Stability Estimates in the Inverse Transmission Scattering Problem

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Abstract
We consider the inverse transmission scattering problem with piecewise constant refractive index. Under mild a priori assumptions on the obstacle we establish logarithmic stability estimates.

1 Introduction
In this paper we consider the scattering of acoustic time-harmonic waves in an inhomogeneous medium. More precisely we shall consider a penetrable obstacle \( D \) and we want to recover information on its location from a knowledge of Cauchy data on the boundary of a region \( \Omega \) containing the obstacle \( D \).

Given a spherical incident wave \( u^i(\cdot, x_0) = \Phi(\cdot, x_0) \), where the point source \( x_0 \) is located on the boundary of a ball \( B \) of radius \( R \), such that \( \Omega \subset B \), and \( \Phi \) denotes the fundamental solution to the Helmholtz equation

\[
\Phi(x, x_0) = \frac{1}{4\pi} \frac{e^{ik|x-x_0|}}{|x-x_0|}, \quad x \in \mathbb{R}^3, \quad x \neq x_0,
\]

we denote by \( G(x, x_0) = u^i(x, x_0) + u^s(x, x_0) \) the Green’s function of the equation

\[
(1.1a) \quad \text{div} (\gamma(x) \nabla G(x, x_0)) + k^2 n(x) G(x, x_0) = -\delta(x-x_0), \quad \text{in} \ \mathbb{R}^3,
\]

where the scattered field \( u^s \) satisfies the Sommerfeld radiation condition

\[
(1.1b) \quad \lim_{|x| \to \infty} |x| \left( \frac{\partial u^s}{\partial r}(x) - iku^s(x) \right) = 0.
\]

Here \( k > 0 \) is the wave number and \( r = |x| \). We shall study equation (1.1a) with piecewise constant coefficients, in particular we shall consider \( \gamma \) and \( n \) to be of the following form

\[
\gamma(x) = 1 + (a-1)\chi_D(x) \quad n(x) = 1 + (b-1)\chi_D(x) \quad a \geq \lambda > 0, \quad b \geq \lambda > 0, \quad (a-1)^2 + (b-1)^2 \geq \delta^2 > 0,
\]
where λ and δ are given constants. We refer to [Co-Kr, Is06] for basic information on scattering problem of this type.

The unique determination of D from a knowledge of the far field data has been established by Isakov [Is90]. The purpose of the present paper is to establish a stability result. Under reasonable mild assumptions on the regularity of ∂D we show that there is a continuous dependance of D on the Cauchy data on ∂Ω with a modulus of continuity of logarithmic type. This rate of continuity appears optimal in view of the recent paper [DC-Ro] indicating the strong ill-posedness of the inverse problem.

The main ideas employed to obtain stability rely on the study of the behavior of G(x,x₀) when x and x₀ get close and the use of unique continuation. These ideas go back to [Is88] where a uniqueness result for the inverse inclusion problem is proved and it has also been used in inverse scattering theory in [Is90]. In order to apply these ideas to stability some further properties on singular solutions and quantitative estimates of unique continuation are needed. We refer to [Al-DC] where similar ideas are developed for studying the stability of the inverse inclusion problem.

The stability issue in inverse scattering theory has been considered by Isakov [Is92, Is93] for the determination of a sound-soft obstacle. Hähner and Hohage [Ha-Ho] considered equation (1.1a) with a = 1 and n(x) smooth. They showed that n depends on G(x,x₀), x,x₀ ∈ ∂B, with a logarithmic rate of continuity. They considered both far field data and near field data. They improve and simplify a previous result of Stefanov [St]. We finally mention a result obtained by Potthast [Po] for impenetrable obstacles which is also based on the use of singular solutions.

The plan of the paper is the following. In Section 2 we give the a priori assumptions we need and we state the stability theorem. In Section 3 the proof of the stability theorem is given based on some auxiliary results whose proofs are collected in Section 4 and Section 5. In particular, in Section 4 we establish some results on singular solutions of equation (1.1a) and in Section 5 we study quantitative estimates of unique continuation.

2 The Main Result

In this section we state the stability theorem. Before doing this we shall give some definitions we need and introduce the a priori assumptions on the regularity of the obstacle. For any x = (x₁, x₂, x₃) ∈ ℝ³ and any r > 0 we denote by Bᵣ(x) the open ball in ℝ³ of radius r centered in the point x, Bᵣ(0) = Bᵣ and for x' = (x₁, x₂) ∈ ℝ² we denote by B'r(x') the open ball in ℝ² of radius r centered in the point x'. In places, we shall denote a point x ∈ ℝ³ by x = (x', x₃) where x' ∈ ℝ², x₃ ∈ ℝ.

Definition 2.1. Let Ω be a bounded domain in ℝ³. Given α, 0 < α ≤ 1, we shall say that a portion S of ∂Ω is of class C¹,α with constants r₀, L > 0 if for any P ∈ S, there exists a rigid transformation of coordinates under which we have P = 0 and

Ω ∩ Bᵣ₀ = \{x ∈ Bᵣ₀ : x₃ > φ(x')\},

where φ is a C¹,α function on B'r₀ ⊂ ℝ² satisfying φ(0) = |∇φ(0)| = 0 and ∥φ∥C¹,α(B'r₀) ≤ Lr₀.
Definition 2.2. We shall say that a portion $S$ of $\partial \Omega$ is of Lipschitz class with constants $r_0, L > 0$ if for any $P \in S$, there exists a rigid transformation of coordinates under which we have $P = 0$ and

$$\Omega \cap B_{r_0} = \{ x \in B_{r_0} : x_3 > \varphi(x') \},$$

where $\varphi$ is a Lipschitz continuous function on $B'_{r_0} \subset \mathbb{R}^2$ satisfying $\varphi(0) = 0$ and $\|\varphi\|_{C^{1,1}(B'_{r_0})} \leq Lr_0$.

Remark 2.1. We use the convention to scale all norms in such a way that they are dimensionally equivalent to their argument. For instance, for any $\psi \in C^{1,\alpha}(B'_{r_0})$ we set

$$\|\psi\|_{C^{1,\alpha}(B'_{r_0})} = \|\psi\|_{L^\infty(B'_{r_0})} + r_0 \|\nabla \psi\|_{L^\infty(B'_{r_0})} + r_0^{1+\alpha} |\nabla \psi|_{\alpha, B'_{r_0}}.$$

Assumptions on the obstacle $D$

For given numbers $r_0, L > 0, 0 < \alpha < 1$, we shall assume there exists a bounded domain $\Omega$ such that the obstacle $D$ satisfies the following conditions:

1. $D \subset \Omega$;
2. $\Omega \setminus \overline{D}$ is connected;
3. $\partial D$ is of class $C^{1,\alpha}$ with constants $r_0, L$.

In the sequel we shall refer to numbers $r_0, L, \alpha, R, a, b$ and $k$ as the a priori data.

