Central Limit Theorem in Disordered Monomer-Dimer Model

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Abstract

We consider the disordered monomer-dimer model on general finite graphs with bounded degree, where both the edges and the vertices are equipped with i.i.d. random weights coming from two possibly different distributions. Under the finite fourth moment assumption on the weight distributions, we prove a Gaussian central limit theorem for the free energy of the associated Gibbs measure and also provide a rate of convergence in the Kolmogorov-Smirnov distance. The central limit theorem continues to hold under a nearly optimal finite $(2+\epsilon)$-moment assumption on the weight distributions if the underlying graphs are further assumed to have a uniformly subexponential volume growth. This generalizes a recent result by Dey and Krishnan [11] who showed a Gaussian central limit theorem in the disordered monomer-dimer model on cylinder graphs. Our proof relies on the idea that the disordered monomer-dimer model exhibits a decay of correlation with high probability.

1 Introduction

The monomer-dimer model was introduced as a simple yet effective model in condensed-matter physics that describes the absorption of monoatomic (monomer) or diatomic (dimer) on certain surfaces [21, 22, 6, 5]. The dimers occupy the pair of adjacent sites (or edges) of a graph, whereas the monomers occupy the rest of the vertices. The key feature of the model is the hard-core interaction among the dimers that excludes two dimers to share a common vertex. In the language of graph theory, the set of dimers forms a (partial) matching of the graph and the monomers can be regarded as the unmatched vertices. The monomer-dimer model is the Gibbs measure on the space of dimer configurations or matching on a finite graph where the energy of a matching is given by the sum of the weights of the edges belonging to that matching (dimer activity) and the weights of the unmatched vertices (monomer activity).

In a seminal paper, Heilmann and Lieb [14] established that the monomer-dimer model does not have a phase transition by studying the zeroes of the partition function. If the monomer weights are taken to be a constant $\nu$, then the partition function can be viewed as a polynomial in $e^{\nu}$. Heilmann and Lieb showed that the all zeroes of this polynomial, known as the matching polynomial, lie on the imaginary axis. This makes the free energy an analytic function of $\nu$, which implies the absence

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of any phase transition. This special localization property of the zeroes of the partition function also yields a central limit theorem for the number of edges in a random matching for a large class of finite graphs \[13, 15, 19\].

One can naturally inject randomness into the monomer-dimer model by considering the underlying graph to be random or (and) by taking the weights to be random. For a sequence of graphs converging locally to a random tree, the limiting free energy of the monomer-dimer model with constant weights can be computed via so-called cavity method and the solution turns out to be a function of the unique fixed point of some distributional recursion relation \[26, 1\]. In \[2\], Alberici et. al. considered the monomer-dimer model on a complete graph with constant edge weights and i.i.d. random vertex weights. Exploiting a Gaussian representation of the partition function, they provided a variational formula for the limiting free energy.

Recently, Dey and Krishnan \[11\] studied the monomer-dimer model with both vertex and edge weights being random (referred to by them as the disordered monomer-dimer model) on cylinder graphs (i.e., the cartesian product of a path graph on \(n\) vertices, and a fixed graph). Among other results, they proved that the free energy has Gaussian fluctuation. The proof of their central limit theorem is based on the dyadic decomposition of the free energy into independent components which heavily relies on the one-dimensional nature of the underlying graph. Such decomposition does not extend immediately to the higher dimensional lattices. The aim of our paper is to show that the free energy of the disordered monomer-dimer model on any bounded degree finite graph is asymptotically normal under a mild moment condition on the weights. Moreover, an explicit error bound in the Kolmogorov-Smirnov distance is provided. Our proof hinges on establishing the correlation decay in the disordered model. In the constant weight case, van den Berg \[24\] introduced a geometric argument involving disagreement percolation to establish decay of correlation and applied it to deduce that the model has no phase transition. We adapt van den Berg’s technique in the random weights setting to obtain the necessary correlation decay result. To show the asymptotic normality of the free energy, we then invoke a central limit result by Chatterjee \[8\] based on a generalized perturbative approach to Stein’s method.

1.1 The model and main results

Let \(G = (V(G), E(G))\) be a finite graph. A (partial) matching \(M\) of \(G\) is a collection of disjoint edges from \(E(G)\) such that no pair of them is incident on the same vertex. We denote the collection of all matchings on \(G\) by \(\mathcal{M}_G\).

We assign i.i.d. weights \((w_e)_{e \in E(G)}\) to the edges and i.i.d. weights \((\nu_x)_{x \in V(G)}\) to the vertices of \(G\). The edge weights are independent of the vertex weights, but their distributions are possibly different. For \(x \in V(G)\) and \(M \in \mathcal{M}_G\), we write \(x \notin M\) to indicate that the vertex \(x\) is unmatched in the matching \(M\). Given the weights, the disordered monomer-dimer model on \(G\) is defined by the following random Gibbs measure \(\mu_G\).

\[
\mu_G(M) = Z_G^{-1} \exp \left( \beta \left( \sum_{e \in M} w_e + \sum_{x \notin M} \nu_x \right) \right), \quad M \in \mathcal{M}_G,
\]

where \(\beta > 0\) is the inverse temperature and \(Z_G\) is the normalization constant, called the partition
function, which is given by
\[ Z_G = \sum_{M \in \mathcal{M}_G} \exp \left( \beta \left( \sum_{e \in M} w_e + \sum_{x \notin M} \nu_x \right) \right). \]

Note that the inverse temperature \( \beta \) can be absorbed into the weights. So, henceforth, we will assume, without loss of generality, that \( \beta = 1 \).

Let \( F = \log Z_G \) be the free energy. Our main result states that under the assumptions that the fourth moments of \( w_e \) and \( \nu_x \) are finite, \( F \) obeys a Gaussian central limit theorem with an explicit rate of convergence, as the size of the graph gets large but its maximum degree stays bounded. Define the volume growth function of \( G \) as
\[ \Psi_G(R) = \max_{x \in V(G)} |V(\mathbb{B}_R^x)|, \]
where \( \mathbb{B}_R^x \) is the subgraph of \( G \) induced by all the vertices that are at most distance \( R \) away from vertex \( x \). Let \( \Phi \) be the cumulative distribution function of the standard Gaussian distribution.

**Theorem 1.1.** Consider the disordered monomer-dimer model on a finite graph \( G \) with maximum degree \( D \). Assume that the vertex and the edge weights have finite fourth moments and the edge weights are non-degenerate. Then there exist constants \( c, C > 0 \), depending only on \( D \) and the distributions of the weights, such that for all \( R \geq 1 \),
\[ \sup_{s \in \mathbb{R}} \left| P \left( \frac{F - EF}{\sqrt{\text{Var}(F)}} \leq s \right) - \Phi(s) \right| \leq C \left( \frac{|V(G)|^{1/4}}{|E(G)|^{1/2}} \Psi_G(3R)^{1/4} + \frac{|V(G)|^{1/2}}{|E(G)|^{1/2}} e^{-cR} + \frac{|V(G)|^{1/2}}{|E(G)|^{3/4}} \right). \]

(1.1)

Assume further that for some constants \( K_1 \) and \( K_2 \), the weight distributions satisfy

(i) \( \max(\mathbb{E}|w_e|^2, \mathbb{E}|\nu_x|^2) \leq K_1 \),

(ii) \( \mathbb{E}|w_e - w_e'| \geq K_2 \) where \( w_e' \) is an i.i.d. copy of \( w_e \).

Then the constant \( C \) in (1.1) can be taken as \( C = C_1 (1 + \mathbb{E}|w_e|^4 + \mathbb{E}|\nu_x|^4)^{3/8} \) and the constants \( c \) and \( C_1 \) can be chosen depending only on \( D, K_1, \) and \( K_2 \).

Let \( (G_n)_{n \geq 1} \) be a sequence of connected graphs with \( |V(G_n)| \to \infty \) as \( n \to \infty \). Assume that the maximum degrees of \( G_n \) are bounded above by some constant \( D \), which implies that \( \Psi_{G_n}(R) \leq CD^R \) for all \( n, R \geq 1 \). Also, the numbers of edges and vertices \( G_n \) are of same order since \( V(G_n) - 1 \leq E(G_n) \leq (D/2)V(G_n) \). We can take \( R = \sqrt{\log |V_n|} \) in (1.1) to derive that the normalized \( F_n \), the free energy associated with \( \mu_{G_n} \), converges in distribution to a standard Gaussian.

