Remarks on perturbation theory for Hamiltonian systems

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Abstract
A comparative discussion of the normal form and action angle variable method is presented in a tutorial way. Normal forms are introduced by Lie series which avoid mixed variable canonical transformations. The main interest is focused on establishing a third integral of motion for the transformed Hamiltonian truncated at finite order of the perturbation parameter. In particular, for the case of the action angle variable scheme, the proper canonical transformations are worked out which reveal the third integral in consistency with the normal form. Details are discussed exemplarily for the Hénon-Heiles Hamiltonian. The main conclusions are generalized to the case of $n$ perturbed harmonic oscillators.

1 Introduction

The following contribution is concerned with finite perturbation series characterized by bounded remainders in properly chosen compact domains of phase space. We are, however, not interested here in estimating the rest terms. As has been known since Poincaré, infinite perturbation series for Hamiltonians, in general, do not converge in compact domains. Or more precisely, if convergence takes place according to the KAM theorem, then it generally occurs in an invariant subset of phase space whose complement is open and dense [1].

Let us assume that the Hamiltonian can be brought into the form $H = h + \epsilon V$ where $h$ refers to $n$ uncoupled harmonic oscillators and $\epsilon$ is the smallness parameter. Then the main differences of a perturbative treatment by normal forms and mixed variable generating functions, respectively, can be characterized as follows. In the latter method one uses action angle variables $I \in \mathbb{R}^n$, $\phi \in \mathbb{T}^n$ and tries to find a canonical transformation $(I, \phi) \rightarrow (J, \psi)$ which makes the transformed Hamiltonian independent of $\psi$ in a certain domain $D$ of phase space. For an elementary introduction into this method, including the main ideas of the proof of the KAM theorem, the textbook [2] is recommended. In the normal form case, on the other hand, one adopts complex canonical variables $(u_\nu, v_\nu)$, and tries to make $H$ canonically equivalent to $n$ harmonic oscillators given by

$$h = -i \sum_{\nu=1}^n \omega_\nu u_\nu v_\nu.$$  (1)

Both strategies fail, because terms with resonance denominators occur to any order, in general, which cannot be transformed away. Thus, even if one accepts a finite cutoff at order $\epsilon^N$, it is not possible, in general, to transform a Hamiltonian into an integrable form. There seems to be at least one advantage with normal forms: they straightforwardly provide us with a third integral. As a consequence, in the case of two degrees of freedom, for instance, the cutoff part of the Hamiltonian which is normalized up to order $N$, is integrable within the definition domain of the normal form transformation. In the case of the Hénon-Heiles Hamiltonian [3] this was first demonstrated by [4]. As a further advantage, the normal form transformation can be carried out very efficiently by Lie series and thus by symbolic computer algebra [5].

As a power series in $\epsilon$, both perturbation schemes should be equivalent. However, it does not seem to be obvious, how the third integral can be detected in the action angle variable picture. We will adopt an iterated transformation scheme where new canonical variables are introduced at each perturba-
tion order. Eventually, an elementary linear canonical transformation will make coming forth the additional integral. For demonstration, the Hénon-Heiles Hamiltonian is considered. The results are generalized to the case of $n$ perturbed harmonic oscillators.

The existence of a third integral up to order $\epsilon^N$ may be useful in Nekhoroshev-like estimates [2,3]. For instance, in the three body problem an approximate integral, in addition to energy and angular momentum, should help to get sharper bounds for the remainders. We make use of this occasion to remark, that a former study by the present author [3] on the three body problem in celestial mechanics, essentially, was a failure because of explicitly and tacitly (eq.(77)) adopted adiabatic assumptions; the rigorous estimates in the Appendices C and D of [3], on the other hand, may be helpful elsewhere for similar problems. The N-body problem of celestial mechanics was recently examined more rigorously [9], on the other hand, may be helpful elsewhere for rigorous estimates in the Appendices C and D of [9].

In particular, inner points of $D_0$ are mapped into inner points of $D_t$, and the boundaries of $D_t$ and $D_0$ are equally smooth. Moreover, by the Liouville theorem, the domain volumes are preserved. Chaotic behaviour, clearly, develops for such systems, if at all, as an asymptotic property. Its observability depends on the adopted degree of resolution.

