Correlation functions of the integrable spin-s chain

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Abstract

We study the correlation functions of $su(2)$ invariant spin-$s$ chains in the thermodynamic limit. We derive nonlinear integral equations for an auxiliary correlation function $\omega$ for any spin $s$ and finite temperature $T$. For the spin-3/2 chain for arbitrary temperature and zero magnetic field we obtain algebraic expressions for the reduced density matrix of two-sites. In the zero temperature limit, the density matrix elements are evaluated analytically and appear to be given in terms of Riemann’s zeta function values of even and odd arguments.

Keywords: spin chains, correlation function, integrable models

1. Introduction

The static correlation functions of quantum integrable models, most notably the spin-1/2 Heisenberg model, have been extensively studied over the years. The first results were presented in terms of multiple integrals [1–4]. In the course of an explicit evaluation of these integrals a factorization in terms of sums over products of single integrals was found [5]. These results were extended to finite temperature in the thermodynamic limit and to zero temperature and finite chains [6, 7]. In this context, a hidden Grassmann structure was identified [8], which made possible to prove the complete factorization of the correlation functions under general conditions [9].

The multiple integral representation of the spin-1/2 chain was successfully used to obtain, from first principles, the long distance asymptotic behavior of the correlation functions, which confirmed at leading order the conformal field theory predictions [10]. Moreover, explicit values for the short range correlations were systematically obtained by direct

$^*$ Dedicated to Professor Rodney Baxter on the occasion of his 75th birthday.

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evaluation of multiple integrals [5] and later by solving recurrently functional equations obtained from the quantum Knizhnik–Zamolodchikov (qKZ) equation [11–13]. These correlation functions are remarkably given in terms of combinations of the zeta function with odd integer arguments [5].

Nevertheless, one still lacks a better understanding of the correlation properties of models based on higher rank algebras and higher-spin representations of the su(2) algebra. For the latter case, the high-spin generalizations, there exist multiple integral representations of the correlation functions of the ground state [14–17] and for finite temperature [18]. However, the expressions are very intricate for explicit evaluations. On the other hand, the explicit computation of the short range correlations of the integrable spin-1 chain [19, 20] has been done relatively recently [21] by means of the solution of discrete functional equations [22]. Surprisingly, the correlation functions for two-sites and three-sites are given in terms of powers of $\pi^2$ or alternatively in terms of the zeta function of only even integer arguments.

In this work, we are interested in applying the approach of discrete functional equations to evaluate the first non-trivial correlation function, for two-sites, in case of the integrable spin-3/2 chain. We use a systematic approach to tackle the high-spin chains in analogy to [21]. This approach uses an auxiliary spin-1/2 nearest neighbor correlation function $\omega$ and fusion principles. In making it systematically work for spin $s > 1$ we found an important auxiliary condition of the function $\omega$ which did not appear yet for $s = 1$. Our procedure yields the solution for the two-sites correlation function at finite and zero temperature, proving that the approach is viable for obtaining explicit results. Amazingly, our explicit results are in contrast to a naive conjecture based on the spin-1/2 and 1 results, that the correlations for half odd integer (integer) spin chains may be given just in terms of zeta function values with odd (even) integer arguments. In fact, our results already for the two-sites of the spin-3/2 chain are a combination of $\pi^2$ and $\log 2$, which can be seen as zeta function values of even and odd arguments.

This paper is organized as follows. In section 2, we outline the integrable Hamiltonians and the associated integrable structure. In section 3, we introduce the physical density matrix and its functional equations. The physical properties of the model are given in terms of nonlinear integral equations in section 4. In section 5, we present the two-site density matrix for the spin-$s$ chains for $s$ up to 3/2. Our conclusions are given in section 6.

2. The model

The Hamiltonians of the integrable spin-$s$ generalization of the Heisenberg model for $s = 1/2$, 1 and 3/2 are given by

$$\mathcal{H}^{(1)} = J \sum_{i=1}^{L} \left[ \frac{1}{4} \mathbf{S}_i \cdot \mathbf{S}_{i+1} \right],$$

$$\mathcal{H}^{(2)} = \frac{J}{4} \sum_{i=1}^{L} \left[ 3 + \mathbf{S}_i \cdot \mathbf{S}_{i+1} - (\mathbf{S}_i \cdot \mathbf{S}_{i+1})^2 \right],$$

$$\mathcal{H}^{(3)} = \frac{J}{532} \sum_{i=1}^{L} \left[ 234 - 27 \mathbf{S}_i \cdot \mathbf{S}_{i+1} + 8(\mathbf{S}_i \cdot \mathbf{S}_{i+1})^2 + 16(\mathbf{S}_i \cdot \mathbf{S}_{i+1})^3 \right],$$

(1)

where $L$ is the number of sites and $\mathbf{S}_i = (\mathbf{S}_i^x, \mathbf{S}_i^y, \mathbf{S}_i^z)$ are the su(2) generators and $J$ is the exchange constant. Here, it is worth to recall that the Hamiltonian $\mathcal{H}^{(2)}$ is obtained from the logarithmic derivative of the row-to-row transfer matrix $t^{(2)}(\lambda) = \text{Tr}_\lambda [\rho^{(2,2)}_{\lambda L}(\lambda) \cdots \rho^{(2,2)}_{\lambda 1}(\lambda)]$, which results into
\[ \mathcal{H}^{[2s]} = \frac{\text{i} J}{2} \frac{d}{d\lambda} \log (R^{[2s]}(\lambda)) \bigg|_{\lambda = 0} = J \sum_{i=1}^{L} h_{i,i+1}, \quad h_{i,i+1} = \frac{i}{\lambda} \frac{d}{d\lambda} R_{i,i+1}^{[2s]}(\lambda) \bigg|_{\lambda = 0}, \]  
(2)

where \( R_{j_2}^{[2s]} = P_{j_2}^{[2s]} R_{j_2}^{[2s]}(\lambda), \) \( P_{j_2}^{[2s]} \) is the permutation operator and \( L \) is the number of sites, \( \mathcal{A} \) is the auxiliary space of dimension \( 2j + 1 \), where \( j = 1/2, 1, \ldots \). The \( R \)-matrix \( R_{j_2}^{[2s]}(\lambda) \) is a solution of the Yang–Baxter equation

\[ R_{j_2}^{[2s]}(\lambda - \mu) R_{j_3}^{[2s]}(\mu) R_{j_3}^{[2s]}(\lambda) = R_{j_2}^{[2s]}(\mu) R_{j_3}^{[2s]}(\mu) R_{j_2}^{[2s]}(\lambda - \mu), \]  
(3)

which has the properties of regularity and unitarity as given below

\[ R_{j_2}^{[2s]}(0) = P_{j_2}^{[2s]}, \]  
(4)

\[ R_{j_2}^{[2s]}(\lambda) R_{j_2}^{[2s]}(-\lambda) = \text{Id}. \]  
(5)

The operator \( R_{j_2}^{[2s]}(\lambda) \) is a rational solution of the Yang–Baxter equation and is obtained, for instance, by the fusion process \([23]\).

\[ R_{j_2}^{[2s]}(\lambda) = \sum_{l=|s_1-s_2|}^{s_1+s_2} f_l(\lambda) \hat{P}_l, \]  
(6)

where \( f_l(\lambda) = \prod_{k=1}^{s_1+s_2} \left( \frac{\lambda - ik}{\lambda + ik} \right) \) and \( \hat{P}_l \) is the usual \( \text{su}(2)_l \) projector

\[ \hat{P}_l = \prod_{i=1}^{s_1+s_2} \left( \frac{\bar{s}_i - s_k}{x_i - x_k} \right), \]  
(7)

with \( x_k = \frac{1}{2} [k(k+1) - s_1(s_1 + 1) - s_2(s_2 + 1)] \).

More generally and compactly, the Hamiltonians can be written as

\[ \mathcal{H}^{[2s]} = \frac{J}{2} \sum_{i=1}^{L} Q_{2s}(\bar{s}_i, \bar{s}_i + 1), \]  
(8)

where

\[ Q_{2s}(x) = \sum_{i=0}^{2s} \left( 2\Psi(i+1) - \Psi(1) - \Psi(2s+1) \right) \prod_{k=1}^{2s} \frac{x - x_k}{x_i - x_k}, \]  
(9)

and \( \Psi(x) \) is the digamma function.

### 3. Density matrix

The formalism to calculate the thermal correlation functions of quantum integrable Hamiltonians derived from the row-to-row transfer matrix was established in \([24]\). This formalism can be understood as a lattice path integral formulation and the evaluation of physical properties by use of the quantum transfer matrix (QTM) \([25]\). The quantum chain at finite temperature is mapped by an integrable version of Trotter–Suzuki relations onto a two-dimensional staggered vertex model on a \( L \times N \) lattice. The statistical operator \( \rho_{NL} \) of the finite temperature quantum chain is then given in terms of row-to-row transfer matrices

\[ \rho_{NL} = e^{-\beta H^{(\alpha)}} = \lim_{N \to \infty} \rho_{N,L} = \left[ t^{(2s)}(-\beta/N) T^{(2s)}(-\beta/N) \right]^{N/2}, \]  
(10)

where \( t^{(2s)}(\lambda) \) and \( T^{(2s)}(\lambda) \) are the usual and the adjoint transfer matrices. For more details see for instance \([24]\). For convenience, the \( \rho_{N,L} \) is rewritten along the quantum direction in terms of the column-to-column monodromy matrix \( T^{(2s)}(\lambda) \),
\[
\rho_{N,L} = \text{Tr}_{N} \cdots \text{Tr}_{1} [T_{L}^{[2]}(0) \cdots T_{L}^{[2]}(0)],
\]
where the trace over spaces 1, \ldots, N takes account of periodic boundary conditions in horizontal direction and
\[
T_{L}^{[2]}(\lambda) = R_{L}^{[2,2]}(\lambda + iu) R_{L}^{[2,2]}(\lambda - iu) \cdots R_{L}^{[2,2]}(\lambda - iu), \quad R_{L}^{[2,2]}(\lambda) = (\lambda + iu) R_{L}^{[2,2]}(\lambda),
\]
where \( u = -J\beta/N \) and at every second matrix the \( t_k \) denotes transposition in the \( k\)th space (which actually is the first space of the concerned matrix).

Taking the trace over the quantum space \( i \in \{1, \ldots, L\} \) realizes periodic boundary conditions in vertical direction and results in
\[
T_{N}^{[2]}(\lambda) = \text{Tr}_{N}[T_{L}^{[2]}(\lambda)],
\]
which is called the QTM. Note that it acts in the auxiliary space of states along the vertical direction. Thanks to the Yang–Baxter equation (3), this transfer matrix forms a family of commuting operators \( [T_{L}^{[2]}(\lambda), T_{L}^{[2]}(\mu)] = 0 \), which implies the QTM eigenvectors \( |\Phi_{\mu}\rangle \) do not depend on the spectral parameter. This matrix can be diagonalized by Bethe ansatz techniques [26, 27] and quite generally it has a non-degenerate largest eigenvalue \( \Lambda_{0}^{[2]}(0) \) split from the rest of the spectrum by a gap [25, 28].

The thermodynamic properties of the quantum Hamiltonian are obtained via the partition function \( Z_{L} = \text{Tr}_{N} \cdots \text{Tr}_{1} [e^{-\beta H}] = \lim_{N \to \infty} \text{Tr}_{N} L[\rho_{N,L}] \). As \( \text{Tr}_{N} L[\rho_{N,L}] = \text{Tr}_{N} \cdots \text{Tr}_{1} [T_{L}^{[2]}(0)]^{2} \), the free energy in the thermodynamic limit is given by the leading eigenvalue \( \Lambda_{0}^{[2]} \) of the QTM \( T_{L}^{[2]} \)
\[
f^{[2]} = -\frac{1}{\beta N} \lim_{N \to \infty} \frac{1}{L} \log Z_{L} = -\frac{1}{\beta N} \lim_{N \to \infty} \log \Lambda_{0}^{[2]}(0). \tag{14}
\]
The leading eigenvalue, especially in the Trotter limit \( N \to \infty \), is obtained [26, 27] in terms of the solution of a set of nonlinear integral equations (see section 4).