The inverse problem we are concerned with is the determination of the obstacle $D$ from the knowledge of the Cauchy data of the singular solutions $G(\cdot, x_0)$ on $\partial \Omega$ for all points source $x_0$ located on $\partial B$.

For two possible obstacles $D_1, D_2$ satisfying (2.2) we shall denote by $G_i, i = 1, 2$, the corresponding solutions to (1.1a) satisfying the Sommerfeld radiation condition (1.1b).

Theorem 2.2. Let $D_1, D_2$ be two obstacles satisfying (2.2). If, given $\varepsilon > 0$, we have

$$\sup_{x \in \partial B} \left( \frac{\|\partial G_1(\cdot, x)\|_{L^2(\partial \Omega)}}{\|\partial G_2(\cdot, x)\|_{L^2(\partial \Omega)}} + \frac{\|G_1(\cdot, x) - G_2(\cdot, x)\|_{L^2(\partial \Omega)}}{\|G_1(\cdot, x) - G_2(\cdot, x)\|_{L^2(\partial \Omega)}} \right) \leq \varepsilon,$$

then

$$d_H(\partial D_1, \partial D_2) \leq \omega(\varepsilon),$$

where $\omega$ is an increasing function on $[0, +\infty)$, which satisfies

$$\omega(t) \leq C |\log t|^{-\eta}, \quad \text{for every} \quad 0 < t < 1$$

and $C, \eta, C > 0, 0 < \eta \leq 1$, are constants only depending on the a priori data.

Remark 2.3. We stress the fact that we don’t need any assumption on $k$. 

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3 Proof of the Stability Theorem

We denote by $G$ the connected component of $\Omega \setminus (D_1 \cup D_2)$ such that $\partial \Omega \subset \overline{G}$ and $\Omega_D = \Omega \setminus G$.

Theorem 2.2 evaluates how close the two inclusions are in terms of the Hausdorff distance $d_H$. We recall a definition of this metric.

$$d_H(D_1, D_2) = \max \left\{ \sup_{x \in D_1} \text{dist}(x, D_2), \sup_{x \in D_2} \text{dist}(x, D_1) \right\}.$$  

In order to deal with the Hausdorff distance we introduce a simplified variation of it which we call modified distance.

**Definition 3.1.** We shall call modified distance between $D_1$ and $D_2$ the number

$$(3.4) \quad d_\mu(D_1, D_2) = \max \left\{ \sup_{x \in \partial D_1 \cap \partial \Omega} \text{dist}(x, D_2), \sup_{x \in \partial D_2 \cap \partial \Omega} \text{dist}(x, D_1) \right\}.$$  

We wish to remark here that such modified distance does not satisfy the axioms of a metric and in general does not dominate the Hausdorff distance (see [Al-Be-Ro-Ve, §3] for related arguments).

**Proposition 3.1.** Let $D_1$, $D_2$ be two obstacles satisfying (2.2). Then

$$(3.5) \quad d_H(\partial D_1, \partial D_2) \leq c d_\mu(D_1, D_2),$$

where $c$ depends only on the a priori assumptions.

**Proof.** See [Al-DC, Proposition 3.1]  

With no loss of generality, we can assume that there exists a point $O$ of $\partial D_1 \cap \partial \Omega_D$, where the maximum in the Definition 3.1 is attained, that is

$$(3.6) \quad d_\mu = d_\mu(D_1, D_2) = \text{dist}(O, D_2).$$

We remark that $G$ is solution to

$$\text{div} (\gamma(x) \nabla G(x, y)) + k^2 n(x) G(x, y) = -\delta(x, y).$$

We shall denote by $G_1$ and $G_2$ Green’s functions when $D = D_1$ and $D_2$ respectively and $\gamma_i, n_i, i = 1, 2$, the corresponding coefficients.

Integrating by parts we have

$$(a - 1) \left\{ \int_{D_1} \nabla G_1(\cdot, y) \cdot \nabla G_2(\cdot, w) - \int_{D_2} \nabla G_1(\cdot, y) \cdot \nabla G_2(\cdot, w) \right\}$$

$$+ k^2 (b - 1) \left\{ \int_{D_1} G_2(\cdot, w) G_1(\cdot, y) - \int_{D_2} G_1(\cdot, y) G_2(\cdot, w) \right\}$$

$$= \int_{\partial \Omega} \left( \frac{\partial G_1(\cdot, y)}{\partial \nu} G_2(\cdot, w) - G_1(\cdot, y) \frac{\partial G_2(\cdot, w)}{\partial \nu} \right)$$

$$= \int_{\partial \Omega} \frac{\partial G_1(\cdot, y)}{\partial \nu} (G_2(\cdot, w) - G_1(\cdot, w))$$

$$(3.7) \quad + \int_{\partial \Omega} G_1(\cdot, y) \left( \frac{\partial G_1(\cdot, w)}{\partial \nu} - \frac{\partial G_2(\cdot, w)}{\partial \nu} \right) \quad \forall y, w \in C_B.$$
Let us define for $y, w \in \mathbb{C}B$

$$S_1(y, w) = (a-1) \int_{D_1} \nabla G_1(\cdot, y) \cdot \nabla G_2(\cdot, w) + k^2(b-1) \int_{D_1} G_1(\cdot, y) G_2(\cdot, w),$$

$$S_2(y, w) = (a-1) \int_{D_2} \nabla G_1(\cdot, y) \cdot \nabla G_2(\cdot, w) + k^2(b-1) \int_{D_2} G_1(\cdot, y) G_2(\cdot, w),$$

$$f(y, w) = S_1(y, w) - S_2(y, w).$$

Thus (3.7) can be rewritten as

$$f(y, w) = \int_{\partial \Omega} \frac{\partial G_1(\cdot, y)}{\partial \nu} (G_2(\cdot, w) - G_1(\cdot, w)) + \int_{\partial \Omega} G_1(\cdot, y) \left( \frac{\partial G_1(\cdot, w)}{\partial \nu} - \frac{\partial G_2(\cdot, w)}{\partial \nu} \right) \forall y, w \in \mathbb{C}B. \quad (3.8)$$

Let us fix $P \in \partial D$. We can assume $P \equiv 0$. We denote by $\nu(P)$ the outer unit normal vector to $\Omega_D$ in $P$ and we rotate the coordinate system in such a way that $\nu(P) = (0, 0, -1)$.

Let us denote by $\chi^+(x)$ the characteristic function of the half-space and by $G_+$ the Green’s function of $\text{div}(1 + (a-1)\chi^+)\nabla) + k^2(1 + (b-1)\chi^+)$. 

