SMALL DOUBLING
IN GROUPS WITH MODERATE TORSION

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ABSTRACT. We determine the structure of a finite subset \( A \) of an abelian group given that \( |2A| < 3(1 - \varepsilon)|A|, \varepsilon > 0 \); namely, we show that \( A \) is contained either in a “small” one-dimensional coset progression, or in a union of fewer than \( \varepsilon^{-1} \) cosets of a finite subgroup.

The bounds \( 3(1 - \varepsilon)|A| \) and \( \varepsilon^{-1} \) are best possible in the sense that none of them can be relaxed without tightened another one, and the estimate obtained for the size of the coset progression containing \( A \) is sharp.

In the case where the underlying group is infinite cyclic, our result reduces to the well-known Freiman’s \((3n - 3)\)-theorem; the former thus can be considered as an extension of the latter onto arbitrary abelian groups, provided that there is “not too much torsion involved”.

1. Introduction and summary of results

For subsets \( A \) and \( B \) of an additively written abelian group, by \( A + B \) we denote the set of all group elements representable as \( a + b \) with \( a \in A \) and \( b \in B \). We abbreviate \( A + A \) as \( 2A \) and define the doubling coefficient of a nonempty set \( A \) to be the quotient \( |2A|/|A| \).

It is a basic folklore fact that if \( A \) is a finite set of integers, then \( |2A| \geq 2|A| - 1 \); more generally, if \( A \) and \( B \) are finite nonempty subsets of a torsion-free abelian group, then \( |A + B| \geq |A| + |B| - 1 \). An extension of this fact onto general abelian groups with torsion is a deep result due to Kneser, discussed in Section 3.

In another direction, Freiman [F62] has established the structure of integer sets \( A \) satisfying \( |2A| \leq 3|A| - 3 \); that is, roughly, sets with the doubling coefficient up to 3. This result, commonly referred to as Freiman’s \((3n - 3)\)-theorem, along with its generalizations onto distinct set summands, can be found in any standard additive combinatorics monograph; see, for instance, [N96, Theorem 1.13].

It is a notoriously difficult open problem to merge together the results of Kneser and Freiman establishing the structure of sets with the doubling coefficient less than 3 in abelian groups with torsion. This paper is intended as a step towards the solution of this problem.

Our main result shows that a small-doubling set is either contained in the union of a small number of cosets of a finite subgroup, or otherwise is densely contained in a coset progression.
**Theorem 1.** Let \( A \) be a finite subset of an abelian group \( G \) such that \( A \) cannot be covered with fewer than \( n \) cosets of a finite subgroup of \( G \), for some real \( n > 0 \). If \( |2A| < 3\left(1 - \frac{1}{n}\right)|A| \), then there exist an arithmetic progression \( P \subseteq G \) of size \( |P| \geq 3 \) and a finite subgroup \( K \leq G \) such that \( A \subseteq P + K \) and \( (|P| - 1)|K| \leq |2A| - |A| \).

Letting \( \tau = |2A|/|A| \), the conclusion \( (|P| - 1)|K| \leq |2A| - |A| \) can be rewritten as \( |P + K| \leq (\tau - 1)|A| + |K| \); it is a way to say that \( A \) is dense in \( P + K \).

We derive Theorem 1 from the following, essentially equivalent, result.

**Theorem 2.** Suppose that the abelian group \( G \) has the direct sum decomposition \( G = \mathbb{Z} \oplus H \) with \( H < G \) finite. Let \( A \subseteq G \) be a finite set, and let \( n \) be number of elements of the image of \( A \) under the projection \( G \to \mathbb{Z} \) along \( H \). If \( |2A| < 3\left(1 - \frac{1}{n}\right)|A| \), then there exist an arithmetic progression \( P \subseteq G \) of size \( |P| \geq 3 \) and a subgroup \( K \leq H \) such that \( A \subseteq P + K \) and \( (|P| - 1)|K| \leq |2A| - |A| \).

The equality \( G = \mathbb{Z} \oplus H \) means that \( G \) is the direct sum of its infinite cyclic subgroup and the subgroup \( H \). To simplify the notation, the former is identified with the group of integers.

The following example shows that Theorem 2 is sharp in the sense that the assumption \( |2A| < 3\left(1 - \frac{1}{n}\right)|A| \) cannot be relaxed, and the conclusion \( (|P| - 1)|K| \leq |2A| - |A| \) cannot be strengthened.

**Example 1.** Let \( P := [0, l] \) and \( A := ([0, n-2] \cup \{l\}) + K \), where \( K \leq H \), and \( l \geq n - 1 \geq 2 \) are integers; thus, \( |A| = n|K| \). If \( l > 2n - 3 \), then \( |2A| = (3n - 3)|K| = 3\left(1 - n^{-1}\right)|A| \), and \( A \) fails to have the structure described in Theorem 3 as \( |2A| - |A| = (2n - 3)|K| < (|P| - 1)|K| \). Thus, to conclude that a set \( A \subseteq \mathbb{Z} \oplus H \) with \( |2A| < 3\left(1 - \varepsilon\right)|A| \) is densely contained in a coset progression, one needs to assume that \( A \) cannot be covered with fewer than \( \varepsilon^{-1} \) cosets of a finite subgroup (or make some other extra assumption).

On the other hand, if \( l \leq 2n - 3 \), then \( |2A| = (l + n)|K| \); therefore, \( |2A| - |A| = (|P| - 1)|K| \).

**Remark 1.** The inequality \( |P| \geq 3 \) in Theorem 2 follows in fact automatically from other assertions of the theorem. We cannot have \( |P| = 1 \) because this would lead to \( n = 1 \), and consequently to \( |2A| < 0 \). We also cannot have \( |P| = 2 \) because this would result in \( n = 2 \) and \( |2A| < \frac{3}{2}|A| \). The latter, in its turn, is known to imply (see, for instance, Lemma 1 below) that \( A \) is contained in a coset of a finite subgroup of \( G \); hence, in an \( H \)-coset. This, however, contradicts the equality \( n = 2 \). The same remark applies to Theorem 1.

**Remark 2.** In the particular case where \( H \) is trivial, and \( A \) is a subset of the infinite cyclic group, Theorem 2 is equivalent to Freiman’s classical \((3n - 3)\)-theorem, see [F62] or [N96, Theorem 1.13]. Theorem 2 thus can be considered as an extension of the \((3n - 3)\)-theorem onto the groups with torsion.
Remark 3. As a corollary of Theorem 2, for any finite set \( A \subseteq \mathbb{Z} \oplus H \), denoting by \( n \) the size of the projection of \( A \) onto \( \mathbb{Z} \) along \( H \), we have \(|2A| \geq (2 - \frac{1}{n})|A|\). This follows by letting \( \tau := |2A|/|A| \) and observing that

\[
(\tau - 1)|A| = |2A| - |A| \geq (|P| - 1)|K| \geq (n - 1)|K| \geq \left(1 - \frac{1}{n}\right)|A|.
\]

We remark that, while the resulting estimate \(|2A| \geq (2 - \frac{1}{n})|A|\) may not be completely trivial, it is not particularly deep either, and can be proved independently of the theorem, with a simple combinatorial reasoning in the spirit of the proof of Lemma 6 (Section 4).

Remark 4. It should be possible to use our method to treat sumsets of the form \( A + B \) with different set summands, and in particular to prove analogues of Theorems 1 and 2 for the difference sets \( A - A \).

Theorem 2 can be compared against the following result of Balasubramanian and Pandey, which is an elaboration on an earlier result of Deshouillers and Freiman [DF86, Theorem 2].

**Theorem 3** (Balasubramanian-Pandey [BP18, Theorem 5]). Let \( d \geq 2 \) be an integer and suppose that \( A \subseteq \mathbb{Z} \oplus (\mathbb{Z}/d\mathbb{Z}) \) is a finite set with \(|2A| < 2.5|A|\). For \( z \in \mathbb{Z} \), let \( A_z := A \cap (z + \mathbb{Z}/d\mathbb{Z}) \), and let \( B := \{z \in \mathbb{Z} : A_z \neq \emptyset\} \). If \(|B| \geq 6 \) and \( \gcd(B - B) = 1 \), then there exists a subgroup \( K \leq \mathbb{Z}/d\mathbb{Z} \) and elements \( x, y \in \mathbb{Z}/d\mathbb{Z} \) such that, letting \( l := \max B - \min B \), we have

- i) \( A \subseteq \{(b, bx + y) : b \in B\} + K; \)
- ii) there exists \( b \in B \) with \(|A_b| \geq \frac{2}{3}|K|; \)
- iii) \( |K| \leq |2A| - |A| \).

Balasubramanian and Pandey also include into the statement the estimate \( l < \frac{3}{2}|B| \), but in fact this estimate follows easily from i) and iii):

\[
l|K| < 2.5|A| - |A| = \frac{3}{2}|A| \leq \frac{3}{2}|B||K|.
\]

Generally, switching back to the notation and settings of Theorem 2, and letting \( \tau := |2A|/|A| \), we have

\[
(|P| - 1)|K| \leq (\tau - 1)|A| \leq (\tau - 1)n|K|
\]

whence \(|P| \leq (\tau - 1)n + 1\). (Incidentally, in view of \( \tau \leq 3(1 - \frac{1}{n}) \) this implies \(|P| \leq 2n - 3\).)

