REDUCTION OF SASAKIAN MANIFOLDS

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Abstract. We show that the contact reduction can be specialized to Sasakian manifolds. We link this Sasakian reduction to Kähler reduction by considering the Kähler cone over a Sasakian manifold. We present examples of Sasakian manifolds obtained by $S^1$ reduction of standard Sasakian spheres.

1. Introduction

Reduction technique was naturally extended from symplectic to contact structures by H. Geiges in [7]. Even earlier, Ch. Boyer, K. Galicki and B. Mann defined in [3] a moment map for 3-Sasakian manifolds, thus extending the reduction procedure for nested metric contact structures. Quite surprisingly, a reduction scheme for Sasakian manifolds (contact manifolds endowed with a compatible Riemannian metric satisfying a curvature condition), was still missing.

In this note we fill the gap by defining a Sasakian moment map and constructing the associated reduced space. We then relate Sasakian reduction to Kähler reduction via the Kähler cone over a Sasakian manifold.

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2. Sasakian manifolds

Let us briefly recall the notion of a Sasakian manifold. The definition we give is not the standard one but is suited for our purpose. For more details, we refer to [2] and [4].

Definition 2.1. A Sasakian manifold is a $(2n+1)$-dimensional Riemannian manifold $(N, g)$ endowed with a unitary Killing vector field $\xi$ such that the curvature tensor of $g$ satisfies the equation:

$$R(X, \xi)Y = \eta(Y)X - g(X, Y)\xi$$

where $\eta$ is the metric dual 1-form of $\xi$: $\eta(X) = g(\xi, X)$.

Let $\phi = \nabla \xi$, where $\nabla$ is the Levi-Civita connection of $g$. The following formulae are then easily deduced:

$$\phi \xi = 0, \quad g(\phi Y, \phi Z) = g(Y, Z) - \eta(Y)\eta(Z).$$

It can be seen that $\eta$ is a contact form on $N$, whose Reeb field is $\xi$ (it is also called the characteristic vector field). Moreover, the restriction of $\phi$ to the contact distribution $\eta = 0$ is a complex structure.

The simplest example is the standard sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$, with the metric induced by the flat one of $\mathbb{C}^{n+1}$. The characteristic Killing vector field is $\xi_p = -i \overrightarrow{p}$, $i$ being the imaginary unit. Other Sasakian structures on the sphere can be obtained by $D$-homothetic transformations (cf. [9]). Also, the unit sphere bundle of any space form is Sasakian.

More generally, the quantization bundle of a compact Kähler manifold naturally carries a Sasakian structure. The converse construction, possible when the characteristic field is regular, is known as the Boothby-Wang fibration. Precisely, the following result (the metric part is due to Morimoto and Hatakeyama) is available (cf. [11] or [4]):

Theorem 2.1. Let $(P, h)$ be a Hodge manifold. There exists a principal circle bundle $\pi : N \to P$ and a connection form $\eta$ in it, with curvature form the pull-back of the Kähler form of $P$, which is a contact form on $S$. Let $\xi$ be the vector field dual to $\eta$ with respect to the metric $g = \pi^* h + \eta \otimes \eta$. Then $(N, g, \xi)$ is Sasakian.

The following equivalent definition puts Sasakian geometry in the framework of holonomy groups. Let $C(N) = N \times \mathbb{R}_+$ be the cone over $(N, g)$. Endow it with the warped-product cone metric $C(g) = r^2 g + dr^2$. Let $R_0 = r \partial r$ and define on $C(N)$ the complex structure $J$ acting like this (with obvious identifications): $JY = \phi Y - \eta(Y)R_0$, $JR_0 = \xi$. We have:
Theorem 2.2. \( (N, g, \xi) \) is Sasakian if and only if the cone over \( N \) \( (C(N), C(g), J) \) is Kählerian.

3. Main results

Theorem 3.1. Let \( (N, g, \xi) \) be a compact \( 2n + 1 \) dimensional Sasakian manifold and \( G \) a compact \( d \)-dimensional Lie group acting on \( N \) by contact isometries. Suppose \( 0 \in g^* \) is a regular value of the associated moment map \( \mu \). Then the reduced space \( M = N//G := \mu^{-1}(0)/G \) is a Sasakian manifold of dimension \( 2(n - d) + 1 \).

Proof. By [7], the contact moment map \( \mu : N \to g^* \) is defined by

\[
< \mu(x), X > = \eta(X)
\]

for any \( X \in g \) and \( X \) the corresponding field on \( N \). We know that the reduced space is a contact manifold, loc. cit. Hence we only need to check that (1) the Riemannian metric is projected on \( M \) and (2) the field \( \xi \) projects to a unitary Killing field on \( M \) such that the curvature tensor of the projected metric satisfies formula (2.1).

