THE JOSEPH IDEAL FOR $\mathfrak{sl}(m|n)$

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ABSTRACT. Using deformation theory, Braverman and Joseph obtained an alternative characterisation of the Joseph ideal for simple Lie algebras, which included even type A. In this note we extend that characterisation to define a remarkable quadratic ideal for $\mathfrak{sl}(m|n)$. When $m - n > 2$ we prove the ideal is primitive and can also be characterised similarly to the construction of the Joseph ideal by Garfinkle.

1. Preliminaries

We use the notation $\mathfrak{g} = \mathfrak{sl}(m|n)$. See [CW] for the definition and more information on $\mathfrak{sl}(m|n)$ and Lie superalgebras. We take the Borel subalgebra $\mathfrak{b}$ to be the space of upper triangular matrices and the Cartan subalgebra $\mathfrak{h}$ diagonal matrices, both with zero supertrace. With slight abuse of notation we will write elements of $\mathfrak{h}^*$ as elements of $\mathfrak{h}^* \oplus \mathfrak{n}^* \oplus \mathfrak{n}^*$. A highest weight vector $v_\lambda$ of a weight module $M$ satisfies $\mathfrak{n}^* \cdot v_\lambda = 0$ and $\mathfrak{h} \cdot v_\lambda = \lambda(\mathfrak{h}) \cdot v_\lambda$. The corresponding weight $\lambda \in \mathfrak{h}^*$ will be called a highest weight. We use the notation $L(\lambda)$ for the simple module with highest weight $\lambda \in \mathfrak{h}^*$. We also set $\rho_0 = \frac{1}{2} \sum_{\alpha \in \Delta_0^+} \alpha$ and $\rho = \rho_0 - \frac{1}{2} \sum_{\gamma \in \Delta_1^+} \gamma$, so concretely

$$\rho = \frac{1}{2} \sum_{i=1}^m (m-n-2i+1) \epsilon_i + \frac{1}{2} \sum_{j=1}^n (n+m-2j+1) \delta_j.$$  

We choose the form $(\cdot, \cdot)$ on $\mathbb{C}^{m+n}$, and on $\mathfrak{h}^*$ by restriction, by setting $(\epsilon_i, \epsilon_j) = \delta_{ij}$, $(\delta_j, \delta_k) = -\delta_{jk}$ and $(\epsilon_i, \delta_j) = 0$.

From now on we consider only weights $\lambda$ which are integral, that is $(\lambda + \rho, \alpha^\vee) \in \mathbb{Z}$ for all $\alpha \in \Delta_0$, with $\alpha^\vee := 2\alpha / (\alpha, \alpha)$. If $\langle \lambda + \rho, \alpha^\vee \rangle > 0$, for all $\alpha \in \Delta_0^+$, we say that the integral weight $\lambda$ is dominant regular.

Denote by $C$ the quadratic Casimir operator. It is an element of the center of $U(\mathfrak{g})$ and it acts on a highest weight vector of weight $\lambda$ by the scalar

$$C \cdot v_\lambda = (\lambda + 2 \rho, \lambda).$$

We denote by $M^\vee$ the dual module of $M$ in category $\mathcal{O}$, see e.g. [Hu chapter 3]. The functor $\vee$ is exact and contravariant, we have that $L(\lambda) \cong L(\lambda^\vee)$ and for finite dimensional modules $(M \otimes N)^\vee \cong M^\vee \otimes N^\vee$.

We set $V = \mathbb{C}^{m|n}$ the natural representation of $\mathfrak{g}$. We will use the notation $A_{ij}^k$ for an element in $V \otimes V^*$ and we have the identification $V \otimes V^* \cong V^* \otimes V$ given by
For Theorem 2.2.

If \( m \neq n \) the supertrace gives a decomposition of \( V \otimes V^* \) in a traceless and a pure trace part. The Lie superalgebra \( \mathfrak{g} \) consist exactly of the traceless elements in \( V \otimes V^* \). We will use the identification \( V \otimes V^* \cong V^* \otimes V \) for taking the supertrace of higher order tensor powers. For example, if \( A \in V \otimes V^* \otimes V \otimes V^* \), then the supertrace over the first and last component is given by

\[
\text{str}_{1,4} : V \otimes V^* \otimes V \otimes V^* \rightarrow V^* \otimes V ; \quad A_{i j}^k l \mapsto \sum_i (-1)^{|i||j|+|k||l|} A_{i j}^k l.
\]

With these conventions \( \text{str} \) always corresponds to a \( \mathfrak{g} \)-module morphism.

