THE COMPLEXITY OF FUZZY LOGIC

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Abstract. We show that the set of valid formulas in Łukasiewicz predicate logic is a complete \( \Pi_2^0 \) set. We also show that the classically valid formulas are exactly those formulas in the classical language whose fuzzy value is 1/2.

Łukasiewicz’ infinite valued logic can be seen as a particular “implementation” of fuzzy logic. The set of possible “truth values” (or, in another interpretation, degrees of certainty) is the real interval \([0,1]\). Minimum, maximum, and truncated addition are the basic operations.

It is well known that the propositional fragment version of Łukasiewicz logic is decidable.

The exact complexity of Łukasiewicz predicate logic was a more difficult problem. For an upper bound, it is known that the set of valid formulas in this logic is a \( \Pi_2^0 \).

(An explicit \( \Pi_2^0 \) representation can be found through the axiomatisation of Novak and Pavelka. See \cite{1} for references.)

For a lower bound, Scarpellini \cite{3} showed that the set of valid formulas is not r.e., and in fact \( \Pi_0^1 \)-hard. He also remarks in a footnote that this set is not \( \Sigma_2^0 \), either.

In his unpublished thesis \cite{3}, Mathias Ragaz showed that the set of valid formulas in Łukasiewicz predicate logic is actually \( \Pi_2^0 \)-complete. The proof of this theorem that we give here was found independently.

Furthermore, we show that if we restrict our attention to classical formulas, the classically valid formulas are exactly those formulas which have value \( \geq 1/2 \) in every fuzzy model.

1. Definitions

For the reader’s convenience, we recall the syntax and semantics of Łukasiewicz’ logic.

1.1. The natural MV-algebra on \([0,1]\): We define a unary operation \( \neg \) and binary operations \( \rightarrow, \land, \lor, \& \) and + on the unit interval \([0,1]\) as follows: \( \neg r = 1 - r \), \( r \lor s = \max(r,s) \), \( r + s = \min(1, r+s) \) (where + on the right side is “true” addition), and \( \land \) and \( \& \) are dual to \( \lor \) and +: \( r \land s = \min(r,s) \), \( 1 - (r \& s) = (1-r) + (1-s) \).

We let \( r \rightarrow s = (\neg r) + s \).

(\& and + are called the “strict conjunction” and “strict disjunction”.)

1.2. Propositional L-formulas and assignments: Propositional Łukasiewicz logic uses the connectives \( \rightarrow, \neg, \land, \lor, \& \) and + (sometimes also written as \( \vee \)). Formulas are built in the usual way from an infinite set of propositional variables.

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An assignment is a map from the variables into $[0, 1]$. Each assignment $s$ naturally induces a map $\bar{s}$ from the set of all formulas into $[0, 1]$: $\bar{s}$ extends $s$, $\bar{s}(\varphi \& \psi) = \bar{s}(\varphi) \& \bar{s}(\psi)$, etc.

We let $\| \varphi \| = \inf \{ s(\varphi) : s \text{ an assignment} \}$, and we call a formula $\varphi$ an L-tautology iff $\bar{s}(\varphi) = 1$ for all assignments $s$, i.e., if $\| \varphi \| = 1$.

### 1.3. Predicate L-logic and models

In Lukasiewicz predicate logic, we have in addition to the connectives also quantifiers $\forall$ and $\exists$.

Let $\sigma$ be a (finite) set of relation symbols and constant symbols, with an “arity” attached to each relation symbol.

The set of terms and formulas in the language $L_\sigma$ for fuzzy predicate calculus over the signature $\sigma$ is defined in the usual way, starting from constants and variables.

A ‘model’ $M$ for the language $L_\sigma$ is given by a nonempty set $M$, together with interpretations $c_M$ and $R_M$ of all constants and relation symbols. Each $c_M$ is an element of $M$, and if $R$ is a $k$-ary relation symbol, then $R_M$ is a map from $M^k$ into $[0, 1]$.

$L_\sigma(M)$ is the language $L_\sigma$ expanded by special constant symbols $c_m$ for every $m \in M$. We will not notationally distinguish between $m$ and $c_m$, and we require that always $c^M_m = m$.

An $M$-formula is a formula in $L(M)$.

