PURELY INFINITE SIMPLE REDUCED C*-ALGEBRAS OF ONE-RELATOR SEPARATED GRAPHS

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Abstract. Given a separated graph $(E, C)$, there are two different C*-algebras associated to it, the full graph C*-algebra $C^*(E, C)$, and the reduced one $C^*_{red}(E, C)$. For a large class of separated graphs $(E, C)$, we prove that $C^*_{red}(E, C)$ either is purely infinite simple or admits a faithful tracial state. The main tool we use to show pure infiniteness of reduced graph C*-algebras is a generalization to the amalgamated case of a result on purely infinite simple free products due to Dykema.

1. Introduction

Separated graphs have been introduced in [3] and [4]. Their associated graph algebras provide generalizations of the usual graph C*-algebras ([25]) and Leavitt path algebras ([1], [5]) associated to directed graphs, although these algebras behave quite differently from the usual graph algebras because the range projections associated to different edges need not commute. One motivation for their introduction was to provide graph-algebraic models for C*-algebraic analogues of the Leavitt algebras of [21]. These had been studied by various authors over the years, notably McClanahan (see [22], [23], [24]). As discussed below, another motivation was to obtain graph algebras whose structure of projections is as general as possible.

Given a finitely separated graph $(E, C)$, two different graph C*-algebras are considered in [4], the full graph C*-algebra $C^*(E, C)$ and the reduced graph C*-algebra $C^*_{red}(E, C)$. These two C*-algebras agree in the classical, non-separated case, by [4, Theorem 3.8(2)], but they differ generally, see [4, Section 4]. In this paper we obtain significant progress on some of the open problems raised in [4]. In particular we completely solve [4, Problem 7.3] for the class of one-relator separated graphs, giving a characterization of the purely infinite simple reduced graph C*-algebras corresponding to this class of separated graphs. As a special case, we deduce that the reduced C*-algebras $C^*_{red}(E(m, n), C(m, n))$, which are certain C*-completions of the Leavitt algebras $L_C(m, n)$ of type $(m, n)$, are purely infinite simple for $m \neq n$ (Corollary 4.5).

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Reduced graph C*-algebras of separated graphs are defined as reduced amalgamated free products of certain usual graph C*-algebras, with respect to canonical conditional expectations, see Section 2. The reader is referred to [27] and [10, 4.7, 4.8] for a quick introduction to the subject of reduced amalgamated free products of C*-algebras. Sufficient conditions for a reduced free product of two C*-algebras to be purely infinite simple were obtained in [16], [11] and [14]. The main tool we use in the present paper to show our results on reduced graph C*-algebras of separated graphs is a generalization to certain amalgamated free products of a result of Dykema [14] (see Theorem 3.3).

Let us recall the definition of a separated graph:

**Definition 1.1.** [4, Definition 1.3] A separated graph is a pair \((E, C)\) where \(E\) is a graph, \(C = \bigsqcup_{v \in E^0} C_v\), and \(C_v\) is a partition of \(s^{-1}(v)\) (into pairwise disjoint nonempty subsets) for every vertex \(v\). (In case \(v\) is a sink, we take \(C_v\) to be the empty family of subsets of \(s^{-1}(v)\).)

If all the sets in \(C\) are finite, we say that \((E, C)\) is a finitely separated graph. This necessarily holds if \(E\) is row-finite.

The set \(C\) is a trivial separation of \(E\) in case \(C_v = \{s^{-1}(v)\}\) for each \(v \in E^0 \setminus \text{Sink}(E)\). In that case, \((E, C)\) is called a trivially separated graph or a non-separated graph.

By its definition (see Section 2), the projections of \(C^*(E, C)\) and \(C^*_\text{red}(E, C)\) satisfy some obvious relations, prescribed by the structure of the separated graph \((E, C)\). These relations can be chosen arbitrarily, and this was one of the main motivations for the work in [3] and [4]. This can be formalized as follows. Let \((E, C)\) be a finitely separated graph, and let \(M(E, C)\) be the abelian monoid given by generators \(a_v\), \(v \in E^0\), and relations \(a_v = \sum_{e \in X} a_{r(e)}\), for \(X \in C_v\), \(v \in E^0\). Then there is a canonical monoid homomorphism \(M(E, C) \rightarrow V(C^*(E, C))\), which is conjectured to be an isomorphism for all finitely separated graphs \((E, C)\) (see Section 6 for a discussion on this problem). Here \(V(A)\) denotes the monoid of Murray-von Neumann equivalence classes of projections in \(M_\infty(A)\), for any C*-algebra \(A\).

Given a presentation \(\langle X \mid R \rangle\) of an abelian conical monoid \(M\), satisfying some natural conditions, it was shown in [3, Proposition 4.4] how to associate to it a separated graph \((E, C)\) such that \(M(E, C) \cong M\). We will now recall this construction for one-relator monoids. Let

\[
\langle a_1, \ldots, a_n \mid \sum_{i=1}^n r_i a_i = \sum_{i=1}^n s_i a_i \rangle
\]

be a presentation of the one-relator abelian conical monoid \(M\), where \(a_1, \ldots, a_n\) are free generators, and \(r_i, s_i\) are non-negative integers such that \(r_i + s_i > 0\) for all \(i\). Let \((E, C)\) be the finitely separated graph constructed as follows:

1. \(E^0 := \{v, w_1, w_2, \ldots, w_n\}\).
2. \(v\) is a source, and all the \(w_i\) are sinks.
3. For each \(i \in \{1, \ldots, n\}\), there are exactly \(r_i + s_i\) edges with source \(v\) and range \(w_i\).
4. \(C = C_v = \{X, Y\}\), where \(X\) contains exactly \(s_i\) edges \(v \rightarrow w_i\) for each \(i\), and \(Y\) contains exactly \(r_i\) edges \(v \rightarrow v_i\) for each \(i\). Thus, \(E^1 = X \sqcup Y\).
We call a separated graph constructed in this way a \textit{one-relator separated graph}. As a particular example, we may consider the presentation $\langle a \mid ma = na \rangle$, with $1 \leq m \leq n$. This gives rise to the separated graph $(E(m, n), C(m, n))$ considered in [4, Example 4.5], with two vertices $v$ and $w$, and $n + m$ arrows from $v$ to $w$, and with $C(m, n) = \{X, Y\}$ where $|X| = n$ and $|Y| = m$. The $C^*$-algebras $C^*(E(m, n), C(m, n))$ and $C^*_{\text{red}}(E(m, n), C(m, n))$ are closely related to the $C^*$-algebras studied in [9, 22, 23, 24], see [4, Sections 4 and 6].

In this paper, we show the following dichotomy for the reduced graph $C^*$-algebras of one-relator separated graphs:

\textbf{Theorem 1.2.} Let $(E, C)$ be the one-relator separated graph associated to the presentation

$$\langle a_1, \ldots, a_n \mid \sum_{i=1}^{n} r_i a_i = \sum_{i=1}^{n} s_i a_i \rangle.$$ 

Set $M = \sum_{i=1}^{n} r_i$ and $N = \sum_{i=1}^{n} s_i$, and assume that $2 \leq M \leq N$. Then $C^*_{\text{red}}(E, C)$ either is purely infinite simple or has a faithful tracial state, and it is purely infinite simple if and only if $M < N$ and there is $i_0 \in \{1, \ldots, n\}$ such that $s_{i_0} > 0$ and $r_{i_0} > 0$. Moreover, if $N + M \geq 5$ and $C^*_{\text{red}}(E, C)$ is finite, then it is simple with a unique tracial state.

When $N + M \leq 4$, the $C^*$-algebra $C^*_{\text{red}}(E, C)$ is also simple, except for a few cases. We analyze the different possibilities in the final part of Section 5. The case where $M = 1$ corresponds to an ordinary graph $C^*$-algebra. Observe that in particular we get that the algebras $C^*_{\text{red}}(E(m, n), C(m, n))$ are purely infinite simple when $1 < m < n$. It was suggested in [24, Example 4.3] that the simple $C^*$-algebras $U_{(m,n),\text{red}}^{nc}$, for $1 < m < n$, might be examples of finite but not stably finite $C^*$-algebras. In view of [4, Proposition 6.1], our main result shows in particular that the $C^*$-algebras $U_{(m,n),\text{red}}^{nc}$ are purely infinite if $m < n$.

It is worth to mention the appearance of some group $C^*$-algebras as graph $C^*$-algebras of separated graphs. As noted in [4], one example of this situation occurs when we consider the separated graph $(E, C)$ with just one vertex and with the sets of the partition reduced to singletons. In this case, the full graph $C^*$-algebra $C^*(E, C)$ is just the full group $C^*$-algebra $C^*(\mathbb{F})$ of a free group $\mathbb{F}$ of rank $|E^1|$, while the reduced graph $C^*$-algebra is precisely the reduced group $C^*$-algebra $C^*_{\text{r}}(\mathbb{F})$. In the present investigation, we discover another situation in which the full and the reduced graph $C^*$-algebras correspond (through a Morita-equivalence) to the full and reduced group $C^*$-algebras of a group, respectively (see Lemma 5.5(2) and Proposition 5.6). The universal unital $C^*$-algebra generated by a partial isometry also appears as a full corner of the full $C^*$-algebra of a one-relator separated graph (see Lemma 5.5(1)).

We now outline the contents of the paper. After a section of preliminaries, we obtain in Section 3 the generalization to certain amalgamated free products of Dykema's result ([14, Theorem 3.1]). Section 4 contains the proof of pure infiniteness and simplicity for a class of reduced graph $C^*$-algebras of one-relator separated graphs (Theorem 4.3). It also contains a direct application of Dykema's Theorem to show that certain reduced free products of Cuntz algebras are purely infinite simple (Proposition 4.1). We study the finite cases in Section 5.
obtaining in particular the proof of our main result (Theorem 1.2). Finally, in Section 6 we briefly discuss an open problem raised in [4] on the non-stable K-theory of $C^*(E, C)$, and we point out some K-theoretic consequences of our results. In particular, we give an example of a one-relator separated graph $(E, C)$ such that the full graph C*-algebra $C^*(E, C)$ is stably finite (indeed residually finite dimensional), while the reduced graph C*-algebra $C^*_{red}(E, C)$ is purely infinite simple (Example 6.7).

2. Preliminaries

Throughout, all graphs will be directed graphs of the form $E = (E^0, E^1, s, r)$, where $E^0$ and $E^1$ denote the sets of vertices and edges of $E$, respectively, and $s, r : E^1 \to E^0$ are the source and range maps. We follow the convention of composing paths from left to right – thus, a path in $E$ is given in the form $\alpha = e_1e_2 \cdots e_n$ where the $e_i \in E^1$ and $r(e_i) = s(e_{i+1})$ for $i < n$. The length of such a path is $|\alpha| := n$. Paths of length 0 are identified with the vertices of $E$.

Definition 2.1. [4, Definition 1.5] For any separated graph $(E, C)$, the full graph C*-algebra of the separated graph $(E, C)$ is the universal C*-algebra with generators $\{v, e \mid v \in E^0, e \in E^1\}$, subject to the following relations:

(V) $vw = \delta_{v,w}v$ and $v = v^*$ for all $v, w \in E^0$,

(E) $s(e)e = er(e) = e$ for all $e \in E^1$,

(SCK1) $e^*f = \delta_{e,f}r(e)$ for all $e, f \in X, X \subset C$, and

(SCK2) $v = \sum_{e \in X} ee^*$ for every finite set $X \subset C_v, v \in E^0$.

In case $(E, C)$ is trivially separated, $C^*(E, C)$ is just the classical graph C*-algebra $C^*(E)$.

We now recall the important abstract characterization of the reduced amalgamated free product of $C^*$-algebras, due to Voiculescu [27].

Definition 2.2. The reduced amalgamated product $(A, \Phi)$ of a nonempty family $(A_i, \Phi_i)_{i \in I}$ of unital C*-algebras containing a unital subalgebra $A_0$ with conditional expectations $\Phi_i : A_i \to A_0$ is uniquely determined by the following conditions:

1. $A$ is a unital C*-algebra, and there are unital *-homomorphisms $\sigma_i : A_i \to A$ such that $\sigma_i|A_0 = \sigma_i'|A_0$ for all $i, i' \in I$. Moreover the map $\sigma_i|A_0$ is injective and we identify $A_0$ with its image in $A$ through this map.

2. $A$ is generated by $\bigcup_{i \in I} \sigma_i(A_i)$.

