SPECTRAL PROPERTIES OF SOME DEGENERATE ELLIPTIC DIFFERENTIAL OPERATORS

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This paper is dedicated to David Edmunds on his 80th birthday and to Desmond Evans on his 70th birthday.

Abstract. In this paper we extend classical criteria for determining lower bounds for the least point of the essential spectrum of second-order elliptic differential operators on domains $\Omega \subset \mathbb{R}^n$ allowing for degeneracy of the coefficients on the boundary. We assume that we are given a sesquilinear form and investigate the degree of degeneracy of the coefficients near $\partial \Omega$ that can be tolerated and still maintain a closable sesquilinear form to which the First Representation Theorem can be applied. Then, we establish criteria characterizing the least point of the essential spectrum of the associated differential operator in these degenerate cases. Applications are given for convex and non-convex $\Omega$ using Hardy inequalities, which recently have been proven in terms of the distance to the boundary, showing the spectra to be purely discrete.

The classical criterion for the least point of the essential spectrum was given by Persson [22] for a Schrödinger operator

$$-\Delta + q(x), \quad x \in \Omega,$$

with the only singularity being at infinity, assuming Dirichlet boundary conditions on $\Omega$ and assuming $q$ to be bounded below at infinity. For $q$ bounded below at infinity and near $\partial \Omega$, Edmunds and Evans [10] extended this result to include singularities on the boundary $\partial \Omega$ showing that “if $q \in L^2_{loc}(\Omega)$ and the negative part of $q$ behaves itself locally, then the essential spectrum” of the Friedrichs extension of the operator “is only influenced by the behaviour of $q$ at $\partial \Omega$ and at infinity in the respective cases.” Conditions (1.3) and (1.6) below give a mathematical description of the requirement that “$q$ behaves itself locally.” Related techniques were used in [20] to establish conditions for a purely discrete spectra of second order elliptic differential operators in weighted $L^2$ spaces including mixed boundary conditions. While still assuming that $q$ is bounded below near singularities on $\partial \Omega$ or at $\infty$, Evans and

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Lewis [13] used techniques developed in [10] to study even-order elliptic differential operators in weighted $L^2_w(\Omega)$ spaces with emphasis upon the criteria for the finiteness or infiniteness of the eigenvalues below the essential spectrum. We refer to that paper for many other references to related work.

Edmunds and Evans [12] study the Neumann operator generated by

$$-\text{div}(d(x)^{2\mu} \nabla) + d(x)^{-2\theta}, \quad \mu, \theta \geq 0,$$

on a proper open subset $\Omega \subset \mathbb{R}^n$ where $d(x) := \text{dist}(x, \partial \Omega)$. They present upper and lower estimates for the eigenvalue counting function as well as examining the embedding properties for associated spaces.

In this paper we study second-order elliptic sesquilinear forms that give rise to differential operators whose coefficients may “blow-up” near parts of $\partial \Omega$ including some cases in which the potential diverges to negative infinity near the boundary. Applications are given when the coefficients are approximated by the distance function $d(x)$ near $\partial \Omega$.

We follow and abbreviate the structure established in [13], but without the introduction of weights or higher-order cases. Those extensions should be clear from [13] and the presentation in this paper.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be open and connected. Throughout this paper $\|u\| := \|u\|_{L^2(\Omega)}$. If $\Omega$ is unbounded then $\infty$ is considered to be on the boundary of $\Omega$ in the sense of a one-point compactification of $\mathbb{R}^n$. The finite points of the boundary are denoted by $\partial \Omega$. Outside some set $S$, which contains the singular part of $\partial \Omega$, we assume that $\partial \Omega$ has a normal in order that certain boundary conditions are met. If $\Omega$ is unbounded then $\{\infty\} \subseteq S$, but the emphasis here is upon the part of $S$ on $\partial \Omega$. The finite part of the singular set $S \setminus \{\infty\}$ is assumed to be a closed subset of $\partial \Omega$. Let the singular and regular parts of the boundary be defined by

$$\Gamma_S := N_S \cap \partial \Omega \quad \text{and} \quad \Gamma_R := \partial \Omega \setminus \Gamma_S$$

where $N_S$ is an open neighborhood of $S \setminus \{\infty\}$ and $N_\infty := \{x : |x| > K\}$ for some large $K$. We may assume that $N_S \cap N_\infty = \emptyset$ for unbounded domains $\Omega$.

For an Hermitian matrix $A(x) = (a_{ij}(x))$, real-valued $q(x)$, $x \in \Omega$, and $\sigma(s)$, $s \in \Gamma_R$, and a function $c(s)$ that assumes either the value 1 or 0 for $s \in \Gamma_R$, we are interested in differential operators of the form $T : D(T) \to L^2(\Omega)$ with

$$Tu = -\sum_{i,j=1}^n \frac{\partial}{\partial x_j} \left( a_{ij}(x) \frac{\partial}{\partial x_i} \right) + q(x), \quad x \in \Omega,$$
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for

\[ \mathcal{D}(T) := \{ u : u = \varphi \mid_{\Omega}, \varphi \in C^\infty_0(\mathbb{R}^n \setminus \Gamma_S), \; Tu \in L^2(\Omega), \quad \text{and } c(s)\frac{\partial \varphi(s)}{\partial \eta} + \sigma(s)\varphi(s) = 0, \; s \in \Gamma_R \} \]

where \( \frac{\partial \varphi}{\partial \eta} := \langle A\eta, \nabla \varphi \rangle \) and \( \eta \) is the unit outward normal on \( \Gamma_R \).

The coefficients \( c(s) \) and \( \sigma(s) \) are assumed not to be simultaneously zero allowing for mixed boundary conditions on \( \Gamma_R \). The case \( \Gamma_S = \partial \Omega \), which requires Dirichlet boundary conditions, is included.

In the case of sufficiently smooth coefficients for a symmetric operator \( T \) that is bounded from below, the sesquilinear form

\[ t[u, v] := (Tu, v), \quad \mathcal{D}(t) := \mathcal{D}(T), \quad (1.1) \]

is closable, Kato \( \text{[19]}, \text{Theorem VI.1.27, p.318} \). In the absence of smooth coefficients, the problem can be interpreted in a weak or variational sense initially involving only a sesquilinear form. In that case consider the form

\[ t[u, v] := \int_{\Omega} \left( \langle A(x)\nabla u, \nabla v \rangle > +quv \right) dx + \int_{\Gamma_R} \sigma(s)u(s)v(s)ds \quad (1.2) \]

with domain

\[ \mathcal{D}(t) := \{ u : u = \varphi \mid_{\Omega}, \varphi \in C^\infty_0(\mathbb{R}^n \setminus \Gamma_S) \}. \]

The value of \( c(s) \) is implicit in (1.2). At points where \( \sigma(s) = 0 \) Neumann conditions are implied so that \( c(s) = 1 \) and at points where \( \sigma(s) \neq 0 \) there are either Dirichlet or mixed conditions. For example see R.E. Showalter \( \text{[24]}, \text{Chapter III, Theorem 3A and Example 4.1} \).

