Blow-up rates for a higher-order semilinear parabolic equation with nonlinear memory term

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ABSTRACT
In this paper, we establish blow-up rates for a higher-order semilinear parabolic equation with nonlocal in time nonlinearity with no positive assumption on the solution. We also give Liouville-type theorem for higher-order semilinear parabolic equation with infinite memory nonlinear term which plays the main tools to prove our blow-up rate result. Finally, we study the well-posedness of mild solutions.

1. Introduction
In this paper, we investigate the higher-order semilinear parabolic equation with nonlocal in time nonlinearity

\[
\begin{align*}
    u_t + (-\Delta)^m u &= \int_0^t (t-s)^{-\gamma} |u|^p \, ds \quad x \in \mathbb{R}^n, \quad t > 0, \\
    u(x, 0) &= u_0(x) \quad x \in \mathbb{R}^n,
\end{align*}
\]

where \( u_0 \in C_0(\mathbb{R}^n) \), \( n \geq 1 \), \( m \in \mathbb{N}^* \), \( 0 < \gamma < 1 \), \( p > 1 \). The space \( C_0(\mathbb{R}^n) \) denotes the space of all continuous functions tending to zero at infinity.

Higher-order semilinear homogeneous equations arise, see e.g. the monograph [1], in numerous problems in applications such as the higher-order diffusion, the phase transition, the flame propagation, and the thin film theory.

When \( m = 1 \) and \( \gamma \to 1 \), using the relation

\[
\lim_{\gamma \to 1} c_\gamma s_+^{-\gamma} = \delta_0(s) \quad \text{in distributional sense with } s_+^{-\gamma} := \begin{cases} 
    s^{-\gamma} & \text{if } s > 0, \\
    0 & \text{if } s < 0,
\end{cases}
\]

with \( c_\gamma = 1/ \Gamma(1 - \gamma) \), and a suitable change of variable, problem (1) reduced to the following semilinear heat equation

\[
\begin{align*}
    u_t - \Delta u &= |u|^p \quad x \in \mathbb{R}^n, \quad t > 0, \\
    u(x, 0) &= u_0(x) \quad x \in \mathbb{R}^n.
\end{align*}
\]
The exponent $p_F = 1 + 2/n$ is known as the critical Fujita exponent of (2). Namely, for $p < p_F$, Fujita [2] proved the nonexistence of nonnegative global-in-time solution for any nontrivial initial condition, and for $p > p_F$, global solutions do exist for any sufficiently small nonnegative initial data. The proof of a blow-up of all nonnegative solutions in the critical case $p = p_F$ was completed in [3].

To understand the behavior of the solution near the finite blow-up time, the first step consists in deriving a bound for the blow-up rate. Giga and Kohn [4] proved that if $(n-2)p < n+2$ and $u$ is a positive solution of (2), then there exists a constant $C > 0$ such that $u(x, t) \leq C (T^* - t)^{-1/(p-1)}$, for all $x \in \mathbb{R}^n$, where $T^*$ is the maximal time of existence.

When $m = 1$ and $0 < \gamma < 1$, Cazenave et al. [5] studied problem (1) and proved that the critical exponent is

$$p_* = \left\{ \frac{1}{\gamma}, 1 + \frac{2(2 - \gamma)}{(n - 2 + 2\gamma)_+} \right\},$$

where $(\cdot)_+$ is the positive part. The study in [5] reveals the surprising fact that for Equation (1) the critical exponent in Fujita’s sense $p_*$ is not the one predicted by scaling. Moreover, Fino and Kirane [6] derived the blow-up rate estimates for the parabolic Equation (1). Namely, they proved that, if $u_0 \in C_0(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, $u_0 \geq 0$, $u_0 \not\equiv 0$ and if $u$ is the blowing-up solution of (1) at the finite time $T^* > 0$, then there are constants $c, C > 0$ such that $c (T^* - t)^{-\alpha_1} \leq \sup_{\mathbb{R}^n} u(\cdot, t) \leq C (T^* - t)^{-\alpha_1}$ for $1 < p \leq 1 + 2(2 - \gamma)/(n - 2 + 2\gamma)_+$ or $1 < p < 1/\gamma$ and all $t \in (0, T^*)$, where $\alpha_1 := (2 - \gamma)/(p - 1)$. They used a scaling argument to reduce the problems of blow-up rate to Fujita-type theorems (it is similar to blow-up analysis in elliptic problems to reduce the problems of a priori bounds to Liouville-type theorems). As far as we know, this method was first applied to parabolic problems by Hu [7], and then was used in various parabolic equations and systems (see [8, 9]). We refer the reader to the excellent paper of Andreucci and Teseev [10] for the blow-up rate by an alternative method.

When $m > 1$ and $\gamma \to 1$, Galaktionov and Pohozaev [11] have shown that $p = 1 + 2m/n$ is the critical exponent of (1). Moreover, Pan and Xing [12] studied this equation and its corresponding system, they derived the blow-up rates of the solution and proved that $\sup_{\mathbb{R}^n} |u(\cdot, t)| \leq C (T^* - t)^{-1/(p-1)}$ for $1 < p \leq 1 + 2m/n$, $m \geq 1$, under some condition on the initial data, where $T^*$ is the maximal time of existence.

When $m > 1$ and $0 < \gamma < 1$, problem (1) has been considered by Sun and Shi [13]. They studied the global existence/nonexistence of solution. Namely, they proved that the critical exponent for (1) is

$$p_* = \left\{ \frac{1}{\gamma}, 1 + \frac{2m(2 - \gamma)}{(n - 2m + 2m\gamma)_+} \right\}.$$

Our main goal is to derive the blow-up rate estimates for the parabolic Equation (1). Our proof is similar to the ones in [6] and [12]. We also prove Liouville-type theorem for (1) with different memory nonlinear term (see (30)), which plays a crucial role to obtain our blow-up rate result. The novelty is that no positive assumption on the solution is needed. Finally, in order to obtain a lower-bound for the blow-up rate, we complete the study of [13] by proving the local existence of mild solutions for (1).

Let us first present our well-posedness result.

**Theorem 1.1 (Local existence):** Given $u_0 \in C_0(\mathbb{R}^n)$, $0 < \gamma < 1$, $m \geq 1$, and $p > 1$. There exist a maximal time $T_{\text{max}} > 0$ and a unique mild solution $u \in C([0, T_{\text{max}}), C_0(\mathbb{R}^n))$ to the problem (1). Moreover, either $T_{\text{max}} = \infty$ or else $T_{\text{max}} < \infty$ and in this case $\|u(t)\|_{L^r(\mathbb{R}^n)} \to \infty$ as $t \to T_{\text{max}}$. In addition, if $u_0 \in C_0(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$, for $1 \leq r < \infty$, then $u \in C([0, T_{\text{max}}), C_0(\mathbb{R}^n) \cap L^r(\mathbb{R}^n))$.

Next, our main result is the following theorem which present the blow-up rate for the blowing-up solutions to the parabolic problem (1).
\textbf{Theorem 1.2:} Let \( u_0 \) satisfies (29) below, and
\[
 p \leq 1 + 2m(2 - \gamma)/(n - 2m + 2m\gamma) \quad \text{or} \quad p < 1/\gamma.
\]
If \( u \) is the blowing-up mild solution of (1) in a finite time \( T_{\max} := T^* \), then there exist two constants \( c, C > 0 \) such that
\[
 c(T^* - t)^{-\alpha_1} \leq \sup_{\mathbb{R}^n} |u(\cdot, t)| \leq C(T^* - t)^{-\alpha_1}, \quad t \in (0, T^*),
\]
where \( \alpha_1 := (2 - \gamma)/(p - 1) \).

