Magnetic Branes in \((n + 1)\)-dimensional Einstein-Maxwell-dilaton gravity

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We construct two new classes of spacetimes generated by spinning and traveling magnetic sources in \((n + 1)\)-dimensional Einstein-Maxwell-dilaton gravity with Liouville-type potential. These solutions are neither asymptotically flat nor (A)dS. The first class of solutions which yields a \((n + 1)\)-dimensional spacetime with a longitudinal magnetic field and \(k\) rotation parameters have no curvature singularity and no horizons, but have a conic geometry. We show that when one or more of the rotation parameters are nonzero, the spinning branes has a net electric charge that is proportional to the magnitude of the rotation parameters. The second class of solutions yields a static spacetime with an angular magnetic field, and have no curvature singularity, no horizons, and no conical singularity. Although one may add linear momentum to the second class of solutions by a boost transformation, one does not obtain a new solution. We find that the net electric charge of these traveling branes with one or more nonzero boost parameters is proportional to the magnitude of the velocity of the branes. We also use the counterterm method and calculate the conserved quantities of the solutions.

I. INTRODUCTION

In many unified theories, including string theory, dilatons appear. The appearance of dilaton changes the asymptotic behavior of the solutions to be neither asymptotically flat nor (anti)-de Sitter \([(A)dS\]. A motivation to investigate nonasymptotically flat, nonasymptotically AdS solutions of Einstein gravity is that these might lead to possible extensions of AdS/CFT correspondence. Indeed, it has been speculated that the linear dilaton spacetimes, which arise as near-horizon limits of dilatonic black holes, might exhibit holography [1]. Another motivation is that such solutions may be used to extend the range of validity of methods and tools originally developed for, and tested in the case of, asymptotically flat or asymptotically AdS black holes. Specifically, we will find that the counterterm method inspired by the AdS/CFT correspondence may be applied successfully to the computation of the conserved quantities of nonasymptotically AdS rotating

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magnetic branes with flat boundary at constant $r$ and $t$. Exact dilaton black hole solutions in the absence of dilaton potential have been constructed by many authors \cite{2,3,4,5}. In the presence of Liouville-type potential, static charged black hole solutions have been discovered with positive \cite{6}, zero or negative constant curvature horizons \cite{7}. Recently, properties of these black hole solutions which are not asymptotically AdS or dS, have been studied \cite{8}. Also, exact spherically symmetric dyonic black hole solutions in four-dimensional and higher dimensional Einstein-Maxwell-dilaton gravity with Liouville-type potentials have been considered \cite{9}. These exact solutions mentioned above \cite{2,3,4,5,6,7,8,9} are all static. Exact rotating solutions to the Einstein equation coupled to matter fields with curved horizons are difficult to find except in a limited number of cases. Indeed, rotating solutions of EMd gravity with curved horizons have been obtained only for some limited values of the coupling constant \cite{10,11,12,13}. For general dilaton coupling, the properties of rotating charged dilaton black holes only with infinitesimally small charge \cite{14} or small angular momentum in four \cite{15} and five \cite{16} dimensions have been investigated. For arbitrary values of angular momentum and charge, only a numerical investigation has been done \cite{17}. When the horizons are flat, charged rotating dilaton black string solutions, in four-dimensional EMd gravity have been constructed \cite{18}. Recently, these solutions have been generalized to the $(n+1)$-dimensional EMd gravity for an arbitrary dilaton coupling and $k$ rotation parameters \cite{19}.

In this paper we want to construct $(n+1)$-dimensional horizonless solutions of EMd gravity. The motivation for constructing these kinds of solutions is that they may be interpreted as cosmic strings. Cosmic strings are topological defects that arise from the possible phase transitions in the early universe, and may play an important role in the formation of primordial structures. There are many papers which are dealing directly with the issue of spacetimes in the context of cosmic string theory \cite{20}. All of these solutions are horizonless and have a conical geometry; they are everywhere flat except at the location of the line source. The spacetime can be obtained from the flat spacetime by cutting out a wedge and identifying its edges. An extension to include the electromagnetic field has also been done \cite{21}. Asymptotically AdS spacetimes generated by static and spinning magnetic sources in three and four dimensional Einstein-Maxwell gravity with negative cosmological constant have been investigated in \cite{22,23}. The generalization of these asymptotically AdS magnetic rotating solutions of the Einstein-Maxwell equation to higher dimensions \cite{24} and higher derivative gravity \cite{25} have been also done. In the context of electromagnetic cosmic string, it has been shown that there are cosmic strings, known as superconducting cosmic strings, that behave as superconductors and have interesting interactions with astrophysical magnetic fields \cite{26}. The properties of these superconducting cosmic strings have been investigated in \cite{27}. Su-
perconducting cosmic strings have also been studied in Brans-Dicke theory [28], and in dilaton gravity [29]. In EMd gravity, the exact magnetic rotating solutions in three dimensions have been presented in [30], while two classes of magnetic rotating solutions in four-dimensional EMd gravity with Liouville-type potential have been constructed by one of us [31]. These solutions [30, 31] are not black holes, and represent spacetimes with conic singularities. The possibility that spacetime may have more than four dimensions is now a standard assumption in high energy physics. Thus, it is worth to generalize the 4-dimensional solutions of Ref. [31] to the case of \((n + 1)\)-dimensional EMd gravity for an arbitrary value of coupling constant and investigate their properties.

