Canonical quantization of a massive particle on $AdS_3$

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Abstract: The classical theory for a massive free particle moving on the group manifold $AdS_3 \cong SL(2, \mathbb{R})$ is analysed in detail. In particular a symplectic structure and two different sets of canonical coordinates are explicitly found, corresponding to the Cartan and Iwasawa decomposition of the group. Canonical quantization is performed in two different ways; either by imposing the future-directed constraint before or after quantization. It is found that this leads to different quantum theories. The Hilbert space of either theory decomposes into the sum of certain irreducible representations of $sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R})$; however, depending on how the constraint is imposed we get different representations. Quantization of the mass occurs, although a continuum exists in the unconstrained theory corresponding to particles that can reverse their direction in time. A representation on function space is worked out in detail, and from this the wavefunctions of $\widetilde{AdS}_3$ and the BTZ black hole are deduced. A quantization in terms of the “chiral” variables of the theory is also carried out giving the same results. Comparisons are made between QFT in $AdS_3$ and the quantum mechanics derived.
1. Introduction

Our present understanding of string theory in curved backgrounds is rather limited. One of the simplest non-trivial backgrounds would appear to be $AdS_3$; this space is actually the group manifold $SL(2, \mathbb{R})$, therefore this allows one to re-express bosonic string theory with a $B$ field as an $SL(2, \mathbb{R})$ WZW model, which arguably is easier to study. It is important to understand strings in $AdS_3$ from the point of view of the $AdS/CFT$ correspondence, as a detailed understanding of the string theory would allow non-trivial tests of the conjecture and maybe even insights into its proof. Now, much work has been done on the subject, for example [9], [10], [22], however a systematic canonical treatment is lacking. An obvious starting point for all this is to study the simpler case of a free particle in such a space, since it should correspond to the $\alpha' \to 0$ limit of the string theory, and of course it is interesting in its own right. The particle Lagrangian will be cast in a form similar to the WZW model, and then analyzed in a group theoretic way, following [3], [8].
In this paper we first study the classical theory for a massive free particle moving on the group manifold $AdS_3 \cong SL(2, \mathbb{R})$. We will make use of the elegant approach to phase space and Poisson brackets introduced by Witten [6] and Zuckerman [5]. This involves the identification of the phase space with the manifold of all classical solutions, and defining a symplectic form on this manifold directly from the Lagrangian. This leads to interesting (quadratic) Poisson brackets in terms of certain “natural” variables. As expected it is found that the current algebras provide a representation of $sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R})$ the Lie algebra of the isometry group of the manifold. Quantization of this system can be done canonically, leading to the result that the Hilbert space of the theory furnishes the direct sum of certain irreducible representations of the quantum current algebra which is still $sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R})$. The irreducible representations turn out to be most of both discrete series, but the exact representations seem to depend on how one imposes the constraint that the particle should be future directed. The consequences of this are that the particle mass becomes quantized (as expected since there is a closed time-like direction). Interestingly $C^0_0$ in the exceptional interval appears in the unconstrained system leading to a continuum of mass states which enjoy the property of being able to reverse their direction in time. We must note that quantization of a particle on $AdS_3$ has already been attempted [15], however our results differ slightly; if we impose the constraint before quantization we do indeed reproduce their results, however quantizing the unconstrained theory we get a quantum correction to the Casimir which changes the allowed representations, and hence the masses, even after the constraint is imposed in the quantum theory.

A Schrodinger representation of the algebra is taken, and all the wavefunctions are worked out, confirming the allowed values of the mass. From this we have also deduced the Hilbert space structure for quantum mechanics on $\widetilde{AdS}_3$ (the universal cover of $AdS_3$) which is a physically more reasonable space since it has no closed timelike curves. Wavefunctions for the $BTZ$ black hole (rotating, massive and non-extremal) were also deduced.

A more interesting quantization, going back to Fadeev et al. [7], [8] and later Goddard et al. [3], [4], is carried out too, involving quantities which correspond to the “chiral” nature of the theory inherited from the bivariance of the metric. This gives the same results, as one of our quantum theories, and provides a strategy which will be useful for quantization of the string.

Finally a section on QFT is presented in which some of the fundamental functions, such as the Wightman function, are calculated. These are compared to the propagator in the quantum mechanics, which can take two different forms, depending on which expression for the Casimir is chosen. The Casimir with the quantum correction leads to a simplification in the formulae and thus appears critical in some sense; further, it is also consistent with the Breitenlohner-Freedman bound in $AdS_3$.

2. The classical theory

2.1 The geometry of $SL(2, \mathbb{R})$

The Lie algebra $sl(2, \mathbb{R})$ consists of all real two dimensional traceless matrices. A convenient basis is given by,
\[
T_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad T_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad T_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

which satisfy \( T_a T_b = \eta_{ab} 1 + \epsilon_{abc} T_c \), where \( (\eta_{ab}) = \text{diag}(1, -1, 1) \) and \( \epsilon_{abc} \) is the usual alternating symbol \( (\epsilon_{123} = 1) \), and \( \epsilon_{ab}^\alpha = \epsilon_{abcd} \eta^{dc} \). Note that the structure constants in this basis are \( 2 \epsilon_{abc} \). The Killing form of a Lie algebra \( g \) is defined as a \((0, 2)\) tensor \( \kappa \), such that

\[
\kappa(x, y) = \text{tr} \left[ \text{ad}_x \text{ad}_y \right]
\]

where \( x, y \in g \). It is easy to show that \( \kappa(T_a, T_b) = 8 \eta_{ab} \), and it is this metric (actually \( \frac{1}{8} \kappa \)) on the Lie algebra which is used to raise and lower Lie algebra indices.

Now, any group element can be written uniquely as

\[
g = e^{uT_2} e^{\rho T_1} e^{vT_2},
\]

where \( t = u + v, \phi = v - u \in [0, 2\pi) \) and \( \rho \geq 0 \), this is called the Cartan decomposition. Any (semi-simple) Lie group possesses a natural left and right invariant metric,

\[
G_{\mu\nu} = \frac{1}{2} \text{tr} \left( g^{-1} \partial_\mu g g^{-1} \partial_\nu g \right).
\]

A calculation then gives,

\[
G = -\cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \, d\phi^2.
\]  

This is the metric for \( AdS_3 \) with the cosmological constant \( \Lambda = -1 \). To get the metric for the universal cover \( \tilde{AdS}_3 \) one simply takes \( t \) to range over \( \mathbb{R} \).

### 2.2 Massive Particles

Let \( g: \mathbb{R} \to SL(2, \mathbb{R}) \) be the curve corresponding to the particle’s worldline. The parameter of the curve \( g \) will be called the proper time and will be assumed to take all real values since \( AdS_3 \) is geodesically complete. The Lagrangian we will use is,

\[
L = \frac{1}{2} \text{tr} (g^{-1} \dot{g})^2.
\]

Hence we see, from (2.1), that the Lagrangian is simply \( L = G_{\mu\nu} \dot{g}^\mu \dot{g}^\nu \), where \( G_{\mu\nu} \) is the metric for \( AdS_3 \), although it must be supplemented with the mass-shell constraint (see Appendix B).

Now, one may wonder why we have chosen to use (2.2) as our particle Lagrangian, and not the more familiar \( L = -2m \sqrt{|G_{\mu\nu} \dot{g}^\mu \dot{g}^\nu|} \), where \( m \) is the mass; this latter Lagrangian is singular (i.e. \( \det \frac{\partial^2 L}{\partial \dot{g}^\mu \partial \dot{g}^\nu} = 0 \) and hence one cannot express \( \dot{g}^\mu \) in terms of the canonical momenta), which will lead to a constraint (the mass-shell condition) which by itself is not so bad. What complicates matters is that one cannot gauge fix the reparameterisation invariance covariantly (see \cite{1} for a good account); breaking spacetime covariance is undesirable, as after quantization one needs to show that the theory is still covariant. Also it must be noted that the “square-root Lagrangian” is actually classically equivalent to \( L = \frac{1}{e} G_{\mu\nu} \dot{g}^\mu \dot{g}^\nu - em^2 \), where \( e \) is an einbein on \( \mathbb{R} \); this Lagrangian can be gauge fixed covariantly, e.g. \( e \approx 1 \), and in fact with this constraint we get that the phase space structure is the same as that of the Lagrangian (2.2) with the mass-shell constraint (see Appendix B).

\footnote{The symbol \( \approx \) means weakly equal in the language of Dirac, in other words equal modulo the constraints.}
It will be useful to find the Hamiltonian formulation for our Lagrangian system. Firstly, the canonical momenta are \( \pi_\mu \equiv \frac{\partial L}{\partial \dot{\phi}_\mu} = 2G^{\mu\nu}(g)\dot{g}^\nu \). The canonical Hamiltonian is then found to be \( H = \frac{1}{2}G^{\mu\nu} \pi_\mu \pi_\nu \). The mass-shell constraint is \( \frac{1}{2}G^{\mu\nu} \pi_\mu \pi_\nu + m^2 \approx 0 \). Now at this point one is tempted to take the phase space as \( T^*(\text{AdS}_3) \) with coordinates \( (g^\mu, \pi_\nu) \), and proceed as usual by defining the standard Poisson bracket, working out the Noether currents and their algebras in preparation for the quantum theory. However this route will turn out to be plagued with pitfalls at the quantization stage, such as operator ordering ambiguities in expressions involving the metric in a general coordinate system. Also, solving the equations of motion covariantly is not really possible in this formalism. Since we are dealing with a group manifold we will employ a more group theoretic approach.

The left invariance of the metric gives the current \( L = -\dot{g}g^{-1} \) and the right invariance gives \( R = g^{-1}\dot{g} \). The equations of motion are simply the current conservation laws \( \dot{L} = \dot{R} = 0 \). A general solution can be easily obtained in this language, \( g(\tau) = e^{-LT}g(0) = g(0)e^{RT} \).

