Deformation Quantization for Actions of Kählerian Lie Groups

Pierre Bieliavsky
University of Louvain, Belgium.
e-mail: pierre.bieliavsky@gmail.com
and
Victor Gayral
University of Reims, France.
e-mail: victor.gayral@univ-reims.fr

Abstract

Let $\mathbb{B}$ be a Lie group admitting a left-invariant negatively curved Kählerian structure. Consider a strongly continuous action $\alpha$ of $\mathbb{B}$ on a Fréchet algebra $\mathcal{A}$. Denote by $\mathcal{A}^\infty$ the associated Fréchet algebra of smooth vectors for the action $\alpha$. In the Abelian case $\mathbb{B} = \mathbb{R}^2n$ and $\alpha$ isometric, Marc Rieffel proved in [19] that Weyl’s operator symbol composition formula yields a deformation through Fréchet algebra structures $\{\star_\theta\}_{\theta \in \mathbb{R}}$ on $\mathcal{A}^\infty$. When $\mathcal{A}$ is a $C^*$-algebra, every deformed algebra $(\mathcal{A}^\infty, \star_\theta)$ admits a compatible pre-$C^*$-structure. In this paper, we prove both analogous statements in the general negatively curved Kählerian group and (non-isometric) “tempered” action case. The construction relies on the one hand on combining a non-Abelian version of oscillatory integral on tempered Lie groups with geometrical objects coming from invariant WKB-quantization of solvable symplectic symmetric spaces, and, on the second hand, in establishing a non-Abelian version of the Calderón-Vaillancourt Theorem. In particular, we give an oscillating kernel formula for WKB-star products on symplectic symmetric spaces that fiber over an exponential Lie group.
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1 Introduction

The general idea of deforming a given theory by use of its symmetries goes back to Drinfel’d. One paradigm being that the data of a Drinfel’d twist based on a bi-algebra acting on an associative algebra $A$, produces an associative deformation of $A$. In the context of Lie theory, one considers for instance the category of module-algebras over the universal enveloping algebra $U(g)$ of the Lie algebra $g$ of a given Lie group $G$. In that situation, the notion of Drinfel’d twist is in a one to one correspondence with the one of left-invariant formal star-product $\star_\nu$ on the space of formal power series $C^\infty(G)[[\nu]]$, see [11]. Disposing of such a twist, every $U(g)$-module-algebra $A$ may then be formally deformed into an associative algebra $A[[\nu]]$.

It is important to observe that, within this situation, the symplectic leaf $B$ through the unit element $e$ of $G$ in the characteristic foliation of the (left-invariant) Poisson structure directing the star-product $\star_e$, always consists in an immersed Lie subgroup of $G$. The Lie group $B$ therefore carries a left-invariant symplectic structure. This stresses the importance of symplectic Lie groups (i.e. connected Lie groups endowed with invariant symplectic forms) as semi-classical approximations of Drinfel’d twists attached to Lie algebras.

In the present work, we address the question of designing non-formal Drinfel’d twists for actions of symplectic Lie groups $B$ that underly negatively curved Kählerian Lie groups, i.e. Lie groups that admit a left-invariant Kählerian structure of negative curvature. These groups exactly correspond to the normal j-algebras defined by Pyatetskii-Shapiro in his work on automorphic forms [17]. In particular, this class of groups contains all Iwasawa factors $AN$ of Hermitian type simple Lie groups $G = KAN$.

Roughly speaking, one looks for a smooth one-parameter family of complex valued smooth two-point functions on the group, $\{K_\theta\}_{\theta \in B} \subset C^\infty(B \times B, \mathbb{C})$, with the property that, for every sufficiently regular action $\alpha$ of $B$ on a Fréchet or a $C^*$-algebra $A$, the following formula

$$a \star^B_\theta b := \int_{B \times B} K_\theta(x, y) \alpha_x(a) \alpha_y(b) \, dx \, dy,$$

defines a one-parameter deformation of the Fréchet or $C^*$-algebra structure on $A$.

The above program was realized by Marc Rieffel in the particular case of the Abelian Lie group $B = \mathbb{R}^{2n}$ in [19]. More precisely, Rieffel proved that for any strongly continuous and isometric action of $\mathbb{R}^{2n}$ on any Fréchet algebra $A$, the associated Fréchet sub-algebra $A^\infty$ of smooth vectors for this action, is deformed by the rule [11], where the two-point kernel there, consists in the Weyl symbol composition kernel:

$$K_\theta(x, y) := \theta^{-2n} \exp \left\{ \frac{i}{\theta} \omega^0(x, y) \right\},$$

associated to an invariant symplectic structure $\omega^0$ on $\mathbb{R}^{2n}$. At the formal level, the associated star-product $\star_\nu$ therefore corresponds here to Moyal’s product. In the special case where the Fréchet algebra $A$ is a $C^*$-algebra, Rieffel also constructed a deformed $C^*$-structure, so that $(A^\infty, \star^B_\theta)$ becomes a pre-$C^*$-algebra, which in turn yields a deformation theory at the level of $C^*$-algebras too. Many further results have been proven then (for example continuity of the field of deformed $C^*$-algebras [19], invariance of the $K$-theory [20],...), and many applications have been found (for instance in locally compact quantum groups [21], quantum fields theory [12, in spectral triples [13],...).

In the present article, we first investigate the deformation theory of Fréchet algebras endowed with an action of a negatively curved Kählerian Lie group. Most of the results we present here are of a pure analytical nature. Indeed, once a family $\{K_\theta\}_{\theta \in B}$ of associative (i.e. such that the associated deformed product [11] is at least formally associative) two-point functions has been found, in order to give a precise meaning of the associated multiplication rule, it makes no doubt that the integrals in [11] need to be interpreted in a suitable (here oscillatory) sense. Indeed, there is no reason to expect the two-point function $K_\theta$ to be integrable: it is typically not even bounded in the non-Abelian case! Thus, already in the case of an isometric action on a $C^*$-algebra, we have to face a serious analytical difficulty. We stress that contrarily to the case of $\mathbb{R}^{2n}$, in the situation of a non-Abelian group action, this is an highly non-trivial feature of our deformation theory.

The paper is organized as follows.

In section 2, we start by introducing non-Abelian and unbounded versions of Fréchet-valued symbol
spaces\footnote{We recently learned that in \cite{15}, G. Lechner and S. Waldmann introduced a similar type of symbol spaces in the Abelian context of actions of $\mathbb{R}^d$ on locally convex algebras.} on a Lie group $G$, with Lie algebra $\mathfrak{g}$:

$$B^{(\mu_j)}(G, \mathcal{E}) := \left\{ f \in C^\infty(G, \mathcal{E}) : \forall X \in \mathcal{U}(\mathfrak{g}), \forall j \in \mathbb{N}, \exists C > 0 : \|X f\|_j \leq C \mu_j \right\},$$

where $\mathcal{E}$ is a Fréchet space, $\{\mu_j\}_{j \in \mathbb{N}}$ is a family of specific positive functions on $G$, called \textit{weights} (see Definition \ref{DF11} affiliated to a countable set of semi-norms $\{\|\cdot\|_j\}_{j \in \mathbb{N}}$ defining the Fréchet topology on $\mathcal{E}$ and where $\bar{X}$ is the left-invariant differential operator on $G$ associated to an element $X \in \mathcal{U}(\mathfrak{g})$. For example, $B^1(G, \mathbb{C})$ consists in the smooth vectors of the right regular representation of $G$ on the space of bounded right-uniformly continuous functions on $G$ (the uniform structures on $G$ are generally not balanced in our non-Abelian situation). We then define a notion of oscillatory integrals on Lie groups $G$ that are endowed with a specific type of smooth function $S \in C^\infty(G, \mathbb{R})$ (see Definitions \ref{DF13} and \ref{DF11}). We call such a pair $(G, S)$ an \textit{admissible tempered pair}. The main result of this section is that associated to an admissible tempered pair $(G, S)$, and given a growth-controlled function $m$, the oscillatory integral

$$F \mapsto \int_G m e^{is} F,$$

canonically extends from the space of smooth compactly supported functions $C^\infty(G, \mathcal{E})$ to our symbol space $B^{(\mu_1)}(G, \mathcal{E})$. This construction is explained in Definition \ref{DF13} which turns out to apply in our situation as a direct consequence of Proposition \ref{PP22} the main technical result of this section.