In the same vein, i) and iii) imply an estimate which is only slightly weaker than ii): namely, by iii) we have \( l|K| \leq (\tau - 1)|A| \); therefore, by averaging, there exists an element \( b \in B \) with

\[
|A_b| \geq \frac{|A|}{|B|} \geq \frac{(\tau - 1)|A|}{(\tau - 1)(l + 1)} \geq \frac{l}{l + 1} \frac{|K|}{\tau - 1} \geq \frac{2}{3} \left(1 - \frac{1}{l + 1}\right)|K|.
\]

To match the Balasubramanian-Pandey estimate \( \max_{b \in B} |A_b| \geq \frac{3}{2}|K| \), we prove in Section 2 the following theorem showing (subject to Theorem 2) that if \( n \) is sufficiently large, then there exists a \( K \)-coset containing at least \( \frac{|K|}{\tau - 1} \) elements of \( A \).
Theorem 4. Suppose that $G$, $H$, $A$, $n$, $P$, and $K$ are as in Theorem 2, and let $\tau := \frac{|2A|}{|A|}$. If $n \geq \frac{4\tau - 6}{(\tau - 2)(3 - \tau)}$, then there exists a $K$-coset containing at least $\frac{|K|}{\tau - 1}$ elements of $A$.

Compared to Theorem 3, our Theorem 2 applies to the groups $\mathbb{Z} \oplus H$ with $H$ not necessarily cyclic and, most importantly, allows the doubling coefficient to be as large as $3 - o(1)$ (instead of $2.5$), which is best possible, as shown above.

The layout of the remaining part of the paper is as follows. In Section 2 we deduce Theorems 1 and 4 from Theorem 2, allowing us to concentrate on the proof of the latter theorem for the rest of the paper. In Section 3 we collect some general results needed for the proof; in particular, we introduce and briefly discuss Kneser’s theorem. In Section 4 we prove some basic estimates related to the particular settings of Theorem 2 (in contrast with Section 3 where the results are of general nature). Section 5 contains a result which, essentially, establishes the special case of Theorem 2 where the set $A$ can be partitioned into two “additively independent” subsets. Finally, we prove Theorem 2 in Section 6.

2. Deduction of Theorems 1 and 4 from Theorem 2

Proof of Theorem 1. Let $A$ be a finite subset of an abelian group $G$ such that $A$ cannot be covered with fewer than $n$ cosets of a finite subgroup of $G$, while

$$|2A| < 3\left(1 - \frac{1}{n}\right)|A|,$$

with a real $n > 0$. We want to prove, assuming Theorem 2, that there exist an arithmetic progression $P \subseteq G$ and a subgroup $K \leq G$ such that $A \subseteq P + K$ and $(|P| - 1)|K| \leq |2A| - |A|$. As explained in Section 1 (Remark 1), the progression $P$ will satisfy $|P| \geq 3$.

Without loss of generality, we assume that $G$ is generated by $A$. By the fundamental theorem of finitely generated abelian groups, there is then an integer $r \geq 0$ and a finite subgroup $H \leq G$ such that $G \cong \mathbb{Z}^r \oplus H$. Indeed, we have $r \geq 1$ as otherwise $G$ would be finite; hence, $A$ would be contained in just one single finite coset (the group $G$ itself), forcing $n \leq 1$ and thus contradicting the small-doubling assumption (1).

Let $G' := \mathbb{Z} \oplus H$. To avoid confusion, throughout the proof we use the direct product notation for the elements of the groups $G$ and $G'$.

Fix an integer $M > 0$ divisible by all positive integers up to $|2A|$, and consider the mapping $\psi: G \rightarrow G'$ defined by

$$\psi(x_1, \ldots, x_r, h) := (x_1 + Mx_2 + \cdots + M^{r-1}x_r, h); \quad x_1, \ldots, x_r \in \mathbb{Z}, \ h \in H.$$

If $M$ is large enough (as we assume below), then different elements of $A$ have different images under $\psi$, and similarly for $2A$; consequently, writing $A' := \psi(A)$, we have $|A'| = |A|$ and $|2A'| = |2A|$, whence

$$|2A'| < 3\left(1 - \frac{1}{n}\right)|A'|.$$
(we implicitly use here the equality $2\psi(A) = \psi(2A)$).

Denote by $m$ the number of elements of the projection of $A$ onto the first (torsion-free) component of $G$. If $M$ is sufficiently large, then this is also the number of elements of the projection of $A'$ onto the first component of $G'$. Since $A$ is not contained in a union of fewer than $n$ cosets, we have $m \geq n$, resulting in

$$|2A'| < 3\left(1 - \frac{1}{m}\right)|A'|.$$  

Applying Theorem 2, we conclude that there exist a finite arithmetic progression $P' \subseteq G'$ and a subgroup $K \leq H$ such that $A' \subseteq P' + K$ and

$$\left(|P'| - 1\right)|K| \leq |2A'| - |A'| = |2A| - |A|.$$  

We assume that $P'$ is the shortest progression possible with $A' \subseteq P' + K$.

Write $N := |P'| - 1$, and let $c \in G'$ and $(d, h) \in G'$ denote the initial term and the difference of the progression $P'$, respectively; thus,

$$P' = c + \{j(d, h) : j \in [0, N]\}; \quad d \in \mathbb{Z}, \ h \in H.$$  

Notice that $d \neq 0$, as otherwise we would have $A' \subseteq P' + K \subseteq c + H$, as a result of which $A'$, and therefore also $A$, would be contained in a single $H$-coset.

Since $P'$ is the shortest possible progression with $A' \subseteq P' + K$, there are elements $(a_1, \ldots, a_r, f), (b_1, \ldots, b_r, g) \in A$ such that $\psi(a_1, \ldots, a_r, f) = c$ and $\psi(b_1, \ldots, b_r, g) = c + N(d, h)$; consequently,

$$(b_1 - a_1) + M(b_2 - a_2) + \cdots + M^{r-1}(b_r - a_r) = Nd.$$  

Since $N = |P'| - 1 \leq |2A| - |A| < |2A|$, and recalling that $M$ was chosen to be divisible by all integers up to $|2A|$, we have $N \mid M$, and therefore $b_1 - a_1$ is a multiple of $N$. Thus

$$d = (b_1 - a_1)N^{-1} + MN^{-1}(b_2 - a_2) + \cdots + M^{r-1}N^{-1}(b_r - a_r),$$  

where all summands in the right-hand side are integers.

We know that for any element $(\alpha_1, \ldots, \alpha_r, \eta) \in A$, there exist $j \in [0, N]$ and $k \in K$ such that

$$(\alpha_1 + \cdots + M^{r-1}\alpha_r, \eta) = c + j(d, h) + (0, k)$$  

$$= (a_1 + \cdots + M^{r-1}\alpha_r, f) + j(d, h) + (0, k).$$  

Recalling (3), we obtain

$$(\alpha_1 - a_1) + \cdots + M^{r-1}(\alpha_r - a_r) = jd = j(b_1 - a_1)N^{-1} + \cdots + jM^{r-1}N^{-1}(b_r - a_r);$$

that is,

$$(\alpha_1 - a_1)N + \cdots + M^{r-1}(\alpha_r - a_r)N = j(b_1 - a_1) + \cdots + jM^{r-1}(b_r - a_r)$$  

with $j \in [0, N]$ depending on $\alpha_1, \ldots, \alpha_r$. (Notice that $N$ depends on $M$, but is bounded: $N \leq N[K] \leq |2A| - |A|$ by (2).) Choosing $M$ sufficiently large, from (4) we get

$$(\alpha_i - a_i)N = j(b_i - a_i), \ 1 \leq i \leq r,$$  

(5)
showing that \((b_i - a_i)j\) is divisible by \(N\). Using again the fact that \(P'\) is the shortest possible progression with \(A' \subseteq P' + K\), we conclude that the possible values of \(j\) that can emerge from different elements \((\alpha_1, \ldots, \alpha_r, \eta) \in A\) are coprime. Hence, there is a linear combination of these values, with integer coefficients, which is equal to 1. Consequently, from (5), all numbers \((b_i - a_i)N^{-1}, 1 \leq i \leq r\), are integers, and then, by (5) again,
\[
(\alpha_1, \ldots, \alpha_r, \eta) = (a_1, \ldots, a_r, f) + j((b_1 - a_1)N^{-1}, \ldots, (b_r - a_r)N^{-1}, h) + (0, \ldots, 0, k).
\]
This shows that \(A \subseteq P + K\), where \(P \subseteq G\) is the \((N + 1)\)-term arithmetic progression with the initial term \((a_1, \ldots, a_r, f)\) and the difference \(((b_1 - a_1)N^{-1}, \ldots, (b_r - a_r)N^{-1}, h)\).

