gamma(H)\}_{\gamma \in \hat{\Gamma}(S)_0}$.

(iii) We say that a subgroup $K$ of $H$ is $I$-characteristic if it is preserved by all elements of $\text{Aut}^I(H)$.

**Remark 3.4.** For $H$ a closed subgroup of $\hat{\Gamma}(S)$, we have $\text{Inn} H \subseteq \text{Aut}^I(H)$. In particular, an $I$-characteristic subgroup of $H$ is normal.

**Proposition 3.5.**

(i) An automorphism of $H$ is an $I$-automorphism if and only if it preserves the subset of all inertia groups $\{\hat{I}_\sigma(H)\}_{\sigma \in \hat{C}(S)}$ of $H$.

(ii) $\text{Aut}^I(\hat{\Gamma}(S))$ preserves the conjugacy classes of profinite braid twists in $\hat{\Gamma}(S)$.

**Proof.** (i): One implication is obvious. For the other, let us observe that, for $\sigma = \{\gamma_0, \ldots, \gamma_k\} \in \hat{C}(S)$, we have $\hat{I}_\sigma(H) = \prod_{i=0}^k \hat{I}_{\gamma_i}(H)$. Moreover, by Corollary 4.12 in [7], a
set of nonperipheral profinite simple closed curves \( \{ \gamma_0, \ldots, \gamma_k \} \) forms a \( k \)-simplex of \( \tilde{C}(S) \) if and only if the associated profinite Dehn twists or, equivalently, some nontrivial powers of them commute. The conclusion is then clear.

(ii): For \( n(S) \) or \( k(S) \geq 2 \) (otherwise the claim is trivial), let \( b_\gamma \in \tilde{\Gamma}(S) \) be the profinite braid twist associated to a simple closed curve \( \gamma \) on \( S \) bounding either a 2-punctured disc or a disc containing two boundary components.

From a given \( f \in \text{Aut}^\circ(\tilde{\Gamma}(S)) \), we can then assume, after composing with an element of \( \text{Inn} \tilde{\Gamma}(S) \), that \( f \) preserves the inertia group \( \hat{I}_\gamma = \tau_\gamma^2 \). In particular, \( f \) preserves the normalizer \( N_{\tilde{\Gamma}(S)}(\hat{I}_\gamma) \) of \( \hat{I}_\gamma \) in \( \tilde{\Gamma}(S) \). From the description of this group (cf. (i) of Corollary 4.12 in [7]), it follows that, for \( (g(S), n(S) + k(S)) \neq (0, 4), (1, 2) \), the center of \( N_{\tilde{\Gamma}(S)}(\hat{I}_\gamma) \) is generated by \( b_\gamma \), which concludes the proof of item (ii) in this case.

For \( (g(S), n(S) + k(S)) = (0, 4) \), a similar argument applies. For \( g(S) = 1 \) and either \( n(S) = 2 \) or \( k(S) = 2 \), the claim follows from (i) of Proposition 3.9.

\( \square \)

**Remarks 3.6.** Let \( U \) be an open subgroup of \( \tilde{\Gamma}(S) \):

(i) By Corollary 4.12 in [7], we have that \( \text{Inn}(U) \subseteq \text{Aut}^I(U) \subseteq \text{Aut}^D(U) \). We denote by \( \text{Out}^I(U) \) (resp. \( \text{Out}^D(U) \)) the group of outer \( I \)-automorphisms (resp. \( D \)-automorphisms) of \( U \).

(ii) From Proposition 7.2 and Theorem 5.5 in [7], it follows that, for either \( S \neq S_{1,2} \) or \( U = \text{P}\tilde{\Gamma}(S_{1,2}) \) and \( U = \tilde{\Gamma}(S_{1,2}) \), an \( I \)-automorphism of \( U \) preserves the topological types of the inertia groups \( \{ \hat{I}_\sigma(U) \}_{\sigma \in C(S)} \).

(iii) There is a natural faithful Galois representation \( \rho_Q : G_Q \to \text{Out}^I(\tilde{\Gamma}(S)) \), where \( G_Q \) is the absolute Galois group of the rationals (cf. Corollary 7.6 in [3]).

Except in the low genera cases, the \( D \)-condition and the \( I \)-condition are equivalent for full and pure procongruence mapping class groups. More precisely, we have:

**Proposition 3.7.**

(i) For \( d(S) > 1 \) and \( S \neq S_{1,2}, S_1^2, S_2 \), there holds:

\[ \text{Out}^I(\tilde{\Gamma}(S)) = \text{Out}^D(\tilde{\Gamma}(S)). \]

(ii) For \( S \neq S_{1,1}, S_1^1, S_2, \) there holds:

\[ \text{Out}^I(P\tilde{\Gamma}(S)) = \text{Out}^D(P\tilde{\Gamma}(S)). \]

(iii) For \( S = S_{1,1}, S_1^1, S_1^2, S_{1,2}, S_2^2 \) or \( S_2 \), there is a natural isomorphism:

\[ \text{Out}^D(\tilde{\Gamma}(S)) \cong \text{Out}^I(\tilde{\Gamma}(S)) \times \{ \pm 1 \}. \]

**Proof.** (i): for \( d(S) > 1 \) and \( S \neq S_{1,2}, S_1^2, S_2, \) by Corollary 4.12 in [7], we have \( Z(\tilde{\Gamma}(S)_\gamma) = \hat{I}_\gamma \) and \( N_{\tilde{\Gamma}(S)}(\hat{I}_\gamma) = \tilde{\Gamma}(S)_\gamma \), for all \( \gamma \in \hat{L}(S)_0 \), and so the conclusion follows.

(ii): The same argument above applies to \( P\tilde{\Gamma}(S) \) in the given cases.

(iii): for \( S = S_{1,1}, S_1^1, S_1^2, S_{1,2}, S_2^2 \) or \( S_2 \), by Lemma 7.4 in [7], there is an exact sequence:

\[ 1 \to \text{Hom}(\tilde{\Gamma}(S)/Z(\tilde{\Gamma}(S)), Z(\tilde{\Gamma}(S))) \to \text{Out}(\tilde{\Gamma}(S)) \to \text{Out}(\tilde{\Gamma}(S)/Z(\tilde{\Gamma}(S))). \]
where the center $Z(\hat{\Gamma}(S))$ is generated by the hyperelliptic involution $\iota$ and the abelianization of $\hat{\Gamma}(S)/Z(\hat{\Gamma}(S))$ is a cyclic group of finite even order (cf. § 5.1.3 in [9]). Therefore we have $\text{Hom}(\hat{\Gamma}(S)/Z(\hat{\Gamma}(S)), Z(\hat{\Gamma}(S))) \cong \{\pm 1\}$. Let us denote by $\phi_5$ its generator.

By (i) and (ii), we have that $\text{Out}^\mathbb{Z}(\hat{\Gamma}(S)/Z(\hat{\Gamma}(S))) = \text{Out}^\mathbb{Z}(\hat{\Gamma}(S)/Z(\hat{\Gamma}(S)))$. Therefore, if $f \in \text{Out}^\mathbb{Z}(\hat{\Gamma}(S))$ is such that $f \notin \text{Out}^\mathbb{Z}(\hat{\Gamma}(S))$, we have $f \circ \phi_5 \in \text{Out}^\mathbb{Z}(\hat{\Gamma}(S))$. Since the intersection between the subgroups $\text{Out}^{\mathbb{Z}}(\hat{\Gamma}(S))$ and $\text{Hom}(\hat{\Gamma}(S)/Z(\hat{\Gamma}(S)), Z(\hat{\Gamma}(S)))$ of $\text{Out}(\hat{\Gamma}(S))$ is trivial and $\phi_5$ centralizes $\text{Out}^{\mathbb{Z}}(\hat{\Gamma}(S))$, the conclusion follows.

\[ \square \]

3.2. **Inertia conditions for procongruence relative mapping class groups.** For a surface $S = S_{g,n}$ with boundary $\partial S = \bigcup_{i=1}^{k} \partial_i$, let $\hat{S} \cong S_{g,n+2k}$ be the surface of Definition 2.5. The natural embedding $S \hookrightarrow \hat{S}$ induces a monomorphism of procongruence mapping class groups $\hat{\Gamma}(S, \partial S) \hookrightarrow \hat{\Gamma}(\hat{S})$ (cf. Section 2.7). Therefore, Definition 3.3 also applies to procongruence relative mapping class groups. Note that $\prod_{i=1}^{k} \tau_{i}^{\pm}$ is the maximal normal inertia subgroup of $\hat{\Gamma}(S, \partial S)$. We call it the normal inertia group of $\hat{\Gamma}(S, \partial S)$. From (iii) of Proposition 2.6, it follows that this subgroup is also $\mathbb{I}$-characteristic. In particular, there is a natural homomorphism $\text{Aut}^{\mathbb{Z}}(\hat{\Gamma}(S, \partial S)) \rightarrow \text{Aut}(\hat{\Gamma}(S))$.

3.3. **The profinite Grothendieck-Teichmüller group.** The inertia conditions of the previous sections play a central role in Grothendieck-Teichmüller theory. Let $\text{Aut}^{\mathbb{Z}}(\hat{\Gamma}_{0,n})$ be the subgroup of $\text{Aut}(\hat{\Gamma}_{0,n})$ consisting of those automorphisms which preserve the set of inertia groups $\{\hat{\Gamma}_{i}\}_{\gamma \in \hat{B}(S_{0,n})}$ and let $\text{Out}^{\mathbb{Z}}(\hat{\Gamma}_{0,n})$ be the corresponding outer automorphism group. The main theorem of Harbater and Schneps in [12] states that, for all $n \geq 5$, there is a natural isomorphism:

\[ \text{Out}^{\mathbb{Z}}(\hat{\Gamma}_{0,n}) \cong \Sigma_n \times \hat{\Gamma}, \]

where $\hat{\Gamma}$ is the profinite Grothendieck-Teichmüller group introduced by Drinfeld (cf. [8]) and the symmetric group $\Sigma_n$ identifies with $\text{Inn}(\hat{\Gamma}_{0,n})/\text{Inn}(\hat{\Gamma}_{0,n})$. The key case of this isomorphism is for $n = 5$ (cf. Theorem 4 in [12]). This also allows to give a more intrinsic definition of the Grothendieck-Teichmüller group letting (note that, for $n = 5$, the $b$ and $\mathbb{I}$-conditions are equivalent):

\[ \hat{\Gamma} := Z_{\text{Out}^{\mathbb{Z}}(\hat{\Gamma}_{0,5})}(\Sigma_5). \]

This characterization was further improved by Hoshi, Minamide and Mochizuki, who proved that $\text{Aut}^{\mathbb{Z}}(\hat{\Gamma}_{0,n}) = \text{Aut}(\hat{\Gamma}_{0,n})$ (cf. Corollary 2.8 in [15]). Hence, for all $n \geq 5$, there is a natural isomorphism $\text{Out}(\hat{\Gamma}_{0,n}) \cong \Sigma_n \times \hat{\Gamma}$ and, in particular, we have, even more intrinsically, that:

\[ \hat{\Gamma} = Z_{\text{Out}(\hat{\Gamma}_{0,5})}(\Sigma_5). \]

Since $\hat{\Gamma}_{0,n}$ is a characteristic subgroup of $\hat{\Gamma}_{0,[n]}$ (cf. (ii) of Proposition 4.1 in [21]), restriction of automorphisms induces a homomorphism $\text{Aut}(\hat{\Gamma}_{0,[n]}) \rightarrow \text{Aut}(\hat{\Gamma}_{0,n})$. We can then reformulate the above isomorphisms in the following way:
Proposition 3.8. For \( n \geq 5 \), restriction of automorphisms induces an isomorphism \( \text{Aut}(\hat{\Gamma}_{0,[n]}) \cong \text{Aut}(\hat{\Gamma}_{0,n}) \). In particular, for \( n \geq 5 \), there is a natural isomorphism:

\[
(5) \quad \text{Out}(\hat{\Gamma}_{0,[n]}) \cong \hat{G}\hat{T}.
\]

Proof. Note that, since the action of \( \hat{\Gamma}_{0,[n]} \) on \( \text{Aut}(\hat{C}(S_{0,n})) \) is induced by its conjugation action on the normal subgroup \( \hat{\Gamma}_{0,n} \), from (ii) of Theorem 7.3 in [7], it follows that the natural homomorphism \( \text{Inn}(\hat{\Gamma}_{0,[n]}) \rightarrow \text{Aut}(\hat{\Gamma}_{0,n}) \) is injective for \( n \geq 5 \). By Lemma 3.3 in [7], the homomorphism \( \text{Aut}(\hat{\Gamma}_{0,[n]}) \rightarrow \text{Aut}(\hat{\Gamma}_{0,n}) \) is then also injective.

Let us consider the short exact sequence:

\[
1 \rightarrow \hat{\Gamma}_{0,n} \rightarrow \hat{\Gamma}_{0,[n]} \rightarrow \Sigma_n \rightarrow 1.
\]

In order to prove the proposition, we have to show that, for \( n \geq 5 \), every automorphism of \( \hat{\Gamma}_{0,n} \) extends to \( \hat{\Gamma}_{0,[n]} \). There is a well established theory to determine when this happens.

For an element \( f \in \text{Aut}(\hat{\Gamma}_{0,n}) \) let us denote by \( \bar{f} \) its image in \( \text{Out}(\hat{\Gamma}_{0,n}) \). Let then \( \text{Comp}(\Sigma_n, \hat{\Gamma}_{0,n}) \) be the closed subgroup of \( \text{Aut}(\Sigma_n) \times \text{Aut}(\hat{\Gamma}_{0,n}) \) formed by the pairs \((\phi, f)\) such that, for all \( \alpha \in \Sigma_n \), there holds (in \( \text{Out}(\hat{\Gamma}_{0,n}) \)):

\[
\bar{f} \rho(\alpha) \bar{f}^{-1} = \rho(\phi(\alpha)),
\]

where \( \rho: \Sigma_n \rightarrow \text{Out}(\hat{\Gamma}_{0,n}) \) is the (faithful) outer representation associated to the above extension of profinite groups.

Since the representation \( \rho \) is faithful and the center of \( \hat{\Gamma}_{0,n} \) is trivial, according to the profinite version of Wells’ exact sequence (cf. Lemma 1.5.5 in [22]), with the above hypotheses, there is a canonical isomorphism:

\[
\text{Aut}(\hat{\Gamma}_{0,[n]}) \cong \text{Comp}(\Sigma_n, \hat{\Gamma}_{0,n}).
\]

This isomorphism sends an element \( \tilde{f} \in \text{Aut}(\hat{\Gamma}_{0,[n]}) \) to the pair \((\phi, f)\), where \( \phi \in \text{Aut}(\Sigma_n) \) is the automorphism induced by \( \tilde{f} \) passing to the quotient by the characteristic subgroup \( \hat{\Gamma}_{0,n} \) and \( f \) is the restriction of \( \tilde{f} \) to \( \hat{\Gamma}_{0,n} \).

The conclusion follows if we show that every \( f \in \text{Aut}(\hat{\Gamma}_{0,n}) \) is part of a compatible pair. A necessary condition is that \( \rho(\Sigma_n) \) is a normal subgroup of \( \text{Out}(\hat{\Gamma}_{0,n}) \). In fact, since \( \rho \) is faithful, this condition is also sufficient. By the isomorphism \( \text{Out}(\hat{\Gamma}_{0,n}) \cong \Sigma_n \times \hat{G}\hat{T} \) (cf. also Lemma 4.6), for \( n \geq 5 \), we have that \( \rho(\Sigma_n) \) is a normal subgroup. Then, the action by conjugation of \( \tilde{f} \) on \( \rho(\Sigma_n) \) induces an automorphism \( \phi_f \) of \( \Sigma_n \) such that \((\phi_f, f)\) is a compatible pair. \( \square \)

The following result is essentially due to Minamide and Nakamura (cf. [21]):

Proposition 3.9. For \( g(S) = 1 \) and, either \( n(S) = 2 \) and \( k(S) = 0 \), or \( n(S) = 0 \) and \( k(S) = 2 \), we have:

(i) \( \text{Out}^1(\hat{\Gamma}(S)) = \text{Out}^0(\hat{P}\hat{T}(S)) = \text{Out}(\hat{P}\hat{T}(S)) \cong \hat{G}\hat{T} \).

(ii) \( \text{Out}(\hat{\Gamma}(S)) \cong \{\pm 1\} \times \hat{G}\hat{T} \).

The conclusion follows if we show that every \( f \in \text{Aut}(\hat{\Gamma}_{0,n}) \) is part of a compatible pair. A necessary condition is that \( \rho(\Sigma_n) \) is a normal subgroup of \( \text{Out}(\hat{\Gamma}_{0,n}) \). In fact, since \( \rho \) is faithful, this condition is also sufficient. By the isomorphism \( \text{Out}(\hat{\Gamma}_{0,n}) \cong \Sigma_n \times \hat{G}\hat{T} \) (cf. also Lemma 4.6), for \( n \geq 5 \), we have that \( \rho(\Sigma_n) \) is a normal subgroup. Then, the action by conjugation of \( \tilde{f} \) on \( \rho(\Sigma_n) \) induces an automorphism \( \phi_f \) of \( \Sigma_n \) such that \((\phi_f, f)\) is a compatible pair. \( \square \)

The following result is essentially due to Minamide and Nakamura (cf. [21]):

Proposition 3.9. For \( g(S) = 1 \) and, either \( n(S) = 2 \) and \( k(S) = 0 \), or \( n(S) = 0 \) and \( k(S) = 2 \), we have:

(i) \( \text{Out}^1(\hat{\Gamma}(S)) = \text{Out}^0(\hat{P}\hat{T}(S)) = \text{Out}(\hat{P}\hat{T}(S)) \cong \hat{G}\hat{T} \).

(ii) \( \text{Out}(\hat{\Gamma}(S)) \cong \{\pm 1\} \times \hat{G}\hat{T} \).
Proof. Since $\Gamma(S_2^n) = \Gamma(S_1,2)$ and $\Gamma(S_2^n) = \Gamma(S_1,2)$, it is not restrictive to assume that $k(S) = 0$ and $n(S) = 2$. By Corollary C in [21], it is enough to prove that we have:

$$\text{Out}^1(\hat{\Gamma}(S)) = \text{Out}^1(P\hat{\Gamma}(S)) \quad \text{and} \quad \text{Out}^1(\hat{\Gamma}(S)) \cong \{\pm 1\} \times \text{Out}^1(\hat{\Gamma}(S)).$$

By Lemma 7.4 in [7], there is an exact sequence:

$$1 \to \text{Hom}(\hat{\Gamma}(S)/Z(\hat{\Gamma}(S)), Z(\hat{\Gamma}(S))) \to \text{Out}(\hat{\Gamma}(S)) \to \text{Out}(\hat{\Gamma}(S)/Z(\hat{\Gamma}(S))),$$

where the center $Z(\hat{\Gamma}(S))$ is generated by the hyperelliptic involution $\iota$. From the description of the image of $\text{Hom}(\hat{\Gamma}(S)/Z(\hat{\Gamma}(S)), Z(\hat{\Gamma}(S))) \cong \{\pm 1\}$ in $\text{Out}(\hat{\Gamma}(S))$, it follows that it has trivial intersection with $\text{Out}^1(\hat{\Gamma}(S))$. Since $\hat{\Gamma}(S) = \langle \iota \rangle \times P\Gamma(S)$ and $Z(\hat{\Gamma}(S)) \cong \hat{P}(S)$, the exact sequence (6) is also right exact and then all the claims in the proposition follow.

3.4. Nonseparating inertia groups. A useful subcomplex of the curve complex $C(S)$ is the nonseparating curve complex $C_0(S)$. It consists of the simplices $\sigma \in C(S)$ such that $S \setminus \sigma$ is connected. The procongruence nonseparating curve complex $\hat{C}_0(S)$ is the abstract simplicial profinite complex whose set of $k$-simplices is the closure of $C_0(S)_k$ in $\hat{C}(S)_k$, for $k \geq 0$. In particular, a 0-simplex of $\hat{C}_0(S)$ is determined by a profinite simple closed curve whose topological type is that of a nonseparating simple closed curve. We denote by $\hat{L}^{ns}(S)$ the closed subset of $\hat{L}(S)$ consisting of such curves. The topological type of a simplex $\sigma \in \hat{C}_0(S)$ is determined by its dimension. Note that, for $g(S) = 0$, we have $C_0(S) = \hat{C}_0(S) = \emptyset$.

Definition 3.10. For $H$ a closed subgroup of the procongruence mapping class group $\hat{\Gamma}(S)$, the group of $I_0$-automorphisms $\text{Aut}^{I_0}(H)$ is the closed subgroup of $\text{Aut}(H)$ consisting of those elements which preserve the set of inertia groups $\{\hat{I}_\gamma(H)\}_{\gamma \in \hat{L}^{ns}(S)}$.

Remark 3.11. The same argument of Proposition 7.2 in [7] shows that an element $f \in \text{Aut}(H)$ is an $I_0$-automorphism, if and only if, $f$ preserves the set of inertia groups $\{I_\sigma(H)\}_{\sigma \in \hat{C}_0(S)}$ of $H$.

For pure procongruence mapping class groups, the $I_0$-condition turns out to be no more restrictive than the $I_0$-condition:

Theorem 3.12. For $d(S) > 1$, we have $\text{Aut}^1(P\hat{\Gamma}(S)) = \text{Aut}^{I_0}(P\hat{\Gamma}(S))$.

Proof. We proceed by induction on the genus of $S$. The base for the induction will be provided by the cases $g(S) = 0,1$, which we will treat separately in the following lemmas.

For $g(S) = 0$, we have a slightly stronger result which improves Corollary 2.8 in [15]:

Lemma 3.13. For $n \geq 5$, we have $\text{Aut}^1(\hat{\Gamma}_{0,n}) = \text{Aut}(\hat{\Gamma}_{0,n}) \cong \text{Aut}(\hat{\Gamma}_{0,[n]}) = \text{Aut}^1(\hat{\Gamma}_{0,[n]})$.

Proof. The isomorphism in the middle is given by Proposition 3.8. It is then enough to show that the left hand identity holds. The proof proceeds by induction on $n$. We have $\text{Aut}^1(\hat{\Gamma}_{0,5}) = \text{Aut}^1(\hat{\Gamma}_{0,5})$, so, for $n = 5$, this is just Corollary 2.8 in [15]. Let us now assume the claim holds for $\hat{\Gamma}_{0,n-1}$, with $n \geq 6$, and let us prove it for $\hat{\Gamma}_{0,n}$.
We have to show that \( \text{Aut}(\widehat{\Gamma}_{0,n}) \) preserves the set of inertia groups \( \{\widehat{I}_\sigma\}_{\sigma \in \hat{C}(S_{0,n})} \) of \( \widehat{\Gamma}_{0,n} \).

We claim that the induction hypothesis implies that \( \text{Aut}(\widehat{\Gamma}_{0,n}) \) preserves the subsets of inertia groups \( \{\widehat{I}_\sigma\}_{\sigma \in \hat{C}(S_{0,n})_k} \), for \( k = n - 4, n - 5 \).

It is enough to prove that, for \( f \in \text{Aut}(\widehat{\Gamma}_{0,n}) \) and \( \sigma \in C(S_{0,n})_k \), for \( k = n - 4, n - 5 \), we have \( f(\widehat{I}_\sigma) = \widehat{I}_{\sigma^n} \), for some \( \sigma^n \in \hat{C}(S_{0,n})_k \). The hypothesis on \( k \) implies that there is a simple closed curve \( \beta \in \sigma \) on \( S_{0,n} \) bounding a 2-punctured disc and such that \( \widehat{I}_\sigma \subset \widehat{\Gamma}_\beta \). Since, by Corollary 2.8 in [15], we already know that \( \text{Aut}(\widehat{\Gamma}_{0,n}) \cong \text{Aut}^b(\widehat{\Gamma}_{0,n}) \), after composing \( f \) with some inner automorphism, we can assume that \( f \) preserves the subgroup \( \widehat{\Gamma}_\beta \). According to (ii) of Theorem 2.3, the latter group is described by the short exact sequence:

\[
1 \to \overline{I}_\beta \to \overline{\Gamma}_\beta \to \overline{\Gamma}_{0,n-1} \to 1.
\]

Moreover, since the center of \( \overline{\Gamma}_{0,n-1} \) is trivial, we have \( Z(\overline{\Gamma}_\beta) = \overline{I}_\beta \). Therefore, \( f \) induces an automorphism \( \overline{f} \) of the quotient group \( \overline{\Gamma}_\beta/\overline{I}_\beta \cong \overline{\Gamma}_{0,n-1} \). Let us denote by \( \overline{I}_\sigma \) the image of \( \widehat{I}_\sigma \) in this quotient. By the inductive hypothesis, we have that \( \overline{f}(\overline{I}_\sigma) = \overline{I}_{\sigma^n} \), for some \( \sigma^n \in \hat{C}(S_{0,n-1}) \). This implies that, for some lift \( \sigma' \in \hat{C}_{k-1}(S_{0,n-1}) \) of \( \sigma^n \), we have \( \sigma' \cup \beta \), which proves the above claim.

