Associated Families of Surfaces in Warped Products and Homogeneous Spaces

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Abstract

We classify Riemannian surfaces admitting associated families in three dimensional homogeneous spaces with four-dimensional isometry groups and in a wide family of (semi-Riemannian) warped products, with an extra natural condition. We prove that, provided the surface is not totally umbilical, such families exist in both cases if and only if the ambient manifold is a product and the surface is minimal. In particular, this shows that there exists no associated families of surfaces with rotating structure vector field in the Heisenberg group.

1 Introduction

Classically, associate families are certain isometric deformations of minimal surfaces in Euclidean three-space, the best known example being the deformation of the catenoid into the helicoid. A well-known property for such isometric immersions $\chi : M \to \mathbb{R}^3$ is namely the existence of a so-called strong associate family, i.e. a one-parameter family $\chi_\theta$, with $\theta \in S^1$ and $\chi_0 = \chi$, of isometric immersions ”rotating the differential” and preserving, at every point, the tangent plane and the Gauß map. More precisely, denoting by $R_\theta : TM \to TM$ the rotation of the tangent plane by an angle $\theta$, a strong associate family of isometric immersions is a smooth family $\chi_\theta$ satisfying $d\chi(R_\theta) = d\chi_\theta$. Moreover it is known that minimality and the existence of an associate family are equivalent conditions.

A generalization to a larger class of surfaces is given by constant mean curvature surfaces (shortly CMC) and can also be characterized by the existence of an associate family $\chi_\theta$ which is defined as follows. Let $\chi_0 : M \to \mathbb{R}^3$ be a CMC surface and let $\chi_\theta : M \to \mathbb{R}^3$ be a smooth family of isometric immersions into $\mathbb{R}^3$ with second fundamental forms $A_\theta$. Then $\chi_\theta$ is an associate family if and only if $A_\theta = R_{-\theta}A_0R_\theta$. If $M$ is minimal, the operator $R_\theta$ anticommutes with $A_0$ and we recover the above strong associate family. It is a well-known fact that there exists an associate family if and only if $M$ is a CMC surface, which is furthermore equivalent to the harmonicity of the Gauß map. More generally an analogous result holds for CMC surfaces in three-dimensional space forms. It is worth pointing out that the notion of (strong) associate family was extended in [1] to Kähler manifolds in $\mathbb{R}^n$ without significant modifications (see also [4]). The existence of an associate family for such manifolds is then equivalent to the pluriharmonicity of the Gauß map.
In recent years, minimal and CMC surfaces in other ambient spaces, such as homogeneous three-spaces and (Lorentzian) warped products, have received a lot of attention. Especially, minimal surfaces in the product spaces $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ have been extensively studied and the existence of an associate minimal family was showed in [3] (see also [5] and [6], and [11] for the Lorentzian case). Nevertheless, no equivalence between the minimality of the surface and the existence of the family had been proven until now. We also mention [7] and [9] where associate families of minimal surfaces in (multi)products of space forms where studied.

Daniel considered in [2] surfaces immersed into the other three-dimensional homogeneous spaces with four-dimensional isometry group. Those spaces usually denoted $\mathbb{E}(\kappa, \tau)$ are well-known to be Riemannian fibration over a 2-dimensional space form where $\kappa$ is the curvature of the base surface of the fibration, and $\tau$ the bundle curvature. Daniel showed the existence of a Lawson-type correspondence of CMC surfaces, the so-called sister surfaces, but the question of the existence of the associate family for minimal or CMC surfaces in such geometries when $\tau \neq 0$ remained open. Of particular interest is the case of the Heisenberg group (see for instance [2], Remark 5.10).

Our aim is the classification of surfaces admitting an associate family in the ambient space $P^3$, where $P^3$ is either a Thurston geometry with four-dimensional isometry group or a semi-Riemannian warped product of the form $\varepsilon I \times_a M_k^\varepsilon(c)$, where $\varepsilon = \pm 1$, $a : I \subset \mathbb{R} \to \mathbb{R}^+$ is the scale factor and $M_k^\varepsilon(c)$ is the semi-Riemannian space form of index $k$ and constant curvature $c$, excluding in our study the well-understood case where the ambient space is a space form. In fact these two cases are similar because of the importance in the Thurston geometry case of the unit vertical Killing vector field $\partial_t$ tangent to the fibers, which corresponds in the case of warped products to the vector in the direction of the factor $\mathbb{R}$. Given a hypersurface $M$, the vector field $\partial_t$ can be decomposed along $M$ in its tangent and normal parts i.e. $\partial_t = T + f\nu$, where $\nu$ is the unit vector field normal to $M$. The compatibility equations depend in both cases not only on the shape operator $A$, but also on the tangent vector field $T$ and the normal component function $f$. Moreover, whereas the Gauß and Codazzi integrability conditions are well-known to be necessary and sufficient for the existence of isometric immersions of hypersurfaces into space forms, it was proved in [2] for the homogeneous case and in [8] for the warped product case that two additional equations involving the covariant derivative of $T$ and $f$ have to be satisfied on $M$ in order for the immersion to exist.

Consequently, considering associate families of surfaces in those spaces, it is necessary to not only rotate the shape operator, but also the vector field $T$. We introduce for that purpose a more general transformation and we call a smooth family $\chi_\theta : M \to P^3$ with second fundamental form $A_\theta$ and vector $T_\theta$ a generalized associate family with rotating structure field $T_\theta$, if and only if there exist smooth real functions $F_1$, $F_2$, $\lambda$ and $\mu$, such that

$$A_\theta = F_1 R_\theta A^a R_\theta + F_2 A^c, \quad T_\theta = \lambda T + \mu JT,$$

where $J$ is the rotation of angle $\frac{\pi}{2}$ on $M$ given by the orientation, and $A^a$ (resp. $A^c$) are the parts of $A$ anticommuting (resp. commuting) with $J$. We are then able to prove that, in the case where $P^3$ is an homogeneous space, there exists a generalized associate family if and only if $P^3$ is a product and $M$ is minimal, and in the case of warped products that the existence of such deformations is also equivalent to either $P^3$ being a product and $M$ minimal, or $M$ being totally umbilical. In all cases, the generalized associate family turns out to be the classical associate family with rotating vector $T_\theta = e^{-2J\theta} T$. In particular we show that there exists no associate family of minimal surfaces in the Heisenberg group.
2 Basics

Let \((M, g)\) be a connected, oriented Riemannian surface. It is well-known that it admits a complex structure, so that it becomes a Riemann surface \((M, g, J)\). We consider an isometric immersion \(\chi : (M, g) \to (N^3, \langle \cdot, \cdot \rangle)\) in a semi-Riemannian 3-dimensional manifold, with second fundamental form \(\sigma\). Let \(\nabla\) be the Levi-Civita connection of \(N^3\). Since \(M\) is an oriented hypersurface in \(N^3\), given a unit normal vector field \(e_3\), with sign \(\varepsilon_3 = \langle e_3, e_3 \rangle = \pm 1\), we have the corresponding shape operator \(A\).

For each \(\theta \in \mathbb{R}\), we define the operator on \(M\) by

\[ R_\theta X = \cos(\theta)X + \sin(\theta)JX = e^{J\theta}X, \]

for any \(X \in T M\).

