On Chow-pure cohomology and Euler characteristics for motives and varieties, and their relation to unramified cohomology and Brauer groups

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Abstract

We study Grothedieck groups of triangulated categories using weight structures, weight complexes, and the corresponding pure (co)homological functors. We prove some general statements on $K_0$ of weighted categories and apply it to Voevodsky motives endowed with so-called Chow weight structures. We obtain certain "motivic substitutes" for smooth compactifications of smooth varieties over arbitrary perfect fields; this enables us to make certain unramified cohomology and Euler characteristic calculations that are closely related to results of T. Ekedahl and B. Kahn.

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Introduction

In this paper we discuss weighted triangulated categories and apply some general results to Voevodsky motives.

In particular, we use the properties of weight-exact localizations to obtain that for any smooth variety $X$ there exists a Chow motif that is "birational" to it. This result is a trivial consequence of the Hironaka’s resolution of singularities if the characteristic $p$ of the base field is zero, but it is new in the case $p > 0$ (we prove it for $\mathbb{Z}[1/p]$-linear motives). Applying certain results of [KaS17, §7] we deduce that this statement can be applied to unramified cohomology calculations. In particular, one may consider the $\mathbb{Z}(\ell)$-component for the (cohomological) Brauer groups of smooth varieties, where $\ell$ is a prime distinct from $p$.

Moreover, we prove that weight complex functors (as introduced in [Bon10a] and [Bon18b]; however, the term originates from [GiS96]) can be used to calculate the Grothendieck groups of (bounded) weighted categories. The corresponding general Theorem 3.2.4 extends Theorem 5.3.1 of [Bon10a]. In particular, it says that if a weight structure $w$ on $\mathcal{C}$ is bounded then any additive functor $F$ from the heart $Hw$ of $w$ into a category $A$ gives a homomorphism $F_{K_0} : K_0^{tr}(\mathcal{C}) \rightarrow K_0^{add}(A)$ (see Definition 1.3.3 below); one may call homomorphisms of this sort Euler characteristic ones. An important observation here that if in the case where $\mathcal{C} = DM^{eff}_{gm,R}$ (the category of $R$-linear geometric Voevodsky motives) one can apply $F_{K_0}$ to motives with

$1^1$These are the triangulated categories endowed with weight structures as introduced by the author and independently by D. Pauksztello.
compact support $M_{c \text{gm}}(-)$ to obtain a function $G$ from the set of $k$-varieties into $K_0^\text{add}(A)$ that satisfies the scissors relation

$$G(X) = G(X \setminus Z) + G(Z)$$

if $Z$ is a closed subvariety of $X$. In Theorem 2.3.4 we take $F = E^i$, where $E^i$ is the obvious extension of $H^i(-, \mathbb{Z}_l)$ to Chow$_\text{eff}^R$ (where $R$ is a localization of $\mathbb{Z}[1/p]$) to obtain results closely related to Ekedahl’s invariant calculations in [Eke09b, Theorem 5.1] (see Remark 2.3.5(3)).

Now we describe the contents of the paper. More details can be found at the beginnings of sections.

In §1 we recall several definitions and results from previous papers; they are mostly related to triangulated categories and weight structures. In particular, in §1.3 we recall some of the theory of (strong) weight complexes along with their relation to the so-called pure (co)homological functors and calculations in Grothendieck groups of triangulated categories; these results are generalized in §3.2.

In §2 we apply the results of previous sections to the study of various categories of motives (including the birational ones introduced in [KaS17]), Chow weight structures on them, and their unramified cohomology. In particular, we study the (pure) unramified Brauer group functor and compute $E^i_{K_0}(M_{c \text{gm}}(X))$ for certain smooth $X/k$ and $E^i$ as above. We also demonstrate that this case of our results is closely related to Theorem 5.1 of [Eke09b].

In §3 we give some detail and generalizations for the results of the previous sections. In particular, Theorem 3.2.4(II.1) calculates the Grothendieck group of an (arbitrary) bounded weighted category, and Proposition 3.1.4 can be used to study weight-exact localizations of ("big") motivic categories.

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1 On weight structures in triangulated categories: reminder

In this section we recall those parts of the theory of weight structures that will be applied below.

In §1.1 we introduce some definitions and notation for (mostly, triangulated) categories.

In §1.2 we give the basic definitions and properties of weight structures; this includes a new statement on weight-exact localizations.
In §1.3 we recall some of the theory of (strong) weight complexes along with their relation to the so-called pure (co)homological functors and calculations in Grothendieck groups of triangulated categories.

1.1 Some definitions for (triangulated) categories

- All products and coproducts in this paper will be small.
- Given a category $C$ and $X,Y \in \text{Obj} C$ we will write $C(X,Y)$ for the set of morphisms from $X$ to $Y$ in $C$.
- For categories $C'$ and $C$ we write $C' \subset C$ if $C'$ is a full subcategory of $C$.
- Given a category $C$ and $X,Y \in \text{Obj} C$, we say that $X$ is a retract of $Y$ if $\text{id}_X$ can be factored through $Y$.
- Below $B$ will always denote some additive category.
- A subcategory $H$ of $B$ is said to be retraction-closed in $B$ if it contains all $B$-retracts of its objects. Moreover, for a class $D$ of objects of $B$ we use the notation $\text{Kar}_B(D)$ for the class of all $B$-retracts of elements of $D$.
- We will say that $B$ is idempotent complete if any idempotent endomorphism gives a direct sum decomposition in it.
- The idempotent completion $\text{Kar}(B)$ (no lower index) of $B$ is the category of “formal images” of idempotents in $B$. Respectively, its objects are the pairs $(B,p)$ for $B \in \text{Obj} B$, $p \in B(B,B)$, $p^2 = p$, and the morphisms are given by the formula

$$\text{Kar}(B)((X,p),(X',p')) = \{ f \in B(X,X') : p' \circ f = f \circ p = f \}.$$ 

The correspondence $B \mapsto (B,\text{id}_B)$ (for $B \in \text{Obj} B$) fully embeds $B$ into $\text{Kar}(B)$, and it is well known that $\text{Kar}(B)$ is essentially the smallest idempotent complete additive category containing $B$.
- We will use the term "exact functor" for a functor of triangulated categories (i.e. for a functor that preserves the structures of triangulated categories).

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2Clearly, if $C$ is triangulated or abelian, then $X$ is a retract of $Y$ if and only if $X$ is its direct summand.
• The symbol $C$ below will always denote some triangulated category; it will often be endowed with a weight structure $w$ (which usually will be bounded). The symbols $C'$ and $D$ will also be used for triangulated categories only.

• For any $A, B, C \in \text{Obj} C$ we will say that $C$ is an extension of $B$ by $A$ if there exists a distinguished triangle $A \to C \to B \to A[1]$.

A class $\mathcal{P} \subset \text{Obj} C$ is said to be extension-closed if it is closed with respect to extensions and contains $0$.

• We will write $\langle \mathcal{P} \rangle$ for the smallest full retraction-closed triangulated subcategory of $C$ containing $\mathcal{P}$; we will call $\langle \mathcal{P} \rangle$ the triangulated subcategory densely generated by $\mathcal{P}$ (in particular, in the case $C = \langle \mathcal{P} \rangle$). Moreover, if $\mathcal{H}$ is the full subcategory of $C$ such that $\text{Obj} \mathcal{H} = \mathcal{P}$ then we will also say that $C$ is densely generated by $\mathcal{H}$.

• The smallest strictly full triangulated subcategory of $C$ containing $\mathcal{P}$ will be called the subcategory strongly generated by $\mathcal{P}$.

• For $X, Y \in \text{Obj} C$ we will write $X \perp Y$ if $C(X, Y) = \{0\}$.

For $D, E \subset \text{Obj} C$ we write $D \perp E$ if $X \perp Y$ for all $X \in D$, $Y \in E$. Given $D \subset \text{Obj} C$ we will write $D^\perp$ for the class

$$\{Y \in \text{Obj} C : X \perp Y \forall X \in D\}.$$ Dually, $^\perp D$ is the class $\{Y \in \text{Obj} C : Y \perp X \forall X \in D\}$.

• We will say that an additive covariant (resp. contravariant) functor from $C$ into an abelian category $A$ is homological (resp. cohomological) if it converts distinguished triangles into long exact sequences.

For a (co)homological functor $H$ and $i \in \mathbb{Z}$ we will write $H_i$ (resp. $H^i$) for the composition $H \circ [-i]$.

• We will write $K(B)$ for the homotopy category of (cohomological) complexes over $B$. Its full subcategory of bounded complexes will be denoted by $K^b(B)$. We will write $M = (M^i)$ if $M^i$ are the terms of the complex $M$.

\[\text{Alternative, } \langle \mathcal{P} \rangle \text{ can be called the thick subcategory of } C \text{ generated by } \mathcal{P}.\]
1.2 Weight structures: basics

Let us recall some basic definitions of the theory of weight structures.

**Definition 1.2.1.** A pair of subclasses $C_{w\leq 0}, C_{w\geq 0} \subset \text{Obj } C$ will be said to define a *weight structure* $w$ on a triangulated category $C$ if they satisfy the following conditions.

(i) $C_{w\geq 0}$ and $C_{w\leq 0}$ are retraction-closed in $C$ (i.e., contain all $C$-retracts of their objects).

(ii) **Semi-invariance with respect to translations.**

$C_{w<0} \subset C_{w\leq 0}[1], C_{w>0}[1] \subset C_{w>0}$.

(iii) **Orthogonality.**

$C_{w<0} \perp C_{w\geq 0}[1].$

(iv) **Weight decompositions.**

For any $M \in \text{Obj } C$ there exists a distinguished triangle

$$LM \to M \to RM \to LM[1]$$

such that $LM \in C_{w<0}$ and $RM \in C_{w>0}[1]$.

Moreover, we will say that a triangulated category $C$ is *weighted* if it is endowed with a fixed weight structure.

We will also need the following definitions.

**Definition 1.2.2.** Let $i, j \in \mathbb{Z}$; assume that a triangulated category $C$ is endowed with a weight structure $w$.

1. The full subcategory $Hw$ of $C$ whose objects are $C_{w=0} = C_{w\geq 0} \cap C_{w\leq 0}$ is called the *heart* of $w$.

2. $C_{w\geq i}$ (resp. $C_{w\leq i}$, resp. $C_{w=i}$) will denote the class $C_{w\geq 0}[i]$ (resp. $C_{w<0}[i]$, resp. $C_{w=0}[i]$).

3. We will call $\cup_{i\in \mathbb{Z}} C_{w\geq i}$ (resp. $\cup_{i\in \mathbb{Z}} C_{w\leq i}$) the class of *$w$-bounded below* (resp., *$w$-bounded above*) objects of $C$; we will write $C^b$ for the full subcategory of $C$ whose objects are bounded both above and below.

4. We will say that $(C, w)$ is *bounded* and $C$ is a *bounded weighted category* if $C^b = C$, i.e., if $\cup_{i\in \mathbb{Z}} C_{w\leq i} = \text{Obj } C = \cup_{i\in \mathbb{Z}} C_{w\geq i}$.

5. Let $C'$ be a triangulated category endowed with a weight structure $w'$; let $F : C \to C'$ be an exact functor.

Then $F$ is said to be *weight-exact* (with respect to $w, w'$) if it maps $C_{w\leq 0}$ into $C'_{w'\leq 0}$ and sends $C_{w\geq 0}$ into $C'_{w'\geq 0}$. 

6. Let $D$ be a full triangulated subcategory of $C$.

We will say that $w$ restricts to $D$ whenever the couple $w_D = (C_w \leq 0 \cap \text{Obj } D, C_w \geq 0 \cap \text{Obj } D)$ is a weight structure on $D$. If this is the case then we will call $(C_w \leq 0 \cap \text{Obj } D, C_w \geq 0 \cap \text{Obj } D)$ the restriction of $w$ to $D$, and say that the weight structure $w_D$ extends to $C$.

7. We will say that the subcategory $H$ is negative (in $C$) if $\text{Obj } H \perp (\cup_{i>0} \text{Obj }(H[i]))^\perp$.

8. We will say that an additive category $B$ is weakly idempotent complete if any split monomorphism $i : X \to Y$ (that is, $\text{id}_X$ equals $p \circ i$ for some $g \in B(Y, X)$) is isomorphic to the monomorphism $\text{id}_X \oplus 0 : X \to X \oplus Z$ for some object $Z$ of $B$.

Remark 1.2.3. 1. A simple (and still quite useful) example of a weight structure comes from the stupid filtration on the homotopy category of cohomological complexes $K(B)$ for an arbitrary additive $B$; it can also be restricted to the subcategory $K^b(B)$ of bounded complexes (see Definition 1.2.2(6)). In this case $K(B)_{w_{\leq 0}}$ (resp. $K(B)_{w_{\geq 0}}$) is the class of complexes that are homotopy equivalent to complexes concentrated in degrees $\geq 0$ (resp. $\leq 0$); see Remark 1.2.3(1) of [BoS18b] for more detail.