Proposition 3.2. Let $D \subset \Omega$ be a bounded open set whose boundary is of class $C^{1, \alpha}$ with constants $r_0, L$. Then there exist constants $c_1, c_2$ depending on $a, \alpha, k$ and $L$ such that

$$|\nabla x G(x, y)| \leq c_1|x - y|^{-2}, \quad (3.9)$$

$$|\nabla x G_+(x, y)| \leq c_2|x - y|^{-2} \quad (3.10)$$

for every $x, y \in \mathbb{R}^3$. 

Proof. (3.9) and (3.10) can be obtained following [Al-DC, Proposition 3.1]. In [Al-DC] the key point is the piecewise regularity of the transmission problem. For a proof of that we refer to [DB-El-Fr] and [Li-Vo].

We shall state now two propositions that describe the behavior of $f(y)$ and $S_1(y)$ when we move the singularity $y$ toward the boundary of the inclusion. We postpone their proofs in the last Section 5.

Proposition 3.3. Let $D_1, D_2$ two obstacles verifying (2.2) and let $y = h \nu(O)$, with $O$ defined in (3.6). If, given $\varepsilon > 0$ we have

$$\sup_{x \in \partial B} \left( \left\| \frac{\partial G_1(\cdot, x)}{\partial \nu} - \frac{\partial G_2(\cdot, x)}{\partial \nu} \right\|_{L^2(\partial \Omega)} + \| G_1(\cdot, x) - G_2(\cdot, x) \|_{L^2(\partial \Omega)} \right) \leq \varepsilon,$$

then for every $h, 0 < h < \tilde{r}r_0$, with $\tilde{r} \in (0, 1)$ depending on $L$,

$$|f(y, y)| \leq c \frac{h^A}{h^A},$$

where $0 < A < 1$ and $c, B, F > 0$ are constants that depend only on the a priori data.
Proposition 3.4. Let $D_1, D_2$ two obstacles verifying (2.2) and let $y = h\nu(O)$, with $O$ defined in (3.6). Then for every $h$, $0 < h < \min\{\tau_2, d_\mu\}$,

$$
|S_1(y, y)| \geq c_1 h^{-2} - c_2 (d_\mu - h)^{-2} + c_3
$$

(3.11)

where $c_1, c_2, c_3$ and $\tau_2$ are positive constants only depending on the a priori data.

Proof of Theorem 2.2. Let $O \in \partial D_1$ as defined (3.6), that is

$$
d_\mu(D_1, D_2) = \text{dist}(O, D_2) = d_\mu.
$$

Then, for $y = h\nu(O)$, with $0 < h < h_1$, where $h_1 = \min\{d_\mu, \tau r_0, \tau_2/2\}$, using (3.9), we have

$$
|S_2(y, y)| \leq c \int_{D_2} \frac{1}{(d_\mu - h)} \frac{1}{(d_\mu - h)} dx = c \frac{1}{(d_\mu - h)^2} |D_2|.
$$

Using Proposition 3.3, we have

$$
|S_1(y, y)| - |S_2(y, y)| \leq |S_1(y, y) - S_2(y, y)| = |f(y, y)| \leq \frac{\epsilon^{Bh^p}}{h^A}.
$$

On the other hand, by Proposition 3.4 and (3.12), there exists $h_0 > 0$, only depending on the a priori data, such that for $h$, $0 < h < h_0$

$$
|S_1(y, y)| - |S_2(y, y)| \geq c_1 h^{-2} - c_4 (d_\mu - h)^{-2}.
$$

Thus we have

$$
c_1 h^{-2} - c_4 (d_\mu - h)^{-2} \leq \frac{\epsilon^{Bh^p}}{h^A}.
$$

Let $h = h(\epsilon)$ where $h(\epsilon) = \min\{\ln \epsilon|\frac{1}{A}, d_\mu\}$, for $0 < \epsilon < \epsilon_1$, with $\epsilon_1 \in (0, 1)$ such that $\exp(-B|\ln \epsilon|^{1/2}) = 1/2$. If $d_\mu \leq |\ln \epsilon|^{-\frac{1}{2}}$, the theorem follows using Proposition 3.1. In the other case we have

$$
c_4 (d_\mu - h)^{-2} \geq c_5 h^{-2} - \frac{\epsilon^{Bh^p}}{h^A} \geq c_5 h^{-2}(1 - \epsilon h^{-p} h_{\tilde{A}}),
$$

where $\tilde{A} = 2 - A$, $\tilde{A} > 0$. Since

$$
\epsilon^{Bh(\epsilon)^p} h(\epsilon)^{-\tilde{A}} \leq \epsilon^{B|\ln \epsilon|^{-1/2}} \leq \exp\left(-B|\ln \epsilon|^{1/2}\right),
$$

for any $\epsilon$, $0 < \epsilon < \epsilon_1$,

$$(d_\mu - h(\epsilon))^{-2} \geq c_6 h(\epsilon)^{-2},$$

that is, solving for $d_\mu$, and recalling that, in this case, $h(\epsilon) = |\ln \epsilon|^{-\frac{1}{2}}$

$$
d_\mu \leq c_7 |\ln \epsilon|^{-\frac{\delta}{2}}
$$

where $\delta = 1/(2F)$. When $\epsilon \geq \epsilon_1$, then

$$
d_\mu \leq \text{diam}\Omega
$$

and, in particular when $\epsilon_1 \leq \epsilon < 1$

$$
d_\mu \leq \text{diam}\Omega \frac{|\ln \epsilon|^{-\frac{1}{2}}}{|\ln \epsilon_1|^{-\frac{1}{2}}}
$$

Finally, using Proposition 3.1, the theorem follows. \square
4 Remarks on Singular Solutions

Proposition 4.1. Let $D \subset \mathbb{R}^3$ be an open set with $C^{1,\alpha}$ boundary with constants $r_0, L$, let $P$ be a point in $\partial D$ and let us denote with $\nu(P)$ the outer normal vector to $D$ in $P$. There exist positive constants $c_3, c_4$ depending on $a$, $k$, $\alpha$ and $L$ such that
\begin{align}
|\mathcal{G}(x, y) - \mathcal{G}_+(x, y)| &\leq \frac{c_3}{r_0^\alpha} |x - y|^{1+\alpha}, \\
|\nabla_x \mathcal{G}(x, y) - \nabla_x \mathcal{G}_+(x, y)| &\leq \frac{c_4}{r_0^{2\alpha}} |x - y|^{-2+\alpha^2},
\end{align}
for every $x \in D \cap B_r(\nu(P))$ and $y = h\nu(P)$, with $0 < r < (\min\{\frac{1}{2}(8L)^{-1/\alpha}, \frac{1}{2}\})r_0 = \tilde{r}_0$, $0 < h < (\min\{\frac{1}{2}(8L)^{-1/\alpha}, \frac{1}{2}\})^\frac{\alpha}{2}$.

