Factor-Based Imputation of Missing Values and Covariances in Panel Data of Large Dimensions

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Abstract

Economists are blessed with a wealth of data for analysis, but more often than not, values in some entries of the data matrix are missing. Various methods have been proposed to handle missing observations in a few variables. We exploit the factor structure in panel data of large dimensions. Our TALL-PROJECT algorithm first estimates the factors from a TALL block in which data for all rows are observed, and projections of unit specific length are then used to estimate the factor loadings. A missing value is imputed by its estimated common component which we show is consistent and asymptotically normal without further iteration. Implications for using imputed data in factor augmented regressions are then discussed. To compensate for the downward bias in sample covariance matrices created by an omitted noise in each imputed value, we overlay the imputed data with re-sampled idiosyncratic residuals many times and use the average of the covariances to estimate the parameters of interest. Simulations show that the procedures have desirable finite sample properties.

Keywords: risk management, covariance structure, matrix completion, incomplete data
JEL Classification: C1, C2.

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1 Introduction

Missing data is a problem that empirical researchers frequently encounter. For example, data can be missing due to attrition in longitudinal surveys such as the PSID and SIPP, to IPOs and bankruptcies in the case of stock returns, and to staggered releases of data by different institutions. Data sampled at a lower frequency will have missing values in an analysis that involves higher frequency data. Whatever is the reason, the way we handle missing data is not innocuous. Dropping rows or columns necessarily entails loss of information, and while the EM algorithm allows imputation of missing values, it is designed for low dimensional data and may not be easily scalable.

In recent years, progress has been made for situations when the missing values occur in large panels of data with a low rank component. Assuming incoherence conditions to ensure that the low rank structure is non-degenerate as in Candes and Recht (2009), the machine learning literature has developed regularization based algorithms to solve matrix completion problems of the Netflix challenge type. On the econometrics front, recent work by Jin et al. (2021), Xiong and Pelger (2019), and Bai and Ng (2021) propose different implementations that permit the entire common (low rank) component to be consistently estimated if the factor structure is strong. While the machine learning literature provides worst-case error bounds, the econometrics literature provides the distribution theory that is needed for inference, and is also the approach that this paper takes.

We introduce a Tall-Project estimator (or TP for short) for imputing missing values in a panel of data with $T$ rows and $N$ columns, where $N$ and $T$ are both large. The factors are estimated from a TALL block consisting of complete data for $N_0 \leq N$ units, while the loadings are obtained from $N$ time series projections with series-specific sample size. We obtain two main results. First, under certain assumptions on the factor structure in the different blocks, we show that the TP estimates are consistent and asymptotically normal for every entry of the low rank component though the convergence rate is series specific. And while iteration is not needed for consistent estimation, one re-estimation can improve the convergence rate. Because of missing data, stronger conditions are needed for the factor estimates to be treated as known in factor-augmented regressions.

Our second result concerns estimation of covariances from imputed data. Covariances play an important role in portfolio analysis and in structural equation (covariance structure) modeling. While the imputed data is unbiased for its mean, the variance of the imputed data is biased because the idiosyncratic noise associated with the missing observations are set to zero. To remedy this problem, we repeatedly overlay the first step estimate of missing values with resampled idiosyncratic residuals before computing sample covariances. An average of these imputed covariances is then used to estimate the objects of interest. In simulations calibrated to CRSP data from 1990-

\footnote{An R package that implements TW and TP is available for download at https://github.com/cykbennie/fbi.}
2018, resampling from the own residuals (ie. without pooling) yields risk measures that compare favorably and sometimes outperform the ones based on a factor-based covariance estimator.

In what follows, let $X$ be a $T \times N$ panel of data, $X_i = (X_{i1}, \ldots X_{iT})'$ be a $T \times 1$ vector of random variables and $X = (X_1, X_2, \ldots, X_N)$ be a $T \times N$ matrix. We use $i = 1, \ldots, N$ to index cross-section units and $t = 1, \ldots, T$ to index time series observations. In practice, $X_i$ is transformed to be stationary, demeaned, and is often standardized. It is assumed that the normalized data

$$Z = \frac{X}{\sqrt{NT}} \text{ has singular value decomposition (SVD)}$$

where $D_r$ is a diagonal matrix of $r$ singular values ordered such that $d_1 \geq d_2 \ldots \geq d_r$, while $U_r, V_r'$ are the corresponding left and right singular vectors respectively. Analogously, $U_{n-r}$ and $V_{n-r}'$ are $n \times (n - r)$ matrices of left and right eigenvectors associated with $d_{r+1}, \ldots, d_n$. The low rank component $U_r D_r V_r'$ can be defined without probabilistic assumptions.

We consider a factor model defined as

$$X = FA' + e$$

where $F$ is a $T \times r$ matrix of common factors, $\Lambda$ is a $N \times r$ matrix of factor loadings, and $e$ is a $T \times N$ matrix of idiosyncratic errors. We will let $(F^0, \Lambda^0)$ be the true values of $(F, \Lambda)$. The common component $C^0 = F^0 \Lambda^0'$ has reduced rank $r$ because $F^0$ and $\Lambda^0$ both have rank $r$. We will estimate the factors and loadings by the method of static asymptotic principal components (APC). The normalization $F^0 F^0' = I_r$ gives the APC estimates that are unique up to a column sign:

$$(\tilde{F}, \tilde{\Lambda}) = (\sqrt{T} U_r, \sqrt{N} V_r D_r).$$

Since $r$ can be consistently estimated, we proceed as though $r$ is known.

We consider a large dimensional approximate factor model in which $T$ and $N$ are large and which satisfies the following assumptions:

**Assumption A:** There exists a constant $M < \infty$ not depending on $N, T$ such that

a. (Factors and Loadings):

(i) $E \|F_i^0\|^4 \leq M, \|\Lambda_i\| \leq M$;

(ii) $\frac{F^0 F^0'}{T} \rightarrow \Sigma_F > 0$, and $\frac{\Lambda^0 \Lambda^0}{N} \rightarrow \Sigma_\Lambda > 0$;

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In this paper, we rely heavily on the results given in Stock and Watson (2002); Bai and Ng (2002); Bai (2003); Bai and Ng (2006).
(iii) the eigenvalues of $\Sigma_F \Sigma_A$ are distinct.

b. (Idiosyncratic Errors): Time and cross-section dependence

(i) $E(e_{it}) = 0, E|e_{it}|^8 \leq M$;
(ii) $E(\frac{1}{N} \sum_{i=1}^{N} e_{it} e_{is}) = \gamma_N(s,t), \sum_{t=1}^{T} |\gamma_N(s,t)| \leq M, \forall s$;
(iii) $E(e_{it}e_{jt}) = \tau_{ij,t}, |\tau_{ij,t}| \leq |\tau_{ij}|$ for some $\tau_{ij} \forall t$, and $\sum_{j=1}^{N} |\tau_{ij}| \leq M, \forall i$;
(iv) $E(e_{it}e_{js}) = \tau_{ij, st}$ and $\frac{1}{NT} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} |\tau_{ij, ts}| < M$;
(v) $E|N^{-1/2} \sum_{i=1}^{N} e_{is} e_{it} - E(e_{is} e_{it})|^4 \leq M$ for every $(t, s)$;
(vi) $E(\| \frac{1}{\sqrt{T}} \sum_{t=1}^{T} F_t^0 e_{it}\|^2) \leq M, \forall i$, and $E(\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=1}^{T} \Lambda_i^0 F_t^0 e_{it}\|^2) \leq M$.

c. (Central Limit Theorems): for each $i$ and $t$, $\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \Lambda_i^0 e_{it} \overset{d}{\to} N(0, \Gamma_t)$ as $N \to \infty$, and $\frac{1}{\sqrt{T}} \sum_{t=1}^{T} F_t^0 e_{it} \overset{d}{\to} N(0, \Phi_i)$ as $T \to \infty$; the two limiting distributions are independent.

Assumption A is used in Bai (2003) to develop an inferential theory for large dimensional factor analysis when $X$ is completely observed. The moment conditions in (b) ensure that the factor structure is strong and can be separated from the idiosyncratic errors which can be weakly correlated, both in the time and the cross-section dimensions. Let $D_r^2$ and $\nabla_r$ be the eigenvalues and eigenvectors of the $r \times r$ matrix $\Sigma_A^{1/2} \Sigma_F \Sigma_A^{1/2}$, respectively. As shown in Bai (2003), $\text{plim}_{N,T \to \infty} D_r^2 = D_r^2$ and $\text{plim}_{N,T \to \infty} \tilde{F}_r^0 = Q_r$, where $Q_r = D_r V_r \Sigma_A^{-1/2}$. The following properties of the APC estimator are given in Bai (2003):

Lemma 1. Suppose that Assumption A holds and let $H = (\Lambda^0 \Lambda^0 / N)(\tilde{F}_r^0 \tilde{F}_r^T)D_r^{-2}$. If $\sqrt{N}/T \to 0$ as $N, T \to \infty$, then

\[ \sqrt{N}(\tilde{F}_r - H^T F_r^0) \overset{d}{\to} N\left(0, D_r^{-2} Q_r \Gamma_q Q_r D_r^{-2}\right) \quad (2a) \]
\[ \sqrt{T}(\tilde{\Lambda}_r - H^{-1} \Lambda_r^0) \overset{d}{\to} N\left(0, (Q_r')^{-1} \Phi_r Q_r^{-1}\right) \quad (2b) \]
\[ \min(\sqrt{N}, \sqrt{T}) \left(\tilde{C}_{it} - C_{it}^0\right) \overset{d}{\to} N(0, 1). \quad (2c) \]

where $\tilde{V}_{\tilde{C}_{it}}(N, T)$ is a consistent estimate of $\tilde{\delta}^2_{IT} \Lambda_r^0 \Sigma_A^{-1} \Gamma_q \Sigma_A^{-1} \Lambda_r^0 + \tilde{\Delta}^2_{IT} \tilde{F}_r^0 \Sigma_F^{-1} \Phi_r \Sigma_F^{-1} \tilde{F}_r^0$.

In what follows, we will obtain results for the factor estimates when $X$ has missing values.
2 Missing Data

A panel is said to be *complete* if it does not have missing values. In practice most data panels have missing values. The data are said to be *missing completely at random* (MCAR) if missingness is independent of the data whether or not they are observed. As the observed and missing data are drawn from the same underlying distribution and hence have no systematic differences, MCAR does not create bias. This is not the case with *missing at random* (MAR), though the systematic differences can be explained by other observables. For example, missing values in financial statements of smaller firms may arise because smaller firms are less regulated, in which case the missing data propensity can be related to market capitalization which is observed for publicly traded firms. But the issue remains that the observed values under MAR do not form a random sample.

To overcome the bias when the data are missing at random, one approach is to re-weigh the data (such as 'inverse probability weighting’ used in survey design). A second approach is to rebalance the panel by amputation which can take one of three forms: (a) listwise (complete case) deletion under which the entire row with missing values is eliminated; (b) variable deletion under which a series with one or more missing values will be eliminated; and (c) pairwise deletion where only the cases with missing data involved will be deleted. Though the solution is simple and does not entail modeling assumptions, amputation leads to significant information loss. Furthermore, the resulting balanced sample may not be representative of the population and may yield biased estimates.

Another way to rebalance the panel is imputation. The simplest procedure is to replace the missing values with the sample mean or median. A more sophisticated procedure is to impute from the unconditional sample first and second moments via the EM algorithm. Schneider (2001) proposes a regularized EM algorithm that is quite popular in analysis of climate data. Imputation on the basis of unconditional moments does not require a model. A fully specified approach is to construct the likelihood using incomplete observed data and iteratively solve for the parameters of the conditional mean function. The estimates would be more efficient if the parametric assumptions were correct. These methods are designed for imputing missing values in a small number of variables/predictors. See Robins et al. (1995), Li et al. (2013) and Raghunathan (2004) for a review.

Factor models provide a simple framework for imputing missing values of many variables. Banbura and Modugno (2014), Jungbacker and Koopman (2011), Jungbacker et al. (2011) consider state-space modeling of a strict factor (i.e., a diagonal idiosyncratic error variance matrix) with missing data and uses non-linear filters to compute the likelihood as $F_t$ and $\Lambda_t$ are both random. Giannone et al. (2008) initializes the missing values with estimates from the balanced panel and uses the Kalman filter to perform updating. Stock and Watson (2016) discusses the issues with
state space estimation of factor models with missing data. For large dimensional approximate factor models, Stock and Watson (1998) suggests to fill missing values in $X$ with the APC estimates of the common component. These are all EM algorithms that use the factor structure to evaluate the conditional mean in the E-step, and principal components estimation in the M step.

Though many factor-based imputation methods have been used for some time, the theoretical properties of the imputed estimates are studied only recently in independent works by Jin et al. (2021), Xiong and Pelger (2019), and Bai and Ng (2021). All three exploit the strong factor structure in a large panel setting to estimate the factors and the loadings from incompletely observed data. Jin et al. (2021) focuses on the case of missing at random and analyzes the EM estimator considered in Stock and Watson (1998). After initializing the missing values to zero, the factors are estimated from data reweighted by the frequency of no missing values. Though these estimates are consistent, they are not asymptotically normal without further iteration. Xiong and Pelger (2019) also re-weights the data in principal components estimation of the loadings, but uses cross-sectional regressions to estimate the factors at every $t$. Similar to Jin et al. (2021), the estimates are consistent and asymptotically normal when iterated until convergence. But the ‘all purpose’ estimator proposed in Xiong and Pelger (2019) allows for mechanisms other than missing at random. Though robustness comes at the cost of larger variances, efficiency can be improved if stronger assumptions on missingness are made.

