MOCK (FALSE) THETA FUNCTIONS AS QUANTUM INVARIANTS

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Contribution to the Special Issue Commemorating the 200th Anniversary of the Birth of Carl Gustav Jacob Jacobi

ABSTRACT. We establish a correspondence between the SU(2) Witten–Reshetikhin–Turaev invariant for the Seifert manifold $M(p_1, p_2, p_3)$ and Ramanujan’s mock theta functions.

1. INTRODUCTION

According to Jacobi, theory of elliptic functions was given birth in December 23, 1751, when Euler was asked to referee Fagnano’s paper including addition formula for the arc length of ellipse and the lemniscate. Since then it has been established and developed by Jacobi, Abel, Gauss and others, and it now becomes fundamental and considerable subjects in both mathematics and physics.

One of mysterious topics of the theta functions is Ramanujan’s mock theta function (see e.g. Refs. 2, 6). It is not modular, but it has a nice asymptotic behavior when $q$ is root of unity. Watson gave a proof for the third order mock theta functions [41], though the systematic understanding and even the meaning of “order” are still missing. In his thesis [46], Zwegers investigated the mock theta functions by use of a real analytic modular form with half-integral weight (see also Ref. 45). He showed that the mock theta functions can be written as a sum of the indefinite theta functions and the Eichler integral of the half-integral weight modular forms. He then obtained the transformation formula for the mock theta functions, which indicates that they have a nearly modular property.

Nearly modular property has also revealed in recent studies of the quantum invariant [29]. Therein the SU(2) Witten–Reshetikhin–Turaev (WRT) invariant $\tau_N(M)$ [38, 43] for the Poincaré homology sphere $M = \Sigma(2, 3, 5)$ was identified with a limiting value of the Eichler integral of the modular form with half-integral weight in $\tau \to 1/N$, and an exact asymptotic expansion in $N \to \infty$ was given by use of the modular transformation. This method is further applied to the colored Jones polynomial for torus knot [25] and torus link [19], and to the WRT invariant for the Seifert manifold [21–23], and topological invariants such as the Chern–Simons invariant, the Casson invariants, the Reidemeister–Ray–Singer torsion, and the Ohtsuki invariant are interpreted from the view point of the modular form.

Date: June 15, 2005.
Purpose of this article is to show that the mock theta functions can be regarded as the WRT invariant $\tau_N(M)$ for the Seifert manifold $M = M(p_1, p_2, p_3)$ (see, e.g., Ref. 34 for the Seifert manifolds). Precisely the WRT invariant can be written in terms of the mock theta functions as a limiting value $\tau \to 1/N$ from lower half of the complex plane. This coincidence was already pointed out in Ref. 29 (see also Ref. 45) for the case of the Poincaré homology sphere which is connected to the fifth order mock theta function, and here we find that such relationship does exist for every order of Ramanujan’s mock theta functions.

An outline of the present article is as follows. In Section 2 we introduce the vector-valued modular form with weight 3/2. We define the Eichler integral thereof, and review a property of the Eichler integral. In Sections 3–7 we discuss separately Ramanujan’s mock theta functions of order 3, 5, 6, 7, 10. Therein we consider the mock theta functions when $q$ is outside the unit circle, and prove that they are the Eichler integral of the vector-valued modular form with weight 3/2, and that a limiting value in $\tau \to 1/N$ gives the WRT invariant for the Seifert manifold which was studied in Ref. 23. The last section is devoted to discussions.

Hereafter we set
\[ q = e^{2\pi i\tau}, \]
with $\tau \in \mathbb{H}$, and we use a standard notation of $q$-calculus;
\[ (x)_n = (x; q)_n = \prod_{k=1}^{n} (1 - x q^{k-1}), \]
\[ (a, b, \cdots ; q)_n = (a; q)_n (b; q)_n \cdots, \]
\[ \left[ \begin{array}{c} n \\ m \end{array} \right]_q = \frac{(q)_n}{(q)_{n-m} (q)_m}. \]

For our convention, we list some identities (see, e.g., Ref. 1);

- $q$-binomial theorem,
\[ (-z)_N = \sum_{m=0}^{N} q^{m(m-1)/2} \left[ \begin{array}{c} N \\ m \end{array} \right]_q z^m, \quad (1.1) \]
- $q$-binomial series,
\[ \frac{1}{(z)_N} = \sum_{m=0}^{\infty} \left[ \begin{array}{c} N + m - 1 \\ m \end{array} \right]_q z^m, \quad (1.2) \]
- $q$-binomial formula,
\[ \sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} z^n = \frac{(az)_\infty}{(z)_\infty}, \quad (1.3) \]
- the Euler identity, which follows from (1.1) in $N \to \infty$,
\[ \sum_{m=0}^{\infty} q^{m(m-1)/2} (q)_m z^m = (-z)_\infty, \quad (1.4) \]
• the Jacobi triple product identity,
\[ \sum_{k \in \mathbb{Z}} (-1)^k q^{k^2/2} z^k = (q, z^{-1} q^{1/2}, z q^{1/2}; q)_\infty. \] (1.5)

2. MODULAR FORM

We define an odd periodic function \( \psi_{2P}^{(a)}(n) \) with modulus 2 \( P \)
\[ \psi_{2P}^{(a)}(n) = \begin{cases} \pm 1, & \text{for } n \equiv \pm a \mod 2P, \\ 0, & \text{otherwise}, \end{cases} \] (2.1)
for \( a \in \mathbb{Z} \). When we introduce the function
\[ \Psi_P^{(a)}(\tau) = \frac{1}{2} \sum_{n \in \mathbb{Z}} n \psi_{2P}^{(a)}(n) q^{n^2 \tau}, \] (2.2)
this becomes a vector-valued modular form with weight 3/2 satisfying
\[ \Psi_P^{(a)}(\tau + 1) = e^{\frac{\pi^2}{2P} \tau} \Psi_P^{(a)}(\tau), \] (2.4)
where \( M(P) \) is a \((P-1) \times (P-1)\) matrix
\[ M(P)^b_a = \sqrt{\frac{2}{P}} \sin \left( \frac{a b \pi}{P} \right). \] (2.5)

The Eichler integral of this set of the modular forms with half-integral weight is then defined as a half-integration of \( \Psi_P^{(a)}(\tau) \) with respect to \( \tau \) by [19, 29]
\[ \tilde{\Psi}_P^{(a)}(\tau) = \sum_{n=0}^{\infty} \psi_{2P}^{(a)}(n) q^{n^2 \tau}. \] (2.6)