Proof. Let us fix $r_1 = \min\{\frac{1}{2}(8L)^{-1/\alpha}r_0, \frac{1}{2}\}$. In the ball $B_{r_1}(P)$ the boundary of $D$ can be represented as the graph of a $C^{1,\alpha}$ function $\varphi$. Let us introduce now the following change of variable that transform in $B_{r_1}(P) \partial D$ in the $x'$-axis. For every $r > 0$, let $Q_r(P)$ be the cube centered at $P$, with sides of length $2r$ and parallel to the coordinates axes. We have that the ball $B_{r_1}(P)$ is inscribed into $Q_r(P)$. We define
$$
\Psi : Q_{2r_1}(P) \to Q_{2r_1}(P) \quad \left( \begin{array}{c} x' \\ x_n \end{array} \right) \mapsto \left( \begin{array}{c} \xi' = x' \\ \xi_n = x_n - \varphi(x')\theta\left(\frac{|x'|}{r_1}\right)\theta\left(\frac{x_n}{r_1}\right) \end{array} \right),
$$
where $\theta \in C^\infty(\mathbb{R})$ be such that $0 \leq \theta \leq 1$, $\theta(t) = 1$, for $|t| < 1$, $\theta(t) = 0$, for $|t| > 2$ and $|\theta'| \leq 2$. Since the $C^{1,\alpha}$ regularity of $\varphi$, it is possible to verify that the following inequalities hold:
\begin{align}
&c^{-1}|x_1 - x_2| \leq |\Psi(x_1) - \Psi(x_2)| \leq c|x_1 - x_2|, \\
&|\Psi(x) - x| \leq \frac{c}{r_0^\alpha}|x|^{1+\alpha} \quad \forall x \in \mathbb{R}^3, \\
&|D\Psi(x) - I| \leq \frac{c}{r_0^\alpha}|x|^{\alpha} \quad \forall x \in \mathbb{R}^3
\end{align}
where $c \geq 1$ depends on $L$ and $\alpha$ only. $\Psi$ is a $C^{1,\alpha}$ diffeomorphism from $\mathbb{R}^3$ into itself. Let us define the cylinder $C_{r_1}$ as $C_{r_1} = \{x \in \mathbb{R}^3 : |x'| < r_1, |x_n| < r_1\}$. For $x, y \in C_{r_1}$, we shall denote
\begin{equation}
\mathcal{G}(x, y) = \mathcal{G}(\Psi^{-1}(x), \Psi^{-1}(y)).
\end{equation}

$\mathcal{G}(x, y)$ is solution of
\begin{equation}
\nabla((1 + (a - 1)\chi^+)B\nabla\mathcal{G}(x, y)) + k^2\zeta(1 - (b - 1)\chi_+(x))B\mathcal{G}(x, y) = -\delta(x - y),
\end{equation}
where $B = \frac{J \partial\Psi}{\partial\xi}$, with $J = \frac{\partial\Psi}{\partial\xi}(\Psi^{-1}(\xi))$, is of class $C^\alpha$, $B(0) = I$ and $\zeta = \det J$. Since $\mathcal{G}_+$ is solution to
\begin{equation}
\nabla((1 + (a - 1)\chi^+)\mathcal{G}_+(x, y)) + k^2(1 - (b - 1)\chi_+(x))\mathcal{G}_+(x, y) = -\delta(x, y),
\end{equation}

subtracting (4.18) to (4.17) we obtain that \( \tilde{R}(x, y) = \tilde{G}(x, y) - G_+(x, y) \) is solution to

\[
\text{(4.19)} \quad \text{div}((1 + (a - 1)\chi^+)\tilde{R}(x, y)) \\
+ k^2(1 + (b - 1)\chi^+)\tilde{R}(x, y) \\
= \text{div}((1 + (a - 1)\chi^+)|\mathcal{B}(x) - I|\nabla \tilde{G}(x, y)) \\
+ k^2(1 - \zeta)(1 + (b - 1)\chi^+)\tilde{G}(x, y).
\]

Let \( \tilde{L} \), depending on the a priori data, be such that \( \Omega \subset \tilde{B}_{\tilde{L}}(0) \), then using the fundamental solution \( G_+ \) we get

\[
- \tilde{R}(x, y) = \int_{\tilde{B}_{\tilde{L}}(0)} (1 + (a - 1)\chi^+)|\mathcal{B}(z) - I|\nabla \tilde{G}(z, y) \cdot \nabla x \tilde{G}_+(z, x) dz \\
+ \int_{\partial \tilde{B}_{\tilde{L}}(0)} |\mathcal{B}(z) - I| \left[ \tilde{R}(x, z) \frac{\partial G_+}{\partial v}(z, y) + G_+(z, y) \frac{\partial \tilde{R}}{\partial v}(x, z) \right] d\sigma(z) \\
+ k^2(1 - \zeta) \int_{\tilde{B}_{\tilde{L}}(0)} (1 + (b - 1)\chi^+)\tilde{G}(z, x)G_+(z, y) dz + \\
k^2(1 - \zeta) \int_{\partial \tilde{B}_{\tilde{L}}(0)} (1 + (a - 1)\chi^+) \left[ \tilde{R}(x, z) \frac{\partial G_+}{\partial v}(z, y) + G_+(z, y) \frac{\partial \tilde{R}}{\partial v}(x, z) \right] d\sigma(z)
\]

Integrals over \( \partial \tilde{B}_{\tilde{L}}(0) \) are bounded by a constant. Let us split \( \tilde{B}_{\tilde{L}}(0) = (\tilde{B}_{\tilde{L}}(0) \setminus C_{r_1}) \cup (\tilde{B}_{\tilde{L}}(0) \cap C_{r_1}) \).

For \(|x|, |y| \leq r_1/2\), in \( \tilde{B}_{\tilde{L}}(0) \setminus C_{r_1} \) we are away from the singularity thus the integrals over \( \tilde{B}_{\tilde{L}}(0) \setminus C_{r_1} \) are bounded. Let us evaluate integrals over \( \tilde{B}_{\tilde{L}}(0) \cap C_{r_1} \). We have

\[
\left| \int_{\tilde{B}_{\tilde{L}}(0) \cap C_{r_1}} (1 + (a - 1)\chi^+)|\mathcal{B}(z) - I|\nabla \tilde{G}(z, y) \cdot \nabla x \tilde{G}_+(z, x) dz \right| \\
\leq c \int_{\tilde{B}_{\tilde{L}}(0) \cap C_{r_1}} |z|^\alpha |z - y|^{-2} |z - x|^{-2} dz = I
\]

where \( c \) depends on \( L, \alpha, a \) and \( n \). We can split \( I = I_1 + I_2 \) where

