On numerical verification methods to construct local Lyapunov functions around non-hyperbolic equilibria for two-dimensional cases

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Abstract
In Terasaka et al. JSIAM Lett. (2020), we have proposed numerical verification methods to construct local Lyapunov functions around non-hyperbolic equilibria using non-linear transformations by up to third degree polynomials. In the present study, we extend these methods by proving that polynomials of an arbitrary degree can be applied to define the transformations and show an example problem where we have to use a fifth-degree polynomial to construct local Lyapunov functions.

Keywords numerical verification, dynamical systems, Lyapunov functions, non-hyperbolic equilibria

Research Activity Group Scientific Computation and Numerical Analysis

1. Introduction

Lyapunov functions are a type of energy function useful for analyzing dynamical systems. In general, the construction of Lyapunov functions is difficult. However, there exist numerical verification methods for constructing local Lyapunov functions in quadratic form around hyperbolic equilibrium [1], as well as around non-hyperbolic equilibrium in two-dimensional cases [3]. The method proposed in [3] is based on the normal form theory of dynamical systems. The outline of the method is as follows:

(1) Consider a non-linear transformation by a cubic polynomial and apply it to the original system.

(2) For the dynamical system obtained by the transformation, construct a local Lyapunov function \( L \) in quadratic form around the equilibrium of the transformed system.

(3) Derive \( L \) from \( L \) as a candidate for a local Lyapunov function around the equilibrium of the original system and verify a domain of \( L \) as a Lyapunov function through verified computation.

Although [3] only mentioned transformation by up to third degree polynomials, in reality we can also apply polynomials of degree four or higher. In this paper, we show that a polynomial of up to an arbitrary degree can be applied to derive a non-linear transformation using the method proposed in [3] for the two-dimensional problems specified below. This indicates that we can construct a local Lyapunov function using the methods in [3] for almost any case where the original system has a local Lyapunov function around the equilibrium.

As a numerical example, we present a problem in which we fail to construct a local Lyapunov function using the transformation by polynomials of degree three, but instead succeed using the transformation by a polynomial of degree five.

The present paper is structured as follows:

- Our problems, the definition of local Lyapunov functions, and the approach to determine transformation are described in Section 2.
- We prove that a polynomial up to an arbitrary degree can be applied as a non-linear transformation in Section 3.
- In Section 4, we show a numerical example that requires the transformation by a polynomial of degree five to successfully construct a local Lyapunov function.

2. Setting and approach

For a vector function \( \mathbf{v} = (v_1(t), v_2(t))^T \in C^1(\mathbb{R}, \mathbb{R}^2) \), consider the following system of equations:

\[
\dot{\mathbf{v}} = J\mathbf{v} + p^{(2)}(\mathbf{v}) + p^{(3)}(\mathbf{v}) + \ldots + p^{(r-1)}(\mathbf{v}) + \mathcal{O}(||\mathbf{v}||^r),
\]

where \( J \) implies \( dv/dt \),

\[
J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

is a two-dimensional matrix, \( p^{(i)}(\mathbf{v}) \in \mathbb{R}^2 (2 \leq i \leq r - 1) \) is a vector whose elements are polynomials of \( v_1, v_2 \) with terms of degree \( i \), and \( \mathcal{O}(||\mathbf{v}||^r) \) implies terms of degree \( r \) and higher. For this system, the origin \( \mathbf{0} \in \mathbb{R}^2 \) is a non-hyperbolic equilibrium. Note that even though (1) is a particular case of two-dimensional problems, it is significant because it represents the perturbation of Hamiltonian systems.
Hereafter, the solution of (1) at time $t$ with an initial value $v_0 \in \mathbb{R}^d$ at $t = 0$ is represented by $\varphi(t, v_0) \in \mathbb{R}^d$. Our definition of the local Lyapunov function is as follows:

**Definition 1.** Consider the following system of ordinary differential equations:

$$\dot{x} = f(x),$$

where $f \in C^k(\mathbb{R}^n, \mathbb{R}^n)$ with $k \geq 1$. Let $U \subset \mathbb{R}^n$ and $x^*$ be an open subset and an equilibrium of (2), respectively. A Lyapunov function $L : U \to \mathbb{R}$ is a $C^1$ function that satisfies the following conditions:

1. $(dL/dt)(\varphi(t, x))|_{t=0} \leq 0$ holds for each solution orbit $\{\varphi(t, x)\}$ for all $x \in U$.
2. $(dL/dt)(\varphi(t, x))|_{t=0} = 0$ implies $\varphi(t, x) \equiv x^* \in U$.

