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P=W conjectures for character varieties with symplectic resolution

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P=W CONJECTURES FOR CHARACTER VARIETIES
WITH SYMPLECTIC RESOLUTION

BY CAMILLA FELISETTI & MIRKO MAURI

Abstract. — We establish P=W and PI=WI conjectures for character varieties with structural group $GL_n$ and $SL_n$ which admit a symplectic resolution, i.e., for genus 1 and arbitrary rank, and genus 2 and rank 2. We formulate the P=W conjecture for a resolution, and prove it for symplectic resolutions. We exploit the topology of birational and quasi-étale modifications of Dolbeault moduli spaces of Higgs bundles. To this end, we prove auxiliary results of independent interest, like the construction of a relative compactification of the Hodge moduli space for reductive algebraic groups, and the projectivity of the compactification of the de Rham moduli space. In particular, we study in detail a Dolbeault moduli space which is a specialization of the singular irreducible holomorphic symplectic variety of type O'Grady 6.

Résumé (Les conjectures P=W pour les variétés de caractères ayant une résolution symplectique)
On établit les conjectures P=W et PI=WI pour les variétés de caractères avec groupe structurel $GL_n$ et $SL_n$ qui admettent une résolution symplectique, c'est-à-dire pour le genre 1 en rang arbitraire, et le genre 2 en rang 2. On formule la conjecture P=W pour une résolution et on la prouve pour les résolutions symplectiques. Pour la démonstration on fait appel à la topologie des modifications birationnelles et quasi-étale des espaces de modules de fibrés de Higgs. Pour cela, on démontre des résultats auxiliaires d’intérêt indépendant, comme la construction d’une compactification relative de l’espace de modules de Hodge pour les groupes algébriques réductifs, ou la théorie de l’intersection de certains cycles lagrangiens singuliers. En particulier, on étudie en détail un espace de modules des fibrés de Higgs qui est une spécialisation de la variété symplectique holomorphe irréductible singulière de type O’Grady 6.

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1. Introduction

Let $X$ be a compact Riemann surface of genus $g$, and let $G$ be a complex reductive algebraic group. The Betti and Dolbeault moduli spaces $M_B(X, G)$ and $M_{Dol}(X, G)$ are central objects in non-abelian Hodge theory. The Betti moduli space, or $G$-character variety of $X$, is the affine GIT quotient

$$M_B(X, G) := \text{Hom}(\pi_1(X), G) / \!\!/ G$$

(1)

$$= \{ (A_1, B_1, \ldots, A_g, B_g) \in G^{2g} \mid \prod_{j=1}^g [A_j, B_j] = 1 \} / \!\!/ G.$$ 

It parametrizes isomorphism classes of semistable representations of the fundamental group of $X$ with value in $G$.

The Dolbeault moduli space $M_{Dol}(X, G)$ instead parametrizes semistable principal $G$-Higgs bundles with vanishing Chern classes; see [78]. For example, we have that:

- a $GL_n$-Higgs bundle is a pair $(E, \phi)$ with $E$ vector bundle of rank $n$ and degree 0, and $\phi \in \text{Hom}(E, E \otimes K_X)$;
- a $GL_n$-Higgs bundle is an $SL_n$-Higgs bundle if in addition the determinant of $E$ is trivial and the trace of $\phi$ vanishes;
- a $PGL_n$-Higgs bundle is an equivalence class of $SL_n$-Higgs bundles under tensorization by an $n$-torsion line bundle on $C$.

Despite the different origin of these moduli spaces, there exists a real analytic isomorphism

$$\Psi : M_{Dol}(X, G) \longrightarrow M_B(X, G)$$

called non-abelian Hodge correspondence; see [79] or Section 3. However, the map $\Psi$ is not an algebraic isomorphism. Indeed, note that the Betti moduli space is an affine variety, while the Dolbeault moduli space admits a projective morphism with connected fibers

$$\chi : M_{Dol}(X, G) \longrightarrow A^{\dim M_{Dol}(X, G)/2},$$

called the Hitchin fibration. The purpose of this paper is to study the behaviour in cohomology of the non-abelian Hodge correspondence in view of the $P=W$ conjecture [11]. In the rest of the paper we will only consider reductive groups of type $A$, i.e., $G = GL_n, SL_n, PGL_n$, unless stated otherwise, e.g. in the formulation of the $P=W$ conjectures or in Section 3.1.

One of the main difficulties while studying the cohomology of these moduli spaces is that they are generally singular. To circumvent this issue, it is customary to slightly
change the moduli problem as follows. Given an integer \( d \) coprime with the rank \( n \) of the group, the twisted Betti moduli space is the GIT quotient

\[
M_{\text{tw}}^{\text{B}}(X, G) := \{ (A_1, B_1, \ldots, A_g, B_g) \in G^{2g} \mid \prod_{j=1}^{g} [A_j, B_j] = e^{2\pi id/n} 1_G \} / G.
\]

On the other hand, the twisted version of Dolbeault moduli, denoted \( M_{\text{tw}}^{\text{Dol}}(X, G) \), parametrizes semistable pairs \((E, \phi)\), with \( E \) vector bundle of rank \( n \) and degree \( d \).

The technical advantage of working with these twisted moduli spaces is that they are smooth varieties and satisfy a non-abelian Hodge theorem as in the untwisted case; see [44].

While studying the weight filtration on \( H^*(M_{\text{tw}}^{\text{B}}(X, G), \mathbb{Q}) \), Hausel and Rodriguez-Villegas discovered a surprising symmetry, that they called curious hard Lefschetz theorem: there exists a class \( \alpha \in H^2(M_{\text{tw}}^{\text{B}}(X, G), \mathbb{Q}) \) which induces the isomorphisms

\[
\bigcup_{k} \alpha^k : \bigoplus_{n=2k} H^{2k}(M_{\text{tw}}^{\text{B}}(X, G), \mathbb{Q}) \xrightarrow{\cong} \bigoplus_{n=2k} H^{n+2k}(M_{\text{tw}}^{\text{B}}(X, G), \mathbb{Q}).
\]

The theorem holds for \( G = \text{GL}_2, \text{SL}_2 \) and \( \text{PGL}_2 \) by [41], and for \( G = \text{GL}_n \) by [61]. To explain this phenomenon, de Cataldo, Hausel and Migliorini conjectured that the non-abelian Hodge correspondence should exchange the weight filtration on the space \( H^*(M_{\text{tw}}^{\text{B}}(X, G), \mathbb{Q}) \) with the perverse (Leray) filtration associated to \( \chi \) on the space \( H^*(M_{\text{tw}}^{\text{Dol}}(X, G), \mathbb{Q}) \); see Definition 2.8. In this way, the curious hard Lefschetz theorem would correspond to the classical relative hard Lefschetz theorem for \( \chi \); see Theorem 2.10.

**Conjecture 1.1 (P=W conjecture for twisted moduli spaces)**

\[
P_k H^*(M_{\text{tw}}^{\text{Dol}}(X, G), \mathbb{Q}) = \Psi^k W_{2k} H^*(M_{\text{tw}}^{\text{B}}(X, G), \mathbb{Q}).
\]

The conjecture holds for \( g \geq 2 \) and \( G = \text{GL}_2, \text{SL}_2 \) and \( \text{PGL}_2 \) by [11], and for \( g = 2 \) and \( G = \text{GL}_n, \text{SL}_p \) with \( p \) prime by [15, 14]. An enumerative approach has been proposed in [19], and other P=W phenomena have been studied in [74, 75, 27, 89, 82, 81, 66, 48, 37, 38, 59]. However, P=W phenomena for the original moduli spaces \( M_B(X, G) \) and \( M_{\text{Dol}}(X, G) \) have not been explored yet. This is then the goal of our paper.

In the singular case, relative and curious hard Lefschetz theorems fail in general for singular cohomology; see Remark 8.6. Nonetheless, it is known that the relative hard Lefschetz theorem for \( \chi \) holds for intersection cohomology \( IH^*(M_{\text{Dol}}(X, G)) \); see Sections 2.2 and 2.4. Moreover, de Cataldo and Maulik proved in [13] that the perverse filtration on intersection cohomology is independent of the complex structure of the curve \( X \), exactly as it happens for the weight filtration. Therefore, they conjectured [13, Quest. 4.1.7].

---

(1) We omitted the dependence of \( M_{\text{tw}}^{\text{B}}(X, G) \) and \( M_{\text{tw}}^{\text{Dol}}(X, G) \) on the degree \( d \) not to burden the notation too much.

(2) Note that we recover the untwisted Dolbeault moduli space for \( d = 0 \).
**Conjecture 1.2 (P=W conjecture).** — Let $G$ be a complex reductive group. Then
\[ P_k H^*(M_{Dol}(X, G), \mathbb{Q}) = \Psi^* W_{2k} H^*(M_B(X, G), \mathbb{Q}). \]

It is also conceivable that one could obtain the P=W conjecture for the singular moduli spaces $M_{Dol}(X, G)$ from the previous conjectures.

**Conjecture 1.3 (P=W conjecture for singular moduli spaces).** — Let $G$ be a complex reductive group. Then
\[ P_k H^*(M_{Dol}(X, G), \mathbb{Q}) = \Psi^* W_{2k} H^*(M_B(X, G), \mathbb{Q}). \]

Alternatively, we may also opt for a desingularization of $M_{Dol}(X, G)$, and continue to work with singular cohomology. We show that a P=W conjecture for symplectic resolution does hold, i.e., for resolutions where a holomorphic symplectic form on the smooth locus of each moduli space extends to a symplectic form on the whole of the resolution.

To this end, we first show how to lift the non-abelian Hodge correspondence to resolutions of $M_{Dol}(X, G)$ and $M_B(X, G)$, up to isotopy, according to Theorem 3.8.

**Theorem 1.4 (Theorem 3.8).** — Let $G$ be a complex reductive group. Then there exist resolutions of singularities $f_{Dol}: \tilde{M}_{Dol}(X, G) \to M_{Dol}(X, G)$ and $f_B: \tilde{M}_B(X, G) \to M_B(X, G)$, and a diffeomorphism $\tilde{\Psi}: \tilde{M}_{Dol}(X, G) \to \tilde{M}_B(X, G)$, such that the following square commutes:
\[ \begin{array}{ccc}
H^*(\tilde{M}_{Dol}(X, G), \mathbb{Q}) & \xleftarrow{\tilde{\Psi}^*} & H^*(\tilde{M}_B(X, G), \mathbb{Q}) \\
\downarrow f_{Dol} & & \downarrow f_B \\
H^*(M_{Dol}(X, G), \mathbb{Q}) & \xleftarrow{\Psi^*} & H^*(M_B(X, G), \mathbb{Q}).
\end{array} \]

The resolutions $f_{Dol}$ and $f_B$ can be taken functorial with respect to smooth algebraic or analytic morphisms, and symplectic if $G=GL_n$ or $SL_n$ with $(g,n)=(1,n)$ or $(2,2)$.

**Conjecture 1.5 (P=W conjecture for symplectic resolution).** — Let $G$ be a complex reductive group. Let $\tilde{\Psi}, f_{Dol}$ and $f_B$ be the diffeomorphism appearing in Theorem 1.4. If $f_{Dol}$ is a symplectic resolution (if it exists!), or equivalently $f_B$ is so, then
\[ P_k H^*(\tilde{M}_{Dol}(X, G), \mathbb{Q}) = \tilde{\Psi}^* W_{2k} H^*(\tilde{M}_B(X, G), \mathbb{Q}). \]

In an earlier version of this paper, we stated the P=W conjecture for resolution without the assumption of the existence of a symplectic resolution, but later the second author proved that the hypothesis is indeed essential at least for $G=GL_n$ and $SL_n$, see [59, §5.6]. Recent results suggest that the existence of a holomorphic symplectic form should be a key ingredient for P=W phenomena, see [12, §4.4], [61] and [37, Th.1.7].

In this paper, we provide the first evidence for the P=W conjectures in the singular context.
Main theorem. — Let $G = \text{GL}_n$ or $\text{SL}_n$. Suppose that $(g,n) = (1,n)$ or $(2,2)$. Then the following conjectures hold:

1. the $P=W$ conjecture;
2. the $PI=WI$ conjecture;
3. the $P=W$ conjecture for a symplectic resolution.

Observe that $M_{\text{Dol}}(X, \text{GL}_n)$ and $M_{\text{Dol}}(X, \text{SL}_n)$ admit a (unique) symplectic resolution if and only if $(g,n) = (1,n)$ or $(2,2)$; see [5] and [29, Th. 2.2]. Under this assumption, the $P=W$ and $PI=WI$ conjectures for $M_{\text{Dol}}(X, \text{PGL}_n)$ hold too; see Remark 4.3.

The expectation is that the $PI=WI$ conjecture holds even in the absence of a symplectic resolution. The second author has provided first evidence of this fact in [59, §5].

Proof of the main theorem. — We first reduce to $G = \text{SL}_n$; see Theorem 4.1.

For $g = 1$, the $P=W$ and $PI=WI$ conjectures follow from Theorem 5.3 and Remark 5.4. Although not presented in these terms, the proof of the $P=W$ conjecture for the symplectic resolution in $g = 1$ is due to [12].

The proof of the conjectures for $M := M_{\text{Dol}}(C, \text{SL}_2)$, with $C$ a curve of genus 2, takes up most of the paper. We first reduce the $P=W$ conjecture for $M$ and $\tilde{M}$ to the $PI=WI$ conjecture; see Theorems 7.1, 7.4 and 7.6. Finally, the $PI=WI$ conjecture follows from Theorems 8.1, 8.8 and 8.17.

Symplectic resolutions. — The Dolbeault moduli spaces which admit a symplectic resolution appear as specialization of (a crepant contraction of) of compact hyperkähler manifolds as shown in the table.

| Special fiber $M_{\text{Dol}}(A, \text{GL}_n)$ | Symplectic resolution of the general fiber |
|-----------------------------------------------|------------------------------------------|
| Hilbert scheme of $n$ points on a K3 surface containing the elliptic curve $A$ |
| $M_{\text{Dol}}(A, \text{SL}_n)$ | generalized Kummer variety of dimension $2(n-1)$ associated to the abelian surface $A \times A$ |
| $M_{\text{Dol}}(C, \text{GL}_2)$ | O'Grady 10-dimensional moduli space $\text{OG10}$ |
| $M := M_{\text{Dol}}(C, \text{SL}_2)$ | O'Grady 6-dimensional moduli space $\text{OG6}$ |

Table 1. Degenerations of compact hyperkähler manifolds to the space $M_{\text{Dol}}(X, G)$; see the appendix. We denote by $A$ and $C$ a compact Riemann surface of genus 1 and 2 respectively.

Even if these degenerations are not strictly used in the proof of the main theorem, they have been our sources of inspiration. For instance, the proof of the $P=W$ conjecture for $g = 1$ is inspired by the description of the cohomology of generalized Kummer varieties in [32], while the alterations in Section 6 are specializations of those exploited by [62] to determine the Hodge numbers of $\text{OG6}$. We included details about
the construction of the degenerations in the appendix for the interested reader. In
the twisted case these degenerations have been exploited in [15, §4] and [14, §4]; see
Proposition A.8 and Remark A.9 for a bizarre difference between the behaviour of
the degenerations in the smooth and singular cases. Analogous degenerations on the
Betti side for $g = 1$ have been considered in [60, §5.2 and §5.3] for the proof of the
geometric P=W conjecture.

**Twisted vs untwisted moduli spaces.** — Let $G = GL_n$ or $SL_n$. The known proofs [11]
and [15] of the P=W conjecture for twisted moduli spaces crucially rely on the fact
that $H^*(M_{Dol}^w(X, G))$ is generated in degree not greater than 4. Further, the generators
are Künneth components of the second Chern class of a universal Higgs bundle
on $M_{Dol}^w(X, G) \times X$, called tautological classes.

In the untwisted case this can fail.

- The cohomology ring of $M_{Dol}(X, G)$ may not be generated in degree $\leq 4$. For
instance, Theorem 6.14 and the second paragraph of the proof of Proposition 8.4
imply that

$$H^*(M, \mathbb{Q}) \simeq \mathbb{Q}[\alpha, \gamma_j]/(\alpha^3, \gamma_j^2, \alpha \cup \gamma_j),$$

with $\deg \alpha = 2$, $\deg \gamma_j = 6$, and $j = 1, \ldots, 16$.

- A universal Higgs bundle $E$ on $M_{Dol}(X, G)^{sm\times X}$ may not exist. Indeed, if $E$
exists on $M^{sm \times C}$, then its restriction to the moduli space of semistable vector bundles
of rank 2 and degree 0 would be a universal vector bundle, which does not exist by [65].

If $g = 2$, we fix this problem by constructing a tautological class $\beta$ on a quasi-étale
cover of $M$, i.e., étale in codimension one; see Section 8.3. However, $\beta$ does not descend
in cohomology, but as an intersection cohomology class. More precisely, $IH^*(M, \mathbb{Q})$
is the $H^*(M, \mathbb{Q})$-module

$$IH^*(M, \mathbb{Q}) \simeq H^*(M, \mathbb{Q})[1, \beta]/(\alpha \cup \beta - \sum_{j=1}^{16} \gamma_j, \alpha^2 \cup \beta, \gamma_j \cup \beta).$$

One can avoid constructing a universal bundle on a quasi-étale cover by appealing
to the Dolbeault moduli stack, and the class $\beta$ can be interpreted as a Chern class of
an orbibundle. However, the construction of the universal bundle on the quasi-étale
cover is interesting in itself; cf. [45, §6.1]. Note also that the existence of this cover is
a special feature of $M$: we show that when $g \geq 2$, $M$ is the only Dolbeault moduli
space which admits a non-trivial quasi-étale cover; see Section 6.2.7.

**1.1. Outline of the paper**

- In Sections 2 and 4 we recall basic notions and theorems used throughout the paper.

- In Section 3.1 we lift the non-abelian Hodge correspondence $\Psi$ to a diffeomor-
phism $\tilde{\Psi}$ between the resolutions of the Betti and Dolbeault moduli spaces; see The-
orem 3.8. To this end, we describe an explicit compactification of the Hodge moduli
space in Theorem 3.2. Note that $\tilde{\Psi}$ is the diffeomorphism which appears in the state-
ment of the P=W conjecture for symplectic resolution. As a by-product, we answer
a question by Simpson about the projectivity of the compactification of the de Rham
moduli space, see Corollary 3.3.
Figure 1. The \((p, \Delta)\)-entry of the table is the dimension of the graded piece
\[
\text{Gr}_p^\Delta H^{\Delta-p}(\tilde{M}_{Dol}(C, \text{SL}_2), \mathbb{Q}),
\]
of the perverse Leray filtration on \(H^\ast(\tilde{M}_{Dol}(C, \text{SL}_2), \mathbb{Q})\), where \(C\) is a compact Riemann surface of genus 2. The sums along the northwest-southeast diagonals give the Betti numbers of \(\tilde{M}_{Dol}(C, \text{SL}_2)\). Relative hard Lefschetz accounts for a symmetry of this perverse diamond, namely a reflection about the horizontal axis placed at middle perversity. The P=W conjecture for resolution implies that the sums along the rows are the coefficients of the E-polynomial of \(\tilde{M}_{B}(C, \text{SL}_2)\), which is computed in (43).

- In Section 4.1 we show that the P=W conjecture for \(\text{SL}_n\) implies the P=W conjecture for \(\text{GL}_n\).
- In Section 5 we prove the P=W conjectures for \(g = 1\).
- The rest of the paper is devoted to the proof of the P=W conjectures for \(M := M_{Dol}(C, \text{SL}_2)\), with \(C\) a curve of genus 2. We describe the geometry of \(M\) in great detail in Section 6: its singularities and its symplectic resolution \(\tilde{M}\) in Section 6.1.1; the fixed loci of the \(G_m\)-action on \(M\) and \(\tilde{M}\) in Section 6.1.4; the (universal) quasi-étale cover \(q: M_{\iota} \rightarrow M\) in Section 6.2.1; a universal Higgs bundle on the smooth locus \(M_{\iota}^{\text{sm}}\) of \(M_{\iota}\) in Section 6.2.5; the zero fiber of the Hitchin fibration in Section 6.2.6.
- In Section 7 we explain the strategy of the proof of the P=W conjecture for \(M\). Ultimately, we reduce the proof of the P=W conjectures for \(M\) and \(\tilde{M}\) to the PI=WI conjecture for \(M\).
- In Section 8.1 we compute the necessary intersection Poincaré and E-polynomials.
- In Section 8.3 we build a tautological class of perversity 2 and weight 4, out of the universal bundle on \(M_{\iota}^{\text{sm}}\). This allows to conclude the proof of the PI=WI conjecture for \(M\) in Section 8.4.
- In the appendix we collected some information about degenerations of compact hyperkähler varieties to Dolbeault moduli spaces.
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2. Preliminaries

In this section we introduce preliminary notions and results which will be useful throughout the paper. For further details we refer to [4, 30, 31, 16]. When omitted, the coefficients of (intersection) cohomology are assumed to be rational.

2.1. Perverse sheaves. — An algebraic variety $X$ is an irreducible separated scheme of finite type over $\mathbb{C}$. Denote by $D^b_c(X)$ the bounded derived category of $\mathbb{Q}$-constructible complexes on $X$. Let $D: D^b_c(X) \rightarrow D^b_c(X)$ be the Verdier duality functor. The full subcategories

$$pD^b_{\leq 0}(X) := \{ K^* \in D^b_c(X) | \dim \text{Supp}(\mathcal{H}^j(K^*)) \leq -j \},$$

$$pD^b_{\geq 0}(X) := \{ K^* \in D^b_c(X) | \dim \text{Supp}(\mathcal{H}^j(DK^*)) \leq -j \},$$

determine a $t$-structure on $D^b_c(X)$, called perverse $t$-structure. The heart $\text{Perv}(X) := pD^b_{\leq 0}(X) \cap pD^b_{\geq 0}(X)$ of the $t$-structure is the abelian category of perverse sheaves. The truncation functors are denoted $p\tau_{\leq k}: D^b_c(X) \rightarrow pD^b_{\leq 0}(X)$, $p\tau_{\geq k}: D^b_c(X) \rightarrow pD^b_{\geq 0}(X)$, and the perverse cohomology functors are

$$p\mathcal{H}^k := p\tau_{\leq k} p\tau_{\geq k}: D'_c(X) \rightarrow \text{Perv}(X).$$

Definition 2.1. — Let $K^*$ be a complex in $D'_c(X)$. The cohomology $H^d(X, K^*)$ is endowed with the perverse filtration defined by

$$P_k H^d(X, K^*) = \text{Im}\{ H^d(X, p\tau_{\leq k}K^*) \rightarrow H^d(X, K^*) \}.$$ 

2.2. Intersection cohomology. — The category $\text{Perv}(X)$ is abelian, artinian, and its simple objects are the intersection cohomology complexes.