A more useful form for a general time-like (it is here that restriction to massive particles occurs) solution can be derived as follows. Firstly note that isometries map time-like geodesics into time-like geodesics. The action of the isometries on geodesics is transitive. Hence given one time-like geodesic, \( g(\tau) = e^{p\tau T_2} \) say \( (p > 0 \text{ for future-directed}) \), all others are given by the action of an isometry on it. Therefore a general time-like geodesic can be written as \( g(\tau) = u_0e^{p\tau T_2}v_0 \), where \( u_0 \) and \( v_0 \) are group elements. It is clear that the map \( u_0 \mapsto u_0h \) and \( v_0 \mapsto h^{-1}v_0 \), with \( h = e^{aT_2} \), leaves \( g(\tau) \) unchanged (in fact for any such map which leaves \( g(\tau) \) unchanged, \( h \) must be of this form); therefore,

\[
g(\tau) = \tilde{u}e^{(q+p\tau)T_2}\tilde{v} \tag{2.3}
\]

where \( \tilde{u} \) and \( \tilde{v} \) belong to \( SL(2,\mathbb{R})/T \) and \( T\backslash SL(2,\mathbb{R}) \) respectively, where \( T \) is the Cartan subgroup given by \( T = \{e^{aT_2}\} = SO(2) \). Note that using (2.3) the Hamiltonian works out to be \( H = -p^2 \approx -m^2 \).

### 2.3 Phase space and Poisson brackets

We will follow the elegant approach of Zuckerman and Witten, in order to get a covariant derivation of the Poisson brackets. This involves defining the phase space \( \mathcal{S} \), as the manifold of classical solutions of the Euler-Lagrange equations of the Lagrangian. Then in order to define Poisson brackets, as is well known, one needs to find a symplectic form on \( \mathcal{S} \); recall this is simply a closed non-degenerate two form. The idea is that one can avoid moving into the Hamiltonian formalism by defining a symplectic form directly from the Lagrangian \( \mathcal{L} \); we sketch how this works in the general case of a one-dimensional field theory. Consider a theory with fields \( \phi : \mathbb{R} \to M \) and an action \( S(\phi) = \int_M \mathcal{L}(\phi) \). Let \( \mathcal{F} \) be the space of fields \( \phi \) and \( \mathcal{S} \) the submanifold of solutions to the variational problem \( \delta S(\phi) = 0 \), where \( \delta \) is the exterior derivative on \( \mathcal{F} \). Now, \( \mathcal{L} \) is a \((0,1)\)-form on \( \mathcal{F} \times \mathbb{R} \); it can be shown that \( \delta\mathcal{L} = E + d\theta \) where \( d \) is the exterior derivative on \( \mathbb{R} \) and the decomposition is unique. Then one defines the two form (on \( \mathcal{S} \)) as \( \omega = \delta\theta(\tau) \) which is the symplectic structure desired; it is clear, since \( d \) and \( \delta \) anticommute on \( \mathcal{F} \times \mathbb{R} \), and that \( E \) vanishes
on $S \times \mathbb{R}$ (this corresponds to the Euler-Lagrange equations as $\delta S(\phi) = \int_\mathbb{R} E(\phi, \delta \phi)$) that $d\omega = 0$ showing that no special choice of $\tau$ has been made.

Applying this procedure to (2.2) gives $\theta(\tau) = -\text{tr}[\delta g \partial_\tau(g^{-1})]$. Therefore a symplectic form for $S$ is given by,

$$\omega = -\frac{1}{2} \text{tr}(\delta g \wedge \delta(\partial_\tau g^{-1}))$$

(2.4)

where the factor of two has been introduced for convenience.

Poisson brackets for the theory are defined in the usual way,

$$\{f, g\} = \omega^{ab} \partial_a f \partial_b g,$$

where $f, g \in C^\infty(S)$ and $\omega_{ac} \omega^{cb} = \delta_a^b$.

Now, one can choose a parameterization of the cosets such that $\tilde{u} = e^{A T_2} e^{B T_1}$ and $\tilde{v} = e^{B T_1} e^{A T_2}$. Note that choosing a different parameterization simply redefines $q$ appearing in (2.3). Using this parameterization, and substituting into (2.4) we get,

$$\omega = \delta \lambda, \quad \text{where} \quad \lambda = p \delta q + M \delta A + \bar{M} \delta \bar{A}$$

(2.5)

and $M = p \cosh 2B$ and $\bar{M} = p \cosh 2\bar{B}$. Hence we have found canonical coordinates on the phase space with only the following non-zero Poisson brackets,

$$\{q, p\} = \{A, M\} = \{\bar{A}, \bar{M}\} = 1.$$ 

(2.6)

It is also interesting to work out Poisson brackets of other variables. Define $u = \tilde{u} e^{q T_2}$ and $v = e^{q T_2} \tilde{v}$, and use the following convenient notation for $a \in \text{Mat}(2, \mathbb{R})$, $a_1 = a \otimes 1$ and $a_2 = 1 \otimes a$. Then it is found that,

$$\{\tilde{u}_1, \tilde{v}_2\} = \{\tilde{u}, p\} = \{\tilde{v}, p\} = 0$$

(2.7)

$$\{u_1, u_2\} = u_1 u_2 r_{12}$$

(2.8)

$$\{v_1, v_2\} = -r_{12} v_1 v_2$$

(2.9)

$$r_{12} = \frac{1}{2p} (T_1 \otimes T_3 - T_3 \otimes T_1).$$

(2.10)

Now, the currents are easily found to be,

$$L = -\tilde{u} p T_2 \tilde{u}^{-1} = -u p T_2 u^{-1}$$

(2.11)

$$R = \tilde{v}^{-1} p T_2 \tilde{v} = v^{-1} p T_2 v$$

(2.12)

which then give the following brackets

$$\{L_1, u_2\} = -C_{12} u_2, \quad \{R_1, v_2\} = v_2 C_{12},$$

(2.13)

where $C_{12} = T^u \otimes T_a$ is the tensor Casimir. This gives us the current algebras,

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$^2$Define $(a \otimes b)_{ik,jl} = (a)_{ij} (b)_{kl}$, and $(u)_{ik,pq} (v)_{pq,jl} = (uv)_{ik,jl}$. Then we get nice laws such as $(a \otimes b)(c \otimes d) = ac \otimes bd$. 

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\{L_1, L_2\} = [L_2, C_{12}], \quad \{R_1, R_2\} = [R_2, C_{12}] \quad (2.14)

Since \(L\) and \(R\) belong to the Lie algebra \(sl(2, \mathbb{R})\), then we can write the current algebras in terms of their components in the basis \(\{T_a\}\). We get,

\{L_a, L_b\} = 2\epsilon_{abc}L_c, \quad \{R_a, R_b\} = 2\epsilon_{abc}R_c \quad (2.15)

which are simply two copies of \(sl(2, \mathbb{R})\). It should be noted that the current algebras can be deduced from equations (2.7)-(2.10) alone without the need to resort to canonical coordinates. A possible sticking point in the derivation of (2.13) appears to be when one gets to,

\{L_1, u_2\} = -u_1u_2C_{12}u_1^{-1}. \quad (2.16)

It seems that the only way to know how \(C_{12}\) moves past the \(u\)'s is to use an explicit parameterization (and hence canonical coordinates). In fact we can avoid working in canonical coordinates as follows. Consider \(P = \frac{1}{2}(I \otimes I + C_{12})\). It is convenient to use the isomorphism \(\text{Mat}(n, \mathbb{R}) \otimes \text{Mat}(n, \mathbb{R}) \cong \text{Mat}(n^2, \mathbb{R})\) defined by,

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \otimes 
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix} \mapsto 
\begin{pmatrix}
 a\alpha & a\beta & b\alpha & b\beta \\
 a\gamma & a\delta & b\gamma & b\delta \\
 c\alpha & c\beta & d\alpha & d\beta \\
 c\gamma & c\delta & d\gamma & d\delta
\end{pmatrix}
\]

for \(n = 2\). Then it is easy to show that,

\[
C_{12} = 
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 2 & 0 \\
0 & 2 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\quad (2.17)
\]

\[
P = 
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\quad (2.18)
\]

Note that \(P^2 = I \otimes I\). If we consider \(a \in \text{Mat}(2, \mathcal{A})\) where \(\mathcal{A}\) is a not necessarily commutative algebra, then it is a straightforward computation to check that,

\[
P a_1 a_2 = a_2 a_1 P
\quad (2.19)
\]

which reduces to \(C_{12} a_1 a_2 = a_1 a_2 C_{12}\) in the case where \(\mathcal{A}\) is commutative. Applying this to (2.16) (where \(\mathcal{A} = \mathbb{R}\)) we get the desired result. Equation (2.19) will be needed in the quantum theory.
2.4 Global coordinate systems and constraints

In the previous section we parameterised the cosets using the Cartan decomposition, leading to a global set of coordinates \((q, p, A, M, \bar{A}, \bar{M})\) for the phase space. Actually, this is not quite true since \(A\) and \(\bar{A}\) are identified modulo \(2\pi\), and hence as functions \(S^1 \to \mathbb{R}\) are not continuous and hence not smooth. This corresponds to the well known fact that one cannot cover a circle with a single chart. Thus, strictly, the formulas involving these coordinates at best are valid in some open region of the circles, for example everywhere except one point, such as \(2\pi\). A set of Poisson brackets which do hold globally are \([\sin A, M] = \cos A\) and \([\cos A, M] = -\sin A\), and these serve as a global replacement of \([A, M] = 1\), which as we have explained can only be satisfied at best everywhere except at a point. Of course all the formulas derived actually involve only \(\sin A\) or \(\cos A\) and hence are all valid globally. See Isham [16] for a good account of such subtleties.

Now it is clear that the future-directed constraint \(p > 0\) will need to be imposed. This can be either done before or after quantization. In the coordinates developed so far this is best done after. This is because restricting the phase space classically to \(p > 0\), which implies \(M > 0\) and \(\bar{M} > 0\), means that we will have to quantize canonical variables of the form \(S^1 \times \mathbb{R}^+\) which is problematic. Instead we will derive a set of coordinates which after the restriction \(p > 0\) still allow an easy quantization.

To do this, we need the Iwasawa decomposition of \(SL(2, \mathbb{R})\). This tells us that any element of the group manifold can be written as \(g = e^{AN+}e^{BT_3}e^{CT_2}\), where \(N_+ = \frac{1}{2}(T_1 - T_2)\), and thus the coset \(G/H\) can be parameterised by \(\tilde{u} = e^{AN+}e^{BT_3}\). If one proceeds to calculate the symplectic form, we get the same expression as before except that \(M = -\frac{1}{2}pe^{-2B}\) and similarly for \(\bar{M}\). However, now the variables \(A, \bar{A}\) are not periodic but take all real values. This means that if one imposes the constraint classically the phase space restricts to three canonical pairs of the form \(\mathbb{R}^+ \times \mathbb{R}\) or \(\mathbb{R}^- \times \mathbb{R}\) which are both straightforward to deal with since both these are symplectomorphic to the standard \(\mathbb{R} \times \mathbb{R}\) phase space. More explicitly, one finds that the current \(L = -\dot{g}g^{-1}\) has the following components in these Iwasawa coordinates,

\[
L^3 = 2AM \tag{2.20}
\]
\[
L^+ = L^1 - L^2 = -\left(\frac{p^2}{2M} + 2A^2M\right) \tag{2.21}
\]
\[
L^- = L^1 + L^2 = 2M \tag{2.22}
\]

and of course it is easy to verify from the canonical Poisson brackets that they satisfy the \(sl(2, \mathbb{R})\) algebra. Analogous expressions hold for the right current \(R\).