The Eichler integral of the integral-weight modular form is known to have a nearly modular property (see, e.g., Ref. 27). To see this nearly modular property in our case, we introduce another Eichler integral following Ref. 29;
\[ \tilde{\Psi}_P^{(a)}(z) = \frac{1}{\sqrt{2P}} \int_{\tau}^{i \infty} \frac{\Psi_P^{(a)}(\tau)}{\sqrt{\tau - z}} d\tau, \] (2.7)
which is defined for \( z \) in the lower half plane, \( z \in \mathbb{H}^- \). We see that the modular property of \( \Psi_P^{(a)}(\tau) \), especially the modular \( S \)-transformation \([23]\), leads a nearly modular property of the Eichler integral as [29] (see also Refs. 19, 46)
\[ \tilde{\Psi}_P^{(a)}(z) + \frac{1}{\sqrt{1 z}} \sum_{b=1}^{P-1} M(P)^b_a \tilde{\Psi}_P^{(b)}(-1/z) = \frac{1}{\sqrt{2P}} \int_{0}^{i \infty} \frac{\Psi_P^{(a)}(\tau)}{\sqrt{\tau - z}} d\tau. \] (2.8)
Then taking a limit \( \tau, z \to 1/N \) for \( N \in \mathbb{Z} \) in the Eichler integrals \( \tilde{\Psi}^{(a)}_P(\tau) \) and \( \tilde{\Psi}^{(a)}_P(z) \), we find both Eichler integrals coincide in this limit, and we obtain an asymptotic expansion in \( N \to \infty \) as

\[
\tilde{\Psi}^{(a)}_P(1/N) + \sqrt{N} \sum_{b=1}^{P-1} M(P)^b \tilde{\Psi}^{(b)}_P(-N) \sim \sum_{k=0}^{\infty} \frac{L(-2k, \psi^{(a)}_{2P})}{k!} \left( \frac{\pi i}{2PN} \right)^k,
\]

where \( L(k, \psi^{(a)}_{2P}) \) denotes the Dirichlet \( L \)-function associated with \( \psi^{(a)}_{2P}(n) \) defined in (2.1). We note that for \( N \in \mathbb{Z} \) and \( 1 \leq a \leq P - 1 \) we have

\[
\tilde{\Psi}^{(a)}_P(1/N) = -\sum_{k=0}^{2PN} \psi^{(a)}_{2P}(k) e^{k^2/2PN} B_1 \left( \frac{k}{2PN} \right),
\]

\[
\tilde{\Psi}^{(a)}_P(N) = \left( 1 - \frac{a}{P} \right) e^{a^2/2PN},
\]

where the Bernoulli polynomial is \( B_1(x) = x - \frac{1}{2} \). We note that the Eichler integral \( \tilde{\Psi}^{(P-1)}_P(1/N) \) is the specific value of the \( N \)-colored Jones polynomial for torus link \( T_{2,2P} \) [19].

The nearly modular property (2.8) resembles with the transformation properties of the mock theta functions [41, 46]. In fact the right hand side of (2.8) can be rewritten in terms of the Mordell integrals [46], and it suggests a connection between the Eichler integrals of the weight 3/2 modular form and the mock theta functions.

### 3. The 5th Order Mock Theta Functions

We start from the fifth order mock theta functions [5]. Ramanujan defined 10 functions, and here we treat 2 functions among them defined by

\[
\chi_0(q) = \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1})_n},
\]

\[
\chi_1(q) = \sum_{n=0}^{\infty} \frac{q^n}{(q^{n+1})_{n+1}},
\]

Other 8 functions can be given by use of these functions and theta functions [16, 42].

As was noticed in Ref. 29, definitions (3.1) and (3.2) can be extended to outside the unit circle \(|q| > 1\), and we define new functions by substituting \(1/q\) in place of \(q\);

\[
\chi_0^*(q) = 2 - \chi_0(1/q) = 2 - \sum_{n=0}^{\infty} (-1)^n \frac{q^{3n^2 - \frac{1}{2}n}}{(q^{n+1})_n},
\]

\[
\chi_1^*(q) = q^{-1} \chi_1(1/q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{3n(n+1)}}{(q^{n+1})_{n+1}}.
\]
Proposition 1.

\[ \chi_0^* (q) = \sum_{n=0}^{\infty} \chi_{60}^{(1,1,1)} (n) q^{\frac{n^2}{120}}, \]  
\[ \chi_1^* (q) = \sum_{n=0}^{\infty} \chi_{60}^{(1,1,2)} (n) q^{\frac{n^2+49}{120}}, \]

where \( \chi_{60}^{(1,1,1)} (n) \) is an odd periodic function with modulus 60 defined by

\[ \chi_{60}^{(1,1,1)} (n) = \psi_{60}^{(1)} (n) + \psi_{60}^{(11)} (n) + \psi_{60}^{(19)} (n) + \psi_{60}^{(29)} (n), \]

\[ \chi_{60}^{(1,1,2)} (n) = \psi_{60}^{(7)} (n) + \psi_{60}^{(13)} (n) + \psi_{60}^{(17)} (n) + \psi_{60}^{(23)} (n). \]

Proof. These can be proved by applying the Baily chain [9] (see also Ref. 4); if for \( n \geq 0 \) the Bailey pair \((\alpha_n, \beta_n)\) satisfies

\[ \sum_{k=0}^{n} \alpha_k \frac{(q)_{n-k} (xq)_{n+k}}{(q)_{n} (xq)_{n+k}} = \beta_n, \]  
\[ \sum_{n=0}^{\infty} \frac{(\rho_1)_n (\rho_2)_n}{(xq)_{\rho_1 \rho_2}_n} \left( \frac{xq}{\rho_1 \rho_2} \right)^n \alpha_n \frac{1}{(q)_{N-n} (xq)_{N+n}} \]

which reduces to

\[ (1-x) \sum_{n=0}^{\infty} \frac{(q)_n}{(x)_n} x^n \alpha_n (-1)^n q^{\frac{3}{2}n(n-1)} = \sum_{n=0}^{\infty} (q)_n x^n \beta_n (-1)^n q^{\frac{3}{2}n(n-1)}, \]

by setting \( \rho_1 = q, \rho_2 \to \infty \) and \( N \to \infty \).

Eqs. (3.5) and (3.6) follow immediately when we use the Bailey pairs, A(5) and A(8) in Slater’s list [40]. See also Ref. 39, where these functions are called the false theta function. \( \square \)

According to Ref. 29, these false theta functions should rather be identified as the Eichler integral. Namely when we set

\[ \Phi_{2,3,5}(\tau) = \begin{pmatrix} q^{\frac{3}{2} \tau}\chi_{60}^{(1)} (q) \\ q^{49} \chi_{60}^{(1)} (q) \\ q^{\frac{49}{2} \tau}\chi_{60}^{(1)} (q) \end{pmatrix}, \]

which has a form like (2.6) due to (3.5) – (3.6), it can be regarded as the Eichler integral of the vector-valued modular form with weight 3/2;

\[ \Phi_{2,3,5}(\tau) = \frac{1}{2} \sum_{n \in \mathbb{Z}} n \begin{pmatrix} \chi_{60}^{(1,1,1)} (n) \\ \chi_{60}^{(1,1,2)} (n) \\ \chi_{60}^{(1,1,1)} (n) \end{pmatrix} q^{\frac{n^2}{120}}. \]  
\[ 5 \]
The modular $S$- and $T$-matrices under $\tau \to -1/\tau$ and $\tau \to \tau + 1$ are respectively given by

$$ S = \frac{2}{\sqrt{5}} \begin{pmatrix} \sin \left( \frac{\pi}{5} \right) & \sin \left( \frac{2\pi}{5} \right) \\ \sin \left( \frac{2\pi}{5} \right) & -\sin \left( \frac{\pi}{5} \right) \end{pmatrix}, \quad T = \begin{pmatrix} e^{\frac{1}{20}\pi i} & e^{-\frac{1}{20}\pi i} \\ e^{-\frac{1}{20}\pi i} & e^{\frac{1}{20}\pi i} \end{pmatrix}. \quad (3.11) $$

This shows that the modular form $[\eta(\tau)]^{-1/5} \cdot \Phi_{2,3,5}(\tau)$ with rational weight $7/5$ is on the principal congruence subgroup $\Gamma(5)$.