$\| \varphi \|^M \in [0, 1]$ is defined by induction for all closed $M$-formulas in the natural way: $\| R(c) \|^M = R^M(c^M)$ for any constant symbol $c$, $\| \neg \varphi \| = \neg \| \varphi \|$, $\| \forall x \varphi(x) \| = \inf_{a \in M} \| \varphi(a) \|^M$, etc.

We let $\| \varphi \|$ be the infimum over $\| \varphi \|^M$, taken over all fuzzy models $M$.

### 1.4. L-validities and L-tautologies

We say that a closed formula $\varphi$ is L-valid if $\| \varphi \| = 1$, i.e., if $\| \varphi \|^M = 1$ for all models $M$.

We call a formula $\varphi$ in $L_\sigma$ (or even in $L_\sigma(M)$) an L-tautology iff there is an L-tautology $\chi$ in the propositional fragment of Lukasiewicz logic and a homomorphism $h$ that assigns formulas in $L_\sigma$ to propositional variables such that $\varphi = h(\chi)$.

Thus, the set of L-tautologies is a (decidable, proper) subset of the set of all L-validities.

### 2. The complexity of L-validities

We now proceed to prove the following theorem.

**Theorem 2.1.** Let $\sigma$ be a sufficiently rich (relational) signature, $L_\sigma$ the set of formulas for this signature in Lukasiewicz logic. Then the set of formulas in $L_\sigma$ which is “valid” is $\Pi^0_2$-complete.

**Definition 2.2.** Let $R \subseteq \mathbb{N} \times \mathbb{N}$ be a primitive recursive relation such that

$$A := \{ m : \exists n (m, n) \in R \}$$

is $\Pi^0_2$-complete.

If $m \notin A$, let $f(m)$ be a positive natural number such that $\forall n \geq f(m) : (m, n) \notin R$. 

Definition 2.3. We start with an extended (purely relational) language $\mathcal{L}_{\sigma_0}$ of Peano arithmetic. The signature $\sigma_0$ contains constant symbols 0, 1, binary relation symbols $<$ and $\preceq$ (read: “is direct successor of”) and finitely many relation symbols (such as ternary $Add$ and $Mul$) intended to code various primitive recursive relations, among them a symbol $R$ intended to be interpreted by $R$.

Our full signature $\sigma$ will have in addition to the symbols in $\sigma_0$ two extra predicates (which are intended to be fuzzy): A binary relation $Q$ and unary relation $P$.

Definition 2.4. Let $\varphi_0$ be an axiom similar to Robinson’s $Q$, coding enough primitive recursive definitions such that in classical logic the following are derivable:

$(+)$ Whenever $(m, n) \in R$, then $\varphi_0 \vdash 0 < 1 < x_2 < \cdots < x_m \land 0 < 1 < y_2 < \cdots < y_n \rightarrow R(x_m, y_n)$

$(-)$ Whenever $(m, n) \notin R$, then $\varphi_0 \vdash 0 < 1 < x_2 < \cdots < x_m \land 0 < 1 < y_2 < \cdots < y_n \rightarrow \neg R(x_m, y_n)$

Moreover, we assume $\varphi_0 \vdash \forall x \exists y \, x \preceq y$

Definition 2.5. The formula $\varphi_1$ is the universal closure of $R(x, y) \& Q(y, y) \rightarrow P(x)$

The formula $\varphi_2$ is the universal closure of $Q(1, y) \leftrightarrow \neg Q(y, y)$

The formula $\varphi_3$ is the universal closure of $x < x' \rightarrow ([Q(x, y) \lor Q(1, y)) \leftrightarrow Q(x', y)]$

Definition 2.6. The standard model $\mathbb{N}$ is defined as follows:

* The universe of $\mathbb{N}$ is the set of natural numbers.
* $Add$, $Mul$, $<$, $\preceq$, $\ldots$, $R$ are interpreted naturally.
* $\|Q(m, n)\|^N = \min(1, m/(n+1))$ for all $m, n \in \mathbb{N}$.
* $\|P(m)\|^N = 1$ if $m \in A$
* $\|P(m)\|^N = 1 - \frac{1}{f(m)}$ if $m \notin A$ (where $f$ is the function defined in 2.2).