3. $\Phi : A \to A_0$ is a conditional expectation such that $\Phi \circ \sigma_i = \Phi_i$ for all $i \in I$.

4. For $(\iota_1, \ldots, \iota_n) \in \Lambda(I)$ and $a_j \in \ker \Phi$, we have $\Phi(\sigma_{\iota_1}(a_1) \cdots \sigma_{\iota_n}(a_n)) = 0$. Here, $\Lambda(I)$ denotes the set of all finite tuples $(\iota_1, \ldots, \iota_n) \in \bigcup_{n=1}^{\infty} I^n$ such that $\iota_i \neq \iota_{i+1}$ for $i = 1, \ldots, n - 1$.

5. If $c \in A$ is such that $\Phi(a^*c^*ca) = 0$ for all $a \in A$, then $c = 0$.

As usual, we will write $A^o = \ker(\Phi)$, where $\Phi$ is a conditional expectation from $A$ onto a unital subalgebra $C$. Given subsets $T_1, T_2$ of an algebra $H$, we will denote by $\Lambda^o(T_1, T_2)$
the set of all elements of $\mathcal{H}$ of the form $a_1a_2\cdots a_r$, where $a_j \in T_{ij}$ and $i_1 \neq i_2$, $i_2 \neq i_3$, ..., $i_{r-1} \neq i_r$.

We can now recall the definition of the reduced graph C*-algebra $C^*_\text{red}(E,C)$ of a finitely separated graph. We only need here the case in which $E^0$ is a finite set. In this case, all the C*-algebras involved in the definition are unital, and this simplifies somewhat the definition. The reader is referred to [4] for the general case.

Let $(E,C)$ be a finitely separated graph with $|E^0| < \infty$. Set $A_0 = C(E^0)$ and, for $X \in C$, $A_X = C^*(E_X)$, where $E_X$ is the subgraph of $E$ with $(E_X)^0 = E^0$ and $(E_X)^1 = X$.

**Definition 2.3.** Let $(E,C)$ be a finitely separated graph, with $|E^0| < \infty$, and let $A_0, A_X$ be as defined above, for $X \in C$. Let $\Phi_X : C^*(E_X) \to C(E^0)$ be the canonical conditional expectations defined in [4, Theorem 2.1]. Then the reduced graph C*-algebra $(C^*_\text{red}(E,C), \Phi)$ associated to $(E,C)$ is the reduced amalgamated product of the family $(C^*(E_X), \Phi_X)_{X \in C}$. Since all the conditional expectations $\Phi_X$ are faithful, it follows from [19, Theorem 2.1] that the canonical conditional expectation $\Phi : C^*_\text{red}(E,C) \to C(E^0)$ is also faithful.

A concrete description in terms of a reduced amalgamated product of finite-dimensional C*-algebras will be given in Section 4 for the reduced graph C*-algebras associated to one-relator separated graphs.

We will make use of the following result and notation:

**Theorem 2.4.** [10, Theorem 4.8.5] Let $1 \in D \subseteq A_i$ and nondegenerate conditional expectations $E_i^A$ from $A_i$ onto $D$ be given, for $i = 1, 2$. Assume $1 \in D \subseteq B_i$ and assume there exist nondegenerate conditional expectations $E_i^B$ from $B_i$ onto $D$. Let $\theta_i : A_i \to B_i$ be u.c.p maps such that $(\theta_i)|_D = id_D$ and $E_i^B \circ \theta_i = E_i^A$. Then, there is a u.c.p. map

$$\theta_1 \ast \theta_2 : (A_1, E_1^A) *_D (A_2, E_2^A) \to (B_1, E_1^B) *_D (B_2, E_2^B)$$

such that $(\theta_1 \ast \theta_2)|_D = id_D$ and

$$(\theta_1 \ast \theta_2)(a_1a_2\cdots a_n) = \theta_{i_1}(a_1)\theta_{i_2}(a_2)\cdots \theta_{i_n}(a_n)$$

for $a_j \in A_{i_j}$ with $i_1 \neq i_2 \neq \cdots \neq i_n$.

3. AN ADAPTATION OF DYKEMA’S THEOREM.

In this section, we adapt Dykema’s result in [14] to the amalgamated case, and indeed we generalize the range of applications, since our hypothesis are a bit different.

Let $(A, \Phi_A)$ and $(B, \Phi_B)$ be C*-algebras with faithful conditional expectations $\Phi_A : A \to C$ and $\Phi_B : B \to C$, where $C$ is a common C*-subalgebra of $A$ and $B$. Consider the C*-algebra reduced amalgamated free product

$$(\mathfrak{A}, \Phi) = (A, \Phi_A) * (B, \Phi_B).$$

Let $P \in C$ be a central projection in $C$ such that $PC = PC$, and let $\gamma : C \to C$ be a faithful state on $C$. Then $\phi := \gamma \circ \Phi$ is a faithful state on $\mathfrak{A}$, since by [19, Theorem 2.1], $\Phi$ is also faithful.
Assume that there exists a partial isometry $v$ in $A$ such that $v^*v = p$ and $vv^* = q$, where $p$ and $q$ are projections in $A$ such that $p \leq P$ and $q \leq 1 - p$ and there is $0 < \lambda$ such that $\phi(vx) = \lambda \phi(xv)$ for all $x \in A$.

Set $A_0 := pAp + (1 - p)A(1 - p)$ and $\mathfrak{A}_0 := C^*(A_0 \cup B)$. We assume that there is $y \in \mathfrak{A}_0$ such that $yy^* = q$, $p_1 := y^*y \leq p$, so that $p_1 \in \mathfrak{A}_0$. Assume moreover there is $0 < \mu$ such that $\phi(y^*x) = \mu \phi(xy^*)$ for all $x \in \mathfrak{A}$, and that $\lambda \mu < 1$.

Define $w := y^*v$, so that $ww^* = p_1$ and $w^*w = p$. Define $p_n := w^n(w^*)^n$ and note that $p_0 \leq p_{n-1}$ and $p_n \in p\mathfrak{A}p$ for all $n \geq 1$ (where $p_0 := p$).

Let $E \colon \mathfrak{A} \to A_0$ be the composition of the conditional expectations $E^A_\mathfrak{A} \colon \mathfrak{A} \to A$ and $E^A_{A_0} \colon A \to A_0$, where $E^A_\mathfrak{A} = \text{id}_A \ast \Phi_B$ is the canonical conditional expectation given by Theorem 2.1 and

$$E^A_{A_0}(a) = pap + (1 - p)a(1 - p)$$

for $a \in A$.

**Lemma 3.1.** With the above notation, we have $\Phi \circ E = \Phi$, and in particular $\phi \circ E = \phi$.

**Proof.** It suffices to show that $\Phi_A(pa(1 - p)) = 0$ for all $a \in A$. For $a \in A$ we have

$$\Phi_A(pa(1 - p)) = \Phi_A(Ppa(1 - p)) = \phi(pa(1 - p))\phi(P)^{-1}P$$

$$= \phi(va(1 - p)\phi(P)^{-1}P$$

$$= \phi(va(1 - p)\phi(P))^{-1}P = 0,$$

showing the result. \hfill \Box

The hypothesis of the theorem of this section involves the following concept.

**Definition 3.2.** In the above situation, let $z$ be a projection in $p\mathfrak{A}p$. Then $z$ is said to be $A$-free if

1. $E(z) \in \mathbb{C} \cdot p$.
2. For any word $a \in \Lambda^0((pAp)^0, C^*(z, p)^0)$ of length $> 1$, we have $E(a) = 0$.

We can state now the following result, which is a generalization of [14, Theorem 3.1] to the amalgamated case.

**Theorem 3.3.** Let $(A, \Phi_A)$ and $(B, \Phi_B)$ be C*-algebras with conditional expectations satisfying the conditions stated above, and let $p, q, p_k = w^k(w^*)^k$, $k \geq 1$, be the projections defined above. Assume that, for each $k \geq 1$, the projections $p_k$ are $A$-free. Suppose in addition that $q\mathfrak{A}q$ contains a unital diffuse abelian C*-subalgebra which is contained in the centralizer of $\phi$ in $\mathfrak{A}$, and that $p$ is full in $\mathfrak{A}$. Then $\mathfrak{A}$ is purely infinite and simple.

Before we prove Theorem 3.3 let us show that it provides a generalization of Dykema’s result.

**Corollary 3.4.** [14, Theorem 3.1] Let $A$ and $B$ be C*-algebras with faithful states $\phi_A$ and $\phi_B$ respectively. Consider the C*-algebra reduced free product

$$(\mathfrak{A}, \phi) = (A, \phi_A) \ast (B, \phi_B).$$
Suppose there is a partial isometry \( v \in A \), whose range projection \( q = vv^* \), and domain projection \( p = v^*v \), are orthogonal, and such that, for some \( 0 < \lambda < 1 \), we have \( \phi_A(vx) = \lambda \phi_A(xv) \) for all \( x \in A \). Let

\[
A_{00} = \mathbb{C}p \oplus \mathbb{C}q \oplus (1 - p - q)A(1 - p - q)
\]

and let \( \mathfrak{A}_{00} = C^*(A_{00} \cup B) \). Suppose that \( q \) is equivalent in the centralizer of the restriction of \( \phi \) to \( \mathfrak{A}_{00} \) to a subprojection of \( p \), and that the centralizer of the restriction of \( \phi \) to \( q\mathfrak{A}_{00}q \) contains a unital abelian subalgebra on which \( \phi \) is diffuse. Suppose that \( p \) is full in \( \mathfrak{A} \).

Then \( \mathfrak{A} \) is simple and purely infinite.

**Proof of Corollary 3.4.** We have to show that the hypothesis of Theorem 3.3 are satisfied. We take \( P = 1 \), and \( \Phi = \phi \).

By hypothesis, there exists an element \( y \) in the centralizer of the restriction of \( \phi \) to \( \mathfrak{A}_{00} \) such that \( yy^* = q \) and \( p_1 := y^*y \leq p \). Note that \( y \in (p + q)\mathfrak{A}_{00}(p + q) \). By (a slight extension of) [14, Proposition 2.8], the algebra \((p + q)\mathfrak{A}(p + q)\) is generated by \((p + q)\mathfrak{A}_{00}(p + q)\) and \((p + q)A(p + q)\), which are free with amalgamation over \( p\mathbb{C} \oplus q\mathbb{C} \). It follows that \( y \) belongs to the centralizer of the restriction of \( \phi \) to \((p + q)\mathfrak{A}(p + q)\). But since \( y \in (p + q)\mathfrak{A}(p + q) \) and \( \phi((p + q)\mathfrak{A}(1 - p - q)) = 0 \), we get that \( y \) belongs to the centralizer of \( \phi \) in \( \mathfrak{A} \).

Also by hypothesis, the centralizer of the restriction of \( \phi \) to \( q\mathfrak{A}_{00}q \) contains a unital abelian subalgebra \( \mathcal{D} \) on which \( \phi \) is diffuse. Observe that, since \( \phi(p\mathfrak{A}_{00}q) = 0 \), we get that \( \mathcal{D} \) is contained in the centralizer of the restriction of \( \phi \) to \((p + q)\mathfrak{A}_{00}(p + q) \). Now the same argument as before shows that \( \mathcal{D} \) is contained in the centralizer of \( \phi \) in \( \mathfrak{A} \).

It only remains to check that all the projections \( p_k \) are \( A \)-free. We will show by induction on \( k \) that \( p_k \) belongs to the set

\[
S := p\mathbb{C} + \sum_{i \geq 0} pB^0(A^0B^o)^i p.
\]

This clearly shows that the projections \( p_k \) are \( A \)-free.

