RICCI CURVATURE AND YAMABE CONSTANTS

JIMMY PETEAN

Abstract. We prove that if $(M^n, g)$ is a closed Riemannian manifold of dimension $n \geq 3$ with volume $V$ and Ricci curvature $\text{Ricci}(g) \geq \rho > 0$ then the Yamabe constant of the conformal class $[g]$ satisfies $Y(M, [g]) \geq n\rho V^{2/n}$; the equality is achieved if $g$ is an Einstein metric (of Ricci curvature $\rho$). This has actually already been proved by S. Ilias [6] in the context of Sobolev inequalities. This implies for instance that if $g_1$ is the Fubini-Study metric on $\mathbb{C}P^2$ and $g$ is any other Riemannian metric on $\mathbb{C}P^2$ with $\text{Ricci}(g) \geq \text{Ricci}(g_1)$ then $\text{Vol}(\mathbb{C}P^2, g) \leq \text{Vol}(\mathbb{C}P^2, g_1)$.

1. Introduction

Let $(M^n, g)$ be a closed Riemannian manifold of dimension $n \geq 3$. Restricting the total scalar curvature functional to the conformal class $[g]$ of $g$ we have the Yamabe functional defined on $L^2_1(M)$ by

$$Y_g(f) = \frac{a_n \int_M \|\nabla f\|^2 dvol_g + \int_M \text{Scal}_g f^2 dvol_g}{(\int_M f^p dvol_g)^{2/p}}.$$  

In the expression, and throughout the article, $a_n = \frac{4(n-1)}{n-2}$, $p = p_n = \frac{2n}{n-2}$, $dvol_g$ is the volume element of $g$ and $\text{Scal}_g$ its scalar curvature.

The Yamabe constant of the conformal class of $g$, $Y(M, [g])$ is the infimum of this functional. A fundamental result proved in several stages by Yamabe [13], Trudinger [12], Aubin [11] and R. Schoen [10] says that there is always a minimizing function $f_0$ which is smooth and positive. The metric $f_0^{-n/(n-2)} g$ then has constant scalar curvature and is called a Yamabe metric.

The metric of constant sectional curvature 1, $g_0$, on the sphere is a Yamabe metric and we will denote $Y_n = Y(S^n, g_0) = n(n-1)V_n^{2/n}$ ($V_n$ is the volume of $(S^n, g_0)$). This value is important in the study of Yamabe constants since Aubin [11] showed that for any conformal class $[g]$ in any closed $n$-dimensional manifold $M$, $Y(M, [g]) \leq Y_n$ (actually the solution of the Yamabe problem comes from showing that the inequality is strict except for the case of $[g_0]$). It is also easy to check that $Y(M, [g])$ is positive if and only if there is a metric of positive scalar curvature on $[g]$ (since the infimum of the Yamabe functional is always realized). One sees that the study of the Yamabe constant of a conformal class depends strongly on whether the invariant is positive or non-positive. In the non-positive case it is particularly useful that the Yamabe constant of the conformal class of a metric $g$ is bounded from below by

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\[ \inf_M \{\text{Scal}_g\} \Vol(M, g)^{\frac{2}{n}}. \] This follows by a simple application of H"older’s inequality to the Yamabe functional and it was first pointed out by O. Kobayashi [7]. This is no longer true in the positive case; by considering Riemannian products one can easily build examples of unit volume Riemannian metrics with scalar curvature constant and very big (\( \gg Y_n \)).

