DECOUPLED PATH INTEGRAL FORMULATION
OF CHIRAL QCD$_2$ WITH $a_{JR} = 2$

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Abstract

We analyse the BRST constraints and corresponding Hilbert-space structure of chiral QCD$_2$ in the decoupled formulation for the case of the Jackiw-Rajaraman parameter $a = 2$. We show that despite formal similarities this theory is not equivalent to QCD$_2$, and that its extension to $U(N)$ does not lead to an infinite vacuum degeneracy.
1 Introduction

The recent formulation of QCD as a perturbed Wess-Zumino-Witten (WZW) theory \cite{1}, \cite{2}, \cite{3} has provided some interesting insight into structural aspects of the theory \cite{2}, \cite{4}, \cite{5}, \cite{6}. In the so-called non-local decoupled formulation \cite{2} the corresponding (“enlarged”) Hilbert space is generated from an effective partition function given in terms of the direct product of non-interacting fermion and ghost sectors, as well as a “massive” interacting sector. The physical Hilbert subspace is obtained by imposing BRST conditions on the states. As was shown in ref. \cite{6}, these conditions correspond in the abelian $U(1)$ case (vector Schwinger model) to the familiar Lowenstein-Swieca \cite{7} conditions requiring that the longitudinal part of the current annihilate the physical states. Analogous conditions have been obtained by Bojanovsky et al. \cite{8} for the case of the chiral Schwinger model \cite{9}, \cite{10}. A corresponding analysis of this anomalous model (chiral QCD) is, however, lacking. The purpose of this paper is to fill this gap. We shall in particular concentrate on the case of the Jakiw-Rajaraman parameter \cite{9} $a$ taking the value $a = 2$, for which the chiral Schwinger model has been claimed \cite{11} to be equivalent to the Schwinger model. We shall reexamine this question in the context of chiral QCD and show that, like in the abelian case \cite{12}, this equivalence does not exist.

In section 2 we show for $a = 2$ that the effective Lagrangian obtained in \cite{8} just corresponds to the abelian counterpart of the non-local decoupled formulation of chiral QCD, and that the conditions imposed in \cite{8} on the physical states correspond to the BRST conditions one systematically derives in the corresponding non-abelian case. This will also serve to streamline the presentation of ref. \cite{8}.

In section 3 we then discuss why chiral QCD for $a = 2$ is not equivalent to QCD by examining in more detail its physical Hilbert space. Section 4 contains our conclusions and some general remarks on the structure of the physical Hilbert space. Considerations showing that the BRST symmetry operating in the “massive” sector does not imply restrictions on the physical states are relegated to the appendix.

2 BRST constraints of chiral QCD

Our starting point is the (Minkowski) partition function of chiral QCD, with left-handed fermions coupled to a $SU(N)$ gauge field:

$$Z = \int \mathcal{D}A_\mu \int \mathcal{D}\psi^{(0)}_1 \mathcal{D}\psi^{(0)}_2 \int \mathcal{D}\bar{\psi}_2 \mathcal{D}\psi_1 e^{iS[A,\psi,\bar{\psi}]}$$ (2.1)
Here \( \Gamma \) variables (2.3)

\[
S[A, \psi, \bar{\psi}] = \int d^2 x \left\{ -\frac{1}{4} tr F^{\mu \nu} F_{\mu \nu} + \psi_1^{(0)} i \partial_+ \psi_1^{(0)} + \psi_2^{(0)} (i \partial_- + e A_-) \psi_2 \right\} (2.2)
\]

Parametrizing \( A_\pm \) as follows

\[
e A_+ = U^{-1} i \partial_+ U, \quad e A_- = V i \partial_- V^{-1}, \quad (2.3)
\]

the Yang-Mills action (2.2) then can be written in the two alternative forms \[13\]

\[
S_{YM}[\Sigma] = \frac{1}{4 e^2} \int tr \frac{1}{2} [\partial_+ (\Sigma i \partial_- \Sigma^{-1})]^2 (2.4)
\]

\[
= \frac{1}{4 e^2} \int tr \frac{1}{2} [\partial_- (\Sigma^{-1} i \partial_+ \Sigma)]^2 (2.5)
\]

with \( \Sigma \) the gauge-invariant variable

\[
\Sigma = UV. \quad (2.6)
\]

Making the change of variables \( A_+ \to U, \quad A_- \to V \) as well as the chiral rotation,

\[
\psi_2 \to \psi_2^{(0)} : \psi_2^{(0)} = V^{-1} \psi_2, \quad (2.7)
\]

and taking due account of the Jacobians in the integration measure \[11\] \[12\], we then arrive, following the procedure of references \[2\], \[4\], at the partition function

\[
Z = Z_F^{(0)} Z_{gh}^{(0)} \hat{Z}, \quad (2.8)
\]

where \( Z_F^{(0)} \) is the partition function of free fermions,

\[
Z_F^{(0)} = \int \mathcal{D} \psi^{(0)} \mathcal{D} \bar{\psi}^{(0)} e^{i \int \bar{\psi}^{(0)} \partial \psi^{(0)}}, \quad (2.9)
\]

\( Z_{gh}^{(0)} \) is the partition function of free ghosts associated with the change of variables (2.3)

\[
Z_{gh}^{(0)} = \int \mathcal{D} b^{(0)} \mathcal{D} c_+^{(0)} e^{i \int b^{(0)} \partial \psi_1^{(0)} + \int \mathcal{D} b_-^{(0)} \mathcal{D} c_-^{(0)} e^{i \int b_-^{(0)} \partial \psi_2^{(0)}}, \quad (2.10)
\]

and where

\[
\hat{Z} = \int \mathcal{D} U \mathcal{D} V \mathcal{D} e^{i S_{YM}[UV]} e^{-i C_V F[UV] - i \Gamma[V]} e^{i S_{JR}}. \quad (2.11)
\]

Here \( \Gamma[g] \) is the WZW functional \[13\],

\[
\Gamma[g] = \frac{1}{8 \pi} \int d^2 x tr \partial_\mu g \partial^\mu g^{-1} + \frac{1}{12 \pi} \int_{s_B} d^3 x e^{ijk} tr [\hat{g}^{-1} \partial_i \hat{g}^{-1} \partial_j \hat{g}^{-1} \partial_k \hat{g}] \quad (2.12)
\]

and

\[
S_{JR} = \frac{a}{24 \pi} \int d^2 x tr [A_+ A_-] = -\frac{a}{24 \pi} \int d^2 x tr [(U^{-1} \partial_+ U)(V \partial_- V^{-1})]. \quad (2.13)
\]
The presence of the last factor in (2.11) reflects the usual regularization ambiguity, with \( a \) the Jackiw-Rajaraman parameter \([9]\).