Fact 2.7. $\|\varphi_1 \land \varphi_2 \land \varphi_3 \land \varphi_4\|^N = 1$.

Definition 2.8. For each formula $\varphi$, let $\varepsilon_\varphi$ be the universal closure of $\varphi \land \neg \varphi$.

Let $\varepsilon$ be the disjunction over all $\varepsilon_\varphi$, where $\varphi$ runs over all subformulas of $\varphi_0$, $\varphi_1$, $\varphi_2$, $\varphi_3$ except the formulas $Q(x, y)$ and $P(x)$.

So $\varepsilon$ measures how close a structure (disregarding $Q$ and $P$) comes to being a crisp model. In the structures that we are interested in the value of $\varepsilon$ will be 0 (or at least close to 0).

Note that for any formula $\varphi(x, y, \ldots)$ that was used in the definition of $\varepsilon$, any structure $\mathcal{M}$, and any $a, b \ldots$ in $\mathcal{M}$ we have: If $e := \|e\|^\mathcal{M}$, then $\|\varphi(a, b, \ldots)\|^\mathcal{M} \in [0, e] \cup [1 - e, 1]$. 
Definition 2.9. For each \( m > 3 \), let \( \psi_m \) be the universal closure of the following formula:
\[
\varphi_0 \& \varphi_1 \& \varphi_2 \& \varphi_3 \& (0 < 1 < x_2 < \cdots < x_m) \rightarrow P(x_m) + 10\varepsilon
\]
where \( 2\varepsilon = \varepsilon + \varepsilon, 3\varepsilon = 2\varepsilon + \varepsilon, \) etc. The formula \( 0 < 1 < \cdots \) is any conjunction (it does not matter if sharp or not) of the formulas \( 0 < 1, 1 < x_2, \ldots \)

Explanation 2.10. In a fuzzy model where \( \varphi_0 \) is true (or at least “sufficiently true”, i.e., has a truth value close to 1), the formula \( \exists x_2 \cdots \exists x_m \ 0 < 1 < x_2 < \cdots < x_m \) says that \( x_m \) is similar to the number \( m \).

In a model where \( \varphi_2 \& \varphi_3 \) is (sufficiently) true, the formula \( Q(y, y) \) expresses the fact that \( y \) is “infinite”, or at least “large”.

In a model of \( \varphi_1 \), \( P(x) \) says that \( R(x, y) \) holds for some large \( y \).

Hence, \( \psi_m \) says that an object that has properties similar to the number \( m \) is in a set that is similar to \( A \).

This is only an approximation, of course. We will now show that this approximation is good enough for our purposes.

Main Claim 2.11. For all \( m > 3 \): \( m \in A \iff \| \psi_m \| = 1 \).

Clearly this claim will imply our theorem.

Proof of the main claim, part 1. First assume that \( m \) is not in \( A \). So for all \( n \geq f(m), (m, n) \notin R \). Evaluate \( \psi_m \) in the standard model \( N \):

Clearly \( \varepsilon \) will have value 0.

Instantiate \( \psi_m \) with \( x_i = i \). Now the antecedent has value 1, whereas the consequent has value < 1, so \( \| \psi_m \| < 1 \), and we are done.

Proof of the main claim, part 2. Now assume that \( m \in A \). Fix a small number \( \delta \) (in particular \( \delta < 1/m \)). We will show that in all fuzzy structures, all instances of \( \psi_m \) have a value \( \geq 1 - \delta \).

So fix a structure \( \mathcal{M} \), and let \( a_2, \ldots, a_m \) be elements of \( \mathcal{M} \). Write \( e \) for the value \( \| \varepsilon \|^{\mathcal{M}} \).

Let \( \psi_m' \) be the instance of \( \psi_m \) obtained by substituting \( a_i \) for \( x_i \), i.e.,
\[
\psi_m' = \varphi_0 \& \varphi_1 \& \varphi_2 \& \varphi_3 \& (0 < 1 < a_2 < \cdots < a_m) \rightarrow P(a_m) + 10\varepsilon
\]
If \( e \geq 0.1 \), then clearly the value of \( \psi_m' \) is 1. So assume that \( e < 0.1 \). Hence all fuzzy relations associated with symbols in the language \( L_0 \) will always take values < 0.1 or > 0.9.