Note that every element in \( p\mathfrak{A}_{00}p \) belongs to \( S \). Now for \( k = 1 \), the result is clear since \( p_1 = y^*v v^* y = y^* q y \in p\mathfrak{A}_{00}p \). Assume that \( p_k \) belongs to \( S \), and let us show that \( p_{k+1} \in S \). Note that \( y^*v \) belongs to the closed linear span of

\[
\sum_{i \geq 0} pB^0(A^0B^o)^i v.
\]

Observe now that \( p_{k+1} = (y^*v)p_k(v^*y) \), so we are led to consider either terms of the form

\[
(pB^0(A^0B^o)^i q)vv^*(qB^0(A^0B^o)^j p)
\]

or terms of the form

\[
(pB^0(A^0B^o)^i v)(pB^0 A^0 \cdots A^0 B^o p)(v^* B^0 (A^0B^o)^j p).
\]

In the former case we get an element in \( p\mathfrak{A}_{00}p \subset S \). In the latter case, since \( v \in A^0 \), we get an element in \( pB^0(A^0B^o)^l p \) for some \( l \geq 1 \). So, in either case, we obtain an element in \( S \), as desired.
Proof of Theorem 3.3. The proof follows the steps of the one of \cite{14} Theorem 3.1. Note that there was an error in the statement and application of \cite{16} Theorem 2.1(i). The word “outer” that appears there should be “multiplier outer,” i.e., outer relative to the multiplier algebra of \( A \), instead of relative to the unitization of \( A \). This led to deficiencies in the proof of \cite{14} Theorem 3.1, which have been corrected in \cite{2}. This involves a change in the order of the different steps of that proof. Here we will outline the main steps of the proof of our result, referring to \cite{14} for the proofs which are identical.

Recall that \( w = y^*v \). Since \( \phi(vx) = \lambda \phi(xv) \) and \( \phi(y^*x) = \mu \phi(xy^*) \) for all \( x \in \mathfrak{A} \), we have \( \phi(wx) = (\lambda \mu) \phi(xw) \) for all \( x \in \mathfrak{A} \). It follows that

\[
\phi(p_k) = (\lambda \mu)^k \phi(p),
\]

so that, recalling that \( \lambda \mu < 1 \), we have that \( \lim_{k \to \infty} \phi(p_k) = 0 \).

Since \( p \) is full in \( \mathfrak{A} \), we need only to show that \( p\mathfrak{A}p \) is purely infinite and simple. By assumption \( q\mathfrak{A}_0q \) contains a unital diffuse abelian subalgebra \( \mathcal{D} \) which is contained in the centralizer of \( \phi \) in \( \mathfrak{A} \). Observe that

\[
q\mathfrak{A}_0q = yy^*\mathfrak{A}_0yy^* = y(y^*\mathfrak{A}_0y)y^* \subseteq y\mathfrak{A}_0y^* \subseteq q\mathfrak{A}_0q,
\]

so that \( q\mathfrak{A}_0q = y\mathfrak{A}_0y^* \), and \( p_1\mathfrak{A}_0p_1 = y^*y\mathfrak{A}_0y^*y = y^*(y\mathfrak{A}_0y^*)y = y^*q\mathfrak{A}_0qy \). It follows that \( y^*Dy \) is a unital diffuse abelian subalgebra of \( p_1\mathfrak{A}_0p_1 \) which is contained in the centralizer of \( \phi \), since for \( d \in \mathcal{D} \) and \( a \in \mathfrak{A} \) we have:

\[
\phi((y^*dy)a) = \mu \phi(dyay^*) = \mu \phi(yay^*d) = \mu \mu^{-1} \phi(ay^*dy) = \phi(a(y^*dy)).
\]

Therefore \( p_1\mathfrak{A}_0p_1 \) contains a unital diffuse abelian subalgebra which is contained in the centralizer of \( \phi \) in \( \mathfrak{A} \).

Note that \( A \) is generated by \( A_0 \cup \{ v \} \), because \( pA(1-p) = v^*vA(1-p) \subseteq v^*(1-p)A(1-p) \). It follows that \( \mathfrak{A} = C^*(A_0 \cup \{ w \}) \), and thus \( p\mathfrak{A}p = C^*(p\mathfrak{A}_0p \cup \{ w \}) \).

Note that since \( p \leq P \) and \( PC = PC \), we have

\[
(p\mathfrak{A}p)^\sigma = \{ x \in p\mathfrak{A}p \mid \Phi(x) = 0 \} = \{ x \in p\mathfrak{A}p \mid \phi(x) = 0 \}.
\]

Let \( \Theta \) be the set of all

\[
x = x_1x_2 \cdots x_n \in \Lambda^\sigma((p\mathfrak{A}_0p)^\sigma, \{ w^k \mid k \geq 1 \} \cup \{ (w^*)^k \mid k \geq 1 \})
\]

such that whenever \( 2 \leq j \leq n-1 \) and \( x_j \in (p\mathfrak{A}_0p)^\sigma \):

if \( x_{j-1} = w^i (i > 0) \) and \( x_{j+1} = (w^*)^j (j > 0) \) then \( x_j \in p\mathfrak{A}_0p \ominus pA_0p \)

if \( x_{j-1} = (w^*)^i (i > 0) \) and \( x_{j+1} = w^j (j > 0) \) then \( x_j \in p_1\mathfrak{A}_0p_1 \ominus y^*(qA_0q)y \).

(Note that \( E \) restricts to a conditional expectation \( p\mathfrak{A}_0p \rightarrow pA_0p \) and \( y^*E(y \cdot y^*)y \) provides one from \( p_1\mathfrak{A}_0p_1 \) onto \( y^*(qA_0q)y \).) Note that \( \text{span} (\{ p \} \cup \Theta) \) is the *-algebra generated by \( p\mathfrak{A}_0p \cup \{ w \} \) (see \cite{14}).
For $x \in \Theta$ of length $\ell(x)$ and $0 \leq j \leq \ell(x)$, let $t_j(x)$ be the number of $w$ minus the number of $w^*$ appearing in the first $j$ letters of $x$. (Here, of course, $t_0(x) = 0$.) For an interval $I$ of $\mathbb{Z}$ containing 0, define

$$
\Theta_I = \{ x \in \Theta \mid t_i(x) = 0 \text{ and } \forall 1 \leq j \leq \ell(x), t_j(x) \in I \}.
$$

Then span($\Theta_I \cup \{p\}$) is a $*$-subalgebra of $p\mathfrak{A}p$. Let $\mathfrak{A}_I = \text{span}(\Theta_I \cup \{p\})$.

Claim 1: $E(x) = 0$ for all $x \in \Theta_{(-\infty, \infty)} \setminus (p\mathfrak{A}_0)p^\circ$ and $\phi(x) = 0$ for all $x \in \Theta_{(-\infty, \infty)}$.

Proof of Claim 1. See the proof of [14, Claim 3.3].

Claim 2: The subalgebras $w^*\mathfrak{A}_{(-\infty,0)}w$ and $\mathfrak{A}_{[0,\infty)}$ are free with amalgamation over $pA_0p$ (with respect to the restrictions of the conditional expectation $E$).

Proof of Claim 2. See the proof of [14, Claim 3.4].

Claim 3: $\mathfrak{A}_{(-\infty,0]}$ is simple.

Proof of Claim 3. $\mathfrak{A}_{(-\infty,0]}$ is generated by $w^*\mathfrak{A}_{(-\infty,0]}w$ and $p\mathfrak{A}_0p$, which by Claim 2 are free with amalgamation over $pA_0p$. Let $\mathfrak{A}_0 = C^* (B \cup (\mathbb{C}p + (1-p)A(1-p)))$. Then $p\mathfrak{A}_0p$ is the $C^*$-algebra generated by $pA_0p$ and $p\mathfrak{A}_0p$, which are free with respect to $\phi$ (cf. [13, 2.8]). By using Claim 2 we get that $\mathfrak{A}_{(-\infty,0]}$ is generated by $w^*\mathfrak{A}_{(-\infty,0]}w$ and $p\mathfrak{A}_0p$, which are free with respect to $\phi$ (with amalgamation over $p\mathfrak{C}$).

But now $w^*\mathfrak{A}_{(-\infty,0]}w$ contains $w^*\mathfrak{A}_0w = w^*p_1\mathfrak{A}_0p_1w$ and, by the same argument applied above to $p_1\mathfrak{A}_0p_1$, we can deduce that $w^*p_1\mathfrak{A}_0p_1w$ contains a unital diffuse abelian subalgebra, which is contained in the centralizer of $\phi$ in $\mathfrak{A}$. So [13, 3.2] gives that $\mathfrak{A}_{(-\infty,0]}$ is simple.

Claim 4: For all $n \geq 0$, the $C^*$-algebra $\mathfrak{A}_{(-\infty,n]}$ is simple.

Proof of Claim 4. Use the same proof as in [14, proof of Claim 3.6].

Claim 5: Let $n \geq 0$, $k \geq 1$. Then $p_{n+1}\mathfrak{A}_{(-\infty,n]}p_{n+1}$ and $\{p_{n+k}\}$ are free with amalgamation over $\mathbb{C}p_{n+1}$) with respect to $\phi$ (after scaling).

Proof of Claim 5. As in [14, proof of Claim 3.7], we may reduce to show that $w^*\mathfrak{A}_{(-\infty,0]}w$ and $\{p_{k-1}\}$ are free (with amalgamation over $\mathbb{C}p_{n+1}$). Obviously, we can assume that $k \geq 2$.

Recall that, by hypothesis, $p_{k-1}$ is $A$-free, so that $E(p_{k-1}) \in \mathbb{C}p$. Set $b := p_{k-1} - \frac{\phi(p_{k-1})}{\phi(p)}p$.

Then $b$ belongs to $\mathfrak{A}_{[0,\infty)} \oplus pA_0p$, because $E(b) = 0$. We have to show that $\phi(x) = 0$ for all $x \in A^\circ((w^*\mathfrak{A}_{(-\infty,0]}w)^\circ, \{b\})$. Since $w^*\mathfrak{A}_{(-\infty,0]}w$ and $\mathfrak{A}_{[0,\infty)}$ are free with respect to $E$, with amalgamation over $pA_0p$ (Claim 2), we are led to show that the words in $A^\circ((pA_0p)^\circ, \{b\})$ of length $> 1$ belong to the kernel of $E$ (and so they belong to $\mathfrak{A}_{[0,\infty)} \oplus pA_0p$). But this follows from the $A$-freeness hypothesis of $p_{k-1}$.

There exists an injective endomorphism $\sigma : \mathfrak{A}_{(-\infty,\infty)} \to \mathfrak{A}_{(-\infty,\infty)}$ given by $\sigma(a) = waw^*$. Since $p\mathfrak{A}p = C^* (\mathfrak{A}_{(-\infty,\infty)} \cup \{w\})$, $p\mathfrak{A}p$ is a quotient of $\mathfrak{A}_{(-\infty,\infty)} \rtimes_{\sigma} \mathbb{N}$. It is enough thus to show that $\mathfrak{A}_{(-\infty,\infty)} \rtimes_{\sigma} \mathbb{N}$ is simple and purely infinite.

Claim 6: For all $m \geq 1$, $\alpha^m$ is multiplier outer in $\mathfrak{A}_{(-\infty,\infty)}$, where $\mathfrak{A}_{(-\infty,\infty)}$ denotes the inductive limit $\lim \mathfrak{A}_{(-\infty,\infty)} \sigma \to \mathfrak{A}_{(-\infty,\infty)} \sigma \to \cdots$.
Proof of Claim 6. This is proved in Lemma 2.3 of \cite{2}.

\textit{Claim 7:} Let $D$ be a nonzero hereditary $C^*$-subalgebra of $\mathfrak{A}_{(-\infty,\infty)}$. Then there is a projection in $D$ that is equivalent in $\mathfrak{A}_{(-\infty,\infty)}$ to $p_n$ for some $n$.

The proof of Claim 7 is exactly the same as the corresponding one in \cite{14} proof of Claim 3.8.

Claim 6 and Claim 7 show that the hypothesis in \cite{16, Theorem 2.1(ii)} are satisfied in our situation, and so we get from this result that $\mathfrak{A}_{(-\infty,\infty)} \rtimes_{\phi} \mathbb{N}$ is simple and purely infinite. (Recall that the hypothesis of $\alpha^n$ being outer in \cite{16} must be interpreted as being multiplier outer.)

\section{Purely infinite simple reduced graph $C^*$-algebras}

We first give an application of Dykema’s result (Corollary 3.4) to reduced free products of Cuntz algebras. It was shown in \cite{4, Proposition 4.2} that the $C^*$-algebras $C^*_{\text{red}}(E, C)$ in the next proposition are simple. By using Dykema’s Theorem, we can now show that they are also purely infinite.

\textbf{Proposition 4.1.} Let $n,m > 1$, and let $(E, C)$ be the separated graph with one vertex $v$ and with $C_v := \{X,Y\}$, where $|X| = n$ and $|Y| = m$. Then the reduced graph $C^*$-algebra $C^*_{\text{red}}(E, C)$ is purely infinite and simple.