Intriguing is that the Eichler integral $\tilde{\Phi}_{2,3,5}(\tau)$ gives the WRT invariant for the Poincaré homology sphere in a limit of $q$ being the $N$-th root of unity as was proved by Lawrence and Zagier [29].

**Theorem 2.**

$$ e^{\frac{2\pi i}{N}} \left( e^{\frac{2\pi i}{N}} - 1 \right) \tau_N (\Sigma(2, 3, 5)) = 1 - \frac{1}{2} \chi_0^\ast (e^{\frac{2\pi i}{N}}). \quad (3.12) $$

Here the WRT invariant $\tau_N(\mathcal{M})$ for 3-manifold $\mathcal{M}$ is normalized to be

$$ \tau_N(S^3) = 1, $$

$$ \tau_N(S^2 \times S^1) = \sqrt{\frac{N}{2}} \frac{1}{\sin (\pi/N)}. $$

We should remark that we have the icosahedral symmetry of the group $\Gamma(5)$ and the fundamental group of the Poincaré sphere. As a consequence of (3.12), we see that the Eichler integral in $\tau \to 1/N$ has a nearly modular property (2.9) replacing $M(P)$ and $\psi_2^{(a)}(n)$ with $S$ (3.11) and $\chi_60^\ast (n)$ respectively, and $P = 30$.

We can obtain an explicit form of the quantum invariant (3.12) as a linear combination of (2.10) by taking a limiting value of the Eichler integral $\tilde{\Phi}_{2,3,5}(\tau)$ in $\tau \to 1/N$ [29]. To derive this invariant in terms of $q$-series, it is useful to rewrite the Eichler integral in the form such that the infinite sum terminates at the finite sum in the case of $q$ being root of unity.

**Proposition 3.**

$$ \chi_0^\ast (q) = 1 + q \sum_{m=0}^{\infty} q^{2m} (q^{m+1})_m = \sum_{n=0}^{\infty} q^n \left( q^n \right)_n, \quad (3.13) $$

$$ \chi_1^\ast (q) = \sum_{n=0}^{\infty} q^n \left( q^n \right)_n. \quad (3.14) $$
Proof. From (3.3) we compute as follows;

\[
\chi^*_0(q) = 1 + \sum_{n=0}^{\infty} (-1)^n q^{\frac{1}{2}(n+1)(3n+2)} (q^{n+2})_{n+1}^{-1}
\]

\[
= 1 + q \sum_{n=0}^{\infty} \frac{(q^{2n+3})_{\infty}}{(q^{n+2})_{\infty}} (-1)^n q^{\frac{3}{2}n^2 + \frac{3}{2}n}
\]

\[
= 1 + q \sum_{n,k=0}^{\infty} (-1)^n q^{\frac{3}{2}n^2 + \frac{3}{2}n} \frac{(q^{n+1})_k}{(q)_k} q^{(n+2)_k}
\]

\[
= 1 + q \sum_{m \geq n \geq 0} q^{2m} q^{(n+1)_m}
\]

\[
= 1 + q \sum_{m=0}^{\infty} q^{2m} (q^{m+1})_m
\]

\[
= 1 + q \sum_{m=0}^{\infty} \frac{q^{2m}}{(q)_m} \frac{1}{(q^{2m+1})_{\infty}}
\]

\[
= 1 + q \sum_{n,m=0}^{\infty} q^n q^{2nm+2m} \frac{(q)_n}{(q)_m}
\]

\[
= 1 + q \sum_{n=0}^{\infty} q^n (q^{n+1})_{n+1}.
\]

(3.17)

Identity (3.4) can be proved in the same manner. \qed

We see that, by definition of the \(q\)-product, the infinite sum in the expression (3.13) terminates at the finite order when \(q\) is root of unity. Furthermore the expression (3.13) exactly coincides with the form given by Le in Refs. 30, 31, where the WRT invariant was computed by use of Habiro’s cyclotomic expansion of the \(N\)-colored Jones polynomial for trefoil [13, 14, 28, 33]

\[
J_{T_{2,3}}(N) = \sum_{k=0}^{\infty} q^{-k(k+2)} (q^{1-N})_k (q^{1+N})_k,
\]

(3.15)

where we have used the normalized colored Jones polynomial s.t. \(J_{\text{unknot}}(N) = 1\). It is known to be rewritten as [14, 20, 30]

\[
J_{T_{2,3}}(N) = q^{1-N} \sum_{k=0}^{\infty} q^{-kN} (q^{1-N})_k,
\]

(3.16)

from which, simply applying \((+1)\)-surgery on this expression following Ref. 38 (see also Ref. 31), we obtain another expression

\[
\chi^*_0(q) = 1 + q \sum_{k,n \geq 0} \frac{k}{n} q^{k(k+1) + \frac{1}{2}n(3n+5) + kn}.
\]

(3.17)
This expression also reduces to finite sum when $q$ is root of unity. We do not have a direct proof of this $q$-series identity at present.

To close this section we comment on the $q$-hypergeometric type generating function of the $L$-function at negative values. Studies on such generating functions have been developed since Ref. 44. See Refs. 8, 17, 18, 32, 36. Based on (3.13), we have the following as a power series in $t$;

\[
e^{-t/120} \sum_{n=0}^{\infty} e^{-nt} (1 - e^{-nt}) (1 - e^{-(n+1)t}) \cdot \cdots (1 - e^{-(2n-1)t}) = \frac{1}{2} \sum_{k=0}^{\infty} L \left(\frac{-2k, \chi_{60}^{(1,1,1)}}{60} \right) \left(\frac{-t}{120}\right)^k,
\]

\[
e^{-49t/120} \sum_{n=0}^{\infty} e^{-nt} (1 - e^{-(n+1)t}) (1 - e^{-(n+2)t}) \cdot \cdots (1 - e^{-2nt}) = \frac{1}{2} \sum_{k=0}^{\infty} L \left(\frac{-2k, \chi_{60}^{(1,1,2)}}{60} \right) \left(\frac{-t}{120}\right)^k,
\]

where

\[
2 \cos(5x) \cos(9x) \cos(15x) = \sum_{k=0}^{\infty} L \left(\frac{-2k, \chi_{60}^{(1,1,1)}}{60} \right) \left(\frac{-t}{120}\right)^k x^{2k},
\]

\[
2 \cos(5x) \cos(3x) \cos(15x) = \sum_{k=0}^{\infty} L \left(\frac{-2k, \chi_{60}^{(1,1,2)}}{60} \right) \left(\frac{-t}{120}\right)^k x^{2k}.
\]

4. THE 3RD ORDER MOCK THETA FUNCTIONS

We next study the third order mock theta functions. Among 4 Ramanujan’s mock theta functions of the third order, we study functions defined by

\[
\phi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2; q^2)^n},
\]

\[
\nu(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q; q^2)^{n+1}}.
\]

Watson gave 3 more functions in Ref. 41, which were in “lost” notebook [37]. Other third order mock theta functions can be given from these functions and the theta functions as was proved in Ref. 41.
Both defining \( q \)-series \( \phi(q) \) and \( \nu(q) \) converge not only inside, but also outside the unit circle as was noticed in Ref. 45, and we define new functions replacing \( q \) by \( 1/q \) as

\[
\phi^*(q) = \phi(1/q) = \sum_{n=0}^{\infty} \frac{q^n}{(-q^2;q^2)_n},
\]

(4.3)

\[
\nu^*(q) = q^{-1}\nu(1/q) = \sum_{n=0}^{\infty} \frac{q^n}{(-q^2;q^2)_{n+1}}.
\]

(4.4)

As in the case of the 5th order mock theta functions, these are the false theta functions in a sense of Rogers as was proved in Ref. 2.

