Polysymplectic Reduction and the Moduli Space of Flat Connections

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Abstract

A polysymplectic structure is a vector-valued symplectic form, that is, a closed nondegenerate 2-form with values in a vector space. We first outline the polysymplectic Hamiltonian formalism with coefficients in a vector space $V$, then apply this framework to show that the moduli space $\mathcal{M}(P)$ of flat connections on a principal bundle $P$ over a compact manifold $M$ is a polysymplectic reduction of the space $\mathcal{A}(P)$ of all connections on $P$ by the action of the gauge group $G$ with respect to a natural $\Omega^2(M)/B^2(M)$-valued symplectic structure on $\mathcal{A}(P)$. This extends to the setting of higher-dimensional base spaces $M$ the process by which Atiyah and Bott identify the moduli space of flat connection on a principal bundle over a closed surface $\Sigma$ as the symplectic reduction of the space of all connections.

Along the way, we establish various properties of polysymplectic manifolds. For example, a Darboux-type theorem asserts that every $V$-symplectic manifold $(M, \omega)$ locally symplectically embeds in a standard polysymplectic manifold $\text{Hom}(TQ, V)$. We also show that both the Arnold conjecture and the well-known convexity properties of the classical moment map fail to hold in the polysymplectic setting.

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1 Introduction

Polysymplectic geometry was introduced by Günther [18, 19] to provide a Hamiltonian counter­part to the Lagrangian formalism of classical field theory. A polysymplectic structure on \( M \) is a nondegenerate 2-form \( \omega \in \Omega^2(M, \mathbb{R}^k) \) for some \( k \geq 1 \). By decomposing \( \omega \) as the direct sum \( \oplus_i \omega_i \) of \( k \) closed 2-forms \( \omega_i \in \Omega^2(M) \), this is seen to be equivalent to the earlier \( k \)-symplectic formalism of Awane [4, 5] in which a \( k \)-symplectic structure \((\omega_1, \ldots, \omega_k)\) on \( M \) consists of \( k \) closed 2-forms \( \omega_i \) with \( \cap_i \ker \omega_i = 0 \). Indeed, the terms \( k \)-symplectic and polysymplectic appear interchangeably throughout the literature.

In addition to its applications to classical field theory [16, 27, 29, 34, 35], polysymplectic geometry has also been the subject of intrinsic mathematical interest [7, 10, 12, 37]. In this regard, we note the independent work of Norris [31–33] on the canonical polysymplectic structure of the frame bundle on a smooth manifold, as well as the more recent appearance of \( k \)-symplectic Lie systems [13, 14]. Additionally, steps have been taken to relate polysymplectic geometry to the field of multisymplectic geometry [15], which similarly arises in field-theoretic contexts [11, 25, 30].

Our approach was developed with the aim of furnishing a setting in which a gauge­theoretic observation of Atiyah and Bott, described below, may be generalized to a broader class of manifolds. This has required a greater degree of attention to the space of coefficients \( V \). As such, we define a \( V \)-symplectic structure on \( M \) to be a nondegenerate 2-form \( \omega \in \Omega^2(M, V) \) with values in the vector space \( V \). This terminology will prove useful as the spaces of coefficients \( V \) often arises naturally and without preferred identifications with \( \mathbb{R}^n \).

In Section 2, we outline the local theory of \( V \)-symplectic vector spaces, which furnish the local models for the \( V \)-symplectic manifolds to follow. A \( V \)-symplectic vector space consists of a vector space \( U \) with a nondegenerate alternating bilinear form \( \omega : U \times U \to V \). Most symplectic constructions extend in a straightforward manner to the \( V \)-symplectic setting, though their properties often differ in important ways. For example, one highly consequential distinction between the linear symplectic and polysymplectic formalisms is that, while the double orthogonal \( A^{\omega \omega} \) of a subspace \( A \leq U \) satisfies \( A^{\omega \omega} = A \) in the symplectic setting, we are only guaranteed to have \( A^{\omega \omega} \geq A \) in the polysymplectic context. The polysymplectic orthogonal will be a primary object of study.

We illustrate the local theory with four characteristic examples:

A. The \( \mathbb{R}^N \)-symplectic vector space \((U, \oplus_i \omega_i)\) consisting of an even-dimensional vector space \( U \) and the sum of \( N \) classical symplectic forms \( \omega_i \).

B. The \( g \)-symplectic vector space \((g, [\cdot, \cdot])\) consisting of a centerless Lie algebra \( g \) with its Lie bracket \([\cdot, \cdot]\).

C. The \( \mathbb{R}^3 \)-symplectic vector space \((\mathbb{R}^3, \times)\), where \( \times \) is the cross product.
D. The $V$-symplectic vector space $(U \oplus \text{Hom}(U,V), \omega)$, for any vector spaces $U$ and $V$ of positive dimension, where $\omega(u + \phi, u' + \phi') = \phi'(u) - \phi(u')$ extends the canonical symplectic structure on $T^*Q$.

We will show the last example to be universal in the sense that every $V$-symplectic vector space $(U, \omega)$ naturally polysymplectically embeds in $U \oplus \text{Hom}(U,V)$.

In Section 3 we present the $V$-symplectic counterpart to the Hamiltonian formalism. As with the theory of $V$-symplectic vector spaces, the constructions of the classical Hamiltonian formalism find a natural $V$-valued equivalents, thought often with a greater variability of behavior. For example, when $\dim V \geq 2$, it is no longer the case that every function $f : M \to V$ is Hamiltonian, or that the reduction $(M_0, \omega_0)$ of a $V$-Hamiltonian system $(M, \omega, G, \mu)$ is necessarily a $V$-symplectic manifold, as the reduced 2-form $\omega_0$ may be degenerate. In addition, we will show that the Arnold conjecture and the convexity properties of the classical moment map do not obtain in the polysymplectic context.

We obtain seven examples of $V$-symplectic manifolds:

A. The $\mathbb{R}^N$-symplectic manifold $(M, \oplus_i \omega_i)$ consisting of an even-dimensional manifold $M$ equipped with the fiberwise sum $\oplus_i \omega_i \in \Omega^2(M, \mathbb{R}^N)$ of a collection of $N$ classical symplectic structures $\omega_i$ on $M$.

B. The $\mathfrak{g}$-symplectic manifold $(G, -d\theta)$ comprising a Lie group $G$ with discrete center $\text{Z}(G)$ with $\mathfrak{g}$-symplectic potential $\theta \in \Omega^1(G, \mathfrak{g})$ the Maurer-Cartan form on $G$.

C. The $\mathbb{R}^3$-symplectic manifold $(Q_B, \omega_B)$ comprising the configuration space $Q_B$ of a rigid body $B$ under rotations about a fixed point in space, with polysymplectic structure $\omega_B$ induced by the principal homogeneous action of $\text{SO}(3)$.

D. The $V$-symplectic manifold $(\text{Hom}(TQ, V), -d\theta)$ for a manifold $Q$ and vector space $V$ of positive dimension, where $\theta$ is the canonical $V$-symplectic potential on $\text{Hom}(TQ, V)$.

E. The $V$-symplectic manifold $(TQ, \omega_L)$ associated to a $V$-mechanical system $(Q, L)$ with configuration space $Q$ and Lagrangian $L : TQ \to V$. Here $\omega_L$ is the pullback of the canonical $V$-symplectic structure above by the fiber derivative $\mathcal{F}L : TQ \to \text{Hom}(TQ, V)$.

F. The $\Omega^2(M)/B^2(M)$-symplectic manifold $(\mathcal{A}(P), \omega)$ comprising the space $\mathcal{A}(P)$ of connections on a principal bundle $P$ over a base space $M$, with a polysymplectic form $\omega$ to be defined.

G. Under suitable conditions, the polysymplectic reduction $(\mathcal{M}(P), \omega_0)$ of $(\mathcal{A}(P), \omega)$ inherits the structure of a $H^2(M)$-symplectic manifold.

The first four are global extensions of the linear examples above, while the final two comprise the central topic of this paper. We will show that every $V$-symplectic manifold $(M, \omega)$ locally polysymplectically embeds in $\text{Hom}(TM, V)$. The embedding is global precisely when $\omega$ is exact. The space $\text{Hom}(TM, V)$, occasionally termed the polymomentum phase space, provided the initial motivation for the polysymplectic formalism through its connection with classical field theory [18].
It is interesting to compare our list of examples to Kirillov’s [20] three sources of classical symplectic manifolds:

i. Algebraic submanifolds of the complex projective space \(\mathbb{C}P^N\).

ii. The coadjoint orbits \(\mathcal{O} \subseteq \mathfrak{g}^*\) of a Lie group \(G\).

iii. The momentum phase space \(T^*Q\) of a smooth manifold \(Q\).

The coadjoint orbits \(\mathcal{O}\) and the phase space \(T^*Q\) find \(V\)-symplectic counterparts in the polysymplectic manifolds \((G, -d\theta)\) and \(\text{Hom}(TQ, V)\). Though we do not investigate them here, we note that under suitable conditions the orbits of the coadjoint action \(\text{Ad}^*: G \curvearrowright \text{Hom}(\mathfrak{g}, V)\) possess natural \(V\)-symplectic forms [18,24] which more directly extend the case of the classical coadjoint orbits in \(\mathfrak{g}^*\). If \(G\) is centerless, then \((G, -d\theta)\) is polysymplectomorphic to the orbit through the identity map \(1_\mathfrak{g} \in \text{Hom}(\mathfrak{g}, \mathfrak{g})\). On the other hand, there does not appear to be a natural candidate for a polysymplectic equivalent to \(\mathbb{C}P^N\).

In Section 4 we apply the \(V\)-symplectic framework in the setting of gauge theory. Atiyah and Bott observed [3] that the space of flat connections on a principal bundle over a closed surface is the symplectic reduction of the space of all connections by the action of the gauge group. The aim of this paper is to exploit the polysymplectic formalism to extend this result beyond the surface case.

Let \(M\) be a compact manifold of dimension at least 2, let \(P\) be a principal bundle on \(M\), let \(\mathcal{A}(P)\) be the space of connections on \(P\), and denote by \(B^2(M) = d\Omega^1(M)\) the space of 2-coboundaries on \(M\). There is a natural polysymplectic form \(\omega\) on \(\mathcal{A}(P)\), with values in the infinite-dimensional vector space \(\Omega^2(M)/B^2(M)\), given by

\[
\omega(\alpha, \beta) = [\alpha \wedge \beta]_{\Omega^2/B^2}
\]

for \(\alpha, \beta \in T_A\mathcal{A}(P)\). The operator \(\wedge: T_A\mathcal{A}(P) \times T_A\mathcal{A}(P) \to \Omega^2(M)\) is induced by the natural isomorphism \(\Omega^1(M, \text{ad}P) \cong \mathcal{A}(P)\). Our primary result is the following.

**Theorem 4.3.** Let \(M\) be either

i. a compact manifold of dimension at least 3, possibly with boundary, or

ii. a closed orientable surface.

Fix a \(G\)-principal bundle \(P\) on \(M\) with connected gauge group \(G\) and let \(\mathcal{A}(P)\) be the space of connections on \(P\). The map

\[
\mu: \mathcal{A}(P) \longrightarrow \text{Hom}(\mathfrak{g}, \Omega^2/B^2)
\]

given by

\[
\mu(A)(f) = [F(A) \wedge f]_{\Omega^2/B^2}
\]

where \(F: \mathcal{A}(P) \to \Omega^2(M, \text{ad}P)\) is the curvature, is a moment map for the action of the gauge group \(G\) on \(\mathcal{A}(P)\) with respect to the polysymplectic structure \(\omega \in \Omega^2(\mathcal{A}(P), \Omega^2(M)/B^2(M))\), defined by

\[
\omega(\alpha, \beta) = [\alpha \wedge \beta]_{\Omega^2/B^2}
\]
for $\alpha, \beta \in \Omega^1(M, \text{ad}P) \cong T_A A(P)$. The reduced space $A(P)_0$ is the moduli space of flat connection $\mathcal{M}(P) = F^{-1}(0)/G$ on $P$, and the reduced form $\omega_0$ takes values in the finite-dimensional vector space $H^2(M)$, that is, $\omega_0 \in \Omega^2(\mathcal{M}(P), H^2(M))$.

As a final application, we apply the polysymplectic reduction procedure to a class of degenerate closed $\Omega^2(M)/B^2(M)$-valued 2-forms arising from Chern-Weil theory. It is interesting to note that in the surface case the moduli space $\mathcal{M}(P)$ arises in the context of Jones-Witten topological quantum field theory [1, 38]. Though we do not do so here, it would be interesting to investigate similar connections with the material in this paper.

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2 The Local Theory

In this section, we introduce the key entity of $V$-symplectic geometry: the $V$-symplectic vector space. Our treatment begins with the basic definitions and culminates in a linear $V$-symplectic reduction theorem.

Throughout this exposition $U$ and $V$ will denote real vector spaces of differing roles. The space $U$ will represent the underlying space on which a vector-valued form $\omega$ is defined, while $V$ represents the space of coefficients. This notation is consistent with that of the following sections, where we consider manifolds modeled on $U$ and vector-valued forms with coefficients in $V$.

2.1 $V$-Symplectic Vector Spaces

We begin with the fundamental construction of this section.