3. The quantum theory

3.1 Canonical quantization

Canonical quantization has a long history and many problems. Schematically, given a phase space \(S\) and canonical coordinates \((q_i, p_i)\), then the quantum theory is constructed
via a correspondence map \( \hat{\cdot} : C^\infty(S) \to \text{End } \mathcal{H} \), such that the quantum observables are Hermitian with respect to the inner product on \( \mathcal{H} \). The canonical coordinates will satisfy \([\hat{q}_i, \hat{p}_j] = i\hbar \delta_{ij}\), and

\[
\lim_{\hbar \to 0} \frac{[\hat{f}, \hat{g}]}{i\hbar} = \{f, g\}
\]

for more general observables. Now this only works if the canonical coordinates take all real values, and is not particularly useful unless they are global coordinates for the phase space. It is also clear that ordering ambiguities arise upon quantization, and in general these have to be resolved case by case.

Before moving on we recall the standard, straightforward example of when the phase space is \( \mathbb{R}^n \times \mathbb{R}^n \). Then we take global coordinates \((q^i, p^i)\) and of course the natural symplectic form \( \omega = \sum dp^i \wedge dq^i \) gives the Poisson brackets \( \{q^i, p^j\} = \delta^i_j \). To quantize the system one takes unitary representations of this algebra, called the Heisenberg algebra, and it is well known that there exists a unique irreducible representation of this realised on \( L^2(\mathbb{R}^n) \) (in fact the more precise statement concerns unitary representations of the Heisenberg group since this will involve strictly bounded operators, see [16]).

### 3.1.1 Cartan coordinates

If we carry this procedure out for the system studied in the previous section then we simply get the following quantum conditions

\[
[q, p] = i\hbar, \quad [e^{iA}, M] = -\hbar e^{iA}, \quad [e^{i\bar{A}}, \bar{M}] = -\hbar e^{i\bar{A}}.
\]

The Hamiltonian will be \( H = -p^2 \), and a general observable, \( O \), will evolve according to Heisenberg’s equation of motion \( i\hbar \dot{O} = [O, H] \), which is equivalent to \( O(\tau) = e^{iH\tau/\hbar} O(0) e^{-iH\tau/\hbar} \). Of course we are now interested in the quantum versions of the current algebras; tentatively we take the left current to be \( L = -\hbar \bar{u} T_2 u^{-1} - i \alpha_L \hbar \) where \( \alpha_L \) is some unknown constant. Now, if we compute the components of the current we find that indeed there are potential ordering problems since we end up with functions of \( A \times M \) functions of \( M \), however they happen to be all Hermitian combinations of the canonical variables, and so we conclude that we have taken an acceptable definition for the current. Note that if we choose \( \alpha_L = 1 \) we get \( \text{tr } L = 0 \) which is a desirable property (but not essential). Anyway, we get

\[
L_2 = M, \quad L_\pm = e^{\pm 2iA} \sqrt{(M \pm \hbar)^2 - p^2}
\]

where \( L_\pm = L_3 \pm iL_1 \). It is easy to the check that \( < L_1, L_2, L_3 > \) provide a unitary representation of the algebra of \( sl(2, \mathbb{R}) \),

\[
[L_2, L_\pm] = \pm 2\hbar L_\pm, \quad [L_+, L_-] = -4\hbar L_2.
\]

The Casimir \( Q_L = \eta^{ab} L_a L_b \) can be also calculated to give,

\[\text{[\text{footnote}]} \]

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\[\text{[\text{footnote}]} \]We make use here of the correct quantization of the angular variables, as discussed in the previous section and in more detail in [14].
\[ Q_L = -p^2 + \hbar^2. \]  

(3.4)

A similar set of calculations for the right algebra gives,

\[ Q_R = -p^2 + \hbar^2. \]  

(3.5)

Therefore we deduce that \( Q_L = Q_R \) which then tells us that the Hilbert space of states for the quantum theory decomposes as follows,

\[ \mathcal{H} = \bigoplus_{\rho} \mathcal{V}^L_\rho \otimes \mathcal{V}^R_\rho \]  

(3.6)

where \( \rho \) is the eigenvalue of the Casimir which labels the irreducible representations, and \( \mathcal{V}^L_\rho \otimes \mathcal{V}^R_\rho \) is the carrier space of an irreducible representation. Explicitly (for the left algebra) we have,

\[ Q_L |\rho, m\rangle = 4\hbar^2 \rho |\rho, m\rangle \]  

(3.7)

\[ L_2 |\rho, m\rangle = 2\hbar m |\rho, m\rangle \]  

(3.8)

\[ L_\pm |\rho, m\rangle = 2\hbar \sqrt{\rho + m(m \pm 1)} |\rho, m \pm 1\rangle \]  

(3.9)

where \( \{ |\rho, m\rangle \} \) is a basis for \( \mathcal{V}^L_\rho \), and \( m \) is either integer or half integer depending on the representation. However not all irreducible representations of \( sl(2, \mathbb{R}) \) are allowed. In fact the only ones allowed are \( D_j^+, D_j^- \) (where \( \rho = -j(j+1), \; j \in \{-1, -3/2, ...\} \) and \( C_\rho^0 \) for \( 0 < \rho < \frac{1}{4} \). This follows from the fact that \( Q_L < \hbar^2 \), and hence \( \rho < \frac{1}{4} \).

We deduce that the allowed values for the mass (in units of \( \hbar \)) \(^4\) are,

\[ \mu = |2j + 1|, \quad j \in \{-1, -3/2, ...\} \]  

(3.10)

\[ \mu = \sqrt{1 - 4\rho}, \quad 0 < \rho < \frac{1}{4}, \]  

(3.11)

that is \( \mu \) is either a positive integer or in the interval \((0, 1)\).

Quantization of the mass is not unexpected since \textit{AdS}$_3$ has closed time-like direction, that is topologically it is \( S^1 \times \mathbb{R}^2 \), where the \( S^1 \) refers to time. The continuum \( 0 < \mu < 1 \) is more interesting.

An explicit basis for \( \mathcal{H} \) can be constructed from a state \( |0\rangle \) satisfying \( p|0\rangle = M|0\rangle = \bar{M}|0\rangle = 0 \). Namely \( \{|\mu, m, \bar{m}\rangle\} \), where

\[ |\mu, m, \bar{m}\rangle = e^{imq} e^{2i\bar{m}\bar{A}} e^{2im\bar{A}} |0\rangle \]  

(3.12)

will span the Hilbert space for appropriate values of \( \mu, m, \bar{m} \). It is in fact an orthogonal basis and normalised if \( \langle 0|0 \rangle = 1 \).

\(^4\)The geodesic \( g(\tau) = e^{(q+p\tau)T_2} \) in the coordinates \( x^\mu = (t, \rho, \phi) \) is \( g^\nu(\tau) = (q + p\tau, 0, \phi_0) \). Hence \( \dot{g}^\mu = (p, 0, 0) \), and thus \( m^2 = \mu^2 \hbar^2 \equiv -G_{\mu\nu} \dot{g}^\mu \dot{g}^\nu \equiv p^2 \). Of course the mass \( \mu \hbar \) is then defined as the positive square root. This is all classical; Appendix B tells us that the same holds quantum mechanically i.e. \( p^2 = m^2 \) on physical states.
We have not yet imposed the constraint \( p > 0 \) corresponding to the future directed condition of the geodesics. Now we need to find a way to impose this condition quantum mechanically. This turns out to be actually fairly easy. We merely note that classically the constraint told us that \( M > 0 \). Hence we see that quantum mechanically a sensible constraint is \( L_2 > 0 \); note the corresponding statement \( R_2 < 0 \) also exists. Thus we need to pick a subspace of our Hilbert space for which these conditions hold. This is easy to do and gives,

\[
\mathcal{H}_{p > 0} = \bigoplus_{j \leq -1} D_j^+ \otimes D_j^-.
\]  

(3.13)

Note in particular that the mysterious continuum has disappeared. This continuum actually corresponds to states which can flip their direction in time and of course have no classical counterpart; naturally they are excluded in the quantum theory after imposing the future directed constraint.

### 3.1.2 Iwasawa coordinates

In these coordinates we have the interesting option of imposing the constraint \( p > 0 \) before or after quantization. Interestingly this leads to slightly different quantum theories.

First we will mimic the previous section and quantize the unconstrained system and then impose the constraint. Of course we hope to reproduce the same results. The quantum conditions are,

\[
[q, p] = i\hbar, \quad [A, M] = i\hbar, \quad [\bar{A}, \bar{M}] = i\hbar,
\]  

(3.14)

and note there are no subtleties due to periodic variables here. Resolving some of the ordering ambiguities in the currents (imposing that the components be self-adjoint suffices) leads to our left current being,

\[
L^3 = AM + MA
\]

(3.15)

\[
L^+ = L^1 - L^2 = -\left(\frac{p^2}{2M} + 2AMA\right)
\]

(3.16)

\[
L^- = 2M
\]

(3.17)

and these satisfy the same \( sl(2, \mathbb{R}) \) algebra as in the previous section, as they should! The Casimir can be computed, and reassuringly we get \( Q_L = -p^2 + \hbar^2 \), which is the same as we got by quantization of the system in the very different Cartan coordinates. Note this wasn’t a priori guaranteed, since it isn’t necessarily obvious that quantizing a system in completely different canonical coordinates is compatible. Of course, from here on we’ll get the same quantum theory as in the previous section before and after the future-directed constraint is imposed.