**Proposition 4.**

\[
\phi^*(q) = \sum_{n=0}^{\infty} \chi_{24}^{(1)}(n) q^{\frac{k}{24}(n^2-1)},
\]

(4.5)

\[
\nu^*(q) = \sum_{n=0}^{\infty} \chi_{24}^{(2)}(n) q^{\frac{k}{24}(n^2-16)},
\]

(4.6)

\[
\phi^*(-q) = \sum_{n=0}^{\infty} \psi_{16}^{(1)}(n) q^{\frac{k}{24}(n^2-1)},
\]

(4.7)

where \( \chi_{24}^{(a)}(n) \) is an odd periodic function with modulus 24 defined by

\[
\chi_{24}^{(1)}(n) = \psi_{24}^{(1)}(n) + \psi_{24}^{(5)}(n) + \psi_{24}^{(7)}(n) + \psi_{24}^{(11)}(n),
\]

\[
\chi_{24}^{(2)}(n) = \psi_{24}^{(4)}(n) + \psi_{24}^{(8)}(n),
\]

and \( \psi_{24}^{(a)}(n) \) is defined in (2.1).

Such false theta functions can be defined based on some of other mock theta functions of the third order. We recall the definitions [37, 41];

\[
f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{[(q^2;q^2)_n]^2},
\]

(4.8)

\[
\omega(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{[(q;q^2)_{n+1}]^2},
\]

(4.9)

\[
\chi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q+q^2)(1-q^2+q^4)\cdots(1-q^n+q^{2n})},
\]

(4.10)

\[
\varrho(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(1+q+q^2)(1+q^3+q^6)\cdots(1+q^{2n+1}+q^{4n+2})}.
\]

(4.11)
As was proved in Ref. 11, we have

\[ f(q) = 2 - \sum_{n=0}^{\infty} (-1)^n \frac{q^n}{(-q)_n}, \quad (4.12) \]

\[ \omega(q) = \sum_{n=0}^{\infty} \frac{q^n}{(q; q^2)_{n+1}}, \quad (4.13) \]

\[ \chi(q) = 1 - e^{2\pi i/3} \sum_{n=1}^{\infty} \frac{e^{2\pi i n} q^n}{(e^{2\pi i/3} q)_n}, \quad (4.14) \]

\[ \varphi(q) = \sum_{n=0}^{\infty} \frac{e^{-2\pi i n} q^n}{(e^{2\pi i/3} q; q^2)_{n+1}}. \quad (4.15) \]

We can extend these defining q-series into \(|q| > 1\), and define new functions;

\[ f^*(q) = f(1/q) = 2 - \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n-1)/2}}{(-q)_n}, \quad (4.16) \]

\[ \omega^*(q) = -q^{-1} \omega(1/q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n+1}}{(q; q^2)_{n+1}}, \quad (4.17) \]

\[ \chi^*(q) = \chi(1/q) = 1 - e^{2\pi i/3} \sum_{n=1}^{\infty} \frac{e^{-2\pi i n} q^{n(n-1)/2}}{(e^{2\pi i/3} q)_n}, \quad (4.18) \]

\[ \varphi^*(q) = \varphi(1/q) = \sum_{n=0}^{\infty} \frac{e^{-2\pi i n} q^{n+1}}{(e^{-2\pi i/3} q; q^2)_{n+1}}. \quad (4.19) \]

By applying the Bailey chain method with the pairs such as C(7) and C(5) in Slater’s list [40], we obtain [39] the following;

**Proposition 5.**

\[ f^*(q) = 2 \sum_{n=0}^{\infty} \psi_6^{(1)}(n) q^{\frac{1}{24} (n^2 - 1)}, \quad (4.20) \]

\[ \omega^*(q) = \sum_{n=0}^{\infty} \left( \psi_6^{(1)}(n) + \psi_6^{(2)}(n) \right) q^{\frac{1}{3} (n^2 - 1)}, \quad (4.21) \]

\[ \chi^*(q) = \sum_{n=0}^{\infty} \psi_6^{(1)}(n) q^{\frac{1}{24} (n^2 - 1)} \left( 1 + e^{-2\pi i n} \right), \quad (4.22) \]

\[ \varphi^*(q) = \sum_{n=0}^{\infty} \left( \psi_6^{(1)}(n) + \psi_6^{(2)}(n) \right) q^{\frac{1}{7} (n^2 - 1)} e^{\frac{2\pi i}{3} (1-n)}. \quad (4.23) \]
We remark that, comparing with (4.6), we find
\[ \omega^*(q^2) = \nu^*(q^2). \tag{4.24} \]

To study the transformation property of these functions using (2.9), it is better to regard these functions as the Eichler integrals. We define
\[ \tilde{\Phi}_{2,3,4}(\tau) = \begin{pmatrix} q^{\frac{1}{12}} \phi^*(q^{1/2}) \\ q^{\frac{1}{4}} \nu^*(q^{1/2}) \\ q^{\frac{1}{12}} \phi^*(-q^{1/2}) \end{pmatrix}. \tag{4.25} \]

Due to (4.5) – (4.7), this is the Eichler integral of the vector-valued modular form with weight 3/2;
\[ \Phi_{2,3,4}(\tau) = \frac{1}{2} \sum_{n \in \mathbb{Z}} n \begin{pmatrix} \lambda_{24}^{(1)}(n) \\ \lambda_{34}^{(2)}(n) \\ \psi^{(1)}_{6}(n) \end{pmatrix} q^{\frac{2}{12} n^2}. \tag{4.26} \]

The modular S- and T-matrices under \( \tau \to -1/\tau \) and \( \tau \to \tau + 1 \) are respectively computed by use of (2.3) and (2.4) as
\[ S = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} e^{\frac{1}{12} \pi i} \\ e^{\frac{1}{4} \pi i} \\ e^{\frac{1}{12} \pi i} \end{pmatrix}, \tag{4.27} \]

which shows that the modular form \( [\Phi_{2,3,4}(\tau)]^2 \) is on \( \Gamma(4) \), and that it has the octahedral symmetry. As a consequence the modular form \( \Phi_{2,3,5}(\tau) \) can be rewritten in terms of the Jacobi theta functions [23].

We then obtain an explicit form of the Eichler integral (4.25) in a limit \( \tau \to 1/N \) in terms of the Bernoulli polynomial as (2.10), and in this case it gives the WRT invariant for the Seifert manifold \( M(2, 3, 4) \) [23], whose fundamental group represents the octahedral group;

**Theorem 6.**
\[ e^{\frac{3 \pi i}{12N}} \left( e^{\frac{2 \pi i}{12N}} - 1 \right) \tau_N (M(2, 3, 4)) = \frac{\sqrt{2}}{4} \left( 1 + (-1)^N \right) \left( 2 - \phi^*(e^{\frac{2 \pi i}{12N}}) \right). \tag{4.28} \]

By definition (4.3) of \( \phi^*(q) \), we can conclude that the WRT invariant is a limiting value from outside the unit circle of the Ramanujan mock theta function of the third order.

