Influence of nonlocal discrete and integral boundary conditions involving Caputo derivative in boundary value problem

P. Duraisamy¹, A.R. Vidhya Kumar², T. Nandha Gopal² and M. Subramanian³*

Department of Mathematics,
¹Gobi Arts and Science College, Gobichettipalayam, Tamilnadu, India
²Sri Ramakrishna Mission Vidyalaya College of Arts and Science, Coimbatore, Tamilnadu, India

*Corresponding author: subramanianmcbe@gmail.com

Abstract. Our primary motive is to provide the readers the idea of existence and uniqueness solutions over a non linear differential equation of fractional order with non local discrete and integral boundary condition. Schaefer fixed point theorem, Leray-Schauder theorem and Arzela-Ascoli theorem are proven to be good for the stated problem. To brace the results appropriate examples are bestowed.

Keywords. Fractional differential equation, Caputo derivative, Nonlocal, Integral boundary conditions, Existence, Fixed point.

1. Introduction

From the past decades, it easy to perceive that for the growth of classical calculus, the contribution of non integer order differential equations along with various boundary conditions is substantial. Besides, certain interpretative results and utilization of fractional calculus have been outlined in their historical context, for instance, see [6,7,9,10,12]. Not long ago, considerable interest has been created in establishing BVPs in a variety of conditions such as classical, nonlocal, multi-point, periodic/anti-periodic, fractional order, and integral boundary conditions has been studied by many researchers, for instance, see [1-5,8,11]. The aspiration of this write-up is applied to the Fractional order derivative of Caputo nature to a differential equation involving non local discrete and integral boundary conditions.

\[ D^\alpha w(\xi) = h(\xi, w(\xi)), \quad \xi \in \mathbb{R} := [0,1], \quad 1 < \alpha \leq 2, \quad 0 < \eta < 1 \]

\[ w(0) = \eta \int_0^1 w(\tau) d\tau, \quad D^\alpha w(1) = \sum_{j=1}^{k-2} \xi_j w(\phi_j), \quad 0 < \xi < 1 \]

where \( D^\alpha, D^\beta \) are the CFD’s, \( h : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is a continuous function, provided \( 0 < \eta < \phi_1 < \phi_2 < \cdot \cdot \cdot < \phi_{k-2} < 1 \), are positive real constants. The subsequent content of the paper is structured as follows: Module 2 is dedicated to some foundational concepts of fractional calculus with basic lemma related to the given problem. The existence and uniqueness results based on by Leray-Schauder degree theory, Nonlinear contractions, Schaefer fixed point theorem and Contraction mapping principle are obtained in Module 3. The validation of the results is done by providing examples in Module 4.
2. Preliminaries

Definition 2.1 The R-L-fractional integral of order $\xi$ is given by

$$\mathcal{I}_t^\xi h(t) = \frac{1}{\Gamma(\xi)} \int_0^t (t - \tau)^{\xi - 1} h(\tau) d\tau, \quad \xi > 0.$$ 

Definition 2.2 For an $(n-1)$-time absolutely continuously function $h : [0, \infty) \to \mathbb{R}$, the CFD of order $\xi$, is described as

$$\mathcal{D}_t^\xi h(t) = \frac{1}{\Gamma(n-\xi)} \int_0^t (t - \tau)^{n-\xi - 1} h^{(n)}(\tau) d\tau, \quad n - 1 < \xi < n, \quad n = \lfloor \xi \rfloor + 1.$$ 

Lemma 2.3 For any $\gamma_1, \gamma_2 \in C(\mathbb{R}, \mathbb{R})$, the following linear FDE

$$\mathcal{D}_t^\xi \varphi(t) = \gamma(t), \quad 1 < \xi \leq 2, \quad \varphi(t) \in \mathbb{R},$$

with the boundary conditions (2), is equivalent to the FIE

$$\varphi(t) = \frac{1}{\Gamma(\xi)} \int_0^t \left[ \frac{1}{\Gamma(2 - \xi)} - \frac{1}{\Gamma(\xi - \xi)} \right] \frac{1}{\Gamma(\xi - \xi)}.$$ 

We come to realize that the system (1)-(2) possess solutions not until the operator equation $w = \mathcal{F}w$ has fixed points.