When the domain $U$ is a bounded neighborhood of $x^*$, we say that $L(x)$ is a local Lyapunov function for $x^*$. Note that our Lyapunov function can take negative values to treat saddle equilibria.

For the system of ODEs (1), consider the following transformation:

$$v = u + \sum_{m=2}^{M} M q^{(m)}(u) (2 \leq M \leq r - 1),$$

where $u = (u_1(t), u_2(t))^T \in C^1(\mathbb{R}, \mathbb{R}^2)$ and $q^{(m)}(u) \in \mathbb{R}^2$ is a vector whose elements are polynomials of $u_1, u_2$ with terms of degree $m$. From (1) and (3), we obtain the following system of ODEs of $u$:

$$\dot{u} = Ju + \sum_{m=2}^{M} M d^{(m)}(u) + O(||u||^{M+1}),$$

where $d^{(m)}(u) \in \mathbb{R}^2$ is a vector whose elements are polynomials of $u_1, u_2$ with terms of degree $m$.

For the system of ODEs (4), we aim to construct the following local Lyapunov function $L$:

$$L(u) = u^T Y u,$$

where $Y \in \mathbb{R}^{2x2}$ is a real symmetric matrix. Differentiating (5) with respect to $t$, we obtain

$$\frac{dL}{dt} = 2u^T Y u$$

$$= 2u^T Y \left( Ju + \sum_{m=2}^{M} M d^{(m)}(u) \right) + O(||u||^{M+2}).$$

If the quadratic term $2u^T Y J u = u^T (Y J + J^T Y)$ on the right-hand side is negative definite, then $L$ is a local Lyapunov function around the origin. Let matrix $Y$ be

$$Y = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Then, the matrix $Y J + J^T Y$ cannot be negative definite because the eigenvalues of $Y J + J^T Y$ are $\lambda = \pm \sqrt{4bc + (c-a)^2}$. Therefore, we have to assign the values $a = c \neq 0$, $b = 0$ in order to eliminate $2u^T Y J u$.

Now, consider

$$2u^T Y d^{(2)}(u) = 2au^T d^{(2)}(u),$$

which is the cubic term of $dL/dt$. This should vanish because it cannot take a definite sign. To specify $d^{(2)}(u)$, we explicitly state the time derivative of $u$:

$$\dot{u} = \dot{v} - Dq^{(2)}(u) \dot{u} + O(||u||^3)$$

$$= Jv + p^{(2)}(v) - Dq^{(2)}(u) \dot{v} + O(||u||^3)$$

$$= Ju + Jq^{(2)}(u) + p^{(2)}(u)$$

$$- Dq^{(2)}(u) J \dot{u} + O(||u||^3),$$

where $Dq^{(2)}(u)$ denotes a $2 \times 2$ real matrix representing the derivative of $q^{(2)}(u)$ with respect to $u$. Therefore, the equation

$$d^{(2)}(u) = Jq^{(2)}(u) - Dq^{(2)}(u) J \dot{u} + p^{(2)}(u)$$

holds.

Note that $p^{(2)}(u)$ is a given term and both $Jq^{(2)}(u)$ and $Dq^{(2)}(u) J \dot{u}$ can be controlled by the transformation. If we can find $q^{(2)}(u)$ such that

$$u^T (Jq^{(2)}(u) - Dq^{(2)}(u) J \dot{u}) = -u^T p^{(2)}(u)$$

for the given term $p^{(2)}(u)$, it satisfies $u^T d^{(2)}(u) = 0$.

