Integrand-Level Reduction of Loop Amplitudes by Computational Algebraic Geometry Methods

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ABSTRACT: We present an algorithm for the integrand-level reduction of multi-loop amplitudes of renormalizable field theories, based on computational algebraic geometry. This algorithm uses (1) the Gröbner basis method to determine the basis for integrand-level reduction, (2) the primary decomposition of an ideal to classify all inequivalent solutions of unitarity cuts. The resulting basis and cut solutions can be used to reconstruct the integrand from unitarity cuts, via polynomial fitting techniques. The basis determination part of the algorithm has been implemented in the Mathematica package, BasisDet. The primary decomposition part can be readily carried out by algebraic geometry softwares, with the output of the package BasisDet. The algorithm works in both $D = 4$ and $D = 4 - 2\varepsilon$ dimensions, and we present some two and three-loop examples of applications of this algorithm.
1 Introduction

The study of higher-loop amplitudes for gauge theories is important for both theoretical and phenomenological reasons. The data analysis of Large Hadron Collider (LHC) physics requires great accuracy of the standard-model cross sections computation. For many channels, not only the next-to-leading order (NLO) amplitudes, but also the next-to-next-to-leading order (NNLO) amplitudes are important in order to control theoretical uncertainties.

The traditional Feynman diagram approach for amplitude calculation becomes very complicated in the higher-loop cases. Integration-by-parts (IBP) identities were used to reduce the number of integrals in loop diagrams [1], and efficiently implemented in Laporta algorithm [2]. The method of Gröbner basis was used to express Feynman integrals as a linear combination of master integrals [3–5].
New methods based on unitarity [6–8] decompose loop amplitudes as the product of tree amplitudes and greatly simplify the computation. There is a particularly convenient method: integrand-level reduction (OPP method) [9], which decomposes the amplitude directly at the integrand level. OPP method can be used to automatically and efficiently calculate one-loop amplitudes with multiple legs [10–15]. The original OPP method can be generalized to $D = 4 - 2\epsilon$ at one loop [16–18]. The polynomial division method was first used in the integrand-level reduction for one-loop amplitudes in ref. [19].

The generalized unitarity method also applies to two-loop amplitudes: Buchbinder and Cachazo calculated two-loop planar amplitudes for $\mathcal{N} = 4$ super-Yang-Mills [20], by generalized unitarity. Gluza, Kajda and Kosower [21] used a Gröbner basis to find IBP relations without double propagators, and then determined the master integrals for two-loop planar diagrams. The IBP relations can also be generated by linear algebra, without using Gröbner basis [22]. With these master integrals, Kosower and Larsen [23] applied maximal unitarity method for two-loop planar diagrams to obtain the coefficients of master integrals from the products of tree amplitudes. Furthermore, Larsen [24] applied this method to two-loop six-point amplitudes, using multidimensional contour integrals. The two-loop double-box diagram maximal-cut solutions can be related to Riemann surfaces, whose geometry uniquely defines the contour integrals [25].

Alternatively, using integrand-level reduction, Mastrolia and Ossola [26] applied OPP-like methods to study two-loop $\mathcal{N} = 4$ super-Yang-Mills amplitudes. Badger, Frellesvig and Zhang [27] then used the Gram-matrix method to find the integrand basis systematically for two-loop amplitudes of general renormalizable theories. They calculated the double-box and crossed-box contributions to two-loop four-point $\mathcal{N} = 0, 1, 2, 4$ (super)-Yang-Mills amplitudes.

It is interesting to generalize and automate the integrand-level reduction to amplitudes with more legs and more than two loops. The main limitations in the previous integrand-level reductions are,

1. The basis for integrand-level reduction grows quickly as the number of loops increases. At three-loop order and the beyond, the Gram-matrix method becomes very complicated and the integrand-level basis is hard to obtain.

2. It is difficult to find and classify all inequivalent unitarity cut solutions for complicated diagrams. It is necessary to find all cut solutions, to reconstruct the integrand. However, for diagrams with many legs or more than two loops, the solutions become complicated. And often two solutions appear to be different, are but actually equivalent after reparametrization [25]. We have to remove this redundancy before reconstructing the integrand.

These difficulties come from the complexity of the algebraic system of cut equations. The ideal approach to deal with these problems is computational algebraic geometry. In this paper, we reformulate these two problems as classic mathematical problems and solve them by powerful mathematical tools,
1. Integrand-level basis is equivalent to the linear basis in the quotient ring, of polynomials in irreducible scalar products (ISPs) modulo the cut equations. Then the integrand basis can be generated automatically using the standard Gröbner basis and polynomial reduction methods [28, Ch. 5].

2. The collection of all cut solutions is an algebraic set. The latter can be uniquely decomposed to a finite number of affine varieties. Each variety is an independent solution of the unitarity cuts, and different varieties are not equivalent by reparametrization. In practice, this decomposition is automatically done by primary decomposition of an ideal [29, Ch. 1]. This classifies all inequivalent unitarity cut solutions. Furthermore, dimension theory in algebraic geometry [29, Ch. 1] can determine the number of free parameters in each solution.

We implement the first part of our algorithm in the Mathematica package BasisDet which can automatically generate the integrand-level basis. It also provides a list of irreducible scalar products (ISP)’s and the ideal $I$ generated by the cut equations. The latter information can be directly used by computational algebraic geometry software, like Macaulay2 [30], to carry out the second part of the algorithm. Once the primary decomposition is done, we get all inequivalent solutions of the unitarity cuts. Furthermore, for each solution, the software will find the number of free parameters.

The package BasisDet has been tested for $D = 4$ and $D = 4 - 2\epsilon$ one-loop box, triangle and bubble diagrams, $D = 4$ two-loop four-point double-box, crossed-box, pentagon-triangle diagrams, $D = 4$ two-loop five-point double-box diagram, pentagon-box diagram, and $D = 4 - 2\epsilon$ two-loop four-point diagram. It has also been tested in two-loop level diagrams beyond maximal unitarity, for example, $D = 4$ two-loop four-point box-triangle, sunset and double-bubble diagrams. The output bases have been verified for all these cases.

We have also used this algorithm to calculate $D = 4$ three-loop triple-box basis, and have verified that terms inside the basis are linearly-independent on the unitarity cuts. We also successfully carried out a primary decomposition on this diagram to find all the inequivalent cut solutions.

This paper is organized as follows. In section 2, we briefly review the known integrand-level reduction for one and two-loop diagrams. The limitation of previous approaches is also pointed out. In section 3, our new algorithm is presented, and its validity is mathematically proven. Then in section 4, several examples are presented for one, two and three-loop diagrams. Finally, our conclusion and discussion on future directions are provided in section 5. The manual for the package BasisDet is given in Appendix A.

The package BasisDet and examples are included in ancillary files of the arXiv version of this paper. The package and its future updates can also be downloaded from the website, http://www.nbi.dk/~zhang/BasisDet.html.

2 Review of integrand-level reduction methods

In this section, we briefly review integrand-level reduction for one and two-loop amplitudes. (See [18] for detailed review of the one-loop integrand reduction.)
2.1 One-loop integrand-level reduction

\[ P_1 = p_{i_1,i_2}, P_2 = p_{i_2,i_3}, P_3 = p_{i_3,i_4}, P_4 = p_{i_4,i_1} \]

Figure 1. One-loop box diagram

Schematically, for \( D = 4 \), an one-loop amplitude must be decomposed as \( [9] \),

\[
A_n^{(1)} = \int \frac{d^4k}{(2\pi)^{4/2}} \sum_{i_1=1}^{n-3} \sum_{i_2=i_1+1}^{n-2} \sum_{i_3=i_2+1}^{n-1} \sum_{i_4=i_3+1}^{n} \frac{\Delta_{4,i_1i_2i_3i_4}(k)}{D_{i_1}D_{i_2}D_{i_3}D_{i_4}} \\
+ \sum_{i_1=1}^{n-2} \sum_{i_2=i_1+1}^{n-1} \sum_{i_3=i_2+1}^{n} \frac{\Delta_{4,i_1i_2i_3}(k)}{D_{i_1}D_{i_2}D_{i_3}} \sum_{i_4=i_1+2}^{n-2} \frac{\Delta_{2,i_1i_2}(k)}{D_{i_1}D_{i_2}}
\]

\[ + \text{-tadpoles, wave-function bubbles and rational terms,} \]

where we define the propagators \( D_{i_x} = (k - p_{i_x,i_x-1})^2 \), \( p_{i,j} = \sum_{k=i}^{j} p_k \) (figure 1) as the sum of external momenta such that \( p_{i_1,i_1-1} = 0 \), and have taken the restriction that all propagators are massless. We must require that \( \Delta_{4,i_1i_2i_3i_4} \) contain no term which is proportional to \( D_{i_x} \), \( x = 1, \ldots, 4 \), otherwise one of the denominator in the integral is cancelled out. Similarly, \( \Delta_{3,i_1i_2i_3} \) must contain no term proportional to \( D_{i_x} \), \( x = 1, \ldots, 3 \) and so on.

