CONSTRUCTION OF THE RESHETIKHIN-TURAEV TQFT FROM
CONFORMAL FIELD THEORY

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ABSTRACT. In [14] we constructed the vacua modular functor based on the sheaf of vacua theory developed in [44] and the abelian analog in [13]. We here provide an explicit isomorphism from the modular functor underlying the skein-theoretic model for the Reshetikhin-Turaev TQFT due to Blanchet, Habeger, Masbaum and Vogel to the vacua modular functor. This thus provides a geometric construction of the TQFT constructed first by Reshetikhin-Turaev from the quantum group $U_q(sl(N))$.

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1. Introduction

This is the main paper in a series of four papers ([13], [14], [15]), where we provide a geometric construction of modular functors and topological quantum field theories (TQFT’s) from conformal field theory building on the constructions of Tsuchiya, Ueno and Yamada in [44] and [49] and Kawamoto, Namikawa, Tsuchiya and Yamada in [31] and further provide an explicit isomorphism of the modular functors underlying the Reshetikhin-Turaev TQFT for \(U_q(sl(N))\) and the vacua modular functors coming from conformal field theory for the Lie algebra \(sl(N)\) based on the constructions of in [44] and [49] and [31]. We use the Skein theory approach to the Reshetikhin-Turaev TQFT of Blanchet, Habegger, Masbaum and Vogel [20], [21] and [19] to set up this isomorphism. Since the modular functor determines the TQFT uniquely, this therefore also provides a geometric construction of the Reshetikhin-Turaev TQFT. That there should be such an isomorphism is a well established conjecture which is due to Witten, Atiyah and Segal (see e.g. [56], [17] and [42]).

Let us now outline our construction of the above mentioned isomorphism between the two theories. In [13] we described how to reconstruct the rank one abelian theory first introduced by Kawamoto, Namikawa, Tsuchiya and Yamada in [31] from the the point of view of [44] and [49]. In [14] we described how one combines the work of Tsuchiya, Ueno and Yamada ([44] and [49]) with [13] to construct the vacua modular functor for each simple Lie algebra and a positive integer call the level. Let us here denote the theory we constructed for the the Lie algebra \(sl(N)\) and level \(K\) by \(V^\dagger_{N,K}\) (we briefly give the outline of this construction in section 3). We recall that a modular functors is a functor from a certain category of extended labeled marked surfaces (see section 2) to the category of finite dimensional vector spaces. The functor is required to satisfy Walker’s topological version [52] of Segal’s axioms for a modular functor [42] (see section 2). Note that we do not consider the duality axiom as part of the definition of a modular functor. We consider the duality axioms as extra data. For modular functors which satisfies the duality axiom, we say that it is a modular functor with duality. In [19] Blanchet constructed a modular tensor category which we will here denote \(H^G_{SU(N)}\) (see section 4). It is constructed using skein theory and one can build a modular functor and a TQFT from this category following either the method of [21] or [45]. We denote the resulting modular functor \(V^G_{SU(N)}\). It is easy to check that these two modular functors have the same label set. In this paper we explicitly construct an isomorphism between these to modular functors.

**Theorem 1.1.** There is an isomorphism of modular functors

\[
I_{N,K} : V^G_{SU(N)} \rightarrow V^\dagger_{N,K},
\]

i.e. for each extended labeled marked surface \((\Sigma, \lambda)\) we have an isomorphism of complex vector spaces

\[
I_{N,K}(\Sigma, \lambda) : V^G_{K}(\Sigma, \lambda) \rightarrow V^\dagger_{N,K}(\Sigma, \lambda),
\]

which is compatible with all the structures of a modular functor.

The main idea behind the construction of \(I_{N,K}\) is to use the GNS construction applied to the infinite Hecke algebra with respect to the relevant Markov traces as was first done by Jones [28] and Wenzl [54]. On the skein theory side, we identify
the usual purification construction in terms of the GNS construction. This allows us to show that the resulting representations of the Hecke algebras are isomorphic to Wenzl’s representations. On the vacua side, we know by the results of Kanie [30] (see also [50]) that the space of vacua modular functor gives a geometric construction of Wenzl’s representations. By an inductive limit construction, we build a representation of the infinite Hecke algebra, which we identify with the one coming from Wenzl’s GNS construction. This allows us to establish the important Theorem 7.4 which provides a unique infinite Hecke algebra isomorphism from the inductive limit of the relevant morphism spaces on the Skein side to this inductive limit of spaces of vacua with a certain normalization property. Analyzing the properties of this isomorphism further we find that it determines isomorphism from all relevant morphisms in \( H_{N,K}^{SU(N)} \) to the corresponding spaces of vacua (up to a choice of scalar for each label of the theory). By following Turaev’s construction of a modular functor from a modular tensor category, we build the modular functor \( V_{N,K}^{SU(N)} \) from the modular category \( H_{N,K}^{SU(N)} \) and the isomorphism from Theorem 7.4 now determines all the needed isomorphism between the vector spaces \( V_{N,K}^{SU(N)} \) and the vector spaces \( V_{N,K}^{\ast} \) associates to all labeled marked surfaces.

We have the following geometric application of our construction.

**Theorem 1.2.** The connections constructed in the bundle of vacua for any holomorphic family of labeled marked curve given in [44] preserves projectively a unitary structure which is projectively compatible with morphism of such families.

This theorem is an immediate corollary of our main Theorem 1.1. By definition \( V_{N,K}^{\ast}(\Sigma, \lambda) \) is the covariant constant sections of the bundle of vacua twisted by a fractional power of a certain abelian theory over Teichmüller space. Using the isomorphism \( I_{N,K} \) from our main Theorem 1.1 we transfer the unitary structure on \( V_{N,K}^{SU(N)}(\Sigma, \lambda) \) to the bundle of vacua over Teichmüller space. Here we have used the preferred section of the abelian theory, to transfer the unitary structure to the bundle of vacua (see [14]). Since the unitary structure on \( V_{N,K}^{SU(N)}(\Sigma, \lambda) \) is invariant under the extended mapping class group, the induced unitary structure on the bundle of vacua will be projectively invariant under the action of the mapping class group. But since the bundle of vacua for any holomorphic family naturally is isomorphism to the pull back of the bundle of vacua over Teichmüller space, we get the stated theorem.

**Theorem 1.3.** The Hitchin connections constructed in the bundle over Teichmüller space, whose fiber over an algebraic curve, representing a point in Teichmüller space, is the geometric quantization at level \( K \) of the moduli space of semi-stable bundles of rank \( N \) and trivial determinant over the curve, projectively preserves a unitary structure which is projectively preserved by the mapping class group.

This is an immediate corollary of Theorem 1.2 and then the theorem by Laszlo in [35], which provides a projective isomorphism of the two bundle with connections over Teichmüller space from Theorem 1.2 and 1.3. We also get the following corollary

**Corollary 1.1.** The projective monodromy of the Hitchin connection is infinite for \( N = 2 \) and \( k \notin \{1, 2, 4, 8\} \).
This is an immediate corollary of Theorem 1.1 and Gregor Masbaum’s corresponding result for the Reshetikhin-Turaev Theory [39] (see also [23]). For a purely algebraic geometric proof of this result see [36].

Further the combination of Laszlo’s theorem from [35] allows us to use the geometric quantization of the moduli space of flat $SU(N)$-connections to study the Reshetikhin-Turaev TQFT’s. We have already provided a number of such applications, see e.g. [3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. In fact the result of this paper is used in [11] as an essential component in the first authors proof that the mapping class group does not have Kazhdan’s property T [11].

The paper is organized as follows. In section 2 we briefly recall the theory of modular functors. In section 3 we will recall the vacua modular functor constructed from our previous paper [14]. In section 4 following the description of Blanchet [19], we set up the needed modular tensor category using skein theory. In section 5 we recall Jones and Wenzl work on the representation theory of the Hecke algebra and establish the needed relation between the skein theory representations and Wenzl’s representations of the Hecke algebra. In the following section 6, we define the modular functor $V_{SU(N)}^K$ which gives the skein-theoretic model for Reshetikhin-Turaev TQFT due to Blanchet, Habegger, Masbaum and Vogel [20], [21]. The genus zero part of the of the isomorphism $I_{N,K}$ of the modular functors $V_{SU(N)}^K$ and $V_{N,K}^+$ is provided in section 7. The extension of the isomorphism to higher genus is provided in section 8. In the appendix we provide some normalizations and notations for the Lie algebra $sl(N,\mathbb{C})$.

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2. Modular functors

2.1. The axioms for a modular functor. We shall in this section give the axioms for a modular functor. These are due to G. Segal and appeared first in [42]. We present them here in a topological form, which is due to K. Walker [52]. See also [25]. We note that similar, but different, axioms for a modular functor are given in [45] and in [18]. It is however not clear if these definitions of a modular functor is equivalent to ours.

Let us start by fixing a bit of notation. By a closed surface we mean a smooth real two dimensional manifold. For a closed oriented surface $\Sigma$ of genus $g$ we have the non-degenerate skew-symmetric intersection pairing

$$(\cdot, \cdot) : H_1(\Sigma, \mathbb{Z}) \times H_1(\Sigma, \mathbb{Z}) \to \mathbb{Z}.$$ 

Suppose $\Sigma$ is connected. In this case a Lagrangian subspace $L \subset H_1(\Sigma, \mathbb{Z})$ is by definition a subspace, which is maximally isotropic with respect to the intersection pairing. - A $\mathbb{Z}$-basis $(\tilde{\alpha}, \tilde{\beta}) = (\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g)$ for $H_1(\Sigma, \mathbb{Z})$ is called a symplectic basis if

$$(\alpha_i, \beta_j) = \delta_{ij}, \quad (\alpha_i, \alpha_j) = (\beta_i, \beta_j) = 0,$$

for all $i, j = 1, \ldots, g$.

If $\Sigma$ is not connected, then $H_1(\Sigma, \mathbb{Z}) = \bigoplus_i H_1(\Sigma_i, \mathbb{Z})$, where $\Sigma_i$ are the connected components of $\Sigma$. By definition a Lagrangian subspace is in this paper a subspace of the form $L = \bigoplus_i L_i$, where $L_i \subset H_1(\Sigma_i, \mathbb{Z})$ is Lagrangian. Likewise a symplectic
basis for $H_1(\Sigma, \mathbb{Z})$ is a $\mathbb{Z}$-basis of the form $((\vec{a}^i, \vec{b}^i))$, where $(\vec{a}^i, \vec{b}^i)$ is a symplectic basis for $H_1(\Sigma_i, \mathbb{Z})$.

For any real vector space $V$, we define $PV = (V - \{0\}) / \mathbb{R}^+$. 

**Definition 2.1.** A pointed surface $(\Sigma, P)$ is an oriented closed surface $\Sigma$ with a finite set $P \subset \Sigma$ of points. A pointed surface is called stable if the Euler characteristic of each component of the complement of the points $P$ is negative. A pointed surface is called saturated if each component of $\Sigma$ contains at least one point from $P$.

**Definition 2.2.** A morphism of pointed surfaces $f : (\Sigma_1, P_1) \rightarrow (\Sigma_2, P_2)$ is an isotopy class of orientation preserving diffeomorphisms which maps $P_1$ to $P_2$. Here the isotopy is required not to change the induced map of the first order Jet at $P_1$ to the first order Jet at $P_2$.

**Definition 2.3.** A marked surface $\Sigma = (\Sigma, P, V, L)$ is an oriented closed smooth surface $\Sigma$ with a finite subset $P \subset \Sigma$ of points with projective tangent vectors $V \in \sqcup_{P \in P \Sigma} PT_P\Sigma$ and a Lagrangian subspace $L \subset H_1(\Sigma, \mathbb{Z})$.

**Remark 2.1.** The notions of stable and saturated marked surfaces are defined just like for pointed surfaces.

**Definition 2.4.** A morphism $f : \Sigma_1 \rightarrow \Sigma_2$ of marked surfaces $\Sigma_i = (\Sigma_i, P_i, V_i, L_i)$ is an isotopy class of orientation preserving diffeomorphisms $f : \Sigma_1 \rightarrow \Sigma_2$ that maps $(P_1, V_1)$ to $(P_2, V_2)$ together with an integer $s$. Hence we write $f = (f, s)$.

**Remark 2.2.** Any marked surface has an underlying pointed surface, but a morphism of marked surfaces does not quite induce a morphism of pointed surfaces, since we only require that the isotopies preserve the induced maps on the projective tangent spaces.

Let $\sigma$ be Wall’s signature cocycle for triples of Lagrangian subspaces of $H_1(\Sigma, \mathbb{R})$ (see [53]).

**Definition 2.5.** Let $f_1 = (f_1, s_1) : \Sigma_1 \rightarrow \Sigma_2$ and $f_2 = (f_2, s_2) : \Sigma_2 \rightarrow \Sigma_3$ be morphisms of marked surfaces $\Sigma_i = (\Sigma_i, P_i, V_i, L_i)$ then the composition of $f_1$ and $f_2$ is

$$f_2 f_1 = (f_2 f_1, s_2 + s_1 - \sigma((f_2 f_1)_L, L_1, f_2, L_2, L_3)).$$

With the objects being marked surfaces and the morphism and their composition being defined as in the above definition, we have constructed the category of marked surfaces.

The mapping class group $\Gamma(\Sigma)$ of a marked surface $\Sigma = (\Sigma, L)$ is the group of automorphisms of $\Sigma$. One can prove that $\Gamma(\Sigma)$ is a central extension of the mapping class group $\Gamma(\Sigma)$ of the surface $\Sigma$ defined by the 2-cocycle $c : \Gamma(\Sigma) \rightarrow \mathbb{Z}$, $c(f_1, f_2) = \sigma((f_1 f_2)_L, L, f_1, L, L)$. One can also prove that this cocycle is equivalent to the cocycle obtained by considering two-framings on mapping cylinders (see [16]).

Notice also that for any morphism $(f, s) : \Sigma_1 \rightarrow \Sigma_2$, one can factor

$$(f, s) = ((\text{Id}, s') : \Sigma_2 \rightarrow \Sigma_2) \circ (f, s - s')$$

$$= (f, s - s') \circ ((\text{Id}, s') : \Sigma_1 \rightarrow \Sigma_1).$$

In particular $(\text{Id}, s) : \Sigma \rightarrow \Sigma$ is $(\text{Id}, 1)^s$. 

Definition 2.6. The operation of disjoint union of marked surfaces is
\[(\Sigma_1, P_1, V_1, L_1) \sqcup (\Sigma_2, P_2, V_2, L_2) = (\Sigma_1 \sqcup \Sigma_2, P_1 \sqcup P_2, V_1 \sqcup V_2, L_1 \oplus L_2).\]

Morphisms on disjoint unions are accordingly \[(f_1, s_1) \sqcup (f_2, s_2) = (f_1 \sqcup f_2, s_1 + s_2).\]

We see that disjoint union is an operation on the category of marked surfaces.

Definition 2.7. Let \(\Sigma\) be a marked surface. We denote by \(-\Sigma\) the marked surface obtained from \(\Sigma\) by the operation of reversal of the orientation. For a morphism \(f = (f, s) : \Sigma_1 \to \Sigma_2\) we let the orientation reversed morphism be given by \(-f = (f, -s) : -\Sigma_1 \to -\Sigma_2\).

We also see that orientation reversal is an operation on the category of marked surfaces. Let us now consider glueing of marked surfaces. Let \((\Sigma, \{p_-, p_+\} \sqcup P, \{v_-, v_+\} \sqcup V, L)\) be a marked surface, where we have selected an ordered pair of marked points with projective tangent vectors \(((p_-, v_-), (p_+, v_+))\), at which we will perform the glueing.

Let \(c : P(T_p, \Sigma) \to P(T_p, \Sigma)\) be an orientation reversing projective linear isomorphism such that \(c(v_-) = v_+\). Such a \(c\) is called a glueing map for \(\Sigma\). Let \(\tilde{\Sigma}\) be the oriented surface with boundary obtained from \(\Sigma\) by blowing up \(p_-\) and \(p_+\), i.e.
\[\tilde{\Sigma} = (\Sigma - \{p_-, p_+\}) \sqcup P(T_p, \Sigma) \sqcup P(T_p, \Sigma),\]
with the natural smooth structure induced from \(\Sigma\). Let now \(\Sigma_c\) be the closed oriented surface obtained from \(\tilde{\Sigma}\) by using \(c\) to glue the boundary components of \(\tilde{\Sigma}\). We call \(\Sigma_c\) the glueing of \(\Sigma\) at the ordered pair \(((p_-, v_-), (p_+, v_+))\) with respect to \(c\).

Let now \(\Sigma'\) be the topological space obtained from \(\Sigma\) by identifying \(p_-\) and \(p_+\). We then have natural continuous maps \(q : \Sigma_c \to \Sigma'\) and \(n : \Sigma \to \Sigma'\). On the first homology group \(n\) induces an injection and \(q\) a surjection, so we can define a Lagrangian subspace \(L_c \subset H_1(\Sigma_c, \mathbb{Z})\) by \(L_c = q_*^{-1}(n_*(L))\). We note that the image of \(P(T_p, \Sigma)\) (with the orientation induced from \(\tilde{\Sigma}\)) induces naturally an element in \(H_1(\Sigma_c, \mathbb{Z})\) and as such it is contained in \(L_c\).