We will give conditions which guarantee that the form is bounded below and closable. In that case the First Representation Theorem (Kato \( \text{[19]}, \text{§VI, Theorem 2.1} \)) guarantees a unique self-adjoint operator \( \tilde{T} \) associated with the closure \( \tilde{t} \) of \( t \) for which \( \mathcal{D}(\tilde{T}) \subset \mathcal{D}(\tilde{t}) \). For forms defined by (1.1), \( \tilde{T} \) is the Friedrichs extension of \( T \). Once we have established that \( t \) is bounded below and closable, we will assume that \( t[u] \geq ||u||^2 \), which can be accomplished by the addition of a positive constant to \( \tilde{T} \) merely translating \( \sigma_e(\tilde{T}) \). In this case, according to the Second Representation Theorem \( \text{[19]}, \text{Theorem VI-2.23} \), \( \tilde{T}^\frac{1}{2} \) exists, \( \mathcal{D}(\tilde{T}^\frac{1}{2}) = \mathcal{D}(\tilde{t}) \), and

\[ \tilde{t}[u, v] = (\tilde{T}^\frac{1}{2}u, \tilde{T}^\frac{1}{2}v) := (u, v); \quad (1.3) \]

In this paper, we will use the Sobolev space \( H^1(G) = W^{1,2}(G) \) for an open set \( G \subset \mathbb{R}^n \), see Lieb and Loss \( \text{[21]}, \text{chapter 7} \).

Let \( \Omega_k, k = 1, 2, \ldots, \) be bounded domains in \( \mathbb{R}^n \) which satisfy

(i) \( \Omega_k \subseteq \Omega_{k+1} \);
(ii) \( \overline{\Omega} \setminus S = \cup_{k=1}^\infty (\overline{\Omega} \cap \overline{\Omega_k}) \);
(iii) there is a \( k_0 \in \mathbb{N} \) such that

\[ \overline{\Omega} \setminus \Omega_k \subset \overline{\Omega} \cap (N_S \cup N_\infty) \quad (1.4) \]
for all \( k \geq k_0 \); and

(iv) the embedding \( H^1(\Omega_k) \to L^2(\Omega_k) \) is compact for each \( k \in \mathbb{N} \).

(Recall the notation \( \overline{\Omega}_k \subset \Omega_{k+1} \), i.e. \( \overline{\Omega}_k \) is compact and \( \overline{\Omega}_k \subset \Omega_{k+1} \).)

This family of domains \( \{\Omega_k\}_{k=1}^\infty \) is an S-admissible family of domains in \( \Omega \) as defined in Edmunds and Evans [11], p.278. Note that (iv) holds provided the Rellich embedding theorem applies, e.g., if \( \partial(\Omega \cap \Omega_k) \) has the segment property, Agmon [1], Theorem 3.8. In most applications considerable flexibility in constructing each \( \Omega_k \) will be available.

Denote the maximum and minimum eigenvalue of \( A(x) \) by \( \nu_A(x) \) and \( \mu_A(x) \) respectively. The notation \( f^- (x) := -\min\{f(x),0\} \) and \( f^+ (x) := f(x) + f^-(x) \) will be used. Assume

**Hypothesis \( (H) \):** For each \( k \), assume that

(a) \( \partial(\Omega \cap \Omega_k) \) is \( C^1 \);
(b) \( \mu_A(x) > 0 \) a.e. on \( \Omega \) and \( \mu_A^{-1} \in L^\infty(\Omega \cap \Omega_k) \);
(c) \( q \in L^\alpha(\Omega \cap \Omega_k) \) with

\[
\alpha \left\{ \begin{array}{l}
= \frac{n}{2}, \quad n > 2, \\
> 1, \quad n = 2;
\end{array} \right.
\]

(d) \( \sigma_-(s) = 0 \) for \( s \in \Gamma_R \setminus \overline{\Omega}_{k_0} \); and
(e) \( \sigma \in L^\beta(\Gamma_R) \) with

\[
\beta \left\{ \begin{array}{l}
= n - 1, \quad n > 2, \\
> 1, \quad n = 2.
\end{array} \right.
\]

The next lemma is a special case of Lemma 1 of [13]. We refer to that paper for the complete proof. It indicates the degree of unbounded behavior of \( q^- \) that is allowed locally.

**Lemma 1.** If \( (H) \) holds, then for \( \epsilon > 0 \) and each \( k \in \mathbb{N} \) there is a \( K(\epsilon, k) > 0 \) such that

\[
\int_{\Omega \cap \Omega_k} q^- |u|^2 \, dx + \int_{\Gamma_R \cap \Gamma_k} |\sigma(s)||u(s)|^2 \, ds \\
\leq \epsilon \int_{\Omega \cap \Omega_k} |A\nabla u, \nabla u| \, dx + K(\epsilon, k) \int_{\Omega \cap \Omega_k} |u|^2 \, dx
\]

for all \( u \in \mathcal{D}(t) \).

**Proof.** The proof follows from the Monotone Convergent Theorem, the H"{o}lder Inequality, and the Sobolev Inequality.

\[\Box\]

2. **The main results**

When we know of the existence of \( \bar{T} \) we let \( \ell_\epsilon = \ell_\epsilon(\bar{T}) \) denote the least point of its essential spectrum. The following Proposition compares with Corollary 7D, Chapter III, of R.E. Showalter [24].
Proposition 1. Assume hypothesis (H), that
\[
\nu_A(x) \in L^\infty(\Omega \cap \Omega_k), \quad k \in \mathbb{N},
\]
and that for all k sufficiently large
\[
t[u] + \alpha_k \|u\|_{L^2(\Omega \cap \Omega_k)}^2 \geq c_k \|u\|_{H^1(\Omega \cap \Omega_k)}^2, \quad u \in \mathcal{D}(t),
\]  
for positive constants \(\alpha_k\) and \(c_k\).

If \(t\) is bounded below and closable, then
\[
\ell_c := \inf \{ \lambda : \lambda \in \sigma_e(T) \} = \lim_{k \to \infty} \inf \{ t[u] : u \in \mathcal{D}(t), \text{supp } u \subset \Omega \setminus \Omega_k \}. \tag{2.2}
\]

Proof. It will suffice to show that the following holds (see p.476 of [11]):
\((A)\) For each \(k \in \mathbb{N}\) large enough and \(\phi \in C_0^\infty(\mathbb{R}^n \setminus \Gamma_S)\) such that
\[
\phi(x) = \begin{cases} 
1, & x \in \Omega_k, \\
0, & x \notin \Omega_{k+1}, 
\end{cases} \tag{2.3}
\]
with \(0 \leq \phi \leq 1\), we have
(i) \(\phi v \in \mathcal{D}(t)\) for every \(v \in \mathcal{D}(t)\) and
(ii) if \(v_\ell \in \mathcal{D}(t)\) with \(\|v_\ell\|_1 = 1\) and \(v_\ell \rightharpoonup 0\) in the Hilbert space
\(H(t) := (\mathcal{D}(t); \|\cdot\|_1)\), then
\[
\|(1 - \phi)v_\ell\|^2_1 \leq 1 + o(1) \quad \text{as } \ell \to \infty.
\]

Part \((A)\) is immediate.

Since \(t\) is bounded below, without loss of generality, we may assume that \(t \geq 1\) on \(\mathcal{D}(t)\) as discussed above. Therefore (1.3) holds.