3. Main Results

Suitable for computation, we represent

$$\Theta = \left( \frac{\phi_2}{\Gamma(\xi - \xi + 1)} + \frac{1}{\Gamma(\xi + 1)} \left( \frac{\phi_1(\gamma \xi + 1)}{\xi + 1} + \phi_2 \sum_{j=1}^{l-2} \epsilon_j \phi_j + 1 \right) \right).$$

In the sequel, we assume that $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function.

(\(g_1\)) There is a constant $\gamma > 0$ such that $|h(3, w_1) - h(3, w_2)| \leq \gamma \|w_1 - w_2\|$, $\forall w_1, w_2 \in \mathbb{R}$.
(6.2) \( \exists \) a constant \( 0 \leq \omega < \frac{1}{2\theta} \), and \( \Theta > 0 \) \( \exists |b(\delta, w)| \leq \omega |w| + \Theta \) \( \forall \delta, w \in \mathbb{R} \times \mathbb{R} \), where \( \Theta \) is described by (6).

(6.3) \( \exists \) a constant \( \theta > 0 \) \( \exists |b(\delta, w)| \leq \omega \), \( \forall \delta \in \mathbb{R} \), \( w \in \mathbb{R} \);

(6.4) \( \exists \) a constant \( \Phi \), \( \exists |b(\delta, u) - b(\delta, v)| \leq \psi(\delta) \frac{|u - v|}{|u + v|} \), \( \forall \delta \in \mathbb{R} \), \( u, v \geq 0 \), where \( \psi : \mathbb{R} \rightarrow \mathbb{R}^+ \) is continuous and \( \Phi \) the constant described by (6).

Theorem 3.1 Let us speculate that the condition (6.2) hold. In addition, it is assumed that \( \phi \Theta < 1 \), where \( \Theta \) is described by (6). Then \( \exists \) at most one solution for the problem (1)-(2) on \( \mathbb{R} \).

**Proof:** Interpret \( sup_{\mathbb{R}} |b(\delta, 0)| = \zeta < \infty \). Nominating \( \rho \geq \frac{\zeta\Theta}{1 - \phi \Theta} \), we demonstrate that \( B_\rho \subset B_\rho \), where \( B_\rho = \{ w \in \mathbb{R} : |w| \leq \rho \} \). For \( w \in B_\rho \), we procure

\[
|b(\delta, w)| \leq \sup_{\mathbb{R}} \left\{ \int_0^1 (1 - \tau)^{\zeta-1} |b(\tau, w(\tau)) - b(\tau, 0)| \, d\tau \right\}
\]

**Proof:** Interpret \( sup_{\mathbb{R}} |b(\delta, 0)| = \zeta < \infty \). Nominating \( \rho \geq \frac{\zeta\Theta}{1 - \phi \Theta} \), we demonstrate that \( B_\rho \subset B_\rho \), where \( B_\rho = \{ w \in \mathbb{R} : |w| \leq \rho \} \). For \( w \in B_\rho \), we procure

\[
|b(\delta, w)| \leq \sup_{\mathbb{R}} \left\{ \int_0^1 (1 - \tau)^{\zeta-1} |b(\tau, w(\tau)) - b(\tau, 0)| \, d\tau \right\}
\]