Throughout this paper, positive constants will be denoted by \( C \) and will change from line to line.

The remainder of this paper is organized as follows: Section 2 concerns preliminaries. In Section 3, we prove the local existence of the mild solution (Theorem 1.1) of (1). We devote to the proof of the main result (Theorem 1.2) in Section 4.

\section{Preliminaries}

In this section, we present some definitions and results that will be used hereafter.

Let us start by giving the solution of the following homogenous equation
\[
\begin{cases}
 u_t + (-\Delta)^m u = 0 & x \in \mathbb{R}^n, \ t > 0, \\
 u(x, 0) = u_0(x) & x \in \mathbb{R}^n.
\end{cases}
\]

Let \( u_0 \in X := C_0(\mathbb{R}^n) \), and \( A = -(-\Delta)^m \). Using [14, Theorem 3.7, p. 217], \( A \) is the infinitesimal generator of an analytic semigroup \( S(t) : X \to X, t \geq 0 \). Therefore, by [14, Theorem 1.3, p. 102] and [14, Corollary 1.5, p. 104], the initial value problem (4) has a unique solution \( u(t) = S(t)u_0, t \geq 0 \), which is continuously differentiable on \([0, \infty[.\) Moreover, the operator \( S(t) \) can be presented as follows (see [11])
\[
S(t) : \ X \to X \\
\varphi \mapsto S(t)\varphi = b(t, \cdot) \ast \varphi,
\]
where \( b(t, \cdot) \) denotes the kernel of the operator \( S(t) \) (the fundamental solution of the parabolic operator \( \partial_t + (-\Delta)^m \)), which is presented by
\[
b(t, x) = \mathcal{F}^{-1}(e^{-|w|^{2m}t}) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \exp^{-|w|^{2m}t - iw \cdot x} \, dw.
\]

Furthermore, by [15, Theorem 3.3], \( S(t) \) satisfies the following \( L^p-L^q \) estimate
\[
\|S(t)\varphi\|_q \leq Ct^{-\frac{m}{2p}(\frac{1}{p} - \frac{1}{q})} \|\varphi\|_p, \quad (5)
\]
for all \( t > 0, \varphi \in L^p(\mathbb{R}^n), 1 \leq p \leq q \leq \infty \), for some positive constant \( C = C(m, n, p, q) \). The kernel \( b(t, \cdot) \) changes sign, when \( m > 1 \), and is oscillatory as \( |x| \to \infty \), and the associated semigroup \( S(t) \) is not order-preserving. So, there is no comparison principle for (4).

Next, we present the tools concerning the fractional integrals and fractional derivatives.

\textbf{Definition 2.1 (Absolutely continuous functions):} A function \( f : [a, b] \to \mathbb{R}, -\infty < a < b < \infty \), is absolutely continuous if and only if there exists a Lebesgue summable function \( \varphi \in L^1(a, b) \) such that
\[
f(t) = f(a) + \int_a^t \varphi(s) \, ds.
\]
The space of these functions is denoted by \( AC[a, b] \). Moreover, we define
\[
AC^2[a, b] := \{ f : [a, b] \to \mathbb{R} \text{ such that } f' \in AC[a, b] \}.
\]
Definition 2.2 (Riemann–Liouville fractional integrals [16, Chapter 1]): Let $f \in L^1(a, b)$, $-\infty < a < b < \infty$. The Riemann–Liouville left- and right-sided fractional integrals of order $\alpha \in (0, 1)$ are, respectively, defined by

$$I_{a+}^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{-(1-\alpha)} f(s) \, ds, \quad t > a, \quad (6)$$

and

$$I_{b-}^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{-(1-\alpha)} f(s) \, ds, \quad t < b, \quad (7)$$

where $\Gamma$ is the Euler gamma function.

Definition 2.3 (Riemann–Liouville fractional derivatives [16, Chapter 1]): Let $f \in AC[a, b]$, $-\infty < a < b < \infty$. The Riemann–Liouville left- and right-sided fractional derivatives of order $\alpha \in (0, 1)$ are, respectively, defined by

$$D^\alpha_{a+} f(t) := \frac{d}{dt} I_{a+}^{1-\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-s)^{-\alpha} f(s) \, ds, \quad t > a, \quad (8)$$

and

$$D^\alpha_{b-} f(t) := -\frac{d}{dt} I_{b-}^{1-\alpha} f(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b (s-t)^{-\alpha} f(s) \, ds, \quad t < b. \quad (9)$$

Proposition 2.4 (Integration by parts formula [16, (2.64) p.46]): Let $\alpha \in (0, 1)$ and $-\infty < a < b < \infty$. The fractional integration by parts formula

$$\int_a^b f(t) D^\alpha_{a+} g(t) \, dt = \int_a^b g(t) D^\alpha_{b-} f(t) \, dt, \quad (10)$$

is valid for every $f \in I_{a+}^\alpha (L^p(a, b))$, $g \in I_{a+}^\alpha (L^q(a, b))$ such that $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$, $p, q > 1$, where

$$I_{a+}^\alpha (L^q(0, T)) := \left\{ f = I_{a+}^\alpha h, h \in L^q(a, b) \right\},$$

and

$$I_{b-}^\alpha (L^p(a, b)) := \left\{ f = I_{b-}^\alpha h, h \in L^p(a, b) \right\}.$$

Remark 2.5: A simple sufficient condition for functions $f$ and $g$ to satisfy (10) is that $f, g \in C[a, b]$, such that $D^\alpha_{a+} f(t), D^\alpha_{b-} g(t)$ exist at every point $t \in [a, b]$ and are continuous.

Proposition 2.6 ([16, Chapter 1]): For $0 < \alpha < 1$, $-\infty < a < b < \infty$, we have the following identities

$$D^\alpha_{a+} I_{a+}^\alpha f(t) = f(t), \quad a.e. \, t \in (a, b), \quad \text{for all } f \in L^r(a, b), \quad 1 \leq r \leq \infty, \quad (11)$$

and

$$-D^\alpha_{b-} f = D_{b-}^{1+\alpha} f, \quad \text{for all } f \in AC^2[a, b], \quad (12)$$

where $D := \frac{d}{dt}$.
Given $T > 0$, let us define the functions $w_1$ and $w_2$ by

$$w_1(t) = (1 - t/T)^\sigma, \quad \text{for all } 0 \leq t \leq T,$$

and

$$w_2(t) = (1 + t/T)^\sigma, \quad \text{for all } -T \leq t \leq 0,$$

where $\sigma \gg 1$ is big enough. Later on, we need the following properties concerning the functions $w_i$, $i = 1, 2$.