Definition 2.2 (Intersection cohomology complex). — Let $L$ be a local system on a smooth Zariski-dense open subset $U \subseteq X$. The intersection cohomology complex $IC_X(L)$ is a complex of sheaves in $D^b_c(X)$ which is uniquely determined up to isomorphism by the following conditions:

- $IC_X(L)|_U \simeq L[\dim X]$;
- $\dim \text{Supp} \mathcal{H}^j (IC_X (L)) < - j$, for all $j > - \dim X$;
- $\dim \text{Supp} \mathcal{H}^j (\mathcal{D}IC_X (L)) < - j$, for all $j > - \dim X$.

When $L = \mathbb{Q}_{X_{\text{sm}}}$, i.e., the constant sheaf on the smooth locus of $X$, we just write $IC_X$ for $IC_X (\mathbb{Q}_{X_{\text{sm}}})$. Further, if $X$ has at worst quotient singularities, then $IC_X \simeq \mathbb{Q}_X [\dim X]$.

**Definition 2.3 (Intersection cohomology).** — The intersection cohomology of $X$ with coefficient in $L$ is its (shifted) cohomology $IH^\ast (X, L) = H^{\ast - \dim X} (X, IC_X (L))$.

Analogously, the intersection cohomology of $X$ with compact support and coefficients in $L$ is $IH_c^\ast (X, L) = H^{\ast - \dim X} (X, DIC_X (L))$. For further details, we refer the interested reader to [50].

There is a natural morphism $H^\ast (X) \rightarrow IH^\ast (X)$, which is an isomorphism when $X$ has at worst quotient singularities. This morphism equips $IH^\ast (X)$ with the structure of $H^\ast (X)$-module, but in general intersection cohomology has no ring structure or cup product.

Moreover, the groups $IH^\ast (X)$ are finite dimensional, satisfy Mayer-Vietoris theorem and Künneth formula. Although they are not homotopy invariant, they satisfy analogues of Poincaré duality, i.e., $IH^\ast (X) \simeq IH^{2 \dim X - \ast} (X)^\vee$ and of the hard Lefschetz theorem. They also carry a mixed Hodge structures.

**Definition 2.5 (Mixed Hodge structure).** — The mixed Hodge structure $(V, F^\ast, W^\ast)$ is the datum of
- a $\mathbb{Q}$-vector space $V$,
- an increasing filtration $W_\ast$ on $V$, called weight filtration,
- a decreasing filtration $F^\ast$ on $V \otimes \mathbb{C}$, called Hodge filtration,

such that the graded pieces $\text{Gr}_k W V \coloneqq W_k V/W_{k-1} V$ admit a pure Hodge structure of weight $k$, induced by $F^\ast$ on $\text{Gr}_k W V \otimes \mathbb{C}$.

An element $v \in V$ has weight $k$ if $v \in W_k V$ but $v \not\in W_{k-1} V$.

**Definition 2.6 (E-polynomial).** — The E-polynomial of $X$ is an additive function on the category of separated $\mathbb{C}$-schemes of finite type given by

$$E(X) = \sum_{p,q,d} (-1)^d \dim (\text{Gr}_p W^q H^d_c (X, \mathbb{C})) u^p v^q.$$ 

Additivity means that if $Z \subset X$ is a closed subscheme, then $E(X) = E(X^{\text{red}}) = E(X \setminus Z) + E(Z)$.

Analogously, we define the intersection E-polynomial as

$$IE(X) = \sum_{p,q,d} (-1)^d \dim (\text{Gr}_p W^q IH^d_c (X, \mathbb{C})) u^p v^q.$$
Note however that the intersection $E$-polynomial is not an additive function, due to the fact that in general the restriction to a closed subscheme $Z \subset X$ of $IC_X$ is not isomorphic to $IC_Z$.

2.3. Decomposition theorem. — In this section we recall in brief the statement of the decomposition theorem for semismall maps.

Definition 2.6. — A morphism of algebraic varieties $f : X \to Y$ is semismall if $\dim X \times_Y X \leq \dim X$.

A stratification of $f$ is a collection of finitely many locally closed subsets $Y_k$ such that $f^{-1}(Y_k) \to Y_k$ are topologically locally trivial fibrations. A stratum $Y_k$ is relevant if $2 \dim f^{-1}(Y_k) - \dim Y_k = \dim X$.

Theorem 2.7 (Decomposition theorem for semismall maps). — Let $f : X \to Y$ be a proper algebraic semismall map from a smooth variety $X$. Then there exists a canonical isomorphism

$$Rf_! Q_X^{\dim X} \cong \bigoplus_{Y_k} IC_{Y_k}(R^{\dim X - \dim Y_k} f_! Q_{f^{-1}(Y_k)})$$

where the summation index runs over all the relevant strata of a stratification of $f$.

2.4. Perverse Leray filtration. — Let $\chi : X \to Y$ be a projective morphism of algebraic varieties of relative dimension $r$. Set $r(\chi) := \dim X \times_Y X - \dim X$.

Definition 2.8. — The perverse Leray filtration associated to $\chi$ is the (shifted) perverse filtration on the cohomology of the complex $R\chi_* IC_X$

$$P_k IH^*(X) = P_k H^{*-\dim X - r(\chi)}(Y, R\chi_* IC_X[\dim X - r(\chi)])$$

When $Y$ is affine, de Cataldo and Migliorini provided an equivalent geometric description of the perverse Leray filtration. Assume for simplicity that $\dim X = 2 \dim Y = 2r(\chi)$. Let $\Lambda^k \subset Y$ be a general $k$-dimensional linear section of $Y \subset A^N$.

Theorem 2.9 (Flag filtration [17, Th. 4.1.1])

$$P_k IH^d(X) = \text{Ker} \left( IH^d(X) \to IH^d(\chi^{-1}(\Lambda^{d-k-1})) \right)$$

This means that the class $\eta \in IH^d(X)$ belongs to $P_k IH^d(X)$ if and only if its restriction to $\chi^{-1}(\Lambda^{d-k-1})$ vanishes, i.e., $\eta|_{\chi^{-1}(\Lambda^{d-k-1})} = 0$.

Most remarkably, the perverse Leray filtration satisfies the relative hard Lefschetz theorem.

Theorem 2.10 (Relative hard Lefschetz). — Let $\chi : X \to Y$ be a proper map of algebraic varieties, and let $\alpha \in H^2(X)$ be the first Chern class of a relatively ample line bundle. Then there exists an isomorphism

$$\alpha^1 : Gr_{r-k}^P IH^*(X) \to Gr_{r+k}^P IH^{*-2k}(X).$$
3. Lifting the non-abelian Hodge correspondence

Let $X$ be a compact Riemann surface, and fix a complex reductive algebraic group $G$. The first cohomology group $H^1(X, G)$ comes in various incarnations (cf. [78] and [79]):

- the Betti moduli space $M_B(X, G)$, also named character variety, parametrizing semistable representations of the fundamental group of $X$ with value in $G$;
- the Dolbeault moduli space $M_{Dol}(X, G)$ of semistable principal $G$-Higgs bundles with vanishing Chern classes;
- the De Rham moduli space $M_{DR}(X, G)$ of semistable principal $G$-bundles with an integrable connection.

All these moduli spaces are homeomorphic to each other. The Riemann-Hilbert correspondence yields a complex analytic isomorphism

$$M_{DR}(X, G)^{an} \simeq M_B(X, G)^{an}. \tag{3}$$

There exists an algebraic fibration (real analytically trivializable)

$$\lambda: M_{Hod}(X, G) \to \mathbb{A}^1, \tag{4}$$

whose fibers are moduli spaces of semi-simple principal $G$-bundles with $\lambda$-connections; see [80]. Hence, the fiber over 0 is $M_{Dol}(X, G)$, and the fibers over $\lambda \neq 0$ are isomorphic to $M_{DR}(X, G)$. The space $M_{Hod}(X, G)$ is called Hodge moduli space. In particular, a continuous trivialization $M_{Hod}(X, G)^{top} \simeq M_{Dol}(X, G) \times \mathbb{A}^1$ gives the homeomorphism

$$M_{Dol}(X, G)^{top} \simeq M_{DR}(X, G)^{top}. \tag{5}$$

The non-abelian Hodge correspondence

$$\Psi: M_{Dol}(X, G)^{top} \to M_B(X, G)^{top}$$

is the composition of the maps (3) and (5) for a choice of a preferred real analytic trivialization; see [80] for details.

3.1. Compactification of Hodge moduli spaces

The Hodge moduli space $M_{Hod}(X, G)$ admits a partial compactification, relative to the morphism

$$\lambda: M_{Hod}(X, G) \to \mathbb{A}^1. \tag{6}$$

We obtain it as a $\mathbb{G}_m$-quotient of the total space of the degeneration of $M_{Hod}(X, G)$ to the normal cone of $\lambda^{-1}(0) \simeq M_{Dol}(X, G)$. The construction is an extension to the singular case of [43, Lem.6.1] or [40, Th.7.2.1]. To this end, we shall use the following results by Simpson.

**Proposition 3.1** ([80, Th.11.2]). — Let $Z$ be a variety over the variety $S$, endowed with a $\mathbb{G}_m$-action covering the trivial $\mathbb{G}_m$-action on $S$. Assume that $Z/S$ carries a relatively ample line bundle admitting a $\mathbb{G}_m$-linearization. Assume that the fixed point set $\text{Fix}(Z) \subseteq Z$ is proper over $S$, and that for any $z \in Z$ the limit $\lim_{t \to 0} t \cdot z$ exist.
in $Z$. Let $U \subset Z$ be the subset of points $z$ such that the limit $\lim_{t \to \infty} t \cdot z$ does not exist. Then $U$ is open in $Z$ and there exists a universal geometric quotient $Z/\mathbb{G}_m$.

This quotient is separated and proper over $S$.

**Theorem 3.2** (Partial compactification of the Hodge moduli space). — There exists a projective morphism

$$\overline{\lambda} : \overline{M}_{\text{Hod}}(X,G) \to \mathbb{A}^1$$

which is a relative compactification of the morphism $\lambda$.

**Proof.** — $M_{\text{Hod}}(X,G)$ is endowed with the $\mathbb{G}_m$-action

$$t \cdot (E, \nabla) \mapsto (E, t \nabla)$$

covering the standard $\mathbb{G}_m$-action on $\mathbb{A}^1$, namely $t \cdot \lambda = t\lambda$. Equip $\mathbb{A}^2$ with the $\mathbb{G}_m$-action given by $t \cdot (x, y) = (x, ty)$. The morphism $\mathbb{A}^2 \to \mathbb{A}^1$, given by $(x, y) \mapsto xy$, is $\mathbb{G}_m$-equivariant. Therefore the fiber product $M_{\text{Hod}}(X,G) \times_{\mathbb{A}^1} \mathbb{A}^2$ is equipped with a $\mathbb{G}_m$-action. We summarize the maps constructed in a diagram: note that the subscripts indicate the coordinatization chosen for the affine spaces.

$$\begin{array}{ccc}
M_{\text{Hod}}(X,G) \times_{\mathbb{A}^1} \mathbb{A}^2 & \to & M_{\text{Hod}}(X,G) \\
\downarrow & & \downarrow \lambda \\
\mathbb{A}^2_{x,y} \to \mathbb{A}^1_{\lambda} \\
\downarrow & & \downarrow \\
\mathbb{A}^1_x \to \mathbb{A}^1_y \\
\chi & & (x, y) \mapsto xy \\
\end{array}$$

Choose a $\lambda'$-ample line bundle $L'$ on $M_{\text{Hod}}(X,G) \times_{\mathbb{A}^1} \mathbb{A}^2$ admitting a $\mathbb{G}_m$-linearization (which exists since $M_{\text{Hod}}(X,G) \times_{\mathbb{A}^1} \mathbb{A}^2$ is normal and because of [63, Cor. 1.6]). Let $\chi(X,G) : M_{\text{Dol}}(X,G) \to \mathbb{A}^{\dim M_{\text{Dol}}(X,G)/2}$ be the Hitchin’s proper map for $M_{\text{Dol}}(X,G)$; see [79, p. 22]. The fixed locus is contained in

$$\chi(X,G)^{-1}(0) \circ \{y = 0\} \subset M_{\text{Dol}}(X,G) \times \{y = 0\} \subset M_{\text{Hod}}(X,G) \times_{\mathbb{A}^1} \mathbb{A}^2,$$

so it is proper over $\mathbb{A}^1_x$. By Proposition 3.1, there exists a universal geometric quotient

$$\overline{M}_{\text{Hod}}(X,G) := (M_{\text{Hod}}(X,G) \times_{\mathbb{A}^1} \mathbb{A}^2 \setminus (\chi(X,G)^{-1}(0) \times \mathbb{A}^1_x))/\mathbb{G}_m$$

and a proper morphism $\overline{\chi} : \overline{M}_{\text{Hod}}(X,G) \to \mathbb{A}^1_x$.

$\overline{M}_{\text{Hod}}(X,G)$ contains an open subset isomorphic to $M_{\text{Hod}}(X,G)$, given by the $\mathbb{G}_m$-quotient of

$$(M_{\text{Hod}}(X,G) \times_{\mathbb{A}^1} \mathbb{A}^2 \setminus \mathbb{A}^1_x \cap \{0\}) \simeq M_{\text{Hod}}(X,G) \times \mathbb{G}_m.$$

We show now that the morphism $\overline{\lambda}$ is projective. Let $\partial M_{\text{Hod}} := \overline{M}_{\text{Hod}}(X,G) \setminus M_{\text{Hod}}(X,G)$ be the Cartier boundary divisor. By [24, Th. 2.3] (or [10, Proof of Prop. 3.2.2]), a power of the line bundle $L'$ descends to a line bundle $L$ on $\overline{M}_{\text{Hod}}(X,G)$. We claim that the line bundle $L \otimes \mathcal{O}(m \cdot \partial M_{\text{Hod}})$ is ample for $m \gg 0$.

To this end, observe that $\overline{\lambda}^{-1}(0) := \overline{M}_{\text{Dol}}(X,G)$ coincides with the projective compactification of $M_{\text{Dol}}(X,G)$ constructed in [10, Th. 3.1.1(1)]. Let $\overline{\chi} : \overline{M}_{\text{Dol}}(X,G) \to \overline{\lambda}$

\[\overline{\chi} \begin{array}{ccc}
M_{\text{Dol}}(X,G) & \to & \overline{M}_{\text{Dol}}(X,G) \\
\overline{\chi} & & \\
\overline{\lambda} & \downarrow & \\
\overline{\lambda} & \downarrow & \\
\mathbb{A}^1_x & \to & \mathbb{A}^1_y \\
\end{array} \]
be also the projective compactification of the Hitchin morphism constructed in [10, Th. 3.1.1(2)]. The restriction of \( \mathcal{L} \) to \( \overline{X}^{-1}(0) \) is \( \overline{x} \)-ample by [10, Prop. 3.2.2], while the restriction of \( \partial M_{Hod} \) is the pullback of an ample divisor on \( \overline{X} \). Therefore, the line bundle \( \mathcal{L} \otimes \mathcal{O}(m \cdot \partial M_{Hod}) \) is ample for \( m > 0 \) when restricted to \( \mathcal{M}_{DR}(X, G) \). By the openness of ampleness [55, Th. 1.2.17], it is \( \overline{x} \)-ample in a neighbourhood of \( \overline{X}^{-1}(0) \). But \( \mathcal{M}_{Hod}(X, G) \times \overline{X}^{-1}(0) \) is isomorphic to the trivial product \( \mathcal{M}_{DR}(X, G) \times (\mathbb{A}^1 \setminus \{0\}) \), where the first factor is Simpson’s compactification of \( M_{DR}(X, G) \); see [80, §11]. Therefore, we conclude that \( \mathcal{L} \otimes \mathcal{O}(m \cdot \partial M_{Hod}) \) is \( \overline{x} \)-ample for \( m > 0 \).

Incidentally, note that Theorem 3.2 answers the question about the projectivity of the compactification of the de Rham moduli space risen in [80, p. 268] and [10, Rem. 3.1.2].

**Corollary 3.3.** — *Simpson’s compactification* \( \mathcal{M}_{DR}(X, G) \) *is projective.*

We now study the local geometry of the morphism of \( \overline{X} \).

**Proposition 3.4.** — *The morphism* \( \overline{X} \) *is locally analytically trivial, i.e., for any* \( p \in \mathcal{M}_{Hod}(X, G) \) *over* \( \lambda_p \in \mathbb{A}^1 \) *there exist analytic neighborhoods* \( p \in U_p \subseteq \mathcal{M}_{Hod}(X, G) \) *and* \( p \in V_p \subseteq \mathcal{M}_{Hod}(X, G) \) *such that* \( U_p \cong V_p \times \mathbb{D} \), *with* \( \mathbb{D} \) *a disk in* \( \mathbb{A}^1 \), *and* \( \lambda \) *corresponds to the second projection* \( V_p \times \mathbb{D} \to \mathbb{D} \).

**Proof.** — *The* \( \mathbb{G}_m \)-*action on* \( M_{Hod}(X, G) \) *extends to* \( \mathcal{M}_{Hod}(X, G) \), *and so*

\[
\mathcal{M}_{Hod}(X, G)_{\lambda_p ^{\lambda^{-1}(0)}} \cong \mathcal{M}_{DR}(X, G) \times \mathbb{G}_m;
\]

*see also [80, p. 232]. By [80, Th. 9.1], \( \lambda \) is locally analytically trivial. Therefore, it is enough to show that \( \overline{X} \) is locally analytically trivial at \( p \in \overline{X}^{-1}(0) \) \( \setminus \lambda^{-1}(0) \).

Let \( p' \in \mathcal{M}_{Hod}(X, G) \times _{\mathbb{A}^1} \mathbb{A}^2 \) *be a lift of* \( p \). *Since* \( \lambda \) *is locally analytically trivial, so* \( \lambda' \) *is. Following the proof of [10, Lem. 3.5.1], we can choose a transverse slice to the \( \mathbb{G}_m \)-*orbit though* \( p' \), *locally isomorphic to an affine variety* \( N_{p'} \times \mathbb{A}^1_{\lambda'} \), *such that* \( \mathcal{M}_{Hod}(X, G) \) *is locally isomorphic at* \( p \) *to* \( N_{p'}/\text{Stab}(p') \times \mathbb{A}^1_{\lambda'} \), *and* \( \overline{X} \) *is the projection onto the second factor. As a result, we obtain that* \( \overline{X} \) *is locally analytically trivial.*

In Theorem 3.8 we show that there exists a diffeomorphism \( \Psi \) *which lifts the isomorphism*

\[
\Psi^*: H^*(M_B(X, G)) \rightarrow H^*(\mathcal{M}_{DR}(X, G))
\]

to an isomorphism between the cohomology of the resolution spaces.

To this purpose we recall that for any noetherian quasi-excellent generically reduced scheme \( X \) over Spec(\( \mathbb{Q} \)) there exists a resolution of singularities \( \mathcal{R}(X) \to X \) functorial with respect to regular morphism \( X' \to X \), in the sense that \( \mathcal{R}(X') \) is isomorphic to \( \mathcal{R}(X) \times_X X' \). See [83] for further details and the definition of quasi-excellent schemes and regular morphisms. Here we just mention that by definition, if \( X \) is excellent, then the completion morphism \( \hat{X}_x := \text{Spec} \mathcal{O}_{X, x} \to X \) is regular for any closed point \( x \in X \). In [83, Th. 5.2.2], Temkin showed also that quasi-compact analytic spaces admit functorial resolutions compatible with smooth analytic morphism. The
following lemma is implicit in [83], and it has been kindly communicated to us by Temkin. For clarity, we distinguish the complex algebraic variety $X$ from its complex analytification $X^\text{an}$, but omit the difference elsewhere in the paper.

**Lemma 3.5.** — If $X$ is a complex algebraic variety, then the analytification of the algebraic functorial resolution is biholomorphic to the analytic functorial resolutions of $X^\text{an}$, i.e., $R(X)^\text{an} \simeq R(X^\text{an})$.

**Proof.** — Without loss of generality suppose that $X = \text{Spec}(B)$ is affine. We briefly recall Temkin’s construction of the analytic functorial resolution; see [83, Th.5.2.2]. Take a covering of $X^\text{an} = \bigcup_i X_i$ by Stein compact domains (e.g. embed locally $X^\text{an}$ in a complex affine space and take intersections of $X^\text{an}$ with closed polydiscs). The ring of functions $A_i := \mathcal{O}^\text{an}_{X_i}$ is excellent, and the functorial resolution of $\text{Spec} A_i$ glue to the analytic functorial resolution $R(X^\text{an})$. Since $B$ and $A_i$ are excellent, the completion morphism $B \to \hat{\mathcal{O}}_{X,x}$ and $A_i \to \hat{\mathcal{O}}^\text{an}_{X,x}$ are regular, so the algebraic and the functorial resolutions $R(X)$ and $R(X^\text{an})$ are compatible with completions. Now, since $\hat{\mathcal{O}}_{X,x} \simeq \hat{\mathcal{O}}^\text{an}_{X,x}$, we have $R(\hat{X}_x) \simeq R(\hat{X}_x^\text{an})$. By functoriality, we obtain that

$$R(X) \times_X \hat{X}_x \simeq R(\hat{X}_x) \simeq R(\hat{X}_x^\text{an}) \simeq R(X^\text{an}) \times_X \hat{X}_x$$

for any closed point $x \in X$. Hence, $R(X)^\text{an} \simeq R(X^\text{an})$. □

**Corollary 3.6.** — A biholomorphism $f : X' \to X$ between complex algebraic varieties (not necessarily algebraizable) lifts to a biholomorphism $R(f) : R(X') \to R(X)$ between their functorial resolutions, which gives a fiber product square.

![Fiber Product Square](image)

**Proof.** — By functoriality in the complex analytic category, $R(X) \times_X X'$ is an analytic functorial resolution, so biholomorphic to $R(X)^\text{an}$ by Lemma 3.5. □

**Lemma 3.7.** — Let $X$ be a normal locally $\mathbb{Q}$-factorial complex variety. Suppose that $X$ admits a symplectic resolution $f : Y' \to X$ with an irreducible exceptional divisor, obtained by blowing-up the singular locus. Then $f$ is functorial.

