Conformal Maps and Dispersionless Integrable Hierarchies

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Abstract

We show that conformal maps of simply connected domains with an analytic boundary to a unit disk have an intimate relation to the dispersionless 2D Toda integrable hierarchy. The maps are determined by a particular solution to the hierarchy singled out by the conditions known as "string equations". The same hierarchy locally solves the 2D inverse potential problem, i.e., reconstruction of the domain out of a set of its harmonic moments. This is the same solution which is known to describe 2D gravity coupled to $c = 1$ matter. We also introduce a concept of the $\tau$-function for analytic curves.

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1 Introduction

By the Riemann mapping theorem, any simply connected domain on the complex plane with a boundary consisting of at least two points can be conformally mapped onto the unit disk. However, the theorem does not say how to construct the map. We show that the map is explicitly given by a solution of an integrable hierarchy of the hydrodynamic type [1].

In this paper, we only consider simply connected domains bounded by an analytic curve. We show that in this case conformal maps are given by a particular solution of the dispersionless 2D Toda hierarchy (see [2, 3] and references therein). Surprisingly, this solution obeys (and is selected by) the string equation familiar in topological gravity and matrix models of 2D gravity [4, 5].

One may characterize a closed analytic curve by a set of harmonic moments of, say the exterior of the domain surrounded by the curve and its area:

\[ C_k = -\iint_{\text{exterior}} z^{-k} \, dx \, dy, \quad k \geq 1; \quad C_0 = \iint_{\text{interior}} dx \, dy \]  

(1.1)

Here \( z = x + iy \) and it is implied that the point \( z = 0 \) is inside the domain, whereas the point \( z = \infty \) is outside. The integrals at \( k = 1, 2 \) are assumed to be properly regularized. We assume this (infinite) set to be given and address two problems:

(1) To find the harmonic moments of the interior,

\[ C_{-k} = \iint_{\text{interior}} z^k \, dx \, dy, \quad k \geq 1, \]  

(1.2)

and to reconstruct the curve. It is known that if the domain is convex, the solution is unique [6]. However, in general the domain can be reconstructed with some discrete ambiguity. Nevertheless, if the domain has a smooth boundary, then a small change of moments uniquely determines a new domain (also with a smooth boundary). Below we address only the latter problem: to find the deformation of a smooth closed curve under a small deformation of the moments of the exterior of the domain bounded by the curve. The integrable hierarchy gives the solution to this problem through the \( \tau \)-function, thus suggesting a new concept - the \( \tau \)-function of the curve.

(2) To construct an invertible conformal map of the exterior of the unit circle onto the exterior of the domain.

Explicitly, the area and the set of moments of the exterior (and their complex conjugate) are identified with the times of the dispersionless 2D Toda hierarchy as follows:

\[ t \equiv t_0 = \frac{C_0}{\pi}, \quad t_k = \frac{C_k}{\pi k}, \quad \tilde{t}_k = \frac{\tilde{C}_k}{\pi k}, \quad k \geq 1. \]  

(1.3)

We prove that the derivatives of the harmonic moments of the interior \( C_{-k} \) with respect to the \( t_j, \tilde{t}_j \) enjoy the symmetry

\[ \frac{\partial C_{-k}}{\partial t_j} = \frac{\partial C_{-j}}{\partial t_k}, \quad \frac{\partial C_{-j}}{\partial \tilde{t}_j} = \frac{\partial C_{-k}}{\partial \tilde{t}_k}. \]
Therefore, the moments of the interior can be expressed as derivatives of a single real function which appears to be the logarithm of the $\tau$-function $\tau(t, \{t_k\}, \{\tilde{t}_k\})$ of the dispersionless 2D Toda hierarchy:

$$v_k \equiv \frac{C_{-k}}{\pi} = \frac{\partial \log \tau}{\partial t_k}, \quad \tilde{v}_k \equiv \frac{\tilde{C}_{-k}}{\pi} = \frac{\partial \log \tau}{\partial \tilde{t}_k}, \quad k \geq 1$$

(1.4)

The equation which determines the $\tau$-function (the dispersionless Hirota equation) and therefore the moments of the interior is given in Sec. 5. Supplemented by a proper initial condition provided by the string equation, it solves the inverse moments problem for small deformations of a simply connected domain with analytic boundary. The $\tau$-function at $t_k = 0, \ k \geq 3$ is given in the Appendix.

In its turn, the conformal map is determined by the dispersionless limit of the Lax-Sato equations of the Toda hierarchy. Let $z(w)$ be a univalent function that provides an invertible conformal map of the exterior of the unit circle $|w| > 1$ to the exterior of the domain. Let us represent it by a Laurent series

$$z(w) = r w + \sum_{j=0}^{\infty} u_j w^{-j}$$

(1.5)

The conditions that $w = \infty$ is mapped to $z = \infty$ and $r$ is real fix the map, so the potentials $r$, $u_j$ are uniquely determined by the domain. The potentials of the conformal map obey an infinite set of differential equations with respect to the times (harmonic moments). They are evolution Lax-Sato equations of the hierarchy. The series $z(w)$ is identified with the Lax function. The Lax-Sato equations derived in Sec.4.