For all \(u \in \mathcal{D}(t)\) and any \(\phi\) satisfying (2.3),
\[
\| (1 - \phi)u \|^2_1 - \int_{\Gamma_R} \sigma|1 - \phi|u|^2ds
= \int_{\Omega \cap \Omega_k} \left\{ \begin{array}{l}
\left< A\nabla(1 - \phi)u, \nabla(1 - \phi)u \right> + q|1 - \phi|u|^2 \\
\left< A\nabla u, \nabla u \right> + q|u|^2 - (2 - \phi)q|\phi u|^2
\end{array} \right\} dx
\]
\[
= \int_{\Omega \cap \Omega_k} \left\{ \begin{array}{l}
(1 - \phi)^2 < A\nabla u, \nabla u > + q|u|^2 - (2 - \phi)q|\phi u|^2 \\
2\int_{(\Omega \cap \Omega_{k+1}) \cap \Omega_k} \text{Re} \left\{ A^{1/2}(1 - \phi)\nabla u, A^{1/2}u\nabla(1 - \phi) \right\}
\end{array} \right\} dx
\]
\[
\leq \int_{\Omega \setminus \Omega_k} \left\{ (1 - \phi)^2 < A\nabla u, \nabla u > + q|u|^2 + (2 - \phi)q|\phi u|^2 \right\} dx
\]
\[
-2\int_{(\Omega \setminus \Omega_{k+1}) \setminus \Omega_k} \text{Re} \left\{ A^{1/2}\nabla u, A^{1/2}\nabla u \right\} dx
\]
\[
+ \int_{(\Omega \setminus \Omega_{k+1}) \setminus \Omega_k} < A\nabla(1 - \phi)u, \nabla(1 - \phi)u > dx
\]
\[
\leq \int_{\Omega} \left< A\nabla u, \nabla u \right> + q|u|^2 dx + \int_{\Omega \setminus \Omega_{k+1}} q-|u|^2 dx
\]
\[
+ \delta \int_{(\Omega \setminus \Omega_{k+1}) \setminus \Omega_k} < A\nabla u, \nabla u > dx
\]
\[
+(1 + \delta^{-1}) \int_{(\Omega \setminus \Omega_{k+1}) \setminus \Omega_k} < A\nabla u, \nabla u > dx
\]
\[
\leq \int_{\Gamma_R} \sigma(1 - \phi)^2|u|^2ds.
\]

for \(\delta > 0\). Similarly,
\[
\int_{\Gamma_R} \sigma|1 - \phi|u|^2ds = \int_{\Gamma_R} \sigma|u|^2ds - \int_{\Gamma_{R \setminus \Omega_{k+1}}} \phi(2 - \phi)\sigma|u|^2ds.
\]
Since \( \nu_A \in L^\infty(\Omega \cap \Omega_k) \) and (1.6) holds for each \( k \), it then follows that
\[
\| (1 - \phi) u \|_t^2 \leq \| u \|_t^2 + \delta'\int_{\Omega \cap \Omega_k} < A\nabla u, \nabla u > dx + C(\delta', k) \int_{\Omega \cap \Omega_k} |u|^2 dx,
\]
for an arbitrarily small \( \delta' \) and \( C(\delta', k) > 0 \). According to the coercivity requirement (2.1) and the fact that \( \nu_A \in L^\infty(\Omega \cap \Omega_k) \) for each \( k \in \mathbb{N} \)
\[
\| (1 - \phi) u \|_t \leq (1 + \frac{\epsilon}{\epsilon_{k+1}})\| u \|_t^2 + (C(\delta', k) + \frac{\alpha_k}{\epsilon_{k+1}})\| u \|_{L^2(\Omega \cap \Omega_{k+1})}^2
\]
for an arbitrarily small \( \epsilon \).

As in (A)(ii) suppose that \( \{v_\ell\} \subset \mathcal{D}(t) \) satisfies \( \|v_\ell\|_t = 1 \) and \( v_\ell \to 0 \) in \( H(t) \). We have that
\[
\| (1 - \phi) v_\ell \|_t^2 \leq 1 + \epsilon' + C'(\delta', k)\| v_\ell \|_{L^2(\Omega \cap \Omega_{k+1})}^2.
\]
By (2.1) and the fact that \( t \geq 1 \), it follows that the embedding \( H(t) \to H^1(\Omega \cap \Omega_{k+1}) \) is continuous. Since \( H^1(\Omega \cap \Omega_{k+1}) \to L^2(\Omega \cap \Omega_{k+1}) \) is compact, then \( \|v_\ell\|_{L^2(\Omega \cap \Omega_{k+1})}^2 = o(1) \) as \( \ell \to \infty \). Hence,
\[
\| (1 - \phi) v_\ell \|_t^2 \leq 1 + o(1)
\]
since \( \epsilon' \) can be chosen arbitrarily small. That completes the proof.

In unbounded domains \( \Omega \) we will assume that \( q \) is bounded below at infinity as in (2.5) below. When we know a priori that \( t \) is bounded below, we may assume without loss of generality that for \( k \) sufficiently large \( q(x) > 0 \) for \( x \in (\Omega \setminus \Omega_k) \cap N_\infty \) as well as \( t[u] \geq \| u \|^2 \), mentioned above, since the addition of a constant only translates the spectrum.

In contrast to [13], [10], and the classical criterion of Persson [22], we are not requiring that the potential \( q \) be bounded below in a neighborhood \( N_S \) of the finite singularities. The next theorem shows that in the case of a coefficient degenerate on \( S \cap \partial \Omega \), the existence of a Hardy-type inequality in a neighborhood of the singularities may be sufficient to ensure that the form is closable and bounded below, i.e., inequality (2.4) replaces the requirement that \( q \) be bounded below on \( \partial \Omega \).

**Theorem 1.** Assume (H) holds and that for some \( \gamma \in (0, 1) \) and \( k_0 \) given in (1.4)
\[
\int_{(\Omega \cap \Omega_k) \cap N_S} [(1 - \gamma) < A\nabla u, \nabla u > -q_-|u|^2] dx \geq 0, \quad u \in \mathcal{D}(t), \quad (2.4)
\]
for all \( k \geq k_0 \) and
\[
\lim_{k \to \infty} \text{ess sup}_{x \in (\Omega \cap \Omega_k) \cap N_\infty} q_-(x) = C_\infty < \infty \quad (2.5)
\]
when \( \Omega \) is unbounded. Then \( t \) is bounded below and closable and (2.1) holds. Furthermore, if
\[
\nu_A \in L^\infty(\Omega \cap \Omega_k), \quad k \in \mathbb{N},
\]
then \( \ell_\epsilon(\mathcal{T}) \) is given by (2.2).

Proof. We give the proof in the case that \( \Omega \) is unbounded. The proof for \( \Omega \) bounded requires only slight modification.

Let

\[
\begin{align*}
t_1[u] & := \int_\Omega [\langle A\nabla u, \nabla u \rangle + q_+|u|^2]dx + \int_{\Gamma_R} \sigma_+(s)|u(s)|^2ds, \\
t'_1[u] & := -\int_\Omega q_-|u|^2dx - \int_{\Gamma_R} \sigma_-(s)|u|^2ds
\end{align*}
\]

with \( \mathcal{D}(t) = \mathcal{D}(t'_1) = \mathcal{D}(t_1) \) and \( t = t_1 + t'_1 \).

We first show that \( t'_1 \) is \( t_1 \)-bounded with \( t_1 \)-bound less than 1. Then, in order to conclude that \( t \) is closable it will suffice to show that \( t_1 \) is closable - see Kato [19], Theorem 1.33, p.320.