Lie series are tied to the time evolution of a function $f(p,q)$ along a trajectory with given initial point, and write therefore $f(p,q) = F(t; p_0, q_0)$ with $F(0; p_0, q_0) = f(p_0, q_0)$. The Taylor expansion in the time interval $t \in (0, \Delta t)$ reads

$$F(t) = \sum_{k=0}^{N} \frac{t^k}{k!} F^{(k)}(t = 0) + O((\Delta t)^{N+1}).$$

Defining the linear operator $L_H$ by the Poisson bracket

$$L_H f := \sum_{k=1}^{n} \left( \frac{\partial H}{\partial p_k} \frac{\partial f}{\partial q_k} - \frac{\partial H}{\partial q_k} \frac{\partial f}{\partial p_k} \right)$$

and making use of the canonical equations, we can write

$$\frac{df}{dt} = \sum_{k=1}^{n} \left( \frac{\partial f}{\partial p_k} \dot{p}_k + \frac{\partial f}{\partial q_k} \dot{q}_k \right) = \sum_{k=1}^{n} \left( \frac{\partial f}{\partial p_k} \left( - \frac{\partial H}{\partial q_k} \right) + \frac{\partial f}{\partial q_k} \frac{\partial H}{\partial p_k} \right) = L_H f. \quad (4)$$

As a consequence, the Taylor expansion can be expressed in terms of the following Lie series

$$F(t) = \sum_{k=0}^{N} \frac{t^k}{k!} (L_H f(p,q))_{(p,q)=(p_0,q_0)} + O((\Delta t)^{N+1}), \quad (5)$$

and in the limit $N \rightarrow \infty$ we can write in compact form

$$F(t) = \exp[t \, L_H] f(p,q)_{(p,q)=(p_0,q_0)}. \quad (6)$$

For small enough time steps $\Delta t$, the time evolution of any dynamical variable, in particular $f \equiv p$ or $f \equiv q$, can be approximated by iterated truncated Lie series. A cutoff at $N = 4$ corresponds to a
fourth order Runge-Kutta integration. If $H$ and $f$ are given analytically, the coefficients of the Lie series can also be determined analytically, e.g. by means of symbolic computer calculators. However, in each step symplecticity is fulfilled only up to an error of order $|\Delta t|^{N+1}$. In numerical problems, it may be more adequate to adopt so-called symplectic integrators which are canonical in every step within the number precision of the computer, see e.g. [11]. The remainder of the truncated Lie series, on the other hand, can be rigorously expressed through (if $f$ is scalar)

$$O(|\Delta t|^{N+1}) = \frac{F^{N+1}(t^*)}{(N+1)!} |\Delta t|^{N+1} \text{ with } t^* \in (0, \Delta t),$$  

or

$$O(|\Delta t|^{N+1}) = \frac{f^{N+1}(p^*, q^*)}{(N+1)!} |\Delta t|^{N+1}$$  

with $(p^*, q^*) = (p(t^*), q(t^*))$. From a priori or a posteriori knowledge on the domain of $(p, q)$, the remainder can be estimated by upper bounds.

### 3 Normal form and third integral

For demonstration, let us consider the Hénon-Heiles Hamiltonian

$$H = h + \epsilon V \quad \text{with} \quad h = \frac{1}{2} (p_1^2 + q_1^2 + p_2^2 + q_2^2),$$

$$V = q_1 q_2^2 - \frac{1}{3} q_2^3$$

where $(p_\nu, q_\nu) \in \mathbb{R}^2$ with $\nu = 1, 2$ are canonical variables. We assume that the variables have been made dimensionless, in particular $H=1$, which implies that the smallness parameter $\epsilon$ is proportional to the square root $\sqrt{E}$ of the energy of a given trajectory. With the aid of the linear canonical transformation $(p_\nu, q_\nu) \rightarrow (u_\nu, v_\nu)$ where

$$u_\nu = (q_\nu - ip_\nu)/\sqrt{2}; \quad v_\nu = i (q_\nu + ip_\nu)/\sqrt{2},$$

we obtain

$$h = -i (u_1 v_1 + u_2 v_2);$$

$$V = \frac{u_2 - i v_2}{2\sqrt{2}} \left[ (u_1 - i v_1)^2 - \frac{1}{3} (u_2 - i v_2)^2 \right],$$

which has the suitable form for being subject to a normal form transformation.

