On the boundary conditions in estimating $\nabla \omega$ by $\text{div} \, \omega$ and $\text{curl} \, \omega$

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Abstract

In this paper we study under what boundary conditions the inequality

$$\| \nabla \omega \|_{L^2(\Omega)}^2 \leq C \left( \| \text{curl} \, \omega \|_{L^2(\Omega)}^2 + \| \text{div} \, \omega \|_{L^2(\Omega)}^2 + \| \omega \|_{L^2(\Omega)}^2 \right)$$

holds true. It is known that such an estimate holds if either the tangential or normal component of $\omega$ vanishes on the boundary $\partial \Omega$. We show that the vanishing tangential component condition is a special case of a more general one. In two dimensions we give an interpolation result between these two classical boundary conditions.

1 Introduction

In this paper we study the estimate

$$\| \nabla \omega \|_{L^2(\Omega)}^2 \leq C \left( \| \text{curl} \, \omega \|_{L^2(\Omega)}^2 + \| \text{div} \, \omega \|_{L^2(\Omega)}^2 + \| \omega \|_{L^2(\Omega)}^2 \right),$$

(1)

where $\omega \in H^1(\Omega)^n$ is a vector field ($n = 2, 3$ in most applications) and $C$ is a constant independent of $\omega$. $H^1(\Omega)^n$ denotes the Sobolev space of vector fields, whose components and all its derivatives are $L^2$ integrable. It is well known that such an estimate holds true if either the tangential or normal component of $\omega$ vanishes on the boundary $\partial \Omega$, which we shall call the classical boundary conditions. More precisely if $\nu$ is the unit exterior normal vector on $\partial \Omega$, then (1) holds true if

$$\nu \times \omega = 0 \quad \text{on} \; \partial \Omega \quad \text{or} \quad \langle \nu; \omega \rangle = 0 \quad \text{on} \; \partial \Omega.$$  

(2)

These boundary conditions have been studied in great detail and the literature on it and its applications to physical systems, mainly Maxwell’s equations and Navier-Stokes equations, is very large.
Our aim is to show that some of these classical boundary conditions can be extended to much more general ones. A particular case of our main result gives in two dimensions an interpolation between the two classical boundary conditions (cf. Remark 5 (ii)).

Let us first mention that inequality (1) cannot hold true without further restrictions on \( \omega \). To see this take any domain \( \Omega \subset \mathbb{R}^2 \) and define for \( n \in \mathbb{N} \)

\[
\omega_n(x) = (e^{nx_1} \cos(nx_2), -e^{nx_1} \sin(nx_2)).
\]

Then one easily verifies that \( \text{div} \omega_n = 0, \text{curl} \omega_n = 0, \)

\[
|\nabla \omega_n(x)|^2 = 2n^2 e^{2nx_1} \quad \text{and} \quad |\omega_n(x)|^2 = e^{2nx_1}.
\]

Hence there can be no constant \( C \) independent of \( n \) such that for all \( n \)

\[
2n^2 \int_\Omega e^{2nx_1} \leq C \int_\Omega e^{2nx_1}.
\]

A similar example works also in higher dimensions.

Some standard references on (1) and its applications are Amrouche-Bernardi-Dauge-Girault [1], Costabel [8], Dautray-Lions [15], and Grisvard [22]. The inequality (1) has also been studied in the more general context of differential forms, where curl is replaced by the \( \delta \) operator, respectively \( \text{div} \) is replaced by \( \delta \). In this setting it is called Gaffney-Friedrichs inequality after Gaffney [17], [18], but for domains with boundary and the classical boundary conditions it is due to Morrey [28], [30] or Friedrichs [16]. Proofs of this general version can also be found in Csató-Dacorogna-Kneuss [13], Iwaniec-Martin [23], Morrey [29], Schwarz [31], Taylor [32]. Therefore we will call also (1) Gaffney inequality henceforth.

The first and simplest generalization of the boundary conditions (cf. Theorem 1) is by mixing the classical ones, namely requiring that on some parts of the boundary the tangential part vanishes and on other parts the normal part vanishes. This result already seems to be known, see for instance Goldstein-Mitrea [20] or Jakab-Mitrea-Mitrea [25] and the references therein. We state and indicate a very simple proof of this result for completeness (cf. Theorem 1), since it does not appear explicitly in the references.

First attempts to give more general boundary conditions have been obtained in Csató-Dacorogna [12], see also Csató [10] for a general version on Riemannian manifolds. There the authors have proven in particular that, in three dimensions, if \( \lambda \) is a given fixed vector field, then there exists a constant \( C = C(\Omega, \lambda) \) such that (1) holds true if

\[
\nu \times \omega = \lambda \langle \nu, \omega \rangle \quad \text{on} \quad \partial \Omega.
\]

This generalizes the classical condition of vanishing tangential component by setting \( \lambda = 0 \).

Our first main result is an even simpler generalization of this classical boundary condition (vanishing tangential component) which additionally has an obvious geometric interpretation. Namely Theorem 4 asserts that (1) holds true if

\[
\lambda \times \omega = 0 \quad \text{on} \quad \partial \Omega,
\]

where again \( C \) will depend on \( \lambda \) and \( \Omega \). Geometrically this means that Gaffney inequality holds true whenever the vector fields \( \omega \) are collinear with a given fixed vector field on \( \partial \Omega \). This time, setting \( \lambda = \nu \) gives the classical boundary condition. We will prove Gaffney inequality under the
condition (4) for Lipschitz domains as long as \( \lambda \) is \( C^1 \). Thus, if \( \Omega \) is not \( C^2 \) (and thus \( \nu \) is not \( C^1 \)), this result does not include the classical boundary condition \( \nu \times \omega = 0 \). However, we will give in the case of domains in \( \mathbb{R}^2 \) a better result which does not even require \( \lambda \) to be globally Lipschitz on \( \partial \Omega \), see Theorem 18. A special case of this theorem is for instance Gaffney inequality on polygonal domains with either of the classical boundary conditions on different parts of the polygon. This is a first step in providing more general Gaffney inequalities, with a simple proof, to be applicable in numerical analysis. We refer to Arnold-Falk-Winther [2] (Section 7.7) and Bonizzoni-Buffa-Nobile [4] for a discussion on vector-valued finite element methods and applications of Gaffney inequality in that setting.

We do not require in any of our results that \( \Omega \) is convex. This is because we assume that our vector fields \( \omega \) are at least in \( H^1(\Omega)^n \). A weaker formulation of the classical Gaffney inequality for Lipschitz domains requires \( \Omega \) to be convex. By the weak formulation we mean that we assume

\[
\omega \in H_T(\text{div}, \text{curl}; \Omega) = \{ \omega \in L^2(\Omega)^n | \text{div} \omega \in L^2(\Omega), \text{curl} \omega \in L^2(\Omega)^n, \nu \times \omega = 0 \text{ on } \partial \Omega \}.
\]

Under this hypothesis on \( \omega \), Gaffney inequality becomes a regularity result and states that \( \omega \in H^1(\Omega)^n \) and satisfies the corresponding estimate (1). The same result holds true if we replace \( H_T \) by \( H_N \), the space with vanishing normal component. The usual approach to prove such regularity results is to use Gaffney inequality for an approximating sequence \( \{ \omega_k \} \) in \( H^1 \). The difficulty consists in establishing \( \nu \times \omega_k = 0 \) on \( \partial \Omega \), using the assumption that \( \nu \times \omega = 0 \) on \( \partial \Omega \) in a weak sense. This approximation fails for nonconvex domains, which are only Lipschitz and the regularity statement does not hold true. See for instance the Remark following the proof of Theorem 5.1 in Mitrea [26]. This is essentially the same example as the one for the Laplace equation: it is well known that the solution \( u \) of \( \Delta u = f, f \in L^2 \), is in general only in \( H^{3/2} \) if \( \Omega \) is a nonconvex polygonal domain, cf. Grisvard [21]. For more details on these approximation theorems and regularity results we refer to Amrouche-Bernardi-Dauge-Girault [1], Belgacem-Bernardi-Costabel-Dauge [5], Ciarlet-Hazard-Lohrengel [6], Costabel [7], Costabel-Dauge [9] and Girault-Raviart [19]. For a different approach in proving the classical Gaffney inequality for nonsmooth domains see Mitrea [26], where the inequality is obtained using existence and regularity of an elliptic boundary value system established in Mitrea [27].

Note that the proof of Theorem 1 (Gaffney inequality with condition (1)) would not simplify if we assumed \( \Omega \) to be smooth.

In view of the condition (4) one might expect that the classical condition \( \langle \nu; \omega \rangle = 0 \) can be generalized too, by replacing \( \nu \) by a nonvanishing vector field \( \lambda \). This is however not true if \( n \geq 3 \) as can be seen by a simple counterexample. It is also not true that condition (4) generalizes to differential forms of higher order. We give these counterexamples at the end of this paper.