If we further assume that the sequence of graphs \( (G_n)_{n \geq 1} \) has uniformly sub-exponential volume growth, that is, for any \( \alpha > 0 \), there exists \( K \) such that
\[ \Psi_{G_n}(R) \leq K \exp(\alpha R) \text{ for all } n, R \geq 1, \]
then we are able to prove a Gaussian central limit theorem under a nearly-optimal finite \((2 + \epsilon)\)-moment assumption.
Theorem 1.2. Let \((G_n)_{n \geq 1}\) be a sequence of finite graphs such that \(|V(G_n)| \to \infty\) as \(n \to \infty\) and \(|E(G_n)| \geq \delta |V(G_n)|\) for all \(n\) large enough for some positive constant \(\delta\). Furthermore, assume that the maximum degrees of \((G_n)_{n \geq 1}\) are bounded above by some constant \(D\) and that \((G_n)_{n \geq 1}\) has uniformly sub-exponential volume growth. Suppose that \(E|w_e|^{2+\epsilon} + E|\nu_x|^{2+\epsilon} < \infty\) for some \(\epsilon > 0\) and \(w_e\) is non-degenerate. Then
\[
\frac{F_n - EF_n}{\sqrt{\text{Var}(F_n)}} \xrightarrow{d} N(0, 1) \quad \text{as } n \to \infty,
\]
where \(F_n\) is the free energy associated with \(\mu_{G_n}\). Here \(\xrightarrow{d}\) implies the convergence in distribution.

Remark. The proof of Theorem 1.2 will show that the hypothesis of uniform subexponential volume growth can be replaced by the following weaker assumption. There exists \(K\) such that
\[
\Psi_{G_n}(R) \leq K \exp(\epsilon R/20) \quad \text{for all } n, R \geq 1.
\]
In above, \(\epsilon\) is the same as one that appears in the moment condition \(E|w_e|^{2+\epsilon} + E|\nu_x|^{2+\epsilon} < \infty\).

The most important examples covered by the above theorem are the growing boxes of \(\mathbb{Z}^d\) for any \(d \geq 1\), which trivially satisfies the uniform subexponential volume growth assumption.

To prove Theorem 1.1, we apply a result by Chatterjee [8], which has turned out to be quite successful in proving central limit theorems with a rate of convergence in a variety of problems in geometric probability, combinatorial optimization, and number theory [8, 10, 9, 4].

Chatterjee’s method quantifies the rough idea that a function \(f(X_1, X_2, \ldots, X_N)\) of independent random variables \(X_1, X_2, \ldots, X_N\) is asymptotically Gaussian if its partial derivatives \(\partial_i f(X_1, X_2, \ldots, X_N)\) are ‘approximately independent’. See Theorem 3.1 for a precise statement. For the free energy, these partial derivatives are given by
\[
\partial_{w_e} F = \langle 1_{\{e \in \mathcal{M}_G\}} \rangle, \quad \partial_{\nu_x} F = \langle 1_{\{x \not\in \mathcal{M}_G\}} \rangle,
\]
where \(\mathcal{M}_G\) denotes a random matching of \(G\) sampled according to \(\mu_G\) and \(\langle \cdot \rangle\) is the Gibbs expectation with respect to \(\mu_G\). To successfully apply Chatterjee’s method, one needs to show that the above derivatives can be well-approximated by some functions that only depend on the randomness of the local neighborhood of \(e\) or \(x\).

Let us restrict our discussion to the edge derivative \(\partial_{w_e} F\) as the vertex derivative \(\partial_{\nu_x} F\) behaves very similarly. For an edge \(e = (xy)\) of \(G\), let \(\mathcal{B}_R^e\) denote the subgraph of \(G\) induced by all the vertices that are at most distance \(R\) away from either \(x\) or \(y\). Also, for a subgraph \(H \subseteq G\), let \(\mathcal{M}_H\) denote a random matching on \(H\) sampled according to the Gibbs measure restricted to the graph \(H\) where the weights of the vertices and edges of \(H\) are kept unchanged from \(G\). The next result confirms that \(\partial_{w_e} F\) can indeed be well-approximated by a local function.

Theorem 1.3. Consider the disordered monomer-dimer model on a finite graph \(G\) with maximum degree \(D\). For any \(p \geq 1\), there exist constants \(c, C > 0\) depending on \(p, D\) and the weight distributions such that the following holds. For any edge \(e\) and for any \(R \geq 1\), we have
\[
E|\langle 1_{\{e \in \mathcal{M}_G\}} \rangle - \langle 1_{\{e \in \mathcal{M}_B^n\}} \rangle|^p \leq Ce^{-cR}.
\]
Note that the above result readily implies the exponential correlation decay in the disordered monomer-dimer model.

\[ E \left| \left< 1_{e \in M_G} \right> - \left< 1_{e \in M_H} \right> \right| \leq C e^{-c \text{dist}(e,e')} \]

In [24], van den Berg used a beautiful geometric argument involving disagreement percolation to control the effect of boundary in the standard monomer-dimer model (where \( w_e \equiv \log \lambda \) and \( \nu_x \equiv 0 \) for some constant \( \lambda > 0 \)). A similar argument can also be found in [25]. This showed exponential strong spatial mixing for the monomer-dimer model at any temperature and gave an alternative proof of the absence of phase transition. To prove Theorem 1.3, we broadly follow van den Berg’s argument. However, in the disordered case, we have random, possibly unbounded weights on the edge and vertices, which leads to additional complications.

Finally, let us mention that the bound on Chatterjee’s theorem requires a finite fourth-moment assumption on the weight distributions. To remove this assumption, we carefully interpolate the centered free energy with the original weights with the one with the appropriately truncated weights in the proof of Theorem 1.2.

The rest of the paper is organized as follows. In Section 2, we prove the correlation decay result for the disordered monomer-dimer model. We then use it to prove the central limit theorems in Section 3. We conclude the paper by listing a few open problems in Section 4.

## 2 Correlation Decay

This section contains the proof of the correlation decay (Theorem 1.3) in the disordered monomer-dimer model. In fact, a slightly stronger version of Theorem 1.3 is required to prove the central limit theorem. If the weights are bounded, we can show exponential correlation decay for all possible realizations of the weights. For unbounded weights, the same conclusion holds only after we throw away a bad set of weights with an exponentially small probability.

Call a function \( \varphi : (0, \infty) \to (0, \infty) \) vanishing if \( \lim_{s \to \infty} \varphi(s) = 0 \). For a vanishing function \( \varphi \), we say a random variable \( X \) to be \( \varphi \)-bounded if

\[ P(|X| \geq t) \leq \varphi(t) \text{ for all } t > 0. \]

For \( U \subseteq V(G) \) and \( F \subseteq E(G) \), let \( \mathcal{F}^{U \times F} \) denote the product \( \sigma \)-algebra generated by the independent random variables \((\nu_x)_{x \in U}, (w_e)_{e \in F}\).

**Proposition 2.1.** Given a vanishing function \( \varphi \) and integer \( D \geq 1 \), there exist constants \( c, C > 0 \) depending only on \( \varphi \) and \( D \) such that the following holds. Consider the disordered monomer-dimer model with \( \varphi \)-bounded edge and vertex weights on any finite graph \( G \) whose maximum degree is bounded above by \( D \). Then for any \( e = (xy) \in E(G) \) and for any \( R \geq 1 \), there exists an event \( A \in \mathcal{F}^{V(B_R \setminus \{x,y\} \times E(B_R \setminus \{e\}) \setminus \{e\}} \) with \( P(A) \geq 1 - Ce^{-cR} \), such that on the event \( A \), we have

\[ \left| \left< 1_{e \in M_G} \right> - \left< 1_{e \in M_H} \right> \right| \leq C e^{-cR}, \]

for any subgraph \( H \) satisfying \( B_R^c \subseteq H \subseteq G \).
The above edge correlation decay result easily yields a similar correlation decay result for vertices.