Note that because of the presence of the factor \( \exp(iS_{JR}) \), our change of variables did not result in a decoupling of the fields, unlike in the case of QCD\(_2\). Nevertheless, for \( a = 2 \) a decoupling of these fields is easily achieved. Indeed, making use of the Polyakov-Wiegmann identity \([16]\),

\[
\Gamma[gh] = \Gamma[g] + \Gamma[h] + \frac{1}{4\pi} \int tr[(g^{-1}g)(h\partial h^{-1})],
\]  

we obtain from (2.8)

\[
Z = Z_F^{(0)} Z_g^{(0)} \int DU D\Sigma e^{iS_{YM}[U]} e^{-i(C_V + \frac{\pi}{2})\Gamma[U]} e^{i\frac{\pi}{4}\Gamma[U]} e^{i(\frac{3}{2} - 1)\Gamma[\Sigma]}. 
\]  

Unlike the case of of QCD\(_2\), the transformations (2.14) and (2.7) have not led to a decoupling of the fields. However, for \( a = 2 \), (2.15) reduces to the decoupled partition function

\[
Z = Z_F^{(0)} Z_g^{(0)} \left( \int DU e^{i\Gamma[U]} \right) \left( \int D\Sigma e^{iS_{YM}[\Sigma]} e^{-i(1 + C_V)\Gamma[\Sigma]} \right). 
\]  

Except for the factor \( \exp\{i\Gamma[U]\} \), which appears to play merely a spectator role, this is the partition function of QCD\(_2\) in the decoupled formulation \([2, 3, 4, 5, 13]\). As we shall see, however, the apparently decoupled field \( U \) plays an important role in the analysis of the physical Hilbert space\(^2\).

Let us check where we stand by specializing to the abelian case. Introducing in that case the parametrization

\[
U = e^{-ie(\chi + \lambda)}, \quad V = e^{-ie(\chi - \lambda)}
\]  

and noting that \( C_V = 0 \) for \( U(1) \), we obtain from (2.16) \( Z_g^{(0)} = 1 \),

\[
Z = \int D\phi D\chi D\lambda e^{i\int L^{(Box)}_{U(1)}}
\]  

with

\[
L^{(Box)}_{U(1)} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}(\Delta \chi)^2 - \frac{1}{2}m^2(\partial_\mu \chi)^2 + \frac{1}{2}(\partial_\mu \varphi)^2,
\]  

where we have set \( \chi + \lambda = \frac{2\sqrt{\pi}}{e^2} \varphi \), \( m^2 = 4e^2/\pi \) and have made use of bosonization formula

\[
\bar{\psi}^{(0)} i\Phi \psi^{(0)} \to \frac{1}{2}(\partial_\mu \phi)^2.
\]  

The Lagrangian (2.18) is just the one obtained in ref. \([8]\) for \( a = 2 \) after a suitable redefinition of the fields. The appearance of fourth-order derivative terms in the Lagrangian (2.18) is already evident from (2.4) and (2.3).
In order to obtain a reduction to second-order derivatives we introduce an auxiliary Lie-algebra-valued field $E$ or $E'$, depending on which form of the Yang-Mills action, eq. (2.4) or (2.5), we choose to work with:

$$e^{iS_{YM}[\Sigma]} = \int \mathcal{D}E e^{i \int \text{tr} \left[ \frac{1}{4} E^2 + \frac{1}{2} [\partial_+ (\Sigma i \partial_+ \Sigma^{-1})] \right]}$$

$$= \int \mathcal{D}E' e^{i \int \text{tr} \left[ \frac{1}{4} E'^2 + \frac{1}{2} [\partial_+ (\Sigma^{-1} i \partial_+ \Sigma)] \right]}.$$  (2.20)

Making the change of variable [5]

$$\partial_+ E = \left( \frac{1 + CV}{2\pi} \right) e^{\beta^{-1} i \partial_+ \beta}$$

and

$$\partial_- E' = \left( \frac{1 + CV}{2\pi} \right) e^{\beta' i \partial_- \beta'^{-1}}$$  (2.22)

one arrives at two alternative “non-local” forms of the decoupled partition function:

$$Z = Z_F^{(0)} Z_{gh}^{(0)} \int \mathcal{D}\tilde{\Sigma} \mathcal{D}U \mathcal{D}E e^{-i \left( \frac{1 + CV}{2\pi} \right)^2 e^{2 \int \text{tr} \left[ \partial_+^{-1} (\beta^{-1} i \partial_+ \beta) \right]^2}}$$

$$\times e^{-i (1 + CV) \Gamma[\tilde{\Sigma}] e^{i \Gamma[U]} e^{i \Gamma[V]} \int \mathcal{D}\tilde{\phi} \mathcal{D}\tilde{\phi} e^{i \int \text{tr} \left[ \partial_+^{-1} \partial_- \partial_+ \partial_- \right]}}$$  (2.24)

$$= Z_F^{(0)} Z_{gh}^{(0)} \int \mathcal{D}\tilde{\Sigma} \mathcal{D}U \mathcal{D}E' e^{-i \left( \frac{1 + CV}{2\pi} \right)^2 e^{2 \int \text{tr} \left[ \partial_-^{-1} (\beta' i \partial_- \beta'^{-1}) \right]^2}}$$

$$\times e^{-i (1 + CV) \Gamma[\tilde{\Sigma}'] e^{i \Gamma[U]} e^{i \Gamma[V]} \int \mathcal{D}\tilde{\phi} \mathcal{D}\tilde{\phi} e^{i \int \text{tr} \left[ \partial_+ \partial_- \right]}}$$  (2.25)

where the changes of variable

$$\tilde{\Sigma} = \beta \Sigma, \quad \tilde{\Sigma}' = \Sigma \beta'$$  (2.26)

have been made and use has been made of the Polyakov-Wiegmann identity (2.14). Again, aside from (the all important factor) $\exp i \Gamma[U]$, this is just the QCD$_2$ partition function before gauge fixing.

As before let us check where we stand by considering the special case of an abelian $U(1)$ group. As one easily verifies, one has in this case $\beta = \beta'$ (and hence $\tilde{\Sigma}' = \tilde{\Sigma}$). Parametrizing $\beta$, $\tilde{\Sigma}$ and $U$ as follows

$$\beta = e^{-2i \sqrt{\pi} \sigma}, \quad \tilde{\Sigma} = e^{-2i \sqrt{\pi} \eta}, \quad U = e^{-2i \sqrt{\pi} \varphi}$$  (2.27)

expressions (2.24) and (2.25) reduce to

$$Z = \int \mathcal{D}\phi \mathcal{D}\varphi \mathcal{D}\sigma \mathcal{D}\eta e^{i \int \hat{g}(B_{\alpha})}$$  (2.28)
with
\[ \tilde{L}^{(\text{Bos})}_{U(1)} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{1}{2}m^2 \sigma^2 - \frac{1}{2}(\partial_\mu \eta)^2 + \frac{1}{2}(\partial_\mu \varphi)^2, \] (2.29)

where use has again been made of the bosonization formula (2.19). Expression (2.29) is identical, after suitable relabelling, with the one obtained in ref. [8] after a series of manipulations. Note that \( \eta \) is a negative metric (unphysical) field, corresponding to the fact that \( \tilde{\Sigma}(\tilde{\Sigma}') \) in eq. (2.24) (eq. (2.25)) is a WZW field of negative level \(-1 + CV\). Except for the last term, the Lagrangian (2.29) coincides with that of the vector Schwinger mode (VSM). In the VSM the gauge invariance ensures that the field \( \varphi \) is a pure gauge excitation and does not appear in the bosonized Lagrangian density [8, 12]. However, in the anomalous chiral model the additional degree of freedom \( \varphi \) is a dynamical field and its presence ensures the existence of fermions in the asymptotic states [12].