In section 3, we consider an arbitrary \textit{normal $j$-group} $\mathbb{B}$ (i.e. a connected simply connected Lie group whose Lie algebra is a normal $j$-algebra—see Definition \ref{DEF3.1}). The main result of this section, Theorem \ref{TH3.33} shows that its square $\mathbb{B} \times \mathbb{B}$ canonically underlies an admissible tempered pair $(\mathbb{B} \times \mathbb{B}, S_{\mathbb{B}}^{\mathbb{S}})$. When elementary, every normal $j$-group has a canonical simply transitive action on a specific solvable symplectic symmetric space. The two-point function $S^{\mathbb{B}}_{\mathbb{S}}$ we consider here comes from an earlier work of one of us. It consists in the sum of the phases $S_{\mathbb{S}}^{\mathbb{B}}$, of the oscillatory kernels associated to invariant star-products on solvable symplectic symmetric space \cite{11,13}, in the Pyatetskii-Shapiro decomposition \cite{17} of a normal $j$-group $\mathbb{B}$ into a sequence of split extensions of elementary normal $j$-factors: $\mathbb{B} = (\ldots(\mathbb{S}_1 \ltimes \mathbb{S}_2) \ltimes \mathbb{S}_3) \ltimes \ldots...) \ltimes \mathbb{S}_N$. The two-point phase function $S^{\mathbb{S}}_{\mathbb{B}}$ in that case, then consists in the symplectic area of the unique geodesic triangle in $\mathbb{S}$ (viewed as a solvable symplectic symmetric space), whose geodesics admit $e, x$ and $y$ as midpoints ($e$ denotes the unit element in $\mathbb{S}$):

$$S^{\mathbb{B}}_{\mathbb{S}}(x_1, x_2) := \text{Area} (\Phi_{\mathbb{S}}^{-1}(e, x_1, x_2)),$$

with

$$\Phi_{\mathbb{S}} : \mathbb{S}^3 \to \mathbb{S}^3, \quad (x_1, x_2, x_3) \mapsto (\text{mid}(x_1, x_2), \text{mid}(x_2, x_3), \text{mid}(x_3, x_1))$$

where $\text{mid}(x, y)$ denotes the geodesic midpoint between $x$ and $y$ (again uniquely defined in our situation).

In section 4, we consider an arbitrary normal $j$-group and define the above-mentioned oscillatory kernels $K_\theta$ simply by tensorizing oscillating kernels found in \cite{7} on elementary $j$-factors. The resulting kernel has the form

$$K_\theta = \theta^{-\dim \mathbb{B}} m^{\mathbb{B}}_{\mathbb{S}} \exp \left\{ \frac{i}{\theta} S^{\mathbb{B}}_{\mathbb{S}} \right\},$$

where $S^{\mathbb{B}}_{\mathbb{S}}$ is the two-point phase mentioned in the description of section 3 above, and $m^{\mathbb{B}}_{\mathbb{S}} = m^{\mathbb{S}}_{\mathbb{S}} \otimes \ldots \otimes m^{\mathbb{S}}_{\mathbb{S}}$, where $m^{\mathbb{S}}_{\mathbb{S}} = \text{Jac} \Phi_{\mathbb{S}}^{-1/2}$ denotes the square root of the Jacobian of the “double triangle” map $\Phi_{\mathbb{S}}^{-1}$.

In particular, it defines an oscillatory integral on every symbol space of the type $B^{(\mu_1)}(\mathbb{B} \times \mathbb{B}, B^{(\mu_1)}(\mathbb{B}, \mathcal{E}))$ (where the $N$ indexes a family of semi-norms on $B^{(\mu_1)}(\mathbb{B}, \mathcal{E})$). When valued in a Fréchet \textit{algebra} $\mathcal{A}$, this yields a non-perturbative and associative star-product $\star_\theta$ on the union of all symbol spaces $B^{(\mu_1)}(\mathbb{B}, \mathcal{A})$.

In section 5, we consider any \textit{tempered} action of a normal $j$-group $\mathbb{B}$ on a Fréchet algebra $\mathcal{A}$. By tempered action we mean a strongly continuous action $\alpha$ of $\mathbb{B}$ by automorphisms on $\mathcal{A}$, such that for every semi-norm
with associated “quantization rule”: 
\[\mu_j^\alpha \text{ such that } \|\alpha_g(a)\|_j \leq \mu_j^\alpha(g) \|a\|_j \]
for all \(a \in A\) and \(g \in B\). In that case, the space of smooth vectors \(A^\infty\) of \(\alpha\) naturally identifies with a subspace of \(B^{(\mu_j)}(B, A^\infty)\), where the \(\mu_j\)'s are affiliated to the \(\mu_j^\alpha\)'s: 
\[\alpha : A^\infty \to B^{(\mu_j)}(B, A^\infty) : a \mapsto [g \mapsto \alpha_g(a)] .\]

We stress that even in the case of an isometric action, and contrarily to the Abelian situation, the map \(\alpha\) always takes values in a symbol space \(B^{(\mu_j)}\), with non-trivial \(\mu_j\)'s, which explains why the non-Abelian framework forces implementing such symbol spaces. Applying the results of section 4 to this situation, we get a new associative product on \(A^\infty\) defined by the formula 
\[a \star^\alpha b := (\alpha(a) \star^\alpha \alpha(b))(e) .\]

Then main result of this section, stated as Theorem 5.8, is the following fact:

**Universal Deformation Formula for Actions of Kählerian Lie Groups on Fréchet Algebras:**

Let \((A, \alpha, B)\) be a Fréchet algebra endowed with a tempered action of a normal \(j\)-group. Then, \((A^\infty, \star^\alpha)\) is an associative Fréchet algebra with jointly continuous product.

The rest of the paper is devoted to defining an operator calculus that represents the above algebras \((A^\infty, \star^\alpha)\) in the case \(A\) is a \(C^*\)-algebra.

In section 6, we define a special class of symplectic symmetric spaces which naturally give rise to explicit WKB-quantizations (i.e. invariant star-products representable through oscillatory kernels) that underlie a unitary operator calculus. Roughly speaking, an **elementary symplectic symmetric space** is a symplectic symmetric space that consists of the total space of a fibration in flat fibers over a Lie group \(Q\) of exponential type. In that case, a variant of Kirillov’s orbit method yields a unitary and self-adjoint representation on an Hilbert-space \(\mathcal{H}\), of the symmetric space \(M\):
\[\Omega : M \to \mathcal{U}_m(\mathcal{H}) ,\]
with associated “quantization rule”:
\[\Omega : L^1(M) \to \mathcal{B}(\mathcal{H}), \quad F \mapsto \Omega(F) := \int_M F(x)\Omega(x) \, dx .\]

Weighting the above mapping by the multiplication by a (growth controlled) function \(m\) defined on the base \(Q\) yields a pair of adjoint maps:
\[\Omega_m : L^2(M) \to L^2(\mathcal{H}) \quad \text{and} \quad \sigma_m : L^2(\mathcal{H}) \to L^2(M) ,\]
where \(L^2(\mathcal{H})\) denotes the algebra of Hilbert-Schmidt operators on \(\mathcal{H}\). Both of the above maps are equivariant under the whole automorphism group of \(M\). Note that this last feature very much contrasts with the usual notion of coherent-state quantization for groups (as opposed to symmetric spaces). The corresponding “Berezin transform” \(B_m := \sigma_m \circ \Omega_m\) is explicitly controlled. In particular, when invertible, the associated star-product \(F_1 \star F_2 := B_m^{-1} \sigma_m(\Omega_m(F_1)\Omega_m(F_2))\) is of oscillatory (WKB) type and its associated kernel is explicitly determined. Note that, because entirely explicit, this section yields a proof of Weinstein’s conjectural form for star-product WKB-kernels on symmetric spaces [27] in the situation considered here. The section ends with considerations on extending the construction to semi-direct products.

Section 7 is entirely devoted to applying the construction of section 6 to the particular case of Kählerian Lie groups with negative curvature. Such a Lie group is always a normal \(j\)-group is the sense of Pyatetskii-Shapiro. Every of its elementary factors admits the structure of an elementary symplectic symmetric space. Accordingly to [3], the obtained non-formal star products coincide with the one described in sections 3-5.

Section 8 deals with the deformation theory for \(C^*\)-algebras. We essentially prove the following statement:

\[^2\text{From the space of } L^2\text{-functions to space of Hilbert-Schmidt operators.}\]
Universal Deformation Formula for Actions of Kählerian Lie Groups on $C^\ast$-Algebras:

Let $(A, \alpha, \beta)$ be a $C^\ast$-algebra endowed with a strongly continuous and isometric action of a normal $\mathfrak{g}$-group. Then, there exists a $C^\ast$-norm on the involute Fréchet algebra $(A^\infty, \ast_0, \ast_0)$.