**Definition 2.1.** Let $U$ and $V$ be vector spaces. A *$V$-symplectic structure* $\omega : U \times U \to V$ on $U$ is a $V$-valued alternating bilinear form which is nondegenerate in the sense that $\iota_u \omega = 0$ for $u \in U$ only if $u = 0$. We call the pair $(U, \omega)$ a *$V$-symplectic vector space*.

Thus, a polysymplectic vector space $(U, \omega)$ is a $V$-symplectic vector space for some $V$. As in the symplectic case, there is a correspondence $\omega \mapsto \iota \omega$ between the $V$-symplectic structures on $U$ and the injective linear maps from $U$ to $\text{Hom}(U, V)$.

**Example 2.2.** A. Every classical symplectic vector space is an $\mathbb{R}$-symplectic vector space.

More generally, for a family $(\omega_i)_{i \leq N}$ of symplectic structures on the even-dimensional vector space $U$, we define the $\mathbb{R}^N$-symplectic form $\bigoplus_i \omega_i : U \times U \to \mathbb{R}^N$ by

$$\bigoplus_i \omega_i(u, u') = \bigoplus_i [\omega_i(u, u')]$$
B. Recall that the *center* $\mathfrak{z}$ of a Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ is the ideal
\[
\mathfrak{z} = \{ \xi \in \mathfrak{g} \mid \text{ad}_\xi = 0 \}
\]
and that $\mathfrak{g}$ is said to be *centerless* if $\mathfrak{z} = 0$. For such a Lie algebra $\mathfrak{g}$, the bracket $[\cdot, \cdot]$ is nondegenerate and thus constitutes a $\mathfrak{g}$-symplectic structure on $\mathfrak{g}$. Since the center $\mathfrak{z}$ is an abelian ideal of $\mathfrak{g}$, this class of examples includes every semisimple Lie algebra.

C. As a more concrete instance of part b., corresponding to $\mathfrak{g} = \text{so}(3)$, we consider the cross product, $\times$, as an $\mathbb{R}^3$-symplectic structure on $\mathbb{R}^3$. For nondegeneracy, we note that for any $X \in \mathbb{R}^3 \backslash \{0\}$ and any orthogonal $Y \in \mathbb{R}^3 \backslash \{0\}$, we have $\|X \times Y\| = \|X\| \cdot \|Y\| > 0$.

D. For vector spaces $U$ and $V$, of strictly positive dimension, the assignment
\[
\omega(u + \phi, u' + \phi') = \phi'(u) - \phi(u')
\]
defines a $V$-symplectic structure on $U \oplus \text{Hom}(U, V)$. To see that $\omega$ is nondegenerate, let $u + \phi \in U \oplus \text{Hom}(U, V)$ be any nonzero element, choose $u' \in U$ and $\phi' \in \text{Hom}(U, V)$ so that precisely one of $\phi(u')$ and $\phi'(u)$ is nonzero, and observe that
\[
\omega(u + \phi, u' + \phi') = \phi'(u) - \phi(u') \neq 0
\]
As a notational convenience, we will identify $U$ and $\text{Hom}(U, V)$ with their images in $U \oplus \text{Hom}(U, V)$.

**Definition 2.3.** Let $(U, \omega)$ and $(U', \omega')$ be $V$ and $V'$-symplectic vector spaces, respectively. A *weak morphism* of polysymplectic vector spaces,
\[
f : (U, \omega) \to (U', \omega')
\]
consists of a pair of linear maps
\[
f_0 : U \to U'
f_1 : V \to V'
\]
such that $f_0^* \omega' = f_1 \circ \omega$.

We distinguish two classes of weak morphisms,

i. If $f_1 = 1_{V'}$ then we call $f$ a *morphism* of $V$-symplectic vector spaces, and we identify $f$ with $f_0 : U \to U'$.

ii. If $f_0 = 1_U$ then we call $f$ a *morphism of coefficients*, and we identify $f$ with $f_1 : V \to V'$. If $f_1 : V \to V'$ is injective (resp. surjective) then we say that $f$ is an *extension* (resp. *reduction*) of coefficients.

**Example 2.4.** A. Let $(U, \omega)$ and $(U', \omega')$ be classical symplectic vector spaces. The space of classical linear symplectic maps from $(U, \omega)$ to $(U', \omega')$ coincides with the space of morphisms of $\mathbb{R}$-symplectic vector spaces from $(U, \omega)$ to $(U', \omega')$. A map $f : (U, \omega) \to (U', \omega')$ is a weak morphism precisely when $f^* \omega' = \lambda \omega$ for some $\lambda \in \mathbb{R}$.

The classical symplectic vector space $(U, \omega_i)$ is obtained by reducing the coefficients of $(U, \oplus_i \omega_i)$ from $\mathbb{R}^N$ to $\mathbb{R}$. The map $f : (M, \oplus_i \omega_i) \to (M', \oplus_i \omega_i')$ is a morphism if and only if it is a classical symplectic map from $(M, \omega_i)$ to $(M', \omega_i')$ for each $i \leq N$. 

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B. Every Lie algebra morphism \( f : \mathfrak{g} \rightarrow \mathfrak{h} \) is a weak morphism with \( f_0 = f_1 = f \).

C. Every rotation about the origin is a weak automorphism of \((\mathbb{R}^3, \times)\). The space of automorphisms is trivial.

D. Every linear automorphism \( \bar{f} : U \rightarrow U \) extends to a polysymplectic automorphism

\[
f : U \oplus \text{Hom}(U, V) \rightarrow U \oplus \text{Hom}(U, V)
\]

\[
u + \phi \mapsto \bar{f}u + \bar{f}_*\phi
\]

In particular, the \( V \)-symplectic structure on \( U \oplus \text{Hom}(U, V) \) is invariant under the induced action of \( \text{Aut} U \).

**Proposition 2.5.** If \((U, \bar{\omega})\) is a \( V \)-symplectic vector space, then the map

\[
i : U \hookrightarrow U \oplus \text{Hom}(U, V)
\]

\[
u \mapsto u - \frac{1}{2}\iota_u \bar{\omega}
\]

is an inclusion of \( V \)-symplectic vector spaces. That is, the graph of \(-\frac{1}{2}\iota \omega : U \rightarrow \text{Hom}(U, V)\) is isomorphic to \((U, \omega)\)

**Proof.** Denote by \( \omega \) the canonical \( V \)-symplectic form on \( U \oplus \text{Hom}(U, V) \). For any \( u, u' \in U \), a direct computation yields

\[
2i^*\omega(u, u') = \omega(u - \iota_u \bar{\omega}, u' - \iota_{u'} \bar{\omega})
\]

\[
= -\bar{\omega}(u', u) + \bar{\omega}(u, u')
\]

\[
= 2\bar{\omega}(u, u')
\]

The result follows as the injectivity of \( i \) is clear.

We will show in Theorem 3.6 that this is the local manifestation of a global phenomenon. Looking ahead to Section 4, we consider the following infinite-dimensional example.

**Example 2.6.** Let \( \Sigma \) be a closed orientable surface. In Subsection 4.1 we consider the following three \( V \)-symplectic structures on the vector space \( \Omega^1(\Sigma) \),

\[
\begin{array}{c|c}
\omega(\alpha, \beta) & \in V \\
\hline
\alpha \wedge \beta & \Omega^2(\Sigma) \\
\alpha \wedge \beta + B^2(\Sigma) & \Omega^2(\Sigma)/B^2(\Sigma) \\
f_\Sigma \alpha \wedge \beta & \mathbb{R}
\end{array}
\]

The spaces \((\Omega^1(\Sigma), \omega)\) are related as follows.

\[
\begin{array}{c}
\Omega^2(\Sigma, \mathbb{R}) \\
|^{\text{extension}} \searrow^{\text{reduction}}
\end{array}
\]

\[
\begin{array}{c}
\Omega^2(\Sigma, \mathbb{R})/B^2(\Sigma, \mathbb{R}) \\
|^{\text{extension}} \searrow^{\text{reduction}}
\end{array}
\]
The first space $\Omega^2(\Sigma)$ is the natural polysymplectic structure on $\Omega^1(\Sigma)$ induced by the wedge product, while the third $\mathbb{R}$ corresponds to the classical symplectic structure defined by Atiyah and Bott [3]. It turns out that it is the intermediate space $\Omega^2(\Sigma)/B^2(\Sigma)$ that will prove most suitable for our purposes.

**Definition 2.7.** A $V$-symplectic vector space $(U, \omega)$ is said to be irreducible if every reduction of coefficients is an isomorphism of $V$-symplectic vector spaces.

**Proposition 2.8.** The $V$-symplectic space $U \oplus \text{Hom}(U, V)$ is irreducible.

**Proof.** Let $f : V \to V'$ be a linear map with $\dim V' < \dim V$, let $\phi_v \in \text{Hom}(U, V)$ denote the function with constant value $v \in \ker f$, and observe that

$$(f \circ \omega)(u + \phi, \phi_v) = f[\phi_v(u) - \phi(0)] = f(v) = 0$$

for all $u + \phi \in U \oplus \text{Hom}(U, V)$. Thus, $\phi_v \in \ker f\omega$ and we deduce that $f\omega$ is not a polysymplectic form. Therefore, $f$ is not a reduction of coefficients. \hfill \Box

### 2.2 The Polysymplectic Orthogonal

**Definition 2.9.** Let $A$ be a subspace of the $V$-symplectic vector space $(U, \omega)$. The polysymplectic orthogonal of $A$ in $U$ is the subspace

$$A^\omega = \{ v \in U | \omega(A, v) = 0 \}$$

We now collect various properties of the polysymplectic orthogonal that will prove useful in the development of the theory.

**Lemma 2.10.** Let $(U, \omega)$ be a $V$-symplectic vector space, with subspaces $A, A_i, B, B_i \leq U$ ($i \leq N$). We have

i. $U^\omega = 0$ and $0^\omega = U$.

ii. If $A \leq B$, then $A^\omega \geq B^\omega$.

iii. $A \leq A^\omega$.

iv. $A^\omega = A^{\omega\omega}$.

v. $\bigcap_i A_i^\omega = (\bigcup_i A_i)^\omega$.

vi. $\bigcup_i A_i^\omega \leq (\bigcap_i A_i)^\omega$.

**Proof.** (i) and (ii) are immediate.

iii. We have

$$u \in A \implies \forall u' \in A^\omega : \omega(u, u') = 0 \implies u \in A^{\omega\omega}$$

iv. Apply (ii) to (iii) to obtain $A^\omega \geq (A^{\omega\omega})^\omega$ and note that (iii) alone provides $A^\omega \leq (A^\omega)^{\omega\omega}$.
v. A direct computation yields

\[ u \in (\sum_i A_i)^\omega \iff \forall u' \in \sum_i A_i : \omega(u, u') = 0 \]
\[ \iff \forall i \leq N : \forall u_i \in A_i : \omega(u, u_i) = 0 \]
\[ \iff \forall i \leq N : u \in A_i^\omega \]
\[ \iff u \in \bigcap_i A_i^\omega \]

vi. Applying (ii) to the inclusion \( \bigcap_i A_i \leq A_j \), we deduce that \( A_j^\omega \leq (\bigcap_i A_i)^\omega \) for all \( j \leq N \), and thus \( \sum_j A_j^\omega \leq (\bigcap_i A_i)^\omega \).

Example 2.11.  

A. Let \((U, \oplus; \omega_i)\) be the \( \mathbb{R}^N \)-symplectic vector space as above. For \( A \leq U \) and \( u \in U \), we have

\[ u \leq A^{\oplus; \omega_i} \iff \forall i \leq N : u \in A_i^\omega \]

from which we conclude

\[ A^{\oplus; \omega_i} = \bigcap_i A_i^\omega \]

B. The polysymplectic orthogonal of a subspace \( a \leq g \) is the centralizer \( c_g(a) = \{ \xi \in g \mid [a, \xi] = 0 \} \) of \( a \) in \( g \).

If \( g \) is semisimple, then there is a decomposition \( g = \bigoplus_{j \leq N} I_j \) where each \( I_j \leq g \) is a simple ideal of \( g \). As ideals are preserved under intersection, \( a \cap I_j \leq I_j \) is an ideal of \( g \). Since \( I_j \) is simple, we have \( a \cap I_j = I_j \) or 0. Consequently, \( a \) is of the form \( \bigoplus_{j \in P} I_j \) for some \( P \subseteq \{1, \ldots, N\} \). It follows that

\[ a^\omega = (\bigoplus_{j \in P} I_j)^\omega = \bigoplus_{j \in P'} I_j \]

where \( P' = \{1, \ldots, N\} \setminus P \). Applying this procedure twice yields \( a^{\omega\omega} = a \). Thus, for semisimple \( g \), we have \( a^{\omega\omega} = a \) if and only if \( a \leq g \) is an ideal.

C. Let \( e_1, e_2, e_3 \) be the standard coordinate basis vectors of the \( \mathbb{R}^3 \)-symplectic vector space \((\mathbb{R}^3, \times)\). Then

\[ (e_1)^\omega = \{ v \in \mathbb{R}^3 \mid v \times e_1 = 0 \} = (e_1) \]

and

\[ (e_1, e_2)^\omega = \{ v \in \mathbb{R}^3 \mid v \times e_1 = v \times e_2 = 0 \} = 0 \]

Thus we have

| \( A \) | \( A^\omega \) |
|-------|--------|
| \( 0 \) | \( \mathbb{R}^3 \) |
| \( \ell \) | \( \ell \) |
| \( w \) | 0 |
| \( \mathbb{R}^3 \) | 0 |
for any 1-dimensional subspace $\ell$ and 2-dimensional subspaces $w$.