Concerning the function \( \omega^*(q^2) \), we see from (4.21) that the Eichler integral (2.6) with \( P = 3 \) is written as
\[ \tilde{\Phi}_{P=3}(\tau) = \frac{1}{2} q^{\frac{1}{12}} \begin{pmatrix} \omega^*(q^{1/4}) + \omega^*(-q^{1/4}) \\ \omega^*(q^{1/4}) - \omega^*(-q^{1/4}) \end{pmatrix}, \tag{4.29} \]

and a result of Ref. 23 indicates the following;
Theorem 7.

\[ e^{\frac{2\pi i}{N}} \left( e^{\frac{2\pi i}{2N}} - 1 \right) \tau_N(M(2, 2, 3)) = 1 - \omega^*(e^{\frac{2\pi i}{N}}) + e^{\frac{2\pi i}{N}} \left( 1 - \omega^*(-e^{\frac{2\pi i}{N}}) \right). \] (4.30)

To rewrite the quantum invariant in terms of the \( q \)-product, it makes sense to give the mock (false) theta functions \( \phi^*(q) \) and \( \nu^*(q) \) in the form which terminates at the finite order in the case of \( q \) being root of unity.

Proposition 8.

\[ \phi^*(q) = 1 + q \sum_{n=0}^{\infty} (q; -q)_n q^n, \] (4.31)

\[ \nu^*(q) = \omega^*(q^2) = \sum_{n=0}^{\infty} (q^2; q^4)_n q^{2n}. \] (4.32)

**Proof.** When we define the \( q \)-hypergeometric function by

\[ F \left( \begin{array}{c} a \\ b \end{array} ; q, t \right) = \sum_{n=0}^{\infty} \frac{(aq)_n}{(bq)_n} t^n, \] (4.33)

we have (see (20.71) and (20.72) in Ref. 11)

\[ \frac{1}{1 + a} F \left( \begin{array}{c} 0 \\ -a \end{array} ; q, a \right) = F \left( \begin{array}{c} q^{-1} \\ 0 \end{array} ; q^2, a^2 \right), \] (4.34)

\[ (1 - a) F \left( \begin{array}{c} -1 \\ 0 \end{array} ; q, a \right) = F \left( \begin{array}{c} 0 \\ a \end{array} ; q^2, a q \right). \] (4.35)

Applying these identities, we obtain the statement. \( \Box \)

Alternative computation of \( q \)-series follows from topological fact that the Seifert manifold \( M(2, 3, 4) \) is constructed from \((-2)\)-surgery on the right-handed trefoil [35]. Applying a surgery formula [26, 38] to the colored Jones polynomial (3.16) for the trefoil, we get

\[ \phi^*(q) = 1 + q \sum_{k \geq n \geq 0} (-1)^n \left[ \begin{array}{c} k \\ n \end{array} \right] q^{n(2n+3)+k^2}. \] (4.36)
To close this section, we note that (4.31) and (4.32) may give formulae as follows as a power series in $t$;

$$e^{-t/24} \left(1 + \sum_{n=0}^{\infty} e^{-(n+1)t} (1 - e^{-t}) (1 + e^{-2t}) \cdots (1 + (-1)^n e^{-nt})\right)$$

$$= \sum_{k=0}^{\infty} \frac{L \left(-2k; \chi_{24}^{(1)}\right)}{k!} \left(-\frac{t}{24}\right)^k,$$

$$e^{-t/3} \sum_{n=0}^{\infty} e^{-nt} (1 - e^{-t}) (1 - e^{-3t}) \cdots (1 - e^{-(2n-1)t}) = \sum_{k=0}^{\infty} \frac{L \left(-2k; \chi_{24}^{(2)}\right)}{k!} \left(-\frac{t}{48}\right)^k,$$

where

$$2 \frac{\cos(3x) \cos(2x)}{\cos(6x)} = \sum_{k=0}^{\infty} \frac{L \left(-2k; \chi_{24}^{(1)}\right)}{(2k)!} (-1)^k x^{2k},$$

$$\frac{\cos(x)}{\cos(3x)} = \sum_{k=0}^{\infty} \frac{L \left(-2k; \chi_{24}^{(2)}\right)}{(2k)!} (-1)^k \left(\frac{x}{2}\right)^{2k}.$$  

### 5. The 7th Order Mock Theta Functions

We continue to study the seventh order mock theta functions [5, 15]. There are 3 Ramanujan’s mock theta functions, and they are read as

$$\mathcal{F}_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^{n+1})_n},$$

$$\mathcal{F}_1(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q^n)_n},$$

$$\mathcal{F}_2(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^{n+1})_{n+1}}.$$  

These $q$-series have also meaning even when $|q| > 1$, and for our convention we define

$$\mathcal{F}_0^*(q) = \mathcal{F}_0(1/q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n+1)}}{(q^{n+1})_n},$$

$$\mathcal{F}_1^*(q) = \mathcal{F}_1(1/q) = \sum_{n=1}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n-1)}}{(q^n)_n},$$

$$\mathcal{F}_2^*(q) = q^{-1} \mathcal{F}_2(1/q) = -\sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n+3)}}{(q^{n+1})_{n+1}}.$$
By applying the method of the Bailey chain \((3.9)\) with the Bailey pair A(3), A(2), and A(4) in Slater’s list \([40]\), we can prove that these are the \textit{false} theta function in the sense of Rogers \([39]\).

**Proposition 9.**

\[
F_0^*(q) = \sum_{n=0}^{\infty} \chi^{(1,1,1)}_{84}(n) q^{\frac{n^2}{168}}, \quad (5.7)
\]

\[
F_1^*(q) = \sum_{n=0}^{\infty} \chi^{(1,1,2)}_{84}(n) q^{\frac{n^2-25}{168}}, \quad (5.8)
\]

\[
F_2^*(q) = \sum_{n=0}^{\infty} \chi^{(1,1,3)}_{84}(n) q^{\frac{n^2-121}{168}}, \quad (5.9)
\]

where \(\chi^{\ell}_{84}(n)\) is an odd periodic function with modulus \(84\) defined by

\[
\chi^{(1,1,1)}_{84}(n) = \psi^{(1)}_{84}(n) - \psi^{(13)}_{84}(n) - \psi^{(29)}_{84}(n) + \psi^{(41)}_{84}(n),
\]

\[
\chi^{(1,1,2)}_{84}(n) = -\psi^{(5)}_{84}(n) - \psi^{(19)}_{84}(n) - \psi^{(23)}_{84}(n) - \psi^{(37)}_{84}(n),
\]

\[
\chi^{(1,1,3)}_{84}(n) = -\psi^{(11)}_{84}(n) - \psi^{(17)}_{84}(n) - \psi^{(25)}_{84}(n) - \psi^{(31)}_{84}(n).
\]

This proposition enables us to define \([21]\)

\[
\tilde{\Phi}_{2,3,7}(\tau) = \begin{pmatrix} q^{1/168} F_0^*(q) \\ q^{25/168} F_1^*(q) \\ q^{121/168} F_2^*(q) \end{pmatrix}, \quad (5.10)
\]

which can be regarded as the Eichler integral of the vector-valued modular form with weight \(3/2\);

\[
\Phi_{2,3,7}(\tau) = \frac{1}{2} \sum_{n \in \mathbb{Z}} \begin{pmatrix} \chi^{(1,1,1)}_{84}(n) \\ \chi^{(1,1,2)}_{84}(n) \\ \chi^{(1,1,3)}_{84}(n) \end{pmatrix} q^{\frac{n^2}{168}}. \quad (5.11)
\]

