GROUPOID COHOMOLOGY AND EXTENSIONS

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ABSTRACT. We show that Haefliger’s cohomology for étale groupoids, Moore’s cohomology for locally compact groups and the Brauer group of a locally compact groupoid are all particular cases of sheaf (or Čech) cohomology for topological simplicial spaces.

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1. Introduction

Let $G$ be a locally compact Hausdorff groupoid with Haar system. In [10], the authors studied the group of Morita equivalence classes of actions of $G$ on continuous fields of $C^*$-algebras over the unit space $G_0$ such that

- Each fiber is isomorphic to the algebra of compact operators on some Hilbert space (depending on the fiber);
They called this group the Brauer group $\text{Br}(G)$ of $G$, and showed that it is naturally isomorphic to the group $\text{Ext}(G, \mathbb{T})$ of Morita equivalence classes of central extensions

$$\mathbb{T} \times G' \to E \to G',$$

where $G'$ is some Morita equivalent groupoid. In the case of discrete groups, it is well-known that central extensions of $G$ by $\mathbb{T}$ are classified by $H^2(G, \mathbb{T})$. Actually, given any locally compact group $G$ and any Polish (i.e. metrizable separable complete) $G$-module $A$, Moore’s cohomology groups $H^2(G, A)$ classify extensions $A \to E \to G$ such that the action of $G$ on $A$ by conjugation is exactly the action of $G$ on the $G$-module $A$. One of the possible definitions of Moore’s cohomology is the following: consider $C^n(G, A)$ the space of all measurable maps $c : G^n \to A$ with the differential

$$\begin{align*}
\delta c(g_1, \ldots, g_n) &= g_1 c(g_2, \ldots, g_n) \\
&\quad + \sum_{k=1}^n (-1)^k c(g_1, \ldots, g_k g_{k+1}, \ldots, g_n) + (-1)^{n+1} c(g_1, \ldots, g_n),
\end{align*}$$

then $H^n(G, A)$ is the $n$-th cohomology group of the complex $C^*(G, A)$.

On the other hand, Haefliger [6], see also [9] defined sheaf cohomology groups $H^*(G, A)$ given any étale groupoid $G$ and any abelian $G$-sheaf $A$ (i.e. an abelian sheaf on $G_0$ endowed with a continuous action of $G$). It was thus natural to expect that a single cohomology theory for groupoids should unify all these: this is the question asked by A. Kumjian in Boulder (1999).

Our approach is to consider the simplicial (topological) space $G_\bullet = (G_n)_{n \in \mathbb{N}}$ and to use sheaf cohomology for simplicial spaces [4] and Čech cohomology (see Section 4). We show:

**Theorem 1.1.** Let $M_\bullet$ be a simplicial space and $A^\bullet$ be an abelian sheaf on $M_\bullet$. Denote by $H^n(M_\bullet; A^\bullet)$ and $\check{H}(M_\bullet; A^\bullet)$ the sheaf and Čech cohomology groups respectively. Then

- (a) $H^n(M_\bullet; A^\bullet) \cong \check{H}(M_\bullet; A^\bullet)$ if $M_n$ is paracompact for all $n$;
- (b) $H^n(G_\bullet; A^\bullet)$ coincides with Haefliger’s cohomology $H^n(G; A^0)$ if $G$ is an étale groupoid and $A^\bullet$ is the sheaf on $G_\bullet$ corresponding to an abelian $G$-sheaf $A^0$;
- (c) $H^n(G_\bullet; A^\bullet) \cong \check{H}(G_\bullet; A^\bullet) \cong H^n_{\text{Moore}}(G, A)$ if $G$ is a locally compact group, $A$ is a Polish $G$-module and $A^\bullet$ is the sheaf on $G_\bullet$ associated to $A$;
- (d) $H^2(G_\bullet; \mathbb{T}) \cong \check{H}(G_\bullet; \mathbb{T})$ is the Brauer group of $G$ if $G$ is a locally compact Hausdorff groupoid with Haar system;
- (e) $H^n(G_\bullet; A^\bullet)$ and $\check{H}^*(G_\bullet; A^\bullet)$ are invariant under Morita equivalence of topological groupoids.
(Actually we prove more than statement (d); see Proposition 5.6.)

It is possible that some parts of this paper are well-known among specialists, but apparently written nowhere in the literature, in particular the definition of Čech cohomology and its relation to sheaf cohomology for simplicial spaces. Besides, a more conceptual approach would have consisted in using Grothendieck’s cohomology of sites [1], as Moerdijk did in [13]. Moreover, it is possible that the present work has non-trivial intersection with Moerdijk’s in [14]. However, we hope that the present approach, being rather direct and elementary, will still be of interest to the reader.

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2. Simplicial spaces and groupoids

2.1. Definition of simplicial spaces. Let us recall some basic facts about simplicial spaces. Let $\Delta$ (resp. $\Delta'$) be the simplicial (resp. semi-simplicial) category, whose objects are the nonnegative integers, and whose morphisms are the nondecreasing (resp. increasing) maps $[m] \to [n]$ (where $[n]$ denotes the interval $\{0, \ldots, n\}$). We denote by $\Delta^{(N)}$ the $N$-truncated simplicial category, i.e. the full sub-category of $\Delta$ whose objects are the integers $\leq N$.

A simplicial (resp. semi-simplicial, $N$-simplicial) topological space is a contravariant functor from the category $\Delta$ (resp. $\Delta'$, resp. $\Delta^{(N)}$) to the category of topological spaces. In the same way, one can define the notion of simplicial (resp. semi-simplicial, $N$-simplicial) manifold. In this paper we shall work with simplicial topological spaces and will use the terminology “simplicial space”, but some of the results can easily be transposed to simplicial manifolds.

In practice, a (semi-)simplicial space is a sequence $M_\bullet=(M_n)_{n\in \mathbb{N}}$ of topological spaces, given with continuous maps $\tilde{f}: M_n \to M_k$ for every morphism $f: [k] \to [n]$, satisfying the relation $\tilde{f} \circ g = g \circ \tilde{f}$ for all composable morphisms $f$ and $g$.

Let $\varepsilon^n_i: [n-1] \to [n]$ be the unique increasing map that avoids $i$, and $\eta^n_i: [n+1] \to [n]$ be the unique non-decreasing surjective map such that $i$ is reached twice ($0 \leq i \leq n$). We will usually omit the superscripts for convenience of notation.

If $M_\bullet=(M_n)_{n\in \mathbb{N}}$ is a simplicial space, then the face maps $\varepsilon^n_i: M_n \to M_{n-1}$, $i=0, \ldots, n$ and the degeneracy maps $\eta^n_i: M_n \to M_{n+1}$, $i=0, \ldots, n$, satisfy the following simplicial identities: $\varepsilon^{n-1}_i \varepsilon^n_j = \varepsilon^{n-1}_{j-1} \varepsilon^n_i$ if $i < j$, $\eta^{n+1}_i \eta^n_j = \eta^{n+1}_{j+1} \eta^n_i$ if $i \leq j$, $\varepsilon^{n+1}_i \eta^n_j = \eta^{n+1}_{j-1} \varepsilon^n_i$ if $i < j$, $\varepsilon^{n+1}_i \eta^n_j = \eta^{n+1}_{j+1} \varepsilon^n_i-1$ if $i > j+1$ and $\varepsilon^{n+1}_j \eta^n_j = \varepsilon^{n+1}_{j+1} \eta^n_j = \text{Id}_{M_n}$.

Conversely, if we are given a sequence $M_\bullet$ of topological spaces and maps satisfying such identities, then there is a unique simplicial structure on $M_\bullet$ such that $\varepsilon^n_i$ are the face maps and $\eta^n_i$ are the degeneracy maps.
2.2. Groupoids. In order to fix notations, we first recall some basic facts about groupoids. For more details, see e.g. [13].

A topological groupoid is given by two topological spaces $G_0$ and $G$, two maps $r$ and $s$ from $G$ to $G_0$, called the range and source maps, a unit map $\eta$: $G_0 \to G$, a partially defined multiplication $G_2 = \{(g,h) \in G^2 | s(g) = t(h)\} \to G$ denoted by $(g,h) \mapsto gh$, and an inversion map $G \to G$ denoted by $g \mapsto g^{-1}$ such that the following identities hold (for $g, h, k \in G$ and $x \in G_0$):

- $r(gh) = r(g), s(gh) = s(h)$;
- $(gh)k = g(hk)$ whenever $s(g) = r(h)$ and $s(h) = r(k)$;
- $s(\eta(x)) = r(\eta(x)) = x$;
- $gr(s(g)) = \eta(r(g))g = g$;
- $r(g^{-1}) = s(g), s(g^{-1}) = r(g), gg^{-1} = \eta(r(g)), g^{-1}g = \eta(s(g))$.

We will usually identify the unit space $G_0$ to a subspace of $G$ by means of the unit map $\eta$.

Standard examples are:
- groups, with $G_0 = pt$;
- spaces $M$, with $G = G_0 = M$, $r = s = \text{Id}$;
- the homotopy groupoid of a space $M$, where $G_0 = M$, $G$ is the set of homotopy classes of paths in $M$, $s(g)$ is the starting point of the path and $r(g)$ is the endpoint.

Here are a few notations that we will use: $G_x = s^{-1}(x), G^x = r^{-1}(x), G^y_x = G_x \cap G^y$.

A left action of a topological groupoid $G$ on a space $Z$ is given by a (continuous) map $p: Z \to G_0$ and a map $G \times s.pZ \to Z$, denoted by $(g, z) \mapsto gz$, such that

- $p(gz) = r(g)$;
- $(gh)z = g(hz)$ whenever $(g, h) \in G_2$ and $s(h) = p(z)$;
- $ez = z$ if $e \in G_0 \subset G$.

We will say that $Z$ is a (left) $G$-space. Given a $G$-space $Z$, we form the crossed-product groupoid $G \times Z := G \times_{s.p} Z$ with unit space $Z$, source and range maps $s(g, z) = z, r(g, z) = gz$, product $(g, z)(h, z') = (gh, z')$ if $z = hz'$, and inverse $(g, z)^{-1} = (g^{-1}, gz)$.

Any topological groupoid $G$ canonically gives rise to a simplicial space as follows [13]: Let

$$G_n = \{(g_1, \ldots, g_n) | s(g_i) = t(g_{i+1}) \forall i\}$$

be the set composable $n$-tuples.

Define the face maps $\varepsilon_i^n: G_n \to G_{n-1}$ by, for $n > 1$

$$\varepsilon_0^n(g_1, g_2, \ldots, g_n) = (g_2, \ldots, g_n)$$
$$\varepsilon_n^n(g_1, g_2, \ldots, g_n) = (g_1, \ldots, g_{n-1})$$
$$\varepsilon_i^n(g_1, \ldots, g_n) = (g_1, \ldots, g_ig_{i+1}, \ldots, g_n), \ 1 \leq i \leq n-1,$$
and for $n=1$ by, $\varepsilon_0^n(g) = s(g)$, $\varepsilon_1^n(g) = r(g)$. Also define the degeneracy maps: $\tilde{\eta}_0^n : G_0 \to G_1$ is the unit map of the groupoid, and $\tilde{\eta}_i^n : G_n \to G_{n+1}$ by:

$$
\tilde{\eta}_0^n(g_1, \ldots, g_n) = (r(g_1), g_1, \ldots, g_n)
$$

$$
\tilde{\eta}_i^n(g_1, \ldots, g_n) = (g_1, \ldots, g_i, s(g_i), g_{i+1}, \ldots, g_n), \quad 1 \leq i \leq n.
$$

Another way to view the simplicial structure of $G_\bullet$ is the following: we note that $G_n$ can be identified with the quotient of $(EG)_n = \{(\gamma_0, \ldots, \gamma_n) \in G^{n+1} \mid r(\gamma_0) = \cdots = r(\gamma_n)\}$ by the left action of $G$, the correspondence being

$$(g_1, \ldots, g_n) = (\gamma_0^{-1} \gamma_1, \ldots, \gamma_{n-1}^{-1} \gamma_n)$$

$$[\gamma_0, \ldots, \gamma_n] = [r(g_1), g_1, g_2, \ldots, g_1 \ldots g_n].$$

Then, for any morphism $f : [k] \to [n]$, $\tilde{f} : G_n \to G_k$ is defined by

$$\tilde{f}[\gamma_0, \ldots, \gamma_n] = [\gamma_f(0), \ldots, \gamma_f(n)].$$

For instance, in the first picture, if $f$ is injective then

$$\tilde{f}(g_1, \ldots, g_n) = (g_{f(0)+1} \cdots g_{f(1)}, g_{f(1)+1} \cdots g_{f(2)}, \ldots, g_{f(k-1)+1} \cdots g_{f(k)}).$$

2.3. Morita equivalence and generalized morphisms. We recall (see for instance [4 [8 [12 [17 [11 [20]) that a generalized morphism between two (topological, or locally compact, or Lie) groupoids $G'$ and $G$ is given by a topological space (or a locally compact space, or a manifold) $Z$, two maps $G' \xleftarrow{\rho} Z \xrightarrow{\sigma} G_0$ such that $Z$ admits a left action of $G'$ with respect to $\rho$, a right action of $G$ with respect to $\sigma$, with the property that the two actions commute and $\rho : Z \to G_0'$ is a locally trivial $G'$-principal bundle.

Topological (or locally compact...) groupoids and generalized morphisms form a category whose isomorphisms are Morita equivalences. Every groupoid morphism naturally defines a generalized morphism.

If $\mathcal{U} = (U_i)_{i \in I}$ is an open cover of $G_0$, define the cover groupoid

$$G[\mathcal{U}] = \{(i, g, j) \in I \times G \times I \mid r(g) \in U_i, s(g) \in U_j\}$$

with unit space $\{(i, x) \in I \times G_0 \mid x \in U_i\}$, source and range maps $s(i, g, j) = (j, s(g))$, $r(i, g, j) = (i, r(g))$ and product $(i, g, j)(j, h, k) = (i, gh, k)$.

Then the canonical morphism $G[\mathcal{U}] \to G$ is a Morita equivalence. Moreover, every generalized morphism $G' \to G$ admits a decomposition $G' \xrightarrow{f} G[\mathcal{U}'] \xrightarrow{\tilde{f}} G$ for some open cover $\mathcal{U}'$ of $G_0'$ and some groupoid morphism $f$.

