Spectral analysis of the interior transmission eigenvalue problem

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Abstract

In this paper we prove some results on interior transmission eigenvalues. First, under reasonable assumptions, we prove that the spectrum is a discrete countable set and that the generalized eigenfunctions span a dense subspace in the range of resolvent. This is a consequence of the spectral theory of Hilbert–Schmidt operators. The main ingredient is the proof of a smoothing property of the resolvent. This allows us to prove that a power of the resolvent is Hilbert–Schmidt. We obtain an estimate of the number of eigenvalues, counted with multiplicities, with modulus less than $t^2$ when $t$ is large. We also prove some estimate on the resolvent near the real axis when the square of the refraction index is not real. Under some assumptions we obtain a lower bound for the resolvent using results obtained by Dencker, Sjöstrand and Zworski on the pseudospectrum.

1. Introduction

In this paper we prove the existence of an infinite number of interior transmission eigenvalues under some condition on the refraction index. We first recall the setting of the problem. Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^n$. Let $n(x)$ be a smooth function defined in $\Omega$, called the refraction index. The problem is to find $k \in \mathbb{C}$ and a pair of functions $(w, v)$ such that

\[
\begin{align*}
\Delta w + k^2 n(x)w &= 0 \quad \text{in } \Omega, \\
\Delta v + k^2 v &= 0 \quad \text{in } \Omega, \\
w &= v \quad \text{on } \partial \Omega, \\
\partial_n w = \partial_n v &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

where $\partial_n$ is the exterior normal derivative to $\partial \Omega$. We consider here the function $n(x)$ complex valued. In physical models, we have $n(x) = n_1(x) + i n_2(x)/k$, where $n_j$ are real valued. Taking $u = w - v$ and $\tilde{v} = k^2 v$, we obtain the following equivalent system:

\[
\begin{align*}
(\Delta + k^2 (1 + m))u + m v &= 0 \quad \text{in } \Omega, \\
(\Delta + k^2) v &= 0 \quad \text{in } \Omega, \\
u = \partial_n u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

where, for simplicity, we have replaced $\tilde{v}$ by $v$ and $n$ by $1 + m$. 

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The assumptions on \( n(x) \) are rather general; essentially we require that the range of \( n(x) \) is contained in a cone of \( \mathbb{C} \). The assumptions on the size of this cone depend on each theorem and will be made precise in the statements. As the method of proof is based on a reduction on a system of pseudo-differential operators on the boundary, we need that \( n(x) \neq 1 \) for \( x \) on the boundary. Under these assumptions, we prove that the associated resolvent is compact on \( H^2(\Omega) \oplus L^2(\Omega) \) (see theorem 3) and we obtain a countable set of \( k_j^2 \) and generalized finite-dimensional eigenspaces \( E_j \) such that \( \bigcup_{j \in \mathbb{N}} E_j \) span a dense subspace in the range of the resolvent (see theorem 5). When \( n(x) \) is real, Päivärinta and Sylvester [25] proved that there exist interior transmission eigenvalues; Cakoni et al [5] proved that the set of \( k_j^2 \) is infinite and discrete. For \( n(x) \) complex valued, Sylvester [27] proved that this set is discrete finite or infinite. Probably because of the application of pseudo-differential operators we do not face issues in the presence of such cavities. We recall that we say that this set is discrete finite or infinite. Probably because of the application of pseudo-differential operators we do not face issues in the presence of such cavities. We recall that we say that there are cavities if \( n(x) = 1 \) in parts of \( \Omega \). Problems with cavities were considered by Cakoni et al [3, 4].

In [15, 16] Hitrik et al studied the same type of problems where the Laplace operator is replaced by an elliptic operator with constant coefficients of order \( m \geq 2 \). In some cases they proved the existence of interior transmission eigenvalues and that the generalized eigenfunctions span a dense space. Their proof uses the property of trace class operators we do not face issues in the presence of such cavities. We recall that we say that this set is discrete finite or infinite. Probably because of the application of pseudo-differential operators we do not face issues in the presence of such cavities. We recall that we say that there are cavities if \( n(x) = 1 \) in parts of \( \Omega \). Problems with cavities were considered by Cakoni et al [3, 4].

We complete our result by providing a weak version of the Weyl asymptotic law for the \( k_j \). If we denote by \( N(t) \) the number of \( |k_j| \), counted with multiplicities, less than \( t \), we prove that \( N(t) \leq Ct^{m+\frac{1}{2}} \) (see theorem 7). In [19–21], Lakshtanov and Vainberg study a problem as (1) where the boundary condition \( \partial_w w = \partial_v w \) is replaced by \( \partial_w w = a(x)\partial_v v \) where \( a(x) \neq 1 \) for all \( x \in \partial \Omega \). In this case the problem is elliptic in the sense that if \( (f, g) \in L^2(\Omega) \) then \( (w, v) \in H^2(\Omega) \) (see (6) for the definition of the system with force terms). In [22] they prove a lower bound on the counting function \( N(t) \), if \( n(x) \) is real and for problem (1). To be precise, they prove \( N(t) \geq Ct^2 \) where \( C > 0 \). Actually they consider only the real eigenvalues but it is not clear that the bound is sharp even for real eigenvalues.

To prove the results on a non-self-adjoint operator, we follow the method given by Agmon [1]. For the reader’s convenience we give this method in section 4. This section may be read without knowledge of the pseudo-differential operator. We need only a priori estimates on resolvent. In this section we prove an estimate of the Hilbert–Schmidt norm by more tractable norms, a method to obtain the density of the space spanned by the generalized eigenfunctions, the relation between the spectrum of a bounded operator and its power, and an estimate of the trace of an operator allowing the estimate on the counting function.

In [9], Colton and Kress prove that if \( \text{Im} n(x) \geq 0 \) for all \( x \in \Omega \) and \( \text{Im} n \neq 0 \), then \( k^2 \) is not real. Here we give an estimate on the resolvent for \( k^2 \in \mathbb{R} \) (see theorem 8). This result is based on Carleman estimates and is obtained similarly as is done in the context of control theory, stabilization and scattering (see for instance [11, 23, 24]).

In some cases (see theorem 9) we give a lower bound on the resolvent using a result obtained by Dencker et al [13] on the pseudospectrum. Even if the bounds obtained by the Carleman method and by the pseudospectrum results are of the same size, we cannot apply both methods for a given particular situation.

The interest of problem (1) is related to theorem 8.9 in [9], first proved by Colton et al [8]. Here, we provide a quick survey of this result. Let \( n(x) \) be defined in \( \Omega \) as in (1) and equal
to 1 in $\mathbb{R}^n \setminus \Omega$. We assume $n(x) \neq 1$ in $\Omega$. For $k \in \mathbb{R}$, let $u$ be the solution of the following problem:

$$\begin{cases}
\Delta u + k^2 n(x)u = 0 \quad \text{in } \mathbb{R}^n, \\
u = u' + u^\prime, \\
\lim_{r \to +\infty} r^{(n-1)/2} \left( \frac{\partial u'}{\partial r} - ik u' \right) = 0,
\end{cases}$$

where $u'$ is a solution to $\Delta u' + k^2 u' = 0$ in $\mathbb{R}^n$, and $r = |x|$. We have $u'(x) = e^{ik|x|} u_\infty(\hat{x}) + O\left(\frac{1}{|x|^{n-1}}\right)$, where $\hat{x} = x/|x|$.

If $u'(x) = e^{ik|x|}$, where $d \in \mathbb{S}^{n-1}$, then we denote the corresponding $u_\infty$ by $u_\infty(\hat{x}, d)$. If we assume that $\Omega$ is connected and contains 0, the space spanned by $u_\infty(\cdot, d)$ for $d \in \mathbb{S}^{n-1}$ is dense in $L^2(\mathbb{S}^{n-1})$ if and only if the space of solutions of (1) is reduced to $\{0\}$. This problem can be interpreted as follows: $u'$ is an incident plane wave and $u_\infty$ is the first relevant term of $u$ created by the perturbation $n$ localized in $\Omega$. This kind of result is interesting in the field of inverse scattering problems. Readers interested in this field can also find more material in [9] and in the recent survey by Cakoni and Haddar [6].

1.1. Results

Let $\Omega$ be a $\mathcal{C}^\infty$-bounded domain in $\mathbb{R}^n$. Let $n(x) \in \mathcal{C}^\infty(\overline{\Omega})$ be complex valued. We set $m(x) = n(x) - 1$. We consider also the case where $n(x) = n_1(x) + in_2(x) / k$ where $n_1$ are real valued and $k$ is the spectral parameter. This case is different from the previous one but can be treated similarly. We assume that for all $x \in \overline{\Omega}$, $n(x) \neq 0$ or $n_1(x) \neq 0$ or equivalently $m(x) \neq -1$. We assume that there exists a neighborhood $W$ of $\partial \Omega$ such that $x \in \overline{W}$, $n(x) \neq 1$ or $n_1(x) \neq 1$ or equivalently $m(x) \neq 0$. Actually if $n(x) \neq 1$ for all $x \in \partial \Omega$, such a neighborhood $W$ exists.

Let $C_\nu$ be the cone in $\mathbb{C}$ defined by

$$C_\nu = \{ z \in \mathbb{C}, \exists x \in \overline{\Omega}, \exists \lambda \geq 0, \text{ such that } z = -\lambda (1 + \overline{n}(x)) \}. \quad (3)$$

In the case where $n = n_1 + in_2 / k$, $C_\nu = (-\infty, 0)$ if $n_1(x) > -1$ for all $x \in \overline{\Omega}$, and $C_\nu = [0, +\infty)$ if $n_1(x) < -1$ for all $x \in \overline{\Omega}$. We try the same $C_\nu$ if we take $m(x) = n_1(x) - 1$ in (3).

Our regularity result will be stated in Sobolev spaces. We use the following notation. The $L^2(\Omega)$ norm will be denoted $\| \cdot \|$. For $s \in \mathbb{R}$, we denote the usual $H^s$ norm in $\mathbb{R}^n$ by $\| u \|_{H^s} = \left( 1 + |\xi|^2 \right)^s \hat{u}(\xi)^2 d\xi$ where $\hat{u}$ is the classical Fourier transform. The $H^s$ space on $\Omega$ will be denoted $\mathcal{H}^s(\Omega)$ and we say that $u$ is a distribution in $\Omega$ if $\mathcal{H}^s(\Omega)$ if there exists $\beta \in H^s$ such that $\beta|_{\partial \Omega} = u$. The norm is given by $\| u \|_{\mathcal{H}^s(\Omega)} = \inf\| \beta \|_{H^s}$, where $\beta|_{\partial \Omega} = u$. For $q \in \mathbb{N}$ it is classical that $\| u \|_{\mathcal{H}^q(\Omega)}$ is equivalent to $\sum_{|\alpha| \leq q} \| \partial^\alpha u \|^2$ (see for instance [18, vol 3, corollary B.2.5]). The space $H^s_0(\Omega)$ is the adherence of $\mathcal{C}_0^\infty(\Omega)$ for the $\mathcal{H}^s(\Omega)$ norm or equivalently $w \in H^s_0(\Omega)$ if and only if $w \in \mathcal{H}^s(\Omega)$ and $u|_{\partial \Omega} = \partial_n u|_{\partial \Omega} = 0$.

Let $z \in \mathbb{C}$. We denote by $B_z(u, v) = (f, g)$ the mapping defined from $H^2_0(\Omega) \oplus \{ v \in L^2(\Omega) \}$ to $L^2(\Omega) \oplus L^2(\Omega)$ by

$$\begin{cases}
\left( \frac{1}{1+m} \Delta - z \right) u + \frac{m}{1+m} v = f \quad \text{in } \Omega \\
(\Delta - z) v = g \quad \text{in } \Omega.
\end{cases} \quad (4)$$
Theorem 4. Assume $C_e \neq \mathbb{C}$, then there exists $z \in \mathbb{C}$ such that $B_z$ is bijective from $H^2_0(\Omega) \oplus \{v \in L^2(\Omega), \; \Delta v \in L^2(\Omega)\}$ to $L^2(\Omega) \oplus L^2(\Omega)$.

In the case $n(x) = n_1(x) + n_2(x)/k$, here $C_e \neq \mathbb{C}$ and we have the same result.

Theorem 5. Assume $C_e \neq \mathbb{C}$ and there exists $z \in \mathbb{C}$ such that the resolvent $R_z$ from $H^2_0(\Omega) \oplus L^2(\Omega)$ to itself is compact.

In particular, applying the Riesz theory, the spectrum is finite or is a discrete countable set. If $\lambda \neq 0$ is in the spectrum, $\lambda$ is an eigenvalue associated with a finite-dimensional generalized eigenspace.

Theorem 6. There exists $k \in \mathbb{C}$ such that $\hat{B}_k$ is bijective from $H^2_0(\Omega) \oplus \{v \in L^2(\Omega), \; \Delta v \in L^2(\Omega)\}$ to $L^2(\Omega) \oplus L^2(\Omega)$.

Remark 1. These results improve Sylvester’s theorem [27, theorem 2] with respect to the geometrical assumption on $m$. Nevertheless, here the regularity assumption made on $m$ is stronger than the assumption made in [27].

Remark 2. Actually if $z_0 \not\in C_e \cup (-\infty, 0]$ for all $\lambda > 0$ large enough, we can take $z = \lambda z_0$ in theorems 1 and 3.

If $k_0^2 \not\in C_e \cup (-\infty, 0]$ for all $\lambda > 0$ large enough, we can take $k = \lambda k_0$ in theorems 2 and 4. Here we estimate the resolvent in the exterior of a conic neighborhood of $C_e \cup (-\infty, 0]$. In particular, the eigenvalues $k^2$, except a finite number, are in all small conic neighborhoods of $(0, +\infty)$. In [17], Hitrik et al prove that the eigenvalues are in a parabolic neighborhood of $(0, +\infty)$.

Remark 3. Actually we can also consider $R_z$ to $L^2(\Omega) \oplus L^2(\Omega)$ into itself. The range is in $H^2_0(\Omega) \oplus L^2(\Omega)$. Then with our regularity results we can prove that $R_z^2$ is a mapping from $L^2(\Omega) \oplus L^2(\Omega)$ to $H^4(\Omega) \oplus H^4(\Omega)$. In particular, $R_z^2$ is compact from $L^2(\Omega) \oplus L^2(\Omega)$ to itself and we can deduce the same properties on the spectrum of $R_z$ as in theorem 3. The same result is true for $\hat{R}_k$. 

In the case where $n = n_1 + in_2/k$ we must change the definition of $B_z$. We define $m_1(x) = n_1(x) - 1$ and $m_2(x) = n_2(x)$. The mapping $\hat{B}_k(u, v) = (f, g)$ is given by

$$
\begin{align*}
\left\{ \begin{array}{l}
\left(\frac{1}{1 + m_1} - k^2 - k \frac{m_2}{1 + m_1}\right) u + \left(\frac{m_1}{1 + m_1} + \frac{m_2}{k(1 + m_1)}\right) v = f \quad \text{in } \Omega \\
(\Delta - k^2)v = g \quad \text{in } \Omega.
\end{array} \right.
\end{align*}
$$

Here we have change the $k$ of (2) in $ik$ such that the principal symbol of $\hat{B}_k$ is essentially the same than the one of $B_z$.
In general, for a non-self-adjoint problem, we cannot claim that the spectrum is non-empty. In the following theorem, with a stronger assumption on \( C_e \), we can prove that the spectrum is non-empty.

We say that \( C_e \) is contained in a sector with angle less than \( \theta \) if there exist \( \theta_1 < \theta_2 \), such that \( C_e \subset \{ z \in \mathbb{C}, \ z = 0 \text{ or } e^{i\theta} \} \), where \( \theta_1 \leq \varphi \leq \theta_2 \), and \( \theta_2 - \theta_1 \leq \theta \).

**Theorem 5.** Assume that \( C_e \) is contained in a sector with angle less than \( \theta \) with \( \theta < 2\pi/p \) and \( \theta < \pi/2 \) where \( 4p > n \). Then there exists \( z \) such that the spectrum of \( R_z \) is infinite and the space spanned by the generalized eigenspaces is dense in \( H_0^2(\Omega) \oplus \{ v \in L^2(\Omega), \ \Delta v \in L^2(\Omega) \} \).