Finally, by (2),
\[
|2A| - |A| = |2A'| - |A'| \geq (|P'| - 1)|K| = (|P| - 1)|K|.
\]

\[\square\]

**Proof of Theorem 4.** Let \(B\) denote the projection of \(A\) onto \(\mathbb{Z}\) along \(H\); thus, \(|P| \geq |B| = n\), with equality if and only if \(B\) is an arithmetic progression. If this is not the case, then \(|P| \geq n + 1\) and, by averaging, there is a \(K\)-coset containing at least
\[
\frac{|A|}{n} \geq \frac{|A|}{|P| - 1} \geq \frac{|K|}{\tau - 1}
\]
elements of \(A\) (the last inequality following directly from the estimate \(|2A| - |A| \geq (|P| - 1)|K|\) of Theorem 2). Suppose thus that \(B\) is an arithmetic progression and, consequently, \(|P| = n\) and \(|2A| - |A| \geq (n - 1)|K|\), whence
\[
|A| \geq \frac{n - 1}{\tau - 1}|K|.
\]

Let
\[
M := \max\{|A \cap (g + K)| : g \in P\}, \quad \mu := |M|/|K|,
\]
\[
P_0 := \{g \in P : |A \cap (g + K)| \leq \frac{1}{2}|K|\}, \quad P_1 := P \setminus P_0, \quad \text{and} \quad m := |P_0|.
\]

Notice that \(M > \frac{1}{2}|K|\) as otherwise we would have
\[
\frac{1}{2}|K| \geq M \geq \frac{|A|}{n} \geq \left(1 - \frac{1}{n}\right) \frac{|K|}{\tau - 1},
\]
which is easily seen to contradict \(\tau < 3(1 - \frac{1}{n})\). Therefore \(P_1\) is nonempty, and \(m < n\).

We want to show that \(\mu > \frac{1}{\tau - 1}\). Suppose for a contradiction that this is wrong. Since \(P + K\) is a union of \(n\) pairwise disjoint \(K\)-cosets, of which \(m\) contain at most \(\frac{1}{2}|K|\) elements of \(A\), and the remaining \(n - m\) contain at most \(M\) elements each, we have
\[
\frac{n - 1}{\tau - 1}|K| \leq |A| \leq m \cdot \frac{1}{2}|K| + (n - m) \cdot M,
\]
leading to
\[
\frac{n - 1}{\tau - 1} \leq \frac{1}{2}m + (n - m)\mu < \frac{1}{2}m + \frac{n - m}{\tau - 1},
\]
which is impossible.
where the last inequality follows from the assumption \( \mu < 1/(\tau - 1) \). This simplifies to the estimate

\[
m < \frac{2}{3 - \tau}
\]  

which we will need shortly.

The set \( 2P_1 + K \) is a union of \(|2P_1| \geq 2|P_1| - 1 = 2(n - m) - 1 \) distinct \( K \)-cosets contained in \( 2A \) by the pigeonhole principle. The set \( P + P_1 + K \) is a union of \(|P + P_1| \geq \vert P \vert + \vert P_1 \vert - 1 = 2n - m - 1 \) distinct \( K \)-cosets, each of them containing at least \( \frac{1}{2} \vert K \vert \) elements of \( 2A \). We thus can find \( 2n - 2m - 1 \) cosets represented by the elements of \( 2P_1 \), and then \( m \) more cosets represented by the elements of \( P + P_1 \). Altogether, we get \( 2n - m - 1 \) cosets containing at least

\[
(2n - 2m - 1)|K| + \frac{1}{2}|K|m = \left(2n - \frac{3}{2}m - 1\right)|K|
\]

elements of \( 2A \). It follows that

\[
2n - \frac{3}{2}m - 1 \leq \frac{|2A|}{|K|} = \tau \frac{|A|}{|K|} \leq \left(\frac{1}{2}m + (n - m)\mu\right)\tau < \left(\frac{1}{2}m + \frac{n - m}{\tau - 1}\right)\tau,
\]

cf. (6). Rearranging the terms gives

\[
\left(1 - \frac{1}{\tau - 1}\right)n < \left(\frac{3}{2} + \frac{\tau}{2} - \frac{\tau}{\tau - 1}\right)m + 1;
\]

that is, using (7),

\[
\frac{\tau - 2}{\tau - 1} n < \frac{\tau^2 - 3}{2(\tau - 1)} m + 1 < \frac{\tau^2 - 3}{(\tau - 1)(3 - \tau)} + 1
\]

leading to

\[
n < \frac{\tau^2 - 3}{(\tau - 2)(3 - \tau)} + \frac{\tau - 1}{\tau - 2} = \frac{4\tau - 6}{(\tau - 2)(3 - \tau)},
\]

and the assertion follows. \( \square \)

The rest of the paper is devoted to the proof of Theorem 2.

3. General results

In this section we collect some general results valid in any abelian group, regardless of the particular settings of Theorem 2.

For a subset \( S \) of an abelian group, let \( \pi(S) \) denote the period (stabilizer) of \( S \); that is, \( \pi(S) \) is the subgroup consisting of all those group elements \( g \) with \( S + g = S \). The set \( S \) is called aperiodic or periodic according to whether \( \pi(S) \) is or is not the zero subgroup.

We start with a basic theorem due to Kneser which is heavily used in our argument.

**Theorem 5** (Kneser, [Kn53, Kn55]; see also [N96, Theorem 4.1]). If \( B \) and \( C \) are finite, non-empty subsets of an abelian group with

\[
|B + C| \leq |B| + |C| - 1,
\]
then letting \( L := \pi(B + C) \) we have

\[
|B + C| = |B + L| + |C + L| - |L|.
\]

We will be referring Theorem 5 as Kneser’s theorem.

Since, in the above notation, we have \(|B + L| \geq |B| \) and \(|C + L| \geq |C| \), Kneser’s theorem shows that \(|B + C| \geq |B| + |C| - |L|\), leading to

**Corollary 1.** If \( B \) and \( C \) are finite, non-empty subsets of an abelian group, such that \(|B + C| < |B| + |C| - 1\), then \( B + C \) is periodic.

The following lemma is well known, but tracing it down to the origin is hardly possible.

**Lemma 1.** Let \( B \) be a finite subset of an abelian group. If \(|2B| < \frac{3}{2}|B|\), then there is a subgroup \( L \) such that \( B - B = L \), and \( 2B \) is an \( L \)-coset (as a result of which \( B \) is contained in a unique \( L \)-coset).

We give a somewhat nonstandard, self-contained proof of the lemma.

*Proof of Lemma 1.* For a group element \( g \), denote by \( r(g) \) the number of representations of \( g \) as a difference of two elements of \( B \). If \( g \in B - B \), then choosing arbitrarily \( b, c \in B \) with \( g = b - c \) we get

\[
r(g) = |(b + B) \cap (c + B)| \geq 2|B| - |2B| > \frac{1}{2}|B|.
\]

By the pigeonhole principle, for any \( g_1, g_2 \in B - B \) there are representations \( g_1 = b_1 - c_1 \), \( g_2 = b_2 - c_2 \) with \( c_1 = c_2 \); consequently, \( g_1 - g_2 = b_1 - b_2 \in B - B \), showing that \( L := B - B \) is a subgroup. Clearly, \( B \) is contained in a unique \( L \)-coset.

As we have shown, for every element \( g \in L = B - B \) we have \( r(g) > \frac{1}{2}|B| \). As a result,

\[
|B|(|B| - 1) = \sum_{g \in L \setminus \{0\}} r(g) > \frac{1}{2}|B| \cdot (|L| - 1),
\]

implying \(|B| > \frac{1}{2}|L|\). Recalling that \( B \) is contained in a unique \( L \)-coset, and using the pigeonhole principle again, we conclude that \( 2B \) is an \( L \)-coset. \( \square \)

**Lemma 2.** Suppose that \( B \) is a subset of an abelian group with \( 0 \in B \). If \( N \geq 3 \) is an integer such that \(|B| = N + 1 \) and \(|2B| = 2N + 1 \) (thus \(|2B \setminus B| = N\)), then one of the following holds:

i) there exist \( b_1, \ldots, c_N \in B \) such that \( 2B \setminus B = \{b_1 + c_1, \ldots, b_N + c_N\} \), and every element of \( B \) appears among \( b_1, \ldots, c_N \) at most \( N \) times;

ii) there is a subgroup \( L \) with \(|L| = N \) and a group element \( g \) with \( 2g \notin L \) such that \( B = L \cup \{g\} \). (In this case there exist \( b_1, \ldots, c_N \in B \) such that \( 2B \setminus B = \{b_1 + c_1, \ldots, b_N + c_N\} \), and every element of \( B \) appears among \( b_1, \ldots, c_N \) exactly once, except that \( 0 \) does not appear at all, and \( g \) appears \( N + 1 \) times.)
iii) $N = 2$ and there is a subgroup $L$ with $|L| = 2$ and a group element $g$ with $2g \notin L$ such that $B = (g + L) \cup \{0\}$.

**Proof.** Leaving the case $N = 2$ to the reader (hint: write $B = \{0, b, g\}$ and consider two cases: $b + g = 0$ and $b + g \neq 0$), we confine ourself to the general case where $N \geq 3$.

Choose $b_1, \ldots, c_N \in B$ arbitrarily to have $2B \setminus B = \{b_1 + c_1, \ldots, b_N + c_N\}$. Since all sums $b_i + c_i$ are distinct, for any $g \in B$ there is at most one index $i \in [1, N]$ with $b_i = c_i = g$. Consequently, if there is an element $g \in B$ which appears at least $N + 1$ times among $b_1, \ldots, c_N$ (as we now assume), then in fact it appears $N + 1$ times exactly: namely, $b_i = c_i = g$ for some $i \in [1, N]$ and, besides, for any $j \neq i$, exactly one of $b_i$ and $c_j$ is equal to $g$. Redenoting, we assume that $b_1 = c_1 = \cdots = c_N = g$.

