DISSIPATIVE MECHANICAL SYSTEMS WITH DELAY

Ion D. ALBU, Mihaela NEAMȚU, Dumitru OPRIȘ

Abstract. The idea of dissipative mechanical system with delay is proposed. The paper studies the phenomenon of dissipation with delay for Euler-Poincaré systems on Lie algebras or equivalently, for Lie-Poisson systems on the duals of Lie algebras. The study was suggested by the work [2] and it is ended with a discussion regarding the stability and the Hopf bifurcations for the free rigid body with delay.

Keywords: delay differential equation, dissipation with delay, stability, Hopf bifurcation.

2000 AMS Mathematics Subject Classification: 34K13, 34K20, 37G15, 37J15, 53D20.

Introduction

In many applications one assumes the system under consideration is governed by a principle of causality; that is, the future state of the system is independent of the past states and is determined solely by the present. If it is also assumed that the system is governed by an equation involving the state and the rate of change of the state, then, generally, one considers either ordinary or partial differential equations. However, under a closer scrutiny, it becomes apparent that the principle of causality is often only a first approximation to the true situation and that a more realistic model would include some of the past states of the system. The simplest type of past dependence is through the state variable and not the derivative of the state variable, the so-called retarded functional differential equations or retarded difference equations or systems with delay. Systems with delay are
studied in many biological research topics, as well as in several branches of engineering, in neural networks, in economics, optimal production decision for an oligopoly with information lags ([9], [11], [12] etc.).

Functional differential equations on finite-dimensional manifolds are considered in [5], where the topological properties of the global attractor of an retarded functional differential equation in terms of limit capacity and Hausdorff dimension are presented.

The purpose of this paper is to study the phenomenon of dissipation with delay for Euler–Poincaré systems on Lie algebras or equivalently, for Lie–Poisson systems on the duals of Lie algebras. The dissipation without delay inducing instabilities for Euler–Poincaré systems is studied in [2]. The dissipation with delay that we construct has the essential feature: the energy is dissipated, but the angular momentum is not. In the context of Euler–Poincaré or Lie–Poisson systems this means that the coadjoint orbits remain invariant but on them the energy is decreasing along orbits. It is interesting the geometry behind the construction of the nonlinear dissipative term with delay, which has a Brockett with delay double bracket form. In fact, this form is well adapted to the study of dissipation with delay on Lie groups since it was constructed as a gradient system and it is well known in other contexts that this formalism plays an important role in the study of integrable systems.

The general equations of motion for dissipative systems with delay that we consider have the following form:

\[ \dot{F} = \{F, H\} - \{\{\tilde{F}, \tilde{H}\}\}_d \]

where \( H \) is the total energy of the system, \( \{F, H\} \) is a skew symmetric bracket which is a Poisson bracket in the usual sense and where \( \{\{\tilde{F}, \tilde{H}\}\}_d \) is a symmetric bracket with delay.

The outline of this paper is as follows. In section 1 the functional differential equations on manifolds are introduced. In section 2 a concrete idea of dissipative mechanism with delay is presented. The same formalism can be applied to other systems as well. Some of the basic and essential facts about dissipative mechanical systems with delay are described in section 3. In section 4 we study Lagrangian systems that are invariant under a group action and we shall add to them, in the sense of section 3, dissipative fields with delay that are equivariant. It is given by a necessary and sufficient condition
that the integral curves of the vector field $Z + Y$, for a vertical $G$–invariant vector field $Y$ on $TQ \times TQ$ and a Lagrangian vector field $Z$ of a $G$–invariant Lagrange function, preserve the inverse images of the coadjoint orbits in $\mathfrak{g}^*$ by the momentum map $J$. In section 5 the Euler–Poincaré and Lie–Poisson equations with delay are given and the double bracket with delay is defined; in the case “without delay” this is defined in [2]. In section 6 is presented the free rigid body with delay if the components $(I_1, I_2, I_3)$ of the moment of the inertial tensor satisfy the conditions $I_1 > I_2, I_1 > I_3$. For the equilibrium state $\Omega_1 = (m/I_1, 0, 0)^T$, $m \neq 0$, the value $\tau_c$ of the delay for which $\Omega_1$ is asymptotically stable and the value $\tau_0$ of the delay for which there is a Hopf bifurcation are determined. They are also determined the local center manifold and the quantities $C_1(0), \mu_2, T_2, \beta_2$ sketching out the direction of the Hopf bifurcation, the stability and the period of the bifurcating periodic solutions. These quantities are calculated for fixed values of $\alpha$ and $m$. Finally, some conclusions are drawn and further research directions are discussed in the last section.

1. Functional differential equations on manifolds

In this section, we begin with examples that will serve as a motivation for the consideration of functional differential equations on manifolds.

Example 1. For any constant $c$, the scalar equation

$$\dot{q}(t) = c \sin(q(t - 1)) \quad (1.1)$$

can be considered as an retarded functional differential equations (RFDE) on the circle $S^1 = \{ x \in \mathbb{R}^2, (x^1)^2 + (x^2)^2 = 1 \}$ by considering $q$ as an angle variable only determined up to a multiple $2\pi$.

Example 2. If $b, c$ are constants then we write the second–order RFDE

$$\ddot{q}(t) + b\dot{q}(t) = c \sin(q(t - 1)) \quad (1.2)$$
as a system of first–order RFDE

$$\dot{q}^1(t) = q^2(t), \quad \dot{q}^2(t) = c \sin(q^1(t - 1)) - bq^2(t). \quad (1.3)$$

By considering $q^1$ as an angle variable only determined up to a multiple $2\pi$, the equation (1.3) is a RFDE on the cylinder $S^1 \times \mathbb{R}$. We remark that
we can take the space of initial data for the solution \((q^1, q^2)\) of (1.3) as \(C([-1, 0], S^1) \times \mathbb{R}\).

The simplest type of past dependence in a differential equation is that in which the past dependence is through the state variable and not the derivative of the state variable, the so-called retarded functional differential equations or retarded differential difference equations. For a discussion of the physical applications of the differential difference equation

\[
\dot{q}(t) = F(t, q(t), q(t - \tau)),
\]

to control problems, see [5], [6].

**Example 3.** The equations involved in the study of vibrating masses attached to an elastic bar are

\[
\ddot{q}^1(t) + \omega_1^2 q^1(t) = \varepsilon f_1(q^1(t), \dot{q}^1(t), q^2(t), \dot{q}^2(t)) + \gamma_1 \dot{q}^2(t - \tau), \\
\ddot{q}^2(t) + \omega_2^2 q^2(t) = \varepsilon f_2(q^1(t), \dot{q}^1(t), q^2(t), \dot{q}^2(t)) + \gamma_2 \dot{q}^1(t - \tau).
\]

(1.4)

**Example 4.** Let \(S^2 = \{(q^1, q^2) \in \mathbb{R}^2, (q^1)^2 + (q^2)^2 + (q^3)^2 = 1\}\) and consider the following system of RFDE:

\[
\dot{q}^1(t) = -q^1(t - \tau)q^2(t) - q^3(t), \\
\dot{q}^2(t) = q^1(t - \tau)q^1(t) - q^3(t), \\
\dot{q}^3(t) = q^1(t) + q^2(t).
\]

(1.5)

If \((q^1(t), q^2(t), q^3(t))\) is a solution of the equation (1.5), it is easy to see that

\[
\sum_{i=1}^{3} q^i(t)\dot{q}^i(t) = 0
\]

for all \(t \geq 0, \forall \tau\). As a consequence, for \(t \geq 0\), \(\sum_{i=1}^{3} (q^i(t))^2 = a^2, a\) constant.

Thus, if an initial condition \(\phi = (\phi^1, \phi^2, \phi^3)\) satisfies \(\phi(\theta) \in S^2\) for all \(\theta \in [-\tau, 0]\), we conclude that the solution \((q^1, q^2, q^3)(t; \phi) \in S^2\) for all \(t \geq 0\).

With this remark, we can define an RFDE on \(S^2\) by the map

\[
F : \phi = (\phi^1, \phi^2, \phi^3) \in C([-\tau, 0], S^2) \to F(\phi) \in TS^2
\]

where \(F(\phi)\) is the tangent vector to \(S^2\) at the point \(\phi(0)\) given by

\[
F(\phi) = (-\phi^1(-\tau)\phi^2(0) - \phi^3(0), \phi^1(-\tau)\phi^1(0) - \phi^3(0), \phi^1(0) + \phi^2(0))
\]
We now formalize the notions in these examples to obtain a RFDE on a $n$–dimensional manifold. Roughly speaking, a RFDE on a manifold $Q$ is a function $F$ mapping each continuous path $\phi$ lying on $Q$, $\phi \in C([-\tau,0],Q)$, into a vector tangent $F(\phi)$ to $Q$ at the point $\phi(0) \in Q$.

Let $Q$ be a separable $C^\infty$ finite $n$–dimensional manifold (configuration manifold), $I = [-\tau,0]$, $\tau \geq 0$, and $C(I,Q)$ the totality of the continuous maps $\phi$ of $I$ into $Q$. The space $C(I,Q)$ is separable and is a $C^\infty$–manifold modeled on a separable Banach space.

If $\rho : C(I,Q) \to Q$ is the evaluation map, $\rho(\phi) = \phi(0)$, then $\rho$ is $C^\infty$, and for each $q \in Q$, $\rho^{-1}(q)$ is a closed submanifold of $C(I,Q)$ of codimension $n = \dim Q$.

A retarded functional differential equation (RFDE) on $Q$ is a continuous function $F : C(I,Q) \to TQ$ such that $\pi_{TQ} \circ F = \rho$. If we want to emphasize the function $F$ defining the RFDE, we write RFDE($F$).

A solution of RFDE($F$) is defined in the obvious way, namely, as a continuous function $q : [-\tau,\alpha) \to Q$, $\alpha > 0$, such that $\dot{q}(t)$ exists and is continuous for $t \in [0,\alpha)$ and $(q(t),\dot{q}(t)) = F(q_t)$, for $t \in [0,\alpha)$ where $q_t(\theta) = q(t + \theta)$, $\theta \in [-\tau,0]$. Locally, if $F(\phi) = (\phi(0),f(\phi))$ for an appropriate function $f$, then this is equivalent to $\dot{q}(t) = f(q_t)$.

The basis theory of existence, uniqueness and continuous dependence on initial data for general RFDE on manifold is the same as the theory when $Q = \mathbb{R}^n$ [4].

Any $C^k$–vector field on $Q$ defines a $C^k$–RFDE on $Q$. In fact, if $X : Q \to TQ$ is a $C^k$–vector field on $Q$, it is easy to see that $F = X \circ \rho$ is a $C^k$–RFDE on $Q$.

To show that the equation considered in Example 2 is an RFDE according to the definition, we need the concept of a second order RFDE on $Q$. Let $\overline{F} : C(I,TQ) \to TQ \times TQ$ be a continuous function that locally has the representation

$$\overline{F}(\phi,\psi) = ((\phi(0),\psi(0)), (\psi(0), f(\phi,\psi))).$$

The solution $(x(t),y(t))$ of the RFDE($\overline{F}$) on $TQ$ satisfies the equations

$$\dot{q}(t) = y(t), \quad \ddot{y}(t) = f(x_t,y_t)$$

where $q(t) \in Q$. If it is possible to perform the differentiations, then we obtain the second–order equation

$$\ddot{q}(t) = f(q_t,\dot{q}_t).$$

(1.6)
If we now return to Example 2, we see that the formulation requires that we consider initial data in the space \( C(I, S^1) \times C(I, \mathbb{R}) \). However, this does not affect the dynamics since the solution will be in space \( C(I, S^1) \times \mathbb{R} \) after one unit of time.

