Four dimensional supersymmetrization of $\mathcal{R}^4$

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Abstract

We review our recent works on the supersymmetrization of the leading string correction (the $\mathcal{R}^4$ term) to $\mathcal{N}=1,2$ supergravity theories in four dimensions. We show that, in the "old minimal" formulations of these theories, when going on-shell in the presence of this correction, the auxiliary fields which come from multiplets with physical fields cannot be eliminated, but those ones that come from compensating multiplets without any physical fields can be eliminated. We conjecture similar results for other versions of these theories.
1 \( \mathcal{N} = 1, 2 \) supergravity: from conformal to Poincaré

\( \mathcal{N} = 1, 2 \) Poincaré supergravities can be obtained from the corresponding conformal theories by consistent couplings to compensating multiplets that break superconformal invariance and local U(\( \mathcal{N} \)). There are different possible choices of compensating multiplets, leading to different formulations of the Poincaré theory. What is special about these theories is the existence of a completely off-shell formalism. This means that, for each of these theories, a complete set of auxiliary fields is known (actually, there exist three known choices for each theory). In superspace this means that, after imposing constraints on the torsions, we can completely solve the Bianchi identities without using the field equations [1, 2], and there is a perfect identification between the superspace and \( x \)-space descriptions.

1.1 The \( \mathcal{N} = 1 \) case

The fields of \( \mathcal{N} = 1 \) conformal supergravity multiplet are the graviton \( e^m_\mu \), the gravitino \( \psi^A_\mu \) and a U(1) gauge field \( A_\mu \). In order to break the superconformal invariance and obtain the ”old minimal” formulation of \( \mathcal{N} = 1 \) Poincaré supergravity [3, 4], one must impose some constraint which restricts a linear combination of the superconformal parameters \( L, L \) to a compensating chiral multiplet [5]. This is achieved by setting the nonconformal superspace torsion constraint

\[
T_{Am} = 0 \quad (1.1)
\]

This formulation of supergravity is described in terms of the superfields \( R, G_m, W_{ABC} \), their complex conjugates and their covariant derivatives. \( R \) and \( W_{ABC} \) are antichiral:

\[
\nabla^A R = 0, \quad \nabla_A W_{AB\dot{C}} = 0 \quad (1.2)
\]

The Bianchi identities and all the torsion constraints imply the following off-shell differential relations between the \( \mathcal{N} = 1 \) supergravity superfields:

\[
\nabla^A G_{AB} = \frac{1}{24} \nabla_B R \quad (1.3)
\]

\[
\nabla^A W_{ABC} = i \left( \nabla_{BA} G_{C}^A + \nabla_{CA} G_{B}^A \right) \quad (1.4)
\]

which imply the relation

\[
\nabla^2 R - \nabla^2 \overline{R} = 96i \nabla^n G_n \quad (1.5)
\]

At \( \theta = 0 \), we have \( G_m \big| = A_m \), now an auxiliary field. The (anti)chirality condition on \( R, \overline{R} \) implies their \( \theta = 0 \) components (resp. the auxiliary fields \( M - iN, M + iN \)) lie in antichiral/chiral multiplets (the compensating multiplets); (1.3) shows the spin-1/2 parts of the gravitino lie on the same multiplets (because \( \nabla_A G_{BB} \big| \) includes the gravitino curl) and, according to (1.5), so does \( \partial^\mu A_\mu \).

The superspace action for this formulation of supergravity is given by [6]:

\[
\mathcal{L}_{SG} = \frac{1}{2k^2} \int E d^4 \theta, \quad E = \text{sdet} E^M_A \quad (1.6)
\]
1.2 The $\mathcal{N} = 2$ case

The $\mathcal{N} = 2$ Weyl multiplet has 24+24 degrees of freedom. Its field content is given by the graviton $e^m_\mu$, the gravitinos $\psi^{\hat{A}a}$, the U(2) connection $\Phi^{ab}_\mu$, an antisymmetric tensor $W_{A\hat{A}B\hat{B}} = 2\varepsilon_{\hat{A}\hat{B}}W_{AB} + 2\varepsilon_{AB}W_{\hat{A}\hat{B}}$, a spinor $\Lambda^a_\hat{A}$ and, as auxiliary field, a dimension 2 scalar $I$.