Notice that $2g = b_1 + c_1 \in 2B \setminus B$ along with $0 \in B$ show that $g \neq 0$. Write $B_0 := B \setminus \{0\}$ and $B_g := B \setminus \{g\}$. Since the sums $b_i + c_i = b_i + g$ are pairwise distinct, so are the elements $b_1, \ldots, b_N \in B$. Moreover, $b_1, \ldots, b_N$ are nonzero in view of $b_i + g = b_i + c_i \notin B$ and $g \in B$, and since $|B_0| = N$, it follows that $\{b_1, \ldots, b_N\} = B_0$; consequently, $2B \setminus B = g + B_0$.

If there exist some $b, c \in B_g$ with $b + c \notin B$, then choosing $i \in [1, N]$ with $b_i + g = b + c$ and replacing $b_i$ with $b$ and $c_i$ with $c$ in the $2N$-tuple $(b_1, \ldots, c_N)$, we get another $2N$-tuple $(b'_1, \ldots, c'_N)$ such that the sums $b'_i + c'_i$ list all elements of $2B \setminus B$. If $i \in [2, N]$, then $g$ appears exactly $N$ times among $b'_1, \ldots, c'_N$, so that no other element of $B$ can appear $N + 1$ or more times. Similarly, if $i = 1$, then in view of $c'_2 = \cdots = c'_N = g$, and since all sums $b'_2 + c'_2, \ldots, b'_N + c'_N$ are pairwise distinct, every element $b \in B_g$ appears at most $3 < N + 1$ times among $b'_1, \ldots, c'_N$. Thus, the assertion holds true in this case.

Suppose therefore that $b, c \in B_g$ with $b + c \notin B$ do not exist; that is, $2B_g \subseteq B$. This gives $|2B_g| \leq |B_g| + 1$; hence, by Lemma 1 and in view of $0 \in B_g$, the set $L := B_g - B_g = 2B_g$ is a subgroup. Furthermore, since $B_g \subseteq 2B_g = L$ and $|B_g| \geq |2B_g| - 1 = |L| - 1$, we have either $B_g = L$, or $B_g = L \setminus \{l\}$ with some $l \in L$, $l \neq 0$. The latter case is in fact impossible as we would have $l \in 2B_g \setminus B$ in this case, contradicting the present assumption $2B_g \subseteq B$. In the former case we have $B = L \cup \{g\}$ and $2B = L \cup (g + L) \cup \{2g\}$, with $2g \notin L$ in view of $|2B| = 2N + 1 = 2|B| - 1 = 2|L| + 1$.

**Lemma 3.** Suppose that $L$ is a subgroup, and that $B$ and $C$ are subsets of an abelian group. Let $\varphi_L$ denote the canonical homomorphism onto the quotient group.

i) We have $\varphi_L(B \cup C) = \varphi_L(B) \cup \varphi_L(C)$.

ii) If at least one of $B + L = B$ and $C + L = C$ holds, then $\varphi_L(B \cap C) = \varphi_L(B) \cap \varphi_L(C)$, and hence $\varphi_L(B \setminus C) = \varphi_L(B) \setminus \varphi_L(C)$.

**Proof.** The first assertion is trivial and is stated for completeness only.

For the second assertion, we assume, for definiteness, that $B + L = B$, and show that $\varphi_L(B) \cap \varphi_L(C) \subseteq \varphi_L(B \cap C)$; the opposite inclusion is trivial. Fix an element $t \in \varphi_L(B) \cap \varphi_L(C)$. Since $t \in \varphi_L(C)$, there exists $c \in C$ with $t = \varphi_L(c)$. Now $\varphi_L(c) = t \in L$.
ϕ_L(B) gives c ∈ B + L = B, showing that c ∈ B ∩ C and therefore t = ϕ_L(c) ∈ ϕ_L(B ∩ C).

The assertion follows.  □

**Corollary 2.** Suppose that L is a subgroup, and that B and C are subsets of an abelian group. If B + L = B, then ϕ_L(B ∩ C) = ϕ_L(B) is equivalent to any of ϕ_L(B) ⊆ ϕ_L(C) and B + L ⊆ C.

**Proof.** Applying the lemma, we get ϕ_L(B ∩ C) = ϕ_L(B) ∩ ϕ_L(C). Thus, ϕ_L(B ∩ C) = ϕ_L(B) is equivalent to ϕ_L(B) ⊆ ϕ_L(C), which is immediately seen to be equivalent to B ⊆ C + L.  □

**Lemma 4.** If G is an abelian group having the direct sum decomposition G = Z ⊕ H with H finite, then every subgroup G' < G is of the form G' = ⟨g⟩ + K with some g ∈ G and K ≤ H; indeed, one can take K := G' ∩ H.

**Proof.** The assertion is immediate if G' ≤ H; assume therefore that G' ∉ H. In this case the projection of G' onto Z along H is a non-zero subgroup of Z; let z' be its generator. For k ∈ Z, the “slice” G'(k) := G' ∩ (k + H) is non-empty if and only if z' | k. Furthermore, for any k_1, k_2 divisible by z', and any fixed d ∈ G'(k_2 - k_1), we have G'(k_1) + d ⊆ G'(k_2). This shows that all slices G'(k) with k divisible by z' are actually translates of each other; hence, each of them is a coset of the subgroup K := G'(0) ≤ H.

Fix arbitrarily g ∈ G'(z'). For any integer k divisible by z', we have (k/z')g ∈ G' ∩ (k + H) = G'(k). It follows that G'(k) = (k/z')g + K for any integer k with z' | k. As a result, G' = ⟨g⟩ + K.  □

We need the following lemma in the spirit of [BP18].

**Lemma 5.** Suppose that B and C are finite, nonempty integer sets, and write m := |B| and B = {b_1, ..., b_m}, where the elements of B are numbered in an arbitrary order. Then there exist c_2, ..., c_m ∈ C such that the sums b_2 + c_2, ..., b_m + c_m are distinct from each other and from the elements of the set b_1 + C.

**Proof.** The proof follows the line of reasoning of [BP18].

Let n := |C| and consider the family of m + n - 1 sets

\[ b_1 + C, ..., b_1 + C \] (n sets)

\[ b_2 + C, ..., b_m + C \] (m - 1 sets).

Following Balasubramanian and Pandey, we use the Hall marriage theorem to show that this set family has a system of distinct representatives; clearly, this will imply the result.

Suppose thus that for some 1 ≤ k ≤ m + n - 1 we are given a subsystem S of k sets from among those listed above, and show, to verify the hypothesis of Hall’s theorem, that |∪_{S ∈ S} S| ≥ k. Let B' ⊆ B consist of all those elements b_i ∈ B (1 ≤ i ≤ m) such that at least one of the sets in S has the form b_i + C. Then ∪_{S ∈ S} = B' + C and we thus want to show that |B' + C| ≥ k. Since |B' + C| ≥ |B'| + |C| - 1, it suffices to show that
\(|B'| + n - 1 \geq k\). Indeed, this inequality is trivial for \(k \leq n\), while for \(k \geq n\) it becomes evident upon writing \(k = n + \varkappa, \varkappa \geq 0\) and observing any \(n + \varkappa\) sets under consideration determine at least \(\varkappa + 1 = k - n + 1\) elements \(b_i\).

**Corollary 3.** Suppose that the abelian group \(G\) has the direct sum decomposition \(G = \mathbb{Z} \oplus H\) with \(H < G\) finite. Let \(B, C\) be finite, nonempty subsets of \(G\). If \(m\) and \(n\) denote the sizes of the images of \(B\) and \(C\), respectively, under the projection \(G \to \mathbb{Z}\) along \(H\), then

\[|B + C| \geq \left(1 + \frac{n - 1}{m}\right)|B|\]

**Proof.** Denote by \(\psi\) the projection in question, and write \(\psi(B) := \{b_1, \ldots, b_m\}\), where \(b_1\) is chosen so that \(|\psi^{-1}(b_1) \cap B| \geq |B|/m\); otherwise, the elements of \(\psi(B)\) are numbered arbitrarily. Let \(B_i := \psi^{-1}(b_i) \cap B\) \((1 \leq i \leq m)\). By Lemma 5 as applied to the sets \(\psi(B)\) and \(\psi(C)\), there are (not necessarily distinct) elements \(c_2, \ldots, c_m \in \psi(C)\) such that all sums \(b_2 + c_2, \ldots, b_m + c_m\) are distinct from each other and from the elements of the set \(b_1 + \psi(C)\). Consequently, the sumsets \(B_2 + (\psi^{-1}(c_2) \cap C), \ldots, B_m + (\psi^{-1}(c_m) \cap C)\) are pairwise disjoint, and they are also disjoint from each of the \(n\) sumsets \(B_1 + (\psi^{-1}(c) \cap C), c \in \psi(C)\). As a result,

\[
|B + C| \geq \sum_{i=2}^{m} |B_i + (\psi^{-1}(c_i) \cap C)| + \sum_{c \in \psi(C)} |B_1 + (\psi^{-1}(c) \cap C)| \\
\geq |B_2| + \cdots + |B_m| + n|B_1| = |B| + (n-1)|B_1| \geq |B| + \frac{n - 1}{m} |B|.
\]

\(\Box\)

4. **Basic Estimates**

We collect in this section several basic estimates used in the proof of Theorem 2.