\textit{Proof.} Set $A := \mathcal{O}_n$ and $B := \mathcal{O}_m$, where as usual $\mathcal{O}_k$ denotes the Cuntz algebra, and identify $A = C^*(E_X)$ and $B = C^*(E_Y)$. Then $(C^*_{\text{red}}(E, C), \phi)$ is the reduced free product of $(\mathcal{O}_n, \phi_n)$ and $(\mathcal{O}_m, \phi_m)$, where we denote by $\phi_k$ the canonical faithful state on $\mathcal{O}_k$ (see \cite{4} Theorem 2.1]). Set $X = \{e_1, \ldots, e_n\}$, $Y = \{f_1, \ldots, f_m\}$, and $p_i = e_i e^*_i$, $r_j = f_j f^*_j$, for $1 \leq i \leq n$, $1 \leq j \leq m$. Observe that $\phi_n(p_i) = 1/n$ and $\phi_m(f_j) = 1/m$. Set $p = p_1$, $q = e_2 e^*_1 e^*_2$, $v = e_2 e^*_1 e^*_2$. Then $vv^* = q$ and $v^*v = p$, and moreover $\phi_n(vx) = \frac{1}{n} \phi_n(xv)$ for all $x \in A$. Clearly $p$ is full in $C^*_{\text{red}}(E, C)$. Note that the centralizer of $\phi_m$ in $B = \mathcal{O}_m$ contains an abelian subalgebra $D$ on which $\phi_m$ is diffuse, namely the diagonal $C^*$-subalgebra $D$ generated by all the elements $\lambda \lambda^*$, with $\lambda$ a path in $E_Y$. (The spectrum of this algebra is a Cantor set.) So the rest of the conditions needed to apply Corollary 3.4 is verified in the same way as in \cite{14} 3.9(iii)].

We now recover the setting of the introduction. Recall that an abelian monoid $M$ is said to be \textit{conical} in case, for $x,y \in M$, $x + y = 0$ implies $x = y = 0$. Let $F$ be the free abelian monoid on free generators $a_1, a_2, \ldots, a_n$. Let

$$x = \sum_{i=1}^{n} r_i a_i, \quad y = \sum_{i=1}^{n} s_i a_i$$

be nonzero elements in $F$. Let $M$ be the abelian conical monoid $F/\sim$, where $\sim$ is the congruence on $F$ generated by $(x,y)$. We shall assume, without loss of generality, that $r_i + s_i > 0$ for $i = 1, \ldots, n$. (Otherwise the $C^*$-algebras we consider will have a finite-dimensional direct summand.)
Let \((E, C)\) be the separated graph associated to the presentation \((a_1, a_2, \ldots, a_n \mid x = y)\). To be precise, \(E\) is a graph with \(n + 1\) vertices \(E^0 = \{v, w_1, w_2, \ldots, w_n\}\) and \(N + M\) arrows, where \(M = \sum_{i=1}^{n} r_i\) and \(N = \sum_{i=1}^{n} s_i\), with \(v\) being a source and all \(w_i\) being sinks. The arrows in \(E\) are labeled as \(\alpha^{(i)}\), for \(1 \leq j \leq s_i\), \(1 \leq i \leq n\); and \(\beta^{(i)}\), for \(1 \leq j \leq r_i\), \(1 \leq i \leq n\). We have \(r(\alpha^{(i)}) = r(\beta^{(i)}) = w_i\), and \(s(\alpha^{(i)}) = s(\beta^{(i)}) = v\). There are two elements \(X, Y\) in \(C = C_v\), given by

\[
X = \{\alpha^{(i)} \mid 1 \leq j \leq s_i, 1 \leq i \leq n\}, \quad Y = \{\beta^{(i)} \mid 1 \leq j \leq r_i, 1 \leq i \leq n\}.
\]

The \(C^*\)-algebra \(C_{\text{red}}^*(E, C)\) turns out to be the amalgamated free product

\[
(C_{\text{red}}^*(E, C), \Phi) = (A, \Phi_A) *_{C^{n+1}} (B, \Phi_B),
\]

where \(A = \prod_{i=1}^{n} M_{s_i+1}(\mathbb{C})\) and \(B = \prod_{i=1}^{n} M_{r_i+1}(\mathbb{C})\). We will denote the canonical matrix units in \(A\) and \(B\) by \(e^{(i)}_{jk}\), \(0 \leq j, k \leq s_i, 1 \leq i \leq n\), and \(f^{(i)}_{jk}\), \(1 \leq j, k \leq r_i, 1 \leq i \leq n\), respectively. With this notation we can describe the unital embeddings \(\iota_A\) and \(\iota_B\) of \(C^{n+1}\) into \(A\) and \(B\) by the formulas

\[
\iota_A(e_i) = e^{(i)}_{00}, \quad \iota_B(e_i) = f^{(i)}_{00}
\]

for \(i = 1, \ldots, n\), where \(e_i, i = 1, \ldots, n+1\) are the minimal projections of \(C^{n+1}\). The conditional expectation \(\Phi_A\) is given as follows

\[
\Phi_A \left( \sum_{i=1}^{n} \sum_{j,k=0}^{s_i} a^{(i)}_{jk} e^{(i)}_{jk} \right) = \left( a^{(1)}_{00}, a^{(2)}_{00}, \ldots, a^{(n)}_{00}, \frac{1}{N} \sum_{i=1}^{n} \sum_{j=1}^{s_i} a^{(i)}_{jj} \right),
\]

with a similar formula holding for \(\Phi_B\).

In several cases we can show that \(A\) is simple, using a generalization of Avitzour’s Theorem ([4, 4.3]) (see [7] for the original result on free products).

**Lemma 4.2.** Let \((E, C)\) be the separated graph associated to the presentation \((a_1, \ldots, a_n \mid \sum_{i=1}^{n} r_i a_i = \sum_{i=1}^{n} s_i a_i)\), as described above. Assume that \(M \geq 2\) and \(N \geq 3\). Then the \(C^*\)-algebra \(C_{\text{red}}^*(E, C)\) is simple.

**Proof.** We shall use [4, 4.3 and 4.4]. Consider the projection in \(C_{\text{red}}^*(E, C)\) corresponding to the vertex \(v\), which is denoted by the same symbol. This projection corresponds to \((0, \ldots, 0, 1) \in C^{n+1}\) in the above picture. Observe that

\[
vAv \cong \prod_{i=1}^{n} M_{s_i}(\mathbb{C}) \quad \text{and} \quad vBv \cong \prod_{i=1}^{n} M_{r_i}(\mathbb{C}),
\]

and the canonical conditional expectations induce the tracial states \(\tau_{vAv}\) and \(\tau_{vBv}\) on \(vAv\) and \(vBv\) given by

\[
\tau_{vAv}(x_1, x_2, \ldots, x_n) = \frac{1}{M} \sum_{i=1}^{n} \text{Tr}_{r_i}(x_i), \quad \tau_{vBv}(x_1, x_2, \ldots, x_n) = \frac{1}{N} \sum_{i=1}^{n} \text{Tr}_{s_i}(x_i)
\]
respectively. Let $D_A$ and $D_B$ denote the canonical maximal commutative subalgebras of $vAv$ and $vBv$. Then \( \dim_C(D_A) = M \geq 2 \) and \( \dim_C(D_B) = N \geq 3 \). We can thus find unitaries \( a \) in $D_A$, and \( b, c \) in $D_B$ such that

\[
\tau_{vAe}(a) = 0, \quad \tau_{vBe}(b) = \tau_{vBe}(c) = \tau_{vBe}(b^*c) = 0.
\]

These unitaries satisfy all the hypothesis required in [4, Proposition 4.4].

In order to show simplicity, it remains to observe that \( v \) is full in $\mathfrak{A}$. To see this, it is enough to show that \( w_i \not\prec v \) in $\mathfrak{A}$ for all \( i \). Now given \( i \in \{1, \ldots, n\} \), either \( r_i \neq 0 \) or \( s_i \neq 0 \) by the hypothesis that \( r_i + s_i > 0 \), so either \( w_i \not\prec v \) in $A$ or \( w_i \not\prec v \) in $B$. In any case we have \( w_i \not\prec v \) in $\mathfrak{A}$, as wanted. We can therefore conclude from [4, Corollary 4.4] that $C^*_\text{red}(E, C)$ is simple.

**Theorem 4.3.** Let $(E, C)$ be the separated graph associated to the presentation \( \langle a_1, \ldots, a_n \mid \sum_{i=1}^n r_i a_i = \sum_{i=1}^n s_i a_i \rangle \), as described above, and put $M = \sum_{i=1}^n r_i$ and $N = \sum_{i=1}^n s_i$. Assume that there is \( i_0 \in \{1, \ldots, n\} \) such that \( s_{i_0} \geq 1 \) and \( r_{i_0} \geq 1 \), and that \( 2 \leq M < N \). Then the $C^*$-algebra $C^*_\text{red}(E, C)$ is purely infinite simple.

**Proof.** Write $\mathfrak{A} = C^*_\text{red}(E, C)$. We have \( (\mathfrak{A}, \Phi) = (A, \Phi_A) *_{C^{n+1}} (B, \Phi_B) \), as described above. By Lemma [4.2] $\mathfrak{A}$ is a simple $C^*$-algebra. Therefore every nonzero projection in $\mathfrak{A}$ is full.

By hypothesis, there exists \( i_0 \in \{1, \ldots, n\} \) such that \( s_{i_0} \geq 1 \) and \( r_{i_0} \geq 1 \). Without loss of generality, we shall assume that \( i_0 = 1 \).

Now consider the faithful state $\gamma$ on $C^{n+1}$ given by

\[
\gamma(x_1, x_2, \ldots, x_{n+1}) = \frac{1}{n+1} \sum_{i=1}^{n+1} x_i,
\]

and write $\phi = \gamma \circ \Phi$, which is a faithful state on $\mathfrak{A}$.

Let $P = (1, 0, \ldots, 0) \in C^{n+1}$, and consider the projections $p = e^{(1)}_{10}$ and $q = e^{(1)}_{11}$ in $A$. Observe that $p = P$ in $\mathfrak{A}$. Let $v = e^{(1)}_{10} \in A$, and observe that $v$ is a partial isometry in $A$ such that $vv^* = q$, $v^*v = p$, and $\phi(vx) = \lambda \phi(vx)$, where $\lambda = 1/N$.

Set $A_0 = pAp + (1-p)A(1-p) \cong \mathbb{C} \times M_{s_1}(\mathbb{C}) \times \prod_{i=2}^n M_{s_i+1}(\mathbb{C})$, and set $A_{00} = C^*(A_0 \cup B)$. We also put $A_{00} = \mathbb{C}p + \mathbb{C}q + (1-p-q)A(1-p-q)$, and $\mathfrak{A}_{00} = C^*(A_{00} \cup B)$. Let $\mathfrak{B}$ be the $C^*$-subalgebra of $(1-p)A_{00}(1-p)$ generated by \( \{e^{(i)}_{jj}, f^{(i)}_{kk} \mid 1 \leq j \leq s_i, 1 \leq k \leq r_i, i = 1, \ldots, n\} \).

Note that we have a natural isomorphism

\[
(C^*_r(\mathbb{Z}_N * \mathbb{Z}_M), \tau) \longrightarrow (\mathfrak{B}, (n+1)\phi)
\]

sending the canonical spectral projections in $C^*_r(\mathbb{Z}_N)$ to $e^{(i)}_{jj}$, $1 \leq j \leq s_i$, $1 \leq i \leq n$, and the canonical spectral projections in $C^*_r(\mathbb{Z}_M)$ to $f^{(i)}_{kk}$, $1 \leq k \leq r_i$, $1 \leq i \leq n$. Since

\[
\phi(q) = \frac{1}{(n+1)N} < \frac{1}{(n+1)M} = \phi(f^{(1)}_{11}),
\]

we get that $|q| < |f^{(1)}_{11}|$ in $K_0(\mathfrak{B})$ by [17, Theorem 2]. Since $M \geq 2$ and $N \geq 3$, it follows from [15, Corollary 3.9] that $C^*_r(\mathbb{Z}_N * \mathbb{Z}_M)$ has stable rank one. Therefore we get that $q \not\prec f^{(1)}_{11}$ in $\mathfrak{B}$. 
Observe that $\mathfrak{B}$ is contained in the centralizer of $\phi$, so $q = zz^*$ and $z^*z \leq f^{(1)}_{11}$ for some $z$ in the centralizer of $\phi$. Now consider $y = zf^{(1)}_{10} \in \mathfrak{A}_{00}$. We have $yy^* = q$ and $y^*y \leq f^{(1)}_{00} = e^{(1)}_{00} = p$. Moreover, since $z$ belongs to the centralizer of $\phi$, we have

$$\phi(y^*x) = \mu \phi(xy^*)$$

for all $x \in \mathfrak{A}$, where $\mu = M$. Note that $\lambda \mu = \frac{M}{N} < 1$. Set $w = y^*v$ and $p_k = w^k(w^*)^k$ for all $k \geq 1$. Then, since $y \in \mathfrak{A}_{00}$, the same proof as in Corollary 3.4 gives that the projections $p_k$ are $A$-free.

