The QCD rotator with a light quark mass

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**Abstract**

The low-lying energy excitations of 2-flavour QCD in the chiral limit confined to a small spatial box ($\delta$-regime) are that of an $O(4)$ rotator. In this work, we calculate the corrections due to the presence of a nonzero quark mass up to NNL order by means of dimensional regularised chiral perturbation theory. The final result for the energy gap of the system only involves the low-energy constants $F, \Lambda_1, \Lambda_2$, and $B$. 
1 Introduction and summary

Quantum chromodynamics, the quantum field theory describing the interactions between quarks and gluons, is non-perturbative at low energies (large distances). An expansion in the coupling constant breaks down, since the latter becomes large (of $O(1)$) at low energies. In order to study QCD properties at low energies, we have to apply other methods.

The lattice provides the only known non-perturbative treatment of QCD in terms of the fundamental fields, the quarks and the gluons. However, lattice QCD calculations are very time-consuming, and so we are restricted to relatively small volumes or relatively large lattice spacings. Thus, the results we obtain from lattice QCD simulations suffer from finite size effects and discretisation errors.

In this work, we will restrict ourselves to two light quark flavours in the isospin limit ($m_u = m_d = m$). QCD is known to be spontaneously broken in case the $u$ and $d$ quarks are massless. The QCD Lagrangian is invariant under the global $SU(2)_L \times SU(2)_R$ transformations, the vacuum of QCD is only invariant under global $SU(2)_V$ transformations. According to Goldstone’s theorem [1], the breaking of a global, continuous symmetry gives rise to massless particles, the Goldstone bosons (GB).

At low energies, QCD can be described by chiral perturbation theory (ChPT) [2, 3, 4], an effective field theory with pions as degrees of freedom. The low-energy coefficients (LECs), associated to the operators, showing up in effective Lagrangian, have to be determined by experiments or by measurements from lattice simulations. Since ChPT can also be applied if the system is enclosed in a finite volume [5, 6], finite volume effects can be calculated analytically.

We consider a volume $V = L_s \times L_s \times L_s \times L_t$, where the temporal extent of the box and the Compton wavelength of the pion are both much larger than the spatial size of the box, $L_t \gg L_s$, $M L_s \ll 1$ respectively. This special regime is called $\delta$-regime ($\delta$-expansion) [7]. Here, $M$ refers to the leading term for the pion mass in infinite volume

$$M^2 = 2mB. \tag{1}$$

Since $SU(2)_L \times SU(2)_R$ is the covering group of $O(4)$, we use a $O(4)$ nonlinear sigma model in $d = 3 + 1$ Euclidean space-time dimensions to describe the effective theory.

Due to the large Compton wavelength of the pion, large compared to the spatial extent of the box, the system exhibits a global mode which slowly rotates in the internal space. These slow modes and the related energy excitations interest us in this work. In the chiral limit ($m \to 0$), the low-lying energy excitations of this system can be described by the spectrum of the $O(4)$ rotator

$$E_\ell = \frac{\ell(\ell + 2)}{2\Theta}, \quad \ell = 0, 1, \ldots, \tag{2}$$

where this form is only valid up to NNL order. Here, $\ell$ denotes the $O(4)$ “angular momentum” (isospin) and $\Theta$ the moment of inertia which is given as $\Theta = F^2 L_s^3$ at leading order [7][8][9]. The low energy constant $F$ is the pion decay constant in the chiral limit. Allowing for higher order terms in the effective Lagrangian the moment of inertia will receive fast mode corrections proportional to $1/(FL_s)^2$ at NL order [10], respectively proportional to $1/(FL_s)^4$ at NNL order [11]. We
will denote the dimensionless expansion parameter as

\[ \delta^2 = \frac{1}{F^2 L_s^2} \]  

throughout this work.

For sufficiently small quark masses \( M \leq \frac{1}{F^2 L_s^2} \), the system is still dominated by the rotator spectrum. The latter then obtains small corrections due to the symmetry breaking terms. The dimensionless expansion parameter related to the symmetry corrections will be denoted by

\[ r = \frac{F^2 L_s^3 M}{L_t^3} \tag{4} \]

The NLO correction to the rotator spectrum coming from the symmetry breaking terms is proportional to \( r^4 \).

In Section 3, we will illustrate how to separate the slow modes from the fast modes. In order to apply perturbation theory, the effective action has to be expanded in terms of the fast modes. A detailed discussion for the symmetry-breaking terms is given in Section 4. In Section 5 we will integrate out the fast modes. The calculations are performed in the zero temperature limit \( (L_t \rightarrow \infty) \), and we use dimensional regularisation (DR). This setup is in accordance with calculations in [11].

The remaining quantum mechanical \( O(4) \) rotator, now in an external “magnetic” field, is characterised by the moment of inertia, and by the symmetry-breaking, external field \( \eta \), where \( \Theta \) and \( \eta \) contain corrections from the fast modes.