**Corollary 2.2.** Assume the same set-up as in Proposition 2.1. Then for any for \( x \in V(G) \) and any \( R \geq 1 \), there exists an event \( A \in \mathcal{F}^{V(B_R)} \times E(B_R) \) with \( P(A) \geq 1 - Ce^{-cR} \), such that on event \( A \), we have

\[
\left| \langle 1_{\{x \notin M_G\}} \rangle - \langle 1_{\{x \notin M_{B_R}^\infty\}} \rangle \right| \leq Ce^{-cR}.
\]

**Proof of Corollary 2.2.** Let \( e_1, e_2, \ldots, e_d \) be the edges incident on \( x \). By assumption, \( d \leq D \). Clearly, for any matching of \( G \) (or \( B_R^\infty \)), \( x \) does not belong to the matching if and only if exactly one of the edges \( e_1, e_2, \ldots, e_d \) belongs to that matching. Consequently,

\[
1_{\{x \notin M_G\}} - 1_{\{x \notin M_{B_R^\infty} \}} = \sum_{i=1}^{d} \left( 1_{\{e_i \in M_G\}} - 1_{\{e_i \in M_{B_R^\infty} \}} \right).
\]

For the edge \( e_i = (xx_i) \), we apply Proposition 2.1 with \( H = B_R^\infty \supseteq B_{R-1}^\infty \) to obtain that

\[
\left| \langle 1_{\{e_i \in M_G\}} \rangle - \langle 1_{\{e_i \in M_{B_R^\infty} \}} \rangle \right| \leq C_1e^{-c_1(R-1)},
\]

on some event \( A_i \) such that \( A_i \in \mathcal{F}^{V(B_R^\infty) \setminus \{x,x_i\}} \times E(B_{R-1}^\infty) \} \subseteq \mathcal{F}^{V(B_R^\infty) \setminus \{x\}} \times E(B_R^\infty) \) and \( P(A_i) \geq 1 - C_1e^{-c_1(R-1)} \). Moreover, the constants \( c_1, C_1 \) can be chosen depending only on \( D \) and \( \varphi \). Take \( A = \bigcap_{i=1}^{d} A_i \) and set \( c = c_1/2 \) and \( C = DC_1 \). Note that \( A \) remains \( \mathcal{F}^{V(B_R^\infty) \setminus \{x\}} \times E(B_R^\infty) \)-measurable. The corollary now follows from the triangle inequality. \( \square \)

The following lemma is an immediate consequence of Proposition 2.1 and Corollary 2.2.

**Lemma 2.3.** Assume the same set-up as in Proposition 2.1. Fix \( p \geq 1 \). Then for any edge \( e_i \), any vertex \( x \), and for any \( R \geq 1 \), we have

\[
E^{\sup_{\nu_x}} \left| \langle 1_{\{e_i \in M_{G}\}} \rangle - \langle 1_{\{e_i \in M_{B_R^\infty} \}} \rangle \right|^p \leq Ce^{-cR},
\]

\[
E^{\sup_{\nu_x}} \left| \langle 1_{\{x \notin M_G\}} \rangle - \langle 1_{\{x \notin M_{B_R^\infty} \}} \rangle \right|^p \leq Ce^{-cR},
\]

where \( c, C \) are constants that depend only on \( p, D \) and \( \varphi \).

The rest of the section is devoted to proving Proposition 2.1.

### 2.1 Markov random fields and path of disagreement

We first recall a coupling idea by van den Berg for a general Markov random field. Fix a finite graph \( H \) and let \( \Sigma = \{0,1\}^{V(H)} \) be the space of binary spins indexed by its vertices. The elements of \( \Sigma \) are called configurations. Let \( \lambda \) be a probability measure on \( \Sigma \) and let \( (\sigma_v)^{v \in V(H)} \) be a random element of \( \Sigma \) drawn according to \( \lambda \). The probability measure \( \lambda \) is called a Markov random field if for any \( U \subseteq V(H) \), and for any configuration \( \eta \in \{0,1\}^{U^c} \),

\[
\lambda(\sigma_v = \cdot, v \in U \ | \ \sigma_v = \eta_v, v \in U^c) = \lambda(\sigma_v = \cdot, v \in U \ | \ \sigma_v = \eta_v, v \in \partial V U),
\]

where the above equality holds if \( \sigma_v = \cdot, v \in U \ | \ \sigma_v = \eta_v, v \in U^c \) is well-defined.
provided \( \lambda(\sigma_v = \eta, v \in \partial_V U) > 0 \). In above, for \( U \subseteq V(H) \), \( \partial_V U \) denotes the outer vertex boundary of \( U \) in \( H \), that is, \( \partial_V U = \{ x \in V(H) \setminus U : (xy) \in E(H) \text{ for some } y \in U \} \).

Given \( U \subseteq V(H) \) and a boundary condition \( \eta \in \{0,1\}^{\partial_V U} \), define the conditional measure of \( \lambda \) on \( \overline{U} := U \cup \partial_V U \) by

\[
\lambda_U^\eta(\sigma_v = \cdot, v \in \overline{U}) := \lambda(\sigma = \cdot \mid \sigma = \eta \text{ on } \partial_V U),
\]

provided \( \lambda(\sigma = \eta \mid \partial_V U) > 0 \), in which case we say \( \eta \) to be admissible.

**Lemma 2.4** ([24], Lemma 11.2.1). Let \( \lambda \) be a Markov random field on \( H \). Let \( v \in U \subseteq V(H) \) and let \( \eta^1, \eta^2 \in \{0,1\}^{\partial_V U} \) be two admissible boundary conditions. Let \( \Pi \) be the product measure \( \lambda_U^\eta_1 \otimes \lambda_U^\eta_2 \) on \( \{0,1\}^{\overline{U}} \times \{0,1\}^{\overline{U}} \). Then

\[
|\lambda_U^\eta_1(\sigma_v = 1) - \lambda_U^\eta_2(\sigma_v = 1)| \leq \Pi \left( \exists \text{ a path of disagreement from } v \text{ to } \partial_V U \right).
\]

We say a pair of spin configurations \( (\alpha^1_u, \alpha^2_u)_{u \in \overline{U}} \) has a path of disagreement from \( v \) to \( \partial_V U \) if there exists a finite path of adjacent vertices \( v_0 = v, v_1, v_2, \ldots, v_\ell \) in \( \overline{U} \) such that \( v_\ell \in \partial_V U \) and \( \alpha^1_{v_i} \neq \alpha^2_{v_i} \) for all \( 1 \leq i \leq \ell \).

We are going to apply the above result to the monomer-dimer model. A matching \( M \in M_G \) can be naturally viewed as a configuration \( \alpha \in \{0,1\}^{E(G)} \) where \( \alpha_e = 1 \) if and only if the edge \( e \) belongs to \( M \).

To use the setting of Lemma 2.4, the monomer-dimer model can be recast as a probability measure on the spins indexed by vertices of the line graph of \( G \), denoted by \( \widehat{G} \), as follows.

\[
\widehat{\mu}(\alpha) \propto \prod_{e \in V(\widehat{G})} e^w_e \alpha_e \prod_{x \in V(G)} \left( 1_{\{e : x \in e \}} + e^{\nu_x} 1_{\{e \notin \partial_G x \}} \right), \quad \alpha \in \{0,1\}^{V(\widehat{G})}.
\]  

The measure \( \widehat{\mu} \) admits a clique factorization in \( \widehat{G} \), since, for each \( x \in V(G) \), the set of all edges of \( G \) incident on \( x \) forms a clique in \( \widehat{G} \). Hence, \( \widehat{\mu} \), for a fixed realization of weights, defines a Markov random field with respect to \( \widehat{G} \).

For \( F \subseteq E(G) \), by the boundary of \( F \), we mean its outer edge boundary in \( G \), which is defined as \( \partial_E F = \{ e \in E(G) \setminus F : \exists e' \in F, e \sim e' \} \), where \( e \sim e' \) means that the edges \( e \) and \( e' \) are adjacent in \( G \). Also, let \( \overline{F} = F \cup \partial_E F \). In the context of matching, a boundary condition \( \eta \in \{0,1\}^{\partial_E F} \) is admissible if \( e \in \partial_E F : \eta_e = 1 \) is a matching in \( G \). In such case, let \( M_F^\eta \) denote the random matching of \( (V(G), \overline{F}) \) drawn from the conditional Gibbs measure \( \widehat{\mu}^\eta_F \). Lemma 2.4, specialized to the monomer-dimer model, now reads as follows.