D. Let $A \leq U$ and $B \leq \text{Hom}(U, V)$, define the subspace

$$I(A) = \{ \phi \mid \phi(A) = 0 \} \leq \text{Hom}(U, V)$$

and let $B^0 \leq U$ be the annihilator of $B$. By noting that $A^{\omega} = U \oplus I(A)$ and $B^{\omega} = B^0 \oplus \text{Hom}(U, V)$, and invoking Lemma 2.10, we obtain

$$A^{\omega \omega} = (U \oplus I(A))^{\omega} = U \cap I(A)^{\omega}$$

$$= U \cap (I(A)^0 \oplus \text{Hom}(U, V)) = I(A)^0 = A$$

On the other hand, it is not generally true that $B^{\omega \omega} = B$.

Parallel to the classical subspace designations, we apply the following terminology for the subspaces $A$ of a $V$-symplectic vector spaces $(U, \omega)$.

| term            | condition               |
|-----------------|-------------------------|
| isotropic       | $A \leq A^{\omega}$    |
| coisotropic     | $A^{\omega} \leq A$    |
| Lagrangian      | $A^{\omega} = A$       |
| polysymplectic  | $A^{\omega} \cap A = 0$|

Example 2.12.  
A. The subspace $A \leq U$ is polysymplectic with respect to $\oplus_i \omega_i$ if it symplectic with respect to each $\omega_i$. This condition, however, is not necessary.

B. The isotropic subspaces of $(g, [,])$ are the precisely the abelian subalgebras. If $g$ is semisimple, then the Lagrangian subspaces are precisely the Cartan subalgebras.

C. The Lagrangian (resp. coisotropic) subspaces of $(\mathbb{R}, \times)$ are precisely the 1-dimensional (resp. 2 and 3 dimensional) subspaces.

D. The Lagrangian subspaces of $U \oplus \text{Hom}(U, V)$ include $U$ and $\text{Hom}(U, V)$, from which it follows that every subspace $A \subseteq U$ and $B \subseteq \text{Hom}(U, V)$ is isotropic. Lemma 2.10 yields

$$(U \oplus B)^{\omega} = U \cap B^{\omega} = B^0$$

from which we deduce

$$(U \oplus B)^{\omega} \cap (U \oplus B) = B^0 \cap (U \oplus B) = B^0$$

Thus, $U \oplus B$ is polysymplectic if and only if the annihilator $B^0 \leq U$ vanishes.

As in the classical situation, Lagrangian subspaces cannot properly contain each other.

Proposition 2.13. If $A \leq B$ are Lagrangian subspaces of $(U, \omega)$, then $A = B$.

Proof. An application of Lemma 2.10 yields $A = A^{\omega} \geq B^{\omega} = B$. \qed
2.3 Reduction of V-Symplectic Vector Spaces

**Theorem 2.14.** If $A$ is a subspace of the V-symplectic vector space $(U, \omega)$, then $\omega$ descends to a bilinear form $\omega_A$ on the quotient $A^\omega/(A \cap A^\omega)$ with kernel $(A^{\omega\omega} \cap A^\omega)/(A \cap A^\omega)$. In particular, $\omega_A$ is polysymplectic if $A^{\omega\omega} = A$.

*Proof.* Let $u, u' \in A^\omega$, put $B = A \cap A^\omega$, and observe that

$$\omega(u + B, u' + B) = \omega(u, u') + \omega(u, B) + \omega(B, u') + \omega(B, B) = \omega(u, u')$$

Thus, $\omega$ descends to a well-defined form on $A^\omega/(A \cap A^\omega)$. If $u \in A^\omega$, then $\omega(u, A \cap A^\omega) = 0$ precisely when $u \in A^{\omega\omega} \cap A^\omega$. It follows that the kernel of the induced form on $A^\omega/(A \cap A^\omega)$ is equal to $(A^{\omega\omega} \cap A^\omega)/(A \cap A^\omega)$. □

The following corollary is immediate.

**Corollary 2.15.** If $A$ is an isotropic subspace of $(U, \omega)$, then $\omega$ descends to a bilinear form $\omega_A$ on $A^\omega/A$ with kernel $A^{\omega\omega}/A$. In particular, $\omega_A$ is polysymplectic if and only if $A^{\omega\omega} = A$.

We call $(A^\omega/(A \cap A^\omega), \omega_A)$ the reduction of $(U, \omega)$ by $A \leq U$, with reduced space $A^\omega/(A \cap A^\omega)$ and reduced form $\omega_A$. In contrast to the classical case, the reduced form $\omega_A$ may be degenerate.

**Example 2.16.**

A. The reduction of $(U, \oplus_i \omega_i)$ by the subspace $A \leq U$ is the intersection $\cap_i A^{\omega_i}/(A \cap A^{\omega_i})$ of the reductions of $(U, \omega_i)$ by $A$ for each $i \leq N$.

B. The reduction of $(g, [\cdot, \cdot])$ by the subspace $a \leq g$ is the quotient $c_g(a)/\mathfrak{z}(a)$ of the centralizer $c_g(a)$ of $a$ by its center $\mathfrak{z}(a) = a \cap c_g(a)$. If $g$ is semisimple and $a$ is an ideal, then the reduction is polysymplectic.

C. The reduction of $(\mathbb{R}^3, \times)$ by any subspace is a point.

D. The reduction of $U \oplus \text{Hom}(U, V)$ by the subspace $A \leq U$ is the sum $U/A \oplus I(A)$. Since $A^{\omega\omega} = A$, Corollary 2.15 ensures that the reduced form $\omega_A$ is always polysymplectic. Indeed, the reduction is naturally isomorphic to $U/A \oplus \text{Hom}(U/A, V)$.

3 The V-Hamiltonian Formalism

Having developed the theory of V-symplectic vector spaces, we turn our attention now to the global setting. Our aim is to arrive at a theory parallel to the classical Hamiltonian formalism. In particular, we would like to arrive at suitable definitions for the notions of Hamiltonian actions and symplectic reduction in the vector-valued context.

3.1 V-Symplectic Manifolds

The fundamental definition of V-symplectic geometry is as follows.
Definition 3.1. Fix a manifold $M$ and a vector space $V$. A $V$-symplectic structure $\omega \in \Omega^2(M, V)$ on $M$ is a closed 2-form which is nondegenerate in the sense that $\iota_X \omega = 0$ only if $X = 0$. We call the pair $(M, \omega)$ a $V$-symplectic manifold.

A polysymplectic manifold is a $V$-symplectic manifold for some vector space $V$.

If $(M, \omega)$ is exact, that is, if $\omega = -d\theta$ for some $\theta \in \Omega^1(M, V)$, then we call $\theta$ a $V$-symplectic potential for $\omega$.

Example 3.2. A. Let $M$ be a smooth and even-dimensional manifold and suppose that $(\omega_i)_{i \leq N}$ is a collection of symplectic forms on $M$. The map $\oplus_i \omega_i \in \Omega^2(M, \mathbb{R}^N)$, given by

$$\left(\oplus_i \omega_i\right)(X, Y) = \oplus_i [\omega_i(X, Y)]$$

is evidently an $\mathbb{R}^N$-symplectic form on $M$.

B. Let $G$ be a Lie group with discrete center $Z(G)$ and denote by $\theta \in \Omega^1(G, \mathfrak{g})$ Maurer-Cartan form, that is, $\theta_g(X) = (\lambda_g^{-1})_* X \in \mathfrak{g}$ for $g \in G$ and $X \in T_g G$, where $\lambda$ is the left regular representation on $G$. Since the Maurer-Cartan identity asserts that $-d\theta_g(X, Y) = [\theta_g X, \theta_g Y]$ and since the center of $\mathfrak{g}$ is trivial, it follows that $-d\theta \in \Omega^2(G, \mathfrak{g})$ is nondegenerate and thus constitutes a $\mathfrak{g}$-symplectic form on $G$.

C. Let $Q_B$ be the configuration manifold of a rigid body $B$ in ambient 3-space $S$ under rotations about a basepoint $O \in S$. We identify $Q_B$ with the space of pointed orientation-preserving isometries from $(S, O)$ to $(\mathbb{R}^3, 0)$. The natural identification of the infinitesimal rotation $X \in T_q Q_B$ with the angular velocity vector $\theta_q(X) \in \mathbb{R}^3 \cong_q S$ constitutes a polysymplectic potential $\theta \in \Omega^1(Q_B, \mathbb{R}^3)$ for $\omega_B = -d\theta = \theta \times \theta$, where $\times$ is the cross product on $\mathbb{R}^3$.

D. Define the canonical 1-form $\theta \in \Omega^1(\text{Hom}(TQ, V), V)$ by

$$\theta_\phi(X) = \phi(\pi_* X)$$

where $\phi \in \text{Hom}(TQ, V)$, $X \in T_\phi \text{Hom}(TQ, V)$, and $\pi : \text{Hom}(TQ, V) \to Q$ is the projection map. By locally identifying the manifold $Q$ with the vector space $U$ on which it is modeled, it is readily shown that $-d\theta$ induces the standard $V$-symplectic form on the vector space

$$T_\phi \text{Hom}(TQ, V) \cong U \oplus \text{Hom}(U, V)$$

for each $\phi \in \text{Hom}(TQ, V)$. In particular, $-d\theta$ is a $V$-symplectic structure on $\text{Hom}(TQ, V)$.

All of these spaces are regular in the sense that every two points have symplectomorphic neighborhoods. This property is similar to what in multisymplectic geometry is known as flatness [36].
Remark 3.3. If $L : TQ \to V$ is a smooth map with nonvanishing second variation along the fibers of $TQ$, then the fiber derivative $\mathbb{F}L : TQ \to \text{Hom}(TQ, V)$ defines an immersion of $TQ$ in $\text{Hom}(TQ, V)$. Moreover, the canonical $V$-symplectic form $\omega$ on $\text{Hom}(TQ, V)$ pulls back to a $V$-symplectic form $\omega_L = \mathbb{F}L^*\omega$ on the velocity phase space $TQ$. Unlike the classical case, when $\text{dim } V \geq 2$ the immersion $\mathbb{F}L$ is never an embedding and may even have compact image.

There are two natural generalizations of the classical notion of a symplectic map.

Definition 3.4. Let $(M, \omega)$ and $(M', \omega')$ be $V$ and $V'$-symplectic manifolds, respectively. A weak polysymplectic map,

$$f : (M, \omega) \to (M', \omega')$$

consists of a diffeomorphism and a linear transformation

$$f_0 : M \to M'$$
$$f_1 : V \to V'$$

such that $f_0^*\omega' = f_1 \circ \omega$. We call $f$ a polysymplectic map when $V = V'$ and $f_1 = 1_{V'}$, and we call $f$ a morphism of coefficients when $M = M'$ and $f_0 = 1_M$. A morphism of coefficients $f$ is said to be an extension (resp. reduction) of coefficients if $f_1$ is injective (resp. surjective).

When there is no room for confusion, we will frequently identify $f$ with $f_0$ or $f_1$.

Example 3.5. A. The diffeomorphism $f : M \to M'$ is a polysymplectic map from $(M, \oplus_{i \leq N} \omega_i)$ to $(M', \oplus_{i \leq N} \omega'_i)$ if and only if it is a classical symplectic map from $(M, \omega_i)$ to $(M', \omega'_i)$ for each $i \leq N$.

B. If $G$ and $G'$ are Lie groups with discrete centers, and if $f : G \to G'$ is any homomorphism, then $f$ is a weak polysymplectic map from $(G, -d\theta)$ to $(G', -d\theta')$.

Fix $g \in G$ and let $\lambda_g : G \to G$ denote left multiplication by $g$. For any $h \in G$ and $X \in T_h G$, we have

$$\lambda^*_g \theta_{gh}(X) = (\lambda_{(gh)^{-1}})_*(\lambda^*_g)_*X = \theta_h(X)$$

from which we deduce $\lambda^*_g \theta = \theta$. We conclude that $\lambda^*_g d\theta = d\theta$, and thus $\lambda_g$ is a polysymplectomorphism of $(G, -d\theta)$. This establishes the regularity of $(G, -d\theta)$.

C. If $B$ and $B'$ are two rigid bodies in $S$ with basepoints $O$ and $O'$, respectively, then the polysymplectic maps from $Q_B$ to $Q_{B'}$ are precisely the pointed orientation-preserving isometries from $(S, O)$ to $(\mathbb{R}^3, O')$. Additionally, the action of $SO(3)$ on $Q_B$ establishes a weak polysymplectomorphism from $(Q_B, \omega_B)$ to $(SO(3), -d\theta)$ which is natural up to the choice of reference configuration $q_0 \in Q_B$.

D. Any diffeomorphism $\tilde{f} : Q \to Q$ extends to a polysymplectomorphism $f : \text{Hom}(TQ, V) \to \text{Hom}(TQ, V)$. In particular, the $V$-symplectic structure of $\text{Hom}(TQ, V)$ is preserved by the action of $\text{Diff } Q$.

