Certain fractional integral operators pertaining to S-function
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Abstract: Fractional integral operators introduced by Saxena and Kumbhat involving Fox’s H-function as kernel are applied, and find new image formulas of S-function and properties are established. Also, by implementing Euler, Whittaker and K-transforms on the resulting formulas. On account of S-function, a number of results involving special functions can be obtained merely by giving particular values for the parameters.

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1. Introduction and preliminaries
In recent years, the fractional calculus has become a significant instrument for the modeling analysis and assumed a significant role in different fields, for example, material science, science, mechanics, power, economy and control theory (see details: Alaria et al. (2019), Berdnikov and Lokhin (2019), Drapaca (2018), and Hammachukiattikul (2019)). In addition, a number of researchers like (Agarwal and Jain (2011), Baleanu (2009), Kalla (1969), and Kilbas (2005)) have studied in-depth level of properties, applications and diverse extensions of a range of operators of fractional calculus. Also, on other analogous topics is very active and extensive around the world. One may refer to the research monographs Kiryakova (1994) and Miller and Ross (1993), and the recent papers Kilbas et al. (2006), Mathai et al. (2010), Samko et al. (1993), and Suthar and Amsalu (2019). Recently, Saxena and Daiya (2015) defined and study a special function called as S-function, its relation with other special functions, which is a generalization of k-Mittag-Leffler function.
K-function, M-series, Mittag-Leffler function and other many special functions. These special functions have recently found essential applications in solving problems in applied sciences, biology, physics and engineering. The S-function is defined for $\rho, \delta, \omega, \tau \in \mathbb{C}$, $\Re(\rho) > 0$, $k \in \mathbb{R}$, $\Re(\rho) > k \Re(\tau)$, $l_i (i = 1, 2, 3, \ldots, p)$, $m_j (j = 1, 2, 3, \ldots, q)$, and $p < q + 1$ as

$$S_{\rho, \delta, \omega, \tau}^{\rho, \delta, \omega, \tau} \left[ l_1, l_2, \ldots, l_p; m_1, m_2, \ldots, m_q; x \right] = \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n (\omega)_n}{(m_1)_n \cdots (m_q)_n} \frac{x^n}{n!}.$$  \hspace{1cm} (1.1)

Here, k-Pochhammer symbol

$$\left( \frac{\rho}{\delta} \right)_n^{(k)} = \left\{ \begin{array}{ll} I_k^{(k)} & k \in \mathbb{R}, \zeta \in \mathbb{C}/\{0\} \\ \zeta^{(n+k)} & (n+k), (n \in \mathbb{N}, \zeta \in \mathbb{C}) \end{array} \right.$$ \hspace{1cm} (1.2)

and the k-gamma function

$$\Gamma_k(\zeta) = k^{\zeta-1} \Gamma(\zeta).$$ \hspace{1cm} (1.3)

where $\zeta \in \mathbb{C}$, $k \in \mathbb{R}$ and $n \in \mathbb{N}$, introduced by Daz and Pariguan (2007) (see also; Romero and Cerutti (2012)).

Some important special cases of S-function are enumerated below:

(i) For $p = q = 0$, the generalized $k$-Mittag-Leffler function from Saxena et al. (2014):

$$E_{\rho, \delta}^{\rho, \delta}(x) = S_{\rho, \delta}^{\rho, \delta, 0, 0} \left[ -1, -1; -x \right] = \sum_{n=0}^{\infty} \frac{(\omega)_n}{(\rho)_n} \frac{x^n}{n!} \Re(\rho/k) > p - q.$$  \hspace{1cm} (1.4)

(ii) Again, for $k = \tau = 1$, the S-function is the generalized K-function, defined by Sharma (2011):

$$K_{\rho, \omega}^{\rho, \omega} \left[ l_1, \ldots, l_p; m_1, \ldots, m_q; x \right] = S_{\rho, \omega, 1, 1}^{\rho, \omega, 1, 1} \left[ l_1, \ldots, l_p; m_1, \ldots, m_q; x \right]$$

$$= \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n (\omega)_n}{(m_1)_n \cdots (m_q)_n} \frac{x^n}{n!} \Re(\rho) > p - q.$$  \hspace{1cm} (1.5)

(iii) For $\tau = k = \omega = 1$, the S-function reduced to generalized M-series defined by Sharma and Jain (2009):

$$M_{\rho, q}^{\rho, q} \left[ l_1, \ldots, l_p; m_1, \ldots, m_q; x \right] = S_{\rho, q, 1, 1}^{\rho, q, 1, 1} \left[ l_1, \ldots, l_p; m_1, \ldots, m_q; x \right]$$

$$= \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n}{(m_1)_n \cdots (m_q)_n} \frac{x^n}{n!} \Re(\rho) > p - q - 1.$$  \hspace{1cm} (1.6)

