Conjugate influence of current relaxation and of current-vortex sheet formation on the magnetorotational instability

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Abstract. The conjugate influence of current relaxation and of current-vortex sheet formation on the magnetorotational instability is reviewed as firstly derived in [13]. It is shown that the relative amplification of the magnetic viscosity from marginal stability to the instability determined by the maximum growth rate is around 924% when resistive effects dominate, while the corresponding quantity is around 220% in the ideal limit. This means that the conjugate influence is much more efficient to amplify the magnetic viscosity than just the effect due to the standard magnetic tension. The results presented here may contribute to the understanding of the various processes that play a significant role in the mechanism of anomalous viscosity observed in Keplerian disks. It is argued that the new effect shall be most relevant in thin accretion disks.

1. Introduction

The magnetorotational instability was first dealt with by Chandrasekhar [1, 2] and Velikhov [3] about half a century ago. It did not attract much attention from the community of plasma physics as a whole in the immediately following decades, except from a few interested in its more basic aspects [4, 5, 6]. However, since the work of Balbus and Hawley [7] about twenty years ago, the magnetorotational instability became a popular starting point for several investigations, mainly in astrophysical plasmas. Actually, thin disks of conducting gases are frequently observed to rotate about forming stars or in binary systems, where they are subjected to stellar fields. They may be found at the center of galaxies as well, where they can be very luminous due to the action of galactic fields. For instance, a conducting disk rotating about a compact object (a neutron star or a black hole) may originate a quasar (an intense source of X-ray) [8]. Such systems are known as accretion disks. Early investigations of the magnetorotational instability were pursued on the assumption of a perfectly conducting fluid (neglecting resistivity). The studies evolved rapidly to include finite resistive effects since they can contribute to the effective viscosity of a conducting fluid. Given the difficulty to deal analytically with the resulting dispersion relation, the first approaches to the problem were conducted numerically [9]. Another important motivation to take the influence of a finite resistivity into account is that the magnetorotational instability can be observable in laboratory plasmas (liquid metals) whose unperturbed motion is determined...
by a Couette flow [10]. Actually, there have been some interesting attempts to tackle the problem analytically, although the essence of the physics remains somewhat obscured [11, 12]. This work reviews how the assumption of the formation of a current-vortex sheet modifies the description of the magnetorotational instability in a conducting fluid as firstly derived in [13]. Particularly, since we describe the dynamics of thin accretion disks and the natural length scale of the problem is given by the characteristic wavelength of the perturbative fields along the width of the system (neglecting compressive effects), we include more rich physical processes. Actually, we require the influence of the inertia of charged species to be also as important. In this case, the current density flowing in the plasma relaxes typically in a finite time scale. Such a situation implies a time-dependent correction to Ohm’s law. As a result, we may state that this work presents the conjugate effect of current relaxation and of current-vortex sheet formation on the magnetorotational instability in a conducting fluid.

2. Dispersion relation

Let us start by considering a fluid element describing a circular orbit with angular speed $\Omega$. We allow $\Omega$ to be a function of the radius $R$ of the circumference. The centripetal acceleration of the fluid element is $-R\Omega^2$, the minus sign indicating its direction toward the center of the circle. For instance, if this acceleration is of gravitational origin, due to a particle of mass $M$ located at the center, then it must be identified with $-GM/R^2$, where $G$ is Newton’s constant. We regard now small departures from the circular orbit due to some disturbing process. Let us attach the origin of Cartesian coordinates $(x, y)$ to the unperturbed orbit determined by the radius, say, $R_0$. At the initial instant, the actual orbit is determined by the distance $x = R_0 + x$, from the center of the unperturbed orbit, and the fluid element moves with an angular speed $\Omega(R)$. We assume that $x$ increases in the same direction of the radius of the unperturbed orbit and that $y$ increases in the same direction of the unperturbed angular speed $\Omega(R_0) = \Omega_0$. Needless to say, the conditions $x \ll R_0$ and $y \ll R_0$ must be satisfied. As time evolves, $x$ and $y$ are determined from the coupled equations $\ddot{x} - 2\Omega_0\dot{y} + R\Omega^2 - R\Omega_0^2 = -\omega_0^2 x - \beta \dot{x}$ and $\ddot{y} + 2\Omega_0\dot{x} = -\omega_0^2 y - \beta \dot{y}$, where a dot denotes total time derivative. The terms $-2\Omega_0\dot{y}$ and $2\Omega_0\dot{x}$ are the $x$ and $y$ components, respectively, of the Coriolis acceleration, $R\Omega^2$ is the centrifugal acceleration, and $\omega_0$ and $\beta$ denote time rates. By Taylor expanding $\Omega^2$ about $R_0$, we get $\Omega^2(R_0 + x) \approx \Omega^2(R_0) + x [\Omega^2]'(R_0)$, where the prime denotes a total derivative with respect to $R$. Then, by assuming that $x$ and $y$ depend on time as $\sim e^{\gamma t}$, we obtain the dispersion relation

