THE BOUNDEDNESS OF FRACTIONAL MAXIMAL OPERATORS ON VARIABLE LEBESGUE SPACES OVER SPACES OF HOMOGENEOUS TYPE

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ABSTRACT. Given a space of homogeneous type $(X, d, \mu)$, we give sufficient conditions on a variable exponent $p(\cdot)$ so that the fractional maximal operator $M_\eta$, $0 \leq \eta < 1$, maps $L^{p(\cdot)}(X)$ to $L^{q(\cdot)}(X)$, where $1/p(\cdot) - 1/q(\cdot) = \eta$. In the endpoint case we also prove the corresponding weak type inequality. As an application we prove norm inequalities for the fractional integral operator $I_\eta$. Our proof for the fractional maximal operator uses the theory of dyadic cubes on spaces of homogeneous type, and even in the Euclidean setting it is simpler than existing proofs. For the fractional integral operator we extend a pointwise inequality of Welland to spaces of homogeneous type. Our work generalizes results in \cite{6, 8} from the Euclidean case and extends recent work by Adamowicz, et al. \cite{1} on the Hardy-Littlewood maximal operator on spaces of homogeneous type.

1. Introduction

In this paper we study the boundedness of the fractional maximal operator on variable Lebesgue spaces defined over spaces $(X, d, \mu)$ of homogeneous type. Variable Lebesgue spaces are Banach function spaces which generalize the classical Lebesgue spaces; intuitively, they consist of all measurable functions that satisfy

$$\int_X |f(x)|^{p(x)} \, d\mu < \infty.$$ 

These spaces have been intensively studied for the past twenty years: see the books \cite{9, 11} for detailed histories and references. Such spaces were first studied on $\mathbb{R}^n$ equipped with the standard Euclidean distance and Lebesgue measure. However, more recently there has been interest in working with variable exponent spaces defined over spaces of homogeneous type. See, for instance, \cite{2, 12, 14, 18, 22, 23}.

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A central problem is to determine sufficient conditions on the exponent \(p(\cdot)\) such that the Hardy-Littlewood and fractional maximal operators are bounded on \(L^{p(\cdot)}(X)\). For a detailed history of this problem in the Euclidean setting, see [9]. This problem was first considered on metric measure spaces in [17]. However, in this and subsequent papers, results were proved with the (somewhat) unnatural restrictions that either \(X\) is bounded or that the exponent \(p(\cdot)\) is constant outside a large ball. Adamowicz, Harjulehto and Hästö [1] were able to remove this restriction and established the boundedness of the maximal operator on \(L^{p(\cdot)}(X)\) on an unbounded quasi-metric space \(X\). While they did not assume that the underlying measure is doubling, they did assume that \(1/p(\cdot)\) satisfies a Diening type condition.

In our paper we consider the same problem for the fractional maximal operator: given \(0 \leq \eta < 1\), define the operator \(M_\eta\) by

\[
M_\eta f(x) = \sup_{B \ni x} \mu(B)^{\eta} \int_B |f| \, d\mu.
\]

When \(\eta = 0\) this reduces to the Hardy-Littlewood maximal operator and we write \(M\) instead of \(M_0\). On classical Lebesgue spaces over \(\mathbb{R}^n\), norm inequalities for \(M_\eta\) are well known. For variable Lebesgue spaces in the Euclidean case, norm inequalities for \(M_\eta\) were first proved in [6] and then subsequently in [8]. Our main results are generalizations of the results in [6, 8] to the setting of spaces of homogeneous type. We state them here, though we defer the statement of precise definitions for all of our results until Section 2.

**Theorem 1.1.** Let \((X, d, \mu)\) be a space of homogeneous type. Given \(0 \leq \eta < 1\), let \(p(\cdot) : X \to [1, \infty]\) be such that \(1/p(\cdot) \in LH\) and \(1 < p_- \leq p_+ \leq 1/\eta\). For each \(x \in X\), define \(q(\cdot)\) pointwise by \(1/p(x) - 1/q(x) = \eta\). Then there exists a constant \(C = C(p(\cdot), \eta, X)\) such that for all \(f \in L^{p(\cdot)}(X)\),

\[
\|M_\eta f\|_{q(\cdot)} \leq C\|f\|_{p(\cdot)}.
\]

Moreover, if \(\mu(X) < +\infty\), then we can replace the hypothesis that \(1/p(\cdot) \in LH\) with \(1/p(\cdot) \in LH_0\).

When \(\eta = 0\), Theorem 1.1 is a particular case of the results in [1]. Unlike in this paper we prefer to work on spaces of homogeneous type. Further, we do not \textit{a priori} assume that \(1/p(\cdot)\) satisfies a Diening type condition: rather, we derive this property as a consequence of the log-Hölder continuity of \(1/p(\cdot)\). Our method of proof is very different from theirs and generalizes an argument given in [8] (see also [9]) that is based on the Calderón-Zygmund decomposition in Euclidean spaces. In order to do so we exploit the theory of dyadic cubes on spaces of homogeneous type, first introduced by Christ [7] and refined by Hytönen and Kaïcrema [20]. We want to emphasize that our results are not simply a translation of this earlier work to the setting of spaces of homogeneous type: even in the Euclidean case our proof is a
significant refinement of the one in [8], and in adapting it to spaces of homogeneous type we needed to overcome various technical obstacles.

We also prove a weak type inequality in the endpoint case \( p_- = 1 \). In the Euclidean case this was first proved in [6]; in the setting of spaces of homogeneous type it is new, even when \( \eta = 0 \).

**Theorem 1.2.** Let \((X, d, \mu)\) be a space of homogeneous type. Given \( 0 \leq \eta < 1 \), let \( p(\cdot) : X \to [1, \infty] \) be such that \( 1/p(\cdot) \in LH \) and \( 1 = p_- \leq p_+ < 1/\eta \). Then there exists a positive constant \( C = C(p(\cdot), \eta, X) \) such that for all \( f \in L^{p(\cdot)}(X) \),

\[
\sup_{t>0} t \left\| \chi_{\{M_{\eta} f > t\}} \right\|_{q(\cdot)} \leq C \| f \|_{p(\cdot)},
\]

where \( 1/p(\cdot) - 1/q(\cdot) = \eta \). Furthermore, if \( \mu(X) < +\infty \), we can replace the hypothesis that \( 1/p(\cdot) \in LH \) with \( 1/p(\cdot) \in LH_0 \).

As an application of Theorems 1.1 and 1.2 we prove strong and weak type norm inequalities for the fractional integral operator (also referred to as the Riesz potential),

\[
I_{\eta} f(x) = \int_X \frac{f(y)}{\mu(B(x, d(x, y)))^{1-\eta}} \, d\mu(y).
\]

These operators have been extensively studied on spaces of homogeneous type for constant exponents: see [21] and the references it contains. In the Euclidean case, strong and weak type inequalities on variable Lebesgue spaces were proved in [6] (but see the references there for earlier, partial results). On spaces of homogeneous type they were considered in [12, 14] when \( \mu(X) < +\infty \). For our results we need to impose an additional condition on the space \((X, d, \mu)\); we will discuss this hypothesis in more detail in Section 2.

**Theorem 1.3.** Let \((X, d, \mu)\) be a reverse doubling space. Given \( 0 < \eta < 1 \), let \( p(\cdot) : X \to [1, +\infty) \) be such that \( p(\cdot) \in LH \) and \( 1 < p_- \leq p_+ < 1/\eta \). Define \( q(\cdot) \) by \( 1/p(\cdot) - 1/q(\cdot) = \eta \). Then there exists \( C = C(p(\cdot), \eta, X) \) such that for all \( f \in L^{p(\cdot)}(X) \),

\[
\| I_{\eta} f \|_{q(\cdot)} \leq C \| f \|_{p(\cdot)}.
\]

Moreover, if \( \mu(X) < +\infty \), we can replace the hypothesis that \( 1/p(\cdot) \in LH \) with \( 1/p(\cdot) \in LH_0 \).

**Theorem 1.4.** Let \((X, d, \mu)\) be a reverse doubling space. Given \( 0 < \eta < 1 \), let \( p(\cdot) : X \to [1, +\infty) \) be such that \( p(\cdot) \in LH \) and \( 1 < p_- \leq p_+ < 1/\eta \). Define \( q(\cdot) \) by \( 1/p(\cdot) - 1/q(\cdot) = \eta \). Then there exists \( C = C(p(\cdot), \eta, X) \) such that for all \( f \in L^{p(\cdot)}(X) \),

\[
\sup_{t>0} t \left\| \chi_{\{|I_{\eta} f| > t\}} \right\|_{q(\cdot)} \leq C \| f \|_{p(\cdot)}.
\]

Moreover, if \( \mu(X) < +\infty \), we can replace the hypothesis that \( 1/p(\cdot) \in LH \) with \( 1/p(\cdot) \in LH_0 \).
As an immediate consequence of Theorem 1.3 we can prove norm inequalities for other variants of the fractional integral operator on an Ahlfors regular space. We say that \((\Omega, d, \mu)\) is Ahlfors regular if there exist constants \(C_1, C_2, Q > 0\) such that for every \(x \in \Omega\) and \(r > 0\)
\[
C_1 r^Q \leq \mu(B(x, r)) \leq C_2 r^Q.
\]
(1.3)

The constant \(Q\) is referred to as the dimension of the space.

Given \(0 < \alpha < Q\), define the operators

\[
I^\ast_\alpha f(x) = \int_X \frac{f(y)}{d(x, y)^{Q-\alpha}} d\mu(y)
\]
and

\[
I^{**}_\alpha f(x) = \int_X \frac{f(y) d(x, y)^\alpha}{\mu(B(x, d(x, y)))} d\mu(y).
\]

These operators have also been extensively studied in spaces of homogeneous type for constant exponents: see [21, 25]. They have applications to the study of Sobolev and Poincaré inequalities over metric spaces: see [16, 24].

If \((\Omega, d, \mu)\) is Ahlfors regular, then it is immediate that these operators are pointwise equivalent to \(I^\eta f\) with \(\eta = \alpha/Q\), and strong and weak type norm inequalities follow from Theorems 1.3 and 1.4. For brevity we only state the strong type inequality with the (implicit) assumption that \(\mu(X) = +\infty\); precise statements of the other results are left to the interested reader.

**Corollary 1.5.** Suppose \((X, d, \mu)\) is an Ahlfors regular space with dimension \(Q\). Given \(0 < \alpha < Q\), let \(p(\cdot) : X \to [1, +\infty)\) be such that \(p(\cdot) \in LH\) and \(1 < p_- \leq p_+ < Q/\alpha\). Define \(q(\cdot)\) by \(1/p(\cdot) - 1/q(\cdot) = \alpha/Q\). Then there exists \(C = C(p(\cdot), Q, \alpha, X)\) such that for all \(f \in L^{p(\cdot)}(X)\),

\[
\|I^\ast_\alpha f\|_{q(\cdot)} \leq C\|f\|_{p(\cdot)}.
\]

The same inequality also holds for \(I^{**}_\alpha\).

**Remark 1.6.** We can actually prove inequalities for \(I^\ast_\alpha\) or \(I^{**}_\alpha\) assuming that the space is either upper or lower Ahlfors regular—i.e., that either the righthand or lefthand inequality in (1.3) holds. Details are left to the interested reader.

The remainder of this paper is organized as follows. In Section 2 we gather together the necessary definitions and a number of preliminary results about spaces of homogeneous type, fractional maximal operators and variable Lebesgue spaces. In Section 3 we prove Theorem 1.1, and in Section 4 we prove Theorem 1.2. Since there are many similarities between the proofs of Theorem 1.1 and Theorem 1.2, we omit overlapping details. Finally, in Section 5 we prove Theorems 1.3 and 1.4. Our proof involves extending a pointwise estimate due to Welland [26] to spaces of homogeneous
type that relates the fractional integral $I_\eta$ to the fractional maximal operator $M_\eta$. This estimate is interesting in its own right and should have applications to other problems on spaces of homogeneous type. (A related estimate was proved in [13] where it was used to derive weighted norm inequalities.)