Assume that $u^T d^{(2)}(u) = 0$ holds. Then, if we can choose $q^{(3)}(u)$ such that $au^T d^{(3)}(u)$ has a negative value for an arbitrary $u \neq 0$, the function $L$ is a local Lyapunov function around 0. If we cannot select such a $q^{(3)}(u)$, we aim to determine $q^{(3)}(u)$ such that $u^T d^{(3)}(u) = 0$ holds. We determine $q^{(3)}(u)$ in a similar manner. Namely:

1. When $m$ is even, determine $q^{(m)}(u)$ such that $u^T d^{(m)}(u) = 0$ holds.
2. When $m$ is odd, check whether we can choose $q^{(m)}(u)$ such that $u^T d^{(m)}(u)$ takes a negative value for an arbitrary $u \neq 0$. If we can, the function $L$ satisfies the conditions of local Lyapunov functions. If we cannot, determine $q^{(m)}(u)$ such that $u^T d^{(m)}(u) = 0$ holds.

Assume that we obtain a local Lyapunov function $L$ by the above procedure. Note that there exists an inverse transformation from $u$ to $v$ such that

$$u = h(v) + O(||v||^{m+1})$$

by virtue of the inverse function theorem [2]. A concrete form of the inverse transformation is described in Section 4. Let $L(v)$ be a candidate for the local Lyapunov function around 0 in the original systems, which is derived by substituting (6) into (5) and omitting terms of degree higher than $m + 1$. A numerical verification method to verify the domain of $L(v)$ as a local Lyapunov function is proposed in [3].

We prove that there exists $q^{(m)}(u)$ such that $u^T d^{(m)}(u) = 0$ holds in the next section.

3. The condition of $u^T d^{(m)}$

The term $d^{(m)}(u)$ is calculated by

$$d^{(m)}(u) = Jq^{(m)}(u) - Dq^{(m)}(u) J u + p^{(m)}(u),$$

where
where
\[
\hat{p}^{(m)}(u) = p^{(m)}(u) + d^{(m)}(u),
\]
\[
\hat{p}^{(m)}(u)
\]
is the sum of terms of degree \( m \) in \( p^{(2)}(u + \sum_{j=2}^{m-1} q^{(j)}(u)), \quad p^{(3)}(u + \sum_{j=2}^{m-2} q^{(j)}(u)), \quad \ldots, \quad \) and \( p^{(m)}(u) \), and
\[
\hat{d}^{(m)}(u) = - \sum_{k=2}^{m-1} Dq^{(k)}(u)d^{(m-(k-1))}(u).
\]

Let \( P^{(m)} \) be a vector space consisting of two-dimensional vectors such that each element is a polynomial of \( u_1 \) and \( u_2 \) with terms of degree \( m \). Define the mapping \( N^{(m)} : P^{(m)} \to P^{(m)} \) by
\[
N^{(m)}(q) = Jq - DqJu, \quad q \in P^{(m)}.
\]

Note that \( N^{(m)} \) is linear with respect to \( q \).

We choose
\[
\phi^{(m)}_i = \begin{cases} (u^{m-i}u_2^i) & (0 \leq i \leq m), \\ (u_1^{2m+1-i}u_2^{i-1}) & (m+1 \leq i \leq 2m+1) \end{cases}
\]
as base functions of \( P^{(m)} \). Setting
\[
q^{(m)}(u) = \sum_{i=0}^{2m+1} q_i \phi^{(m)}_i(u), \quad q_i \in \mathbb{R},
\]
we have
\[
u^T d^{(m)}(u) = u^T N^{(m)} \left( \sum_{i=0}^{2m+1} q_i \phi^{(m)}_i(u) \right) + u^T \hat{p}^{(m)}(u)
\]
from (7), (8), and the linearity of \( N^{(m)} \). Note that \( u^T N^{(m)}(\phi^{(m)}_i(u)) \) is a polynomial of \( u_1, u_2 \) with terms of degree \( m+1 \), and all polynomials of \( u_1, u_2 \) with terms of degree \( m+1 \) create an \( m+2 \) dimensional vector space, which we denote as \( P^{(m)} \) hereafter. The vector space \( Q^{(m)} \subset P^{(m)} \) is defined as
\[
Q^{(m)} = \text{span} \{ u^T N^{(m)}(\phi^{(m)}_0(u)), u^T N^{(m)}(\phi^{(m)}_1(u)), \ldots, u^T N^{(m)}(\phi^{(m)}_{2m+1}(u)) \}.
\]

If the dimension of \( Q^{(m)} \) is equal to \( m+2 \), we can choose \( q_i \) such that
\[
\sum_{i=0}^{2m+1} q_i u^T N^{(m)}(\phi^{(m)}_i(u)) = -u^T \hat{p}^{(m)}(u).
\]
Namely, \( u^T d^{(m)}(u) = 0 \).