We consider now smooth one-parameter families \(\chi_\theta : (M, g) \to (N, \langle \cdot, \cdot \rangle)\) of isometric immersions, with their corresponding unit normal vector fields \(e_3^\theta\), signs \(\varepsilon_3^\theta = \langle e_3^\theta, e_3^\theta \rangle = \pm 1\) and shape operators \(A_\theta\). For this family to be smooth, \(\varepsilon_3 = \varepsilon_3^\theta\) for any \(\theta \in \mathbb{R}\).

**Definition 1** Such a family will be called an associate family with \(\chi\) if their shape operators satisfy

\[ A_\theta = e^{-J\theta}A e^{J\theta}. \]

The classical definition states that the second fundamental forms \(\sigma_{\chi_\theta}\) satisfy \(\sigma_{\chi_\theta}(X, Y) = \sigma(R_\theta X, R_\theta Y)\), which is clearly equivalent to the above definition. For further references see for example [4]. The shape operator \(A\) can be decomposed as \(A = A^c + A^a\) in such a way that \(A^c J = JA^c\) and \(A^a J = -JA^a\). This decomposition is unique.

**Definition 2** A smooth map \(\tilde{\chi} : \mathbb{R} \times M \to N\) will be called a generalized associate family with \(\chi\) if the following conditions hold:

1. \(\tilde{\chi}(0, \cdot) = \chi\);

2. for each \(\theta \in \mathbb{R}\), \(\chi_\theta = \tilde{\chi}(\theta, \cdot) : (M, g) \to (N, \langle \cdot, \cdot \rangle)\) is a smooth isometric immersion;

3. there exist two smooth functions \(F_1, F_2 : \mathbb{R} \to \mathbb{R}\) satisfying \(F_1(0) = F_2(0) = 1\) and the shape operators satisfy \(A_\theta = F_1 e^{-J\theta} A^c e^{J\theta} + F_2 A^a\) for any \(\theta \in \mathbb{R}\).

Note that whenever \(F_1 = F_2 = 1\), we recover the action \(A_\theta = e^{-J\theta}A e^{J\theta}\).

Let \(\nabla\) be the Levi-Civita connection of \(M\). We call \(\sigma_\theta\) the second fundamental form of \(\chi_\theta\). Next, we recall the Gauß formula:

\[ \nabla_X Y = \nabla_X Y + \sigma_\theta(X, Y) = \nabla_X Y + \varepsilon_3^\theta \langle A_\theta X, Y \rangle e_3^\theta, \]

for any \(X, Y \in T M\). In addition, let \(H = \text{tr}(A)/2\), and \(H_\theta = \text{tr}(A_\theta)/2\) be the mean curvatures of the immersions \(\chi\) and \(\chi_\theta\), respectively.

**Lemma 1** For any \(X \in TM\), \(JAX + AJX = 2HJX\) holds.

*Proof:* Since \(\dim M = 2\), for any unit vector (field) \(X\) on the surface, this yields \(2H = \langle AX, X \rangle + \langle AJX, JX \rangle\). We readily obtain the lemma. \(\square\)
2.1 3-dimensional Homogeneous Manifolds with 4-dim Isometry Group

We now consider surfaces immersed in a 3-dim homogeneous Riemannian manifold $\mathbb{E}$ whose isometry group has dimension 4. The classification of such manifolds is well-known and depends on two parameters, namely the curvature $\kappa$ of the base of the fibration and the bundle curvature $\tau$, where $\kappa$ and $\tau$ are real numbers and $\kappa \neq 4\tau^2$. B. Daniel gave in [3] a fundamental theorem for surfaces in such spaces, which we recall.

**Theorem 1** [3] Let $M$ be a simply connected oriented Riemannian surface with connection $\nabla$, $J$ the rotation angle $\frac{\pi}{2}$ on $TM$, $A$ a self-adjoint $(1, 1)$-tensor, $T$ a vector field on $M$, and $f$ a smooth real valued function such that $\|T\|^2 + f^2 = 1$. Let $\kappa$ and $\tau$ be real numbers such that $\kappa \neq 4\tau^2$. Then there exists an isometric immersion $\chi : M \to \mathbb{E}$ if and only if the data $(A, T, f)$ satisfy the following structure equations, where $K$ is the Gauss curvature of $M$.

\[
\begin{align*}
K &= \det A + \tau^2 + (\kappa - 4\tau^2)(1 - \|T\|^2) \quad (1) \\
(\nabla_X A)Y - (\nabla_Y A)X &= (\kappa - 4\tau^2)f(\langle Y, T \rangle X - \langle X, T \rangle Y) \quad (2) \\
\nabla_X T &= f(AX - \tau JX) \quad (3) \\
X(f) &= -\langle AX, T \rangle + \langle \tau JX, T \rangle \quad (4)
\end{align*}
\]

The operator $A$ turns out to be the shape operator of the immersion and $T$ is the part of $\partial_t$ tangent to the surface $M$.

2.2 Warped products

The authors proved in [8] a fundamental theorem for hypersurfaces in some warped products, which we recall for the case of surfaces. We choose numbers $\varepsilon, \varepsilon_0, \varepsilon_3 \in \{1, 1\}$ and $c \in \{-1, 0, 1\}$ such that either $c = \varepsilon_0 = \pm 1$ or $c = 0, \varepsilon_0 = 1$. We define $M_k^2(c), g_o$ as the 2-dimensional space form of index $k \in \{0, 1\}$ and sectional curvature $c$. Given a smooth positive function $a : M_k^2 \to (0, \infty)$, we consider the warped product $(P^3 = I \times M_k^2(c), (,) = \varepsilon dt^2 + a^2 g_o)$, with projection $\pi_1 : P^3 \to I$.

**Theorem 2** [8] Let $M$ be a simply connected oriented Riemannian surface with Levi-Civita connection $\nabla$, Gauss curvature $K$, a self-adjoint $(1, 1)$-tensor $A$, a vector field $T$ on $M$, and two smooth real valued function $f$, $\pi$ on $M$ such that $\|T\|^2 + \varepsilon_3 f^2 = \varepsilon$ and $T = \varepsilon \text{grad}(\pi)$. Then, there exists an isometric immersion $\chi : M \to P^3$ such that $\pi_1 \circ \chi = \pi$ if and only if data $(K, A, T, f)$ satisfy the following structure equations:

\[
\begin{align*}
K &= \varepsilon_3 \det A - \varepsilon \left( \frac{(a')^2}{a^2} - \frac{\varepsilon \varepsilon_0}{a^2} \right) - \left( \frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{\varepsilon \varepsilon_0}{a^2} \right) \|T\|^2, \quad (5) \\
(\nabla_X A)Y - (\nabla_Y A)X &= \varepsilon_3 \left( \frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{\varepsilon \varepsilon_0}{a^2} \right) f(\langle Y, T \rangle X - \langle X, T \rangle Y), \quad (6) \\
\nabla_X T &= fAX + \frac{a'}{a} (X - \varepsilon \langle X, T \rangle T), \quad (7) \\
X(f) &= -\langle AX, T \rangle - \frac{\varepsilon a'}{a} f\langle X, T \rangle, \quad (8)
\end{align*}
\]

for any $X, Y \in TM$, where $a \equiv a \circ \pi$, $a' \equiv a' \circ \pi$. 