Finally, we apply perturbation theory in quantum mechanics in order to determine the corrections to the unperturbed rotator spectrum. Since the two expansion parameters enter at NLO as \( \delta^2 \), and \( r^4 \) respectively, we consider these two combinations to be of the same order \( \delta^2 \sim r^4 = \mathcal{O}(\delta^2) \). We are interested in corrections up to NNLO, therefore we have to take into account any terms up to \( \mathcal{O}(\delta^4) \).

We quote the final result for the energy gap \( E_{L_s} \) up to NNLO order

\[ E_{L_s} = \frac{3}{2 \Theta} \left[ 1 + \frac{(\Theta \eta)^2}{15} - \frac{193}{120} \left( \frac{(\Theta \eta)^4}{15^2} \right) \right] \tag{5} \]

where \( \Theta \) is given in [11] as

\[ \Theta = F^2 L_s^3 \left[ 1 - \frac{2 \hat{G}^*}{F^2 L_s^3} \left( 1 - \frac{1}{3\pi^2} \frac{1}{4} \log(A_1 L_s)^2 + \log(A_2 L_s)^2 \right) \right] + \partial_0^2 \hat{G}^* \left( 1 - \frac{3 \hat{G}^*}{2F^2 L_s^3} \right) \right] \tag{6} \]

and

\[ \eta = F^2 L_s^3 M^2 \left( 1 - \frac{3 \hat{G}^*}{2F^2 L_s^3} \right) \tag{7} \]

\( \hat{G}^* \) and \( \partial_0^2 \hat{G}^* \) are given by the following numbers

\[ \hat{G}^* = -0.2257849591 \tag{8} \]

\[ \partial_0^2 \hat{G}^* = -0.8375369106 \tag{9} \]

The low-energy constants \( l_1 \) and \( l_2 \) enter [9] over their intrinsic scales \( \Lambda_1 \), respectively \( \Lambda_2 \) [12].
2 The effective action up to $O(p^4)$

The low-energy properties of QCD with two nearly massless flavours, considered in infinite volume, can be covered by an effective field theory in terms of the pseudo-Goldstone bosons, the pions. We choose to describe the GB dynamics by an $O(4)$ nonlinear sigma model. The effective Lagrangian is ordered in a systematic way by the increasing number of derivatives and increasing power of the symmetry-breaking field $\tilde{H}$. Derivatives are counted as $O(p)$ and the symmetry-breaking field is counted as $O(p^2)$. Using the convention which identifies the term $A_{(d,h)}$ in the effective action containing operators with $d$ derivatives and the field $\tilde{H}$ to the power $h$, we write

$$A_{\text{eff}} = A_{(2,0)} + A_{(0,1)} + A_{(4,0)} + A_{(2,1)} + A_{(0,2)} + \ldots ,$$

for the effective action up to $O(p^4)$. The explicit expressions for these terms are

$$A_{(2,0)} = \int dx \frac{F^2}{2} \partial_\mu \vec{S}(x) \partial_\mu \vec{S}(x),$$

$$A_{(0,1)} = -\int dx \Sigma \tilde{H} \vec{S}(x),$$

$$A_{(4,0)} = \int dx \frac{1}{4} g_4^{(2)} \left( \partial_\mu \vec{S}(x) \partial_\mu \vec{S}(x) \right)^2 + \frac{1}{4} g_4^{(3)} \left( \partial_\mu \vec{S}(x) \partial_\nu \vec{S}(x) \right)^2,$$

$$A_{(2,1)} = \int dx k_1 \frac{\Sigma}{F^2} \left( \tilde{H} \vec{S}(x) \right) \left( \partial_\mu \vec{S}(x) \partial_\mu \vec{S}(x) \right),$$

$$A_{(0,2)} = -\int dx k_2 \frac{\Sigma^2}{F^4} \left( \tilde{H} \vec{S}(x) \right)^2.$$

$\vec{S}(x)$ is a 4-component vector of unit length, $\vec{S}(x)^2 = 1$, and we choose the external field $\tilde{H}$ to point in the zeroth direction $\tilde{H} = (H, 0, \ldots, 0)$. Terms which do not depend on $x$ at all have been omitted, since they will enter as an overall factor in the path integral representation. We have used the same conventions as in [13] for the low-energy constants.

Due to the explicit symmetry breaking, the pions acquire a mass. If we consider the external field $H$ to be small, the pseudo-Goldstone boson mass is given by [13]

$$M^2 = \frac{\Sigma H}{F^2},$$

at leading order in $H$. In the following, we will replace the combination $\Sigma H$ by $M^2 F^2$. By comparing Eq. (15) with Eq. (1), we can identify the external symmetry breaking parameter $H$ with the quark mass $m$ and $\Sigma$ with $BF^2$.