Despite the factorized form (2.24) (or (2.25)) of the partition function, the physical Hilbert space is not just the direct product of decoupled sectors, but is restricted by a set of BRST condition, which can be derived from first principles, following the procedure of ref. [18]. These conditions ensure in particular that negative norm states associated with \( \tilde{\Sigma}(\tilde{\Sigma}') \) are absent in \( H_{\text{phys}} \). Following the steps outlined in refs. [3] and [13], one thus systematically discovers the existence of three nilpotent transformations. They are associated with the changes of variable 1) \( A_- \to V \), 2) \( A_+ \to U \) and 3a) \( E \to \beta \), or 3b) \( E' \to \beta' \), respectively. According to the discussion in the appendix only the first two imply non-trivial restrictions on the physical states. They are derived from (2.23) [13] and represent nilpotent symmetry transformations of the local partition function (2.28):

1) \[ \begin{align*}
\delta U &= 0, \quad \delta V^{-1}V = c_+^{(0)} \\
\delta \psi_1^{(0)} &= 0, \quad \delta \psi_2^{(0)} = c_+^{(0)} \psi_2^{(0)} \\
\delta c_-^{(0)} &= 0, \quad \delta c_+^{(0)} = \frac{1}{2}\{c_-^{(0)}, c_+^{(0)}\} \\
\delta b_+^{(0)} &= \frac{1}{4e^2} \Sigma^{-1} [\partial_2^2 (\Sigma i \partial_\Sigma^{-1})] \Sigma \\
&- \frac{1 + CV}{4\pi} \Sigma^{-1} i \partial_\Sigma + \psi_2^{(0)} \psi_2^{(0)\dagger} + \{b_+^{(0)}, c_+^{(0)}\} \quad (2.30)
\end{align*} \]

and

2) \[ \begin{align*}
\delta V &= 0, \quad \delta U^{-1}U = c_-^{(0)} \\
\delta \psi_1^{(0)} &= c_-^{(0)} \psi_1^{(0)}, \quad \delta \psi_2^{(0)} = 0 \\
\delta c_-^{(0)} &= 0, \quad \delta c_+^{(0)} = \frac{1}{2}\{c_-^{(0)}, c_-^{(0)}\}
\end{align*} \]
\[
\begin{align*}
\delta b_+^{(0)} &= 0 \\
\delta b_-^{(0)} &= \frac{1}{4e^2} \Sigma [\partial^2 (\Sigma^{-1} i \partial_+ \Sigma)] \Sigma^{-1} \\
&- \frac{1 + C V}{4\pi} \Sigma i \partial_+ \Sigma^{-1} + \frac{1}{4\pi} U i \partial_- U^{-1} + \{b_-^{(0)}, c_-^{(0)}\}.
\end{align*}
\] (2.31)

The BRST transformations associated with the changes of variable 3a) and 3b) are of exactly the same form as in the case of QCD \[5\]. As we show in the appendix, they however imply no restriction on the physical states.

Going through the usual Noether construction, it is straightforward to show that these symmetry transformations imply the existence of the following conserved charges:

\[
Q_\pm = \int dx^1 trc_\pm^{(0)} (\Omega_\pm - \frac{1}{2} \{b_\pm^{(0)}, c_\pm^{(0)}\})
\] (2.32)

where

\[
\Omega_\pm = \delta b_\pm^{(0)}.
\] (2.33)

Canonical quantization (see later) shows that \(\Omega_\pm = trt^a \Omega_\pm\) are weakly first-class operators. As a result, \(Q_\pm\) are nilpotent \[17\].

The nilpotency of the charges, together with the condition that they annihilate the physical states, implies that \(\Omega_\pm\) must vanish on such states. Let us express \(\Omega_\pm\) in terms of the variables of the non-local formulation. Following ref. \[5\] we make use of the equation of motion

\[
\begin{align*}
E &= -\frac{1}{2e} \partial_+ (\Sigma i \partial_- \Sigma^{-1}) \\
E' &= -\frac{1}{2e} \partial_- (\Sigma^{-1} i \partial_+ \Sigma)
\end{align*}
\] (2.34)

as well as of (2.22) and (2.23) in the expressions (2.30) and (2.31) for \(\delta b_\pm^{(0)}\), which allows us to reduce \(\Omega_\pm\) to the simple form

\[
\begin{align*}
\Omega_+ &= -\frac{1 + C V}{4\pi} \tilde{\Sigma}^{-1} i \partial_+ \tilde{\Sigma} + \psi_2^{(0)} \psi_2^{(0)*} + \{b_+^{(0)}, c_+^{(0)}\} \\
\Omega_- &= -\frac{1 + C V}{4\pi} \tilde{\Sigma}^{-1} i \partial_- \tilde{\Sigma}^{-1} + \frac{1}{4\pi} U i \partial_- U^{-1} + \{b_-^{(0)}, c_-^{(0)}\}.
\end{align*}
\] (2.35)

Let us check once more where we stand by considering the abelian case. As before we have in this case \(\beta = \beta', \tilde{\Sigma} = \tilde{\Sigma}'\). With the parametrizations (2.19) and (2.27), the conditions \(\Omega_\pm \approx 0\) reduce to

\[
\begin{align*}
\Omega_+ &= -\frac{1}{2\sqrt{\pi}} \partial_+ \phi - \frac{1}{2\sqrt{\pi}} \partial_+ \eta \approx 0, \\
\Omega_- &= +\frac{1}{2\sqrt{\pi}} \partial_- \eta - \frac{1}{2\sqrt{\pi}} \partial_- \varphi \approx 0,
\end{align*}
\] (2.36)
which may be summarized by

\[
\Omega_\mu = -\frac{1}{4\sqrt{\pi}}(\partial_\mu + \epsilon_{\mu\nu}\partial^\nu)\phi - \frac{1}{4\sqrt{\pi}}(\partial_\mu - \epsilon_{\mu\nu}\partial^\nu)\varphi - \frac{1}{4\sqrt{\pi}}\epsilon_{\mu\nu}\partial^\nu \eta \approx 0. \tag{2.37}
\]

Except for a trivial relabelling, these are precisely the conditions obtained in ref. [8] for the case of the chiral Schwinger model, from another point of view.