The organization of our paper is as follows: In Sec II we have a brief review of the field equations and general formalism of calculating the conserved quantities. In Sec III we present the \((n + 1)\)-dimensional magnetic solutions of EMd gravity with longitudinal and angular magnetic fields, and investigate their properties. We finish our paper with some concluding remarks.

II. FIELD EQUATIONS AND CONSERVED QUANTITIES

The action of Einstein-Maxwell dilaton gravity with one scalar field \(\Phi\) with Liouville-type potential in \((n + 1)\) dimensions can be written as

\[
I_G = -\frac{1}{16\pi} \int_M d^{n+1}x \sqrt{-g} \left( R - \frac{4}{n-1} (\nabla \Phi)^2 - 2\Lambda e^{4\alpha\Phi/(n-1)} - e^{-4\alpha\Phi/(n-1)} F^2 \right) - \frac{1}{8\pi} \int_{\partial M} d^n x \sqrt{-\gamma} \Theta(\gamma),
\]

where \(R\) is the Ricci scalar curvature, \(\Phi\) is the dilaton field and \(\Lambda\) is a constant which may be referred to as the cosmological constant, since in the absence of the dilaton field (\(\Phi = 0\)) the action \(I_G\) reduces to the action of Einstein-Maxwell gravity with cosmological constant. \(\alpha\) is a constant determining the strength of coupling of the scalar and electromagnetic fields, \(F^2 = F_{\mu\nu}F^{\mu\nu}\), where \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\) is the electromagnetic tensor field and \(A_\mu\) is the vector potential. The manifold \(M\) has metric \(g_{\mu\nu}\) and covariant derivative \(\nabla_\mu\). \(\Theta\) is the trace of the extrinsic curvature \(\Theta^{ab}\) of any boundary(ies) \(\partial M\) of the manifold \(M\), with induced metric \(\gamma_{ij}\). The first integral of Eq. (1) does not have a well-defined variational principle, since one encounters a total derivative that produces a surface integral involving the derivative of \(\delta g_{\mu\nu}\) normal to the boundary. These normal derivative terms do not vanish by themselves, but as in the case of Einstein gravity, they are canceled by the variation of the Gibbons-Hawking surface term (second integral).

The equations of motion can be obtained by varying the action \(I_G\) with respect to the gravita-
tional field $g_{\mu\nu}$, the dilaton field $\Phi$ and the gauge field $A_\mu$ which yields the following field equations

$$R_{\mu\nu} = \frac{4}{n-1} \left( \partial_\mu \Phi \partial_\nu \Phi + \frac{\Lambda}{2} e^{4\alpha\Phi/(n-1)} g_{\mu\nu} \right) + 2 e^{-4\alpha\Phi/(n-1)} \left( F_{\mu\eta} F^{\eta}_\nu - \frac{g_{\mu\nu}}{2(n-1)} F^2 \right),$$

\[ (2) \]

$$\nabla^2 \Phi = \Lambda \alpha e^{4\alpha\Phi/(n-1)} - \frac{\alpha}{2} e^{-4\alpha\Phi/(n-1)} F^2,$$

\[ (3) \]

$$\nabla_\mu \left( e^{-4\alpha\Phi/(n-1)} F^{\mu\nu} \right) = 0.$$ \[ (4) \]

The conserved mass and angular momentum of the solutions of the above field equations can be calculated through the use of the substraction method of Brown and York \[32\]. Such a procedure causes the resulting physical quantities to depend on the choice of reference background. For asymptotically (A)dS solutions, the way that one deals with these divergences is through the use of counterterm method inspired by (A)dS/CFT correspondence \[33\]. However, in the presence of a non-trivial dilaton field, the spacetime may not behave as either dS ($\Lambda > 0$) or AdS ($\Lambda < 0$). In fact, it has been shown that with the exception of a pure cosmological constant potential, where $\alpha = 0$, no AdS or dS static spherically symmetric solution exist for Liouville-type potential \[5\]. But, as in the case of asymptotically AdS spacetimes, according to the domain-wall/QFT (quantum field theory) correspondence \[34\], there may be a suitable counterterm for the stress energy tensor which removes the divergences. In this paper, we deal with spacetimes with zero curvature boundary $[R_{abcd}(\gamma) = 0]$, and therefore the counterterm for the stress energy tensor should be proportional to $\gamma^{ab}$. Thus, the finite stress-energy tensor in $(n+1)$-dimensional Einstein-dilaton gravity with Liouville-type potential may be written as

$$T^{ab} = \frac{1}{8\pi} \left[ \Theta^{ab} - \Theta \gamma^{ab} + \frac{n-1}{l_{\text{eff}}} \gamma^{ab} \right],$$

\[ (5) \]

where $l_{\text{eff}}$ is given by

$$l_{\text{eff}}^2 = \frac{(n-1)(\alpha^2 - n)}{2\Lambda} e^{-4\alpha\Phi/(n-1)}.$$ \[ (6) \]