\[
I_1 = \int_{\{|z| < 4h\} \cap C_{r_1}} |z|^\alpha |x - z|^{-2} |y - z|^{-2} dz, \\
I_2 = \int_{\{|z| > 4h\} \cap C_{r_1}} |z|^\alpha |x - z|^{-2} |y - z|^{-2} dz.
\]
Now
\[ I_1 \leq \int_{|w| < 4} h^\alpha |w|^{\alpha} |h^{-2} \frac{x}{h} - w|^{-2} |h^{-2} \frac{y}{h} - w|^{-2} dw \]
\[ = h^{\alpha-1} \int_{|w| < 4} |w|^\alpha \left| \frac{x}{h} - w \right|^{-2} \left| \frac{y}{h} - w \right|^{-2} dw \]
\[ \leq h^{\alpha-1} F(\xi, \eta), \]
where \( h = |x - y| \) and
\[ F(\xi, \eta) = 4^\alpha \int_{|w| < 4} |\xi - w|^{-2} |\eta - w|^{-2} dw \]
and \( \xi = x/h \) and \( \eta = y/h \). From standard bounds (see, for instance, [Mi, Ch. 2, § 11]), it is not difficult to see that
\[ F(\xi, \eta) \leq \text{const.} < \infty, \]
for all \( \xi, \eta \in \mathbb{R}^3, |\xi - \eta| = 1 \). Thus
\[ I_1 \leq c|x - y|^{\alpha-1}. \]
Let us consider now \( I_2 \). Since \( |y| = -y_n \leq |x - y| = h \), we can deduce \( |z| \leq \frac{3}{4} |y - z| \) and \( |z| \leq 2|x - z| \) and thus obtain that
\[ I_2 \leq c \int_{|z| > 4h} |z|^{\alpha+1-n+1-\eta} dz \leq c h^{\alpha-1}. \]
Then we conclude
\[ (4.20) \]
\[ |\tilde{R}(x, y)| \leq c|x - y|^{-1+\alpha}, \]
for every \( |x|, |y| \leq r_1/2 \), where \( c \) depends on \( L, \alpha, k \) and \( a \) only.

We observe that if \( x \in \Psi^{-1}(B_{r_1/2}^\circ(0)) \) and \( y = c_3 y_3 \), with \( y_3 \in (-r_1/2, 0) \) then
\[ (4.21) \]
\[ c^{-1} |x| \leq |\Psi(x)| \leq |\Psi(x) - y| \leq c|x - y|. \]
From (4.20) and (4.21) we can conclude
\[ (4.22) \]
\[ |\tilde{R}(x, y)| \leq c|x - y|^{-1+\alpha}. \]
Now, since
\[ G(x, y) - G_+(x, y) = G(x, y) - G_+(x, y) + G_+(\Psi(x), \Psi(y)) - G_+(\Psi(x), \Psi(y)) = \tilde{R}(\Psi(x), \Psi(y)) + G_+(\Psi(x), y) - G_+(x, y), \]
using Theorem 4.1 of [Li-Vo], the properties of \( \Psi \) and (4.22) we obtain
\[ |G(x, y) - G_+(x, y)| \leq \frac{c}{\epsilon_0} |x - y|^{\alpha-1} + \frac{c}{\epsilon_0} \|
abla G_+(\cdot, y)\|_{L^\infty(Q_1^\circ)} |x - \Psi(x)| \]
\[ \leq \frac{c}{\epsilon_0} |x - y|^{\alpha-1} + \frac{c'}{\epsilon_0} |x - y|^{1+\alpha} h^{-2} \]
\[ \leq \frac{c''}{\epsilon_0} |x - y|^{\alpha-1}, \]
where \( c' \) depends on \( k, \alpha \) and \( L \) only.

We estimate now the first derivative of \( \tilde{R} \). To estimate the first derivative of \( \tilde{R} \) let us consider a cube \( Q \subset B^+_t(x) \) of side \( ct/4 \), with \( 0 < c < 1 \), such that \( x \in \partial Q \). The following interpolation inequality holds:

\[
\| \nabla \tilde{R}(\cdot, y) \|_{L^\infty(Q)} \leq c \| \tilde{R}(\cdot, y) \|_{L^\infty(Q)}^{1-\delta} \| \nabla \tilde{R}(\cdot, y) \|_{L^\infty(Q)}^\delta,
\]

where \( \delta = \frac{1}{1+\alpha} \), \( c \) depends on \( L \) only and

\[
|\nabla \tilde{R}|_{\alpha,Q} = \sup_{x,x' \in Q, x \neq x'} \frac{|\nabla \tilde{R}(x, y) - \nabla \tilde{R}(x', y)|}{|x - x'|^{\alpha}}.
\]

Since, from the piecewise Hölder continuity of \( \nabla G \) and of \( \nabla G_+ \), we have that

\[
|\nabla \tilde{R}(x, y)|_{\alpha,Q} \leq |\nabla \tilde{G}(x, y)|_{\alpha,Q} + |\nabla G_+(x, y)|_{\alpha,Q} \leq ch^{-\alpha-2},
\]

where \( c \) depends on \( L \) only, thus we conclude

\[
|\nabla_x \tilde{R}(x, y)| \leq \frac{c}{r^0} h^{(\alpha-1)(1-\delta)} h^{(-\alpha-2)\delta} = \frac{c}{r^0} h^{-2+\eta},
\]

where \( \eta = \frac{\alpha^2}{1+\alpha} \). Thus

\[
(4.23) \quad |\nabla_x \tilde{R}(x, y)| \leq \frac{c}{r^0} |x - y|^{\eta-2},
\]

where \( \eta = \frac{\alpha^2}{1+\alpha} \) and \( c \) depends on \( L \) only. Concerning \( G_+ \) we have

\[
|\nabla_x G_+(\Psi(x), y) - \nabla_x G_+(x, y)|
\]

\[
= |D\Psi(x)^T \nabla G_+(\cdot, y)|_{\Psi(x)} - \nabla_x G_+(x, y)|
\]

\[
\leq |(D\Psi(x)^T - I) \nabla G_+(\cdot, y)|_{\Psi(x)}| + |\nabla G_+(\cdot, y)|_{\Psi(x)} - \nabla_x G_+(x, y)|
\]

\[
\leq \frac{c}{r^0} \|\nabla G_+(\cdot, y)\|_{L^\infty(Q_{1,\alpha})} |x - \Psi(x)| + |\nabla G_+(\cdot, y)|_{\alpha,Q} |\Psi(x) - x|^\alpha
\]

\[
\leq \frac{c}{r^0} h^{1+\alpha} h^{-2} + \frac{c}{r^0} h^{-\alpha-2} h^{(1+\alpha)\alpha}
\]

\[
\leq \frac{c}{r^0} h^{-2+\alpha^2},
\]

where \( c \) depends on \( k, \alpha \) and \( L \) only.

Let us denote by \( G^0_+ \) the Green’s function of the operator \( \text{div}((1 + (a-1)\chi_+)^\nabla) \).

