Stability of symmetric tops via one variable calculus

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Abstract

We study the stability of symmetric trajectories of a particle on the Lie group $SO(3)$ whose motion is governed by an $SO(3) \times SO(2)$ invariant metric and an $SO(2) \times SO(2)$ invariant potential. Our method is to reduce the number of degrees of freedom at singular values of the $SO(2) \times SO(2)$ momentum map and study the stability of the equilibria of the reduced systems as a function of spin. The result is an elementary analysis of the fast/slow transition in the Lagrange and Kirchhoff tops. More generally, since an $SO(2) \times SO(2)$ invariant potential on $SO(3)$ can be thought of as $\mathbb{Z}_2$ invariant function on a circle, we get a condition on the second and fourth derivatives of the potential at the symmetric points that guarantees that the corresponding system gains stability as the spin increases.

1 Introduction

The motivation for writing this paper is two-fold. The first one is to see the stable/unstable transition in the Lagrange top without resorting to the machinery of equivariant singularity theory as in [C], [CvdM] let alone the more sophisticated techniques of [LRSM]. The idea is to exploit the $SO(2) \times SO(2)$ symmetry and use reduction instead. The resulting system can then be analyzed by looking at the signs of two derivatives of a function of one variable at one particular point. The second motivation is to explain why it is that both the Lagrange top and the Kirchhoff top [BZ] undergo the Hamiltonian Hopf bifurcation as their spin varies.

I will show that by carrying out reduction at certain singular values of the $SO(2) \times SO(2)$ momentum map one can understand the bifurcation in both systems (in fact in an infinite family of systems) as a phenomenon of the following sort. Consider a particle on a line whose behavior is governed by a 1-parameter family of potentials

$$f_\lambda(x) = \frac{1}{6}x^4 - x^2 + \lambda^2 x^2,$$

where $\lambda$ is the parameter. Then for $|\lambda| < 1$ the origin is a local maximum of the potential, hence an unstable equilibrium of the system. Global minima occur at $x = \pm(1 - \lambda^2)^{1/2}$. As $|\lambda|$ increases, these stable points move closer to the origin until they collide at $|\lambda| = 1$; the origin becomes a

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a global minimum (hence a stable equilibrium). In other words, the reduced system undergoes a

generic Hamiltonian $Z_2$ invariant bifurcation — the figure eight bifurcation (cf. [GoSt]).

There is another interpretation of the results of this paper. Namely, the systems that are

being studied are three degrees of freedom completely integrable systems. Thus the nondegenerate

singularities of the corresponding Lagrangian foliation are, in Eliasson’s classification [E], of elliptic

or of hyperbolic type. It turns out that both types of singularities are present in our systems

and that the closures of the sets of elliptic and hyperbolic points intersect at some degenerate

singular point. This fact was already observed by Eliasson in the case of the Lagrange top (op.

cit.). The point I would like to make is that the Hamiltonian Hopf bifurcation in these systems

can be described as a transition from an elliptic to a hyperbolic singularity through a degenerate

singularity. It would be interesting to see if such transitions are in some sense generic for the

three degrees of freedom completely integrable systems. In higher dimensions many more types of

nondegenerate singularities of the Lagrangian foliation can occur. This suggests that transitions

between these singularities should give rise to interesting bifurcations.

2 The set-up and the first $SO(2)$ reduction

The configuration space of the systems we will study is the Lie group $SO(3)$ and the Hamiltonians
$H$ are of the form Kinetic Energy + Potential Energy, where the kinetic term comes from a left
$SO(3)$ and right $SO(2)$ invariant metric, i.e. the moments of inertia satisfy $I_1 = I_2 \neq I_3$, and the
potential is left-right $SO(2)$ invariant. Here $SO(2)$ is identified with the isotropy group of $e_3$, the
third vector in the standard basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$. For example $W(A) = (Ae_3, e_3)$ is the potential
of the Lagrange top and $W(A) = (Ae_3, e_1)^2 + (Ae_3, e_2)^2 + c(Ae_3, e_3)^2$, $c$ a positive constant, is the potential of the Kirchoff top.