The heart of the weight structure $w_{st}$ is the retraction-closure of $B$ in $K(B)$; hence it is equivalent to $\text{Kar}(B)$ (since both $K^-(B)$ and $K^+(B)$ are idempotent complete).

2. In the current paper we use the “homological convention” for weight structures; it was previously used in [Wil09], [BoS18b], [Bon18a], and in [BoK18], whereas in [Bon10a] and [Bon10b], the “cohomological convention” was used. In the latter convention the roles of $C_{w_{\leq 0}}$ and $C_{w_{\geq 0}}$ are essentially interchanged, that is, one considers $C_{w_{\leq 0}} = C_{w_{\geq 0}}$ and $C_{w_{\geq 0}} = C_{w_{\leq 0}}$.

We also recall that D. Pauksztello has introduced weight structures independently in [Pau08], he called them co-t-structures.

3. Till section 3 the reader may assume that all weight structures we consider are bounded. Note that weight structures of this sort have an easy description in terms of negative subcategories; see Proposition 1.2.4(6, 7, 3).

Proposition 1.2.4. Let $m \leq n \in \mathbb{Z}$, $M, M' \in \text{Obj } C$, $g \in C(M, M')$.  

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4In several papers (mostly, on representation theory and related matters) a negative subcategory satisfying certain additional assumptions was said to be silting; this notion generalizes the one of tilting.

5It appears that this term was introduced in [Büh10, Definition 7.2], whereas the notion itself probably originates from [Fre66], where it was said that retracts have complements in $B$. 

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1. The axiomatics of weight structures is self-dual, i.e., for $C' = C^{op}$ (so Obj$C' = \text{Obj } C$) there exists the (opposite) weight structure $w'$ for which $C_{w' \leq 0} = C_{w \geq 0}$ and $C_{w' \geq 0} = C_{w \leq 0}$.

2. $C_{w \leq 0}$, $C_{w \geq 0}$, and $C_{w = 0}$ are (additive and) extension-closed.

3. $C^b$ is the triangulated subcategory of $C$ strongly generated by $C_{w = 0}$.

4. $C_{w \geq 0} = (C_{w \leq -1})^{\perp}$ and $C_{w \leq 0} = C_{w \geq 1}^{\perp}$.

5. $Hw$ is weakly idempotent complete.

6. Any full subcategory of $Hw$ is a negative subcategory of $C$.

7. Assume that $C$ is densely generated by its negative additive subcategory $B$.

Then there exists a unique weight structure $w_B$ on $C$ whose heart contains $B$. Moreover, this weight structure is bounded, $C_{w_B = 0} = \text{Kar}_C(\text{Obj } B)$, and $C_{w_B \geq 0}$ (resp. $C_{w_B \leq 0}$) is the smallest class of objects that contains $\text{Obj } B[i]$ for all $i \geq 0$ (resp. $i \leq 0$), and is also extension-closed and retraction-closed in $C$.

8. Let $C'$ be a triangulated category endowed with a weight structure $w'$; let $F : C \rightarrow C'$ be an exact functor.

If $F$ is weight-exact then it sends $C_{w = 0}$ into $C_{w' = 0}'$. Moreover, if $w$ is bounded then the converse implication is valid is well.

Proof. Assertions 1–5 were essentially established in [Bon10a] (yet cf. Remark 1.2.3(4) of [BoS18b], pay attention to Remark 1.2.3(2) above, and see [BoV20] for some more detail on assertion 5).

Assertion 6 follows from the orthogonality axiom (iii) in Definition 1.2.1 immediately.

Assertion 7 is given by Corollary 2.1.2 of [BoS18b].

The first implication in assertion 8 is obvious. The converse implication easily follows from assertion 3; see also Lemma 2.7.5 of [Bon10b].

Let us now recall some statements on weight-exact localizations; more information can be found in Proposition 3.1.4 and Remark 3.1.5(3) below.

**Proposition 1.2.5.** Assume that $C$ is endowed with a bounded weight structure $w$, $D \subset C$ is a triangulated subcategory strongly generated by a class $P \subset C_{w = 0}$, and $\pi$ is the localization functor $\pi : C \rightarrow C' = C / D$. 

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1. Then $w$ restricts to $D$, and there exists a bounded weight structure $w'$ on $C'$ such that $\pi$ is weight-exact and the class $C_{w'=0}$ essentially equals $\pi(C_{w=0})$.

2. Moreover, for any $M \in C_{w\leq 0}$ and $N \in C_{w\geq 0}$ the homomorphism $C(M,N) \to C'((\pi(M),\pi(N)))$ is surjective.

Consequently, if $\pi(M) \cong \pi(N')$ for some $N' \in C_{w=0}$ then there exists $f \in C(M,N')$ such that $\pi(f)$ is an isomorphism.

**Proof.** Proposition [1.2.3(6)] implies that $w$ restricts to $D$ indeed. By Proposition 3.3.3(1) of [BoS18c] there exists a weight structure $w'$ on $C$ such that $\pi$ is weight-exact, and the class $C_{w'=0}$ essentially equals $\pi(C_{w=0})$. Lastly, Proposition [1.2.3(3)] implies that $w'$ is bounded; see also Proposition 8.1.1(3) of [Bon10a].

Proposition 1.8(I.3) of [Bon18a] gives the first part of the assertion.

Next, if $\pi(M) \cong \pi(N')$ for $N' \in C_{w=0}$ then $N'$ also belongs to $C_{w\geq 0}$. Thus we can apply the first part for $N' = N$ to lift the isomorphism $\pi(M) \rightarrow \pi(N')$ to a $C$-morphism $f$. 

1.3 On (strong) weight complexes, pure functors, and Grothendieck groups of weighted categories

Now we recall the theory of so-called "strong" weight complex functors. Note here that this version of the theory is less general than the "weak" one that we will recall in §3.1 below. However, both versions of the theory are fine for the purposes of the current paper, and we start from the strong one that it easier to describe.

**Proposition 1.3.1.** Assume that $C$ possesses an $\infty$-enhancement (see §1.1 of [Sos19] for the corresponding references), and is endowed with a bounded weight structure $w$.

Then there exists an exact functor $t^M : C \rightarrow K^b(Hw)$, $M \mapsto (M^M)$, that enjoys the following properties.

1. The composition of the embedding $Hw \rightarrow C$ with $t^M$ is isomorphic to the obvious embedding $Hw \rightarrow K^b(Hw)$.

2. Let $C'$ be a triangulated category that possesses an $\infty$-enhancement as well and is endowed with a bounded weight structure $w'$; let $F : C \rightarrow C'$ be a weight-exact functor that lifts to $\infty$-enhancements. Then the composition $t^{t^M \circ F}$ is isomorphic to $K^b(HF) \circ t^{t^M}$, where $t^{t^M}$ is the weight complex functor corresponding $w'$, and the functor $K^b(HF)$ :
\[ K^b(Hw) \to K^b(Hw') \] is the obvious \( K^b(-) \)-version of the restriction \( HF : Hw \to Hw' \).

Moreover, the obvious modification of this statement corresponding to contravariant weight-exact functors (cf. Proposition 1.2.4(1)) is valid as well.

3. If \( M \in C_{w \leq \eta} \) (resp. \( M \in C_{w \geq \eta} \)) then \( t^\eta(M) \) belongs to \( K(Hw)_{w, \leq \eta} \) (resp. to \( K(Hw)_{w, \geq \eta} \)).

4. Assume that \( \mathcal{A} \) is an additive covariant (resp., contravariant) functor from \( Hw \) into an abelian category \( \mathcal{A} \). Then the functor \( H^\mathcal{A} \) (resp. \( H_\mathcal{A} \)) that sends \( M \) into the zeroth homology of the complex \( \mathcal{A}(M^i) \) (resp. \( \mathcal{A}(M^{-i}) \)) is (co)homological. Moreover, this is the only (co)homological functor (up to a unique isomorphism) whose restriction to \( Hw \) equals \( \mathcal{A} \) and whose restrictions to \( Hw[i] \) for \( i \neq 0 \) vanish.

**Proof.** Assertions 1 and 2 easily follow from Corollary 3.5 of [Sos19] (along with its proof which is essentially self-dual); see also §6.3 of [Bon10a] for the case where \( C \) possesses a differential graded enhancement (and that is sufficient for our purposes below).

The easy assertion 3 is given by Proposition 1.3.4(10) of [Bon18b] (note however that to apply the results of ibid. one should recall that \( t^\eta \) is compatible with the weak weight complex functor as defined in loc. cit.; see Remark 1.3.5(3) of ibid. and Corollary [Sos19] for this statement, whereas the "weak" theory itself is recalled in Proposition 3.2.2 below).

The only non-trivial statement in assertion 4 is the uniqueness one, that is given by Theorem 2.1.2(2,3) of [Bon18b].

**Remark 1.3.2.** The term "weight complex" originates from [GiS96]; yet the domains of the ("concrete") weight complex functors considered in that paper were not triangulated.

Now let us relate weight complexes to certain Grothendieck groups.

**Definition 1.3.3.** Let \( B \) be an essentially small additive category, whereas \( C \) is an essentially small triangulated category.

1. Then the (split) Grothendieck group \( K^\text{add}_0(B) \) of \( B \) is the abelian group whose generators are the isomorphism classes of objects of \( B \), and the relations are of the form \( [B] = [A] + [C] \) for all \( A, B, C \in \text{Obj} B \) such that \( B \cong A \oplus C \).

2. The Grothendieck group \( K^\text{tr}_0(C) \) is the abelian group whose generators are the isomorphism classes of objects of \( C \), and the relations are of the form \( [B] = [A] + [C] \), where \( A \to B \to C \to B[1] \) is a \( C \)-distinguished triangle.
Now we will formulate a rather general statement on Grothendieck groups; we will generalize it in Theorem 3.2.4 below.

**Theorem 1.3.4.** Adopt the assumptions of Proposition 1.3.1; assume $N$ is an object of $C$ and $F : Hw \rightarrow A$ is an additive functor (consequently, $A$ is additive as well).

1. Then the correspondence $M \mapsto \sum_{i \in \mathbb{Z}} (-1)^i [F(M')]$ (where $\text{tst}(M) = (M')$) gives a well-defined homomorphism $F_{K_0} : K_0^b(C) \rightarrow K_0^{\text{add}}(A)$.

2. Assume that $A$ is an abelian category, and there exists a $C$-morphism $h$ either from $N'$ to $N$ or vice versa such that $F_{K_0}([\text{Cone}(h)]) = 0$ and $N' \in C_{w=0}$. Then $F_{K_0}([N]) = [F(N')]$.

Moreover, if $H_j^F(\text{Cone}(h)) = 0$ for $j = 0, 1$ and the homological functor $H^F$ given by Proposition 1.3.1(4), then $F_{K_0}([N]) = [H^F(N)]$.

3. Assume that $A$ is abelian semi-simple. Then $F_{K_0}(N) = \sum_{i \in \mathbb{Z}} (-1)^i [H^F(N)]$.

4. Assume that $F = \text{id}_{Hw}$. Then the corresponding homomorphism $\text{id}_{Hw,K_0} : K_0^b(C) \rightarrow K_0^{\text{add}}(Hw)$ is an isomorphism.

**Proof.** 1. Obviously $\text{id} \circ K^b(F)$ gives a well-defined homomorphism $K_0^b(C) \rightarrow K_0^b(A)$. It remains to recall that $K_0^b(A) \cong K_0^{\text{add}}(A)$ by Theorem 1 of [Ros11]; see also Theorem 3.2.4(II.1) and Remark 3.2.5(1) below.

2. Immediately from the definition of $K_0^b(C)$, we have $F_{K_0}([N]) = F_{K_0}([N'])$. Next, assertion 1 implies that $F_{K_0}([N]) = [H^F(N')] = F_{K_0}([N'])$.

Lastly, if $H^F(\text{Cone}(h)) = 0 = H^F(\text{Cone}(h)[-1])$ then we have $H^F(N') = H^F(N)$, since the functor $H^F$ is homological.

3. This statement is an immediate consequence of our definitions.

4. The embedding $Hw \rightarrow C$ obviously gives a homomorphism $K_0^{\text{add}}(B) \rightarrow K_0^b(C)$; it is surjective since $Hw$ strongly generates $C$ according to Proposition 1.2.4(3). Hence $\text{id}_{Hw,K_0}$ is an isomorphism, since its composition with $F_{\text{id}}$ obviously equals $\text{id}_{K_0^{\text{add}}(B)}$.

**Remark 1.3.5.** Assume that we have a "tensor product" bi-functor $\nu : C \times C \rightarrow C$ that becomes and exact functor if one of the arguments is fixed. It clearly defines a bi-additive operation $K_0^b(C) \times K_0^b(C) \rightarrow K_0^b(C)$.

