THE ARITHMETIC EXTENSIONS OF A NUMERICAL SEMIGROUP

I. OJEDA AND J.C. ROSALES

Abstract. In this paper we introduce the notion of extension of a numerical semigroup. We provide a characterization of the numerical semigroups whose extensions are all arithmetic and we give an algorithm for the computation of the whole set of arithmetic extension of a given numerical semigroup. As by-product, new explicit formulas for the Frobenius number and the genus of proportionally modular semigroups are obtained.

Introduction

Let \( \mathbb{N} \) be the set of non-negative integers. A subset \( M \) of \( \mathbb{N} \) is called a submonoid of \( (\mathbb{N},+) \) if \( M \) contains \( 0 \in \mathbb{N} \) and \( M \) is closed under the sum in \( \mathbb{N} \). A numerical semigroup is a submonoid \( S \) of \( (\mathbb{N},+) \) such that \( \mathbb{N} \setminus S \) is finite.

If \( A \) is a non-empty subset of \( \mathbb{N} \), then we will write \( \langle A \rangle \) for the submonoid of \( (\mathbb{N},+) \) generated by \( A \), that is,

\[
\langle A \rangle = \left\{ \sum_{i=1}^{n} u_{i}a_{i} \mid n \in \mathbb{N} \setminus \{0\}, \{a_{1}, \ldots, a_{n}\} \subseteq A \text{ and } \{u_{1}, \ldots, u_{n}\} \subseteq \mathbb{N} \right\}.
\]

It is a well-known fact that \( \langle A \rangle \) is a numerical semigroup if and only if \( \gcd(A) = 1 \) (see, for instance, [13, Lemma 2.1]).

If \( M \) is a submonoid of \( (\mathbb{N},+) \) such that \( M = \langle A \rangle \) for some \( A \subseteq \mathbb{N} \), then we will say that \( A \) is system of generators of \( M \). Furthermore, if \( M \neq \langle B \rangle \) for every \( B \subsetneq A \), then we will say that \( A \) is a minimal system of generators of \( M \). In [13] Corollary 2.8, it is proved that every submonoid of \( (\mathbb{N},+) \) has a unique minimal system of generators, which in addition is finite. We will write \( \text{msg}(M) \) for the minimal system of generators of \( M \).

If \( S \) is a numerical semigroup, the elements in \( \mathbb{N} \setminus S \) are called gaps of \( S \). The cardinality of \( \mathbb{N} \setminus S \) is the called genus of \( S \) and is denoted \( g(S) \). The smallest integer in \( S \setminus \{0\} \) is called the multiplicity of \( S \) and is

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denoted \( m(S) \). The greatest integer that does not belong to \( S \) is called the **Frobenius number** of \( S \) and is denoted \( F(S) \).

Let \( S \) and \( T \) be numerical semigroups. We say that \( T \) is an **extension** of \( S \) if \( S \subseteq T \) and we say that \( T \) is an **arithmetic extension** of \( S \) if there exist positive integer numbers \( d_1, d_2, \ldots, d_n \) such that

\[
T = \{ x \in \mathbb{N} \mid d_i x \in S, \ i = 1, \ldots, n \}.
\]

Not all the extensions of a numerical semigroup is arithmetic. In Theorem 8, we characterize the numerical semigroups whose extensions are all arithmetic.

If \( S \) is a numerical semigroup and \( d \) is a positive integer, then

\[
\frac{S}{d} := \{ x \in \mathbb{N} \mid d x \in S \}
\]

is a numerical semigroup, too (see [13, Proposition 5.1]). The numerical semigroup \( \frac{S}{d} \) is called the **quotient of \( S \)** by \( d \). The quotients of numerical semigroups by an element have been studied by different authors; see, for instance, [11, 14] and more recently [2].

Notice that \( \frac{S}{d} = \mathbb{N} \) if and only if \( d \in S \), and observe that \( \frac{S}{d} \) is an arithmetic extension of \( S \), and that an extension \( T \) of \( S \) is arithmetic if and only if there exists \( \{d_1, \ldots, d_n\} \subseteq \mathbb{N} \setminus S \) such that

\[
T = \frac{S}{d_1} \cap \frac{S}{d_2} \cap \ldots \cap \frac{S}{d_n}.
\]

Indeed, by definition, \( x \in T \) if and only if there exist positive integers \( d_1, \ldots, d_n \) such that \( d_i x \in S \), for all \( i \in \{1, \ldots, n\} \); equivalently, \( x \in \frac{S}{d_i} \) for every \( i \in \{1, \ldots, n\} \).