Let \( g : Q \times Q \to TQ \) be such that \((\pi_{TQ} \circ g)(q, \bar{q}) = q\) and let \( d : C(I, Q) \to Q \times Q \) be defined by \( d(\phi) = (\phi(0), \phi(-\tau)) \). The function \( F = g \circ d \) is a RFDE on \( Q \) which can be written locally as

\[
\dot{q}(t) = \overline{g}(q(t), q(t - \tau))
\]  

(1.7)

where \( g(q, \bar{q}) = (q, \overline{g}(q, \bar{q})) \). The equations (1.7) is a delay differential equation (DDE) on \( Q \).

A \( C^k \)-vector field with delay is given by the mappings \( g : Q \times Q \to T^*Q \), such that \((\pi_{T^*Q} \circ g)(q_1, q_2) = q_1\), and \( d : C(I, Q) \to Q \times Q \), defined by \( d(\phi) = (\phi(0), \phi(-\tau)) \). The function \( G = g \circ d \) is a covector field with delay on \( Q \) that can be written locally as

\[
\omega(t) = f_i(q(t), q(t - \tau)) dq^i
\]  

(1.8)

where \( g(q, \bar{q}) = (q, f(q, \bar{q})) \).

The topological properties of the RFDE on manifolds are discussed in [5].

2. Motivating examples

To get a concrete idea of the type of dissipative mechanisms with delay we have in mind, we now give a simple example of it for perhaps the most basic of Euler–Poincaré or Lie–Poisson systems, namely the rigid body. Here, the Lie algebra in question is that of the rotation group; that is, the Euclidean space \( \mathbb{R}^3 \) interpreted as the space of body angular velocities \( \Omega \) equipped with the cross product as the Lie bracket. On this space, we put the standard kinetic energy Lagrangian \( L(\Omega) = \frac{1}{2}(I \cdot \Omega) \) [where \( I = \text{diag}(I_1, I_2, I_3) \) is the moment of inertial tensor] so that the general Euler–Poincaré equations become the standard rigid body equations for a freely spinning rigid body:

\[
I \dot{\Omega} = (I \Omega) \times \Omega
\]  

(2.1)

or, in terms of the body angular momentum \( M = I \Omega \),

\[
\dot{M} = M \times \Omega.
\]  

(2.2)
In this case, the energy equals the Lagrangian; $E(\Omega) = L(\Omega)$ and the energy is conserved by the solutions of (2.1). Now we modify the equations by adding a term with delay

$$\dot{M} = M \times \Omega + \alpha M \times (\tilde{M} \times \Omega) \quad (2.3)$$

where $\alpha$ is a positive constant, $\tilde{M}(t) = M(t - \tau)$, for $t \geq 0$, $\tau \geq 0$ and $\tilde{M}(t) = \varphi(t)$, for $t \in [-\tau, 0]$.

A related example is the Laudau–Lifshitz equations with delay for the magnetization vector $M$ in a given magnetic field $B$

$$\dot{M} = \gamma M \times B + \frac{\lambda}{\|M\|^2 \cos \theta} \left( M \times (\tilde{M} \times \tilde{B}) \right) \quad (2.4)$$

where $\gamma$ is the magneto–mechanical ratio, $\lambda$ is the damping coefficient due to domain walls and $\theta$ is the angle between $M$ and $\tilde{M}$.

One checks in each case that the addition of the dissipative term with delay has a member of interesting properties. First of all, this dissipation with delay is derivable from a $SO(3)$–invariant force field. However, it is induced by a dissipation function with delay in the following restricted sense. It is a gradient when is restricted to each momentum sphere (coadjoint orbit) where each sphere carries a special metric (later to be called the normal metric). Namely, the extra dissipative term with delay in (2.3) equals the negative gradient of the Hamiltonian with respect to the following metric on the sphere. Each vector $v \in \mathbb{R}^3$ can be orthogonally decomposed with respect to the standard metric on $\mathbb{R}^3$ into a component tangent at $M$ to the sphere $\|M\|^2 = c^2$ and a component on $\tilde{M}$, where $\|\tilde{M}\| = c^2$, $\tilde{M} \neq M$:

$$v = \frac{M \cdot v}{c^2 \cos \theta} \tilde{M} - \frac{1}{c^2 \cos \theta} \left[ M \times (\tilde{M} \times v) \right] \quad (2.5)$$

where $\theta$ is the angle from $M$ and $\tilde{M}$. The metric on the sphere is chose to be $(c^2 \cos \theta)^{-2} \alpha$ times the standard inner product of the components tangent to the sphere in the case of the rigid body model with delay and just $\lambda$ times the standard metric in the case of the Laudau–Lifshitz equations with delay.

Secondly, the dissipation with delay to the equations has the obvious form of a repeated Lie bracket, i.e. a double bracket, and it has the properties that the conservation law

$$\frac{d}{dt} \|M\|^2 = 0 \quad (2.6)$$
is preserved by the dissipation with delay (since the extra force is orthogonal to $M$) and the energy is strictly monotone except at relative equilibria. In fact, we have:

$$\frac{d}{dt}E = -\alpha \|\tilde{M} \times \tilde{\Omega}\|^2$$

for the rigid body and

$$\frac{d}{dt}E = -\frac{\lambda}{\|M\|^2 \cos \theta} \|\tilde{M} \times \tilde{B}\|^2$$

in the case of the Laudau–Lifschitz equations, so that the trajectories on the angular momentum sphere converge to the minimum (for $\alpha$ and $\lambda$ positive) of the energy restricted to the sphere, apart from the set of measure zero consisting of orbits that are relative equilibria or are the stable manifolds of the perturbed saddle point.

Another interesting feature of the dissipations with delay is that they can be derived from a bracket in the same way that the Hamiltonian equations can be derived from a skew symmetric Poisson bracket. For the case of the rigid body with delay, this bracket is

$$\{\{F, K\}\} = \alpha (M \times \nabla \tilde{F}) \cdot (\tilde{M} \times \nabla \tilde{K}).$$

As we have already indicated, the same formalism can be applied to other systems as well. In fact, later in the paper we develop an abstract constructions for dissipative with delay terms with the same general properties as the above examples.

3. Dissipative systems with delay

For later use, it will be useful some of the basic and essential facts about dissipative mechanical systems with delay. Let $Q$ be a manifold, $L : TQ \to \mathbb{R}$ be a smooth function and let $\pi : TQ \to Q$ be the tangent bundle projection. Let $FL : TQ \to T^*Q$ be the fibre derivative of $L$; recall that it is defined by

$$< FL(v), w > = \frac{d}{d\varepsilon}|_{\varepsilon=0} L(v + \varepsilon w).$$

Where $< , >$ denotes the pairing between the tangent and cotangent spaces. We also recall that the vertical lift of a vector $w \in T_qQ$ along $v \in T_qQ$ is
defined by:

\[
\text{ver}_v(w) = \frac{d}{d\varepsilon}|_{\varepsilon=0}(v + \varepsilon w) \in T_v(TQ).
\] (3.2)

The action and energy of \( L \) are defined by

\[
A(v) = \langle FL(v), v \rangle
\] (3.3)

and

\[
E(v) = A(v) - L(v).
\] (3.4)

Let \( \Omega_L = (FL)^*\Omega \) denote the pull back of the canonical sympletic form on \( T^*Q \) by the fibre derivative of \( L \).

A vector field \( Z \) on \( TQ \) is called a Lagrangian vector field of \( L \) if

\[
i_Z\Omega_L = dE.
\] (3.5)

In this generality, \( Z \) need not exist, nor be unique. However, we shall assume throughout that \( Z \) is a second order equation; that is \( T\pi \circ Z \) is the identity on \( TQ \). A second order equation is a Lagrangian vector field if and only if the Euler–Lagrange equations hold in local charts. We note that, by skew symmetry of \( \Omega_L \), the energy is always conserved; that is, \( E \) is constant along an integral curve of \( Z \). We also recall that the Lagrangian is called regular if \( \Omega_L \) is a (weak) sympletic form; it is nondegenerate. This is equivalent to the second fibre derivative of the Lagrangian being, in local charts, also weakly nondegenerate. In the regular case, if the Lagrangian vector field exists, it is unique, and is given by the Hamiltonian vector field with energy \( E \) relative to the sympletic form \( \Omega_L \). If, in addition, the fibre derivative is a global diffeomorphism, then \( Z \) is the pull back by the fibre derivative of the Hamiltonian vector field on the cotangent bundle with Hamiltonian \( H = E \circ (FL)^{-1} \). It is well known how one can pass back and forth between the Hamiltonian and Lagrangian pictures in the hyperregular case [2].

Consider a general Lagrangian vector field \( Z \) for a (not necessarily regular) Lagrangian on \( TQ \). A map \( Y : TQ \times TQ \to TQ \) is called a dissipative vector field with delay if it is vertical, i.e. \( T\pi \circ Y = 0 \) and if at each point of \( TQ \times TQ \)

\[
< dE, Y > \leq 0.
\] (3.6)

If the inequality is pointwise strict at each nonzero \( v \in TQ, \bar{v} \in TQ, v \neq \bar{v} \), then we say that the map \( Y \) is dissipative. A dissipative Lagrangian system
on $TQ$ is a vector field of the form $X = Z + Y$, where $Z$ is a (second order) Lagrangian vector field and $Y$ is a dissipative vector field with delay. Define the 1–form $\Delta^Y$ on $TQ \times TQ$ by

$$\Delta^Y = -i_Y \Omega_L$$

and the force field with delay $F^Y : TQ \times TQ \to T^*Q$ given by

$$< F^Y (v, \bar{v}), w > = \Delta^Y (v, v) \cdot V_v = -\Omega_L(v) (Y (v, \bar{v}), V_v)$$

where $T\pi(V_v) = w$, and $V_v \in T_v(TQ)$.

**Proposition 3.1.** A vertical vector field $Y : TQ \times TQ \to TQ$ is dissipative with delay if and only if the induced force field with delay $F^Y$ satisfies

$$< F^Y (v, \bar{v}), v > < 0$$

for all nonzero $v \in TQ$ ($\leq 0$ for the weakly dissipative with delay).

**Proof.** Let $Y$ be a vertical vector field $Y : TQ \times TQ \to TQ$ with $T\pi \circ Y = 0$, $\Delta^Y$ the form on $TQ \times TQ$ given by (3.7) and $F^Y$ the force field with delay given by (3.8). If $Z$ denotes the Lagrangian system defined by $L$, we get

$$(dE \cdot Y) (v, \bar{v}) = (i_Z \Omega_L)(Y) (v, \bar{v}) = \Omega_L(Z, Y) (v, \bar{v}) =$$

$$= -\Omega_L(v) (Y (v, \bar{v}), Z(v)) =$$

$$= < F^Y (v, \bar{v}), T_v \pi(Z(v)) > = < F^Y (v, \bar{v}), v > ,$$

since $Z$ is a second–order equation. We conclude that $< dE, Y > < 0$ if and only if $< F^Y (v, \bar{v}), v > < 0$, for all $(v, \bar{v}) \in TQ \times TQ$, $v \neq \bar{v}$, which gives the result.

Treating $\Delta^Y$ as the exterior force with delay 1–form acting on a mechanical system with a Lagrangian $L$, we now shall write the governing equation of motion. The basic principle is of course the Lagrange–d’Alembert principle.