Now, we recall the Saxena and Kumbhat (1974) operators involving Fox’s H-function as kernel, by means of the following equations:

$$R_{\rho, x, t}^{\rho, x, t} \left[ \frac{f(t)}{x^\nu} \right] = \gamma x^{\nu-\rho-1} \int_0^x t^\rho (x^\nu - t^\nu)^{\rho-1} H_{\rho, \nu}^{\rho, \nu} \left[ \frac{\nu}{\alpha, \beta} \right] f(t) dt,$$  \hspace{1cm} (1.4)

$$K_{\rho, x, t}^{\rho, x, t} \left[ \frac{f(t)}{x^\nu} \right] = \gamma x^{\nu} \int_x^\infty (t^\nu - x^\nu)^{\nu-1} H_{\rho, \nu}^{\rho, \nu} \left[ \frac{\nu}{\alpha, \beta} \right] f(t) dt.$$  \hspace{1cm} (1.5)

where $U$ and $V$ represent the expressions

$$\left( \frac{x^\nu}{x^\nu} \right)^\nu \left( 1 - \frac{x^\nu}{x^\nu} \right)^\nu.$$
respectively with \( \xi, \nu > 0 \). The sufficient conditions of above said operators are given below:

(1) \( 1 \leq c, d < \infty, c^{-1} + d^{-1} = 1 \);

(2) \( \Re(\eta + \chi \xi(b_j/B_j)) > -d^{-1}; \Re(\sigma + \rho \xi(b_j/B_j)) > -d^{-1} \);

(3) \( f(x) \in L_p(0, \infty) \);

(4) \( |\arg \lambda| < \frac{\pi}{2} , \theta > 0 \).

where \( \theta = \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{d} B_j + \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{c} A_j > 0 \).

where, Fox H-function Fox (1961), in operator (1.4) and (1.5) defined in terms of Mellin-Barnes type contour integral as:

\[
H_{c,d}^{m,n}(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{\ell(s)z^s}{s} ds,
\]

where

\[
\ell(s) = \frac{\prod_{i=1}^{m} \Gamma(b_i + B_j \phi) \prod_{i=1}^{n} \Gamma(1 - a_i - A_j \phi)}{\prod_{i=n+1}^{c} \Gamma(a_i + A_j \phi) \prod_{j=m+1}^{d} \Gamma(1 - b_j - B_j \phi)}.
\]

Here, \( m, n, c, d \in \mathbb{N}_0 \) with \( 1 \leq m \leq d, 0 \leq n \leq c, \ a_i, b_j \in \mathbb{R} \) or \( \mathbb{C} \), \( A_i, B_j \in \mathbb{R}_+ \), \( i = 1, 2, \ldots, c \), \( j = 1, 2, \ldots, d \).

For the convergence conditions together with the conditions of analytical continuations of H-function, one can see Mathai and Saxena (1978) and Mathai et al. (2010). Throughout this paper, we assume that the above conditions are fulfilled by the said function.

The Euler transform (Sneddon (1978)) of a function \( f(z) \) is defined as:

\[
B(f(z); g, h) = \int_0^1 z^{\theta-1}(1-z)^{h-1} f(z) \, dz, \quad g, h \in \mathbb{C}, \Re(h) > 0, \Re(g) > 0.
\]

Due to Whittaker transform (Whittaker and Watson (1996)), the following result true:

\[
\int_0^\infty e^{-t \zeta^{-1}} W_{\chi, \theta}(t) dt = \frac{\Gamma((1/2) + \theta + \zeta) \Gamma((1/2) - \theta + \zeta)}{\Gamma(1 - \chi + \zeta)},
\]

where \( \Re(\theta \pm \zeta) > -1/2 \) and \( W_{\chi, \theta}(t) \) is the Whittaker confluent hypergeometric function.

\[
W_{\chi, \theta}(z) = \frac{\Gamma(-2\theta)}{\Gamma((1/2) + \chi - \theta)} M_{\chi, \theta}(z) + \frac{\Gamma(2\theta)}{\Gamma((1/2) + \chi + \theta)} M_{\chi, -\theta}(z),
\]

where \( M_{\chi, \theta}(z) \) is defined by

\[
M_{\chi, \theta}(z) = z^{1/2 + \theta} e^{-1/2z^2} I_1((1/2) + \theta, 2\theta + 1; z).
\]

The following integral equation defined in term of K-transform (Erđélyi et al. (1954)) as:
where $\Re(h) > 0; K_\nu(x)$ is the Bessel function of the second kind defined by (Srivastava et al. 1982, p. 332)

$$K_\nu(z) = \left( \frac{\pi}{2z} \right)^{1/2} W_{0,\nu}(2z),$$

where $W_{0,\nu}(.)$ is the Whittaker function defined in equation (1.9).

$$\int_0^\infty \nu^{-1} K_\nu(ax) dx = 2\nu^{-2} a^{-\alpha} \Gamma(\mu + \nu/2); \Re(\sigma) > 0; \Re(\mu + \nu) > 0. \quad (1.10)$$

The above result given in (Mathai et al. 2010), pp. 54, Eq. 2.37) will be used in evaluating the integrals.