**Lemma 2.5.** Consider a disordered monomer-dimer model on a finite graph \( G \) with a fixed realization of the weights. Let \( e \in F \subseteq E(G) \) and let \( \eta^1, \eta^2 \) be two admissible boundary conditions. Suppose that the matchings \( M_F^\eta_1 \) and \( M_F^\eta_2 \) are drawn independently from \( \widehat{\mu}^\eta_F \) and \( \widehat{\mu}^\eta_F \) respectively and set \( M_1 = M_F^\eta_1 \) and \( M_2 = M_F^\eta_2 \) for brevity. Let \( \Pi \) be the distribution \( (M_1, M_2) \). Then

\[
|\{1_{\{e \in M_1\}} - (1_{\{e \in M_2\}})| \leq \Pi(\exists \text{ a path of disagreement from } e \text{ to } \partial_E F).
\]

A path of disagreement from \( e \) to \( \partial_E F \) means a finite sequence of edges \( e_0 = e \sim e_1 \sim \cdots \sim e_\ell \) such that \( e_\ell \in \partial_E F \) and \( e_i \in M_1 \oplus M_2 \), the symmetric difference of \( M_1 \) and \( M_2 \), for each \( 1 \leq i \leq \ell \).
For matching, disagreement paths have a special structure as they can never intersect. Indeed, for a pair of matchings \( M_1 \) and \( M_2 \) of \( G \), \( M_1 \oplus M_2 \) is the union of disjoint self-avoiding paths (or cycles). If not, then we can find three distinct edges in \( M_1 \oplus M_2 \) who share a common vertex. But then at least two of these edges must belong to either \( M_1 \) or \( M_2 \), which is impossible. As a result, any disagreement path is (vertex) self-avoiding and there can be at most two disagreement paths from \( e \) to \( \partial_E F \), originating at either endpoint of \( e \). Moreover, the consecutive edges on any disagreement path must belong alternatively to \( M_1 \) and \( M_2 \).

### 2.2 Proof of Proposition 2.1

Fix \( e = (xy) \in E(G), R \geq 1 \) and let \( H \) be a subgraph containing \( \mathbb{B}_e^c \). Let \( \partial H \) be the outer edge-boundary of the edge set \( E(H) \) in \( G \). That is, \( \partial H = \partial_E E(H) \).

If \( \partial \mathbb{B}_e^c = \emptyset \), then \( B_e^c = H = G \) and the proposition holds trivially as the difference between the two Gibbs expectations is identically zero. So, we assume that \( \partial \mathbb{B}_e^c \neq \emptyset \). For notational simplicity, let us write \( M^\eta_H \) for \( M^\eta_{E(H)} \), where \( \eta \in \{0,1\}^{\partial H} \) is an admissible boundary condition.

By Markov property, we have

\[
\langle 1_{e \in M_G} \rangle = \sum_\eta \langle 1_{e \in M_H^\eta} \rangle \mu_{\partial H}(\eta),
\]

where \( \mu_{\partial H} \) is the marginal of \( \mu_G \) on \( \partial H \) and the sum is over all possible admissible boundary conditions \( \eta \). It suffices to show that there exists an event \( A \in \mathcal{F}^{V(\mathbb{B}_e^c) \setminus \{x,y\}} \times \mathcal{E}(\mathbb{B}_e^c) \setminus \{e\} \), independent of the choice of \( \eta \), with \( P(A) \geq 1 - Ce^{-cR} \), such that on the event \( A \),

\[
\max_\eta |\langle 1_{e \in M_H^\eta} \rangle - \langle 1_{e \in M_H^0} \rangle| \leq Ce^{-cR},
\]

and the constants \( c \) and \( C \) can be chosen depending only on \( D \) and \( \varphi \). To achieve this, we fix an admissible boundary condition \( \eta \) on \( \partial H \). Let \( \Pi \) be the independent coupling of \( (M_H^\eta, M_H^0) \). By Lemma 2.5, we deduce that

\[
|\langle 1_{e \in M_H^\eta} \rangle - \langle 1_{e \in M_H^0} \rangle| \leq \Pi(\exists \text{ a path of disagreement w.r.t. } (M_H^\eta, M_H^0) \text{ from } e \text{ to } \partial H)
\leq \Pi(\exists \text{ a path of disagreement w.r.t. } (M_H^\eta, M_H^0) \text{ from } e \text{ to } \partial \mathbb{B}_e^c).
\]

Let \( a > 0 \) be sufficiently large. We call an edge \( f \) of \( \mathbb{B}_e^c \) to be ‘good’ if (i) \( f \) is at least distance 3 away from \( e \) and \( \partial \mathbb{B}_e^c \) and (ii) the weights on \( f \) and all the edges adjacent to \( f \) are bounded above by \( a \), and the weights on the end-vertices of these edges are bounded below by \( -a \). Let \( c_1 > 0 \) be a constant to be chosen later and let \( A \) be the event that all self-avoiding paths from \( e \) to \( \partial \mathbb{B}_e^c \) contain at least \( c_1 R \) many good edges. From the definition of good edges, it is clear that \( A \) is measurable with respect to \( \mathcal{F}^{V(\mathbb{B}_e^c) \setminus \{x,y\}} \times \mathcal{E}(\mathbb{B}_e^c) \setminus \{e\} \).

We claim that for \( a > 0 \) sufficiently large and for \( c_1 > 0 \) sufficiently small, there exist constants \( c, C > 0 \), depending only on \( D \) and \( \varphi \), such that

\[
P(A) \geq 1 - Ce^{-cR}.
\]
To prove this claim, let us fix a self-avoiding path from $e$ to $\partial B^c_R$ of length of $\ell \geq R$. By a greedy search in the subgraph $B^c_{R-3} \setminus B^c_3$, one can find a deterministic set of edges $S$ of size at least $\ell/C_1$ on that path, where $C_1 \geq 1$ depends only on $\Gamma$, such that the 2-neighborhoods of the edges in $S$ are pairwise disjoint and do not intersect with $e$ and $\partial B^c_R$. Note that the events \{f is good\}$_{f \in S}$ are independent and for any $\delta > 0$, there exists $a_0 > 0$, depending only on $\Gamma$ and $\varphi$, such that $\mathbb{P}(f \text{ is good}) \geq 1 - \delta$ for all $f \in S$ and $a \geq a_0$. From Chernoff bound, for any constant $C_3 > 0$, we can choose $\delta > 0$ sufficiently small which can be guaranteed by choosing $a$ sufficiently large, such that

$$
\mathbb{P} \left( \text{at least half of the edges in } S \text{ are good} \right) \geq 1 - C_2 e^{-C_3 |S|} \geq 1 - C_2 e^{-C_3/(C_1) \ell},
$$

for some absolute constant $C_2$. Choose $C_3$ so that (2.4) is at least $1 - C_2 D^{-3\ell}$ and then take $c_1 = 1/(2C_1)$ in the definition of the event $A$. Since there are at most $2D^\ell$ self-avoiding paths from $e$ of length $\ell$, by a union bound, we have

$$
\mathbb{P}(A^c) \leq \sum_{\ell=R}^{\infty} 2D^\ell \cdot C_2 D^{-3\ell} \leq C e^{-cR},
$$

for appropriate choices of $c$ and $C$.

Next, we claim that if $A$ holds, then there exist at least $c_1 R$ edge-disjoint cut-sets consisting of good edges which separate $e$ from $\partial B^c_R$. To argue that let us introduce the first-passage distance between any two vertices $u,v$ as the minimum number of good edges on a (self-avoiding) path between $u$ and $v$. Now for $k \geq 0$, let $B_k$ be the set of vertices with the first-passage distance at most $k$ from either of the endpoints of $e$. Clearly, $B_k \subseteq B_{k+1}$ and on the event $A$, we have $B_{c_1 R-1} \subseteq V(B^c_k)$. For each $1 \leq k \leq c_1 R$, let $C_k$ be set of the boundary edges of $B_{k-1}$, i.e., all edges of the form $(uv)$ with $u \in B_{k-1}$ and $v \notin B_{k-1}$. Obviously, $C_k$ is a cut-set since any path escaping from $B_{k-1}$ to outside must use one of the edges in $C_k$. Moreover, all edges of $C_k$ must be good. To see this, fix an edge $(uv) \in C_k$ with $u \in B_{k-1}$ and $v \notin B_{k-1}$. If this edge is not good, we can reach $v$ from $e$ using only $k-1$ good edges contradicting the fact $v \notin B_{k-1}$.