The above statement follows from a non-Abelian generalization (see Theorem 8.19) of the Calderon-Vainlancourt Theorem in the context of the usual Weyl pseudo-differential calculus on $\mathbb{R}^{2n}$. Our result asserts that the element $\Omega_m(F)$ associated to a function $F$ in $\mathcal{B}^1(\mathbb{R}, A)$ naturally consists of elements of the spacial tensor product of $A$ by an external $C^\ast$-algebra. Its proof relies of a combination of a resolution of the identity obtained from wavelet analysis considerations (see subsection 8.1) and further properties of our oscillatory integral defined in section 2 We also prove that $K$-theory is an invariant of the deformation.

Notations and conventions

Given a Lie group $G$, with Lie algebra $\mathfrak{g}$, we denote by $d_G(g)$ a left invariant Haar measure. In the non-unimodular case, we consider the modular function $\Delta_G$:

$$d_G(g) \Delta_G(g) := d_G(g^{-1})$$

Otherwise specified, $L^p(G), \ p \in [1, \infty)$, will always denote the Lebesgue $p$-space associated with the choice of a left-invariant Haar measure made above. We also denote by $\mathcal{D}(G)$ the space of smooth compactly supported functions on $G$ and by $\mathcal{D}'(G)$ the dual space of distributions.

We use the notations $L^r$ and $R^r$, for the left and right regular actions:

$$L^r_g f(g') := f(g^{-1}g'), \quad R^r_g f(g') := f(g'g).$$

By $\tilde{X}$ and $X$, we mean the left-invariant and right-invariant vector fields on $G$ associated to the elements $X$ and $-X \in \mathfrak{g}$:

$$\tilde{X} := \left. \frac{d}{dt} \right|_{t=0} R^r_{e^{tx}}, \quad X := \left. \frac{d}{dt} \right|_{t=0} L^r_{e^{tx}}.$$

Given an element $X$ of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$, we adopt the same notations $\tilde{X}$ and $X$ for the associated left- and right-invariant differential operator on $G$. Let $\Delta_G$ be the ordinary co-product of $\mathcal{U}(\mathfrak{g})$. We make use of the Sweedler’s notation:

$$\Delta_G(X) = \sum_{(X)} X(1) \otimes X(2) \in \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}), \quad X \in \mathcal{U}(\mathfrak{g}),$$

and accordingly, for $f_1, f_2 \in C^\infty(G)$ and $X \in \mathcal{U}(\mathfrak{g})$, we write

$$\tilde{X}(f_1 f_2) = \sum_{(X)} (\tilde{X}(1)) f_1 (\tilde{X}(2)) f_2, \quad X(f_1 f_2) = \sum_{(X)} (X(1)) f_1 (X(2)) f_2. \quad (2)$$

To a fixed ordered basis $\{X_1, \ldots, X_m\}$ of the Lie algebra $\mathfrak{g}$, we associate a PBW basis of $\mathcal{U}(\mathfrak{g})$:

$$\{X^\beta, \beta \in \mathbb{N}^m\}, \quad X^\beta := X^\beta_1 X^\beta_2 \ldots X^\beta_m. \quad (3)$$

This induces a filtration

$$\mathcal{U}(\mathfrak{g}) = \bigcup_{k \in \mathbb{N}} \mathcal{U}_k(\mathfrak{g}), \quad \mathcal{U}_k(\mathfrak{g}) \subset \mathcal{U}_l(\mathfrak{g}), \quad k \leq l,$$

in terms of the subsets

$$\mathcal{U}_k(\mathfrak{g}) := \left\{ \sum_{|\beta| \leq k} C_\beta X^\beta, \ C_\beta \in \mathbb{R} \right\}, \quad k \in \mathbb{N}, \quad (4)$$

where $|\beta| := \beta_1 + \cdots + \beta_m$. For $\beta, \beta_1, \beta_2 \in \mathbb{N}^m$, we define the ‘structure constants’ $\omega^\beta_1, \beta_2 \in \mathbb{R}$ of $\mathcal{U}(\mathfrak{g})$, by

$$X^\beta_1 X^\beta_2 = \sum_{|\beta| \leq |\beta_1| + |\beta_2|} \omega^\beta_1, \beta_2 X^\beta \in \mathcal{U}_{|\beta_1| + |\beta_2|}(\mathfrak{g}). \quad (5)$$
We endow the finite dimensional vector space \( U_k(\mathfrak{g}) \), with the \( \ell^1 \)-norm \( |.|_k \) within the basis \{ \( X^\beta, |\beta| \leq k \) \:

\[
|X|_k := \sum_{|\beta| \leq k} |C_\beta| \quad \text{if} \quad X = \sum_{|\beta| \leq k} C_\beta X^\beta \in U_k(\mathfrak{g}).
\]

We observe that the family of norms \{ \( ., |.|_k \}_{k \in \mathbb{N}} \) is compatible with the filtered structure of \( U(\mathfrak{g}) \), in the sense that if \( X \in U_k(\mathfrak{g}) \), then \( |X|_k = |X|_l \) whenever \( l \geq k \). Considering a subspace \( V \subset \mathfrak{g} \), we also denote by \( U(V) \) the unital sub-algebra of \( U(\mathfrak{g}) \) generated by \( V \):

\[
U(V) = \text{span}\{ X_1X_2\ldots X_n : X_j \in V, n \in \mathbb{N} \},
\]

that we may filtrate using the induced filtration of \( U(\mathfrak{g}) \). We also observe that the co-product preserves the latter sub-algebras, in the sense that \( \Delta_U(U(V)) \subset U(V) \otimes U(V) \).

Regarding the uniform structures on a locally compact group \( G \), we say that a function \( f : G \to \mathbb{C} \) is right (respectively left) uniformly continuous if for all \( \varepsilon > 0 \), there exists \( U \), an open neighborhood of the neutral element \( e \), such that for all \( (g, h) \in G \times G \) we have

\[
|f(g) - f(h)| \leq \varepsilon, \quad \text{whenever} \quad g^{-1}h \in U \quad \text{(respectively} \quad hg^{-1} \in U).\]

Last, we call a Lie group \( G \) (with Lie algebra \( \mathfrak{g} \)) exponential, if the exponential map \( \exp : \mathfrak{g} \to G \) is a global diffeomorphism.

### 2 Oscillatory integrals

#### 2.1 Symbol spaces

In this preliminary subsection, we consider a non-Abelian, weighted and Fréchet-valued version of the Laurent Schwartz space \( \mathcal{B} \) of smooth functions that, together with all of their derivatives, are bounded. For reasons that will become clear latter, we refer to such function spaces as symbol spaces. They are constructed out of a family of specific functions on a Lie group \( G \), that we call weights. The prototype of a weight for a non-Abelian Lie group is constructed in Example 2.3. The key properties of these symbol spaces are established in Lemmas 2.6 and 2.8. In Lemma 2.10, we show on an example, how such spaces naturally appear in the context of non-Abelian Lie group actions.

**Definition 2.1** Consider a connected real Lie group \( G \) with Lie algebra \( \mathfrak{g} \). An element \( \mu \in C^\infty(G, \mathbb{R}^*_+) \) is called a weight if it satisfies the following properties:

(i) For every element \( X \in U(\mathfrak{g}) \), there exist \( C_L, C_R > 0 \) such that

\[
|X.\mu| \leq C_L \mu \quad \text{and} \quad |\overline{X}.\mu| \leq C_R \mu.
\]

(ii) There exist positive integers \( L, R \in \mathbb{N} \) and a constant \( C > 0 \) such that for all \( g, h \in G \):

\[
\mu(gh) \leq C \mu(g)^L \mu(h)^R.
\]

A pair \( (L, R) \in \mathbb{N}^2 \) as in item (ii) is called a sub-multiplicative degree of the weight \( \mu \). A weight with sub-multiplicative degree \((1, 1)\) is called a sub-multiplicative weight.

**Remark 2.2** For \( \mu \in C^\infty(G) \), we set \( \mu^\vee(g) := \mu(g^{-1}) \). Then, from the relation \( \overline{X}.\mu^\vee = (X.\mu)^\vee \) for all \( X \in U(\mathfrak{g}) \), we see that \( \mu \) is a weight of sub-multiplicative degree \((L, R)\) if and only if \( \mu^\vee \) is a weight of sub-multiplicative degree \((R, L)\). Moreover, a product of two weights is a weight and a (positive) power of a weight is a weight.