In section 2, from the time evolution $(p_0, q_0) \rightarrow (p(t), q(t))$, we had derived as generator of a canonical transformation the operator $\exp[t L_H]$. Clearly, any function $H(p, q)$ which does not explicitly depend on time, gives rise to such a generator. Moreover, the variable $t$ in $\exp[t L_H]$ does not need to be identified as time, it can be any real parameter. This is seen e.g., when symplecticity is inferred from the Poisson bracket $L_{p,q}$ calculated with respect to $(p_0, q_0)$. It is therefore legitimate to adopt $\epsilon$ as parameter of the generating function $\tilde{H}$. The point of view adopted here is to generate, at any given time $t$, a canonical transformation $(p(t; \epsilon = 0), q(t; \epsilon = 0)) \rightarrow (p(t; \epsilon), q(t; \epsilon))$, which is parametrized with respect to the interaction parameter $\epsilon$. The transformed Hamiltonian is written as a power series

$$H(p(t; \epsilon), q(t; \epsilon)) = h(p(t; 0), q(t; 0)) + \sum_{k=1}^{\infty} H_k(p(t; 0), q(t; 0)) \epsilon^k$$

with $h$ being the unperturbed Hamiltonian. Clearly, such canonical transformations can be achieved by means of arbitrary scalar functions. In the following it is convenient to write the generating function in the form

$$\chi'(p(t; \epsilon), q(t; \epsilon)) = \frac{d}{d\epsilon} \chi(p(t; \epsilon), q(t; \epsilon))$$

$$= \frac{\partial \chi}{\partial p} \frac{dp}{d\epsilon} + \frac{\partial \chi}{\partial q} \frac{dq}{d\epsilon}$$

where we assume

$$\frac{dp}{d\epsilon} = -\tilde{h}_p; \quad \frac{dq}{d\epsilon} = \tilde{h}_q$$

for some scalar function $\tilde{h}(p, q)$ which we do not need to specify.

The canonical transformation of an arbitrary scalar function $g$ is defined by the constituent equation (we omit writing the time parameter)

$$\frac{d}{d\epsilon} g(p(\epsilon), q(\epsilon)) = L_{\chi'} g(p(\epsilon), q(\epsilon)),$$
which gives rise to the power series representation

\[
g(p(\epsilon), q(\epsilon)) = \exp[\epsilon L_{\chi_0}] g(p(0), q(0))
\]  
with \(\chi' = \chi(p(0), q(0))\).

When \(\chi\), too, is expanded in a power series

\[
\chi(p(\epsilon), q(\epsilon)) = \sum_{k=1}^{\infty} \epsilon^k \chi_k(p(0), q(0)),
\]  
then the arbitrary functions \(\chi_k\) will be at our disposition to simplify the Hamiltonian coefficients \(H_k\). Henceforth we will write simply \((p, q)\) for the phase space variables \((p(\epsilon = 0), q(\epsilon = 0))\). As is shown in Appendix A, the transformed terms, \(H_k\), can be determined recursively as follows:

\[
H_0 = h; \quad L_h \chi_1 + H_1 = V_0;
\]

\[
L_h \chi_k + H_k = \frac{1}{k} V_{k-1} + \sum_{j=1}^{k-1} \frac{j}{k} L_{\chi_j} H_{k-j},
\]

\[k = 2, 3, ..., \] where \(V_j\) is defined through the power series of the transformed potential, namely \(\exp[\epsilon L_{\chi}] V = \sum_{j=0}^{\infty} V_j \epsilon^j\).

Let us start with the term \(k = 1\). Then we have, with \(V_0 \equiv V\),

\[
L_h \chi_1 + H_1 = V_1
\]
and we try to set \(H_1 = 0\) with the implication that \(\chi_1\) has to fulfil the relation \(L_h \chi_1 = V\). To discuss, whether \(V\) is in the range of the homology operator \(L_h\), we adopt the canonical variables (10) together with the representation (1) of \(h\). Furthermore, we exploit the fact that \(V\) is a linear combination of monomials of the form \(U^m := u_1^{m_1} v_1^{m_2} u_2^{m_2} v_2^{m_4}\) with \(|m| := m_1 + m_2 + m_3 + m_4 = 3\) and \(m_j \in \mathbb{N}_0\) for \(j = 1, 2, 3, 4\). Now, each monomial is an eigenfunction of \(L_h\), because

\[
L_h U^m = \sum_{\nu=1,2} \left( \frac{\partial h}{\partial u_\nu} \frac{\partial U^m}{\partial v_\nu} - \frac{\partial h}{\partial v_\nu} \frac{\partial U^m}{\partial u_\nu} \right)
= i(m_1 - m_2 + m_3 - m_4) U^m.
\]