**Lemma 2.7 ([16, (2.45), p. 40]):** Let $T > 0$, $0 < \alpha < 1$. For all $t \in [0, T]$, we have

$$D^{\alpha}_{t|T} w_1(t) = \frac{\Gamma(\sigma + 1)}{\Gamma(\sigma + 1 - \alpha)} T^{-\alpha} (1 - t/T)^{\sigma - \alpha},$$

and

$$D^{1+\alpha}_{t|T} w_1(t) = \frac{\Gamma(\sigma + 1)}{\Gamma(\sigma - \alpha)} T^{-(1+\alpha)} (1 - t/T)^{\sigma - \alpha - 1}.$$  

**Lemma 2.8 ([16, (2.45), p. 40]):** Let $T > 0$, $0 < \alpha < 1$. For all $t \in [-T, 0]$, we have

$$D^{\alpha}_{t|0} w_2(t) = \frac{\Gamma(\sigma + 1)}{\Gamma(\sigma + 1 - \alpha)} T^{-\alpha} (1 + t/T)^{\sigma - \alpha},$$

and

$$D^{1+\alpha}_{t|0} w_2(t) = \frac{\Gamma(\sigma + 1)}{\Gamma(\sigma - \alpha)} T^{-(1+\alpha)} (1 + t/T)^{\sigma - \alpha - 1}.$$  

**Lemma 2.9:** Let $T > 0$, $0 < \alpha < 1$, $p > 1$, we have

$$\int_0^T (w_1(t))^{-1/(p-1)} |D^{\alpha}_{t|T} w_1(t)|^{p/(p-1)} dt = C T^{1-\alpha p/(p-1)},$$

and

$$\int_0^T (w_1(t))^{-1/(p-1)} |D^{1+\alpha}_{t|T} w_1(t)|^{p/(p-1)} dt = C T^{1-(1+\alpha)p/(p-1)},$$

for some $C > 0$.

**Proof:** The proof of this lemma can be found, e.g. in Furati and Kirane [17]. To make this paper self-contained, we will present the proof in details. First, we prove (19), while the identity (20) can be done similarly. Using Lemma 2.7, we have

$$\int_0^T (w_1(t))^{-1/(p-1)} |D^{\alpha}_{t|T} w_1(t)|^{p/(p-1)} dt = C T^{-\alpha} \int_0^T (w_1(t))^{-1/(p-1)} (w_1(t))^{p/(p-1)} dt$$

$$= C T^{-\alpha} \int_0^T (1 - t/T)^{\sigma - \alpha} \frac{1}{T^{\sigma - \alpha - 1}} \frac{1}{T^{\sigma - \alpha - 1}} dt$$

$$= C T^{1-\alpha} \int_0^1 (1 - s)^{\sigma - \alpha} \frac{1}{T^{\sigma - \alpha - 1}} ds$$

$$= C T^{1-\alpha} \frac{1}{T^{\sigma - \alpha - 1}}.$$  

Similarly, we have

**Lemma 2.10:** Let \( T > 0, \alpha < 1, p > 1, \) we have

\[
\int_{-T}^{0} (w_2(t))^{-1/(p-1)} |D^\alpha_{\xi}w_2(t)|^{p/(p-1)} \, dt = C T^{1-\alpha p/(p-1)},
\]

and

\[
\int_{-T}^{0} (w_2(t))^{-1/(p-1)} |D^{1+\alpha}_{\xi}w_2(t)|^{p/(p-1)} \, dt = C T^{1-(1+\alpha)p/(p-1)},
\]

for some \( C > 0. \)

On the other hand,

**Lemma 2.11 ([18, Lemma 8.18] C∞ Urysohn Lemma):** If \( K \subset \mathbb{R}^n \) is compact and \( U \) is an open set containing \( K, \) there exists \( f \in C^\infty_c(\mathbb{R}^n) \) such that \( 0 \leq f \leq 1, f = 1 \) on \( K, \) and \( \text{supp} f \subset U. \)

Using Lemma 2.11, there exists a function \( \phi \in C^\infty_c(\mathbb{R}^n) \) such that

\[
\phi(x) = \phi(|x|) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| \geq 2. \end{cases}
\]

An explicit example of this function can be found in [19, Chapter 1, p.40].

**Lemma 2.12:** Let \( m \geq 1, \ell > 2m, \) and \( \phi \) is defined in (23). Then, the following estimate holds:

\[
|\Delta^m(\phi^\ell)| \leq C\phi^{\ell-2m},
\]

for some \( C = C(m, \ell) > 0. \)

**Proof:** First, we recall the following formula of derivatives of composed functions for \(|\alpha| \geq 1:

\[
\partial^\alpha x h(f(x)) = \sum_{k=1}^{\lfloor |\alpha| \rfloor} h^{(k)}(f(x)) \left( \sum_{\gamma_1 + \cdots + \gamma_k \leq \alpha, |\gamma_1| + \cdots + |\gamma_k| = |\alpha|, |\gamma_i| \geq 1} (\partial^{\gamma_1} x f(x)) \cdots (\partial^{\gamma_k} x f(x)) \right),
\]

where \( h = h(z) \) and \( h^{(k)}(z) = \frac{d^k h(z)}{dz^k}. \) Applying this formula with \( h(z) = z^\ell \) and \( f(x) = \phi(x), 1 \leq |x| \leq 2, \) we obtain

\[
|\partial^\alpha x (\phi(x))^\ell| \leq \sum_{k=1}^{\lfloor |\alpha| \rfloor} \ell(\ell - 1) \cdots (\ell - k + 1) (\phi(x))^{\ell-k} \left( \sum_{|\gamma_1 + \cdots + \gamma_k \leq \alpha, |\gamma_i| = |\alpha|, |\gamma_i| \geq 1} |\partial^{\gamma_1} x \phi(x)| \cdots |\partial^{\gamma_k} x \phi(x)| \right)
\]

Using \( \phi \in C^\infty_c(\mathbb{R}^n), \) we have

\[
|\partial^{\gamma_i} x \phi(x)| \leq C_{\alpha}, \quad \text{for all } 1 \leq i \leq k, 1 \leq |x| \leq 2,
\]

for some constant \( C_{\alpha} = C(\alpha) > 0, \) which implies,

\[
|\partial^\alpha x (\phi(x))^\ell| \leq \tilde{C}_{\alpha} \sum_{k=1}^{\lfloor |\alpha| \rfloor} \ell(\ell - 1) \cdots (\ell - k + 1) (\phi(x))^{\ell-k},
\]
for some constant $\tilde{C}_\alpha > 0$. Therefore, as $\phi \leq 1$ and $\ell - k \geq \ell - |\alpha|$ for all $1 \leq k \leq |\alpha|$, we conclude that

$$\left| \partial_x^\alpha (\phi(x)) \right| \leq C_{\alpha, \ell} (\phi(x))^{\ell - |\alpha|}, \quad \text{for all} \ 1 \leq |x| \leq 2$$

Finally, as

$$|\Delta^m (\phi^\ell)| \leq m \sum_{|\alpha|=m} \left| \partial_x^{2\alpha} (\phi(x))^\ell \right|,$$

the proof is complete. ■

**Lemma 2.13:** Let $m \geq 1$, $R > 0$, $\ell > 2mp/(p-1)$, and $p > 1$. Then, the following estimate holds

$$\int_{\mathbb{R}^n} (\phi_R(x))^{-\frac{1}{p-1}} \left| (-\Delta)^m \phi_R(x) \right|^{\frac{p}{p-1}} \, dx \leq CR^{-\frac{2mp}{p-1} + n},$$

for some $C > 0$, where $\phi_R(x) := \phi^\ell (x/R)$ and $\phi$ is given in (23).

**Proof:** Using the change of variables $\tilde{x} = x/R$, we have

$$(-\Delta)^m \phi_R(x) = R^{-2m} (-\Delta)^m \phi(\tilde{x}).$$

Therefore, by Lemma 2.12, we conclude that

$$\int_{\mathbb{R}^n} (\phi_R(x))^{-\frac{1}{p-1}} \left| (-\Delta)^m \phi_R(x) \right|^{\frac{p}{p-1}} \, dx \leq CR^{-\frac{2mp}{p-1} + n} \int_{|\tilde{x}| \leq 2} (\phi(\tilde{x}))^{-2m} \tilde{x} \, d\tilde{x} \leq CR^{-\frac{2mp}{p-1} + n}. $$

■

**3. Local existence**

This section is dedicated to proving the local existence and uniqueness of mild solutions to the problem (1). Let us start by the

**Definition 3.1 (Mild solution):** Let $u_0 \in C_0(\mathbb{R}^n)$, $0 < \gamma < 1$, $m \geq 1$, $p > 1$ and $T > 0$. We say that $u \in C([0, T], C_0(\mathbb{R}^n))$ is a mild solution of problem (1) if $u$ satisfies the following integral equation

$$u(t) = S(t)u_0 + C_\alpha \int_0^t S(t-s) I_{0,s}^\alpha (|u(s)|^p) \, ds, \quad t \in [0, T],$$

(24)

where $\alpha := 1 - \gamma \in (0, 1)$ and $C_\alpha = \Gamma(\alpha)$.