In the following, we construct a canonical and non-trivial weight for non-Abelian Lie groups. This specific weight is an important object as it will naturally and repeatedly appear in all our analysis.
Example 2.3 Choosing a Euclidean structure $|.|$ on $\mathfrak{g}$, for $x \in G$, we let $|\text{ad}_x|$ be the operator norm of the adjoint action of $G$ on $\mathfrak{g}$. The function
$$\delta : G \to \mathbb{R}_+^*, \ x \mapsto \sqrt{1 + |\text{ad}_x|^2 + |\text{ad}_{x^{-1}}|^2},$$
is a sub-multiplicative weight on $G$. Indeed, from the relations for $X \in \mathfrak{g}$ and $x \in G$,

$$\begin{align*}
\bar{X} \cdot |\text{ad}_x|^2 &= 2 \sup_{Y \in \mathfrak{g}, |Y|=1} \langle \text{ad}_x \circ \text{ad}_X(Y), \text{ad}_x(Y) \rangle, \\

\bar{X} \cdot |\text{ad}_{x^{-1}}|^2 &= -2 \sup_{Y \in \mathfrak{g}, |Y|=1} \langle \text{ad}_{x^{-1}} \circ \text{ad}_x(Y), \text{ad}_{x^{-1}}(Y) \rangle, \\

X \cdot |\text{ad}_x|^2 &= -2 \sup_{Y \in \mathfrak{g}, |Y|=1} \langle \text{ad}_{x^{-1}} \circ \text{ad}_X(Y), \text{ad}_{x^{-1}}(Y) \rangle, \\

X \cdot |\text{ad}_{x^{-1}}|^2 &= 2 \sup_{Y \in \mathfrak{g}, |Y|=1} \langle \text{ad}_x \circ \text{ad}_X(Y), \text{ad}_x(Y) \rangle,
\end{align*}$$

we get by induction and for every $X \in \mathcal{U}(\mathfrak{g})$ of strictly positive homogeneous degree:

$$|\bar{X} \circ \delta(x)|, |X \circ \delta(x)| \leq |\text{ad}_x| \frac{|\text{ad}_x|^2 + |\text{ad}_{x^{-1}}|^2}{\sqrt{1 + |\text{ad}_x|^2 + |\text{ad}_{x^{-1}}|^2}} \leq |\text{ad}_x| \delta(x),$$

where, for $X \in \mathcal{U}(\mathfrak{g})$, we denote by $|\text{ad}_x|$ the operator norm of the adjoint action of $\mathcal{U}(\mathfrak{g})$ on $\mathfrak{g}$. The sub-multiplicativity follows from a direct check. The element $\delta$ is called the modular weight of $G$.

Also, the modular function $\Delta_G$ is a sub-multiplicative weight. Indeed the multiplicativity property implies that for every $X \in \mathcal{U}(\mathfrak{g})$ and $x \in G$:

$$\bar{X} \cdot \Delta_G(x) = (\bar{X} \cdot \Delta_G)(e) \cdot \Delta_G(x), \quad (X \cdot \Delta_G)(x) = (X \cdot \Delta_G)(e) \cdot \Delta_G(x).$$

The next notion will play a key role to establish density results for our symbol spaces. We assume from now on the Lie group $G$ to be non-compact.

Definition 2.4 Given two weights $\mu$ and $\mu'$, we say that $\mu$ dominates $\mu'$, which we denote by $\mu \succ \mu'$, if

$$\lim_{g \to \infty} \frac{\mu'(g)}{\mu(g)} = 0.$$ 

Remark 2.5 We stress that it is the modular weight $\delta$, and not the modular function $\Delta_G$, which plays a fundamental role. Not only because it appears everywhere in our estimates, but also because for negatively curved Kählerian Lie groups it has the crucial property to dominate the constant weight $1$.

We now let $\mathcal{E}$ be a complex Fréchet space with topology underlying a countable family of semi-norms $\{|.|_j\}_{j \in \mathbb{N}}$. A given weight $\mu$, we first consider the following space of $\mathcal{E}$-valued functions on $G$:

$$\mathcal{B}^\mu(G, \mathcal{E}) := \left\{ F \in C^\infty(G, \mathcal{E}) : \forall X \in \mathcal{U}(\mathfrak{g}), \forall j \in \mathbb{N}, \exists C > 0 : \|\bar{X} F\|_j \leq C \mu \right\}.$$ 

When $\mathcal{E} = \mathbb{C}$ (respectively when $\mu = 1$, respectively when $\mathcal{E} = \mathbb{C}$ and $\mu = 1$), we denote $\mathcal{B}^\mu(G, \mathcal{E})$ by $B^\mu(G)$ (respectively by $B(G, \mathcal{E})$, respectively by $B(G)$). We endow the space $\mathcal{B}^\mu(G, \mathcal{E})$ with the natural topology associated to the following semi-norms:

$$\|F\|_{j,k,\mu,\infty} := \sup_{X \in \mathcal{U}_k(\mathfrak{g})} \sup_{g \in G} \left\{ \|\bar{X} F(g)\|_j / \mu(g) |X|_k \right\}, \quad j, k \in \mathbb{N}, \quad (8)$$

where $\mathcal{U}(\mathfrak{g}) = \bigcup_{k \in \mathbb{N}} \mathcal{U}_k(\mathfrak{g})$ is the filtration described in (1) and $|.|_k$ is the norm on $\mathcal{U}_k(\mathfrak{g})$ defined in (2). Note that for $X = \sum_{|\beta| \leq k} C_\beta X^\beta \in \mathcal{U}_k(\mathfrak{g})$, we have

$$\frac{\|\bar{X} F(g)\|_j}{|X|_k} \leq \sum_{|\beta| \leq k} |C_\beta| \|\bar{X}^\beta F(g)\|_j \leq \max_{|\beta| \leq k} |\bar{X}^\beta F(g)\|_j,$$

and hence

$$\|F\|_{j,k,\mu,\infty} \leq \max_{|\beta| \leq k} \sup_{g \in G} \|\bar{X}^\beta F(g)\|_j / \mu(g) = \max_{|\beta| \leq k} \|\bar{X}^\beta F\|_{j,0,\mu,\infty}, \quad (9)$$

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which shows in particular that the semi-norms [9] are well defined on \( \mathcal{B}^\mu(G, \mathcal{E}) \). When \( \mathcal{E} = \mathbb{C} \) (respectively when \( \mu = 1 \), respectively when \( \mathcal{E} = \mathbb{C} \) and \( \mu = 1 \)), we denote the semi-norms [9] by \( \| \cdot \|_{k,\mu,\infty} \), \( k \in \mathbb{N} \). Let \( \mathcal{C}_b(G, \mathcal{E}) \) be the Fréchet space of \( \mathcal{E} \)-valued continuous and bounded functions on \( G \). The topology we consider on the latter is the one associated to the semi-norms \( \| F \|_{j,\infty} := \sup_{g \in G} \| F(g) \|_j, j \in \mathbb{N} \). This space carries an action of \( G \) by right-translations. This action is of course isometric but not necessarily strongly continuous. Consider therefore its closed subspace \( \mathcal{C}_{ru}(G, \mathcal{E}) \) constituted by the right-uniformly continuous functions. The results we establish in the next lemma are essentially standard.

**Lemma 2.6** Let \((G, \mathcal{E})\) as above and let \( \mu \) and \( \mu' \) be two weights on \( G \).

(i) The right regular action \( R^* \) of \( G \) on \( \mathcal{C}_{ru}(G, \mathcal{E}) \) is isometric and strongly continuous.

(ii) Let \( \mathcal{C}_{ru}(G, \mathcal{E})^\infty \) be the subspace of \( \mathcal{C}_{ru}(G, \mathcal{E}) \) of smooth vectors for the right regular action. Then \( \mathcal{C}_{ru}(G, \mathcal{E})^\infty \) identifies with \( \mathcal{B}(G, \mathcal{E}) \) as topological vector spaces. In particular, \( \mathcal{B}(G, \mathcal{E}) \) is Fréchet.

(iii) The left regular action \( L^* \) of \( G \) on \( \mathcal{B}(G, \mathcal{E}) \) is isometric.

(iv) The map

\[
\mathcal{B}^\mu(G, \mathcal{E}) \to \mathcal{B}(G, \mathcal{E}), \quad F \mapsto \mu^{-1} F,
\]

is an homeomorphism. In particular, the space \( \mathcal{B}^\mu(G, \mathcal{E}) \) is Fréchet as well.