**Theorem 6.** There exists \( k \) such that the spectrum of \( \hat{R}_k \) is infinite and the space spanned by the generalized eigenspaces is dense in \( H_0^2(\Omega) \oplus \{ v \in L^2(\Omega), \ \Delta v \in L^2(\Omega) \} \).

**Remark 4.** These results are based on the theory given in Agmon [1] and using the spectral results on Hilbert–Schmidt operators. In this theory we deduce that the spectrum is infinite from the proof that the generalized eigenspaces form a dense subspace in the closure of the range of \( R_z \) [resp. \( \hat{R}_k \)]. Here we prove that \( R_k \) [resp. \( \hat{R}_k \)] is a Hilbert–Schmidt operator if \( 4p > n \). We can deduce the spectral decomposition of \( R_z \) [resp. \( \hat{R}_k \)] from that of \( R_k \) [resp. \( \hat{R}_k \)].

We can prove a weak Weyl asymptotic law. Let \( z_j \) be the elements of the spectrum of \( R_z \) [resp. \( \hat{R}_k \)] and \( E_j \) the generalized associated eigenspace. We denote \( N(t) = \sum_{|z_j|^{1+1/p} \leq t} \dim E_j \).

**Theorem 7.** Under the same assumption in theorem 5 or 6, there exists \( C > 0 \) such that \( N(t) \leq Ct^{4/n} \).

**Remark 5.** I do not know if this result is optimal. The estimate is weaker than the usual Weyl law provide a bound of order \( t^n \). This is due to the estimates obtained here on the resolvent which are different to that used to prove the usual Weyl law. Motivated by the analytic hypoellipticity (see [7] and [14]) Robert [26] has proved an estimate, by a power of the spectral parameter, on the resolvent (see lemma 2.4) and deduced an estimate on the counting function different from the Weyl law (see proposition 3.4). In this case the estimate is optimal with respect to the power.

When the problem is elliptic, for instance if the boundary condition \( \partial_n w = \phi, \ \partial_w v = a(x) \partial_u \) with \( a \neq 1 \) in (1), Lakshtanov and Vainberg proved an optimal Weyl law. Nevertheless, for the problem studied here they prove in [22] a lower bound on the counting function. In the context treated here the precise asymptotic for the counting function is an open problem.

Here we give some ideas on how to obtain theorems 5, 6 and 7 using the method given in Agmon [1]. First we prove a regularity result, that is, we consider the iterates of \( R_z \), and we find that \( R_k \) is bounded from \( H^2 \oplus L^2 \) to \( H^{2p+2} \oplus H^{2p} \). This implies that \( R_k \) is a Hilbert–Schmidt operator if \( p \) is large enough and we can use the spectral theory for this operator class. The main issue is the proof that the regularity result is at the boundary. To address this issue we reduce the problem to the boundary using pseudo-differential calculus. It is well known that for a second-order elliptic problem, one can find a relation between the two traces of a solution at the boundary. These two relations, for \( u \) and \( \nu \) in (4) and the assumption on the two traces of \( u \) allow one to compute the trace of \( v \) from the data. Actually the coupling is very weak because it involves a lower order term; consequently we obtain a weak estimate.

In the context of stabilization or control for the wave equation, there are a lot of results on the decay of energy obtained by Carleman estimates (see for instance [11, 23, 24]). This method allows us to give quantitative results related to the uniqueness result. We use here the same method to prove an estimate on the resolvent near the real axis for a complex index of
refraction. The theorem below is a quantitative version of theorem 8.12 given by Colton and Kress [9]. Here it is more convenient to use the variables introduced in (1). Let \((w, v)\) be the solution of

\[
\begin{align*}
\Delta w + k^2 n(x)w &= f & \text{in } \Omega \\
\Delta v + k^2 v &= g & \text{in } \Omega \\
w &= v & \text{on } \partial \Omega \\
\partial_n w &= \partial_n v & \text{on } \partial \Omega,
\end{align*}
\]

where \((f, g) \in L^2(\Omega) \oplus L^2(\Omega)\). We denote \(\hat{R}_k; (f, g) = (w, v)\). We remark that \(\hat{R}_k\) exists except for a discrete set of values of \(k^2\). Indeed, we can check that \(R_{-k^2}(f - g, k^2 g) = (w - v, k^2 v)\), which gives the existence of \((w, v)\) if \((-k^2)^{-1}\) is not in the spectrum of \(R_0\).

Using \(\hat{R}_k\) instead of \(R_{-k^2}\) we obtain the same result in the case \(n(x) = n_1(x) + n_2(x)/k\).

**Theorem 8.** We assume that \(\text{Im } n \geq 0\) and \(\text{Im } n \neq 0\) or if \(n(x) = n_1(x) + in_2(x)/k\), \(n_2(x) \geq 0\) and \(n_2 \neq 0\). Then there exist \(C_1 > 0\) and \(C_2 > 0\) such that \(\|\hat{R}_k\| \leq C_1 e^{C_2 |k|}\) for all \(k \in \mathbb{R}\).

Here \(\| \cdot \|\) denotes the norm of the operator from \(L^2(\Omega) \oplus L^2(\Omega)\) into itself.

**Remark 6.** We have the same result if we assume \(\text{Im } n \leq 0\) and \(\text{Im } n \neq 0\) or if \(n(x) = n_1(x) + i n_2(x)/k\), \(n_2(x) \leq 0\) and \(n_2 \neq 0\).

In the context of non-self-adjoint operators, the spectrum is not the most relevant notion. Actually Davies [12] introduced the notion of pseudospectrum. Roughly speaking this set is defined by the points \(z\) where the resolvent is large. This notion is related to the ill-conditioned problems for the matrices. Here we use the result proved by Dencker et al [13] to obtain a lower bound on the norm of the resolvent.

**Theorem 9.** Assume that there exist \(x_0 \in \mathbb{R}^n\) and \(x_0 \in \Omega\) such that \(\text{Im } (\Pi(x_0))(x_0, \partial_n(x_0)) \neq 0\). Then for all \(N > 0\), \(\sup_{r > 0} |k|^N \|\hat{R}_k\|, k^2 = r(n(x_0))^{-1}, r > 0\) is \(+\infty\). Moreover, if \(n\) is an analytic function in a neighborhood of \(x_0\) there exists \(C > 0\) such that \(\sup \{e^{-C|k|} \|\hat{R}_k\|, k^2 = r(n(x_0))^{-1}, r > 0\} = +\infty\).

**Remark 7.** Even if the lower bound, in the analytic case, is of the same type of upper bound obtained in theorem 8, we cannot apply both theorems for the same direction \(z\). Actually \(k^2\) is not real general. If we want to apply theorem 8, we need \(n(x_0) \in \mathbb{R}\), that is, \(\text{Im } n(x_0) = 0\), as \(\text{Im } n(x) \geq 0\). Then we cannot have \(\text{Im } \partial_n n(x_0) \neq 0\) to apply theorem 9. It is may be possible to give a result in the case \(n(x) = n_1(x) + in_2(x)/k\) but it is not a straightforward application of [13].

### 1.2. Outline

In section 2, we prove the main technical results. Roughly speaking if the data \((f, g)\) are more regular in the \(H^r\) norm, we prove that the solution \((u, v)\) is also more regular. More precisely, we prove for \(p \geq 0\) that if \((f, g) \in H^{2p+4}(\Omega) \oplus H^{2p}(\Omega)\), then \(R_p(f, g) = (u, v) \in H^{2p+4}(\Omega) \oplus H^{2p+2}(\Omega)\). This proves first that \(R_p\) from \(H^1(\Omega) \oplus L^2(\Omega)\) to itself is compact and \(R_p\) is an operator from \(H^2(\Omega) \oplus L^2(\Omega)\) to \(H^{2p+2}(\Omega) \oplus H^{2p}(\Omega)\). This implies that \(R_p\) is a Hilbert–Schmidt operator on \(H^2(\Omega) \oplus L^2(\Omega)\) if \(4p > n\). To prove the regularity results, first we prove an estimate on \(u\) in subsection 2.1. It is an easy estimate to obtain as \(u|_{\partial \Omega} = 0\), \(u\) satisfies a classical Dirichlet problem. Second, we prove in subsection 2.2 the regularity of \(v\) in all compact in \(\Omega\). As \(v\) satisfies an elliptic equation, far away from the boundary of \(\Omega\), it is a classical result. In subsection 2.3, we prove the regularity result on \(v\) in a neighborhood of \(\partial \Omega\). The idea to do this is to explain \(v\) by the unknown traces of \(v\). This description allows one to obtain a relation between \(v_{|\partial \Omega}\) and \(\partial_n v_{|\partial \Omega}\). Then we can use this formula on \(v\) in the equation
on \( u \). The fact that \( u|_{\partial \Omega} = 0 \) and \( \partial_{\nu} u|_{\partial \Omega} = 0 \) gives another relation between \( v|_{\partial \Omega} \) and \( \partial_{\nu} v|_{\partial \Omega} \). These relations allow one to determine \( v|_{\partial \Omega} \) and \( \partial_{\nu} v|_{\partial \Omega} \) from \((f, g)\). This explicit formula, in the sense of pseudo-differential calculus, allows one to prove the regularity result. Following the same approach, we also prove an estimate of the \( L^2 \) norm of \( v \) by the \( L^2 \) norm of \( f \). This implies a weak convergence result. Actually the problem is that \( v \) has the same regularity as \( f \). In particular, if we consider the resolvent as an operator from \( L^2(\Omega) \oplus L^2(\Omega) \) to itself, we cannot prove that the resolvent is compact. Here, we avoid this problem by the assumption that \( f \in \mathcal{H}^1(\Omega) \).

In section 4, we recall some result proved in [1] and we apply this to prove theorems 3, 5 and 7.

In section 5, we prove some \( a \) \( p r i o r i \) bound on the resolvent. In subsection 5.1, we prove an upper bound on the resolvent near the real axis when the imaginary part of the refraction index has a sign and does not vanish identically. The main tool is an interpolation estimate that follows from a Carleman estimate. In subsection 5.2, we use the result obtained by Dencker et al [13] on the pseudospectra to obtain a lower bound on the resolvent. Roughly speaking, this result says that when the operator is not elliptic in the semi-classical sense, even if a point is not in the spectrum, the resolvent is generically large.

As we use deeply semi-classical pseudo-differential calculus in section 2, in the appendix we fix the notation used in the rest of paper, we recall the classical results used, we give a proof for the action of pseudo-differential operators on \( \mathcal{H}^1(\Omega) \) spaces and we give some computations on integrals to obtain some explicit formulae used in subsection 2.3. This allows us to give the explicit first term of the resolvent in the sense of semi-classical pseudo-differential calculus.

We need only some minor changes to treat the case \( \tilde{R}_k \) instead of \( R_k \).

2. Regularity results

We describe now the idea of the proof. As we want to prove an estimate when \( |z| \) is large we will compute the resolvent in the semi-classical framework. We multiply equations (4) by \( \hbar^2 \) and denote \( \mu = -\hbar^2 z \), where \( \mu \) belongs to a bounded domain of \( \mathbb{C} \), \( a = 1/(1+m) \) and \( V = m/(1+m) \). We change \((f, g)\) in \((-f, -g)\).

We recall the assumption made on \( m \); we have \( m(x) \neq -1 \) for all \( x \in \Omega \) and \( m(x) \neq 0 \) for \( x \) in a neighborhood of \( \partial \Omega \).

Thus following (4), we obtain the system

\[
\begin{align*}
(-a\hbar^2 \Delta - \mu)u - \hbar^2 V v &= \hbar^2 f & \text{in } \Omega \\
(-\hbar^2 \Delta - \mu)v &= \hbar^2 g & \text{in } \Omega \\
0 &= \partial_{\nu} u & \text{on } \partial \Omega.
\end{align*}
\]  

(7)

The goal of this section is to prove the following estimates if \( s \geq 0 \). The result is given using the semi-classical \( H^s \) norm; see the appendix for the definition of these spaces.

In the case where \( n(x) = n_1(x) + n_2(x)/k \), we multiply equations (5) by \( \hbar^2 \) and denote \( \mu = -\hbar^2 k^2 \), \( a = 1/(1+m_1) \), \( V = m_1/(1+m_1) + hm_2/(v + v_0) \), where \( v = \hbar k \). We change \((f, g)\) in \((-f, -g)\). Thus, following (5) we obtain the system

\[
\begin{align*}
(-a\hbar^2 \Delta + h W_0 - \mu)u - \hbar^2 V v &= \hbar^2 f & \text{in } \Omega \\
(-\hbar^2 \Delta - \mu)v &= \hbar^2 g & \text{in } \Omega \\
0 &= \partial_{\nu} u & \text{on } \partial \Omega,
\end{align*}
\]  

(8)

where \( W_0 = -vm_2/(1 + m_2) \). In particular, the principal semi-classical symbol of \(-a\hbar^2 \Delta + h W_0 - \mu \) is \(-a|z|^2 - \mu \) and the principal semi-classical symbol of \( V \) is \( m_1/(1+m_1) \). In what follows only the principal symbols of \(-a\hbar^2 \Delta + h W_0 - \mu \) and \( W \) must be taken into
account. For simplicity, we write the proof for the system (7); the case of the system (8) may be treated in the same way.

**Theorem 10.** We assume that for all \( x \in \Omega \) and all \( \xi \in \mathbb{R}^n \), \( a(x)|\xi|^2 - \mu \neq 0 \) and \( |\xi|^2 - \mu \neq 0 \). Let \( s \geq 0 \), there exists \( h_0 > 0 \) such that for \( f \in H^{2,s}_{m} (\Omega) \), \( g \in H^{1,s}_{m} (\Omega) \), \( u \in H^{1,s}_{m} (\Omega) \cap H^{2,s}_{m} (\Omega) \) and \( v \in H^{1,s}_{m} (\Omega) \) solutions of system (7) then \( u \in H^{2,1}_{m} (\Omega) \), \( v \in H^{2,1}_{m} (\Omega) \) and for \( h \in (0, h_0) \) we have

\[
\begin{align*}
\|u\|_{\pi^{s}_{m} (\Omega)} & \lesssim h^2 \|f\|_{\pi^{s}_{m} (\Omega)} + h^3 \|g\|_{\pi^{s}_{m} (\Omega)} \\
\|v\|_{\pi^{s}_{m} (\Omega)} & \lesssim \|f\|_{\pi^{s}_{m} (\Omega)} + h^2 \|g\|_{\pi^{s}_{m} (\Omega)}.
\end{align*}
\]

First we prove an estimate on \( u \). For this, we work globally in \( \Omega \). The estimate on \( v \) is more difficult to obtain. In a first step we prove an estimate in the interior by usual pseudo-differential tools. In a second step we prove the estimate in a neighborhood of the boundary \( \partial \Omega \) and we finish the proof. This will be done in the following three sections.

In the proof, we use semi-classical pseudo-differential calculus. We provide in the appendix the results used, trace formula, action of pseudo-differential operators on Sobolev space and parametrices.

### 2.1. Estimate on \( u \)

The goal of this section is to prove a weak version of (9).

**Lemma 2.1.** We assume that for all \( x \in \Omega \), and all \( \xi \in \mathbb{R}^n \), \( a(x)|\xi|^2 - \mu \neq 0 \). There exists \( h_0 > 0 \) such that for \( s \geq 0 \), for all \( f \in H^{s}_{m} (\Omega) \), \( v \in H^{1,s}_{m} (\Omega) \), and \( u \in H^{2,s}_{m} (\Omega) \) solution of

\[
\begin{align*}
(-ah^2 \Delta - \mu)u &= h^2 V v + h^2 f \quad \text{in } \Omega \\
u &= \partial_{\nu} u = 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

then \( u \in H^{2,s}_{m} (\Omega) \) and for \( h \in (0, h_0) \),

\[
\|u\|_{\pi^{s}_{m} (\Omega)} \lesssim h^2 \|f\|_{\pi^{s}_{m} (\Omega)} + h^3 \|v\|_{\pi^{s}_{m} (\Omega)}.
\]

**Proof.** As \( u \in H^{2}_{m} (\Omega) \), we can extend \( u \) by 0 in the exterior of \( \Omega \) and \( u \) satisfies the same equation in the whole space. Here we extend also \( v \) and \( f \) by 0; this makes sense at least in \( L^2 \). We have

\[
(-ah^2 \Delta - \mu)u = h^2 V v + h^2 f \quad \text{in } \mathbb{R}^n,
\]

where we denote, for \( w \in L^2 (\Omega) \),

\[
w(x) = \begin{cases} w(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \notin \Omega. \end{cases}
\]

Let \( N \geq s + 2 \) and we choose a parametrix \( Q \) of \( -ah^2 \Delta - \mu \), which is possible because we assume for all \( x \in \Omega \), \( a(x)|\xi|^2 - \mu \neq 0 \). We have \( Q(-ah^2 \Delta - \mu) = \chi + hK \) where \( \chi \in C_0^{\infty} (\mathbb{R}^n) \), \( \chi = 1 \) in a neighborhood of \( \Omega \), \( K \) is of order \( -N \) and \( Q \) is of order \(-2 \).