In such case, the operator $A$ turns out to be the shape operator of the immersion, $\varepsilon_3$ is the causal character of the normal vector field along $\chi$, and $T$ is the part of $\partial_t$ tangent to the manifold. In addition, notice that equation (5) can be rewritten as

$$K = \varepsilon_3 \det A - \varepsilon \frac{a''}{a} + \left( \frac{a''}{a^2} - \frac{a'^2}{a^3} + \varepsilon 0 \right) (\varepsilon - \|T\|^2).$$

As an additional remark, the authors showed in [8] that if we set $\eta(X) = \langle X, T \rangle$, for any $X \in TM$, we get $d\eta = 0$, where $\eta = \varepsilon d\pi$. In other words, one can replace the condition of the choice of $T$ in Theorem 2 by the choice of the function $\pi$. In this paper, we choose to use $T$.

**Lemma 2** The (maximal) solutions to the equation $a''a - (a')^2 + \varepsilon_0 = 0$ are the following:

1. If $\varepsilon\varepsilon_0 = -1$, then $a(t) = C_1^{-1} \cosh(C_1 t + C_2)$, for some $C_1, C_2 \in \mathbb{R}$, $C_1 \neq 0$.
2. If $\varepsilon\varepsilon_0 = 1$, then $a(t) = \pm t + C_2$ or $a(t) = C_1^{-1} \sinh(C_1 t + C_2)$ or $a(t) = C_1^{-1} \sin(C_1 t + C_2)$, for some $C_1, C_2 \in \mathbb{R}$, $C_1 \neq 0$.

Among the warped products we are considering, it is well-known that those associated with these solutions are isometric to (open subsets) of space forms. Indeed, $-\mathbb{R} \times_{\cosh(t)} S^2 \cong dS^3$, the three-dimensional De Sitter space, $\mathbb{R}^+ \times_{\sinh(t)} \mathbb{H}^2 \cong \mathbb{H}^3 \cong \mathbb{R} \times_{\cosh(t)} \mathbb{H}^2$, the three-dimensional hyperbolic space, $(0, \pi) \times_{\sin(t)} S^2 \cong S^3 \setminus \{ North, South \}$, $\mathbb{R}^+ \times_{t} S^2 \cong \mathbb{R}^3 \setminus \{ 0 \}$ and $\mathbb{R}^+ \times_{t} \mathbb{H}^2 \cong \mathbb{L}^3$, the three-dimensional Minkowski space. Since surfaces in space forms are well understood, we will exclude them in the following discussion.

### 3 Generalized associate Families

Let $(M, \langle \cdot, \cdot \rangle, J)$ be a Riemann surface. We consider $(\mathbb{E}, \langle \cdot, \cdot \rangle)$ a 3-dimensional manifold. Assume that there exists an isometric immersion $\chi : (M, \langle \cdot, \cdot \rangle) \to (\mathbb{E}, \langle \cdot, \cdot \rangle)$. Let $A$ be the shape operator of the immersion. By Lemma 1, it is very easy to check that

$$A^c = H1, \quad A^a = A - H1,$$

where $1$ is the identity map on $TM$, and $H = \text{tr}(A)/2$ is the mean curvature function.

**Lemma 3** Definition 2 is equivalent to

$$A_\theta = F_1 e^{-2J\theta}(A - H1) + F_2 H1,$$

where $\theta \in \mathbb{R}$.

We also need a family of vector fields, $T_\theta \in \mathfrak{X}(M)$, $\theta \in \mathbb{R}$.

**Definition 3** We will say that the family of immersions $\chi_\theta : M \to P$ has rotating structure vector field if there exists a smooth map $\mathbb{R} \times M \to TM$, $(\theta, p) \mapsto T_\theta(p)$ satisfying:

1. For each $\theta \in \mathbb{R}$, the restriction $T_\theta \in \mathfrak{X}(M)$;
2. There exist two smooth functions $\lambda, \mu : \mathbb{R} \to \mathbb{R}$ such that $\lambda(0) = 1$, $\mu(0) = 0$, and $T_\theta = \lambda T + \mu J T$.  
In the following we use the notation $\lambda T + \mu JT = (\lambda \mathbf{1} + \mu J)T$, where $\mathbf{1}$ is the identity map on $TM$. We also need to construct the corresponding family of functions $f_{\theta} : M \rightarrow \mathbb{R}$, $\theta \in \mathbb{R}$, from a map $\mathbb{R} \times M \rightarrow \mathbb{R}$, $(\theta, p) \mapsto f_{\theta}(p)$ such that for each $\theta \in \mathbb{R}$, the restriction $f_{\theta}$ satisfies the conditions

$$1 = \|T_{\theta}\|^2 + f_{\theta}^2, \quad (\lambda^2 + \mu^2)(1 - f^2) = 1 - f_{\theta}^2, \quad \theta \in \mathbb{R}, \quad f_0 = f,$$

in the homogeneous case and

$$\varepsilon = \|T_{\theta}\|^2 + \varepsilon_3 f_{\theta}^2, \quad (\lambda^2 + \mu^2)(\varepsilon - \varepsilon f^2) = \varepsilon - \varepsilon_3 f_{\theta}^2, \quad \theta \in \mathbb{R}, \quad f_0 = f,$$

in the warped product case. Note that in either case, the map $\mathbb{R} \times M \rightarrow \mathbb{R}$, $(\theta, p) \mapsto f_{\theta}(p)$ is always continuous, and smooth whenever it is different from zero. However, we can assume without loss of generality that each $f_{\theta}$ is always smooth.

**Lemma 4** $\det A_{\theta} = F_1^2 \det A + (F_2^2 - F_1^2)H^2$.

**Proof:** Since the determinant of matrices is invariant under rotations, we have

$$\det A_{\theta} = \det \left( F_1 e^{-2J\theta} (A - H\mathbf{1}) + F_2 H\mathbf{1} \right) = \det \left( e^{-J\theta} \left( F_1 (A - H\mathbf{1}) + F_2 H\mathbf{1} \right) e^{J\theta} \right) = \det \left( F_1 (A - H\mathbf{1}) + F_2 H\mathbf{1} \right).$$

But using the fact that $H = \frac{1}{2} \text{tr}(A)$ we get easily

$$\det \left( F_1 (A - H\mathbf{1}) + F_2 H\mathbf{1} \right) = F_1^2 \det A + 2F_1 (F_2 - F_1)H^2 + (F_2 - F_1)^2 H^2$$

$$= F_1^2 \det A + (F_2 - F_1)(2F_1 + F_2 - F_1)H^2 = F_1^2 \det A + (F_2^2 - F_1^2)H^2.$$

\[\square\]

**Lemma 5** $H_{\theta} = F_2 H$

**Proof:** $2H_{\theta} = \text{tr}(A_{\theta}) = \text{tr} \left( F_1 e^{-2J\theta} (A - H\mathbf{1}) + F_2 H\mathbf{1} \right) = F_1 \text{tr}(A - H\mathbf{1}) + 2F_2 H = 2F_2 H$. \[\square\]

Let $(E_1, E_2)$ be a parallel local orthonormal frame of $M$. Then, we recall that the divergence of an operator $\mathcal{T}$ is given by $\delta \mathcal{T} = \text{tr}(\nabla \mathcal{T})$, or in other words,

$$\langle \text{tr}(\nabla \mathcal{T}), X \rangle = \langle \delta \mathcal{T}, X \rangle = \sum_i \langle (\nabla_{E_i} \mathcal{T}) E_i, X \rangle.$$