The set of all thermal correlation functions with finite separation of the local operators is encoded in the reduced density matrix on a finite chain segment of some length \( n \),
\[
D_{[1,n]}^{[2]} = \lim_{N \to \infty} \frac{\text{Tr}_{N} \cdots \text{Tr}_{n+1} \cdots \text{Tr}_{L} [T_{L}^{[2]}(0) \cdots T_{L}^{[2]}(0)]}{\text{Tr}_{N} \cdots \text{Tr}_{n+1} \cdots \text{Tr}_{L}[\rho_{N,L}]}, \tag{15}
\]
Introducing in (15) a complete set of eigenstates of the quantum transfer matrix we obtain [24],
\[
D_{[1,n]}^{[2]} = \lim_{N \to \infty} \frac{\text{Tr}_{N} \cdots \text{Tr}_{n+1} \cdots \text{Tr}_{L} [T_{L}^{[2]}(0) \cdots T_{L}^{[2]}(0)]}{\text{Tr}_{N} \cdots \text{Tr}_{n+1} \cdots \text{Tr}_{L}[\rho_{N,L}]},
\]
\[
= \lim_{N \to \infty} \frac{\text{Tr}_{n} \cdots \text{Tr}_{L} [T_{L}^{[2]}(0) \cdots T_{L}^{[2]}(0)] \langle \Phi_{n} | T_{L}^{[2]}(0) \cdots T_{L}^{[2]}(0) | \Phi_{n} \rangle}{\langle \Phi_{n} | T_{L}^{[2]}(0) \cdots T_{L}^{[2]}(0) | \Phi_{n} \rangle}, \tag{16}
\]
where in the end the thermodynamic limit was taken by dropping the sub-dominant states with \( m > 0 \). This implies that all the static correlation functions at finite temperature are determined by the dominant eigenvector \( |\Phi_{0}\rangle \). Note the difference in typesetting, \( D \) and \( D \) for the density matrix for finite Trotter number \( N \) and in the limit \( N \to \infty \).
For technical reasons, it is convenient to introduce mutually distinct spectral parameters \((\xi_1, \ldots, \xi_n)\) along the vertical lines corresponding to the quantum states \(1, \ldots, n\) in the expression for the reduced density matrix, such that

\[
D_n^{(2)}(\xi_1, \ldots, \xi_n) = \langle \Phi_0 | T_n^{(2)}(\xi_1) \cdots T_n^{(2)}(\xi_n) | \Phi_0 \rangle.
\]

which is called the inhomogeneous density matrix with finite Trotter number. This object enjoys interesting relations as a function of the variables \((\xi_1, \ldots, \xi_n)\). These relations will be used for the explicit computations. The final physically interesting result is obtained by taking the homogeneous limit (and \(N \to \infty\))

\[
D_{[1,n]}^{(2)} = \lim_{N \to \infty} \lim_{\xi_n \to 0} D_n^{(2)}(\xi_1, \ldots, \xi_n).
\]

### 3.1. Discrete functional equations

The general framework to obtain discrete functional equations for the reduced density matrix of integrable models on semi-infinite \((N \times \infty)\) lattices was established in [21, 22].

The general idea consists in starting with a slightly more general density matrix, where different spectral parameters \(u_i\) are introduced on the \(N\) horizontal lines. The derivation of the discrete functional equations relies on the standard unitarity, regularity and crossing symmetry of the \(R\)-matrix, which allows to perform the lattice manipulations described in [22]. The lattice surgery is allowed when one of the spectral parameters \(\xi_1, \ldots, \xi_n\), let us say the last \(\xi_n\), is set identical to any of the spectral parameters along the horizontal lines. This results in a mapping of \(D_n^{(2)}(\xi_1, \ldots, \xi_n)\) to \(D_n^{(2)}(\xi_1, \ldots, \xi_n - 2i)\) by a linear map \(A\) acting on the space of reduced density matrices. The resulting equations are the discrete qKZ functional equations

\[
D_n^{(2)}(\xi_1, \ldots, \xi_n - 2i) = A_n^{(2)}(\xi_1, \ldots, \xi_n)[D_n^{(2)}(\xi_1, \ldots, \xi_n)],
\]

where the linear operator \(A_n^{(2)}\) can be written as [22],

\[
A_n^{(2)}(\xi_1, \ldots, \xi_n)[B] := \text{Tr}_n[R_{1,2}^{(2,2)}(-\xi_1, -\xi_n) \cdots R_{n-1,2}^{(2,2)}(-\xi_{n-1}, -\xi_n) | P_n\rangle \langle P_n | \otimes B]_{n+1}.
\]

Here we use the short-hand \(\xi_{ij} = \xi_i - \xi_j\) and \(P_n = (2s + 1)P_0\) is a (not normalized) projector onto the two-site singlet and the partial trace is taken in the \(n\)th vertical space. The graphical depiction of the functional equation (19) is given in figure 1. Moreover, we have the asymptotic condition

\[
\lim_{\xi_n \to \infty} D_n^{(2)}(\xi_1, \ldots, \xi_n) = D_n^{(2)}(|\xi_1, \ldots, \xi_{n-1}) \otimes \text{Id}.
\]

Besides, the under-determinacy of the functional equations for the density matrix as a function of a single variable is resolved by exploiting the full dependence on all variables, fusion and using the intertwining symmetry relations

\[
R_{k,k+1}^{(2,2)}(\xi_k - \xi_{k+1})D_n^{(2)}(\xi_1, \ldots, \xi_k, \xi_{k+1}, \ldots, \xi_n)[R_{k,k+1}^{(2,2)}(\xi_k - \xi_{k+1})]^{-1} = D_n^{(2)}(\xi_1, \ldots, \xi_{k+1}, \xi_k, \ldots, \xi_n).
\]

We avoid the explicit application of the intertwining symmetry by an approach already used in [21]. By viewing each of the spin-\(s\) horizontal lines as a result of \(2s\) many spin-1/2 lines in a suitable limit, it was argued in [21] that the famous factorization expressions of arbitrary
spin-1/2 correlators in terms of nearest-neighbor spin-1/2 correlators $\omega$ hold literally as in [12, 13, 29]. Then in principle, by taking the fusion of $2s$ many vertical spin-1/2 lines to produce vertical spin-$s$ lines, the results for the spin-$s$ correlators can be derived. We do not follow this procedure literally. Instead of using the explicit coefficients of multilinear expressions in $\omega$ as worked out for the spin-1/2 case by [12, 13], we use their principle of the analytical structure of the coefficients and apply this directly to the high spin case.

In the next section, the function $\omega$ will be calculated for arbitrary spin $s$ and $T > 0$. The computation of the coefficients of the multilinear expressions in $\omega$ for the spin-$s$ case will be summarized in section 5.

4. Bethe ansatz and nonlinear integral equations

In order to evaluate the free energy (14) and the density matrix (17) knowledge of the leading eigenvalue $\Lambda^{(2i)}_0(\lambda)$ and the associated eigenvector $|\Phi_0\rangle$ is required.

As we are going to use fusion relations for the QTM, it is convenient to consider additional staggered monodromy matrices with arbitrary spin-$j$ in the vertical space, $N$ spin-$s$ spaces and additional two spin-1/2 spaces with spectral parameter $\mu$ and $\mu + \delta$ along the horizontal direction as given by

$$
\mathcal{T}_i^{[2j]}(\lambda) = R_{iN}^{[2j]}(\lambda - \mu)R_{N+1,i}^{[1,2j]}(\mu + \delta - \lambda) 
\cdot R_{i,N}^{[2j]}(\lambda + in)R_{N-1,i}^{[2,2j]}(in - \lambda) \cdots R_{i,2}^{[2j]}(\lambda + in)R_{l,1}^{[2,2j]}(in - \lambda).
$$

The two spin-1/2 spaces will be used for generating the auxiliary spin-1/2 nearest neighbor correlator $\omega$ by means of the derivative with respect to the parameter $\delta$ which later will be sent to zero, see [21, 22] and the construction below.

The trace of the monodromy matrix (23) results in the generalized QTM

$$
T_\delta^{[2j]}(\lambda) = Tr_i[\mathcal{T}_i^{[2j]}(\lambda)].
$$

Please note that the transfer matrix without the first two $R$-matrices on the rhs of (23) is related to $T_\delta^{[2j]}(\lambda)$ in a simple way. For instance, the largest eigenvalues of these two transfer matrices are identical up to a trivial factor and the corresponding nonlinear integral equations are strictly identical. As from now on we are going to use $T_\delta^{[2j]}(\lambda)$ only, and for simplicity of notation, we refer to $T_\delta^{[2j]}(\lambda)$ as just $T^{[2j]}(\lambda)$. 

![Figure 1. Graphical illustration of the functional equation (19).](image-url)
The transfer matrix (24) is known to satisfy the fusion hierarchy, which reads [23, 27]

\[
T^{[2]}(\lambda) T^{[1]}(\lambda + i(2j + 1)) = T^{[2]+1}(\lambda + i) + \chi(\lambda + i(2j)) T^{[2]-1}(\lambda - i),
\]

T^{[0]}(\lambda) = \text{Id}, \quad j = 1/2, 1, 3/2, ...,

(25)

where \( \chi(\lambda) = \frac{\phi_0(\lambda + i(2j)) \phi_0(\lambda - i(2j)) \varphi(\lambda + i) \varphi(\lambda - i)}{\phi_0(\lambda - i(2j)) \phi_0(\lambda + i(2j)) \varphi(\lambda - i) \varphi(\lambda + i)} \) is the quantum determinant and \( \phi_0(\lambda) = (\lambda \pm i)^{N/2} \) and \( \varphi(\lambda) = \lambda - \mu, \quad \varphi(\lambda) = \lambda - \mu - \delta \). The fusion hierarchy is important to the analysis of the spectra and the derivation of auxiliary functional relations.

The eigenvalues \( \Lambda^{[2]}(\lambda) \) associated to \( T^{[2]}(\lambda) \) also satisfy the functional relations (25), due to the commutativity property among transfer matrices. Therefore, we can obtain the eigenvalues at any fusion level in terms of the first level eigenvalue through the iteration of the relations (25) [27]. The eigenvalues of the QTM (24) are given in the form

\[
\Lambda^{[2]}(\lambda) = \sum_{m=1}^{2j+1} \lambda^{(j)}_m(\lambda),
\]

(26)

\[
\lambda^{(j)}_m(\lambda) = e^{2i\hbar(j+1-m)} t^{(j)}_{+m}(\lambda) r^{(j)}_{-m}(\lambda) \frac{q\left(\lambda + 2i\left(\frac{1}{2} + j\right)\right) q\left(\lambda - \frac{1}{2} + j\right)}{q\left(\lambda + 2i\left(\frac{3}{2} + j - m\right)\right) q\left(\lambda + 2i\left(\frac{1}{2} + j - m\right)\right)},
\]

(27)

where \( t^{(j)}_{+m}(\lambda) = \frac{\psi\left(\lambda + 2i\left(j + \frac{1}{2} - m\right)\right)}{\psi\left(\lambda + 2i\left(j + \frac{1}{2}\right)\right)} \prod_{l=0}^{j} \frac{\phi_0(\lambda - 2i\left(l + \frac{1}{2}\right))}{\phi_0(\lambda + 2i\left(l + \frac{1}{2}\right))} \) and \( q(\lambda) = \prod_{l=1}^{n} (\lambda - \lambda_l) \). The corresponding Bethe ansatz equations can be written as

\[
e^{2i\hbar} \psi_+\left(\lambda_j + 2i\left(s + \frac{1}{2}\right)\right) \psi_+\left(\lambda_j - 2i\left(s - \frac{1}{2}\right)\right) = \prod_{j=1}^{n} \frac{\lambda_j - \lambda_j + 2i}{\lambda_j - \lambda_j - 2i}.
\]

(28)

These relations hold for all states. However, for describing the physics of the model in the thermodynamic limit only the leading eigenstate is required. The corresponding sector has quantum number \( n = sN + 1 \) (remember we are dealing with \( N \) copies of spin \( s \) spaces and two spin 1/2 spaces). However, one needs to take the Trotter limit \( N \to \infty \) and this cannot be done by a straightforward numerical analysis of the Bethe ansatz equations. The standard approach allowing for taking the Trotter limit employs suitable algebraic and analytical properties, in particular auxiliary functions are used that originate from the analysis of the fusion hierarchy and from its closure at finite level. Also analytical properties regarding the location of zeros and poles are exploited in order to encode the information of the Bethe ansatz roots in a set of nonlinear integral equations. This has been done for several models and especially for the case of high-spin chains [21, 27]. Therefore, we present here only the final set of nonlinear integral equations adjusted to the generalized quantum transfer matrix (24) (see appendix A for details).
\[
\begin{pmatrix}
\log y^{[1]}(\lambda) \\
\log y^{[2]}(\lambda) \\
\vdots \\
\log y^{[2s-1]}(\lambda) \\
\log b(\lambda) \\
\log \bar{b}(\lambda)
\end{pmatrix} = 
\begin{pmatrix}
d_1(\lambda) \\
0 \\
\vdots \\
0 \\
d(\lambda) + \beta \frac{\pi}{2} \\
d(\lambda) - \beta \frac{\pi}{2}
\end{pmatrix} + \widehat{\mathcal{K}} \ast 
\begin{pmatrix}
\log y^{[1]}(\lambda) \\
\log y^{[2]}(\lambda) \\
\vdots \\
\log y^{[2s-1]}(\lambda) \\
\log b(\lambda) \\
\log \bar{b}(\lambda)
\end{pmatrix},
\]  
(29)

where the symbol \( \ast \) denotes the convolution \( f \ast g(\lambda) = \int_{-\infty}^{\infty} f(\lambda - \mu)g(\mu) d\mu \) and

\[
d_1(\lambda) = \log \left[ \frac{\tanh \left( \frac{\pi}{4} (\lambda - \mu + i) \right)}{\tanh \left( \frac{\pi}{4} (\lambda - \mu - \delta + i) \right)} \right] \approx -i \frac{\pi}{2 \cosh \left( \frac{\pi}{4} (\lambda - \mu) \right)} \cdot \delta, \quad (30)
\]

\[
d(\lambda) = -\beta \frac{\pi}{2 \cosh \left( \frac{\pi}{4} (\lambda) \right)}. \quad (31)
\]