Remark 2.3. If we have two glueing maps \(c_i : P(T_p, \Sigma) \to P(T_p, \Sigma), i = 1, 2,\) we note that there is a diffeomorphism \(f\) of \(\Sigma\) inducing the identity on \((p_-, v_-) \sqcup (p_+, v_+) \sqcup (P, V)\) which is isotopic to the identity among such maps, such that \((df_{c_1})^{-1}c_2 df_{c_2} = c_1\). In particular \(f\) induces a diffeomorphism \(f : \Sigma_1 \to \Sigma_2\) compatible with \(f : \Sigma \to \Sigma\), which maps \(L_{c_1}\) to \(L_{c_2}\). Any two such diffeomorphisms of \(\Sigma\) induce isotopic diffeomorphisms from \(\Sigma_1\) to \(\Sigma_2\).

Definition 2.8. Let \(\Sigma = (\Sigma, \{p_-, p_+\} \sqcup P, \{v_-, v_+\} \sqcup V, L)\) be a marked surface. Let
\[c : P(T_{p_1}, \Sigma) \to P(T_{p_1}, \Sigma)\]
be a glueing map and \(\Sigma_c\) the glueing of \(\Sigma\) at the ordered pair \(((p_-, v_-), (p_+, v_+))\) with respect to \(c\). Let \(L_c \subset H_1(\Sigma_c, \mathbb{Z})\) be the Lagrangian subspace constructed above from \(L\). Then the marked surface \(\Sigma_c = (\Sigma_c, P, V, L_c)\) is defined to be the glueing of \(\Sigma\) at the ordered pair \(((p_-, v_-), (p_+, v_+))\) with respect to \(c\).

We observe that glueing also extends to morphisms of marked surfaces which preserves the ordered pair \(((p_-, v_-), (p_+, v_+))\), by using glueing maps which are compatible with the morphism in question.

We can now give the axioms for a 2 dimensional modular functor.
Definition 2.9. A label set \( \Lambda \) is a finite set furnished with an involution \( \lambda \mapsto \hat{\lambda} \) and a trivial element \( 0 \) such that \( \hat{0} = 0 \).

Definition 2.10. Let \( \Lambda \) be a label set. The category of \( \Lambda \)-labeled marked surfaces consists of marked surfaces with an element of \( \Lambda \) assigned to each of the marked point and morphisms of labeled marked surfaces are required to preserve the labelings. An assignment of elements of \( \Lambda \) to the marked points of \( \Sigma \) is called a labeling of \( \Sigma \) and we denote the labeled marked surface by \( (\Sigma, \lambda) \), where \( \lambda \) is the labeling.

We define a labeled pointed surface similarly.

Remark 2.4. The operation of disjoint union clearly extends to labeled marked surfaces. When we extend the operation of orientation reversal to labeled marked surfaces, we also apply the involution \( \hat{\cdot} \) to all the labels.

Definition 2.11. A modular functor based on the label set \( \Lambda \) is a functor \( V \) from the category of labeled marked surfaces to the category of finite dimensional complex vector spaces satisfying the axioms MF1 to MF5 below.

MF1. Disjoint union axiom: The operation of disjoint union of labeled marked surfaces is taken to the operation of tensor product, i.e. for any pair of labeled marked surfaces there is an isomorphism

\[
V((\Sigma_1, \lambda_1) \sqcup (\Sigma_2, \lambda_2)) \cong V(\Sigma_1, \lambda_1) \otimes V(\Sigma_2, \lambda_2).
\]

The identification is associative.

MF2. Glueing axiom: Let \( \Sigma \) and \( \Sigma_c \) be marked surfaces such that \( \Sigma_c \) is obtained from \( \Sigma \) by glueing at an ordered pair of points and projective tangent vectors with respect to a glueing map \( c \). Then there is an isomorphism

\[
V(\Sigma_c, \lambda) \cong \bigoplus_{\mu \in \Lambda} V(\Sigma, \mu, \hat{\mu}, \lambda),
\]

which is associative, compatible with glueing of morphisms, disjoint unions and it is independent of the choice of the glueing map in the obvious way (see remark 2.3). This isomorphism is called the glueing isomorphism and its inverse is called the factorization isomorphism.

MF3. Empty surface axiom: Let \( \emptyset \) denote the empty labeled marked surface. Then

\[
\dim V(\emptyset) = 1.
\]

MF4. Once punctured sphere axiom: Let \( \Sigma = (S^2, \{p\}, \{v\}, 0) \) be a marked sphere with one marked point. Then

\[
\dim V(\Sigma, \lambda) = \begin{cases} 
1, & \lambda = 0 \\
0, & \lambda \neq 0.
\end{cases}
\]

MF5. Twice punctured sphere axiom: Let \( \Sigma = (S^2, \{p_1, p_2\}, \{v_1, v_2\}, \{0\}) \) be a marked sphere with two marked points. Then

\[
\dim V(\Sigma, (\lambda, \mu)) = \begin{cases} 
1, & \lambda = \hat{\mu} \\
0, & \lambda \neq \hat{\mu}.
\end{cases}
\]

In addition to the above axioms one may has extra properties, namely
**MF-D. Orientation reversal axiom:** The operation of orientation reversal of labeled marked surfaces is taken to the operation of taking the dual vector space, i.e. for any labeled marked surface $(\Sigma, \lambda)$ there is a pairing

$$\langle \cdot, \cdot \rangle : V(\Sigma, \lambda) \otimes V(-\Sigma, \hat{\lambda}) \to \mathbb{C},$$

compatible with disjoint unions, glueings and orientation reversals (in the sense that the induced isomorphisms $V(\Sigma, \lambda) \cong V(-\Sigma, \hat{\lambda})$ and $V(-\Sigma, \hat{\lambda}) \cong V(\Sigma, \lambda)$ are adjoints).

and

**MF-U. Unitarity axiom**
Every vector space $V(\Sigma, \lambda)$ is furnished with a hermitian inner product

$$(\cdot, \cdot) : V(\Sigma, \lambda) \otimes \overline{V(\Sigma, \lambda)} \to \mathbb{C},$$

so that morphisms induces unitary transformation. The hermitian structure must be compatible with disjoint union and glueing. If we have the orientation reversal property, then compatibility with the unitary structure means that we have a commutative diagrams

$$
\begin{align*}
V(\Sigma, \lambda) & \xrightarrow{\cong} V(-\Sigma, \hat{\lambda})^* \\
\downarrow{\cong} & \Downarrow{\cong} \\
\overline{V(\Sigma, \lambda)^*} & \xrightarrow{\cong} \overline{V(-\Sigma, \hat{\lambda})},
\end{align*}
$$

where the vertical identifications come from the hermitian structure and the horizontal from the duality.

### 3. The Vacua Modular Functors

In this section we recall the construction of the vacua modular functor given in [14].

#### 3.1. Affine Lie Algebra and Sheaf of Vacua.
Let $\mathfrak{g}$ be a simple Lie algebra over the complex numbers $\mathbb{C}$. The normalized Cartan-Killing form $(\cdot, \cdot)$ is defined to be a constant multiple of the Cartan-Killing form such that

$$(\theta, \theta) = 2.$$ 

for the longest root $\theta$. By $\mathbb{C}[[\xi]]$ and $\mathbb{C}((\xi))$ we mean the ring of formal power series in $\xi$ and the field of formal Laurent power series in $\xi$, respectively. The affine Lie algebra $\hat{\mathfrak{g}}$ over $\mathbb{C}((\xi))$ associated with $\mathfrak{g}$ is defined to be

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}((\xi)) \oplus \mathbb{C}c$$

where $c$ belongs to the center of $\hat{\mathfrak{g}}$ and the Lie algebra structure is given by

$$[X \otimes f(\xi), Y \otimes g(\xi)] = [X, Y] \otimes f(\xi)g(\xi) + (X, Y) \text{Res}_{\xi=0}(g(\xi)d f(\xi)) \cdot c$$

for $X, Y \in \mathfrak{g}, f(\xi), g(\xi) \in \mathbb{C}((\xi))$.

Let us fix a positive integer $K$ and put

$$P_K = \{ \lambda \in p_+ | 0 \leq \langle \theta, \lambda \rangle \leq K \}$$
where \( p_+ \) is the set of dominant integral weights. The involution
\[
\dagger : \quad P_K \rightarrow P_K
\]
\[
\lambda \mapsto \lambda^\dagger
\]
is defined by
\[
(2) \quad \lambda^\dagger = -w(\lambda)
\]
where \( w \) is the longest element of the Weyl group of the simple Lie algebra \( \mathfrak{g} \). Then for each \( \lambda \in P_K \) there exists a unique left \( \mathfrak{g} \)-module \( \mathcal{H}_\lambda \) (called the integrable highest weight \( \mathfrak{g} \)-module of level \( K \)) characterized by the following properties (see \([29]\)):

1. \( V_\lambda = \{ |v\rangle \in \mathcal{H}_\lambda \mid \mathbf{g}_+ |v\rangle = 0 \} \) is the irreducible left \( \mathfrak{g} \)-module with highest weight \( \lambda \) where \( \mathbf{g}_+ = \mathfrak{g} \otimes \mathbb{C}[[\xi]] \xi \subset \mathfrak{g} \).

2. The central element \( c \) acts on \( \mathcal{H}_\lambda \) as \( K \cdot \text{id} \).

3. \( \mathcal{H}_\lambda \) is generated by \( V_\lambda \) over \( \mathbf{g}_- \) with only one relation
\[
(X_\theta \otimes \xi^{-1}) K^{-(\theta, \lambda) + 1} |\lambda\rangle = 0
\]
where \( \mathbf{g}_- = \mathfrak{g} \mathbb{C}[[\xi^{-1}]][\xi^{-1}] \subset \mathfrak{g} \), \( X_\theta \in \mathfrak{g} \) is the element corresponding to the maximal root \( \theta \) and \( |\lambda\rangle \in V_\lambda \) is the highest weight vector.

The dual space \( \mathcal{H}_\lambda^\dagger \) of \( \mathcal{H}_\lambda \) is defined to be
\[
\mathcal{H}_\lambda^\dagger = \text{Hom}_\mathbb{C}(\mathcal{H}_\lambda, \mathbb{C}).
\]
For \( \vec{\lambda} = (\lambda_1, \ldots, \lambda_m) \in (P_K)^m \) put
\[
\mathcal{H}_\vec{\lambda} = \mathcal{H}_{\lambda_1} \otimes \mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_m}
\]
\[
\mathcal{H}_{\vec{\lambda}}^\dagger = \mathcal{H}_{\lambda_1}^\dagger \otimes \mathcal{H}_{\lambda_1}^\dagger \otimes \cdots \otimes \mathcal{H}_{\lambda_m}^\dagger
\]
where \( \otimes \) is the complete tensor product.

An \( m \)-pointed Riemann surface \((\mathcal{C}; q_1, q_2, \ldots, q_m)\) consists of a compact Riemann surface \( \mathcal{C} \) of genus \( g \) with \( m \) distinct points on it. For each positive integer \( n \) an \( n \)-th order neighbourhood \( \eta^{(n)} \) of \( C \) at a point \( q \in \mathcal{C} \) is a \( \mathbb{C} \)-algebra isomorphism
\[
\eta^{(n)} : \mathcal{O}_{C,q}/m_q^n \simeq \mathbb{C}[[\xi]]/(\xi^n + 1)
\]
where \( \mathcal{O}_{C,q} \) is the stalk at the point \( q \) of the sheaf \( \mathcal{O}_\mathcal{C} \) of germs of holomorphic functions on \( \mathcal{C} \) and \( m_q \) is the maximal ideal of \( \mathcal{O}_{C,q} \) consisting of germs of holomorphic functions vanishing at \( q \). A formal neighbourhood \( \eta \) of \( \mathcal{C} \) at a point \( q \in \mathcal{C} \) is a \( \mathbb{C} \)-algebra isomorphism
\[
\eta : \hat{\mathcal{O}}_{C,q} = \varprojlim_{n \to \infty} \mathcal{O}_{C,q}/m_q^{n+1} \simeq \varprojlim_{n \to \infty} \mathbb{C}[[\xi]]/(\xi^{n+1}) = \mathbb{C}[[\xi]].
\]
A family of \( m \)-pointed Riemann surfaces with formal coordinates \( \mathfrak{f} = (\pi : \mathcal{C} \rightarrow \mathcal{B}; s_1, \ldots, s_m; \eta_1, \ldots, \eta_m) \) consists of the following data:

1. Complex manifolds \( \mathcal{C} \) and \( \mathcal{B} \) with \( \dim_{\mathbb{C}} \mathcal{C} = \dim_{\mathbb{C}} \mathcal{B} + 1 \) and \( \pi : \mathcal{C} \rightarrow \mathcal{B} \) is a proper holomorphic map of maximal rank at each point.

2. Holomorphic maps \( s_j : \mathcal{B} \rightarrow \mathcal{C} \) such that \( \pi \circ s_j \) is the identity map.
(3) Each fiber $\pi^{-1}(x), x \in B$ is a Riemann surface of genus $g$ and
\[
(\pi^{-1}(x), s_1(x), \ldots, s_m(x))
\]
is an $m$-pointed Riemann surface.

(4) Maps $\eta_j$ which are $\mathcal{O}_B$-module isomorphisms
\[
\eta_j : \mathcal{O}_{C/B} \to \mathcal{O}_{C/kS}
\]
with the following commutation relation, which is
\[
\hat{\mathcal{O}}(\mathfrak{g}) = \mathcal{O}_C/\mathcal{O}_B[\xi]/(\xi^{n+1}) = \mathcal{O}_B[[\xi]].
\]
where $I_1$ is the $\mathcal{O}_B$-ideal sheaf consisting of germs of holomorphic functions vanishing along $s_j(B)$.

The sheaf $\hat{\mathfrak{g}}_m(B)$ of affine Lie algebra over $B$ is a sheaf of $\mathcal{O}_B$-module
\[
\hat{\mathfrak{g}}_m(B) = \mathfrak{g} \otimes_{\mathcal{O}_C} \left( \bigoplus_{j=1}^m \mathcal{O}_B((\xi_j)) \right) \oplus \mathcal{O}_B : c
\]
with the following commutation relation, which is $\mathcal{O}_B$-bilinear.
\[
\left[ (X_1 \otimes f_1, \ldots, X_m \otimes f_m), (Y_1 \otimes g_1, \ldots, Y_m \otimes g_m) \right] =\left[ (X_1, Y_1) \otimes (f_1 g_1, \ldots, (X_m, Y_m) \otimes (f_m g_m)))) \right] \oplus c : \sum_{j=1}^m (X_j, Y_j) \mathfrak{Res}_{\xi_j=0} (g_j d f_j)
\]
where $X_j, Y_j \in \mathfrak{g}, f_j, g_j \in \mathcal{O}_B((\xi_j))$ and we require $c$ to be central.

Put
\[
\hat{\mathfrak{g}}(\hat{\mathfrak{s}}) = \mathfrak{g} \otimes_{\mathcal{O}_C} \mathfrak{π}_s(\mathcal{O}_C(*S))
\]
where $S = \sum_{j=1}^m s_j(B)$ and $\mathfrak{π}_s(\mathcal{O}_C(*S))$ is the inductive limit of $\{ \mathfrak{π}_s(\mathcal{O}_C(kS)) \}$ and $\mathfrak{π}_s(\mathcal{O}_C(kS))$ is the direct image sheaf of $\mathcal{O}_C(kS)$ by the holomorphic map $\mathfrak{π} : C \to B$. Here $\mathcal{O}_C(kS)$ is a sheaf of germs of meromorphic functions on $C$ having poles of order at most $k$ along $s_j(B)$ and holomorphic outside $S$.