Since the left action of the group $SO(2)$ on the cotangent bundle of $SO(3)$ is free, we may carry
out the Marsden-Weinstein-Meyer reduction. It amounts to fixing the rate of rotation of the top
about its axis of symmetry. A routine calculation shows the reduced spaces, as manifolds, are the
cotangent bundle of the two sphere $S^2$ with the reduced symplectic structures of the form

$$\omega_\lambda = \omega_{T^*S^2} + \lambda \nu,$$

where $\lambda$ is the value of the $SO(2)$ momentum map, $\omega_{T^*S^2}$ is the standard symplectic form on the
cotangent bundle of the sphere and $\nu$ is the standard area form on the sphere (pulled up to the
cotangent bundle). For example, to compute these reduced spaces one could use Kummer’s results
[K] or the fact that $SO(3)$ and the Euclidean group $E(3)$ form a dual pair. The form $\nu$ is often
referred to as the magnetic form. This calculation will not be carried out in the paper. The reader
may simply assume that we are studying a family of Hamiltonian systems on $(T^*S^2, \omega_\lambda)$, $\lambda \in \mathbb{R}$.

There is one action of $SO(2)$ left on the reduced space. If we identify the sphere with the
standard round sphere in $\mathbb{R}^3$ then the action under the identifications made above is the lift of
the rotation about the $e_3$ axis. The reduced Hamiltonians are again of the form Kinetic Energy +
Potential Energy, where the kinetic term comes from a round metric on the sphere and the potential
$V$ is an $SO(2)$ invariant function. For example the Lagrange top potential is $V(x) = x_3|_{S^2}$ and the
Kirchhoff top potential is $V(x) = (x_1^2 + x_2^2 + cx_3)|_{S^2} = (1 + (c - 1)x_3^2)|_{S^2}$. Here $(x_1, x_2, x_3)$ are the
standard coordinates on $\mathbb{R}^3$.

The momentum map $\Phi_\lambda : (T^*S^2, \omega_\lambda) \rightarrow \mathbb{R}$ is the sum of two momentum maps: one coming
from the lifted action of $SO(2)$ on $(T^*S^2, \omega_{T^*S^2})$ and the other coming from the action of $SO(2)$
on $(S^2, \lambda \nu)$. We fix the constant in the definition of the momentum maps by requiring the map to
be 0 at the North pole.
We think of the reduced systems \((T^*S^2, \omega_\lambda, h)\) as two degrees of freedom systems depending on one real parameter \(\lambda\), the spin. The action of \(SO(2)\) fixes the North and South poles. Consequently these points are critical for our Hamiltonians \(h\). We would like to understand how the behavior of the systems near the symmetric points changes as we vary the parameter \(\lambda\). We will consider the North pole. The arguments regarding the South pole are exactly the same. In fact the Hamiltonian for the Kirchhoff top is invariant under the \(\mathbb{Z}_2\) symmetry interchanging the poles.

To understand the behavior of the system near the North pole we reduce with respect to the remaining symmetry at \(\Phi_\lambda = 0\), the value of the normalized momentum map at the North pole. Since the North pole is a fixed point, the standard Marsden-Weinstein-Meyer procedure does not apply. We carry out the reduction in the next section.

### 3 Reduction at a critical value of the momentum map

Let us briefly summarize the procedure that we are going to follow. A full account is given in \cite{Sl}. Our point of view is that given a Hamiltonian action of a compact Lie group \(K\) on a symplectic manifold \((M, \omega)\) with momentum map \(F : M \to k^*\), the reduced space at a coadjoint orbit \(O_\alpha = K \cdot \alpha \subset k^*\) is the Hausdorff topological space

\[ M_\alpha := F^{-1}(O_\alpha)/K. \]

Smooth \(K\) invariant functions on \(M\) descend to continuous functions on \(M_\alpha\); we denote these functions by \(C^\infty(M_\alpha)\). Algebraically

\[ C^\infty(M_\alpha) = C^\infty(M)^K/I_\alpha, \]

where \(C^\infty(M)^K\) denotes the algebra of smooth \(K\) invariant functions and \(I_\alpha\) is the ideal in \(C^\infty(M)^K\) consisting of functions that vanish on \(F^{-1}(O_\alpha)\). It is not hard to see that \(I_\alpha\) is a Poisson ideal \cite{ACQ}. Indeed, the Hamiltonian flows of \(K\) invariant functions preserve the set \(F^{-1}(O_\alpha)\), so for any \(f \in C^\infty(M)^K\) and any \(h \in I_\alpha\) we have \(\{f, h\} = 0\). Therefore \(C^\infty(M_\alpha)\) is a Poisson algebra. The Poisson bracket allows us to associate flows to functions. It also makes it possible for us to say when two reduced spaces are isomorphic.