In view of the effectiveness and extraordinary significance of the fractional integral operators given by Saxena and Kumbhat in specific issues, the authors establish the image formulas and derive certain properties of $S$-function. The results obtained here involve special functions like $k$-Mittag-Leffler function, $K$-function and $M$-series, due to their general nature and usefulness in the theory of integral operators and relevant part of computational mathematics.

2. Images of $S$-function under the fractional integral operators

In this part, we obtain the images of $S$-function under the generalized fractional integral operators defined in 1.4 and 1.5.

Theorem 1. Let $\rho, \delta, \omega, \tau \in \mathbb{C}, \Re(\rho) > 0, \Re(\delta) > 0, k \in \mathbb{R}, \Re(\rho) > k \Re(\tau), x > 0$, the fractional integration $R_{\gamma}^\nu$ of $S$-function exists, under the condition

$$c_1 + d_1 = 1; \Re(\eta + \gamma b_j) > -d_1;$$

then there holds the result:

$$R_{\gamma}^\nu \left( \int_{\eta}^{\sigma} x^{\rho-1} S_{\rho,\sigma,\kappa,\ell} (t^\epsilon) \right)(x)$$

$$= x^{\rho-1} k^{1-(\delta/k)} \frac{1}{\Gamma(\omega/k)} \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n}{(m_1)_n \cdots (m_q)_n} \frac{(k \Gamma(\nu/k)x)^n}{n!}$$

$$\times \frac{\Gamma((\omega/k) + n)}{\Gamma((\omega/k) + (\delta/k))} \prod_{n=0}^{m-1} \left[ \frac{\alpha_0 \cdots \alpha_n}{\beta_0 \cdots \beta_n} \right] \left[ \frac{\alpha_0 \cdots \alpha_n}{\beta_0 \cdots \beta_n} \right]$$

$$\times \frac{1}{\Gamma((n+1)k)} \int_{\eta}^{\sigma} \frac{(t^\epsilon)^n}{(n^\ell + 1)\ell!} d\ell$$

$$\quad \times \left. \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n}{(m_1)_n \cdots (m_q)_n} \frac{(\omega)_{n+k}}{\Gamma_k(\gamma/(n+1)k)} \frac{(t^\epsilon)^n}{n!} \right|_{t=0}^{t=1}.$$  \quad (2.1)

Proof. Let $\mathbb{C}$ be the left-hand side of (2.1), using (1.1) and (1.4), we have

$$\mathbb{C} = \gamma x^{-\rho-\sigma-1} \int_0^{\infty} e^{\sigma-x^\epsilon} (t^\epsilon)^\sigma$$

$$\times \frac{1}{2\pi} \int_0^{2\pi} \epsilon(\phi)(\phi)^d d\phi \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n}{(m_1)_n \cdots (m_q)_n} \frac{(\omega)_{n+k}}{\Gamma_k(\gamma/(n+1)k)} \frac{(t^\epsilon)^n}{n!}.$$ 

$$\quad \times \left. \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n}{(m_1)_n \cdots (m_q)_n} \frac{(\omega)_{n+k}}{\Gamma_k(\gamma/(n+1)k)} \frac{(t^\epsilon)^n}{n!} \right|_{t=0}^{t=1}\right.$$  \quad (2.2)
Further, the substitution $t'/x' = z$, then $t = xz^{1/n}$ in above term, we get

$$
\Xi = x^{\rho-1} \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n}{(m_1)_n \cdots (m_q)_n} \frac{(\omega)^{n+1}}{\Gamma(n\rho + \delta)n!} \int_0^1 \frac{\xi(\phi) \phi^n}{\Gamma(1+\sigma)(1+\omega \sigma)} \phi d\phi.
$$

(2.3)

Applying Beta function for (2.3), the inner integral becomes

$$
\Xi = x^{\rho-1} \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n}{(m_1)_n \cdots (m_q)_n} \frac{(\omega)^{n+1}}{\Gamma(n\rho + \delta)n!} \times \frac{1}{2\pi i} \int_\mathcal{C} \phi^{n+\lambda+\gamma+\phi'(1/n)} d\phi,
$$

(2.4)

Interpreting the right-hand side of (2.4), in view of the definition $H$-function under (1.2), (1.3) and (1.6), we reached at the desired result (2.1).

