Monogamy is one of crucial properties of entanglement. It is essential in quantum cryptography. A simple example is the Bell state $|\Phi\rangle_{AB} = 1/\sqrt{2}(|00\rangle + |11\rangle)$ shared between Alice and Bob. Monogamy of the pure entangled state $|\Phi\rangle_{AB}$ excludes any possibility that another party including the potential eavesdropper Eve could correlate. The monogamous property of the pure entangled state is extended to the un-sharable property for the mixed state in a recent paper \cite{1,2,3}. In this paper, we give a simple proof of the monogamy of entanglement and provide an upper bound of the number of parties.

**Definition 1** A bipartite state $\rho_{AB}$ is said to be $n$-sharable when it is possible to find a quantum state $\rho_{AB_1B_2\ldots B_n}$ such that $\rho_{AB_1} = \rho_{AB_2} = \cdots = \rho_{AB_n} = \rho_{AB}$ where $\rho_{AB_n} = tr_B^n\rho_{AB_1B_2\ldots B_n}$. If such state exists, $\rho_{AB_1B_2\ldots B_n}$ is called as an $n$-extension of $\rho_{AB}$.

**Theorem 1** \cite{2,3} A bipartite state is $n$-sharable for any $n$ if and only if it is separable.

**Proof.** For a separable state $\rho_{AB}$, there always exists a separable decomposition

$$\rho_{AB} = \sum_i p_i (|\phi_i\rangle_A \langle \phi_i|_A \otimes |\psi_i\rangle_B \langle \psi_i|_B). (1)$$

It is explicit that

$$\rho_{AB_1B_2\ldots B_n} = \sum_i p_i (|\phi_i\rangle_A \langle \phi_i|_A \otimes |\psi_i\rangle_B \langle \psi_i|_B) \otimes^n_{i=1} \rho_{B_i}$$

is a valid $n$-extension of $\rho_{AB}$ for any $n$.

Next we prove that for any entangled state $\rho_{AB}$, there always exists a finite $N$ such that no valid $n$-extension can be found for any $n > N$. Recall that the duality relation of a pure tripartite state $\phi_{ABC}$ is in \cite{6}

$$S(\rho_A) = E_f(\rho_{AB}) + C_-(\rho_{AC}). (3)$$

Here $S(\rho_A) = -tr\rho_A \log \rho_A$ is the von Neumann entropy of $\rho_A$. $E_f(\rho_{AB}) = \min \sum_i p_i E(\phi_{AB}^i)$ is the entanglement of formation (EoF) \cite{7}, where the minimum is taken over all pure ensembles $\{p_i, |\phi^i\rangle_{AB}\}$ satisfying $\rho_{AB} = \sum_i p_i (|\phi^i\rangle\langle \phi^i|)_{AB}$, and entanglement for pure state $\phi_{AB}$ is $E(\phi_{AB}) = S(tr_B(|\phi\rangle \langle \phi|)_{AB})$. $C_-(\rho_{AC}) = \max_{C_1^i} S(\rho_A) - \sum_i p_i S(\rho_A^i)$ is the classical correlation of bipartite state $\rho_{AC}$ \cite{8}, where $\{C_1^i\}$ is a positive operator-valued measurement (POVM) performed on subsystem $C$, $\rho_A^i = tr_C(I \otimes C_i \rho_{AC} I \otimes C_i^\dagger)$ is the remaining state of $A$ after obtaining the outcome $i$ on $C$, and $p_i = tr_{AC}(I \otimes C_i \rho_{AC} I \otimes C_i^\dagger)$ is the probability to obtain outcome $i$.

Suppose the optimal decomposition of EoF for a mixed tripartite state $\rho_{ABC}$ is $\{p_i, |\phi^i\rangle_{ABC}\}$, we have

$$E_f(\rho_{A:BC}) = \sum_i p_i S(\rho_A^i)$$

(4)

$$= \sum_i p_i (E_f(\rho_{A:B}^i) + C_-(\rho_{AC}^i))$$

(5)

$$\geq E_f(\rho_{A:B}) + G_-(\rho_{AC}).$$

(6)

The inequality \cite{8} comes from the convexity of EoF $\sum_i p_i E_f(\rho_{A:B}^i) \geq E_f(\sum_i p_i \rho_{A:B}^i)$ \cite{8} and $G_-(\rho_{AC}) = \min \sum_i p_i C_-(\rho_{AC}^i)$ \cite{8}, where the minimum is taken over all mixed ensembles $\{p_i, \rho_{AC}^i\}$ satisfying $\rho_{AC} = \sum_i p_i \rho_{AC}^i$. 

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A simple proof of monogamy of entanglement

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Monogamy of entanglement means that an entangled state cannot be shared with many parties. The more parties, the less entanglement between them. In this paper, we give a simple proof of this property and provide an upper bound of the number of parties.

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Now iteratively applying the inequality (6) to the n-extension state $\rho_{AB_1B_2\ldots B_n}$ of an entangled state $\rho_{AB}$ and further noticing $E_f(\rho_{A:B}) \geq G_{\rho}(\rho_{A:B})$ by definition of $G_{\rho}$ [8], we obtain

$$
E_f(\rho_{A:B_1B_2\ldots B_n}) \geq E_f(\rho_{A:B_2\ldots B_n}) + G(\rho_{A:B_1}) \\
\geq E_f(\rho_{A:B_3\ldots B_n}) + G(\rho_{A:B_2}) + G(\rho_{A:B_1}) \\
= E_f(\rho_{A:B_3\ldots B_n}) + 2G(\rho_{A:B}) \\
\geq \ldots \\
\geq E_f(\rho_{A:B}) + (n-1)G(\rho_{A:B}) \\
\geq nG(\rho_{A:B}).
$$

Employing the explicit relation $S(\rho_A) \geq E_f(\rho_{A:B_1B_2\ldots B_n})$ and an important property of $G_{\rho}$ asserting that $G_{\rho}(\rho_{A:B}) > 0$ if and only if $\rho_{AB}$ is entangled [8], we get the upper bound of n-extension of an entangled state $\rho_{AB}$,

$$
n \leq N = \left\lfloor \frac{S(\rho_A)}{G_{\rho}(\rho_{AB})} \right\rfloor,
$$

where $\lfloor x \rfloor$ is the maximal integer not larger than $x$. Thus we proved that for entangled state, there exists a finite number $N$ such that no n-extension can be found for $n > N$. Physically, entanglement in a given state can not be sharable in arbitrarily many parties.

For the system of many identical particles, the state $\rho_{A_1A_2\ldots A_n}$ has the symmetry of permutation. It holds for any state that entanglement between any pair particles tends to zero as $n \to \infty$.

In summary, we give a simple proof of monogamy of entanglement and provide an upper bound of the number of parties beyond which entanglement cannot be shared.

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[1] B. M. Terhal, quant-ph/0307120
[2] R. F. Werner, Lett. Math. Phys. 17, 359 (1989).
[3] A. C. Doherty, P. A. Parrilo, and F. M. Spedalieri, Phys. Rev. A 69, 022308 (2004).
[4] J. Bae and A. Acin, quant-ph/0603078
[5] M. Koashi and A. Winter, Phys. Rev. A 69, 022309 (2004).
[6] L. Henderson and V. Vedral J. Phys. A 34, 6899 (2001).
[7] C.H. Bennett, D.P. DiVincenzo, J.A. Smolin, W.K. Wootters, Phys. Rev. A 54, 3824 (1996).
[8] D. Yang, M. Horodecki, R. Horodecki, and B. Synak-Radtke, Phys. Rev. Lett. 95, 190501 (2005).