Now, as we’ve mentioned in these coordinates we can actually quantize the constrained classical system. To do this, we briefly explain how one would quantize the classical system with phase space \( \mathbb{R} \times \mathbb{R}^+ \), see [14]. Let \((x, \pi)\) be global coordinates. There is a nice way
of mapping this to the standard $\mathbb{R} \times \mathbb{R}$ phase space; simply define the diffeomorphism $(x, \pi) \to (x\pi, \log \pi) \equiv (\hat{x}, \hat{\pi})$. It then follows that $\omega = d\pi \wedge dx = d\hat{\pi} \wedge d\hat{x}$ showing that the phase spaces are symplectomorphic as previously asserted. Therefore one can choose $(\hat{x}, \hat{\pi})$ as canonical coordinates and simply quantize as usual via a Heisenberg algebra, $[\hat{x}, \hat{\pi}] = i\hbar$. Note since $\pi = e^{\hat{\pi}}$ it follows that $[\hat{x}, \pi] = i\hbar\pi$, which can be taken as our fundamental quantum condition for quantum mechanics on $\mathbb{R} \times \mathbb{R}^+$. Now, let's apply this to our system. Expressing the classical current $L$ in components in the coordinates $(\hat{q} = qp, \hat{A}, M = -e^\hat{M})$ gives,

$$L^3 = 2\hat{A}$$

$$L^+ = L_1 - L_2 = -\left(\frac{p^2}{2M} + \frac{2\hat{A}^2}{M}\right)$$

$$L^- = L_1 + L_2 = 2M.$$  

Quantization then involves imposing the conditions,

$$[\hat{q}, p] = i\hbar p, \quad [\hat{A}, M] = i\hbar M, \quad [\hat{A}, \hat{M}] = i\hbar \hat{M},$$

and quantization of the currents requires simply resolving the order ambiguity in $L^+$ which we do by writing $L^+ = -\left(\frac{p^2}{2M} + 2\hat{A}M^{-1}\hat{A}\right)$. Then, once again it is an easy exercise showing that we have the $sl(2, \mathbb{R})$ algebra satisfied using the quantum conditions. Interestingly computing the Casimir gives $Q_L = -p^2$ in contrast to the previous quantizations. Now, the Hilbert space decomposes into irreducible representations of $sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R})$, and it is easy to see that $L_2 > 0$ as an operator (since $M < 0$). Thus since $Q < 0$ (the right algebra will have the same Casimir) we see that the Hilbert space decomposes as,

$$\mathcal{H}_{p>0} = \bigoplus_{j \leq -\frac{1}{2}} D^+_j \otimes D^-_j.$$  

Note this differs from our other quantum theory by the absence of $D^+_1 \otimes D^-_1$ in the Hilbert space; of course we get a different mass spectrum, namely $\mu = \sqrt{j(j+1)}$.

### 3.2 Representation on function space

We would like to find a representation of the quantum mechanics on the space of functions on $AdS_3$. As usual, the wavefunction of a state $|\psi\rangle$ in the position representation is $\psi(g) = \langle g | \psi \rangle$, and a basis for these functions is desirable. It would be nice to derive how the currents act on the position eigenstates from the canonical formalism developed. This would involve calculating matrix elements of the operator matrix $\hat{q}$ in some basis. Instead we cheat slightly and take a shortcut. To do this we note that the algebra of the Killing vectors is $sl(2, \mathbb{R}) \oplus sl(2, \mathbb{R})$ and of course this acts naturally on functions on $AdS_3$. Hence all we need to do is find the Killing vectors as these give what we want automatically. One can check that all the following are Killing vectors,
\[ L_2 = i\hbar \frac{\partial}{\partial v} \quad R_2 = -i\hbar \frac{\partial}{\partial u} \]  
(3.23)

\[ L_\pm = i\hbar e^{\pm 2i\upsilon} \left( \coth 2\rho \frac{\partial}{\partial v} - \cosech 2\rho \frac{\partial}{\partial u} \mp i \frac{\partial}{\partial \rho} \right) \]  
(3.24)

\[ R_\pm = i\hbar e^{\pm 2i\upsilon} \left( \coth 2\rho \frac{\partial}{\partial u} - \cosech 2\rho \frac{\partial}{\partial v} \pm i \frac{\partial}{\partial \rho} \right) \]  
(3.25)

and that they satisfy the following algebra

\[ [L_a, L_b] = 2i\hbar \epsilon^{abc} L_c \quad [R_a, R_b] = 2i\hbar \epsilon^{abc} R_c \quad [L_a, R_b] = 0 \]  
(3.26)

where \( L_\pm = L_3 \pm iL_1 \). The Casimirs \( Q_L = \eta^{ab} L_a L_b \) and \( Q_R \), turn out to be

\[ Q_L = Q_R = -\hbar^2 \Box \]  
(3.27)

where \( \Box = \frac{1}{\sqrt{g}} \partial_{\mu} (\sqrt{g} g^{\mu\nu} \partial_{\nu}) \) is the Laplacian of the metric \( g \). Now, one can easily solve for the eigenfunctions of \( L_2, R_2, Q \). It consists of the following set of differential equations,

\[ i\hbar \partial_v \psi = 2m\hbar \psi, \quad -i\hbar \partial_u \psi = 2\bar{m}\hbar \psi, \quad \Box \psi = -4g\psi \]  
(3.28)

the first two of which can be solved to give,

\[ \psi(u, \rho, v) = e^{-2imv} e^{2i\bar{m}u} \chi(\rho) = e^{-iEt} e^{-ik\phi} \chi(\rho) \]  
(3.29)

where \( E = m - \bar{m} \) and \( k = m + \bar{m} \). Note that single valuedness of \( \psi \) requires \( E, k \in \mathbb{Z} \).

The third equation, with the help of (3.4) (we choose this Casimir since it leads to simpler mass dependence in the wavefunctions, however it is easy to convert between the two), can be rewritten as \( \Box \psi = (\mu^2 - 1)\psi \), which is the Klein-Gordon equation with shifted mass. Substituting into this leads to a differential equation for \( \chi \), which takes the form,

\[ \frac{1}{\sinh 2\rho} \partial_{\rho} (\sinh 2\rho \partial_{\rho} \chi) + \left( \frac{E^2}{\cosh^2 \rho} - \frac{k^2}{\sinh^2 \rho} - (\mu^2 - 1) \right) \chi = 0. \]  
(3.30)

Now, change variables to \( x = \tanh^2 \rho \), and \( \chi(x) = x^{\frac{|k|}{2}} (1 - x)^{\frac{1}{2}(1+\mu)} y(x) \). The result is,

\[ x(x-1)y''(x) + [(1 + a + b) x - c] y'(x) + aby(x) = 0 \]  
(3.31)

with \( a = \frac{1}{2}(1 + \mu + E + |k|) \) and \( b = \frac{1}{2}(1 + \mu + |k| - E) \) and \( c = 1 + |k| \), which is the hypergeometric equation.

For \( c \notin \mathbb{Z} \) two independent solutions about \( x = 0 \) are,

\[ y_1(x) = {}_2F_1(a, b, c, x) \]  
(3.32)

\[ y_2(x) = x^{1-c} {}_2F_1(a + 1 - c, b + 1 - c, 2 - c, x) \]  
(3.33)
and for $c - a - b \notin \mathbb{Z}$ we have two independent solutions about $x = 1$,

$$y_3(x) = \frac{1}{2}F_1(a, b, a + b + 1 - c, 1 - x)$$  \hspace{1cm} (3.34)

$$y_4(x) = (1 - x)^{c-a-b} \frac{1}{2}F_1(c - a, c - b, c + 1 - a - b, 1 - x).$$  \hspace{1cm} (3.35)

Now, as in ordinary quantum mechanics, we would like to restrict wavefunctions to be square integrable. However since we are dealing with a non-compact space this might be too restrictive and we want to account for the divergence arising from infinite volume. Also, arguably, a more natural norm on the wavefunctions might be the one from the Klein-Gordon inner product. Instead we’ll find the larger class of bounded functions on $AdS_3$, which will include finite norm functions in either norm.

Now, $c = 1 + |k|$ and $c - a - b = -\mu$. As $x \to 0$ we have $\chi_1(x) \sim x^{|k|/2}$, and using the Wronskian one can show that the solution independent of $\chi_1$ goes as $x^{-|k|/2}$ for $k \neq 0$ and as $\log x$ for $k = 0$ as $x \to 0$. Therefore the solution independent of $\chi_1$ is not allowed. The behaviour of the solutions must be studied as $x \to 1$ as well; $\chi_3(x) \sim (1 - x)^{(1+\mu)/2}$, and solutions independent of this behave as $(1 - x)^{(1-\mu)/2}$ for $\mu \neq 0$ and as $(1 - x)^{1/2} \log(1 - x)$ for $\mu = 0$. Therefore for $0 < \mu \leq 1$ we get $\chi_1(x)$. Also for $\mu > 1$ we see that solutions independent of $\chi_3(x)$ are disallowed; hence $\chi_1(x)$ and $\chi_3(x)$ must be dependent which occurs if and only if either $-a$ or $-b \in \mathbb{N}_0$ (see [14] p.79).

Therefore, we deduce that for $\mu > 1$ either $-a$ or $-b \in \mathbb{N}_0$ which causes the mass $\mu$ to be quantised; for $0 < \mu \leq 1$ the mass can take any value. Note that this agrees exactly with the results of the previous section. More explicitly we have the following possibilities:

1. A continuum $0 < \mu \leq 1$,

$$\psi_{\mu E k}(t, \rho, \phi) = \frac{1}{\sqrt{2}} N_{\mu E k} e^{-i\rho t} e^{-i k \phi} x \frac{\tanh|k|}{\cosh \rho} \left( 1 + \mu + E + |k|, 1 + \mu + |k| - E, 1 + |k|, \tanh^2 \rho \right)$$  \hspace{1cm} (3.36)

where $E, k \in \mathbb{Z}$.

2. A discrete series, with $\mu \in \mathbb{N}$ and $n \in \mathbb{N}_0$,

$$\psi_{\mu E k}(t, \rho, \phi) = \frac{1}{\sqrt{2}} N_{\mu E k} e^{-i\rho t} e^{-i k \phi} x \frac{\tanh|k|}{\cosh \rho} \left( 1 + \mu + 2n - |k| - 1, \tanh^2 \rho \right),$$  \hspace{1cm} (3.37)

where either $E < -2n - |k| - 1$ and $\mu = -2n - E - |k| - 1$, or where $E > 2n + |k| + 1$ and $\mu = -2n + E - |k| - 1$, and $k \in \mathbb{Z}$ in both cases. Note that the case corresponding to both $-a$ and $-b \in \mathbb{N}_0$ is not allowed since $\mu > 0$.