**Proof of Theorem 1.1:** For arbitrary $T > 0$, let

$$E_T := \left\{ u \in C([0, T], C_0(\mathbb{R}^n)); \|u(t)\|_\infty \leq 2\|u_0\|_\infty, \text{for all} \ t \in [0, T] \right\},$$

where $\|\cdot\|_\infty := \|\cdot\|_{L_\infty(\mathbb{R}^n)}$, and we equip $E_T$ with the following metric generated by the norm of $C([0, T], C_0(\mathbb{R}^n))$

$$d(u, v) = \max_{t \in [0, T]} \|u(t) - v(t)\|_\infty, \quad \text{for all} \ u, v \in E_T.$$ 

Since $C([0, T], C_0(\mathbb{R}^n))$ is a Banach space, $(E_T, d)$ is a complete metric space. Next, for all $u \in E_T$, we define

$$\Psi(u)(t) := S(t)u_0 + C_\alpha \int_0^t S(t-s) I_{0,s}^\alpha (|u(s)|^p) \, ds.$$

We prove the local existence by the Banach fixed point theorem.
• \( \Psi : E_T \to E_T \): Let \( u \in E_T \), using (5), we obtain

\[
\| \Psi(u)(t) \|_\infty \leq \| u_0 \|_\infty + \int_0^t \int_0^s (s-\sigma)^{-\gamma} \| u(\sigma) \|_\infty^\beta \, d\sigma \, ds
\]

\[
= \| u_0 \|_\infty + \int_0^t \int_0^s (s-\sigma)^{-\gamma} \| u(\sigma) \|_\infty^\beta \, d\sigma \, ds
\]

\[
\leq \| u_0 \|_\infty + \frac{T^{2-\gamma} 2^p \| u_0 \|_\infty^{p-1}}{(1-\gamma)(2-\gamma)} \| u_0 \|_\infty,
\]

for all \( t \in [0, T] \). Now, if we choose \( T \) small enough such that

\[
\frac{T^{2-\gamma} 2^p \| u_0 \|_\infty^{p-1}}{(1-\gamma)(2-\gamma)} \leq 1,
\]

we conclude that \( \| \Psi(u)(t) \|_\infty \leq 2 \| u_0 \|_\infty \), for all \( t \in [0, T] \). Therefore, using the fact that \( S(t) : C_0(\mathbb{R}^n) \to C_0(\mathbb{R}^n) \), and the continuity of the semigroup \( S(t) \), we get \( \Psi(u) \in E_T \).

• \( \Psi \) is a contraction: For \( u, v \in E_T \), using again (5), we have

\[
\| \Psi(u)(t) - \Psi(v)(t) \|_\infty \leq \int_0^t \int_0^s (s-\sigma)^{-\gamma} \| u(\sigma) \|_\infty^\beta - \| v(\sigma) \|_\infty^\beta \, d\sigma \, ds
\]

\[
= \int_0^t \int_\sigma^t (s-\sigma)^{-\gamma} \| u(\sigma) \|_\infty^\beta - \| v(\sigma) \|_\infty^\beta \, d\sigma \, ds
\]

\[
\leq \frac{C(p) 2^p \| u_0 \|_\infty^{p-1} T^{2-\gamma}}{(1-\gamma)(2-\gamma)} d(u, v)
\]

\[
\leq \frac{1}{2} d(u, v),
\]

for all \( t \in [0, T] \), thanks to the following inequality

\[
\| u \|^p - \| v \|^p \leq C(p) |u-v| (\| u \|^{p-1} + \| v \|^{p-1});
\]

(26)

\( T \) is chosen such that

\[
\frac{T^{2-\gamma} 2^p \| u_0 \|_\infty^{p-1} \max(2C(p), 1)}{(1-\gamma)(2-\gamma)} \leq 1.
\]

(27)

Then, by the Banach fixed point theorem, see e.g. [20, Theorem 1.1.1], there exists a mild solution \( u \in E_T \), to problem (1).

• **Uniqueness:** If \( u, v \) are two mild solutions in \( E_T \) for some \( T > 0 \), using (5) and (26), we obtain

\[
\| u(t) - v(t) \|_\infty \leq C(p) 2^p \| u_0 \|_\infty^{p-1} \int_0^t \int_0^s (s-\sigma)^{-\gamma} \| u(\sigma) - v(\sigma) \|_\infty \, d\sigma \, ds
\]

\[
= C(p) 2^p \| u_0 \|_\infty^{p-1} \int_0^t \int_\sigma^t (s-\sigma)^{-\gamma} \| u(\sigma) - v(\sigma) \|_\infty \, d\sigma \, ds
\]

\[
= \frac{C(p) 2^p \| u_0 \|_\infty^{p-1}}{1-\gamma} \int_0^t (t-\sigma)^{1-\gamma} \| u(\sigma) - v(\sigma) \|_\infty \, d\sigma,
\]

for all \( t \in [0, T] \). So the uniqueness follows from Gronwall’s inequality (cf. [20]).
Next, using the uniqueness of solutions, we conclude the existence of a maximal solution

\[ u \in C([0, T_{\text{max}}), C_0(\mathbb{R}^n)). \]

where

\[ T_{\text{max}} := \sup\{T > 0; \text{there exist a mild solution } u \in E_T \text{ to } (1)\} \leq +\infty. \]

Moreover, if \( 0 \leq t \leq t + \tau < T_{\text{max}} \), using (24), we can write

\[
\begin{align*}
    u(t + \tau) &= S(\tau)u(t) + \int_0^\tau S(\tau - s) \int_0^s (s - \sigma)^{-\gamma} |u(t + \sigma)|^p \, d\sigma \, ds \\
    &\quad + \int_0^\tau S(\tau - s) \int_0^t (t + s - \sigma)^{-\gamma} |u(\sigma)|^p \, d\sigma \, ds.
\end{align*}
\]

(28)

To prove that \( \|u(t)\|_\infty \to \infty \) as \( t \to T_{\text{max}} \), whenever \( T_{\text{max}} < \infty \), we proceed by contradiction. Suppose that \( u \) is a solution of (24) on some interval \([0, T]\) with \( \|u\|_{L^\infty([0,T]\times\mathbb{R}^n)} < \infty \) and \( T_{\text{max}} < \infty \). Using the fact that the last term in (28) depends only on the values of \( u \) in the interval \((0,t)\) and using again a fixed-point argument, we conclude that \( u \) can be extended to a solution on some interval \([0, T')\) with \( T' > T \). If we repeat this iteration, we obtain a contradiction with the fact that the maximal time \( T_{\text{max}} \) is finite.