that is to say \( B_\rho \subset B_\rho \). Now, for \( w_1, w_2 \in B_\rho \), and for each \( \delta \in \mathbb{R} \), we derive
\begin{align*}
|\langle \mathcal{F}w_1 \rangle(\xi) - \langle \mathcal{F}w_2 \rangle(\xi) | & \leq \int_0^1 (1 - r)^{\xi-1} |b(r,w_1(r)) - b(r,w_2(r))| \, dr \\
& \leq \frac{\eta}{\Gamma(\xi)} \int_0^\theta \left( \int_0^s (r - \theta)^{\xi-1} |b(r,w_1(\theta)) - b(r,w_2(\theta))| \, d\theta \right) \, dr \\
& \quad + \phi_2(\xi) \left\{ \phi_j \int_0^1 \left[ \frac{\phi_j - \tau}{\Gamma(\xi - \eta)} \right] \, d\tau \right\} \\
& \quad + \int_0^1 \left[ \frac{1 - (1 - r)^{\xi-1} |b(r,w_1(r)) - b(r,w_2(r))| \, dr}{\Gamma(\xi - \eta)} \right] \\
& \leq \nu \|w_1 - w_2\| \sup_{\xi \in \mathbb{R}} \left\{ \int_0^\xi (1 - \theta)^{\xi-1} |b(r,w(\theta))| \, d\theta \right\} \\
& \quad + \phi_2(\xi) \left\{ \phi_j \int_0^1 \left[ \frac{\phi_j - \tau}{\Gamma(\xi - \eta)} \right] \, d\tau \right\} \\
& \quad + \int_0^1 \left[ \frac{1 - (1 - r)^{\xi-1} |b(r,w(r))| \, dr}{\Gamma(\xi - \eta)} \right] \\
& \leq \nu \|w_1 - w_2\| \sup_{\xi \in \mathbb{R}} \left\{ \int_0^\xi (1 - \theta)^{\xi-1} |b(r,w(\theta))| \, d\theta \right\} \\
& \quad + \phi_2(\xi) \left\{ \phi_j \int_0^1 \left[ \frac{\phi_j - \tau}{\Gamma(\xi - \eta)} \right] \, d\tau \right\} \\
& \quad + \int_0^1 \left[ \frac{1 - (1 - r)^{\xi-1} |b(r,w(r))| \, dr}{\Gamma(\xi - \eta)} \right] \\
& \leq \omega \left\{ \frac{1}{\phi_j} \left[ \frac{\phi_j}{\Gamma(\xi + 1)} \right] + \phi_2(\xi) \left\{ \phi_j \int_0^1 \left[ \frac{\phi_j - \tau}{\Gamma(\xi - \eta)} \right] \, d\tau \right\} + \int_0^1 \left[ \frac{\phi_2}{\Gamma(\xi - \eta)} \right] \right\} \leq \omega \Theta.
\end{align*}

Thus, \( \|\mathcal{F}w\| \leq \omega \Theta \). We will continue to demonstrate that the operator \( \mathcal{F} \) maps bounded sets into equicontinuous sets of \( \mathcal{C}(\mathbb{R}, \mathbb{R}) \). At the point, we procure
degree, it follows that following Leray-Schauder degrees are well described and by the homotopy invariance of topological

Let us speculate that the condition

Theorem 3.3

∴

boundedness of the set of all solutions to equations \( \mathcal{B} = \{ w \in \mathcal{S} : w = \varphi \mathcal{G}(w), 0 < \epsilon < 1 \} \). Let \( w \) be a solution. Then, for \( \beta \in \mathbb{R} \), and using the computations in proving that \( \mathcal{B}(\beta) \) is bounded, we procure

\[
||w|| \leq \alpha \left( \frac{1}{\Gamma(\xi + 1)} \left[ \frac{\varphi_1(\eta \epsilon^{\xi + 1})}{(\xi + 1)} + \frac{\varphi_2}{\Gamma(\xi - \epsilon)} \sum_{j=1}^{\xi - 1} \varphi_j \right] + \left[ \frac{\varphi_2}{\Gamma(\xi - \epsilon)} + 1 \right] \right) \leq \omega \theta.
\]

\( \therefore \mathcal{B}(\beta) \) is bounded. Hence, the BVP (1)-(2) has at least one solution on \( \mathbb{R} \).

Theorem 3.3 Let us speculate that the condition (62) hold. Then the BVP (1)-(2) has at least one solution on \( \mathbb{R} \).