The Laurent expansions with respect to the formal neighbourhoods $\eta_j$'s gives an $\mathcal{O}_B$-module inclusion:
\[
\bar{\iota} : \mathfrak{π}_s(\mathcal{O}_C(*S)) \hookrightarrow \bigoplus_{j=1}^m \mathcal{O}_B((\xi_j))
\]
and we may regard $\hat{\mathfrak{g}}(\hat{\mathfrak{s}})$ as a Lie subalgebra of $\hat{\mathfrak{g}}_m(B)$. For any $\bar{\lambda} = (\lambda_1, \ldots, \lambda_m) \in (P_K)^m$, put
\[
\mathcal{H}^\dagger_{\bar{\lambda}}(B) = \mathcal{O}_B \otimes_{\mathcal{O}_C} \mathfrak{H}^\dagger_{\bar{\lambda}}.
\]
The sheaf of affine Lie algebra $\hat{\mathfrak{g}}_m(B)$ acts on $\mathcal{H}^\dagger_{\bar{\lambda}}(B)$ by
\[
F \otimes (\psi)((X_1 \otimes \sum_{n \in \mathbb{Z}} a_n^{(1)} \xi_1^n), \ldots, (X_m \otimes \sum_{n \in \mathbb{Z}} a_n^{(m)} \xi_m^n)) = \sum_{j=1}^m \sum_{n \in \mathbb{Z}} (a_n^{(j)} F) \otimes (\psi|_{\bar{\lambda}}(X_j(n)), F \in \mathcal{O}_B, \ \psi \in \mathcal{H}^\dagger_{\bar{\lambda}}
\]
where $\psi|_{\bar{\lambda}}(X_j(n))$ means that $X_j(n) = X_j \otimes \xi_j^n$ acts on the $j$-th component $\mathcal{H}^\dagger_{\bar{\lambda}}$ of $\mathcal{H}^\dagger_{\bar{\lambda}}$ from the right. Similarly $\hat{\mathfrak{g}}_m(B)$ acts on $\mathcal{H}^\dagger_{\bar{\lambda}}(B)$ from the left.
Definition 3.1. The sheaves of vacua $\mathcal{V}^f_{\lambda}(\mathfrak{F})$ and covacua $\mathcal{V}^\psi_{\lambda}(\mathfrak{F})$ attached to the family $\mathfrak{F} = (\pi : \Sigma \to \mathbb{C}; s_1, \ldots, s_m; \eta_1, \ldots, \eta_m)$ is defined as

$$\mathcal{V}^f_{\lambda}(\mathfrak{F}) = \{ \langle \Psi \rangle \in H^1_{\lambda}(\mathfrak{F}) \mid \langle \Psi \rangle a = 0 \text{ for any } a \in \mathfrak{F}(\mathfrak{F}) \},$$

$$\mathcal{V}^\psi_{\lambda}(\mathfrak{F}) = H^1_{\lambda}(\mathfrak{F}) / \mathfrak{F}(\mathfrak{F}) H^1_{\lambda}(\mathfrak{F}).$$

If the base space $B$ is a point $p$, that is $\mathfrak{F} = (\pi : \Sigma \to \{p\}; q_1, \ldots, q_m : \eta_1, \ldots, \eta_m)$, $\mathcal{V}^f_{\lambda}(\mathfrak{F})$ and covacua $\mathcal{V}^\psi_{\lambda}(\mathfrak{F})$ are called the space of vacua and the space of covacua, respectively and written as $\mathcal{V}^f_{\lambda}(\mathfrak{F})$, $\mathcal{V}^\psi_{\lambda}(\mathfrak{F})$ where $\mathfrak{F} = (\Sigma, \eta_1, \ldots, \eta_m)$.

Theorem 3.1 ([44]). The sheaves $\mathcal{V}^f_{\lambda}(\mathfrak{F})$ of vacua and covacua $\mathcal{V}^\psi_{\lambda}(\mathfrak{F})$ are locally free $\mathcal{O}_B$-modules of finite rank and dual to each other. Moreover for any point $b \in B$ we have

$$\mathcal{V}^f_{\lambda}(\mathfrak{F}) \otimes_{\mathcal{O}_B} \mathcal{O}_b = \mathcal{V}^\psi_{\lambda}(\mathfrak{F})$$

where $\mathcal{O}_b = \mathcal{O}_{B,b}/m_b$ and $\mathcal{F}_b = (\pi^{-1}(b); s_1(b), \ldots, s_m(b); \eta_1|_{\pi^{-1}(b)}, \ldots, \eta_m|_{\pi^{-1}(b)})$.

Theorem 3.2 ([44], Theorem 5.6 [14]). The sheaf $\mathcal{V}^f_{\lambda}(\mathfrak{F})$ of vacua carries a family of projectively flat connection parametrized by bidifferentials on $\mathfrak{F}$.

3.2. Teichmüller space and the bundle of Vacua. Let us first review some basic Teichmüller theory.

Definition 3.2. A marked Riemann surface $\Sigma$ is a Riemann surface $\Sigma$ with a finite set of marked points $Q$ and non-zero tangent vectors $W \in T_\Sigma C = \bigcup_{q \in Q} T_q C$.

Note that a non-zero tangent vector $v \in T_q C$ determines uniquely the first order neighbourhood at the point $q$ by

$$\eta : \mathcal{O}_{\Sigma,q}/m_q^2 \ni f \mod m_q^2 \mapsto f(q) + v(f)\zeta \in \mathbb{C}[[\zeta]]$$

and conversely a first order neighbourhood $\eta$ determines uniquely the tangent vector $v \in T_q C$ by the above formula.

Definition 3.3. A morphism between marked Riemann surface is a biholomorphism of the underlying Riemann surface which induces a bijection between the two sets of marked points and tangent vectors at the marked points.

The notions of stable and saturated is defined just like for pointed surfaces (see Definition 2.1). Let $\Sigma$ be a closed oriented smooth surface and let $P$ be finite set of points on $\Sigma$.

Definition 3.4. A complex structure on $(\Sigma, P)$ is a marked Riemann surface $\Sigma = (\Sigma, Q, W) = (\Sigma, Q, W)$ together with an orientation preserving diffeomorphism $\phi : \Sigma \to \Sigma$ mapping the points $P$ onto the points $Q$. Two such complex structures $\phi_1 : (\Sigma, P) \to \Sigma_i = (\Sigma_i, Q, W)$ are equivalent if there exists a morphism of marked Riemann surfaces $\Phi : \Sigma_1 \to \Sigma_2$

such that $\phi_2^{-1}\Phi\phi_1 : (\Sigma, P) \to (\Sigma, P)$ is isotopic to the identity through maps inducing the identity on the first order neighbourhood of $P$.

We shall often in our notation suppress the diffeomorphism, when we denote a complex structure on a surface.

Definition 3.5. The Teichmüller space $T_{(\Sigma, P)}$ of the pointed surface $(\Sigma, P)$ is by definition the set of equivalence classes of complex structures on $(\Sigma, P)$.
We note there is a natural projection map from $\mathcal{T}_{(\Sigma, P)}$ to $T_p\Sigma = \sqcup_{p \in P} T_p\Sigma$, which we call $\pi_P$.

**Theorem 3.3 (Bers).** There is a natural structure of a finite dimensional complex analytic manifold on Teichmüller space $\mathcal{T}_{(\Sigma, P)}$. Associated to any morphism of pointed surfaces $f : (\Sigma_1, P_1) \to (\Sigma_2, P_2)$ there is a biholomorphism $f^* : \mathcal{T}_{(\Sigma_1, P_1)} \to \mathcal{T}_{(\Sigma_2, P_2)}$ which is induced by mapping a complex structure $C = (C, Q, W)$, $\phi : \Sigma_1 \to C$ to $\phi \circ f^{-1} : \Sigma_2 \to C$. Moreover, compositions of morphisms go to compositions of induced biholomorphisms.

There is an action of $R^*_+$ on $\mathcal{T}_{(\Sigma, P)}$ given by scaling the tangent vectors. This action is free and the quotient $\mathcal{T}^{(r)}_{(\Sigma, P)} = \mathcal{T}_{(\Sigma, P)} / R^*_+$ is a smooth manifold, which we call the reduced Teichmüller space of the pointed surface $(\Sigma, P)$. Moreover the projection map $\pi_P$ descend to a smooth projection map from $\mathcal{T}^{(r)}_{(\Sigma, P)}$ to $\sqcup_{p \in P} P(T_p\Sigma)$, which we denote $\pi^{(r)}_P$. We denote the fiber of this map over $V \in \sqcup_{p \in P} P(T_p\Sigma)$ by $\mathcal{T}^{(r)}_{(\Sigma, P, V)}$. Teichmüller space of a marked surface $\Sigma = (\Sigma, P, V, L)$ is by definition $\mathcal{T}_\Sigma = \mathcal{T}^{(r)}_{(\Sigma, P, V)}$ which we call the Teichmüller space of the marked surface. Morphisms of marked surfaces induce diffeomorphism of the corresponding Teichmüller spaces of marked surfaces, which also behaves well under composition. We observe that the self-morphism $(\text{Id}, s)$ of a marked surface acts trivially on the associated Teichmüller space for all integers $s$. General Teichmüller theory implies that

**Theorem 3.4.** The Teichmüller space $\mathcal{T}_\Sigma$ of any marked surface $\Sigma$ is contractible.

Since there are many ways to extend the universal family of marked Riemann surfaces over the Teichmüller space $\mathcal{T}_\Sigma = \mathcal{T}_{(\Sigma, P, V)}$ of a marked surface $\Sigma = (\Sigma, P, V, L)$ to the family of pointed Riemann surfaces with formal coordinates, careful consideration is necessary to construct the bundle of vacua over the Teichmüller space $\mathcal{T}_\Sigma$. This was done in our previous paper [14] §5.3. We recall the crucial parts of the construction.

Let $\mathcal{F} = (\pi : \mathcal{C} \to B; \vec{s}; \vec{\eta})$ be a family of pointed Riemann surfaces with normal neighbourhoods and assume we have a smooth fiber preserving diffeomorphism $\Phi_\mathcal{F}$ from $Y$ to $\mathcal{C}$ taking the marked points to the sections $\vec{s}$ and inducing the identity on $B$. This data induces a unique holomorphic map $\Psi_\mathcal{F}$ from $B$ to the Teichmüller space $\mathcal{T}_{(\Sigma, P)}$ of the surface $(\Sigma, P)$ by the universal property of Teichmüller space.

**Definition 3.6.** If a family $\mathcal{F} = (\pi : \mathcal{C} \to B; \vec{s}; \vec{\eta})$ of pointed Riemann surfaces with formal neighbourhoods on $(\Sigma, P)$, as above, has the properties, that the base $B$ is biholomorphic to an open ball and that the induced map $\Psi_\mathcal{F}$ is a biholomorphism onto an open subset of Teichmüller space $\mathcal{T}_{(\Sigma, P)}$ then the family is said to be good.

Note that if a family of pointed Riemann surfaces with formal neighbourhoods on $(\Sigma, P)$ is versal around some point $b \in B$, in the sense of Definition 1.24 in [50], then there is an open ball around $b$ in $B$, such that the restriction of the family to this neighbourhood is good.

**Proposition 3.1.** For a stable and saturated pointed surface $(\Sigma, P)$ the Teichmüller space $\mathcal{T}_{(\Sigma, P)}$ can be covered by images of such good families.

This follows from Theorem 1.28 in [50].
Using these good families we can construct the bundle of vacua over the Teichmüller space $T_{(\Sigma, P)}$ (see Definition 5.2 [14] and the arguments after the definition).

**Theorem 3.5.** There exists a unique holomorphic vector bundle $V_1^\dagger = V_1^\dagger(\Sigma, P)$ over Teichmüller space $T_{(\Sigma, P)}$ which is specified to be the bundle $(\Psi_\mathfrak{g})_+^\dagger V_\lambda(\mathfrak{g})$ over $\Psi_B(\mathfrak{b})$ for any good families $\mathfrak{g} = (\pi : C \to B; \mathfrak{s}, \mathfrak{t})$ of complex structures on $(\Sigma, P)$, where $\Psi_\mathfrak{g} : B \to T_{(\Sigma, P)}$ is the canonical mapping.

3.3. The bundle of Abelian Vacua. To a family $\mathfrak{g} = (\pi : C \to B; \mathfrak{s}, \mathfrak{t})$ of pointed Riemann surfaces with formal neighbourhoods we can associate the line bundle of abelian vacua $V_{ab}^\dagger(\mathfrak{g})$ which is isomorphic to the determinant bundle of the relative canonical sheaf $\omega_C/B$ (Theorem 3.2, [13]).

**Theorem 3.6 ([13], Theorem 4.2).** The line bundle $V_{ab}^\dagger(\mathfrak{g})$ carries a family of projectively flat connection parametrized by bidifferentials on $C$.

To compare the curvatures of the connections in the abelian vacua with that for the non-abelian ones, we needed an explicit description of the curvature form, hence we constructed the line bundle $V_{ab}^\dagger(\mathfrak{g})$ of abelian vacua by using the fermion Fock space and fermion operators in [13].

Also by the similar method to the one for non-abelian vacua we can construct the line bundle of abelian vacua $V_{ab}^\dagger = V_{ab}^\dagger(\Sigma, P)$ over Teichmüller space $T_{(\Sigma, P)}$ of a marked surface $(\Sigma, P)$ (section 8 in [14]).

The relation of curvature forms of projectively flat connections of the bundle of vacua and the line bundle of abelian vacua is given the following theorem. The theorem plays the crucial role to construct our modular functor.

**Theorem 3.7 ([14], Corollary 9.1).** Let $\mathfrak{g}$ be a family of stable and saturated pointed Riemann surfaces with formal neighbourhoods on $(\Sigma, P)$. If we use the same bidifferential $\omega$ to define the connections on the bundle of vacua and the line bundle of abelian vacua on $\mathfrak{g}$, then we have

$$R^\omega(X, Y) = \frac{c_\nu}{2} R(X, Y) \otimes \text{Id},$$

where $R^\omega$ is the curvature form of the connection of the bundle of vacua, $R$ is the curvature form of the connection of the line bundle of abelian vacua constructed by using the bidifferential $\omega$ and

$$c_\nu = \frac{K \cdot \dim G}{g^* + K},$$

where $g^*$ is the dual Coxeter number of the simple Lie algebra $g$ (for example, $g^* = n$ for $g = \mathfrak{sl}(n, \mathbb{C})$).

To define $c_\nu/2$-th root of line bundle $V_{ab}^{\dagger b'}$, we need a non-vanishing holomorphic section of this bundle (see [14] and the following section). To this end we have the following theorem.

**Theorem 3.8 ([14], Theorem 10.8).** Let $\Sigma$ be a closed oriented surface and let $(a, \beta) = (a_1, \ldots, a_g, \beta_1, \ldots, \beta_\mathfrak{s})$ be a symplectic basis of $H_1(\Sigma, \mathbb{Z})$. Then there is a unique non-vanishing holomorphic section $s(a, \beta) = s_{a, \beta}(\Sigma, z)$ in the bundle $V_{ab}^\dagger(\Sigma) \otimes^2$ over $T_{\Sigma}$ which behaves well under an orientation preserving diffeomorphism of surfaces which maps the
symplectic basis \((\tilde{a}(1), \tilde{b}(1))\) of \(H_1(\Sigma_1, \mathbb{Z})\) for some surface \(\Sigma_1\) to the symplectic basis \((\tilde{a}(2), \tilde{b}(2))\) of \(H_1(\Sigma_2, \mathbb{Z})\) for some other surface \(\Sigma_2\).

3.4. The geometric construction of the modular functor. For the bundle of vacua over the Teichmüller space we summarize its properties in the following theorem.

**Theorem 3.9.** Let \((\Sigma, P, \lambda)\) be a stable and saturated labeled pointed surface.

1. There exists a vector bundle \(V^\dagger_\lambda = V^{\dagger}_\lambda(\Sigma, P)\) over the Teichmüller space \(T_{(\Sigma, P)}\) of \((\Sigma, P)\) whose fiber at a complex structure \(C\) on \((\Sigma, P)\) is identified with the space of vacua \(V^\dagger_\lambda(C)\).

2. For each symplectic basis of \(H_1(\Sigma, \mathbb{Z})\), we get induced a connection in \(V^\dagger_\lambda\) over \(T_{(\Sigma, P)}\). Any two of these connections differ by a global scalar-value 1-form on \(T_{(\Sigma, P)}\).

3. Each of these connections is projectively flat.

4. There is a natural lift of morphisms of pointed surfaces to these bundles covering induced biholomorphisms between Teichmüller spaces, which preserves compositions.

5. A morphism of pointed surfaces transforms these connections according to the way it transforms symplectic bases of the first homology. Moreover, if we choose a Lagrangian subspace \(L\) of \(H_1(\Sigma, \mathbb{Z})\) and constrain the symplectic basis \((\tilde{a}, \tilde{b}) = (a_1, \ldots, a_q, b_1, \ldots, b_q)\) of \(H_1(\Sigma, \mathbb{Z})\) such that \(L = \text{Span}\{\beta_i\}\) then we get a connection in \(V^\dagger_\lambda\) which depends only on \(L\).

For the line bundle of abelian vacua over the Teichmüller space we summarize its properties in the following theorem.

**Theorem 3.10.** Let \(\Sigma\) be a closed oriented surface.

1. There exists a line bundle \(V^\dagger_{ab} = V^{\dagger}_{ab}(\Sigma)\) over the Teichmüller space \(T_{\Sigma}\) of \(\Sigma\), whose fiber at a complex structure \(C\) on \(\Sigma\) is identified with the space of abelian vacua \(V^{\dagger}_{ab}(C)\).

2. For each symplectic basis of \(H_1(\Sigma, \mathbb{Z})\), we get induced a holomorphic connection in \(V^\dagger_{ab}\) over \(T_{\Sigma}\). The difference between the connections associated to two different basis’s is the global scalar-value 1-form on \(T_{\Sigma}\). If we choose a Lagrangian subspace \(L\) of \(H_1(\Sigma, \mathbb{Z})\) and constrain the symplectic basis \((\tilde{a}, \tilde{b})\) of \(H_1(\Sigma, \mathbb{Z})\) such that \(L = \text{Span}\{\beta_i\}\) then we see that the connection in \((V^\dagger_{ab})^\otimes 2\) only depends on \(L\).

3. The curvature of each of these connections are related to the corresponding connection in the non-abelian bundle of vacua by formula (3).