- **Regularity:** If \( u_0 \in L^r(\mathbb{R}^n) \cap C_0(\mathbb{R}^n) \), for \( 1 \leq r < \infty \), then by repeating the fixed point argument in the metric space

\[
E_{T,r} := \{ u \in C([0, T], C_0(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)); \|u(t)\|_\infty \leq 2\|u_0\|_{L^r}, \|u(t)\|_r \leq 2\|u_0\|_{L^r}, \text{ for all } t \in [0, T]\},
\]

equipped with

\[
d_r(u, v) = \max_{t \in [0, T]} (\|u(t) - v(t)\|_\infty + \|u(t) - v(t)\|_r), \quad \text{for all } u, v \in E_{T,r},
\]

instead of \((E_T, d)\), where \( \| \cdot \|_r := \| \cdot \|_{L^r(\mathbb{R}^n)} \), and by estimating \( \|u^p\|_{L^r(\mathbb{R}^n)} \) by \( \|u\|_{L^\infty(\mathbb{R}^n)} \) in the contraction mapping argument, using (5), we obtain a unique solution in \( E_{T,r} \), and therefore we conclude that

\[ u \in C([0, T_{\text{max}}), C_0(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)). \]

**Remark 3.2:** If \( T_{\text{max}} = \infty \), the solution \( u \) is said to be global in time, while \( u \) is said to blow up in a finite time when \( T_{\text{max}} < \infty \), and in this case we have \( \|u(t)\|_{L^\infty(\mathbb{R}^n)} \to \infty \) as \( t \to T_{\text{max}} \).

### 4. Blow-up rate

In this section, we prove the blow-up rate for the blowing-up solutions of problem (1), namely Theorem 1.2. We take the solution of (1) with an initial condition satisfying

\[ u_0 \in L^1(\mathbb{R}^n) \cap C_0(\mathbb{R}^n), \quad \int_{\mathbb{R}^n} u_0(x) \, dx > 0. \]

(29)

The following Lemma will be used in the proof of Theorem 1.2. The test function method (see [21–25] and the references therein) is the key to prove this Lemma.
Lemma 4.1: Let \( \nu \) be a bounded classical solution of
\[
v_t + (-\Delta)^m \nu = \int_{-\infty}^{t} (t-s)^{-\gamma}|\nu(s)|^p \, ds \quad \text{in} \; \mathbb{R} \times \mathbb{R}^n,
\]
m \geq 1, p > 1. Then \( \nu \equiv 0 \) whenever
\[
p \leq 1 + 2m(2-\gamma)/(n-2m+2m\gamma) \quad \text{or} \quad p < 1/\gamma.
\]

Proof: Suppose that \( \nu \) is a bounded classical solution to (30) on \( \mathbb{R}^n \times \mathbb{R} \). Then
\[
v_t + (-\Delta)^m \nu = \int_{-\infty}^{t} (t-s)^{-\gamma}|\nu(s)|^p \, ds \quad \text{in} \; [0, \infty) \times \mathbb{R}^n,
\]
and
\[
v_t + (-\Delta)^m \nu = \int_{-\infty}^{t} (t-s)^{-\gamma}|\nu(s)|^p \, ds \quad \text{in} \; (-\infty, 0] \times \mathbb{R}^n.
\]
Let \( w_i, i = 1, 2, \) and \( \phi \) be the functions defined, respectively, in (13)–(14) and (23). For \( T, R \gg 1 \), let us define our test functions as follows:
\[
\varphi_1(t,x) = D^{\alpha}_{t,T}(\tilde{\varphi}_1(t,x)), \quad (t,x) \in [0, T] \times \mathbb{R}^n,
\]
and
\[
\varphi_2(t,x) = D^{\beta}_{t,0}(\tilde{\varphi}_2(t,x)), \quad (t,x) \in [-T, 0] \times \mathbb{R}^n,
\]
where \( \alpha = 1 - \gamma, \tilde{\varphi}_i(t,x) = w_i(t)\phi_R(x), i = 1, 2, \) and \( \phi_R(x) = \phi^\ell(x/R), \ell > 2mp/(p-1) \).