The aim of this article is to prove that in the positive case there is a similar lower bound for the Yamabe constant using the infimum of the Ricci curvature instead of the scalar curvature. Namely, we will prove:

**Theorem A:** Let \((M^n, g)\) be a closed Riemannian manifold with Ricci curvature \(\text{Ricci}(g) \geq n - 1\) and volume \(V_0\). Then

\[ Y(M, [g]) \geq n(n - 1) V_0^{\frac{2}{n}} = \left( \frac{V_0}{V_n} \right)^{\frac{2}{n}} Y_n. \]

The author was informed by Guofang Wang that the inequality in this Theorem has already been proved by S. Ilias [6]. Actually the proof given in this article goes along the same lines as Ilias’ original proof. Note that if \(g\) is an Einstein metric (of constant Ricci curvature \(n-1\)) then it is known to be a Yamabe metric and \(Y(M, [g]) = n(n - 1)V_0^{\frac{2}{n}} = \left( \frac{V_0}{V_n} \right)^{\frac{2}{n}} Y_n\). The inequality is therefore optimal. C. Böhm, M. Wang and W. Ziller [3] have shown that for \(\delta\) close to 1 and \(g_0\) the round metric on \(S^2\) the Riemannian metric \(\delta g_0 \times g_0\) on \(S^2 \times S^2\) is a Yamabe metric: when \(\delta \neq 1\) this is probably the simplest case where inequality in Theorem A is strict.

The Yamabe invariant of \(M\) was introduced by R. Schoen [11] and O. Kobayashi [7] as:

\[ Y(M) = \sup_{[g]} Y(M, [g]), \]

the supremum of the Yamabe constants over the space of all conformal classes of metrics on \(M\). Knowledge of the Yamabe invariant and Theorem A produce some restriction between Ricci curvature and volume of any Riemannian metric on the given manifold. For instance C. LeBrun [8] (and M. Gursky and C. LeBrun [5] by more elementary methods) have shown that the Yamabe invariant of \(\mathbb{CP}^2\) is realized by the conformal class of the (Kähler-Einstein) Fubini-Study metric \(g_1\). Therefore we obtain:
Theorem B: For any Riemannian metric \( g \) on \( \mathbb{CP}^2 \) with Ricci curvature \( \text{Ricci}(g) \geq \text{Ricci}(g_1) \) we have \( \text{Vol} (\mathbb{CP}^2, g) \leq \text{Vol} (\mathbb{CP}^2, g_1) \).

As another application one recalls that for a Riemannian 4-manifold \((M, g)\) the space of self-dual 2-forms gives a polarization of \( M \); namely, a maximal linear subspace of \( H^2(M) \) where the intersection form is positive definite \( \mathbb{C} \). Now if \( g_K \) is a positive Kähler-Einstein metric on \( M \) and \( g \) is any other Riemannian metric on \( M \) which defines the same polarization as \( g_K \) then C. LeBrun proved that \( Y(M, [g]) \leq Y(M, [g_K]) \), and then we have again that if \( \text{Ricci}(g) \geq \text{Ricci}(g_K) \) then \( \text{Vol}(M, g) \leq \text{Vol}(M, g_K) \).

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2. Spherical rearrangements and isoperimetric inequalities

In this section we recall a few results we will need for the proof of Theorem A. Fix a smooth positive function \( f \) on a closed Riemannian manifold \((M, g)\) of volume \( V_0 \). The spherical rearrangement of \( f \) is the radially symmetric positive function \( f_* \) on \( S^n \) such that if we renormalize \( S^n \) to have volume \( V_0 \) (and constant sectional curvature) then \( \mu(\{f > t\}) = \mu(\{f_* > t\}) \), for all \( t \in \mathbb{R} \). Here and throughout the article \( \mu \) means the measure corresponding to the volume element of the corresponding Riemannian metric.

Note that for any positive number \( q \)

\[
\int_M f^q = \int_{S^n_{V_0}} f_*^q.
\]

Also recall the coarea formula:

\[
\int_M \|\nabla f\|^2 = \int_0^\infty \left( \int_{f^{-1}(t)} \|\nabla f\| d\sigma_t \right) dt,
\]

and if \( t_0 \) is a regular value of \( f \) then the function \( t \to \mu(f < t) \) is smooth at \( t_0 \) and

\[
\frac{d}{dt}\mu(f < t)(t_0) = \int_{f^{-1}(t_0)} \|\nabla f\|^{-1} d\sigma_t
\]

(\( d\sigma_t \) means the volume element coming from the induced Riemannian metric).

