A COMBINATORIAL PROOF OF THE KRONECKER–WEBER THEOREM IN POSITIVE CHARACTERISTIC

JULIO CESAR SALAS–TORRES, MARTHA RZEDOWSKI–CALDERÓN, AND GABRIEL VILLA–SALVADOR

Abstract. In this paper we present a combinatorial proof of the Kronecker–Weber Theorem for global fields of positive characteristic. The main tools are the use of Witt vectors and their arithmetic developed by H. L. Schmid. The key result is to obtain, using counting arguments, how many $p$-cyclic extensions exist of fixed degree and bounded conductor where only one prime ramifies are there. We then compare this number with the number of subextensions of cyclotomic function fields of the same type and verify that these two numbers are the same.

1. Introduction

The classical Kronecker–Weber Theorem establishes that every finite abelian extension of $\mathbb{Q}$, the field of rational numbers, is contained in a cyclotomic field. Equivalently, the maximal abelian extension of $\mathbb{Q}$ is the union of all cyclotomic fields. In 1974, D. Hayes [2], proved the analogous result for rational congruence function fields. Hayes constructed first cyclotomic function fields as the analogue to classical cyclotomic fields. Indeed, the analogy was developed in the first place by L. Carlitz in the 1930’s. The union of all these cyclotomic function fields is not the maximal abelian extension of the rational congruence function field $k = \mathbb{F}_q(T)$ since all these extensions are geometric and the infinite prime is tamely ramified. Hayes proved that the maximal abelian extension of $k$ is the composite of three linearly disjoint fields: the first one is the union of all cyclotomic fields; the second one is the union of all constant extensions and the third one is the union of all the subfields of the corresponding cyclotomic function fields, where the infinite prime is totally wildly ramified. The proof of this theorem uses Artin–Takagi’s reciprocity law in class field theory.

In the classical case, possibly the simplest proof of the Kronecker–Weber Theorem uses ramification groups. The key tool in the proof is that there is only one cyclic extension of $\mathbb{Q}$ of degree $p$, $p$ an odd prime, where $p$ is the only ramified prime. Indeed, this field is the unique subfield of degree $p$ of the cyclotomic field of $p^2$-roots of unity. In the case of function fields the situation is quite different. There exist infinitely many cyclic extensions of $k$ of degree $p$ where only one fixed prime divisor is ramified.

In this paper we present a proof of the Kronecker–Weber Theorem analogue for rational congruence function fields using counting arguments for the case of wild...
ramification. First, similarly to the classical case, we prove that a finite abelian tamely ramified extension of \( k \) is contained in the composite of a cyclotomic function field and a constant extension (see [4]). Next, the key part, is to show that every cyclic extension of \( p \)-power degree, where there is only one ramified prime and it is fully ramified, and the infinite prime is fully decomposed, is contained in a cyclotomic function field. The particular case of an Artin–Schreier extension was completely solved in [5] using counting techniques. In this paper we present another proof for the Artin–Schreier case that also uses counting techniques but is suitable of generalization for cyclic extensions of degree \( p^n \). Once the latter is proven, the rest of the proof follows easily. We use the arithmetic of Witt vectors developed by Schmid in [6] to give two proofs, one by induction and the other is direct.

2. The result

We give first some notation and some results in the theory of cyclotomic function fields developed by D. Hayes [2]. See also [7]. Let \( k = \mathbb{F}_q(T) \) be a congruence rational function field, \( \mathbb{F}_q \) denoting the finite field of \( q = p^s \) elements, where \( p \) is the characteristic. Let \( R_T = \mathbb{F}_q[T] \) be the ring of polynomials, that is, \( R_T \) is the ring of integers of \( k \). For \( N \in R_T \setminus \{0\} \), \( \Lambda_N \) denotes the \( N \)-torsion of the Carlitz module and \( k(\Lambda_N) \) denotes the \( N \)-th cyclotomic function field. The \( R_T \)-module \( \Lambda_N \) is cyclic. For any \( m \in \mathbb{N} \), \( C_m \) denotes a cyclic group of order \( m \). Let \( K_T := \bigcup_{M \in R_T} k(\Lambda_M) \) and \( F_\infty := \bigcup_{m \in \mathbb{N}} \mathbb{F}_q^m \).

We denote by \( p_\infty \) the pole divisor of \( T \) in \( k \). In \( k(\Lambda_N)/k \), \( p_\infty \) has ramification index \( q - 1 \) and decomposes into \( \mathbb{F}_q^{2^n} \) different prime divisors of \( k(\Lambda_N) \) of degree 1, where \( G_N := \text{Gal}(k(\Lambda_N)/k) \). Furthermore, with the identification \( G_N \cong (R_T/(N))^\times \), the inertia group \( I \) of \( p_\infty \) is \( \mathbb{F}_q^\times \subseteq (R_T/(N))^\times \), that is, \( I = \{ \sigma_a \mid a \in \mathbb{F}_q^\times \} \), where for \( A \in R_T \) we use the notation \( \sigma_A(\lambda) = \lambda^A \) for \( \lambda \in \Lambda_N \). We denote by \( R_T^+ \) the set of monic irreducible polynomials in \( R_T \). The primes that ramify in \( k(\Lambda_N)/k \) are \( p_\infty \), and the polynomials \( P \in R_T^+ \) such that \( P \mid N \), except in the extreme case \( q = 2 \), \( N \in \{ T, T + 1, T(T + 1) \} \) because in this case we have \( k(\Lambda_N) = k \).

We set \( L_n \) to be the largest subfield of \( k(\Lambda_1/T^{n+1}) \) where \( p_\infty \) is fully and purely wildly ramified, \( n \in \mathbb{N} \). That is, \( L_n = k(\Lambda_1/T^{n+1})^{p_\infty} \). Let \( L_\infty := \bigcup_{n \in \mathbb{N}} L_n \).

For any prime divisor \( p \) in a field \( K \), \( v_p \) will denote the valuation corresponding to \( p \).

The main goal of this paper is to prove the following result.

**Theorem 2.1** (Kronecker–Weber, [2], [7, Theorem 12.8.31]). The maximal abelian extension \( A \) of \( k \) is \( A = KT\mathbb{F}_\infty L_\infty \). □

To prove Theorem 2.1 it suffices to show that any finite abelian extension of \( k \) is contained in \( k(\Lambda_N)\mathbb{F}_q^m L_n \) for some \( N \in R_T \) and \( m, n \in \mathbb{N} \).

Let \( L/k \) be a finite abelian extension. Let \( G := \text{Gal}(L/k) \cong C_{n_1} \times \cdots \times C_{n_l} \times C_{p^s_1} \times \cdots \times C_{p^s_h} \) where \( \gcd(n_i, p) = 1 \) for \( 1 \leq i \leq l \) and \( a_j \in \mathbb{N} \) for \( 1 \leq j \leq h \). Let \( S_i \subseteq L \) be such that \( \text{Gal}(S_i/k) \cong C_{n_i} \), \( 1 \leq i \leq l \) and let \( R_j \subseteq L \) be such that \( \text{Gal}(R_j/k) \cong C_{p^s_j} \), \( 1 \leq j \leq h \). To prove Theorem 2.1 it is enough to show that each \( S_i \) and each \( R_j \) are contained in \( k(\Lambda_N)\mathbb{F}_q^m L_n \) for some \( N \in R_T \) and \( m, n \in \mathbb{N} \).
In short, we may assume that \( L/k \) is a cyclic extension of degree \( h \) where either \( \gcd(h, p) = 1 \) or \( h = p^n \) for some \( n \in \mathbb{N} \).

### 3. Geometric tamely ramified extensions

In this section, we prove Theorem 2.1 for the particular case of a tamely ramified extension. Let \( L/k \) be an abelian extension. Let \( P \in R_T, d := \deg P \).

**Proposition 3.1.** Let \( P \) be tamely ramified in \( L/k \). If \( e \) denotes the ramification index of \( P \) in \( L \), we have \( e \mid q^d - 1 \).

**Proof:** First we consider in general an abelian extension \( L/k \). Let \( G_{-1} = D \) be the decomposition group of \( P \), \( G_0 = I \) be the inertia group and \( G_i, i \geq 1 \) be the ramification groups. Let \( \mathfrak{P} \) be a prime divisor in \( L \) dividing \( P \). Then if \( \mathcal{O}_\mathfrak{P} \) denotes the valuation ring of \( \mathfrak{P} \), we have

\[
U^{(i)} = 1 + \mathfrak{P}^i \subseteq \mathcal{O}_\mathfrak{P} = \mathcal{O}_\mathfrak{P} \setminus \mathfrak{P}, i \geq 1, U^{(0)} = \mathcal{O}_\mathfrak{P}^*.
\]

Let \( l(\mathfrak{P}) := \mathcal{O}_\mathfrak{P}/\mathfrak{P} \) be the residue field at \( \mathfrak{P} \). The following are monomorphisms:

\[
G_i/G_{i+1} \xrightarrow{\varphi} U^{(i)}/U^{(i+1)} \cong \begin{cases} l(\mathfrak{P})^*, i = 0 \\ \mathfrak{P}^i/\mathfrak{P}^{i+1} \cong l(\mathfrak{P}), i \geq 1. \end{cases}
\]

where \( \pi \) denotes a prime element for \( \mathfrak{P} \).