Let \( \mathcal{G}(\widehat{\Gamma}_{0,n}) \) be the (profinite) set of closed subgroups of \( \widehat{\Gamma}_{0,n} \). As we observed in Section 4 of [7], for every \( 0 \leq k \leq n - 4 \), there is a \( \widehat{\Gamma}_{0,n} \)-equivariant continuous embedding:

\[
\mathcal{I}_k: \hat{C}_k(S_{0,n}) \hookrightarrow \mathcal{G}(\widehat{\Gamma}_{0,n}),
\]

defined by the assignment \( \sigma \mapsto \hat{I}_\sigma \). Thus we can identify the procongruence curve complex \( \hat{C}(S_{0,n}) \) with the simplicial complex \( \hat{C}_k(S_{0,n}) \) whose set of \( k \)-simplices is the set of closed subgroups \( \{\hat{I}_\sigma\}_{\sigma \in \hat{C}_k(S_{0,n})} \).

Let \( \hat{C}_k^*(S_{0,n}) \) be the dual graph of \( \hat{C}_k(S_{0,n}) \) (cf. Definition 3.9 in [7]). The claim proved above implies that there is a natural continuous action of \( \text{Aut}(\widehat{\Gamma}_{0,n}) \) on \( \hat{C}_k^*(S_{0,n}) \), but then, by Lemma 3.10 in [7], this implies that there is also a natural continuous action of \( \text{Aut}(\widehat{\Gamma}_{0,n}) \) on \( \hat{C}_k^*(S_{0,n}) \), which completes the induction step and so the proof of the lemma. \( \square \)

**Lemma 3.14.** For \( n \geq 2 \), we have \( \text{Aut}^3(\widehat{\Gamma}_{1,n}) = \text{Aut}^{50}(\widehat{\Gamma}_{1,n}) \).

**Proof.** The case \( n = 2 \) follows from (ii) of Proposition 3.9 and the observation that, in the short exact sequence:

\[
1 \to \text{Hom}(\widehat{\Gamma}_{1,2}/Z(\widehat{\Gamma}_{1,2}), Z(\widehat{\Gamma}_{1,2})) \to \text{Aut}(\widehat{\Gamma}_{1,2}) \to \text{Aut}(\widehat{\Gamma}_{1,2}/Z(\widehat{\Gamma}_{1,2})),
\]

the image of \( \text{Hom}(\widehat{\Gamma}_{1,2}/Z(\widehat{\Gamma}_{1,2}), Z(\widehat{\Gamma}_{1,2})) \) has trivial intersection with \( \text{Aut}^{30}(\widehat{\Gamma}_{1,2}) \).

For \( n \geq 3 \), it is enough to prove that, for any \( f \in \text{Aut}^1(\widehat{\Gamma}_{1,n}) \) and \( \gamma \in \mathcal{L}(S)_0 \subset \hat{\mathcal{L}}(S)_0 \) separating, we have \( f(\overline{I}_\gamma) = \overline{I}_{\gamma'}, \) for some \( \gamma' \in \hat{\mathcal{L}}(S)_0 \). Let \( \alpha \) be a nonseparating simple closed curve on \( S \) such that \( \tau_\gamma \in \Gamma_\alpha \). After composing \( f \) with a suitable inner automorphism, we can assume that \( f \) preserves the stabilizer \( \Gamma_\alpha \). According to (ii) of
Theorem 2.3, this group is described by the short exact sequence:

\[ 1 \to \hat{I}_\alpha \to \hat{\Gamma} \to \hat{\Gamma}_{0,n+2} \to 1. \]

Therefore, \( f \) induces an automorphism \( \hat{f} \) of the quotient group \( \hat{\Gamma}/\hat{I}_\alpha \cong \hat{\Gamma}_{0,n+2} \). Let us denote by \( \hat{I}_\gamma \) the image of \( \hat{I}_\gamma \) in this quotient. By Lemma 3.13, we have that \( \hat{f}(\hat{I}_\gamma) = \hat{I}_{\gamma'} \), for some profinite simple closed curve \( \gamma' \). Let \( \gamma' \in \hat{\mathcal{L}}(S_{1,n}) \) be a lift of \( \gamma' \). Then, \( \gamma' \) is also of separating type and there is an identity \( f(\tau) = \tau^h r^k \), for some \( h \in \hat{\mathbb{Z}}^* \) and \( k \in \hat{\mathbb{Z}} \). The kernel of the natural epimorphism \( \hat{\Gamma}_{1,n} \to \hat{\Gamma}_{1,1} \) is topologically generated by elements of the form \( \tau_1^h \tau_2^{-h} \), with \( \beta_1, \beta_2 \in \hat{\mathcal{L}}_{ns}(S) \) and \( h \in \hat{\mathbb{Z}}^* \), and is therefore preserved by \( f \). In particular, \( f(\tau) \) belongs to this kernel. By projecting the identity \( f(\tau) = \tau^h r^k \) to \( \hat{\Gamma}_{1,1} \), we then see that \( k = 0 \), which concludes the proof of the lemma.

We now proceed in the proof of Theorem 3.12 by induction on the genus. The base for the induction is provided by Lemma 3.14. So let us assume that the statement of Theorem 3.12 holds in genus \( g - 1 \) and let us prove it for genus \( g = g(S) \). The proof follows the pattern of the proof of Lemma 3.13. For simplicity, let us denote by \( \{\hat{I}_\sigma\}_{\sigma \in C(S)} \) the set of inertia groups of \( \hat{\mathbf{P}}(S) \). The key point is the observation that the induction hypothesis implies that \( \text{Aut}^{\hat{\mathbf{I}}} (\hat{\mathbf{P}}(S)) \) preserves the subsets of inertia groups \( \{\hat{I}_\sigma\}_{\sigma \in C(S), k} \), for \( k = d(S) - 1 \) and \( d(S) - 2 \). As in the proof of Lemma 3.13, this follows from the fact that, from the hypothesis \( g(S) \geq 2 \), it follows that a \( k \)-simplex \( \sigma \in C(S) \), for \( k = d(S) - 1 \) and \( d(S) - 2 \), contains at least a nonseparating curve. The rest of the proof is then identical to that of Lemma 3.13 and shows that the action of \( \text{Aut}^{\hat{\mathbf{I}}} (\hat{\mathbf{P}}(S)) \) preserves the set of all inertia groups of \( \hat{\mathbf{P}}(S) \).

4. Extending \( \mathbb{I} \)-automorphisms

The pure procongruence mapping class group \( \hat{\mathbf{P}}(S) \) is topologically generated by Dehn twists. It is then an \( \mathbb{I} \)-characteristic subgroup of the full procongruence mapping class group \( \hat{\Gamma}(S) \). Therefore, restriction of \( \mathbb{I} \)-automorphisms from \( \hat{\Gamma}(S) \) to \( \hat{\mathbf{P}}(S) \), defines a natural homomorphism \( \text{Aut}^{\hat{\mathbf{I}}} (\hat{\mathbf{P}}(S)) \to \text{Aut}^{\hat{\mathbf{I}}} (\hat{\Gamma}(S)) \). Let \( S = S_{g,n}^k \). Then, we have:

**Lemma 4.1.** For \((g, n + k) \neq (0, 4)\), the homomorphism \( \text{Aut}^{\hat{\mathbf{I}}} (\hat{\Gamma}(S)) \to \text{Aut}^{\hat{\mathbf{I}}} (\hat{\mathbf{P}}(S)) \) is injective.

**Proof.** The claim is trivial for \( n(S) \leq 1 \), while, for \( g(S) = 1 \) and \( n(S) = 2 \), \( k(S) = 0 \), or \( n(S) = 0 \), \( k(S) = 2 \), it follows from (i) of Proposition 3.9. By Theorem 4.14 in [7], we can thus assume that the center of \( \hat{\Gamma}(S) \) is trivial and, since \((g, n + k) \neq (0, 4)\), also that the natural homomorphism \( \text{Inn}(\hat{\Gamma}(S)) \to \text{Aut}^{\hat{\mathbf{I}}}(\hat{\mathbf{P}}(S)) \) is injective. By Lemma 3.3 in [7], the homomorphism \( \text{Aut}^{\hat{\mathbf{I}}}(\hat{\Gamma}(S)) \to \text{Aut}^{\hat{\mathbf{I}}}(\hat{\mathbf{P}}(S)) \) is then also injective.

In particular, from the above lemma and Theorem 3.12, it immediately follows:

**Corollary 4.2.** For \( d(S) > 1 \), we have \( \text{Aut}^{\hat{\mathbf{I}}} (\hat{\Gamma}(S)) = \text{Aut}^{\hat{\mathbf{I}}}(\hat{\Gamma}(S)) \).
The main result of this section is the following:

**Theorem 4.3.** Let $S = S_{g,n}^k$ be a hyperbolic surface with $d(S) > 1$. Let us recall the notation $\hat{S} := S \setminus \partial S$. We then have:

(i) Restriction of automorphisms induces an isomorphism $\text{Aut}^1(\hat{\Gamma}(\hat{S})) \cong \text{Aut}^1(\hat{P}(\hat{S}))$.

(ii) For $(g, n + k) \neq (1, 2)$, there is a natural isomorphism:

$$\text{Out}^1(\hat{P}(\hat{S})) \cong \Sigma_{n+k} \times \text{Out}^1(\hat{\Gamma}(\hat{S})).$$

(iii) For $(g, n + k) = (1, 2)$, there holds $\text{Out}^1(\hat{P}(\hat{S})) = \text{Out}^1(\hat{\Gamma}(\hat{S}))$.

4.1. **A key lemma.** Thanks to Theorem 8.1 in [7], there is a useful criterion to determine whether an $\mathbb{I}$-automorphism of $\hat{P}(\hat{S})$ is induced by an inner automorphism of $\hat{\Gamma}(\hat{S})$. Let us recall that, as done in the proof of Lemma 3.13, the procongruence curve complex $\hat{C}(S)$ can be identified with the simplicial profinite complex $\hat{C}_\mathbb{I}(S)$ with simplices the inertia groups $\{I_\sigma\}_{\sigma \in \hat{C}(S)}$. The vertices of the procongruence pants complex $\hat{C}_p(S)$ are then identified with the set $\{I_\sigma\}_{\sigma \in \hat{C}(S)_{d(S)-1}}$ of inertia subgroups of maximal rank. We have the following key lemma:

**Lemma 4.4.** Let $\phi \in \text{Aut}^1(\hat{P}(\hat{S}))$ be an element which, via the natural representation $\text{Aut}^1(\hat{P}(\hat{S})) \rightarrow \text{Aut}(\hat{C}_\mathbb{I}(\hat{S}))$, acts trivially on the simplices corresponding to the vertices of a profinite Farey subgraph $\hat{F}_\mu$ of $\hat{C}_p(S)$ (cf. Definition 6.3 in [7]). Then, $\phi \in \text{Inn}(\hat{\Gamma}(\hat{S}))$.

**Proof.** It is clearly not restrictive to assume $\partial S = \emptyset$. We proceed by induction on $d(S) \geq 1$. For $d(S) = 1$ the claim is trivial so let us proceed with $d(S) = 2$. For $g(S) = 0$ and $n(S) = 5$, there is only one topological type of 0-simplices in $\hat{C}(S)$ and so there is a single $\hat{\Gamma}(\hat{S})$-orbit of edges in $\hat{C}_p(S)$. Therefore, given an edge $\{\alpha_0, \alpha_1\} \in \hat{C}_p(S)$, there is an edge $\{\beta_0, \beta_1\} \in \hat{F}_\mu$ such that, for some $x \in \hat{\Gamma}(\hat{S})$, there holds $\alpha_i = x \cdot \beta_i \cdot x^{-1}$, for $i = 0, 1$.

By Proposition 3.8, we can extend $\phi$ to an automorphism of $\hat{\Gamma}(\hat{S})$, so that we have:

$$\phi(\alpha_i) = \phi(x \cdot \beta_i \cdot x^{-1}) = \phi(x) \cdot \beta_i \cdot \phi(x)^{-1}, \quad \text{for } i = 0, 1,$$

and then $\{\phi(\alpha_0), \phi(\alpha_1)\} = \text{inn} \phi(x)(\{\beta_0, \beta_1\}) \in \hat{C}_p(S)$. Therefore, the continuous action of the automorphism $\phi$ on the vertices of $\hat{C}_p(S)$ induces a continuous action on the procongruence pants complex $\hat{C}_p(S)$. Since every profinite Farey subgraph of $\hat{C}_p(S)$ is in the $\hat{\Gamma}(\hat{S})$-orbit of the profinite Farey subgraph $\hat{F}_\mu \subset \hat{C}_p(S)$, on which $\phi$ acts by the identity map, it follows that $\phi \in \text{Aut}^1(\hat{C}_p(S)) = \text{Inn}(\hat{\Gamma}(\hat{S}))$ (cf. Theorem 8.1, Lemma 8.6 and remarks in Section 8.7 of [7]).

For $g(S) = 1$ and $n(S) = 2$, let us observe that $\hat{P}(S_{1,2})$ identifies with a finite index subgroup of $\hat{\Gamma}(S_{0,5})$ which contains $\hat{P}(S_{0,5})$. From Proposition 3.8 and Lemma 3.13, it follows that every $\mathbb{I}$-automorphism of $\hat{P}(S_{1,2})$ extends to an $\mathbb{I}$-automorphism of $\hat{\Gamma}(S_{0,5})$. Since $\hat{C}_\mathbb{I}(S_{1,2}) = \hat{C}_\mathbb{I}(S_{0,5})$ and $\hat{C}_p(S_{1,2}) = \hat{C}_p(S_{0,5})$, the same argument of the case $g(S) = 0$ and $n(S) = 5$ then applies and implies Lemma 4.4 also in this case.

Let us now deal with the case $g(S) = 0$ and $d(S) > 2$. Note that every $(d(S) - 2)$-simplex in $\hat{C}(S)$ contains at least a simple closed curve bounding a 2-punctured disc. Let then $\gamma$
be one such curve contained in \( \mu \). After composing with an element of \( \text{Inn}(\tilde{\Gamma}(S)) \), we can assume that the given automorphism \( \phi \in \text{Aut}(\tilde{\Gamma}(S)) \), besides acting trivially on the simplices corresponding to the vertices of the profinite Farey subgraph \( \tilde{\Gamma}_\mu \), preserves the inertia group \( I_\gamma \). Hence, \( \phi \) preserves the link of \( \gamma \) in \( \tilde{\Gamma}(S) \) which is naturally isomorphic to \( \tilde{\gamma}(S \setminus D_\gamma) \), where \( D_\gamma \) is the 2-punctured disc in \( S \) with boundary \( \gamma \) (cf. Remark 4.7 in [3]). From the induction hypothesis, it follows that \( \phi \) acts on \( \tilde{\Gamma}(S \setminus D_\gamma) \) through an element of \( \text{Inn}(\tilde{\Gamma}(S \setminus D_\gamma)) \) and then it acts on \( \text{Link}(\gamma) \) through an element of \( \text{Inn}(\tilde{\Gamma}(S)_\gamma) \).

Since the set of \( (d(S) - 2) \)-simplices in \( C(S) \) containing \( \gamma \) comprises all topological types, after composing with an element of \( \text{Inn}(\tilde{\Gamma}(S)) \), we can assume that \( \phi \) restricts to the identity on the vertex set of any given profinite Farey subgraph of \( \tilde{\Gamma}_P(S) \). Therefore, \( \phi \) preserves the edge set of the procongruence pants complex \( \tilde{\Gamma}_P(S) \), thus inducing an action on it which moreover preserves the orientations of its profinite Farey subgraph. Therefore, by Theorem 8.1 in [7], we conclude that \( \phi \in \text{Inn}(\tilde{\Gamma}(S)) \).

The next case to consider is then \( g(S) = 1 \) and \( d(S) > 2 \). Let us consider first the case when \( g(S \setminus \mu) = 1 \). Then, \( \mu \) contains at least a profinite simple closed curve \( \gamma \) of the topological type of a curve bounding a 2-punctured disc. It is not restrictive to assume that \( \gamma \in \mathcal{L}(S) \subset \tilde{\Gamma}(S) \) and, as above, that \( \phi \) preserves the inertia group \( I_\gamma \). Then, \( \phi \) preserves the subcomplex \( \text{Link}(\gamma) \cong \tilde{\gamma}(S \setminus D_\gamma) \) and, as above, from the induction hypothesis, we conclude that \( \phi \) acts on \( \text{Link}(\gamma) \) through an element of \( \text{Inn}(\tilde{\Gamma}(S)_\gamma) \).

Let then \( \delta \) be a nonseparating simple closed curve \( \delta \) on \( S \) disjoint from \( \gamma \). The link of the \( 1 \)-simplex \( \{ \gamma, \delta \} \) in \( \tilde{\Gamma}(S) \) identifies with a subcomplex of both links of \( \gamma \) and \( \delta \), so that, after composing with an element of \( \text{Inn}(\tilde{\Gamma}(S)) \), we can assume that \( \phi \) preserves the inertia group \( I_\delta \) and, moreover, restricts to the identity on the vertex set of some profinite Farey subgraph of \( \tilde{\Gamma}_P(S \setminus \delta) \). From the induction hypothesis, it then follows, as above, that \( \phi \) acts on \( \text{Link}(\delta) \) through an element of \( \text{Inn}(\tilde{\Gamma}(S)_\delta) \).

In conclusion, we see that, after composing with an element of \( \text{Inn}(\tilde{\Gamma}(S)) \), we can assume that \( \phi \) restricts to the identity on the vertex set of any given profinite Farey subgraph of \( \tilde{\Gamma}_P(S) \). As remarked above, by Theorem 8.1 in [7], this implies that \( \phi \in \text{Inn}(\tilde{\Gamma}(S)) \).

The case when \( g(S \setminus \mu) = 0 \) can be treated similarly. We only need to invert the roles of the simple closed curves \( \gamma \) and \( \delta \) above.

For \( g(S) \geq 2 \), every \( (d(S) - 2) \)-simplex in \( C(S) \) contains at least a nonseparating simple closed curve. Induction and the same argument above then yield the conclusion in this case too.

The following lemma will be essential for the proof of Theorem 4.3:

**Lemma 4.5.** For \( d(S) > 1 \), the natural monomorphism \( \text{Aut}(\tilde{\Gamma}(S)) \hookrightarrow \text{Aut}(\tilde{\Gamma}_P(S)) \) identifies \( \text{Inn}(\tilde{\Gamma}(S)) \) with a normal subgroup of \( \text{Aut}(\tilde{\Gamma}_P(S)) \).

**Proof.** It is not restrictive to assume \( \partial S = \emptyset \). Since \( \tilde{\Gamma}(S) \) is generated by its normal subgroup \( \tilde{P}(S) \) and braid twists, it is enough to show that, given a braid twist \( b_\gamma \), for \( \gamma \in \mathcal{L}^b(S) \), for all \( f \in \text{Aut}(\tilde{P}(S)) \), we have \( \phi := f \circ \text{inn} b_\gamma \circ f^{-1} \in \text{Inn}(\tilde{\Gamma}(S)) \). The key observation is that, since \( \text{inn} b_\gamma \) restricts to the identity on the stabilizer \( \tilde{P}(S)_\gamma \), the
automorphism $\phi$ restricts to the identity on the stabilizer $\hat{\Gamma}(S)_{f(\gamma)}$. Since, for every $\sigma \in \hat{\mathcal{C}}(S)_{d(S)-2}$ such that $\gamma \in \sigma$, the vertices of the profinite Farey subgraph $\hat{\mathcal{F}}_{f(\sigma)} \subset \hat{\mathcal{F}}(S)$ are subgroups of the stabilizer $\hat{\Gamma}(S)_{f(\gamma)}$, the action of $\phi$ restricts to the identity on the vertices of all these profinite Farey subgraphs. By Lemma 4.4, we conclude that $\phi \in \text{Inn}(\hat{\Gamma}(S))$. \qed

Note that for $S = S^k_{g,n}$ and $(g, n + k) \neq (1, 2)$, the quotient $\text{Inn}(\hat{\Gamma}(\hat{S})) / \text{Inn}(\hat{\Gamma}(S))$ identifies with the quotient $\hat{\Gamma}(\hat{S}) / \hat{\Gamma}(S) \cong \Sigma_{n+k}$. From Lemma 4.5, it then follows that $\Sigma_{n+k}$ identifies with a normal subgroup of $\text{Out}^1(\hat{\Gamma}(S))$. We have:

**Lemma 4.6.** For $S = S^k_{g,n}$ a hyperbolic surface such that $(g, n + k) \neq (1, 2)$, restriction of inner automorphisms induces an epimorphism $\text{Out}^1(\hat{\Gamma}(S)) \to \text{Inn} \Sigma_{n+k}$.

**Proof.** We can assume that $n + k \geq 2$ and, since $(g, n + k) \neq (1, 2)$, in particular, that the center of $\hat{\Gamma}(S)$ is trivial. It is enough to show that an automorphism of $\Sigma_{n+k}$ induced by $\text{Inn}(\text{Out}^1(\hat{\Gamma}(S)))$ sends a transposition to another transposition (cf. [26]). This follows from the fact that, by (ii) of Proposition 3.5, the conjugacy action of $\text{Out}^1(\hat{\Gamma}(S))$ on its normal subgroup $\text{Inn} \hat{\Gamma}(S) \cong \hat{\Gamma}(S)$ sends a profinite braid twist to another profinite braid twists, since it preserves conjugacy classes of profinite Dehn twists. \qed

In particular, for $n + k > 2$, there is a natural epimorphism $\text{Out}^1(\hat{\Gamma}(S)) \to \Sigma_{n+k}$ which is a left inverse of the inclusion $\Sigma_{n+k} \hookrightarrow \text{Out}^1(\hat{\Gamma}(S))$. This is also true for $n + k = 2$, thanks to the following lemma:

**Lemma 4.7.** For $S = S^k_{g,n}$ a hyperbolic surface such that $g \geq 1$ and $(g, n + k) \neq (1, 2)$, there is an epimorphism $\text{Aut}(\hat{\mathcal{C}}(S)) \to \Sigma_{n+k}$, such that its composition with the homomorphism $\hat{\Gamma}(\hat{S}) \to \text{Aut}(\hat{\mathcal{C}}(S))$ is the natural epimorphism $\hat{\Gamma}(\hat{S}) \to \Sigma_{n+k}$.

**Proof.** As usual, we can assume $\partial S = \emptyset$. Let us then fix a set of labels $B$ for the punctures on $S$. For a 1-simplex $\{\gamma_0, \gamma_1\} \in \hat{\mathcal{C}}(S)$, such that the two curves $\gamma_0, \gamma_1$ bound a 1-punctured annulus on $S$, we define the marked topological type of $\{\gamma_0, \gamma_1\}$ as the data of the topological type of $\{\gamma_0, \gamma_1\}$ plus the label $b \in B$ which marks the puncture on the annulus. Note that the hypothesis $S \neq S_{1,2}$ makes sure that this is well defined. The marked topological type of $\{\gamma_0, \gamma_1\}$ is preserved by the action of the pure mapping class group $\text{PT}(S)$ but, for $n(S) > 1$, not by the action of the full mapping class group $\Gamma(S)$.

For a 1-simplex $\{\gamma_0, \gamma_1\} \in \hat{\mathcal{C}}(S)$ with the topological type of a 1-punctured annulus on $S$, we define its marked topological type as the marked topological type of a 1-simplex $\{\gamma'_0, \gamma'_1\} \in C(S)$ in the $\hat{\Gamma}(S)$-orbit of $\{\gamma_0, \gamma_1\}$. By Corollary 7.6 in [5] and the remark above, this is well defined.

Let us now fix a simplex $\sigma \in C(S)$ such that every puncture of $S$ is contained in a 1-punctured annulus bounded by curves in $\sigma$ (this exists by the hypothesis $g(S) > 0$). For $f \in \text{Aut}(\hat{\mathcal{C}}(S))$, by Theorem 5.5 in [7] and the hypothesis $S \neq S_{1,2}$, the simplex $f(\sigma) \in \hat{\mathcal{C}}(S)$ has the same (unmarked) topological type of $\sigma$. In particular, for a puncture marked by $b \in B$ and contained in the annulus bounded by a 1-simplex $\{\gamma_0, \gamma_1\} \subset \sigma$, the simplex $\{f(\gamma_0), f(\gamma_1)\} \in \hat{\mathcal{C}}(S)$ has the marked topological type of a 1-punctured annulus
marked by some other \( b' \in B \). Letting \( f(b) := b' \) then defines an action of \( \text{Aut}(\hat{C}(S)) \) on \( B \) and so a homomorphism \( \Phi_\sigma: \text{Aut}(\hat{C}(S)) \to \Sigma_B \). The definition of \( \Phi_\sigma \) implies that this homomorphism is compatible with the natural action of \( \hat{\Gamma}(S) \) on \( \hat{C}(S) \) and with the natural epimorphism \( \hat{\Gamma}(S) \to \Sigma_{n+k} \). In particular, \( \Phi_\sigma \) is surjective. \( \square \)

**Remark 4.8.** It is not difficult to see that \( \Phi_\sigma \) only depends on the topological type of \( \sigma \). It is likely that \( \Phi_\sigma \) does not depend from the choice of \( \sigma \) at all. However, we were not able to find a simple proof of this fact which, in any case, we will not need in the sequel.