As a consequence, the set of resonance monomials defined by

\[
\{ U^m \mid m_1 + m_3 = m_2 + m_4 = 0; \ m_i \in \mathbb{N}_0 \}
\]
are not in the range of \(L_h\), and therefore cannot be removed by the generating function \(\chi_1\). Clearly, the resonance case is possible for monomials of even order \(|m|\) only. Since \(V\), according to (11), consists of third order terms, eq.(20) is solvable for \(\chi_1\) with \(H_1\) set equal to zero. The general solution includes an arbitrary part of the kernel of \(L_h\) consisting of resonance monomials. If, as usual, this kernel part is set equal to zero, \(\chi_1\) is uniquely given by a linear combination of the monomials occurring in \(V\). Furthermore, \(V_1 = L_{\chi_1} V\) is now determined in terms of 4-th order monomials.

We examine the next iteration, which will be sufficient to reveal the general structure of the normalized Hamiltonian:

\[
L_h \chi_2 + H_2 = \frac{1}{2} V_1 + \frac{1}{2} L_{\chi_1} H_1 = \frac{1}{2} V_1.
\]

Here, \(V_1\) contains both types of monomials, nonresonant ones which are in the range of the operator \(L_h\) and resonant monomials. The latter must be compensated by \(H_2\), while the nonresonant terms are transformed away by the proper choice of \(\chi_2\). This is typical of all orders. Thus, an optimal simplification is achieved when the generating function is disposed of in such a way that the transformed Hamiltonian terms \(H_k\) contain resonant monomials only.

When this normalization is carried out up to order \(N\), then the truncated Hamiltonian

\[
\hat{H}^{(N)} := \sum_{k=0}^{N} \epsilon^k H_k
\]

is a constant of motion up to a rest term of the order \(\epsilon^{N+1}\). Moreover, by the definition (22) of the resonance monomials and because of (21), we have the property

\[
L_h \hat{H}^{(N)} = 0
\]
which tells that \(h\) is in involution with \(\hat{H}^{(N)}\) and therefore a further constant of motion. As should be remarked, the remainder \(R_{N+1}\) in general is finite within properly chosen domains of phase space. [12]

Let \(E\) and \(\hbar\) be the integral constants of a given trajectory. Going back to the original variables and choosing as Poincaré surface of section the plane
$q_2 = 0$, one eliminates the variable $p_2$ from the energy integral through

$$p_2 = p_2(p_1, q_1, q_2 = 0; E),$$  \(26\)

and inserts $p_2$ into the third integral

$$h = h(p_1, q_1, p_2(p_1, q_1, E)).$$  \(27\)

The latter equation implicitly defines one-dimensional manifolds $M(p_1, q_1; E, h) = 0$ which for constant energy $E$ and different values $h$ were first plotted in reference [4]. The manifolds turned out as closed curves corresponding to the intersection of 2-tori with the Poincaré plane and thus demonstrating the integrability of the approximated Hénon-Heiles Hamiltonian. For small enough energies $E \equiv \epsilon^2$, as is well known, this picture is confirmed by numerical integration of the model. A compact symmetrized form of the normalized Hénon-Heiles Hamiltonian can be found in [3]

4 Mixed variable generating function

In terms of action angle variables $(I_\nu, \phi_\nu) \leftrightarrow (p_\nu, q_\nu)$ defined by

$$p_\nu = \sqrt{2I_\nu} \cos(\phi_\nu); \quad q_\nu = \sqrt{2I_\nu} \sin(\phi_\nu),$$  \(28\)

$\nu = 1, 2$, the Hénon-Heiles Hamiltonian reads

$$H = h + \epsilon V; \quad h = I_1 + I_2,$$

$$V = V(I_1, I_2, \phi_1, \phi_2).$$  \(29\)

In order to reveal a third integral in the truncated part of the perturbatively transformed Hamiltonian, we stepwise introduce generating functions as follows

$$F^{(n)}(J_1, J_2, \phi_1, \phi_2) = J_1 \phi_1 + J_2 \phi_2 + \epsilon^n S^{(n)}(J_1, J_2, \phi_1, \phi_2),$$  \(30\)