4. For each symplectic basis of \(H_1(\Sigma, \mathbb{Z})\), we also get a preferred non-vanishing section of \((V^\dagger_{ab})^\otimes 2\). If we choose a Lagrangian subspace \(L\) of \(H_1(\Sigma, \mathbb{Z})\) and constrain the symplectic basis \((\tilde{a}_i, \tilde{b}_i)\) of \(H_1(\Sigma, \mathbb{Z})\) such that \(L = \text{Span}\{\beta_i\}\) then the preferred non-vanishing section only depends on \(L\) (Theorem 10.8, 11).

5. There is a natural lift of morphisms of surfaces to these bundles covering induced biholomorphisms between Teichmüller space, which preserves compositions.

6. A morphism of surfaces transforms these connections and the preferred sections according to the way it transforms symplectic bases of the first homology.

As we already mentioned the preferred section in Theorem 3.8 enable us to define a fractional power of the line bundle \(V^\dagger_{ab}\) which has the following properties.
Theorem 3.11. For any marked surface \( \Sigma = (\Sigma, L) \) there exists a line bundle, which we denoted \( (V^\dagger_{ab})^{-\frac{1}{2}c_v}(L) = (V^\dagger_{ab})^{-\frac{1}{2}c_v}(\Sigma) \), over \( T_\Sigma \) that satisfies the following:

1. \( (V^\dagger_{ab})^{-\frac{1}{2}c_v} \) is a functor from the category of marked surfaces to the category of line bundles over Teichmüller spaces of closed oriented surfaces.
2. If we choose a symplectic basis of \( H_1(\Sigma, \mathbb{Z}) \) for a marked surface \( \Sigma \) then we get induced a connection in \( (V^\dagger_{ab})^{-\frac{1}{2}c_v}(L) \), whose curvature is \(-\frac{1}{2}c_v \) times the curvature of the corresponding connection in \( V^\dagger_{ab} \). The difference between the connections associated to two different bases is \(-\frac{1}{2}c_v \) times the global scalar-value 1-form on \( T_\Sigma \) given in (69) of [14].

By virtue of Theorem 3.7 we can now make the following definition.

Definition 3.7. For a stable and saturated labeled marked surface \( (\Sigma, \lambda) \) we define the vector bundle \( V^\dagger_\lambda(\Sigma) \) with its flat connection \( \nabla(\Sigma, \lambda) \) as the push forward of the bundle \( V^\dagger_\lambda \otimes (V^\dagger_{ab})^{-\frac{1}{2}c_v}(L) \) to the reduced Teichmüller space \( T^{(r)}(\Sigma, P) \) followed by restriction to the fiber \( T_\Sigma \).

Then we have the following theorem.

Theorem 3.12 (Theorem 11.3, [14]). The above construction gives a functor from the category of labeled marked surfaces to the category of vector bundles with flat connections over Teichmüller spaces of marked surfaces.

Finally the modular functor is defined as follows.

Definition 3.8 (Definition 11.3, [14]). The functor \( V^K_\lambda \). Let \( K \) be a positive integer. Let \( P_K \) be the finite set defined in (1) with the involution \( \dagger \) as defined by (2). Let \( (\Sigma, \lambda) = (\Sigma, P, V, L, \lambda) \) be a labeled marked surface using the label set \( P_K \). The functor \( V^K_\lambda \) is by definition the composite of the functor, which assigns to \( (\Sigma, \lambda) \) the flat vector bundle \( V^\dagger_\lambda(\Sigma) \) over \( T_\Sigma \), and the functor, which takes covariant constant sections.

Theorem 3.13 (Theorem 13.1 [14]). The functor \( V^K_\lambda \) satisfies all the axioms of a modular functor.

The main part of the proof (given in [14]) of this theorem is the construction of the glueing isomorphisms. We shall not repeat this construction here, since we will not need details of it in this paper. We refer to sections 11 and 12 in [14] for the details.

4. The Skein Theory Construction of Modular Categories

Let us review Blanchet's [19] constructions of the Hecke-category and its associated modular tensor categories. This construction is really a generalization of the BHMV-construction of the \( U_q(sl_2(\mathbb{C})) \)-Reshetikhin-Turaev TQFT [21] to the \( U_q(sl_N(\mathbb{C})) \)-case. We give a slightly more direct construction of this category and its associated modular functor, which implements skein theoretically some of the abstract categorical constructions presented in [19] and [45]. This is done in complete parallel to the \( N = 2 \) case treated in [21].

Throughout we will fix integers, \( N \geq 2 \) and \( K \geq 1 \). Let \( q \) be the following primitive \( (N + K) \)'th root of 1 in \( \mathbb{C} \), \( q = e^{2\pi i/(K+N)} \). We will also need the following
roots of \( q, q^{1/2N} = e^{2\pi i/(2N(K+N))} \) and \( q^{1/2} = e^{2\pi i/(2(K+N))} \). We observe that the quantum integers

\[
[j] = \frac{q^{j/2} - q^{-j/2}}{q^{1/2} - q^{-1/2}}
\]

are invertible if \( 1 \leq j < N + K \).

4.1. **The Hecke algebra and Jones-Wenzl idempotents.** Let \( B_n \) be the braid group on \( n \) strands. The standard generators of \( \sigma_i \in B_n, i = 1, \ldots, n - 1 \) are given by the braids on \( n \) strands where the \( i \)'th strand is crossing over the \( (i + 1) \)'th strand

\[
\sigma_i = \begin{array}{c}
\ldots \\
\uparrow \\
\ldots \\
\end{array}
\]

The Hecke algebra \( H_n \) is the following quotient of the group ring of \( B_n \).

**Definition 4.1.** The Hecke algebra \( H_n \) is

\[
H_n = C[B_n] / \langle q^{-1/2N} \sigma_i - q^{1/2N} \sigma_i^{-1} = (q^{1/2} - q^{-1/2}) \text{Id} \mid i = 1, \ldots, n - 1 \rangle.
\]

The Jones-Wenzl idempotent of \( H_n \) are given explicitly as follows:

\[
g_n = \frac{1}{[n]!} q^{-n(n-1)/4} \sum_{\pi \in S_n} (-q)^{1-N}(\ell(\pi)/2Nw_\pi}
\]

\[
f_n = \frac{1}{[n]!} q^{n(n-1)/4} \sum_{\pi \in S_n} q^{-1}(\ell(\pi)/2Nw_\pi}
\]

where \( w_\pi \) is the positive braid associated to \( \pi \) and \( \ell(\pi) \) is the length of \( \pi \) and the quantum factorial

\[
[n]! = \prod_{j=1}^n [j]
\]

is assumed to be invertible. These idempotents have the following properties

\[
\sigma_i f_n = q^{1-N}/2N f_n = f_n \sigma_i \text{ and } \sigma_i g_n = -q^{1+N}/2N g_n = g_n \sigma_i.
\]

Following further Jones and Wenzl, we introduce the idempotents \( e_i \in H_n, i = 1, 2, \ldots, n - 1 \) given by

\[
e_i = \frac{q + q^{(N-1)/2N}\sigma_i}{q + 1}.
\]

These idempotents satisfies the relations (H1) and (H2) from [54] and they clearly also generate \( H_n \). One can define a *-structure on \( H_n \) by the assignment \( e_i^* = e_i, i = 1, 2, \ldots, n - 1 \).

4.2. **Skein theory.** Let \( \Sigma \) be a compact oriented surface.

**Definition 4.2.** A framed set of points \( \ell \) is a finite set of points \( P \subset \Sigma - \partial \Sigma \) together with oriented directions

\[
v_P \in P(T_p \Sigma)^{\times P}
\]

and signs \( \varepsilon_p \) attached to each point \( p \in P \). For a framed set of points \( \ell \) in \( \Sigma \), we denote by \( -\ell \) the same framed set of points, but with all signs negated.
Let $M$ be a compact oriented 3-manifold with boundary $\partial M$, which might be empty. Let $\ell$ be a frame set of points in $\partial M$.

For the definition of an oriented ribbon graph $R$ in $M$ with $\partial R = \ell$, see pp. 31–35 in Turaev’s book [45]. We just recall here that the signs at the boundary indicates the direction of the band, positive for outgoing and negative for ingoing.

We are only interested in the equivalence class of the ribbon graphs in $M$ up to the action of orientation preserving diffeomorphisms of $M$, which are the identity on the boundary and isotopic to the identity among such.

**Definition 4.3.** A ribbon graph $R$ in $M$ is called special if it only contains coupons of the following type:

\[
\begin{array}{c}
1^N \\
\end{array}
\quad \quad
\begin{array}{c}
\text{1}^N \\
\end{array}
\]

which has $N$ incoming or $N$ outgoing bands.

The label $1^N$ on these coupons is as such immaterial, but will be justified by the relations on them introduced below.

Let $\mathcal{H}(M, \ell)$ be the free complex vector space generated by special ribbon graphs in $M$, whose boundary is $\ell$, modulo the following local relations

\[
q^{-1/2} \begin{array}{c}
\circ L \quad N \\
\end{array} - q^{1/2} \begin{array}{c}
\circ N \quad L \\
\end{array} = (q^{1/2} - q^{-1/2}) \begin{array}{c}
\circ \quad N \\
\end{array} = q^{(N^2-1)/2N} \begin{array}{c}
\circ \quad N \\
\end{array} = q^{-(1+N^2)/2N} \begin{array}{c}
\circ \quad N \\
\end{array},
\]

plus the two coupon relations:

\[
L \cup \bigcirc = [N]L
\]

Suppose now $(M_i, \ell_i), i = 1, 2$ are two compact oriented 3-manifolds and $\ell_i$ is a framed set of points in $\partial M_i$. Assume further that we have components

\[(\Sigma_i, \ell'_i) \subset (\partial M_i, \ell_i)\]

and an orientation preserving diffeomorphism $f : (\Sigma_i, \ell'_i) \to (-\Sigma_2, -\ell'_2)$. Let $(M, \ell)$ be obtained by glueing $(M_i, \ell_i)$ to $(M_i, \ell_i)$ along $f$

\[(M, \ell) = (M_1 \cap_f M_2, \ell)\]

where $\ell$ is obtained from $\ell_1 \cup \ell_2$ by erasing $\ell'_1 \cup \ell'_2$. Glueing along $f$ induces a complex bilinear map

\[\mathcal{H}(M_1, \ell_1) \otimes \mathcal{H}(M_2, \ell_2) \to \mathcal{H}(M, \ell),\]

which we simply write as composition

\[(u, v) \mapsto uv.\]

For any closed oriented surface $\Sigma$, $\mathcal{H}(\Sigma \times [0, 1], -\ell, \ell)$ gets the structure of an algebra and for any compact oriented 3-manifold $M$ with boundary $\partial M = \Sigma$, $\mathcal{H}(M, \ell)$ becomes a module over the algebra $\mathcal{H}(\Sigma \times [0, 1], -\ell, \ell)$. 
Suppose $\ell$ is a framed set of points in $\partial M$. We observe that if the algebraic number of points in $\ell$ is not a non-zero multiple of $N$, then all coupons of any special ribbon graph in $(M, \ell)$ can be reduced away by the first of the coupon relations, so certainly $\mathcal{H}(M, \ell)$ is in this case generated by ribbon links. By Proposition 1.1 in [19] it also follows that no extra relations is obtained from the second coupon relation, so that $\mathcal{H}(M, \ell)$ is in this case isomorphic to the Homfly skein module of $M$ in the $sl_N(C)$-specialization.

We have the following fundamental theorem.

**Theorem 4.1.** The framed version of the Homfly polynomial which by definition satisfies the relations (4.3) induces by evaluation an isomorphism

$$\langle \cdot \rangle : \mathcal{H}(S^3) \longrightarrow C$$

The Homfly polynomial was first introduced in [24].

### 4.3. The Hecke category $H$.

The Hecke category $H$ is defined as follows. The objects are pairs $\alpha = (D^2, \ell)$, where $\ell$ is a framed set of points in the interior of the 2-disk $D^2$. The morphisms $\text{Hom}(\alpha, \beta) = H(\alpha, \beta)$, between two objects $\alpha = (D^2, \ell_0)$ and $\beta = (D^2, \ell_1)$ are

$$H(\alpha, \beta) = \mathcal{H}(D^2 \times [0, 1], \ell_0 \times \{0\} \sqcup \ell_1 \times \{1\})$$

There is a trace $\text{tr}_{N,K}^N$ on $H_{\alpha} = H(\alpha, \alpha).

$$^\wedge : H_{\alpha} \longrightarrow \mathcal{H}(D^2 \times S^1) \longrightarrow \mathcal{H}(S^3) \longrightarrow C$$

The first map is obtained by glueing the bottom and top disk of $D^2 \times [0, 1]$. The second map is induced by the standard inclusion of $D^2 \times S^1$ into $S^3$ and the last is induced by the framed Homfly polynomial.

Define

$$\langle \cdot, \cdot \rangle : H(\alpha, \beta) \times H(\beta, \alpha) \rightarrow C$$

by

$$\langle f, g \rangle = \hat{f} \hat{g}.$$

The tensor product of two objects

$$\alpha_i = (D^2, \ell_i), \quad i = 1, 2$$

is

$$\alpha_1 \otimes \alpha_2 = (D^2, j_{-1}(\ell_1) \sqcup j_{+1}(\ell_2))$$

where $j_\varepsilon : D^2 \longrightarrow D^2$, $\varepsilon \pm 1$, are the embeddings $j_\varepsilon(z) = z/4 + \varepsilon/2.$
The tensor product of morphisms are similarly defined. There is a preferred braiding morphism in \( H(\alpha_1 \otimes \alpha_2, \alpha_2 \otimes \alpha_1) \) as indicated in the following figure.

The twist \( \theta \in H(\alpha) \) is indicated here:

For an object \( \alpha = (D^2, \ell) \) the dual object is \( \alpha^* = (D^2, \ell^*) \), where \( \ell^* = -\ell_1 \) and \( \ell_1 \) is obtained from \( \ell \) by applying the map \( z \mapsto -\bar{z} \) to \( \ell \). We define \( b_\alpha \in H(0, \alpha \otimes \alpha^*) \) and \( d_\alpha \in H(\alpha^* \otimes \alpha, 0) \) by

It is trivial to check that

**Proposition 4.1.** The category \( H \) with the above structure is a ribbon category.

The object consisting of the points \(-(n-1)/n, -(n-3)/n, \ldots, (n-1)/n\) framed along the positive real axis we denote simply \( n \). We observe that \( H_n \) is naturally isomorphic to \( H(n,n) \). The trace \( tr^{N,K}_n \) defined above induces therefore a trace on \( H_n \).
4.4. Young symmetrizers. Let us now for each Young diagram define an object in the Hecke category and the Aiston-Morton realization of the corresponding Young symmetrizer. For a Young diagram $\lambda$, let $|\lambda| = n$. Let $\bigtriangleup$ be the object in $H$ obtained by “putting $\lambda$ over $D^2$,” i.e., put a point at $(k+i)/(n+1)$ if $\lambda$ has a cell at $(n, l)$, where we index by (row, column). Let $F_\lambda \in H_{\bigtriangleup}$ be obtained by putting $[\lambda_i]!f_\lambda$ along row $i$ in $\lambda$, $i = 1, \ldots, p$ and $G_\lambda \in H_{\bigtriangleup}$ be obtained by putting $[\lambda_j]!g_\lambda$ along column $j$, $j = 1, \ldots, p^\vee$. Then $\tilde{y}_\lambda = F_\lambda G_\lambda$ is a quasi-idempotent, since by Proposition 1.6 in [19]

$$\tilde{y}_\lambda^2 = [hl(\lambda)]\tilde{y}_\lambda,$$

where $hl(\lambda) = \text{hook-length of } \lambda$. So if $hl(\lambda)$ is non-zero we define

$$y_\lambda = [hl(\lambda)]^{-1}\tilde{y}_\lambda,$$

which is an idempotent. By Proposition 1.8 in [19], we have that

$$\mu \neq \lambda \Rightarrow y_\lambda H(\bigtriangleup, \bigtriangleup_\mu) y_\mu = 0$$

and

$$y_\lambda H_{\bigtriangleup} y_\lambda = C y_\lambda.$$

4.5. $\mathcal{C}$-completed Hecke category $H^\mathcal{C}$. We now introduce the $\mathcal{C}$-completed Hecke category, where $\mathcal{C}$ is the set of Young diagrams $\lambda$, such that $[hl(\lambda)]$ is non-zero. The objects of $H^\mathcal{C}$ are triples $\alpha = (D^2, \ell, \lambda)$ where $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(m)})$, $\lambda^{(i)} \in \mathcal{C}$, and $\ell = (\ell_1, \ldots, \ell_m)$ being a framed set of points in the interior of $D^2$. We have an expansion operation $E$ which maps objects of $H^\mathcal{C}$ to objects of $H$. For an object of $H^\mathcal{C}$, we let $E(\alpha) = (D^2, E(\ell))$ be the object in $H$, where $E(\alpha)$ is obtained by embed $\bigtriangleup_{\lambda^{(i)}}$ in a neighborhood of $\ell_i$ according to the tangent vector of $\ell_i$. Then $\pi_\alpha = y_{\lambda^{(1)}} \otimes \cdots \otimes y_{\lambda^{(m)}}$ defines an idempotent in $H_{E(\alpha)}$ and we let $H^\mathcal{C}(\alpha, \beta) = \pi_\alpha H(E(\alpha), E(\beta)) \pi_\beta$. By simply associating the Young diagram $\bigtriangleup$ to all points, we have a natural inclusion of the objects of $H$ into the objects of $H^\mathcal{C}$. Moreover, for such objects, $E$ is just the identity and the home spaces between such are identical for $H$ and $H^\mathcal{C}$.