Polysymplectic spaces that are locally isomorphic to $\text{Hom}(TQ, V)$ for some $V$ are referred to in the literature as standard [18,24].
Recall that the classical Darboux theorem asserts that every symplectic manifold \((M, \omega)\) is locally isomorphic to a cotangent bundle \(T^*Q\). In the \(V\)-symplectic setting, we obtain a weaker result.

**Theorem 3.6.** Every \(V\)-symplectic manifold \((M, \bar{\omega})\) locally polysymplectically embeds in \(\text{Hom}(TM, V)\).

**Proof.** Let \(O \subseteq M\) be any open set on which \(\omega\) is exact, choose \(\bar{\theta} \in \Omega^1(O, V)\) so that \(\omega|_O = -d\bar{\theta}\), and observe that the section \(\bar{\theta} : O \to \text{Hom}(TM, V)\)

\[
x \mapsto \bar{\theta}_x
\]

is a smooth embedding. For any \(x \in O\) and \(X \in T_xM\), the image \(\bar{\theta}_xX\) is tangent to \(\text{Hom}(TM, V)\) at \(\bar{\theta}_x\) so that

\[
\theta(\bar{\theta}_xX) = \bar{\theta}_x(\pi_*\bar{\theta}_xX) = \bar{\theta}_x(X)
\]

were we have used the fact that \(\bar{\theta}\) is a section of \(\pi : \text{Hom}(TM, V) \to M\). We conclude that the embedding \(\theta\) is polysymplectic. \(\Box\)

**Remark 3.7.** There is an analogue of the classical symplectic volume on a \(2n\)-dimensional \(V\)-symplectic manifold \((M, \omega)\),

\[
\text{vol } M = \frac{1}{n!} \int_M \omega^n \in V^\otimes n
\]

The designation is purely formal however, as \(\text{vol } M\) does not in general behave like a volume. Consider, for example, a Lie group \(G\) of even dimension \(2n\). The polysymplectic volume of \((G, -d\theta)\) is

\[
\text{vol } G = \frac{1}{n!} \int_G (-d\theta)^n = 0
\]

More generally, for any even-dimensional compact \(V\)-symplectic manifold \((M^{2n}, \omega)\), the closedness of \(\omega\) implies that \(\text{vol } M = 0\) if and only if \(\omega^n \in \Omega^{2n}(M, V^\otimes n)\) is exact. The polysymplectic volume of an odd manifold vanishes by default.

Let us turn briefly to describe certain special classes of submanifolds.

**Definition 3.8.** Fix a \(V\)-symplectic manifold \((M, \omega)\). A smooth submanifold \(N \subseteq M\) is said to be *polysymplectic* (resp. *isotropic*, *coisotropic*, *Lagrangian*) when the subspace \(T_xN \subseteq T_xM\) is polysymplectic (resp. isotropic, coisotropic, Lagrangian) at every point \(x \in N\).

Equivalently, \(N \subseteq M\) is polysymplectic (resp. isotropic) when the restriction of \(\omega\) to \(N\) is a \(V\)-symplectic structure (resp. the zero form) on \(N\).

**Example 3.9.**

A. The submanifold \(N \subseteq M\) is polysymplectic with respect to \(\oplus i \omega_i\) if it symplectic with respect to each \(\omega_i\). However, this is only a sufficient condition.

B. If \(T \leq G\) is a maximal torus, then \(T\) is a Lagrangian submanifold of \((G, -d\theta)\).
C. The Lagrangian (resp. coisotropic) submanifolds of $Q_B$ are precisely the 1-dimensional (resp. 2 and 3-dimensional) submanifolds.

D. The fibers of Hom($TQ, V$) are Lagrangian submanifolds.

**Proposition 3.10.** Suppose that $(M, \omega)$ is a $V$-symplectic manifold and that $\omega' \in \Omega^2(M, V')$ is an extension of coefficients of $\omega \in \Omega^2(M, V)$. If $N \subseteq M$ is polysymplectic (resp. coisotropic) with respect to $\omega$, then $N$ is polysymplectic (resp. coisotropic) with respect to $\omega'$.

**Proof.** Fix $x \in N$. Since $\omega'$ refines $\omega$, it follows that

$$T_x N \omega' \leq T_x N \omega$$

Consequently, if $N$ is polysymplectic with respect to $\omega$, then

$$T_x N \cap T_x N \omega' \leq T_x N \cap T_x N \omega = 0$$

and $N$ is polysymplectic with respect to $\omega'$. If $N$ is coisotropic with respect to $\omega$, then

$$T_x N \omega' \leq T_x N \omega \leq T_x N$$

and thus $N$ is coisotropic with respect to $\omega$. □

**Proposition 3.11.** If $N$ is a Lagrangian submanifold of a $V$-symplectic manifold $(M, \omega)$, then $N$ is not contained in any Lagrangian manifold of strictly greater dimension.

**Proof.** This is an immediate consequence of Proposition 2.13. □

### 3.2 Polysymplectic and Hamiltonian Actions

We continue the parallel development with the classical theory with the introduction of polysymplectic and Hamiltonian actions.

**Definition 3.12.** Let $(M, \omega)$ be a $V$-symplectic manifold. A **polysymplectic action** $\lambda : G \curvearrowright M$ is one which preserves $\omega$, that is, $\lambda(g)^* \omega = \omega$ for all $g \in G$. A **polysymplectic vector field** $X \in \mathfrak{X}(M)$ is one which is induced by a polysymplectic action.

Thus, $X$ is polysymplectic precisely when $\mathcal{L}_X \omega = 0$. As in the classical context, we require a strengthening of this definition.

**Definition 3.13.** Let $(M, \omega)$ be a $V$-symplectic manifold and suppose that $f \in C^\infty(M, V)$ and $X \in \mathfrak{X}(M)$ satisfy

$$-\iota_X \omega = df$$

Then $X$ is called the **Hamiltonian vector field** of $f$, and $f$ is called the **Hamiltonian function** of $X$. We also call $X$ the **polysymplectic gradient** of $f$ and denote it by $s\text{-grad} f$. More generally, $f$ (resp. $X$) is said to be **Hamiltonian** if it possesses a Hamiltonian vector field (resp. Hamiltonian function).
We will denote by \( C^\infty_H(M, V) \) the space of Hamiltonian functions on \((M, \omega)\). Observe that \( C^\infty_H(M, V) \) is a \( C^\infty_H(M, V) \)-symplectic vector space. We note that our Hamiltonian functions are termed *currents* in the Günther’s original paper [18].

In contrast with the classical case, it is not true in general that every function \( f \in C^\infty(M, V) \) is Hamiltonian. This is apparent by observing that the assignment 

\[
\iota_\omega : T_xM \to T_x^*M \otimes V, \quad x \in M
\]

\[
X \mapsto \iota_X \omega_x
\]

is never an isomorphism when \( \dim V \geq 2 \). A function \( f \in C^\infty(M, V) \) is Hamiltonian if and only if \( df_x \) lies in the image of \( \iota_\omega : T_xM \hookrightarrow \text{Hom}(T_xM, V) \) at every point \( x \in M \). The vector field \( X \in \mathfrak{X}(M) \) is Hamiltonian when \((\iota_\omega)^{-1}X \in \Omega^1(M, V)\) is exact.

**Example 3.14.**  
A. The function \( f = (f_i) \in C^\infty(M, \mathbb{R}^N) \) is Hamiltonian with respect to \( \oplus_1 \omega_i \), with Hamiltonian vector field \( X \in \mathfrak{X}(M) \), if and only if \( X \) is the Hamiltonian vector field of \( f_i \) with respect to \( \omega_i \) for each \( i \leq N \).

B. Fix \( \xi \in \mathfrak{g} \) and let \( \bar{\xi} \in \mathfrak{X}(G) \) be the *right* invariant extension of \( \xi \) to \( \mathfrak{X}(G) \). Since the integral flow of \( \xi \) is realized by left multiplication by \( \exp(t\xi) \), and since we have shown left multiplication to preserve \( \theta \), it follows that \( \mathcal{L}_\xi \theta = 0 \). Therefore,

\[
-i_{\bar{\xi}} \omega = i_{\bar{\xi}} d\theta = -d\theta(\bar{\xi})
\]

Now, for every \( g \in G \),

\[
\theta_g(\bar{\xi}) = (\lambda_{g^{-1}} \rho_g)_* \xi = \text{Ad}^{-1}_g \xi
\]

and thus the function

\[
\text{Ad}^{-1}_g : G \longrightarrow \mathfrak{g}
\]

\[
g \mapsto \text{Ad}^{-1}_g \xi
\]

is Hamiltonian, with associated Hamiltonian vector field \( -\bar{\xi} \).

C. The steady rotation of \( B \) about a fixed axis \( \ell \subseteq S \) induces a Hamiltonian vector field on \( Q_B \) with Hamiltonian function the angular velocity of \( B \) in the frame \( q \in Q_B \). We defer to Example 3.20 for further details.

D. The Hamiltonian functions on \( \text{Hom}(TQ, V) \) include the lifts \( \pi^* f : \text{Hom}(TQ, V) \) of smooth functions \( f : Q \to V \), with associated Hamiltonian vector fields precisely the vertical vector fields on the bundle \( \text{Hom}(TQ, V) \) over \( Q \).

**Definition 3.15.** The *Poisson bracket*

\[
\{ , \} : C^\infty_H(M, V) \times C^\infty_H(M, V) \to C^\infty_H(M, V)
\]

is defined on the space of Hamiltonian functions \( C^\infty_H(M, V) \) by

\[
\{ f, f' \} = -\omega(X_f, X_{f'})
\]

for \( f, f' \in C^\infty_H(M) \).
The Poisson bracket is a Lie bracket on $C^\infty_H(M, V)$, with respect to which the polysymplectic gradient map is a Lie algebra anti-homomorphism.

**Definition 3.16.** Let $\lambda$ be a polysymplectic action of a Lie group $G$ on the $V$-symplectic manifold $(M, \omega)$. A *weak comoment map* is any linear map

$$\tilde{\mu} : \mathfrak{g} \to C^\infty_H(M, V)$$

that lifts the fundamental vector fields of $\lambda$ to the space of Hamiltonian functions $C^\infty_H(M, V)$, as indicated in the following diagram.

$$\begin{array}{ccc}
C^\infty_H(M, V) & \xrightarrow{\text{s-grad}} & \mathfrak{g} \\
\tilde{\mu} \downarrow & & \downarrow \lambda_* \\
\mathfrak{g} & \xrightarrow{\lambda_*} & \mathfrak{X}(M)
\end{array}$$

If $\tilde{\mu}$ is additionally a morphism of Lie algebras, then it is called a *comoment map*. The (weak) moment map associated to a (weak) comoment map $\tilde{\mu}$ is the smooth function

$$\mu : M \to \text{Hom}(\mathfrak{g}, V)$$

given by

$$\mu(x)(\xi) = \tilde{\mu}(\xi)(x)$$

for $x \in M$ and $\xi \in \mathfrak{g}$. When the action of $G$ admits a moment map $\mu$, the action is said to be *Hamiltonian* and the quadruple $(M, \omega, G, \mu)$ is called a *$V$-valued Hamiltonian system* or a *$V$-Hamiltonian system*.

**Proposition 3.17.** If the action of a Lie group $G$ on an exact $V$-symplectic manifold $(M, -d\theta)$ preserves the polysymplectic potential $\theta \in \Omega^1(M, G)$, then the function $\mu_\theta : M \to \text{Hom}(\mathfrak{g}, V)$ given by

$$\mu_\theta(x)(\xi) = \theta_x(\xi)$$

for $x \in M$ and $\xi \in \mathfrak{g}$ is a moment map.

**Proof.** Since $G$ preserves $\theta$, we have

$$-\iota_\xi(-d\theta) = \mathcal{L}_\xi \theta - d\iota_\xi \theta = d[-\theta(\xi)]$$

for all $\xi \in \mathfrak{g}$, and it follows that $-\theta(\xi) \in C^\infty(M, V)$ is a Hamiltonian function for the vector field $\xi \in \mathfrak{X}(G)$. Since

$$\theta([\xi, \eta]) = -\mathcal{L}_{\xi \eta} \theta = -\iota_{\xi \eta} d\theta$$

$$= \iota_{\xi \eta} \theta = -d\theta(\xi, \eta) = \{\theta(\xi), \theta(\eta)\}$$

we deduce that the assignment $\xi \mapsto \iota_\xi \theta$ is a comoment map. \qed
We catalog the foregoing constructions beside their classical counterparts in the table below.

|                      | classical          | V-valued            |
|----------------------|--------------------|---------------------|
| symplectic form      | $\omega \in \Omega^2(M,\mathbb{R})$ | $\omega \in \Omega^2(M,V)$ |
| Hamiltonian function | $f \in C^\infty(M)$ | $f \in C^\infty(M,V)$ | $\omega(\cdot, X_f) = df$ |
| comoment map         | $\bar{\mu} : \mathfrak{g} \to C^\infty(M)$ | $\bar{\mu} : \mathfrak{g} \to C^\infty(M,V)$ |
| moment map           | $\mu : M \to \mathfrak{g}^*$ | $\mu : M \to \text{Hom}(\mathfrak{g}, V)$ |

**Definition 3.18.** The coadjoint action $\text{Ad}^* : G \curvearrowright \text{Hom}(\mathfrak{g}, V)$ is given by

$$(\text{Ad}^*_g \alpha)(\xi) = \alpha(\text{Ad}_g^{-1}\xi)$$

for $g \in G$, $\alpha \in \text{Hom}(\mathfrak{g}, V)$, and $\xi \in \mathfrak{g}$.

