Non inertial dynamics and holonomy on the ellipsoid

José A. Vallejo
Departament de Matemàtica Aplicada IV
Universitat Politècnica de Catalunya
Escola Politècnica Superior de Castelldefels
Av. Canal Olímpic s/n
08860 Castelldefels (Spain)
e-mail: jvallejo@ma4.upc.edu

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Abstract
Traditionally, the discussion about the geometrical interpretation of inertial forces is reserved for General Relativity handbooks. In these notes an analysis of the effect of such forces in a classical (newtonian) context is made, as well as a study of the relation between the precession of the Foucault pendulum and the holonomy on a surface, including some critical remarks about the common statement “precession of Foucault pendulum equals holonomy on the sphere”.

1 Introduction
These notes intend to analyze a classical problem in Physics, that of inertial forces, with the methods of elementary differential geometry of curves and surfaces. Inertial forces appear in the study of the dynamics of non inertial reference systems, and the problem they pose is that of their interpretation in classical (Newtonian) terms.

In Newton’s theory, the inertial forces are a consequence of the action of absolute space on physical bodies; however, Mach and Einstein eradicated the privileged rôle of this absolute space as a theoretical element, and this fact led them to a reinterpretation of inertial forces (see the discussion in [Rin 77], pgs 10 – 11). Following them, we could say that Newtonian conceptions have a strong unilateral character (in Einstein’s words “In the first place, it is contrary to the mode of thinking in science to conceive a thing (absolute space) which acts itself, but which cannot be acted upon. This is the reason why E. Mach was led to make the attempt to eliminate space as an active cause in the system of mechanics”). See [Ein 56] pages 55 – 56) and this reflects on the fact that, for
inertial forces, the (third) Newtonian law of action-reaction is not satisfied. To summarize, the conceptual difficulty regarding inertial forces is that they are not real forces, with origin in some known physical interaction (this motivates the term “fictitious forces”, sometimes used to describe them).

The solution offered by Einstein is well known: he proposed a geometrical interpretation of accelerations (thus also of forces) expressed in his famous “equivalence principle”, according to which the inertial and gravitational forces can be absorbed into a non Euclidean structure of spacetime (see [Lan 87], [Wei 72], Section 1.3). However, much care must be taken when doing this translation from inertial forces to geometrical (non euclidean) structures. For example, it is a common place (see [Mar 92], [Sha 88]) that Newton equations in the non inertial reference system associated to the Earth, applied to the Foucault pendulum motion are equivalent to the holonomy on the sphere, a typical phenomenon of non Euclidean geometry. However, sometimes this fact is presented as something miraculous which calls for an explanation: geometry determines dynamics. But, how is it possible that the geometry of the surface on which it moves influence the plane of oscillations of the pendulum? We will argue that this kind of questions must be taken “with a grain of salt”: the holonomy on the sphere is not responsible of the pendulum precession, rather (as a result of some physical approximations), the plane of oscillations of the pendulum is a parallel vector field on the sphere, in such a way that a physical effect, its total precession in a day, coincides with the holonomy (we could say that both effects agree at first order). But if we modify slightly the model to study the motion of the pendulum, by considering the more realistic case of an ellipsoid, this coincidence is no longer true.

In brief, the whole plan of this paper is as follows. The physical analysis of the Foucault pendulum fits into a wider class of problems, called dynamics in non inertial systems; in Section 2 we present a résumé of the general ideas concerning it. Next follows a reminder of the basic notions of Differential Geometry that will be used and, finally, we will discuss how to apply these results to the Foucault pendulum.

Let us mention that there exist in the literature studies with a similar spirit. Specifically, the excellent paper of J. Oprea (see [Opr 95]) has been a stimulus for this work. However, the development that can be found there can be somewhat confusing from a physical point of view, because it makes an explicit use of the inertial “forces” and, by contrast, it does not account for real forces as the string tension, so it can be hardly seen as a correct physical analysis. Indeed, this influences the crucial idea that the field of pendulum oscillations is parallel, as if the tension is taken into account it is no longer possible to directly conclude that “on the plane of oscillations of the pendulum there are no tangential forces” (cfr. [Opr 95], pg 521).

