Similarity transformations and linearization for a family of
dispersionless integrable PDEs

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Abstract

We apply the theory of Lie point symmetries for the study of a family of partial differential equations which are integrable by the hyperbolic reductions method and are reduced to members of the Painlevé transcendents. The main results of this study is that from the application of the similarity transformations provided by the Lie point symmetries all the members of the family of the partial differential equations are reduced to second-order differential equations which are maximal symmetric and can be linearized.

Keywords: Lie symmetries; invariants; similarity transformations; linearization

1 Introduction

In [1], Ferapontov et al. classified the partial differential equations of the form

\[(A (u))_{xx} + (B (u))_{yy} + (C (u))_{yy} + 2 \left( (P (u))_{xy} + (Q (u))_{xt} + (P (u))_{yt} \right) = 0, \tag{1}\]

which are integrable under the method of hydrodynamic reductions [2] and can be reduced into an Painlevé equation [3]. There are five partial differential equations of the form (1) which are integrable by the hydrodynamic reductions method [1]

\[\mathcal{H}_A \equiv u_{xx} + u_{yy} - (\ln (e^u - 1))_{yy} - (\ln (e^u - 1))_{tt} = 0, \tag{2}\]

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\[ H_B \equiv u_{xx} + u_{yy} - (e^u)_{tt} = 0, \]  
\[ H_C \equiv (e^u - u)_{xx} + 2u_{xy} + (e^u)_{tt} = 0, \]  
\[ H_D \equiv u_{xt} - (uu_x)_x - u_{yy} = 0, \]  
\[ H_E \equiv (u^2)_{xx} + u_{yy} + 2u_{xt} = 0. \]  

Equations (3) and (5) are the Boyer-Finley and dispersionless Kadomtsev-Petviashvili equations respectively. For these two equations it is known that they are reduced into the Painlevé transcendent by applying the central quadric ansatz. The hydrodynamic reductions method was found to provide more general solutions. Indeed, by studying the dispersionless Kadomtsev-Petviashvili with the hydrodynamic reductions and the central quadric ansatz, it was found that the solutions coming from the later method form a subclass of two-phase solutions provided by the hydrodynamic reductions approach.  

As far as the reduction of equations (2)-(6) into a Painlevé equation, is concerned, it was found that equation \( H_A \) reduces to the Painlevé \( PV \) equation, the Boyer-Finley equation \( H_B \) reduces into the \( PV \) equation reducible to \( P_{11} \). Moreover, \( H_C \) reduces to \( PV \), the dispersionless Kadomtsev-Petviashvili is related with the \( P_{11} \) with a reduction to \( P_1 \), while the fifth equation \( H_E \) is reduced to \( P_{IV} \). For extensions of the results of \( P_1 \) and a connection of the hydrodynamic reductions method with the conformal structure of Einstein-Weyl geometry we refer the reader to \( P_1 \).  

In this work we apply the Lie symmetry analysis in order to investigate the algebraic properties and the similarity transformations for the five partial differential equations (2)-(6). The method of Lie symmetries of differential equations established by Sophus Lie at the end of the 19th century, and provides a systematic approach for the study and determination of solutions and conservation laws for nonlinear differential equations.  

The novelty of symmetry analysis is that invariant functions can be determined for a given differential equation. From the invariant functions we can define similarity transformations which are necessary to simplify the differential equation. The similarity transformations are used to reduce the given differential equation into an equivalent equation with less dynamical variables. In the case of partial differential equations the independent variables are reduces, while in the case of ordinary differential equations, the order of the equation, that is, the dependent variables, are reduced. There is a plethora of applications in the literature on the symmetry analysis of various dynamical systems. The method of symmetry analysis is applied in various systems of fluid dynamics in the studies. The Burgers-heat system is investigated by applying the symmetry analysis in the studies. A recent application of the Lie symmetry approach on time-fractional systems is presented in. However, Lie symmetries are very useful and for the study of ordinary differential equations. Some studies on the symmetry analysis on the geodesic equations in curved spaces are presented in. Finally, in a discussion is given on the novelty on the application of the Lie symmetry analysis in gravitational physics and cosmology.  

Another important application of the Lie symmetry approach is the classification scheme of differential equations according to the admitted group of symmetries, and to the construction of equivalent transformation
which transform a given differential equation into another differential equation of the same order, when the admitted Lie symmetries form the same Lie algebra \[33–35\]. Recently, in \[36\] the authors investigated which of the six ordinary differential equations of the Painlevé transcendents admit nontrivial Lie point symmetries. It was found that equations $P_{III}$, $P_{V}$ and $P_{V I}$ have nontrivial symmetries for special values of the free parameters. On the other hand, in \[37\] the method of Jacobi last multiplier applied in order to determine generalized symmetries for a particular case of the $P_{XIV}$ equation. By using generalized-hidden symmetries, the linearization of the Painleve-Ince equation was proved in \[38\]. The plan of the paper in as follows.