### 3 The physical Hilbert space

In order to get a more detailed understanding of the implications of the constraints \(\Omega_\pm \approx 0\) on the physical states, we must go over to phase space. The canonical quantization proceeds as in [5]. In terms of phase-space variables, \(\Omega_\pm\) and then takes the form

\[
\Omega_+ = J_+(\tilde{\Sigma}) + \psi_2^{(0)}\psi_2^{(0)\dagger} + \{b_+^{(0)}, c_+^{(0)}\} \tag{3.1}
\]

\[
\Omega_- = J_-(\tilde{\Sigma}') + J_-(U) + \{b_-^{(0)}, c_-^{(0)}\} \tag{3.2}
\]

where (see ref. [5] for notation and more details; the superscript “T” denotes “transpose”)

\[
J_+(\tilde{\Sigma}) = -i\tilde{\Pi}_T^{\dagger}\tilde{\Sigma} - \frac{1 + CV}{4\pi}\tilde{\Sigma}^{-1}i\partial_1\tilde{\Sigma} \tag{3.3}
\]

\[
J_-(\tilde{\Sigma}') = i\tilde{\Sigma}'\tilde{\Pi}_T^{\dagger} + \frac{1 + CV}{4\pi}\tilde{\Sigma}'i\partial_1\tilde{\Sigma}'^{-1} \tag{3.4}
\]

\[
J_-(U) = iU\tilde{\Pi}_U^{\dagger} - \frac{1}{4\pi}Ui\partial_1U^{-1} \tag{3.5}
\]

The WZW-currents \(J_+^{(\tilde{\Sigma})}, J_-^{(\tilde{\Sigma}')}\) and \(J_-^{(U)}\) satisfy a Kac-Moody algebra\(^3\)

\[
(J_\pm^a = trt^a J_\pm)
\]

\[
\{J_\pm^a(h(x)), J_\pm^b(h(y))\}_P = -f^{abc}J_\pm^a(h(x))\delta(x^1 - y^1) + \frac{K}{2\pi}\partial_1 \delta^{ab}\delta(x^1 - y^1)
\]

\[
\{J_\pm^a(h(x)), J_\pm^b(h(y))\}_P = 0 \tag{3.6}
\]

of level \(\kappa = -(1 + CV), -(1 + CV)\) and 1, respectively.

We next show that the fields \(\psi_1^{(0)}, \psi_2\) and \(A_\mu\) commute with the operators \(\{3.1\}-\{3.2\}\), and hence represent (physical) observables of the theory. To this end we first rewrite these fields in terms of the fields of the decoupled non-local formulation \(U, \tilde{\Sigma}, \beta\) and their canonical conjugates:

\[
\psi_2 = V\psi_2^{(0)} = U^{-1}\tilde{\Sigma}\psi_2^{(0)} = U^{-1}\beta^{-1}\tilde{\Sigma}\psi_2^{(0)} \tag{3.7}
\]

\[
e A_+ = U^{-1}i\partial_+ U = 4\pi J_+(U) \tag{3.8}
\]
\[ eA_- = Vi\partial_- V^{-1} = (U^{-1} \beta^{-1} \Sigma) i\partial_- (\Sigma^{-1} \beta U) \] (3.9) \\
\[ = -\frac{4\pi}{1 + C_V} (\beta U)^{-1} J_- (\Sigma)(\beta U) - U^{-1} J_-(U)U + U^{-1} (\beta^{-1} i\partial_- \beta)U \]

or in terms of \( \tilde{\Sigma}' \),

\[ eA_- = U^{-1} \tilde{\Sigma}' (\beta^{-1} i\partial_- \beta') \tilde{\Sigma}'^{-1} U \]
\[ - \frac{4\pi}{1 + C_V} U^{-1} J_-(\tilde{\Sigma}')U - U^{-1} J_-(U)U. \] (3.10)

i) First of all, \( \psi^{(0)}_1 \) commutes with all constraints, since they do not involve \( \psi^{(0)}_1 \). Hence, \( \psi^{(0)}_1 \) is physical.

ii) We have (different sectors commute with each other)

\[ \{ J_+^a (\tilde{\Sigma}(x)), U^{-1} \beta^{-1} \tilde{\Sigma} \psi^{(0)}_2 (y) \}_P = iU^{-1} \beta^{-1} \tilde{\Sigma} t^a \psi^{(0)}_2 (x) \delta(x^1 - y^1) \] (3.11)

and

\[ \{ tr(t^a \psi^{(0)}_2 \psi^{(0)}_2^\dagger (x)), U^{-1} \beta^{-1} \tilde{\Sigma} \psi^{(0)}_2 (y) \}_P = -iU^{-1} \beta^{-1} \tilde{\Sigma} t^a \psi^{(0)}_2 (x) \delta(x^1 - y^1) \] (3.12)

Hence \( \{ \Omega_+(x), \psi_2(y) \}_P = 0 \). In a similar way one verifies that

\( \{ \Omega_-(x), \psi_2(y) \}_P = 0 \). Hence \( \psi_2 \) is physical.

iii) \( eA_+ = U^{-1} i\partial_+ U \) evidently commutes with \( \Omega^a_+ \). As for \( \Omega^a_- \) we have

\[ \{ \Omega_-(x), A_+(y) \}_P = 4\pi \{ J_-(U(x)), J_+(U(y)) \}_P = 0 \] (3.13)

Hence \( A_+ \) is physical.

iv) Similar considerations show that \( A_- \) also commutes with \( \Omega_\pm \). The vanishing of the Poisson bracket \( \{ \Omega^a_+(x), A_-(y) \}_P \) follows from the commutativity of \( J_+ \) with \( J_- \). Furthermore, making use of the Poisson brackets

\[ \{ J_+^a (h(x)), h(x) \}_P = -i(t^a h(x)) \delta(x^1 - y^1) \]
\[ \{ J_-^a (h(x)), h^{-1}(y) \}_P = i(h^{-1}(x)t^a) \delta(x^1 - y^1) \] (3.14)

and

\[ \{ J_+^a (h(x)), h^{-1}Qh(y) \}_P = i(h^{-1} [t^a, Q] h) \delta(x^1 - y^1) \]
\[ + h^{-1} t^b h(x) \{ J_+^a (h(x)), Q^b (y) \}_P \] (3.15)

one finds that all contributions (including the central terms) to \( \{ \Omega^a_-(x), A_-(y) \}_P \) cancel, so that \( A_- \) is physical, as well.

Summarizing, the BRST conditions just tell us, that the physical Hilbert space is constructed from \( \{ U, V, \psi^{(0)}_1, \psi^{(0)}_2 \} \) as combinations of the basic fields of the original Lagrangian, as expected.
4 Conclusions and Final Remarks

We have presented a decoupled path integral formulation of the anomalous chiral QCD for the case $a = 2$. In this section we summarize our conclusions and also make some general remarks on the structure of the physical Hilbert space of the anomalous chiral model, which are crucial to distinguish it from the gauge invariant case.

In order to construct the Hilbert space associated with the Wightman functions that define the theory we only use the field algebra generated from the intrinsic irreducible set of field operators of the theory. The local field algebra $\mathfrak{I}$ intrinsic to the chiral model is generated from the irreducible set of field operators $\{\psi_1^{(0)}, \psi_2^{(0)}, A_\mu\}$ [12, 20]. These field operators constitute the intrinsic mathematical description of the model and serve as a kind of building material in terms of which the theory is formulated and whose Wightman functions define the model. The field algebra $\mathfrak{I}$ is represented on the Hilbert space $\mathcal{H}$.