To this end, we first describe the metric geometry of the Riemannian submanifold \( \mu^{-1}(0) \).

Let \( \{X_1, ..., X_d\} \) be a basis of \( g \) and let \( \{X_1, ..., X_a\} \) be the corresponding vector fields on \( N \). Since \( 0 \) is a regular value of \( \mu \), \( \{X_1\} \) is a linearly independent system in each \( T_x\mu_X^{-1}(0) \). From the very definition of the moment map we have \( \eta_p(X_i) = \mu(p)(X_i) = 0 \) hence \( X_i \perp \xi \). As \( G \) acts by contact isometries, we have

(3.1) \[
\mathcal{L}_{X_i}g = 0, \quad \mathcal{L}_{X_i}\eta = 0 \quad i = 1, ..., d.
\]

Note that these also imply \([X_i, \xi] = \mathcal{L}_{X_i}\xi = 0\).

Observe that \( \mu^{-1}(0) \) is an isometrically immersed submanifold of \( N \) (we denote the induced metric also with \( g \)) whose tangent space in each point is described by: \( Y \in T_x\mu^{-1}(0) \) if and only if \( d\mu_x(Y) = 0 \). Hence, by the definition of the moment map, the vector fields \( \xi \) and \( X_i \) are tangent to \( \mu^{-1}(0) \). Moreover, for any \( Y \) tangent to \( \mu^{-1}(0) \), one has \( g(\varphi X_i, Y) = d\eta(Y, X_i) = d\mu(Y) = 0 \), hence the vector fields \( \{X_i\} \) produce a local basis (not necessarily orthogonal) of the normal bundle of \( \mu^{-1}(0) \). The shape operators \( A_i := A_{\varphi X_i} \) of this submanifold in \( N \) are computed as follows (we let \( \nabla, \nabla^N \) be the Levi Civita covariant
derivatives of $\mu^{-1}(0)$, resp. $\mathcal{N}$):

\begin{equation}
(3.2) \quad g(A_i Y, Z) = -g(\nabla^N Y (\|X_i\|^{-1} \varphi X_i), Z) = \\
= -g(Y (\|X_i\|^{-1}) \varphi X_i, Z) - g(\|X_i\|^{-1} \nabla^N Y (\varphi X_i), Z) = \\
= -\|X_i\|^{-1} g(\nabla^N Y (\varphi X_i), Z) = \\
= -\|X_i\|^{-1} g(\nabla^N Y (\varphi) X_i + \varphi \nabla^N Y X_i, Z) = \\
= -\|X_i\|^{-1} g(\eta(X_i) Y - g(X_i, Y) \xi + \varphi \nabla^N Y X_i, Z) = \\
= \|X_i\|^{-1} \{g(X_i, Y) \eta(\xi) - g(\varphi \nabla^N Y X_i, Z)\}.
\end{equation}

In particular, for the corresponding quadratic second fundamental forms we get:

\begin{equation}
(3.3) \quad h_i(Y, \xi) = \|X_i\|^{-1} g(X_i, Y), \quad h_i(\xi, \xi) = 0.
\end{equation}

Consequently, one easily obtains: the restriction of the vector field $\xi$ is Killing on $\mu^{-1}(0)$ too.

Using the Gauss equation of a submanifold

\[ R^N(X, Y, Z, W) = R^{\mu^{-1}(0)}(X, Y, Z, W) + g(h(X, Z), h(Y, W)) - g(h(X, W), h(Y, Z)) \]

and the formula \((3.2)\) we now compute the needed part of the curvature tensor of $\mu^{-1}(0)$ at a fixed point $p \in \mu^{-1}(0)$. We take $X, Y, Z$ orthogonal to $\xi_p$ and obtain:

\begin{equation}
(3.4) \quad g(R^{\mu^{-1}(0)}(X, \xi) Y, Z) - g(R^N(X, \xi) Y, Z) = \\
= -\sum_{i=1}^{d} \|X_i\|^{-2} \{h_i(X, Y) h_i(\xi, Z) - h_i(X, Z) h_i(\xi, Y)\} = \\
= -\sum_{i=1}^{d} \|X_i\|^{-2} \{g(X_i, Z) g(\nabla^N X_i, Y) - g(X_i, Y) g(\nabla^N X_i, \varphi Z)\}
\end{equation}

(Note that $\nu_i = \|X_i\|^{-1} \varphi X_i$ are chosen to be orthonormal in $p$; this is always possible pointwise by appropriate choice of the initial $X_i$).