$P_{n}^{(a,b)}(x)$ are the Jacobi polynomials.
Note that to extend this to quantum mechanics on $\tilde{AdS}_3$ one simply removes the identification $t \sim t + 2\pi$; this implies that $E$ is not quantised and hence neither is the mass $\mu$ in the region $\mu > 1$. This is entirely expected as time now is topologically $\mathbb{R}$. Note that in field theory quantization of the frequencies is restored by imposing conservation of energy, and two different stress tensors (minimal and conformal) give two different types of modes as is well known [13], [21].

Now, of course, we need to think about imposing the constraint $p > 0$. As explained this amounts to $L_2 > 0$ and $R_2 < 0$, which tells us that $m > 0$ and $\bar{m} < 0$. This allows us to deduce that $E > 0$, as one would want for future directed particles. Note that this is actually consistent only with $E > 2n + |k| + 1$ of the discrete series. Only the discrete series describes the time directed theory, and these functions are actually all strictly square integrable on $AdS_3$. Note that on the cover $\tilde{AdS}_3$ these wavefunctions will not be square integrable since time is not compact.

### 3.3 Deductions for the BTZ black hole

It is well known that one can get the BTZ black hole from $AdS_3$ as the quotient $\tilde{AdS}_3/Z$, where $Z$ is a discrete subgroup generated by a particular Killing vector [12]. In other words BTZ is an orbifold of $AdS_3$. Therefore we can deduce quantum mechanics in the BTZ background from the quantum mechanics we have already worked out for $AdS_3$ by simply restricting the Hilbert space of wavefunctions to ones which are consistent with the identification procedure. Unfortunately the coordinate system $(t, \rho, \phi)$ used in the previous section is inconvenient for expressing the identification. Therefore we introduce the coordinates $(\hat{t}, \hat{r}, \hat{\phi})$ defined by,

\[
\begin{align*}
    x_0 &= \sqrt{(\hat{r}^2 - 1)} \sinh \hat{t}, \quad x_1 = \hat{r} \cosh \hat{\phi}, \quad x_2 = \hat{r} \sinh \hat{\phi}, \\
    x_3 &= \sqrt{(\hat{r}^2 - 1)} \cosh \hat{t}, \quad -x_0^2 - x_1^2 + x_2^2 + x_3^2 = -1 \\
    |\hat{t}| &< \infty, \quad |\hat{\phi}| < \infty, \quad \hat{r}^2 > 1.
\end{align*}
\]

Note that these coordinates do not cover the whole of $AdS_3$, but after the identification,

\[
\hat{t} \sim \hat{t} - 2\pi r_-, \quad \hat{\phi} \sim \hat{\phi} + 2\pi r_+
\]  

one has a coordinate system which covers the black hole in the exterior region $r > r_+$. Note to see this one needs to make the coordinate change,

\[
\hat{r}^2 = \left( \frac{r^2 - r_+^2}{r_+^2 - r_-^2} \right)
\]

\[
\hat{t} = r_+ t - r_- \phi, \quad \hat{\phi} = -r_- t + r_+ \phi
\]

which gives the metric for BTZ in the exterior region,
\[ g_{\text{BTZ}} = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2} dt^2 + \frac{r^2}{(r^2 - r_+^2)(r^2 - r_-^2)} dr^2 + r^2(d\phi - \frac{r+r-}{r^2} dt)^2 \] (3.41)

with \( r \geq 0 \), \( |t| < \infty \) and \( \phi \sim \phi + 2\pi \); note that \( t, \phi \) are different to the ones in the previous section. The mass of the black hole is \( M = (r_+^2 + r_-^2) \) and the angular momentum is \( J = 2r_+r_- \). It was shown that any wavefunction \( \psi \) on \( \text{AdS}_3 \) must satisfy \( \Box \psi = (\mu^2 - 1) \psi \); this result then follows for wavefunctions on \( \text{BTZ} \). The metric in the coordinates (3.38) reads,

\[ g = -(r^2 - 1)dt^2 + \frac{dr^2}{(r^2 - 1)} + r^2d\phi^2. \] (3.42)

Then solving one gets \( \psi(\hat{t}, \hat{r}, \hat{\phi}) = e^{-i\omega \hat{t}} e^{-im \hat{r} \hat{\phi}} (\hat{r}^2 - 1)^{i\omega/2} \eta(\hat{r}^2) \), where \( \omega, m \) are constants and \( \eta \) satisfies the hypergeometric equation with \( a = \frac{1}{2}(1 + i\omega + im + \mu) \), \( b = \frac{1}{2}(1 + i\omega + im - \mu) \) and \( c = 1 + im \). Then, imposing the identifications (3.38), it follows that \( \omega = n/r_- \) and \( m = l/r_+ \) with \( n, l \in \mathbb{Z} \). For \( a - b \notin \mathbb{Z} \) we have two linearly independent solutions about \( x = \infty \),

\[ y_1(x) = (-x)^{-a} \, 2F_1(a, a + 1 - c, a - b + 1, \frac{1}{x}) \] (3.43)
\[ y_2(x) = (-x)^{-b} \, 2F_1(b, b + 1 - c, b - a + 1, \frac{1}{x}), \] (3.44)

and for \( c - a - b \notin \mathbb{Z} \) we have two linearly independent solutions about \( x = 1 \),

\[ y_3(x) = 2F_1(a, a + b + 1 - c, 1 - x) \] (3.45)
\[ y_4(x) = (1 - x)^{c-a-b} \, 2F_1(c - a, c - b, c + 1 - a - b, 1 - x). \] (3.46)

Now, \( a - b = \mu \) and \( c - a - b = -i\omega = -in/r_- \). Therefore for \( n \neq 0 \) a general solution can be written as \( \eta(x) = A y_1(x) + B y_2(x) \sim A + B(1-x)^{c-a-b} \) and hence \( |\eta(x)|^2 \sim \text{const} \) as \( x \to 1^+ \), which imposes no conditions. The large \( x \) limit is more interesting; note that for \( \mu \notin \mathbb{N} \) a general solution can be written as \( \eta(x) = A y_1(x) + B y_2(x) \sim A(-x)^{-a} + B(-x)^{-b} \) as \( x \to \infty \). Now, \( |x^{-a}|^2 = x^{-(1+\mu)} \) and \( |x^{-b}|^2 = x^{-(1-\mu)} \) which implies that for \( 0 < \mu < 1 \) the general solution is in \( L^2(\text{BTZ}) \), whereas for \( \mu > 1 \) only \( y_1(x) \) belongs to \( L^2(\text{BTZ}) \). The special cases \( n = 0 \) and \( \mu \in \mathbb{N} \) need attention. Firstly it can be shown, using the Wronskian, that as \( x \to \infty \) a solution independent of \( y_1(x) \) behaves as \( (-x)^{-b} \) for \( a \neq b \) and as \( (-x)^{-a} \) as \( x \to 1^+ \) if \( a = b \). From this we deduce that if \( \mu = 1 \) two linearly independent solutions are allowed, and for \( \mu > 1 \) only one is allowed. It can also be shown that a solution independent of \( y_3(x) \) behaves as \( (1-x)^{c-a-b} \) for \( c - a - b \neq 0 \), and as \( \log(1-x) \) for \( c - a - b = 0 \) as \( x \to 1 \). From this we deduce that if \( 0 < \mu \leq 1 \) then only one solution exists for \( \omega = 0 \), and for \( \mu > 1 \) solutions exist with \( \omega = 0 \) only when \( y_1(x) \) and \( y_3(x) \) are linearly related; this occurs when \( -a \in \mathbb{N}_0 \), or when \( -(a-c+1) \) and \( -b \in \mathbb{N}_0 \), which are both impossible.

To summarise the allowed wavefunctions are:
1. For $0 < \mu \leq 1$ with $n, l \in \mathbb{Z}$ and $n \neq 0$,

$$
\psi^{(1)}_{nl\mu}(\hat{t}, \hat{r}, \hat{\phi}) = e^{-i\mu/\hat{r}_+} e^{-i\hat{\phi}/\hat{r}_+} \hat{\phi}^{l/\hat{r}_+} (\hat{r}^2 - 1) \frac{\sin}{\hat{r}} y_1(\hat{r}^2),
$$

$$
\psi^{(2)}_{nl\mu}(\hat{t}, \hat{r}, \hat{\phi}) = e^{-i\mu/\hat{r}_+} e^{-i\hat{\phi}/\hat{r}_+} \hat{\phi}^{l/\hat{r}_+} (\hat{r}^2 - 1) \frac{\sin}{\hat{r}} y_2(\hat{r}^2)
$$

(3.47) (3.48)

where $\psi^{(1)}$ and $\psi^{(2)}$ are linearly independent; for $n = 0$ a wavefunction linearly independent of $\psi^{(1)}$ can be constructed and will be of the form $y_1(x) \log x$ plus an analytic function; however this does not belong to $L^2(BTZ)$, and so we lose a solution in this case.

2. For $\mu > 1$ with $n, l \in \mathbb{Z},$

$$
\psi_{nl\mu}(\hat{t}, \hat{r}, \hat{\phi}) = e^{-i\mu/\hat{r}_+} e^{-i\hat{\phi}/\hat{r}_+} \hat{\phi}^{l/\hat{r}_+} (\hat{r}^2 - 1) \frac{\sin}{\hat{r}} y_1(\hat{r}^2).
$$

(3.49)

where for $n = 0$ we have no solution.

Therefore, we see that for quantum mechanics in a $BTZ$ background, a “larger” portion of the Hilbert space occurs for $0 < \mu \leq 1$, i.e. it is more likely that a particle will have its mass in that region. We have not yet imposed any constraint corresponding to future-directed motion of the particle. A simple way of doing this might be to impose that the energy of the eigenfunction, that is the eigenvalue of $ih\partial_t$, be positive. Doing this gives the condition $n > l \left(\frac{r_+}{r_-}\right)^2$.