Let \( \mathcal{M}' \) be the structure obtained from \( \mathcal{M} \) by keeping the values of \( Q \) and \( P \), but rounding all other basic relations to 0 or 1.

We will write \( r \sim s \) if \( |r - s| < 0.1 \).

Claim 2.12. Whenever \( \chi \) is a closed \( M \)-formula which is a substitution instance of some subformula of \( \psi' \) (except for \( Q(\cdot, \cdot) \) and \( P(\cdot) \)), we have \( \| \chi \|^M \sim \| \chi \|^{\mathcal{M}'} \).

The proof of the claim is by induction on the complexity of \( \chi \), using the fact that [by the definition of \( \varepsilon \)] we must have \( \| \chi \|^{\mathcal{M}} \in [0, e] \cup [1 - e, 1] \), and \( e + e < 1 - e \).

We continue the proof of part 2 of the main claim:
If any of the formulas \( \varphi_0, \varphi_1, \varphi_2, \varphi_3 \), 0 < 1, 1 < a_2, a_i < a_{i+1} has \((M)\) a value \( \leq e \), then \( \|\psi'_m\|^{M} = 1 \). So assume that all these values are \( \geq 1 - e \). By the above claim, \( \|\varphi_0\|^{M} = 1 \).

Choose some large enough \( n \) (specifically, let \( n > 1/\delta \), and if \( e > 0 \) then also \( n > 1/e \)), such that \((m, n) \in R\).

Working in \( M' \), we can use the fact that \( M'|L_0 \) satisfies \( \varphi_0 \) in the classical sense:
1. We can find \( a_{m+1}, \ldots, a_n \) such that for all \( i < n: M' |\models a_i < a_{i+1} \).
2. \( M' \models R(a_m, a_n) \).
3. Therefore, \( \|R(a_m, a_n)\|^{M} \geq 1 - e \).

We now claim that
\[
(**) \quad \|Q(a_n, a_n)\|^{M} \geq \min(1 - 4e, 1 - \delta)
\]

Using \((**)*\) and the fact that \( \|R(a_m, a_n)\&Q(a_n, a_n) \rightarrow P(a_m)\| \geq \|\varphi_1\|^{M} \geq 1 - e \), we can then conclude
\[
\|P(a_m)\|^{M} \geq \min(1 - 6e, 1 - \delta - 2e),
\]
hence \( \|\psi'_m\|^{M} \geq 1 - \delta \), so \((**)*\) would finish the proof of the main claim.

So it remains to prove \((**)*\).

Let us abbreviate \( q_1 := Q^M(1, a_n) \) and \( q_k := Q^M(a_k, a_n) \) for \( k = 2, \ldots, n \).

Then we have for \( k = 1, \ldots, n \):
\[
\|a_k < a_{k+1}\|^{M} = 1 - e, \quad \text{where} \quad \|Q(a_k, a_n) + Q(1, a_n) \rightarrow Q(a_{k+1}, a_n)\|^{M} \geq 1 - e.
\]

Since also \( \|a_k < a_{k+1}\|^{M} \geq 1 - e \), we get
\[
\|(Q(a_k, a_n) + Q(1, a_n)) \rightarrow Q(a_{k+1}, a_n)\|^{M} \geq 1 - 2e,
\]
so \( q_{k+1} \geq (q_k + q_1) - 2e \). Using induction on \( k \) we can show for all \( k \):
\[(*1)_k \quad q_k \geq \min(k \cdot (q_1 - 2e), 1 - 2e).
\]

In particular, we get
\[(*1) \quad q_n \geq \min(n \cdot (q_1 - 2e), 1 - 2e).
\]

We also have \( \|\neg Q(1, a_n) \rightarrow Q(a_n, a_n)\|^{M} \geq 1 - e \), so
\[(*2) \quad q_n \geq 1 - q_1 - e.
\]

In the proof of \((**)*\) we distinguish 3 cases:
1. \( e = 0 \).
2. \( e > 0, Q^M(1, a_m) \leq 3e \).
3. \( e > 0, Q^M(1, a_m) > 3e \).