Next, if $\mu$ restricts to a bi-functor $Hw \times Hw \rightarrow Hw$, and the latter is compatible (via $F$) to a certain bi-additive functor $A \times A \rightarrow A$ then the homomorphism $F_{K_0}$ clearly becomes a ring one (with respect to the corresponding operations).

2. The idea for parts 2 and 3 of our theorem is that in some cases the value of $F_{K_0}([M])$ can be computed by means of the homological functor $H^F$.

We will discuss concrete examples of these calculations in Theorem 2.3.4 and Remark 2.3.5(1) below.
2 Applications to motives and unramified cohomology

Now we apply the results of previous sections to motives, Chow weight structures, and unramified cohomology.

In §2.1 we recall some facts on Voevodsky’s effective motivic categories $DM_{\text{eff}}^R \subset DM_{\text{eff}}$ and sheaves with transfers.

In §2.2 we define the weight structure $w_{\text{Chow}}^{eff,gm}$ on $DM_{\text{gm}}^R$ and relate the corresponding pure functors to unramified cohomology. In particular, we study the unramified Brauer group sheaf with transfers.

In §2.3 we recall some properties of motives with compact support that demonstrate that applying functors of the sort $F_{K_0}$ (see Theorem 1.3.4(2)) to $M_{\text{gm}}^c(-)$ yields (Euler characteristic) homomorphisms from the Grothendieck group of varieties. Next we apply our results in the case where $F$ is given by $\mathbb{Z}(\ell)$-adic étale cohomology. These statements are closely related to Theorem 5.1 of [Eke09b].

2.1 On Voevodsky motives and sheaves with transfers

First we introduce some notation and recall some basics on Voevodsky motives.

$k$ will be our perfect base field of characteristic $p$ (possibly zero). We introduce the following convention: in the case $p = 0$ the notation $\mathbb{Z}[1/p]$ will mean just the ring $\mathbb{Z}$.

We will write $R$ for the coefficient ring of our motivic categories; respectively, $R$ is a unital commutative associative ring. The reader may assume that $R = \mathbb{Z}[1/p]$ since this case is the most interesting one for the results of this paper. On the other hand, when we will write $1/p \not\in R$ we will always assume (in addition) that $p > 0$.

$\text{Var} \supset \text{SmVar} \supset \text{SmPrVar}$ will denote the set of all (not necessarily connected) varieties over $k$, resp. of smooth varieties, resp. of smooth projective varieties. The category of smooth $k$-varieties will be denoted by $\text{Sm}$.

We will not define any categories of motives in this paper. The reader can find these definitions in [BeV08] and [Deg11]; cf. also [Voe00], [MVW06], and [BoK18]. Instead, we list some well-known properties of motives (that mostly originate from [Voe00]) and introduce some notation.

Proposition 2.1.1. Let $X, Y \in \text{SmVar}$.

1. The category $DM_{\text{eff}}^R$ of (unbounded) $R$-linear motivic complexes is triangulated, $R$-linear, and equipped with a functor $M_{\text{gm}}^R$ (of the $R$-linear
motif) from \text{Sm} into it. Moreover, $M^R_{\text{gm}}(\mathbb{A}^1 \times X \to X)$ is an isomorphism.

We will write $\text{DM}^{\text{eff}}_{\text{gm}, R}$ for the subcategory of $\text{DM}^{\text{eff}}_R$ densely generated by $M^R_{\text{gm}}(\text{SmVar})$; it is essentially small. Moreover, we will write \text{Sm}' for the full subcategory of $\text{DM}^{\text{eff}}_{\text{gm}, R} \subset \text{DM}^{\text{eff}}_R$ whose object class is $M^R_{\text{gm}}(\text{SmVar})$.

2. $\text{DM}^{\text{eff}}_{\text{gm}, R}$ is a symmetric monoidal triangulated category. Moreover, the functor $M^R_{\text{gm}}$ converts products of varieties into tensor products, and sends disjoint unions into direct sums.

3. Let us write $\text{pt}$ for $\text{Spec} \ k$, $R$ for $M^R_{\text{gm}}(\text{pt})$, and $R(1)$ for the "complement" of $R$ to $M^R_{\text{gm}}(\mathbb{P}^1)$ corresponding to the morphisms $\text{pt} \to \mathbb{P}^1 \to \text{pt}$ (here we send $\text{pt}$ into $0 \in \mathbb{P}^1(k)$).

Then the functor $-\langle 1 \rangle = - \otimes R(1)$ is fully faithful on $\text{DM}^{\text{eff}}_{\text{gm}, R}$ (and on $\text{DM}^{\text{eff}}_R$ as well).

4. Let $Z \in \text{SmVar}$ be an equicodimensional closed subvariety of codimension $n$ in $X$ (i.e., all of the connected components of $Z$ are of the same codimension $n \geq 0$ in $X$). Then there exists a Gysin distinguished triangle

$$M^R_{\text{gm}}(X \setminus Z) \to M^R_{\text{gm}}(X) \to M^R_{\text{gm}}(Z) \langle n \rangle \to M^R_{\text{gm}}(X \setminus Z)[1]$$

in $\text{DM}^{\text{eff}}_{\text{gm}, R}$, where $-\langle n \rangle$ is the $n$th iteration of $-\langle 1 \rangle$.

5. Let us write $\text{HI}_R$ for the heart of the homotopy $t$-structure $t^R_{\text{hom}}$ on $\text{DM}^{\text{eff}}_R$ (see §3.1 below); consequently, $\text{HI}_R$ is a full abelian subcategory of $\text{DM}^{\text{eff}}_R$. We will call the objects of $\text{HI}_R$ (Nisnevich homotopy invariant $R$-linear) sheaves with transfers.

Then the correspondence sending an a sheaf with transfers $N$ into the functor $\text{DM}^{\text{eff}}_R(M^R_{\text{gm}}(-), N) : \text{Sm}^{\text{op}} \to \text{Ab}$ gives a faithful exact functor For from $\text{HI}_R$ into the abelian category of Nisnevich sheaves on $\text{Sm}$ (however, it will not probably cause any misunderstanding if one will write $N(X)$ instead of $\text{DM}^{\text{eff}}_R(M^R_{\text{gm}}(X), N)$).

6. Furthermore, for any $N \in \text{Obj} \ \text{HI}_R$ and $i \in \mathbb{Z}$ the group $\text{DM}^{\text{eff}}_R(M^R_{\text{gm}}(X), N[i])$ is naturally isomorphic to $H^i_{\text{Nis}}(X, \text{For}(N))$ (i.e., to the $i$th Nisnevich cohomology of $X$ with coefficients in $\text{For}(N)$; this is certainly zero if $i < 0$).

\footnote{Actually, these Nisnevich cohomology groups are canonically isomorphic to the corresponding Zariski ones.}
7. Moreover, for any dense embedding \( f \) of smooth \( k \)-varieties the homomorphism \( \text{For}(N)(f) \) is injective.

8. The subcategory \( \text{Chow}^{\text{eff}}_R \) whose objects are the \( DM^{\text{eff}}_R \)-retracts of elements of \( M^{\text{R}}_{\text{gm}}(\text{SmPrVar}) \) is negative in \( DM^{\text{eff}}_{\text{gm},R} \), and it is naturally equivalent to the category of \( R \)-linear effective Chow motives. Moreover, if \( R \) is a \( \mathbb{Z}[1/p] \)-algebra then \( \text{Chow}^{\text{eff}}_R \) densely generates \( DM^{\text{eff}}_{\text{gm},R} \) (cf. Theorem 2.2.1(1) below).

9. Assume that \( R \) is a localization of \( \mathbb{Z}[1/p] \). Then there exists an object \( RG_m \) of \( DM^{\text{eff}}_R \) such that the functors \( DM^{\text{eff}}_R (M^{\text{R}}_{\text{gm}}(-), RG_m[i]) \) give the étale cohomology \( H^i_{\text{et}}(-, G_m \otimes R) \) (here \( G_m \) is the unit group sheaf) for all \( i \geq 0 \). Moreover, for \( S^{\text{Br}}_m = H^1_{\text{hom}}(RG_m) = H^0_{\text{hom}}(RG_m[2]) \) (see Definition 3.1.1 below) the restriction of \( DM^{\text{eff}}_R (-, S^{\text{Br}}_m) \) to the category \( \text{Sm}' \subset DM^{\text{eff}}_R \) (see assertion 1) is isomorphic to the same restriction of \( DM^{\text{eff}}_R (M^{R}_{\text{gm}}(-), RG_m[2]) \), and the sheaf \( \text{For}(S^{\text{Br}}_m) \) is the \( R \)-linear cohomological Brauer group functor \( \text{Br} : X \mapsto H^2_{\text{et}}(X, G_m \otimes R) \).

10. Assume that \( f : Y \to X \) is a finite flat \( \text{Sm} \)-morphism. Then the morphism \( \deg(f) \cdot \text{id}_{M^{\text{R}}_{\text{gm}}(X)} \) factors through the motif \( M^{\text{R}}_{\text{gm}}(Y) \).

Proof. §2.3, §4.4, Theorem 3.3, and Proposition 6.3.1 of [Be V08], and §1.3 and Lemma 1.1.1 of [BoK18] give assertions 1–6.

Next, assertion 7 immediately follows from the existence of Gersten resolutions for objects of \( H_{\text{RI}} \); see Theorem 24.11 of [MVW06].

The first part of assertion 8 immediately follows from Corollary 6.7.3 of [Be V08]. Next, the category \( \text{Chow}^{\text{eff}}_R \) densely generates \( DM^{\text{eff}}_{\text{gm},R} \) if \( R \) is a \( \mathbb{Z}[1/p] \)-algebra since this statement is true in the case \( R = \mathbb{Z}[1/p] \) (here one can apply Proposition 1.3.3 of [BoK18]), whereas in this case the statement is given by Proposition 5.3.3 of [Kel17] and also by Theorem 2.2.1(1) of [Bon11] (cf. Corollary 3.5.5 of [Voe00]).

The existence of \( RG_m \) is well-known (as well) and easily follows from Example 6.22 of [MVW06]; see also the introduction of [KaS18].

The calculation of \( S^{\text{Br}}_m \) is mentioned in Lemma 5.2 of ibid. We will give some detail for the proof of this statement and compare the restrictions of \( DM^{\text{eff}}_R (-, S^{\text{Br}}_m) \) and \( DM^{\text{eff}}_R (M^{\text{R}}_{\text{gm}}(-), RG_m[2]) \) to \( \text{Sm}' \) in §3.1 below.

Lastly, assertion 10 easily follows from Lemma 2.3.5 of [SnV00]; cf. the proof of [Bon11] Corollary 1.2.2(1).

\[ \square \]

\[ ^7 \text{I am deeply grateful to prof. D.-Ch. Cisinski for teaching me this argument.} \]
2.2 On Chow weight structures and unramified cohomology

Starting from this moment we will assume that $R$ is a $\mathbb{Z}[1/p]$-algebra.

**Theorem 2.2.1.** Assume that $X \in \text{SmVar}$. Then the following statements are valid.

1. The subcategory $\text{Chow}^{\text{eff}}_R$ of $DM^{\text{eff}}_{gm,R}$ actually strongly generates $DM^{\text{eff}}_{gm,R}$. Moreover, there exists a bounded weight structure $w_{\text{Chow}}^{\text{eff, gm}}$ on $DM^{\text{eff}}_{gm,R}$ whose heart equals $\text{Chow}^{\text{eff}}_R$, and $M_R^{gm}(X) \in DM^{\text{eff}}_{gm,Rw_{\text{Chow}}^{\text{eff, gm}} \leq 0}$.

   Furthermore, if $X$ is of dimension $n > 0$ then $M_R^{gm}(X)$ belongs to the subcategory of $DM^{\text{eff}}_{gm,R}$ densely generated by motives of projective varieties of dimension at most $n$.

2. The twist functor $-\langle 1 \rangle$ is weight-exact with respect to $w_{\text{Chow}}^{\text{eff, gm}}$. Consequently, $w_{\text{Chow}}^{\text{eff, gm}}$ restricts to the subcategory $DM^{\text{eff}}_{gm,R} \langle 1 \rangle$ (see Proposition 2.1.1(3)), and there exists a weight structure $w_{gm}^{\alpha}$ on the localization $DM^{\text{eff}}_{gm,R} = DM^{\text{eff}}_{gm,R}/DM^{\text{eff}}_{gm,R} \langle 1 \rangle$ such that the localization functor $\pi : DM^{\text{eff}}_R \to DM^{\text{eff}}_{gm,R} \langle 1 \rangle$ is weight-exact.