The kernel matrix is given explicitly by

\[
\widehat{\mathcal{K}}(\lambda) = \begin{pmatrix}
0 & \mathcal{K}(\lambda) & 0 & \cdots & 0 & 0 & 0 \\
\mathcal{K}(\lambda) & 0 & \mathcal{K}(\lambda) & \cdots & \vdots & \vdots & \vdots \\
0 & \mathcal{K}(\lambda) & 0 & \cdots & 0 & \mathcal{K}(\lambda) & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & \mathcal{K}(\lambda) & 0 & \mathcal{K}(\lambda) \\
0 & 0 & \cdots & 0 & 0 & \mathcal{K}(\lambda) & -\mathcal{F}(\lambda - 2i) \\
0 & 0 & \cdots & 0 & 0 & \mathcal{K}(\lambda) & \mathcal{F}(\lambda)
\end{pmatrix}, \quad (32)
\]

where \( \mathcal{K}(\lambda) = \frac{1}{4 \cosh(\pi \lambda/2)} \) and

\[
\mathcal{F}(\lambda) = \int_{-\infty}^{\infty} \frac{e^{-|k| - ik\lambda}}{2 \cosh k} \frac{dk}{2\pi} = \frac{1}{8\pi} \left\{ \psi \left( -\frac{i}{4} \lambda \right) + \psi \left( \frac{i}{4} \lambda \right) - \psi \left( \frac{1}{2} - \frac{i}{4} \lambda \right) - \psi \left( \frac{1}{2} + \frac{i}{4} \lambda \right) \right\}. \quad (33)
\]

We give the main steps towards the derivation of these equations in appendix A.

Next we are interested in

\[
\frac{\partial}{\partial \delta} \log \Lambda^{[t]}(\lambda) \bigg|_{\delta = 0} \text{ for which we derive an integral expression in terms of functions satisfying a set of linear integral equations. These functions are the derivatives of } \log y^{[k]}(\lambda), \log b(\lambda), \log \bar{b}(\lambda) \text{ with respect to } \delta, \text{ where } k = 2j = 1, 2, \ldots, 2s - 1. \text{ We introduce the functions} \quad (34)
\]

\[
G_y^{[k]}(\lambda, \mu) = \frac{\partial}{\partial \delta} \log y^{[k]}(\lambda) \bigg|_{\delta = 0}, \quad G_b(\lambda, \mu) = \frac{\partial}{\partial \delta} \log b(\lambda) \bigg|_{\delta = 0} = [1 + b^{(-1)}(\lambda)] |_{\delta = 0} G_b(\lambda, \mu), \quad \text{and} \\
G_{\bar{b}}(\lambda, \mu) = \frac{\partial}{\partial \delta} \log \bar{b}(\lambda) \bigg|_{\delta = 0} = [1 + \bar{b}^{(-1)}(\lambda)] |_{\delta = 0} G_{\bar{b}}(\lambda, \mu). \quad (34)
\]
These satisfy the set of linear integral equations

\[
\begin{pmatrix}
[1 + (y^{11}(\lambda))^{-1}]G^{[1]}(\lambda, \mu) \\
[1 + (y^{21}(\lambda))^{-1}]G^{[2]}(\lambda, \mu) \\
\vdots \\
[1 + (y^{2n-11}(\lambda))^{-1}]G^{[2n-1]}(\lambda, \mu) \\
[1 + (b(\lambda))^{-1}]G_b(\lambda, \mu) \\
[1 + (\bar{b}(\lambda))^{-1}]G_{\bar{b}}(\lambda, \mu)
\end{pmatrix}
= \begin{pmatrix}
\frac{\bar{\omega}}{\cosh\left(\frac{\lambda}{2}(\mu - \lambda)\right)} \\
0 \\
\vdots \\
0 \\
0
\end{pmatrix}
+ \tilde{K}^* \begin{pmatrix}
G^{[1]}(\lambda, \mu) \\
G^{[2]}(\lambda, \mu) \\
\vdots \\
G^{[2n-1]}(\lambda, \mu) \\
G_b(\lambda, \mu) \\
G_{\bar{b}}(\lambda, \mu)
\end{pmatrix}.
\tag{35}
\]

Next we obtain an explicit expression of the eigenvalue’s derivative in terms of the \(G\) functions

\[
\frac{\partial}{\partial \delta} \log \lambda^{[1]}(\lambda) \bigg|_{\delta=0} = \int_{-\infty}^{\infty} dx \frac{1}{4 \cosh\left(\frac{x}{2}(\lambda - \chi)\right)} \left\{ G^{[1]}(\chi, \mu) + \frac{2i}{(\chi - \mu)^2 + 1} \right\},
\]

\[
= \int_{-\infty}^{\infty} dx \frac{1}{4 \cosh\left(\frac{x}{2}(\lambda - \chi)\right)} G^{[1]}(\chi, \mu) + 2\pi i \mathcal{F}(\lambda - \mu),
\tag{36}
\]

for \(|\lambda(\lambda)| < 1\).

For any high-spin chain \([21]\), an auxiliary density matrix with two neighboring spin-1/2 quantum spaces (on vertical lines in the statistical mechanics language) in a sea of \(L (\to \infty)\) many spin-\(s\) objects is defined. The nearest-neighbor spin-1/2 correlators are given by a function \(\omega\), see \((B.9)\). For details see \([21, 22]\) where a scheme was introduced to derive the two-site function \(\omega\) from a generating function like

\[
\omega^{[2]}(\lambda, \mu) = \frac{1}{2} - \frac{(\lambda - \mu)^2 + 4}{2i} \frac{\partial}{\partial \delta} \log \lambda^{[1]}(\lambda) \bigg|_{\delta=0}.
\tag{37}
\]

With our result \((36)\) we have explicitly

\[
\omega^{[2]}(\lambda, \mu) = \frac{1}{2} - \frac{(\lambda - \mu)^2 + 4}{2i} \left\{ \int_{-\infty}^{\infty} dx \frac{G^{[1]}(\chi, \mu)}{4 \cosh\left(\frac{x}{2}(\lambda - \chi)\right)} + 2\pi i \mathcal{F}(\lambda - \mu) \right\}.
\tag{38}
\]

### 5. Computation of the two-site density matrix

Due to the \(su(2)\) invariance, the two-sites reduced density matrix can be written as a superposition of the projectors \((7)\), likewise the \(R\)-matrix \((6)\) is given in terms of \(su(2)\) projectors. Therefore, one can write

\[
D^{[2]}(\xi_1, \xi_2) = \sum_{k=0}^{2n} \rho^{[2]}(\xi_1, \xi_2) \tilde{P}_k.
\tag{39}
\]

Besides that, thanks to the intertwining symmetry, the density matrix is a symmetric function \(D^{[2]}(\xi_1, \xi_2) = D^{[2]}(\xi_2, \xi_1)\).

Using the above representation for the density matrix \((39)\) in the functional equation qKZ \((19)\) specialized to the two-site case, implies the following system of equations
where from now on $\xi = \xi_1 - \xi_2$ and $A^{[2s]}(\xi)$ is a $(2s + 1) \times (2s + 1)$ matrix which results from the action of the linear operator $A^{[2s]}(\xi, \xi_2)$ (20) on the density matrix (39). For the two-site case, this matrix is a function of the difference of the parameters. We give the explicit form of $A^{[2s]}(\xi)$ for $s$ up to 3/2.

5.1. The spin-1/2 case

The two-site density matrix for the spin-1/2 chain is the simplest case as the density matrix is a linear superposition of projectors onto total spin 0 and spin 1. It can be written explicitly as

$$D^{[1]}_2(\xi_1, \xi_2) = \begin{pmatrix} \rho^{[1]}_0(\xi_1, \xi_2) & 0 & 0 & 0 \\ 0 & \frac{\rho^{[1]}_0(\xi_1, \xi_2) + \rho^{[1]}_0(\xi_2, \xi_1)}{2} & \frac{\rho^{[1]}_0(\xi_1, \xi_2) - \rho^{[1]}_0(\xi_2, \xi_1)}{2} & 0 \\ 0 & \frac{\rho^{[1]}_0(\xi_1, \xi_2) - \rho^{[1]}_0(\xi_2, \xi_1)}{2} & \frac{\rho^{[1]}_0(\xi_1, \xi_2) + \rho^{[1]}_0(\xi_2, \xi_1)}{2} & 0 \\ 0 & 0 & 0 & \rho^{[1]}_0(\xi_1, \xi_2) \end{pmatrix}.$$  (41)

Alternatively, the two-site density matrix has been written in the literature in terms of the identity and permutation operators, which is the sum and the difference of the projectors onto the total spin 1 and 0. The trace condition of the density matrix leaves only one non-trivial parameter to be determined which is just the $\omega$ function.

The final result for the physical correlations on two-sites is even simpler and can be obtained directly from the ground state energy [30]. And this, of course, agrees with the homogeneous limit of the inhomogeneous density matrix.

So to say for pedagogical reasons, for setting up notations and the way of reasoning we work out the functional equation satisfied by the two-site density matrix for the spin-1/2 case. The matrix $A^{[1]}(\xi)$ in (40) is readily obtained

$$A^{[1]}(\xi) = \begin{pmatrix} \frac{\xi - 4\lambda}{2(\xi + 2\lambda)} & -\frac{\xi}{2(\xi + 2\lambda)} \\ -\frac{\xi}{2(\xi + 2\lambda)} & \frac{\xi - 4\mu}{2(\xi + 2\mu)} \end{pmatrix}. $$

The solution for the coefficients $\rho^{[1]}(\xi_1, \xi_2)$ is given by

$$\rho^{[1]}_0(\xi_1, \xi_2) = \frac{1}{4} - \frac{\omega^{[1]}(\xi_1, \xi_2)}{2},$$  (42)

$$\rho^{[1]}_1(\xi_1, \xi_2) = \frac{1}{4} + \frac{\omega^{[1]}(\xi_1, \xi_2)}{6}. $$  (43)

Here the $\omega^{[1]}(\lambda, \mu)$ (38) is given in terms of the solution of the integral equations. From (29) and (35) we read off the zero-temperature limit $T \to 0$ of the auxiliary functions, which allows us to write down the zero-temperature limit of the $\omega$ function

\[\text{J. Phys. A: Math. Theor. 49 (2016) 254001 G A P Ribeiro and A Klümper}\]
\[
\lim_{\tau \to 0} \omega^{(1)}(\lambda, \mu) = \frac{1}{2} - \frac{(\lambda - \mu)^2}{8} + 4 \left\{ \Psi\left( -\frac{1}{4} (\lambda - \mu) \right) + \Psi\left( \frac{1}{4} (\lambda - \mu) \right) - \Psi\left( \frac{1}{2} - \frac{1}{4} (\lambda - \mu) \right) - \Psi\left( \frac{1}{2} + \frac{1}{4} (\lambda - \mu) \right) \right\},
\]

which is the transcendental part of the correlation functions [29]. The computation of the density function for \( n \geq 2 \) was successfully done up to \( n = 8 \). These results can be obtained by means of a suitable ansatz for the coefficients of the transcendental function \( \omega^{(1)}(\lambda, \mu) \) and its products [13].

