Supersymmetric Intersecting Domain Walls in Massive Hyper-Kähler Sigma Models

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ABSTRACT

The general scalar potential of D-dimensional massive sigma-models with eight supersymmetries is found for $D = 3, 4$. These sigma models typically admit $1/2$ supersymmetric domain wall solutions and we find, for a particular hyper-Kähler target, exact $1/4$ supersymmetric static solutions representing a non-trivial intersection of two domain walls. We also show that the intersecting domain walls can carry Noether charge while preserving $1/4$ supersymmetry. We briefly discuss an application to the D1-D5 brane system.

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1 Introduction

Supersymmetric four-dimensional (D=4) field theories with multiple isolated supersymmetric vacua typically admit 1/2 supersymmetric (BPS) domain wall solutions that interpolate between these vacua. The Wess-Zumino model provides a simple example with N=1 supersymmetry; the critical points of its superpotential are isolated supersymmetric vacua and the minimal tension domain walls that separate them are BPS [1, 2]. Examples with N=2 supersymmetry are provided by certain massive sigma models; these have no superpotential but they do have a potential proportional to the square of a tri-holomorphic Killing vector field (KVF) of the hyper-Kähler (HK) target space [3]. Fixed points of this KVF are supersymmetric vacua and models with multiple fixed points admit 1/2 supersymmetric domain walls [4]. Recently it has been discovered that N=1 models for which the superpotential has at least three critical points typically admit intersecting domain wall solutions preserving 1/4 supersymmetry [5, 6, 7, 8, 9]. The aim of this paper is to show that there is a class of massive N=2 supersymmetric sigma models that similarly admit 1/4 supersymmetric intersecting domain wall solutions.

To see why such configurations might be expected, consider a massive sigma model with a target space that is the direct product of two 4-dimensional HK target spaces, each admitting a tri-holomorphic KVF. We then have two non-interacting sigma models, each with its potential proportional to a tri-holomorphic KVF, and a domain wall solution of one model can be superposed with a domain wall solution of the other model. Since each domain wall separately preserves 1/2 supersymmetry, experience suggests that some superposition will preserve 1/4 supersymmetry, and we shall confirm this intuition. Of course, this example is a trivial one because the intersection is purely geometrical. We are principally interested in massive sigma models with irreducible target spaces for which simple superposition fails; in this case any geometrical intersection must also be a physical one in the sense that each domain wall must deform the other near the point of intersection. Nevertheless, the trivial case suggests that we should consider HK target spaces admitting at least two linearly independent triholomorphic KVFs, which requires the HK target space to be at least 8-dimensional.

We shall consider the class of toric HK 4n-metrics, which admit n linearly independent commuting KVFs. In coordinates (ψI, XJ) (I = 1, ..., n), these metrics are determined by an n × n symmetric matrix U with entries UIJ that are functions only of the 3n coordinates XI. The metric takes the form

\[ ds^2 = U_{IJ} dX^I \cdot dX^J + U^{IJ}(d\psi_I + A_I)(d\psi_J + A_J), \]  

(1)

where \( U^{IJ} \) are the entries of \( U^{-1} \) and

\[ A_I = dX^J \cdot \omega_{JI}, \quad \nabla_J (\omega_K I) = \nabla_{JI} U_{KI}. \]  

(2)

This last equation implies that \( \nabla_J U_{KI} = 0 \). The triplet of Kahler 2-forms is

\[ \Omega = (d\psi_I + A_I) dX^I - \frac{1}{2} U_{IJ} dX^I \times dX^J, \]  

(3)

where the wedge product of forms is implicit. It follows that the n Killing vector fields

\[ k^I = \partial / \partial \psi_I \]  

(4)

are tri-holomorphic.

For n = 1 we get the well-known multi-centre 4-metrics; the simplest two-centre case is the Eguchi-Hanson 4-metric, and in this case the kink solution of the corresponding massive sigma model is known explicitly [10]. Consider now the n = 2 HK 8-metrics.
If the $2 \times 2$ matrix $U$ is diagonal then the 8-metric is a direct product of two HK 4-metrics; otherwise, it is irreducible. In either case, a potential equal to the square of a tri-holomorphic KVF must take the form

$$V = \frac{1}{2} |k^I|^2$$

(5)

for some constants $\lambda_I$. When $U$ is diagonal the KVFs $k^1$ and $k^2$ are orthogonal, so the potential can be written as

$$V = \frac{1}{2} \lambda_1^2 |k^1|^2 + \frac{1}{2} \lambda_2^2 |k^2|^2,$$

(6)

as expected for a Lagrangian that is the sum of two non-interacting massive sigma-models with 4-dimensional HK target spaces. For this special case, the potential (5) is the most general one compatible with the $\mathbb{N}=2 \, D=4$ supersymmetry. As discussed above, it leads to a model that allows a trivial, but $1/4$ supersymmetric, intersection of kink domain walls. One might hope that a sigma-model with an irreducible HK 8-metric and a potential of the form (5) might admit a supersymmetric solution representing a genuinely physical intersection of domain walls, but this seems not to be the case.

Fortunately, when $U$ is non-diagonal the potential (5) is not the most general one allowed by $\mathbb{N}=2 \, D=4$ supersymmetry. As we shall show, the general potential is the the sum of squares of two independent tri-holomorphic KVFs. For the toric HK models this means that

$$V = \frac{1}{2} \lambda_1^2 |k^1|^2 + \frac{1}{2} \rho_I \rho J |k^J|^2$$

(7)

where $\lambda_I$ and $\rho_I$ are two constant $n$-vectors. Equivalently,

$$V = \frac{1}{2} \mu_{IJ} U^{IJ}$$

(8)

where

$$\mu_{IJ} = \lambda_I \lambda_J + \rho_I \rho J.$$  

(9)

In other words, the constants $\mu_{IJ}^2$ are the entries of a constant symmetric mass-squared matrix $\mu^2$ of maximal rank 2. For $n = 2$, this means that $\mu^2$ is an arbitrary constant symmetric matrix, so the general scalar potential of a sigma model with a generic 8-dimensional HK target space depends on three parameters. For the special case of a product HK 8-manifold the number of parameters is reduced to two; these can be taken to be the constants $\lambda_1$ and $\lambda_2$ in (5), which is therefore the general potential. In contrast, for an irreducible 8-manifold the potential (5) is the special case of (8) for which the matrix $\mu^2$ has non-maximal rank.