Remark 2.1. The simplicial space $G[\mathcal{U}]_\bullet$ is isomorphic to the sub-simplicial space of $(I^{n+1} \times G_n)_{n \in \mathbb{N}}$ such that $G[\mathcal{U}]_n$ consists of $(2n+1)$-tuples $(i_0, \ldots, i_n, g_1, \ldots, g_n)$ satisfying the condition

$$r(g_1) \in U_{i_0}, \quad s(g_1) \in U_{i_1}, \ldots, s(g_n) \in U_{i_n}.$$

Finally, we will need the following

Proposition 2.2. For any functor $F$ from the category of topological groupoids to any category, the following are equivalent:
(i) $F$ is invariant under Morita-equivalence;
(ii) $F$ factors through the category whose objects are groupoids and whose morphisms are generalized morphisms;
(iii) For any groupoid $G$ and any open cover $\mathcal{U}$ of $G_0$, the canonical map $G[\mathcal{U}] \to G$ induces an isomorphism $F(G[\mathcal{U}]) \xrightarrow{\sim} F(G)$.

Proof. See for instance [20, Proposition 2.5].

3. Sheaves on simplicial spaces

3.1. Basic definitions. Recall [4] that if $u: X \to Y$ is continuous, $\mathcal{A}$ is a sheaf on $X$ and $\mathcal{B}$ is a sheaf on $Y$, then a $u$-morphism from $\mathcal{B}$ to $\mathcal{A}$ is by definition an element of $\text{Hom} (\mathcal{B}, u_* \mathcal{A}) \cong \text{Hom} (u^* \mathcal{B}, \mathcal{A})$. A sheaf on a simplicial (resp. semi-simplicial) space $M_\bullet$ is a sequence $\mathcal{A}^\bullet=(\mathcal{A}^n)_{n \in \mathbb{N}}$ such that $\mathcal{A}^n$ is a sheaf on $M_n$, and such that for each morphism $f: [k] \to [n]$ in the category $\Delta$ (resp. $\Delta'$) we are given $\tilde{f}$-morphisms

$$\tilde{f}^*: \mathcal{A}^k \to \mathcal{A}^n$$

such that $\tilde{f}^* \tilde{g}^* = \tilde{f} \circ \tilde{g}^*$ if $g: [\ell] \to [k]$.

In practice, given open sets $U \subset M_n$ and $V \subset M_k$ such that $\tilde{f}(U) \subset V$ we have a restriction map $\tilde{f}^*: \mathcal{A}^k(V) \to \mathcal{A}^n(U)$ such that $\tilde{f}^* \circ \tilde{g}^* = \tilde{f} \circ \tilde{g}^*: \mathcal{A}^\ell(W) \to \mathcal{A}^n(U)$ whenever $\tilde{g}(V) \subset W$.

A fundamental example is given by $G$-sheaves. In the definition below, recall that a map $f: X \to Y$ is said to be étale if it is a local homeomorphism, i.e. every point $x \in X$ has an open neighborhood $U$ such that $f(U)$ is open and $f$ induces a homeomorphism from $U$ onto $f(U)$. We will also say that $X$ is an étale space over $Y$. A groupoid is étale if the range (equivalently the source) map is étale. A morphism $\pi_\bullet: E_\bullet \to M_\bullet$ is étale if each $\pi_n: E_* \to M_*$ is étale. Finally, recall that a sheaf over a space $X$ can be considered as a (not necessarily Hausdorff) étale space over $X$.

Definition 3.1. [6] Let $G$ be a topological groupoid. Then a $G$-sheaf is an étale space $E_0$ over $G_0$, endowed with a continuous action of $G$.

Of course, an abelian $G$-sheaf is a $G$-sheaf $E_0$ such that $E_0$ is an abelian sheaf on $G_0$ and such that for each $g \in G$, the action $\alpha_g: (E_0)_{s(g)} \to (E_0)_{r(g)}$ is a group morphism.

Example 3.2. If $G$ is a group then a $G$-sheaf is just a space endowed with an action of $G$.

To show that any $G$-sheaf defines a sheaf over $M_\bullet$, we need some preliminaries:

Definition 3.3. Let $\pi_\bullet: E_\bullet \to M_\bullet$ a morphism of simplicial spaces. We say that $\pi_\bullet$ is reduced if for all $k, n$ and all $f \in \text{Hom}_\Delta(k, n)$, the map $\tilde{f}$ induces an isomorphism $E_n \cong M_n \times_{f, \pi_k} E_k$. In this case, we will say that $E_\bullet$ is a reduced simplicial space over $M_\bullet$. 
Definition 3.4. Let \( \mathcal{A}^\bullet \) be a sheaf over the simplicial space \( M_\bullet \). We will say that \( \mathcal{A}^\bullet \) is reduced if for all \( k, n \) and all \( f \in \text{Hom}_\Delta(k, n) \), the morphism \( \tilde{f}^* \in \text{Hom}(\tilde{f}^* \mathcal{A}^k, \mathcal{A}^n) \) is an isomorphism.

Lemma 3.5. There is a one-to-one correspondence between reduced sheaves over \( M_\bullet \) and reduced étale simplicial spaces over \( M_\bullet \).

Proof. The proof is easy. Let us just explain the construction of the sheaf \( \mathcal{A}^\bullet \) out of the reduced simplicial space \( E_\bullet \) over \( M_\bullet \).

Let \( \mathcal{A}^n(U) \) be the space of continuous sections over \( U \) of the projection map \( \pi_n : E_n \to M_n \). If \( f : [k] \to [n] \) is a morphism in \( \Delta \) and \( \tilde{f}(U) \subset V \), then for any section \( \sigma \in \mathcal{A}^k(V) \) we define \( \tilde{f}^* \sigma \in \mathcal{A}^n(U) \) by \( (\tilde{f}^* \sigma)(x) = (x, \sigma(\tilde{f}(x))) \in M_n \times \tilde{f}^* p_k E_k \approx E_n \).

Lemma 3.6. Any reduced simplicial space over \( M_\bullet \), étale or not, determines a sheaf over \( M_\bullet \).

Proof. The proof is the same. Note that it is not clear whether all sheaves can be constructed this way.

Corollary 3.7. Let \( G \) be a topological groupoid, then any \( G \)-space determines a sheaf on \( G_\bullet \). If the \( G \)-space is étale then it determines a reduced sheaf on \( G_\bullet \).

Proof. Suppose \( Z \) is a \( G \)-space and let \( \pi_n \) be the first projection \( (G \ltimes Z)_n = G_n \times \tilde{p}_n, p Z \to G_n \), where \( \tilde{p}_n(g_1, \ldots, g_n) = s(g_n) \). Then \( \pi_n \) is clearly a simplicial map \( (G \ltimes Z)_\bullet \to G_\bullet \).

Corollary 3.8. Every \( G \)-sheaf canonically defines a reduced sheaf over the simplicial space \( G_\bullet \).

Another example is given by \( G \)-modules:

Definition 3.9. Let \( G \) be a topological groupoid. A \( G \)-module is a topological groupoid \( A^\bullet \), with source and range maps equal to a map \( p : G_0 \), such that

- \( A^x \) is an abelian group for all \( x \);
- As a space, \( A \) is endowed with a \( G \)-action \( G \times_s, p A \to A \);
- for each \( g \in G \), the map \( \alpha_g : A_{s(g)} \to A_{r(g)} \) given by the action is a group morphism.

By Corollary 3.7, any \( G \)-module defines a sheaf \( \mathcal{A}^\bullet \) which is clearly abelian. More explicitly, the simplicial structure on \( (G \ltimes A)_\bullet \) is defined as follows: for all \( f \in \text{Hom}_\Delta(k, n) \),

\[
\tilde{f}([\gamma_0, \ldots, \gamma_n], a) = ([\gamma_f(0), \ldots, \gamma_f(k)], \gamma_f^{-1}(k) \gamma_n a).
\]

Then \( \mathcal{A}^n \) is the sheaf of germs of continuous sections of \( (G \ltimes A)_n \to G_n \), i.e. sections are continuous maps \( \varphi(g_1, \ldots, g_n) \in A_{s(g_n)} \). However, to recover the usual formulas like (1.1), it is better to work with the maps

\[
c(g_1, \ldots, g_n) = g_1 \cdots g_n \varphi(g_1, \ldots, g_n) \in A_{r(g_1)}
\]
and this is what we shall usually do.

Note that for all \( \vec{g} = (g_1, \ldots, g_n) \in G_n \), the stalk \( A_{\vec{g}} \) maps to \( A_{\tilde{p}_n(\vec{g})} \). This map is surjective iff \( p: A \to G_0 \) has enough cross-sections; for injectivity, it is enough that \( p \) be an étale map.

If \( A = G_0 \times B \) has constant fibers (\( B \) being a topological abelian group with no action of \( G \)), then the corresponding sheaf is called the constant sheaf and is again (abusively) denoted by \( B \).

When \( G \) is a group, a \( G \)-module is just a topological abelian group \( A \) endowed with a continuous action \( G \to \text{Aut}(A) \) and the sheaf \( A^n \) is just the constant sheaf \( A \) on \( G_n \).

3.2. \( G \)-sheaves and sheaves over simplicial spaces. Recall (Corollary \( \text{[3.8]} \) that any \( G \)-sheaf defines a sheaf over \( G^\bullet \). In this subsection, which can safely be omitted by the reader, we examine the converse. More precisely:

**Proposition 3.10.** Let \( G \) be a topological groupoid. There is a one-to-one correspondence between:

(i) \( G \)-sheaves;
(ii) reduced sheaves over \( G^\bullet \);
(iii) reduced étale spaces over \( G^\bullet \).

**Proof.** (ii) \( \iff \) (iii) was proved in Lemma \( \text{[3.5]} \) and (i) \( \implies \) (ii) is given by Corollary \( \text{[3.8]} \). For (iii) \( \implies \) (i), the proposition below will allow us to conclude. \( \square \)

**Proposition 3.11.** Let \( G \) be a topological groupoid. There is a one-to-one correspondence between reduced morphisms of simplicial spaces \( \pi^\bullet : E^\bullet \to G^\bullet \) and \( G \)-spaces. Under this correspondence, étale spaces over \( G^\bullet \) are mapped onto \( G \)-spaces which are étale over \( G_0 \).

**Proof.** The only difficulty is to show that \( E_0 \) is endowed with an action of \( G \) such that \( (G \times E_0)^\bullet \cong E_\bullet \). Consider the map

\[
\phi_n : E_n \to G_n \times E_0^{n+1}
\]

defined by \( \phi_n = (\pi_n, \tilde{p}_0, \ldots, \tilde{p}_n) \) where \( p_i : [0] \to [n] \) is the map \( p_i(0) = i \).

One can view \( (E_0^{n+1})_{n \in \mathbb{N}} \) as a simplicial space, with

\[
\tilde{f}(\xi_0, \ldots, \xi_n) = (\xi_{f(0)}, \ldots, \xi_{f(k)}) \quad \forall f \in \text{Hom}_\Delta(k, n).
\]

It is not hard to show, using the relation \( \tilde{p}_j \circ \tilde{f} = \tilde{p}_f(j) \) (which holds because \( f \circ p_j = p_{f(j)} \)), that \( \phi_\bullet = (\phi_n)_{n \in \mathbb{N}} \) is a morphism of simplicial spaces. Since \( E_\bullet \) is reduced, \( (\pi_n, \tilde{p}_n) \) is a homeomorphism from \( E_n \) to \( G_n \times \tilde{p}_n, \pi_0 E_0 \) and in particular \( \phi_n \) is injective.

Define the action of \( G \) on \( E_0 \) to be the composition of maps

\[
G \times_{s, p} E_0 \xrightarrow{(\pi_1, \tilde{p}_1)^{-1}} E_1 \xrightarrow{\tilde{p}_0} E_0,
\]

i.e.

\[
(3.3) \quad \xi' = g \xi \iff \exists x \in E_1, \phi_1(x) = (g, \xi', \xi).
\]
Let $x \in E_n$, let $(g_1, \ldots, g_n) = \pi_n(x)$ and $\xi = \tilde{p}_n(x)$. Let us show that
\begin{equation}
\phi_n(x) = (\pi_n(x), g_1 \cdots g_n \xi, g_2 \cdots g_n \xi, \ldots, \xi),
\end{equation}
where $g_1 \cdots g_n \xi = g_1(g_2(\cdots (g_n \xi) \cdots))$ (we don’t know yet that $(g, \xi) \mapsto g \xi$ is an action).

Consider the map $f : [1] \to [n]$ such that $f(0) = j - 1$ and $f(1) = j$. Let $y = \tilde{f}(x) \in E_1$, then
\[ \phi_1(y) = \phi_1(\tilde{f}(x)) = \tilde{f}(\phi_n(x)) = (g_j, \tilde{p}_{j-1}(x), \tilde{p}_j(x)), \]
hence by (3.3), $\tilde{p}_{j-1}(x) = g_j \tilde{p}_j(x)$.

Now, let $(g, h, \xi) \in G_2 \times \tilde{p}_2 \cdot E_0$. We want to show that $g(h \xi) = (gh) \xi$. Let $x = (\pi_2, \tilde{p}_2)^{-1}(g, h, \xi) \in E_2$. By Equation (3.4) above, $\phi_2(x) = (g, h, g(h \xi), h \xi, x)$. It follows that $\phi_1(\xi_1(x)) = \xi_1(\phi_2(x)) = (gh, g(h \xi), x)$. Using again (3.3), we get $(gh) x = g(h \xi)$.

To show that $p(\xi) \xi = \xi$ for all $\xi \in E_0$, let $x = \tilde{\eta}_0(\xi) \in E_1$, then
\[ \phi_1(x) = (\pi_1(x), \tilde{p}_0(x), \tilde{p}_1(x)) = (p(\xi), \xi, \xi), \]
therefore, by (3.3), we get $p(\xi) \xi = \xi$.

We have shown that $G$ acts on $E_0$. To show that the simplicial spaces $(G \ltimes E_0)_*$ and $E_*$ are isomorphic, let $\phi'_n : G_n \times \tilde{p}_n \cdot E_0 = (G \ltimes E_0)_n \to G_n \times E_0^{n+1}$ be defined like $\phi$, then $\phi'_n$ is also injective for all $n$ and $\phi'_n$ is a morphism of simplicial spaces. By Equation (3.4), we have a commutative diagram
\[
\begin{array}{ccc}
E_n & \xrightarrow{(\pi_n, \tilde{p}_n)} & G_n \times \tilde{p}_n \cdot E_0 \\
\downarrow & & \downarrow \\
G_n \times E_0^{n+1} & \xrightarrow{\phi'} & G_n \times E_0^{n+1}
\end{array}
\]
where $\phi_n$ and $\phi'_n$ are injective and $(\pi_n, \tilde{p}_n)$ is a homeomorphism. Therefore, $(\pi_n, \tilde{p}_n)_{n \in \mathbb{N}}$ is an isomorphism of simplicial spaces. \hfill $\square$

4. ČECH COHOMOLOGY

4.1. Covers of simplicial spaces.

**Definition 4.1.** An open cover of a semi-simplicial space $M_*$ is a sequence $\mathcal{U}_* = (\text{U}_n)_{n \in \mathbb{N}}$ such that $\text{U}_n = (\text{U}_i^n)_{i \in I_n}$ is an open cover of the space $M_n$.