**Theorem 2.** Let $\rho, \sigma, \omega, \gamma \in \mathbb{C}, \Re(\rho) > 0, \Re(\lambda) < 1, k \in \mathbb{R}, \Re(\rho) > k \Re(\tau), \kappa > 0$, the fractional integration $K^{s,\gamma}_{x,\sigma}$ of $S$-function exists, under the condition $c + d = 1$; $\Re(\eta + \phi(\lambda_i/\lambda_j)) > -d - 1$; $\Re(\tau + \phi(\lambda_i/\lambda_j)) > -c - 1; (j = 1, \ldots, m)$. Then there holds the result:

$$
K^{s,\gamma}_{x,\sigma}(t^{-1}S^{s,\alpha,\lambda,k}(t^{-\kappa}))(x)
$$

$$
= x^{-\varphi} k^{-1}(s/k) \frac{1}{\Gamma(\omega/k)} \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n}{(m_1)_n \cdots (m_q)_n} \frac{(k^{-1}x^{-\kappa})^n}{n!} \times \frac{\Gamma((\omega/k) + nr)}{\Gamma((n\omega/k) + (\gamma/r))} \prod_{j=1}^{\infty} \left(1 - (\sigma_j - (\omega + \nu)/\gamma, -\phi(\lambda_i/\lambda_j)), (\omega, -\nu/n/r, \lambda_i, \lambda_j)\right).
$$

(2.5)

**Proof.** Assume $\varphi$ be the left-hand side of (2.5), using (1.1) and (1.5), we have

$$
\varphi = \gamma x^\varphi \int_x^\infty t^{-\nu-\kappa+1}(t^{-\kappa})(t^{-\nu})^\lambda
$$

$$
\times \frac{1}{2\pi i} \int_\mathcal{C} \phi^{\nu+\lambda+\phi'(1/n)} d\phi \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n (\omega)^{n+1}}{(m_1)_n \cdots (m_q)_n \Gamma(n\rho + \delta)n!} (t^{-\kappa})^n.
$$

Changing the order of the integration under the valid condition provided in the theorem statement, we get

$$
\varphi = \gamma x^\varphi \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n (\omega)^{n+1}}{(m_1)_n \cdots (m_q)_n \Gamma(n\rho + \delta)n!} \int_0^1 \frac{\xi(\phi) \phi^n}{\Gamma(1+\sigma)(1+\omega \sigma)} \phi d\phi.
$$

Therefore, we have

$$
\varphi = \gamma x^\varphi \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n (\omega)^{n+1}}{(m_1)_n \cdots (m_q)_n \Gamma(n\rho + \delta)n!} \int_0^1 \frac{\xi(\phi) \phi^n}{\Gamma(1+\sigma)(1+\omega \sigma)} \phi d\phi.
$$

(2.6)
Let the substitution \(x'/t' = u\), then \(t = x/u^{1/\gamma}\) in above term and applying beta function, we get

\[
\nu = X^{-\alpha} \sum_{n=0}^{\infty} \frac{(l_1)_n \ldots (l_p)_n (omega)_{a,n,k} x^{-\nu n}}{(m_1)_n \ldots (m_p)_n \Gamma(n\nu + \delta)n!}
\]

\[
\times \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma\left((\epsilon + \nu + \nu n)/\gamma + 1 - \sigma \phi\right) \Gamma\left(\sigma + 1 - \sigma \phi\right)}{\Gamma\left(\epsilon + \nu + \nu n/\gamma + 1 - \sigma \phi\right) \Gamma\left(\sigma + 1 - \sigma \phi\right)} d\phi,
\]

Interpreting the right-hand side of (2.6), in view of the definition (1.2), (1.3) and (1.6), we arrive at the result (2.5).

3. Special cases

(i) If we put \(p = q = 0\), in Theorem 1 and Theorem 2, then we find the following interesting results on the right which are known as generalized k-Mittag-Leffler function.

Corollary 1. Eq. (2.1) reduces in the following form:

\[
R_{k,\tau} (t^{\sigma-1} S_{p,\nu,1,k}(t^{\gamma})) (x)
\]

\[
= x^{\nu-1} S_{k,\delta}(x^{\gamma}) H_{\epsilon, 2, d+1}^m \left[ \left\{ a_c, a_c, 1 - (\eta - \sigma - \nu n - 1)/\gamma, \epsilon, (-\sigma, v), (\epsilon, \epsilon, \epsilon), (b_c, b_c) \right\} \right].
\]

Corollary 2. Eq. (2.5) reduces in the following form:

\[
K_{k,\tau} (t^{\sigma-1} S_{p,\nu,1,k}(t^{\gamma})) (x)
\]

\[
= x^{\nu-1} S_{k,\delta}(x^{\gamma}) H_{\epsilon, 2, d+1}^m \left[ \left\{ a_c, a_c, 1 - (\eta - \sigma - \nu n + 1)/\gamma, \epsilon, (-\sigma, v), (\epsilon, \epsilon, \epsilon), (b_c, b_c) \right\} \right].
\]

(ii) For putting \(k = \tau = 1\) in Theorem 1 and Theorem 2, then we get the following interesting results on the right is known as K-function.