Consider \( \Delta_{4,i_1i_2i_3i_4} \) first. There exists a vector \( \omega \) perpendicular to \( p_{i_x,i_x-1} \), \( x = 2, 3, 4 \). Of all the scalar products involving loop momenta, only \( k \cdot \omega \) is not a polynomial in denominators \( D_{i_1}, D_{i_2}, D_{i_3} \) and \( D_{i_4} \). We call such scalar products irreducible scalar products (ISPs) and the other scalars products reducible scalar products (RSPs). \( \Delta_{4,i_1i_2i_3i_4} \) should be a function of ISPs, i.e. \( (k \cdot \omega) \) only.

Furthermore, we find that \( \Delta_{4,i_1i_2i_3i_4} \) is at most linear in \( k \cdot \omega \),

\[
\Delta_{4,i_1i_2i_3i_4} = c_0 + c_1(k \cdot \omega).
\]

\( c_0 \) and \( c_1 \) are constants independent of the loop momentum. Higher-order terms in \( (k \cdot \omega) \) are absent, because

\[
(k \cdot \omega)^2 = \text{linear combination of } D_{i_1}, D_{i_2}, D_{i_3} \text{ and } D_{i_4}.
\]
The coefficients $c_0$ and $c_1$ can be calculated from quadruple cuts, $D_{i_1}(k^{(s)}) = D_{i_2}(k^{(s)}) = D_{i_3}(k^{(s)}) = D_{i_4}(k^{(s)}) = 0$, where $s = 1, 2$ since there are two cut solutions. The integrand at the two cut solutions determined the two coefficients $c_0$ and $c_1$. Note that although the term $c_1(k \cdot \omega)$ integrates to zero, it is necessary to keep it for the triple-cut calculation. We call the set $\{1, (k \cdot \omega)\}$ the integrand basis for $D = 4$ one-loop quadruple cut and $(k \cdot \omega)$ the spurious term.

Similarly, $\Delta_{3,i_1i_2i_3}$ can be reconstructed from triple cuts. We have two vectors $\omega_1$ and $\omega_2$, which are in perpendicular to the external momenta. There are two ISPs, $k \cdot \omega_1$ and $k \cdot \omega_2$. The expansion over integrand basis reads,

$$
\Delta_{3,i_1i_2i_3}(k) = c_{00} + c_{10}(k \cdot \omega_1) + c_{01}(k \cdot \omega_2) + c_{11}(k \cdot \omega_1)(k \cdot \omega_2)
$$

$$
+ c_{12}(k \cdot \omega_1)(k \cdot \omega_2)^2 + c_{21}(k \cdot \omega_1)^2(k \cdot \omega_2) + c_{20,02}((k \cdot \omega_1)^2 - (k \cdot \omega_2)^2).
$$

(2.4)
The basis contains 7 terms, of which 6 are spurious. There are two cut solutions for the triple cut,

$$
D_{i_1}(k^{(s)}(\tau)) = D_{i_2}(k^{(s)}(\tau)) = D_{i_3}(k^{(s)}(\tau)) = 0,
$$

(2.5)
with $s = 1, 2$, but each of them contains one complex free parameter $\tau$. The original numerator of the integral at triple cuts, with all $\Delta_{4,i_1i_2i_3i_4}$ subtracted, becomes a Laurent series in $\tau$. The corresponding Laurent coefficients determine all the 7 "c" coefficients of eq. (2.4).

The one-loop integrand-level reduction also works for $D = 4 - 2\epsilon$ [18]. The loop momenta contains both the four-dimensional part and the extra-dimensional part $^1$,

$$
k = k^{[4]} + k^\perp, \quad (k^\perp)^2 \equiv -\mu^2.
$$

(2.6)
For the quadruple cut, the basis has larger size than that of the $D = 4$ case. Instead of (2.3), we have

$$
(k \cdot \omega)^2 - \mu^2 = \text{linear combination of } D_{i_1}, D_{i_2}, D_{i_3} \text{ and } D_{i_4}.
$$

(2.7)
So we can remove either $(k \cdot \omega)^2$ or $\mu^2$ to obtain an integrand basis. One convenient choice is

$$
\Delta_{4,i_1i_2i_3i_4}^{4-2\epsilon} = c_0 + c_1(k \cdot \omega) + c_2\mu^2 + c_3(k \cdot \omega)\mu^2 + c_4\mu^4.
$$

(2.8)
The $D = 4 - 2\epsilon$ quadruple cut has only one solution. This solution depends on one complex free parameter $\tau$. Note that geometrically, this solution is complex one-dimensional, and contains the two $D = 4$ box quadruple cut solutions (zero-dimensional) as two isolated points. The Taylor expansion in $\tau$ of the integrand, at the quadruple cut, determined the coefficients $c_0$, $c_1$, $c_2$, $c_3$ and $c_4$.

$^1$Throughout this paper, we use the scheme that all external momenta and polarization vectors have no $(-2\epsilon)$-dimensional components.
2.2 Two-loop integrand-level reduction

For the one-loop cases considered above, it is relatively easy to determine the integrand basis and find the unitarity cut solutions. However, in the two-loop cases, it is much harder to find the integrand basis and the unitarity cut solutions are more complicated.

Mastrolia and Ossola [26] applied the OPP-like method for two-loop $N = 4$ super Yang-Mills amplitudes. Two-loop four-point amplitudes for general renormalizable theories were calculated in [27]. To show clearly new features of two-loop integrand-level reduction, we review the Gram-matrix method presented in ref. [27].

For example, consider the two-loop four-point planar diagram (Figure 2). The integrand-level reduction reads,

$$A_4^{[\text{dbox}]}(1, 2; 3, 4;) = \int \int \frac{d^4k}{(2\pi)^{4/2}} \frac{d^4q}{(2\pi)^{4/2}} \left( \frac{\Delta_{7,12\times34^*}^{\text{dbox}}(k, q)}{D_1 D_2 \ldots D_7} \right) + \ldots$$  

(2.9)

where ... stands for the integrals with less than 7 propagators. Our aim is to reconstruct the double box function $\Delta_{7,12\times34^*}^{\text{dbox}}$ from hepta-cuts (maximal cut for diagrams with 7 propagators). Again, there exists one vector $\omega$ which is perpendicular to all the external momenta. There are four ISPs: $(k \cdot p_4)$, $(q \cdot p_1)$, $(k \cdot \omega)$ and $(q \cdot \omega)$. The integrand basis

![Figure 2. Four-point two-loop planar diagram](image-url)

consists of terms with the form,

$$(k \cdot p_4)^m (q \cdot p_1)^n (k \cdot \omega)^\alpha (q \cdot \omega)^\beta,$$

(2.10)

where $m$, $n$, $\alpha$ and $\beta$ are non-negative integers. For renormalizable theories, power counting requires that, $m + n + \alpha + \beta \leq 6$, $m + \alpha \leq 4$ and $n + \beta \leq 4$. Furthermore, it is easy to see that $(k \cdot \omega)^2$, $(q \cdot \omega)^2$ and $(k \cdot \omega)(q \cdot \omega)$ are linear combinations of the seven denominators. Hence $\alpha \leq 1$, $\beta \leq 1$ and $\alpha \cdot \beta = 0$. The above analysis is similar to that of one-loop cases, and the integrand basis appears to contain 56 terms.