We now fix a realization of the weights such that $A$ holds and proceed to bound the probability in (2.2). As mentioned before, there can be at most two paths of disagreement from $e = (xy)$ to $\partial B^c_R$. For definiteness, let us fix one of them. Denote it by $\gamma$, where $\gamma(0) = e \sim \gamma(1) = e_1 \sim \gamma(2) = e_2 \sim \cdots$ is the enumeration of adjacent edges of $\gamma$ starting from $e$. Define the random times $\tau_0 = 0 < \tau_1 < \tau_2 < \cdots$ such that $\tau_k = \inf\{i : \gamma(i) \in C_k\} \in \mathbb{Z}_+ \cup \{\infty\}$. Clearly, to reach $\partial B^c_R$, the path $\gamma$ needs to cross all the cutsets $C_1, C_2, \ldots, C_{c_1 R}$ at least once, which implies that $\tau_k < \infty$ for each $1 \leq k \leq c_1 R$. Therefore,

$$
\Pi(\gamma \text{ reaches } \partial B^c_R) \leq \prod_{k=0}^{c_1 R-1} \Pi(\tau_{k+1} < \infty \mid \tau_k < \infty).
$$

To bound $\Pi(\tau_{k+1} < \infty \mid \tau_k < \infty)$, we condition on a finite value of $\tau_k$, the edges $\gamma(i), 0 \leq i \leq \tau_k$ such that $\gamma$ enters $C_k$ at time $\tau_k$ for the first time, and on the event whether $e \in M^\gamma_H$ or $e \in M^0_H$ (which, in turn, determines whether $\gamma(i) \in M^\gamma_H$ or $\gamma(i) \in M^0_H$ for each $i \geq 1$ by the alternating
property of the path of disagreement) and we seek to bound the conditional probability that $\gamma$ can be extended one step further after $\tau_k$.

For definiteness, suppose that $e_{\tau_k} \in M_H^0 \setminus M_H^1$. Let $z$ be the end vertex of $e_{\tau_k}$, which is not shared by $e_{\tau_k-1}$. Let $f_1 = (zz_1), \ldots, f_d = (zz_d)$ be the set of those edges incident to $z$ which do not share a common vertex with any of the edges $\gamma(i), 0 \leq i \leq \tau_k - 1$. In particular, $e_{\tau_k}$ is excluded.

If $d = 0$, then $\gamma$ cannot be extended and the conditional probability is zero. So, assume that $1 \leq d \leq D$. Since $e_{\tau_k}$ is a good edge, each $f_i$ lies inside $B_R^\varphi$. To be able to extend $\gamma$ one step further, one of the edges $f_1, f_2, \ldots, f_d$ must belong to $M_H^1$ and none of them must belong to $M_H^0$. We further condition on the information whether $f \in M_H^1$ and $f \in M_H^0$ for every edge $f \in E(H) \setminus \{f_1, f_2, \ldots, f_d\}$ that are not on $\gamma(i), 0 \leq i \leq \tau_k$ (we have already conditioned on them before). Let $Q \subseteq \{1, 2, \ldots, d\}$ be the set of the indices of the vertices among $z_1, z_2, \ldots, z_d$ which remain unmatched in $M_H^0 \setminus \{f_1, f_2, \ldots, f_d\}$ after the conditioning. Since $M_H^0$ and $M_H^1$ are independent, the conditional probability in question is bounded above by the probability that one of these edges $f_1, f_2, \ldots, f_d$ belongs to $M_H^1$ after we condition on the information whether $f \in M_H^1$ for all edges $f$ of $H$ except $f_1, f_2, \ldots, f_d$. This is given by

$$\frac{\sum_{i \in Q} \exp(w_{f_i} + \sum_{j \in Q \setminus \{i\}} \nu_{z_j})}{\exp(\nu_z + \sum_{j \in Q} \nu_{z_j}) + \sum_{i \in Q} \exp(w_{f_i} + \sum_{j \in Q \setminus \{i\}} \nu_{z_j})} = \frac{\sum_{i \in Q} \exp(w_{f_i} - \nu_{z_i} - \nu_z)}{1 + \sum_{i \in Q} \exp(w_{f_i} - \nu_{z_i} - \nu_z)}. \quad (2.5)$$

Since $e_{\tau_k}$ is good, the edge and vertex weights of $f_1, \ldots, f_d$ are all bounded above and below by $a$ and $-a$ respectively, which implies that (2.5) is bounded above by $De^{3a}(1 + De^{3a})^{-1}$. Hence,

$$\Pi(\tau_{k+1} < \infty \mid \tau_k < \infty) \leq \frac{De^{3a}}{1 + De^{3a}},$$

and consequently, on the event $A$,

$$\Pi(\gamma \text{ reaches } \partial B_R^\varphi) \leq \left(\frac{De^{3a}}{1 + De^{3a}}\right)^{c_1R} \leq Ce^{-cR},$$

for some constants $c, C > 0$ which can be chosen depending only on $D$ and $\varphi$, as promised. This completes the proof of the proposition. \hfill \Box

### 3 Proofs of the central limit theorems

In this section, we prove Theorem 1.1 and Theorem 1.2. Throughout the section, $c, c_0$ and $C, C_0, C_1, C_2$ will be positive constants that depend only on $D, K_1$, and $K_2$, unless mentioned otherwise. However, the values of $c$ and $C$ may vary from line to line.

#### 3.1 Proof of Theorem 1.1

The proof of the central limit theorem will be based on a result of Chatterjee [8], which we state below. Let $X = (X_1, X_2, \ldots, X_N)$ be a vector of independent random variables and let $g : \mathbb{R}^N \to \mathbb{R}$
be a measurable function. Suppose that \( X' = (X'_1, X'_2, \ldots, X'_N) \) is an independent copy of \( X \). For any subset \( S \subseteq \{1, 2, \ldots, N\} \), define the random vector \( X^S \) as
\[
X^S_i = \begin{cases} 
X'_i & \text{if } i \in S, \\
X_i & \text{if } i \notin S.
\end{cases}
\]

For each \( i \) and \( S \subseteq \{1, 2, \ldots, N\} \) with \( i \notin S \), set
\[
\Delta_i g = g(X) - g(X^{(i)}), \quad \Delta_i g^S = g(X^S) - g(X^{S \cup \{i\}}).
\]

Finally, let \( \sigma^2 = \text{Var}(g(X)) \).

**Theorem 3.1** (Corollary 3.2 of [7]). For \( i, j \in \{1, 2, \ldots, N\} \), let \( c(i, j) \) be a constant such that for all \( S \subseteq \{1, 2, \ldots, N\} \setminus \{i\} \) and \( T \subseteq \{1, 2, \ldots, N\} \setminus \{j\} \), one has
\[
\text{Cov}(\Delta_i g \Delta_j g^S, \Delta_j g \Delta_j g^T) \leq c(i, j).
\]

Then
\[
\sup_{s \in \mathbb{R}} \left| P \left( \frac{g(X) - E g(X)}{\sigma} \leq s \right) - \Phi(s) \right| \leq \frac{\sqrt{2}}{\sigma} \left( \sum_{i,j=1}^N c(i, j) \right)^{1/4} + \frac{1}{\sigma^{3/2}} \left( \sum_{i=1}^N E |\Delta_i g|^3 \right)^{1/2}.
\]

We apply the above theorem to the free energy \( F = \log Z \), viewed as a function of independent random variables \( (\nu_x)_{x \in V(G)} \) and \( (w_e)_{e \in E(G)} \). Let \( I = V(G) \cup E(G) \) be the union of the set of vertices and edges of \( G \), which serves as the common index set of all random weights. Clearly, \(|V(G)| \leq |I| \leq (1 + D/2)|V(G)|\). As before, let \( (\nu'_x)_{x \in V(G)} \) and \( (w'_e)_{e \in E(G)} \) denote the independent resamples of the vertex and edge weights. The discrete derivatives \( \Delta_i F \) are given as follows.

\[
\Delta_e F = \int_{w'_e}^{w_e} \langle 1_{\{e \in M_G\}} \rangle |w_e = t| \, dt, \quad \text{if } i = e \text{ is an edge}, \quad (3.1)
\]
\[
\Delta_x F = \int_{w'_x}^{w_x} \langle 1_{\{x \in \partial M_G\}} \rangle |w_x = t| \, dt, \quad \text{if } i = x \text{ is a vertex}. \quad (3.2)
\]

For \( S \subseteq I \setminus \{i\} \), we have a similar expression for \( \Delta_i F^S \), but the Gibbs expectation is now taken after the weights with indices belonging to \( S \) are resampled.