$n = 1, 2,...$, which implicitly define new canonical torus variables $(I_1, I_2, \phi_1, \phi_2) \to (J_1, J_2, \psi_1, \psi_2)$ through the relations

$$I_\nu = J_\nu + \epsilon^n \frac{\partial S^{(n)}}{\partial \phi_\nu}; \quad \psi_\nu = \phi_\nu + \epsilon^n \frac{\partial S^{(n)}}{\partial J_\nu}.$$  \(31\)

In the first step, one substitutes the old action variables in terms of the new ones as usual to obtain

$$H^{(1)} = J_1 + J_2 + \epsilon \left[ \frac{\partial S^{(1)}}{\partial \phi_1} + \frac{\partial S^{(1)}}{\partial \phi_2} \right] +$$

$$\epsilon V(J_1 + \epsilon \frac{\partial S^{(1)}}{\partial \phi_1}, J_2 + \epsilon \frac{\partial S^{(1)}}{\partial \phi_2}, \phi_1, \phi_2).$$  \(32\)

Now we try to remove the potential term $V$ to first order in $\epsilon$ by choosing the Fourier components of $S = \sum_{n_1, n_2} S_{n_1 n_2}(J_1, J_2) \exp(i n_1 \phi_1 + i n_2 \phi_2)$ as follows

$$S_n^{(1)}(J_1, J_2) = \frac{V_{n_1 n_2}(J_1, J_2)}{n_1 \omega_1 + n_2 \omega_2}.$$  \(33\)

Here, with the unperturbed oscillator frequencies $\omega_1 = \omega_2 = 1$, this is possible, because resonance components of $V$ with $n_1 + n_2 = 0$ do not exist. With this, the transformed Hamiltonian reads

$$H^{(1)} = J_1 + J_2 + \epsilon^2 V^{(2)}(J_1, J_2, \psi_1, \psi_2; \epsilon)$$  \(34\)

where, due to the elimination of the old angle variables $\phi_\nu$ in terms of $\psi_\nu$, the potential $V^{(2)}$ now is an infinite power series in $\epsilon$.

Proceeding to second order, with the canonical transformation $(J_1, J_2, \psi_1, \psi_2) \to (\tilde{J}_1, \tilde{J}_2, \tilde{\psi}_1, \tilde{\psi}_2)$ defined by $F^{(2)}(\tilde{J}_1, \tilde{J}_2, \tilde{\psi}_1, \tilde{\psi}_2)$, we obtain the transformed Hamiltonian (omitting the tilde, for simplicity)

$$H^{(2)} = J_1 + J_2 + \epsilon^2 h_2(J_1, J_2) +$$

$$\epsilon^2 R^{(2)}(J_1, J_2, \psi_1 - \psi_2) +$$

$$\epsilon^3 V^{(3)}(J_1, J_2, \psi_1, \psi_2; \epsilon)$$  \(35\)

where

$$h_2(J_1, J_2) = \frac{1}{(2\pi)^2}$$

$$\int_0^{2\pi} d\phi_1 \int_0^{2\pi} d\phi_2 V^{(2)}(J_1, J_2, \phi_1, \phi_2; \epsilon = 0).$$  \(36\)

It is important to realize that the angle dependence of the resonance term $R^{(2)}$ is special and given through the difference $\psi_1 - \psi_2$, because it contains only Fourier components with $n_1 + n_2 = 0$. 

5
As a consequence, if we truncate at second order in $\epsilon$, we can apply the linear canonical transformation $J_1, J_2, \psi_1, \psi_2 \rightarrow J'_1, J'_2, \psi'_1, \psi'_2$ with
\begin{equation}
J'_1 := J_1 + J_2; \quad J'_2 := J_2; \quad \psi'_1 := \psi_1; \quad \psi'_2 := \psi_2 - \psi_1
\end{equation}
to arrive at an effectively one-dimensional Hamiltonian with $J'_1$ being a constant of motion. With respect to the remaining degree of freedom, $(J'_2, \psi'_2)$, it is standard to achieve the integrable form, see e.g. [14]. The corresponding canonical transformation is given in Appendix B.