The coefficients of the conformal map have an equivalent description in terms of the $\tau$-function: the inverse map $w(z)$ is explicitly given by the formula

$$\log w = \log z - \left( \frac{1}{2} \frac{\partial^2}{\partial t^2} + \sum_{k \geq 1} \frac{z^{-k}}{k} \frac{\partial^2}{\partial t \partial \tilde{t}_k} \right) \log \tau$$

(1.6)

A comment is in order. Many equations of Secs. 3-6 can be found in the literature on dispersionless hierarchies [2, 3]. We adopt their proofs for conformal maps.

2 Analytic curves and the Schwarz function

Let a closed analytic curve be the boundary of a simply connected domain on the complex plane with coordinates $z = x + iy, \bar{z} = x - iy$. The equation of the curve can be written in the form

$$\bar{z} = S(z)$$

(2.1)

Thus the curve determines the function $S(z)$ which is defined also outside the curve. Moreover it is an analytic function within some domain containing the curve. We call the function $S(z)$ the Schwarz function of the curve (see e.g. [7]). Its singularities encode the information about the curve.
The Schwarz function may not be an arbitrary function of \( z \). It obeys the unitarity condition; as is seen from (2.1), its complex conjugate function\(^1\) is equal to the inverse function, i.e.,
\[
\tilde{S}(S(z)) = z \tag{2.2}
\]

Being an analytic function in a strip containing the curve, the Schwarz function can be decomposed into a sum of two functions \( S(z) = S^{(+)}(z) + S^{(-)}(z) \): one, \( S^{(+)} \), is holomorphic in the interior of the domain, the second, \( S^{(-)} \), is holomorphic in the exterior. Their expansions around \( z = 0 \) and \( z = \infty \) determine the harmonic moments in the exterior \((1.1)\) and interior \((1.2)\) respectively:
\[
S^{(+)}(z) = \frac{1}{\pi} \sum_{k=1}^{\infty} C_{k} z^{k-1}, \quad S^{(-)}(z) = \frac{1}{\pi} \sum_{k=0}^{\infty} C_{-k} z^{-k-1} \tag{2.3}
\]
This follows from the contour integral representation of the moments
\[
C_{k} = \frac{1}{2i} \oint_{\text{curve}} z^{-k} S(z) dz, \quad -\infty < k < \infty \tag{2.4}
\]
obtained from \((1.1), (1.2)\) with the help of the Green formula.

The Schwarz function is closely related to conformal maps. Consider the conformal map \((1.5)\) of the exterior of the unit circle to the exterior of the domain. The inverse map sends a point \( z \) to a point \( w(z) \). Let us invert this point with respect to the circle, \( w \to (\bar{w})^{-1} \) and map it back (It is known that the conformal can be extended into some domain through the curve, We assume that the inverted point belongs to it. ). This operation is carried out by the conjugate Schwarz function \( \tilde{S}(z) = z((\bar{w})^{-1}) \) or \( S(z) = \bar{z}(w^{-1}) \). Obviously, if \( z \) belongs to the curve, then \( S(z) = \bar{z} \). Therefore, we can write \( S(z(w)) = \bar{z}(w^{-1}) \).

Summing up, we have two formal Laurent series (see \((1.3), (1.4)\) for the notation):
\[
S(z) = \sum_{k=1}^{\infty} k t_{k} z^{k-1} + \frac{t}{z} + \sum_{k=1}^{\infty} v_{k} z^{-k-1} \tag{2.5}
\]
\[
\tilde{S}(z) = \sum_{k=1}^{\infty} k \tilde{t}_{k} z^{k-1} + \frac{t}{z} + \sum_{k=1}^{\infty} \tilde{v}_{k} z^{-k-1} \tag{2.6}
\]
They are connected by the unitarity condition \((2.2)\). The latter is resolved by the conformal maps
\[
z(w) = \tilde{S}(\bar{z}(w^{-1})) \tag{2.7}
\]
\[
\bar{z}(w^{-1}) = S(z(w)) \tag{2.8}
\]
In their turn, the conformal maps are given by the series\(^2\)
\[
z(w) = rw + \sum_{j=0}^{\infty} u_{j} w^{-j} \tag{2.9}
\]
\(^1\)A comment on the notation: given a Laurent series \( f(z) = \sum_{j} f_{j} z^{j} \), we set \( \tilde{f}(z) = \sum_{j} \tilde{f}_{j} z^{-j} \).
\(^2\)To avoid confusion, let us stress that the functions \( z(w) \) and \( \bar{z}(w^{-1}) \) are complex conjugate only on the curve, i.e., at \( |w| = 1 \).
\[ z(w^{-1}) = r w^{-1} + \sum_{j=0}^{\infty} \tilde{u}_j w^j \]  
\[ (2.10) \]

They and the unitarity condition (2.7) establish relations between harmonic moments of the exterior \( t_k \), harmonic moments of the interior \( v_k \) and the coefficients of the conformal map \( u_j \). Below we find an infinite set of differential equations, which determine the evolution of the potentials and the moments of the interior in times – moments of the interior.