In the homogeneous limit \( \xi_i \to 0 \), we obtain
\[
\rho^{(1)}_0(0, 0) = \log 2,
\]
\[
\rho^{(1)}_1(0, 0) = \frac{1 - \log 2}{3}.
\]

5.2. The spin-1 case

In the context of general spin-\( s \) chains, the first case for which the physical correlations even for two-sites cannot be obtained just by \( su(2) \) symmetry is the spin-1 chain [21]. The two-site density matrix is a superposition of projectors onto total spin 0, 1, and 2. Here we have at hand the trace condition and a simple correlator (that can be shown to be a linear combination of \( \omega^s \)'s) which are not sufficient to fix the three coefficients. In this case, the matrix \( A^{[2]}(\xi) \) is given by

\[
A^{[2]}(\xi) = \begin{pmatrix}
\frac{(\xi - 4i)(\xi - 6i)}{3(\xi + 2i)(\xi + 4)} & -\frac{(\xi - 4i)(\xi - 6i)}{3(\xi - 2i)(\xi + 4)} & \frac{5(\xi - 6i)}{3(\xi - 2i)} \\
\xi - 2i & \xi - 6i & 6i(\xi - 2i)(\xi - 4i) \\
\frac{\xi}{3(\xi + 4)} & \frac{\xi(\xi + 2i)}{2(\xi - 2i)(\xi + 4)} & \frac{\xi(\xi + 2i)}{6(\xi - 2i)(\xi - 4i)} \\
\end{pmatrix}.
\]

In [21] solutions to these functional equations were constructed in the manner of the spin-1/2 factorized form [13]. According to the fusion principle, the spin-1 2-point function is a special spin-1/2 4-point function with spectral parameters \( \xi_1 \pm i \) and \( \xi_2 \pm i \) in the four local quantum spaces. For finite Trotter number, the correlations are polynomials in \( \xi_1, \xi_2 \) divided by \( N^{[2]}(\xi_1)N^{[2]}(\xi_2) \). The multi-linear expressions in terms of \( \omega \)-functions will be polynomials divided by \( N^{[2]}(\xi_1 + i)N^{[2]}(\xi_1 - i)N^{[2]}(\xi_2 + i)N^{[2]}(\xi_2 - i) \). Hence, a scale factor \( N^{[2]}(\xi) := N^{[2]}(\xi)/N^{[2]}(\xi + i)N^{[2]}(\xi - i) \) has to be introduced when attempting to satisfy the functional equations and the analyticity conditions simultaneously. An explicit expression for \( N^{[2]}(\xi) \) in terms of the \( \omega \) function is given in the appendix, see (B.10).

For satisfying the functional equation (40), in [21] an ansatz for \( \rho^{[2]}_k(\xi_1, \xi_2) \) in terms of multi-linear expressions in \( \omega \) with rational coefficients in \( \xi_1 - \xi_2 \) was used. The success of this ansatz depends on the fact that the \( \omega \) function itself satisfies a functional equation closely related to (40). This equation was derived by use of the fusion hierarchy [21]. It has the structure of a three-point equation involving the \( \omega \) function, which reads
\[
C^{[2]}(\xi_1 + i, \xi_2) + C^{[2]}(\xi_1 - i, \xi_2) = 0,
\]

where
\[
C^{[2]}(\xi_1, \xi_2) = \frac{\Omega^{[2]}(\xi_1 - i, \xi_2) + \Omega^{[2]}(\xi_1 + i, \xi_2) - \sigma(\xi)}{N^{[2]}(\xi_1)}.
\]
\[ \Omega^{[2]}(\xi_1, \xi_2) = 2i \frac{\omega^{[2]}(\xi_1, \xi_2) + \frac{1}{2}}{\xi^2 + 4}, \]  
\[ N^{[2]}(\xi) = \frac{3}{4} + \frac{1}{2} \omega^{[2]}(\xi - i, \xi + i), \]  
\[ \sigma(\xi) = \frac{2i(\xi^2 - 3)}{(\xi^2 + 9)(\xi^2 + 1)}. \]

The final result of [21] is

\[ \rho^{[2]}_0(\xi_1, \xi_2) = \frac{1}{N^{[2]}(\xi_1)N^{[2]}(\xi_2)} \left\{ \frac{1}{16} + \frac{(16 + \xi^2)\omega(\xi_1 - i, \xi_1 + i)}{24\xi^2} \right. \\
- \left. \frac{(4 + \xi^2)\omega(\xi_1 - i, \xi_2 + i)}{6\xi^2} - \frac{(4 + \xi^2)\omega(\xi_1 + i, \xi_2 - i)}{6\xi^2} \right. \\
\left. + \frac{(16 + \xi^2)\omega(\xi_1 - i, \xi_1 - i)\omega(\xi_1 + i, \xi_2 + i)}{36\xi^2} \right. \\
\left. + \frac{(16 + \xi^2)\omega(\xi_2 - i, \xi - 2 + i)}{24\xi^2} \right\}, \]

\[ \rho^{[2]}_1(\xi_1, \xi_2) = \frac{1}{N^{[2]}(\xi_1)N^{[2]}(\xi_2)} \left\{ \frac{1}{16} + \frac{(-4 + \xi^2)(16 + \xi^2)\omega(\xi_1 - i, \xi_1 + i)}{24\xi^2(4 + \xi^2)} \right. \\
- \left. \frac{(16 + \xi^2)\omega(\xi_1 - i, \xi_2 - i)}{12(4 + \xi^2)} + \frac{(2 - i\xi)\omega(\xi_1 - i, \xi_2 + i)}{3\xi^2} \right. \\
\left. + \frac{(4 + \xi^2)(8 + \xi^2)\omega(\xi_1 - i, \xi_2 + i)\omega(\xi_1 + i, \xi_2 - i)}{72\xi^2} \right. \\
\left. - \frac{(16 + \xi^2)\omega(\xi_1 + i, \xi_3 + i)}{12(4 + \xi^2)} \right. \\
\left. - \frac{(8 + \xi^2)(16 + \xi^2)\omega(\xi_1 - i, \xi_3 - i)\omega(\xi_1 + i, \xi_2 + i)}{72(4 + \xi^2)} \right. \\
\left. + \frac{(-4 + \xi^2)(16 + \xi^2)\omega(\xi_2 - i, \xi - 2 + i)}{24\xi^2(4 + \xi^2)} \right. \\
\left. + \frac{(-4 + \xi^2)(16 + \xi^2)\omega(\xi_1 - i, \xi_1 + i)\omega(\xi_2 - i, \xi - 2 + i)}{36\xi^2(4 + \xi^2)} \right\}. \]
\[
\rho^{[2]}_{2}(\xi_1, \xi_2) = \frac{1}{N^{[2]}(\xi_1)N^{[2]}(\xi_2)} \left\{ \frac{1}{16} + \frac{(128 - 20\xi^2 + 5\xi^4)\omega(\xi_1 - i, \xi_1 + i)}{120\xi^2(4 + \xi^2)} 
+ \frac{(16 + \xi^2)\omega(\xi_1 - i, \xi_2 - i)}{20(4 + \xi^2)} + \frac{(2i + \xi)(4i + \xi)\omega(\xi_1 - i, \xi_2 + i)}{30\xi^2} 
+ \frac{(-2i + \xi)(-4i + \xi)\omega(\xi_1 + i, \xi_2 - i)}{30\xi^2} 
- \frac{(4 + \xi^2)(16 + \xi^2)\omega(\xi_1 - i, \xi_2 + i)\omega(\xi_1 + i, \xi_2 - i)}{360\xi^2} \right\}.
\]

(54)

Taking the zero-temperature limit of the \(\omega\)-function results in

\[
\lim_{T \to 0} \omega^{[2]}(\lambda, \mu) = \frac{1}{2} - \frac{(\lambda - \mu)^2 + 4}{8} \left\{ \left( \psi\left( -\frac{i}{4}(\lambda - \mu) \right) + \psi\left( \frac{i}{4}(\lambda - \mu) \right) \right) 
- \psi\left( \frac{1}{2} - \frac{i}{4}(\lambda - \mu) \right) - \psi\left( \frac{1}{2} + \frac{i}{4}(\lambda - \mu) \right) \right\}.
\]

(55)

Taking furthermore the homogeneous limit and using the zero-temperature \(\omega\)-function, the coefficients of the projection operators are obtained

\[
\rho^{[2]}_{0}(0, 0) = -\frac{8}{9} + \frac{4}{27}\pi^2,
\]

\[
\rho^{[2]}_{1}(0, 0) = \frac{14}{9} - \frac{4}{27}\pi^2,
\]

\[
\rho^{[2]}_{2}(0, 0) = -\frac{5}{9} + \frac{8}{135}\pi^2.
\]

(56)

In this case, only \(\pi^2\) appears which is related to the zeta function value \(\zeta(2) = \frac{\pi^2}{6}\). For \(s = 1\), it was also possible to deal with the three site density matrix [21], whose results are given in terms of \(\pi^2\), \(\pi^4\) and \(\pi^6\) or alternatively \(\zeta(2)\), \(\zeta(4)\) and \(\zeta(6)\).

5.3. The spin-3/2 case

The case of \(s = 3/2\) is the subject of our new work. This case is non-trivial as it is computationally much more involved than the lower spin cases. This is because the ansatz for \(\rho^{[2]}_{2}(\xi_1, \xi_2)\) is built upon the fusion of spin-1/2 objects such that the spin-\(s\) case with \(n\) sites is as involved as the spin-1/2 case with \(2n\) sites. Here, the matrix \(A^{[3]}(\xi)\) of the functional equation (40) of the two-site density operator reads
This equation is obtained from the fusion hierarchy and holds true at certain values where the internal energy nearest neighbor correlators after the homogeneous limit is taken. For instance, in the case of \( J = 1 \) we have

\[
\mathcal{A}^3(\xi) = \begin{pmatrix}
-\frac{(\xi - 4)(\xi - 6)(\xi - 8)}{4(\xi + 2)(\xi + 4)(\xi + 6)} & \frac{3(\xi - 4)(\xi - 6)(\xi - 8)}{4(\xi - 2)(\xi + 4)(\xi + 6)} & \frac{-5(\xi - 6)(\xi - 8)}{4(\xi - 2)(\xi + 6)} & \frac{7(\xi - 8)}{4(\xi - 2)} \\
\frac{(\xi - 4)(\xi - 6)(\xi - 8)}{4(\xi + 2)(\xi + 4)(\xi + 6)} & -\frac{11(\xi - 4)(\xi - 6)(\xi - 8)}{4(\xi - 2)(\xi + 4)(\xi + 6)} & \frac{4(\xi - 2)(\xi - 4)(\xi + 6)}{4(\xi + 2)(\xi + 4)(\xi + 6)} & \frac{21(\xi - 8)}{4(\xi - 2)} \\
\frac{(\xi - 4)(\xi - 6)(\xi - 8)}{4(\xi + 2)(\xi + 4)(\xi + 6)} & \frac{20(\xi - 2)(\xi + 4)(\xi + 6)}{4(\xi - 2)(\xi + 4)(\xi + 6)} & \frac{3(\xi + 2)(\xi - 8)}{4(\xi + 2)(\xi + 4)(\xi + 6)} & \frac{20(\xi - 2)(\xi - 4)(\xi - 8)}{4(\xi - 2)(\xi + 4)(\xi + 6)} \\
\frac{(\xi - 4)(\xi - 6)(\xi - 8)}{4(\xi + 2)(\xi + 4)(\xi + 6)} & \frac{20(\xi - 2)(\xi + 4)(\xi + 6)}{4(\xi - 2)(\xi + 4)(\xi + 6)} & \frac{3(\xi + 2)(\xi - 8)}{4(\xi + 2)(\xi + 4)(\xi + 6)} & \frac{20(\xi - 2)(\xi - 4)(\xi - 8)}{4(\xi - 2)(\xi + 4)(\xi + 6)}
\end{pmatrix}.
\]

The analogue of the functional equation (47) is given by

\[
C^3(\xi_1 + i, \xi_2) + C^3(\xi_1 - i, \xi_2) = \frac{1}{2} \left( \frac{1}{\xi + i} - \frac{1}{\xi - i} \right) + \left( \frac{1}{\xi + 5i} - \frac{1}{\xi - 5i} \right),
\]

where

\[
C^3(\xi_1, \xi_2) = \left\{ \begin{array}{c}
-\frac{3}{2} + \omega^3(\xi_1, \xi_2) \Omega^3(\xi_1 - 2i, \xi_2) \\
2\Omega^3(\xi_1, \xi_2) - \left( \frac{3}{2} + \omega^3(\xi_1 - 2i, \xi_1) \right) \Omega^3(\xi_1 + 2i, \xi_2) \\
+ \frac{1}{2} \left[ -\frac{1}{\xi + 1} + \frac{1}{\xi - 2i} - \frac{1}{\xi - 4i} + \frac{1}{\xi + 2i} \right] \omega^3(\xi_1 - 2i, \xi_1) \\
- \frac{1}{2} \left[ -\frac{1}{\xi + 1} + \frac{1}{\xi - 2i} - \frac{1}{\xi + 4i} + \frac{1}{\xi + 2i} \right] \omega^3(\xi_1, \xi_1 + 2i) \\
+ \frac{2i}{\xi^2 + 16} / N^3(\xi_1),
\end{array} \right. \]

\[
\Omega^3(\xi_1, \xi_2) = 2i \frac{\omega^3(\xi_1, \xi_2) + \frac{1}{2}}{\xi^2 + 4},
\]

\[
N^3(\xi) = 1 + \omega^3(\xi, \xi + 2i) + \omega^3(\xi - 2i, \xi).
\]

This equation is obtained from the fusion hierarchy and holds true at certain values where \( \phi(\xi) = 0 \). Another equation that is essential for consistently satisfying the functional equations is (B.16) which has no counterpart in the spin-1 case. We give some details of the derivations in appendix B.