For any $\lambda \in \mathcal{C}$, we simply write $\lambda$ for the object in $H^\mathcal{C}$ given by $(D^2, (0, v, +1), \lambda)$, where $v$ is the direction of the positive real line through $0$. The category $H^\mathcal{C}$ inherits the structure of a ribbon category from $H$. 
4.6. The modular category $H^\text{SU(N)}_K$. Consider the following subsets of Young diagrams

$$\Gamma_{N,K} = \{ (\lambda_1, \ldots, \lambda_p) \mid \lambda_1 \leq K, p \leq N \}$$
$$\Gamma^\circ_{N,K} = \{ (\lambda_1, \ldots, \lambda_p) \mid \lambda_1 \leq K, p < N \}$$

and

$$\mathcal{G}_{N,K} = \{ (\lambda_1, \ldots, \lambda_p) \mid \lambda_1 + \lambda_1^\vee \leq N + K \}$$

We define the category $H^\text{SU(N)}_K$ to have the objects of $H^\mathcal{G}_{N,K}$ and the morphisms given by

$$H^\text{SU(N)}_K(\alpha, \beta) = \frac{H^\mathcal{G}_{N,K}(\alpha, \beta)}{\mathcal{N}^\mathcal{G}_{N,K}(\alpha, \beta)}$$

where

$$\mathcal{N}^\mathcal{G}_{N,K}(\alpha, \beta) = \left\{ f \in H^\mathcal{G}_{N,K}(\alpha, \beta) \mid \langle f, g \rangle = 0, \forall g \in H^\mathcal{G}_{N,K}(\beta, \alpha) \right\}.$$

This construction on the morphisms is called “purification”, where one removes “negligible” morphisms $\mathcal{N}^\mathcal{G}_{N,K}(\alpha, \beta)$. We observe that $H^\mathcal{G}_{N,K}(\alpha, \beta)$ is a sub-quotient of $H(\alpha, \beta)$.

Let $I^N_l$ be the Young diagram with $l$ columns containing $N$ cells. The object $1 \otimes \ldots \otimes 1$ ($l$ factors) in $H^\text{SU(N)}_K$ will simply be denoted $l$. We further use the notation $\mathcal{N}^\mathcal{G}_{N,K} = \mathcal{N}^\mathcal{G}_{N,K}(n, n)$.

For a Young diagram $\lambda$ in $\Gamma_{N,K}$ we define $\lambda^\dagger \in \Gamma_{N,K}$ to be the Young diagram obtained from the skew-diagram $(\lambda)^N_1/\lambda$ by rotation as indicated in the figure below.

According to Theorem 2.11 in [19] we have that.

**Theorem 4.2.** The category $H^\text{SU(N)}_K$ with simple objects $\lambda \in \Gamma_{N,K}$ and duality involution $\lambda \mapsto \lambda^\dagger$ is modular.

Let $n \in \mathbb{N}$ and consider the decomposition

$$H^\text{SU(N)}_K(n, n) = \bigoplus_{\lambda \in \Gamma_{N,K}^n} H^\text{SU(N)}_K(n, \lambda) \otimes H^\text{SU(N)}_K(\lambda, n),$$

which we have since $H^\text{SU(N)}_K$ is a modular tensor category. Let

$$\Gamma_{N,K}^n = \{ \lambda \in \Gamma_{N,K} \mid n \geq |\lambda| \text{ and } \mathcal{N}|(n - |\lambda|) \}. $$

Let $\lambda \in \Gamma_{N,K}^n$. Then $H^\text{SU(N)}_K(n, \lambda) \neq 0$ and we let $z_\lambda \in H^\text{SU(N)}_K(n, \lambda) \otimes H^\text{SU(N)}_K(\lambda, n)$ be such that

$$1_n = \sum_{\lambda \in \Gamma_{N,K}^n} z_\lambda.$$
We recall that there is a one to one correspondence between irreducible representations of a finite dimensional algebra over the complex numbers and its minimal central idempotents. Each subalgebra $H^{SU(N)}_K(n, \lambda) \otimes H^{SU(N)}_K(\lambda, n)$ is the full matrix algebra, since $H^{SU(N)}_K(\lambda, n)$ is the dual of $H^{SU(N)}_K(n, \lambda)$ (see Corollary 1.10 in [19]), so we conclude that the $z_\lambda, \lambda \in \Gamma_{N,K}^u$ are the minimal central idempotents of $H^{SU(N)}_K(n, n)$ and $H^{SU(N)}_K(n, \lambda), \lambda \in \Gamma_{N,K}^u$, are the irreducible modules of $H^{SU(N)}_K(n, n)$. We have thus proved the following proposition.

**Proposition 4.2.** We have that $H^{SU(N)}_K(n, \lambda), \lambda \in \Gamma_{N,K}^u$, are the irreducible modules of $H^{SU(N)}_K(n, n)$. Moreover $H^{SU(N)}_K(n, \lambda)$ corresponds to the minimal central idempotent $z_\lambda$ for all $\lambda \in \Gamma_{N,K}^u$.

**Definition 4.4.** Suppose $n \in \mathbb{N}$ and let $\lambda \in \Gamma_{N,K}^u$. Then we define

$$\Gamma_{N,K}^{u-1,\lambda} = \left\{ \lambda' \in \Gamma_{N,K} \mid (\lambda' < \lambda \text{ and } |\lambda' / \lambda| = 1) \text{ or } (\lambda < \lambda', n > |\lambda| \text{ and } \lambda' / \lambda = 1^{N-1}) \right\}.$$

We observe that $\Gamma_{N,K}^{u-1,\lambda} \subseteq \Gamma_{N,K}^{u,\lambda}$.

**Proposition 4.3.** Suppose $\lambda \in \Gamma_{N,K}$ and $n \in \mathbb{N}$ such that $N$ divides $n - |\lambda|$ and let $\lambda' \in \Gamma_{N,K}^{u-1,\lambda}$. Then the $H_{n-1}$-module $H^{SU(N)}_K(n - 1, \lambda')$ is isomorphic to a submodule of the $H_{n-1}$-module $H^{SU(N)}_K(n, \lambda)$.

**Proof.** Let us consider first the case where $\lambda' < \lambda$. Let us observe that

$$H^{SU(N)}_K(\lambda' \otimes 1, \lambda) \neq 0$$

by Corollary 1.10 in [19], so we choose $e \in H^{SU(N)}_K(\lambda' \otimes 1, \lambda) - \{0\}$. By the same Corollary we see that there then must exist $e' \in H^{SU(N)}_K(\lambda', \lambda' \otimes 1)$, such that $e \circ e' = 1_\lambda$. The morphism

$$f \mapsto e \circ (f \otimes 1) \in H^{SU(N)}_K(n, \lambda)$$

for $f \in H^{SU(N)}_K(n - 1, \lambda')$ must be non-zero, since if it is possible to choose $f$ such that there exists $f' \in H^{SU(N)}_K(\lambda', n - 1)$ which satisfies $f \circ f' = 1_{\lambda'}$. But then

$$e \circ (f \otimes 1) \circ (f' \otimes 1) e' = 1_\lambda$$

by Corollary 1.10 in [19]. Since $H^{SU(N)}_K(n - 1, \lambda')$ is an irreducible $H_{n-1}$-module this morphism must be injective and hence an isomorphism onto its image. Now we consider the case where $\lambda < \lambda', n > |\lambda|$ and $\lambda' / \lambda = 1^{N-1}$. First we fix an isomorphism $e \in H^{SU(N)}_K(1^N \otimes \lambda, \lambda)$. Now let $\bar{\lambda} \in \Gamma_{N,K}$ be the young diagram obtain from $\lambda$ by putting $1^N$ to the immediate left of $\lambda$. Again by Corollary 1.10 of [19], we have that there exist

$$g_1 \in H^{SU(N)}_K(\lambda' \otimes 1, \bar{\lambda}) - \{0\} \text{ and } g_2 \in H^{SU(N)}_K(\bar{\lambda}, 1^N \otimes \lambda) - \{0\},$$

for which there exists

$$g'_1 \in H^{SU(N)}_K(\bar{\lambda}, \lambda' \otimes 1) - \{0\} \text{ and } g'_2 \in H^{SU(N)}_K(1^N \otimes \lambda, \bar{\lambda}) - \{0\},$$

such that

$$g_1 g'_1 = 1_\lambda, g'_2 g_2 = 1_\lambda.$$
Let \( g = g_2 g_1 \). Now consider the morphism

\[
f \mapsto e \circ g \circ (f \otimes 1) \in H^\text{SU}(N)_{K}(n, \lambda),
\]

for \( f \) in \( H^\text{SU}(N)_{K}(n - 1 + N, \lambda') \). Again by Corollary 1.10 in [19] we may choose \( f \) such that there exist \( f' \in H^\text{SU}(N)_{K}(\lambda', n - 1 + N) \), such that \( f \circ f' = 1_{\lambda'} \). But then we see that

\[
g_2' \circ e^{-1} \circ e \circ g \circ (f \otimes 1) \circ (f' \otimes 1) \circ g_1' = 1_{\tilde{\lambda}}.
\]

hence the above morphism is non-zero and therefore again an isomorphism onto its image.

\[\square\]

5. The Representation Theory of the Hecke Algebra

Following Jones [28] and Wenzl [54] we will now consider the tower of algebras \( H_1 \subset H_2 \subset \ldots \) and consider the towers of representations corresponding to Markov traces on this tower, first constructed by Wenzl in [54]. We will see that the modular category \( H^\text{SU}(N)_{K} \) constructed above yields another realization of Wenzl’s representations via the GNS construction. These representations are central to Wenzl’s construction of subfactors from the Hecke-algebras in [54].

Let \( H_{\infty} \) by the inductive limit of the algebras

\[
H_1 \subset H_2 \subset \ldots
\]

A trace \( \text{Tr} \) on \( H_{\infty} \) is a linear functional \( \text{Tr} : H_{\infty} \rightarrow \mathbb{C} \), such that \( \text{Tr}(xy) = \text{Tr}(yx) \) for all \( x, y \in H_{\infty} \) and such that \( \text{Tr}(1) = 1 \).

**Definition 5.1.** A trace \( \text{Tr} \) on \( H_{\infty} \) is a Markov trace if there is an \( \eta \in \mathbb{C} \) such that

\[
\text{Tr}(xe_n) = \eta \text{Tr}(x)
\]

for all \( x \in H_n \) and all \( n \in \mathbb{N} \).

**Lemma 5.1.** A Markov trace is uniquely determined by \( \eta = \text{Tr}(e_1) \).

For an argument for this lemma see [28]. In [28] Jones also proved the theorem, that there exist a Markov trace for any \( \eta \in \mathbb{C} \). Let now

\[
\eta_{N,K} = (q - q^N)/(1 + q)(1 - q^N),
\]

and let \( \text{Tr}^{N,K} \) be the corresponding Markov trace.

**Theorem 5.1** (Wenzl). The GNS construction applied to the state \( \text{Tr}^{N,K} \) on the \(*\)-algebra \( H_{\infty} \) gives a representation \( \pi^{N,K+N} \) of \( H_{\infty} \) with the property\(^1\) that

\[
\pi^{N,K+N}_{\Lambda|H_n} \cong \bigoplus_{\lambda \in \Lambda^{N,K+N}} \pi^{N,K+N}_{\Lambda,\lambda},
\]

where \( \Lambda^{N,K+N} \) is the set of Young diagrams defined in Definition 2.4 in [54] and the \( \pi^{N,K+N}_{\Lambda,\lambda}'s \) are Wenzl’s representations

\[
\pi^{N,K+N}_{\Lambda,\lambda} : H_n \rightarrow B(V^{N,K+N}_{\Lambda,\lambda})
\]

constructed right after Definition 2.4 in [54].

\(^1\)Please see the discussion following this theorem for the proper interpretation of this property.
Theorem 5.3. For any $n$, we have that
\[ \mathcal{N}_{n}^{N,K} = \{ x \in H_{n} \mid \text{tr}^{N,K}(x^{*}x) = 0 \} , \]
and let $H_{\infty}^{N,K} = H_{\infty}/\mathcal{N}_{\infty}^{N,K}$. Then $\pi^{N,K,N : H_{\infty} \to B(H_{\infty}^{N,K})$ is just given by the left action of $H_{\infty}$ on $H_{\infty}^{N,K}$. Here $H_{\infty}^{N,K}$ gets a pre-Hilbert space structure induced from the trace by the formula
\[ (x, y)_{N,K} = \text{Tr}^{N,K}(y^{*}x) . \]
The Cauchy-Schwarz inequality shows that this is indeed a well-defined inner product on $H_{\infty}^{N,K}$. The identity $1 \in H_{\infty}$ projects in $H_{\infty}^{N,K}$ to the required cyclic vector for
\[ \pi^{N,K} : H_{\infty} \to B(H_{\infty}^{N,K}) . \]

The Cauchy-Schwarz inequality shows that this is indeed a well-defined inner product on $H_{\infty}^{N,K}$. The identity $1 \in H_{\infty}$ projects in $H_{\infty}^{N,K}$ to the required cyclic vector for
\[ \pi^{N,K} : H_{\infty} \to B(H_{\infty}^{N,K}) . \]

The isomorphism in Theorem 5.1 should be understood in the following sense. Let $\tilde{H}_{n}^{N,K}$ be the image of $H_{n}$ in $H_{\infty}^{N,K}$. Then the action of $H_{n}$ on $H_{\infty}^{N,K}$ preserves $\tilde{H}_{n}^{N,K}$ and it is this representation of $H_{n}$ on $\tilde{H}_{n}^{N,K}$ which is isomorphism to the direct sum of Wenzl’s representations given in the right hand side of (6). We will use the notation
\[ \pi_{n}^{N,K} : H_{n} \to B(\tilde{H}_{n}^{N,K}) \]
for this representation.

We now introduce the following trace on $H_{n}$
\[ \text{tr}^{N,K}(x) = \text{tr}^{N,K}(x)/\text{tr}^{N,K}(\text{Id})^{n} , \]
for all $x \in H_{n}$. We observe that $\text{tr}^{N,K}(x) = \text{tr}^{N,K}(x \otimes 1)$ for all $x \in H_{n}$, hence $\text{tr}^{N,K}$ is a well-defined trace on $H_{\infty}$.

Theorem 5.2. The traces $\text{tr}^{N,K}$ and $\text{Tr}^{N,K}$ coincide.

Proof. From the Skein theory it is clear that $\text{tr}^{N,K}$ is a Markov trace and an explicitly simple computation show that $\text{tr}^{N,K}(e_{1}) = \eta_{N,K}$.

□

Theorem 5.3. For any $n$, we have that
\[ \mathcal{N}_{n}^{N,K} = \{ x \in H_{n} \mid \text{tr}^{N,K}(x^{*}x) = 0 \} . \]

Proof. It is clear that the left hand side is contained in the right hand side. The other inclusion follows from the observation that $\text{tr}^{N,K}(yx) = \text{tr}^{N,K}((y^{*})x) = 0$ for all $x \in \mathcal{N}_{\infty}^{N,K}$ by Cauchy-Schwarz.

□

By using similar easy arguments one finds that $\mathcal{N}_{\infty}^{N,K}$ is two-sided ideal in $H_{\infty}$, from which we conclude that $\pi_{n}^{N,K,N : H_{\infty}/\mathcal{N}_{\infty}^{N,K} \to B(H_{\infty}^{N,K})$. A simple argument shows that this representation is injective. Moreover, if we consider $\pi_{n}^{N,K,N : H_{n} \cap \mathcal{N}_{\infty}^{N,K}$, it clearly also factors through $H_{n} \cap \mathcal{N}_{\infty}^{N,K}$, and the same argument again shows that also this representation is injective on $H_{n}/H_{\infty} \cap \mathcal{N}_{\infty}^{N,K}$ which by the above lemma is naturally isomorphic to $H_{K}^{SU(N)}(n,n)$. From this we have the following theorem as an immediate consequence.
Theorem 5.4. For all $n$ we have a canonical $H_n$ isomorphism
\[ \Psi^n_{n,K} : H^\text{SU}(N)_K(n,n) \cong \pi^{N,K+N}_n(H_n). \]

We now consider the following inductive limit.

Definition 5.2. We let
\[ H^\text{SU}(N)_K(\infty, \infty) = \lim_{n \to \infty} H^\text{SU}(N)_K(n,n). \]

From the above discussion we see that there is a natural $H_\infty$-isomorphism from $H^\text{SU}(N)_K(\infty, \infty)$ to $H^\text{SU}(N)_K(\infty, \infty)$.

Let $\lambda \in \Gamma_{N,K}$. Let $n$ be an integer such that $n - |\lambda|$ is a non-negative multiple, say $l$, of $N$, thus $\lambda \in \Gamma_{N,K}^l$. Let $(\lambda)_n$ be obtained from $\lambda$ by attaching $l$ column of $N$ boxes to the right of $\lambda$. This provides a bijection between $\Gamma_{N,K}^l$ and Wenzl’s $\Lambda_{n,K}^N$.