It remains to check that $q\mathfrak{A}_{00}q$ contains a unital diffuse abelian subalgebra contained in the centralizer of $\phi$. For this it is enough to see that $qC^*_r(\mathbb{Z}_N \ast \mathbb{Z}_M)q \cong q\mathfrak{B}q$ contains a unital diffuse abelian subalgebra, where $q$ is identified with a minimal projection in $C^*_r(\mathbb{Z}_N)$. Let $r$ be a minimal projection in $C^*_r(\mathbb{Z}_M)$. Let $\mathfrak{B}'$ be the $C^*$-subalgebra of $C^*_r(\mathbb{Z}_N \ast \mathbb{Z}_M) = (C^*_r(\mathbb{Z}_N), \tau_N) \ast (C^*_r(\mathbb{Z}_M), \tau_M)$ generated by $q, 1 - q, r, 1 - r$ (where $\tau_i$ is the canonical trace on $C^*_r(\mathbb{Z}_i)$). By [13, 2.7], we have

$$(\mathfrak{B}', \tau|_{\mathfrak{B}'}) = \left( \mathbb{C}_0 \oplus C([a, b], M_2(\mathbb{C})) \oplus \mathbb{C} \right)_{1 - \frac{q}{r} - \frac{1}{M}}$$

for some $0 < a < b < 1$. In this picture $q$ corresponds to the projection $0 \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \oplus 0$ and $\tau$ is given by the indicated weights on the projections $(1 - q) \wedge r$ and $(1 - q) \wedge (1 - r)$, together with an atomless measure whose support is $[a, b]$. It follows that $q\mathfrak{B}'q$ contains a unital diffuse abelian subalgebra, and the same will be true for $q\mathfrak{B}q$.

**Remarks 4.4.** (i) We remark that the above proof, combined with the proof of Theorem 3.3, gives the well-known description of $\mathcal{O}_n$ as a crossed product when applied to the presentation $\langle a \mid a = na \rangle$.

(ii) Theorem 4.3 does not hold if the hypothesis that there is $i_0 \in \{1, \ldots n\}$ such that $r_{i_0} > 0$ and $s_{i_0} > 0$ is suppressed, see Proposition 5.3.

It is worth to state explicitly the following particular case. Recall that the $C^*$-algebras $C^*_\text{red}(E(m, n), C(m, n))$ provide higher dimensional generalizations of Cuntz algebras. Indeed, we have $C^*_\text{red}(E(1, n), C(1, n)) \cong M_2(\mathcal{O}_n)$ (see [4, Example 4.5]). The representation $\mathcal{O}_2 \cong M_2(\mathcal{O}_2) \cong M_2(\mathbb{C}) \ast_{C^2} M_3(\mathbb{C})$ goes back to Choi ([12, Theorem 2.6]).

**Corollary 4.5.** Assume that $1 \leq m < n$. Then the $C^*$-algebra $C^*_\text{red}(E(m, n), C(m, n))$ is purely infinite simple.

5. The finite case

We will use the following well-known result for the existence of tracial states on an amalgamated free product.

**Lemma 5.1.** Let $(\mathfrak{A}, \Phi) = (A, \Phi_A) *_C (B, \Phi_B)$ be an amalgamated free product with respect to faithful conditional expectations $\Phi_A$ and $\Phi_B$. Then there is a faithful tracial state on $\mathfrak{A}$ if...
and only if there is a faithful state $\tau$ on $C$ such that both $\tau \circ \Phi_A$ and $\tau \circ \Phi_B$ are tracial states on $A$ and $B$ respectively. In this case the tracial state on $\mathfrak{A}$ is defined by $\theta = \tau \circ \Phi$.

We now show the existence of a faithful tracial state in the balanced case, as follows:

**Proposition 5.2.** Let $(E, C)$ be the separated graph associated to the presentation $\langle a_1, \ldots, a_n \mid \sum_{i=1}^n r_i a_i = \sum_{i=1}^n s_i a_i \rangle$, and put $M = \sum_{i=1}^n r_i$ and $N = \sum_{i=1}^n s_i$. Assume that $N = M$. Then the $C^*$-algebra $C^*_\text{red}(E, C)$ has a faithful tracial state. Moreover, if in addition $N = M > 2$, then $C^*_\text{red}(E, C)$ is simple and has a unique tracial state.

**Proof.** We put $\mathfrak{A} = C^*_\text{red}(E, C)$ and use the notation introduced in Section 4. Define the faithful state $\tau$ on $\mathbb{C}^{n+1}$ by

$$\tau(a_1, a_2, \ldots, a_{n+1}) = \frac{1}{N + n} \sum_{i=1}^n a_i + \frac{N}{N + n} a_{n+1}.$$ 

Since $N = M$, it is easily checked that $\tau \circ \Phi_A$ and $\tau \circ \Phi_B$ are tracial states of $A$ and $B$ respectively. So $\theta = \tau \circ \Phi$ is a faithful tracial state on $\mathfrak{A}$ by Lemma 5.1. $\square$

We now study the case where $\{i \in \{1, \ldots, n\} \mid r_i \neq 0\} \cap \{i \in \{1, \ldots, n\} \mid s_i \neq 0\} = \emptyset$. This case corresponds (by a Morita-equivalence) to an ordinary free product of finite-dimensional $C^*$-algebras, with respect to faithful tracial states.

**Proposition 5.3.** Assume that $\{i \in \{1, \ldots, n\} \mid s_i \neq 0\} \cap \{i \in \{1, \ldots, n\} \mid r_i \neq 0\} = \emptyset$. Then $C^*_\text{red}(E, C)$ admits a faithful tracial state. Moreover $C^*_\text{red}(E, C)$ is simple if $2 \leq M < N$.

**Proof.** Set $I_1 = \{i \in \{1, \ldots, n\} \mid s_i > 0\}$ and $I_2 = \{i \in \{1, \ldots, n\} \mid r_i > 0\}$. Then, by hypothesis $\{1, \ldots, n\}$ is the disjoint union of $I_1$ and $I_2$. Set $n_i := |I_i|$ for $i = 1, 2$, and $K := n_1 M + n_2 N + N M$, where as usual $N = \sum_{i=1}^n s_i$ and $M = \sum_{i=1}^n r_i$. Define a faithful state $\tau$ on $\mathbb{C}^{n+1}$ by

$$\tau(a_1, \ldots, a_n, a_{n+1}) = \frac{1}{K} \left( M \sum_{i \in I_1} a_i + N \sum_{i \in I_2} a_i + N M a_{n+1} \right).$$

Then $\tau \circ \Phi_A$ and $\tau \circ \Phi_B$ are tracial states on $A$ and $B$ respectively. We check this for $\tau \circ \Phi_A$:

$$(\tau \circ \Phi_A) \left( \sum_{i=1}^n \sum_{j,k=0}^{s_i} a_{ijk}^{(i)} e_{jj}^{(i)} \right) = \tau \left( a_{00}^{(1)}, \ldots, a_{00}^{(n)}, \frac{1}{N} \sum_i \sum_{j=1}^{s_i} a_{jj}^{(i)} \right)$$

$$= \frac{1}{K} \left( M \sum_{i \in I_1} a_{00}^{(i)} + N \sum_{i \in I_2} a_{00}^{(i)} + N M \sum_i \sum_{j=1}^{s_i} a_{jj}^{(i)} \right)$$

$$= \frac{M}{K} \sum_{i \in I_1} \sum_{j=0}^{s_i} a_{jj}^{(i)} + \frac{N}{K} \sum_{i \in I_2} a_{00}^{(i)},$$

which is a trace on $A$. Similarly $\tau \circ \Phi_B$ is a trace on $B$. By Lemma 5.1 it follows that $\tau \circ \Phi$ is a faithful trace on $C^*_\text{red}(E, C)$. $\square$
We can now provide the proof of theorem 1.2.

**Proof of Theorem 1.2** Assume that $2 \leq M \leq N$. Set $I_1 = \{i \in \{1, \ldots, n\} \mid s_i > 0\}$ and $I_2 = \{i \in \{1, \ldots, n\} \mid r_i > 0\}$. If $M < N$ and $I_1 \cap I_2 \neq \emptyset$, then $C^*_{\text{red}}(E, C)$ is purely infinite simple by Theorem 4.3. If $M < N$ and $I_1 \cap I_2 = \emptyset$, then $C^*_{\text{red}}(E, C)$ admits a faithful tracial state by Proposition 5.3. If $N = M$, then $C^*_{\text{red}}(E, C)$ admits a faithful tracial state by Proposition 5.2.

Assume now that $2 \leq M \leq N$, $N + M \geq 5$ and $C^*_{\text{red}}(E, C)$ is finite. Then $\mathfrak{A}$ is simple by Lemma 4.2 and by the dichotomy showed before, there exists a tracial state on $\mathfrak{A}$. If $\phi_1$ and $\phi_2$ are two tracial states on $\mathfrak{A} := C^*_{\text{red}}(E, C)$, then since, by [4. Proposition 4.3]

$$\Phi(x) \in \overline{co}\{u^*xu : u \text{ unitary in } v\mathfrak{A}v\}$$

for all $x \in v\mathfrak{A}v$, we get that $\phi_1$ and $\phi_2$ agree on $v\mathfrak{A}v$. Since $v$ is a full projection in $\mathfrak{A}$, we obtain that $\phi_1 = \phi_2$. Thus there is exactly one tracial state on $\mathfrak{A}$, as desired. \qed

In order to have a complete description of the graph $C^*$-algebras $C^*_{\text{red}}(E, C)$ in the one-relator case, it remains to study the cases where $M = 1$, and also the cases where $N = M = 2$ in terms of simplicity and uniqueness of the trace. All cases are easy to analyze, except two. We first collect the easy cases in a lemma.

**Lemma 5.4.** Let $(E, C)$ be the separated graph associated to the presentation $\langle a_1, \ldots, a_n \mid \sum_{i=1}^n r_i a_i = \sum_{i=1}^n s_i a_i, \rangle$, and put $M = \sum_{i=1}^n r_i$ and $N = \sum_{i=1}^n s_i$.

1. If $M = 1$, then we have $C^*_{\text{red}}(E, C) \cong M_2(C^*(F))$, where $F$ is the graph obtained from $E$ by collapsing $v$ and $r(\beta_1)$ and eliminating the arrow $\beta_1$.

2. If $M = N = 2$, then there is a faithful tracial state on $C^*_{\text{red}}(E, C)$, and there are several cases:

   a. If $n = 4$, then $C^*_{\text{red}}(E, C)$ is Morita-equivalent to $C^*_r(\mathbb{Z}_2 \ast \mathbb{Z}_2) \cong C^*(\mathbb{Z}_2 \ast \mathbb{Z}_2)$, and so it is non-simple.

   b. If $n = 3$, and either $r(\alpha_1) = r(\alpha_2)$ or $r(\beta_1) = r(\beta_2)$, then $C^*_{\text{red}}(E, C)$ is Morita-equivalent to $(M_2(\mathbb{C}), \text{Tr}_2) \ast \left(1/2 \oplus 1/2\right)$. It is simple with a unique trace.

   c. If $n = 2$ and $r(\alpha_1) \neq r(\alpha_2)$ or $r(\beta_1) = r(\beta_2)$, then $C^*_{\text{red}}(E, C)$ is Morita-equivalent to $(M_2(\mathbb{C}), \text{Tr}_2) \ast (M_2(\mathbb{C}), \text{Tr}_2)$. It is simple with a unique trace.

   d. If $n = 2$ and $r(\alpha_1) \neq r(\alpha_2)$ and $r(\beta_1) = r(\beta_2)$, then $C^*_{\text{red}}(E, C)$ is also simple with a unique trace.

   e. If $n = 1$, then $C^*_{\text{red}}(E, C)$ is simple with a unique trace.