The *Lagrangian force* associated with a given Lagrangian $L$ and a given second–order vector field $X$ is the horizontal 1–form on $TQ$ defined by

$$\phi_L(X) = i_X \Omega_L - dE.$$  (3.9)

Given a horizontal 1–form $\omega$ (referred to 1–form as the exterior force with delay), the local Lagrange–d’Alembert principle states that

$$\phi_L(X) + \omega = 0.$$  (3.10)
It is easy to check that $\phi_L(X)$ is indeed horizontal if $X$ is of second order. Conversely, if $L$ is regular and if $\phi_L(X)$ is horizontal, then $X$ is of second order. One can also formulate an equivalent principle in variational form.

Given a Lagrangian $L$ and a force field with delay (as defined in Proposition 1) the integral Lagrange–d’Alembert principle with delay for a curve $q(t)$ in $Q$ is

$$\delta \int_a^b L(q(t), \dot{q}(t)) \, dt + \int_a^b F\left( (q(t), \dot{q}(t)), \left( \tilde{q}(t), \tilde{\dot{q}}(t) \right) \right) \cdot \delta q \, dt = 0,$$

where the variation is given by the usual expression

$$\delta \int_a^b L(q(t), \dot{q}(t)) \, dt = \int_a^b \left( \frac{\partial L}{\partial q^i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) \right) \delta q^i \, dt \quad (3.12)$$

for a given variation $\delta q$ (vanishing at the endpoints).

The two forms of the Lagrange–d’Alembert principle are equivalent. This follows from the fact that both give the Euler–Lagrange equations with forcing with delay in local coordinates (provided that $Z$ is of second order). We shall see this in the following development.

**Proposition 3.2.** Let $\omega$ be the delay exterior force 1–form associated to a vertical vector field with delay $Y$, i.e. $\omega = \Delta Y = -i_Y \Omega_L$. Then $X = Z + Y$ satisfies the local Lagrange–d’Alembert principle with delay. Conversely, if, in addition, $L$ is regular, the only second-order vector field $X$ satisfying the local Lagrange–d’Alembert principle with delay is $X = Z + Y$.

**Proof.** For the first part, the equality $\phi_L(X) + \omega = 0$ is a simple verification. For the converse, we already know that $X$ is a solution and the uniqueness is guaranteed by regularity.

To develop the differential equations associated to $X = Z + Y$, we take $\omega = -i_Y \Omega_L$ and note that, in a coordinate chart, $Y\left( (q, \dot{q}), \left( \tilde{q}, \tilde{\dot{q}} \right) \right) = Y^i \left( (q, \dot{q}), \left( \tilde{q}, \tilde{\dot{q}} \right) \right) \frac{\partial}{\partial q^i}$ and the equation (3.10) is given by

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} Y^j \left( (q^k, \dot{q}^k), \left( \tilde{q}^k, \tilde{\dot{q}}^k \right) \right). \quad (3.13)$$

The force 1–form with delay $\Delta Y$ is therefore given by

$$\Delta Y \left( (q^k, \dot{q}^k), \left( \tilde{q}^k, \tilde{\dot{q}}^k \right) \right) = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} Y^j \left( (q^k, \dot{q}^k), \left( \tilde{q}^k, \tilde{\dot{q}}^k \right) \right) \, dq^i \quad (3.14)$$
and the corresponding force field with delay is given by
\[ F_i^Y = \left( q^k, \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} Y^j \left( (q^k, \dot{q}^k), (\tilde{q}^k, \dot{\tilde{q}}^k) \right) \right). \] (3.15)

Thus, the condition for an integral curve takes the form of the Euler–Lagrange equations with force with delay
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = F_i^Y \left( (q^k, \dot{q}^k), (\tilde{q}^k, \dot{\tilde{q}}^k) \right). \] (3.16)

Since the integral Lagrange–d’Alembert principle with delay gives the same equations, it follows that the two principles are equivalent. From now on, we shall refer to either one as simple the Lagrange–d’Alembert principle with delay.

**Example 1.** The inertial neuron with time delay is described by the equations
\[ \ddot{q}^i = -a \dot{q}^i - bq^i + cf(q^i - h \tilde{q}^i) + d \sum_{j=1}^{n} \sum_{j \neq i} f \left( q^j - h \tilde{q}^j \right), \quad i = 1, n, \] (3.17)

where \( a, b, c, d > 0, h \geq 0 \) are constants and \( \tilde{q}^i(t) = q^i(t-\tau) \) is the time delay.

For \( n = 1 \) the model is discussed in [1]. For \( L(\dot{q}, q) = \frac{1}{2} \sum_{i=1}^{n} (\dot{q}^i)^2 - b \sum_{i=1}^{n} q^i \) and the force field with delay given by
\[ Y^i \left( \dot{q}, \tilde{q} \right) = -a \dot{q}^i + cf(q^i - h \tilde{q}^i) + d \sum_{j=1}^{n} \sum_{j \neq i} f \left( q^j - h \tilde{q}^j \right), \] (3.18)

the equations (3.17) are the Euler–Lagrange equations with force with delay. The model for \( n = 2 \) is analyzed in [9]. The force field with delay (3.18) is not dissipative.

**Example 2.** The simplest mechanical model of the regenerative machine tool vibration in the case of the so–called orthogonal cutting is given by the equation
\[ \ddot{q} + 2k \alpha \dot{q} + \alpha^2 q = \frac{1}{m} f(\dot{q}, \beta), \] (3.19)
where \( \alpha = \sqrt{s/m} \) is the natural angular frequency of the undamped free oscillating system, \( k \) is the so-called relative damping factor and \( f(\dot{q}, \beta) \) is the cutting force as a function of technological parameters and as a function of the chip thickness \( \beta \) which depends on the position \( q \) of the tool edge. In [12] the function \( f \) is given by

\[
f(\tilde{q}, \dot{q}, \beta) = -\frac{2\pi k_1}{8\beta \Omega m} \left[ (\dot{q}_1 - \tilde{q}_1) + \frac{5}{\beta} (q_1 - \tilde{q}_1)^3 \right],
\]

(3.20)

where \( k_1 = \frac{3}{4} aw\beta^{-\frac{1}{2}} \) is the parameter depending on further technological parameters and considered to be constant, \( w \) is the width of the chip, \( \Omega \) is the constant angular velocity of rotating work–piece (or tool), the delay \( \tau = \frac{2\pi}{\Omega} \) is the time period of one revolution and \( \tilde{q}_1(t) = q^1(t - \tau) \). For

\[
L(q, \dot{q}) = \frac{1}{2} \dot{q}^2 - \frac{1}{2} \alpha \dot{q}^2 \quad \text{and the force filed with delay given by}
\]

\[
Y(q, \dot{q}) = -2k\alpha \dot{q} + f(q, \dot{q}, \beta),
\]

(3.21)

the equation (3.19) is the Euler–Lagrange equation with force with delay. The model was analysed in [12]. The force field given by (3.21) is not dissipative.

4. Equivariant dissipation with delay

In this section we study Lagrangian systems that are invariant under a group action and we shall add to them, in the sense of the preceding sections, dissipative fields with delay that are equivariant. This invariance property will yield dissipative mechanisms with delay that preserve the basic conserved quantities, yet dissipate energy, as we shall see.

Let \( G \) be a Lie group acting on the configuration manifold \( Q \) and assume that the lifted action leaves the Lagrangian \( L \) invariant. In this case, the fibre derivative \( \mathbb{I}L : TQ \to T^*Q \) is equivariant with respect to this action on \( TQ \) and the dual action on \( T^*Q \). The action \( A \), the energy \( E \) and the Lagrangian 2–form \( \Omega_L \) are all invariant under the action of \( G \) on \( TQ \). Let \( Z \) be the Lagrangian vector field for the Lagrangian \( L \), which we assume to be regular. Because of regularity, the vector field \( Z \) is also invariant under \( G \). If the action is free and proper, so that \( (TQ)/G \) is a manifold, then the vector field and its flow \( F_t \) drop to a vector field \( Z^G \) and a flow \( F^G_t \) on \( (TQ)/G \), [2].
Let \( J : TQ \to \mathfrak{g}^* \) be the momentum map associated with the \( G \)-action, given by:

\[
J(v_q) \cdot \xi = \langle \mathbb{F} L(v_q), \xi_Q(q) \rangle
\]

for \( v_q \in T_qQ \) and for \( \xi \in \mathfrak{g} \), where \( \xi_Q \) denotes the infinitesimal generator for the action on \( Q \). The infinitesimal generator for the action on the tangent bundle will be likewise denoted by \( \xi_{TQ} \) and for later use, we note the relation \( T\pi \circ \xi_{TQ} = \xi_Q \circ \pi \). If \( v(t) \) denotes an integral curve of the vector field with an equivariant dissipation term with delay \( Y \) added as in the preceding section and if \( J^\xi(v) = \langle J(v), \xi \rangle \) is the \( \xi \)-component of the momentum mapping, then we have:

\[
\frac{d}{dt}J^\xi(v(t)) = dJ^\xi(v(t)) \cdot Z(v(t)) + dJ^\xi(v(t))Y(v(t), \bar{v}(t)).
\]

The first term vanishes by conservation of the momentum map for the Lagrangian vector field \( Z \). From (2.8) and the definition of the momentum map, we get:

\[
dJ^\xi(v) \cdot Y(v, \bar{v}) = (i_{\xi_{TQ}}\Omega_L)(Y)(v, \bar{v}) = -(i_Y\Omega_L)(\xi_{TQ})(v, \bar{v})
\]

\[
= \langle F^Y(v, \bar{v}), T_{\xi_{TQ}}\pi(v) \rangle = \langle F^Y(v, \bar{v}), \xi_Q(\pi(v)) \rangle
\]

and therefore

\[
\frac{d}{dt}J^\xi(v(t)) = \langle F^Y(v(t), \bar{v}(t)), \xi_QT\pi(v(t)) \rangle.
\]

We summarize this discussion as follows.

**Proposition 4.1.** The momentum map \( J : TQ \to \mathfrak{g}^* \) is conserved under the flow of a \( G \)-invariant dissipative vector field with delay \( Z + Y \) if and only if \( \langle F^Y, \xi_Q \circ \tau \rangle = 0 \) for all Lie algebra elements \( \xi \in \mathfrak{g} \).

In this paper we shall consider dissipative vector fields with delay for which the flow drops to the reduced spaces. Thus a first requirement is that \( Y \) be a vertical \( G \)-invariant vector field on \( TQ \times TQ \). A second requirement is that all integral curves \( v(t) \) of \( Z + Y \) preserve the sets \( J^{-1}O \), where \( O \) is an arbitrary coadjoint orbit in \( \mathfrak{g}^* \). Under these hypotheses the vector field \( Z + Y \) induces a vector field \( Z^G + Y^G \) on \((TQ \times TQ)/G \times G\) that preserves the symplectic leaves of this Poisson manifold, namely all reduced spaces \( J^{-1}O/G \).
The condition that $v(t) \in J^{-1}O$ is equivalent to $J(v(t)) \in O^\ast$, i.e., to the existence of an element $\eta(t) = \eta(v(t)) \in g$ such that $dJ(v(t))/dt = ad_{\eta(t)}^*J(v(t))$ or
\[
\frac{dJ^\xi(v(t))}{dt} = J[\eta(t), \xi](v(t))
\]
for all $\xi \in g$. In view of (4.4), we get the following

**Corollary 4.2.** The integral curves of the vector field $Z + Y$, for $Y$ a vertical $G$–invariant vector field on $TQ \times TQ$ and $Z$ the Lagrangian vector field of a $G$–invariant Lagrangian function $L : TQ \to \mathbb{R}$, preserve the inverse images of the coadjoint orbits in $g^\ast$ by the momentum map $J$ if and only if for each $(v, \tilde{v}) \in TQ \times TQ$ there is some $\eta(v, \tilde{v}) \in g$ such that
\[
<F^Y(v, \tilde{v}), (\xi_Q \circ T\pi)(v) > = J[\eta(v, \tilde{v}), \xi](v)
\]
for all $\xi \in g$. As before, $F^Y$ denotes the force field with delay induced by $Y$.