We will make use only of the machinery of the basic Differential Geometry of curves and surfaces, in particular, no manifold theory or Riemannian geometry will be needed. Suitable references are [DoC 77], [Gray 98] or [Dub 98].
2 Non inertial Newtonian dynamics

A more or less standard statement of Newton’s Laws could be:

LAW I: Free (not interacting) particles follow rectilinear uniform motion (including the limit case of being at rest).

LAW II: The acceleration of a body is inversely proportional to its mass, and directly proportional to the external force which acts on it:

\[ \vec{F} = m \cdot \ddot{\vec{r}}. \]  

(1)

LAW III: If a body A acts upon another body B with a certain force, the body B will act upon A with an equal in magnitude but opposed force along the line joining the bodies.

Newton himself states that his laws are applicable just in inertial frames. A frame is basically a collection of physical and geometrical objects which allows us to assign a set of numerical coordinates to any event (Einstein described them intuitively as a coordinate system or reference and a clock in each point of space). Newton considered that such a frame with an absolute character exists (a very interesting discussion is given in Rin 77, pgs. 2 and 14 – 15) and defined the inertial frames as those moving with a uniform rectilinear velocity with respect to the absolute frame, an idea already present in Galileo.

From an operational point of view, the inertial frames can be distinguished with the aid of the first law, but what happens with non inertial frames (NIF) as in the case of two observers with uniform rotational relative motion?. In these systems the preceding laws are no longer valid. The classical approach tries to continue making use of the inertial frames machinery and forces the validity of the second law, at the cost of introducing the so called inertial forces. Thus, in a non inertial frame the second law is not but

LAW II’: The acceleration of a body is inversely proportional to its mass, and directly proportional to the total force which acts on it, including external (real) and inertial forces, \( \vec{F}_{ext} \) and \( \vec{F}_{in} \):

\[ \vec{F}_{in} = m \cdot \ddot{\vec{r}} - \vec{F}_{ext}. \]  

(2)

Example 1: In the particular case of a frame with uniform rotational motion with respect to another fixed one (this one considered as inertial), \( \vec{F}_{in} \) takes the form of the Coriolis “force”:

\[ -\vec{\omega} \wedge (\vec{\omega} \wedge \vec{r}) - 2\vec{\omega} \wedge \vec{v}_{rot}, \]

where \( \vec{\omega} \) is the rotation velocity of the rotating system with respect to the fixed one, \( \vec{r} \) is the vector joining the origins of both systems and \( \vec{v}_{rot} \) is the velocity with respect to the rotating system.

The appearance of the inertial “forces” can be seen as the consequence of making a change of coordinates, from a set of Euclidean coordinates (in the
example those of the fixed-inertial system) for which the Christoffel symbols equal zero, to another set of arbitrary ones without this property (see \cite{Kop92}, pg. 113). These terms involving the $\Gamma^i_{jk}$ are precisely those generating $\vec{F}_{in}$. On the other hand, the Christoffel symbols are intimately related to intrinsical magnitudes such as curvature, so it seems plausible to ask if the effect of inertial forces can be understood in terms of these intrinsical magnitudes. Intuitively, we could say that the idea is to replace equation (2) by

$$\vec{F}_{in} = "\text{geometrical effect}".$$  

As we have remarked, for the Foucault pendulum it has been suggested that the geometrical effect is the holonomy on the sphere. However, as we will see, this must be carefully analyzed.

3 Geodesic curvature and holonomy

It is well known that in the case of curves in the Euclidean space the Frenet frame gives an $\mathbb{R}^3$ basis (a reference) which is adapted to the geometry of the curve. Now, we will work with a curve contained in a regular surface and so we will consider a similar object, that is, an adapted basis not only to the curve but also to the surface.