Besides a theoretical interest, a major motivation for the study of arithmetic extensions of numerical semigroups relies in the following two examples.

In [2] it is shown that quotients of Arf numerical semigroups (see [10]) by a positive integer are Arf numerical semigroups. Since, the intersection of finitely many Arf numerical semigroups is an Arf numerical semigroup (see, for instance, [13, Proposition 3.22]), we conclude that arithmetic extensions are stable for the Arf property. Arf numerical semigroups are a relevant family of numerical semigroups as they are semigroup of values of Arf rings (see [6]).

In [9] the notion of proportionally modular numerical semigroup (PM-semigroup, for short) was introduced. In [3] the authors characterize the numerical semigroups that can be written as an intersection of finitely many PM-semigroups, these semigroups are called SPM-semigroups. On the other hand, in [15] Toms introduced a family of numerical semigroups, that we call T-semigrups, which are \( K_0 \)-groups of certain \( C^* \)-algebras (see [15, Theorem 1.1]). In [12] it is shown that the T-semigroups are precisely the SPM-semigroups. As a consequence of [7, Corollary 4.1] we have that a numerical semigroup is SPM-semigroup if and only if the exist positive integer
numbers $a, d_1, \ldots, d_n$ such that
\[
S = \frac{\langle a, a + 1 \rangle}{d_1} \cap \ldots \cap \frac{\langle a, a + 1 \rangle}{d_n}.
\]

Thus, we conclude that the T-semigroups are the arithmetic extensions of the numerical semigroups generated by two consecutive integers.

One of the main results in this paper is Theorem 14 which determines the Apéry set of the quotient of a numerical semigroup by a positive integer. As a consequence, explicit formulas for the genus and the Frobenius number of PM-semigroups are obtained.

To conclude the introduction we outline how this note is organized: in Section 1 we study the first properties of the set of arithmetic extensions of a numerical semigroup and we characterize the numerical semigroups whose extensions are all arithmetic. In Section 2 we see that if $S$ is a numerical semigroup and $n$ is an element in $S$ different from zero, then the Apéry set of $\frac{S}{n}$ with respect $n$ can be easily computed in terms of the Apéry set of $S$ with respect $n$. This fact will provide us nice explicit formulas for the Frobenius number and the genus of the numerical semigroup $\frac{\langle a, a + 1 \rangle}{d}$ (see Remark 19) and consequently of any T-semigroup by Proposition 16. Finally, in the last section, we take advantage of the results in the previous sections to formulate an algorithm for the computation of all the arithmetic extensions of a numerical semigroup. We finish the paper by giving a simple GAP implementation of our algorithm; this implementation requires the GAP package NumericalSgps [4].

1. The set of arithmetic extensions of a numerical semigroup

Let $\Delta$ be a numerical semigroup and let $d_1, d_2, \ldots, d_n$ be positive integer numbers. We write $\Delta(\{d_1, d_2, \ldots, d_n\}) = \{x \in \mathbb{N} \mid d_ix \in \Delta, \ i = 1, \ldots, n\}$. Clearly,

\[
\Delta(\{d_1, d_2, \ldots, d_n\}) = \bigcap_{i=1}^{n} \frac{\Delta}{d_i}
\]

is an arithmetic extension of $\Delta$. By convention, we assume that $\Delta(\emptyset) = \mathbb{N}$.

In the following, we will denote by $\mathcal{F}(\Delta)$ the set of arithmetic extensions of $\Delta$. The proof of the following result is straightforward.

**Proposition 1.** If $\Delta$ is a numerical semigroup, then
\[
\mathcal{F}(\Delta) = \{\Delta(X) \mid X \subseteq \mathbb{N} \setminus \Delta\}.
\]

**Theorem 2.** If $\Delta$ is a numerical semigroup, then
\[
\mathcal{F}(\Delta) = \{\Delta(X) \mid X \subseteq \mathbb{N} \setminus \Delta \text{ and } X = \text{msg}(\langle X \rangle)\}.
\]

