Sigma-Point Filtering Based Parameter Estimation in Nonlinear Dynamic Systems

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Abstract—We consider approximate maximum likelihood parameter estimation in non-linear state-space models. We discuss both direct optimization of the likelihood and expectation–maximization (EM). For EM, we also give closed-form expressions for the maximization step in a class of models that are linear in parameters and have additive noise. To obtain approximations to the filtering and smoothing distributions needed in the likelihood-maximization methods, we focus on using Gaussian filtering and smoothing algorithms that employ sigma-points to approximate the required integrals. We discuss different sigma-point schemes based on the third, fifth, seventh, and ninth order uncentered transforms and Gauss–Hermite quadrature rule. We compare the performance of the methods in two simulated experiments: a univariate toy model as well as tracking of a maneuvering target. In the experiments, we also compare against approximate likelihood estimates obtained by particle filtering and extended Kalman filtering based methods. The experiments suggest that the higher-order uncentered transforms may in some cases provide more accurate estimates.

I. INTRODUCTION

This paper is an extended version of our article [1] where we considered parameter estimation in state-space models using expectation–maximization (EM) algorithms based on sigma-point and particle smoothers. In this paper, we extend our interest from EM algorithms to so called direct maximum likelihood based parameter estimation methods, where instead of using the EM algorithm, the marginal likelihood of the parameters is directly approximated using non-linear filtering methods. In particular, we focus our interest to sigma-point filters which use high-order uncentered Kalman filters and Gauss–Hermite Kalman filters to approximate the likelihood surface.

We consider state-space models of the following form:

\[
\begin{align*}
x_k &= f(x_{k-1}, \theta) + q_{k-1}, \\
y_k &= h(x_k, \theta) + r_k,
\end{align*}
\]

where \(x_k \in \mathbb{R}^n\) is the discrete-time state sequence with an initial distribution \(x_0 \sim \mathcal{N}(x_0 | m_0, P_0)\). \(y_k \in \mathbb{R}^d\) is the measurement sequence, \(q_k \sim \mathcal{N}(0, Q)\) is the Gaussian process noise sequence, and \(r_k \sim \mathcal{N}(0, R)\) is the Gaussian measurement error sequence. Typically, one is interested in computing the posterior distribution of the state \(x_k\) given measurements up to time \(k\), \(p(x_k | y_1, \ldots, y_k)\), known as the filtering problem, or computing the posterior distribution of the state \(x_k\) given all measurements, \(p(x_k | y_1, \ldots, y_T)\), known as the smoothing problem. In the general case, analytical expressions do not exist and we have to resort to approximative algorithms such as the sigma-point methods. See, for example, [2] for a general overview of Bayesian filtering and smoothing.

In this paper, our interest lies in estimating the static parameters \(\theta\) of Equation (1) by maximum-likelihood estimation, which is based on maximizing the marginal likelihood of parameters \(p(y_1, \ldots, y_T | \theta)\). Formally, the marginal likelihood can be computed by marginalizing out the states from the joint distribution of the measurements and states with help of non-linear filtering equations and the prediction error decomposition (see, e.g., [3], [2]). However, since the state variables \(x\) cannot in general be marginalized out analytically, one needs to employ approximative methods. In the so called direct likelihood methods, the likelihood is approximated directly using approximative non-linear filtering methods (see, e.g., [4], [5], [6], [2], [1]) and its maximum is found via non-linear optimization. An alternative is the expectation–maximization (EM) algorithm, which consists of iterating the expectation (E) step where a bound of the log-likelihood is computed using the current parameter estimates, and the maximization (M) step where the bound is maximized with respect to the parameters. The evaluation of the bound in the E-step requires solving the smoothing problem, which cannot be solved exactly, but can be approximated with non-linear smoothing algorithms (see, e.g., [7], [8], [9], [10], [11]).

The aim of this paper is to extend the results of our paper [1] by showing how high-order (i.e., third, fifth, seventh, and ninth order) uncentered transforms and Gauss–Hermite integration based sigma-point methods can be used for approximate direct likelihood and EM-based parameter estimation in non-linear state space models. For EM, we also give closed-form expressions for the maximization step in a class of models that are linear in parameters and have additive noise. We compare the methods to linearization-based extended Kalman filter algorithms and Monte Carlo based particle filtering algorithms. We also provide an algorithm for computing the gradients required by the gradient-based optimization methods. Although we focus on maximum likelihood estimation, the provided algorithms can be easily extended to computation of maximum a posteriori estimates by including a prior distribution to the objective function.

II. SIGMA-POINT FILTERING AND SMOOTHING

Under our interpretation, sigma-point filtering and smoothing is derived by assuming Gaussian approximations for the
state distributions, which enables the use of a Kalman filter like filtering recursion and a Rauch–Tung–Striebel backward pass for the smoothing distributions. The Gaussian filtering and smoothing equations contain expectations over Gaussian distributions which cannot be generally evaluated in closed-form. The sigma-points arise from approximating these Gaussian integrals by weighted sums determined by some cubature (multi-dimensional quadrature) formula. Hence, we interpret the different sigma-point methods as incarnations of different integral approximations.

In the following, we first present the assumed Gaussian density filtering and smoothing framework. Then, we discuss various different cubature rules for approximating the Gaussian integrals. Finally, we show the filtering and smoothing equations explicitly in the sigma-point form.

A. General Gaussian Filtering and Smoothing

Assumed density Gaussian filtering (see [11], [12], [2]) is based on assuming that the filtering distributions are approximately Gaussian, that is, assuming means $m_{k|k}$ and covariances $P_{k|k}$ such that

$$p(x_k \mid y_{1:k}) \approx N(x_k \mid m_{k|k}, P_{k|k})$$

as well as means $m_{k|k+1}$ and covariances $P_{k|k+1}$ such that

$$p(x_{k+1} \mid y_{1:k}) \approx N(x_{k+1} \mid m_{k+1|k}, P_{k+1|k}).$$

The resulting equations are

$$m_{k|k-1} = E[f(x_{k-1})],
$$

$$P_{k|k-1} = E[(f(x_{k-1}) - m_{k|k-1}) (f(x_{k-1}) - m_{k|k-1})^T] + Q,$$

where the expectations are taken with respect to the distribution $x_{k-1} \sim N(m_{k-1|k}, P_{k-1|k}).$

In the corresponding update step, we assume a Gaussian density $p(x_k \mid y_{1:k-1}) = N(x_k \mid m_{k|k-1}, P_{k|k-1})$ and compute the state mean and covariance for the distribution $p(x_k \mid y_{1:k}).$ The resulting equations are

$$
\begin{align*}
\mu_k &= E[h(x_k)], \\
S_k &= E[(h(x_k) - \mu_k) (h(x_k) - \mu_k)^T] + R, \\
C_k &= E[(x_k - m_{k|k-1}) (h(x_k) - \mu_k)^T], \\
K_k &= C_k S_k^{-1}, \\
m_{k|k} &= m_{k|k-1} + K_k (y_k - \mu_k), \\
P_{k|k} &= P_{k|k-1} - K_k S_k K_k^T,
\end{align*}