In Section 2 we present the basic properties and definitions for the Lie symmetry analysis of differential equations. In Sections 3-7 we determine the Lie point symmetries for the five equations of our analysis. We determine the commutators and the Adjoint representation such that to derive, when it is feasible, the one-dimensional optimal system. Equation $\mathcal{H}_A$ admits a finite dimensional Lie algebra of dimension four, in particular the $A_{4,5}$ in the Patera-Winternitz classification scheme \[39\]. However, the rest of the equations admit infinity Lie point symmetries. We were able to define four-dimensional Lie subalgebras. The main observation of this work is that the application of the Lie invariants define similarity transformation where the partial differential equations reduce to maximal symmetric second-order equations. That is an important result because we are able to investigate the integrability properties of equations (2)-(6) by using the symmetry analysis. In Section 8 we summarize our results and we draw our conclusions. The main result of this analysis is given in a proposition where we show that equations (2)-(6) can be linearized with the application of Lie invariants.

2 Preliminaries

Assume the infinitesimal one-parameter point transformation

\begin{align*}
t' &= t + \varepsilon \xi^t (t, x, y, u), \\
x' &= x + \varepsilon \xi^x (t, x, y, u), \\
y' &= y + \varepsilon \xi^y (t, x, y, u), \\
u' &= u + \varepsilon \eta (t, x, y, u),
\end{align*}

where $\varepsilon$ is the infinitesimal parameter, $\varepsilon^2 \to 0$, and infinitesimal generator

\[X = \xi^t (t, x, y, u) \partial_t + \xi^x (t, x, y, u) \partial_x + \xi^y (t, x, y, u) \partial_y + \eta (t, x, y, u) \partial_u.\]

We define the second extension $X^{[2]}$ of $X$ in the jet space \{t, x, y, u, u_t, u_x, u_y, u_{tx}, u_{xy}, u_{yy}, u_{txy}\} as follows

\[X^{[2]} = \xi^\mu \partial_\mu + \eta^A \partial_A + \eta^{[1]} \partial_A \mu + \eta^{[2]} \partial_A \mu \nu,\]

in which $\eta^{[1]}$, $\eta^{[2]}$ are defined as

\[\eta^{[n]} = D_\mu \eta^{[n-1]} - u_{\mu_1 \mu_2 \ldots \mu_{n-1}} D_{\mu} (\xi^\mu),\]
where $\mu = (t, x, y)$.

By definition a partial differential equation $H = H(t, x, y, u, u_t, u_x, u_y, u_{tx}, u_{xx}, u_{yy}, u_{tx}, u_{xy})$ is invariant under the action of the one-parameter point transformation with infinitesimal generator the vector field $X$ if and only if there exist a function $\lambda$ such that

$$L_X (H) = \lambda H,$$

in which $L_X$ is the Lie derivative with respect to the vector field $X$.

Lie symmetries are mainly applied for the construction of similarity transformations. The latter are necessary in order to simplify a given differential equation by mean of reduction. The exact and analytic solutions which are determined by the application of the Lie symmetries are known as similarity solutions.

In order to perform a complete derivation of all the possible similarity solutions we should find the admitted one-dimensional optimal system. Consider the $n$-dimensional Lie algebra $G_n$ with elements $X_1, X_2, \ldots, X_n$ admitted by the differential equation $H$.

The vector fields

$$Z = \sum_{i=1}^{n} a_i X_i, \quad W = \sum_{i=1}^{n} b_i X_i, \quad a_i, b_i \text{ are constants.}$$

are equivalent if and only if $W = \text{Ad}(\exp(\varepsilon X_i)) Z$ or $W = cZ$ where $c$ is a constant.

Operator $\text{Ad}(\exp(\varepsilon X_i)) X_j = X_j - \varepsilon [X_i, X_j] + \frac{1}{2} \varepsilon^2 [X_i, [X_i, X_j]] + \ldots$ is known as the adjoint representation. The derive all the independent similarity transformations for a given differential equation the adjoint representation of the admitted Lie algebra should be determined. This lead to the construction of the one-dimensional optimal system.

### 3 Lie symmetry analysis for equation $H_A$

The first equation of our analysis, equation $H_A$, admits the Lie point symmetries

$$X_1^A = \partial_t, \quad X_2^A = \partial_x, \quad X_3^A = \partial_y, \quad X_4^A = t\partial_t + x\partial_x + y\partial_y$$

with commutators and Adjoint representation as presented in Tables 1 and 2. The admitted Lie algebra is the $A_{4,5}$. The one-dimensional system for the finite Lie algebra is consisted by the one-dimensional Lie algebras:

$$\{ X_1^A \}, \{ X_2^A \}, \{ X_3^A \}, \{ X_1^A + \alpha X_2^A \}, \{ X_1^A + \alpha X_2^A + \beta X_3^A \}, \{ X_2^A + \alpha X_3^A \}, \{ X_1^A + \alpha X_2^A + \beta X_3^A \}.$$

Application of $\{ X_1^A + \alpha X_2^A \}$, $\{ X_1^A + \beta X_3^A \}$ provides the reduced equation

$$\left( \beta^2 (e^u - 1)^2 - \alpha^2 (e^u - 1) (1 + \beta^2 e^u) \right) U_{\sigma\sigma} + \alpha^2 (\beta^2 + 1) e^U (U_{\sigma})^2 = 0, \quad (7)$$

with $u = U(\sigma)$ and $\sigma = y - \beta t + \frac{\beta}{\alpha} x$. 