4.2. **Proof of Theorem 4.3.** We can assume \( n + k \geq 2 \), otherwise the theorem is trivial. For the proof of item (i) and (ii) of the theorem, it is not restrictive to assume \( \partial S = \emptyset \).

(i): For \( S = S_{1,2} \), this is just (i) of Proposition 3.9. Therefore, we can also assume \( S \neq S_{1,2} \) and then, by (i) of Theorem 4.14 in [7], that the centralizer of \( \hat{P}(S) \) in \( \hat{\Gamma}(S) \) is trivial.

We have to show that the natural monomorphism \( \text{Aut}\hat{\hat{\Gamma}}(S) \hookrightarrow \text{Aut}\hat{\hat{\Gamma}}(S) \) is surjective. For this, as in the proof of Proposition 3.8, we consider the short exact sequence:

\[ 1 \to \text{P}\hat{\hat{\Gamma}}(S) \to \hat{\Gamma}(S) \to \Sigma_n \to 1. \]

Let \( \rho: \Sigma_n \to \text{Out}(\text{P}\hat{\hat{\Gamma}}(S)) \) be the associated outer representation.

Since \( Z(\hat{\Gamma}(S))(\text{P}\hat{\hat{\Gamma}}(S)) = \{1\} \), the representation \( \rho \) is faithful and \( \rho(\Sigma_n) \) coincides with the image of \( \text{Inn}(\hat{\Gamma}(S)) \) in \( \text{Out}(\text{P}\hat{\hat{\Gamma}}(S)) \). By Lemma 4.5, we then have that \( \rho(\Sigma_n) \) is a normal subgroup of \( \text{Out}^1(\text{P}\hat{\hat{\Gamma}}(S)) \). By the same argument of the proof of Proposition 3.8, this implies that every automorphism \( f \in \text{Aut}^1(\text{P}\hat{\hat{\Gamma}}(S)) \) extends to \( \hat{\Gamma}(S) \), thus completing the proof of the first item.

(ii): By the previous item of the theorem, there is a short exact sequence:

\[ 1 \to \text{Inn}(\hat{\Gamma}(S))/\text{Inn}(\text{P}\hat{\hat{\Gamma}}(S)) \to \text{Aut}^1(\text{P}\hat{\hat{\Gamma}}(S))/\text{Inn}(\text{P}\hat{\hat{\Gamma}}(S)) \to \text{Aut}^2(\hat{\Gamma}(S))/\text{Inn}(\hat{\Gamma}(S)) \to 1. \]

Since, by hypothesis, \( n \neq 2 \) for \( g = 1 \), there holds \( Z(\hat{\Gamma}(S)) = Z(\text{P}\hat{\hat{\Gamma}}(S)) = \{1\} \) and so we have \( \text{Inn}(\hat{\Gamma}(S))/\text{Inn}(\text{P}\hat{\hat{\Gamma}}(S)) \cong \hat{\Gamma}(S)/\text{P}\hat{\hat{\Gamma}}(S) \cong \Sigma_n \). By Lemma 4.7, the monomorphism \( \Sigma_n \hookrightarrow \text{Out}^1(\text{P}\hat{\hat{\Gamma}}(S)) \) has a left inverse \( \text{Out}^1(\text{P}\hat{\hat{\Gamma}}(S)) \to \Sigma_n \) which provides the splitting.

(iii): For \( S = S_{1,1} \), we have \( \text{P}\hat{\hat{\Gamma}}(S) \cong \hat{\Gamma}(S) \) and the statement is trivial. For \( S = S_{1,2} \) or \( S_1^2 \), this is just the first item of Proposition 3.9.

4.3. **Automorphisms of procongruence relative mapping class groups.** For a surface \( S = S_{g,n}^k \) with boundary \( \partial S = \bigcup_{i=1}^k \delta_i \), let \( \tilde{S} \cong S_{g,n+2k} \) be the surface defined in Definition 2.5. By definition of procongruence relative mapping class group, the natural embedding \( S \hookrightarrow \tilde{S} \) induces a monomorphism \( \text{P}\hat{\hat{\Gamma}}(S,\partial S) \hookrightarrow \text{P}\hat{\hat{\Gamma}}(\tilde{S}) \) (cf. Section 2.7).

**Definition 4.9.** For an open subgroup \( U \) of \( \hat{\Gamma}(S,\partial S) \), we have already defined \( \text{Aut}^1(U) \) to be the subgroup of \( \text{Aut}(U) \) consisting of those automorphisms which preserve the inertia groups \( \{\hat{I}_\gamma(U)\}_{\gamma \in \hat{L}(\tilde{S})} \) (cf. Section 3.2). We then let \( \text{Aut}^{1,0}(U) \) to be the subgroup of \( \text{Aut}(U) \) consisting of those automorphisms which preserve the set of inertia groups \( \{\hat{I}_\gamma(U)\}_{\gamma \in \hat{L}^{\text{ns}}(S)} \cup \{\hat{I}_{\delta_i}(U)\}_{i=1,\ldots,k} \) (note that \( \hat{L}^{\text{ns}}(S) \equiv \hat{L}^{\text{ns}}(\tilde{S}) \)).
From (iii) of Proposition 2.6, it follows that the normal inertia group \( A := U \cap \prod_{\gamma=1}^{k} \tau_{\gamma} \) is \( \mathbb{I} \)-characteristic in \( U \) and, by definition, it is also \( \mathbb{I}_0 \)-characteristic. Thus, if we let \( \overline{U} := U/A \), there are natural homomorphisms \( \text{Aut}^\delta(U) \to \text{Aut}^\delta(\overline{U}) \) and \( \text{Aut}^{\delta_0}(U) \to \text{Aut}^{\delta_0}(\overline{U}) \). We say that a group \( G \) is virtually (topologically) generated by a subset \( T \), if \( G \) contains a finite index subgroup (topologically) generated by this subset. We then have:

**Lemma 4.10.**

(i) Let \( U \) be an open subgroup of \( \bar{\Gamma}(S, \partial S) \) virtually topologically generated by the inertia groups \( \{ \bar{I}_{\gamma}(U) \}_{\gamma \in \hat{\mathcal{E}}(S)} \). Then, the natural homomorphisms:

\[
\text{Aut}^\delta(U) \to \text{Aut}^\delta(\overline{U}) \times \text{Aut}(A) \quad \text{and} \quad \text{Out}^\delta(U) \to \text{Out}^\delta(\overline{U}) \times \text{Aut}(A)
\]

are injective.

(ii) Let \( U \) be an open subgroup of \( \bar{\Gamma}(S, \partial S) \) virtually topologically generated by the inertia groups \( \{ \bar{I}_{\gamma}(U) \}_{\gamma \in \hat{\mathcal{E}}(S)} \cup \{ \bar{I}_{\delta}(U) \}_{i=1,\ldots,k} \). Then, the natural homomorphisms:

\[
\text{Aut}^{\delta_0}(U) \to \text{Aut}^{\delta_0}(\overline{U}) \times \text{Aut}(A) \quad \text{and} \quad \text{Out}^{\delta_0}(U) \to \text{Out}^{\delta_0}(\overline{U}) \times \text{Aut}(A)
\]

are injective.

**Proof.** It is enough to prove the first item since the proof of the second is basically the same. By Lemma 7.4 in [7], the kernel of the homomorphism \( \text{Aut}(U)_A \to \text{Aut}(\overline{U}) \times \text{Aut}(A) \) (resp. \( \text{Out}(U)_A \to \text{Out}(\overline{U}) \times \text{Aut}(A) \)) identifies with the (free) abelian group \( \text{Hom}(\overline{U}, A) \).

The restriction of a homomorphism \( \phi \in \text{Hom}(\overline{U}, A) \cap \text{Aut}^\delta(U) \) to the finite index subgroup of \( \overline{U} \) topologically generated by inertia groups is trivial. Since \( A \) is torsion free, this implies that \( \phi \) itself is trivial and hence that \( \text{Hom}(\overline{U}, A) \cap \text{Aut}^\delta(U) = \{ 1 \} \).

Let us observe that \( \overline{\Gamma}(S, \partial S) \), for \( g(S) \geq 1 \), is topologically generated by nonseparating Dehn twists, so that \( \bar{\Gamma}(S, \partial S) \) is virtually topologically generated by them. An immediate consequence of Lemma 4.10 and Theorem 3.12 is then:

**Proposition 4.11.** For \( g(S) \geq 1 \), there holds \( \text{Aut}^\delta(\overline{\Gamma}(S, \partial S)) = \text{Aut}^{\delta_0}(\overline{\Gamma}(S, \partial S)) \) and \( \text{Aut}^\delta(\bar{\Gamma}(S, \partial S)) = \text{Aut}^{\delta_0}(\bar{\Gamma}(S, \partial S)) \).

The subgroup \( \overline{\Gamma}(S, \partial S) \) of \( \bar{\Gamma}(S, \partial S) \) is \( \mathbb{I} \)-characteristic. There is then a natural homomorphism \( \text{Aut}^\delta(\bar{\Gamma}(S, \partial S)) \to \text{Aut}^\delta(\overline{\Gamma}(S, \partial S)) \). From Lemma 4.1 and (i) of Lemma 4.10, we have:

**Lemma 4.12.** For \( (g, n + k) \neq (0, 4) \), the natural homomorphism \( \text{Aut}^\delta(\bar{\Gamma}(S, \partial S)) \to \text{Aut}^\delta(\overline{\Gamma}(S, \partial S)) \) is injective.

For \( (g, n + k) \neq (0, 4) \), let us identify \( \text{Inn}(\bar{\Gamma}(S, \partial S)) \) with a subgroup of \( \text{Aut}^\delta(\overline{\Gamma}(S, \partial S)) \) by means of the above monomorphism. We then have:

**Lemma 4.13.** For \( d(S) > 1 \), there holds \( \text{Inn}(\bar{\Gamma}(S, \partial S)) \triangleleft \text{Aut}^\delta(\overline{\Gamma}(S, \partial S)) \).
Proof. By means of the natural monomorphism in (i) of Lemma 4.10, let us identify \( \text{Inn}(\tilde{\Gamma}(S, \partial S)) \) with a subgroup of \( \text{Aut}^1(P\tilde{\Gamma}(S)) \times \text{Aut}(\prod_{i=1}^{k} \tau_{\delta_i}^2) \) and let us observe that:

\[
\text{Inn}(\tilde{\Gamma}(S, \partial S)) / \text{Inn}(P\tilde{\Gamma}(S, \partial S)) \cong \text{Inn}(\tilde{\Gamma}(S)) / \text{Inn}(P\tilde{\Gamma}(S)) \cong \Sigma_n \times \Sigma_k.
\]

From Lemma 4.5, it then easily follows that the image of the group \( \text{Inn}(\tilde{\Gamma}(S, \partial S)) \) inside \( \text{Aut}^1(P\tilde{\Gamma}(S)) \times \text{Aut}(\prod_{i=1}^{k} \tau_{\delta_i}^2) \) is normalized by the image of \( \text{Out}^1(P\tilde{\Gamma}(S, \partial S)) \).

We also have:

**Lemma 4.14.** For \( S = S_{g,n}^k \neq S_{1,2} \) a hyperbolic surface, there is a natural epimorphism \( \text{Aut}^1(P\tilde{\Gamma}(S, \partial S)) \to \Sigma_n \times \Sigma_k \), such that its composition with the homomorphism \( \text{inn}: \tilde{\Gamma}(S, \partial S) \to \text{Aut}^1(P\tilde{\Gamma}(S, \partial S)) \) is the natural epimorphism \( \tilde{\Gamma}(S, \partial S) \to \Sigma_n \times \Sigma_k \).

**Proof.** For \( n = 0 \) and \( k > 0 \), we consider the action of \( \text{Aut}^1(P\tilde{\Gamma}(S, \partial S)) \) on the normal inertia subgroup \( \prod_{i=1}^{k} \tau_{\delta_i}^2 \) induced by restriction of inner automorphisms. This action permutes the procyclic subgroups \( \tau_{\delta_i}^2 \) and defines an epimorphism \( \text{Aut}^1(P\tilde{\Gamma}(S, \partial S)) \to \Sigma_k \) with the desired properties.

For \( n > 0 \), we consider the natural homomorphism \( \text{Aut}^1(P\tilde{\Gamma}(S, \partial S)) \to \text{Aut}^1(P\tilde{\Gamma}(S)) \). We then get, for \( g = 0 \), by the isomorphism (4), and, for \( g > 0 \), \( (g, n + k) \neq (1, 2) \), by Lemma 4.7, a homomorphism \( \text{Aut}^1(P\tilde{\Gamma}(S, \partial S)) \to \Sigma_{n+k} \) compatible with the natural homomorphisms \( \text{Inn}(\tilde{\Gamma}(S, \partial S)) \to \text{Out}^1(P\tilde{\Gamma}(S, \partial S)) \) and \( \text{Inn}(\tilde{\Gamma}(S, \partial S)) \to \Sigma_n \times \Sigma_k \).

Let \( \{\gamma_0, \gamma_1\} \in C(S) \subset \hat{C}(S) \) be a 1-simplex bounding an annulus in \( S \) containing a boundary component of the surface. From the description of the normalizer in \( P\tilde{\Gamma}(S, \partial S) \) of the closed subgroup topologically generated by \( \tau_{\gamma_0} \) and \( \tau_{\gamma_1} \) (cf. Theorem 2.8), it easily follows that an element of \( \text{Aut}^1(P\tilde{\Gamma}(S, \partial S)) \) maps the simplex \( \{\gamma_0, \gamma_1\} \) to another simplex of \( \hat{C}(S) \) which has the same topological type, that is to say which is in the \( \tilde{\Gamma}(S) \)-orbit of a 1-simplex in \( C(S) \) consisting of curves which also bound an annulus in \( S \).

Therefore, the action of \( \text{Aut}^1(P\tilde{\Gamma}(S, \partial S)) \) on the set of all punctures and boundary components of \( S = S_{g,n}^k \) preserves the partition of this set into punctures and boundary components. This means that the image of the homomorphism \( \text{Aut}^1(P\tilde{\Gamma}(S, \partial S)) \to \Sigma_{n+k} \) defined above is the subgroup \( \Sigma_n \times \Sigma_k \), which stabilizes this partition. The second claim of the lemma is also clear.

The analogue of Theorem 4.3 then holds:

**Theorem 4.15.** Let \( S = S_{g,n}^k \) be a hyperbolic surface such that \( d(S) > 1 \).

(i) Restriction of automorphisms induces an isomorphism:

\[
\text{Aut}^1(\tilde{\Gamma}(S, \partial S)) \cong \text{Aut}^1(P\tilde{\Gamma}(S, \partial S)).
\]

(ii) For \( S \neq S_{1,2} \), there is a natural isomorphism:

\[
\text{Out}^1(P\tilde{\Gamma}(S, \partial S)) \cong \Sigma_n \times \Sigma_k \times \text{Out}^1(\tilde{\Gamma}(S, \partial S)).
\]

**Proof.** (i): We can assume \( k > 0 \) and then, since \( d(S) > 1 \), that the centers of \( P\tilde{\Gamma}(S, \partial S) \) and \( \tilde{\Gamma}(S, \partial S) \) do not contain a hyperelliptic involution. We have to show that the natural
monomorphism \( \text{Aut}^\dagger(\hat{\Gamma}(S, \partial S)) \hookrightarrow \text{Aut}^\dagger(\check{\Gamma}(S, \partial S)) \) is surjective. For this, we consider the short exact sequence:

\[ 1 \rightarrow \text{P}\hat{\Gamma}(S, \partial S) \rightarrow \hat{\Gamma}(S, \partial S) \rightarrow \Sigma_n \times \Sigma_k \rightarrow 1 \]

and then argue as in the proof of (i) of Theorem 4.3.

(ii): The claim follows from the first item of the theorem and the same argument of (ii) of Theorem 4.3.

\[ \square \]

5. HOMOMORPHISMS BETWEEN OUTER AUTOMORPHISM GROUPS OF PROCONGRUENCE MAPPING CLASS GROUPS

In this section, for simplicity, we will assume that \( S \) is a hyperbolic surface with empty boundary.

5.1. Functorial homomorphisms. Let \( \gamma \) be a nonseparating simple closed curve on \( S \). By (ii) of Theorem 2.3, there is a short exact sequence:

\[ 1 \rightarrow \tau_{\gamma} \rightarrow \text{P}\check{\Gamma}(S) \rightarrow \text{P}\check{\Gamma}(S \setminus \gamma) \rightarrow 1. \]

Since the group \( \text{P}\check{\Gamma}(S \setminus \gamma) \) is center free (cf. Corollary 6.2 in [3]), we have \( Z(\text{P}\check{\Gamma}(S)_{\gamma}) = \tau_{\gamma} \).

We want to construct a natural homomorphism \( \text{Aut}^\dagger(\text{P}\check{\Gamma}(S)) \rightarrow \text{Out}^\dagger(\text{P}\check{\Gamma}(S \setminus \gamma)). \) The first step is to construct a homomorphism \( \text{Aut}^\dagger(\text{P}\check{\Gamma}(S)) \rightarrow \text{Out}^\dagger(\text{P}\check{\Gamma}(S)_{\gamma}). \)

**Remark 5.1.** Note that, for \( U \) an open subgroup of \( \check{\Gamma}(S)_{\gamma} \) (e.g. \( \text{P}\check{\Gamma}(S)_{\gamma} \)), the nontrivial inertia groups of \( U \) are parameterized by the subcomplex \( \text{Star}(\gamma) \subset \check{C}(S) \) while the nontrivial inertia groups of the image of \( U \) in \( \check{\Gamma}(S \setminus \gamma) \) are parameterized by the subcomplex \( \text{Link}(\gamma) \cong \check{C}(S \setminus \gamma) \subset \check{C}(S) \) (cf. Remark 4.7 in [3]).

We then proceed as in the proof of Theorem 7.4 in [3]. Let us choose an orientation for every element of \( \check{\Gamma}(S) \) (cf. Section 4 in [3]). By definition, an element \( f \in \check{\Gamma}(S)_{\gamma} \) preserves the orientation chosen for \( \gamma \) if and only if \( f \in \check{\Gamma}(S)_{\gamma}. \)

For \( f \in \text{Aut}^\dagger(\text{P}\check{\Gamma}(S)) \), by item (ii) of Remarks 3.6, there is an element \( x \in \text{P}\check{\Gamma}(S) \) such that \( \text{inn} x \circ f \) preserves the inertia group \( I_{\gamma} \) and, by (i) of Corollary 4.12 in [7], also the decomposition group \( \text{P}\check{\Gamma}(S)_{\gamma}. \) Let us then observe that \( \text{P}\check{\Gamma}(S)_{\gamma} \) is a \( \mathbb{L} \)-characteristic subgroup of \( \text{P}\check{\Gamma}(S)_{\gamma}, \) because it is (topologically) generated by the profinite Dehn twists contained in \( \text{P}\check{\Gamma}(S)_{\gamma}. \) Hence, by restriction, \( \text{inn} x \circ f \) induces an automorphism of \( \text{P}\check{\Gamma}(S)_{\gamma}. \)

Since there is an element of \( \text{P}\check{\Gamma}(S)_{\gamma} \) which reverses the orientation of \( \gamma, \) we can choose the element \( x \in \text{P}\check{\Gamma}(S) \) in a way that it sends the orientation fixed on \( \gamma \) to the orientation fixed on \( x(\gamma). \) If \( y \in \text{P}\check{\Gamma}(S) \) is another such element, then \( xy^{-1} \in \text{P}\check{\Gamma}(S)_{\gamma}. \) In this way, the automorphism \( \text{inn} x \circ f \) is determined by \( f \) modulo an inner automorphism of \( \text{P}\check{\Gamma}(S)_{\gamma}, \) so that we get a natural homomorphism:

\[ \text{PR}_{\gamma}: \text{Aut}^\dagger(\text{P}\check{\Gamma}(S)) \rightarrow \text{Out}^\dagger(\text{P}\check{\Gamma}(S)_{\gamma}). \]
Since \( Z(\hat{\Gamma}(S)_{\tilde{\gamma}}) = \tau_{\tilde{\gamma}}^2 \) is a characteristic subgroup of \( \hat{\Gamma}(S)_{\tilde{\gamma}} \), there is also a natural homomorphism \( \text{Out}^1(\hat{\Gamma}(S)_{\tilde{\gamma}}) \to \text{Out}^1(\hat{\Gamma}(S)_{\tilde{\gamma}}/\tau_{\tilde{\gamma}}^2) \cong \text{Out}^1(\hat{\Gamma}(S \setminus \gamma)) \). Composing with the homomorphism (7), we get the natural homomorphism:

\[
PR_{\gamma}: \text{Aut}^1(\hat{\Gamma}(S)) \to \text{Out}^1(\hat{\Gamma}(S \setminus \gamma)).
\]

**Remark 5.2.** Note that (10) and (11) (again by Corollary 4.3) a homomorphism:

\[
\tau_{\delta}^2: \text{Out}^1(\hat{\Gamma}(S)) \to \text{Out}^1(\hat{\Gamma}(S \setminus \gamma)).
\]

Let \( \delta \) be a simple closed curve on \( S \) bounding a 2-punctured closed disc \( D_\delta \). There is then a short exact sequence (cf. (ii) of Theorem 2.3):

\[
1 \to \tau_{\delta}^2 \to \hat{\Gamma}(S)_{\delta} \to \hat{\Gamma}(S \setminus D_\delta) \to 1.
\]

For \( S \neq S_{1,2} \), we have that \( Z(\hat{\Gamma}(S)_{\delta}) = \tau_{\delta}^2 \), otherwise, the center of \( \hat{\Gamma}(S)_{\delta} \) is generated by \( \tau_{\delta}^2 \) and a lift of the hyperelliptic involution \( \iota \in \hat{\Gamma}(S \setminus D_\delta) \) (cf. Corollary 6.2 in [3]).

The same construction used above for a nonseparating curve \( \gamma \) (except that we do not need orientations) shows that there is a natural homomorphism:

\[
P\tilde{R}_{\delta}: \text{Aut}^1(\hat{\Gamma}(S)) \to \text{Out}^1(\hat{\Gamma}(S)_{\delta}),
\]

whose kernel contains \( \text{Inn}(\hat{\Gamma}(S)) \) and thus descends to a homomorphism:

\[
PR_{\delta}: \text{Out}^1(\hat{\Gamma}(S)) \to \text{Out}^1(\hat{\Gamma}(S)_{\delta}).
\]

Since \( \text{Out}^1(\hat{\Gamma}(S)_{\delta}) \) preserves the central subgroup \( \tau_{\delta}^2 \), we also get a homomorphism:

\[
PR_{D_\delta}: \text{Out}^1(\hat{\Gamma}(S)) \to \text{Out}^1(\hat{\Gamma}(S \setminus D_\delta)),
\]

and (again by Corollary 4.3) a homomorphism:

\[
R_{D_\delta}: \text{Out}^1(\hat{\Gamma}(S)) \to \text{Out}^1(\hat{\Gamma}(S \setminus D_\delta)).
\]

For \( d(S) > 1 \) and \( \gamma \) a simple closed curve on \( S \) which is either nonseparating or bounds a 2-punctured disc, \( \text{Inn} \hat{\Gamma}(S)_{\tilde{\gamma}} \) acts trivially on the central subgroup \( \hat{\Gamma}_{\tilde{\gamma}} \). So, the homomorphism \( P\tilde{R}_{\gamma} \) (cf. (7) and (9)) induces a representation \( \tilde{X}(S)_{\gamma}: \text{Aut}^1(\hat{\Gamma}(S)) \to \text{Aut}(\hat{\Gamma}_{\tilde{\gamma}}) \) such that the kernel of \( \tilde{X}(S)_{\gamma} \) contains the normal subgroup \( \text{Inn} \hat{\Gamma}(S) \). Hence, by Theorem 4.3, we also get a natural character which only depends on the topological type of \( \gamma \):

\[
X(S)_{\gamma}: \text{Out}^1(\hat{\Gamma}(S)) \to \text{Aut}(\hat{\Gamma}_{\tilde{\gamma}}).
\]
The following theorem is one of the key results of the paper:

**Theorem 5.3.** Let us assume that \( d(S) > 1 \). With the above definitions, we have:

(i) For \( g(S) \geq 1 \), let \( \gamma \) be a nonseparating simple closed curve on \( S \). Then, there is a natural monomorphism:

\[
R_\gamma : \text{Out}^\sharp(\hat{\Gamma}(S)) \hookrightarrow \text{Out}^\sharp(\hat{\Gamma}(S \setminus \gamma)).
\]

(ii) For \( n(S) \geq 2 \), let \( \delta \) be a simple closed curve bounding a 2-punctured closed disc \( D_\delta \) on \( S \). Then, there is a natural monomorphism:

\[
R_{D_\delta} : \text{Out}^\sharp(\hat{\Gamma}(S)) \hookrightarrow \text{Out}^\sharp(\hat{\Gamma}(S \setminus D_\delta)).
\]

For the proof of Theorem 5.3, we need a series of lemmas.