However, there are more constraints. For four dimension momenta, the determinants of $5 \times 5$ Gram matrices are zero,

$$\det G \begin{pmatrix} 1 & 2 & 4 & k & q \\ 1 & 2 & 4 & k & q \end{pmatrix} = 0, \quad \det G \begin{pmatrix} 1 & 2 & 4 & k & q \\ 1 & 2 & 4 & \omega & k \end{pmatrix} = 0, \quad \det G \begin{pmatrix} 1 & 2 & 4 & k & q \\ 1 & 2 & 4 & \omega & q \end{pmatrix} = 0. \quad (2.11)$$

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For example, on the hepta-cut, the first Gram-matrix relation reads,

$$0 = 4(k \cdot p_4)^2(q \cdot p_1)^2 + 2s(k \cdot p_4)^2(q \cdot p_1) + 2s(k \cdot p_4)(q \cdot p_1)^2 - st(k \cdot p_4)(q \cdot p_1) \tag{2.12}$$

which means that either $m \leq 2$ or $n \leq 2$. Finally, the integrand basis for the double box amplitude is,

$$\Delta_{7;12\times 34}^{\text{dbox}}(k, q) = \sum_{mna \beta} c_{mn(a+2\beta)}(k \cdot p_4)^m(q \cdot p_1)^n(k \cdot \omega)^a(q \cdot \omega)^\beta. \tag{2.13}$$

There are 16 non-spurious terms,

$$(c_{000}, c_{010}, c_{100}, c_{020}, c_{110}, c_{002}, c_{102}, c_{022}, c_{112}, c_{003}, c_{103}, c_{030}, c_{120}, c_{210}, c_{300}). \tag{2.14}$$

and 16 spurious terms,

$$(c_{001}, c_{011}, c_{101}, c_{021}, c_{111}, c_{003}, c_{103}, c_{023}, c_{113}, c_{032}, c_{122}, c_{132}). \tag{2.15}$$

We comment that, alternatively, (2.12) can be obtained using the elimination method on the 7 cut equations. However, its computation is quite long and not systematic, comparing with the Gram-matrix method. In two-loop cases, the Gram-matrix method provides a very efficient way to determine the basis.

Again, we can determine the 32 coefficients from the hepta-cuts solutions. The solutions are much more complicated than one-loop cut solutions: there are 6 solutions, and each of which depends on a free parameter, $\tau$. From the Taylor or Laurent expansion of the integrand at hepta-cuts, we can solve for the 32 coefficients, in cases of Yang-Mills and $\mathcal{N} = 1, 2, 4$ super-Yang-Mills theories. Then IBP relations can further reduce the 16 non-spurious integrals to two master integrals. However, to get the lower cut functions like the hexa-cut case, we have to subtract all the 32 terms first, not only the two master integrals.

The four-point non-planar function $\Delta_{7;11\times 23;4}^{\text{xbox}}$ has been determined by the same method [27].

The Gram-matrix method becomes more complicated as we attempt to add more loops and legs. Furthermore, it is not easy to automate the Gram-matrix method: once a Gram-matrix relation is found, we need to determine which monomial inside the relation should be removed from the basis. For diagrams with many legs or more than two loops, it is also complicated to classify all the cut solutions. Hence a new automatic algorithm is needed, to carry out integrand-level reduction for higher-loop and many-leg amplitudes.

3 The algorithm

We present an automatic algorithm for integrand-level basis determination for generalized unitarity, based on the techniques of computational algebraic geometry. The goal is (1) to find the integrand basis by Gröbner basis method (2) to classify all inequivalent unitary cut solutions by primary decomposition and find the dimension of each solution.
3.1 Setup

We parametrize the loop momenta using scalar products. This is a variation of van Neerven-Vermaseren basis [31]. This parameterization has the following advantages:

- It does not depend on spinor helicity formalism or particular basis choices.
- The cut equations in terms of scalar products have a particularly simple form. This makes it convenient to carry out primary decomposition later. It is also easier to apply polynomial fitting techniques to reconstruct the integrand.

Consider an $L$-loop $n$-point diagram. The dimension is $D = d$ or $D = d - 2\varepsilon$. $d$ is an integer which stands for the dimension of the physical spacetime, while $-2\varepsilon$ is the number of extra dimensions introduced by dimensional regularization. In most examples, we consider $d = 4$.

Let $(l_1, \ldots, l_L)$ be the loop momenta and $(p_1, \ldots, p_n)$ be the external momenta. We use the scheme that all extra momenta and polarization vectors have no extra-dimensional components. The momenta $p_j$ can be either massless or massive.

We choose a basis $\{e_1, \ldots, e_d\}$ for the physical spacetime. Each $e_i$ is either an external momentum or an $\omega_j$, that is a momentum perpendicular to all the external legs. We define the $d \times d$ Gram matrix,

$$G_d = G\left(\begin{array}{c} e_1, \ldots, e_d \\ e_1, \ldots, e_d \end{array}\right),$$

where $G_d$ is nonsingular, as it should be.

For the case $D = d - 2\varepsilon$, we decompose the loop momenta into physical and extra-dimensional components,

$$l_i = l_i^{[d]} + l_i^\perp,$$

and we define $\mu_{ij} \equiv -l_i^\perp \cdot l_j^\perp$. For the case $D = d$, we simply set $l_i^\perp = 0$ and all the $\mu_{ij}$ are absent.

We parametrize $l_i^{[d]}$ using scalar products, $(l_i \cdot e_j)$, $1 \leq j \leq D$,

$$l_i^{[d]} = (e_1, \ldots, e_d) G_d^{-1} \begin{pmatrix} (l_i \cdot e_1) \\ \vdots \\ (l_i \cdot e_d) \end{pmatrix}. \quad (3.3)$$

We define the set of (fundamental-)scalar products (SPs) to be

$$\mathcal{SP} = \{(l_i \cdot e_j) | 1 \leq i \leq L, 1 \leq j \leq d\} \cup \{\mu_{ij} | 1 \leq i \leq L, i \leq j \leq L\}, \quad D = d - 2\varepsilon, \quad (3.4)$$

or

$$\mathcal{SP} = \{(l_i \cdot e_j) | 1 \leq i \leq L, 1 \leq j \leq d\}, \quad D = d. \quad (3.5)$$

All the other scalar products, like $(l_i \cdot u)$, $l_i^2$, $(l_i \cdot l_j)$, where $u$ is a constant vector in the physical dimension, can be written as polynomial functions of (fundamental-)scalar products.
products, using the Gram matrix $G_d$. For example,

$$l_i \cdot u = ((l_i \cdot e_1), ..., (l_i \cdot e_d)) G^{-1}_d \begin{pmatrix} (u \cdot e_1) \\ \vdots \\ (u \cdot e_d) \end{pmatrix}, \quad (3.6)$$

$$l_i \cdot l_j = ((l_i \cdot e_1), ..., (l_i \cdot e_d)) G^{-1}_d \begin{pmatrix} (l_j \cdot e_1) \\ \vdots \\ (l_j \cdot e_d) \end{pmatrix} - \mu_{ij}. \quad (3.7)$$

Next, we consider the $m$-fold unitarity cut of the amplitude, i.e., $m$ propagators are set to zero.

$$D_1(l_1, \ldots, l_L) = \ldots = D_m(l_1, \ldots, l_L) = 0, \quad (3.8)$$

Using the Gram matrix $G_d$, these cut equations can be expressed as polynomial equations in the SPs. We denote the polynomial ring of SPs, i.e. the collection of all polynomials in SPs, as $R'$. Then we introduce the concept of an ideal in a ring [28, Ch. 1]: in general, an ideal $J$ generated by several polynomials $f_1, \ldots, f_k$ in a ring $S$, is the subset of $S$,

$$J = \langle f_1, \ldots, f_k \rangle = \{ a_1 f_1 + \ldots + a_k f_k | \forall a_i \in S, 1 \leq i \leq k \}, \quad (3.9)$$

where $a_i$s are arbitrary polynomials in $S$. Here we define,

$$I' = \langle D_1, \ldots, D_m \rangle, \quad (3.10)$$

which is the ideal generated by all the cut equations in terms of SPs.

Some scalar products’ values are uniquely determined at all cut solutions, i.e., they are polynomials of propagators,

$$x = \text{const} + O(D_1, \ldots, D_m). \quad (3.11)$$

We may call these scalar products reducible scalar products (RSPs) and all the other scalar products in SP irreducible scalar products (ISPs).