We will also use the Killing form

\[
\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C} \quad \langle A, B \rangle = 2(m-n) \sum_{i,j} (-1)^{|i|} A_{i j} B_{i j},
\]

which satisfies \( \langle A, B \rangle = \text{str}_2(\text{ad}_A \text{ad}_B) \). This is an invariant, even, supersymmetric form. If \( m-n \neq 0 \) it is non-degenerate. We introduce the corresponding \( \mathfrak{g} \)-module morphism \( \mathcal{K} = 2(m-n) \text{str}_{1,4} \circ \text{str}_{2,3} : V \otimes V^* \otimes V \otimes V^* \rightarrow \mathbb{C} ; \quad A_{i j}^k l \mapsto 2(m-n) \sum_{i,j} (-1)^{|i|} A_{i j}^k l.
\]

In particular, for \( A, B \) in \( \mathfrak{g} \) we have \( \mathcal{K}(A \otimes B) = \langle A, B \rangle \).

2. Second tensor power of the adjoint representation for \( \mathfrak{sl}(m|n) \)

In this section we will always set \( \mathfrak{g} = \mathfrak{sl}(m|n) \) with \( m \neq n \). We will also always assume \( m \neq 0 \neq n \). In case \( m = 1 \) one needs to replace all \( \varepsilon_2 \) occurring in formulæ by \( \delta_1 \) and for \( n = 1 \) one replaces \( \delta_{n-1} \) by \( e_1 \). Furthermore \( V \) will be the natural \( \mathfrak{sl}(m|n) \) module and we identify \( \mathfrak{sl}(m|n) \) with the corresponding tensors in \( V \otimes V^* \).

**Theorem 2.1.** For \( \mathfrak{g} = \mathfrak{sl}(m|n) \) with \( |m-n| > 2 \), the second tensor power of the adjoint representation \( \mathfrak{g} \otimes \mathfrak{g} \cong \mathfrak{g} \circ \mathfrak{g} \oplus \mathfrak{g} \wedge \mathfrak{g} \) decomposes as

\[
\mathfrak{g} \circ \mathfrak{g} \cong L_{2\varepsilon_1 - \delta_{n-1} - \delta_n} \oplus L_{\varepsilon_1 + \varepsilon_2 - 2\delta_n} \oplus L_{\varepsilon_1 - \delta_n} \oplus L_0,
\]

\[
\mathfrak{g} \wedge \mathfrak{g} \cong L_{2\varepsilon_1 - 2\delta_n} \oplus L_{\varepsilon_1 + \varepsilon_2 - \delta_{n-1} - \delta_n} \oplus L_{\varepsilon_1 - \delta_n}.
\]

We define the Cartan product \( \mathfrak{g} \circ \mathfrak{g} \) as the direct summand of \( \mathfrak{g} \otimes \mathfrak{g} \) isomorphic to \( L_{2\varepsilon_1 - \delta_{n-1} - \delta_n} \).

To give an explicit expression for the decomposition of the symmetric part we will use a projection operator \( \chi : \mathfrak{g} \circ \mathfrak{g} \rightarrow \mathfrak{g} \circ \mathfrak{g} \) given by \( \chi := \phi \circ \text{str}_{2,3} \), where \( \phi \) is the \( \mathfrak{g} \)-module morphism \( \phi : V \otimes V^* \rightarrow V \otimes V^* \otimes V \otimes V^* \) defined in Lemma 2.4.

**Theorem 2.2.** According to the decomposition of \( \mathfrak{g} \circ \mathfrak{g} \) in Theorem 2.1 respecting that order, a tensor \( A \in \mathfrak{g} \circ \mathfrak{g} \) decomposes as \( A = B + C + D + E \), where

- \( B_{i j}^k l = \frac{1}{2}(A_{i j}^k l - \chi(A)_{i j}^k l) \quad \chi = \phi \circ \text{str}_{2,3} \),
- \( C = A - \chi(A) = -B \),
- \( E = (2(m-n))^{\frac{1}{2}} \mathcal{K} \phi(\delta) \),
- \( D = \chi(A) - E \).
By construction \( \text{str}_{2,3}(B) = 0 = \text{str}_{2,3}(C) \) and \( \mathcal{K}(D) = 0 \).

The explicit formula for \( \varphi(\delta) \), where \( \delta = \delta^i_j \) is the Kronecker delta, is given by

\[
(\varphi(\delta))^k_i = ((m-n)^2 - 1)^{-1} \left( (-1)^{|k|}(m-n)\delta^i_j\delta^k_j - \delta^i_j\delta^k_j \right).
\]

The remainder of this section is devoted to the proof of these theorems.

**Lemma 2.3.** The possible highest weights of the \( g \)-module \( g \otimes g \) are

\[ 2\varepsilon_1 - 2\delta_n, 2\varepsilon_1 - \delta_{n-1} - \delta_n, \varepsilon_1 + \varepsilon_2 - 2\delta_n, \varepsilon_1 - \delta_n, 0. \]

The space of highest weight vectors for \( \varepsilon_1 - \delta_n \) has at most dimension 2 and for the other weights at most 1.