The cover is said to be semi-simplicial if $I_* = (I_n)_{n \in \mathbb{N}}$ is a semi-simplicial set such that for all $f \in \text{Hom}_\Delta(k, n)$ and for all $i \in I_n$ one has $f(\text{U}_i^n) = \text{U}_{f(i)}^k$. In the same way, one defines the notions of simplicial cover and of $N$-simplicial cover.

The reason why we need to introduce this terminology is that, even when $M_*$ is a simplicial space, there may not exist sufficiently fine simplicial covers. However, given a cover $\mathcal{U}_*$, we can form the semi-simplicial cover $\sigma \mathcal{U}_*$ defined as follows.

Let $\mathcal{P}_n = \bigcup_{k=0}^n \mathcal{P}_n^k$, where $\mathcal{P}_n^k = \text{Hom}_\Delta(k, n)$. Note that $\mathcal{P}_n$ can be identified with the set of nonempty subsets of $[n]$. Let $\Lambda_n$ (or $\Lambda_n(I)$ if there is a risk of confusion) be the set of maps

\[(4.1) \quad \lambda: \mathcal{P} \to \bigcup_k I_k \text{ such that } \lambda(\mathcal{P}_n^k) \subset I_k.\]

For all $\lambda \in \Lambda_n$, we let

\[U^n_\lambda = \bigcap_{k \leq n} \bigcap_{f \in \mathcal{P}_n^k} \tilde{f}^{-1}(U^k_{\lambda(f)}).\]

It is clear that $(U^\lambda)_{\lambda \in \Lambda_n}$ is an open cover of $M_n$.

The semi-simplicial structure on $\Lambda_n$ is defined in an obvious way: for all $g \in \text{Hom}_\Delta(n, n')$, $\tilde{g}: \Lambda_n \to \Lambda_n$ is the map

\[(\tilde{g}\lambda)(f) = \lambda'(g \circ f).\]

In the same way, for all integers $n \leq N$, let

\[(4.2) \quad (\sigma_N \mathcal{U})_n = (U^n_\lambda)_{\lambda \in \Lambda^n_N},\]

where $\Lambda^n_N$ is the set of all maps $\lambda: \bigcup_{k \leq n} \text{Hom}_\Delta(k, n) \to \bigcup_{k \leq n} I_k$ which satisfy

$\lambda(\text{Hom}_\Delta(k, n)) \subset I_k$, and

\[U^n_\lambda = \bigcap_{k \leq n} \bigcap_{f \in \text{Hom}_\Delta(k, n)} \tilde{f}^{-1}(U^k_{\lambda(f)}).\]

The $N$-simplicial structure on $\Lambda^n_N$ is defined as follows: for all integers $n, n' \leq N$ and all $g \in \text{Hom}_\Delta(n, n')$, $\tilde{g}: \Lambda^n_n \to \Lambda^n_{n'}$ is the map $(\tilde{g}\lambda')(f) = \lambda'(g \circ f)$.

Then $\sigma_N \mathcal{U}_n = (\sigma_N \mathcal{U}_n)_n \leq N$ is a $N$-simplicial cover of the $N$-skeleton of $M_\bullet$.

**Convention 4.2.** We will also (abusively) denote by $\sigma_N \mathcal{U}_n$ the open cover which coincides with $\sigma_N \mathcal{U}_n$ for $n \leq N$ and with $\mathcal{U}_n$ for $n \geq N + 1$.

**Example 4.3.** Let $M_\bullet = (M)_{n \in \mathbb{N}}$ be the constant simplicial space associated to a topological space $M$, and suppose $\mathcal{U}_0 = (U^0_i)_{i \in I_0}$ is an open cover of $M$. Define $I_n = I_0^{n+1}$, then $\mathcal{I}_n = (I_n)_{n \in \mathbb{N}}$ is endowed with a simplicial structure by $\tilde{f}(i_0, \ldots, i_n) = (i_{f(i_0)}, \ldots, i_{f(i_n)})$ for all $f \in \text{Hom}_\Delta(k, n)$. Let $U^n_{i_0, \ldots, i_n} = U^0_{i_0} \cap \cdots \cap U^0_{i_n}$, and let $\mathcal{U}_n = (U^n_{i})_{i \in I_n}$, then $\mathcal{U}_n$ is a simplicial cover of $M_\bullet$.

The “set” of covers of a simplicial space $M_\bullet$ is endowed with a partial preorder. Suppose $\mathcal{U}_n$ and $\mathcal{V}_n$ are open covers of $M_n$, with $\mathcal{U}_n = (U^n_i)_{i \in I_n}$ and $\mathcal{V}_n = (V^n_j)_{j \in J_n}$. We say that $\mathcal{V}$ is finer than $\mathcal{U}$ if for all $n$ there exists $\theta_n: J_n \to I_n$ such that $\theta_n(V^n_j) \subset U^n_{\theta(j)}$ for all $j$. The map $\theta_n(\mathcal{n} \in \mathbb{N})$ is required to be semi-simplicial (resp. $N$-simplicial) if $\mathcal{U}$ and $\mathcal{V}$ are semi-simplicial (resp. $N$-simplicial).
4.2. Čech cohomology. Let \( U_\bullet \) be a semi-simplicial open cover of \( M_\bullet \) and let \( \mathcal{A}^\bullet \) be a semi-simplicial abelian sheaf. Define a complex

\[
C^m_{ss}(U_\bullet; \mathcal{A}^\bullet) = \prod_{i \in I_n} \mathcal{A}^n(U^n_i),
\]

i.e. \( C^n_{ss}(U_\bullet; \mathcal{A}^\bullet) \) is the space of global sections of the pull-back of \( \mathcal{A}^n \) on \( \coprod_{i \in I_n} U^n_i \).

Define the differential \( d: C^n_{ss}(U_\bullet; \mathcal{A}^\bullet) \to C^{n+1}_{ss}(U_\bullet; \mathcal{A}^\bullet) \) by

\[
(d\epsilon)_I = \sum_{k=0}^{n+1} (-1)^k \epsilon_k \sigma_k(i),
\]

where \( \epsilon_k \sigma_k(i) \) is the “restriction” of \( \epsilon_k \sigma_k(i) \in \mathcal{A}^n(U^n_{\sigma_k(i)}) \) to a section in \( \mathcal{A}^{n+1}(U^{n+1}_I) \).

It is immediate to check that \( d^2 = 0 \), hence we may define the cohomology groups \( H^n_{ss}(U_\bullet; \mathcal{A}^\bullet) \).

Example 4.4. In Example 4.3, suppose \( \mathcal{A} \) is an abelian sheaf on \( M \) and that \( \mathcal{A}^n = A \) for all \( n \). Then \( H^n_{ss}(U_\bullet; \mathcal{A}^\bullet) \) is identical to the usual cohomology group \( H^n(U_0; \mathcal{A}) \).

Let \( U_\bullet \) be any open cover of \( M_\bullet \). We denote

\[
\begin{align*}
C^n(U_\bullet; \mathcal{A}^\bullet) &= C^n_{ss}(\sigma U_\bullet; \mathcal{A}^\bullet) \\
H^n(U_\bullet; \mathcal{A}^\bullet) &= H^n_{ss}(\sigma U_\bullet; \mathcal{A}^\bullet).
\end{align*}
\]

Now, we want to define Čech cohomology. The idea is to define \( H^n(M_\bullet; \mathcal{A}^\bullet) \) as the inductive limit over \( U_\bullet \) of the groups \( H^n(U_\bullet; \mathcal{A}^\bullet) \). The problem is that if \( \theta : J_\bullet \to I_\bullet \) is a refinement, then \( \theta \) indeed defines a restriction map

\[
\begin{align*}
\theta^* : C^n(U_\bullet; \mathcal{A}^\bullet) &\to C^n(V_\bullet; \mathcal{A}^\bullet) \\
(\theta^* \varphi)_j &= \text{restriction to } V^n_j \text{ of } \varphi_{\theta_e(j)}
\end{align*}
\]

which commutes with the differentials, and thus \( \theta^* \) defines a map

\[
\theta^* : H^n(U_\bullet; \mathcal{A}^\bullet) \to H^n(V_\bullet; \mathcal{A}^\bullet).
\]

However, that map may depend on the choice of \( \theta \). On the other hand we have the

Lemma 4.5. Let \( N \in \mathbb{N} \). Suppose that \( U_\bullet \) and \( V_\bullet \) are open covers of \( M_\bullet \) such that \( V_\bullet \) admits a \( N \)-simplicial structure. Suppose that \( V_\bullet \) is finer than \( U_\bullet \) and that \( \theta_0, \theta_1 : U_\bullet \to V_\bullet \) are two refinements. Then for all \( n \leq N \) there exists \( H : C^n(U_\bullet; \mathcal{A}^\bullet) \to C^{n-1}(V_\bullet; \mathcal{A}^\bullet) \) (with the convention \( C^{-1} = \{0\} \)) such that \( \theta_1^* - \theta_0^* = dH + Hd \).

(In the lemma above, we say that \( V_\bullet \) has a \( N \)-simplicial structure if the \( N \)-skeleton \( (V_n)_{n \in \mathbb{N}} \) has a \( N \)-simplicial structure.)
Proof. Define for all $\varphi \in C^n(\mathcal{U}_*; \mathcal{A}^*)$ and for all $\lambda \in \Lambda_{n-1}(J)$ (recall notation (4.1)):

$$\eta \in \mathcal{U}_{\alpha_k}(\lambda),$$

where as usual $\eta_k: [n] \to [n - 1]$ is the $k$-th degeneracy map, and $\alpha_k$ is defined as follows: for all $f \in \text{Hom}_{\Delta'}(r, n)$, let

$$\alpha_k(\lambda)(f) = \begin{cases} 
\theta_0(\lambda(\eta_k \circ f)) & \text{if } \{k, k+1\} \not\subset f([r]) \text{ and } f(0) \leq k \\
\theta_1(\lambda(\eta_k \circ f)) & \text{if } \{k, k+1\} \not\subset f([r]) \text{ and } f(0) \geq k + 1 \\
\theta_0(\eta_k(\lambda(f')) & \text{if } \{k, k+1\} \subset f([r])
\end{cases}$$

where in the third line, $k'$ is the integer such that $f(k') = k$ and $f'$ is the unique morphism in $\text{Hom}_{\Delta'}(r - 1, n - 1)$ such that the following diagram commutes:

$$\begin{array}{ccc}
[r] & \xrightarrow{f} & [n] \\
\downarrow{\eta_k'} & & \downarrow{\eta_k} \\
[r - 1] & \xrightarrow{f'} & [n - 1]
\end{array}$$

i.e. $f'(i) = f(i)$ for $i \leq k'$ and $f'(i) = f(i + 1) - 1$ for $i \geq k' + 1$.

Let us first check that formula (4.7) makes sense, i.e. that $V^{n-1}_\lambda \subset \tilde{\eta}_k^{-1}(U_{\alpha_k}(\lambda))$ for all $k$.

Since $U_{\alpha_k}(\lambda) = \bigcap_{r=0}^{n} \bigcup_{f \in \text{Hom}_{\Delta'}(r, n)} \tilde{f}^{-1}(U_{\alpha_k}(\lambda)(f))$, we need to show that

$$V^{n-1}_\lambda \subset \tilde{\eta}_k^{-1}(U_{\alpha_k}(\lambda)).$$

If $\{k, k+1\} \not\subset f([r])$ and $f(0) \leq k$ then

$$V^{n-1}_\lambda \subset (\eta_k \circ \tilde{f})^{-1}(V^{r-1}_\lambda(\eta_k \circ f)) \text{ by definition of } V^{n-1}_\lambda$$

$$= \tilde{\eta}_k^{-1}(\tilde{f}^{-1}(V^{r-1}_\lambda(\eta_k \circ f)))$$

$$\subset \tilde{\eta}_k^{-1}(\tilde{f}^{-1}(U_{\alpha_k}(\lambda)(f))) \text{ since } \theta_0: J_0 \to I_0 \text{ is a refinement}$$

$$= \tilde{\eta}_k^{-1}(U_{\alpha_k}(\lambda)(f)).$$

If $\{k, k+1\} \not\subset f([r])$ and $f(0) \geq k + 1$ the proof of (4.9) is the same, except that $\theta_0$ is replaced by $\theta_1$.

If $\{k, k+1\} \subset f([r])$ then

$$V^{n-1}_\lambda \subset (\tilde{f})^{-1}(V^{r-1}_\lambda(\tilde{f}))$$

$$\subset \tilde{f}^{-1}(\tilde{\eta}_{k'}^{-1}(V^{r}_\lambda(\eta_k(f')))) \text{ (recall } V_\lambda \text{ is } N\text{-simplicial})$$

$$\subset \tilde{\eta}_k^{-1}(\tilde{f}^{-1}(U_{\alpha_k}(\lambda)(f))) \text{ by (4.8)},$$

thus (4.9) is proved.
Let us show that

\[(4.10) \quad dH + Hd = \theta_1^* - \theta_0^*.
\]

We have

\[(Hd\varphi)_{\lambda} = \sum_{k=0}^{n} \sum_{\ell=0}^{n-1} A_{k,\ell}, \quad (dH\varphi)_{\lambda} = \sum_{k=0}^{n} \sum_{\ell=0}^{n-1} B_{k,\ell},
\]

where \(A_{k,\ell} = (-1)^{k+\ell+1} \eta_{k} \varphi_{\lambda}(\alpha(\eta_{k}))\) and \(B_{k,\ell} = (-1)^{k+\ell} \xi_{k} \varphi_{\lambda}(\alpha(\eta_{k}))\).

We have \(A_{\ell,\ell} = \eta_{\ell} \varphi_{\lambda}(\alpha(\eta_{\ell}))\), and for all \(f \in \mathcal{P}_n\),

\[\tilde{\epsilon}(\alpha(\eta_{\ell}))(f) = \alpha(\lambda)(\epsilon_{\ell} \circ f) = \theta_{j}(\lambda(\eta_{\ell} \circ \epsilon_{\ell}(f))) = \theta_{j}(\lambda(f)),\]

where \(j = 0 \iff \epsilon_{\ell} \circ f(0) \leq \ell \iff f(0) \leq \ell - 1, \) and \(j = 1\) otherwise.