Suppose that \(A\) is a finite subset of the group \(G = \mathbb{Z} \oplus H\), where \(H < G\) is finite abelian. For each \(z \in \mathbb{Z}\), let \(A_z := A \cap (z + H)\), and write \(B := \{z \in \mathbb{Z}: A_z \neq \emptyset\}\); that is, \(B\) is the image of \(A\) under the projection of \(G\) onto \(\mathbb{Z}\) along \(H\). Suppose, furthermore, that \(\min B = 0, \max B = l > 0, 0 \in A_0,\) and \(\delta \in A_l\). Finally, write \(n := |B|, \sigma := |A_0| + |A_l|\), and \(A^* := A_0 \cap (A_l - \delta)\).

**Lemma 6.** We have \(|2A| + |A^*| \geq \sigma n\).

**Proof.** Considering the projections of the “slices” \(A_0\) onto \(\mathbb{Z}\), we get

\[
|2A| \geq \sum_{z \in B} |A_0 + A_z| + |A_0 + A_l| + \sum_{z \geq 0} |A_z + A_l| \\
\geq (n - 1)|A_0| + |A_0 + A_l| + (n - 1)|A_l|.
\]
To estimate the sum $A_0 + A_l$ we notice that both $A_0 + \delta$ and $A_l$ are subsets of $A_0 + A_l$, whence

$$|A_0 + A_l| \geq |(A_0 + \delta) \cup A_l| = (|A_0| + |A_l|) - |A^*|.$$ 

Combining these estimates yields the sought inequality. □

**Corollary 4.** Let $\tau := |2A|/|A|$. If $\tau < 3(1 - \frac{1}{n})$, then

$$(3 - \tau)(|A| + |A^*|) > 3\sigma,$$

$$3|A| - |2A| > \sigma, \quad (8)$$

and

$$|2A| < 3|A| - 2|A^*|. \quad (9)$$

**Proof.** To prove (8), we multiply the inequality of the lemma by the inequality $3 - \tau > \frac{3}{n}$ following from $\tau < 3(1 - \frac{1}{n})$, and then substitute $|2A| = \tau|A|$.

For (9), we use (8) to get

$$3|A| - |2A| = (3 - \tau)|A| > \frac{1}{\tau}(3\sigma - (3 - \tau)|A^*|) = \frac{3}{\tau}\sigma - \left(\frac{3}{\tau} - 1\right)|A^*| \geq \frac{3}{\tau}\sigma - \left(\frac{3}{\tau} - 1\right) \cdot \frac{\sigma}{2} = \frac{1}{2} \left(\frac{3}{\tau} + 1\right)\sigma > \sigma.$$ 

Finally, (10) follows from

$$|2A| < 3|A| - \sigma \leq 3|A| - 2|A^*|.$$ 

□

5. The two-coset case

In this section we prove a result which is easily seen to imply the special case of Theorem 2 where the set $A$ is contained in a union of two cosets of a subgroup $F < G$ (but not contained in a single coset of either $F$ or a subgroup containing $F$ as an index-2 subgroup).

**Proposition 1.** Suppose that the abelian group $G$ has the direct sum decomposition $G = Z \oplus H$ with $H < G$ finite. Let $A_1, A_2 \subset G$ be finite, nonempty subsets of $G$, and for $i \in \{1, 2\}$ let $n_i := |\psi(A_i)|$, where $\psi: G \to Z$ is the projection along $H$. Then

$$|2A_1| + |A_1 + A_2| + |2A_2| \geq 3\left(1 - \frac{1}{n_1 + n_2}\right)(|A_1| + |A_2|).$$

**Example 2.** If, for $i \in \{1, 2\}$, we let $A_i = P_i + K$, where $P_i$ are arithmetic progressions with the same difference not contained in $H$, and where $K \leq H$, then $n_i = |P_i|$ and

$$|2A_1| + |A_1 + A_2| + |2A_2| = 3(|P_1| + |P_2| - 1)|K| = 3\left(1 - \frac{1}{n_1 + n_2}\right)(|A_1| + |A_2|).$$

This shows that the estimate of the proposition is best possible.
Proof of Proposition 1. Recall that for a subset $S$ of an abelian group, by $\pi(S)$ we denote the period of $S$; see Section 3.

For $i \in \{1, 2\}$, we have $\pi(2A_i) \leq H$ (as $2A_i$ are finite), and $|\psi(2A_i)| \geq 2n_i - 1$, whence
\[ |2A_i| \geq (2n_i - 1)|\pi(2A_i)|. \]

On the other hand, by Kneser’s theorem (Section 3)
\[ |2A_i| \geq 2|A_i| - |\pi(2A_i)|. \]

Multiplying the latter inequality by $2n_i - 1$ and adding the former to the result (to cancel out the term $|\pi(2A_i)|$) we get
\[ |2A_i| \geq \left(2 - \frac{1}{n_i}\right)|A_i|. \]

Similarly, letting $n := n_1 + n_2$ and observing that
\[ |\psi(A_1 + A_2)| = |\psi(A_1) + \psi(A_2)| \geq n_1 + n_2 - 1 = n - 1, \]
we get $|A_1 + A_2| \geq (n - 1)|\pi(A_1 + A_2)|$ and $|A_1 + A_2| \geq |A_1| + |A_2| - |\pi(A_1 + A_2)|$, implying
\[ |A_1 + A_2| \geq \left(1 - \frac{1}{n}\right)(|A_1| + |A_2|). \]

In view of (11), it suffices to show that
\[ |A_1 + A_2| \geq \left(1 + \frac{1}{n_1} - \frac{3}{n}\right)|A_1| + \left(1 + \frac{1}{n_2} - \frac{3}{n}\right)|A_2|. \]

Assuming for definiteness that $n_1 \leq n_2$, we distinguish two cases.

If $|A_1|/n_1 \leq |A_2|/n_2$, then we apply (12), reducing the inequality to prove to
\[ \left(1 - \frac{1}{n}\right)(|A_1| + |A_2|) \geq \left(1 + \frac{1}{n_1} - \frac{3}{n}\right)|A_1| + \left(1 + \frac{1}{n_2} - \frac{3}{n}\right)|A_2|. \]

This can be rewritten as
\[ \frac{1}{n_1} |A_1| + \frac{1}{n_2} |A_2| \leq 2 \left(\frac{|A_1| + |A_2|}{n_1 + n_2}\right) \]
and, furthermore, as
\[ (n_1 - n_2)\left(\frac{|A_1|}{n_1} - \frac{|A_2|}{n_2}\right) \geq 0, \]
which is true by our present assumptions $n_1 \leq n_2$ and $|A_1|/n_1 \leq |A_2|/n_2$.

Assume now that, in addition to $n_1 \leq n_2$, we have
\[ |A_1|/n_1 > |A_2|/n_2. \]

By Corollary 3 (applied with $B = A_1$ and $C = A_2$),
\[ |A_1 + A_2| \geq \frac{n - 1}{n_1}|A_1|. \]

Substituting this estimate into (13), we see that it suffices to prove that
\[ \frac{n - 1}{n_1}|A_1| \geq \left(1 + \frac{1}{n_1} - \frac{3}{n}\right)|A_1| + \left(1 + \frac{1}{n_2} - \frac{3}{n}\right)|A_2|; \]
that is,
\[
\left(\frac{n-2}{n_1} - 1 + \frac{3}{n}\right) |A_1| \geq \left(1 + \frac{1}{n_2} - \frac{3}{n}\right) |A_2|.
\]
In view of (14), this will follow from
\[
\left(\frac{n-2}{n_1} - 1 + \frac{3}{n}\right) n_1 \geq \left(1 + \frac{1}{n_2} - \frac{3}{n}\right) n_2
\]
which is easily verified to hold (as an equality, in fact). \square

6. Proof of Theorem 2

Recall that we have a finite subset \(A\) of the abelian group \(G = \mathbb{Z} \oplus H\), where \(H \leq G\) is finite. We assume that \(|2A| < 3(1 - \frac{1}{n})|A|\), where \(n\) is the number of elements in the image of \(A\) under the projection \(G \to \mathbb{Z}\) along \(H\), and we want to show that there exist an arithmetic progression \(P \subseteq G\) and a subgroup \(K \leq H\) such that \(A \subseteq P + K\) and \((|P| - 1)|K| \leq |2A| - |A|\). (As shown in Section 1, the inequality \(|P| \geq 3\) follows automatically and we thus disregard it for the rest of the proof.) We write \(|2A| = \tau|A|\); thus, \(\tau < 3(1 - \frac{1}{n})\).

Let \(\psi: G \to \mathbb{Z}\) be the projection mentioned in the previous paragraph. Without loss of generality we assume that \(0 \in A\) and \(\min A = 0\), and we let \(l := \max \psi(A)\); thus, \(A \cap (z + H) = \emptyset\) for \(z < l\) and also for \(z > l\), while the sets \(A_0 := A \cap H\) and \(A_l := A \cap (l + H)\) are nonempty.

Fix arbitrarily an element \(\delta \in A_l\), and let \(A^* := A_0 \cap (A_l - \delta)\) and \(\sigma := |A_0| + |A_l|\). Notice that \(0 \in A^*\), \(\sigma \geq 2|A^*|\), and \(|A_0 \cup (A_l - \delta)| = \sigma - |A^*|\).