**Proof.** A highest weight vector \( v_\lambda \) in \( g \otimes g \) is of the form

\[ v_\lambda = X_{\varepsilon_1} - \delta_n \otimes A + \cdots, \]

where \( A \in g \).

Thus the highest weight \( \lambda \) is of the form \( \lambda = \varepsilon_1 - \delta_n + \mu \) with \( \mu \in \Delta \cup \{0\} \). Since it also has to be regular dominant we have the following possibilities for \( \lambda \):

\[
2\varepsilon_1 - 2\delta_n, 2\varepsilon_1 - \delta_{n-1} - \delta_n, \varepsilon_1 + \varepsilon_2 - 2\delta_n, \varepsilon_1 - \delta_n, \varepsilon_1 - \delta_n, 0,
\]

and

\[
2\varepsilon_1 - \varepsilon_m - \delta_n, \varepsilon_1 + \varepsilon_2 - \varepsilon_m - \delta_n, \varepsilon_1 - \varepsilon_m, 0.
\]

A corresponding highest weight vector \( v_\lambda \) has to satisfy \([X, v_\lambda] = 0\) for all \( X \in \mathfrak{n}^+ \).

Writing out this condition for all possible simple roots vectors, we deduce that there are no highest weight vectors corresponding to the weights in (3) and that the dimension of the space of highest weight vectors for \( \varepsilon_1 - \delta_n \) is at most 2. The fact that for the other possibilities the dimension is at most 1 follows from the dimension of the corresponding root space in \( g \), which is always 1. \( \square \)

We want to construct a \( g \)-module morphism \( \varphi : V \otimes V^* \to V \otimes V^* \otimes V \otimes V^* \), such that its image is in \( g \otimes g \) and \( \text{str}_{2,3} \circ \varphi = \text{id} \). Thus this morphism has to satisfy the following properties for all \( B \in V \otimes V^* \):

1. \( \text{str}_{1,2} \varphi(B) = 0 \)
2. \( \varphi(B)^{ij} = (-1)^{|i|+|j|}\varphi(B)^{ij} \)
3. \( \text{str}_{2,3} \varphi(B) = B \).

**Lemma 2.4.** Consider the map \( \varphi : V \otimes V^* \to V \otimes V^* \otimes V \otimes V^* \) given by

\[
\varphi(B)^{ij} = a \left( (-1)^{|k|}B^{ij}\delta^k_j + (-1)^{|i|+|j|}\delta^i_j \right) \delta^k_j + \frac{-2}{m-n}\delta^i_j.
\]

For the constants \( a = \frac{(m-n)^2 - 1}{(m-n)^2 - 2}, c_1 = \frac{(m-n)^2 + 2}{(m-n)^2 - 2}, \) and \( c_2 = \frac{3}{(m-n)^2 - 1} \), the map \( \varphi \) is a \( g \)-module morphism satisfying conditions 1.-2.-3. above.

**Proof.** One can easily see that \( \varphi(B) \) is supersymmetric for the indices \( (i, j) \) and \( (k, l) \), hence it satisfies the second condition. The first condition leads to

\[ c_1 + (m-n)c_2 - 2(m-n)^{-1} \]

while the third condition gives us the following two equations:

\[ a((m-n) - 4(m-n)^{-1}) = 1 \quad \text{and} \quad 1 + (m-n)c_1 + c_2 = 0. \]
This system of equations has as solution the constants given in the lemma. One can also check directly that \( \varphi \) is indeed a \( g \)-module morphism. \( \square \)

**Proof of Theorem 2.2.** Define the \( g \)-module morphism \( \chi : g \to g \) by \( \varphi = \varphi \circ \text{str} \). Since \( \text{str}_{2,3} \circ \varphi = \text{id} \), we have \( \chi^2 = \chi \). This implies that the representation splits up into \( \ker \chi = \text{im}(1 - \chi) \) and \( \ker \chi = \text{ker}(1 - \chi) \). Hence

\[
g \circ g = \ker \chi \oplus \text{im} \chi.
\]

We have \( \text{im} \chi = \ker \chi = V \otimes V^* \), since \( \varphi \) is injective. From Section 1, we know that \( V \otimes V^* \cong L_{e_1 - \delta_1} \oplus L_0 \), where this decomposition is based on the supertrace.

Let \( q \in \text{End}_g(\ker \chi) \) denote the super symmetrisation in the upper indices, so \( q^2 = q \) and hence \( \ker \chi = \ker q \oplus \text{im} q \).

In the proof of Theorem 2.1, we will show that \( g \wedge g \) has three direct summands. From Lemma 2.3, we know that \( g \otimes g \) contains at most seven highest weight vectors, of which thus three are already contained in \( g \wedge g \). Therefore \( \ker q \) and \( \text{im} q \) each contain exactly one highest weight vector. Since \( \ker q \oplus \text{im} q \) is self-dual in category \( \mathcal{O} \), this implies that they are both simple modules. Therefore \( g \circ g = \ker q \oplus \text{im} q \oplus L_{e_1 - \delta_1} \oplus L_0 \) is a decomposition in simple modules. One can verify, by tracking the highest weights of the respective subspaces, that \( \ker q = L_{e_1 + e_2 - \delta_1} \), and that \( \text{im} q = L_{2e_1 - \delta_1 - \delta_2} \).