(5)

where the expectations are taken with respect to the distribution $x_k \sim N(m_{k|k-1}, P_{k|k-1}).$

The smoothing distributions $p(x_k \mid y_{1:T})$ are obtained from a backward pass, that is, starting from $k = T$ and iterating backwards in time. On each step, the smoothing density of $x_{k+1}$ is assumed to be Gaussian: $p(x_{k+1} \mid y_{1:T}) = N(x_{k+1} \mid m_{k+1|T}, P_{k+1|T}).$ The mean and covariance for $p(x_k \mid y_{1:T})$

are then computed from the previous Gaussian smoothing density and the Gaussian filtering densities as follows.

$$
\begin{align*}
m_{k+1|k} &= E[f(x_{k+1})], \\
P_{k+1|k} &= E[(f(x_{k+1}) - m_{k+1|k}) (f(x_{k+1}) - m_{k+1|k})^T] + Q, \\
D_{k+1} &= E[(x_k - m_{k|k}) (f(x_k) - m_{k+1|k})^T], \\
G_k &= D_{k+1} P_{k+1|k}^{-1}, \\
m_{k|T} &= m_{k|k} + G_k (m_{k+1|T} - m_{k+1|k}), \\
P_{k|T} &= P_{k|k} + G_k (P_{k+1|T} - P_{k+1|k}) G_k^T,
\end{align*}

(6)

where the expectations are taken with respect to the distribution $x_k \sim N(m_{k|k}, P_{k|k}).$ The pairwise joint smoothing distributions $p(x_k, x_{k-1} \mid y_{1:T})$ are also of interest since they are used in the expectation–maximization algorithm (see Section III-B). Gaussian approximations for these distributions are obtained as a by-product of the smoothing backward pass results as follows.

$$
\begin{align*}
p(x_k, x_{k-1} \mid y_{1:T}) &= N\left(\begin{pmatrix} x_k \\ x_{k-1} \end{pmatrix} \mid \begin{pmatrix} m_{k|T} \\ m_{k-1|T} \end{pmatrix}, \begin{pmatrix} P_{k|T} & P_{k|T} G_{k-1}^T \\ G_{k-1} P_{k|T} & P_{k-1|T} \end{pmatrix}\right).
\end{align*}

(7)

B. Approximating the Gaussian Integrals

As we saw in the previous section, during the evaluation of the prediction and update steps of the Gaussian filter and smoother, we need to solve a set of Gaussian integrals on each step. These integrals are of the following form:

$$E[g(x)] = \int_{\mathbb{R}^n} g(x) N(x \mid m, P) \, dx,$$

where $g : \mathbb{R}^n \to \mathbb{R}^d$ is the integrand and the weighting function $N(x \mid m, P)$ is a multi-dimensional Gaussian density with mean $m$ and covariance matrix $P.$ In this paper, these integrals are computed by using multi-dimensional generalizations of Gaussian quadratures—also referred to as Gaussian cubatures [14]. They give approximations of the form

$$E[g(x)] \approx \sum_i w_i g(x_i),$$

(9)

where the weights $w_i$ and sigma-points $x_i$ are functions of the mean $m$ and covariance $P$ of the Gaussian weighting function.

The sigma-points are positioned as follows:

$$x_i = m + L \xi_i,$$

(10)

where $\xi_i$ are method specific unit sigma-points, and $L$ is a matrix square-root factor such that $P = LL^T$ (e.g., the Cholesky decomposition of $P$). The differences in the methods come from different choices of weights and unit sigma-points.

In the following we briefly introduce a number of schemes for choosing the weights and sigma-points. The difference between these schemes stems from a trade-off between the number of sigma-points (required function evaluations) versus precision in the approximation. The degree of approximation is quantified by the highest polynomial order, $p,$ for which the method is exact.
Unscented transform. The unscented transform (UT, \[15\], [16]) uses a set of \(2n+1\) cubature points located in the center and on the surface of an \(n\)-sphere. The radius and the weights can be controlled using a set of parameters. The cubature points are given by:

\[
\xi_0 = 0,
\]

\[
\xi_i = \begin{cases} \\
\sqrt{\lambda + n} e_i, & i = 1, \ldots, n, \\
-\sqrt{\lambda + n} e_{i-n}, & i = n + 1, \ldots, 2n,
\end{cases}
\]

where \(e_i\) denotes a unit vector to the direction of coordinate axis \(i\), and the weights are defined as follows:

\[
w^{(0)} = \left\{ \frac{\lambda}{n+\lambda}, \quad \text{for mean terms}, \right.
\]

\[
w^{(i)} = \frac{1}{2(n+\lambda)}, \quad i = 1, \ldots, 2n,
\]

where \(\lambda = \alpha^2(n+\kappa) - n\) and \(\alpha, \beta, \kappa\) are parameters of the method.

Symmetric, 3rd order. A widely applicable sigma-point scheme is constructed by setting the unscented transform parameters to \(\alpha = \pm 1, \beta = 0, \kappa = 0\) [13]. This is also known as the 3rd order symmetric spherical–radial cubature method (CKF, [17]; see [13] for the explicit connection). This method utilizes a scaled and rotated set of \(2n\) points, which are selected to be at the intersections of an \(n\)-sphere and the coordinate axes:

\[
\xi_i = \begin{cases} \\
\sqrt{n} e_i, & i = 1, 2, \ldots, n, \\
-\sqrt{n} e_{i-n}, & i = n + 1, \ldots, 2n.
\end{cases}
\]

The weights are defined as \(w_i = 1/(2n)\) for \(i = 1, 2, \ldots, 2n\). The number of evaluation points is a linear function of the state dimension. The corresponding sigma-point filter is referred to as UKF 3.

Symmetric, 5th order. Building upon the work of McNamee and Stenger [19], it is possible to find explicit fully symmetric integration formulas of higher order than three. These integration schemes are exact for symmetric polynomials up to a given order \(p\). For order \(p = 5\), the number of required sigma-points is \(2n^2 + 1\). The corresponding sigma-point filter is referred to as UKF 5.

Symmetric, 7th order. For order \(p = 7\), the number of required sigma-points is \(\frac{1}{2}(4n^3 + 8n + 3)\), meaning that they scale cubicly with the number of state dimensions.

The corresponding sigma-point filter is referred to as UKF 7.

Symmetric, 9th order. For order \(p = 9\), the number of required sigma-points is \(\frac{1}{4}(2n^4 - 4n^3 + 22n^2 - 8n + 3)\). The corresponding sigma-point filter is referred to as UKF 9. If required, even higher order methods can be constructed in the spirit of [19].

Gauss–Hermite. The \(n\)-dimensional Gauss–Hermite quadrature method forms the sigma-points as a Cartesian product of the one-dimensional Gauss–Hermite quadratures, and the weights are simply products of the one-dimensional weights [14], [11], [13]. The disadvantage of this method is that with a \(p\)th order GH approximation (exact for polynomials up to order \(p\)), the required number of evaluation points is \(p^n\), the number growing exponentially with state dimension \(n\). The corresponding filter is referred to as GHKF.