**Proof.** Let $\mathcal{A} = \{\Delta(X) \mid X \subseteq \mathbb{N} \setminus \Delta \text{ and } X = \text{msg}(\langle X \rangle)\}$. By Proposition 14, $\mathcal{A} \subseteq \mathcal{F}(\Delta)$. Let us prove the opposite inclusion. To this end, we consider $S \in \mathcal{F}(\Delta)$. By definition, we have that there exists $X = \{x_1, x_2, \ldots, x_n\} \subseteq \mathbb{N} \setminus \Delta$
such that $S = \Delta(X) = \frac{\Delta}{x_1} \cap \frac{\Delta}{x_2} \cap \ldots \cap \frac{\Delta}{x_n}$. Let $Y = \{y_1, y_2, \ldots, y_m\} = \text{msg}(X)$. Since $Y \subseteq X$, we have that

$$\Delta(X) \subseteq \Delta(Y).$$

On the other hand, if $y \in \Delta(Y)$, then $y y_j \in \Delta$, for every $j = 1, \ldots, m$ and $y x_i \in \Delta$, because $x_i \in \{y_1, y_2, \ldots, y_m\}$, for every $i = 1, \ldots, n$. Thus $y \in \Delta(X)$. Therefore $\Delta(X) = \Delta(Y) \in \mathcal{A}$. \hfill $\Box$

In general, not all the extensions of numerical semigroups are arithmetic, as we will see later on. One of the aims of this section is to characterize the numerical semigroups $\Delta$ such that $\mathcal{F}(\Delta)$ agrees with the set of extension of $\Delta$.

If $S$ is a numerical semigroup. An element $x \in \mathbb{N} \setminus S$ is said to be a fundamental gap of $S$ if $\{2x, 3x\} \subset S$ (see Section 5 of Chapter 3 in [13] or [8]). The set of fundamental gaps of $S$ is denoted by $\text{FG}(S)$. Observe that $F(S) \in \text{FG}(S)$, provided that $S \neq \mathbb{N}$.

The proof of the following result is trivial.

**Lemma 3.** Let $S$ be a numerical semigroup and let $x$ be a gap of $S$. Then $x \in \text{FG}(S)$ if and only if $\{k x \mid k \in \mathbb{N} \setminus \{1\}\} \subseteq S$.

The proof of the next result is straightforward, too.

**Lemma 4.** If $S \neq \mathbb{N}$ is a numerical semigroup, then

(a) $\frac{S}{d} = \mathbb{N}$ if and only if $d \in S$.

(b) $\frac{S}{d} = \langle 2, 3 \rangle$ if and only if $d \in \text{FG}(S)$. In particular, $\frac{S}{F(S)} = \langle 2, 3 \rangle$,

(c) $\frac{S}{d} \cap \frac{S}{d} = S \cup \text{FG}(S)$,

(d) if $d \in \mathbb{N} \setminus \{0, 1\}$, then $S \cup \text{FG}(S) \subseteq \frac{S}{d}$.

Notice that if $S$ is a numerical semigroup, then $\mathbb{N} \setminus \text{F}(S)$ and $S \cup \text{FG}(S)$ are (non-necessarily different) arithmetic extension of $S$. Furthermore, the first and the third ones are proper extensions of $S$ and the second one is proper if and only if $S \neq \langle 2, 3 \rangle$.

**Example 5.**

1. The only proper extension of $\langle 2, 3 \rangle$ is $\mathbb{N}$.

2. The proper extensions of $\langle 3, 4, 5 \rangle$ are $\mathbb{N}$ and $\langle 2, 3 \rangle = \langle 3, 4, 5 \rangle$.

3. The proper extensions of $\langle 2, 5 \rangle$ are $\mathbb{N}$ and $\langle 2, 3 \rangle = \langle 2, 5 \rangle$.

4. The proper extensions of $\langle 3, 5, 7 \rangle$ are $\mathbb{N}$, $\langle 2, 3 \rangle = \langle 3, 5, 7 \rangle$ and $\langle 3, 4, 5 \rangle = \langle 3, 5, 7 \rangle$.

5. The only proper extensions of $\langle 4, 5, 7 \rangle$ are $\mathbb{N}$, $\langle 2, 3 \rangle = \langle 4, 5, 7 \rangle$, $\langle 3, 4, 5 \rangle = \langle 4, 5, 7 \rangle$ and $\langle 2, 5 \rangle = \langle 4, 5, 7 \rangle$.