**Lemma 3.19.** Let $(M,\omega,G,\mu)$ be a $V$-Hamiltonian system.

i. If $\alpha \in \text{Hom}(\mathfrak{g}, V)$, then the assignment

$$\mu + \alpha : M \to \text{Hom}(\mathfrak{g}, V)$$

$$x \mapsto \mu(x) + \alpha$$

is a weak moment map. In particular, the set of weak moment maps compatible with $(M,\omega,G)$ is a $\text{Hom}(\mathfrak{g}, V)$-affine space.

If additionally $\alpha$ vanishes on commutators $[\xi,\eta] \in \mathfrak{g}$ ($\xi,\eta \in \mathfrak{g}$) then $\alpha$ is a moment map. Consequently, the set of moment maps is a $[\mathfrak{g},\mathfrak{g}]^0$-affine space, where $[\mathfrak{g},\mathfrak{g}]^0$ denotes the annihilator of $[\mathfrak{g},\mathfrak{g}] \leq \mathfrak{g}$ in $\text{Hom}(\mathfrak{g}, V)$.

ii. If $G$ is connected, then the map $\mu : M \to \text{Hom}(\mathfrak{g}, V)$ is a moment map for the action of $G$ on $(M,\omega)$ precisely when

$$\langle \mu_* X, \xi \rangle = \omega(X, \xi_x)$$

for all $x \in M$, $X \in T_xM$, and $\xi \in \mathfrak{g}$. Here $\langle \cdot, \cdot \rangle : \text{Hom}(\mathfrak{g}, V) \times \mathfrak{g} \to V$ denotes the natural pairing and $\xi_x$ is the value of the action-induced vector field for $\xi$ at $x$.

iii. If $G$ is connected, then $\mu$ intertwines the action of $G$ on $M$ with the coadjoint action of $G$ on $\text{Hom}(\mathfrak{g}, V)$.

iv. If $H \leq G$ is a Lie subgroup, and if the map $\mu|_h$ is given by

$$\mu|_h : M \to \text{Hom}(\mathfrak{h}, V)$$

$$x \mapsto \mu(x)|_h$$

then $(M,\omega,H,\mu|_h)$ is a $V$-Hamiltonian system.

The proof of these assertions are so similar to their classical analogues that we omit the proofs and refer instead to the corresponding symplectic literature [9, 28].
Example 3.20. A. The action of $G$ on $(M, \oplus_i \omega_i)$ is Hamiltonian if and only if it is Hamiltonian with respect to each $\omega_i$ for $i \leq N$. In this case, a moment map is given by $\oplus_i \mu_i : M \rightarrow (\mathfrak{g}^*)^N \cong \text{Hom}(\mathfrak{g}, \mathbb{R}^N)$.

B. Since the fundamental vector fields of the left regular representation of $G$ are the right invariant vector fields on $G$, Proposition 3.17 implies that the map

$$\mu : G \rightarrow \text{End} \mathfrak{g}$$

given by

$$\mu(g)(\xi) = \theta_g(\xi_g) = (\lambda_g^{-1})_* (\rho_g)_* \xi = \text{Ad} g^{-1} \xi$$

is a moment map for the left regular representation of $G$. Here we denote by $\lambda$ and $\rho$ the left and right regular representations, respectively.

C. Let $\ell \subseteq S$ be a line containing the basepoint $O$, and consider the action of the circle $T$ on $Q_B$ corresponding to the rotation of $B$ about $\ell$. This action is Hamiltonian with moment map $\mu : Q_B \rightarrow \text{Hom}(t, \mathbb{R}^3)$ given by the angular velocity $\mu(q)(\xi) = \theta_q(\xi_q) = q(\xi_q) \in \mathbb{R}^3$.

D. The induced action on $\text{Hom}(TQ, V)$ of a subgroup $G \leq \text{Aut} Q$ is Hamiltonian with canonical moment map $\mu : \text{Hom}(TQ, V) \rightarrow V$ given by $\mu(\phi)(\xi) = \theta_\phi(\xi_\phi) = \phi(\xi_q)$ where $q = \pi \phi \in Q$.

We recall the Arnold conjecture for compact classical symplectic manifolds $(M, \omega)$.

**Conjecture 1** (Arnold [26]). A symplectomorphism that is generated by a time-dependent Hamiltonian vector field should have at least as many fixed points as a Morse function on the manifold must have critical points.

From Example 3.20 we deduce the following.

**Theorem 3.21.** The Arnold conjecture fails in the $V$-symplectic setting.

*Proof.* Consider the $\mathfrak{g}$-symplectic manifold $(G, -d\theta)$, where $G$ is a compact semisimple Lie group and $\theta \in \Omega^1(G, \mathfrak{g})$ is the Maurer-Cartan form. Let $T \leq G$ be a 1-torus, let $\xi \in \mathfrak{g}$ be a generator of $T$ with $\exp(\xi) = 1$, and let $\bar{\xi} \in \mathfrak{X}(G)$ be the right invariant extension of $\xi$. The 1-periodic family of polysymplectomorphisms

$$\phi_t : G \rightarrow G$$
$$g \mapsto e^{t\bar{\xi}} g$$

is generated by the Hamiltonian vector fields $t\bar{\xi}$. When $e^{t\bar{\xi}} \neq 1$ the transformation $\phi_t$ is fixed-point free. However, any nondegenerate function on the compact space $G$ has at least two critical points. \hfill $\square$

**Remark 3.22.** Let $(M, \omega)$ be a $V$-symplectic manifold and let $(\text{Symp} M, -d\theta)$ be the group of symplectomorphisms of $(M, \omega)$. Then

$$-d\theta(X, Y) = [X, Y]$$
for $X, Y \in \mathcal{X}(M) \cong T_1\text{Symp} M$, where $[,]$ is the Lie bracket on $\mathcal{X}(M)$. We may also equip the space $\text{Symp} M$ with the $\text{Symp} M$-invariant $C^\infty(M, V)$-symplectic form $\tilde{\omega}$, defined on $T_1\text{Symp} M$ by $\omega(X, Y)$. However, $(\text{Symp} M, \tilde{\omega})$ is not generally a reduction of coefficients of $(\text{Symp} M, -d\theta)$ since the assignment

$$\mathcal{X}(M) \to C^\infty(M, V)$$

$$[X, Y] \mapsto \omega(X, Y)$$

is not well-defined. The polysymplectic actions of $G$ on $(M, \omega)$ are in correspondence with the weak polysymplectic maps from $(G, -d\theta)$ to $(\text{Symp} M, \tilde{\omega})$.

### 3.3 $V$-Hamiltonian Reduction

We are ready to present the fundamental theorem of $V$-Hamiltonian systems.

**Theorem 3.23** (Vector-Valued Hamiltonian Reduction). Let $(M, \omega, G, \mu)$ be a $V$-Hamiltonian systems and fix $\alpha \in \text{Hom}(\mathfrak{g}, V)$. If the stabilizer subgroup $G_\alpha$ of $\alpha$ under the coadjoint action is connected, and if $M_\alpha = \mu^{-1}(\alpha)/G_\alpha$ is smooth, then there is a unique $V$-valued 2-form $\omega_\alpha \in \Omega^2(M_\alpha, V)$ such that

$$\pi^* \omega_\alpha = i^* \omega$$

where $i : \mu^{-1}(\alpha) \hookrightarrow M$ is the inclusion and $\pi : \mu^{-1}(\alpha) \to M_\alpha$ is the projection. The form $\omega_\alpha$ is closed and is nondegenerate at $\pi x$ if and only if $\mathfrak{g}_\alpha \cap \mathfrak{g}_x$.

**Proof.** First note that the equivariance of $\mu$ ensures that the action of $G_\alpha$ preserves $\mu^{-1}(\alpha)$, and thus that the quotient $\mu^{-1}(\alpha)/G_\alpha$ exists as a topological space.

Fix $x \in \mu^{-1}(\alpha)$. Lemma 3.19 implies that

$$X \in \mathfrak{g}_x^\omega \iff \omega(X, \mathfrak{g}_x) = \langle \mu_* X, \mathfrak{g} \rangle = 0 \iff \mu_* X = 0$$

for all $X \in T_x M$, so that

$$\mathfrak{g}_x^\omega = T_x \mu^{-1}(\alpha)$$

Therefore, Theorem 2.14 implies that $\omega_x$ descends to a bilinear form on

$$T_x \mu^{-1}(\alpha)/\mathfrak{g}_x \cong \mathfrak{g}_x^\omega/(\mathfrak{g}_x \cap \mathfrak{g}_x^\omega)$$

with kernel $(\mathfrak{g}_x^\omega \cap \mathfrak{g}_x)/\mathfrak{g}_x$.

Since $T_{\pi x} M_\alpha \cong T_{T_x \mu^{-1}(\alpha)}/\mathfrak{g}_{\pi x}$, we obtain a 2-form $\omega_\alpha \in \Omega^2(M_\alpha, V)$ with $\pi^* \omega_\alpha = i^* \omega$.

As $\pi$ is surjective, the induced map $\pi^*$ is injective and $\omega_\alpha$ is unique. Closedness follows by the injectivity of $\pi^*$ and the equality $\pi^* d\omega_\alpha = d\pi^* \omega_\alpha = 0$. \hfill $\square$

We call $(M_\alpha, \omega_\alpha)$ the reduction of $(M, \omega, G, \mu)$ at level $\alpha$, with reduced space $M_\alpha$ and reduced 2-form $\omega_\alpha$.

**Remark 3.24.** When $\alpha = 0$, the distribution $\mathfrak{g}_0 = \mathfrak{g}$ is isotropic along $\mu^{-1}(0)$ and the condition for the nondegeneracy of $\omega_0$ at $\pi x \in M_0$ becomes $\mathfrak{g}_x = \mathfrak{g}_x^\omega$. 

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**Remark 3.25.** There is another approach to reduction, which is equivalent to ours in the classical case, but which diverges for more general coefficients $V$. Given a $V$-Hamiltonian system $(M, \omega, G, \mu)$ and a level $\alpha \in \text{Hom}(\mathfrak{g}, V)$, the restriction $i^*\omega$ of $\omega$ to $\mu^{-1}(\alpha)$ is a closed form. Now, the kernel distribution of any closed form $\sigma \in \Omega^*(M, V)$ is integrable, as $X, Y \in \ker \sigma$ implies that

$$i_{[X,Y]}\sigma = (\mathcal{L}_X\iota_Y - \iota_Y\mathcal{L}_X)\sigma = (\mathcal{L}_X\iota_Y - \iota_Y\iota_X d - \iota_Y d\iota_X)\sigma = 0$$

When it is smooth, the leaf space $\tilde{M}_\alpha = \mu^{-1}(\alpha)/\ker i^*\omega$ naturally inherits a $V$-symplectic structure $(\tilde{M}_\alpha, \tilde{\omega}_\alpha)$, where $\tilde{\omega}_\alpha$ is the unique 2-form on $\tilde{M}_\alpha$ satisfying $\tilde{\pi}^*\tilde{\omega}_\alpha = i^*\omega$. From the proof of Theorem 3.23, it is apparent that $\tilde{M}_\alpha$ is a quotient of the reduced space $M_\alpha$. When the reduced 2-form $\omega_\alpha$ is polysymplectic, as is always the case in the classical symplectic setting, the spaces $(\tilde{M}_\alpha, \tilde{\omega}_\alpha)$ and $(M_\alpha, \omega_\alpha)$ coincide.

We refer to [24] for further details.

**Example 3.26.** A. The reduced space of the $\mathbb{R}^N$-Hamiltonian system $(M, \oplus_i \omega_i, G, \oplus_i \mu_i)$ is the intersection of the reduction of each $(M, \omega_i, G, \mu_i)$. That is,

$$M_\alpha = \left( \bigcap_i \mu_i^{-1}(\alpha) \right)/G_\alpha = \bigcap_i \left( \mu_i^{-1}(\alpha)/G_\alpha \right)$$

B. Let $G$ be a Lie group with discrete center, let $\theta \in \Omega^1(G, \mathfrak{g})$ be the Maurer-Cartan form, and $H \leq G$ be a connected Lie subgroup of $G$. Since the left regular action of $H$ on $G$ is Hamiltonian with moment map $\mu = \text{Ad}^{-1}|_\mathfrak{h} : G \to \text{Hom}(\mathfrak{h}, \mathfrak{g})$. Since the adjoint representation acts by automorphisms, the preimage of $0 \in \text{Hom}(\mathfrak{h}, \mathfrak{g})$ under $\mu$ is empty, and thus the reduced space $G_0$ is empty as well.