Throughout this paper our notation is standard and will be defined as needed. Hereafter, $C, c$ will denote constants whose values depend only on “universal” parameters and whose value may change from line to line. In particular, constants may depend on the underlying triple $(X, d, \mu)$. We will use the convention that $1/\infty = 0$ and $1/0 = \infty$.

2. Preliminary results

Spaces of homogeneous type. We begin with a definition. For more information, see [4, 19].

Definition 2.1. Given a set $X$ and a function $d : X \times X \to [0, \infty)$, we say that $(X, d)$ is a quasi-metric space if $d$ satisfies the following conditions:

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, y) \leq A_0(d(x, z) + d(y, z))$ for all $x, y, z \in X$ and some constant $A_0 \geq 1$.

Property (3) is called the quasi-triangle inequality and $A_0$ the quasi-metric constant.

Definition 2.2. Given a quasi-metric space $(X, d)$ and a positive measure $\mu$ that is defined on the $\sigma$-algebra generated by quasi-metric balls and open sets, we say that $(X, d, \mu)$ is a space of homogeneous type if there exists a constant $C \mu \geq 1$ such that for any $x \in X$ and any $r > 0$,

$$\mu(B(x, 2r)) \leq C \mu \mu(B(x, r)),$$

where $B(x, r)$ is the ball centred at $x$ with radius $r$. To avoid trivial measures we will always assume that $0 < \mu(B) < +\infty$ for every ball $B$.

A measure $\mu$ that satisfies the property in Definition 2.2 is called doubling. The next lemma gives a consequence of this property referred to as the lower mass bound. The proof is well-known and we omit it.

Lemma 2.3. Let $(X, d, \mu)$ be a space of homogeneous type. Then there exists a positive constant $C = C(C_\mu, A_0)$ such that for all $x \in X$, $0 < r < R$ and $y \in B(x, R)$,

$$\frac{\mu(B(y, r))}{\mu(B(x, R))} \geq C \left(\frac{r}{R}\right)^{\log_2 C_\mu}.$$

The next lemma characterizes spaces of homogeneous type with finite measure. We refer the reader to [5] for a proof.
Lemma 2.4. Let \((X, d, \mu)\) be a space of homogeneous type. Then \(\mu(X) < +\infty\) if and only if
\[
\text{diam}(X) = \sup_{x, y \in X} d(x, y) < +\infty.
\]

On Euclidean spaces dyadic cubes play a fundamental role in harmonic analysis. In particular they let us define dyadic versions of various operators. Christ \cite{Christ1} constructed a system of sets on a space of homogeneous type which satisfy many of the essential properties of a system of dyadic cubes in Euclidean space. His construction was further refined by Hytönen and Kairema \cite{Hytokairema} and an equivalent formulation was given in \cite{Christ2}. We will use the version from \cite{Christ2}.

Theorem 2.5. Let \((X, d, \mu)\) be a space of homogeneous type. There exist constants \(C > 0\), \(0 < \delta, \epsilon < 1\) which depend on \(X\), a family of sets \(D = \bigcup_{k \in \mathbb{Z}} D_k\), and a collection of points \(\{x_c(Q)\}_{Q \in D}\) that satisfy the following properties:

1. For every \(k \in \mathbb{Z}\) the cubes in \(D_k\) are pairwise disjoint and \(X = \bigcup_{Q \in D_k} Q\). We will refer to the cubes in \(D_k\) as cubes in the \(k\)-th generation;
2. If \(Q_1, Q_2 \in D\), then either \(Q_1 \cap Q_2 = \emptyset\), \(Q_1 \subseteq Q_2\) or \(Q_2 \subseteq Q_1\);
3. For any \(Q_1 \in D_k\) there exists at least one \(Q_2 \in D_{k+1}\), which is called a child of \(Q_1\), such that \(Q_2 \subseteq Q_1\) and there exists exactly one \(Q_3 \in D_{k-1}\), which is called a parent of \(Q_1\), such that \(Q_1 \subseteq Q_3\);
4. If \(Q_2\) is a child of \(Q_1\), then \(\mu(Q_2) \geq \epsilon \mu(Q_1)\);
5. For every \(k\) and \(Q \in D_k\), \(B(x_c(Q), \delta^k) \subseteq Q \subseteq B(x_c(Q), C\delta^k)\).

The collection \(D\) is referred to as a dyadic grid on \(X\) and the sets \(Q \in D\) as dyadic cubes. The last property in Theorem 2.5 permits a comparison between a dyadic cube and quasi-metric balls; however, we will also need a way to compare a quasi-metric ball with dyadic cubes. For this reason it is important to have a finite family of dyadic grids such that an arbitrary quasi-metric ball is contained in a dyadic cube from one of these grids. Such a finite family of dyadic grids is referred to as an adjacent system of dyadic grids.

Theorem 2.6. Let \((X, d, \mu)\) be a space of homogeneous type. There exists a positive integer \(K = K(X)\), a finite constant \(C = C(X)\), and a finite collection of dyadic grids, \(D^t, 1 \leq t \leq K\), such that given any ball \(B = B(x, r) \subseteq X\) there exists \(t\) and a dyadic cube \(Q \in D^t\) such that \(B \subseteq Q\) and \(\text{diam} Q \leq Cr\).

Reverse doubling and Ahlfors regular spaces. For our results on fractional integral operators we need to impose an additional condition on our underlying space.

Definition 2.7. Given a space of homogeneous type \((X, d, \mu)\), we say that it is a reverse doubling space if there exists a constant \(0 < \gamma < 1\) such that for every \(x \in X\)
and \( r > 0 \) such that \( B(x, r) \subsetneq X \),
\[
\mu(B(x, r/2)) \leq \gamma \mu(B(x, r)).
\]
If this condition holds we also say that the measure \( \mu \) is reverse doubling.

On Euclidean space any doubling measure is reverse doubling; the same is true on any metric (as opposed to quasi-metric) space that is connected: see [4]. More generally, it holds on any space of homogeneous type that satisfies a non-empty annuli condition: for a precise definition and proof, see [10].

Reverse doubling spaces do not have atoms: this is the content of the next lemma.

**Lemma 2.8.** If \((X, d, \mu)\) is a reverse doubling space, then for all \( x \in X \), \( \mu(\{x\}) = 0 \).

**Proof.** If \( \mu(X) = +\infty \) then for any \( x \) by the definition of reverse doubling
\[
\mu(\{x\}) = \lim_{i \to \infty} \mu(B(x, 2^{-i})) \leq \lim_{i \to \infty} \gamma^i \mu(B(x, 1)) = 0.
\]
Now assume that \( \mu(X) < +\infty \) and let \( x \in X \). Choose
\[
0 < r < \frac{\text{diam}(X)}{8A_0}.
\]
By definition there exist points \( y, z \in X \) such that \( 2^{-1} \text{diam}(X) < d(y, z) \). If both \( y \) and \( z \) belong to \( B(x, r) \), then an application of the quasi-triangle inequality gives
\[
\frac{1}{2} \text{diam}(X) < d(y, z) \leq A_0(d(x, y) + d(x, z)) < 2A_0 r < \frac{1}{4} \text{diam}(X),
\]
which is a contradiction. Therefore, \( B(x, r) \subsetneq X \) and we may replace the balls \( B(x, 2^{-i}) \) with the balls \( B(x, 2^{-i}r) \) and repeat the previous argument in order to get \( \mu(\{x\}) = 0 \).

\[\square\]

**Remark 2.9.** Macias and Segovia showed that on any space of homogeneous type \((X, d, \mu)\) there exists an equivalent quasi-metric \( \rho \) such that the quasi-metric balls with respect to \( \rho \) are open. Therefore we could have assumed from the outset that our \( \sigma \)-algebra is the Borel algebra and that \( \mu \) is a positive Borel measure which is doubling. The definition of the reverse doubling condition would need to be changed slightly: there exist constants \( C > 0 \) and \( 0 < \gamma < 1 \) such that for any ball \( B(x, r) \subsetneq X \) and any \( i \geq 1 \)
\[
\mu(B(x, 2^{-i}r)) \leq C \gamma^i \mu(B(x, r)).
\]
For further details on this perspective, see [15]. The proofs we give Section 5 go through with essentially no change using this definition of reverse doubling and we leave the details to the interested reader.
Remark 2.10. If the space \((X, d, \mu)\) is Ahlfors regular, then it is immediate that the measure \(\mu\) is doubling. It need not be reverse doubling, but it does satisfy the weaker condition in Remark 2.9. Given this, the proof of Corollary 1.5 is a straightforward modification of the proof of Theorem 1.3 and so is omitted.

**Fractional maximal operators.** We begin by restating the definition given in the Introduction. Given a set \(E, \mu(E) > 0\), we will use the notation

\[
\int_E f \, d\mu = \frac{1}{\mu(E)} \int_E f \, d\mu.
\]

**Definition 2.11.** Given a space of homogeneous type \((X, d, \mu)\) and \(0 \leq \eta < 1\), define the fractional maximal operator of order \(\eta\) acting on \(f \in L^1_{\text{loc}}(X)\) by

\[
M^\eta f(x) = \sup_{B \ni x} \mu(B)^\eta \int_B |f| \, d\mu,
\]

where the supremum is taken over all balls which contain the point \(x\). When \(\eta = 0\) we write \(M\) instead of \(M_0\).

**Definition 2.12.** Given a space of homogeneous type \((X, d, \mu)\), a dyadic grid \(D\) on \(X\) and \(0 \leq \eta < 1\), the dyadic fractional maximal operator of order \(\eta\) with respect to \(D\) is defined by

\[
M^\eta_D f(x) = \sup_{x \in Q} \mu(Q)^\eta \int_Q |f| \, d\mu,
\]

where the supremum is taken over all \(Q \in D\). When \(\eta = 0\) we write \(M^D\) instead of \(M^D_0\).

Theorem 2.6 yields a pointwise comparison between the fractional maximal operator and the dyadic fractional maximal operators associated with an adjacent system of dyadic cubes. The following result is proved in [20, 21].

**Proposition 2.13.** Given a space of homogeneous type \((X, d, \mu)\), let \(\{D^t\}\) be the adjacent dyadic system from Theorem 2.6. Fix \(0 \leq \eta < 1\). Then there exists a constant \(C = C(\eta) \geq 1\) such that for all \(f \in L^1_{\text{loc}}(X)\),

\[
M^\eta_{D^t} f(x) \leq M^\eta f(x) \quad \text{and} \quad M^\eta f(x) \leq C \sum_{t=1}^{K} M^\eta_{D^t} f(x).
\]

We will need a variant of the classical Calderon-Zygmund decomposition adapted to spaces of homogeneous type. The proof is essentially the same as in the Euclidean case and we refer the reader to [3] for further details.

**Lemma 2.14.** Given a space of homogeneous type \((X, d, \mu)\) such that \(\mu(X) = +\infty\), let \(D\) be a dyadic grid on \(X\). Fix \(0 \leq \eta < 1\). Let \(f \in L^1_{\text{loc}}(X)\) be a function such that
\[ \mu(Q)^n f_Q |f| \, d\mu \to 0 \text{ as } \mu(Q) \to \infty \text{ where } Q \in \mathcal{D}. \] Then for each \( \lambda > 0 \), there exists a set of pairwise disjoint dyadic cubes \( \{Q_j\} \) and a constant \( C = C(X, \mathcal{D}) > 1 \) such that
\[ \{x \in X : M^D_\eta f(x) > \lambda\} = \bigcup_j Q_j, \]
and
\[ \lambda < \mu(Q_j)^n \int_{Q_j} |f| \, d\mu \leq C\lambda. \]

If \( \mu(X) < +\infty \), then the same conclusion holds for all \( \lambda > \mu(X)^n f_X |f| \, d\mu \).

Finally, we need two results which were proved in [8] in the Euclidean case; the proofs in spaces of homogeneous type are identical and so we omit them. The first is a pointwise approximation theorem.