Notice that we have shown that all the extensions of the following numerical semigroups: $\langle 2, 3 \rangle$, $\langle 3, 4, 5 \rangle$, $\langle 2, 5 \rangle$, $\langle 3, 5, 7 \rangle$ and $\langle 4, 5, 7 \rangle$ are arithmetic.

Notice that the set $\mathcal{F}(\Delta)$ is partially ordered by inclusion. Moreover, we have the following:
Proposition 6. If $\Delta \neq \mathbb{N}$ is a numerical semigroup and $\mathcal{F}(\Delta)$ is the set of arithmetic extensions of $\Delta$, then

(a) $\max(\mathcal{F}(\Delta)) = \mathbb{N}$,
(b) $\min(\mathcal{F}(\Delta)) = \Delta$,
(c) $\max(\mathcal{F}(\Delta) \setminus \{\mathbb{N}\}) = \langle 2, 3 \rangle$,
(d) $\min(\mathcal{F}(\Delta) \setminus \{\Delta\}) = \Delta \cup \text{FG}(\Delta)$.

Proof. (a) Notice that that $\frac{\Delta}{d} = \mathbb{N}$ for every $d \in \Delta$ and that all numerical semigroups are submonoids of $\mathbb{N}$.

(b) If $\Delta \neq \mathbb{N}$, then $1 \notin \Delta$ and $\frac{\Delta}{d} = \Delta$. Now, it suffices to observe that $\Delta \subseteq \frac{\Delta}{d}$, for every $d \in \mathbb{N} \setminus \{0, 1\}$, to conclude that every arithmetic extension of $\Delta$ contains $\Delta$.

(c) By part (b) of Lemma 4 we have that $\langle 2, 3 \rangle \in \mathcal{F}(\Delta)$. Now, it suffices to observe that if $S \neq \mathbb{N}$ is a numerical semigroup, then $1 \notin S$ and consequently $S \subseteq \langle 2, 3 \rangle$.

(d) By part (c) of Lemma 4 and the remark before Proposition 6 we have that $\Delta \cup \text{FG}(\Delta) \subseteq \mathcal{F}(\Delta) \setminus \{\Delta\}$. If $S \in \mathcal{F}(\Delta) \setminus \{\Delta\}$, then there exists $\{d_1, \ldots, d_n\} \subseteq (\mathbb{N} \setminus S) \setminus \{1\}$ such that $S = \frac{\Delta}{d_1} \cap \ldots \cap \frac{\Delta}{d_n}$. By part (d) of Lemma 4 we have that $\Delta \cup \text{FG}(\Delta) \subseteq \frac{\Delta}{d_i}$ for every $i = 1, \ldots, n$. So, $\Delta \cup \text{FG}(\Delta) \subseteq S$.

The next example shows (among other things) that there exist extensions of a numerical semigroup that are not arithmetic.

Example 7. Let $S = \langle 5, 7, 9 \rangle$. By direct checking, one can see that $\text{FG}(S) = \{6, 8, 11, 13\}$. Clearly, $S \cup \{13\}$ is an extension $S$ and $S \subseteq S \cup \{13\} \subseteq S \cup \text{FG}(S)$. Now, by part (d) of Proposition 6 we have the $S \cup \{13\}$ cannot be an arithmetic extension of $S$.

Now, we can take advantage of Proposition 6 to characterize the numerical semigroups whose extensions are all arithmetic.

Theorem 8. The only numerical semigroups whose extensions are all arithmetic are $\mathbb{N}$, $\langle 2, 3 \rangle$, $\langle 3, 4, 5 \rangle$, $\langle 2, 5 \rangle$, $\langle 3, 5, 7 \rangle$, and $\langle 4, 5, 7 \rangle$.

Before proving the previous theorem, we need some lemmas.

Lemma 9. If $S$ is a numerical semigroup and the cardinal of $\text{FG}(S)$ is greater than 1, then $S$ has non-arithmetic extensions.

Proof. Since $S \cup \{\text{F}(S)\}$ is a proper extension of $S$ and $\{\text{F}(S)\} \subseteq \text{FG}(S)$, it follows that $S \cup \{\text{F}(S)\} \subseteq S \cup \text{FG}(S)$. Thus, by part (d) of Proposition 6 $S \cup \{\text{F}(S)\}$ cannot be an arithmetic extension of $S$.