We will prove that if \( G_{-1}/G_1 = D/G_1 \) is abelian, then

\[
\varphi = \varphi_0: G_0/G_1 \rightarrow U^{(0)}/U^{(1)} \cong (\mathcal{O}_\mathfrak{P}/\mathfrak{P})^*
\]

satisfies that \( \text{im} \, \varphi \subseteq \mathcal{O}_P/(P) \cong R_T/(P) \cong \mathbb{F}_{q^d} \). In particular it will follow \( |G_0/G_1| \mid |\mathbb{F}_{q^d}| = q^d - 1 \).

To prove this statement, note that

\[
\text{Aut}((\mathcal{O}_\mathfrak{P}/\mathfrak{P})/(\mathcal{O}_P/(P))) \cong \text{Gal}((\mathcal{O}_\mathfrak{P}/\mathfrak{P})/(\mathcal{O}_P/(P))) = D/I = G_{-1}/G_0
\]

(see [7, Corollary 5.2.12]).

Let \( \sigma \in G_0 \) and \( \varphi(\sigma) = \varphi(\sigma \text{ mod } G_1) = [\alpha] \in (\mathcal{O}_\mathfrak{P}/\mathfrak{P})^* \). Therefore \( \sigma \pi \equiv \alpha \pi \text{ mod } \mathfrak{P}^2 \).

Let \( \theta \in G_{-1} = D \) be arbitrary and let \( \pi_1 := \theta^{-1} \pi \). Then \( \pi_1 \) is a prime element for \( \mathfrak{P} \). Since \( \varphi \) is independent of the prime element, it follows that \( \sigma \pi_1 \equiv \alpha \pi_1 \text{ mod } \mathfrak{P}^2 \), that is \( \sigma \theta^{-1} \pi \equiv \alpha \theta^{-1} \pi \text{ mod } \mathfrak{P}^2 \). Since \( G_{-1}/G_1 \) is an abelian group, we have

\[
\sigma \pi = (\theta \sigma \theta^{-1})(\pi) \equiv (\alpha \pi) \pi \text{ mod } \mathfrak{P}^2.
\]

Thus \( \sigma \pi \equiv \theta(\alpha) \pi \text{ mod } \mathfrak{P}^2 \) and \( \sigma \pi \equiv \alpha \pi \text{ mod } \mathfrak{P}^2 \). It follows that \( \theta(\alpha) \equiv \alpha \text{ mod } \mathfrak{P} \) for all \( \theta \in G_{-1} \).

If we write \( \bar{\theta} = \theta \text{ mod } G_0, \bar{\theta}[\alpha] = [\alpha] \), that is, \( [\alpha] \) is a fixed element under the action of the group \( G_{-1}/G_0 \cong \text{Gal}((\mathcal{O}_\mathfrak{P}/\mathfrak{P})/(\mathcal{O}_P/(P))) \). We obtain that \( [\alpha] \in \mathcal{O}_P/(P) \). Therefore \( \text{im} \, \varphi \subseteq (\mathcal{O}_P/(P))^* \) and \( |G_0/G_1| \mid (\mathcal{O}_P/(P))^* \mid = q^d - 1 \).

Finally, since \( L/k \) is abelian and \( P \) is tamely ramified, \( G_1 = \{1\} \), it follows that \( e = |G_0| = |G_0/G_1| \mid q^d - 1 \). \( \Box \)

Now consider a finite abelian tamely ramified extension \( L/k \) where \( P_1, \ldots, P_r \) are the finite ramified primes. Let \( P \in \{P_1, \ldots, P_r\} \), \( \deg P = d \). Let \( e \) be the ramification index of \( P \) in \( L \). Then by Proposition 3.1 we have \( e \mid q^d - 1 \). Now \( P \)
is totally ramified in $k(\Lambda_P)/k$ with ramification index $q^d - 1$. In this extension $p_\infty$ has ramification index equal to $q - 1$.

Let $k \subseteq E \subseteq k(\Lambda_P)$ with $[E : k] = e$. Set $\mathfrak{P}$ a prime divisor in $LE$ dividing $P$. Let $q := \mathfrak{P}|_E$ and $\mathfrak{P} := \mathfrak{P}|_L$.

We have $e = e_{L/k}(\mathfrak{P}|P) = e_{E/k}(q|P)$. By Abhyankar's Lemma [7, Theorem 12.4.4], we obtain

$$e_{LE/k}(\mathfrak{P}|P) = \text{lcm}[e_{L/k}(\mathfrak{P}|P), e_{E/k}(q|P)] = \text{lcm}[e, e] = e.$$ 

Let $H \subseteq \text{Gal}(LE/k)$ be the inertia group of $\mathfrak{P}/P$. Set $M := (LE)^H$. Then $P$ is unramified in $M/k$. We want to see that $L \subseteq Mk(\Lambda_P)$. Indeed we have $[LE : M] = e$ and $E \cap M = k$ since $P$ is totally ramified in $E/k$ and unramified in $M/k$. It follows that $[ME : k] = [M : k][E : k]$. Therefore

$$[LE : k] = [LE : M][M : k] = e\frac{[ME : k]}{[E : k]} = e\frac{[ME : k]}{e} = [ME : k].$$

Since $ME \subseteq LE$ it follows that $LE = ME = EM \subseteq k(\Lambda_P)M$. Thus $L \subseteq k(\Lambda_P)M$.

In $M/k$ the finite ramified primes are $\{P_2, \ldots, P_r\}$. In case $r - 1 \geq 1$, we may apply the above argument to $M/k$ and we obtain $M_2/k$ such that at most $r - 2$ finite primes are ramified and $M \subseteq k(\Lambda_{P_2})M_2$, so that $L \subseteq k(\Lambda_{P_1})M \subseteq k(\Lambda_{P_1})k(\Lambda_{P_2})M_2 = k(\Lambda_{P_1P_2})M_2$.

Performing the above process at most $r$ times we have

$$(3.1) \quad L \subseteq k(\Lambda_{P_1P_2}\cdots P_r)M_0$$

where in $M_0/k$ the only possible ramified prime is $p_\infty$.

We also have

**Proposition 3.2.** Let $L/k$ be an abelian extension where at most one prime divisor $p_0$ of degree 1 is ramified and the extension is tamely ramified. Then $L/k$ is a constant extension.

**Proof:** By Proposition 3.1 we have $e := e_{L/k}(p_0)|q - 1$. Let $H$ be the inertia group of $p_0$. Then $|H| = e$ and $p_0$ is unramified in $E := L^H/k$. Therefore $E/k$ is an unramified extension. Thus $E/k$ is a constant extension.
the degree of $\mathcal{P}_0$ is 1 (see [7, Theorem 6.2.1]). Therefore $\mathcal{P}_0$ is the only prime divisor ramified in $L/E$ and it is of degree 1 and totally ramified. Furthermore $[L : E] = e | q - 1 = |\mathbb{F}_q^*|$. The $(q - 1)$-th roots of unity belong to $\mathbb{F}_q \subseteq k$. Hence $k$ contains the $e$-th roots of unity and $L/E$ is a Kummer extension, say $L = E(y)$ with $y^e = \alpha \in E = k\mathbb{F}_{q^m} = \mathbb{F}_{q^m}(T)$. We write $\alpha$ in a normal form as prescribed by Hasse [1]:

$$(\alpha)_E = \mathbb{F}_{q^a}^*, \quad 0 < a < e.$$ Now since $\deg(\alpha)_E = 0$ it follows that $\deg_E a$ or $\deg_E b$ is not a multiple of $e$. This contradicts that $\mathfrak{p}_0$ is the only ramified prime. Therefore $L/k$ is a constant extension.

As a corollary to (3.1) and Proposition 3.2 we obtain

**Corollary 3.3.** If $L/k$ is a finite abelian tamely ramified extension where the ramified prime divisors are $P_1, \ldots, P_r$, then

$$L \subseteq k(\Lambda_{P_1, \ldots, P_r})\mathbb{F}_{q^m},$$

for some $m \in \mathbb{N}$. \qed

4. Reduction steps

As a consequence of Corollary 3.3, Theorem 2.1 will follow if we prove it for the particular case of a cyclic extension $L/k$ of degree $p^n$ for some $n \in \mathbb{N}$. Now, this kind of extensions are given by a Witt vector:

$$K = k(\bar{y}) = k(y_1, \ldots, y_n) \quad \text{with} \quad \bar{y}^p - \bar{y} = \bar{\beta} = (\beta_1, \ldots, \beta_n) \in W_n(k)$$

where for any field $E$ of characteristic $p$, $W_n(E)$ denotes the ring of Witt vectors of length $n$ with components in $E$.