The exact formulas for the higher-order methods become lengthy and have been omitted here for brevity (see, [19], [20], [13], for implementation details and discussion). Figure 1 gives a pictorial example of how the points and weights are placed in two dimensions (\(n = 2\)) for each of the methods.

For higher state dimensions, Figure 2 shows how the number of required points scale in each of the schemes. The exponentially growing number of evaluation points for Gauss–Hermite is apparent from the figure. In the UKFs, the number of evaluation points grow polynomially. McNamee and Stenger provide the following bound for the number of evaluation points for the fully symmetric integration formulas of arbitrary degree \(p = 2k + 1\) in \(n\)-space: \(O((2n)^k/k!)\).

C. Sigma-Point Filtering and Smoothing

The following sigma-point filtering and smoothing equations are obtained by selecting a cubature rule, for example, one of the rules discussed in Section II-B and substituting it in place of the expectations in the Gaussian filtering and smoothing equations (Sec. II-A; Eq. 4-6).

In the following equations, we denote the lower triangular matrix square-root (Cholesky) factor of a covariance matrix \(P\) by \(L\) so that for example \(P_{k|k-1} = L_k k-1 L_k^T k-1\). The
The Rauch–Tung–Striebel smoother equations are

\[ \begin{align*}
\mathbf{m}_{k|k-1} & = \sum_i w_i \mathbf{f} \left( \mathbf{m}_{k-1|k-1} + \mathbf{L}_{k-1|k-1} \xi_i \right), \\
\mathbf{P}_{k|k-1} & = \sum_i \left\{ w_i \left( \mathbf{f} \left( \mathbf{m}_{k-1|k-1} + \mathbf{L}_{k-1|k-1} \xi_i \right) - \mathbf{m}_{k|k-1} \right) \right. \\
& \quad \left. \times \left( \mathbf{f} \left( \mathbf{m}_{k-1|k-1} + \mathbf{L}_{k-1|k-1} \xi_i \right) - \mathbf{m}_{k|k-1} \right)^T \right\} + \mathbf{Q}.
\end{align*} \] (11)

and the update step is

\[ \begin{align*}
\mathbf{\mu}_k & = \sum_i w_i \mathbf{h} \left( \mathbf{m}_{k|k-1} + \mathbf{L}_{k|k-1} \xi_i \right), \\
\mathbf{S}_k & = \sum_i \left\{ w_i \left( \mathbf{h} \left( \mathbf{m}_{k|k-1} + \mathbf{L}_{k|k-1} \xi_i \right) - \mathbf{\mu}_k \right) \right. \\
& \quad \left. \times \left( \mathbf{h} \left( \mathbf{m}_{k|k-1} + \mathbf{L}_{k|k-1} \xi_i \right) - \mathbf{\mu}_k \right)^T \right\} + \mathbf{R}, \\
\mathbf{C}_k & = \sum_i \left\{ w_i \mathbf{L}_{k|k-1} \xi_i \right. \\
& \quad \left. \times \left( \mathbf{h} \left( \mathbf{m}_{k|k-1} + \mathbf{L}_{k|k-1} \xi_i \right) - \mathbf{\mu}_k \right)^T \right\}, \\
\mathbf{K}_k & = \mathbf{C}_k \mathbf{S}_k^{-1}, \\
\mathbf{m}_{k|k} & = \mathbf{m}_{k|k-1} + \mathbf{K}_k \left( \mathbf{y}_k - \mathbf{\mu}_k \right), \\
\mathbf{P}_{k|k} & = \mathbf{P}_{k|k-1} - \mathbf{K}_k \mathbf{S}_k \mathbf{K}_k^T.
\end{align*} \]

The Rauch–Tung–Striebel smoother equations are

\[ \begin{align*}
\mathbf{m}_{k+1|k} & = \sum_i \left\{ w_i \mathbf{f} \left( \mathbf{m}_{k|k} + \mathbf{L}_{k|k} \xi_i \right) \right\}, \\
\mathbf{P}_{k+1|k} & = \sum_i \left\{ w_i \left( \mathbf{f} \left( \mathbf{m}_{k|k} + \mathbf{L}_{k|k} \xi_i \right) - \mathbf{m}_{k+1|k} \right) \right. \\
& \quad \left. \times \left( \mathbf{f} \left( \mathbf{m}_{k|k} + \mathbf{L}_{k|k} \xi_i \right) - \mathbf{m}_{k+1|k} \right)^T \right\} + \mathbf{Q}, \\
\mathbf{D}_{k+1} & = \sum_i \left\{ w_i \mathbf{L}_{k|k} \xi_i \left( \mathbf{f} \left( \mathbf{m}_{k|k} + \mathbf{L}_{k|k} \xi_i \right) - \mathbf{m}_{k+1|k} \right) \right\}, \\
\mathbf{G}_k & = \mathbf{D}_{k+1} \left[ \mathbf{P}_{k+1|k} \right]^{-1}, \\
\mathbf{m}_{k|T} & = \mathbf{m}_{k|k} + \mathbf{G}_k \left( \mathbf{m}_{k+1|T} - \mathbf{m}_{k+1|k} \right), \\
\mathbf{P}_{k|T} & = \mathbf{P}_{k|k} + \mathbf{G}_k \left( \mathbf{P}_{k+1|T} - \mathbf{P}_{k+1|k} \right) \mathbf{G}_k^T.
\end{align*} \] (13)

III. PARAMETER ESTIMATION

In this section, we consider methods for estimating the static parameters \( \mathbf{\theta} \) of the state-space model (1). All methods discussed target the maximum likelihood solution, that is, aim to maximize \( p(\mathbf{y}_{1:T} \mid \mathbf{\theta}) \), or equivalently the log-likelihood:

\[ \mathbf{\theta}_{\text{ML}} = \arg \max_{\mathbf{\theta}} \log p(\mathbf{y}_{1:T} \mid \mathbf{\theta}). \] (14)

Since the state variables \( \mathbf{x}_{0:T} \) cannot in general be marginalized in closed-form, approximative numeric methods are needed.

Here, we focus on three approaches where sigma-point filtering and smoothing is used to approximate the likelihood. First, we consider a so-called direct-likelihood approach, where the sigma-point algorithm is used to directly approximate the log-likelihood and its gradient, which are then used in numeric optimization algorithms such as conjugate-gradient optimization. Second, the expectation–maximization (EM) algorithm which is based on a lower bound for the log-likelihood and iterating optimization of parameters with respect to the lower bound and updating the lower bound with new parameters. The third approach is a modification of the direct-likelihood optimization where Fisher’s identity is used to express the gradient of the log-likelihood using the same lower bound function that appears in the EM algorithm. Each of these three approaches may be used in combination with any of the sigma-point rules discussed in Section II-B.