As for the dimension of \( Q^{(m)} \), the following theorem holds:

**Theorem 2.** If \( m \) is even, the dimension of \( Q^{(m)} \) is \( m + 2 \). If \( m \) is odd, the dimension of \( Q^{(m)} \) is \( m + 1 \).

**Proof** Calculating \( u^T N^{(m)}(\phi^{(m)}_i(u)) \) concretely:
\[
u^T N^{(m)}(\phi^{(m)}_i(u)) = (m+1)u_1^m u_2^i,
\]
holds for \( i = 0, 2m+1, \) respectively, and
\[
u^T N^{(m)}(\phi^{(m)}_i(u)) = (m-i+1)u_1^{m-i}u_2^{i+1} - iu_1^{m-i+2}u_2^{-1}
\]
holds for \( 1 \leq i \leq m \). This indicates
\[
\text{span}\{u^T N^{(m)}(\phi^{(m)}_0(u)), \ldots, u^T N^{(m)}(\phi^{(m)}_{2m+1}(u))\} = \text{span}\{u^T N^{(m)}(\phi^{(m)}_0(u)), \ldots, u^T N^{(m)}(\phi^{(m)}_m(u)), u^T N^{(m)}(\phi^{(m)}_{2m+1}(u))\}.
\]

Let us prove the linear independence of \( u^T N^{(m)}(\phi^{(m)}_0(u)), \ldots, u^T N^{(m)}(\phi^{(m)}_m(u)) \). Assume
\[
\sum_{i=0}^{m} q_i u^T N^{(m)}(\phi^{(m)}_i(u)) = 0.
\]

Then, the coefficients of each term \( u_1^{m+1}, u_1^m u_2, u_1^{m-1}u_2^2, \ldots, u_1^2 u_2^m, u_2^{m+1} \) on the left-hand side are equal to 0, and we obtain the following simultaneous linear equations:
\[
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
m+1 & 0 & -2 & 0 & \cdots \\
m & 0 & 0 & \cdots & 0 \\
2 & 0 & \cdots & 0 & 1
\end{pmatrix}
\]
holds for \( m+1 \leq i \leq m \). This indicates \( \text{span}\{u^T N^{(m)}(\phi^{(m)}_0(u)), \ldots, u^T N^{(m)}(\phi^{(m)}_{2m+1}(u))\} = \text{span}\{u^T N^{(m)}(\phi^{(m)}_0(u)), \ldots, u^T N^{(m)}(\phi^{(m)}_m(u)), u^T N^{(m)}(\phi^{(m)}_{2m+1}(u))\} \).
From Theorem 2, there exists $q^{(m)}(u) \in P^{(m)}$ such that $u^T d^{(m)}(u) = 0$ when $m$ is even.

When $m$ is odd, the following theorem holds:

**Theorem 3.** Assume that $m$ is odd. If $u^T \tilde{p}^{(m)}(u) \notin Q^{(m)}$, then we can determine $q^{(m)}(u) \in P^{(m)}$ and $a \in \mathbb{R}$ such that $au^T d^{(m)}(u) < 0$ holds for any $u \neq 0$. On the contrary, if $u^T \tilde{p}^{(m)}(u) \in Q^{(m)}$, then we can determine $q^{(m)}(u)$ such that $u^T d^{(m)}(u) = 0$ holds.

**Proof.** Setting $q^{(m)}(u) = \sum_{j=0}^{2m+1} q_j \phi_j^{(m)}$, we have

$$u^T d^{(m)}(u) = \sum_{j=1}^{2m+1} a_{q_j} u^T N^{(m)}(\phi_j^{(m)}) + au^T \tilde{p}^{(m)}(u).$$

Note that the dimension of $P^{(m)}$ is $m+2$ and the dimension of $Q^{(m)} \subset P^{(m)}$ is $m+1$.