Let \( F_{[i, R]} \) be the free energy associated with the Gibbs measure \( \mu_{\mathbb{E}_R^e} \) or \( \mu_{\mathbb{E}_R^e} \) depending on whether \( i = e \), an edge or \( i = x \), a vertex. For \( i \notin S \), we approximate the discrete derivative \( \Delta_i F^S \) by the discrete derivative \( \Delta_i F^S_{[i, R]} \) of the local function \( F_{[i, R]} \). We apply (3.1) separately for \( G \) and \( \mathbb{E}_R^e \) and use Lemma 2.3 to obtain that
\[
E \left| \Delta_e F^S - \Delta_e F^S_{[e, R]} \right|^4 \leq E |w_e - w'_e|^4 E \sup_{t} \left| \langle 1_{\{e \in M_G\}} \rangle |w_e = t| - \langle 1_{\{e \in \mathbb{E}_R^e\}} \rangle |w_e = t| \right|^4
\]
\[
\leq C_0 E |w_e|^4 e^{-cR}. \quad (3.3)
\]
Similarly, we have
\[ \mathbb{E} \left| \Delta_x F^S - \Delta_x F^S_{[x,R]} \right|^4 \leq C_0 \mathbb{E} |\nu_x|^4 e^{-cR}. \] (3.4)

Note that if we assume that \( \mathbb{E}|w_e|^2, \mathbb{E}|\nu_x|^2 \leq K_1 \), then the random weights are all \( \varphi \)-bounded with \( \varphi(t) = K_1 t^{-2} \). Therefore, Lemma 2.3 ensures that the constants \( c \) and \( C_0 \) in (3.3) and (3.4) can be chosen depending only on \( D \) and \( K_1 \). Also, (3.1) and (3.2) yield the following trivial bound
\[ |\Delta_x F^S| \leq |w_e - w_e'| \quad \text{and} \quad |\Delta_x F^S| \leq |\nu_x - \nu_x'|. \]

Consequently, \( \mathbb{E}|\Delta_x F^S|^4 \leq 16(\mathbb{E}|w_e|^4 + \mathbb{E}|\nu_x|^4) \). The same argument also implies that \( \mathbb{E}|\Delta_i F^S_{[i,R]}|^4 \leq 16(\mathbb{E}|w_e|^4 + \mathbb{E}|\nu_x|^4) \).

To bound \( c(i,j) \), let us define the error terms
\[ E_{[i,R]} = \Delta_i F - \Delta_i F_{[i,R]}, \quad E^S_{[i,R]} = \Delta_i F^S - \Delta_i F^S_{[i,R]} \quad \text{for} \ i \not\in S. \]

We can then write, for \( i \not\in S \) and \( j \not\in T \),
\[
\text{Cov}(\Delta_i F \Delta_i F^S, \Delta_j F \Delta_j F^T) = \text{Cov} \left( (\Delta_i F_{[i,R]} + E_{[i,R]})(\Delta_i F^S_{[i,R]} + E^S_{[i,R]}), (\Delta_j F_{[j,R]} + E_{[j,R]})(\Delta_j F^T_{[j,R]} + E^T_{[j,R]}) \right).
\]

By the bilinearity of the covariance, the above expression can be expanded as the sum of 16 terms. The first term is given by
\[
\text{Cov}(\Delta_i F_{[i,R]} \Delta_i F^S_{[i,R]}, \Delta_j F_{[j,R]} \Delta_j F^T_{[j,R]}).
\]

The key observation is that \( F_{[i,R]} \) and \( F^S_{[i,R]} \) (and hence \( \Delta_i F_{[i,R]} \) and \( \Delta_i F^S_{[i,R]} \) as well) only depend on the weights (either original or resampled) of the vertices and edges that are distance at most \( R \) from \( i \). As a result, the above covariance vanishes if \( \text{dist}(i,j) > 2R + 1 \). Meanwhile, if \( \text{dist}(i,j) \leq 2R + 1 \), then it can be bounded above by \( 32(\mathbb{E}|w_e|^4 + \mathbb{E}|\nu_x|^4) \).

Each of the remaining 15 terms involves at least one error term and they all can be bounded by a similar approach. For example, let us consider a term of the form \( \text{Cov}(E_{[i,R]}|W, Y Z) \) where \( W, Y, Z \) are appropriate discrete derivatives. By Hölder’s inequality,
\[
|\text{Cov}(E_{[i,R]}|W, Y Z)| \leq \mathbb{E}|E_{[i,R]}|W|Y Z| + \mathbb{E}|E_{[i,R]}|W|\mathbb{E}|Y Z|
\leq (\mathbb{E}|E_{[i,R]}|^4 \mathbb{E}|W|^4 \mathbb{E}|Y|^4 \mathbb{E}|Z|^4)^{1/4} + (\mathbb{E}|E_{[i,R]}|^2 \mathbb{E}|W|^2)^{1/2} (\mathbb{E}|Y|^2 \mathbb{E}|Z|^2)^{1/2}
\leq 2(\mathbb{E}|E_{[i,R]}|^4 \mathbb{E}|W|^4 \mathbb{E}|Y|^4 \mathbb{E}|Z|^4)^{1/4}.
\]

The fourth moments of \( W, Y, \) and \( Z \) are bounded above by \( 16(\mathbb{E}|w_e|^4 + \mathbb{E}|\nu_x|^4) \). On the other hand, \( \mathbb{E}|E_{[i,R]}|^4 \leq C_0(\mathbb{E}|w_e|^4 + \mathbb{E}|\nu_x|^4)e^{-cR} \) by (3.3) and (3.4). Consequently,
\[
|\text{Cov}(E_{[i,R]}|W, Y Z)| \leq 32C_0(\mathbb{E}|w_e|^4 + \mathbb{E}|\nu_x|^4)e^{-cR}.
\]

Summing up the 16 covariance terms, we can take, as long as \( \text{dist}(i,j) > 2R + 1 \),
\[
c(i,j) \leq C_1(\mathbb{E}|w_e|^4 + \mathbb{E}|\nu_x|^4)e^{-cR}.
\]

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Next, we estimate the upper bound which is an easy consequence of Efron-Stein inequality, see \[ \text{Proof.} \]

(b) Assume that there exist positive constants \( K_1 \) and \( K_2 \) such that \( \max(\mathbb{E}|w_e|^2, \mathbb{E}|\nu_x|^2) \leq K_1 \) and \( \mathbb{E}|w_e - w'_e| \geq K_2 \) where \( w'_e \) is an i.i.d. copy of \( w_e \). Then there exists a constant \( c_0 > 0 \) depending only on \( D, K_1, \) and \( K_2 \) such that

\[
\mathbb{E}|\nu_x|^2 |V(G)|.
\]

To apply Theorem 3.1, it remains to find a suitable lower bound on the variance of \( F \). In the following lemma, we will show that \( \sigma^2 := \text{Var}(F) \geq c_0 |E(G)| \). Combining these estimates, we conclude that

\[
\frac{\sqrt{2}}{\sigma} \left( \sum_{i,j \in I} c(i,j) \right)^{1/4} + \frac{1}{\sigma^{3/2}} \left( \sum_{i \in I} \mathbb{E}|\Delta_i F|^3 \right)^{1/2}
\leq \frac{\sqrt{2}}{\sigma} \left( \sum_{i,j \in I} c(i,j) \right)^{1/4} + \frac{1}{\sigma^{3/2}} \left( \sum_{i \in I} \mathbb{E}|\Delta_i F|^3 \right)^{1/2}
\leq C_2(\mathbb{E}|w_e|^4 + \mathbb{E}|\nu_x|^4)^{1/4} \left( \frac{|V(G)|^{1/4}}{|E(G)|^{1/2}} \Psi_G(3R)^{1/4} + \frac{|V(G)|^{1/2}}{|E(G)|^{1/2}} e^{-cR} \right) + C_2(\mathbb{E}|w_e|^4 + \mathbb{E}|\nu_x|^4)^{3/8} \frac{|V(G)|^{1/2}}{|E(G)|^{3/4}}.
\]

which completes the proof of Theorem 1.1. \( \square \)

Lemma 3.2. (a) We have

\[
\text{Var}(F) \leq 2(\mathbb{E}|w_e|^2 |E(G)| + \mathbb{E}|\nu_x|^2 |V(G)|).
\]

(b) Assume that there exist positive constants \( K_1 \) and \( K_2 \) such that \( \max(\mathbb{E}|w_e|^2, \mathbb{E}|\nu_x|^2) \leq K_1 \) and \( \mathbb{E}|w_e - w'_e| \geq K_2 \) where \( w'_e \) is an i.i.d. copy of \( w_e \). Then there exists a constant \( c_0 > 0 \) depending only on \( D, K_1, \) and \( K_2 \) such that

\[
\text{Var}(F) \geq c_0 |E(G)|.
\]