**Proof.** — By [29, Th.2.2], any symplectic resolution of $X$ is isomorphic to $f$. Let $h : X' \to X$ be any smooth morphism. The blow-up $Y' \to X'$ of the singular locus of $X'$ is smooth and symplectic since $h$ is smooth. Then $X'$ satisfies all the hypotheses of Lemma 3.7 with symplectic resolution $Y'' := Y \times_X X'$, so $Y' = Y''$ by [29, Th.2.2], i.e., the resolution is functorial for smooth morphisms, and also for regular morphism following [7, Th.1.2, Cor. 4.6]. □

---

(3) This means that for any closed point $x \in X$ the analytic local ring $\mathcal{O}^\text{an}_{X,x}$ are $\mathbb{Q}$-factorial, that is some multiple of every Weil divisor is Cartier.
Theorem 3.8 (Lift of the non-abelian Hodge correspondence \( \Psi \)). There exist resolutions of singularities \( f_{\text{Dol}} : \widetilde{M}_{\text{Dol}}(X,G) \rightarrow M_{\text{Dol}}(X,G) \) and \( f_{\text{B}} : \widetilde{M}_{\text{B}}(X,G) \rightarrow M_{\text{B}}(X,G) \), and a diffeomorphism
\[
\widetilde{\Psi} : \widetilde{M}_{\text{Dol}}(X,G) \rightarrow \widetilde{M}_{\text{B}}(X,G),
\]
such that the square
\[
\begin{array}{ccc}
\widetilde{M}_{\text{Dol}}(X,G) & \xrightarrow{\widetilde{\Psi}} & \widetilde{M}_{\text{B}}(X,G) \\
f_{\text{Dol}} \downarrow & & \downarrow f_{\text{B}} \\
M_{\text{Dol}}(X,G) & \xrightarrow{\Psi} & M_{\text{B}}(X,G)
\end{array}
\]
commutes up to an isotopy of \( M_{\text{B}}(X,G) \). In particular, the following square in cohomology commutes
\[
\begin{array}{ccc}
H^*(\widetilde{M}_{\text{Dol}}(X,G)) & \xleftarrow{\widetilde{\Psi}^*} & H^*(\widetilde{M}_{\text{B}}(X,G)) \\
f_{\text{Dol}}^* \uparrow & & \uparrow f_{\text{B}}^* \\
H^*(M_{\text{Dol}}(X,G)) & \xleftarrow{\Psi^*} & H^*(M_{\text{B}}(X,G)).
\end{array}
\]
The resolutions \( f_{\text{Dol}} \) and \( f_{\text{B}} \) can be taken functorial with respect to smooth algebraic or analytic morphisms, and symplectic if \( G = \text{GL}_n \) or \( \text{SL}_n \) with \((g,n)=(1,n)\) or \((2,2)\).

\textbf{Proof.} Let \( f_{\text{Hod}} : \mathcal{R}(\overline{M}_{\text{Hod}}(X,G)) \rightarrow \overline{M}_{\text{Hod}}(X,G) \) be the functorial resolution of \( \overline{M}_{\text{Hod}}(X,G) \), equivalently in the analytic or algebraic category by Lemma 3.5. Since \( \overline{X} \) is locally analytically trivial by Proposition 3.4, \( f_{\text{Hod}} \) is a simultaneous resolution of \( \overline{M}_{\text{Hod}}(X,G) \); see for instance [33, Lem. 4.2]. Note also that any vector field on the smooth locus of \( \overline{M}_{\text{Hod}}(X,G) \) can be lifted to a vector field on \( \mathcal{R}(\overline{M}_{\text{Hod}}(X,G)) \) by [34, Cor. 4.7].

For such a resolution, Proposition 5.2 in [1] holds: the family \( \overline{X} \circ f_{\text{Hod}} \) admits a real analytic Ehresmann connection such that the corresponding flow of diffeomorphisms preserves the exceptional locus of \( f_{\text{Hod}} \), and moreover it does so fiberwise over its image in \( M_{\text{Hod}}(X,G) \). The same proof as that of [1, Prop. 5.2] shows that we can further suppose that the flow preserves \( \partial M_{\text{Hod}}(X,G) \cong \partial M_{\text{Dol}}(X,G) \times \mathbb{A}^1 \) and its inverse image in \( \mathcal{R}(\overline{M}_{\text{Hod}}(X,G)) \). Hence, there exists a resolution of singularities
\[
\overline{M}_{\text{Hod}}(X,G) := f_{\text{Hod}}^{-1}(M_{\text{Hod}}(X,G))
\]
of \( M_{\text{Hod}}(X,G) \) such that the following square commutes
\[
\begin{array}{ccc}
\overline{M}_{\text{Dol}}(X,G) := \overline{M}_{\text{Hod}}(X,G)_{0} & \rightarrow & \overline{M}_{\text{Hod}}(X,G)_{\varepsilon} =: \overline{M}_{\text{DR}}(X,G) \\
f_{\text{Dol}} := f_{\text{Hod},0} \downarrow & & \downarrow f_{\text{DR}} =: f_{\text{Hod},\varepsilon} \\
M_{\text{Dol}}(X,G) := M_{\text{Hod}}(X,G)_{0} & \rightarrow & M_{\text{Hod}}(X,G)_{\varepsilon} =: M_{\text{DR}}(X,G),
\end{array}
\]
where the horizontal arrows are stratified diffeomorphisms, and \( \varepsilon \neq 0 \).

\[\text{J.E.P.} - \text{M., 2022, tome 9}\]
Since the Riemann-Hilbert correspondence is a smooth analytic map, the map $f_{\text{DR}}$ is obtained via base change from the functorial resolution $f_B: \tilde{M}_B(X,G) := \mathcal{R}(M_B(X,G)) \to M_B(X,G)$ by functoriality. Therefore, we obtain the commutative square

$$
\begin{array}{ccc}
\tilde{M}_{\text{Dol}}(X,G) & \xrightarrow{\Psi} & \tilde{M}_B(X,G) \\
\downarrow f_{\text{Dol}} & & \downarrow f_B \\
M_{\text{Dol}}(X,G) & \xrightarrow{\Psi'} & M_B(X,G)
\end{array}
$$

Since $\Psi'$ and the non-abelian Hodge correspondence $\Psi$ are induced by trivialization of $M_{\text{Hod}}(X,G)$, the square (6) commutes up to a stratified isotopy of $M_{\text{Hod}}(X,G)$. Since stratified isotopy are trivial in cohomology, the square (7) commutes too.

We now show that the functorial resolutions $f_{\text{Dol}}$ and $f_B$ are symplectic if $G = \text{GL}_n$ or $\text{SL}_n$, with $(g,n) = (1,n)$ or $(2,2)$. Indeed, in this case $M_{\text{Dol}}(X,G)$ and $M_B(X,G)$ are normal complex varieties which admit a symplectic resolution obtained by blowing-up the singular locus; see Sections 5.1 and 6.1.1, or [5, Th.1.8]. Note that the results of [5] are stated for $M_B(X,G)$, but they extend to $M_{\text{Dol}}(X,G)$ by the isosingularity principle, see [79, Th.10.6] or [59, §2.4 and the first paragraph of §3.2]. Further, the analytic neighborhoods of the singularities of these varieties are $\mathbb{Q}$-factorial. Indeed, the singularities of $M_{\text{Dol}}(X,G)$ and $M_B(X,G)$ are either quotient singularities or the nilpotent cone in $\mathfrak{sp}(4)$, which is a cone over a projective variety with quotient singularities and Picard number one; see the last paragraph of the proof of [5, Th.1.3] and references therein, and [62, Lem.1.3] or [59, §3.4]. By [53, Prop.5.15] and [52, Prop.7.4] these singularities are analytically $\mathbb{Q}$-factorial. Hence, the last statement of Theorem 3.8 follows from Lemma 3.7.

\begin{remark}
In this paper, functorial resolutions are used only for the following purposes: to lift vector fields and group actions to resolutions, and for the compatibility with respect to the Riemann–Hilbert correspondence; see proof of Theorem 3.8 and Section 4.1. If $G = \text{GL}_n$ or $\text{SL}_n$ with $(g,n) = (1,n)$ or $(2,2)$, the symplectic resolutions of $M_{\text{Dol}}(X,G)$ and $M_B(X,G)$ are indeed functorial by Lemma 3.7 but these properties can be shown more directly. The resolutions are obtained by blowing up the singular locus, which is invariant with respect to any group action on the varieties and preserved by the Riemann–Hilbert correspondence. Further, the liftability of vector fields follows easily for instance from [1, Lem.5.3].
\end{remark}

\section{Moduli spaces for $\text{GL}_n$ vs $\text{SL}_n$}

Let $\Gamma := \text{Pic}^0(X)[n] \simeq \mathbb{Z}/n\mathbb{Z}^{2g}$ be the group of $n$-torsion line bundles on the Riemann surface $X$ of genus $g$ and canonical line bundle $K_X$. We review the relation between the moduli spaces $M_{\text{Dol}}(X,G)$ and $M_B(X,G)$ for $G = \text{GL}_n$ and $\text{SL}_n$; see also [46, 79, 78].

\textit{J.É.P. – M., 2022, tome 9}
Recall that $M_{\text{Dol}}(X, \text{GL}_n)$ parametrizes semistable Higgs bundles $(E, \phi)$, where $E$ is a vector bundle on $X$ of rank $n$ and degree $0$, and $\phi \in \text{Hom}(E, E \otimes K_X)$.

The fiber of the isotrivial morphism

$$\text{alb}: M_{\text{Dol}}(X, \text{GL}_n) \to \text{Pic}^0(X) \times H^0(X, K_X)$$

(8)

$$(E, \phi) \mapsto (\det E, \text{tr} \phi)$$

is isomorphic to $M_{\text{Dol}}(X, \text{SL}_n)$. In particular, the monodromy of alb is the group $\Gamma$. Indeed, the étale cover

$$M_{\text{Dol}}(X, \text{SL}_n) \times \text{Pic}^0(X) \times H^0(X, K_X) \to M_{\text{Dol}}(X, \text{GL}_n)$$

(9)

$$(E, \phi), L, s \mapsto (E \otimes L, \phi + (s/n)\text{id}_E)$$

has Galois group $\Gamma$, which acts on the domain diagonally by tensorisation

$$\Gamma \times M_{\text{Dol}}(X, \text{SL}_n) \times \text{Pic}^0(X) \times H^0(K) \to M_{\text{Dol}}(X, \text{SL}_n) \times \text{Pic}^0(X) \times H^0(K)$$

$$(L, \gamma, (E, \phi), L, s) \mapsto (L, \gamma, (E \otimes L, \phi), L \otimes L^{-1}, s).$$

Therefore, when we take cohomology, we obtain

$$H^*(M_{\text{Dol}}(X, \text{GL}_n)) \simeq H^*(M_{\text{Dol}}(X, \text{SL}_n) \times \text{Pic}^0(X) \times H^0(X, K_X))^\Gamma$$

(10)

$$\simeq H^*(M_{\text{Dol}}(X, \text{SL}_n))^\Gamma \otimes H^*(\text{Pic}^0(X)),$$

where the former equality follows from an observation of Grothendieck in [36], and the latter from the fact that $\Gamma$ acts trivially on $H^*(\text{Pic}^0(X))$, since it is a restriction to a subgroup of the action of the connected group $\text{Pic}^0(X)$.

The *Hitchin map*

$$\chi(X, \text{GL}_n): M_{\text{Dol}}(X, \text{GL}_n) \to \bigoplus_{i=1}^n H^0(X, K_X^{\otimes i})$$

is a projective fibration sending $(E, \phi)$ to the characteristic polynomial of $\phi$. It is Lagrangian with respect to $\omega$, i.e., the holomorphic symplectic form of the canonical hyperkähler metric on the smooth locus of $M_{\text{Dol}}(X, \text{GL}_n)$; see [46, §6]. The map $\chi(X, \text{GL}_n)$ restricts on $M_{\text{Dol}}(X, \text{SL}_n)$ to

$$\chi(X, \text{SL}_n): M_{\text{Dol}}(X, \text{SL}_n) \to \bigoplus_{i=2}^n H^0(X, K_X^{\otimes i}).$$

The map $\chi(X, \text{SL}_n)$ is $\Gamma$-equivariant, covering the trivial $\Gamma$-action of the codomain. In particular, there exists a commutative diagram

$$M_{\text{Dol}}(X, \text{SL}_n) \times \text{Pic}^0(X) \times H^0(K_X) \to M_{\text{Dol}}(X, \text{GL}_n)$$

(11)

$$\bigoplus_{i=2}^n H^0(X, K_X^{\otimes i}) \times H^0(X, K_X) \to \bigoplus_{i=1}^n H^0(X, K_X^{\otimes i})$$

with $S_{\text{Pic}^0(X)}: \text{Pic}^0(X) \to \text{pt}$.

Via the non-abelian Hodge correspondence $\Psi$, the action of $\Gamma$ on $M_{\text{Dol}}(X, \text{SL}_n)$ corresponds to the algebraic action of the group of characters $\text{Hom}(\pi_1(C), \mathbb{Z}/n\mathbb{Z})$, which
acts on $M_B(X, SL_n)$ by multiplication (changing the signs of the matrices $A_j, B_j$ as in (1)).

The multiplication map $SL_n \times G_m \to GL_n$ induces the étale cover

$$M_B(X, SL_n) \times (\mathbb{C}^*)^{2g} \to M_{Dol}(X, GL_n)$$

with Galois group $\Gamma$. Therefore, the analogue of (10) holds

$$H^*(M_B(X, GL_n)) \simeq H^*(M_B(X, SL_n) \times (\mathbb{C}^*)^{2g})^\Gamma$$

$$\simeq H^*(M_{Dol}(X, SL_n))^\Gamma \otimes H^*((\mathbb{C}^*)^{2g}).$$

4.1. P=W for $SL_n$ implies P=W for $GL_n$. — In this section we show that the P=W conjectures for $SL_n$ imply the corresponding statements for $GL_n$. In the twisted case, this is proved in [11, §2.4]; see also [14, §1]. In view of Theorem 4.1, starting from Section 5, we will focus our attention on the $SL_n$ case exclusively.

Fix $\Gamma$-equivariant resolutions of singularities

$$f_{Dol}(X, SL_n): \tilde{M}_{Dol}(X, SL_n) \to M_{Dol}(X, SL_n),$$

$$f_B(X, SL_n): \tilde{M}_B(X, SL_n) \to M_B(X, SL_n),$$

which satisfy Theorem 3.8. Note that the functorial resolutions in the proof of Theorem 3.8 are actually $(\Gamma \times G_m)$-equivariant; see [51, Prop. 3.9.1]. By the isotriviality of

$$alb_{Hod}: M_{Hod}(X, GL_n) \to M_{Hod}(X, G_m)$$

$$(E, \nabla_\lambda) \mapsto (\det E, tr \nabla_\lambda)$$

(which extends the morphism $alb$ defined in (8)), the resolutions $f_{Dol}(X, SL_n)$ and $f_B(X, SL_n)$ extend to resolutions

$$f_{Dol}(X, GL_n): \tilde{M}_{Dol}(X, GL_n) \to M_{Dol}(X, GL_n),$$

$$f_B(X, GL_n): \tilde{M}_B(X, GL_n) \to M_B(X, GL_n),$$

such that the square

$$\begin{array}{ccc}
\tilde{M}_{Dol}(X, SL_n) \times T^* Pic^0(X) & \xrightarrow{\tilde{\Psi}(X, SL_n) \times \tilde{\Psi}(X, G_m)} & \tilde{M}_B(X, SL_n) \times (\mathbb{C}^*)^{2g} \\
/\Gamma & & /\Gamma \\
\tilde{M}_{Dol}(X, GL_n) & \xrightarrow{\tilde{\Psi}(X, GL_n)} & \tilde{M}_B(X, GL_n)
\end{array}$$

(13)

and the diagrams in Theorem 3.8 commute.

**Theorem 4.1. — In the notation above, if the P=W conjecture for the resolution $f_{Dol}(X, SL_n)$ holds, then it holds for $f_{Dol}(X, GL_n)$.**

**Proof. —** Cohomologically, the Hitchin fibration

$$\chi(X, GL_n) \circ f_{Dol}(X, GL_n): \tilde{M}_{Dol}(X, GL_n) \to \bigoplus_{i=1}^{n} H^0(X, K_X^i)$$


behaves like the product of the fibration $\chi(X, \text{SL}_n) \circ \text{f}_\text{Dol}(X, \text{SL}_n)$ and $S_{\text{Pic}^0(X)}$: Pic$^0(X) \to \text{pt}$, by lifting (11) to the resolution. Hence, the perverse filtration associated to $\chi(X, \text{GL}_n) \circ \text{f}_\text{Dol}(X, \text{GL}_n)$ is the convolution of the $\Gamma$-invariant part of the perverse filtrations associated to $\chi(X, \text{SL}_n) \circ \text{f}_\text{Dol}(X, \text{SL}_n)$ and $S_{\text{Pic}^0(X)}$ (the latter being trivial); compare with [11, §2.4]. In symbols, we write

$$P_k H^d(\widetilde{M}_\text{Dol}(X, \text{GL}_n)) \simeq \bigoplus_{j \geq 0} P_{k-j} H^{d-j}(\widetilde{M}_\text{Dol}(X, \text{SL}_n))^\Gamma \otimes H^j(\text{Pic}^0(C)).$$

By the $\Gamma$-equivariance of $\text{f}_B(X, \text{SL}_n)$, the map

$$\text{M}_B(X, \text{SL}_n) \times (\mathbb{C}^*)^{2g} \to \text{M}_\text{Dol}(X, \text{GL}_n)$$

lifts to the resolutions, and so there exists an isomorphism of mixed Hodge structures

$$H^*(\text{M}_B(X, \text{GL}_n)) \simeq H^*(\text{M}_B(X, \text{SL}_n))^\Gamma \otimes H^*((\mathbb{C}^*)^{2g}),$$

as in (12). Explicitly, we write

$$W_k H^d(\widetilde{M}_B(X, \text{GL}_n)) \simeq \bigoplus_{j \geq 0} W_{k-2j} H^{d-j}(\widetilde{M}_B(X, \text{SL}_n))^\Gamma \otimes H^j((\mathbb{C}^*)^{2g}),$$

since $H^j((\mathbb{C}^*)^{2g})$ has weight $2j$.

Assume now that

$$P_k H^*(\widetilde{M}_\text{Dol}(X, \text{SL}_n)) = \widetilde{\Psi}(X, \text{SL}_n)^* W_{2k} H^*(\widetilde{M}_B(X, \text{SL}_n)).$$

Then by the commutativity of (13), together with (14) and (16), we conclude that

$$P_k H^*(\widetilde{M}_\text{Dol}(X, \text{GL}_n)) = \widetilde{\Psi}(X, \text{GL}_n)^* W_{2k} H^*(\widetilde{M}_B(X, \text{GL}_n)). \quad \square$$

Remark 4.2. — With obvious change, the analogues of Theorem 4.1 for the PI=W conjectures hold.

Remark 4.3. — Since $M_{\text{Dol}}(X, \text{PGL}_n)$ is the quotient of $M_{\text{Dol}}(X, \text{SL}_n)$ by the $\Gamma$-action, the PI=W conjecture for $M_{\text{Dol}}(X, \text{SL}_n)$ (or $M_{\text{Dol}}(X, \text{GL}_n)$) implies the PI=W conjecture for $M_{\text{Dol}}(X, \text{PGL}_n)$.

5. P=W conjectures for genus 1

Let $A$ be a compact Riemann surface of genus 1. The construction of the moduli spaces $M_{\text{Dol}}(A, \text{SL}_n)$ and $M_B(A, \text{SL}_n)$ agrees formally with that of a generalized Kummer variety in [2, §7]. It is possible to make this analogy more precise by showing that $M_{\text{Dol}}(A, \text{SL}_n)$ and $M_B(A, \text{SL}_n)$ are specializations of generalized Kummer varieties; see Example 5.1 and also [60, §5.3].

Following [32], we describe a stratification of these Kummer-like varieties in Section 5.1, from which we deduce the P=W conjecture in genus 1 (Theorem 5.3).
5.1. Kummer-like varieties. — Let \(X\) be a complex algebraic group of dimension 2. We denote by \(X^{[n]}\) and \(X^{[n]}\) the \(n\)-fold symmetric product of \(X\) and the Hilbert scheme of \(n\)-points on \(X\); see [2, §6] for an overview of their construction. Recall that the Hilbert-Chow morphism \(f : X^{[n]} \to X^{(n)}\) is a desingularization of \(X^{(n)}\).

Consider the addition map \(a_n : X^{(n)} \to X\), given by \(a_n(x_1, \ldots, x_n) = \sum_{i=1}^n x_i\). For any \(g \in \mathbb{Z}_{\geq 0}\), denote by \(X(g)\) the set of \(g\)-torsion points in \(X\). Let \(P(n)\) be the set of partitions of \(n\). We write \(\alpha \in P(n)\) as \(n = \alpha_1 \cdot 1 + \cdots + \alpha_\ell \cdot \ell\), and put \(|\alpha| = \sum \alpha_i\) and \(g(\alpha) := \gcd\{\nu \mid \alpha_\nu \neq 0\}\).

Following [32], we describe a stratification of the fiber of \(a_n\).

- The variety \(K^{[n]}\) is the fiber \(f^{-1} a_n^{-1}(0)\) of the composition \(X^{[n]} \overset{f}{\to} X^{(n)} \overset{a_n}{\to} X\).

When necessary, we emphasize the dependence on \(X\) by writing \(K^{[n]}(X)\).

- The fiber \(K^{(n)} := a_n^{-1}(0)\) can be described as the set of maps from \(X\) to \(\mathbb{Z}_{\geq 0}\) of total sum \(n\)

\[
K^{(n)} = \{ h \in \text{Hom}_{\text{sets}}(X, \mathbb{Z}_{\geq 0}) \mid \sum_{x \in X} h(x) = n \}.
\]

We say that \(K^{(n)}\) is Kummer-like.

- There exists a stratification

\[
K^{(n)} = \bigsqcup_{\alpha \in P(n)} K^{(n)}_{\alpha},
\]

with \(K^{(n)}_{\alpha} = \{ h \in K^{(n)} \mid \# h^{-1}(x) = \alpha_\nu, \forall \nu \}\).