**Case 1:** \( e = 0 \).

Assume \( q_n < 1 \). By \((*2)\), \( q_1 \geq 1 - q_n \). By \((*1)\), \( q_n \geq n \cdot q_1 \geq n \cdot (1 - q_n) \). So \( q_n \geq n/(n + 1) > 1 - \delta \).

**Case 2:** \( e > 0, q_1 \leq 3e \).

By \((*2)\), we have \( q_n \geq 1 - 4e \).

**Case 3:** \( e > 0, q_1 > 3e \).

Recall that \( n \) was chosen so large that \( n \cdot e > 1 \), so \( n \cdot (q_1 - 2e) > 1 \), so \((1)\) implies \( q_n \geq 1 - 2e \).
3. A restricted language

In the previous section we have made good use of the fact that addition is hardwired into the semantics of our particular brand of fuzzy logic. In this section we show that if we restrict the language to the lattice operations $\lor$, $\land$, together with $\neg$, then the computation of $\|\varphi\|$ can be reduced to the problem of deciding classically valid formulas, and conversely.

Let $L^\text{class}$ be the “classical” propositional language, using only the connectives $\land$, $\lor$ and $\neg$, and let $L^\text{class}_\sigma$ be the classical predicate language over the signature $\sigma$.

We mention some easy (and well-known) fact:

Observation 3.1. If $\varphi$ is a propositional formula in $L^\text{class}$, then

(a) For any $\varphi \in L^\text{class}$, $\|\varphi\| \leq \frac{1}{2}$.
(b) If $\varphi$ is a classical tautology, then $s(\varphi) \geq 0.5$ under any assignment $s$.
(c) If $\varphi$ is not a classical tautology, then $\|\varphi\| = 0$.
(d) The following are equivalent for $\varphi \in L^\text{class}$:
   - $\varphi$ is a classical tautology
   - $\|\varphi\| = \frac{1}{2}$
   - $\|\varphi\| > 0$.
   - $\|\neg \varphi \rightarrow \varphi\| = 1$.
   - $\|p \land \neg p \rightarrow \varphi\|$, where $p$ is any propositional variable not appearing in $\varphi$.
(e) The set of fuzzy tautologies is co-NP-complete.

Proof. (a): Assign the value $\frac{1}{2}$ to every propositional variable.

(c) is clear, and (d) is a reformulation of (a)–(c). (e) follows from (d).

It remains to prove (b):

Let us call a formula $\varphi$ a “literal” if $\varphi = p$ or $\varphi = \neg p$ for some propositional variable $p$. A “clause” will be a nonempty conjunction of literals. We say that $\psi$ is in “normal form” if $\psi$ is a nonempty disjunction of clauses.

Using the distributive law, as well as de Morgan’s laws and cancelling of double negations, we can find a formula $\psi$ which is equivalent (classically as well as in propositional $L$-logic) to $\varphi$ and is in normal form.

Now, if $\psi$ is a classical tautology, then each clause of $\psi$ contains some variable $p$ both in negated and unnegated form. Hence, under any fuzzy assignment $s$, $s(\psi) \geq 0.5$.

Theorem 3.2. Let $\varphi$ be a formula in Łukasiewicz predicate logic which in $L^\text{class}_\sigma$.

Then:

(a) There is a model $M$ such that $\|\varphi\|^M = 0.5$.
(b) If $\varphi$ is classically valid, then $\|\varphi\| = 0.5$.
(c) If $\varphi$ is not classically valid, then $\|\varphi\| = 0$.

Proof. (a) and (c) are clear.

Instead of (b), it is enough to show the following:

If $\neg \varphi$ is classically valid, then $\|\varphi\| \leq 0.5$.

Without loss of generality we may assume that $\varphi$ is in prenex form (since a transformation to prenex form preserves the classical truth value as well as $\|\varphi\|$), say $\varphi = (\forall x_1)(\exists y_1)(\forall x_2) \cdots (\exists y_n)\psi(x_1, y_1, \ldots, x_n, y_n)$, where $\psi$ is quantifier-free.
Let \( \bar{\psi} \) be the Skolemization of \( \psi \). That is, let \( g_i \) be an \( i \)-ary function symbol for \( i = 1, \ldots, n \), and let

\[
\bar{\psi}(x_1, \ldots, x_n) := \psi(x_1, g_1(x_1), g_2(x_1, x_2), \ldots, g_n(x_1, \ldots, x_n))
\]

and let \( \bar{\varphi} = \forall x_1 \forall x_2 \cdots \forall x_n \bar{\psi}(x_1, \ldots, x_n) \).