**Lemma 5.4.** For \( g(S) \geq 1 \), let \( \gamma \) be a nonseparating simple closed curve on \( S \). Then, the group \( \text{Inn}(\hat{\Gamma}(S)_\gamma) \) identifies with a subgroup of \( \text{Aut}^\sharp(\hat{\Gamma}(S)_{\hat{\gamma}}) \) and there is a natural homomorphism:

\[
\text{Aut}^\sharp(\hat{\Gamma}(S)_{\hat{\gamma}}) \to \text{Out}^\sharp(\hat{\Gamma}(S \setminus \gamma)) \times \text{Aut}(\hat{I}_\gamma)
\]

with kernel \( \text{Inn}(\hat{\Gamma}(S)_\gamma) \).

**Proof.** The first statement is clear since \( \hat{\Gamma}(S)_{\hat{\gamma}} \) is a normal subgroup of \( \hat{\Gamma}(S)_\gamma \).

By Lemma 7.4 in [7], the short exact sequence \( 1 \to \hat{I}_\gamma \to \hat{\Gamma}(S)_{\hat{\gamma}} \to \hat{\Gamma}(S \setminus \gamma) \to 1 \) determines an exact sequence:

\[
1 \to \text{Hom}(\hat{\Gamma}(S \setminus \gamma), \hat{I}_\gamma) \to \text{Aut}(\hat{\Gamma}(S)_{\hat{\gamma}}) \to \text{Aut}(\hat{\Gamma}(S \setminus \gamma)) \times \text{Aut}(\hat{I}_\gamma).
\]

By the explicit description of the image of \( \text{Hom}(\hat{\Gamma}(S \setminus \gamma), \hat{I}_\gamma) \) in \( \text{Aut}(\hat{\Gamma}(S)_{\hat{\gamma}}) \) given in the proof of Lemma 7.4 in [7], we see that \( \text{Aut}^\sharp(\hat{\Gamma}(S)_{\hat{\gamma}}) \) has trivial intersection with it. Hence, the natural homomorphism \( \text{Aut}(\hat{\Gamma}(S)_{\hat{\gamma}}) \to \text{Aut}(\hat{\Gamma}(S \setminus \gamma)) \times \text{Aut}(\hat{I}_\gamma) \) restricts to a monomorphism \( \text{Aut}^\sharp(\hat{\Gamma}(S)_{\hat{\gamma}}) \to \text{Aut}^\sharp(\hat{\Gamma}(S \setminus \gamma)) \times \text{Aut}(\hat{I}_\gamma) \).

Since \( \text{Inn}(\hat{\Gamma}(S)_{\hat{\gamma}}) \cong \text{Inn}(\hat{\Gamma}(S \setminus \gamma)) \) and \( \hat{I}_\gamma \) is a central subgroup, there is then a natural monomorphism:

\[
\text{Out}^\sharp(\hat{\Gamma}(S)_{\hat{\gamma}}) \hookrightarrow \text{Out}^\sharp(\hat{\Gamma}(S \setminus \gamma)) \times \text{Aut}(\hat{I}_\gamma).
\]

By (i) of Theorem 4.3, there is a natural epimorphisms \( \text{Out}^\sharp(\hat{\Gamma}(S \setminus \gamma)) \to \text{Out}^\sharp(\hat{\Gamma}(S \setminus \gamma)) \) with kernel \( \text{Inn}(\hat{\Gamma}(S \setminus \gamma))/\text{Inn}(\hat{\Gamma}(S \setminus \gamma)) \).

For \( n(S) = 0 \), the image of \( \text{Inn}(\hat{\Gamma}(S)_\gamma)/\text{Inn}(\hat{\Gamma}(S)_{\hat{\gamma}}) \) in \( \text{Out}^\sharp(\hat{\Gamma}(S \setminus \gamma)) \) identifies with this kernel, which implies the lemma in this case.

For \( n(S) \geq 1 \), by (ii) of Theorem 4.3, there is a natural isomorphism:

\[
\text{Out}^\sharp(\hat{\Gamma}(S \setminus \gamma)) \cong \Sigma_{n+2} \times \text{Out}^\sharp(\hat{\Gamma}(S \setminus \gamma)).
\]

We claim that, via this isomorphism, the image of \( \text{Out}^\sharp(\hat{\Gamma}(S)_{\hat{\gamma}}) \) in \( \text{Out}^\sharp(\hat{\Gamma}(S \setminus \gamma)) \) intersects \( \Sigma_{n+2} \) in the stabilizer \( \Sigma_n \times \Sigma_2 \) of the partition of the \( n+2 \) punctures of \( S \setminus \gamma \) into those which have no boundary in \( S \) and those bounded by \( \gamma \). This claim implies the lemma, since this stabilizer identifies with the image of \( \text{Inn}(\hat{\Gamma}(S)_\gamma) \) in \( \text{Out}^\sharp(\hat{\Gamma}(S \setminus \gamma)) \).
By Lemma 4.6, the homomorphism \( \text{Out}^i(\hat{\Gamma}(S)_{\gamma}) \to \Sigma_{n+2} \) is determined by the natural homomorphism \( \text{Aut}^i(\hat{\Gamma}(S)_{\gamma}) \to \text{Aut}^i(\hat{\Gamma}(S \setminus \gamma)) \) and the inner action of \( \text{Aut}^i(\hat{\Gamma}(S \setminus \gamma)) \) on \( \text{Inn} \hat{\Gamma}(S \setminus \gamma) \). An element of \( \text{Aut}^i(\hat{\Gamma}(S)_{\gamma}) \), modulo \( \text{Inn} \hat{\Gamma}(S)_{\gamma} \), fixes any Dehn twist about a simple closed curve bounding an unpunctured genus 1 subsurface of \( S \) which contains \( \gamma \). Therefore, an element of \( \text{Aut}^i(\hat{\Gamma}(S \setminus \gamma)) \) in the image of \( \text{Out}^i(\hat{\Gamma}(S)_{\gamma}) \), modulo \( \text{Inn} \hat{\Gamma}(S \setminus \gamma) \), fixes any Dehn twist about a simple closed curve bounding a disc containing only the two punctures of \( S \setminus \gamma \) which are bounded by \( \gamma \). In particular, by the proof of (ii) of Proposition 3.5, the inner action of such an element on \( \text{Inn} \hat{\Gamma}(S \setminus \gamma) \) preserves the braid twist about the same curve, which proves the claim above. 

\begin{lemma}
For \( d(S) > 1 \), \( g(S) \geq 0 \), \( n(S) \geq 2 \) and \( \delta \) a simple closed curve bounding a 2-punctured closed disc \( D_{\delta} \) on \( S \), the group \( \text{Inn}(\hat{\Gamma}(S)_{\delta}) \) identifies with a subgroup of \( \text{Aut}^i(\hat{\Gamma}(S)_{\delta}) \) and there is a natural homomorphism:
\[
\text{Aut}^i(\hat{\Gamma}(S)_{\delta}) \to \text{Out}^i(\hat{\Gamma}(S \setminus D_{\delta})) \times \text{Aut}(\hat{I}_{\delta})
\]
with kernel \( \text{Inn}(\hat{\Gamma}(S)_{\delta}) \).
\end{lemma}

\begin{proof}
The proof is essentially the same as that of Lemma 5.4 but simpler.
\end{proof}

\begin{lemma}
For \( d(S) > 1 \), suppose that one of the following hypotheses is satisfied:
\begin{enumerate}
\item \( g(S) \geq 1 \) and \( \gamma \) is a nonseparating simple closed curve on \( S \);
\item \( n(S) \geq 2 \) and \( \gamma \) is a simple closed curve on \( S \) bounding a 2-punctured disc.
\end{enumerate}

Then, the kernel of the natural homomorphism:
\[
\text{Aut}^i(\hat{\Gamma}(S)) \to \text{Aut}^i(\hat{\Gamma}(S)_{\gamma})/\text{Inn}(\hat{\Gamma}(S)_{\gamma})
\]
is \( \text{Inn}(\hat{\Gamma}(S)) \).
\end{lemma}

\begin{proof}
Let \( f \in \ker(\text{Aut}^i(\hat{\Gamma}(S)) \to \text{Aut}^i(\hat{\Gamma}(S)_{\gamma})/\text{Inn}(\hat{\Gamma}(S)_{\gamma})) \). After composing \( f \) with an element of \( \text{Inn}(\hat{\Gamma}(S)) \), we can assume that \( f \in \text{Aut}^i(\hat{\Gamma}(S)_{\gamma}) \). Composing again by an element of \( \text{Inn}(\hat{\Gamma}(S)_{\gamma}) \subset \text{Inn}(\hat{\Gamma}(S)) \), we can then also assume that \( f \) restricts to the identity on the subgroup \( \hat{\Gamma}(S)_{\gamma} \).

Since, for every \( \sigma \in \hat{C}(S)_{d(S)-2} \) such that \( \gamma \in \sigma \), the vertices of the profinite Farey subgraph \( \hat{\Gamma}_\sigma \subset \hat{C}_P(S) \) identify with subgroups of the stabilizer \( \hat{\Gamma}(S)_{\gamma} \), the action of \( f \) on the vertices of the procongruence pants complex \( \hat{C}_P(S) \), induced by the natural representation \( \text{Aut}^i(\hat{\Gamma}(S)) \to \text{Aut}(\hat{C}_T(S)) \), trivially extends to the identity map on the vertices of all profinite Farey subgraphs \( \hat{\Gamma}_\sigma \subset \hat{C}_P(S) \) such that \( \gamma \in \sigma \).

Since \( d(S) > 1 \), for \( \gamma \) nonseparating, we have that \( d(S \setminus \gamma) \geq 1 \) (resp. \( d(S \setminus D_{\gamma}) \geq 1 \) for \( \gamma \) separating). Therefore, the set of profinite Farey subgraphs \( \hat{\Gamma}_\sigma \subset \hat{C}_P(S) \) such that \( \gamma \in \sigma \) is non-empty and, from Lemma 4.4, it follows that \( f \in \text{Inn}(\hat{\Gamma}(S)) \).
\end{proof}

\begin{proof}
of Theorem 5.3. (i): By Lemma 5.4 and (i) of Lemma 5.6, the kernel of the homomorphism: \( (\hat{R}_{\gamma}, X(S)_{\gamma}) : \text{Out}^1(\hat{\Gamma}(S)) \to \text{Out}^1(\hat{\Gamma}(S \setminus \gamma)) \times \text{Aut}(\hat{I}_{\gamma}) \) is \( \text{Inn}(\hat{\Gamma}(S)) \), so that, by Theorem 4.3, the natural homomorphism:
\[
(\hat{R}_{\gamma}, X(S)_{\gamma}) : \text{Out}^1(\hat{\Gamma}(S)) \to \text{Out}^1(\hat{\Gamma}(S \setminus \gamma)) \times \text{Aut}(\hat{I}_{\gamma})
\]
is injective.

For $g(S) \geq 2$, the conclusion of the theorem then follows observing that the character \((12)\) $X(S)_{\gamma} : \text{Out}^1(\hat{\Gamma}(S)) \to \text{Aut}(\hat{I}_\gamma)$ has the same kernel of the composition of the homomorphism $R_{\gamma} : \text{Out}^1(\hat{\Gamma}(S)) \to \text{Out}^1(\hat{\Gamma}(S \setminus \gamma))$ with $X(S \setminus \gamma)_{\gamma} : \text{Out}^1(\hat{\Gamma}(S \setminus \gamma)) \to \text{Aut}(\hat{I}_\gamma)$, where $\gamma'$ is any nonseparating simple closed curve on $S \setminus \gamma$.

For $g(S) = 1$ and $n(S) = 2$, through the natural isomorphism $\text{Out}^1(\hat{\Gamma}(S)) \cong \hat{\Gamma}T$ (cf. (i) of Proposition 3.9), the character \((12)\) $X(S)_{\gamma} : \text{Out}^1(\hat{\Gamma}(S)) \to \text{Aut}(\hat{I}_\gamma)$ identifies with the natural character $\chi : \hat{\Gamma}T \to \hat{\mathbb{Z}}^s$ on the Grothendieck-Teichmüller group defined by the assignment $(f, \lambda) \mapsto \lambda$. The latter clearly factors through the homomorphism $R_{\gamma} : \text{Out}^1(\hat{\Gamma}(S)) \to \text{Out}^1(\hat{\Gamma}(S \setminus \gamma))$ and the natural isomorphism $\text{Out}^1(\hat{\Gamma}(S \setminus \gamma)) \cong \hat{\Gamma}T$ (cf. Proposition 3.8) and the conclusion follows as above.

For $g(S) = 1$ and $n(S) > 2$, through a series of natural homomorphisms of type \((11)\), we can reduce to the case $g(S) = 1$ and $n(S) = 2$ treated above.

(ii): From Lemma 5.5 and (ii) of Lemma 5.6, it follows that the homomorphism:
\[(R_\delta, X(S)_\delta) : \text{Out}^1(\hat{\Gamma}(S)) \to \text{Out}^1(\hat{\Gamma}(S \setminus D_\delta)) \times \text{Aut}(\hat{I}_\delta)\]
is injective. The conclusion of the theorem then follows from a similar argument as above. \hfill \Box

5.2. Injective functorial homomorphisms. There is a third natural homomorphism between outer $\mathbb{I}$-automorphism groups of procongruence mapping class groups which we need to consider. For $S$ a closed hyperbolic surface of genus $\geq 2$ and $P \in S$, let us consider the associated procongruence Birman short exact sequence (cf. Corollary 4.7 in \cite{5}):
\[1 \to \hat{\pi}_1(S, P) \to \hat{\Gamma}(S \setminus P) \to \hat{\Gamma}(S) \to 1,\]
where a simple generator $\gamma$ of $\pi_1(S, P) \subset \hat{\pi}_1(S, P)$ is sent by the map $\hat{\pi}_1(S, P) \to \hat{\Gamma}(S \setminus P)$ to the bounding pair map $\tau_{\gamma_1} \tau_{\gamma_2}^{-1}$, where $\gamma_1$ and $\gamma_2$ are the boundary components of a tubular neighborhood of $\gamma$ in $S$. In particular, the group $\hat{\pi}_1(S, P)$ is (topologically) generated in $\hat{\Gamma}(S \setminus P)$ by such bounding pair maps.

Since, by Theorem 5.5 in \cite{7}, the action of $\text{Aut}^1(\hat{\Gamma}(S \setminus P))$ on the procongruence curve complex $\hat{C}(S \setminus P)$ preserves topological types of simplices, it follows, in particular, that the action of $\text{Aut}^1(\hat{\Gamma}(S \setminus P))$ on $\hat{\Gamma}(S \setminus P)$ preserves the topological type of bounding pair maps associated to annuli on $S$ containing the puncture and so it preserves the image of $\hat{\pi}_1(S, P)$ in $\hat{\Gamma}(S \setminus P)$. An element of $\text{Aut}^1(\hat{\Gamma}(S \setminus P))$ then induces an automorphism on the quotient $\hat{\Gamma}(S \setminus P)/\hat{\pi}_1(S, P) \cong \hat{\Gamma}(S)$.

Therefore, there is a natural homomorphism $\tilde{B}_P : \text{Aut}^1(\hat{\Gamma}(S \setminus P)) \to \text{Aut}^1(\hat{\Gamma}(S))$ and then a natural homomorphism:
\[(14) \quad B_P : \text{Out}^1(\hat{\Gamma}(S \setminus P)) \to \text{Out}^1(\hat{\Gamma}(S)).\]

In conclusion, we have constructed three types of natural homomorphisms between outer $\mathbb{I}$-automorphism groups of procongruence mapping class groups:

(i) For $g(S) \geq 1$, a homomorphism $R_\gamma : \text{Out}^1(\hat{\Gamma}(S)) \to \text{Out}^1(\hat{\Gamma}(S \setminus \gamma))$.  

(ii) For \(n(S) \geq 2\), a homomorphism \(R_{D_4} \colon \text{Out}^1(\hat{\Gamma}(S)) \to \text{Out}^2(\hat{\Gamma}(S \setminus D_3))\).

(iii) For a closed surface, a homomorphism \(B_P \colon \text{Out}^2(\hat{\Gamma}(S \setminus P)) \to \text{Out}^3(\hat{\Gamma}(S))\).

Let \(S\) and \(S'\) be surfaces such that \(g(S) \geq 2\) and \(\chi(S) \leq \chi(S') < 0\). Cutting \(S\) along \(g(S) - g(S')\) nonseparating simple closed curves and then filling in \(\chi(S') - \chi(S)\) punctures, we obtain a surface homeomorphic to \(S'\). Composing \(g(S) - g(S')\) maps of type (ii) with \(\chi(S') - \chi(S)\) maps of type (iii), if \(n(S') > 0\), or \(\chi(S') - \chi(S) - 1\) maps of type (ii) and a map of type (iii), if \(n(S') = 0\), we obtain a homomorphism:

\[\mu_{S,S'} \colon \text{Out}^1(\hat{\Gamma}(S)) \to \text{Out}^1(\hat{\Gamma}(S')).\]

It is not difficult to check that changing the order of the compositions between the maps of the various types gives the same homomorphism and so also that the collection of natural homomorphisms \(\{\mu_{S,S'}\}\) thus obtained is functorial in the sense that, given surfaces \(S, S'\) and \(S''\) such that \(g(S) \geq g(S') \geq g(S'')\) and \(\chi(S) \leq \chi(S') \leq \chi(S'') < 0\), there holds

\[\mu_{S,S''} = \mu_{S',S''} \circ \mu_{S,S'}\]

By Theorem 5.3, Corollary 4.2 and the above discussion, we have:

**Theorem 5.7.** Let \(S\) and \(S'\) be surfaces such that \(g(S) \geq g(S')\), \(\chi(S) \leq \chi(S')\) and \(d(S') \geq 1\). Then, there is a natural and functorial monomorphism:

\[(15) \quad \mu_{S,S'} \colon \text{Out}^{1o}(\hat{\Gamma}(S)) \hookrightarrow \text{Out}^{1o}(\hat{\Gamma}(S')).\]

**Proof.** We only have to show that the homomorphism \(B_P \colon \text{Out}^1(\hat{\Gamma}(S \setminus P)) \to \text{Out}^2(\hat{\Gamma}(S))\) is injective. This follows from the fact that, for \(\gamma\) a nonseparating simple closed curve on \(S \setminus P\), the composition of \(B_P\) with the map \(R_\gamma \colon \text{Out}^2(\hat{\Gamma}(S)) \to \text{Out}^3(\hat{\Gamma}(S \setminus \gamma))\) equals the composition of the map \(R_\delta \colon \text{Out}^1(\hat{\Gamma}(S \setminus P)) \to \text{Out}^2(\hat{\Gamma}((S \setminus P) \setminus \gamma))\) with the map \(R_{D_4} \colon \text{Out}^1(\hat{\Gamma}((S \setminus P) \setminus \gamma)) \to \text{Out}^1(\hat{\Gamma}(S \setminus \gamma))\), where \(\delta\) is a simple closed curve on \((S \setminus P) \setminus \gamma\) bounding a disc containing two punctures, and we already know (cf. Theorem 5.3) that the latter two homomorphisms are injective. \(\square\)

6. **Procongruence Grothendieck-Teichmüller theory**

6.1. **The automorphism group of the procongruence Grothendieck-Teichmüller tower.** In the landmark paper [8], Drinfeld, following Grothendieck’s *Esquisse d’un Programme* [11], speculated that the profinite Grothendieck-Teichmüller group, which is defined taking into account only the first two levels of the genus 0 stage of the profinite Grothendieck-Teichmüller tower, naturally acts on the full tower and that, moreover, this action induces an isomorphism:

**Conjecture 6.1** (Drinfeld-Grothendieck). \(\hat{\Gamma} \cong \text{Aut}(\hat{\mathcal{F}}^\text{out})\).

Combining Conjecture 6.1 with Conjecture 2.10 yields the Grothendieck-Teichmüller conjecture:

**Conjecture 6.2.** \(\hat{\Gamma} \cong G_{\mathbb{Q}}\).

In this section, we will prove the procongruence version of Conjecture 6.1:

**Theorem 6.3.** There is a natural isomorphism \(\hat{\Gamma} \cong \text{Aut}(\hat{\mathcal{F}}^\text{out})\).
Of course, assuming the congruence subgroup property, the profinite and procongruence versions of the Drinfeld-Grothendieck conjecture are just equivalent. Even if this is still an open problem, the next result shows that, in any case, there are no additional restrictions coming from the profinite Grothendieck-Teichmüller tower:

**Theorem 6.4.** There is a natural faithful representation $\hat{\rho}_{\Gamma_T}: \hat{\Gamma}_T \to \text{Aut}(\hat{\mathcal{F}}^\text{out})$. In particular, Conjecture 2.10 implies Conjecture 6.2.

We will deduce Theorem 6.3 and Theorem 6.4 from somewhat stronger results. The first one is a procongruence analogue of a theorem by Ivanov (see the introduction). Let us recall that, by definition, the profinite Grothendieck-Teichmüller group comes with a natural character $\chi_\lambda: \hat{\Gamma}_T \to \hat{\mathbb{Z}}^*$ defined by the assignment $(f, \lambda) \mapsto \lambda$ and such that its composition with the embedding $G_Q \subseteq \hat{\Gamma}_T$ is the standard cyclotomic character. We have:

**Theorem 6.5.** Let $S = S_{g,n}^k$ be a hyperbolic surface with $d(S) > 1$:

(i) There is a natural isomorphism:

\[
\text{Out}^\Sigma_0(\hat{\Gamma}(S)) \cong \hat{\Gamma}_T.
\]

(ii) For $(g, n + k) \neq (1, 2)$, there is a natural isomorphism:

\[
\text{Out}^\Sigma_0(\hat{\Gamma}(S)) \cong \Sigma_{n+k} \times \hat{\Gamma}_T.
\]

(iii) There is a natural isomorphism:

\[
\text{Out}^\Sigma(\hat{\Gamma}(S)) \cong \hat{\Gamma}_T.
\]

(iv) The outer action of $\hat{\Gamma}_T$ on $\hat{\Gamma}(S, \partial S)$ of item (iii) restricts to a genuine action on the normal inertia group $\prod_{i=1}^k \hat{\tau}_{\delta_i}$ with the property that each procyclic subgroup $\tau_{\delta_i}$ is preserved and acted upon through the character $\chi_\lambda: \hat{\Gamma}_T \to \hat{\mathbb{Z}}^*$.

(v) For $S \neq S_{1,2}$, there is a natural isomorphism:

\[
\text{Out}^\Sigma(\hat{\Gamma}(S, \partial S)) \cong \Sigma_n \times \Sigma_k \times \hat{\Gamma}_T.
\]

**Remark 6.6.** In the above as in the next statements, "natural" means that the isomorphisms are compatible with the homomorphisms $\mu_{S,S'}$ introduced in Section 5.2.

6.2. **The proof of Theorem 6.5.** By (ii) of Theorem 4.3 (cf. also Theorem 3.12 and Corollary 4.2) and (ii) of Theorem 4.15, we have that (i)$\Rightarrow$(ii) and (iii)$\Rightarrow$(v). Therefore, it is enough to prove items (i), (iii) and (iv) of Theorem 6.5. For the proof, we will need the fundamental lemma:

**Lemma 6.7.** For a hyperbolic surface $S$, there is a natural representation:

\[
\Psi_{(S,\partial S)}: \hat{\Gamma}_T \to \text{Out}^\Sigma(\hat{\Gamma}(S, \partial S)).
\]

In the next two subsections, we will show how Theorem 6.5 follows from Lemma 6.7.
6.3. **Lemma 6.7 implies (i) of Theorem 6.5.** Let us assume first that $S$ has empty boundary. By Theorem 5.7, for $d(S) > 1$, there is then a natural monomorphism:

$$
\mu_{S,S_{0,5}} : \text{Out}^{{}^0}(\hat{\Gamma}(S)) \hookrightarrow \text{Out}^{{}^0}(\hat{\Gamma}(S_{0,5})) = \widehat{\Gamma T}.
$$

By Lemma 6.7 and Corollary 4.2, there is a homomorphism $\Psi_{(S,\emptyset)} : \widehat{\Gamma T} \rightarrow \text{Out}^{{}^0}(\hat{\Gamma}(S))$ so that we get a series of natural homomorphisms:

$$
\widehat{\Gamma T} \rightarrow \text{Out}^{{}^0}(\hat{\Gamma}(S)) \hookleftarrow \widehat{\Gamma T},
$$

which proves that $\mu_{S,S_{0,5}}$ is surjective and then (i) of Theorem 6.5, for $S = \hat{S}$, follows.