Corollary 3. Eq.(2.1) reduces in the following form:

\[
R_{k,\tau} (t^{\sigma-1} S_{p,\nu,1,k}(t^{\gamma})) (x)
\]

\[
= x^{\nu-1} K_{p,q} (x^{\gamma}) H_{\epsilon, 2, d+1}^m \left[ \left\{ a_c, a_c, 1 - (\eta - \sigma - \nu n - 1)/\gamma, \epsilon, (-\sigma, v), (\epsilon, \epsilon, \epsilon), (b_c, b_c) \right\} \right].
\]

Corollary 4. Eq.(2.5) reduces in the following form:

\[
K_{k,\tau} (t^{\sigma-1} S_{p,\nu,1,k}(t^{\gamma})) (x)
\]

\[
= x^{\nu-1} K_{p,q} (x^{\gamma}) H_{\epsilon, 2, d+1}^m \left[ \left\{ a_c, a_c, 1 - (\eta - \sigma - \nu n + 1)/\gamma, \epsilon, (-\sigma, v), (\epsilon, \epsilon, \epsilon), (b_c, b_c) \right\} \right].
\]
For taking $r = k = \omega = 1$ in Theorem 1 and Theorem 2, then we obtain the following results on the right is known as M-series.

**Corollary 5.** Eq.(2.1) reduces in the following form:

$$R_{n,k}^{(\varphi)}(t^{r-1}S_{[\varphi]}^{\omega,\alpha,1,1}(t^r))(x)$$

$$= x^{\varphi-1}M_{\rho}^{(\varphi)}(x^{\varphi})H_{c,2,d-1}^n \left[ \lambda \left( a_c, A_c, (1 - (\eta - \vartheta - \nu n - 1)/r, \xi, -\sigma, v) \right) 
\left( -\sigma - (\eta + \vartheta + \nu n + 1)/r, \xi + \nu, (b_\delta, B_\delta) \right) \right].$$

**Corollary 6.** Eq.(2.5) reduces in the following form:

$$K_{n,k}^{(\varphi)}(t^{r-1}S_{[\varphi]}^{\omega,\alpha,1,1}(t^r))(x)$$

$$= x^{\varphi-1}M_{\rho}^{(\varphi)}(x^{\varphi})H_{c,2,d-1}^n \left[ \lambda \left( a_c, A_c, (1 - (\epsilon + \vartheta + \nu n)/r, \xi, -\sigma, v) \right) 
\left( -\sigma - (\epsilon + \vartheta + \nu n)/r, \xi + \nu, (b_\delta, B_\delta) \right) \right].$$

### 4. Integral transforms of S-function involving fractional integral operators

In this part, the results established in Theorems 1 and Theorems 2 have been obtained in terms of Euler, Whittaker and K-transforms.

**Theorem 3.** Let $\rho, \sigma, \omega, r \in \mathbb{C}$, $\Re(\rho) > 0$, $\Re(\sigma) > 0$, $k \in \mathbb{N}$, $\Re(\rho) > k\Re(r)$, $x > 0$, $c^{-1} + d^{-1} = 1$; $\Re(\eta + \gamma (b_j / B_j)) > -d^{-1}$; $\Re(\sigma + \gamma \nu (b_j / B_j)) > -d^{-1}$; ($j = 1, \ldots, m$). Then

$$B \left( R_{n,k}^{(\varphi)}(t^{r-1}S_{[\varphi]}^{\omega,\alpha,1,1}(t^r)); g, h \right)$$

$$= \Gamma(h) \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n (\omega)_n k}{(m_1)_n \cdots (m_q)_n \Gamma(k(n\rho + \delta)n)} \Gamma(g + \vartheta + 1 + \nu n)$$

$$H_{c,2,d+1}^n \left[ \lambda \left( a_c, A_c, (1 - (\eta - \vartheta - \nu n - 1)/r, \xi, -\sigma, v) \right) 
\left( -\sigma - (\eta + \vartheta + \nu n + 1)/r, \xi + \nu, (b_\delta, B_\delta) \right) \right].$$

**Proof.** Using (2.1) and (1.7), it gives

$$B \left( R_{n,k}^{(\varphi)}(t^{r-1}S_{[\varphi]}^{\omega,\alpha,1,1}(t^r)); g, h \right) = \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n (\omega)_n k}{(m_1)_n \cdots (m_q)_n \Gamma(k(n\rho + \delta)n)}$$

$$\times H_{c,2,d+1}^n \left[ \lambda \left( a_c, A_c, (1 - (\eta - \vartheta - \nu n - 1)/r, \xi, -\sigma, v) \right) 
\left( -\sigma - (\eta + \vartheta + \nu n + 1)/r, \xi + \nu, (b_\delta, B_\delta) \right) \right]$$

$$\times \int_0^1 z^{\eta + \vartheta + n - 1 - (1 - 2)^{\nu n - 1}} (1 - z)^{h-1} \, dz,$$

$$= \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n (\omega)_n k}{(m_1)_n \cdots (m_q)_n \Gamma(k(n\rho + \delta)n)} \Gamma(g + \vartheta + \nu n - 1) \Gamma(h)$$

$$\times H_{c,2,d+1}^n \left[ \lambda \left( a_c, A_c, (1 - (\eta - \vartheta - \nu n - 1)/r, \xi, -\sigma, v) \right) 
\left( -\sigma - (\eta + \vartheta + \nu n + 1)/r, \xi + \nu, (b_\delta, B_\delta) \right) \right].$$
Now, we get the result (4.1).