In U(2) $\mathcal{N} = 2$ superspace there is an off-shell solution to the Bianchi identities. The torsions and curvatures can be expressed in terms of superfields $Y_{AB}$, $U_{\hat{A}a}$, $X_{ab}$ and the previously mentioned $W_{AB}$, their complex conjugates and their covariant derivatives. In order to obtain the Poincaré theory, the first step is to couple to the conformal theory an abelian vector multiplet (with central charge), which includes a vector $A_\mu$ described, in superspace, by a 1-form $A_\mu$. The field strength $F_{\mu\nu}$ satisfies its own Bianchi identities $\nabla_{[\mu}F_{\nu\rho]} = 0$. After imposing conventional and conformal-breaking constraints in some of its components, we solve the Bianchi identities for the fermionic derivatives $P, \rho_a$. Although the algebra closes with this multiplet, it does not admit a consistent lagrangian because of the higher-dimensional scalar $I$.

In order to obtain the "old minimal" Poincaré theory, we break the remaining local SU(2) invariance, by restricting its parameter $L^{ab}$ to a compensating nonlinear multiplet [7]. This is achieved by imposing the following constraint on the fermionic SU(2) connection:

$$\Phi^{abc}_A = 2\varepsilon^{ab}\varepsilon^c_A$$

(1.7)

This constraint requires introducing a new fermionic superfield $\rho^a_\hat{A}$. We also introduce its fermionic derivatives $P$ (a complex scalar) and $H_m$. The previous SU(2) connection $\Phi^{ab}_\mu$ is now an unconstrained auxiliary field. The divergence of the vector field $H_m$ is constrained by

$$I = 4R - 6\nabla_{\hat{A}}H_{\hat{A}\hat{A}} - 24X^{ab}X_{ab} - 12W^{AB}Y_{AB} - 12W^{\hat{A}\hat{B}}Y_{\hat{A}\hat{B}} + 3\mathcal{P}\mathcal{T} + \frac{3}{2}H_{\hat{A}\hat{A}}H_{\hat{A}A} - 12\Phi^{\hat{A}\hat{A}}\Phi^{ab}_{\hat{A}\hat{A}} - 12U^{\hat{A}\hat{A}}U_{\hat{A}A} + 16i\rho^a_\hat{A}\Lambda^a_\hat{A}$$

$$- 16i\rho^a_\hat{A}\Lambda^A_{\hat{A}} - 48\rho^a_\hat{A}W_{AB}^a - 48\rho^a_\hat{A}W_{\hat{A}\hat{B}}^a + 48\rho^a_\hat{A}U_{\hat{A}a}^a - 48i\rho^a_\hat{A}\rho^a_\hat{A}W_{AB} - 48i\rho^a_\hat{A}U_{\hat{A}a}^a - 48i\rho^a_\hat{A}\nabla_{\hat{A}}\rho^a_\hat{A}$$

$$+ 48i\rho^a_\hat{A}\nabla_{\hat{A}}\rho^a_\hat{A} + 96i\rho^a_\hat{A}\Phi^{ab}_{\hat{A}\hat{A}}\rho^b_\hat{A}$$

(1.8)

which is equivalent to saying that $I$ is no longer an independent field. This constraint implies that only the longitudinal part of $H_m$ belongs to the nonlinear multiplet; its divergence lies in the original Weyl multiplet. One has off-shell identities relating the covariant derivatives of $\rho^a_\hat{A}$ and the other auxiliary fields [8, 9].

Altogether, these component fields form then the "old minimal" $\mathcal{N} = 2$ 40+40 multiplet [10]:

$$e^m_\mu, \psi^{\hat{A}a}, A_\mu, \Phi^{ab}_\mu, Y_{mn}, U_m, \Lambda^a_\hat{A}, X_{ab}, H_m, P, \rho^a_\hat{A}$$

(1.9)

The final lagrangian of "old minimal" $\mathcal{N} = 2$ supergravity may be written, in superspace, as ($\epsilon$ is the chiral density) [11]:

$$\mathcal{L}_{SG} = -\frac{3}{4\kappa^2} \int \epsilon d^4\bar{\theta} + \text{h.c.}$$

(1.10)
2 Leading string corrections to supergravity

For many different reasons, it is important to know the quantum corrections to supergravity originated from string theory. The fourth power of the Riemann tensor \( R^4 \) shows up in all effective actions for all string theories and M-theory, with nonzero coefficients [12]. In this sense, it is the most general quantum correction to supergravity one has. Therefore, it is of interest to know it, and the supersymmetric invariants to which it belongs, better.