(v) The bilinear map:

\[
\mathcal{B}^\mu(G) \times \mathcal{B}^{\mu'}(G, \mathcal{E}) \to \mathcal{B}^{\mu+\mu'}(G, \mathcal{E}), \quad (u, F) \mapsto \{ g \in G \mapsto u(g) F(g) \in \mathcal{E} \},
\]

is jointly continuous.

(vi) For every \( X \in \mathcal{U}(g) \), the associated left invariant differential operator \( \tilde{X} \) acts continuously on \( \mathcal{B}^\mu(G, \mathcal{E}) \).

(vii) If there exists \( C > 0 \) such that \( \mu' \leq C \mu \), then \( \mathcal{B}^\mu'(G, \mathcal{E}) \subset \mathcal{B}^\mu(G, \mathcal{E}) \), continuously.

(viii) Assume that \( \mu \succ \mu' \). Then the closure of \( \mathcal{D}(G, \mathcal{E}) \) in \( \mathcal{B}^\mu(G, \mathcal{E}) \) contains \( \mathcal{B}^\mu'(G, \mathcal{E}) \). In particular, the space \( \mathcal{D}(G, \mathcal{E}) \) is a dense sub-set of \( \mathcal{B}^\mu'(G, \mathcal{E}) \) for the induced topology of \( \mathcal{B}^\mu(G, \mathcal{E}) \).

**Proof.** (i) Recall that \( G \) being locally compact and countable at infinity, the space \( \mathcal{C}_b(G, \mathcal{E}) \) is Fréchet (by the same argument as in the proof of [23, Proposition 44.1 and Corollary 1]). The subspace \( \mathcal{C}_{ru}(G, \mathcal{E}) \) is then closed as a uniform limit of right-uniformly continuous functions is right-uniformly continuous. Thus \( \mathcal{C}_{ru}(G, \mathcal{E}) \) endowed with the induced topology is a Fréchet space as well.

Being isometric on \( \mathcal{C}_b(G, \mathcal{E}) \), the right action is consequently isometric on \( \mathcal{C}_{ru}(G, \mathcal{E}) \) too. Moreover, for any converging sequence \( \{ g_n \} \subset G \), with limit \( g \in G \), and any \( F \in \mathcal{C}_{ru}(G, \mathcal{E}) \), we have \( \| (R^*_g - R^*_g) F \|_{j,\infty} = \sup_{g \in G} \| F(g_0 g_n) - F(g_0 g) \|_j \), which tends to zero due to the right-uniform continuity of \( F \). Hence the right regular action \( R^* \) is strongly continuous on \( \mathcal{C}_{ru}(G, \mathcal{E}) \).

(ii) Note that an element \( F \in \mathcal{C}_{ru}(G, \mathcal{E})^\infty \) is such that the function \( g \mapsto R^*_g F \) is smooth as a \( \mathcal{C}_{ru}(G, \mathcal{E}) \)-valued function on \( G \). In particular, for every \( X \in \mathcal{U}(g) \), \( \tilde{X} F \) is bounded and smooth. This clearly gives the inclusion \( \mathcal{C}_{ru}(G, \mathcal{E})^\infty \subset \mathcal{B}(G, \mathcal{E}) \).

Reciprocally, \( G \) acts on \( \mathcal{B}(G, \mathcal{E}) \) via the right regular representation. Indeed, for all \( g \in G \) and \( X \in \mathcal{U}(g) \), we have \( \tilde{X} R^*_g = R^*_g (\text{Ad}_{g^{-1}} X) \) and hence for \( j, k \in \mathbb{N} \) and \( F \in \mathcal{B}(G, \mathcal{E}) \), we deduce

\[
\| R^*_g F \|_{j, k, \infty} = \sup_{X \in \mathcal{U}(g)} \sup_{g' \in G} \left\{ \frac{\| (\text{Ad}_{g^{-1}} X)^* F(g') \|_j}{|X|_k} \right\}
\]

\[
= \sup_{X \in \mathcal{U}(g)} \sup_{g' \in G} \left\{ \frac{\| (\text{Ad}_{g^{-1}} X)^* F(g') \|_j}{|X|_k} \right\} \leq |\text{Ad}_{g^{-1}}|_k \| F \|_{j, k, \infty},
\]
where \(|\text{Ad}_g|_k\) denotes the operator norm of the adjoint action of \(G\) on the (finite dimensional) Banach space \((\mathcal{U}_k(\mathfrak{g}), |.|_k)\). Now we have the inclusion \(\mathcal{B}(G, \mathcal{E}) \subset C_{ru}(G, \mathcal{E})\). Indeed, for \(F \in \mathcal{B}(G)\), \(g \in G\) and for fixed \(X \in \mathfrak{g}\), one observes that

\[
|F(g \exp(tX)) - F(g)| = \left| \int_0^t \frac{d}{d\tau}(F(g \exp(\tau X))) \, d\tau \right| = \left| \int_0^t \tilde{X} F(g \exp(\tau X)) \, d\tau \right| \leq |X|_1 \|F\|_{1,\infty} |t| ,
\]

hence the right-uniform continuity of \(F\). To show that \(F \in \mathcal{B}(G)\) is a differentiable vector for the right-action, we observe that

\[
\left| \frac{1}{t}(F(g \exp(tX)) - F(g)) - (\tilde{X} F)(g) \right| \leq \int_0^1 \left| (\tilde{X} F)(g \exp(t\tau X)) - (\tilde{X} F)(g) \right| \, d\tau \\
\leq \int_0^1 \int_0^t \left| (\tilde{X}^2 F)(g \exp(\tau' X)) \right| \, d\tau' \, d\tau \\
\leq |t| \sup_{g \in G} \left\{ |\tilde{X}^2 F|(g) \right\} \leq |X|^2 \|F\|_{2,\infty} |t| ,
\]

which tends to zero together with \(t\). This yields differentiability at the unit element. One gets it everywhere else by observing that

\[
\tilde{X}_g(R_g^*F) = R_g^*(\tilde{X} F) , \quad \forall X \in \mathcal{U}(\mathfrak{g}) , \quad \forall g \in G , \quad \forall F \in \mathcal{B}(G) .
\]

An induction on the order of derivation implies \(\mathcal{B}(G) \subset C_{ru}(G)^\infty\). The \(\mathcal{E}\)-valued case is entirely similar. The assertion concerning the topology follows from the definition of the topology on smooth vectors \([26]\).

(iii) The fact that \(G\) acts isometrically on \((\mathcal{B}(G, \mathcal{E})\) via the left regular representation, follows from

\[
\|L_g^*F\|_{j,k,\infty} = \sup_{X \in \mathcal{U}_k(\mathfrak{g})} \|\tilde{X} L_g^*F\|_j = \sup_{X \in \mathcal{U}_k(\mathfrak{g})} \|L_g\tilde{X} F\|_j = \sup_{X \in \mathcal{U}_k(\mathfrak{g})} \|L_g\tilde{X} F\|_j = \|F\|_{j,k,\infty} .
\]

(iv) Since \(\mu \in \mathcal{B}^\mu(G)\), we see that for every \(X \in \mathcal{U}(\mathfrak{g})\), there exists \(C > 0\) such that \(|\tilde{X}(\mu^{-1})| \leq C\mu^{-1}\). Thus, the Leibniz rule entails that the map \(F \mapsto \mu^{-1}F\) is continuous with continuous inverse, from \(\mathcal{B}^\mu(G, \mathcal{E})\) to \(\mathcal{B}(G, \mathcal{E})\).