Let \( k \geq k_0 \) in (1.6) recalling that \( \sigma_-(s) = 0 \) for \( s \in \Gamma_R \setminus \Omega_{k_0} \) according to (H). Without loss of generality, we may assume that for \( \delta > 0 \),

\[
q_-(x) < C_\infty + \delta, \quad x \in (\Omega \setminus \Omega_{k_0}) \cap N_\infty.
\]

Then it follows from (2.4) and (1.6) that for all \( u \in \mathcal{D}(t) \), \( \epsilon \leq (1 - \gamma) \), and \( \alpha(\epsilon, k) \geq \max\{K(\epsilon, k), C_\infty + \delta\} + 1, \)

\[
|t'_1[u]| \leq (1 - \gamma) \int_{(\Omega \setminus \Omega_k) \cap N_\infty} A(x) \nabla u, \nabla u \rangle dx + \int_{(\Omega \setminus \Omega_k) \cap N_\Gamma} q_-|u|^2dx + \int_{\Gamma_R} \sigma_+ |u|^2ds
\leq (1 - \gamma)t_1[u] + (C_\infty + \delta) \int_{(\Omega \setminus \Omega_k) \cap N_\infty} |u|^2dx + K(\epsilon, k) \int_{\Omega \setminus \Omega_k} |u|^2dx
\leq (1 - \gamma)t_1[u] + \alpha(\epsilon, k)||u||_{L^2(\Omega)}^2.
\]

Therefore, \( t'_1 \) has \( t_1 \)-bound less than 1.

Note that (2.6) implies the inequality

\[
t[u] + \alpha(\epsilon, k)||u||_{L^2(\Omega)}^2 \geq \gamma t_1[u] \geq 0.
\]

Therefore, \( t \) is bounded below.

To show that \( t_1 \) is closable in \( L^2(\Omega) \), choose \( \{\varphi_n\} \subset \mathcal{D}(t) \) such that

\[
t_1[\varphi_n - \varphi_m] \to 0, \quad ||\varphi_n|| \to 0 \quad \text{as} \quad m, n \to \infty
\]

i.e., \( \{\varphi_n\} \) is \( t_1 \)-convergent to 0. Then, we must show that \( t_1[\varphi_n] \to 0 \) as \( n \to \infty \). First, note that (2.6) implies that

\[
\int_\Omega < A(x) \nabla (\varphi_n - \varphi_m), \nabla (\varphi_n - \varphi_m) > dx \to 0, \quad m, n \to \infty
\]

It follows as in (3.13) of [13] that

\[
\int_\Omega < A(x) \nabla \varphi_n, \nabla \varphi_n > dx \to 0, \quad n \to \infty.
\]

Since

\[
t_1[u] + \alpha(\epsilon, k_0)||u||^2 \geq \int_\Omega q_+|u|^2dx + ||u||^2
\]
then \( \{ \varphi_n \} \) must be a Cauchy sequence in \( L^2_{q+1}(\Omega) \). Since this space is complete, we must have that \( \varphi_n \to \psi \) for some \( \psi \in L^2_{q+1}(\Omega) \). But, \( \varphi_n \to 0 \) in \( L^2(\Omega) \) implies that \( \psi \equiv 0 \).

Consequently, we have shown that
\[
\int_\Omega [ < A \nabla \varphi_n, \nabla \varphi_n > + q_+ |\varphi_n|^2] \, dx \to 0.
\]

We need to show that
\[
\int_{\Gamma_R} \sigma_+(s)|\varphi_n(s)|^2 \, ds \to 0
\]
in order to complete the proof. An analysis similar to (3.17) in [13] applies here as well since
\[
|t_1(\varphi_n - \varphi_m)| \geq \int_{\Gamma_R} \sigma_+(s)(\varphi_n - \varphi_m)^2 \, ds \geq 0.
\]

Hence, \( \{ \varphi_n \} \) is Cauchy in \( L^2_{\sigma_+ +1}(\Gamma_R) \) and converges to a \( v \in L^2_{\sigma_+ +1}(\Gamma_R) \). By (1.6) and (2.9) we conclude that \( t \geq 1 \) and \( q > 0 \) in \( (\Omega \setminus \Omega_k) \cap N_{\infty} \) for \( k \) large.

Since \( q_-(x) = 0 \) for \( k \) large and \( x \in (\Omega \setminus \Omega_k) \cap N_{\infty} \), it follows from (2.6) that
\[
|t_1[u]| \leq (1 - \gamma)t_1[u] + K(\epsilon, k) \int_{\Omega \setminus \Omega_k} |u|^2 \, dx, \quad u \in D(t),
\]
which implies that
\[
t_1[u] + K(\epsilon, k)\| u \|^2_{L^2(\Omega \setminus \Omega_k)} \geq \gamma t_1[u] \geq \gamma \int_{\Omega \setminus \Omega_k} < A \nabla u, \nabla u > \, dx, \quad u \in D(t).
\]

Since \( \mu_A^{-1} \in L^\infty(\Omega \cap \Omega_k) \), then (2.1) holds. If we know that
\[
\nu_A \in L^\infty(\Omega \cap \Omega_k), \quad k \in \mathbb{N},
\]
then it follows from Proposition [11] that (2.2) holds. \( \square \)

Note that if \( q \) is bounded below by \( B < 0 \) on \( (\Omega \setminus \Omega_{k_0}) \cap N_{S} \) as assumed in earlier work, e.g., [10], [11], and [13], then we may apply Theorem [1] to the form \( t[u] + |B| \int_{\Omega \chi_{(\Omega \setminus \Omega_{k_0}) \cap N_{S}}} |u|^2 \, dx \).

It may be advantageous to need only show that the inequality in (2.4) holds for \( u \in H^1_0((\Omega \setminus \Omega_{k_0}) \cap N_{S}) \). The next Theorem shows that is allowed. However, we will see in the applications below that in some cases it is best to use (2.4) directly avoiding certain convexity requirements.
Theorem 2. Assume hypothesis (H), that
\[ \nu_A \in L^\infty(\Omega \cap \Omega_k), \quad k \in \mathbb{N}, \]
and (2.3) for \( \Omega \) not bounded. If for all \( \varphi \in H^1_0((\Omega \setminus \Omega_{k_0}) \cap N_\delta) \)
\[ \int_{(\Omega \setminus \Omega_{k_0}) \cap N_\delta} [(1 - \gamma) < A(x)\nabla \varphi, \nabla \varphi > - q_-(x)|\varphi|^2] dx \geq 0 \tag{2.10} \]
for some \( \gamma \in (0, 1) \), then (2.2) holds.

Proof. Recall that \( N_\delta \) is an open neighborhood of the finite singularities, \( S \setminus \{\infty\} \), with \( \Omega \setminus \Omega_k \subset \Omega \setminus (N_\delta \cup N_\infty) \) for \( k \geq k_0 \). We employ a simple IMS localization formula - see [8], p.28. Choose \( k_2 > k_1 \geq k_0 \). There exists \( \phi_1 \in C^\infty(\mathbb{R}^n) \) for which
\[ \phi_1(x) = \begin{cases} \vphantom{\sum} 1, & x \in (\Omega \setminus \Omega_{k_2}) \cap N_\delta, \\ 0, & x \in \Omega \setminus \Omega_k, \end{cases} \]
(with the support of \( \phi \) extending into \( \mathbb{R}^n \setminus \overline{\Omega} \) as needed) and \( \phi_2 \) such that
- \( \phi_j(x) \in [0, 1] \) for \( j = 1, 2 \), and all \( x \in \mathbb{R}^n \);
- \( \phi_1(x) + \phi_2(x) \equiv 1 \) for all \( x \in \mathbb{R}^n \);
- \( \phi_j \in C^\infty(\mathbb{R}^n) \); and
- \( \sup_{x \in \mathbb{R}^n} |\nabla \phi_1(x)|^2 + |\nabla \phi_2(x)|^2 < \infty \).