Let now $S = S_{g,n}^k$, with $k \geq 1$. By Proposition 3.9 and Corollary 4.2, we already know that $\text{Out}^{{}^0}(\hat{\Gamma}(S_{1}^2)) \cong \text{Out}^{{}^0}(\hat{\Gamma}(S_{1,1}^1)) \cong \widehat{\Gamma T}$, so that we can assume $(g,n+k) \neq (1,2)$. Let us also observe that, since $\hat{P}\Gamma(S) \cong \hat{P}\Gamma(\hat{S})$, by the case $S = \hat{S}$, just proved, of (i) of Theorem 6.5 and (ii) of Theorem 4.3, we have that $\text{Out}^{{}^0}(\hat{P}\Gamma(S)) \cong \Sigma_{n+k} \times \widehat{\Gamma T}$. Let us then consider the short exact sequence:

$$
1 \rightarrow \hat{P}\Gamma(S) \rightarrow \hat{\Gamma}(S) \rightarrow \Sigma_n \times \Sigma_k \rightarrow 1.
$$

By the above assumptions, we have that $Z(P\Gamma(S)) = Z(\hat{\Gamma}(S)) = \{1\}$ and, for $n,k \neq 2$, there also holds $Z(\Sigma_n \times \Sigma_k) = \{1\}$. Therefore, from Lemma 4.2 in [21] and Lemma 4.6, it follows that there is a natural isomorphism:

$$
\text{Out}^{{}^0}(\hat{\Gamma}(S)) \cong Z_{\text{Out}^{{}^0}(P\Gamma(S))}(\text{Inn}\hat{\Gamma}(S)/\text{Inn}\hat{P}\Gamma(S)),
$$

where $\text{Inn}\hat{\Gamma}(S)/\text{Inn}\hat{P}\Gamma(S)$ is identified, via the isomorphism $\text{Out}^{{}^0}(\hat{P}\Gamma(S)) \cong \Sigma_{n+k} \times \widehat{\Gamma T}$, with the subgroup $\Sigma_n \times \Sigma_k$ of $\Sigma_{n+k}$. It then follows that:

$$
Z_{\text{Out}^{{}^0}(P\Gamma(S))}(\text{Inn}\hat{\Gamma}(S)/\text{Inn}\hat{P}\Gamma(S)) \cong \widehat{\Gamma T},
$$

proving that $\text{Out}^{{}^0}(\hat{\Gamma}(S)) \cong \widehat{\Gamma T}$, for $n,k \neq 2$.

For $n = 2$ and $k \neq 2$, an argument similar to the proof of Lemma 4.2 in [21] and Lemma 4.6 imply that there is a short exact sequence:

$$
1 \rightarrow \Sigma_2 \rightarrow Z_{\text{Out}^{{}^0}(P\Gamma(S))}(\text{Inn}\hat{\Gamma}(S)/\text{Inn}\hat{P}\Gamma(S)) \rightarrow \text{Out}^{{}^0}(\hat{\Gamma}(S)) \rightarrow 1.
$$

But in this case we also have that $Z_{\text{Out}^{{}^0}(P\Gamma(S))}(\text{Inn}\hat{\Gamma}(S)/\text{Inn}\hat{P}\Gamma(S)) \cong \Sigma_2 \times \widehat{\Gamma T}$ with the isomorphism identifying the image of $\Sigma_2$ in the above short exact sequence with the copy of $\Sigma_2$ appearing on the left hand side of the isomorphism. Therefore, we conclude that we have $\text{Out}^{{}^0}(\hat{\Gamma}(S)) \cong \widehat{\Gamma T}$ also in this case.

For $n \neq 2$ and $k = 2$, we argue exactly in the same way. For $n = k = 2$, we get a short exact sequence:

$$
1 \rightarrow \Sigma_2 \times \Sigma_2 \rightarrow Z_{\text{Out}^{{}^0}(P\Gamma(S))}(\text{Inn}\hat{\Gamma}(S)/\text{Inn}\hat{P}\Gamma(S)) \rightarrow \text{Out}^{{}^0}(\hat{\Gamma}(S)) \rightarrow 1,
$$

and an isomorphism $Z_{\text{Out}^{{}^0}(P\Gamma(S))}(\text{Inn}\hat{\Gamma}(S)/\text{Inn}\hat{P}\Gamma(S)) \cong \Sigma_2 \times \Sigma_2 \times \widehat{\Gamma T}$, so that we again conclude that $\text{Out}^{{}^0}(\hat{\Gamma}(S)) \cong \Gamma T$. 

6.4. **Lemma 6.7 implies (iii) and (iv) of Theorem 6.5.** Let us first observe that Theorem A in [21] and its proof imply the following special case of items (iii) and (iv) of Theorem 6.5:

**Lemma 6.8.** For \( S = S^1_{0,n} \) with \( n \geq 4 \), there is a natural isomorphism:

\[
\Out^1(\tilde{\Gamma}(S, \partial S)) \cong \widehat{\Gamma}T,
\]

such that \( \widehat{\Gamma}T \) acts on the center \( \tau_{\delta_i}^2 \) of \( \tilde{\Gamma}(S, \partial S) \) through the character \( \chi_\lambda : \widehat{\Gamma}T \to \mathbb{Z}^* \).

**Proof.** Note that, in the notation of [21], we have \( \tilde{\Gamma}(S, \partial S) = \hat{B}_n \) and \( \tilde{\Gamma}(S) = \hat{B}_n \). Since \( \tau_{\delta_i}^2 \) is a characteristic subgroup of \( \tilde{\Gamma}(S, \partial S) \), the natural epimorphism \( \tilde{\Gamma}(S, \partial S) \to \tilde{\Gamma}(S) \) induces a homomorphism \( \Aut(\tilde{\Gamma}(S, \partial S)) \to \Aut(\tilde{\Gamma}(S)) \) which, by Theorem 4.6 in [21], is surjective and whose kernel, as described in Lemma 4.5 of [21], has trivial intersection with \( \Aut^\delta(\tilde{\Gamma}(S, \partial S)) \). Therefore, there is a natural isomorphism \( \Aut^1(\tilde{\Gamma}(S, \partial S)) \cong \Aut^1(\tilde{\Gamma}(S)) \).

Since, by Proposition 2.2 and Proposition 3.1 in [21], \( \tilde{\Gamma}(S) \) is a characteristic subgroup of \( \tilde{\Gamma}(S) \), there is also a natural homomorphism \( \Aut(\tilde{\Gamma}(S)) \to \Aut(\tilde{\Gamma}(S)) \) which, since the centralizer of \( \tilde{\Gamma}(S) \) in \( \tilde{\Gamma}(S) \) is trivial (cf. for instance, Theorem 4.14 in [7]), is injective.

By Lemma 3.13, we have that \( \Aut^1(\tilde{\Gamma}(S)) = \Aut(\tilde{\Gamma}(S)) \) which then implies that we have \( \Aut^\delta(\tilde{\Gamma}(S)) = \Aut(\tilde{\Gamma}(S)) \) as well. By (i) of Theorem 4.3 in [21], there is a natural isomorphism \( \Out(\tilde{\Gamma}(S)) \cong \Gamma T \). Combined with the previous isomorphisms, this yields the isomorphism, claimed in the lemma: \( \Out^\delta(\tilde{\Gamma}(S, \partial S)) \cong \widehat{\Gamma}T \).

The last claim of the lemma then follows from the explicit description of this action given, for instance, in the appendix of [12]. \( \square \)

Let us now observe that, given a nonseparating simple closed curve \( \gamma \) on \( S \), by the same construction of the homomorphism (8), where we use Theorem 2.7 and Theorem 2.8, instead of Theorem 2.3 and Corollary 6.2 in [3], and Theorem 4.15 instead of Theorem 4.3, we get a natural homomorphism:

\[
R_\gamma : \Out^1(\tilde{\Gamma}(S, \partial S)) \to \Out^1(\tilde{\Gamma}(S \setminus \gamma, \partial S)).
\]

For \( S = S^k_{g,n} \), by composing \( g \) of such homomorphisms, we get a natural homomorphism:

\[
R_0 : \Out^1(\tilde{\Gamma}(S, \partial S)) \to \Out^1(\tilde{\Gamma}(S^k_{0,n+2g}, \partial S)).
\]

**Lemma 6.9.** For \( d(S) > 1 \), the homomorphism \( R_0 \) (16) is injective.

**Proof.** There is a natural commutative diagram:

\[
\begin{array}{ccc}
\Out^1(\tilde{\Gamma}(S, \partial S)) & \to & \Out^1(\tilde{\Gamma}(S)) \times \Aut(A) \\
\downarrow_{R_0} & & \downarrow \\
\Out^1(\tilde{\Gamma}(S^k_{0,n+2g}, \partial S)) & \to & \Out^1(\tilde{\Gamma}(S^k_{0,n+2g})) \times \Aut(A),
\end{array}
\]

where \( A := \prod_{i=1}^k \tau_{\delta_i}^2 \) identifies with the normal inertia subgroups of both groups \( \tilde{\Gamma}(S, \partial S) \) and \( \tilde{\Gamma}(S^k_{0,n+2g}, \partial S) \). By (i) of Lemma 4.10, we have that both horizontal homomorphisms are injective. From the isomorphisms \( \Out^1(\tilde{\Gamma}(S)) \cong \widehat{\Gamma}T \) and \( \Out^1(\tilde{\Gamma}(S^k_{0,n+2g})) \cong \widehat{\Gamma}T \) (cf.
Section 6.3), it then follows that also the righthand vertical homomorphism is injective and so the lefthand vertical homomorphism $R_0$ is injective.

Let $R_{\otimes}$: $\text{Out}^1(\hat{\Gamma}(S, \partial S)) \to \text{Out}^3(\hat{\Gamma}(S))$ be the natural homomorphism defined in Section 4.3. We have:

**Lemma 6.10.** For $S = S_{g}^{k}$ with $n + k \geq 5$ and $k \geq 1$, the natural homomorphism $R_{\otimes}$: $\text{Out}^1(\hat{\Gamma}(S, \partial S)) \to \text{Out}^3(\hat{\Gamma}(S))$ is injective.

**Proof.** By (ii) of Theorem 4.15, the group $\text{Out}^1(\hat{\Gamma}(S, \partial S))$ identifies with the centralizer of $\text{Inn}(\hat{\Gamma}(S, \partial S)) / \text{Inn}(\hat{\Gamma}(S, \partial S)) \cong \Sigma_n \times \Sigma_k$ in $\text{Out}^3(\hat{\Gamma}(S, \partial S))$. This implies that the action of $\text{Out}^1(\hat{\Gamma}(S, \partial S))$ on $\prod_{i=1}^{k} \tau_{\alpha_i}$ preserves the procyclic subgroups $\tau_{\alpha_i}$, for $i = 1, \ldots, k$. Moreover, since these subgroups are all conjugated by the action of $\text{Inn}(\hat{\Gamma}(S, \partial S))$, one of this action, say the one on $\tau_{\alpha_1}$, determines them all. From (i) of Lemma 4.10, it then follows that there is a natural monomorphism:

$$\text{(17)} \quad \text{Out}^1(\hat{\Gamma}(S, \partial S)) \hookrightarrow \text{Aut}(\tau_{\alpha_1}) \times \text{Out}^1(\hat{\Gamma}(S)).$$

By the remarks above, the outer action of $\text{Out}^1(\hat{\Gamma}(S, \partial S))$ on $\hat{\Gamma}(S, \partial S)$ (cf. item (ii) of Theorem 4.15) preserves the kernel of the epimorphism $\hat{\Gamma}(S, \partial S) \to \hat{\Gamma}(S', \partial S')$, where we let $S' := S_{g,k-1}$. Therefore, the latter epimorphism induces a homomorphism $\text{Out}^1(\hat{\Gamma}(S, \partial S)) \to \text{Out}^3(\hat{\Gamma}(S', \partial S'))$ and then, by (ii) of Theorem 4.15, a homomorphism:

$$\text{(18)} \quad \text{Out}^3(\hat{\Gamma}(S, \partial S)) \to \text{Out}^3(\hat{\Gamma}(S', \partial S')).$$

Let us observe that the component $\text{Out}^1(\hat{\Gamma}(S, \partial S)) \to \text{Aut}(\tau_{\alpha_1})$ of the monomorphism (17) factors through the homomorphism (18). By Lemma 6.8, this is then determined by the character $\chi_{\lambda}: \hat{\Gamma} \to \hat{Z}$. Hence, since, by the proof of item (i) of Theorem 6.5 in Section 6.3, we already know that $\text{Out}^1(\hat{\Gamma}(S)) \cong \hat{\Gamma}$, it follows that the component $\text{Out}^1(\hat{\Gamma}(S, \partial S)) \to \text{Out}^3(\hat{\Gamma}(S))$ of the monomorphism (17) is injective.

For $S = S_{g,n}^{k}$ and $d(S) > 1$, by Lemma 6.9, Lemma 6.10 and the results of Section 6.3, there is a natural monomorphism:

$$R_{\otimes} \circ R_0: \text{Out}^2(\hat{\Gamma}(S, \partial S)) \hookrightarrow \text{Out}^1(\hat{\Gamma}(S_{g,n+2g})) \cong \hat{\Gamma}.$$

Thus, by Lemma 6.7, we get a series of natural monomorphisms:

$$\hat{\Gamma} \hookrightarrow \text{Out}^1(\hat{\Gamma}(S, \partial S)) \hookrightarrow \hat{\Gamma},$$

from which we conclude that $\text{Out}^2(\hat{\Gamma}(S, \partial S)) \cong \hat{\Gamma}$. Item (iv) of Theorem 6.5 is also clear from the proof above.
6.5. **The proof of Lemma 6.7 and Theorem 6.4.** In the following sections, we will first prove a version of Lemma 6.7 for profinite mapping class groups in genus ≤ 2. By the congruence subgroup property in genus ≤ 2, this will complete the proof of Lemma 6.7 and thus of Theorem 6.5 for the genus ≤ 2 case.

For higher genus, we will need first to answer an old open question in Grothendieck–Teichmüller theory, namely whether there exist natural \( \hat{\Gamma} \)-actions on profinite mapping class groups for all genera.

Let \( \text{Aut}_{0}(\hat{\Gamma}(S)) \) be the closed subgroup of \( \text{Aut}(\hat{\Gamma}(S)) \) consisting of those elements which preserve the conjugacy class of a procyclic subgroup (topologically) generated by a Dehn twists about a nonseparating simple closed curve and by \( \text{Out}_{0}(\hat{\Gamma}(S)) \) its quotient by the subgroup of inner automorphisms. We will prove that:

**Theorem 6.11.** For \( S \) a hyperbolic surface such that \( d(S) > 1 \), there is a natural faithful representation:

\[
\hat{\rho}_{\hat{\Gamma}} : \hat{\Gamma} \hookrightarrow \text{Out}_{0}(\hat{\Gamma}(S, \partial S)).
\]

This outer representation restricts to an action by automorphisms on the central inertia group \( \prod_{i=1}^{k} \tau_{\delta_{i}}^{2} \) of \( \hat{\Gamma}(S, \partial S) \) with the property that each procyclic subgroup \( \tau_{\delta_{i}}^{2} \) is preserved and acted upon through the character \( \chi_{\lambda} : \hat{\Gamma} \to \mathbb{Z}^{*} \), for \( i = 1, \ldots, k \).

An immediate consequence of Theorem 6.11 is Theorem 6.4. The proof of Lemma 6.7 will instead be obtained as a corollary of Theorem 6.11 (or better a corollary of its proof).

6.6. **Profinite hyperelliptic mapping class groups.** A hyperelliptic hyperbolic surface \((S, \upsilon)\) is the datum of a hyperbolic surface \( S \) and a hyperelliptic involution \( \upsilon \) on \( S \), that is to say, an involution such that the quotient surface \( S_{\upsilon} := S/\langle \upsilon \rangle \) has genus 0. The **hyperelliptic mapping class group** \( \Upsilon(S) \) is then defined to be the centralizer of \( \upsilon \) in the mapping class group \( \Gamma(S) \). For a finite subset of \( S \) and \( \hat{\mathcal{P}} \) the same subset with a fixed order of its elements, we let \( \Gamma(S, \mathcal{P}) \cong \Gamma(S \setminus \mathcal{P}) \) and \( \Gamma(S, \hat{\mathcal{P}}) \cong \text{PT}(S \setminus \mathcal{P}) \) be, respectively, the mapping class group and the pure mapping class group of the marked surface \((S, \mathcal{P})\).

The **hyperelliptic mapping class groups** \( \Upsilon(S, \mathcal{P}) \) and \( \Upsilon(S, \hat{\mathcal{P}}) \) are, respectively, the inverse images of \( \Upsilon(S) \) by the epimorphisms \( \Gamma(S, \mathcal{P}) \to \Gamma(S) \) and \( \Gamma(S, \hat{\mathcal{P}}) \to \Gamma(S) \).

The **profinite hyperelliptic mapping class groups** \( \hat{\Upsilon}(S, \mathcal{P}) \) and \( \hat{\Upsilon}(S, \hat{\mathcal{P}}) \) are then, respectively, the profinite completions of \( \Upsilon(S, \mathcal{P}) \) and \( \Upsilon(S, \hat{\mathcal{P}}) \). The congruence topology on \( \Gamma(S, \mathcal{P}) \cong \Gamma(S \setminus \mathcal{P}) \) induces a profinite topology on \( \Upsilon(S, \mathcal{P}) \) and \( \Upsilon(S, \hat{\mathcal{P}}) \) via the embeddings \( \Upsilon(S, \hat{\mathcal{P}}) \subseteq \Upsilon(S, \mathcal{P}) \subseteq \Gamma(S, \mathcal{P}) \) also called the congruence topology. By Theorem 7.2 in [6], this topology coincides with the full profinite topology. Hence, in the sequel, we will identify profinite and procongruence hyperelliptic mapping class groups.

6.7. **The complex of symmetric profinite curves.** A symmetric multicurve on a marked hyperelliptic surface \((S, \upsilon, \mathcal{P})\) is a multicurve on \( S \setminus \mathcal{P} \) whose isotopy class in \( S \) is preserved by the hyperelliptic involution \( \upsilon \). The **complex of symmetric curves** \( C(S, \upsilon, \mathcal{P}) \) is the (abstract) simplicial complex whose simplices are the symmetric multicurves on \((S, \upsilon, \mathcal{P})\) (cf. Definition 2.8 in [6]). This is clearly a full subcomplex of the curve complex.
$C(S \setminus \mathcal{P})$. We then define the \textit{complex of symmetric profinite curves} $\hat{C}(S, v, \mathcal{P})$ on $(S, v, \mathcal{P})$ to be the (abstract) simplicial profinite complex whose (profinite) set of $k$-simplices is the closure of $C(S, v, \mathcal{P})_k$ inside $\hat{C}(S \setminus \mathcal{P})_k$ (cf. Definition 7.11 in [6], where this complex is denoted by $L(\hat{\Pi}(S \setminus \mathcal{P}), v)$).

Simplices of $C(S, v, \mathcal{P})$ parameterize the multitwists of the hyperelliptic mapping class group $\mathcal{Y}(S, \mathcal{P})$ as well as the abelian subgroups generated by Dehn twists. In Section 7.7 of [6], we showed that the simplices of $\hat{C}(S, v, \mathcal{P})$ parameterize profinite multitwists in the profinite hyperelliptic mapping class group $\hat{\mathcal{Y}}(S, \mathcal{P})$. In particular, for $U$ an open subgroup of $\hat{\mathcal{Y}}(S, \mathcal{P})$ (which is then a closed subgroup of $\hat{\Gamma}(S \setminus \mathcal{P})$), the nontrivial decomposition and inertia groups $U_\sigma$ and $\hat{I}_\sigma(U)$ of $U$ (cf. Definition 3.1 and Definition 3.3) are parameterized by $\hat{C}(S, v, \mathcal{P})$. The same arguments of the proofs of Proposition 7.2 and (ii) of Theorem 7.3 in [7] show that there is a natural continuous homomorphism $\text{Aut}^D(U) \to \text{Aut}(\hat{C}(S, v, \mathcal{P}))$ whose kernel identifies with the abelian group $\text{Hom}(U/Z(U), Z(U))$. The usual argument then shows that this has trivial intersection with the subgroup $\text{Aut}^\text{I}(U)$ of $\text{Aut}^D(U)$, so that there is a monomorphism $\text{Aut}^\text{I}(U) \hookrightarrow \text{Aut}(\hat{C}(S, v, \mathcal{P}))$.

It is likely that a version of Theorem 5.5 in [7] holds in the hyperelliptic case. This would show, in particular, that the $\mathbb{I}$-automorphisms of $U$ preserve the topological types of the inertia groups $\hat{I}_\sigma(U)$ for $\sigma \in \hat{C}(S, v, \mathcal{P})$. Since we will not need this result in the sequel, we instead give the following definition:

\textbf{Definition 6.12.} For $H$ a closed subgroup of $\hat{\mathcal{Y}}(S, B)$, we let $\text{Aut}^\text{I}_\text{top}(H)$ be the closed subgroup of $\text{Aut}^\text{I}(H)$ consisting of those elements which preserve the topological types of the inertia groups of $H$. We then say that a subgroup $K$ of $H$ is $\mathbb{I}_\text{top}$-\textit{characteristic} if it is preserved by $\text{Aut}^\text{I}_\text{top}(H)$. Note that $\text{Inn}(U) \subseteq \text{Aut}^\text{I}_\text{top}(U)$, for $U$ an open subgroup of $\hat{\mathcal{Y}}(S, \mathcal{P})$, so that, in particular, an $\mathbb{I}_\text{top}$-\textit{characteristic} subgroup of $U$ is normal.

For the closed surface case, we have the following result:

\textbf{Proposition 6.13.} For $S$ a closed surface of genus $g \geq 2$, endowed with a hyperelliptic involution $v$, there is a natural isomorphism $\text{Out}(\hat{\mathcal{Y}}(S)) \cong \{\pm 1\} \times \text{Out}^\text{I}(\hat{\mathcal{Y}}(S))$ and a series of natural isomorphisms:

$$\text{Out}^\text{I}(\hat{\mathcal{Y}}(S)) = \text{Out}^\text{I}_\text{top}(\hat{\mathcal{Y}}(S)) \cong \text{Out}(\hat{\Gamma}_0, [2g+2]) \cong \hat{\Gamma}T.$$  

In particular, we have $\text{Aut}^\text{I}(\hat{\mathcal{Y}}(S)) = \text{Aut}^\text{I}_\text{top}(\hat{\mathcal{Y}}(S))$.

\textbf{Proof.} By Lemma 7.4 in [7], there is an exact sequence:

$$1 \to \text{Hom}(\hat{\mathcal{Y}}(S)/Z(\hat{\mathcal{Y}}(S)), Z(\hat{\mathcal{Y}}(S))) \to \text{Out}(\hat{\mathcal{Y}}(S)) \to \text{Out}(\hat{\mathcal{Y}}(S)/Z(\hat{\mathcal{Y}}(S))),$$

where $\hat{\mathcal{Y}}(S)/Z(\hat{\mathcal{Y}}(S)) \cong \hat{\Gamma}_0, [2g+2]$ and the center $Z(\hat{\mathcal{Y}}(S))$ is generated by the hyperelliptic involution $v$. Since the abelianization of $\hat{\Gamma}_0, [2g+2]$ is isomorphic to $\mathbb{Z}/2(2g+1)$ (cf. § 5.1.3 in [9]), we have that $\text{Hom}(\hat{\mathcal{Y}}(S)/Z(\hat{\mathcal{Y}}(S)), Z(\hat{\mathcal{Y}}(S))) \cong \{\pm 1\}$.