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Table 1: Commutators for the Lie point symmetries of equation \((2)\).

| \([X_I, X_J]\) | \(X^A_1\) | \(X^A_2\) | \(X^A_3\) | \(X^A_4\) |
|-----------------|----------|----------|----------|----------|
| \(X^A_1\)      | 0        | 0        | 0        | \(X^A_1\) |
| \(X^A_2\)      | 0        | 0        | 0        | \(X^A_2\) |
| \(X^A_3\)      | 0        | 0        | 0        | \(X^A_3\) |
| \(X^A_4\)      | \(-X^A_1\) | \(-X^A_2\) | \(-X^A_3\) | 0        |

Table 2: Adjoint representation for the Lie point symmetries of equation \((2)\).

| \(Ad(e^{\epsilon X_i}) X_j\) | \(X^A_1\) | \(X^A_2\) | \(X^A_3\) | \(X^A_4\) |
|-----------------------------|----------|----------|----------|----------|
| \(X^A_1\)                  | \(X^A_1\) | \(X^A_2\) | \(X^A_3\) | \(X^A_1 - \epsilon X^A_1\) |
| \(X^A_2\)                  | \(X^A_1\) | \(X^A_2\) | \(X^A_3\) | \(X^A_1 - \epsilon X^A_2\) |
| \(X^A_3\)                  | \(X^A_1\) | \(X^A_2\) | \(X^A_3\) | \(X^A_1 - \epsilon X^A_3\) |
| \(X^A_4\)                  | \(e^\epsilon X^A_1\) | \(e^\epsilon X^A_2\) | \(e^\epsilon X^A_3\) | \(X^A_4\) |

Equation \((7)\) is a second-order ordinary differential equation of the form \(U_{\sigma\sigma} + L(U) U_{\sigma} = 0\), which means that is maximally symmetric. It admits eight Lie point symmetries which form the \(sl(3, R)\) algebra. Thus, according to the main theorem of S. Lie theorem, equation \((7)\) can be linearized \([8–11]\). Indeed the transformation which linearized the differential equation is of the form \(V = \int e^{\int L(U) dU} dU\). We remark that any reduction of equation \((2)\) with any Lie symmetries provided by the optimal system of the Abelian Lie subalgebra \(\{X^A_1, X^A_2, X^A_3\}\) provides a similar result.

On the other hand, reduction with \(\{X^A_4\}\) provides the partial differential equation
\[
0 = (e^U - 1) (U_{\zeta\zeta} (\zeta^2 e^U + 1) + U_{\omega\omega} (e^U (\omega^2 - 1) + 1) + 2\omega \zeta e^U U_{\zeta\omega}) \\
+ e^U (U_{\zeta} + U_{\omega})^2 - 2 (e^U - 1) (U_{\zeta} + U_{\omega}) + U_{\zeta\zeta}^2, 
\]
(8)
with \(u = U (\zeta, \omega)\), \(\zeta = \frac{\eta}{\rho}\) and \(\omega = \frac{\eta}{\rho}\), is the similarity transformation. Equation \((8)\) does not possess any Lie point symmetry, thus we can not reduce further the differential equation.

Moreover, reduction with the symmetry vectors \(\{X^A_2 + \alpha X^A_3\}\) and later with the reduced symmetry of \(\{X^A_4\}\), provides the ordinary differential equation
\[
(\alpha^2 (e^U - 1) - 1 - e^U \rho^2) U_{\rho\rho} + \frac{e^U}{(e^U - 1)} (\rho^2 + 1) (U_{\rho})^2 - 2\rho e^U U_{\rho} = 0, 
\]
(9)
where \(u = U (\rho)\) and \(\rho = \frac{\eta - \alpha \eta}{\rho}\).
Without loss of generality we assume $\alpha^2 = 1$. Then equation (9) is written in the equivalent form

$$\frac{U_{\rho \rho}}{U_\rho} + \frac{e^U (\rho^2 + 1)}{(e^U - 1)(e^U (1 - \rho^2) - 2)} (U_\rho) - \frac{2e^U}{(e^U (1 - \rho^2) - 2)} = 0,$$

that is

$$\frac{U_{\rho \rho}}{U_\rho} + \frac{e^U}{e^U - 1} U_\rho + \frac{U_\rho e^U \rho^2 + 2e^U \rho - U_\rho e^U}{e^U (\rho^2 - 1) + 2} = 0.$$