The four-dimensional supersymmetrization of \( R^4 \) had never been worked out. A term like that was first considered as a possible quantum correction to supergravity [13]; its study is the purpose of this work.

2.1 The \( \mathcal{N} = 1 \) case

The lagrangian we will be considering is

\[
\mathcal{L}_{SG} + \mathcal{L}_{R^4} = \frac{1}{2\kappa^2} \int E \left( 1 + \alpha W^2 \right) d^4\theta
\]

\( \alpha \) is a coupling constant of mass dimension -6. The \( \alpha \) term represents the supersymmetrization of one combination of the fourth power of the Weyl tensor, more precisely \([14] W^2_+ W^2_- \).

To compute the field equations from this lagrangian, we need the constrained variation of \( W_{ABC} \) [6]. We presented elsewhere [15] the details of this calculation and the final result for \( \int \delta [E \left( 1 + \alpha W^2 \right)] d^4\theta \), and we do not reproduce them here again. From this result, the \( R, \overline{R} \) field equations are immediately read:

\[
R = 6\alpha \frac{W^2 \nabla^2 W^2}{1 - 2\alpha W^2 \overline{W}^2} = 6\alpha \overline{W}^2 \nabla^2 W^2 + 12\alpha^2 W^4 \overline{W}^4 \nabla^2 W^2
\]

From (1.5), we can easily determine \( \nabla^n G_n \). This way, auxiliary fields belonging to the compensating chiral multiplet can be eliminated on-shell. This is not the case for the auxiliary fields which come from the Weyl multiplet \((A_m)\), as we obtained [15] a complicated differential field equation for \( G_m \).

2.2 The \( \mathcal{N} = 2 \) case

To (1.10) we are adding a correction given by

\[
\mathcal{L}_{R^4} = \alpha \kappa^4 \int \epsilon \phi d^4\theta + \text{h.c.}
\]

where we have defined the chiral superfield \((W^2 \overline{W}^2 \) is acted by the chiral projector\)

\[
\phi = \left( \nabla^A_a \nabla^b_A \left( \nabla^B_b \nabla^B_b + 16X_{ab} \right) - \nabla^A_a \nabla_A \left( \nabla^b_b \nabla^B_B - 16iY^B_{AB} \right) \right) W^2 \overline{W}^2
\]

\( \alpha \) is now a numerical constant. The term \( \epsilon \nabla^A_a \nabla^B_b [\nabla_{Ab}, \nabla_{Ba}] \phi \) + h.c. contains \( \epsilon W^2_+ \overline{W}^2_- \).
We then proceeded with the calculation of the components of $\phi$ and analysis of its field content [9]. For that, we used the differential constraints from the solution to the Bianchi identities and the commutation relations. The process is straightforward but lengthy. The results can be summarized as follows: with the correction (2.3), auxiliary fields $X_{ab}$, $\Lambda_{\hat{c} c}$, $Y_{A\hat{B}}$, $U_m$ and $\Phi_{ab m}$ get derivatives, and the same should be true for their field equations; therefore, these superfields cannot be eliminated on-shell. We also fully checked that superfields $P$ and $H_m$ do not get derivatives with this correction (with the important exception of $\nabla^m H_m$) and, therefore, have algebraic field equations which should allow for their elimination on shell. The only obscure case is the auxiliary field $\rho^a_A$. We did not analyze its derivatives because that would require computing a big number of terms and, for each term, a huge number of different contributions. This is probably because $\rho^a_A$ belongs to a nonlinear multiplet. We believe that its derivatives should cancel, though, because derivatives of the other fields from the nonlinear multiplet cancel.

### 2.3 Conclusions and a conjecture

A careful analysis shows that, in the cases we studied, the auxiliary fields that can be eliminated come from multiplets which, on-shell, have no physical fields; while the auxiliary fields that get derivatives come from multiplets with physical fields on-shell (the graviton, the gravitino(s) and, in $N = 2$, the vector). Our general conjecture for $R^4$ supergravity, which is fully confirmed in the "old minimal" $N = 1$ case, can now be stated: the auxiliary fields which come from multiplets with on-shell physical fields cannot be eliminated, but the ones that come from compensating multiplets that, on shell, have no physical fields, can. This analysis should also be extended to the other different versions of these supergravity theories.

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