By construction, the expressions for projections on simple summands follow. \( \square \)

**Proof of Theorem 2.7.** We have already dealt with the symmetric part in the proof of Theorem 2.2. For the antisymmetric part, we remark that \( \text{str}_{1,4} \circ \text{str}_{2,3}(A) = 0 \) for all \( A \in g \wedge g \). Thus \( \text{str}_{2,3} \) is a \( g \)-module morphism from \( g \wedge g \to g \cong L_{e_1 - \delta_1} \). Consider the \( g \)-module morphism \( \psi : g \to g \wedge g \) given by

\[
B \mapsto (m - n)^{-1} \left( (-1)^{|k|} B^k \delta^k - (-1)^{|(k, l)|} (|k| + |l|) B^k \delta^l \right).
\]

For this morphism it holds that \( \text{str}_{2,3} \circ \psi = \text{id} \). Denote by \( q \) again the symmetrisation in the upper indices. Then we find in the same way as for the symmetrical part

\[
g \wedge g = \ker q \oplus \text{im} q \oplus \text{im} \psi,
\]

and \( \ker q \cong L_{e_1 + e_2 - \delta_1 - \delta_2} \), \( \text{im} q \cong L_{2e_1 - \delta_1} \), and \( \text{im} \psi \cong L_{e_1 - \delta_1} \). \( \square \)

### 3. The Joseph Ideal for \( \mathfrak{s}(m|n) \)

In this section we define and characterise the Joseph ideal for \( \mathfrak{g} = \mathfrak{s}(m|n) \), where from now on we always assume \( |m - n| > 2 \). Similar results for \( \mathfrak{o}(m|2n) \) have been obtained in [CSS].

We define a one-parameter family \( \mathcal{J}_\lambda \mid \lambda \in \mathbb{C} \) of quadratic two-sided ideals in the tensor algebra \( T(\mathfrak{g}) = \bigoplus_{j \geq 0} \otimes^j \mathfrak{g} \), where \( \mathcal{J}_\lambda \) is generated by

\[
\{ X \otimes Y - X \otimes Y - \frac{1}{2} [X, Y] - \lambda (X, Y) \mid X, Y \in \mathfrak{g} \} \subset \mathfrak{g} \otimes \mathfrak{g} \oplus \mathfrak{g} \oplus \mathbb{C} \subset T(\mathfrak{g}).
\]

By construction there is a unique ideal \( J_\lambda \) in the universal enveloping algebra \( U(\mathfrak{g}) \), which satisfies \( T(\mathfrak{g}) / \mathcal{J}_\lambda \cong U(\mathfrak{g}) / J_\lambda \). Now we define \( J^\lambda := -1/(8(m - n + 1)) \).

**Theorem 3.1.** (i) For \( \lambda \neq \lambda^c \), the ideal \( J_\lambda \) has finite codimension, more precisely \( J_\lambda = U(\mathfrak{g}) \) for \( \lambda \in \{0, \lambda^c\} \) and \( J_\lambda = g U(\mathfrak{g}) \) for \( \lambda = 0 \).

(ii) For \( \lambda = \lambda^c \), the ideal \( J_\lambda \) has infinite codimension.
From now on we call the ideal $J_{\lambda}$ the Joseph ideal. If $m - n > 2$, we give another characterisation of the Joseph ideal, which generalises the characterisation in [Ga] to type A (super and classical). The classical case, $n = 0$, was already obtained through different methods in the proof of Proposition 3.1 in [AB]. For this we need the canonical antiautomorphism $\tau$ of $U(\mathfrak{g})$, defined by $\tau(X) = -X$ for $X \in \mathfrak{g}$.

**Theorem 3.2.** Let $\mathfrak{g} = \mathfrak{sl}(m|n)$ with $m - n > 2$. Any two-sided ideal $\mathcal{R}$ in $U(\mathfrak{g})$ of infinite codimension, with $\tau(\mathcal{R}) = \mathcal{R}$, such that the graded ideal $\text{gr}(\mathcal{R})$ in $\mathfrak{g}$ satisfies

$$\left(\text{gr}(\mathcal{R}) \cap \mathfrak{g} \otimes \mathfrak{g}\right) \oplus \mathfrak{g} \otimes \mathfrak{g} = \mathfrak{g}^2,$$

is equal to the Joseph ideal $J_{\lambda}$.

In the remainder of this section we will prove both theorems.

