Improved lower bound on the size of Kakeya sets over finite fields

Shubhangi Saraf∗

Madhu Sudan†

August 22, 2008

Abstract

In a recent breakthrough, Dvir showed that every Kakeya set in \( F^n \) must be of cardinality at least \( \frac{n}{|F|^n} \), where \( c_n \approx \frac{1}{n!} \). We improve this lower bound to \( \beta |F|^n \) for a constant \( \beta > 0 \). This pins down the growth of the leading constant to the right form as a function of \( n \).

Let \( F \) be a finite field of \( q \) elements.

Definition 1 (Kakeya Set) A set \( K \subseteq F^n \) is said to be a Kakeya set in \( F^n \), if for every \( b \in F^n \), there exists a point \( a \in F^n \) such that for every \( t \in F \), the point \( a + t \cdot b \in K \).

We show:

Theorem 2 There exist constants \( c_0, c_1 > 0 \) such that for all \( n \), if \( K \) is a Kakeya set in \( F^n \) then \( |K| \geq c_0 \cdot (c_1 \cdot q)^n \).

Remark Our proofs give some tradeoffs on the constants \( c_0, c_1 \) that are achievable. We comment on the constants at the end of the paper.

The question of establishing lower bounds on the size of Kakeya sets was posed in Wolff [7]. Till recently, the best known lower bound on the size of Kakeya sets was of the form \( q^{\alpha n} \) for some \( \alpha < 1 \). In a recent breakthrough Dvir [1] showed that every Kakeya set must have cardinality at least \( c_n q^n \) for \( c_n = (n!)^{-1} \). (Dvir’s original bound achieved a weaker lower bound of \( c_n \cdot q^{n-1} \), but [1] includes the stronger bound of \( c_n \cdot q^n \), with the improvements being attributed to Alon and Tao.) Our improvement now shows that \( c_n \) remains bounded from below by \( \beta^n \) for some fixed \( \beta > 0 \). While this improvement in the lower bound on size of Kakeya sets is quantitatively small (say, compared to the improvement of Alon and Tao over Dvir’s original bound) it is qualitatively significant in that it does determine the growth of the leading constant \( c_n \), up to the determination of the right constant \( \beta \). In particular, it compares well with known upper bounds.

Previously, it was known there exists a constant \( \beta < 1 \) such that there are Kakeya sets of cardinality at most \( \beta^n q^n \), for every odd \( q \). A bound of \( \beta \leq \frac{1}{\sqrt{2}} \) follows from Mockenhaupt and Tao [6] and the fact that products of Kakeya sets are Kakeya sets (in higher dimension). The best known constant has \( \beta \rightarrow \frac{1}{2} \) due to Dvir [2]. We include his proof in the appendix (see Appendix A), complementing it with a similar construction and bound for the case of even \( q \) as well (so now the upper bounds work for all large fields).

Our proof follows that of Dvir [1]. Given a Kakeya set \( K \) in \( F^n \), we show that there exists an \( n \)-variate polynomial, whose degree is bounded from above by some function of \( |K| \), that vanishes at all of \( K \). Looking at restrictions of this polynomial on lines yields that this polynomial has too many zeroes, which in turn yields a lower bound on the size of \( K \). Our main difference is that we look for polynomials that vanish with “high multiplicity” at each point in \( K \). The requirement of high multiplicity forces the degree of the \( n \)-variate polynomial to go up slightly, but yields more zeroes when this polynomial is restricted to lines. The resulting tradeoff turns out to yield an improved bound. (We note that this is similar to the techniques used for “improved list-decoding of Reed-Solomon codes” by Guruswami and Sudan [3].) We now give a formal proof of Theorem 2.

∗MIT CSAIL. shibs@mit.edu.
†MIT CSAIL. madhu@mit.edu.
1 Preliminaries

For \( x = (x_1, \ldots, x_n) \), let \( \mathbb{F}[x] \) denote the ring of polynomials in \( x_1, \ldots, x_n \) with coefficients from \( \mathbb{F} \). We recall the following basic fact on polynomials.