**Lemma 2.15.** Given a space of homogeneous type \((X, d, \mu)\), let \( \mathcal{D} \) be a dyadic grid on \( X \). Fix \( 0 \leq \eta < 1 \). Let \( f_N \) be a sequence of non-negative functions that increases pointwise a.e. to a function \( f \). Then the functions \( M^D_\eta f_N \) increase to \( M^D_\eta f \) pointwise. The same is true if we replace \( M^D_\eta \) by \( M_\eta \).

The second will let us compare the fractional maximal operator to the Hardy-Littlewood maximal operator.

**Lemma 2.16.** Fix \( 0 \leq \eta < 1 \) and suppose \( r \) and \( s \) satisfy \( 1 < r < 1/\eta \) and \( 1/r - 1/s = \eta \). Then for every set \( E \) of finite measure and for every non-negative function \( f \),
\[ \mu(E)^n \int_E f \, d\mu \leq \left( \int_E f^r \, d\mu \right)^{\frac{1 - \frac{1}{s}}{r - \frac{1}{r}}} \left( \int_E f^r \, d\mu \right)^{\frac{1}{r}}. \]

**Remark 2.17.** Fix a dyadic grid \( \mathcal{D} \) and \( 0 < \eta < 1 \). Given \( f \in L^r(X) \), where \( r \) and \( s \) are as in Lemma 2.16, let \( x \in X \) and \( Q \in \mathcal{D} \) be such that \( x \in Q \). If we let \( E = Q \) in inequality (2.2) and take the supremum over all such dyadic cubes, we get
\[ M^D_\eta f(x)^s \leq \|f\|_{s-r}^s M^D f(x)^r. \]
If we further assume that \( f \in L^\infty(X) \), then for every \( x \in X \)
\[ M^D_\eta f(x)^s \leq \|f\|_{s-r}^s \|f\|_{\infty}^r < +\infty. \]

We can actually say more. By Lemma 2.15 and Marcinkiewicz interpolation, a standard argument shows that \( M^D : L^1(X) \to L^{1,\infty}(X) \) and \( M^D \) is bounded operator on \( L^r(X) \) when \( r > 1 \). In this latter case we immediately have that
\[ \|M^D_\eta f\|_{s} \leq \|f\|_{s-r}^s \|M^D f\|_{r} \leq C\|f\|_{s}. \]
In other words, \( M^D_\eta : L^r(X) \to L^s(X) \) is a bounded operator.
When $r = 1$ we have that $M^D_\eta : L^r(X) \to L^{s,\infty}(X)$. (If $r > 1$ this follows at once from Chebyshev’s inequality and the strong type inequality.) If $\eta = 0$, this was noted above. If $\eta > 0$, then by Lemma 2.14 there exist disjoint dyadic cubes $\{Q_j\}$ such that

$$
\mu(\{M^D_\eta f > t\})^{1-\eta} = \left(\sum_j \mu(Q_j)\right)^{1-\eta} \leq \sum_j \mu(Q_j)^{1-\eta},
$$

where

$$
\mu(Q_j)^{1-\eta} < \frac{1}{t} \int_{Q_j} |f| \, d\mu.
$$

Since $r = 1$ and $s = 1/(1-\eta)$ it follows that

$$
\sup_{t>0} t \|\chi_{\{M^D_\eta f > t\}}\|_s \leq \|f\|_r.
$$

**Variable Lebesgue spaces.** We now give the definition and some basic properties of variable Lebesgue spaces. For complete details see [9, 11]. Given a space of homogeneous type $(X, d, \mu)$, let $p(\cdot) : X \to [1, \infty]$ be a measurable function and define the set $\Omega^p_\infty = \{x \in X : p(x) = \infty\}$. The variable Lebesgue space $L^{p(\cdot)}(X)$ is the set of measurable functions such that for some $\lambda > 0$,

$$
\rho_{p(\cdot)}(f/\lambda) = \int_{X \setminus \Omega^p_\infty} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} \, d\mu(x) + \lambda^{-1} \|f\|_{L^\infty(\Omega^p_\infty, p(\cdot))} < \infty.
$$

$L^{p(\cdot)}(X)$ is a Banach function space when equipped with the Luxemburg norm

$$
\|f\|_{p(\cdot)} = \inf\{\lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \leq 1\}.
$$

For the fractional maximal operator to be bounded, we need to impose some restrictions on the exponent function $p(\cdot)$. To state them, we will need a simple measure of the oscillation of $p(\cdot)$. Given a set $E \subset X$, we define

$$
p_+(E) = \operatorname{ess sup}_{x \in E} p(x), \quad p_-(E) = \operatorname{ess inf}_{x \in E} p(x).
$$

For brevity we write $p_+ = p_+(X)$ and $p_- = p_-(X)$. We will also need to control the continuity of $p(\cdot)$ locally and at infinity. Our hypothesis is the same log-Hölder continuity condition that has played an important role in the Euclidean case: see [9] for details and further references.

**Definition 2.18.** Given a function $r(\cdot) : X \to [0, \infty)$, we say that $r(\cdot)$ is locally log-Hölder continuous and write $r(\cdot) \in LH_0$ if there exists a constant $C_0$ such that

$$
|r(x) - r(y)| \leq \frac{-C_0}{\log d(x, y)},
$$
where \(x, y \in X\) and \(d(x, y) < 1/2\). The constant \(C_0\) is called the \(LH_0\) constant of \(r(\cdot)\).

**Definition 2.19.** Given a function \(r(\cdot) : X \to [0, \infty]\), we say that \(r(\cdot)\) is log-Hölder continuous at infinity with respect to a base point \(x_0 \in X\), and write \(r(\cdot) \in LH_\infty\), if there exist constants \(C_\infty, r_\infty\) such that

\[
|r(x) - r_\infty| \leq \frac{C_\infty}{\log(e + d(x, x_0))},
\]

for every \(x \in X\). The constant \(C_\infty\) is called the \(LH_\infty\) constant of \(r(\cdot)\).

When \(r(\cdot) \in LH_0 \cap LH_\infty\) we say it is globally log-Hölder continuous and we define \(LH = LH_0 \cap LH_\infty\).

**Remark 2.20.** Since we wish to allow for the possibility of unbounded exponents \(p(\cdot)\), we will apply the \(LH_0\), \(LH_\infty\) and \(LH\) conditions to the function \(1/p(\cdot)\) instead of applying them to \(p(\cdot)\).

The definition of the \(LH_\infty\) condition assumes the existence of a base point \(x_0\) which is taken to be the origin in the Euclidean case. On a general space of homogeneous type, there may not be such a distinguished point; however, the choice of the base point is immaterial as the next lemma shows. We refer the reader to [1] for a proof.

**Lemma 2.21.** Let \(r(\cdot) \in LH_\infty\) with respect to the base point \(x_0 \in X\). Given any \(y_0 \in X\), we have that \(r(\cdot) \in LH_\infty\) with respect to \(y_0\) with a possibly different \(LH_\infty\) constant.

In the calculations to follow, we will need to estimate certain integrals that appear as error terms. The following result was proved in [1]; we include the short proof for completeness.

**Lemma 2.22.** Let \((X, d, \mu)\) be a space of homogeneous type. If \(N > \log_2 C_\mu\), then for any \(x_0 \in X\),

\[
\int_X \frac{1}{(e + d(x, x_0))^N} d\mu < +\infty.
\]

**Proof.** Fix \(x_0 \in X\), define \(B_n = B(x_0, 2^n)\) for \(n \geq 0\) and let \(B_{-1} = \emptyset\). Then \(X = \bigcup_{n \geq 0} (B_n \setminus B_{n-1})\) and so we have that

\[
\int_X \frac{1}{(e + d(x, x_0))^N} d\mu = \sum_{n \geq 0} \int_{B_n \setminus B_{n-1}} \frac{1}{(e + d(x, x_0))^N} d\mu
\]

\[
\leq \frac{1}{e^N} \mu(B_0) + \sum_{n \geq 1} \frac{1}{(e + 2^{n-1})^N} \mu(B_n \setminus B_{n-1})
\]
\[ \leq \frac{1}{e^N} \mu(B_0) + \sum_{n \geq 1} C^n \frac{\mu(B_0)}{2(2n-1)^N} \]
\[ \leq \mu(B_0) \left( 1 + C \sum_{n \geq 0} \left( \frac{C}{2N} \right)^n \right). \]

Since \( \log_2 C_\mu < N \) the final summation converges. \( \square \)

**Remark 2.23.** Below we will use Lemma 2.22 as follows: fix \( 0 < \Gamma < 1 \). Then we may write
\[ \int_X \Gamma C_{6^{-1}}^\infty \log(e + d(x, x_0)) \, d\mu(x) = \int_X \frac{1}{(e + d(x, x_0))^{C_{6^{-1}}^\infty \log(1/\Gamma)}} \, d\mu(x). \]

If \( C_{6^{-1}}^\infty \log(1/\Gamma) > \log_2 C_\mu \), then by Lemma 2.22 the integral on the right converges. Therefore, by the dominated convergence theorem,
\[ \lim_{\Gamma \to 0} \int_X \Gamma C_{6^{-1}}^\infty \log(e + d(x, x_0)) \, d\mu(x) = 0. \]

In particular, we can always fix a constant \( 0 < \Gamma < 1 \) so that the integral of \( \Gamma C_{6^{-1}}^\infty \log(e + d(\cdot, x_0)) \) over \( X \) is smaller than any given positive number.

We will use the \( LH_\infty \) condition to replace variable exponents with constant ones. The following lemma was proved in [8] in the Euclidean case. We include the short proof for completeness.

**Lemma 2.24.** Let \((X, d, \mu)\) be a space of homogeneous type. Given \( r(\cdot) \in LH_\infty \), suppose \( r_\infty > 0 \). Fix \( x_0 \in X \) and define \( R(x) = (e + d(x, x_0))^{-N} \), where \( Nr_\infty > \log_2 C_\mu \). Then given any measurable set \( E \subset X \) and any measurable function \( F \) such that \( 0 \leq F(y) \leq 1 \) for a.e. \( y \in E \),
\[ \int_E F(y)^{r(y)} \, d\mu(y) \leq e^{NC_\infty} \int_E F(y)^{r_\infty} \, d\mu(y) + e^{NC_\infty} \int_E R(y)^{r_\infty} \, d\mu(y), \]
\[ \int_E F(y)^{r_\infty} \, d\mu(y) \leq e^{NC_\infty} \int_E F(y)^{r(y)} \, d\mu(y) + \int_E R(y)^{r_\infty} \, d\mu(y). \]

**Proof.** We shall prove inequality (2.8); the proof of (2.7) is similar. Write
\[ \int_E F(y)^{r_\infty} \, d\mu(y) = \int_{E_1} F(y)^{r_\infty} \, d\mu(y) + \int_{E_2} F(y)^{r_\infty} \, d\mu(y), \]
where
\[ E_1 = \{ y \in E : F(y) \leq R(y) \}, \quad E_2 = \{ y \in E : F(y) > R(y) \}. \]
On the set \( E_1 \), \( F(y)^{r_\infty} \leq R(y)^{r_\infty} \); hence,
\[ \int_{E_1} F(y)^{r_\infty} \, d\mu(y) \leq \int_{E_1} R(y)^{r_\infty} \, d\mu(y). \]
To estimate the integral over the set $E_2$, note that since $0 \leq F(y) \leq 1$ for a.e. $y \in E_2$, $F(y)^{r_\infty - r(y)} \leq R(y)^{-|r(y) - r_\infty|}$. Thus

$$\int_{E_2} F(y)^{r_\infty} \, d\mu(y) = \int_{E_2} F(y)^{r_\infty} F(y)^{r_\infty - r(y)} \, d\mu(y) \leq \int_{E_2} F(y)^{r(y)} R(y)^{-|r(y) - r_\infty|} \, d\mu(y).$$

Since $r(\cdot) \in LH_\infty$,

$$R(y)^{-|r(y) - r_\infty|} = e^{N|r(y) - r_\infty| \log(e + d(y,x_0))} \leq e^{NC_\infty}.$$

If we combine all these estimates, we get (2.8).