For any massive sigma model with an 8-dimensional HK target space the supersymmetric vacua for generic mass-squared matrix $\mu^2$ are in 1-1 correspondence with a finite set of points in $\mathbb{E}^3 \times \mathbb{E}^3$ at which the $2 \times 2$ matrix $U^{-1}$ equals the zero matrix. These vacua will be interconnected via $1/2$ supersymmetric domains. As we shall see, the additional parameter in the potential for irreducible target spaces allows the construction of models admitting $1/4$ supersymmetric domain wall intersections. We shall discuss a particular one-parameter family of such models in detail, obtaining an explicit and exact $1/4$ supersymmetric solution representing a non-trivial intersection of two domain walls.

The above result for the general potential compatible with $D=4 \, N=2$ supersymmetry is a special case of an analogous result for the general scalar potential of a D-dimensional sigma model with eight supersymmetries. This potential is the sum of squares of $(6-D)$

1This is true both classically and in the quantum theory because instanton solutions interpolating between these vacua are Euclidean kinks with an even number of fermion zero modes.
linearly independent commuting tri-holomorphic KVFs. This was conjectured by Bak et al. \cite{10} on the basis of their result that the D=1 sigma model, i.e. quantum mechanics, admits a potential equal to the sum of squares of five independent commuting tri-holomorphic KVFs, and the observation that this might be explained by a five-fold non-trivial dimensional reduction on T^5. Indeed, a non-trivial dimensional reduction of the, necessarily massless, D=6 supersymmetric sigma model is known to yield a D=5 supersymmetric sigma model with a superpotential equal to the square of a tri-holomorphic KVF \cite{11}, but it was not previously appreciated that a further non-trivial dimensional reduction using other tri-holomorphic KVFs might yield more general potentials. Here we establish, by a direct determination of the conditions implied by N=2 D=4 and N=4 D=3 supersymmetry that this procedure indeed yields the general scalar potential compatible with eight supersymmetries.

Of course, a potential can be constructed from \((6 - D)\) independent commuting tri-holomorphic KVFs only in those models for which the space spanned by a maximally commuting set of tri-holomorphic KVFs has a dimension of at least \((6 - D)\). For the 4n-dimensional toric HK manifolds this means that we need \(n \geq (6 - D)\), but it is reasonable to suppose that most of the physics is captured by the minimal \(n = 6 - D\) case. This means that we should consider \(n = 2\) in D=4, as we have been doing, but that we should consider \(n = 3\) in D=3. One might suppose that there will now be triple intersections of domain walls preserving only 1/8 supersymmetry, but the reduced space dimensionality does not allow it. Instead, one can use the additional freedom to find charged 1/4-supersymmetric intersecting domain wall solutions. These have a D=4 interpretation as intersecting walls with a wave along the string intersection; the momentum along the string becomes the charge on reduction to D=3. These charged D=3 solutions are stationary rather than static; i.e. they are time-dependent but such that the energy density is time independent. They include, as a special case, new stationary charged domain wall solutions preserving 1/4 supersymmetry (in contrast to the Q-kinks which preserve 1/2 supersymmetry). In D=3 one can thus break 1/2 supersymmetry to 1/4 supersymmetry either by intersection with another domain wall or by the addition of charge, or both.

We begin with a derivation of the scalar potentials allowed in supersymmetric HK sigma models. We then focus on the toric HK manifolds and derive the BPS equations satisfied by sigma model configurations that preserve some fraction of supersymmetry. We then apply these results to find both static and stationary intersecting brane configurations. In the concluding section we briefly discuss the relevance of these results to the D1-D5 brane system in type IIB string theory.

2 Hyper-Kähler supersymmetric sigma models

Massless supersymmetric hyper-Kahler sigma models exist in all spacetime dimensions \(D \leq 6\). All such models in \(D \geq 3\) have eight supercharges. This is obvious for \(D = 5, 6\) because the minimal spinor in these dimensions has eight real components. In D=3 the minimal spinor has 2 components, but each of the three complex structures yields an additional supersymmetry, leading to an N=4 D=3 supersymmetric sigma model, which again has eight supercharges. In D=4 we have an N=2 model with two 4-component spinor supercharges.

A D=6 supersymmetric sigma model is necessarily massless as the chirality of the hypermultiplet spinor allows neither a mass term, nor a Yukawa coupling to the scalar fields. This argument does not apply in D=5 however, and a ‘massive’ D=5 supersymmetric sigma model can be found by non-trivial dimensional reduction of the D=6 model\footnote{This type of dimensional reduction was introduced by Scherk and Schwarz \cite{12} but their idea was to use it to obtain supersymmetry breaking mass terms, whereas we are using it here to obtain supersym-}.
if the HK target space admits a tri-holomorphic KVF \( \zeta \). Specifically, given \( 4n \) real fields \( \phi^X \), and a mass parameter \( \mu \), one sets
\[
\frac{\partial \phi^X}{\partial x^5} = \mu \zeta^X \quad (X = 1, \ldots, 4n)
\] (10)
where \( x^5 \) is the coordinate of the 5th spatial direction on which we reduce\(^3\). This produces a scalar potential proportional to the square of the norm of \( \zeta \). Given a further tri-holomorphic KVF we could repeat this procedure (as suggested in [10]) to obtain a potential in D=4 that is the sum of squares of two tri-holomorphic KVFs, and then in D=3 to obtain a potential that is the sum of three tri-holomorphic KVFs, but one should expect supersymmetry to require some restriction on the sets of tri-holomorphic KVFs that can be used for this purpose. Here we shall show, by a direct construction of the general massive D=3 and D=4 supersymmetric sigma models with eight supercharges, that the tri-holomorphic KVFs used in this procedure must commute. The same calculation also shows that this procedure yields the general scalar potential.

We shall first review some features of HK manifolds that will be needed in the construction, and the corresponding D=6 massless sigma models. We then turn to the construction of the D=4 massive HK sigma models, followed by the D=3 case.