3. The weight structure $w_{gm}^{\alpha}$ extends (see Definition 1.2.2(6)) to a bounded weight structure $w_{gm}^{\alpha}$ on the category $DM^{\text{eff}}_{gm,R} = \text{Kar}(DM^{\text{eff}}_{gm,R})$ whose heart $Hw_{gm}^{\alpha} = \text{Chow}^{\text{eff}}_R$ consists of the retracts of elements of $M^{\text{eff}}_{gm}(\text{SmVar})$, where $M^{\text{eff}}_{gm} = \pi \circ M_{gm}$.

4. If $X \in \text{SmVar}$ then there exists $M \in \text{Obj Chow}^{\text{eff}}_R = DM^{\text{eff}}_{gm,Rw_{\text{Chow}}^{\text{eff, gm}} = 0}$ and $h \in DM^{\text{eff}}_{gm,R}(M^{\text{eff}}_{gm}(X), M)$ such that $\pi(h)$ is an isomorphism.

**Proof.** Applying Proposition 1.2.4(1) along with the negativity of Chow$^{\text{eff}}_R$ in $DM^{\text{eff}}_{gm,R}$ provided by Proposition 2.1.1(8) we obtain the existence of a bounded weight structure $w_{\text{Chow}}^{\text{eff, gm}}$ on $DM^{\text{eff}}_{gm,R}$ whose heart equals Chow$^{\text{eff}}_R$, note that Chow$^{\text{eff}}_R$ is retraction-closed in $DM^{\text{eff}}_{gm,R}$ by definition. Applying Proposition 1.2.4(3) we also obtain that Chow$^{\text{eff}}_R$ strongly generates $DM^{\text{eff}}_{gm,R}$. Next, Proposition 6.7.3 of [BeV08] implies that $M^{R}_{gm}(X) \perp DM^{\text{eff}}_{gm,Rw_{\text{Chow}}^{\text{eff, gm}} \geq 1}$; hence $M^{R}_{gm}(X) \in DM^{\text{eff}}_{gm,Rw_{\text{Chow}}^{\text{eff, gm}} \leq 0}$ by Proposition 1.2.4(4) (alternatively, see §2.1 of [BoK18]).

Lastly, the aforementioned arguments of [Bon11] and [Kel17] easily yield that $M^{R}_{gm}(X)$ belongs to the subcategory of $DM^{\text{eff}}_{gm,R}$ densely generated by motives of projective varieties of dimension at most $\dim X$ indeed.
The twist functor $-\langle 1 \rangle$ obviously sends the subcategory $\text{Chow}_{\text{eff}}$ into itself. Applying Proposition 1.2.5(1) we obtain that $w^{\text{eff}, \text{gm}}_{\text{Chow}}$ restricts to the subcategory $DM^\text{eff}_{\text{gm}, R}(1)$, and there exists a weight structure $w^\text{gm}_{\text{eff}}$ on $DM^\text{eff}_{\text{gm}, R} = DM^\text{eff}_{\text{gm}, R}/DM^\text{eff}_{\text{gm}, R}(1)$ such that the localization functor is weight-exact.

According to Proposition 1.2.4(6,7), $w^\text{gm}_{\text{eff}}$ extends to a weight structure $w^\text{gm}$ on $DM^\text{gm}_{\text{eff}}$ indeed. On the other hand, Theorem 5.2.1 of [BoS18a] gives a bounded weight structure $w$ on $DM^\text{gm}_{\text{eff}}$. Since $\pi(\text{Chow}_{\text{eff}}^\text{eff}) \subset \text{Chow}_{\text{eff}}$, Proposition 1.2.4(7) yields that $w = w^\text{gm}_{\text{eff}}$.

Since $M^\text{gm}_{\text{eff}}(X)$ is an object of $DM^\text{eff}_{\text{gm}}$, the previous assertion implies that $M^\text{gm}_{\text{eff}}(X) \in DM^\text{eff}_{\text{gm}, R} w^{\text{gm}}_{\text{eff}} = 0$. Moreover, $M^R_{\text{gm}}(X) \in DM^\text{eff}_{\text{gm}, R} w^{\text{eff}, \text{gm}}_{\text{Chow}} \leq 0$ by assertion 1. Thus we can apply Proposition 1.2.5 (for $C = DM^\text{eff}_{\text{gm}, R}$ and $\mathcal{P} = \text{Obj Chow}_{\text{eff}}^\text{eff}(1)$) to obtain the existence of $M$ and $h$ as desired.

Remark 2.2.2. The proof of the aforementioned Proposition 5.3.3 of [Kel17] and Theorem 2.2.1(1) of [Bon11] relied on Gabber’s resolution of singularities (see Theorem X.2.1 of [ILO14]). Thus if some "$p$-adic version" of loc. cit. was available (in the case $p > 0$) then one would be able to extend all the statements of this section to the case of an arbitrary coefficient ring $R$.

On the other hand, if $p = 0$ then Hironaka’s resolution of singularities gives a smooth compactification $P$ for (every) $X \in \text{SmVar}$, and the Gysin distinguished triangle in Proposition 2.1.1(4) implies that $\text{Cone}(M^R_{\text{gm}}(X) \to M^R_{\text{gm}}(P)) \in \text{Obj} DM^\text{eff}_{\text{gm}, R}(1)$. This makes our results much less interesting in the case $p = 0$.

Now we will relate motivic categories to unramified cohomology. We will use several results of [KaS17] where only the case $R = \mathbb{Z}$ was considered; note however that the proofs generalize to the case of an arbitrary coefficient ring $R$ without any problems.

**Theorem 2.2.3.** Let $N$ be an object of $DM^\text{eff}_{R}$, $S$ is a sheaf with transfers (see Proposition 2.1.1(5)), and $X \in \text{SmVar}$.

1. There exists an exact fully faithful functor $i^0$ that is right adjoint to $\pi$.

   Moreover, $N$ belongs to the essential image of $i^0$ if and only if the homomorphism $DM^\text{eff}_{R}(M^R_{\text{gm}}(f), N[i])$ is bijective for any $i \in \mathbb{Z}$ and any open dense embedding $f$ of smooth $k$-varieties; if this is the case then we will say that $N$ is birational.

2. $S$ is birational if and only if the homomorphism $DM^\text{eff}_{R}(M^R_{\text{gm}}(f), S) \simeq \text{For}(S)(f)$ (see Proposition 2.1.1(5) once again) is bijective for any $f$ as
above. Moreover, if this is the case then $DM^\text{eff}_R(M^R_{gm}(X), S[i]) = \{0\}$ for any $i \neq 0$.

3. The subcategory $\text{HI}_R^0$ of birational sheaves is a Serre subcategory of $\text{HI}_R$, and the embedding $\text{HI}_R^0 \to \text{HI}_R$ possesses a right adjoint functor $R^0_{nr}$ that sends a sheaf with transfers into its maximal birational subsheaf.

Moreover, for the counit homomorphism $c(S) : R^0_{nr}(S) \to S$ the image of $\text{For}(c(S))(X)$ gives the unramified part of $\text{For}(S)(X)$ in the sense of [KaS17, Definition 7.2.1].

4. Assume that $M \in \text{Obj Chow}^\text{eff}_R$ and for $h \in DM^\text{eff}_{gm,R}(M^R_{gm}(X), M)$ the morphism $\pi(h)$ is an isomorphism. Then the homomorphism $DM^\text{eff}_R(h, S)$ is injective, and its image coincides with the aforementioned unramified subgroup of $DM^\text{eff}_R(M^R_{gm}(X), S) \cong \text{For}(S)(X)$.

5. The restriction of the functor $DM^\text{eff}_R(\mathcal{\cdot}, R^0_{nr}(S))$ to $DM^\text{eff}_{gm,R}$ is isomorphic to $H_{S_{\text{Chow}^\text{eff}_R}}$ (see Proposition 1.3.1(4)), where $S_{\text{Chow}^\text{eff}_R}$ is the restriction of $DM^\text{eff}_R(\mathcal{\cdot}, S)$ to $\text{Chow}^\text{eff}_R$.

6. A cohomological functor from $DM^\text{eff}_{gm,R}$ into an abelian category $A$ is birational (i.e., $H(M^R_{gm}(f)[-i])$ is an isomorphism any open dense embedding $f$ of smooth $k$-varieties and all $i \in \mathbb{Z}$) if and only if $H$ annihilates $DM^\text{eff}_{gm,R}(1)$; these conditions are fulfilled if and only if $H(M^R_{gm}(P)[1][-i]) = 0$ for all $P \in \text{SmPrVar}$ and $i \in \mathbb{Z}$.

In particular, $N$ is birational (in the sense of assertion 1) if and only if $M^R_{gm}(P)[1] \perp N[i]$ for any $i \in \mathbb{Z}$ and $P \in \text{SmPrVar}$, and for an additive functor $F : \text{Chow}^\text{eff}_R \to A$ the corresponding functor $H_F$ (see Proposition 1.3.1(4)) is birational if and only $F$ kills all $M^R_{gm}(P)[1]$.

**Proof.**

1. The existence of $i^0$ is an immediate consequence of well-known abstract nonsense; see Theorem 4.3.5 of [KaS17]. The second part of the assertion is obvious.

2. Easy from Theorem 4.2.2(e) of ibid. (along with Proposition 2.1.1(4)).

3. $\text{HI}_R^0$ a Serre subcategory of $\text{HI}_R$ according to Proposition 2.6.2 of ibid. The right adjoint functor $R^0_{nr}$ exists according to Proposition 2.6.3 of ibid., and it gives the maximal birational subsheaf since $\text{HI}_R^0$ is a Serre subcategory. Lastly, $c(S)$ gives the corresponding unramified subgroup according to Theorem 7.3.1 of ibid.

4. Assume that $M$ is a retract of $M^R_{gm}(P_M)$ for $P_M \in \text{SmPrVar}$. Then the homomorphism $DM^\text{eff}_{gm,R}(M^R_{gm}(P_M), c(S))$ is bijective by Corollary 7.3.2
of ibid. Thus we have a commutative square
\[
\begin{array}{ccc}
DM^\text{eff}_R(M, R^0_{nr}(S)) & \rightarrow & DM^\text{eff}_R(M^R_{gm}(X), R^0_{nr}(S)) \\
\downarrow \cong & & \downarrow \\
DM^\text{eff}_R(M, S) & \rightarrow & DM^\text{eff}_R(M^R_{gm}(X), S)
\end{array}
\]
that clearly gives the result in question.

To apply Proposition 1.3.1[4] we should verify that the restriction of \(DM^\text{eff}_R(\cdot, R^0_{nr}(S))\) to \(\text{Chow}^\text{eff}_R[i] \subset DM^\text{eff}_R\) is zero if \(i \neq 0\) and is isomorphic to \(H^1_{\text{Chow}^\text{eff}_R}\) for \(i = 0\).

The first of these statements immediately follows from assertion 2 and the second one is an easy consequence of assertion 4 (as well as of \([\text{Eke09b}],[\text{KaS17}],\) Corollary 7.3.2).

If \(f\) is a smooth dense embedding then Proposition 2.1.1[(4)] easily implies that \(\text{Cone}(M^R_{gm}(f)) \in \text{Obj} \ DM^\text{eff}_{gm,R}(1)\). On the other hand, if \(X \in \text{SmVar}\) then \(f_X : \mathbb{A}^1 \times X \rightarrow \mathbb{P}^1 \times X\) is a dense embedding, and \(\text{Cone}(M^R_{gm}(f_X)) \cong M^R_{gm}(X)(1)\). The first equivalence in the assertion follows easily.

Conversely, if \(f\) is a smooth dense embedding then Proposition 2.1.1[(4)] easily implies that \(\text{Cone}(M^R_{gm}(f)) \in \text{Obj} \ DM^\text{eff}_{gm,R}(1)\). Theorem 2.2.1[(1)] implies that the set of all \(M^R_{gm}(P)(1)\) densely generates the category \(DM^\text{eff}_{gm,R}(1)\); this gives the converse implication.

The "in particular" statements are easy as well; we only note that the functor \(H_F\) annihilates \(\text{Chow}^\text{eff}_R[i]\) for all \(i \neq 0\) (see Proposition 1.3.1[(4)]).

Now let us apply Theorem 2.2.3 to the study of Brauer groups. We will also relate it to Theorem 5.1 of [Eke09] later.

**Corollary 2.2.4.** Assume that \(R\) is a localization of \(\mathbb{Z}[1/p]\), and set \(P^\text{Br} = R^0_{nr}(S^\text{Br})\) (see Proposition 2.1.1[(9)] for the definition of \(S^\text{Br}\) along with \(RG_m\)).

1. Then \(P^\text{Br}\) is a birational sheaf with transfers (consequently, it annihilates \(\text{Chow}^\text{eff}_R(1)\)), and \(\text{For}(P^\text{Br})\) is the *unramified Brauer group sheaf*, that is, it sends a connected smooth \(k\)-variety \(Y\) into the unramified part of the group \(\text{Br}(\text{Spec}(k(Y)))\).