Applying \( Q \) to equation (12) we obtain

\[
u + hK u = h^2 Q f + h^2 Q(V v).
\]

As \( Q \) is a mapping on Sobolev spaces (see (A.4)) we obtain

\[
\|u\|_{\pi^{s}_{m} (\Omega)} \lesssim h\|u\|_{L^2 (\Omega)} + h^2 \|f\|_{\pi^{s}_{m} (\Omega)} + h^2 \|v\|_{\pi^{s}_{m} (\Omega)}.
\]

We can absorb the term \( h\|u\|_{L^2 (\Omega)} \) by the left-hand side taking \( g \) sufficiently small and this implies (11).
2.2. Estimate on $v$ in the interior of $\Omega$

To prove an estimate on $v$ in the interior of $\Omega$, we follow essentially the approach taken in the proof of the estimate on $u$ given in the previous section, except that we cannot extend $v$ in the exterior of $\Omega$ but we use semi-classical pseudo-differential calculus in relatively compact open sets in $\Omega$. The estimate proved is given in the following lemma.

Lemma 2.2. We assume that for all $\xi \in \mathbb{R}^n$, we have $|\xi|^2 - \mu \neq 0$. Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be supported in $U$ relatively compact in $\Omega$, and $s \geq 0$; there exists $h_0 > 0$ such that for $g \in \mathcal{H}_w(\Omega)$ and $v \in \mathcal{H}_{w, 2}^{s + 1}(\Omega)$ solution of

$$(−h^2\Delta − \mu)v = h^2g \quad \text{in } \Omega,$$

then $v \in \mathcal{H}_{w, 2}^{s + 1}(U)$ and for $h \in (0, h_0)$ we have,

$$\|\chi v\|_{\mathcal{H}_w^{s + 1}} \lesssim h\|v\|_{\mathcal{H}_w^{s + 1}(\Omega)} + h^2\|g\|_{\mathcal{H}_w^{s}(\Omega)}.$$

(13)

Proof. In what follows we can take $\chi$ supported in $U$ or $\chi = 1$ on $U$ and supported in a compact of $\Omega$. We can essentially repeat the proof of lemma 2.1 given to estimate $u$.

As we have assumed that $|\xi|^2 - \mu \neq 0$, we can take a parametrix $Q$ of $(-h^2\Delta - \mu)$ defined globally in $\mathbb{R}^n$, such that we have $Q((-h^2\Delta - \mu)) = \chi + hK$ where $K$ is of order $-1$ and $Q$ is of order $-2$. Let $\chi_j \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $j = 1, 2$, be supported in a compact of $\Omega$ and $\chi_1 = 1$ on the support of $\chi$ and $\chi_2 = 1$ on support of $\chi_1$. By pseudo-differential calculus we have $Q\chi_1(-h^2\Delta - \mu) = \chi + hK_{-1}$ where $K_{-1}$ is of order $-1$. Now we have $Q\chi_1(-h^2\Delta - \mu)\chi_2 = \chi + hK_{-1}\chi_2$. We can localize the equation on $v$ and we have $\chi_1(-h^2\Delta - \mu)\chi_2v = h^2\chi_2g$. Applying $\tilde{Q}$ to this equation we obtain

$$\chi v + hK_{-1}(\chi_2 v) = h^2\tilde{Q}(\chi_1 g).$$

(14)

Taking the $H_{w, 2}^{s + r}$ norm of $\chi v$ we obtain (13). \hfill \square

2.3. Estimate of $v$ in a neighborhood of the boundary

Proof of theorem 10. Taking into account lemmas 2.1 and 2.2, to achieve the proof, we need an estimate of $v$ near the boundary $\partial \Omega$. It is well known that we can find in a neighborhood $W$ of the boundary $\partial \Omega$ a system of coordinates such that the Laplacian can be written as $\partial_x^2 + R(x, \partial_x) + \alpha(x)\partial_{x_n}$, where $x'$ are the coordinates on the manifold $\partial \Omega$, $x_n \in (0, \varepsilon)$, $\Psi(W) = \partial \Omega \times (0, \varepsilon)$. $\Psi$ is the change of coordinates and $R$ is a differential operator on $\partial \Omega$ of order 2 depending on the parameter $x_n$.

We keep the notation $u, v, a$ in the coordinates $x$ instead of $u \circ \Psi$, etc. Equations (7) become

$$\begin{cases}
(a(D_{x_n}^2 + R(x, D') + \alpha D_{x_n}) - \mu)u - h^2Vv = h^2f & \text{in } \partial D \times (0, \varepsilon) \\
(D_{x_n}^2 + R(x, D') + \alpha D_{x_n} - \mu)v = h^2g & \text{in } \partial D \times (0, \varepsilon) \\
u = \partial_x u = 0 & \text{on } \partial \Omega \times [0].
\end{cases}$$

(15)

We have taken the semi-classical notation, $D_{x_n} = h^\frac{1}{2}\partial_x, D' = (D_{x_1}, \ldots, D_{x_{n-1}})$. Actually $R$ and $V$ may depend explicitly on $h$ but this introduces no problem in the estimates if $V_{|_{h=0}} \neq 0$. This is the case if $n = n_1 + n_2/k$. We have $V_{|_{h=0}}(x) = (1 - n_1)/n_1 \neq 0$ by assumption.

Let $w \in L^2(x_n > 0)$. We denote

$$w = 1_{x_n > 0}w = \begin{cases}
w & \text{if } x_n > 0 \\
0 & \text{if } x_n < 0.
\end{cases}$$

(16)
Usually, we use \( w \) but sometimes it is more convenient to use \( 1_{x_n > 0} w \). We have

\[
D_{x_n} w = 1_{x_n > 0} D_{x_n} w + \frac{\hbar}{i} w_{|x_n=0} \otimes \delta_{x_n=0} \nonumber
\]

\[
D^2_{x_n} w = 1_{x_n > 0} D^2_{x_n} w + \frac{\hbar}{i} D_{x_n} w_{|x_n=0} \otimes \delta_{x_n=0} + \frac{\hbar}{i} w_{|x_n=0} \otimes D_{x_n} \delta_{x_n=0} \nonumber.
\]

Here and in what follows, for simplicity we denote by \( w_{|x_n=0} \) the limit, when \( x_n \) goes to 0 with \( x_n > 0 \), of \( w(x', x_n) \), if the limit exists. Here, if the distributions \( u \) and \( v \) are solutions of elliptic equations, then the limits exist in a space of distributions.

From (15), we obtain

\[
\begin{cases}
(a(D^2_{x_n} + R(x, D')) + h\alpha D_{x_n} - \mu) u - h^2 V u = h^2 f \quad & \text{in } \Omega \times (-\varepsilon, \varepsilon) \\
(D^2_{x_n} + R(x, D') + h\alpha D_{x_n} - \mu) v = h^2 g + \frac{\hbar}{i} \gamma_0 \otimes \delta_{x_n=0} + \frac{\hbar}{i} \gamma_1 \otimes D_{x_n} \delta_{x_n=0} \quad & \text{in } \partial \Omega \times (-\varepsilon, \varepsilon),
\end{cases}
\]

(17)

where \( \gamma_0 = D_{x_n} v_{|x_n=0} + ah v_{|x_n=0} \) and \( \gamma_1 = v_{|x_n=0} \). We can consider these equations for \( x_n \in (-\varepsilon, \varepsilon) \); indeed, the coefficients of \( R \) are smooth up the boundary, and we can extend \( R \) in a neighborhood of the boundary for \( x_n < 0 \). The functions \( u \) and \( v \) vanish for \( x_n < 0 \) so the equations are relevant only to take into account the boundary terms. We remark that in the first equation, as the traces of \( u \) vanish, there are no boundary terms.

The main goal of this section, is to obtain estimates on \( \gamma_0 \) and \( \gamma_1 \).

Now we search, using equations (17), two relations between the traces of \( v \). First we localize \( v \) in a neighborhood of the boundary. We denote \( w = \chi_0 v \) where \( \chi_0 \in C^\infty(\mathbb{R}) \), \( \chi_0(x_n) = 1 \) in a neighborhood of the boundary, for instance, if \( |x_n| \leq \varepsilon/4 \), and \( \chi_0(x_n) = 0 \) if \( |x_n| \geq \varepsilon/2 \). From the second equation of (17) we obtain

\[
(D^2_{x_n} + R(x, D') + h\alpha D_{x_n} - \mu) w = h^2 \chi_0 g + \frac{\hbar}{i} \gamma_0 \otimes \delta_{x_n=0} + \frac{\hbar}{i} \gamma_1 \otimes D_{x_n} \delta_{x_n=0} \quad \text{on } \partial \Omega \times \mathbb{R},
\]

(18)

where \( K \) is a first-order differential operator coming from the commutators between \( D^2_{x_n} \) or \( D_{x_n} \) and \( \chi_0 \).

Let \( \chi_1 \in C^\infty(\mathbb{R}) \) such that \( \chi_1 \chi_0 = \chi_0 \); for instance, \( \chi_1(x_n) = 1 \) if \( |x_n| \leq \varepsilon/2 \) and \( \chi_0(x_n) = 0 \) if \( |x_n| \geq 3\varepsilon/4 \). By assumption, we have \( \xi_n^2 + R(x, \xi') - \mu \neq 0 \), then by semi-classical pseudo-differential calculus there exists \( \tilde{Q} \) of order \(-2\) such that \( \tilde{Q}(D^2_{x_n} + R(x, D') + h\alpha D_{x_n} - \mu) = \chi_1 + hK \) where \( K \) is of order \(-N\) where \( N \geq s + 2 \).

Applying \( \tilde{Q} \) to (18), we obtain

\[
\chi_1 w = \tilde{Q} \left( \frac{\hbar}{i} \gamma_0 \otimes \delta_{x_n=0} + \frac{\hbar}{i} \gamma_1 \otimes D_{x_n} \delta_{x_n=0} \right) + g_1 \nonumber
\]

where

\[
g_1 = -hK w + h^2 \tilde{Q}(\chi_0 g) + h\tilde{Q}K u \nonumber
\]

and thus

\[
\|g_1\|_{\mathcal{P}^{-1}_{\tilde{Q}}(\Omega)} \lesssim h\|v\|_{\mathcal{P}^{-1}_Q(\Omega)} + h^2\|g\|_{\mathcal{P}^{-1}_Q(\Omega)}. \nonumber
\]

(19)

Actually we can estimate \( K w \) because \( w \in L^2(\mathbb{R}^n) \) and \( K \) is smoothing. By this trick we need not check that \( K \) is a mapping on the \( \mathcal{P}^{-1}_Q(\Omega) \). In the appendix, estimate (A.4), we have proved that a parametrix \( \tilde{Q} \) is a mapping on the \( \mathcal{P}^{-1}_Q(\Omega) \).

By ellipticity assumption on \( \xi_n^2 + R(x, \xi') - \mu \), there exist \( \rho_1(x, \xi') \) and \( \rho_2(x, \xi') \) with \( \text{Im} \rho_1 > 0 \) and \( \text{Im} \rho_2 < 0 \) such that \( \xi_n^2 + R(x, \xi') - \mu = (\xi_n - \rho_1(x, \xi'))(\xi_n - \rho_2(x, \xi')) \) (see section A.2 in the appendix).
From (A.7) and lemma A.1 we have
\[
\hat{Q} \left( h^{\gamma} \otimes D_{\alpha}^{s} \delta_{x_{n}=0} \right) = \text{op}(\tilde{q}) \gamma
\]
where
\[
\tilde{q} = \frac{1}{2\pi i} \int_{2} \frac{\xi_{n}^{k}}{(\xi_{n} - \rho_{1})(\xi_{n} - \rho_{2})} \mathrm{d}\xi_{n} + r_{-2+k},
\]
and we denote here and in what follows \( r_{j} \) an operator of order \( j \).

From lemma A.2, if we restrict (19) on \( \{ x_{n} = 0 \} \) and as \( u|_{x_{n}=0} = \gamma_{1} \), we obtain
\[
\gamma_{1} = \text{op} \left( \frac{1}{\rho_{1} - \rho_{2}} \right) \rho_{0} + \text{op} \left( \frac{\rho_{1}}{\rho_{2} - \rho_{1}} \right) \gamma_{1} + h \text{op}(r_{-2}) \gamma_{0} + h \text{op}(r_{-1}) \gamma_{1} + (g_{1})|_{x_{n}=0}.
\]
Thus, we obtain
\[
\text{op} \left( \frac{\rho_{2}}{\rho_{2} - \rho_{1}} \right) \gamma_{1} + \text{op} \left( \frac{1}{\rho_{2} - \rho_{1}} \right) \gamma_{0} = (g_{1})|_{x_{n}=0} + h \text{op}(r_{-2}) \gamma_{0} + h \text{op}(r_{-1}) \gamma_{1}.
\]

Then applying \( \text{op}(\rho_{2} - \rho_{1}) \) on both sides of (23), using estimate (19) to estimate \( g_{1} \) and the trace formula (see formula (A.1)), and by pseudo-differential calculus we obtain
\[
\text{op}(\rho_{2}) \gamma_{1} + \gamma_{0} = g_{2} \text{ where } g_{2} = \text{op}(\rho_{2} - \rho_{1}) (g_{1})|_{x_{n}=0} + h \text{op}(r_{-1}) \gamma_{0} + h \text{op}(r_{0}) \gamma_{1}.
\]
and
\[
|g_{2}|_{\mathcal{H}^{-1/2}_{\text{sc}}(\Omega)} \lesssim h^{1/2} \| v \|_{\mathcal{H}^{-1/2}_{\text{sc}}(\Omega)} + h^{3/2} \| g \|_{\mathcal{H}^{-3/2}_{\text{sc}}(\Omega)} + h^{1} \| \gamma_{0} \|_{\mathcal{H}^{-1/2}_{\text{sc}}(\Omega)} + h^{1} \| \gamma_{1} \|_{\mathcal{H}^{-1/2}_{\text{sc}}(\Omega)}.
\]

To obtain a second equation on the traces, we use the first equation of (17). As before, there exist \( Q \) of order \(-2\), \( K \) of order \(-N-4\), \( \chi_{2} \) such that \( \chi_{2} \chi_{0} = \chi_{0} \) and such that \( Q(aD_{\alpha}^{s} + R + h\alpha D_{\alpha}) - \mu = \chi_{2} + hK \).