- The normalization of the closure of the stratum \(K^{(n)}_{\alpha}\) in \(K^{(n)}\), denoted \(\overline{K^{(n)}_{\alpha}}\), is the disjoint union

\[
K^{(n)} = \bigsqcup_{y \in X(\overline{\mu(\alpha)})} \overline{K^{(n)}_{\alpha}}(y),
\]

where \(K^{(n)}_{\alpha} = \{ h = (h_1, \ldots, h_\ell) \in K^{(n)} \mid \sum_{x \in X} (\nu/g(\alpha)) h_{\nu}(x) \cdot x = y \}\).

- Let \(\tau_z : X \to X\) be the translation by \(z \in X\). The finite map \(g_{\alpha}^{(n)} : X \times K^{(n)}_{\alpha} \to X^{(n)}\), given by \(g_{\alpha}^{(n)}(z, h_1, \ldots, h_\ell) = (h_1 \circ \tau_z, \ldots, h_\ell \circ \tau_z)\), induces the isomorphism of mixed Hodge structures \(H^*(X \times K^{(n)}_{\alpha}) \cong H^*(X^{(n)});\) see [32, p. 243].

All these facts implies the following theorem due to Göttsche and Soergel, that we state without proof.

**Theorem 5.1 ([32, Th. 7]).** — Denote by \(f_0\) the birational map \(f_0 \colonequals (\text{id}_X, f|_{K^{[n]}}) : X \times K^{[n]} \to X \times K^{(n)}\). Let \(\kappa_y^{(n)} : K^{(n)}_{\alpha} \to K^{(n)}\) be the composition

\[
K^{(n)}_{\alpha} \hookrightarrow K^{(n)} \twoheadrightarrow \overline{K^{(n)}_{\alpha}} \quad \hookrightarrow \quad K^{(n)}.
\]

Then there exists a distinguished splitting isomorphism

\[
(f_0)_*(-\mathbb{Q}_{X \times K^{[n]}(\overline{\mu(\alpha)})}) \cong \bigoplus_{\alpha \in P(n)} \bigoplus_{y \in X(\overline{\mu(\alpha)})} \left( \text{id}_X \times \kappa_y^{(n)} \right)_* \left( \mathbb{Q}_{X \times K^{(n)}_{\alpha}}(\overline{\mu(\alpha)}) \right).
\]

The splitting induces a canonical isomorphism of mixed Hodge structures (recall that a Tate twist \((-k)\) increases the weights by \(2k\)):

\[
H^{d+2|\alpha|}(X \times K^{[n]}(\overline{\mu(\alpha)})) \cong \bigoplus_{\alpha \in P(n)} \bigoplus_{y \in X(\overline{\mu(\alpha)})} H^{d+2|\alpha|}(X^{(n)})(\overline{\mu(\alpha)}).
\]
A morphism $\chi : X \to \mathbb{C}$ yields the commutative diagram
\[
X \times K(n) \quad \xymatrix{ & X \times K_y(n) \ar[r]^{q_y(n)} & X(n) \\
\chi_0 \downarrow & \id_X \times \kappa_y(n) \downarrow^{\chi_y(n)} & \id_X \times \kappa_y(n-1) \downarrow \chi_y(n) & \chi(n) \downarrow^{\chi(n)} \ar[l]_{\kappa_y(n-1)} & \mathbb{C} \times \mathbb{C}^n(n-1) \ar[l]_{\id_X \times \kappa_y(n-1)} & \mathbb{C} \times \mathbb{C}^n(n-1) \ar[l]_{\id_X \times \kappa_y(n-1)} & \mathbb{C} \times \mathbb{C}^n(n-1) \ar[l]_{\id_X \times \kappa_y(n-1)}}
\]

The perverse filtration associated with $\chi_0 := \id_X \times \chi(n)|_{K(n)}$ can be written in terms of the perverse filtration associated with $\chi(n)$.

**Theorem 5.2.** — **The perverse filtration associated with** $\chi_0$ **can be expressed as**
\[
P_k H^{d+2n}(X \times K(n)) \simeq \bigoplus_{\alpha \in P(n)} \bigoplus_{y \in X(g(n))} P_k H^{d+2\alpha}(X(n))(\alpha),
\]
**where** $P_k H^*(X(n))$ **is the perverse filtration associated with** $\chi(n)$.

**Proof.** — By Theorem 5.1 and the $t$-exactness of finite morphisms, we obtain
\[
\mathcal{P}_k((\chi_0 \circ f)_* Q_{X \times K(n)}[n]) \simeq \bigoplus_{\alpha \in P(n)} \bigoplus_{y \in X(g(n))} \mathcal{P}_k((\id_X \times \kappa_y(n))((\chi_y(n)) Q_{X \times K_y(n)}[n])]
\]
\[
\bigoplus_{\alpha \in P(n)} \bigoplus_{y \in X(g(n))} \mathcal{P}_k((\id_X \times \kappa_y(n))((\chi_y(n)) Q_{X \times K_y(n)}[n])]
\]

This means that the isomorphism (18) is filtered strict with respect to the perverse filtration associated with $\chi_0$ and $\chi(n)$.

**5.2. The proof of the conjecture**

**Theorem 5.3.** — **The PI=W conjectures for** $M_{\text{Dol}}(A, \text{SL}_n)$ **and the P=W conjectures for its symplectic resolutions hold.**

**Proof.** — The moduli space $M_{\text{Dol}}(A, \text{SL}_n)$ parametrizes semistable Higgs bundles on the elliptic curve $A$, and it is isomorphic to $K(n)(A \times \mathbb{C})$; see for instance [28, Th. 4.27(v)], which actually holds for any $n$, not only for $n \geq 4$, or [35]. The character variety $M_B(A, \text{SL}_n)$ instead is isomorphic to $K(n)(\mathbb{C}^* \times \mathbb{C}^*)$ (cf. [60, Proof of Th. 5.3.2]), and in suitable coordinates the non-abelian Hodge correspondence is induced by the symmetric product of the exponential map
\[
A \times \mathbb{C} \longrightarrow \mathbb{C} \times \mathbb{C}^*
\]
\[
\quad (\theta_1, \theta_2, r_1, r_2) \quad \longrightarrow \quad (\exp(-2r_1 + i\theta_1), \exp(2r_2 + i\theta_2));
\]
see [77, Ex. after Prop. 1.5]. By Theorem 5.1 and 5.2, the P=W conjecture for the symplectic resolution $K(n)(A \times \mathbb{C})$ is equivalent to
\[
P_k H^*((A \times \mathbb{C})(\alpha)) = W_{2k} H^*((A \times \mathbb{C})(\alpha))
\]
for any partition $\alpha \in P(n)$. The identity (19) has already been proved in [12, Lem. 3.1.1 & 3.2.2].

**Remark 5.4.** — Since $M_{\text{Dol}}(A, \text{SL}_n)$ has at worst quotient singularities, the P=W conjecture for $M_{\text{Dol}}(A, \text{SL}_n)$ is equivalent to the PI=W conjecture for $M_{\text{Dol}}(A, \text{SL}_n)$.

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6. The moduli space of Higgs bundles $M$ and its alterations

Here and in the following $C$ is a compact Riemann surface of genus 2. We denote by $\iota : C \to C$ the hyperelliptic involution, and by $K_C$ the canonical bundle of $C$.

For the sake of notational simplicity, we denote

- the Dolbeault moduli space $\check{M}_{\text{Dol}}(C, \text{SL}_2)$ simply by $\check{M}$;
- the desingularization $\check{M}_{\text{Dol}}(C, \text{SL}_2)$ in Proposition 6.1 by $\check{M}$;
- the character variety $\check{M}_B(C, \text{SL}_2)$ by $\check{M}_B$;
- the resolution $f_{\text{Dol}}(C, \text{SL}_2)$ by $f : \check{M} \to M$;
- the Hitchin map $\chi(C, \text{SL}_2)$ by $\chi : M \to H^0(C, K_C \otimes ^2)$.

6.1. Symplectic resolution of $M$

6.1.1. Singularities of $M$ and its resolution. — We briefly recall the description of the singular locus of $M$ and the construction of the resolution. A key aspect is the local isomorphism between the singularities of $M$ and those of the celebrated O’Grady six dimensional example of irreducible holomorphic symplectic variety. We refer to [26] for more details. Via the non-abelian Hodge correspondence, we obtain an analogous description of the singularities of $M_B$.

There exists a Whitney stratification of $M$

$$\Omega = \text{Sing}(\Sigma) \subset \Sigma = \text{Sing}(M) \subset M,$$

where

$$\Sigma \simeq \{(E, \phi) \simeq (L, \varphi) \oplus (L^{-1}, -\varphi) \mid L \in \text{Pic}^0(C), \text{ and } \varphi \in H^0(C, K_C)\},$$

$$\Omega \simeq \{(E, \phi) \simeq (L, 0) \oplus (L, 0) \mid L \in \text{Pic}^0(C) \text{ s.t. } L^2 \simeq \mathcal{O}_C\}.$$  

Note that $\Sigma$ is isomorphic to the quotient of $\text{Pic}^0(C) \times H^0(C, K_C)$ by the involution $(L, \varphi) \mapsto (L^{-1}, -\varphi)$, hence it has dimension 4. The locus $\Omega$ instead is the branch locus of the quotient map $\text{Pic}^0(C) \times H^0(C, K_C) \to \Sigma$, and consists of 16 points $\Omega_j$, with $j = 1, \ldots, 16$.

A transverse slice to $\Sigma$ at a point in $\Sigma \smallsetminus \Omega$ has a quotient surface singularity of type $A_1$. An analytic neighbourhood of a point of $\Omega$ is more complicated, and it was described in detail in [56]. The singularities are symplectic, and a symplectic resolution can be constructed simply by blowing-up $M$ along $\Sigma$.

Proposition 6.1 ([26, Prop. 4.2]). — Let $f : \check{M} \to M$ be the blow-up of $M$ along $\Sigma$. Then $f$ is a symplectic resolution, and we have that:

- $f$ is an isomorphism over $M \smallsetminus \Sigma$;
- $f^{-1}(p) \simeq \mathbb{P}^1$ for all $p \in \Sigma \smallsetminus \Omega$;
- $f^{-1}(\Omega_j) \simeq \tilde{\Omega}_j$, where $\tilde{\Omega}_j$ is the Grassmannian of Lagrangian planes in a symplectic 4-dimensional vector space, which is isomorphic to a smooth quadric in $\mathbb{P}^4$. 

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Via the non-abelian Hodge correspondence $\Psi$, the stratification of $M$ in (20) induces the stratification of $M_B$ given by

\begin{equation}
\Omega_B = \text{Sing}(\Sigma_B) = \Psi(\Omega) \subset \Sigma_B = \text{Sing}(M_B) = \Psi(\Sigma) \subset M_B,
\end{equation}

where

\begin{equation}
\Sigma_B := \{(A_1, A_2, B_1, B_2) \in (\mathbb{C}^*)^4 \subset \text{SL}_2^4 \} \parallel \text{SL}_2 \simeq (\mathbb{C}^*)^4/(\mathbb{Z}/2\mathbb{Z});
\end{equation}

\begin{equation}
\Omega_B := \{(A_1, A_2, B_1, B_2) \in (\pm \text{id})^4 \subset \text{SL}_2^4 \} = \bigcup_{j=1}^{16} \Omega_{B,j}.
\end{equation}

By Theorem 1.4, $M_B$ admits a symplectic resolution, and its fibers can be described as in Proposition 6.1.

6.1.2. Attracting and repelling sets

Definition 6.2. — Let $X$ be a complex variety with a $\mathbb{G}_m$-action, and $F$ be a subset of its fixed locus. We denote by

\[ \text{Attr}(F) = \{ x \in X \mid \lim_{\lambda \to 0} \lambda \cdot x \in F \} \]

the attracting set of $F$, and by

\[ \text{Repell}(F) = \{ x \in X \mid \lim_{\lambda \to \infty} \lambda \cdot x \in F \} \]

the repelling set of $F$.

The tangent space of any fixed point $p \in \text{Fix}(X)$ decomposes into the direct sum of weights spaces

\[ T_p X = \bigoplus_{m \in \mathbb{Z}} T_p X_m, \]

where $T_p X_m = \{ v \in T_p X \mid \lambda \cdot v = \lambda^m v \text{ for all } \lambda \in \mathbb{G}_m \}$.

Definition 6.3. — The sequences of integers $m_1, m_2, \ldots$ such that $\lambda^{m_1}, \lambda^{m_2}, \ldots$ are eigenvalues of the linear operator induced by the $\mathbb{G}_m$-action on $T_p X$ are called weights of the $\mathbb{G}_m$-action at the fixed point $p$.

Let $X^{\text{sm}}$ be the smooth locus of $X$, and denote a connected component of the fixed locus $\text{Fix}(X^{\text{sm}})$ simply by $F$. Note that the function of weights

\[ \text{Fix}(X^{\text{sm}}) \longrightarrow \mathbb{Z}^{(\dim X)} \]

\[ p \mapsto (m_1(p), m_2(p), \ldots) \]

is locally constant.

In particular, the following identities hold:

\begin{align*}
T_p \text{Attr}(p) &= \bigoplus_{m>0} T_p X_m, & T_p \text{Repell}(p) &= \bigoplus_{m<0} T_p X_m, \\
T_p \text{Attr}(F) &= \bigoplus_{m \geq 0} T_p X_m, & T_p \text{Repell}(F) &= \bigoplus_{m \leq 0} T_p X_m.
\end{align*}
6.1.3. Białynicki–Birula decomposition. — We briefly recall the celebrated Białynicki–Birula decomposition.

**Definition 6.4 ([42, Def. 1.1.1]).** A semiprojective variety is a complex quasi-projective algebraic variety $X$ with a $\mathbb{G}_m$-action such that:
- the fixed point set $\text{Fix}(X)$ is proper;
- for every $x \in X$ the limit $\lim_{\lambda \to 0} \lambda \cdot x$ exists.

**Theorem 6.5 (Białynicki–Birula decomposition).** Let $X$ be a normal semiprojective variety. Then the following facts hold:

1. $X$ admits a decomposition into $\mathbb{G}_m$-invariant locally closed subsets $X = \bigsqcup_{F \in \pi_0(\text{Fix}(X))} \text{Attr}(F)$;
2. the limit map $\text{Attr}(F) \to F : x \mapsto \lim_{x \to 0} \lambda \cdot x$ is an algebraic map, and it is an affine bundle if $F \subset X^{\text{sm}}$;
3. the connected components of the fixed locus $\text{Fix}(X^{\text{sm}})$ are smooth.

**Proof.** See [6, Th. 4.3] in the smooth projective case; [42, §1.2] and [58, Lem. 3.2.4] in the smooth semiprojective case; [85, Cor. 4] in the normal complete case. 

The cohomology of a semiprojective variety can be expressed in terms of the cohomology of the components of the fixed locus.

**Theorem 6.6 (Local-to-global spectral sequence, [86, §4.4]).** Let $X$ be a normal semiprojective variety. Fix an ordering $F_0, F_1, \ldots$ of the connected components of $\text{Fix}(X)$ such that if $F_i < F_j$ then $\dim \text{Attr} F_i \geq \dim \text{Attr} F_j$. Then the following facts hold.

1. The Białynicki–Birula decomposition yields the spectral sequence
   \[ E^{i,j}_1 = H^{i+j}(\text{Attr}(F_i), u_i^!\mathbb{Q}_X) \Rightarrow H^{i+j}(X, \mathbb{Q}), \]
   where $u_i : \text{Attr}(F_i) \to X$ is the inclusion.
2. If $X$ is smooth and $\text{Attr}(F_i)$ are smooth subvarieties of codimension $c_j$, then we can rewrite the spectral sequence (25) as
   \[ E^{i,j}_1 = H^{i+j-2c_j}(F_i, \mathbb{Q}) \Rightarrow H^{i+j}(X, \mathbb{Q}) \]
   - The spectral sequence (26) degenerates at the first page, and the Poincaré polynomial $P_t(X) := \sum_{k=0}^{\dim X} (-1)^n \dim H^k(X, \mathbb{Q})$ can be written
   \[ P_t(X) = \sum P_t(F_i)t^{2c_i}. \]
6.1.4. Torus action on $M$ and $\tilde{M}$. — The multiplicative group $\mathbb{G}_m$ acts on $M$ by rescaling the Higgs field

$$\lambda \cdot (E, \phi) = (E, \lambda \phi).$$

The Hitchin map $\chi: M \to H^0(C, K_C^{\otimes 2})$ is $\mathbb{G}_m$-equivariant, where $\mathbb{G}_m$ acts linearly on $H^0(C, K_C^{\otimes 2})$ with weight $(2, 2, 2)$. In particular, the fixed locus of $M$ is contained in the nilpotent cone $\chi^{-1}(0)$. Therefore, $M$ is semiprojective. Since the singular locus $\Sigma$ of $M$ is $\mathbb{G}_m$-invariant, the action lifts to $\tilde{M}$, and $\tilde{M}$ is semiprojective as well.

The goal of this section is to describe the fixed locus of the $\mathbb{G}_m$-action on $M$, $\Omega_j$ and $\tilde{M}$, and to compute the weights of the action.

**Proposition 6.7 ([46, Ex.3.13]).** — A vector bundle $E$ underlying a semistable Higgs bundle $(E, \phi) \in M$ satisfies one of the following property:

1. $E$ is a stable vector bundle;
2. $E \simeq L \oplus L^{-1}$ with $L \in \text{Pic}^0(C)$ and $L^2 \not\simeq \mathcal{O}_C$, i.e., $(E, \phi) \in \Sigma \setminus \Omega$;
3. $E \simeq L \oplus L^{-1}$ with $L^2 \simeq \mathcal{O}_C$, i.e., $(E, \phi) \in \Omega$;
4. $E$ is a non-trivial extension of $L$ by $L^{-1}$ with $L^2 \simeq \mathcal{O}_C$;
5. $E$ is an unstable vector bundle isomorphic to $\theta_j^{-1} \oplus \theta_j$, where $\theta_j$ is a theta-characteristic, i.e., a line bundle such that $\theta_j^2 = K_C$.

**Theorem 6.8 (Fixed locus of $M$).** — The fixed locus of the $\mathbb{G}_m$-action on $M$ is

$$\text{Fix}(M) = N \sqcup \Theta = N \sqcup \bigcup_{j=1}^{16} \Theta_j,$$

where

1. $N$ is the moduli space of semistable Higgs bundles $(E, \phi)$ with $\phi = 0$, equivalently the moduli space of semistable vector bundles of rank 2 and degree 0, which is isomorphic to $\mathbb{P}^3$;
2. $\Theta$ is the set of 16 points in $M$ corresponding to the Higgs bundles

$$\Theta_j := \begin{pmatrix} \theta_j^{-1} \oplus \theta_j, & (0 \ 1 \\ 0 \ 0) \end{pmatrix},$$

**Proof.** — It is clear that $N$ and $\Theta$ are fixed by the $\mathbb{G}_m$-action. Hence, we just need to show that they are the only components of $\text{Fix}(M)$.

To this end, recall that by Proposition 6.7 the vector bundle $E$ underlying a semistable Higgs bundle $(E, \phi) \in M$ is

1. either a semistable vector bundle,
2. or an unstable vector bundle, isomorphic to $\theta_j^{-1} \oplus \theta_j$ for some $\theta_j$.

In the former case, the limit of the one-parameter subgroup $(E, \lambda \cdot \phi)$ is $(E, 0)$ (or $(L \oplus L^{-1}, 0)$ in case (4) of Proposition 6.7), and so it lies in $N$, which is isomorphic to $\mathbb{P}^3$ by [64]. In the latter case, $(E, \lambda \cdot \phi)$ is isomorphic to

$$\left( \theta_j^{-1} \oplus \theta_j, \lambda \cdot \begin{pmatrix} 0 \ 1 \\ u \ 0 \end{pmatrix} \right) \simeq \left( \theta_j^{-1} \oplus \theta_j, \begin{pmatrix} 0 \ \lambda^2 u \ 0 \end{pmatrix} \right)$$
for some $u \in \text{Hom}(\theta_j^{-1}, \theta_j \otimes K_C)$, after normalizing with the group of diagonal automorphisms of $E$; see [46, §11]. Therefore, the locus of $G_m$-fixed Higgs bundles with underlying unstable vector bundles is given by $\Theta$ (which corresponds to $u = 0$).  □

In Proposition 6.1 we mentioned that the $G_m$-invariant fiber $f^{-1}(\Omega_j) \subset \tilde{M}$ over $\Omega_j \cong (L \oplus L, 0)$, with $L^2 \cong O_C$, is the Grassmannian of Lagrangian subspaces of the 4-dimensional symplectic vector space $(V, \omega_V)$.

The deformation theory of Higgs bundles gives the identification of $(V, \omega_V)$ with the space of Higgs bundles extensions of $(L, 0)$ by itself, namely

$$\text{Ext}^1_{\text{Higgs}}(L, L) \cong H^0(C, K_C) \oplus H^1(C, O_C) \cong H^1(C, \mathbb{C}),$$

equipped with the symplectic form given by cup product. For further details, we refer the interested reader to [26, §3.2.2]. We just observe that $H^0(C, K_C)$ parametrizes deformations of $L$ with fixed underlying line bundle, while $H^1(C, O_C)$ parametrizes deformations of $L$ with fixed underlying Higgs field. Therefore, the rescaling action of Higgs fields yields the $G_m$-action on $\text{Ext}^1_{\text{Higgs}}(L, L)$ defined by $\lambda \cdot (v, \pi) = (\lambda v, \pi)$, where $v \in H^0(C, K_C)$ and $\pi \in H^1(C, O_C)$. This in turn induces the $G_m$-action on $\tilde{\Omega}_j$, whose fixed loci are described in the next Proposition 6.9.

**Proposition 6.9 (Fixed locus of $\tilde{\Omega}_j$).** — The fixed locus of the $G_m$-action on $\tilde{\Omega}_j$ is

$$\text{Fix}(\tilde{\Omega}_j) = t_j \cup s_j^+ \cup T_j,$$

where

1. the points $t_j$ and $T_j$ correspond to the Lagrangian subspaces $H^0(C, K_C)$ and $H^1(C, O_C)$;
2. the curve $s_j^+$ parametrizes Lagrangian subspaces generated by $v_1 \in H^0(C, K_C)$ and $v_2 \in H^1(C, O_C)$, and it is isomorphic to $\mathbb{P}^1$.