In practice, we can extend the definition of RSPs. For example, if two scalar products $x_1$ and $x_2$, satisfy the relation,

$$\alpha_1 x_1 + \alpha_2 x_2 = \text{const} + O(D_1, \ldots, D_m), \quad (3.12)$$

where $\alpha_1$ and $\alpha_2$ are nonzero constant. We may pick up one of the two scalar products as RSP, say $x_1$, and write it as a linear function of $x_2$ on the multiple cut.

Hence we have the following formal definition of reducible scalar products (RSP) and and all the other scalar products irreducible scalar products (ISP):

**Definition 1.** The set ISP of irreducible scalar products is a minimal subset of SP, such that all the other scalar products can be expressed as linear functions in ISPs on the unitarity cut.
This definition minimizes the number of ISPs, so the following calculation will be simpler. The choice of ISP is not unique but different choices are equivalent. We have the following decomposition:

$$\text{ISP} = \text{RSP} \cup \text{ISP}. \quad (3.13)$$

To simplify notations, we label the ISPs by $x_1, \ldots, x_{n_I}$.

We can eliminate all the RSPs from the cut equations to obtain a new set of algebraic equations $F$ in ISPs. With an abuse of notations,

$$F = \{D_k(x_1, \ldots, x_{n_I}) = 0|1 \leq k \leq m\}, \quad (3.14)$$

where $D_k$ is the polynomial in ISPs obtained from rewriting the $k$-th propagators in terms of ISPs, after RSPs are eliminated.

We denote the polynomial ring of ISPs as $R$, and the ideal generated by $D_\alpha$'s as $I$,

$$I = \langle D_1, \ldots, D_m \rangle \subset R, \quad (3.15)$$

where $\langle \ldots \rangle$ stands for an ideal generated by several polynomials.

It is easy to identify the RSPs and ISPs by hand for one and two-loop diagrams. However, this calculation becomes messy for more complicated diagrams. In practice, the identification of the RSP and ISP can be done quickly and systematically using Gröbner basis method, as described in appendix B.

The algebraic equation system $F$ in ISPs plays the central role in our algorithm. We will see that it contains all the information on cut solutions and the integrand basis.

### 3.2 Algorithm for integrand basis determination

In this section, we present an automatable algorithm for the determination of the integrand basis.

From the previous section, we see that all Lorentz invariants can be reduced to polynomials of (fundamental-)scalar products. Furthermore, RSPs can be reduced to constants or linear functions of ISPs. Hence, schematically, on $m$-fold unitarity cuts of a $L$-loop amplitude, the numerator of the integrand is reduced to a polynomial of ISPs, like $\Delta_{L_{\text{loop}}}$, $\Delta_{\text{topology}}$ and $\Delta_{\text{two-loop}}$,

$$\Delta_{L_{\text{loop}} m} = \sum_{(\alpha_1, \ldots, \alpha_{n_I}) \in S} c_{\alpha_1 \ldots \alpha_{n_I}} x_1^{\alpha_1} \ldots x_{n_I}^{\alpha_{n_I}}. \quad (3.16)$$

Here the tuple $(\alpha_1, \ldots, \alpha_{n_I})$ groups together the non-negative integer powers of the ISPs. We further require that $\Delta_{L_{\text{loop}} m}$ have no dependence in the propagators $D_i$, $i = 1, \ldots, m$. In previous examples, this was achieved by using cut equations directly for one-loop topologies, or Gram-matrix method for two-loop topologies. The finite set $S$ contains all the power tuples for the reduction. The constant coefficients $c_{\alpha_1 \ldots \alpha_{n_I}}$ are independent of loop momenta. They can be fitted from tree amplitudes by unitarity as in previous examples.

The set $B$ of the monomials $x_1^{\alpha_1} \ldots x_{n_I}^{\alpha_{n_I}}$ appearing in (3.16), is called the integrand basis. The goal is to determine this basis, or equivalently, the finite set $S$. We translate the requirement that $\Delta_{L_{\text{loop}} m}$ have no dependence in the propagators $D_i$, $i = 1, \ldots, m$ into mathematical language,
Proposition 1. The monomials in the integrand basis must be linearly independent in the quotient ring $R/I$.

Proof. Otherwise, there exist constant coefficients $d_{\alpha_1 \ldots \alpha_n}$, $(\alpha_1, \ldots, \alpha_n) \in S$ such that

$$\sum_{(\alpha_1, \ldots, \alpha_n) \in S} d_{\alpha_1 \ldots \alpha_n} x_1^{\alpha_1} \ldots x_n^{\alpha_n} = \sum_{k=1}^{m} f_k D_k$$

(3.17)

where $f_k$’s are polynomials in ISPs. Suppose that one coefficient $d_{\beta_1 \ldots \beta_n}$ is not zero. We define a subset $\tilde{S} = S - \{ (\beta_1, \ldots, \beta_n) \}$. Then $\Delta_m^{L\text{-loop}}$ can be reduced even further,

$$\Delta_m^{L\text{-loop}} = \sum_{(\alpha_1, \ldots, \alpha_n) \in S} c_{\alpha_1 \ldots \alpha_n} x_1^{\alpha_1} \ldots x_n^{\alpha_n}$$

$$= \sum_{(\alpha_1, \ldots, \alpha_n) \in \tilde{S}} \left( c_{\alpha_1 \ldots \alpha_n} - \frac{c_{\beta_1 \ldots \beta_n}}{d_{\beta_1 \ldots \beta_n}} d_{\alpha_1 \ldots \alpha_n} \right) x_1^{\alpha_1} \ldots x_n^{\alpha_n} + \frac{c_{\beta_1 \ldots \beta_n}}{d_{\beta_1 \ldots \beta_n}} \sum_{k=1}^{m} f_k D_k$$

(3.18)

Thus $\Delta_m^{L\text{-loop}}$ still depends on $D_1, \ldots, D_m$. We can redefine the first term in (3.18) as $\tilde{\Delta}_m^{L\text{-loop}}$, $\tilde{S}$ as the new power set for the basis, and move the second term in (3.18) to fewer-propagator integrals. The size of the basis decreases by one, so the reduction is not complete.

There is a classic method to find the complete linearly independent basis in $R/I$: Buchberger’s algorithm [28, Ch. 5]. (See [28, Ch. 2] for a review of Gröbner basis.)

1. Define a monomial ordering in $R$ and calculate the corresponding Gröbner basis $G(I)$ of $I$. Denote $LT(K)$ as the collection of leading terms of all polynomials in a set $K$, according to this monomial ordering.

2. Compute $LT(G(I))$, the leading terms of all polynomials in $G(I)$. Obtain $\langle LT(G(I)) \rangle$, the ideal generated by $LT(G(I))$. By the properties of Gröbner basis, $\langle LT(I) \rangle = \langle LT(G(I)) \rangle$, where $\langle LT(I) \rangle$ is the ideal generated by all leading terms in $I$.

3. Then the linear basis of $R/I$ is $\hat{B}$, which is the set of all monomials which are not in $\langle LT(I) \rangle$.

$$\hat{B} = \{ x_1^{\alpha_1} \ldots x_n^{\alpha_n} \mid x_1^{\alpha_1} \ldots x_n^{\alpha_n} \notin \langle LT(I) \rangle \}. \quad (3.19)$$

This method is fast. However, the basis generated by Buchberger’s method usually contains an infinite number of terms, since the renormalizability conditions have not been imposed. We find that after the ring $R$ is reduced to $\hat{B}$, it is not easy to impose renormalizability conditions.

Hence, we propose the following alternative algorithm for basis determination, based on multivariate synthetic division,

1. Define a monomial ordering in $R$ and calculate the corresponding Gröbner basis $G(I)$ of $I$. 

— 11 —
2. Generate the set $A$ of all monomials in ISPs, which satisfy renormalizablity conditions. $A$ must be a finite set.

3. For each monomial $a_j(x_1 \ldots x_{n_I})$ in $A$, $1 \leq j \leq |A|$,
   - Carry out the multivariate synthetic division of $a_j$ by the Gröbner basis $G(I)$.
     \[ a_j(x_1 \ldots x_{n_I}) = g_j(x_1 \ldots x_{n_I}) + r_j(x_1 \ldots x_{n_I}), \quad g_j(x_1 \ldots x_{n_I}) \in I \]  
     (3.20)
     where $r_j(x_1 \ldots x_{n_I})$ is the remainder of multivariate synthetic division. Given the Gröbner basis $G(I)$, $r_j(x_1 \ldots x_{n_I})$ is uniquely determined.
   - Decompose $r_j(x_1 \ldots x_{n_I})$ as monomials and collect them in a set $B_j$.