We apply \( Q \) to the first equation of (17) and obtain
\[
\chi_{2} u = h^{2} Q(Vw) + g_{3},
\]
where
\[
g_{3} = -hKu + h^{2} Q(V(1 - \chi_{0})w) + h^{2} Qf,
\]
which satisfies
\[
\| g_{3} \|_{\mathcal{H}^{s}_{\text{sc}}(\Omega)} \lesssim h \| u \|_{\mathcal{H}^{s}_{\text{sc}}(\Omega)} + h^{2} \| (1 - \chi_{0})w \|_{\mathcal{H}^{s}_{\text{sc}}(\Omega)} + h^{5} \| f \|_{\mathcal{H}^{s}_{\text{sc}}(\Omega)}.
\]
Using lemma 2.2 we can estimate the term \((1 - \chi_{0})w\) and using lemma 2.1 we can estimate \( \| u \|_{\mathcal{H}^{s}_{\text{sc}}(\Omega)} \), and we obtain
\[
\| g_{3} \|_{\mathcal{H}^{s}_{\text{sc}}(\Omega)} \lesssim h^{3} \| v \|_{\mathcal{H}^{s}_{\text{sc}}(\Omega)} + h^{4} \| g \|_{\mathcal{H}^{s}_{\text{sc}}(\Omega)} + h^{5} \| f \|_{\mathcal{H}^{s}_{\text{sc}}(\Omega)}.
\]

We substitute the value of \( w \) given by formula (19), we take the trace on \( x_{n} = 0 \), and as \( u|_{x_{n}=0} = 0 \), we obtain
\[
\left[ Q \left( V \left( Q \left( h^{\gamma} \otimes \delta_{x_{n}=0} + \frac{1}{h} \gamma_{1} \otimes D_{\alpha} \delta_{x_{n}=0} \right) \right) \right) \right]|_{x_{n}=0} = g_{4},
\]
where
\[
g_{4} = -h^{-2} (g_{3})|_{x_{n}=0} - [QV (g_{1})]|_{x_{n}=0},
\]
satisfying
\[
|g_{4}|_{\mathcal{H}^{-1/2}_{\text{sc}}(\Omega)} \lesssim h^{1/2} \| v \|_{\mathcal{H}^{1/2}_{\text{sc}}(\Omega)} + h^{3/2} \| g \|_{\mathcal{H}^{-3/2}_{\text{sc}}(\Omega)} + h^{-1/2} \| f \|_{\mathcal{H}^{-1/2}_{\text{sc}}(\Omega)}.
\]
As \( a(x)(\xi_{n}^{2} + R(x, \xi_{n}')) - \mu \) is elliptic, the polynomial in \( \xi_{n} \) has two roots \( \lambda_{j} \), \( j = 1, 2 \). One satisfies \( \text{Im} \lambda_{1} > 0 \) and the other \( \text{Im} \lambda_{2} < 0 \) (see section A.2 in the appendix). We have
We set $a(x)(\xi_n^2 + R(x, \xi)) - \mu = a(x_\star)(\xi_n - \lambda_1)(\xi_n - \lambda_2)$. The principal symbol of $Q$ is
\[
\frac{\eta}{a(x_\star)(\xi_n - \lambda_1)(\xi_n - \lambda_2)}.
\]
The principal symbol of $QV\hat{Q}$ is
\[
\frac{\eta}{a(x_\star)(\xi_n - \lambda_1)(\xi_n - \lambda_2)(\xi_n - \mu)}.
\]

Following the same method used to obtain (23) from (19), we have by (A.7), lemmas A.1 and A.3, as $x_\star = 1$ in the neighborhood of $\partial \Omega$,
\[
\begin{align*}
\text{op} \left( \frac{V}{a} V(\lambda_2 - \lambda_1 - \rho_2) V(\rho_2) (\rho_1 - \rho_2) (\rho_1 - \rho_2) \right) \gamma_0 & + \text{op} \left( \frac{V}{a} V(\rho_2) (\rho_1 - \rho_2) V(\rho_2) (\rho_1 - \rho_2) \right) \gamma_1 = g_5,
\end{align*}
\]
where $g_5 = g_4 + h \text{op}(r_{-4}) \gamma_0 + h \text{op}(r_{-3}) \gamma_1$. As $V \neq 0$ in a neighborhood of $\partial \Omega$, we can apply $\text{op}(\frac{V}{a} V(\lambda_2 - \lambda_1 - \rho_2) (\rho_1 - \lambda_2) V(\rho_1 - \rho_2))$, and we obtain
\[
\begin{align*}
\text{op} (\lambda_2 - \lambda_1 + \rho_2 - \rho_1) \gamma_0 & + \text{op} ((\rho_2 - \rho_1) (\rho_1 - \rho_2)) \gamma_1 = g_6,
\end{align*}
\]
where
\[
g_6 = \text{op} \left( \frac{V}{a} (\lambda_2 - \lambda_1 - \rho_2) (\rho_1 - \lambda_2) (\rho_1 - \rho_2) \right) g_5 + h \text{op}(r_0) \gamma_0 + h \text{op}(r_1) \gamma_1
\]
satisfies
\[
|g_6|_{H^{1/2}(\partial \Omega)} \lesssim h^{1/2} \|v\|_{H^1(\Omega)} + h^{3/2} \|g\|_{H^1(\Omega)} + h^{-1/2} \|f\|_{H^1(\Omega)}
\]
\[
+ h|\gamma_0|_{H^{1/2}(\partial \Omega)} + h|\gamma_1|_{H^{1/2}(\partial \Omega)}.
\]

Now we have two equations on the traces $\gamma_0$ and $\gamma_1$. Substituting the value of $\gamma_0$ given by (24) in (28), we obtain by pseudo-differential calculus
\[
\text{op}(\rho_2 \lambda_2 - \lambda_1 \rho_1) \gamma_1 - \text{op}((\lambda_2 - \lambda_1 + \rho_2 - \rho_1) \rho_2) \gamma_1 = g_6 - \text{op}(\lambda_2 - \lambda_1 + \rho_2 - \rho_1) g_2 = g_7.
\]

This implies,\[\text{op}((\rho_2 - \rho_1) (\lambda_1 - \lambda_2)) \gamma_1 = g_7,\]
with
\[
|g_7|_{H^{1/2}(\partial \Omega)} \lesssim h^{1/2} \|v\|_{H^1(\Omega)} + h^{3/2} \|g\|_{H^1(\Omega)} + h^{-1/2} \|f\|_{H^1(\Omega)}
\]
\[
+ h|\gamma_0|_{H^{1/2}(\partial \Omega)} + h|\gamma_1|_{H^{1/2}(\partial \Omega)}.
\]

As $\text{Im} \lambda_1 > 0$, $\text{Im} \rho_1 > 0$ and $\text{Im} \rho_2 < 0$, the symbol $(\rho_2 - \rho_1)(\lambda_1 - \lambda_2)$ is elliptic and by inversion we obtain
\[
|\gamma_1|_{H^{1/2}(\partial \Omega)} \lesssim h^{1/2} \|v\|_{H^1(\Omega)} + h^{3/2} \|g\|_{H^1(\Omega)} + h^{-1/2} \|f\|_{H^1(\Omega)}
\]
\[
+ h|\gamma_0|_{H^{1/2}(\partial \Omega)} + h|\gamma_1|_{H^{1/2}(\partial \Omega)}.
\]
Using (24), we obtain
\[
|\gamma_0|_{H^{1/2}(\partial \Omega)} \lesssim h^{1/2} \|v\|_{H^1(\Omega)} + h^{3/2} \|g\|_{H^1(\Omega)} + h^{-1/2} \|f\|_{H^1(\Omega)}
\]
\[
+ h|\gamma_0|_{H^{1/2}(\partial \Omega)} + h|\gamma_1|_{H^{1/2}(\partial \Omega)}.
\]

Summing (30) and (31) we have, for $h_0$ small enough,
\[
|\gamma_1|_{H^{1/2}(\partial \Omega)} + |\gamma_0|_{H^{1/2}(\partial \Omega)} \lesssim h^{1/2} \|v\|_{H^1(\Omega)} + h^{3/2} \|g\|_{H^1(\Omega)} + h^{-1/2} \|f\|_{H^1(\Omega)}.
\]

From (19) and from estimate (A.5) obtained in the appendix, we find
\[
\|
\begin{align*}
\|v\|_{H^1(\Omega)} & \lesssim g_4 \|\gamma_0\|_{H^{1/2}(\partial \Omega)} + h^{1/2} (|\gamma_0|_{H^{1/2}(\partial \Omega)} + |\gamma_1|_{H^{1/2}(\partial \Omega)})
\end{align*}
\]
\[
\lesssim h\|v\|_{H^1(\Omega)} + h^2 \|g\|_{H^1(\Omega)} + \|f\|_{H^1(\Omega)}.
\]
We can now estimate $v$. By (32) and lemma 2.2 we have
\[
\|v\|_{H_s^{−1}(\Omega)} \leq h\|v\|_{H_{s+1}(\Omega)} + \|(1−χ_\delta)v\|_{H_{s+1}(\Omega)}
\]
\[
\leq h\|v\|_{H_{s+1}(\Omega)} + h^2\|g\|_{H_{s}(\Omega)} + \|f\|_{H_{s}(\Omega)}.
\]
This implies
\[
\|v\|_{H_s^{−1}(\Omega)} \leq h^2\|g\|_{H_{s}(\Omega)} + \|f\|_{H_{s}(\Omega)},
\]
if $h_0$ is small enough. Using this estimate and (11) we obtain
\[
\|u\|_{H_{s+1}(\Omega)} \leq h^2\|f\|_{H_{s}(\Omega)} + h^4\|g\|_{H_{s}(\Omega)}.
\]
These two last estimates imply the result of theorem 10. □

**Remark 8.** Actually in this proof we have assumed $v \in H_{s+1}^1(\Omega)$. To prove the result with $v \in H_{s}^1(\Omega)$, we argue in two steps; first, following the same proof we can obtain $v \in H_{s+1}^1(\Omega)$ and second, the proof given above gives $v \in H_{s+1}^2(\Omega)$.

The estimate proved for $v$ does not suffice for our purpose. In fact, we need to estimate the $H^s$ norm of $v$ with respect to the norm of $f$.

**Proposition 2.3.** We assume that for all $x \in \Omega$ and all $ξ \in \mathbb{R}^n$, $a(\alpha)|ξ|^2 − ρ \neq 0$ and $|ξ|^2 − ρ \neq 0$. There exist $h_0 > 0$ and $C > 0$ such that for all $δ > 0$, and $χ_δ \in C^\infty_{\text{c}}$ supported in a $δ$-neighborhood of $\partial \Omega$, there exists $C_0 > 0$ such that for $h \in (0, h_0)$, $f \in H_{s}^1(\Omega)$, $g = 0$, we have $u \in H_{s+1}^1(\Omega) \cap H_{s}^1(\Omega)$, $v \in H_{s}^1(\Omega)$ and $\Delta v \in L^2(\Omega)$, solutions of system (4) satisfying the estimate
\[
\|v\|_{H_s^{−1}(\Omega)} \leq C\|χ_δf\|_{H_s^{−1}(\Omega)} + C_1h\|f\|_{H_s^{−1}(\Omega)}.
\]

**Proof.** The proof follows that of theorem 10. We only highlight differences. From (18) we obtain (19) with the estimate
\[
\|g\|_{H_s^{−1}(\Omega)} \leq h\|v\|_{H_s^{−1}(\Omega)}.
\]
Thus, we obtain (23) and (24) with the estimate
\[
\|g\|_{H_s^{−1}(\Omega)} \leq \|\text{op}(r_1)(g_1)\|_{H_s^{−1}(\Omega)} + h\|\gamma_0\|_{H_{s+2}^1(\Omega)} + h\|\gamma_1\|_{H_{s+1}^2(\Omega)}.
\]
We must modify (25) to obtain the term $χ_δ$. We take the same $Q$ as in (25) and we apply $Qχ_δ$ to the first equation in (17). We obtain
\[
Qχ_δ(a(D^2_x + R(x, D') + hαD_{\lambda}u) − ρ)u − h^2Qχ_δV_w = h^2Qχ_δf \quad \text{in} \quad \hat{Ω} \times (−\varepsilon, \varepsilon).
\]
We have $χ_δ(a(D^2_x + R(x, D') + hαD_{\lambda}u) − ρ) = a(D^2_x + R(x, D') + hαD_{\lambda}u) − ρ)χ_δ + L_1$ where $L_1$ is a differential operator of order 1 depending on $\delta$. As $Q(a(D^2_x + R + hαD_{\lambda}u) − ρ) = χ_δ + L_1$, we have
\[
χ_δu = h^2Q(χ_δV_w) + g_3
\]
where
\[
g_3 = −hQL_1u + hK_{−N}u + h^2Q(χ_δf)
\]
satisfying
\[
\|g_3\|_{H_s^{−1}(\Omega)} \leq C_1h\|u\|_{H_s^{−1}(\Omega)} + C_2h\|f\|_{H_s^{−1}(\Omega)}.
\]
We have used that $Q$ and $L_1$, a differential operator, act on $\mathcal{H}_{s_1}^\delta$. We can estimate $u$ by (11), which gives
\[ \|g_1\|_{\mathcal{H}_{s_1}^\delta(\Omega)} \leq C_1 h^3 \|f\|_{\mathcal{H}_{s_1}^\delta(\Omega)} + C_1 h^3 \|v\|_{\mathcal{H}_{s_1}^\delta(\Omega)} + C_1 h^3 \|\chi_\delta f\|_{\mathcal{H}_{s_1}^\delta(\Omega)}. \]
We substitute the value of $w$ by its value given by formula (19) in estimate (33). We obtain (26) with
\[ \|g_1\|_{\mathcal{H}_{s_1}^\delta(\Omega)} \leq C_1 h^{1/2} \|f\|_{\mathcal{H}_{s_1}^\delta(\Omega)} + C_1 h^{1/2} \|v\|_{\mathcal{H}_{s_1}^\delta(\Omega)} + C_1 h^{1/2} \|\chi_\delta f\|_{\mathcal{H}_{s_1}^\delta(\Omega)}. \] (35)
If we compare this estimate with (27) we see that the ‘bad’ power of $h$ in front of $f$ now only concerns $f$ localized in a neighborhood of the boundary.
Following the proof of theorem 10, we obtain (28) where the estimate on $g_6$ is now
\[ \|g_6\|_{H^{s_2}(\Omega)} \leq C_1 h^{1/2} \|f\|_{\mathcal{H}_{s_1}^\delta(\Omega)} + C_1 h^{1/2} \|v\|_{\mathcal{H}_{s_1}^\delta(\Omega)} + C_1 h^{1/2} \|\chi_\delta f\|_{\mathcal{H}_{s_1}^\delta(\Omega)} + C h |\gamma_0|_{H^{s_2}(\Omega)} + C h |\gamma_1|_{H^{s_2}(\Omega)}. \]
We have formula (29) where $g_3$ from (34) is estimated by
\[ \|g_3\|_{H^{s_2}(\Omega)} \leq C_1 h^{1/2} \|f\|_{\mathcal{H}_{s_1}^\delta(\Omega)} + C_1 h^{1/2} \|v\|_{\mathcal{H}_{s_1}^\delta(\Omega)} + C_1 h^{1/2} \|\chi_\delta f\|_{\mathcal{H}_{s_1}^\delta(\Omega)} + C h |\gamma_0|_{H^{s_2}(\Omega)} + C h |\gamma_1|_{H^{s_2}(\Omega)}. \]
By ellipticity and formula (29) we obtain
\[ |\gamma_1|_{H^{s_2}(\Omega)} \leq C_1 h^{1/2} \|f\|_{\mathcal{H}_{s_1}^\delta(\Omega)} + C_1 h^{1/2} \|v\|_{\mathcal{H}_{s_1}^\delta(\Omega)} + C_1 h^{1/2} \|\chi_\delta f\|_{\mathcal{H}_{s_1}^\delta(\Omega)} + C h |\gamma_0|_{H^{s_2}(\Omega)} + C h |\gamma_1|_{H^{s_2}(\Omega)}, \]
and by (24) where $g_2$ satisfies (34), we have
\[ |\gamma_0|_{H^{s_2}(\Omega)} \leq C_1 h^{1/2} \|f\|_{\mathcal{H}_{s_1}^\delta(\Omega)} + C_1 h^{1/2} \|v\|_{\mathcal{H}_{s_1}^\delta(\Omega)} + C_1 h^{1/2} \|\chi_\delta f\|_{\mathcal{H}_{s_1}^\delta(\Omega)} + C h |\gamma_0|_{H^{s_2}(\Omega)} + C h |\gamma_1|_{H^{s_2}(\Omega)}. \]
Summing the previous estimates, and for $h$ small enough, we obtain
\[ |\gamma_1|_{H^{s_2}(\Omega)} + |\gamma_0|_{H^{s_2}(\Omega)} \leq C_1 h^{1/2} \|f\|_{\mathcal{H}_{s_1}^\delta(\Omega)} + C_1 h^{1/2} \|v\|_{\mathcal{H}_{s_1}^\delta(\Omega)} + C_1 h^{1/2} \|\chi_\delta f\|_{\mathcal{H}_{s_1}^\delta(\Omega)}. \]
From (19) with $g_1$ satisfying (33), we have by (A.5)
\[ \|u\|_{\mathcal{H}_{s_1}^\delta(\Omega)} \leq C \|g_1\|_{\mathcal{H}_{s_1}^\delta(\Omega)} + C h^{1/2} \|\gamma_0|_{H^{s_2}(\Omega)} + C h^{1/2} \|\gamma_1|_{H^{s_2}(\Omega)} \] \[ \leq C_1 h \|f\|_{\mathcal{H}_{s_1}^\delta(\Omega)} + C h \|v\|_{\mathcal{H}_{s_1}^\delta(\Omega)} + C h \|\chi_\delta f\|_{\mathcal{H}_{s_1}^\delta(\Omega)}. \] (36)
Using formula (14) in the proof of lemma 2.2 with $g = 0$, we obtain $\|v\|_{\mathcal{H}_{s_1}^\delta(\Omega)} \leq C \|v\|_{\mathcal{H}_{s_1}^\delta(\Omega)}$. This estimate and (36) give
\[ \|v\|_{\mathcal{H}_{s_1}^\delta(\Omega)} \leq C_1 h \|f\|_{\mathcal{H}_{s_1}^\delta(\Omega)} + C h \|v\|_{\mathcal{H}_{s_1}^\delta(\Omega)} + C h \|\chi_\delta f\|_{\mathcal{H}_{s_1}^\delta(\Omega)}. \]
This estimate also holds true for a fixed $\delta$; then we have for $h \in (0, h_0)$, $h_0$ small enough,
\[ \|v\|_{\mathcal{H}_{s_1}^\delta(\Omega)} \leq C \|f\|_{\mathcal{H}_{s_1}^\delta(\Omega)}. \]
This with the previous estimate implies the result of proposition 2.3.