Let us also recall the following definition introduced by Bérard-Besson-Gallot \([2]\)

**Definition 2.1.** For any \( \beta \in (0, 1) \) let \( W_\beta = \{ \Omega \subset M : \Omega \text{ is open with smooth boundary and } \text{Vol}(\Omega) = \beta V_0 \} \). The isoperimetric function of \((M, g)\) is
$h_{(M,g)}(\beta) = h(\beta) = \inf \left( \frac{\mu(\partial \Omega)}{V_0} : \Omega \in W_\beta \right)$. 

Bérard-Besson-Gallot proved that if the Ricci curvature of $(M, g)$, $\text{Ricci}(g) \geq n - 1$ and $d$ is the diameter then $h(\beta) \geq A(d) h_0(\beta)$; where $h_0$ is the isoperimetric function of the round sphere of curvature 1 and

$$A(d) = \left( \int_0^\frac{\pi}{2} \cos^{n-1}(t) \, dt \right)^\frac{1}{n} \left( \int_0^\frac{\pi}{2} \cos^{n-1}(t) \, dt \right)^{-\frac{1}{n}}.$$

Note that $A(d) \geq 1$ by Myers theorem. This is an improvement on M. Gromov’s estimate in [3, Appendix C]. Actually, Gromov’s estimate (which does not contain the factor $A(d)$) would be enough for the proof of Theorem A. It is well-known that $h_0(\beta) V_n$ is the area of the $(n-1)$-sphere which bounds a geodesic ball of volume $\beta V_n$. Note also that if $\lambda$ is a positive constant then the isoperimetric functions of $g$ and $\lambda g$ are related by $h_{\lambda g} = \frac{1}{\sqrt{\lambda}} h_g$.

3. PROOF OF THEOREM A

Proof. Let $f$ be a positive smooth function on $M$ with only non-degenerate (and therefore finite) critical points. We will consider the spherical rearrangement $f_*$ of $f$. We will think of $f_*$ as defined in the round sphere $S^n_{V_0}$ of volume $V_0$ and therefore for any $t \in \mathbb{R}$, $\mu\{f > t\} = \mu\{f_* > t\}$. Note that $S^n_{V_0}$ is obtained by multiplying the round metric of curvature 1 by $(\frac{V_0}{V_n})^{\frac{2}{n}}$. One can put the maximum of $f_*$ in the south pole $q_0$ of $S^n_{V_0}$. Then if $r$ is the distance in $S^n_{V_0}$ to $q_0$ then $f_*$ is a function of $r$ and $f_*(r) = t$ if and only if the volume of the geodesic ball of radius $r$ in $S^n_{V_0}$ equals $\mu\{f > t\}$. It follows that if $t$ is a regular value of $f$ then $f_*$ is differentiable at $r$ and $t$ is a regular value of $f_*$. Note in this case that $\|\nabla f_*\|$ is constant along $f_*^{-1}(t)$ since $f_*$ is radially symmetric. Then we can write

$$\int_{f_*^{-1}(t)} \|\nabla f_*\| d\sigma_t = \left( \mu(f_*^{-1}(t)) \right)^2 \left( \int_{f_*^{-1}(t)} \|\nabla f_*\|^{-1} d\sigma_t \right)^{-1}.$$

We now want to compare the $L^2$-norms of the gradients of $f$ and $f_*$. By the coarea formula

$$\int_M \|\nabla f\|^2 = \int_0^\infty \left( \int_{f^{-1}(t)} \|\nabla f\| d\sigma_t \right) dt.$$

But from Hölder’s inequality (write $1 = \|\nabla f\|^{-1/2} \|\nabla f\|^{1/2}$)

$$\int_{f_*^{-1}(t)} \|\nabla f\| d\sigma_t \geq \left( \mu(f^{-1}(t)) \right)^2 \left( \int_{f_*^{-1}(t)} \|\nabla f\|^{-1} d\sigma_t \right)^{-1}.$$