The following result was proved in [3]. It “separates” the ramified prime divisors.

**Theorem 4.1.** Let $K/k$ be a cyclic extension of degree $p^n$ where $P_1, \ldots, P_r \in R_T$ and possibly $p_\infty$, are the ramified prime divisors. Then $K = k(\bar{y})$ where

$$\bar{y}^p - \bar{y} = \bar{\beta} = \delta_1 + \cdots + \delta_r + \bar{\mu},$$

with $\delta_{ij} = \frac{Q_{ij}}{P_{ij}}$, $e_{ij} \geq 0$, $Q_{ij} \in R_T$ and

(a) if $e_{ij} = 0$ then $Q_{ij} = 0$;

(b) if $e_{ij} > 0$ then $p | e_{ij}$, $\gcd(Q_{ij}, P_i) = 1$ and $\deg(Q_{ij}) < \deg(P_i^{e_{ij}})$,

and $\mu_j = f_j(T) \in R_T$ with

(c) $p \nmid \deg f_j$ when $f_j \notin \mathbb{F}_q$ and

(d) $\mu_j \notin \varphi(\mathbb{F}_q) := \{a^p - a \mid a \in \mathbb{F}_q\}$ when $\mu_j \in \mathbb{F}_q^*$. \qed

Consider the field $K = k(\bar{y})$ as above, where precisely one prime divisor $P \in R_T$ ramifies, with

$$\beta_i = \frac{Q_i}{P_i}, \quad Q_i \in R_T \quad \text{such that} \quad \lambda_i \geq 0,$$

\begin{equation}
(4.1) \quad \text{if} \quad \lambda_i = 0 \text{ then } Q_i = 0,
\end{equation}

\begin{equation}
\text{if} \quad \lambda_i > 0 \text{ then } \gcd(\lambda_i, p) = 1, \quad \gcd(Q_i, P) = 1 \text{ and } \deg Q_i < \deg P^{\lambda_i}, \quad \lambda_i > 0.
\end{equation}

A particular case of Theorem 4.1 suitable for our study is given in the following proposition.
Proposition 4.2. Assume that every extension $K_1/k$ that meets the conditions of (4.1) satisfies that $K_1 \subseteq k(\Lambda_1)$ for some $\alpha \in \mathbb{N}$. Let $K/k$ be the extension defined by $K = k(\bar{y})$ where $\phi(\bar{y}) = \bar{y}$ for $\beta = (\beta_1, \ldots, \beta_n)$, $\beta_i$ given in normal form: $\beta_i \in \mathbb{F}_q$ or $\beta_i = \frac{Q_i}{P_i}$, $Q_i \in R_T$ and $\lambda_i > 0$, $\gcd(\lambda_i, p) = 1$, $\gcd(Q_i, P_i) = 1$ and $\deg Q_i \leq \deg P_i$. Then $K \subseteq \mathbb{F}_q \cdot k(\Lambda_{1})$ for some $\alpha \in \mathbb{F}_q$.

Proof: From Theorem 4.1 we have that we can decompose the vector $\bar{\beta}$ as $\bar{\beta} = \bar{v} + \bar{\gamma}$ with $\varepsilon_i \in \mathbb{F}_q$ for all $1 \leq i \leq n$ and $\gamma_i = 0$ or $\gamma_i = \frac{\lambda_i}{Q_i}$, $Q_i \in R_T$ and $\lambda_i > 0$, $\gcd(\lambda_i, p) = 1$, $\gcd(Q_i, P_i) = 1$ and $\deg Q_i < \deg P_i$.

Let $\gamma_1 = \cdots = \gamma_r = 0$, and $\gamma_{r+1} \notin \mathbb{F}_q$. We have $K \subseteq k(\bar{v})k(\bar{\gamma})$. Now $k(\bar{v}) \subseteq \mathbb{F}_q \cdot k(\Lambda)$ and $k(\bar{\gamma}) = k(0, \ldots, 0, \gamma_{r+1}, \ldots, \gamma_n)$.

For any Witt vector $\bar{x} = (x_1, \ldots, x_n)$ we have the decomposition given by Witt himself

$$\bar{x} = (x_1, 0, 0, \ldots, 0) \uparrow (0, x_2, 0, \ldots, 0) \uparrow \cdots \uparrow (0, \ldots, 0, x_j, 0, \ldots, 0)$$

for each $0 \leq j \leq n - 1$. It follows that $k(\bar{\gamma}) = k(\gamma_{r+1}, \ldots, \gamma_n)$. Since this field fulfills the conditions of (4.1), we have $k(\bar{\gamma}) \subseteq k(\Lambda_{1})$ for some $\alpha \in \mathbb{N}$. The result follows.

Remark 4.3. The prime $p_\infty$ can be handled in the same way. The conditions (4.1) for $p_\infty$ are the following. Let $K = k(\bar{\mu})$ with $\mu_j = f_j(T) \in R_T$, with $f_j(0) = 0$ for all $j$ and either $f_j(T) = 0$ or $f_j(T) \neq 0$ and $p \nmid \deg f_j(T)$. The condition $f_j(0) = 0$ means that the infinite prime for $T' = 1/T$ is either decomposed or ramified in each layer, that is, its inertia degree is 1 in $K/k$. In this case with the change of variable $T' = 1/T$ the hypotheses in Proposition 4.2 say that any field meeting these conditions satisfies that $K \subseteq k(\Lambda_{1}) = k(\Lambda_{1})$ for some $m \in \mathbb{N}$. However, since the degree of the extension $K/k$ is a power of $p$ we must have that $K \subseteq k(\Lambda_{1}) = k(\Lambda_{1})$.

With the notation of Theorem 4.1 we obtain that if $\bar{\zeta}_i = \bar{\mu}_i$, $1 \leq i \leq r$ and if $\bar{v} = \bar{\nu}$, then $L = k(\bar{y}) \subseteq k(\bar{z}_1, \ldots, \bar{z}_r, \bar{v}) = k(\bar{z}_1) \cdots k(\bar{z}_r)k(\bar{v})$. Therefore if Theorem 2.1 holds for each $k(\bar{z}_i)$, $1 \leq i \leq r$ and for $k(\bar{v})$, then it holds for $L$.

From Theorem 4.1, Proposition 4.2 and the remark after this proposition, we obtain that to prove Theorem 2.1 it suffices to show that any field extension $K/k$ meeting the conditions of (4.1) satisfies that either $K \subseteq k(\Lambda_{1})$ for some $\alpha \in \mathbb{N}$ or $K \subseteq L_m$ for some $m \in \mathbb{N}$.

Next we study the behavior of $p_\infty$ in an arbitrary cyclic extension $K/k$ of degree $p^n$.

Consider first the case $K/k$ cyclic of degree $p$. Then $K = k(y)$ where $y^p - y = \alpha \in k$. The equation can be normalized as:

$$y^p - y = \alpha = \sum_{i=1}^r \frac{Q_i}{P_i^{e_i} - f(T)} = \frac{Q}{P_1^{e_1} - f(T)},$$

where $P_i \in R_T^+, Q_i \in R_T$, $\gcd(P_i, Q_i) = 1$, $e_i > 0$, $p \nmid e_i$, $\deg Q_i < \deg P_i^{e_i}$, $1 \leq i \leq r$, $\deg Q < \sum_{i=1}^r \deg P_i^{e_i}$, $f(T) \in R_T$, with $p \nmid \deg f$ when $f(T) \notin \mathbb{F}_q$ and $f(T) \notin \phi(\mathbb{F}_q)$ when $f(T) \in \mathbb{F}_q^*$. 

We have that the finite primes ramified in $K/k$ are precisely $P_1, \ldots, P_r$ (see [1]). With respect to $p_\infty$ we have the following well known result. We present a proof for the sake of completeness.

**Proposition 4.4.** Let $K = k(y)$ be given by (4.2). Then the prime $p_\infty$ is

(a) decomposed if $f(T) = 0$.
(b) inert if $f(T) \in \mathbb{F}_q$ and $f(T) \notin \varphi(\mathbb{F}_q)$.
(c) ramified if $f(T) \notin \mathbb{F}_q$ (thus $p \nmid \deg f$).