**Proof.** (1) It is a straightforward computation.

(2) There is a faithful trace by Proposition 5.2. In cases (b), (c), (d), (e) one can use [4 Corollary 4.4] to show simplicity and uniqueness of the trace, because both $v\mathfrak{A}v$ and $v\mathfrak{B}v$ are at least 2-dimensional, and at least one of them is a matrix algebra of size at least $2 \times 2$. \qed

The two cases remaining to analyze when $N = M = 2$ are:

- $n = 3$ and $r(\alpha_i) = r(\beta_j)$ for some $i, j$. 


• \( n = 2 \) and \( r(\alpha_i) = r(\beta_i) \) for \( i = 1, 2 \).

We now study these two cases, in which we cannot apply the generalization of Avitzour’s Theorem. It is enough to pay attention to the structure of the full corner \( vC^*_\text{red}(E,C)v \). This is what turns out to be a significant \( C^* \)-algebra in these examples.

![Figure 1. The separated graph \((E,C)\)](image1)

![Figure 2. The separated graph \((F,D)\)](image2)

We start by analyzing the full \( C^* \)-algebras.

**Lemma 5.5.** (1) Let \((E,C)\) be the separated graph associated to the presentation \( \langle a,b,c \mid a+b = a+c \rangle \) (see Figure 1). Then the corner \( vC^*(E,C)v \) is the universal unital \( C^* \)-algebra generated by a partial isometry.

(2) Let \((F,D)\) be the separated graph associated to the presentation \( \langle a,b \mid a+b = a+b \rangle \) (see Figure 2). Then

\[
vC^*(F,D)v \cong C^*((\ast_z\mathbb{Z}_2) \rtimes \mathbb{Z})
\]

where \( \mathbb{Z} \) acts on \( \ast_z\mathbb{Z}_2 \) by shifting the factors of the free product. Moreover \( vC^*(E,C)v \) is a \( C^* \)-subalgebra of \( vC^*(F,D)v \).

**Proof.** (1) We have \( r(\alpha_1) = w_1 = r(\beta_1) \) and \( r(\alpha_2) = w_2, r(\beta_2) = w_3 \). Let \( s = \beta_1\alpha_1^* \in vC^*(E,C)v \). Then \( s^*s = \alpha_1\alpha_1^* =: p \) and \( ss^* = \beta_1^*\beta_1^* =: q \). Let \( \mathfrak{G} \) be the universal unital \( \ast \)-homomorphism
$G \to vC^*(E, C)v$ sending $w$ to $s$. It is not difficult, using the universal property of $C^*(E, C)$, to build an inverse of this homomorphism.

(2) Here we have $r(\alpha_i) = r(\beta_i) = w_i$ for $i = 1, 2$. Consider the unitary $u = \beta_1\alpha_1^* + \beta_2\alpha_2^*$ in $vC^*(F, D)v$, and the projection $p_0 := \alpha_1\alpha_1^*$. Let $\mathcal{U} = C^*(u', p')$ be the universal C*-algebra generated by a unitary $u'$ and a projection $p'$. There exists a unique *-homomorphism $\mathcal{U} \to vC^*(F, D)v$ sending $u'$ to $u$ and $p'$ to $p$. It is easily seen (again using the universal property of $C^*(F, D)$) that this map is an isomorphism. So we will identify $u'$ with $u$ and $p'$ with $p$, and we shall write $\mathcal{U} = vC^*(F, D)v$.

We will now show that $\mathcal{U} \cong C^*((\ast Z\mathbb{Z}_2) \rtimes \mathbb{Z})$. Write $p_0 = p$ and $p_n = u^np(u^*)^n$. Let $(u_n)_{n \in \mathbb{Z}}$ denote the unitaries in $C^*((\ast Z\mathbb{Z}_2) \rtimes \mathbb{Z})$ corresponding to the generators of the different copies of $\mathbb{Z}_2$, and let $z$ be the unitary implementing the action of $\mathbb{Z}$ on $\ast Z\mathbb{Z}_2$. We may define a *-homomorphism $\varphi: \mathcal{U} \to C^*((\ast Z\mathbb{Z}_2) \rtimes \mathbb{Z})$ by sending $u$ to $z$ and $p$ to $\frac{1-wn}{2}$. On the other hand, we have a unitary representation of $(\ast Z\mathbb{Z}_2) \rtimes \mathbb{Z}$ on $\mathcal{U}$ obtained by sending $u_n$ to $1 - 2p_n$ and $z$ to $u$. It is straightforward to check that $\varphi$ and $\psi$ are mutually inverse.

Now we show that the canonical homomorphism $\eta: \mathcal{G} \to \mathcal{U}$ sending $s$ to $up$ is an isometry. There is a faithful representation $\rho$ of $\mathcal{G}$ on a separable Hilbert space $H$ such that both $(1 - \rho(p))H$ and $(1 - \rho(q))H$ are infinite-dimensional. Therefore there is a unitary $U$ on $H$ extending $\rho(s)$, so that $U\rho(p) = \rho(s)$. It follows that the *-homomorphism $\rho: \mathcal{G} \to B(H)$ factors through $\mathcal{U}$, that is there is a *-homomorphism $\varphi: \mathcal{U} \to B(H)$ such that $\rho = \varphi \circ \eta$. Since $\rho$ is faithful, we see that $\eta$ is injective, and so it is an isometry.

The universal non-unital C*-algebra $\mathcal{G}'$ generated by an isometry has recently been studied by Brenken and Niu in [8]. The C*-algebra $\mathcal{G} = vC^*(E, C)v$ of Lemma 5.5(1) is just the unitization of $\mathcal{G}'$. Some properties of $C^*(E, C)$ can thus be derived from [8], for instance we see from [8, Corollary 1] that $C^*(E, C)$ is a non-exact C*-algebra.

The following result was obtained in collaboration with Ken Goodearl.

**Proposition 5.6.** Let $(F, D)$ be the separated graph associated to the presentation $\langle a, b \mid a + b = a + b \rangle$. Then

$vC^*_{\text{red}}(F, D)v \cong C^*_r((\ast Z\mathbb{Z}_2) \rtimes \mathbb{Z}) \cong (C^*_r(\ast Z\mathbb{Z}_2)) \rtimes_r \mathbb{Z}$.

**Proof.** We follow with the notation introduced in the proof of Lemma 5.5. We have

$$(\mathfrak{A}, \Phi) = (C^*_{\text{red}}(F, D), \Phi) = (A, \Phi_A) \ast (B, \Phi_B),$$

where $A$ and $B$ are the usual graph C*-algebras of the graphs corresponding to the edges $\{\alpha_1, \alpha_2\}$ and $\{\beta_1, \beta_2\}$ respectively, and the maps $\Phi_A$ and $\Phi_B$ are the canonical conditional expectations onto $\mathbb{C}v + \mathbb{C}w_1 + \mathbb{C}w_2$, as defined in [4, Theorem 2.1].

Put $T = M_2(C^*_r((\ast Z\mathbb{Z}_2) \rtimes \mathbb{Z}))$, and denote by $e_{ij}$, $1 \leq i, j \leq 2$ the canonical matrix units in $T$. Set

$$p_i^+ = \frac{1 ± u_i}{2}, \quad i \in \mathbb{Z}.$$
We define unital \(*\)-homomorphisms \(\sigma_A: A \to T\) and \(\sigma_B: B \to T\) by
\[
\sigma_A(v) = \sigma_B(v) = e_{11}, \quad \sigma_A(w_1) = \sigma_B(w_1) = p_0^+ e_{22}, \quad \sigma_A(w_2) = \sigma_B(w_2) = p_0^- e_{22}, \quad \sigma_A(\alpha_1) = p_0^+ e_{12}, \quad \sigma_A(\alpha_2) = p_0^- e_{12}, \quad \sigma_B(\beta_1) = zp_0^+ e_{12}, \quad \sigma_B(\beta_2) = zp_0^- e_{12}.
\]
Let \(\Theta: T \to \mathbb{C}e_{11} + \mathbb{C}p_0^+ + \mathbb{C}p_0^-\) be the conditional expectation given by
\[
\Theta\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) = \tau(a_{11})e_{11} + \tau(p_0^+ a_{22} p_0^+) p_0^+ + \tau(p_0^- a_{22} p_0^-)p_0^-,
\]
where \(\tau\) is the canonical faithful trace on the reduced group \(\text{C}^*\)-algebra \(C_r^*((\ast_z \mathbb{Z}_2) \rtimes \mathbb{Z})\). In order to check that \((\mathfrak{A}, \Phi) \cong (T, \Theta)\), it suffices to show that the conditions (1)–(5) in Definition 22 are satisfied. All conditions are easily verified, with the exception of (4).

To show (4), we compute the kernels of \(\Theta|_{\sigma_A(A)}\) and \(\Theta|_{\sigma_B(B)}\) to be \(Z_0 = \mathbb{C}u_0 e_{11} + A_0 e_{12} + A_0 e_{21}\) and \(Z_1 = \mathbb{C}u_1 e_{11} + A_1 z e_{12} + A_0 z^* e_{21}\) respectively, where \(A_i = \mathbb{C}1 + \mathbb{C}u_i\) is the subalgebra of \(C_r^*((\ast_z \mathbb{Z}_2) \rtimes \mathbb{Z})\) generated by \(u_i\). Then we have to show that \(\Theta(\alpha) = 0\) for every \(\alpha \in \Lambda^0(Z_0, Z_1)\). This is shown by induction on the length of \(\alpha\). In order to prove it, we introduce the following notation. For \(i \in \mathbb{Z}\), denote by \(F_i\) the linear span of the set of reduced words in \(\{u_n : n \in \mathbb{Z}\}\) ending in \(u_i\). We will write \(F_{\leq i} = \sum_{j \leq i} F_j\) and \(\widetilde{F}_{\leq i} = \mathbb{C}1 + F_{\leq i}\), and similarly for \(F_{\geq i}\) and \(\widetilde{F}_{\geq i}\).

One shows by induction on the length of \(\alpha \in \Lambda^0(Z_0, Z_1)\) that if \(\alpha\) ends in \(Z_0\), then
\[
\alpha \in L^{(0)}_{11} e_{11} + L^{(0)}_{12} e_{12} + L^{(0)}_{21} e_{21} + L^{(0)}_{22} e_{22},
\]
where
\[
L^{(0)}_{11} = F_{\leq 0} + \sum_{i=1}^{n} [\widetilde{F}_{\leq i}] z^i + \sum_{j=1}^{n} [F_{\leq -j}] (z^*)^j,
\]
\[
L^{(0)}_{22} = [F_{\geq 1}] A_0 + \sum_{i=1}^{n-1} [F_{\geq i+1}] A_i z^i + \sum_{j=1}^{n} [\widetilde{F}_{\geq -j+1}] A_j (z^*)^j,
\]
and \(L^{(0)}_{21} = A_0 + L^{(0)}_{11}, L^{(0)}_{12} = A_0 + L^{(0)}_{22}\), for suitable \(n \geq 0\).

Correspondingly, if \(\alpha\) ends in \(Z_1\), then
\[
\alpha \in L^{(1)}_{11} e_{11} + L^{(1)}_{12} e_{12} + L^{(1)}_{21} e_{21} + L^{(1)}_{22} e_{22},
\]
where
\[
L^{(1)}_{11} = L^{(1)}_{21} = F_{\geq 1} + \sum_{i=1}^{n} [F_{\geq i+1}] z^i + \sum_{j=1}^{n+1} [\widetilde{F}_{\geq -j+1}] (z^*)^j,
\]
and
\[
L^{(1)}_{22} = L^{(1)}_{12} = [F_{\leq -1}] A_0 + \sum_{i=1}^{n+1} [\widetilde{F}_{\leq i-1}] A_i z^i + \sum_{j=1}^{n-1} [F_{\leq -j-1}] A_j (z^*)^j.
\]

This concludes the proof.\(\square\)
Corollary 5.7. Let \((F, D)\) be the separated graph associated to the presentation \(\langle a, b \mid a+b = a+b \rangle\). Then \(C^*_{\text{red}}(F, D)\) is a simple \(C^*\)-algebra.