Let $S \subset \mathbb{R}^3$ be an oriented regular surface and $c : I \to S$ with $c : t \mapsto c(t)$ a curve on it. We will call an intrinsic Frenet frame a reference along $c$, denoted $\{E_1, E_2, E_3\}$, and defined as follows (the point in $\dot{c}$ denotes derivation with respect to $t$, and $N$ is the unitary vector in $\mathbb{R}^3$ normal to the surface. The symbol $\wedge$ denotes the cross product in $\mathbb{R}^3$):

$$\begin{align*}
E_1(t) &= \frac{\dot{c}(t)}{|\dot{c}(t)|}, \\
E_2(t) &= N(c(t)) \wedge E_1(t) \\
E_3(t) &= N(c(t)).
\end{align*}$$

By construction, $\{E_1, E_2\}$ is an orthonormal basis of $T_{c(t)}S$, the tangent plane to $S$ at $c(t)$. If we represent by $\langle \cdot, \cdot \rangle$ the Euclidean scalar product in $\mathbb{R}^3$ and by $\frac{DV}{dt}(t)$ the covariant derivative of the vector field $V(t)$ on $S$ (that is, the projection of $\dot{V}(t)$ on $T_{c(t)}S$ with respect to the Euclidean scalar product of the ambient space), then it is clear that

$$\left\langle E_1(t), \frac{DE_1}{dt}(t) \right\rangle = \left\langle E_1(t), \dot{E}_1(t) \right\rangle = 0,$$

so for each $t \in I$, $\frac{DE_1}{dt}(t)$ must be colinear with $E_2(t)$; thus

$$\frac{DE_1}{dt}(t) = \left\langle E_2(t), \frac{DE_1}{dt}(t) \right\rangle E_2(t).$$
The geodesic curvature of $c$ at $t$ is the proportionality factor between $\frac{DE_1}{dt}(t)$ and $E_2(t)$. More precisely, let $S \subset \mathbb{R}^3$ be an oriented regular surface and $c : I \to S$ a curve on it. Its geodesic curvature is the function

$$\kappa_g(t) = \frac{1}{|\dot{c}(t)|} \left\langle E_2(t), \frac{DE_1}{dt}(t) \right\rangle.$$  \hspace{1cm} (3)

Geometrically, $\kappa_g$ expresses the angle between the acceleration of the curve and the normal to the surface. When this angle is 0, the acceleration is tangent to the surface and the curve is said to be a geodesic. It turns out that, for the sphere, the geodesics are given by the intersection with planes passing through the origin, that is, the great circles.

Let $W(t)$ be a vector field on $S$ parallel along $c$, that is a vector field such that $\frac{DW}{dt}(t) = 0$ for all $t \in I$. Let us call $\phi$ the angle it defines, at each point, with the tangent to the curve $c$:

$$\phi(t) = \angle (\dot{c}(t), W(t)).$$

Under these assumptions, we have a basic result (cfr. [DoC 77], pg. 252 or [One 97], pg 338) telling us that

$$\dot{\phi}(t) = -\kappa_g(t) |\dot{c}(t)|,$$

that is, the variation of the angle $\phi$ is a characteristic of the curve $c$. This expression, is particularly suited to the practical computation of the parallel transport. Let $t_1, t_2 \in I$ be two instants, then the parallel transport along $c$ between $t_1$ and $t_2$ mapping, denoted $\tau(c)|_{t_1}^{t_2} : T_{c(t_1)}S \to T_{c(t_2)}S$, carries $v \in T_{c(t_1)}S$ to

$$\tau(c)|_{t_1}^{t_2}(v) = |v| \left( E_1(t_2) \cos \phi(t_2) + E_2(t_2) \sin \phi(t_2) \right),$$

where

$$\phi(t_2) = \phi(t_1) - \int_{t_1}^{t_2} \kappa_g(t) |\dot{c}(t)| \, dt.$$  

According to this formula, if we have a closed curve on $S$ (parametrized in $[0, 2\pi]$ without loss of generality) with $c(0) = c(2\pi) = p$, when parallel transporting a tangent vector $v$ in $p$, making a complete loop around $c$, we will not recover $v$ but a vector defining a certain angle with it; this phenomenon is known as the holonomy.