Proposition 4.2. Let \( G_+ \) and \( G^0_+ \) as above, then there exist positive constants \( c_5, c_6 \) depending on the a priori data such that for every \( x, y \in \mathbb{R}^3 \) we have

\[
(4.24) \quad |G_+(x, y) - G^0_+(x, y)| \leq c_5 |x - y|
\]

\[
(4.25) \quad |\nabla_x G_+(x, y) - \nabla_x G^0_+(x, y)| \leq c_6 |x - y|^{-1}
\]

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Proof. Defining $R(x, y) = G_+(x, y) - G^0_+(x, y)$, we have that

$$\text{div}((1 + (b - 1)\chi_{+})\nabla R(x, y)) = -k^2(1 + ((b - 1)\chi_{+})G_+(x, y).$$

Thus

$$-R(x, y) = k^2 \int_{\Omega} (1 + (b - 1)\chi_{+})G_+(z, y)G^0_+(x, z)dz.$$

Hence for [Li-St-We] we have

$$|R(x, y)| \leq C \int_{\Omega} |x - z|^{-1}|y - z|^{-1}dz.$$

Let decompose $\Omega = B_{\frac{|x-y|}{3}}(x) \cup B_{\frac{|x-y|}{3}}(y) \cup G$.

For $z \in B_{\frac{|x-y|}{3}}(x)$ we have that

$$|y - z| \geq |y| - |z| \geq |y| - |z - y| - |x| \geq |x - y| - \frac{|x - y|}{3} = \frac{2}{3}|x - y|.$$

Thus

$$\int_{B_{\frac{|x-y|}{3}}(x)} |x - z|^{-1}|y - z|^{-1}dz \leq \frac{2}{3}|x - y|^{-1} \int_{0}^{\frac{|x-y|}{3}} \rho d\rho \leq c|x - y|^2.$$

Similarly it can be evaluated the integral over $B_{\frac{|x-y|}{3}}(y)$.

Let us consider now the integral over $G$. For $z \in G$ we have that $|z - y| \geq \frac{|x - z|}{3}$, then we obtain

$$\int_{G} |x - z|^{-1}|y - z|^{-1}dz \leq c \int_{G} |x - z|^{-1}|x - z|^{-1}dz \leq c \int_{G \setminus B_{\frac{|x-y|}{3}}(x)} |x - z|^{-1}|x - z|^{-1}dz \leq c \int_{G \setminus B_{\frac{|x-y|}{3}}(x)} |x - z|^{-1}dz \leq c \int_{\frac{|x-y|}{3}}^{2\tilde{L}} \rho d\rho \leq c_1|x - y|^{-2} + c_2.$$

Let us prove now (4.25). We use the interpolation inequality

$$\|\nabla R(\cdot)\|_{L^\infty(Q)} \leq \|R(\cdot)\|^{1-\delta}_{L^\infty(Q)} \|\nabla R(\cdot, y)\|_{\alpha, Q}^\delta.$$

As in Proposition 4.1, since

$$|\nabla R(\cdot, y)|_{\alpha, Q} \leq h^{-\alpha - 2},$$

we obtain

$$|\nabla R(x, y)| \leq ch^{-2+n} \leq ch^{-1}.$$ 

$\square$
5 Proof of Proposition 3.3 and 3.4

Proof of Proposition 3.3. Let us consider $f(y, \varpi)$, where $\varpi$ is a fixed point in $\overline{CB}$. Since $f$, as a function of $y$, is a radiating solution of

$$L_y f = \Delta_y f + k^2 f = 0 \quad \text{in} \, \Omega_D,$$

then by [Co-Kr, Theorem 2.14], for $y \in \overline{CB}$ we have

$$f(y, \varpi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m h_{n}^{(1)}(k|y|) Y_{n}^{m}(\hat{y}),$$

where $\hat{y} = y/|y|$, $Y_{n}^{m}$ is a spherical harmonic of order $n$ and $h_{n}^{(1)}$ is a spherical Hankel function of the first kind of order $n$. Let us consider $y$ such that $R < R_1 < |y| < R_2$. For an integer $N$, using Schwarz inequality and the asymptotic behavior of Hankel function (see [Co-Kr, (2.38) pg. 28]) we have

$$\left| \sum_{n=0}^{N} \sum_{m=-n}^{n} a_n^m h_{n}^{(1)}(k|y|) Y_{n}^{m}(\hat{y}) \right|^2 \leq \sum_{n=0}^{N} \left| h_{n}^{(1)}(k|y|) \right|^2 \sum_{n=0}^{N} \sum_{m=-n}^{n} \left| a_n^m \right|^2 \left| h_{n}^{(1)}(kR) \right|^2 \left| Y_{n}^{m}(\hat{y}) \right|^2,$$

$$\leq c \sum_{n=0}^{N} \sum_{m=-n}^{n} \left| a_n^m \right|^2 \left| h_{n}^{(1)}(kR) \right|^2 \left| Y_{n}^{m}(\hat{y}) \right|^2,$$

for some constant $c$ depending on $R$, $R_1$ and $R_2$. Thus, taking the limit as $N \to +\infty$, we can conclude that

$$|f(y, \varpi)|^2 \leq c|f(\cdot, \varpi)|_{\partial B}^2, \quad \forall y \in B_{R_2} \setminus \overline{B}_R,$$

where $c$ depends on $R$, $R_1$ and $R_2$. Analogous considerations can be carried on fixing $y$ and varying $w$. Thus, we can conclude that for all $(y, w) \in [B_{R_2} \setminus \overline{B}_R]^2$

$$|f(y, w)| \leq |f_{\partial B \times \partial B}| \leq c\varepsilon.$$

For $y \in G^h$, where $G^h = \{ x \in G : \text{dist}(x, \Omega_D) \geq h \}$,

$$|S_1(y, \varpi)| \leq c \int_{D_1} |x - y|^{-2} \leq c h^{-2},$$

where $c = c(L, R)$. Similarly $|S_2(y, \varpi)| \leq c h^{-2}$. Then we conclude that

$$|f(y, \varpi)| \leq c h^{-2} \quad \text{in} \, \, G^h. \tag{5.27}$$

At this stage we shall make use iteratively of the three spheres inequality (see [La, Ku]). Let $u$ be a solution of $Lu = 0$ in $G$, let $x \in G$. There exist $r_1$, $r$, $r_2$, $0 < r_1 < r < r_2 < R$ and $\tau \in (0, 1)$ such that

$$|u|_{L^\infty(B_r(x))} \leq c |u|_{L^\infty(B_{r_1}(x))} |u|_{L^\infty(B_{r_2}(x))}^{1-\tau}, \tag{5.28}$$

$$12$$
where \( c \) and \( \tau \) depend on \( R, r/r_2, r_1/r_2 \) and \( L \). Applying (5.28) to \( u(\cdot) = f(\cdot, \overline{w}) \), with \( x = \overline{w} \in B_{4R} \cap \overline{B}_{3R}, r_1 = r_0/2, r = 3r_0/2 \) and \( r_2 = 2r_0 \) we obtain

\[
\|f\|_{L^\infty(B_{3r_0/2}(\overline{w}))} \leq c \|f\|_{L^\infty(B_{r_0/2}(\overline{w}))} \|f\|_{L^\infty(B_{2r_0}(\overline{w}))}^{1-\tau}.
\]