Proof. The upper bound is an easy consequence of Efron-Stein inequality, see [11, Lemma 3.3]. For the lower bound, we will follow the ideas of [11, Lemma 3.6]. However, we need to be careful in keeping track of the dependence of the constant \( c_0 \) on the distribution of the weights. Let \( m = |E(G)| \). We enumerate the edges in \( E(G) \) as \( \{1, 2, \ldots, m\} \), and their respective weights are denoted as \( \{w_1, \ldots, w_m\} \). Consider the edge-revealing martingale \( (\mathbb{E}[F | \mathcal{F}_j])_{0 \leq j \leq m} \), where \( \mathcal{F}_j = \sigma(w_1, \ldots, w_j) \). The variance of \( F \) can then be written as

\[
\text{Var}(F) = \sum_{j=1}^m \mathbb{E} \left( \mathbb{E}[F | \mathcal{F}_j] - \mathbb{E}[F | \mathcal{F}_{j-1}] \right)^2.
\]
Let $F(j)$ be obtained from $F$ after replacing $w_j$ with an i.i.d. copy $w'_j$. Then the above martingale difference is equal to

$$E[F \mid F_j] - E[F \mid F_{j-1}] = E[F - F(j) \mid F_j].$$

Therefore, we write

$$\text{Var}(F) = \sum_{j=1}^{m} E\left[ E[F - F(j) \mid F_j]^2 \right] \geq \sum_{j=1}^{m} E\left[ E[F - F(j) \mid w_j]^2 \right] = \frac{1}{2} \sum_{j=1}^{m} E_{w_j, w'_j} \left[ E'(F - F(j))^2 \right], \quad (3.6)$$

where $E'$ is the conditional expectation given $w_j$ and $w'_j$. For $e = (xy)$, let $\alpha_e$ and $\beta_e$ denote the partition functions associated with the subgraphs of $G$ obtained by removing the edge $e$ and the vertices $x, y$ respectively. Clearly, $\alpha_e$ and $\beta_e$ does not depend on $w_e$. We have $Z_G = \alpha_e j + \beta_e j \exp(w_j)$, and

$$F(j) - F = \int_{w_j}^{w'_j} \frac{\beta_e e^t}{\alpha_e + \beta_e e^t} \, dt.$$

By monotonicity of $t \mapsto \beta_e e^t(\alpha_e + \beta_e e^t)^{-1}$, we obtain

$$|E'(F(j) - F)| \geq |w'_j - w_j| \cdot E'_j \cdot \frac{\beta_e e^{\min(w_j, w'_j)}}{\alpha_e + \beta_e e^{\min(w_j, w'_j)}}.$$

Let $e_j = (x_j y_j)$ and $E_j$ be the set of all edges adjacent to $e_j$. Observe that

$$\alpha_{e_j} \leq e^{\nu_{x_j} + \nu_{y_j}} \cdot \beta_{e_j} \prod_{e=(uv) \in E_j} (1 + e^{w_e - \nu_u - \nu_v}).$$

From the above inequality, we deduce that

$$|E'(F(j) - F)| \geq |w'_j - w_j| E' \left(1 + e^{\nu_{x_j} + \nu_{y_j} - \min(w_j, w'_j)} \prod_{e=(uv) \in E_j} (1 + e^{w_e - \nu_u - \nu_v})\right)^{-1} \geq |w'_j - w_j| 1_{\{\min(w_j, w'_j) \geq -a\}} \cdot E \left(1 + e^{\nu_{x_j} + \nu_{y_j} + a} \prod_{e=(uv) \in E_j} (1 + e^{w_e - \nu_u - \nu_v})\right)^{-1},$$

for any $a > 0$. Note that

$$E|w'_j - w_j|^2 1_{\{\min(w_j, w'_j) \geq -a\}} \geq \left(E|w'_j - w_j|^2 1_{\{\min(w_j, w'_j) \geq -a\}}\right)^2 \quad (3.7)$$
\[ E|w'_j - w_j|1_{\{\min(w_j,w'_j) \geq -a\}} \geq E|w'_j - w_j| - E|w'_j - w_j|1_{\{\max(|w_j|,|w'_j|) > a\}} \]

\[ \geq K_2 - (E|w'_j - w_j|^2 \mathbb{P}(\max(|w_j|,|w'_j|) > a)^1/2 \]

\[ \geq K_2 - 4K_1(2K_1/a^2)^1/2 \geq K_2/2, \]

if \( a = a_0 \) is chosen sufficiently large depending on \( K_1 \) and \( K_2 \). On the other hand, the random variable \( \left( 1 + e^{|w_j|+a_0} \prod_{e=\{(uv)\in E\}} (1 + e^{|w_e|-V(\nu_e)}) \right)^{-1} \) is independent of \( w_j \) and \( w'_j \) and its expectation can be bounded below by a positive constant \( c_1 \) that depends only on \( D, a_0 \), and \( K_1 \), but not on \( j \) or \( m \). So finally, by (3.7),

\[ E_{w_j,w'_j} \left[ |E'(F - F^{(j)})|^2 \right] \geq c_1^2 E|w'_j - w_j|^2 1_{\{\min(w_j,w'_j) \geq -a_0\}} \geq c_1^2 (K_2/2)^2. \]

The desired lower bound on \( \text{Var}(F) \) now follows from (3.6). \( \square \)

### 3.2 Proof of Theorem 1.2

Without loss of generality, assume that \( |V(G_n)| = n \). By hypothesis, we have \( |E(G_n)| \geq \delta n \). Let \( L = n^\kappa \) for some small positive constant \( \kappa > 0 \) (\( \kappa = 1/10 \) suffices). For \( t \in [0,1] \), define the weights

\[ w'_e = w_e 1_{\{|w_e| \leq L\}} + tw_e 1_{\{|w_e| > L\}}, \]

\[ \nu'_e = \nu_e 1_{\{|\nu_e| \leq L\}} + t\nu_e 1_{\{|\nu_e| > L\}}, \]

which interpolate between the weights truncated at \( L \) at \( t = 0 \) and the original weights at \( t = 1 \). Let \( F_{n,t} \) be the free energy of the disordered monomer-dimer model on \( G_n \) with the weights \((w'_e)_{e \in E(G_n)}, (\nu'_e)_{e \in V(G_n)}\) and write \( \langle \cdot \rangle_{n,t} \) for the corresponding Gibbs expectation. For a random variable \( W \), we write \( \overline{W} = W - EW \) for its centered version.

It follows from Theorem 1.1 that

\[ \frac{F_{n,0}}{\sqrt{\text{Var}(F_{n,0})}} \xrightarrow{d} N(0,1). \]

Indeed, by assumption, \( E|w_e^0|^2 \) and \( E|\nu_e^0|^2 \) are bounded above by \( K_1 := \max(E|w_e|^2, E|\nu_e|^2) < \infty \). For large \( n \),

\[ E|w_e^0 - (w_e^0)'| \geq K_2 := \frac{1}{2} E|w_e - w'_e| > 0. \]

Moreover, we have the trivial bound \( E|w_e^0|^4, E|\nu_e^0|^4 \leq n^{4\kappa} \). Therefore, Theorem 1.1 yields that

\[ \sup_{s \in \mathbb{R}} \left| \mathbb{P} \left( \frac{F_{n,0}}{\sqrt{\text{Var}(F_{n,0})}} \leq s \right) - \Phi(s) \right| \leq C n^{3\kappa/2} \left( n^{-1/4} \Psi_{G_n}(3R)^{1/4} + e^{-cR} + n^{-1/4} \right) \]

for any \( R \geq 1 \), where \( c \) and \( C \) depend on \( \delta, K_1, K_2 \), and \( D \) only. For \( R = 2c^{-1} \log n \), the uniformly sub-exponential volume growth implies that there exists \( K \) such that