Lemma 10. If $S$ is a numerical semigroup and $\text{FG}(S) = \{\text{F}(S)\}$, then the set gaps of $S$ is $\{d \in \mathbb{N} \mid d \text{ divides } \text{F}(S)\}$.

Proof. Let $d$ be a gap of $S$, that is to say, $d \in \mathbb{N} \setminus S$. Since $\mathbb{N} \setminus S$ is finite, there exists $n \in \mathbb{N} \setminus \{0\}$ such that $nd \in \mathbb{N} \setminus S$, and $\{2(nd), 3(nd)\} \subset S$. So,
\(nd\) is a fundamental gap. Then \(nd \in FG(S) = \{F(S)\}\) by hypothesis, and we conclude that \(d\) divides \(F(S)\). Conversely, if \(d\) divides \(F(S)\), say \(F(S) = nd\), then \(d \notin S\); otherwise, \(F(S) = nd \in S\) which is impossible. \(\square\)

**Lemma 11.** Let \(F\) be a positive integer number. Then 

\[
\mathbb{N} \setminus \{d \in \mathbb{N} \mid d \text{ divides } F\}
\]

is a numerical semigroup if and only if \(F \in \{1, 2, 3, 4, 6\}\).

**Proof.** Since \(\{d \in \mathbb{N} \mid d \text{ divides } F\}\) is a finite set, we have that the set \(\mathbb{N} \setminus \{d \in \mathbb{N} \mid d \text{ divides } F\}\) is a numerical semigroup if and only if given two positive integers \(x\) and \(y\) not dividing \(F\), then their sum does not divides \(F\) either. By [8, Lemma 2], it is easy to prove this is only possible if, and only if, \(F \in \{1, 2, 3, 4, 6\}\). \(\square\)

**Proof of Theorem 8.** Let \(S \neq \mathbb{N}\) be a numerical semigroup whose extensions are all arithmetic. By Lemma 9, \(FG(S) = \{F(S)\}\). Then, by Lemma 10,

\[
S = \mathbb{N} \setminus \{d \in \mathbb{N} \mid d \text{ divides } F(S)\}.
\]

Now, since \(S\) is a numerical semigroup, by Lemma 11 we have that \(F(S) \in \{1, 2, 3, 4, 6\}\). Therefore \(S\) is equal to one of the following numerical semigroups \(\mathbb{N} \setminus \{1\} = \langle 2, 3 \rangle\), \(\mathbb{N} \setminus \{1, 2\} = \langle 3, 4, 5 \rangle\), \(\mathbb{N} \setminus \{1, 3\} = \langle 2, 5 \rangle\), \(\mathbb{N} \setminus \{1, 2, 4\} = \langle 3, 5, 7 \rangle\) or \(\mathbb{N} \setminus \{1, 2, 3, 6\} = \langle 4, 5, 7 \rangle\).

Finally, since all the extensions of the five above numerical semigroups are arithmetic, as it was shown in Example 5 we are done. \(\square\)

2. The Apéry Set of the Quotient of a Numerical Semigroup

If \(S\) is a numerical semigroup, then the **Apéry set of \(S\) with respect to \(n \in S \setminus \{0\}\)** is \(\text{Ap}(S, n) := \{x \in S \mid x - n \notin S\}\). It is known (see, for instance, [B13, Lemma 2.4]) that

\[
\text{Ap}(S, n) = \{0 = \omega(0), \omega(1), \ldots, \omega(w - 1)\}
\]

where \(\omega(i)\) is the least element in \(S\) such that \(\omega(i) \equiv i \mod n\), for each \(i = 0, \ldots, n - 1\). In particular, we have that the cardinality of \(\text{Ap}(S, n)\) is \(n\).

**Remark 12.** Notice that given \(x \in \mathbb{N}\) we have that \(x \in S\) if and only if there exists \((k, \omega) \in \mathbb{N} \times \text{Ap}(S, n)\) such that \(x = kn + \omega\). Therefore, \((\text{Ap}(S, n) \setminus \{0\}) \cup \{n\}\) is system of generators of \(S\).

The Apéry sets of \(S\) provides formulas for the computation of \(F(S)\) and \(g(S)\):

**Proposition 13.** If \(S\) is a numerical semigroup and \(n \in S \setminus \{0\}\), then

(a) \(F(S) = \max \text{Ap}(S, n) - n\).

(b) \(g(S) = \frac{1}{n} \sum_{\omega \in \text{Ap}(S, n)} \omega - \frac{n - 1}{2}\).