(v) Let \(u \in \mathcal{B}^\mu(G)\) and \(F \in \mathcal{B}^\mu(G, \mathcal{E})\). Using Sweedler’s notation \([2]\), we have for \(j, k \in \mathbb{N}\):

\[
\|uF\|_{j,k,\mu,\infty} = \sup_{X \in \mathcal{U}_k(\mathfrak{g})} \sup_{\mu'(g) \in \mathcal{U}_k(\mathfrak{g})} \|\tilde{X} (uF)(g)\|_j \leq \sup_{X \in \mathcal{U}_k(\mathfrak{g})} \sum_{(\gamma)} \|\tilde{X} (1)u\| (\tilde{X} (2)F)(g) \|_{j} \|X\|_k \\
\leq \left( \sup_{X \in \mathcal{U}_k(\mathfrak{g})} \sum_{(\gamma)} |X(1)k| X(2)k |X|_k \right) \|u\|_{k,\mu,\infty} \|F\|_{j,k,\mu,\infty} .
\]

Now, for \(X = \sum_{|\beta| \leq k} C_{\beta} X^\beta \in \mathcal{U}_k(\mathfrak{g})\), expanded in the PBW basis \([3]\), we have

\[
\Delta_{\mathcal{U}}(X) = \sum_{|\beta| \leq k} C_{\beta} \sum_{\gamma \leq \beta} \frac{\beta}{\gamma} ) X^\gamma \otimes X^{\beta - \gamma} ,
\]

which, with \(m\) the dimension of \(\mathfrak{g}\), implies that

\[
\sum_{(\gamma)} |X(1)k| X(2)k k \leq \sum_{|\beta| \leq k} |C_\beta| \sum_{\gamma \leq \beta} \frac{\beta}{\gamma} \leq 2^{mk} \sum_{|\beta| \leq k} |C_\beta| = 2^{mk} |X|_k .
\]

Hence we get

\[
\|uF\|_{j,k,\mu,\infty} \leq 2^{mk} \|u\|_{k,\mu,\infty} \|F\|_{j,k,\mu,\infty} ,
\]
proving separate continuity. Joint continuity follows then by a generic property of Fréchet spaces.

(vi) and (vii) are obvious.

(viii) Choose an increasing sequence \( \{C_n\}_{n \in \mathbb{N}} \) of relatively compact open sub-sets in \( G \), such that \( \lim_n C_n = G \). Pick \( 0 \leq \psi \in \mathcal{D}(G) \) of \( L^1(G, d_G) \)-norm one and define

\[
e_n := \int_G \psi(g) \, R_g^*(\chi_n) \, d_G(g),
\]

where \( \chi_n \) denotes the characteristic function of \( C_n \). It is clear that \( e_n \) is an increasing family of smooth compactly supported functions, which by Lebesgue dominated convergence, converges point-wise to the unit function. Moreover, for all \( F \in \mathcal{B}^{\mu'}(G, \mathcal{E}) \), we have

\[
\| (1 - e_n) F \|_{j, 0, \mu, \infty} = \sup_{g \in G} \left\{ \frac{1}{\mu(g)} (1 - e_n(g)) \| F(g) \|_j \right\} \leq \| F \|_{j, 0, \mu, \infty} \sup_{g \in G} \left\{ \frac{\mu'(g)}{\mu(g)} (1 - e_n(g)) \right\},
\]

which converges to zero when \( n \) goes to infinity, since \( \mu'/\mu \to 0 \) when \( g \to \infty \) and for fixed \( g \in G \), \( 1 - e_n(g) \) decreases to zero when \( n \to \infty \). We need to show that the same property holds true for all the semi-norms \( \| . \|_{j, k, \mu, \infty} \), \( k \geq 1 \). We use an induction. First note that if \( X \in \mathfrak{g} \), then we have

\[
\tilde{X} e_n = \frac{d}{dt} \bigg|_{t=0} R_{e^t X}^*(e_n) = \frac{d}{dt} \bigg|_{t=0} \int_G \psi(g) \, R_{e^t X}^*(R_g^*(\chi_n)) \, d_G(g) = \int_G \left( X \psi \right)(g) \, R_{e^t X}^*(\chi_n) \, d_G(g).
\]

A routine inductive argument then gives

\[
\tilde{X} e_n = \int_G (X \psi)(g) \, R_{e^t X}^*(\chi_n) \, d_G(g), \quad \forall X \in \mathcal{U}(\mathfrak{g}),
\]

which entails

\[
\| \tilde{X} e_n \|_\infty \leq \| X \psi \|_1 < \infty, \quad \forall X \in \mathcal{U}(\mathfrak{g}).
\]

This means that the sequence \( \{e_n\}_{n \in \mathbb{N}} \) belongs to \( \mathcal{B}(G) \), uniformly in \( n \).

Now, assume that \( \| (1 - e_n) F \|_{j, k, \mu, \infty} \to 0, \; n \to \infty \), for a given \( k \in \mathbb{N} \), for all \( F \in \mathcal{B}^{\mu'}(G, \mathcal{E}) \) and all \( j \in \mathbb{N} \). From the same reasoning as those leading to (11) and with \( X^\beta \) the element of the PBW basis of \( \mathcal{U}(\mathfrak{g}) \) defined in (3), we see that

\[
\| (1 - e_n) F \|_{j, k, \mu, \infty} \leq \| (1 - e_n) F \|_{j, k, \mu, \infty} + \max_{|\beta|=k+1} \| \tilde{X}^\beta (1 - e_n) F \|_{j, 0, \mu, \infty}.
\]

We only need to show that the second term in the inequality above goes to zero when \( n \to \infty \), as the first does by induction hypothesis. Writing \( X^\beta = X^\gamma X \), with \( |\gamma| = k \) and \( X \in \mathfrak{g} \), by virtue of the Liebniz rule, we get

\[
\tilde{X}^\gamma \tilde{X} (1 - e_n) F = -\tilde{X}^\gamma ((\tilde{X} e_n) F) + \tilde{X}^\gamma ((1 - e_n) \tilde{X} F) = -\tilde{X}^\gamma ((\tilde{X} e_n) F) + \tilde{X}^\gamma ((1 - e_n) \tilde{X} F)\).
\]

Note that

\[
\| \tilde{X}^\gamma (1 - e_n) \tilde{X} F \|_{j, 0, \mu, \infty} \leq \| (1 - e_n) \tilde{X} F \|_{j, k, \mu, \infty},
\]

which converges to zero when \( n \to \infty \) by induction hypothesis, since \( \tilde{X} F \in \mathcal{B}^{\mu'}(G, \mathcal{E}) \) and \( |\gamma| = k \). Regarding the first term, we have using Sweedler’s notations (2) and for a finite sum:

\[
\tilde{X}^\gamma ((\tilde{X} e_n) F) = \sum_{(\tilde{X}^\gamma)} (\tilde{X}^\gamma_{(1)} \tilde{X} e_n) (\tilde{X}^\gamma_{(2)} F).
\]

Note that \( \int P X \psi \, d_G = 0 \) for any \( P \in \mathcal{U}(\mathfrak{g}) \), \( X \in \mathfrak{g} \) any \( \psi \in \mathcal{D}(G) \). Indeed, this follows from an inductive argument starting with

\[
\int_G X \psi(g) \, d_G(g) = \frac{d}{dt} \bigg|_{t=0} \int_G L_{e^t X}^*(\psi)(g) \, d_G(g) = \frac{d}{dt} \bigg|_{t=0} \int_G \psi(g) \, d_G(g) = 0, \quad \forall X \in \mathfrak{g}.
\]
Using \[13\], we arrive at
\[
\tilde{X}^\gamma((\tilde{X}e_n)F) = \sum_{(X^\gamma)} \left( \int_G (X^\gamma_1(X^\psi)(g(R^*_g(e_n) - 1))d_G(g) \right) \tilde{X}^\gamma(2)F,
\]
which converges to zero in the norms \(\|\cdot\|_{j,0,\mu,\infty}\), \(j \in \mathbb{N}\), since it is a finite sum of terms of the form \((1 - e_n)F\) (with possibly re-defined \(F\)'s in \(B^{\mu'}(G, \mathcal{E})\), \(e_n\)'s and \(\psi\)'s in \(D(G)\)).

\[ \blacksquare \]

**Remark 2.7** On \(B(G, \mathcal{E})\), the left regular action is generally not strongly continuous and the right regular action is never isometric unless \(G\) is Abelian.

We now generalize the spaces \(B^\mu(G, \mathcal{E})\), by allowing a certain behavior at infinity of the \(\mathcal{E}\)-valued functions on \(G\), which is not necessarily uniform with respect to the semi-norm index. So, we still consider a Fréchet space \(\mathcal{E}\) associated to a family of semi-norms \(\{\|\cdot\|_j\}_{j \in \mathbb{N}}\), but we let now \(\{\mu_j\}_{j \in \mathbb{N}}\) be a countable family of weights on \(G\). We then define
\[
B^{(\mu_j)}(G, \mathcal{E}) := \left\{ F \in \mathcal{C}^\infty(G, \mathcal{E}) : \forall X \in U(\mathfrak{g}), \forall j \in \mathbb{N}, \exists C > 0 : \|\tilde{X}F\|_j \leq C \mu_j \right\}.
\]

We endow the latter space with the following set of the semi-norms:
\[
\|F\|_{j,k,\mu_j,\infty} := \sup_{x \in U_\Delta(\mathfrak{g})} \sup_{g \in G} \left\{ \frac{\|\tilde{X}F(g)\|_j}{\mu_j(g)} \right\}, \quad j, k \in \mathbb{N}, \tag{14}
\]
As expected, the space \(B^{(\mu_j)}(G, \mathcal{E})\) is Fréchet for the topology induced by the semi-norms \(14\) and most of the properties of Lemma 2.6 remain true.

