TORIC CUBES ARE CLOSED BALLS

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Abstract. We prove that toric cubes, which are images of \([0,1]^d\) under monomial maps, are the closures of graphs of monotone maps, and in particular semi-algebraically homeomorphic to closed balls.

1. Introduction

In [3] Engström, Hersh and Sturmfels introduced a class of compact semi-algebraic sets which they call toric cubes.

The following definition is adapted from [3].

Definition 1.1. Let \(A = \{a_1, \ldots, a_n\} \subset \mathbb{N}^d\), and \(f_A : [0,1]^d \to [0,1]^n\) be the map \(t = (t_1, \ldots, t_d) \mapsto (t^{a_1}, \ldots, t^{a_n})\),

where \(t^{a_i} := t_1^{a_{i,1}} \cdots t_d^{a_{i,d}}\) for \(a_i = (a_{i,1}, \ldots, a_{i,d})\). The image of \(f_A\) is called a toric cube.

We call the image of the restriction of \(f_A\) to \((0,1)^d\) an open toric cube. The closure of an open toric cube is a toric cube. Note that an open toric cube is not necessarily an open subset of \(\mathbb{R}^n\), and need not be contained in \((0,1)^n\) (if some \(a_i = 0\)).

In [1, 2] the authors introduced a certain class of definable subsets of \(\mathbb{R}^n\) (called semi-monotone sets) and definable maps \(f : X \to \mathbb{R}^k\) (called monotone maps), where \(X \subset \mathbb{R}^n\) is a semi-monotone set. Here “definable” means “definable in an o-minimal structure over \(\mathbb{R}\)”, for example, real semi-algebraic.

These objects are meant to serve as building blocks for obtaining a conjectured cylindrical cell decomposition of definable sets into topologically regular cells, without changing the coordinate system in the ambient space \(\mathbb{R}^n\) (see [1, 2] for a more detailed motivation behind these definitions).

The main result of this note is the following theorem.

Theorem 1.2. An open toric cube \(C \subset \mathbb{R}^n\) is the graph of a monotone map.

As a result we obtain

Corollary 1.3. An open toric cube \(C \subset [0,1]^n\), with \(\dim(C) = k\), is semi-algebraically homeomorphic to a standard open ball. The pair \((C,C)\) is semi-algebraically homeomorphic to the pair \(([0,1]^k, (0,1)^k)\), in particular, a toric cube is semi-algebraically homeomorphic to a standard closed ball.

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Remark 1.4. Note that the first statement in Corollary 1.3 is also proved in [3 Proposition 1]. In conjunction with Theorem 2 in [3], Corollary 1.3 implies that any CW-complex in which the closures of each cell is a toric cube, must be a regular cell complex, and this answers in the affirmative the Conjecture 1 in [3].

2. Proof of Theorem 1.2 and Corollary 1.3

We begin with a few preliminary definitions.

Definition 2.1. Let \( L_{j,\sigma,c} := \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_j \sigma c \} \) for \( j = 1, \ldots, n \), \( \sigma \in \{<,=,>\} \), and \( c \in \mathbb{R} \). Each intersection of the kind \( C := L_{j_1,\sigma_1,c_1} \cap \cdots \cap L_{j_m,\sigma_m,c_m} \subset \mathbb{R}^n \), where \( m = 0, \ldots, n \), \( 1 \leq j_1 < \cdots < j_m \leq n \), \( \sigma_1, \ldots, \sigma_m \in \{<,=,>\} \), and \( c_1, \ldots, c_m \in \mathbb{R} \), is called a coordinate cone in \( \mathbb{R}^n \).

Each intersection of the kind \( S := L_{j_1,=,c_1} \cap \cdots \cap L_{j_m,=,c_m} \subset \mathbb{R}^n \), where \( m = 0, \ldots, n \), \( 1 \leq j_1 < \cdots < j_m \leq n \), and \( c_1, \ldots, c_m \in \mathbb{R} \), is called an affine coordinate subspace in \( \mathbb{R}^n \).

In particular, the space \( \mathbb{R}^n \) itself is both a coordinate cone and an affine coordinate subspace in \( \mathbb{R}^n \).

Definition 2.2 ([I]). An open (possibly, empty) bounded set \( X \subset \mathbb{R}^n \) is called semi-monotone if for each coordinate cone the intersection \( X \cap C \) is connected.