**Definition 3.1** Two reduced spaces \(X\) and \(Y\) are isomorphic if there exists a homeomorphism \(\phi : X \to Y\) such that the induced map \(\phi^* : C^\infty(Y) \to C^\infty(X)\) is a Poisson isomorphism.

The following example will be very important for us. In fact, thanks to the equivariant Darboux theorem, this is precisely the situation that we are interested in. Consider the standard lifted action of \(SO(2)\) on the cotangent bundle \(T^*\mathbb{R}^2\). Let \((x, y)\) be the coordinates on \(\mathbb{R}^2\) and \((x, y, p_x, p_y)\) the canonical coordinates on \(T^*\mathbb{R}^2\). The standard symplectic form \(\omega\) is given in these coordinates by \(\omega = dx \wedge dp_x + dy \wedge dp_y\) and the momentum map \(F\) by

\[ F(x, y, p_x, p_y) = xp_y - yp_x. \]

The map \(\psi : T^*\mathbb{R} \to T^*\mathbb{R}^2, (u, p_u) \mapsto (u, 0, p_u, 0)\) is a symplectic embedding whose image lies entirely in the zero level set \(F^{-1}(0)\) of \(F\). Moreover, all the orbits of \(SO(2)\) in \(F^{-1}(0)\) (except \(\{0\}\)) intersect the image in exactly two points: \((u, 0, p_u, 0)\) and \((-u, 0, -p_u, 0)\). Consequently \(\psi\) descends to a homeomorphism \(\phi : T^*\mathbb{R}/\mathbb{Z}_2 \to F^{-1}(0)/SO(2) \cong (T^*\mathbb{R}^2)_0\) where \(\mathbb{Z}_2\) acts on \(T^*\mathbb{R}\) by the map \((u, p_u) \mapsto (-u, -p_u)\).

Note that since any \(SO(2)\) invariant function on \(T^*\mathbb{R}^2\) pulls back via \(\psi\) to a \(\mathbb{Z}_2\) invariant function on \(T^*\mathbb{R}\), \(\phi\) pulls back smooth functions on the reduced space \((T^*\mathbb{R}^2)_0\) to smooth functions.
on the orbifold \( T^*\mathbb{R}/\mathbb{Z}_2 \). Since \( \psi \) is a symplectic embedding the pull-back \( \phi^* \) is Poisson. It is easy to see that \( \phi^* \) is injective: \( \phi \) is a homeomorphism. To prove surjectivity of the pull-back we use a theorem of G. Schwarz on invariant smooth functions [Sch]. The \( \mathbb{Z}_2 \) invariant polynomials on \( T^*\mathbb{R} \) are generated by three polynomials: \( u^2, p_u^2 \) and \( u p_u \). Schwarz’s theorem in this case says that any smooth \( \mathbb{Z}_2 \) invariant function is a composition of a smooth function on \( \mathbb{R}^3 \) with the map \( T^*\mathbb{R} \rightarrow \mathbb{R}^3 \), \((u, p_u) \mapsto (u^2, p_u^2, u p_u)\), i.e., if \( f \in C^\infty(T^*\mathbb{R}) \), \( f(u, p_u) = \bar{f}(u^2, p_u^2, u p_u) \).

Now \( u^2 = \psi^*(x^2 + y^2) \), \( p_u^2 = \psi^*(p_x^2 + p_y^2) \) and \( u p_u = \psi^*(xp_x + yp_y) \). Therefore any function \( f \in C^\infty(T^*\mathbb{R}/\mathbb{Z}_2) \) is in the image of \( \phi^* \).