Let \(\lambda^{(p)}(f) = \left\{ \begin{array}{ll} \theta_{0}(\lambda(f)) & \text{if } f(0) \leq p \\ \theta_{1}(\lambda(f)) & \text{if } f(0) \geq p + 1 \end{array} \right., \) then

\[(4.11) \quad A_{\ell,\ell} = \varphi_{\lambda}^{\ell - 1}.
\]

Similarly, we have \(A_{\ell+1,\ell} = - \eta_{\ell} \varphi_{\lambda+1} \varphi_{\lambda}(\alpha(\eta_{\ell})) = - \varphi_{\lambda+1}(\alpha(\eta_{\ell})), \) and for all \(f \in \mathcal{P}_n,\)

\[\tilde{\epsilon}_{\ell+1}(\alpha(\eta_{\ell}))(f) = \alpha(\lambda)(\epsilon_{\ell+1} \circ f) = \theta_{j}(\lambda(\eta_{\ell} \circ \epsilon_{\ell+1}(f))) = \theta_{j}(\lambda(f)),\]

where \(j = 0 \iff \epsilon_{\ell+1} \circ f(0) \leq \ell \iff f(0) \leq \ell, \) and \(j = 1\) otherwise. We thus get

\[(4.12) \quad A_{\ell+1,\ell} = - \varphi_{\lambda}^{\ell}.
\]

From \((4.11)\) and \((4.12)\) we obtain \(\sum_{\ell \leq k \leq \ell + 1} A_{k,\ell} = \varphi_{\lambda}(-1) - \varphi_{\lambda}(\eta) = \theta_1^* \varphi - \theta_0^* \varphi.\)

Let us examine the other terms. We have \(\sum_{k \leq \ell - 1} A_{k,\ell} = \sum_{\ell \leq k \leq n-1} A_{k,\ell+1}\) and \(\sum_{k \geq \ell + 2} A_{k,\ell} = \sum_{0 \leq k \leq n} A_{k,\ell+1}.\) To complete the proof of \((4.10)\) it suffices to show that \(A_{k,\ell+1} + B_{k,\ell} = 0\) for \(k \leq \ell, \) and that \(A_{k,\ell+1} + B_{k,\ell} = 0\) for \(k \geq \ell + 1.\) Noting that \(\eta_{\ell+1} \epsilon = \epsilon_{\ell} \eta_{\ell} \) for \(k \leq \ell \) and that \(\eta_{\ell} \epsilon_{k+1} = \epsilon_{k} \eta_{\ell} \) for \(k \geq \ell + 1,\) it suffices to show that

1. \(\tilde{\epsilon}_{k}(\alpha_{k+1}(\lambda)) = \alpha_{\ell}(\tilde{\epsilon}_{k}(\lambda)) \) for \(k \leq \ell,\)
2. \(\tilde{\epsilon}_{k+1}(\alpha_{k}(\lambda)) = \alpha_{\ell}(\tilde{\epsilon}_{k}(\lambda)) \) for \(k \geq \ell + 1.\)

Let us show (a). Suppose that \(f \in \text{Hom}_{\Delta}(r, n)\) and let us first treat the case \(\{\ell, \ell + 1\} \not\subseteq f([r]).\) Then, letting \(j = 0\) for \(\epsilon_{\ell} \circ f(0) \leq \ell + 1 \iff f(0) \leq \ell,\) and \(j = 1\) otherwise, we have

\[\tilde{\epsilon}_{k}(\alpha_{k+1}(\lambda))(f) = \alpha_{\ell+1}(\lambda)(\epsilon_{\ell} \circ f) = \theta_{j}(\lambda(\eta_{\ell+1} \circ \epsilon_{k} \circ f)) = \theta_{j}(\lambda(\epsilon_{\ell} \circ \eta_{k} \circ f)) = \alpha_{\ell}(\tilde{\epsilon}_{k}(\lambda))(f).
\]

Let us treat the case \(\{\ell, \ell + 1\} \subseteq f([r]).\) Let \(\ell' \) such that \(f(\ell') = \ell, \) and let \(f': [r - 1] \rightarrow [n - 1] \) be the increasing map such that the diagram

\[(4.13) \quad \begin{array}{ccc}
r & \xrightarrow{f} & n \\
\downarrow \eta & & \downarrow \eta' \\
r' - 1 & \xrightarrow{f'} & n - 1
\end{array}
\]
commutes. Since \( \varepsilon_k \circ \eta = \eta \circ \varepsilon_k : [n] \to [n] \), the diagram

\[
\begin{array}{ccc}
[r] & \xrightarrow{\varepsilon_k \circ f} & [n + 1] \\
\downarrow \eta & & \downarrow \eta' \\
[r - 1] & \xrightarrow{\varepsilon_k \circ f'} & [n] \\
\end{array}
\]

commutes. We thus see that

\[
\tilde{\varepsilon}_k (\alpha_{\ell + 1}(\lambda))(f) = \alpha_{\ell + 1}(\lambda)(\varepsilon_k \circ f) = \theta_0(\tilde{\eta}_e(\lambda(\varepsilon_k \circ f'))) \quad \text{by (4.14)}
\]

This completes the proof of (a).

Let us show (b): the method is similar. If \( \{\ell, \ell + 1\} \not\subset f([r]) \) then

\[
\tilde{\varepsilon}_{k + 1}(\alpha_{\ell}(\lambda))(f) = \alpha_{\ell}(\lambda)(\varepsilon_{k + 1} \circ f) = \theta_j(\lambda(\varepsilon_k \circ f')) = \theta_j((\tilde{\varepsilon}_k \lambda)(\varepsilon_k \circ f)).
\]

If \( \{\ell, \ell + 1\} \subset f([r]) \), let \( \ell' \) such that \( f(\ell') = \ell \) and let \( f' \) such that

\[
\begin{array}{ccc}
[r] & \xrightarrow{f} & [n] \\
\downarrow \eta & & \downarrow \eta \\
[r - 1] & \xrightarrow{f'} & [n - 1] \\
\end{array}
\]

commutes. Then since \( \varepsilon_k \circ \eta = \eta \circ \varepsilon_{k + 1} : [n] \to [n] \), the diagram

\[
\begin{array}{ccc}
[r] & \xrightarrow{\varepsilon_{k + 1} \circ f} & [n + 1] \\
\downarrow \eta' & & \downarrow \eta \\
[r - 1] & \xrightarrow{\varepsilon_k \circ f'} & [n] \\
\end{array}
\]

commutes, therefore

\[
\tilde{\varepsilon}_{k + 1}(\alpha_{\ell}(\lambda))(f) = \alpha_{\ell}(\lambda)(\varepsilon_{k + 1} \circ f) = \theta_0(\tilde{\eta}_e(\lambda(\varepsilon_k \circ f'))) \quad \text{by (4.15)}
\]

This completes the proof of (b) and hence of (4.10).

Let us now define

\[
\tilde{H}^n(M; A^\bullet) = \varprojlim H^n(U; A^*)
\]
where \( \mathcal{U}_* \) runs over open covers of \( M_* \) whose \( N \)-skeleton admits a \( N \)-simplicial structure for some \( N \geq n+1 \). (Recall that \( H^n(M_*; A^*) \) was defined by equation (4.4).)

To avoid set-theoretic difficulties (since the collection of open covers is not a set), we can restrict ourselves to open covers indexed by sets of cardinality \( \leq \sum_n \# M_n \).

By Lemma 4.3 above, if \( \mathcal{V}_* \) is finer than \( \mathcal{U}_* \) and if \( \mathcal{V}_* \) has a \( N \)-simplicial structure, then there is a canonical map \( H^*(\mathcal{U}_*; A^*) \to H^*(\mathcal{V}_*; A^*) \) defined by Eqn. (4.6) (\( \theta \) is not required to respect the \( N \)-simplicial structures).

Since every open cover of \( M_* \) admits a \( N \)-simplicial refinement (see Convention 4.2), the inductive limit is well-defined and is an abelian group.

Moreover, for every open cover \( \mathcal{U}_* \) of \( M_* \), \( N \)-simplicial or not, there is a canonical map \( H^n(\mathcal{U}_*; A^*) \to H^n(M_*; A^*) \) obtained by mapping \( H^n(\mathcal{U}_*; A^*) \) to \( H^n(\mathcal{V}_*; A^*) \) using (4.6), where \( \mathcal{V}_* \) is any refinement admitting a \( N \)-simplicial structure for some \( N \geq n+1 \).

It is clear that any element \([\varphi]\) of \( H^n(\mathcal{U}_*; A^*) \) maps to 0 in \( \tilde{H}^n(M_*; A^*) \) if and only if there exists a refinement (\( N \)-simplicial or not) such that \([\varphi]\) maps to 0 in \( H^n(\mathcal{V}_*; A^*) \). Thus, in some sense, we can say that \( \tilde{H}^n(M_*; A^*) \) is the inductive limit of \( H^n(\mathcal{U}_*, A^*) \) where \( \mathcal{U}_* \) runs over all open covers of \( M_* \).

**Example 4.6.** Consider the case of a discrete group \( G \), and suppose that \( A^* \) is the sheaf associated to a \( G \)-module \( A \) (Definition 3.9). Then, from (3.2) and below, we see that

\[
(dc)_{\lambda}(g_1, \ldots, g_{n+1}) = g_1c_{\xi_0\lambda}(g_2, \ldots, g_{n+1}) + \\
+ \sum_{k=1}^{n} (-1)^k c_{\xi_{k}\lambda}(g_1, \ldots, g_{k}g_{k+1}, \ldots, g_{n+1}) + \\
+ (-1)^{n+1} c_{\xi_{n+1}\lambda}(g_1, \ldots, g_n).
\]

(Compare with (1.1).) Considering the maximal open cover \((U^n_x)_{x \in G^n}\) where \( U^n_x = \{x\} \), one easily sees that Čech cohomology coincides with usual group cohomology.

**Remark 4.7.** As in [5], \( \tilde{H}^n(M_*; A^*) \) can be seen as the \( n \)-th cohomology group of a canonical Čech complex. Indeed, let \( \mathcal{R}(M_*) \) be the set of covers of the form \( \mathcal{U}_n = (U^n_x)_{x \in M_n} \). If \( \mathcal{U}_n \) and \( \mathcal{V}_n \) are in \( \mathcal{R}(M_*) \), let us say that \( \mathcal{V}_n \) is finer than \( \mathcal{U}_n \) if for all \( n \) and all \( x \in M_n \) we have \( V^n_x \subset U^n_x \). Given \( \mathcal{U}_n \) in \( \mathcal{R}(M_*) \), denote by \( \sigma_N \mathcal{U}_n \) the associated \( N \)-simplicial cover (see Convention 4.2) and let

\[ C^n_N(\mathcal{U}_n; A^*) = \lim_{\mathcal{U}_n} C^n(\sigma_N \mathcal{U}_n; A^*) = \lim_{\mathcal{U}_n} C^n_{ss}(\sigma_N \mathcal{U}_n; A^*), \]

where \( \mathcal{U}_n \) runs over open covers in \( \mathcal{R}(M_*) \).

Then \( \tilde{H}^n(M_*; A^*) \) is the cohomology of \( C^n_N(\mathcal{U}_n; A^*) \) whenever \( N \geq n+1 \).
4.3. Compatibility with usual Čech cohomology for spaces. Let $M$ be a space and $A$ an abelian sheaf on $M$. Denote by $M_\ast$ be the constant simplicial space associated to $M$ and by $A^\ast$ the sheaf on $M_\ast$ corresponding to $A$.

We want to show the

**Proposition 4.8.** With the above assumptions, the usual Čech cohomology groups $\tilde{H}^n(M; A)$ are isomorphic to the Čech cohomology groups $\check{H}^n(M_\ast; A^\ast)$.

**Proof.** To determine $\check{H}^n(M_\ast; A^\ast)$ we can restrict ourselves to covers $\mathcal{U}_\ast$ of the form $\mathcal{U}_\ast=\{U^n_i\}_{i \in I_\ast}$ where $I_\ast \subseteq I_\ast^{n+1}$ and $U^n_{i_0, \ldots, i_n}=U^n_{i_0} \cap \cdots \cap U^n_{i_n}$. Let us show that $H^\ast(\mathcal{U}_\ast; A^\ast)=H^\ast(\mathcal{U}_0; A)$. It is not obvious that these two groups are isomorphic, since $C^\ast(\mathcal{U}_\ast; A^\ast)=C^\ast_s(\mathcal{U}_\ast; A^\ast)$ while $C^\ast(\mathcal{U}_0; A)=C^\ast_s(\mathcal{U}_0; A^\ast)$. However, we show that these two complexes are homotopically equivalent:

First, there is an obvious map

$$q: C^\ast(\mathcal{U}_\ast; A^\ast) \to C^\ast(\mathcal{U}_0; A)$$

defined by $(q\varphi)_{i_0, \ldots, i_n}=\varphi_{\lambda(n)}$, where

$$\lambda(i)(f)=(i_{f(0)}, \ldots, i_{f(r)}) \quad \text{for all } f \in \text{Hom}_\Delta(r, n).$$

In the other direction, define $\iota: C^\ast(\mathcal{U}_0, A) \to C^\ast(\mathcal{U}_\ast; A^\ast)$ by

$$(\iota c)_\lambda=c_{\lambda_0, \ldots, \lambda_n},$$

where $\lambda_k$ denotes $\lambda(p_k)$ and $p_k:\{0\} \to [n]$ denotes the map $p_k(0)=k$.

We have $q \circ \iota=\text{Id}$. Indeed, $((q \circ \iota)(c))_{i_0, \ldots, i_n}=((\iota c)_\lambda)=c_{\lambda_0, \ldots, \lambda_n}$.

Conversely, we don’t have $\iota \circ q=\text{Id}$ since $(\iota \circ q)(\varphi)(\lambda)=\varphi_{\lambda'}$, where

$$\lambda'(f)=(\lambda(f(0), \ldots, \lambda(f(r))) \quad \text{for all } f \in \text{Hom}_\Delta(r, n).$$

However, $\iota \circ q$ and $\text{Id}$ are homotopic. Indeed, define $H: C^n(\mathcal{U}_\ast; A^\ast) \to C^{n-1}(\mathcal{U}_\ast; A^\ast)$ by

$$(H \varphi)_\lambda=\sum_{k=0}^{n-1}(-1)^k\check{n}_k^\ast \varphi \alpha_k(\lambda)$$

where

$$\alpha_k(\lambda)(f)=\begin{cases} 
\lambda(\eta_k \circ f) & \text{if } \{k, k+1\} \not\subseteq f([r]) \text{ and } f(0) \leq k \\
\lambda(\eta_k \circ f) & \text{if } \{k, k+1\} \not\subseteq f([r]) \text{ and } f(0) \geq k+1 \\
\check{n}_k'(\lambda(f')) & \text{if } \{k, k+1\} \subseteq f([r])
\end{cases}$$

($f(k')=k$ and $f'$ is defined as in the proof of Lemma 4.5, also recall that $\check{n}_k'(i_0, \ldots, i_{r-1})=(i_0, \ldots, i_k', i_{k'}, \ldots, i_{r-1})$.)