As usual, the image of $\text{Hom}(\hat{\mathcal{Y}}(S)/Z(\hat{\mathcal{Y}}(S)), Z(\hat{\mathcal{Y}}(S)))$ in $\text{Out}(\hat{\mathcal{Y}}(S))$ has trivial intersection with the subgroups $\text{Out}^\text{I}(\hat{\mathcal{Y}}(S))$ and $\text{Out}^\text{I}_\text{top}(\hat{\mathcal{Y}}(S))$. Since $\hat{\mathcal{Y}}(S)/Z(\hat{\mathcal{Y}}(S)) = \hat{\Gamma}_0, [2g+2]$,
by Lemma 3.13, the image of \( \text{Out}(\hat{\Upsilon}(S)) \) in \( \text{Out}(\hat{\Upsilon}(S)/\mathcal{Z}(\hat{\Upsilon}(S))) \) identifies with both \( \text{Out}^1(\hat{\Upsilon}(S)) \) and \( \text{Out}^{\text{iso}}(\hat{\Upsilon}(S)) \). The isomorphism \( \text{Out}(\hat{\Upsilon}(S)) \cong \{\pm 1\} \times \text{Out}^1(\hat{\Upsilon}(S)) \) and the identity \( \text{Out}^1(\hat{\Upsilon}(S)) = \text{Out}^{\text{iso}}(\hat{\Upsilon}(S)) \) then follow.

Let \((S^\circ, v)\) be the hyperelliptic surface with boundary obtained from the hyperelliptic surface \((S, v)\) by replacing one of the Weierstrass points with a circle. The Birman-Hilden theorem states that there is a natural isomorphism \( \mathcal{Z}_{\Gamma(S^\circ, \partial S^\circ)}(v) \cong B_{2g+1} \), where \( B_{2g+1} \) is Artin braid group. Moreover, if \( z \) is the standard generator of the (cyclic) center of \( B_{2g+1} \), this isomorphism induces an isomorphism \( \Upsilon(S) \cong B_{2g+1}/\langle z^2 \rangle \) (cf. Chapter 9 in [9]).

In Appendix 4 of [18], Ihara showed that there is an action of the profinite Grothendieck-Teichmüller group \( \mathcal{G}_T \) on the profinite completion \( \hat{B}_n \) of the Artin braid group \( B_n \), for all \( n \geq 3 \), which preserves the conjugacy class of the procyclic subgroups generated by the standard braids. The Birman-Hilden theorem then implies that there is a representation \( \mathcal{G}_T \to \text{Out}^{\text{iso}}(\hat{\Upsilon}(S)) \) compatible with the natural monomorphism \( \text{Out}^{\text{iso}}(\hat{\Upsilon}(S)) \hookrightarrow \text{Out}(\hat{\Upsilon}(S)/\mathcal{Z}(\hat{\Upsilon}(S))) \) and the isomorphism \( \text{Out}(\hat{\Upsilon}(S)/\mathcal{Z}(\hat{\Upsilon}(S))) \cong \mathcal{G}_T \). This yields the series of isomorphisms in the statement of the proposition. The last claim then follows from the isomorphism \( \text{Out}^1(\hat{\Upsilon}(S)) \cong \text{Out}(\hat{\Upsilon}_{0,[2g+2]}) \) and Lemma 3.13.

### 6.8. Modular subgroups of the hyperelliptic mapping class group.

The definition of hyperelliptic mapping class group makes sense also when \((S, v, \mathcal{P})\) is a marked disconnected hyperelliptic surface (cf. sections 2.8 and 2.9 in [6]). For instance, for \( \gamma \) a symmetric simple closed curve on \((S, v, \mathcal{P})\), the marked hyperelliptic surface \((S \setminus \gamma, v, \mathcal{P})\) is a, possibly, disconnected marked hyperelliptic surface. Let us label by \( Q_+, Q_- \) the pair of punctures on \( S \setminus \gamma \) bounded by \( \gamma \) and denote by \( \Upsilon(S \setminus \gamma, \mathcal{P}) \) the stabilizer of this pair of punctures for the action of \( \Upsilon(S \setminus \gamma, \mathcal{P}) \) on the set of all punctures of \( S \setminus \gamma \). The stabilizer \( \Upsilon(S, \mathcal{P}) \) for the action of the hyperelliptic mapping class group \( \Upsilon(S, \mathcal{P}) \) on the set \( C(S, v, \mathcal{P})_0 \) is then described by the short exact sequence (cf. the short exact sequence (14) in [6]):

\[
1 \to \tau_{\gamma}^Z \to \Upsilon(S, \mathcal{P}) \to \Upsilon(S \setminus \gamma, \mathcal{P})_{\{Q_+, Q_-\}} \to 1.
\]

A similar short exact sequence also describes the stabilizer \( \Upsilon(S, \hat{\mathcal{P}}) \).

In Section 7.6 of [6], the above description of stabilizers is extended to profinite hyperelliptic mapping class group. More precisely, for \( \gamma \) a symmetric simple closed curve on \((S, v, \mathcal{P})\), which we now regard as an element of the profinite set \( \hat{C}(S, v, \mathcal{P})_0 \), the stabilizer \( \hat{\Upsilon}(S, \mathcal{P}) \) for the action of \( \hat{\Upsilon}(S, \mathcal{P}) \) on \( C(S, v, \mathcal{P})_0 \) is described by the short exact sequence (cf. Corollary 7.8 in [6]):

\[
1 \to \tau_{\gamma}^Z \to \hat{\Upsilon}(S, \mathcal{P}) \to \hat{\Upsilon}(S \setminus \gamma, \mathcal{P})_{\{Q_+, Q_-\}} \to 1,
\]

where \( \hat{\Upsilon}(S, \mathcal{P}) \) and \( \hat{\Upsilon}(S \setminus \gamma, \mathcal{P})_{\{Q_+, Q_-\}} \) are naturally isomorphic to the profinite completions of the groups \( \Upsilon(S, \mathcal{P}) \) and \( \Upsilon(S \setminus \gamma, \mathcal{P})_{\{Q_+, Q_-\}} \), respectively. Again, a similar description holds for the stabilizer \( \hat{\Upsilon}(S, \hat{\mathcal{P}}) \).
For the applications in this paper, we are more interested in the case when \( \gamma \) is a nonseparating simple closed curve and the set of marked points \( \mathcal{P} \) is ordered. Therefore, we will give a more precise description of the stabilizer in this case.

For \( S \) a closed surface and \( \gamma \) a nonseparating symmetric simple closed curve on \((S, v, \mathcal{P})\), let \( S_{\gamma} := S \setminus \gamma \) and let \( \Upsilon(S_{\gamma}, \vec{\mathcal{P}})_{0} \) be the index 2 subgroup of \( \Upsilon(S_{\gamma}, \vec{\mathcal{P}}) \) consisting of those elements which do not swap the punctures of \( S_{\gamma} \) labeled by \( Q_{+} \) and \( Q_{-} \) (this is also described as the kernel of the natural representation \( \Upsilon(S_{\gamma}, \vec{\mathcal{P}}) \to \Sigma_{(Q_{+}, Q_{-})} \)). The stabilizer \( \Upsilon(S, \vec{\mathcal{P}})_{\gamma} \) of the oriented simple closed curve \( \vec{\gamma} \) is described by the short exact sequence (cf. Theorem 3.1 in [6]):

\[
1 \to \tau_{\gamma}^{\mathbb{Z}} \to \Upsilon(S, \vec{\mathcal{P}})_{\gamma} \to \Upsilon(S_{\gamma}, \vec{\mathcal{P}})_{0} \to 1.
\]

Let \( N_{\gamma} \) be the normal subgroup of \( \Upsilon(S_{\gamma}, \vec{\mathcal{P}})_{0} \) generated by the Dehn twists about symmetric separating simple closed curves on \((S_{\gamma}, v, \mathcal{P})\) bounding a disc which contains no marked points and only the two punctures labeled by \( Q_{+} \) and \( Q_{-} \). We will give a more precise description of the stabilizer in this case.

From the above results, it follows:

**Proposition 6.14.** Let \( (S, v, \mathcal{P}) \) be a marked hyperelliptic hyperbolic surface and \( \gamma \) a symmetric nonseparating simple closed curve on \((S, v, \mathcal{P})\). With the above notations, we have:

(i) The stabilizer \( \hat{\Upsilon}(S, \vec{\mathcal{P}})_{\gamma} \) for the action of \( \hat{\Upsilon}(S, \vec{\mathcal{P}}) \) on the profinite set \( \hat{\mathcal{C}}(S, v, \mathcal{P})_{0} \) is described by the two short exact sequences:

\[
1 \to \hat{\Upsilon}(S, \vec{\mathcal{P}})_{\gamma} \to \hat{\Upsilon}(S, \vec{\mathcal{P}})_{\gamma} \to \{\pm 1\} \to 1
\]

and

\[
1 \to \tau_{\gamma}^{\mathbb{Z}} \to \hat{\Upsilon}(S, \vec{\mathcal{P}})_{\gamma} \to \Upsilon(S_{\gamma}, \vec{\mathcal{P}})_{0} \to 1,
\]

where \( \epsilon \) is the orientation character associated to the curve \( \gamma \) and \( \hat{\Upsilon}(S_{\gamma}, \vec{\mathcal{P}})_{0} \) is the kernel of the natural representation \( \hat{\Upsilon}(S_{\gamma}, \vec{\mathcal{P}}) \to \Sigma_{(Q_{+}, Q_{-})} \).

(ii) The profinite group \( \hat{\Upsilon}(S_{\gamma}, \vec{\mathcal{P}})_{0} \) fits in the short exact sequence:

\[
1 \to \overline{N}_{\gamma} \to \hat{\Upsilon}(S_{\gamma}, \vec{\mathcal{P}})_{0} \to \hat{\Upsilon}(\overline{S}_{\gamma}, \vec{\mathcal{P}} \cup \overline{Q}_{+}) \to 1,
\]

where \( \overline{N}_{\gamma} \) is the closure of the group \( N_{\gamma} \) in the profinite group \( \hat{\Upsilon}(S_{\gamma}, \mathcal{P}) \).

### 6.9. Homomorphisms between outer automorphism groups of profinite hyperelliptic mapping class groups.

Let \( S \) and \( \gamma \) be as in the statement of Proposition 6.14. The subgroup \( \mathcal{P} \Upsilon(S, B)_{\gamma} \) of \( \mathcal{P} \Upsilon(S, B) \) is \( \mathbb{I}_{op} \)-characteristic. By the same construction of the homomorphism (7), there is then a natural homomorphism:

\[
\text{Aut}^{\text{top}}(\hat{\Upsilon}(S, \vec{\mathcal{P}})) \to \text{Out}^{\text{top}}(\hat{\Upsilon}(S, \vec{\mathcal{P}})_{\gamma}).
\]
By (i) of Proposition 6.14, there is also a natural homomorphism:

$$\text{Out}^{\text{top}}(\hat{\mathfrak{Y}}(S, \overline{\mathcal{P}})) \to \text{Out}^{\text{top}}(\hat{\mathfrak{Y}}(S_\gamma, \overline{\mathcal{P}}))$$

By composing the two above homomorphisms, we get the homomorphism:

$$\text{Aut}^{\text{top}}(\hat{\mathfrak{Y}}(S, \overline{\mathcal{P}})) \to \text{Out}^{\text{top}}(\hat{\mathfrak{Y}}(S_\gamma, \overline{\mathcal{P}})). \quad (19)$$

The subgroup $\overline{N}_\gamma$ of $\text{Out}^{\text{top}}(\hat{\mathfrak{Y}}(S_\gamma, \overline{\mathcal{P}}))$ (cf. (ii) of Proposition 6.14) is (topologically) generated by the profinite Dehn twists about symmetric profinite curves on $(S_\gamma, \nu, \mathcal{P})$ of topological type a simple closed curve bounding a disc which only contains the two punctures labeled by $Q_+$ and $Q_-$. Therefore, $\overline{N}_\gamma$ is an $I_{\text{top}}$-characteristic subgroup of $\text{Out}^{\text{top}}(\hat{\mathfrak{Y}}(S_\gamma, \overline{\mathcal{P}}))$ and there is a natural homomorphism $\text{Out}^{\text{top}}(\hat{\mathfrak{Y}}(S_\gamma, \overline{\mathcal{P}})) \to \text{Out}^{\text{top}}(\hat{\mathfrak{Y}}(S_\gamma, \overline{\mathcal{P}} \cup Q_+))$. Composing the latter with the homomorphism (19), we get a natural homomorphism:

$$\text{Out}^{\text{top}}(\hat{\mathfrak{Y}}(S, \overline{\mathcal{P}})) \to \text{Out}^{\text{top}}(\hat{\mathfrak{Y}}(S_\gamma, \overline{\mathcal{P}} \cup Q_+)).$$

It is not difficult to see that the kernel of this homomorphism contains $\text{Inn}(\hat{\mathfrak{Y}}(S, \overline{\mathcal{P}}))$. Therefore, there is a natural homomorphism:

$$\text{Out}^{\text{top}}(\hat{\mathfrak{Y}}(S, \overline{\mathcal{P}})) \to \text{Out}^{\text{top}}(\hat{\mathfrak{Y}}(S_\gamma, \overline{\mathcal{P}} \cup Q_+)). \quad (20)$$

**Proposition 6.15.** For $(S, \nu, \mathcal{P})$ a marked hyperelliptic closed surface of genus $g \geq 2$, there is a natural representation:

$$\Phi_{(S, \overline{\mathcal{P}})} : \hat{\Gamma} \to \text{Out}^{\text{top}}(\hat{\mathfrak{Y}}(S, \overline{\mathcal{P}})).$$

**Proof.** Let $n := \# \mathcal{P}$ and let $\sigma := \left\{ \gamma_1, \ldots, \gamma_n \right\}$ be a set of disjoint symmetric closed curves on a hyperelliptic closed surface $(S', \nu)$ of genus $g + n$ such that $S'_\sigma := S' \setminus \sigma$ is connected. Then, $(S'_\sigma, \nu)$ is a hyperelliptic surface endowed with $n$ pairs of symmetric punctures (with respect to the hyperelliptic involution $\nu$). Let $(S, \nu)$ be the hyperelliptic closed surface obtained filling the $2n$ punctures on $S'_\sigma$ with the set of points $\mathcal{P} \cup \nu(\mathcal{P})$. Then, composing $n$ homomorphism of type (20) (note that each one adds a marked point out of the set $\mathcal{P}$), by Proposition 6.13, we get the representation:

$$\Phi_{(S, \overline{\mathcal{P}})} : \hat{\Gamma} \cong \text{Out}^{\text{top}}(\hat{\mathfrak{Y}}(S')) \to \text{Out}^{\text{top}}(\hat{\mathfrak{Y}}(S, \overline{\mathcal{P}})).$$

□

**Corollary 6.16.** For $S = S_{2,n}$ and all $n \geq 0$, there is a natural representation:

$$\Psi_{S} : \hat{\Gamma} \to \text{Out}^{\text{top}}(\hat{\mathfrak{Y}}(S)).$$

**Proof.** Note that, for $S'$ a closed surface of genus 2 and $\mathcal{P}$ a set of distinct $n$ points on $S'$, we have $\hat{\mathfrak{Y}}(S' \setminus \mathcal{P}) \cong \hat{\mathfrak{Y}}(S', \overline{\mathcal{P}})$. Let then $S := S' \setminus \mathcal{P}$ and compose the homomorphism $\Phi_{(S', \overline{\mathcal{P}})}$ of Proposition 6.15 with the homomorphism $\text{Out}^{\text{top}}(\hat{\mathfrak{Y}}(S)) = \text{Out}^{\text{top}}(\hat{\mathfrak{Y}}(S)) \to \text{Out}^{\text{top}}(\hat{\mathfrak{Y}}(S))$ (cf. Theorem 4.3).

□
6.10. **Proof of Lemma 6.7** for $g(S) \leq 2$ and $\partial S = \emptyset$. The genus 0 case, for $n(S) \geq 5$, follows from Proposition 3.8 and Lemma 3.13. The case $g(S) = 0$ and $n(S) = 4$ then follows from (ii) of Theorem 5.3. The genus 2 case was already treated in Corollary 6.16.

For the genus 1 case, let $S := S_{2,n}$ and $S' := S_{1,n+1}$, with $n \geq 0$. By Theorem 5.7, there is then a natural injective homomorphism:

$$\mu_{S,S'}: \text{Out}^1(\widehat{\Gamma}(S)) \hookrightarrow \text{Out}^1(\widehat{\Gamma}(S')),$$

which, precomposed with $\Psi_S$ yields the genus 1 case of Lemma 6.7 for $\partial S = \emptyset$.

6.11. **Proof of Lemma 6.7** for $g(S) \leq 2$ and $\partial S \neq \emptyset$. Let $S \subset \widetilde{S} \cong S_{g,n+2k}$ and $\sigma = \{\delta_1, \ldots, \delta_k\}$ be as in Definition 2.5. The same construction as that of the homomorphism (10) provides a natural homomorphism:

$$PR_{\sigma}: \text{Out}^0(\widehat{\Gamma}(\widetilde{S})) \rightarrow \text{Out}^0(\widehat{\Gamma}(\widetilde{S})_{\sigma}).$$

After identifying $\Gamma(\partial S)$ with $\Gamma(\widetilde{S})_{\sigma}$, we then get a natural homomorphism:

$$PR_{(S,\partial S)}: \text{Out}^1(\widehat{\Gamma}(\widetilde{S})) \rightarrow \text{Out}^1(\widehat{\Gamma}(S, \partial S)).$$

Composing $PR_{(S,\partial S)}$ with the natural homomorphism $\text{Out}^1(\widehat{\Gamma}(S, \partial S)) \rightarrow \text{Out}^0(\widehat{\Gamma}(S, \partial S))$ (cf. Theorem 4.15), we get a homomorphism $\text{Out}^1(\widehat{\Gamma}(S)) \rightarrow \text{Out}^1(\widehat{\Gamma}(S, \partial S))$, whose kernel contains $\text{Inn}(\Gamma(S)) / \text{Inn}(\Gamma(\widetilde{S})) = \text{Inn}(\widehat{\Gamma}(S)) / \text{Inn}(\widehat{\Gamma}(\widetilde{S}))$ and thus, by Theorem 4.3, induces a natural homomorphism:

$$R_{(S,\partial S)}: \text{Out}^0(\widehat{\Gamma}(\widetilde{S})) \rightarrow \text{Out}^0(\widehat{\Gamma}(S, \partial S)).$$

For $g(\widetilde{S}) \leq 2$, precomposing $R_{(S,\partial S)}$ with the representation $\Psi_{(\widetilde{S},\emptyset)}$ obtained in Section 6.10, we finally get the representation:

$$\Psi_{(S,\partial S)}: \widehat{\Gamma} \rightarrow \text{Out}^0(\widehat{\Gamma}(S, \partial S)).$$

6.12. **Proof of Theorem 6.11** for $g(S) \leq 2$. This, in particular, follows from the case $g(S) \leq 2$ of Theorem 6.5, which, as we showed in Section 6.4, in its turn, follows from the case $g(S) \leq 2$ of Lemma 6.7 proved in Section 6.10 and Section 6.11 above.

6.13. **A presentation for the profinite relative mapping class group of a surface.** In [10], Gervais gave a finite presentation of the pure mapping class group of a surface with boundary. In this section, we will show that this result implies that the profinite pure mapping class group of an open surface admits a presentation as a quotient of a profinite amalgamated product of stabilizers of oriented nonseparating simple closed curves by a few simple commutator relations.

For a finite set of groups $G_1, \ldots, G_k$ and monomorphisms $H \hookrightarrow G_i$, for $i = 1, \ldots, k$, we denote by $\bigoplus_{H}^{i=1,\ldots,k} G_i$ their amalgamated free product with amalgamated subgroup $H$. If the groups $G_1, \ldots, G_k, H$ are profinite and the monomorphisms $H \hookrightarrow G_i$ continuous, we denote by $\bigcup_{H}^{i=1,\ldots,k} G_i$ their amalgamated free profinite product with amalgamated subgroup $H$ (cf. Section 9.2 in [25]). With the above notations, we have:
Corollary 6.19. Let $S = S^k_g$ with $k \geq 1$ and $g \geq 3$, let $\sigma = \{\gamma_1, \ldots, \gamma_{g-1}\}$ be a set of disjoint simple closed curves on $S$ such that $S \setminus \sigma$ is connected and $\sigma' = \{\beta_1, \ldots, \beta_{g-1}\}$ a set of disjoint simple closed curves such that $S \setminus \sigma'$ is connected and the intersection $\beta_i \cap \gamma_j$ is a single point for $i = j$ and empty otherwise. Let $\sigma_i := \sigma \setminus \{\gamma_i\}$, for $i = 1, \ldots, g - 1$.

There is then a natural epimorphism:

$$\Theta: \bigotimes_{i=1, \ldots, g-1} \Gamma(S, \partial S)_{\sigma_i} \to \Gamma(S, \partial S),$$

whose kernel is normally generated by the commutators $[\tau_{\beta_i}, \tau_{\beta_j}]$, for $i, j = 1, \ldots, g - 1$.

Proof. The proposition is an immediate consequence of Theorem 1 in [10]. Let us observe indeed that the curves in $\sigma \cup \sigma'$ are part of the set of curves denoted by $\mathcal{G}_{g,n}$ in [10], where the curves $\beta_1, \ldots, \beta_{g-1}$ correspond to the curves denoted in the same way in [10] and the curves $\gamma_1, \ldots, \gamma_{g-1}$ are contained in the set of curves $(\gamma_{i,j})_{1 \leq i,j \leq g+n-2, i \neq j}$ of [10]. Note also that the set of Dehn twists about the curves in $\mathcal{G}_{g,n}$ contains sets of generators for the subgroups $\Gamma(S, \partial S)_{\sigma}$ and $\Gamma(S, \partial S)_{\sigma_i}$, for $i = 1, \ldots, g - 1$, of $\Gamma(S, \partial S)$.

It is then enough to observe that, with the exception of the ”disjointness relations” $\tau_{\beta_i} \tau_{\beta_j} = \tau_{\beta_j} \tau_{\beta_i}$, for $1 \leq i < j \leq g - 1$, all relations of the presentation given in Theorem 1 in [10] are supported on some subsurface $S \setminus \sigma_i$, for $i = 1, \ldots, g - 1$, of $S$. \qed

Remark 6.18. After a careful reading of the paper [10], I noted a gap in the proof of Lemma 8 in [10] where the author claims: ”Pasting a pair of pants to $\gamma_{2g+n-3,1}$ allows us to view $\Sigma_{g,n-1}$ as a subsurface of $\Sigma_{g,n}$”. But this makes sense only for $n > 1$, which means that the proof of Theorem 1 in [10] is complete only for $n \geq 1$. This is the reason why, in Proposition 6.17, we assume $k \geq 1$.

By the congruence subgroup property in genus $\leq 2$ and (ii) of Theorem 2.3, the closures of the subgroups $\Gamma(S, \partial S)_{\sigma}$ and $\Gamma(S, \partial S)_{\sigma_i}$, for $i = 1, \ldots, g - 1$, in the profinite mapping class group $\hat{\Gamma}(S)$ identify with their respective profinite completions. Therefore, by the universal property of amalgamated free profinite products (cf. Section 9.2 in [25]), we have:

Corollary 6.19. With the notations of Proposition 6.17, there is a natural continuous epimorphism:

$$\hat{\Theta}: \bigotimes_{i=1, \ldots, g-1} \hat{\Gamma}(S, \partial S)_{\sigma_i} \to \hat{\Gamma}(S, \partial S),$$

whose kernel is (topologically) normally generated by the commutators $[\tau_{\beta_i}, \tau_{\beta_j}]$, for $i, j = 1, \ldots, g - 1$.

6.14. Proof of Theorem 6.11 for $S = S^k_g$, with $k \geq 1$ and $g \geq 3$. Let $\sigma = \{\gamma_1, \ldots, \gamma_{g-1}\}$ and $\sigma' = \{\beta_1, \ldots, \beta_{g-1}\}$ be as in the statement of Proposition 6.17 and let $S_\sigma$ be the compact surface of genus 1 obtained replacing with boundary circles the punctures of $S \setminus \sigma$. Let us denote by $\gamma_i^+$ and $\gamma_i^-$ the boundary circles which replace the two punctures obtained removing $\gamma_i$ from $S$, for $i = 1, \ldots, g - 1$. By the genus 1 case of (v) of Theorem 6.5, there
is a natural isomorphism:

\[
\text{Out}^f(\hat{\text{P}}(S_\sigma, \partial S_\sigma)) \cong \Sigma_{k+2g-2} \times \hat{\Gamma}. 
\]

Let \(\sigma'' = \{\alpha_1, \ldots, \alpha_{g-1}\}\) be a set of disjoint separating simple closed curves on \(S\), with empty intersection with the curves in \(\sigma\) and \(\sigma'\), such that \(\alpha_i\) is the only boundary component of a compact genus 1 subsurface \(S_{\alpha_i}\) of \(S\) which contains \(\gamma_i\) and \(\beta_i\), for \(i = 1, \ldots, g - 1\). Let then \(S_{\sigma''} := S \setminus \cup_{i=1}^{g-1} S_{\alpha_i}\) be the compact genus 1 subsurface of \(S_\sigma\) with boundary the union of the curves \(\alpha_1, \ldots, \alpha_{g-1}, \delta_1, \ldots, \delta_k\). By the genus 1 case of (v) of Theorem 6.5, there is a natural isomorphism:

\[
\text{Out}^f(\hat{\text{P}}(S_{\sigma''}, \partial S_{\sigma''})) \cong \Sigma_{k+g-1} \times \hat{\Gamma}. 
\]

A given \(f \in \hat{\Gamma}\) then defines an element of \(\text{Out}^f(\hat{\text{P}}(S_{\sigma''}, \partial S_{\sigma''}))\) which extends to an element of \(\text{Out}^f(\hat{\text{P}}(S_\sigma, \partial S_\sigma))\) and we also denote by \(f\). This element centralizes \(\Sigma_{k+g-1}\) in \(\text{Out}^f(\hat{\text{P}}(S_{\sigma''}, \partial S_{\sigma''}))\) and \(\Sigma_{k+2g-2}\) in \(\text{Out}^f(\hat{\text{P}}(S_\sigma, \partial S_\sigma))\). Therefore, \(f\) preserves the procyclic subgroups generated by the Dehn twists \(\tau_{\alpha_i}^\pm\), for \(i = 1, \ldots, g - 1\), and \(\tau_{\gamma_i^+}, \tau_{\gamma_i^-}\), for \(i = 1, \ldots, g - 1\), and, by the genus \(\leq 2\) case of (iv) of Theorem 6.5, acts on these through the character \(\chi_\lambda: \hat{\Gamma} \to \mathbb{Z}^*\). There is then a lift \(\breve{f} \in \text{Aut}^f(\hat{\text{P}}(S_\sigma, \partial S_\sigma))\) of \(f\) with the same properties.