The effective action (10)-(14) is based on conventions from condensed matter physics. In chiral perturbation theory, the pion fields are usually parametrised by $SU(2)$ matrices, and the low-energy constants in the $O(p^4)$ effective Lagrangian are labelled by $l_1, l_2, l_3, l_4$ and $h_1$. We determine the relations between the LECs by comparing the $O(p^4)$ effective action in [13] with the terms (10)-(14)

$$g_4^{(2)} = -4l_1,$$

$$g_4^{(3)} = -4l_2,$$

$$k_1 = l_4,$$

$$k_2 = l_3 + l_4,$$

$$k_3 = h_1.$$

3
3 Separating the fast modes from the slow modes

Chiral perturbation theory can also be applied in finite volume [5, 6]. In the \( \delta \)-regime (\( L_t \gg L_s \) and \( M L_s \ll 1 \)), the fields \( \vec{S}(x) \) on a given time slice are strongly correlated and exhibit a net “magnetisation”

\[
\vec{m}(t) = \frac{1}{V_s} \int d\vec{x} \vec{S}(t, \vec{x}) , \quad \vec{m}(t) = m(t) \vec{e}(t) . \tag{16}
\]

The direction \( \vec{e}(t) \) of the net "magnetisation" performs a slow rotation in the internal space. These slow modes have to be treated non-perturbatively. The fluctuations (fast modes) around direction of the “magnetisation” can be integrated out in perturbation theory. Therefore, the fast modes have to be separated from the slow modes.

We incorporate the collective behaviour of the variables \( \vec{S} \) by introducing

\[
1 = \prod_t \int d\vec{m}(t) \delta^{(N)} \left[ \vec{m}(t) - \frac{1}{V_s} \int d\vec{x} \vec{S}(t, \vec{x}) \right]
\]

into the partition function of the system. The partition function then reads as

\[
Z = \prod_x \int d\vec{S}(x) \delta \left[ \vec{S}^2(x) - 1 \right] \\
\prod_t \int d\vec{m}(t) \delta^{(N)} \left[ \vec{m}(t) - \frac{1}{V_s} \int d\vec{x} \vec{S}(t, \vec{x}) \right] e^{-A_{\text{eff}}(\vec{S})} . \tag{17}
\]

By choosing appropriate field redefinitions, which have been worked out in [11], \( Z \) can be written as

\[
Z = \prod_t \int d\vec{e}(t) \prod_x \int d\vec{\pi}(x) \delta^{(N-1)} \left[ \frac{1}{V_s} \int d\vec{x} \vec{\pi}(t, \vec{x}) \right] e^{-A_{\text{eff}}(\Omega V^T \vec{R})} . \tag{18}
\]

It is indicated that in the effective action the variable \( \vec{S}(x) \) will be replaced by the combination

\[
\vec{S}(t, \vec{x}) = \Omega(t) V^T(t) \vec{R}(t, \vec{x}) . \tag{19}
\]

The 4-component vector \( \vec{R} \) of unit length is parametrised as

\[
\vec{R}(x) = \left( \sqrt{1 - \vec{\pi}^2(x)}, \vec{\pi}(x) \right) . \tag{20}
\]

Eqs. (20) and (18) ensure that the slow modes are not a part of the \( \pi \)-fields. In fact the \( k = (k_0, \vec{k} = 0) \) modes have to be left out when we calculate the Green’s functions. Furthermore, it follows that

\[
\frac{1}{V_s} \int d\vec{x} \pi_i(t, \vec{x}) = 0 , \quad i = 1, \ldots, N - 1 . \tag{21}
\]

\( \Omega(t) \) and \( V(t) \) are \( O(N) \) matrices. The first column of the matrix \( \Omega(t) \) is

\[
\vec{e}_\alpha(t) = \Omega_{\alpha 0}(t) , \quad \alpha = 0, \ldots, N - 1 , \tag{22}
\]

\[\text{In paper [11], the notation } \Sigma(t) \text{ was introduced to our } \hat{V}(t) . \text{ We decided for this change, since the quark condensate is denoted, here (and in many other works), by } \Sigma.\]
and the matrix $\hat{V}(t)$ has the following structure

$$
\hat{V}(t) = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & & & \\
\vdots & & \ddots & \hat{V}(t) \\
0 & \cdots & & 0
\end{pmatrix}.
$$

(23)

The partition function (18) is expressed as a path integral over the slow modes $\vec{e}(t)$ and as a path integral over the fast modes $\vec{\pi}(x)$. The small fluctuations (fast modes) can be integrated out in perturbation theory. Therefore, we have to expand the effective action in terms of the $\vec{\pi}$-fields, up to the desired order. The $\vec{\pi}$-fields enter by replacing the variables $\vec{S}$ according to the parametrisations defined in Eq. (19) and Eq. (20).

We have not defined the two matrices $\Omega(t)$ and $\hat{V}(t)$ completely. For the symmetry-breaking terms the unknown (undetermined) parts of $\Omega(t)$ and $\hat{V}(t)$ will drop out. Hence, the fixing of these two matrices will not be discussed in this work. A detailed treatment of this issue is given in [11].