Let us compute the reduction of $(G, -d\theta)$ at the inclusion $i : \mathfrak{h} \hookrightarrow \mathfrak{g}$. We have

$$g \in \mu^{-1}(i) \iff \forall \xi \in \mathfrak{h} : \text{Ad}_{g^{-1}} \xi = \xi \iff g \in C_G(H)$$

where $C_G(H)$ is the centralizer of $H$ in $G$, and where the second equivalence follows as $H$ is connected. Denote by $H_i \leq H$ the stabilizer subgroup of $i$ under the coadjoint action of $H$ on $\text{Hom}(\mathfrak{h}, \mathfrak{g})$. It follows that

$$h \in H_i \iff \forall \xi \in \mathfrak{h} : \text{Ad}_h \xi = \xi \iff h \in Z(H)$$

where $Z(H) \leq H$ is center of $H$. We conclude that the reduced space is

$$G_i = \mu^{-1}(i)/H_i = C_G(H)/Z(H)$$

Since $Z(H)$ is a central subgroup of $C_G(H)$, it follows that

$$G_i = C_G(H)/Z(H) = C_G(H)/(H \cap C_G(H))$$

as a normal quotient of groups. Let $\bar{\theta}_i \in \Omega^1(G_i, \mathfrak{g}(\mathfrak{h})/\mathfrak{z}(\mathfrak{h}))$ denote the Maurer-Cartan form on $G_i$. When $G_i$ has discrete center, $-d\bar{\theta}_i$ is a $\mathfrak{c}_\mathfrak{g}(\mathfrak{h})/\mathfrak{z}(\mathfrak{h})$-symplectic form, and is the image of a reduced potential $\theta_i \in \Omega^1(G_i, \mathfrak{g})$, i.e. $\omega_i = -d\bar{\theta}_i$, under a reduction of coefficients $f : \mathfrak{g} \to \mathfrak{c}_\mathfrak{g}(\mathfrak{h})/\mathfrak{z}(\mathfrak{h})$. In particular, the reduction $(G_i, \omega_i)$ is polysymplectic when $C_G(H)/Z(H)$ has discrete center.
C. Using the fact that \((Q_B, \omega_B)\) and \((\text{SO}(3), -d\theta)\) are weakly symplectomorphic, it follows that the reduction of \((Q_B, \omega_B, T, \mu)\) at any level \(\alpha \in \text{Hom}(t, \mathbb{R}^3)\) is either empty or a point.

D. The reduction of \(\text{Hom}(TQ, V)\) by a subgroup \(G \leq \text{Diff} Q\) at level \(0 \in \text{Hom}(g, V)\) is naturally isomorphic to \(\text{Hom}(T(Q/G), V)\). In particular, the reduction is polysymplectic.

**Remark 3.27.** In contrast with the classical situation \([2, 17, 21]\), Example 3.26.B. shows that the image of a moment map \(\mu: M \to \text{Hom}(t, V)\) for the Hamiltonian action of a torus \(T \leq G\) is not necessarily convex.

**Proposition 3.28.** If \((M, \omega, G, \mu)\) is a \(V\)-Hamiltonian system, and if the reduced 2-form \(\omega_0\) vanishes on \(M_0\), then the regular part of \(\mu^{-1}(0)\) is a Lagrangian submanifold of \(M_0\).

**Proof.** Let \(x \in \mu^{-1}(0)\). Since \(\omega_x\) descends to zero on \(T_x\mu^{-1}(0)\), it follows that \(\omega_x\) vanishes on \(T_x\mu^{-1}(0)\), from which \(T_x\mu^{-1}(0) \leq T_x\mu^{-1}(0)\omega\). Taking the polysymplectic orthogonal of both sides of the inclusion

\[
\mathfrak{g}_x \leq \mathfrak{g}_x^\omega = T_x\mu^{-1}(0)
\]

yields

\[
T_x\mu^{-1}(0) = \mathfrak{g}_x^\omega \geq \mathfrak{g}_x^{\omega\omega} = T_x\mu^{-1}(0)^{\omega}
\]

Thus, \(T_x\mu^{-1}(0) = T_x\mu^{-1}(0)^{\omega}\). \(\square\)

We complete this chapter with a result on the topology of the reduced space.

**Theorem 3.29.** Let \((M, \omega, G, \mu)\) be a \(V\)-Hamiltonian system with compact Lie group \(G\), and suppose that \(\alpha \in \text{Hom}(g, V)\) is a regular value of the moment map \(\mu: M \to \text{Hom}(g, V)\). Then the reduced space \(M_\alpha\) has at most orbifold singularities.

**Proof.** Since \(\alpha\) is a regular value, \(\mu^{-1}(\alpha) \subseteq M\) is a smooth manifold. Fix \(x \in \mu^{-1}(\alpha)\) and let \(G_x \leq G\) be the stabilizer subgroup of \(x\), with Lie algebra \(\mathfrak{g}_x\). Since \(\alpha\) is a regular value of \(\mu\) it follows that \(\mu_x T_x M = \text{Hom}(g, V)\) and thus

\[
\langle \text{Hom}(g, V), \mathfrak{g}_x \rangle = \omega(\mu_x T_x M, \mathfrak{g}_x) = 0
\]

Consequently, \(\mathfrak{g}_x = 0\) and \(G_x\) is discrete. We conclude that the stabilizer subgroup \((G_\alpha)_x \leq G_x\) is discrete as well and the quotient \(\mu^{-1}(\alpha)/G_\alpha\) has at most orbifold singularities. \(\square\)

We note that in the finite-dimensional classical symplectic situation the converse is also true: the level \(\alpha \in \text{Hom}(g, V)\) is a regular value of \(\mu\) if and only if \(M_\alpha\) possesses an orbifold structure.

4 Gauge Theory in Higher Dimensions

Atiyah and Bott observed that, for a compact connected group \(G\) with \(\text{Ad}\)-invariant metric \(\langle \cdot, \cdot \rangle_g\) on the Lie algebra \(g\), the moduli space of flat connections on a \(G\)-principal bundle \(P\) over a surface \(\Sigma\) is obtained as the symplectic reduction of the space of all connections
\( A(P) \) by the action of the gauge group \( G = \Gamma(\Sigma, \text{Ad}P) \) \[3\]. More precisely, there is a natural symplectic structure on \( A(P) \) given by
\[
\omega_A(\alpha, \beta) = \int_\Sigma \alpha \wedge \beta
\]
where \( A \in A(P), \alpha, \beta \in \Omega^1(\Sigma, \text{ad}P) \cong T_A A(P), \text{ad}P = P \times_{\text{Ad}} g \) is the adjoint bundle of \( P \), and the wedge product is defined to be the composition
\[
\wedge : \Omega^1(\Sigma, \text{ad}P) \times \Omega^1(\Sigma, \text{ad}P) \longrightarrow \Omega^2(M, \text{ad}P \otimes \text{ad}P) \xrightarrow{\langle \cdot , \cdot \rangle_{\text{ad}P}} \Omega^2(\Sigma)
\]
Here the metric \( \langle \cdot , \cdot \rangle_{\text{ad}P} \) on \( \text{ad}P \) is induced by the \( \text{Ad} \)-invariant metric on \( g \). Writing \( g \) for the Lie algebra of \( G \), a moment map \( \mu : A(P) \rightarrow g^* \) for the induced action of \( G \) on \( A(P) \) is given by
\[
\mu(A)(f) = \int_\Sigma F_A \wedge f
\]
where \( F_A \in \Omega^2(\Sigma, \text{ad}P) \) is the curvature of \( A \in A(P) \), and \( f \in \Omega^0(\Sigma, \text{ad}P) \cong g \). They reduction of the Hamiltonian system \((A(P), \omega, G, \mu)\) at the level \( 0 \in g^* \) is the moduli space \( M(P) \) of flat connections on \( P \).

The main result of this paper is that there is a similar polysymplectic characterization of the moduli space of flat connections over a higher dimensional manifold \( M \). For clarity of exposition, we first examine the relatively simple case given by the first cohomology \( \Omega^1(M) \) of \( M \), which we will equip with a \( \Omega^2(M) \)-symplectic structure. We then proceed to establish the main result. We complete this section with an application of the polysymplectic reduction procedure to a family of degenerate 2-forms relating to Chern-Weil.

### 4.1 The Model Space \((\Omega^1(M), \wedge)\)

Observe that the vector space \( \Omega^1(\Sigma) \) carries a natural classical symplectic structure: namely,
\[
\omega(\alpha, \beta) = \int_\Sigma \alpha \wedge \beta, \quad \alpha, \beta \in \Omega^1(\Sigma)
\]
Our present aim is to adapt this symplectic form to the case in which \( \dim M \geq 3 \).

The most natural polysymplectic structure on \( \Omega^1(M) \) is the wedge product \( \wedge \). The following proposition establishes that \( \wedge \) is indeed a \( \Omega^2(M) \)-symplectic form on \( \Omega^1(M) \).

**Proposition 4.1.** Let \( M \) be a manifold of dimension at least 2, possibly with boundary. The wedge product
\[
\wedge : \Omega^1(M) \times \Omega^1(M) \longrightarrow \Omega^2(M)
\]
is an \( \Omega^2(M) \)-symplectic structure on the vector space \( \Omega^1(M) \).

**Proof.** As \( \wedge \) is clearly a skew-symmetric \( \Omega^2(M) \)-valued form on \( \Omega^1(M) \), we have only to show that it is nondegenerate. Thus let \( \alpha \in \Omega^1(M) \) and suppose that \( \alpha \wedge \beta = 0 \) for all \( \beta \in \Omega^1(M) \). Let \( (x^i)_{i \leq n} \) \((n = \dim M)\) be a system of coordinates on a neighborhood \( U \subseteq M \), and let \( \alpha_i \in C^\infty(U) \) be given by
\[
\alpha = \sum_i \alpha_i \, dx^i
\]
For each $k \leq n$,
\[ 0 = \alpha \wedge dx^k = \sum_i \alpha_i dx^i \wedge dx^k. \]

Since $n \geq 2$, for each $i \leq n$ there is a $k \leq n$ with $k \neq i$, and thus $dx^i \wedge dx^k \neq 0$ so that $\alpha_i = 0$. Since our choice of $U$ was arbitrary, we conclude that $\alpha = 0$. \hfill \Box

It turns out that this polysymplectic structure is too fine for our purposes. By this we mean that action of $C^\infty(M)$ on $\Omega^1(M)$ given by
\[ f \cdot \alpha = df + \alpha \]

is not in general Hamiltonian with respect to polysymplectic structure $\omega$ obtained by lifting $\wedge$ to the fibers of $T\Omega^1(M)$. The issue is resolved by reducing the space of coefficients from $\Omega^2(M)$ to $\Omega^2(M)/B^2(M)$, where $B^2(M) = d\Omega^1(M)$ is the space of 2-coboundaries on $M$.

To show this, we first establish a technical lemma.

**Lemma 4.2.**

i. Let $U$ be a vector space with $\dim U \geq 3$ and let $w \in \Lambda^2 U$. If $u \wedge w = 0$ for all $u \in U$ then $w = 0$.

ii. Let $M$ be a manifold with $\dim M \geq 3$. If $\theta \in \Omega^2(M)$ satisfies $d(f\theta) = 0$ for all $f \in C^\infty(M)$, then $\theta = 0$.

**Proof.**

i. Fix a basis $\{e_i\}_{i \leq n}$ of $U$ and choose coefficients $w^{ij} \in \mathbb{R}$ so that
\[ w = \sum_{i,j \leq n} w^{ij} e_i \wedge e_j. \]

For each $k \leq n$, we have
\[ 0 = e_k \wedge \omega = \sum_{i,j \leq n} w^{ij} e_k \wedge e_i \wedge e_j. \]

Since $n \geq 3$, for every pair of distinct $i, j \leq n$ we can find a $k \leq n$ with $k \neq i, j$. Consequently, $e_k \wedge e_i \wedge e_j \neq 0$ and thus $w^{ij} = 0$.

ii. As $d(1 \cdot \theta) = 0$,
\[ df \wedge \theta = d(f\theta) = 0 \]

for all $f \in C^\infty(M)$. Fix $p \in M$ and observe that $\alpha \wedge \theta_p = 0 \in \Lambda^3(T^*_p M)$ for all $\alpha = df_p \in T^*_p M$. Now part i. yields $\theta_p = 0$. \hfill \Box

**Proposition 4.3.** Let $M$ be a compact manifold of dimension at least 3, possibly with boundary. The assignment
\[ \omega : \Omega^1(M) \times \Omega^1(M) \rightarrow \Omega^2(M)/B^2(M) \]

defined by
\[ \omega(\alpha, \beta) = \alpha \wedge \beta + B^2(M), \quad \alpha, \beta \in \Omega^1(M) \]

is an $\Omega^2(M)/B^2(M)$-symplectic structure on $\Omega^1(M)$ if and only if
i. the dimension of $M$ is at least 3, or

ii. $M$ is a closed orientable surface.

Proof. The cases in which $\dim M = 0$ or 1 are clear. Suppose first that $\dim M \geq 3$, let $\alpha \in \Omega^1(M)$, and assume that $\alpha \wedge \gamma \in B^2(M)$ for all $\gamma \in \Omega^1(M)$. Let $\beta \in \Omega^1(M)$ and observe that

$$d(\alpha \wedge f\beta) = d[f(\alpha \wedge \beta)] \in B^2(M)$$

for all $f \in C^\infty(M)$. Thus Lemma 4.2 implies that $\alpha \wedge \beta = 0$. Since our choice of $\beta$ was arbitrary, the nondegeneracy of the wedge product on $\Omega^1(M)$ yields $\alpha = 0$.