Then the same proof as in Lemma 4.5 shows that $dH+Hd=\iota \circ q-\text{Id}$. We leave out details; anyway we will show later that Čech cohomology coincides with sheaf cohomology for paracompact simplicial spaces, so (at least in the paracompact case) this will provide a second proof that sheaf cohomology for spaces coincides with sheaf cohomology for the associated constant simplicial space.

$\square$
Let us introduce some notation:

**Notation 4.9.** For any simplicial space $M_*$ and any open cover $U_*$, let us write elements $\lambda \in \Lambda_n$ (see (4.11)) as $(2^{n+1} - 1)$-tuples $(\lambda_S)_{\emptyset \neq S \subset [n]}$, where subsets $S$ are ordered first by cardinality, then by lexicographic order. For instance, the triple $(\lambda_0, \lambda_1, \lambda_{01})$ represents the element $\lambda \in \Lambda_1$ such that $\lambda(\{0\}) = \lambda_0$, $\lambda(\{1\}) = \lambda_1$, $\lambda(\{0,1\}) = \lambda_{01}$. A cochain in $C^1(U_*;A^*)$ is thus a family $(\varphi_{\lambda_0\lambda_1\lambda_{01}})$.

Then, we can write (4.18) more explicitly. For instance, the formulas for $n=1$ and $n=2$ are respectively

\[
(H\varphi)_{\lambda_0} = \varphi_{\lambda_0\lambda_0\lambda_0}, \quad (H\varphi)_{\lambda_0\lambda_1\lambda_{01}} = \varphi_{\lambda_0\lambda_1\lambda_{01}} - \varphi_{\lambda_0\lambda_1\lambda_{01}} - \varphi_{\lambda_0\lambda_1\lambda_{01}}\lambda_{11}^{11},
\]

where $\lambda'_{i_0\ldots i_r}$ denotes the $r + 1$-tuple $(\lambda_{i_{r-1}}, \ldots, \lambda_{i_r}) \in I_r = I_r$.\[\]

4.4. **Long exact sequences in Čech cohomology.** In this section, most proofs are almost identical to [5], thus we will only sketch them.

**Proposition 4.10.** If $0 \rightarrow A^* \rightarrow A^* \rightarrow A''^* \rightarrow 0$ is an exact sequence of abelian presheaves then the functor $A \mapsto C^*_N(M_*;A^*)$ (see Remark 4.7) maps the above exact sequence to an exact sequence of complexes.

**Proof.** $0 \rightarrow C^*_N(\sigma \sigma_N U_*;A^*) \rightarrow C^*_N(\sigma \sigma_N U_*;A^*) \rightarrow C^*_N(\sigma \sigma_N U_*;A''^*) \rightarrow 0$ is exact for every open cover $U_*$. \[\]

Let us say that a simplicial space $M_*$ is paracompact if each $M_n$ is paracompact.

**Proposition 4.11.** [5] theorem 5.10.2] If $A^*$ is an abelian presheaf on a paracompact simplicial space $M_*$ such that $A^*$ induces the zero sheaf then $\check{H}^n(M_*;A^*)=0$ for all $n \geq 0$.

**Proof.** Using paracompactness, every cohomology class is represented by a cocycle in $C^*_N(U_*;A^*)$ with $U_n$ locally finite $\forall n$. Then, using the fact that each $A^n$ induces the zero sheaf, every cochain of that cover becomes zero once restricted to a suitable finer cover. \[\]

**Corollary 4.12.** If an abelian presheaf $A^*$ over a paracompact simplicial space $M_*$ induces the sheaf $\check{A}^*$ then $\check{H}^n(M_*;A^*) \cong \check{H}^n(M_*;\check{A}^*)$.

**Proof.** There are exact sequences of presheaves

\[
0 \rightarrow N^* \rightarrow A^* \rightarrow J^* \rightarrow 0, \quad 0 \rightarrow J^* \rightarrow A^* \rightarrow Q^* \rightarrow 0,
\]

where $N^*$ and $Q^*$ induce the zero sheaf: $N^n(U)$ is the set of sections in $A^n(U)$ whose germ at every point is zero, and $\check{A}^n(U) = \{((\sigma_i)_{i \in I})/\sim$, where $\sigma_i \in J^n(U_i)$ for some open cover $(U_i)_{i \in I}$ of $U$ and the equivalence relation
~ is defined by \((\sigma_i)_{i \in I} \sim (\sigma'_j)_{j \in J}\) if and only if, for all \(i, j\), \(\sigma_i|_{U_i \cap U'_j} = \sigma'_j|_{U_i \cap U'_j}\). The conclusion follows from Propositions 4.10 and 4.11 above.

\[\square\]

**Corollary 4.13.** If \(0 \to A'^* \to A^* \to A''^*\) is an exact sequence of sheaves over a paracompact simplicial space \(M_\bullet\), then there is a natural long exact sequence

\[
0 \to \tilde{H}^0(M_\bullet; A'^*) \to \tilde{H}^0(M_\bullet; A^*) \to \tilde{H}^0(M_\bullet; A''^*) \overset{\partial}{\to} \tilde{H}^1(M_\bullet; A^*) \to \cdots
\]

**Proof.** Follows from Proposition 4.10 and Corollary 4.12 \(\square\)

5. Low dimensional Čech cohomology

5.1. The group \(\tilde{H}^0\). Consider a simplicial space \(M_\bullet\). Let \(U_\bullet\) be an open cover of \(M_\bullet\), then, using Notation 4.10, a 0-cocycle is given by a family \((c_0)_{\lambda_0 \in I_0}\), with \(c_0 \in \mathcal{A}^0(U_{\lambda_0})\), and

\[
0 = (dc)_{\lambda_0 \lambda_1 \lambda_0} = \tilde{c}^*_{\lambda_1} - \tilde{c}^*_0 c_0
\]

on \(U_{\lambda_0}^1 = \bigcap \tilde{c}^{-1}_0(U_{\lambda_1}^0) \cap \tilde{c}^{-1}_1(U_{\lambda_1}^0)\). Therefore, \(\tilde{c}_{\lambda_1}^* = \tilde{c}_{\lambda_0}^* c_0\) on \(\tilde{c}_0^{-1}(U_{\lambda_0}^0) \cap \tilde{c}_1^{-1}(U_{\lambda_1}^0)\) for all \(\lambda_0, \lambda_1 \in I_0\). Applying \(\tilde{c}_0^*\) to both sides, we find that \(c_0 = c_{\lambda_1}\) on \(U_{\lambda_0}^0 \cap U_{\lambda_1}^0\). Since \(\mathcal{A}^0\) is a sheaf, there exists a global section \(\varphi \in \mathcal{A}^0(M_0)\) such that \(c_0\) is the restriction of \(\varphi\) to \(U_{\lambda_0}^0\) for all \(\lambda_0 \in I_0\). Now, Equation 5.1 is equivalent to \(\tilde{\lambda}_1^* \varphi = \tilde{\lambda}_0^* \varphi\). We have thus proved:

**Proposition 5.1.** Let \(A^*\) be an abelian sheaf on a simplicial space \(M_\bullet\) and let \(U_\bullet\) be an open cover of \(M_\bullet\). Then

\[
\tilde{H}^0(M_\bullet; A^*) = H^0(U_\bullet; A^*) = \Gamma_{\text{inv}}(A^*) := \ker (\mathcal{A}^0(M_0) \to \mathcal{A}^1(M_1)).
\]

(Of course, in the case of a groupoid and an abelian \(G\)-sheaf, a section in \(\mathcal{A}^0(G_0)\) is in \(\Gamma_{\text{inv}}(A^*)\) if and only if it is an invariant section in the usual sense, i.e. under the action of \(G\)).

5.2. The group \(\tilde{H}^1\). Consider a groupoid \(G\). The cocycle relation in degree 1 is

\[
\tilde{c}_{\lambda_1}^* c_{\lambda_2 \lambda_1 \lambda_2 \lambda_0} - \tilde{c}_{\lambda_1}^* c_{\lambda_0 \lambda_2 \lambda_0} + \tilde{c}_{\lambda_0}^* c_{\lambda_0 \lambda_1 \lambda_0} = 0
\]

on \(U_{\lambda_0 \lambda_1 \lambda_2 \lambda_0 \lambda_2 \lambda_1 \lambda_2 \lambda_0 \lambda_2 \lambda_1 \lambda_0 \lambda_2 \lambda_1 \lambda_2 \lambda_0 \lambda_2 \lambda_1 \lambda_0}^1\). Exactly the same method as in the preceding paragraph shows that \(c_{\lambda_0 \lambda_1 \lambda_0}\) does not depend on the choice of \(\lambda_0\), hence there exists a section \(\varphi_{\lambda_0} \in \mathcal{A}^1(\tilde{c}_0^{-1}(U_{\lambda_0}^0) \cap \tilde{c}_1^{-1}(U_{\lambda_1}^0))\) such that \(c_{\lambda_0 \lambda_1 \lambda_0}\) is the restriction to \(U_{\lambda_0 \lambda_1 \lambda_0}^1\) of \(\varphi_{\lambda_0} \lambda_1\). The cocycle relation becomes

\[
\tilde{c}_{\lambda_0}^* \varphi_{\lambda_0} \lambda_1 - \tilde{c}_{\lambda_1}^* \varphi_{\lambda_0} \lambda_2 + \tilde{c}_{\lambda_0}^* \varphi_{\lambda_0} \lambda_1 = 0.
\]

Coboundaries are cocycles of the form \(\varphi_{\lambda_0} \lambda_1 = \tilde{c}_{\lambda_0}^* \alpha_{\lambda_0} - \tilde{c}_{\lambda_1}^* \alpha_{\lambda_1}\).

**Proposition 5.2.** Let \(G\) be a topological groupoid and \(A\) be a \(G\)-module. Denote by \(A^*\) the associated sheaf on \(G_\bullet\). Then \(\tilde{H}^1(G_\bullet; A^*)\) is the group of \(G\)-equivariant locally trivial \(A\)-principal bundles over \(G_0\).
Proof. This is well-known. Let us give the proof for completeness. Suppose we are given such a principal bundle \( E \rightarrow G_0 \). Choose an open cover \( U_0=(U_i^0)_{i \in I_0} \) of \( G_0 \) such that for all \( i \) there exists a (not necessarily equivariant) section \( x \mapsto \sigma_i(x) \) of the bundle \( E \rightarrow G_0 \) over the open subset \( U_i^0 \). Let us define \( \varphi \)(We have denoted by “+” the action of \( \text{ant} \) section \( x \))(5.3) \[ \sigma_{\lambda_0}(r(g)) = g\sigma_{\lambda_1}(s(g)) + \varphi_{\lambda_0\lambda_1}(g). \]

(We have denoted by “+” the action of \( A \) on the \( A \)-principal bundle \( E \).) If \( (g,h) \in G_2 \), then we get

\[
\begin{align*}
\sigma_{\lambda_1}(r(h)) &= h\sigma_{\lambda_2}(s(h)) + \varphi_{\lambda_1\lambda_2}(h) \\
\sigma_{\lambda_0}(r(gh)) &= gh\sigma_{\lambda_2}(s(gh)) + \varphi_{\lambda_0\lambda_2}(gh).
\end{align*}
\]

Substitute (5.4) in (5.3) and compare to (5.5) to get the cocycle relation

\[
\varphi_{\lambda_0\lambda_2}(gh) = g\varphi_{\lambda_1\lambda_2}(h) + \varphi_{\lambda_0\lambda_1}(g)
\]

(which is exactly (5.2)).

If \( \sigma'_i \) is another section, let \( \alpha_i(x)=\sigma_i(x) - \sigma'_i(x) \), then from (5.3) and its analogue for \( \sigma' \), we get by substraction \( \alpha_{\lambda_0}(r(g)) = g\alpha_{\lambda_1}(s(g)) + \varphi_{\lambda_0\lambda_1}(g) - \varphi'_{\lambda_0\lambda_1}(g) \), i.e. \( \varphi' - \varphi = da \).

The above shows that any \( G \)-equivariant \( A \)-principal bundle which is trivial over each open set \( U_i^0 \) defines an element of \( \tilde{H}^1(U_i;\mathcal{A}) \). Moreover, passing to a finer cover obviously doesn’t change the Čech cohomology class, hence any bundle as above defines a cohomology class in \( \tilde{H}^1(G;\mathcal{A}) \).

Conversely, suppose we are given \( (\varphi_{\lambda_0\lambda_1}) \) satisfying the cocycle relation (5.6). Define

\[ E = (\coprod_{i \in I_0} U_i^0 \times_{G_0} A) / \sim \]

with the identifications \( (\lambda_0, x, a) \sim (\lambda_1, x, a + \varphi_{\lambda_0\lambda_1}(x)) \) and action

\[ g \cdot [(\lambda_1, s(g), a)] = [(\lambda_0, r(g), a - \varphi_{\lambda_0\lambda_1}(g))].\]

It is elementary to check that \( E \) is a locally trivial \( G \)-equivariant \( A \)-principal bundle, and that the associated 1-cocycle is indeed \( \varphi \). \( \square \)

5.3. The group \( \tilde{H}^2 \), extensions and the Brauer group. Let \( G \) be a topological groupoid, and let \( A \) be a \( G \)-module. Let us denote by \( \text{ext}(G, A) \) the set of extensions of the form

\[ A \xrightarrow{i} E \xrightarrow{\pi} G \]

such that the unit spaces of the groupoids \( A, E \) and \( G \) are all equal to \( G_0 \), the maps \( i \) and \( \pi \) are the identity map on \( G_0 \), and such that for all \( \gamma \in E \) and all \( a \in A_{s(\gamma)} \) we have

\[ \gamma a \gamma^{-1} = \pi(\gamma) \cdot a. \]

In \( \pi(\gamma) \cdot a \), the dot denotes the action of \( G \) on the \( G \)-module \( A \).
Two such extensions \( A \to E \to G \) and \( A \to E' \to G \) are considered equivalent if there is commutative diagram

\[
\begin{array}{ccc}
A & \longrightarrow & E \\
\downarrow \text{Id} & & \downarrow \text{Id} \\
A & \longrightarrow & E'
\end{array}
\]

such that the map \( E \to E' \) is a groupoid isomorphism.