In order to see that this property continues to higher orders, it will be sufficient to go one perturbative step further. With the aid of the generating function $F^{(3)}(\bar{J}_1, \bar{J}_2, \psi_1, \psi_2)$ we obtain in terms of mixed variables
\begin{align}
H^{(3)} &= \bar{J}_1 + \bar{J}_2 + \\
&\quad \epsilon^2 \left[ h_2(\bar{J}_1, \bar{J}_2) + O(\epsilon^3) \right] + \\
&\quad \epsilon^2 \left[ R^{(2)}(\bar{J}_1, \bar{J}_2, \psi_1 - \psi_2) + O(\epsilon^3) \right] + \\
&\quad \epsilon^3 \left[ \frac{\partial S^{(3)}}{\partial \psi_1} + \frac{\partial S^{(3)}}{\partial \psi_2} + \\
&\quad V^{(3)}(\bar{J}_1, \bar{J}_2, \psi_1, \psi_2; \epsilon = 0) + \\
&\quad O(\epsilon) \right].
\end{align}
In the last bracket the remainder of order $\epsilon$ stems from the expansion of $V^{(3)}(\bullet; \epsilon)$ as a power series in $\epsilon$. The decisive point is that, by the chosen $\epsilon$-dependence of the generating functions (30), the frequencies remain unrenormalized. This is also the case in the normal form method. As a consequence, we have the same resonance condition $n_1 + n_2 = 0$. Taking into account that $\psi_\nu = \bar{\psi}_\nu + O(\epsilon^3)$, we obtain the third order transformed Hamiltonian (once more omitting the tilde) in the form
\begin{align}
H^{(3)} &= J_1 + J_2 + \epsilon^2 h_2(J_1, J_2) + \epsilon^3 h_3(J_1, J_2) + \\
&\quad \epsilon^2 R^{(2)}(J_1, J_2, \psi_1 - \psi_2) + \\
&\quad \epsilon^3 R^{(3)}(J_1, J_2, \psi_1 - \psi_2) + \\
&\quad \epsilon^4 V^{(4)}(J_1, J_2, \psi_1, \psi_2; \epsilon)
\end{align}
with the resonance terms $R^{(2)}, R^{(3)}$ depending on the angle difference as claimed. This property, obviously, carries to the higher orders, and thus leads to an integrable truncated Hénon-Heiles Hamiltonian in agreement with the normal form.

In every perturbation step one has to keep track of the definition domain of the new action variables. For instance, if the original variable $I_1$ is defined in the positive interval $[0, d_1]$, then by (31) $J_1 + \epsilon^3 \frac{\partial S^{(n)}}{\partial \psi_1}$ is confined to the same domain. As a consequence, we have to restrict $J_1$ to $J_1 \in [0, d_1 - \delta^*]$ where
\begin{equation}
\delta^* = \epsilon^n \max_{\psi_1 \in [0,2\pi]} \frac{\partial S^{(n)}}{\partial \psi_1} \quad \text{if} \quad \delta^* \leq d_1; \quad (40)
\end{equation}
If $\delta^* > d_1$, then the transformation is ill defined.

5 Generalization

The above reasoning can be immediately extended to the case of $n$ perturbed harmonic oscillators. We first discuss the normal form method. With the generalized multi-index notation $U^m := u_1^{m_1} u_1^{m_2} \cdots u_1^{m_n} u_1^{m'}$, the eigenvalue relation (21) becomes
\begin{align}
L_h U^m &\equiv \sum_{\nu=1}^{n} \left( \frac{\partial h}{\partial u_\nu} \frac{\partial U^m}{\partial v_\nu} - \frac{\partial h}{\partial v_\nu} \frac{\partial U^m}{\partial u_\nu} \right) \\
&= i U^m \sum_{\nu=1}^{n} \omega_\nu (m_\nu - m'_\nu), \quad (41)
\end{align}
which gives rise to the resonance monomials
\begin{equation}
\{ U^m \mid \sum_{\nu=1}^{n} \omega_\nu (m_\nu - m'_\nu) = 0; \quad m_\nu \in \mathbb{N}_0 \}. \quad (42)
\end{equation}
As is remarked, even if the frequencies are all rationally independent, there are possible resonances with $m_\nu = m'_\nu$ for $\nu = 1, 2, \ldots, n$. Since, by the normal form method, the truncated normalized Hamiltonian $H^{(N)}$ consists of resonance monomials only, eq.(41) implies the commutation of the Poisson bracket, namely $L_h H^{(N)} = 0$, and thus establishes a third integral $h$ in addition to the energy $H^{(N)}$.