$$\gamma^4 + \left[ 2 \left( \omega_0^2 + \beta \gamma \right) + \kappa^2 \right] \gamma^2 + \left[ \left( \omega_0^2 + \beta \gamma \right) - 2q\Omega_0^2 \right] \left( \omega_0^2 + \beta \gamma \right) = 0, \quad (1)$$

where we have introduced the usual epicyclic frequency $\kappa$ and shear parameter $q$, which are defined through the formulae $\kappa^2 = 4\Omega_0^2 + R_0 [\Omega^2]'(R_0)$ and $-2q\Omega_0^2 = R_0 [\Omega^2]'(R_0)$.

3. Current relaxation and current-vortex sheet

In the presence of a magnetic field $\vec{B}$, a (electrically neutral) perfectly conducting fluid (neglecting resistivity), moving with velocity $\vec{v}$, shields itself against the Lorentz force per unit charge of carriers, $\vec{v} \times \vec{B}$, by producing the internal electric field $\vec{E} = -\vec{v} \times \vec{B}$. But, if heating processes cannot be neglected, this ideal limit does not provide a satisfactory description of the plasma dynamics. Then, the usual approach to circumvent this issue is to simply add the term $\eta \vec{j}$ to $-\vec{v} \times \vec{B}$ on the right hand side of the equation above, where $\eta$ denotes the resistivity and $\vec{j}$, the current density. However, even this standard procedure fails to provide a satisfactory description of dissipative phenomena occurring at sufficiently fast rates in the plasma. The reason is that it completely ignores inertial effects due to charged species. Actually, when inertial effects are taken into account, the balance of field inside the plasma shall be determined...
appropriately through [14, 15]

\[ \vec{E} - \eta \tau \partial_t \vec{J} = -\vec{v} \times \vec{B} + \eta \vec{J}, \]

where \( \tau \) is a time constant. Note that Eq. (2) contains the two opposite limiting situations mentioned above. On one hand, in the perfect conductor approximation (the resistivity \( \eta \) becomes vanishingly small), the ideal law mentioned above. On the other hand, if \( \vec{B} \) and \( \vec{E} \) are suddenly removed from the presence of the conducting fluid, (of course, provided that \( \eta \) is finite) then \( \vec{J} = J_0 e^{-t/\tau} \). This means that any initial current \( J_0 \) dissipates in a time scale of the order of \( \tau \) inside the plasma. Thus, \( \tau \) must be interpreted properly as the relaxation time of the current density. If \( \tau \) were ignored, then \( J_0 \) would quench instantaneously inside the plasma. Clearly, such a situation could be justifiable only if the other rates that are relevant to the problem were slow enough. Thus, inertial effects due to charged species should be negligible. As it appears, Eq. (2) provides an extension of Ohm’s law for a resistive plasma, \( \vec{E} = -\vec{v} \times \vec{B} + \eta \vec{J} \), when fast enough rates are involved. Additionally, let us assume that the current density \( \vec{J} \) can be expressed in terms of the usual flow vorticity \( \nabla \times \vec{v} \) through

\[ \vec{J} = -\omega \left( \frac{\rho_0}{\mu_0} \right)^{1/2} \nabla \times \vec{v} + \nabla f, \]

where \( \mu_0 \) denotes the vacuum magnetic permeability and \( \rho_0 \), the (constant and uniform) mass density, and \( f \) is a (generally complex) function and \( a > 0 \), a constant. Note that if \( f \) is uniform, then \( \vec{J} \) and \( \nabla \times \vec{v} \) become aligned. In any case, \( f \) is a solution of Laplace’s equation. Notice also that \( a \) measures the strength of \( \vec{B} \) with respect to that of \( \vec{v} \). Finally, we remark that \( \vec{v} \) lags \( \vec{B} \) in time by \( \pi/2 \). An expression such as Eq. (3) is known as a (local) current-vortex sheet.