**CONCLUSION:** the reduced space \( (T^*\mathbb{R}^2)_0 \) is isomorphic to the orbifold \( T^*\mathbb{R}/\mathbb{Z}_2 \) (the cotangent bundle of a line reduced by a finite group \( \mathbb{Z}_2 \)).

### 3.1 Stability

Marsden and Weinstein [MW] proved that stable equilibria of reduced systems correspond to stable relative equilibria of the original system. More precisely they considered a Lie group \( G \) acting on a symplectic manifold \((M, \omega)\) in a Hamiltonian fashion with corresponding momentum map \( \Phi: \rightarrow g^* \) and an invariant Hamiltonian \( h \in C^\infty(M)^G \). For a regular value \( \alpha \) of the momentum map let \( h_\alpha \) denote the corresponding reduced Hamiltonian on the reduced space \( M_\alpha := \Phi^{-1}(\alpha)/G \). Marsden and Weinstein showed that if \( x \in M_\alpha \) is a critical point of \( h_\alpha \) and the Hessian of \( h_\alpha \) at \( x \) is definite then the trajectory of the original Hamiltonian \( h \) through a point \( m \in \Phi^{-1}(\alpha) \) that corresponds to \( x \in M_\alpha \) is stable relative to the group action. In fact the result holds under a slightly weaker hypothesis, namely that the action of the group \( G \) is free at \( m \), i.e., the reduced space \( M_\alpha \) is smooth near \( x \). Thus \( \alpha \) being a regular value is not really needed.

If a point \( m \) is symmetric, i.e., has a nontrivial isotropy, then it is automatically a critical point of any invariant Hamiltonian. Since it is also a critical point of the momentum map Marsden and Weinstein’s result does not apply. However, for the action of \( SO(2) \) on the cotangent bundle of \( \mathbb{R}^2 \) that we are considering, the correspondence still holds.

**Lemma 3.2** Suppose a function \( h \in C^\infty(T^*\mathbb{R}^2) \) is \( SO(2) \) invariant. Then if the Hessian of \( h_0 := h|_{(u,0,p_u,0) \in T^*\mathbb{R}^2} \) is definite at the origin, the origin \( 0 = (0,0,0,0) \in T^*\mathbb{R}^2 \) is \( SO(2) \) stable, i.e., given any \( SO(2) \) invariant neighborhood \( U \) of \( 0 \) in the cotangent bundle \( T^*\mathbb{R}^2 \) there is an \( SO(2) \) invariant neighborhood \( V \) of \( 0 \) such that for all \( x \in V \) the trajectory of the Hamiltonian flow of \( h \) through \( x \) stays in \( U \).

**Proof.** It is no loss of generality to assume that the Hessian \( \mathcal{H}(h_0) \) of \( h_0 \) at \( 0 \) is of the form

\[
\mathcal{H}(h_0) = au^2 + bp_u^2; \quad a, b > 0.
\]

Indeed, any quadratic form on \( \mathbb{R}^2 \) can be diagonalized by an orthogonal change of coordinates,

\[
\begin{pmatrix}
u \\
p_u
\end{pmatrix} \mapsto \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} u \\
p_u
\end{pmatrix},
\]

where \( c^2 + s^2 = 1 \). Moreover, this change of coordinates is induced by an \( SO(2) \) equivariant change of coordinates on \( T^*\mathbb{R}^2 \); it is given by the matrix

\[
\begin{pmatrix}
c & 0 & s & 0 \\
0 & c & 0 & s \\
-s & 0 & c & 0 \\
0 & -s & 0 & c
\end{pmatrix}.
\]
Since the Hessian of $h$ at 0 is $SO(2)$ invariant and since $\mathcal{H}(h)|_{\{(u,0,p_u,0)\in T^*\mathbb{R}^2\}} = \mathcal{H}(h_0)$, the Hessian of $h$ has to be of the form

$$\mathcal{H}(h) = a(x^2 + y^2) + b(p_x^2 + p_y^2) + cF;$$

where $c$ is a constant and $F = xp_y - yp_x$ is the momentum map. It cannot have any more terms for then the Hessian of the reduced Hamiltonian $\mathcal{H}(h_0)$ would have more terms.