### 3 Symplectic structure of conformal maps and generating function

Deformations of the domain and, therefore, of the conformal map reveal the symplectic structure. In this section we show that the pairs \( \log w, t \) and \( z(w, t), \bar{z}(w^{-1}, t) \) are canonical:

\[ \{ z(w, t), \bar{z}(w^{-1}, t) \} = 1 \]  
\[ (3.1) \]

where \( \{ , \} \) is the Poisson bracket with respect to \( w \) and the area \( t \) (all moments of the exterior \( t_k, \bar{t}_k \) are kept fixed). For any two functions \( f(w, t), g(w, t) \) the Poisson bracket is defined by

\[ \{ f, g \} = w \frac{\partial f}{\partial w} \frac{\partial g}{\partial t} - w \frac{\partial g}{\partial w} \frac{\partial f}{\partial t} \]  
\[ (3.2) \]

We refer to (3.1) as the string equation\(^3\). This equation suggests that deformations of conformal maps form a group with respect to the composition. To prove it, let us rewrite the l.h.s. in two equivalent forms with the help of (2.7), (2.8). First, let us compute the derivatives of \( \bar{z}(w^{-1}, t) \) in (3.1) treating it as the composition \( S(z(w, t), t) \) as is in (2.8):

\[ \frac{\partial \bar{z}(w^{-1}, t)}{\partial t} = \frac{\partial S(z, t)}{\partial t} + \frac{\partial S(z, t)}{\partial z} \frac{\partial z(w, t)}{\partial t}, \quad \frac{\partial \bar{z}(w^{-1}, t)}{\partial w} = \frac{\partial S(z, t)}{\partial z} \frac{\partial z(w, t)}{\partial w} \]  
\[ (3.3) \]

As a result, we obtain:

\[ \{ z(w, t), \bar{z}(w^{-1}, t) \} = w \frac{\partial z(w, t)}{\partial w} \frac{\partial S(z, t)}{\partial t} \]  
\[ (3.4) \]

Similarly, treating \( z(w, t) \) in (3.1) as the composition \( \bar{S}(\bar{z}(w^{-1}, t), t) \) we get

\[ \{ z(w, t), \bar{z}(w^{-1}, t) \} = -w \frac{\partial \bar{z}(w^{-1}, t)}{\partial w} \frac{\partial \bar{S}(\bar{z}, t)}{\partial t} \]  
\[ (3.5) \]

In the r.h.s. of these equations the derivatives in \( t \) is taken at fixed \( z \) or \( \bar{z} \) and then understood as functions of \( w \).

Now, using the series (2.5) and (2.9), we conclude that the r.h.s. of (3.4) is 1 plus positive powers in \( w \). However, the series (2.6) and (2.10) tell that the r.h.s. of (3.5) is 1 plus negative powers in \( w \). This prompts us to (3.1).

\(^3\) See Sec. 7 for the history of this equation and references.
The rest follows from the symplectic structure by treating deformations of the conformal map (2.9) along the lines of the multi-time Hamilton-Jacobi formalism [2, 8, 9].

Let us introduce the generating function \( \Omega(z, t) \) of the canonical transformation \((\log w, t) \to (z, S)\). Its differential, \( d\Omega = S dz + \log w dt \), implies that

\[
S = \frac{\partial \Omega(z, t)}{\partial z}, \quad \log w = \frac{\partial \Omega(z, t)}{\partial t}
\]  

Using the Laurent series for the Schwarz function (2.5), we get

\[
\Omega(z, t) = \sum_{k=1}^{\infty} t_k z^k + t \log z - \frac{1}{2} v_0(t) - \sum_{k=1}^{\infty} \frac{v_k(t)}{k} z^{-k},
\]

where \(v_0\) obeys

\[
\partial_t v_0 = 2 \log r.
\]

Similarly to the Schwarz function, the generating function can be represented as the sum \(\Omega(z) = \Omega^{(+)}(z) + \Omega^{(-)}(z) - v_0/2\) of functions, which derivatives \(S^{(\pm)}(z) = \partial_z \Omega^{(\pm)}(z)\) are analytical in the exterior and the interior of the domain. They have a simple electrostatic interpretation. Say, \(\Omega^{(+)}(z)\), is the (complex) 2D Coulomb potential in the interior of the domain created by a homogeneously distributed charge in the exterior. In its turn, the \(\Omega^{(-)}(z)\) is the complex Coulomb potential in the exterior created by a homogeneous charge in the exterior:

\[
\Omega^{(+)}(z) = \frac{1}{\pi} \iint_{\text{exterior}} \log \left(1 - \frac{z}{z'}\right) dx' dy' = \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{C_k}{k} z^{-k}
\]

\[
\Omega^{(-)}(z) = \frac{1}{\pi} \iint_{\text{interior}} \log \left(z - z'\right) dx' dy' = \frac{C_0}{\pi} \log z - \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{C_k}{k} z^{-k}
\]

The imaginary part of the generating function treated as a function of a point \(z\) on the curve also has a geometric interpretation: \(\frac{1}{2} \Im \left(\Omega(z, t) - \Omega(z_1, t)\right)\) is the area of the domain (counted modulo \(\pi t\)) bounded by the rays \(\arg z_1\) and \(\arg z_2\).