4 The symmetry-breaking terms up to NNLO

The terms $\left[10\right]-\left[14\right]$ of the effective action are still expressed in the field variable $\vec{S}$. As mentioned before, we have to expand the effective action in Eq. (18) in the $\vec{\pi}$-fields up to the desired order (NNL). In this paper, the focus is set only on the symmetry-breaking terms, i.e. the terms $A_{(0,1)}$, $A_{(2,1)}$ and $A_{(0,2)}$. The expansion of the symmetric terms has been covered in [11].

4.1 $A_{(0,1)}$ up to NNLO

The symmetry-breaking term of the $\mathcal{O}(p^2)$ effective action is given by

$$
A_{(0,1)}(\vec{S}) = -F^2 M^2 \int dx \, S_0(x).
$$

(24)

Using the relations (19), (20), (22), (23) to express $\vec{S}$ in terms of $\vec{\pi}$-fields, $S_0$ can be written as

$$
S_0(x) = \Omega_{0\alpha}(t)\hat{V}_{\alpha\beta}(t)R_\beta(x) = c_0(t)R_0(x) + \Omega_{0\alpha}(t)\hat{V}_{\alpha\beta}(t)\pi_\beta(x)
$$

We use the convention that Greek indices run from 0 to $N - 1$, whereas Latin indices run from 1 to $N - 1$, and we will stick to this convention throughout this paper, as long as nothing else is mentioned.

The term proportional to $\pi_j(x)$ contains some unknown elements of the matrices $\hat{V}$ and $\Omega$. According to (21) this term will, however, vanish when $S_0$ is plugged into Eq. (24). After having expanded $R_0(x)$ in terms of the $\vec{\pi}$-fields, $A_{(0,1)}$ reads as

$$
A_{(0,1)} = -F^2 M^2 \int dx \, c_0(t) \left( 1 - \frac{1}{2} \pi^2(x) + \ldots \right).
$$

(25)
The dots indicate that there are terms of higher order in the $\pi$-fields. However, we can truncate the expansion already at this order, since the leading contribution from the symmetry breaking will enter already as a NLO correction in the rotator spectrum.

4.2 $A_{(0,2)}$ and $A_{(2,1)}$ up to NNLO

The symmetry-breaking terms entering at $O(p^4)$ are given by Eq. (14), and Eq. (13) respectively. We will give arguments why contributions from these two terms will only enter beyond NNLO.

In order to get a crude estimate for the sizes of the corrections coming from these two terms, we consider the system in a simplified form. We neglect the fast mode corrections. The variable $\vec{S}(x)$ can then be replaced by $\vec{e}(t)$, and the leading contribution from the symmetric term simplifies to

$$A_{(2,0)} \approx F^2 L_s^3 \int dt \frac{1}{2} \dot{\vec{e}}(t) \vec{e}(t)$$

which can be interpreted as the action of the $O(4)$ rotator. The energy eigenvalues of this system are proportional to $1/F^2 L_s^3$.

Under the same assumptions ($\pi \to 0$), the symmetry breaking term $A_{(0,2)}$ is reduced to the following form

$$A_{(0,2)} \approx -\int dt k_2 M^4 L_s^3 e_0(t)^2,$$

This term can now be regarded as a perturbation to the symmetric rotator ($M$ is small). The field $e_0$ is quantity of $O(1)$, naively the spectrum of the rotator will receive a correction proportional to $\sim M^4 L_s^3\delta^6$. The relative correction to the unperturbed rotator energy is then proportional to $F^2 L_s^3 M^5$. The latter combination, however, is of $O(\delta^7)$. This can be shown easily by expressing the strength of the perturbation in terms of the dimensionless expansion parameters $\delta$ and $r$

$$F^2 L_s^3 M^3 = r^4 \delta^6 = O(\delta^7).$$

This expression can be considered again as a small perturbation to the unperturbed rotator spectrum. Assuming the field $e_0$, which is of $O(1)$, to be a constant, the action reduces to the symmetric case. The corrections to the moment of inertia is then proportional to $M^4 L_s^3$. The latter combination is of $O(\delta^7)$, and therefore also these corrections are beyond NNLO. In addition, we have completely neglected the fact that this term would not contribute in the first order of the expansion, due to odd number of fields involved. Thus, the actual corrections would be even smaller than $O(\delta^7)$.
5 Integrating out the fast modes