Recall the pointwise identity that gives rise to the IMS localization formula: for \( u \in \mathcal{D}(t) \) and \( j = 1, 2 \),
\[ < A\nabla (\phi_j u), \nabla (\phi_j u) > = \phi_j^2 < A\nabla u, \nabla u > + < A\nabla \phi_j, \nabla \phi_j > |u|^2 \]
+ \( \Re < A\nabla \phi_j^2, \overline{u} \nabla u > \). \tag{2.11}

Summing over \( j = 1, 2 \), and integrating yields the identity
\[ t[u] = \sum_{j=1}^2 \int_{\Omega} < A(x)\nabla (\phi_j u), \nabla (\phi_j u) > + q|\phi_j u|^2 - < A\nabla \phi_j, \nabla \phi_j > |u|^2 dx \]
+ \( \int_{\Gamma_R} \sigma(s)|u(s)|^2 ds \)

since \( \phi_2(s) = 1 \) on \( \Gamma_R \). Then \( \phi_1 u \in C^\infty(\Omega \setminus \Omega_{k_1}) \).

It follows from the pointwise identity (2.11) that
\[ \int_{(\Omega \setminus \Omega_{k_1}) \cap N_\delta} < A\nabla u, \nabla u > dx \]
\[ = \sum_{j=1}^2 \int_{(\Omega \setminus \Omega_{k_1}) \cap N_\delta} [< A\nabla (\phi_j u), \nabla (\phi_j u) > - < A\nabla \phi_j, \nabla \phi_j > |u|^2] dx \]
\[ \geq \int_{(\Omega \setminus \Omega_{k_1}) \cap N_\delta} < A(x)\nabla (\phi_1 u), \nabla (\phi_1 u) > dx \]
\[ - C_{k_2} \int_{(\Omega_2 \setminus \Omega_{k_1}) \cap N_\delta} |u|^2 dx \]

for
\[ C_{k_2} := \sup_{x \in (\Omega_{k_2} \setminus \Omega_{k_1}) \cap N_\delta} \sum_{j=1}^2 < A\nabla \phi_j, \nabla \phi_j > < \infty \]
since \( \nu_A \in L^\infty(\Omega \cap \Omega_k), \ k \in \mathbb{N} \).
Since (2.10) holds for $\gamma \in (0, 1)$,
\[ \int_{(\Omega \setminus \Omega_{k1}) \cap N_S} A \nabla u, \nabla u > dx + C_{k2} \int_{(\Omega \setminus \Omega_{k1}) \cap N_S} |u|^2 dx \]
\[ \geq (1 - \gamma)^{-1} \int_{(\Omega \setminus \Omega_{k1}) \cap N_S} q_-|\phi_1|^2 dx \]
\[ = (1 - \gamma)^{-1} \int_{(\Omega \setminus \Omega_{k1}) \cap N_S} q_-|u|^2 dx \]
\[ - \int_{(\Omega_{k2} \setminus \Omega_{k1}) \cap N_S} q_-|\phi_2|^2 dx. \]

As in Lemma 1 we have that for any $\epsilon > 0$ there is a positive constant $K(\epsilon, k_2)$ such that
\[ \int_{(\Omega_{k2} \setminus \Omega_{k1}) \cap N_S} q_-|\phi_2|^2 dx \leq \epsilon \int_{(\Omega_{k2} \setminus \Omega_{k1}) \cap N_S} A \nabla u, \nabla u > dx \]
\[ + K(\epsilon, k_2) \int_{(\Omega_{k2} \setminus \Omega_{k1}) \cap N_S} |u|^2 dx \]
(see (2.9) of [13]) which implies that
\[ (1 + \epsilon) \int_{(\Omega \setminus \Omega_{k1}) \cap N_S} A \nabla u, \nabla u > dx + C(\epsilon, k_2) \int_{(\Omega_{k2} \setminus \Omega_{k1}) \cap N_S} |u|^2 dx \]
\[ \geq (1 - \gamma)^{-1} \int_{(\Omega \setminus \Omega_{k1}) \cap N_S} q_-|u|^2 dx \]
for $C(\epsilon, k_2) := C_{k3} + K(\epsilon, k_2)$. Then, for $\epsilon$ chosen sufficiently small $(1 + \epsilon)(1 - \gamma) \in (0, 1)$. Since $k_1$ is an arbitrary integer greater than or equal to $k_0$, the hypothesis of Theorem 1 holds for
\[ h[u] := t[u] - (1 - \gamma)C(\epsilon, k_2) \int_{\Omega} \chi_{(\Omega_{k2} \setminus \Omega_{k1}) \cap N_S} |u|^2 dx, \quad u \in \mathcal{D}(t), \]
implying that $h[u]$ is bounded below and closable and, as shown in the proof of Theorem 1 that (2.1) holds for $h$. But, this implies that $t$ is bounded below and closable (cf. (2.5)) and (2.1) holds as well for $t$. The conclusion follows from Proposition 1. \[ \square \]

With appropriate conditions required of the coefficients, inequality (2.10) is associated with the existence of a nonnegative solution of the Dirichlet problem for
\[ -(1 - \gamma) \text{div}(A(x) \nabla u) - q_-(x)u = 0 \]
on $(\Omega \setminus \Omega_{k0}) \cap N_S$, the absence of nodal domains, and the finiteness of the negative spectrum ([2, 22, 23]).

**Corollary 1.** Assume the hypothesis of Theorem 2 and for $k \geq k_0$ define
\[ L_S[u; k] := \int_{(\Omega \setminus \Omega_{k}) \cap N_S} [< A(x) \nabla u, \nabla u > + q_+(x)|u|^2] dx, \quad u \in \mathcal{D}(t). \]
Then, for $\Omega$ bounded
\[ \ell_e \geq \lim_{k \to \infty} \inf_{||u||=1} L_S[u; k] \]
with the infimum taken over all $u \in \mathcal{D}(t)$ with $\text{supp } u \subset (\Omega \setminus \Omega_{k}) \cap N_S$. If $\Omega$ is unbounded and (2.1) holds then
\[ \ell_e \geq \lim_{k \to \infty} \inf_{||u||=1} L_S[u; k] - C_{\infty}. \]
Proof. We give the proof for the case in which Ω is unbounded. The adaptation for Ω bounded is straightforward. According to Theorem 2, for \( k \geq k_0 \) and \( \varphi := u/\|u\| \) for \( u \in \mathcal{D}(t) \) with supp \( u \subset \Omega \setminus \Omega_k \)

\[
\ell_e(t) = \lim_{k \to \infty} \inf_{\varphi} t[\varphi] = \lim_{k \to \infty} \inf_{\varphi} L_S[\varphi; k] - C_\infty.
\]

\[\square\]

3. Applications using Hardy inequalities in \( d(x) \).

In this section we explore applications of Theorems 1 & 2 with some of the more recent results for Hardy inequalities given in terms of the distance to the boundary of the domain, i.e., \( d(x) := \text{dist}(x, \partial \Omega) \).