**Proof of Theorem 3.2.** Similarly to the proof of Theorem 2.1 in [ESS] for $\mathfrak{gl}(m)$, to which we refer for more details, we construct a special tensor $S$ in $\otimes^3 \mathfrak{g}$, which we will reduce inside $T(\mathfrak{g}) / \mathcal{J}_\lambda$ in two different ways. This will show that for $\lambda$ different from $\lambda^c$, the ideal $\mathcal{J}_\lambda$ contains $\mathfrak{g}$. Note that the existence of the tensor $S$ in the setting of [ESS] was already non-constructively proved in [BHI].

Consider $T \in \mathfrak{g}$ and define the tensor $S$ as

$$S^a_{\ e}b_{\ d}f = (-1)^{|a||d|} \delta^e_a \delta^f_d T^a_{\ b} - \frac{1}{m-n} \delta^e_a \delta^f_d T^a_{\ b}$$

$$- (-1)^{|b|+|a|+|b|} \langle a|+|d|\rangle \delta^e_a \delta^f_d T^a_{\ d} + \frac{1}{m-n} \delta^e_a \delta^f_d T^a_{\ d}$$

$$+ (-1)^{|b|+|a|+|b|} \langle c|+|d|\rangle \delta^e_a \delta^f_d T^a_{\ d} - \frac{1}{m-n} (-1)^{|d|+|a|+|b|+|d|} \delta^e_a \delta^f_d T^a_{\ d}$$

$$- (-1)^{|c|+|d|+|a|+|b|+|d|} \delta^e_a \delta^f_d T^a_{\ d} + \frac{1}{m-n} (-1)^{|b|} \delta^e_a \delta^f_d T^a_{\ d}.$$

One can calculate that $\text{str}_{1,2} S = \text{str}_{3,4} S = \text{str}_{5,6} S = 0$, hence $S \in \otimes^3 \mathfrak{g}$. Remark that we also defined $S$ so that it is antisymmetric in the indices $(a,b)$ and $(c,d)$, hence $S \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$. Since the Cartan product lies in $\mathfrak{g} \otimes \mathfrak{g}$, the Cartan part with respect to the first four indices $a,b,c,d$ vanishes. Now we consider (for each $a,b$), the tensor in $\otimes^3 \mathfrak{g}$ corresponding to the indices $c,d,e,f$. First we symmetrise, to find a tensor in $\mathfrak{g} \otimes \mathfrak{g}$. When we apply $1 - \chi$ to that tensor and then symmetrise in the upper indices, we obtain zero. Theorem 2.2 thus shows that $S$ also has no part lying in $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$.

Now, on the one hand, we can reduce $S$ using the fact that the Cartan part vanishes with respect to the first four indices $a,b,c,d$. Then we find

$$S \simeq -\frac{1}{2} (m-n)(m-n-2) T \mod \mathcal{J}_\lambda.$$

If, on the other hand, we reduce $S$ using the fact that the Cartan part vanishes with respect to the last four indices $c,d,e,f$, we find

$$S \simeq (m-n)(m-n-2)(2\lambda(m-n+1) - \frac{1}{4}) T \mod \mathcal{J}_\lambda.$$

Therefore, if $\lambda \neq \lambda^c$, then $T$ is an element of $\mathcal{J}_\lambda$. Hence, we have proven that $\mathfrak{g} \subset \mathcal{J}_\lambda$ for $\lambda \neq \lambda^c$. This also implies for $\lambda \neq 0$, by equation (4), that $\mathfrak{C} \subset \mathcal{J}_\lambda$. 

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Hence \( \mathfrak{g}_\lambda = T(\mathfrak{g}) \) for \( \lambda \notin \{0, \lambda^c\} \) and \( \mathfrak{g}_0 = \oplus_{k>0} \otimes^k \mathfrak{g} \). This proves part (i) of Theorem 3.1. Part (ii) will follow from the construction in Section 4. \( \square \)

To prove Theorem 3.2, we will need two lemmata. First we define \( I_2 \) as the complement representation of \( g \otimes g \) in \( g \otimes g \) and recursively

\[
I_k = I_{k-1} \otimes g + g \otimes I_{k-1} \quad \text{for} \quad k > 2.
\]

Denote by \( \lambda^k \) the highest weight occurring in \( \otimes^k g \), then

\[
\lambda^k = \begin{cases} 
  k \epsilon_1 - \delta_{n-k+1} - \delta_{n-k+2} - \cdots - \delta_{n-1} - \delta_n & \text{for} \quad k \leq n, \\
  k \epsilon_1 - (k-n) \epsilon_m - \delta_1 - \delta_2 - \cdots - \delta_{n-1} - \delta_n & \text{for} \quad k > n.
\end{cases}
\]

**Lemma 3.3.** Let \( g = sl(m|n) \) with \( m-n > 2 \). Then \( \otimes^k g \cong L(\lambda^k) \oplus I_k \).

**Proof.** Set \( \beta_2 = g \otimes g \) and define the submodule \( \beta_k \) of \( \otimes^k g \) by

\[
\beta_k := \beta_{k-1} \otimes g \cap g \otimes \beta_{k-1}, \quad \text{for} \quad k > 2.
\]

We will show by induction that \( \beta_k = L(\lambda^k) \) and that this is a direct summand in \( \otimes^k g \). This holds for \( k = 2 \) by definition. Now we assume that it holds for \( k \) and start by proving that all highest weight vectors in \( \beta_{k+1} \) are in the 1 dimensional subspace of \( \otimes^k g \) of the vectors with the highest occurring weight \( \lambda^{k+1} \).