### 2.1 Hyper-Kahler manifolds

As above, we let \( \phi^X \) be the real coordinates of a \( 4n \)-dimensional HK manifold. The HK metric can be written in terms of a vielbein \( f^X_{ia} \) transforming in the \( (2, 2n) \) representation of \( Sp_1 \times Sp_n \), as
\[
g_{XY} = f^X_{ia} f^Y_{jb} \Omega_{ij} \varepsilon_{ab} \quad (i = 1, \ldots, 2n; \ a = 1, 2)
\] (11)
where \( \Omega_{ij} \) is the real antisymmetric \( Sp_n \)-invariant tensor (numerically equal to \( \Omega^{ij} \)) and \( \varepsilon_{ab} \) is the real antisymmetric invariant tensor of \( Sp_1 \cong SU(2) \). The vielbein obeys the pseudo-reality condition
\[
(f^X_{ia})^* = f_{X^{ia}} = f^X_{ib} \Omega_{ji} \varepsilon_{ba}.
\] (12)
We may define
\[
f^X_{ia} = g^{XY} f^Y_{ia}
\] (13)
where \( g^{XY} \) is the inverse of the HK metric, in which case
\[
f^X_{ia} f^X_{ib} = \delta^i_j \delta^a_b.
\] (14)

Note the further identity
\[
f^X_{ia} f^Y_{ib} = \frac{1}{2} \left[ \delta^X_Y \delta^a_b + i \sigma^a \cdot I^Y \right]
\] (15)
where \( \sigma \) is the triplet of Hermitian Pauli matrices and \( I \) is the triplet of complex structures. Given [14] this identity can be established by showing that
\[
I^X = -if^X_{ia} f^Y_{ib} \sigma^a \cdot I^Y
\] (16)
A simple computation using this formula shows that
\[
[n \cdot I][m \cdot I] = -n \cdot m + n \times m \cdot I
\] (17)

\(^3\)The corresponding \( x^5 \)-dependence of the fermion fields can be found by interpreting this equation as a superfield equation.
Taking \( \mathbf{n} \) and \( \mathbf{m} \) to be any two of three orthonormal vectors one recovers the algebra of quaternions.

Note that the triplet of Kähler 2-forms associated with this quaternionic structure is

\[
\Omega = -\frac{i}{2} d\phi^X \wedge d\phi^Y f^a_X f^b_Y \sigma_a \sigma_b .
\]  

A tri-holomorphic KVF \( \zeta \) is one for which

\[
\mathcal{L}_\zeta \Omega = 0 .
\]

Given the Killing condition \( \zeta(X;Y) = 0 \), this is equivalent to the constraint

\[
\zeta_{[i a; j b]} = 0
\]

where \( \zeta^a = \zeta^X f^a_X \). Together with the Killing condition, this implies that the only non-vanishing component of \( \zeta_{X;Y} \) is the symmetric second-rank \( Sp_n \) tensor \( \zeta_{ia;jb} \).

Finally, we observe that the spin-connection, defined by requiring

\[
\mathcal{D}[X f^a_Y]_{ia} = 0
\]

takes values in \( Sp_n \) for a HK manifold, and therefore has the form

\[
\omega_X^{ia;jb} = \omega_X^{ij} \delta^{a}_b
\]

with \( \omega_X^{i,j} = 0 \). It follows that the curvature tensor takes the form

\[
R_{ia;jb;kc;ld} = R_{ijkl} \epsilon^{a}_{ab} \epsilon^{c}_{cd} ;
\]

where \( R_{ijkl} \) is a totally-symmetric \( Sp_n \) tensor.

### 2.2 D=6 HK sigma models

The \( D=6 \) sigma model has a Lagrangian given, in \( SU^*(4) \) spinor notation, by

\[
\mathcal{L} = \frac{1}{8} g_{XY} \partial^\alpha \phi^X \partial\alpha \phi^Y - i \chi_{a i} \mathcal{D}^{\alpha \beta} \chi^X_{\alpha} - \frac{1}{12} R_{ijkl} \chi^X_{\alpha} \chi^X_{j} \chi^X_{k} \chi^X_{l} \epsilon^{\alpha \beta \gamma \delta}
\]

where \( \epsilon^{\alpha \beta \gamma \delta} \) is the \( SU^*(4) \) invariant antisymmetric tensor, and we take the Minkowski metric to have ‘mostly plus’ signature. The supersymmetry transformations are given by

\[
\delta \phi^X = i f^a_X \epsilon^{a}_{a} \chi^i_{a}
\]

\[
\delta \chi^i_{\alpha} = f^a_X \partial_{\alpha \beta} \phi^X \epsilon^a_{\beta} - \delta \phi^X \omega_X^{i,j} \chi^j_{\alpha}
\]

where \( \epsilon^a_{\beta} \) is a constant \( Sp_1 \times SU^*(4) \) spinor parameter.

For later use we derive the conditions for a bosonic configuration of the \( D=6 \) supersymmetry sigma model to preserve supersymmetry. We require the supersymmetry variations of the \( D=6 \) fermion fields to vanish. Written in standard Dirac matrix form, with Lorentz spinor indices suppressed, this condition becomes

\[
\Gamma^m f^a_X \partial_m \phi^X \epsilon^a = 0 ,
\]

where \( \Gamma^m \) \( (m = 0, 1, \ldots, 5) \) are the \( D=6 \) Dirac matrices and \( \epsilon^a \) is an \( Sp_n \)-Majorana and Weyl spinor; the latter condition implies that

\[
\Gamma^{012345} \epsilon = \epsilon .
\]
Multiplying (27) by \( f^Y_{ib} \), and using the identity (15), we find that this is equivalent to
\[
\Gamma^m \left[ \partial_m \phi^X + \partial_m \phi^Y i \sigma \cdot I^X_Y \right] \epsilon = 0 \tag{29}
\]
where we now suppress the \( Sp_1 \) indices. Equivalently
\[
\Gamma^m \epsilon \partial_m \phi^Y g_{XY} = i \Gamma^m \epsilon \cdot \Omega_{XY} \partial_m \phi^Y. \tag{30}
\]
The dimension of the space of solutions for constant \( \epsilon \) is the number of supersymmetries preserved.