We shall see in Section 5 how to construct such a force field with delay in the case $Q = G$.

5. Dissipation with delay for Euler–Poincaré and Lie–Poisson equations

A key step in the reduction of the Euler–Lagrange equations from the tangent bundle $TG$ of a Lie group $G$ to its Lie algebra $g$ is to understand how to drop the variational principle to the quotient space. The formulation of the Euler–Poincaré equations and the reduced variational principle is given by

**Theorem 5.1.** Let $G$ be a Lie group and $L : TG \to \mathbb{R}$ a left invariant Lagrangian. Let $l : g \to \mathbb{R}$ be its restriction to the tangent space at the identity. For a curve $g(t) \in G$, let $\xi(t) = Tg(t)Lg(t)^{-1} \dot{g(t)}$. Then the followings are equivalent:

i) $g(t)$ satisfies the Euler–Lagrange equations for $L$ on $G$.

ii) The variational principle
\[
\delta \int_a^b L(g(t), \dot{g(t)}) \, dt = 0
\]
holds for variations with fixed endpoints.
iii) The Euler–Poincaré equations hold,

\[
\frac{d}{dt} \left( \frac{\partial l}{\partial \xi} \right) = ad_\xi^* \left( \frac{\partial l}{\partial \xi} \right). \tag{5.2}
\]

iv) The variational principle
\[
\delta \int_a^b l(\xi(t)) dt = 0 \tag{5.3}
\]
holds on \( g \), using variations of the form \( \delta \xi = \dot{\eta} + [\xi, \eta] \) where \( \eta \) vanishes at the endpoints.

In coordinates, the Euler–Poincaré equations are
\[
\frac{d}{dt} \left( \frac{\partial l}{\partial \xi^d} \right) = C_{ad}^b \frac{\partial L}{\partial \xi^b} \xi^a \tag{5.4}
\]
where \( C_{ad}^b \) are the structure constants of \( g \) relative to a given basis \( \xi^a \) are the components of \( \xi \) relative to this basis.

Since the Euler–Lagrange and the Hamilton equations on \( TQ \) and \( T^*Q \) are equivalent if the fibre derivative of \( L \) is a diffeomorphism from \( TQ \) to \( T^*Q \), it follows that the Lie–Poisson and the Euler–Poincaré equations are also equivalent under similar hypotheses. To see this directly, we make the following Legendre transformation from \( g \) to \( g^* \):
\[
\mu = \frac{\partial l}{\partial \xi}, \quad h(\mu) = \langle \mu, \xi \rangle - l(\xi) \tag{5.5}
\]
and assume that \( \xi \rightarrow \mu \) is a diffeomorphism. Note that \( \frac{\partial h}{\partial \mu} = \xi \) and so it is now clear that the Euler–Poincaré equations are equivalent to the Lie–Poisson equations on \( g^* \), namely
\[
\frac{d\mu}{dt} = ad_{\mu}^* \mu. \tag{5.6}
\]

Now we are ready to synthesize our discussions on forces with delay and on the Euler–Poincaré equations and to transfer this forcing with delay to the Lie–Poisson equations by means of the Legendre transform. We begin with a formulation of the Lagrange–d’Alembert principle with delay.
Theorem 5.2. Let $G$ be a Lie group, $L : TG \to \mathbb{R}$ a left invariant Lagrangian, and $F : TG \times TG \to T^*G$ a force field with delay equivariant relative to the canonical left actions of $G$ on $TG \times TG$ and $T^*G$ respectively. Let $l : g \to \mathbb{R}$ and $f : g \times g \to g^*$ be the restriction of $L$ and $F$ to $T_gG = g$. For a curve $g(t) \in G$, let $\xi(t) = T_{g(t)}L_{g(t)}^{-1}\dot{g}(t)$. Then the followings are equivalent:

i) $g(t)$ satisfies the Euler–Lagrange equations with forcing with delay for $L$ on $G$.

ii) The integral Lagrange–d'Alembert principle with delay

$$\delta \int_a^b L(g(t), \dot{g}(t))dt = \int_a^b F((g(t), \dot{g}(t)), (\tilde{g}(t), \tilde{\dot{g}}(t))) \cdot \delta g(t)dt$$ (5.7)

holds for all variations $\delta g(t)$ with fixed endpoints.

iii) The Euler–Poincaré equations with forcing with delay are valid

$$\frac{d}{dt} \left( \frac{\partial l}{\partial \xi} \right) - ad^*_{\xi} \frac{\partial l}{\partial \xi} = f(\xi, \tilde{\xi})$$ (5.8)

iv) The variational principle

$$\delta \int_a^b l(\xi(t))dt = \int_a^b f(\xi(t), \tilde{\xi}(t)) \cdot \delta \xi(t)dt$$ (5.9)

holds on $g$ using variations of the form $\delta \xi = \dot{\eta} + [\xi, \eta]$ where $\eta$ vanishes at the endpoints.

Proof. We have already seen that i) and ii) are equivalent for any configuration manifold $Q$ in Section 2. Next we prove that ii) and iv) are equivalent. First, note that $l : g \to \mathbb{R}$ and $f : g \times g \to g^*$ determine uniquely a function $L : TG \to \mathbb{R}$, and a function $F : TG \times TG \to TG$ by left translation of the argument and conversely. Thus, the equivalence of ii) and iv) comes down to proving that all variations $\delta g(t) \in TG$ of $g(t)$ with fixed endpoints induce and are induced by variations $\delta \xi(t)$ of $\xi(t)$ of the form $\delta \xi = \dot{\eta} + [\xi, \eta]$ where $\eta(t)$ vanishes at the endpoints. But this is precisely the matter of the proposition 5.1 in [2].

The Euler–Poincaré equations with forcing with delay have the following expression in local coordinates:

$$\frac{d}{dt} \left( \frac{\partial l}{\partial \xi^a} \right) - C_{ba}^d \frac{\partial l}{\partial \xi^d} = f_a(\xi, \tilde{\xi})$$ (5.10)
where \( C_{ba}^d \) are the structure constants of the Lie algebra \( g \).

The condition that the integral curves of the dissipative vector field with delay preserve the inverse images of coadjoint orbits by momentum map and hence the integral curves of (5.8) preserve the coadjoint orbits of \( g^* \) is given by (4.4), Section 4. Since \( \xi_G(g) = T_e R_g(\xi) \) and \( J(v_g) = T^{*}_e R_g \mathbb{F}L(v_g) \), we get

\[
< F, \xi_G \circ \pi > (v_g, \tilde{v}_g) =< F(v_g, \tilde{v}_g), T_e R_g(\xi) >= T^{*}_e R_g F(v_g, \tilde{v}_g) \cdot \xi
\]

and

\[
J[\eta(v_g, \tilde{v}_g), \xi](v_g) = T^{*}_e R_g \mathbb{F}L(v_g) \cdot [\eta[v_g, \tilde{v}_g], \xi] = (ad^{*}_{\eta(v_g, \tilde{v}_g)} \circ T^{*}_e R_g \circ \mathbb{F}L)(v_g) \cdot \xi.
\]

Since \( F \) and \( \mathbb{F}L \) are equivariant,

\[
T^{*}_e R_g F(v_g, \tilde{v}_g) = Ad^{*}_{g^{-1}} F(T_g L_{g^{-1}} v_g, T_{\tilde{g}} L_{g^{-1}} \tilde{v}_g)
\]

and

\[
(ad^{*}_{\eta(v_g, \tilde{v}_g)} \circ T^{*}_e R_g \circ \mathbb{F}L)(v_g) = (ad^{*}_{\eta(v_g, \tilde{v}_g)} \circ Ad^{*}_{g^{-1}} \circ \mathbb{F}l)(T_g L_{g^{-1}} v_g).
\]

Because \( Ad_{g^{-1}} \circ ad_{\eta(v_g, \tilde{v}_g)} = ad_{Ad_{g^{-1}} \eta(v_g, \tilde{v}_g)} \circ Ad_{g^{-1}} \) we get

\[
J[\eta(v_g, \tilde{v}_g), \xi](v_g) = Ad^{*}_{g^{-1}} \circ ad^{*}_{Ad_{g^{-1}} \eta(v_g, \tilde{v}_g)} \circ \mathbb{F}L(T_g L_{g^{-1}} v_g)
\]

and the identity (4.4), Section 4 thus becomes

\[
F(T_g L_{g^{-1}} v_g, T_{\tilde{g}} L_{g^{-1}} \tilde{v}_g) = (ad^{*}_{Ad_{g^{-1}} \eta(v_g, \tilde{v}_g)} \circ \mathbb{F}L)(T_g L_{g^{-1}} v_g).
\]

Letting \( \xi = T_g L_{g^{-1}} v_g, \tilde{\xi} = T_{\tilde{g}} L_{g^{-1}} \tilde{v}_g \), this becomes

\[
f(\xi, \tilde{\xi}) = ad^{*}_{Ad_{g^{-1}} \eta(v_g, \tilde{v}_g)} \frac{\partial l}{\partial \xi}(\xi), \quad (5.11)
\]

The left hand side is independent of \( g \) and thus the right hand side must be also \( g \)-independent. Thus taking \( g = \tilde{g} = e \) the criterion (4.4), Section 4 becomes: for every \( \xi, \tilde{\xi} \in g \) there is some \( \eta(\xi, \tilde{\xi}) \in g \) such that

\[
f(\xi, \tilde{\xi}) = ad^{*}_{\eta(\xi, \tilde{\xi})} \frac{\partial l}{\partial \xi}(\xi), \quad (5.12)
\]
In other words, the force field with delay \( f \) (and hence \( F \)) is completely determined by an arbitrary map \( \eta : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \) via formula (5.11) and we conclude the following.

**Corollary 5.3.** The solutions of the Euler–Poincaré equations with forcing with delay (5.8) preserve the coadjoint orbits of \( \mathfrak{g}^* \) provided the force field with delay \( f \) is given by (5.12) for some smooth map \( \eta : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \).

Transforming the Euler–Poincaré equations with forcing with delay by means of the Legendre transformation (5.5), the equations (5.8) with the force field with delay (5.12) become

\[
\frac{d\mu}{dt} - ad_{\eta(\mu, \bar{\mu})}^* \mu = -ad^*_{\eta(\mu, \bar{\mu})} \mu
\]

(5.13)

where \( \eta : \mathfrak{g}^* \times \mathfrak{g}^* \to \mathfrak{g} \). The requirement on the map \( \eta \) is that the right hand side of (5.13) be a gradient relative to a certain metric on the orbit.

To generalize the metric defined by Killing form [3] to coadjoint orbits of the dual \( \mathfrak{g}^* \) of a general Lie algebra \( \mathfrak{g} \), we introduce a symmetric positive definite bilinear form.

Let a symmetric positive definite bilinear form \( \tilde{\Gamma} : \mathfrak{g}^* \times \mathfrak{g}^* \to \mathbb{R} \) and denote by \( \Gamma : \mathfrak{g}^* \to \mathfrak{g} \) the induced map given by \( \tilde{\Gamma}(\alpha, \beta) = < \beta, \Gamma \alpha > \) for all \( \alpha, \beta \in \mathfrak{g}^* \), where \( <,> : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R} \) is the pairing between \( \mathfrak{g}^* \) and \( \mathfrak{g} \).