Multiplying (32) by \( \varphi_1(t,x) \) (resp. (33) by \( \varphi_2(t,x) \)) and integrating over \([0, T] \times \mathbb{R}^n\) (resp. over \([-T, 0] \times \mathbb{R}^n\)), we get
\[
\int_0^T \int_{\mathbb{R}^n} \int_{-\infty}^{t} (t-s)^{-\gamma}|\nu(s)|^p \, ds \varphi_1(t,x) \, dx \, dt = \int_0^T \int_{\mathbb{R}^n} \nu(t,x)(-\Delta)^m \varphi_1(t,x) \, dx \, dt \\
+ \int_0^T \int_{\mathbb{R}^n} \varphi_1(t,x) \varphi_1(t,x) \, dx \, dt,
\]
and
\[
\int_{-T}^0 \int_{\mathbb{R}^n} \int_{-\infty}^{t} (t-s)^{-\gamma}|\nu(s)|^p \, ds \varphi_2(t,x) \, dx \, dt = \int_{-T}^0 \int_{\mathbb{R}^n} \nu(t,x)(-\Delta)^m \varphi_2(t,x) \, dx \, dt \\
+ \int_{-T}^0 \int_{\mathbb{R}^n} \varphi_1(t,x) \varphi_2(t,x) \, dx \, dt,
\]
where we have used the Green’s identity several times. Using integration by parts, (15), and (17), we have
\[
\int_0^T \int_{\mathbb{R}^n} \int_{-\infty}^{t} (t-s)^{-\gamma}|\nu(s)|^p \, ds \varphi_1(t,x) \, dx \, dt + C_{\alpha,\sigma} \int_{\mathbb{R}^n} \nu(0,x)\phi_R(x) \, dx \\
= \int_0^T \int_{\mathbb{R}^n} \nu(t,x)(-\Delta)^m \varphi_1(t,x) \, dx \, dt - \int_0^T \int_{\mathbb{R}^n} \varphi_1(t,x) \partial_t \varphi_1(t,x) \, dx \, dt,
\]
and
\[
\int_{-\infty}^{0} \int_{\mathbb{R}^n} \int_{-T}^{t} (t-s)^{-\gamma} |v(s)|^p \, dG(t,x) \, dx \, dt - C_{\alpha,\sigma} \int_{\mathbb{R}^n}^{T} v(0,x) \phi_R(x) \, dx
\]
\[
= \int_{-\infty}^{0} \int_{\mathbb{R}^n} \int_{-T}^{t} v(t,x)(-\Delta)^m \varphi_2(t,x) \, dx \, dt - \int_{-\infty}^{0} \int_{\mathbb{R}^n} \int_{-T}^{t} v(t,x) \partial_t \varphi_2(t,x) \, dx \, dt,
\]
where \( C_{\alpha,\sigma} = \Gamma(\sigma + 1)/\Gamma(\sigma + 1 - \alpha) \). Using again (15), and (17), we can see that \( \varphi_i \geq 0, i = 1, 2 \), then
\[
\int_{0}^{T} \int_{\mathbb{R}^n} \int_{0}^{t} (t-s)^{-\gamma} |v(s)|^p \, dG(t,x) \, dx \, dt \leq \int_{0}^{T} \int_{\mathbb{R}^n} \int_{-\infty}^{t} (t-s)^{-\gamma} |v(s)|^p \, dG(t,x) \, dx \, dt,
\]
and
\[
\int_{-T}^{0} \int_{\mathbb{R}^n} \int_{-T}^{t} (t-s)^{-\gamma} |v(s)|^p \, dG(t,x) \, dx \, dt \leq \int_{-T}^{0} \int_{\mathbb{R}^n} \int_{-\infty}^{t} (t-s)^{-\gamma} |v(s)|^p \, dG(t,x) \, dx \, dt,
\]
which implies
\[
\int_{0}^{T} \int_{\mathbb{R}^n} \int_{0}^{t} (t-s)^{-\gamma} |v(s)|^p \, dG(t,x) \, dx \, dt + C_{\alpha,\sigma} \int_{\mathbb{R}^n}^{T} v(0,x) \phi_R(x) \, dx \leq \int_{0}^{T} \int_{\mathbb{R}^n} \int_{0}^{t} v(t,x)(-\Delta)^m \varphi_1(t,x) \, dx \, dt - \int_{0}^{T} \int_{\mathbb{R}^n} \int_{t}^{T} v(t,x) \partial_t \varphi_1(t,x) \, dx \, dt,
\]
and
\[
\int_{-T}^{0} \int_{\mathbb{R}^n} \int_{-T}^{t} (t-s)^{-\gamma} |v(s)|^p \, dG(t,x) \, dx \, dt - C_{\alpha,\sigma} \int_{\mathbb{R}^n}^{T} v(0,x) \phi_R(x) \, dx \leq \int_{-T}^{0} \int_{\mathbb{R}^n} \int_{-T}^{t} v(t,x)(-\Delta)^m \varphi_2(t,x) \, dx \, dt - \int_{-T}^{0} \int_{\mathbb{R}^n} \int_{t}^{T} v(t,x) \partial_t \varphi_2(t,x) \, dx \, dt,
\]
that is
\[
\Gamma(\alpha) \int_{0}^{T} \int_{\mathbb{R}^n} \int_{0}^{t} I_{0}^{\alpha} |v|^p D_{1,T}^{\alpha} \varphi_1(t,x) \, dx \, dt + C_{\alpha,\sigma} \int_{\mathbb{R}^n}^{T} v(0,x) \phi_R(x) \, dx \leq \int_{0}^{T} \int_{\mathbb{R}^n} \int_{0}^{t} v(t,x)(-\Delta)^m \varphi_1(t,x) \, dx \, dt - \int_{0}^{T} \int_{\mathbb{R}^n} \int_{t}^{T} v(t,x) \partial_t \varphi_1(t,x) \, dx \, dt,
\]
(34)
and
\[
\Gamma(\alpha) \int_{-T}^{0} \int_{\mathbb{R}^n} \int_{-T}^{t} I_{-T}^{\alpha} |v|^p D_{1,0}^{\alpha} \varphi_2(t,x) \, dx \, dt - C_{\alpha,\sigma} \int_{\mathbb{R}^n}^{T} v(0,x) \phi_R(x) \, dx \leq \int_{-T}^{0} \int_{\mathbb{R}^n} \int_{-T}^{t} v(t,x)(-\Delta)^m \varphi_2(t,x) \, dx \, dt - \int_{-T}^{0} \int_{\mathbb{R}^n} \int_{t}^{T} v(t,x) \partial_t \varphi_2(t,x) \, dx \, dt,
\]
(35)
where \( I_{0}^{\alpha} \) and \( I_{-T}^{\alpha} \) are defined in (6). Adding (34) with (35), and using (10)–(11), we may obtain
\[
\Gamma(\alpha) I(v) + \Gamma(\alpha) J(v)
\]
\[
\leq \int_{0}^{T} \int_{\mathbb{R}^n} \int_{0}^{t} v(t,x)(-\Delta)^m \varphi_1(t,x) \, dx \, dt - \int_{0}^{T} \int_{\mathbb{R}^n} \int_{t}^{T} v(t,x) \partial_t \varphi_1(t,x) \, dx \, dt
\]
\[
+ \int_{-T}^{0} \int_{\mathbb{R}^n} \int_{-T}^{t} v(t,x)(-\Delta)^m \varphi_2(t,x) \, dx \, dt - \int_{-T}^{0} \int_{\mathbb{R}^n} \int_{t}^{T} v(t,x) \partial_t \varphi_2(t,x) \, dx \, dt,
\]
(36)
where

\[ I(\nu) = \int_0^T \int_{\mathbb{R}^n} |\nu(t,x)|^{p} \tilde{\varphi}_1(t,x) \, dx \, dt \quad \text{and} \quad J(\nu) = \int_{-T}^0 \int_{\mathbb{R}^n} |\nu(t,x)|^{p} \tilde{\varphi}_2(t,x) \, dx \, dt. \]

Using (12), we get

\[
\Gamma(\alpha)I(\nu) + \Gamma(\alpha)J(\nu) \leq \int_0^T \int_{\mathbb{R}^n} |\nu(t,x)|D_{t|T}^\alpha w_1(t)|\Delta^m \phi_R(x)| \, dx \, dt \\
+ \int_0^T \int_{\mathbb{R}^n} |\nu(t,x)|\phi_R(x)|D_{t|T}^{1+\alpha} w_1(t)| \, dx \, dt \\
+ \int_{-T}^0 \int_{\mathbb{R}^n} |\nu(t,x)|D_{t|0}^\alpha w_2(t)|\Delta^m \phi_R(x)| \, dx \, dt \\
+ \int_{-T}^0 \int_{\mathbb{R}^n} |\nu(t,x)|\phi_R(x)|D_{t|0}^{1+\alpha} w_2(t)| \, dx \, dt \\
=: I_1 + I_2 + J_1 + J_2.
\]

(37)

We start to estimate \( I_1 \). Using Hölder’s estimate, we have

\[
I_1 = \int_0^T \int_{|x|>R} |\nu(t,x)|\tilde{\varphi}^{1/p} \tilde{\varphi}^{-1/p} D_{t|T}^\alpha w_1(t)|\Delta^m \phi_R(x)| \, dx \, dt \\
\leq (\tilde{I}(\nu))^{1/p} \left( \int_0^T \int_{\mathbb{R}^n} (\tilde{\varphi}(t,x))^{-\frac{1}{p-1}} (D_{t|T}^\alpha w_1(t))^{\frac{p}{p-1}} |\Delta^m \phi_R(x)|^{\frac{p}{p-1}} \, dx \, dt \right)^{\frac{p-1}{p}} \\
= (\tilde{I}(\nu))^{\frac{1}{p}} \left( \int_0^T (w_1(t))^{-\frac{1}{p-1}} (D_{t|T}^\alpha w_1(t))^{\frac{p}{p-1}} \, dt \int_{\mathbb{R}^n} (\phi_R(x))^{-\frac{1}{p-1}} |\Delta^m \phi_R(x)|^{\frac{p}{p-1}} \, dx \right)^{\frac{p-1}{p}}
\]

where

\[
\tilde{I}(\nu) = \int_0^T \int_{|x|>R} |\nu(t,x)|^{p} \tilde{\varphi}(t,x) \, dx \, dt.
\]

Using Lemmas 2.9 and 2.13, we obtain

\[
I_1 \leq C \left( \tilde{I}(\nu) \right)^{\frac{1}{p}} T^{\frac{p-1}{p} - \alpha} \frac{n(p-1)}{R^{p-1}} - 2m.
\]

(38)

Similar, we estimate \( I_2 \) as follows:

\[
I_2 = \int_0^T \int_{|x|\leq2R} |\nu(t,x)|\tilde{\varphi}^{1/p} \tilde{\varphi}^{-1/p} \phi_R(x)|D_{t|T}^{1+\alpha} (w_1(t))| \, dx \, dt \\
\leq (I(\nu))^{\frac{1}{p}} \left( \int_0^T \int_{|x|\leq2R} (w_1(t))^{-\frac{1}{p-1}} |D_{t|T}^{1+\alpha} w_1(t)|^{\frac{p}{p-1}} \phi_R(x) \, dx \, dt \right)^{\frac{p-1}{p}} \\
= (I(\nu))^{\frac{1}{p}} \left( \int_0^T (w_1(t))^{-\frac{1}{p-1}} |D_{t|T}^{1+\alpha} w_1(t)|^{\frac{p}{p-1}} \, dt \int_{|x|\leq2R} \phi_R(x) \, dx \right)^{\frac{p-1}{p}}.
\]