What about the regions $0 < r < r_-$ and $r_- < r < r_+$ of the black hole? Well, a similar analysis can be carried out as before, however we can take a short cut. First we note that $(\hat{t}, \hat{r}, \hat{\phi})$ will be a coordinate system for $r_- < r < r_+$ if we make the formal replacements $x^0 \rightarrow -ix^3$, $x^1 \rightarrow x^1$, $x^2 \rightarrow x^2$, $x^3 \rightarrow -ix^0$ and $0 < \hat{r} < 1$. Therefore we get the same metric and differential equation (except for the range of $\hat{r}$). Solutions near $\hat{r} = 1$ have already been studied above and we found two solutions for $\omega \neq 0$ and just one for $\omega = 0$. The behaviour at the origin must also be examined; now since $c = 1 + il/r_+$, we see that for $l \neq 0$ we get two well-behaved solutions, whereas for $l = 0$ only one solution is allowed. The special case $n = l = 0$ is allowed only when $\mu$ is an odd integer. So dependence on mass does not strongly feature here. Finally, the region $0 < r < r_-$ can also be studied in the same way by letting $x^0 \rightarrow -ix^3$, $x^1 \rightarrow ix^2$, $x^2 \rightarrow ix^1$, $x^3 \rightarrow -ix^0$ and $-\frac{r^2}{r_- r_+} < \hat{r}^2 < 0$.

Under the mapping $\hat{r}^2 \rightarrow \hat{r}^2/(\hat{r}^2 - 1)$ the region $(-\frac{r^2}{r_- r_+}, 0)$ is mapped in the interval $(0, r_-^2/r_+^2)$, and therefore using Pfaff’s transformation \(^5\) only the behaviour at the origin needs examining, which we have done above telling us that we have two solutions for $l \neq 0$ and just one for $l = 0$, and hence the mass does not feature here at all.

\(^5\) This is a simple identity regarding hypergeometric functions, namely $\sum F_1(a, b, c, x) = (1 - x)^{-a} \sum F_1(a, c - b, c, \frac{x}{a - 1})$. 

3.4 “Chiral” quantization

In this section we aim to show how a subtly different kind of quantization can be done, involving quantities which relate more directly to the symmetries of the system. The proposed quantization [6], [8], consists of quantizing \( q, p \) to Hermitian operators as before, and quantizing \( u, v \) to operator matrices which are unitary in the sense \( u_{ab}u_{bc}^\dagger = \delta_{ac} \). The quantum conditions are

\[
[q, p] = i\hbar, \quad [\bar{u}_1, \bar{v}_2] = 0, \quad [\bar{u}, p] = [\bar{v}, p] = 0
\]  
(3.50)

\[
u_1\nu_2 = u_2u_1B_{12}, \quad \nu_1\nu_2 = B_{12}^{-1}v_2v_1
\]  
(3.51)

where \( \bar{u} = u e^{-qT_2} \) and \( \bar{v} = e^{-qT_2}v \). Quantum consistency requires that

\[
B = \mathbb{1} \otimes \mathbb{1} + i\hbar r_{12} + O(\hbar^2), \quad \text{Classical limit (3.52)}
\]

\[
B^{-1} = B^\dagger = \mathbb{P} B \mathbb{P}, \quad \text{Unitarity and antisymmetry (3.53)}
\]

\[
B_{23}(\tilde{p}_1)B_{13}(p)B_{12}(\tilde{p}_3) = B_{12}(p)B_{13}(\tilde{p}_2)B_{23}(p), \quad \text{Associativity (3.54)}
\]

\[
[B, e^{q((T_2)^2+T_2)v}] = 0 \quad \text{Locality (3.55)}
\]

where \( \tilde{p} = p - i\hbar T_2 \) (note that \( pu = u\tilde{p} \)). Equations (3.53) follow from \( u \) being unitary and equations (3.51). Equation (3.54) comes from associativity \( u_1(u_2u_3) = (u_1u_2)u_3 \), where \( u_1 = u \otimes \mathbb{1} \otimes \mathbb{1} \) and similarly for the others. Equation (3.53) deserves some attention; it is equivalent to \( [g_1(\tau), g_2(\tau)] = 0 \) hence ensuring locality in the quantum theory. We will show this at \( \tau = 0 \), the general result follows from Heisenberg’s equations of motion. First note that \( g(0) = \tilde{u}Q\tilde{v} = uQ^{-1}v = u\tilde{v} = \tilde{uv} \), where \( Q = e^{qT_2} \). Now, the argument runs as follows:

\[
g_1(0)g_2(0) = u_1\tilde{u}_1\tilde{v}_2v_2 = u_1\tilde{u}_2\tilde{v}_1v_2 = u_1u_2Q_2^{-1}Q_1^{-1}v_1v_2
\]

\[
= u_2u_1B_{12}Q_2^{-1}Q_1^{-1}B_{12}^{-1}v_2v_1 = u_2u_1Q_2^{-1}Q_1^{-1}v_2v_1
\]

\[
= u_2\tilde{u}_1\tilde{v}_2v_1 = u_2\tilde{v}_2\tilde{u}_1v_1 = g_2(0)g_1(0)
\]

where (3.55) was used in the fifth equality and the quantum conditions were used in the other steps. It is easy to see, from the algebra above, that requiring \( [g_1(0), g_2(0)] = 0 \) implies (3.55), hence completing the equivalence.

Now, an explicit formula for \( B \) the braiding matrix is wanted. If we start from the ansatz,

\[
B(p) = \exp \left( - \sum_{\alpha \in \{\pm 1\}} \theta_\alpha(p)E_\alpha \otimes E_{-\alpha} \right)
\]  
(3.56)

\(^6\text{Note that unitarity of} u \text{ and} v \text{ is consistent with canonical quantization if, for example, the choices} B^\dagger = -B \text{ and} B^\dagger = -B \text{ are made for the Cartan coordinates. This is allowed since the canonical variables} M \text{ and} \tilde{M} \text{ are even functions of} B \text{ and} \tilde{B} \text{ respectively, and hence their Hermiticity is unaffected by such a choice.}\)
where $E_\pm = \frac{1}{2}(T_3 \pm iT_1)$, and then impose the conditions above we get $\theta_{-\alpha} = -\theta_\alpha$ (from (3.53),(3.55) independently), and that $\theta_\alpha$ can be any analytic function of $p$ or $p^{-1}$ (from (3.54)). A computation gives,

$$B(p) = l \otimes l - \sin^2(\theta/2)(1 \otimes 1 + T_2 \otimes T_2) - \sum_\alpha \sin \theta_\alpha E_\alpha \otimes E_{-\alpha} \quad (3.57)$$

where $\theta \equiv \theta_{+1}$. Imposing the classical limit implies,

$$\sin \theta_\alpha = \frac{\hbar}{p_\alpha} + O(\hbar^2). \quad (3.58)$$

If we make the choice $\sin \theta_\alpha = \frac{\hbar}{p_\alpha}$ then we get,

$$B(p) = l \otimes l + i\hbar r(p) - \frac{1}{2} \left( 1 - \sqrt{1 - \frac{\hbar^2}{p^2}} \right)(l \otimes l + T_2 \otimes T_2). \quad (3.59)$$

Now we come to the problems of ordering of the quantum variables. In particular how do we define the currents $L, R$ in the quantum theory. Motivated by the canonical quantization we define the quantum currents to be,

$$L = -\tilde{u}pT_2\tilde{u}^{-1} - i\alpha_L \hbar l \quad (3.60)$$

$$R = \tilde{v}^{-1}pT_2\tilde{v} + i\alpha_R \hbar l \quad (3.61)$$

where $\alpha_{L,R}$ are unknown constants. This now allows one to calculate the following brackets,

$$[L_1, u_2] = -i\hbar C_{12}u_2, \quad [R_1, v_2] = i\hbar v_2C_{12} \quad (3.62)$$

$$[L_1, L_2] = i\hbar [L_2, C_{12}], \quad [R_1, R_2] = i\hbar [R_2, C_{12}] \quad (3.63)$$

which reduce to the algebras of $sl(2, \mathbb{R})$ when one expands $L = L^a T_a + kl$, and similarly for $R$. What is remarkable is that when one computes the Casimirs $Q_L = \eta^{ab} L_a L_b$ and $Q_R = \eta^{ab} R_a R_b$, one finds that they are independent of $\alpha_L$ and $\alpha_R$, and we get,

$$Q_L = -p^2 + \hbar^2, \quad Q_R = -p^2 + \hbar^2 \quad (3.64)$$

However to do this the result $\text{tr}(\tilde{u}pT_2\tilde{u}^{-1}) = -2i\hbar$ is needed, which appears only to be calculable using canonical coordinates. Note that to show equations (3.62) the explicit form of the braiding matrix (3.59) is required. Straightforward manipulations lead to,

$$[L_1, u_2] = u_1 u_2 [B_{12}^{-1} - l \otimes l, p(T_2)_1] u_1^{-1} + i\hbar u_1 u_2 (T_2 \otimes T_2) B_{12}^{-1} u_1^{-1} \quad (3.65)$$

and from equation (3.59) it is easily verified that,

$$[B_{12}^{-1} - l \otimes l, p(T_2)_1] = -i\hbar(T_1 \otimes T_1 + T_3 \otimes T_3) \quad (3.66)$$

$$(T_2 \otimes T_2)(B_{12}^{-1} - 1 \otimes 1) = B_{12}^{-1} - l \otimes l \quad (3.67)$$
which when substituted into (3.65) give,
\[
[L_1, u_2] = -2i\hbar u_2 P u_1^{-1} + i u_2.
\] (3.68)

Now using equation (2.19) where \(\mathcal{A} = \text{End} \mathcal{H}\) we get the first of equation (3.62) as required, by the sole use of the quantum conditions (3.50), (3.51) and (3.53).

### 3.5 Comparison to Quantum Field Theory

In the previous sections we have built up a quantum theory in \(\text{AdS}_3\), however we found that there are (at least) two possible theories depending on how one imposes one of the constraints. Of course one can also investigate quantum field theory in this space. It may prove fruitful to compare results here with the quantum mechanics in order to gain insight into which is the “true” quantum theory, or even if this question is meaningful. In order to make such a comparison we need to calculate some quantity computable in both theories. One obvious such candidate is the amplitude for the transition of a particle between two points in \(\text{AdS}_3\). In the QFT this will correspond to the two-point function, or the so-called Wightman function.