4. The integrand basis $B$ is then,
   \[ B = \bigcup_j B_j. \]  
   (3.21)
   The validity of this algorithm can be verified as follows,
   - The monomials in $B$ are linearly independent in $R/I$. Multivariate synthetic division by Gröbner basis ensures that all monomials in $B_j$ are not in $\langle LT(I) \rangle$, therefore $B$ is a subset of $\hat{B}$. So by a corollary of Buchberger’s method, linear independence is proven.
   - The basis $B$ is big enough for integrand-level reduction. From step 3, we see that every renormalizable term in the numerator of the integrand is reduced to monomials in $B$. In other words, it is a sum of a linear combination of monomials in $B$ and other terms vanishing on the unitarity cut.

We implement this part of our algorithm in the Mathematica package BasisDet. The Gröbner basis calculation and multivariate synthetic division are done by the functions in Mathematica. The monomial order is chosen as “degree lexicographic” (“deglex” in mathematica language, see [28, Ch. 2] for a review of monomial orderings.) and the coefficient field can be chosen as rational functions for analytic computation, or rational numbers for numeric computation.

### 3.3 Primary decomposition of cut solutions

Given the cut equations in ISP variables, or equivalently, the ideal $I$, the following questions naturally arise:

- How many inequivalent cut solutions are there?
- For each cut solution, how many free parameters are needed to parametrize it? In the other world, which is the dimension of each cut-solution?
These questions can be studied systematically using algebraic geometry. We again translate these problems to mathematical language. Consider the affine space \( A^n \) defined by

\[
A^n_{\mathbb{F}} = (x_1, \ldots, x_n). \tag{3.22}
\]

The ideal \( I \) defines an affine algebraic set \( Y \) in \( A^n \),

\[
Y \equiv \mathbb{Z}(I) = \{(z_1, \ldots, z_n) | D_k(z_1, \ldots, z_n) = 0, \ \forall k\} \tag{3.22}
\]

which is the collection of all cut solutions in term of ISPs.

In general, \( Y \) can always be decomposed uniquely to the union of a finite number of irreducible components \([29, \text{Ch. 1}],\)

\[
\mathcal{Y}_a = \bigcup_{a=1}^{n_{\text{sol}}} \mathcal{Y}_a, \quad \mathcal{Y}_a \nsubseteq \mathcal{Y}_b, \text{ if } a \neq b. \tag{3.23}
\]

where each \( \mathcal{Y}_a \) is an affine variety. Here, \( n_{\text{sol}} \) is the number of irreducible components. Different components are not related by parameter redefinition. Each irreducible component corresponds to a cut solution. So we have the following proposition:

**Proposition 2.** The inequivalent cut solutions are in one-to-one correspondence with the irreducible components of the algebraic set \( Y \). In particular, the number of inequivalent cut solutions equals the number of the irreducible components of the algebraic set \( Y \).

This decomposition can be achieved easily by the algebraic method, primary decomposition of an ideal \([29, \text{Ch. 1}].\) Since \( R \) (the polynomial ring of ISPs) is a Noetherian ring, the primary decomposition of \( I \) uniquely exists (Lasker-Noether theorem) \([32],\)

\[
I = \bigcap_{a=1}^{s} I_a. \tag{3.24}
\]

where \( s \) is a finite integer and each \( I_a \) is a primary ideal. Furthermore, the primary decomposition guaranteed that,

\[
\sqrt{I_a} \neq \sqrt{I_b}, \quad \text{if } a \neq b. \tag{3.25}
\]

\[
I_a \nsubseteq \bigcap_{b \neq a} I_b, \ \forall a. \tag{3.26}
\]

where \( \sqrt{I_a} \) is the radical of \( I_a \).\(^2\) Because \( I_a \) is primary, \( \sqrt{I_a} \) is a prime ideal.

Hence we have the corresponding decomposition of \( Y \). Define \( \mathbb{Z}(I_a) \) to be the zero-locus (set of all solution points) of the ideal \( I_a \),

\[
Y = \mathbb{Z}(I) = \bigcup_{a=1}^{s} \mathbb{Z}(I_a) = \bigcup_{a=1}^{s} \mathbb{Z}(\sqrt{I_a}). \tag{3.27}
\]

Since \( \sqrt{I_a} \) is prime, \( \mathbb{Z}(I_a) = \mathbb{Z}(\sqrt{I_a}) \) is an affine variety \([29, \text{Ch. 1}]\) which is irreducible. Then we define \( n_{\text{sol}} = s \) and \( \mathcal{Y}_a = \mathbb{Z}(\sqrt{I_a}) \), and the decomposition is done.

\(^2\)The radical of an ideal \( J \) is the set of all elements \( a, \) such that \( a^n \in J, \) where \( n \) is some positive integer. \([28, \text{Ch. 4}]\)
The dimension of each component is given by dimension theory in commutative algebra [29, Ch. 1],
\[ \dim Y_a = n_I - \text{height}(\sqrt{I_a}). \]  
(3.28)

where \( \text{height}(\sqrt{I_a}) \) is the height of the prime ideal \( \sqrt{I_a} \), which is defined to be the largest integer \( N \), for all possible series of prime ideals \( \langle 0 \rangle = p_0 \subset p_1 \subset \ldots \subset p_N = \sqrt{I_a} \). Recall that \( \mathbb{S}^P \) is the set of (fundamental-)scalar products, which is defined in (3.4). Note the \( \dim Y_a \) may not equal \( |\mathbb{S}^P| - m \), the difference between the number of (fundamental-)scalar products and the number of cut equations, because of the possible redundancy in the cut equations. Furthermore, for \( a \neq b \), \( \dim Y_a \) may not equal \( \dim Y_b \), since they are independent components.

Once all the irreducible components are obtained, we can parametrize each inequivalent solution. Together with the RSPs, the explicit form for loop momenta \( l_i \) at each cut solution can be recovered.

The primary decomposition (3.24) and dimension (3.28) can be calculated using computational algebraic geometry software, for example, by standard built-in functions in Macaulay2 [30] by Daniel Grayson and Michael Stillman. Alternatively, if we only need the number of irreducible components, then a numeric algebraic geometry approach could be applied, as described in [33].

4 Examples

We implemented the basis determination part of our algorithm in the Mathematica package BasisDet. The only required inputs are the kinematic relations for external legs, a list of propagators and the renormalization conditions. The output is the integrand basis. It also provides \( I \), as in (3.15), the ideal generated by the cut equations in terms of ISPs. Then we can carry out the primary decomposition and dimension theory calculation in the computational algebraic geometry program, Macaulay2, with the ideal \( I \) obtained from BasisDet. Here we list several examples of application of our algorithm. All computations were done on a laptop with an Intel core i7 CPU.

4.1 \( D = 4 - 2\epsilon \) one-loop four-point box topology

Take \( D = 4 - 2\epsilon \), and consider the one-loop contribution with box topology to four-point-all-massless amplitude. The BasisDet package takes 0.05 seconds to generate the basis in the analytic mode (see the appendix A for the modes of the package),
\[ \Delta_{4-2\epsilon}^{4-2\epsilon} = c_0 + c_1(k \cdot \omega) + c_2\mu^2 + c_3(k \cdot \omega)\mu^2 + c_4\mu^4. \]  
(4.1)

which is exactly the same basis as (2.8), which was obtained in ref. [18]. The package automatically find the two ISPs \( (k \cdot \omega) \) and \( \mu^2 \). The cut equations, after all RSP are eliminated, become one equation,
\[ 4(k \cdot \omega)^2 - \frac{4tu}{s}\mu^2 - t^2 = 0, \]  
(4.2)
where $s, t, u$ are Mandelstam variables. It is clear that this equation defines an irreducible parabola in the parameter space $(k \cdot \omega, \mu^2)$. So there is only one solution, with the dimension $1$. As a trivial test, we can also see that from primary decomposition. Let,

$$I = \langle 4(k \cdot \omega)^2 - \frac{4tu}{s} \mu_1^2 - t^2 \rangle.$$  \hspace{1cm} (4.3)

Macaulay2 determines that $I$ itself is primary, so no decomposition is needed. It also automatically finds that $\dim I = 1$, which means there is one free parameter for the solution.