2. The restriction \(H^0_{nr}\) of the functor \(DM^\text{eff}_R(\cdot, P^\text{Br})\) to \(DM^\text{eff}_{gm,R}\) is isomorphic to the functor \(H^0_{\text{Chow}^\text{eff}_R} \cong H^0_{RG_m[2], \text{Chow}^\text{eff}_R}\), where \(S^\text{Br}_{\text{Chow}^\text{eff}_R}\) (resp. \(RG_m[2]_{\text{Chow}^\text{eff}_R}\)) is the restriction to \(\text{Chow}^\text{eff}_R\) of the functor \(DM^\text{eff}_R(\cdot, S^\text{Br})\) (resp. \(DM^\text{eff}_R(\cdot, RG_m[2])\)).

Moreover, if \(N = M^R_{gm}(X)\) for some \(X \in \text{SmVar}\) and \(h : N \rightarrow M \in DM^\text{eff}_{gm, R^e_{gm,R_{\text{Chow}^\text{eff}_R}}(f, g_m)}\) is the morphism provided by Theorem 2.2.1[(4)] then \(H^0_{nr}(N) \cong \text{Im}(DM^\text{eff}_{gm,R}(h, S^\text{Br})) \cong \text{Im}(DM^\text{eff}_{gm,R}(h, RG_m[2]))\).
Proof. All the statements easily follow from Theorem 2.2.3 combined with Proposition 2.1.1(9). \qed

2.3 On the relation to motivic Euler characteristics of varieties

Starting from this moment we will assume that \( R \) is a \( \mathbb{Z}[1/p] \)-algebra; the main case is just \( R = \mathbb{Z}[1/p] \).

Let us recall some properties of motives with compact support.

**Proposition 2.3.1.** Denote by \( M^c_{gm} \) the \( R \)-linear motif with compact support functor from the category \( \text{SchPr} \) of \( k \)-varieties with morphisms being proper morphisms of varieties into \( DM^\text{eff}_R \); this functor is essentially provided by §4.1 of [Voe00] along with §5.3 of [Kel17] (see the proof below).

Then \( M^c_{gm} \) satisfies the following properties.

1. We have \( M^c_{gm}(P) = M^R_{gm}(P) \) whenever \( P \in \text{SmPrVar} \).

2. If \( i : Z \to X \) is a closed embedding of \( k \)-varieties and \( U = X \setminus Z \) then there exists a distinguished triangle

\[
M^c_{gm}(Z) \xrightarrow{M^c_{gm}(i)} M^c_{gm}(X) \to M^c_{gm}(U) \to M^c_{gm}(Z)[1].
\]  

(2.3.1)

3. If \( X, Y \in \text{Var} \) then \( M^c_{gm}(X \times Y) \cong M^c_{gm}(X) \otimes M^c_{gm}(Y) \).

4. Consequently, the function sending \( X \in \text{Var} \) into the class of \( M^c_{gm}(X) \) in \( K^0_{\text{gm}}(DM^\text{eff}_{gm,R}) \) sends products of varieties into products in \( K^0_{\text{gm}}(DM^\text{eff}_{gm,R}) \) (see Remark 1.3.5(3)) and satisfies the scissors relation \([X] = [U] + [Z]\) under the assumptions of assertion 2.

5. Thus for any additive functor \( F : \text{Chow}^\text{eff}_R \to \underline{\mathcal{A}} \), where \( \underline{\mathcal{A}} \) is an additive category, the correspondence \( X \mapsto F_{K_0}([M^c_{gm}(X)]) \) (see Theorem 1.3.4(1)) satisfies the scissors relation as well.

Proof. In Definition 5.3.1, Lemma 5.3.6, Proposition 5.3.12(1) (combined with Theorems 5.2.20, 5.2.21, and 5.3.14), Proposition 5.3.5, Proposition 5.3.8, and Corollary 5.3.9 of [Kel17], respectively, the obvious \( \mathbb{Z}[1/p] \)-linear analogues of assertions 1–3 were justified.

The \( R \)-linear results in questions can either be proved similarly or deduced from the results of ibid. using the properties of the obvious connecting functor

\[
- \otimes_{\mathbb{Z}[1/p]} R : DM^\text{eff}_{Z[1/p]} \to DM^\text{eff}_R
\]

given by Proposition 1.3.3 of [BoK18].

Lastly, assertions 4 and 5 follow from assertions 2 and 3 immediately. \( \Box \)

8More generally, \( M^c_{gm}(X) \in \text{Obj } DM^\text{eff}_{gm,R} \) for any \( X \in \text{Var} \). However, we will not consider motives of singular varieties in this paper.

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Remark 2.3.2. If $k$ is characteristic 0 field then Hironaka’s resolution of singularities easily implies that any function from $\text{SmVar}$ into an abelian group $A$ that satisfies the scissors relation is uniquely determined by its values on smooth projective varieties.

Using this observation one can obtain an extension of [Eke09a, Theorem 3.4(i)] to fields of positive characteristics.

We will discuss other functions of this sort in Theorem 2.3.4 and Remark 2.3.5; see also Proposition 4.2 of [Sca20].

In order to relate motives with compact support with earlier results we recall some basics on the twist-stable motivic category $DM^R_{\text{gm}}$. Proposition 2.1.1(3) implies that the obvious functor from $DM^R_{\text{eff}}$ into the 2-colimit category $\lim_{\rightarrow} \cdots$ is a full exact embedding (of triangulated categories); cf. §2.1 of [Voe00] or §6.1 of [BeV08]. Consequently, we will assume that $DM^R_{\text{eff}}$ is a full strict subcategory of $DM^R_{\text{gm}}$. Moreover, $DM^R_{\text{gm}}$, a monoidal category (see loc. cit.), this embedding is monoidal, and for any $n \geq 0$ the functor $-\langle n \rangle_{DM^R_{\text{gm}}} = - \otimes R\langle n \rangle$ has an obvious inverse that we will denote by $-\langle -n \rangle$.

Proposition 2.3.3. Assume that $X, Y \in \text{SmVar}$, and $\dim X \leq n$ (for some $n \geq 0$).

1. $DM^R_{\text{gm}}$ is a rigid category, i.e., all its objects are dualizable.

2. If the connected components of $X$ are of dimension $n$ then $M^c_{\text{gm}}(X) \cong D(M^R_{\text{gm}}(X))\langle n \rangle$, where $D = D_{DM^R_{\text{gm}}}$ is the duality functor.

3. There exists a unique bounded weight structure $w_{\text{Chow}}$ on $DM^R_{\text{gm}}$ whose heart is the category $\text{Chow}_R = \text{Chow}^\text{eff}_R[(-1)]$ (i.e., this is the full strict subcategory of $\text{Chow}^\text{eff}_R$ whose object class equals $\bigcup_{n \geq 0} \text{Obj Chow}^\text{eff}_R(-n)$).

4. The auto-equivalences $-\langle n \rangle$ are weight-exact with respect to $w_{\text{Chow}}$ for all $n \in \mathbb{Z}$.

5. The duality functor $D_{DM^R_{\text{gm}}}$ is weight-exact (as well), i.e., it sends $DM^R_{\text{gm}}w_{\text{Chow}} \leq 0$ into $DM^R_{\text{gm}}w_{\text{Chow}} \geq 0$ and vice versa (cf. Proposition 1.2.4(1)).

6. Duality restricts to the subcategory $\text{Chow}_R$ of $DM^R_{\text{gm}}$, and for the weight complex functor $t^{st} : DM^R_{\text{gm}} \rightarrow K^b(\text{Chow}_R)$ we have $D_{K^b(\text{Chow}_R)} \circ t^{st} \cong t^{st} \circ D_{DM^R_{\text{gm}}}$.

7. $K^0_{\text{ct}}(DM^R_{\text{gm}}) \cong K_0(DM^R_{\text{gm}})\langle R\langle 1 \rangle \rangle^{-1} \cong K^\text{add}_{\text{ct}}(\text{Chow}_R) \cong K^\text{add}_{\text{ct}}(\text{Chow}^\text{eff}_R)\langle R\langle 1 \rangle \rangle^{-1}$.  

20
8. Denote the category of finitely generated $\mathbb{Z}_\ell$-modules by $\mathbb{Z}_\ell^{-\text{mod}}$. Then $K_0^\text{add}(\mathbb{Z}_\ell^{-\text{mod}})$ is the free abelian group generated by the classes of $\mathbb{Z}_\ell$ and of $\mathbb{Z}_\ell/\ell^i\mathbb{Z}_\ell$ for all $i > 0$.

9. Assume that $k$ is an algebraically closed field, $R$ is a localization of $\mathbb{Z}[1/p]$, and a prime $\ell$ belongs to $\mathbb{P} \setminus \{p\}$ and is not invertible in $R$.

Then there exists an étale realization functor $RH_{et,\mathbb{Z}_\ell} : DM^R_{\text{gm}} \to D^b(\mathbb{Z}_\ell)$ that possesses the following properties:

$$RH_{et,\mathbb{Z}_\ell}(M_{gm}(X)) \cong RH(X_{et,\mathbb{Z}_\ell}), \quad RH_{et,\mathbb{Z}_\ell} \circ - \langle n \rangle \cong RH_{et,\mathbb{Z}_\ell} \circ [2n]$$

and

$$RH_{et,\mathbb{Z}_\ell} \circ D_{DM^R_{\text{gm}}} \cong D_{D^b(\mathbb{Z}_\ell)} \circ RH_{et,\mathbb{Z}_\ell}.$$
more general) Theorem 7.2.11 of [CD16] as well as from Remark 4.8 of [Ivo07].

Now let us apply this statement to the calculation of certain $K_0$-classes.

**Theorem 2.3.4.** Adopt the notation and the assumptions of Proposition 2.3.39 (in particular, $R$ is a localization of $\mathbb{Z}[1/p]$, and one may assume that $R = \mathbb{Z}(\ell)$ if $R = \mathbb{Z}[1/p]$); assume that all the connected components of $X \in \text{SmVar}$ have dimension $d$, and let $N$ be an object of $DM_{gm,R}^{eff}$.

For $n \geq 0$ take the functors $E^n$, $H^n$, $F^n$, and $G^n$ to be the restrictions to $\text{Chow}^{eff}_R$ of the functors $H_{et,Z_n}^{2d-n}$, $H_{et,Z_n}$, Tor $H^n$, and $H^n / \text{Tor} H^n$, respectively; here we assume that the target of these functors equals $\mathbb{Z}_{\ell} - \text{mod}^p$.

Then the following statements are valid.

1. The class $E^n_{K_0}([M_{gm}^R(X)]) \in K_0^{add}(\mathbb{Z}_{\ell} - \text{mod}^p) = K_0^{add}(\mathbb{Z}_{\ell} - \text{mod})$ equals $F^{n+1}_{K_0}([M_{gm}^R(X)]) + G^n_{K_0}([M_{gm}^R(X)])$.

2. $G^n_{K_0}([N]) = m[\mathbb{Z}]$, where $m \in \mathbb{Z}$ satisfies the equality $G^n \otimes Q_{K_0}([N]) = m[Q_{\ell}] \in K_0(Q_{\ell} - \text{mod})$. Moreover, we have $G^n \otimes Q_{K_0}([N]) = \sum_{j=-\infty}^{\infty} (-1)^j [H^n_{G^n \otimes Q}(N)]$; here $H^n_{G^n \otimes Q}$ is the cohomological functor from $DM_{gm,R}^{eff}$ into $Q_{\ell}$-vector spaces that corresponds to $G_n \otimes Q$ according to Proposition 1.3.1(4).

3. Let $N = M_{gm}^R(X)$.

Then $H^n_{G^n \otimes Q}(N) = \{0\}$ if $i > 0$, and $H^0_{G^n \otimes Q}(N) \cong Q^c_{\ell}$, where $c$ is the number of connected components of $X$.

Furthermore, if $k$ is of finite transcendence degree over its prime subfield then $H^n_{G^n \otimes Q}(N) \cong Gr^n_{W}(H^{i+n}_{et}(X, Q_{\ell}))$; that is, we take the factors of the Deligne’s weight filtration (for this purpose we take a field of definition $k' \subset k$ of $X$ such that $k'$ is finitely generated over its prime subfield). Consequently, $m = \sum_{j=0}^{2d} (-1)^j \dim_{Q_{\ell}} Gr^n_{W}(H^{j}_{et}(X, Q_{\ell}))$.

4. $F^0 = F^1 = 0$, whereas the functors $F^2$, $F^3$, $G^0$, and $G^1$ kill $\text{Chow}^{eff}_R(1)$.

Consequently, the functor $H_A$ is birational (in the sense of Theorem 2.2.3(6)) and $A_{K_0}([M_{gm}^R(X)]) = [H_{A}(M_{gm}^R(X))] = [A(M)]$ if $A$ is one of the functors $F^2$, $F^3$, $G^0$, or $G^1$, and $h : N \to M \in \text{Obj Chow}^{eff}_R$ is the morphism provided by Theorem 2.2.1(4).