Figure 1: Missing Data with a Well-Organized Structure

| Block Missing | Semi-Block Missing |
|---------------|--------------------|
| $X_1$ | $X_{N_o}$ | $X_{N_o+1}$ | $\ldots$, $X_{N-1}$ | $X_N$ | $X_1$ | $X_{N_o}$ | $X_{N_o+1}$ | $\ldots$, $X_{N-1}$ | $X_N$ |
| x | x | x | | | x | x | x |
| x | x | x | | | x | x | x |
| x | x | x | | | x | x | x |

While Jin et al. (2021) and Xiong and Pelger (2019) reweigh the data in APC estimation, Bai and Ng (2021) implicitly re-organizes the data into blocks. This is partly motivated by the concern that the missing at random assumption may at times be inappropriate. Examples of microeconomic applications are discussed in Athey et al. (2013). Furthermore, for macroeconomic panels such as FRED-MD where the data are collected from different sources, assuming that the missing values are due to the same missing mechanism seems restrictive.

Consider the left panel of Figure 1. This systematic (non-random) missing pattern can arise if,
for example, weekly data for related variables (such as inventories and orders) are released in the same week of every month. The right panel shows a case of ‘semi-block missing’. This can arise if, for example in a survey, respondent \( N \) dropped out early, respondent \( N - 1 \) did not respond in period \( T - 1 \), while \( N_0 + 1 \) did not respond in period \( T \). The observations can be missing for different reasons likely unknown to the researcher.

While the missing data mechanism can be difficult to verify, we do observe the missing data pattern, and organizing the data into blocks offers a new way of thinking about imputation. The insight in [Bai and Ng (2021)](https://doi.org/10.1038/s41598-021-92757-2) is that the quantities needed for imputation can be obtained from two completely observed blocks. Their TALL-WIDE (or TW for short) procedure uses the TALL block consisting of complete data for \( N_0 < N \) units to estimate the factors, and a WIDE block in which data for all units are available over \( T_0 < T \) periods to estimate the factor loadings. To align the space spanned by the estimated factors and loadings, one then estimates a new rotation matrix by regressing the \( N_0 \times r \) matrix of loadings estimated from the TALL block on a sub-block of \( N_0 \times r \) matrix of loadings estimated from the WIDE block. The procedure produces consistent and asymptotically normal estimates without iteration. Re-estimation using the completed (imputed) data will improve efficiency of the estimates in the balance block where the TALL and WIDE blocks intersect to the fastest rate possible, which is the rate obtained in the complete data case.

Figure 2: Missing Data with a Less Structured Pattern

| X1 | … | XN₀ | XN₀+1 | … | XN₋₁ | XN |
|----|---|-----|-------|---|------|-----|
| x  | x | x   |       |   |      |     |
|    |   |     | x     |   |      |     |
| x  |   |     | x     |   |      |     |
|    |   |     | x     |   |      |     |

| X1 | … | XN₀ | XN₀+1 | … | XN₋₁ | XN |
|----|---|-----|-------|---|------|-----|
|    |   |     | x     |   |      |     |
| x  |   |     |       |   |      |     |
| x  |   |     |       |   |      |     |
|    |   |     | x     |   |      |     |
|    |   |     | x     |   |      |     |

The TW algorithm takes as given that the TALL and WIDE blocks are 'big enough' for consistent estimation. Because the size of the missing data block is defined by number of time periods and units with complete cases, the size of the missing block is the same for both examples in Figure 1. The algorithm uses information available efficiently when the missing pattern is homogeneous. But for the example in the right panel, more information could have been used.

More concerning are situations when a few series significantly reduce \( N_0 \) and \( T_0 \). As an example, consider Figure 2. The left panel shows a case of reverse monotone missing. In multi-country panels, this pattern will arise if data for less developed countries are only available at a later date than developed countries. The right panel shows a case where the sample size is much smaller for unit
$N$ than for units $N-1$ and $N-2$. In both examples, the missing block can contain many observed values.

3 A Projection Based Procedure for Imputing $X$

In this section, we will develop a new procedure to obtain consistent estimates without iteration as in $\text{tw}$, but can accommodate flexible missing data patterns while making more efficient use information available. The point of departure is to partition the $T \times N$ matrix $X$ in a different way. We assume that there are $N_o$ series with no missing values so there is a $T \times N_o$ block labeled TALL, and a $T \times (N - N_o)$ block of partially observed data labeled INCOMPLETE. There is no need to reorganize the data in practice, but Figure 3 shows the data with the $N_o$ variables ordered first to help visualize the idea.

Figure 3: The Tall-Incomplete Representation of the Data

|   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|
| ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| ✓ | tall | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| ✓ | $T \times N_o$ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |
| ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ |

It will be helpful to define two locator sets as follows:

$J^t := \{j : X_{jt} \text{ observed, i.e. units with data in period } t\}$

$J_i := \{s : X_{is} \text{ observed, i.e. periods with data for unit } i\}$

Let $N_o$ be the number of units observed at time $t$ and $T_o$ be the number of rows observed for series $i$. Then $J^t$ is a $N_o \times 1$ vector that keeps track of the units observed at $t$, and $J_i$ is a $T_o \times 1$ vector that keeps track of the periods that unit $i$ is observed. For a set $\mathcal{A}$, let $|\mathcal{A}|$ be its cardinality.

In this notation, $N_o = |\bigcap_t J^t|$, $T_o = |\bigcap_i J_i|$, and $T_o_i = T$ for every unit in TALL.

Algorithm Tall-Project (TP):

i. Estimate the $T \times r$ matrix $\tilde{F}$ from the TALL block by APC and let $\tilde{F}_t$ of dimension $r \times 1$ be the $t$-th row of $\tilde{F}$.  

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ii. For each $i$, regress the $T_{o_i} \times 1$ vector consisting of the observed values of $X_i$ on the corresponding $T_{o_i} \times r$ submatrix of $\tilde{F}$ to obtain $\tilde{\Lambda}_i$.

iii. For each $(i, t)$, let $\tilde{C}_{it} = \tilde{F}_t' \tilde{\Lambda}_i$ and $\tilde{e}_{it} = \tilde{X}_{it} - \tilde{C}_{it}$ where

$$\tilde{X}_{it} = \begin{cases} X_{it} & \text{if } X_{it} \text{ observed} \\ \tilde{C}_{it} & \text{if } X_{it} \text{ missing.} \end{cases}$$

Step (i) of TP is the same as TW since both procedures estimate $F$ from the TALL block. If such a block does not exist, then neither TW or TP will be appropriate. Fortunately, for macroeconomic panels, a TALL block is often available. Algorithm TP accommodates staggered and irregular patterns of missingness by estimating $\Lambda_i$ using a customized sample for each series. The two algorithms should be numerically identical if $T_{o_i} = T_o$ for all $i$, but since $N_{o_i} \geq N_o$ and $T_{o_i} \geq T_o$, TP will utilize more information in general. The difference can be significant if $T_o$ is dictated by a few series with many missing observations. Note, however, that TP estimates the $H\Lambda_i$ matrix directly, while TW estimates $H$ and $\Lambda_i$ separately. Furthermore, the number of factors in Algorithm TP is determined by the TALL block so Algorithm TW is more flexible in this regard.

We will need additional assumptions to analyze the properties of TP.

**Assumption B:** \( \frac{\sqrt{N}}{\min(N_o, T_o)} \to 0 \) and \( \frac{\sqrt{T}}{\min(N_o, T_o)} \to 0 \) as $N \to \infty$ and $T \to \infty$.

**Assumption C:** There exists $M < \infty$ such that for all $N$ and $t$, $N_{o_t}(N - N_{o_t}) / N N_{o_t} \leq M$.

**Assumption D:** For each $i$, \( \left( \frac{1}{T - T_{o_i}} \sum_{s \in J_i} F_s^0 F_s^0 \right) \left( \frac{1}{T_{o_i}} \sum_{s \in J_i} F_s^0 F_s^0 \right)^{-1} \xrightarrow{p} I_r, \) \( \frac{1}{\sqrt{T_{o_i}}} \sum_{s \in J_i} F_s^0 e_{is} \xrightarrow{d} N(0, \Phi_i). \)

For each $t$, \( \left( \frac{1}{N - N_{o_t}} \sum_{k \in J} \Lambda_k^0 \Lambda_k^0 \right) \left( \frac{1}{N_{o_t}} \sum_{k \in J \cap \Theta_t} \Lambda_k^0 \Lambda_k^0 \right)^{-1} \xrightarrow{p} I_r, \) \( \frac{1}{\sqrt{N_{o_t}}} \sum_{k \in J} \Lambda_k^0 e_{kt} \xrightarrow{d} N(0, \Gamma_t). \)

Assumption B essentially puts a lower bound on the observed number of rows and columns. Assumption C puts an upper bound on the extent of missing values at any $t$ and can be understood as a noise-to-signal constraint. Assumption C allows $N_{o_t} / N \to 0$, and especially $N_o / N \to 0$. For example, if $N_{o_t} = N_o$, then the ratio is bounded by 1. For the central limit theorems in Assumption D to hold, $T_{o_i}$ should be independent of $(F_s^0, e_s)$, and $N_{o_t}$ independent of $(\Lambda_k^0, e_k)$. The assumption is satisfied if the missing data are unrelated to the intrinsic properties of the series. For example, missing data due to mergers are allowed if the events are unrelated to $F_t^0$. However, missing data due to bankruptcies would not be allowed if the bankruptcy probability depends on $F_t^0$. These conditions are stronger than stationarity and need to be justified on an application by application basis. All factor-based imputation procedure requires a similar assumption because without some commonality between the observed and missing units, imputation would not be possible.
Lemma 2. (First Pass Estimation) Let $N_o$ be the number of units in the tall block and $T_o$ be number of periods that unit $i$ is observed. Let $H_{tall}$ be a rotation matrix based on the tall block of the data. Under Assumptions A-D, the TP estimates $(\hat{F}, \hat{\Lambda})$ have the following properties:

i. $\sqrt{N_o}(\hat{F}_t - H_{tall}^T F_o^0) \xrightarrow{d} N\left(0, \Sigma^{-2} \Omega_i \Sigma^{-1} \Gamma_t \Omega_t^0 \Sigma^{-2}\right)$, for $t \in [1, T]$, if $\frac{\sqrt{T}}{N_o} \to 0$

ii. (a) $\sqrt{T}(\hat{\Lambda}_i - H_{tall}^T \Lambda^0_i) \xrightarrow{d} N\left(0, (Q_t^i)^{-1} \Phi_t \Omega_t^{-1}\right)$, for $i \leq N_o$, if $\frac{\sqrt{T}}{N_o} \to 0$

(b) $\sqrt{T_o}(\hat{\Lambda}_i - H_{tall}^T \Lambda^0_i) \xrightarrow{d} N\left(0, (Q'_t)^{-1} \Phi_t \Omega_t^{-1}\right)$, for $i > N_o$, if $\frac{\sqrt{T_o}}{N_o} \to 0$

iii. Let $\tilde{V}_{it}(N_o, T_o)$ be a consistent estimate of $V_{it} = \frac{\delta_{N_o, T_o}}{N_o} \Lambda^0_t \Sigma^{-1} \Gamma_t \Sigma^{-1} \Lambda^0_t + \frac{\delta_{N_o, T_o}}{T_o} F^0_t \Sigma^{-1} \Phi_t \Sigma^{-1} F^0_t$

where $\delta_{N_o, T_o} = \min(N_o, T_o)$. Then

$$\min\left(\sqrt{N_o}, \sqrt{T_o}\right) \left(\frac{\hat{C}_{it} - C_{it}^0}{\sqrt{\tilde{V}_{it}(N_o, T_o)}}\right) \xrightarrow{d} N(0, 1).$$

Part i and ii(a) are implied by Lemma 1 because $\hat{F}_t$ and $\hat{\Lambda}_i$ are estimated from the tall block, which is a complete matrix of $T \times N_o$ dimension. Result (b) of part ii arises because the factor loadings for unit $i$ in the incomplete block are estimated from a regression with $T_o$ observations. Part (iii) shows that each $\hat{C}_{it}$ has its own convergence rate that depends on $T_o$, the number of observed entries in $X_t$. The result is based on the fact for those with $X_{it}$ that are missing,

$$\hat{C}_{it} - C_{it}^0 = \Lambda^0_t \left(\frac{1}{N_o} \sum_{k \in \gamma_i} \Lambda_k^0 \Lambda_k^0 \right)^{-1} \frac{1}{N_o} \sum_{k \in \gamma_i} \Lambda_k^0 \epsilon_{kt} + F_t^0 \left(\frac{1}{T_o} \sum_{s \in J_i} F_s^0 F_s^0 \right)^{-1} \frac{1}{T_o} \sum_{s \in J_i} F_s^0 \epsilon_{is} + r_{it}$$

$$\triangleq u_{it} + v_{it} + r_{it}. \tag{3}$$

The error due to estimating $F$ from the tall block appears as the first term on the right hand side and is denoted $u_{it}$, while the error due to estimating $\Lambda_i$ by projections is summarized by the second term and is denoted $v_{it}$. The term $r_{it} = O_p(\delta_{T_o, N_o}^2)$ uniformly in $i$ and $t$ represents high order errors when estimating the factor and factor loadings. These quantities depend on $N$ and $T$ but the notation is suppressed for simplicity. The representation (shown in the Appendix) is useful in understanding how $N$ and $T$ affect factor-based imputation.