Remark 2.3. In fact, in Definition 2.2 above, it suffices to consider intersections with only affine coordinate subspaces (see [2, Theorem 4.3] or Theorem 2.5 below).

Notice that any convex open subset of \( \mathbb{R}^n \) is semi-monotone.

The definition of monotone maps is given in [2] and is a bit more technical. We will not repeat it here but recall a few important properties of monotone maps that we will need. In particular, Theorem 2.5 below, which appears in [2], gives a complete characterization of monotone maps. For the purposes of the present paper this characterization can be taken as the definition of monotone maps.

Definition 2.4 ([2], Definition 1.4). Let a bounded continuous quasi-affine map \( f = (f_1, \ldots, f_k) \) defined on an open bounded non-empty set \( X \subset \mathbb{R}^n \) have the graph \( F \subset \mathbb{R}^{n+k} \). We say that \( f \) is quasi-affine if for any coordinate subspace \( T \subset \mathbb{R}^{n+k} \), the projection \( \rho_T : F \to T \) is injective if and only if the image \( \rho_T(F) \) is \( n \)-dimensional.

The following theorem is proved in [2].

Theorem 2.5 ([2], Theorem 4.3). Let a bounded continuous quasi-affine map \( f = (f_1, \ldots, f_k) \) defined on an open bounded non-empty set \( X \subset \mathbb{R}^n \) have the graph \( F \subset \mathbb{R}^{n+k} \). The following three statements are equivalent.

(i) The map \( f \) is monotone.

(ii) For each affine coordinate subspace \( S \subset \mathbb{R}^{n+k} \) the intersection \( F \cap S \) is connected.

(iii) For each coordinate cone \( C \subset \mathbb{R}^{n+k} \) the intersection \( F \cap C \) is connected.

Remark 2.6. In view of Theorem 2.5 it is natural to identify any semi-monotone set \( X \subset \mathbb{R}^n \) with the graph of an identically constant function \( f \equiv c \) on \( X \), where \( c \) is an arbitrary real.
Definition 2.7. A definable bounded open set $U \subset \mathbb{R}^n$ is called (topologically) regular cell if $U$ is definably homeomorphic to a closed ball, and the frontier $\overline{U} \setminus U$ is definably homeomorphic $(n-1)$-sphere. In other words, the pair $(\overline{U}, U)$ is definably homeomorphic to the pair $([0,1]^n, (0,1)^n)$.

Theorem 2.8 ([2], Theorem 5.1). The graph $F \subset \mathbb{R}^{n+k}$ of a monotone map $f : X \to \mathbb{R}^k$ on a semi-monotone set $X \subset \mathbb{R}^n$ is definably homeomorphic to a regular cell.

Proof of Theorem 1.2. Let $C \subset [0,1]^n$ be an open toric cube and suppose that $C = f_A((0,1)^d)$ for a monomial map $f_A$ (see Definition 1.1).

Make the coordinate change $z_i = \log(t_i)$ for every $i = 1, \ldots, d$, and take the logarithm of every component of the map $f_A$ expressed in coordinates $z_i$. Denote the resulting map by $\log f_A$. Then $\log f_A$ is the restriction of a linear map, namely $\log f_A : (-\infty,0)^d \to (-\infty,0)^n$,

defined by $z = (z_1, \ldots, z_d) \mapsto (a_1 \cdot z, \ldots, a_n \cdot z)$.

Observe that $\log$ (the component-wise logarithm) maps the open cube, $(0,1)^d$ (resp. $(0,1)^n$) homeomorphically onto $(-\infty,0)^d$ (resp. $(-\infty,0)^n$). It follows that the fiber of the orthogonal projection of $C$ to any $k$-dimensional coordinate subspace is the pre-image under the log map of an affine subset of $(-\infty,0)^n$, and is a single point if it is zero-dimensional. Hence $C$ is a graph of a quasi-affine map (choose any set of $k$ coordinates such that the image of $C$ under the orthogonal projection to the coordinate subspace of those coordinates is full dimensional).

Similarly, the intersection of $C$ with any affine coordinate subspace is the pre-image under the log map, of an affine subset of $(-\infty,0)^n$ and hence connected.

We proved that $C$ satisfies the conditions of Theorem 2.5, hence $C$ is the graph of a monotone map. □

Proof of Corollary 1.3. Immediate consequence of Theorem 1.2 and Theorem 2.8. □

References
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