The present techniques used to treat QFT models, such as bosonization and Faddeev-Popov prescription, require the use of a larger field algebra, which includes non-observable bosonic fields as well as ghosts. The partition function of the resulting effective theory, for example in the local decoupled formulation, is given in terms of the set of fields $\{\psi_1^{(0)}, \psi_2^{(0)}, U, V, gh\}$, which defines an enlarged redundant field algebra $\mathfrak{I}^c$. The field algebra $\mathfrak{I}^c$ is represented on the Hilbert space $\mathcal{H}^c$, which is the direct product of decoupled sectors appearing in the effective partition function (2.16). The fields $\{\psi_1^{(0)}, \psi_2^{(0)}, U, V, gh\}$ should not be considered as elements of the field algebra intrinsic to the model.

The field algebra $\mathfrak{I}$ is a subalgebra of $\mathfrak{I}^c$. Not all functional of the fields $\{\psi_1^{(0)}, \psi_2^{(0)}, U, V, gh\}$ belong to the algebra $\mathfrak{I}$ of the local fields, nor all vectors of $\mathcal{H}^c$ belong to the state space $\mathcal{H}$ of the model. The set of states corresponding to the largely redundant field algebra $\mathfrak{I}^c$, contains elements which are not intrinsic to the model. Of course, the Hilbert space $\mathcal{H}$ is a subspace of $\mathcal{H}^c$.

The physical states of $\mathcal{H}$ are required to satisfy the BRST subsidiary conditions

$$Q_\pm |\Psi\rangle = 0 , \quad \forall |\Psi\rangle \in \mathcal{H}_{phys},$$

which ensure that unphysical fields violating norm-positivity appear in the physical subspace only as zero-norm combinations. This condition also ensures that the physical Hilbert space cannot be decomposed as a direct product of decoupled sectors appearing in the effective partition function (2.24), (2.25). As a consequence, the dependence on the "apparently decoupled field" $U$ cannot be factorized from the completion of states. In this way, the partition function of the anomalous chiral model cannot be factorized as a direct product of a coset model and a massive model [3], and thus cannot
be identified with the partition function of QCD$_2$.

The algebra of the physical operators $\mathcal{Z}^{\text{phys}}_\text{phys}$ must be identified as the subalgebra $\mathcal{Z}^{\text{ir}}$ of $\mathcal{Z}$ which obeys the constraint condition in a proper Hilbert space completion of states. As we have shown in the previous section, it is a peculiarity of the anomalous chiral model that the algebra $\mathcal{Z}^{\text{phys}}_\text{phys} \equiv \mathcal{Z}$, since all operators belonging to the intrinsic field algebra $\mathcal{Z}$ commute with the BRST constraints and thus represent physical observables of the theory. In analogy with the abelian case and in agreement with the conclusions of refs. [8, 12, 19], this is expected in the anomalous case since the quantum theory has lost the local gauge invariance. In contrast to the QCD$_2$, the field algebra $\mathcal{Z}$ is physical and is represented in the state space $\mathcal{H}^{\text{phys}}_\text{phys} \equiv \mathcal{H}$.

The fact that the irreducible set of field operators $\{\psi_1^{(0)}, \psi_2, A_\mu\}$ represents physical observables of the chiral QCD$_2$ is a peculiar structural aspect of the anomalous theory which allows for two isomorphic formulations, i.e., the gauge invariant and gauge noninvariant formulations [12, 19]. This property of the algebra of observables enables a basic structural distinction between the anomalous chiral theory and the gauge invariant one and also corroborates to support the conclusion that in analogy with the abelian case (chiral QED$_2$) [13], chiral QCD$_2$ for the JR parameter $a = 2$ is not equivalent to QCD$_2$.

The structural properties of the model must be analyzed taking a careful control on the Hilbert space associated with the Wightman functions that provide a representation of the field algebra generated from the intrinsic irreducible set of field operators of the model. This rigorous control on the construction of the Hilbert space together with the BRST constraints, constitute the necessary and sufficient requirements to identify correctly the physical operators in the effective decoupled formulation of the anomalous theory.

In order to show that by relaxing this rigorous control on the construction of the physical Hilbert space of the anomalous theory some misleading conclusions can arise, as for example the existence of an infinite number of states which are degenerate in energy with the vacuum, it is interesting to examine the effect of extending the group $SU(N)$ to $U(N)$. In the case of a true gauge theory, we expect in this case an infinite degeneracy of the ground state as it is known to occur in the Schwinger model.

Let us first specialize to the $U(1)$ case, that is, the chiral Schwinger model with $a = 2$. This particular case has generated some confusion in the literature since the operator algebra exhibits non-trivial and delicate features [12] which might lead to misleading conclusions [11] about structural properties of the model. As stressed in ref. [12], one can construct a field subalgebra which commutes with the constraints (2.37) and is isomorphic to the field subalgebra of the vector Schwinger model (QED$_2$), but does not belong to the intrinsic field algebra of the anomalous chiral model.
The physical Hilbert space is constrained by the requirement, that the operators (2.36) annihilate the physical states. The physical fermion field \( \psi_2 \) is expressed in terms of the free field according to eqs. (2.6), (2.7), (2.26) and (2.27) as

\[
\psi_2 = U^{-1} \beta^{-1} \Sigma \psi^{(0)}_2 =: e^{2i \sqrt{\pi} (\sigma - \eta + \varphi)} : \psi^{(0)}_2. \tag{4.1}
\]

We bosonize the free fermion in terms of \( \phi \) and consider the physical composite operator \( \psi_2 \psi^{(0)}_1 \)

\[
: \psi_2 \psi^{(0)}_1 : = e^{2i \sqrt{\pi} (\sigma - \eta + \varphi + \phi)} :. \tag{4.2}
\]

Defining the new fields \( \phi' = \phi(x^+) + \varphi(x^-) \) and \( \phi' = \varphi(x^+) + \phi(x^-) \) we see that, in the case \( a = 2 \), the field \( \phi' \) decouples from the longitudinal current (2.37) and there is a broader class of operators belonging to \( \mathfrak{I}^e \) that satisfy the constraint conditions. The composite operator (4.2) can be factorized as the product of two “physical” exponential operators that separately commute with the constraint

\[
: \psi_2 \psi^{(0)*}_1 : = e^{2i \sqrt{\pi} (\sigma - \eta + \varphi')} : e^{2i \sqrt{\pi} \phi'} :. \tag{4.3}
\]

The first exponential factor appearing in (4.3) is the Lowenstein-Swieca solution of the Schwinger model leading for instance to \( \theta \)-vacua parametrization [7]. As stressed in [12] the operator (4.3) cannot be reduced by extracting the exponential of the field \( \phi' \), since the corresponding exponential operator separately does not belongs to the intrinsic field algebra of the anomalous chiral model. Although the operator \( \exp\{2i \sqrt{\pi} \phi'\} \in \mathfrak{I}^e \) commutes with the constraint (2.37), it cannot be defined on the Hilbert space \( \mathcal{H} \). This follows from the fact that some charges carried by the field \( \phi' \) get trivialized in the restriction from \( \mathcal{H}^e \) to \( \mathcal{H} \) [20, 12]. In this way, the equivalence of vector Schwinger model and chiral Schwinger model with \( a = 2 \) and the need for a \( \theta \)-vacuum parametrization in the anomalous model, as suggested in ref. [11], cannot be established in terms of the intrinsic field algebra and are consequence of an improper factorization of the completion of states [12].