4. Magnetorotational instability

By assuming that during an infinitesimal interval of time \( \delta t \) the resulting infinitesimal displacement \( (\vec{v}) \delta t \) and magnetic field \( \left( \partial_t \vec{B} \right) \delta t \) both vary harmonically in space as \( \sim e^{ikz} \), the combination of Eqs. (2) and (3) with Maxwell’s equations imply the term between parentheses in Eq. (1) may be expressed as

\[ \omega_0^2 + \beta \gamma = k^2 v_A^2 \left[ \frac{1 + (kv_A T)(a)}{1 + (kv_A T)(kv_A \tau)} \right], \]

where we have introduced the usual Alfvén speed \( v_A = B_0/\sqrt{\mu_0 \rho_0} \) and time \( T = \eta \rho_0 / B_0^2 \) of decay of the flow on the plane \( (x,y) \) perpendicular to the equilibrium magnetic field \( \vec{B}_0 \) [16]. In astrophysics, one is often concerned with a conducting gas, which is supported by a differential rotation of angular speed \( \Omega_0 \) against the gravitational attraction to a central object of mass \( M \). The identification of the centripetal acceleration as being of gravitational origin leads to \( \Omega_0^2 = GM/R_0^3 \), at a distance \( R_0 \) from the object, where \( G \) is Newton’s constant. Such a result is nothing but Kepler’s third law. The system is known as a Keplerian disk. According to Sec. 2, the shear parameter \( q = 3/2 \). Since \( q \) is greater than 1, if the rotating gas is subjected to a perpendicular constant and uniform magnetic field, then Eq. (1) shows that \( \gamma > 0 \) and the disk becomes magnetorotationally unstable, provided that the condition \( \omega_0^2 + \beta \gamma < 3\Omega_0^2 \) is satisfied in accordance with Eq. (4). An useful notion in the study of plasma instabilities is that of marginal stability, the (still stable) state of the fluid which is determined by the minimum value attained by the growth rate (actually, by requiring that \( \gamma = 0 \)). According to Eq. (1), a Keplerian disk is marginally stable if the condition \( \omega_0^2 + \beta \gamma = 3\Omega_0^2 \) is satisfied. In this case, such a condition implies the wavelength \( \lambda = 2\pi/k \) is given by

\[ \lambda_{id} (\gamma = 0) = \frac{2\pi v_A}{\Omega_0 \sqrt{3}}, \quad \lambda_{res} (\gamma = 0) = \frac{2\pi a v_A}{3 \Omega_0^2 \mu_0 \tau}, \]

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where the suffix id (res) stands for ideal (resistive). Needless to say, the opposite limiting situation for which the growth rate acquires its possible maximum value should be of interest as well. In accordance with Eq. (1), such a value is \( \gamma = 3\Omega_0/4 \), which occurs if the condition \( \omega_0^2 + \beta \gamma = 15\Omega_0^2/16 \) is satisfied. In this case, such a condition leads to

\[
\lambda_{id} (\gamma = 3\Omega_0/4) = \frac{8\pi v_A}{\Omega_0 \sqrt{15}}, \quad \lambda_{res} (\gamma = 3\Omega_0/4) = \frac{32\pi v_A}{15\Omega_0^2 \tau} \tag{6}
\]

We arrive now at the main result of this work. As mentioned in [16], if \( L \) is a characteristic length scale of the disk, then \( L^2 B_0^2/\eta \) may be interpreted as the magnetic viscosity, where \( B_0 \) denotes the strength of the magnetic field in the state of equilibrium of the gas. Thus, by assuming that \( L \sim \lambda \), Eqs. (5) and (6) show that the relative amplification of the magnetic viscosity from marginal stability to the instability determined by the maximum growth rate is given by

\[
\frac{\lambda_{id}^2 (\gamma = 3\Omega_0/4) - \lambda_{id}^2 (\gamma = 0)}{\lambda_{id}^2 (\gamma = 0)} = 220\%, \quad \frac{\lambda_{res}^2 (\gamma = 3\Omega_0/4) - \lambda_{res}^2 (\gamma = 0)}{\lambda_{res}^2 (\gamma = 0)} = 924\%. \tag{7}
\]

Quite interestingly, we see that the conjugate influence of current relaxation and of current-vortex sheet formation is much more efficient to amplify the magnetic viscosity than just the effect due to the standard magnetic tension in the ideal limit.

5. Conclusion

The main result of this work is expressed by the contrast of Eqs. (7). The former provides the relative amplification of the magnetic viscosity from marginal stability to the instability determined by the maximum growth rate in the ideal limit and the latter, the corresponding quantity when resistive effects dominate. We have found that the conjugate influence of current relaxation and of current-vortex sheet formation is much more efficient to amplify the magnetic viscosity than just the effect due to the standard magnetic tension in the ideal limit. The results presented here may contribute to the understanding of the various processes that play a significant role in the mechanism of anomalous viscosity observed in Keplerian disks. Since the wavelength \( \lambda \) is in the direction perpendicular to the plane of rotation of the gas and inertial effects are important at short wavelengths, the new effect shall be appreciable in thin accretion disks.

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