If $au^T \tilde{p}^{(m)}(u) \notin Q^{(m)}$
holds, $au^T N^{(m)}(\phi_0^{(m)})$, $au^T N^{(m)}(\phi_1^{(m)})$, $\ldots$, $au^T N^{(m)}(\phi_m^{(m)})$, $u^T \tilde{p}^{(m)}(u)$ can be chosen as the base functions of $P^{(m)}$. Then, we can choose $q_j$ and $a$ such that $au^T d^{(m)}(u) < 0$ holds.

Now, assume that

$$au^T \tilde{p}^{(m)}(u) \in Q^{(m)}.$$

Then, we can choose $q_j$ such that

$$\sum_{j=1}^{2m+1} a_{q_j} u^T N^{(m)}(\phi_j^{(m)}) = -au^T \tilde{p}^{(m)}(u)$$
holds, which implies $au^T d^{(m)}(u) = 0$.

(QED)

Then we determine

$$q^{(3)} = \begin{pmatrix}
\frac{1}{4} u_1^2 + u_1^2 u_2 + u_2^3 \\
-\frac{1}{4} u_2^3
\end{pmatrix}$$

such that $u^T d^{(3)} = 0$ holds. Using polynomials of degree four and five, we determine

$$q^{(4)} = \begin{pmatrix}
\frac{1}{4} u_4^4 - 2u_1^2 u_2 + u_1^2 u_2^2 - 3u_1 u_2^3 + \frac{1}{2} u_2^5 \\
-\frac{1}{4} u_2^6
\end{pmatrix}$$

such that $u^T d^{(4)} = 0$.

$$q^{(5)} = \begin{pmatrix}
-\frac{1}{60} u_4^4 + \frac{1}{6} u_1^2 u_2 + u_1^2 u_2^2 + \frac{1}{6} u_1^2 u_2^2 + 2u_1 u_2^3 + u_2^6
\end{pmatrix}$$

with $a = 2$ such that $u^T d^{(5)} < 0$, respectively.

Using them, we obtain

$$dL/dt = -u_1^6 - u_2^6.$$  (12)

Setting the inverse transformation

$$u = v - q^{(5)}(v) - q^{(4)}(v - q^{(2)}(v)) - q^{(3)}(v - q^{(3)}(v - q^{(2)}(v))) - q^{(4)}(v - q^{(4)}(v - q^{(4)}(v - q^{(2)}(v)))) - q^{(5)}(v - q^{(5)}(v - q^{(4)}(v - q^{(3)}(v)))) + O(||v||^6)$$

to (5) and omitting terms of degree higher than six, we obtain

$$L(v) = \frac{130}{3} v_1^6 - 35v_1^4 v_2 + 20v_1^2 v_2^2 + 36v_1^4 v_2^2 - 12v_1^4 v_2^2 + 9v_1^4$$

$$- \frac{139}{3} v_2^6 + 32v_2^4 v_2^2 - 4v_2^4 v_2^2 + 4v_3^6 + 46v_1^2 v_2^4$$

$$- 12v_1^2 v_2^4 + 12v_1^2 v_2^4 + 2v_1^2 - 11v_1 v_2^4 + 12v_1 v_2^4$$

$$- 4v_1 v_2^4 + 4v_1 v_2^4 + 10v_2^6 + 3v_1^4 + 2v_2^4.$$  (13)

By applying the numerical verification method proposed in [3] to the time derivative of $L(v)$ in (13), we have verified that $[-0.001, 0.001]^2$ is the domain of $L(v)$ as a local Lyapunov function for the equilibrium $v = 0$.

5. Conclusion

We prove that polynomials up to an arbitrary degree can be applied as non-linear transformations using the method proposed in [3] for particular two-dimensional problems with $J$ in (1). Furthermore, we apply a polynomial of degree five as the transformation using our method and construct a local Lyapunov function in our numerical example.

One of our future works is to apply our method to three or higher dimensional problems.

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