Returning to the nonabelian case the same analysis applies to the \( U(1) \) piece of the \( U(N) \) model. To this end let us separate the fields into \( U(1) \times SU(N) \) [21, 24]. Within our present decoupling method we consider the Wess-Zumino-Witten fields now taking values on \( U(N) \). Instead of (2.30) the fields are expressed in terms of an \( U(1) \) factor and \( SU(N) \)-valued field operators as:

\[
\hat{\beta} = e^{-2i \sqrt{\pi} \sigma} \beta, \quad \hat{\Sigma} = e^{-2i \sqrt{\pi} \eta} \Sigma, \quad \hat{U} = e^{-2i \sqrt{\pi} \varphi} U. \tag{4.4}
\]

All the decoupling and the analysis of the previous sections are the same with the \( C_V \) factors appearing in the actions multiplying only the \( SU(N) \)
WZW fields. Factoring out the $U(1)$ dependence of the free fermion in terms of the scalar field $\phi$ \[13, 23\], instead of (4.3) we obtain

\[
\hat{\psi}_2 \hat{\psi}_1^{(0)\dagger} = e^{2i\sqrt{\pi} (\sigma - \eta + \phi')} : e^{2i\sqrt{\pi} \phi'} : \left[ U^{-1} \beta^{-1} \tilde{\Sigma} \left( \hat{\psi}_2^{(0)} \hat{\psi}_1^{(0)\dagger} \right) \right]. \tag{4.5}
\]

Once again the first exponential factor, excluding the massive field $\sigma$, is a spurious operator with zero scale dimension and spin that generates constant Wightman functions \[24\]. It is tempting to extract from the operator (4.5) the dependence on the field $\phi'$, since for $a = 2$ it commutes with the BRST constraints. With this procedure one would erroneously conclude for the need of the $\theta$-vacua parametrization in the $U(N)$ model. Within the formulation in terms of the intrinsic field algebra generated from the irreducible set of field operators \[12\], the spurious operators and the operator $: \exp\{2i\sqrt{\pi} \phi'\} :$ do not exist separately and cannot be defined on $\mathcal{H}$.

It would be interesting to generalize the method here used to the case $a \neq 2$, in order to seek a transformation decoupling the fields in this case and, eventually obtain a complete decoupled formulation for the chiral anomalous QCD$_2$.

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**Appendix**

In this appendix we show that the Lowenstein-Swieca type constraints are the only ones to be imposed, and that the additional BRST symmetry relating the conformal to the massive $\beta$ sector in the non-local formulation implies no restriction on the physical states. We do this in the context of the Schwinger model, where the results may be familiar to the reader. Our conclusions will however equally apply to the chiral Schwinger model and chiral QCD$_2$. For future reference we make the discussion self-contained, at the expense of some redundancy.

To begin with we recall the general procedure for identifying the BRST symmetry and charges associated with a change of variables. Consider the generic partition function

\[
Z = \int [d\phi] e^{iS[\phi]}, \tag{A.1}
\]
where $\phi$ stands for a generic set of fields. We now consider a change of
variables $\phi \to \sigma$ given by $\phi = f(\sigma)$. We implement the change of variable
$\phi \to \sigma$ by introducing in (A.1) the identity

$$1 = \int [d\sigma] \det \left( \frac{\delta f}{\delta \sigma} \right) \delta(\phi - f(\sigma)). \quad (A.2)$$

Using the Fourier representation of the delta–function, and representing the
functional determinant in terms of ghosts, we arrive at the partition function

$$Z = \int [d\phi] [d\rho] \int [db] [dc] e^{iS[\phi]} e^{i \int \rho(\phi - f(\sigma)) + i \int b \frac{\delta f}{\delta \sigma} c}, \quad (A.3)$$

where summation over the indices labelling the fields is understood. As is
well known [18] there is a BRST symmetry associated with the change of
variable which is readily read off from (A.3):

$$\delta \rho = \delta \phi = \delta c = 0 \quad \delta \sigma = c, \quad \delta b = \rho \quad (A.4)$$

This symmetry is off–shell nilpotent. In terms of the graded variation $\delta$, the
effective action $S_1$ in (A.3) can be written as $S[\phi]$ plus a $\delta$ exact term:

$$S_1 = S[\phi] + \delta(b(\phi - f(\sigma))). \quad \text{Hence, in order to have equivalence of the}
\text{extended description (A.3) with the original one as given by (A.1), we must}
\text{require that the transformation (A.4) be a symmetry of the physical states,}
\text{and of any operator acting on them.}
$$

To implement the change of variables we integrate over $\rho$ and $\phi$ leaving
us with the extended action

$$S_1 = S[f(\sigma)] + \int b \frac{\delta f}{\delta \sigma} c, \quad (A.5)$$

and the transformation (A.4) now reads

$$\delta \sigma = c, \quad \delta c = 0, \quad \delta b = - \left( \frac{\delta S[\phi]}{\delta \phi} \right)_{\phi=f(\sigma)} \quad (A.6)$$

One readily checks that this transformation is a symmetry of (A.5). This
symmetry is required to be a symmetry of the physical states and operators.

Now, the equation of motion for $\sigma$ reads

$$\left[ \left( \frac{\delta S}{\delta \phi_\alpha} \right)_{\phi=f(\sigma)} \right] \left( \frac{\delta f_\alpha}{\delta \sigma_\beta} \right) = 0. \quad (A.7)$$

Hence, as long as the transformation $\phi = f(\sigma)$ is invertible (one-to-one mapping), $\delta b = 0$ on shell, so that the BRST symmetry implies no constraint
on the states. This shows, that non trivial constraints on the states are associated with mappings which are not one-to-one. In that case the BRST symmetry of the states insures that the formal identity introduced in order to realize the desired change of variables does indeed act as the identity in the space of BRST invariant functionals, also in the case where the mapping is not one-to-one.