**Fact 3** Let \( P \in \mathbb{F}[x] \) be a polynomial of degree at most \( q - 1 \) in each variable. If \( P(a) = 0 \) for all \( a \in \mathbb{F}^n \), then \( P \equiv 0 \).

For real \( \alpha \geq 0 \), let \( N_q(n, m) \) denote the number of monomials in \( n \) variables of total degree less than \( m \cdot q \) and of individual degree at most \( q - 1 \) in each variable.

We say that a polynomial \( g \in \mathbb{F}[x] \) has a zero of multiplicity \( m \) at a point \( a \in \mathbb{F}^n \) if the polynomial \( g_a(x) = g(x + a) \) has no support on monomials of degree strictly less than \( m \). Note that the coefficients of \( g_a \) are (homogenous) linear forms in the coefficients of \( g \) and thus the constraint \( g \) has a zero of multiplicity \( m \) at \( a \) yields \( \binom{m+n-1}{n} \) homogenous linear constraints on the coefficients of \( g \). As a result we conclude:

**Proposition 4** Given a set \( S \subseteq \mathbb{F}^n \) satisfying \( \binom{m+n-1}{n} \cdot |S| < N_q(n, m) \), there exists a non-zero polynomial \( g \in \mathbb{F}[x] \) of total degree less than \( mq \) and degree at most \( q - 1 \) in each variable such that \( g \) has a zero of multiplicity \( m \) at every point \( a \in S \).

**Proof** The number of possible coefficients for \( g \) is \( N_q(n, m) \) and the number of (homogenous linear) constraints is \( \binom{m+n-1}{n} \cdot |S| < N_q(n, m) \). Since the number of constraints is strictly smaller than the number of unknowns, there is a non-trivial solution.

For \( g \in \mathbb{F}[x] \) we let restriction \( g_{a,b}(t) = g(a + t \cdot b) \) denote its restriction to the “line” \( \{a + t \cdot b | t \in \mathbb{F}\} \).

We note the following facts on the restrictions of polynomials to lines.

**Proposition 5** If \( g \in \mathbb{F}[x] \) has a root of multiplicity \( m \) at some point \( a + t_0 b \) then \( g_{a,b} \) has a root of multiplicity \( m \) at \( t_0 \).

**Proof** By definition, the fact that \( g \) has a zero of multiplicity \( m \) at \( a + t_0 b \) implies that the polynomial \( g(x + (a + t_0 b)) \) has no support on monomials of degree less than \( m \). Thus, under the homogenous substitution \( x \leftarrow t \cdot b \), we get no monomials of degree less than \( m \) either, and thus we have \( t^m \) divides \( g(t b + (a + t_0 b)) = g(a + (t + t_0) b) = g_{a,b}(t + t_0) \). The final form implies that \( g_{a,b} \) has a zero of multiplicity \( m \) at \( t_0 \).

**Proposition 6** (II) Let \( g \in \mathbb{F}[x] \) be a non-zero polynomial of total degree \( d \) and let \( g_0 \) be the (unique, non-zero) homogenous polynomial of degree \( d \) such that \( g = g_0 + g_1 \) for some polynomial \( g_1 \) of degree strictly less than \( d \). Then \( g_{a,b}(t) = g_0(b)t^d + h(t) \) where \( h \) is a polynomial of degree strictly less than \( d \).

2 Proof of Theorem [2]

**Lemma 7** If \( K \) is a Kakeya set in \( \mathbb{F}^n \), then for every integer \( m \geq 0 \), \( |K| \geq \left( \frac{n}{m} \right) \cdot N_q(n, m) \).

**Proof** Assume for contradiction that \( |K| < \left( \frac{n}{m} \right) \cdot N_q(n, m) \). Let \( g \in \mathbb{F}[x] \) be a non-zero polynomial of total degree less than \( mq \) and degree at most \( q - 1 \) in each variable that has a zero of multiplicity \( m \) for each \( x \in K \). (Such a polynomial exists by Proposition [4].) Let \( d < mq \) denote the total degree of \( g \) and let \( g = g_0 + g_1 \) where \( g_0 \) is homogenous of degree \( d \) and \( g_1 \) has degree less than \( d \). Note that \( g_0 \) is also non-zero and has degree at most \( q - 1 \) in every variable.