The modular \(S\)- and \(T\)-matrices under \(\tau \to -1/\tau\) and \(\tau \to \tau + 1\) are respectively given by

\[
S = -\frac{2}{\sqrt{7}} \begin{pmatrix} \sin(\frac{\pi}{7}) & \sin(\frac{2\pi}{7}) & \sin(\frac{3\pi}{7}) \\ \sin(\frac{2\pi}{7}) & -\sin(\frac{3\pi}{7}) & \sin(\frac{4\pi}{7}) \\ \sin(\frac{3\pi}{7}) & \sin(\frac{4\pi}{7}) & -\sin(\frac{5\pi}{7}) \end{pmatrix},
\]

\[
T = \begin{pmatrix} e^{\frac{1}{84}\pi i} & e^{\frac{25}{84}\pi i} & e^{\frac{47}{84}\pi i} \\ e^{\frac{25}{84}\pi i} & e^{\frac{47}{84}\pi i} & e^{\frac{1}{84}\pi i} \end{pmatrix}. \quad (5.12)
\]

Result in Ref. 21 thus proves that the mock (false) theta function \(F_0^*(q)\) gives the WRT invariant for the Brieskorn homology sphere \(\Sigma(2, 3, 7)\) in a limit that \(\tau\) goes to the \(N\)-th root of unity;
Theorem 10.

\[ e^{-\frac{1}{2\text{π}^2}} \left( e^{\frac{2\text{π}i}{N}} - 1 \right) \tau_N (\Sigma(2, 3, 7)) = \frac{1}{2} F^*_0(e^{\frac{2\text{π}i}{N}}). \] (5.13)

An explicit form of the right hand side follows from (2.10). Correspondingly an exact asymptotic expansion of the WRT invariant in \( N \to \infty \) can be computed from the nearly modular property (2.9) \[21\].

By the correspondence with the quantum invariant which is originally defined for \( q \) being root of unity, it is tempting to get the \( q \)-series expression which terminates at the finite order when \( q \) is root of unity. We obtain the following from (5.4);

\[ F^*_0(q) = 1 - q \sum_{k \geq n \geq 0} (-1)^n \binom{k}{n} q^{(n+2)k - \frac{1}{2}n(n+1)}. \] (5.14)

Another construction of \( q \)-series follows from topological understanding that the Brieskorn homology sphere \( \Sigma(2, 3, 7) \) is constructed by \((-1)\)-surgery on the right-handed trefoil \[35\]. Applying a surgery formula to the colored Jones polynomial for the trefoil (3.16) we obtain

\[ F^*_0(q) = 1 - \sum_{k \geq n \geq 0} (-1)^n \binom{k}{n} q^{k+\frac{1}{2}n(n-1)+(k+n+1)^2}. \] (5.15)

A simpler expression was given in Ref. 30 where used was the cyclotomic expansion of the colored Jones polynomial for trefoil (3.15).

6. THE 6TH ORDER MOCK THETA FUNCTIONS

There are 7 sixth order mock theta functions in Ramanujan’s “lost” notebook \[37\], and we treat 3 functions among them defined by

\[ \varphi(q) = \sum_{n=0}^{\infty} (-1)^n q^{n^2} \frac{(q; q^2)_n}{(-q)_{2n}}, \] (6.1)

\[ \psi(q) = \sum_{n=0}^{\infty} (-1)^n q^{(n+1)^2} \frac{(q; q^2)_n}{(-q; q)_{2n+1}}, \] (6.2)

\[ \rho(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} \frac{(-q)_n}{(q; q^2)_{n+1}}. \] (6.3)
Other functions can be written in terms of these functions and theta functions as was proved in Ref. 7. These $q$-series converge also for $|q| > 1$, and for our convention we define

$$
\varphi^* (q) = \varphi (1/q) = \sum_{n=0}^{\infty} q^n \frac{(q; q^2)_n}{(-q)_{2n}}, \quad (6.4)
$$

$$
\psi^* (q) = \psi (1/q) = \sum_{n=0}^{\infty} q^n \frac{(q; q^2)_n}{(-q)_{2n+1}}, \quad (6.5)
$$

$$
\rho^* (q) = -q^{-1} \rho (1/q) = \sum_{n=0}^{\infty} (-1)^n q^n \frac{(-q)_n}{(q; q^2)_{n+1}}. \quad (6.6)
$$

These new functions are also the false theta functions à la Rogers as follows;

**Proposition 11.**

$$
\varphi^* (q) = \sum_{n=0}^{\infty} \left( \psi^{(1)}_{12} (n) + \psi^{(5)}_{12} (n) \right) q^{\frac{n^2-1}{24}}, \quad (6.7)
$$

$$
\psi^* (q) = \sum_{n=0}^{\infty} \psi^{(3)}_{12} (n) q^{\frac{n^2-9}{24}}, \quad (6.8)
$$

$$
\rho^* (q) = \psi^* (q^2) = \sum_{n=0}^{\infty} \psi^{(6)}_{24} (n) q^{\frac{n^2-36}{48}}, \quad (6.9)
$$

where the odd periodic function $\psi^{(a)}_{2p} (n)$ is defined in (2.1).

**Proof.** We do not have a simple proof. Anyway we recall the transformation formula of the $q$-hypergeometric functions as (see, e.g., Ref. 3)

$$
\sum_{n=0}^{\infty} \frac{\left( \alpha; q^2 \right)_n \left( \beta \right)_n \left( \gamma \right)_n}{\left( q^2; q^2 \right)_n \left( \gamma \right)_n} z^n = \frac{\left( \beta \right)_\infty}{\left( \gamma \right)_\infty} \left( \alpha z; q^2 \right)_\infty \sum_{m=0}^{\infty} \frac{\left( z \right)_m \left( z; q^2 \right)_m}{\left( q \right)_m \left( \alpha z; q^2 \right)_m} \beta^m, \quad (6.10)
$$

and another identity (see, e.g., (25.96) in Ref. 11)

$$
\sum_{m=0}^{\infty} \frac{\left( \alpha q \right)_{2m} \left( \beta q \right)_{2m}}{\left( \alpha q \right)_m \left( q \right)_m} z^m = \frac{\left( \beta z \right)_\infty \left( \alpha z \right)_\infty}{\left( z \right)_\infty} \sum_{k=0}^{\infty} \frac{\left( \beta q \right)_k \left( \alpha z q \right)_k}{\left( q \right)_k \left( \beta z q \right)_k} (-\alpha z)^k q^{\frac{k}{2} (3k+1)}. \quad (6.11)
$$
To prove (6.8), we compute as follows;

\[
\psi^*(q) = \frac{1}{1 + q} \sum_{n=0}^{\infty} q^n \frac{(q)_{2n}}{(q^2; q^2)_n (-q^2)_{2n}} \\
= \frac{(q^2; q^2)_\infty}{(-q)_{\infty}} \sum_{m=0}^{\infty} q^m \frac{(-q)_m (q; q^2)_m}{(q)_m} \quad \text{(by (6.10) with } \alpha = 0, \beta = q, \gamma = -q^2, z = q) \\
= (q)_\infty \sum_{m=0}^{\infty} q^m \frac{(q^2)_{2m}}{[(q)_m]^2} \\
= \sum_{k=0}^{\infty} (-q)^k \frac{q^2}{q^2} k^{(k+1)} \quad \text{(by (6.11) with } \alpha = 1, \beta = 0, z = q)
\]