5. Assume that $N = M_{gm}^R(X)$. If there exists a finite flat morphism $Y \to X$, where $Y = A^d \setminus Z$, then $H^n_{G^n \otimes Q}(N) = \{0\}$ for any $i \in \mathbb{Z}$ if $0 < n < 2 \text{codim}_{A^d}(Z)$, and $H^n_{G^n \otimes Q}(N) = \{0\}$ for $i \neq 0$ as well.

---

\(^{10}\) One can also take $n$ to be negative here; yet all the classes mentioned in this proposition vanish in this case.
More generally, if \( X \) is unirational then \( H_{\text{et}} \otimes \mathbb{Q}(N) = \{0\} \), and also \( H_{F^3}(N) \cong H_{\text{et}}^0(N) \otimes \mathbb{Z} \cong DM_{\text{eff}}^R(N, RG_m(2)) \otimes \mathbb{Z} (\text{see Corollary 2.2.4}); \) thus \( H_{F^3}(N) \) is the \( \ell \)-adic part of the unramified Brauer group of \( k(X) \).

**Proof.**  
1. Applying Proposition 1.3.1(2) one easily obtains that the assertion is an easy application of duality provided by Proposition 2.3.3(2.5) along with properties of twists mentioned in parts 3 and 2 of the proposition.

2. Obviously, there does exist \( m \in \mathbb{Z} \) such that \( G^m_{K_0}([N]) = m[\mathbb{Z}] \), and we also have \( G^n \otimes \mathbb{Q}_{K_0}([M_{\text{eff}}^R(X)]) = m[\mathbb{Q}] \).

Next, \( G^n \otimes \mathbb{Q}_{K_0}([N]) = \sum_{i=-\infty}^{\infty} (-1)^i[H^i_{G^n \otimes \mathbb{Q}}(N)] \) by Theorem 1.3.4(3).

3. \( H^i_{G^n \otimes \mathbb{Q}}(N) = 0 \) for \( i < 0 \) since \( t^{\text{et}}(N) \) is homotopy equivalent to a complex concentrated in non-negative degrees; see Proposition 1.3.1(3) along with Theorem 2.2.1(1).

Next, Proposition 1.3.1(1) implies that \( H^0_{G^n \otimes \mathbb{Q}}(N) \) is the only cohomological functor from \( DM_{\text{eff}}^{gm,R} \) that restricts to 0 on \( \text{Chow}_{\text{et}}^{\text{eff}}[i] \) for \( i \neq 0 \) and whose restriction to \( \text{Chow}_{\text{et}}^{\text{eff}} \) gives the functor \( H^0 \otimes \mathbb{Q} \). On the other hand, tensoring the restriction of the zeroth cohomology of \( RH_{\text{et}, \mathbb{Z}} \) (see Proposition 2.3.3(9)) to \( DM_{\text{eff}}^{gm,R} \) by \( \mathbb{Q} \) one obtains a functor that possesses these properties as well. Hence \( H^0_{G^n \otimes \mathbb{Q}}(N) \cong \mathbb{Q}_{\ell}^{\text{et}} \) indeed.

The remaining calculation is an easy application of the theory of weight spectral sequences that was introduced in \( \text{[Bon10a]} \). Recall that for any cohomological functor \( H \) from \( DM_{\text{eff}}^{gm,R} \) into an abelian category \( \mathcal{A} \) and any \( N \in \text{Obj} \ DM_{\text{eff}}^{gm,R} \) there exists a spectral sequence \( T = T_n(H, N) \) with \( E^2_p = H^q(N^{-p}) \), such that \( N^i \) and the boundary morphisms of \( E_1(T) \) come from \( t^{\text{et}}(N) \); it converges to \( H^{p+q}(N) \) (see Proposition 1.4.1(1) of \( \text{[Bon18b]} \) or Theorem 2.4.2 of \( \text{[Bon10a]} \); note that we have convergence for any \( H \) since \( w_{\text{Chow}}^{\text{eff}} \) is bounded). Now, easy weight arguments described in Remark 2.4.3 of ibid. and (especially) in Remark 3.5.2(3) of \( \text{[BoST14]} \) yield that this spectral sequence degenerates at \( E_2 \) if \( H = H^0_{\text{et}, \mathbb{Q}_{\ell}} \). Moreover, the corresponding filtration on \( H^{p+q}(N_i) \) is the Deligne’s weight one up to a shift (that is described in loc. cit.) if one takes \( N_i = M_{\text{gm}}^R(X)[i] \) for \( i \in \mathbb{Z} \), computes \( H \) via the functor \( H^0_{\text{et}, \mathbb{Q}_{i}, k'} : DM_{\text{eff}}^{gm,R}(k') \to \mathbb{Q}_{i}[G] \) – mod. \( N^i \mapsto H^0_{\text{et}}(N^i \otimes_{\text{Spec } k'} \text{Spec } k) \), where \( G = \text{Gal}(k') \), and uses the isomorphism \( H^0_{\text{et}, \mathbb{Q}_{i}, k'}(M_{\text{gm}, k'}(X_{k'})[i]) \cong H^i(X_{\text{et}}, \mathbb{Q}_{i}) \); cf. Proposition 4.1.6(1) of ibid. Combining these observations we obtain the result easily.

4. Obviously, \( F^0 = 0 \). Next, the vanishing of \( F^1 \) can be easily checked using Poincare duality; cf. the proof of assertion 1.

Hence applying Proposition 2.3.3(9) we obtain that the functors \( G^0, G^1, \) \( F^2 \), and \( F^3 \) kill \( \text{Chow}^{\text{eff}}_1 \).
Lastly, Theorem 2.2.3[6] implies that $H_A$ is birational for any $A$ as above. Thus applying Theorem 1.3.4(2) we obtain that $A_{K_0}([M^R_{gm}(X)]) = [H_A(M^R_{gm}(X))] = [A(M)]$ indeed.

3 First we study the cohomology of $N' = M^R_{gm}(Y)$ (for $Y = k^d \setminus Z$).

It is easily seen that we can assume the field $k$ to be of finite transcendence degree over its prime subfield when computing $H^i_{G^q \otimes Q}(N')$; cf. the proof of [Bon15, Theorem 2.5.4(II.1)] for a closely related argument. Thus one can use the Gysin exact sequence for étale cohomology to check that if $n < 2 \text{codim } Z$ then $Gr^3_{nr}(H^i_{nr}(Y, \mathbb{Q}_l)) = \{0\}$ unless $(n, j) = (0, 0)$. Hence applying assertion 3 we obtain the corresponding vanishing of $H^i_{G^q \otimes Q}(N')$.

Next recall that the morphism $\deg f \cdot \text{id}_N$ factors through $N'$ whenever $f$ is a finite flat morphism $Y \to X$; see Proposition 2.1.1[10]. Thus if $n < 2 \text{codim } Z$ then $H^i_{G^q \otimes Q}(N) = \{0\}$ unless $(n, i) = (0, 0)$ indeed.

Now let $X$ be an arbitrary (smooth) unirational $k$-variety.

Corollary 2.2.4(2) implies that the functor $H^i_{nr} \otimes \mathbb{Z}(l)$ is birational, and the functors $H^i_{G^q \otimes Q}$ and $H^i_{F^3}$ also are according to the previous assertion. Hence it suffices to consider the case where there exists a finite flat morphism $Y \to X$ such that $Y$ is an open (dense) subvariety of $k^d$. As we have just proved, this implies $H^i_{G^q \otimes Q}(N) = \{0\}$ for all $i \in \mathbb{Z}$. Next, $H^i_{nr}(N') \cong H^i_{nr}(M^R_{gm}(k^d)) \cong H^i_{nr}(R) = \{0\}$; thus $H^i_{nr}(N)$ is of finite exponent by the same factorization argument as above.

Recall also that for $h : N \to M$ as in the previous assertion we have $H^3_{F^3}(N) \cong H^3_{F^3}(M)$ and $H^i_{nr}(N) \cong H^i_{nr}(M)$ (since Cone($h$) $\in \text{Obj } DM^\text{eff}_{gm,R}(1)$). Thus $H^i_{nr}(M) \cong DM^\text{eff}_R(M, S^\text{Br})$ is of finite exponent as well. Since $M$ is a retract of $M^R_{gm}(P_M)$ for some smooth projective $P_M$, Proposition 2.1.1[9] implies that $H^i_{nr}(M)$ is also isomorphic to $DM^\text{eff}_R(M, RG_m[2])$ in the notation on this proposition.

Now we note that by passing to the direct limit with respect to $n$ in the long exact sequences given by this proposition one can obtain an exact sequence

$$
\{0\} \to (\mathbb{Q}_l/\mathbb{Z}_l)^c \to DM^\text{eff}_R(M, RG_m[2]) \otimes \mathbb{Z}(l) \to F^3(M) \to \{0\};
$$

(2.3.3)

this is the summand corresponding to $M$ of the exact sequence (4.3) of [CT19] for the variety $P_M$. Since the group $DM^\text{eff}_R(M, RG_m[2])$ is of finite exponent, $c_M = 0$, and we obtain $DM^\text{eff}_R(M, RG_m[2]) \otimes \mathbb{Z}(l) \cong F^3(M)$.

Thus $H^3_{F^3}(N) \cong DM^\text{eff}_R(M, RG_m[2]) \otimes \mathbb{Z}(l) \cong H^i_{nr}(N) \otimes \mathbb{Z}(l)$ indeed.  

\[\Box\]

\[11\] Alternatively, one may obtain this fact (along with the similar vanishing of $H^i_{G^q}(N')$) more directly by using the Gysin distinguished triangle for motives along with the fact that effective Chow motives have no étale cohomology of negative degrees.
Remark 2.3.5. 1. Putting assertions 1 and 5 together we obtain that for any unirational variety $X$ we have $E^2_{K_0}([M_{gm}^c(X)]) = [H^{Br}_{nr}(M_{gm}^R(X))] = \mathbb{Z}$. Moreover, for all $n > 0$ the classes $E^n_{K_0}([M_{gm}^c(X)])$ are torsion, that is, they belong to the subgroup of $K^{add}_0(\mathbb{Z}_\ell \mod)$ generated by $[\mathbb{Z}_\ell/\ell^i\mathbb{Z}_\ell]$ for all $i > 0$; see Proposition 2.3.3(8).

2. Furthermore, our arguments demonstrate that $E^2_{K_0}([M_{gm}^c(X)])$ is closely related to $H^{Br}_{nr}(M_{gm}^R(X)) \otimes \mathbb{Z}_\ell$ for a general smooth $X$ as well; see (2.3.3). Moreover, it can make sense to consider the class of $H^{Br}_{nr}(M_{gm}^R(X)) \otimes \mathbb{Z}_\ell$ in $K^{add}_0(A)$, where $A$ is the category of $\mathbb{Z}_\ell$-modules of cofinite type, that is, of modules of the form $T \bigoplus (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^c$, where $c \geq 0$ and $T$ is finite.

3. Now recall that in [Eke09a] and [Eke09b] certain Grothendieck group of stacks was studied by means of Euler characteristic homomorphisms. Moreover, if $G$ is a finite group then an image is the class $[BG]$ with respect to a homomorphism of this sort coincides with that of $[X]$, where $X = Y/G$, and $Y = \mathbb{A}^d \setminus Z$ is an open subvariety such that $Z$ is of codimension large enough, $G$ acts linearly on $\mathbb{A}^d$, and this action restricts to a free action on $Y$. As demonstrated in Proposition 4.2 of [Sca20] (cf. Remark 2.3.2), this Euler characteristic of $BG$ can be computed as $H_{K_0}(M_{gm}^c(X))$, where $H$ is a certain functor from Chow$_{eff}$ to $A^d \setminus Z$ is an open subvariety such that $Z$ is of codimension large enough, $G$ acts linearly on $A^d$, and this action restricts to a free action on $Y$.

Thus our calculations above (see also part 1 of this remark) enable one to extend most of [Eke09b] Theorem 5.1 to fields of arbitrary characteristics. Note here that if $X = Y/G$ as above, and $G$ is a (finite) group of order prime to $\text{char} k = p$, the unramified Brauer group of $X$ has a rather easy description solely in terms of $G$; see Theorem 12 of [Sal90] that generalizes the corresponding Bogomolov’s calculation (made in the case $k = \mathbb{C}$). Respectively, it is known in several cases that $E^2_{K_0}([M_{gm}^c(X)]) \neq 0$ for $X$ of this sort.