In [2, 4] the transition from the fermionic formulation of $QCD_2$ to the so called "non-local" formulation involved two sequential changes of variable. In reality, however, only one set of transformations is involved. This becomes clear if we choose to write from the start the Yang-Mills Lagrangian of $QCD_2$ in a "first order" form by making use of the identity (compare with (2.20))

$$e^i \frac{1}{2} \text{tr} F^2 = \int d[E] e^i \int \text{tr} \left[ \frac{1}{2} E^2 - EF \right]$$  \hspace{1cm} (A.8)

where

$$F = \frac{1}{2} \epsilon^{\mu \nu} F_{\mu \nu}$$  \hspace{1cm} (A.9)

and then performing the change of variable (2.3) and (2.4), i.e

$$eA_+ = U^{-1} i \partial_+ U, \quad eA_- = V i \partial_- V^{-1}, \quad \partial_+ E = e \frac{(1 + cV)}{2\pi} \beta^{-1} i \partial_+ \beta$$  \hspace{1cm} (A.10)

The above mappings are evidently not one-to-one, so that we can expect non-trivial BRST constraints. As we wish to show now, there actually exist only two nontrivial such constraints associated with the first two transformations involving the gauge field. Let us illustrate this for the case of the Schwinger model.

With the parametrization

$$U = e^{-i c \mu}, \quad V = e^{i c \nu}, \quad \beta = e^{-i 2\sqrt{\pi} \sigma}$$  \hspace{1cm} (A.11)

the transformations (A.10) read

$$A_+ = \partial_+ \mu, \quad A_- = \partial_- \nu, \quad \partial_+ E = m \partial_+ \sigma$$  \hspace{1cm} (A.12)

where we have set $m = e / \sqrt{\pi}$. It is clear that the above transformations do not represent a one-to-one mapping.

Corresponding to (A.2) we realize the change of variables by introducing the identity

$$1 = \int [d\rho][d\bar{\rho}][d\bar{\rho}] \int [d\mu][d\nu] \int [d(ghosts)] e^i \int \rho (A_+ - \partial_+ \mu) + i \int b_- i \partial_+ c_- \times e^i \int \rho (A_- - \partial_- \nu) + i \int b_+ i \partial_- c_+ e^i \int \rho (\partial_+ E - m \partial_+ \sigma) + i \int b_- i \partial_+ \bar{c}_-$$  \hspace{1cm} (A.13)
If zero modes of the light-cone derivatives are present, then (A.13) is not truly an identity. In this case BRST symmetries have to be respected by the correlators, in order to turn (A.13) into an identity on the space of BRST invariant functionals. The integral in (A.13) exhibits the following BRST symmetries

\[
\begin{align*}
(A) & \quad \delta \bar{\rho} = \delta c_- = 0, \quad \delta \mu = ic_-, \quad \delta b_- = \bar{\rho} \\
(B) & \quad \delta \rho = \delta c_+ = 0, \quad \delta \nu = ic_+, \quad \delta b_+ = \rho \\
(C) & \quad \delta \hat{\rho} = \delta \hat{c}_- = 0, \quad \delta \sigma = i\hat{c}_-, \quad \delta \hat{b}_- = m\hat{\rho}
\end{align*}
\] (A.14)

All these transformations are off-shell nilpotent and commute with each other. The resulting effective action can be written as the sum of the original fermionic action plus three BRST exact parts:

\[
S_{\text{eff}}[A, E, \psi^\dagger, \psi] + \Delta_A + \Delta_B + \Delta_C
\] (A.15)

where

\[
\begin{align*}
\Delta_A & = \delta_A(b_-(A_+ - \partial_+ \mu)), \\
\Delta_B & = \delta_B(b_+(A_- - \partial_- \nu)), \\
\Delta_C & = \delta_C(\hat{b}_-(\partial_+ E - m\partial_+ \sigma))
\end{align*}
\] (A.16)

Hence the three BRST symmetries are to be imposed on the states in order to insure that (A.13) really acts like the identity.

We now show that under suitable asymptotic conditions the third of these BRST symmetries implies no restriction on shell. To this end we integrate over \(A_\pm, \rho, \bar{\rho}\), as well as \(\hat{\rho}, E\). The integration over the first set of variables is unproblematic, and yields for the corresponding b-ghost transformations,

\[
\begin{align*}
\delta_A b_- & = \frac{1}{2} \partial_- E - e\psi_1^\dagger \psi_1 \\
\delta_B b_+ & = -\frac{1}{2} \partial_+ E - e\psi_2^\dagger \psi_2.
\end{align*}
\] (A.17)

The integration over \(E\) requires a special consideration, since it involves the light-cone derivative of \(E\), implying an invariance of the effective action under translations of \(E\) by functions involving the light-cone coordinate \(x^+\) only (zero modes of \(\partial_+\)). From the equation of motion (see (A.8)), \(E = F\), and \(F = \Box \chi\), with \(\chi\) the gauge invariant combination \(\chi = \frac{1}{2}(\mu - \nu)\), it follows that \(E\) does not contain zero modes. Hence \(E\) is uniquely determined as a function of \(\sigma : E(x^+, x^-) = \sigma(x^+, x^-) - \sigma(-\infty, x^-)\), and corresponds to the inverse of \(\partial_+\) being defined by \(< x|\partial_+^{-1}|y> = \theta(x^+ - y^+)\). Excluding zero modes in \(\sigma\) then implies the (in the abelian case trivially) one-to-one mapping \(E = m\sigma\). Under these conditions the transformation for the b-ghosts read:

\[
\delta_A b_- = \frac{m}{2} \partial_- \sigma - e\psi_1^\dagger \psi_1
\]
We now perform the chiral transformation
\[ \psi_1 = U^{-1}\psi_1^{(0)}, \quad \psi_2 = V\psi_2^{(0)} \]  
(A.19)

Taking account of the usual chiral anomaly, we have under this change of variable
\[ e\psi_1^\dagger\psi_1 = e\psi_1^{(0)}\psi_1^{(0)} - \frac{1}{2}m\partial_-(m\chi) \]
\[ e\psi_2^\dagger\psi_2 = e\psi_2^{(0)}\psi_2^{(0)} + \frac{1}{2}m\partial_+(m\chi). \]  
(A.20)

The free fermionic bilinears admit the familiar [7, 25] bosonic representation
\[ \psi_1^{(0)}\partial_+\psi_1^{(0)} + \psi_2^{(0)}\partial_-\psi_2^{(0)} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \]
\[ \psi_1^{(0)}\psi_1^{(0)} = -\frac{1}{2\sqrt{\pi}}\partial_-\phi, \quad \psi_1^{(0)}\psi_1^{(0)} = \frac{1}{2\sqrt{\pi}}\partial_+\phi \]  
(A.21)

In the partition function the chiral transformation (A.19) contributes an anomalous term \( \frac{m^2}{2}\chi\Box\chi \) to the action.