Theorem 5.5. Let $n \in \mathbb{N}$. The canonical $H_n$ isomorphism $\Psi^n_{n,K}$ from the previous theorem takes the decomposition
\[ H^\text{SU}(N)_K(n,n) \cong \bigoplus_{\lambda \in \Gamma_{N,K}^l} H^\text{SU}(N)_K(n,\lambda) \otimes H^\text{SU}(N)_K(\lambda,n) \]

Thus, we have the following two possible diagrams: $\lambda(2)$ consisting of a row of two boxes and $\lambda(1,1)$ consisting of a column consisting of two boxes. A simple calculation with the two minimal idempotents corresponding to the two labels $\lambda(2)$ and $\lambda(1,1)$ immediately gives the conclusion of the theorem for $n = 2$.

Assume that we have proved the theorem for all integers less than $n$. We will now deduce the second half of the theorem for the integer $n$. Suppose $\lambda \in \Gamma_{N,K}^l$. By Corollary 2.5 in [54] and the discussion following the proof of the Corollary,
we have that $V_{\nu}^{N,K+n}$ is the irreducible representation of $\pi_{N,K}(H_n)$ which corresponds to $z_{(\lambda)\nu}$. By Proposition 1.2 we have that $H_{K}^{\text{SU}(N)}(n,\lambda)$ is the irreducible representation of $H_{K}^{\text{SU}(N)}(n,n)$ corresponding to $z_{\lambda}$. We now consider these to representations as representations of $H_{n-1}$. By the explicit construction of the representation $V_{\nu}^{N,K+n}$ in [54] we see that it decomposes as follows under the action of $H_{n-1}$

$$V_{\nu}^{N,K+n}(\lambda)_{\nu} = \bigoplus_{\lambda' \in \Gamma_{N,K}^{\nu-1}} V_{\nu}^{N,K+n}(\lambda')_{\nu-1}.$$ 

Then by induction we get an $H_{n-1}$ module isomorphism

$$V_{\nu}^{N,K+n}(\lambda)_{\nu} \cong \bigoplus_{\lambda' \in \Gamma_{N,K}^{\nu-1}} H_{K}^{\text{SU}(N)}(n-1,\lambda').$$

But since $V_{\nu}^{N,K+n}(\lambda)_{\nu}$ is an irreducible $H_{K}^{\text{SU}(N)}(n,n)$-module, we know there is a unique $\mu(\lambda) \in \Gamma_{N,K}^{\nu}$ such that

$$V_{\nu}^{N,K+n}(\lambda)_{\nu} \cong H_{K}^{\text{SU}(N)}(n,\mu(\lambda)).$$

We observe that the map $\lambda \mapsto \mu(\lambda)$ is a bijection from $\Gamma_{N,K}^{\nu-1}$ to itself. We also know that

$$H_{K}^{\text{SU}(N)}(n,\mu(\lambda)) \cong \bigoplus_{\lambda' \in \Gamma_{N,K}^{\nu-1}} H_{K}^{\text{SU}(N)}(n-1,\lambda').$$

Hence we conclude that $\Gamma_{N,K}^{\nu-1,\mu(\lambda)} \subset \Gamma_{N,K}^{\nu-1,\nu}$. 

**Claim 5.0.1.** We have that $\mu(\lambda) = \lambda$ for all $\lambda \in \Gamma_{N,K}^{\nu}$. 

First we observe that $\mu(\emptyset) = \emptyset$ (in the case $N \mid n$). Second we will consider the special case where $\lambda = l^{m}$ for some $l, m \in \mathbb{N}$. Then $|\Gamma_{N,K}^{\nu-1,\mu(\lambda)}| = 1$. But then $|\Gamma_{N,K}^{\nu-1,\mu(\lambda)}| = 1$ and we must have that $\Gamma_{N,K}^{\nu-1,\mu(\lambda)} = \Gamma_{N,K}^{\nu-1,\nu}$. But then we see immediately that in this case $\mu(\lambda) = \lambda$. 

We split the rest of the proof into four cases. In each case we assume that $\lambda \neq \mu(\lambda)$ and then we derive a contradiction. We further recall that we can assume that $n > 2$. We will just write $\mu = \mu(\lambda)$.

**Case 1.** $n = |\lambda| = |\mu|$. We observe that $\Gamma_{N,K}^{\nu-1,\mu} \neq \emptyset$, so we choose $\mu' \in \Gamma_{N,K}^{\nu-1,\mu}$. Then $\mu'$ is obtained from $\lambda$ by removing one box say $b_{\lambda}$. Since $\lambda \neq \mu$, we see there is unique Yon diagram $\nu$ such that $|\nu| = n + 2$ and such that $\lambda \prec \nu$ and $\mu \prec \nu$ and

$$\nu = \lambda \cup b_{\mu}, \quad \nu = \mu \cup b_{\lambda}.$$ 

If $\mu$ has another box $b$ different from $b_{\mu}$ such that $\mu - b$ is a Yon diagram, then $\mu - b \in \Gamma_{N,K}^{\nu-1,\mu}$, but $\mu - b \notin \Gamma_{N,K}^{\nu-1,\mu}$, hence this is a contradiction. But then $|\Gamma_{N,K}^{\nu-1,\mu}| = 1$. This mean that there exists $l, m \in \mathbb{N}$ such that $\mu = l^{m}$. But then we have a contradiction, since we are assuming that $\mu \neq \lambda$, jet we have established in this special case for $\mu$, that $\mu = \lambda$, since $\lambda \mapsto \mu(\lambda)$ is a bijection. 

**Case 2.** $n > |\lambda|$ and $n = |\mu|$. By counting boxes, we see that we cannot have the situation of equation (8), hence we can assume that

$$\nu = \lambda \cup b_{\mu}, \quad \nu = \mu \cup b_{\lambda}.$$ 

If $\mu$ has another box $b$ different from $b_{\mu}$ such that $\mu - b$ is a Yon diagram, then $\mu - b \in \Gamma_{N,K}^{\nu-1,\mu}$, but $\mu - b \notin \Gamma_{N,K}^{\nu-1,\mu}$, hence this is a contradiction. But then $|\Gamma_{N,K}^{\nu-1,\mu}| = 1$. This mean that there exists $l, m \in \mathbb{N}$ such that $\mu = l^{m}$. But then we have a contradiction, since we are assuming that $\mu \neq \lambda$, jet we have established in this special case for $\mu$, that $\mu = \lambda$, since $\lambda \mapsto \mu(\lambda)$ is a bijection. 

**Case 3.** $n \prec |\lambda|$ and $n = |\mu|$. By counting boxes, we see that we cannot have the situation of equation (8), hence we can assume that

$$\nu = \lambda \cup b_{\mu}, \quad \nu = \mu \cup b_{\lambda}.$$ 

If $\mu$ has another box $b$ different from $b_{\mu}$ such that $\mu - b$ is a Yon diagram, then $\mu - b \in \Gamma_{N,K}^{\nu-1,\mu}$, but $\mu - b \notin \Gamma_{N,K}^{\nu-1,\mu}$, hence this is a contradiction. But then $|\Gamma_{N,K}^{\nu-1,\mu}| = 1$. This mean that there exists $l, m \in \mathbb{N}$ such that $\mu = l^{m}$. But then we have a contradiction, since we are assuming that $\mu \neq \lambda$, jet we have established in this special case for $\mu$, that $\mu = \lambda$, since $\lambda \mapsto \mu(\lambda)$ is a bijection.
But then we get the same contradiction as before.

**Case 3.** \( n = |\lambda| \) and \( n > |\mu| \). Then we know that \( \lambda \) has a removable box \( b_\lambda \) such that
\[
\lambda - b_\lambda = 1^{N-1} \cup \mu,
\]
which implies that \(|\lambda| = N + |\mu|\). But if \( \mu \neq \emptyset \), then \( \mu \) has a box \( b_\mu \) which can be removed, so \( \lambda \) must have another box \( b'_\lambda \) such that
\[
\lambda - b'_\lambda = \mu - b_\mu,
\]
which contradicts the above count of sizes. Hence we must have that \( \mu = \emptyset \). From this we have an immediate contradiction by the above.

**Case 4.** \( n > |\lambda| \) and \( n > |\mu| \). We cannot have that \( 1^{N-1} \cup \lambda = 1^{N-1} \cup \mu \), since \( \mu \neq \lambda \). Hence \( \lambda \) must have a removable box \( b_\lambda \) and
\[
\lambda - b_\lambda = 1^{N-1} \cup \mu.
\]
Hence \(|\lambda| = N + |\mu|\). But if \( \mu \neq \emptyset \) then it has a removable box \( b_\mu \) such that
\[
1^{N-1} \cup \lambda = \mu - b_\mu \implies |\lambda| = |\mu| - N
\]
or for some other removable box \( b'_\lambda \) from \( \lambda \)
\[
\lambda - b'_\lambda = \mu - b_\mu \implies |\lambda| = |\mu|,
\]
which contradicts \(|\lambda| = N + |\mu|\). So we must have that \( \mu = \emptyset \). Again this gives an immediate contradiction.

\[\square\]

6. **The Reshetikhin-Turaev modular functor via skein theory**

Let us briefly recall Turaev’s construction of a modular functor from a modular tensor category in [45]. Since we are interested in the modular tensor category \( H_{K}^{SU(N)} \), we will at the same time apply it to Blanchet’s category, the construction of which we reviewed in section 4. Let \( \Sigma = (\Sigma, P, V, L) \) be a marked surface and \( \lambda \) a labeling of it by labels from the \( \Gamma_{N,K} \). Hence \( \lambda : P \to \Gamma_{N,K} \). Let \( \Sigma_0 = (\Sigma, P_0, V_0, L_0) \) be the standard surface (see section 1.2 Chapter IV in [45]) of the same type as \( \Sigma \). Let the genus of \( \Sigma \) be \( g \). We recall the following definition.

**Definition 6.1.** The modular functor \( \mathcal{V}_{K}^{SU(N)}(\phi) \) obtained by applying the Reshetikhin-Turaev construction to the modular tensor category \( H_{K}^{SU(N)} \) associates to any labeled marked surface \( (\Sigma, \lambda) \) the vector space \( \mathcal{V}_{K}^{SU(N)}(\Sigma, \lambda) \) which is uniquely determined by the following property. For any morphism \( \phi : \Sigma_0 \to \Sigma \) of marked surfaces, there is a unique isomorphism

\[
\mathcal{V}_{K}^{SU(N)}(\phi) : \bigoplus_{\mu \in \Gamma_{N,K}} H_{K}^{SU(N)}(\lambda_1 \otimes \ldots \otimes \lambda_p \otimes \left( \bigotimes_{i=1}^{g} \mu_i \otimes \mu_i^\dagger \right), 0) \to \mathcal{V}_{K}^{SU(N)}(\Sigma, \lambda)
\]

where \( (\lambda_1, \ldots, \lambda_p) \) is the labeling of \( P_0 \) induced from \( \lambda \) via \( \phi \), such that if \( \phi' : \Sigma_0 \to \Sigma \) is another parametrization, then we get the formula

\[
\mathcal{V}_{K}^{SU(N)}((\phi')^{-1} \circ \phi) = \phi(\phi', \phi)
\]

where \( \phi(\phi', \phi) \) is defined in section 6.3 in Chapter IV of [45].
We remark that \( \varphi(\phi', \phi) \) is constructed by producing a ribbon graph presentation of the mapping cylinder of \((\phi')^{-1} \circ \phi \) and then computing the TQFT morphism determined by \( H^{SU(N)}_K \) for this ribbon graph. In case \( g = 0 \), we observe that this mapping torus is just determined by a spherical braid on \( p \) strands connecting \( P_0 \) to itself. For further details on this please see section 2 of Chapter IV in \[45\].

The construction of the gluing morphism for the modular functor is described in Section 4 of Chapter V. This morphism is also constructed by producing an explicit 3-dimensional cobordism between the unglued and the glued marked surface. This cobordism is described in detail in section 4.5 of Chapter V in \[45\]. Let us now consider a special case of this gluing construction, which we make use of in section\[74\].

Let \( \ell = (\ell_1, \ldots, \ell_t) \) be a framed set of points in the interior of \( D^2 \). Suppose that \( \gamma \) is an arbitrary simple closed curve in \( D^2 - \ell \), with a preferred point \( p \) on \( \gamma \). Assume that there are \( l_+ \) points from \( \ell \) inside \( \gamma \) and \( l_- \) points outside \( \gamma \), such that \( l = l_- + l_+ \). We will assume the labeling is such that \( \ell_- = (\ell_1, \ldots, \ell_{l_-}) \) is the subset of \( \ell \) which is contained in the interior of \( \gamma \). Let \( \ell_+ = (\ell_{l_-+1}, \ldots, \ell_t) \). Pick a diffeomorphism \( \phi_- \) from \( D^2 \) to the interior of \( \gamma \) union \( \gamma \), such that 1 in the unit disc goes to \( p \), such that \( \ell_- \) is mapped to \( \ell_- \). Let \( p_0 \) be the left most point of the object \( l_+ + 1 \). Now pick a diffeomorphism \( \phi_+ \) from \( D^2 - \{p_0\} \) onto the exterior of \( \gamma \), which is the identity on the unit circle and which maps the tangent direction at \( p_0 \) to the point \( p \) on \( \gamma \) and the remaining framed set of points in \( l_+ + 1 \) is mapped to \( \ell_+ \). Let \( \Sigma_- = \Sigma_{l_-}^\gamma \) and \( \Sigma_+ = \Sigma_{l_+} \) be the marked surfaces as defined above from the objects \( l_- \) and \( l_+ + 1 \). The two diffeomorphisms \( \phi_- \) and \( \phi_+ \) induces a diffeomorphism \( \phi \) from \( \Sigma_{\ell} \) to \( (\Sigma_- \cup \Sigma_+)_{\ell} \), where we glue \( \Sigma_- \) at \((\infty, v_\infty)\) to \( \Sigma_+ \) at \((p_0, v_0)\). Let \( \lambda_- \) be the labeling of the points \( \ell_- \) by boxes and let \( \lambda_+ \) be the labeling of the points \( \ell_+ \) by boxes. Further let \( \lambda \) be the box labeling of all points in \( \ell \).

By pushing a copy of the interior of \( \gamma \) into \( D^2 \times I \) and by writing the identity in \( H^{SU(N)}_K(\ell_-, \ell_-) \) as a sum of minimal central idempotents in analogy with \[5\] and inserting this sum in this embedded disc in \( D^2 \), we get an isomorphism

\[
H^{SU(N)}_K(\ell, 0) \cong \bigoplus_{\mu \in \Gamma_{N,K}} H^{SU(N)}_K(\ell_-, \mu) \otimes H^{SU(N)}_K(\mu \otimes \ell_+, 0).
\]

Since the category has duality, we have a preferred isomorphism

\[
H^{SU(N)}_K(\ell_-, \mu) \cong H^{SU(N)}_K(\ell_- \otimes \mu^+, 0).
\]

**Proposition 6.1.** We have the following commutative diagram

\[
\begin{array}{ccc}
H^{SU(N)}_K(\ell, 0) & \longrightarrow & \bigoplus_{\mu \in \Gamma_{N,K}} H^{SU(N)}_K(\ell_- \otimes \mu^+, 0) \otimes H^{SU(N)}_K(\mu \otimes \ell_+, 0) \\
\varphi_k^{SU(N)}(\phi) \downarrow & & \bigoplus_{\mu \in \Gamma_{N,K}} \varphi_k^{SU(N)}(\phi_-) \otimes V^{SU(N)}_K(\phi_+) \\
V^{SU(N)}_K(\Sigma, \lambda) & \longrightarrow & \bigoplus_{\mu \in \Gamma_{N,K}} V^{SU(N)}_K(\Sigma_-, \lambda_-, \mu^+) \otimes V^{SU(N)}_K(\Sigma_+, \mu, \lambda_+)
\end{array}
\]

where top horizontal arrow is the composite of the isomorphism discussed just above this theorem and the lower horizontal arrow is the factorization isomorphism for the modular functor \( V^{SU(N)}_K \).
Proof. As discussed above, the glueing isomorphism for the modular functor $V_{K}^{SU(N)}$ is obtained by applying the TQFT functor associated to $H_{K}^{SU(N)}$ to a certain cobordism as discussed in detail in section 2 of Chapter IV in [45]. We observe that by pushing a copy of the interior of $\gamma$ into $D^2 \times I$, we see this cobordism inside $D^2 \times I$ and by following through various identifications in the construction of the morphism for this cobordism, we immediately get the needed commutativity of the diagram in the Proposition.

\[ \square \]

7. THE GENUS ZERO ISOMORPHISM

7.1. The label sets and the action of the Hecke algebra. The vacua modular functor $V_{N,K}^{SU}$ for $\mathfrak{sl}(N,\mathbb{C})$ at level $K$ has the labeling set $\Gamma_{N,K}$ specified by (1). We observe that the defining representation of $SU(N)$ corresponds to the Young diagram $\square \in \Gamma_{N,K}$.