4. In contrast to Theorem 5.1 of [Eke09b], we cannot say much on the class $E^2_{K_0}([M_{gm}^c(X)]) = F^2_{K_0}([M_{gm}^R(X)])$ (for $X = Y/G$ as above). This appears to be related to the fact that $H^2_{et}(X', \mathbb{Z}_\ell)$ does not have to be torsion-free for a smooth projective unirational $X'$ over an algebraically closed field of characteristic $p \neq 0, \ell$. 25
5. The author wonders whether it suffices to assume that $X$ is rationally (chain) connected in part 5 of our theorem. Note that if $p = 0$ then $X$ possesses a smooth compactification $X'$ (by Hironaka’s resolution of singularities); then $X'$ is rationally (chain) connected as well. Next, and one can take $M = M^{\text{gm}}_X(X')$ in our argument (see Remark 2.2.2), and $H^{nM}_c(M)$ is torsion (see Lemma 2.6 of [Kah17]). However, the author does not know how to extend this observation to the case $p > 0$.

6. In the case $p = 0$ and $X \in \text{SmVar}$ the classes $[M^{\text{gm}}_c(X)] \in K_0^{\text{add}}(\text{Chow}^\text{eff}_Z)$ and $F_{K_0}([M^{\text{gm}}_c(X)])$ for certain additive functors from $\text{Chow}^\text{eff}_Z$ were essentially studied in §§3.2–3.3 of [GiS96]; see Proposition 6.6.2 of [Bon09] for a justification of this claim.

3 Supplements

In §3.1 we prove Proposition 2.1.1[9] and discuss some ideas for extending our results to the case where $p$ is (positive and) not invertible in $R$. For this purpose we also give some (more) properties of weight-exact localizations.

In §3.2 we describe a certain generalization of the results of §1.3 to the case where no $\infty$-model is available for $C$ (and $w$ is not necessarily bounded). In particular, we calculate the Grothendieck group of an (arbitrary) bounded weighted category; this statement generalizes Theorem 5.3.1 of [Bon10a].

3.1 More on the Brauer group sheaf and motives

We start from recalling some basics on $t$-structures. Here we use the so-called homological convention for them; note that it is "opposite" to the cohomological convention that was applied in [BBD82] and in several previous papers of the author.

**Definition 3.1.1.** 1. A couple of subclasses $C_{t \leq 0}, C_{t \geq 0} \subset \text{Obj } C$ will be said to be a $t$-structure $t$ on $C$ if they satisfy the following conditions:

(i) $C_{t \leq 0}$ and $C_{t \geq 0}$ are strict, i.e., contain all objects of $C$ isomorphic to their elements.

(ii) $C_{t \leq 0} \subset C_{t \leq 0}[1]$ and $C_{t \geq 0}[1] \subset C_{t \geq 0}.

(iii) $C_{t \geq 0}[1] \perp C_{t \leq 0}.$

(iv) For any $M \in \text{Obj } C$ there exists a $t$-decomposition distinguished triangle

$$L_tM \to M \to R_tM \to L_tM[1] \quad (3.1.1)$$

such that $L_tM \in C_{t \geq 0}$ and $R_tM \in C_{t \leq 0}[-1].$
2. \( Ht \) is the full subcategory of \( C \) whose object class is \( C_{t=0} = C_{t<0} \cap C_{t>0} \).

3. We will say that a class \( P \subset \text{Obj} C \) generates \( t \) if \( C_{t \leq 0} = (\bigcup_{i>0} P[i])^\perp \).

Let us recall some properties of \( t \)-structures.

**Proposition 3.1.2.** Let \( t \) be a \( t \)-structure on \( C \). Then the following statements are valid.

1. The triangle (3.1.1) is canonically and functorially determined by \( M \). Moreover, the functor \( L_t \) is right adjoint to the embedding \( C_{t \geq 0} \rightarrow C \) (if we consider \( C_{t \geq 0} \) as a full subcategory of \( C \)), and the composition \( t_{\geq 0} = [1] \circ R_t \circ [-1] \) is left adjoint to the embedding \( C_{t \leq 0} \rightarrow C \).

2. \( Ht \) is an abelian category with short exact sequences corresponding to distinguished triangles in \( C \).

3. For any \( n \in \mathbb{Z} \) we will use the notation \( t_{\geq n} \) for the functor \([n] \circ L_t \circ [-n] \), and \( t_{\leq n} = [n + 1] \circ R_t \circ [-1 - n] \).

Then there is a canonical isomorphism of functors \( t_{\leq 0} \circ t_{\geq 0} \cong t_{\geq 0} \circ t_{\leq 0} \) (if we consider these functors as endofunctors of \( C \)), and the composite functor \( H^t = H^t_0 \) actually takes values in the subcategory \( Ht \) of \( C \).

Furthermore, this functor \( H^t : C \rightarrow Ht \) is homological.

4. \( C_{t \leq 0} = C_{t \geq 1}^\perp \) and \( C_{t > 0} = (C_{t \leq -1})^\perp \); hence these classes are retraction-closed and extension-closed in \( C \).

Moreover, if \( t \) is generated by \( P \) then \( P \subset C_{t \geq 0} \).

**Proof.** All of these statements were essentially established in §1.3 of [BBD82].
corresponding \( t^R_{\text{hom}} \)-truncation of \( RG'_m \) we obtain for any \( X \in \text{SmVar} \) an exact sequence as follows:

\[
H^2_{\text{Nis}}(X, G_m) \to DM^\text{eff}_R(M^\text{gm}_m(X), RG'_m[2])(= \mathcal{B}r(X)) \to \text{For}(S^{\mathcal{B}r})(X) \to H^3_{\text{Nis}}(X, G_m).
\]

Since the groups \( H^2_{\text{Nis}}(X, G_m) \) and \( H^3_{\text{Nis}}(X, G_m) \) are well-known to be zero (see (6.2.2) and (6.4.2) of [Be V08]), we obtain that \( \text{For}(S^{\mathcal{B}r}) = \mathcal{B}r \) indeed.

Remark 3.1.3. 1. Let us now describe a possible approach for studying unramified cohomology in the case where \( 1/p \notin R \) (recall that this also means \( p > 0 \)); certainly, one may just take \( R = \mathbb{Z} \). Firstly, we note that the arguments relying on [KaS17] extend to this setting without any problem.

2. Recall that \( DM^\text{eff}_R \supset DM^\text{eff}_{\text{gm}, R} \) is a symmetric monoidal category as well. In Theorem 3.1.1(1) of [BoK18] it was proved that \( w^\text{eff}_{\text{Chow}} \) restricts to the subcategory \( DM^\text{eff}_R(1) \) of \( DM^\text{eff}_R \). In Proposition 3.2.1 of ibid. it was deduced that there exists a weight-exact localization functor \( DM^\text{eff}_R \to DM^\text{eff}_R = DM^\text{eff}_R/DM^\text{eff}_R(1) \). Moreover, the corresponding weight structure \( w^\rho \) is an extension of \( w^\rho_{\text{gm}} \); see Remark 3.2.2(1) of ibid.

For this reason we formulate some more properties of weight-exact localizations in Proposition 3.1.4 below; note also that part 1 of that proposition can be vastly generalized (see §3 of [BoS19]).

3. Another relevant observation is that Theorem 3.2.3(1) of [Bon19b] easily yields the existence a \( t \)-structure \( t_{\text{Chow}} \) on \( DM^\text{eff}_R \) that is right adjacent to \( w^\text{eff}_{\text{Chow}} \) (i.e., \( DM^\text{eff}_R t_{\text{Chow}} \geq 0 = DM^\text{eff}_R w^\text{eff}_{\text{Chow}} \geq 0 \)); see Remark 3.2.4(1) of ibid.

The author is going to study the relation of \( t \)-structures obtained this way to unramified cohomology in detail in a forthcoming paper.

**Proposition 3.1.4.** Assume that \( C \) is endowed with a weight structure \( w \) that restricts to a triangulated subcategory \( D \subset C \), and \( \pi \) is the localization functor \( \pi : C \to C' = C/D \).

1. Then there exists a weight structure \( w' \) on \( C' \) such that the localization functor \( \pi : C \to C' \) is weight-exact with respect to \( (w',w') \).

Moreover, \( C'_{w<0} = \text{Kar}_{C'}(\pi(C_{w<0})) \), \( C'_{w>0} = \text{Kar}_{C'}(\pi(C_{w>0})) \), and \( C'_{w=0} = \text{Kar}_{C'}(\pi(C_{w=0})) \).
Furthermore, the obvious functor $\frac{Hw}{Hw_D}$ → $Hw'$ induced by $\pi$ is a full embedding; here $\frac{Hw}{Hw_D}$ is the heart of the restriction of $w$ to $D$, whereas $\frac{Hw}{Hw_D}$ is the category whose object class equals $\text{Obj } Hw$ and $\frac{Hw}{Hw_D}(X,Y) = Hw(X,Y)/(\sum_{Z\in D, p=0} Hw(Z,Y) \circ Hw(X,Z))$ for any $X, Y \in \text{Obj } Hw$.

2. Assume that $N \oplus \pi(M') \cong \pi(M)$ then we can lift the corresponding monomorphism $i : \pi(M') \rightarrow \pi(M)$ and the projection $p' : \pi(M) \rightarrow \pi(M')$ to $C$-morphisms $M' \overset{i}{\rightarrow} M \overset{p'}{\rightarrow} M'$; see the previous assertion. Moreover, since the restriction of $\pi$ to $Hw$ is the composition of the "factorization" of $Hw$ by $Hw_D$ with a full embedding, the morphism $\text{id}_{M'} - p \circ i$ factors through an object $D$ of $Hw_D$. Hence $M'$ is a retract of $M \oplus D$, and applying Proposition 1.2.4(5) we obtain the following: there exists $M'' \in C_{w=0}$ such that $M' \oplus M'' \cong M \oplus D$ and the composition of the corresponding morphisms $M \rightarrow M' \rightarrow M$ equals $i \circ p$. Thus we can apply $\pi$ to the corresponding direct sum diagram to obtain that $\pi(M') \oplus \pi(M'') \cong \pi(M)$ and the composition of the corresponding morphisms $\pi(M) \rightarrow \pi(M') \rightarrow \pi(M)$ equals $\text{id}_M - i \circ p'$. Thus $\pi(M') \cong N$ indeed.

Now recall that $Hw'$ is weakly idempotent complete by Proposition 1.2.4(5). Hence assertion 1 implies that for any $N_0 \in C'_{w'=0}$ there exist $N_0' \in C'_{w'=0}$ and $M_0 \in C_{w=0}$ such that $\pi(M_0) \cong N_0 \bigoplus N_0'$. Thus it remains to verify that if $Hw'$ and $\pi(C_{w=0})$ are closed with respect to countable $Hw'$-coproducts then we can take $N_0'$ to belong to $\pi(C_{w'=0})$. For this purpose we apply the following well-known Eilenberg swindle argument: we have $\bigsqcup \pi(M_0) \cong \bigsqcup N_0 \bigoplus \bigsqcup N'_0 \cong N_0 \bigoplus \bigsqcup N_0 \bigoplus \bigsqcup N'_0 \cong N_0 \bigoplus \bigsqcup \pi(M_0)$; see the main result (Proposition) of [Pre66] for more detail.

Remark 3.1.5. 1. The author is deeply grateful to Vladimir Sosnilo for telling him the main point of Proposition 3.1.4(2).
2. Possibly, the argument above can be used for certain explicit calculations in the case where a decomposition of the sort \( N \oplus \pi(M') \cong \pi(M) \) is known.

3. It could make sense to apply the "weight lifting" statements of \([\text{BoS18c}]\) to motives. In particular, note that Theorem 3.3.1 of ibid. easily implies that in the setting of Proposition 1.2.5 the class \( C'_{w'\geq 0} \) (resp. \( C'_{w'\leq 0} \)) essentially equals \( \pi(C_{w\geq 0}) \) (resp. \( \pi(C_{w\leq 0}) \)); here one should also invoke Remark 3.3.2 of ibid. and categorical duality.

### 3.2 On weak weight complexes and their relation to Grothendieck groups

Now let us discuss a certain generalization of Theorem 1.3.4. To drop the assumption that \( C \) possesses an \( \infty \)-enhancement (and \( w \) is bounded) we recall some of the theory of so-called weak weight complex functors.

**Definition 3.2.1.** Let \( m_1, m_2 : A \to B \) be morphisms of \( B \)-complexes, where \( B \) is an additive category. Then we will write \( m_1 \sim m_2 \) if \( m_1 - m_2 = d_B h + j d_A \) for some collections of arrows \( j^* : A^* \to B^{*-1} \).

We will call this relation the **weak homotopy one**.

To omit other technical definitions needed for the theory of weight complexes (cf. §1.3 of \([\text{Bon18b}]\)), we will formulate the main properties of weight complex functors somewhat axiomatically.

**Proposition 3.2.2.** I. Let \( B \) be an additive category.

1. Then factoring morphisms in \( K(B) \) by the weak homotopy relation yields an additive category \( K_w(B) \). Moreover, the corresponding full functor \( K(B) \to K_w(B) \) is (additive and) conservative.