**Lemma 2.8** Let \((G, \mathcal{E}, \{\|\cdot\|_j\}_{j \in \mathbb{N}})\) as above and let \(\{\mu_j\}_{j \in \mathbb{N}}\) and \(\{\mu'_j\}_{j \in \mathbb{N}}\) be two families of weights on \(G\). 

(i) The space \(B^{(\mu_j)}(G, \mathcal{E})\) is Fréchet.

(ii) Assume that \(\mu_j\) has sub-multiplicative degree \((L_j, R_j)\). Then, for every \(g \in G\) the left-translation \(L^*_g\) defines a continuous map from \(B^{(\mu_j)}(G, \mathcal{E})\) to \(B^{(\mu'_j)}(G, \mathcal{E})\).

(iii) The bilinear map:
\[
B^{(\mu_j)}(G) \times B^{(\mu'_j)}(G, \mathcal{E}) \to B^{(\mu_j,\mu'_j)}(G, \mathcal{E}), \quad (u, F) \mapsto [g \in G \mapsto u(g)F(g) \in \mathcal{E}],
\]
is jointly continuous.

(iv) For every \(X \in U(\mathfrak{g})\), the left invariant differential operator \(\tilde{X}\) acts continuously on \(B^{(\mu_j)}(G, \mathcal{E})\).

(v) If for every \(j \in \mathbb{N}\), there exists \(C_j > 0\) such that \(\mu'_j \leq C_j \mu_j\), then \(B^{(\mu'_j)}(G, \mathcal{E}) \subset B^{(\mu_j)}(G, \mathcal{E})\).

(vi) Assume that \(\mu_j \sim \mu'_j\) for every \(j \in \mathbb{N}\). Then, the closure of \(D(G, \mathcal{E})\) in \(B^{(\mu_j)}(G, \mathcal{E})\) contains \(B^{(\mu_j)}(G, \mathcal{E})\).

In particular, \(D(G, \mathcal{E})\) is a dense sub-set of \(B^{(\mu_j)}(G, \mathcal{E})\) for the induced topology of \(B^{(\mu_j)}(G, \mathcal{E})\).

**Proof.** (i) For each \(j \in \mathbb{N}\), define \(\|\cdot\|_j^\gamma := \sum_{k=0} \|\cdot\|_k\). Clearly, the topologies on \(\mathcal{E}\) associated with the families of semi-norms \(\{\|\cdot\|_j\}_{j \in \mathbb{N}}\) and \(\{\|\cdot\|_j^\gamma\}_{j \in \mathbb{N}}\) are equivalent. Thus, we may assume without loss of generality that the family of semi-norms \(\{\|\cdot\|_j\}_{j \in \mathbb{N}}\) is increasing. We start by recalling the standard realization of the Fréchet space \((\mathcal{E}, \{\|\cdot\|_j\}_{j \in \mathbb{N}})\) as a projective limit. One considers the null spaces \(V_j := \{v \in \mathcal{E} \mid \|v\|_j = 0\}\) and form the normed quotient spaces \(\hat{\mathcal{E}}_j := \mathcal{E}/V_j\). Denoting by \(\tilde{E}_j\) the Banach completion of the latter, the family of semi-norms being increasing, one gets, for every pair of indices \(i \leq j\), a natural continuous linear mapping \(g_{ji} : \tilde{E}_j \to \tilde{E}_i\). The Fréchet space \(\mathcal{E}\) is then isomorphic to the subspace \(\tilde{\mathcal{E}}\) of the product space \(\prod_j \tilde{E}_j\) constituted by the elements \((x) \in \prod_j \tilde{E}_j\) such that \(x_i = g_{ji}(x_j)\). Within this setting, the subspace \(\tilde{\mathcal{E}}\) is endowed with the projective topology associated with the family of maps \(\{f_j : \tilde{\mathcal{E}} \to \tilde{E}_j : (x) \mapsto x_j\}\) (i.e. the coarsest topology that renders continuous each of the \(f_j\)'s— see e.g. [22] pp. 50-52)).
Within this context, we then observe that the topology on \( B^{(\mu)}(G, \mathcal{E}) \) consists in the projective topology associated with the mappings \( \phi_j : B^{(\mu)}(G, \mathcal{E}) \to B^{\mu_j}(G, \mathcal{E}_j) : F \mapsto f_j \circ F \). Next we consider a Cauchy sequence \( \{F_n\}_{n \in \mathbb{N}} \) in \( B^{(\mu)}(G, \mathcal{E}) \). Since every space \( B^{\mu_j}(G, \mathcal{E}_j) \) is Fréchet, each sequence \( \{f_j \circ F_n\}_{n \in \mathbb{N}} \) converges in \( B^{\mu_j}(G, \mathcal{E}_j) \) to an element denoted by \( F^j \). Moreover, for every \( g \in G \), one has

\[
\|g_{ji}(F^j(g)) - F^i(g)\|_i = \|g_{ji}(F^j(g)) - f_i F_n(g) + f_i F_n(g) - F^i(g)\|_i \\
\leq \|g_{ji}(F^j(g) - f_j F_n(g))\|_i + \|f_i F_n(g) - F^i(g)\|_i ,
\]

which can be rendered as small as we want since every \( g_{ji} \) is continuous. Hence \( g_{ji}(F^j) = F^i \) which amounts to say that \( B^{(\mu)}(G, \mathcal{E}) \) is complete.

(ii) Let \( F \in B^{(\mu)}(G, \mathcal{E}) \) and \( g \in G \). We have for \( j, k \in \mathbb{N} \):

\[
\|L^*_g F\|_{j, k, \mu_j, R_j, \infty} = \sup_{X \in U_k(g)} \sup_{g' \in G} \frac{\|\tilde{X}(L^*_g F)(g')\|_j}{\mu_j(g')^{R_j} |X|_k} = \sup_{X \in U_k(g)} \sup_{g' \in G} \frac{\|L^*_g (\tilde{F})(g')\|_j}{\mu_j(g')^{R_j} |X|_k} \\
= \sup_{X \in U_k(g)} \sup_{g' \in G} \frac{\|\tilde{F}(g^{-1}g')\|_j}{\mu_j(g')^{R_j} |X|_k} \leq \mu_j(g^{-1}) L_j \|F\|_{j, k, \mu_j, \infty}.
\]

Items (iii), (iv), (v) and (vi) are proven in the same way as to their counterparts in Lemma 2.10.

In Lemma 2.10, we show how the notion of \( B \)-spaces for families of weights, naturally appears in the context of non-Abelian Lie group actions. We start by a preliminary result. We fix a Euclidean structure on \( g \), such that the basis \( \{X_1, \ldots, X_m\} \) (from which we have constructed the PBW basis \( [5] \) is orthonormal.

**Lemma 2.9** For \( g \in G \) and \( k \in \mathbb{N} \), denote by \( |\text{Ad}_g|_k \) the operator norm of the adjoint action \( \text{ad} \) of \( G \) on the Banach space \( (U_k(g), |.|_k) \). Then, for each \( k \in \mathbb{N} \), there exists a constant \( C_k > 0 \), such that

\[
|\text{Ad}_g|_k \leq C_k \delta(g)^k .
\]

where \( \delta \in C^\infty(G) \) is the modular weight (defined in Example 2.2).