3.1 Re-estimation using $\tilde{X}$

As $\tilde{F}$ is estimated using the tall block alone, re-estimation of the factors and the loadings from the completed (imputed) matrix $\tilde{X}$ appears desirable. But imputation error in $\tilde{X}$ must be taken
inflated from \( N \). Under Assumption C, the best rate possible (the same as in complete data). Third, the asymptotic variance of \( \tilde{X}_it \) into account. Using (3), we have

\[
\tilde{X}_it = \Lambda^0_i F^0_t + e_it, \quad \text{if } X_it \text{ observed} \\
\tilde{X}_it = \Lambda^0_i F^0_t + u_it + v_it + r_it \quad \text{if } X_it \text{ missing.}
\]

A quantity that will play a role in APC estimation from \( \tilde{X} \) is

\[
B_i^t = \begin{cases} 
\frac{N-o_i}{N} I_r + \frac{o_i}{N} \left( \frac{N-N_o}{N_o} \right) \left( \frac{1}{N-N_o} \sum_{k \in J_t} \Lambda^0_k \Lambda^0_k \right)^{-1} & i \in J^1 \cap \cdots \cap J^T \\
\frac{N-o_i}{N} I_r & i \in J^t \setminus \left( J^1 \cap \cdots \cap J^T \right)
\end{cases}
\]

Under Assumption C, \( B_i^t \) is bounded. For units in the tall block (i.e., \( i \in \cap_s J^s \)), \( B_i^t \) is roughly inflated from \( \frac{N-o_i}{N} I_r \) by \( o_i(N-N_o)/(N N_o) \) which can be thought of as the noise to signal ratio.

**Proposition 1.** Let \( (\tilde{F}^+, \tilde{\Lambda}^+) = (\sqrt{T \tilde{U}_r}, \sqrt{N \tilde{V}_r D_r}) \) where \( \tilde{U}_r, \tilde{V}_r \) are the \( r \) left and right singular vectors of \( \tilde{X} \). Let \( H^+ \) be a rotation matrix. Under Assumptions A – D, the factors and loadings estimated from \( \tilde{X} \) have the following properties:

\[ i \quad \sqrt{N-o_i}(\tilde{F}_i^+ - H^+ F^0) \xrightarrow{d} N(0, \mathbb{I}_r^{-1} \mathbb{Q}_r^T \mathbb{Q}_r \mathbb{I}_r^{-1}) \]

\[ ii \quad \sqrt{T_o_i}(\tilde{\Lambda}_i^+ - (H)^{-1} \Lambda_0^0) \xrightarrow{d} N(0, (\mathbb{Q}_r^0)^{-1} \Phi_1(\mathbb{Q}_r^0)^{-1}) \]

\[ iii \quad \text{Suppose that } e_it \text{ is cross-sectionally uncorrelated. Let } \tilde{C}_{it}^+ = \tilde{\Lambda}_i^+ \tilde{F}_i^+. \text{ Then, for all } (i, t), \]

\[
\min(\sqrt{N-o_i}, \sqrt{T_o_i}) \left( \frac{\tilde{C}_{it}^+ - C_{it}^0}{\sqrt{\tilde{V}_{it}^+ (N-o_i, T_o_i)}} \right) \xrightarrow{d} N(0, 1)
\]

where \( \tilde{V}_{it}^+ (N_o, T_o) \) is a consistent estimate of

\[
\tilde{V}_{it}^+ = \frac{\delta^2_{N-o_i, T_o_i}}{N-o_i} \Lambda_0^0 \Sigma^{-1}_\Lambda T_t \Sigma^{-1}_\Lambda \Lambda_0^0 + \frac{\delta^2_{N-o_i, T_o_i}}{T_o_i} F_t^0 \Sigma^{-1}_F \Phi_1 \Sigma^{-1}_F F_t^0,
\]

\[
\delta^2_{N-o_i, T_o_i} = \min(N_o, T_o_i) \text{ and } \Gamma^* = \text{plim} \frac{1}{N-o_i} \sum_{i \in J_t} B_i^t \Lambda_0^0 B_i^t \delta^2_{N-o_i, T_o_i}.
\]

The proposition establishes the convergence rate of the factors and loadings constructed from \( \tilde{X} \). There are three changes due to re-estimation. First, there is only one (instead of many) rotation matrices for the factor loadings. This is a consequence of the fact that the factors are now estimated from \( \tilde{X} \) in its entirety, instead of sub-blocks. Second, re-estimation generates efficiency gains. The convergence rate for \( \tilde{F}_i^+ \) is improved from \( \sqrt{N_o} \) to \( \sqrt{N-o_i} \) or to \( \sqrt{N} \). Though the rate for \( \tilde{\Lambda}^+ \) is unchanged, the convergence rate of \( \tilde{C}_{it}^+ \) is now \( \min(\sqrt{N-o_i}, \sqrt{T_o_i}) \) instead of \( \min(\sqrt{N-o_i}, \sqrt{T_o_i}) \). If \( N_o = N \) and \( T_o_i = T \) for a given pair \((i,t)\), the convergence rate for \( \tilde{C}_{it}^+ \) is \( \min(\sqrt{N}, \sqrt{T}) \), the best rate possible (the same as in complete data). Third, the asymptotic variance of \( \tilde{C}_{it}^+ \) depends

10
Consider the infeasible regression with observed covariates $W_t$ and a latent predictor $F_t$:

$$
y_{t+h} = \alpha F_t + \beta W_t + \epsilon_{t+h}$$

$$= \delta z_t + \epsilon_{t+h}$$

where $z_t = (F_t', W_t')'$ and $\delta = (\alpha', \beta')'$. Suppose that $F_t$ is replaced by an estimate based on a small number of predictors. It is known that in general, the sampling uncertainty in a generated regressor will inflate standard errors in subsequent regressions. A useful result in Bai and Ng (2006) is that if $F$ is estimated by APC from a completely observed panel $X$, then $\hat{F}$ can be used in a second step regression without the need for standard error adjustments if certain conditions on the sample size are satisfied. Our foregoing analysis suggests that even if $X$ has missing values, $\hat{F}^+$ estimated by TP is still consistent for the space spanned by $F$ albeit at a slower convergence rate.

**Lemma 3.** Suppose Proposition 4 holds. Then (i) $\frac{1}{T} \sum_{i=1}^{T} \| \hat{F}^+_{i t} - H_{NT} F_i \|^2 = O_p(\min[NT, T_o]^{-1})$. Let $\hat{z}_i = (\hat{F}^+_{i t}, W_{i t}')'$ be used in place of $z_i$ in the factor-augmented regression (6), and denote $\delta^0 = (\alpha'H_{NT}^{-1}, \beta')'$. If $\sqrt{T}/N_o \to 0$ and $\sqrt{T}/T_o \to 0$, then (ii)

$$\sqrt{T}(\hat{\delta} - \delta^0) \xrightarrow{d} N(0, J^{-1} \Sigma_{zz}^{-1} \Sigma_{z\delta} \Sigma_{\delta z}^{-1} J^{-1}).$$

where $J$ is the probability limit of $J_{NT} = \text{diag}(H_{NT}'I_{dim(W)})$.

Part (i) is based on Lemma 3 of Bai and Ng (2021) obtained for TW which can be used as a worse case rate for TP since $T_o$ in TW cannot exceed $\min T_{ae}$ in TP. A consequence of (i) is that

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \hat{z}_t (F^0_{t t}H_{NT} - \hat{F}^+_t) = O_p\left(\frac{\sqrt{T}}{\min(N_o, T_o)}\right),$$

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \epsilon_{t+h} \hat{F}^+_t = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \epsilon_{t+h} H_{NT} F^0_t + \frac{1}{\sqrt{T}} \sum_{t=1}^{T} (\hat{F}^+_t - H_{NT} F^0_t) \epsilon_{t+h}$$

$$= \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \epsilon_{t+h} H_{NT} F^0_t + O_p\left(\frac{\sqrt{T}}{\min(N_o, T_o)}\right).$$
These results are used to obtain (ii). In particular, let
$$S_{zz} = \frac{1}{T} \sum_{t=1}^{T} \hat{z}_t \hat{z}_t'$$
Then
$$\sqrt{T}(\hat{\delta} - \delta_0) = S_{zz}^{-1} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \hat{z}_t \epsilon_{t+h} + S_{zz}^{-1} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \hat{z}_t \alpha' H'_{NT} (H'_{NT} F^0_t - \tilde{F}^+_{t})'$$
provided $\frac{T}{N_o} \to 0$, $\frac{T}{T_o} \to 0$. If, in addition, $\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \hat{z}_t \epsilon_{t+h} \overset{d}{\to} N(0, \Sigma_{zz, \epsilon})$, the asymptotic variance of $\hat{\delta}$ is of the sandwich form $A^{-1}BA^{-1}$ with $A = J\Sigma_{zz}J$ and $B = J\Sigma_{zz, \epsilon}J'$, where $J$ is the probability limit of $J_{NT}$. Simplifying yields (ii). The result implies that as in Bai and Ng (2006), the estimated factors can be treated as though $F$ were observed albeit under more stringent conditions than the complete data case which only requires that $\sqrt{T}/N \to 0$. For macroeconomic analysis when the factors are estimated from panels with large $T_o$ and $N_o$, one can still expect $\hat{\delta}$ to be precisely estimated up to a rotation.

4 Factor Based Estimation of Covariance Matrices

Consider portfolio analysis where $X$ is a matrix of returns and $\Sigma_X$ is its covariance. The population weights of a minimum variance portfolio are given by $w_p = \Sigma^{-1} \frac{1}{1^{\prime} \Sigma^{-1} 1}$ where $1$ is the unit vector. If $X$ was completely observed and $x_i = X_i - \bar{X}_i$, the sample moments (SM) estimator of the $N \times N$ matrix is
$$\hat{\Sigma}_X = \frac{1}{T} \sum_{t=1}^{T} x_t x_t' \quad \text{(SM)}$$
It is well known that the sample covariance matrix is singular when $N > T$ and has large sampling uncertainty when $N$ and $T$ are large. Nonetheless, when the data admit a factor structure so that the decomposition $\Sigma_X = \Lambda \Sigma_F \Lambda' + \Sigma_e$ holds, a factor-based covariance estimator can be defined as
$$\hat{\Sigma}_X = \tilde{\Lambda} \hat{\Sigma}_F \hat{\Lambda}' + \hat{\Sigma}_\tilde{e}$$
where $\hat{\Sigma}_\tilde{e}$ is the sample covariance of $\tilde{e}$, which is never exactly diagonal even when $\Sigma_e$ is diagonal. As argued in Chamberlain and Rothschild (1983), an approximate factor model that allows for some correlation in the idiosyncratic errors is usually a better characterization of returns data. To distinguish a strict from an approximate factor structure, let $\Psi_e$ be a diagonal matrix whose $i$-th entry is $E[e_{it}^2]$ and which can be consistently estimated by
$$\tilde{\Psi}_\tilde{e} = \text{diag} \left( \frac{1}{T} \sum_{t=1}^{T} e_{it}^2, i = 1, 2, ..., N \right). \quad (7)$$
The strict-factor (SF) covariance estimator is defined as

\[ \tilde{\Sigma}_X = \tilde{\Lambda} \tilde{\Sigma}_F \tilde{\Lambda}^\prime + \tilde{\Psi}_\varepsilon \]

(SF)

Consistency of this estimator is studied by Fan et al. (2011), among others, for large \( N \) and large \( T \), assuming that \( X \) is completely observed.

We now turn to the incomplete data case. Missing data presents a challenge for risk management because the portfolio weights depend on \( \Sigma_X \). Variable deletion is not an option when the variances and covariances of the missing variables are the objects of interest. Practitioners often resort to listwise deletion of the sample data, deleting the entire record from the analysis if any single value is missing. The covariance estimates can be unstable when the reduction in the sample size is sufficiently large. Though it is possible to calculate the sample covariance matrix with pairwise-complete observations, there is no guarantee that the resulting matrix will be positive definite. This is problematic because a singular covariance matrix must have at least one eigenvalue that is zero, implying that it would be possible to construct an eigenportfolio that has zero volatility (risk) by using the corresponding eigenvector as the weights, making it possible to have an infinite ex-ante Sharpe ratio. Not only is this unrealistic, the hedges implied by the eigenportfolio weights are spurious and would fail out of sample. For these reasons, singular covariance matrices are of little use in portfolio construction in practice. Most risk applications require an estimate of a non-singular covariance matrix. The EM algorithm is one possibility, but it is likelihood based and requires parametric assumptions.

Unfortunately, successful factor-based imputation of the level of the data is not enough for precise estimation of their covariances. Even if the strict factor structure is correctly imposed, both the sample-covariance estimator and factor-based covariance estimator based on the imputed data will not be consistent. The problem with both estimators is that \( \varepsilon_{it} \) is set to zero when \( X_{it} \) is not observed, so the sample variance of series \( i \) is biased whenever the series has missing values. We can replace \( \tilde{\Psi}_\varepsilon \) in (7) by

\[ \Psi_{\varepsilon} = \text{diag} \left( \frac{1}{T_{o_i}} \sum_{t \in J_i} \tilde{\varepsilon}_{it}^2, i = 1, 2, \ldots, N \right) \]

to recognize that some \( \tilde{\varepsilon}_{it} \) are zero, and the estimate \( \Psi_{\varepsilon,ii} = \frac{1}{T_{o_i}} \sum_{t \in J_i} \tilde{\varepsilon}_{it}^2 \) is consistent for \( \Psi_{e,ii} \) if \( T_{o_i} \) is sufficiently large. Define the strict-factor adjusted (SFA) covariance matrix estimator as

\[ \hat{\Sigma}_X = \hat{\Lambda} \hat{\Sigma}_F \hat{\Lambda}^\prime + \hat{\Psi}_\varepsilon \]

(SFA)

**Lemma 4.** Let \( \hat{\Sigma}_X \) be the SFA estimate of \( \Sigma_X \). Under Assumptions A, B, exponential tail distribution for \( e_{it} \), and that \( \Sigma_e \) is a diagonal matrix,

\[ \| \hat{\Sigma}_X - \Sigma_X \|_\infty = \max_{1 \leq i, j \leq N} | \hat{\Sigma}_{X,ij} - \Sigma_{X,ij} | = o_p(1). \]
where for a matrix \( \mathbf{A} \) with \( A_{ij} \) as its \((i, j)\) entry, \( \|\mathbf{A}\|_\infty = \max_i \max_j |A_{ij}| \).