So first we briefly outline QFT in \(\text{AdS}_3\), also see [13] for a good detailed account where boundary conditions are discussed which we will not do here. In a curved space a free scalar field satisfies the following equation of motion,

\[
(-\Box + m^2 + \xi R)\phi = 0
\] (3.69)

where \(\xi\) is a constant. Minimal coupling corresponds to \(\xi = 0\) and conformal coupling to \(\xi = 1/8\) (in three dimensions). We will use the standard static metric for \(\text{AdS}_3\),

\[
ds^2 = l^2 (-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\phi^2).
\] (3.70)

Then we have \(R = -6/l^2\). We proceed by writing the general solution to the Klein-Gordon equation as,

\[
\phi(x) = \sum_i u_i(x) a_i + \bar{u}_i(x) a_i^\dagger
\] (3.71)

where \(\{u_i(x), \bar{u}_i(x)\}\) is a complete basis for functions satisfying (3.69). One can derive a finite norm basis in the static coordinates introduced, and we find,

\[
u_{\beta Ek}(t, \rho, \phi) = N_{\beta Ek} e^{-iEt} e^{-ik\phi} \frac{\tanh |k| (\rho)}{\cosh^{1+\beta}(\rho)} P_{n}(|k|, \beta)(1 - 2 \tanh^2 \rho),
\] (3.72)

where

\[
N_{\beta Ek}^2 = \frac{1}{2\pi} \frac{(\beta + |k| + n)!n!}{(\beta + n)!(|k| + n)!}, \quad E = 2n + \beta + |k| + 1 \quad (3.73)
\]

and

\[
\beta = \sqrt{l^2 m^2 + 1 - 6\xi},
\] (3.74)

for \(\beta \geq 1\). The normalisation of these functions is with respect to the Klein-Gordon inner product \((\phi_1, \phi_2) = i \int_{\Sigma} dS^\mu (\bar{\phi}_1 \partial_\mu \phi_2 - \phi_2 \partial_\mu \bar{\phi}_1)\), where \(\Sigma\) is a spacelike hypersurface. Also
note that $n \in \mathbb{N}_0$ and $E, k \in \mathbb{Z}$. Note $\beta \geq 1$ implies $\beta \in \mathbb{N}$. Now we are ready to compute the Wightman function. From the canonical commutation relations $[a_i, a_j^\dagger] = \delta_{i,j}$ it is easy to see that,

$$G^+(x, x') = \sum_{\beta E k} u_{\beta E k}(t, \rho, \phi) \bar{u}_{\beta E k}(t', \rho', \phi')$$  \hspace{1cm} (3.75)$$

If we choose $x' = 0$ then since $u_{\beta E k}(0, 0, 0) = \frac{1}{2\pi} \delta_{k,0}$, the sum collapses to,

$$G^+(x, 0) = \frac{1}{2\pi} e^{-i(1+\beta)t} (\text{sech} \rho)^{1+\beta} \sum_{n=0}^{\infty} e^{-2i\text{nt}} P_n^{(0, \beta)}(1 - 2\tanh^2 \rho)$$  \hspace{1cm} (3.76)$$

which can be evaluated using well known generating functions for the Jacobi polynomials [14] to give,

$$G^+(x, 0) = \frac{1}{2^{3+2\pi(1 + \sigma/l^2)^{1+\beta}} \sqrt{|\sigma|}} \times _2F_1 \left( \frac{1+\beta}{2}, \frac{\beta+2}{2}, 1+\beta, \frac{1}{(1 + \sigma/l^2)^2} \right)$$  \hspace{1cm} (3.77)$$

where $\sigma(x, x') = \frac{1}{2} \eta^{(4)}_{\mu\nu} (x - x')^\mu (x - x')^\nu$ and thus $\sigma(x, 0) = l^2 (-1 + \cosh \rho \cos t)$ is the invariant distance on AdS$_3$. Note that the expression obtained for the Wightman function is only well-defined for $\sigma > 0$ (i.e outside the “lightcone”), since there are branch points at $\sigma/l^2 = 0, -2$ and a cut for $-2 \leq \sigma/l^2 \leq 0$. Now, it is clear that $t \to t - i\epsilon$ improves the convergence of the sum over modes and still gives a solution to the homogeneous equation. Note that under this transformation $\sigma/l^2 \to \sigma/l^2 + i\epsilon \sin t + O(\epsilon^2)$. Thus a suitable expression for the Wightman function is

$$G^+(x, 0) = \lim_{\epsilon \to 0} \frac{1}{2^{3+2\pi(1 + \sigma/l^2 + i\epsilon \sin t)^{1+\beta}}}$$

$$\times _2F_1 \left( \frac{1+\beta}{2}, \frac{\beta+2}{2}, 1+\beta, \frac{1}{(1 + \sigma/l^2 + i\epsilon \sin t)^2} \right).$$  \hspace{1cm} (3.78)$$

Now we can deduce some other interesting functions closely related to the Wightman function. The commutator is given by the jump across the branch cut, that is $G^+(x, 0) - \tilde{G}^+(x, 0)$. Thus, for $\sigma \leq 0$,

$$[\phi(x), \phi(0)] = -\frac{i}{2\pi} \text{sgn} \sin t \text{ } P \left( \frac{1}{(1 + \sigma/l^2)^{1+\beta}} \sqrt{\frac{(\sigma + l^2)^2}{\sigma(\sigma + 2l^2)}} \right)$$

$$\times _2F_1 \left( \frac{1+\beta}{2}, \frac{\beta}{2}, \frac{1}{2}, \frac{(\sigma + 2l^2)}{(\sigma + l^2)^2} \right)$$  \hspace{1cm} (3.79)$$

which can be deduced from well known hypergeometric identities [14] and the result $1/(x - i0)^n = P(1/x^n) + i\pi(-1)^{n-1}\delta^{(n-1)}(x)/(n-1)!$. Of course for $\sigma > 0$ the commutator is zero, as required by causality. Hadamard’s elementary function can be computed in a similar manner, yielding for $\sigma \leq 0$
\[ G^{(1)}(x,0) = \{\{\phi(x),\phi(0)\}\} = \frac{(-1)^{\beta+1}}{2^\beta \beta!} \delta^{(\beta)}(1 + \sigma/l^2) \sqrt{\frac{(\sigma + l^2)^2}{\sigma(\sigma + 2l^2)}} \times_2 F_1 \left( \frac{1 + \beta - \beta_1}{2}, \frac{\beta + 1}{2}, \frac{1 + \beta}{1 + \sigma/l^2 + i0} \right), \tag{3.80} \]

and for \( \sigma > 0 \) it is simply twice the real part of equation (3.77). Now we can move on to the Feynmann function which satisfies the inhomogeneous equation. In the mode sum (3.76) if we replace \( t \rightarrow |t| \) we get the Feynman function, and \( |t| \rightarrow |t| - i\epsilon \) improves convergence. This will correspond to \( \sigma/l^2 \rightarrow \sigma/l^2 + i0 \) thus giving

\[ iG_F(x,0) = \frac{1}{2^{\beta+2} \pi (1 + \sigma/l^2 + i0)^{1+\beta}} \times_2 F_1 \left( \frac{1 + \beta - \beta_1}{2}, \frac{\beta + 1}{2}, \frac{1}{1 + \sigma/l^2 + i0} \right) \tag{3.81} \]

We should note the following identity [14],

\[ \frac{1}{2^{\beta+2} \pi (1 + \sigma/l^2)^{1+\beta}} \times_2 F_1 \left( \frac{1 + \beta - \beta_1}{2}, \frac{\beta + 1}{2}, \frac{1}{1 + \sigma/l^2} \right) = \frac{1}{4\pi \sqrt{\sigma/l^2(\sigma/l^2 + 2)}(\sigma/l^2 + 1 + \sqrt{\sigma/l^2(\sigma/l^2 + 2)})^\beta} \tag{3.82} \]

allows to us see more clearly the behaviour of the function, the positions of the branch points for instance.

Now we will explore a more direct method for deriving these various functions. The operator \( l^2(-\Box + m^2 + \xi R) \) acting on functions of the invariant distance \( \sigma \) becomes,

\[ \sigma(\sigma + 2l^2) \frac{d^2}{d\sigma^2} + 3(\sigma + l^2) \frac{d}{d\sigma} - (m^2l^2 - 6\xi) \tag{3.83} \]

and since the Wightman function satisfies the homogeneous Klein-Gordon equation, we can write it in terms of two independent solutions to the corresponding ODE, namely \((-\sigma/2l^2)^{-(1+\beta)}_2 F_1(1 + \beta, 1/2 + \beta, 1 + 2\beta, -2l^2/\sigma) \) and \((-\sigma/2l^2)^{(-1+\beta)}_2 F_1(1 - \beta, 1/2 - \beta, 1 - 2\beta, -2l^2/\sigma) \). Inspecting these solutions, we see that the first goes to zero as \( \sigma \rightarrow -\infty \) and the second diverges, we therefore discard the second solution. Upon comparison \cite{7} with the Wightman function derived with the mode expansion we can get the proportionality factor so,

\[ \text{2For purely mathematical interest we should note that we have deduced a quadratic transformation formula for hypergeometric functions, namely,} \]

\[ _2 F_1 \left( \frac{1 + n}{2}, \frac{1 + n}{2}, 1 + n, \frac{1}{1 - 2x} \right) = \left( 1 - \frac{1}{2x} \right)^{1+n} _2 F_1 \left( 1 + n, \frac{1}{2} + n, 1 + 2n, \frac{1}{x} \right). \tag{3.84} \]
\[ G^+(\sigma) = (-\sigma/l^2)^{-(1+\beta)}_2F_1(1 + \beta, 1/2 + \beta, 1 + 2\beta, -2l^2/\sigma). \quad (3.85) \]

The \(i\epsilon\) prescription can be invoked by letting \(\sigma/l^2 \to \sigma/l^2 + i\epsilon\sin t\) as before.