**Proof:** First consider the case $f(T) = 0$. Then $v_{p_\infty}(\alpha) = \deg(P_1^{e_1} \cdots P_r^{e_r}) - \deg Q > 0$. Therefore $p_\infty$ is unramified. Now $y^p - y = \prod_{i=0}^{p-1} (y - i)$. Let $\mathfrak{P}_\infty \mid p_\infty$. Then

$$v_{\mathfrak{P}_\infty}(y^p - y) = \sum_{i=0}^{p-1} v_{\mathfrak{P}_\infty}(y - i) = e(\mathfrak{P}_\infty|p_\infty)v_{p_\infty}(\alpha) = v_{p_\infty}(\alpha) > 0.$$ 

Therefore, there exists $0 \leq i \leq p - 1$ such that $v_{p_\infty}(y - i) > 0$. Without loss of generality we may assume that $i = 0$. Let $\sigma \in \text{Gal}(K/k) \setminus \{\text{Id}\}$. Assume that $\mathfrak{P}_\infty^\sigma = \mathfrak{P}_\infty$. We have $y^\sigma = y - j, j \neq 0$. Thus, on the one hand

$$v_{\mathfrak{P}_\infty}(y - j) = v_{\mathfrak{P}_\infty}(y^\sigma) = v_{\sigma(\mathfrak{P}_\infty)}(y) = v_{\mathfrak{P}_\infty}(y) > 0.$$ 

On the other hand, since $v_{\mathfrak{P}_\infty}(y) > 0 = v_{\mathfrak{P}_\infty}(j)$, it follows that

$$v_{\mathfrak{P}_\infty}(y - j) = \min\{v_{\mathfrak{P}_\infty}(y), v_{\mathfrak{P}_\infty}(j)\} = 0.$$ 

This contradiction shows that $\mathfrak{P}_\infty^\sigma \neq \mathfrak{P}_\infty$ so that $p_\infty$ decomposes in $K/k$.

Now we consider the case $f(T) \neq 0$. If $f(T) \notin \mathbb{F}_q$, then $p_\infty$ ramifies since it is in the normal form prescribed by Hasse [1].

The last case is when $f(T) \in \mathbb{F}_q$, $f(T) \notin \varphi(\mathbb{F}_q)$. Let $b \in \mathbb{F}_{q^p}$ with $b^p - b = a = f(T)$. Since $\deg p_\infty = 1$, $p_\infty$ is inert in the constant extension $k(b)/k$ ([7, Theorem 6.2.1]). Assume that $p_\infty$ decomposes in $k(y)/k$. We have the following diagram

$$\begin{array}{ccc}
 k(y) & \overset{p_{\infty}}{\text{inert}} & k(y, b) \\
 \text{p_{\infty} decomposes} \downarrow & & \downarrow \\
 k & \overset{p_{\infty}}{\text{inert}} & k(b)
\end{array}$$

The decomposition group of $p_\infty$ in $k(y, b)/k$ is $\text{Gal}(k(y, b)/k(y))$. Therefore $p_\infty$ is inert in every field of degree $p$ over $k$ other than $k(y)$. Since the fields of degree $p$ are $k(y + ib), k(b), 0 \leq i \leq p - 1$, in $k(y + b)/k$ we have

$$(y + b)^p - (y + b) = (y^p - y) + (b^p - b) = \alpha - a = \frac{Q}{P_1^{e_1} \cdots P_r^{e_r}}$$

with $\deg(\alpha - a) < 0$. Hence, by the first part, $p_\infty$ decomposes in $k(y + b)/k$ and in $k(y)/k$ which is impossible. Thus $p_\infty$ is inert in $k(y)/k$. \hfill \Box

The general case for the behavior of $p_\infty$ in a cyclic $p$-extension is given in [3, Proposition 5.6] and it is a consequence of Proposition 4.4.

**Proposition 4.5.** Let $K/k$ be given as in Theorem 4.1. Let $\mu_1 = \cdots = \mu_s = 0, \mu_{s+1} \in \mathbb{F}_q^s, \mu_{s+1} \notin \varphi(\mathbb{F}_q)$ and finally, let $t + 1$ be the first index with $f_{t+1} \notin \mathbb{F}_q$ (and therefore $p \nmid \deg f_{t+1}$). Then the ramification index of $p_\infty$ is $p^{n-t}$, the inertia degree of $p_\infty$ is $p^{1-s}$ and the decomposition number of $p_\infty$ is $p^s$. More precisely,
Proposition 4.6. If $T_f$ ramified and $\infty$ is the only ramified prime, it is fully decomposed.

Similarly, if $K = k(\bar{v})$ where $v_i = f_i(T) \in R_T$, $f_i(0) = 0$ for all $1 \leq i \leq n$ and $f_1(T) \notin \mathbb{F}_q$, $p \nmid \deg f_1(T)$, then $p_{\infty}$ is the only ramified prime in $K/k$, it is fully ramified and the zero divisor of $T$ which is the infinite prime in $R_1/T$, is fully decomposed.

We have reduced the proof of Theorem 2.1 to prove that any extension of the type given in Proposition 4.6 is contained in either $k(\Lambda_{p^n})$ for some $\alpha \in \mathbb{N}$ or in $L_m$ for some $m \in \mathbb{N}$. The second case is a consequence of the first one with the change of variable $T' = 1/T$.

Let $n, \alpha \in \mathbb{N}$. Denote by $v_n(\alpha)$ the number of cyclic groups of order $p^n$ contained in $(R_T/(P^n))^*$, where $n \in \mathbb{N}$ and $n \neq 0$. We have that $v_n(\alpha)$ is the number of cyclic field extensions $K/k$ of degree $p^n$ and $K \subseteq k(\Lambda_{p^n})$. Every such extension satisfies that its conductor $\mathfrak{c}_K$ divides $P^n$.

Let $\alpha \in \mathbb{N}$. Denote by $v_\alpha(\alpha)$ the number of cyclic groups of order $p^n$ contained in $(R_T/(P^n))^*$, where $n \in \mathbb{N}$ and $n \neq 0$. We have that $v_n(\alpha)$ is the number of cyclic field extensions $K/k$ of degree $p^n$ and $K \subseteq k(\Lambda_{p^n})$. Every such extension satisfies that its conductor $\mathfrak{c}_K$ divides $P^n$.

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We prove $v_n(\alpha) \leq v_\alpha(\alpha)$ for all $n, \alpha \in \mathbb{N}$.

5. Wildly ramified extensions

In this section we prove (4.3) by induction on $n$ and as a consequence we obtain our main result, Theorem 2.1. First we compute $v_n(\alpha)$ for all $n, \alpha \in \mathbb{N}$.

Proposition 5.1. The number $v_n(\alpha)$ of different cyclic groups of order $p^n$ contained in $(R_T/(P^n))^*$ is

$$v_n(\alpha) = \frac{q(d(\alpha - \lfloor \frac{\alpha}{p^n} \rfloor) - q(\alpha - \lfloor \frac{\alpha}{p^n-1} \rfloor))}{p^{n-1}(p-1)} = \frac{q(d(\alpha - \lfloor \frac{\alpha}{p^n} \rfloor) - q(\alpha - \lfloor \frac{\alpha}{p^n-1} \rfloor))}{p^{n-1}(p-1)},$$

where $\lfloor x \rfloor$ denotes the ceiling function, that is, $\lfloor x \rfloor$ denotes the minimum integer greater than or equal to $x$.

Proof: Let $P \in R_T^+$ and $\alpha \in \mathbb{N}$ with $\deg P = d$. First we consider how many cyclic extensions of degree $p^n$ are contained in $k(\Lambda_{p^n})$. Since $p_{\infty}$ is tamely ramified in $k(\Lambda_{p^n})$, if $K/k$ is a cyclic extension of degree $p^n$, $p_{\infty}$ decomposes fully in $K/k$ ([7, Theorem 12.4.6]). We have $\text{Gal}(k(\Lambda_{p^n})/k) \cong (R_T/(P^n))^*$ and the exact sequence

\[
0 \longrightarrow D_{p,p^n} \longrightarrow (R_T/(P^n))^* \stackrel{\varphi}{\longrightarrow} (R_T/(P))^* \longrightarrow 0,
\]
where

\[ \varphi: \left( R_T/(P^\alpha) \right)^* \to \left( R_T/(P) \right)^* \]

\[ A \mod P^\alpha \mapsto A \mod P \]

and \( D_{P,P^\alpha} = \{ N \mod P^\alpha \mid N \equiv 1 \mod P \} \). We safely may consider \( D_{P,P^\alpha} = \{ 1 + hP \mid h \in R_T, \deg h < \deg P^\alpha = d\alpha \} \).