In the particular case $\alpha = 0$, the effective $l_{\text{eff}}^2$ of Eq. \[6\] reduces to $l^2 = -n(n-1)/2\Lambda$ of the AdS spacetimes. The first two terms in Eq. \[5\] are the variation of the action \[11\] with respect to $\gamma^{ab}$, and the last term is the counterterm which removes the divergences. One may note that the counterterm has the same form as in the case of asymptotically AdS solutions with zero curvature boundary, where $l$ is replaced by $l_{\text{eff}}$. To compute the conserved charges of the spacetime, one should choose a spacelike surface $B$ in $\partial M$ with metric $\sigma_{ij}$, and write the boundary metric in ADM (Arnowitt-Deser-Misner) form:

$$\gamma_{ab} dx^a dx^b = -N^2 dt^2 + \sigma_{ij} \left( d\varphi^i + V^i dt \right) \left( d\varphi^j + V^j dt \right),$$
where the coordinates $\varphi^i$ are the angular variables parameterizing the hypersurface of constant $r$ around the origin, and $N$ and $V^i$ are the lapse and shift functions, respectively. When there is a Killing vector field $\xi$ on the boundary, then the quasilocal conserved quantities associated with the stress tensors of Eq. (5) can be written as

$$Q(\xi) = \int_B d^{n-1}x \sqrt{\sigma} T_{ab} n^a \xi^b,$$

where $\sigma$ is the determinant of the metric $\sigma_{ij}$, $\xi$ and $n^a$ are the Killing vector field and the unit normal vector on the boundary $B$. For boundaries with timelike ($\xi = \partial/\partial t$), rotational ($\varsigma_i = \partial/\partial \varphi^i$) and translational Killing vector fields ($\zeta_i = \partial/\partial x^i$), one obtains the quasilocal mass, components of total angular and linear momenta as

$$M = \int_B d^{n-1}x \sqrt{\sigma} T_{ab} n^a \xi^b,$$  

$$J_i = \int_B d^{n-1}x \sqrt{\sigma} T_{ab} n^a \varsigma_i^b,$$  

$$P_i = \int_B d^{n-1}x \sqrt{\sigma} T_{ab} n^a \zeta_i^b,$$

provided the surface $B$ contains the orbits of $\varsigma$. These quantities are, respectively, the conserved mass, angular and linear momenta of the system enclosed by the boundary $B$. Note that they will both be dependent on the location of the boundary $B$ in the spacetime, although each is independent of the particular choice of foliation $B$ within the surface $\partial M$.

III. (N+1)-DIMENSIONAL MAGNETIC ROTATING SOLUTIONS

In this section, we obtain the $(n+1)$-dimensional horizonless solutions of Eqs. (2)-(4). First, we construct a spacetime generated by a magnetic source which produces a longitudinal magnetic field. Second, we obtain a spacetime generated by a magnetic source that produces angular magnetic fields along the $\varphi^i$ coordinates.

A. Longitudinal magnetic field solutions

Here we want to obtain the $(n+1)$-dimensional solutions of Eqs. (2)-(4) which produce longitudinal magnetic fields in the Euclidean submanifold spanned by the $x^i$ coordinates ($i = 1, ..., n-2$). We assume that the metric has the following form:

$$ds^2 = \frac{\rho^2}{l^2} R^2(\rho) dt^2 + \frac{d\rho^2}{f(\rho)} + l^2 f(\rho) d\varphi^2 + \frac{\rho^2}{l^2} R^2(\rho) dX^2,$$  

(11)
where \( dX^2 = \sum_{i=1}^{n-2} (dx^i)^2 \). Note that the coordinates \( x^i \) have the dimension of length, while the angular coordinate \( \phi \) is dimensionless as usual and ranges in \( 0 \leq \phi < 2\pi \). The motivation for this metric gauge \([g_{tt} \propto -\rho^2\) and \((g_{\rho\rho})^{-1} \propto g_{\phi\phi}\)] instead of the usual Schwarzschild gauge \([(g_{\rho\rho})^{-1} \propto g_{tt}\) and \(g_{\phi\phi} \propto \rho^2\)] comes from the fact that we are looking for a string solution with conic singularity.