There is a natural epimorphism \(\hat{\text{P}}(S_\sigma, \partial S_\sigma) \to \hat{\text{P}}(S, \partial S)\), whose kernel is topologically generated by the set of multitwists \(\tau_{\gamma_i^+} \tau_{\gamma_i^-}^{-1}\), for \(i = 1, \ldots, g - 1\). Since \(\breve{f}\) preserves all the procyclic subgroups generated by the Dehn twists \(\tau_{\gamma_i^+}, \tau_{\gamma_i^-}\) and act on these through the character \(\chi_\lambda\), it follows that \(\breve{f}\) descends to to an automorphism \(\breve{f} \in \text{Aut}^f(\hat{\text{P}}(S, \partial S)\), and so \(f\) descends to an outer automorphism \(f \in \text{Out}(\hat{\text{P}}(S, \partial S))\) such that \(\breve{f}\) lifts \(f\) and both preserve the procyclic subgroups \(\tau_{\alpha_i}^\pm\), for \(i = 1, \ldots, g - 1\).

As in Proposition 6.17, let \(\sigma_i := \sigma \setminus \{\gamma_i\}\), for \(i = 1, \ldots, g - 1\). We have:

**Lemma 6.20.** Let \(\breve{f} \in \text{Aut}^f(\hat{\text{P}}(S, \partial S)\) be the element defined above. Then, \(\breve{f}\) admits a unique extension \(\breve{f}_i \in \text{Aut}^f(\hat{\text{P}}(S_i, \partial S_i)\), for all \(i = 1, \ldots, g - 1\).

**Proof.** Let \(S_{\sigma_i}\) be the compact surface of genus 2 obtained from \(S_\sigma\) identifying the boundary circles \(\gamma_i^+\) and \(\gamma_i^-\), so that \(\hat{S}_{\sigma_i} = S \setminus \sigma_i\). By the genus 2 case of (v) of Theorem 6.5, there is a natural isomorphism, for \(i = 1, \ldots, g - 1:\)

\[
\text{Out}^f(\hat{\text{P}}(S_{\sigma_i}, \partial S_{\sigma_i})) \cong \Sigma_{k+2g-4} \times \hat{\Gamma}. 
\]

There is a natural epimorphism \(\hat{\text{P}}(S_{\sigma_i}, \partial S_{\sigma_i}) \to \hat{\text{P}}(S_{\sigma_i}, \partial S_{\sigma_i})_{\gamma_i}\), for \(i = 1, \ldots, g - 1\), whose kernel, topologically generated by the multitwist \(\tau_{\gamma_i^+} \tau_{\gamma_i^-}^{-1}\), by the same argument above, is preserved by the given element \(f \in \hat{\Gamma} \subset \text{Out}^f(\hat{\text{P}}(S_\sigma, \partial S_\sigma))\). Hence, \(f\) descends to an outer automorphism of \(\text{P} \hat{\text{P}}(S_{\sigma_i}, \partial S_{\sigma_i})\), which, by the isomorphism (24), admits a unique extension \(f_i \in \hat{\Gamma} \subset \text{Out}^f(\hat{\text{P}}(S_{\sigma_i}, \partial S_{\sigma_i}))\).
There is a natural epimorphism $P\hat{\Gamma}(S_{\sigma_i}, \partial S_{\sigma_i}) \to P\hat{\Gamma}(S, \partial S)_{\sigma_i}$, for $i = 1, \ldots, g - 1$, whose kernel, topologically generated by the multitwists $\tau_{j_i}^{-1}$ for $j \neq i$, by the usual argument, is preserved by $f_i$. Hence, $f_i$ descends to $\bar{f}_i \in \text{Out}(P\hat{\Gamma}(S, \partial S)_{\sigma_i})$, for $i = 1, \ldots, g - 1$.

Let $\bar{f}_i \in \text{Aut}(P\hat{\Gamma}(S, \partial S)_{\sigma_i})$ be a lift of the element $f_i$ defined above. The same construction of the homomorphism (7) then provides a natural homomorphism:

$$P\hat{\Gamma}_{\sigma_i} : \text{Aut}(P\hat{\Gamma}(S, \partial S)_{\sigma_i}) \to \text{Out}(P\hat{\Gamma}(S, \partial S)_{\sigma_i}),$$

such that $P\hat{\Gamma}_{\sigma_i}(\bar{f}_i) = \bar{f}_i$, for $i = 1, \ldots, g - 1$. Let $x_i \in P\hat{\Gamma}(S, \partial S)_{\sigma_i}$ be an element such that $\text{inn} x_i \circ \bar{f}_i$ preserves the inertia group $\bar{I}_{\gamma_i}$ and $x_i$ fixes a chosen orientation on $\gamma_i$. As it is shown in the definition of the homomorphism (7), with these choices, $\text{inn} x_i \circ \bar{f}_i$ restricts to an automorphism of $P\hat{\Gamma}(S, \partial S)_{\sigma}$ which differs from $\bar{f}$ by an inner automorphism $\text{inn} y_i \in \text{Inn}(P\hat{\Gamma}(S, \partial S)_{\sigma})$. It follows that the automorphism $\bar{f}_i := \text{inn}(y_i x_i) \circ \bar{f}_i$ of the group $P\hat{\Gamma}(S, \partial S)_{\sigma_i}$ is an extension of $\bar{f}_i$, for all $i = 1, \ldots, g - 1$.

In order to conclude the proof of Lemma 6.20, we have to prove that this extension is unique, that is to say that, given an automorphism $\bar{f}_i' \in \text{Aut}(P\hat{\Gamma}(S, \partial S)_{\sigma_i})$ which preserves the subgroup $P\hat{\Gamma}(S, \partial S)_{\sigma_i}$ and restricts there to $\bar{f}_i$, we have $\bar{f}_i' = \bar{f}_i$, for $i = 1, \ldots, g - 1$.

By Theorem 2.7 and Theorem 2.8, the center $C$ of $P\hat{\Gamma}(S, \partial S)_{\sigma_i}$ is the direct sum:

$$C = \prod_{j=1}^{k} \hat{Z}_{\delta_j} \oplus \prod_{l=1,\ldots,g-1} \hat{Z}_{\gamma_i}.$$

In particular, the quotient of $P\hat{\Gamma}(S, \partial S)_{\sigma_i}$ by its center is isomorphic to $P\hat{\Gamma}(S \setminus \sigma_i)$ and so there is a natural homomorphism, for $i = 1, \ldots, g - 1$:

$$(25) \quad \text{Aut}(P\hat{\Gamma}(S, \partial S)_{\sigma_i}) \to \text{Aut}(P\hat{\Gamma}(S \setminus \sigma_i)).$$

Lemma 6.21. The homomorphism (25) is injective.

Proof. Let us consider the wreath products $\hat{Z}^* \wr \Sigma_k$ and $\hat{Z}^* \wr \Sigma_{g-2}$ and their respective canonical primitive actions on $\prod_{j=1}^{k} \hat{Z}_{\delta_j}$ and $\prod_{l=1,\ldots,g-1} \hat{Z}_{\gamma_i}$.

The natural action of $\text{Aut}(P\hat{\Gamma}(S, \partial S)_{\sigma_i})$ on $C$ factors through the above canonical actions and a natural homomorphism:

$$\text{Aut}(P\hat{\Gamma}(S, \partial S)_{\sigma_i}) \to (\hat{Z}^* \wr \Sigma_k) \times (\hat{Z}^* \wr \Sigma_{g-2}).$$

Let us then observe that this homomorphism can be recovered from:

- a natural character $\text{Aut}(\hat{\Gamma}(S, \partial S)_{\sigma_i}) \to \text{Aut}(\hat{\Gamma}(S, \partial S)_{\sigma_i})$, which, by the above construction and the case $g \leq 2$ of Theorem 6.5, is determined by the character $\chi_\lambda : \hat{\Gamma} \to \hat{Z}^*$;

- a natural character $\text{Aut}(\hat{\Gamma}(S, \partial S)_{\sigma_i}) \to \text{Aut}(\hat{\Gamma}(S, \partial S)_{\sigma_i})$, which is determined by the homomorphism (25) and a natural character $\text{Aut}(\hat{\Gamma}(S \setminus \sigma_i)) \to \text{Aut}(\hat{\Gamma}(S \setminus \sigma_i))$, where $\gamma$ is a nonseparating simple closed curve on $S \setminus \sigma_i$;
• a natural epimorphism \( \text{Aut}^i(\widehat{\Gamma}(S, \partial S)) \to \Sigma_k \times \Sigma_{g-2} \) which can be recovered from the homomorphism (25) and the epimorphism \( \text{Aut}^i(\widehat{\Gamma}(S \setminus \sigma_i)) \to \Sigma_{k+2g-4} \) (cf. (ii) of Theorem 4.3).

The above remarks imply that the action of \( \text{Aut}^i(\widehat{\Gamma}(S, \partial S)) \) on \( C \) is determined by the homomorphism (25). Since, by Lemma 7.4 in [7] and the same argument of the proof of Lemma 4.10, the natural homomorphism:

\[
\text{Aut}^i(\widehat{\Gamma}(S, \partial S)) \to \text{Aut}(C) \times \text{Aut}^i(\widehat{\Gamma}(S \setminus \sigma_i))
\]

is injective, it follows that the homomorphism (25) is already injective. \( \square \)

Let us now denote, respectively, by \( \tilde{f}_i \) and \( f_i^\prime \) the images in \( \text{Aut}^i(\widehat{\Gamma}(S \setminus \sigma_i)) \) of the elements \( f_i \) and \( f_i^\prime \) defined above by the monomorphism (25).

By hypothesis, \( f_i^\prime \cdot f_i^{-1} \) restricts to the identity on the subgroup \( \widehat{\Gamma}(S, \partial S) \) and so induces the identity automorphism on the quotient \( \widehat{\Gamma}(S \setminus \sigma_i)_{\bar{\gamma}_i} \) of this subgroup. From (i) of Lemma 5.6, it then follows that \( f_i^\prime \cdot f_i^{-1} \in \text{Inn}(\widehat{\Gamma}(S \setminus \sigma_i)) \).

By construction, \( f_i^\prime \cdot f_i^{-1} \) preserves the inertia group \( \hat{I}_{\gamma_i} \). Hence, \( f_i^\prime \cdot f_i^{-1} \in \text{Inn}(\widehat{\Gamma}(S \setminus \sigma_i)_{\bar{\gamma}_i}) \), where the latter group is identified with a subgroup of \( \text{Inn}(\widehat{\Gamma}(S \setminus \sigma_i)) \subset \text{Aut}^i(\widehat{\Gamma}(S \setminus \sigma_i)) \).

Above, we observed that \( f_i^\prime \cdot f_i^{-1} \) induces the identity on \( \widehat{\Gamma}(S \setminus \sigma_i)_{\bar{\gamma}_i} \) and so \( f_i^\prime \cdot f_i^{-1} \) identifies with an inner automorphism of \( \widehat{\Gamma}(S \setminus \sigma_i)_{\bar{\gamma}_i} \) which restricts to the identity on \( \widehat{\Gamma}(S \setminus \sigma_i)_{\bar{\gamma}_i} \). By Theorem 4.14 in [7], this implies that \( f_i^\prime \cdot f_i^{-1} = 1 \). Since the homomorphism (25) is injective, it follows that \( f_i^\prime \cdot f_i^{-1} = 1 \) and then \( f_i^\prime = f_i \).

The automorphisms \( \tilde{f}_i \), for \( i = 1, \ldots, g - 1 \), then glue to an automorphism:

\[
F \in \text{Aut}( \prod_{\text{PP}(S, \partial S)_{\bar{\sigma}_i}}^{i=1, \ldots, g-1} \text{PP}(S, \partial S)).
\]

Let us recall that \( S_{\bar{\alpha}_i} \) is the compact genus 1 subsurface of \( S \) with boundary \( \alpha_i \), for \( i = 1, \ldots, g - 1 \). By (i) of Theorem 2.3 and the subgroup congruence property for mapping class groups in genus \( \leq 2 \), there are natural monomorphisms \( \text{PP}(S_{\bar{\alpha}_i}, \alpha_i) \hookrightarrow \text{PP}(S, \partial S)_{\bar{\sigma}_i} \).

Let us identify \( \text{PP}(S_{\bar{\alpha}_i}, \alpha_i) \) with its image in \( \text{PP}(S, \partial S)_{\bar{\sigma}_i} \), for \( i = 1, \ldots, g - 1 \).

We defined the automorphism \( \tilde{f}_i \) of \( \text{PP}(S, \partial S)_{\bar{\sigma}_i} \) in such a way that it preserves the procyclic subgroup \( \tau_{\alpha_i}^{\pm 1} \). Thus, \( \tilde{f}_i \) also preserves the normalizer of \( \tau_{\alpha_i}^{\pm 1} \) in \( \text{PP}(S, \partial S)_{\bar{\sigma}_i} \). By Theorem 2.8, \( \tilde{f}_i \) then preserves the subgroup \( \text{PP}(S_{\bar{\alpha}_i}, \alpha_i) \), for \( i = 1, \ldots, g - 1 \).

Since, inside \( \text{PP}(S, \partial S) \), for \( i \neq j \), all elements of the subgroup \( \text{PP}(S_{\bar{\alpha}_i}, \alpha_i) \) commute with all the elements of the subgroup \( \text{PP}(S_{\bar{\alpha}_j}, \alpha_j) \), it follows that \( F \) sends the closed normal subgroup of \( \prod_{\text{PP}(S, \partial S)_{\bar{\sigma}_i}}^{i=1, \ldots, g-1} \text{PP}(S, \partial S)_{\bar{\sigma}_i} \) (topologically) generated by the commutators \([\tau_{\beta_i}, \tau_{\beta_j}]\), for \( i, j = 1, \ldots, g - 1 \), to itself.

By Corollary 6.19 and the Hopfian property of (topologically) finitely generated profinite groups, this implies that \( F \) descends to an automorphism \( F \in \text{Aut}(\text{PP}(S, \partial S)) \). Moreover,
by definition, $\bar{F}$ preserve the inertia groups $\hat{I}_{\gamma_i}$, for $i = 1, \ldots, g - 1$. In particular, we have that $\bar{F} \in \text{Aut}^0(\widehat{P}(S, \partial S))$.

The only choice involved in the definition of $F$ is that of the lift $\tilde{f} \in \text{Aut}^0(\widehat{P}(S_\sigma, \partial S_\sigma))$ of the given $f \in \widehat{G}$, which we had identified with an element of $\text{Out}^0(\widehat{P}(S_\sigma, \partial S_\sigma))$. Therefore, if we let $F$ be the image of $\bar{F}$ in $\text{Out}^0(\widehat{P}(S, \partial S))$, the assignment $f \mapsto F$ is well defined and defines a representation $\widehat{G}T \to \text{Out}^0(\widehat{P}(S, \partial S))$ as claimed in Theorem 6.11. That this representation is faithful, follows considering its restriction to the subgroup $\widehat{P}(S, \partial S)_{\bar{\sigma}}$. The last part of the statement of Theorem 6.11 about the action on the central inertia group is already clear by the construction above.

6.15. **Proof of Theorem 6.11 for $S = S_{g,n}^k$ and $n + k \neq 0$.** This case of Theorem 6.11 follows from the one considered in the previous section. In fact, the kernel of the natural epimorphism $\widehat{P}(S_{g,n}^{k+n}, \partial S_{g,n}^{k+n}) \to \widehat{P}(S_{g,n}^k, \partial S_{g,n}^k)$ is preserved by the outer action of $\widehat{G}T$ on $\widehat{P}(S_{g,n}^{k+n}, \partial S_{g,n}^{k+n})$ as defined in Section 6.14 above. Hence, we get a representation $\hat{\rho}_{\widehat{G}T}: \widehat{G}T \to \text{Out}^0(\widehat{P}(S, \partial S))$, for $S = S_{g,n}^k$, with the properties claimed in Theorem 6.11.

6.16. **Proof of Theorem 6.11 for $S = S_g$.** The closed surface case of Theorem 6.11 is instead a consequence of the following considerations, which will be fundamental also for the proof of the case $g(S) \geq 3$ of Lemma 6.7 given in Section 6.17.

With the notations of Section 6.14, the automorphism $\bar{F}$ restricts on the subgroup $\widehat{P}(S, \partial S)_{\bar{\sigma}}$ of $\widehat{P}(S, \partial S)$ to the automorphism $\hat{f} \in \text{Aut}^1(\widehat{P}(S, \partial S))$. We then have:

**Lemma 6.22.** Let $\gamma, \gamma'$ be simple nonseparating curves on $S \setminus \sigma$ which, together with the boundary curve $\delta_1$, bound a 3-holed genus 0 subsurface of $S$. Then, the automorphism $\hat{f}$ preserves the conjugacy class in $\widehat{P}(S, \partial S)_{\bar{\sigma}}$ of the procyclic subgroup generated by $\tau_{\gamma, \gamma'}$.

**Proof.** Let us observe again that, by the subgroup congruence property in genus $\leq 2$, we have $\widehat{P}(S \setminus \sigma) = \widehat{P}(S \setminus \sigma)$ and $\widehat{P}(S, \partial S)_{\bar{\sigma}} = \widehat{P}(S, \partial S)_{\bar{\sigma}}$.

The group $\text{Aut}^1(\widehat{P}(S, \partial S)_{\bar{\sigma}})$ acts on the complex of profinite curves $\mathcal{C}(S \setminus \sigma)$ through the natural homomorphism $\text{Aut}^1(\widehat{P}(S, \partial S)_{\bar{\sigma}}) \to \text{Aut}^1(\widehat{P}(S \setminus \sigma))$. From Theorem 5.5 and Proposition 7.2 in [7], it then follows that the topological type of the bounding pair $\{\gamma, \gamma'\} \in \mathcal{C}(S \setminus \sigma)$ is preserved by the action of $\hat{f}$, which means that $\hat{f}$ maps the pair $\{\gamma, \gamma'\}$ to some other bounding pair $\{\hat{f}(\gamma), \hat{f}(\gamma')\}$ of the same topological type, that is to say, in the $\text{Inn}(\widehat{P}(S, \partial S)_{\bar{\sigma}})$-orbit of $\{\gamma, \gamma'\}$.

Since $\hat{f}$ preserves the procyclic subgroup generated by $\tau_{\delta_1}$, from Theorem 2.7 and Theorem 2.8, it follows that it also preserves the marked topological type of the bounding pair $\{\gamma, \gamma'\}$, that is to say, the bounding pair $\{\hat{f}(\gamma), \hat{f}(\gamma')\}$ is in the $\text{Inn}(\widehat{P}(S, \partial S)_{\bar{\sigma}})$-orbit of $\{\gamma, \gamma'\}$. By the construction of the automorphism $\hat{f}$, we also know that it acts on a nonseparating profinite Dehn twist in $\widehat{P}(S, \partial S)_{\bar{\sigma}}$ by conjugation twisted by the character $\chi_\lambda: \mathcal{G} \to \hat{\mathbb{Z}}^*$. These facts then imply that $\hat{f}$ preserves the conjugacy class in $\widehat{P}(S, \partial S)_{\bar{\sigma}}$ of the procyclic subgroup generated by $\tau_{\gamma, \gamma'}$. $\square$
Let $\mathcal{S}$ be the surface obtained gluing a disc $D$ to the boundary component $\delta_1$ of $S$ and let us fix a base point $P \in D$. There is then a profinite Birman exact sequence:

$$1 \to \hat{\pi}_1(\mathcal{S}, P) \times \tau^2_{\delta_1} \to \hat{\Pi}(S, \partial S) \to \hat{\Pi}(\mathcal{S}, \partial \mathcal{S}) \to 1,$$

where the conjugacy class of $\tau^{-1}_{\gamma} \tau^\gamma$ in $\hat{\Pi}(\mathcal{S}, \partial \mathcal{S})$ (topologically) generates the normal subgroup $\hat{\pi}_1(\mathcal{S}, P)$. By Lemma 6.22, the automorphism $\hat{F}$ preserves the conjugacy class in $\hat{\Pi}(S, \partial S)$ of the procyclic subgroup generated by $\tau^{-1}_{\gamma} \tau^\gamma$. Therefore, the representation $\hat{\rho}_{\mathcal{S}}: \hat{\Gamma} \to \text{Out}^\infty(\hat{\Pi}(\mathcal{S}, \partial \mathcal{S}))$ preserves the normal subgroup $\hat{\pi}_1(\mathcal{S}, P) \times \tau^2_{\delta_1}$ of $\hat{\Pi}(S, \partial S)$ and so induces a representation on the quotient:

$$\hat{\rho}_{\mathcal{S}}: \hat{\Gamma} \to \text{Out}^\infty(\hat{\Pi}(\mathcal{S}, \partial \mathcal{S})).$$

That this representation is also faithful follows from its compatibility with the restriction of $\hat{\rho}_{\mathcal{S}}$ to modular subgroups of $\hat{\Pi}(\mathcal{S}, \partial \mathcal{S})$. Note that, for $k = 1$, the surface $\mathcal{S}$ is closed and $\partial \mathcal{S} = \emptyset$, so that, in particular, we get the closed surface case of Theorem 6.11.

6.17. **Proof of Lemma 6.7 for $g(S) \geq 3$.** The image of the natural representation:

$$\hat{\rho}_{\mathcal{S}}: \hat{\Pi}(S, \partial S) \to \text{Aut}(\hat{\pi}_1(\mathcal{S}, P)),$$

induced by restriction of inner automorphisms to the normal subgroup $\hat{\pi}_1(\mathcal{S}, P)$ appearing in the profinite Birman exact sequence (26), is precisely the procongruence mapping class group $\hat{\Pi}(S)$ (cf. Corollary 4.7 in [5]).

Let us then show that $\hat{\rho}_{\mathcal{S}}: \hat{\Gamma} \to \text{Out}^\infty(\hat{\Pi}(\mathcal{S}, \partial \mathcal{S}))$ induces a representation:

$$\hat{\rho}_{\mathcal{S}}: \hat{\Gamma} \to \text{Out}^\infty(\hat{\Pi}(S)).$$

This will follow from the lemma:

**Lemma 6.23.** The outer automorphisms of $\hat{\Pi}(S, \partial S)$ in the image of $\hat{\rho}_{\mathcal{S}}$ preserve the normal subgroup $\ker \hat{\rho}_{\mathcal{S}}$ of $\hat{\Pi}(S, \partial S)$.