Let now $\pi : \mu^{-1}(0) \rightarrow M$ and endow $M$ with the projection $g^M$ of the metric $g$ such that $\pi$ becomes a Riemannian submersion. This is possible because $G$ acts by isometries. In this setting, the vector fields $X_i$ span the vertical distribution of the submersion, whilst $\xi$ is horizontal and projectable (because $\mathcal{L}_{X_i} \xi = 0$). Denote with $\zeta$ its projection on $M$. $\zeta$ is obviously unitary. To prove that $\zeta$ is Killing on $M$, we just observe that $\mathcal{L}_{\xi} g(Y, Z) = \mathcal{L}_{\xi} g(Y^h, Z^h)$, where $Y^h$ denotes
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the horizontal lift of of $Y$. Finally, to compute the values $R^M(X, \zeta)Y$ of the curvature tensor of $g^M$, we use O'Neill formula (cf. \[1\], (9.28f))

$$g^M(R^M(X, \zeta)Y, Z) = g(R^{\mu-1(0)}(X^h, \xi)Y^h, Z^h) + g(A(X^h, \xi), A(Y^h, Z^h))$$

$$- g(A(\xi, Y^h), A(X^h, Z^h)) + g(A(X^h, Z^h), A(\xi, Z^h))$$

where $X, Y, Z$ are unitary, normal to $\zeta$ and the O'Neill (1, 2) tensor $A$ is defined as: $A(Z^h, Y^h) = \text{vertical part of } \nabla_{Z^h}X^h$. Using Gauss formula and \[3,3\], we obtain

$$g(\nabla_{Z^h}\xi, X_i) = g(\varphi Z^h, X_i) = -g(Z^h, \varphi X_i) = 0$$

hence $\nabla_{Z^h}\xi$ has no vertical part and $A(Z^h, \xi) = 0$. Thus

$$R^M(X, \zeta)Y = R^{\mu-1(0)}(X^h, \xi)Y^h = R^N(X^h, \xi)Y^h$$

because of \[3,4\] and the fact that $X^h, Y^h$ are normal to all $X_i$. Hence

$$R^M(X, \zeta)Y = g(\xi, Y^h)X^h - g(X^h, Y^h)\xi = g^M(\zeta, Y)X - g^M(X, Y)\zeta$$

which proves that $(M, g^M, \zeta)$ is a Sasakian manifold.

\[\square\]

In the following we relate Sasakian reduction to Kähler reduction by using the cone construction. Roughly speaking, we prove that reduction and taking the cone are commuting operations.

Let $\omega = dr^2 \wedge \eta + r^2 dq$ be the Kähler form of the cone $C(N)$ over a Sasakian manifold $(N, g, \xi)$. If $\rho_t$ are the translations acting on $C(N)$ by $(x, r) \mapsto (x, tr)$, then the vector field $R_0 = r \partial r$ is the one generated by $\{\rho_t\}$. Moreover, the following two relations are useful:

(3.5) \hspace{1cm} \mathcal{L}_{R_0}\omega = \omega, \hspace{1cm} \rho_t^*\omega = t\omega.

Suppose a compact Lie group $G$ acts on $C(N)$ by holomorphic isometries, commuting with $\rho_t$. This ensures a corresponding action of $G$ on $N$. In fact, we can consider $G \cong G \times \{1d\}$ acting as $(g, (x, r)) \times (gx, r)$.

Suppose that a moment map $\Phi : C(N) \rightarrow \mathfrak{g}$ exists.

As above, let $\{X_1, \ldots, X_d\}$ be a basis of $\mathfrak{g}$ and let $\{X_1, \ldots, X_d\}$ be the corresponding vector fields on $C(N)$. We see that $X_i$ are independent on $r$, hence can be considered as vector fields on $N$. Furthermore, the commutation of $G$ with $\rho_t$ implies

(3.6) \hspace{1cm} \Phi(\rho_t(p)) = t\Phi(p).

Now imbed $N$ in the cone as $N \times \{1\}$ and let $\mu := \Phi|_{N \times \{1\}}$. This is the moment map of the action of $G$ on $N$. To see this, recall the definition of the symplectic moment map $\Phi = (\Phi_1, \ldots, \Phi_d)$: $\Phi_i$ is given up to constant by $d\Phi_i(Y) = \omega(X_i, Y)$. Here we uniquely determine $\Phi_i$ by imposing the condition $\eta(X_i) = \Phi|_{N \times \{1\}}$. This immediately implies
that the Reeb field of $N$ is orthogonal to the vector fields $X_i$ since $g(\xi, X_i) = \eta(X_i) = 0$. As $G$ acts by isometries on $C(N)$, we may project the cone metric to a metric on $N'//G \times \mathbb{R}_+$ which we denote by $g_0$. Then $g_0(Y, Z) = C(g)(Y^h, Z^h)$, where $Y^h, Z^h$ are the unique vector fields on $\Phi^{-1}(0)$ orthogonal to all of $X_i$ which project on $Y, Z$ (we call them horizontal).