2 Mixed classical boundary conditions

If \( \Omega \) is a bounded \( C^{1,1} \) open set with unit exterior normal \( \nu \) on its boundary \( \partial \Omega \) and \( \omega \) is some vector field, we shall decompose it as

\[
\omega = \omega_T + \omega_N, \text{ where } \omega_N = \langle \omega; \nu \rangle \nu \text{ and } \omega_T = \omega - \omega_N.
\]
Throughout this paper for vectors fields $\omega, \lambda$ in $\mathbb{R}^n$, the curl and cross product are defined as vectors in $\mathbb{R}^{(n^2)}$ defined by

$$(\text{curl} \omega)_{ij} = \frac{\partial \omega_j}{\partial x_i} - \frac{\partial \omega_i}{\partial x_j} \quad \text{and} \quad (\omega \times \lambda)_{ij} = \omega_i \lambda_j - \omega_j \lambda_i, \quad 1 \leq i < j \leq n.$$ 

We now state a theorem whose proof is essentially the same as the one presented in Csató-Dacorogna-Kneuss [13] for the classical Gaffney inequality.

**Theorem 1** Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded open $C^{1,1}$ set with exterior unit normal $\nu$ on $\partial \Omega$. Then there exists a constant $C = C(\Omega)$ such that

$$\|\nabla \omega\|_{L^2(\Omega)}^2 \leq C \left( \|\text{curl} \omega\|_{L^2(\Omega)}^2 + \|\text{div} \omega\|_{L^2(\Omega)}^2 + \|\omega\|_{L^2(\Omega)}^2 \right),$$

for all $\omega \in H^1(\Omega)^n$ satisfying

$$\omega_T = 0 \quad \text{or} \quad \omega_N = 0 \quad \text{on} \quad \Gamma_i, \quad \partial \Omega = \bigcup_{i=1}^M \Gamma_i,$$

and $\Gamma_i$ are open sets in $\partial \Omega$ and $M \in \mathbb{N}$.

**Remark 2** If $M = 1$ (classical boundary conditions, $\Gamma_1 = \partial \Omega$) and $\Omega$ is contractible, then one easily obtains the better estimate

$$\|\nabla \omega\|_{L^2(\Omega)}^2 \leq C \left( \|\text{curl} \omega\|_{L^2(\Omega)}^2 + \|\text{div} \omega\|_{L^2(\Omega)}^2 \right),$$

see Csató-Dacorogna-Kneuss [13] Theorem 6.5 and Theorem 6.7 (Step 1 of the proof). A precise treatment of the optimal topological assumptions on the domain for such an estimate to hold true is carried out in von Wahl [33].

**Proof** We will not give a detailed proof. The result follows from [13] Theorem 5.7 (see also [22] Theorem 3.1.1.1) in the same way as the classical Gaffney inequality: Indeed as in the proof of Theorem 5.16 in [13] one obtains that

$$\int_\Omega ((|\text{curl} \omega|^2 + |\text{div} \omega|^2) \geq \int_\Omega |\nabla \omega|^2 - C \int_{\partial \Omega} |\omega|^2,$$

and one concludes similarly. The above mentioned references treat $C^2$ domains but remain valid without any change for $C^{1,1}$ domains. See also Step 3 in the first proof of Proposition 6.

3 The $\lambda \times \omega = 0$ condition

We now state our first main result. We will distinguish the case $n = 2$ as we will give in Section 4 in the two dimensional case an improvement of the theorem by weakening the regularity assumptions. To state the theorem we need the following definition (which we will use actually only for $C^{\gamma,\alpha} = C^{1,0}$ or $C^{0,1}$).
Then there exists a constant \( C = C(\Omega, \lambda) \) such that
\[
\| \nabla \omega \|_{L^2(\Omega)}^2 \leq C \left( \| \text{curl} \omega \|_{L^2(\Omega)}^2 + \| \text{div} \omega \|_{L^2(\Omega)}^2 + \| \omega \|_{L^2(\Omega)}^2 \right),
\]
for all \( \omega \in H^1(\Omega)^n \) which satisfy
\[
\lambda \times \omega = 0 \quad \text{on} \ \partial \Omega.
\]
If \( n = 2 \) then the same conclusion holds under the weaker regularity assumptions \( \lambda \in C^{0,1}(\partial \Omega)^2 \).

**Remark 5** (i) Note that if \( \Omega \) is a \( C^2 \) set, then the unit outward normal vector \( \nu \) is \( C^1 \) and the
Theorem implies the classical boundary condition \( \nu \times \omega = 0 \).

(ii) If \( n = 2 \), then this theorem interpolates between the two classical boundary conditions \( \omega_T = 0 \), respectively \( \omega_N = 0 \). To see this take \( \lambda = \nu = (\nu_1, \nu_2) \), respectively \( \lambda = (\nu_2, -\nu_1) \).

(iii) Recall (see Remark[2]) that if \( \Omega \) is contractible and \( \lambda = \nu \), then in the above theorem the
inequality can be replaced by
\[
\| \nabla \omega \|_{L^2(\Omega)}^2 \leq C \left( \| \text{curl} \omega \|_{L^2(\Omega)}^2 + \| \text{div} \omega \|_{L^2(\Omega)}^2 \right),
\]
This is not true for general \( \lambda \). To see this just notice that one can take \( \lambda \in C^\infty(\overline{\Omega})^n \) equal to a
harmonic field (i.e. \( \text{curl} \lambda = 0 \) and \( \text{div} \lambda = 0 \)) that never vanishes on the boundary. Then \( \omega = \lambda \)
trivially satisfies \( \lambda \times \omega = 0 \) on \( \partial \Omega \). Such non-constant harmonic fields exist, for example take
\( \omega = (x_2, x_1) \) and a domain \( \Omega \subset \mathbb{R}^2 \) such that \( 0 \not\in \partial \Omega \), so that \( \lambda = \omega \neq 0 \) on the boundary.

(iv) If \( \lambda \) is constant then \( C = 1 \), see Lemma[12] or proof of Proposition[6]. \( C = 1 \) also if \( \lambda = \nu \) is
the normal to \( \partial \Omega \) and \( \Omega \) is convex (actually \( n - 1 \) convex is sufficient), see[14], but this requires a
different proof.

**Proof of Theorem 4** We first prove the result for \( C^1 \) vector fields \( \omega \), respectively Lipschitz vector
fields if \( n = 2 \) (cf. Proposition[9]). Theorem[4] will then follow by approximation (cf. Proposition[17]).

**Proposition 6** Let \( n \geq 2 \), \( \Omega \subset \mathbb{R}^n \) be a bounded open Lipschitz set and \( \lambda \in C^{0,1}(\partial \Omega)^n \) be such
that \( \lambda \neq 0 \) on \( \partial \Omega \). Then there exists a constant \( C = C(\Omega, \lambda) \) such that
\[
\| \nabla \omega \|_{L^2(\Omega)}^2 \leq C \left( \| \text{curl} \omega \|_{L^2(\Omega)}^2 + \| \text{div} \omega \|_{L^2(\Omega)}^2 + \| \omega \|_{L^2(\Omega)}^2 \right),
\]
for all \( \omega \in C^1(\overline{\Omega})^n \) which satisfy \( \lambda \times \omega = 0 \) on \( \partial \Omega \). If \( n = 2 \), the same holds true if \( \omega \in C^{0,1}(\overline{\Omega})^2 \).

**Remark 7** Note that in this proposition we require that \( \lambda \) is only Lipschitz. The loss of regularity
compared to the main Theorem[4] arises in the approximation, see Proposition[17].
We give two proofs of this proposition. The first one is simpler, following the ideas of Csató-Dacorogna [12]. However, we do not use the identity established in [12] and which is used in establishing the classical Gaffney inequality, respectively Theorem 1. The second proof that we give is a generalization of Morrey’s original proof of Gaffney inequality (see Morrey [28], Morrey-Eells [30] or Iwaniec-Scott-Stroffolini [24] for an $L^p$ version) for the boundary condition $\nu \times \omega = 0$. It is longer, but several of the intermediate steps are of interest on their own right, cf. Lemma 12, and also Lemmas 10 and 14 which are independent of the boundary conditions. In the first proof we will use the following abbreviation, $f$ being a function defined on a neighborhood of $\partial \Omega$:

$$\partial_{ij}[f] := \nu_j \frac{\partial f}{\partial x_i} - \nu_i \frac{\partial f}{\partial x_j},$$

where $\nu = (\nu_1, \ldots, \nu_n)$ is the outward unit normal vector on $\partial \Omega$. It can be easily seen that $\partial_{ij}[f]$ is a tangential derivative and depends only on the values of $f$ on $\partial \Omega$. Therefore, if $f$ is Lipschitz then $\partial_{ij}[f]$ is well defined $H^{n-1}$ almost everywhere on any Lipschitz boundary $\partial \Omega$, see for instance Lemma 8 equations (10)-(11) for the case $n = 2$ (if $n \geq 3$, the argument is similar by composing $f$ with a local parametrization of $\partial \Omega$). Moreover by the product rule of derivation:

$$\partial_{ij}[fg] = \partial_{ij}[f]g + f\partial_{ij}[g].$$

(5)

Throughout the proof we will frequently use that any Lipschitz function defined on a subset of $\mathbb{R}^n$ can be extended to a Lipschitz function on the whole space, and conversely, that restrictions of Lipschitz functions to any subset are still Lipschitz.