We have \( \left( R_T/(P^\alpha) \right)^* \cong \left( R_T/(P) \right)^* \times D_{P,P^\alpha} \) and \( \left( R_T/(P) \right)^* \cong C_{q^d-1} \). First we compute how many elements of order \( P^\alpha \) belong to \( D_{P,P^\alpha} \). These elements belong to \( D_{P,P^\alpha} \). Let \( A = 1 + hP \in D_{P,P^\alpha} \) of order \( P^\alpha \). We write \( h = gP^n \) with \( g \in R_T, \gcd(g,P) = 1 \) and \( \gamma \geq 0 \). We have \( A = 1 + gP^{1+\gamma} \). Since \( A \) is of order \( P^n \), it follows that

\[ A^{P^n} = 1 + g^{P^n} P^n(1+\gamma) \equiv 1 \mod P^\alpha \]

and

\[ A^{P^{n-1}} = 1 + g^{P^{n-1}} P^{n-1}(1+\gamma) \neq 1 \mod P^\alpha. \]

From (5.2) and (5.3) it follows that

\[ P^{n-1}(1+\gamma) < \alpha \leq P^n(1+\gamma), \]

which is equivalent to

\[ \left( \frac{\alpha}{P^n} \right) - 1 \leq \gamma < \left( \frac{\alpha}{P^{n-1}} \right) - 1. \]

Observe that for the existence of at least one element of order \( P^n \) we need \( \alpha > p^{n-1}. \)

Now, for each \( \gamma \) satisfying (5.4) we have \( \gcd(g,P) = 1 \) and \( \deg g + d(1+\gamma) < d\alpha \), that is, \( \deg g < d(\alpha - \gamma - 1) \). Thus, there exist \( \Phi(P^{\alpha-\gamma-1}) \) such \( g \)'s, where for any \( N \in R_T, \Phi(N) \) denotes the order of \( \left( R_T/(N) \right)^* \).

Therefore the number of elements of order \( P^n \) in \( D_{P,P^\alpha} \) is

\[ \sum_{\gamma = \left[ \frac{\alpha}{P^n} \right] - 1}^{\alpha - \left[ \frac{\alpha}{P^{n-1}} \right]} \Phi(P^{\alpha-\gamma-1}) \]

(5.6)

\[ \sum_{\gamma = \left[ \frac{\alpha}{P^n} \right] - 1}^{\alpha - \left[ \frac{\alpha}{P^{n-1}} \right] + 1} \Phi(P^{\gamma}). \]

Note that for any \( 1 \leq r \leq s \) we have

\[ \sum_{i=r}^{s} \Phi(P^i) = \sum_{i=r}^{s} q^{d(i-1)}(q^d - 1) = (q^d - 1)q^{d(r-1)} \sum_{j=0}^{s-r} q^{dj} \]

\[ = (q^d - 1)q^{d(r-1)} \frac{q^{d(s-r+1)} - 1}{q^d - 1} = q^{ds} - q^{d(r-1)}. \]

Hence (5.6) is equal to

\[ q^{\left( \alpha - \left[ \frac{\alpha}{P^n} \right] \right)} - q^{\left( \alpha - \left[ \frac{\alpha}{P^{n-1}} \right] \right)} = q^{\left( \alpha - \left[ \frac{\alpha}{P^n} \right] \right)} (q^{\left( \left[ \frac{\alpha}{P^n} \right] - \left[ \frac{\alpha}{P^{n-1}} \right] \right) - 1). \]

Since each cyclic group of order \( P^n \) has \( \varphi(P^n) = P^{n-1}(p-1) \) generators, we obtain the result. \( \square \)

Note that if \( K/k \) is any field contained in \( k(\Lambda_{P^\alpha}) \) then the conductor \( \frak{s}_K \) of \( K \) is a divisor of \( P^\alpha \).
During the proof of Proposition 5.2 we compute $t_1(\alpha)$, that is, the number of cyclic extensions $K/k$ of degree $p$ such that $P$ is the only ramified prime (it is fully ramified), $p_\infty$ decomposes in $K/k$ and $\mathfrak{f}_K \mid P^n$ and we obtain (4.3) for the case $n = 1$. We have already solved this case in [5]. Here we present another proof, which is suitable of generalization to the case of cyclic extensions of degree $p^n$.

**Proposition 5.2.** Every cyclic extension $K/k$ of degree $p$ such that $P$ is the only ramified prime, $p_\infty$ decomposes in $K/k$ and $\mathfrak{f}_K \mid P^n$ is contained in $k(\Lambda_{p^n})$.

**Proof:** From the Artin–Schreier theory (see (4.2)) and Proposition 4.4, we have that the field $K$ is given by $K = k(y)$ with the Artin–Schreier equation of $y$ normalized as prescribed by Hasse [1]. That is

$$y^p - y = \frac{Q}{P^\lambda},$$

where $P \in R^+_T$, $Q \in R_T$, $\gcd(P, Q) = 1$, $\lambda > 0$, $p \mid \lambda$, $\deg Q < \deg P^\lambda$. Now the conductor $\mathfrak{f}_K$ satisfies $\mathfrak{f}_K = P^{\lambda+1}$ so $\lambda \leq \alpha - 1$.

We have that if $K = k(z)$ with $z^p - z = a$ then there exist $j \in \mathbb{F}_p^*$ and $c \in k$ such that $z = jy + c$ and $a = j\frac{Q}{P^\lambda} + \varphi(c)$ where $\varphi(c) = c^p - c$. If $a$ is also given in normal form then $c = \frac{h}{P^\gamma}$ with $\gamma \leq \lambda$ (indeed, $\gamma \leq \lambda$ since $\gcd(\lambda, p) = 1$) and $\deg h < \deg P^\gamma$ or $h = 0$. Let $\gamma_0 := \lceil \frac{h}{P^\gamma} \rceil$. Then any such $c$ can be written as $c = \frac{h}{P^{\gamma_0}}$. Therefore $c \in G := \left\{ \frac{h}{P^\gamma} \mid h \in R_T, \deg h < \deg P^{\gamma_0} = d_{\gamma_0} or h = 0 \right\}$.

If $c \in G$ and $j \in \{1, 2, \ldots, p - 1\}$ we have

$$a = j \frac{Q}{P^\lambda} + \varphi(c) = j \frac{Q}{P^\lambda} + h \frac{h}{P^{\gamma_0} + P^{\gamma_0}} = \frac{Q_1}{P^\lambda},$$

with $\deg Q_1 < \deg P^\lambda$. Since $\lambda - \gamma_0 > 0$ and $\lambda - \gamma_0 > 0$, we have $\gcd(Q_1, P) = 1$. Therefore $a$ is in normal form.

It follows that the same field has $|\mathbb{F}_p^*|\varphi(G)|$ different representations in standard form. Now $G$ and $\varphi(G)$ are additive groups and $\varphi : G \to \varphi(G)$ is a group epimorphism with kernel $\ker \varphi = G \cap \{c \mid \varphi(c) = c^p - c = 0\} = G \cap \mathbb{F}_p = \{0\}$. We have $|\varphi(G)| = |G| = |R_T/(P^{\gamma_0})| = q^{d_{\gamma_0}}$.

From the above discussion we obtain that the number of different cyclic extensions $K/k$ of degree $p$ such that the conductor of $K$ is $\mathfrak{f}_K = P^{\lambda+1}$ is equal to

$$\Phi(P_{\lambda+1}) = \frac{q^{d(\lambda-1)(q^d-1)}}{(p-1)q^{d_{\gamma_0}}} = \frac{q^{d(\lambda-\frac{\lambda}{p})-1}(q^d-1)}{p-1} = \frac{1}{p-1} \Phi(P^{\lambda-\frac{\lambda}{p}}).$$

Therefore the number of different cyclic extensions $K/k$ of degree $p$ such that the conductor $\mathfrak{f}_K$ of $K$ is a divisor of $P^\alpha$ is given by $t_1(\alpha) = \frac{w(\alpha)}{p-1}$ where

$$w(\alpha) = \sum_{\lambda = 1}^{\alpha-1} \Phi(P^{\lambda-\frac{\lambda}{p}}).$$

To compute $w(\alpha)$ write $\alpha - 1 = pt_0 + r_0$ with $t_0 \geq 0$ and $0 \leq r_0 \leq p - 1$. Now $\{\lambda \mid 1 \leq \lambda \leq \alpha - 1, \gcd(\lambda, p) = 1\} = A \cup B$ where

$A = \{pt + r \mid 0 \leq t \leq t_0 - 1, 1 \leq r \leq p - 1\}$ and $B = \{pt_0 + r \mid 1 \leq r \leq r_0\}$. 

Then
\[ w(\alpha) = \sum_{\lambda \in A} \Phi(P^{\lambda - \frac{[\lambda]}{p}}) + \sum_{\lambda \in B} \Phi(P^{\lambda - \frac{[\lambda]}{q}}) \]

where we understand that if a set, \( A \) or \( B \) is empty, the respective sum is 0.