\[ \Psi_{G_n}(3R) \leq K \exp \left( (1/10) \log n \right). \]
After plugging in, the RHS of (3.9) becomes
\[ Cn^{3n/2} \left( n^{-1/4} \exp((1/40) \log n) + e^{-2\log n} + n^{-1/4} \right) \to 0 \quad \text{as } n \to \infty, \]
and (3.8) follows. From Lemma 3.2, the variance of the free energy is of order \( n \), namely,
\[ cn \leq \text{Var}(F_{n,0}) \leq Cn, \quad cn \leq \text{Var}(F_{n,1}) \leq Cn. \tag{3.10} \]
Our main claim is as follows.
\[ n^{-1} \mathbb{E}[F_{n,0} - F_{n,1}]^2 \to 0. \tag{3.11} \]
Once the claim is established, it follows from the variance upper bounds in (3.10) and from the Cauchy-Schwarz inequality that
\[ n^{-1} |\text{Var}(F_{n,0}) - \text{Var}(F_{n,1})| \leq n^{-1} \left( \text{Var}(F_{n,0}) + \text{Var}(F_{n,1}) \right)^{1/2} \left( \mathbb{E}|F_{n,0} - F_{n,1}|^2 \right)^{1/2} \leq C \left( n^{-1} \mathbb{E}|F_{n,0} - F_{n,1}|^2 \right)^{1/2} \to 0. \]
Coupled with the lower bound on the variance provided in (3.10), the above implies that
\[ \frac{\text{Var}(F_{n,0})}{\text{Var}(F_{n,1})} \to 1. \tag{3.12} \]
Finally, we write
\[ \frac{F_{n,1}}{\sqrt{\text{Var}(F_{n,1})}} = \frac{F_{n,0}}{\sqrt{\text{Var}(F_{n,0})}} \cdot \frac{\sqrt{\text{Var}(F_{n,0})}}{\text{Var}(F_{n,1})} \cdot \frac{1}{\sqrt{n}} \cdot \frac{\text{Var}(F_{n,1})}{n}, \]
and conclude that
\[ \frac{F_{n,1}}{\sqrt{\text{Var}(F_{n,1})}} \xrightarrow{d} N(0, 1) \]
from (3.8), (3.12), (3.11), and (3.10).

It remains to show the claim (3.11). Note that
\[ F_{n,1} - F_{n,0} = \int_0^1 \frac{d}{dt} F_{n,t} \, dt = \int_0^1 \sum_{e \in E(G_n)} \langle 1_{\{e \in M_{G_n}\}} \rangle_{n,t} w_e 1_{\{|w_e| > L\}} \, dt + \int_0^1 \sum_{x \in V(G_n)} \langle 1_{\{x \notin M_{G_n}\}} \rangle_{n,t} \nu_x 1_{\{|\nu_x| > L\}} \, dt. \]
For notational simplicity, write
\[ X_{n,t}^e = \sum_{e \in E(G_n)} \langle 1_{\{e \in M_{G_n}\}} \rangle_{n,t} w_e 1_{\{|w_e| > L\}}, \quad X_{n,t}^x = \sum_{x \in V(G_n)} \langle 1_{\{x \notin M_{G_n}\}} \rangle_{n,t} \nu_x 1_{\{|\nu_x| > L\}}. \]
By Jensen,
\[ \mathbb{E}|F_{n,0} - F_{n,1}|^2 \leq 2 \int_0^1 \mathbb{E}|X_{n,t}^e|^2 \, dt + 2 \int_0^1 \mathbb{E}|X_{n,t}^x|^2 \, dt. \]
To tackle the first integral above, let \( B_{n,R} \) be the ball of radius \( R \) around \( e \) in \( G_n \). Define
\[
\gamma_{n,t}(e, R) = \langle 1_{\{e \in M \cap B_{n,R}\}} \rangle_{n,t}, \quad \text{and} \quad Y_{n,t}^e = \sum_{e \in E(G_n)} \gamma_{n,t}(e, R) w_e 1_{\{|w_e| > L\}}.
\]

By Lemma 2.3, there exist constants \( C, c > 0 \) (depending only on \( D \) and \( K_1 \)) such that for all \( t \in [0, 1] \),
\[
E \sup_{w_e} |\langle 1_{\{e \in M \cap G_n\}} \rangle_{n,t} - \gamma_{n,t}(e, R)|^2 \leq Ce^{-cR} = Cn^{-2},
\]
which implies that
\[
E \left[ (\langle 1_{\{e \in M \cap G_n\}} \rangle_{n,t} - \gamma_{n,t}(e, R) w_e)^2 \right] \leq C \sup_{w_e} |\langle 1_{\{e \in M \cap G_n\}} \rangle_{n,t} - \gamma_{n,t}(e, R)|^2 \cdot E|w_e|^2 \leq Cn^{-2}.
\]

Therefore, by Cauchy-Schwarz,
\[
\sup_{t \in [0, 1]} E|X_{n,t}^e - Y_{n,t}^e|^2 \leq C. \tag{3.13}
\]

Note that
\[
E|Y_{n,t}^e|^2 = \sum_{e, e' \in E(G_n)} \text{Cov}(\gamma_{n,t}(e, R) w_e 1_{\{|w_e| > L\}}, \gamma_{n,t}(e', R) w_{e'} 1_{\{|w_{e'}| > L\}}).
\]

If the distance between \( e \) and \( e' \) is greater than \( 2R + 1 \), \( \gamma_{n,t}(e, R) \) and \( \gamma_{n,t}(e', R) \) are independent and the covariance vanishes. Else, by Cauchy-Schwarz and Hölder’s inequalities, the absolute value of the covariance is bounded above by
\[
E|w_e 1_{\{|w_e| > L\}}|^2 \leq \frac{E|w_e|^{2+\epsilon}}{L^\epsilon} \leq Cn^{-\kappa \epsilon}.
\]

Due to uniform subexponential volume growth assumption on \( G_n \), for any \( \alpha > 0 \), the number of pairs \( (e, e') \) with distances at most \( 2R + 1 \) is bounded above by \( Cne^{\alpha \log n} \) for \( C \) sufficiently large. Consequently, after choosing \( \alpha = \kappa \epsilon / 2 \), we have
\[
\sup_{t \in [0, 1]} n^{-1} E|Y_{n,t}^e|^2 \leq n^{-1} \cdot Cne^{\alpha \log n} \cdot Cn^{-\kappa \epsilon} \to 0.
\]

This, together with (3.13), implies that
\[
\sup_{t \in [0, 1]} n^{-1} E|X_{n,t}^e|^2 \to 0. \tag{3.14}
\]

An exact similar argument shows that
\[
\sup_{t \in [0, 1]} n^{-1} E|X_{n,t}^x|^2 \to 0. \tag{3.15}
\]

The claim (3.11) now follows from (3.14) and (3.15) and the proof of the theorem is now complete. \(\square\)
4 Open questions

We conclude the paper by listing a few open problems.

- Is it possible to remove the subexponential growth assumption on the underlying graphs and replace the finite \((2 + \epsilon)\)-moment assumption with just the finite second-moment assumption in Theorem 1.2?

- Consider the disordered monomer-dimer model with \(\beta = \infty\), the so-called “zero-temperature” case. The free energy then reduces to the weight of the maximum weight matching on \(G\), which is unique almost surely if the weights are assumed to be continuous. It is natural to ask whether this model exhibits a decay of correlation on any bounded degree graphs, say, on the finite boxes in \(\mathbb{Z}^d\)? The case when \(G\) is a sparse Erdős-Rényi graph or a random regular graph was studied in Gamarnik et. al. [12], and a correlation decay was shown for exponentially distributed edge weights. Relying on this correlation decay result, Cao [4] was able to prove the central limit theorem for the weight of the maximum weight matching in sparse Erdős-Rényi graph with exponential edge weights. Beyond that, any correlation decay result is unavailable for bounded degree graphs including non-regular trees, and the central limit theorem for the weight of maximum matching remains an open problem.

- Another related interesting model to look at would be the disordered pure dimer model on a finite box on \(\mathbb{Z}^d\) with an even side-length. The configuration space here is the set of all perfect matchings (no monomers). Does this model have the correlation decay at any temperature and at any dimension? What about the central limit theorem for the free energy? It might be worthwhile to mention that for \(d = 2\) or more generally for finite planar graphs, the weighted dimer model is known to be exactly solvable due to Kasteleyn [16, 17] and independently Temperley and Fisher [23]. Indeed, its partition function has an explicit formula in terms of Pfaffian (or determinant in case of bipartite graphs) of the Kasteleyn matrix. The formulas for correlation functions of the edges are also available due to Kenyon [18]. For \(d = 2\), the dimer model with random weights (under some ellipticity assumption on the weights) falls under the general framework considered by Berestycki, Laslier, and Ray [3] (see also [20]), where they showed that the fluctuation of the height function of a random dimer configuration converges to a Gaussian free field for a wide class of planar graphs. Their work implies a correlation decay for the dimer model.

Acknowledgement

We thank Gourab Ray for bringing the references [3, 20] to our attention.

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