**First Proof of Proposition 6.** Step 1. Let us assume first that $\omega \in C^2(\overline{\Omega})^n$. A direct calculation gives the identity

$$|\text{curl}\omega|^2 + |\text{div}\omega|^2 - |\nabla\omega|^2 = 2 \sum_{i<j} \left( \frac{\partial \omega_i}{\partial x_j} \frac{\partial \omega_j}{\partial x_i} - \frac{\partial \omega_j}{\partial x_j} \frac{\partial \omega_i}{\partial x_i} \right).$$

So we obtain by partial integration that

$$\int_{\Omega} (|\text{curl}\omega|^2 + |\text{div}\omega|^2 - |\nabla\omega|^2) = -\sum_{i<j} \int_{\partial \Omega} \omega_i \partial_{ij}[\omega_j] + \sum_{i<j} \int_{\partial \Omega} \omega_j \partial_{ij}[\omega_i].$$

(6)

Note that (6) involves only first derivatives of $\omega$. Therefore by approximation one directly deduces that (6) remains true for any $\omega \in C^4(\overline{\Omega})^n$. To see this note that standard convolution in the whole space works, since the derivatives of $\omega$ are uniformly continuous, and the derivatives of the approximating sequence will converge also uniformly on $\partial \Omega$. If $n = 2$ we apply Lemma 8 to obtain that (6) remains true if $\omega \in C^{0,1}(\overline{\Omega})^2$.

Step 2. Since $\lambda = (\lambda_1, \ldots, \lambda_n) \neq 0$ on $\partial \Omega$, there exist open sets $W_1, \ldots, W_M$, integers $1 \leq k(1), \ldots, k(M) \leq n$ and $\epsilon > 0$ such that

$$\partial \Omega \subset \bigcup_{l=1}^M W_l \quad \text{and} \quad |\lambda_{k(l)}| \geq \epsilon \text{ in } W_l \text{ for } 1 \leq l \leq M.$$

We now define inductively

$$S_1 = W_1 \cap \partial \Omega, \quad S_2 = (W_2 \cap \partial \Omega) \setminus S_1, \ldots, \quad S_j = (W_j \cap \partial \Omega) \setminus \left( \bigcup_{m=1}^{j-1} S_m \right),$$

for $j = 2, \ldots, M$. Then $S_j$ is an open set, disjoint from $S_l$ for $l < j$, and $S_M \cap \partial \Omega = \emptyset$. Therefore $S_M$ is a sequence of sets that converges to $\partial \Omega$ in the Hausdorff metric. Since $\nu \times \omega = 0$ on $\partial \Omega$, we have $\partial_{ij}[f]$ well defined $H^{n-1}$ almost everywhere on $\partial \Omega$.
for \( j = 1, \ldots, M \). Thus the \( S_j \) form a disjoint union of \( \partial \Omega \) and we can write

\[
\int_{\partial \Omega} (-\omega_i \partial_{ij} [\omega_j] + \omega_j \partial_{ij} [\omega_i]) = \sum_{l=1}^{M} \int_{S_l} (-\omega_i \partial_{ij} [\omega_j] + \omega_j \partial_{ij} [\omega_i]), \tag{7}
\]

for any \( i < j \). We now claim that for each \( l = 1, \ldots, M \) and each \( i < j \), there exists a constant \( C = C(\Omega, \lambda) > 0 \) such that

\[
\left| \int_{S_l} (-\omega_i \partial_{ij} [\omega_j] + \omega_j \partial_{ij} [\omega_i]) \right| \leq C \int_{S_l} |\omega|^2 \tag{8}
\]

for any \( \omega \) satisfying \( \lambda \times \omega = 0 \) on \( \partial \Omega \). Indeed fix \( l \) and assume without loss of generality that \( k(l) = 1 \). Then we obtain from the boundary condition on \( \omega \) that \( \lambda_1 \omega_i - \lambda_1 \omega_1 = 0 \) for \( i = 1, \ldots, n \) on \( \partial \Omega \). Thus we first obtain that for \( i = 1, \ldots, n \)

\[
\omega_i = \mu_i \omega_1 \quad \text{and where} \quad \mu_i = \frac{\lambda_i}{\lambda_1} \in C^{0,1}(S_l).
\]

This gives, using (5), that on \( S_l \) we have

\[
(-\omega_i \partial_{ij} [\omega_j] + \omega_j \partial_{ij} [\omega_i]) = -\omega_i^2 \sum_{\mu_j} \mu_j (\mu_i \partial_{ij} [\mu_j] - \mu_j \partial_{ij} [\mu_i]).
\]

From this identity we obtain (8).

Step 3. From (5), (7) and (8) it follows that

\[
\int_{\Omega} (|\text{curl}\omega|^2 + |\text{div}\omega|^2 - |\nabla \omega|^2) \geq -C_1 \int_{\partial \Omega} |\omega|^2
\]

for some constant \( C_1 = C_1(\Omega, \lambda) > 0 \). We now recall that there exists a constant \( C_2 = C_2(\Omega) \) such that (see for instance [22] Theorem 1.5.1.10 or [13] Proposition 5.15) for any \( 0 < \epsilon < 1 \)

\[
\int_{\partial \Omega} |\omega|^2 \leq \epsilon \int_{\Omega} |\nabla \omega|^2 + \frac{C_2}{\epsilon} \int_{\Omega} |\omega|^2.
\]

Choose \( \epsilon \) such that \( \epsilon C_1 \leq 1/2 \) and then the theorem follows.

We have used in the proof of Proposition 6 in the case \( n = 2 \), the following lemma. In this case one cannot prove (5) for Lipschitz vectors by approximation, since standard convolution by some smoothing kernels \( \{\eta_k\}_{k \in \mathbb{N}} \) in the whole space does not imply any kind of convergence of \( \{\eta_k * \partial \omega_i / \partial x_j\}_{k \in \mathbb{N}} \) on \( \partial \Omega \) to the required function.

Lemma 8 Let \( \Omega \subset \mathbb{R}^2 \) be a bounded open Lipschitz set with unit outward normal \( \nu \) and assume that \( \omega_1, \omega_2 \in W^{1,\infty}(\Omega) \). Then the following identity holds

\[
\int_{\partial \Omega} \omega_1 \left( \frac{\partial \omega_2}{\partial x_2} \nu_1 - \frac{\partial \omega_2}{\partial x_1} \nu_2 \right) = \int_{\Omega} \left( \frac{\partial \omega_1}{\partial x_1} \frac{\partial \omega_2}{\partial x_2} - \frac{\partial \omega_2}{\partial x_1} \frac{\partial \omega_1}{\partial x_2} \right). \tag{9}
\]

Proof Step 1. Clearly (9) holds true for \( (\omega_1, \omega_2) \in C^2(\Omega)^2 \), by partial integration. Let us first show that (5) holds true if \( \omega_i \in C^2(\Omega) \) and \( \omega_2 \) is Lipschitz. Let us first assume that \( \partial \Omega \) is connected and hence there exists a Lipschitz curve \( \varphi \) and some interval \( [0, L] \) such that

\[
\varphi : [0, L] \to \partial \Omega, \quad \varphi(0) = \varphi(L) \tag{10}
\]
is a parametrization of $\partial \Omega$. We obtain that $\omega_2 \circ \varphi \in W^{1,\infty}([0, L])$, as it is the composition of two Lipschitz functions, and it is differentiable almost everywhere in $[0, L]$ with

$$
\frac{d}{dt}(\omega_2 \circ \varphi)(t) = \frac{\partial \omega_2}{\partial x_1}(\varphi(t))\varphi'_1(t) + \frac{\partial \omega_2}{\partial x_2}(\varphi(t))\varphi'_2(t) = \left(\frac{\partial \omega_2}{\partial x_1} \nu_1 - \frac{\partial \omega_2}{\partial x_2} \nu_2\right)(\varphi(t))|\varphi'(t)|.
$$

We have assumed here that $\varphi$ turns around the domain counterclockwise. Thus we obtain, using that $\varphi(0) = \varphi(L)$, $\omega_1 \in C^2(\Omega)$

This proves the claim of the present step, in case $\partial \Omega$ is connected. If $\partial \Omega$ is not connected then we first show that on each connected component $S_i$ of $\partial \Omega$ ($i = 1, \ldots, K$ for some $K \in \mathbb{N}$)

$$
\int_{S_i} \omega_1 \left(\frac{\partial \omega_2}{\partial x_1} \nu_1 - \frac{\partial \omega_2}{\partial x_2} \nu_2\right) = -\int_{S_i} \omega_2 \left(\frac{\partial \omega_1}{\partial x_1} \nu_1 - \frac{\partial \omega_1}{\partial x_2} \nu_2\right),
$$

as before, taking periodic parameterizations $\varphi_i$ of $S_i$. Then we take the sum over these integrals and can proceed in the same way. This proves the claim of Step 1.