We consider again the partition function (18) and insert the effective action, which has been expanded in terms of the fast modes, in the exponent \( \exp(-W_{\text{eff}}) \). While keeping the kinetic terms for \( \vec{e} \) and \( \vec{\pi} \) in the exponent, the remaining terms are expanded in a Taylor series and will be treated as interactions

\[
Z = \prod_t \int d\vec{e}(t) \prod_x \int d\pi(x) \prod_{i=1}^{N-1} \delta \left[ \frac{1}{V_s} \int \pi_i(x) \right]
\cdot \exp \left[ -\int_t \frac{F^2 L^3}{2} \vec{e}(t) \dot{\vec{e}}(t) \right] \cdot \exp \left[ -\int_x \frac{F^2}{2} \partial_\mu \pi(x) \partial_\mu \pi(x) \right]
\cdot \exp \left[ \int_t \frac{F^2}{2} \vec{e}(t) \dot{\vec{e}}(t) \pi^2(x) + \ldots \right]
\cdot \exp \left[ -\int_x \frac{2}{\epsilon^2} Q_{ij}(t) \pi_i(x) \pi_j(x) + \ldots \right]
\cdot \exp \left[ \int_x \frac{1}{\epsilon^2} Q_{ij}(t) Q_{j0}(t') \left[ \frac{1}{2} \pi^2(x) \partial_0 \pi_i(x) - \left( \pi(x) \partial_0 \pi(x) \right) \pi_i(x) \right]
\cdot \left[ \frac{1}{2} \pi^2(y) \partial_0 \pi_i(y) - \left( \pi(y) \partial_0 \pi(y) \right) \pi_i(y) \right] + \ldots \right]
\cdot \exp \left[ -\int_x \frac{2}{\epsilon^2} \left( \pi(x) \partial_\mu \pi(x) \right) \left( \pi(x) \partial_\mu \pi(x) \right) + \ldots \right]
\cdot \exp \left[ -\int_x g_4^{(2)} \left[ \frac{1}{2} \vec{e}(t) \dot{\vec{e}}(t) \partial_\mu \pi(x) \partial_\mu \pi(x) \right.ight.
\left. + \frac{1}{\epsilon^2} Q_{ij}(t) Q_{j0}(t) \partial_0 \pi_i(x) \partial_0 \pi_j(x) \right] + \ldots \right]
\cdot \exp \left[ -\int_x g_4^{(3)} \left[ \frac{1}{2} \vec{e}(t) \dot{\vec{e}}(t) \partial_0 \pi_i(x) \partial_0 \pi_j(x) + \frac{1}{\epsilon^2} Q_{ij}(t) Q_{j0}(t)
\cdot \left( \partial_0 \pi_i(x) \partial_0 \pi_j(x) + \frac{1}{2} \sum_{k=1}^3 \partial_k \pi_i(x) \partial_k \pi_j(x) \right) \right] + \ldots \right]
\cdot \exp \left[ \int_x F^2 M^2 \epsilon_0(t) \left[ 1 - \frac{1}{2} \pi^2(x) + \ldots \right]
\right.\left. + \frac{1}{2} \int_x \int_x F^4 M^4 \epsilon_0(t) \epsilon_0(t') \left[ 1 - \frac{1}{2} \pi^2(x) - \frac{1}{2} \pi^2(y) \right] + \ldots \right].
\tag{29a}
\tag{29b}
\tag{29c}
\tag{29d}
\tag{29e}
\tag{29f}
\tag{29g}
\]

At this point the fast modes can be integrated out in a systematic way, keeping only terms up to \( O(\delta^4) \). In Eq. (29) we used the convention \( x = (t, \vec{x}) \), and \( y = (t', \vec{y}) \) respectively. Furthermore, it is assumed that after having integrated out the fast modes, the limit \( \epsilon \to 0 \) has to be taken. The symmetric terms (29a) - (29f) have been discussed in [11], where also the origin of the \( \epsilon \) is explained in...
more detail. In the following, the focus is set on the additional symmetry-breaking part (29g).

The pairing of two $\pi$-field components which live at the same space time point $x$ will give a contribution of the form

$$\langle \pi_i(x)\pi_j(x) \rangle \propto \frac{\delta_{ij}}{F^2}D^*(0).$$

$D^*(0)$ denotes the finite volume propagator evaluated at $x = 0$. The * indicates that the $k = (k_0, \vec{k} = 0)$ modes have to be left out when calculating the propagator. In $d = 4$ dimensions, the finite volume propagator is proportional to $L_s^{-2}$. The pairing two $\pi$-fields results in a contribution $\sim \delta^2 F^2 D^*(0) = 1 \bar{G}^* L_s^{-2}$.

Let us consider expression (29g) separately, that means we ignore cross terms with any of the symmetric interactions for now. Again, we integrate out the fast modes, and we keep only terms up to $O(\delta^4)$. Hence, the partition function can be written as

$$Z \propto \prod_t \int d\tilde{\epsilon}(t)e^{-\frac{1}{2}(\frac{\delta^2}{F^2})\tilde{\epsilon}(t)} \left\{ 1 + \int dt F^2 L_s^3 M^2 \left( 1 - \frac{3 D^*(0)}{F^2} \right) e_0(t) \ight. \\
+ \frac{1}{2!} \int dt \int dt' (F^2 L_s^3 M^2)^2 \left( 1 - 3 \frac{D^*(0)}{F^2} \right) e_0(t)e_0(t') + \ldots \right\}. \quad (31)$$

where, up to NNL order $\Theta$ is given as quoted in Eq. (6). The expression inside the curly brackets in Eq. (31), can be written as an exponential again

$$Z \propto \prod_t \int d\tilde{\epsilon}(t) \exp \left[ -\int dt \frac{\Theta}{2} \tilde{\epsilon}(t)\tilde{\epsilon}(t) - \eta e_0(t) \right]. \quad (32)$$

Here, we introduced the new expression $\eta$ which we have already given in Eq. (7) as

$$\eta = F^2 L_s^3 M^2 \left( 1 - \frac{3\bar{G}^*}{2F^2 L_s^2} \right).$$

Since the leading contribution from the symmetry-breaking interactions is a $O(\delta^2)$ correction to the rotator spectrum, the corrections to $\eta$ are only considered up to $(FL_s)^{-2}$.