Now fix a “direction” \( b \in \mathbb{F}^n \). Since \( K \) is a Kakeya set, there exists \( a \in \mathbb{F}^n \) such that \( a + tb \in K \) for every \( t \in \mathbb{F} \). Now consider the restriction \( g_{a,b} \) of \( g \) to the line through \( a \) in direction \( b \). \( g_{a,b} \) is a univariate polynomial of degree at most \( d < mq \). At every point \( t_0 \in \mathbb{F} \) we have that \( g_{a,b} \) has a zero of multiplicity \( m \) (Proposition [6]). Thus counting up the zeroes of \( g_{a,b} \) we find it has \( mq \) zeroes \( (m \text{ at every } t_0 \in \mathbb{F}) \) which is
more than its degree. Thus $g_{a,b}$ must be identically zero. In particular its leading coefficient must be zero. By Proposition 6 this leading coefficient is $g_0(b)$ and so we conclude $g_0(b) = 0$.

We conclude that $g_0$ is zero on all of $\mathbb{F}^n$ which contradicts the fact (Fact 3) that it is a non-zero polynomial of degree at most $q - 1$ in each of its variables. ■

**Proof of Theorem 2** Theorem 2 now follows by choosing $m$ appropriately. Using for instance $m = n$, we obtain that $|K| \geq \left(\frac{1}{n^2}\right)^2 q^n \geq (q/4)^n$, establishing the theorem for $c_0 = 1$ and $c_1 = 1/4$.

A better choice is with $m = \lceil n/2 \rceil \leq (n + 1)/2$. In this case $N_q(n, m) \geq \frac{1}{2} q^n$ (since at least half the monomials of individual degree at most $q - 1$ have degree at most $nq/2$). This leads to a bound of $|K| \geq \left(\frac{1}{2^{(3/2)n}}\right)^2 q^n \geq \frac{1}{2} (q/2.6)^n$, yielding the theorem for $c_0 = 1/2$ and $c_1 = 1/2.6$. ■

To improve the constant $c_1$ further, one could study the asymptotics of $N_q(n, m)$ closer. Let $\tau_\alpha$ denote the quantity $\liminf_{n \to \infty} \{\liminf_{q \to \infty} \frac{1}{q} \cdot N_q(n, \alpha \cdot n)^{1/n}\}$. I.e., for sufficiently large $n$ and sufficiently larger $q$, $N_q(n, \alpha n) \to \tau_\alpha^n \cdot q^n$. Lemma 7 can be reinterpreted in these terms as saying that for every $\alpha \in [0, 1)$, every Kakeya set has size at least $c_0(c_\alpha \cdot q)^n - o(q^n)$ for some $c_0 > 0$, where $c_\alpha = \tau_\alpha/2^{(1+\alpha)H(1/(1+\alpha))}$ (where $H(x) = -x \log_2 x - (1-x) \log_2 (1-x)$ is the binary entropy function). The best estimate on $\tau_\alpha$ we were able to obtain does not have a simple closed form expression. As $q \to \infty$, $\tau_\alpha^n$ equals the volume of the following region in $\mathbb{R}^n$: \{$(x_1, x_2, \ldots, x_n) \in [0, 1]^n | \sum_{i=1}^n x_i \leq \alpha \cdot n$\}. This volume can be expressed in terms of Eulerian numbers (See [5], §4.3). [5] [6] gives some asymptotics for Eulerian numbers and using their estimates $\alpha = 0.398$, it seems one can reduce $c_\alpha$ to something like $\frac{1}{2.58}$. This still remains bounded away from the best known upper bound which has $c_1 \to 1/2$.

**Remark** While the main theorem only gives the limiting behavior of Kakeya sets for large $n, q$, Lemma 7 can still be applied to specific choices and get improvements over [1]. For example, for $n = 3$, using $m = 2$ we get a lower bound of $\frac{7}{3} q^3$ as opposed to the bound of $\frac{8}{3} q^3$ obtainable from [1].