**Remark 2.25.** If $\mu(E) < \infty$, then the integral of $R(y)$ is always finite. If $\mu(E) = \infty$, then by Lemma 2.22 the assumption that $Nr_\infty > \log_2 C_\mu$ ensures

$$\int_E R(y)^{r_\infty} \, d\mu(y) \leq \int_X \left(\frac{1}{e + d(y,x_0)}\right)^{Nr_\infty} \, d\mu(y) < +\infty.$$

We will use the LH condition to get estimates on the measure of cubes. The following result is referred to as a Diening type estimate: see [1] for a proof and the history of this important condition. In the Euclidean case this estimate is equivalent to the LH$_0$ condition; in the more general setting of spaces of homogeneous type we seem to require a stronger hypothesis.

**Lemma 2.26.** Given a space of homogeneous type $(X, d, \mu)$, let $p(\cdot) : X \to [1, \infty]$ be such that $1/p(\cdot) \in LH$. Then there is a positive constant $C$ such that for any ball $B$

1. $\mu(B)^{\frac{1}{p_+(B)} - \frac{1}{p_-(B)}} \leq C$;

2. for all $x \in B$, $\mu(B)^{\frac{1}{p(X)} - \frac{1}{p_-(B)}} \leq C$ and $\mu(B)^{\frac{1}{p_+(B)} - \frac{1}{p_+(B)}} \leq C$.

We will actually need the analog of the Diening estimate for dyadic cubes.

**Corollary 2.27.** Given a space of homogeneous type $(X, d, \mu)$, let $p(\cdot) : X \to [1, \infty]$ be such that $1/p(\cdot) \in LH$ and let $\mathcal{D}$ be a dyadic grid on $X$. Then inequalities (1) and (2) of Lemma 2.26 hold with $B$ replaced by any dyadic cube $Q \in \mathcal{D}$. If $\mu(X) < +\infty$, then the same conclusion holds if $1/p(\cdot) \in LH_0$.

**Proof.** Let $Q$ be a dyadic cube. If $\mu(Q) \geq 1$, then the result follows trivially. Assume $\mu(Q) < 1$. From Theorem 2.5 it follows that there are balls $B_1$ and $B_2$ with the same centers and comparable radii such that $B_1 \subseteq Q \subseteq B_2$. By the lower mass bound (2.1), there exists a constant $K \geq 1$ such that $1/K \leq \mu(B_1)/\mu(B_2) \leq \mu(Q)/\mu(B_2)$. Since $Q \subseteq B_2$ it follows that $1/p_-(Q) - 1/p_+(Q) \leq 1/p_-(B_2) - 1/p_+(B_2)$. Since $K \geq 1$ and $\mu(Q) < 1$, by Lemma 2.26 we have that

$$\left(\frac{1}{\mu(Q)}\right)^{\frac{1}{p_-(Q)} - \frac{1}{p_+(Q)}} \leq K \left(\frac{1}{\mu(B_2)}\right)^{\frac{1}{p_-(B_2)} - \frac{1}{p_+(B_2)}} \leq KC.$$
This proves (1). The proof of (2) is essentially the same.

If \( \mu(X) < +\infty \), then by Lemma 2.4, \( \text{diam}(X) < +\infty \). Therefore, there exists \( x_0 \in X \) and \( R > 0 \) such that \( X \subset B(x_0, R) \). Let \( p_\infty = p_- \). Then for any \( x \in X \),

\[
\frac{1}{p(x)} - \frac{1}{p_\infty} \leq \frac{2}{p_- \log(e + d(x, x_0))} \leq \frac{2 \log(e + R)}{\log(e + d(x, x_0))}.
\]

Hence \( 1/p(\cdot) \) satisfies the \( LH_\infty \) condition with \( C_\infty = 2 \log(e + R) \). \( \square \)

Finally, we will need a version of the monotone convergence theorem for variable Lebesgue spaces. For a proof in the Euclidean case, see [9]; the same proof holds without change on spaces of homogeneous type.

**Lemma 2.28.** Given a space of homogeneous type \( (X, d, \mu) \) and a non-negative function \( f \in L^{p(\cdot)}(X) \), suppose that the sequence \( \{ f_N \} \) of non-negative functions increases pointwise to \( f \) almost everywhere. Then \( \| f_N \|_{p(\cdot)} \) increases to \( \| f \|_{p(\cdot)} \).

3. **Proof of the boundedness of \( M_\eta \)**

In this section we prove Theorem 1.1. We will first assume that \( \mu(X) = \infty \). The proof when \( \mu(X) < \infty \) is similar but shorter, and we will prove it at the end of the section.

We begin the proof with some reductions. First, we may assume that \( p_- < \infty \). If \( p_- = \infty \), then \( p(\cdot) = \infty \) a.e. and hence \( \eta = 0 \). In this case Theorem 1.1 reduces to the elementary fact that the maximal operator is bounded on \( L^\infty(X) \).

Second, by Proposition 2.13 it will suffice to prove inequality (1.1) with \( M_\eta \) replaced by \( M_\eta^D \) for an arbitrary dyadic grid \( D \) on \( X \).

Third, by the definitions of variable Lebesgue norms and dyadic fractional maximal operators, we may assume without loss of generality that \( f \) is non-negative. Moreover, we may assume that \( f \) is a bounded function with bounded support. If inequality (1.1) holds for such functions, then given an arbitrary non-negative function \( f \in L^{p(\cdot)}(X) \) and a base point \( x_0 \), then it holds for the functions \( f_n(x) = \min\{f(x), n\} \chi_{B(x_0, n)}(x) \) and by Lemmas 2.15 and 2.28 we have

\[
\| M_\eta^D f \|_{q(\cdot)} = \lim_{n \to \infty} \| M_\eta^D f_n \|_{q(\cdot)} \leq C \lim_{n \to \infty} \| f_n \|_{p(\cdot)} = \| f \|_{p(\cdot)}.
\]

Finally, by the homogeneity of the norm we may assume that \( \| f \|_{p(\cdot)} = 1 \).

Fix such a function \( f \) and write \( f = f_1 + f_2 \), where \( f_1 = f \chi_{\{f > 1\}} \) and \( f_2 = f \chi_{\{f \leq 1\}} \).

Then

\[
\| M_\eta^D f \|_{q(\cdot)} \leq \| M_\eta^D f_1 \|_{q(\cdot)} + \| M_\eta^D f_2 \|_{q(\cdot)},
\]

and for \( i = 1, 2 \), \( \| f_i \|_{p(\cdot)} \leq \| f \|_{p(\cdot)} = 1 \). Therefore, it will suffice to find positive constants \( \lambda_i \) independent of \( D \) such that \( \| M_\eta^D f_i \|_{q(\cdot)} \leq \lambda_i \) for \( i = 1, 2 \). By the
definition of variable Lebesgue space norms, this is equivalent to finding \( \lambda_i \) such that 
\[
\rho_{p_i}(\lambda_i^{-1} M^D_{\eta} f_i) \leq 1 \quad \text{for } i = 1, 2.
\]

We will also use the fact that since \( \|f\|_{p_i} = 1 \) for \( i = 1, 2 \), we have that \( \rho_{p_i}(f_i) \leq \rho_{p_i}(f) \leq \|f\|_{p_i} = 1 \). This follows from the definition of the norm: see [9, 11] for details.

The estimate for \( f_1 \). This part of the argument closely follows the proof given in [8]. Since \( \rho_{p_i}(f_1) \leq 1 \), by the definition of the modular, \( \|f_1\|_{L^\infty(\Omega^p_\infty)} \leq 1 \). Hence, \( f_1 \leq 1 \) almost everywhere on \( \Omega^p_\infty \). But by definition \( f_1(x) > 1 \) or \( f_1(x) = 0 \), so \( f_1 = 0 \) a.e. on \( \Omega^p_\infty \). In other words, up to a set of measure zero \( \text{supp}(f_1) \subset X \setminus \Omega^p_\infty \).

We want to show that there exists a constant \( \lambda_1 > 1 \) such that 
\[
\rho_{q_i}(\lambda_1^{-1} M^D_{\eta} f_1) \leq 1.
\]

For the argument below it is convenient to write \( \lambda_1^{-1} = \alpha_1 \beta_1 \gamma_1 \), where we will assume \( 0 < \alpha_1, \beta_1, \gamma_1 < 1 \). In fact, we will show that these constants can be chosen so that 
\[
\int_{X \setminus \Omega^p_\infty} (\alpha_1 \beta_1 \gamma_1 M^D_{\eta} f_1(x))^{q_i(x)} d\mu(x) \leq \frac{1}{2}
\]
and 
\[
\alpha_1 \beta_1 \gamma_1 \|M^D_{\eta} f_1\|_{L^\infty(\Omega^p_\infty)} \leq \frac{1}{2}.
\]

We first prove inequality (3.1). If \( \eta = 0 \), then for every \( x \in X \), \( M^D_{\eta} f_1(x) \leq \|f_1\|_\infty < +\infty \). If \( 0 < \eta < 1 \) and \( 1 < r < 1/\eta \), then \( f_1 \in L^r(X) \cap L^\infty(X) \) because \( f_1 \) is bounded and has bounded support. Hence, by Remark 2.17 and inequality (2.4), \( M^D_{\eta} f_1(x) < +\infty \).

Let \( C \) be the constant in Lemma 2.14. For each integer \( k \) define the set
\[
\Omega_k = \{ x \in X : M^D_{\eta} f_1(x) > C^k \}.
\]

Since \( M^D_{\eta} f_1 \) is finite and positive everywhere,
\[
X = \bigcup_k (\Omega_k \setminus \Omega_{k+1}).
\]

Since \( f_1 \) satisfies the hypotheses of Lemma 2.14 we can write
\[
\Omega_k = \{ x \in X : M^D_{\eta} f_1(x) > C^k \} = \bigcup_j Q_j^k,
\]
where the dyadic cubes \( Q_j^k \) satisfy
\[
\mu(Q_j^k)^{p_i} \int_{Q_j^k} f_1 \, d\mu > C^k.
\]
It follows by the properties of dyadic cubes that if we define \( E^k_j = Q^k_j \setminus \Omega_{k+1} \), then the sets \( E^k_j \) are pairwise disjoint. (See [3] where this fact is given using somewhat different terminology.)

We can now estimate as follows:

\[
\int_{X \setminus \Omega_{\infty}^{(q)}} (\alpha_1 \beta_1 \gamma_1 M^n f_1(x))^q(x) \, d\mu(x)
\]

\[
= \sum_k \int_{(\Omega_k \setminus \Omega_{k+1}) \setminus \Omega_{\infty}^{(q)}} (\alpha_1 \beta_1 \gamma_1 M^n f_1(x))^q(x) \, d\mu(x)
\]

\[
\leq \sum_k \int_{(\Omega_k \setminus \Omega_{k+1}) \setminus \Omega_{\infty}^{(q)}} (\alpha_1 \beta_1 \gamma_1 C^{k+1})^q(x) \, d\mu(x).
\]

If we choose \( 0 < \alpha_1 \leq 1/C \), then

\[
(3.3)
\]

\[
\leq \sum_{k,j} \int_{E^k_j \setminus \Omega_{\infty}^{(q)}} \left( \beta_1 \gamma_1 \mu(Q^k_j)^{q/\eta} \int_{Q^k_j} f_1(y) \, d\mu(y) \right)^q(x) \, d\mu(x).
\]

If \( k \) and \( j \) are such that \( \mu(E^k_j \setminus \Omega_{\infty}^{(q)}) = 0 \), then this term in the sum is zero. Hence, we may disregard those terms and assume that \( \mu(E^k_j \setminus \Omega_{\infty}^{(q)}) > 0 \). Since \( E^k_j \subseteq Q^k_j \), we have that the exponents \( p_{jk} = p_-(Q^k_j) \) and \( d_{jk} = q_-(Q^k_j) \) are both finite and satisfy \( 1 < p_{jk} < (1/\eta) \) and \( (1/p_{jk}) - (1/d_{jk}) = \eta \).