Proof: We characterize the operator \( \mathcal{G} : \mathcal{S} \rightarrow \mathcal{S} \) as in (5). In perspective of the fixed point problem \( \mathcal{G} \) is completely continuous. Coming immediately, we demonstrate that the boundedness of the set of all solutions to equations \( \mathcal{B} = \{ w \in \mathcal{S} : w = \varphi \mathcal{G}(w), 0 < \epsilon < 1 \} \). Let \( w \) be a solution. Then, for \( \beta \in \mathbb{R} \), and using the computations in proving that \( \mathcal{B}(\beta) \) is bounded, we procure

\[
||w|| \leq \alpha \left( \frac{1}{\Gamma(\xi + 1)} \left[ \frac{\varphi_1(\eta \epsilon^{\xi + 1})}{(\xi + 1)} + \frac{\varphi_2}{\Gamma(\xi - \epsilon)} \sum_{j=1}^{\xi - 1} \varphi_j \right] + \left[ \frac{\varphi_2}{\Gamma(\xi - \epsilon)} + 1 \right] \right) \leq \omega \theta.
\]

\( \therefore \mathcal{B}(\beta) \) is bounded. Hence, the BVP (1)-(2) has at least one solution on \( \mathbb{R} \).

Theorem 3.3 Let us speculate that the condition (62) hold. Then the BVP (1)-(2) has at least one solution on \( \mathbb{R} \).

Proof: We characterize the operator \( \mathcal{G} : \mathcal{S} \rightarrow \mathcal{S} \) as in (5). In perspective of the fixed point problem \( \mathcal{G} \) is completely continuous. Coming immediately, we demonstrate that the boundedness of the set of all solutions to equations \( \mathcal{B} = \{ w \in \mathcal{S} : w = \varphi \mathcal{G}(w), 0 < \epsilon < 1 \} \). Let \( w \) be a solution. Then, for \( \beta \in \mathbb{R} \), and using the computations in proving that \( \mathcal{B}(\beta) \) is bounded, we procure

\[
||w|| \leq \alpha \left( \frac{1}{\Gamma(\xi + 1)} \left[ \frac{\varphi_1(\eta \epsilon^{\xi + 1})}{(\xi + 1)} + \frac{\varphi_2}{\Gamma(\xi - \epsilon)} \sum_{j=1}^{\xi - 1} \varphi_j \right] + \left[ \frac{\varphi_2}{\Gamma(\xi - \epsilon)} + 1 \right] \right) \leq \omega \theta.
\]

\( \therefore \mathcal{B}(\beta) \) is bounded. Hence, the BVP (1)-(2) has at least one solution on \( \mathbb{R} \).

Theorem 3.3 Let us speculate that the condition (62) hold. Then the BVP (1)-(2) has at least one solution on \( \mathbb{R} \).

Proof: We characterize the operator \( \mathcal{G} : \mathcal{S} \rightarrow \mathcal{S} \) as in (5). In perspective of the fixed point

problems \( \mathcal{G}(w) = \mathcal{G}(w) \).

(7)
\[ |(\overline{g}\omega)(\xi)| \leq \int_0^1 (3 - \tau)^{\xi-1} |b(t, \omega(t))| \, dt + \phi_1(\xi) \left[ \eta \int_0^\eta \left( \int_0^\eta (\tau - \theta)^{\xi-1} |b(\theta, \omega(\theta))| \, d\theta \right) \, d\theta \right] + \phi_2(3) \left[ \sum_{j=0}^{k-2} \left( \frac{\eta}{\Gamma(\xi - \zeta + 1)} \right)^{\xi-1} \right] \]

\[ \leq \sup_{\eta \in \mathbb{R}} \left( \int_0^1 (3 - \tau)^{\xi-1} (\omega |w| + \Omega) \, dt + \phi_1(\xi) \left[ \eta \int_0^\eta \left( \int_0^\eta (\tau - \theta)^{\xi-1} (\omega |w| + \Omega) \, d\theta \right) \, d\theta \right] + \phi_2(3) \left[ \sum_{j=0}^{k-2} \left( \frac{\eta}{\Gamma(\xi - \zeta + 1)} \right)^{\xi-1} \right] \right) \]

which, on taking norm \( \sup_{\eta \in \mathbb{R}} |w(\eta)| = ||w|| \) and solving for \( ||w|| \), yields \( ||w|| \leq \frac{\phi \Omega}{1 - \omega \phi} \).

If \( \mathcal{B} \leq \frac{\phi \Omega}{1 - \omega \phi} + 1 \), inequality (8) holds.