More precisely, with the preceding conditions the angle of holonomy of $c$ in $p$ is defined as the angle between $v$ and $\tau(c)|_{0}^{2\pi}(v)$,

$$h = \angle (v, \tau(c)|_{0}^{2\pi}(v)).$$

Example 2: Let us compute the holonomy of a parallel on the sphere $S^2$ (of unit radius). Consider the geographical parametrization, given by

$$X : \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \times [0, 2\pi] \to S^2$$

$$(u, v) \mapsto X(u, v) = (\cos u \cos v, \cos u \sin v, \sin u),$$
and the parallel of latitude $\theta_0$, $c(t) = (\cos \theta_0 \cos t, \cos \theta_0 \sin t, \sin \theta_0)$; it is easy to see that its geodesic curvature is constant with value (note that the sign depends on the orientation of $S^2$ chosen)

$$\kappa_g(t) = -\tan \theta_0,$$

while

$$|\dot{c}(t)| = \cos \theta_0,$$

so

$$\h = \int_0^{2\pi} \kappa_g(t)|\dot{c}(t)|\,dt = \int_0^{2\pi} \sin \theta_0\,dt = 2\pi \sin \theta_0. \quad (4)$$

**Example 3**: A model for Earth’s surface more realistic than a sphere is an ellipsoid of revolution $E_{a,b}$ (with distinct semiaxes $a$ and $b$). It can also be parametrized geographically:

$$X : [-\pi, \pi] \times [0, 2\pi] \to E_{a,b}$$

$$(u, v) \mapsto X(u, v) = (a \cos u \cos v, a \cos u \sin v, b \sin u),$$

and a computation as in Example 2 gives for the parallel on the ellipsoid $c(t) = (a \cos \theta_0 \cos t, a \cos \theta_0 \sin t, b \sin \theta_0)$ the results

$$\kappa_g(t) = -\frac{\tan \theta_0}{\sqrt{a^2 \sin^2 \theta_0 + b^2 \cos^2 \theta_0}}$$

and

$$\h = \frac{2\pi a \sin \theta_0}{\sqrt{a^2 \sin^2 \theta_0 + b^2 \cos^2 \theta_0}}. \quad (5)$$

Of course, when $a = b$ these coincide with the corresponding to the sphere.

4 Parallel vector fields on the sphere and the ellipsoid

Consider a curve $c : I \to S \subset \mathbb{R}^3$ and $X : U \subset \mathbb{R}^2 \to S$ a parametrization of the surface $S$, so $c(t) = X(u(t), v(t))$. Let $V$ be a vector field along $c$ such that in the reference $\{X_u(t), X_v(t)\}$ of $T_{c(t)}S$ has coordinates $(V^1(t), V^2(t))$, that is,

$$V(t) = V^1(t)X_u(t) + V^2(t)X_v(t) \in T_{c(t)}S. \quad (6)$$

For a vector field with that expression, it is known that the covariant derivative along $c$ is given by

$$\frac{DV}{dt} = \left(\dot{\nu}^1 + \Gamma_{11}^1 \nu^1 \dot{\nu} + \Gamma_{12}^1 \nu^1 \dot{\nu} + \Gamma_{12}^1 \nu^1 \dot{\nu} + \Gamma_{12}^1 \nu^2 \dot{\nu}\right) X_u + \left(\dot{\nu}^2 + \Gamma_{11}^2 \nu^1 \dot{\nu} + \Gamma_{12}^2 \nu^1 \dot{\nu} + \Gamma_{12}^2 \nu^2 \dot{\nu}\right) X_v.$$
In particular, if we consider \( S = S^2 \) (the unit sphere), \( X \) the geographical parametrization of Example 2 and \( c(t) \) the parallel of latitude \( \theta_0 \), we get

\[
\begin{align*}
  u &= \theta_0, \quad \dot{u} = 0 \\
  v &= t \quad \dot{v} = 1.
\end{align*}
\]

Also, for the Christoffel symbols:

\[
\begin{align*}
  \Gamma^1_{11} &= 0, & \Gamma^1_{12} &= 0, & \Gamma^1_{22} &= \sin u \cos u \\
  \Gamma^2_{11} &= 0, & \Gamma^2_{12} &= -\tan u, & \Gamma^2_{22} &= 0,
\end{align*}
\]

so

\[
\frac{DV}{dt}(t) = \left( \dot{V}^1(t) + \sin \theta_0 \cos \theta_0 V^2(t) \right) X_u + \left( \dot{V}^2(t) - \tan \theta_0 V^1(t) \right) X_v.
\]