Also note that
\[
\int_{f^{-1}(t)} \|\nabla f\|^{-1} d\sigma_t = -\frac{d}{dt}(\mu\{f > t\}) = \int_{f^{-1}(t)} \|\nabla f\|^{-1} d\sigma_t.
\]

On the other hand, \(\{f > t\}\) is a domain in \(M\) with volume \(\mu\{f > t\}\) and boundary \(f^{-1}(t)\). By the definition of the isoperimetric function

\[
\mu(f^{-1}(t)) \geq V_0 h_{(M,g)} \left( \frac{\mu\{f > t\}}{V_0} \right).
\]

If we let \(h_0\) be the isoperimetric function for the sphere then the estimate of Bérard-Besson-Gallot says that

\[
h \geq h_0 A(d).
\]

The isoperimetric function on the round sphere is realized by round balls. Therefore

\[
\mu(f^{-1}(t)) \geq V_0 h_0 \left( \frac{\mu\{f > t\}}{V_0} \right) A(d) = V_0 \left( \frac{V_0}{V_n} \right)^{\frac{1}{n}} h_{S^{n}_0} \left( \frac{\mu\{f > t\}}{V_0} \right) A(d)
\]

\[
= \left( \frac{V_0}{V_n} \right)^{\frac{1}{n}} \mu(f^{-1}(t)) A(d).
\]

And finally,

\[
\int_M \|\nabla f\|^2 \geq \int_0^\infty (\mu(f^{-1}(t)))^2 \left( \int_{f^{-1}(t)} \|\nabla f\|^{-1} d\sigma_t \right)^{-1} dt
\]

\[
\geq \left( \frac{V_0}{V_n} \right)^{\frac{2}{n}} (A(d))^2 \int_0^\infty (\mu(f^{-1}(t)))^2 \left( \int_{f^{-1}(t)} \|\nabla f\|^{-1} d\sigma_t \right)^{-1} dt
\]

\[
= \left( \frac{V_0}{V_n} \right)^{\frac{2}{n}} (A(d))^2 \int_{S^{n}_0} \|\nabla f\| d\sigma_t dt
\]

\[
= \left( \frac{V_0}{V_n} \right)^{\frac{2}{n}} (A(d))^2 \int_{S^{n}_0} \|\nabla f\|^2
\]

(by the coarea formula).

Therefore

\[
Y_g(f) = \frac{a_n \int_M \|\nabla f\|^2 + \int_M s_g f^2}{(\int_M f^p)^{\frac{2}{p}}} \geq \frac{a_n \int_M \|\nabla f\|^2 + n(n-1) \int_M f^2}{(\int_M f^p)^{\frac{2}{p}}},
\]

since \(\text{Ricci}_g \geq n - 1\). And then from the previous discussion
\[ Y_g(f) \geq \frac{a_n V_0^2 V_n^{-2} (A(d))^2 \int_{S_{V_0}} \| \nabla f_* \|^2 + V_0^2 V_n^{-2} \int_{S_{V_0}} \text{Scal}_{S_{V_0}} f_*^2}{\left( \int_{S_{V_0}} f_*^p \right)^{\frac{2}{p}}}. \]

And since \( A(d) \geq 1, \)

\[ Y_g(f) \geq \left( \frac{V_0}{V_n} \right)^{\frac{2}{n}} Y_{S_{V_0}} (f_*) \geq \left( \frac{V_0}{V_n} \right)^{\frac{2}{n}} Y_n = V_0^{\frac{2}{n}} n(n - 1). \]

Since every non-negative function \( f \in L^2_1(M) \) can be approximated (in \( L^2_1(M) \)) by a positive Morse function, Theorem A follows by taking the infimum for all \( f \in L^2_1(M). \)

□

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