### 4.2 Two-loop examples

First, we consider $D = 4$ four-massless-particle amplitude with two-loop double-box topology (Figure 2). The BasisDet package takes 0.95 seconds to generate the basis in the analytic mode, or 0.43 seconds to generate the same basis in the numeric mode.

$$\Delta_{dbox}^{7;12;34}(k, q) = \sum_{m\alpha\beta} c_{m(\alpha+2\beta)} \langle k \cdot p_4 \rangle^m \langle q \cdot p_1 \rangle^n \langle k \cdot \omega \rangle^\alpha \langle q \cdot \omega \rangle^\beta.$$  \hspace{1cm} (4.4)

with 16 non-spurious terms,

$$(c_{000}, c_{010}, c_{100}, c_{020}, c_{110}, c_{200}, c_{030}, c_{120}, c_{210}, c_{300}, c_{040}, c_{130}, c_{220}, c_{310}, c_{400}, c_{140}, c_{320}),$$

which are exactly the same as (2.14). There are also 16 spurious terms,

$$(c_{001}, c_{011}, c_{101}, c_{201}, c_{021}, c_{031}, c_{002}, c_{012}, c_{022}, c_{032}, c_{102}, c_{112}, c_{122}, c_{132}, c_{202}, c_{230}, c_{202}).$$  \hspace{1cm} (4.6)

Note that although the number of terms is the same as (2.15), some terms are different from (2.15). It means we get a different but equivalent integrand basis. Even for the one loop $D = 4 - 2\epsilon$ box quadruple cut, there are already several different choices of basis. We can check explicitly that the difference between (4.6) and (2.15) is proportional to the seven propagators, so it does not change the double-box contribution to the amplitude.

There are four ISPs, $(l_1 \cdot p_4)$, $(l_2 \cdot p_1)$, $(l_1 \cdot \omega)$ and $(l_2 \cdot \omega)$. The cut equations in ISPs read,

$$f_1 \equiv -t^2 + 4t(l_1 \cdot p_4) - 4(l_1 \cdot p_4)^2 + 4(l_1 \cdot \omega)^2 = 0,$$  \hspace{1cm} (4.7)

$$f_2 \equiv -t^2 + 4t(l_2 \cdot p_1) - 4(l_2 \cdot p_1)^2 + 4(l_2 \cdot \omega)^2 = 0,$$  \hspace{1cm} (4.8)

$$f_3 \equiv s(-2(l_1 \cdot p_4)(l_2 \cdot p_1) + (l_1 \cdot \omega)^2 + 2(l_1 \cdot \omega)(l_2 \cdot \omega) - (l_2 \cdot p_1)^2 + (l_2 \cdot \omega)^2) - 4t(l_1 \cdot p_4)(l_2 \cdot p_1) = 0.$$  \hspace{1cm} (4.9)

In this case, the ideal $I = \langle f_1, f_2, f_3 \rangle$ is quite complicated. It is not easy to find the inequivalent solutions by hand or by elementary analytic geometry. We use primary decomposition to find inequivalent solutions automatically, for example, in Macaulay2, in just a couple of seconds,

$$I = \bigcap_{i=1}^{6} I_i,$$  \hspace{1cm} (4.10)
where $I_i$s are six primary ideals:

\begin{align}
I_1 &= \langle l_1 \cdot p_4, 2(l_2 \cdot \omega) - 2(l_2 \cdot p_1) + t, 2(l_1 \cdot \omega) - t \rangle \quad (4.11)
\end{align}

\begin{align}
I_2 &= \langle l_1 \cdot p_4, 2(l_2 \cdot \omega) + 2(l_2 \cdot p_1) - t, 2(l_1 \cdot \omega) \rangle \quad (4.12)
\end{align}

\begin{align}
I_3 &= \langle l_2 \cdot p_1, 2(l_2 \cdot \omega) + t, 2(l_1 \cdot \omega) + 2(l_1 \cdot p_4) - t \rangle \quad (4.13)
\end{align}

\begin{align}
I_4 &= \langle l_2 \cdot p_1, 2(l_2 \cdot \omega) - t, 2(l_1 \cdot \omega) - 2(l_1 \cdot p_4) + t \rangle \quad (4.14)
\end{align}

\begin{align}
I_5 &= \langle l_2 \cdot \omega - 2(l_2 \cdot p_1) + t, 2(l_1 \cdot \omega) - 2(l_1 \cdot p_4) + t,
4(l_1 \cdot p_4)(l_2 \cdot p_1) + 2(l_1 \cdot p_4)s + 2(l_2 \cdot p_1)s - st \rangle \quad (4.15)
\end{align}

\begin{align}
I_6 &= \langle l_2 \cdot \omega + 2(l_2 \cdot p_1) - t, 2(l_1 \cdot \omega) + 2(l_1 \cdot p_4) - t,
4(l_1 \cdot p_4)(l_2 \cdot p_1) + 2(l_1 \cdot p_4)s + 2(l_2 \cdot p_1)s - st \rangle. \quad (4.16)
\end{align}

So there are 6 inequivalent unitarity cut solutions, consistent with [23, 26]. Furthermore, Macaulay2 automatically finds that every solution of $I_i$ has dimension 1.

Note that all $I_i$’s are generated by simple polynomials, so it is straightforward to solve them for ISPs. Then using the Gram matrix relation, (3.3), we can rewrite the solutions in terms of the loop momenta $l_1$ and $l_2$ and find the one-to-one correspondence with the six solutions in ref. [23]. However, this step is not necessary since we can fit the coefficients $c_{mn(\alpha+2\beta)}$ directly from the solution for ISPs, as described in ref. [27].

Similarly, we can apply the same method on other two-loop diagrams using BasisDet and Macaulay2. Several examples are listed in Table 1.

| Diagram                  | #ISP | $n_{NS}$ | $n_S$ | $n_{basis}$ | #Solution |
|--------------------------|------|----------|-------|-------------|-----------|
| Box-triangle             | 5    | 18       | 51    | 69          | 4         |
| Five-point double-box    | 4    | 32       | 0     | 32          | 6         |
| Sunset                   | 8    | 12       | 30    | 42          | 1         |
| Double-bubble            | 6    | 8        | 48    | 56          | 1         |

Table 1. Several examples of the integrand-level reduction of $D = 4$ two-loop diagrams: All external legs are massless. “#ISP” is the number of ISPs. $n_{NS}$ and $n_S$ are the numbers of non-spurious and spurious terms in the integrand basis, respectively. $n_{basis} = n_{NS} + n_S$ is the total number of terms. “#Solution” is the number of inequivalent solutions. The explicit expression of the integrand basis can be obtained by running the code “example.nb” with BasisDet.
4.3 \( D = 4 \) three-loop triple-box topology

Consider \( D = 4 \), four-massless-particle diagram with three-loop triple-box topology (figure 3). The package uses about 42 seconds in numeric mode or 4 minutes in analytic mode, to generate the same integrand basis. It contains 199 non-spurious terms and 199 spurious terms.

Furthermore we used Macaulay2 to find the inequivalent cut solutions by primary decomposition. It takes about 2 minutes to get,

\[
I = \bigcap_{i=1}^{14} I_i, \tag{4.17}
\]

so there are 14 inequivalent solutions. And,

\[
\dim I_i = 2, \quad i = 1, \ldots, 14. \tag{4.18}
\]

Every solution thus depends on two free parameters. These solutions have been verified both analytically and numerically. Furthermore, by the explicit solutions, we can check that the \( 398 = 199 + 199 \) terms in the basis are linearly independent on the unitarity cuts. This validates the basis.

With the integrand basis and all inequivalent solutions, we can reconstruct the triple-box contribution to three-loop amplitude for any renormalizable theory, via the polynomial fitting techniques.

5 Conclusions and future directions

In the paper, we have presented a new method for integrand-level reduction, based on computational algebraic geometry. It applies (1) a Gröbner basis to find the basis for integrand-level reduction, (2) a primary decomposition of ideals to classify all inequivalent solutions of unitarity cuts. The first part is realized in our Mathematica package BasisDet, which automatically generates the basis from the propagator information. This package
also generates the ideal of the cut equations, which can be used as the input of primary
decomposition. Then computational algebraic geometry software, like Macaulay2 [30] can
classify all inequivalent cut solutions and determine the dimension of each solution.