### 2.3 D=4 HK sigma models

The fermion fields of a D=4 HK sigma model consist of a set of 2n complex 2-component \( SL(2; \mathbb{C}) \) spinors \( \chi^i_\alpha \) (\( \alpha = 1, 2 \)) and their complex conjugates \( \bar{\chi}^i_{\dot{\alpha}} \). The general massive sigma model has a Lagrangian of the form
\[
\mathcal{L} = \frac{1}{4} g_{XY} \partial_{\alpha} \phi^X X^Y \partial_{\dot{\alpha}} \bar{\phi}^Y + \frac{i}{4} \chi^i_\alpha D^X \bar{\chi}^i_{\dot{\alpha}} + \frac{i}{4} \chi^i_{\dot{\alpha}} M_{ij} - \frac{i}{4} \bar{\chi}^i_{\dot{\alpha}} \bar{\chi}^j_{\dot{\beta}} M_{ij} - V \tag{31}
\]
where we assume a ‘mostly-plus’ Minkowski signature and use conventions for which
\[
\varepsilon_{\beta^\gamma} \varepsilon_{\alpha^\gamma} = \delta^\alpha_\beta, \quad \partial^\alpha \partial^\beta = -2 \Box. \tag{32}
\]
The complex Yukawa coupling tensor \( M_{ij} \), and the potential \( V \) are expressed in terms of a complex vector field \( N \) appearing in the supersymmetry transformations,
\[
\delta \phi^X = \frac{1}{2} g_{XY} N^X \bar{N}^Y, \quad \delta \chi^i_\alpha = \frac{1}{2} g_{XY} N^X \bar{N}^Y. \tag{33}
\]
Specifically, one finds that
\[
M_{ij} = N_{ia} N^a_{j}, \quad V = \frac{1}{2} g_{XY} N^X \bar{N}^Y. \tag{35}
\]
Supersymmetry implies that \( N \) is tri-holomorphic, so that \( M_{ij} \) is the only non-vanishing component of \( N^X \), and \( M_{ij} = M_{ji} \). The only other condition required by supersymmetry is that the vector fields \( N \) and \( \bar{N} \) commute.

To obtain these results it is convenient to begin by supposing \( M_{ij} \), \( N^X \) and \( V \) to be dimensionless tensors on the target space and to introduce a mass parameter \( \mu \) (set to unity at the end of the calculation) to take care of the dimensions. Only variations that vanish when \( \mu = 0 \) need be considered because the invariance of the massless model is long-established. Cancellation of the terms linear in \( \mu \) determines \( M_{ij} \) and establishes that the complex vector field \( N \) must be both Killing and tri-holomorphic. Cancellation of terms quadratic in \( \mu \) determines \( V \) and requires that \( [N, \bar{N}] = 0 \). Thus, we have now shown that the scalar potential of the general ‘massive’ N=2 D=4 sigma model is
\[
V = \frac{1}{2} g_{XY} (N^X_1 N^Y_1 + N^X_2 N^Y_2) \tag{36}
\]
where \( g \) is the HK metric and \( N_1, N_2 \) are two real commuting tri-holomorphic KVF's. Note that this potential is precisely of the form that one gets by a non-trivial dimensional reduction from D=6 with
\[
\partial_4 \phi = N_1, \quad \partial_5 \phi = N_2. \tag{37}
\]
2.4 D=3 HK sigma models

The D=3 hypermultiplet fermions \( \chi^A \) transform in the \((2n,2)\) representation of \( Sp_n \times Sp_1' \), with \( Sp_1 \times Sp_1' \) being the automorphism group of the supersymmetry algebra. The massive N=4 D=3 sigma model has the Lagrangian (with Lorentz spinor indices suppressed)

\[
\mathcal{L} = -\frac{1}{2}g_{XY} \partial^\mu \phi^X \partial_\mu \phi^Y - i \frac{1}{2} \bar{\chi} A^i_\mu \chi^i A + i \frac{1}{4} \bar{\chi} A^i_\lambda \chi^i A M_{ijAB} - V
\]

where \( \gamma^\mu (\mu = 0,1,2) \) are the D=3 Dirac matrices and

\[
\bar{\chi}^i A = (\chi^i A) T^0 \gamma^0 = \Omega_{ij}^{AB} \chi^j B \gamma^0.
\]

The supersymmetry transformations are

\[
\delta \phi^X = i f_{ia} \epsilon^a \chi^i A
\]

\[
\delta \chi_A^i = f_{ia} \left[ \gamma^\mu \partial_\mu \phi^X \epsilon_{aA} + N^{XAB} \epsilon_{aB} \right] - \delta \phi^X \omega_X i j \chi_A^j.
\]

Supersymmetry fixes the Yukawa tensor to be

\[
M_{ijAB} = (N_{AB})_{ia,j}^a
\]

and the potential to be

\[
V = \frac{1}{4} N^{XAB} N_{XAB}.
\]

We can write

\[
N^X = i (\sigma)_A^B \cdot k^X
\]

where \( k \) is a triplet of vector fields. Supersymmetry implies that they are Killing, triholomorphic, and mutually commuting. Thus, the general sigma model potential compatible with N=4 D=3 supersymmetry is

\[
V = \frac{1}{2} g_{XY} k^X \cdot k^Y
\]

where \( g \) is the HK metric and \( k \) is a triplet of mutually commuting tri-holomorphic KVFs. This is again precisely of the form that one would get by non-trivial dimensional reduction from D=6.