**Step 2.** Let us now assume that $\omega_1, \omega_2$ are both Lipschitz. Take a sequence $\{\omega_1^k\} \in C^\infty(\Omega)$, $k \in \mathbb{N}$, such that

$$
\omega_1^k \to \omega \quad \text{in } W^{1,2}(\Omega) \quad \text{for } k \to \infty.
$$

By Step 1 we have for each $k$

$$
\int_{\partial \Omega} \omega_1^k \left(\frac{\partial \omega_2}{\partial x_1} \nu_1 - \frac{\partial \omega_2}{\partial x_2} \nu_2\right) = \int_{\Omega} \left(\frac{\partial \omega_1^k}{\partial x_1} \frac{\partial \omega_2}{\partial x_2} - \frac{\partial \omega_2}{\partial x_1} \frac{\partial \omega_1^k}{\partial x_2}\right).
$$

By the trace theorem $\omega_1^k \to \omega_1$ in $L^2(\partial \Omega)$, and by $\text{(11)}$

$$
\left(\frac{\partial \omega_2}{\partial x_1} \nu_1 - \frac{\partial \omega_2}{\partial x_2} \nu_2\right) \in L^\infty(\partial \Omega).
$$

So by letting $k \to \infty$ we obtain $\text{(11)}$. □

We split the second proof of Proposition $\text{(6)}$ (which requires $\lambda$ to be $C^{1,1}$) into several intermediate steps. We first recall the definition of the pushforward of a vector field.

**Definition 9** Let $U, V \subset \mathbb{R}^n$ be two open sets and $\Phi \in \text{Diff}^1(U; V)$. Then for any $\omega \in C(U)^n$ we define its pushforward $\Phi_*(\omega) \in C(V)^n$ by

$$
\Phi_*(\omega)(x) = \nabla \Phi \left(\Phi^{-1}(x)\right) \omega \left(\Phi^{-1}(x)\right),
$$

where $A \mathbf{b}$ is the usual multiplication of a (column) vector $\mathbf{b}$ by a matrix $A$. 
We will use several times the following elementary properties: \((\Phi \circ \Psi)_*(\omega) = \Phi_* (\Psi_* (\omega))\) and
\[
\alpha \times \beta = 0 \quad \text{at} \quad x \iff \Phi_* (\alpha) \times \Phi_* (\beta) = 0 \quad \text{at} \quad \Phi (x).
\] (12)

The proof of the next lemma is a straightforward algebraic calculation. The analogous result for the pullback of general \(k\)-forms can be found in [11], Lemma B.13. However, in the present case of vector fields, the proof is much simpler. \(O(n)\) shall denote the set of orthogonal matrices.

**Lemma 10** Let \(U, V \subset \mathbb{R}^n\) be open sets, \(A \in O(n), b \in \mathbb{R}^n\), and \(\psi : U \to V = \psi(U)\) defined by \(\psi(u) = Au + b\). Then for all \(\omega \in C^{0,1}(U)^n\) and almost every \(u \in U\) the following three identities hold true:
\[
|\nabla \omega(u)|^2 = |\nabla (\psi_\ast (\omega))|^2 (\psi(u)) \quad \text{(13)}
\]
\[
|\text{curl} \omega(u)|^2 = |\text{curl} (\psi_\ast (\omega))|^2 (\psi(u)) \quad \text{(14)}
\]
\[
|\text{div} \omega(u)|^2 = |\text{div} (\psi_\ast (\omega))|^2 (\psi(u)) \quad \text{(15)}
\]

**Remark 11** This lemma holds true by the specific algebraic properties of \(\nabla\), curl and div and is not valid in general for an arbitrary linear combination of derivatives of \(\omega\). In case of div we have actually something stronger: \(\text{div} \omega(u) = (\text{div} \psi_\ast(\omega)) (\psi(u))\) for any invertible matrix \(A\).

**Proof** We first prove (13). Let \(a_{ij}\) denote the entries of the matrix \(A\). Since \(A^t = A^{-1}\) we have that for any \(k, l = 1, \ldots, n\),
\[
\sum_{i=1}^n a_{ik} a_{il} = \delta_{kl}.
\] (16)
We can assume that \(b = 0\). Let \(x = \psi(u) = Au\), and hence \(\psi_\ast (\omega)(x) = A \omega (A^t x)\). So the components of \(\psi_\ast (\omega)\), respectively their derivatives are
\[
(\psi_\ast (\omega))_i (x) = \sum_{k=1}^n a_{ik} \omega_k (A^t x) \quad \text{and} \quad \frac{\partial (\psi_\ast (\omega))_i}{\partial x_j} (x) = \sum_{k,l=1}^n a_{ik} a_{jl} \frac{\partial \omega_k}{\partial u_l} (A^t x).
\]
We therefore obtain
\[
|\nabla (\psi_\ast (\omega))(u)|^2 = \sum_{i,j=1}^n \left( \sum_{k,l=1}^n a_{ik} a_{jl} \frac{\partial \omega_k}{\partial u_l} (u) \right)^2 = \sum_{i,j=1}^n a_{ik} a_{jl} \frac{\partial \omega_k}{\partial u_l} (u) \frac{\partial \omega_l}{\partial u_s}(u).
\]
Using now (16) gives the desired result. To prove (14) we use that
\[
|\text{curl} (\psi_\ast (\omega))(u)|^2 = \sum_{i<j} \left( \frac{\partial (\psi_\ast (\omega))_i}{\partial x_j} - \frac{\partial (\psi_\ast (\omega))_j}{\partial x_i} \right)^2 = \frac{1}{2} \sum_{i,j=1}^n \left( \frac{\partial (\psi_\ast (\omega))_i}{\partial x_j} - \frac{\partial (\psi_\ast (\omega))_j}{\partial x_i} \right)^2 \]
\[
= \frac{1}{2} \sum_{i,j=1}^n \sum_{k,l=1}^n a_{ik} a_{jl} \left( \frac{\partial \omega_k}{\partial u_l} - \frac{\partial \omega_l}{\partial u_k} \right)^2
\]
and proceed as in the proof of (13). The proof of (15) is very similar. ■

We start proving Proposition 10 in a special case.
Lemma 12 Let $\Omega \subset \mathbb{R}^n$ be a bounded open Lipschitz set and let $\lambda \in \mathbb{R}^n$ be a nonzero constant vector. Then the equality
\[
\int_{\Omega} |\nabla \omega|^2 = \int_{\Omega} (|\text{curl} \omega|^2 + |\text{div} \omega|^2)
\]
holds true for all $\omega \in C^1(\Omega)^n$ which satisfy $\lambda \times \omega = 0$ on $\partial \Omega$.

Remark 13 We will only use this lemma for $\lambda = e_1$, and will therefore only prove that case. The result for general $\lambda$ follows easily from this particular case, Lemma 10 and 12.

Proof As remarked, we only prove the lemma in the case when $\lambda = e_1 = (1,0,\ldots,0)$. In this case the boundary condition $\lambda \times \omega = 0$ is equivalent with $\omega_2 = \cdots = \omega_n = 0$ on $\partial \Omega$. Recall that, see (6),
\[
\int_{\Omega} (|\text{curl} \omega|^2 + |\text{div} \omega|^2 - |\nabla \omega|^2) = - \sum_{i<j} \int_{\partial \Omega} \omega_i \partial_{ij} [\omega_j] + \sum_{i<j} \int_{\partial \Omega} \omega_j \partial_{ij} [\omega_i].
\]
The right hand side of the previous equality cancels since, for any $i \neq j$, $\omega_i \partial_{ij} [\omega_j]$ is pointwise zero on $\partial \Omega$; indeed either $\omega_i = 0$ on $\partial \Omega$ or
\[
\partial_{ij} [\omega_j] = \frac{\partial \omega_j}{\partial x_i} \nu_j - \frac{\partial \omega_i}{\partial x_j} \nu_i = 0
\]
on $\partial \Omega$ recalling that $\partial_{ij} [\omega_j]$ is a tangential derivative. ■

The main statement of the next Lemma (Part (ii)), states that the change of the $L^2$ norms of $\nabla \omega$, curl $\omega$ and div $\omega$ under the pushforward of $\Phi$ can be estimated appropriately, if $\nabla \phi \in SO(n)$ at some point and if a neighborhood is taken small enough near that point.