The proof is given in the Appendix.

4.1 SM Estimation and Double Imputation

This subsection considers a sample covariance estimator as an alternative to the SFA covariance estimator just described. Motivated by the fact that the variance of an imputed variable is downward biased, we use a second imputation to rectify this problem.

Algorithm Residual Overlay

Let \( \tilde{e} = \tilde{X} - \tilde{C} \) where \( \tilde{X} \) is produced by Algorithm TP. For residual resampling scheme \( j \) (\( j = 1, 2, 3, 4 \)) to be discussed below, repeat for \( s = 1, \ldots, S \):

a. For each \( i \) with missing data, replace those \( \tilde{e}_{it} = 0 \) with a \( \tilde{e}_{it}(s, j) \) randomly sampled from those non-zero \( \tilde{e}_{it} \) associated with observed \( X_{it} \).

b. Define \( \hat{e}_{it}(s, j) = \begin{cases} \tilde{e}_{it} & \text{if } X_{it} \text{ observed} \\ \tilde{e}_{it}(s, j) & \text{if } X_{it} \text{ is missing} \end{cases} \).

c. Let \( \hat{X}_{it}(s, j) = \hat{X}_t \hat{F}_t + \hat{e}_{it}(s, j) \) and \( \hat{x}_{it}(s, j) = \hat{X}_{it}(s, j) - \bar{\bar{X}}(s, j) \). Estimate the covariance of \( \hat{X}(s, j) \) as \( \hat{\Sigma}_{\tilde{X}}(s, j) = \frac{1}{S} \sum_{t=1}^{T} \hat{x}(s, j) \hat{x}(s, j)' \).

d. The \( N \times N \) covariance estimator obtained from \( \tilde{X} \) imputed by TP and overlaid with residuals sampled using scheme \( j \) is

\[
\hat{\Sigma}_{\tilde{X}}(j) = \frac{1}{S} \sum_{s=1}^{S} \hat{\Sigma}_{\tilde{X}}(s, j). \tag{SM+j}
\]

The term residual overlay is motivated by the fact that errors are added to \( \tilde{X} \). The algorithm injects randomness to \( \hat{X}_{it} \) whenever \( X_{it} \) is missing. It uses stochastic simulations to compensate for the lost variability, an idea briefly considered in Enders (2010, Chapter 5) in a fixed \( N \) or \( T \) setting. Even though each \( \hat{\Sigma}_{\tilde{X}}(s, j) \) is large in dimension and has rank \( \min(N, T) \), in our experience, the average \( \bar{\bar{\Sigma}}_{\tilde{X}} \) is always full rank, making it a viable alternative estimator in the case when \( N > T \).

The method of multiple imputation typically estimates the object of interest each time an imputed set of data is obtained, and produces as output the average over the multiple estimates. In contrast, we average the multiple imputed covariances to obtain \( \Sigma_{\tilde{X}} \) and compute the object of interest (for example, portfolio weights) from it. Averaging has the advantage of reducing sampling uncertainty especially when \( N \) and \( T \) are both large.
Step a: Four Sampling Schemes

- (SM1) Let \( u \) be a \( \sum_{i=1}^{N} T_{oi} \) vector, obtained by stacking up all \( \tilde{e}_{it} \) associated with the observed \( X_{it} \). Resample \( \sum_{i=1}^{N} (T - T_{oi}) \) observations from \( u \) with replacement and randomly assign them to \( \hat{e}_{it} \) whenever \( \tilde{e}_{it} = 0 \).

- (SM2) For each \( i \), let \( u_i \) be \( T_{oi} \) sub-vector of non-zero entries of \( \hat{e}_{i} \). Sample with replacement \( T - T_{oi} \) errors from \( u_i \) and assign them to \( \hat{e}_{it} \) whenever \( \tilde{e}_{it} = 0 \).

- (SM3) Compute \( \hat{\sigma}_u \) from the \( \sum_{i=1}^{N} T_{oi} \) vector of non-zero estimated errors as in Method 1. If \( \tilde{e}_{it} = 0 \), replace it by \( \hat{e}_{it} := u_{it} \hat{\sigma}_u \), where \( u_{it} \sim N(0, 1) \).

- (SM4) Compute \( \hat{\sigma}_{u,i} \) from the \( T_{oi} \) non-zero estimated errors as in Method 2. If \( \tilde{e}_{it} = 0 \), replace it by \( \hat{e}_{it} := u_{it} \hat{\sigma}_{u,i} \), where \( u_{it} \sim N(0, 1) \).

All four methods assume that the returns data are covariance stationary. Methods SM1 and SM2 are non-parametric, while method SM3 and SM4 calibrate the first two moments to the estimated errors associated with the observed data. Methods SM1 and SM3 are better suited for homogeneous data and since \( \hat{e}_{it} \) are sampled from the stacked up vector of estimated errors. Methods SM2 and SM4 are better suited for heterogeneous data as they sample from the errors of the corresponding series. Methods SM1 and SM2 do not make distributional assumptions about the errors. As many financial time series tend to have fat tails and skewed distributions, other parametric distributions can be used in place of the normal distribution in SM3 and SM4. The four sampling schemes do not account for cross-correlation in the errors, which implicitly shrinks \( \Sigma_e \) towards zero. This can be desirable when estimating high dimensional covariance matrices.

Methods SM+1, SM+2, SM+3, SM+4 are likewise defined with \( \tilde{e}_{it} \) replaced by \( \hat{e}_{it}^{+} = \tilde{X}_{it} - \tilde{F}_{i}^{+} \tilde{\Lambda}_{i}^{+} \), where \( \tilde{F}^{+} \) and \( \tilde{\Lambda}^{+} \) are obtained from the principal components of the imputed data \( \tilde{X} \). In principle, we can also compute a SF estimator \( \tilde{X}(j) = \tilde{\Lambda} \tilde{\Sigma}_{e} \tilde{\Lambda}' + \frac{1}{T} \sum_{s=1}^{S} \tilde{\Psi}_{e}(s,j) \) with \( \tilde{\Psi}_{e}(s) = \frac{1}{T} \sum_{t=1}^{T} \tilde{e}_{it}^{2}(s,j) \), but there seems no advantage over SFA which already gives a good estimate of the variance of \( e_{it} \) when \( T_{oi} \) is sufficiently large. This is indeed the case in simulations, and hence not considered.

Though injecting noise to imputed values is not new, overlaying the noise to \( \tilde{X} \) imputed using SM appears to be new. The resampling involves bootstrap draws. Bootstrapping large dimensional matrices is not a trivial exercise. Our set up is different, as we only need to bootstrap the noise corresponding to the missing data, which is much smaller in dimension. However, we have to pre-estimate the factors and the loadings, and bootstrapping the factor estimates is also not a trivial

\(^3\)If the variances and correlations change over time, the covariance matrix can be calculated by estimating GARCH models, for example.
problem as seen in Goncalves and Perron (2020). The proof of consistency of this estimator is thus quite delicate and is beyond the scope of this analysis. However, we can use simulations to evaluate if the idea holds promise to be worthy of further investigation.

5 Simulations

In this section, we consider four experiments: the first and second assess the accuracy of the asymptotic approximations for \( \tilde{C}_{it} \) and \( \tilde{C}_{it}^+ \) given in Lemma 2 and Proposition 1. The third evaluates the residual overlay procedures assuming a strict factor structure, while the fourth uses observed SP 500 returns as complete data to mimic an approximate factor structure.

5.1 Finite Sample Properties for TP

The first experiment compares the performance of TP with and without re-estimation. The design of the monte-carlo is identical to the one used in Bai and Ng (2021). Data are generated from \( F \sim N(0, D_r) \) and \( \Lambda \sim N(0, D_r) \) with \( r = 2 \), the diagonal entries in \( D_r \) are equally spaced between 1 and \( 1/r \), and \( e_{it} \sim N(0,1) \). For each replication, the error in estimating \( C_{it} \) is computed for four locations of \((i, t)\). As benchmark, we consider the infeasible case of complete data labeled complete. Also reported are results for the EM algorithm in Stock and Watson (2016) that uses the estimates from the balanced panel as initial values. The algorithm repeatedly regresses \( X \) on \( F \) and then \( X \) on \( \Lambda \) till convergence, but the converged estimates may not be mutually orthogonal. We consider three versions of each estimator: one applied to the raw data \( X \), one to the demeaned data, one to the standardized data. These are labeled TP2, TP1, TP0 in the tables reported. The means and standard deviations are computed using the observations available for each series.

Table 1 reports the root-mean-square-error for four chosen \((i, t)\) pairs, one in each of the four blocks: tall is \( T \times N_o \), wide is \( T_o \times N \), bal is \( T_o \times N_o \), and miss is \( (T - T_o) \times (N - N_o) \). With \( N = T = 200 \), results are reported for four configurations of missingness. Evidently, the error depends on observability of \( X_{it} \). The estimation error is largest if \( X_{it} \) is in the miss block and smallest when \( X_{it} \) is in the bal block. For a given block, the estimator errors are smaller when the factors are re-estimated from \( \tilde{X} \). This is consistent with the theory. Results for \((N, T) = (300, 500)\) and \((500, 300)\) are similar.

In the second experiment, we generate data from a model with two factors to assess the adequacy of the asymptotic approximations. Two configurations of \((T, N)\) are considered: \((300, 500)\) and \((500, 300)\) with \( T_o = .4T \) and \( N_o = .6N \). In both cases, about 15% of the observations are missing. The factors and the loadings are drawn from the standard normal distribution once and held fixed. In each of the 5000 replications, a new batch of idiosyncratic errors are drawn from the normal
distribution, so $X$ varies across replications but the locations of the missing values do not change. We then evaluate estimates of $C_{it}$ at four different $(i,t)$ chosen in the neighborhood of $N_o, T_o$ so that they come from four blocks as defined above.

Row 1 of Table 2 reports the mean estimate of $C_{it}$ for (i) when all data are observed, (ii) $TWO$ (no re-estimation), (ii) $TW+$ (re-estimation), (iv) $TP0$, and (v) $TP+$. All five estimates are close to the true values, showing that $TWO$ and $TP0$ are consistent without iteration. The second row, which gives the standard deviation of the estimates in the monte-carlo, shows that the $TW+$ and $TP+$ estimates are slightly less variable than $TW$ and $TP$, showing efficiency gains. The third row gives the mean of the estimated asymptotic standard errors which are quite similar to row two, showing that the asymptotic approximation is quite accurate. The next two rows labeled $q_{05}$ and $q_{95}$ present the 5 and 95 percentage points of the empirical distribution of the standardized estimates. They are quite close to the quantiles from the normal distribution of $\pm 1.64$. The last row shows that coverage is generally satisfactory and reinforces the adequacy of the asymptotic normal approximation. In summary, the proposed TP estimates are already consistent but the convergence rate of $C_{it}$ depends on the position of all $(i,t)$ as given in Proposition 1. The updated estimates make use of additional information and have improved statistical properties.

5.2 Results for Covariance Estimation

Having shown that the properties of $\tilde{X}$ are satisfactory, we proceed to evaluate the covariance estimates using economically meaningful benchmarks. From a $T \times N^*$ panel of complete data $X^*$ treated as stock returns, we first calculate the “true” $N^* \times N^*$ covariance matrix $\Sigma_{X^*}$. Using the $N^* \times 1$ vector of portfolio weights defined from $\Sigma_{X^*}$, we compute $r^*_{pt} = R^*_t w^*_p$ and treat it as the “true” portfolio return at time $t$. We then randomly select $N$ stocks and assume missing values in the south-west block of the returns matrix $R^*$. Hence the data matrix for analysis is of dimension $(T, N)$.

For each method, the (in-sample) bias and root-mean-squared error are computed for the following performance measures over $B = 1000$ replications.

i. PVOL: Portfolio volatility defined as $\text{RISKP} = \sqrt{\frac{1}{T} \sum_{t=1}^{T} (r^*_{pt} - \bar{r}^*_{p})^2}$.

ii $\text{PVAR}_\alpha$: Portfolio value-at-risk at confidence level $\alpha$ defined as $\text{PR}(r_p < -\text{VaR}_\alpha) = 1 - \alpha$.

iii. CALL options price assuming that the current and strike price of each security are one-dollar, a risk-free rate of 2%, and time to maturity of one year.

iv VAR: the variance of returns
v COVAR: the covariance of returns.

Of these measures, the first two are based on equally weighted portfolios while the last three are based on returns data. Portfolio volatility is often used as a risk measure when the benchmark is cash. Portfolio value-at-risk is used as a measure of downside risk.

For CALL, we price plain vanilla European call option prices written on SP500 stocks calculated using Black-Scholes formula. The call price, variance, and covariance measures are calculated only for returns with incomplete data. Since we have \( N_m \) series with missing values, we compute \( N_m \) call prices and variances, and \( N_o N_m + (N_m - 1)/2 \) covariances at each iteration.

The following notation is used Tables 3 to 5.