According to that expression, a vector field along a parallel of latitude \( \theta_0 \) on the sphere is parallel if and only if its components with respect to the basis \( \{X_u(t), X_v(t)\} \) (recall that \( X \) is the geographic parametrization) verify

\[
\begin{align*}
  \dot{V}^1(t) + \sin \theta_0 \cos \theta_0 V^2(t) &= 0 \\
  \dot{V}^2(t) - \tan \theta_0 V^1(t) &= 0
\end{align*}
\]

(7)

If we write \( \beta = \sin \theta_0, x = V^1, y = \cos \theta_0 V^2 \) this system is equivalent to

\[
\begin{align*}
  \dot{x} + y &= 0 \\
  \dot{y} - \beta x &= 0,
\end{align*}
\]

which has solution

\[
\begin{align*}
  y(t) &= B \sin \beta t + A \cos \beta t \\
  x(t) &= -A \sin \beta t + B \cos \beta t.
\end{align*}
\]

Thus, the more general expression for a parallel vector field along the curve \( c(t) = (\cos \theta_0 \cos t, \cos \theta_0 \sin t, \sin \theta_0) \) (in the geographical parametrization) on the sphere \( S^2 \) is

\[
V(t) = (-A \sin \beta t + B \cos \beta t) X_u(t) + \frac{1}{\cos \theta_0} (B \sin \beta t + A \cos \beta t) X_v(t).
\]

(8)

Given the Christoffel symbols for the ellipsoid \( E_{a,b} \):

\[
\begin{align*}
  \Gamma^1_{11} &= \frac{2 \sin u \cos u (b^2 - a^2)}{a^2 \sin^2 u + b^2 \cos^2 u}, & \Gamma^1_{12} &= 0, & \Gamma^1_{22} &= \frac{a^2 \sin u \cos u}{a^2 \sin^2 u + b^2 \cos^2 u} \\
  \Gamma^2_{11} &= 0, & \Gamma^2_{12} &= -\tan u, & \Gamma^2_{22} &= 0,
\end{align*}
\]

a similar analysis shows that the more general expression for a parallel vector field along the curve \( c(t) = (\cos \theta_0 \cos t, \cos \theta_0 \sin t, \sin \theta_0) \) (in the geographical
parametrization) on the ellipsoid is given by a vector $V(t) = V^1(t)X_u(t) + V^2(t)X_v(t)$ whose components satisfy a system of equations similar to (7):

\[
\begin{align*}
\dot{V}^1(t) + \frac{a^2 \sin \theta_0 \cos \theta_0}{a^2 \sin^2 \theta_0 + b^2 \cos^2 \theta_0} V^2(t) &= 0 \\
\cos \theta_0 \dot{V}^2(t) - \sin \theta_0 V^1(t) &= 0.
\end{align*}
\]

(9)

5 Pendula, Earth rotation and holonomy

To carry on our analysis, we begin with two basic assumptions. The first, that the Earth surface is spherical (and, as a consequence, the gravitational field it creates). The second, that the Cartesian frame with origin the center of the Earth sphere is (approximately) inertial. In order to do explicit computations we will consider the geographical parametrization of the sphere, and we will choose a non inertial frame associated to the parallel $u = \theta_0$ as the intrinsic Frenet frame, that is: at each point $c(t = v)$ the “$X$ axis” is the direction of $E_1(t)$, the “$Y$ axis” is the direction of $E_2(t)$ and the “$Z$ axis” is the direction of $E_3(t)$.

Figure 1: The Frenet frame.

Imagine the daily motion of the Foucault pendulum as the result of displacing the pendulum along a parallel on an Earth which stand still. Then, this non inertial frame will be rotating jointly with the pendulum. For the sake of simplicity, let us suppose that the mass of the bob is $m = 1$ and that we set
the time scale in such a way that 1 day equals $2\pi$ units of time, that is, the
module of the angular velocity of the pendulum is $\omega = 1$. If $\vec{\omega}$ is the vector
giving the position of the origin of the rotating system in the Cartesian one and
$\vec{v}_f$, $\vec{a}_f$, $\vec{v}_{\text{rot}}$, $\vec{a}_{\text{rot}}$ are the velocity and acceleration (respectively) with respect
to the fixed Cartesian axes and the rotating system then, for an observer in this
rotating system the effective force on the pendulum is (see [Cho 95], Sec. 10.4
for the following development)

$$F_{\text{ef}} = \vec{a}_{\text{rot}} = \vec{a}_f + \vec{\omega} \wedge (\vec{\omega} \wedge \vec{v}) - 2\vec{\omega} \wedge \vec{v}_{\text{rot}}.$$ 