Hence, we can write easily the later equation as follows

$$\frac{d}{d\rho} \left( \ln (U_\rho) - \ln (e^U - 1) + \ln (e^U (\rho^2 - 1) + 2) \right) = 0.$$ 

Therefore, the conservation law is

$$\frac{U_\rho (e^U (\rho^2 - 1) + 2)}{e^U - 1} = I_0. \quad (10)$$

We observe that equation (10) can be written in a linear form after the change of the independent variable $d\kappa = \frac{e^U - 1}{(e^U (\rho^2 - 1) + 2)} d\rho$, and $U = U(\kappa)$. Thus, equation (10) becomes $U_\kappa = I_0$ which is nothing else than the conservation law for the maximal symmetric second-order ordinary differential equation $U_{\kappa \kappa} = 0$.

Last but not least we remark that find similar result if we apply first the reduction of any one-dimensional Lie algebra of the three-dimensional Abelian subalgebra and then we consider the $\{X_A^4\}$.

4 Lie symmetry analysis for equation $\mathcal{H}_B$

In order to proceed with the analysis for equation $\mathcal{H}_B$ we select the new set of independent variables $(z, \bar{z}) = \frac{1}{2} (u + v, i (u - v))$, such that equation (3) to be written as follows

$$u_{z\bar{z}} - (e^u)_{tt} = 0. \quad (11)$$

Application of the Lie symmetry conditions indicates that equation (11) admits the Lie symmetry vectors

$$X_1^B = \partial_t, \ X_2^B = t\partial_t + u\partial_u,$$

$$X_3^B = \Phi (z) \partial_z - \Phi (\bar{z}) \partial_{\bar{z}}, \ X_4^B = \Psi (\bar{z}) \partial_z - \Psi (z) \partial_{\bar{z}}.$$

The vector fields $X_3^B, X_4^B$ indicates the infinity number of solutions for the Laplace operator $u_{z\bar{z}}$. The commutators and the Adjoint representation for the admitted symmetry vectors are presented in Tables 3 and 4 respectively. The Lie point symmetries, from the finite Lie algebra $A_{2,1}$ plus the infinity algebra consisted by the vector fields $X_3^B$ and $X_4^B$. In Tables 3 and 4 we assumed that functions $\Phi$ and $\Psi$ are specific and not arbitrary. Because in general it holds $[X_3^B (\Phi_1), X_3^B (\Phi_2)] = X_3^B (\Phi_3)$.

Hence, from Tables 3 and 4 it follows that one-dimensional optimal system is consisted by the following one-dimensional Lie algebras, $\{X_1^B\}, \{X_2^B\}, \{X_3^B\}, \{X_4^B\}, \{X_3^B + \alpha X_4^B\}, \{X_3^B + \alpha X_4^B\}, \{X_1^B + \alpha X_4^B\}$.
Table 3: Commutator table for the Lie point symmetries of equation (3).

| $[X_i, X_j]$ | $X_i^B$ | $X_j^B$ | $X_i^B$ | $X_i^B$ |
|---------------|--------|--------|--------|--------|
| $X_i^B$       | 0      | $X_i^B$ | 0      | 0      |
| $X_i^B$       | $-X_i^B$ | 0      | 0      | 0      |
| $X_i^B$       | 0      | 0      | 0      | 0      |
| $X_i^B$       | 0      | 0      | 0      | 0      |

Table 4: Adjoint representation for the Lie point symmetries of equation (3).

| $\text{Ad} \left(e^{iX_i}\right) X_j$ | $X_i^B$ | $X_i^B$ | $X_i^B$ | $X_i^B$ |
|--------------------------------------|--------|--------|--------|--------|
| $X_i^B$                              | $X_i^B$ | $X_i^B$ | $X_i^B$ | $X_i^B$ |
| $X_i^B$                              | $e^{iX_i^B}$ | $X_i^B$ | $X_i^B$ | $X_i^B$ |
| $X_i^B$                              | $X_i^B$ | $X_i^B$ | $X_i^B$ | $X_i^B$ |
| $X_i^B$                              | $X_i^B$ | $X_i^B$ | $X_i^B$ | $X_i^B$ |

\{X_2^B + \alpha X_3^B\}, \{X_2^B + \alpha X_4^B\}, \{X_3^B + \alpha X_2^B + \beta X_4^B\} and \{X_2^B + \alpha X_3^B + \beta X_4^B\}. We proceed with the reduction of the equation and the determination of similarity solutions.