Note that all the algorithms presented in this section are easily extended to maximum a posteriori estimation since maximizing the posterior density is equivalent to maximizing the (unnormalized) log-posterior. That is, the sum of log-likelihood and log-prior:

\[ \mathbf{\theta}_{\text{MAP}} = \arg \max_{\mathbf{\theta}} \log p(\mathbf{y}_{1:T} \mid \mathbf{\theta}) + \log p(\mathbf{\theta}). \] (15)

Since the log-prior is known, approximations of the unnormalized log-posterior as well as its gradient and lower bounds are immediately obtained from the corresponding approximations for the log-likelihood.

A. Direct Likelihood Based Parameter Estimation

The marginal log-likelihood function can be formulated as the following sum

\[ \mathcal{L}_T(\mathbf{\theta}) = \sum_{k=1}^{T} \log p(\mathbf{y}_k \mid \mathbf{y}_{1:k-1}, \mathbf{\theta}). \] (16)

Furthermore, the terms of the sum on the right hand side may in principle be evaluated by

\[ p(\mathbf{y}_k \mid \mathbf{y}_{1:k-1}, \mathbf{\theta}) = \int p(\mathbf{y}_k \mid \mathbf{x}_k, \mathbf{\theta}) p(\mathbf{x}_k \mid \mathbf{y}_{1:k-1}, \mathbf{\theta}) \, d\mathbf{x}_k, \] (17)

that is, integrating the measurement model \( p(\mathbf{y}_k \mid \mathbf{x}_k, \mathbf{\theta}) \) over the predicted state distribution \( p(\mathbf{x}_k \mid \mathbf{y}_{1:k-1}, \mathbf{\theta}) \) which is computed during the Bayesian filtering recursion. In assumed density Gaussian filtering is used, we get the approximation

\[ p(\mathbf{y}_k \mid \mathbf{y}_{1:k-1}, \mathbf{\theta}) \approx \mathcal{N}(\mathbf{y}_k \mid \mathbf{\mu}_k, \mathbf{S}_k), \] (18)
whence the marginal log-likelihood expression in Equation (16) evaluates to

\[ \mathcal{L}_T(\theta) = \log p(y_{1:T} | \theta) \approx -\frac{1}{2} \sum_{k=1}^{T} \log |2\pi S_k| \]

\[ -\frac{1}{2} \sum_{k=1}^{T} (y_k - \mu_k)^T S_k^{-1} (y_k - \mu_k), \quad (19) \]

where the quantities \( \mu_k \) and \( S_k \) are evaluated during the filtering recursion, in the case of sigma-point methods by Equation (12).

To enable use of gradient-based optimization algorithms, we also need a method for evaluating the gradients of the marginal log-likelihood. This is based on the so-called sensitivity equations (31), (2). The gradient is based on the recursion

\[ \frac{\partial L_k(\theta)}{\partial \theta_i} = \frac{\partial L_{k-1}(\theta)}{\partial \theta_i} - \frac{1}{2} \text{tr} \left( S_k^{-1}(\theta) \frac{\partial S_k(\theta)}{\partial \theta_i} \right) \]

\[ -v_k^T S_k^{-1}(\theta) \frac{\partial v_k(\theta)}{\partial \theta_i} \]

\[ + \frac{1}{2} v_k^T S_k^{-1}(\theta) \frac{\partial S_k(\theta)}{\partial \theta_i} S_k^{-1}(\theta) v_k(\theta), \quad (20) \]

where the derivatives \( \frac{\partial S_k(\theta)}{\partial \theta_i} \) and \( \frac{\partial v_k(\theta)}{\partial \theta_i} \) are obtained by the following recursion computed along the filtering pass. First, the derivatives of the predicted mean and covariance in the prediction step are obtained as follows.

\[ \frac{\partial m_{k|k-1}}{\partial \theta_i} = \sum_j \left\{ w_j \left[ F_x \left( m_{k-1|k-1} + L_{k-1|k-1} \xi_j, \theta \right) \right. \right. \]

\[ \times \left. \left( \frac{\partial m_{k-1|k-1}}{\partial \theta_i} + \frac{\partial L_{k-1|k-1}}{\partial \theta_i} \xi_j \right) \right. \]

\[ \left. + \frac{\partial f}{\partial \theta_i} \left( m_{k-1|k-1} + L_{k-1|k-1} \xi_j, \theta \right) - \frac{\partial m_{k|k-1}}{\partial \theta_i} \right\} \left. \frac{\partial S_k}{\partial \theta_i} \right. \]

\[ \frac{\partial P_{k|k-1}}{\partial \theta_i} = \sum_j \left\{ w_j \left[ F_x \left( m_{k-1|k-1} + L_{k-1|k-1} \xi_j, \theta \right) \right. \right. \]

\[ \times \left. \left( \frac{\partial m_{k-1|k-1}}{\partial \theta_i} + \frac{\partial L_{k-1|k-1}}{\partial \theta_i} \xi_j \right) \right. \]

\[ \left. + \frac{\partial f}{\partial \theta_i} \left( m_{k-1|k-1} + L_{k-1|k-1} \xi_j, \theta \right) - \frac{\partial m_{k|k-1}}{\partial \theta_i} \right\} \frac{\partial S_k}{\partial \theta_i} \]

\[ \times \left[ f \left( m_{k-1|k-1} + L_{k-1|k-1} \xi_j, \theta \right) - m_{k|k-1} \right]^T \]

\[ + f \left( m_{k-1|k-1} + L_{k-1|k-1} \xi_j, \theta \right) \] \[
\times \left( \frac{\partial m_{k-1|k-1}}{\partial \theta_i} + \frac{\partial L_{k-1|k-1}}{\partial \theta_i} \xi_j \right) \]

\[ + \frac{\partial f}{\partial \theta_i} \left( m_{k-1|k-1} + L_{k-1|k-1} \xi_j, \theta \right) - \frac{\partial m_{k|k-1}}{\partial \theta_i} \right\}^T \]

\[ + \frac{\partial Q}{\partial \theta_i} \right\}, \quad (21) \]

where \( F_x \) is the Jacobian of \( f(x, \theta) \) as a function of \( x \). The derivatives in the update step are obtained as follows.