The field \( \chi \) contains zero mass modes, which are disentangled by defining a new (zero mass) field \( \eta = m\chi + \sigma \). Summarizing our results for \( S_{eff} \) and the BRST transformations we have
\[ S_{eff} = \int \left[ -\frac{1}{2}[\phi\Box\phi - \eta\Box\eta + \sigma(\Box + m^2)\sigma] + \int [b_+i\partial_+c_+ + b_-i\partial_-c_- + \tilde{b}_-i\partial_+\tilde{c}_-] \right] \]
(A.22)

as well as (we have rescaled the transformation laws by dividing through by \( m \))
\[ \delta_A b_- = \partial_-(\eta + \phi), \quad \delta_A \eta = -\delta_A \phi = ic_- \]
\[ \delta_B b_+ = -\partial_+(\eta + \phi), \quad \delta_B \eta = -\delta_B \phi = ic_+ \]
\[ \delta_C \tilde{b}_- = \partial_+^{-1}(\Box + m^2)\sigma - \Box\eta), \quad \delta_C \eta = \delta_C \sigma = ic_- \]  
(A.23)

with all the remaining variations vanishing. In the usual case, BRST constraints correspond to the requirement that \( \delta(b - \text{ghost}) = 0 \) on the states. We see that in the first two cases this leads to the familiar Lowenstein-Swieca conditions defining the physical Hilbert space. On the other hand, because of the existence of the inverse of the operator \( \partial_+ \) as discussed before, \( \delta_C \tilde{b}_- = 0 \) is identically satisfied on mass shell, and thus represents no additional constraint on the states. This is completely in line with what is known about the Schwinger model, and is consistent with the observation that in the case
of the Schwinger model the change of variable $E \to \sigma$ was superfluous, since by the simple change of variable $\eta = m\chi + \frac{1}{m}E$ one already succeeded in decoupling the massive ($E$) and massless ($\eta$) degrees of freedom. This is no longer so in the case of $QCD_2$, in which case the change of variable (A.10) required to achieve this decoupling takes one from the Lie-algebra valued field $E$ to the group-valued field $\beta$:

$$
\beta^{-1}(x^+, x^-) = Pe^{\int_{-\infty}^{x^+} dy^+ \partial_+ E} \beta^{-1}(-\infty, x^-)
$$

By requiring $\beta(-\infty, x^-) = 1$ we again exclude zero modes and the mapping becomes one-to-one.

Our previous considerations indicate that only the conformal sector of $QCD_2$ is constrained by BRST conditions. This is in conflict with the claims made in [5]. To gain further insight into the problem let us review in the context of the Schwinger model the steps followed in [5], which led to a phase space representation of $\delta \hat{C} \hat{b}_-$. They correspond to rewriting $\frac{1}{2}m^2\sigma$ in (A.22) as follows:

$$
- \int \frac{m^2}{2} \sigma^2 \to \int \left[ \frac{1}{2}(\partial_+ B - m\sigma)^2 - \frac{m^2}{2} \sigma^2 \right] = \int \left[ \frac{1}{2}(\partial_+ B)^2 - m\sigma \partial_+ B \right].
$$

(A.25)

Assuming $\partial_+ B$ to have no zero modes, we obtain upon integrating once the equation of motion $\partial_+^2 B = m\partial_+ \sigma$ (since $\sigma$ is a massive field, we may set $\sigma(-\infty, x^+) = 0$)

$$
\partial_+ B = m\sigma.
$$

(A.26)

Consider once more the operator $\hat{\Omega}_- = \partial_+^{-1} \left[ (\Box + m^2)\sigma - \Box \eta \right]$ appearing on the right hand side of (A.23). Noting that $\partial_+^{-1}$ stands for $\int_{x^+}^{\infty}$ and making use of (A.26) we obtain

$$
\hat{\Omega}_- = \left[ \partial_-(\sigma - \eta) + mB \right] - \left[ \partial_-(\sigma - \eta) + mB \right]_{x^+ = -\infty}
$$

(A.27)

Hence $\hat{\Omega}_- = 0$ implies

$$
\partial_-(\sigma - \eta) + mB = 0.
$$

(A.28)

Hence for this equation to be satisfied, we must allow for a chiral zero mode $\partial_\eta$ in $B$. This zero mode is right moving so that $\partial_+ B$ has no zero mode in agreement with the assumption made above. In phase space, eq. (A.28) reads (see (A.23) and (A.25))

$$
\pi_\sigma + \pi_\eta - \partial_1 \sigma + \partial_1 \eta + mB = 0.
$$

(A.29)

From (A.25) one obtains for the momentum conjugate to $B$

$$
\pi_B = \partial_+ B - m\sigma.
$$

(A.30)
Hence the equation of motion (A.26) actually represents the constraint \( \pi_B = 0 \)!
Since this constraint does not commute with \( \tilde{\Omega}_- = 0 \) we implement these two constraints strongly. Setting \( \Phi_1 = \tilde{\Omega}_- \) and \( \Phi_2 = \pi_B \) we have

\[
\{ \Phi_i(x), \Phi_j(y) \} = m\epsilon_{ij}\delta(x^1 - y^1). \tag{A.31}
\]

Because of the \( \epsilon \)-tensor one finds that the Poisson brackets of the constraints \( \Omega_\pm \) remain unchanged with respect to the new Dirac-brackets. The whole procedure thus shows that the constraint \( \tilde{\Omega}_- = 0 \), now implemented strongly, just serves to determine the field \( B \) as a function of the remaining fields. This is in accordance with the point of view of ref. [4] and shows once more that the third BRST symmetry in (A.23) does not imply a constraint on the states. The constraints thus only operate in the conformal (massless) sector.

The non-abelian case actually corresponds to writing (A.22) in the partially integrated form

\[
- \int \frac{m^2}{2} \sigma^2 \rightarrow \int \left[ \frac{1}{2} \partial_+ B - m\sigma \right]^2 = \int \left[ \frac{1}{2} \left( \partial_+ B \right)^2 - m\sigma \partial_+ B \right]. \tag{A.32}
\]

Following the same argumentation as above, we are lead to the two constraints

\[
\begin{align*}
\Phi_1 &= \pi_\sigma + \pi_\eta - \partial_1 (\sigma - \eta) = 0 \\
\Phi_2 &= \pi_B - m\sigma = 0. \tag{A.33}
\end{align*}
\]

We again have the property (A.31), showing that these constraints are to be implemented strongly. In terms of the corresponding Dirac brackets, the constraints \( \Omega_\pm \approx 0 \) are first class, and are thus to be implemented on the states.

In the non abelian case the constraints \( \Phi_2 \approx 0 \) take a non-local form in phase space, so that an implementation a la Dirac is not possible. The general ideas however remain valid. It is clear that the same type of argument also applies to chiral QCD$_2$. 

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1) Our conventions are: $A_{\pm} = A_0 \pm A_1$, $\partial_{\pm} = \partial_0 \pm \partial_1$, $\epsilon^{01} = 1$. We follow here the notation of refs. [4], [5].

2) See also the abelian case discussed in Ref. [12].

3) Note the – sign on the r.h.side. Indeed, our definitions (3.3)-(3.4) differ from the conventional ones by a – sign.

4) Instead of (A.25) we could have first considered the replacement

$$- \int \frac{m^2}{2} \sigma^2 \to \int \left[ \frac{1}{2} (G - m\sigma)^2 - \frac{1}{2} m^2 \sigma^2 \right]$$

(A.34)

and then have made the change of variable $G = \partial_+ B$. Following the method of ([18]), eq. (A.26) would then emerge as the BRST constraint associated with this change of variable.
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