Let $\rho$ be a a compactly supported smooth function on the cotangent bundle of $\mathbb{R}^2$ which is identically 1 near 0. Then near the origin the Poisson bracket $\{h, \rho cF\}$ is zero. The Hessian of $\bar{h} := h - \rho cF$ is positive definite at 0, hence the origin is a stable equilibrium of $\bar{h}$. On the other hand, since the flows of $\bar{h}$ and of $\rho cF$ commute near the origin, the flow of the original Hamiltonian is the composition of the flows of $\bar{h}$ and of $cF$. Therefore $0 \in T^*\mathbb{R}^2$ is $SO(2)$ stable for the flow of $h$.

3.2 Reduction of the cotangent bundle of the 2-sphere

Consider coordinates on the upper hemisphere of $S^2$ induced by the projection $(x, y, z) \mapsto (x, y)$. The inverse map $\phi$ is given by $\phi(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$. These coordinates allow us to compute the most interesting part of the reduced system. We don’t need the coordinates to compute the reduced space per se, rather the coordinates are convenient for computing the reduced Hamiltonians. Since

$$\phi^*(xdy \wedge dz + ydz \wedge dx + zdx \wedge dy) = \frac{1}{z}dx \wedge dy,$$

where $z = \sqrt{1 - x^2 - y^2}$, the symplectic form $\omega_\lambda$ in the canonical coordinates corresponding to $(x, y)$ is

$$\omega_\lambda = dx \wedge dp_x + dy \wedge dp_y + \frac{\lambda}{z}dx \wedge dy.$$

The momentum map $\Phi_\lambda$ in these coordinates is then

$$\Phi_\lambda(x, y, p_x, p_y) = yp_x - xp_y - \lambda z + \lambda.$$

Note that the North pole in these coordinates is the origin and that $\Phi(0) = 0$.

The metric $g$ by assumption is the one that comes from the embedding of $S^2$ as a round sphere in $\mathbb{R}^3$. We now normalize the metric by setting the radius of the sphere to 1. Then the sphere is cut out by the equation

$$x^2 + y^2 + z^2 = 1,$$  \hspace{1cm} (1)

In coordinates the metric $g$ is given by

$$g = \begin{pmatrix}
1 + \frac{x^2}{z^2} & \frac{xy}{z^2} \\
\frac{xy}{z^2} & 1 + \frac{y^2}{z^2}
\end{pmatrix}.$$

Consequently

$$g^{-1} = z^2 \begin{pmatrix}
1 + \frac{x^2}{z^2} & -\frac{xy}{z^2} \\
-\frac{xy}{z^2} & 1 + \frac{y^2}{z^2}
\end{pmatrix}.$$

We are now in position to compute the reduced Hamiltonian. From the discussion in the beginning of the section we know that locally the reduced space is the cotangent bundle of the interval $(-1, 1)$ divided by the involution $(u, p_u) \mapsto (-u, -p_u)$. Since our coordinates are not Darboux we proceed as follows. Set $y = 0$. Then $\Phi_\lambda(x, 0, p_x, p_y) = 0$ if and only if

$$p_y = \lambda(1 - z)/x.$$
Note that for $y = 0$,

$$1 - z = 1 - \sqrt{1 - x^2} = 1 - \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 + \mathcal{O}(x^6)\right) = \frac{1}{2}x^2 + \frac{1}{8}x^4 + \mathcal{O}(x^6),$$

so

$$p_y(x) = \frac{\lambda}{2}(x + \frac{1}{8}x^3 + \mathcal{O}(x^5))$$

is smooth at $x = 0$. Consider now a map $\psi : T^*(-1, 1) \to \Phi^{-1}_\lambda(0), (u, p_u) \mapsto (u, 0, p_u, \lambda(1 - z)/u)$ where now $z = \sqrt{1 - u^2}$. As in the example considered above, the composition

$$T^*(-1, 1) \to \Phi^{-1}_\lambda(0) \to \Phi^{-1}_\lambda(0)/SO(2)$$

is a branched double cover. Therefore, we will simply work on $T^*(-1, 1)$ keeping the involution in mind. Now the pull-back by $\psi$ of the kinetic energy part of the Hamiltonian is