Bearing this in mind, we see

$$\langle (\nabla_{E_1} \mathcal{T}) E_2, E_1 \rangle - \langle (\nabla_{E_2} \mathcal{T}) E_1, E_1 \rangle = \langle \delta \mathcal{T}, E_2 \rangle - \langle (\nabla_{E_2} \mathcal{T}) E_2, E_2 \rangle - \langle (\nabla_{E_2} \mathcal{T}) E_1, E_1 \rangle = \langle \delta \mathcal{T}, E_2 \rangle - E_2 \text{tr}(\mathcal{T}) \rangle.$$

Similarly we get $\langle (\nabla_{E_1} \mathcal{T}) E_2, E_2 \rangle - \langle (\nabla_{E_2} \mathcal{T}) E_1, E_2 \rangle = (\delta \mathcal{T})(E_1) - E_1(\text{tr}(\mathcal{T}))$. Therefore, we obtain $\langle (\nabla_{E_1} \mathcal{T}) E_2 - (\nabla_{E_2} \mathcal{T}) E_1, X \rangle = \langle \delta \mathcal{T}, X \rangle - X(\text{tr}(\mathcal{T}))$, for any $X \in TM$. Without the vector field, we have

$$d^\nabla \mathcal{T} = \delta \mathcal{T} - \nabla \text{tr}(\mathcal{T}).$$

We can apply this formula to $A$ and $A_{\theta}$.

We need to discuss three extreme cases: $T = \partial_t$, $T = 0$ and $0 \neq T \neq \partial_t$ everywhere.
4 The case \( T = \partial_t \)

This condition is equivalent to \( f = 0 \) everywhere. All these surfaces are known as vertical cylinders. The reason is that there exists a curve \( \alpha \) on \( \mathbb{M}^2(\kappa) \) such that \( M = \pi^{-1}(\alpha) \), where \( \pi : \mathbb{E} \to \mathbb{M}^2(\kappa) \) is the natural projection on the fiber. By equation (11) we get \( \delta A = \nabla H \).

4.1 Homogeneous spaces

By equation (4), we get that \( A \) has the form \[
\begin{pmatrix}
0 & -\tau \\
-\tau & 2H
\end{pmatrix}
\]. Notice that \( \text{det} A = -\tau^2 \). Since \( f = 0 \), then \( \nabla_X T = 0 \) by equation (3). Consequently
\[
0 = (\lambda \mathbf{1} + \mu J) \nabla_X T = [1 - (\lambda^2 + \mu^2)](A\theta X - \tau JX),
\]
and either \( A\theta = \tau J \), or \( 1 = \lambda^2 + \mu^2 \). By (3), \( 1 = \lambda^2 + \mu^2 \) is equivalent to \( f\theta = 0 \). If \( A\theta = \tau J \), since \( A\theta \) is symmetric and \( J \) skew-symmetric, then \( A\theta = 0 \) and \( \tau = 0 \). The three equations reduce to
\[
K\theta = \kappa f\theta^2, \quad 0 = f\theta(\langle T\theta, Y \rangle X - \langle T\theta, X \rangle Y), \quad X(f\theta) = 0,
\]
for any \( X, Y \in T M \). But then, for each \( \theta \), \( f\theta \) is a constant function. If for some \( \theta \), \( f\theta \neq 0 \), then \( 0 = \langle T\theta, Y \rangle X - \langle T\theta, X \rangle Y \), which implies \( T\theta = 0 \), and since \( \partial_t = T\theta + f\theta N_\theta \), then \( f\theta = \pm 1 \). Since the map \( (\theta, p) \mapsto f\theta(p) \) is continuous and \( f\theta = f = 0 \), we get a contradiction. Therefore, \( f\theta = 0 \) for any \( \theta \). This means that each immersion can be recovered as the pre-image of a curve on the base, as pointed out at the beginning of this section. Moreover, \( T\theta = \partial_t \) for any \( \theta \). But now, \( A\theta \equiv \begin{pmatrix} 0 & -\tau \\ -\tau & 2H \end{pmatrix} \), for any \( \theta \). This means
\[
-\tau JT\theta = -\tau JT = A\theta T\theta = A\theta T = (F_1 e^{-2J\theta}(A - H \mathbf{1}) + F_2 H \mathbf{1}) T = H ((F_2 - F_1 \cos(2\theta)) T + F_1 \sin(2\theta)JT),
\]
and from this,
\[
0 = H(F_2(\theta) - F_1(\theta) \cos(2\theta)), \quad -\tau = HF_1(\theta) \sin(2\theta),
\]
for any \( \theta \) and everywhere on \( M \). If \( H_\theta \neq 0 \) for some \( p \in M \), then \( F_1(\theta) \sin(2\theta) = -\tau / H \). By taking two different values of \( \theta \), we see \( \tau = 0 \), and then \( F_1 = 0 \), which is a contradiction. Thus, we arrive to \( H = 0 \), and then \( \tau = 0 \). Similarly, by using \( JT\theta \), we obtain \( H\theta = 0 \) for any \( \theta \). We therefore proved the following result.

**Theorem 3** Assume that the immersion \( \chi : M \to \mathbb{E} \) satisfies \( T = \partial_t \) everywhere. Then, \( \chi \) admits a generalized associate family with rotating structure vector field if, and only if, \( \tau = 0 \) and \( \chi \) is a vertical cylinder over a geodesic in \( \mathbb{M}^2(\kappa) \). In such case, all other immersions \( \chi_\theta \) are constructed in the same way.

4.2 Warped Product Spaces

By equation (5), we get that \( \langle AT, X \rangle = 0 \) for any \( X \) tangent to \( M \), thus \( A \) has to have the form \[
\begin{pmatrix}
0 & 0 \\
0 & 2H
\end{pmatrix}
\]. Notice then that \( A^a = \begin{pmatrix} H & 0 \\ 0 & -H \end{pmatrix} \) and \( \text{det} A = 0 \).

Firstly, we assume that \( a' \neq 0 \). From equation (7) we get
\[
(\lambda \mathbf{1} + \mu J) \nabla_X T = (\lambda \mathbf{1} + \mu J) \frac{a'}{a}(X - \varepsilon \langle X, T \rangle T) = \frac{a'}{a}(X - \varepsilon \langle X, T\theta \rangle T). \]
By inserting $X = T$, we have $(\lambda 1 + \mu J)\frac{d}{d\theta}(T - \varepsilon(T, T)T) = 0$, and consequently since $a' \neq 0$, $0 = T - \varepsilon(T, T)T = (1 - \lambda^2)T - \lambda \mu JT$, which readily implies $\lambda = 1, \mu = 0$ and $T = T_\theta$ for any $\theta$. From here, $f_\theta = 0$ for any $\theta$, so that we repeat the computations to get $A_\theta T_\theta = 0$. By using the expression of $A_\theta$, we obtain $0 = A_\theta T_\theta = -F_1 H (\cos(2\theta) - \sin(2\theta)JT) + F_2 HT$, which means $F_1 H \cos(2\theta) = 0$. Clearly, $H = 0$ and by Lemma 5 then $H_\theta = 0$ for any $\theta$. In other words, $A_\theta = 0$ for any $\theta$.

Secondly, we assume there is a connected open subset $U$ of $M$ such that $a' \circ \pi_I(p) = 0$ for any $p \in U$. By shrinking $U$ if necessary, we have that $\chi(U) \subset \{t_o\} \times \mathbb{M}^2_k(c)$, that is to say, $U$ is mapped onto a slice. In such case, on $U$, $T = \partial_t$ is normal to the surface, which is a contradiction.