\[ \frac{\partial \mu_k}{\partial \theta_i} = \sum_j \left\{ w_j \left[ H_x \left( m_{k|k-1} + L_{k|k-1} \xi_j, \theta \right) \right. \right. \]

\[ + \frac{\partial h}{\partial \theta_i} \left( m_{k|k-1} + L_{k|k-1} \xi_j, \theta \right) \] \[
\times \left( \frac{\partial m_{k|k-1}}{\partial \theta_i} + \frac{\partial L_{k|k-1}}{\partial \theta_i} \right) \]

\[ + \frac{\partial h}{\partial \theta_i} \left( m_{k|k-1} + L_{k|k-1} \xi_j, \theta \right) - \frac{\partial \mu_k}{\partial \theta_i} \right\} \] \[
\times \left( \frac{\partial m_{k|k-1}}{\partial \theta_i} + \frac{\partial L_{k|k-1}}{\partial \theta_i} \right) \]

\[ + \frac{\partial h}{\partial \theta_i} \left( m_{k|k-1} + L_{k|k-1} \xi_j, \theta \right) - \frac{\partial \mu_k}{\partial \theta_i} \right\} \}

\[ + \frac{\partial v_k}{\partial \theta_i} = -\frac{\partial \mu_k}{\partial \theta_i}, \quad (24) \]

\[ \frac{\partial S_k}{\partial \theta_i} = \sum_j \left\{ w_j \left[ H_x \left( m_{k|k-1} + L_{k|k-1} \xi_j, \theta \right) \right. \right. \]

\[ \times \left( \frac{\partial m_{k|k-1}}{\partial \theta_i} + \frac{\partial L_{k|k-1}}{\partial \theta_i} \right) \]

\[ + \frac{\partial h}{\partial \theta_i} \left( m_{k|k-1} + L_{k|k-1} \xi_j, \theta \right) - \frac{\partial \mu_k}{\partial \theta_i} \right\} \} \]

\[ + \frac{\partial h}{\partial \theta_i} \left( m_{k|k-1} + L_{k|k-1} \xi_j, \theta \right) - \frac{\partial \mu_k}{\partial \theta_i} \right\} \right\} \}

\[ \left. + \frac{\partial h}{\partial \theta_i} \left( m_{k|k-1} + L_{k|k-1} \xi_j, \theta \right) - \frac{\partial \mu_k}{\partial \theta_i} \right\} \}

\[ + \frac{\partial R}{\partial \theta_i} \right\}, \quad (25) \]

\[ \frac{\partial C_k}{\partial \theta_i} = \sum_j \left\{ w_j \left[ H_x \left( m_{k|k-1} + L_{k|k-1} \xi_j, \theta \right) \right. \right. \]

\[ \times \left( \frac{\partial m_{k|k-1}}{\partial \theta_i} + \frac{\partial L_{k|k-1}}{\partial \theta_i} \right) \]

\[ + \frac{\partial h}{\partial \theta_i} \left( m_{k|k-1} + L_{k|k-1} \xi_j, \theta \right) - \frac{\partial \mu_k}{\partial \theta_i} \right\} \} \]

\[ \left. + \frac{\partial h}{\partial \theta_i} \left( m_{k|k-1} + L_{k|k-1} \xi_j, \theta \right) - \frac{\partial \mu_k}{\partial \theta_i} \right\} \]

\[ \left. + \frac{\partial h}{\partial \theta_i} \left( m_{k|k-1} + L_{k|k-1} \xi_j, \theta \right) - \frac{\partial \mu_k}{\partial \theta_i} \right\} \}

\[ \left. + \frac{\partial h}{\partial \theta_i} \left( m_{k|k-1} + L_{k|k-1} \xi_j, \theta \right) - \frac{\partial \mu_k}{\partial \theta_i} \right\} \}

where \( H_x \) is the Jacobian of \( h(x, \theta) \) as a function of \( x \). Algorithms for computing the derivatives of the Cholesky factors \( L \) such that \( P = LL^T \) are omitted here (see (2)).

\[ B. \text{Expectation–Maximization Based Parameter Estimation} \]

Expectation–maximization (EM) is an iterative algorithm for finding maximum likelihood parameter estimates in settings with some unobserved variables, such as the state variables \( x \) in the state-space context. The motivation is that the so-called full-data log-likelihood of the observed and unobserved variables is easier to compute, and a lower bound for the marginal likelihood of the observed variables may be obtained based on expected full-data log-likelihood. In the following,
we present the EM algorithm following the formulation by Neal and Hinton [22] and the notation of Schönh et al. [10].

The EM algorithm is based on the following lower bound of the log-likelihood:

$$\log p(y_{1:T} \mid \theta) \geq \int q(x_{0:T}) \log \frac{p(x_{0:T}, y_{1:T} \mid \theta)}{q(x_{0:T})} \; dx_{0:T},$$

where $q$ is an arbitrary probability density over the states $x_{0:T}$. The idea is to iteratively maximize this lower bound with respect to $q$ (holding $\theta$ fixed) and with respect to $\theta$ (holding $q$ fixed). Furthermore, when $\theta = \theta^{(n)}$ is fixed, the maximum with respect to $q$ is obtained by

$$q(x_{0:T}) := p(x_{0:T} \mid y_{1:T}, \theta^{(n)}).$$

By substituting this into Equation (30), the bound becomes

$$\int p(x_{0:T} \mid y_{1:T}, \theta^{(n)}) \log \frac{p(x_{0:T}, y_{1:T} \mid \theta)}{p(x_{0:T} \mid y_{1:T}, \theta^{(n)})} \; dx_{0:T}
= \int p(x_{0:T} \mid y_{1:T}, \theta^{(n)}) \log p(x_{0:T} \mid y_{1:T} \mid \theta) \; dx_{0:T}
- \int p(x_{0:T} \mid y_{1:T}, \theta^{(n)}) \log p(x_{0:T} \mid y_{1:T}, \theta^{(n)}) \; dx_{0:T}.$$

The latter term is independent of $\theta$ and may thus be omitted when maximizing the lower bound with respect to $\theta$. The first term is the conditional expectation of $\log p(y_{1:T}, x_{0:T} \mid \theta)$ conditional on $\theta^{(n)}$ and $y_{1:T}$. Thus, the step of maximizing the lower bound (Eq. [30]) may be replaced by computing the following function:

$$Q(\theta, \theta^{(n)}) = \mathbb{E}[\log p(x_{0:T}, y_{1:T} \mid \theta) \mid y_{1:T}, \theta^{(n)}].$$

The EM algorithm in its general form thus consists of initializing the parameters to $\theta^{(0)}$ and for $n = 0, 1, \ldots$ iterating the following two steps:

- **E-step**: compute $Q(\theta, \theta^{(n)})$.
- **M-step**: $\theta^{(n+1)} \leftarrow \arg \max_{\theta} Q(\theta, \theta^{(n)})$.

In state-space models, the $Q$-function can be decomposed by employing the Markov property of the state sequence and the conditional independence of the measurements:

$$Q(\theta, \theta^{(n)}) = I_1(\theta, \theta^{(n)}) + I_2(\theta, \theta^{(n)}) + I_3(\theta, \theta^{(n)}),$$

where the terms are

$$I_1(\theta, \theta^{(n)}) = \mathbb{E}[\log p(x_0) \mid y_{1:T}, \theta^{(n)}],$$

$$I_2(\theta, \theta^{(n)}) = \sum_{k=1}^{T} \mathbb{E}[\log p(x_k \mid x_{k-1}, \theta) \mid y_{1:T}, \theta^{(n)}],$$

$$I_3(\theta, \theta^{(n)}) = \sum_{k=1}^{T} \mathbb{E}[\log p(y_k \mid x_k, \theta) \mid y_{1:T}, \theta^{(n)}].$$