Symmetry of \( \tilde{\Gamma} \) is equivalent to symmetry of \( \Gamma \), i.e. \( \Gamma^* = \Gamma \). We introduce the following new inner product on \( \mathfrak{g} \):

\[
< \xi, \eta >_{\Gamma^{-1}} = < \Gamma^{-1} \eta, \xi >
\]

(5.14)

for all \( \xi, \eta \in \mathfrak{g} \) and call it the \( \Gamma^{-1} \)-inner product.

Let \( \mathfrak{g}_\mu = \{ \xi \in \mathfrak{g} \mid ad_\xi^* \mu = 0 \} \) denote the coadjoint isotropy subalgebra of \( \mu \in \mathfrak{g} \) and denote by \( \mathfrak{g}_\mu^* \) the orthogonal complement of \( \mathfrak{g}_\mu \) relative to the \( \Gamma^{-1} \)-inner product. For an element \( \xi \in \mathfrak{g} \) we denote by \( \xi_\mu \) and \( \xi_\mu^* \) the components of \( \xi \) in the orthogonal direct sum decomposition \( \mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{g}_\mu^* \). For all \( g \in G, \xi \in \mathfrak{g} \) and \( Ad_g^* : \mathfrak{g}^* \to \mathfrak{g}^*, ad_\xi^* : \mathfrak{g}^* \to \mathfrak{g}^*, Ad_g : \mathfrak{g} \to \mathfrak{g} \) we have

\[
\Gamma(Ad_g^* \mu) = Ad_g \Gamma \mu, \\
ad_\xi^*(Ad_g^* \mu) = Ad_g(\ad_{Ad_g^* \xi}(Ad_g^* \mu)), \quad \mu \in \mathfrak{g}^*.
\]

(5.15)

Let \( \mathcal{O}_{\mu_0}^\mu \) be the coadjoint orbit through \( \mu_0 \in \mathfrak{g}^* \) and \( \mu, \bar{\mu} \in \mathcal{O}_{\mu_0}^\mu \). There exists \( g \in G \) such that \( \mu = Ad_{g^{-1}} \bar{\mu} \) and \( \mathfrak{g}_\mu = Ad_{g^{-1}} \mathfrak{g}_{\bar{\mu}}, \mathfrak{g}_\mu^* = Ad_{g^{-1}} \mathfrak{g}_{\bar{\mu}}^* \).
Let $C$ be a positive Casimir function on $\mathfrak{g}^*$ and let $\omega$ be the coadjoint orbit symplectic structure defined by

$$
\omega(\mu)(\xi_\ast(\mu), \eta_\ast(\mu)) = -<\mu, [\xi, \eta]>
$$

for all $\mu \in O_{\mu_0}$, $\xi, \eta \in \mathfrak{g}$ and $\xi_\ast(\mu), \eta_\ast(\mu) \in T O_{\mu_0}$. If $\mu, \tilde{\mu} \in O_{\mu_0}$ then $ad^*_\xi \mu \in T_{\mu} O_{\mu_0}$, $ad^*_\xi \tilde{\mu} \in T_{\mu} O_{\mu_0}$. Since $\mu = Ad_{g^{-1}} \tilde{\mu}$ it results

$$
ad^*_\xi \mu = Ad_{g^{-1}} \left( ad^*_\xi \tilde{\mu} \right).
$$

We define the $(C, \Gamma^{-1})$–normal metric on $O_{\mu_0}$ with respect to $\mu$ and $\tilde{\mu}$ by

$$
<ad^*_\xi \tilde{\mu}, ad^*_\eta \tilde{\mu}>_N (\mu) = C(\tilde{\mu}) <\Gamma^{-1}\eta \tilde{\mu}, \xi \tilde{\mu}> = -
$$

$$
- \frac{1}{C(\mu)} \omega(\Gamma \xi) ((\Gamma \mu)_\ast, (\Gamma \tilde{\mu})_\ast) \cdot \omega(\Gamma \eta) ((\Gamma \mu)_\ast, (\Gamma \tilde{\mu})_\ast)
$$

for all $\xi, \eta \in \mathfrak{g}$.

We shall regard $C$ and $\Gamma$ as fixed in the following discussion and just refer to this metric as the normal metric. If $\mu = \tilde{\mu}$ then the normal metric is given in [2].

Let $k : \mathfrak{g}^* \to \mathbb{R}$ be a smooth function. We shall compute the gradient vector of $k|_{O_{\mu_0}}$ with respect to the normal metric. For this purpose we denote by $\frac{\partial k}{\partial \mu} \in \mathfrak{g}$ the derivative of $k$ at $\mu$ and by grad $k(\mu)$ the gradient of $k|_{O_{\mu_0}}$.

Since grad $k(\mu) \in T_{\mu} O_{\mu_0}$ we can write grad $k(\mu) = ad^*_\eta \mu$ for some $\eta \in \mathfrak{g}$. Since $\xi_\mu$ and $\eta^\mu$ are orthogonal in the $\Gamma^{-1}$–inner product, we get

$$
-<ad^*_{\tilde{\mu}} \mu, \xi > (\mu) = \left< \mu, \left[ \xi, \frac{\partial k}{\partial \mu} \right] \right> (\mu) = \left< ad^*_{\tilde{\mu}} \mu, \frac{\partial k}{\partial \mu} \right> (\mu) =
$$

$$
= <\text{grad} k(\mu), ad^*_{\tilde{\mu}} \mu >_N (\mu) = <ad^*_\eta \mu, ad^*_{\tilde{\mu}} \mu >_N (\mu) = C(\tilde{\mu}) <\Gamma^{-1}\xi \tilde{\mu}, \eta \tilde{\mu}> -
$$

$$
- \frac{1}{C(\mu)} <\Gamma \xi, [\Gamma \mu, \Gamma \tilde{\mu}] > \cdot <\Gamma \eta, [\Gamma \mu, \Gamma \tilde{\mu}] > = C(\tilde{\mu}) <\Gamma^{-1} \xi \tilde{\mu} + \xi \tilde{\mu}, \eta \tilde{\mu}> -
$$

$$
- \frac{1}{C(\mu)} <\Gamma^{-1} [\Gamma \mu, \Gamma \tilde{\mu}], \xi > \cdot <\Gamma^{-1} [\Gamma \mu, \Gamma \tilde{\mu}], \eta > = C(\tilde{\mu}) <\Gamma^{-1} \eta \tilde{\mu}, \xi > -
$$

$$
- \frac{1}{C(\mu)} <\Gamma^{-1} [\Gamma \mu, \Gamma \tilde{\mu}], \eta > \cdot <\Gamma^{-1} [\Gamma \mu, \Gamma \tilde{\mu}], \xi >
$$

20
for any $\xi \in \mathfrak{g}$. Therefore

$$C(\tilde{\mu}) \Gamma^{-1} \eta^\mu - \frac{1}{C(\mu)} \Gamma^{-1} [\Gamma \mu, \Gamma \tilde{\mu}]^\mu < \Gamma^{-1} [\Gamma \mu, \Gamma \tilde{\mu}], \eta > = -ad^*_\eta \mu. \quad (5.19)$$

Because $\eta^\mu = Ad_g \eta^\mu$ it results

$$C(Ad_g^* \mu) Ad_g \eta^\mu - \frac{1}{C(\mu)} \left\langle \Gamma \left( \frac{\partial k}{\partial \mu} \right), [\Gamma \mu, \Gamma \tilde{\mu}] \right\rangle = -\Gamma \left( ad^* \frac{\partial \mu}{\partial \mu} \right) \quad (5.20)$$

and

$$\eta^\mu = -\frac{1}{C(\tilde{\mu})} Ad_{\tilde{\mu}} - \Gamma \left( ad^* \frac{\partial \mu}{\partial \mu} \right) + \frac{1}{C(\mu) C(\tilde{\mu})} \left\langle \Gamma \left( \frac{\partial k}{\partial \mu} \right), [\Gamma \mu, \Gamma \tilde{\mu}] \right\rangle \cdot [\Gamma \mu, \Gamma \tilde{\mu}]^\mu$$

or

$$\eta^\mu = -\frac{1}{C(\tilde{\mu})} \Gamma \left( ad^* \frac{\partial \mu}{\partial \mu} \right) + \frac{1}{C(\mu) C(\tilde{\mu})} \left\langle \Gamma \left( \frac{\partial k}{\partial \mu} \right), [\Gamma \mu, \Gamma \tilde{\mu}] \right\rangle \cdot [\Gamma \mu, \Gamma \tilde{\mu}]^\mu. \quad (5.21)$$

Thus

$$\text{grad} k(\mu) = ad^* \frac{\partial \mu}{\partial \mu} = -\frac{1}{C(\tilde{\mu})} \Gamma \left( ad^* \frac{\partial \mu}{\partial \mu} \right) +$$

$$+ \frac{1}{C(\mu) C(\tilde{\mu})} \left\langle \Gamma \left( \frac{\partial k}{\partial \mu} \right), [\Gamma \mu, \Gamma \tilde{\mu}] \right\rangle \cdot [\Gamma \mu, \Gamma \tilde{\mu}]^\mu \quad (5.22)$$

and the equation of the gradient vector field in $\mu \in \mathcal{O}_{\mu_0}$ relative to the normal metric on $\mathcal{O}_{\mu_0}$ is

$$\frac{d\mu}{dt} = -\frac{1}{C(\tilde{\mu})} \Gamma \left( ad^* \frac{\partial \mu}{\partial \mu} \right) \tilde{\mu} + \frac{1}{C(\mu) C(\tilde{\mu})} \left\langle \Gamma \left( \frac{\partial k}{\partial \mu} \right), [\Gamma \mu, \Gamma \tilde{\mu}] \right\rangle \cdot [\Gamma \mu, \Gamma \tilde{\mu}]^\mu. \quad (5.23)$$

Therefore in (5.23) we put $\eta(\mu, \tilde{\mu}) = -\eta^\mu$ and the Lie–Poisson equations with delay forcing (5.6) become

$$\frac{d\mu}{dt} = \frac{1}{C(\tilde{\mu})} \Gamma \left( ad^* \frac{\partial \mu}{\partial \mu} \right) \mu - \frac{1}{C(\mu) C(\tilde{\mu})} \left\langle \Gamma \left( \frac{\partial k}{\partial \mu} \right), [\Gamma \mu, \Gamma \tilde{\mu}] \right\rangle \cdot ad^*_\eta \mu. \quad (5.24)$$

If $\mathfrak{g}$ is a compact algebra with the bi-invariant inner product $< \cdot, \cdot >$ on $\mathfrak{g}$ and $\mathfrak{g}$ is also semisimple, then we could let $< \cdot, \cdot > = -k(\cdot, \cdot)$ where $-k(\cdot, \cdot)$
is the Killing form. In these conditions the inner product identifies $g$ with its dual $g^*$, coadjoint orbits with adjoint orbits so that $ad^*_{\xi} \mu = [\mu, \xi]$ and $\frac{\partial k}{\partial \mu} = \nabla k(\mu)$, where $\nabla k(\mu)$ is the gradient of $k$ on $g$ at $\mu$ relative to the bi-invariant inner product $\langle \cdot, \cdot \rangle$. The formula for the gradient vector field on the adjoint orbit $O_{\mu_0}$ relative to $\mu$ and $\tilde{\mu}$ becomes

$$
\frac{d\mu}{dt} = -\frac{1}{C(\mu)}[\mu, \Gamma[\tilde{\mu}, \nabla k(\tilde{\mu})]] - \frac{1}{C(\mu)C(\tilde{\mu})}\left\langle \Gamma \left( \frac{\partial k}{\partial \mu} \right), [\Gamma \mu, \Gamma \tilde{\mu}] \right\rangle \cdot [\mu, [\Gamma \mu, \Gamma \tilde{\mu}]^\mu],
$$

where $\Gamma : g \to g$ defines the symmetric positive definite bilinear form $(\xi, \eta) \mapsto \langle \Gamma \xi, \eta \rangle$. Thus in this case the Lie–Poisson equations with delay forcing become

$$
\frac{d\mu}{dt} = -[\nabla h(\mu), \mu] + \frac{1}{C(\mu)}[\mu, \Gamma[\tilde{\mu}, \nabla k(\tilde{\mu})]] - \frac{1}{C(\mu)C(\tilde{\mu})}\left( \Gamma(\nabla k(\mu)), [\Gamma \mu, \Gamma \tilde{\mu}] \right) \cdot [\mu, [\Gamma \mu, \Gamma \tilde{\mu}]^\mu].
$$

Taking $C(\mu) = 1$ and $\Gamma$ to be the identity, the dissipative term with delay in (5.26) is the Brockett double bracket. The condition that the delay forcing term be dissipative is $\frac{dh}{dt} < 0$ and this imposes some conditions on the choice of the function $k : g^* \to \mathbb{R}$.