The Maxwell equation (4) can be integrated immediately to give

\[
F_{\phi\rho} = \frac{q e^{4\alpha \Phi/(n-1)}}{(\rho R)^{n-1}}
\]  

(12)

where \( q \) is an integration constant related to the electric charge of the brane. In order to solve the system of equations (12) and (13) for three unknown functions \( f(\rho) \), \( R(\rho) \) and \( \Phi(\rho) \), we make the ansatz

\[
R(\rho) = e^{2\alpha \Phi/(n-1)}.
\]  

(13)

Using (13), the Maxwell fields (12) and the metric (11), one can easily show that Eqs. (12) and (13) may be written as

\[
f \Phi'' + f' \Phi' + 2\alpha f \Phi'^2 + (n-1)\rho^{-1} f \Phi' - \alpha \Delta e^{4\alpha \Phi/(n-1)} \\
+ \alpha q^2 \rho^{2-2n} e^{-4\alpha \Phi(n-2)/(n-1)} = 0,
\]  

(14)

\[
(n-1)f'' + (n-1)^2 \rho^{-1} f' + 2(n-1)\alpha f' \Phi' + 4\Lambda e^{4\alpha \Phi/(n-1)} \\
+ 4(n-2)q^2 \rho^{2-2n} e^{-4\alpha \Phi(n-2)/(n-1)} = 0,
\]  

(15)

\[
(n-1)\rho^{-1} f' + 2\alpha f' \phi' + 2\alpha f \phi'' + 4\alpha(n-1)\rho^{-1} f \Phi' + 4\alpha^2 f \Phi'^2 \\
+ (n-1)(n-2)\rho^{-2} f + 2\Lambda e^{4\alpha \Phi/(n-1)} - 2q^2 \rho^{2-2n} e^{-4\alpha \Phi(n-2)/(n-1)} = 0,
\]  

(16)

\[
(n-1)f'' + (n-1)^2 \rho^{-1} f' + 2\alpha(n-1) f \phi' + 4\alpha(n-1) f \phi'' + 8\alpha(n-1)\rho^{-1} f \Phi' \\
+ 8(\alpha^2 + 1)f \Phi'^2 + 4\Lambda e^{4\alpha \Phi/(n-1)} + 4(n-2)q^2 \rho^{2-2n} e^{-4\alpha \Phi(n-2)/(n-1)} = 0,
\]  

(17)

where the prime denotes a derivative with respect to the \( \rho \) coordinate. The above equations (14)-(17) are coupled differential equations for the unknown functions \( f(\rho) \) and \( \Phi(\rho) \). Combining Eqs. (15) and (17), one can find an uncoupled differential equation for \( \Phi(\rho) \) as

\[
(n-1)\alpha (\rho \Phi'' + 2\Phi') + 2(1 + \alpha^2)\rho \Phi'^2 = 0,
\]  

(18)

with the solution

\[
\Phi(\rho) = \frac{(n-1)\alpha}{2(1 + \alpha^2)} \ln \left( \frac{\rho}{b} \right),
\]  

(19)
where \( b \) is an arbitrary constant. Substituting \( \Phi(\rho) \) given by Eq. (19) in the field equations (14)-(17) one finds the function \( f(\rho) \) as

\[
f(\rho) = \frac{2\Lambda(\alpha^2 + 1)b^{2\gamma}}{(n-1)(\alpha^2 - n)}\rho^{2(1-\gamma)} + \frac{m}{\rho^{(n-1)(1-\gamma)-1}} - \frac{2q^2(\alpha^2 + 1)b^{-2(n-2)\gamma}}{(n-1)(\alpha^2 + n-2)\rho^{2(n-2)(1-\gamma)}},
\]

where \( m \) is an arbitrary constant and \( \gamma = \alpha^2/(\alpha^2 + 1) \).

In order to study the general structure of these solutions, we first look for curvature singularities. It is easy to show that the Kretschmann scalar \( R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa} \) diverges at \( \rho = 0 \) and therefore one might think that there is a curvature singularity located at \( \rho = 0 \). However, as we will see below, the spacetime will never achieve \( \rho = 0 \). The function \( f(\rho) \) is negative for \( \rho < r_+ \) and positive for \( \rho > r_+ \), where \( r_+ \) is the largest root of \( f(\rho) = 0 \). Indeed, \( g_{\rho\rho} \) and \( g_{\phi\phi} \) are related by \( f(\rho) = g_{\rho\rho}^{-1} = l^2g_{\phi\phi} \), and therefore when \( g_{\rho\rho} \) becomes negative (which occurs for \( \rho < r_+ \)) so does \( g_{\phi\phi} \). This leads to apparent change of signature of the metric from \( (n-1)^+ \) to \( (n-2)^+ \) as one extends the spacetime to \( \rho < r_+ \). This indicates that we are using an incorrect extension. To get rid of this incorrect extension, we introduce the new radial coordinate \( r \) as

\[
r^2 = \rho^2 - r_+^2 \Rightarrow d\rho^2 = \frac{r^2}{r^2 + r_+^2}dr^2.
\]