Let \( v_\mu \) be a highest weight vector in \( \beta_{k+1} \). Then

\[
v_\mu = X \otimes v_{\lambda^k} + \cdots,
\]

where \( v_{\lambda^k} \) is a highest weight vector in \( \beta_k = L(\lambda^k) \) and \( X \in g \) is a Cartan element or a root vector. It follows that \( \mu = \alpha + \lambda^k \) for \( \alpha \in \Delta \) or \( \mu = \lambda^k \).

First assume \( \mu = \lambda^k \). Equation (2) implies that \( C v_\mu = (\lambda^k, \lambda^k + 2\rho) v_\mu \), for \( C \) the Casimir operator. Similarly to Lemma 4.5 in [CSS] it follows that \( C \) acts on \( \beta_{k+1} \) through \( (\lambda^{k+1}, \lambda^{k+1} + 2\rho) \). A highest weight vector \( v_\mu \) in \( \beta_{k+1} \) hence implies

\[
(\lambda^k, \lambda^k + 2\rho) = (\lambda^{k+1}, \lambda^{k+1} + 2\rho).
\]

Using (1) it follows that \( (\lambda^k, \lambda^k + 2\rho) = 2k(k+m-n-1) \), so the displayed condition is equivalent to \( 2k = -m+n \). As this contradicts \( m-n > 2 \), we conclude that there is no highest weight vector in \( \beta_{k+1} \) with weight \( \mu = \lambda^k \).

Now assume \( \mu = \lambda^k + \alpha \) for \( \alpha \in \Delta \). We will consider the case \( k \geq n \), the case \( k < n \) being similar. Since \( \mu \) has to be dominant regular, the possibilities for \( \alpha \) are

\[
\epsilon_1 - \epsilon_m, \epsilon_1 - \epsilon_{m-1}, \epsilon_2 - \epsilon_m, \epsilon_1 - \delta_n, \epsilon_2 - \delta_n,
\]

\[
(6) \quad \epsilon_2 - \epsilon_{m-1}, \epsilon_m - \delta_n, \delta_1 - \delta_n, -\epsilon_1 + \epsilon_2, -\epsilon_1 + \epsilon_m, \delta_1 - \epsilon_1, \delta_1 - \epsilon_m, \delta_1 - \epsilon_{m-1}.
\]

Observe that for example \( \epsilon_2 - \epsilon_{m-1} \) can not occur since applying \( X_{\epsilon_1 - \epsilon_2} \) to the highest weight vector should be zero, but the result would contain a term with the factor \( X_{\epsilon_1 - \epsilon_{m-1}} \) which can not be compensated for. By choosing the appropriate simple root vector, we can eliminate all the possibilities in (6).

For the root \( \epsilon_1 - \delta_n \) the Casimir operator acts on \( v_\mu \) by

\[
(\lambda^k + \epsilon_1 - \delta_n, \lambda^k + \epsilon_1 - \delta_n + 2\rho) = 2k(k+m-n) + 2(m-n-1).
\]
Since this is different from \((\lambda^{k+1}, \lambda^{k+1} + 2\rho) = 2(k+1)(k+m-n)\), this excludes 
\(\epsilon_1 - \delta_n\). Similarly for \(\epsilon_2 - \delta_n\), \(\epsilon_1 - \epsilon_{m-1}\) and \(\epsilon_2 - \epsilon_m\) we get
\[
(\lambda^k + \epsilon_2 - \delta_n, \lambda^k + \epsilon_2 - \delta_n + 2\rho) = 2k(k+m-n-1) + 2(m-n-2),
\]
\[
(\lambda^k + \epsilon_1 - \epsilon_{m-1}, \lambda^k + \epsilon_1 - \epsilon_{m-1} + 2\rho) = 2k(k+m-n) + 2m-2(1),
\]
\[
(\lambda^k + \epsilon_2 - \epsilon_m, \lambda^k + \epsilon_2 - \epsilon_m + 2\rho) = 2k(k+m-n) + 2(m-n-1).
\]
Because \(k \geq n\), these expressions are different from \((\lambda^{k+1}, \lambda^{k+1} + 2\rho)\). Hence there exists no \(\nu\) in \(\beta_{k+1}\) for these roots.