3 Static supersymmetric solitons

We now focus on the toric HK manifolds. Their essential features were summarized in the introduction. We take \( n \geq 2 \) and consider a scalar potential of the form \( \mathcal{V} \). As we have just seen, this potential is compatible with both D=4 N=2 and D=3 N=4 supersymmetry. For generic mass-squared matrix \( \mu^2 \) the potential vanishes only when the matrix \( U^{IJ} \) vanishes. This yields a set of discrete supersymmetric vacua. We are interested in the kinks that interpolate between these vacua, which we expect to preserve 1/2 supersymmetry. In D=3,4, kinks are domain walls and, as we shall see, we can have intersecting walls that preserve 1/4 supersymmetry.
3.1 Energy bounds

Consider the case of time-independent fields. The D=3 energy density, which is the same as the D=4 energy density for fields that are translational-invariant in the 3-direction, is then

\[ E = \frac{1}{2} U_{IJ} \nabla_1 X^I \cdot \nabla_1 X^J + \frac{1}{2} U_{IJ} \nabla_2 X^I \cdot \nabla_2 X^J + \frac{1}{2} \mu_{IJ} U^{IJ} \]

where

\[ D_\mu \psi_I = \nabla_\mu \psi_I + \nabla_\mu X^K \cdot \omega_{KI} \]

We can rewrite this as

\[ E = \frac{1}{2} U_{IJ} (\nabla_1 X^I - U^{IK} n_K) (\nabla_1 X^J - U^{KL} n_L) + \nabla_1 X^I \cdot n_I \]

\[ + \frac{1}{2} U_{IJ} (\nabla_2 X^I - U^{IK} p_K) (\nabla_2 X^J - U^{KL} p_L) + \nabla_2 X^I \cdot p_I \]

\[ + \frac{1}{2} U_{IJ} D_\psi_1 D_\psi_1 + \frac{1}{2} U_{IJ} D_\psi_2 D_\psi_2 \]

(48)

where \( n_I \) and \( p_I \) are constants such that

\[ n_I \cdot n_J + p_I \cdot p_J = \mu^2_{IJ}. \]

(49)

Note that the integrals \( \int dx^1 \nabla_1 X^I \) and \( \int dx^2 \nabla_2 X^I \) are topological charges carried by domain wall orthogonal to the 1-direction and the 2-direction, respectively. From the above expression for the energy density we deduce the bound

\[ E \geq \nabla_1 X^I \cdot n_I + \nabla_2 X^I \cdot p_I. \]

(50)

This is saturated by solutions of

\[ \nabla_1 X^I = U^{IJ} n_J, \quad \nabla_2 X^I = U^{IJ} p_J, \]

(51)

together with

\[ \nabla_1 \psi_I = -U^{JK} \omega_{JI} \cdot n_K, \quad \nabla_2 \psi_I = -U^{JK} \omega_{JI} \cdot p_K, \]

(52)

which can be solved for \( \psi_I \) given a solution of (51). Consistency of these equations requires that

\[ n_I \times p_J = 0. \]

(53)

3.2 Supersymmetry

We wish to determine the fraction of supersymmetry preserved by solutions of (51) and (52). This could be done directly from the D=3,4 supersymmetry transformation laws, but it can also be done from the supersymmetry transformations of the D=6 massless sigma model, by making use of the fact that the bosonic part of the massive D=4 model is obtained by the reduction ansatz

\[ \partial_4 \phi^X = (\lambda_I k^I)^X, \quad \partial_5 \phi^X = (\rho_I k^I)^X. \]

(54)

This is just (37) with \( N_1 = \lambda_I k^I \) and \( N_2 = \rho_I k^I \). A further trivial dimensional reduction leads to the D=3 model with the same scalar potential.
Substituting (54) into the D=6 supersymmetry preservation condition (30), and using the explicit form of the metric and Kahler 2-forms for toric HK manifolds, we obtain the two conditions

\[ \Gamma^m \sigma \cdot \partial_m X^I + i U^{IJ} \Gamma^m \partial_m \psi_J \epsilon = 0 \] (55)

and

\[ \Gamma^m \partial_m X^I - i \Gamma^m \sigma \times \partial_m X^I + i U^{IJ} \Gamma^m \sigma D_m \psi_I \epsilon = 0 \] . (56)

The second of these conditions is actually implied by the first, so that all we need consider is (55). Using (54) this becomes

\[ \Gamma^\mu [\sigma \cdot \partial^\mu X^I + i U^{IJ} \Gamma^\mu D^\mu \psi_J] \epsilon = -i U^{IJ} (\lambda J \Gamma^4 + \rho J \Gamma^5) \epsilon \] (57)

where, \( \mu = 0, 1, 2, 3 \). This is the condition for partial preservation of supersymmetry for the massive sigma models under consideration. When applied to time-independent, and \( x^3 \)-independent, configurations satisfying the BSTPS equations (51) and (52), the condition (57) becomes

\[ (\Gamma^1 \sigma \cdot n_J + \Gamma^2 \sigma \cdot p_J) \epsilon = -i (\lambda J \Gamma^4 + \rho J \Gamma^5) \epsilon \] (58)

Using (53), we deduce the consistency condition

\[ \sigma \cdot (n_1 \times n_2 + p_1 \times p_2) \epsilon - i \Gamma^{12} (n_1 \cdot p_2 - n_2 \cdot p_1) \epsilon - i \Gamma^{45} (\lambda_1 \rho_2 - \lambda_2 \rho_1) \epsilon = 0 . \] (59)

There are various ways to ensure that both this condition and (53) are satisfied. One is to set

\[ n_I = n \lambda_I , \quad p_I = n \rho_I \] (60)

for unit 3-vector \( n \). In this case (53) is identically satisfied and (59) implies either that \( \lambda_I \) and \( \rho_I \) are proportional, or that

\[ \Gamma^{1245} \epsilon = \epsilon . \] (61)

We shall make the choice (60) in what follows, in which case the equations (51) and (52) reduce to

\[ \nabla_1 X^I = n U^{IJ} \lambda_J , \quad \nabla_2 X^I = n U^{IJ} \rho_J . \] (62)

and

\[ \nabla_1 \psi_I = -U^{JK} \lambda_K \omega_{JI} \cdot n , \quad \nabla_2 \psi_I = -U^{JK} \rho_K \omega_{JI} \cdot n . \] (63)

It is straightforward to show that generic solutions of these equations preserve 1/4 supersymmetry. We simply note that the choice (60) reduces (58) to

\[ \Gamma^4 \lambda_I [1 + i \Gamma^{14} (\sigma \cdot n)] \epsilon + \Gamma^5 \rho_I [1 + i \Gamma^{25} (\sigma \cdot n)] \epsilon = 0 . \] (64)

