Monopole and Magnetic String Solutions of the Kaluza-Klein Theory

Ahmad M. Farahani* and Nematollah Riazi†
Department of Physics, Shahid Beheshti University, G.C., Evin, Tehran 19839, Iran
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Abstract

We consider a new Kaluza-Klein string solution in five dimensions. We figure out that there is no event horizon in the four-dimensional spacetime. In the far region, the electromagnetic fields approach a magnetic and electric dipole for special cases of parameters which appear in the solution. The extension to the Kaluza-Klein monopole of Gross-Perry-Sorkin solution is briefly discussed.

1 Introduction

The idea of extra dimensions made its first appearance in physics by Kaluza-Klein theory in 1926[1, 2], which attempted to unify gravitation and electromagnetism using five space-time dimensions. The theory was first suggested by Theodor Kaluza as a higher dimensional theory of gravity, and ultimately led to the eleven-dimensional supergravity theories and the ten-dimensional superstring[3]. Like Einstein, was in quest of what we call "the unified theory", that there is a theory that may explain all of the fundamental forces. He endeavored to represent the electromagnetic force in a similar way, such as gravity in general relativity. The existence of one extra spatial dimension with a couple of special features was suggested. The idea was this: if we want to explain one more force, maybe we need an extra dimension. Therefore, Kaluza imagined that the universe had four instead of three dimensions of space, and accordingly his theory was formulated. Since the extra dimension is not observed, if one day extra dimensions are discovered in our universe, they are probably to be compact along the lines of Kaluza-Klein theory. Several exact solutions of Kaluza-Klein equations have been discovered since the introduction of this theory[2, 4].

As it is well-known, Dirac indicated a magnetic monopole with a strong singularity extending from the particle’s position to infinity[5]. Afterwards, more magnetic monopole solutions have been developed[6].

Gross and Perry[7], and Sorkin[8], simultaneously obtained a special group of solutions of the five-dimensional Kaluza-Klein theory related to a magnetic monopole. Their solutions explained a string singularity if the spatial extra dimension is compactified. Likewise, Gegenberg and Kunstatter introduced another magnetic monopole solution[9]. According to [10] the Kaluza-Klein monopole has an important role in M/String theory.

*Electronic address: ah.mazidabadi@mail.sbu.ac.ir
†Electronic address: n_riazi@sbu.ac.ir
Gross-Perry-Sorkin (GPS) magnetic monopole is a well-known solution of the Kaluza-Klein theory which is a generalization of the self-dual Euclidean Taub-NUT solution. This procedure can be applied to the other configurations. In a similar way, the Kaluza-Klein magnetic dipoles was represented by choosing the Euclidean Kerr solution. In this paper, we consider a string solution of the Kaluza-Klein theory in five-dimensional spacetime.

The structure of this paper is as follows. In section 2, we briefly bring up the Kaluza-Klein formalism and mention the main properties of the Kaluza-Klein magnetic monopole of GPS. Subsequently, we examine the boosted solution. We will then present a new vacuum solution of the Kaluza-Klein theory and investigate its physical properties in section 3. The section 4 is devoted to summary and discussion.

2 Kaluza-Klein Theory and the GPS Solution

As we have already said, in this section we briefly study the Kaluza-Klein theory and then review the Kaluza-Klein magnetic monopole of GPS solution. Next, the GPS solution will be boosted.

2.1 Kaluza-Klein Theory: a brief review

The Kaluza-Klein theory postulates the five-dimensional spacetime and the dimensional reduction of the vacuum spacetime, which leads to four-dimensional gravity coupled to a U(1) Maxwell field and a scalar dilaton field. It is also assumed that the 5D energy-momentum tensor vanishes and thus

\[ \tilde{G}_{AB} = 0 \]

where

\[ \tilde{G}_{AB} = \tilde{R}_{AB} - \frac{1}{2} \tilde{g}_{AB}, \]