6. Free rigid body with delay

Let $G = SO(3)$ and the usual identification $(so(3), [\cdot, \cdot]) \cong (\mathbb{R}^3, \times)$ implies $so(3)^* \cong \mathbb{R}^3$ via the natural pairing given by the Euclidean inner product. Consider $O(M_0)$ the coadjoint orbit through $M_0 \in \mathbb{R}^3$. The infinitesimal generator of the coadjoint action is given by $\xi_{so(3)}(M) = \xi \times M$ for $M \in O(M_0)$ and $\xi \in so(3)$. For $M, \tilde{M} \in O(M_0)$ let the tangent vectors $\xi_{\mathbb{R}^3}(M) = \xi \times M \in T_M O(M_0)$ and $\xi_{\mathbb{R}^3}(\tilde{M}) = \xi \times \tilde{M} \in T_{\tilde{M}} O(M_0)$; the coadjoint orbit symplectic structure becomes

$$
\omega \left( \xi_{\mathbb{R}^3}(M), \xi_{\mathbb{R}^3}(\tilde{M}) \right) = M \cdot \left( \tilde{M} \times \xi \right).
$$

(6.1)
The normal metric on $O(M_0)$ with respect to $M$ and $\tilde{M}$, given by (5.18), is

$$\langle \tilde{M} \times \xi, \tilde{M} \times \eta \rangle_N(M) = \frac{1}{c^4} \langle \tilde{M} \times (\tilde{M} \times \xi), \tilde{M} \times (\tilde{M} \times \eta) \rangle - \frac{1}{c^4} \langle (M \cdot (\tilde{M} \times \xi)) \cdot (M \cdot (\tilde{M} \times \eta)) \rangle =$$

$$= \langle \tilde{M} \times \xi, \tilde{M} \times \eta \rangle_N(\tilde{M}) - \frac{1}{c^4} \langle (M \cdot (\tilde{M} \times \xi)) \cdot (M \cdot (\tilde{M} \times \eta)) \rangle$$

(6.2)

with $c = C(\tilde{M}) = C(M)$.

The normal metric at $\tilde{M}$ on two tangent vectors $\tilde{M} \times \xi$, $\tilde{M} \times \eta$ to the sphere of radius $c$ is given by

$$\langle \tilde{M} \times \xi, \tilde{M} \times \eta \rangle_N(\tilde{M}) = \frac{1}{c^4} \langle \tilde{M} \times (\tilde{M} \times \xi), \tilde{M} \times (\tilde{M} \times \eta) \rangle$$

(6.3)

where the inner product of the right hand side is the standard inner product in $\mathbb{R}^3$.

The normal metric at $M$ with $M \neq \tilde{M}$, on two tangent vectors $\tilde{M} \times \xi$, $\tilde{M} \times \eta$ to the sphere of radius $c$ at $\tilde{M}$ is given by

$$\langle \tilde{M} \times \xi, \tilde{M} \times \eta \rangle_N(M) = \frac{1}{c^4} \langle M \times (\tilde{M} \times \xi), M \times (\tilde{M} \times \eta) \rangle$$

(6.4)

where the inner product of the right hand side is the standard inner product in $\mathbb{R}^3$. From (6.3) and (6.4) it results (6.2).

From (5.26) where $k(M) = h(M) = \frac{1}{2} \|M\|^2$ it results the Lie–Poisson equation for the rigid body with delay:

$$\dot{M}(t) = M \times \Omega + \frac{\alpha}{c^2} M \times (\tilde{M} \times \tilde{\Omega}).$$

(6.5)

We shall discuss the stability of the equilibrium states for a free rigid body with delay.

Let the free rigid body with delay given by the equation

$$\dot{M} = M \times \Omega + \alpha M \times (\tilde{M} \times \tilde{\Omega}),$$

(6.6)

where $M = I\Omega = (I_1x(t), I_2y(t), I_3z(t))^T$, $\Omega = (x(t), y(t), z(t))^T$, $\tilde{M} = I\tilde{\Omega} = (I_1x(t-\tau), I_2y(t-\tau), I_3z(t-\tau))^T$, $\tilde{\Omega} = (x(t-\tau), y(t-\tau), z(t-\tau))^T$, $I_1 > 0$, $I_2 > 0$, $I_3 > 0$, $\alpha$ a constant and $\tau \geq 0$. 23
It is not hard to see that the equilibrium states of our system are \( \Omega_1 = (m/I_1, 0, 0)^T, \Omega_2 = (0, m/I_2, 0)^T, \Omega_3 = (0, 0, m/I_3), m \in \mathbb{R}^* \).

**Proposition 6.1.** The equilibrium state \( \Omega_1 \) has the following behavior:

(i) The corresponding linear system is given by
   \[
   \delta \dot{\Omega} = A \delta \Omega + \alpha G \tilde{\Omega}
   \]  
   (6.7)

where
   \[
   A = \begin{pmatrix}
   0 & 0 & 0 \\
   0 & 0 & \frac{I_3 - I_1}{I_1 I_2} m \\
   0 & \frac{I_1 - I_3}{I_1 I_3} m & 0
   \end{pmatrix},
   \quad
   G = \begin{pmatrix}
   0 & \frac{I_3 - I_1}{I_1 I_2} m^2 & 0 \\
   0 & 0 & \frac{I_3 - I_1}{I_1 I_3} m^2
   \end{pmatrix};
   \]  
   (6.8)

(ii) The characteristic equation is
   \[
   \lambda^2 - \frac{\alpha m^2}{I_1} \left( \frac{I_2 - I_1}{I_2} + \frac{I_3 - I_1}{I_3} \right) \lambda e^{-\tau \lambda} + \frac{\alpha^2 m^4}{I_1^2 I_2 I_3} (I_2 - I_1)(I_3 - I_1)e^{-2\tau \lambda} - \frac{(I_1 - I_2)(I_3 - I_1)}{I_1^2 I_2 I_3} m^2 = 0;
   \]  
   (6.9)

(iii) On the tangent space at \( \Omega_1 \) to the sphere of radius \( m^2 \) the linear operator given by the linearized vector field has the characteristic equation
   \[
   \lambda^2 - \frac{\alpha m^2}{I_1} \left( \frac{I_2 - I_1}{I_2} + \frac{I_3 - I_1}{I_3} \right) \lambda e^{-\tau \lambda} + \frac{\alpha^2 m^4}{I_1^2 I_2 I_3} (I_2 - I_1)(I_3 - I_1)e^{-2\tau \lambda} - \frac{(I_1 - I_2)(I_3 - I_1)}{I_1^2 I_2 I_3} m^2 = 0;
   \]  
   (6.10)

(iv) If \( I_1 > I_2, I_1 > I_3 \) for \( 0 \leq \tau < \tau_c \), where
   \[
   \tau_c = \frac{I_1 [I_3(I_1 - I_2) + I_2(I_1 - I_3)]}{3|\alpha| m^2 (I_1 - I_2)(I_1 - I_3)},
   \]  
   (6.11)

then the equilibrium state \( \Omega_1 \) is asymptotically stable.

**Proof.** (i), (ii), (iii) result from the definitions of the linearized and the characteristic equation by calculus. For (iv) consider \( I_1 > I_2, I_1 > I_3 \). If \( \tau = 0 \) then the characteristic equation (6.9) has eigenvalues with the real parts negative and \( \Omega_1 \) is asymptotically stable. Following [9] and [12] it
results that for $0 < \tau \leq \tau_c$ the equilibrium state $\Omega_1$ remains asymptotically stable.

In the following we study the existence of Hopf bifurcations for the free rigid body with delay (6.6) by choosing the delay $\tau$ as a bifurcation parameter. First we would like to know when the equation (6.10) has purely imaginary roots $\pm i\omega_0$ ($\omega_0 > 0$) at $\tau = \tau_0$. Note that $\lambda = i\omega_0$ is a root of (6.1) if
\[
\begin{align*}
\omega_0^2 - c - a\omega_0 \sin \omega_0 \tau_0 - b \cos 2\omega_0 \tau_0 &= 0, \\
a\omega_0 \cos \omega_0 \tau_0 - b \sin 2\omega_0 \tau_0 &= 0
\end{align*}
\]
(6.12)

with $a, b, c$ given by
\[
a = \frac{\alpha m^2}{I_1^2 I_2 I_3} [I_3(I_1 - I_2) + I_2(I_1 - I_3)], \quad b = \frac{\alpha^2 m^4}{I_1^2 I_2 I_3} (I_1 - I_2)(I_1 - I_3),
\]
\[
c = \frac{m^2}{I_1^2 I_2 I_3} (I_1 - I_2)(I_1 - I_3).
\]
(6.13)

We deduce the following

**Proposition 6.2.** (i) If $|m| < \frac{1}{|\alpha|}$ then $\lambda = i\omega_0$ is a simple root of (6.10) and
\[
\omega_0 = \frac{a + \sqrt{a^2 - 4(b - c)}}{2}, \quad \tau_0 = \frac{\pi}{2\omega_0}, \quad \tau_0 > \tau_c.
\]
(6.14)

(ii) If $|m| > \frac{1}{|\alpha|}$ then $\lambda = i\omega_0$ is a simple root of (6.10) and
\[
\omega_0 = \frac{-a + \sqrt{a^2 - 4(b - c)}}{2}, \quad \tau_0 = \frac{3\pi}{2\omega_0}, \quad \tau_0 > \tau_c.
\]
(6.15)

We proceed to calculate $\text{Re} \left( \frac{d\lambda}{d\tau} \right)$ at $\tau = \tau_0$. By differentiating the equation (6.10) implicitly with respect to $\tau$, we obtain:
\[
\frac{d\lambda}{d\tau} = \frac{a\lambda^2 e^{-\tau\lambda} + 2b\lambda e^{-2\tau\lambda}}{2\lambda + a(1 - \tau\lambda)e^{-\tau\lambda} - 2b\tau e^{-2\tau\lambda}}.
\]
(6.16)

It is then evaluated at $\lambda = i\omega_0$ and $\tau = \tau_0$ given by (6.14), (6.15), yielding
\[
\text{Re} \left( \frac{d\lambda}{d\tau} \right)_{\lambda = i\omega_0, \tau = \tau_0} = \frac{\omega_0(\omega_0 + a)(a - 2b)}{\tau_0(a\omega_0 - 2b)^2 + (\omega_0 + a)^2}.
\]
(6.17)
From the standard Hopf bifurcation theory we have the following result.  

**Proposition 6.3.** If \( I_1 > I_2, \ I_1 > I_3 \) and \( \omega_0, \tau_0 \) are given by (6.14), (6.15) with (6.13), then \( \Re \left( \frac{d\lambda}{d\tau} \right)_{\lambda = \omega_0, \tau = \tau_0} \neq 0 \) and a Hopf–type bifurcation occurs at \( \Omega_1 \) when \( \tau \) passes through \( \tau_0 \).