To evaluate this expression, one needs the smoothing distributions $p(x_{0:t} \mid y_{1:T}, \theta^{(n)})$ and the joint smoothing distributions of consecutive states $p(x_{k}, x_{k+1} \mid y_{1:T}, \theta^{(n)})$. Sigma-point approximations to the EM algorithm are then obtained by replacing the expectations over the smoothing distributions by their sigma-point smoother approximations. The Gaussian smoother approximation for $Q$ is

$$Q(\theta, \theta^{(n)}) \approx -\frac{1}{2} \log |2\pi P_0| - \frac{T}{2} \log |2\pi Q| - \frac{T}{2} \log |2\pi R|$$

$$- \frac{1}{2} \text{tr} \left\{ P_0^{-1} \left[ P_{0/T} + (m_{0/T} - m_0) (m_{0/T} - m_0)^T \right] \right\}$$

$$- \frac{1}{2} \sum_{k=1}^{T} \text{tr} \left\{ Q^{-1} \mathbb{E} \left[ (x_k - f(x_{k-1})(x_k - f(x_{k-1})^T \mid y_{1:T}) \right] \right\}$$

$$- \frac{1}{2} \sum_{k=1}^{T} \text{tr} \left\{ R^{-1} \mathbb{E} \left[ (y_k - h(x_k)(y_k - h(x_k)^T \mid y_{1:T}) \right] \right\},$$

where $P_0, Q, R$ and the model functions $f(\cdot), h(\cdot)$ depend on the parameters $\theta$. The smoothing distribution means and covariances $m_{k/T}, P_{k/T}$ are obtained during the smoothing backward pass. The expectations over the smoothing distribution in the latter two terms are evaluated by using the sigma-point approximations for Gaussian integrals as follows. The second expectation depends only on the smoothing distribution $N(x_k \mid m_{k/T}, P_{k/T})$ and is computed as follows:

$$\mathbb{E}[(y_k - h(x_k)(y_k - h(x_k)^T \mid y_{1:T})$$

$$\approx \sum_i w_i (y_k - h(m_{k/T}) + L_{k/T} \xi_i) \times (y_k - h(m_{k/T}) + L_{k/T} \xi_i)^T.$$ (38)

The first expectation depends on the pairwise joint smoothing distribution $p(x_k, x_{k-1} \mid y_{1:T})$ (cf. Sec. [10], Eq. 7). Thus, to evaluate it we need a 2n-dimensional sigma-point representation of the distribution:

$$\begin{pmatrix} x_k^{(i)} \\ x_{k-1}^{(i)} \end{pmatrix} = \begin{pmatrix} m_{k/T}^{(i)} \\ m_{k-1/T}^{(i)} \end{pmatrix} + \sqrt{D_k} \xi_i,$$ (39)

where $\xi_i$ are 2n-dimensional unit sigma-points and $D_k$ is the corresponding cross-covariance matrix from the solution. Then, the expectation is approximated as

$$\mathbb{E}[(x_k - f(x_{k-1})(x_k - f(x_{k-1})^T \mid y_{1:T})$$

$$\approx \sum_i w_i (x_k^{(i)} - f(x_{k-1}^{(i)})(x_k^{(i)} - f(x_{k-1}^{(i)}))^T, (40)

In general, maximizing $Q$ in the M-step requires numerical optimization. In the following, we present closed-form solutions for the special case where the model functions are linear combinations of the parameters where the parameters appear as coefficients of the linear combinations and/or the covariances. That is, we consider models that can be represented as follows:

$$x_k = A \tilde{f}(x_{k-1}) + q_k,$$

$$y_k = H \tilde{h}(x_k) + r_k,$$

where $\tilde{f}(\cdot)$ and $\tilde{h}(\cdot)$ are functions containing the nonlinearities and the parameters are a subset of $\{A, H, Q, R, m_0, P_0\}$. 
For these models, the expression for $Q$ can be written as
\[
Q(\theta, \theta^{(n)}) = -\frac{1}{2} \log \left[ 2\pi P_0 \right] - \frac{T}{2} \log \left[ 2\pi Q \right] - \frac{T}{2} \log \left[ 2\pi R \right] - \frac{1}{2} \text{tr} \left\{ P_0^{-1} \left[ \dot{P}_0[T] + (m_{0[T]} - m_0)(m_{0[T]} - m_0)^T \right]\right\}
\]
\[
- \frac{T}{2} \text{tr} \left\{ Q^{-1} \left[ \Sigma - C A^T - A C^T + A \Phi A^T \right]\right\}
\]
\[
- \frac{T}{2} \text{tr} \left\{ R^{-1} \left[ D - B H^T - B H^T + H \Theta H^T \right]\right\},
\]
where the model parameters to be optimized are some subset of $\{A, H, Q, R, m_0, P_0\}$ and $\Sigma, \Phi, \Theta, B, C, D$ can be evaluated based on the latest E-step sigma-point smoother results as follows:
\[
\Sigma = \frac{1}{T} \sum_{k=1}^{T} P[k|T] + m[k|T][m[k|T]^T],
\]
\[
\Phi = \frac{1}{T} \sum_{k=1}^{T} E[\dot{f}(x_{k-1})\dot{f}^T(x_{k-1}) \mid y_{1:T}],
\]
\[
\Theta = \frac{1}{T} \sum_{k=1}^{T} E[\dot{h}(x_k)\dot{h}^T(x_k) \mid y_{1:T}],
\]
\[
B = \frac{1}{T} \sum_{k=1}^{T} y_k E[\dot{h}^T(x_k) \mid y_{1:T}],
\]
\[
C = \frac{1}{T} \sum_{k=1}^{T} E[x_k \dot{f}^T(x_{k-1}) \mid y_{1:T}],
\]
\[
D = \frac{1}{T} \sum_{k=1}^{T} y_k y_k^T.
\]
Using these values, the optimal parameters in the M-step are
- When $\theta = A$, we get
  \[
  A^* = C \Phi^{-1}.
  \]
- When $\theta = H$, we get
  \[
  H^* = B \Theta^{-1}.
  \]
- When $\theta = Q$, we get
  \[
  Q^* = \Sigma - C A^T - A C^T + A \Phi A^T.
  \]
- When $\theta = R$, we get
  \[
  R^* = D - B H^T - B H^T + H \Theta H^T.
  \]
- When $\theta = m_0$, we get
  \[
  m_0^* = m_{0[T]}.
  \]
- Finally, the maximum with respect to the initial covariance $\theta = P_0$ is
  \[
  P_0^* = P_{0[T]} + (m_{0[T]} - m_0)(m_{0[T]} - m_0)^T.
  \]

C. Evaluating the Gradient Based on Fisher’s Identity

The expected log-likelihood that appears in the EM algorithm may also be used as a basis of an alternative approach for evaluating the gradient in direct optimization. Based on Fisher’s identity, the gradient of the marginal log-likelihood may be expressed as
\[
\frac{\partial L_T(\theta)}{\partial \theta} = \frac{\partial Q(\theta, \theta^{(n)})}{\partial \theta} \bigg|_{\theta^{(n)}=\theta},
\]
where $Q$ is the function defined in the EM algorithm (Eq. 43). When the $Q$-function is approximated with sigma-point smoothers, we obtain an alternative approximation of the gradient of the marginal log-likelihood that may be used in place of the approximation derived in Section A. For linear state-space models, this approach was suggested by Segal and Weinstein [23] and later by Olsson et al. [24] who called the approach the ‘easy gradient recipe’. See [3, 2] for discussions of the nonlinear case.