A similar argument can be used to neglect any cross terms between the symmetric interactions and the symmetry-breaking interactions (on the level of the fast modes). The only possible cross terms which would connect fast modes from the symmetric part with fast modes from the symmetry-breaking part will enter only beyond $O(\delta^4)$. Thus, the cross terms factorise in a simple way and we can write the symmetric part and the symmetry-breaking part as an exponential (32).

The original problem, described by an effective field theory in $d = 4$ Euclidean space-time dimensions, has been reduced to a 1-dimensional system
described by the action
\[ A = \int dt \frac{\Theta}{2} \dddot{\vec{e}}(t) \dddot{\vec{e}}(t) - \eta e_0(t), \tag{33} \]
where the variables \( \dddot{\vec{e}}(t) \) satisfy the constraint
\[ \dddot{\vec{e}}(t) \dddot{\vec{e}}(t) = 1. \]

In the chiral limit (\( \eta = 0 \)), we identify the system \(^{(33)}\) as that of a symmetric \( O(4) \) quantum mechanical rotator, where up to NNLO, the spectrum is given by \(^{(2)}\). In the explicitly broken case, the rotator is considered in an external “magnetic” field which points along the zeroth direction. \( \eta \) is assumed to be small, so that the symmetry-breaking potential can be treated as a perturbation. The corrections to the unperturbed rotator are then calculated in perturbation theory in quantum mechanics. Considering again corrections up to \( O(\delta^4) \), requires perturbation theory in quantum mechanics up to 4th order.

6 The quantum mechanical \( O(4) \) rotator

In this section, we will do the discussion for general \( N \geq 3 \), although we are explicitly interested in case \( N = 4 \), finally. The system described by Eq. \(^{(33)}\) can be identified as an \( O(N) \) rotator in a small external “magnetic” field. The Hamilton operator for this constraint system is given by
\[ H = \frac{1}{\Theta} \left( \frac{L^2}{2} - \lambda e_0 \right), \tag{34} \]
where \( L \) can be considered as the “angular momentum” operator in the \( N \)-dimensional internal space. Again, \( \Theta \) is the moment of inertia and \( \lambda \), given by
\[ \lambda = \eta \Theta, \tag{35} \]
is the small, dimensionless parameter which denotes the strength of the external “magnetic” field.

The energy gap of the system is defined as the difference between the energy of the first excited state and the ground state energy. In the chiral limit (\( \lambda \to 0 \)), the ground state energy is zero and the energy gap is simply given by the energy of first excited state. Due to the presence of the “magnetic” field, the energy levels of the first excited state split up. For \( N = 4 \) the first excited state splits up into a singlet and a triplet. The triplet provides the lower energy difference and is identified as the energy gap. Up to \( O(\lambda^2) \) the energy gap has already been calculated in \(^{[7]}\), but without taking into account fast modes corrections.

Besides taking into account also fast modes corrections, we are interested in corrections up to \( O(\lambda^4) \). Since \( \lambda \) is assumed to be small, we write the energy spectrum of \(^{(34)}\) as a power series in \( \lambda \) up to \( O(\lambda^4) \).
\[ E_{\ell,k}(\lambda) = \frac{1}{\Theta} \left( \epsilon_{\ell,k}^{(0)} + \sum_{i=1}^{4} \lambda^i \epsilon_{\ell,k}^{(i)} + O(\lambda^5) \right). \tag{36} \]
Here, $\varepsilon_{\ell,k}^{(0)}$ denotes the spectrum in the chiral limit ($\lambda \to 0$) which for general $N \geq 3$ is given by

$$\varepsilon_{\ell,k}^{(0)} = \varepsilon_{\ell}^{(0)} = \frac{1}{2} \ell(\ell + N - 2), \quad \ell = 0, 1, \ldots .$$ (37)

In order to determine the coefficients $\varepsilon_{\ell,k}^{(i)}$ we apply standard perturbation theory in quantum mechanics.