For a subgroup \(L \leq G\), by \(\varphi_L\) we denote the canonical homomorphism of \(G\) onto the quotient group \(G/L\). Let \(\Delta := \langle \delta \rangle \leq G\). We adopt a special notation for the homomorphism \(\varphi_{\Delta}\), which is particularly important for our argument: whenever \(s\) denotes an element of \(G\), by \(\overline{s}\) we denote the image of \(s\) under \(\varphi_{\Delta}\), and similarly for sets: \(\overline{S} = \varphi_{\Delta}(S)\), \(S \subseteq G\). Thus, for instance, \(\overline{A} = \varphi_{\Delta}(A)\) and \(\varphi_{\Delta}(2A) = 2\overline{A}\).

To make the proof easier to follow, we split it into several parts.

6.1. Deficiency and the induction framework. We use induction on \(|H|\), the base case \(|H| = 1\) being Freiman’s \((3n - 3)\)-theorem (see Section 1). Suppose that \(|H| \geq 2\).

Given a subset \(S \subseteq G\) and a subgroup \(L \leq G\), both finite, we define the deficiency of \(S\) on a coset \(g + L \subseteq G\) by
\[
d(S, g + L) := \begin{cases} 
|\{(g + L) \setminus S\}| & \text{if } S \cap (g + L) \neq \emptyset, \\
0 & \text{if } S \cap (g + L) = \emptyset;
\end{cases}
\]
notice that in the first case we can also write \(d(S, g + L) = |L| - |(g + L) \cap S|\). The total deficiency of \(S\) with respect to \(L\) is
\[
D(S, L) := |(S + L) \setminus S|;
\]
equivalently,
\[ D(S, L) = \sum_{g+L} d(S, g+L), \]
where the sum extends over all \( L \)-cosets having a nonempty intersection with \( S \).

Suppose that \( L \leq H \) is a nonzero finite subgroup with
\[ D(2A, L) \leq D(A, L). \quad (15) \]
Then, letting \( T := 3(1 - \frac{1}{n}) \),
\[ |2(A + L)| \leq |A + L| + |2A| - |A| < |A + L| + (T - 1)|A| \leq T|A + L|; \]
that is, writing \( \tilde{G} := G/L \cong (H/L) \oplus \mathbb{Z} \), \( \tilde{A} := \varphi_L(A) \), and \( 2\tilde{A} := \varphi_L(2A) \), we have \( |2\tilde{A}| < 3(1 - \frac{1}{n})|\tilde{A}| \). Applying the induction hypothesis to the subset \( \tilde{A} \subseteq \tilde{G} \), we conclude that there are an arithmetic progression \( \tilde{P} \subseteq \tilde{G} \) and a subgroup \( \tilde{K} \subseteq \tilde{H} := H/L \) such that \( A \subseteq \tilde{P} + \tilde{K} \) and \( (|\tilde{P}| - 1)|\tilde{K}| \leq |2\tilde{A}| - |\tilde{A}| \). Let \( K := \varphi_L^{-1}(\tilde{K}) \); thus, \( L \leq K \leq H \) and \( |K| = |L||\tilde{K}| \). Also, it is easily seen that \( \varphi_L^{-1}(\tilde{P}) = P + L \) where \( P \subseteq G \) is an arithmetic progression with \( |P| \leq |\tilde{P}| \). From \( A \subseteq \tilde{P} + \tilde{K} \) we derive then that \( A \subseteq P + K \), and from \( (|\tilde{P}| - 1)|\tilde{K}| \leq |2\tilde{A}| - |\tilde{A}| \) we get
\[ (|P| - 1)|K| \leq (|\tilde{P}| - 1)|\tilde{K}| |L| \leq (2|\tilde{A}| - |\tilde{A}|)|L| \]
\[ = |2A + L| - |A + L| = |2A| + D(2A, L) - |A| - D(A, L) \leq |2A| - |A|, \]
completing the induction step.

Of particular interest is the situation where \( L \) is a nonzero, finite subgroup satisfying
\[ D(A, L) \leq |L| - 1. \quad (16) \]
Let in this case \( m \) denote the number of \( L \)-cosets on which \( A \) has positive deficiency, and fix \( a_1, \ldots, a_m \in A \) such that \( a_1 + L, \ldots, a_m + L \) list all these cosets. It follows easily from (16) that there is at most one pair of indices \( 1 \leq i \leq j \leq m \) such that \( d(A, a_i + L) + d(A, a_j + L) \geq |L| \), and if such a pair exists, then in fact \( i = j \). By the pigeonhole principle, we have then \( d(2A, g + L) = 0 \) for every coset \( g + L \), with the possible exception of one single \( L \)-coset which is then of the form \( 2a + L \), with some \( a \in A \). This yields
\[ D(2A, L) = d(2A, 2a + L) \leq d(A, a + L) \leq D(A, L). \]
Clearly, the resulting estimate
\[ D(2A, L) \leq D(A, L) \]
remains valid also if there are no exceptional \( L \)-cosets.

Thus, once we are able to find a nonzero finite subgroup \( L < H \) satisfying either (15) or (16), we can complete the proof applying the induction hypothesis.

As a result, we can assume that for any nonempty subsets \( A', A'' \subseteq A \) with \( A = A' \cup A'' \),
\[ |A' + A''| \geq |A'| + |A''| - 1; \]
for if this fails to hold, then letting \( L := \pi(A' + A'') \), by Kneser’s theorem we have \( |L| \geq 2 \) and \(|A' + L| + |A'' + L| - |L| = |A' + A''| \leq |A'| + |A''| - 2 \), whence
\[
D(A, L) \leq D(A', L) + D(A'', L) \leq |L| - 2
\]
(for the first inequality, notice that \( d(A, g + L) \leq d(A', g + L) + d(A'', g + L) \) for any coset \( g + L \), which follows from the assumption \( A', A'' \subseteq A = A' \cup A'' \)).

In particular, we assume that \(|A + S| \geq |A| + |S| - 1\) for any nonempty subset \( S \subseteq A \).

As an important special case,
\[
|A + A^*| \geq |A| + |A^*| - 1.
\]  \hspace{1cm} (17)

6.2. The set \( \overline{A} \) has small doubling. The quantity \(|A^*|\) can be interpreted as the number of representations of \( \delta \) as a difference of two elements of \( A \). Generally, for a set \( S \subseteq G \) and an element \( g \in G \), denote by \( r_S(g) \) the number of representations of \( g \) as a difference of two elements of \( S \); thus, for instance, \(|A^*| = r_A(\delta)\). Clearly, every \( \Delta \)-coset intersects \( A \) by at most two elements, and if the intersection contains exactly two elements, then the two elements differ by \( \delta \). It follows that
\[
|A| = |\overline{A}| + r_A(\delta) = |\overline{A}| + |A^*|.
\]  \hspace{1cm} (18)

Similarly, since \( s_1 = s_2 \) for any \( s_1, s_2 \in 2A \) with \( s_2 - s_1 = \delta \), we have \(|2A| \geq |2\overline{A}| + r_{2A}(\delta)\). Furthermore, \( r_{2A}(\delta) \geq |A + A^*| \) as to any \( a \in A \) and \( a^* \in A^* \) there corresponds the representation \((a^* + \delta) + a\) - \((a^* + a) = \delta\), and the sum \( a + a^* \) is uniquely determined by this representation. Therefore,
\[
|2A| \geq |2\overline{A}| + |A + A^*|.
\]  \hspace{1cm} (19)

We now claim that
\[
|2\overline{A}| < 2|\overline{A}| - 1.
\]  \hspace{1cm} (20)

In view of \(|2\overline{A}| \leq |2A| - |A + A^*| \leq \tau|A| - |A| - |A^*| + 1 \) and \(|\overline{A}| = |A| - |A^*|\) (cf. (19), (17), and (18)), it suffices to show that
\[
\tau|A| - |A| - |A^*| + 1 < 2|A| - 2|A^*| - 1;
\]
that is,
\[
(3 - \tau)|A| > |A^*| + 2.
\]  \hspace{1cm} (21)

To this end we notice that, by (8) and in view of \(|A^*| \leq \min\{|A_0|, |A_l|\}\),
\[
(3 - \tau)(\tau|A| + |A^*|) > 3(|A_0| + |A_l|) \geq 6|A^*|.
\]
Consequently,
\[
(3 - \tau)|A| > \left( \frac{3}{\tau} + 1 \right) |A^*| > 2|A^*|,
\]
which proves (21) in the case where \(|A^*| \geq 2\). In the remaining case \(|A^*| = 1\), we obtain (21) as an immediate corollary of \(|A| \geq n \) and \( \tau < 3 \left( 1 - \frac{1}{n} \right) \).
Thus, (20) is established, and from Kneser’s theorem it follows that the period $\bar{F} := \pi(2\bar{A})$ is a nonzero subgroup of the quotient group $G/\Delta$, and also, in view of $2|\bar{A} + F| - |\bar{F}| = 2|\bar{A}| \leq 2|A| - 2$, that

$$D(\bar{A}, \bar{F}) \leq \frac{1}{2}|\bar{F}| - 1. \quad (22)$$

We let $F := \varphi^{-1}_\Delta(\bar{F})$, so that $\bar{F} = \varphi_\Delta(F)$ and $\Delta \leq F \leq G$.

Observing that $0 \in A$ implies $\bar{A} + F \subseteq 2\bar{A} + F = 2\bar{A}$, we denote by $N$ the number of $\bar{F}$-cosets contained in $2\bar{A}$, but not in $\bar{A} + \bar{F}$; that is,

$$N = (|2\bar{A}| - |\bar{A} + \bar{F}|)/|\bar{F}|.$$

Combining $|2\bar{A}| - |\bar{A} + \bar{F}| = N|\bar{F}|$ and $|2\bar{A}| = 2|\bar{A} + \bar{F}| - |\bar{F}|$, we get

$$|\bar{A} + \bar{F}| = (N + 1)|\bar{F}| \text{ and } |2\bar{A}| = (2N + 1)|\bar{F}|. \quad (23)$$

Let $K := F \cap H$.