We use an ansatz for the solution in terms of multilinear expressions in \( \omega \) functions with rational coefficients. The expressions contain up to trilinear combinations of \( \omega \)-functions with arguments \( \xi_1, \xi_1 \pm 2i, \xi_2, \xi_2 \pm 2i \). We use a scale factor \( N^3(\xi) = \mathcal{N}^3(\xi)/\mathcal{N}^3(\xi + 2i)\mathcal{N}^3(\xi - 2i) \mathcal{N}^3(\xi - 2i) \) that can be expressed just in terms of \( \omega \) functions, see (B.14).

By use of (40), (57) and (58) and the above described ansatz for the solution, we obtain the coefficients \( \rho_{ij}^3(\xi_j, \xi_2) \) which are listed in the appendix C.

Based on the knowledge of the coefficients \( \rho_{ij}^3(\xi_j, \xi_2) \), we can easily calculate any nearest neighbor correlators after the homogeneous limit is taken. For instance, in the case of the internal energy \( e = \langle \mathcal{H}^3 \rangle = \text{Tr} [\mathcal{H}^3 D^3_{ij}(0, 0)] = -T^2 \frac{\partial}{\partial T} (f/T) \) at zero magnetic field, we have
At zero temperature, one has the transcendental function given by

\[
\lim_{T \to 0} \omega^{[3]}(\lambda, \mu) = \frac{1}{2} - \frac{(\lambda - \mu)^2 + 4}{8} \left\{ \left( \psi\left(\frac{i}{4}(\lambda - \mu)\right) + \psi\left(\frac{i}{4}(\lambda + \mu)\right) \right) - \psi\left(\frac{1}{2} + \frac{i}{4}(\lambda - \mu)\right) - \psi\left(\frac{1}{2} - \frac{i}{4}(\lambda - \mu)\right) \right\} - 2\pi \sinh\left(\frac{\pi}{4}(\lambda - \mu)\right) \sinh\left(\frac{\pi}{4}(\lambda + \mu)\right),
\]

which in the homogeneous limit produces the following results

\[
\rho^{[3]}_0(0, 0) = \left( 3 - \frac{3}{10}\pi^2 \right) + \left( -\frac{21}{4} + \frac{3}{5}\pi^2 \right) \log 2,
\]
\[
\rho^{[3]}_1(0, 0) = \left( -\frac{183}{20} + \frac{47}{50}\pi^2 \right) + \left( \frac{291}{20} - \frac{37}{25}\pi^2 \right) \log 2,
\]
\[
\rho^{[3]}_2(0, 0) = \left( -\frac{429}{40} - \frac{27}{25}\pi^2 \right) + \left( -\frac{309}{20} + \frac{39}{25}\pi^2 \right) \log 2,
\]
\[
\rho^{[3]}_3(0, 0) = \left( -\frac{161}{40} + \frac{72}{175}\pi^2 \right) + \left( \frac{111}{20} - \frac{99}{175}\pi^2 \right) \log 2.
\]

It is worth noting that for \( s = 3/2 \) and two-sites \( \pi^2 \) and \( \log 2 \) appear. This has a structure which is different from the above results for \( s = 1/2, 1 \). For the cases \( s = 1/2 \) and \( s = 1 \) all known data indicate that the results are given by combinations of \( \log 2, \zeta(3), \zeta(5), \ldots \) or in powers of \( \pi^2 \) (zeta function values of even integer arguments). In the present half-odd integer spin case, we do not find expressions just in terms of zeta function values of odd integer arguments, we find a mixture, and this already for the two-site correlation functions.

At last, in order to compute the energy at zero temperature we insert (64) into (62). The result is simply given as

\[
e_{T=0} = J\left(\frac{1}{2} + \log 2\right),
\]

which is in agreement with the literature [20].

We can also compute spin correlators, e.g. at zero temperature we find

\[
\langle S_i^z S_{i+1}^z \rangle = \left( -\frac{105}{8} + \frac{13}{10}\pi^2 \right) + \left( 15 - \frac{8}{5}\pi^2 \right) \log 2 \approx -0.843 048 \ldots
\]

Due to isotropy, one has \( \langle S_i^+ S_{i+1}^z \rangle = \langle S_i^- S_{i+1}^z \rangle = \langle S_i^{-} S_{i+1}^{+} \rangle \) implying \( \langle S_i^+ S_{i+1}^{-} \rangle = 3 \langle S_i^z S_{i+1}^z \rangle \approx -2.529 144 \ldots \)

6. Conclusion

We have exploited the new approach developed in [21] to obtain further results for correlation functions of high-spin \( su(2) \) chains. It is based on the discrete functional relation of qKZ type and the fusion procedure.
We obtained the general two-site correlation function for the integrable spin-$3/2$ chain. Surprisingly, the result is given in terms of log $2$ and $\pi^2$, which can be seen as zeta function values of even and odd arguments. This structure is very different from that of the famous results for the spin-$1/2$ case [11–13] and from that of the recently studied spin-1 chain [21]. For the spin-$1/2$ case the result is given only in terms of zeta function values with odd arguments and for the spin-1 case only zeta function values of even arguments appear.

Although having shown that this approach is viable for an explicit computation of the correlation functions, the direct application to more spins becomes quickly cumbersome. Finding an alternative, elegant computational tool would be highly desirable. Also, the generalization of our approach to higher rank spin chains would be very interesting. We hope to come back to these scientific issues in the near future.

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Appendix A. Algebraic relations and auxiliary functions

For our purposes it is convenient to deal with polynomials rather than with rational functions. For this reason, we slightly change the normalization of the $R$-matrix, which naturally results in transfer matrix eigenvalues with different normalization. The relation among the eigenvalues with different normalization is given by

$$
A_2(\lambda) = \prod_{l=1}^{2j} \prod_{\sigma=\pm} \phi_{\sigma}(\lambda + i2\sigma(s - j + l)) \varphi_\sigma(\lambda + i2\sigma(\frac{1}{2} - j + l)) \hat{V}^{(2)}(\lambda). 
$$

(A.1)

Using this new normalization, we have the following modified expressions:

Fusion hierarchy

$$
T_{2j}(\lambda)T_j(\lambda + i(2j + 1)) = T_{2j+1}(\lambda + i) + \hat{\chi}(\lambda + i(2j))T_{2j-1}(\lambda - i),
$$

$$
T_0(\lambda) = \text{Id, } j = 1/2, 1, 3/2 ..., 
$$

(A.2)

where

$$
\hat{\chi}(\lambda) = \phi(\lambda - i(2s + 1))\phi(\lambda + i(2s + 1))\varphi(\lambda - 2i)\varphi(\lambda + 2i),
$$

$$
\phi(\lambda) = \phi_\sigma(\lambda + i)\phi_\sigma(\lambda - i),
$$

$$
\varphi(\lambda) = \varphi_\sigma(\lambda + i)\varphi_\sigma(\lambda - i).
$$

(A.3)

The $T$-system

$$
T_{2j}(\lambda + i)T_{2j}(\lambda - i) = T_{2j-1}(\lambda)T_{2j+1}(\lambda) + f_j(\lambda)\text{Id, } j = 1/2, 1, 3/2 ... 
$$

(A.4)

where

$$
f_j(\lambda) = \prod_{l=1}^{j} \prod_{\sigma=\pm} \phi(\lambda + i\sigma(2s + 2l))\varphi(\lambda + i\sigma(1 + 2l)).
$$
Eigenvector expressions

\[
\Lambda_{2j}(\lambda) = \sum_{m=1}^{2j+1} \hat{\lambda}_m^{(j)}(\lambda),
\]

(A.5)

\[
\hat{\lambda}_m^{(j)}(\lambda) = e^{2jb(j+1-m)} i_{j,m}^{(j)}(\lambda) j_{j,m}^{(j)}(\lambda) \frac{q\left(\lambda + 2i\left(\frac{j}{2} + j\right)\right) q\left(\lambda - 2i\left(\frac{j}{2} + j\right)\right)}{q\left(\lambda + 2i\left(\frac{j}{2} + j - m\right)\right) q\left(\lambda - 2i\left(\frac{j}{2} + j - m\right)\right)},
\]

(A.6)

where \( i_{j,m}^{(j)}(\lambda) = \prod_{l=m-j+1}^{2j+1-m} \phi(\lambda \pm i(2l - 2s - 1)) \varphi(\lambda \pm i(2l - 2s)) \) and \( q(\lambda) = \prod_{s=1}^{n} (\lambda - \lambda_i) \). The corresponding Bethe ansatz equations can be written as

\[
e^{2jb(\lambda + i(2s))} = \prod_{s=1}^{n} \frac{\lambda_i - \lambda_j + 2i}{\lambda_i - \lambda_j - 2i}.
\]

(A.7)

This allows us to define a suitable set of auxiliary functions as

\[
y^{(2j)}(\lambda) = \frac{\Lambda_{2j-1}(\lambda) \Lambda_{2j+1}(\lambda)}{f_j(\lambda)} , j = \frac{1}{2}, ..., s - \frac{1}{2}.
\]

(A.8)

and

\[
b(\lambda) = \frac{\hat{\lambda}_2^{(j)}(\lambda + i) + \cdots + \hat{\lambda}_{2j}^{(j)}(\lambda + i)}{\hat{\lambda}_j^{(j)}(\lambda + i)},
\]

(A.9)

\[
\bar{b}(\lambda) = \frac{\hat{\lambda}_2^{(j)}(\lambda - i) + \cdots + \hat{\lambda}_{2j}^{(j)}(\lambda - i)}{\hat{\lambda}_j^{(j)}(\lambda - i)}.
\]

(A.10)

In addition to this, we define \( B(\lambda) := 1 + b(\lambda) \), \( \bar{B}(\lambda) := 1 + \bar{b}(\lambda) \) and \( y^{(2j)}(\lambda) := 1 + y^{(2j)}(\lambda) \).

According the previous definition, we note that \( B(\lambda) = \frac{\Lambda_2(\lambda + i)}{\Lambda_1(\lambda + i)} \) and \( \bar{B}(\lambda) = \frac{\Lambda_2(\lambda - i)}{\Lambda_1(\lambda - i)} \) with product \( B(\lambda) \bar{B}(\lambda) = Y^{(2j)}(\lambda) \). This implies for the first \((2s - 1)\) functional relations

\[
y^{(2j)}(\lambda + i)y^{(2j)}(\lambda - i) = Y^{(2j-1)}(\lambda) Y^{(2j+1)}(\lambda) \text{ for } j = \frac{1}{2}, 1, ..., s - 1,
\]

(A.11)

\[
y^{(2s-1)}(\lambda + i)y^{(2s-1)}(\lambda - i) = Y^{(2s-2)}(\lambda) B(\lambda) \bar{B}(\lambda).
\]

(A.12)

Therefore, we end up in the following set of algebraic relations

\[
b(\lambda) = \frac{q(\lambda + i(2s + 2)) e^{-i2bj(2s + 1)} \phi(\lambda) \varphi(\lambda + i(2s - 1)) \Lambda_{2s-1}(\lambda)}{q(\lambda + i2s) \prod_{j=1}^{2s} \phi(\lambda + i2j) \varphi(\lambda + i(2s - 2j + 3))},
\]

(A.13)

\[
\bar{b}(\lambda) = \frac{q(\lambda - i(2s + 2)) e^{i2bj(2s + 1)} \phi(\lambda) \varphi(\lambda - i(2s - 1)) \Lambda_{2s-1}(\lambda)}{q(\lambda - i2s) \prod_{j=1}^{2s} \phi(\lambda - i2j) \varphi(\lambda - i(2s - 2j + 3))}.
\]

(A.14)

In this way, it is evident that \( b(\lambda), \bar{b}(\lambda) \) are related to \( \Lambda_{2s-1}(\lambda) \).

Moreover, \( \Lambda_{2s-1}(\lambda) \) is related to \( Y^{(2s-1)}(\lambda) \) through the definition of the \( y \)-functions. This relation can be written as
Now we have all the ingredients to derive the nonlinear integral equations [21, 27, 31].

The main idea is to compute the Fourier transform of the logarithm of the above defined relations (A.11)–(A.15). This allows us to get rid of the Bethe ansatz roots by eliminating the function \( q(\lambda) \). After a long but straightforward calculation, where we take the inverse Fourier transform of the transformed auxiliary functions, we finally obtain the nonlinear integral equations (29).