Let $P = \Phi^{-1}(0)/G$ be the reduced Kähler manifold. The key remark is that because of (3.6), $\Phi^{-1}(0)$ is the cone $N' \times \mathbb{R}_+$ over $N' = \{ x \in N : (x, 1) \in \Phi^{-1}(0) \}$. Moreover, since the actions of $G$ and $p_t$ commute, one has an induced action of $G$ on $N'$. Then

$$\Phi^{-1}(0)/G \cong (N' \times \mathbb{R}_+)/G \cong N'/G \times \mathbb{R}_+$$

The manifold $N'//G \times \mathbb{R}_+$ is Kähler, as reduction of a Kähler manifold, but we still have to check that this Kähler structure is a cone one. For the more general, symplectic case, this was done in [3]. Let $g_0$ be the reduced Kähler metric and $g'$ be the Sasakian reduced metric on $N'//G$. It is easily seen that the lift of $g_0$ to $\Phi^{-1}(0)$ coincides with the lift of the cone metric $r^2 g' + dr^2$ on horizontal fields. This implies that the cone metric coincides with $g_0$.

Summing up we have proved:

**Theorem 3.2.** Let $(N, g, \xi)$ be a Sasakian manifold and let $(C(N), C(g), J)$ be the Kähler cone over it. Let a compact Lie group $G$ act by holomorphic isometries on $C(N)$ and commuting with the action of the 1-parameter group generated by the field $R_0$. If a moment map with regular value 0 exists for this action, then a moment map with regular value 0 exists also for the induced action of $G$ on $N$. Moreover, the reduced space $C(N)//G$ is the Kähler cone over the reduced Sasakian manifold $N//G$.

The advantage of defining the Sasakian reduction via Kähler reduction, as done in [3] for 3-Sasakian manifolds, is the avoiding of curvature computations.

4. **Examples: $S^1$ actions on Sasakian spheres**

**Example 4.1.** Start with $S^7 \subset \mathbb{C}^4$ with its standard Sasakian structure. Let the complex coordinates of $\mathbb{C}^4$ be $(z_0, ..., z_3)$, with $z_j = x_j + iy_j$. The contact form on $S^7$ can then be written

$$\eta = \sum_{j=0}^{3} (x_jdy_j - y_jdx_j)$$
and its Reeb field is
\[ \xi = \sum_{j=0}^{3} (x_j \partial y_j - y_j \partial x_j). \]

Let \( S^1 \) act on \( S^7 \) by \( e^{it} \mapsto (e^{-it}z_0, e^{it}z_1, e^{it}z_2, e^{it}z_3) \). The associated field of this action is (in real coordinates)
\[ X_0 = -(x_0 \partial y_0 - y_0 \partial x_0) - (x_1 \partial y_1 - y_1 \partial x_1) + (x_2 \partial y_2 - y_2 \partial x_2) + (x_3 \partial y_3 - y_3 \partial x_3). \]

The moment map \( \mu : S^7 \to \mathbb{R} \) reads:
\[ \mu(z) = \eta_z(X_0) = -|z_0|^2 - |z_1|^2 + |z_2|^2 + |z_3|^2 \]
with zero level set
\[ \{ z \in S^7 : |z_0|^2 + |z_1|^2 = |z_2|^2 + |z_3|^2 \} = S^3(\frac{1}{\sqrt{2}}) \times S^3(\frac{1}{\sqrt{2}}). \]
Clearly \( \mu \) is nondegenerate on \( \mu^{-1}(0) \).

The reduced space can be identified with \( S^3 \times S^3 / S^1 \), which, by [10], is diffeomorphic with \( S^2 \times S^3 \). (In this case, one can also avoid the topological arguments in [10] and identify the reduced space by observing that the following diffeomorphism of \( S^3 \times S^3 \): \((z_0, z_1, z_2, z_3) \mapsto (z_1z_4 + \overline{z}_2\overline{z}_3, z_1z_3 - \overline{z}_2\overline{z}_4, z_3, z_4)\) is equivariant with respect to the previous \( S^1 \) action which restricted to the second factor of the product is the usual action inducing the Hopf fibration; \textit{mille grazie} to Rosa Gini and Maurizio Parton for letting us know it, [6]).