**Acknowledgments**

Thanks to Zeev Dvir for explaining the Kakeya problem and his solution to us, for detailed answers to many queries, and for his permission to include his upper bound on the size of Kakeya sets here (see Appendix A). Thanks also to Swastik Kopparty for helping us extend Dvir’s proof to even characteristic. Thanks to Chris Umans and Terry Tao for valuable discussions.

**References**

[1] Zeev Dvir. On the size of Kakeya sets in finite fields. *Journal of the American Mathematical Society*, (to appear), 2008. Article electronically published on June 23, 2008.

[2] Zeev Dvir. Personal communication, August 2008.

[3] Eldar Giladi and Joseph B. Keller. Eulerian number asymptotics. *Proceedings of the Royal Society of London, Series A*, 445(1924):291–303, 1994.

[4] Venkatesan Guruswami and Madhu Sudan. Improved decoding of Reed-Solomon and algebraic-geometric codes. *IEEE Transactions on Information Theory*, 45:1757–1767, 1999.

[5] Jean-Luc Marichal and Michael J. Mossinghoff. Slices, slabs, and sections of the unit hypercube. *Online Journal of Analytic Combinatorics*, 3, 2008. 11 pages. Earlier version appears as eprint math/0607715 (2006).

[6] Gerd Mockenhaupt and Terence Tao. Restriction and Kakeya phenomena for finite fields. *Duke Mathematics Journal*, 121(1):35–74, 2004.
A An upper bound on Kakeya sets

We include here Dvir’s proof [2] giving a non-trivial upper bound on the size of Kakeya sets in fields of odd characteristic. The proof is based on the construction of Mockenhaupt and Tao [8]. For the case of even characteristic we complement their results by using a variation (obtained with Swastik Kopparty) of their construction.

Theorem 8 (2) For every \( n \geq 2 \), and field \( \mathbb{F} \), there exists a Kakeya set in \( \mathbb{F}^n \) of cardinality at most \( 2^{-(n-1)} \cdot q^n + O(q^{n-1}) \).

Proof We consider two cases depending on whether \( \mathbb{F} \) is of odd or even characteristic.

Odd characteristic: Let \( D_n = \{ (\alpha_1, \ldots, \alpha_{n-1}, \beta) | \alpha, \beta \in \mathbb{F}, \alpha + \beta^2 \text{ is a square} \} \). Now let \( K_n = D_n \cup (\mathbb{F}^{n-1} \times \{0\}) \) where \( \mathbb{F}^{n-1} \times \{0\} \) denotes the set \( \{ (a, 0) | a \in \mathbb{F}^{n-1} \} \). We claim that \( K_n \) is a Kakeya set of the appropriate size.

Consider a direction \( \mathbf{b} = (b_1, \ldots, b_n) \). If \( b_n = 0 \), for \( \mathbf{a} = (0, \ldots, 0) \) we have that \( \mathbf{a} + \mathbf{t} \mathbf{b} \in \mathbb{F}^{n-1} \times \{0\} \subseteq K_n \). The more interesting case is when \( b_n \neq 0 \). In this case let \( \mathbf{a} = (b_1/(2b_n))^2, \ldots, (b_{n-1}/(2b_n))^2, 0 \). The point \( \mathbf{a} + \mathbf{t} \mathbf{b} \) has coordinates \( (\alpha_1, \ldots, \alpha_{n-1}, \beta) \) where \( \alpha_i = (b_i/(2b_n))^2 + t b_i \) and \( \beta = t b_n \). We have \( \alpha_i + \beta^2 = (b_i/(2b_n) + t b_n)^2 \) which is a square for every \( i \) and so \( \mathbf{a} + \mathbf{t} \mathbf{b} \in D_n \subseteq K_n \). This proves that \( K_n \) is indeed a Kakeya set.

Finally we verify that the size of \( K_n \) is as claimed. First note that the size of \( D_n \) is exactly \( q \cdot ((q + 1)/2)^{n-1} = 2^{-(n-1)}q^n + O(q^{n-1}) \) (q choices for \( \beta \) and \((q + 1)/2 \) choices for each \( \alpha_i + \beta^2 \)). The size of \( K_n \) is at most \( |D_n| + q^{n-1} = 2^{-(n-1)}q^n + O(q^{n-1}) \) as claimed.