Therefore, by Lemma 2.16 we have

\[
\mu(Q^k_j)^{q/\eta} \int_{Q^k_j} f_1 \, d\mu \leq \left( \int_{Q^k_j} f_1^{p_{jk}} \, d\mu \right)^{p_{jk}/p_-} \left( \int_{Q^k_j} f_1(y) \, d\mu(y) \right)^{p_{jk}/d_{jk}}.
\]

Since \( \text{supp}(f_1) \subset X \setminus \Omega_{\infty}^{(q)} \) (up to a set of measure zero), and since \( f_1 \) is either 0 or greater than 1, it follows that the first integral on the right-hand side is dominated by \( \rho_{(p_-)}(f_1) \) and so it is less than or equal to 1. Therefore, by Hölder’s inequality with respect to the exponent \( p_{jk}/p_- \) we get

\[
\mu(Q^k_j)^{q/\eta} \int_{Q^k_j} f_1(y) \, d\mu(y) \leq \mu(Q^k_j)^{-q/\eta} \left( \int_{Q^k_j} f_1(y) \, d\mu(y) \right)^{p_{jk}/d_{jk}}.
\]

If we substitute this inequality into inequality (3.3) and rearrange terms, we get

\[
(3.4) \int_{X \setminus \Omega_{\infty}^{(q)}} (\alpha_1 \beta_1 \gamma_1 M^n f_1(x))^q(x) \, d\mu(x)
\]
\[ \leq \sum_{k,j} \int_{E_j^k \setminus \Omega_{\infty}} \beta_1^\mu(Q_j^k)^{-\frac{p}{\gamma_{jk}}} \gamma_1 \left( \int_{Q_j^k} f_1(y)^{\frac{p}{\gamma_{jk}}} \, d\mu(y) \right)^{\frac{p}{\gamma_{jk}}} q(x) \, d\mu(x). \]

Since \(1/\beta < \beta_1 \leq 1/C^{p_\infty}\), by Corollary 2.27 there exists a constant \(C\) such that for any \(x \in Q_j^k\),

\[ \beta_1^\mu(Q_j^k)^{-\frac{p}{\gamma_{jk}}} \leq \beta_1 C^{p_\infty} \mu(Q_j^k)^{-\frac{p}{\alpha x^j}}. \]

If we choose \(0 < \beta_1 \leq 1/C^{p_\infty}\), then

\[ \beta_1^\mu(Q_j^k)^{-\frac{p}{\gamma_{jk}}} \leq \mu(Q_j^k)^{-\frac{p}{\alpha x^j}} \]

for any \(x \in E_j^k \subset Q_j^k\). If we substitute this estimate into (3.4), we get

\[ \int_{X \setminus \Omega_{\infty}^{p(x)}} (\alpha_{\beta_1} \gamma_1 M^{p}\left( f_1(x) \right))^{q(x)} \, d\mu(x) \]

\[ \leq \sum_{k,j} \int_{E_j^k \setminus \Omega_{\infty}^{p(x)}} \mu(Q_j^k)^{-p_-} \gamma_1 \left( \int_{Q_j^k} f_1(y)^{p(y)} \, d\mu(y) \right)^{\frac{p}{\gamma_{jk}}} q(x) \, d\mu(x). \]

Since \(q_{jk} \geq p_{jk} \geq p_-\) and \(\gamma_1 \leq \gamma_{1jk}^{p_-}\), we have that

\[ \leq \sum_{k,j} \int_{E_j^k \setminus \Omega_{\infty}^{p(x)}} \mu(Q_j^k)^{-p_-} \gamma_1 \left( \int_{Q_j^k} f_1(y)^{p(y)} \, d\mu(y) \right)^{p_-} \gamma_{1jk}^{p_-} \, d\mu(x). \]

Since \(f_1 \geq 1\) or \(f_1 = 0\), \(\text{supp}(f_1) \subset X \setminus \Omega_{\infty}^{p(x)}\) (up to a set of measure zero), \(\rho_{p(x)}(f_1) \leq 1\) and \(p_- > 1\). Hence, the quantity inside the parentheses is less than or equal to 1. Since \(q(x) \geq q_{jk}\), we get

\[ \leq \sum_{k,j} \int_{E_j^k \setminus \Omega_{\infty}^{p(x)}} \mu(Q_j^k)^{-p_-} \gamma_1 \left( \int_{Q_j^k} f_1(y)^{p(y)} \, d\mu(y) \right)^{p_-} \, d\mu(x) \]

\[ = \sum_{k,j} \int_{E_j^k \setminus \Omega_{\infty}^{p(x)}} \gamma_1 \left( \int_{Q_j^k} f_1(y)^{p(y)} \, d\mu(y) \right)^{p_-} \, d\mu(x) \]

\[ \leq \int_{X \setminus \Omega_{\infty}^{p(x)}} \gamma_1^{p_-} M^{p(x)}(f_1(y))^{p_-} \, d\mu(x). \]
Since $p_+ > 1$, $M^D$ is bounded on $L^{p_-}$ and there exists a constant $C = C(p_-, X)$ such that

$$
\leq \gamma_1^{p_-} C \int_{X \setminus \Omega^{p_-}(\cdot)} f_1(x)^{p(x)} \, d\mu(x)
\leq \gamma_1^{p_-} C \rho_{p(\cdot)}(f_1)
\leq \gamma_1^{p_-} C.
$$

Thus, inequality (3.1) holds if we choose $0 < \gamma_1 \leq 2C^{-1/p_-}$.

We now prove inequality (3.2). We will show that

$$
\frac{1}{2} \|M^D f_1\|_{L^{\infty}(\Omega^{p_-})} \leq C;
$$

this gives us inequality (3.2) if we choose $\alpha_1, \beta_1, \gamma_1$ such that $0 < \alpha_1 \beta_1 \gamma_1 < 1/(4 \max\{1, C\})$.

Without loss of generality we may assume that $\|M^D f_1\|_{L^{\infty}(\Omega^{p_-})} > 0$. By definition there exists $x \in \Omega^{p_-}$ and a dyadic cube $Q$ such that $x \in Q$ and

$$
\frac{1}{2} \|M^D f_1\|_{L^{\infty}(\Omega^{p_-})} < \mu(Q) \int_Q f_1 \, d\mu.
$$

Since $f_1 = 0$ almost everywhere on $\Omega^{p_-}$ it follows that $\mu(Q \setminus \Omega^{p_-}) > 0$. This implies that $1 < p_-(Q) < +\infty$. We consider three cases:

1. $p_-(Q) = 1/\eta$ and $0 < \eta < 1$,
2. $1 < p_-(Q) < 1/\eta$ and $0 < \eta < 1$,
3. $1 < p_-(Q) < +\infty$ and $\eta = 0$.

If (1) holds, an application of Hölder’s inequality with respect to the exponent $1/\eta$, together with the facts that $f_1$ is either 0 or greater than 1, $\text{supp}(f_1) \subset X \setminus \Omega^{p_-}$ (up to a set of measure zero) and $\rho_{p(\cdot)}(f_1) \leq 1$, gives

$$
\frac{1}{2} \|M^D f_1\|_{L^{\infty}(\Omega^{p_-})} \leq \mu(Q)^{1/p_-(Q)} \left( \int_Q f_1^{p_-(Q)} \, d\mu \right)^{1/p_-(Q)} \leq \left( \int_Q f_1^{p_-(x)} \, d\mu \right)^{1/p_-(Q)} \leq 1.
$$

If (2) holds, then by Lemma 2.16 and arguing as in the first case, we have that

$$
\frac{1}{2} \|M^D f_1\|_{L^{\infty}(\Omega^{p_-})} < \mu(Q) \int_Q f_1 \, d\mu
\leq \left( \int_Q f_1^{p_-(Q)} \, d\mu \right)^{1/p_-(Q)} \left( \int_Q f_1 \, d\mu \right)^{1/q_-(Q)} \leq \left( \frac{1}{\mu(Q)} \right)^{1/q_-(Q)}.
$$
Finally, when (3) holds we get the same estimate as in Case (2); to prove it simply replace $M^D$ with $M^P$ and $q_-(Q)$ with $p_-(Q)$ and argue as before.

To complete the proof we must show that the last term in (3.5) is uniformly bounded. If $\mu(Q) \geq 1$, this is immediate. Now assume $\mu(Q) < 1$ and let $m$ be an arbitrary positive integer. Since $1/q(\cdot) \in LH$ and $x \in \Omega^\eta_{\infty}^p$ it follows that the set

$$\left\{ y \in X : \frac{1}{q(y)} < \frac{1}{m} \right\}$$

is non-empty and open. Thus there exists a $\tau > 0$ such that the ball $B(x, \tau)$ is contained in it.

Let the constants $C$ and $\delta$ be the same as in Theorem 2.5. Choose an integer $k$ such that $2A_0 C \delta^k < \tau$. Since the dyadic cubes in the $k$-th generation cover $X$ it follows that there exists a dyadic cube $\tilde{Q}$ in the $k$-th generation which contains $x$. By choosing $k$ large enough we may assume that $\tilde{Q} \subset Q$. Then by the quasi-triangle inequality,

$$x \in \tilde{Q} \subseteq B(x_c(\tilde{Q}), C \delta^k) \subseteq B(x, 2A_0 C \delta^k) \subseteq B(x, \tau);$$

in other words,

$$\tilde{Q} \subseteq \left\{ y \in X : \frac{1}{q(y)} < \frac{1}{m} \right\}.$$ 

Hence, $q_+(Q) \geq q_+(\tilde{Q}) \geq m$. Since the positive integer $m$ was arbitrary, it follows that $q_+(Q) = +\infty$.

Now let $n$ be an arbitrary positive integer such that $q_-(Q) < n$. Since $q_+(Q) = +\infty$, there exists a point $x_n \in Q$ such that $q_-(Q) < n < q(x_n) < q_+(Q) = +\infty$. Because $1/q(\cdot) \in LH$, by Corollary 2.27,

$$\mu(Q) \frac{1}{q_+(Q)} \frac{1}{q_-(Q)} \leq C.$$ 

Therefore, since $\mu(Q) < 1$, we have that

$$\frac{1}{2} \|M^D_\eta f_1\|_{L^\infty(\Omega^\eta_{\infty}^p)} \leq C \left( \frac{1}{\mu(Q)} \right) \frac{1}{q_+(Q)} \frac{1}{q_-(Q)} \leq C \left( \frac{1}{\mu(Q)} \right)^{\frac{1}{n}}.$$ 

If we take the limit as $n \to \infty$, we get the desired bound. This completes the proof of estimate (3.2).

**The estimate for $f_2$.** Recall that the function $f_2$ has bounded support, $0 \leq f_2 \leq 1$ and $\rho_{p(\cdot)}(f_2) \leq 1$. We want to prove that there exists $\lambda_2 > 1$ such that $\rho_{\eta(\cdot)}(\lambda_2^{-1} M^D_\eta f_2) \leq 1$. Let $\alpha_2 = \lambda_2^{-1}$. We will show that there exists $0 < \alpha_2 < 1$ such that
we get
\[ (3.6) \quad \int_{X \setminus \Omega_{\infty}^{(\cdot)}} (\alpha_2 M^D_\eta f_2(x))^{q(x)} \, d\mu(x) \leq \frac{1}{2} \]
and
\[ (3.7) \quad \alpha_2 \| M^D_\eta f_2 \|_{L^\infty(\Omega_{\infty}^{(\cdot)})} \leq \frac{1}{2}. \]

To prove both of these inequalities we first give a pointwise estimate for $M^D_\eta f_2$. Let $x \in X$ and let $Q$ be any dyadic cube containing $x$. If $\mu(Q \cap \Omega_{\infty}^{(\cdot)}) > 0$, then $\eta = 0$ because $p_+(Q) = +\infty$ and $p_+(Q) \leq 1/\eta$. Hence,
\[ \mu(Q)^\eta \int_Q f_2(y) \, d\mu(y) = \int_Q f_2(y) \, d\mu(y) \leq 1. \]

If $\mu(Q \cap \Omega_{\infty}^{(\cdot)}) = 0$ and $\eta = 0$, it is immediate that the same inequality holds. Now suppose that $\mu(Q \cap \Omega_{\infty}^{(\cdot)}) = 0$ and $0 < \eta < 1$. In this case Hölder’s inequality with respect to the exponent $1/\eta$ gives
\[ \left( \int_Q f_2(y) \, d\mu(y) \right)^{1/\eta} \leq \int_Q f_2(y)^{1/\eta} \, d\mu(y) \]
\[ = \int_{Q \setminus \Omega_{\infty}^{(\cdot)}} f_2(y)^{1/\eta} \, d\mu(y) \leq \int_{Q \setminus \Omega_{\infty}^{(\cdot)}} f_2(y)^{p(y)} \, d\mu(y) \leq 1. \]
The last inequalities hold since $p(y) \leq p_+ \leq 1/\eta$, $f_2 \leq 1$ and $p_{p(\cdot)}(f_2) \leq 1$. Therefore, in every case the fractional average of $f_2$ is at most 1 and so $M^D_\eta f_2(x) \leq 1$.