Weighted Hardy inequalities in \( L^2(G) \), which best suit our purposes, are of the following form: for an open connected set \( G \subset \mathbb{R}^n \) and \( u \in H^1_0(G) \)

\[
\int_G d(x)^\beta |\nabla u(x)|^2 dx \geq \kappa(\beta) \int_G \frac{|u(x)|^2}{d(x)^{2-\beta}} dx + \lambda(G) \int_G d(x)^\alpha |u(x)|^2 dx
\]

(3.1)

with \( \beta < 1 \) and \( \alpha > (\beta - 2) \). Here, \( \kappa(\beta) \) is assumed to be positive for each \( \beta < 1 \) and \( \lambda(G) \geq 0 \) depends upon certain geometric properties of \( G \), e.g., the diameter of \( G \), the volume of \( G \), etc. Several results of this type are discussed below.

Corollary 2. Assume hypothesis (H), \( \nu_A \in L^\infty(\Omega \cap \Omega_k) \) for all \( k \), and that for some \( \beta < 1 \)

\[
\mu_A(x) \geq d(x)^\beta, \quad x \in (\Omega \setminus \Omega_{k_0}) \cap N_S.
\]

(3.2)

For \( \Omega \) unbounded assume that \( q_- \) is bounded below at infinity as in (2.3). Finally, assume that (2.1) holds for some \( \beta < 1 \) and for \( G = (\Omega \setminus \Omega_{k_0}) \cap N_S \). If for some \( \gamma \in (0, 1) \)

\[
q_-(x) \leq (1 - \gamma) \left[ \frac{\kappa(\beta)}{d(x)^{2-\beta}} + \lambda(G)d(x)^\alpha \right], \quad x \in G,
\]

(3.3)

then \( t \) is bounded below and closable and the spectrum of \( \tilde{T} \) is purely discrete.

Proof. The fact that \( t \) is bounded below and closable follows from Theorem 1. By (3.1) and (3.3) the hypothesis of Theorem 2 holds. We
may apply Corollary 1. For $k > k_0$

$$L_S[u; k] \geq \int_{(\Omega \setminus \Omega_k)^c} \gamma d(x)^\beta |\nabla u|^2 \, dx \geq \gamma \kappa(\beta) \int_{(\Omega \setminus \Omega_k)^c} \frac{|u|^2}{d(x)^{2-\beta}} \, dx$$

according to (3.2) followed by (3.1). According to property (ii) of the S-admissible family of domains $\{\Omega_k\}_{k=1}^\infty$ we may assume that $d(x) < \frac{1}{k}$ for $x \in (\Omega \setminus \Omega_k) \cap N_S$ and $k \geq k_0$. Since the infimum in Corollary 1 is taken over all $u \in D(t)$ with support in $(\Omega \setminus \Omega_k) \cap N_S$, then for $k \geq k_0$

$$\inf_{\|u\|=1} L_S[u; k] \geq \gamma \kappa(G) k^{2-\beta}, \quad \beta < 1.$$ 

Letting $k \to \infty$, we conclude that $\ell_e = \infty$ implying that the spectrum is purely discrete. \hfill \Box

Corollary 2 indicates that if a Hardy inequality (3.1) holds, the form can be bounded below and closable even though all coefficients are degenerate at parts of the boundary $\partial \Omega$. We review some of the earlier results in which (3.1) holds.

For $\alpha = \beta = 0$, (3.1) reduces to

$$\int_\Omega |\nabla u(x)|^2 \, dx \geq 1 \int_\Omega \frac{|u(x)|^2}{d(x)^2} \, dx + \lambda(\Omega) \int_\Omega |u(x)|^2 \, dx. \quad (3.4)$$

Recent results for this inequality were motivated by work of Brezis and Marcus in [7] who showed that for $\Omega$ convex with $\partial \Omega \in C^2$, $\lambda(\Omega) \geq \frac{1}{4 d(\Omega)^2}$ with $D(\Omega)$ denoting the usual diameter of $\Omega$. For the “interior diameter” defined by $D_{int}(\Omega) := 2 \sup_{x \in \Omega} d(x)$, Filippas, Maz’ya, and Tertikas [15] showed that for $\Omega$ convex, $\lambda(\Omega) \geq \frac{3}{D_{int}(\Omega)^2}$. Subsequently, Avkhadiev and Wirths [5] have shown that $\lambda(\Omega) \geq \frac{4 \lambda_0}{D_{int}(\Omega)^2}$ where $\lambda_0 \geq 0.94$. Using methods of Davies [9], M. Hoffmann-Ostenhof, T Hoffmann-Ostenhof, and Laptev [18] answered a question posed by Brezis and Marcus showing that for convex domains $\lambda(\Omega) \geq \frac{3 K(n)}{4 D_{int}(\Omega)^2}$, $K(n) := n^{1-2/n} |S^{n-1}|^{2/n}$, in which $|\Omega|$ denotes the volume of $\Omega$. Using similar methods, Evans and Lewis [14] showed that $\lambda(\Omega) \geq \frac{3 K(n)}{2 D_{int}(\Omega)^2}$.

Since a ball of diameter $D_{int}(\Omega)$ must be contained in $\Omega$, it follows that for $n = 2, 3$, the results for $\lambda(\Omega)$ in the paper of Filippas, Maz’ya, and Tertikas [15] are comparable to those in terms of the volume improving the inequality in the paper of M. Hoffmann-Ostenhof, T Hoffmann-Ostenhof, and Laptev [18]. Also, there is some advantage in the fact that the inequalities of [5], [7] and [15] do not require $|\Omega|$ to be finite, e.g., $\Omega = \omega \times \mathbb{R}$ with $\omega \subset \mathbb{R}^{n-1}$ convex. In that case $|\Omega| = \infty$, but $D_{int}(\Omega) < \infty$ if $D_{int}(\omega) < \infty$.

While applying some of these inequalities in Corollary 2 convexity may be required, but that requirement is diminished by the fact it is needed only on $(\Omega \setminus \Omega_{k_0}) \cap N_S$ and not necessarily on $\Omega$. In addition, a certain degree of flexibility is available in constructing the family.
\{\Omega_k\}_{k=1}^\infty \text{ in } N_S. \text{ Nevertheless, we will also be interested in inequalities not requiring convexity.}

In a domain \(\Omega \subset \mathbb{R}^n\) the distance function \(d(x)\) is uniformly Lipschitz continuous (cf. Gilbarg and Trudinger [17], §14.6) and consequently, differentiable almost everywhere according to Rademacher’s theorem. Moreover, if \(\Omega\) is bounded and \(\partial \Omega \in C^k\), \(k \geq 2\), then for some \(\delta > 0\) sufficiently small, \(d \in C^k(\Omega_\delta)\) in which \(\Omega_\delta := \{x \in \Omega : d(x) < \delta\}\) – Lemma 14.6 of [17]. If \(\Omega\) is convex, then the distance function is superharmonic, i.e., \(-\Delta d(x)\) is a nonnegative measure. (See Lemma 3 of [4] for a short proof). For dimension \(n = 2\), \(-\Delta d \geq 0\) implies that \(\Omega\) is convex, but not for \(n > 2\). Armitage and Kuran [3] give an example of a torus in dimension greater than 2, which is (obviously) not convex, but \(-\Delta d(x) \geq 0\).