Thirdly, we can assume that $a' = 0$ (on an open interval). From the structure equations we obtain $\nabla_X T_\theta = f_\theta A_\theta X$, for any $\theta$ and any $X \in TM$. In particular, for $\theta = 0$, we obtain $\nabla_X T = 0 = \nabla_X JT$. This means $f_\theta A_\theta X = \nabla_X T_\theta = \lambda \nabla_X T + \mu \nabla_X T = 0$. Next, $f_\theta X(f_\theta) = -f_\theta \langle A_\theta X, T_\theta \rangle = 0$, which implies $X(f_\theta) = 0$. This shows that $f_\theta$ is a constant function. As in the homogeneous case, by the continuity of $(\theta, p) \mapsto f_\theta(p)$, we obtain that $f_\theta = 0$ for any $\theta$. Next, from (8), we see $A_\theta T_\theta = 0$. We repeat the computations as in the case $a' \neq 0$ to obtain $A_\theta = 0$ for any $\theta$.

**Theorem 4** Let $\chi : M \to (P^3 = I \times \mathbb{M}^2_k(c), \langle \cdot, \cdot \rangle_1 = \varepsilon dt^2 + a^2 g_o)$, be an immersion, where $\varepsilon = \pm 1$. Assume that it satisfies $T = \partial_t$ everywhere. Then, $\chi$ admits a generalized associate family with rotating structure vector field if, and only if, $\alpha$ is a constant function and $\chi$ is a vertical cylinder over a geodesic in $\mathbb{M}^2_k(c)$. In such case, all other immersions $\chi_\theta$ are constructed in the same way.

## 5 The case $0 \neq T \neq \partial_t$ everywhere

Note that we have $f \neq 0$ everywhere. Since the map $(\theta, p) \in \mathbb{R} \times M \mapsto f_\theta(p)$ is continuous and $f_\theta \neq 0$, there exist an interval $\bar{I} \subset \mathbb{R}$ and an open subset $U \subset M$ such that $f_\theta(p) \neq 0$ for any $(\theta, p) \in \bar{I} \times U$. In addition, since $F_1(0) = 1$, we can also assume that $F_1(\theta) \neq 0$ for any $\theta \in \bar{I}$. Then, we work on this subset $\bar{I} \times U$.

### 5.1 Homogeneous Spaces

**Lemma 6** If $f \neq 0$, then the structure equations are equivalent to

\begin{align}
\det A - \det A_\theta &= (\kappa - 4\tau^2)(1 - (\lambda^2 + \mu^2))(1 - f^2) \\
f(\delta A_\theta - 2\nabla H_\theta) &= f_\theta(\lambda 1 + \mu J)(\delta A - 2\nabla H) \\
(\lambda 1 + \mu J)\nabla_X T &= f_\theta(F_1 e^{-2\lambda}(A - H 1)X + F_2 HX - \tau JX)
\end{align}

**Proof:** Formulae (12) and (14) are a direct consequence of (9) and Theorem 4. Next, we write $Y = kX + mJX$, for some smooth functions $k$ and $m$ defined on open subsets. An easy computation shows that the right hand side is given by

\begin{align}
\langle Y, T_\theta \rangle X - \langle X, T_\theta \rangle Y &= k\lambda[\langle JX, T \rangle X - \langle X, T \rangle JX] + m\mu[\langle X, T \rangle X + \langle JX, T \rangle JX] \\
&= m(\lambda 1 + \mu J)[\langle JX, T \rangle X - \langle X, T \rangle JX] \\
&= (\lambda 1 + \mu J)[(Y, T)X - \langle X, T \rangle Y].
\end{align}
In this way, by (11), we have
\[ f(\delta A_0 - 2\nabla H_0) = f(d^\nabla A_0) = f f_0(\kappa - 4\tau^2)(\langle E_2, T_0 \rangle E_1 - \langle E_1, T_0 \rangle E_2) \]
\[ = f f_0(\kappa - 4\tau^2)(\lambda_1 + \mu J)(\langle E_2, T \rangle E_1 - \langle E_1, T \rangle E_2) \]
\[ = f_0(\lambda_1 + \mu J)(d^\nabla A)(X) = f_0(\lambda_1 + \mu J)(\delta A - 2\nabla H). \]

\[ \square \]

**Lemma 7** If \( f \neq 0 \), the three equations of Lemma 6 are equivalent to
\[ (1 - F^2_1)(K - \tau^2) - (F^2_2 - F^2_1)H^2 = (\kappa - 4\tau^2)
\left(1 - (\lambda^2 + \mu^2) + (\lambda^2 + \mu^2 - F_1)f^2\right), \tag{15} \]
\[ F_1 e^{-2J^\theta} \delta A^a - F_2 \nabla H = (\lambda_1 + \mu J) \frac{f_0}{f}(\delta A^a - \nabla H), \tag{16} \]
\[ \left(f(\lambda_1 + \mu J) - f_0 F_1 e^{-2J^\theta}\right)AX = f_0(F_2 - F_1 e^{-2J^\theta})HX + (f(\lambda_1 + \mu J) - f_0)\tau JX. \tag{17} \]

**Proof:** By inserting Lemma 6 in (12), we get
\[ (1 - F^2_1) \det A - (F^2_2 - F^2_1)H^2 = (\kappa - 4\tau^2)(1 - (\lambda^2 + \mu^2))(1 - f^2). \]
Using again the original Codazzi equation \( K = \det A + \tau^2 + (\kappa - 4\tau^2)(f^2) \), we have then
\[ (1 - F^2_1)(K - \tau^2) - (F^2_2 - F^2_1)H^2 = (\kappa - 4\tau^2)
\left(1 - (\lambda^2 + \mu^2)(1 - f^2) + (1 - F_1)f^2\right) \]
\[ = (\kappa - 4\tau^2)
\left(1 - (\lambda^2 + \mu^2) + (\lambda^2 + \mu^2 - F_1)f^2\right), \]
From (13) we immediately get the second equation. Finally, by using equation (14),
\[ \nabla_X T_0 = f_0(F_1 e^{-2J^\theta}(A - H^1)X + F_2 HX - \tau JX) \]
\[ = f_0 F_1 e^{-2J^\theta}AX + f_0 H(F_2 - F_1 e^{-2J^\theta})X - f_0 \tau JX \]
\[ = f_0 F_1 e^{-2J^\theta}(\frac{1}{f} \nabla_X T + \tau JX) + f_0 H(F_2 - F_1 e^{-2J^\theta})X - f_0 \tau JX \]
\[ = \frac{f_0}{f} F_1 e^{-2J^\theta} \nabla_X T + f_0 H(F_2 - F_1 e^{-2J^\theta})X + f_0 \tau (F_1 e^{-2J^\theta} - 1)JX \]
and we have
\[ \left((\lambda_1 + \mu J) - \frac{f_0}{f} F_1 e^{-2J^\theta}\right)\nabla_X T = f_0 H(F_2 - F_1 e^{-2J^\theta})X + f_0 \tau (F_1 e^{-2J^\theta} - 1)JX \]
or equivalently
\[ \left(f(\lambda_1 + \mu J) - f_0 F_1 e^{-2J^\theta}\right)AX = f_0 H(F_2 - F_1 e^{-2J^\theta})X + \tau (f(\lambda_1 + \mu J) - f_0)1JX. \tag{18} \]
\[ \square \]