- SM (sample moments-based) covariance estimators:
  - sm0: single imputation.
  - sm+0: single imputation followed by one re-estimation.
  - smj: double imputation after sm0 using overlay method \( j = 1, \ldots, 4 \).
  - sm+j: double imputation after sm+ using overlay method \( j = 1, \ldots, 4 \).

- factor based covariance estimators:SF and SF+

**Strict Factor Model:** \( X^* \) is simulated from a strict factor model:

\[
X^*_{it} = \lambda'_i F_t + e_{it}, \quad i = 1, ..., N^*, \quad t = 1, ..., T
\]

\[
F_t \sim iid \ N(0, \sigma_F^2 I_r), \quad \lambda_i \sim iid \ N(0, \sigma^2 \Lambda_r)
\]

\[
e_{it} \sim iid \ N \left( 0, 1 - \frac{R^2_i}{R^2_t} \sum_{q=1}^{r} \lambda^2_{iq} \sigma^2_F \right)
\]

where \( I_r \) is the \( r \times r \) identity matrix, \( \sigma^2_F \) and \( \sigma^2 \Lambda \) are the variances of each loading and factor, respectively, and \( R^2_i \) is the percentage contribution of the systematic component to total variance (i.e. coefficient of determination) for series \( i \). Note that setting the variance of \( e_{it} \) in the way above guarantees that the coefficient of determination of each series is exactly equal to \( R^2_i \). Since \( r \) is assumed known, we set \( r = 5 \) without loss of generality. We set \( R^2_i \) to 0.6, \( \sigma^2 \Lambda \) to 1, and \( \sigma^2_F \) to 0.035, respectively. These give average volatility of 9.6%, similar to monthly SP 500 returns of 9.4%. The DGP abstracts from time varying loadings because the covariance estimators and will be affected by omitted time variation in the same way. Here, we focus on estimating the covariance of \( X^* \).

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*Portfolio VaR is generally quoted as a positive number (ie, as a loss).*
Table 3 reports results over 1000 replications for \((T, N) = (339, 100)\) with 15\% of the values missing (60\% of rows and 40\% of columns). Whether we use \(TW\) or \(TP\) seems to make little difference for \(\text{SM}_{ij}\) estimation. This can be due to the fact that either way, the missing idiosyncratic errors are set to zero and does not make a big difference for covariance estimation. Re-estimation is, however, beneficial for \(\text{SM}\) estimation especially when the first step is based on \(TW\). This makes sense because \(TW\) uses a more restricted sample in the first step estimation and stands to gain more from re-estimation. Of the four methods, \(\text{SM}+2\) and \(\text{SM}+4\) which resample from the non-zero \(\tilde{e}_{it}\) of series \(i\) alone have smaller bias and root-mean-squared error.

Turning to the factor-based estimators, since the idiosyncratic errors are mutually uncorrelated by design, we had expected \(\text{SFA}\) and \(\text{SFA+}\) which impose the strict factor structure to give better estimates than the \(\text{SM}\) counterparts. While \(\text{SFA+}\) is comparable to \(\text{SM}2\) and \(\text{SM}4\), it is inferior to \(\text{SM}+2\) and \(\text{SM}+4\). This can be due to the fact that our implementation of \(\text{SM}\) is an average of \(S\) imputed covariances which has a noise reducing property. It is also possible that the \(\text{SM}\) estimators impose sparsity through univariate re-sampling, and as mentioned above, residual cross-section dependence can only come from the non-zero \(\tilde{e}_{it}\) associated with the non-missing data. As these are estimates of errors that are uncorrelated by design, the \(\text{SM}\) estimators should return sample covariances that are approximately diagonal.

As robustness check, the top panel of Table 4 presents results for 30\% missing (80\% of rows and 60\% of columns). The relative performance of the estimation methods are similar across levels of missingness, and as expected, estimation accuracy of a given model is higher when the missingness is lower. The results so far are presented for \(N < T\). Results for \(T = 200, N = 250\) are shown in the bottom panel Table 4. Note that with this design, each \(\tilde{\Sigma}_{X}(s, j)\) is verified to be singular. However, the averaged estimator \(\tilde{\Sigma}_{X}(j)\) is not. Similar to results in Table 3, \(\text{SM}+2\) and \(\text{SM}+4\) perform well and often better than \(\text{SFA+}\).

**Monte Carlo Calibrated to SP500 Returns**  
The final exercise is a Monte-Carlo that takes \(X^*\) to be SP500 monthly returns between 1990 and 2018 to capture the approximate factor structure. The first factor in \(X^*\) of dimension \(348 \times 339\) explains over 26.2\% of the variations, the second explains 4.1\%. The next three factors explain 3.8, 2.7, and 2.1\% of the variation, respectively. As in the previous monte carlo, we random select \(N = 100\) stocks in each replication and set some values in the south-west block to missing.

The results based on 1000 replications are reported in Table 5. Imputation leads to some reduction in bias and variance, but the errors are significantly smaller than no adjustment upon comparing \(\text{SM}1-4\) with \(\text{SM}0\). Though double imputation is generally better than single imputation, the precise gain depends on the risk measure and the estimation sample. As in the strict factor case,
TW and TP give similar risk-performance errors. Also as in the strict factor case, SFA+ is comparable to SM+1 and SM+3, but has larger errors than SM+2 and SM+4. However, improvements from re-estimation both in terms of bias and RMSE are larger for this DGP that mimics the approximate factor structure.

Overall, we find that single imputation is strongly preferred over no imputation but is inferior to double imputation. Ultimately, the appropriate method depends on the data generating process and we offer several alternatives worthy of consideration. Averaging the covariances of imputed data overlaid with resampled errors has two desirable effects that we had not anticipated: it reduces noise, and the averaged estimator is full rank. This interesting finding and the role of resampling in covariance estimation is an interesting problem that warrants further investigation.

6 Conclusion

This paper provides three sets of results. The first is a TP algorithm that can consistently estimate the entire low rank matrix without iteration and a distribution theory for the estimates is provided. The second result makes precise the conditions under which factors estimated from incomplete data can be treated as known in factor-augmented regressions. The third pertains to estimation of covariances from incomplete data. We consider different schemes to compensate for an omitted error in the level estimates. Implications for using imputed factors in augmented regressions are also discussed.
Table 1: Root Mean-Squared-Error of $\tilde{C}_{it}$ at four $(i, t)$ pairs

| case (i, t) in | (0) | (1) | (2) | (0) | (1) | (2) | (0) | (1) | (2) |
|---------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1 tall (200, 200) | 0.26 | 0.25 | 0.23 | 0.29 | 0.29 | 0.29 | 0.27 | 0.27 | 0.25 |
| 1 wide (120, 200) | 0.26 | 0.26 | 0.24 | 0.32 | 0.31 | 0.31 | 0.29 | 0.29 | 0.29 |
| 1 bal (200, 120) | 0.26 | 0.25 | 0.22 | 0.29 | 0.29 | 0.29 | 0.26 | 0.25 | 0.23 |
| 1 miss (120, 120) | 0.26 | 0.26 | 0.23 | 0.32 | 0.31 | 0.31 | 0.31 | 0.31 | 0.27 |
| 2 tall (80, 80) | 0.26 | 0.25 | 0.23 | 0.30 | 0.29 | 0.29 | 0.29 | 0.28 | 0.25 |
| 2 wide (200, 200) | 0.26 | 0.26 | 0.23 | 0.38 | 0.38 | 0.37 | 0.35 | 0.35 | 0.30 |
| 2 bal (120, 200) | 0.26 | 0.25 | 0.23 | 0.29 | 0.29 | 0.29 | 0.27 | 0.26 | 0.23 |
| 2 miss (200, 60) | 0.26 | 0.26 | 0.23 | 0.38 | 0.37 | 0.37 | 0.37 | 0.36 | 0.32 |
| 3 tall (120, 60) | 0.26 | 0.25 | 0.22 | 0.36 | 0.35 | 0.35 | 0.32 | 0.31 | 0.28 |
| 3 wide (80, 140) | 0.26 | 0.25 | 0.23 | 0.39 | 0.38 | 0.37 | 0.29 | 0.29 | 0.26 |
| 3 bal (200, 200) | 0.26 | 0.26 | 0.23 | 0.38 | 0.36 | 0.36 | 0.27 | 0.26 | 0.24 |
| 3 miss (60, 200) | 0.26 | 0.26 | 0.23 | 0.39 | 0.38 | 0.37 | 0.35 | 0.34 | 0.31 |
| 4 tall (200, 120) | 0.26 | 0.25 | 0.23 | 0.38 | 0.36 | 0.36 | 0.35 | 0.34 | 0.32 |
| 4 wide (60, 120) | 0.26 | 0.25 | 0.23 | 0.46 | 0.45 | 0.44 | 0.38 | 0.37 | 0.32 |
| 4 bal (140, 80) | 0.26 | 0.26 | 0.23 | 0.37 | 0.36 | 0.36 | 0.30 | 0.28 | 0.25 |
| 4 miss (200, 200) | 0.26 | 0.26 | 0.23 | 0.45 | 0.44 | 0.42 | 0.42 | 0.42 | 0.37 |

Note: DGP: The $T \times N$ data matrix $X$ is generated by $X = FA' + e$, $F \sim N(0, D_r)$, $\Lambda \sim (0, D_r)$ with $r = 2$, $e \sim N(0, 2.5)$, and $\text{diag}(D_r) = [1; .5]$. $(N, T)$ is the number of columns and rows in the block. Four configurations of missing data are considered. Case 1 has the smallest miss block and case 4 has the largest. Reported are the root-mean-squared error over 5000 replications.
|                  | APC     | TW      | TW+     | TP      | TP+     | APC     | TW      | TW+     | TP      | TP+     |
|------------------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| \(T, N\) = (300, 500), \((T_o, N_o) = (120, 300)\) |         |         |         |         |         |         |         |         |         |         |
| Mean             | -1.066  | -1.067  | -1.066  | -1.066  | -1.066  | 1.017   | 1.016   | 1.018   | 1.016   | 1.018   |
| SD               | 0.157   | 0.160   | 0.157   | 0.160   | 0.157   | 0.090   | 0.103   | 0.091   | 0.103   | 0.090   |
| ASE              | 0.145   | 0.155   | 0.132   | 0.154   | 0.145   | 0.079   | 0.123   | 0.077   | 0.119   | 0.079   |
| q0.05            | -1.769  | -1.692  | -1.951  | -1.698  | -1.776  | -1.965  | -1.505  | -1.467  | -1.960  | -1.660  |
| q0.95            | 1.797   | 1.715   | 1.978   | 1.721   | 1.784   | 1.827   | 1.297   | 1.382   | 1.843   |         |
| Coverage         | 0.930   | 0.941   | 0.898   | 0.940   | 0.930   | 0.912   | 0.981   | 0.903   | 0.980   | 0.911   |
| \(C_{115,200} = -1.066\) |         |         |         |         |         |         |         |         |         |         |
|                  | APC     | TW      | TW+     | TP      | TP+     | APC     | TW      | TW+     | TP      | TP+     |
| \(T, N\) = (500, 300), \((T_o, N_o) = (300, 1200)\) |         |         |         |         |         |         |         |         |         |         |
| Mean             | 1.083   | 1.083   | 1.083   | 1.083   | 1.083   | -1.429  | -1.429  | -1.428  | -1.429  | -1.428  |
| SD               | 0.101   | 0.106   | 0.104   | 0.106   | 0.104   | 0.100   | 0.112   | 0.107   | 0.112   | 0.107   |
| ASE              | 0.079   | 0.089   | 0.084   | 0.089   | 0.077   | 0.088   | 0.113   | 0.093   | 0.113   | 0.080   |
| q0.05            | -2.157  | -2.070  | -2.122  | -2.065  | -2.306  | -1.786  | -1.565  | -1.567  | -2.102  |         |
| q0.95            | 2.048   | 1.887   | 1.987   | 1.889   | 2.178   | 1.936   | 1.719   | 1.725   | 2.337   |         |
| Coverage         | 0.873   | 0.900   | 0.883   | 0.899   | 0.852   | 0.919   | 0.952   | 0.907   | 0.952   | 0.858   |
| \(C_{125,200} = 1.084\) |         |         |         |         |         |         |         |         |         |         |
|                  | APC     | TW      | TW+     | TP      | TP+     | APC     | TW      | TW+     | TP      | TP+     |
| \(C_{190,165} = 1.017\) |         |         |         |         |         |         |         |         |         |         |
| \(C_{115,325} = -3.260\) |         |         |         |         |         |         |         |         |         |         |
| \(C_{195,165} = 0.923\) |         |         |         |         |         |         |         |         |         |         |
| \(C_{140,325} = 2.451\) |         |         |         |         |         |         |         |         |         |         |
| \(C_{220,205} = -1.115\) |         |         |         |         |         |         |         |         |         |         |
| Mean             | -3.255  | -3.245  | -3.249  | -3.242  | -3.252  | 0.922   | 0.917   | 0.920   | 0.917   | 0.921   |
| SD               | 0.174   | 0.257   | 0.212   | 0.216   | 0.207   | 0.092   | 0.129   | 0.106   | 0.115   | 0.104   |
| ASE              | 0.162   | 0.289   | 0.175   | 0.220   | 0.190   | 0.080   | 0.107   | 0.089   | 0.125   | 0.085   |
| q0.05            | -1.750  | -1.407  | -1.960  | -1.525  | -1.789  | -1.953  | -2.108  | -2.070  | -1.601  | -2.109  |
| q0.95            | 1.832   | 1.538   | 2.066   | 1.718   | 1.828   | 1.843   | 1.826   | 1.835   | 1.414   | 1.889   |
| Coverage         | 0.932   | 0.972   | 0.889   | 0.951   | 0.922   | 0.910   | 0.895   | 0.899   | 0.967   | 0.895   |
| \(C_{140,325} = 2.451\) |         |         |         |         |         |         |         |         |         |         |
| \(C_{220,205} = -1.115\) |         |         |         |         |         |         |         |         |         |         |
| Mean             | 2.448   | 2.442   | 2.445   | 2.440   | 2.444   | -1.120  | -1.110  | -1.115  | -1.112  | -1.116  |
| SD               | 0.145   | 0.210   | 0.187   | 0.188   | 0.183   | 0.138   | 0.214   | 0.184   | 0.179   | 0.177   |
| ASE              | 0.129   | 0.225   | 0.153   | 0.173   | 0.135   | 0.127   | 0.180   | 0.133   | 0.158   | 0.148   |
| q0.05            | -1.874  | -1.653  | -2.120  | -1.911  | -2.349  | -1.847  | -1.884  | -2.209  | -1.841  | -1.984  |
| q0.95            | 1.837   | 1.452   | 1.951   | 1.692   | 2.181   | 1.772   | 2.032   | 2.322   | 1.901   | 1.971   |
| Coverage         | 0.916   | 0.963   | 0.885   | 0.922   | 0.847   | 0.922   | 0.899   | 0.847   | 0.915   | 0.897   |