Thus, the equation of motion becomes

$$\vec{a}_{\text{rot}} = \vec{g} + \vec{T} - 2\vec{\omega} \wedge \vec{v}_{\text{rot}},$$

(10)

where $\vec{T}$ is the string tension and $\vec{g}$ is the gravitational force.

**Remark:** Note that the term $-\vec{\omega} \wedge (\vec{\omega} \wedge \vec{v})$ can be included in $\vec{g}$, as $\vec{g}$ is only defined through the measures we make. Incidentally, this is the cause of the fact that the direction of $\vec{g}$ actually differs from the local vertical. However, we will assume that $\vec{g}$ is directed along $E_3(t)$.

Here comes our third assumption: if the displacements of the bob have little
amplitude, so the approximation $\sin \zeta \simeq \zeta$ is valid (where $\zeta$ is the angle that
the string determines with the vertical), and $l$ is the string longitude, we have

$$T_z \simeq -T x/l$$
$$T_y \simeq -T y/l$$
$$T_z \simeq T$$

Also, by the symmetry of the gravitational field (we are supposing the Earth
spherical),

$$\begin{cases}
g_x = 0 \\
g_y = 0 \\
g_z = -g
\end{cases}
\begin{cases}
\omega_x = -\cos \theta_0 \\
\omega_y = 0 \\
\omega_z = \sin \theta_0
\end{cases}
\begin{cases}
(\vec{v}_{\text{rot}})_x = \dot{x} \\
(\vec{v}_{\text{rot}})_y = \dot{y} \\
(\vec{v}_{\text{rot}})_z = 0.
\end{cases}$$

Then, computing $\vec{\omega} \wedge \vec{\omega}$ we get

$$\begin{cases}
(\vec{\omega} \wedge \vec{\omega}_{\text{rot}})_x = -\dot{y} \sin \theta_0 \\
(\vec{\omega} \wedge \vec{\omega}_{\text{rot}})_y = \dot{x} \sin \theta_0 \\
(\vec{\omega} \wedge \vec{\omega}_{\text{rot}})_z = -\dot{y} \cos \theta_0,
\end{cases}$$

and the equations we are interested in are:

$$\begin{cases}
(\vec{a}_{\text{rot}})_x = \ddot{x} = -T x/l + 2\dot{y} \sin \theta_0 \\
(\vec{a}_{\text{rot}})_y = \ddot{y} = -T y/l - 2\dot{x} \sin \theta_0.
\end{cases}$$
For displacements of little amplitude, the tension and the weight are approximately equal $T \simeq g$. If we set

$$\alpha^2 = \frac{T}{l} = \frac{g}{l},$$

and

$$\beta = \sin \theta_0,$$

we arrive at the (approximate) equations

$$\begin{cases} \ddot{x} + \alpha^2 x = 2\beta \dot{y} \\ \ddot{y} + \alpha^2 y = -2\beta \dot{x}, \end{cases}$$

a system of coupled second order ordinary differential equations that are the mathematical model for the Foucault pendulum within the approximation of little amplitude displacements. It can be solved adding to the first equation the second multiplied by $i$, so (writing $q = x + iy$) we get

$$\ddot{q} + 2i\beta \dot{q} + \alpha^2 q = 0,$$

with solution

$$q(t) \simeq \exp(-i\beta t)(A \exp(-t\sqrt{\beta^2 - \alpha^2}) + B \exp(-t\sqrt{-\beta^2 - \alpha^2})), $$