Lemma 14 Let $x_0 \in \mathbb{R}^n$ and $\lambda$ be a $C^{1,1}$ vector field defined in a neighborhood of $x_0$, such that $|\lambda(x_0)| = 1$.

(i) Then there exist open sets $O,W \subset \mathbb{R}^n$, $x_0 \in O$, $0 \in W$, and a diffeomorphism $\Phi \in \text{Diff}^{1,1}(\overline{O},\overline{W})$ such that $\Phi(x_0) = 0$,
\[
\Phi_*(\lambda) = e_1 \quad \text{in } W \quad \text{and} \quad \nabla \Phi(x_0) \in SO(n).
\]

(ii) Moreover for any $0 < \epsilon \leq 1$, up to taking $O$ and $W$ smaller, there exists a constant $C = C(\Phi)$ satisfying the following three inequalities:
\[
\left| \int_O |\nabla \omega|^2 \right| - \int_W |\nabla (\Phi_*(\omega))|^2 \leq \epsilon \int_O |\nabla \omega|^2 + \frac{C}{\epsilon} \int_O |\omega|^2 \quad (17)
\]
\[
\left| \int_O |\text{curl} \omega|^2 \right| - \int_W |\text{curl} (\Phi_*(\omega))|^2 \leq \epsilon \int_O |\nabla \omega|^2 + \frac{C}{\epsilon} \int_O |\omega|^2 \quad (18)
\]
\[
\left| \int_O |\text{div} \omega|^2 \right| - \int_W |\text{div} (\Phi_*(\omega))|^2 \leq \epsilon \int_O |\nabla \omega|^2 + \frac{C}{\epsilon} \int_O |\omega|^2 \quad (19)
\]
for all $\omega \in C^{0,1}(\overline{O})^n$.

Remark 15 The proof will actually show that (17)-(19) remain valid with the same constant $C$ replacing $O$ by any of its own open subsets $V$ and replacing $W$ by $U = \Phi(V)$.
Proof Without loss of generality we can assume that $x_0 = 0$.

Step 1. We first prove (i). Let $\tilde{\Psi}(t,x)$ be the solution of
\[
\frac{\partial \tilde{\Psi}}{\partial t} = \lambda(\tilde{\Psi}) \quad \text{and} \quad \tilde{\Psi}(0,x) = Ax,
\]
where $A \in SO(n)$ is such that its first column is equal to $\lambda(x_0)$. Then define $\Psi(x) = \tilde{\Psi}(x_1,0,x_2,\cdots,x_n)$. It can be easily verified that $\Phi = \Psi^{-1}$ has all the desired properties.

Step 2. We now prove (ii). We will only do the proof for (17). The proof for (18) and (19) is very similar. Let $\Phi \in Diff^{1,1}(\mathcal{O},\mathcal{W})$ be as in (i) and $\Psi = \Phi^{-1}$. Throughout the proof $C_1,C_2,C_3$ and $C_4$ will denote constants depending only on $\Phi$. Let us write
\[
\nabla(\Phi^*(\omega))(y) = \sum_{k=1}^n S^k(\Phi,y)\omega_k(\Psi(y)) + \nabla\Phi(\Psi(y))\nabla\omega(\Psi(y))\nabla\Psi(y),
\]
where $S^k(\Phi,y), k = 1,\ldots,n$, are matrix valued functions depending only on derivatives of at most second order of $\Phi$. Its entries shall be denoted by $S^k_{ij}(\Phi,y)$.

Fix $0 < \epsilon \leq 1$. Using the inequality $2ab \leq a^2/\epsilon + b^2\epsilon$ and the fact that $\Phi$ is $C^{1,1}$, one immediately obtains
\[
E(y) \leq C_1 \epsilon |\nabla\omega|^2(\Psi(y)) + \frac{C_1}{\epsilon} |\omega|^2(\Psi(y)) \quad \text{and} \quad F(y) \leq C_2 |\omega|^2(\Psi(y)) \quad \text{for all } y \in \mathcal{O}.
\]
Changing the variables we therefore get
\[
\int_{\mathcal{W}} E \leq \int_{\mathcal{O}} \left( C_1 \epsilon |\nabla\omega|^2(x) + \frac{C_1}{\epsilon} |\omega|^2(x) \right) \det \nabla\Phi(x) dx \leq \int_{\mathcal{O}} \left( C_3 \epsilon |\nabla\omega|^2 + \frac{C_3}{\epsilon} |\omega|^2 \right)
\]
and similarly
\[
\int_{\mathcal{W}} F \leq \int_{\mathcal{O}} C_4 |\omega|^2.
\]
Combining (20), (21) and (22) it is enough to estimate
\[
\left| \int_{\mathcal{W}} D - \int_{\mathcal{O}} |\nabla\omega|^2 \right|
\]
to prove (17). By the change of variables formula we get
\[
\int_W D = \int_O |\nabla \Phi(x) \nabla \omega(x) (\nabla \Phi(x))^{-1}|^2 \det \nabla \Phi(x) dx.
\]

Thus
\[
\int_W D - \int_O |\nabla \omega|^2 = \int_O \left( |\nabla \Phi(x) \nabla \omega(x) (\nabla \Phi(x))^{-1}|^2 \det \nabla \Phi(x) - |\nabla \Phi(0) \nabla \omega(x) (\nabla \Phi(0))^{-1}|^2 \right) dx
+ \int_O \left( |\nabla \Phi(0) \nabla \omega(x) (\nabla \Phi(0))^{-1}|^2 - |\nabla \omega(x)|^2 \right) dx.
\]

It follows from (13) that the integrand in the second integral of the right-hand side of the previous equation is pointwise 0 in \(O\). To see this, fix \(x \in O\), set \(A = \nabla \Phi(0) \in SO(n)\) and apply the map \(\psi(u) = Au\) to Lemma [10] then [13] evaluated at \(u = x\) gives
\[
|\nabla \omega(x)|^2 = |\nabla (\psi_*(\omega))|^2(\psi(x)) = |A \nabla \omega(x) A^t|^2 = |\nabla \Phi(0) \nabla \omega(x) (\nabla \Phi(0))^{-1}|^2.
\]

Hence, recalling that \(\det \nabla \Phi(0) = 1\), it follows from continuity of \(\nabla \Phi\) that, taking \(O\) smaller (and consequently \(W\) as well) if necessary, that
\[
\left| \int_W D - \int_O |\nabla \omega|^2 \right| \leq \epsilon \int_O |\nabla \omega|^2.
\]

This concludes the proof of the lemma since the estimates on \(E\) and \(F\) remain valid for the new smaller open sets \(O\) and \(W\) with the same constants \(C_1, C_2, C_3\) and \(C_4\).

We now prove Proposition [5] in the special case when the vector fields \(\omega\) have compact support in a sufficiently small neighborhood of a boundary point \(x_0 \in \partial \Omega\).

**Lemma 16** Let \(\Omega \subset \mathbb{R}^n\) be a bounded open Lipschitz set and \(\lambda \in C^{1,1}(\partial \Omega)^n\) be such that \(\lambda \neq 0\) on \(\partial \Omega\) and assume that \(x_0 \in \partial \Omega\). Then there exists an open set \(O \in \mathbb{R}^n\), \(x_0 \in O\) and a constant \(C = C(\Omega, O, \lambda)\) such that
\[
\int_V |\nabla \omega|^2 \leq C \int_V (|\text{curl} \omega|^2 + |\text{div} \omega|^2 + |\omega|^2),
\]

where \(V = \Omega \cap O\), for all \(\omega \in C^1(\overline{\Omega})\) which satisfy
\[
\lambda \times \omega = 0 \quad \text{on} \ \partial \Omega \quad \text{and} \quad \text{supp}(\omega) \subset O.
\]

**Proof** The proof follows from Lemmas [12] and [13]. With no loss of generality we can assume that \(|\lambda(x_0)| = 1\). We claim that \(O\) given by Lemma [13] will have the desired property and we shall use the notation of that lemma. If \(\lambda \times \omega = 0\) on \(\partial \Omega\), we get that, using [12],
\[
e_1 \times \Phi_*(\omega) = 0 \quad \text{on} \ \Phi(\partial \Omega \cap O)
\]

Also, since \(\omega\) has compact support in \(O\), then \(\Phi_*(\omega)\) has compact support in \(\Phi(O)\). We conclude that, setting \(U = \Phi(V)\),
\[
e_1 \times \Phi_*(\omega) = 0 \quad \text{all over} \ \partial U
\]
for any \( \omega \) satisfying the assumptions of the lemma. We thus conclude from Lemma 12 that
\[
\int_{U} |\nabla(\Phi_\ast(\omega))|^2 = \int_{U} (|\text{curl}(\Phi_\ast(\omega))|^2 + |\text{div}(\Phi_\ast(\omega))|^2).
\]
Finally using (14)–(16) (and Remark 15) with \( \epsilon = 1/6 \) and the previous equality we obtain that
\[
\int_{V} |\nabla \omega|^2 \leq \epsilon \int_{V} |\nabla \omega|^2 + C \int_{V} |\omega|^2 + \int_{U} |\nabla(\Phi_\ast(\omega))|^2 \\
\leq 3\epsilon \int_{V} |\nabla \omega|^2 + 3C \epsilon \int_{V} |\omega|^2 + \int_{V} (|\text{curl} \omega|^2 + |\text{div} \omega|^2) \\
= \frac{1}{2} \int_{V} |\nabla \omega|^2 + 18C \int_{V} |\omega|^2 + \int_{V} (|\text{curl} \omega|^2 + |\text{div} \omega|^2),
\]
which proves the lemma. 