If \( \lambda_I \) and \( \rho_I \) are not proportional we must impose both (61) and

\[ -i \Gamma^{14} (\sigma \cdot n) \epsilon = \epsilon . \] (65)

These two constraints are compatible and imply preservation of 1/4 supersymmetry. If \( \lambda_I \) and \( \rho_I \) are proportional then the constraint (61) is not needed and (64) implies preservation of 1/2 supersymmetry. An example is the kink domain wall with \( \rho = 0 \) but \( \lambda \neq 0 \), in which case the only constraint on \( \epsilon \) is (53).
3.3 Intersecting domain walls

We now aim to show that the equations (62) and (63) admit solutions that can be interpreted as intersecting domain walls. When considering the toric HK 8-metrics it is convenient to set

\[ X^I = (X, Y). \] (66)

We shall focus here on the special case in which the \( 2 \times 2 \) matrix function \( U \) takes the form

\[ U_{IJ} = \begin{pmatrix} U(X) & c \\ c & U(Y) \end{pmatrix} \] (67)

for constant \( c \) (such that \( \det U \neq 0 \)). For this choice

\[ A_1 = dX \cdot A(X) \quad A_2 = dY \cdot A(Y) \] (68)

with \( \nabla \times A = \nabla U \). It follows that \( U \) is a harmonic function. We shall choose

\[ U(Z) = a + \frac{1}{2} \left[ \frac{1}{|Z - n|} + \frac{1}{|Z + n|} \right] \] (69)

for \( Z = (X, Y) \) and unit 3-vector \( n \). The factor of \( 1/2 \) ensures that the metric is complete if the angles \( \psi_I \) have period \( 2\pi \). When \( c = 0 \) the matrix \( U \) is diagonal and the 8-metric is the direct sum of two identical two-centre HK 4-metrics, but the 8-metric is irreducible when \( c \neq 0 \).

If we identify the unit 3-vector \( n \) with the unit 3-vector appearing in (60) and choose \( n \cdot A = 0 \), as is possible \( [13] \), then (63) becomes

\[ \nabla_x \psi_I = \nabla_y \psi_I = 0 \] (70)

where we have renamed the two space coordinates \((x^1, x^2)\) as \((x, y)\). These equations imply that the angles \( \psi_I \) are constants, which play no further role in the solution. After setting

\[ X = nX, \quad Y = nY, \] (71)

the remaining BPS equations (62) can be written as

\[ \begin{pmatrix} U(X)\nabla_x X + c\nabla_x Y \\ U(X)\nabla_y X + c\nabla_y Y \end{pmatrix} = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \rho_1 & \rho_2 \end{pmatrix} \] (72)

These equations can be solved by setting

\[ X = \tanh u, \quad Y = \tanh v, \] (73)

and the solution is then given implicitly by

\[ u + a \tanh u + c \tanh v = \lambda_1 x + \rho_1 y \]
\[ v + a \tanh v + c \tanh u = \lambda_2 x + \rho_2 y \] (74)

Equivalently, \( X(x, y) \) and \( Y(x, y) \) are given implicitly by

\[ X = \tanh [\lambda_1 x + \rho_1 y - aX - cY] \]
\[ Y = \tanh [\lambda_2 x + \rho_2 y - aY - cX] \] (75)

This is the advertised exact solution. If \( \lambda_1 \) and \( \rho_1 \) are proportional then \( X \) and \( Y \) are independent of one linear combination of \( x \) and \( y \), so the solution is a domain wall (shown
above to preserve 1/2 supersymmetry). Otherwise we have a 1/4 supersymmetric solution representing the intersection of two domain walls.

For simplicity we shall now set \(a = 0\) and choose

\[
\lambda_I = (1, 0), \quad \rho_I = (0, 1),
\]

(76)
corresponding to the potential

\[
V = \frac{1}{2} (|k_1|^2 + |k_2|^2).
\]

(77)

In this case (75) reduces to

\[
X = \tanh(x - cY), \quad Y = \tanh(y - cX)
\]

(78)

For \(c = 0\) this reduces further to \(X = \tanh x\) and \(Y = \tanh y\), i.e., two orthogonally intersecting planar domain walls. The intersection is trivial, as expected, because for \(c = 0\) we have the sum of two massive sigma models, each with an identical Eguchi-Hansen target space 4-metric. For \(c \neq 0\) the two domain walls are asymptotically planar but are deformed near the intersection, as we shall now show.

The energy density for the solution (78) is

\[
\mathcal{E}(x, y) = \frac{\cosh^2 u + \cosh^2 v}{\cosh^2 u \cosh^2 v - c^2}.
\]

(79)

This has a maximum at \(x = y = 0\), which we may take to be the ‘point’ of intersection of the domain walls. This maximum occurs at \(u = v = 0\), and

\[
\mathcal{E}_{\text{max}} = \frac{2}{1 - c^2}.
\]

(80)

As the non-vanishing of \(\det U\) requires \(|c| < 1\) (when \(a = 0\), as we are now assuming) this maximum is finite and positive. The angle \(\theta\) between the domain walls at the point of intersection is given by

\[
\cos \theta = \frac{\nabla X \cdot \nabla Y}{|\nabla X||\nabla Y|}|_{u=v=0} = \frac{\nabla u \cdot \nabla v}{|\nabla u||\nabla v|}|_{u=v=0},
\]

(81)

which yields

\[
\cos \theta = -2 \frac{c}{1 + c^2}.
\]

(82)

Note that \(\theta\) is real, as it should be, because \(|c| < 1\).