In particular, $t_j$, $s_j^+$ and $T_j$ have weights $(1, 1, 1)$, $(-1, 0, 1)$ and $(-1, -1, -1)$ respectively.

**Proof.** — The Plücker polarization $H_j$ embeds $\tilde{\Omega}_j$ as a smooth quadric in the linear system $|H_j| = \mathbb{P}(W) \cong \mathbb{P}^4 \subset \mathbb{P}(\lambda^2 V)$. The $G_m$-action on $\tilde{\Omega}_j$ induces an action on $W$ with weights $(0, 1, 1, 1, 2)$ in suitable coordinates $(x_0, \ldots, x_4)$. In these coordinates, $\tilde{\Omega}_j$ is defined by the equation $x_1^2 + x_2 x_3 + x_0 x_4 = 0$.

Since the Plücker embedding is $G_m$-equivariant, the fixed loci of $\tilde{\Omega}_j$ are the intersections of $\tilde{\Omega}_j$ with the isotypic components of the $G_m$-representation $W$, i.e.,

$$t_j = [1 : 0 : 0 : 0 : 0],$$

$$s_j^+ = [0 : x_1 : x_2 : x_3 : 0] \cap \tilde{\Omega}_j = \{ x_1^2 + x_2 x_3 = 0 \} \cong \mathbb{P}^1,$$

$$T_j = [0 : 0 : 0 : 0 : 1].$$

Moreover, the tangent space

$$T_{t_j} \tilde{\Omega}_j \cong T_{t_j} \mathbb{P}(W)/N_{\tilde{\Omega}_j/\mathbb{P}(W), t_j} \cong \text{Hom}(t_j, W/(t_j, T_j))$$
has weight $(1,1,1)$. Analogously, if $p = [0 : 0 : 1 : 0 : 0] \in S^+_j$, then $T_p\widetilde{\Omega}_j \simeq \text{Hom}(p, (t_j, \partial x_i, T_j))$ has weight $(-1,0,1)$, while $T_m\widetilde{\Omega}_j \simeq \text{Hom}(T_j, W/(t_j, T_j))$ has weight $(-1,-1,1).$ \hfill \square

**Proposition 6.10** (Fixed locus of $\widetilde{M}$). — The fixed locus of the $\mathbb{G}_m$-action on $\widetilde{M}$ is

$$\text{Fix}(\widetilde{M}) = \tilde{N} \cup \tilde{S}^+ \cup \tilde{\Theta} \cup \bigcup_{j \in \mathbb{Z}} T_j,$$

where

1. $\tilde{N} := f^{-1}_*N$ is the strict transform of $N$, isomorphic to $\mathbb{P}^3$;
2. $\tilde{S}^+$ is a Kummer surface;
3. $\tilde{\Theta} := f^{-1}(\Theta)$;
4. $T_j$ are points lying on the Lagrangian Grassmannians $\tilde{\Omega}_j = f^{-1}(\Omega_j)$.

**Proof.** — First, observe that $\text{Fix}(\widetilde{M})$ lies over $\text{Fix}(M)$, and so

$$\tilde{N} \cup \tilde{\Theta} = f^{-1}_*\text{Fix}(M) \subset \text{Fix}(\widetilde{M}) \subset f^{-1}(\text{Fix}(M)).$$

The component of $\text{Fix}(\widetilde{M})$ not contained in $f^{-1}_*\text{Fix}(M)$ lies over $\text{Fix}(M) \cap \Sigma := S \subset N$, which is isomorphic to $\text{Pic}^0(C)/(\mathbb{Z}/2\mathbb{Z})$, i.e., the singular Kummer surface associated to $\text{Pic}^0(C)$.

The fiber of $f$ over $p \in S \setminus \Omega$ is isomorphic to $\mathbb{P}^1$, and $\mathbb{G}_m$ acts with nontrivial weight on it by Proposition 6.11. Therefore, the $\mathbb{P}^1$-bundle $f^{-1}(S \setminus \Omega)$ has two $\mathbb{G}_m$-fixed sections. We denote their closure by $\tilde{S}^-$ and $\tilde{S}^+$. Since the restriction of $f$ to $\tilde{N}$ is an isomorphism, one of the two sections, say $\tilde{S}^-$, lies in $\tilde{N}$. The same holds for one of the two fixed points in each $\tilde{\Omega}_j$, namely $t_j$ because of the weight considerations in Proposition 6.9 and Proposition 6.11.

The other section $\tilde{S}^+$ must be the union of a copy of $S \setminus \Omega$ and the rational curve $s_j$, with $j = 1, \ldots, 16$, thus isomorphic to the nonsingular Kummer surface associated to $\text{Pic}^0(C)$. Indeed, by construction $\tilde{S}^+ \cap \tilde{\Omega}_j$ is a non-empty component of $\text{Fix}(\Omega_j)$ different from a point; otherwise $\tilde{S}^+$ would be singular, which is a contradiction since $\tilde{S}^+$ is a fixed locus of a $\mathbb{G}_m$-action on a smooth manifold. Therefore, $\tilde{S}^+ \cap \tilde{\Omega}_j = s_j$ by Proposition 6.9. \hfill \square

**Proposition 6.11** (Weights of $\widetilde{M}$)

1. $\tilde{N}$ has weight $(0,0,0,1,1,1)$;
2. $\tilde{S}^+$ has weight $(-1,0,0,1,1,2)$;
3. $\tilde{T}_j$ and $\Theta_j$ have weight $(-1,-1,-1,2,2,2)$;
4. $\tilde{T}_j$ has weight $(-1,-1,-1,2,2,2)$.

**Proof.** — Let $\tilde{\omega}$ be the holomorphic symplectic form on the symplectic resolution $\widetilde{M}$ extending the canonical holomorphic symplectic form $\omega$ on the smooth locus of $M$. As in [46, Prop. 7.1], the $\mathbb{G}_m$-action rescales the holomorphic symplectic form $\tilde{\omega}$

$$\lambda^*\tilde{\omega} = \lambda\tilde{\omega}.$$
Let \( p \in \text{Fix}(\tilde{M}) \), and \( W \) be a Lagrangian subspace of \( T_p\tilde{M} \) with weights \((a, b, c)\). Then the isotropy condition yields an isomorphism

\[ W^* \simeq T_p\tilde{M}/W, \]

and the weights of the action on \( W \) become

\[ \lambda(\lambda^{-a}, \lambda^{-b}, \lambda^{-c}) = (\lambda^{-a+1}, \lambda^{-b+1}, \lambda^{-c+1}) \]
on \( W^* \). As a result the torus action at a fixed point has weights

\[(a, b, c, -a + 1, -b + 1, -c + 1).\]

In this way, the weights of the \( \mathbb{G}_m \)-action at \( \tilde{N}, \tilde{S}^+ \) and \( \tilde{T}_j \) follow immediately from the computations in Proposition 6.9, by observing that \( t_j \in \tilde{N} \) and \( s_j^+ \in \tilde{S}^+ \). For the weights at \( \tilde{\Theta}_j \), instead, note that the locus of semistable Higgs bundles with underlying vector bundle \( \theta^{-1} j \oplus \theta j \) is Lagrangian by definition of \( \omega \) (cf. [46, Lem.6.8]), and has weight \((2, 2, 2)\).

\[ \square \]

**Corollary 6.12.** — \( \tilde{N} \) and \( \tilde{\Omega}_j \) intersect transversely at the point \( t_j = \tilde{N} \cap \tilde{\Omega}_j \).

**Proof.** — By Proposition 6.9, the tangent space \( T_{t_j} \tilde{\Omega}_j \) has weight one, while \( T_{t_j} \tilde{N} \) has weight zero. \[ \square \]

**Corollary 6.13.** — The attracting sets \( \text{Attr}(N) \), \( \text{Attr}(\tilde{S}^+) \), \( \text{Attr}(\tilde{\Theta}_j) \) and \( \text{Attr}(\tilde{T}_j) \) have codimension \( 0, 1, 3, 3 \) respectively.

**Proof.** — It is an immediate corollary of (24) and Proposition 6.11. \[ \square \]

### 6.1.5. Poincaré polynomials of \( M \) and \( \tilde{M} \)

**Theorem 6.14 (Cohomology of \( M \) and \( \tilde{M} \)).** — The Poincaré polynomials of \( M \) and \( \tilde{M} \) are

\[
P_t(M) := \sum_k (-1)^k \dim H^k(M) t^k = 1 + t^2 + t^4 + 17 t^6,
\]

\[
P_t(\tilde{M}) := \sum_k (-1)^k \dim H^k(\tilde{M}) t^k = 1 + 2t^2 + 23t^4 + 34t^6.
\]

**Proof.** — Since \( \text{Attr}(N) \) is an open subset of \( M \), we have \( H^*(\text{Attr}(N), u\mathbb{Q}_M) = H^*(\text{Attr}(N)) = H^*(\mathbb{P}^3) \). The spectral sequence (25) gives

\[
P_t(M) = P_t(N) + P_t(\Theta)t^6 = 1 + t^2 + t^4 + 17 t^6.
\]

See [21, Th.1.5] for an alternative proof.
Similarly, by Theorem 6.6, Proposition 6.10 and Corollary 6.13, we obtain

\[ P_t(\tilde{M}) = P_t(\tilde{N}) + P_t(\tilde{S}^+)^2 + P_t(\tilde{\Theta})^2 + \sum_{j=1}^{16} P_t(T_j)^2 \]

\[ = (1 + t^2 + t^4 + t^6) + (1 + 22t^2 + t^4)^2 + 16t^6 + 16t^6 \]

\[ = 1 + 2t^2 + 23t^4 + 34t^6. \]

\[ \square \]

6.2. Moduli space of equivariant Higgs bundles \( M_i \)

6.2.1. Equivariant Higgs bundle and the forgetful map \( q \). — Recall that \( \iota: C \to C \) is the hyperelliptic involution of the curve \( C \) of genus 2.

**Definition 6.15.** — A \((\iota,\)-equivariant Higgs bundle over \( C \) is a triple \((E,h,\phi)\) such that:

1. \( E \) is an \( \iota \)-invariant vector bundle, i.e., \( \iota^*E \cong E \);
2. \( h: E \to \iota^*E \) is a lift of the \( \iota \)-action on \( E \) such that \( \iota^*h \circ h = \text{id}_E \);
3. \( \phi \in \text{Hom}(E,E \otimes K_C) \) is an \( \iota \)-invariant Higgs field, i.e., a \( O_C \)-linear morphism which makes the following diagram commutative:

\[
\begin{array}{c}
E \quad \phi \\
\downarrow h \\
\iota^*E \quad \iota^* \phi \quad h \otimes \text{id}_{K_C}
\end{array}
\]

A morphism between two equivariant Higgs bundles \((E_1,h_1,\phi_1)\) and \((E_2,h_2,\phi_2)\) is a homomorphism of vector bundles \( \psi \in \text{Hom}(E_1,E_2) \) such that the following diagrams commute:

\[
\begin{array}{ccc}
E_1 & \xrightarrow{h_1} & \iota^*E_1 \\
\downarrow \psi & & \downarrow \psi \\
E_2 & \xrightarrow{h_2} & \iota^*E_2,
\end{array}
\quad
\begin{array}{ccc}
E_1 & \xrightarrow{\phi_1} & E_1 \otimes K_C \\
\downarrow \psi & & \downarrow \psi \\
E_2 & \xrightarrow{\phi_2} & E_2 \otimes K_C.
\end{array}
\]

The slope of a vector bundle \( E \) over a curve \( C \) is defined by

\[ \mu(E) := \deg(E) / \text{rank}(E). \]

**Definition 6.16.** — An equivariant Higgs bundle \((E,h,\phi)\) is semistable or stable if for any proper equivariant Higgs subbundle \( F \subset E \), the inequality \( \mu(F) \leq \mu(E) \) holds, respectively \( \mu(F) < \mu(E) \).

Let \( W = \{w_1, \ldots, w_6\} \) be the set of all Weierstrass points, i.e., the fixed points of \( \iota \). For every \( w \in W \), \( h_w: E_w \to E_w \) is an involution of the fiber \( E_w \).

**Definition 6.17.** — The normal quasi-projective variety \( M_i \) (respectively \( M_i^* \)) is the coarse moduli space of semistable (respectively stable) equivariant Higgs bundle \((E,h,\phi)\) of rank 2 over \( C \) with trivial determinant and \( \text{tr}(h_w) = 0 \) for all \( w \in W \).
The existence of $M_s$ follows from the work of Seshadri [73] and Nitsure [67]. In Section 6.2.4 we review the construction. Here we first describe $M_s$ as a quasi-étale cover of $M$. This cover appears also in [45, §6.3] and references therein.

**Definition 6.18 (Quasi-étale morphism).** A morphism $f : X \to Y$ between normal varieties is quasi-étale if $f$ is quasi-finite, surjective and étale in codimension one, i.e., there exists a closed, subset $Z \subseteq X$ of codimension $\text{codim } Z \geq 2$ such that $f|_{X \setminus Z} : X \setminus Z \to Y$ is étale.

**Remark 6.19.** By the purity of the branch locus, a quasi-étale morphism induces an étale cover of the smooth locus of the codomain.

**Proposition 6.20.** The forgetful map $q : M_s \longrightarrow M$ 

$(E, h, \phi) \longrightarrow (E, \phi)$

is well-defined, quasi-étale of degree two, and branched along the singular locus $\Sigma$ of $M_s$.

**Proof.** The forgetful map $q$ is well-defined, because an equivariant Higgs bundle $(E, h, \phi)$ is semistable if and only if the Higgs bundle $(E, \phi)$ is semistable in the usual sense (the same proof of [8, Lem.2.7] applies). The map $q$ is also surjective: any semistable Higgs bundles $(E, \phi)$ admits a lift of the $\iota$-action on $E$ conjugating $\phi$ and $i^*\phi$ by [45, Chap.6,p.74, & Th.2.1].

We show now that $q$ is quasi-étale. To this end, we closely follow the proof of [54, Th.2.1]. Given two equivariant Higgs bundles $(E, h_1, \phi)$ and $(E, h_2, \phi)$, there exists an automorphism $A \in \text{Aut}(E)$ such that $h_2 = h_1 \circ A$ and $\phi = A^{-1} \phi A$.

If $(E, \phi)$ is stable, then the only automorphisms which fix the Higgs field are scalars. Then $h_2 = \pm h_1$, and so there are only two non-equivalent equivariant Higgs bundles $(E, h_1, \phi)$ and $(E, -h_1, \phi)$ over $(E, \phi)$. Hence, $q$ is generically 2 : 1.

If $(E, \phi)$ is strictly semistable, i.e., $(E, \phi) \in \Sigma$, then $E \simeq L \oplus L^{-1}$ with $L \in \text{Pic}^0(C)$, and any two lifts are equivalent. Hence, $q$ is quasi-finite and branched along $\Sigma$. □

**6.2.2. Non-abelian Hodge correspondence.** Let $C \to \mathbb{P}^1$ be the quotient of $C$ via the hyperelliptic involution, and let $\underline{W}$ be the critical divisors on $\mathbb{P}^1$, i.e., the projection of the Weierstrass points.

The moduli space $M_s$ is isomorphic to the moduli space of parabolic Higgs bundle of rank 2 on $\mathbb{P}^1$ with parabolic weight $1/2$ at all points of $\underline{W}$ and parabolic degree zero; see [9, Th.3.5].

The topological space underlying $M_s$ parametrizes also representations of the orbifold fundamental group

$\pi_1^{\text{orb}}(C/\iota) \simeq \langle \gamma_1, \ldots, \gamma_6 \mid \gamma_1^2 = \cdots = \gamma_6^2 = 1 \text{ and } \gamma_1 \cdots \gamma_6 = 1 \rangle.$
Theorem 6.21 (Non-abelian Hodge correspondence). — There exists a commutative square

\[
\begin{array}{ccc}
M & \xrightarrow{\Psi_i} & M_B(2, \text{SL}_2, i) \\
\downarrow q & & \downarrow q^{\text{top}} \\
M & \xrightarrow{\Psi} & M_B(2, \text{SL}_2) = \text{Hom}(\pi_1(C), \text{SL}_2) \sslash \text{PGL}_2.
\end{array}
\]

where the horizontal arrows are real analytic isomorphisms, and the vertical arrows are quasi-étale covers.

Proof. — Identify \(M_i\) with a moduli space of parabolic Higgs bundles as above. The correspondences \(\Psi\) and \(\Psi_i\) have been constructed by Hitchin [46] and Simpson [76] respectively. By construction, the square commutes. □

6.2.3. Singularities of \(M_i\)

Notation 6.22. — We fix the following notation:

- \(\text{Bun}^{\text{sa}}(C/i)\) is the moduli space of semistable \(i\)-equivariant vector bundles \((E, h)\). It is the inverse image of the moduli space of semistable vector bundles \(N\) via \(q\);
- the inverse images of \(\Omega\) via \(q\) consists of the 16 points \(\Omega_i\);
- the inverse images of \(\Theta\) via \(q\) consists of the 32 points \(\Theta_i\).

Proposition 6.23 (Singularities of \(M_i\))

1. \(\Omega_i\) is the singular locus of \(M_i\).
2. The smooth locus of \(M_i\), denoted \(M_i^{\text{sm}}\), is the moduli space of stable equivariant Higgs bundles \(M_i^s\).

Proof. — The local isomorphism type of the singularities of \(M_i\) coincides with the model described in [62, Lem. 3.1]. This yields the first statement. For the second statement, it is enough to show that

\[
M_i^{\text{sm}} = q^{-1}(M \setminus \Sigma) \cup q^{-1}(\Sigma \setminus \Omega) \subseteq M_i^s.
\]

Any Higgs bundle \((E, \phi) \in M \setminus \Sigma\) is stable, and so the equivariant Higgs bundles in \(q^{-1}(M \setminus \Sigma)\) are stable too. If \((E, \phi) \in \Sigma \setminus \Omega\) with \(E \simeq L \oplus L^{-1}\), then the only line sub-bundles of \(E\) are \(L\) and \(L^{-1}\), but since they are not \(i\)-invariant, \(q^{-1}(\Sigma \setminus \Omega) \subseteq M_i^s\). □

6.2.4. Construction of \(M_i\). — The moduli space \(M_i\) is constructed in the following way. All the ingredients have already appeared in [73, 67, 39].

Let \((E, h, \phi)\) be a stable equivariant Higgs bundle of rank 2 over \(C\) with trivial determinant and \(\text{tr}(h_w) = 0\) for all \(w \in W\). Fix an equivariant ample line bundle \(\Theta_C(1)\) on \(C\). Choose an integer \(m \in \mathbb{Z}\) such that \(H^1(C, E(m)) = 0\) and \(E(m)\) is globally generated.

The quotient scheme \(Q\) parametrizes all quotient sheaves of \(H^0(C, E(m)) \otimes \Theta_C\) with the Hilbert polynomial of \(E(m)\). Let \(H^0(C, E(m)) \otimes p_C^* \Theta_C \rightarrow \delta_Q \otimes p_C^* \Theta_C(m)\) be the universal quotient bundle on \(Q \times C\), with the natural projection \(p_C: Q \times C \rightarrow C\).
Let $R \subset Q$ be the subset of all $q \in Q$ for which $\mathcal{E}_q$ is locally free and the map $H^0(C, E(m)) \to H^0(C, \mathcal{E}_q(m))$ is an isomorphism.

By [67, Prop. 3.6], there exists a locally universal family of semistable Higgs bundles $\mathcal{E}_s \xrightarrow{\Phi} \mathcal{E}_s \otimes p_C^* K_C$ on $F_{ss} \times C$, where $F_{ss}$ is an open subset of a linear $R$-scheme $F \to R$ together with a family of Higgs bundles $\mathcal{E}_F \xrightarrow{\Phi} \mathcal{E}_F \otimes p_C^* K_C$.

The involution $\iota^*$ on $H^0(C, E(m))$ induces a natural lift $j_0$ of the $\iota$-action on the trivial bundle $C \times H^0(C, E(m))$, and so an $\iota$-action on $F_{ss}$ with fixed locus $\text{Fix}_{\iota}(F_{ss})$. In particular, $j_0$ descends to a lift $h_0$ of the $\iota$-action on $\mathcal{E}_q$, for any $q \in \text{Fix}_{\iota}(F_{ss})$.

Call $F_{ss, \iota}$ the connected component of $\text{Fix}_{\iota}(F_{ss})$ consisting of the equivariant Higgs bundles $(\mathcal{E}_q, h_0, \Phi_q)$ with $\text{tr}(h_{q,w}) = 0$; see [73, Chap. II, Prop. 6(iv)] and [73, Chap. II, Prop. 5 & Rem. 2].

Let $H$ be the group of automorphisms of the trivial bundle which commute with $j_0$, and $PH := H/\mathbb{G}_m$ the quotient of $H$ modulo scalar matrices. The moduli spaces $M_s$ and $M_{ss}^i$ are the quotients $F_{ss,i} / PH$ and $F_{ss,i} / PH$ respectively, where $F_{ss,i}$ is the subset of stable equivariant bundles in $F_{ss,i}$.

### 6.2.5 Universal bundles

We show the existence of a universal bundle on $M_{ss}^i \times C$ (cf. [39, §5]).

**Definition 6.24.** Let $Z$ be a subset of $M_{ss}^i$. A **universal Higgs bundle** on $Z \times C$ is a rank two Higgs bundle $(\mathbb{E}, \Phi)$ such that $[(\mathbb{E}, \Phi)]_{|(E,h,\phi)} \times C \simeq (E, \phi)$ for all $(E, h, \phi) \in Z$.

**Remark 6.25.** Let $(\mathbb{E}_1, \Phi_1)$ and $(\mathbb{E}_2, \Phi_2)$ be universal Higgs bundles on $Z \times C$. Then there exists a line bundle $\mathcal{L} \in \text{Pic}(Z)$ such that $[(\mathbb{E}_1, \Phi_1)] \simeq (\mathbb{E}_2 \otimes p_C^* \mathcal{L}, \Phi_2)$, with $p_C : Z \times C \to C$ the natural projection. In particular, $\mathbb{P}(\mathbb{E}_1) \simeq \mathbb{P}(\mathbb{E}_2)$ is canonical. See [44, 4.2].

We adopt the notation of Section 6.2.4. In addition, we define $F_i^0$ as being the open subset of $F_{ss,i}$ parametrizing stable equivariant Higgs bundle whose underlying vector bundle is either stable or isomorphic to $L \oplus \iota^* L$ with $L \in \text{Pic}^0(C)$ with $L^2 \neq \mathcal{O}_C$.