6.3. The case where $N = 0$. If $N = 0$, then $\bar{A} + \bar{F} = 2\bar{A}$. Adding $\bar{A}$ to both sides we get $2\bar{A} = 2\bar{A} + \bar{A}$, showing that $\bar{A} \subseteq \pi(2\bar{A}) = \bar{F}$. Combining this with $\bar{A} + \bar{F} = 2\bar{A}$, we conclude that $2\bar{A} = \bar{F}$. Thus, $2A + \Delta = F$ and, by Lemma 4,

$$A \subseteq 2A + \Delta = F = \langle g \rangle + K \quad (24)$$

with some $g \in G$. Notice that $g \notin H$, as otherwise we would have $A \subseteq H$, and hence $n = 1$.

Let $P := \langle g \rangle \cap \psi^{-1}([0, l])$, so that $A \subseteq P + K$. Since $\psi^{-1}([0, l])$ contains exactly one representative out of every $\Delta$-coset, we also have

$$|2\bar{A}| = |\varphi_\Delta(2A)|$$
$$= |\varphi_\Delta(2A + \Delta)|$$
$$= |(2A + \Delta) \cap \psi^{-1}([0, l])|$$
$$= |(\langle g \rangle + K) \cap \psi^{-1}([0, l])|$$
$$= |\langle g \rangle \cap \psi^{-1}([0, l])||K|$$
$$= |\langle g \rangle \cap \psi^{-1}([0, l])||K| - |\langle g \rangle \cap \psi^{-1}(l)||K|$$
$$= (|P| - 1)|K|,$$

the middle equality following from (24), and the last equality from

$$\varnothing \neq A \cap \psi^{-1}(l) \subseteq ((\langle g \rangle + K) \cap \psi^{-1}(l) = (\langle g \rangle \cap \psi^{-1}(l)) + K$$

and the resulting $\langle g \rangle \cap \psi^{-1}(l) \neq \varnothing$. Consequently, (19) yields $(|P| - 1)|K| \leq |2A| - |A|$, completing the proof in the present case.

We thus assume for the remaining part of the argument that $N > 0$; that is

$$\bar{A} + \bar{F} \subsetneq 2\bar{A}.$$
Therefore, $2\overline{A}$ is not a subgroup (if it were, we would have $F = \pi(2\overline{A}) = 2\overline{A}$ implying $\overline{A} + F \supseteq 2\overline{A}$).

6.4. The case where $N = 1$. If $N = 1$, then $\overline{A} + F$ is a union of exactly two $F$-cosets, and $2\overline{A}$ is a union of exactly three $F$-cosets. Since $0 \in A$, we derive that $A = A_1 \cup (g + A_2)$, where $A_1, A_2 \subseteq F$ are nonempty and finite, and where $g \in G$ satisfies $2g \notin F$, as a result of $2\overline{A}$ being a union of three $F$-cosets. Write $n_i := |\psi^{-1}(A_i)|$, $i \in \{1, 2\}$, so that $n := |\psi^{-1}(A)| \leq n_1 + n_2$. By Proposition 1, we have then

\[
|2A| = |2A_1| + |A_1 + A_2| + |2A_2| \\
\geq 3 \left(1 - \frac{1}{n_1 + n_2}\right)(|A_1| + |A_2|) \\
\geq 3 \left(1 - \frac{1}{n}\right)|A|,
\]

a contradiction.

Let $\overline{H} := \varphi_\Delta(H)$ and, following our standard convention, write $\overline{K} := \varphi_\Delta(K)$. We split the remaining case $N \geq 2$ into two further subcases: that where $\overline{K}$ is a proper subgroup of $\overline{F}$ (which, by Corollary 2, is equivalent to any of $\overline{F} \nless \overline{H}$ and $F \nless H + \Delta$, and that where $\overline{K} = \overline{F}$ (equivalently, $\overline{F} \leq \overline{H}$, $F \leq H + \Delta$, or $F = K \oplus \Delta$).

6.5. The case where $N \geq 2$ and $\overline{K} \nless \overline{F}$. We show that in this case

\[
|2A \setminus A| \geq 2|\overline{A}|;
\]  

(25)

in view of (18) and (10), this will give

\[
|2A| - |A| \geq 2|\overline{A}| = 2|A| - 2|A^*| > 2|A| - (3 - \tau)|A| = (\tau - 1)|A|,
\]

a contradiction.

To prove (25), we partition the elements $s \in 2A \setminus A$ into two groups, according to whether $\overline{s} = \varphi_\Delta(s)$ lies in $\overline{A} + F$.

For the first group we have the estimate

\[
|\{s \in 2A \setminus A: \overline{s} \in \overline{A} + F\}| \geq |\overline{A} + F|;
\]

for, $\overline{A} + F \subseteq 2\overline{A}$ shows that for every element $\overline{s} \in \overline{A} + F$, the set $\{s \in 2A: \varphi_\Delta(s) = \delta s\}$ is nonempty, and the (unique) element of this set with the largest value of $\psi(s)$ does not lie in $A$ as $s \in A$ implies $s + \delta \in 2A$, because of $\delta \in A$. (This argument shows that, indeed, for any subset $\overline{S} \subseteq 2\overline{A}$ there are at least $|\overline{S}|$ elements $s \in 2A \setminus A$ such that $\overline{s} \in \overline{S}$.)

To complete the treatment of the case $\overline{K} \nsubseteq F$, we show that

\[
T := |\{s \in 2A \setminus A: \overline{s} \notin \overline{A} + F\}| \geq 2|\overline{A}| - |\overline{A} + F|.
\]

(Notice that the trivial estimate would be $T \geq 2|\overline{A}| - |\overline{A} + F|$.)

The set $(2\overline{A}) \setminus (\overline{A} + F)$ is a union of $F$-cosets, and we find $a_1, \ldots, a_N, b_1, \ldots, b_N \in A$ such that the cosets in question are $\overline{a_i} + \overline{b_i} + F$, $i \in [1, N]$. 

Let
\[ A_i := A \cap (a_i + F) \text{ and } B_i := A \cap (b_i + F), \hspace{1em} i \in [1, N]. \]
By Corollary 2 we have \( \mathcal{A}_i = \mathcal{A} \cap (\overline{a_i} + F) \) and \( \mathcal{B}_i = \mathcal{A} \cap (\overline{b_i} + F) \), and it follows that
\[ |A_i| \geq |\mathcal{A}_i| = |\overline{a_i} + F| - |(\overline{a_i} + F) \setminus \mathcal{A}| = |F| - d(\mathcal{A}, \overline{a_i} + F), \hspace{1em} i \in [1, N] \tag{26} \]
and, similarly,
\[ |B_i| \geq |\mathcal{B}_i| = |\overline{b_i} + F| - |(\overline{b_i} + F) \setminus \mathcal{A}| = |F| - d(\mathcal{A}, \overline{b_i} + F), \hspace{1em} i \in [1, N]. \tag{27} \]
Since \( \varphi(\overline{A} + B_i) = \mathcal{A}_i + \mathcal{B}_i \subseteq \overline{a_i} + \overline{b_i} + F \subseteq (2\mathcal{A}) \setminus (\mathcal{A} + F) \) by the choice of \( a_i \) and \( b_i \), we have
\[ T = \sum_{i=1}^{N} |\{ s \in 2A \setminus A : \sigma \in \overline{a_i} + \overline{b_i} + F \}| \geq \sum_{i=1}^{N} |A_i + B_i|. \tag{28} \]

By Lemma 2 as applied to the subset \( \tilde{A} := (\overline{A} + F)/F \) of the quotient group \( \overline{G}/F \), we can assume that each \( F \)-coset from \( \overline{A} + F \) appears among the \( 2N \) cosets \( \overline{a_1} + F, \ldots, \overline{b_N} + F \) at most \( N \) times, except if there is a subgroup \( \overline{L} \leq \overline{G}/F \) and an element \( \tilde{c} \in \overline{G}/F \) with \( 2\tilde{c} \notin \overline{L} \) such that either \( \overline{A} = \overline{L} \cup \{ \tilde{c} \} \), or \( \overline{A} = (\overline{c} + \overline{L}) \cup \{ 0 \} \). In this exceptional situation \( A \) meets exactly two cosets of the subgroup \( L = \varphi^{-1}(\overline{L}) \), while \( 2A \) meets exactly three cosets of this subgroup. As a result, we can apply Proposition 1, exactly as in the case \( N = 1 \) considered above, to get \( |2A| \geq 3 \left( 1 - \frac{1}{2} \right) |A| \).

Addressing now case i) of Lemma 2, assume that each \( F \)-coset from \( \overline{A} + F \) appears among \( \overline{a_1} + F, \ldots, \overline{b_N} + F \) at most \( N \) times.