Let $\Sigma = S^2 = \mathbb{C} \cup \{\infty\}$ with the marked points $P = \{1, \ldots, n, \infty\}$. Choose $V$ to consist of the direction along the positive real axis at the points in $P$. We label the points $\{1, \ldots, n\}$ by $\square$ and $\infty$ by an $\lambda \in \Gamma_{N,K}$. Let $\Sigma = (\Sigma, P, V)$ and $\mu_n = (\square, \ldots, \square)$.

We have a group homomorphism from the braid group on $n$ stands $B_n$ to the mapping class group of $\Sigma$, which we denote $\Gamma_{\Sigma}$:

\[ f : B_n \to \Gamma_{\Sigma}. \]

This group homomorphism induces an algebra morphism

\[ Y : \mathbb{C}[B_n] \to \text{End}(V_{N,K}(\Sigma, \mu_n, \lambda)) \]

by the assignment

\[ Y(b) = V_{N,K}^{SU}(f(b)) \]

for all $b \in B_n$.

For this action Kanie proved in [30] the following formula. See also [50].

**Theorem 7.1 (Kanie).** We have the following skein relation for $Y$

\[ q^{\frac{1}{2}}Y(\sigma_1) - q^{-\frac{1}{2}}Y(\sigma_1^{-1}) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\text{Id}. \]
This means that $V$ factors to the Hecke algebra $H_n$, and so we see that $V_{N,K}^{\dagger}(\Sigma, \mu_n, \lambda)$ becomes a module over $H_n$.

**Theorem 7.2 (Kanie).** The representations

$$Y : H_n \to B(V_{N,K}^{\dagger}(\Sigma, \mu_n, \lambda))$$

and

$$\pi_{N,K+N}^{\dagger(\lambda)n} : H_n \to B(V_{\lambda}^{N,K+N})$$

are isomorphic for all $\lambda \in \Gamma_{N,K}^{\dagger}$.

This theorem is proved in [30]. See also Theorem 6.29 in [50].

7.2. **The morphism from the Hecke module to the space of vacua.** Denote by $\Sigma_n$ the surface with the $2n$-points $\{-n, -(n-1), \ldots, -2, -1, 1, 2, \ldots, n\}$ with all tangent vectors along the real axis, thought of as marked points on $\Sigma = \mathbb{R} \cup \{\infty\}$. We let $\lambda^{(n)}_\pm$ denote the labeling of the $n$ points along the negative/positive real axis by $\Box$ respectively $\Box^\dagger$. Let $\lambda^{(n)}$ be resulting labeling of $\Sigma_n$. We observe that $H_n$ acts on $V_{N,K}^{\dagger}(\Sigma_n, \lambda^{(n)})$ via the above construction.

By factorization in a circle in $\Sigma$ centered in $n$ of radius $n$, we have an isomorphism

$$V_{N,K}^{\dagger}(\Sigma_n, \lambda^{(n)}) = \bigoplus_{\mu \in \Gamma_{N,K}} V_{N,K}^{\dagger}(\Sigma_n^+, \lambda^{(n)}_+, \mu) \otimes V_{N,K}^{\dagger}(\Sigma_n^-, \lambda^{(n)}_-, \mu^\dagger).$$

of $H_n$-modules. We recall that $V_{N,K}^{\dagger}(\Sigma_n^+, \lambda^+, \mu)$ is an irreducible representation of the Hecke algebra $H_n$, hence the above decomposition is the decomposition of $V_{N,K}^{\dagger}(\Sigma_n, \lambda^{(n)})$ into irreducible $H_n$-modules. We also use the notation

$$Y : H_n \to B(V_{N,K}^{\dagger}(\Sigma_n, \lambda^{(n)}))$$

Using the isomorphism (9) and the fact that

$$\dim V_{N,K}^{\dagger}(\Sigma_n^+, \lambda^{(n)}_+, \mu) = \dim V_{N,K}^{\dagger}(\Sigma_n^-, \lambda^{(n)}_-, \mu^\dagger),$$

which is proved in [44], we conclude that

**Proposition 7.1.** The representation

$$Y : H_n \to B(V_{N,K}^{\dagger}(\Sigma_n, \lambda^{(n)}))$$

is isomorphic to

$$\pi_{N,K+N}^{\dagger} : H_n \to \pi_{N,K+N}(H_n).$$

Fix an element $\xi \in V_{N,K}^{\dagger}(\Sigma_1, \lambda^{(1)}) \setminus \{0\}$. For each $n \geq 1$ we now choose a simple smooth curve $\gamma$ in the lower half plane from $-(n+1) \in C$ to $n+1 \in C$, which is tangential to the real axis to all orders at both $-n-1$ and $n+1$, and which comes in along the positive direction of the real line at $-n-1$. Let $\tilde{\Sigma}_n$ be $\Sigma_n$ blown up at $\infty$. Then we choose a smooth map from $\tilde{\Sigma}_1$ to $\{z \in C \mid |z| > n+1/2\}$, which maps the interval $[-1, 1]$ onto $\gamma$ and the rest of the real axis to the real axis. Now pick a diffeomorphism from $\tilde{\Sigma}_n$ to the complement of the image of $\tilde{\Sigma}_1$, which is the identity on $[-n-1/4, n+1/4]$ and which maps the point on the boundary of $\tilde{\Sigma}_n$ corresponding to the negative real axis direction to the point on the boundary of $\tilde{\Sigma}_1$ which also corresponds to the negative real axis. These diffeomorphism induces a diffeomorphism from $\tilde{\Sigma}_1 \cup \tilde{\Sigma}_n$ to $\Sigma_{n+1}$. Let now $\Sigma_n^\infty$ be the marked surface.
obtained from \( \Sigma_n \) by further marking \( \infty \) and selected the direction of the negative real axis at \( \infty \). By the glueing isomorphism we get now the isomorphism

\[
\mathcal{V}_{N,K}^+(\Sigma_{n+1}, \lambda^{(n+1)}) \cong \bigoplus_{\mu \in \Gamma_{N,K}} \mathcal{V}_{N,K}^+(\Sigma_n, \lambda^{(n)}; \mu) \otimes \mathcal{V}_{N,K}^+(\Sigma_1, \lambda^{(1)}, \mu^+) \]

By using the isomorphism \( \mathcal{V}_{N,K}^+(\Sigma_n^\infty, \lambda^{(n)}; 0) \cong \mathcal{V}_{N,K}^+(\Sigma_n, \lambda^{(n)}) \) and tensoring with \( \zeta \), we get inclusions maps

\[
I_n : \mathcal{V}_{N,K}^+(\Sigma_n, \lambda^{(n)}) \rightarrow \mathcal{V}_{N,K}^+(\Sigma_{n+1}, \lambda^{(n+1)})
\]

for all \( n \).

**Definition 7.1.** Let

\[
\mathcal{V}_{N,K}^+(\infty) = \lim_{\mu} \mathcal{V}_{N,K}^+(\Sigma_n, \lambda^{(n)}).
\]

We see that \( \mathcal{V}_{N,K}^+(\infty) \) is an \( H_\infty \) left module. By Proposition \( \mathbb{Z}_A \) we conclude that

**Theorem 7.3.** The representation

\[
\mathcal{Y} : H_\infty \rightarrow B(\mathcal{V}_{N,K}^+(\infty))
\]

is isomorphic to

\[
\pi^{N,K+N} : H_\infty \rightarrow B(\mathcal{H}_{\mathbb{Z}_A}^{N,K}).
\]

But since Wenzl’s representation \( \pi^{N,K+N} \) is generated from the vacuum vector \( 1 \in \mathcal{H}_{\mathbb{Z}_A}^{N,K} \), we get the following theorem.

**Theorem 7.4.** There is a unique \( H_\infty \)-isomorphism

\[
I_{N,K}^\infty : H_{K}^{SU(N)}(\infty, \infty) \rightarrow \mathcal{V}_{N,K}^+(\infty)
\]

such that \( I_{N,K}^\infty(1) = \xi \).

From the above it now follows that the isomorphism \( I_{N,K}^\infty \) must take the subspace \( H_{K}^{SU(N)}(n, n) \) to the subspace \( \mathcal{V}_{N,K}^+(\Sigma_n, \lambda^{(n)}) \) and it must take the decomposition

\[
H_{K}^{SU(N)}(n, n) = \bigoplus_{\mu \in \Gamma_{N,K}} H_{K}^{SU(N)}(n, \mu) \otimes H_{K}^{SU(N)}(\mu, n)
\]

to the decomposition

\[
\mathcal{V}_{N,K}^+(\Sigma_n, \lambda^{(n)}) = \bigoplus_{\mu \in \Gamma_{N,K}} \mathcal{V}_{N,K}^+(\Sigma_n^+, \lambda^{(n)}_+, \mu) \otimes \mathcal{V}_{N,K}^+(\Sigma_n^-, \lambda^{(n)}_-, \mu^+),
\]

since

\[
H_{K}^{SU(N)}(n, \mu) \cong \mathcal{V}_{N,K}^{N,K+N} \cong \mathcal{V}_{N,K}^+(\Sigma_n^+, \lambda^{(n)}_+, \mu)
\]

for \( \mu \in \Gamma_{N,K} \) and these are all the non-isomorphic irreducible \( H_\infty \)-modules. But this means we have obtained the following Theorem

**Theorem 7.5.** There are isomorphisms

\[
I_{N,K}(n, \mu) : H_{K}^{SU(N)}(n, \mu) \rightarrow \mathcal{V}_{N,K}^+(\Sigma_n^+, \lambda^{(n)}_+, \mu)
\]

and

\[
I_{N,K}^\prime(\mu, n) : H_{K}^{SU(N)}(\mu, n) \rightarrow \mathcal{V}_{N,K}^+(\Sigma_n^-, \lambda^{(n)}_-, \mu^+).
\]
The isomorphism \( I_{N,K}(n,\mu) \) is uniquely determined by equation (10) up to multiplication by a scalar for each \( \mu \in \Gamma_{N,K} \).

We remark that if we scale the isomorphism \( I_{N,K}(n,\mu) \) by \( s_\mu \in \mathbb{C}^* \), then we must scale \( I'_{N,K}(\mu,n) \) by \( s_\mu^{-1} \). From now on, we will assume that each of the \( I_{N,K}(n,\mu) \) have been fixed.

7.3. The isomorphism for general box-labeled object. Let \( \alpha \) be an object from the category \( H \), which is also an objects of \( H_k^{SU(N)} \). Let \( n \) be the number of marked points in \( \alpha \). Let \( \Sigma_\alpha \) be marked surface obtained from \( \alpha \) by including the unit disc into \( \Sigma = \mathbb{C} \cup \{\infty\} \) and marking \( \infty \) as well with a tangent vector along the real negative axis. Let \( \lambda_\alpha \) be the labeling of all the marked points in \( \alpha \) by \( \square \) and \( \infty \) by \( 0 \).

If we now further have another object \( \beta \) which also has \( n \) marked points and \( T_{\alpha,\beta} \) is a tangle in \( D^2 \times [0,1] \) representing an element in \( H_k^{SU(N)}(\alpha,\beta) \), then \( T_{\alpha,\beta} \) induces a diffeomorphism of the disc \( \phi_{T_{\alpha,\beta}} \) with the property that \( \phi_{T_{\alpha,\beta}}(\alpha) = \beta \).

**Theorem 7.6.** For all objects \( \alpha \) in the category \( H \), there is a unique isomorphism

\[
I_{N,K}(\alpha) : H_k^{SU(N)}(\alpha,0) \rightarrow \mathcal{Y}_{N,K}^+(\Sigma_\alpha,\lambda_\alpha)
\]

such that

\[
I_{N,K}(n) = I_{N,K}(n,0)
\]

and which makes the following diagram commutative

\[
\begin{array}{ccc}
H_k^{SU(N)}(\alpha,0) & \xrightarrow{I_{N,K}(\alpha)} & \mathcal{Y}_{N,K}^+(\Sigma_\alpha,\lambda_\alpha) \\
\downarrow_{T_{\alpha,\beta}} & & \downarrow_{\mathcal{Y}_{N,K}^+(\phi_{T_{\alpha,\beta}})} \\
H_k^{SU(N)}(\beta,0) & \xrightarrow{I_{N,K}(\beta)} & \mathcal{Y}_{N,K}^+(\Sigma_\beta,\lambda_\beta)
\end{array}
\]

for all pairs of objects \( \alpha,\beta \) in \( H \) and all tangles \( T_{\alpha,\beta} \) as above.

From the above theorem, it is clear that \( H_k^{SU(N)}(\alpha,\mu) \) and \( \mathcal{Y}_{N,K}^+(\Sigma_\alpha,\lambda_\alpha,\mu) \) are isomorphic irreducible \( H_k^{SU(N)}(\alpha,\mu) \)-module and that all of this algebra’s irreducible modules are of this form.

7.4. The isomorphism for general labels. Let \( \lambda_i \in \Gamma_{N,K} \). Let \( n_i = |\lambda_i|, i = 1,\ldots,l \) and let \( n = n_1 + \ldots + n_l \). Let \( \ell = (\ell_1,\ldots,\ell_l) \) be a framed set of points in the interior of \( D^2 \). Further let \( \lambda \) be object in \( H_k^{SU(N)} \) obtained by labeling \( \ell_i \) by \( \lambda_i \).

Now let \( \Sigma_\ell \) be the marked curve induced by including \( \ell \) into the unit disc inside \( \mathbb{C} \cup \{\infty\} \). Similarly let \( \Sigma_{\ell_i} \) be the marked curve induced by including \( \ell_i \) into the unit disc inside \( \mathbb{C} \cup \{\infty\} \) and let \( \Sigma^{\infty}_{\ell_i} \) be obtained from \( \Sigma_{\ell_i} \) by also marking infinity and selecting the tangent direction along the negative real axis at \( \infty \). Now let \( \lambda = E(\lambda) \), where \( E(\lambda) \) is the object in \( H \) defined in Section 4.5. Let \( \Sigma_\lambda \) be the corresponding marked surface. We also denote the labeling of all points in \( \lambda \) simply also by \( \lambda \). Further let \( \lambda_i \) denote the object \( E(\ell_i,\lambda_i) \) and also the labeling of all points in \( \ell_i \) by boxes.
From Theorem 7.6 we have the isomorphism

\[ I_{N,K}(\bar{\lambda}) : H^\text{SU(N)}_K(\bar{\lambda}, 0) \rightarrow \mathcal{V}^\dagger_{N,K}(\Sigma_{\ell}, \bar{\lambda}). \]

By definition

\[ H^\text{SU(N)}_K(\lambda, 0) = \pi_\lambda H^\text{SU(N)}_K(\bar{\lambda}, 0). \]

By factorization in the boundaries of small discs inside the unit disc, which contains each \( \ell_i \), we get the isomorphism

\[ \mathcal{V}^\dagger_{N,K}(\lambda, \mu) \cong \bigoplus_{\mu \in \Gamma^{\dagger}_{\lambda, K}} \mathcal{V}^\dagger_{N,K}(\lambda, \mu) \otimes \mathcal{V}^\dagger_{N,K}(\lambda, \mu). \]

We observe that this is a \( H^\text{SU(N)}_K(\lambda, \mu) \times \ldots \times H^\text{SU(N)}_K(\lambda, \mu) \)-module isomorphism. By writing the identity in \( H^\text{SU(N)}_K(\lambda, \mu) \) as a sum of minimal central idempotents in analogy with (3) we also get the decomposition

\[ H^\text{SU(N)}_K(\lambda, 0) \cong \bigoplus_{\mu \in \Gamma^{\dagger}_{\lambda, K}} H^\text{SU(N)}_K(\mu, 0) \otimes \mathcal{V}^\dagger_{N,K}(\lambda, \mu) \]

which is also a \( H^\text{SU(N)}_K(\lambda, \mu) \times \ldots \times H^\text{SU(N)}_K(\lambda, \mu) \)-module isomorphism. Hence we see that the isomorphism (11) must preserve these decompositions, hence we have proved.