is the Einstein tensor. \( \tilde{R}_{AB}, \tilde{R} \) and \( \tilde{g}_{AB} \) are the five-dimensional Ricci tensor, scalar, and metric tensor, respectively. The indices \( A, B, ... \) run over 0, 1, 2, 3, 4, and five-dimensional quantities are denoted by hats. The equations of motion can be derived by varying the five-dimensional Einstein-Hilbert action

\[ S = -\frac{1}{16\pi \hat{G}} \int \hat{R} \sqrt{-\hat{g}} \, d^4x \, dy, \]

where \( \hat{G} \) is the five-dimensional gravitational constant. Consequently, the Kaluza-Klein field equations in four dimensions read

\[ G_{\alpha \beta} = \frac{\kappa^2 \phi^2}{2} T^{EM}_{\alpha \beta} - \frac{1}{\phi} \left[ \nabla_{(\alpha} (\partial_{\beta)} \phi) - g_{\alpha \beta} \Box \phi \right], \]

\[ \nabla^\alpha F_{\alpha \beta} = -3 \frac{\partial^\alpha \phi}{\phi} F_{\alpha \beta}, \]

\[ \Box \phi = \frac{\kappa^2 \phi^3}{4} F_{\alpha \beta} F^{\alpha \beta}, \]

where \( T^{EM}_{\alpha \beta} = \frac{1}{2} g_{\alpha \beta} F_{\mu \nu} F^{\mu \nu} - F_{\alpha \mu} F^\mu_{\beta} \) is the electromagnetic energy-momentum tensor, and \( F_{\alpha \beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \) is the field strength. \( \phi \) and \( \kappa \) are the scalar field and coupling constant for the electromagnetic potential \( A_\alpha \), respectively. (Throughout this paper, Greek indices \( \alpha, \beta, ... \) run over 0, 1, 2, 3).
2.2 Gross-Perry-Sorkin Magnetic Monopole

In what follows, we review the Kaluza-Klein monopole, known also as the GPS solution. Actually, it is a generalization of the self-dual Euclidean Taub-NUT solution\(^{18, 19}\). The Taub-NUT solution was discovered by Taub, followed by Newman, Tamburino, and Unti who considered it as a simple generalization of the Schwarzschild spacetime\(^{11, 16}\). According to Gross and Perry\(^{7}\) and Sorkin\(^{17}\) the solution is obtained by embedding the Taub-NUT gravitational instanton\(^{11}\). The solution is discussed in\(^{7}\).

The Kaluza-Klein monopole of GPS solution is represented by the following metric

\[
    ds^2 = -dt^2 + V(dx^5 + 4m(1 - \cos \theta)d\phi)^2 + \frac{1}{V}(dr^2 + r^2d\theta^2 + r^2 \sin^2 \theta d\phi^2),
\]

where

\[
    V = 1 + \frac{4m}{r}.
\]

For this solution the electromagnetic potential is \(A_\phi = 4m(1 - \cos \theta)\), so the magnetic field is \(B = \frac{4m}{r}\), which is manifestly that of a monopole\(^{20}\). The coordinate singularity is located at \(r = 0\), which is called NUT singularity\(^{21}\). The magnetic charge of the monopole is \(g = m/\sqrt{\pi G}\), and the total magnetic flux is constant. Moreover, the mass of the solution is given by \(M = m/G\).

2.2.1 The Boosted Kaluza-Klein Magnetic Monopole of GPS Solution

From the five-dimensional standpoint, we implement a boost to the Kaluza-Klein magnetic monopole of GPS solution. The proposed boost is along the extra spatial dimension \(y\). We determine the boosted coordinates as \((t', r, \theta, \phi, y)\), and consider metric (7) with coordinate renamed as \((t', r', \theta', \phi', y')\). Accordingly we apply the following transformations

\[
    t' = t \cosh \alpha - y \sinh \alpha, \quad y' = y \cosh \alpha - t \sinh \alpha,
\]

where \(\alpha\) is the boosted parameter. So, the transformed metric is given by