We give the second proof of the main proposition under the more restrictive hypothesis that \( \lambda \in C^{1,1}(\partial \Omega)^n \).

**Second Proof (Proposition 6).** Since \( \partial \Omega \) is compact, we can cover it by open neighborhoods \( O_i \subset \mathbb{R}^n, i = 1, \ldots, M \) which satisfy the conclusion of Lemma 10. Moreover let us choose a further open set \( O_0 \supset \overline{O_0} \subset \Omega \) such that \( \Omega \subset \bigcup_{i=0}^{M} O_i \). Let \( \{\xi_i\}_{i=0}^{M} \) be a partition of unity subordinate to the \( O_i \):
\[
0 \leq \xi_i \leq 1, \quad \text{supp}(\xi_i) \subset O_i \quad \text{and} \quad \sum_{i=0}^{M} \xi_i = 1 \quad \text{in} \ \overline{\Omega}.
\]
Let now \( \omega \in C^1(\overline{\Omega})^n \) be a vector field such that \( \lambda \times \omega = 0 \) on \( \partial \Omega \). Then using Lemma 12 for \( i = 0 \), respectively Lemma 14 for \( i = 1, \ldots, M \), we obtain that
\[
\int_{V_i} |\nabla (\xi_i \omega)|^2 \leq C_i \int_{V_i} (|\text{curl}(\xi_i \omega)|^2 + |\text{div}(\xi_i \omega)|^2 + |\xi_i \omega|^2),
\]
for some constants \( C_i = C_i(\Omega, \lambda) \), where \( V_i = \Omega \cap O_i \). Note that
\[
\int_{\Omega} |\nabla \omega|^2 = \int_{\Omega} \left| \nabla \left( \sum_{i=0}^{M} \xi_i \omega \right) \right|^2 \leq M \sum_{i=0}^{M} \int_{\Omega} |\nabla(\xi_i \omega)|^2 = M \sum_{i=0}^{M} \int_{V_i} |\nabla(\xi_i \omega)|^2.
\]
Thus combining (24) and (23) one gets
\[
\int_{\Omega} |\nabla \omega|^2 \leq C_1 \sum_{i=0}^{M} \int_{V_i} (|\text{curl} \xi_i \omega|^2 + |\text{div} \xi_i \omega|^2 + |\xi_i \omega|^2) \leq C_2 \int_{\Omega} (|\text{curl} \omega|^2 + |\text{div} \omega|^2 + |\omega|^2),
\]
for some constants \( C_1 \) and \( C_2 \) depending only on \( \Omega \) and \( \lambda \). 

To extend Proposition 6 to \( H^1 \) vector fields we need to show that a vector field \( \omega \in H^1(\Omega)^n \) which satisfies \( \lambda \times \omega = 0 \) on the boundary can be approximated by \( C^1 \) vector fields also satisfying the same boundary condition. This is possible according to the next proposition.
Proposition 17 Let \( n \geq 2, r \geq 0 \) be an integer and \( 0 \leq \alpha \leq 1 \), with \( r + \alpha \geq 1 \). Suppose \( \Omega \subset \mathbb{R}^n \) is a bounded open Lipschitz set and \( \lambda \in C^{r,\alpha} (\partial \Omega)^n \) be such that
\[
\lambda \neq 0 \quad \text{on } \partial \Omega.
\]
Suppose \( \omega \in H^1 (\Omega)^n \) is such that \( \lambda \times \omega = 0 \) on \( \partial \Omega \). Then there exists a sequence \( \{ \omega^k \}_{k \in \mathbb{N}} \subset C^{r,\alpha} (\overline{\Omega})^n \) such that for \( k \to \infty \)
\[
\omega^k \to \omega \in H^1 (\Omega)^n \quad \text{and} \quad \lambda \times \omega^k = 0 \quad \text{on } \partial \Omega \text{ for all } k.
\]

Proof Step 1. We first prove the following claim: For every \( x_0 \in \partial \Omega \) there exists a neighborhood \( W \subset \mathbb{R}^n \) of \( x_0 \) such that for all \( \omega \in H^1 (\Omega) \) satisfying
\[
\text{supp}(\omega) \subset W \quad \text{and} \quad \lambda \times \omega = 0 \quad \text{on } \partial \Omega,
\]
there exists a sequence \( \{ \omega^k \}_{k \in \mathbb{N}} \subset C^{r,\alpha} (\overline{\Omega \cap W})^n \) such that
\[
\omega^k \to \omega \quad \text{in } H^1 (\Omega \cap W)^n \quad \text{and} \quad \lambda \times \omega^k = 0 \quad \text{on } \partial \Omega \cap W \text{ for all } k.
\]

We extend \( \lambda \) to a \( C^{r,\alpha} \) vector field in \( \mathbb{R}^n \), see Definition \( 3 \). Since \( \lambda \) does not vanish on the boundary we can assume with no loss of generality that \( \lambda_1 \neq 0 \) in \( W \) where \( W \) is a small enough neighborhood of \( x_0 \). Let us define
\[
\alpha_i = \lambda_1 \omega_i - \lambda_i \omega_1 = (\lambda \times \omega)_i.
\]
Note that by the additional assumptions (25) the support of \( \omega \) is contained in \( W \), and in particular vanishes on \( \partial W \). Therefore \( \alpha_i \in H^1_0 (\Omega \cap W) \) and hence there exists a sequence \( \alpha^k_i \) with the properties
\[
\{ \alpha^k_i \}_{k \in \mathbb{N}} \subset C^\infty (\Omega \cap W), \quad \alpha^k_i \to \alpha_i \quad \text{in } H^1 (\Omega \cap W).
\]

Moreover we choose a sequence \( \{ \beta^k \} \subset C^\infty (\overline{\Omega \cap W}) \) such that \( \beta^k \to \omega_1 \) in \( H^1 (\Omega \cap W) \). We finally define \( \omega^k = (\omega^k_1, \ldots, \omega^k_n) \) by
\[
\omega^k_1 = \beta^k \quad \omega^k_i = \frac{\alpha^k_i + \lambda_i \beta^k}{\lambda_1} \quad \text{for } i = 2, \ldots, n.
\]

Using that \( \alpha^k_i = 0 \) on \( \partial \Omega \cap W \) we obtain that for any \( i, j \in \{ 1, \ldots, n \} \)
\[
\lambda_j \omega^k_i - \lambda_i \omega^k_j = \frac{\lambda_j}{\lambda_1} \lambda_i \beta^k - \frac{\lambda_i}{\lambda_1} \lambda_j \beta^k = 0 \quad \text{on } \partial \Omega \cap W.
\]
and thus \( \omega^k \) has all the desired properties claimed in Step 1.

Step 2. Using that \( \partial \Omega \) is compact, we can cover it by a finite number of open sets \( W_1, \ldots, W_L \) with the properties given by Step 1. Clearly we can add \( W_0 \) such that \( W_0 \) is also open, \( \overline{\Omega} \subset \bigcup_{l=0}^L W_l \) and any \( \omega^0 \in H^1 (W_0) \) with compact support in \( W_0 \) can be approximated by smooth vector fields \( \omega^{0,k} \) with compact support in \( W_0 \). In particular \( \lambda \times \omega^{0,k} = 0 \) on \( \partial \Omega \) for all \( k \). Let \( \eta_l \) be a smooth partition of unity subordinate to this covering such that
\[
\sum_{l=0}^L \eta_l^2 = 1 \quad \text{in } \overline{\Omega}.
\]
Define $\omega^l = \eta_l \omega$. Using Step 1 there exists for each $l = 1, \ldots, L$ sequences $\{\omega^{l,k}\}_{k \in \mathbb{N}}$ of $C^{r, \alpha}$ vector fields such that for $k \to \infty$

$$\omega^{l,k} \to \omega^l \quad \text{in } H^1(\Omega \cap W) \quad \text{and} \quad \lambda \times \omega^{l,k} = 0 \quad \text{on } \partial \Omega \cap W \quad \text{for all } k.$$ 

Then $\eta_l \omega^{l,k} \in C^{r, \alpha}(\overline{\Omega})^n$ is well defined and $\omega^k = \sum_{l=0}^L \eta_l \omega^{l,k}$ has all the desired properties.