The reduced Sasakian structure obtained in this way on \( S^2 \times S^3 \) is easily checked to be Einstein and to project on the Kähler Einstein metric of \( \mathbb{C}P^1 \times \mathbb{C}P^1 \) making the fibre map be a Riemannian submersion. As by [10] such an Einstein metric is unique, our reduced Sasakian structure coincides with the Sasakian structure found in [8] viewing \( S^2 \times S^3 \) as minimal submanifold of \( S^7 \), total space of the pull-back over \( \mathbb{C}P^1 \times \mathbb{C}P^1 \) of the Hopf bundle \( S^7 \to \mathbb{C}P^3 \). The same Einstein-Sasakian metric on \( S^2 \times S^3 \) also appears in [9], constructed by a different approach.

\textbf{Example 4.2.} Consider again \( S^7 \) as starting Sasakian manifold, but let \( S^1 \) act by: \( e^{it} \mapsto (e^{-kit}z_0, e^{it}z_1, e^{it}z_2, e^{it}z_3), k \in \mathbb{Z}_+ \). Now \( \mu^{-1}(0) \cong S^1(\sqrt{\frac{k}{k+1}}) \times S^5(\sqrt{\frac{1}{k+1}}) \). In order to identify the reduced space, consider the \( k : 1 \) mapping
\[ S^1 \times S^5 \ni (z_0, z_1, z_2, z_3) \mapsto ((z_0)^{-k}, z_1, z_2, z_3) \in S^1 \times S^5. \]
It induces a $k : 1$ map from $M = S^1 \times S^5/S^1$, where $S^1$ acts diagonally, to the reduced space $\mu^{-1}(0)/S^1$ with the action given above. As in $[3]$, the map

$$(z_0, ..., z_3) \mapsto (z_0, \overline{z_0}z_1, \overline{z_0}z_2, \overline{z_0}z_3)$$

is an equivariant diffeomorphism of $S^1 \times S^5$, equivariant with respect to the diagonal action of $S^1$ and the action of $s^1$ on the first factor. Hence $M$ is diffeomorphic to $S^5$ and the reduced Sasakian space is $S^5/Z_k$.

**Example 4.3.** In general, consider the weighted action of $S^1$ on $S^{2n-1} \subset \mathbb{C}^n$ by:

$$(e^{it}, (z_0, ..., z_{n-1})) \mapsto (e^{\lambda_0 it}z_0, ..., e^{\lambda_{n-1} it}z_{n-1})$$

where $(\lambda_0, ..., \lambda_{n-1}) \in \mathbb{Z}^n$. The associated moment map

$$\mu(z) = \lambda_0|z_0|^2 + ... + \lambda_n|z_{n-1}|^2$$

is regular on $\mu^{-1}(0)$ for any $(\lambda_0, ..., \lambda_{n-1})$ such that $\lambda_0 ... \lambda_{n-1} \neq 0$, $(\lambda_0, ..., \lambda_{n-1}) = 1$ and at least two $\lambda$’s have different signs (compare with the 3-Sasakian case where the weights obey to more restrictions, cf. $[3]$).

Now take $\lambda_0 = ... = \lambda_k = a$ and $\lambda_{k+1} = ... = \lambda_{n-1} = -b$, $a, b \in \mathbb{Z}_+$ relatively prime. Then $\mu^{-1}(0) \cong S^{2k+1}(\sqrt{\frac{a}{a+b}}) \times S^{2(n-k)-1}(\sqrt{\frac{b}{a+b}})$. Note that the induced metric on $\mu^{-1}(0)$ coincides with the product metric of the standard metrics of the two factors. We then see that the reduced space is diffeomorphic with an $S^1$ factor of the above product of spheres given by the following action:

$$(e^{it}, (x, y)) \mapsto (e^{iat}x, e^{-ibt}y).$$

One can now adapt the arguments of $[10]$, Cor. 2.2 and prove that the reduced spaces are $S^1$ bundles over $\mathbb{C}P^k \times \mathbb{C}P^{n-k-1}$ and, for $1 \leq k$, $4 < n$, they are not homeomorphic to each other in general.

However, for $k = 1$, $n = 4$, the reduced space is always diffeomorphic with $S^2 \times S^3$. Hence, one obtains an infinite family of Sasakian structures on $S^2 \times S^3$.

Note also that if $n$ is even, choosing like in the first example, the first half of the $\lambda$’s to be $-1$, the rest of them 1, the reduced Sasakian metric is *Einstein*, again according to $[10]$.

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