Given this estimate, we can immediately prove (3.7): choose $0 < \alpha_2 \leq 1/2$; then
\[ \alpha_2 \| M^D_\eta f_2 \|_{L^\infty(\Omega_{\infty}^{(\cdot)})} \leq \frac{1}{2}. \]

To prove (3.6) we will consider two cases: $q_\infty < +\infty$ and $q_\infty = +\infty$. First suppose that $q_\infty < +\infty$. Since $1/q(\cdot) \in LH$, by Lemma 2.24 we get
\[ (3.8) \quad \int_{X \setminus \Omega_{\infty}^{(\cdot)}} (\alpha_2 M^D_\eta f_2(x))^{q(x)} \, d\mu(x) = \int_{X \setminus \Omega_{\infty}^{(\cdot)}} ((\alpha_2 M^D_\eta f_2(x))^{q(x)q_\infty})^{1/q_\infty} \, d\mu(x) \]
\[ \leq e^{NC_\infty} \int_{X \setminus \Omega_{\infty}^{(\cdot)}} (\alpha_2 M^D_\eta f_2(x))^{q(\cdot)q_\infty} \, d\mu(x) + \int_{X \setminus \Omega_{\infty}^{(\cdot)}} R(x)^{1/q_\infty} \, d\mu(x), \]
where $R(x) = (e + d(x, x_0))^{-N}$. By Lemma 2.22 and the dominated convergence theorem we may choose $N$ large enough that
\[ (3.9) \quad \int_X R(x)^{1/q_\infty} \, d\mu(x) \leq \frac{1}{4} \quad \text{and} \quad \int_X R(x)^{1/q_\infty} \, d\mu(x) \leq \frac{1}{4}. \]
Since \( p_{\infty} \geq p_- > 1 \) it follows from Remark 2.17 that \( M^p_\eta : L^{p_\infty}(X) \to L^{p_\infty}(X) \) is a bounded operator. Thus, there is a constant \( C = C(p_{\infty}, \eta) \) such that

\[
\int_X M^p_\eta f_2(x)^{q_\infty} \, d\mu(x) \leq C \left( \int_X f_2(x)^{p_\infty} \, d\mu(x) \right)^{\frac{q_\infty}{p_\infty}}.
\]

(3.10)

If we combine (3.8), (3.9) and (3.10), we get

\[
\int_{\Omega^{\rho}_{\infty}(\eta)} (\alpha_2 M^p_\eta f_2(x))^{q(x)} \, d\mu(x) \leq e^{NC_{\infty}} C \left( \int_X (\alpha_2 f_2(x))^{p_\infty} \, d\mu(x) \right)^{\frac{q_\infty}{p_\infty}} + \frac{1}{4}.
\]

(3.11)

To estimate the integral on the righthand side, we divide it into two pieces. Since \( 0 \leq f_2 \leq 1 \), \( \rho_{p(\cdot)}(f_2) \leq 1 \) and \( 1/p(\cdot) \in LH \), by Lemma 2.24 and (3.9) we have that

\[
\int_{X \setminus \Omega^{\rho}_{\infty}(\eta)} (\alpha_2 f_2(x))^{p_\infty} \, d\mu(x)
\]

\[
= \alpha_2^{p_\infty} \int_{X \setminus \Omega^{\rho}_{\infty}(\eta)} (f_2(x)^{p_\infty})^{\frac{1}{p(x)}} \, d\mu(x)
\]

\[
\leq \alpha_2^{p_\infty} e^{NC_{\infty}} \int_{X \setminus \Omega^{\rho}_{\infty}(\eta)} f_2(x)^{p(x)} \, d\mu(x) + \alpha_2^{p_\infty} e^{NC_{\infty}} \int_{X \setminus \Omega^{\rho}_{\infty}(\eta)} R(x)^{\frac{1}{p(x)}} \, d\mu(x)
\]

\[
\leq \frac{5}{4} \alpha_2^{p_\infty} e^{NC_{\infty}}.
\]

On the other hand, \( p_{\infty} \geq C_{\infty}^{-1} \log(e + d(x, x_0)) \) for every \( x \in \Omega^{\rho}_{\infty}(\eta) \) because \( 1/p(\cdot) \in LH \). Since \( 0 < \alpha_2 < 1 \) and \( 0 \leq f_2 \leq 1 \) we have that

\[
\int_{\Omega^{\rho}_{\infty}(\eta)} (\alpha_2 f_2(x))^{p_\infty} \, d\mu(x) \leq \int_X \alpha_2^{C_{\infty}^{-1} \log(e + d(x, x_0))} \, d\mu(x).
\]

(3.13)

If we combine (3.11), (3.12) and (3.13), we get

\[
\int_{X \setminus \Omega^{\rho}_{\infty}(\eta)} (\alpha_2 M^p_\eta f_2(x))^{q(x)} \, d\mu(x)
\]

\[
\leq e^{NC_{\infty}} C \left( \frac{5}{4} \alpha_2^{p_\infty} e^{NC_{\infty}} + \int_X \alpha_2^{C_{\infty}^{-1} \log(e + d(x, x_0))} \, d\mu(x) \right)^{\frac{q_\infty}{p_\infty}} + \frac{1}{4}.
\]

By Remark 2.23 the integral on the righthand side can be made arbitrarily small by choosing \( \alpha_2 \) sufficiently small. In particular, if we choose \( \alpha_2 \) such that

\[
0 < \alpha_2 \leq \left( \frac{2}{5e^{NC_{\infty}}} \right)^{\frac{1}{p_{\infty}}} \left( \frac{1}{4e^{NC_{\infty} C}} \right)^{\frac{1}{q_{\infty}}},
\]

and

\[
\int_X \alpha_2^{C_{\infty}^{-1} \log(e + d(x, x_0))} \, d\mu(x) \leq \frac{1}{2} \left( \frac{1}{4e^{NC_{\infty} C}} \right)^{\frac{p_{\infty}}{q_{\infty}}},
\]

(3.14)
then we get inequality (3.6).

Finally, suppose \( q_\infty = +\infty \). In this case, by Remark 2.23 we can choose \( \alpha_2 \) sufficiently small so that
\[
\int_{X \setminus \Omega_{k_0}^\infty} (\alpha_2 M^D \eta f_2(x))^{q(x)} \, d\mu(x) \leq \int_X \alpha_2^{-\frac{1}{\log(e+d(x,x_0))}} \, d\mu(x) \leq \frac{1}{2},
\]
because \( 0 < \alpha_2 < 1 \), \( 0 \leq M^D \eta f_2 \leq 1 \) and \( 1/q(\cdot) \in LH \). This completes the proof.

### Spaces with finite measure.
We now consider the case when \( \mu(X) < +\infty \). By Lemma 2.4, \( \text{diam}(X) < +\infty \). Hence, by Corollary 2.27, \( 1/p(\cdot) \in LH_0 \) implies that \( 1/p(\cdot) \in LH \). Therefore, by Lemma 2.26 \( 1/p(\cdot) \) satisfies a Diening type estimate. Thus, the proof of inequality (3.2) for \( f_1 \) carries over without any changes. The proof of (3.1) must be modified to let us apply Lemma 2.14. We sketch the changes, using the same notation as before. Let \( k_0 \) be the smallest integer such that
\[
C^{k_0} > \mu(X)^{\eta} \int_X |f| \, d\mu.
\]
Then for all all \( k \geq k_0 \) we can apply Lemma 2.14. Therefore, we can modify the estimate immediately before (3.3), replacing the righthand term by
\[
\sum_{k \geq k_0} \int_{(\Omega_k \setminus \Omega_{k+1}) \setminus \Omega_{k_0}^\infty} (\alpha_1 \beta_1 \gamma_1 \gamma_{1} C^{k+1})^{q(x)} \, d\mu(x) + \int_{(X \setminus \Omega_{k_0}^\infty)} (\alpha_1 \beta_1 \gamma_1 \gamma_{1} C^{k_0})^{q(x)} \, d\mu(x).
\]
The estimate for the first sum proceeds as in the original argument except that we choose our constants so that it is less than \( 1/4 \). To estimate the second integral, choose \( \alpha_1 < C^{-k_0} \); then it is bounded by \( \beta_1 \gamma_1 \mu(X) \), and by modifying our choices of \( \beta_1 \) and \( \gamma_1 \) we can make this term smaller than \( 1/4 \).

The estimate for \( f_2 \) can be greatly simplified. Since \( M^D \eta f_2(x) \leq 1 \), for \( 0 < \alpha_2 < 1 \) we get
\[
\int_{X \setminus \Omega_{k_0}^\infty} (\alpha_2 M^D \eta f_2(x))^{q(x)} \, d\mu(x) + \alpha_2 \| M^D \eta f_2 \|_{L^\infty(\Omega_{k_0}^\infty)} \leq \alpha_2^{q-} \mu(X) + \alpha_2 \leq \alpha_2 (\mu(X) + 1).
\]
Therefore, by choosing \( \alpha_2 \) sufficiently small we get \( \rho_{q(\cdot)}(\alpha_2 M^D \eta f_2) \leq 1 \).
4. Weak type inequalities

In this section we state and prove Theorem 1.2; we omit those details which are similar to the proof of Theorem 1.1. As before, we will first consider the case when \( \mu(X) = \infty \), and we will describe the changes to the proof which are needed when \( \mu(X) < \infty \) at the end of the section.

We begin the proof with the same reductions as before: it will suffice to prove inequality (1.2) with \( M_\eta \) replaced by \( M_\eta^D \) for a fixed dyadic grid \( D \), and for a function \( f \) that is non-negative, bounded and with bounded support, and such that \( \|f\|_{p(\cdot)} = 1 \).

We will write \( f = f_1 + f_2 \), where \( f_1 = f \chi_{\{f > 1\}} \) and \( f_2 = f \chi_{\{f \leq 1\}} \). Then for \( i = 1,2 \), the functions \( f_i \) satisfy

\[
\rho_{p(\cdot)}(\|\chi_{\{M_\eta^D f_i > t\}}\|_{q(\cdot)}) \leq \rho_{p(\cdot)}(\|f\|_{p(\cdot)}) = 1.
\]

By the definition of the variable Lebesgue norm and because

\[
sup_{t>0} t \|\chi_{\{M_\eta^D f_i > t\}}\|_{q(\cdot)} \leq 2 \left( \sup_{t>0} t \|\chi_{\{M_\eta^D f_1 > t\}}\|_{q(\cdot)} + \sup_{t>0} t \|\chi_{\{M_\eta^D f_2 > t\}}\|_{q(\cdot)} \right),
\]

it will suffice to find \( \lambda_i > 1 \) such that

\[
(4.1) \quad \sup_{t>0} \rho_{q(\cdot)}(\lambda_i^{-1} t \chi_{\{M_\eta^D f_i > t\}}) \leq 1.
\]

We will prove each inequality in turn.