### Appendix B. Functional equations for the basic functions for \( s = 3/2 \)

From the fusion hierarchy (A.2) we have the following explicit relations

\[
\Lambda_2(\lambda) = \Lambda_1(\lambda - i) \Lambda_1(\lambda + i) - \phi(\lambda - 4i)\phi(\lambda + 4i)\varphi(\lambda - 2i)\varphi(\lambda + 2i),
\]

(B.1)

\[
\Lambda_3(\lambda) = \Lambda_2(\lambda - i) \Lambda_1(\lambda + 2i) - \phi(\lambda - 3i)\phi(\lambda + 5i)\varphi(\lambda - i)\varphi(\lambda + 3i)\Lambda_1(\lambda - 2i).
\]

(B.2)

Besides that, from the eigenvalue expression (A.5) we see that

\[
\Lambda_3(\lambda - i)\Lambda_3(\lambda + i) = G(\lambda) [\varphi(\lambda)\varphi(\lambda - 2i)\varphi(\lambda + 2i)\varphi(\lambda + 4i)\varphi(\lambda - 4i)] \\
+ \phi(\lambda)\tilde{\Lambda}(\lambda),
\]

(B.3)

where \( G(\lambda) = \prod_{l=1}^{3} \prod_{\sigma = \pm} \phi(\lambda + \sigma i2l) \) is independent of \( \delta \).

We divide all of the above equations by the \( \varphi \)-function with the respective arguments on the right

\[
\frac{\Lambda_2(\lambda)}{\varphi(\lambda - 2i)\varphi(\lambda + 2i)} = \frac{\Lambda_1(\lambda - i)\Lambda_1(\lambda + i)}{\varphi(\lambda - 2i)\varphi(\lambda + 2i)} - \phi(\lambda - 4i)\phi(\lambda + 4i),
\]

(B.4)

\[
\frac{\Lambda_3(\lambda)}{\varphi(\lambda - i)\varphi(\lambda + 3i)} = \frac{\Lambda_2(\lambda - i)\Lambda_1(\lambda + 2i)}{\varphi(\lambda - i)\varphi(\lambda + 3i)} - \phi(\lambda - 3i)\phi(\lambda + 5i)\Lambda_1(\lambda - 2i),
\]

(B.5)

\[
\frac{\Lambda_3(\lambda - i)}{\varphi(\lambda - 2i)\varphi(\lambda + 2i)} \cdot \frac{\Lambda_3(\lambda + i)}{\varphi(\lambda)\varphi(\lambda + 4i)} \cdot \frac{1}{\varphi(\lambda - 4i)} \\
= G(\lambda) + \frac{\phi(\lambda)\tilde{\Lambda}(\lambda)}{\varphi(\lambda)\varphi(\lambda - 2i)\varphi(\lambda + 2i)\varphi(\lambda + 4i)\varphi(\lambda - 4i)}.
\]

(B.6)

From the last equation we derive

\[
\frac{\partial}{\partial \delta} \log \frac{\Lambda_3(\lambda - i)}{\varphi(\lambda - 2i)\varphi(\lambda + 2i)} + \frac{\partial}{\partial \delta} \log \frac{\Lambda_3(\lambda + i)}{\varphi(\lambda)\varphi(\lambda + 4i)} \\
+ \frac{\partial}{\partial \delta} \log \frac{1}{\varphi(\lambda)\varphi(\lambda - 4i)} = 0, \quad \text{if} \quad \phi(\lambda) = 0,
\]

(B.7)

because for \( \phi(\lambda) = 0 \) the right-hand side of (B.6) is completely independent of \( \delta \).

We next replace (B.4) in (B.5), which is eventually substituted in (B.7) and results in an equation for \( \frac{\partial}{\partial \delta} \log \Lambda_1(\lambda) \). Using the relation between \( \Lambda_1(\lambda) \) and \( \Lambda^{[1]}(\lambda) \) (A.1), we can re-write the equation (B.7) in terms of the \( \omega \)-function (38) or its simply related \( \Omega \)-function given by [21].
\[ \Omega^{[3]}(\lambda, \mu) = 2i \frac{\omega^{[3]}(\lambda, \mu) + \frac{1}{3}}{(\lambda - \mu)^2 + 4} = -\frac{\partial}{\partial \phi} \log N^{[1]}(\lambda; \mu) \big|_{\phi=0} + \frac{2i}{(\lambda - \mu)^2 + 4}. \] (B.8)

After a long but straightforward calculation, we obtain from (B.7) the equation (58), where we identified \((\lambda, \mu)\) with \((\xi_1, \xi_2)\).

We need another equation for the \(\omega\)-function with arguments differing by \(2i\). These special arguments appear in the expressions of the scale factors \(N^{[2]}(\lambda)\) like \(N^{[3]}(\lambda)\). The rhs of this relation can be expressed as the expectation value of the projector \(P_1\) onto triplet states in the tensor product of two spin-1/2 objects with respect to the density matrix

\[ D^{[1]}(\lambda, \mu) = \left( \frac{1}{4} - \frac{1}{6} \omega(\lambda, \mu) \right) \text{Id} + \frac{1}{3} \omega(\lambda, \mu) P_1, \] (B.9)

with \(\lambda \rightarrow \lambda - i, \mu \rightarrow \lambda + i\) yielding

\[ \frac{\Lambda_2(\lambda)}{\Lambda_4(\lambda + i)\Lambda_4(\lambda - i)} = \frac{3}{4} + \frac{1}{2} \omega(\lambda - i, \lambda + i). \] (B.10)

Hence with \(\omega = \omega^{[2]}\) we find \(N^{[2]}(\lambda) = \frac{3}{4} + \frac{1}{2} \omega^{[2]}(\lambda - i, \lambda + i)\). Note that (B.9) and (B.10) hold for all \(\omega = \omega^{[2]}\) with arbitrary \(s\).

For \(N^{[3]}(\lambda)\) we find an expression linear in \(\omega\) functions by use of fusion relations like above, however without the \(\phi\) factors stemming from the insertion of the two auxiliary spin-1/2 spaces in (23) which now have to be dropped

\[ \begin{align*}
\Lambda_2(\lambda) &= \Lambda_1(\lambda - i)\Lambda_1(\lambda + i) - \phi(\lambda - 4i)\phi(\lambda + 4i), \\
\Lambda_4(\lambda) &= \Lambda_2(\lambda - i)\Lambda_4(\lambda + 2i) - \phi(\lambda - 3i)\phi(\lambda + 5i)\Lambda_1(\lambda - 2i).
\end{align*} \] (B.11)

From (B.11) and (B.10) we obtain

\[ \frac{\phi(\lambda - 4i)\phi(\lambda + 4i)}{\Lambda_1(\lambda - i)\Lambda_1(\lambda + i)} = \frac{1}{4} - \frac{1}{2} \omega(\lambda - i, \lambda + i). \] (B.13)

Dividing (B.12) by \(\Lambda_1(\lambda - 2i)\Lambda_1(\lambda)\Lambda_4(\lambda + 2i)\) and applying (B.10) and (B.13)

\[ \begin{align*}
\frac{\Lambda_4(\lambda)}{\Lambda_1(\lambda - 2i)\Lambda_1(\lambda)\Lambda_4(\lambda + 2i)} &= \frac{\Lambda_2(\lambda - i)}{\Lambda_1(\lambda - 2i)\Lambda_1(\lambda)} - \frac{\phi(\lambda - 3i)\phi(\lambda + 5i)}{\Lambda_1(\lambda)\Lambda_4(\lambda + 2i)}, \\
&= \frac{1}{2} [1 + \omega(\lambda - 2i, \lambda) + \omega(\lambda, \lambda + 2i)].
\end{align*} \] (B.14)

Taking the analogue of (B.3), i.e. without \(\phi\) factors, at \(\lambda\) values for which \(\phi(\lambda) = 0\) we find

\[ \begin{align*}
\Lambda_1(\lambda - i) &= \frac{\Lambda_1(\lambda - i)\Lambda_1(\lambda + i)}{\Lambda_1(\lambda - i)\Lambda_1(\lambda + i)}, \\
\Lambda_4(\lambda + i) &= \frac{\Lambda_4(\lambda - i)\Lambda_4(\lambda + i)}{\Lambda_4(\lambda - i)\Lambda_4(\lambda + i)}, \\
&= \frac{\phi(\lambda - 6i)\phi(\lambda + 2i)}{\Lambda_1(\lambda - 3i)\Lambda_1(\lambda - i)} \frac{\phi(\lambda - 4i)\phi(\lambda + 4i)}{\Lambda_4(\lambda - i)\Lambda_4(\lambda + i)} \frac{\phi(\lambda - 2i)\phi(\lambda + 6i)}{\Lambda_4(\lambda + i)\Lambda_4(\lambda + 3i)}.
\end{align*} \] (B.15)

Inserting (B.14) on the lhs and (B.13) on the rhs we find after some simple transformations

\[ \omega^{[3]}(\lambda - 3i, \lambda - i) = -\frac{1}{2} \cdot 8 \frac{\omega^{[3]}(\lambda - i, \lambda + i) + 5 + 6 \omega^{[3]}(\lambda + i, \lambda + 3i)}{3 + 2 \omega^{[3]}(\lambda + i, \lambda + 3i)}, \] (B.16)

which will be important for consistently solving the functional equations in the spin-3/2 case.

Here we have set \(\omega = \omega^{[3]}\). Note that this equation does not hold for \(\omega^{[2]}\).
Appendix C. Coefficients $\tilde{\rho}_i(\xi_1, \xi_2)$

In this appendix we list the coefficients $\tilde{\rho}_i$, where $\tilde{\rho}_i = N^{(3)}(\xi_1)N^{(3)}(\xi_2)\tilde{\rho}_i^{(3)}(\xi_1, \xi_2)$.

$$\tilde{\rho}_0(\xi_1, \xi_2) = \frac{1}{16} \left( 1 + \omega(\xi_1, \xi_1) + \omega(\xi_1, \xi_1^+) \right) \left( 1 + \omega(\xi_2, \xi_2) + \omega(\xi_2, \xi_2^+) \right)$$

$$= \frac{1}{4} \left( \xi^2 + 16 \right) \omega(\xi_1, \xi_2)$$

$$+ \frac{3}{2} \left[ \omega(\xi_1, \xi_1^+) + \omega(\xi_2, \xi_2^+) + \omega(\xi_2, \xi_2^+) + \omega(\xi_1, \xi_2^+) \right]$$

$$- \frac{3}{16} \left[ \frac{\xi + 2i}{\xi - 2i} \omega(\xi_1, \xi_2^+) + \frac{\xi - 2i}{\xi + 2i} \omega(\xi_2, \xi_1^+) \right]$$

$$- \frac{3}{5} \left[ \frac{(3\xi^3 + 2\xi^2 - 8\xi - 48)i}{\xi_1(\xi + 2i)} \omega(\xi_1, \xi_1^+) \omega(\xi_2, \xi_2^+) \right]$$

$$+ \frac{9}{5} \left( \xi^2 + 4 \right) \left[ \omega(\xi_1, \xi_1^+) \omega(\xi_2, \xi_2^+) + \omega(\xi_1, \xi_1^+) \omega(\xi_2, \xi_2^+) \right]$$

$$- \frac{3}{80} \left( \xi^2 + 16 \right) \left( \xi - 6i \right) \left[ \omega(\xi_1, \xi_2^+) \omega(\xi_2, \xi_2^+) + \omega(\xi_1, \xi_2^+) \omega(\xi_2, \xi_2^+) \right]$$

$$- \frac{3}{80} \left( \xi^2 + 16 \right) \left( \xi + 6i \right) \left[ \omega(\xi_1, \xi_2^+) \omega(\xi_2, \xi_2^+) + \omega(\xi_1, \xi_2^+) \omega(\xi_2, \xi_2^+) \right]$$

$$- \frac{3}{80} \left( \xi^2 + 36 \right) \left( \xi^2 + 16 \right) \left[ \omega(\xi_1, \xi_1^+) \omega(\xi_2, \xi_2^+) + \omega(\xi_1, \xi_1^+) \omega(\xi_2, \xi_2^+) \right]$$

$$+ \frac{1}{1280} \left( \xi^2 + 16 \right)^2 \omega(\xi_1, \xi_2^+) \omega(\xi_2, \xi_2^+)$$

$$- \frac{1}{1280} \left( \xi^2 + 36 \right) \left( \xi - 2i \right)^2 \left( \xi + 4i \right)^2 \left[ \omega(\xi_1, \xi_2^+) \omega(\xi_1, \xi_2^+) + \omega(\xi_1, \xi_2^+) \omega(\xi_1, \xi_2^+) \right]$$

$$- \frac{1}{1280} \left( \xi^2 + 36 \right) \left( \xi + 2i \right)^2 \left( \xi - 4i \right)^2 \left[ \omega(\xi_1, \xi_2^+) \omega(\xi_1, \xi_2^+) + \omega(\xi_1, \xi_2^+) \omega(\xi_1, \xi_2^+) \right]$$