The energy eigenstates of the unperturbed system can be represented by associated Jacobi polynomials

$$|\ell, k_\ell, k_\ell\rangle \sim P_{\ell,k}^{(N)}(z).$$ (38)

The polynomials $P_{\ell,k}^{(N)}(z)$ are defined as

$$P_{\ell,0}^{(N)}(z) = \frac{(-1)^N (2k)^{\ell + 1}}{2N^\ell} \frac{d}{dz} \left(1 - z^2\right)^{\ell/2} P_{\ell}^{(0)}(z),$$ (39)

and have been normalised to $P_{\ell,0}^{(N)}(1) = 1$. $k$ denotes the $O(N - 1)$ “angular momentum” and runs from $0, \ldots, \ell$. The variable $z$ takes values in the interval $[-1, 1]$ and can be identified with the direction of the “magnetic” field in the internal $N$-dimensional space. For fixed $N$ and $k$, two of these polynomials are orthogonal in the interval $[-1, 1]$ with respect to the weighting function $w(z) = (1 - z^2)^{(N-3)/2}$

$$\int_{-1}^{1} dz (1 - z^2)^{(N-3)/2} P_{\ell,k}^{(N)}(z) P_{\ell',k}^{(N)}(z) = \delta_{\ell,\ell'} \delta_{k,k'}. $$ (41)

In perturbation theory matrix elements of the following form have to be calculated repeatedly

$$V_{\ell k}^{k} = \langle \ell, k | z | \ell', k \rangle = \frac{\langle z P_{\ell,k}^{(N)}(z) P_{\ell',k}^{(N)}(z) \rangle}{\langle P_{\ell,k}^{(N)}(z) P_{\ell,k}^{(N)}(z) \rangle},$$ (42)

where only matrix elements of the form $V_{\ell,k}^{k}$ for $k = 0, \ldots, \ell$, and $V_{\ell,k}^{k}$ for $k = 0, \ldots, \ell - 1$ are nonzero. As a consequence, the corrections to the orders $O(\lambda)$ and $O(\lambda^2)$ vanish

$$\varepsilon_{\ell,k}^{(1)} = \varepsilon_{\ell,k}^{(3)} = 0,$$ (43)

and at a given order of $\lambda^{2n}$, $n = 1, 2, \ldots, \ell$ only a finite number of terms survive the (in general) infinite sums for the coefficients $\varepsilon_{\ell,k}^{(2)}$, $\varepsilon_{\ell,k}^{(4)}$ respectively

$$\varepsilon_{\ell,k}^{(2)} = \frac{V_{\ell,k}^{k} V_{\ell,k}^{k}}{\varepsilon_{\ell}^{(0)} - \varepsilon_{\ell-1}^{(0)}},$$ (44)

$$\varepsilon_{\ell,k}^{(4)} = \frac{V_{\ell,k}^{k} V_{\ell,k}^{k}}{\varepsilon_{\ell}^{(0)} - \varepsilon_{\ell-1}^{(0)}} \left(\frac{\varepsilon_{\ell}^{(0)} - \varepsilon_{\ell-2}^{(0)}}{\varepsilon_{\ell}^{(0)} - \varepsilon_{\ell-1}^{(0)}}\right)^2 + \frac{V_{\ell,k}^{k} V_{\ell,k}^{k}}{\varepsilon_{\ell}^{(0)} - \varepsilon_{\ell+1}^{(0)}} \left(\frac{\varepsilon_{\ell}^{(0)} - \varepsilon_{\ell+2}^{(0)}}{\varepsilon_{\ell}^{(0)} - \varepsilon_{\ell+1}^{(0)}}\right)^2,$$ (45)
Using Eqs. (37)-(45), the energies \( E_{\ell,k} \) can be calculated for arbitrary \( N \geq 3 \).

7 The energy gap for \( N = 4 \)

The energy gap for \( N = 4 \) is defined as the difference between the energy of first excited state \((\ell = 1, k = 0,1)\) (singlet, triplet) and the ground state \((\ell = 0, k = 0)\). In Tab. 1 we quote the coefficients for the ground state energy correction and for the first excited state energy correction. The triplet \((\ell = 1, k = 1)\) provides the lower energy difference, and therefore the energy gap is defined as

\[
E_{L_s} = E_{1,1}(\lambda) - E_{0,0}(\lambda). \tag{46}
\]

Up to \( \mathcal{O}(\lambda^4) \), the explicit result for energy gap reads as

\[
E_{L_s} = \frac{3}{2\Theta} \left[ 1 + \frac{\lambda^2}{15} - \frac{193}{120} \frac{\lambda^4}{15^2} \right]. \tag{47}
\]

Inserting the results for \( \Theta \) and \( \eta \), according to Eq. (6), and Eq. (7) respectively, we recover the final result in Eq. (4). The leading term in \( \lambda \) is given by the dimensionless expansion parameter \( r^2 \), and up to \( \mathcal{O}(\delta^4) \) \( \lambda \) reads as

\[
\lambda^2 = F^8 L_s^{12} M^4 \left[ 1 - \frac{7G^*}{(FL_s)^2} \right] + \mathcal{O}(\delta^6). \tag{48}
\]

| \( N = 4 \) | \( \varepsilon_{\ell,k}^{(0)} \) | \( \varepsilon_{\ell,k}^{(2)} \) | \( \varepsilon_{\ell,k}^{(4)} \) |
|----------|--------------|--------------|----------------|
| \( \ell = 0 \) | \( k = 0 \) | 0 | \(-\frac{1}{6}\) | \(\frac{5}{452}\) |
| \( \ell = 1 \) | \( k = 0 \) | \(\frac{3}{2}\) | \(\frac{1}{15}\) | \(-\frac{317}{27000}\) |
| \( k = 1 \) | \(\frac{3}{2}\) | \(-\frac{1}{15}\) | \(\frac{23}{27000}\) |