4 Lax-Sato equations for conformal maps

Let us now vary higher harmonic moments \(t_k\). The differential of the generating function changes accordingly:

\[
d\Omega = S dz + \log w dt + \sum_{k=1}^{\infty} (H_k dt_k - \bar{H}_k d\bar{t}_k)
\]

where

\[
H_j = \left(\frac{\partial \Omega}{\partial t_j}\right)_z, \quad \bar{H}_j = -\left(\frac{\partial \Omega}{\partial \bar{t}_j}\right)_z
\]

are the Hamiltonians generating the higher flows in \(t_k, \bar{t}_k\). Here the derivatives in \(t_k\) are taken at a fixed \(z\). The Hamiltonians, being treated as functions of \(z\) and \(t\) obey the integrability condition [2]

\[
\left(\frac{\partial H_i}{\partial \bar{t}_j}\right)_z = \left(\frac{\partial H_j}{\partial t_i}\right)_z
\]
Computing the derivative in (4.2), we obtain:

\[ H_j(z) = z^j - \frac{1}{2} \frac{\partial v_0}{\partial t_j} - \sum_{k=1}^{\infty} \frac{\partial v_k z^{-k}}{\partial t_j} \]  

(4.4)

and similarly for \( \bar{H}_k \).

The Hamiltonians \( H_j \) written in terms of the canonical variables \( w, t \) determine the evolution of the conformal map \( z(w) \) and \( \bar{z}(w^{-1}) \) with respect to the hierarchical times \( t_k \) (harmonic moments):

\[ \frac{\partial z(w)}{\partial t_j} = \{ H_j, z(w) \} \]  

(4.5)

\[ \frac{\partial \bar{z}(w^{-1})}{\partial t_j} = \{ H_j, \bar{z}(w^{-1}) \} \]  

(4.6)

Note that these formulas can be extended to \( j = 0 \): \( H_0 = \log w \) generates the flow \( t_0 = t \). Consistency of these equations acquires the form of the "zero curvature" condition \( \partial_j H_i - \partial_i H_j + \{ H_i, H_j \} = 0 \). It is equivalent to (4.3) rewritten in the variable \( w \).

Now we are ready to prove that the Hamiltonians \( H_i \) and \( \bar{H}_i \) have the form

\[ H_j(w) = \left( z^j(w) \right)_+ + \frac{1}{2} \left( z^j(w) \right)_0 \]  

(4.7)

\[ \bar{H}_j(w) = \left( \bar{z}^j(w^{-1}) \right)_- + \frac{1}{2} \left( \bar{z}^j(w^{-1}) \right)_0 \]  

(4.8)

The symbols \( (f(w))_{\pm} \) mean a truncated Laurent series, where only terms with positive (negative) powers of \( w \) are kept, \( (f(w))_0 \) is a constant part \( (w^0) \) of the series.

To this end, let us differentiate \( H_j \) by \( w \) and \( t \), and express the result in terms of derivatives of \( z \) and \( \bar{S} \). We begin with the formula

\[ \frac{\partial H_j}{\partial w} = \frac{\partial z(w)}{\partial w} \frac{\bar{z}(w^{-1})}{\partial t_j} - \frac{\partial \bar{z}(w^{-1})}{\partial t_j} \frac{\partial \bar{z}(w)}{\partial w} \]

which is a simple consequence of the definition (4.2). Replacing here \( z(w) \) by \( \bar{S}(\bar{z}) \) as before, we get:

\[ \frac{\partial H_j}{\partial w} = \frac{\partial \bar{z}}{\partial w} \frac{\partial \bar{S}(\bar{z})}{\partial t_j} \]  

(4.9)

Using (2.6), we find that \( \partial_j \bar{S}(\bar{z}) = \sum_{k=1}^{\infty} \partial_j v_k \bar{z}^{j-k-1} \) is a regular function of \( w \) at \( w = 0 \) and its Taylor expansion starts from \( w^2 \). Thus \( H_j \) is also a regular function in \( w \). Moreover, from the (4.4) we find that \( H_j \) is a polynomial in \( w \) of the degree \( j \). Thus, being so, we have \( H_j = (z^j)_+ + (z^j)_0 - \frac{1}{2} \partial_j v_0 \). To complete the proof, let us find the \( w^0 \)-term in the Laurent series

\[ \frac{\partial H_j}{\partial t} = \frac{\partial \bar{S}(\bar{z})}{\partial t} \frac{\partial \bar{z}(w)}{\partial t_j} - \frac{\partial \bar{z}(w)}{\partial t} \frac{\partial \bar{S}(\bar{z})}{\partial t_j} \]  