\[
    ds^2 = - \left( \cosh^2 \alpha - \frac{\sinh^2 \alpha}{1 + \frac{4m}{r}} \right) dt'^2 + \left( 1 + \frac{4m}{r} \right) (dr'^2 + r'^2d\theta'^2) + \left( \frac{16m^2(1 - \cos \theta)^2}{1 + \frac{4m}{r}} \right) d\phi'^2 + \left( \frac{\sinh 2\alpha - \sinh 2\alpha/1 + \frac{4m}{r}}{1 + \frac{4m}{r}} \right) dt'dy' + 8m(1 - \cos \theta)\cosh \alpha d\phi dy - \frac{8m(1 - \cos \theta)}{1 + \frac{4m}{r}} \sinh \alpha dt'd\phi.
\]

Because of the \(dt'd\phi\) term the metric becomes a stationary solution. The transformed scalar and gauge fields resulting from the above metric are given by

\[
    \phi'^2 = \frac{\cosh^2 \alpha}{1 + \frac{4m}{r}} - \sinh^2 \alpha, \quad A_{t'} = \frac{1}{\kappa} \frac{4m \sinh 2\alpha}{(r - 4m \sinh^2 \alpha)},
\]
\[ A_\phi = \frac{1}{\kappa} \frac{8mr \cosh \alpha (1 - \cos \theta)}{(r - 4m \sinh^2 \alpha)}. \]  

(14)

Thus, the electromagnetic fields are driven

\[ E_r = -\frac{1}{\kappa} \frac{4m \sinh 2\alpha}{(r - 4m \sinh^2 \alpha)^2}, \]  

(15)

\[ B_r = \frac{1}{\kappa} \frac{8m \cosh \alpha (r - 4m \sinh^2 \alpha)}{r (r - 4m \sinh^2 \alpha)^2}, \]  

(16)

\[ B_\theta = \frac{8m \cosh \alpha (r - 4m \sinh^2 \alpha)}{r (r - 4m \sinh^2 \alpha)^2}. \]  

(17)

The boosted solution can be reduced to the previous metric (11) by setting \( \sinh \alpha = 0 \).

By applying a Kaluza-Klein reduction, the four-dimensional metric for the boosted GPS solution (11) becomes

\[ ds^2_{(4D)} = - \left( \frac{r - 4m + 12m^2 \sin^2 \theta}{(r - 4m \sinh^2 \alpha)^2} \right) dt^2 + \left( 1 + \frac{4m}{r} \right) dr^2 + r^2 \left( 1 + \frac{4m}{r} \right) d\theta^2 
+ \frac{(r - 4m \sin^2 \alpha) \left( 16m^2 r (1 - \cos \theta)^2 + r^3 \sin^2 \theta \left( 1 + \frac{4m}{r} \right)^2 \right)}{(r + 4m)} d\phi^2 
- \frac{8m \sinh \alpha (1 - \cos \theta) (r - 4m \sin^2 \alpha) - 32m^2 r (1 - \cos \theta) \cosh \alpha \sinh ^2 \alpha}{(r + 4m)} (r - 4m \sinh^2 \alpha) dtd\phi \]  

(18)

3 The Solution

Here, we introduce a metric which is a vacuum five-dimensional solution, having some properties in common with the string solution. The proposed static metric is given by

\[ ds^2 = - (1 - C^2 r^2 \sin^2 \theta) dt^2 + (1 + B^2 (r)^2 \sin^2 \theta) dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 
+ (1 + A^2 r^2 \sin^2 \theta) dy^2 + 2CB(r)r^2 \sin^2 \theta dtdr + 2Cr^2 \sin^2 \theta dtd\phi + 2CAr^2 \sin^2 \theta dtdy 
+ 2B(r)r^2 \sin^2 \theta dtd\phi + 2AB(r)r^2 \sin^2 \theta drdy + 2Ar^2 \sin^2 \theta d\phi dy, \]  

(19)

where, the extra spatial coordinate is represented by \( y \), and \( A \) and \( C \) are constant. \( B(r) \) is a function of \( r \). The coordinates are given by \( r, \theta, \phi \) with usual ranges \( r \geq 0, 0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi \) and \( 0 \leq y \leq 2\pi \). The metric has the signature \((- + + + +)\). We have constructed the metric (19) in such a way that it satisfies the vacuum Einstein equations in five dimensions while keeping all metric parameters, although some are obviously superficial.