Theorem 4. Let \( \rho, \delta, \omega, \tau \in \mathbb{C}, \ \Re(\rho) > 0, \ \Re(\delta) < 1, \ k \in \mathbb{R}, \ \Re(\rho) > k \Re(\tau), \ x > 0, \ c^{-1} + d^{-1} = 1; \ \Re(\eta + \rho \Re(\beta_j/B_j)) > d^{-1}; \ \Re(e + \sigma + \eta \Re(\beta_j/B_j)) > c^{-1}; \ (j = 1, \ldots, m). \) Then

\[
B \left\{ K_{\rho, \delta} \left( \frac{r^{-1} e^{x_0, \omega, r \Re(\rho)}}{K_{\rho, \delta}}(x); g, h \right) \right\}
\]

\[
= \Gamma(h) \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n (\omega)^{n+k}}{(m_1)_n \cdots (m_q)_n \Gamma(n \rho + \delta)n!} \Gamma(g - \nu - \nu n) \]

\[
\times H_{c+2,d-1} \left[ \lambda \left( (\alpha, A_c), \frac{1 - (e + \delta + vn)/y}{y} \xi, (-\sigma, \nu), \left( \beta_d, B_d \right) \right) \right]. \tag{4.2}
\]

Proof. In similar method of proof of Theorem 3, we get the result (4.2).

Theorem 5. Follow stated Theorem 1 for conditions on parameters, along with \( \Re(\theta \pm (\phi + \zeta + vn - 1)) > 1/2. \) Then the subsequent result true:

\[
\int_0^{\infty} e^{-\frac{\theta}{2}t^2} t^{-1} W_{x, \phi}(\theta t) \left\{ K_{\rho, \delta} \left( \frac{r^{1-1} e^{x_0, \omega, r \Re(\rho)}}{K_{\rho, \delta}}(t) \right) \right\} dt
\]

\[
= \frac{\phi^{-1} \Gamma(\theta + \phi + \zeta + vn - 1/2) \Gamma(\theta + \phi + \zeta + vn - 1/2)}{\Gamma(\theta - y + \zeta + vn)} \]

\[
\times H_{c+2,d-1} \left[ \lambda \left( (\alpha, A_c), \frac{1 - (\eta - \phi - vn - 1)}{y} \xi, (-\sigma, \nu), \left( \beta_d, B_d \right) \right) \right]. \tag{4.3}
\]

Proof. Using (2.1) and (1.8), it gives

\[
= \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n (\omega)^{n+k}}{(m_1)_n \cdots (m_q)_n \Gamma(n \rho + \delta)n!}
\]

\[
\times H_{c+2,d-1} \left[ \lambda \left( (\alpha, A_c), \frac{1 - (\eta - \phi - vn - 1)}{y} \xi, (-\sigma, \nu), \left( \beta_d, B_d \right) \right) \right] \]

\[
\times \int_0^{\infty} e^{-\frac{\theta}{2}t(t^2 + \nu + 1)} W_{x, \phi}(\theta t) dt.
\]

Assume that \( \theta t = k, \ \Rightarrow dt = dk/\phi, \) we get

\[
= \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_p)_n (\omega)^{n+k}}{(m_1)_n \cdots (m_q)_n \Gamma(n \rho + \delta)n!}
\]

\[
\times H_{c+2,d-1} \left[ \lambda \left( (\alpha, A_c), \frac{1 - (\eta - \phi - vn - 1)}{y} \xi, (-\sigma, \nu), \left( \beta_d, B_d \right) \right) \right] \]

\[
\times \phi^{-1} \int_0^{\infty} e^{-k/2} k^{(1-\nu) + 1/2} W_{x, \phi}(k) dk. \tag{4.4}
\]
Interpreting the right-hand side of (4.4), in view of H-function (1.8), we arrive at the result (4.3).

Theorem 6. Follow stated Theorem 2 for conditions on parameters, along with $\Re(\theta + (-\vartheta + \zeta - vn - 1) > 1/2$. Then the subsequent result true:

$$\int_0^\infty \exp^{-\vartheta/2} \mathcal{W}_\mathcal{E}^{\mathcal{E}}(\varphi t) \left\{ \mathcal{K}^{\mathcal{E}}_{\mathcal{E}} \left( t^{-\delta} \mathcal{S}^{\mathcal{E}}_{\mathcal{E}} \right) \right\} dt$$

$$= \varphi^{\theta - \vartheta} \sum_{n=0}^\infty \frac{(l_1)_n \ldots (l_p)_n (\omega)_n \Gamma(n + \delta)n!}{(m_1)_n \ldots (m_q)_n \Gamma(n + \delta)n!} \frac{\Gamma(-\vartheta + \zeta - vn + 1/2)}{\Gamma(1 - \vartheta - \zeta - vn)}$$

$$\times H_{\ell + 1, d - 1} \left[ \lambda_1 \left( \frac{(a, A, c, 1 - (\vartheta - \vartheta + vn)}{y, \xi, (-\sigma, v)}, (b_d, B_d) \right) \right] . \tag{4.5}$$

Proof. In same direction, proof of Theorem 5, we get the result (4.5).