In the following we obtain some conditions which guarantee that the free rigid body with delay undergoes a Hopf bifurcation at \( \tau = \tau_0 \). The method we use is based on the normal form theory and the center manifold theorem introduced in [7].

With the translation \( V = \Omega - \Omega_1, \ U = M - M_1 \) the equation (6.6) becomes

\[
\dot{U} = IAV + \alpha IG\tilde{V} + F(U,V,\tilde{U},\tilde{V}), \tag{6.18}
\]

where \( A, G \) are given by (6.8), \( I = \text{diag} (I_1, I_2, I_3) \) and

\[
F(U,V,\tilde{U},\tilde{V}) = U \times V + \alpha \left[ U \times (\tilde{U} \times \Omega_1) + U \times (M_1 \times \tilde{V}) + \right.
\]

\[
+ M_1 \times (\tilde{U} \times \tilde{V}) \bigg] + \alpha U \times (\tilde{U} \times \tilde{V}). \tag{6.19}
\]

From (6.18) it results

\[
\dot{V} = AV + \alpha G\tilde{V} + N(V,\tilde{V}), \tag{6.20}
\]

where \( N(V,\tilde{V}) = I^{-1}F(I^{-1}V,V,I^{-1}\tilde{V},\tilde{V}) \).

For \( \phi \in C^1([-\tau_0, 0], \mathbb{R}^3) \) we define an operator \( \mathcal{A} \) by

\[
\mathcal{A}\phi(\theta) = \begin{cases} \frac{d\phi}{d\theta}, \quad &\theta \in [-\tau_0, 0) \\ \mathcal{A}\phi(0) + \alpha G\phi(-\tau_0), \quad &\theta = 0 \end{cases} \tag{6.21}
\]

and for \( \psi \in C^1([0, \tau_0], \mathbb{R}^{3*}) \) we define the adjoint operator \( \mathcal{A}^* \) of \( \mathcal{A} \) by

\[
\mathcal{A}^*\psi(s) = \begin{cases} -\frac{d\psi}{ds}, \quad &s \in [0, \tau_0) \\ \psi(0)A + \alpha \psi(\tau_0)G, \quad &s = \tau_0; \end{cases} \tag{6.22}
\]

\( \mathcal{A} \) and \( \mathcal{A}^* \) are adjoint operators with respect to the bilinear form

\[
<\psi, \phi> = \overline{\psi}(0)\phi(0) - \alpha \int_{-\tau_0}^{0} \int_{\xi=0}^{\theta} \overline{\psi}(\xi - \theta)G\phi(\xi)d\xi d\theta. \tag{6.23}
\]
\( \phi \in C^1([\tau_0, 0], \mathbb{R}^3), \psi \in C^1([0, \tau_0], \mathbb{R}^{3\ast}). \)

Let now \( \lambda_1 = i\omega, \lambda_2 = -i\omega \) be eigenvalues of \( A \), where \( \omega \) is given by (6.14). They are also eigenvalues of \( A^\ast \). We can easily obtain that

\[
\phi(\theta) = (0, v_2, v_3)^T e^{\lambda_1 \theta}, \quad \theta \in [-\tau_0, 0],
\]

(6.24)

where \( v_2 = (I_3 - I_1)m, v_3 = \lambda_1 I_1 I_2 - (I_2 - I_1)m^2 \alpha e^{-\lambda_1 \tau_0}, \) is an eigenvector of \( A \) corresponding to \( \lambda_1 \) and

\[
\psi(s) = (0, w_2, w_3)e^{\lambda_1 s}, \quad s \in [0, \tau_0],
\]

(6.25)

where \( w_2 = I_2(I_1 - I_2)m, w_3 = (\lambda_1 I_1 I_2 - (I_2 - I_1)m^2 \alpha e^{\lambda_1 \tau_0})I_3, \) is an eigenvector of \( A^\ast \) corresponding to \( \lambda_1 \).

From (6.23), (6.24), (6.25) it results

\[
a_{11} = < \psi, \phi > = v_2 w_2 + v_3 \bar{w}_3 - \frac{\alpha m^2}{I_1 I_2 I_3} \cdot \frac{\alpha \tau_0}{\lambda_2} (e^{\lambda_2 \tau_0} + \lambda_2 e^{\lambda_2 \tau_0} - 1)[I_3(I_2 - I_1)v_2 w_2 + I_2(I_3 - I_1)v_3 \bar{w}_3],
\]

\[
a_{12} = < \psi, \bar{\theta} > = v_2 w_2 + \bar{v}_3 w_3 - \frac{\alpha m^2}{I_1 I_2 I_3} \cdot \frac{\alpha}{2 \lambda_2} (2 - e^{-\lambda_2 \tau_0} - e^{\lambda_2 \tau_0})[I_3(I_2 - I_1)v_2 w_2 + I_2(I_3 - I_1)\bar{v}_3 \bar{w}_3],
\]

\[
a_{21} = < \psi, \phi > = < \bar{\psi}, \bar{\phi} > = \bar{a}_{12}, \quad a_{22} = < \psi, \bar{\phi} > = \\
< \bar{\psi}, \phi > = \bar{a}_{11}.
\]

(6.26)

Let \( d = a_{11} \bar{a}_{11} - a_{12} \bar{a}_{12} \) and \( b_{11} = \frac{a_{11}}{d}, b_{12} = -\frac{a_{12}}{d} \). The vector field given by \( \bar{\psi}(s) = b_{11} \psi(s) + b_{12} \bar{\psi}(s), s \in [0, \tau_0] \) is an eigenvector of \( A^\ast \) satisfying the relations:

\[
< \bar{\psi}, \theta > = 1, \quad < \bar{\psi}, \bar{\theta} > = < \bar{\psi}, \theta > = 0, \quad < \bar{\psi}, \bar{\theta} > = 1.
\]

(6.27)

Then we have

\[
\bar{\psi}(s) = (0, \bar{w}_2, \bar{w}_3) e^{\lambda_1 s}, \quad s \in [0, \tau_0],
\]

(6.28)

where

\[
\bar{w}_2 = (b_{11} + b_{12}) w_2, \quad \bar{w}_3 = b_{11} w_3 + b_{12} \bar{w}_3.
\]

(6.29)

We shall discuss the existence of a local center manifold around the equilibrium point of the equation (6.20).
Let $B = C^1([−τ_0, 0], \mathbb{R}^3)$ and $\Delta$ the vectorial space, span of the eigenvectors $\phi(\theta), \overline{\phi(\theta)}$ corresponding to $\lambda_1 = i\omega_0, \lambda_2 = -i\omega_0$. For a given neighborhood $\mathcal{V}$ of $0 \in B$, a local center manifold $W^c_{loc}(0) = W^c(0, \mathcal{V})$ of the equilibrium point $O(0, 0, 0)$ of (6.20) is a $C^1$–submanifold that is a graph over $\mathcal{V} \cap \Delta$ in $B$, tangent to $\Delta$ at $O$ and locally invariant under the flow defined by the equation (6.20). In other words,

$$W^c_{loc}(0) = \left\{ \varphi^c \in B \mid \varphi^c = u\phi + \overline{w(\varphi)}, \varphi \in \mathcal{V} \cap \Delta \right\}, \quad (6.30)$$

where $w: \Delta \rightarrow B$ is a $C^1$–mapping with $\varphi(0) = 0$, $D_\varphi w(0) = 0$ and $\langle \overline{\psi}, w \rangle = 0$. Moreover, every orbit that begins on $W^c_{loc}(0)$ remains in this set as long as it stays in $\mathcal{V}$.

The basic result on the existence of the local center manifold for the delay differential equations is given in [6]. From the definition of the local center manifold it results that

$$W^c_{loc}(0) \cap V_1 = \left\{ \varphi^c \in B \mid \varphi^c = u\phi + \overline{w(\varphi)}, \quad u = u_1 + iu_2, \right\}, \quad (6.31)$$

where $w: [-\tau, 0] \times \mathbb{C}^2 \rightarrow \mathbb{R}^3$ is given by $w(\theta, u, \overline{u}) = w \left( u\phi(\theta) + \overline{\phi(\theta)} \right)$. Following [6], $\varphi^c \in W^c_{loc}(0) \cap V_1$ and the function $w$ is the solution of the partial derivative system

$$\frac{\partial w}{\partial t} (\theta, u(t), \overline{u}(t)) + g(u(t), \overline{u}(t)) \phi(\theta) + g(u(t), \overline{u}(t)) \cdot \overline{\phi(\theta)} = \frac{\partial w}{\partial \theta} (\theta, u(t), \overline{u}(t)) \quad (6.32)$$

with

$$\frac{\partial w}{\partial t} (0, u(t), \overline{u}(t)) + g(u(t), \overline{u}(t)) \phi(0) + g(u(t), \overline{u}(t)) \cdot \overline{\phi(0)} =$$

$$= Aw (0, u(t), \overline{u}(t)) + \alpha Gw (-\tau_0, u(t), \overline{u}(t)) + N \left( u(t)\phi(0) + \overline{u(t)}\phi(0) \right) +$$

$$+ w(0, u(t), \overline{u}(t)) + u(t)\phi(-\tau_0) + \overline{u(t)}\phi(-\tau_0) + w(-\tau_0, u(t), \overline{u}(t)), \quad (6.33)$$

where $u(t)$ is a solution of the ordinary differential equation

$$\dot{u}(t) = \lambda_1 u(t) + g(u(t), \overline{u}(t)) \quad (6.34)$$
and
\[
g(u, \overline{w}) = \overline{\psi}(0) N \left( u\phi(0) + u\overline{\phi}(0) + w(0, u, \overline{w}) \right) + w(-\tau_0, u, \overline{w}).
\]

For \( \varphi^c \in W^c_0(0) \cap V_1 \) the solution of (6.20) is given by
\[
V(t)(\theta) = u(t)\phi(\theta) + \overline{\alpha}(t)\overline{\varphi}(\theta) + w(\theta, u(t), \overline{\alpha}(t)), \quad \theta \in [-\tau_0, 0].
\]

Because the equation (6.20) has a Casimir function (conservation laws) \( \|IV\| = m \), for the solution given by (6.36) we have \( IV(t) \cdot \dot{IV}(t) = 0 \).