With this new coordinate, the metric (11) is

\[
ds^2 = -\frac{r^2 + r_+^2}{l^2}R(r)dt^2 + l^2f(r)d\phi^2 + \frac{r^2}{(r^2 + r_+^2)f(r)}dr^2 + \frac{r^2 + r_+^2}{l^2}R(r)dX^2,
\]

where the coordinates \( r \) assumes the values \( 0 \leq r < \infty \), and \( f(r) \), \( R(r) \) and \( \Phi(r) \) are now given as

\[
f(r) = \frac{2\Lambda(\alpha^2 + 1)b^{2\gamma}}{(n-1)(\alpha^2 - n)}(r^2 + r_+^2)^{(1-\gamma)} + \frac{m}{(r^2 + r_+^2)^{(n-1)(1-\gamma)-1/2}} - \frac{2q^2(\alpha^2 + 1)b^{-2(n-2)\gamma}}{(n-1)(\alpha^2 + n-2)(r^2 + r_+^2)^{(n-2)(1-\gamma)}},
\]

\[
R(r) = \frac{b^{\gamma}}{(r^2 + r_+^2)^{\gamma/2}}, \quad \Phi(r) = \frac{(n-1)\alpha}{4(1 + \alpha^2)} \ln \left( \frac{b^2}{r^2 + r_+^2} \right).
\]

The metric (21) is neither asymptotically flat nor (anti)-de Sitter. One can easily show that the Kretschmann scalar does not diverge in the range \( 0 \leq r < \infty \). However, the spacetime has a conic geometry and has a conical singularity at \( r = 0 \). In fact, using a Taylor expansion, the metric (21) is written as

\[
ds^2 = -b^{\gamma}r^{2-\gamma}_{+}dt^2 + G^{-1}_{+}dr^2 + l^2Gr^2d\phi^2 + b^{\gamma}r^{2-\gamma}_{+}dX^2,
\]
where
\[ G = \frac{\Lambda(\alpha^2 + 1) b^{2\gamma} ((n + 1)(1 - \gamma) - 1)}{(n - 1)(\alpha^2 - n) \gamma^2} \]
\[ + \frac{q^2 (\alpha^2 + 1)^2 b^{(4-2n)\gamma} ((n - 3)(1 - \gamma) + 1)}{(n - 1)(\alpha^2 + n - 2) \gamma^2} \].

(25)

Indeed, there is a conical singularity at \( r = 0 \) since:
\[ \lim_{r \to 0} \frac{1}{r} \sqrt{\frac{g_{\varphi\varphi}}{g_{rr}}} \neq 1. \]

(26)

That is, as the radius \( r \) tends to zero, the limit of the ratio “circumference/radius” is not \( 2\pi \) and therefore the spacetime has a conical singularity at \( r = 0 \). The canonical singularity can be removed if one identifies the coordinate \( \varphi \) with the period
\[ \text{Period}_\varphi = 2\pi \left( \lim_{r \to 0} \frac{1}{r} \sqrt{\frac{g_{\varphi\varphi}}{g_{rr}}} \right)^{-1} = 2\pi (1 - 4\mu), \]

(27)

where \( \mu \) is given by
\[ \mu = \frac{1}{4} \left[ 1 - \left( \frac{1}{2} \frac{ml(\alpha^2 + n - 2)}{\alpha^2 + 1} r^{(n-1)(\gamma-1)} + \frac{2(1 + \alpha^2)}{(\alpha^2 - n) \Lambda b^{2\gamma + 1 - 2\gamma}} \right)^{-1} \right]. \]

(28)

From Eqs. (21)-(28), one concludes that near the origin \( r = 0 \), the metric (21) describes a spacetime which is locally flat but has a conical singularity at \( r = 0 \) with a deficit angle \( \delta \varphi = 8\pi \mu \). Since near the origin the metric (21) for \( n = 3 \) is identical to the spacetime generated by a cosmic string, by use of Vilenkin procedure [35], one can show that \( \mu \) of Eq. (28) can be interpreted as the mass per unit length of the string.

Now we investigate the casual structure of the spacetime. As one can see from Eq. (22), there is no solution for \( \alpha = \sqrt{n} \) with a Liouville potential (\( \Lambda \neq 0 \)). The cases with \( \alpha > \sqrt{n} \) and \( \alpha < \sqrt{n} \) should be considered separately. For \( \alpha > \sqrt{n} \), as \( r \) goes to infinity the dominant term in Eq. (22) is the second term, and therefore the function \( f(r) \) is positive in the whole spacetime, despite the sign of the cosmological constant \( \Lambda \), and is zero at \( r = 0 \). Thus, the solution given by Eqs. (21) and (22) exhibits a spacetime with conic singularity at \( r = 0 \). For \( \alpha < \sqrt{n} \), the dominant term for large values of \( r \) is the first term, and therefore the function \( f(r) \) given in Eq. (22) is positive in the whole spacetime only for negative values of \( \Lambda \). In this case the solution represents a spacetime with conic singularity at \( r = 0 \). The solution is not acceptable for \( \alpha < \sqrt{n} \) with positive values of \( \Lambda \), since the function \( f(r) \) is negative for large values of \( r \). Of course, one may ask for the completeness of the spacetime with \( r \geq 0 \) (or \( \rho \geq r_+ \)) [23, 36]. It is easy to see that the spacetime described by Eq. (21) is both null and timelike geodesically complete. In fact, we can show that
every null or timelike geodesic starting from an arbitrary point can either extend to infinite values of the affine parameter along the geodesic or end on a singularity at $r = 0$. Using the geodesic equation, one obtains

$$i = \frac{l^2}{b^2\gamma(r^2 + r_+^2)^{1-\gamma}}, \quad \dot{x}^i = \frac{l^2}{b^2\gamma(r^2 + r_+^2)^{1-\gamma}} P^i, \quad \dot{\phi} = \frac{1}{l^2f(r)} L,$$