IV. EXPERIMENTS

In this section, we demonstrate the different sigma-point schemes and different parameter estimation algorithms with two example models. First, we use a one-dimensional toy model (the univariate nonstationary growth model, UNGM, [25, 26]) to illustrate the approximate likelihood curves obtained by different methods. Second, we compare the performance of different algorithms with simulated data in a problem of tracking a maneuvering target with bearings-only measurements. In this example, we focus on estimating the sensor variances and compare the variance estimates as well as the actual tracking error.

A. Simple

We simulated a realization with $T = 100$ data from the following model:
\[
x_{k+1} = a x_k + b \frac{x_k}{1 + x_k} + c \cos(1.2k) + q_k,
\]
\[
y_k = d x_k + r_k.
\]
with $a = 0.5$, $b = 25$, $c = 8$, $d = \sqrt{0.05}$, $q_k \sim N(0, 10)$, $r_k \sim N(0, 0.01)$. This is the univariate nonstationary growth model [25, 26] except that we changed the measurement model to linear as the model with the typically used quadratic measurement model is known to be challenging for sigma-point algorithms [27].

First, we estimated the likelihood of parameter $a$, holding other parameters fixed at their ground-truth values. Likelihood curves obtained by direct likelihood estimation with various sigma-point rules as well as the EM lower bounds for two iterations are shown in Figure 5. For comparison, a likelihood estimate obtained by particle filtering is also shown. The EM iterations seem to converge toward the maximum of the likelihood curve and the second EM bound is rather close to the particle filter likelihood estimate. The EM lower bounds are mostly below the sigma-point likelihood curve as expected, except that the first EM lower bound slightly exceeds the sigma-point likelihood approximation in the vicinity of the
likelihood approximation. The dashed lines correspond to the sigma-point EM filter log-likelihood estimate, while the solid line is the sigma-point filter log-univariate estimation of parameter $\theta$. Fig. 3. Visualization of the one-step evolution of the EM algorithm for the univariate estimation of parameter $\alpha$. The dotted line represents the particle filter log-likelihood estimate, while the solid line is the sigma-point filter log-likelihood approximation. The dashed lines correspond to the sigma-point EM bounds for iterations $n$ and $n+1$.

![Diagram showing the visualization of the one-step evolution of the EM algorithm for the univariate estimation of parameter $\alpha$.](image)

initial parameter. However, both the evaluation of $Q$ and the sigma-point estimate of the likelihood are approximations.

Second, to compare the different sigma-point rules, we considered estimation of the parameter $\beta$ with other parameters fixed and parameter $c$ with other parameters fixed, using a grid of parameter values with close proximity to the maximum likelihood values. These are shown in Figure 4. The estimate obtained by the Gauss–Hermite rule and the estimates obtained by the higher-order UKFs are rather close to each other while the estimate by the 3rd order UKF is farther in both parameters.

![Diagram showing log-likelihood curves for parameters $b$ and $c$ evaluated by five different sigma-point methods.](image)

**B. Coordinated-Turn Model**

In this section, we compare the performance of the parameter estimation methods discussed in this article using a more practical example. The problem is tracking a target maneuvering according to the coordinated turn model [28], [29], [72], [18] with bearings-only sensor measurements. The state is 5-dimensional:

$$\mathbf{x} = (x_1 \ x_2 \ \dot{x}_1 \ \dot{x}_2 \ \omega)^T,$$

(53)

where $(x_1, x_2)$ is the location of the target in 2-dimensional Cartesian coordinates, $(\dot{x}_1, \dot{x}_2)$ is the corresponding speed, and $\omega$ is the turn rate. The dynamic model is

$$\mathbf{x}_{k+1} = \begin{pmatrix}
1 & 0 & \sin(\omega \Delta t) & -\cos(\omega \Delta t) & 0 \\
0 & 1 & -\cos(\omega \Delta t) & -\sin(\omega \Delta t) & 0 \\
0 & 0 & \cos(\omega \Delta t) & \sin(\omega \Delta t) & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix} \mathbf{x}_k + \mathbf{q}_k.$$

(54)

The process noise $\mathbf{q}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{Q})$, where

$$\mathbf{Q} = \begin{pmatrix}
q_c \Delta t^3 / 3 & 0 & q_c \Delta t^2 / 2 & 0 & 0 \\
0 & q_c \Delta t^3 / 3 & 0 & q_c \Delta t^2 / 2 & 0 \\
0 & 0 & q_c \Delta t^2 / 2 & 0 & q_c \Delta t \\
0 & 0 & 0 & 0 & q_\omega \Delta t
\end{pmatrix}.$$

(55)

The measurements are angles from the sensors with additive Gaussian noise:

$$\mathbf{y}_k = \mathbf{h}(\mathbf{x}_k) + \mathbf{r}_k,$$

(56)

where the measurement noise is $\mathbf{r}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R})$. The covariance matrix $\mathbf{R}$ is naturally assumed diagonal, as the measurement errors of separate sensors should be independent. For each sensor $i$ at location $s_i$ the measurement is given by

$$h_i(\mathbf{x}_k) = \arctan2(x_{2,k} - s_{2,i}, x_{1,k} - s_{1,i}),$$

(57)

where $\arctan2$ is the four-quadrant inverse tangent. We focus on estimating the measurement noise variances while keeping other parameters fixed. That is, the sensor locations and dynamic model covariance are assumed to be known and the initial state distribution fixed.

The parameters of the process noise covariance were set to $q_c = 0.1$, $q_\omega = 0.1$ and the time step $\Delta t = 0.1$. The ground-truth measurement noise covariance was $\mathbf{R} = \text{diag}(0.05^2, 0.1^2)$. The two sensors were located at $s_1 = (-1, 0.5)$ and $s_2 = (1, 1)$. The parameters of the initial distribution were $\mathbf{m}_0 = (2, 0, 0, 0, 0)^T$ and $\mathbf{P}_0 = \text{diag}(0.5^2 0.5^2, 0.1^2, 0.1^2, 1^2)$. We simulated 100 different trajectories with $T = 50$ timesteps from this model.

To compare performance of the different sigma-point schemes, we performed direct maximum likelihood estimation of the sensor noise standard deviation of the first sensor, keeping the noise of the second sensor as well as other parameters fixed at their ground truth values. The sigma-point schemes used were UKF 3, UKF 5, and UKF 7, as well as GHKF 3, GHKF 5, and GHKF 7. The 9th order schemes were omitted since the number of sigma-points is already quite high as the state is 5-dimensional. In addition to the sigma-point methods, we also compared against maximum likelihood estimation based on the extended Kalman filter (EKF, our implementation based on [2]).
The optimization was performed without gradient information using the Matlab optimization toolbox. Furthermore, we investigated how the estimation performance varies as a function of uncertainty of the target’s initial location. This was done by using an additional parameter for the per-coordinate standard deviation ($\sigma \in (0,0.5)$). The first two diagonal components of $P_0$ were set to $\sigma^2$. Furthermore, the first two components of $m_0$ were interpolated between the original $m_0$ and the simulated $x_0$ to keep the uncertainty of the initial location consistent with the prior.