Even characteristic: This case is handled similarly with minor variations in the definition of \( K_n \). Specifically, we let \( K_n = E_n = \{ (\alpha_1, \ldots, \alpha_{n-1}, \beta) | \alpha, \beta \in \mathbb{F}, \exists \gamma_i \in \mathbb{F} \text{ such that } \alpha_i = \gamma_i^2 + \gamma_i \beta \} \). (As we see below \( E_n \) contains \( \mathbb{F}^{n-1} \times \{0\} \) and so there is no need to set \( K_n = E_n \cup \mathbb{F}^{n-1} \times \{0\} \).)

Now consider direction \( \mathbf{b} = (b_1, \ldots, b_n) \). If \( b_n = 0 \), then let \( \mathbf{a} = 0 \). We note that \( \mathbf{a} + \mathbf{t} \mathbf{b} = (tb_1, \ldots, tb_n, 0) = (\gamma_1^2 + \beta \gamma_1, \ldots, \gamma_{n-1}^2 + \beta \gamma_{n-1}, \beta) \) for \( \beta = 0 \) and \( \gamma_i = \sqrt{tb_i} = (tb_i)^{q/2} \). We conclude that \( \mathbf{a} + \mathbf{t} \mathbf{b} \in E_n \) for every \( t \in \mathbb{F} \) in this case. Now consider the case where \( b_n \neq 0 \). Let \( \mathbf{a} = (b_1/b_n)^2, \ldots, (b_{n-1}/b_n)^2, 0 \). The point \( \mathbf{a} + \mathbf{t} \mathbf{b} \) has coordinates \( (\alpha_1, \ldots, \alpha_{n-1}, \beta) \) where \( \alpha_i = (b_i/b_n)^2 + tb_i \) and \( \beta = tb_n \). For \( \gamma_i = (b_i/b_n) \), \( \gamma_i^2 + \gamma_i \beta = (b_i/b_n)^2 + tb_i = \alpha_i \). Hence \( \mathbf{a} + \mathbf{t} \mathbf{b} \in E_n = K_n \).

It remains to compute the size of \( E_n \). The number of points of the form \( (\alpha_1, \ldots, \alpha_{n-1}, 0) \in E_n \) is exactly \( q^{n-1} \). We now determine the size of \( \{ (\alpha_1, \ldots, \alpha_{n-1}, \beta) \in E_n \text{ for fixed } \beta \neq 0 \} \). We first claim that the set \( \{ \gamma^2 + \beta \gamma | \gamma \in \mathbb{F} \} \) has size exactly \( q/2 \). This is so since for every \( \gamma \in \mathbb{F} \), we have \( \gamma^2 + \beta \gamma = \tau^2 + \beta \tau \) for \( \tau = \gamma + \beta \neq \gamma \), and so the map \( \gamma \mapsto \gamma^2 + \beta \gamma \) is a 2-to-1 map on its image. Thus, for \( \beta \neq 0 \), the number of points of the form \( (\alpha_1, \ldots, \alpha_{n-1}, \beta) \in E_n \) is exactly \( (q/2)^{n-1} \). We conclude that \( E_n \) has cardinality \( (q - 1) \cdot (q/2)^{n-1} + q^{n-1} = 2^{-(n-1)}q^n + O(q^{n-1}) \).}

We remark that for the case of odd characteristic, one can also use a recursive construction, replacing the set \( \mathbb{F}^{n-1} \times \{0\} \) by \( K_{n-1} \times \{0\} \). This would reduce the constant in the \( O(q^{n-1}) \) term, but not alter the leading term. Also we note that the construction used in the even case essentially also works in the odd characteristic case. Specifically the set \( E_n \cup \mathbb{F}^{n-1} \times \{0\} \) is a Kakeya set also for odd characteristic. Its size can also be argued to be \( 2^{-(n-1)} \cdot q^n + O(q^{n-1}) \).