Consider the function \( w \) given by
\[
w(\theta, u, \overline{w}) = \frac{1}{2}w_{02}(\theta)u^2 + w_{11}(\theta)u\overline{w} + \frac{1}{2}w_{02}(\theta)u^2,
\]
with \( w_{02}(\theta) = \overline{w}_{02}(0), w_{11}(\theta) = \overline{w}_{11}(\theta), \theta \in [-\tau_0, 0] \). From (6.20) it results that the components of \( N(V, \overline{V}) \) for \( V = (x^1, x^2, x^3)^T, \overline{V} = (\overline{x}^1, \overline{x}^2, \overline{x}^3)^T \) are the followings:
\[
N^1(V, \overline{V}) = \frac{I_2 - I_3}{I_1}x^2x^3 + \alpha m \left[ \frac{I_2(I_1 - I_2)}{I_1}x^2x^3 - \frac{I_3(I_3 - I_1)}{I_1}x^2x^3 \right] + \\
+ \alpha \left[ \frac{I_2(I_1 - I_2)}{I_1}x^2x^3 - \frac{I_3(I_3 - I_1)}{I_1}x^2x^3 \right],
\]
\[
N^2(V, \overline{V}) = \frac{I_3 - I_1}{I_2}x^1x^3 + \alpha m \left[ \frac{I_3(I_2 - I_3)}{I_2}x^1x^3 - \frac{I_1(I_1 - I_2)}{I_2}x^1x^3 \right] + \\
+ \alpha \left[ \frac{I_3(I_2 - I_3)}{I_2}x^1x^3 - \frac{I_1(I_1 - I_2)}{I_2}x^1x^3 \right],
\]
\[
N^3(V, \overline{V}) = \frac{I_1 - I_2}{I_3}x^1x^2 + \alpha m \left[ \frac{I_1(I_3 - I_1)}{I_3}x^1x^2 - \frac{I_2(I_2 - I_3)}{I_3}x^1x^2 \right] + \\
+ \alpha \left[ \frac{I_1(I_3 - I_1)}{I_3}x^1x^2 - \frac{I_2(I_2 - I_3)}{I_3}x^1x^2 \right].
\]

From (6.38) with \( V(t), \overline{V}(t) = V(t)(-\tau_0) \), given by (6.36) and \( \phi(\theta), \overline{\varphi}(\theta) \) given by (6.24), it results
\[
N(V(t), \overline{V}(t)) = \frac{1}{2}F_{00} + F_{11}u(t)\overline{w}(t) + \frac{1}{2}F_{02}(t)^2 + \frac{1}{2}F_{21}(t)^2\overline{w}(t),
\]

29
with \( F_{20} = (F_{20}^1, F_{20}^2, F_{20}^3)^T, F_{11} = (F_{11}^1, F_{11}^2, F_{11}^3)^T, F_{02} = (F_{02}^1, F_{02}^2, F_{02}^3)^T, F_{21} = (F_{21}^1, F_{21}^2, F_{21}^3)^T \), where

\[
F_{20}^1 = \frac{2(I_2 - I_3)}{I_1} v_2 v_3 + \frac{\alpha m}{I_1} [I_2(I_1 - I_2)v_2^2 - I_3(I_3 - I_1)v_3^2] e^{\lambda_1 \tau_0},
\]
\[
F_{20}^2 = F_{20}^3 = 0,
\]
\[
F_{11}^1 = \frac{I_3 - I_1}{I_1} v_3 (v_3 + \overline{v}_3) + \frac{\alpha m}{I_1} [I_2(I_1 - I_2)v_2^2 - I_3(I_3 - I_1)v_3^2] (e^{\lambda_1 \tau_0} + e^{\lambda_2 \tau_0}),
\]
\[
F_{11}^2 = F_{11}^3 = 0,
\]
\[
F_{02}^1 = \frac{2(I_2 - I_3)}{I_1} v_2 \overline{v}_3 + \frac{\alpha m}{I_1} [I_2(I_1 - I_2)v_2^2 - I_3(I_3 - I_1)v_3^2] e^{\lambda_1 \tau_0},
\]
\[
F_{02}^2 = F_{02}^3 = 0,
\]
\[
F_{21}^1 = \frac{I_2 - I_3}{I_1} [v_2\{2w_{11}^3(0) + w_{20}^3(0)\} + 2v_3 w_{11}^2(0) + \overline{v}_3 w_{20}^2(0)] + 
+ \frac{2\alpha m I_2(I_1 - I_2)}{I_1} \left[2v_2 w_{11}^2(0)e^{-\lambda_1 \tau_0} + \frac{1}{2} v_2 w_{20}^2(e^{\lambda_1 \tau_0} + e^{\lambda_2 \tau_0})\right] - 
- \frac{2\alpha m I_3(I_3 - I_1)}{I_1} \left[v_3 w_{11}^2(0)e^{-\lambda_1 \tau_0} + \frac{1}{2} \overline{v}_3 w_{20}^2(0)(e^{\lambda_1 \tau_0} + e^{\lambda_2 \tau_0})\right],
\]
\[
F_{21}^2 = \frac{I_3 - I_1}{I_2} \overline{v}_3 w_{20}^1(0) - \frac{2\alpha m I_1(I_1 - I_2)}{I_2} v_2 w_{20}^1(0)e^{\lambda_1 \tau_0},
\]
\[
F_{21}^3 = \frac{I_1 - I_2}{I_3} v_2 w_{20}^1(0) - \frac{2\alpha m I_1(I_1 - I_3)}{I_3} \overline{v}_3 w_{20}^1(0)e^{\lambda_1 \tau_0}.
\]

From (6.35) with \( \tilde{\psi}(0) \) given by (6.28) and \( N(\bar{V}(t), \tilde{V}(t)) \) given by (6.39) we obtain

\[
g(u(t), \overline{\omega}(t)) = \frac{1}{2} g_{21} u(t)^2 \overline{\omega}(t),
\]

where

\[
g_{21} = \overline{w}_2 F_{21}^2 + \overline{w}_3 F_{21}^3.
\]

Taking into account of (6.32) it results that \( w_{20}(\theta), w_{11}(\theta) \) verify the differential equations

\[
\dot{w}_{20}(\theta) = 2\lambda_1 w_2(\theta), \quad \dot{w}_{11}(\theta) = 0, \quad \theta \in [-\tau_0, 0].
\]
From (6.43), (6.33) and $IV(t) \cdot I \dot{V}(t) = 0$ it results
\[ w_{20}(\theta) = E_1 e^{2\lambda_1 \theta}, \quad w_{11}(\theta) = 0, \quad \theta \in [-\tau_0, 0], \quad (6.44) \]
where $E_1$ is the solution of the linear system of equations
\[ \left( A + \alpha e^{-\lambda_1 \tau_0} G - 2\lambda_1 E \right) E_1 = -F_{20}. \quad (6.45) \]

From (6.45) and (6.40) we deduce
\[ w_{20}^1(\theta) = \frac{1}{2\lambda_1} F_{20}^1 e^{2\lambda_1 \theta}, \quad w_{20}^2(\theta) = w_{20}^3(\theta) = 0, \quad \theta \in [-\tau_0, 0], \quad (6.46) \]

obtaining the following

**Proposition 6.4.** The solution of the equation (6.6) near upon the stationary state $\Omega_1 = \left( \frac{m}{T_1}, 0, 0 \right)$ is
\[ \dot{x}(t) = \frac{m}{T_1} + \text{Re} \left( w_{20}^1(0) u^2(t) \right), \]
\[ \dot{y}(t) = 2v_2 \text{Re} \left( u(t) \right), \]
\[ \dot{z}(t) = 2\text{Re} \left( v_3 u(t) \right), \quad (6.47) \]
where $u(t)$ is a solution of the equation
\[ \dot{u}(t) = \frac{1}{2} g_{21} u(t)^2 \varpi(t) \quad (6.48) \]
and $w_{20}^1(0) = \frac{1}{2\lambda_1} F_{20}^1$, $v_2, v_3$ are given by (6.24), $g_{21}$ is given by (6.42).

Based on the above analysis and calculi, we can see that $g_{21}, w_{20}^1(0), v_2, v_3$ are determined by the parameters and the delay of (6.6). Thus we can explicitly compute the following quantities:
\[ C_1(0) = \frac{1}{2} g_{21}, \quad \mu_2 = -\frac{\text{Re} \left( C_1(0) \right)}{\text{Re} \left( \frac{d}{dt} \lambda = i\omega_0 \right)} \]
\[ \lambda = \tau_0 \]
\[ I\text{m}(C_1(0)) + \mu_2 I\text{m} \left( \frac{d}{dt} \lambda = i\omega_0 \right) = \frac{T_2}{\omega_0}, \quad \beta_2 = 2\text{Re} \left( C_1(0) \right). \quad (6.49) \]
In summary, this leads to the following result.

**Proposition 6.5.** In the formulas (6.49) \( \mu_2 \) determines the direction of the Hopf bifurcation: if \( \mu_2 > 0 \) (respectively \( \mu_2 < 0 \)) then the Hopf bifurcation is supercritical (respectively subcritical) and the bifurcating periodic solutions exist for \( \tau > \tau_0 \) (respectively \( \tau < \tau_0 \)); \( \beta_2 \) determines the stability of the bifurcation periodic solutions: the solutions are orbitally stable (respectively instable) if \( \beta_2 < 0 \) (respectively \( \beta_2 > 0 \)); \( T_2 \) determines the periods of the bifurcating periodic solutions: the periods increase (respectively decrease) if \( T_2 > 0 \) (respectively \( T_2 < 0 \)). For \( I_1 = 0.8, I_2 = 0.5, I_3 = 0.4, \alpha = 0.3, m = 1.5 \) and \( \omega_0, \tau_0 \) given by the formulas (6.14) we obtain \( \omega_0 = 3.20631, \tau_0 = 0.88154, \mu_2 = 0.00958, T_2 = 0.00057, \beta_2 = -0.00139 \). The limit cycle is supercritical with the period \( T_2 \). For \( I_1 = 0.8, I_2 = 0.5, I_3 = 0.4, \alpha = 0.3, m = 1.8 \) and \( \omega_0, \tau_0 \) given by the formulas (6.15) we obtain \( \omega_0 = 0.68547, \tau_0 = 0.88154, \mu_2 = 0.00344, T_2 = 0.00050, \beta_2 = 0.00097 \). The limit cycle is supercritical with the period \( T_2 \).

7. Conclusions and comments

In this paper we have given a general method to construct a dissipative mechanism with delay preserving the symplectic leaves of the reduced space and dissipating the energy. The most important case is that of the dual of a Lie algebra when the dissipative term with delay is shown to have a double bracket with delay form. This theory applies to a number of interesting examples from ferromagnetics, ideal fluid flow and plasma dynamics in which the previous state of the phenomenon is important. In the future we would like to analyze the systems given by Landau–Lifschitz equations with delay, the Heavy Top with delay etc.

References

[1] I.D. Albu, D. Opris, *Local stability, Hopf bifurcations for a system of harmonic oscillators with time delay*, Italian J. of Pure and Applied Math., vol. XXIX, 2003, 203–220.
[2] A. Bloch, P.S. Krishnaprasad, J.E. Marsden, T.S. Ratiu, *The Euler–Poincaré Equations and Double Bracket Dissipation*, Comm. Math. Phys., 175 (1996), 1–42.

[3] R. W. Brockett, *Differential geometry and the design of gradient algorithms*, Proc. Symp. Pure. Math. AMS 54 (I), 1993, 69–92.

[4] A. Halanay, *Systems a retardment. Resultats et problèmes*, Third Conference on Nonlinear Vibrations, Ed. Berliner Academic Verlag, Berlin, 1965 (Survey Article).

[5] J.K. Hale, L. Magalhães, W. Oliva, *An introduction to infinite dimensional dynamical systems–geometric theory*, Springer–Verlag, 1984.

[6] J.K. Hale, S.M. Verduyn Lunel, *Introduction to functional differential equations*, Springer–Verlag, 1995.

[7] B.D. Hassard, N.D. Kazarinoff, Y.H. Wan, *Theory and applications of Hopf bifurcation*, Cambridge University Press, Cambridge, 1981.

[8] Y.A. Kuznetsov, *Elements of applied bifurcation theory*, Springer Verlag, 1995.

[9] X. Liao, G. Chen, *Local stability, Hopf and resonant codimension–two bifurcation in a harmonic oscillator with two time delays*, International Journal of Bifurcation and Chaos, Vol. 11, No. 8(2001), 2105–2121.

[10] J.E. Marsden, M. Mc Cracken, *The Hopf bifurcation and its applications*, Springer–Verlag, 1976.

[11] D. Opriș, C. Udriste, *Pole shifts explained by Dirac delay in a Stefanescu Magnetic Flow*, Analele Univ. București, vol. XXV, tom 2, 2004, 203–229.

[12] G. Stepan, *Retarded dynamical systems, stability and characteristic functions*, Longman Scientific and Technical, England, 1989.