(29)

$$r^2\dot{r}^2 = (r^2 + r_+^2)f(r) \left[ \frac{l^2(E^2 - P^2)}{b^2\gamma(r^2 + r_+^2)^{1-\gamma}} - \eta \right] - \frac{r^2 + r_+^2}{l^2} L^2,$$

(30)

where the dot denotes the derivative with respect to an affine parameter and $\eta$ is zero for null geodesics and +1 for timelike geodesics. $E$, $L$, and $P^i$ are the conserved quantities associated with the coordinates $t$, $\phi$, and $x^i$ respectively, and $P^2 = \sum_{i=1}^{n-2}(P^i)^2$. Notice that $f(r)$ is always positive for $r > 0$ and zero for $r = 0$.

First we consider the null geodesics ($\eta = 0$). (i) If $E^2 > P^2$ the spiraling particles ($L > 0$) coming from infinity have a turning point at $r_{tp} > 0$, while the nonspiraling particles ($L = 0$) have a turning point at $r_{tp} = 0$. (ii) If $E = P$ and $L = 0$, whatever is the value of $r$, $\dot{r}$ and $\dot{\phi}$ vanish and therefore the null particles move in a straight line in the $(n-2)$-dimensional submanifold spanned by $x^1$ to $x^{n-2}$. (iii) For $E = P$ and $L \neq 0$, and also for $E^2 < P^2$ and any value of $L$, there is no possible null geodesic.

Second, we analyze the timelike geodesics ($\eta = +1$). Timelike geodesics are possible only if $l^2(E^2 - P^2) > b^2\gamma r_+^2(1-\gamma)$. In this case the turning points for the nonspiraling particles ($L = 0$) are $r^1_{tp} = 0$ and $r^2_{tp}$ given as

$$r^2_{tp} = \sqrt{[b^{-2\gamma}l^2(E^2 - P^2)]^{1/(1-\gamma)} - r_+^2},$$

(31)

while the spiraling ($L \neq 0$) timelike particles are bound between $r^a_{tp}$ and $r^b_{tp}$ given by

$$0 < r^a_{tp} \leq r^b_{tp} < r^2_{tp}.$$

(32)

Therefore, we have confirmed that the spacetime described by Eq. (21) is both null and timelike geodesically complete.

### B. Longitudinal magnetic field solutions with all rotation parameters

Our aim here is to obtain the $(n+1)$-dimensional longitudinal magnetic field solutions with a complete set of rotation parameters. The rotation group in $n+1$ dimensions is $SO(n)$ and therefore the number of independent rotation parameters is $[n/2]$, where $[x]$ is the integer part of $x$. We now
generalize the above metric given in Eq. (21) with \( k \leq \lfloor n/2 \rfloor \) rotation parameters. This generalized solution can be written as

\[
\begin{align*}
    ds^2 &= -\frac{r^2 + r^2_+}{l^2} R^2(r) \left( \Xi dt - \sum_{i=1}^{k} a_i d\phi^i \right)^2 + f(r) \left( \sqrt{\Xi^2 - 1} dt - \frac{\Xi}{\sqrt{\Xi^2 - 1}} \sum_{i=1}^{k} a_i d\phi^i \right)^2 \\
    &+ \frac{r^2 dr^2}{(r^2 + r^2_+) f(r)} + \frac{r^2 + r^2_+}{l^2 (\Xi^2 - 1)} R^2(r) \sum_{i<j} (a_i d\phi_j - a_j d\phi_i)^2 + \frac{r^2 + r^2_+}{l^2} R^2(r) dX^2,
\end{align*}
\]

(33)

where \( \Xi = \sqrt{1 + \sum_i a^2_i / l^2} \), \( dX^2 \) is the Euclidean metric on the \((n-k-1)\)-dimensional submanifold and \( f(r) \) and \( R(r) \) are the same as given in Eq. (22). The gauge potential is

\[
A_\mu = \frac{q b^{(3-n)\gamma}}{\Gamma (r^2 + r^2_+)^{\gamma/2}} \left( \sqrt{\Xi^2 - 1} \delta^t_\mu - \frac{\Xi}{\sqrt{\Xi^2 - 1}} a_i \delta^\mu_i \right) \quad \text{(no sum on } i) \tag{34}
\]

where \( \Gamma = (n-3)(1-\gamma) + 1 \). Again this spacetime has no horizon and curvature singularity. However, it has a conical singularity at \( r = 0 \). One should note that these solutions reduce to those discussed in [24], in the absence of dilaton field \((\alpha = \gamma = 0)\) and those presented in [31] for \( n = 3 \).