### 4 Formulation in $\mathbb{R}^2$ for discontinuous $\lambda$

In two dimensions we improve Theorem 11: we no longer require $\lambda$ to be continuous on the whole boundary, but still Lipschitz on different pieces of $\partial \Omega$. More precisely we make the following assumption.

**Assumption 1** Assume that $\Omega \subset \mathbb{R}^2$ is a bounded open Lipschitz set, such that for some integer $N$

$$\partial \Omega = \bigcup_{i=1}^N \Gamma_i$$

and $\Gamma_i \cap \Gamma_{i+1} = \{S_i\}$ for $i = 1, \ldots, N$,

where $\Gamma_i$ are disjoint open sets in $\partial \Omega$ (with the convention that $\Gamma_{N+1} = \Gamma_1$) and the $S_i$ are $N$ different points on the boundary, called vertices. Let $\lambda_i \in C^{0,1}(\overline{\Gamma_i})^2$ for $i = 1, \ldots, N$ and define

$$\lambda : \bigcup_{i=1}^N \Gamma_i \to \mathbb{R}^2,$$

by $\lambda^i = \lambda$ on $\Gamma_i$. We also assume that

$$\lambda_i \neq 0 \quad \text{on } \Gamma_i.$$

Note that we allow that at a vertex $S_i$ the segments $\Gamma_i$ and $\Gamma_{i+1}$ can meet at an angle $\pi$. In this setting we have the following theorem.

**Theorem 18** Let $\Omega$ and $\lambda$ be as in Assumption 1. Then there exists a constant $C = C(\Omega, \lambda)$ such that

$$\|\nabla \omega\|_{L^2(\Omega)}^2 \leq C \left( \|\operatorname{curl} \omega\|_{L^2(\Omega)}^2 + \|\operatorname{div} \omega\|_{L^2(\Omega)}^2 + \|\omega\|_{L^2(\Omega)}^2 \right),$$

(26)

for all $\omega \in H^1(\Omega)^2$ which satisfy

$$\lambda \times \omega = 0 \quad \text{on } \partial \Omega,$$

where the last equality is understood as $\lambda_i \times \omega = 0$ on $\Gamma_i$ for each $i = 1, \ldots, N$.

**Example 19** As a special case we obtain Gaffney inequality with the classical boundary conditions in polygonal domains.

The proof of Theorem 18 is essentially the same as the corresponding result for globally Lipschitz $\lambda$: only the approximation result, i.e. the analogy to Proposition 17 has to be adapted. This is done in the next proposition.
Proposition 20  Let $\Omega$ and $\lambda$ be as in Assumption 1. Suppose $\omega \in H^1(\Omega)^2$ is such that $\lambda \times \omega = 0$ on $\partial \Omega$. Then there exists a sequence $\{\omega^k\}_{k \in \mathbb{N}} \subset C^{0,1}(\overline{\Omega})^2$ such that for $k \to \infty$

$$\begin{align*}
\omega^k \to &\omega \in H^1(\Omega)^2 \quad \text{and} \quad \lambda \times \omega^k = 0 \quad \text{on} \quad \partial \Omega \quad \text{for all} \quad k.
\end{align*}$$

Proof Step 1. We first prove the following claim: For every $x_0 \in \partial \Omega$ there exists a neighborhood $W \subset \mathbb{R}^2$ of $x_0$ such that for all $\omega \in H^1(\Omega)$ satisfying

$$\begin{align*}
supp(\omega) \subset W \quad \text{and} \quad \lambda \times \omega = 0 \quad \text{on} \quad \partial \Omega,
\end{align*}$$

there exists a sequence $\{\omega^k\}_{k \in \mathbb{N}} \subset C^{0,1}(\overline{\Omega \cap W})^2$ such that

$$\begin{align*}
\omega^k \to &\omega \quad \text{in} \quad H^1(\Omega \cap W)^2 \quad \text{and} \quad \lambda \times \omega^k = 0 \quad \text{on} \quad \partial \Omega \cap W \quad \text{for all} \quad k.
\end{align*}$$

The proof of this claim is the same as the proof of Proposition 17 if $x_0$ is not a vertex and so we can assume that $x_0 = \Gamma_1 \cap \Gamma_{i+1}$ is a vertex. Then we distinguish two cases.

Case 1. We assume that $\lambda^i(x_0)$ and $\lambda^{i+1}(x_0)$ are linearly dependent. Since the boundary condition $\lambda \times \omega = 0$ is invariant under scaling or sign change of $\lambda$, and neither $\lambda^i$ nor $\lambda^{i+1}$ vanish, we can assume that $\lambda^i(x_0) = \lambda^{i+1}(x_0)$. But then $\lambda$ is Lipschitz in $\Gamma_i \cup \Gamma_{i+1}$ and thus we can again proceed as in Proposition 17.

Case 2. We assume that $\det(\lambda^i(x_0)|\lambda^{i+1}(x_0)) \neq 0$. In this case we extend both $\lambda^i$ and $\lambda^{i+1}$ separately to $C^{0,1}$ vector fields defined in $\mathbb{R}^2$. By continuity, there exists a neighborhood $W$ of $x_0$ such that $\det(\lambda^i|\lambda^{i+1}) \neq 0$ in $\overline{W}$. Let $\omega$ be a vector field satisfying (27). Define the two functions

$$\begin{align*}
p = &\lambda^i \times \omega \in H^1(\Omega), \quad \text{and} \quad q = \lambda^{i+1} \times \omega \in H^1(\Omega),
\end{align*}$$

which can also be written in the matrix form

$$\begin{align*}
\begin{pmatrix} p \\ q \end{pmatrix} &= \begin{pmatrix} -\lambda_2^i & \lambda_1^i \\ -\lambda_2^{i+1} & \lambda_1^{i+1} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = M \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.
\end{align*}$$

By an extension theorem (see Bernard [3] or Theorem 1.6.1 of Grisvard [21] for polygonal domains) there exists a sequence $\{p^k\}_{k \in \mathbb{N}}, \{q^k\}_{k \in \mathbb{N}} \subset C^1(\overline{\Omega})$ such that both $p^k$ (respectively $q^k$) converges to $p$ (respectively $q$) in $H^1(\Omega)$ and

$$\begin{align*}
p^k = 0 \quad \text{on} \quad \Gamma_i \quad \text{and} \quad q^k = 0 \quad \text{on} \quad \Gamma_{i+1} \quad \text{for all} \quad k \in \mathbb{N}.
\end{align*}$$

Since $\det(\lambda^i|\lambda^{i+1}) \neq 0$ on $\overline{W}$, we can define $\omega^k \in C^{0,1}(\overline{W \cap \Omega})$ by

$$\begin{align*}
\omega^k = M^{-1} \begin{pmatrix} p^k \\ q^k \end{pmatrix}.
\end{align*}$$

Note that $\lambda^i \times \omega^k = p^k$, respectively $\lambda^{i+1} \times \omega^k = q^k$. It straightforward to check that $\omega^k$ has all the desired properties claimed by Step 1.

Step 2. We finally conclude exactly as in Step 2 of the proof of Proposition 17.

We now prove the main theorem of this section.
Proof of Theorem 18. Since $\Omega$ is a Lipschitz domain we can use partial integration and obtain that

$$\int_{\Omega} (|\text{curl}\omega|^2 + |\text{div}\omega|^2 - |\nabla\omega|^2) = \int_{\partial\Omega} \omega_1 \left( \nu_1 \frac{\partial \omega_2}{\partial x_2} - \nu_2 \frac{\partial \omega_2}{\partial x_1} \right) - \int_{\partial\Omega} \omega_2 \left( \nu_1 \frac{\partial \omega_1}{\partial x_2} - \nu_2 \frac{\partial \omega_1}{\partial x_1} \right)$$

$$= \sum_{i=1}^N \left[ \int_{\Gamma_i} \omega_1 \left( \nu_1 \frac{\partial \omega_2}{\partial x_2} - \nu_2 \frac{\partial \omega_2}{\partial x_1} \right) - \int_{\Gamma_i} \omega_2 \left( \nu_1 \frac{\partial \omega_1}{\partial x_2} - \nu_2 \frac{\partial \omega_1}{\partial x_1} \right) \right]$$

holds for any $\omega \in C^{0,1}(\Omega)^2$, where the first equality is exactly as in Step 1 of the first proof of Proposition 6. We now proceed as in Step 2 of the first proof of Proposition 6, working on each $\Gamma_i$ separately: Using that each $\Gamma_i$ is a $C^{0,1}$ curve and that $\lambda_i$ does not vanish on $\Gamma_i$, one obtains that there exists a constant $C_1 = C_1(\Omega, \lambda) > 0$ such that

$$\int_{\Omega} (|\text{curl}\omega|^2 + |\text{div}\omega|^2 - |\nabla\omega|^2) \geq -C_1 \int_{\partial\Omega} |\omega|^2$$

for all $\omega \in C^{0,1}(\Omega)^2$ satisfying $\lambda \times \omega = 0$ on $\partial\Omega$. This proves the Theorem for $C^{0,1}$ vector fields $\omega$.