Proof. By Proposition 5.6, it is enough to show that \(C_r((\ast_Z \mathbb{Z}2) \rtimes \mathbb{Z})\) is simple. First observe that

\[
(C_r((\ast_Z \mathbb{Z}2), \tau) = \ast_{i \in \mathbb{Z}}(C^*_r(\mathbb{Z}2), \tau_i),
\]

that is \(C^*_r(\ast_Z \mathbb{Z}2)\) is the reduced crossed product of countably many group \(C^*\)-algebras \(C^*_r(\mathbb{Z}2)\), with respect to their canonical tracial states \(\tau_i\), and so it is simple by an application of Avitzour’s Theorem. By Kishimoto’s Theorem [20, Theorem 3.1], in order to show that \(C^*_r((\ast_Z \mathbb{Z}2) \rtimes \mathbb{Z})\) is simple, it is enough to show that the action of each non-trivial element of \(\mathbb{Z}\) on \(C_r((\ast_Z \mathbb{Z}2) \rtimes \mathbb{Z})\) is outer. To show this, we first show that the relative commutant of \(\mathfrak{B} := C^*_r(\ast_Z \mathbb{Z}2)\) in \(\mathfrak{C} := C_r((\ast_Z \mathbb{Z}2) \rtimes \mathbb{Z})\) is trivial. For this, we use an argument similar to the one in [11, proof of Claim 3]. We put \(G := (\ast_Z \mathbb{Z}2) \rtimes \mathbb{Z}\).

Suppose that \(x \in \mathfrak{C}\) and \(x\) commutes with \(\mathfrak{B}\). We will show that \(x_0 = x - \tau(x)1\) is zero. Suppose, to obtain a contradiction, that \(x_0 \neq 0\). Since \(\tau\) is faithful, \(\|x_0\|_2 = \tau(x_0^*x_0)^{1/2} > 0\). Choose \(\epsilon > 0\) so that \(0 < \epsilon < \frac{\|x_0\|_2}{3}\). There is an element \(y\) in the group algebra \(\mathbb{C}G\) such that

\[
y = \sum_{g \in F} \lambda_g g,
\]

where \(F\) is a finite subset of \(G \setminus \{1\}\), \(\lambda_g \in \mathbb{C} \setminus \{0\}\), and \(\|x_0 - y\| < \epsilon\). We consider the canonical expression of elements in \(G\), as \(wz^j\), where \(w\) is a reduced word in the \(u_i\)'s and \(j\) is an integer. Let \(I\) be the finite subset of \(\mathbb{Z}\) consisting of those integers \(n\) such that \(u_n\) is involved in the canonical expression of some of the elements of \(F\). Let \(J\) be the (finite) set of integers that appear as powers of \(z\) in the canonical expression of the elements of \(F\). Take \(N \in \mathbb{N}\) big enough so that \(N > \max\{i, i' - j \mid i, i' \in I, j \in J\}\), and write \(v = u_N\).

We claim that \(vy\) and \(yv\) are orthogonal with respect to the inner product on \(\mathfrak{C}\) induced by \(\tau\), that is, \(\tau(v^*y^*vy) = 0\). Indeed we have

\[
\tau(v^*y^*vy) = \tau\left( \sum_{g,h \in F} \lambda_g \lambda_h u_N g^{-1} u_N h \right) = 0,
\]

because of the choice of \(N\). Now, by orthogonality of \(vy\) and \(yv\), we have \(\|vy - yv\| \geq \|vy\|_2 + \|yv\|_2 = 2\|y\|_2\), and thus

\[
\|vx_0 - x_0x\| \geq \|vy - yv\| - 2\epsilon > \|y\|_2 - 2\epsilon \geq \|x_0\|_2 - 3\epsilon > 0,
\]

contradicting the fact that \(x_0\) centralizes \(\mathfrak{B}\).

It is now easy to see that, for every non-zero integer \(m\), the action \(\alpha^m\) is outer on \(\mathfrak{B}\). Indeed, if \(\alpha^m\) is an inner automorphism of \(\mathfrak{B}\), induced by a unitary \(d\) in \(\mathfrak{B}\), then there is \(\lambda \in \mathbb{T}\) such that \(d = \lambda \alpha^m\) in \(\mathfrak{C} = \mathfrak{B} \rtimes \mathbb{Z}\), which is a contradiction. Finally, Kishimoto’s Theorem [20, Theorem 3.1] gives that \(\mathfrak{C}\), and so \(C^*_{\text{red}}(F, D)\), is a simple \(C^*\)-algebra. \(\square\)

Finally, we show that the embedding of \(vC^*(E, C)v\) into \(vC^*(F, D)v\) established in Lemma 5.6(2) extends to the reduced setting.
Proposition 5.8. Adopt the notation of Lemma 5.5. Then there is a trace-preserving embedding $vC^*_r(E,C)v \hookrightarrow vC^*_r(F,D)v$.

Proof. Consider the subalgebra $\mathcal{D} = (1 \oplus w_2)M_2(C^*_r(F,D))(1 \oplus w_2)$ of $M_2(C^*_r(F,D))$. Set $C = Cv \oplus Cw_1 \oplus Cw_2 \oplus Cv_3$. Define $\Phi: \mathcal{D} \rightarrow C$ by

$$\Phi\left(\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\right) = (\Phi(x_{11}), \tau_{w_2}(x_{22})w_3),$$

where $\Phi: C^*_r(F,D) \rightarrow Cv \oplus Cw_1 \oplus Cw_2$ is the canonical conditional expectation and $\tau_{w_2}$ is the state of $w_2C^*_r(F,D)w_2$ given by $\Phi(x) = \tau_{w_2}(x)w_2$ for $x \in w_2C^*_r(F,D)w_2$. We look at $C$ as a $C^*$-subalgebra of $\mathcal{D}$ by sending $v, w_1, w_2$ to the corresponding vertices in $(1 \oplus 0)M_2(C^*_r(F,D))(1 \oplus 0)$ and sending $w_3$ to $0 \oplus w_2$. With this embedding in mind, $\Phi$ is a faithful conditional expectation from $\mathcal{D}$ onto $C$.

Write $C^*_r(E,C) = A_1 \ast_C A_2$, where $A_1 = C^*(C, \alpha_1, \alpha_2)$ and $A_2 = C^*(C, \beta_1, \beta_2)$. We define $\ast$-homomorphisms $\sigma_i: A_i \rightarrow \mathcal{D}$ by sending canonically $C$ to $\mathcal{D}$ as above, and putting

$$\sigma_1(\alpha_1) = \alpha_1 \oplus 0, \quad \sigma(\alpha_2) = \alpha_2 \oplus 0, \quad \sigma(\beta_1) = \beta_1 \oplus 0, \quad \sigma(\beta_2) = \begin{pmatrix} 0 & \beta_2 \\ 0 & 0 \end{pmatrix}.$$

In order to show that these maps define an isomorphism from $C^*_r(E,C)$ onto $C^*(\sigma(1) \cup \sigma(A_2))$, it suffices to check conditions (1)-(5) of Definition 2.2. All properties are obvious with the exception of (4). In order to check (4), set

$$Z_1 = \{1 - 2\alpha_1 \alpha_2, \alpha_1, \alpha_2, \alpha_2^*\}, \quad Z_2 = \{1 - 2\beta_1 \beta_2, \beta_1, \beta_2, \beta_2^*\},$$

and let $a_1a_2 \cdots a_n \in L^0(Z_1, Z_2)$, where $a_i \in Z_{\iota_i}$, with $\iota_i \neq \iota_{i+1}$ for $i = 1, \cdots, n - 1$. Then we have to prove that $\Phi(\sigma_1(a_1)\sigma_2(a_2) \cdots \sigma_n(a_n)) = 0$. This is obvious if all the letters in $a_1a_2 \cdots a_n$ are different from $\beta_2$ and $\beta_2^*$. The nonzero expressions involving $\beta_2$ or $\beta_2^*$ give terms in $\mathcal{D}$ which have one of the following forms:

$$\begin{pmatrix} 0 & a \beta_2 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ \beta_2^*a & 0 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 0 & 0 \\ 0 & \beta_2^*a \beta_2 \end{pmatrix},$$

with $\Phi(a\beta_2) = 0$, $\Phi(\beta_2^*a) = 0$, and $\Phi(\beta_2^*a\beta) = 0$ in the respective cases. Consequently, $\Phi(\sigma_1(a_1)\sigma_2(a_2) \cdots \sigma_n(a_n)) = 0$ in all cases, as desired.

It remains an open problem to determine whether the $C^*$-algebra $C^*_r(E,C)$ considered in the above result is simple.

6. Some remarks on K-theory

We recall from [4] the following conjecture:

Conjecture 6.1. [4, 7.6] Let $(E, C)$ be a finitely separated graph. Let $M(E, C)$ be the abelian monoid with generators $\{a_v | v \in E^0\}$ and relations given by $a_v = \sum_{e \in X} a_{r(e)}$ for all $v \in E^0$ and all $X \subseteq C_v$. Then the natural map $M(E, C) \rightarrow \mathcal{V}(C^*(E, C))$ is an isomorphism.
Let $L(E, C)$ be the dense $*$-subalgebra of $C^*(E, C)$ generated by the canonical generators of $C^*(E, C)$. It was shown in [3, Theorem 4.3] that there is a natural isomorphism $M(E, C) \to \mathcal{V}(L(E, C))$, sending $a_v$ to $[v] \in \mathcal{V}(L(E, C))$, where $\mathcal{V}(L(E, C))$ is the abelian monoid of isomorphism classes of finitely generated projective right modules over $L(E, C)$. So the above conjecture is equivalent to the question of whether the natural induced map $\mathcal{V}(L(E, C)) \to \mathcal{V}(C^*(E, C))$ is an isomorphism. The answer is positive in the non-separated case [5, Theorem 7.1].

We now make two weaker conjectures, which can be checked in various situations of interest.

**Conjecture 6.2.** Let $(E, C)$ be a finitely separated graph. Then the natural map

$$M(E, C) \to \mathcal{V}(C^*(E, C))$$

is injective.

**Conjecture 6.3.** Let $(E, C)$ be a finitely separated graph. Let $(G(E, C), G(E, C)^+)$ be the Grothendieck group of $M(E, C)$, with the canonical pre-ordered structure given by taking $G(E, C)^+ = \iota(M(E, C))$, where $\iota: M(E, C) \to G(E, C)$ is the canonical map (note that $\iota$ does not need to be injective). Then we have a natural homomorphism

$$(G(E, C), G(E, C)^+) \to (K_0(C^*(E, C)), K_0(C^*(E, C)^+))$$

of partially pre-ordered abelian groups. We conjecture that this map is an isomorphism. By [4, Theorem 5.2] the map $G(E, C) \to K_0(C^*(E, C))$ is an isomorphism, so the conjecture is that the order structure in $K_0(C^*(E, C))$ is the one given by $G(E, C)^+$.

Let us show that Conjecture 6.1 implies a positive answer to a question of Rørdam and Villadsen [26, Question 2.1(a)]. Recall that an ordered group is a pair $(G, G^+)$ where $G$ is an abelian group, $G^+ \subseteq G$, and

$$G^+ + G^+ \subseteq G^+, \quad G^+ - G^+ = G, \quad G^+ \cap -G^+ = \{0\}.$$

**Remark 6.4.** If Conjecture 6.1 holds, then for every ordered abelian group $(G, G^+)$ there is a stably finite C*-algebra $A$ such that $(K_0(A), K_0(A)^+) \cong (G, G^+)$. Indeed, since $G^+$ is a conical abelian monoid, we may take a presentation $\langle \mathcal{X} | \mathcal{R} \rangle$ of $G^+$ as indicated in the proof of [3, Proposition 4.4]. Then, with $(E, C)$ being the separated graph associated to this presentation, we have $M(E, C) \cong G^+$ ([3, 4.4]). Since the Grothendieck group of $G^+$ is $G$ we obtain from Conjecture 6.1 that $(G, G^+) \cong (K_0(A), K_0(A)^+)$, where $A = C^*(E, C)$. It remains to verify that $A$ is stably finite. But since $\mathcal{V}(A) \cong G^+$ embeds into a group, it is clear that all projections in $M_\infty(A)$ are finite, as desired.

However the validity of Conjecture 6.1 seems to be known in very few cases in the non-separated case.