The scalar field \( \phi \), and the gauge field \( A_\mu \) deduced from the metric (19) are

\[ \phi^2 = 1 + A^2 r^2 \sin^2 \theta, \]  

(20)

and

\[ A_t = \frac{1}{\kappa} \frac{CAr^2 \sin^2 \theta}{1 + A^2 r^2 \sin^2 \theta}, \]  

(21)
\[ A_r = \frac{1}{\kappa} \frac{AB(r)r^2 \sin^2 \theta}{(1 + A^2 r^2 \sin^2 \theta)}, \]  
\[ A_\phi = \frac{1}{\kappa} \frac{A r^2 \sin^2 \theta}{(1 + A^2 r^2 \sin^2 \theta)}, \]  
respectively. Thus the components of the electromagnetic fields are given by
\[ F_{tr} = -\frac{1}{\kappa} \frac{2ACr \sin^2 \theta}{(1 + A^2 r^2 \sin^2 \theta)^2} = E_r, \]
\[ F_{t\theta} = -\frac{1}{\kappa} \frac{2ACr^2 \sin \theta \cos \theta}{(1 + A^2 r^2 \sin^2 \theta)^2} = r E_\theta, \]
\[ F_{\theta\phi} = \frac{1}{\kappa} \frac{2Ar^2 \sin \theta \cos \theta}{(1 + A^2 r^2 \sin^2 \theta)^2} = -r \sin \theta B_r, \]
\[ F_{r\phi} = \frac{1}{\kappa} \frac{2Ar \sin^2 \theta}{(1 + A^2 r^2 \sin^2 \theta)^2} = r \sin \theta B_\theta, \]
\[ F_{r\theta} = -\frac{1}{\kappa} \frac{2AB(r)r^2 \sin^2 \theta}{(1 + A^2 r^2 \sin^2 \theta)^2} = -r B_\phi. \]

In which \( E_r \) and \( E_\theta \) are similar to an electric dipole when \( r \to \infty \). \( B_\phi \) resembles a string magnetic field. If \( B(r) \) is a well-behaved function of \( r \), it is also easy to see that \( B_\phi \) is similar to a magnetic dipole when \( r \to \infty \). The three-dimensional electric and magnetic fields lines diagram are shown in Figs. (1) and (2), respectively. If we convert the magnetic fields from spherical coordinates to a cartesian one and then if \( B(r) = 0 \), there is a magnetic field along the \( z \) axis. An association of the magnetic field components \( B_r, B_\theta \) and \( B_\phi \) are given by
\[ B_r^2 + B_\theta^2 = B_z^2 |_{B(r)=0} = \frac{4A^2}{\kappa^2 \theta^2}. \]

We can easily show that
\[ \vec{\nabla} \cdot \vec{B} = 0. \]  

The four-dimensional spacetime is described by the following metric which is obtained by performing a Kaluza Klein reduction
\[ ds_{4D}^2 = -\left(1 - \frac{C^2 r^2 \sin^2 \theta}{1 + A^2 r^2 \sin^2 \theta}\right) dt^2 + \left(1 + \frac{B^2(r)r^2 \sin^2 \theta}{1 + A^2 r^2 \sin^2 \theta}\right) dr^2 + r^2 d\theta^2 + \frac{r^2 \sin^2 \theta}{1 + A^2 r^2 \sin^2 \theta} d\phi^2 + \frac{C B(r)r^2 \sin^2 \theta}{1 + A^2 r^2 \sin^2 \theta} dt dr + \frac{C r^2 \sin^2 \theta}{1 + A^2 r^2 \sin^2 \theta} dt d\phi + \frac{B(r)r^2 \sin^2 \theta}{1 + A^2 r^2 \sin^2 \theta} dr d\phi, \]  