Now we calculate conserved quantities of these solutions. Denoting the volume of the hypersurface boundary at constant \( t \) and \( r \) by \( V_{n-1} = (2\pi)^k \Sigma_{n-k-1} \), the mass and angular momentum per unit volume \( V_{n-1} \) of the branes \((\alpha < \sqrt{n})\) can be calculated through the use of Eqs. (8) and (9). We find

\[
\begin{align*}
    M &= \frac{b^{(n-1)\gamma}}{16\pi l^{n-2}} \frac{(n - \alpha^2) \Xi^2 - (n - 1)}{1 + \alpha^2} m, \tag{35} \\
    J_i &= \frac{b^{(n-1)\gamma}}{16\pi l^{n-2}} \frac{n - \alpha^2}{1 + \alpha^2} \Xi m a_i. \tag{36}
\end{align*}
\]

For \( a_i = 0 \ (\Xi = 1) \), the angular momentum per unit volume vanishes, and therefore \( a_i \)'s are the rotational parameters of the spacetime. Of course, one may note that in the particular case \( n = 3 \), these conserved charges reduce to the conserved charges of the magnetic rotating black string obtained in Ref. [31], and in the absence of dilaton field \((\alpha = \gamma = 0)\) they reduce to those of Ref. [24].

C. The Angular Magnetic Field Solutions

In subsection III A, we found a spacetime generated by a magnetic source which produces a longitudinal magnetic field along \( x^i \) coordinates. Now, we want to obtain a spacetime generated by a magnetic source that produces angular magnetic fields along the \( \phi^i \) coordinates. Following the steps of Subsection III A but now with the roles of \( \phi \) and \( x \) interchanged, we can directly write
the metric and vector potential satisfying the field equations (14)-(17) as

\[ ds^2 = -\frac{r^2 + r_+^2}{l^2}R^2(r)dt^2 + \frac{r^2 dr^2}{(r^2 + r_+^2)f(r)} + f(r)dx^2 + (r^2 + r_+^2)R^2(r)d\Omega^2, \]

where \( d\Omega^2 = \sum_{i=1}^{n-2}(d\phi^i)^2 \) and \( f(r) \) and \( R(r) \) are given in Eq. (22). The angular coordinates \( \phi^i \)'s range in \( 0 \leq \phi^i < 2\pi \). The gauge potential is now given by

\[ A_\mu = \frac{q b^{(3-n)\gamma}}{\Gamma(r^2 + r_+^2)^{1/2}} \delta^x_\mu, \]

The Kretschmann scalar does not diverge for any \( r \) and therefore there is no curvature singularity. The spacetime (37) is also free of conic singularity. In addition, it is notable to mention that the radial geodesic passes through \( r = 0 \) (which is free of singularity) from positive values to negative values of the coordinate \( r \). This shows that the radial coordinate in Eq. (37) can take the values \(-\infty < r < \infty\). This analysis may suggest that one is in the presence of a traversable wormhole with a throat of dimension \( r_+ \). However, in the vicinity of \( r = 0 \), the metric (37) can be written as

\[ ds^2 = -b^{\gamma} \frac{r^2 - \gamma}{l^2}dt^2 + \frac{G^{-1}}{r^2_+}dr^2 + Gr^2 dx^2 + b^{\gamma} r_+^{2-\gamma}d\Omega^2, \]

where \( G \) is given in Eq. (25). Equation (39) shows that, at \( r = 0 \), the \( x \) direction collapses and therefore we have to abandon the wormhole interpretation.

To add linear momentum to the spacetime along the coordinate \( x_i \), we perform the boost transformation

\[ t \mapsto \Xi t - (v_i/l)x^i; \quad x^i \mapsto \Xi x^i - (v_i/l)t \quad \text{(no sum on } i). \]

in the \( t - x_i \) plane, where \( v_i \) is a boost parameter and \( \Xi = \sqrt{1 + \sum_i v_i^2/l^2} \). One obtains

\[ ds^2 = -\frac{r^2 + r_+^2}{l^2}R^2(r) \left( \Xi dt - l^{-1} \sum_{i=1}^\kappa v_i dx^i \right)^2 + f(r) \left( \sqrt{\Xi^2 - 1} dt - \frac{\Xi}{l\sqrt{\Xi^2 - 1}} \sum_{i=1}^\kappa v_idx^i \right)^2 + \frac{r^2 + r_+^2}{l^4(\Xi^2 - 1)}R^2(r) \sum_{i<j} (v_i dx_j - v_j dx_i)^2 + \frac{r^2 dr^2}{(r^2 + r_+^2)f(r)} + (r^2 + r_+^2)R^2(r)d\Omega^2, \]

The gauge potential is given by

\[ A_\mu = \frac{q b^{(3-n)\gamma}}{\Gamma(r^2 + r_+^2)^{1/2}} \left( \sqrt{\Xi^2 - 1} \delta^i_\mu - \frac{\Xi}{l\sqrt{\Xi^2 - 1}} v_i \delta^i_\mu \right) \quad \text{(no sum on } i). \]
This boost transformation is permitted globally since \( x^i \) is not an angular coordinate. Although the boosted solution (40) generates an electric field, it is not a new solution. The conserved quantities of the spacetime (40) are the mass and linear momentum. The mass and linear momentum per unit volume \( V_{n-1} \) of the branes (\( \alpha < \sqrt{n} \)) can be calculated through the use of Eqs. (8) and (10).