We conclude that the only possibility is \(\epsilon_1 - \epsilon_m\) for \(k \geq n\). For \(k < n\) we find similarly that only \(\epsilon_1 - \delta_{n-k}\) is possible. Therefore \(\beta_{k+1}\) contains only one highest weight vector, up to multiplicative constant, namely \(\nu_{k+1}\). The submodule of \(\beta_{k+1}\) (which is also a submodule of \(\otimes^{k+1}g\)) generated by such a highest weight vector must therefore be isomorphic to \(L(\lambda^{k+1})\). Since \(\otimes^{k+1}g\) is self-dual for \(\forall\), \(L(\lambda^{k+1})\) must also appear as a quotient of \(\otimes^{k+1}g\). However, as the weight \(\lambda^{k+1}\) appears with multiplicity one in \(\otimes^{k+1}g\), we find \([\otimes^{k+1}g : L(\lambda^{k+1})] = 1\) and \(L(\lambda^{k+1})\) must be a direct summand.

In particular \(L(\lambda^{k+1})\) has a complement inside \(\beta_{k+1}\). By the above, the latter complement is a finite dimensional weight module which has no highest weight vectors, implying that it must be zero, so \(\beta_{k+1} \cong L(\lambda^{k+1})\). Hence we find indeed that for all \(k \geq 2\) we have \(\beta_k \cong L(\lambda^k)\) and that this is a direct summand in \(\otimes^k g\).

We have a non-degenerate form on \(\otimes^k g\) such that \(\beta_k^\perp = I_k\) (see Section 4 in [CSS]). Hence \(\dim \otimes^k g = \dim \beta_k + \dim I_k\). Since \(I_k \cap L(\lambda^k) = 0\) we conclude \(\otimes^k g = L(\lambda^k) \oplus I_k\), which finishes the proof of the lemma. □

Any two sided ideal \(\mathcal{L}\) in \(T(g)\) is a submodule for the adjoint representation. Set \(T_{\leq k}(g) = \oplus_{j \leq k} \otimes^j g\) and define the modules \(\mathcal{L}_k \subseteq \otimes^k g\) by
\[
\mathcal{L}_k = ((\mathcal{L} + T_{\leq k-1}(g)) \cap T_{\leq k}(g))/T_{\leq k-1}(g).
\]
One can easily prove that if there is a strict inclusion \(\mathcal{L}_1 \subset \mathcal{L}_2\), then there must be some \(k\) for which \(\mathcal{L}_k^1 \subseteq \mathcal{L}_k^2\), see e.g. the proof of Theorem 5.4 in [CSS].

**Lemma 3.4.** Let \(g = sl(m|n)\) with \(m-n > 2\). Consider a two-sided ideal \(\mathfrak{R}\) in \(U(g)\). If \(\mathfrak{R}\) contains \(J_{k^*}\) and has infinite codimension, then \(\mathfrak{R} \supseteq J_{k^*}\).

**Proof.** Let \(J_{k^*}\) be as defined in [4] and denote by \(\mathcal{H}\) the kernel of the composition. \(T(g) \rightarrow U(g) \rightarrow U(g)/\mathfrak{R}\). We have that \((J_{k^*})_k = I_k\) with \(I_k\) as defined in [5]. Since \(J_{k^*} \subset \mathfrak{R}\), also \((J_{k^*})_k \subset \mathcal{H}_k\) holds. If \(\mathcal{H}\) would be strictly bigger than \(J_{k^*}\), then for some \(k\), \(\mathcal{H}_k\) would be bigger than \((J_{k^*})_k = I_k\). Lemma [3,3] would then imply that \(\mathcal{H}_k = \otimes^k g\) and thus also \(\mathcal{H}_l = \otimes^l g\) for all \(l \geq k\), since \(\mathcal{H}\) is a two-sided ideal. This is a contradiction with the infinite codimension of \(\mathfrak{R}\). Therefore we conclude that \(\mathcal{H} = J_{k^*}\) and thus \(\mathfrak{R} = J_{k^*}\). □

**Proof of Theorem 3.2** From the assumed property of \(gr(\mathfrak{R})\) follows that for each \(X, Y \in g\), we have
\[
(7) \quad XY + (-1)^{|X||Y|}YX - 2X \otimes Y + Z(X,Y) + c(X,Y) \in \mathfrak{R},
\]
where \(Z(X,Y) \in g\) and \(c(X,Y) \in \mathbb{C}\). Since \(\mathfrak{R}\) is a two-sided ideal, we can interpret \(Z\) and \(c\) as \(g\)-module morphism from \(g \otimes g\) to \(g\) and to \(\mathbb{C}\) respectively. Furthermore we assumed \(\mathfrak{R}\) to be invariant under the canonical automorphism \(\tau\). So applying \(\tau\) to \((7)\) and subtracting we get that \(2Z(X,Y)\) is in \(\mathfrak{R}\). If \(Z\) would be a morphism different
from zero, then it follows from the simplicity of \( g \) under the adjoint operation that \( Z \) is surjective. Hence \( g \subset \mathfrak{h} \), a contradiction with the infinite codimension of \( \mathfrak{h} \). From Theorem 3.1 it also follows that \( c(X, Y) = \lambda \langle X, Y \rangle \) for some constant \( \lambda \). This implies that \( J_\lambda \subset \mathfrak{h} \). Since \( \mathfrak{h} \) has infinite codimension, Theorem 3.1 and Lemma 3.4 imply that \( \lambda = \lambda^c = -\frac{1}{8(m-n+1)} \) and \( \mathfrak{h} = J_{\lambda^c} \). \( \square \)