$$\frac{z^2}{2} \left(p_u^2 + \left(1 + \frac{x^2}{z^2}\right) \left(\lambda \frac{1 - z}{u}\right)^2\right) = \frac{1}{2}G(u)p_u^2 + \frac{\lambda^2}{2} \left(\frac{1 - z}{u}\right)^2,$$

where $G(u) = (1 - u^2)$. The potential energy part pulls back to an even function of $u$, hence to a function of $u^2$. For example, for the Lagrange top the corresponding function is

$$V(u^2) = \sqrt{1 - u^2}$$

and for the Kirchhoff top

$$V(u^2) = 1 + (c - 1)(1 - u^2).$$

The critical points of the reduced Hamiltonian

$$H_\lambda(u, p_u) = \frac{1}{2}G(u)p_u^2 + \frac{1}{2}\lambda^2 \left(1 - \sqrt{1 - u^2}\right)^2 + V(u^2)$$

are in one-to-one correspondence with the critical points of the effective potential

$$U_\lambda(u) = \frac{1}{2}\lambda^2 \left(\frac{1 - \sqrt{1 - u^2}}{u}\right)^2 + V(u^2).$$

Moreover, relative maxima of $U_\lambda$ correspond to unstable equilibria of $H_\lambda$ and relative minima to stable equilibria.

**Remark** Since we are working on the branched double cover of the reduced system, any pair of critical points of $U_\lambda$ of the form $\pm u$, $u \neq 0$ correspond to the same critical point of the reduced system. Therefore only the non-negative critical points of the potential need be considered.

### 3.3 Critical values of the effective potential $U_\lambda(u)$, $u \geq 0$

Since

$$\left(\frac{1 - \sqrt{1 - u^2}}{u}\right)^2 = \frac{u^2}{4} \left(1 + \frac{1}{2}u^2 + \mathcal{O}(u^4)\right)$$

there exists a change of variables $u = \tau(v)$ so that

$$v^2 = \left(\frac{1 - \sqrt{1 - \tau(v)^2}}{\tau(v)}\right)^2.$$
Note that $\tau(v)$ is an odd function of the form

$$\tau(v) = 2v - 2v^3 + O(v^5).$$

Then the transformed reduced potential $V(\tau(v)^2)$ is even in $v$ and so has to be of the form $V(\tau(v)^2) = f(v^2)$ for some smooth function $f$. Therefore the study of the bifurcation of the critical points of the potential $U_\lambda(u)$ for $u$ small and non-negative is reduced to studying the bifurcation of the critical points of $F_\lambda(v) = \frac{\lambda^2}{2}v^2 + f(v^2)$ for $v$ small and nonnegative.

It will be useful to express the derivatives $f'(0)$ and $f''(0)$ in terms of the derivatives $V'(0)$ and $V''(0)$. Since $\tau(v)^2 = 4v^2 - 8v^4 + O(v^6)$ and $\tau(v)^4 = 16v^4 + O(v^6)$, we have

$$V(\tau(v)^2) = V(0) + V'(0)\tau(v)^2 + V''(0)\tau(v)^4 + O(\tau(v)^6)$$

$$= V(0) + 4V'(0)v^2 + 8(V''(0) - V'(0))v^4 + O(v^6).$$

Therefore

$$f'(0) = 4V'(0) \quad \text{and} \quad f''(0) = 8(V''(0) - V'(0)). \quad (2)$$

### 3.4 Bifurcation of critical points of $F_\lambda(v) = \frac{\lambda^2}{2}v^2 + f(v^2)$, $v \geq 0$

It is no loss of generality to assume that $\lambda > 0$. We also make the genericity assumptions: $f'(0) \neq 0$ and $f''(0) \neq 0$.

$$\frac{d}{dv}F_\lambda(v) = v(\lambda^2 + 2f'(v^2))$$

Thus if $f'(0) > 0$ then $v = 0$ is the only critical point for all values of $\lambda$, i.e., no bifurcations occur. Physically this means that if a top in a straight up position with zero spin is stable, then it is stable in this position for all values of spin.