Since this method has no dependence on the spacetime dimension, the number of loops
or the number of external legs, it works for general multi-loop diagrams of renormalizable
theories.

We applied this method to many one-loop and two-loop topologies. We have also used
it to generate the correct basis and cut solutions for the three-loop triple-box topology. This
method presented in this paper can be used to calculate many two-loop and higher-loop
amplitudes, via polynomial fitting techniques.

In future, it would be interesting to work on the following directions,

• Symmetries in the diagram. It is interesting to find a monomial ordering to keep
  symmetries in the diagram manifest, for the basis determination part of our algo-
  rithm. Then this algorithm could be sped up considerably if all symmetries could be
  made manifest.

• Automatic parametrization of each cut solutions. We would like to find an auto-
  matic way of parametrizing each cut solution, after the dimension of each solution
  is obtained. It would be helpful for the polynomial fit process, to reconstruct the
  integrand from unitarity cuts.

Acknowledgements

YZ would like to thank Simon Badger, Emil Bjerrum-Bohr, Hjalte Frellesvig and Valery
Yundin for helpful discussions and comments. Especially, YZ wants to express the gratitude
to Simon Badger for his testing of the package BasisDet, and his careful reading of this
paper at the draft stage. YZ also acknowledges Pierpaolo Mastrolia for the communication
with him on using Gröbner basis methods, and his comments on the draft of this paper.
YZ is supported by Danish Council for Independent Research — Natural Science (FNU)
grant 11-107241.

A Manual for the package BasisDet 1.01

This current code is powered by Mathematica with its embedded Gröbner basis function.
The latest version of the package is available on the website, http://www.nbi.dk/~zhang/BasisDet.htm

A.1 Set up

The main program is “BasisDet-a-b.m”, where $a.b$ is the version number of the package.
It should be executed as,

```
<<’/path/BasisDet-a-b.m’
```

where “/path/” is the path for the package.
A.2 Input for loop diagrams

The following variables need to be defined for the basis determination,

- **L.** It is the number of loops in the diagram.
- **Dim.** It is the dimension of the spacetime, which should be $d$ or $d-2\epsilon$. $d$ is a positive integer and in most cases $d=4$.
- **n.** It is the number of external legs.
- **ExternalMomentaBasis.** It is a list of external momenta in the basis for physical spacetime. Note that for $n$-point amplitude, because of momentum conservation, we can pick up at most $n-1$ external momenta for the basis. In summary,
  - If $n < d + 1$, we need to put $n - 1$ external momenta in “ExternalMomentaBasis”. The program will automatically name the $d - n + 1$ spurious vectors as $\omega_1, \ldots, \omega_{d-n+1}$.
  - If $n \geq d + 1$, we need to put $d$ external momenta in “ExternalMomentaBasis”.
- **Kinematics.** This is the list for replacement rules, from the kinematics. Note that only the scalar products of vectors in the basis need to be defined. To ensure that an kinematic constraints are resolved, only the independent set of $s_{ij}$, $s_{ij} = 2p_i \cdot p_j$, can appear in this list. For example, for a four-point diagram, we can only use two variables of the three Mandelstam variables.
- **numeric.** It is an optional variable for the basis calculation. When “numeric” is given and the numerical calculation in the GenerateBasis function is enabled, all the Gröbner basis calculation is done numerically. It will speed up the computation by $2 \sim 5$ times. However, the numeric calculation has the risk of meeting kinematic singularities (like infrared limit and collinear limit). The numeric values should be rational numbers, otherwise the result depends on the floating-point tolerance inside the Gröbner basis computation.
- **Props.** This is the list for the propagator momenta. No specific order for the propagators is necessary. The direction of the propagator momenta is also irrelevant. In this version, the propagators are set to be massless.
- **RenormalizationCondition.** This variable define the constraints from the renormalizablity condition. Each constraint on the power of the loop momenta is expressed as a linear inequality. For example, when $L = 3$, the loop momenta are $l_1, l_2, l_3$ and the corresponding powers for the loop momenta are $\alpha_1, \alpha_2, \alpha_3$. The constraint

  \[
  \alpha_1 + \alpha_2 \leq 6, \quad (A.1)
  \]

  should be given as an item \{\{1, 1, 0\}, 6\} in "RenormalizationCondition". \{1, 1, 0\} is the list of the coefficients of $\alpha_1, \alpha_2, \alpha_3$ and 6 is the upper bound.

The program will name the (fundamental-) scalar products $(l_i \cdot e_j)$ as “xij".
A.3 Computation and the output

Once the input is given, all the basis determination computation is done by one command GenerateBasis:

- \texttt{GenerateBasis[1]}. This calculates the basis analytically.
- \texttt{GenerateBasis[0]}. This calculates the basis with numeric coefficients.

The outputs are stored in the following variables,

- \texttt{ISP}. This is the list for irreducible scalar products.
- \texttt{RSPSolution}. This is the solutions for reducible scalar products at the unitarity cut.
- \texttt{CutEqnISP}. This is the list for the cut equations in terms of ISPs, after all RSPs are eliminated. Depending on if the numeric mode is enabled, the coefficient can be either numeric or analytic.
- \texttt{Basis}. This is the list for the terms in the basis. The output form for each term is \((\alpha_1, \ldots, \alpha_n)\), where \(\alpha_i\) is the power of the \(i\)-th ISP.
- \texttt{SpuriousBasis}. This is the subset of the basis which contains all spurious terms.
- \texttt{NSpuriousBasis}. This is the subset of the basis which contains all non-spurious terms.
- \texttt{Integrand}. This is the integrand after the reduction, which is an expansion over the integrand-level basis. \(cc[\alpha_1, \ldots, \alpha_n]\) stands for the coefficient \(c_{\alpha_1\ldots\alpha_n}\) for the term \(x_1^{\alpha_1} \cdots x_n^{\alpha_n}\), as described in (3.16).
- \texttt{Gr}. This is the Gröbner basis for the ideal \(I\), generated by cut equations in ISPs. It can be exported for other purposes.

The lists “ISP” and “CutEqnISP” can be readily used for primary decomposition and then the dimension theory part of our algorithm, in softwares like Macaulay2 [30].

A.4 Example, integrand basis for two-loop double-box diagram

```
<< "/path/Basis-050712.m"
L=2;
Dim=4;
ExternalMomentaBasis={p1,p2,p4};
Kinematics={p1^2->0,p2^2->0,p4^2->0,p1 p2->s/2,p1 p4->t/2,
[p2 p4->-(s+t)/2,\[Omega]1^2->-t(s+t)/s];
numeric={s->11,t->3};
Props={l1-p1,l1,l1-p1-p2,l2-p3-p4,l2,l2-p4,l1+l2};
RenormalizationLoopMomenta={{1,0},{0,1},{1,1}};
RenormalizationPower={4,4,6}
GenerateBasis[1]
```
It takes about 0.95 second to generate the basis, with analytic calculation. A typical output is,

Physical spacetime basis is \{p1, p2, p4, [Omega]1\}

Number of irreducible scalar products: 4

Irreducible Scalar Products: \{x14, x24, x13, x21\}

Cut equations for ISP are listed in the variable 'CutEqnISP'

Possible renormalizable terms: 160

The basis contains 32 terms, which are listed in the variable 'Basis'

The explicit form of the integrand is listed in the variable 'Integrand'

Number of spurious terms: 16, listed in the variable 'SpuriousBasis'

Number of non-spurious terms: 16, listed in the variable 'NSpuriousBasis'

Time used: 0.955934 seconds

The we can obtain the basis information from the variables, “CutEqnISP”, “Basis”, “Integrand”, “SpuriousBasis” and “NSpuriousBasis”.

More examples are included in the Mathematica notebook, “examples.nb”.

B The algorithm of identifying the ISPs

We have the following simple algorithm to find the ISPs, which is embedded in the package BasisDet,

- Calculate the Gröbner basis $G(I')$ for the ideal $I'$ generated by cut equations in terms of SPs, in the polynomial order of “deglex”.
- Obtain $LT(G(I'))$, the set of the leading terms in $G(I')$. The linear terms in $LT(G(I'))$ are the RSPs.