The quotient $M^i := F_i^0 / PH$ is the attracting set of $\text{Bun}^{ss}(C/\iota) \times \Omega$. Thus, according to Proposition 6.7, the complement $M_s - M^i$ parametrizes stable equivariant Higgs bundles whose underlying vector bundle is unstable or a non-trivial extension of $L$ by $L$ with $L^2 \simeq \mathcal{O}_C$, and so it has codimension 2 by [46, Ex. 3.13(iv) & (v)]; see also [49, Lem. 3.4]. In particular, $F_{ss,i} \setminus F_i^0$ has codimension 2.

**Proposition 6.26.** A universal Higgs bundle on $M_{ss}^i \times C$ does exist.

**Proof.** Let $\mathcal{E}$ be the restriction of the universal Higgs bundle $\mathcal{E}_F$ to $F_{ss,i} \times C$, and denote by $p_F : F_{ss,i} \times C \to F_{ss,i}$ and $p_C : F_{ss,i} \times C \to C$ the two projections.

The natural lift of the $H$-action is such that the subgroup of scalar matrices acts by homotheties. Suppose that there exists an $H$-equivariant line bundle $\lambda(\mathcal{E})$ over $F_{ss,i}$ with the same property, i.e., that the center of $H$ acts by homotheties. Then, the center of $H$ acts trivially on $\mathcal{E} \otimes p_F^* \lambda(\mathcal{E})^{-1}$. By Kempf’s descent lemma [24, Th. 2.3],
the $PH$-equivariant bundle $\mathcal{E} \otimes p_F^* \lambda(\mathcal{E})^{-1}$ descends to a vector bundle on $M_{\text{sm}} \times C$, and since the section $\Phi$ is invariant, it also descends.

Here is how to construct $\lambda(\mathcal{E})$. For any $(E, h, \phi) \in F^0$, $h$ acts on $H^0(C, E \otimes K_C)$, and induces a splitting

$$H^0(C, E \otimes K_C) = H^0(C, E \otimes K_C)^+ \oplus H^0(C, E \otimes K_C)^-$$

into one-dimensional eigenspaces (relative to eigenvalues $\pm 1$ respectively); see [45, Prop. 4.1]. The lift $j_0$ induces an involution on $p_F^*(\mathcal{E} \otimes p_C^* K_C)$.

Set

$$\lambda(\mathcal{E}) := p_{F_*}^* (\mathcal{E} \otimes p_C^* K_C)^+. $$

By semicontinuity, $\lambda(\mathcal{E})$ is a line bundle on $F^0$ with fiber $H^0(C, E \otimes K_C)^+$. The multiplication by a scalar in $E$ induces multiplication in $H^0(C, E \otimes K_C)^+$ too, and so in $\lambda(\mathcal{E})$.

Now let $i_{F_*} : F^0 \hookrightarrow F$ be the natural inclusion, and define

$$\lambda(\mathcal{E}) = i^*_{F_*} \lambda(\mathcal{E})^0.$$

Since $F_{s,t}$ is smooth and $F_{s,t} \smallsetminus F^0$ has codimension 2, $\lambda(\mathcal{E})$ is a line bundle on $F_{s,t}$ with the right $H$-linearization.

6.2.6. Nilpotent cone. — In this section we describe the components of the nilpotent cone of $M$, i.e., the zero fiber of the Hitchin fibration $\chi : M \to H^0(C, K_C^2)$.

We show that $\chi^{-1}(0)$ has 17 irreducible components, one of them being the moduli space of semistable vector bundle $N$. By [65, Main Th., §3] there is no universal Higgs bundle over any Zariski open set of $N$. On the other hand, we construct a universal bundle on the normalization of the other components; see Proposition 6.28 and Lemma 6.29.

**Proposition 6.27.** — The nilpotent cone of $M$ is a compact union of 3-dimensional manifolds:

$$\chi^{-1}(0) = N \sqcup \bigcup_{j=1}^{16} N_j,$$

where $N_j$ is isomorphic to the vector space $\text{Ext}^1(\theta_j, \theta_j^{-1})$, where $\theta_j$ runs over the 16 theta-characteristics $\theta_j^2 = K_C$.

**Proof.** — We adapt the proof of [84, Prop. 19]; see also [68, §2]. Since $N \subset M$ is the locus of semistable Higgs bundles with trivial Higgs field, we see that $N \subset \chi^{-1}(0)$. However, there are also stable Higgs bundles $(E, \phi) \in \chi^{-1}(0)$ with $\phi \neq 0$.

Under this assumption, $\phi$ has generically rank one: denote by $A$ the line bundle $\text{Im } \phi \subset E \otimes K_C$. Then $E$ sits in the following diagram

$$\begin{array}{cccc}
0 & \longrightarrow & A^{-1} & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\
\phi \downarrow & & & & & & & & \\
0 & \longrightarrow & A \otimes K_C & \longrightarrow & E \otimes K_C & \longrightarrow & A^{-1} \otimes K_C & \longrightarrow & 0.
\end{array}$$

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Since $\text{tr}(\phi) = 0$, the composition $A \to E \otimes K_C \to A \otimes K_C$ is zero, and the inclusion $A \to E \otimes K_C$ factors through $u: A \to A^{-1} \otimes K_C$. The stability of $E$ implies $-\deg A < \deg E = 0$, and since $u \in H^0(C, K_C \otimes A^{\otimes(-2)})$ is non-zero, we conclude that $A$ is a theta-characteristic.

Therefore, the Higgs bundle $(E, \phi)$ is determined by the triple $(\theta_j, v, u)$ given by

- the theta-characteristic $\theta_j$,
- the extension class $v \in \text{Ext}^1(\theta_j, \theta_j^{-1})$ giving the exact sequence $\theta_j^{-1} \to E \to \theta_j$,
- the non-zero scalar $u \in H^0(C, \text{Hom}(\theta_j, \theta_j^{-1} \otimes K_C)) \simeq H^0(C, \mathcal{O}_C)$,

modulo the $\mathbb{G}_m$-action

$$c \cdot (\theta_j, v, u) = (\theta_j, cv, cu).$$

The equivalence class $(\theta_j, v, u)$ under rescaling is denoted $[\theta_j, v, u]$, and we identify the Higgs bundle $(E, \phi) \in N_j$ with $[\theta_j, v, u]$. In particular, the irreducible components of $\chi^{-1}(0)$ different from $N$ are

$$N_j := \mathbb{P}(\text{Ext}^1(\theta_j, \theta_j^{-1}) \oplus H^0(C, \mathcal{O}_C)) \smallsetminus \{u = 0\} \simeq \text{Ext}^1(\theta_j, \theta_j^{-1}).$$

Alternative proof. — The nilpotent cone on $M$ is the union of the repelling sets of all the fixed loci

$$\chi^{-1}(0) = \text{Repell}(N) \cup \text{Repell}(\Theta) = N \cup \bigcup_{j=1}^{16} \text{Repell}(\Theta_j).$$

By Theorem 6.5(2), $\text{Repell}(\Theta_j)$ is isomorphic to a 3-dimensional vector space. However, we rely on the previous proof for a modular interpretation of $\text{Repell}(\Theta_j)$.

Let $R_j$ be the total space of the projective bundle $\mathbb{P}(\text{Ext}^1(\theta_j, \theta_j^{-1}) \oplus R^0(C, \mathcal{O}_C))$ with hyperplane bundle $\mathcal{O}_{R_j}(1)$. As we observed above, there is a natural decomposition $R_j = N_j \cup \mathbb{P}(\text{Ext}^1(\theta_j, \theta_j^{-1})).$ The inclusion $N_j \hookrightarrow \chi^{-1}(0)$ extends to a bijective and algebraic morphism

$$r_j: R_j \hookrightarrow \chi^{-1}(0)$$

$$[v : u] \mapsto [\theta_j, v, u] = (E, \phi),$$

whose image is the closure $\overline{N}_j$ of $N_j$ in $\chi^{-1}(0)$; see Theorem 6.5(2) and also [84, Prop. 24].

**Proposition 6.28.** — There exists a universal bundle $\mathcal{E}_{R_j}$ on $R_j \times C$ which sits in the following exact sequence

$$0 \to p_{R_j}^* \mathcal{O}_{R_j}(1) \otimes p_C^* \theta_j^{-1} \to \mathcal{E}_{R_j} \to p_{R_j}^* \theta_j \to 0,$$

where $p_{R_j}: R_j \times C \to R_j$ and $p_C: R_j \times C \to C$ are the natural projections.

**Proof.** — Mutatis mutandis, the same argument as in [84, p. 22] works.

Consider now the quasi-étale cover $q: M \to M$. Since $N_j$ is simply connected, $q^{-1}(N_j)$ breaks into two irreducible components, say $N_j^+$ and $N_j^-$. In particular,
6.29. — is an algebraic bijection. Let \( r \) restricts to an isomorphism between \( M^\circ \cap N^+ \) (equivalently \( M^\circ \cap N^- \)) and \( \overline{\mathcal{N}}_j \setminus (\Omega \cup \Theta_j) \). If we set \( R^j = R_j \setminus (r_j^{-1}(\Omega) \cup r_j^{-1}(\Theta_j)) \), then the product map
\[
\tau_j := (r_j \circ q^{-1}, \text{id}) : R^j \times C \longrightarrow (M^\circ \cap N^+) \times C
\]
is an algebraic bijection. Let \( \mathcal{E} \) be the universal bundle on \( M^\circ \times C \).

**Lemma 6.29.** — The \( \mathbb{P}^1 \)-bundles \( \mathbb{P}(\tau_j^* \mathcal{E}) \) and \( \mathbb{P}(\mathcal{E}_{R_j}) \) on \( R^j \times C \) are isomorphic.

**Proof.** — The vector bundles \( \tau_j^* \mathcal{E} \) and \( \mathcal{E}_{R_j} \) are both universal on \( R^j \times C \). The result follows from Remark 6.25.

6.2.7. Quasi-étale covers of \( M_{\text{Dol}}(X,SL_n) \). — In this section we show that the quasi-étale cover \( \iota \) is a special feature of the moduli space \( M = M_{\text{Dol}}(C,SL_2) \), which is not shared by any other space \( M_{\text{Dol}}(X,SL_n) \), \( g \geq 2 \).

**Proposition 6.30.** — The smooth locus \( M_{\text{Dol}}^{sm}(X,SL_n) \) of \( M_{\text{Dol}}(X,SL_n) \) is simply-connected for \( g \geq 2 \) and \((g,n) \neq (2,2)\). In particular,
\[
\pi_1(M_{\text{Dol}}^{sm}) = \mathbb{Z}/2\mathbb{Z}.
\]

**Proof.** — \( M_{\text{Dol}}^{sm}(X,SL_n) \) contains a Zariski open subset which can be identified with the cotangent bundle of the moduli space \( \text{Bun}^r(X,n) \) of stable vector bundles of rank \( r \) and trivial determinant over \( X \). Therefore, the fundamental group of \( M_{\text{Dol}}^{sm}(X,SL_n) \) is a quotient of \( \pi_1(\text{Bun}^r(X,n)) \), which is trivial by [20, Th.3.2(i)], for \( g \geq 2 \) and \((g,n) \neq (2,2)\).

Consider now \( M \). The forgetful map \( q \) induces the following exact sequence in homotopy
\[
1 \longrightarrow \pi_1(M_{\text{Dol}}^{sm}) \longrightarrow \pi_1(M_{\text{Dol}}^{sm}) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1.
\]
As before, \( M_{\text{Dol}}^{sm} \) contains a Zariski open subset isomorphic to the cotangent bundle of the moduli space \( \text{Bun}^r(C/\iota) \) of \( \iota \)-equivariant bundles of rank \( 2 \) over \( C \) with trivial determinant and \( \text{tr}(h_w) = 0 \) for all \( w \in W \); see for instance [45, Chap. 6, p. 73]. Thus, we obtain that \( \pi_1(M_{\text{Dol}}^{sm}) \) is a quotient of \( \pi_1(\text{Bun}^r(C/\iota)) \).

The space \( \text{Bun}^r(C/\iota) \) is the smooth locus of the double cover \( \text{Bun}^{ss}(C/\iota) \) of \( \mathbb{P}^3 \) branched along a singular Kummer quartic. The singular locus of \( \text{Bun}^{ss}(C/\iota) \) consists of 16 ordinary double points, which are known to admit a small resolution, i.e., the exceptional locus has codimension \( \geq 2 \). This implies that \( \pi_1(\text{Bun}^r(C/\iota)) \) coincides with the fundamental group of a (small) resolution of \( \text{Bun}^{ss}(C/\iota) \). Further, \( \text{Bun}^{ss}(C/\iota) \) is rational by [54, Th. 2.2] or [18, Th. 1.3]; see also [45, §5.4.2 & §5.5], where a small resolution of \( \text{Bun}^{ss}(C/\iota) \) is denoted \( \text{Bun}^{ss}_{1/1,2/3}(C/\iota) \). Since the fundamental group is a birational invariant of smooth proper varieties, we observe that \( \text{Bun}^{ss}_{1/1,2/3}(C/\iota) \) is simply-connected, since the projective space is so.

To summarize, we have shown that
\[
1 = \pi_1(\mathbb{P}^3) \simeq \pi_1(\text{Bun}^{ss}_{1/1,2/3}(C/\iota)) \simeq \pi_1(\text{Bun}^r(C/\iota)) \longrightarrow \pi_1(M_{\text{Dol}}^{sm}).
\]
By the exact sequence (31), we conclude that \( \pi_1(M_{\text{Dol}}^{sm}) \simeq \mathbb{Z}/2\mathbb{Z} \).
The following corollary is an immediate consequence of Remark 6.19 and Proposition 6.30.

**Corollary 6.31.** — There are no non-trivial quasi-étale cover of $M_{\text{Dol}}(X,\text{SL}_n)$ for $g \geq 2$ and $(g,n) \neq (2,2)$. The forgetful map $q$ is the only non-trivial quasi-étale cover of $M$.

### 7. P=W conjectures for $M$

In this section we reduce the proof of the $\text{P=W}$ conjecture for $M$ and $\widetilde{M}$ to $\text{P=W}$ phenomena for the summands of the decomposition theorem for $f: \widetilde{M} \to M$; see Theorem 7.1 and Theorem 7.4. The exchange of the perverse and weight filtrations for the summands supported on a subvariety strictly contained in $M$ is proved in Theorem 7.6. Therefore, the ultimate goal of this section is to reduce the proof of the $\text{P=W}$ conjecture for $M$ and $\widetilde{M}$ to the $\text{PI=WI}$ conjecture.

We first show that the $\text{PI=WI}$ conjecture for $M$ implies the $\text{P=W}$ conjecture for $M$.

Actually, this first statement does not require the decomposition theorem.

**Theorem 7.1.** — If the $\text{PI=WI}$ conjecture for $M$ holds, then the $\text{P=W}$ conjecture for $M$ holds.

**Proof.** — The fixed locus of the $\mathbb{G}_m$-action on $M$ can be identified with the (disjoint) union of connected components of the fixed locus of the $\mathbb{G}_m$-action on $\widetilde{M}$; see Proposition 6.8 and Proposition 6.10. This induces an injective morphism between the local-to-global spectral sequences (25) for $M$ and $\widetilde{M}$. Therefore, $f^*: H^*(M) \to H^*(\widetilde{M})$ is an injective map, and so is the natural map $H^*(M) \to H^*(\widetilde{M})$, since $f^*: H^*(M) \to H^*(\widetilde{M})$ factors as $H^*(M) \to IH^*(M) \to H^*(\widetilde{M})$. The statement now follows from the fact that the injective map $H^*(M) \to IH^*(M)$ preserves the perverse and weight filtrations.

With a slight abuse of notation, we denote by $f$ both the symplectic resolutions $f_{\text{Dol}}(C,\text{SL}_2): \widetilde{M} \to M$ and $f_{\text{B}}(C,\text{SL}_2): \widetilde{M}_B \to M_B$. By [47, Lem.2.11], any symplectic resolution is semismall. Therefore, the decomposition theorem (Theorem 2.7) provides canonical isomorphisms:

$$
\text{Rf}_*\mathbb{Q}_{\widetilde{M}}[6] \simeq IC_M \oplus \mathbb{Q}_\Sigma[4](-1) \oplus \mathbb{Q}_\Omega(-3),
$$

$$
\text{Rf}_*\mathbb{Q}_{\widetilde{M}_B}[6] \simeq IC_{M_B} \oplus \mathbb{Q}_{\Sigma_B}[4](-1) \oplus \mathbb{Q}_{\Omega_B}(-3).
$$

Thus, in cohomology we have:

$$
H^*(\widetilde{M}) \simeq IH^*(M) \oplus H^{*-2}(\Sigma)(-1) \oplus H^{*-6}(\Omega)(-3),
$$

$$
H^*(\widetilde{M}_B) \simeq IH^*(M_B) \oplus H^{*-2}(\Sigma_B)(-1) \oplus H^{*-6}(\Omega_B)(-3).
$$

These decompositions split the perverse and weight filtration, as shown in the following lemmas.
Lemma 7.2. — We have
\[ P_k^* H^* (\tilde{M}) = P_k^* H^* (M) \oplus P_{k-1}^* H^{*-2} (\Sigma) \oplus P_{k-2}^* H^{*-6} (\Omega), \]
where \( P_k^* H^* (M) \), \( P_k^* H^* (\Sigma) \) and \( P_k^* H^* (\Omega) \) denote the pieces of the perverse filtration associated to the maps \( \chi \circ f \), \( \chi \rvert \Sigma \) and \( \chi \rvert \Omega \) respectively.

Proof. — Apply \( \chi^* \) to the splitting (32) and notice that perverse truncation functors \( \tau_{\leq i} \) are exact. \( \square \)

Lemma 7.3
\[ W_{2k}^* H^* (\tilde{M}_B) = W_{2k}^* H^* (M_B) \oplus W_{2k-2}^* H^{*-2} (\Sigma_B) \oplus W_{2k-6}^* H^{*-6} (\Omega_B). \]

Proof. — As the decomposition theorem is an isomorphism of mixed Hodge structures, we have
\[ W_{2k}^* H^* (\tilde{M}_B) = W_{2k}^* H^* (M_B) \oplus W_{2k}^* H^{*-2} (\Sigma_B)(-1) \oplus W_{2k}^* H^{*-6} (\Omega_B)(-3). \]

Recalling that Tate shifts \((-k)\) increase weights of \( 2k \), the result follows by including them in the grading of the weight filtration. \( \square \)

Theorem 7.4. — The P=W conjecture for \( \tilde{M} \) is equivalent to the following two statements:

1. \( P=W \) conjecture for \( M \);
2. \( P=W \) conjecture for \( \Sigma \) and \( \Omega \), i.e.,
\[ P_k^* H^* (\Sigma) = \Psi^*|_{\Sigma} W_{2k}^* H^* (\Sigma_B), \quad P_k^* H^* (\Omega) = \Psi^*|_{\Omega} W_{2k}^* H^* (\Omega_B). \]

Proof. — Let \( \Psi : M \to M_B \) be the non-abelian Hodge correspondence, and \( \hat{\Psi} : \tilde{M} \to \tilde{M}_B \) be the diffeomorphism lifting \( \Psi \) in the sense of Theorem 3.8. By the commutativity of the diagram (7), and since the map \( \Psi \) preserves the stratifications (20) and (21), the map \( \hat{\Psi}^* : H^* (M_B) \to H^* (\tilde{M}) \) splits on the summands of the decomposition theorem. More precisely, \( \hat{\Psi}^* \) is given by the product map
\[ (\hat{\Psi}^*, \Psi^*|_{\Sigma}^{-2}, \Psi^*|_{\Omega}^{-6}) : IH^* (M_B) \oplus H^{*-2} (\Sigma_B)(-1) \oplus H^{*-6} (\Omega_B)(-3) \to IH^* (M) \oplus H^{*-2} (\Sigma)(-1) \oplus H^{*-6} (\Omega)(-3). \]

The statement then follows by Lemma 7.2 and Lemma 7.3. \( \square \)

Remark 7.5. — The product map (36) suggests that it is possible to define the isomorphism in cohomology \( \hat{\Psi}^* \) without constructing the diffeomorphism \( \hat{\Psi} \). This is indeed the approach of [12]. However, the virtue of Theorem 3.8 is to establish that the isomorphism between the cohomology rings of \( \tilde{M} \) and \( \tilde{M}_B \) which realizes the exchange of perverse and weight filtration has a geometric origin.

Theorem 7.6 (P=W for singular loci). — The P=W conjecture for \( \Sigma \) and \( \Omega \) holds.
Proof: — Since $\Omega$ is a collection of points, the perverse and the weight filtrations are all concentrated in degree zero, and so the P=W conjecture for $\Omega$ trivially holds.

We show now that the P=W conjecture for $\Sigma$ holds. To this end, note that the map $\chi|_{\Sigma}$ factors as follows:

$$\chi|_{\Sigma}: \Sigma \cong (\text{Pic}^0(C) \times H^0(K_C))/\langle\mathbb{Z}/2\mathbb{Z}\rangle \longrightarrow H^0(K_C)/\langle\mathbb{Z}/2\mathbb{Z}\rangle \subset H^0(K_C^\otimes 2).$$

Equivalently, $\chi|_{\Sigma}$ can be identified with the quotient of the projection

$$\text{Pic}^0(C) \times H^0(C,K_C) \longrightarrow H^0(C,K_C)$$

via the involution $(L,s) \mapsto (L^{-1},-s)$. Therefore, the general fiber $\chi|_{\Sigma}^{-1}(s)$, with $s \in H^0(K_C)/\langle\mathbb{Z}/2\mathbb{Z}\rangle$, is isomorphic to $\text{Pic}^0(C)$. The zero fiber $\chi|_{\Sigma}^{-1}(0)$ instead is isomorphic to the singular Kummer surface associated to $\text{Pic}^0(C)$, denoted by $S$ as in the proof of Proposition 6.10. Since $\Sigma$ is attracted by $S$ via the flow of the $G_m$-action, $\Sigma$ retracts on $S$. In particular, we obtain that

$$H^*(\Sigma) \cong H^*(S) \cong H^*(\chi|_{\Sigma}^{-1}(s))^{\mathbb{Z}/2\mathbb{Z}},$$

and the restriction $H^*(\Sigma) \rightarrow H^*(\chi|_{\Sigma}^{-1}(s))$ is injective. Hence, by Theorem 2.9, we conclude that $H^d(\Sigma)$ has top perversity $d$.