2. Let \( A : B \to A \) be an additive functor, where \( A \) is an abelian category. Then for any \( B, B' \in \text{Obj}(K(B)) \) any pair of weakly homotopic morphisms \( m_1, m_2 \in C(Hw)(B, B') \) induce equal morphisms of the homology \( H_*(A(B^i)) \to H_*(A(B'^i)) \).

   Hence the correspondence \( N \mapsto H_0(A(N^i)) \) gives a well-defined functor \( K_w(B) \to A \).

3. Applying an additive functor \( F : B \to B' \) to complexes termwise one obtains an additive functor \( K_w(F) : K_w(B) \to K_w(B') \).

II. Assume that \( C \) is endowed with a weight structure \( w \).

Then there exists an additive functor \( t : C \to K_w(Hw) \) that enjoys the following properties.
1. The composition of the embedding $Hw \to C$ with $t$ is isomorphic to the obvious embedding $Hw \to K_m(Hw)$.

2. Let $n \in \mathbb{Z}$. Then $t \circ [n] \cong [n]K_m(Hw) \circ t$, where $[n]K_m(Hw)$ is the obvious shift by $[n]$ (invertible) endofunctor of the category $K_m(Hw)$.

3. If $M \in C_{w \le n}$ (resp. $M \in C_{w \ge n}$) then $t(M)$ belongs to $K(Hw)_{w \le n}$ (resp. to $K(Hw)_{w \ge n}$).

4. Let $M, M' \in \text{Obj } C$, $g : C \to C'$ (where $C$ is endowed with a weight structure $w$), and $h : M' \to \text{Cone}(g)$. Then there exists a lift of the $K(Hw)$-morphism chain $t(M) \xrightarrow{t(g)} t(M') \xrightarrow{t(h)} t(\text{Cone}(g))$ to two sides of a distinguished triangle in $K(Hw)$.

5. Let $C'$ be a triangulated category endowed with a weight structure $w'$; let $F : C \to C'$ be a weight-exact functor. Then the composition $t' \circ F$ is isomorphic to $K_m(HF) \circ t$, where $t'$ is a weight complex functor corresponding to $w'$ and $HF : Hw \to Hw'$ if the restriction of $F$ to hearts (see also assertion I.3 for the definition of $K(Hw)$).

6. Let $A : Hw \to A$ be an additive functor, where $A$ is an abelian category. Then the composition $H^A = H_0 \circ K_m(A) \circ t$ (that sends an object $M$ of $C$ into the zeroth homology of the complex $A(M')$, where $(M') = t(M)$) is a homological functor. Moreover, if $w$ is bounded then this is the only homological functor (up to a unique isomorphism) whose restriction to $Hw$ equals $A$ and whose restrictions to $Hw[i]$ for $i \neq 0$ vanish.

Proof. I. These statements easily follow from the results of [Bon10a, §3.1]; see Lemma 1.3.2 of [Bon19a].

II. All these statements are stated in [Bon18b]; see Proposition 1.3.4(6,7,8,12) and Theorem 2.1.2 of ibid. \qed

Remark 3.2.3. 1. $t$ can be "enhanced" to an exact functor $t^* : C \to K(Hw)$ at least in the case where $C$ possesses an $\infty$-category model and either $w$ is bounded (see Proposition 3.1.1 for this case) or $w$ is purely compactly generated in the sense of [Bon18b, Remark 3.2.3(4)]; see Corollary 3.5 and Remark 3.6 of [Sos19]. Thus the functor $t^*$ exists for all "motivic" weight structures whose hearts consist of certain Chow motives. On the other hand, it is currently not clear whether $t^*$ exists for the weight structure $w^\text{eff}_{\text{Chow}}$ mentioned in Remark 3.1.3 (in the case $1/p \notin R$).
2. In §1.3 of [Bon18b] a (weak) weight complex functor was defined as a canonical additive functor $C_w \to K_m(Hw)$, where $C_w$ is a category canonically equivalent to $C$. Hence to define a functor $t$ as in Proposition 3.2.2 one should choose a splitting $C \to C_w$ for this equivalence. Applying the conservativity of the projection $K(Hw) \to K_m(Hw)$ we obtain that for any object $M$ of $C$ the $K(Hw)$-isomorphism class of $t(M)$ does not depend on any choices.

3. The weak homotopy equivalence relation was introduced in §3.1 of [Bon10a] independently from the earlier and closely related notion of absolute homology, cf. Theorem 2.1 of [Bar05].

Now we are able to generalize Theorem 1.3.4 significantly (recall that the corresponding $K_0$-groups are defined in Definition 1.3.3); certainly, the proofs have much in common. One of the most important cases is $C' = C$ and $B = Hw$; yet cf. Remark 3.2.5(3) below.

**Theorem 3.2.4.** Assume that $C$ is endowed with a weight structure $w$, $C'$ is a triangulated subcategory of $C$, $B$ is an additive subcategory of $Hw$. $N$ is an object of $C'$, $F : B \to A$ is an additive functor (and $A$ is additive), and for any object $M$ of $C'$ the complex $t(M)$ is homotopy equivalent to an object $(M')$ of $K(B)$ such that $F(M') = 0$ for almost all $i \in \mathbb{Z}$.

I. 1. Then the correspondence $M \mapsto \sum_{i \in \mathbb{Z}} (-1)^i[F(M')]$ gives a well-defined homomorphism $F_{K_0} : K_0(C') \to K_0^\text{add}(A)$.

2. Assume that $B = Hw \subset C'$, $A$ is an abelian category, and there exists a $C'$-morphism $h$ either from $N'$ to $N$ or vice versa such that $F_{K_0}([\text{Cone}(h)]) = 0$ and $N' \in C_w=0$. Then $F_{K_0}([N]) = [F(N')]$.

Moreover, if $H^F_j(\text{Cone}(h)) = 0$ for $j = 0, 1$, then $F_{K_0}([N]) = [H^F(N')]$; here the homological functor $H^F$ is the one given by Proposition 3.2.2(ii).[6]

3. Assume that $A$ is abelian semi-simple. Then $F_{K_0}(N) = \sum_{i \in \mathbb{Z}} (-1)^i[H^F_i(N)]$.

II. Assume that $C' = C$ and $B$ strongly generates $C$.

1. Then the corresponding homomorphism $\text{id}_{K_0} : K_0(C') \to K_0^\text{add}(B)$ is an isomorphism.

In particular, this is the case if $B = Hw$.

2. Assume that $B = Hw$, $A$ is abelian, $F = F' \circ H_G \circ i_w$, where $i_w$ is the embedding $Hw \to C$, $H_G : Hw \to Hv$ is the restriction of a weight-exact functor $G : (C, w) \to (D, v)$ to hearts, $F' : Hv \to A$ is an additive functor, and for some $j \in \mathbb{Z}$ the object $G(N)$ belongs to $D_{w=j}$. Then we have $F_{K_0}([N]) = (-1)^j[H^F_j(N)]$.

**Proof.** 1.1. First we check that our correspondence gives a function $\text{Obj} C' \to K_0^\text{add}(A)$ whose values on isomorphic objects are equal. For this purpose it
suffices to verify for $\mathcal{B}$-complexes $(M^i) \cong_{K_w(H_w)} (N^i)$ that \( \sum_{i \in \mathbb{Z}} (-1)^i [F(M^i)] = \sum_{i \in \mathbb{Z}} (-1)^i [F(N^i)] \) if almost all of the summands in both parts are zero. Hence for the (contractible) cone $C = (\mathcal{C}^i)$ of the corresponding $K(\mathcal{B})$-isomorphism (see Remark 1.3.2(2)) we should prove that \( \sum_{i \in \mathbb{Z}} (-1)^i [F(C^i)] = 0 \).

Note now that $F$ canonically extends to a functor $\text{Kar}(F) : \text{Kar}(\mathcal{B}) \to \text{Kar}(\mathcal{A})$. Next, Proposition 10.9 of [Büh10] says that $C$ splits as a $\text{Kar}(\mathcal{B})$-complex, i.e., $C$ is $\text{Kar}(\mathcal{B})$-isomorphic to $\bigoplus \text{id}_{O_1} [-i]$ for certain $O_1^i \in \text{Obj} \text{Kar}(\mathcal{B})$. Our assumptions obviously imply the existence of $N_0 > 0$ such that $\text{Kar}(F)(O_1^i) = 0$ if $i < -N_0$ or $i > N_0$. Hence \[
\bigoplus_{-j \leq i \leq j} F(C^{2i}) \cong \bigoplus_{-j \leq i \leq j} F(C^{2i+1})
\]
for $j \gg 0$, and we obtain the existence and the $\mathcal{C}$-isomorphism invariance of the function in question.

Lastly, Proposition 3.2.2(b) implies that for any $\mathcal{C}'$-morphism $M \to M'$ there exists a $K(h_w)$-distinguished triangle $(M^i) \to (M'^i) \to t(\text{Cone}(f)) \to (M^i)[1]$, where $(M^i)$ and $(M'^i)$ are any $\mathcal{B}$-complexes that are isomorphic to $t(M)$ and $t(M')$, respectively. The existence of (a group homomorphism) $F_{K_0}$ follows immediately.

2. Immediately from the definition of $K_0^\mathcal{B}(\mathcal{C}')$, we have $F_{K_0}([N]) = F_{K_0}([N'])$. Next, ("the correctness part" of) assertion I.1 implies that $F_{K_0}([N]) = [H^F(N')] = F_{K_0}([N'])$.

Lastly, if $H^F(\text{Cone}(h)) = 0 = H^F(\text{Cone}(h)[-1])$ then we have $H^F(N') = H^F(N)$, since the functor $H^F$ is homological.

3. This statement is an immediate consequence of our definitions.

II.1. The embedding $\mathcal{B} \to \mathcal{C}$ obviously gives a homomorphism $K_0^{\text{add}}(\mathcal{B}) \to K_0^{\mathcal{C}}(\mathcal{C})$; it is surjective since $\mathcal{B}$ strongly generates $\mathcal{C}$. Hence $\text{id}_{\mathcal{B} \to K_0}$ is an isomorphism, since its composition with $F_{\text{id}}$ obviously equals $\text{id}_{K_0^{\text{add}}(\mathcal{B})}$.

Lastly, one can apply this statement if $\mathcal{B} = H_w$ according to Proposition 1.2.4(3); note here that $w$ is bounded according to Proposition 1.2.4(7).

2. Proposition 3.2.2(b) easily implies that $F_{K_0}(N) = F_{K_0}(wG(N))$. Hence we can assume that $D = \mathcal{C}'$, $v = w$, $G$ is the identity on $\mathcal{C}'$, $F = F'$, and $N \in \mathcal{C}_{w=j}$. Then $N' = N[-j] \in \mathcal{C}_{w=0}$; hence for $t(N) = (N^i)$ the complex $(F(N^i))$ is $K(A)$-isomorphic to the one-term complex $(F(N')^i)[j]$. The result in question follows immediately; cf. the proof of assertion I.2.

Remark 3.2.5. 1. Part II.1 of our theorem generalizes Theorem 5.3.1 of [Bon10a], where it was assumed that $\mathcal{B}$ is idempotent complete and equals $H_w$. 33
Moreover, our theorem clearly generalizes Theorem 1 of [Ros11] where \( \mathcal{C} = K^b(\mathcal{B}) \) was considered. On the other hand, our arguments essentially "involve" Proposition 1 of ibid., and the formulation was partially inspired by ibid.

2. Obviously, the homomorphism \( F_{K_0} \) naturally extends to the triangulated subcategory of \( \mathcal{C} \) strongly generated by \( \text{Obj} \mathcal{C}' \cup \text{Obj} \mathcal{B} \).

Next, if \( \mathcal{B} \subset \mathcal{C}' \) then for any object \( M \) of \( \mathcal{C}' \) that belongs to its subcategory strongly generated by \( \mathcal{B} \) and any homomorphism \( h \) from \( K^\text{tr}_0(\mathcal{C}') \) into an abelian group the element \( h(M) \) is clearly canonically determined by the values of \( h \) at the set \( \{[M], M \in \text{Obj} \mathcal{B}\} \subset K_0(\mathcal{C}') \).

In particular, the isomorphism provided by part II.1 of our theorem is canonical.

3. Our motivation for considering the case \( \mathcal{C}' \neq \mathcal{C} \) comes from [BoK18]; see Remark 3.1.3. In this context one can take \( \mathcal{C} = \text{DM}_{\text{eff}}^R \), but \( \mathcal{C}' \) should be smaller since \( K^\text{tr}_0(\text{DM}_{\text{eff}}^R) \) is easily seen to be zero.

4. Probably, one can apply Theorem 1.3.4(II.2) to obtain a proof of Theorem 2.3.4 that will not depend on Theorem 2.2.1(4). For this purpose one may take \( G \) to be the natural functor \( \text{DM}_{\text{gm},R}^\text{eff} \to \text{DM}_{\text{gm},R}^o \).

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