Consider now reduction with the use of the symmetry vector $\{X_3^B\}$, then it follows $u = -\ln \Phi (z) + \ln U (t, \bar{z})$ with reduced equation the

$$U_{tt} = 0 , \quad U (t, \bar{z}) = U_1 (\bar{z}) t + U_0 (\bar{z}).$$

We remark that the reduced equation is that of the free particle and it is maximal symmetric, thus it admits eight Lie point symmetries which form the $sl(3, R)$ Lie algebra. A similar result it follows if we assume reduction with respect to the field $X_4^B$.

Let us assume now reduction with the field $\{X_1^B + X_3^B\}$. The reduced equation is found to be

$$U_{zT} + (e^U)_{TT} = 0 \quad (12)$$

where $T = t - \int \frac{dz}{\Phi(z)}$ and $u = -\ln \left(\Phi (\bar{z})\right) + U (T, x)$. Equation (12) admits the symmetry vectors $\bar{X}_1^B = \partial_T$, $\bar{X}_2 = T \partial_T + \partial_U$ and $\bar{X}_3^B = \Phi (z) \partial_z - \Phi (z) \partial_U$. Hence, application for the field $\bar{X}_1^B + \bar{X}_3^B$ gives the similarity transformation $U (T, x) = -\ln (\Phi (z)) + V (\tau)$, $\tau = T - \int \frac{dz}{\Phi (z)}$, with reduced equation the maximal symmetric ordinary differential equation

$$V_{\tau \tau} - \frac{e^V}{(1 - e^V)} (V_{\tau})^2 = 0.$$
Moreover, application of the vector field $\bar{X}_2 + \bar{X}_3^B$ in (12) provides the reduced equation

$$V_{\bar{\lambda}} (e^V - \lambda) + e^V (V_{\bar{\lambda}})^2 - V_{\bar{\lambda}} = 0,$$

where $\lambda = T e^{-S(z)}$, $M(z) = \frac{1}{\sqrt{u}}$, and $U(T, z) = S(z) + \ln (S(z)z) + V (T e^{-S(z)})$.

Equation (13) is not maximal symmetric, however it can be integrated and it can written in the equivalent form

$$\tilde{V}_{\bar{\lambda}} = \frac{e^{\lambda} (1 + \tilde{V}) \tilde{V}}{e^{\lambda - 1}}, \quad \bar{\lambda} = V - \ln \lambda, \quad \tilde{V} = (V_{\bar{\lambda}} \lambda - 1)^{-1}.$$

Therefore, if we do the change of variables $d\lambda = I_0 \frac{e^{\lambda}(1 + \tilde{V})\tilde{V}}{e^{\lambda - 1}} d\lambda$, the latter differential equation becomes $\tilde{V}_{\lambda\lambda} = I_0$, which is the conservation law for the maximal symmetric differential equation $\tilde{V}_{\lambda\lambda} = 0$.

5 Lie symmetry analysis for equation $\mathcal{H}_C$

As far as the Lie symmetries of $\mathcal{H}_C$ are concerned, they are calculated

$$X_1^C = \partial_t, \quad X_2^C = \partial_y, \quad X_3^C = t\partial_t + x\partial_x + y\partial_y,$$

$$X_4^C (Z(y)) = Z(y) (\partial_x - 2\partial_y) + 2Z(y)_y \partial_u.$$

Hence, we can infer that equation $\mathcal{H}_C$ admits infinity Lie symmetries. The nonzero commutators are

$$[X_2^C, X_3^C] = X_2^C, \quad [X_2^C, X_4^C (Z(y))] = X_4^C \left(Z(y)_y\right),$$

$$[X_3^C, X_4^C (Z(y))] = X_4^C \left(yZ(y)_y - Z(y)\right).$$

and

$$[X_4^C (Z(y)), X_4^C (W(y))] = 2X_4^C (Y(y)) \text{ with } Y(y) = W(y) Z(y)_y - Z(y) W(y)_y.$$

We observe that for $Z(y)_y = 0$, a finite dimensional Lie algebra exist, the four-dimensional Lie algebra $A_{4,5}$ of equation $\mathcal{H}_A$.

Consider the application of the Lie point symmetries $\{X_1^C + \alpha X_2^C, X_1^C + \alpha \partial_x\}$, then, the reduced equation is derived

$$0 = (\alpha (e^U - 1) + 2\beta + \alpha \beta e^U) U_{\sigma\sigma} + (1 + \beta^2) \alpha \beta e^U (U_{\sigma})^2,$$

where now $u = U(\sigma), \sigma = y - \alpha t + \frac{\beta}{\alpha} x$. We observe that equation (14) is a maximal symmetric second-order ordinary differential equation.