Finally, suppose that $M = \Sigma$ is a closed compact orientable surface and equip $\Sigma$ with an orientation and a Riemannian structure $g$. Let $*: \Omega^1(\Sigma) \to \Omega^1(\Sigma)$ denote the Hodge star operator and let $\alpha \in \Omega^1(\Sigma)$ be arbitrary. If

$$\alpha \wedge \beta \in B^2(\Sigma)$$

for all $\beta \in B^2$, then, in particular,

$$\|\alpha\|^2 \, d\text{vol} = \alpha \wedge *\alpha \in B^2(\Sigma)$$

from which

$$\int_{\Sigma} \|\alpha\|^2 \, d\text{vol} = 0$$

so that $\alpha = 0$, as required.

If $M = \Sigma$ is a connected surface which is nonorientable, noncompact, or has nonempty boundary, then the space of coefficients $\Omega^2(\Sigma)/B^2(\Sigma) = 0$ is trivial and $\omega$ is the zero form. The case for disconnected $\Sigma$ is similar. \qed

4.2 Cohomology as $\Omega^2(M)/B^2(M)$-Symplectic Reduction

Let $M$ be a smooth manifold with boundary, let $\omega$ be the $\Omega^2(M)/B^2(M)$-valued 2-form on $\Omega^1(M)$ be given by

$$\omega_A(\alpha, \beta) = \alpha \wedge \beta + B^2(M), \quad A \in \Omega^1(M), \ \alpha, \beta \in \Omega^1(M) \cong TA\Omega^1(M)$$

and let $C^\infty(M)$ act on $\Omega^1(M)$ by

$$f \cdot \alpha = df + \alpha$$

The aim of this section is to show that the polysymplectic reduction of $(\Omega^1(M), \omega)$ is the first cohomology $H^1(M)$ with the wedge product $\wedge_{H^1}$.

Proposition 4.4. The space $(\Omega^1(M), \omega)$ is an $\Omega^2(M)/B^2(M)$-symplectic manifold.

Proof. The form $\omega$ is closed as it is constant on $\Omega^2(M)$. For every $A \in \Omega^1(M)$, Proposition 4.3 ensures that the restriction of $\omega$ to the fiber $\Omega^1(M) \cong TA\Omega^1(M)$ of the tangent bundle $T\Omega^1(M)$ is nondegenerate. \qed
Theorem 4.5. The action of $C^\infty(M)$ on $\Omega^1(M)$, given by $f \cdot \alpha = df + \alpha$, is Hamiltonian with respect to $\omega$. A moment map

$$
\mu : \Omega^1(M) \longrightarrow \text{Hom}(C^\infty(M), \Omega^2(M)/B^2(M))
$$

is given by

$$
\mu(A)(f) = dA \land f + B^2(M)
$$

The reduced space is $(H^2(M), \omega_0)$ where, for each cohomology class $\bar{A} = A + B^1(M) \in H^1(M)$, the 2-form

$$
\omega_0 : T_{\bar{A}}H^2(M) \times T_{\bar{A}}H^2(M) \longrightarrow H^2(M) \cong \Omega^2(M)/B^2(M)
$$

is given by

$$
\omega_0(\bar{\alpha}, \bar{\beta}) = \bar{\alpha} \land \bar{\beta}, \quad \bar{\alpha}, \bar{\beta} \in T_{\bar{A}}H^2(M) \cong H^2(M)
$$

Proof. For $\alpha \in \Omega^1(M)$, $f \in C^\infty(M)$, the equality

$$
d(\alpha \land f) = d\alpha \land f - \alpha \land df
$$

implies that

$$
[d\alpha \land f]_{\Omega^2/B^2} = [\alpha \land df]_{\Omega^2/B^2}
$$

Since $d : \Omega^1(M) \rightarrow \Omega^2(M)$ is linear, the induced map $d_* : T\Omega^1(M) \rightarrow T\Omega^2(M)$ is given by

$$
d_*\alpha_A = (d\alpha)_{dA} \in T_{dA}\Omega^2(M)
$$

for every $A \in \Omega^1(M)$ and $\alpha \in \Omega^1(M) \cong T_A\Omega^1(M)$. Thus,

$$
\langle \mu_*\alpha_A, f \rangle = d\alpha \land f + B^2(M)
$$

$$
= \alpha \land df + B^2(M)
$$

$$
= \omega(\alpha_A, f_{A})
$$

and it follows that $\mu$ is a moment map for the action of $C^\infty(M)$ on $\Omega^1(M)$. Since

$$
\mu(A) = 0 \iff dA \land C^\infty(M) + B^2 = 0 \iff dA = 0
$$

we conclude that the reduced space is $\mu^{-1}(0)/C^\infty(M) = Z^2(M)/B^2(M) = H^2(M)$. \qed

4.3 The Reduction of the Space of Connections

Let us briefly review the language of principal bundles and curvature. We refer to [22] for further details.

Let $M$ be a compact connected manifold, possibly with boundary, of dimension at least 2, let $G$ be a Lie group with Ad-invariant metric $\langle \cdot, \cdot \rangle_g$ on the Lie algebra $\mathfrak{g}$, let $P$ be a $G$-principal bundle on $M$ with connected gauge group $\text{Ad}P$, and let $\mathcal{A}(P)$ be the $\Omega^1(M, \text{ad}P)$-affine space of connections on $P$. Explicitly, we identify the gauge group $\text{Ad}P$ with the representation product $P \times_c G$, where $c: G \cong G$ is the action of conjugation, and the Lie algebra $\text{ad}P$ with $P \times_{\text{Ad}} \mathfrak{g}$. 26
At each connection \( A \in \mathcal{A}(P) \), we invoke the identification
\[
\Omega^1(M, \text{ad}P) \xrightarrow{\sim} T_A \mathcal{A}(P)
\]
given by
\[
\alpha_A = \frac{d}{dt} A + t\alpha|_{t=0}, \quad A \in \mathcal{A}(P), \alpha \in \Omega^1(M, \text{ad}P)
\]
where \(+\) denotes the action of \( \Omega^1(M, \text{ad}P) \) on \( \mathcal{A}(P) \). That is, we identify the 1-form \( \alpha \in \Omega^1(M, \text{ad}P) \cong T_0 \Omega^1(M, \text{ad}P) \) with the vector field \( \overline{\alpha} \in \mathfrak{X}(\mathcal{A}(P)) \) induced by the affine action of \( \Omega^1(M, \text{ad}P) \) on \( \mathcal{A}(P) \).

The invariant metric \( \langle \cdot, \cdot \rangle_\text{g} \) induces a metric \( \langle \cdot, \cdot \rangle_{\text{ad}P} \) on the fibers of \( \text{ad}P \), with which we define the wedge product
\[
\wedge : \Omega^1(M, \text{ad}P) \times \Omega^1(M, \text{ad}P) \xrightarrow{\wedge} \Omega^2(M, \text{ad}P \otimes \text{ad}P) \xrightarrow{\langle \cdot, \cdot \rangle_{\text{ad}P}} \Omega^2(M)
\]
The exterior covariant derivative \( d_A : \Omega^k(M, \mathfrak{g}) \rightarrow \Omega^{k+1}(M, \mathfrak{g}) \) is given by
\[
d_A \sigma(X_1, \ldots, X_{k+1}) = d\sigma(h_A X_1, \ldots, h_A X_{k+1})
\]
where \( h_A : TP \rightarrow A \) is the fiberwise horizontal projection induced by the splitting \( A \oplus V(P) \), where \( V(P) \) is the vertical tangent bundle of \( P \). Since \( d_A \) preserves the space of tensorial forms in \( \Omega^k(P, \mathfrak{g}) \) of type \( \text{Ad}P \), we also consider the exterior covariant derivative as a map
\[
d_A : \Omega^k(M, \text{ad}P) \rightarrow \Omega^{k+1}(M, \text{ad}P).
\]

The wedge product and the exterior covariant derivative are related [3] by the property that
\[
d(\alpha \wedge \beta) = d_A\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d_A\beta
\]

Finally, we define the 2-form \( \omega \in \Omega^2(\mathcal{A}(P), \Omega^2(M)/B^2(M)) \) by
\[
\omega_A(\alpha, \beta) = [\alpha \wedge \beta]|_{\Omega^2/B^2} \in \Omega^2(M)/B^2(M)
\]
for \( A \in \mathcal{A}(P) \) and \( \alpha, \beta \in \Omega^2(M, \text{ad}P) \cong T_A \mathcal{A}(P) \).

**Theorem 4.6.** The form \( \omega \) is a \( \Omega^2(M)/B^2(M) \)-symplectic structure on \( \mathcal{A}(P) \) if and only if \( \dim M = 0, \dim M \geq 3, \) or \( M \) is a closed compact orientable surface.

**Proof.** Closedness follows from the fact that \( \omega \) is constant on \( \mathcal{A}(P) \). It remains to prove \( \omega \) is nondegenerate. Again, the cases with \( \dim M = 0, 1 \) are clear.

First suppose \( \dim M \geq 3 \). Fix \( A \in \mathcal{A}(P) \) and assume that \( \alpha \in \Omega^1(M, \text{ad}P) \cong T_A \mathcal{A}(P) \) is nonzero at \( x \in M \). Let \( N \subseteq M \) be a closed ball containing \( x \), so that \( N \) is a submanifold with boundary and
\[
\Omega^1(N, \text{ad}P) \cong \Omega^1(N, \mathfrak{g}) \cong \Omega^1(N)^{\dim \mathfrak{g}}
\]
As \( \alpha \) is nonzero on \( N \), Proposition 4.3 yields a \( \beta \in \Omega^1(N, \text{ad}P) \) with
\[
\alpha|_N \wedge \beta \notin B^2(N)
\]
Since \( N \) is closed, we may smoothly extend \( \beta \) to \( M \), in which case
\[
\alpha \wedge \beta \notin B^2(M)
\]

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and we conclude that \( \omega \) is nondegenerate.

Now suppose \( M = \Sigma \) is a closed compact orientable surface. Equip \( \Sigma \) with an orientation and a Riemannian structure \( g \). Let \( * : \Omega^1(\Sigma, \text{ad} P) \to \Omega^1(\Sigma, \text{ad} P) \) denote the Hodge star operator determined by \( g \) and \( \langle \cdot, \cdot \rangle_{\text{ad} P} \). If \( \alpha \in \Omega^2(\Sigma, \text{ad} P) \cong T_A \mathcal{A}(P) \) satisfies

\[
\alpha \wedge \beta \in B^2(\Sigma)
\]

for all \( \beta \in \Omega^2(\Sigma, \text{ad} P) \cong T_A \mathcal{A}(P) \), then

\[
\int_{\Sigma} \| \alpha \|^2 \, d\text{vol} = \int_{\Sigma} \alpha \wedge * \alpha = 0
\]

and consequently \( \alpha = 0 \). It follows that \( \omega \) is nondegenerate.

Now suppose that \( M = \Sigma \) is a surface which is nonorientable, noncompact, or has nonempty boundary. Thus, \( \Sigma \) possesses at least one connected component \( \Sigma_0 \) with \( H^2(\Sigma_0) = 0 \) and it follows that

\[
\alpha \in \ker \omega \in \Omega^2(\mathcal{A}(P), \Omega^2 / B^2)
\]

for any \( \alpha \in \Omega^1(\Sigma_1, \text{ad} P) \) with support on \( \Sigma_1 \). In particular, \( \ker \omega \) is nonempty.

**Theorem 4.7.** Let \( M \) be either

i. a compact manifold of dimension at least 3, possibly with boundary, or

ii. a closed orientable surface.

Fix a \( G \)-principal bundle \( P \) on \( M \) with connected gauge group \( G \) and let \( \mathcal{A}(P) \) be the space of connections on \( P \). The map

\[
\mu : \mathcal{A}(P) \to \text{Hom}(\mathcal{g}, \Omega^2 / B^2)
\]

given by

\[
\mu(A)(f) = [F(A) \wedge f]_{\Omega^2 / B^2}
\]

where \( F : \mathcal{A}(P) \to \Omega^2(M, \text{ad} P) \) is the curvature, is a moment map for the action of the gauge group \( \mathcal{G}(P) \) on \( \mathcal{A}(P) \) with respect to the polysymplectic structure \( \omega \in \Omega^2(\mathcal{A}(P), \Omega^2(M) / B^2(M)) \), defined by

\[
\omega(\alpha, \beta) = [\alpha \wedge \beta]_{\Omega^2 / B^2}
\]

for \( \alpha, \beta \in \Omega^1(M, \text{ad} P) \cong T_A \mathcal{A}(P) \). The reduced space \( \mathcal{A}(P)_0 \) is the moduli space of flat connection \( \mathcal{M}(P) = F^{-1}(0) / \mathcal{G} \) on \( P \), and the reduced form \( \omega_0 \) takes values in the finite-dimensional vector space \( H^2(M) \), that is, \( \omega_0 \in \Omega^2(\mathcal{M}(P), H^2(M)) \).