$\alpha$ being the pendulum’s pulse of oscillation. Of course, this is much greater than the rotational velocity of the Earth, so $\alpha \gg \beta$ and

$$q(t) = \exp(-i\beta t)(A \exp(i\alpha t) + B \exp(-i\alpha t)), $$

Figure 2: Forces on the pendulum.
or, if we denote by \( q_0(t) \) the solution corresponding to the absence of rotation (a static pendulum),

\[
q(t) \approx q_0(t) \exp(-i\beta t).
\]

Equating real and imaginary parts:

\[
\begin{pmatrix}
  x(t) \\
  y(t)
\end{pmatrix} = 
\begin{pmatrix}
  \cos(\beta t) & \sin(\beta t) \\
  -\sin(\beta t) & \cos(\beta t)
\end{pmatrix}
\begin{pmatrix}
  x_0(t) \\
  y_0(t)
\end{pmatrix}
\]

where \((x_0(t), y_0(t))\) is the “static” solution for the pendulum in \( c(0) \). As is well known, this oscillation would always take place in a plane determined by the initial conditions; what (11) is telling us is that this plane precesses when the pendulum is displaced along the parallel \( c(t) \), as was noted by Foucault.

Let us check that the dynamics of the problem determines a vector field (associated to the pendulum oscillations) which is parallel along \( c \), so the total angle accumulated after a complete loop will coincide with the holonomy of \( c \), given by (4).

As we have already mentioned, the initial oscillation defines a unitary vector through the intersection of the plane in which it is contained (in \( \mathbb{R}^3 \)) and the tangent plane to the sphere in \( c(0) \), let us call it

\[
\begin{pmatrix}
  A \\
  B
\end{pmatrix} \in T_{c(0)}S^2,
\]

and what we have is that \((x_0(t), y_0(t))\) is parallel to \( (A, B) \):

\[
\begin{pmatrix}
  x_0(t) \\
  y_0(t)
\end{pmatrix} \propto \begin{pmatrix}
  A \\
  B
\end{pmatrix}.
\]

Now, the vector field associated to the oscillations of the pendulum (defined by the intersection of the oscillation plane with the tangent plane at each point) is

\[
V : [0, 2\pi] \to TS \\
t \mapsto V(t) = x(t)E_1(t) + y(t)E_2(t),
\]

where, from (11),

\[
\begin{cases}
x(t) = B \sin \beta t + A \cos \beta t \\
y(t) = -A \sin \beta t + B \cos \beta t,
\end{cases}
\]

so, as time goes by, the plane of oscillation actually precesses with respect to the plane of the initial oscillations.

Let us note that in the geographical parametrization the following relations hold:

\[
E_1(t) = \frac{X_1(\theta_0, t)}{\|X_1(\theta_0, t)\|} = \frac{X_\theta(\theta_0, t)}{\cos \theta_0}
\]

\[
E_2(t) = X_\theta(t),
\]

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so we can write

\[
V(t) = (B \sin \beta t + A \cos \beta t) \frac{1}{\cos \theta_0} X_v(t) + (-A \sin \beta t + B \cos \beta t) X_u(t).
\]

Comparing with (8), we see that the vector field of the oscillations is parallel transported, and the precession it experiments after a day (a whole loop around \(c\)) is \(2 \pi \sin \theta_0\), according to (4).

**Remark:** As a curious observation, at the equator we would not find any holonomy. From a geometrical point of view this is a consequence of the fact that the equator is the only one among the parallels of the sphere that is a geodesic (a great circle) and, consequently, has \(\kappa_g(t) = 0\) (recall the comments following (3)).

6 The ellipsoidal Earth

The purpose of this Section is to analyze what happens when we change the sphere model of the Earth. In order to understand it, we recall some elementary facts about Geodesy.

The theoretical shape of the Earth is described by means of its normal equipotential surface, which coincides with the mean level sea surface (in firm land, the surface of a water channel) and is called the geoid. The geoid includes the variations in gravitational potential due, among other factors, to the non homogeneity of the crust; this is the fundamental reason that makes the description of the geoid only approximate.