In order to accommodate weights, we give a small extension of Theorem 3.1 of Filippas, Maz’ya, and Tertikas [15] requiring only a modification of their change of variable. Rather than assuming convexity of \(\Omega\) it suffices (here and in the proof of Theorem 3.1 of [15]) to assume the weaker condition that \(-\Delta d(x) \geq 0\) in \(\Omega\).

**Theorem 3.** If \(-\Delta d(x) \geq 0\) in a domain \(\Omega\), then for all \(u \in H^1_0(\Omega), \beta < 1\), and \(\alpha > \beta - 2\)

\[
\int_\Omega d(x)^\beta |\nabla u|^2 dx - \frac{(1 - \beta)^2}{4} \int_\Omega \frac{|u|^2}{d(x)^{2-\beta}} dx \geq C_{\alpha,\beta} D^{\alpha+2-\beta}_{int} \int_\Omega d(x)^\alpha |u|^2 dx
\]

for a constant

\[
C_{\alpha,\beta} := 2^{\alpha-\beta} \cdot \begin{cases} 
(\alpha + 2 - \beta)^2 & \alpha \in (\beta - 2, -1) \\
(1 - \beta)(2\alpha + 3 - \beta) & \alpha \in [-1, \infty)
\end{cases}.
\]

**Proof.** It will suffice to show the inequality for real-valued \(u \in C^\infty_0(\Omega).\)

Let \(u = d^{1-\beta} v\). Since \(|\nabla d|^2 = 1\), it follows that

\[
\int_\Omega d^\beta |\nabla u|^2 dx - \frac{(1 - \beta)^2}{4} \int_\Omega d^{\beta-2} u^2 dx = \frac{1 - \beta}{2} \int_\Omega (-\Delta d) v^2 dx + \int_\Omega d |\nabla v|^2 dx.
\]

After noting the identity

\[
\int_\Omega d^\alpha u^2 dx = \int_\Omega d^{\alpha+1-\beta} v^2 dx
\]

we estimate the integral on the right-hand side for \(\alpha > \beta - 2\) following a path similar to that of [15] to arrive at their inequality (3.4) and see that for this case

\[
(\alpha + 2 - \beta - \delta) \int_\Omega d^{\alpha+1-\beta} v^2 dx \leq R^{\alpha+2-\beta}_{int} \left( \frac{1}{\delta} \int_\Omega d |\nabla v|^2 dx + \int_\Omega (-\Delta d) v^2 dx \right).
\]

Here \(R_{int} := \frac{1}{2} D_{int}(\Omega)\). Choose \(\delta \leq \min\{\frac{1-\beta}{2}, \frac{\alpha+2-\beta}{2}\}\) and the conclusion follows. □
If we know that \(G\) in Corollary 2 is convex, then \(-\Delta d(x)\) is a positive measure and we may apply Theorem 3.

**Corollary 3.** Assume the hypothesis of Corollary 2. If for \(\gamma \in (0, 1)\) and \(\alpha > \beta - 2\)

\[
q_-(x) \leq (1 - \gamma)\left[\frac{(1 - \beta)^2}{4d(x)^2} + \lambda(G)d(x)^\alpha\right], \quad x \in G,
\]

for \(G = (\Omega \setminus \Omega_{\delta_0}) \cap N_S\) convex and \(\lambda(G) = C_{\alpha, \beta}/D_{\text{int}}^{\alpha-(\beta-2)}(G)\), then \(t\) is bounded below and closable and the spectrum of \(\tilde{T}\) is purely discrete.

**Proof.** The proof follows from Corollary 2 and Theorem 3. \(\square\)

In [16] Filippas, Maz’ya, and Tertikas prove a Hardy-Sobolev inequality in a tubular domain \(\Omega_\delta := \{x \in \Omega : d(x) < \delta\}\) for some \(\delta > 0\). Here, we adapt some of those ideas to use as an application of Corollary 1. The next Lemma allows application for the case in which \(-\Delta d(x) \geq 0\) in the whole of a non-convex \(\Omega\), but \(G\) in Corollary 2 is not convex and \(d\) is not superharmonic in \(G\). The prototype for \(\Omega\) in this case is the torus studied by Armitage and Kuran [3].

It’s important to note that in the next Lemma, \(d(x) = d(x; \Omega)\), the distance from \(x\) to \(\partial \Omega\) as before, as opposed to the distance from \(x\) to \(\partial \Omega_{\delta}, d(x; \Omega_\delta)\). We will use this additional notation in some cases below to avoid confusion.

**Lemma 2.** Assume that \(\Omega\) is a bounded domain with a \(C^2\) boundary and \(-\Delta d \geq 0\) in \(\Omega_\delta\) for all \(\delta > 0\) sufficiently small. Let \(\beta < 1\) and \(\alpha > (\beta - 3)/2\). If \(0 < \delta \leq \frac{1-\beta}{2}\) then for all \(u \in C_0^\infty(\Omega)\)

\[
\int_{\Omega_\delta} d^\beta |\nabla u|^2 dx - \frac{(1-\beta)^2}{4} \int_{\Omega_\delta} d^{\beta-2} |u|^2 dx \geq C(\alpha, \beta)\delta \int_{\Omega_\delta} d^\alpha |u|^2 dx
\]

for a positive constant

\[
C(\alpha, \beta) := \frac{2^{\alpha-\beta+1}(2\alpha - \beta + 3)}{(1 - \beta)^{\alpha-\beta+2}}.
\]

**Proof.** Since \(\Omega\) is bounded and \(\partial \Omega \in C^2\), then \(d \in C^2(\Omega_\delta, \cap \Omega)\) for some \(\delta_0\) (Lemma 14.16 of [17]). We may assume without loss of generality that \(\delta_0 = \delta \in (0, \frac{1-\beta}{2})\). It will suffice to prove the inequality for functions \(u \in C_0^\infty(\Omega)\) that are real-valued and nonnegative (Lieb & Loss [21], pp.176-177). For \(u \in C_0^\infty(\Omega)\) and \(u = d^{\frac{1-\beta}{2}}v\) it follows from integrating by parts that

\[
\int_{\Omega_\delta} d^\beta |\nabla u|^2 dx = \frac{(1-\beta)^2}{4} \int_{\Omega_\delta} d^{-1}v^2 dx + \frac{1-\beta}{2} \int_{\Omega_\delta} (-\Delta d)v^2 dx + \frac{1-\beta}{2} \int_{\partial \Omega_\delta} (\nabla d \cdot \nu)v^2 ds + \int_{\Omega_\delta} d|\nabla v|^2 dx
\]
since $|\nabla d| = 1$ where $\nu$ is the unit outward normal from $\Omega_\delta$ on $\partial \Omega_\delta \cap \Omega = \partial \Omega_\delta^c$. Since $\nabla d \cdot \nu = 1$ on $\partial \Omega_\delta^c$ we have that
\[
\int_{\Omega_\delta} d^\beta |\nabla u|^2 \, dx - \frac{(1-\beta)^2}{4} \int_{\Omega_\delta} d^{\beta-2} u^2 \, dx = \frac{1-\beta}{2} \int_{\Omega_\delta} (-\Delta d) v^2 \, dx + \int_{\Omega_\delta} d |\nabla v|^2 \, dx + \frac{1-\beta}{2} \int_{\partial \Omega_\delta} v^2 \, ds.
\] (3.8)