Theorem 7. Follow stated Theorem 1 for conditions on parameters, with $\Re(r) > 0; \Re((\mu + \vartheta + vn - 1) \pm u) > 0$. Then the subsequent result true:

$$\int_0^\infty t^{\theta-1} K_\mathcal{E}(rt) \left\{ \mathcal{R}^{\mathcal{E}}_{\mathcal{E}} \left( t^{-\delta} \mathcal{S}^{\mathcal{E}}_{\mathcal{E}} \right) \right\} dt$$

$$= 2^{\mu + \vartheta - 3} \rho(1 - u - \vartheta) \sum_{n=0}^\infty \frac{(l_1)_n \ldots (l_p)_n (\omega)_n \Gamma(n + \delta)n!}{(m_1)_n \ldots (m_q)_n \Gamma(n + \delta)n!} \frac{\Gamma((\mu + \vartheta + vn - 1) \pm u)}{2}$$

$$\times \frac{(r/2)^{\ell + 1, d + 1} H_{\ell + 1, d + 1} \left[ \lambda_1 \left( a, A, c, 1 - (\vartheta - \vartheta + vn - 1)/y, \xi, (-\sigma, v) \right) \right] (b_d, B_d) \right] . \tag{4.6}$$

Proof. Using (2.1) and (1.10), it gives

$$= \sum_{n=0}^\infty \frac{(l_1)_n \ldots (l_p)_n (\omega)_n \Gamma(n + \delta)n!}{(m_1)_n \ldots (m_q)_n \Gamma(n + \delta)n!}$$

$$\times H_{\ell + 1, d + 1} \left[ \lambda_1 \left( a, A, c, 1 - (\vartheta - \vartheta + vn - 1)/y, \xi, (-\sigma, v) \right) \right] (b_d, B_d) \right]$$

$$\times \int_0^\infty t^{\mu + \vartheta + vn - 1} K_\mathcal{E}(rt) dt,$$

we get

$$= \sum_{n=0}^\infty \frac{(l_1)_n \ldots (l_p)_n (\omega)_n \Gamma(n + \delta)n!}{(m_1)_n \ldots (m_q)_n \Gamma(n + \delta)n!}$$

$$\times H_{\ell + 1, d + 1} \left[ \lambda_1 \left( a, A, c, 1 - (\vartheta - \vartheta + vn - 1)/y, \xi, (-\sigma, v) \right) \right] (b_d, B_d) \right]$$

$$\times 2^{\mu + \vartheta + vn - 3} \rho(1 - u - \vartheta - vn) \frac{\Gamma((\mu + \vartheta + vn - 1) \pm u)}{2} . \tag{4.7}$$
simplification on right-hand side of (4.7), we obtain at the result (4.6).

Theorem 8. Follow stated Theorem 2 for conditions on parameters, along with 
\(\Re(r)>0\); \(\Re((\mu - \vartheta - vn) \pm u)>0\). Then the subsequent result true:

\[
\int_0^\infty t^{\nu-1}K_\nu(rt) \left\{ K_{\nu,r} \left( t^{-\frac{\nu}{\nu-\nu}S^{0,\beta,\vartheta,\alpha+1}_x(t^{-\nu})} \right) \right\} dt
\]

\[= 2^{\nu-2}r^{\nu-\nu} \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_\nu)_n (\omega)_n}{(m_1)_n \cdots (m_\nu)_n \Gamma(n\rho + \delta)n!} \Gamma \left( \frac{\mu - \vartheta - vn + u}{2} \right)
\]

\[\times \left( \frac{r}{2} \right)^n \Gamma^{\nu+n+2} \left[ A \left( \alpha_c, \alpha_c, (1 - (\vartheta + vn)/y, \xi, (-\sigma, \nu)) \right), \left( \sigma - (\vartheta + vn)/y, \xi + \nu, (b_d, \nu) \right) \right]. \tag{4.8}\]

Proof. According to the solution of Theorem 7, we get the result (4.8).

5. Properties of integral operators

Here, we established some properties of the operators concerning with Theorem 1 and Theorem 2. These properties are given in the compositions of power function.