**Theorem 3.4** Let us speculate that the condition \( (\Phi_1) \) hold. Then \( \exists \) at most one solution for the problem (1)-(2) on \( \mathbb{R} \).

**Proof:** In any case the operator \( \overline{f} : \mathbb{R} \rightarrow \mathbb{R} \) as in (5), and the continuous nondecreasing function \( \chi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) by

\[ \chi(\lambda) = \frac{\Phi_1}{\Phi_1 + \lambda}, \quad \forall \lambda \geq 0. \]

Bearing in mind that the function \( \chi \) satisfies \( \chi(0) = 0 \) and \( \chi(\lambda) < \lambda, \forall \lambda > 0 \). For any \( w_1, w_2 \in \mathbb{R} \), furthermore for each \( \xi \in \mathbb{R} \), we get

\[ |(\overline{g}\omega_1)(\xi) - (\overline{g}\omega_2)(\xi)| \]

\[ \leq \int_0^1 (3 - \tau)^{\xi-1} |b(t, \omega_1(t)) - b(t, \omega_2(t))| \, dt + \phi_1(\xi) \left[ \eta \int_0^\eta \left( \int_0^\eta (\tau - \theta)^{\xi-1} |b(\theta, \omega_1(\theta)) - b(\theta, \omega_2(\theta))| \, d\theta \right) \, d\theta \right] + \phi_2(3) \left[ \sum_{j=0}^{k-2} \left( \frac{\eta}{\Gamma(\xi - \zeta + 1)} \right)^{\xi-1} \right] \]

\[ \leq \chi(||w_1 - w_2||) \left( \int_0^1 (3 - \tau)^{\xi-1} \, dt + \phi_1(\xi) \left[ \eta \int_0^\eta \left( \int_0^\eta (\tau - \theta)^{\xi-1} \, d\theta \right) \, d\theta \right] + \phi_2(3) \left[ \sum_{j=0}^{k-2} \left( \frac{\eta}{\Gamma(\xi - \zeta + 1)} \right)^{\xi-1} \right] \right) \]

\[ \leq \chi(||w_1 - w_2||). \]
\[ \| \mathfrak{F} w_1 - \mathfrak{F} w_2 \| \leq \chi \| w_1 - w_2 \|, \text{ and } \mathfrak{F} \text{ is a nonlinear contraction. Hence, the operator } \mathfrak{F} \text{ has a unique fixed point which is the unique solution of the problem (1)-(2).} \]

4. Examples

**Example 4.1** Consider the following BVP of FDE given by

\[
\mathcal{D}^\alpha w(t) = \frac{1}{3 + 1} + \frac{(e^t)}{(3^2 + 9)} \left( \frac{|w(t)|}{1 + |w(t)|} \right), \quad 3 \in \mathbb{R}, \quad 1 < \xi \leq 2, \quad (9)
\]

\[
w(0) = \eta \int_0^t w(r) dr, \quad \mathcal{D}^\eta w(1) = \sum_{j=1}^{b-2} \epsilon_j w(\varphi_j), \quad 0 < \zeta < 1 \quad (10)
\]

**Solution** : Here, \( \xi = 3, \ t = 6, \eta = \frac{1}{3}, \ \varphi = \frac{1}{2}, \ \epsilon_1 = \frac{11}{15}, \ \epsilon_2 = \frac{3}{20}, \ \epsilon_3 = \frac{11}{50}, \ \epsilon_4 = \frac{1}{4}, \ \varphi_1 = \frac{1}{5}, \ \varphi_2 = \frac{3}{5}, \ \varphi_3 = \frac{17}{20}. \) We able acquire those values by utilizing the specified information \( \delta_1 = 0.833333, \delta_2 = 0.0416667, \gamma_1 = 0.73, \gamma_2 = 0.655124, \Lambda = 0.51552, \Phi_1 = 2.68685, \Phi_2 = 1.69731, \ \Theta = 3.48671. \) Since \( b(3, w) = \frac{1}{3+1} + \frac{(e^t)}{(3^2+9)} \left( \frac{|w|}{1+|w|} \right), \) it is clear that \( |b(3, w)| \leq \frac{1}{9}. \) Thus, the presumptions of Theorem 3.2 hold and consequently the BVP (8)-(9) has at least one solution on \( \mathbb{R}. \)