The inverse metric tensor is given by
\[ g^{\alpha \beta} = \begin{pmatrix} -1 & 0 & 0 & C \\ 0 & 1 & 0 & -B(r) \\ C & -B(r) & 0 & H(r, \theta) \end{pmatrix}. \]  

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Figure 1: The three-dimensional electric field lines for $A = C = 1$

Figure 2: The three-dimensional magnetic field lines for $A = C = 1$. 
where
\[ H(r, \theta) = \frac{1}{r^2 \sin^2 \theta} + A^2 + B^2(r) - C^2. \]

According to the Ricci scalar \( R \), and the nontrivial quadratic curvature invariant \( R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu} \), the curvature singularities are determined
\[
\begin{align*}
    r_1 &= \frac{1}{A} \sqrt{\frac{2(\cos^2 \theta - 1)}{\cos^4 \theta - 2 \cos^2 \theta + 1}}, \\
    r_2 &= -\frac{1}{A} \sqrt{\frac{2(\cos^2 \theta - 1)}{\cos^4 \theta - 2 \cos^2 \theta + 1}}, \\
    r_3 &= \left( -\frac{\cos^{12} \theta - 6 \cos^{10} \theta + 6 \cos^8 \theta + 16 \cos^6 \theta - 39 \cos^4 \theta + 30 \cos^2 \theta - 8}{(\cos^6 \theta - 3 \cos^4 \theta + 3 \cos^2 \theta - 1)^3} \right)^{1/3} \\
    &\quad + \frac{1}{3} \left( \frac{3 \cos^4 \theta - 6 \cos^2 \theta + 3}{\cos^6 \theta - 3 \cos^4 \theta + 3 \cos^2 \theta - 1} \right)^{1/3}.
\end{align*}
\]

Note that \( r_2(\theta) \) is irrelevant since it is negative, thus not physical. In general, \( r_1 \) is imaginary in \( \theta \in (0, \pi) \), and if \( \theta = 0, \pi \) then \( r_1 \to +\infty \). It is easy to understand that \( r_3 \) is a negative function of coordinate \( \theta \). However, if \( \theta = 0, \pi \), then \( r_3 \to +\infty \). Therefore, a string singularity is present.

The four-dimensional metric has at least two Killing vector \( \partial_t \) and \( \partial_\phi \). In order to obtain the Killing horizon, we can use the condition \( \xi^2 = 0 \), where \( \xi \) is an otherside timelike Killing vector, therefore we have
\[
g_{\mu\nu} \xi^\mu \xi^\nu = -1 + \frac{C^2 r^2 \sin^2 \theta}{1 + A^2 r^2 \sin^2 \theta} = 0,
\]
which gives
\[ r = \frac{1}{\sin \theta \sqrt{C^2 - A^2}}, \]
that is similar to the infinite red-shift surface \( g_{tt} = 0 \). Furthermore, the event horizon can be obtained by \( g^{rr} = 0 \). However, for metric (31) \( g^{rr} = 1 \), which means there is no event horizon.

According to the radial electric and magnetic fields, we can calculate the net electric and magnetic fluxes through any two-dimensional surface. The electric flux could be calculated through
\[ Q_E = -\int_{\partial \Sigma} d^{n-2} z \sqrt{|g^{\partial \Sigma}|} n_\mu \sigma_\nu F^{\mu\nu}, \]
where \( n_\mu \) and \( \sigma_\nu \) are the unit normal vectors, \( \Sigma \) is a hypersurface of constant \( t \) and \( r \) and, \( |g^{\partial \Sigma}| \) is the determinant of the induced metric on \( \partial \Sigma \). After some calculation, we take the limit \( r \to \infty \), the electric flux will be zero
\[ Q_E = 0. \]

The magnetic flux for metric (31) is as follows
\[ \Phi_B = \oint_{S^2} \frac{1}{2} F^{\mu\nu} d_s{\mu\nu} = \oint_{S^2} \frac{1}{2} |g^{(2)}| g^{\theta\phi} g^{\phi\theta} F_{\theta\phi} d\theta d\phi, \]