**Proof.** For a proof, see for instance [B13, Proposition 2.12]. \(\square\)
As usual $\lfloor r \rfloor$ and $\lceil r \rceil$ stand for the integer part and the upper integer part of the rational number $r$, respectively. If $a$ and $b$ are integer numbers, then we write $(a \mod b)$ for the remainder of the Euclidean of $a$ by $b$, in symbols, $(a \mod b) = a - \lfloor \frac{a}{b} \rfloor b$.

**Theorem 14.** Let $S$ be a numerical semigroup and $n \in S \setminus \{0\}$. If $\text{Ap}(S, n) = \{0 = \omega(0), \omega(1), \ldots, \omega(n-1)\}$ and $a \in \mathbb{N} \setminus S$, then

$$\text{Ap} \left( \frac{S}{a}, n \right) = \{0, n \kappa_1 + 1, n \kappa_2 + 2, \ldots, n \kappa_{n-1} + n - 1\},$$

where $\kappa_i = \lceil \omega \left( \frac{ai \mod n}{a} \right) - ai \rceil$, $i = 1, \ldots, n - 1$.

**Proof.** Let $i \in \{1, \ldots, n - 1\}$ and $k \in \mathbb{N}$. Then $kn + i \in \frac{S}{a}$ if and only if $a(kn + i) \in S$. Thus, by Remark 12, we have that $a(kn + ai) \in S$ if and only if $a(kn + ai) \geq \omega(ai \mod n)$. Therefore, $kn + i \in \frac{S}{a}$ if and only if

$$k \geq \frac{\omega(ai \mod n) - ai}{an}.$$ 

Hence, the smallest element in $\frac{S}{a}$ that is congruent with $i$ modulo $n$ is $\left\lceil \frac{\omega(ai \mod n) - ai}{an} \right\rceil n + i$. Now, by (2), we are done. $\square$

Given a numerical semigroup $S$, we can use the above result to compute the Apéry set of the quotient of $S$ by $b$, and thereafter use Proposition 13, to compute the Frobenius number and the genus of $\frac{S}{b}$. Let us illustrate these computations through an example.

**Example 15.** Let $S = \langle 7, 8 \rangle$. Then $\text{Ap}(S, 7) = \{0 = \omega(0), \omega(1) = 8, \omega(2) = 16, \omega(3) = 24, \omega(4) = 32, \omega(5) = 40, \omega(6) = 48\}$. If $\overline{S} = \frac{S}{3}$, then, by Theorem 14, we have that

$$\text{Ap} \left( \frac{\overline{S}}{7}, 7 \right) = \{0 \} \cup \{7 \kappa_i + i \mid i = 1, \ldots, 6\},$$

where $\kappa_i = \lceil \frac{\omega \left( \frac{3i \mod 7}{3} \right) - 3i}{21} \rceil$, $i = 1, \ldots, 6$. In this case, $\kappa_1 = 1, \kappa_2 = 2, \kappa_3 = 1, \kappa_4 = 2, \kappa_5 = 0$ and $\kappa_6 = 1$. Therefore, $\text{Ap} \left( \frac{\overline{S}}{7}, 7 \right) = \{0, 8, 16, 10, 18, 5, 13\}$ and, consequently,

$$F(\overline{S}) = 18 - 7 = 11$$

and

$$g(\overline{S}) = \frac{1}{7} \left( 8 + 16 + 10 + 18 + 5 + 13 \right) - \frac{7 - 1}{2} = 10 - 3 = 7$$

by Proposition 13.

Clearly, if $S$ and $T$ are numerical semigroups, then $S \cap T$ is a numerical semigroup, too. Therefore, from expression (2) we deduce the following result.
Proposition 16. Let $S$ and $T$ be numerical semigroups and $n \in (S \cap T) \setminus \{0\}$. If $\text{Ap}(S, n) = \{0 = \omega(0), \omega(1), \ldots, \omega(n - 1)\}$ and $\text{Ap}(S', n) = \{0 = \omega'(0), \omega'(1), \ldots, \omega'(n - 1)\}$, then $\text{Ap}(S \cap T, n) = \{0 = \bar{\omega}(0), \bar{\omega}(1), \ldots, \bar{\omega}(n - 1)\}$, where $\bar{\omega}(i) = \max(\omega(i), \omega'(i))$, para todo $i = 1, \ldots, n - 1$.