**Proof.** Fix \( A \in \mathcal{A}(P) \), \( \alpha \in \Omega^1(M, \text{ad} P) \cong T_A \mathcal{A}(P) \) and \( f \in \Omega^0(M, \text{ad} P) \cong \mathcal{g}(P) \). Since

\[
d(\alpha \wedge \beta) = d_A \alpha \wedge \beta - \alpha \wedge d_A \beta
\]

we have

\[
[d_A \alpha \wedge f]_{\Omega^2 / B^2} = [\alpha \wedge d_A f]_{\Omega^2 / B^2}
\]
As $F \ast \alpha = d_A \alpha$ and $f_A = d_A f$, we obtain

$$\langle \mu \ast \alpha, f \rangle = \omega_A(\alpha, f_A)$$

Therefore, $F$ is a moment map. For any $A \in \mathcal{A}(P)$, we have

$$\mu(A) = 0 \iff F(A) \wedge \Omega^0(M, \text{ad}P) \in B^2(M) \iff F(A) = 0$$

and we conclude that $\mu^{-1}(0)/\mathcal{G} = F^{-1}(0)/\mathcal{G}$.

If $A \in \mu^{-1}(0)$ and $\alpha, \beta \in \Omega^1(M, \text{ad}P) \cong T_A \mathcal{A}$ are tangent to $\mu^{-1}(0)$, then $d_A \alpha = F \ast \alpha = 0$ and so $\alpha \wedge \beta \in Z^2(M)$ is a cocycle. It follows that the reduced form $\omega_0$ on $\mathcal{M}(P)$ takes values in $Z^2(M)/B^2(M) = H^2(M)$.

**Remark 4.8.** Suitable adjustments are in order when the base space $M$ is noncompact. In this case, the construction of $\mathcal{A}(P)$ as Banach manifold encounters complications. This difficulty can be overcome by, for example, replacing $\mathcal{A}(P)$ with the subspace of asymptotically flat connections on $P$. We will not address such considerations in this exposition and, in the following, we will assume $M$ to be compact.

**Corollary 4.9.** If $H^2(M) = 0$, then the regular part of $\mathcal{M}(P)$ is a Lagrangian submanifold of $\mathcal{A}(P)$.

**Proof.** If the second cohomology $H^2(M)$ vanishes, then the reduced form $\omega_0 \in \Omega^2(\mathcal{M}(P), H^2(M))$ is necessarily zero, and Proposition 3.28 implies that $\mu^{-1}(0)$ is a Lagrangian submanifold of $\mathcal{A}(P)$. The result follows as $F^{-1}(0) = \mu^{-1}(0)$. □

**Remark 4.10.** There is a linear multisymplectic form $\omega$ of degree $n$ on the cohomology $\Omega^1(M)$ of an $n$-dimensional manifold $M$ given by

$$\omega(\alpha_1, \ldots, \alpha_n) = \int_M \alpha_1 \wedge \cdots \wedge \alpha_n$$

where the wedge of $n$ forms $\alpha_1, \ldots, \alpha_n \in \Omega^1(M, \text{ad}P)$ is appropriately defined. Given a $G$-principal bundle $P$ as above, we obtain a multisymplectic form of degree $n$ on the space $\mathcal{A}(P)$ of connections on $P$. A derivation similar to that above can be found in [8].

### 4.4 Characteristic Forms of Degree 2 and Ricci Curvature

Let $M$ be a closed manifold with dim $3 \geq 2$, let $G$ be a Lie group, and let $P$ be a $G$-principal bundle on $M$. In this section, we apply the polysymplectic reduction procedure in the context of a degenerate $\Omega^2(M)/B^2(M)$-valued 2-form on the space of connections $\mathcal{A}(P)$. First we recall Lemma II.5.5 of [22].

**Lemma 4.11.** Let $A \in \mathcal{A}(P)$ be a connection, let $\eta \in \Omega^1(P, \mathfrak{g})$ be the connection 1-form for $A$, and let $\alpha \in \Omega^1(P, \mathfrak{g})$ be a tensorial 1-form of type $\text{Ad}G$. Then

$$d_A \alpha(X, Y) = d\alpha(X, Y) + \frac{1}{2} [\alpha(X), \eta(Y)] + \frac{1}{2} [\eta(X), \alpha(Y)]$$

for $X, Y \in T_u P$, $u \in P$.

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We leverage this fact to establish the following result.

**Lemma 4.12.** Suppose that $\alpha \in \Omega^1(M, \text{ad}P)$. If $\phi \in \mathfrak{g}^*$ is invariant under the coadjoint action of $G$, then

i. the assignment

$$\text{ad}P \to \mathbb{R}$$

$$[u, \xi] \mapsto \phi(\xi)$$

is well-defined and fiberwise linear. We denote this assignment, as well as the induced maps $\Omega^k(M, \text{ad}P) \to \Omega^k(M)$, by $\phi$.

ii. $\phi(d\alpha) = d(\phi \alpha)$

iii. $d(\phi \alpha \wedge \phi \beta) = \phi d\alpha \wedge \phi \beta - \phi \alpha \wedge \phi d\beta$

**Proof.**

i. Let $(u, Y)$ and $(u', Y') \in P \times \mathfrak{g}$ represent the same element in $\text{ad}P = P \times \text{Ad} G$.

That is, we suppose there is a $g \in G$ with $u' = ug^{-1}$ and $\xi' = \text{Ad}_g \xi$. Since $\phi$ is $\text{Ad}^*$-invariant, $\phi(\xi) = \phi(\xi')$ and $\phi$ is well defined.

ii. Let $\xi, \eta \in \mathfrak{g}$ be arbitrary and observe that

$$\phi(\xi, \eta) = \frac{d}{dt} \phi(\text{Ad}_{\exp(t\xi)} \eta) \bigr|_{t=0} = 0$$

Thus, by Lemma 4.11 and the linearity of $\phi$, we obtain

$$\phi(d\alpha) = \phi(d\alpha) + \frac{1}{2} \phi[\alpha, \omega] + \frac{1}{2} \phi[\omega, \alpha] = d(\phi \alpha)$$

as required.

iii. By part (ii), we have

$$d(\phi \alpha \wedge \phi \beta) = d\phi \alpha \wedge \phi \beta - \phi \alpha \wedge d\phi \beta$$

$$= \phi d\alpha \wedge \phi \beta - \phi \alpha \wedge \phi d\beta$$

**Remark 4.13.** For any connection $A \in \mathcal{A}(P)$, the image of the curvature $F_A$ under the induced map $\phi : \Omega^2(M, \text{ad}P) \to \Omega^2(M)$ represents in $H^2(M)$ the characteristic class corresponding to the $\text{Ad}$-invariant multilinear map $\phi : \mathfrak{g} \to \mathbb{R}$. That is, $[\phi F_A]_{H^2}$ is the image of $\phi : \mathfrak{g} \to \mathbb{R}$ under the Chern-Weil homomorphism. We will call $\phi F_A$ the characteristic form of $A$ associated to $\phi$.

**Corollary 4.14.** Let $V$ be a real (resp. complex) vector space, $M$ a smooth manifold, $E$ a $V$-vector bundle over $M$ with structure group $\text{GL}(V)$, and $\text{tr} : \mathfrak{gl}(V) \to \mathbb{R}$ (resp. $\text{tr} : \mathfrak{gl}(V) \to \mathbb{C}$) the trace map on $\mathfrak{gl}(V) \cong \text{End}(V)$. We have

$$\text{tr}(d\alpha) = d(\text{tr} \alpha)$$

and

$$d(\text{tr} \alpha \wedge \text{tr} \beta) = \text{tr} d\alpha \wedge \text{tr} \beta - \text{tr} \alpha \wedge \text{tr} d\beta$$
The main result of this section is as follows.

**Theorem 4.15.** Let $M$ be a manifold with $\dim M \geq 3$, $G$ a Lie group with $\dim G \geq 2$, $P$ a $G$-principal bundle on $M$ with connected gauge group $\mathcal{G}$, and suppose that $\phi \in g^*$ is nonzero and $\text{Ad}^*$-invariant. The assignment

$$(\omega_\phi)_A(\alpha, \beta) = [\phi \alpha \wedge \phi \beta]_{\Omega^2/B^2}$$

for $\alpha, \beta \in \Omega^1(M, \text{ad}P) \cong T_A \mathcal{A}(P)$, defines a closed 2-form

$$\omega_\phi \in \Omega^2(\mathcal{A}(P), \Omega^2(M)/B^2(M))$$

on the space $\mathcal{A}(P)$ of connections on $P$, with kernel distribution

$$\ker (\omega_\phi)_A = \ker [\phi : \Omega^1(M, \text{ad}P) \to \Omega^1(M)], \quad A \in \mathcal{A}(P)$$

Moreover, the action of the gauge group $\mathcal{G}$ on $(A, \omega_\phi)$ is Hamiltonian, with moment map

$$\mu_\phi : \mathcal{A}(P) \to \Omega^2(M)/B^2(M)$$

given by

$$\mu_\phi(A)Y = [\phi F(A) \wedge Y]_{\Omega^2/B^2}, \quad X \in \Omega^0(M, \text{ad}P)$$

and the reduced space is

$$\mathcal{A}(P)_0 = (\phi F)^{-1}(0)/\mathcal{G}$$

**Proof.** Closedness follows as $\omega_\phi$ is constant on $\mathcal{A}(P)$.

Fix a connection $A \in \mathcal{A}(P)$ and a tangent vector $\alpha \in \Omega^1(M, \text{ad}P) \cong T_A \mathcal{A}(P)$. If $\alpha \in \ker \phi$, then it immediately follows that $\alpha \in \ker \omega_\phi$. If, on the other hand, $\alpha \in \ker \omega_\phi$, then

$$\phi \alpha \wedge \phi \beta \in B^2(M)$$

for all $\beta \in \Omega^1(M, \text{ad}P) \cong T_A \mathcal{A}(P)$. Since $\phi : \Omega^1(M, \text{ad}P) \to \Omega^1(M)$ is surjective and $\wedge \Omega^1(M)$ is nondegenerate, we deduce that $\phi \alpha = 0$. Thus, $\ker (\omega_\phi)_A = \ker \phi$.

For $f \in \Omega^0(M, \text{ad}P) \cong g$ and $\alpha \in \Omega^1(M, \text{ad}P)$, Lemma 4.12 implies that

$$\langle (\mu_\phi)_* \alpha_A, f \rangle = \phi d_A \alpha \wedge \phi f + B^2(M)$$

$$= \phi \alpha \wedge \phi d_A f + B^2(M)$$

$$= \omega(\alpha_A, f_A)$$

and thus $\mu_A$ is a moment map for the action of $\mathcal{G}$ on $\mathcal{A}(P)$. Finally,

$$\mu_\phi(A) = 0 \iff \forall f \in \Omega^0(M, \text{ad}P) : \phi F(A) \wedge \phi f \in B^2 \iff F(A) \in \ker \phi$$

so that $\mu_\phi^{-1}(0)/\mathcal{G} = (\phi F)^{-1}(0)/\mathcal{G}$. \hfill \Box

Consider a complex manifold $M$ and a holomorphic vector bundle $E$ over $M$. Recall that the first Chern class $c_1(E)$ is represented by the form

$$c_1(A) = \frac{-1}{2\pi i} \text{tr} F_A \in \Omega^2(M)$$
where \( \text{tr} \) denotes the complex trace of \( F_A \in \Omega^2(M, \text{End}_C E) \), and where \( A \) is any connection on the holomorphic frame bundle \( PE \). We will call \( c_1(A) \) the first Chern form of \( A \). If \( A \) is the Chern connection of a Hermitian structure \( h : E \otimes \overline{E} \to \mathbb{C} \), then \( c_1(A) \) is proportional to the Ricci form \( \rho(h) \) of \( h \) [23]. This motivates the following terminology,

**Definition 4.16.** We call the connection \( A \in \mathcal{A}(E) \) **Ricci flat** if \( c_1(A) = 0 \).

The following corollary follows immediately from

**Corollary 4.17.** Let \( M \) be a complex manifold and let \( E \) be a holomorphic vector bundle over \( M \) with \( c_1(E) = 0 \). The moduli space of Ricci flat connections is the polysymplectic reduction of the space of connections \( \mathcal{A}(E) \) equipped with the polysymplectic form \( \omega_{\text{tr}} \) and moment map given by \( A \mapsto \text{tr} F_A \), where \( \text{tr} \) represents the fiberwise complex trace of \( F_A \in \Omega^2(M, \text{End} T M^C) \).

**Remark 4.18.** Consider the map \( f : \text{Met}(E) \to \mathcal{A}(PE) \) from the space of Hermitian structure on \( E \) to the space of connections on \( PE \), which sends a Hermitian structure \( h \) to its Chern connection \( f(h) \). Then \( f \) is equivariant under the action of the gauge group, \( f^* \omega_{\text{tr}} \) is an \( \Omega^2(M)/B^2(M) \)-valued 2-form on \( \text{Met}(E) \), and the polysymplectic reduction of \( (\text{Met}(E), f^* \omega_{\text{tr}}) \) with respect to the moment map \( f^* \mu_{\text{tr}} \) is the moduli space of Ricci flat Hermitian structures on \( E \).

In the case that \( E = TM^C \) is the complexified tangent bundle, then the reduced space is the moduli space of Ricci flat Kähler metrics on \( M \).

**Remark 4.19.** It is significant in the preceding material that \( \text{tr} \) denotes the complex trace. Indeed, the argument cannot be adapted to Riemannian structures as \( \text{tr}_{\mathbb{R}} F_A = 0 \) for any metric connection \( A \).

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