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which gives
\[ \Phi_B = -\frac{2\pi A}{\kappa} \int_0^\pi \frac{r^4 \sin^3 \theta \cos \theta}{(1 + A^2 r^2 \sin^2 \theta)^3} \left( \frac{1}{r^2 \sin^2 \theta} + A^2 + B^2(r) - C^2 \right) d\theta, \tag{40} \]
so that the integral becomes
\[ \Phi_B = 0. \tag{41} \]
We conclude that, the solution is a magnetic dipole.

Because of the existence of two Killing vectors \( \xi^\mu = \delta^\mu_0 \) and \( \xi^\mu = \delta^\mu_\phi \) we can get the mass \( M \) and the angular momentum \( J \), which correspond to the time translation and the axial symmetry, respectively. To calculate the conserved quantities we use the following integral\[23\]
\[ I = \frac{1}{8\pi G} \int_s \nabla^m \xi^n d^2\Sigma_{mn}, \tag{42} \]
where \( d^2\Sigma_{mn} \) is a two-dimensional surface. By using
\[ d^2\Sigma_{mn} = \epsilon_{mn0\phi} r^2 \sin \theta d\theta d\phi, \tag{43} \]
we can easily show that the relevant integration measure for the time translation is as follows
\[ \nabla^n \xi^m d^2\Sigma_{mn} = 2\nabla^r \xi^r r^2 \sin \theta d\theta d\phi. \tag{44} \]
Substituting into the integral we obtain
\[ M = \frac{1}{8\pi G} \int 2\nabla^r \xi^r r^2 \sin \theta d\theta d\phi = 0. \tag{45} \]
In the case of axial symmetry the integral will give \( J = 0 \).

3.1 Investigation of Special Cases

We now examine the solution for special cases of constant parameters.

By applying \( A = 0 \), we see that there are no electric and magnetic fields and the scalar diaton field would be constant. The event horizon wouldn’t exist, therefore the four-dimensional spacetime singularity is naked. The infinite red-shift surface is as follows
\[ r = \frac{1}{C \sin^2 \theta}. \tag{46} \]
We conclude that, there is no magnetic string and the curvature singularity is a string singularity along \( \theta = 0, \pi \).

Assuming a well-behaved \( B(r) \), we can look for \( B(r_0) = 0 \), in which \( r_0 \) is a root of the function. The electric and magnetic fields are \( [24], \) \( [25], \) \( [26] \) and \( [27] \), and the scalar dilaton field is \( [20] \). The null horizon is at \( r = 0 \) and the infinite red-shift surface is as follows
\[ r = \frac{1}{\sqrt{C^2 - A^2 \sin^2 \theta}}. \tag{47} \]
By comparing this case with the boosted Kaluza-Klein magnetic monopole of GPS solution \[18\], we find that these spacetimes are similar to each other, and maybe have the same physical properties. The curvature singularity is still a string singularity.

In the \( C = 0 \) case, the electric field is zero and there are three components of the magnetic field in the spherical coordinates. The scalar dilaton field is given by eq. \[20\]. The event horizon and the infinite red-shift surface do not exist. Therefore, we figure out that in this case there is a magnetic monopole with naked string singularity.
4 Conclusion

We consider a new Kaluza-Klein string solution in five dimensions. The substantial contribution of this paper was the introduction of this solution and investigating the physical properties of the represented solution. The gravitational mass was calculated and it was shown to vanish. We computed the magnetic charge and demonstrated that the magnetic flux of the solution would be constant, which means that there is no extended magnetized source. The three-dimensional electric and magnetic fields lines were shown. In general, it was pointed out that the curvature singularity is not covered by a horizon. It was also shown that the infinite red-shift surface is associated with the $A$ and $C$ parameters. As a special case by applying $C = 0$, we figured out that there is no infinite red-shift surface. It will be interesting to investigate this solution for special cases of parameters which appear in it.

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