3. Existence and compactness

In this section we prove theorems 1, 2, 3 and 4.
3.1. Proof of theorems 1 and 2

**Proof.** We give in detail the proof of theorem 1; the proof of theorem 2 follows the same way. We follow the proof given by Sylvester [27, proposition 10]. We prove that the range of \( B_1 \) is closed and dense.

To prove that the range is closed we apply the *a priori* estimates proven in section 2. We recall that \( a = 1/(1 + m) \) and \( V = m/(1 + m) \).

We remark that if we have \( C_r \neq \emptyset \) then \( C_r \cup (\infty, 0] \neq \emptyset \). Indeed, as \( \overline{\Omega} \) is compact, \( C_r \) is closed. If \( C_r \cup (\infty, 0] = \emptyset \) then \( C_r \neq \emptyset \). As \( C_r \subset \emptyset \) as \( C_r \subset (\infty, 0] \), \( C_r \subset \emptyset \) is dense in \( \emptyset \), we have \( C_r = \bar{C} \). Let \( z_0 \) be such that \( z_0 \notin C_r \cup (\infty, 0]. \) We can choose \( |z_0| = 1 \). Letting \( z = h^{-2}z_0 \), we have \( \mu = -z_0 \). First we can estimate \( \|v\|_{L^2(\Omega)} \) by proposition 2.3 with \( s = 0 \) and \( \delta \) fixed if \( g = 0 \) and by theorem 10 if \( f = 0 \). There exists \( C > 0 \) such that, for all \( |z| \) large enough,

\[
\|v\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)} + \frac{C}{|z|^2}\|g\|_{L^2(\Omega)}.
\]  

(37)

Applying lemma 2.1 with \( s = 0 \), we obtain with the previous estimate on \( v \),

\[
|z|^2\|u\|_{L^2(\Omega)} + \|u\|_{\mathcal{V}(\Omega)}^2 + \|u\|_{\mathcal{V}(\Omega)} \leq C\|f\|_{L^2(\Omega)} + \frac{C}{|z|^2}\|g\|_{L^2(\Omega)}.
\]  

(38)

Clearly, these estimates prove that the range of \( B_1 \) is closed where the norm on the domain of the operator is given by the \( H^2 \) norm for \( u \) and by \( \|v\| + \|\Delta v\| \) for \( v \).

To prove the density of the range of \( B_1 \), we prove that the orthogonal of the range is \( \{0\} \). We recall the Green formula; if \( v \) and \( q \) are smooth functions in \( \Omega \), we have

\[
(v|\Delta q) - (\Delta v|q) = (v|\partial_\nu q)_{\partial\Omega} - (\partial_\nu v|q)_{\partial\Omega},
\]  

(39)

where \((\cdot|\cdot)\) is the inner product on \( \Omega \), \((\cdot|\cdot)_{\partial\Omega}\) is the inner product on \( \partial\Omega \) and \( \partial_\nu \) is the exterior normal derivative on \( \partial\Omega \). Actually (39) is true if \( v \) is smooth, \( q \in L^2(\Omega) \) and \( \Delta q \in L^2(\Omega) \). Indeed, in this case it is well known that \( q_{\partial\Omega} \in H^{-1/2}(\partial\Omega) \) and \( \partial_\nu q_{\partial\Omega} \in H^{-3/2}(\partial\Omega) \). Then we can find \( f_n \) and \( g_n \) sequences of smooth functions such that \( f_n \) converges to \( \Delta q \) in \( L^2(\Omega) \) and \( g_n \) converges to \( q_{\partial\Omega} \). Letting \( q_n \) be the solution of \( \Delta q_n = f_n \) in \( \Omega \) and \( q_{\partial\Omega} = g_n \), \( q_n \) is a smooth function and by continuity \((\partial_\nu q_n)_{\partial\Omega} \) converges to \((\partial_\nu q)_{\partial\Omega} \) in \( H^{-3/2}(\partial\Omega) \). Then we can pass to the limit in (39).

Let \( p, q \in L^2(\Omega) \) and \( u \) and \( v \) be smooth functions in \( \Omega \). If \((p, q)\) is in the orthogonal of the range, we have

\[
(\Delta u - z(1 + m)u + mv|p) + (\Delta v - zv|q) = 0.
\]  

(40)

We take \( u, v \in H^\infty_0(\Omega) \) in (40). Integrating by parts in the sense of distribution we find

\[
\Delta p - z(1 + \overline{m})p = 0 \quad \text{in} \; \Omega
\]  

(41)

\[
\Delta q - z\overline{q} + \overline{m}p = 0 \quad \text{in} \; \Omega.
\]  

(42)

In particular \( \Delta p \) and \( \Delta q \) are in \( L^2(\Omega) \), then we can apply (39) to integrate by part in (40) if now \( u \) and \( v \) are smooth functions up the boundary with \( u_{\partial\Omega} = \partial_\nu u_{\partial\Omega} = 0 \). Using (41) and (42), we have

\[
(v|\partial_\nu q)_{\partial\Omega} - (\partial_\nu v|q)_{\partial\Omega} = 0.
\]  

As \( u_{\partial\Omega} \) and \( \partial_\nu u_{\partial\Omega} \) are arbitrary, we obtain \( q_{\partial\Omega} = \partial_\nu q_{\partial\Omega} = 0 \). By (42), \( q \) satisfies a Dirichlet boundary value problem and \( \partial_\nu q_{\partial\Omega} = 0 \); then \( q \in \mathcal{V}(\Omega) \). By (41), \( \Delta p \in L^2(\Omega) \) and \( p \in L^2(\Omega) \). We deduce that \((q, p) \in H^2_0(\Omega) \oplus \{v \in L^2(\Omega), \; \Delta v \in L^2(\Omega) \} \) satisfies the same kind of equation as \((u, v)\). Then inequalities (37) and (38) prove that \( p = q = 0 \). This concludes the proof of theorem 1. \( \square \)
3.2. Proof of theorems 3 and 4

Proof. We choose the same \( z \) as in the proof of theorem 1. Apply theorem 10 with \( s = 0 \). We obtain (with classical norms)

\[
|z|\|u\|_{L^2(\Omega)} + |z|^{1/2}\|u\|_{H^1(\Omega)} + \|u\|_{H^1(\Omega)} + \frac{1}{|z|^{1/2}}\|u\|_{H^1(\Omega)} + |z|^{-1}\|u\|_{H^1(\Omega)} \leq C\|f\|_{H^1(\Omega)} + \frac{C}{|z|}\|g\|_{L^2(\Omega)}.
\]

This proves that \( R : \mathcal{H}^2(\Omega) \oplus L^2(\Omega) \to \mathcal{T}^1(\Omega) \oplus \mathcal{H}^2(\Omega) \), then \( R \) is compact from \( \mathcal{H}^2(\Omega) \oplus L^2(\Omega) \) to itself. We can then apply the Riesz theory.

For the case \( n(x) = n_1(x) + in_2(x)/k \), we just have to replace \( |z| \) by \( k^2 \) in (43) and we obtain theorem 4.

\( \square \)

4. Spectral results

Here we prove how the regularity results obtained in section 2 allow one to prove the spectral results. Actually the result obtained in theorem 3 is not enough to prove that the spectrum is a countable set. The theory given by Agmon [1] is based on the spectral decomposition of Hilbert–Schmidt operators. We adapt two results given by Agmon to our case, in lemma 4.1 and proposition 4.2. With these two results we shall then prove theorems 5 and 7.

Let \( T \) be a Hilbert–Schmidt operator from \( \mathcal{H}^2(\Omega) \oplus L^2(\Omega) \) to itself. We denote by \( |||T||| \) the Hilbert–Schmidt norm. Let \( \{\phi_j\}_{j \in \mathbb{N}} \) be a Hilbert basis on \( \mathcal{H}^2(\Omega) \) and \( \{\psi_k\}_{k \in \mathbb{N}} \) be a Hilbert basis on \( L^2(\Omega) \); then \( \{(\phi_j, 0)\}_{j \in \mathbb{N}}, \{(0, \psi_k)\}_{k \in \mathbb{N}} \) is a Hilbert basis on \( \mathcal{H}^2(\Omega) \oplus L^2(\Omega) \). We set \( T(\phi_j, 0) = u_j = (u_{j1}, u_{j2}^0) \) and \( T(0, \psi_k) = v_k = (v_{k1}, v_k^0) \). With this notation, we have

\[
|||T|||^2 = \sum_{j=0}^{\infty} (|u_{j2}^0|^2_{H^2(\Omega)} + |v_{k1}|^2_{H^2(\Omega)} + |u_{j1}|^2_{L^2(\Omega)} + |v_{k1}|^2_{L^2(\Omega)}).
\]

We denote by \( ||T|| \) the operator norm from \( \mathcal{H}^2(\Omega) \oplus L^2(\Omega) \to \mathcal{H}^{1/2}(\Omega) \oplus \mathcal{H}^2(\Omega) \), where \( \mathcal{H}^0(\Omega) = L^2(\Omega) \).

Lemma 4.1. Let \( m > n/2 \). There exists \( C > 0 \) such that if \( T \) is a bounded operator from \( \mathcal{H}^m(\Omega) \oplus L^2(\Omega) \to \mathcal{H}^{m+1/2}(\Omega) \oplus \mathcal{H}^m(\Omega) \), then \( T \) is a Hilbert–Schmidt operator and

\[
|||T||| \leq C\|T\|_{m/2(\text{mod})} \|T\|_{1-n/2(\text{mod})}.
\]

Proof. We follow the proof given by Agmon [1, theorem 13.5].

Let \( u = T \left( \sum_{j=0}^{N} a_j (\phi_j, 0) + \sum_{j=0}^{N} b_j (0, \psi_j) \right) = \sum_{j=0}^{N} a_j u_j + \sum_{j=0}^{N} b_j v_j = (u^0, u^1) \). We have \( u_j = (u_{j2}^0, u_{j1}) \) and \( v_j = (v_{k1}, v_k^0) \). We consider the term \( u^0 \).

If \( m > n/2 \), \( \mathcal{H}^m(\Omega) \subset L^\infty(\Omega) \) and (see [1, lemma 13.2]), for \( \alpha \in \mathbb{N}^d, |\alpha| \leq 2 \), there exists \( C > 0 \) such that

\[
\|\partial^\alpha u\|_{L^\infty(\Omega)} \leq C\|v\|_{m/2(\text{mod})} \|v\|_{1-n/2(\text{mod})} \|u\|_{\mathcal{H}^m(\Omega)}.
\]

With the assumption made on \( T \), we have

\[
\|u^0\|_{\mathcal{H}^{1/2}(\Omega)}^2 \leq C\|T\|^2 \sum_{j=0}^{N} (|a_j|^2 + |b_j|^2) \quad \text{and} \quad \|u^0\|_{\mathcal{H}^m(\Omega)}^2 \leq C\|T\|^2 \sum_{j=0}^{N} (|a_j|^2 + |b_j|^2).
\]
Let $K = \|T\|_{m}^{n/2m} \|T\|_{0}^{1-n/2m}$. We have for $x \in \Omega$, and for $\alpha \in \mathbb{N}^{n}$, $|\alpha| \leq 2$, $|\partial^{\alpha}u^{0}(x)|^{2} \leq CK^{2} \sum_{j=0}^{N} (|a_{j}|^{2} + |b_{j}|^{2})$. We have $\partial^{\alpha}u^{0}(x) = \sum_{j=0}^{N} (a_{j}\partial^{\alpha}u^{0}_{j}(x) + b_{j}\partial^{\alpha}v^{0}_{j}(x))$. We take in the previous inequality $a_{j} = \partial^{\alpha}u^{0}_{j}(x)$ and $b_{j} = \partial^{\alpha}v^{0}_{j}(x)$, we sum on $\alpha$, and we obtain for all $x \in \overline{\Omega}$

$$
\sum_{|\alpha| \leq 2} \left( \sum_{j=0}^{N} (|\partial^{\alpha}u^{0}_{j}(x)|^{2} + |\partial^{\alpha}v^{0}_{j}(x)|^{2}) \right)^{2} \leq CK^{2} \sum_{|\alpha| \leq 2} \sum_{j=0}^{N} (|\partial^{\alpha}u^{0}_{j}(x)|^{2} + |\partial^{\alpha}v^{0}_{j}(x)|^{2}).
$$

Thus, $\sum_{|\alpha| \leq 2} (|\partial^{\alpha}u^{0}(x)|^{2} + |\partial^{\alpha}v^{0}(x)|^{2}) \leq CK^{2}$; integrating this on $\Omega$ (which is bounded) we find $\sum_{j=0}^{N} (\|u^{0}_{j}\|^{2}_{H^{2}(\Omega)} + \|v^{0}_{j}\|^{2}_{H^{2}(\Omega)}) \leq CK^{2}$. As the right-hand side does not depend on $N$, we can let $N$ go to infinity. We can treat by the same method the terms $\sum_{j=0}^{N} (\|u^{0}_{j}\|^{2}_{L^{2}(\Omega)} + \|v^{0}_{j}\|^{2}_{L^{2}(\Omega)})$.

It suffices to repeat the previous argument without the derivative terms. This means that $|||T|||$ is bounded by $CK = C\|T\|_{m}^{n/2m} \|T\|_{0}^{1-n/2m}$.

We give here a small improvement of theorem 16.4 in [1].

We introduce some notation. The inner product in $H^{2}(\Omega) \oplus L^{2}(\Omega)$ will be denoted $(\cdot, \cdot)$. Let $T$ be an operator from $H^{2}(\Omega) \oplus L^{2}(\Omega)$ into itself. If $\lambda^{-1}$ is in the resolvent set of $T$, we set $T_{\lambda} = (I - \lambda T)^{-1}$. We remark that if $T$ is the resolvent of $P$, that is $PT = I$, then $T_{\lambda}$ is the resolvent of $P - \lambda I$. Indeed,

$$(P - \lambda I)T_{\lambda} = (P - \lambda I)(I - \lambda T)^{-1} = (I - \lambda T)^{-1} - \lambda T(I - \lambda T)^{-1} = (I - \lambda T)^{-1}(I - \lambda T) = I.$$

**Proposition 4.2.** Let $T$ be a Hilbert–Schmidt operator on $H^{2}(\Omega) \oplus L^{2}(\Omega)$. We assume that there exists $0 \leq \theta_{1} < \theta_{2} < \cdots < \theta_{N} < 2\pi$ such that $\theta_{k} - \theta_{k-1} < \pi/2$ for $k = 2, \ldots, N$ and $2\pi - \theta_{N} + \theta_{1} < \pi/2$ satisfying the condition that there exist $r_{0} > 0$, $C > 0$ such that $sup_{r > r_{0}} \|T_{\lambda}\| \leq C$, for $k = 1, \ldots, N$. Moreover we assume that there exists $(\lambda_{j})$ such that $|\lambda_{j}| \rightarrow +\infty$ and for all $(f, g)$ in $H^{2}(\Omega) \oplus L^{2}(\Omega)$, $(T_{\lambda_{j}} f|g) \rightarrow 0$. Then the space spanned by the nonzero generalized eigenfunctions of $T$ is dense in the closure of the range of $T$.