Now plugging in \( T \) and \( JT \) for \( X \) in (17), we get
\[
\begin{cases}
(f(\lambda_1 + \mu J) - f_0 F_1 e^{-2J^\theta})AT = f_0 H(F_2 - F_1 e^{-2J^\theta})T + \tau (f(\lambda_1 + \mu J) - f_0)JT, \\
(f(\lambda_1 + \mu J) - f_0 F_1 e^{-2J^\theta})AJT = f_0 H(F_2 - F_1 e^{-2J^\theta})JT - \tau (f(\lambda_1 + \mu J) - f_0)T, \\
(f(\lambda_1 + \mu J) - f_0 F_1 e^{-2J^\theta})AT = f_0 H(F_2 - F_1 e^{-2J^\theta})T + \tau (f(\lambda_1 + \mu J) - f_0)JT \\
(f(\lambda_1 + \mu J) - f_0 F_1 e^{-2J^\theta})JAJT = -f_0 H(F_2 - F_1 e^{-2J^\theta})T - \tau (f(\lambda_1 + \mu J) - f_0)JT.
\end{cases}
\]
Hence, by adding the two equations, we obtain
\[
\left(f(\lambda 1 + \mu J) - f_\theta F_1 e^{-2J\theta}\right)\left(AT + AJT\right) = 0.
\] (19)

Note that this equation holds for any \(\theta\).

Let us put \(V = AT + AJT\) and define the operator \(B = f(\lambda 1 + \mu J) - f_\theta F_1 e^{-2J\theta}\).

**Lemma 8** If \(V = 0\) on an open subset \(U\) of \(M\), then \(V\) is totally umbilical.

**Proof:** According to Lemma 1, we know \(0 = AT + AJT\) and \(JAT + AJT = 2HJT\). From here, a simple computation shows \(AT = HT\) and \(AJT = HJT\). \(\Box\)

In [3], the classification of such surfaces is obtained. As a result, among 3-dim homogeneous manifolds with 4-dim isometry group, only the products \(S^2 \times \mathbb{R}\) and \(\mathbb{H}^2 \times \mathbb{R}\) admit totally umbilical surfaces. In that paper, the author obtained a full classification, as well as local coordinates of all such surfaces. As a fast description, either they are totally geodesic or invariant by 1-dim isometry subgroups which also leave invariant the slices of \(S^2 \times \mathbb{R}\) and \(\mathbb{H}^2 \times \mathbb{R}\).

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Next, we assume that \(V \neq 0\) on \(U\). In fact, \(\text{Span}\{V\} \subset \ker B\). Then, we know \(f(\lambda 1 + \mu J)V = f_\theta F_1 e^{-2J\theta}V\), which is equivalent to \(f\lambda V + f\mu JV = f_\theta F_1 \cos(2\theta)V - f_\theta F_1 \sin(2\theta)JV\).

Since \(V\) and \(JV\) are linearly independent, we have \(f\lambda - f_\theta F_1 \cos(2\theta) = 0\) and \(f\mu + f_\theta F_1 \sin(2\theta) = 0\). Since we are assuming \(f \neq 0\), we get the following expressions for \(\lambda\) and \(\mu\):
\[
\lambda = \frac{f_\theta}{f} F_1 \cos(2\theta), \quad \mu = -\frac{f_\theta}{f} F_1 \sin(2\theta).
\] (20)

However, by inserting these formulae in \(B\), we obtain
\[
B = f(\lambda 1 + \mu J) - f_\theta F_1 e^{-2J\theta} = f\lambda 1 + f\mu J - f_\theta F_1 \cos(2\theta)1 + f_\theta F_1 \sin(2\theta)J = 0.
\]

Equation (18) becomes \(0 = BAX = f_\theta (F_2 - F_1 e^{-2J\theta})HX + (f_\theta F_1 e^{-2J\theta} - f_\theta 1)\tau JX\), and since \(f_\theta\) is not 0, we get \((F_2 - F_1 e^{-2J\theta})HX + (F_1 e^{-2J\theta} - 1)\tau JX = 0\), and therefore
\[
\begin{align*}
(F_2 - F_1 \cos(2\theta))H + F_1 \sin(2\theta)\tau &= 0, \\
F_1 \sin(2\theta)H + (F_1 \cos(2\theta) - 1)\tau &= 0.
\end{align*}
\] (21)

These two equations hold for any \(\theta \in \bar{I}\). By taking two different values for \(\theta\), but close enough, we obtain that \(H = \tau = 0\). Coming back to (20), we see that for any \(X \in TM\),
\[
X\left(\frac{f_\theta}{f}\right) F_1 \cos(2\theta) = 0 = X\left(\frac{f_\theta}{f}\right) F_1 \sin(2\theta).
\]

Since \(F_1 \neq 0\), there exists a function \(b(\theta)\) defined for \(\theta \in \bar{I}\) such that \(f_\theta = bf\). This means that \(\lambda = bF_1 \cos(2\theta)\) and \(\mu = -bF_1 \sin(2\theta)\), which leads to \(T_\theta = bF_1 e^{-2J\theta} T\). But now, \(b^2 f^2 = f_\theta^2 = 1 - |T_\theta|^2 = 1 - b^2 F_1^2 |T|^2 = 1 - b^2 F_1^2 (1 - f^2) = 1 - b^2 F_1^2 + b^2 F_1^2 f^2\). This means that \(f\) is constant on \(U\), and so is \(f_\theta\) for each \(\theta\), or \(b = F_1 = 1\). Again, in the first case, we see by [3] that \(A_\theta T_\theta = 0\). Since \(M\) is a surface, we have \(A_\theta JT_\theta = 2H_\theta JT_\theta = 2HF_2 JT_\theta = 0\). In particular, \(A = 0\) and \(M\) is totally geodesic. In the second case, the associate family is the minimal family discussed by Daniel in [3]. It is not a problem to include the totally geodesic surfaces among the totally umbilical ones. Thus, we have proved the following

**Theorem 5** Assume that the immersion \(\chi : M \to \mathbb{R}\) satisfies \(0 \neq T \neq \partial_t\) everywhere. Then, \(\chi\) admits a generalized associate family with rotating structure vector field \(if\); and only if, \(\tau = 0\), and \(M\) is minimal or \(M\) is totally umbilical.  

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5.2 Warped Products

With computations similar to the case of homogeneous spaces we obtain easily the analog of Lemma 9 in the case of warped products.