Thus, combining Theorem 14 and Proposition 16, we can compute the Apéry set of all the arithmetic extension of $\Delta$.

Example 17. By direct computation, one can see that the Apéry set of $\langle 4, 5, 7 \rangle$ with respect to 4 is $\{0, 5, 10, 7\}$. By Theorem 14, we have that the Apéry set of $\langle 4, 5, 7 \rangle^2 = \langle 2, 5 \rangle$ and $\langle 4, 5, 7 \rangle^3 = \langle 3, 4, 5 \rangle$ with respect to 4 are $\{0, 5, 2, 7\}$ and $\{0, 5, 6, 3\}$, respectively. Therefore, the Apéry set of $\langle 4, 5, 7 \rangle^2 \cap \langle 4, 5, 7 \rangle^3 = \langle 4, 5, 6, 7 \rangle$ is $\{0, 5, 6, 7\}$.

Using the notation introduced in [9], a proportionally modular Diophantine inequality is an expression of the form
\[
ax \mod b \le cx,
\]
where $a, b$ and $c$ are positive integer numbers. Given $a, b$ and $c \in \mathbb{N} \setminus \{0\}$, the set of positive integer solutions of (3) constitutes a numerical semigroup ([9, Theorem 13]). These semigroups are called proportionally modular numerical semigroups (PM-semigroups, for short). In [7, Corollary 3.5] the following characterization of PM-semigroup was given.

Proposition 18. If $S$ is a PM-semigroup, there exist $a$ and $b \in \mathbb{N} \setminus \{0\}$ such that $S = \langle a, a + 1 \rangle^b$.

Remark 19. It is a challenging problem to find formulas for the Frobenius number or the genus of $\langle a, a + 1 \rangle^b$ in terms of $a$ and $b$. However, since $\text{Ap}(\langle a, a + 1 \rangle, a) = \{\omega(0) = 0, \omega(1) = a + 1, \ldots, \omega(a - 1) = (a - 1)a + (a - 1)\}$, we can use Proposition 13 and Theorem 14 to conclude that
\[
F\left(\frac{\langle a, a + 1 \rangle}{b}\right) = \max_{i=1, \ldots, a-1} \left(\left\lceil \left(\frac{b(i \mod a)(a + 1) - b i}{a b}\right) a + i \right\rceil \right) - a
\]
and
\[
g\left(\frac{\langle a, a + 1 \rangle}{b}\right) = \frac{1}{a} \left(\sum_{i=1}^{a-1} \left\lceil \left(\frac{b(i \mod a)(a + 1) - b i}{a b}\right) a + i \right\rceil \right) - \frac{a - 1}{2}.
\]

Thus, by Proposition 18 we have obtained formulas for the Frobenius number and the genus of any PM-semigroup.

Observe that from Proposition 16 and Remark 19 we can deduce a formula for the Frobenius number of any $T$-semigroup (see expression (11)).
3. An Algorithm to Compute the Arithmetic Extensions of a Numerical Semigroup

Let $\Delta$ be a numerical semigroup. Our main aim in this section is to give an algorithm for the computation of $\mathcal{F}(\Delta)$.

**Definition 20.** Let $S$ be a numerical semigroup and let $n \in S \setminus \{0\}$. If $\text{Ap}(S, n) = \{0 = \omega(0), \omega(1) = \kappa_1 n + 1, \ldots, \omega(n - 1) = \kappa_{n-1} n + n - 1\}$, then the vector

$$\varphi_n(S) := (\kappa_1, \ldots, \kappa_{n-1})$$

is called the $n$-th Kunz-coordinates vector of $S$.

The Kunz-coordinates vector of a numerical semigroup where introduced by V. Blanco and J. Puerto in [1].

Observe that Theorem 14 provides a formula for $\varphi_n(\Delta_a)$ in terms of $a, n$ and $\text{Ap}(\Delta, n)$.

**Notation.** Given $x = (x_1, x_2, \ldots, x_{n-1})$ and $y = (y_1, y_2, \ldots, y_{n-1}) \in \mathbb{N}^{n-1}$, we write $x \lor y := (\max(x_1, y_1), \max(x_2, y_2), \ldots, \max(x_{n-1}, y_{n-1}))$.

As an immediate consequence of Proposition 16 we have the following result.