**Proof.** As by Agmon [1, p 284] we define $F(\lambda) = (T_{\lambda} f|g)$ where $g$ is orthogonal to the generalized eigenfunctions. The goal is to prove that $F(0) = 0$. As by Agmon, we can prove that $F(\lambda)$ is analytic in $\mathbb{C}$ and bounded. Then $F(\lambda)$ is constant by the Liouville theorem and as $(T_{\lambda_{j}} f|g) \rightarrow 0$ this implies that $F(\lambda) = 0$.

**Proof of theorem 5.** Before the proof, we give some results on the connection between the spectral decomposition of $S$ and $S^{p}$, where $S$ is a bounded operator on $H^{2}(\Omega) \oplus L^{2}(\Omega)$.

Let $\omega_{j}$ be the roots of $z^{p} = 1$, for $j = 1, \ldots, p$. We have $z^{p} - 1 = \prod_{j=1}^{p} (z - \omega_{j})$. In particular, for $z = 0$ we have $z^{p} - 1 = \prod_{j=1}^{p} (-\omega_{j})$. Thus, we find

$$z^{p} - 1 = \prod_{j=1}^{p} (z - \omega_{j}) = \prod_{j=1}^{p} (-\omega_{j}) \prod_{j=1}^{p} (1 - \omega_{j}^{-1}z) = - \prod_{j=1}^{p} (1 - \omega_{j}z), \quad (44)$$

as $\{\omega_{j}, j = 1, \ldots, p\} = \{\omega^{-1}_{j}, j = 1, \ldots, p\}$.

Applying (44) to $zS$, we obtain

$$(1 - z^{p}S) = \prod_{j=1}^{P} (1 - \omega_{j}zS). \quad (45)$$

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If \((I - \omega_j z S)\) is invertible for all \(j\), this implies that \((I - z^p S^\theta)\) is invertible. If for a fixed \(j\), \((I - \omega_j z S)\) is not invertible, either \(\ker(I - \omega_j z S) \neq \{0\}\) implying \(\ker(I - z^p S^\theta) \neq \{0\}\) or the range is not \(\overline{H^2}(\Omega) \oplus L^2(\Omega)\) implying that the range of \((I - z^p S^\theta)\) is not \(\overline{H^2}(\Omega) \oplus L^2(\Omega)\). We deduce that

\[ I - \omega_j z S \text{ is invertible for all } j \iff I - z^p S^\theta \text{ is invertible.} \]

If \(S^\theta\) is compact and \((I - z^p S^\theta)\) is not invertible then by the Riesz theorem, \(z^{-p}\) is an eigenvalue of \(S^\theta\) and there exists \(k\) such that \(\ker(I - z^p S^\theta)^{k-1} \neq \ker(I - z^p S^\theta)^k = \ker(I - z^p S^\theta)^{k+1}\) and the dimension of \(\ker(I - z^p S^\theta)^k\) is finite.

We shall prove that all the eigenvalues of \(S\) have the form \(\omega_j z^{-1}\). Indeed, \(S\) is an operator on \(\ker(I - z^p S^\theta)^k\). Then \(S\) admits a spectral decomposition on \(\ker(I - z^p S^\theta)^k\). Let \(u \neq 0\) and \(\lambda\) be such that \(u = \lambda Su\).

We prove the following formula:

\[ (I - z^p S^\theta)^k = (I - (I - z \omega_j S))^p = \left( p(I - z \omega_j S) + \sum_{\mu=2}^{p} C_\mu (I - z \omega_j S)^\mu \right)^k = p^k (I - z \omega_j S)^k \left( I + \sum_{\mu=1}^{p-1} C_\mu (I - z \omega_j S)^\mu \right)^k. \]

This implies \((I - z^p S^\theta)^k u = p^k (I - \omega_j z S)^k u = 0\), which is the claim.

Using the previous remark and estimate (43) we find that \(\|S^{\omega_0}\| = \|R_{z+\omega_0}\|_0\) is bounded uniformly with respect to \(r\), for \(r\) large enough, if \(\theta \neq 0\) and \(re^{i\theta} \notin C_{\epsilon}\). We prove the following formula:

\[ pe^{p-1} T_r = \sum_{k=1}^{p} \omega_k S_{\omega_0 z}. \]

Indeed, taking the inverse of (45) when the formula makes sense, we have

\[ (1 - z^p S^\theta)^{-1} = \prod_{j=1}^{p} (1 - \omega_j z S)^{-1}. \]

Differentiating (44) with respect to \(z\), we obtain

\[ pe^{p-1} = \sum_{k=1}^{p} \omega_k \prod_{j=1, j \neq k}^{p} (1 - \omega_j z). \]
Applying this formula to \( zS \), we obtain

\[
pe^{\bar{p}-1}S^{p-1} = \sum_{k=1}^{\rho} o_k \prod_{j=1, j \neq k}^{\rho} (1 - o_j zS).
\] (49)

We multiply term by term (49) and (48) and multiplying the result by \( S \) yields (47).

If \( S_{\infty}z \) is bounded uniformly for \( r \) large and for all \( k \), by (47) we have \( \| T_{r} \| \leq \frac{C}{r^{\frac{p}{2}}}. \)

If we assume that \( C \) is contained in a sector with angle less than \( \theta \) with \( \theta < \pi/2 \) and \( \theta < 2\pi/p \), the union of \( C \) and \( C \) rotated by angle \( 2k\pi/p \) does not yield \( C \) and formula (47) proves that we can find the \( \theta_j \)s satisfying the assumption of proposition 4.2. If \( p \geq 2 \), the estimate on \( \| T_{r} \|_0 \) is stronger than weak convergence. In the case \( p = 1 \), we have by theorem 10 and proposition 2.3 with the notation \( \bar{\mu} \) where

\[
\| R_{f}f_{\| T_{0}^{(2)} \| \leq L^{2}(\Omega)} \| H^{2}_{1}(\Omega) \cap \| v \in L^{2}(\Omega), \Delta v \in L^{2}(\Omega) \). \]

Let \((u, v) \in H^{2}_{1}(\Omega) \cap \{ v \in L^{2}(\Omega), \Delta v \in L^{2}(\Omega) \). We have \( R_{f}(u, v) = (f, g) \in L^{2}(\Omega) \oplus L^{2}(\Omega) \). Let \((f_{n}, g_{n}) \in \overline{T}(\Omega) \oplus L^{2}(\Omega) \) be such that \((f_{n}, g_{n}) \to (f, g) \) in \( L^{2}(\Omega) \oplus L^{2}(\Omega) \). We can take for instance \( f_{n} \) and \( g_{n} \) in \( \Omega(\Omega) \). We have by continuity

\[
\| R_{f}f_{\| T_{0}^{(2)} \| \leq L^{2}(\Omega)} \| H^{2}_{1}(\Omega) \cap \| v \in L^{2}(\Omega), \Delta v \in L^{2}(\Omega) \).
\]

\[
\| v \|_{L^{2}(\Omega)} + \| \Delta v \|_{L^{2}(\Omega)}\] on \( \{ v \in L^{2}(\Omega), \Delta v \in L^{2}(\Omega) \}\) and the usual norm on \( H^{2}_{1}(\Omega) \).

\( \square \)

**Proof of theorem 7.** Using (46) as \( z \) is fixed, estimating the number of eigenvalues less than \( r^{2} \) is equivalent to estimating the number of eigenvalues \( \lambda \) such that \( \lambda + \lambda \) is less than \( r^{2} \). In what follows, we estimate the number of \( \lambda \) less than \( r^{2} \) such that \( \lambda^{-1} \) is an eigenvalue of \( S = R_{f} \).

We have shown in the proof of theorem 5 that \( \| T_{r} \|_{0} \leq \frac{C}{r^{\frac{p}{2}}+1} \) if \( o_{k}z \) is on a radial line \( d(r_{0}, \theta) = \{ z \in \mathbb{C}, \ z = re^{\theta}, r \geq r_{0} \} \subset C \). As \( (1 - z^{2})^{-1} = 1 + z^{2}T_{r} \), we obtain for \( \| z \|_{1} \geq 1 \), \( \| (1 - z^{2})^{-1} \|_{0} \leq 1 + \| |z|^{2}/|T_{r}||_{0} \leq |Cz| \).

We have \( \| T_{r} \|_{2p} \leq \| T_{r} \|_{2p}(1 - z^{2})^{-1} \|_{0} \leq C |z| \). We obtain from lemma 4.1

\[
\| |T_{r}| \| \leq C \| T_{r} \|_{2p}^{1-n/(4p)} \| T_{r} \|_{0}^{1-n/(4p)} \leq C |z|^{1-p+n/4}.
\]

We recall [1, theorem 12.14] that if \( T \) is Hilbert–Schmidt, we have \( \sum |\mu_{j}|_{0}^{2} \leq \| |T||^{2} \) where \( \mu \neq 0 \) are the eigenvalues counted with multiplicities.

Let \( \lambda \) be such that \( \lambda^{-1} \) is an eigenvalue of \( S \); we find that \( 1/\lambda^{-1} \) is an eigenvalue of \( T_{r} \). We obtain

\[
\sum_{j} \frac{1}{|\lambda_{j}^{p} - z^{p}|}^{2} \leq \| |T_{r}| \|_{0}^{2} \leq C |z|^{2-2p+n/2}.
\]

If \( |\lambda_{j}| \) \( r^{2} \) and taking \( z \in d(r_{0}, \theta) \) satisfying \( |z| = r^{2} \), we have \( |\lambda_{j}^{p} - z^{p}| \leq 2r^{2p} \). Then we have

\[
\sum_{|\lambda_{j}| \leq r^{2}} \frac{1}{4^{4p}} \leq \sum_{|\lambda_{j}^{p} - z^{p}|^{2} \leq \| |T_{r}| \|_{0}^{2} \leq C r^{4-4p+n}. \]

Then we obtain \( N(t) \leq C t^{4+n} \).
5. Estimate on the resolvent

5.1. Upper bound

In this section, we prove theorem 8. We recall the well-known Green’s formula. For regular functions \(u\) and \(v\), we have
\[
\int_{\Omega} (u \Delta v - v \Delta u) \, dx = \int_{\partial \Omega} (u \partial_{\nu} v - v \partial_{\nu} u) \, ds,
\]
where \(\partial_{\nu}\) is the exterior normal derivative on \(\partial \Omega\) and \(ds\) is the surface measure on \(\partial \Omega\). Here we work with smooth functions. As the problem is well posed by theorem 1, we can apply the estimate for non-smooth functions by passing to the limit in the estimate:

\[
\begin{align*}
\Delta w + k^2 n(x) w &= f \quad \text{in } \Omega, \\
\Delta v + k^2 v &= g \quad \text{in } \Omega, \\
w &= v \quad \text{on } \partial \Omega, \\
\partial_{\nu} w &= \partial_{\nu} v \quad \text{on } \partial \Omega.
\end{align*}
\]

(50)

By Green’s formula and (50) we have
\[
\int_{\Omega} (w \Delta \tilde{w} - \tilde{w} \Delta w) \, dx = \int_{\Omega} (w \partial_{\nu} \tilde{w} - \tilde{w} \partial_{\nu} w) \, ds
\]
\[
= \int_{\Omega} [w(f - k^2 \tilde{n} \tilde{w}) - \tilde{w}(f - k^2 n w)] \, dx
\]
\[
= \int_{\Omega} [w \tilde{f} - \tilde{w} f - 2ik \text{Im } n |w|^2] \, dx,
\]
and
\[
\int_{\Omega} (v \Delta \tilde{v} - \tilde{v} \Delta v) \, dx = \int_{\Omega} (v \partial_{\nu} \tilde{v} - \tilde{v} \partial_{\nu} v) \, ds
\]
\[
= \int_{\Omega} [v(g - k^2 \tilde{v}) - \tilde{v}(g - k^2 v)] \, dx
\]
\[
= \int_{\Omega} [v \tilde{g} - \tilde{v} g] \, dx.
\]

Using the boundary conditions in (50), we obtain
\[
\int_{\Omega} (w \tilde{f} - \tilde{w} f) \, dx - \int_{\Omega} (v \tilde{g} - \tilde{v} g) \, dx = 2i \int_{\Omega} k^2 |w|^2 \text{Im } n \, dx.
\]

Thus, we deduce
\[
\delta \int_{\omega} k^2 |w|^2 \leq \|v\| \|g\| + \|w\| \|f\|, \tag{51}
\]

where \(\omega = \{x \in \Omega, \text{ Im } n(x) \geq \delta\} \).

Remark 9. In the case where \(n = n_1 + in_2/k\), we have \(\text{Im } n = n_2/k\), and in the previous computations we must change the left-hand side of (51) to \(\delta \int_{\omega} k |w|^2\) where \(\omega = \{x \in \Omega, n_2(x) \geq \delta\}\). We leave the reader to check that the rest of the proof does not change with this new estimate. Indeed, the powers of \(k\) do not play any role with respect to the estimates by \(e^{Ck}\).

We recall the interpolation estimate. We can find this type of estimate in [23, section 3, formulas (1) and (2)], [24, theorem 3] and [11, proposition 1.2]. Estimate (52) below does not appear in these references, but we can prove it by following the same approach. Indeed,
in the Carleman estimate used to prove the interpolation estimates, we estimate also the boundary terms but in the previous mentioned paper we did not need the boundary term in the interpolation estimates.

Let \( X = (-3, 3) \times \Omega, Y = (-2, 2) \times \Omega, \) and \( \Omega = (-1, 1) \times \omega. \) We set \( \partial Y = (-2, 2) \times \partial \Omega. \) Then there exist \( \delta > 0 \) and \( C > 0 \) such that for all \( W \in \mathcal{H}^1(X) \) so that \( \partial^2_\omega W + \Delta W \in L^2(X), \) \( W|_{\partial Y} \in H^1(\partial Y), \) \( \partial_\nu W|_{\partial Y} \in L^2(\partial Y), \) we have

\[
\parallel W \parallel^2_{\mathcal{H}^1(Y)} + \| W \|_{\partial^2_\omega L^2(\partial Y)} + \| \partial_\nu W \|_{L^2(\partial Y)} \leq C(\| \partial^2_\omega W + \Delta W \|_{L^2(X)} + \| W \|_{L^2(\partial Y)}),
\]

(52)

\[
\parallel W \parallel^2_{\mathcal{H}^1(Y)} \leq C(\| \partial^2_\omega W + \Delta W \|_{L^2(X)} + \| W \|_{L^2(\partial Y)}),
\]

(53)

where \( s \) is an additional variable. This variable allows us to give an estimate uniform with respect to the large parameter \( k. \) We shall see that in the following.