In the action angle variable picture the Hamiltonian (1) reads
\begin{equation}
h = \omega_1 I_1 + \omega_2 I_2 + \ldots \omega_n I_n. \quad (43)
\end{equation}
Now, after the elementary canonical transformation
\[ I_\nu = J_\nu / \omega_\nu; \quad \phi_\nu = \psi_\nu / \omega_\nu \quad \nu = 1, 2, ... n, \quad (44) \]
we obtain \( h = J_1 + J_2 + ... J_n \), and the Fourier representations
\[ S^{(k)} = \sum_{(\mu_1, \mu_2, ..., \mu_n) \in \mathbb{Z}^n} S^{(k)}_{\mu_1, ..., \mu_n} (J_1, ..., J_n) \times \exp(i \mu_1 \omega_1 \psi_1 + ... + i \mu_n \omega_n \psi_n); \quad \psi_\nu \in [0, 2\pi / \omega_\nu). \]
\[ \text{(45)} \]
From the resonance conditions \( \mu_1 + \mu_2 + ... + \mu_n = 0 \), we may eliminate e.g. \( \mu_1 = -\mu_2 - ... - \mu_n \) in the phases of the resonance terms of the transformed Hamiltonian with the result that these terms depend on the following \( n-1 \) differences only
\[ \omega_2 \psi_2 - \omega_1 \psi_1, \quad \omega_3 \psi_3 - \omega_1 \psi_1, \quad ... \omega_n \psi_n - \omega_1 \psi_1. \]
\[ \text{(46)} \]
Now, after the elementary canonical transformation \((J_\nu, \psi_\nu) \rightarrow (J'_\nu, \psi'_\nu)\) with
\[ (J'_1, \psi'_1) = \left( \sum_{\nu=1}^{n} \frac{\omega_1}{\omega_\nu} J_\nu, \psi_1 \right) \]
\[ (J'_\nu, \psi'_\nu) = (J_\nu, \psi_\nu - \psi_1 \omega_1 / \omega_\nu), \quad (47) \]
\( \nu = 2, 3, ... n \), the resonance terms do not depend on the new angle variable \( \psi'_1 \). Therefore \( J'_1 \) is a constant of motion of the truncated transformed Hamiltonian in consistency with the normal form method.

As a final remark, the Hamiltonian of the three-body problem in celestial mechanics (and straightforwardly also the \( N \)-body case) can be expressed in terms of suitable action angle variables which avoid (chart dependent) singularities at small inclinations and eccentricities, see e.g. [1]. It would be interesting to find out, whether a third integral can be worked out in a finite order perturbation procedure. This may be helpful in estimating upper bounds over finite time intervals of the order of the age of the planetary system.

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**Appendix A: Recursive Lie series**

We prove here the recursion relations (18) and (19) in a different way as compared with reference [13]. First we show that the coefficients \( g_k \) of an arbitrary function \( g(p,q) = \sum_k g_k \epsilon^k \) can be determined by the following recursive system [1]

\[ g_0 = g(p,q); \quad g_{-n} = 0 \quad \text{for} \quad n = 1, 2, ...; \]
\[ g_n = \sum_{j=1}^{n} \frac{j}{n} L_{X_j} g_{n-j} \quad \text{for} \quad n = 1, 2, ... \quad (48) \]

where the expansion coefficients \( g_k \) and \( \chi_k \) have to be taken at the point \((p,q) := (p(\epsilon = 0), q(\epsilon = 0))\). To show that (15) follows from the recursion system, we multiply with \( \epsilon^n \) and sum over \( n \)

\[ \sum_{n=0}^{\infty} \epsilon^n g_n = g_0 + \sum_{n=1}^{\infty} \epsilon^n \sum_{j=1}^{n} j L_{X_j} g_{n-j}. \quad (49) \]