We proceed with the second reduction approach, where we apply the Lie symmetries $\{X_1^C + \alpha X_2^C, X_1^C + \alpha X_3^C\}$. Hence, equation $\mathcal{H}_C$ is reduced to the second-order ordinary differential equation

$$0 = \left( (\alpha^2 + \omega^2) e^U - \omega (\omega + 2) \right) U_{\omega\omega} + (\alpha^2 + \omega^2) e^U (U_{\omega})^2 + 2(\omega (e^U - 1) - 1) (U_{\omega}),$$
where \( u = U(\omega) \) and \( \omega = \frac{u + at}{2} \). The latter equation can be integrated as follows

\[
U_\omega \left((\alpha^2 + \omega^2) e^U - \omega (\omega + 2)\right) = I_0
\]

(15)

which can be written in the equivalent form

\[
U_\omega = I_0.
\]

Let us assume reduction with respect to the Lie symmetry vector \( \{X_1^G + X_4^G (Z(y))\} \). The similarity transformation is \( u = U(T, X) \) with \( T = t - \int \frac{dx}{Z(y) + 2x - 2\chi} \) and \( X = y + 2x \), while the reduced equation is

\[
\left(U_{TT} + (U_T)^2 + 4 \left(U_{YY} + (U_Y)^2\right)\right) e^U + 2U_{TY} = 0.
\]

(16)

The latter equation admits the reduced symmetry vectors \( \{\partial_T, \partial_Y, T\partial_T + Y\partial_Y\} \). It follows that reduction with respect to the symmetry vector \( \{\partial_T + \beta\partial_Y\} \) gives

\[
\left((4 + \beta^2) e^V - 2\beta\right) V_{zz} + (\beta^2 + 4) e^V (V_z)^2 = 0.
\]

On the other hand, reduction of equation (16) with respect to the similarity transformation provided by \( T\partial_T + Y\partial_Y \) gives

\[
\left((4\lambda^2 + 1) e^V - 2\lambda\right) V_{\lambda\lambda} + (4\lambda^2 + 1) e^V (V_\lambda)^2 + (8\lambda e^V - 2) V_\lambda = 0,
\]

that is

\[
V_\lambda \left((4\lambda^2 + 1) e^V - 2\lambda\right) = I_0,
\]

which can be written as a maximal symmetric second-order differential equation.

### 6 Lie symmetry analysis for equation \( \mathcal{H}_D \)

We proceed our analysis with the derivation of the Lie symmetry vectors for equation \( \mathcal{H}_D \). The application of the Lie symmetry condition provides that equation \( \mathcal{H}_D \) admits infinity number of Lie symmetries as they described by the following families of vector fields

\[
X_1^D (\Phi(t)) = \Phi(t) \partial_t + \left(\frac{1}{6} \Phi_{tt} y^2 + \frac{1}{3} x \Phi\right) \partial_x + \frac{2}{3} y \Phi_1 \partial_y - \frac{1}{3} \left(\Phi_{tt} x + \frac{1}{2} \Phi_{ttt} y^2 - 2 \Phi_t u\right) \partial_u,
\]

\[
X_2^D (\Psi(t)) = \Psi(t) \partial_x - \Psi(t) \partial_u,
\]

\[
X_3^D (\Sigma(t)) = \frac{1}{2} \Sigma t y \partial_x + \Sigma \partial_y - \frac{1}{2} \Sigma tt y \partial_u,
\]

and

\[
X_4^D = 2x \partial_x + \partial_y + 2u \partial_u.
\]
The nonzero commutators of the admitted Lie symmetries are

\[
[X_1^D(\Phi), X_2^D(\Psi)] = X_2^D \left( \Phi \Psi_t - \frac{1}{3} \Phi_t \Psi \right),
\]

\[
[X_1^D(\Phi), X_3^D(\Sigma)] = X_3^D \left( \frac{2}{3} \Sigma \Phi_t - 3 \Phi \Sigma_t \right),
\]

\[
[X_2^D(\Psi), X_4^D] = 2X_2^D(\Psi), \quad [X_3^D(\Sigma), X_1^D] = X_3^D(\Sigma),
\]

\[
[X_1^D(\Phi(t)), X_1^D(\Phi(t))] = X_1^D \left( \Phi \Phi_t - \Phi_t \Phi \right),
\]

\[
[X_3^D(\Sigma(t)), X_3^D(\Sigma(t))] = X_3^D \left( \Sigma \Sigma_t - \Sigma_t \Sigma \right).
\]

For \( \Phi(t) = 1 \), \( \Psi(t) = 1 \) and \( \Sigma(t) = 1 \), we find the four-dimensional subalgebra

\[
\hat{X}_1^D = \partial_t, \quad \hat{X}_2^D = \partial_x, \quad \hat{X}_3^D = \partial_y, \quad \hat{X}_4^D,
\]

which form the \( A_{3,3} \otimes A_1 \) Lie algebra. The commutators and the Adjoint representation of the four dimensional Lie algebra \( \{ \hat{X}_1^D, \hat{X}_2^D, \hat{X}_3^D, \hat{X}_4^D \} \) are presented in Tables 5 and 6.