Since \( \varphi \) is a classical contradiction, also \( \bar{\varphi} \) has no classical model. By Herbrand’s theorem, there is a finite conjunction \( \bar{\psi}_1 \land \cdots \land \bar{\psi}_k \) of closed instances of \( \bar{\psi} \) which is a contradiction in the sense of classical propositional logic.

[More formally, there is a propositional formula \( \chi \in \mathcal{L}^{\text{class}} \) which is a classical propositional contradiction, and a homomorphism \( h \) such that \( \bar{\psi}_1 \land \cdots \land \bar{\psi}_k = h(\chi) \).]

Let \( \bar{\psi}_i = \bar{\psi}(t^1_i, \ldots, t^n_i) \), where all the \( t^j_i \) are closed terms (involving the function symbols \( g_1, \ldots, g_n \) and some constant symbols from the original signature).

Now let \( \mathcal{M} \) be a fuzzy model, and assume, towards a contradiction, than \( \| \varphi \|^\mathcal{M} > 0.5 \). For each \( m_1 \in \mathcal{M} \) choose \( f_1(m_1) \in M \) such that

\[
\|(\forall x_2)(\exists y_2)\cdots(\exists y_n)\psi(m_1, f_1(m_1), x_2, y_2, \ldots, x_n, y_n)\|^\mathcal{M} > \frac{1}{2}
\]

Now for each \( m_1, m_2 \) in \( M \) choose \( f_2(m_1, m_2) \in M \) such that

\[
\|(\forall x_3)\cdots(\exists y_n)\psi(m_1, f_1(m_1), m_2, f_2(m_1, m_2), \ldots, y_n)\|^\mathcal{M} > \frac{1}{2}
\]

and continue by induction. We thus get functions \( f_1, \ldots, f_n \) such that for any \( a_1, \ldots, a_n \in \mathcal{M} \):

\[
\|\psi(a_1, f_1(a_1), a_2, f_2(a_1, a_2), \ldots, a_n, f_n(a_1, a_2, \ldots, a_n))\|^\mathcal{M} > \frac{1}{2}
\]

Now \( \bar{\mathcal{M}} := (M, f_1, \ldots, f_n, c^\mathcal{M} : c \in \sigma) \) is a (classical) structure for the signature \( (g_1, \ldots, g_n, c : c \in \sigma) \).

For any (quantifier-free) closed \( \bar{\mathcal{M}} \)-formula \( \chi \) let \( \chi^{\bar{\mathcal{M}}} \) be obtained by replacing each atomic subformula \( \bar{\psi}(t_1, \ldots) \) by \( \bar{\psi}(t_1^{\mathcal{M}}, \ldots) \), where \( t^{\mathcal{M}} \) is the value of the closed term \( t \).

Now recall that \( \bar{\psi}_1 \& \cdots \) was a classical contradiction. So also \( (\bar{\psi}_1 \& \cdots)^{\bar{\mathcal{M}}} \) is a classical contradiction. Note that this formula does not contain any function symbols any more, so we can compute its value in our fuzzy structure \( \mathcal{M} \). By [3.1], this value is at most 1/2, so wlog \( \|\bar{\psi}_1^{\mathcal{M}}\|^\mathcal{M} \leq 1/2 \).

Let \( a_j := t_j^{\mathcal{M}} \). Now

\[
\|\bar{\psi}_1^{\mathcal{M}}\|^\mathcal{M} = \|\bar{\psi}(t_1^{\mathcal{M}}, \ldots, t_n^{\mathcal{M}})^{\mathcal{M}}\|^\mathcal{M} = \|\psi(t_1^{\mathcal{M}}, g_1(t_1^{\mathcal{M}}), t_2^{\mathcal{M}}, \ldots, g_n(t_1^{\mathcal{M}}, \ldots, t_n^{\mathcal{M}})^{\mathcal{M}}\|^\mathcal{M} = \|\psi(a_1, f_1(a_1), a_2, \ldots)\|^\mathcal{M} > 1/2
\]

a contradiction.

\[\square\]

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