Although the domain walls are not orthogonal near the intersection point, they are asymptotically orthogonal. One can get an idea of the shape of the solution by rewriting (78) as

\[
\begin{align*}
u + c \tanh (y - c \tanh u) & = x \\
v + c \tanh (x - c \tanh v) & = y
\end{align*}
\]

(83)
The curves in the \((x, y)\) plane corresponding to \(u = 0\) and \(v = 0\) (which are the curves on which \(\mathcal{E}\) is a maximum for fixed \(v\) and \(u\), respectively) are

\[
x = c \tanh y, \quad y = c \tanh x,
\]

(84)
respectively. For \(c = 0\) these are just the \((x, y)\) coordinate axes, as expected, but for \(c \neq 0\) they each shift by \(2c\) in passing through the origin, just so as to intersect at the angle \(\theta\) given by (82).
4 Stationary supersymmetric solitons

We shall say that a non-static field configuration is stationary if its energy density is time-independent. The Q-kinks of \[4\] provide simple examples of stationary but non-static domain walls in models with 4-dimensional HK target spaces. These can be generalized to 1/2 supersymmetric domain wall solutions of models with 4\(n\)-dimensional toric HK target spaces, as we shall see below, but our principal aim in this section is to exploit the new features provided by higher-dimensional target spaces to find essentially different stationary generalizations of the intersecting domain wall solutions discussed in the previous section.

In the case of intersecting domain walls of the N=1 D=4 Wess-Zumino model it was found to be possible to add a wave along the intersection, preserving 1/4 supersymmetry \[5\]. One can ask whether the same is true of the N=2 D=4 sigma model domain walls. Given such a configuration, its reduction to D=3 would be expected to yield a charged intersecting domain wall. We shall now show that such D=3 configurations, preserving 1/4 supersymmetry indeed exist, provided the target space admits at least three linearly independent tri-holomorphic KVFs. For this we need at least a 12-dimensional toric HK target space. Our starting point will be a massive D=3 N=4 supersymmetric sigma model with potential

\[ V = \frac{1}{2} \mu_{IJ}^2 U^{IJ} \]  

(85)

where

\[ \mu_{IJ}^2 = (\lambda_I \lambda_J + \rho_I \rho_J + q_I q_J) . \]  

(86)

This potential can be obtained by reduction from D=6 with

\[ \partial_4 \psi_I = \lambda_I, \quad \partial_5 \psi_I = \rho_I, \quad \partial_3 \psi_I = q_I \]  

(87)

where \(\lambda, \rho, q\) are three 3-vectors.

4.1 Charged intersecting domain walls

The energy density of the above model can be written in the form

\[ \mathcal{E} = \frac{1}{2} U_{IJ} (\nabla_1 X^I - U^{IK}\lambda_K \mathbf{n}) (\nabla_1 X^J - U^{KL}\lambda_L \mathbf{n}) + \nabla_1 X^I \lambda_I \]

\[ + \frac{1}{2} U_{IJ} (\nabla_2 X^I - U^{IK}\rho_K \mathbf{n}) (\nabla_2 X^J - U^{KL}\rho_L \mathbf{n}) + \nabla_2 X^I \rho_I \]

\[ + \frac{1}{2} U^{IJ} D_1 \psi_I D_1 \psi_J + \frac{1}{2} U^{IJ} D_2 \psi_I D_2 \psi_J + \frac{1}{2} U_{IJ} \dot{X}^I \cdot \dot{X}^J \]

\[ + \frac{1}{2} U^{IJ} (D_t \psi_I \mp q_I)(D_t \psi_J \pm q_J) \pm U^{IJ} D_t \psi_I q_J \]  

(88)

where \(\mathbf{n}\) is a constant unit 3-vector and \(X = X \cdot \mathbf{n}\). Noting that \(\int d^2 x \nabla X^I\) are topological kink charges and \(\int d^2 x U^{IJ} D_t \psi_J\) are the Noether charges corresponding to the isometries generated by the Killing vector fields \(k^I\), we deduce the bound

\[ \mathcal{E} \geq \nabla_1 X^I \lambda_I + \nabla_2 X^I \rho_I + U^{IJ} D_t \psi_I q_J , \]  

(89)

which is saturated when

\[ \dot{X}^I = 0, \quad \dot{\psi}_I = \pm q_I , \]  

(90)

together with

\[ D_1 \psi^I = D_2 \psi^I = 0 , \]  

(91)
and
\[ \nabla_1 X^I = U^{IJ} \lambda_I n \quad \nabla_2 X^I = U^{IJ} \rho_J n. \]  

The simplest way to solve these equations is to suppose the target space to be a direct product of the 8-dimensional HK manifold chosen previously with a 4-dimensional HK manifold for which \( k^3 = \partial/\partial \psi_3 \) is the tri-holomorphic KVF. If one then chooses \( q_1 = q_2 = 0 \) (which one can do without loss of generality) and \( \lambda_3 = \rho_3 = 0 \) then the solution is identical to the previous one but with \( \psi_3 = q_3 t \). It seems likely that this is a limiting solution of a more general one for irreducible 12-dimensional HK target spaces, but we shall not pursue this here. Assuming that we have a solution, we now turn to the determination of the fraction of supersymmetry it preserves.

The supersymmetry preservation condition for the massive D=3 models under consideration is
\[ [\sigma \cdot n \Gamma^\mu \partial_\mu X^I] \epsilon = -iU^{IJ} [\lambda_I \Gamma^4 + \rho_I \Gamma^5 + q_I \Gamma^3 + \Gamma^\mu D_\mu \psi_J] \epsilon. \]  

where, now, \( \mu = 0, 1, 2 \). For solutions of the first-order equations (90)-(92) this becomes
\[ \left[ (\Gamma^1 \sigma \cdot n + i \Gamma^4) \lambda_I + (\Gamma^2 \sigma \cdot n + i \Gamma^5) \rho_I + i (\Gamma^3 \pm \Gamma^0) q_I \right] \epsilon = 0. \]  

If \( q \) vanishes but \( \lambda \) and \( \rho \) are not proportional then supersymmetry imposes the conditions
\[ i \Gamma^{51} \sigma \cdot n \epsilon = \epsilon, \quad i \Gamma^{42} \sigma \cdot n \epsilon = \epsilon, \]  

which preserve 1/4 supersymmetry. Because of the chirality condition (23), these conditions imply that \( \Gamma^{03} \epsilon = \epsilon \), so that 1/4 supersymmetry is maintained when \( q_I \) is non-zero and \( \psi_I = q_I \) but all supersymmetries are broken if \( \psi_I = -q_I \).