Data are generated from a model with two factors. The factors and the loadings are drawn from the normal distribution once and hence fixed. In each of the 5000 replications, new idiosyncratic errors are resampled. The location of the missing values are fixed throughout. The APC column is the infeasible case when the data are completely observed. The columns TW and TP are one step estimators. A '+' indicates one re-estimation.
|       | Bias         | RMSE        |
|-------|--------------|-------------|
|       | pvol pVaR call var covar | pvol pVaR call var covar |
| TW    |              |             |
| sm0   | -0.037 -0.078 -1.105 -0.167 0.000 | 0.041 0.088 1.231 0.212 0.031 |
| sm1   | -0.003 -0.011 0.068 -0.019 0.000 | 0.017 0.040 0.815 0.131 0.031 |
| sm2   | 0.003 -0.001 0.038 0.005 0.000 | 0.016 0.037 0.388 0.072 0.031 |
| sm3   | -0.003 -0.011 0.068 -0.019 0.000 | 0.017 0.040 0.815 0.131 0.031 |
| sm4   | 0.003 -0.000 0.044 0.006 0.000 | 0.016 0.037 0.388 0.072 0.031 |
| sfa   | 0.003 -0.000 0.100 0.015 0.000 | 0.023 0.049 0.535 0.099 0.042 |
| sm+1  | -0.003 -0.009 0.080 -0.015 0.000 | 0.016 0.034 0.748 0.116 0.029 |
| sm+2  | -0.003 -0.008 -0.070 -0.013 0.000 | 0.015 0.033 0.369 0.069 0.029 |
| sm+3  | -0.003 -0.009 0.080 -0.015 0.000 | 0.016 0.034 0.748 0.116 0.029 |
| sm+4  | -0.003 -0.008 -0.066 -0.012 0.000 | 0.015 0.033 0.368 0.069 0.029 |
| sf+a  | -0.009 -0.020 -0.135 -0.025 -0.000 | 0.019 0.041 0.427 0.083 0.030 |
| TP    |              |             |
| sm0   | -0.038 -0.079 -1.129 -0.170 0.000 | 0.042 0.088 1.239 0.212 0.028 |
| sm1   | -0.004 -0.013 0.040 -0.024 0.000 | 0.016 0.037 0.795 0.128 0.028 |
| sm2   | 0.001 -0.002 0.003 0.000 0.000 | 0.015 0.035 0.349 0.064 0.028 |
| sm3   | -0.004 -0.013 0.040 -0.024 0.000 | 0.016 0.037 0.795 0.128 0.028 |
| sm4   | 0.001 -0.002 0.008 0.001 0.000 | 0.015 0.035 0.348 0.064 0.028 |
| sfa   | -0.001 -0.007 0.011 0.001 0.000 | 0.017 0.039 0.348 0.064 0.029 |
| sm+0  | -0.036 -0.074 -1.053 -0.158 0.000 | 0.040 0.082 1.152 0.194 0.028 |
| sm+1  | -0.004 -0.010 0.074 -0.016 0.000 | 0.015 0.033 0.741 0.114 0.028 |
| sm+2  | -0.003 -0.009 -0.077 -0.014 0.000 | 0.015 0.033 0.359 0.067 0.028 |
| sm+3  | -0.004 -0.010 0.074 -0.016 0.000 | 0.015 0.033 0.740 0.114 0.028 |
| sm+4  | -0.003 -0.009 -0.073 -0.013 0.000 | 0.015 0.033 0.358 0.067 0.028 |
| sf+a  | -0.010 -0.022 -0.152 -0.027 -0.000 | 0.018 0.040 0.398 0.077 0.028 |
Table 4: Additional TP Results: Strict Factor Model

|       | Bias  | RMSE  |
|-------|-------|-------|
|       | pvol  | pVaR  | call | var  | covar | pvol  | pVaR  | call | var  | covar  |
|       |       |       |      |      |       |       |       |      |      |       |
| 30% Missing, $(T, N) = (339, 100)$  |       |       |      |      |       |       |       |      |      |       |
| sm0   | -0.082| -0.163| -1.601| -0.236| 0.000 | 0.086| 0.174| 1.739| 0.289| 0.040 |
| sm1   | -0.010| -0.023| 0.003| -0.037| 0.000 | 0.026| 0.060| 1.045| 0.170| 0.040 |
| sm2   | 0.003 | 0.003 | 0.003| 0.001| 0.000 | 0.023| 0.055| 0.447| 0.081| 0.040 |
| sm3   | -0.010| -0.023| 0.003| -0.037| 0.000 | 0.026| 0.060| 1.045| 0.170| 0.040 |
| sm4   | 0.003 | 0.003 | 0.011| 0.002| 0.000 | 0.023| 0.055| 0.447| 0.081| 0.040 |
| sfa   | 0.004 | 0.005 | 0.016| 0.003| 0.000 | 0.026| 0.060| 0.447| 0.081| 0.042 |
| sm+0  | -0.080| -0.157| -1.516| -0.223| 0.000 | 0.084| 0.167| 1.642| 0.270| 0.038 |
| sm+1  | -0.013| -0.026| -0.015| -0.036| -0.000 | 0.026| 0.054| 0.970| 0.157| 0.038 |
| sm+2  | -0.010| -0.020| -0.167| -0.028| 0.000 | 0.024| 0.050| 0.478| 0.088| 0.038 |
| sm+3  | -0.013| -0.026| -0.015| -0.036| -0.000 | 0.026| 0.054| 0.970| 0.157| 0.038 |
| sm+4  | -0.009| -0.019| -0.159| -0.027| 0.000 | 0.024| 0.050| 0.475| 0.088| 0.038 |
| sf+a  | -0.016| -0.032| -0.259| -0.044| 0.000 | 0.028| 0.058| 0.536| 0.101| 0.039 |
| 15% Missing, $(T, N) = (200, 250)$ |       |       |      |      |       |       |       |      |      |       |
| sm0   | -0.015| -0.034| -0.607| -0.092| 0.000 | 0.018| 0.041| 0.906| 0.153| 0.029 |
| sm1   | -0.001| -0.006| 0.092| -0.005| 0.000 | 0.009| 0.022| 0.780| 0.122| 0.029 |
| sm2   | 0.000| -0.005| -0.007| -0.001| 0.000 | 0.009| 0.021| 0.349| 0.061| 0.029 |
| sm3   | -0.001| -0.006| 0.093| -0.005| 0.000 | 0.009| 0.022| 0.780| 0.122| 0.029 |
| sm4   | 0.000| -0.005| -0.002| -0.001| 0.000 | 0.009| 0.021| 0.349| 0.061| 0.029 |
| sfa   | -0.001| -0.007| 0.001| -0.000| 0.000 | 0.009| 0.020| 0.349| 0.061| 0.028 |
| sm+0  | -0.015| -0.032| -0.589| -0.089| -0.000 | 0.018| 0.038| 0.881| 0.148| 0.029 |
| sm+1  | -0.001| -0.005| 0.102| -0.003| -0.000 | 0.009| 0.020| 0.762| 0.118| 0.029 |
| sm+2  | -0.001| -0.005| -0.023| -0.004| -0.000 | 0.009| 0.020| 0.349| 0.061| 0.029 |
| sm+3  | -0.001| -0.005| 0.102| -0.003| -0.000 | 0.009| 0.020| 0.762| 0.118| 0.029 |
| sm+4  | -0.001| -0.005| -0.018| -0.003| -0.000 | 0.009| 0.020| 0.349| 0.061| 0.029 |
| sm+d  | -0.002| -0.008| -0.033| -0.006| -0.000 | 0.009| 0.019| 0.352| 0.062| 0.028 |

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|       |       |       |     |     |       |       |     |     |     |     |
|-------|-------|-------|-----|-----|-------|-------|-----|-----|-----|-----|
|       |       |       |     |     |       |       |     |     |     |     |
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|       |       |       |     |     |       |       |     |     |     |     |
|       |       |       |     |     |       |       |     |     |     |     |
|       |       |       |     |     |       |       |     |     |     |     |
| sm0   | -0.006 | -0.025 | -1.399 | -0.211 | 0.001 | 0.074 | 0.147 | 2.131 | 0.396 | 0.068 |
| sm1   | 0.004  | -0.006 | 0.392  | 0.012  | 0.001 | 0.073 | 0.145 | 1.794 | 0.335 | 0.068 |
| sm2   | 0.007  | -0.000 | 0.532  | 0.080  | 0.001 | 0.073 | 0.144 | 1.647 | 0.333 | 0.068 |
| sm3   | 0.004  | -0.006 | 0.392  | 0.012  | 0.001 | 0.073 | 0.145 | 1.794 | 0.335 | 0.068 |
| sm4   | 0.007  | -0.000 | 0.540  | 0.081  | 0.001 | 0.073 | 0.144 | 1.650 | 0.334 | 0.068 |
| sfa   | 0.029  | 0.043  | 0.788  | 0.122  | -0.000 | 0.116 | 0.225 | 2.049 | 0.410 | 0.106 |
| sm+   | 0.010  | 0.026  | -1.393 | -0.206 | 0.003 | 0.059 | 0.117 | 2.050 | 0.378 | 0.062 |
| sm+1  | 0.018  | 0.043  | 0.332  | 0.008  | 0.003 | 0.061 | 0.122 | 1.673 | 0.317 | 0.062 |
| sm+2  | 0.019  | 0.045  | 0.266  | 0.032  | 0.003 | 0.061 | 0.122 | 1.438 | 0.289 | 0.062 |
| sm+3  | 0.018  | 0.043  | 0.333  | 0.008  | 0.003 | 0.061 | 0.122 | 1.673 | 0.317 | 0.062 |
| sm+4  | 0.019  | 0.045  | 0.273  | 0.033  | 0.003 | 0.061 | 0.122 | 1.440 | 0.289 | 0.062 |
| sfa+  | 0.065  | 0.135  | 0.237  | 0.022  | 0.006 | 0.098 | 0.197 | 1.540 | 0.304 | 0.082 |

|       |       |       |     |     |       |       |-----|-----|-----|-----|
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|       |       |       |     |     |       |       |     |     |     |     |
|       |       |       |     |     |       |       |     |     |     |     |
|       |       |       |     |     |       |       |     |     |     |     |
|       |       |       |     |     |       |       |     |     |     |     |
|       |       |       |     |     |       |       |     |     |     |     |
|       |       |       |     |     |       |       |     |     |     |     |
|       |       |       |     |     |       |       |     |     |     |     |
| sm0   | -0.019 | -0.047 | -1.553 | -0.220 | -0.001 | 0.049 | 0.100 | 2.076 | 0.380 | 0.052 |
| sm1   | -0.010 | -0.028 | 0.231  | -0.011 | -0.001 | 0.046 | 0.093 | 1.582 | 0.303 | 0.052 |
| sm2   | -0.007 | -0.023 | 0.323  | 0.048  | -0.001 | 0.046 | 0.091 | 1.381 | 0.286 | 0.052 |
| sm3   | -0.010 | -0.028 | 0.231  | -0.011 | -0.001 | 0.046 | 0.093 | 1.582 | 0.303 | 0.052 |
| sm4   | -0.007 | -0.023 | 0.331  | 0.049  | -0.001 | 0.046 | 0.091 | 1.384 | 0.287 | 0.052 |
| sfa   | 0.004  | -0.001 | 0.336  | 0.050  | -0.003 | 0.045 | 0.088 | 1.385 | 0.287 | 0.068 |
| sm+   | 0.003  | 0.015  | -1.458 | -0.213 | 0.002 | 0.044 | 0.088 | 1.998 | 0.363 | 0.053 |
| sm+1  | 0.011  | 0.031  | 0.273  | -0.001 | 0.002 | 0.046 | 0.092 | 1.551 | 0.294 | 0.053 |
| sm+2  | 0.012  | 0.033  | 0.198  | 0.022  | 0.002 | 0.046 | 0.092 | 1.315 | 0.266 | 0.053 |
| sm+3  | 0.011  | 0.031  | 0.273  | -0.001 | 0.002 | 0.046 | 0.092 | 1.551 | 0.294 | 0.053 |
| sm+4  | 0.012  | 0.033  | 0.205  | 0.023  | 0.002 | 0.046 | 0.092 | 1.317 | 0.266 | 0.053 |
| sfa+  | 0.052  | 0.111  | 0.115  | 0.005  | 0.004 | 0.069 | 0.142 | 1.329 | 0.269 | 0.068 |
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Appendix

We observe $X$, but not $F^0$ or $\Lambda^0$. As $F$ and $\Lambda$ are not separately identifiable. The method of asymptotic principal components (APC) uses the normalization $\mathcal{E}_F^F = I_r$ and $\Lambda^t \Lambda$ being diagonal to produce estimates

$$(\tilde{F}, \tilde{\Lambda}) = (\sqrt{T}U_r, \sqrt{N}V_rD_r).$$

For each $t \in [1, T]$ and for each $i \in [1, N]$, $(\tilde{F}_t, \tilde{\Lambda}_i)$ consistently estimate $(F_{it}^0, \Lambda_{it}^0)$ up to a rotation matrices $H$. This rotation matrix is not unique, and for the present analysis, let

$$H = \left( \frac{\Lambda^0 \Lambda^0}{N} \right) \left( \frac{F^0 \tilde{F}}{T} \right) D_r^{-2}$$

denote the rotation matrix for complete data, and let

$$H_{tall} = \left( \frac{\Lambda^0 \Lambda^0}{N_o} \right) \left( \frac{F^0 \tilde{F}_{tall}}{T} \right) D_{r,tall}^{-2}$$

denote the rotation matrix for the tall block of data, where $\Lambda^0_{it}$ is the $N_o \times r$ matrix consisting of factor loadings of $\Lambda^0_k$ for all $k \in i \cap J^s$, and $\tilde{F}_{tall}$ is $T \times r$, and $D_{r,tall}^2$ is $r \times r$.