Then
\[
w(\alpha) = \sum_{0 \leq t \leq t_0-1} \sum_{1 \leq r \leq p-1} q^{d(pt+r-t-1)}(q^d - 1) + \sum_{r=1}^{q_0} q^{d(pt_0+r-t_0-1)}(q^d - 1)
\]
\[
= (q^d - 1) \left( \sum_{t=0}^{t_0-1} q^{d(p-1)t} \right) \left( \sum_{r=1}^{p-1} q^{d(r-1)} \right) + (q^d - 1)q^{d(p-1)t_0} \sum_{r=1}^{q_0} q^{d(r-1)}
\]
\[
= (q^d - 1)(q^{d(p-1)t_0} - 1)q^{d(p-1)} - 1 q^{d - 1} + (q^d - 1)q^{d(p-1)t_0} q^{d-1} - 1
\]
\[
= q^{d(p-1)t_0+r_0} - 1 = q^{d(pt_0+r_0-t_0)} - 1 = q^{d\left(\alpha - \left\lfloor \frac{\alpha-1}{p} \right\rfloor \right)} - 1.
\]

Therefore, the number of different cyclic extensions \( K/k \) of degree \( p \) such that \( P \) is the only ramified prime, \( \mathfrak{S}_K \mid P^\alpha \) and \( p_\infty \) decomposes, is
\[
t_1(\alpha) = \frac{w(\alpha)}{p-1} = \frac{q^{d\left(\alpha - \left\lfloor \frac{\alpha-1}{p} \right\rfloor \right)} - 1}{p-1}.
\]

To finish the proof of Proposition 5.2 we need the following

**Lemma 5.3.** For any \( \alpha \in \mathbb{Z} \) and \( s \in \mathbb{N} \) we have
\[
\begin{align*}
(\alpha) & \quad \left\lfloor \frac{\alpha}{p^s} \right\rfloor = \left\lfloor \frac{\alpha}{p^{s+1}} \right\rfloor = \left\lfloor \frac{\alpha}{p^{s+1}} \right\rfloor . \\
(\frac{\alpha}{p^s}) & \quad \left\lfloor \frac{\alpha}{p^s} \right\rfloor = \left\lfloor \frac{\alpha}{p^{s+1}} \right\rfloor + 1.
\end{align*}
\]

**Proof:** For (a), we prove only \( \left\lfloor \frac{\alpha}{p^s} \right\rfloor = \left\lfloor \frac{\alpha}{p^{s+1}} \right\rfloor \), the other equality is similar.

Note that the case \( s = 0 \) is clear. Set \( \alpha = tp^{s+1} + r \) with \( 0 \leq r \leq p^{s+1} - 1 \). Let \( r = lp^s + r' \) with \( 0 \leq r' \leq p^s - 1 \). Note that \( 0 \leq l \leq p-1 \). Hence \( \alpha = tp^{s+1} + lp^s + r' \),
\[
0 \leq r' \leq p^s - 1 \quad \text{and} \quad 0 \leq l \leq p - 1. \text{ Therefore } \left\lfloor \frac{\alpha}{p^s} \right\rfloor = tp + l, \text{ and } \left\lfloor \frac{\alpha}{p^{s+1}} \right\rfloor = t + \frac{l}{p},
\]
\[
0 \leq l \leq p - 1. \text{ Therefore } \left\lfloor \frac{\alpha}{p^s} \right\rfloor = t = \left\lfloor \frac{\alpha}{p^{s+1}} \right\rfloor .
\]

For (b) write \( \alpha = p^st + r \) with \( 0 \leq r \leq p^s - 1 \). If \( p^s \mid \alpha \) then \( r = 0 \) and \( \left\lfloor \frac{\alpha}{p^s} \right\rfloor = t, \)
\[
\left\lfloor \frac{\alpha}{p^{s+1}} \right\rfloor = \left\lfloor \frac{p^st-r}{p^s} \right\rfloor = \left\lfloor \frac{t}{p^s} \right\rfloor = t - 1 = \left\lfloor \frac{\alpha}{p^s} \right\rfloor - 1.
\]

If \( p^s \nmid \alpha \), then \( 1 \leq r \leq p^s - 1 \) and \( \alpha - 1 = p^st + (r - 1) \) with \( 0 \leq r - 1 \leq p^s - 2 \).

Thus \( \left\lfloor \frac{\alpha}{p^s} \right\rfloor = \left\lfloor \frac{t + r}{p^s} \right\rfloor = t + 1 \) and \( \left\lfloor \frac{\alpha-1}{p^s} \right\rfloor = \left\lfloor \frac{t + r - 1}{p^s} \right\rfloor = t = \left\lfloor \frac{\alpha}{p^s} \right\rfloor - 1. \) This finishes the proof of Lemma 5.3. \( \square \)
From Lemma 5.3 (b) we obtain that (5.10) is equal to
\[
(5.11) \quad t_1(\alpha) = \frac{w(\alpha)}{p-1} = \frac{q^{d(\alpha-\frac{[\alpha]}{p})} - 1}{p-1} = \frac{q^{d(\alpha-\frac{[\alpha]}{p})} - 1}{p-1} = v_1(\alpha).
\]

As a consequence of (5.11), we have Proposition 5.2. \(\square\)

Proposition 5.2 proves (4.3) for \(n = 1\) and all \(\alpha \in \mathbb{N}\).

Now consider any cyclic extension \(K_n/k\) of degree \(p^n\) such that \(P\) is the only ramified prime, it is fully ramified, \(p_{\infty}\) decomposes fully in \(K_n/k\) and \(\mathfrak{P}_K | P^n\). We want to prove that \(K_n \subseteq k(\Lambda_{P^n})\), that is, (4.3): \(t_n(\alpha) \leq v_n(\alpha)\). This will be proved by induction on \(n\). The case \(n = 1\) is Proposition 5.2. We assume that any cyclic extension \(K_{n-1}\) of degree \(p^{n-1}\), \(n \geq 2\) such that \(P\) is the only ramified prime, \(p_{\infty}\) decomposes fully in \(K_{n-1}/k\) and \(\mathfrak{P}_{K_{n-1}} | P^\delta\) is contained in \(k(\Lambda_{P^\delta})\) where \(\delta \in \mathbb{N}\).

Let \(K_n\) be any cyclic extension of degree \(p^n\) such that \(P\) is the only ramified prime and it is fully ramified, \(p_{\infty}\) decomposes fully in \(K_n/k\) and \(\mathfrak{P}_K | P^n\). Let \(K_{n-1}\) be the subfield of \(K_n\) of degree \(p^{n-1}\) over \(k\). Now we consider \(K_{n-1}/k\) generated by the Witt vector \(\vec{\lambda} = (\beta_1, \ldots, \beta_{n-1}, \beta_n)\), that is, \(\varphi(y) = \vec{y}^P - \vec{y} = \vec{\lambda}\), and we assume that \(\vec{\lambda}\) is in the normal form described by Schmid (see Theorem 4.1, [6]). Then \(K_{n-1}/k\) is given by the Witt vector \(\vec{\gamma} = (\beta_1, \ldots, \beta_{n-1})\).

If \(\lambda := (\lambda_1, \lambda_{n-1}, \lambda_n)\) is the vector of Schmid’s parameters, that is, each \(\beta_i\) is given by
\[
\beta_i = \frac{Q_i}{P^{\lambda_i}}, \text{ where } Q_i = 0 \text{ (that is, } \beta_i = 0) \text{ and } \lambda_i = 0 \text{ or gcd}(Q_i, P) = 1, \text{deg } Q_i < \text{deg } P^{\lambda_i}, \lambda_i > 0 \text{ and } \gcd(\lambda_i, p) = 1.
\]

Since \(P\) is fully ramified we have \(\lambda_1 > 0\).

Now we compute how many different extensions \(K_n/K_{n-1}\) can be constructed by means of \(\beta_n\).

**Lemma 5.4.** For a fixed \(K_{n-1}\) the number of different fields \(K_n\) is less than or equal to
\[
(5.12) \quad \frac{1 + w(\alpha)}{p} = \frac{1}{p} q^{d(\alpha-\frac{[\alpha]}{p})}.
\]

**Proof:** For \(\beta_n \neq 0\), each equation in normal form is given by
\[
(5.13) \quad g^n - y_n = z_{n-1} + \beta_n,
\]
where \(z_{n-1}\) is the element in \(K_{n-1}\) obtained by the Witt generation of \(K_{n-1}\) by the vector \(\vec{\gamma}\) (see [6, page 161]). In fact \(z_{n-1}\) is given, formally, by
\[
z_{n-1} = \sum_{i=1}^{n-1} \frac{1}{p^{n-i}} [g_i^{P^{n-1}} + \beta_i^{P^{n-1}} - (y_i + \beta_i + z_{i-1})^{P^{n-1}}],
\]
with \(z_0 = 0\).

As in the case \(n = 1\) we have that there exist \(\Phi(P^{\lambda_n})\) different \(\beta_n\) with \(\lambda_n > 0\). With the change of variable \(y_n \rightarrow y_n + c, c \in G_{\lambda_n} := \{ \frac{h}{p^{\lambda_n}} | h \in R_T, \deg h < \deg P^{\gamma_n} = d\gamma_n \text{ or } h = 0 \}\) where \(\gamma_n = \frac{[\lambda_n]}{p}\), we obtain \(\beta_n \rightarrow \beta_n + \varphi(c)\) also in normal form. Therefore the number of different elements \(\beta_n\) which provide the same field \(K_n\) with this change of variable is \(q^{d(\frac{[\lambda_n]}{p})}\). Therefore we obtain at most
\(\Phi \left( P^{\lambda_n - \left[ \frac{\lambda_n}{p} \right]} \right)\) possible fields \(K_n\) for each \(\lambda_n > 0\) (see (5.7)). More precisely, if for each \(\beta_n\) with \(\lambda_n > 0\) we set \(\overline{\beta_n} := \{\beta_n + \wp(c) \mid c \in \mathcal{G}_{\lambda_n}\}\), then any element of \(\overline{\beta_n}\) gives the same field \(K_n\).