We conclude that 1/4 supersymmetric charged intersecting domain walls are possible in this D=3 model. They are stationary but not static. Intersecting domain walls fitting this description can also be found rather more easily by replacing one or both of the kink domain walls by a Q-kink domain wall, but the solution just found is of a different type that is possible only when the toric HK manifold has dimension 4n with \( n \geq 3 \). It has the feature that the charge can be interpreted in D=4 as a wave with the speed of light along the intersection. This follows directly from the fact that the D=4 fields \( \psi_I \) satisfy
\[ (\partial_t - \partial_3) \psi_I = 0, \]  

since the D=3 model is obtained from D=4 by the ansatz \( \partial_3 \psi_I = q_I \). Note that a wave in the opposite direction breaks all the supersymmetry, so the intersection is chiral. It can viewed as a (2,0) supersymmetric string.

### 4.2 Special Cases

The mass-squared matrix \( \mu^2 \) in the models just considered has maximal rank three and the generic examples will occur when the rank is precisely three. We now wish to consider some special cases of stationary supersymmetric solitons that arise when \( \mu^2 \) has rank less than three. Consider the rank two case, which we can arrange by choosing \( \rho = 0 \). We again make the ansatz \( X^I = X^I n \), and we may now set \( \nabla_y X^I = 0 \). We are then left with the equations
\[ \nabla_x X^I = U^{IJ} \lambda_J, \quad \dot{\psi}_I = \pm q_I \]  

to be solved for \( X^I(x) \) and (trivially) for \( \psi_I(t) \). The supersymmetry preservation condition is now
\[ \lambda_I \left[ i \Gamma^4 + \Gamma^1 (\sigma \cdot n) \right] \epsilon + iq_I \left[ \Gamma^0 \pm \Gamma^3 \right] \epsilon = 0, \]  

where
which implies preservation of 1/4 supersymmetry when $\mu^2$ has rank two. Solutions of (97) thus yield new 1/4 supersymmetric charged domain walls, and since $\rho = 0$ these are also solutions of the D=4 massive sigma model.

If instead $\mu^2$ has rank one then (98) implies preservation of 1/2 supersymmetry. To see this we set
\[
q_I = \nu_I, \quad \lambda_I = \sqrt{1 - v^2} \nu_I, \tag{99}
\]
where $\nu_I$ are constants (not all zero) and $v$ is another constant with $|v| < 1$. In this case (98) is equivalent to
\[
\Gamma_\epsilon = \epsilon \tag{100}
\]
with
\[
\Gamma = i \Gamma^{1 \hat{1}} \sigma \cdot n + \frac{v}{\sqrt{1 - v^2}} (\Gamma^{0 \hat{4}} \pm \Gamma^{3 \hat{4}}) \tag{101}
\]
Since $\Gamma$ is traceless and satisfies $\Gamma^2 = 1$, this condition implies preservation of 1/2 supersymmetry. The equations that yield these 1/2 supersymmetric stationary domain walls are
\[
\nabla_x X^I = \sqrt{1 - v^2} U^{I J} \nu_J, \quad \dot{\psi}_I = \pm v \nu_I. \tag{102}
\]
Solutions of these equations generalize the Q-kinks of [4] to higher-dimensional HK target spaces.

5 Discussion

We have shown that massive hyper-Kähler D=4 supersymmetric sigma models can admit non-trivial static intersecting domain wall solutions that preserve 1/4 of the supersymmetry, provided that the target space is at least 8-dimensional and admits at least two linearly independent commuting tri-holomorphic Killing vector fields. Such models also admit new stationary charged domain wall solutions that also preserve 1/4 supersymmetry. Given three linearly independent tri-holomorphic Killing vector fields, which requires a target space of at least 12 dimensions, there exist 1/4 supersymmetric stationary intersecting domain wall solutions in D=4 with a wave along the string intersection. Reduction on the string direction yields a charged intersecting domain wall solution of a massive D=3 sigma-model, again preserving 1/4 supersymmetry. We have found exact solutions for a class of sigma models with particular toric hyper-Kähler target spaces. It is unlikely that exact solutions can be found in general, but we expect intersecting domain wall solutions to exist for many other toric HK 8-manifolds too, e.g. the Calabi manifolds that we discuss below.

Whereas 1/4 supersymmetry is the minimal possible fraction of N=1 D=4 supersymmetry it is twice the minimal possible fraction of N=2 D=4 (or N=4 D=3) supersymmetry. One wonders whether there are other intersecting brane configurations that preserve only 1/8 supersymmetry. It is not difficult to find such solutions in models with factorizable target spaces because one can just superpose solutions within each factor. It would be interesting to know if these can be generalised to non-trivial examples.

Although we have focussed on solutions of D=3,4 models, we note that the supersymmetric charged domain wall solutions discussed in section 4.2 give rise to supersymmetric kink solutions in D=2. We now conclude with a brief discussion of a possible applications of these solutions to IIB superstring theory. Consider a single D1-brane in the presence of k parallel D5-branes. The low-energy effective dynamics of the D-string is given by a two dimensional quantum field theory which, for coincident D5-branes, has both a Coulomb and a Higgs branch. Switching on a constant NS B-field parallel to the D5-branes corresponds to switching on Fayet–Illiopoulos terms in the two-dimensional quantum field theory on the D-string, and this lifts the Coulomb branch. In this case the
Higgs branch is the charge-1 non-commutative instanton moduli space (after ignoring the centre of mass). In other words, the low-energy dynamics of the Higgs branch is given by a D=2 supersymmetric HK sigma model with a 4(k−1)-dimensional Calabi space as target [14]. This space has k−1 mutually commuting tri-holomorphic Killing vector fields. Separating the D5-branes along an axis leads to the addition of a supersymmetric preserving potential constructed from a single tri-holomorphic Killing vector field. The 1/2 supersymmetric kink and Q-kink solutions were interpreted in [15], following [16], as (1,Q) strings interpolating from one D5-brane to another. For k=3, when the D5-branes are not co-linear, the potential is constructed from two tri-holomorphic Killing vector fields. It seems likely that the 1/4 supersymmetric charged kink solutions found here can be interpreted as a string that interpolates from one D5-brane to two D5-branes via a string junction.

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