Let \( W(s, x) = e^{sk}w(x) \) where \( \Delta w + k^2w = f \) in \( \Omega. \) We have \( \partial^2_\omega W + \Delta W = e^{sk}f \) and we can obtain the following estimates for a \( C > 0: \)

\[
\parallel w \parallel_{\mathcal{H}^1(\Omega)} \leq C\parallel W \parallel_{\mathcal{H}^1(Y)},
\]

\[
\| w \|_{\partial^2_\omega L^2(\partial \Omega)} \leq C\| W \|_{\partial^2_\omega L^2(\partial Y)},
\]

\[
\| \partial_\nu W \|_{L^2(\partial Y)} \leq C\| \partial_\nu W \|_{L^2(\partial Y)},
\]

\[
\| \partial^2_\omega W + \Delta W \|_{L^2(X)} \leq Ce^{2k}\| f \|_{L^2(\partial \Omega)},
\]

\[
\| W \|_{L^2(\partial \Omega)} \leq Ce^{k}\| w \|_{L^2(\partial \Omega)},
\]

\[
\| W \|_{\mathcal{H}^1(\Omega)} \leq Ce^{k}\| w \|_{\mathcal{H}^1(\partial \Omega)}.
\]

(54)

By the interpolation estimate (52), there exists \( C > 0 \) such that for all \( w \in \mathcal{H}^1(\Omega), \) satisfying \( \partial_\nu w|_{\partial \Omega} \in L^2(\partial \Omega) \) and \( w|_{\partial \Omega} \in H^1(\partial \Omega) \) solution of \( \Delta w + k^2w = f \) in \( \Omega, \) we have

\[
\| (\partial_\nu w)|_{\partial \Omega} \|_{L^2(\partial \Omega)} + \| w|_{\partial \Omega} \|_{H^1(\partial \Omega)} \leq Ce^{2k}(\| w\|_{L^2(\partial \Omega)} + \| f \|).
\]

Using (51) and (54) and \( Ce^{2k}\| w\|_{\mathcal{H}^1(\partial \Omega)} \leq (1/2)\| w\|^2 + C^2e^{2k}\| f\|^2, \) we have for a \( C > 0, \)

\[
\| w\|^2_{\mathcal{H}^1(\partial \Omega)} \leq Ce^{2k}(\| f\|^2 + \| w\|_{\mathcal{H}^1(\partial \Omega)}).
\]

(55)

Following the same way, we denote \( W(s, x) = e^{sk}v(x) \) and apply the interpolation estimate (53). We then obtain on \( v \) the estimate

\[
\parallel v \parallel_{\mathcal{H}^1(\Omega)} \leq Ce^{2k}(\| g \| + \| v\|_{H^1(\partial \Omega)} + \| \partial_\nu v|_{\partial \Omega} \|_{L^2(\partial \Omega)}).
\]

(56)

Taking into account the boundary condition in (50) and (54) we have

\[
\| v\|^2_{\mathcal{H}^1(\Omega)} \leq Ce^{2k}(\| g \|^2 + \| v\|_{H^1(\partial \Omega)} + \| \partial_\nu v|_{\partial \Omega} \|^2_{L^2(\partial \Omega)})
\]

\[
\leq Ce^{2k}(\| g \|^2 + \| f\|^2 + \| v\|_{\mathcal{H}^1(\partial \Omega)} + \| W\|^2_{\mathcal{H}^1(\partial \Omega)})
\]

(57)

This estimate and (55) give

\[
\parallel v \parallel_{\mathcal{H}^1(\Omega)} + \| w\|_{\mathcal{H}^1(\partial \Omega)} \leq Ce^{2k}(\| f\| + \| g \|).
\]

This implies the estimate on the \( L^2 \) norm on \( v \) and \( w, \) which gives theorem 8 by density.
5.2. Lower bound

Here we use the results proved first by Davies [12] in one dimension, by Zworski [28] for Schrödinger operators in the $n$ dimension and by Dencker et al [13] for more general subelliptic operators. This allows one to obtain a lower bound on the resolvent.

We recall here the theorem given by Dencker, Sjöstrand and Zworski.

**Theorem 11** (theorem 1.1 [13]). Let $V \in \mathcal{C}^\infty(\mathbb{R}^n)$. Then, for any $z \in \{\xi^2 + V(x), \ (x, \xi) \in \mathbb{R}^n, \ \text{Im}(\xi, \partial_x V(x)) \neq 0\}$, there exists $h_0 > 0$ such that for all $h \in (0, h_0)$, there exists $u(h) \in L^2(\mathbb{R}^n)$ with the property

$$
\| (-h^2 \Delta + V(x) - z) u(h) \| = O(h^\infty) \| u(h) \|.
$$

In addition, $u(h)$ is microlocally localized at a point $(x_0, \xi_0)$ in phase with $\xi_0^2 + V(x_0) = z$.

More precisely, $WF_h(u) = \{(x_0, \xi_0)\}$, where $WF_h(u)$ is the semi-classical wave front set. If the potential is real analytic, then we can replace $h^\infty$ by $\exp(-1/Ch)$.

As a consequence of the microlocal localization of $u$, we can cut off $u$ such that its support is in a neighborhood of $x_0$. Theorem 11 implies that if $z$ is in the resolvent set, $\| (-h^2 \Delta + V(x) - z)^{-1} \| \geq C_N h^{-N}$ for all $N$ in the $\mathcal{C}^\infty$ case and $\| (-h^2 \Delta + V(x) - z)^{-1} \| \geq C \exp(C/\hbar)$ in the analytic case.

**Proof of theorem 9.** We set $z_0 = -(1 + m(x_0))^{-1}\vert \xi_0 \vert^2$ and $V(x) = z_0(1 + m(x))$. We have $\| \xi_0 \|^2 + V(x_0) = 0$ and

$$
\text{Im} (\xi_0 \partial_x V(x_0)) = \text{Im} (z_0 \xi_0 \partial_x m(x_0)) = -|1 + m(x_0)|^{-2} \xi_0^2 \text{Im} ((1 + m(x_0))\xi_0 \partial_x m(x_0)) = -|1 + m(x_0)|^{-2} \xi_0^2 \text{Im} (\xi_0 \partial_x n(x_0)) \neq 0,
$$

by assumption. By theorem 11 there exists $u(h)$ such that $\| (-h^2 \Delta + V(x)) u(h) \| = O(h^\infty) \| u(h) \|$ or $= O(e^{-C/(\hbar)}) \| u(h) \|$ if $m$ is analytic. Setting $f = \Delta u(h) - h^{-2}z_0(1 + m(x))u(h)$, we have $R_{Z_0} (f, 0) = (u(h), 0)$ with $k_2 = -h^{-2}z_0$. We remark that $u(h)$ is localized in a neighborhood of $x_0$, in particular $u(h)$ is null in a neighborhood of $\partial \Omega$, then $(u(h), 0)$ satisfies the boundary conditions. This implies the result of theorem 9. \qed

### Appendix. Notation and basic properties of pseudo-differential calculus

#### A.1. Sobolev spaces and pseudo-differential operators

We introduce some notation for Sobolev spaces.

We denote the semi-classical $H^s$ norm by $\| u \|_{H^s} = \int (1 + h^2 |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi$. On a compact manifold, we define the semi-classical $H^s$ using local coordinates. To distinguish the norm on spaces of dimension $n$ and dimension $n - 1$, we denote the semi-classical $H^s$ norm on $\partial \Omega$ by $\| \cdot \|_{H^s(\partial \Omega)}$. Let $u$ be a distribution on $\Omega$, we denote $\| u \|_{H^s(\Omega)} = \inf \{ \| \beta \|_{H^s(\Omega)} : \beta|_{\partial \Omega} = u \}$.

We recall that we denote $D = (h/\iota) \partial$, and if $s$ is an integer, the quantity $\sum_{|\alpha| \leq s} \| D^\alpha u \|^2_{L^2(\Omega)}$ is equivalent to $\| u \|_{H^s(\Omega)}^2$, uniformly with respect to $h$.

In the context of semi-classical spaces $H^s$, we have the following trace formula, for $s > 0$, for all $w \in \overline{H^{s+1/2}}(\Omega)$:

$$
|w|_{\partial \Omega} n_{H^s(\partial \Omega)} \lesssim h^{-1/2} \| u \|_{H^{s+1/2}(\partial \Omega)},
$$

where $|w|_{\partial \Omega}(x_0)$ means the limit of $w(x)$ when $x \in \Omega$ goes to $x_0$. This definition makes sense if $w$ is a $\mathcal{C}^\infty$ function and (A.1) allows one to extend this definition for function in $\overline{H^{s+1/2}}(\Omega)$. 

$22$
We recall some facts on pseudo-differential operators. Let \( a(x, \xi) \) be in \( \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n) \); we say that \( a \) is a symbol of order \( m \) if for all \( \alpha, \beta \in \mathbb{N}^n \), there exist \( C_{\alpha, \beta} > 0 \) such that
\[
|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha, \beta} |\xi|^{m-|\beta|}.
\]
where \( |\xi|^2 = 1 + |\xi|^2 \). In particular, a polynomial in \( \xi \) of order \( m \) with coefficients in \( \mathcal{C}^\infty(\mathbb{R}^n) \) with bounded derivatives of all orders, is a symbol of order \( m \).

With a symbol we can associate a semi-classical operator by the following formula:
\[
\text{Op}(a)u = a(x, D)u = \frac{1}{(2\pi)^n} \int e^{i\xi \cdot x} a(x, \xi) \hat{u} (\xi) \, d\xi = \frac{1}{(2\pi)^n} \int e^{ix \cdot h} a(x, \xi) \hat{u}(\xi/h) \, d\xi.
\]
This formula makes sense for \( u \in \mathcal{S}'(\mathbb{R}^n) \) and we can extend it to \( u \in H^s \) for all \( s \). For \( a \), a symbol of order \( m \), there exists \( C > 0 \) such that for all \( u \in H^s \),
\[
\|a(x, D)u\|_{H^{s-m}} \leq C\|u\|_{H^s}.
\]

We can compose pseudo-differential operators and we have the following result. Let \( a \) be a symbol of order \( m \) and \( b \) be a symbol of order \( k \), there exists \( c \) a symbol of order \( m + k \) such that \( a(x, D) \circ b(x, D) = c(x, D) \). Moreover there exists a symbol \( d \) of order \( m + k - 1 \) such that \( c(x, D) = (ab)(x, D) + hd(x, D) \). This means the composition of two operators is the operators associated with the product of symbols, up to a remainder with a factor one order lower and with an additional factor \( h \).

We can invert elliptic symbols. More precisely, let \( a \) be a symbol of order \( m \) satisfying for some \( C > 0 \), \( |a(x, \xi)| \geq C|\xi|^m \) for all \( (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \). Then for all \( N > 0 \) there exist \( b \) a symbol of order \(-m\) and \( r \) a symbol of order \(-N\) such that \( b(x, D) \circ a(x, D) = I + h^N r(x, D) \). We can localize this result. Let \( K \) be a closed set of \( \mathbb{R}^n \), and let us assume that there exists \( C > 0 \) such that for all \( (x, \xi) \in K \times \mathbb{R}^n \), \( |a(x, \xi)| \geq C|\xi|^m \). Then for all \( \chi \in \mathcal{C}^\infty_0(\mathbb{R}^n) \) supported in \( K \), there exists \( b \) a symbol of order \(-m\) and \( r \) a symbol of order \(-N\) such that \( b(x, D) \circ a(x, D) = \chi(x) + h^N r(x, D) \). In both cases we say that \( b \) is a parametrix for \( a \).

We can also define pseudo-differential operators on a smooth compact manifold without boundary. We shall use freely the result on \( \mathbb{R}^n \) in the context of manifolds. To distinguish both cases, we denote by \( \text{Op} \) the operators on \( \mathbb{R}^n \) and by \( \text{op} \) the operators on a manifold of dimension \( n - 1 \) or on \( \mathbb{R}^{n-1} \supseteq \{x \in \mathbb{R}^n, x_n = 0\} \).

We use also spaces \( \overline{H}_{sc}^s \) and the pseudo-differential calculus on these spaces. In general, this requires the introduction of the delicate notion of ‘transmission condition’ (see [2]). To avoid this difficulty we follow Hörmander’s strategy (see [18, appendix B]) adapted for parametrices which are particular cases of operators satisfying the ‘transmission condition’. We recall some results proved by Hörmander in the context of classical \( \overline{H} \) spaces. The adaptation to the \( \overline{H}_{sc} \) spaces is easy and we give here only the results and some ideas of proof. Here we give the result in a half space \( \mathbb{R}^n_+ = \{x \in \mathbb{R}^n, x_n > 0\} \). For simplicity we denote \( \overline{H}_{sc}^s = \overline{H}_{sc}^s(\mathbb{R}^n_+) \).

In the proof we need to introduce a space \( \overline{H}_{sc}^{m,s} \). First we say that \( u \in \overline{H}_{sc}^{m,s}(\mathbb{R}^n) = \overline{H}_{sc}^{m,s} \) if \( \|u\|_{\overline{H}_{sc}^{m,s}}^2 = \int (\xi^2)^m (\xi^2)^s |\hat{u}(\xi)|^2 \, d\xi < \infty \) where \( \xi = (\xi', \xi_n) \). As for the \( H^s \) space, we say that \( u \in \overline{H}_{sc}^{m,s} \) where \( u \) is a distribution in \( \mathcal{D}'(\mathbb{R}^n_+) \) if there exists \( v \in \overline{H}_{sc}^{m,s} \) such that \( u = v_{|x_n>0} \) and we denote \( \|u\|_{\overline{H}_{sc}^{m,s}}^2 = \inf \|v\|_{\overline{H}_{sc}^{m,s}}^2 \), where \( v_{|x_n>0} = u \).

We can easily see that if \( u \in \overline{H}_{sc}^{0,s} \) then \( u \in \overline{H}^{0,s} \) and \( \|u\|_{\overline{H}_{sc}^{m,s}} = \|u\|_{\overline{H}^{m,s}} \) (see (16) for the definition of \( \mu \)).

From the definition we have \( \|u\|_{\overline{H}_{sc}^{m-1,s+1}} \leq \|u\|_{\overline{H}_{sc}^{m,s}} \). The inverse estimate is false in general but we can extend theorem B.2.3 given by Hörmander [18], and we have
\[
\begin{align*}
 u \in \overline{H}_{sc}^{m,s} & \iff D_n u \in \overline{H}_{sc}^{m-1,s} \quad \text{and} \quad u \in \overline{H}_{sc}^{m-1,s+1} \\
 & \iff D_n^2 u \in \overline{H}_{sc}^{m-2,s} \quad \text{and} \quad D_n u \in \overline{H}_{sc}^{m-2,s+1} \quad \text{and} \quad u \in \overline{H}_{sc}^{m-2,s+2}. \quad (A.2)
\end{align*}
\]
Of course the natural norms on these spaces are equivalent.
We can use theorem B.2.9 from [18] in the following form adapted to our context. We denote by \( P = D^2_{x_4} + R(x, D') + \alpha(x) D_{x_4} \) a differential operator of second order. We have for all \( k \in \mathbb{R}, \)
\[
u \in \mathcal{T}^{-k,s+k}_{sc} \quad \text{and} \quad Pu \in \mathcal{T}^{-s, s}_{sc} \Rightarrow u \in \mathcal{T}^{-m,s}_{sc}. \tag{A.3}
\]
Of course this is trivial if \( k \leq 0 \). For \( k > 0 \), we can prove this by induction on \( k \). The idea is the following: we can write \( D^2_{x_4} u = Pu - R(x, D') u - \alpha D_{x_4} u \) and for \( k = 1 \) this formula implies that \( D^2_{x_4} u \in \mathcal{T}^{-m,2}_{sc} \). Then we can apply (A.2) to obtain the result in this case. The induction is then straightforward.

The previous results are useful to prove that the parametrix of an elliptic operator is a mapping on the \( \mathcal{T}_{sc} \) space.