Let \(v_p\) denote the valuation at \(P\) and

\[
\mathcal{A}_{\lambda_n} := \left\{ \overline{\beta_n} \mid v_p(\beta_n) = -\lambda_n \right\},
\]

\[
\mathcal{A} := \bigcup_{\lambda_n = 1, \gcd(\lambda_n, p) = 1} \mathcal{A}_{\lambda_n}.
\]

Then any field \(K_n\) is given by \(\beta_n = 0\) or \(\overline{\beta_n} \in \mathcal{A}\). From (5.9) we have that the number of fields \(K_n\) containing a fixed \(K_{n-1}\) that we obtain in (5.13) is less than or equal to

\[
1 + |\mathcal{A}| = 1 + w(\alpha) = q^d(\alpha-1-\left[\frac{\alpha-1}{p}\right]) = q^d(\alpha-1-\left[\frac{\alpha}{p}\right]+1) = q^d(\alpha-\left[\frac{\alpha}{p}\right]).
\]

Now with the substitution \(y_n \to y_n + jy_1, j = 0, 1, \ldots, p-1\), in (5.13) we obtain

\[
(y_n + jy_1)^p - (y_n + jy_1) = y_n^p - y_n + j(y_1^p - y_1) = z_n - 1 + \beta_n + j\beta_1.
\]

Therefore each of the extensions obtained in (5.13) is repeated at least \(p\) times, that is, for each \(\beta_n\), we obtain the same extension with \(\beta_n, \beta_n + \beta_1, \ldots, \beta_n + (p-1)\beta_1\). We will prove that different \(\beta_n + j\beta_1\) correspond to different elements of \(\{0\} \cup \mathcal{A}\).

Fix \(\beta_n\). We modify each \(\beta_n + j\beta_1\) into its normal form: \(\beta_n + j\beta_1 + \wp(c_{\beta_n, j})\) for some \(c_{\beta_n, j} \in k\). Indeed \(\beta_n + j\beta_1\) is always in normal form with the possible exception that \(\lambda_n = \lambda_1\) and in this case it holds for at most one index \(j \in \{0, 1, \ldots, p-1\}\): if \(\lambda_n \neq \lambda_1\),

\[
v_p(\beta_n + j\beta_1) = \begin{cases} -\lambda_n & \text{if } j = 0 \\ -\max\{-\lambda_n, -\lambda_1\} & \text{if } j \neq 0. \end{cases}
\]

When \(\lambda_n = \lambda_1\) and if \(v_p(\lambda_n + j\lambda_1) = u > -\lambda_n = -\lambda_1\) and \(p|u\), then for \(i \neq j\), \(v_p(\beta_n + i\beta_1) = v_p(\beta_n + j\beta_1 + (i - j)\beta_1) = -\lambda_n = -\lambda_1\). In other words \(c_{\beta_n, j} = 0\) with very few exceptions.

Each \(\mu = \beta_n + j\beta_1 + \wp(c_{\beta_n, j})\), \(j = 0, 1, \ldots, p-1\) satisfies that either \(\mu = 0\) or \(\overline{\mu} \in \mathcal{A}\). We will see that all these elements give different elements of \(\{0\} \cup \mathcal{A}\).

If \(\beta_n = 0\), then for \(j \neq 0\), \(v_p(j\beta_1) = -\lambda_1\), so \(j\beta_1 \in \mathcal{A}\). Now if \(j\beta_1 = i\beta_n\), then

\[
j\beta_1 = \beta_n + \wp(c_1) \quad \text{and} \quad i\beta_1 = \beta_n' + \wp(c_2)
\]

for some \(\beta_n' \neq 0\) and some \(c_1, c_2 \in \mathcal{G}_{\lambda_1}\). It follows that \((j - i)\beta_1 = \wp(c_2 - c_1) \in \wp(k)\). This is not possible by the choice of \(\beta_1\) unless \(i = j\).

Let \(\beta_n \neq 0\). The case \(\beta_n + j\beta_1 = 0\) for some \(j \in \{0, 1, \ldots, p-1\}\) has already been considered in the first case. Thus we consider the case \(\beta_n + j\beta_1 + \wp(c_{\beta_n, j}) \neq 0\) for all \(j\). If for some \(i, j \in \{0, 1, \ldots, p-1\}\) we have \(\overline{\beta_n + j\beta_1 + \wp(c_{\beta_n, j})} = \overline{\beta_n + i\beta_1 + \wp(c_{\beta_n, i})}\) then there exists \(\beta_n'\) and \(c_1, c_2 \in k\) such that

\[
\beta_n + j\beta_1 + \wp(c_{\beta_n, j}) = \beta_n' + \wp(c_1) \quad \text{and} \quad \beta_n + i\beta_1 + \wp(c_{\beta_n, i}) = \beta_n' + \wp(c_2).
\]

It follows that \((j - i)\beta_1 = \wp(c_2 - c_1 - c_{\beta_n, i} - c_{\beta_n, j}) \in \wp(k)\) so that \(i = j\).

Therefore each field \(K_n\) is represented by at least \(p\) different elements of \(\{0\} \cup \mathcal{A}\). The result follows. \(\Box\)
Now, according to Schmid [6, page 163], the conductor of $K_n$ is $P^{M_n+1}$ where $M_n = \max\{pM_{n-1}, \lambda_n\}$ and $P^{M_n-1}$ is the conductor of $K_{n-1}$. Since $\mathfrak{F}_{K_n} | P^\alpha$, we have $M_n \leq \alpha - 1$. Therefore $pM_{n-1} \leq \alpha - 1$ and $\lambda_n \leq \alpha - 1$. Hence $\mathfrak{F}_{K_n-1} | P^\delta$
with $\delta = \left[\frac{\alpha - 1}{p}\right] + 1$.

**Proposition 5.5.** We have

$$\frac{v_n(\alpha)}{v_{n-1}(\delta)} = \frac{q^d(\alpha - [\frac{\alpha}{p}])}{p} ,$$

where $\delta = \left[\frac{\alpha - 1}{p}\right] + 1$.

**Proof:** From Proposition 5.1 we obtain

$$v_n(\alpha) = \frac{q^d(\alpha - [\frac{\alpha}{p^n-2}])}{p^{n-1}(p-1)} \left(q^d(\left[\frac{\alpha}{p^n-2}\right] - [\frac{\alpha}{p^n-1}] - 1)\right),$$

and

$$v_{n-1}(\delta) = \frac{q^d(\frac{\delta}{p^{n-2}})}{p^{n-2}(p-1)} \left(q^d(\left[\frac{\delta}{p^{n-2}}\right] - [\frac{\delta}{p^{n-1}}] - 1)\right).$$

From Lemma 5.3 we have

$$\left[\frac{\delta}{p^{n-2}}\right] - \left[\frac{\delta}{p^{n-1}}\right] = \left(\left[\frac{\alpha - 1}{p^{n-2}}\right] + 1\right) - \left(\left[\frac{\alpha - 1}{p^{n-1}}\right] + 1\right)$$

$$= \frac{\alpha - 1}{p^{n-2}} - \frac{\alpha - 1}{p^{n-1}} = [\left[\frac{\alpha - 1}{p}\right]] - [\left[\frac{\alpha - 1}{p^n-1}\right]]$$

$$= \frac{\alpha - 1}{p^{n-2}} - \frac{\alpha}{p^n} = \left(\frac{\alpha - 1}{p^{n-2}} + 1\right) - \left(\frac{\alpha}{p^n} + 1\right)$$

Therefore

$$v_{n-1}(\delta) = \frac{q^d(\left[\frac{\alpha}{p}\right] - [\frac{\alpha - 1}{p^n-2}] - 1)\right)\right).$$
Thus, again by Lemma 5.3

\[ \frac{v_n(\alpha)}{v_{n-1}(\delta)} = \frac{\frac{1}{q^d \left( \alpha - \left\lfloor \frac{\alpha}{p} \right\rfloor \right)}{p^{n-1}(p-1)} (q^d \left( \left\lfloor \frac{\alpha}{p^n} \right\rfloor - \left\lfloor \frac{\alpha}{p^n} \right\rfloor \right) - 1)}{\frac{1}{q^d \left( \alpha - \left\lfloor \frac{\alpha}{p} \right\rfloor \right)}{p^{n-2}(p-1)} (q^d \left( \left\lfloor \frac{\alpha}{p^{n-1}} \right\rfloor - \left\lfloor \frac{\alpha}{p^{n-1}} \right\rfloor \right) - 1)} \]

\[ = \frac{1}{p^d} q^d \left( \alpha - \left\lfloor \frac{\alpha}{p^n} \right\rfloor \right) (q^d \left( \left\lfloor \frac{\alpha}{p^n} \right\rfloor - \left\lfloor \frac{\alpha}{p^n} \right\rfloor \right) - 1) \]

This proves the result. \( \square \)

Hence, from Proposition 5.5, Lemma 5.4 (5.12) and since by the induction hypothesis, \( t_{n-1}(\delta) = v_{n-1}(\delta) \), we obtain

\[ t_n(\alpha) \leq t_{n-1}(\delta) \left( \frac{1}{p^d} q^d \left( \alpha - \left\lfloor \frac{\alpha}{p^n} \right\rfloor \right) \right) = v_{n-1}(\delta) \left( \frac{1}{p^d} q^d \left( \alpha - \left\lfloor \frac{\alpha}{p^n} \right\rfloor \right) \right) = v_n(\alpha). \]

This proves (4.3) and Theorem 2.1.