On the Betti side, $\Sigma_B$ is isomorphic to $(\mathbb{C}^*)^4/(\mathbb{Z}/2\mathbb{Z})$ (cf. (22)). This means that

$$H^*(\Sigma_B) \cong H^*((\mathbb{C}^*)^4)^{\mathbb{Z}/2\mathbb{Z}} \subset H^*((\mathbb{C}^*)^4).$$

In particular, $H^d(\Sigma_B)$ has only even cohomology of top weight $2d$, since $H^d((\mathbb{C}^*)^4)$ does. Since both the perverse and the weight filtrations are supported in top degree, the P=W conjecture for $\Sigma$ holds.

Having proved the second item in Theorem 7.4, Section 8 will be devoted to the proof of the PI=WI conjecture.

8. PI=WI CONJECTURE FOR $M$

8.1. Action of the 2-torsion of the Jacobian. — The action of $\Gamma = \text{Pic}^0(C)[2]$ induces the splitting

$$(37) \quad IH^*(M) = IH^*(M)^\Gamma \oplus IH^*_\text{var}(M),$$

where $IH^*(M)^\Gamma$ is fixed by the action of $\Gamma$, and $IH^*_\text{var}(M)$ is the variant part, i.e., the unique $\Gamma$-invariant complement of $IH^*(M)^\Gamma$ in $IH^*(M)$. Note that the decomposition (37) induces a splitting of the perverse filtration. This follows from the exactness of the perverse truncation functors $p_{\tau_{\leq i}}$, applied to the character decomposition $\chi_*\underline{\mathbb{Q}}_M \cong \chi_*\underline{\mathbb{Q}}^\Gamma_M \oplus \chi_*\underline{\mathbb{Q}}_{M,\text{var}}$.

In a similar way there exists an isomorphism of mixed Hodge structures

$$IH^*(M_B) = IH^*(M_B)^\Gamma \oplus IH^*_\text{var}(M_B).$$

This implies the following theorem.
Theorem 8.1. — The PI=WI conjecture for $M$ is equivalent to the following two statements:

1. (PI = WI conjecture for the invariant intersection cohomology)

$$P_k IH^\ast(M)^\Gamma = \Psi^* W_{2k} IH^\ast(M_B)^\Gamma, \quad k \geq 0.$$  

2. (PI = WI conjecture for the variant intersection cohomology)

$$P_k IH^\ast(M)_{\text{var}} = \Psi^* W_{2k} IH^\ast(M_B)_{\text{var}}, \quad k \geq 0.$$

We continue with the computation of the intersection Poincaré polynomial of $M$ and the intersection E-polynomial of $M_B$.

Proposition 8.2. — The intersection Poincaré polynomials are

$$IP_t(M) := \sum_k \dim IH^k(M) t^k = 1 + t^2 + 17t^4 + 17t^6,$$

$$IP_t(M)^\Gamma := \sum_k \dim IH^k(M)^\Gamma t^k = 1 + t^2 + 2t^4 + 2t^6,$$

$$IP_{t,\text{var}}(M) := \sum_k \dim IH^k_{\text{var}}(M) t^k = 15t^4 + 15t^6.$$

Proof. — By (29) and (35) we have

$$IP_t(M) = P_t(\tilde{M}) - P_t(\Sigma) t^2 - P_t(\Omega) t^6$$

$$= (1 + 2t^2 + 23t^4 + 34t^6) - (1 + 6t^2 + 1) t^2 - 16t^6$$

$$= 1 + t^2 + 17t^4 + 17t^6;$$

see also [26, Th. 6.1].

Since the differentials of the local-to-global spectral sequence (26) are $\Gamma$-equivariant, we obtain

$$P_{t,\text{var}}(\tilde{M}) = P_{t,\text{var}}(\tilde{N}) + P_{t,\text{var}}(\tilde{S}^+) t^2 + P_{t,\text{var}}(\tilde{\Theta}) t^6 + P_{t,\text{var}}(\bigcup_{j=1}^{16} T_j) t^6,$$

in the notation of Theorem 6.14. The group $\Gamma$ acts trivially on $H^\ast(\tilde{N})$ and $H^\ast(\Sigma) \simeq H^\ast(S) \subset H^\ast(\tilde{S}^+)$, and as the regular representation on the 16-dimensional vector spaces

$$\bigoplus_{j=1}^{16} \mathbb{Q}[s_j^+] \subset H^\ast(\tilde{S}^+), \quad H^0(\tilde{\Theta}), \quad \bigoplus_{j=1}^{16} \mathbb{Q}[T_j], \quad H^0(\Omega).$$

Again by (35), we get

$$IP_{t,\text{var}}(M) = P_{t,\text{var}}(\tilde{M}) - P_{t,\text{var}}(\Sigma) t^2 - P_{t,\text{var}}(\Omega) t^6$$

$$= (\dim(\bigoplus_{j=1}^{16} \mathbb{Q}[s_j^+] - 1) t^4 + \dim H^0(\bigcup_{j=1}^{16} T_j) - 1) t^6$$

$$+ (\dim H^0(\tilde{\Theta}) - 1) t^6 - (\dim H^0(\Omega) - 1) t^6$$

$$= 15t^4 + 15t^6.$$

Finally, $IP_t(M)^\Gamma = IP_t(M) - IP_{t,\text{var}}(M) = 1 + t^2 + 2t^4 + 2t^6$. 

\hfill\Box
Proposition 8.3. — The intersection E-polynomial of $M_B$ is
\[
IE(M_B) := \sum_{p,q,d} (-1)^d \dim(Gr^W_{p+q} H^d_c(M_B, \mathbb{C}))^{p,q} \cdot q^d
\]
\[
= \sum_{k,d} \dim Gr^W_{2k} IH^d(M_B)q^k = 1 + 17q^2 + 17q^4 + q^6,
\]
with $q = uv$. In particular, $\dim Gr^W_{2k+1} IH^d(M_B) = 0$ for all $k,d \in \mathbb{N}$.

Proof. — The analogue of Lemma 7.3 for compactly supported cohomology yields
\[
IE(M_B) = E(\tilde{M}_B) - E(\Sigma_B)q - E(\Omega_B)q^3.
\]
In order to compute $E(\tilde{M}_B)$, consider the stratification of $\tilde{M}_B$:
\[
\tilde{M}_B = M_B^{\text{sm}} \cup \tilde{\Sigma}_B \cup \tilde{\Omega}_B \cup \tilde{\Omega}_B,
\]
where $\tilde{\Sigma}_B \cup \tilde{\Omega}_B := f^{-1}(\Sigma_B \cup \Omega_B)$ and $\tilde{\Omega}_B := f^{-1}(\Omega_B)$. It is proved in [57, §8.2.3] that the E-polynomial of $M_B$ is $E(M_B) = 1 + q^2 + 17q^4 + q^6$. This implies that
\[
E(M_B^{\text{sm}}) = E(M_B) - E(\Sigma_B)
\]
\[
= (1 + q^2 + 17q^4 + q^6) - (1 + 6q^2 + q^4) = -5q^2 + 16q^4 + q^6,
\]
where the second equality follows from the fact that the weight filtration on $H^*_c(\Sigma_B)$ is concentrated in top degree; see Theorem 7.6. Since $\tilde{\Sigma}_B \cup \tilde{\Omega}_B$ is a $\mathbb{P}^1$-bundle over $\Sigma_B \cup \Omega_B$, we obtain
\[
E(\tilde{\Sigma}_B \cup \tilde{\Omega}_B) = E(\mathbb{P}^1) \cdot E(\Sigma_B \cup \Omega_B) = (q + 1)(1 + 6q^2 + q^4 - 16).
\]
Observe that $\tilde{\Omega}_B$ is the disjoint union of 16 smooth quadric 3-folds $\tilde{\Omega}_{B,j}$, so that
\[
E(\tilde{\Omega}_B) = \sum_{j=1}^{16} E(\tilde{\Omega}_{B,j}) = 16(1 + q + q^2 + q^3).
\]
Adding up the E-polynomials (41), (8.1) and (42), we get
\[
E(\tilde{M}_B) = 1 + q + 17q^2 + 22q^3 + 17q^4 + q^5 + q^6.
\]
Finally, from (43) and (40) we obtain
\[
IE(M_B) = 1 + 17q^2 + 17q^4 + q^6.
\]

By the vanishing of the odd intersection cohomology (cf. Proposition 8.2), every non-trivial component $(Gr^W_{p+q} H^d_c(M_B, \mathbb{C}))^{p,q}$ will contribute with non-negative coefficient to $IE(M_B)$. Therefore, there is no cancellation and by (44) any non-trivial $(Gr^W_{p+q} H^d_c(M_B, \mathbb{C}))^{p,q}$ has type $(p,p)$, i.e., the mixed Hodge structure on $IH^d_c(M_B, \mathbb{C})$ is of Hodge-Tate type. In symbols, we write
\[
IE(M_B) = \sum_{p,q,d} \dim(Gr^W_{p+q} H^d_c(M_B, \mathbb{C}))^{p,q} \cdot q^d
\]
\[
= \sum_{k,d} \dim(Gr^W_{2k} IH^d_c(M_B, \mathbb{C}))^{k,k} q^k = \sum_{k,d} \dim Gr^W_{2k} IH^d(M_B, \mathbb{C})q^k.
\]
where the last equality follows from Poincaré duality and the fact that the polynomial in (44) is palindromic. ∎

**Proposition 8.4.** — The intersection E-polynomials are

\[
IE(M_B) = \sum_{k,d} \dim Gr^{W}_k IH^d(M_B)q^k = 1 + 2q^2 + 2q^4 + q^6
\]

\[
IE_{\text{var}}(M_B) = \sum_{k,d} \dim Gr^{W}_{k,d} IH^{a}_{\text{var}}(M_B)q^k = 15q^2 + 15q^4.
\]

**Proof:** — The solution of the linear system

\[
\begin{align*}
\text{dim} Gr^W_k IH^d(M_B) &\leq \text{dim} Gr^W_k IH^d(\tilde{M}_B) = 0 \text{ for } k < d \quad \text{[71, Prop. 4.20]} \\
\text{dim} Gr^W_{2k+1} IH^d(M_B) &\leq 0 \quad \text{Proposition 8.3} \\
\sum_k \text{dim} Gr^W_{k} IH^d(M_B)q^k = 1 + 17q^2 + 17q^4 + q^6 \quad \text{Proposition 8.3} \\
\sum_k \text{dim} IH^k(M_B)t^k = 1 + t^2 + 17t^4 + 17t^6 \quad \text{Proposition 8.2}
\end{align*}
\]

is given by

\[
\begin{align*}
(45) \quad \text{dim} Gr^W_{4d} IH^d(M_B) &= 1 \quad \text{for } d = 0, 1, 2, 3, \\
(46) \quad \text{dim} Gr^W_k IH^4(M_B) &= \text{dim} Gr^W_s IH^6(M_B) = 16.
\end{align*}
\]

The terms in this list are all the non-zero graded pieces of the mixed Hodge structure on \(IH^*(M_B)\).

Note that the top graded pieces \(\text{Gr}^W_{2d} IH^d(M_B)\) are generated by \(\alpha^d\), where \(\alpha\) is a \((\Gamma\text{-invariant})\) generator of \(IH^2(M_B)\). The class \(\alpha\) corresponds via the non-abelian Hodge correspondence to the first Chern class of a \(\chi\)-ample (or \(\chi\)-anti-ample) divisor on \(M\). In particular, \(\alpha^2\) and \(\alpha^3\) are non-zero and \(\Gamma\text{-invariant}\). This implies that

\[
\begin{align*}
IH^4_{\text{var}}(M_B) &\subset W_4 IH^4(M_B) \simeq \text{Gr}^W_4 IH^4(M_B), \\
IH^6_{\text{var}}(M_B) &\subset W_8 IH^6(M_B) \simeq \text{Gr}^W_8 IH^6(M_B).
\end{align*}
\]

Together with Proposition 8.2 and Proposition 8.3, we conclude that

\[
\begin{align*}
IE_{\text{var}}(M_B) &= \text{dim} Gr^W_k IH^4_{\text{var}}(M_B)q^2 + \text{dim} Gr^W_s IH^6_{\text{var}}(M_B)q^4 \\
&= \text{dim} IH^4_{\text{var}}(M_B)q^2 + \text{dim} IH^6_{\text{var}}(M_B)q^4 = 15q^2 + 15q^4 \\
IE(M_B) &= IE(M_B) - IE_{\text{var}}(M_B) \\
&= (1 + 17q^2 + 17q^4 + q^6) - (15q^2 + 15q^4) = 1 + 2q^2 + 2q^4 + q^6. \quad \Box
\end{align*}
\]

As a result, an analogue of [41, Cor. 4.5.1] holds for \(M_B\).

**Corollary 8.5.** — The intersection form on \(H^0_c(M_B) = IH^0_c(M_B)\) is trivial. Equivalently, the forgetful map \(H^6_c(M_B) \to H^6(M_B)\) is zero.

**Proof:** — By (45), (46) and Poincaré duality, the weight filtrations on \(IH^6_c(M_B)\) and \(IH^6_{\text{var}}(M_B)\) are concentrated in degree [8, 12] and [0, 4]. Since the forgetful map is a morphism of mixed Hodge structures, it has to vanish. □
Remark 8.6 (Failure of curious hard Lefschetz). — By (28) and the proof of Proposition 8.4, we have
\[ \sum_{k,d} \dim \text{Gr}^{W}_{2k} H^{d}(M_{B}) q^{k} = 1 + q^{2} + 17q^{4} + q^{6}. \]

The fact that the polynomial (47) is not palindromic implies that curious hard Lefschetz (2) fails for $H^{*}(M_{B})$. Analogously, one can show that relative hard Lefschetz fails for $H^{*}(M)$.

8.2. The variant part of $IH^{*}(M)$. — The goal of this section is to show that the PI = WI conjecture for the variant part of $IH^{*}(M)$ holds. As we will explain in the proof of Theorem 8.8, it is enough to prove it in degree 4 and 6.

Proposition 8.7. — $IH^{4}_{\text{var}}(M) \subset P_{2}IH^{4}(M)$.

Proof. — The argument of [11, §4.4] and [14, Prop.1.4] works with few changes.

The endoscopic locus $\mathcal{A}_{e} \subset H^{0}(C, K_{C} \otimes 2)$ is the subset of sections $s' \in H^{0}(C, K_{C} \otimes 2)$ such that the Prym variety associated to the corresponding spectral curve $C_{s'}$ is not connected (cf. [11, §4.4]). It is the union of 15 lines, obtained as images of the squaring map $i_{L}: H^{0}(C, K_{C} \otimes L) \to H^{0}(C, K_{C} \otimes 2)$, $i_{L}(a) = a \otimes a$, where $L \in \Gamma \setminus \{0\}$. In particular, a general affine line $\Lambda^{1}$ in $H^{0}(C, K_{C} \otimes 2)$ does not intersect $\mathcal{A}_{e}$. It is important to remark that $\Gamma$ acts trivially on $H^{*}(\chi^{-1}(s))$ for any $s \in \Lambda^{1}$: the proof in [11, §4.4] is independent of the choice of the degree of the Higgs bundles, and so it holds also in the untwisted case. This implies

$H^{*}(\chi^{-1}(\Lambda^{1}))^{\Gamma} = H^{*}(\chi^{-1}(\Lambda^{1})) = IH^{*}(\chi^{-1}(\Lambda^{1}))$,

where the last equality follows from the fact that $\chi^{-1}(\Lambda^{1})$ has quotient singularities.

We conclude by Theorem 2.9 that

$IH^{4}_{\text{var}}(M) \subset \text{Ker}\{IH^{4}(M) \to IH^{4}(\chi^{-1}(\Lambda^{1})) = IH^{4}(\chi^{-1}(\Lambda^{1}))^{\Gamma}\} = P_{2}IH^{4}(M)$,

because the restriction map is $\Gamma$-equivariant. \hfill \square

Theorem 8.8. — The PI = WI conjecture for the variant intersection cohomology of $M$ (39) holds.

Proof. — The variant Poincaré polynomial in Proposition 8.2 shows that $IH^{*}_{\text{var}}(M)$ is concentrated in degree 4 and 6.

By relative hard Lefschetz, we can write

$Gr^{0}_{0} IH^{4}(M) \simeq Gr^{0}_{0} IH^{10}(M),$ \hspace{1cm} $Gr^{P}_{1} IH^{4}(M) \simeq Gr^{P}_{5} IH^{8}(M),$

which both vanish by Proposition 8.2. Together with Proposition 8.7 and the proof of Proposition 8.4, this implies

$P_{2}IH^{4}_{\text{var}}(M) = IH^{4}_{\text{var}}(M) = \Psi^{*} IH^{4}_{\text{var}}(M_{B}) = \Psi^{*} W_{4} IH^{4}_{\text{var}}(M_{B}).$

This proves the PI=WI conjecture for the variant part in degree 4.
Again by relative hard Lefschetz, there exists a \( \chi \)-ample \( \alpha \in H^2(M) \) such that the cup product \( \cup \alpha \) induces the isomorphism

\[
\cup \alpha : H^4_{\text{var}}(M) \simeq \text{Gr}^P_4 H^4_{\text{var}}(M) \longrightarrow \text{Gr}^P_4 H^6_{\text{var}}(M).
\]

By Proposition 8.2 we obtain that

\[
15 = \dim H^4_{\text{var}}(M) = \dim \text{Gr}^P_4 H^6_{\text{var}}(M) \leq \dim H^6_{\text{var}}(M) = 15.
\]

This implies that the cup product

\[
\cup \alpha : H^4_{\text{var}}(M) \longrightarrow H^6_{\text{var}}(M)
\]

is an isomorphism, which preserves the perverse and weight filtrations; see [11, Lem.1.4.4]. Therefore, the PI=WI conjecture for the variant part holds in degree 6 as well.

\section{8.3. A tautological class.}

We show now that \( H^4(M)^\Gamma \) is generated by the square of the relatively ample class \( \alpha \), and a class of perversity 2 and weight 4. As usual, we adopt the notation of the previous sections, and in particular of Section 6.2.

Consider the forgetful map \( q : M_\iota \rightarrow M \). The action of \( \Gamma \) on \( M \) lifts to \( M_\iota \), and together with the deck transformation of \( q \), we obtain a group of symmetries of order 32, denoted \( \Gamma_\iota \).

**Proposition 8.9.** — \( H^4(M)^\Gamma = H^4(M_\iota^{\text{sm}})^{\Gamma_\iota} \).

**Proof.** — Since \( M_\iota \) has isolated singularities by Proposition 6.23, we have that \( H^4(M_\iota) = H^4(M_\iota^{\text{sm}}) \); see [31, §1.7] or [25, Lem.1]. The proof of [32, Prop.3] implies that

\[
H^4(M)^\Gamma = H^4(M_\iota)^{\Gamma_\iota} = H^4(M_\iota^{\text{sm}})^{\Gamma_\iota}.
\]

Fix a base point \( c \in C \). Recall that \( E \) is a universal bundle on \( M_\iota^{\text{sm}} \times C \); see Section 6.2.5.

**Definition 8.10.** — The space \( R \) is the total space of the projective bundle \( \mathbb{P}(E)|_{M_\iota^{\text{sm}} \times \{c\}} \). Its associated principal \( \text{PGL}_2 \)-bundle parametrizes equivariant Higgs bundles \( (E, h, \phi) \) together with a frame for the fiber \( E_c \), up to rescaling.

The second Chern class of a \( \mathbb{P}^1 \)-bundle is the pull-back of a generator of \( H^2(\text{BPGL}_2) \cong \mathbb{Q} \) via the classifying map. In particular, if the \( \mathbb{P}^1 \)-bundle is a projectivization of the rank-two vector bundle \( E \), then

\[
c_2(\mathbb{P}(E)) = c_1^2(E) - 4c_2(E).
\]

**Proposition 8.11.** — The second Chern class \( c_2(R) \) of the projective bundle \( R \) and the square of the \( \chi \)-ample class \( \alpha \) generate \( H^4(M)^\Gamma \)

\[
H^4(M)^\Gamma = \mathbb{Q} \alpha^2 \oplus \mathbb{Q} c_2(R).
\]

**Proof.** — The proposition is a consequence of the following facts:

1. \( c_2(R) \in H^4(M_\iota^{\text{sm}})^{\Gamma_\iota} = H^4(M)^{\Gamma_\iota} \), since the \( \Gamma_\iota \)-action lifts to \( R \).
2. \( \alpha^2 \in H^4(M)^\Gamma \subset H^4(M)^\Gamma \).
Lemma 6.29 and Proposition 6.28 give

\[ c_2(R) \neq 0 \] by Lemma 8.12.

(4) \( \alpha^2 \) and \( c_2(R) \) are linearly independent, because \( \alpha^2 \) has top perversity by Theorem 2.9, while \( c_2(R) \in \text{Pic}^4(M) \); see Lemma 8.14.

(5) \( \dim \text{IH}^4(M)^{\Gamma} = 2 \) by Proposition 8.2.

We now prove the lemmas used in the proof above.

**Lemma 8.12.** \( c_2(R) \neq 0 \).

**Proof.** Let \( r_j : R_j^s \times C \to (M_j^s \cap N_j^s) \times C \) be the algebraic bijection defined in 6.2.6(30). Lemma 6.29 and Proposition 6.28 give

\[ r_j^* c_2(R) = c_2(\text{Pic}(E_{|R_j^s \times \{c\}})) \]

\[ = (c_1(p_{R_j}^{*}\theta_j^{1}(1) \otimes p_{R_j}^{*}\theta_j^{-1}) - c_1(p_{C}^{*}\theta_j))|_{R_j^s \times \{c\}} = c_1(\theta_R^j(1))^2. \]

In particular, \( 0 \neq c_1(\theta_R^j(1))^2 \in \text{IH}^4(R^s_j) \simeq \text{IH}^4(\mathbb{P}^3) \).

**Lemma 8.13.** \( c_2(R) \in \text{Pic}^4(M) \).

**Proof.** Fix \( s \) a generic point in \( \text{H}^0(C, K_C^{\otimes 2}) \), and let \( p_s : C_s \to C \) the corresponding spectral curve, i.e., the double cover of \( C \) ramified along the zeroes of \( s \); see for instance [3, §3]. We denote the product map \( p_s \times \text{id} : C_s \times \text{Pic}^0(C_s) \to C \times \text{Pic}^0(C_s) \) simply by \( p_s \). A universal bundle on \( \chi^{-1}(s) \times C \simeq C \times \text{Pic}^0(C_s) \) does exist, and it is isomorphic to \( p_s \times \mathcal{P} \) where \( \mathcal{P} \) is the Poincaré line bundle over \( C_s \times \text{Pic}^0(C_s) \).