There is a canonical extension in \( \text{ext}(G,A) \): let \( E=A \times_{p,r} G=\{(a,g) \in A \times G | p(a)=r(g)\} \). The source and range maps in \( E \) are \( s(a,g)=s(g), \ r(a,g)=r(g) \). The product is \( (a,g)(b,h)=(a+g \cdot b,gh) \) (defined whenever \( s(g)=r(g) \); the product in \( A \) is written additively). The inclusion \( A \hookrightarrow E \) is \( i(a)=(a,p(a)) \) and the projection \( \pi: E \to G \) is \( \pi(a,g)=g \).

Let us call this extension the strictly trivial extension.

**Proposition 5.3.** Let \( A \xhookrightarrow{i} E \xrightarrow{\pi} G \) be an element of \( \text{ext}(G,A) \). The following are equivalent:

(i) the extension is strictly trivial;
(ii) there exists a groupoid morphism \( \sigma: G \to E \) which is a section of \( \pi \);
(iii) there exists \( \varphi: E \to A \) such that \( \varphi(a\gamma)=a\varphi(\gamma) \) for all \( (a,\gamma) \in A \times_{p,r} E \) and \( \varphi(\gamma_1\gamma_2)=\varphi(\gamma_1)+\pi(\gamma_1) \cdot \varphi(\gamma_2) \) for all composable pairs \( (\gamma_1,\gamma_2) \in E^2 \).

**Proof.** Let us sketch the easy proof.

(i) \( \implies \) (ii): take \( \sigma(g)=(r(g),g) \).

For (ii) \( \implies \) (iii), let \( \varphi(\gamma)=\gamma[\sigma \circ \pi(\gamma)]^{-1} \).

To prove (iii) \( \implies \) (i), the map \( \gamma \mapsto (\varphi(\gamma),\pi(\gamma)) \) is a groupoid isomorphism from \( E \) onto \( A \times_{p,r} G \). \( \square \)

The set \( \text{ext}(G,A) \) is an abelian group. The (“Baer”) sum \( E_1 \oplus E_2 \) of two extensions \( A \to E_1 \to G \) is given by the extension \( A \to E \to G \) with

\[
E=\{(\gamma_1,\gamma_2) \in E_1 \times E_2 | \pi_1(\gamma_1)=\pi_2(\gamma_2)\}/\sim
\]

where \( (a\gamma_1,\gamma_2) \sim (\gamma_1,a\gamma_2) \) if \( \pi_1(\gamma)=\pi_2(\gamma_2) \) and \( p(a)=r(\gamma_1) \). The map \( \pi: E \to G \) is given by \( \pi(\gamma_1,\gamma_2)=\pi_1(\gamma_1)=\pi_2(\gamma_2) \), and the inclusion \( i: A \to E \) is

\[
i(a)=(i_1(a),p(a)) \sim (p(a),i_2(a)).
\]

Finally, the inverse of the extension is

\[
A \xhookrightarrow{i^t} \bar{E} \xrightarrow{\pi} G
\]

where \( \bar{E}=E \) as a groupoid, but \( i^t(a)=-i(a) \). Denoting by \( \bar{\gamma} \) the element in \( \bar{E} \) which is the same as the element \( \gamma \in E \), this means that \( \pi(\gamma)=-\bar{\gamma} \). To check that \( E \oplus \bar{E} \) is strictly trivial, just note that for all \( g \in G \), the element \( (\gamma,\bar{\gamma}) \in E \oplus \bar{E} \), does not depend on the choice of \( \gamma \in E \) such that \( \pi(\gamma)=g \), since \( (a\gamma,\bar{\gamma})=(\gamma,(-a)\bar{\gamma}) \sim (\gamma,\bar{\gamma}) \). Therefore, \( g \mapsto \sigma(g)=(\gamma,\bar{\gamma}) \) defines a cross-section of \( \pi \).
Example 5.4. When $A = G_0 \times \mathbb{T}$, and $G$ does not act on $\mathbb{T}$, we obtain the group $T \! w(G)$ of twists of $G$ \cite{10}.

It is clear that $\text{ext}(G, A)$ is covariant with respect to $G$-module morphisms. It is also contravariant with respect to groupoid morphisms. Indeed, let $f : G' \to G$ be a groupoid morphism, and let $A' = f^* A = \{(x, a) \in G'_0 \times A | f(x) = p(a)\}$. Then $A'$ is a $G'$-module with respect to the action $g' \cdot (x, a) = (r(g'), f(g') \cdot a)$, and there is a “pull-back” morphism

$$f^* : \text{ext}(G, A) \to \text{ext}(G', f^* A)$$

defined as follows. Let $A \to E \to G$ be an element of $\text{ext}(G, A)$, then its pull-back by $f$ is the extension

$$A \cong E' \xrightarrow{\pi'} G'$$

where $E' = \{(\gamma, g') \in E \times G' | \pi(\gamma) = f(g')\}$, $\pi'(\gamma, g') = g'$, $i'(a) = (i(a), p(a))$. The groupoid structure on $E'$ is the one induced from the product groupoid $E \times G'$.

More generally, suppose that $G'_0 \xleftarrow{\rho} Z \xrightarrow{\sigma} G_0$ is a generalized morphism from $G'$ to $G$ (see \cite{2.3}). Put $A' = Z \times_{G'} A := \{(z, a) \in Z \times A | \sigma(z) = p(a)\} / \sim$, where $(zg, g^{-1}a) \sim (z, a)$ for all triples $(z, a, g) \in Z \times A \times G$ such that $\sigma(a) = p(a) = r(g)$. It is obvious that $A'$ is a $G'$-module with sum $(z, a) + (z, b) = (z, a + b)$ and left $G'$-action $g'(z, a) = (g'z, a)$.

The slight defect of the group $\text{ext}(G, A)$ is that it is not invariant by Morita equivalence. To remedy this, let us define

Definition 5.5.

$$\text{Ext}(G, A) = \lim\sup_{\mathcal{U}} \text{ext}(G[\mathcal{U}], A[\mathcal{U}])$$

where $\mathcal{U}$ runs over open covers of $G_0$ (see notation \cite{2.2}).

By construction, the group $\text{Ext}(G, A)$ is invariant under Morita equivalence (see Proposition \cite{2.2}).

Let us now come to the relation between 2-cohomology and extensions:

Proposition 5.6. Let $G$ be a topological groupoid, $A$ a $G$-module, and $A^*$ the associated sheaf over $G^*$.

(a) For each open cover $\mathcal{U}_*$ of $G^*$, there is a canonical group isomorphism

$$(5.7) \quad \text{ext}_{\mathcal{U}}(G[\mathcal{U}_0], A[\mathcal{U}_0]) \cong H^2(\mathcal{U}_*; A^*),$$

where $\text{ext}_{\mathcal{U}}(G[\mathcal{U}_0], A[\mathcal{U}_0])$ denotes the subgroup of elements of $\text{ext}(G[\mathcal{U}_0], A[\mathcal{U}_0])$ consisting of extensions $A[\mathcal{U}_0] \to E \xrightarrow{\pi} G[\mathcal{U}_0]$ such that $\pi$ admits a continuous lifting over each open set $U^1_{\lambda} \subset G_1$ ($\lambda \in \Lambda_1$).

(b) $(5.7)$ induces an isomorphism

$$\text{Ext}(G, A) \cong H^2(\mathcal{U}_*; A^*).$$
Proof. As in the previous subsection, one easily sees that a 2-cocycle in $Z^2(\mathcal{U}_*; \mathcal{A}^*)$ is given by a family

$$\varphi = (\varphi_{\lambda_0\lambda_1\lambda_2\lambda_0\lambda_1\lambda_2})$$

such that each term is a continuous function $(g, h) \mapsto \varphi_{\lambda}(g, h) \in A_{r(g)}$, defined on the set of pairs $(g, h)$ such that $r(g) \in U_{\lambda_0}^0, s(g) \in U_{\lambda_1}^0, s(h) \in U_{\lambda_2}^0, \ g \in U_{\lambda_0}^1, gh \in U_{\lambda_0}^1, h \in U_{\lambda_2}^1$. The $\varphi$’s satisfy the cocycle identity

$$\varphi_{\lambda_1\lambda_2\lambda_3\lambda_1\lambda_3\lambda_1\lambda_2\lambda_3}(h, k) - \varphi_{\lambda_2\lambda_2\lambda_3\lambda_0\lambda_0\lambda_2\lambda_3\lambda_1}(gh, k) + \varphi_{\lambda_0\lambda_1\lambda_2\lambda_0\lambda_1\lambda_0\lambda_2\lambda_1}(g, h) = 0$$

(5.8)

Let us consider a cover $\mathcal{V}_*$ of $G[\mathcal{U}]$, $\mathcal{V}_n = (V^n_j)_{j \in J_n}$, such that

- $J_0 = \{\text{pt}\}$ and $\mathcal{V}_0$ is the cover consisting of the unique open set $\prod_{i \in I_0} U_i^0$;
- $J_1 = J_0 \times I_0 \times I_1$ with $V^1_{ijk} = \{(i, g, j) | r(g) \in U_i^0, \ s(g) \in U_j^0, \ g \in U_k^1\};$
- $V^n$ arbitrary $\forall n \geq 2$.

Consider the group $Z^2(\mathcal{V}_*; \mathcal{A}^*)$, where $\mathcal{A}^*$ is the pull-back of the sheaf $\mathcal{A}^*$ by $G[\mathcal{U}] \to G$. As above, it consists of families $\psi_{\mu_1\mu_2\mu_3\mu_4}$ satisfying the cocycle identity

$$g\psi_{\mu_1\mu_2\mu_3\mu_4}(h, k) - \psi_{\mu_2\mu_3\mu_4\mu_5}(gh, k) + \psi_{\mu_1\mu_2\mu_3\mu_5}(g, h) = 0$$

(5.9)

We show that $Z^2(\mathcal{U}_*; \mathcal{A}^*) \cong Z^2(\mathcal{V}_*; \mathcal{A}^*)$, where $\mathcal{A}^*$ is the pull-back of the sheaf $\mathcal{A}^*$ by $G[\mathcal{U}] \to G$.

In one direction, let $\psi \in Z^2(\mathcal{V}_*; \mathcal{A}^*)$. For all $\lambda=(\lambda_0, \lambda_1, \lambda_2, \lambda_0, \lambda_0, \lambda_1)$ in $I_0^3 \times I_1^3$, define

$$\mu_{01} = (\lambda_0, \lambda_1, \lambda_0) \quad \mu_{02} = (\lambda_0, \lambda_2, \lambda_0) \quad \mu_{12} = (\lambda_1, \lambda_2, \lambda_1)$$

and $\varphi_\lambda = \psi_{\mu_{01}\mu_{02}\mu_{12}}$.

In the other direction, if we are given a 2-cocycle $\varphi \in Z^2(\mathcal{U}_*; \mathcal{A}^*)$, we want to define a 2-cocycle $\psi \in Z^2(\mathcal{V}_*; \mathcal{A}^*)$. Given $\mu = (\mu_{01}, \mu_{02}, \mu_{12}) \in J_1$, write $\mu_{ab} = (i_{ab}, j_{ab}, k_{ab})$. Then $V^1_{\mu} \neq \emptyset$ implies that $j_{01} = i_{12}, i_{01} = i_{02}, j_{02} = j_{12}$, hence there exists

$$\lambda=(\lambda_0, \lambda_1, \lambda_2, \lambda_0, \lambda_0, \lambda_1) \in I_0^3 \times I_1^3$$

such that (5.10) holds. We then define $\psi_\mu = \varphi_\lambda$.

Comparing (5.8) and (5.9), we see that $Z^2(\mathcal{U}_*; \mathcal{A}^*) \cong Z^2(\mathcal{V}_*; \mathcal{A}^*)$. Moreover, it is not hard to check that this induces an isomorphism

$$H^2(\mathcal{U}_*; \mathcal{A}^*) \cong H^2(\mathcal{V}_*; \mathcal{A}^*)$$

(5.11)

To prove the first part of the proposition, then, we can (after passing to the groupoid $G[\mathcal{U}_0]$), suppose that $\mathcal{U}_0$ consists of the unique open set $G_0$. 
Consider an extension in \( \text{ext}_U(G, A) \)
\[
A \to E \to G.
\]
For each \( i \in I_1 \), consider a continuous section \( \sigma_i : U_i^1 \to E \). Define a cochain \( \varphi \) by the equation

\[
\sigma_{\lambda_{01}}(g)\sigma_{\lambda_{12}}(h) = \varphi_{\lambda_{01}\lambda_{02}\lambda_{12}}(g, h)\sigma_{\lambda_{02}}(gh).
\]
To see that it is indeed a cocycle, just write

\[
(\sigma_{\lambda_{01}}(g)\sigma_{\lambda_{12}}(h))\sigma_{\lambda_{23}}(k) = \sigma_{\lambda_{01}}(g)(\sigma_{\lambda_{12}}(h)\sigma_{\lambda_{23}}(k))
\]
and substitute relations like (5.12) to obtain

\[
g\varphi_{\lambda_{12}\lambda_{13}\lambda_{23}}(h, k) - \varphi_{\lambda_{02}\lambda_{03}\lambda_{23}}(gh, k)
+ \varphi_{\lambda_{01}\lambda_{03}\lambda_{13}}(g, hk) - \varphi_{\lambda_{01}\lambda_{02}\lambda_{12}}(g, h) = 0.
\]

Suppose that \( \sigma' \) is another continuous lifting and let \( \alpha_i : U_i^1 \to A \) such that

\[
(\sigma'_{\lambda_{01}}(g)\sigma'_{\lambda_{12}}(h)) = \alpha_i(g)\sigma_i(g).
\]
Define \( \varphi' \) by

\[
\alpha_{\lambda_{01}\lambda_{02}\lambda_{12}}(g, h) = \varphi'_{\lambda_{01}\lambda_{02}\lambda_{12}}(g, h)\alpha'_{\lambda_{02}}(gh).
\]
Substituting (5.14) in (5.15) and comparing with (5.12), we find

\[
(\varphi' - \varphi)_{\lambda_{01}\lambda_{02}\lambda_{12}}(gh) = g\alpha_{\lambda_{12}}(h) - \alpha_{\lambda_{02}}(gh) + \alpha_{\lambda_{01}}(g),
\]
i.e. \( \varphi' - \varphi = d\alpha \). This proves that an extension in \( \text{ext}_U(G, A) \) determines a unique cohomology class in \( H^2(U_*; A^*) \).

Conversely, given a cocycle \( \varphi_{\lambda_{01}\lambda_{02}\lambda_{12}} \), we want to construct an extension \( A \to E \to G \).