**Example 4.2** Consider the following BVP of FDE given by

\[
\mathcal{D}^\alpha w(t) = \frac{4\epsilon_1}{(3^2 + 8)^2} \left( \frac{|w^2(t)|}{1 + |w(t)|} \right) + \frac{1}{(3 + 1)^2}, \quad 3 \in \mathbb{R}, \quad 1 < \xi \leq 2, \quad (11)
\]

supplemented with the boundary conditions of Example 4.1

**Solution** : We able acquire those values by utilizing the specified information \( \delta_1 = 0.833333, \delta_2 = 0.0416667, \gamma_1 = 0.73, \gamma_2 = 0.655124, \Lambda = 0.51552, \Phi_1 = 2.68685, \Phi_2 = 1.69731, \ \Theta = 3.48671. \) Since \( |b(3, w_1) - b(3, w_2)| \leq \frac{1}{16} |w_1 - w_2|, \) (\( \vartheta_1 \)) is satisfied with \( \nu = \frac{1}{16}. \) We procure \( \nu \Theta \geq 0.217919 < 1, \) the presumptions of Theorem 3.1 hold and hence the BVP (11) with (10) has at most one solution on \( \mathbb{R}. \)

**Conclusion**

On perceiving the above results, the problem of non linear differential equation of fractional order with non local integral and multi point boundary condition holds good for existence and uniqueness conditions. Thus the problem described in the article becomes viable. Besides, the reader shall further evolve the problem with abundant ideas with certain persistent estimates of the parameter associated with the problem.

**References**

[1] Ahmad, B, Alsaedi, A, Garout, D (2016), Existence results for Liouville-Caputo type fractional differential equations with nonlocal multi-point and substrips boundary conditions, *Computers and Mathematics with Applications*, https://doi.org/10.1016/j.camwa.2016.04.015.

[2] Ahmad, B, and Ntouyas, S.K. (2014), On higher-order sequential fractional differential inclusions with nonlocal three-point boundary conditions, *Abstract Analysis and Applications*, 2014, 10 pages.
[3] Cabada, A and Wang, G (2012), Positive solutions of nonlinear fractional differential equations with integral boundary value conditions, *Journal of Mathematics Analysis and Applications*, **389**, 403-411.

[4] Boyd, D. W. and Wong, J.S.W. (1969), On nonlinear contractions, Proc. Amer. Math. Soc, **20**, 458-464.

[5] Duraisamy, P, and Nandhagopal, T (2018), Existence and Uniqueness of Solutions for a Coupled System of Higher Order Fractional Differential Equations with Integral Boundary Conditions, *Discontinuity, Nonlinearity and Complexity*, **7**(1), 1-14.

[6] Lakshimikantham, V, Leela, S, and Devi, J.V (2009), Theory of Fractional Dynamic Systems, Cambridge Academic Publishers, Cambridge.

[7] Kilbas, A.A, Srivastava, H. M, and Trujillo, JJ (2006), Theory and applications of fractional differential equations, Amsterdam, Boston, Elsevier.

[8] Ntouyas, S.K. and Etemad, S (2015), On the existence of solutions for fractional differential inclusions with sum and integral boundary conditions, *Applied Mathematics and Computation*, **266**, 235-243.

[9] Valerio, D, Machado, J.A.T, and Kiryakova, V (2014), Some pioneers of the applications of fractional calculus, *Fractional Calculus & Applied Analysis*, **17**(2), 552-578.

[10] Podlubny, I (1999), Fractional Differential Equations, Academic Press, San Diego-Boston-New York-London-Tokyo-Toronto.

[11] Cui, J and Yan, L (2011), Existence result for fractional neutral stochastic integro-differential equations with infinite delay, *Journal of Physics A : Mathematical and Theoretical*, **44**, 16 pages.

[12] Agarwal, O.P (2007), Analytical schemes for new class of fractional differential equations, *Journal of Physics A : Mathematical and Theoretical*, **40**, 5469-5477.