The abelian variety \( (\chi \circ q)^{-1}(s) \) parametrizes line bundles of \( C_s \) decorated with a lift of the hyperelliptic involution \( \iota : C \to C \). This implies that the restriction of \( E \) to \( (\chi \circ q)^{-1}(s) \times C \) is isomorphic to \( q^* (p_s, \text{id}), \mathcal{P} \), up to tensorization by a line bundle in \( \text{Pic}((\chi \circ q)^{-1}(s)) \).

As a result, we have that

\[ c_2(R)|_{\chi^{-1}(s)} = c_1^2(\mathcal{P}(\chi \circ q)^{-1}(s) \times \{c\}) - 4c_2(\mathcal{P}(\chi \circ q)^{-1}(s) \times \{c\}) \]

\[ = q^* (c_1^2((p_s, \mathcal{P})|_{\chi^{-1}(s) \times \{c\}}) - 4c_2((p_s, \mathcal{P})|_{\chi^{-1}(s) \times \{c\}})) = 0, \]

where the last equality follows from [84, §4] or [11, Eq. (5.1.10) & (5.1.11)]. This implies that \( c_2(R) \) does not have top perversity by Theorem 2.9.

**Lemma 8.14.** \( c_2(R) \in \text{Pic}^4(M) \).

**Proof.** By Lemma 8.13, it is enough to show that the projection \([c_2(R)]\) in the graded piece \( \text{Gr}^R_2 \text{IH}^4(M)^{\Gamma} \) vanishes. Suppose on the contrary that \([c_2(R)] \neq 0 \). Then Proposition 8.2 would imply

\[ \dim \text{Gr}^R_2 \text{IH}^4(M)^{\Gamma} \leq \dim \text{IH}^4(M)^{\Gamma} - \dim(\mathbb{Q} \alpha^2 \oplus \mathbb{Q} c_2(R)) = 2 - 2 = 0. \]

By relative hard Lefschetz, \( \text{Gr}^R_4 \text{IH}^6(M)^{\Gamma} \) would be trivial. Analogously,

\[ \text{Gr}^R_2 \text{IH}^6(M)^{\Gamma} \approx \text{Gr}^R_2 \text{IH}^2(M)^{\Gamma} = 0. \]
Again by Proposition 8.2 we would conclude that
\[ \dim \text{Gr}_3^P \mathcal{H}^6(M)^\Gamma = \dim \mathcal{H}^6(M)^\Gamma - \sum_{k=4}^6 \dim \text{Gr}_k^P \mathcal{H}^6(M)^\Gamma \]
\[ = \dim \mathcal{H}^6(M)^\Gamma - \dim \mathbb{Q} \alpha^3 = 2 - 1 = 1. \]

However, this is a contradiction by Corollary 8.5. \(\square\)

**Lemma 8.15.** — \(\text{P}_3 \mathcal{H}^6(M)^\Gamma = 0.\)

**Proof.** — Lemma 7.2 gives the splitting
\[ P_3 \mathcal{H}^6(\tilde{M})^\Gamma = P_3 \mathcal{H}^6(\tilde{M})^\Gamma \oplus P_2 \mathcal{H}^4(\Sigma)^\Gamma \oplus \mathcal{H}^0(\Omega)^\Gamma. \]

The P=W conjecture for \(\Sigma\) gives \(P_2 \mathcal{H}^4(\Sigma) = 0.\) Moreover, we have that \(\mathcal{H}^0(\Omega)^\Gamma \simeq \mathbb{Q}[\Omega];\) see the proof of Proposition 8.2. Therefore, we get
\[ P_3 \mathcal{H}^6(\tilde{M})^\Gamma = P_3 \mathcal{H}^6(\tilde{M})^\Gamma \oplus \mathcal{Q}[\Omega]. \]

Up to a different numbering convention, [16, Th. 2.1.10] says that the dimension of \(P_3 \mathcal{H}^6(\tilde{M})^\Gamma\) is not greater than the rank of the intersection form on \(\mathcal{H}^0(\tilde{M})^\Gamma.\) Therefore, Corollary 8.5 implies
\[ \dim P_3 \mathcal{H}^6(\tilde{M})^\Gamma \leq 1. \]

We conclude that \(\dim P_3 \mathcal{H}^6(M)^\Gamma = 0.\) \(\square\)

We conclude the section by showing that the class \(c_2(R)\) has weight 4.

**Lemma 8.16.** — \(c_2(R) \in W_4 \mathcal{H}^4(M_B).\)

**Proof.** — The principal PGL₂-bundle \(\mathcal{S} \to M^\text{sm}_B := \Psi_1(M^\text{sm}_i)\) is the restriction of the quotient \(\text{Hom}(\pi_1^{\text{orb}}(C/\iota), \text{SL}_2) \to M_B(2, \text{SL}_2, \iota).\) It parametrizes \(\iota\)-equivariant local systems \(E'\) on \(C\) together with a frame for the fiber \(E'_c\) over \(c \in C,\) i.e., the base point of \(\pi_1^{\text{orb}}(C/\iota) = \pi_1^{\text{orb}}(C/\iota, c),\) up to rescaling.

By construction, the non-abelian Hodge correspondence \(\Psi_1: M^\text{sm}_i \to M^\text{sm}_B; (\text{Theorem 6.21})\) extends to a diffeomorphism between the principal PGL₂-bundle associate to \(R\) and \(\mathcal{S}.\) This implies that \(c_2(R) = (\Psi_1^{-1})^*c_2(\mathcal{S}),\) and \(c_2(\mathcal{S})\) has weight 4 by [22, Th. 9.1.1, Prop. 9.1.2]. \(\square\)

**8.4. The invariant part of \(\mathcal{H}^*(M)\)**

**Theorem 8.17.** — *The invariant PI=W1 conjecture (38) holds for \(M.\)*

**Proof.** — The statement is obvious in degree 0 and 2, because \(\mathcal{H}^0(M)^\Gamma\) and \(\mathcal{H}^2(M)^\Gamma\) have dimension one by Proposition 8.2 and Proposition 8.4.

Now we have
\[ P_2 \mathcal{H}^4(M)^\Gamma \simeq W_4 \mathcal{H}^4(M_B)^\Gamma, \]
\[ \text{Gr}_3^P \mathcal{H}^4(M)^\Gamma \simeq \text{Gr}_3^W \mathcal{H}^4(M_B)^\Gamma = 0, \]
\[ P_3 \mathcal{H}^4(M)^\Gamma = \mathcal{H}^4(M)^\Gamma \simeq \mathcal{H}^4(M_B)^\Gamma = W_4 \mathcal{H}^4(M_B)^\Gamma, \]

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due to Proposition 8.11, Lemma 8.16 and Proposition 8.4. This proves the invariant $P = \mathcal{W}$ conjecture in degree 4.

By relative hard Lefschetz, the cup product with the $\chi$-ample $\alpha \in H^2(M)$ induces the isomorphisms
\[
\cup \alpha : Gr^P H^4(M)^\Gamma \to Gr^P H^6(M)^\Gamma,
\]
\[
\cup \alpha : Gr^P H^4(M) \simeq \mathbb{Q}[\alpha^2] \to Gr^P H^6(M) \simeq \mathbb{Q}[\alpha^3].
\]
Note that $Gr^P H^4(M)^\Gamma$ and $Gr^P H^4(M)$ are the only non-trivial pieces of the perverse filtration on $H^4(M)^\Gamma$, and by Proposition 8.2 we have that $\dim H^4_{\var}(M)^\Gamma = \dim H^6_{\var}(M)^\Gamma$. This implies that the cup product
\[
\cup \alpha : H^4(M)^\Gamma \to H^6(M)^\Gamma
\]
is an isomorphism which preserves both perverse and weight filtration (cf. [11, Lem. 1.4.4]). Therefore, the invariant $P = \mathcal{W}$ conjecture holds in degree 6, as well. □

Appendix. Degenerations of hyperkähler varieties

In this appendix we describe degenerations of compact hyperkähler manifolds to (non-compact) symplectic resolutions of Dolbeault moduli spaces. Instances of these constructions can be found in [23], [15], [14]. Here a degeneration is a flat (not necessarily proper) morphism of normal algebraic varieties, typically over a curve.

The compact hyperkähler manifolds appearing in these degenerations are Mukai moduli spaces of sheaves on a K3 surface or an abelian surface $S$. Given an effective Mukai vector\(^{(4)}\) $v \in H^2_{\text{alg}}(S, \mathbb{Z})$, we denote by $M(S, v)$ the moduli space of Gieseker semistable sheaves on $S$ with Mukai vector $v$ for a sufficiently general polarization $H$ (which we will typically omit in the notation); see [78, §1]. Further, if $S$ is an abelian variety with dual $\hat{S}$, and $\dim M(S, v) \geq 6$, then the Albanese morphism $\text{alb}_S : M(S, v) \to S \times S$ is isotrivial, and we set $K(S, v) := \text{alb}_S^{-1}(0_S, 0_S)$. By [70], the moduli space $M(S, v)$ of sheaves on the K3 surface $S$ and the moduli space $K(S, v)$ of sheaves on the abelian surface $S$ are irreducible holomorphic symplectic varieties, in brief IH Sv.

A.1. Deformation to the normal cone: $\text{GL}_n$ case. — Let $j : X \hookrightarrow S$ be the embedding of a smooth projective curve\(^{(5)}\) of genus $g$ into a K3 surface $S$. The deformation to the normal cone of $j : X \hookrightarrow S$ is the family
\[
\mathcal{F} = (\text{Bl}_{X \times 0} S \times \mathbb{A}^1) \setminus ((S \times 0) \to \mathbb{A}^1).
\]
The central fiber $\mathcal{F}_0$ is isomorphic to $T^* X$, while the restriction to $\mathbb{A}^1 \setminus \{0\}$ is a trivial fibration $S \times (\mathbb{A}^1 \setminus \{0\}) \to \mathbb{A}^1 \setminus \{0\}$.

---

\(^{(4)}\) i.e., there exists a coherent sheaf $\mathcal{F}$ on $S$ such that $v = (\text{rk}(\mathcal{F}), c_1(\mathcal{F}), \chi(\mathcal{F}) - c(\text{rk}(\mathcal{F})))$, with $c(S) = 1$ if $S$ is K3, and $0$ if $S$ is abelian.

\(^{(5)}\) In [23] $X$ is a very ample divisor, but this assumption can be dropped.
For all \( t \in A^1 \), let \( \beta_t = n[X] \in H_2(\mathcal{S}, \mathbb{Z}) \) with \( n > 0 \). Take a relative compactification \( \mathcal{S} \subset \overline{\mathcal{S}} \) over \( A^1 \). Then

\[
M \longrightarrow A^1
\]

is the coarse relative moduli space of one-dimensional Gieseker semistable sheaves \( \mathcal{F} \) whose support is proper and contained in \( \mathcal{S} \subset \overline{\mathcal{S}} \) with \( \chi(\mathcal{F}) = n(1 - g) \) and \( [\text{Supp}\mathcal{F}] = \beta_t \); see [78, Th.1.21]. The central fiber recovers the Dolbeault moduli space

\[
M_0 \simeq M_{\text{Dol}}(X, \text{GL}_n).
\]

Indeed, the moduli space of Higgs bundles on \( X \) of rank \( n \) and degree 0 can be realized as the moduli space of one-dimensional Gieseker-semistable sheaves \( \mathcal{F} \) on \( T^*X \) with \( \chi(\mathcal{F}) = n(1 - g) \) and \( [\text{Supp}\mathcal{F}] = \beta_0 \), via the BNR-correspondence [3]. The general fiber is isomorphic to

\[
M_t \simeq M(S,v)
\]

with Mukai vector \( v = (0, nX, n(g - 1)) \).

**Example A.1 (genus one: K3\(^{(n)}\)).** — If \( g = 1 \), then the degeneration \( \mathcal{M} \to A^1 \) is the relative \( n \)-fold symmetric product of \( \mathcal{S} \). The relative Hilbert-Chow morphism \( \tilde{\mathcal{M}} \to \mathcal{M} \) is a desingularization of \( \mathcal{M} \). The composition \( \tilde{\mathcal{M}} \to \mathcal{M} \to A^1 \) is a family whose general fiber is the compact hyperkähler manifold \( S^{[n]} \) and whose central fiber is \( (T^*X)^{[n]} \), i.e., the symplectic resolution of \( M_{\text{Dol}}(X, \text{GL}_n) \simeq (T^*X)^{(n)} \).

**Example A.2 (genus two and rank two: O’Grady 10).** — If \( (g, n) = (2, 2) \), then the blow-up \( \tilde{\mathcal{M}}_t \) of the singular locus of \( \mathcal{M}_t \simeq M(S,v) \) is a smooth compact hyperkähler manifold deformation equivalent to OG10; see for instance [69]. Analogously, the blow-up \( \tilde{\mathcal{M}}_0 \) of the singular locus of \( \mathcal{M}_0 \simeq M_{\text{Dol}}(X, \text{GL}_2) \) gives the symplectic resolution of \( \mathcal{M}_0 \). Note that the proof of [69, Prop.2.16] shows that the degeneration \( \mathcal{M} \to T \) is locally analytically trivial. Therefore, the blow-up \( \tilde{\mathcal{M}} \) of the singular locus of \( \mathcal{M} \) is a smooth family over \( A^1 \) whose general member is deformation equivalent to OG10 and whose central fiber is the symplectic resolution of \( M_{\text{Dol}}(X, \text{GL}_2) \).

**Remark A.3.** — Taking schematic supports via Fitting ideals defines a Lagrangian morphism \( M(S,v) \to [nX] \), called the Mukai system. It is classically known that the Mukai system degenerates to the Hitchin fibration, see [23].

**Remark A.4.** — If the Mukai vector \( (0, X, g - 1) \) is primitive (e.g. if \( \text{Pic}(S) = \mathbb{Z}X \)), then the second author observed in [59, Rem.2.5] that the degeneration \( \mathcal{M} \to A^1 \) is locally analytically trivial. Therefore, the functorial resolution \( \mathcal{R}(\mathcal{M}) \to \mathcal{M} \) gives a simultaneous resolution of \( \mathcal{M}_t \) for any \( t \in A^1 \); see for instance [33, Lem.4.2].

**A.2. Deformation to the normal cone: SL\(_n\) case.** — Suppose now that \( X \) is a smooth projective curve embedded in an abelian surface \( S \). To avoid confusion, we relabel \( S \) by \( A \). \(^{(6)}\) As in the previous section, there exists a degeneration \( \mathcal{M} \to A^1 \)

\(^{(6)}\)In this section we denote by \( A \) an abelian surface, and not a curve of genus one as in the rest of the paper.
from the moduli space $M(A, v)$ to $M_{Dol}(X, GL_n)$. In this case, however, $M(A, v)$ is no longer an IHSv because of the Albanese morphism $\text{alb}_S: M(A, v) \to \tilde{A} \times A$.

In genus one and two it is possible to slice $\mathcal{M}$ to obtain a degeneration of the IHSv $K(A, v)$ to $M_{Dol}(X, SL_n)$.

**Example A.5 (genus one: $K^{[n]}(A)$).** — If $g = 1$, then the family
\[ \mathcal{A} = (\text{Bl}_{C \times 0} A \times \mathbb{A}^1) \setminus (A \times 0) \to \mathbb{A}^1 \]
is a group scheme, and the degeneration $\mathcal{M} \to \mathbb{A}^1$ is the relative $n$-fold symmetric product $\mathcal{A}^{(n)} \to \mathbb{A}^1$ whose general fiber is $A^{(n)}$, and whose central fiber is $(T^*C)^{(n)}$.

Consider now the relative addition map $a_n: \mathcal{A}^{(n)} \to \mathcal{A}$, given by $a_n(x_1, \ldots, x_n) = \sum_{i=1}^n x_i$. The inverse image of the identity section of $\mathcal{A} \to \mathbb{A}^1$ under the addition map is a degeneration
\[ \tilde{\mathcal{K}} \to \mathbb{A}^1 \]
whose general fiber is the singular generalized Kummer variety $K(A, v) \simeq K^{(n)}(A)$ and whose central fiber is $M_{Dol}(X, SL_n) \simeq K^{(n)}(T^*C)$. The inverse image of the identity section of $\mathcal{A} \to \mathbb{A}^1$ under the composition $\mathcal{A}^{(n)} \to \mathcal{A}$ is a degeneration
\[ \tilde{\mathcal{K}} \to \mathcal{K} \to \mathbb{A}^1 \]
whose general fiber is the generalized Kummer manifold $K^{[n]}(A)$ and whose central fiber is the symplectic resolution of $M_{Dol}(X, SL_n)$.

**Example A.6 (genus two).** — If $g = 2$, the Albanese map [88]
\[ \text{alb}_S: M(A, v) \to \tilde{A} \times A \]
degenerates to the map
\[ \text{alb}: M_{Dol}(X, GL_n) \to \text{Pic}^0(X) \times H^0(X, K_X) \simeq \tilde{A} \times \mathbb{A}^g, \]
defined in (8); see [14, §4]. Taking fibers over the identity, one obtains a family $\mathcal{K} \to \mathbb{A}^1$ such that the central fiber is $M_{Dol}(X, SL_n)$ and the general fiber is the IHSv $K(A, v)$.

**Example A.7 (genus two and rank two: O’Grady 6).** — The symplectic resolution $f_A: \tilde{K}(A, v) \to K(A, v)$, with $v = (0, 2X, 2)$ and $g = 2$, is a compact hyperkähler manifold of OG6 type. Let $\tilde{\mathcal{K}}$ be the blow-up of the singular locus of the variety $\mathcal{K}$ obtained in Example A.6, with $(g, n) = (2, 2)$. Then $\tilde{\mathcal{K}} \to \mathbb{A}^1$ is a degeneration of $\tilde{K}(A, v)$ to the Dolbeault moduli space $M$ in §6. Further, as in Example A.2, $\tilde{\mathcal{K}}$ is a smooth family over $\mathbb{A}^1$ whose general member is the compact hyperkähler manifold $\tilde{K}(A, v)$ of OG6 type and whose central fiber is the symplectic resolution $\tilde{M}$ of $M$.

We observe that the cohomology of $\tilde{K}(A, v)$ governs the cohomology of $\tilde{M}$ in the following sense.

**Proposition A.8.** — The specialization morphism [15, (86)]
\[ \text{sp}^1: H^*(\tilde{K}(A, v)) \to H^*(\tilde{M}) \]
is a surjection.
Proof: — The following facts hold:

- The Mukai system $\chi_A: \widetilde{K}(A, v) \rightarrow |2X|$ specializes to the Hitchin fibration $\chi \circ f: \widetilde{M} \rightarrow H^0(X, K_X^{2d})$. In particular, a $\chi_A$-ample line bundle on $\widetilde{K}(A, v)$ specializes to a generator of $H^2(\widetilde{M})$.

- The fiber $\chi_A^{-1}(2X)$ consists of 34 irreducible components which specialize to the irreducible components of the nilpotent cone of $\widetilde{M}$ that generate $H^0(\widetilde{M})$; see [87, Prop. 3.0.3].

- Denote by $\Sigma_A$ and $\Sigma$ the singular locus of $K(A, v)$ and $M$, isomorphic to $(A \times \hat{A})/\pm 1$ and $(\hat{A}^2 \times \hat{A})/\pm 1$ respectively. As in (35), $H^{*+2}(\Sigma_A)$ is a direct summand of $H^*(K(A, v))$. By definition of $sp^i$ in [15, (86)], the restriction of the specialization map to $H^{*+2}(\Sigma_A)$ is the pullback

$$(\hat{A}^2 \times \hat{A})/\pm 1 \hookrightarrow P(T^* \hat{A} \oplus \Theta_A)/\pm 1 \hookrightarrow (\text{Bl}_{0 \times \hat{A}} \times 0 \times A \times \hat{A} \times \hat{A}^1)/\pm 1 \rightarrow (A \times \hat{A} \times \hat{A}^1)/\pm 1 \rightarrow (A \times \hat{A})/\pm 1.$$

So, given the inclusion $j: (0 \times \hat{A})/\pm 1 \hookrightarrow (A \times \hat{A})/\pm 1$, we have

$$sp^i(\gamma) = j^*\gamma \in H^*(\hat{A}/\pm 1) \cong H^*((\hat{A}^2 \times \hat{A})/\pm 1)$$

for $\gamma \in H^*((A \times \hat{A})/\pm 1)$, which is a surjection.

We conclude that

$$\text{Im}(sp^i) \supset H^2(\widetilde{M}) \oplus H^6(\widetilde{M}) \oplus H^{*+2}(\Sigma_A).$$

By the description of $H^*(\widetilde{M})$ (cf. Fig. 1) and relative hard Lefschetz, this suffices to show that $\text{Im}(sp^i)$ equals the whole $H^*(\widetilde{M})$. □

Remark A.9. — Recall that for any odd number $d$ the twisted Dolbeault moduli space $M^{tw}(X, SL_2, d)$ parametrizes semistable $SL_2$-Higgs bundles of degree $d$ on the curve $X$. It is curious that the analogue of Proposition A.8 fails for $M^{tw}(X, SL_2, d)$ and $g = 2$: there is no degeneration of compact hyperkähler manifolds to $M^{tw}(X, SL_2, d)$ such that the specialization map $sp^i$ is surjective; see [14, Prop. 4.3].

Example A.10 (genus > 2). — There is no degeneration from $K(A, v)$ with Mukai vector $v = (0, nX, n(g - 1))$ with $g > 2$ to $M_{\text{Dol}}(X, SL_n)$ for dimensional reason. However, $K(A, v)$ and $M_{\text{Dol}}(X, SL_n)$ have the same type of singularities: they are stably isosingular in the sense of [59, Def. 2.6 & Th. 2.11]. Therefore, it is natural to ask the following.

Question. — Does there exist a degeneration of compact symplectic varieties equipped with a Lagrangian fibration in Prym varieties to the Hitchin fibration

$$\chi(X, SL_n): M_{\text{Dol}}(X, SL_n) \rightarrow \bigoplus_{i=2}^n H^0(X, K_X^{2i})$$

for $g > 2$?

Note that the question is answered positively in [72] if we replace the special linear group $SL_n$ with the symplectic group $Sp_n$. 

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