**Lemma 9** If \( f \neq 0 \), then the structure equations are equivalent to

\[
\begin{align*}
\det A - \det A_\theta &= \varepsilon_3 \left( \frac{a''}{a} - \frac{a'^2}{a^2} + \frac{\varepsilon \varepsilon_0}{a^2} \right) (1 - (\lambda^2 + \mu^2)) (\varepsilon - \varepsilon_3 f^2) \\
f(\delta A_\theta - 2\nabla H_\theta) &= f_\theta(\lambda 1 + J\mu) (\delta A - 2\nabla H) \\
(\lambda 1 + \mu J) \nabla X T &= f_\theta(F_1 e^{-2 J \theta} (A - H 1) X + F_2 H X) + \frac{a'}{a} (X - \varepsilon (X, T_\theta) T_\theta) 
\end{align*}
\]  

(22)\hspace{1cm} (23)\hspace{1cm} (24)

Using similar arguments, Lemma 7 becomes then

**Lemma 10**

\[
\begin{align*}
(1 - F_1^2) \left( K + \varepsilon \frac{a''}{a} \right) + \varepsilon_3 (F_2^2 - F_1^2) H^2 \\
&= \left( \frac{a''}{a} - \frac{a'^2}{a^2} + \frac{\varepsilon \varepsilon_0}{a^2} \right) \left( \varepsilon (1 - (\lambda^2 + \mu^2)) + (\lambda^2 + \mu^2 - F_1^2) \varepsilon_3 f^2 \right) \\
F_1 e^{-2 J \theta} \delta A^a - F_2 \nabla H &= (\lambda 1 + \mu J) \frac{f_\theta}{f}(\delta A^a - \nabla H) \\
\left( f(\lambda 1 + J \mu) - f_\theta F_1 e^{-2 J \theta} \right) A X &= f_\theta(F_2 - F_1 e^{-2 J \theta}) H X + \frac{a'}{a} (X - \varepsilon (X, T_\theta) T_\theta) - (\lambda 1 + \mu J) (X - \varepsilon (X, T) T).
\end{align*}
\]  

(25)\hspace{1cm} (26)\hspace{1cm} (27)

where for equation (25) we used the fact that \( K = \varepsilon_3 \det A - \varepsilon \frac{a''}{a} + \left( \frac{a''}{a} - \frac{a'^2}{a^2} + \frac{\varepsilon \varepsilon_0}{a^2} \right) (\varepsilon_3 f^2) \) and for equation (27) we got from equation (24)

\[
(\lambda 1 + \mu J) \left( f A X + \frac{a'}{a} (X - \varepsilon (X, T) T) \right) = f_\theta(F_1 e^{-2 J \theta} (A - H 1) X + F_2 H X) + \frac{a'}{a} (X - \varepsilon (X, T_\theta) T_\theta) - (\lambda 1 + \mu J) (X - \varepsilon (X, T) T) 
\]

Hence in other words,

\[
\begin{align*}
\left( f(\lambda 1 + J \mu) - f_\theta F_1 e^{-2 J \theta} \right) A X &= f_\theta(F_2 - F_1 e^{-2 J \theta}) H X + \frac{a'}{a} \left( (X - \varepsilon (X, T_\theta) T_\theta) - (\lambda 1 + \mu J) (X - \varepsilon (X, T) T) \right) 
\end{align*}
\]

(28)

Now, plugging in \( T \) and \( JT \) for \( X \) in the last equation we get

\[
\begin{align*}
\begin{cases}
(f(\lambda 1 + J \mu) - f_\theta F_1 e^{-2 J \theta}) A T &= f_\theta(F_2 - F_1 e^{-2 J \theta}) H T \\
&+ \frac{a'}{a} \left( (1 - (\lambda 1 + \mu J) T - \varepsilon \lambda \| T \|^2 (\lambda 1 + \mu J) T + (\lambda 1 + \mu J) (\varepsilon \| T \|^2 T) \right) \\
(f(\lambda 1 + J \mu) - f_\theta F_1 e^{-2 J \theta}) A J T &= f_\theta(F_2 - F_1 e^{-2 J \theta}) H J T \\
&+ \frac{a'}{a} \left( (1 - (\lambda 1 + \mu J) J T - \varepsilon \mu \| T \|^2 (\lambda 1 + \mu J) T \right)
\end{cases}
\end{align*}
\]  

(29)\hspace{1cm} (30)

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and consequently
\[
\begin{align*}
\left\{ \begin{aligned}
(f(\lambda 1 + \mu J) - f_\theta F_1 e^{-2J\theta}) JAT &= f_\theta(F_2 - F_1 e^{-2J\theta}) HJT \\
+ \frac{a'}{a} \left( \left( 1 - (\lambda 1 + \mu J) + \epsilon(1-\lambda)(\lambda 1 + \mu J) \right) \|T\|^2 \right) JT,
\end{aligned} \right.
\end{align*}
\]
Moreover
\[
\begin{align*}
\left\{ \begin{aligned}
(f(\lambda 1 + \mu J) - f_\theta F_1 e^{-2J\theta}) AJT &= f_\theta(F_2 - F_1 e^{-2J\theta}) HJT \\
+ \frac{a'}{a} \left( \left( 1 - (\lambda 1 + \mu J) + \epsilon J \mu (\lambda 1 + \mu J) \right) \|T\|^2 \right) JT.
\end{aligned} \right.
\end{align*}
\]
Subtracting these formulas we get
\[
\left( f(\lambda 1 + \mu J) - f_\theta F_1 e^{-2J\theta} \right) (JAT - AJT) = \frac{a'}{a} \left( (\lambda 1 + \mu J) \epsilon (1 - \lambda - J \mu) \|T\|^2 \right) JT.
\]
(28)
And adding them,
\[
2 \left( f(\lambda 1 + \mu J) - f_\theta F_2 \right) HJT = \frac{a'}{a} \left( 2(1 - (\lambda 1 + \mu J) JT + \epsilon \left( (\lambda 1 + \mu J) - (\lambda^2 + \mu^2) \right) \|T\|^2 JT \right).
\]
Hence
\[
\mu \left( \frac{a'}{a} \left( 2 - \epsilon \|T\|^2 \right) + 2fH \right) = 0.
\]
Moreover
\[
2(f\lambda - f_\theta F_2) H = \frac{a'}{a} \left( 2(1 - \lambda) + \epsilon (\lambda - (\lambda^2 + \mu^2) \|T\|^2 \right)
\]
\[
= \frac{a'}{a} \left( (1 - \lambda) \left[ 2 - \epsilon \|T\|^2 \right] + \epsilon \left[ 1 - (\lambda^2 + \mu^2) \right] \|T\|^2 \right)
\]
and finally we get the two equations
\[
\mu \left( \frac{a'}{a} \left( 2 - \epsilon \|T\|^2 \right) + 2fH \right) = 0
\]
\[
2(f\lambda - f_\theta F_2) H = \frac{a'}{a} \left( (1 - \lambda) \left[ 2 - \epsilon \|T\|^2 \right] + \epsilon \left[ 1 - (\lambda^2 + \mu^2) \right] \|T\|^2 \right),
\]
(29)
(30)
5.2.1 Case $\mu \neq 0$
If $\mu$ is not 0, then $2fH = -\frac{a'}{a} \left( 2 - \epsilon \|T\|^2 \right)$ and $2(f - f_\theta F_2) H = \frac{a'}{a} \epsilon (1 - \lambda^2 - \mu^2) \|T\|^2$, and consequently writing $z_\theta := \lambda 1 + \mu J$, we have $2f_\theta F_2 H = -\frac{a'}{a} \left( 2 - \epsilon |z_\theta|^2 \right) \|T\|^2$, and so
\[
ad' \left( 2 - \epsilon \langle T, T \rangle \right) f_\theta F_2 = a' \left( 2 - \lambda^2 - \mu^2 \right) \langle T, T \rangle.
\]
As already seen, the case $a' = 0$ means that either $M$ is contained in a slice or $a$ is constant.

Assuming $a' \neq 0$, we have $(2 - \epsilon \langle T, T \rangle) f_\theta F_2 = f(2 - \lambda^2 - \mu^2) \langle T, T \rangle$. Since we are assuming $T \neq 0$, we suppose for a moment that $2 - \epsilon \langle T, T \rangle = 0$ at some point of $\mathcal{U}$. Thus, we get $2 - \epsilon (\lambda^2 + \mu^2) = 0$, which implies $\epsilon = 1$ and $2 = \lambda^2 + \mu^2$, which is a contradiction for $\theta = 0$, since $\lambda(0) = 1$, $\mu(0) = 0$. In addition, by the previous section, we can discard the case $f_\theta = 0$.