A proof of (6.7) follows in the same manner;

\[
\varphi^*(q) = \sum_{n=0}^{\infty} q^n \frac{(q)_{2n}}{(q^2; q^2)_n (-q)_{2n}} \\
= \frac{(q)_{\infty}}{(-q)_{\infty}} \sum_{m=0}^{\infty} q^m \frac{(-1)_m (q^2; q^2)_m}{(q)_m} \quad \text{(by (6.10) with } \alpha = 0, \beta = q, \gamma = -q, z = q) \\
= (q)_{\infty} \left(1 + 2q \sum_{m=0}^{\infty} q^m \frac{(q^2)_{2m}}{(q)_m (q^2)_m} \right) \\
= \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}n(3n-1)} + 2 \sum_{k=0}^{\infty} (-1)^k q^{\frac{1}{2}(k+1)(3k+2)} \quad \text{(by (1.5) and (6.11) with } \alpha = q, \beta = 0, z = q)
\]

Identity (6.9) can be proved as follows;

\[
\rho^*(q) = \frac{1}{1 - q} \sum_{n=0}^{\infty} (-q)^n \frac{(q^2; q^2)_n}{(q)_n (q^2; q^2)_n} \\
= (q^2; q^2)_\infty \sum_{n=0}^{\infty} q^{2n} \frac{(q^2; q^2)_n (-q)_{2n}}{(q^2; q^2)_n} \quad \text{(by (6.10) with } \alpha = q, \beta = -q, \gamma = 0, z = q^2) \\
= (q^2; q^2)_\infty \sum_{n=0}^{\infty} q^{2n} \frac{(-q^2; q^2)_n (q^2; q^4)_n}{(q^2; q^2)_n} \\
= (-q^4; q^2)_\infty (q^2; q^4)_\infty \sum_{n=0}^{\infty} q^{2n} \frac{(q^2; q^2)_{2n}}{(q^4; q^4)_n (-q^4; q^2)_{2n}} \quad \text{(by (6.10) with } q \to q^2 \text{ and } \alpha = 0, \beta = q^2, \gamma = -q^4, z = q^2) \\
= \sum_{n=0}^{\infty} q^{2n} \frac{(q^2; q^4)_n}{(-q^2; q^2)_{2n+1}},
\]

which completes the proof. \(\square\)
Looking at (6.7) and (6.8), we set the vector-valued functions as

$$\Phi_{2,3,3}(\tau) = \frac{1}{2} \sum_{n \in \mathbb{Z}} n \left( \frac{1}{\sqrt{2}} \left( \psi^{(1)}_{12}(n) + \psi^{(5)}_{12}(n) \right) \psi^{(3)}_{12}(n) \right) q^{\frac{n^2}{24}}. \quad (6.12)$$

By the Poisson summation formula we find that this is a modular form with weight $3/2$, and that $S$- and $T$-matrices are respectively given by

$$S = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}, \quad T = \begin{pmatrix} e^{\frac{\pi i}{12}} & e^{\frac{\pi i}{3}} \\ e^{\frac{\pi i}{3}} & e^{\frac{\pi i}{12}} \end{pmatrix}. \quad (6.13)$$

The modular form $[\eta(\tau)]^{-1} \cdot \Phi_{2,3,3}(\tau)$ is on $\Gamma(3)$, which has the symmetry of the tetrahedral group. By definition, the mock theta functions can be regarded as the Eichler integral of this modular form;

$$\tilde{\Phi}_{2,3,3}(\tau) = \begin{pmatrix} q^{\frac{\pi i}{12}} \varphi^*(q) \\ q^{\frac{\pi i}{3}} \psi^*(q) \end{pmatrix}. \quad (6.14)$$

The Eichler integral $\tilde{\Phi}_{2,3,3}(\tau)$ has appeared as the WRT invariant for the Seifert manifold $M(2, 3, 3)$ [23] whose fundamental group is the tetrahedral group;

**Theorem 12.**

$$e^{\frac{\pi i}{N}} \left( e^{\frac{2\pi i}{N}} - 1 \right) \tau_N (M(2, 3, 3)) = \frac{1 + 2 e^{\frac{2\pi i N}{3}}}{\sqrt{3}} \left( 1 - \frac{1}{2} \varphi^*(e^{\frac{2\pi i}{N}}) \right) - \frac{1 - e^{\frac{2\pi i N}{3}}}{\sqrt{3}} e^{\frac{2\pi i}{3 N}} \psi^*(e^{\frac{2\pi i}{N}}). \quad (6.15)$$

As was noticed in Ref. 23, the WRT invariants for the manifold $M(2, 2, 6)$ and $M(2, 3, 3)$ are related to each other, and we also obtain the WRT invariant for $M(2, 2, 6)$ as

**Theorem 13.**

$$e^{\frac{2\pi i}{N}} \left( e^{\frac{2\pi i}{N}} - 1 \right) \tau_N (M(2, 2, 6)) = 2 \left( 1 - \varphi^*(e^{\frac{2\pi i}{N}}) \right) \quad (6.16)$$

The function $q^{1/8} \rho^*(q^{1/6}) = q^{1/8} \psi^*(q^{1/3})$ is also regarded as the Eichler integral of the weight-3/2 modular form $[\eta(\tau)]^3$, which, by the Jacobi triple product formula (1.5), is written in an infinite sum as

$$[\eta(\tau)]^3 = \sum_{n=0}^{\infty} (-1)^n (2n + 1) q^{\frac{1}{3} (2n+1)^2}. \quad (6.17)$$

Recalling a result in Ref. 23, we can conclude that it gives the WRT invariant for the Seifert manifold $M(2, 2, 2)$.

**Theorem 14.**

$$e^{\frac{2\pi i}{N}} \left( e^{\frac{2\pi i}{N}} - 1 \right) \tau_N (M(2, 2, 2)) = 2 \left( 1 - 2 \rho^*(e^{\frac{\pi i}{3 N}}) \right). \quad (6.18)$$
7. The 10th Order Mock Theta Functions

Remaining Ramanujan’s mock theta functions are the tenth order. There are 4 functions in Ref. 37 (see also Ref. 10), and they are defined by

\[ \Phi(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q; q^2)_{n+1}}, \]  
\( \Psi(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)/2}}{(q; q^2)_{n+1}}, \)  
\[ X(q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n^2}}{(-q)_{2n}}, \]  
\[ \chi(q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{(n+1)^2}}{(-q)_{2n+1}}. \]

All these defining \( q \)-series converge also for \(|q| > 1\), and as before we define our functions as follows:

\[ \Phi^*(q) = -q^{-1} \Phi(1/q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+3)/2}}{(q; q^2)_{n+1}}, \]  
\[ \Psi^*(q) = -\Psi(1/q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)/2}}{(q; q^2)_{n+1}}, \]  
\[ X^*(q) = X(1/q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(-q)_{2n}}, \]  
\[ \chi^*(q) = \chi(1/q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(-q)_{2n+1}}. \]

We have the following;
Proposition 15.

\[
\Phi^*(q) = \sum_{n=0}^{\infty} \left( \psi^{(2)}_{10}(n) + \psi^{(3)}_{10}(n) \right) q^{\frac{1}{2}(n^2-4)}, \tag{7.9}
\]

\[
\Psi^*(q) = \sum_{n=0}^{\infty} \left( \psi^{(1)}_{10}(n) + \psi^{(4)}_{10}(n) \right) q^{\frac{1}{2}(n^2-1)}, \tag{7.10}
\]

\[
X^*(q) = \sum_{n=0}^{\infty} \psi^{(1)}_{10}(n)q^{\frac{1}{10}(n^2-1)}, \tag{7.11}
\]

\[
\chi^*(q) = \sum_{n=0}^{\infty} \psi^{(3)}_{10}(n)q^{\frac{1}{40}(n^2-9)}. \tag{7.12}
\]

\textbf{Proof.} All these identities follow from (3.9) with the Bailey pair, C(4), C(3), G(3), and G(2) in Slater’s list \[40\] respectively.