(4.10)

It comes from the first term of the r.h.s. of this expression and, together with (3.7), gives the desired result: \( 2(\partial_j H_j)_0 = 2\partial_j \log r = \partial_j \partial_k v_0 = \partial_t (z^j)_0 \). The dynamics of the conformal map with respect to \( t_k \) can be obtained from (4.5), (4.6) by the complex
conjugation. Note that the Poisson bracket changes the sign as \( w \to \bar{w} = w^{-1} \) that is just consistent with the minus sign in (4.2). Hence \( \hat{H}_j = \bar{H}_j(w^{-1}) \) as is in (4.8).

The Lax-Sato equations with the Hamiltonians (4.7), (4.8) imply that the coefficients of the conformal map obey an infinite set of non-linear differential equations known as the dispersionless 2D Toda hierarchy. The first and the most familiar equation of the hierarchy is the long wave limit of the Toda lattice equation: \( \partial^2 (r^2)/\partial t^2 = \partial^2 \log (r^2)/\partial t \partial \bar{t} \).

We also mention another relation between the conformal map and the Hamiltonians:

\[
\begin{align*}
\bar{z}(w) &= H_1 + \frac{1}{2} \mathcal{I}_1 + \sum_{k=2}^{\infty} k \mathcal{I}_k \bar{H}_{k-1} \\
(\text{and a similar equation for } \bar{z}(w^{-1})).
\end{align*}
\]

(4.11)

It can be immediately obtained from the unitarity condition (2.7), (2.8) by comparing the positive and constant parts of the Laurent series in \( w \) and using (4.7), (4.8). These formulas illustrate an important relation between harmonic moments and conformal maps. If all moments \( t_k = 0 \) for \( k > N \), then the Laurent series of the conformal map (1.5) is truncated: \( u_j = 0 \) for \( j \geq N \). This, in particular, gives another proof of Sakai’s theorem [10]: if all but the first three moments \( t, t_1, t_2 \) of the complement of a simply connected domain on the complex plane are zero, then the domain is an ellipse.

The Lax-Sato equations (4.5), (4.6) supplemented by the string equation (3.1) give the complete set of differential equations for the potentials \( u_j \) of the conformal map as functions of moments \( t, t_k, \bar{t}_k \). They uniquely determine small deformations of the map and eventually the curve.

5 Symmetry of moments and the \( \tau \)-function of conformal maps and curves

The integrable hierarchy suggests a concept of \( \tau \)-function for curves and conformal maps. The \( \tau \)-function solves the problem of moments, i.e., restoration of the moments of the interior \( v_k \) out of the moments of the exterior \( \{t_k\} \) and the area \( t \). We define the \( \tau \)-function \( \tau(t; \{t_i\}, \{\bar{t}_i\}) \) as a real function, which determines the moments of the exterior by the formulas

\[
\begin{align*}
v_k &= \partial \log \tau / \partial t_k, \\
\bar{v}_k &= \partial \log \tau / \partial \bar{t}_k
\end{align*}
\]

(5.1)

The very existence of the \( \tau \)-function is due to the fundamental symmetry of harmonic moments:

\[
\frac{\partial v_i}{\partial t_k} = \frac{\partial v_k}{\partial t_i}, \quad \frac{\partial v_i}{\partial \bar{t}_k} = \frac{\partial \bar{v}_k}{\partial \bar{t}_i}, \quad \frac{\partial v_0}{\partial t_k} = \frac{\partial v_k}{\partial t_0}
\]

(5.2)

which we prove below. The proof follows the lines of Ref. [3]. To prove the first symmetry relation, we notice from (4.4) that \( \partial_j v_k \) is the constant \( (z^0) \) part of the Laurent series \( z^{k+1} \partial_z H_j \) in \( z \), i.e., the residue \( \text{res}(z^k dH_j) \). Then, using the well known property of residues, we find that this is equal to the constant part \( (w^0) \) of the Laurent series \( z^k w \partial_w H_j \) in \( w \). Then, using the equation (4.7) we get:

\[
\frac{\partial v_j}{\partial \bar{t}_k} = (z^k w \partial_w H_j)_0 = (z^k w \partial_w (z^j)_0)_0 =
\]
This chain of equalities is due to the identity \( (f_\omega w \partial_w g_\omega)_0 = - (g_\omega w \partial_w f_\omega)_0 = \lambda (g_\omega w \partial_w f_\omega)_0 \) for residues of Laurent series. The proof of the other two symmetry relations is similar.