Proof of Lemma 2 2

By construction, the first stage estimated factors are obtained from the observed $T \times N_o$ data matrix, giving rise to $\tilde{F}_{tall}$. For $(i, t)$ in the tall block, the regression method and the principal components estimator are the same as the complete data case explained in the text. So Lemma 2, and ii(a) follow from Lemma 1. We consider the remaining claims.

For each $i > N_o$ of the reorganized data (or $i \not\in i \cap J^s$ of the original data), we observe $T_{oi}$ observations. Here for notational simplicity, we assume the first consecutive $T_{oi}$ observations are available. The true model is $X_{it} = F_{it}^0 \Lambda_{it}^0 + e_{it}$. To estimate the factor loadings $\Lambda_{it}^0$, consider the regression

$$X_{it} = \tilde{F}_i^a \Lambda_{it} + a_{it}, \quad t = 1, 2, ..., T_{oi}$$

where $\tilde{F}_i^a$ is the $t$th row of $\tilde{F}_{tall}$, $a_{it}$ is an error term. Then, OLS gives

$$\tilde{\Lambda}_{it} = (\tilde{F}_{oi}^a \tilde{F}_{oi})^{-1} \tilde{F}_{oi}^a X_{oi}$$

where $X_{oi}$ is $T_{oi} \times 1$ vector such that $X_{oi} = F^0_{oi} \Lambda_{oi}^0 + e_{oi}$, and $F_{oi}^0$ is $T_{oi} \times r$, and $e_{oi}$ is $T_{oi} \times 1$; $\tilde{F}_{oi}$ stacks $\tilde{F}_i$. It follows that

$$\tilde{\Lambda}_{it} = (\tilde{F}_{oi}^a \tilde{F}_{oi})^{-1} \tilde{F}_{oi}^a F_{oi}^0 \Lambda_{oi}^0 + (\tilde{F}_{oi}^a \tilde{F}_{oi})^{-1} \tilde{F}_{oi}^a e_{oi}.$$  

Let $G_i = (\tilde{F}_{oi}^a \tilde{F}_{oi})^{-1} \tilde{F}_{oi}^a F_{oi}^0$ and rewrite $F_{oi} = \tilde{F}_{oi}H_{tall}^{-1} - [\tilde{F}_{oi} - F_{oi}H_{tall}]H_{tall}^{-1}$, so that

$$G_i = H_{tall}^{-1} - (\tilde{F}_{oi}^a \tilde{F}_{oi})^{-1} \tilde{F}_{oi}^a [\tilde{F}_{oi} - F_{oi}H_{tall}]H_{tall}^{-1}.$$  

Since $(\tilde{F}_{oi}^a \tilde{F}_{oi}/T_{oi})^{-1} = O_p(1)$, and $\frac{1}{T_{oi}} \tilde{F}_{oi}^a [\tilde{F}_{oi} - F_{oi}H_{tall}] = O_P(\delta_{N_o,T_{oi}}^{-2})$, we obtain

$$G_i = H_{tall}^{-1} + O_P(\delta_{N_o,T_{oi}}^{-2}).$$
Thus,

$$\tilde{\Lambda}_i - H_{\text{tall}}^{-1}\Lambda_0^i = (\tilde{F}_{o_i}^t \tilde{F}_{o_i})^{-1} \tilde{F}_{o_i} e_{o_i} + O_p(1/\delta_{N_o T_{o_i}}^2)$$

$$= H_{\text{tall}}^{-1}(F_{o_i}^{0'}F_{o_i}^{0})^{-1} F_{o_i}^{0'} e_{o_i} + O_p(1/\delta_{N_o T_{o_i}}^2).$$

Multiply by $\sqrt{T_{o_i}}$ and use the limit for $H_{\text{tall}}$ (see Bai and Ng (2021)), we obtain part ii(b) of Lemma 2. Here we use the assumption that $\frac{\sqrt{T_{o_i}}}{N_o} \to 0$.

Consider the estimated common components

$$\tilde{C}_{it} = \tilde{F}_t^t \tilde{\Lambda}_i = \tilde{F}_t^t (\tilde{F}_{o_i}^t \tilde{F}_{o_i})^{-1} \tilde{F}_{o_i}^t F_{o_i}^0 \Lambda_0^i + \tilde{F}_t^t (\tilde{F}_{o_i}^t \tilde{F}_{o_i})^{-1} \tilde{F}_{o_i} e_{o_i}.$$  

Similar to Bai and Ng (2021), we can show, for all missing entries $(i, t)$,

$$\tilde{C}_{it} = C_{it}^0 + \Lambda_0^i (\Lambda_0^0/\Lambda_0^o/\Lambda_0)\sum_{k=1}^{N_o} \Lambda_0^0 \Lambda_0^k \epsilon_{kt} + F_{o_i}^0 (F_{o_i}^0/F_{o_i}^0/T_{o_i})^{-1} \frac{1}{T_{o_i}} \sum_{s=1}^{T_{o_i}} F_{o_i}^0 \epsilon_{is} + O_p(1/\delta_{N_o T_{o_i}}^2).$$

where $\Lambda_0^o$ is $N_o \times r$. Here and for the remainder of the proof, we assume the first $N_o$ units have no missing observations so that $\sum_{k=1}^{N_o} \Lambda_0^0 \Lambda_0^k \epsilon_{kt}$ is identical to $\sum_{k \in J^*} \Lambda_0^0 \Lambda_0^k \epsilon_{kt}$. Let

$$u_{it} = \Lambda_0^o (\Lambda_0^0/\Lambda_0^o/\Lambda_0)\sum_{k=1}^{N_o} \Lambda_0^0 \Lambda_0^k \epsilon_{kt}$$

$$v_{it} = F_{o_i}^0 (F_{o_i}^0/F_{o_i}^0/T_{o_i})^{-1} \frac{1}{T_{o_i}} \sum_{s=1}^{T_{o_i}} F_{o_i}^0 \epsilon_{is}$$

Then

$$\tilde{C}_{it} = C_{it}^0 + u_{it} + v_{it} + O_p(1/\delta_{N_o T_{o_i}}^2).$$

This implies that

$$\left( \frac{1}{N_o} A_{NT} + \frac{1}{T_{o_i}} B_{NT} \right)^{-1/2} (\tilde{C}_{it} - C_{it}^0) \xrightarrow{d} N(0, 1)$$

where $A_{NT} = \Lambda_0^o (\Lambda_0^0/\Lambda_0^o/\Lambda_0)\sum_{k=1}^{N_o} \Lambda_0^0 \Lambda_0^k \epsilon_{kt}$ and $B_{NT} = F_{o_i}^0 (F_{o_i}^0/F_{o_i}^0/T_{o_i})^{-1} \Phi_i (F_{o_i}^0/F_{o_i}^0/T_{o_i})^{-1} F_{o_i}^0$.

This gives part iii of Lemma 2.

### 6.1 Proof of Proposition 1

Consider the principal components estimator of factors and factor loadings based on $\tilde{X}$ (see equations (4a) and (4b)). Let $\tilde{F}^+$ and $\tilde{\Lambda}^+$ denote the PCA estimator so that

$$\frac{1}{NT} \tilde{X} \tilde{X}' \tilde{F}^+ = \tilde{F}^+ \tilde{D}_r$$

with $\tilde{F}^+ \tilde{F}^+ / T = I_r$, and $\tilde{\Lambda} = \frac{1}{NT} \tilde{X} \tilde{X}'; \tilde{D}_r^2$ is a diagonal matrix consisting of the first $r$ largest eigenvalues of $\tilde{X} \tilde{X}^T/(NT)$. Define

$$H^+ = (\Lambda^0 \Lambda_0^0 / N) (F^0 / T) \tilde{D}_r^{-2}$$

Now we assume $T_{o_i} \geq T_o$ for some $T_o$. We also assume $N_{o_i} \geq N_o$, where $T_o$ and $N_o$ satisfy Assumption B. Then Lemma 3 in Bai and Ng (2021) holds. That is
Lemma 5. Suppose that $N_o \geq N_o$ and $T_o \geq T_o$ for all $t$ and $i$. Under Assumptions A and B, we have (a) $\frac{1}{T} \sum_{t=1}^{T} \left\| \tilde{F}_t^+ - H^t F_0^0 \right\|^2 = O_p(\delta_{N_o, T_o}^2)$, and (b) $\text{plim } \tilde{D}_r^2 \rightarrow \mathbb{D}_r^2$ and $\tilde{F}^t F^0 / T \rightarrow Q_r$, where $\mathbb{D}_r$ and $Q_r$ are the same as in complete data.

The proof of this lemma follows the same argument as in Bai and Ng (2021). The details are omitted. We focus on obtaining the asymptotic distribution for the estimated factor and factor loadings, define

$$\tilde{e}_{it} = \begin{cases} e_{it} & \text{if } X_{it} \text{ is observed} \\ u_{it} + v_{it} & \text{if } X_{it} \text{ is missing} \end{cases}$$

Because of the preceding lemma, the asymptotic representation for $\tilde{F}_t^+$ is (see Bai (2003) and Bai and Ng (2021)),

$$\tilde{F}_t^+ - H^t F_0^0 = \tilde{D}^{-2}(\tilde{F}^t F^0 / T) \left( \frac{1}{N} \sum_{i=1}^{N} \Lambda^0_i \tilde{e}_{it} + O_p(1/\delta_{N_o, T_o}^2) \right). \quad (13)$$

For a given $t$, if all individual variables $(X_{1t}, X_{2t}, ..., X_{N_t})$ are observable, then $\tilde{e}_{it} = e_{it}$ for all $i$, and

$$\sqrt{N}(\tilde{F}_t^+ - H^t F_0^0) = \tilde{D}^{-2}(\tilde{F}^t F^0 / T) \left( \frac{1}{N} \sum_{i=1}^{N} \Lambda^0_i e_{it} + O_p(1/\delta_{N_o, T_o}^2) \right) \quad (14)$$

$$\xrightarrow{d} N(0, \mathbb{D}_r^{-2} \omega_r t Q_r^T \mathbb{D}_r^{-2}).$$

For a given $t$, suppose that (the first) $N_{o_t}$ series are available, denoted by $(X_{1t}, X_{2t}, ..., X_{N_{o_t}t})$, with $N_{o_t} \geq N_o$. Then $\tilde{e}_{it} = e_{it}$ for $i \leq N_{o_t}$ and $\tilde{e}_{it} = u_{it} + v_{it}$ for $i > N_{o_t}$, so (13) becomes

$$\tilde{F}_t^+ - H^t F_0^0 = \tilde{D}^{-2}(\tilde{F}^t F^0 / T) \left( \frac{1}{N} \sum_{i=1}^{N_{o_t}} \Lambda^0_i e_{it} + \frac{1}{N} \sum_{i=N_{o_t} + 1}^{N} \Lambda^0_i (u_{it} + v_{it}) \right) + O_p(1/\delta_{N_o, T_o}^2).$$

Note that $\frac{1}{N} \sum_{i=N_{o_t} + 1}^{N} \Lambda^0_i v_{it}$ is negligible (dominated by $\frac{1}{N} \sum_{i=N_{o_t} + 1}^{N} \Lambda^0_i u_{it}$). This follows from the definition of $v_{it}$ in (10),

$$\frac{1}{N} \sum_{i=N_{o_t} + 1}^{N} \Lambda^0_{i,j} v_{it} = O_p(1) \left( \frac{1}{NT_o} \sum_{i=N_{o_t} + 1}^{N} \sum_{s=1}^{T_o} \Lambda^0_{i,j} F^0 e_{is} \right) = O_p(1/\sqrt{NT_o}) = O_p(1/\delta_{N_o, T_o}^2).$$

where $\Lambda^0_{i,j}$ is the $j$th component of $\Lambda^0_i$ ($j = 1, 2, ..., r$).