Theorem 9: Given conditions in Theorem 1 with 
\(\Re(\beta + \vartheta)>0\). Then the subsequent result holds true:

\[
x^\alpha R_{k,r}^{\alpha,\beta,\vartheta,\alpha+1}_x \left[ S^{0,\beta,\vartheta,\alpha+1}_x(t^{-\nu}) \right] (x) = R_{k,r}^{\alpha,\beta,\vartheta,\alpha+1}_x \left[ S^{0,\beta,\vartheta,\alpha+1}_x(t^{-\nu}) \right] (x). \tag{5.1}\]

Proof. From (2.1), the left hand side of Eq. (5.1), we have

\[
x^\alpha R_{k,r}^{\alpha,\beta,\vartheta,\alpha+1}_x \left[ S^{0,\beta,\vartheta,\alpha+1}_x(t^{-\nu}) \right] (x) = \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_\nu)_n (\omega)_n}{(m_1)_n \cdots (m_\nu)_n \Gamma(n\rho + \delta)n!} \Gamma^{\nu+n+1}
\]

\[\times H^{\nu+n+2} \left[ A \left( \alpha_c, \alpha_c, (1 - (\vartheta + vn - 1)/y, \xi, (-\sigma, \nu)) \right), \left( \sigma - (\vartheta + vn)/y, \xi + \nu, (b_d, \nu) \right) \right]. \tag{5.2}\]

again by (2.1), the right hand of (5.1) follows as

\[
R_{k,r}^{\alpha,\beta,\vartheta,\alpha+1}_x \left[ S^{0,\beta,\vartheta,\alpha+1}_x(t^{-\nu}) \right] (x) = \sum_{n=0}^{\infty} \frac{(l_1)_n \cdots (l_\nu)_n (\omega)_n}{(m_1)_n \cdots (m_\nu)_n \Gamma(n\rho + \delta)n!} \Gamma^{\nu+n+1}
\]

\[\times H^{\nu+n+2} \left[ A \left( \alpha_c, \alpha_c, (1 - (\vartheta + vn - 1)/y, \xi, (-\sigma, \nu)) \right), \left( \sigma - (\vartheta + vn)/y, \xi + \nu, (b_d, \nu) \right) \right]. \tag{5.3}\]

It seems that Theorem (5.1) readily follow due to (5.2) and (5.3).

Theorem 10: Given conditions in Theorem 2 with \(\Re(\beta + \vartheta)>0\). Then the subsequent result holds true:

\[
x^\alpha K_{k,r}^{\alpha,\beta,\vartheta,\alpha+1}_x \left[ S^{0,\beta,\vartheta,\alpha+1}_x(t^{-\nu}) \right] (x) = K_{k,r}^{\alpha,\beta,\vartheta,\alpha+1}_x \left[ S^{0,\beta,\vartheta,\alpha+1}_x(t^{-\nu}) \right] (x). \tag{5.4}\]

Proof. From (2.2), the left hand side of eq. (5.4), we have
again by (2.2), the right hand of (5.4) follows as

\[
x^{\alpha-k}K_{\alpha,k}[\Gamma_{\alpha}(1-\nu)y][x]\left(\frac{z}{y}\right) = \sum_{m=0}^{\infty} \frac{\left(1+\nu\right)_{m}}{\left(\alpha+\nu\right)_{m}} K_{\alpha,k}\left(\Gamma_{\alpha}(1-\nu)y\left[\frac{z}{y}\right]\right)
\]

\[
\times \frac{\left(\alpha\right)_{m} \alpha \left(\alpha\right)_{m} \left(\alpha\right)_{m} \left(\alpha\right)_{m} \left(\alpha\right)_{m}}{\left(\alpha\right)_{m} \left(\alpha\right)_{m} \left(\alpha\right)_{m} \left(\alpha\right)_{m} \left(\alpha\right)_{m}} \frac{\left(\alpha\right)_{m} \alpha \left(\alpha\right)_{m} \left(\alpha\right)_{m} \left(\alpha\right)_{m} \left(\alpha\right)_{m}}{\left(\alpha\right)_{m} \left(\alpha\right)_{m} \left(\alpha\right)_{m} \left(\alpha\right)_{m} \left(\alpha\right)_{m}} \left(\alpha\right)_{m} \alpha \left(\alpha\right)_{m} \left(\alpha\right)_{m} \left(\alpha\right)_{m} \left(\alpha\right)_{m}
\]

It seems that Theorem (5.4) readily follow due to (5.5) and (5.6).

6. Conclusions

In the present paper, we have studied the properties of S-function under the extension of generalized fractional integral operators given by Saxena and Kumbhat and developed some new images. The results obtained here involve special functions like \( k \)-Mittag-Leffler function, K-function and M-series, due to their general nature and usefulness in the theory of integral operators and relevant part of computational mathematics. They may have an important place in the literature (see, e.g., Amsalu and Suthar (2018), Purohit et al. (2011), Purohit et al. (2010), Saxena et al. (2009), Suthar and Habenom (2017), and Suthar et al. (2018)). Also, the special functions involved here can be reduced in simpler functions, those have a variety of applications in different domains of science and technology and can be observed as special cases, those we have not mentioned here explicitly.

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