**Theorem 7.7.** For all objects \( \lambda \) in \( H^\text{SU(N)}_K(\lambda, 0) \) as above, there is a unique isomorphism

\[ I_{N,K}(\lambda) : H^\text{SU(N)}_K(\lambda, 0) \rightarrow \mathcal{V}^\dagger_{N,K}(\lambda, \mu) \]

and unique isomorphisms

\[ I_{N,K}(\lambda, \mu) : H^\text{SU(N)}_K(\lambda, \mu) \rightarrow \mathcal{V}^\dagger_{N,K}(\lambda, \mu) \]

for all \( \mu \in \Gamma_{N,K} \) and \( i = 1, \ldots, l \) which for \( \lambda_i = \mu \) agrees with the isomorphism from Theorem 7.5 are compatible with diffeomorphisms of the unit disc, that inducing the identity on the boundary and such that under the above identifications

\[ I_{N,K}(\lambda) = \bigoplus_{\mu \in \Gamma^{\dagger}_{\lambda, K}} I_{N,K}(\mu) \otimes \mathcal{V}^\dagger_{N,K}(\lambda, \mu). \]

Let again \( \ell = (\ell_1, \ldots, \ell_l) \) be a framed set of points in the interior of \( D^2 \). Suppose now that \( \gamma \) is an arbitrary simple closed curve in \( D^2 - \ell \), with a preferred point \( p \) on \( \gamma \). Assume that there are \( l_- \) points from \( \ell \) inside \( \gamma \) and \( l_+ \) points outside \( \gamma \), such that \( l = l_- + l_+ \). We will assume the labeling is such that \( \ell_- = (\ell_1, \ldots, \ell_{l_-}) \) is the subset of \( \ell \) which is contained in the interior of \( \gamma \). Let \( \ell_+ = (\ell_{l_-+1}, \ldots, \ell_l) \). Pick an isomorphism from \( D^2 \) to the interior of \( \gamma \) union \( \gamma \), such that \( 1 \) in the unit disc goes to \( p \), such that \( \ell_- \) is mapped to \( l_- \). Let \( p_0 \) be the left most point of the object \( l_+ + 1 \). Now pick a diffeomorphism from \( D^2 - \{p_0\} \) onto the exterior of \( \gamma \), which is the identity on the unit circle and which maps the tangent direction at \( p_0 \) to the point \( p \) on \( \gamma \). Let \( \Sigma_- = \Sigma_\ell^{\dagger}_\ell \) and \( \Sigma_+ = \Sigma_{\ell_+1}^{\dagger} \) be the marked surfaces as defined above from the objects \( l_- \) and \( l_+ + 1 \). Using these two diffeomorphisms it is clear that we can identify \( \Sigma \ell \) with \( (\Sigma_- \cup \Sigma_+) \), where we glue \( \Sigma_- \) at \( \infty \) to \( \Sigma_+ \) at \( (p_0, \infty) \). Let \( \lambda_- \) be the labeling of the points \( \ell_- \) by boxes and let \( \lambda_+ \) be the labeling of the points \( \ell_+ \) by boxes. Further let \( \lambda \) be the box labeling of all points in \( \ell \).
By pushing a copy of the interior of $\gamma$ into $D^2 \times I$ and by writing the identity in $H^\text{SU}(N)(\ell, \ell)$ as a sum of minimal central idempotents in analogy with \(^5\) and inserting this sub in this embedded disc in $D^2$, we get an isomorphism

$$H^\text{SU}(N)(\ell, 0) \cong \bigoplus_{\mu \in \Gamma_{N,K}} H^\text{SU}(N)(\ell, \mu) \otimes H^\text{SU}(N)(\mu^+ \otimes \lambda_+, 0)$$

just as it was discussed in section \(^6\) Likewise by the glueing isomorphism combined with the above diffeomorphisms we get an isomorphisms

$$V^\dagger_{N,K}(\Sigma, \lambda) \cong \bigoplus_{\mu \in \Gamma_{N,K}} V^\dagger_{N,K}(\Sigma, \lambda, \mu) \otimes V^\dagger_{N,K}(\Sigma+, \mu^+, \lambda_+).$$

**Theorem 7.8.** We get the following commutative diagram

$$\begin{array}{ccc}
H^\text{SU}(N)(\ell, 0) & \longrightarrow & \bigoplus_{\mu \in \Gamma_{N,K}} H^\text{SU}(N)(\ell, \mu) \otimes H^\text{SU}(N)(\mu^+ \otimes \lambda_+, 0) \\
I_{N,K}(\lambda) \downarrow & & \downarrow \bigoplus_{\mu \in \Gamma_{N,K}} I_{N,K}(\lambda, \mu) \otimes \tilde{I}_{N,K}(\mu^+ \otimes \lambda_+) \\
V^\dagger_{N,K}(\Sigma, \lambda) & \longrightarrow & \bigoplus_{\mu \in \Gamma_{N,K}} V^\dagger_{N,K}(\Sigma, \lambda, \mu) \otimes V^\dagger_{N,K}(\Sigma+, \mu^+, \lambda_+)
\end{array}$$

**Proof.** By the way that the isomorphism for general labels is constructed (Theorem \(^7\)) and the way the isomorphisms $I_{N,K}(\lambda, \mu)$ are constructed, this commutativity is immediate.

\[\square\]

### 7.5. The isomorphism for arbitrary genus zero marked surfaces.

Let

$$\Sigma = (\Sigma, P, V, L)$$

be a marked connected surface of genus zero and $\lambda$ a labeling of it by labels from the $\Gamma_{N,K}$, hence $\lambda : P \rightarrow \Gamma_{N,K}$. Let $\Sigma_0 = (\Sigma_0, P_0, V_0, L_0)$ be standard model surface of the same type as $\Sigma$, say where $p_i, i = 1, \ldots, |P_0|$ is the points of $P_0$ ordered as they appear along the real line. Choose a morphism $\phi$ from $\Sigma_0$ to $\Sigma$. This induces a labeling say $\lambda_0$ of $P_0$ which under $\phi|_{P_0}$ matches up with $\lambda$. Further let

$$\bar{\lambda}_0 = \bigotimes_{i=1}^{|P_0|} \lambda_0(p_i).$$

As was explained in section \(^6\) the modular functor $\mathcal{V}_K^\text{SU}(N)$ assigns the vector space $\mathcal{V}_K^\text{SU}(N)(\Sigma, \lambda)$ represented by $H^\text{SU}(N)(\bar{\lambda}_0, 0)$ at the parametrization $(\Sigma, \phi)$.

**Theorem 7.9.** There is a unique isomorphism

$$I_{N,K}(\Sigma, \lambda) : \mathcal{V}^\dagger_{K}^\text{SU}(N)(\Sigma, \lambda) \rightarrow \mathcal{V}^\dagger_{N,K}(\Sigma, \lambda)$$

which for any parametrization $(\Sigma, \lambda)$ is represented by the isomorphism

$$\mathcal{V}^\dagger_{N,K}(\phi) \circ I_{N,K}(\lambda_0) : H^\text{SU}(N)(\bar{\lambda}_0, 0) \rightarrow \mathcal{V}^\dagger_{N,K}(\Sigma, \lambda).$$

**Proof.** We just need to show that $I_{N,K}(\Sigma, \lambda)$ is well defined. Suppose $(\Sigma, \phi_1)$ and $(\Sigma, \phi_2)$ are two parametrizations. Then $\phi_1^{-1} \circ \phi_2$ is a morphism from $\Sigma_0$ to it self,
hence it can be represented by a tangle $T_{\phi_1^{-1}\circ \phi_2}$ which represents an element in $H_K^{SU(N)}(\bar{\lambda}_0, \tilde{\lambda}_0)$. Now we have the commutative diagram

$$
\begin{align*}
H_K^{SU(N)}(\bar{\lambda}_0, 0) & \xrightarrow{I_{N,K}(\lambda_0)} V^+_{N,K}(\Sigma_0, \lambda_0) \xrightarrow{V^+_{N,K}(\phi_1)} V^+_{N,K}(\Sigma, \lambda) \\
\downarrow T_{\phi_1^{-1}\circ \phi_2} & \downarrow V^+_{N,K}(\phi_1^{-1}\circ \phi_2) \\
H_K^{SU(N)}(\bar{\lambda}_0, 0) & \xrightarrow{I_{N,K}(\lambda_0)} V^+_{N,K}(\Sigma_0, \lambda_0) \xrightarrow{V^+_{N,K}(\phi_2)} V^+_{N,K}(\Sigma, \lambda)
\end{align*}
$$

which shows the induced isomorphism with respect to $\phi_1$ is the same as the one induced from $\phi_2$.

\[ \square \]

We extend the isomorphism $I_{N,K}$ to disconnected surfaces of genus zero, by taking the tensor product of the isomorphisms for each component.

Suppose that $(\tilde{P}_p, \tilde{V}_p)$ is obtained from $\lambda(p)$ by the expansion $E$ introduced in section 4.5 for each $p \in P$. We let

$$
(\tilde{P}, \tilde{V}) = \bigsqcup_{p \in P} (\tilde{P}_p, \tilde{V}_p)
$$

Let $\tilde{\lambda}_p$ be the corresponding labeling of all points in $\tilde{P}_p$ by $\square$ and similarly $\tilde{\lambda}$ assigns $\square$ to all points in $\tilde{P}$. Let $\tilde{\Sigma} = (\Sigma, \tilde{P}, \tilde{V}, L)$ and $\Sigma_p = (\Sigma_p, \tilde{P}_p \cup \{\infty\}, \tilde{V}_p \cup \{v_\infty\}, L)$, where $v_\infty$ is the direction of the negative real axis at infinity.

**Theorem 7.10.** We have the following commutative diagram

$$
\begin{align*}
V^{	ext{SU}(N)}_{K}(\Sigma, \tilde{\lambda}) & \longrightarrow \bigoplus_{p \in P \rightarrow \Gamma_{N,K}} V^{	ext{SU}(N)}_{K}(\Sigma, \mu) \otimes \otimes_{p \in P} V^{	ext{SU}(N)}_{K}(\Sigma_p, \tilde{\lambda}_p, \mu(p)) \\
I_{N,K}(\Sigma, \lambda) & \downarrow \bigoplus_{p \in P \rightarrow \Gamma_{N,K}} I_{N,K}(\Sigma, \mu) \otimes \otimes_{p \in P} I_{N,K}(\Sigma_p, \lambda_p, \mu(p)) \\
V^+_{N,K}(\Sigma, \tilde{\lambda}) & \longrightarrow \bigoplus_{p \in P \rightarrow \Gamma_{N,K}} V^+_{N,K}(\Sigma, \mu) \otimes \otimes_{p \in P} V^+_{N,K}(\Sigma_p, \tilde{\lambda}_p, \mu(p))
\end{align*}
$$

where the horizontal arrows are the factorization isomorphisms.

**Proof.** The proof of this theorem is completely parallel to the proof of Theorem 7.7. We just have to observe that if one follows through Turaev’s definition of the glueing isomorphism of two disjoint genus zero marked surfaces, as defined on page 271 – 272 in [45] (see also section 6), which is the inverse of the factorization isomorphism, then one see that the factorization isomorphism is the one induced from the top row in the commutative diagram in Theorem 7.8. But then commutativity of the diagram in Theorem 7.8 implies the commutativity of the diagram in Theorem 7.10.

\[ \square \]

### 7.6. Compatibility with gluing in genus zero

Let $\Sigma = \Sigma^+ \sqcup \Sigma^-$, where

$$
\Sigma^\pm = (\Sigma^\pm, \{p^\pm\} \sqcup P^\pm, \{v^\pm\} \sqcup V^\pm, L)
$$

are marked surfaces of genus zero. Let $\lambda^\pm : P^\pm \rightarrow \Gamma_{N,K}$ be a labeling of $P^\pm$. Further let $\Sigma_c$ be the glueing of $\Sigma$ with respect to $(p^\pm, v^\pm)$ and $\lambda_c$ the corresponding labeling of $P = P^- \sqcup P^+$. 


**Theorem 7.11.** We have the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{V}_K^{SU(N)}(\Sigma, \lambda) & \longrightarrow & \bigoplus_{\mu \in \Gamma} \mathcal{V}_K^{SU(N)}(\Sigma^+, \mu, \lambda^+) \\
I_{N,K}(\Sigma, \lambda) & \downarrow & \bigoplus_{\mu \in \Gamma} I_{N,K}(\Sigma^+, \mu, \lambda^+ \lambda) \\
\mathcal{V}_N^{+, \mu} K^{SU(N)}(\Sigma, \lambda) & \longrightarrow & \bigoplus_{\mu \in \Gamma} \mathcal{V}_N^{+, \mu} K^{SU(N)}(\Sigma^+, \mu, \lambda^+) \\
\end{array}
\]

where the horizontal arrows are the factorization isomorphisms.

**Proof.** This Theorem follows by first applying the expansion \(E\) to all the involved labeled marked surfaces. One tracks back to the definition of the vector space associated to the resulting marked surface for the modular functor \(\mathcal{V}_K^{SU(N)}\). This identifies this vector space with the hom-space of the appropriate object to the zero object of the category \(H_K^{SU(N)}\). Now one applies Theorem 12 to obtain the factorization on the extended marked curve. But then one can apply Theorem 7.10 to get the conclusion of this theorem.

\[\square\]

8. THE S-MATRICES AND THE HIGHER GENUS ISOMORPHISM

First we recall the main theorem from [15].

**Theorem 8.1.** Suppose \(\mathcal{V}_i, i = 1, 2\) are modular functors and we have isomorphisms

\[I(\Sigma, \lambda) : \mathcal{V}_1(\Sigma, \lambda) \rightarrow \mathcal{V}_2(\Sigma, \lambda)\]

for all genus zero labeled marked surfaces \((\Sigma, \lambda)\), which is compatible with disjoint union and gluing within genus zero labeled marked surfaces, then there exists a unique extension of \(I\) to all labeled marked surfaces of all genus, which gives a full isomorphism

\[I : \mathcal{V}_1 \rightarrow \mathcal{V}_2\]

of modular functors.

We recall that this Theorem is proved by showing that the two S-matrices of the two theorems \(\mathcal{V}_i, i = 1, 2\) agree. Once we have this it is clear that there is a unique isomorphism \(I(\Sigma, \lambda)\), which is compatible with obtaining \(\Sigma\) as the gluing of trinions, up to morphism of labeled marked surfaces.

Since we have already in the previous sections established that \(I_{N,K}\) is an isomorphism of genus zero modular functors, our main Theorem 1.1 follows now directly from Theorem 8.1.

9. APPENDIX

**Basic notations and normalizations for the Lie algebra sl(N,C)**

In this section we recall notation and basic facts on the representation of \(g = sl(N,C)\). The Lie algebra \(g = sl(N,C)\) consists of the traceless \(N \times N\)-matrices with the usual bracket operation.

Let \(E_{ij}, 1 \leq i, j \leq N\) be an \(N \times N\)-matrix whose \((i, j)\)-component is 1 and other components are 0. We also let \(\varepsilon_i, i = 1, 2, \ldots, N\) be an element of the dual vector space of \(\bigoplus_{i=1}^N CE_{ii}\) defined by

\[\varepsilon_i(E_{jj}) = \delta_{ij}, \quad 1 \leq i, j \leq N.\]
Put $r = N - 1$. Also put $H_i = E_{ii} - E_{i+1,i+1}$, $i = 1, 2, \ldots, r = N - 1$, $\mathfrak{h} = \bigoplus_{i=1}^r \mathfrak{CH}_i$.

Then, $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. The dual space $\mathfrak{h}^*$ is written as

$$\mathfrak{h}^* = \{ \sum_{i=1}^N n_i \epsilon_i \mid \sum_{i=1}^N n_i = 0 \}.$$ 

Denote by $\Delta$ the root system of $(\mathfrak{g}, \mathfrak{h})$. A root $\gamma$ of $(\mathfrak{g}, \mathfrak{h})$ has the form $\alpha_{ij} = \epsilon_i - \epsilon_j$, $i \neq j$, $1 \leq i, j \leq N$. Choose $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $i = 1, 2, \ldots, r$ as simple positive roots.

For a root $\alpha_{ij} = \alpha_i - \alpha_j$, the root space $\mathfrak{g}_{\alpha_{ij}} = \{ X \in \mathfrak{g} \mid \text{ad}(H)X = \alpha_{ij}(H)X, \forall H \in \mathfrak{h} \}$ is spanned by a matrix $E_{ij}$. Hence, the root space decomposition $\mathfrak{g}$ is given by

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{i \neq j, i,j=1}^r \mathfrak{CE}_{ij}.$$

Let $(\quad, \quad)$ be a constant multiple of the Cartan-Killing form of $\mathfrak{g}$ defined by

$$(X, Y) = \text{Tr}(XY), \quad X,Y \in \mathfrak{g}.$$ 

Since the bilinear form $(\quad, \quad)$ is positive definite, in the following we identify $\mathfrak{h}^*$ with $\mathfrak{h}$ by the bilinear form $(\quad, \quad)$. For any root $\gamma$ define $H_\gamma \in \mathfrak{h}$ in such a way that

$$(\gamma, H) = (H, H_\gamma)$$

holds for any $H \in \mathfrak{h}$. Then for a simple root $\alpha_i$ we have $H_{\alpha_i} = H_i$. The bilinear form $(\quad, \quad)$ on the dual vector space $\mathfrak{h}^*$ is defined by

$$(\alpha, \beta) = (H_\alpha, H_\beta), \quad \alpha, \beta \in \Delta.$$ 

Then, for simple roots $\alpha_i$ we have

(13) \quad $(\alpha_i, \alpha_i) = 2$, \quad $(\alpha_i, \alpha_{i\pm 1}) = -1$, \quad $(\alpha_i, \alpha_j) = 0$ \quad if $|i - j| \geq 2$.

The longest root $\theta$ of $\mathfrak{g}$ is given by

(14) \quad $\theta = \alpha_1 + \alpha_2 + \cdots + \alpha_r = \epsilon_1 - \epsilon_N$ 

and we have $(\theta, \theta) = 2$. Hence our inner product $(\quad, \quad)$ is the normalized Cartan-Killing form.

The fundamental weight $\Lambda_i \in \mathfrak{h}^*$, $i = 1, 2, \ldots, r$ are defined by

(15) \quad $$(\Lambda_i, H_j) = \delta_{ij}, \quad 1 \leq i, j \leq r$$

and they are given by

$$\Lambda_i = \epsilon_1 + \cdots + \epsilon_i - \frac{i}{N} \sum_{j=1}^N \epsilon_j, \quad i = 1, 2, \ldots, r.$$
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