Our functions are regarded as the Eichler integral (2.6) of the modular form (2.2) with \(P = 5\).

A result in Ref. 23 proves that these give the WRT invariant for the Seifert manifold \(M(2, 2, 5)\).

Theorem 16.

\[
e^{\frac{2\pi i}{2N}} \left( e^{\frac{2\pi i}{N}} - 1 \right) \tau_N (M(2, 2, 5)) = 1 + e^{-\frac{N}{2} \pi i} - \left( 1 - e^{-\frac{N}{2} \pi i} \right) \Psi^*(e^{\frac{-2\pi i}{N}}) - 2 e^{-\frac{N}{2} \pi i} X^*(e^{\frac{4\pi i}{N}}). \tag{7.13}
\]

By applying (1.2) to the defining \(q\)-series, it is straightforward to get the infinite \(q\)-series, which terminates at the finite order for root of unity, of the quantum invariants.

8. Discussions

We have shown that the Ramanujan mock theta functions are related to the quantum invariant for the Seifert manifolds \(M(p_1, p_2, p_3)\). We see that some of the mock theta functions can be defined also even when \(q\) is outside the unit circle, and that, by replacing \(q\) by \(1/q\), they become the \textit{false} theta functions in a sense of Rogers. This substitution \(q \to 1/q\) seems to correspond to study, not directly the Eichler integral \(\tilde{\Psi}^{(a)}_P(\tau)\) (2.6), but the Eichler integral \(\hat{\Psi}^{(a)}_P(z)\) (2.7) defined in the lower half plane, which has a nice transformation property (2.8). These \textit{false} theta functions have already appeared in studies of the WRT invariant for the Seifert manifold as the Eichler integral of the half-integral weight modular form. Namely a limiting value of the Eichler integral in \(\tau \to 1/N\) for \(N \in \mathbb{Z}_{>0}\) gives the WRT invariant \(\tau_N(M)\). Combining these results, we can deduce a remarkable connection between the Ramanujan mock theta functions and the quantum invariants. We summarize this correspondence in Table \[1\] Unfortunately we do not find such
Table 1: Mock theta functions as the SU(2) WRT invariant $\tau_N(M)$ for the Seifert manifolds $M = M(p_1, p_2, p_3)$. Functions of order 2, 4, and 8 are our proposal based on the WRT invariants.

correspondence for the eighth order mock theta functions proposed in Ref. 12. Motivated from Ref. 39, 40, we may expect that the functions defined by

\begin{align}
D_5(q) &= \sum_{n=0}^{\infty} q^n \frac{(-q)_n}{(q^2;q^2)_{n+1}}, \\
D_6(q) &= \sum_{n=0}^{\infty} q^n \frac{(-q^2; q^2)_n}{(q^{n+1})_{n+1}}, \\
I_{12}(q) &= \sum_{n=0}^{\infty} q^{2n} \frac{(-q; q^2)_n}{(q^{n+1})_{n+1}}, \\
I_{13}(q) &= \sum_{n=0}^{\infty} q^n \frac{(-q; q^2)_n}{(q^{n+1})_{n+1}},
\end{align}
will be related to the 2nd/4th/8th order mock theta functions. We can introduce the \( q \)-series from above definitions by replacing \( q \) with \( 1/q \) as

\[
D^*_5(q) = -q^{-1} D_5(1/q) = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} \frac{(-q)_n}{(q; q^2)_{n+1}}, \tag{8.5}
\]

\[
D^*_6(q) = -q^{-1} D_6(1/q) = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} \frac{(-q^2; q^2)_n}{(q^{n+1})_{n+1}}, \tag{8.6}
\]

\[
I^*_{12}(q) = -q^{-1} I_{12}(1/q) = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} \frac{(-q; q^2)_n}{(q^{n+1})_{n+1}}, \tag{8.7}
\]

\[
I^*_{13}(q) = -q^{-1} I_{13}(1/q) = \sum_{n=0}^{\infty} (-1)^n q^{n(n+3)/2} \frac{(-q^2; q^2)_n}{(q^{n+1})_{n+1}}, \tag{8.8}
\]

and the Bailey pairs, \( D(5), D(6), I(12), \) and \( I(13) \), in Slater’s list prove the following;

**Proposition 17.**

\[
D^*_5(q) = \sum_{n=0}^{\infty} \psi_4^{(1)}(n) q^{\frac{1}{4}(n^2-1)}, \tag{8.9}
\]

\[
D^*_6(q) = \sum_{n=0}^{\infty} \left( \psi_8^{(1)}(n) + \psi_8^{(3)}(n) \right) q^{\frac{1}{4}(n^2-1)}, \tag{8.10}
\]

\[
I^*_{12}(q) = \sum_{n=0}^{\infty} \left( \psi_{16}^{(1)}(n) + \psi_{16}^{(7)}(n) \right) q^{\frac{1}{16}(n^2-1)}, \tag{8.11}
\]

\[
I^*_{13}(q) = \sum_{n=0}^{\infty} \left( \psi_{16}^{(3)}(n) + \psi_{16}^{(5)}(n) \right) q^{\frac{1}{16}(n^2-9)}. \tag{8.12}
\]

This proves

\[
\rho^*(q) = \psi^*(q^2) = D^*_5(q^2)
\]

where \( \rho^*(q) \) and \( \psi^*(q) \) are from the 6-th order \((6.8), (6.9)\). From computations in Ref. 23, these give the WRT invariants as follows;

**Theorem 18.**

\[
\left( e^{\frac{2\pi i}{N}} - 1 \right) \tau_N(M(2, 2, 2)) = 2 \left( 1 - 2 D^*_5(e^{\frac{2\pi i}{N}}) \right) \tag{8.13}
\]

\[
\left( e^{\frac{2\pi i}{N}} - 1 \right) \tau_N(M(2, 2, 4)) = (1 + (-1)^N) \left( 1 - D^*_6(e^{\frac{2\pi i}{N}}) \right), \tag{8.14}
\]

\[
e^{\frac{3\pi i}{2N}} \left( e^{\frac{2\pi i}{N}} - 1 \right) \tau_N(M(2, 2, 8)) = (1 + (-1)^N) \left( 1 - I^*_{12}(e^{\frac{2\pi i}{N}}) \right). \tag{8.15}
\]
As the WRT invariant for wide class of the Seifert manifolds has a nearly modular property [21–24], we may derive mock (false) theta functions from explicit form of these invariants. In such process, the colored Jones polynomial for torus knots [20] and the twist knots [33] would be helpful.

ACKNOWLEDGMENTS

The author would like to thank Thang Le for explaining his construction of the quantum invariants. He also thanks T. Takata for useful discussions. This work is supported in part by the Grant-in-Aid for Young Scientists from the Ministry of Education, Culture, Sports, Science and Technology of Japan.

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