The idea is to set

\[
E = \prod_{i \in I_1} \langle (a, g, i) | a \in A, g \in U_i^1, p(a) = r(g) \rangle / \sim,
\]
with the product law

\[
[a, g, \lambda_{01}] [b, g, \lambda_{12}] = [a + g \cdot b + \varphi_{\lambda_{01}\lambda_{02}\lambda_{12}}(g, h), gh, \lambda_{02}].
\]
To determine the correct equivalence relation in (5.16), we note that if \( [a, x, i] \) represents a unit element in the groupoid \( E \), then from the product law (5.17) we necessarily have \( [a, x, i] = [a, x, i][a, x, i] = [2a + \varphi_{\lambda_{01}}(x, x), x, i] \), thus \( [-\varphi_{\lambda_{01}}(x, x), x, i] \) must be the unit element. Using again (5.17), we necessarily have

\[
[-\varphi_{\lambda_{01}}(r(g), r(g)), r(g), i][a, g, k] = [-\varphi_{\lambda_{01}}(r(g), r(g)) + a + \varphi_{\lambda_{02}}(g, g), g, j]
\]
thus we necessarily have

\[
(a, g, k) \sim (-\varphi_{\lambda_{01}}(r(g), r(g)) + a + \varphi_{\lambda_{02}}(g, g), g, j)
\]
Conversely, we want to show that (5.18) defines an equivalence relation. We claim that
\[ \psi_{kj}(g) = -\varphi_{iii}(r(g), r(g)) + \varphi_{ijk}(r(g), g) \]
does not depend on the choice of \( i \).

Let us denote \( x = r(g) \). Apply (5.13) to \((x, x, g)\) instead of \((g, h, k)\):
\[
\varphi_{ijk}(x, g) - \varphi_{lmk}(x, g) + \varphi_{nmj}(x, g) - \varphi_{nli}(x, x) = 0.
\]
Taking \( g = x \) and \( j = k = \ell = m = n \) we find
\[
\varphi_{ijj}(x, x) = \varphi_{jji}(x, x) = \varphi_{iii}(x, x).
\]
Taking \( k = j \) and \( n = \ell \) in (5.19) and using (5.20), we get
\[
\varphi_{ijj}(x, g) - \varphi_{\ell\ell i}(x, x) = \varphi_{iii}(x, x).
\]
This proves our claim that \( \psi_{kj} \) is well-defined. Moreover, taking \( n = \ell = i \) in (5.19) we get
\[
\psi_{jk}(g) - \psi_{mk}(g) + \psi_{mj}(g) = 0.
\]
It follows that
\[
\begin{align*}
\psi_{jj} &= 0 \quad \text{(use (5.22) for } k = j) \\
\psi_{kj} &= -\psi_{jk} \quad \text{(use (5.22) for } m = k) \\
\psi_{jm} &= \psi_{jk} + \psi_{km}.
\end{align*}
\]
Therefore, (5.18) defines an equivalence relation.

It is then elementary to check that (5.17) endows \( E \) with a groupoid structure such that the obvious extension
\[
A \to E \xrightarrow{\pi} G
\]
is an element of \( \text{ext}(G, A) \), and that \( \pi \) admits a continuous lifting \( \sigma_i : U_i^1 \to G \) defined by \( \sigma_i(g) = [0, g, i] \); and that the associated cocycle is precisely \( \varphi \). We leave these easy verifications to the reader.

To prove the second part of the proposition, we first pass to the inductive limit over all open covers \( U_i \) of \( G \) (leaving \( U_0 \) fixed) to find that
\[
\text{ext}(G[U_0], A[U_0]) = \lim_{\longleftarrow} H^2(U_i; A^*)
\]
and then take the inductive limit over \( U_0 \).

\textbf{Remark 5.7.} By the same method we used to show (5.11), one can show that for each open cover \( U_i \) of \( G_i \), and each sheaf \( A^* \) over \( G_i \), the canonical morphism \( f : G[U_0] \to G \) induces an isomorphism
\[
H^n(U_i; A^*) \cong H^n(V_i; A^*)
\]
where $\mathcal{V}_0$ is the cover consisting of the unique open set $\coprod_{i \in I} U^0_i$ and the open cover $\mathcal{V}_n = (V^n_{i_0, \ldots, i_n, j})_{i_0, \ldots, i_n, j \in I_0^{n+1} \times I_n}$ of $G[U]_n$ is defined by

$$V^n_{i_0, \ldots, i_n, j} = \{(i_0, \ldots, i_n, g_1, \ldots, g_n) \mid r(g_1) \in U^0_{i_0}, s(g_1) \in U^0_{i_1}, \ldots, s(g_n) \in U^0_{i_n}, (g_1, \ldots, g_n) \in U^n_j\}$$

(see Remark 2.1).

**Remark 5.8.** For completeness, it would remain to examine the relation between the sheaf 2-cohomology groups and extensions of non-paracompact groupoids, since it is not obvious whether $H^2$ is equal or not to $\check{H}^2$ in this case. However, we won’t develop this, due to lack of applications.

**Corollary 5.9.** If $G$ is a locally compact Hausdorff groupoid with Haar system then $\check{H}^2(G_*; \mathbb{T}) \cong \text{Ext}(G, G_0 \times \mathbb{T})$ is the Brauer group of $G$.

**Proof.** Use for instance Proposition 2.13 and remarks preceding Proposition 2.29 of [20]. □

6. Comparison with Moore’s cohomology

Recall [16] that if $G$ is a locally compact group and $A$ is a Polish abelian group (i.e., a topological space, $A$ admits a separable complete metric), then $A$ is a $G$-module if $G$ acts (continuously) by automorphisms on $A$.

Given a Polish $G$-module $A$, let $I(A)$ be the set of $\mu$-measurable functions from $G$ to $A$ (where $\mu$ is the Haar measure), modulo equality almost everywhere. Then $I(A)$ is again a Polish $G$-module, with action $(\gamma \cdot f)(x) = \gamma f(\gamma^{-1}x)$ (caution: our definition is different but isomorphic to Moore’s definition of $I(A)$).

The $G$-module $A$ embeds in $I(A)$ via the obvious map

$$i_A : A \to I(A), \quad (i_A(a))(x) = a.$$  

Let $U(A) = I(A)/A$. Then, using measurable cocycles, Moore defined cohomology groups $H^n(G, A)$ which are are characterized by the proposition below, where $I(A)$ is defined as above and $F(A) = A^G$ (the sub-module of $G$-fixed points).

**Proposition 6.1.** Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two abelian categories. Suppose that $F$ is a left-exact functor from $\mathcal{C}_1$ to $\mathcal{C}_2$, that $I : \mathcal{C}_1 \to \mathcal{C}_2$ is a functor and $i_A : A \to I(A)$ is a natural injection. Then

(a) there exists, up to isomorphism, at most one sequence of functors $H^n : \mathcal{C}_1 \to \mathcal{C}_2$ such that

1) $H^0 = F$

2) Any exact sequence $0 \to A' \to A \to A'' \to 0$ induces a natural long exact sequence $0 \to H^0(A') \to H^0(A) \to H^0(A'') \xrightarrow{\partial} H^1(A') \to \ldots$

3) $H^n(I(A)) = 0$ for all $A$ and for all $n \geq 1$. 

(b) If $I$ is an exact functor and $I(i_A)=i_{I(A)}$ for all $A$, then 3) may be replaced by 3') $H^1(I(A))=0$ for all $A$.
(c) If moreover $F(I(A))\to F(U(I(A)))$ is surjective for all $A$, where $U(A)=I(A)/A$, then there exists a sequence of functors satisfying 1), 2) and 3).

Proof. a) This is essentially [2, Theorem 1] or [16, Theorem 2]: using the long exact sequence associated to $0\to A^i\to I(A)\to U(A)\to 0$, one gets $H^n(A)\cong H^{n-1}(U(A))$ for $n\geq 2$ and $H^1(A)=\ker (F(I(A))\to F(U(A)))$, thus $H^n$ is uniquely determined by induction on $n$.

b) If $I$ is exact then
\[0\to I(A)\xrightarrow{i_{I(A)}} I(I(A))\to I(U(A))\to 0\]
is exact, and
\[0\to I(A)\xrightarrow{i_{I(A)}} I(I(A))\to U(I(A))\to 0\]
is exact by definition. Therefore, the assumption $I(i_A)=i_{I(A)}$ implies $I(U(A))=U(I(A))$ canonically. Thus, 3') implies that for $n\geq 2$,
\[H^n(I(A))=H^1(U^{n-1}(I(A)))=H^1(IU^{n-1}(A))=0.\]

c) Define a resolution $0\to A\to A_0\to A_1\to \cdots$ by $A_0=I(A)$, and $A_{n+1}=I(A_n/A_{n-1})$. The map $A_n\to A_{n+1}$ is the composition $A_n\to A_n/A_{n-1}\xrightarrow{i} I(A_n/A_{n-1})$. Define $H^n(A)$ to be the cohomology of the complex $F(A_0)\to F(A_1)\to \cdots$ and let us check that properties 1), 2) and 3') hold.

1) $H^0(A)=\ker (F(A)\to F(I(I(A)/A)))=\ker (F(I(A))\to F(I(A)/A))$ since $F$ preserves injectivity of morphisms, and $I(A)/A\to I(I(A)/A)$ is injective. Using left exactness of $F$, we see that $H^0(A)=F(\ker (I(A)\to I(A)/A)=F(A))$.

2) Since $I$ is an exact functor, we have an exact sequence of complexes $0\to A'_n\to A_n\to A''_n\to 0$, hence the conclusion by the Snake lemma.

3') $H^1(I(A))=F(I(I(A)))/F(U(I(A)))=0$. \hfill \Box

For instance, in part a) of the proposition, if $I(A)$ is an injective object for all $A$ then $H^n$ are the right derived functors of $F$.

We are now ready to prove:

**Proposition 6.2.** Let $G$ be a locally compact group. Let $A$ be a Polish $G$-module and let $\mathcal{A}^\bullet$ be the associated sheaf on $G$ (see Definition 3.10 and below). Then $\check{H}^\bullet(G;\mathcal{A}^\bullet)\cong \check{H}^\bullet(G,A)$.

Proof. We just need to check that $\check{H}^\bullet(G;\mathcal{A}^\bullet)$ satisfies the conditions 1)–3) of Proposition 6.1.

1) $\check{H}^0(G;\mathcal{A}^\bullet)=A^G$ was proved in Proposition 5.1 and 2) is Corollary 4.13.

$I$ is an exact functor [16, Proposition 9], and it is obvious that $I(i_A)=i_{I(A)}$. It thus remains to show that $\check{H}^1(G;\mathcal{B}^\bullet)=0$ if $\mathcal{B}^\bullet$ is the sheaf on $G$ associated
to the $G$-module $I(A)$. Recall that a 1-cocycle $\varphi$ is a continuous function $\varphi: G \to I(A)$ satisfying

$$g_1 \varphi(g_2) - \varphi(g_1 g_2) + \varphi(g_1) = 0$$

(see (5.6)), hence $\varphi$ is a cocycle in the Moore complex. But $H^1_{\text{Moore}}(G, I(A)) = 0$, hence there exists $\psi \in I(A)$ such that $\varphi(g) = g \psi - \psi$ for all $g \in G$. Therefore, $\varphi = 0$ in $H^1(U; B^*)$.

\begin{remark}
One can easily construct explicitly the isomorphism $\tilde{H}^n(G; A^*) \to H^n_{\text{Moore}}(G, A)$. Take $\varphi \in Z^n(U; A^*)$ where $U$ is an open cover. Choose measurable maps $\theta_k: G^k \to I_k$ such that $x \in U_{\theta_k(x)}$ for all $x \in G^k$, and define

$$c(g_1, \ldots, g_n) = \varphi_{\lambda(g_1, \ldots, g_n)}(g_1, \ldots, g_n)$$

where $\lambda(g_1, \ldots, g_n) \in \Lambda_n$ is defined by $\lambda(g_1, \ldots, g_n)(f) = \theta_r(f(g_1, \ldots, g_n)) \in I_k$ for every $f \in \text{Hom} \Delta(k, n)$.

\end{remark}

\begin{remark}
One might wonder if it is possible to define, for every locally compact groupoid and every (say, locally compact) $G$-module $A$, an analogue of the Moore complex, using measurable cochains $c(g_1, \ldots, g_n) \in A_{r(g_1)}$. In order to get the usual cohomology groups when $G$ is a space, one should probably use the sheaf over $G_0$ of functions $c(g_1, \ldots, g_n) \in A_{r(g_1)}$ which are measurable in the “leaf” direction and continuous in the “transverse” direction. Since this approach doesn’t seem simpler or more useful than \v{C}ech or sheaf cohomology of simplicial spaces, we won’t develop this further.

\section{Comparison with sheaf cohomology and Haefliger’s cohomology}

Let $M_*$ be a simplicial space and $A^*$ an abelian sheaf on $M_*$. By definition [4], the cohomology groups $H^n(M_*; A^*)$ are the derived functors of the functor $\Gamma_{\text{inv}}(M_*; A^*) = \text{Ker}(A^0(M_0) \to A^1(M_1))$, thus they coincide with Haefliger’s cohomology groups in the case of \v{e}tale groupoids [6].

A practical way of calculating the cohomology groups is to take a resolution $(L^{p,q})_{q \in \mathbb{N}}$ of $A^*$ such that $H^n(M_p; L^{p,q}) = 0$ for all $n \geq 1$, and take the cohomology of the double complex $(L^{p,q}(M_p))$, where the first differential is $d' = \sum_{k=0}^{p+1} (-1)^k \varepsilon^*_k$ and the second differential $d''$ is the differential in the resolution

$$A^p \to \mathcal{L}^{p,0} \to \mathcal{L}^{p,1} \to \cdots$$

(See [4, §5.2.3] in the general case, [3, §2.7] or [6] in the case of \v{e}tale groupoids).

In this section, we show:

\begin{proposition}
Let $M_*$ be a paracompact simplicial space, and $A^*$ an abelian sheaf on $M_*$. Then $H^*(M_*; A^*) \cong \tilde{H}^*(M_*; A^*)$. In particular, $\tilde{H}^*(G_1; A^*)$ are Haefliger’s cohomology groups if $G$ is an \v{e}tale paracompact groupoid and $A^*$ is an abelian $G$-sheaf.
\end{proposition}
We will again use Proposition 8.1. Consider $I(\mathcal{A})^\bullet$ the sheaf such that $I(\mathcal{A})^n(U)$ consists of maps $f$ (continuous or not) from $\tilde{\varepsilon}_0^{-1}(U)$ to $\mathcal{A}^{n+1}$ such that $f(x) \in \mathcal{A}_{x}^{n+1}$ for all $x \in \tilde{\varepsilon}_0^{-1}(U)$. We will need the

**Lemma 7.2.** For any open cover $\mathcal{U}_*$ and all $n \geq 1$, $H^n(\mathcal{U}_*; I(\mathcal{A}))^\bullet=\{0\}$.