The \( \tau \)-function determines the Hamiltonians as functions of \( z \). Eq. (4.4) reads

\[
H_j = z^j - \left( \frac{1}{2} \frac{\partial^2}{\partial t \partial t_j} + \sum_{k \geq 1} \frac{z^{-k}}{k} \frac{\partial^2}{\partial t_j \partial t_k} \right) \log \tau
\]

and the inverse conformal map \( w(z) \) (see (1.6) in the Introduction).

The \( \tau \)-function itself obeys the dispersionless limit of the Hirota equation (a leading term of the differential Fay identity [8], [3]).

\[
(z - \zeta) \exp \left( \sum_{n,m \geq 1} \frac{v_{nm}}{nm} z^{-n} \zeta^{-m} \right) = z \exp \left( - \sum_{k \geq 1} \frac{v_{0k}}{k} z^{-k} \right) - \zeta \exp \left( - \sum_{k \geq 1} \frac{v_{0k}}{k} \zeta^{-k} \right)
\] (5.3)

This is an infinite set of relations between the second derivatives \( v_{nm} = \partial_{t_n} \partial_{t_m} \log \tau \), \( v_{0m} = \partial_{t_0} \partial_{t_m} \log \tau \) of the \( \tau \)-function. The equations appear while expanding the both sides of (5.3) in powers of \( z \) and \( \zeta \). Supplemented by proper initial data, satisfying the unitarity condition, these equations formally solve the local problem of moments, i.e., reconstruction of the Laurent expansion of the \( \mathcal{S} \). The proof of (5.3) is similar to the one known in the context of the KP hierarchy [8]: multiply the series for the inverse conformal map \( w(z) \) by \( \zeta^{-k} z^{k-1} \), take the polynomial part in \( w \) and sum over \( k \).

6 Conformal maps as a reduction of the dispersionless Toda lattice hierarchy

The reader familiar with integrable hierarchies of non-linear differential equations is able to identify the dynamical system for conformal maps (4.5,4.6) with the dispersionless limit of the Toda lattice hierarchy [2, 3]. The latter is related with the Whitham hierarchy - the theory of solitons with distinct fast and slow variables. The Whitham hierarchy appears after averaging over fast variables (see [2, 12] and references therein). The dispersionless limit emerges as the genus-zero Whitham hierarchy. Formally, it is a semiclassical limit \( \hbar \to 0 \) of pseudo-differential (or difference) operators. For the Lax operator of the 2D Toda lattice,

\[
L = r(t)e^{\hbar \frac{\partial}{\partial t}} + \sum_{j=0}^{\infty} u_j(t)e^{-j\hbar \frac{\partial}{\partial t}}
\] (6.1)

one should replace the difference operator \( e^{\hbar \frac{\partial}{\partial t}} \) by the canonical variable \( w \) with the Poisson bracket \( \{ \log w, t \} = 1 \). The Lax operator then becomes the Lax function \( L(w) \) given by a formal Laurent series in \( w \). The Lax function is identified with the conformal map \( z(w) \) (1.5). The derivatives of the \( z(w) \) with respect to the times \( t_k \) are given by (4.5) which is nothing else than the dispersionless limit of the Lax-Sato equation \( \hbar \partial_j \partial t_j L = [H_j, L] \), where \( H_j = (L^j)_+ + \frac{1}{2}(L^j)_0 \). (The coefficient \( 1/2 \) is due to a particular choice of gauge in (6.1), where the coefficient in front of the first term is not fixed to be
1. The mathematical theory of the dispersionless hierarchies constrained by the string equations has been developed in Refs. [2, 9] and extended to the Toda hierarchy in Ref. [3]. For a comprehensive review see, e.g., [12].

In the Toda theory, there are two sets of independent times, $t_i$ and $\tilde{t}_i$, and two sets of potentials, $u_j$ and $\tilde{u}_j$. The identification with conformal maps requires them to be complex conjugate: $\tilde{t}_i = \overline{t}_i$, $\tilde{u}_j = \overline{u}_j$ and the function $r$ to be real. Under the reality conditions the semiclassical limit of the second Lax operator,

$$\tilde{L} = r(t - \hbar) e^{-\frac{\pi}{\hbar t}} + \sum_{j=0}^{\infty} \tilde{u}_j(t) e^{j\frac{\pi}{\hbar t}}$$

is identified with $\tilde{z}(w^{-1})$. The reality condition is consistent with the 2D Toda hierarchy and selects a class of solutions relevant to conformal maps.

This class is reduced to a unique solution by imposing the string equation $\{z, \tilde{z}\} = 1$, which in the dispersionful case would be $[L, \tilde{L}] = \hbar$. To clarify the origin of the string equation, one needs two more operators. These are the Orlov-Shulman operators [11]

$$M = \sum_{k=1}^{\infty} kt_k L^k + t + \sum_{k=1}^{\infty} v_k L^{-k}$$

$$\tilde{M} = \sum_{k=1}^{\infty} k\tilde{t}_k \tilde{L}^k + t + \sum_{k=1}^{\infty} \tilde{v}_k \tilde{L}^{-k}$$

obeying the conditions $[L, M] = \hbar L$; $[\tilde{L}, \tilde{M}] = -\hbar \tilde{L}$. Then the string equation follows from the relations $[4] \tilde{L} = L^{-1} M$, $\tilde{M} = M$. From Secs. 2, 3 it follows that the dispersionless limit of the operator $L^{-1} M$ enjoys a simple geometric interpretation: it is the Schwarz function $S(z)$ of an analytic curve.