For every \( \overline{y} \in \mathcal{G}^h \), we denote by \( \gamma \) a simple arc in \( \mathcal{G} \) joining \( \overline{x} \) to \( \overline{y} \). Let us define \( \{x_i\}, i = 1, \ldots, s \) as follows \( x_1 = \overline{x}, x_{i+1} = \gamma(t_i) \), where \( t_i = \max \{t : |\gamma(t) - x_i| = r_0\} \) if \( |x_i - \overline{y}| > r_0 \), otherwise let \( i = s \) and stop the process. By construction, the balls \( B_{r_0/2}(x_i) \) are pairwise disjoint, \( |x_i+1 - x_i| = r_0 \) for \( i = 1, \ldots, s-1 \), \( |x_s - \overline{y}| \leq r_0 \). There exists \( \beta \) such that \( s \leq \beta \). An iterated application of the three spheres inequality (5.28) for \( f \) (see for instance [Al-Be-Ro-Ve, pg. 780], [Al-DB, Appendix E]) gives that for any \( r_0, 0 < r < r_0 \)

\[
\|f\|_{L^\infty(B_{r/2}(\overline{y}))} \leq c \|f\|_{L^\infty(B_{r_0}(\overline{y}))} \|f\|_{L^\infty(\mathcal{G})}^{1-\tau},
\]

We can estimate the right hand side of (5.29) by (5.27) and obtain for any \( r_0, 0 < r < r_0 \)

\[
\|f\|_{L^\infty(B_{r/2}(\overline{y}))} \leq c(h^{-\frac{1}{4}} - \varepsilon)^{1-\tau} \leq ch^{-A} \varepsilon^{\tilde{\beta}},
\]

where \( \tilde{\beta} = \beta \) and \( A = 2(1 - \beta) \). Let \( O \in \partial D_1 \) as defined in (3.6), that is

\[
d(O, D_2) = d_\nu(D_1, D_2).
\]

There exists a \( C^{1,\alpha} \) neighborhood \( U \) of \( O \) in \( \partial \Omega_D \) with constants \( r_0 \) and \( L \). Thus there exists a non-tangential vector field \( \nu \), defined on \( U \) such that the truncated cone

\[
(C(O, \nu(O), \theta, r_0) = \left\{ x \in \mathbb{R}^3 : \frac{(x - O) \cdot \nu(O)}{|x - O|} > \cos \theta, |x - O| < r_0 \right\}
\]

satisfies

\[
C(O, \nu(O), \theta, r_0) \subset \mathcal{G},
\]

where \( \theta = \arctan(1/L) \). Let us define

\[
\lambda_1 = \min \left\{ \frac{r_0}{1 + \sin \theta}, \frac{r_0}{3 \sin \theta} \right\}, \quad \theta_1 = \arcsin \left( \frac{\sin \theta}{4} \right), \quad \rho_1 = \lambda_1 \sin \theta_1.
\]

We have that \( B_{\rho_1}(G_1) \subset C(O, \nu(O), \theta_1, r_0) \), \( B_{2\rho_1}(G_1) \subset C(O, \nu(O), \theta, r_0) \). Let \( \mathcal{G} = G_1 \), since \( \rho_1 \leq r_0/2 \), we can use (5.30) in the ball \( B_{\rho_1}(\mathcal{G}) \) and we can approach \( O \in \partial D_1 \) by constructing a sequence of balls contained in the cone \( C(O, \nu(O), \theta_1, r_0) \). We define, for \( k \geq 2 \)

\[
G_k = O + \lambda_k \nu, \quad \lambda_k = \chi \lambda_{k-1}, \quad \rho_k = \chi \rho_{k-1}, \quad \text{with } \chi = \frac{1 - \sin \theta_1}{1 + \sin \theta_1}.
\]

Hence \( \rho_k = \chi^{k-1} \rho_1, \lambda_k = \chi^{k-1} \lambda_1 \) and

\[
B_{\rho_{k+1}}(G_{k+1}) \subset B_{\rho_k}(G_k) \subset B_{\rho_{k+1}}(G_k) \subset C(O, \nu, \theta, r_0).
\]
Denoting \( d(k) = |G_k - O| - \rho_k = \lambda_k - \rho_k \), we have \( d(k) = \chi^{k-1}d(1) \), with \( d(1) = \lambda_1(1 - \sin \theta) \). For any \( r, 0 < r \leq d(1) \), let \( k(r) \) be the smallest integer such that \( d(k) \leq r \), that is
\[
\frac{|\log \frac{r}{d(1)}|}{|\log \chi|} \leq k(r) - 1 \leq \frac{|\log \frac{r}{d(1)}|}{|\log \chi|} + 1.
\]

By an iterated application of the three spheres inequality over the chain of balls \( B_{\rho_1}(G_1), \ldots, B_{\rho_k(r)}(G_k(r)) \), we have
\[
(5.32) \quad \|f(\cdot, \bar{\nu})\|_{L^\infty(B_{\rho_k(r)}(G_k(r)))} \leq c h^{-A(1 - \tau^k(r) - 1)} e^{\beta \tau^k(r) - 1} \leq c h^{-A} e^{\beta \tau^k(r) - 1},
\]
for \( 0 < r < c\rho_0 \), where \( c, 0 < c < 1 \), depends on \( L \). Let us consider now \( f(y, w) \) as a function of \( w \). First we observe that
\[
L_w f = 0 \quad \text{in} \quad C\Omega_D, \quad \text{for all} \quad y \in C\Omega_D.
\]
For \( y, w \in \mathcal{G}^b, \ y \neq w \), using (3.9)
\[
|S_1(y, w)| \leq c \int_{B_1} |x - y|^{-2} |x - w|^{-2} dx \leq c h^{-4}.
\]
Similarly for \( S_2 \). Therefore
\[
|f(y, w)| \leq c h^{-4} \quad \text{with} \quad y, w \in \mathcal{G}^b.
\]