**Proof.** Let us first explain the simplicial structure on the sheaf $I(\mathcal{A})^\bullet$. Given $f \in \text{Hom}_{\Delta}(k,n)$, $U \subset M_k$, $V \subset M_n$ such that $\tilde{f}(V) \subset U$ and a section $\varphi$ of $I(\mathcal{A})^k$ over $U$, we have to produce a section $\tilde{f}\varphi \in \Gamma(V, I(\mathcal{A})^n)$.

Define $f' \in \text{Hom}_{\Delta}(k+1,n+1)$ such that

$$(7.1) \quad f'(0)=0 \quad \text{and} \quad \varepsilon_0 \circ f = f' \circ \varepsilon_0 : [k] \rightarrow [n+1].$$

Since $\tilde{f}(V) \subset U$ we have $\tilde{f}'(\tilde{\varepsilon}_0^{-1}(V)) \subset \tilde{\varepsilon}_0^{-1}(U)$, thus we get a section

$$x \in \tilde{\varepsilon}_0^{-1}(V) \mapsto \varphi(\tilde{f}'(x)) \in \mathcal{A}_{\tilde{f}'(x)}^{k+1} \mapsto (\tilde{f}'\varphi)(\tilde{f}'(x)) \in \mathcal{A}_{x}^{n+1}.$$

Now, let us show that the complex $C^n(\mathcal{U}_*; I(\mathcal{A}))^\bullet$ is homotopically trivial. First, for all $n \in \mathbb{N}$ and all $x \in M_n$, let us choose $\theta(x) \in I_n$ such that $x \in U_{\theta(x)}$. We define a homotopy

$$H : C^n(\mathcal{U}_*; I(\mathcal{A}))^\bullet \rightarrow C^{n-1}(\mathcal{U}_*; I(\mathcal{A}))^\bullet$$

by $(H\varphi)_\lambda(x)=\tilde{\eta}_0^* \varphi \chi_x(\tilde{\eta}_0(x))$, $\forall x \in \tilde{\varepsilon}_0^{-1}(U_\lambda) \subset M_n$, $\forall \lambda \in \Lambda_{n-1}$, and $\chi_x \in \Lambda_n$ is defined as follows: for all $f \in \text{Hom}_{\Delta}(k,n)$, let

$$\chi_x^k(f) = \begin{cases} \lambda(\eta_0 \circ f) & \text{if } f(0) \neq 0 \\ \theta(\tilde{f}(x)) & \text{if } f(0)=0 \end{cases} \in I_k.$$  

Let $\varphi \in C^n(\mathcal{U}_*; I(\mathcal{A}))^\bullet$. Let us compute $(dH + Hd)\varphi$ and compare it with $\varphi$. We have

$$(d\varphi)_\lambda(x) = \sum_{k=0}^{n+1} (-1)^k \tilde{\varepsilon}_k^* \varphi \chi_x(\lambda)(\tilde{\varepsilon}_k'(x)) \quad \in \mathcal{A}_x^{n+2}.$$  

The meaning of this formula is the following: we take $x \in M_{n+2}$, then its image $\tilde{\varepsilon}_k'(x)$ (see notation 7.1) is in $M_{n+1}$. Its image $\varphi \chi_x(\lambda)(\tilde{\varepsilon}_k'(x))$ belongs to $\mathcal{A}_{\chi_x}^{n+1}(x)$ restricted (see 3.1) by $\tilde{\varepsilon}_k : M_{n+2} \rightarrow M_{n+1}$ to an element of $\mathcal{A}_x^{n+2}$.
Actually, we have $\varepsilon'_k = \varepsilon_{k+1}$ since $\varepsilon_0 \circ \varepsilon_k = \varepsilon_{k+1} \circ \varepsilon_0 : [n] \to [n+2]$, hence

$$(d\varphi)_\lambda(x) = \sum_{k=0}^{n+1} (-1)^k \varepsilon_{k+1}^*(\varphi \varepsilon_k(\lambda)(\varepsilon_{k+1}(x)))$$

and

$$(dH\varphi)_\lambda(x) = \sum_{k=0}^{n} (-1)^k \varepsilon_{k+1}^*(H\varphi)\varepsilon_k(\lambda)(\varepsilon_{k+1}(x))$$

$$= \sum_{k=0}^{n} (-1)^k \varepsilon_{k+1}^*\bar{\eta}_0^*\varphi \varepsilon_k(\lambda)\varepsilon_{k+1}(x)(\bar{\eta}_0 \circ \varepsilon_{k+1}(x))$$

$$= \sum_{k=0}^{n+1} (-1)^k \bar{\eta}_0^*\varepsilon_{k+1}^*\varphi \varepsilon_k(\lambda)\varepsilon_{k+1}(x)(\bar{\eta}_0(\varepsilon_{k+1}(x))).$$

In the last sum, for $k=0$ we get $\varphi \varepsilon_{k+1}(x)$. Now, $(\varepsilon_0(\lambda_x))(f)=\lambda_x(\varepsilon_0 \circ f)=\lambda(\eta_0 \circ \varepsilon_0 \circ f)=\lambda(f)$, thus the term for $k=0$ is just $\varepsilon_\lambda(x)$.

To show that the other terms in the sum $dH + Hd$ cancel out, we just need to check that for all $k \geq 1$,

(a) $\eta_0 \circ \varepsilon_{k+1} = \varepsilon_k \circ \eta_0$, and

(b) $\varepsilon_k(\lambda_x) = \varepsilon_{k-1}(\lambda)\varepsilon_k(x)$.

Assertion (a) is straightforward. Let us prove (b).

If $f(0) \neq 0$ then $\varepsilon_k(\lambda_x)(f) = \lambda_x(\varepsilon_k \circ f) = \lambda(\eta_0 \circ \varepsilon_k \circ f) = \lambda(\varepsilon_{k-1} \circ \eta_0 \circ f) = (\varepsilon_{k-1}(\lambda))\varepsilon_k(x)(f)$.

If $f(0) = 0$ then $\varepsilon_k(\lambda_x)(f) = \lambda_x(\varepsilon_k \circ f) = \theta(\varepsilon_k \circ f(x)) = (\varepsilon_{k-1}(\lambda))\varepsilon_k(x)(f)$.

Remark 7.3. In the case of an étale groupoid $G$ and cohomology groups with coefficients in $G$-sheaves, Haefliger [6], following Atiyah and Wall in the case of discrete groups [2], characterized the cohomology groups as the unique sequence of functors $H^n$ such that

(a) $H^0 = \Gamma_{\text{inv}}$,

(b) $H^n$ admits long exact sequences, and

(c) $H^n(G; I(A)) = 0$ for all $n \geq 1$ and for each $G$-sheaf $I(A)$.

Let us now prove Proposition 7.1. First, we note that $\mathcal{A}^\bullet \hookrightarrow I(A)^\bullet$ canonically: if $c \in \mathcal{A}^n(U)$, then $\varphi(x) = \varepsilon_0^*[c(\varepsilon_0(x))] \in \mathcal{A}_{x+1}^n$ is a section of $I(A)^n$ over $U$.

Using the uniqueness part in Proposition 5.1, it suffices to show that

$$\tilde{H}^n(M^\bullet; I(A)^\bullet) = H^n(M^\bullet; I(A)^\bullet) = 0 \quad \forall n \geq 1.$$

This is true for $\tilde{H}^n$ thanks to Lemma 7.2.

Define inductively a resolution

$$(7.2) \quad 0 \to I(A)^\bullet \to \mathcal{L}^\bullet,0 \to \mathcal{L}^\bullet,1 \to \ldots$$
by \( \mathcal{L}^{p,0} = I(I(\mathbb{A}))^p \) and \( \mathcal{L}^{p,q+1} = I(\mathcal{L}^{p,q}/\mathcal{L}^{p,q-1}) \). Since \( \mathcal{L}^{p,q} \) is flabby for all \( p, q \geq 0 \), the double complex \( K = (\mathcal{L}^{p,q}(M_p)) \) computes \( H^*(G_*, I(\mathbb{A}))^p \) (see above the introduction of this section).

The \( E_2 \)-term with respect to the first filtration is \( E_2^{p,q} = H^p H^q(K) \). Since

\[
0 \to I(\mathbb{A})^p \to \mathcal{L}^{p,0} \to \mathcal{L}^{p,1} \to \cdots
\]

is an exact sequence of flabby sheaves,

\[
0 \to \Gamma(M_p; I(\mathbb{A})^p) \to \Gamma(M_p; \mathcal{L}^{p,0}) \to \cdots
\]

is exact, hence \( E_2^{p,q} = 0 \) for \( q \geq 1 \) and \( E_2^{0,p} = \Gamma(M_p; I(\mathbb{A})^*) \). Using again Lemma \ref{lem:doublecomplex} for the cover \( U_n = \{M_n\} \), we get \( E_2^{0,p} = 0 \) for \( p \geq 1 \). Finally, \( H^n(M_p; I(\mathbb{A}))^* = 0 \) for all \( n \geq 1 \).

8. **Invariance by Morita equivalence**

Let \( G \) be a topological groupoid and \( \mathbb{A}^* \) be an abelian sheaf on \( G_* \). We will show that \( H^*(G_*; \mathbb{A}^*) \) and \( \check{H}^*(G_*; \mathbb{A}^*) \) are invariant under Morita equivalence.

More precisely, if \( G' \) is another groupoid and \( \mathbb{A}'^* \) is a sheaf on \( G'_* \), we say that \( (G, \mathbb{A}^*) \) is Morita equivalent to \( (G', \mathbb{A}'^*) \) if there exists a groupoid \( G'' \), a sheaf \( \mathbb{A}''^* \) on \( G''_* \) and (continuous) groupoid morphisms

\[
G \xleftarrow{f} G'' \xrightarrow{f'} G'
\]

such that \( f \) and \( f' \) are Morita equivalences and \( \mathbb{A}''^* \cong f^* \mathbb{A}^* \cong f'^* \mathbb{A}'^* \). Then

**Proposition 8.1.** With the above assumptions, \( f \) and \( f' \) induce isomorphisms in sheaf and Čech cohomology, thus

\[
H^*(G_*; \mathbb{A}^*) \cong H^*(G'_*; \mathbb{A}'^*) \quad \text{and} \quad \check{H}^*(G_*; \mathbb{A}^*) \cong \check{H}^*(G'_*; \mathbb{A}'^*).
\]

**Proof.** By standard arguments (compare with Proposition \ref{prop:cech}), it suffices to show that for any open cover \( \mathcal{U} = (U_i)_{i \in I} \) of \( G_0 \), the canonical morphism \( f : G[\mathcal{U}] \to G \) induces isomorphisms \( H^*(G_*; \mathbb{A}^*) \cong H^*(G[\mathcal{U}]_*; f^* \mathbb{A}^*) \) and \( \check{H}^*(G_*; \mathbb{A}^*) \cong \check{H}^*(G[\mathcal{U}]_*; f^* \mathbb{A}^*) \). Below, we will abusively write \( \check{H}^*(G[\mathcal{U}]_*; f^* \mathbb{A}^*) \) instead of \( H^*(G[\mathcal{U}]_*; f^* \mathbb{A}^*) \).

For Čech cohomology, using Remark \ref{rem:cech}, we have

\[
H^n(G_*; \mathbb{A}^*) = \lim_{\mathcal{V} \to \mathcal{W}} \lim_{\mathcal{W} \to \mathcal{V}} H^n(\mathcal{W}_*; \mathbb{A}^*)
\]

where \( \mathcal{V} = (V_j)_{j \in J} \) runs over open covers of \( G_0 \) and \( \mathcal{W} \) runs over open covers of \( G[\mathcal{V}] \) such that \( \mathcal{W}_0 \) consists of the single open set \( \coprod V_j \).

Similarly,

\[
\check{H}^n(G[\mathcal{U}]_*; \mathbb{A}^*) = \lim_{\mathcal{V}' \to \mathcal{W}'} \lim_{\mathcal{W}' \to \mathcal{V}'} H^n(\mathcal{W}'_*; \mathbb{A}^*)
\]

where \( \mathcal{V}' = (V'_j)_{j \in J'} \) runs over open covers of \( G[\mathcal{U}]_0 \) and \( \mathcal{W}' \) runs over open covers of \( G[\mathcal{U}'][\mathcal{V}'] \) such that \( \mathcal{W}'_0 \) consists of the single open set \( \coprod V'_j \).

Now, note that if \( \mathcal{V}' \) is an open cover of \( G[\mathcal{U}]_0 \) which is finer than the cover \( (\{i\} \times U_i)_{i \in I} \), then there exists an open cover \( \mathcal{V} \) of \( G_0 \) such that
G[\mathcal{V}] \cong G[\mathcal{U}][\mathcal{V}'] (the elementary proof is left to the reader). Therefore, in the right hand sides of (8.1) and (8.2), the terms \( \lim_{W} H^n(\mathcal{W}_i; \mathcal{A}^*) \) and \( \lim_{W'} H^n(\mathcal{W}'_i; \mathcal{A}^*) \) are identical. It follows that the right hand sides of (8.1) and (8.2) are equal, hence Čech cohomology is invariant by Morita equivalence.

From the above, we already find that sheaf cohomology is invariant under Morita-equivalence when the groupoid is paracompact. In fact, this holds for a general topological groupoid. Let us sketch the proof for completeness.

Consider the resolution

\[ \mathcal{A}^* \to \mathcal{L}^{\ast,0} \to \mathcal{L}^{\ast,1} \to \cdots \]

constructed like (7.2). Since \( \mathcal{L}^{p,q} \) is flabby for all \( p, q \), the double complex \( (\mathcal{L}^{p,q}(G_p)) \) computes \( H^*(G^*_s; \mathcal{A}^*) \) and since the lines are exact (7.2), \( H^*(G^*_s; \mathcal{A}^*) \) is the cohomology of the complex \( (\Gamma_{\text{inv}}(\mathcal{L}^{\ast,q}))_{q \in \mathbb{N}} \).

Similarly, \( H^*(G[\mathcal{U}]; \mathcal{A}^*) \) is the cohomology of the complex \( (\Gamma_{\text{inv}}(f^* \mathcal{L}^{\ast,q}))_{q \in \mathbb{N}} \). Now, it is elementary to check that for every sheaf \( \mathcal{B}^* \), \( \Gamma_{\text{inv}}(\mathcal{B}^*) \) is isomorphic to \( \Gamma_{\text{inv}}(f^* \mathcal{B}^*) \).

\[ \square \]

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