Let

$$A_t = I_r + \left( \frac{N - N_{o_t}}{N_o} \right) \left( \frac{1}{N - N_{o_t}} \sum_{i=N_{o_t} + 1}^{N} \Lambda^0_i \Lambda^0_0 (A^0_0 A_0^0 / N_o)^{-1} \right).$$

We can rewrite the representation as

$$\tilde{F}_t^+ - H^t F_0^0 = \tilde{D}^{-2}(\tilde{F}^t F^0 / T) \left( \frac{1}{N} \sum_{i=1}^{N_{o_t}} A_t \Lambda^0_i e_{it} + \frac{1}{N} \sum_{i=N_{o_t} + 1}^{N} \Lambda^0_i e_{it} \right) + O_p(1/\delta_{N_o, T_o}^2),$$

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For the special case that \( N_{ot} = N_o \), and \( \left( \frac{1}{N - N_{ot}} \sum_{i=N_{ot}+1}^{N} \Lambda_i^0 \Lambda_i^0 \right) \left( \Lambda_o^0 \Lambda_o^0 / N_o \right)^{-1} \rightarrow I_p \) we have \( A_i \sim N/N_o \), the above representation coincides with the TW presentation in Bai and Ng (2021).

Consider the more general case of \( N_{ot} \) with \( N_{ot} \geq N_o \). Define

\[
B_t^i = \begin{cases} 
\frac{N}{N} \Lambda_i & i \leq N_o \\
\frac{N}{N} I_r & N_o < i \leq N_{ot}
\end{cases}
\]

We can further rewrite the representation as

\[
\tilde{F}^+ - H^+ F^0 = D^{-2}(\tilde{F}^+ F^0 / T) \left( \frac{1}{N_{ot}} \sum_{i=1}^{N_{ot}} B_t^i \Lambda_i^0 e_{it} \right) + O_p(1 / \delta^2_{N_o T_o}).
\]  

(15)

Under Assumption C, \( B_t^i \) is bounded. Then the convergence rate is \( \sqrt{N_{ot}} \), and

\[
\sqrt{N_{ot}}(\tilde{F}^+ - H^+ F^0) = D^{-2}(\tilde{F}^+ F^0 / T) \frac{1}{\sqrt{N_{ot}}} \sum_{i=1}^{N_{ot}} B_t^i \Lambda_i^0 e_{it} + o_p(1)
\]

\[
\rightarrow^d N(0, D_r^{-2} \Gamma_t^* Q_r' D_r^{-2}),
\]

where

\[
\Gamma_t^* = \text{plim} \frac{1}{N_{ot}} \sum_{i=1}^{N_{ot}} B_t^i \Lambda_i^0 \Lambda_i^0 B_t^i e_{it}^2
\]

(16)

(assuming cross-sectional uncorrelation for \( e_{it} \)).

The result includes \( N_{ot} = N \) as a special case. When \( N_{ot} = N \), we have \( B_t^i = I_r \) for all \( i \), and \( \Gamma_t^* = \Gamma_t \). The convergence is \( \sqrt{N} \), and the limit is in (13).

**Asymptotic distribution for the estimated factor loadings:** We have the asymptotic representation

\[
\tilde{\Lambda}_i^+ - (H^+)^{-1} \Lambda_i^0 = H^+ \frac{1}{T} \sum_{t=1}^{T} F_t^0 e_{it} + \tilde{\eta}_{NT,i}
\]

where \( \tilde{\eta}_{NT,i} = O_p(\delta^{-2}_{N_o T_o}) \) is uniformly in \( i \) and \( t \).

For a given \( i \), if all \( T \) observations are available, then \( \tilde{e}_{it} = e_{it} \) for all \( t \), so that

\[
\sqrt{T} (\tilde{\Lambda}_i^+ - (H^+)^{-1} \Lambda_i^0) = H^+ \frac{1}{\sqrt{T}} \sum_{t=1}^{T} F_t^0 e_{it} + o_p(1) \rightarrow^d N(0, Q_r^{-1} \Phi_i Q_r^{-1})
\]

where we used the fact that the limit of \( H^+ \) is \( Q_r^{-1} \).

Suppose that for the given \( i \), there are \( T_{oi} \) observations available. Again, for notational simplicity, we assume the first \( T_{oi} \) are observable, the rest are missing. Then \( \tilde{e}_{it} = e_{it} \) for \( t \leq T_{oi} \) and \( \tilde{e}_{it} = u_{it} + v_{it} \) for \( t > T_{oi} \). Thus

\[
\tilde{\Lambda}_i^+ - (H^+)^{-1} \Lambda_i^0 = H^+ \left( \frac{1}{T} \sum_{t=1}^{T_{oi}} F_t^0 e_{it} + \frac{1}{T} \sum_{t=T_{oi} + 1}^{T} F_t^0 (u_{it} + v_{it}) \right) + \tilde{\eta}_{NT,i}
\]

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Note that \( \frac{1}{T} \sum_{t=T_{o_i}+1}^{T} F_t^0 u_{it} \) is negligible, and

\[
\frac{1}{T} \sum_{t=T_{o_i}+1}^{T} F_t^0 v_{it} = \left( \frac{1}{T} \sum_{t=T_{o_i}}^{T} F_t^0 F_t^{0 \prime} \right) \left( \frac{1}{T} \sum_{t=T_{o_i}+1}^{T} F_t^0 F_t^{0 \prime} \right)^{-1} \frac{1}{T \sum_{s=1}^{T_{o_i}}} F_s^0 e_{is}.
\]

Under

\[
\left( \frac{1}{T} \sum_{t=T_{o_i}}^{T} F_t^0 F_t^{0 \prime} \right) \left( \frac{1}{T} \sum_{t=T_{o_i}+1}^{T} F_t^0 F_t^{0 \prime} \right)^{-1} \xrightarrow{p} I_r
\]

the asymptotic representation can be written as

\[
\tilde{\Lambda}_i^+ - (H^+)^{-1} \Lambda_i^0 = H^+ \left( \frac{1}{T} \sum_{t=1}^{T_{o_i}} F_t^0 e_{it} \right) + O_p(\delta_{N_o,T_o}^{-2}).
\] (17)

Then by Assumption D,

\[
\sqrt{T_{o_i}} (\tilde{\Lambda}_i^+ - (H^+)^{-1} \Lambda_i^0) \xrightarrow{d} N(0,Q_{\tilde{\Phi}}^t,\Sigma_{\tilde{\Phi}}^t).
\]

**Asymptotic distribution for the estimated common components:** Let \( \tilde{C}_{it}^+ = \tilde{F}_t^+ \tilde{\Lambda}_i^+ \).

Using the asymptotic representations in [15] and [17], we can show as in [Bai and Ng 2021]

\[
\tilde{C}_{it}^+ - C_{it}^0 = \Lambda_i^0 (\Lambda^0 \Lambda^0 / N)^{-1} \left( \frac{1}{N_{o_i}} \sum_{i=1}^{N_{o_i}} B_i^t \Lambda_i^0 e_{it} \right) + F_{it}^0 (F_{it}^0 F_{it}^{0 \prime} / T)^{-1} \left( \frac{1}{T} \sum_{t=1}^{T_{o_i}} F_t^0 e_{it} \right) + O_p(\delta_{N_o,T_o}^{-2}).
\]

The above representation for \( \tilde{C}_{it}^+ - C_{it}^0 \) implies

\[
\left( \frac{1}{N_{o_i}} V_{it}^* + \frac{1}{T_{o_i}} W_{it} \right)^{-1/2} (\tilde{C}_{it}^+ - C_{it}^0) \xrightarrow{d} N(0,1),
\] (18)

where

\[
V_{it}^* = \Lambda_i^0 \Sigma_{\Lambda}^{-1} \Gamma_t \Sigma_{\Lambda}^{-1} \Lambda_i^0, \quad W_{it} = F_{it}^0 (\Sigma_{\Phi}^{-1} \Phi_t \Sigma_{\Phi}^{-1}) F_{it}^0
\]

and \( \Gamma_t^* \) is defined in [16]. Note that we assume the two terms in the representation are asymptotically independent so that the variance is the sum of the two variances.

A useful special case occurs, if for a given entry \((i,t)\), the corresponding row and column are observable (i.e., if \(X_{i1}, X_{i2}, \ldots, X_{iT}\)) and \(X_{1t}, X_{2t}, \ldots, X_{Nt}\) are observable), then

\[
\left( \frac{1}{N} V_{it} + \frac{1}{T} W_{it} \right)^{-1/2} (\tilde{C}_{it}^+ - C_{it}^0) \xrightarrow{d} N(0,1),
\]

where

\[
V_{it} = \Lambda_i^0 \Sigma_{\Lambda}^{-1} \Gamma_t \Sigma_{\Lambda}^{-1} \Lambda_i^0. \quad W_{it} = F_{it}^0 (\Sigma_{\Phi}^{-1} \Phi_t \Sigma_{\Phi}^{-1}) F_{it}^0
\]

This follows from \(N_{o_i} = N, B_i^t = I_r, \Gamma_t^* = \Gamma_t, \) and \(T_{o_i} = T\).

We remark that using the convergence rate for \( \tilde{C}_{it}^+ - C_{it}^0 \), we can also show, under the assumption that \(N_{o_i} \geq N_o, \) and \(T_{o_i} \geq T_o\), the Frobenius norm of the matrix \( \tilde{C}^+ - C^0 \) is bounded by

\[
\frac{\|\tilde{C}^+ - C^0\|}{\sqrt{NT}} = O_p(\frac{1}{\sqrt{N}}) + O_p(\frac{1}{\sqrt{T}}) + \sqrt{(1 - p_N)(1 - p_T)} \left[ O_p(\frac{1}{\sqrt{NP_N}}) + O_p(\frac{1}{\sqrt{TP_T}}) \right]
\] (19)

where \(p_N = N_0/N\) and \(p_T = T_0/T\).
Proof of Lemma 4: Omitting the subscript $X$, we write $\Sigma_{ij}$ for $\Sigma_{X,ij}$, and similarly for the estimated counterpart. For $i \neq j$,

$$
\hat{\Sigma}_{ij} - \Sigma_{ij} = \hat{\Lambda}_i \hat{\Sigma}_F \hat{\Lambda}_j - \Lambda_0^\prime \Sigma_F \Lambda_0^j
$$

$$
= (\hat{\Lambda}_i - G \Lambda_0^j) \hat{\Sigma}_F \hat{\Lambda}_j + \Lambda_0^\prime G' \hat{\Sigma}_F G G^{-1} \hat{\Lambda}_j - \Lambda_0^\prime \Sigma_F \Lambda_0^j
$$

$$
= (\hat{\Lambda}_i - G \Lambda_0^j) \hat{\Sigma}_F \hat{\Lambda}_j + \Lambda_0^\prime (G' \hat{\Sigma}_F G - \Sigma_F) G^{-1} \hat{\Lambda}_j + \Lambda_0^\prime \Sigma_F (G^{-1} \hat{\Lambda}_j - \Lambda_0^j),
$$

where $G$ is either $H_{tall}^{-1}$ or $(H^+)^{-1}$ depending on the choice of $\hat{\Lambda}$. Thus

$$
\max_{ij} |\hat{\Sigma}_{ij} - \Sigma_{ij}| \leq \max_i \|\hat{\Lambda}_i - G \Lambda_0^i\| \cdot \|\hat{\Sigma}_F\| \max_j \|\hat{\Lambda}_j\|
$$

$$
+ (\max_i \|\Lambda_0^i\|)(\max_j \|\hat{\Lambda}_j\|)\|G' \hat{\Sigma}_F G - \Sigma_F\|\|G^{-1}\|
$$

$$
+ \max_i \|\Lambda_0^i\|\|\Sigma_F G^{-1}\| \max_j \|\hat{\Lambda}_j - G \Lambda_0^i\|.
$$

By Assumption A, $\max_i \|\Lambda_0^i\|$ is bounded. Under exponential tails for the idiosyncratic errors $e_{it}$ (e.g., Fan et al., 2011), then it can be shown that

$$
\max_i \|\hat{\Lambda}_i - G \Lambda_0^i\| \leq \frac{\log(\max\{T, N\})}{\sqrt{\min\{N, T\}}} O_p(1).
$$

By adding and subtracting terms, we have $\max_j \|\hat{\Lambda}_j\| = O_p(1)$. Note $\Sigma_F$ is a fixed dimensional matrix, we have $\|G' \hat{\Sigma}_F G - \Sigma_F\| = o_p(1)$. In summary, for $i \neq j$,

$$
\max_{ij} |\hat{\Sigma}_{ij} - \Sigma_{ij}| = o_p(1).
$$

For $i = j$, we need to show further that the idiosyncratic variance estimator is uniformly consistent over $i$,

$$
\max_i |\hat{\Psi}_{e,ii}^2 - \Psi_{e,ii}^2| = o_p(1),
$$

where $\Psi_{e,ii}^2 = \text{var}(e_{it})$, and $\hat{\Psi}_{e,ii}^2$ is its estimator. From $\bar{e}_{it} = e_{it} + \tilde{C}_{it} - C_{it}$, we have

$$
\max_i |\hat{\Psi}_{e,ii}^2 - \Psi_{e,ii}^2| \leq \max_i \frac{1}{T_{o_1}} \sum_s e_{is}^2 - \Psi_{ii}^2 + 2 \max_i \frac{1}{T_{o_1}} \sum_s |\tilde{C}_{is} - C_{is}| |e_{is}| + \max_i \frac{1}{T_{o_1}} \sum_s (\tilde{C}_{is} - C_{is})^2
$$

where the sum is based on $T_{o_1}$ number of entries. Using the convergence of $\tilde{C}_{it}$, and the exponential tails of $e_{it}$, we can show that each of the right hand side term is $o_p(1)$, implying (20). Lemma 4 is obtained by combining results.