For \( w \in B_{4R} \setminus B_{3R} \) and \( y \in \mathcal{G}^b \), using (5.32), we have
\[
|f(y, w)| \leq c h^{-A} e^{\beta \tau^k(r) - 1}.
\]
Proceeding as before, let us fix \( y \in \mathcal{G} \) such that \( \text{dist}(y, \Omega_D) = h \) and \( \bar{w} \in B_{4R} \setminus B_{3R} \) such that \( \text{dist}(\bar{w}, \partial B_R) = R/2 \). Taking \( r = R/2, \ r_1 = 3r, \ r_2 = 4r, \ w_1 = O + \lambda_1 \nu \) and using iteratively the three spheres inequality, we have
\[
\|f(y, w)\|_{L^\infty(B_{R/2}(w_1))} \leq \|f(y, w)\|_{L^\infty(B_{R/2}(\bar{w}))} \leq c h^{-A} e^{\beta \tau^k(r) - 1},
\]
where \( \tau \) and \( s \) are as above. Therefore
\[
\|f(y, w)\|_{L^\infty(B_{R/2}(w_1))} \leq c(h^{-4})^{1-\tau^s} h^{-A\tau^s} (e^{\beta \tau^k(r) - 1})^{\tau^s} \leq c(h^{-4})^{1-\gamma} h^{-A\tau^s} (e^{\beta \tau^k(r) - 1})^{\gamma} \leq c h^{-A'} (e^{\beta \tau^k(r) - 1})^{\gamma},
\]
where \( \gamma = \tau^\beta \), with \( \beta \) as above, so \( 0 < \gamma < 1 \) and \( A' = A\tau^s - 4 + \gamma \). Once again, let us apply the three spheres inequality over a chain of balls contained in a cone with vertex in \( O \), choosing \( y = w = \nu(0) \) we obtain
\[
(5.33) \quad |f(y, y)| \leq c h^{-A'} (e^{\beta \tau^k(r) - 1})^{\gamma} e^{\gamma\tau^k(r) - 1}.
\]
We observe that, for \( 0 < h < c\rho_0 \), where \( 0 < c < 1 \) depends on \( L \), \( k(h) \leq |\log h| = -c|\log h| \), so we can write
\[
\tau^k(h) = e^{-c|\log h| \log \tau} = h^{-c|\log \tau|} = h^{1/\beta},
\]
where \( f\in C^{\tau^k(h)} \).
with $F = c\log \tau$. Therefore

$$|f(y,y)| \leq e^{-A'\log h + B'h^F \log \epsilon}$$

Then in (5.33) we obtain

$$|f(y,y)| \leq e^{-A'\log h + B'h^F \log \epsilon} = \frac{\epsilon B'h^F}{h^{A'}}.$$

**Proof of Proposition 3.4.** Let us define $r_2 = \min\{r_0, r_2\}$, where $r_0$ is the one of Proposition 4.1 and $r_2$ will be fixed later. For every $x, y$ such that $|x - y| < r$, with $0 < r < r_2$, the following asymptotic formula holds (cf. Proposition 4.1)

$$|G_1(x, y) - G_2(x, y)| \leq c|x - y|^{-1+\alpha}.$$  

We now distinguish two situations:

1) $x \in B_r \cap (D_1 \cap D_2)$;
2) $x \in B_r \cap (D_1 \setminus D_2)$.

If case 1) occurs then the asymptotic formula (4.14) holds also for $G_2$ since the hypothesis of Proposition 4.1 are met. From [Al, Lemma 3.1] there exists $r_2$, depending on the a priori data, such that

$$\nabla G_1(x, y) \cdot \nabla G_2(x, y) \geq c|x - y|^{-2}.$$  

Let us consider case 2). In $B_r \cap (D_1 \setminus D_2)$ we consider a smaller ball $B_\rho(0)$ with radius $\rho$ where $0 < \rho < \min\{d_\mu, r_2\}$. Since the definition of $d_\mu$ we have $B_\rho \cap D_2 = \emptyset$. If $x$ and $y$ are in $B_\rho$ and denoting by $L = \Delta + k^2$ we have

$$L(G_2(x, y) - \Phi(x, y)) = 0 \quad \text{in } B_\rho,$$

where $\Phi$ is the fundamental solution of the Helmholtz equation, with the boundary condition

$$[G_2(x, y) - \Phi(x, y)]|_{\partial B_\rho} \leq c\rho^{-1}.$$  

Thus by maximum principle

$$|G_2(x, y) - \Phi(x, y)| \leq c_1\rho^{-1} \quad \forall x, y \in B_\rho$$

and by interior gradient bound

$$|\nabla G_2(x, y) - \nabla \Phi(x, y)| \leq c_2\rho^{-2} \quad \forall x \in B_{\rho/2}, \forall y \in B_\rho.$$

Thus using Lemma 3.1 of [Al], in $B_{\rho/2}(O)$ we obtain the formula formula

$$\nabla G_1(x, y) \cdot \nabla G_2(x, y) \geq c|x - y|^{-2} - c_4\rho^{-2}.$$  

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Let us consider \( h < r_2/2 \) and \( 0 < r < r_2 \). Then we have

\[
\left| \int_{D_1} \nabla G_1(x, y) \cdot \nabla G_2(x, y) \, dx \right|
\]

\[
= \left| \int_{D_1 \cap B_r(O)} \nabla G_1(x, y) \cdot \nabla G_2(x, y) \, dx \right| + \left| \int_{D_1 \setminus B_r(O)} \nabla G_1(x, y) \cdot \nabla G_2(x, y) \, dx \right|
\]

\[
\geq \left| \int_{D_1 \cap B_r(O)} \nabla G_1(x, y) \cdot \nabla G_2(x, y) \, dx \right| - \left| \int_{D_1 \setminus B_r(O)} \nabla G_1(x, y) \cdot \nabla G_2(x, y) \, dx \right|
\]

The first integral can be estimated as follows

\[
\left| \int_{D_1 \cap B_r(O)} \nabla G_1(x, y) \cdot \nabla G_2(x, y) \, dx \right|
\]

\[
= \left| \int_{(D_1 \cap D_2) \cap B_r(O)} \nabla G_1(x, y) \cdot \nabla G_2(x, y) \, dx \right|
\]

\[
+ \left| \int_{(D_1 \setminus D_2) \cap B_r(O)} \nabla G_1(x, y) \cdot \nabla G_2(x, y) \, dx \right|
\]

\[
\geq \left| \int_{[(D_1 \cap D_2) \cap B_r(O)] \cup [(D_1 \setminus D_2) \cap B_r]} \nabla G_1(x, y) \cdot \nabla G_2(x, y) \, dx \right|
\]

\[
- \left| \int_{[(D_1 \setminus D_2) \cap B_r(O)] \setminus B_r} \nabla G_1(x, y) \cdot \nabla G_2(x, y) \, dx \right|
\]

In conclusion, choosing \( \rho = h \) and using (5.34), (5.35) and (3.9) we obtain

\[
|S_1(y)| \geq c_1 \int_{[(D_1 \cap D_2) \cap B_R(O)] \cup [(D_1 \setminus D_2) \cap B_R]} |x - y|^{-2} \, dx
\]

\[
- c_2 \int_{[(D_1 \setminus D_2) \cap B_R(O)] \setminus B_R} |x - y|^{-1} \, dx - c_3 \int_{D_1 \setminus B_r(O)} |x - y|^{-1} |x - y|^{-1} \, dx
\]

\[
\geq c_4 h^{-2} - c_5 d_\mu^{-2} - c_6.
\]

\[\square\]

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