**The estimate for** \( f_1 \). Let \( \lambda_1 = \beta_1 \gamma_1, 0 < \beta_1, \gamma_1 < 1 \). To prove the modular estimate for \( f_1 \) we will prove that for all \( t > 0 \),

\[
(4.2) \quad \int_{X \setminus \Omega_{\infty}^{q(\cdot)}} (\beta_1 \gamma_1 t \chi_{\{M_\eta^D f_1 > t\}}(x))^{q(x)} \, d\mu(x) \leq \frac{1}{2}
\]

and

\[
(4.3) \quad \beta_1 \gamma_1 t \|\chi_{\{M_\eta^D f_1 > t\}}\|_{L^\infty(\Omega_{\infty}^{q(\cdot)})} \leq \frac{1}{2}.
\]

We first prove (4.2). Assume that \( \mu(\{M_\eta^D f_1 > t\}) > 0 \); otherwise there is nothing to prove. Since \( f_1 \) is bounded and has bounded support, \( f_1 \in L^1(X) \) and so by Lemma 2.14,

\[
\{M_\eta^D f_1 > t\} = \bigcup_j Q_j,
\]

where the dyadic cubes \( Q_j \) are disjoint and satisfy

\[
t < \mu(Q_j)^\eta \int_{Q_j} f_1(y) \, d\mu(y) \leq C t.
\]

Hence,

\[
(4.4) \quad \int_{X \setminus \Omega_{\infty}^{q(\cdot)}} (\beta_1 \gamma_1 t \chi_{\{M_\eta^D f_1 > t\}}(x))^{q(x)} \, d\mu(x)
\]
\[
\leq \sum_j \int_{Q_j \setminus \Omega^{(j)}_\infty} \left( \beta_1 \gamma_1 \mu(Q_j)^n \int_{Q_j} f_1(y) \, d\mu(y) \right)^{q(x)} \, d\mu(x).
\]

To estimate the integral inside the parentheses on the righthand side of (4.4) we will argue much as we did after inequality (3.3). We will consider the case \(0 < \eta < 1\); the case \(\eta = 0\) is essentially the same, bearing in mind that when \(\eta = 0\), \(p(\cdot) = q(\cdot)\).

Assume without loss of generality that \(\mu(Q_j \setminus \Omega^{(j)}_\infty) > 0\) for every \(j\); otherwise this term contributes nothing to the sum. When \(p_-(Q_j) > 1\), by Lemma 2.16 and an argument similar to that between (3.3) and (3.4) it follows that for every \(x \in Q_j \setminus \Omega^{(j)}_\infty\),

(4.5) \[
\left( \beta_1 \gamma_1 \mu(Q_j)^n \int_{Q_j} f_1(y) \, d\mu(y) \right)^{q(x)} \leq \left( \beta_1 \mu(Q_j)^{-\frac{1}{q_-(Q_j)}} \right)^{q(x)} \left( \gamma_1 \int_{Q_j} f_1(y)^{p(y)} \, d\mu(y) \right)^{\frac{q(x)}{p_-(Q_j)}}.
\]

Since \(1/q(\cdot) \in LH\), by Corollary 2.27 we may choose \(0 < \beta_1 < \min\{1, (1/C)\}\) so that the first term on the righthand side of (4.5) is dominated by \(\mu(Q_j)^{-1}\). Since \(supp(f_1) \subset X \setminus \Omega^{(j)}_\infty\) up to a set of measure zero, the integral of \(f_1(\cdot)^{p(\cdot)}\) on \(Q_j\) is dominated by \(\rho_{p(\cdot)}(f_1) \leq 1\). Since \(0 < \gamma_1 < 1\), the second term on the righthand side of (4.5) is dominated by

\[
\gamma_1 \int_{Q_j} f_1(y)^{p(y)} \, d\mu(y),
\]

and hence,

(4.6) \[
\left( \beta_1 \gamma_1 \mu(Q_j)^n \int_{Q_j} f_1(y) \, d\mu(y) \right)^{q(x)} \leq \gamma_1 \int_{Q_j} f_1(y)^{p(y)} \, d\mu(y).
\]

A similar and simpler argument shows that (4.6) also holds when \(p_-(Q_j) = 1\).

Therefore, if we combine all of these estimates with (4.4), we get

\[
\int_{X \setminus \Omega^{(j)}_\infty} (\beta_1 \gamma_1 t \chi_{\{M^p f_1 > t\}}(x))^{q(x)} \, d\mu(x)
\]

\[
\leq \sum_j \int_{Q_j \setminus \Omega^{(j)}_\infty} \left( \gamma_1 \int_{Q_j} f_1(y)^{p(y)} \, d\mu(y) \right) \, d\mu(x)
\]

\[
\leq \sum_j \frac{\mu(Q_j \setminus \Omega^{(j)}_\infty)}{\mu(Q_j)} \int_{Q_j} f_1(y)^{p(y)} \, d\mu(y)
\]
\[
\leq \gamma_1 \int_{X \setminus \Omega_{\infty}^{(1)}} f_1(y)^{p(y)} \, d\mu(y) \\
\leq \gamma_1.
\]

Hence, if we fix \(0 < \gamma_1 \leq 1/2\), then inequality (4.2) holds with \(\beta_1\) and \(\gamma_1\) which are independent of \(t\).

We will now prove inequality (4.3). If \(t > \|M_{\eta}^D f_1\|_{L^\infty(\Omega_{\infty}^{(1)})}\), then we have that \(\|\chi_{\{M_{\eta}^D f_1 > t\}}\|_{L^\infty(\Omega_{\infty}^{(1)})} = 0\) and so there is nothing to prove. On the other hand, if we fix \(t \leq \|M_{\eta}^D f_1\|_{L^\infty(\Omega_{\infty}^{(1)})}\), then
\[
t \beta_1 \gamma_1 \|\chi_{\{M_{\eta}^D f_1 > t\}}\|_{L^\infty(\Omega_{\infty}^{(1)})} \leq \beta_1 \gamma_1 \|M_{\eta}^D f_1\|_{L^\infty(\Omega_{\infty}^{(1)})}.
\]

The argument we used to prove inequality (3.2) did not depend on the fact that \(q_- > 1\) and continues to hold with minor modifications when \(q_- = 1\). Thus, we can find constants \(\beta_1\) and \(\gamma_1\) such that inequality (4.3) holds for all \(t\).

**The estimate for** \(f_2\). To prove the modular estimate (4.1), let \(\lambda_2^{-1} = \alpha_2\), \(0 < \alpha_2 < 1\).

We will show that we can find \(\alpha_2\) sufficiently small so that for all \(t\),
\[
\int_{X \setminus \Omega_{\infty}^{(1)}} (\alpha_2 t \chi_{\{M_{\eta}^D f_2 > t\}}(x))^{q(x)} \, d\mu(x) \leq \frac{1}{2}
\]
and
\[
(4.7) \quad \alpha_2 t \|\chi_{\{M_{\eta}^D f_2 > t\}}\|_{L^\infty(\Omega_{\infty}^{(1)})} \leq \frac{1}{2}.
\]

We begin with two observations. First, in Section 3 we showed that \(M_{\eta}^D f_2(\cdot) \leq 1\), and the proof did not rely on the fact that \(p_- > 1\). Second, since \(f_2\) is bounded and has bounded support, \(f_2 \in L^p(X)\) for all \(p > 1\), and so by the weak \((p, q)\) inequality for \(M_{\eta}^D\) (see Remark 2.17), \(\mu(\{M_{\eta}^D f_2 > t\}) < +\infty\) for all \(t > 0\).

We first prove inequality (4.7). There are two cases: \(q_\infty < +\infty\) and \(q_\infty = +\infty\). First assume that \(q_\infty < +\infty\). Since \(M_{\eta}^D f_2(\cdot) \leq 1\), we may assume \(t < 1\) since otherwise there is nothing to prove. Then by Lemma 2.24 we have that
\[
\int_{X \setminus \Omega_{\infty}^{(1)}} (\alpha_2 t \chi_{\{M_{\eta}^D f_2 > t\}}(x))^{q(x)} \, d\mu(x)
\]
\[
\leq e^{NC_\infty} \int_{X \setminus \Omega_{\infty}^{(1)}} (\alpha_2 t \chi_{\{M_{\eta}^D f_2 > t\}}(x))^{q_\infty} \, d\mu(x) + \int_X R(x)^{\frac{1}{q_\infty}} \, d\mu(x)
\]
\[
\leq e^{NC_\infty} \alpha_2^{q_\infty} t^{q_\infty} \mu(\{M_{\eta}^D f_2 > t\}) + \int_X R(x)^{\frac{1}{q_\infty}} \, d\mu(x),
\]
where \( R(x) = (e + d(x, x_0))^N \). By Lemma 2.22 and the dominated convergence theorem we can choose \( N \) large enough so that

\[
\int_X R(x)^{\frac{1}{2}} d\mu(x) \leq \frac{1}{4} \quad \text{and} \quad \int_X R(x)^{\frac{1}{2}} d\mu(x) \leq \frac{1}{4}.
\]

Again by Remark 2.17 we know that \( M_\mu^P : L^{p_\infty}(X) \to L^{q_\infty, \infty}(X) \). If we combine this fact with inequalities (4.9) and (4.10), and then repeat the argument used to prove (3.12) and (3.13) (which holds without change), we get

\[
\int_{X \setminus \Omega_{t, 2}^{\gamma_1}(\cdot)} \left( \alpha_2 t \chi_{\{M_\mu^P f_2 > t\}}(x) \right)^{q(x)} d\mu(x) \leq e^{NC_\infty} C(p_\infty, X) \left( \int_X \left( \alpha_2 f_2(y) \right)^{p_\infty} d\mu(y) \right)^{\frac{q_\infty}{p_\infty}} + \frac{1}{4},
\]

\[
\leq e^{NC_\infty} C(p_\infty, X) \left( \frac{5}{4} \alpha_2^{p_\infty} e^{NC_\infty} + \int_X \alpha_2^{-1} \log(e + d(x, x_0)) d\mu(x) \right)^{\frac{q_\infty}{p_\infty}} + \frac{1}{4}.
\]

By Remark 2.23 the integral on the righthand side can be made arbitrarily small by choosing \( \alpha_2 \) sufficiently small. Therefore, we can choose \( 0 < \alpha_2 < 1 \) such that inequality (4.7) holds for all \( t \).

Now suppose \( q_\infty = +\infty \). In this case, since \( 1/q(\cdot) \in LH \), \( 0 < t < 1 \) and \( 0 \leq \chi_{\{M_\mu^P f_2 > t\}}(\cdot) \leq 1 \), by Remark 2.23 we can choose \( 0 < \alpha_2 < 1 \) sufficiently small so that for all \( t > 0 \),

\[
\int_{X \setminus \Omega_{t, 2}^{\gamma_1}(\cdot)} \left( \alpha_2 t \chi_{\{M_\mu^P f_2 > t\}}(x) \right)^{q(x)} d\mu(x) \leq \int_X \alpha_2^{-1} \log(e + d(x, x_0)) d\mu(x) \leq \frac{1}{2}.
\]

We now prove (4.8). If \( t \geq 1 \) there is nothing to prove since the lefthand side is 0. On the other hand, if \( t < 1 \), then by choosing \( 0 < \alpha_2 \leq 1/2 \) we get

\[
\alpha_2 t \| \chi_{\{M_\mu^P f_2 > t\}} \|_{L^{\infty}(\Omega_{t, 2}^{\gamma_1}(\cdot))} \leq \alpha_2 \leq 1/2,
\]

and this establishes inequality (4.8) for all \( t > 0 \). This completes the estimate for \( f_2 \) and also the proof of Theorem 1.2.

**Spaces with finite measure.** Finally we consider the case when \( \mu(X) < +\infty \). By Lemma 2.4 and Corollary 2.27, \( 1/p(\cdot) \in LH \), and so by Lemma 2.26 \( 1/p(\cdot) \) satisfies a Diening type condition. Therefore, the estimate for \( f_1 \) is unchanged when \( t > \mu(X)^\gamma \int_X |f| d\mu \). When \( t \) is smaller than this bound we choose \( \beta_1 \) accordingly and bound the lefthand side of (4.2) by \( \gamma_1 \mu(X) \); by the appropriate choice of \( \gamma_1 \) this can be made as small as desired.