On the left hand side we have \( g \). Differentiating with respect to \( \epsilon \), transforming the double sum on the right hand side and making use of the fact that \( L_\chi \) is linear in \( \chi \), we obtain

\[ \frac{dg}{d\epsilon} = \sum_{n=1}^{\infty} \epsilon^{n-1} \sum_{j=1}^{n} j L_{X_j} g_{n-j} \]
\[ = \sum_{n=1}^{\infty} \epsilon^{n-1} \sum_{j=1}^{n} j L_{X_j} g_{n-j} \]
\[ = \sum_{m=0}^{\infty} \sum_{j=1}^{m} \epsilon^{m+j-1} j L_{X_j} g_m \]
\[ = \sum_{j=1}^{\infty} \epsilon^{j-1} j L_{X_j} \sum_{m=0}^{\infty} \epsilon^m g_m \]
\[ = \sum_{j=1}^{\infty} \epsilon^{j-1} j L_{X_j} g = L_{\chi' \epsilon} g \equiv L_{\chi'} g, \quad (50) \]

which is (15) as was claimed.
For the final step we transform the Hamiltonian as follows
\[ \exp[\epsilon L_{\chi'}] H = \sum_{k=0,1,\ldots} H_k \epsilon^k, \] (51)
and on the other hand
\[ \exp[\epsilon L_{\chi'}] H \equiv \exp[\epsilon L_{\chi'}](h + \epsilon V) = \sum_{k=0,1,\ldots} h_k \epsilon^k + \sum_{k=1,2,\ldots} V_{k-1} \epsilon^k. \] (52)
Comparing coefficients we obtain
\[ H_0 = h_0 = h; \quad H_k = h_k + V_{k-1} \quad \text{for} \quad k = 1, 2, \ldots \] (53)
Making use of the recursion formulas (48), we can write
\[ H_k = \sum_{j=1}^{k} \frac{j}{k} L_{\chi_j} h_{k-j} + \sum_{j=1}^{k-1} \frac{j}{k-1} L_{\chi_j} V_{k-1-j} \] (54)
for \( k = 2, 3, \ldots \) taking out the summand \( j = k \) from the first sum, and combining the remaining sums, we find
\[ H_k = L_{\chi_k} h + \sum_{j=1}^{k-1} \frac{j}{k} L_{\chi_j} \left( h_{k-j} + \frac{k}{k-1} V_{k-1-j} \right) \]
\[ = L_{\chi_k} h + \sum_{j=1}^{k-1} \frac{j}{k} L_{\chi_j} \left( h_{k-j} + \frac{1}{k-1} V_{k-1-j} \right) \]
\[ = L_{\chi_k} h + \frac{1}{k} V_{k-1} + \sum_{j=1}^{k-1} \frac{j}{k} L_{\chi_j} \left( h_{k-j} + V_{k-1-j} \right) \] (55)
where the second term of the last equation is a consequence of the recursive system for the coefficients \( V_j \). The term in the last bracket is just \( H_{k-1} \). In view of the commutator property
\[ [L_{\chi_k}, h] = -L h \chi_k \] (56)
we arrive at the desired recursion system (19)
\[ L h \chi_k + H_k = \frac{1}{k} V_{k-1} + \sum_{j=1}^{k-1} \frac{j}{k} L_{\chi_j} \chi_{k-j}; \quad k = 2, 3, \ldots \] (57)

**Appendix B: Integrable second order form of the Hénon-Heiles Hamiltonian**

We start from the transformed Hamiltonian (35), neglect the remainder \( V^{(3)} \), and write at first the resulting Hamiltonian \( H^{(2)}_{\text{trunc}} \) in terms of action angle variables as defined in (37). We will abbreviate the constant of motion \( J'_1 \) by \( J \). When the action angle variables \( (J'_2, \psi'_2) \) are expressed by (28) in terms of cartesian symplectic magnitudes \( (p, q) \), we can write after some efforts
\[ H^{(2)}_{\text{trunc}} = J + \epsilon^2 \left[ -\frac{5}{96} J^2 + \frac{7}{48} J q^2 - \frac{7}{96} q^2 (p^2 + q^2) \right] \] (58)
This is one-degree of freedom Hamiltonian which can be brought into integrable form in a standard way, see e.g. [13]. The corresponding, exact canonical transformation from \( (p, q) \) to action angle variable \( (I, \Phi) \) is found as
\[ p = -2 \cos(\Phi) \sqrt{\frac{(J-I)I}{J-2 \sin(\Phi) \sqrt{I(J-I)}}} \]
\[ q = \frac{2I-J}{\sqrt{J-2 \sin(\Phi) \sqrt{I(J-I)}}}; \quad J \geq I. \] (59)
With this we achieve the integrable form
\[ H^{(2)}_{\text{trunc}} = J + \epsilon^2 \left( 14 J^2 - 14 I J + J^2 \right) \] (60)
in terms of the action variables \( J \) and \( I \).

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