Since \( A_i + B_i \) is contained in an \( F \)-coset, we have \( \pi(A_i + B_i) \leq F \), and since \( A_i + B_i \) is finite, \( \pi(A_i + B_i) \leq H \); as a result, \( \pi(A_i + B_i) \leq F \cap H = K \). Consequently, by (28), Kneser's theorem, (26), and (27),
\[ T \geq 2N|F| - \sum_{i=1}^{N} (d(\mathcal{A}, \overline{a_i} + F) + d(\mathcal{A}, \overline{b_i} + F)) - |K||N|. \]

Recalling that each \( F \)-coset from \( \overline{A} + F \) appears at most \( N \) times among \( \overline{a_1} + F, \ldots, \overline{b_N} + F \), we get
\[ T \geq 2N|F| - N(D(\mathcal{A}, F) - |K||N) \geq \left( \frac{3}{2} |F| - D(\mathcal{A}, F) \right) N \]
(as \( K \leq F \) yields \( |K| = |K| \leq \frac{1}{2}|F| \)). Therefore, by (22), (23), and the definition of the total deficiency,
\[ T \geq (N + 1)|F| - 2D(\mathcal{A}, F) + (N - 2) \left( \frac{1}{2} |F| - D(\mathcal{A}, F) \right) \]
\[ \geq (N + 1)|F| - 2D(\mathcal{A}, F) \]
\[ = |\mathcal{A}| - D(\mathcal{A}, F) \]
\[ = 2|\mathcal{A}| - |\mathcal{A} + F|. \]

As explained above, this leads to a contradiction.
6.6. **The case where** \( N \geq 2 \) **and** \( \overline{K} = \mathcal{F} \). As shown above, in this case \( \mathcal{F} \leq \mathcal{H} \), \( F \leq H + \Delta \), and \( F = K \oplus \Delta \); notice that this implies \( |\mathcal{F}| = |\overline{K}| = |K| \).

Let \( A^\circ := A \setminus (A_0 \cup A_l) \); loosely speaking, \( A^\circ \) is the “middle part” of \( A \).

**Claim 1.** We have \( A_0 \subseteq K \) and \( A_l \subseteq \delta + K \); that is, each of the sets \( A_0 \) and \( A_l \) is contained in a single \( K \)-coset.

**Proof.** From (19), Kneser’s theorem, (17), and (18), we have
\[
|2A| \geq (2|A + F| - |\mathcal{F}|) + (|A| + |A^*| - 1) \\
\geq 2|A| - |\mathcal{F}| + |A| + |A^*| - 1 = 3|A| - |A^*| - |\mathcal{F}| - 1.
\]
Combining this estimate with (9), we get
\[
\sigma \leq |A^*| + |\mathcal{F}|.
\] (29)

On the other hand,
\[
|H \cap (A + \Delta)| \geq |\varphi_\Delta(H \cap (A + \Delta))| = |\varphi_\Delta(H) \cap \varphi_\Delta(A + \Delta)| = |\mathcal{H} \cap \overline{A}|
\]
by Lemma 3. Observing that the left-hand side is
\[
|(H \cap A) \cup (H \cap (A - \delta))| = |A_0 \cup (A_l - \delta)| = \sigma - |A^*|,
\]
and using (29), we obtain
\[
|\mathcal{H} \cap \overline{A}| \leq \sigma - |A^*| \leq |\mathcal{F}|.
\]

Assuming now for a contradiction that, say, \( A_0 \) intersects more than one \( K \)-coset, fix \( a_1, a_2 \in A_0 \) which are distinct modulo \( K \). Since \( \overline{a}_1, \overline{a}_2 \in \overline{H} \) are then distinct modulo \( \overline{K} = \overline{F} \), in view of (22) and the assumption \( \mathcal{F} \leq \mathcal{H} \) we get
\[
\frac{1}{2} |\mathcal{F}| > D(\overline{A}, \mathcal{F}) \\
\geq d(\overline{A}, \overline{a}_1 + \mathcal{F}) + d(\overline{A}, \overline{a}_2 + \mathcal{F}) \\
= 2|\mathcal{F}| - (|\overline{a}_1 + \mathcal{F}| \cap \overline{A}) + (|\overline{a}_2 + \mathcal{F}| \cap \overline{A}) \\
\geq 2|\mathcal{F}| - |(\overline{H} + \mathcal{F}) \cap \overline{A}| \\
= 2|\mathcal{F}| - |\overline{H} \cap \overline{A}| \\
\geq |\mathcal{F}|,
\]
the contradiction sought. \( \square \)

**Claim 2.** We have \( 2A^\circ + K = 2A^\circ \). Moreover, if \( |A_0| \geq |A_l| \), then \( A^\circ + K \subseteq 2A \), and if \( |A_l| \geq |A_0| \), then \( A^\circ + \delta + K \subseteq 2A \).

**Proof.** To prove the first assertion, we fix \( a_1, a_2 \in A^\circ \) and show that \( a_1 + a_2 + K \subseteq 2A^\circ \).

For \( i \in \{1, 2\} \), let \( A_i := (a_i + F) \cap A \); notice that \( A_i \subseteq a_i + F = a_i + K + \Delta \) whence, indeed, \( A_i \subseteq a_i + K \). Write \( S := A_1 + A_2 \subseteq a_1 + a_2 + K \) so that \( \overline{S} = \overline{A}_1 + \overline{A}_2 = \overline{a}_1 + \overline{a}_2 + \mathcal{F} \).
in view of (22). As a result, \( |S| \geq |\overline{S}| = |\overline{\pi_1 + \pi_2 + F}| = |\overline{F}| = |\overline{K}| = |K| \), leading to 
\( S = a_1 + a_2 + K \); thus, \( a_1 + a_2 + K = A_1 + A_2 \subseteq 2A^\circ \).

Addressing the second assertion, we fix \( a^\circ \in A^\circ \) and show that then either \( a^\circ + K \subseteq 2A \), or \( a^\circ + \delta + K \subseteq 2A \), according to the relation between \( |A_0| \) and \( |A_1| \). Write \( B_0 := A \cap F \) and \( B^\circ := A \cap (a^\circ + F) \); equivalently, \( B_0 = A_0 \cup A_1 \) by Claim 1, and \( B^\circ = A \cap (a^\circ + K) \).

Letting \( S := B_0 + B^\circ \), in view of \( B_0 \subseteq F \) and \( B^\circ \subseteq a^\circ + F \) we have then \( S \subseteq 2A \cap (a^\circ + F) \) and \( \overline{S} = \overline{B_0} + \overline{B^\circ} \). Furthermore, from

\[
d(\overline{A}, \overline{F}) = |\overline{F}| - |\overline{A} \cap \overline{F}| = |\overline{F}| - |\overline{B_0}|
\]

and

\[
d(\overline{A}, \overline{\pi^\circ + F}) = |\overline{F}| - |\overline{A} \cap (\overline{\pi^\circ + F})| = |\overline{F}| - |\overline{B^\circ}|
\]

recalling (22) we get

\[
|B_0| + |B^\circ| = 2|F| - (d(\overline{A}, \overline{F}) + d(\overline{A}, \overline{\pi^\circ + F})) \geq 2|F| - D(\overline{A}, \overline{F}) > \frac{3}{2} |F|.
\]

From \( B_0 = A_0 \cup A_1 \) we now derive

\[
|A_0| + |A_1| + |B^\circ| \geq |B_0| + |B^\circ| \geq |B_0| + |B^\circ| > \frac{3}{2} |F|.
\]

Also, we have

\[
|B^\circ| \geq |\overline{B^\circ}| = |\overline{A} \cap (\overline{\pi^\circ + F})| = |\overline{F}| - d(\overline{A}, \overline{\pi^\circ + F}) \geq |\overline{F}| - D(\overline{A}, \overline{F}) > \frac{1}{2} |\overline{F}|.
\]

Therefore,

\[
\max\{|A_0|, |A_1|\} + |B^\circ| \geq \frac{1}{2} (|A_0| + |A_1| + |B^\circ|) + \frac{1}{2} |B^\circ| > \frac{3}{4} |\overline{F}| + \frac{1}{4} |\overline{F}| = |\overline{F}| = |K|.
\]

Since \( B^\circ \subseteq a^\circ + K \), \( A_0 \subseteq K \), and \( A_1 \subseteq \delta + K \), from the pigeonhole principle we conclude that if \( |A_0| > |A_1| \), then \( A_0 + B^\circ = a^\circ + K \), while if \( |A_1| \geq |A_0| \), then \( A_1 + B^\circ = a^\circ + \delta + K \). The assertion follows in view of \( A_0 + B^\circ \subseteq 2A \) and \( A_1 + B^\circ \subseteq 2A \).

We can, eventually, complete the proof. Assuming \( |A_0| \leq |A_1| \) for definiteness, by Claim 2 we have \( 2A^\circ + K \subseteq 2A \) and also \( A^\circ + A_1 + K \subseteq 2A \); that is, the set \( 2A \) has zero deficiency on all \( K \)-cosets with the possible exception of the cosets contained in \( A_0 + A + K \); that is, cosets of the form \( a + K \) with \( a \in A \). On the other hand, in view of

\[
A \cap (a + K) + A_0 \subseteq 2A \cap (a + K)
\]

and \( |2A \cap (a + K)| \geq |A \cap (a + K)| \) resulting from it, we have

\[
d(2A, a + K) \leq d(A, a + K).
\]

Taking the sum over the elements \( a \in A \) representing the \( K \)-cosets contained in \( A \) we get

\[
D(A, K) = \sum_a d(A, a + K) \geq \sum_a d(2A, a + K) = D(2A, K).
\]

As noticed in Section 6.1, this completes the proof by appealing to the induction.
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