The validity of the weaker conjectures 6.2, 6.3 can be checked in several cases. For instance, it holds for the separated graph $(E, C)$ with just one vertex and the sets in the partition $C$ reduced to singletons, because then $C^*(E, C)$ is just the full group C*-algebra $C^*(\mathbb{F}_n)$, where $n$ is the number of edges, and for the C*-algebras $C^*(E(n, n), C(n, n))$ (see the proof of [4].
Theorem 6.3]). For the C*-algebras \( \mathcal{A} = M_k(\mathbb{C}) \ast M_l(\mathbb{C}) \), with \( \gcd(k, l) = 1 \) and at least one of \( k \) or \( l \) prime, it follows from \([26]\) Theorem 3.6] that \( (K_0(\mathcal{A}), K_0(\mathcal{A})^+) \cong (\mathbb{Z}, \langle k, l \rangle) \), with the integers \( k, l \) represented in \( K_0(\mathcal{A}) \) by the classes of the minimal projections in \( M_l(\mathbb{C}) \) and \( M_k(\mathbb{C}) \) respectively. The algebra \( \mathcal{A} \) is Morita-equivalent to the C*-algebra of the separated graph associated to the presentation \( \langle a, b \mid ka = lb \rangle \), and so both Conjectures 6.2 and 6.3 hold. Note that the integers \( a, b \) treated in Proposition 4.1. In this case \( k, l \) values of \( (\text{by } [4, \text{ Theorem 5.2}]) \), and so both Conjectures 6.2 and 6.3 hold. Note that \( C^*_\text{red}(E, C) \) is purely infinite simple by 4.1 and that \( K_0(\mathcal{A}) \cong K_0(C^*_\text{red}(E, C)) \) by Germain’s Theorem \([18]\), so we obtain the very precise information \( \mathcal{V}(C^*_\text{red}(E, C)) = \langle a \mid a = (d + 1)a \rangle \) for the reduced graph C*-algebra.

As another interesting example, we consider the C*-algebra \( \mathcal{A} = \mathcal{O}_n \ast \mathcal{O}_m = C^*(E, C) \) treated in Proposition 4.1. In this case \( K_0(\mathcal{A}) = K_0(\mathcal{A})^+ \cong \mathbb{Z}_d \), where \( d = \gcd(n - 1, m - 1) \) (by \([4\text{, Theorem 5.2}])\), and

\[
M(E, C) = \langle a \mid a = na = ma \rangle = \langle a \mid a = (d + 1)a \rangle
\]

so both Conjectures 6.2 and 6.3 hold. Note that \( C^*_\text{red}(E, C) \) is purely infinite simple by 4.1 and that \( K_0(\mathcal{A}) \cong K_0(C^*_\text{red}(E, C)) \) by Germain’s Theorem \([18]\), so we obtain the very precise information \( \mathcal{V}(C^*_\text{red}(E, C)) = \langle a \mid a = (d + 1)a \rangle \) for the reduced graph C*-algebra.

In contrast, the natural map \( M(E, C) \to \mathcal{V}(C^*_\text{red}(E, C)) \), defined by composing the natural map \( M(E, C) \to \mathcal{V}(C^*(E, C)) \) with the homomorphism induced by the natural surjection \( C^*(E, C) \to C^*_\text{red}(E, C) \), fails to be injective or surjective in many cases. Using our main result, we will show that it may even happen that \( M(E, C) \) is stably finite (i.e. \( x + y = x \implies y = 0 \ \forall \ x, y \in M(E, C) \)) but \( C^*_\text{red}(E, C) \) is purely infinite simple. To see this, we need a monoid-theoretic lemma.

**Lemma 6.5.** Let \( F \) be the free abelian monoid on free generators \( a_1, a_2, \ldots, a_n \). Let

\[
x = \sum_{i=1}^n r_ia_i, \quad y = \sum_{i=1}^n s_ia_i
\]

be nonzero elements in \( F \). Let \( M \) be the conical abelian monoid \( F/\sim \) where \( \sim \) is the congruence on \( F \) generated by \( (x, y) \). Then \( M \) contains infinite elements if and only if either \( x < y \) or \( y < x \) in the usual order of \( F \).

**Proof.** If \( x < y \) or \( y < x \) then clearly \([x]\) and \([y]\) are infinite elements in \( M \).

Observe that \( \sim \) agrees with the congruence on \( F \) defined as follows: For \( a, b \in F \), set \( a \sim b \) in case there exists a sequence \( z_0, z_1, \ldots, z_r, r \geq 0 \), of elements of \( F \) such that \( a = z_0, b = z_r \), and for each \( i = 0, \ldots, r - 1 \), either \( z_i = y_i + x \) and \( z_{i+1} = y_i + y \) for some \( y_i \in F \), or \( z_i = y_i + y \) and \( z_{i+1} = y_i + x \) for some \( y_i \in F \). Hence, if \( a \sim b \) then there is \( t \in \mathbb{Z} \) such that \( b = a + t(y - x) \) in the free abelian group \( \mathbb{Z}^n \). Thus if \( a \sim a + c \) with \( c \in F \setminus \{0\} \), then there is \( t \in \mathbb{Z} \) such that \( c = t(y - x) \), and this implies either \( x < y \) or \( y < x \).

It is interesting to observe that the conditions giving that \( M(E, C) \) is stably finite are the same as the ones giving that \( C^*(E, C) \) is stably finite, as the following proposition and the above lemma show. This provides further evidence to the validity of Conjecture 6.1. Recall that a C*-algebra \( A \) is termed *residually finite dimensional* if it admits a separating family of finite-dimensional *-representations.
Proposition 6.6. Let \((E, C)\) be the separated graph associated to the presentation \(\langle a_1, \ldots, a_n \mid \sum_{i=1}^{n} r_i a_i = \sum_{i=1}^{n} s_i a_i \rangle\). Consider the nonzero vectors \(r = (r_1, \ldots, r_n)\) and \(s = (s_1, \ldots, s_n)\) in \(\mathbb{Z}^n\). Then the following conditions are equivalent:

(i) \(C^*(E, C)\) is residually finite dimensional.

(ii) \(C^*(E, C)\) admits a faithful tracial state.

(iii) \(C^*(E, C)\) is stably finite.

(iv) \(C^*(E, C)\) is finite.

(v) \(r \not< s\) and \(s \not< r\) in the usual order of \(\mathbb{Z}^n\).

Proof. The implications \((i) \implies (ii) \implies (iii) \implies (iv)\) are general, well-known facts.

\((iv) \implies (v)\). If \(r < s\) or \(s < r\) in the usual order of \(\mathbb{Z}^n\) then one can easily see that the projection

\[ v \sim \bigoplus_{i=1}^{n} r_i \cdot w_i, \quad w_i \sim \bigoplus_{i=1}^{n} s_i \cdot w_i \]

is infinite in \(C^*(E, C)\).

\((v) \implies (i)\). Assume that \(r \not< s\) and \(s \not< r\) in the usual order of \(\mathbb{Z}^n\). Then we shall show that \(C^*(E, C)\) is residually finite dimensional by applying [6, Theorem 4.2], which asserts that a full amalgamated free product \(A \ast_C B\) of finite dimensional \(C^*\)-algebras is residually finite dimensional if and only if there are faithful tracial states \(\tau_A\) on \(A\) and \(\tau_B\) on \(B\) whose restrictions to \(C\) agree.

Since \(r \not< s\) and \(s \not< r\), either \(r = s\) or there are indices \(i, j\) such that \(r_i < s_i\) and \(s_j < r_j\). In the former case, it follows from Proposition 5.2 that there is a faithful state \(\tau\) on \(C = \mathbb{C}^{n+1}\) such that \(\tau_A = \tau \circ \Phi_A\) and \(\tau_B = \tau \circ \Phi_B\) are faithful tracial states on \(A\) and \(B\) respectively. So [6, Theorem 4.2] gives the result. In the latter case, without loss of generality, we shall assume that \(r_1 < s_1\) and \(s_2 < r_2\).

We will use the concrete representations of \(A = C^*(EX)\) and \(B = C^*(EY)\) as finite products of matrix algebras introduced in Section 4 so that \(A = \prod_{i=1}^{n} M_{s_i+1}(\mathbb{C})\), \(B = \prod_{i=1}^{n} M_{r_i+1}(\mathbb{C})\) and \(C = \mathbb{C}^{n+1}\), and the embeddings \(t_A : C \to A\) and \(t_B : C \to B\) are as specified in Section 4.

We want to define faithful tracial states \(\tau_A = \sum_{i=1}^{n} \gamma_i \text{Tr}_{s_i+1}\) on \(A\) and \(\tau_B = \sum_{i=1}^{n} \delta_i \text{Tr}_{r_i+1}\) on \(B\), with \(\gamma_i > 0\) and \(\delta_i > 0\) for all \(i\), and \(\sum_{i=1}^{n} \gamma_i = \sum_{i=1}^{n} \delta_i = 1\), such that the restrictions of \(\tau_A\) and \(\tau_B\) to \(C = \mathbb{C}^{n+1}\) agree. This is equivalent to the identities:

\[ \delta_i = \left( \frac{r_i + 1}{s_i + 1} \right) \gamma_i, \quad i = 1, \ldots, n. \quad (6.1) \]

For \(i = 2, \ldots, n\), put

\[ \Delta_i = \left( \frac{r_i + 1}{s_i + 1} \right) - \left( \frac{r_1 + 1}{s_1 + 1} \right). \]

Set \(\Gamma = (r_2 + 1)(s_1 + 1) - (s_2 + 1)(r_1 + 1)\), and observe that \(\Gamma > 0\) by our hypothesis that \(r_1 < s_1\) and \(s_2 < r_2\). Set

\[ \gamma'_1 = \frac{(s_1 + 1)(r_2 - s_2)}{\Gamma}, \quad \gamma'_2 = \frac{(s_2 + 1)(s_1 - r_1)}{\Gamma}, \quad (6.2) \]
and observe that $\gamma'_1 > 0$, $\gamma'_2 > 0$ and $\gamma'_1 + \gamma'_2 = 1$. Now define $\gamma_1, \gamma_2$ by

\begin{equation}
\gamma_1 = \gamma'_1 + \sum_{i=3}^{n} \left( \frac{\Delta_i}{\Delta_2} - 1 \right) \gamma_i, \quad \gamma_2 = \gamma'_2 - \sum_{i=3}^{n} \left( \frac{\Delta_i}{\Delta_2} \right) \gamma_i,
\end{equation}

where $\gamma_3, \ldots, \gamma_n$ are chosen to be positive numbers small enough to make $\gamma_1$ and $\gamma_2$ both positive. With this choice of $\gamma_1, \ldots, \gamma_n$ one has that $\gamma_i > 0$ for all $i$ and that $\sum_{i=1}^{n} \gamma_i = 1$. Now, the equations (6.1) define positive numbers $\delta_i$, and it is easily checked that $\sum_{i=1}^{n} \delta_i = 1$.

Hence, the formulas $\tau_A = \sum_{i=1}^{n} \gamma_i \text{Tr}_{s_i+1}$ and $\tau_B = \sum_{i=1}^{n} \delta_i \text{Tr}_{r_i+1}$ define faithful tracial states on $A$ and $B$ respectively, such that the restrictions of $\tau_A$ and $\tau_B$ to $C = \mathbb{C}^n$ agree. It follows from [6, Theorem 4.2] that $C^*(E, C) = A \ast_C B$ is residually finite dimensional. \hfill $\Box$

We observe that, in view of Lemma 5.5 the above result incorporates [8, Theorem 2.2].

**Example 6.7.** There exists separated graphs $(E, C)$ such that $M(E, C)$ is a stably finite monoid and $C^*(E, C)$ is a stably finite C*-algebra, but $C^*_{\text{red}}(E, C)$ is purely infinite simple, and moreover the natural map $M(E, C) \to \mathcal{M}(C^*_{\text{red}}(E, C))$ is not injective.

**Proof.** Take for instance the separated graph associated to the one-relator monoid $\langle a, b \mid 3a + 2b = 2a + 4b \rangle$. By Lemma 6.5 $M(E, C)$ is a stably finite monoid and, by Proposition 6.6 $C^*(E, C)$ is a stably finite C*-algebra. However, by Theorem 4.3 we have that $C^*_{\text{red}}(E, C)$ is purely infinite simple. In particular we obtain that $\mathcal{V}(C^*_{\text{red}}(E, C)) \setminus \{0\} = K_0(C^*_{\text{red}}(E, C))$ is cancellative. Thus $a \neq 2b$ in $M(E, C)$ but $a = 2b$ in $\mathcal{V}(C^*_{\text{red}}(E, C))$. This shows the result. \hfill $\Box$

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