We proceed with the application of the Lie symmetries for the finite Lie algebra \( A_{3,3} \otimes A_1 \) for the reduction of the partial differential equation \( H_D \). From Table 6 we derive the one-dimensional optimal system, which is consisted by the one-dimensional Lie algebras \( \{ \hat{X}_1^D \}, \{ \hat{X}_2^D \}, \{ \hat{X}_3^D \}, \{ \hat{X}_4^D \}, \{ \hat{X}_1^D + \alpha \hat{X}_2^D \}, \{ \hat{X}_1^D + \alpha \hat{X}_3^D \}, \{ \hat{X}_1^D + \alpha \hat{X}_4^D \}, \{ \hat{X}_2^D + \alpha \hat{X}_3^D \}, \{ \hat{X}_2^D + \alpha \hat{X}_4^D \}, \{ \hat{X}_3^D + \alpha \hat{X}_4^D \}, \{ \hat{X}_1^D + \alpha \hat{X}_2^D + \beta \hat{X}_3^D \}. 

Table 5: Commutators for the elements which form the finite Lie algebra of equation (5).

| \([X_i, X_j]\) | \(\hat{X}_1^D\) | \(\hat{X}_2^D\) | \(\hat{X}_3^D\) | \(\hat{X}_4^D\) |
|----------------|-----------|-----------|-----------|-----------|
| \(\hat{X}_1^D\) | 0         | 0         | 0         | 0         |
| \(\hat{X}_2^D\) | 0         | 0         | 0         | \(2\hat{X}_2^D\) |
| \(\hat{X}_3^D\) | 0         | 0         | 0         | \(\hat{X}_3^D\) |
| \(\hat{X}_4^D\) | 0         | \(-2\hat{X}_2^D\) | \(-\hat{X}_3^D\) | 0         |

Table 6: Adjoint representation for the elements which form the finite Lie algebra (5).

| \(Ad(e^{iX_j^D})X_i\) | \(\hat{X}_1^D\) | \(\hat{X}_2^D\) | \(\hat{X}_3^D\) | \(\hat{X}_4^D\) |
|----------------|-----------|-----------|-----------|-----------|
| \(\hat{X}_1^D\) | \(\hat{X}_1^D\) | \(\hat{X}_2^D\) | \(\hat{X}_3^D\) | \(\hat{X}_4^D\) |
| \(\hat{X}_2^D\) | \(\hat{X}_1^D\) | \(\hat{X}_2^D\) | \(\hat{X}_3^D\) | \(\hat{X}_4^D - 2\varepsilon \hat{X}_2^D\) |
| \(\hat{X}_3^D\) | \(\hat{X}_1^D\) | \(\hat{X}_2^D\) | \(\hat{X}_3^D\) | \(\hat{X}_4^D - \varepsilon \hat{X}_3^D\) |
| \(\hat{X}_4^D\) | \(\hat{X}_1^D\) | \(e^{2\varepsilon} \hat{X}_2^D\) | \(e^{\varepsilon} \hat{X}_3^D\) | \(\hat{X}_4^D\) |
Therefore, by applying the Lie symmetries \( \{ \bar{X}^1_D + \alpha \bar{X}^2_D \} \), \( \{ \bar{X}^1_D + \beta \bar{X}^3_D \} \) for the reduction of equation (5) we end with the second-order ordinary differential equation

\[
(\alpha^2 + \beta^2 U + \alpha \beta^2) U_{\sigma\sigma} + \beta^2 (U_\sigma)^2 = 0, \tag{17}
\]

with \( u = U(\sigma) \) and \( \sigma = y - \beta t + \frac{\alpha}{\beta} x \). Equation (17) is maximal symmetric and can be linearized.

On the other hand, from the Lie symmetry \( \{ \bar{X}^2_D + \bar{X}^3_D \} \) we find the second-order partial differential equation

\[
U_{YY} + U_{Y} + U_{YY} + (U_Y)^2 = 0. \tag{18}
\]

where \( u = U(t, Y), Y = y - x \). Equation (18) admits infinity Lie symmetries consisted by the vector fields \( Y_1^D = \partial_t \), \( Y_2^D = t \partial_t + (U - 1) \partial_U \), \( Y_3^D = Y \partial_Y + (U + 1) \partial_U \), \( Y_4^D = t^2 + t Y \partial_t + (Y - t(U + 1)) \partial_U \) and \( Y_5^D = \zeta(t) \partial_Y + \zeta(t)_x \partial_U \). Vector fields \( \{ Y_1^D, Y_2^D, Y_4^D \} \) form the \( sl(2, R) \) Lie algebra.