$$- \frac{1}{1280} \left( \xi^3 - 6i\xi^2 - 24\xi - 96i \right) \left( \xi^2 + 4 \right) \left( \xi - 6i \right) \left( \xi + 4i \right) \omega(\xi_1, \xi_2^+) \omega(\xi_1, \xi_2^+)$$

$$- \frac{1}{1280} \left( \xi^3 + 6i\xi^2 - 24\xi + 96i \right) \left( \xi^2 + 4 \right) \left( \xi - 6i \right) \left( \xi + 4i \right) \omega(\xi_1, \xi_2^+) \omega(\xi_1, \xi_2^+)$$

$$+ \frac{1}{1280} \left( \xi^3 - 6i\xi^2 - 24\xi - 96i \right) \left( \xi^2 + 16 \right) \left( \xi - 4i \right) \omega(\xi_1, \xi_2^+) \omega(\xi_1, \xi_2^+)$$

$$\frac{\xi (\xi - 2i)}{\xi (\xi - 2i)}$$
\[ + \frac{1}{1280} \frac{(\xi^2 + 36)(\xi^2 + 16)(\xi - 2i)(\xi + 6i)}{\xi^2} \times [\omega(\xi_1^+, \xi_2)^\omega(\xi_1^+, \xi_2^*) + \omega(\xi_1^+, \xi_2)\omega(\xi_1^+, \xi_2^*)] + \frac{1}{1280} \frac{(\xi^2 + 36)(\xi^2 + 16)(\xi + 6i)}{\xi^2} \times [\omega(\xi_1^+, \xi_2)\omega(\xi_1^+, \xi_2^*) + \omega(\xi_1^+, \xi_2)\omega(\xi_1^+, \xi_2^*)] \]

\[ + \frac{1}{1280} \frac{(\xi^2 + 36)(\xi^2 + 16)}{\xi^2 + 4} \times [\omega(\xi_1^+, \xi_2)\omega(\xi_1^+, \xi_2^*) + \omega(\xi_1^+, \xi_2)\omega(\xi_1^+, \xi_2^*) + \omega(\xi_1^+, \xi_2)\omega(\xi_1^+, \xi_2^*)] - \frac{1}{20} \frac{(\xi^2 + 36)(\xi - 4i)}{\xi^2} \times [\omega(\xi_1^+, \xi_2^*)\omega(\xi_1^+, \xi_2) + \omega(\xi_1^+, \xi_2)\omega(\xi_1^+, \xi_2^*)] - \frac{1}{20} \frac{(\xi^2 + 36)(\xi + 4i)}{\xi^2} \times [\omega(\xi_1^+, \xi_2^*)\omega(\xi_1^+, \xi_2) + \omega(\xi_1^+, \xi_2)\omega(\xi_1^+, \xi_2^*)] - \frac{1}{40} \frac{(\xi^2 + 16)(\xi^2 + 16)}{\xi^2}(\xi + 2i) \times [\omega(\xi_1^+, \xi_2^*)\omega(\xi_1^+, \xi_2) + \omega(\xi_1^+, \xi_2)\omega(\xi_1^+, \xi_2^*)] - \frac{1}{40} \frac{(\xi^2 + 36)(\xi^2 + 16)}{\xi^2} \times [\omega(\xi_1^+, \xi_2)^\omega(\xi_1^+, \xi_2^*) + \omega(\xi_1^+, \xi_2)\omega(\xi_1^+, \xi_2^*)] + \frac{1}{3840} \frac{(\xi^2 + 6)(\xi^2 + 4)}{\xi^2} \times [\omega(\xi_1^+, \xi_2^*)\omega(\xi_1^+, \xi_2) + \omega(\xi_1^+, \xi_2)\omega(\xi_1^+, \xi_2^*) + \omega(\xi_1^+, \xi_2)\omega(\xi_1^+, \xi_2^*)] + \frac{1}{3840} \frac{(\xi^2 + 36)(\xi^2 + 16)(\xi^2 + 6)(\xi^2 + 4)}{\xi^2} \times [\omega(\xi_1^+, \xi_2^*)\omega(\xi_1^+, \xi_2) + \omega(\xi_1^+, \xi_2)\omega(\xi_1^+, \xi_2^*) + \omega(\xi_1^+, \xi_2)\omega(\xi_1^+, \xi_2^*)] + \frac{1}{3840} \frac{(\xi^2 + 16)(\xi^2 + 6)}{\xi^2} \times [\omega(\xi_1^+, \xi_2^*)\omega(\xi_1^+, \xi_2) + \omega(\xi_1^+, \xi_2)\omega(\xi_1^+, \xi_2^*) + \omega(\xi_1^+, \xi_2)\omega(\xi_1^+, \xi_2^*)]. \]
where for convenience we abbreviated $\omega(\xi, \xi) = \omega^3(\xi, \xi)$ and $\xi^\pm = \xi \pm 2i$.

\[
\tilde{p}_1(\xi_1, \xi_2) = \frac{1}{16} (1 + \omega(\xi_1, \xi_1) + \omega(\xi_1, \xi_1^*)) (1 + \omega(\xi_2, \xi_2) + \omega(\xi_2, \xi_2^*)) \\
+ \frac{1}{60} \left( \frac{\xi^2 + 196}{\xi^2 + 4} \right) \omega(\xi, \xi) \\
- \frac{1}{10} \left( \frac{\xi^2 + 22i\xi + 96}{\xi(\xi - 2i)^2} \right) [\omega(\xi_1, \xi_1) + \omega(\xi_2, \xi_2^*)] \\
- \frac{1}{10} \left( \frac{\xi^2 - 22i\xi + 96}{\xi(\xi + 2i)^2} \right) [\omega(\xi_1, \xi_1^*) + \omega(\xi_2, \xi_2)] \\
- \frac{1}{10} \left( \frac{\xi - 6i}{\xi^2} \right) [\omega(\xi_1, \xi_2) + \omega(\xi_1, \xi_2^*)] \\
+ \frac{\xi + 6i}{\xi^2} \omega(\xi_1^+, \xi_2^-) \omega(\xi_1^-, \xi_2) + \frac{\xi - 4i}{\xi^2} \omega(\xi_1, \xi_2^-) \\
- \frac{3}{80} \left( \frac{\xi + 2i}{\xi(\xi - 2i)^2} \right) [\omega(\xi_1^+, \xi_2^-) + \omega(\xi_1, \xi_2^-)] \\
+ \frac{\xi - 2i}{\xi(\xi + 2i)^2} [\omega(\xi_1^+, \xi_2^-) + \omega(\xi_1, \xi_2^-)] \\
+ \frac{p_1(\xi)}{25} \omega(\xi_1, \xi_1^+) \omega(\xi_2, \xi_2) + \frac{p_1(\xi)}{400} \left( \frac{\xi^2 + 16}{\xi^2 + 4} \right) [\omega(\xi_1^+, \xi_2^-) + \omega(\xi_1, \xi_2^-)] \\
+ \frac{3}{25} \left( \frac{11\xi^4 - 122\xi^2 + 912}{\xi^2(\xi^2 + 4)^2} \right) [\omega(\xi_1, \xi_1) \omega(\xi_2, \xi_2) + \omega(\xi_1, \xi_1^*) \omega(\xi_2, \xi_2^*)] \\
- \frac{1}{400} \left( \frac{11\xi^2 - 148}{\xi^2 + 4} \right) \left( \frac{\xi^2 + 16}{\xi^2 + 4} \right) \left( \frac{\xi - 6i}{\xi + 2i} \right) [\omega(\xi_1^+, \xi_2^-) + \omega(\xi_1, \xi_2^-)] \\
+ \frac{\xi^2 + 36}{\xi^2 + 4} [\omega(\xi_1^+, \xi_2^-) \omega(\xi_1^+, \xi_1) + \omega(\xi_2^-, \xi_2)] \\
+ \frac{\xi - 4i}{\xi^2} \omega(\xi_1^+, \xi_2^-) + \omega(\xi_1, \xi_2^-)] \\
- \frac{1}{300} \left( \frac{11\xi^2 - 148}{\xi^2 + 4} \right) \left( \frac{\xi^2 + 36}{\xi^2 + 4} \right) [\omega(\xi_1, \xi_1) \omega(\xi_1^+, \xi_2^-) + \omega(\xi_1, \xi_2^-) \omega(\xi_1^+, \xi_2^-)] \\
+ \frac{1}{300} \left( \frac{11\xi^2 - 148}{\xi^2 + 4} \right) \left( \frac{\xi^2 + 36}{\xi^2 + 4} \right) \left( \frac{\xi - 4i}{\xi^2} \right) [\omega(\xi_1, \xi_1) \omega(\xi_1^+, \xi_2^-) + \omega(\xi_1, \xi_2^-) \omega(\xi_1^+, \xi_2^-)] \\
+ \frac{1}{6400} \left( \frac{3\xi^2 + 76}{\xi^2 + 4} \right) \left( \frac{\xi^2 + 36}{\xi^2 + 4} \right) \left( \frac{\xi - 4i}{\xi^2} \right) \omega(\xi_1, \xi_2) \omega(\xi_1, \xi_2^*) \\
+ \frac{1}{6400} \left( \frac{\xi^2 + 16}{\xi^2 + 4} \right) \left( \frac{\xi - 6i}{\xi^2} \right) \omega(\xi_1, \xi_2) \omega(\xi_1, \xi_2^*) \\
+ [\omega(\xi_1, \xi_2) \omega(\xi_1, \xi_2^*)].
\]
\[- \frac{1}{6400} \frac{\xi^2 + 16}{\xi (\xi^2 + 4)} \left[ \frac{p_2(\xi)(\xi - 4i)}{(\xi - 2i)} \omega(\xi_1^-, \xi_2^+) \omega(\xi_1^+, \xi_2^-) \right] + \frac{p_2(\xi)}{6400} \frac{(\xi + 4i)}{(\xi + 2i)} \omega(\xi_1, \xi_2) \omega(\xi_1^+, \xi_2^-) \] 

\[- \frac{1}{6400} \frac{p_3(\xi)(\xi + 6i)(\xi + 2i)(\xi - 4i)^2}{\xi^2 (\xi - 2i)} \left[ \omega(\xi_1, \xi_2^-) \omega(\xi_1^+, \xi_2^+) \right] + \frac{1}{6400} \frac{p_3(\xi)(\xi - 6i)(\xi - 2i)(\xi + 4i)^2}{\xi^2 (\xi + 2i)} \left[ \omega(\xi_1, \xi_2^+ \omega(\xi_1^+, \xi_2^-) \right] \] 

\[- \frac{1}{6400} \frac{p_3(\xi)^2 + 36)(\xi^2 + 16)}{\xi^2 (\xi + 2i)} \left[ \omega(\xi_1^+, \xi_2^-) \omega(\xi_1^-, \xi_2^+) \right] - \frac{1}{6400} \frac{p_3(\xi)^2 + 36)(\xi^2 + 16)}{\xi^2 (\xi - 2i)} \left[ \omega(\xi_1^- \xi_2^+) \omega(\xi_1^+, \xi_2^-) \right] \] 

\[- \frac{1}{19200} \frac{(3\xi^4 + 54\xi^2 + 296)}{(\xi - 6i)(\xi - 2i)(\xi + 4)^2} \left[ \frac{(\xi^2 + 36)(\xi^2 + 16)^2}{(\xi^2 + 4)^2} \omega(\xi_1, \xi_2) \omega(\xi_1^+, \xi_2^-) \right] + \frac{(\xi - 6i)(\xi + 2i)(\xi - 4i)^2}{\xi^2} \omega(\xi_1^+, \xi_2^-) \omega(\xi_1^+, \xi_2^-) \omega(\xi_1^+, \xi_2^-) \] 

\[- \frac{(\xi^2 + 36)(\xi^2 + 16)}{\xi^2} \left[ \omega(\xi_1^+, \xi_2^-) \omega(\xi_1^+, \xi_2^-) \right] \left[ \frac{(\xi^2 + 16)^2}{(\xi^2 + 4)^2} \omega(\xi_1, \xi_2) \omega(\xi_1^+, \xi_2^-) \right] \] 

\[- \frac{1}{600} \frac{(11\xi^2 - 148)(\xi^2 + 16)}{\xi^2 (\xi^2 + 4)} \left[ \frac{(\xi^2 + 36)}{(\xi^2 + 4)^2} \omega(\xi_1, \xi_2) \omega(\xi_1^+, \xi_2^+) \omega(\xi_1^+, \xi_2^-) \right] + \omega(\xi_1, \xi_1^+) \omega(\xi_2, \xi_2^+) \omega(\xi_2, \xi_2^+) \] 