It is easy to show that this algorithm gives the correct ISPs according to the definition.

Proof. Suppose that this algorithm generates $\{y_1, \ldots, y_{n_R}\}$ as the list of the RSPs in the polynomial ordering, while $\{x_1, \ldots, x_{n}\}$ as the list of the ISPs in the polynomial ordering. First, we can prove that $y_{n_R}$ is a linear function of ISPs on the cut. $G(I')$ must contain a linear polynomial,

$$\alpha y_{n_R} + \sum_{i}^{n} \beta_i x_i + \gamma \in I'$$

where $\alpha$, $\beta_i$ and $\gamma$ are constants and $\alpha \neq 0$. This polynomial cannot contain other $y_j$’s, because $y_j \succ y_{n_R}$ for $j < n_R$. Here “$\succ$” stands for the given monomial ordering. Thus $y_{n_R}$ is a linear function of the ISPs on the unitarity cut.

Second, by induction, all $y_j$ are linear functions of ISPs on the cut.

Third, we can prove that the ISP set is minimal. If some $x_i$ can be represented by a linear function of other ISPs at the cut, then

$$x_i - \sum_{j \neq i}^{n} \alpha_j x_j + \beta \in I'$$
Then the leading term of this polynomial is an ISP, say $x_k$. By the property of Gröbner basis, $\langle \text{LT}(I') \rangle = \langle \text{LT}(G(I')) \rangle$. Because $x_k \in \text{LT}(I')$, $x_k \in \langle \text{LT}(G(I')) \rangle$. Furthermore, since $x_k$ has degree one, it is generated by degree-one monomials in $\text{LT}(G(I'))$: \{y_1, \ldots, y_n\}.

$$x_k = \sum_i \gamma_i y_i \quad \text{(B.3)}$$

while $\gamma_i$’s are constants. This contradicts the assumption of ring structure. The ISP set is thus minimal.

References

[1] K. Chetyrkin and F. Tkachov, *Integration by Parts: The Algorithm to Calculate beta Functions in 4 Loops*, Nucl.Phys. B192 (1981) 159–204.

[2] S. Laporta, *High precision calculation of multiloop Feynman integrals by difference equations*, Int.J.Mod.Phys. A15 (2000) 5087–5159, [hep-ph/0102033].

[3] O. Tarasov, *Computation of Grobner bases for two loop propagator type integrals*, Nucl.Instrum.Meth. A534 (2004) 293–298, [hep-ph/0403253].

[4] A. Smirnov and V. A. Smirnov, *Applying Grobner bases to solve reduction problems for Feynman integrals*, JHEP 0601 (2006) 001, [hep-lat/0509187].

[5] A. Smirnov, *Algorithm FIRE – Feynman Integral REduction*, JHEP 0810 (2008) 107, [arXiv:0807.3243].

[6] Z. Bern, L. J. Dixon, D. C. Dunbar, and D. A. Kosower, *One loop n point gauge theory amplitudes, unitarity and collinear limits*, Nucl.Phys. B425 (1994) 217–260, [hep-ph/9403226].

[7] Z. Bern, L. J. Dixon, D. C. Dunbar, and D. A. Kosower, *Fusing gauge theory tree amplitudes into loop amplitudes*, Nucl.Phys. B435 (1995) 59–101, [hep-ph/9409265].

[8] R. Britto, F. Cachazo, and B. Feng, *Generalized unitarity and one-loop amplitudes in N=4 super-Yang-Mills*, Nucl.Phys. B725 (2005) 275–305, [hep-th/0412103].

[9] G. Ossola, C. G. Papadopoulos, and R. Pittau, *Reducing full one-loop amplitudes to scalar integrals at the integrand level*, Nucl.Phys. B763 (2007) 147–169, [hep-ph/0609007].

[10] G. Ossola, C. G. Papadopoulos, and R. Pittau, *CutTools: A Program implementing the OPP reduction method to compute one-loop amplitudes*, JHEP 0803 (2008) 042, [arXiv:0711.3596].

[11] R. Ellis, W. Giele, and Z. Kunszt, *A Numerical Unitarity Formalism for Evaluating One-Loop Amplitudes*, JHEP 0803 (2008) 003, [arXiv:0708.2398].

[12] P. Mastrolia, G. Ossola, T. Reiter, and F. Tramontano, *Scattering AMplitudes from Unitarity-based Reduction Algorithm at the Integrand-level*, JHEP 1008 (2010) 080, [arXiv:1006.0710].

[13] S. Badger, B. Biedermann, and P. Uwer, *NGluon: A Package to Calculate One-loop Multi-gluon Amplitudes*, Comput.Phys.Commun. 182 (2011) 1674–1692, [arXiv:1011.2900].

[14] V. Hirschi, R. Frederix, S. Frixione, M. V. Garzelli, F. Maltoni, et. al., *Automation of one-loop QCD corrections*, JHEP 1105 (2011) 044, [arXiv:1103.0621].
[15] G. Bevilacqua, M. Czakon, M. Garzelli, A. van Hameren, A. Kardos, et al., HELAC-NLO, arXiv:1110.1499.

[16] W. T. Giele, Z. Kunszt, and K. Melnikov, Full one-loop amplitudes from tree amplitudes, JHEP 0804 (2008) 049, [arXiv:0801.2237].

[17] R. K. Ellis, W. T. Giele, Z. Kunszt, and K. Melnikov, Masses, fermions and generalized D-dimensional unitarity, Nucl.Phys. B822 (2009) 270–282, [arXiv:0806.3467].

[18] R. Ellis, Z. Kunszt, K. Melnikov, and G. Zanderighi, One-loop calculations in quantum field theory: from Feynman diagrams to unitarity cuts, arXiv:1105.4319.

[19] P. Mastrolia, E. Mirabella, and T. Peraro, Integrand reduction of one-loop scattering amplitudes through Laurent series expansion, arXiv:1203.0291.

[20] E. I. Buchbinder and F. Cachazo, Two-loop amplitudes of gluons and octa-cuts in N=4 super Yang-Mills, JHEP 0511 (2005) 036, [hep-th/0506126].

[21] J. Gluza, K. Kajda, and D. A. Kosower, Towards a Basis for Planar Two-Loop Integrals, Phys.Rev. D83 (2011) 045012, [arXiv:1009.0472].

[22] R. M. Schabinger, A New Algorithm For The Generation Of Unitarity-Compatible Integration By Parts Relations, JHEP 1201 (2012) 077, [arXiv:1111.4220].

[23] D. A. Kosower and K. J. Larsen, Maximal Unitarity at Two Loops, Phys.Rev. D85 (2012) 045017, [arXiv:1108.1180]. 42 pages, 9 figures.

[24] K. J. Larsen, Global Poles of the Two-Loop Six-Point N=4 SYM integrand, arXiv:1205.0297.

[25] S. Caron-Huot and K. J. Larsen, Uniqueness of two-loop master contours, arXiv:1205.0801.

[26] P. Mastrolia and G. Ossola, On the Integrand-Reduction Method for Two-Loop Scattering Amplitudes, JHEP 1111 (2011) 014, [arXiv:1107.6041].

[27] S. Badger, H. Frellesvig, and Y. Zhang, Hepta-Cuts of Two-Loop Scattering Amplitudes, JHEP 1204 (2012) 055, [arXiv:1202.2019].

[28] D. Cox, J. Little, and D. O’Shea, Ideals, varieties, and algorithms. Undergraduate Texts in Mathematics. Springer, New York, third ed., 2007. An introduction to computational algebraic geometry and commutative algebra.

[29] R. Hartshorne, Algebraic geometry. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[30] D. R. Grayson and M. E. Stillman, “Macaulay2, a software system for research in algebraic geometry.” Available at http://www.math.uiuc.edu/Macaulay2/.

[31] W. van Neerven and J. Vermaseren, LARGE LOOP INTEGRALS, Phys.Lett. B137 (1984) 241.

[32] M. F. Atiyah and I. G. Macdonald, Introduction to commutative algebra. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[33] D. Mehta, Y.-H. He, and J. D. Hauenstein, Numerical Algebraic Geometry: A New Perspective on String and Gauge Theories, arXiv:1203.4235.