Assume now $f'(0) < 0$. Then $f'(0) = -a^2/2$ for some $a > 0$. Consider $\phi(t, \lambda) = \lambda^2 + 2f'(t)$. Since $(f')'(0) \neq 0$, by the implicit function theorem there exists a smooth function $g(\lambda)$ defined near $\lambda = a$ such that $g(a) = 0$ and $\phi(g(\lambda), \lambda) = 0$, i.e.,

$$\frac{1}{2}\lambda^2 + f'(g(\lambda)) = 0. \quad (3)$$

Note that since

$$0 = \frac{d}{d\lambda} \bigg|_{\lambda=a} \phi(g(\lambda), \lambda) = [\lambda + f''(g(\lambda))g'(\lambda)]_{\lambda=a}$$

we have

$$f''(0)g'(a) = -a < 0.$$
we have, using (3),

$$\frac{d^2}{dv^2} \bigg|_{v=g(\lambda)^{1/2}} F_\lambda(v) = 4|g(\lambda)|f''(g(\lambda)) \quad \text{which is} \quad \begin{cases} > 0 & \text{if } f''(0) > 0 \\ < 0 & \text{if } f''(0) < 0 \end{cases}$$

and

$$\frac{d^2}{dv^2} \bigg|_{v=0} F_\lambda(v) = 2(\lambda^2 + f'(0)) \quad \text{which is} \quad \begin{cases} > 0 & \text{if } \lambda > \sqrt{-2f''(0)} \\ < 0 & \text{if } \lambda < \sqrt{-2f''(0)} \end{cases}$$

**Conclusions**
Assume that $f'(0) < 0$.

If $f''(0) > 0$ then as $\lambda$ decreases below $\lambda_0 = \sqrt{-2f''(0)}$, $v = 0$ changes from a local minimum to a local maximum and a one parameter family of local minima $v = g(\lambda)^{1/2}$ bifurcate from $v = 0$.

If $f''(0) < 0$ then the situation is reversed. For $\lambda < \lambda_0$ and $v$ small, the origin is a unique critical point, and it is a local maximum. As $\lambda$ increases past the critical value, $v = 0$ becomes a local minimum and a one parameter family $v = g(\lambda)^{1/2}$ of local maxima bifurcate from it.
These results easily translate into a description of the bifurcation of nonnegative critical points of the potential

\[ U_\lambda(u) = \frac{\lambda^2}{2} \left( \frac{1 - \sqrt{1 - u^2}}{u} \right)^2 + V(u^2), \]

where without loss of generality we shall assume that \( \lambda \geq 0 \). Recall that

\[ f'(0) = 4V'(0) \quad \text{and} \quad f''(0) = 8(V''(0) - V'(0)). \]

It follows that if \( V'(0) > 0 \) then \( u = 0 \) is a local minimum for all values of \( \lambda \) and no bifurcations occur. We now assume that \( V'(0) < 0 \). Then there are two alternatives.

1. If \( V''(0) > V'(0) \) then as \( \lambda \) decreases below \( \lambda_0 = \sqrt{-V'(0)} \), \( u = 0 \) changes from a local minimum to a local maximum of the potential \( U_\lambda \) and a 1-parameter family \( u = \rho(\lambda) \) of local minima bifurcate off from \( u = 0 \). Here \( \rho(\lambda) \) is smooth for \( \lambda < \lambda_0 \) and H"older 1/2 at \( \lambda_0 \). Consequently the corresponding system on the cotangent bundle of the two-sphere undergoes a Hamiltonian Hopf bifurcation.

2. If \( V''(0) < V'(0) \) then for \( 0 \leq \lambda < \lambda_0 \) and \( u \) small, the point \( u = 0 \) is a unique local maximum. As \( \lambda \) increases past \( \lambda_0 \), \( u = 0 \) becomes a local minimum and a 1-parameter family \( u = \rho(\lambda) \) of local maxima bifurcate from it.

For the Lagrange top \( V(u^2) = \sqrt{1 - u^2} \), so the first alternative holds with \( \lambda_0 = 2 \) (this agrees with [E]). For the Kirchhoff top \( V(u^2) = 1 + (c - 1)(1 - u^2) \). So if \( c < 1 \) there is no bifurcation and if \( c > 1 \) the first alternative holds.

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