\[- \frac{(\xi - 6i)^2}{(\xi + 2i)} \omega(\xi_1, \xi_1^+) \omega(\xi_2, \xi_2^+) \omega(\xi_1^+, \xi_2^+) \] 

\[- \frac{(\xi + 6i)^2}{(\xi - 2i)} \omega(\xi_1, \xi_1^+) \omega(\xi_2, \xi_2^+) \omega(\xi_1^+, \xi_2^-) \right] \right\}. \quad \text{(C.2)} \] 

where \( \bar{p}_i(\xi) \) are the complex conjugates of \( p_i(\xi) \) which are defined by

\[
p_1(\xi) = (33\xi^5 - 22i\xi^4 - 116\xi^3 + 824i\xi^2 - 224\xi - 7104i), \]

\[
p_2(\xi) = (3\xi^5 - 18i\xi^4 + 4\xi^3 - 104i\xi^2 - 1824\xi - 4736i), \]

\[
p_3(\xi) = (3\xi^5 - 18i\xi^4 - 4\xi^3 - 296i). \quad \text{(C.3)} \]
\[
\hat{\rho}_2(\xi_1, \xi_2) = \frac{1}{16}(1 + \omega(\xi_1, \xi_1) + \omega(\xi_1, \xi_1^*))(1 + \omega(\xi_2, \xi_2) + \omega(\xi_2, \xi_2^*))
\] 
\[- \frac{1}{16} \left( \frac{\xi^2 + 100}{\xi^2 + 4i} \right) \omega(\xi_1, \xi_2)
\] 
\[- \frac{3(i \xi + 6i)(\xi - 4i)}{5 \xi^2 (\xi + 4i)} [\omega(\xi_1^*, \xi_2) + \omega(\xi_2, \xi_2^*)]
\] 
\[- \frac{3(i \xi - 6i)(\xi + 4i)}{5 \xi^2 (\xi - 4i)} [\omega(\xi_1^*, \xi_2) + \omega(\xi_2, \xi_2^*)]
\] 
\[+ \frac{3}{10} \frac{(\xi^2 - 2i \xi - 16)(\xi^2 - 2i \xi - 24)}{\xi^2 (\xi^2 + 4)(\xi - 2i)(\xi - 4i)} [\omega(\xi_1^*, \xi_1) + \omega(\xi_2, \xi_2^*)]
\] 
\[+ \frac{3}{10} \frac{(\xi^2 + 2i \xi - 16)(\xi^2 + 2i \xi - 24)}{\xi^2 (\xi^2 + 4)(\xi + 2i)(\xi + 4i)} [\omega(\xi_1^*, \xi_1) + \omega(\xi_2, \xi_2^*)]
\] 
\[- \frac{3}{40} \frac{(\xi^2 + 36)}{\xi^2 (\xi + 2i)^2} [\omega(\xi_1^*, \xi_2) + \omega(\xi_1^*, \xi_2^*)]
\] 
\[+ \frac{3}{80} \frac{((\xi + 2i)(\xi + 10i)\omega(\xi_1^*, \xi_2) + (\xi - 4i)(\xi - 2i)(\xi + 10i)\omega(\xi_1^*, \xi_2^*)}{\xi^2 (\xi + 2i)^2}
\] 
\[+ \frac{1}{64} \frac{q_1(\xi)\omega(\xi_1^*, \xi_1)\omega(\xi, \xi_1^*)}{\xi^2 (\xi^2 + 4)(\xi - 4i)} + \frac{\tilde{q}_1(\xi)\omega(\xi_1, \xi_1^*)\omega(\xi, \xi_2)}{\xi^2 (\xi^2 + 4)(\xi + 4i)}
\] 
\[+ \frac{1}{64} \frac{-3391488}{25} + \frac{377856}{25} \frac{\omega(\xi_1^*, \xi_1)\omega(\xi_2, \xi_2) + \omega(\xi_1^*, \xi_2)\omega(\xi_2, \xi_2^*)}{\xi^2 (\xi^2 + 16)(\xi^2 + 4)^2}
\] 
\[+ \frac{1}{6400} \frac{(\xi^4 + 84\xi^2 + 1472)(\xi^2 + 36)(\xi^2 + 16)}{\xi^2 (\xi^2 + 4)^2}
\] 
\[\times [\omega(\xi_1^*, \xi_2)\omega(\xi_1^*, \xi_2^*) + \omega(\xi_1^*, \xi_2^*)\omega(\xi_1^*, \xi_2)]
\] 
\[+ \frac{1}{6400} \frac{(\xi^4 + 84\xi^2 + 1472)(\xi^2 + 36)}{\xi^2}[\omega(\xi_1^*, \xi_2)\omega(\xi_1^*, \xi_2) + \omega(\xi_1^*, \xi_2)\omega(\xi_1^*, \xi_2^*)]
\] 
\[+ \frac{1}{6400} \frac{(\xi^4 + 84\xi^2 + 1472)(\xi^2 + 16)}{\xi^2}[\omega(\xi_1^*, \xi_2)\omega(\xi_1^*, \xi_2) + \omega(\xi_1^*, \xi_2)\omega(\xi_1^*, \xi_2^*)]
\] 
\[+ \frac{1}{6400} \frac{q_2(\xi)(\xi - 6i)\omega(\xi_1^*, \xi_2)\omega(\xi_1^*, \xi_2^*)}{\xi^2 (\xi - 4i)}
\] 
\[+ \frac{\tilde{q}_2(\xi)(\xi + 6i)\omega(\xi_1^*, \xi_2)\omega(\xi_1^*, \xi_2^*)}{\xi^2 (\xi + 4i)}
\] 
\[+ \frac{1}{6400} \frac{(\xi^2 + 16)}{\xi (\xi^2 + 4)} \frac{g_2(\xi)(\xi + 4i)\omega(\xi_1^*, \xi_2)\omega(\xi_1^*, \xi_2)}{(\xi - 2i)}
\]
\[
+ \frac{q_4(\xi)(\xi - 4i)\omega(\xi_1, \xi_2)\omega(\xi_1^+, \xi_2^+)}{(\xi + 2i)} \bigg] \\
- \frac{1}{6400} \frac{q_4(\xi)(\xi + 6i)(\xi - 4i)(\xi + 2i)}{\xi^2(\xi - 2i)} \left[ \omega(\xi_1^+, \xi_2^+\omega(\xi_1, \xi_2) + \omega(\xi_1^+, \xi_2^+)\omega(\xi_1^-, \xi_2^-) \right] \\
- \frac{1}{6400} \frac{q_4(\xi)(\xi - 6i)(\xi + 4i)(\xi - 2i)}{\xi^2(\xi + 2i)} \left[ \omega(\xi_1^+, \xi_2^+)\omega(\xi_1, \xi_2^+) + \omega(\xi_1^-, \xi_2)^\omega(\xi_1^+, \xi_2^+) \right] \\
+ \frac{1}{6400} \frac{q_4(\xi)(\xi^2 + 36)(\xi + 4i)}{\xi^2(\xi - 2i)} \left[ \omega(\xi_1^+, \xi_2^+)\omega(\xi_1, \xi_2^+) + \omega(\xi_1^+, \xi_2^+)\omega(\xi_1^-, \xi_2^-) \right] \\
+ \frac{1}{6400} \frac{q_4(\xi)(\xi^2 + 36)(\xi - 4i)}{\xi^2(\xi + 2i)} \left[ \omega(\xi_1^+, \xi_2^+)\omega(\xi_1, \xi_2^+) + \omega(\xi_1^+, \xi_2^+)\omega(\xi_1^-, \xi_2^-) \right] \\
- \frac{3}{400} \frac{(\xi^4 - 76\xi^2 + 832)(\xi - 6i)}{\xi^2(\xi^2 + 4)(\xi + 2i)} \left[ \omega(\xi_1^+, \xi_2^+)\omega(\xi_1, \xi_2^+) + \omega(\xi_1^+, \xi_2^+)\omega(\xi_2^+, \xi_2^+) \right] \\
- \frac{3}{400} \frac{(\xi^4 - 76\xi^2 + 832)(\xi + 6i)}{\xi^2(\xi^2 + 4)(\xi - 2i)} \left[ \omega(\xi_1^+, \xi_2^+)\omega(\xi_1, \xi_2^+) + \omega(\xi_1^-, \xi_2^+)\omega(\xi_2^+, \xi_2^+) \right] \\
- \frac{3}{400} \frac{(\xi^4 - 76\xi^2 + 832)(\xi^2 + 36)}{\xi^2(\xi^2 + 4)} \left[ \omega(\xi_1^+, \xi_2^+)\omega(\xi_2^+, \xi_2^+) + \omega(\xi_1^+, \xi_2^+)\omega(\xi_2^-, \xi_2^-) \right] \\
+ \omega(\xi_1^+, \xi_2^+)\omega(\xi_2^+, \xi_2^+) + \omega(\xi_1^+, \xi_2^+)\omega(\xi_2^-, \xi_2^-) \right] \\
- \frac{1}{100} \frac{(\xi^4 - 76\xi^2 + 832)(\xi^2 + 36)}{\xi^2(\xi^2 + 4)(\xi + 2i)} \left[ \omega(\xi_1^+, \xi_2^+)\omega(\xi_1, \xi_2^+) + \omega(\xi_1^+, \xi_2^+)\omega(\xi_2^+, \xi_2^+) \right] \\
- \frac{1}{100} \frac{(\xi^4 - 76\xi^2 + 832)(\xi^2 + 36)}{\xi^2(\xi^2 + 4)(\xi - 4i)} \left[ \omega(\xi_1^-, \xi_2^+)\omega(\xi_1, \xi_2^+) + \omega(\xi_1^-, \xi_2^+)\omega(\xi_2^+, \xi_2^+) \right] \\
- \frac{1}{200} \frac{(\xi^4 - 76\xi^2 + 832)}{\xi^2(\xi^2 + 4)} \left[ \omega(\xi_1^+, \xi_2^+)\omega(\xi_1^+, \xi_2^+) + \omega(\xi_1^+, \xi_2^+)\omega(\xi_2^-, \xi_2^-) \right] \\
+ \omega(\xi_1^+, \xi_2^+)\omega(\xi_2^+, \xi_2^+) + \omega(\xi_1^+, \xi_2^+)\omega(\xi_2^-, \xi_2^-) \right] \\
+ \frac{1}{19200} \frac{(\xi^4 + 34\xi^2 + 312)}{\xi^2(\xi^2 + 4)} \left[ \omega(\xi_1^+, \xi_2^+)\omega(\xi_2^+, \xi_2^+) + \omega(\xi_1^+, \xi_2^+)\omega(\xi_2^+, \xi_2^+) \right] \\
+ \frac{(\xi^2 + 36)(\xi^2 + 16)}{\xi^2} \left[ \omega(\xi_1^+, \xi_2^+)\omega(\xi_1^-, \xi_2^+) + \omega(\xi_1^+, \xi_2^+)\omega(\xi_2^+, \xi_2^+) \right] \\
- \frac{1}{\xi^2} \left[ \omega(\xi_1^+, \xi_2^+)\omega(\xi_2^-, \xi_2^-) \right] \\
- \frac{1}{\xi^2} \left[ \omega(\xi_1^+, \xi_2^+)\omega(\xi_2^-, \xi_2^-) \right] \\
- \frac{1}{\xi^2} \left[ \omega(\xi_1^+, \xi_2^+)\omega(\xi_2^-, \xi_2^-) \right] \\
- \left( \frac{\xi^2 + 36)(\xi^2 + 16)}{\xi^2} \omega(\xi_1^+, \xi_2^+)\omega(\xi_1^+, \xi_2^+) \right] \\
\right).
where $\bar{q}_i(\zeta)$ are the complex conjugates of $q_i(\zeta)$ which are defined by

$$q_1(\zeta) = \left( -\frac{1916928}{25} - \frac{86016}{5} i \zeta + \frac{362496}{25} \zeta^2 + \frac{13824}{5} i \zeta^3 - \frac{5376}{5} \zeta^4 - \frac{384}{5} i \zeta^5 + \frac{576}{25} \zeta^6 \right),$$

$$q_2(\zeta) = (\zeta^5 - 14i\zeta^4 - 36\zeta^3 + 104i\zeta^2 - 288\zeta + 4992i),$$

$$q_3(\zeta) = (\zeta^4 - 10i\zeta^3 - 36\zeta^2 - 200\zeta - 1248).$$

Due to the normalization property of the density matrix $\text{Tr} D^{(2)}_1(\xi, \zeta) = 1$, we can write the fourth coefficient in terms of the previous ones. Therefore we have $\rho^{(3)}_3(\xi, \zeta) = \frac{1}{3} (1 - \rho^{(3)}_0(\xi, \zeta) - 3\rho^{(3)}_1(\xi, \zeta) - 5\rho^{(3)}_2(\xi, \zeta)).$

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