Hence, from the vector field \( \{ Y_1^D + \alpha Y_3^D \} \) it follow the similarity transformation \( U = -1 + V(\kappa)e^{\alpha t}, \kappa = Ye^{-\alpha t} \), with reduced equation the differential equation \( V_{\kappa\kappa} + \frac{1}{V_{\kappa}} (V_{\kappa})^2 = 0 \), which can be integrated further

\[
N_\lambda = \frac{N}{\Lambda} (1 + \alpha N)^2 \text{ with } N = (V_{\kappa} - \alpha)^{-1} \text{ and } \kappa = V - \kappa \alpha. \tag{19}
\]

Easily, equation (19) can be written as \( N_{\lambda\lambda} = 0 \). A similar result follows, if we perform the reduction of equation (18) with the rest of the symmetry vectors.

### 7 Lie symmetry analysis for equation \( H_E \)

The fifth equation of our study, namely equation \( H_E \) admits the following Lie symmetries

\[
X_1^E(\Phi(t)) = \Phi(t) \partial_t + \frac{1}{2} y \Phi t \partial_y + \left( \frac{1}{4} \Phi_{tt} y - \frac{1}{2} \Phi_{tt} u \right) \partial_u,
\]

\[
X_2^E = \partial_x, \quad X_3^E(\Sigma(t)) = \Sigma(t) \partial_y + \frac{1}{2} \Sigma t \partial_u,
\]

\[
X_4^E = 2 x \partial_x + y \partial_y + u \partial_u.
\]

The Lie symmetries form an infinity Lie algebra, with nonzero commutators

\[
[X_1^E(\Phi(t)), X_3^E(\Sigma(t))] = X_3^D(\Sigma \Phi_t - 2 \Phi \Sigma_t), \quad [X_2^E, X_4^E] = 2 X_2^E,
\]

\[
[X_3^E(\Sigma(t)), X_4^E] = X_3^E(\Sigma(t)),
\]

and

\[
[X_1^E(\Phi(t)), X_1^E(\Phi(t))] = X_1^E(\Phi \Phi_t - \Phi_\Phi t).
\]

The four-dimensional finite algebra \( A_{3,3} \otimes A_1 \) follows for \( \Phi(t) = 1 \) and \( \Sigma(t) = 1 \), that is, \( A_{3,3} \otimes A_1 \) is consisted by the Lie symmetry vectors \( \{ \bar{X}_1^E, X_2^E, \bar{X}_3^E, X_4^E \} \). We proceed with the application of the Lie symmetry vectors and the determination of the similarity transformations.
From the symmetry vectors \( \{ \bar{X}^E_1 + \alpha X^E_2 \} \), \( \{ \bar{X}^E_1 + \beta \bar{X}^E_3 \} \) we find the similarity transformation \( u = U(\sigma) \), \( \sigma = y - \beta t + \frac{\alpha}{2} x \) with reduced equation the maximal symmetric equation

\[
(2U\beta + \alpha - \beta^2) U_{\sigma\sigma} + 2\beta (U_{\sigma})^2 = 0.
\]

On the other hand, reduction with the symmetry vector \( \{ X^E_2 + \bar{X}^E_3 \} \) provides the second-order partial differential equation

\[
V_{Yt} - V_{YY} + 2VV_{YY} + 2 (V_Y)^2 = 0,
\]

where \( u = V(t, Y), Y = y - x \). Equation (20) is of the form of equation (18). Indeed if we replace in (20) \( V = -\frac{1}{2} U \) and \( t \rightarrow -t \) equation (20) is written in the form of equation (18).

### 8 Conclusions

We applied the Lie symmetry analysis for a family of five partial differential equations of the form (1) which are integrable with the method of hydrodynamic reductions. In particular, we determined the Lie point symmetries and we studied the algebraic properties of the admitted symmetries. Moreover, from the invariant functions provided by the Lie symmetries we defined similarity transformations which were used to reduce the number of independent variables for the differential equations. With the application of two different similarity transformations we were able to reduce the partial differential equations into a second-order ordinary differential equation.

We summarize this result in the following proposition.

**Proposition 1:** The five partial differential equations \( \mathcal{H}_A, \mathcal{H}_B, \mathcal{H}_C, \mathcal{H}_D \) and \( \mathcal{H}_E \) which are integrable with the method of hydrodynamic reductions can be linearized with the use of similarity transformations given by the Lie point symmetries.

Equation \( \mathcal{H}_A \) admits a finite Lie algebra of dimension four, while the rest differential equations, \( \mathcal{H}_B, \mathcal{H}_C, \mathcal{H}_D \) and \( \mathcal{H}_E \) admit infinity Lie point symmetries which however are constructed by four generic vector fields. The application of the Lie point symmetries for these equations indicates that these five equations possess a common feature, they are reduced to a maximal symmetric ordinary differential equation which can be linearized. We show that this is possible not only when we investigate for “travel-wave” solutions but also for more general reductions.

In a future study we plan to investigate by applying the theory of Lie symmetries and other differential equations which are integrable by the method of hydrodynamic reductions.

**Data availability statement**

*Data sharing not applicable to this article as no datasets were generated or analysed during the current study.*
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