Let \( P \) be a differential operator of second order, elliptic, i.e. there exists \( C > 0 \) such that \( \forall x \in \mathbb{R}^n, \forall \xi \in \mathbb{R}^n, |\rho(x, \xi)| \geq C(\xi)^2 \), and let \( Q \) be a parametrix such that \( PQ = Id + h^N K \) where \( K \) is of order \(-N\) where \( N > 0 \). For a distribution \( w \) in \( \mathbb{R}^n \), we denote \( rw = w_{|t > 0} \). The action of \( Q \) on \( \mathcal{T}_{sc} \) is given by the formula \( rQw \) which makes sense if \( u \) makes sense (it is the case if \( u \) is in \( L^2(\Omega) \)). We have the following result. If \( s \in [0, N] \), then there exists \( C > 0 \) such that for all \( u \in \mathcal{T}_{sc}^s \) and we have
\[
\|rQu\|_{\mathcal{T}_{sc}^{-s}} \leq C\|u\|_{\mathcal{T}_{sc}^s}. \tag{A.4}
\]
We remark that here \( u \) is at least in \( L^2 \) so \( rQu \) makes sense. First we prove that \( rQu \in \mathcal{T}_{sc}^{1/2-k \kappa} \) where \( k \geq s \). It is enough to prove that for all \( \alpha \in \mathbb{N}^{n-1}, |\alpha| \leq k, D^\alpha \mathcal{T}_{sc}^s \leq \mathcal{T}_{sc}^{1/2-k \kappa} \). By pseudodifferential calculus we have \( D'Q = QD' + [D', Q] \) where \([D', Q]\) is a pseudo-differential operator of order \(-2\). By induction we have \( D^\alpha \mathcal{T}_{sc}^s = \sum_{|\beta| \leq |\alpha|} Q^\beta D^\beta \mathcal{T}_{sc}^s \), where \( Q^\beta \) is of order \(-2\). We have
\[
\|rD^\alpha \mathcal{T}_{sc}^s \leq \sum_{|\beta| \leq |\alpha|} \|Q^\beta D^\beta \mathcal{T}_{sc}^s \leq C \sum_{|\beta| \leq |\alpha|} \|D^\beta \mathcal{T}_{sc}^s \leq \|u\|_{\mathcal{T}_{sc}^s} \leq C\|u\|_{\mathcal{T}_{sc}^s}.
\]

It is well known that we also have \( P \mathcal{T}_{sc}^s = \mathcal{T}_{sc}^s \) where \( K \) is of order \(-N\). Indeed, there exists a \( \tilde{Q} \) such that \( P = \tilde{Q} + hK \) where \( K \) is of order \(-N\) and it is easy to prove that \( Q = \tilde{Q} + hK \) where \( K \) is of order \(-N\). Thus we have, as \( P \) is a differential operator, \( PrQu = rPQu = ru + hrKu = u + hrKu \). As \( \|ru\|_{\mathcal{T}_{sc}^s} \leq C\|u\|_{\mathcal{T}_{sc}^s} \leq \|u\|_{\mathcal{T}_{sc}^s} \), we obtain for \( s \in [0, N] \), \( \|rPrQu\|_{\mathcal{T}_{sc}^s} \leq C\|u\|_{\mathcal{T}_{sc}^s} \). As we have \( rQu \in \mathcal{T}_{sc}^{1/2-k \kappa} \) and \( PrQu \in \mathcal{T}_{sc}^s \), then from (A.3) this implies \( \|rQu\|_{\mathcal{T}_{sc}^{1/2-k \kappa}} \leq C\|u\|_{\mathcal{T}_{sc}^s} \).

We need also regularity results for \( rQ(\gamma \otimes D^k \delta_{x_4 = 0}) \), where \( \gamma \in H^s_{sc} (\mathbb{R}^{n-1}) \). First we remark \( h^{2k} |\xi_x|^2 (1 + h^2 |\xi_x|^2 + |\xi|^2)^{\gamma} d\xi_x \leq C(\xi)^{2k+2s+1} / h \) if \( \sigma + k < -1/2 \), then by direct computation we have for all \( \gamma \in H^s_{sc} (\mathbb{R}^{n-1}) \), \( \gamma \otimes D^k \delta_{x_4 = 0} \|_{H^{s+k+1/2} \mathcal{T}_{sc}^s} \leq \frac{C}{h^{1/2}} \|\gamma\|_{H^{s} (\mathbb{R}^{n-1})} \), if \( v < 0 \). For \( j \in \mathbb{N} \), we have if \( s < j \), following the same computation as for computing \( rQu \),
\[
\|Q(\gamma \otimes D^k \delta_{x_4 = 0})\|_{H^{s-j+1/2} \mathcal{T}_{sc}^s} \leq C \|\gamma \otimes D^k \delta_{x_4 = 0}\|_{H^{s-j+1/2} \mathcal{T}_{sc}^s} \leq C \|\gamma\|_{H^{s} (\mathbb{R}^{n-1})}.
\]
As \( r(\gamma \otimes D^k \delta_{x_4 = 0}) = 0 \) we have \( PrQ(\gamma \otimes D^k \delta_{x_4 = 0}) = hrK(\gamma \otimes D^k \delta_{x_4 = 0}) \). We deduce,
\[
\|PrQ(\gamma \otimes D^k \delta_{x_4 = 0})\|_{\mathcal{T}_{sc}^{s-j+1/2} \mathcal{T}_{sc}^s} \leq C \|\gamma \otimes D^k \delta_{x_4 = 0}\|_{H^{s-j+1/2} \mathcal{T}_{sc}^s} \leq C \|\gamma\|_{H^{s} (\mathbb{R}^{n-1})},
\]
if \( s < k < N \). Thus from (A.3) we obtain, if \( s - k < N \),
\[
\|rQ(\gamma \otimes D^k \delta_{x_4 = 0})\|_{\mathcal{T}_{sc}^{s-k+1/2} \mathcal{T}_{sc}^s} \leq C \|\gamma\|_{H^{s} (\mathbb{R}^{n-1})}, \tag{A.5}
\]
In section 2.3, we apply the previous result to a local parametrix. Indeed, we can construct the local parametrix with a global parametrix. We can extend $P$ to have a global elliptic operator $\hat{P}$ such that $\hat{P} = P$ in a domain $W$ where $P$ is elliptic. Let $Q$ be a parametrix of $P$ such that $\hat{Q}P = I + dK$ where $K$ is an operator of order $-N$. Let $\chi_1$ and $\chi_2$ be functions in $\mathcal{C}^\infty$ compactly supported in $W$ such that $\chi_2 = 1$ on the support of $\chi_1$. By pseudo-differential calculus, we have $\chi_1 \hat{Q}_2 = \chi_1 Q + hK'$ where $K'$ is an operator of order $-N - 2$. Then we have $\chi_1 \hat{Q} = \chi_1 + h\chi_1 K$ and $\chi_1 \hat{Q}P = \chi_1 \hat{Q}_2 P = -hK'\hat{P}$. As $\hat{P} = P$ on the support of $\chi_2$ we have $\chi_1 \hat{Q}_2 P = \chi_1 + hK''$ where $K''$ is an operator of order $-N$. Then $\chi_1 \hat{Q}_2$ is a local parametrix of $P$. It is easy to see that we can replace $Q$ by $\chi_1 \hat{Q}_2$ in (A.4) and in (A.5).

A.2. Properties on the roots and parametrices

We use some properties of the roots of $\xi^2 + R(x, \xi') - \mu$ and $A(\xi^2 + R(x, \xi')) - \mu$. By assumption these polynomials have no real roots and it is easy to see that for $\xi'$ large enough the imaginary parts have different signs. In particular, the roots are simple thus smooth and the roots are symbols of order 1. Actually, for instance for $\xi^2 + R(x, \xi') - \mu$, (the proof for $A(\xi^2 + R(x, \xi')) - \mu$ is similar and left to the reader) the roots have, for $|\xi'|$ large enough, the following form $\pm i\sqrt{R(x, \xi') + z_\pm(\mu, 1/\sqrt{R(x, \xi')})}$, where $z_\pm$ is a solution to $z_\pm \mp iz_\pm^2 s/2 + i\mu s/2 = 0$ in a neighborhood of $s = 0$. This expression implies that the roots are symbols of order 1.

The parametrices used, denoted $Q$ and $\hat{Q}$, have the particular structure we give here. The symbol of $P$ is a polynomial having the following form, $p_2 + hp_1$, where $p_j$ are polynomials of degree $j$. We seek a parametrix with symbol given formally by $Q = q_{-2} + hq_{-1} + h^2 q_{-2} + \cdots$, where $q_j$ are symbols of order $j$. If we denote by $q \circ p$ the asymptotic expansion of the symbol of $\text{Op}(q)\text{Op}(p)$, we have

$$q \circ p = q_{-2} + \sum_{k=0}^{\infty} \frac{h^{k-j+|\alpha|}}{\alpha!|\alpha|} \partial^\alpha_q q_{-k} p_j,$$

where in the sum we have $j = 2$ or $1$, $k \geq 2$, $\alpha \in \mathbb{N}^n$ and $|\alpha| \geq 1$ or $j = 1$. In particular, we have $k - j + |\alpha| \geq 1$. We choose $q_{-2} = 1/p_2$, and to cancel the terms with the same power in $h$ we have

$$q_{-v-2} = \frac{1}{p_2} \sum_{\alpha \in \mathbb{N}^n} \frac{1}{|\alpha|!} \partial^\alpha_q q_{-\alpha} p_j,$$

where $v \geq 1$, $k = j + |\alpha| = v$, $j = 2$, $k \geq 2$, $\alpha \in \mathbb{N}^n$, $|\alpha| \geq 1$ or $j = 1$. In particular, the sum is finite and $k \leq v + 1$. We claim that

$$q_{-v} = S_{v-6}/p_{v+3},$$

for $v \geq 2$,

where $S_{\mu}$ is a polynomial of degree $\mu$.

Clearly this is true for $v = 2$. We check that for $k \leq v + 1$, $\partial^\alpha_q q_{-\alpha} = S_{v-\alpha}/p_{v+\alpha}$, where $S_{\mu}$ is a polynomial of degree $\mu$. The parameters satisfy $j \leq 2$, $k - j + |\alpha| = v$ and $k \leq v + 1$, then the power of $p_2$ in (A.6) is $2k - 2 + |\alpha| = v + k - 2 \leq 2v + j - 1 \leq 2v + 1 = 2(v + 2) - 3$. The degree of the numerator is $3k - 6 + |\alpha| + 2j = v + 2k - 6 + 2j \leq 3v - 4 + 2j \leq 3v = 3(v + 2) - 6$. This gives the claim.

We need to compute for $\gamma(x')$,

$$\left[ Q \left( \frac{h}{i} \gamma \otimes D_x \delta_{\epsilon=0} \right) \right]_{\epsilon=0} = \frac{1}{(2\pi h)^{2\alpha}} \int e^{i\xi' x/h} \tilde{q}(x', \xi') \hat{\gamma}(\xi'/h) \, d\xi',$$

(7)

where formally $\tilde{q}(x', \xi') = \left( \frac{1}{2\pi h} \int_\mathbb{R} e^{ix' x/h} q(x, \xi) \xi^\alpha \, d\xi \right)_{\epsilon=0}$. It is not clear that $\tilde{q}$ is well defined in general but in the following lemma we prove this is true if $q$ is a rational function, and in this case (A.7) make sense.
Lemma A.1. Let \( v \in \mathbb{N}^* \), let \( S_v(x, \xi) \) be a polynomial of order \( v \) with respect to \( \xi \) and we assume that the coefficient of \( \xi^v \) is a symbol in \( \xi' \) of order \( v - j \). Let \( p \) be a polynomial of degree \( d \) in \( \xi \) and a symbol of order \( d \). We assume that \( p(x, \xi) = \xi^d + \sum_{j=0}^{d} \xi^d_a a_d^{(j)} (x, \xi') \) where \( a_d^{(j)} \) are polynomials of order \( d - j \). Moreover, we assume there exists \( \delta > 0 \) such that \( \forall x \in \mathbb{R}^n, \forall \xi \in \mathbb{R}^m, |p(x, \xi)| \geq \delta |\xi|^d \).

Then
\[
\left( \int_{\mathbb{R}} e^{i \xi_n} S_v(x, \xi) \frac{d\xi_n}{p(x, \xi)} \right)_{|\xi_n=0} \,,
\]
is a symbol of order \( v - d + 1 \).

Proof. The integral \( \int_{\mathbb{R}} e^{i \xi_n} S_v(x, \xi) \frac{d\xi_n}{p(x, \xi)} \) converges for \( x_n > 0 \). For \( (x, \xi') \) fixed, we can change the integration contour by \( \Gamma = \{ -D(\xi')D(\xi') \} \cup \{ z \in \mathbb{C}, |z| = D(\xi') \} \). \( \Im z > 0 \), where \( D \) will be chosen below. Indeed, the integral does not depend on \( D \) if \( D \) is large enough and the integral on \( \{ z \in \mathbb{C}, |z| = D(\xi') \} \), \( \Im z > 0 \) goes to \( 0 \) as \( D \) goes to \( +\infty \). Now we integrate on a compact set and we can take the limit \( x_n \to 0^+ \). We have the following quantity to control:
\[
A = \left( \int_{\Gamma} S_v(0, x', \xi) \frac{d\xi_n}{p(0, x', \xi)} \right) .
\]
On \( \Gamma \) we have \( |S_v(0, x', \xi)| \leq C |\xi'|^v \). For \( \xi \in \{ z \in \mathbb{C}, |z| = D(\xi') \} \), \( \Im z > 0 \) we have \( |p(0, x', \xi)| \geq |\xi|^d + \sum_{j=0}^{d-1} |\xi^d_a| |a_d^{(j)}(0, x', \xi')| \geq D^d(\xi')^d(1 - D^{-1}C) \geq |\xi'|^d D^d / 2 \), where we have \( |a_d^{(j)}(0, x', \xi')| \leq C |\xi'|^{d-j} \) and we choose \( D \) such that \( D \geq \max(1, 2dC) \). Then by assumption for all \( \xi \in \Gamma \) we have \( |p(0, x', \xi)| \geq \delta |\xi'| \). As the length of \( \gamma \) is less than \( K(\xi') \), we obtain \( A \leq K'(\xi')^v \delta^{d+1} \). We can obtain the estimates on the derivative by the same approach because we can differentiate \( A \) and we obtain the same type of quantities to estimate.

Now we compute the boundary symbol obtained in (23) and (26).

Lemma A.2. Let \( k = 0, 1 \) and \( \Im \rho_1 > 0, \Im \rho_2 < 0 \), we have
\[
\left( \int_{\mathbb{R}} e^{i \xi_n} \frac{\xi_n^k}{(\xi_n - \rho_1)(\xi_n - \rho_2)} \frac{d\xi_n}{p(0, x', \xi)} \right)_{|\xi_n=0} = 2i\pi \frac{\rho_1^k}{\rho_1 - \rho_2} .
\]

Proof. As in the proof of lemma A.1, we can integrate on \( \Gamma \) and this integral is equal to \( 2i\pi \) times the residue at \( \rho_1 \). It is easy to see that the residue is \( \frac{\rho_1^k}{\rho_1 - \rho_2} \).\qed

Lemma A.3. Let \( k = 0, 1 \) and \( \Im \lambda_1 > 0, \Im \lambda_2 < 0, \Im \rho_1 > 0, \Im \rho_2 < 0 \), we have
\[
\left( \int_{\mathbb{R}} e^{i \xi_n} \frac{\xi_n^k}{(\xi_n - \lambda_1)(\xi_n - \lambda_2)(\xi_n - \rho_1)(\xi_n - \rho_2)} \frac{d\xi_n}{p(0, x', \xi)} \right)_{|\xi_n=0} = 2i\pi A_k .
\]

where
\[
A_k = \begin{cases} 
\frac{\rho_2 - \rho_1 + \lambda_2 - \lambda_1}{(\lambda_1 - \lambda_2)(\lambda_1 - \rho_1)(\rho_1 - \rho_2)}, & \text{if } k = 0 \\
\frac{\lambda_2 \rho_2 - \lambda_1 \rho_1}{(\lambda_1 - \lambda_2)(\lambda_1 - \rho_1)(\rho_1 - \rho_2)}, & \text{if } k = 1 .
\end{cases}
\]

Proof. Clearly, both sides of the equality are continuous with respect to \( (\lambda_1, \rho_1) \); then it is sufficient to prove the case \( \lambda_1 \neq \rho_1 \).
As in the proof of lemma A.1, we can integrate on $\Gamma'$ and the result is $2\pi \sigma$ times the sum of the residues in the half plane $\text{Im } z > 0$. We obtain

$$\frac{\lambda_k^1}{(\lambda_1 - \lambda_2)(\lambda_1 - \rho_1)(\lambda_1 - \rho_2)} + \frac{\rho_k^1}{(\rho_1 - \lambda_1)(\rho_1 - \lambda_2)(\rho_1 - \rho_2)} = \frac{\lambda_k^1(\rho_1 - \lambda_2)(\rho_1 - \rho_2) - \rho_k^1(\lambda_1 - \lambda_2)(\lambda_1 - \rho_2)}{(\lambda_1 - \lambda_2)(\lambda_1 - \rho_1)(\lambda_1 - \rho_2)(\rho_1 - \lambda_2)(\rho_1 - \rho_2)}. $$

Clearly, the numerator vanishes if $\rho_1 = \lambda_1$. By a straightforward computation, if $k = 0$, the numerator is $(\lambda_1 - \rho_1)(\rho_2 - \rho_1 + \lambda_2 - \lambda_1)$. If $k = 1$, the numerator is $(\lambda_1 - \rho_1)(\lambda_2 \rho_2 - \lambda_1 \rho_1)$.

This gives the result of the lemma. $\square$

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