The simplest approximation considers the geoid as a sphere (the ideas introduced by holonomy are based on this assumption, as we have seen), but another -much more precise- one is that of an ellipsoid of revolution. Indeed, conventionally, the gravimetric reductions of experimental data are carried on to a so called reference ellipsoid defined by the experimental values founded for some parameters (such as the equatorial radius of the Earth \(R\), its total mass \(M\) or the flattening coefficient \(f\)) and the requirement that the ellipsoid surface be an equipotential surface. That is: starting from experimental data, we construct a mathematical model for the gravitational potential of the Earth, in such a way that the reference ellipsoid is an equipotential surface. The precise form of this potential is not of interest for us (see [Men 90] or [Tor 91] for details), but what we will need is to retain the fact that in the ellipsoidal model the gravitational field \(\tilde{g}\) is still normal at each point of the surface. This will be of fundamental importance in what follows.

Let us return for a moment to the computations for the sphere case in Section 4. We can see that the construction of the Frenet frame \(\{E_1(t), E_2(t), E_3(t)\}\) is identical in the case of the ellipsoid, as it is equation of motion (10) and the approximations following it (which are based on the fact that \(\tilde{g}\) is directed along \(E_3(t)\)). Thus, for the motion of the Foucault’s pendulum along a parallel \(c\) on
the ellipsoid we get again the equations

\[ V(t) = (B \sin \beta t + A \cos \beta t)E_1(t) + (-A \sin \beta t + B \cos \beta t)E_2(t). \]

As for the parallel \( c(t) = X(\theta_0, t) = (a \cos \theta_0 \cos t, a \cos \theta_0 \sin t, b \sin t) \) on the ellipsoid we have

\[
E_1(t) = \frac{X_v(\theta_0,t)}{\|X_v(\theta_0,t)\|} = \frac{X_v(t)}{a \cos \theta_0}, \\
E_1(t) = \frac{X_u(t)}{\sqrt{a^2 \sin \theta_0^2 + b^2 \cos \theta_0^2}},
\]

we have, instead of (12):

\[
V(t) = (B \sin \beta t + A \cos \beta t)\frac{1}{a \cos \theta_0}X_v(t) \\
+ (-A \sin \beta t + B \cos \beta t)\frac{1}{\sqrt{a^2 \sin \theta_0^2 + b^2 \cos \theta_0^2}}X_u(t).
\]

This is to be compared with (9), which gives the expression for a parallel vector field along \( c \) on the ellipsoid. What we see, is that the components

\[
V^1(t) = \frac{-A \sin \beta t + B \cos \beta t}{\sqrt{a^2 \sin \theta_0^2 + b^2 \cos \theta_0^2}}
\]

and

\[
V^2(t) = \frac{B \sin \beta t + A \cos \beta t}{a \cos \theta_0}
\]

do not satisfy (9). For instance

\[
tg\theta_0V^1(t) = \frac{\tan \theta_0}{\sqrt{a^2 \sin \theta_0^2 + b^2 \cos \theta_0^2}}(-A \sin \beta t + B \cos \beta t)
\]

while (recall \( \beta = \sin \theta_0 \))

\[
V^2(t) = \frac{\tan \theta_0}{a}(-A \sin \beta t + B \cos \beta t),
\]

so \( \cos \theta_0 V^2(t) - \sin \theta_0 V^1(t) \neq 0 \). However, we see that the equation is true when \( a = b \), as it should be.

To summarize: for the ellipsoidal Earth the vector field associated with the displacement of the Foucault’s pendulum is no longer parallel, so even it does not make sense to speak about its holonomy.

7 Conclusions

We have seen that the holonomy can not be considered as the responsible for the Foucault pendulum precession, as changing slightly the spherical model of
the Earth by an ellipsoid $E_{a,b}$ (which can be taken as “round” as wanted, just making $a \to b$) the numerical values of both effects no longer coincide (indeed, it is nonsense to speak about holonomy in this case), this only happens in the limit case $a = b$. From the analysis in the preceding Sections, we can see explicitly that the holonomy only intervenes in the precession of the pendulum as a first order effect: that the equation describe a parallel vector field is true for the sphere after making a set of physical approximations, as considering only little amplitudes or a spherically symmetric gravitational field. The more realistic situation of an ellipsoidal Earth can be seen as a perturbation of the spherical case, giving a first order term in which the holonomy appears, but not as the definitive effect. Of course, it would be very interesting to study which geometrical effects appear at higher orders.

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