By the change of variable: \( \tilde{x} = x/R \), we have

\[
\int_{|x|\leq2R} \phi_R(x) \, dx = \int_{|\tilde{x}|\leq2} \phi(\tilde{x}) R^n \, d\tilde{x} = C R^n.
\]
Therefore, using Lemma 2.9, we conclude that

\[ I_2 \leq C \left( I(v) \right)^{\frac{1}{p}} T^{\frac{p-1}{p} - 1 - \alpha} R^{-\frac{n(p-1)}{p}}. \]

By \( \varepsilon \)-Young’s inequality

\[ ab \leq \varepsilon a^p + C_\varepsilon b^{\frac{p}{p-1}} \]

where \( a > 0, \ b > 0, \ p > 1, \)

the following estimation holds

\[ I_2 \leq \varepsilon I(v) + C T^{1-(1+\alpha)\frac{p}{p-1}} R^n. \quad (39) \]

Similarly, using Lemma 2.10 instead of Lemma 2.9, we get

\[ J_1 \leq C \left( \tilde{J}(v) \right)^{\frac{1}{p}} T^{\frac{p-1}{p} - \alpha} R^{-\frac{n(p-1)}{p} - 2m}, \quad (40) \]

and

\[ J_2 \leq \varepsilon J(v) + C T^{1-(1+\alpha)\frac{p}{p-1}} R^n, \]

where

\[ \tilde{J}(v) = \int_{-T}^0 \int_{|x| > R} |v(t, x)|^p \tilde{\varphi}(t, x) \, dx \, dt. \quad (41) \]

Insert (38), (39), (40) and (41) in (37), and choose \( \varepsilon < \Gamma(\alpha), \) we get

\[ I(v) + J(v) \leq C T^{1-(1+\alpha)\frac{p}{p-1}} R^n + C \left( \tilde{I}(v) \right)^{\frac{1}{p}} T^{\frac{p-1}{p} - \alpha} R^{-\frac{n(p-1)}{p} - 2m} + C \left( \tilde{J}(v) \right)^{\frac{1}{p}} T^{\frac{p-1}{p} - \alpha} R^{-\frac{n(p-1)}{p} - 2m}. \quad (42) \]

At this stage, we have to distinguish three cases:

- The case \( p < p_\gamma. \) Take \( R = T^{\frac{2m}{p}}, \) we obtain

\[ I(v) \leq C T^{-\delta} + C \left( \tilde{I}(v) \right)^{\frac{1}{p}} T^{-\delta \frac{(p-1)}{p}}. \]

where \( \delta = -1 + (1 + \alpha)\frac{p}{p-1} - \frac{n}{2m}. \) Using the fact that \( \tilde{I}(v) \leq I(v), \) \( \varepsilon \)-Young’s inequality, we infer that

\[ I(v) + J(v) \leq C T^{-\delta} + \frac{1}{2} I(v) + \frac{1}{2} J(v), \]

i.e.

\[ I(v) + J(v) \leq C T^{-\delta}. \quad (43) \]

As \( p < p_\gamma \) implies \( \delta > 0, \) after passing to the limit as \( T \to \infty, \) using the monotone convergence theorem, the continuity of \( v \) in time and space, we get

\[ \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} |v(t, x)|^p \, dx \, dt = 0. \]

Therefore \( v = 0 \) in \( \mathbb{R} \times \mathbb{R}^n. \)
The case $p = p_r$. On the one hand, from (43), we have

\[ v \in L^p((0, \infty), L^p(\mathbb{R}^n)) \quad \text{and} \quad v \in L^p((-\infty, 0), L^p(\mathbb{R}^n)) \]

which implies that

\[ \tilde{I}(v), \tilde{J}(v) \to 0, \quad \text{as } R, T \to \infty. \quad (44) \]

On the other hand, take $R = \frac{T^{2m} K^{-\frac{1}{2m}}}{1}$, where $1 \leq K < T$ is large enough such that when $T \to \infty$ we don't have $K \to \infty$ at the same time. From (42) and $p = p_r$, we obtain

\[ I(v) + J(v) \leq CK^{-\frac{n}{2m}} + C \left( \tilde{I}(v) \right)^{\frac{1}{p}} K^{-\frac{n(p-1)}{2mp}+2} + C \left( \tilde{J}(v) \right)^{\frac{1}{p}} K^{-\frac{n(p-1)}{2mp}+2}. \]

Letting the limit as $T \to \infty$, using (44), and the monotone convergence theorem, we get

\[ \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} |v(t, x)|^p \, dx \, dt \leq CK^{-\frac{n}{2m}}. \]

Taking the limit as $K \to \infty$, we conclude as above that $v = 0$ in $\mathbb{R} \times \mathbb{R}^n$.

- The case $p < 1/\gamma$. In this case, we choose $R \in [1, T)$ large enough such that when $T \to \infty$ we don’t have $R \to \infty$ at the same time. From (42), and the fact that $\tilde{I}(v) \leq I(v), \tilde{J}(v) \leq J(v)$, $\varepsilon$-Young’s inequality, we infer that

\[ I(v) + J(v) \leq CT^{1-(1+\alpha)} \frac{p}{p-1} R^n + \frac{1}{2} I(v) + \frac{1}{2} J(v) + CT^{1-\alpha} \frac{p}{p-1} R^{n-2m} \frac{p}{p-1}, \]

i.e.

\[ I(v) + J(v) \leq CT^{1-(1+\alpha)} \frac{p}{p-1} R^n + CT^{1-\alpha} \frac{p}{p-1} R^{n-2m} \frac{p}{p-1}, \]

Letting the limit as $T \to \infty$, and the fact that $p < 1/\gamma \Rightarrow 1 - \alpha \frac{p}{p-1} < 0$, we get

\[ \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} |v(t, x)|^p \, dx \, dt = 0. \]

Therefore $v = 0$ in $\mathbb{R} \times \mathbb{R}^n$.

This completes the proof.

\[ \blacksquare \]

**Proof of Theorem 1.2**: The proof is in two parts:

- The upper blow-up rate estimate. Let

\[ M(t) := \sup_{\mathbb{R}^n \times (0, t]} |u|, \quad t \in (0, T^*). \]

Clearly, $M$ is positive, continuous, nondecreasing in $(0, T^*)$, and $\lim_{t \to T^*} M(t) = \infty$. Then for all $t_0 \in (0, T^*)$, we can define

\[ t_0^+ := t^+(t_0) := \max\{t \in (t_0, T^*) : M(t) = 2M(t_0)\}. \]

Choose $A \geq 1$ and let

\[ \lambda(t_0) := \left( \frac{1}{2A} M(t_0) \right)^{-1/(2m\alpha_1)}. \quad (45) \]

we claim that

\[ \lambda^{-2m}(t_0)(t_0^+ - t_0) \leq D, \quad t_0 \in \left( \frac{T^*}{2}, T^* \right), \quad (46) \]

where $D > 0$ is a positive constant which does not depend on $t_0$. 