Now we would like to derive the corresponding expression in our quantum theories. More precisely we look for position eigenstates \(|g\rangle\) satisfying \(\hat{g}|g\rangle = g|g\rangle\), where we have put a hat on the operator for clarity. Note that it is a nontrivial fact that we can simultaneously diagonalise the matrix elements of \(\hat{g}\), but in fact one can since they all commute amongst each other which is the locality condition addressed in the “Chiral quantization” section. The quantity we are interested in then is the amplitude \(\langle g|e\rangle\) (\(e\) the identity corresponds to the origin in the static coordinate system). This can be computed using any convenient basis \(\{|i\rangle\}\) for the Hilbert space \(\mathcal{H}\), as \(\langle g|e\rangle = \sum_i \langle g|i\rangle \langle i|e\rangle\). One could use the basis we mentioned earlier \(|\mu, m, \bar{m}\rangle\) and compute the matrix elements of the operator matrix \(\hat{g}\); in principle this would allow one to deduce \(\langle \mu, m, \bar{m}|g\rangle\). It would be nice to perform such a calculation, however instead, as before, we take a shortcut. Just to recap, in a position representation we seek eigenfunctions of the current algebra which can be represented by the differential operators,

\[
L_2 = i\hbar \frac{\partial}{\partial v} \quad R_2 = -i\hbar \frac{\partial}{\partial u} \quad (3.86)
\]

\[
L_\pm = i\hbar e^{\mp 2iu} \left( \coth 2\rho \frac{\partial}{\partial v} - \operatorname{cosech} 2\rho \frac{\partial}{\partial u} \pm \frac{i}{2} \frac{\partial}{\partial \rho} \right) \quad (3.87)
\]

\[
R_\pm = i\hbar e^{\pm 2iu} \left( \coth 2\rho \frac{\partial}{\partial u} - \operatorname{cosech} 2\rho \frac{\partial}{\partial v} \mp \frac{i}{2} \frac{\partial}{\partial \rho} \right). \quad (3.88)
\]

It is straightforward to check that they satisfy the correct algebra

\[
[L_a, L_b] = 2i\hbar \varepsilon_{ab}^c L_c \quad [R_a, R_b] = 2i\hbar \varepsilon_{ab}^c R_c \quad [L_a, R_b] = 0 \quad (3.89)
\]

where \(L_\pm = L_3 \pm iL_1\). The Casimirs \(Q_L = \eta^{ab} L_a L_b\) and \(Q_R\), turn out to be

\[ Q_L = Q_R = -\hbar^2 \Box. \quad (3.90) \]

Thus in an irreducible representation the eigenfunctions of \(L_2, R_2, Q\) satisfy the Klein-Gordon equation with a mass that depends on which expression for the Casimir is used. Therefore we can use the basis \(\{u_{\varepsilon \beta \xi}\}\) and thus end up with an expression for \(\langle g|e\rangle\) which equals \(G^+(x,0)\) with \(\xi = 1/6\) for \(Q = -p^2 + h^2\) and \(\xi = 0\) for \(Q = -p^2\). It is interesting to note that \(\xi = 1/6\) appears to correspond to some sort of critical coupling since \(\beta = lm\) in this case. It should be noted that the well-known Brietenlohner-Freedman bound for the Klein-Gordon equation in \(AdS_3\) is \(m^2 + \xi R \geq -\hbar^2\) (in these conventions) [21, 22]; thus we see this corresponds to \(Q \leq \hbar^2\) which agrees with the Casimir \(Q = -p^2 + h^2\). Hence it appears that the quantum mechanics with the quantum correction to the Casimir is consistent with the field theory, despite the fact that the definition of mass is somewhat arbitrary.
4. Conclusions

We studied the motion of a free massive particle moving on the group manifold $AdS_3$ both classically and quantum mechanically in a covariant canonical formalism. We derived two different quantum theories depending on whether the constraint that the motion be future directed is imposed classically or quantum mechanically. The allowed values of the mass of a particle quantum mechanically are quantized in either case. Only certain representations of the current algebra were allowed in the Hilbert space, namely (most of) the discrete series. Interestingly in the unconstrained theory the continuous series in the exceptional interval is present leading to a small continuum of mass states; these states have the property that they can flip between moving forward or backward in time, and of course have no classical counterpart.

A function representation was derived and all the wavefunctions were found; from this we deduced such a representation for $\hat{AdS}_3$ and the BTZ black hole. It would be desirable to derive the function representation directly from the canonical quantizations discussed, rather than the more ad-hoc method of finding differential operators which satisfy the correct algebra.

A quantization in terms of “chiral” variables of the theory was carried out, which amounts to determining something called a braiding matrix, and this gave the same results as canonical quantization of the unconstrained theory.

Upon comparison with QFT we found that one of the quantum theories corresponds to a critical coupling (neither minimal or conformal), which is consistent with the Breitenlohner-Freedman bound. It would also be interesting to see the connection between the different quantum theories and boundary conditions in $AdS_3$ which must feature since it is not globally hyperbolic.

Possible extensions of this work include a similar treatment for massless particles, and of course attacking the problem of a canonical quantization of string theory in $AdS_3$.

Acknowledgments

I would like to thank Malcolm Perry for many discussions and useful comments. This work was supported by EPSRC.

A. Some representation theory

In this section we summarise the unitary irreducible representations of $SL(2, \mathbb{R})$ and its universal covering group $SL(2, \mathbb{R})$, see [13] [9]. Both have the same algebras, namely $sl(2, \mathbb{R})$. Let $\{t_a : a = 1, 2, 3\}$ be a basis for the algebra satisfying $[t_a, t_b] = \epsilon_{ab}^c t_c$, where $\epsilon_{ab}^c = \epsilon_{abd} \eta^{dc}$ and $\eta$ is the diagonal metric with $\eta_{11} = \eta_{22} = -\eta_{33} = 1$. Consider a unitary representation $R : t_a \mapsto -i J_a$, so $J_a^\dagger = J_a$. Therefore we have,

$$[J_a, J_b] = i \epsilon_{ab}^c J_c.$$  \hfill (A.1)
Define $J_\pm = J_1 \pm iJ_2$. The Casimir operator is $Q = \eta_{ab}J_aJ_b = J_1^2 + J_2^2 - J_3^2$. Now, let's examine the spectrum of $J_3$ in an irreducible unitary representation; necessarily the Casimir will be diagonal.

\[
Q|j, m\rangle = -j(j + 1)|j, m\rangle \quad \text{(A.2)}
\]

\[
J_3|j, m\rangle = m|j, m\rangle \quad \text{(A.3)}
\]

where $j \equiv -\frac{1}{2} - \sqrt{\frac{1}{4} - q}$ and $q$ is the eigenvalue of $Q$. Since $J_a$ is Hermitian, $\{|j, m\rangle\}$ can be chosen to be an orthonormal basis for the carrier space to the representation, also $m$ must be real. It is easy to show that,

\[
J_\pm|j, m\rangle = \sqrt{m(m + 1) - j(j + 1)}|j, m \pm 1\rangle \quad \text{(A.4)}
\]

So far everything stated applies equally to $SL(2, \mathbb{R})$ and $\widetilde{SL}(2, \mathbb{R})$. The differences arise from the differences in the spectrum of $J_3$. By considering the representation induced on the enveloping algebra it is easy to show that $m$ is either integer or half-integer in the case of $SL(2, \mathbb{R})$, whereas this restriction fails for $\widetilde{SL}(2, \mathbb{R})$.

1. The principal discrete representations correspond to highest and lowest weight representations. More explicitly, $D_j^+$ is defined by $J_+|j, -j\rangle = 0$, and thus $m = -j, -j + 1, \ldots$ and $j < 0$. Similarly $D_j^-$ is defined by $J_-|j, j\rangle = 0$, and therefore $m = j, j - 1, \ldots$ and $j < 0$. For $SL(2, \mathbb{R})$ $j$ is not restricted further, whereas for $SL(2, \mathbb{R})$ $2j \in -\mathbb{N}$.

2. The principal continuous representations correspond to no highest or lowest weight, so the spectrum of $J_3$ is unbounded. Also, $j = -\frac{1}{2} + i\kappa$ and $m = \alpha, \alpha \pm 1, \ldots$, where $\kappa \in \mathbb{R} - \{0\}$ and $0 \leq \alpha < 1$. These representations are labelled by $C_j^\alpha$, and $\forall \alpha \in [0, 1)$ correspond to irreducible representations of $SL(2, \mathbb{R})$, and for $\alpha = 0, \frac{1}{2}$ gives two inequivalent irreducible representations of $SL(2, \mathbb{R})$.

3. There is another set of representations for which the spectrum of $J_3$ is unbounded. These are the exceptional representations, for which $-1 < j < -\frac{1}{2} (0 < q < \frac{1}{4})$ and $m = \alpha, \alpha \pm 1, \ldots$, and will be denoted by $E_j^\alpha$. Note these are often also called $C_j^\alpha$ or $C_q^\alpha$, and there is no confusion with the representations above since we have different values of $j$. Here, $\alpha = 0$ corresponds to representations of $SL(2, \mathbb{R})$.

B. Reparameterization invariance at the quantum level

Here we provide justification for why gauge fixing the Lagrangian $\mathcal{L} = \frac{1}{2} e^2 - m^2 e$ by letting $e = 1$ does not spoil reparameterization invariance quantum mechanically; see also [2] for a different argument. So to begin we note that the Lagrangian is singular, leading to the primary constraint $\phi_1 = p_e \equiv \frac{\partial \mathcal{L}}{\partial e} \approx 0$. We want to also impose the constraint $\phi_2 = e - 1 \approx 0$. If we look for secondary constraints we find $\phi_3 = \frac{1}{4} \pi^2 + m^2 \approx 0$, where $\pi_\mu$
is conjugate to \( x^\mu \), and no more occur. It is easy to see that \( \phi_3 \) is first-class, and \( \phi_1, \phi_2 \) are second class. Now, we replace Poisson brackets on the phase space with canonical variables \((x, \pi, e, p_e)\), with the Dirac bracket defined by \( \{A, B\}_D = \{A, B\} - \{A, \phi_a\}C_{ab}^{-1}\{\phi_b, B\} \) for any two observables \( A, B \), where \( C_{ab} = \{\phi_a, \phi_b\} \) and \( a, b \in \{1, 2\} \). Now, using the Dirac brackets instead of the Poisson brackets, allows us to set \( \phi_1 \) and \( \phi_2 \) strongly to zero; therefore \( \{A, \phi_a\} = 0 \) for any observable \( A \), since once the second class constraints are set strongly to zero \( A \) will only be a function of \((x, \pi)\). This allows us to deduce that \( \{A, B\}_D = \{A, B\} \). Hence the Lagrangian \( \mathcal{L} = x^2 \) (the constant \( m^2 \) isn’t important and will be dropped) with the constraint \( \phi_3 \) will give rise to the same phase space and equations of motion as the manifestly reparameterization invariant Lagrangian. Also, the constraint is easy to implement quantum mechanically giving rise to the BRST operator \( Q = \phi_3 c \) which is nilpotent (since \( c \) is a ghost). The physical state conditions are \( Q|\psi\rangle = 0 \) and \( b|\psi\rangle = 0 \), where \( b \) is the antighost and \( \{b, c\} = 1 \); this implies \( \phi_3|\psi\rangle = 0 \), which becomes \( p^2|\psi\rangle = m^2|\psi\rangle \).

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