We find

\[
M = \frac{b^{(n-1)\gamma}}{16\pi l^{n-2}} \left( \frac{(n-\alpha^2)\Xi^2 - (n-1)}{1 + \alpha^2} \right) m, \tag{42}
\]

\[
P_i = \frac{b^{(n-1)\gamma}}{16\pi l^{n-2}} \left( \frac{n-\alpha^2}{1 + \alpha^2} \right) \Xi mv_i. \tag{43}
\]

Finally, we calculate the electric charge of the solutions (33) and (40) obtained in this section. To determine the electric field we should consider the projections of the electromagnetic field tensor on special hypersurfaces. The normal to such hypersurfaces is

\[
u^0 = \frac{1}{N}, \quad \nu^r = 0, \quad \nu^i = -\frac{V^i}{N},
\]

and the electric field is \( E^\mu = g^{\mu\rho} \exp \left[ -4\alpha\phi/(n-1) \right] F_{\rho\nu} \nu^\nu \). Then the electric charge per unit volume \( V_{n-1} \) can be found by calculating the flux of the electric field at infinity, yielding

\[
Q = \frac{\sqrt{\Xi^2 - 1} q}{4\pi l^{n-2}}. \tag{44}
\]

Note that the electric charge of the system per unit volume is proportional to the magnitude of rotation parameters or boost parameters, and is zero for the case of a static solution. This result is expected since now, besides the magnetic field along the \( \phi^i \) (\( x^i \), for the case of angular magnetic field) coordinates, there is also a radial electric field (\( F_{tr} \neq 0 \)). To give a physical interpretation for the appearance of the net electric charge, we first consider the static spacetime. The magnetic field source can be interpreted as composed of equal positive and negative charge densities, where one of the charge density is at rest and the other one is spinning (travelling, in the case of angular magnetic field). Clearly, this system produce no electric field since the net electric charge density is zero, and the magnetic field is produced by the rotating (travelling) electric charge density. Now, we consider the rotating (travelling) solutions. From the point of view of an observer at rest relative to the source (\( S \)), the two charge densities are equal, while from the point of view of an observer \( S' \) that follows the intrinsic rotation (translation) of the spacetime, the positive and negative charge densities are not equal, and therefore the net electric charge of the spacetime is not zero.
IV. CONCLUDING REMARKS

The counterterm method inspired by the AdS/CFT correspondence has been widely used for the computation of the finite action and conserved quantities of asymptotically AdS solutions of Einstein gravity. Testing the validity of the counterterm method for nonasymptotically AdS spacetimes needs new solutions which are not asymptotically AdS. In this paper, we obtained two new classes of \((n+1)\)-dimensional exact magnetic rotating solutions of Einstein-Maxwell dilaton gravity in the presence of Liouville-type potential, which are neither asymptotically flat nor AdS. These solutions which are ill-defined for \(\alpha = \sqrt{n}\) reduce to the horizonless rotating solutions of [24] for \(\alpha = 0\), while for \(n = 3\), they reduce to the four-dimensional magnetic dilaton string presented in [31]. The first class of solutions represent a \((n+1)\)-dimensional spacetime with a longitudinal magnetic field. We found that these solutions have no curvature singularities and no horizons, but have conic singularity at \(r = 0\). We also confirmed that these solutions are both null and timelike geodesically complete. In fact, we showed that every null or timelike geodesic starting from an arbitrary point can either extend to infinite values of the affine parameter along the geodesic or end on a singularity at \(r = 0\). In these spacetimes, when all the rotation parameter are zero (static case), the electric field vanishes, and therefore the brane has no net electric charge. For the spinning brane, when one or more rotation parameters are nonzero, the brane has a net electric charge density which is proportional to the magnitude of rotation parameter given by \(\sum a_i^2\). The second class of solutions represent a spacetime with angular magnetic field. These solutions have no curvature singularity, no horizon, and no conic singularity. Although a boost transformation gives a solution with linear momentum which generates an electric field, it is not a new solution. This is due to the fact that the boost transformation is permitted globally. We also showed that, for the case of traveling brane with one or more nonzero boost parameters, the net electric charge of the brane is proportional to the magnitude of the velocity of the brane \((\sum v_i^2)\). Finally, we used the counterterm method and calculated the conserved quantities of the two classes of solutions which are nonasymptotically AdS. These calculations show that one may use the counterterm method for calculating the conserved quantities of these nonasymptotically AdS spacetimes.
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