We arrive to
\[
F_2 = \frac{f}{f_\theta} \frac{(2 - \epsilon |z_\theta|^2) \|T\|^2}{(2 - \epsilon \|T\|^2)}.
\]
Now we want if a multiple of its conjugate, and the second factor is a multiple of its conjugate if and only

\[ P \text{ irreducible factorization} \]

polynomial is identically 0, or it is easy to check that the polynomial must have the following

\[ P \]

Let now \( W := JAT - AJT \).

If \( W = 0 \) on an open subset \( V \) of \( M \), then by Lemma \[1\] \( V \) is totally umbilical. We notice that totally umbilical surfaces which are neither vertical nor horizontal in (warped) products of the form \( M^n \times_f I \), with \( M^n \) a Riemannian manifold, have been studied and classified in \[12\]. In particular the authors prove that such surfaces exist if and only if \( M^n \) has locally the structure of a warped product, which ensures their existence in our case.

Next, we assume that \( \mathcal{U} \) is free of umbilical points. Then, there exist two smooth functions \( \alpha, \beta \) defined on \( \mathcal{U} \) such that \( (\alpha \mathbf{1} + \beta \mathbf{J}) W = JT \). Then, by \[25\] we have

\[
 f_0 f_1 e^{-2J\theta} \delta A^a = -\frac{a'}{a} \left( z_\theta \varepsilon (1 - z_\theta) ||T||^2 \right) (\alpha \mathbf{1} + \beta \mathbf{J}) \delta A^a + f z_\theta \delta A^a
\]

and therefore

\[
 f^2 \frac{(2 - \varepsilon ||z_\theta||^2)}{2 - \varepsilon ||T||^2} \nabla H + \varepsilon \varepsilon_3 z_\theta (1 - \varepsilon ||z_\theta||^2) (\delta A^a - \nabla H)
\]

which is a cubic polynomial of the form

\[
 P(z_\theta, \bar{z}_\theta) = c_0 + c_1 z\theta + c_2 |z_\theta|^2 + c_3 \bar{z}_\theta + c_4 |z_\theta|^2 = 0,
\]

Now we want \( z_\theta \) to be a smooth family of solutions. In order for the solution set \( z_\theta \) to contain a curve, \( P \) needs to share a common factor with its conjugate. But, in that case either the polynomial is identically 0, or it is easy to check that the polynomial must have the following irreducible factorization \( P(z_\theta, \bar{z}_\theta) = (z_\theta + d_0) (d_1 + d_2 z_\theta + d_3 |z_\theta|^2) \). The first factor cannot be a multiple of its conjugate, and the second factor is a multiple of its conjugate if and only if \( d_3 = 0 \), and consequently the curve is a circle of radius \( r = \sqrt{-\frac{d_2}{d_3}} \) centered at the origin.

Notice that the polynomial could also be quadratic, but then the same term in \( z_\theta^2 \) has to vanish, additionally to the terms in \( z_\theta \) and \( |z_\theta|^2 \), so that we can reduce our study to the previous case. Now by equation \[32\]

\[
 d_2 = \frac{a'f}{a} \varepsilon ||T||^2 (\alpha \mathbf{1} + \beta \mathbf{J}) \delta A^a = 0.
\]

But this is satisfied if and only if one of the following two cases hold.

1. \( a' = 0 \) and the ambient manifold is in fact a product. Moreover, by equation \[21\], we have that \( H = 0 \). In this case we get from equations \[22\] and \[25\] that \( A = \det A_\theta \), which by Lemma \[1\] is equivalent to having \( F_1 = 1 \) and finally by equations \[26\] and \[27\] we can conclude that there exists a family if and only if \( f_\theta = f \) and \( \lambda \mathbf{1} + \mu \mathbf{J} = e^{-2J\theta} \), which is exactly the usual associate family used by Eschenburg. Conversely we see easily that if \( A_\theta = e^{-J\theta} A \mathbf{e}^{J\theta} \) and \( T_\theta = e^{-2J\theta} T \) and \( H = 0 \), the structure equations are all satisfied, recovering hence Daniel’s minimal family when the warped product is Riemannian (see \[3\]) and Roth’s result when the warped product is Lorentzian (see \[11\]).
2. \( \delta A^a = 0 \), then by equation (26) \( F_2 \nabla H = (\lambda I + \mu J) \frac{f_0}{T} \nabla H \), which means that \( H \) is constant since \( \mu \neq 0 \) and \( f \neq 0 \). Hence it is easy to see that \( d^\nabla A = 0 \). By the Codazzi equation in Theorem 2 this holds if and only if \( (\frac{\partial}{\partial x} - \frac{\partial}{\partial y} + \varepsilon \varepsilon_0) \frac{f}{\partial} = 0 \) or \( \|T\| = 0 \). Since \( T \neq 0 \), the remaining situation is when \( a''a - (a')^2 + \varepsilon \varepsilon_0 = 0 \), that is to say, the case of space forms that we excluded at the beginning.

5.2.2 Case \( \mu = 0 \)

In this case
\[
\lambda \nabla_X T = \frac{a'}{a} (X - \varepsilon (X, T) T) = \frac{a'}{a} (X - \varepsilon (X, T_\theta) T_\theta) = \frac{a'}{a} (X - \varepsilon \lambda^2 (X, T) T).
\]

Hence plugging in \( T \) and \( JT \) for \( X \) we get
\[
(\lambda - 1) \frac{a'}{a} T = \frac{a'}{a} \varepsilon \lambda (1 - \lambda) \|T\|^2 T, \quad \frac{a'}{a} \lambda JT = \frac{a'}{a} JT
\]
and consequently either \( a' = 0 \) and the ambient manifold is a product or \( \lambda = 1 \). But in both cases we get from equation (28) that either the vector field \( W = AJT - AJT = 0 \), which implies that \( M \) is totally umbilical, or \( \lambda f = f_\theta F_1 \cos(2 \theta) \) and \( 0 = f_\theta F_1 \sin(2 \theta) \) for all angles \( \theta \) in an interval around zero, which is a contradiction.

Consequently we proved the following

**Theorem 6** Consider the 3-dim Semi-Riemannian warped product \( P^3 = I \times M^2_k(c), \langle \cdot, \cdot \rangle_1 = \varepsilon dt^2 + a^2 g_o \), where \( \varepsilon = \pm 1 \), and assume that it does not contain any open subset with constant sectional curvature. Consider an isometric immersion \( \chi : M \to P^3 \) such that \( 0 \neq T \neq \partial_t \) everywhere. Then, \( \chi \) admits a generalized associate family with rotating structure vector field if, and only if, one of the two following cases hold

1. The ambient manifold is a product and the surface is minimal.
2. The surface is totally umbilical.

6 The case \( T = 0 \) everywhere

In the warped product case, \( M \) is a slice, which is a totally umbilical surface.

In the homogeneous case, then \( f^2 = 1 \), so by (3), we see \( 0 = AX - \tau JX \) for any \( X \in TM \). Since \( A \) is symmetric and \( J \) is skew-symmetric, then \( A = 0 \) and \( \tau = 0 \). This means that we are in the product case. Then, \( M \) is either \( S^2 \) or \( H^2 \) embedded in the ambient space as a totally geodesic slice.

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