The general case follows from Proposition 20.

5 Counterexamples

In view of Theorems 4, 18 and the classical boundary condition $\langle \nu; \omega \rangle = 0$, one could expect that if $n \geq 3$ we also have a Gaffney inequality under the boundary condition $\langle \lambda; \omega \rangle = 0$ on $\partial\Omega$, if $\lambda$ does not vanish on $\partial\Omega$. This is however not true as shown by the following simple example.

Example 21 Let $\Omega \subset \mathbb{R}^3$ be any bounded open smooth set and $\lambda = (0, 0, 1)$. Then there exists no constant $C = C(\Omega, \lambda)$ such that

$$\int_{\Omega} |
abla\omega|^2 \leq C \int_{\Omega} \left( |\text{curl}\omega|^2 + |\text{div}\omega|^2 + |\omega|^2 \right)$$

for all $\omega \in C^2(\Omega; \mathbb{R}^3)$ satisfying $\langle \lambda; \omega \rangle = 0$ on $\partial\Omega$. To see this take

$$\omega(x) = (e^{nx_1} \cos(nx_2), -e^{nx_1} \sin(nx_2), 0).$$

Then one easily verifies that $\text{div}\omega = 0$, $\text{curl}\omega = 0$, $|\nabla\omega(x)|^2 = 2n^2 e^{2nx_1}$ and $|\omega(x)|^2 = e^{2nx_1}$. Hence, as in [8], Gaffney inequality cannot hold.

The question also arises whether Theorem 4 generalizes to differential forms of higher order (identifying vector fields with 1-forms). This is also not true. More precisely we have the following counterexample for 2-forms.

Example 22 Let $n \geq 3$, $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set. Then there exists no constant $C = C(\Omega)$ such that

$$\int_{\Omega} |\nabla\omega|^2 \leq C \int_{\Omega} \left( |\omega|^2 + |\delta\omega|^2 + |\omega|^2 \right)$$
for all \( \omega \in C^2(\Omega; \Lambda^2) \) such that \( dx^3 \wedge \omega = 0 \) on \( \partial \Omega \). To see this take

\[
\omega = e^{nx_1} \cos(nx_2) dx^1 \wedge dx^3 + e^{nx_1} \sin(nx_2) dx^2 \wedge dx^3.
\]

One can verify that \( d\omega = 0 \) and \( \delta\omega = 0 \). Thus one concludes exactly as in Example 21.

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**References**

[1] Amrouche C., Bernardi C., Dauge M. and Girault V., Vector potentials in three-dimensional non-smooth domains, *Math. Methods Appl. Sci.*, 21 (1998), 823–864.

[2] Arnold N., Falk S. and Winther R., Finite element exterior calculus, homological techniques, and applications, *Acta Numer.*, 15 (2006), 1–155.

[3] Bernard J.M., Density results in Sobolev spaces whose elements vanish on a part of the boundary, *Chin. Ann. Math. Ser. B* 32 (2011), no. 6, 823–846.

[4] Bonizzoni F., Buffa A. and Nobile F., Moment equations for the mixed formulation of the Hodge Laplacian with stochastic loading term, *IMA J. Numer. Anal*, 34 (2014), no. 4, 1328–1360.

[5] Ben Belgacem F., Bernardi C., Costabel M. and Dauge M., Un résultat de densité pour les équations de Maxwell, *C. R. Acad. Sci. Paris Sr. I Math.*, 324 (1997), no. 6, 731–736.

[6] Ciarlet P., Hazard C. and Lohrengel S, Les équations de Maxwell dans un polyèdre: un résultat de densité, *C. R. Acad. Sci. Paris Sr. I Math.* 326 (1998), no. 11, 1305–1310.

[7] Costabel M., A remark on the regularity of solutions of Maxwell’s equations on Lipschitz domains, *Math. Methods Appl. Sci.*, 12 (1990), no. 4, 365–368.

[8] Costabel M., A coercive bilinear form for Maxwell’s equations, *J. Math. Anal. Appl.*, 157 (1991), 527–541.

[9] Costabel M. and Dauge M., Un résultat de densité pour les équations de Maxwell régularisées dans un domaine lipschitzien, *C. R. Acad. Sci. Paris Sr. I Math.* 327 (1998), no. 9, 849–854.

[10] Csató G., On an integral formula for differential forms and its applications on manifolds with boundary, *Analysis*, 33 (2013), 349–366.

[11] Csató G, Some boundary value problems for differential forms, Ph.D Thesis, EPFL Lausanne, 2012.
[12] Csató G. and Dacorogna B., An identity involving exterior derivatives and applications to Gaffney inequality, *Discrete Continuous Dynam. Syst., Series S*, 5 (2012), 531-544.

[13] Csató G., Dacorogna B. and Kneuss O., *The pullback equation for differential forms*, Birkhäuser, Boston, 2012.

[14] Csató G., Dacorogna B. and Sil S., On the best constant in Gaffney inequality, to appear.

[15] Dautray R. and Lions J.L., *Analyse mathématique et calcul numérique*, Masson, Paris, 1988.

[16] Friedrichs K. O., Differential forms on Riemannian manifolds, *Comm. Pure Appl. Math. 8* (1955), 551–590.

[17] Gaffney M.P., The harmonic operator for exterior differential forms, *Proc. Nat. Acad. of Sci. U. S. A.*, 37 (1951), 48–50.

[18] Gaffney M.P., Hilbert space methods in the theory of harmonic integrals, *Trans. Amer. Math. Soc*, 78 (1955), 426–444.

[19] Girault V. and Raviart P.A., *Finite element approximation of the Navier-Stokes equations*, Lecture Notes in Math. 749, Springer-Verlag, Berlin, 1979.

[20] Gol’dshtein V., Mitrea I. and Mitrea M, Hodge decompositions with mixed boundary conditions and applications to partial differential equations on Lipschitz manifolds, *Problems in mathematical analysis* No. 52. J. Math. Sci. (N. Y.) 172 (2011), no. 3, 347–400.

[21] Grisvard, P. *Singularities in boundary value problems*, Recherches en Mathématiques Appliquées, 22. Masson, Paris, Springer-Verlag, Berlin, 1992.

[22] Grisvard P., *Elliptic problems in nonsmooth domains*, Monographs and Studies in Mathematics, 24. Pitman (Advanced Publishing Program), Boston, MA, 1985.

[23] Iwaniec T. and Martin G., *Geometric function theory and non-linear analysis*, Oxford University Press, Oxford, 2001.

[24] Iwaniec T., Scott C. and Stroffolini B., Nonlinear Hodge theory on manifolds with boundary, *Annali Mat. Pura Appl.*, 177 (1999), 37–115.

[25] Jakab T., Mitrea I. and Mitrea M., On the regularity of differential forms satisfying mixed boundary conditions in a class of Lipschitz domains, *Indiana Univ. Math. J.* 58 (2009), no. 5, 2043–2071.

[26] Mitrea M., Dirichlet integrals and Gaffney-Friedrichs inequalities in convex domains, *Forum Math.* 13 (2001), no. 4, 531-567.

[27] Mitrea D. and Mitrea M., Finite energy solutions of Maxwell’s equations and constructive Hodge decompositions on nonsmooth Riemannian manifolds, *J. Funct. Anal.* 190 (2002), no. 2, 339–417.

[28] Morrey C.B., A variational method in the theory of harmonic integrals II, *Amer. J. Math.*, 78 (1956), 137–170.
[29] Morrey C.B., *Multiple integrals in the calculus of variations*, Springer-Verlag, Berlin, 1966.

[30] Morrey C.B. and Eells J., A variational method in the theory of harmonic integrals, *Ann. of Math.*, 63 (1956), 91–128.

[31] Schwarz G., *Hodge decomposition - A method for solving boundary value problems*, Lecture Notes in Math. 1607, Springer-Verlag, Berlin, 1995.

[32] Taylor M.E., *Partial differential equations*, Vol. 1, Springer-Verlag, New York, 1996.

[33] Von Wahl W., Estimating $\nabla u$ by $\text{div} u$ and $\text{curl} u$, *Math. Methods Appl. Sci.*, 15 (1992), 123–143.