**Corollary 21.** If $S$ and $T$ are numerical semigroups and $n \in (S \cap T) \setminus \{0\}$, then $\varphi_n(S \cap T) = \varphi_n(S) \lor \varphi_n(T)$.

The following result is one of the key point in our algorithm.

**Corollary 22.** If $\Delta \neq \mathbb{N}$ is numerical semigroups and $n \in S \setminus \{0\}$, then set of $n$-th Kunz-coordinates vectors of the arithmetic extensions of $\Delta$ different from $\mathbb{N}$ is

$$K := \left\{ \varphi_n\left( \frac{\Delta_a}{a} \right) \mid X \subseteq \mathbb{N} \setminus \Delta, X \neq \emptyset \text{ and } X = \text{msg}(\langle X \rangle) \right\}$$

**Proof.** By Theorem 2, the set of arithmetic extensions, $\mathcal{F}(\Delta)$, of $\Delta$ is equal to

$$\left\{ \bigcap_{a \in X} \frac{\Delta}{a} \mid X \text{ is a non-empty subset of } \mathbb{N} \setminus \Delta \text{ and } X = \text{msg}(\langle X \rangle) \right\} \cup \mathbb{N}.$$ 

Thus, by Corollary 21, our claim follows. \qed

Now, we are in condition to give an algorithm for the computation of $\mathcal{F}(\Delta)$.

**Algorithm 23.**

**Input:** A numerical semigroup $\Delta$.

**Output:** $\mathcal{F}(\Delta)$.

1. Set $\mathcal{F}(\Delta) = \{\mathbb{N}, \langle 2, 3 \rangle, \Delta\}$ and $S := \Delta \cup \text{FG}(\Delta)$.
2. Compute $m := m(S)$ and $G := \mathbb{N} \setminus S$.
3. For each $a \in G$ compute $v_a := \varphi_m\left( \frac{\Delta}{a} \right)$.
4. Compute the set $\mathcal{X} = \{X \subseteq \mathbb{N} \setminus \Delta, X \neq \emptyset \text{ and } X = \text{msg}(\langle X \rangle)\}$.
5. Compute \( K := \{ \bigvee_{a \in X} v_a \mid X \in \mathcal{X} \} \).

6. For each \((\kappa_1, \ldots, \kappa_{m-1}) \in K\), append the numerical semigroup generated by \(\{m, \kappa_1 m + 1, \ldots, \kappa_{m-1} m + m - 1\}\) to \(\mathcal{F}(\Delta)\).

The following GAP [5] function is a rude implementation of Algorithm 23. This function requires the GAP package NumericalSgps [4].

\begin{verbatim}
ArithmeticExtensions:=function(D)
    local FD,G,S,m,v,a,pow,C,x,sg;
    FD:=[NumericalSemigroup([1]),NumericalSemigroup(2,3), D];
    G:=Gaps(D);
    S:=NumericalSemigroupByGaps(Difference(G,FundamentalGaps(D)));
    m:=Multiplicity(S);
    G:=GapsOfNumericalSemigroup(S);
    v:=[];
    for a in G do
        Append(v,[KunzCoordinates(S/a,m)]);
        od;
    v:=Set(v);
    pow:=Combinations(v);
    pow:=pow{[2..Length(pow)]};
    C:=[];
    for x in pow do
        x:=TransposedMat(x);
        Append(C,[List(x,i->Maximum(i))]);
        od;
    C:=Set(C);
    for x in C do
        sg:=Concatenation([m],List(x,i->m*i)+[1..Length(x)]);
        Append(FD,[NumericalSemigroup(sg)]);
        od;
    return Set(FD);
end;
\end{verbatim}

**Example 24.** We can compute the arithmetical extension of \(\Delta = \langle 4, 6, 7 \rangle\) by using the following commands

\begin{verbatim}
LoadPackage("NumericalSgps");
D:=NumericalSemigroup(4,6,7);
ArithmeticExtensions(D);
\end{verbatim}

By this way, we obtain that \(\Delta\) has four arithmetic extensions, say \(\mathbb{N}, \langle 2, 3 \rangle, \langle 2, 5 \rangle\) and \(\langle 4, 6, 7 \rangle\).
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Departamento de Matemáticas, Universidad de Extremadura (Spain)
E-mail address: ojedamc@unex.es

Departamento de Álgebra, Universidad de Granada (Spain)
E-mail address: jrosales@ugr.es