## 7 More connections and equivalences

**Analytic curves and dispersionless hierarchies.** Thus one particular solution of the dispersionless 2D Toda hierarchy describes evolution of the univalent conformal map of a domain bounded by an analytic curve to the exterior of the unit circle. The set of times of the hierarchy appears to be equivalent to the set of harmonic moments of the domain, whereas the conformal map itself is the dispersionless limit of the Lax operator.

This proposition can be reversed: conformal maps of the exterior of the unit circle generate a solution of the dispersionless Toda hierarchy. The solution is selected by the string equation (3.1).

Other solutions of the dispersionless Toda hierarchy are selected by more general string equations. The latter are characterized by any two functions $f$ and $g$ forming a canonical pair: $\{f, g\} = f$. Then the general string equations consistent with the hierarchy are $\tilde{L} = f^{-1}(L, M)$, $\tilde{M} = g(L, M)$ (see [3]). We expect that some of them are also relevant to the conformal maps of simply connected domains. It is likely that other types of string equations describe mappings to or from domains other than the exterior of the unit circle and also nonunivalent maps.

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Moreover, we expect that other integrable hierarchies of the hydrodynamic type, for instance KP hierarchy and its reductions, also describe certain classes of conformal maps and curves others than analytic. We plan to address this question elsewhere.

The \( \tau \)-function for curves. All this suggests to introduce a general notion of the \( \tau \)-function for curves. For smooth curves, this is a universal function which determines the curve by means of the Laurent series

\[
x^2 + y^2 - t = \sum_{k=1}^{\infty} R e \left( k t_k (x + iy)^k + (x + iy)^{-k} \frac{\partial}{\partial t_k} \log \tau \right)
\]

(7.1)

The dispersionless Hirota equation (5.3) provides a set of differential equations for the \( \tau \)-function. Again, it has many solutions. The proper solution is selected by the unitarity condition (2.7) and initial data. For instance, if the curve can be reached by small deformations of a circle, one sets \( \partial_{t_k} \log \tau = 0 \) at all \( t_k = 0 \).

Moving curves and the \( c = 1 \) string theory. There is another intriguing relation (in fact equivalence) between deformations of analytic curves and the genus-0 topological sector of 2D gravity coupled to \( c = 1 \) matter [4]. This follows from the known equivalence between the latter and the dispersionless Toda hierarchy restricted by the so-called \( W_{1+\infty} \)-constraints [4, 2, 12]. The interpretation of the objects of the genus-0 string theory in terms of analytic curves is straightforward. The positive (negative) momentum tachyon one-point functions \( < T_n > \) are moments of the interior of the domain \( v_n (\bar{v}_n) \), the partition function of the genus-0 string is the \( \tau \)-function of analytic curves, the Schwarz function \( S(z) \) is the superpotential and, finally, the \( W_{1+\infty} \)-constraints are essentially equivalent to the string equation (3.1) or the relation (4.11). In fact, they are nothing else than the unitarity condition (2.7).

In its turn, the \( c = 1 \) genus-0 string theory, as well as the relevant solution to the dispersionless hierarchy, is known to be equivalent to the planar limit of the two-matrix model (see [5] and references therein). This gives a representation of the \( \tau \)-function of analytic curves as \( N \to \infty \) limit of the \( N \times N \) two-matrix integral \( \tau = \int D M D \bar{M} \exp tr W \), with the potential \( W = \sum_k (t_k M^k + \bar{t}_k \bar{M}^k) + M \bar{M} \). We plan to discuss this subject elsewhere.

The interpretation of the genus-0 string theory as a simply minded classical theory of potential may turned out to be fruitful for both subjects. It is likely that the 2D gravity coupled to \( c < 1 \) matter, being described by various reductions of the dispersionless KP hierarchy, also enjoys a geometric interpretation.

Laplacian growth problem. This paper has been stimulated by our interest in the Laplacian growth problem (see [13] for a review). This problem (a source of interesting mathematics and a great deal of important application) may give further insights in the matters we discuss in the paper. It seems to be the place for a short introduction. This is the problem of a moving interface between two incompressible liquids with different viscosities on the plane. Let, say an exterior of a simply connected domain be occupied by a viscous fluid (oil), while less viscous liquid (water) occupies the interior. Water is supplied by a source at \( z = 0 \), while oil sinks to a source at the infinity, so the interface moves. Experiments and numerical simulations suggest that any smooth interface, regardless of its initial shape, develops a finger-like pattern with a universal fractal characteristics. The hydrodynamics of an ideal interface (with zero surface tension) is
described by the Darcy law