6. ALTERNATIVE PROOF OF (4.3)

We keep the same notation as in previous sections. Let \( K/k \) be an extension satisfying the conditions (4.1) and with conductor a divisor of \( P^n \). We have \( \mathfrak{S}_K = P^{M_n+1} \) where

\[ M_n = \max\{p^{n-1}\lambda_1, p^{n-2}\lambda_2, \ldots, p\lambda_{n-1}, \lambda_n\}, \]

see [6]. Therefore

\[ \mathfrak{S}_K | P^n \iff M_n + 1 \leq \alpha \iff p^{n-i}\lambda_i \leq \alpha - 1, \quad i = 1, \ldots, n. \]

Thus \( \lambda_i \leq \left\lfloor \frac{\alpha - 1}{p^{n-i}} \right\rfloor \). These conditions give all cyclic extensions of degree \( p^n \)

where \( P \in R_f^+ \) is the only ramified prime, it is fully ramified, \( p_\infty \) decomposes fully

and its conductor divides \( P^n \). Now we estimate the number of different forms of

generating \( K \).

Let \( K = k(\bar{y}) \). First, note that with the change of variable \( y_i \) for \( y_i + c_i \) for each

\( i, c_i \in k \) we obtain the same extension. For these new ways of generating \( K \) to

satisfy (4.1), we must have:

(a) If \( \lambda_i = 0, c_i = 0 \).

(b) If \( \lambda_i > 0 \), then \( c_i \in \left\{ \frac{h}{p^{n-i}} \mid h \in R_f, \deg h < \deg P^{\gamma_i} = d\gamma_i \text{ or } h = 0 \right\} \), where

\[ \gamma_i = \left\lfloor \frac{\lambda_i}{p} \right\rfloor \]. Therefore we have at most \( \Phi\left( P^{\lambda_i - \left\lfloor \frac{\lambda_i}{p} \right\rfloor} \right) \) extensions for this \( \lambda_i \) (see

(5.7)). Since \( 1 \leq \lambda_i \leq \left\lfloor \frac{\alpha - 1}{p^{n-i}} \right\rfloor \) and \( \gcd(\lambda_i, p) = 1 \), if we let \( \delta_i := \left\lfloor \frac{\alpha - 1}{p^{n-i}} \right\rfloor + 1 \),

from (5.8) and (5.9) we obtain that we have at most

\[ w(\delta_i) = \sum_{\substack{\lambda_i = 1 \\ \gcd(\lambda_i, p) = 1}}^{\delta_i - 1} \Phi\left( P^{\lambda_i - \left\lfloor \frac{\lambda_i}{p} \right\rfloor} \right) = q^d \left( \delta_i - 1 - \left\lfloor \frac{\alpha - 1}{p^{n-i}} \right\rfloor \right) \]
different expressions for all possible $\lambda_i > 0$.

Now by Lemma 5.3 we have

$$\delta_i - 1 - \left[ \frac{\delta_i - 1}{p} \right] = \left[ \frac{\alpha - 1}{p^{n-1}} \right] - \left[ \frac{\alpha - 1}{p^{n-2}} \right] = \left[ \frac{\alpha - 1}{p^{n-1}} \right] - \left[ \frac{\alpha - 1}{p^{n-2}} \right].$$

Therefore

\begin{equation}
(6.2) \quad w(\delta_i) = q^d(\left[ \frac{\alpha - 1}{p^{n-1}} \right] - \left[ \frac{\alpha - 1}{p^{n-2}} \right]) - 1.
\end{equation}

When $\lambda_i = 0$ is allowed, we have at most $w(\delta_i) + 1$ extensions with parameter $\lambda_i$. Therefore, since $\lambda_1 > 0$ and $\lambda_i \geq 0$ for $i = 2, \ldots, n$, we have that the number of extensions satisfying (4.1) and with conductor a divisor of $P^n$ is at most

$$s_n(\alpha) := w(\delta_1) \cdot \prod_{i=2}^{n} (w(\delta_i) + 1).$$

From (6.1) and (6.2), we obtain

$$s_n(\alpha) = \left( q^d(\left[ \frac{\alpha - 1}{p^{n-1}} \right] - \left[ \frac{\alpha - 1}{p^{n-2}} \right]) - 1 \right) \cdot \prod_{i=2}^{n} q^d(\left[ \frac{\alpha - 1}{p^{n-1}} \right] - \left[ \frac{\alpha - 1}{p^{n-2}} \right]).$$

Therefore $\prod_{i=2}^{n} (w(\delta_i) + 1) = q^{d\mu}$ where

$$\mu = \sum_{i=2}^{n} \left( \left[ \frac{\alpha - 1}{p^{n-1}} \right] - \left[ \frac{\alpha - 1}{p^{n-2}} \right] \right) = \sum_{i=2}^{n} \left[ \frac{\alpha - 1}{p^{n-1}} \right] - \sum_{j=1}^{n-1} \left[ \frac{\alpha - 1}{p^{n-j}} \right]$$

$$\quad = \left[ \frac{\alpha - 1}{p^{n-1}} \right] - \left[ \frac{\alpha - 1}{p^{n-1}} \right] - \left[ \frac{\alpha - 1}{p^{n-1}} \right] = \alpha - 1 - \left[ \frac{\alpha - 1}{p^{n-1}} \right].$$

Hence

$$s_n(\alpha) = \left( q^d(\left[ \frac{\alpha - 1}{p^{n-1}} \right] - \left[ \frac{\alpha - 1}{p^{n-2}} \right]) - 1 \right) \cdot q^d(\alpha - 1 - \left[ \frac{\alpha - 1}{p^{n-2}} \right])$$

$$\quad = q^d(\alpha - \left[ \frac{\alpha - 1}{p^{n-1}} \right]) - q^d(\alpha - 1 - \left[ \frac{\alpha - 1}{p^{n-2}} \right]).$$

From Lemma 5.3 (b) we obtain

$$\alpha - 1 - \left[ \frac{\alpha - 1}{p^n} \right] = \alpha - \left[ \frac{\alpha}{p^n} \right] \quad \text{and} \quad \alpha - 1 - \left[ \frac{\alpha - 1}{p^{n-1}} \right] = \alpha - \left[ \frac{\alpha}{p^{n-1}} \right].$$

Thus

$$s_n(\alpha) = q^{(\alpha - \left[ \frac{\alpha}{p^n} \right])} - q^{(\alpha - \left[ \frac{\alpha}{p^{n-1}} \right])} = p^{n-1}(p - 1)v_n(\alpha).$$

Finally, the change of variable $\vec{y} \rightarrow \vec{y} \times \vec{\beta}$ with $\vec{y} \in W_n(\mathbb{F}_p)^\ast \cong (\mathbb{Z}/p^n\mathbb{Z})^\ast$ gives the same field and we have $\vec{\beta} \rightarrow \vec{y} \times \vec{\beta}$. Therefore

$$t_n(\alpha) \leq \frac{s_n(\alpha)}{\varphi(p^n)} = \frac{s_n(\alpha)}{p^n(p - 1)} = v_n(\alpha).$$

This proves (4.3) and Theorem 2.1.
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Universidad Autónoma de la Ciudad de México, Academia de Matemáticas. Plantel San Lorenzo Tezontle, Prolongación San Isidro No. 151 Col. San Lorenzo, Iztapalapa, C.P. 09790, México, D.F.

E-mail address: jcstorres88@hotmail.com, torresjcesar0@gmail.com

Departamento de Control Automático, Centro de Investigación y de Estudios Avanzados del I.P.N.

E-mail address: mrzedowski@ctrl.cinvestav.mx

Departamento de Control Automático, Centro de Investigación y de Estudios Avanzados del I.P.N.

E-mail address: gillasalvador@gmail.com, gvilla@ctrl.cinvestav.mx