**Proof.** Let $\hat{\phi} \in \text{Out}^\infty(\hat{\Pi}(\mathcal{S}, \partial \mathcal{S}))$ be a lift of an element $\phi \in \text{Im} \hat{\rho}_{\mathcal{S}} \subset \text{Out}^\infty(\hat{\Pi}(S, \partial S))$. By Lemma 6.22, we have that $\hat{\phi}(\hat{\pi}_1(\mathcal{S}, P)) = \hat{\pi}_1(\mathcal{S}, P)$. Let $x \in \ker \hat{\rho}_{\mathcal{S}}$, that is to say an element $x \in \hat{\Pi}(S, \partial S)$ such that the restriction of $\text{inn} x$ to $\hat{\pi}_1(\mathcal{S}, P)$ is trivial. Then, also $\hat{\phi} \circ \text{inn} x \circ \hat{\phi}^{-1}$ restricts to the trivial automorphism on $\hat{\pi}_1(\mathcal{S}, P)$ and, from the identity $\hat{\phi} \circ \text{inn} x \circ \hat{\phi}^{-1} = \text{inn}(\hat{\phi}(x))$, it follows that $\hat{\phi}(x) \in \ker \hat{\rho}_{\mathcal{S}}$. \hfill $\Box$

Note that the representation $\hat{\rho}_{\mathcal{S}}$ (27) is defined for every nonclosed surface. Composing with the natural homomorphism $\text{Out}^\upper(\hat{\Pi}(S)) \to \text{Out}^\upper(\hat{\Gamma}(\mathcal{S}))$ (cf. Theorem 3.12 and Theorem 4.3), we then get the natural representation:

$$\Psi(\mathcal{S}, \emptyset): \hat{\Gamma} \to \text{Out}^\upper(\hat{\Gamma}(\mathcal{S})).$$

If $\mathcal{S} = S' \setminus P$, where $S'$ is a closed surface, we can further compose $\Psi(\mathcal{S}, \emptyset)$ with the homomorphism (14) $B_P: \text{Out}^\upper(\hat{\Gamma}(\mathcal{S})) \to \text{Out}^\upper(\hat{\Gamma}(S'))$, thus obtaining the representation:

$$\Psi(\mathcal{S}', \emptyset): \hat{\Gamma} \to \text{Out}^\upper(\hat{\Gamma}(S')).$$
By construction, these representations are functorial with respect to the restriction to subsurfaces of $S$. This proves Lemma 6.7 for all the cases when $g(S) \geq 3$ and $\partial S = \emptyset$.

The representation $\Psi(S, S)$, for $g(S) \geq 3$ and $\partial S \neq \emptyset$, is then obtained composing $\Psi(S, S)$, for $\tilde{S} = S_{g,n+2k}$ as in Remark 2.4, with the homomorphism $R(S, S)$ (21).

6.18. **Proof of Theorem 6.3.** Let us observe that there are natural homomorphisms:

$$\hat{\tau}_S: \text{Aut}(\hat{\Sigma}^{\text{out}}) \to \text{Out}(P\hat{\Gamma}(S, S)) \quad \text{and} \quad \hat{\tau}_S: \text{Aut}(\hat{\Sigma}^{\text{out}}) \to \text{Out}(P\hat{\Gamma}(S, S)),$$

defined restricting an automorphism of the full tower to one of its components.

**Lemma 6.24.** There holds $\text{Im} \hat{\tau}_S \subseteq \text{Out}^I(P\hat{\Gamma}(S, S))$ and $\text{Im} \hat{\tau}_S \subseteq \text{Out}^I(P\hat{\Gamma}(S, S))$.

**Proof.** The proof is exactly the same for the profinite and the procongruence case. Let us then prove the lemma for the procongruence Grothendieck-Teichmüller tower which is the case we are going to use later.

Let us show first that, for $\partial S = \bigcup_{i=1}^k \delta_i \neq \emptyset$, the image of $\text{Aut}(\hat{\Sigma}^{\text{out}})$ in the group $\text{Out}^I(P\hat{\Gamma}(S, S))$ preserves each procyclic subgroup $\tau_{\delta_i}^{\hat{S}}$, for $i = 1, \ldots, k$. Let $\hat{S}_i$ be the surface obtained attaching a 1-punctured disc to the boundary component $\delta_i$ of $S$. The map $S \to \hat{S}_i$ then induces the homomorphism of procongruence pure relative mapping class groups $P\hat{\Gamma}(S, S) \to P\hat{\Gamma}(\hat{S}_i, \partial \hat{S}_i)$, in the procongruence Grothendieck-Teichmüller tower. An element of $\text{Aut}(\hat{\Sigma}^{\text{out}})$ is compatible with this homomorphism and so preserves its kernel which is precisely the procyclic subgroup $\tau_{\delta_i}^{\hat{S}}$, for $i = 1, \ldots, k$.

In order to show that the image of $\text{Aut}(\hat{\Sigma}^{\text{out}})$ in $\text{Out}^I(P\hat{\Gamma}(S, S))$ preserves the conjugacy class of the procyclic subgroup generated by the Dehn twist $\tau_\gamma$, where $\gamma$ is any nonperipheral simple closed curve on $S$, it is enough to observe that there is an embedding $S' \hookrightarrow S$ in $S$ such that $\gamma$ is in the image of a boundary component of $S'$. From the previous case, it then follows that an automorphism of $\text{Aut}(\hat{\Sigma}^{\text{out}})$ is compatible with the associated homomorphism $P\hat{\Gamma}(S', \partial S') \to P\hat{\Gamma}(S, \partial S)$ only if its image in $\text{Out}(P\hat{\Gamma}(S, \partial S))$ preserves the conjugacy class of $\tau_{\gamma}^{\hat{S}}$.

**Remark 6.25.** Lemma 6.24 shows, in particular, that the so called ”inertia conditions” of Grothendieck-Teichmüller theory are intrinsic to the whole Grothendieck-Teichmüller tower, even though they might not be intrinsic to the single components (the conjecture, however, is that, for $d(S) > 1$, they are).

From Theorem 6.5, it follows that there is a natural monomorphism $\hat{\Gamma} \hookrightarrow \text{Aut}(\hat{\Sigma}^{\text{out}})$. In order to prove that $\text{Aut}(\hat{\Sigma}^{\text{out}}) \cong \hat{\Gamma}$, it is then enough to determine the image of $\hat{\tau}_S$, for $S = S_g^k$ and $d(S) > 2$.

By (v) of Theorem 6.5, for $S = S_g^k$ and $d(S) > 2$, we have $\text{Out}^I(P\hat{\Gamma}(S, \partial S)) \cong \Sigma_k \times \hat{\Gamma}$ so that, by Lemma 6.24, to prove Theorem 6.3, it is enough to show that $\text{Im} \hat{\tau}_S$ has trivial projection to $\Sigma_k$, for $S \neq S_{1,2}$. This claim follows considering the natural maps $S \to \ol{S}_i$, where $\ol{S}_i$ is the surface obtained glueing a disc $D$ to the boundary component $\delta_i$ of $S = S_g^k$, for $i = 1, \ldots, k$. In fact, an element of $\text{Out}^I(P\hat{\Gamma}(S, \partial S))$, which projects to a nontrivial element of the symmetric group $\Sigma_k \cong \text{Inn}(\hat{\Gamma}(S, \partial S))/\text{Inn}(P\hat{\Gamma}(S, \partial S))$, does not preserve
the kernel \( \hat{\pi}_1(\overline{S}_i, P) \times \tau_{\delta_i}^e \) of the homomorphism \( \text{P}\hat{\Gamma}(S, \partial S) \to \text{P}\hat{\Gamma}(\overline{S}_i, \partial \overline{S}_i) \), for \( i = 1, \ldots, k \) (cf. (26)).

7. The automorphism group of the arithmetic procongruence mapping class group

In this section, we assume that \( S \) is a hyperbolic surface without boundary.

7.1. The arithmetic profinite mapping class group. By Grothendieck theory of the \( \acute{e}tale \) fundamental group, for a geometric base point \( \xi \) of the moduli stack \( \mathcal{M}(S)_\mathbb{Q} \), the structural morphism \( \mathcal{M}(S)_\mathbb{Q} \to \text{Spec}(\mathbb{Q}) \) induces the short exact sequence of \( \acute{e}tale \) fundamental groups:

\[
1 \to \pi_1^{et}(\mathcal{M}(S)_{\overline{\mathbb{Q}}}, \overline{\xi}) \to \pi_1^{et}(\mathcal{M}(S)_\mathbb{Q}, \overline{\xi}) \to G_\mathbb{Q} \to 1.
\]

The left hand term \( \pi_1^{et}(\mathcal{M}(S)_{\overline{\mathbb{Q}}}, \overline{\xi}) \) of this short exact sequence is the geometric \( \acute{e}tale \) fundamental group of \( \mathcal{M}(S)_\mathbb{Q} \). It is naturally isomorphic to the profinite completion of the topological fundamental group, with base point the image of \( \overline{\xi}_\mathbb{C} \), of the complex analytic stack associated to the complex DM stack \( \mathcal{M}(S)_\mathbb{C} \).

Thus, \( \pi_1^{et}(\mathcal{M}(S)_{\overline{\mathbb{Q}}}, \overline{\xi}) \) can be identified with the profinite completion \( \hat{\Gamma}(S) \) of the mapping class group \( \Gamma(S) \) associated to the surface \( S \). To be more precise, this identification is determined by the isotopy class of a diffeomorphism \( S \to (\mathcal{C}(S)_\xi \times \mathbb{C})^{an} \) compatible with the orientations.

A (tangential) rational point on \( \mathcal{M}(S)_\mathbb{Q} \) determines a splitting of the short exact sequence (28). There is then a (faithful) representation \( \hat{\rho}_\mathbb{Q}: G_\mathbb{Q} \to \text{Aut}(\hat{\Gamma}(S)) \) which induces an isomorphism of profinite groups:

\[
\pi_1^{et}(\mathcal{M}(S)_\mathbb{Q}, \overline{\xi}) \cong \hat{\Gamma}(S) \rtimes \hat{\rho}_\mathbb{Q} G_\mathbb{Q}.
\]

It is then natural to denote by \( \hat{\Gamma}(S)_\mathbb{Q} \) the profinite group \( \pi_1^{et}(\mathcal{M}(S)_\mathbb{Q}, \overline{\xi}) \) which we will call the arithmetic profinite mapping class group. Let \( \text{P}\hat{\Gamma}(S) \) be the profinite completion of the pure mapping class group \( \text{P}\Gamma(S) \) associated to the surface \( S \). After the identification of \( \pi_1^{et}(\text{P}\mathcal{M}(S)_{\overline{\mathbb{Q}}}, \overline{\xi}) \) with \( \text{P}\hat{\Gamma}(S) \), we will then also put \( \text{P}\hat{\Gamma}(S)_\mathbb{Q} := \pi_1^{et}(\text{P}\mathcal{M}(S)_\mathbb{Q}, \overline{\xi}) \).

7.2. The arithmetic procongruence mapping class group. Let \( \mathcal{C}(S) \to \mathcal{M}(S) \) be the universal punctured curve over the moduli stack \( \mathcal{M}(S) \) and \( \text{P}\mathcal{C}(S) \to \text{P}\mathcal{M}(S) \) be the universal curve over the moduli stack \( \mathcal{M}(S) \).

Let \( \mathcal{C}(S)_\overline{\xi} \) be the fiber over the geometric point \( \overline{\xi} \in \mathcal{M}(S)_\mathbb{Q} \) of the universal curve \( \mathcal{C}(S) \to \mathcal{M}(S) \) and \( \overline{\xi} \in \mathcal{C}(S)_\overline{\xi} \) a geometric point. There is then a short exact sequence of algebraic fundamental groups:

\[
1 \to \pi_1^{et}(\mathcal{C}(S)_\overline{\xi}, \overline{\xi}) \to \pi_1^{et}(\mathcal{C}(S)_\mathbb{Q}, \overline{\xi}) \to \pi_1^{et}(\mathcal{M}(S)_\mathbb{Q}, \overline{\xi}) \to 1.
\]

The associated outer representation

\[
\rho_\mathbb{Q}^{et}: \pi_1^{et}(\mathcal{M}(S)_\mathbb{Q}, \overline{\xi}) \to \text{Out}(\pi_1^{et}(\mathcal{C}(S)_\overline{\xi}, \overline{\xi}))
\]

is called the universal \( \acute{e}tale \) monodromy representation.
After identifying $\pi^e_1(\mathcal{M}(S)_{\overline{q}}, \overline{\xi})$ with $\hat{\Gamma}(S)$ and $\pi^e_1(\mathcal{M}(S)_Q, \overline{\xi})$ with $\hat{\Gamma}(S)_Q$, we then let
\[ \hat{\Gamma}(S) := \rho_S^e(\hat{\Gamma}(S)) \quad \text{and} \quad \hat{\Gamma}(S)_Q := \rho_S^e(\hat{\Gamma}(S)_Q) \]
and call them, respectively, the procongruence and the arithmetic procongruence mapping class groups. The pure procongruence and arithmetic procongruence mapping class groups are then defined by:
\[ \mathcal{P}\hat{\Gamma}(S) := \rho_S^e(\mathcal{P}\hat{\Gamma}(S)) \quad \text{and} \quad \mathcal{P}\hat{\Gamma}(S)_Q := \rho_S^e(\mathcal{P}\hat{\Gamma}(S)_Q). \]

Hoshi and Mochizuki in [14] (see also Corollary 7.10 in [3]) showed that the kernel of $\rho_S^e$ identifies with the congruence kernel, i.e. the kernel of the natural epimorphism $\hat{\Gamma}(S) \to \hat{\Gamma}(S)$. Therefore, the short exact sequence (28) gives rise to a short exact sequence:
\begin{equation}
1 \to \hat{\Gamma}(S) \to \hat{\Gamma}(S)_Q \to G_Q \to 1,
\end{equation}
such that the associated outer representation $\rho_Q : G_Q \to \text{Out}(\hat{\Gamma}(S))$ is faithful.

The splitting of the short exact sequence (28) determined by a tangential base point also determines a splitting of the short exact sequence (29). There is then a (faithful) representation $\rho_Q : G_Q \to \text{Aut}(\hat{\Gamma}(S))$ which induces an isomorphism of profinite groups:
\begin{equation}
\hat{\Gamma}(S)_{Q} \cong \hat{\Gamma}(S) \rtimes_{\rho_Q} G_Q.
\end{equation}

7.3. The automorphism group of the arithmetic procongruence mapping class group. Let us define inertia and decomposition groups for the arithmetic procongruence mapping class group $\hat{\Gamma}(S)_Q$:

**Definition 7.1.** Let $U$ be an open subgroup of $\hat{\Gamma}(S)_Q$. For $\sigma = \{\gamma_0, \ldots, \gamma_k\} \in \check{C}(S)$, the **inertia group** $\check{I}_\sigma(U)$ is the closed abelian subgroup of $U$ topologically generated by the powers of the profinite Dehn twists parameterized by $\sigma$ and contained in $U$. The **decomposition group** $D_\sigma(U)$ is then the normalizer of $\check{I}_\sigma(U)$ in $U$.

**Remark 7.2.** Since all closed strata of $\mathcal{M}(S)$ are defined over $\mathbb{Q}$ and contain $\mathbb{Q}$-rational points, for every $\sigma \in \check{C}(S)$, there is a splitting of the short exact sequence (29) such that the action $\check{\rho}_Q : G_Q \to \text{Aut}(\hat{\Gamma}(S))$ preserves the stabilizer $\hat{\Gamma}(S)_\sigma$ and so the associated isomorphism (30) induces an isomorphism $D_\sigma(\hat{\Gamma}(S)_Q) \cong \hat{\Gamma}(S)_\sigma \rtimes_{\check{\rho}_Q} G_Q$.

**Definition 7.3.** For $U$ an open subgroup of $\hat{\Gamma}(S)_Q$, we define:
(i) $\text{Aut}^D(U)$ is the closed subgroup of $\text{Aut}(U)$ consisting of those elements which preserve the set of decomposition groups $\{D_\gamma(U)\}_{\gamma \in \check{E}(S)_0}$.
(ii) $\text{Aut}^I(U)$ is the closed subgroup of $\text{Aut}(U)$ consisting of those elements which preserve the set of inertia groups $\{\check{I}_\gamma(U)\}_{\gamma \in \check{E}(S)_0}$.
(iii) $\text{Aut}^{I_0}(U)$ is the closed subgroup of $\text{Aut}^I(U)$ consisting of those elements which preserve the set of inertia groups $\{\check{I}_\gamma(U)\}_{\gamma \in \check{E}_{\text{iso}}(S)}$.

**Remark 7.4.** A result similar to (i) of Proposition 3.5 holds for all the groups of automorphisms defined above.
With the above notation, Theorem 9.16 in [7] states, in particular, that, for $U$ an open normal subgroup of $\tilde{\Gamma}(S)_Q$ and $S \neq S_{1,2}, S_2$, we have:

$\text{Aut}^D(U) = \text{Inn}(\tilde{\Gamma}(S)_Q) \cong \tilde{\Gamma}(S)_Q$.  

(31)

Thanks to the results of Section 3.4, we then get the following generalization of Corollary C in [17]:

**Corollary 7.5.** For $d(S) > 1$, we have:

$\text{Aut}^I_0(\tilde{\Pi}(S)_Q) \cong \text{Aut}^I_0(\tilde{\Gamma}(S)_Q) = \text{Inn}(\tilde{\Gamma}(S)_Q)$.

**Proof.** For $S \neq S_{1,2}$ and $S_2$, the centralizer of $\tilde{\Pi}(S)_Q$ in $\tilde{\Gamma}(S)_Q$ is trivial. Therefore, since $\tilde{\Pi}(S)_Q$ is an $\mathbb{I}_0$-characteristic subgroup of $\tilde{\Gamma}(S)_Q$, by Lemma 3.3 in [7], Theorem 3.12, (i) of Proposition 3.7 and the identity (31), for $S \neq S_{1,2}$ and $S_2$, there is a series of natural monomorphisms

$\text{Inn}(\tilde{\Gamma}(S)_Q) \hookrightarrow \text{Aut}^I_0(\tilde{\Gamma}(S)_Q) \hookrightarrow \text{Aut}^I_0(\tilde{\Pi}(S)_Q) \hookrightarrow \text{Aut}^D(\tilde{\Pi}(S)_Q) = \text{Inn}(\tilde{\Gamma}(S)_Q)$

and the conclusion follows.

The cases $S = S_{1,2}$ and $S_2$ can be reduced to the above observing that $\text{Aut}^I_0(\tilde{\Gamma}(S)_Q) = \text{Aut}^I_0(\tilde{\Gamma}(S)_Q/Z(\tilde{\Gamma}(S)_Q))$ (cf. the proofs of (ii) of Proposition 3.9 and of Proposition 6.13).

\[ \square \]

**7.4. The extended mapping class group.** The *extended mapping class group* $\Gamma(S)_R$ is the group of isotopy classes of all diffeomorphisms of $S$. If $\chi_R: \Gamma(S)_R \to \{\pm 1\}$ denotes the orientation character, there is a short exact sequence:

$1 \to \Gamma(S) \to \Gamma(S)_R \xrightarrow{\chi_R} \{\pm 1\} \to 1$.  

(32)

Note that the choice of any antiholomorphic involution on $S$ splits this sequence.

The reason we denoted the extended mapping class group by $\Gamma(S)_R$ comes from the following construction. The moduli stack $\mathcal{M}(S)_R$ is a real model of the complex moduli stack $\mathcal{M}(S)_C$. Hence, the latter carries an antiholomorphic involution $\iota: \mathcal{M}(S)_C \to \mathcal{M}(S)_C$ associated to such real model. There is a notion of equivariant fundamental group associated to $(\mathcal{M}(S)_C, \iota)$ (cf. Section 3 in [16]) and, for a base point $[C] \in \mathcal{M}(S)_R$, we let

$\pi_1^R(\mathcal{M}(S)_R, [C]) := \pi_1^{\text{equiv}}((\mathcal{M}(S)_C/\langle \iota \rangle), [C])$.

The equivariant fundamental group $\pi_1^R(\mathcal{M}(S)_R, [C])$ then identifies with the extended mapping class group $\Gamma(S)_R$ and the short exact sequence (32) identifies with the associated short exact sequence (cf. Proposition 3.2 in [16]). Let us also observe that, in this setting, the orientation group $\{\pm 1\}$ identifies with the absolute Galois group of the reals $G_R$.

**7.5. Automorphisms of extended mapping class group.** Let us denote respectively by $\text{Aut}^I_0(\Gamma(S))$ and $\text{Aut}^I_0(\Gamma(S)_R)$ the group of automorphisms of $\Gamma(S)$ and $\Gamma(S)_R$ which preserve the conjugacy class of a cyclic subgroup generated by a Dehn twist about a nonseparating simple closed curve.
A classical result by Ivanov and McCarthy (cf. Theorem 2 in [19] and Theorem 1 in [20]) asserts that, for \( d(S) > 1 \), there holds (here, the \( I_0 \)-condition is only needed for the exceptional low genus cases):

\[
\text{Aut}^{I_0}(\Gamma(S)) \simeq \text{Aut}^{I_0}(\Gamma(S)_\mathbb{R}) = \text{Inn}(\Gamma(S)_\mathbb{R}).
\]

This implies that, for \( d(S) > 1 \), there is also a natural isomorphism \( \text{Out}^{I_0}(\Gamma(S)) \simeq G_\mathbb{R} \).

### 7.6. Comparing automorphism groups.

As observed in the proof of Theorem 9.16 in [7], \( \hat{\Gamma}(S) \) is a characteristic subgroup of \( \hat{\Gamma}(S)_\mathbb{Q} \). By Lemma 9.18 in [7], there is then a natural monomorphism \( \text{Aut}^{I_0}(\hat{\Gamma}(S)_\mathbb{Q}) \hookrightarrow \text{Aut}^{I_0}(\hat{\Gamma}(S)) \) (for the case \( Z(\hat{\Gamma}(S)_\mathbb{Q}) \neq \{1\} \), as in the proof of Corollary 7.5, one just need to observe that \( \text{Aut}^{I_0}(\hat{\Gamma}(S)_\mathbb{Q}) = \text{Aut}^{I_0}(\hat{\Gamma}(S)_\mathbb{Q}/Z(\hat{\Gamma}(S)_\mathbb{Q})) \)).

A natural question is whether, in analogy with the extended mapping class group case, this is an isomorphism for \( d(S) > 1 \):

**Question 7.6.** For \( d(S) > 1 \), is there a natural isomorphism:

\[
\text{Aut}^{I_0}(\hat{\Gamma}(S)_\mathbb{Q}) \simeq \text{Aut}^{I_0}(\hat{\Gamma}(S))?
\]

Corollary 7.5 and (i) of Theorem 6.5 then imply:

**Corollary 7.7.** A positive answer to Question 7.6 implies that, for \( d(S) > 1 \), there is a natural isomorphism

\[
\text{Out}^{I_0}(\hat{\Gamma}(S)) \simeq \hat{\Gamma} \simeq G_\mathbb{Q}.
\]

**Proof.** The series of isomorphisms \( \text{Aut}^{I_0}(\hat{\Gamma}(S)_\mathbb{Q}) \simeq \text{Aut}^{I_0}(\hat{\Gamma}(S)) \simeq \text{Inn}(\hat{\Gamma}(S)_\mathbb{Q}) \) implies the conjecture after dividing out by the normal subgroup \( \text{Inn}(\hat{\Gamma}(S)) \). \( \square \)

### 7.7. Automorphisms of the procongruence curve complex.

The automorphism group \( \text{Aut}^{I_0}(\hat{\Gamma}(S)_\mathbb{Q}) \) acts on the procongruence curve complex \( \hat{C}(S) \) through the natural homomorphism \( \text{Aut}^{I_0}(\hat{\Gamma}(S)_\mathbb{Q}) \to \text{Aut}^{I_0}(\hat{\Gamma}(S)) = \text{Aut}(\hat{\Gamma}(S)) \) considered in Section 7.6.

**Proposition 7.8.** For \( d(S) > 1 \), there is a natural faithful representation:

\[
\psi: \text{Aut}^{I_0}(\hat{\Gamma}(S)_\mathbb{Q}) \hookrightarrow \text{Aut}(\hat{C}(S)).
\]

**Proof.** As observed in Section 7.6, the homomorphism \( \text{Aut}^{I_0}(\hat{\Gamma}(S)_\mathbb{Q}) \to \text{Aut}^{I_0}(\hat{\Gamma}(S)) \) is injective. The conclusion then follows from (ii) of Theorem 7.3 in [7]. \( \square \)

Ivanov’s fundamental theorem on automorphisms of curve complexes can be formulated, in our terminology, saying that, for \( d(S) > 1 \), there is a natural isomorphism (cf. [19], Theorem 1):

\[
\text{Aut}^{I_0}(\Gamma(S)_\mathbb{R}) \simeq \text{Aut}(C(S)),
\]

where, again, the \( I_0 \)-condition is only needed for the exceptional low genus cases. A similar question can then be asked for the arithmetic procongruence mapping class group:

**Question 7.9.** For \( d(S) > 1 \), is there a natural isomorphism:

\[
\text{Aut}^{I_0}(\hat{\Gamma}(S)_\mathbb{Q}) \simeq \text{Aut}(\hat{C}(S))?\]
By (ii) of Theorem 7.3 in [7] and Proposition 7.8, with the above hypotheses, there is a series of monomorphisms:
\[ \text{Aut}^I_0(\hat{\Gamma}(S)) \hookrightarrow \text{Aut}^I_0(\hat{\Gamma}(S)) \hookrightarrow \text{Aut}(\hat{C}(S)). \]
Therefore, a positive answer to Question 7.9 implies a positive answer to Question 7.6.

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