Since GHKF 7 is the highest-order sigma-point scheme amongst those used in this experiment, we assume it is the most accurate and compare against it. Figure 5 shows comparison of median (over the 100 trajectories) absolute deviation of the MLE estimates obtained by the various filtering schemes compared to the ones obtained by GHKF 7. GHKF 5 is closest, while EKF and UKF 3 are farthest from the baseline. UKF 7, GHKF 3 and UKF 5 have similar performance. UKF 5 and GHKF 3 are essentially identical. This is explained by the observation that in 5 dimensions, all UKF 5 sigma-points are present in the GHKF 3 sigma-point set and the sum of the GHKF 3 weights of these points is 0.79. The contribution of the remaining sigma-points that have total 0.21 weight apparently has a negligible contribution at least with this model. In addition, we also look at track estimation errors using the final parameter estimates by each sigma-point scheme. Figure 6 shows the mean RMSE over the 100 trajectories, that is, for each simulated trajectory, we computed the smoother RMSE and then took the average.

For one simulated trajectory with $\sigma = 0.5$ we also tried EM parameter estimation of $\sqrt{R_{1,1}}$ using UKF 3, UKF 5, and UKF 7. The results are shown in Figure 7. The EM algorithm practically converges in a couple of steps with all three sigma-point schemes, and the final parameter estimates are rather close to each other and to the direct MLE estimates.

V. Conclusion and Discussion

In this paper together with the complementing conference article [5], we have considered various probabilistic point estimation approaches for parameter estimation in nonlinear system identification. We discussed direct likelihood maximization as well as the expectation–maximization (EM) algorithm coupled with various filtering and smoothing algorithms, namely, sigma-point filters, particle filters, and extended Kalman filters as well as the corresponding smoothers. In this paper, we focused on the differences between different sigma-point filters based on unscented transforms of third, fifth, seventh and ninth orders, and the Gauss–Hermite cubature rules.

In diminishing order of computational complexity and theoretical exactness, the filtering methods would rank as follows: particle filter, sigma-point filter, extended Kalman filter based direct likelihood approximation. In theory, particle filters converge to the exact filtering solution as the number of particles increases, while the other methods considered are based on assuming a Gaussian density and using the Kalman filter equations. However, especially in high-dimensional cases, the computational cost may prohibit the use of particle filters. In practice, the assumed density Gaussian filtering approach may have satisfactory performance if the nonlinearity is not too high. In principle, all sigma-point filters are based on assuming Gaussian density, and a higher-order cubature rule should lead to more accurate approximation of the Gaussian integrals at the cost of higher computational burden. However, typically in the literature (e.g. [29]) it has been claimed that when the Gaussian density approximation is already inaccurate, more accurate computation of the Gaussian integrals is not beneficial.

We also tested the methods in two simulated case studies. In the univariate nonstationary growth model, maximum likelihood estimates produced by different sigma-point schemes were similar. However, the estimates obtained by higher-

\[1\] MATLAB version R2014b, the `fminunc` function, default settings, initialized with $\sqrt{R_{1,1}} = 0.1$. 

Fig. 5. Median absolute error of the parameter estimates compared to GHKF 7 (median taken over the 100 simulated trajectories) as a function of the initial location prior standard deviation. UKF 5 is essentially indistinguishable from GHKF 3.

Fig. 6. Mean RMSE of smoothed location (mean taken over the 100 trajectories) using the MLE estimated noise variance of the first sensor. UKF 5 is essentially indistinguishable from GHKF 3.
order unscented schemes were closer to the Gauss–Hermite (order 16) baseline than the conventional 3rd order unscented transform. This suggests that the higher order methods may indeed have some utility.

In the target tracking experiment, we compared the estimates of the noise standard deviation of one of the sensors as a function of prior uncertainty of the target’s location using each of the sigma-point schemes. With higher prior uncertainty, there were more differences amongst the methods. This is reasonable because when there is less uncertainty in the model, all the methods obtain more accurate parameter estimates. Furthermore, the nonlinearity of the model has a stronger effect when the state variance is larger. Since no exact maximum likelihood estimate was available, we compared to the highest-order sigma-point scheme, namely, GHKF 7. Compared to that, the Gauss–Hermite schemes were closer than the unscented transform based schemes and higher order schemes were closer than lower order schemes. Thus, the results are consistent with an assumption that the higher-order sigma-point methods produce better approximations to the Gaussian filtering result.

The sigma-point integration schemes are derived by assuming exact integration results for polynomials of certain degrees. Thus, it should be noted that it is not guaranteed that a higher-order integration rule produces a more accurate result, even though it is accurate for higher-order polynomials. Furthermore, it is not guaranteed that a better approximation to the Gaussian filtering result produces a better approximation to the exact maximum likelihood result. On the other hand, there is no reason why in general a lower-order approximation to the Gaussian filtering integrals would produce more accurate approximations to the exact filtering results.

We also compared the actual tracking performance in terms of the smoother root mean square errors of the target locations using the smoother results obtained with the maximum likelihood parameter estimate of each sigma-point scheme. There was no clear differences between the different sigma-point schemes in terms of the tracking error in this experiment. The tracking error of EKF increased more rapidly as a function of the initial location uncertainty, which demonstrates the local linearization nature of EKF.

In five dimensions, the UKF 5 sigma-point scheme approximates the GHKF 3 scheme in the sense that all UKF 5 sigma-points are GHKF 3 sigma-points as well, and more than half of total weight is contributed by these points. The target-tracking experiment demonstrated that these two schemes indeed produce almost equal results. However, most GHKF 3 sigma-points are not used in the UKF 5 scheme, and thus evaluating an integral by UKF 5 requires considerably fewer function evaluations. These results suggest that there is no reason to use GHKF 3 in five dimensions as UKF 5 produces essentially same results with fewer computations. It is possible that similar computationally lighter close approximations exist for other Gauss–Hermite based sigma-point schemes as well.

In the target-tracking experiment, we also investigated how the performance of the EM algorithm varies with the sigma-point scheme used. As a function of EM iterations, the evolution of the parameter estimate was not affected by the choice of sigma-point scheme. However, the number of sigma-points in the UKF 3 rule is a small fraction of the number of sigma-points with the higher order rules. Thus, measured by model function evaluations, EM with the UKF 3 rule converged faster. This suggests that even when the interest lies in obtaining as accurate parameter estimates as possible, a reasonable computational approach would be to first use EM with a low-order sigma-point scheme, such as UKF 3, to obtain a ballpark estimate. Then, if accuracy is desired, the initial estimate could be refined using a more accurate sigma-point scheme combined with a direct optimization algorithm.

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