We proceed by contradiction. If (46) were false, then there would exist a sequence \( t_n \rightarrow T^* \) such that

\[
\lambda_n^{-2m}(t_n^+ - t_n) \rightarrow \infty,
\]

where \( \lambda_n = \lambda(t_n) \) and \( t_n^+ = t^+(t_n) \). For each \( t_n \) choose

\[
(\hat{x}_n, \hat{t}_n) \in \mathbb{R}^n \times (0, t_n) \quad \text{such that} \quad |u(\hat{x}_n, \hat{t}_n)| \geq \frac{1}{2} M(t_n).
\]

(47)

Obviously, \( M(t_n) \rightarrow \infty \); hence, \( \hat{t}_n \rightarrow T^* \). Next, rescale the function \( u \) as

\[
\varphi^\lambda_n(y, s) := \lambda_n^{2m\alpha_1} u(\lambda_n y + \hat{x}_n, \lambda_n^{2m\alpha_1} s + \hat{t}_n), \quad (y, s) \in \mathbb{R}^n \times I_n(T^*),
\]

where \( I_n(t) := (-\lambda_n^{-2m\alpha_1} \hat{t}_n, \lambda_n^{-2m\alpha_1}(t - \hat{t}_n)) \) for all \( t > 0 \). Then \( \varphi^\lambda_n \) is a mild solution of

\[
\varphi + (-\Delta)^m \varphi = \int_{-\lambda_n^{-2m\alpha_1} \hat{t}_n}^s (s - r)^{-\gamma} |\varphi(r)|^p \, dr \quad \text{in} \quad \mathbb{R}^n \times I_n(T^*).
\]

(49)

On the other hand, \( |\varphi^\lambda_n(0, 0)| \geq A \), and

\[
|\varphi^\lambda_n| \leq \lambda_n^{2m\alpha_1} M(t_n^+) = \lambda_n^{2m\alpha_1} 2M(t_n) = 4A \quad \text{in} \quad \mathbb{R}^n \times I_n(t_n^+),
\]

thanks to (45) and the definition of \( t_n^+ \).

Moreover, as

\[
\varphi^\lambda_n \in C([-\lambda_n^{-2m\alpha_1} t_n, T], C_0(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)) \quad \text{for all} \quad T \in I_n(T^*),
\]

so, as in Lemma [6, Lemma 4.2], \( \varphi^\lambda_n \) is a weak solution of (49).

By the maximal regularity theory [26, Theorem 2], we have

\[
\varphi^\lambda_n \in W^{1,q}((-\lambda_n^{-2m\alpha_1} t_n, T); L^q(\mathbb{R}^N)) \cap L^q((-\lambda_n^{-2m\alpha_1} t_n, T); W^{2m,q}(\mathbb{R}^N)),
\]

for any \( q \in (1, \infty) \). Therefore, from the uniform interior Schauder’s estimates (see [27]), the \( C^{2m+\mu,1+\mu/2m}_{loc}(\mathbb{R}^n \times \mathbb{R}) \)-norm of \( \varphi^\lambda_n \) is uniformly bounded, for some \( \mu \in (0, 1) \). Hence, we obtain a subsequence converging in \( C^{2m+\mu,1+\mu/2m}_{loc}(\mathbb{R}^n \times \mathbb{R}) \) to a solution \( \varphi \) of

\[
\varphi + (-\Delta)^m \varphi = C_\alpha I^\mu_{-\infty,\infty}(|\varphi|^p) \quad \text{in} \quad \mathbb{R}^n \times (-\infty, +\infty),
\]

such that \( |\varphi(0, 0)| \geq A \) and \( |\varphi| \leq 4A \) in \( \mathbb{R}^n \times \mathbb{R} \). Whereupon, using Lemma 4.1, we infer that \( \varphi \equiv 0 \) in \( \mathbb{R}^n \times (-\infty, +\infty) \). Contradiction with the fact that \( |\varphi(0, 0)| \geq A \geq 1 \). This proves (46).

Next we use an idea from Hu [7]. From (45) and (46) it follows that

\[
(t_n^+ - t_0) \leq D(2A)^{1/\alpha_1} M(t_0)^{-1/\alpha_1} \quad \text{for any} \quad t_0 \in \left( \frac{T^*}{2}, T^* \right).
\]
Fix $t_0 \in (T^*/2, T^*)$ and denote $t_1 = t_0^+, t_2 = t_1^+, t_3 = t_2^+, \ldots$. Then

$$t_{j+1} - t_j \leq D(2A)^{1/\alpha_1} M(t_j)^{-1/\alpha_1},$$

$$M(t_{j+1}) = 2M(t_j),$$

$j = 0, 1, 2, \ldots$. Consequently,

$$T^* - t_0 = \sum_{j=0}^{\infty} (t_{j+1} - t_j) \leq D(2A)^{1/\alpha_1} \sum_{j=0}^{\infty} M(t_j)^{-1/\alpha_1}$$

$$= D(2A)^{1/\alpha_1} M(t_0)^{-1/\alpha_1} \sum_{j=0}^{\infty} 2^{-j/\alpha_1}. $$

Finally, we conclude that

$$|u(x, t_0)| \leq M(t_0) \leq C(T^* - t_0)^{-\alpha_1}, \quad \forall t_0 \in (0, T^*)$$

where

$$C = 2A \left( D \sum_{j=0}^{\infty} 2^{-j/\alpha_1} \right)^{\alpha_1};$$

so

$$\sup_{\mathbb{R}^n} |u(\cdot, t)| \leq C(T^* - t)^{-\alpha_1}, \quad \forall t \in (0, T^*).$$

• The lower blow-up rate estimate. If we repeat the proof of the local existence of Theorem 1.1, by taking $\|u\|_1 \leq \theta$ instead of $\|u\|_1 \leq 2\|u_0\|_\infty$ in the space $E_T$ for all positive constant $\theta > 0$ and all $0 < t < T$, then the condition (25) of $T$ will be:

$$\|u_0\|_\infty + CT^* \gamma \|u\|_\infty \theta^p \leq \theta, \quad (50)$$

and then, like before, we infer that $\|u(t)\|_\infty \leq \theta$ for (almost) all $0 < t < T$. Consequently, if $\|u_0\|_\infty + CT^* \gamma \|u\|_\infty \theta^p \leq \theta$, then $\|u(t)\|_\infty \leq \theta$. Applying this to any point in the trajectory, we see that if $0 \leq s < t$ and

$$(t - s)^{2-\gamma} \leq \frac{\theta - \|u(s)\|_\infty}{C\theta^p}, \quad (51)$$

then $\|u(t)\|_\infty \leq \theta$, for all $0 < t < T$.

Moreover, if $0 \leq s < T^*$ and $\|u(s)\|_\infty < \theta$, then:

$$(T^* - s)^{2-\gamma} > \frac{\theta - \|u(s)\|_\infty}{C\theta^p}. \quad (52)$$

Indeed, arguing by contradiction and assuming that for some $\theta > \|u(s)\|_\infty$ and all $t \in (s, T^*)$ we have

$$(t - s)^{2-\gamma} \leq \frac{\theta - \|u(s)\|_\infty}{C\theta^p}. $$

Then, using (51), we infer that $\|u(t)\|_\infty \leq \theta$ for all $t \in (s, T^*)$; this contradicts the fact that $\|u(t)\|_\infty \to \infty$ as $t \to T^*$. 
Next, for example, by setting $\theta = 2 \|u(s)\|_\infty$ in (52), we see that for $0 < s < T^*$ we have:

$$(T^* - s)^{2-\gamma} > C \|u(s)\|_\infty^{1-p},$$

and by the continuity of $u$ we get

$$c(T^* - s)^{-\alpha_1} < \sup_{x \in \mathbb{R}^n} |u(x, s)|, \quad \forall \ s \in (0, T^*). \quad (53)$$

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