The estimate for \( f_2 \) can be greatly simplified. Since \( \mu(\{M_\mu^P f_2 > t\}) \geq 0 \) only when \( t < 1 \), we have that
\[
\int_{X \setminus \Omega_X^{(\alpha)}} (\alpha_2 t \chi_{\{M_{\eta}^\alpha f > t\}}(x))^q(x) \, d\mu(x) + \alpha_2 t \| \chi_{\{M_{\eta}^\alpha f > t\}} \|_{L^{\infty}(\Omega_X^{(\alpha)})} \\
\leq \alpha_2^q \mu(X) + \alpha_2 \leq \alpha_2 (\mu(X) + 1),
\]
if \(0 < \alpha_2 < 1\). Hence, if we choose \(\alpha_2\) sufficiently small, we get the desired inequality.

5. Fractional integral operators

In this section we prove Theorems 1.3 and 1.4. This requires a pointwise estimate that relates the fractional integral operator to the fractional maximal operator. We prove this estimate in Proposition 5.1. Given this inequality, the actual proofs follow exactly as they do in the Euclidean case, and we refer the reader to [6] for complete details.

**Proposition 5.1.** Let \((X, d, \mu)\) be a reverse doubling space. Given \(0 < \eta < 1\), fix \(\varepsilon, 0 < \varepsilon < \min\{\eta, 1 - \eta\}\). Then there exists a constant \(C = C(\eta, \varepsilon, X)\) such that for every \(f \in L^1_{loc}(X)\) and for every \(x \in X\),

\[|I_{\eta} f(x)| \leq C (M_{\eta-\varepsilon} f(x) M_{\eta+\varepsilon} f(x))^{\frac{q}{2}}.\]

**Remark 5.2.** It is only in the proof of this result that we use the assumption that \(\mu\) is a reverse doubling measure. We are not certain if Proposition 5.1 remains true without this hypothesis.

**Proof.** We first assume that \(\mu(X) = +\infty\); we will describe the changes to the proof when \(\mu(X) < +\infty\) afterwards. Given \(f \in L^1_{loc}(X)\) and \(x \in X\) define

\[I_1 f(x) = \int_{B(x, \delta)} f(y) \frac{d\mu(y)}{\mu(B(x, d(x, y)))^{1-\eta}},\]

\[I_2 f(x) = \int_{X \setminus B(x, \delta)} f(y) \frac{d\mu(y)}{\mu(B(x, d(x, y)))^{1-\eta}},\]

where the precise value of \(\delta > 0\) will be fixed below. Clearly \(|I_{\eta} f(x)| \leq |I_1 f(x)| + |I_2 f(x)|\) and so we will estimate each term on the righthand side separately.

We first estimate \(|I_1 f(x)|\). For \(i \geq 0\) define

\[R_i = \{y \in X : 2^{-i-1}\delta \leq d(x, y) < 2^{-i}\delta\}.\]

Since the measure \(\mu\) is both doubling and reverse doubling, we have that

\[|I_1 f(x)| \leq \sum_{i \geq 0} \int_{R_i} \frac{|f(y)|}{\mu(B(x, d(x, y)))^{1-\eta}} \, d\mu(y)\]
\( \leq \sum_{i \geq 0} \left( \frac{\mu(B(x, 2^{-i}\delta))}{\mu(B(x, 2^{-i-1}\delta))} \right)^{1-\eta} \frac{\mu(B(x, 2^{-i}\delta))^{\epsilon}}{\mu(B(x, 2^{-i-1}\delta))^{1-\eta+\epsilon}} \int_{B(x, 2^{-i}\delta)} |f(y)| \, d\mu(y) \)

\( \leq \sum_{i \geq 0} (CC_\mu)^{1-\eta} (\gamma^\epsilon)^i \mu(B(x, \delta))^{\epsilon} M_{\eta-\epsilon} f(x) \)

(5.1)

\( \leq \frac{(CC_\mu)^{1-\eta}}{1 - \gamma^\epsilon} \mu(B(x, \delta))^{\epsilon} M_{\eta-\epsilon} f(x), \)

where the constant \( C \) comes from the lower mass bound which is satisfied by a doubling measure. The last inequality holds because \( 0 < \gamma < 1 \) ensures that the geometric series converges.

We now estimate \( |I_2 f(x)| \) in a similar fashion. For \( i \geq 1 \) we now define

\[ R_i = \{ y \in X : 2^{i-1}\delta \leq d(y, x) < 2^i \delta \}; \]

then essentially the same argument shows that

\[ |I_2 f(x)| \]

\( \leq \sum_{i \geq 1} \left( \frac{\mu(B(x, 2^i\delta))}{\mu(B(x, 2^{i-1}\delta))} \right)^{1-\eta} \frac{\mu(B(x, 2^i\delta))^{-\epsilon}}{\mu(B(x, 2^{i-1}\delta))^{1-\eta-\epsilon}} \int_{B(x, 2^i\delta)} |f(y)| \, d\mu(y) \)

\( \leq \sum_{i \geq 0} (CC_\mu)^{1-\eta} \mu(B(x, \delta))^{-\epsilon} M_{\eta+\epsilon} f(x) (\gamma^\epsilon)^i \)

(5.2)

\( \leq \frac{(CC_\mu)^{1-\eta}}{1 - \gamma^\epsilon} \mu(B(x, \delta))^{-\epsilon} M_{\eta+\epsilon} f(x). \)

To complete the proof we will show that there exists \( \delta > 0 \) such that

(5.3) \( \frac{1}{C_\mu} \left( \frac{M_{\eta+\epsilon} f(x)}{M_{\eta-\epsilon} f(x)} \right)^{\frac{1}{2\epsilon}} < \mu(B(x, \delta)) \leq \left( \frac{M_{\eta+\epsilon} f(x)}{M_{\eta-\epsilon} f(x)} \right)^{\frac{1}{2\epsilon}}. \)

If we assume this for the moment and substitute (5.3) into inequalities (5.1) and (5.2), we get

\[ |I_{\eta} f(x)| \leq C (M_{\eta+\epsilon} f(x) M_{\eta-\epsilon} f(x))^{\frac{1}{4}}, \]

which completes the proof when \( \mu(X) = +\infty. \)

To prove (5.3) define the set

\[ \left\{ r > 0 : \mu(B(x, r)) \leq \left( \frac{M_{\eta+\epsilon} f(x)}{M_{\eta-\epsilon} f(x)} \right)^{\frac{1}{2\epsilon}} \right\}. \]

If this set were empty, then because \( \mu \) is non-trivial this would imply that \( \mu(B(x, r)) \) is greater than the ratio on the righthand side. By choosing a sequence of radii such
that \( r_i > 0, r_i > r_{i+1} \) and \( r_i \to 0 \) as \( i \to \infty \), we would get that \( \mu(\{x\}) > 0 \) and this contradicts Lemma 2.8. Therefore, the given set is non-empty. Furthermore, since we can exclude the trivial cases where either \( f = 0 \) a.e. or \( |f| = +\infty \) on a set of positive measure, it follows that the given set is bounded above.

Hence,

\[
\delta_0 = \sup \left\{ r > 0 : \mu(B(x, r)) \leq \left( \frac{M_{\eta+\varepsilon} f(x)}{M_{\eta-\varepsilon} f(x)} \right)^{\frac{1}{2\varepsilon}} \right\}
\]

is well-defined. Fix \( \delta \) such that \( \delta_0/2 < \delta < \delta_0 \); then

\[
\mu(B(x, \delta)) \leq \left( \frac{M_{\eta+\varepsilon} f(x)}{M_{\eta-\varepsilon} f(x)} \right)^{\frac{1}{2\varepsilon}} < \mu(B(x, 2\delta)) \leq C_\mu(B(x, \delta)),
\]

and inequality (5.3) follows at once.

Now assume that \( \mu(X) < +\infty \); we again ignore the trivial cases where \( f = 0 \) a.e. or \( |f| = +\infty \) on a set of positive measure. Let \( x \in X \) and let \( B \) be an arbitrary ball which contains \( x \); then we have that

\[
\mu(B)^{\eta+\varepsilon} \int_B |f| \, d\mu \leq \mu(X)^{2\varepsilon} \mu(B)^{\eta-\varepsilon} \int_B |f| \, d\mu \leq \mu(X)^{2\varepsilon} M_{\eta-\varepsilon} f(x).
\]

If we take the supremum over all such balls, we get

\[
\left( \frac{M_{\eta+\varepsilon} f(x)}{M_{\eta-\varepsilon} f(x)} \right)^{\frac{1}{2\varepsilon}} \leq \mu(X) < +\infty.
\]

Define the set

\[
\left\{ r > 0 : \mu(B(x, r)) \leq \frac{1}{2} \left( \frac{M_{\eta+\varepsilon} f(x)}{M_{\eta-\varepsilon} f(x)} \right)^{\frac{1}{2\varepsilon}} \right\}.
\]

If this set were empty, then \( \mu(B(x, r)) \) would be greater than the ratio on the righthand side for every \( r > 0 \). By choosing radii \( r_i \) such that \( B(x, r_i) \subseteq X, r_i > 0, r_i > r_{i+1} \) and \( r_i \to 0 \) as \( i \to \infty \) we get that \( \mu(\{x\}) > 0 \) which contradicts Lemma 2.8. Therefore, the set is non-empty and bounded above and its supremum, which we denote by \( \delta_0 \), is well-defined. Thus, for \( 0 < \delta < \delta_0 < 2\delta \) we have

\[
\frac{1}{2C_\mu} \left( \frac{M_{\eta+\varepsilon} f(x)}{M_{\eta-\varepsilon} f(x)} \right)^{\frac{1}{2\varepsilon}} \leq \mu(B(x, \delta)) \leq \frac{1}{2} \left( \frac{M_{\eta+\varepsilon} f(x)}{M_{\eta-\varepsilon} f(x)} \right)^{\frac{1}{2\varepsilon}}.
\]

The upper bound on \( \mu(B(x, \delta)) \) ensures that \( B(x, \delta) \subseteq X \) and we may apply the reverse doubling condition to the balls \( B(x, 2^{-i}\delta) \) for \( i \geq 0 \). Thus the estimate for \( I_1 f(x) \) remains unchanged.
However, we need to be more careful with the estimate for $I_2f(x)$ because this involves positive dilations of the ball $B(x, \delta)$ and the reverse doubling condition only applies to balls which are strictly contained in $X$. Let $k$ be the smallest integer such that $2^k\delta > \text{diam}(X)$. If we define $R_i$ as before, then for $i > k$, $R_i = \emptyset$. Hence, estimating as before, we have that

$$|I_2f(x)| \leq \sum_{i=1}^{k} \left( \frac{\mu(B(x, 2^i\delta))}{\mu(B(x, 2^{i-1}\delta))} \right)^{1-\eta} \frac{\mu(B(x, 2^i\delta))^{-\varepsilon}}{\mu(B(x, 2^{i-1}\delta))^{1-\eta-\varepsilon}} \int_{B(x, 2^i\delta)} |f(y)| \, d\mu(y)$$

Let $a$ be the smallest integer such that $A_0 \leq 2^a$. Then for $1 \leq i \leq k$,

$$2^{i-a-4}\delta \leq (8A_0)^{-1}2^{k-1}\delta \leq (8A_0)^{-1}\text{diam}(X).$$

In the proof of Lemma 2.8 we showed that balls with radii strictly less than $(8A_0)^{-1}\text{diam}(X)$ are strictly contained in $X$, and so the reverse doubling condition holds for the balls $B(x, 2^{i-a-4}\delta)$. Hence,

$$\mu(B(x, 2^i\delta))^{-\varepsilon} \leq \mu(B(x, 2^{i-a-4}\delta))^{-\varepsilon} \leq (\gamma^\varepsilon)^i \mu(B(x, 2^{i-a-4}\delta))^{-\varepsilon} \leq C^{(a+4)\varepsilon}(\gamma^\varepsilon)^i \mu(B(x, \delta))^{-\varepsilon}.$$

If we substitute this into the estimate for $I_2f(x)$, we get

$$|I_2f(x)| \leq C \frac{1-\eta}{1-\gamma^\varepsilon} \mu(B(x, \delta))^{-\varepsilon} M_{\eta+\varepsilon}f(x),$$

where $C$ is the constant from the lower mass bound which is satisfied by a doubling measure. We may now use the lower bound for $\mu(B(x, \delta))$ in order to complete the proof of Proposition 5.1 when $\mu(X) < +\infty$.

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