In order to estimate $\int_{\Omega_\delta} d^\alpha u^2 \, dx$ for $\alpha > \frac{3}{2} > \beta - 2$, we make the substitution $u = d^{\frac{1-\beta}{2}} v$ again and use the identity
\[
\text{div}(d^{\alpha-\beta+2} \nabla d) = (\alpha - \beta + 2) d^{\alpha-\beta+1} + d^{\alpha-\beta+2} \Delta d
\]
in $\Omega_\delta$. Multiply by $v^2$ and integrate by parts to see that
\[
\begin{align*}
(\alpha - \beta + 2) \int_{\Omega_\delta} d^{\alpha-\beta+1} v^2 \, dx &= -2 \int_{\Omega_\delta} d^{\alpha-\beta+2} v \nabla d \cdot \nabla v \, dx + \delta^{\alpha-\beta+2} \int_{\partial \Omega_\delta} v^2 \, ds \\
&= \int_{\Omega_\delta} d^{\alpha-\beta+2} (-\Delta d) v^2 \, dx.
\end{align*}
\] (3.9)

Applying the Cauchy-Schwarz inequality to $I_1$ we have that
\[
I_1 \leq \delta \int_{\Omega_\delta} d^{\alpha-\beta+1} v^2 \, dx + \delta^{\alpha-\beta+2} \int_{\partial \Omega_\delta} v^2 \, ds
\] (3.10)
since $d(\mathbf{x}) \in (0, \delta)$ in $\Omega_\delta$. It follows from (3.9) and (3.10) that
\[
[(\alpha - \beta + 2) - \delta] \int_{\Omega_\delta} d^{\alpha-\beta+1} v^2 \, dx \leq \delta^{\alpha-\beta+2} \int_{\partial \Omega_\delta} v^2 \, ds
\] (3.11)
since $-\Delta d(\mathbf{x}) \geq 0$ in $\Omega_\delta$.

Since $\delta \leq \frac{1-\beta}{2}$ in (3.11) we will have that
\[
\frac{\alpha - \beta + 2 - \delta}{\delta^{\alpha-\beta+1}} \int_{\Omega_\delta} d^{\alpha-\beta+1} v^2 \, dx \leq \frac{1-\beta}{2} \int_{\Omega_\delta} (-\Delta d) v^2 \, dx + \int_{\Omega_\delta} d |\nabla v|^2 \, dx + \frac{1-\beta}{2} \int_{\partial \Omega_\delta} v^2 \, ds
\]
according to (3.8). Finally, use the fact that
\[
C(\alpha, \beta) \delta \leq \frac{\alpha - \beta + 2 - \delta}{\delta^{\alpha-\beta+1}}
\]
to complete the proof.

Next, we present a corollary to Lemma 2 in which we can use Theorem 1 directly avoiding a convexity assumption for $\Omega \setminus \Omega_{k_0}$. Then, we follow with an application on a torus in $\mathbb{R}^3$ in which $-\Delta d(\mathbf{x}) \geq 0$.

**Corollary 4.** Let $\Omega$ be a bounded domain with a $C^2$-boundary and let $\mathfrak{h}[u, v]$ be given by (1.2) with $\sigma \equiv 0$ and $\mathcal{D}(\mathfrak{h}) = C^\infty_0(\Omega)$. Set $S = \partial \Omega$ and define the $S$-admissible family of domains by
\[
\Omega_k := \Omega \setminus \overline{\Omega_{\delta_k}}, \quad \delta_k = \frac{1}{k}
\]
for $k \in \mathbb{N}$. 

\[\square\]
Assume (H)(a),(b), and (c); for some $\beta < 1$

$$\mu_A(x) \geq d(x)^{\beta}, \quad x \in \Omega_{\delta_k};$$
and $\nu_A \in L^\infty(\Omega \cap \Omega_k)$ for $k$ sufficiently large. Suppose for $\gamma \in (0, 1)$ and $\alpha$ satisfying $2\alpha - \beta + 3 > 0$

$$q_-(x) \leq (1 - \gamma)[(1 - \beta)^2 + \delta C(\alpha, \beta) d(x)^{\alpha}], \quad x \in \Omega_{\delta_k};$$
and $-\Delta d(x) \geq 0$ in $\Omega_{\delta_k}$ for $k$ sufficiently large and $C(\alpha, \beta)$ defined in (3.7). Then, $h$ is bounded below and closable. The self-adjoint operator associated with $h$ has a purely discrete spectrum.

Proof. It follows from Theorem 1 and Lemma 2 that $h$ is bounded below and closable as well as the fact that (2.2) holds. For $k \geq k_0$, $u \in D(t)$ with $\text{supp } u \subset \Omega \setminus \Omega_k$ and $\varphi = u/\|u\|$

$$t[\varphi] \geq \gamma \int_{\Omega_k} A(x) \nabla \varphi, \nabla \varphi > dx \geq \gamma \int_{\Omega_k} [\frac{(1-\beta)^2}{4d(x)^{2-\beta}} + \delta C(\alpha, \beta) d(x)^{\alpha}]|\varphi|^2 dx.$$

Since $d(x) \leq \frac{1}{k}$ for $x \in \Omega_{\delta_k}$,

$$\ell_e = \lim_{k \to \infty} \inf_{\varphi} t[\varphi] = \infty$$

implying that the spectrum of the operator associated with the closure of $h$ is discrete.

Example 1. Let $\Omega \subset \mathbb{R}^3$ be the torus obtained by rotating the disc $\omega = \{(0, y, z) : (y-c)^2 + z^2 < R\}$, $c > 2R$, about the $z$-axis. Armitage and Kuran [3] have shown that the distance function $d_\Omega$ on the whole of $\Omega$ is superharmonic, i.e., $-\Delta d_\Omega(x) \geq 0$ in $\Omega$ although $\Omega$ is not convex. Assuming the hypothesis of Corollary 3, the operator associated with the Dirichlet form $h$ on the torus $\Omega$ has a purely discrete spectrum.

Of course, the Example 1 can be extended to the image of any unitary transformation of the torus described there since the spectrum is preserved under such transformations. Note that the distance function $d_{\Omega_\delta}(x)$ in $\Omega_{\delta}$ for small $\delta > 0$

$$d_{\Omega_\delta}(x) = \begin{cases} d_\Omega(x), & x \in \Omega_{\delta/2}, \\ \delta - d_\Omega(x), & x \in \Omega_{\delta} \setminus \Omega_{\delta/2} \end{cases}$$

is not superharmonic. Corollary 3 does not apply to the torus of Example 1 since $\Omega_\delta$ for $\delta > 0$ is not convex and $d_{\Omega_\delta}$ is not superharmonic.

Finally, we refer the reader to recent results in [6] where Hardy inequalities are given which exploit the interesting connection between $\Delta d(x)$ and the principal curvatures at the near point $y \in \partial \Omega$ of $x$. These new Hardy inequalities allow for applications of the results here to far more general non-convex domains such as the torus discussed above. Using a representation of $\Delta d$ in terms of principal curvatures, a new proof is given of Armitage and Kuran’s result discussed in Example 1.
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