Table 1: Here, the coefficients \( \varepsilon_{\ell,k}^{(n)} \) \((n = 0, 2, 4)\) are given for the ground state \((\ell = k = 0)\), for the singlet \((\ell = 1, k = 0)\), and for the triplet \((\ell = k = 1)\).

8 The constraints on \( L_s \) and \( M \)

The formula for the energy gap Eq. (5) involves two different expansions. The requirement that these two expansion are applicable will put some constraints on the values of \( L_s \) and \( M \). These values can be considered as a rough estimate on the domain of \( L_s \) and \( M \), where the approximation used above is valid.

\[\text{The NL contribution is in agreement with [7], as long as we neglect the fast mode corrections, i.e. } E_{L_s} = \frac{3}{(2F^2L_s^3)} \left[ 1 + \frac{(F^2L_s^{12} M^4)}{15} + \ldots \right].\]
Table 2: This table shows the NLO ($\theta_{NL}$) and the NNLO ($\theta_{NNL}$) corrections to the moment of inertia for some selected values of $L_s$. $\hat{M}_0$ is simply $\hat{M}_1 = 1/L_s$, and $\hat{M}_2$ is defined according to (51). As numerical input we used $F = 86.2$ MeV, $\Lambda_1 = 120$ MeV and $\Lambda_2 = 1200$ MeV, obtained from [15].

| $L_s$ [fm] | $\theta_{NL}$ | $\theta_{NNL}$ | $\hat{M}_1$ [MeV] | $\hat{M}_2$ [MeV] | $\hat{M}_2L_s$ |
|------------|---------------|----------------|------------------|------------------|----------------|
| 2.0        | 0.59          | -0.10          | 99               | 213              | 2.16           |
| 2.5        | 0.38          | -0.05          | 80               | 109              | 1.38           |
| 3.0        | 0.26          | -0.03          | 66               | 63               | 0.96           |

What is the constraint on $L_s$? The moment of inertia $\Theta$ receives corrections proportional to $\delta^2$ at NLO, and proportional to $\delta^4$ at NNLO respectively

$$\Theta = F^2L_s^2 \left[1 + \theta_{NL} + \theta_{NNL}\right]. \quad (49)$$

In order to have this expansion working properly, the corrections $\theta_{NL}$ and $\theta_{NNL}$ should be small. In fact, we require that $\theta_{NL}$ should be roughly 50% or smaller. This requirement puts a lower bound on $L_s$. An estimate for the lower bound of the box size can be obtained from Tab.2. There, the size of the corrections at NLO and at NNLO are given for some selected values of $L_s$. For $L_s = 2.0$ fm the NLO corrections turn out to be rather large, about 60%. Going towards $L_s = 2.5$ fm the NLO corrections decrease to 40%. Thus, we conclude that volumes about $L_s \gtrsim 2.5$ fm should be considered.

What are the constraints on $M$? The $\delta$-regime requires the mass to be much smaller than the inverse box size. We define $\hat{M}_1$ to be

$$\hat{M}_1 = \frac{1}{L_s}. \quad (50)$$

This relation gives a first, upper bound on the mass for a given $L_s$. The corresponding values are quoted in Tab.2. Taking $L_s = 2.5$ fm as a reference value for the lower bound on $L_s$, the mass should be smaller than $M = 80$ MeV.

However, this constraint does not account for the fact that the expansion for the energy gap [5] can brake down. This issue can be circumvented requiring the first correction to the energy gap ($r^4/15$) to be small, e.g. 50% or smaller. We define $\hat{M}_2$ as the second upper bound satisfying the relation

$$\frac{F^8L_s^{12}\hat{M}_2^4}{15} = \frac{1}{2}. \quad (51)$$

for a given $L_s$. It turns out that this constraint only becomes relevant for larger values of $L_s$, i.e. around $L_s = 3$ fm.

From the considerations above we conclude that an appropriate choice for $L_s$ and $M$ is crucial, in order to be in the domain where the two expansions work properly, and the formula for the energy gap is valid. We defined the lower bound on $L_s$ to be about 2.5 fm. This forces the leading order term in the pion mass $M$ to be smaller than 80 MeV, and therefore, we have to use quarks with masses below the physical quark masses. Since $M$ scales roughly as $\sim 1/L_s^3$, the upper bound for the mass decreases even faster with increasing $L_s$. 

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