4. A minimal realisation and primitivity of the Joseph ideal

In [BC] the authors construct polynomial realisations for \( \mathbb{Z} \)-graded Lie algebras. Consider the 3-grading on \( \mathfrak{gl}(m/n) \) by the eigenspaces of \( \text{ad} H_\lambda \). We consider the corresponding 3-grading \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \), inherited by the subalgebra \( \mathfrak{g}_0 = \mathfrak{sl}(m/n) \).

The procedure in [BC] Section 3 then gives realisations of \( \mathfrak{g} \) as (complex) polynomial differential operators on a real flat supermanifold with same dimensions as \( \mathfrak{g}_u \), so on \( \mathbb{R}^{m-1} \). We choose coordinates \( x_i \) with corresponding partial differential operators \( \partial_i \), for \( 2 \leq i \leq m+n \), both are even for \( i \leq m \) and odd otherwise.

As \( \mathfrak{g}_0 \cong \mathfrak{gl}(m-1/n) \), the space of characters \( \mathfrak{g}_0 \to \mathbb{C} \) is in bijection with \( \mathbb{C} \). If we apply the construction in [BC] Section 3 to the character corresponding to \( \mu \in \mathbb{C} \), we find a realisation \( \pi_\mu \) satisfying

\[
\pi_\mu(X_j) = x_j \quad \text{and} \quad \pi_\mu(X_{\ell_j - \ell_i}) = (\mu - \mathbb{E})\partial_j \quad \text{for} \quad 2 \leq j \leq m+n,
\]

with \( \mathbb{E} = \sum_{i=2}^{m+n} x_i \partial_i \). The other expressions for \( \pi_\mu \) follow from the above and the fact that, since \( \pi_\mu \) is a realisation, we have for all \( X, Y \) in \( \mathfrak{g} \)

\[
\pi_\mu(X)\pi_\mu(Y) - (-1)^{|X||Y|}\pi_\mu(Y)\pi_\mu(X) = \pi_\mu([X, Y]).
\]

Furthermore for \( A \in \mathfrak{g} \oplus \mathfrak{g} \), let \( A = B + C + D + E \) be the decomposition given in Theorem 2.2. If we choose \( \mu = \frac{A - m}{2} \), then we can calculate

\[
\pi_{\frac{A - m}{2}}(C) = 0 = \pi_{\frac{A - m}{2}}(D) \quad \text{and} \quad \pi_{\frac{A - m}{2}}(E) = \lambda^c \mathcal{H}(A), \quad \text{with} \quad \lambda^c = -\frac{1}{8(m-n+1)}.
\]

Therefore we conclude

\[
\left( \pi_{\frac{A - m}{2}}(X \otimes Y) - \pi_{\frac{A - m}{2}}(X \otimes Y) - \frac{1}{2}\pi_{\frac{A - m}{2}}([X, Y]) - \lambda^c \pi_{\frac{A - m}{2}}(\langle X, Y \rangle) \right) = 0.
\]

Now we interpret \( \pi_\mu \) as a representation of \( \mathfrak{g} \) on the space of polynomials, \( \text{i.e., on } S(\mathfrak{g}_u) \). Equation 2 then implies that the annihilator ideal of the representation \( \pi_{\frac{A - m}{2}} \) contains the Joseph ideal \( J_{\lambda^c} \). Since the representation is infinite dimensional, the Joseph ideal must have infinite codimension, which proves part (ii) of Theorem 3.1. For \( m-n > 2 \) it follows from Lemma 3.4 that the Joseph ideal is even equal to the annihilator ideal. Furthermore in this case, it follows clearly from equation (8) that the representation is simple.

In conclusion, we find that for \( m-n > 2 \), the Joseph ideal \( J_{\lambda^c} \) is primitive.

**Acknowledgment.** SB is a PhD Fellow of the Research Foundation - Flanders (FWO). KC is supported by the Research Foundation - Flanders (FWO) and by Australian Research Council Discover-Project Grant DP140103239. The authors thank Jean-Philippe Michel for raising the question which led to the study in Theorem 4.
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