\[ V_n = -\frac{\partial p}{\partial n} \tag{7.2} \]

where \( V_n \) is the velocity of the interface and \( \frac{\partial p}{\partial n} \) is the normal derivative of the pressure on the interface. The pressure is constant in the water domain and on the interface, while in the oil domain it obeys the Laplace equation \( \nabla^2 p = 0 \) with the asymptotic behavior at the infinity \( p \rightarrow -1/2 \log |z| \). The latter indicates a sink at the infinity. The Darcy law implies [14] that all harmonic moments \( t_k \) of the oil domain are not changed while the interface moves, but the area of the water domain grows linearly in time and thus can be identified with the time. (For connections with the inverse potential problem see [15].) The problem then becomes: find the evolution of the domain as a function of the area \( t \) at fixed moments \( t_k \). This evolution is described by the string equation (3.1). This equation has a long history. It appeared in 1945 in Ref. [16] or even earlier in the mathematical theory of oil hydrodynamics. In our approach, this equation is the basis of the symplectic structure of the conformal maps. Application of integrable hierarchies to the Laplacian growth problem is addressed in Ref. [17].

During the completion of the manuscript E.Ferapontov informed us that J.Gibbons and S.Tsarev discussed a relation between Benney equations and conformal maps of slit domains [18]. We would like to thank J.Gibbons and S.Tsarev for informing us about their recent paper [19].

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Appendix: Ellipse growing from a circle

Here we demonstrate how the Lax-Sato equations describe an ellipse growing from a circle. It is the simplest but nevertheless instructive example which in particular allows one to compute the \( \tau \)-function at all \( t_k = 0 \) but \( t, t_1, \bar{t}_1, t_2, \bar{t}_2 \). Consider an ellipse with
half-axes $a,b$ centered at $z_0 = x_0 + iy_0$ and rotated by the angle $\alpha$:

\[
\frac{(\cos \alpha (x - x_0) - \sin \alpha (y - y_0))^2}{a^2} + \frac{(\sin \alpha (x - x_0) + \cos \alpha (y - y_0))^2}{b^2} = 1
\]

The Schwarz function of the ellipse is

\[
S(z) = e^{2i\alpha} \frac{a^2 + b^2}{a^2 - b^2} (z - z_0) + z_0 - \frac{2ab e^{2i\alpha}}{a^2 - b^2} \sqrt{(z - z_0)^2 - e^{-2i\alpha} (a^2 - b^2)}
\]

The Laurent series of the Schwarz function (2.5)

\[
S(z) = 2t_2 z + t_1 + \frac{t}{z} + \frac{v_1}{z^2} + \frac{v_2}{z^3} + \ldots
\]
gives the moments of the exterior and the interior. The only nonzero moments of the exterior are $t_1, t_2$ and their complex conjugate:

\[
2t_2 = e^{2i\alpha} \frac{a - b}{a + b}, \quad t_1 = z_0 - 2t_2 z_0, \quad t = ab
\]

Contrary, none of moments of the interior vanish. The first two are

\[
v_1 = \frac{t(\bar{t}_1 + 2t_1 \bar{t}_2)}{1 - 4t_2 \bar{t}_2}, \quad v_2 = \frac{t(\bar{t}_1 + 2t_1 \bar{t}_2)^2}{(1 - 4t_2 \bar{t}_2)^2} + \frac{2t^2 \bar{t}_2}{1 - 4t_2 \bar{t}_2}
\]

Adding $\partial_t v_0 = 2 \log r$, one may check the symmetry relations (5.2) and find the $\tau$-function for the ellipse:

\[
\log \tau = \frac{1}{2} t^2 \log t - \frac{3}{4} t^2 - \frac{1}{2} t^2 \log (1 - 4t_2 \bar{t}_2) + t \frac{t_1 \bar{t}_1 + t_2 \bar{t}_2 + \bar{t}_2 t_2}{1 - 4t_2 \bar{t}_2}
\]

The Laurent series for the conformal map from the exterior of the unit circle to the exterior of the ellipse is truncated:

\[
z(w) = rw + u_0 + u_1 w^{-1},
\]

The coefficients of the conformal map are:

\[
r^2 = \frac{1}{4} (a + b)^2 = \frac{t}{1 - 4t_2 \bar{t}_2}, \quad u_1^2 = \frac{1}{4} e^{-4i\alpha} (a - b)^2 = \frac{4t \bar{t}_2}{1 - 4t_2 \bar{t}_2},
\]

\[
u_0 = z_0 = \frac{\bar{t}_1 + 2t_1 \bar{t}_2}{1 - 4t_2 \bar{t}_2}
\]

The first two Hamiltonians are: $H_1 = rw + \frac{1}{2} u_0, H_2 = r^2 w^2 + 2ru_0 w + ru_1 + \frac{1}{2} u_0^2$. Higher flows deform the ellipse. The Lax-Sato equations plus the string equation (3.1) and their conjugate describe how the ellipse grows from the circle.
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