**Proof.** Note first that for all \( k \in \mathbb{N} \), there exists a constant \( \omega_k > 0 \) such that for all \( X \in U_{k_1}(g) \) and \( Y \in U_{k_2}(g) \), we have

\[
|XY|_{k_1+k_2} \leq \omega_{k_1+k_2} |X|_{k_1} |Y|_{k_2} .
\] (15)

Indeed, observe that if

\[
X = \sum_{|\beta| \leq k_2} C^1_\beta X^\beta \in U_{k_1}(g) \quad \text{and} \quad Y = \sum_{|\beta| \leq k_2} C^2_\beta X^\beta \in U_{k_2}(g) ,
\]

we have

\[
XY = \sum_{|\beta_1| \leq k_1, |\beta_2| \leq k_2} C^1_{\beta_1} C^2_{\beta_2} \sum_{|\beta| \leq |\beta_1| + |\beta_2|} \omega_{\beta_1, \beta_2} X^\beta ,
\]

where the constants \( \omega_{\beta_1, \beta_2} \) are defined in [5]. The sub-additivity of the norm \( |.|_{k_1+k_2} \) then entails that

\[
|XY|_{k_1+k_2} \leq \sum_{|\beta| \leq |\beta_1| + |\beta_2|} |C^1_{\beta_1}| |C^2_{\beta_2}| \sum_{|\beta| \leq |\beta_1| + |\beta_2|} |\omega_{\beta_1, \beta_2}| .
\]

Thus, it leads to defining

\[
\omega_{k_1+k_2} := \sup_{|\beta_1|+|\beta_2| \leq k_1+k_2} \sum_{|\beta| \leq |\beta_1| + |\beta_2|} |\omega_{\beta_1, \beta_2}| ,
\]

and the inequality (15) is proven. Next, for

\[
X = \sum_{|\beta| \leq k} C_\beta X^\beta_1 \ldots X^\beta_m \in U_k(g) ,
\]
we have
\[ \text{Ad}_g(X) = \sum_{|\beta| \leq k} C_\beta \left( \text{Ad}_g(X_1) \right)^{\beta_1} \cdots \left( \text{Ad}_g(X_m) \right)^{\beta_m} \in \mathcal{U}_k(\mathfrak{g}), \]
and thus by the previous considerations, we deduce
\[ |\text{Ad}_g(X)|_k \leq \sum_{|\beta| \leq k} |C_\beta| \left( \prod_{j=2}^{|\beta|} \omega_j \right) |\text{Ad}_g(X_1)|_1^{\beta_1} |\text{Ad}_g(X_2)|_1^{\beta_2} \cdots |\text{Ad}_g(X_m)|_1^{\beta_m}. \]

As the restriction of the norm \(|.\)_1 from \(\mathcal{U}_1(\mathfrak{g})\) to \(\mathfrak{g}\) coincides with the \(\ell^1\)-norm of \(\mathfrak{g}\) within the basis \(\{X_1, \ldots, X_m\}\), we deduce for \(j = 1, \ldots, m\) and with \(|\text{Ad}_g|\) the operator norm of \(\text{Ad}_g\) with respect to the Euclidean structure of \(\mathfrak{g}\) chosen:
\[ |\text{Ad}_g(X_j)|_1 \leq \sqrt{m} |\text{Ad}_g(X_j)| \leq \sqrt{m} |\text{Ad}_g| |X_j|_\mathfrak{g} = \sqrt{m} |\text{Ad}_g|, \]
as \(X_j \in \mathfrak{g}\) belongs to the unit sphere of \(\mathfrak{g}\) for the Euclidean norm \(|.|_\mathfrak{g}\). This implies
\[ |\text{Ad}_g(X)|_k \leq m^{k/2} \left( \sup_{|\beta| \leq k} \prod_{j=2}^{|\beta|} \omega_j \right) |\text{Ad}_g|^k, \]
and the result follows from the definition of the modular weight \(\mathfrak{d}\) (see Example 2.3).

**Lemma 2.10** Let \(\{\mu_j\}\) be a family of weights on \(G\) with sub-multiplicativity degrees \(\{(L_j, R_j)\}\). Then the linear mapping
\[ \mathcal{R} := \left[ F \in C^\infty(G, \mathcal{E}) \mapsto [g \mapsto R^*_g F] \in C^\infty(G, C^\infty(G, \mathcal{E})) \right], \]
is continuous from \(\mathcal{B}^{(\mu_j)}_{j \in \mathbb{N}}(G, \mathcal{E})\) to \(\mathcal{B}^{(\mu_j L_j)}_{j \in \mathbb{N}}(G, \mathcal{B}^{(\mu_j)}_{L_{j \leq \infty}}(G, \mathcal{E}))\), where \(\mathfrak{d}\) denotes the modular weight. More precisely, labeling by \((j, k) \in \mathbb{N}^2\) the semi-norm \(\|\|_{j, k, \mu_j L_j, \infty}\) of \(\mathcal{B}^{(\mu_j L_j, \infty)}(G, \mathcal{E})\), for each \((j, k, k') \in \mathbb{N}^3\), there exists a constant \(C > 0\), such that for all \(F \in \mathcal{B}^{(\mu_j)}(G, \mathcal{E})\), we have
\[ \|\mathcal{R}(F)\|_{(j, k), k', \mu_j L_j, \infty} \leq C \|F\|_{j, k+k', \mu_j, \infty}. \]

**Proof.** Using the relation (10), we obtain for \(X \in \mathcal{U}_k(\mathfrak{g}), F \in \mathcal{B}^{(\mu_j)}_{j \in \mathbb{N}}(G, \mathcal{E})\) and \(g \in G:\)
\[ \|\tilde{X}_g R^*_g(F)\|_{j, k, \mu_j L_j, \infty} = \|R^*_g(\tilde{X}F)\|_{j, k, \mu_j L_j, \infty} = \sup_{Y \in \mathcal{U}_k(\mathfrak{g})} \sup_{x \in G} \|\tilde{Y}_g R^*_g(\tilde{X}F)(x)\|_j. \]
Moreover, since for any \(Y \in \mathcal{U}(\mathfrak{g})\) and \(g \in G\), we have \(R^*_g \tilde{Y}_g R^*_g = \text{Ad}_{g^{-1}} Y\) and since \(F \in \mathcal{B}^{(\mu_j, L_j)}(G, \mathcal{E})\) and \(\mu_j\) is sub-multiplicative with degree \((L_j, R_j)\), we get
\[ \|\tilde{X}_g R^*_g(F)\|_{j, k, \mu_j L_j, \infty} = \sup_{Y \in \mathcal{U}_k(\mathfrak{g})} \sup_{x \in G} \frac{\|\text{Ad}_{g^{-1}} Y \tilde{X} F(xg)\|_j}{\mu_j L_j(x) |Y|_k} \leq \|F\|_{j, k+k', \mu_j, \infty} |\text{Ad}_{g^{-1}} Y|_k \sup_{x \in G} \frac{\mu_j L_j(xg)}{\mu_j L_j(x)} \leq \|F\|_{j, k+k', \mu_j, \infty} |\text{Ad}_{g^{-1}} Y|_k \sup_{x \in G} \frac{\mu_j L_j(xg)}{\mu_j L_j(x)} |X|_k \mu_j R_j^*(g), \]
and one concludes using Lemma 2.3. \(\blacksquare\)
2.2 Tempered pairs

In this subsection, we establish the main technical result the first part of this article (Proposition 2.25), on which the construction of our oscillatory integral (and thus of our universal deformation formula for Fréchet algebras) essentially relies. To this aim, we start by introducing the class of tempered Lie groups (Definition 2.12) and the sub-class of tempered pairs (Definition 2.17). In Lemma 2.16, we give simple but important consequences for the modular weight, modular function and Haar measure, when the group is tempered. The rest of this subsection is devoted to the proof of Proposition 2.25.

The following result is extracted from [4], but for the sake of completeness, we reproduce the proof here.

**Lemma 2.11** Let \( G \) be a connected real Lie group and \( \psi : \mathbb{R}^m \to G \) be a global diffeomorphism. Then the multiplication and inverse operations seen through \( \psi \) are tempered functions\(^3\) (in the ordinary sense of \( \mathbb{R}^m \)) if and only if for every element \( A \in \mathcal{U}(\mathfrak{g}) \) their derivatives along \( \tilde{A} \) is bounded by a function which is polynomial within the chart \( \psi \).

**Proof.** Denote \( \mu(x,y) = \mu_x(y) = \psi^{-1}((\psi(x) \cdot \psi(y)) \) and \( \iota(x) = \psi^{-1}(\psi(x)^{-1}) \) the multiplication and inverse of \( G \) seen through \( \psi \in \text{Diff}(\mathbb{R}^m, G) \), and for \( X \in \mathbb{R}^m \) denote \( \tilde{X}_x^\psi = \mu_x^{-1}(X) = \psi^{-1} \cdot (\psi_0 \cdot X) \psi(x) \) the left invariant vector field corresponding to \( \psi_0 \cdot X \in \mathfrak{g} \).

Assume \( \mu \) and \( \iota \) are tempered in the usual sense. Then for \( X \in \mathbb{R}^m \), by definition

\[
\tilde{X}_x^\psi = \left. \frac{d}{dt} \mu(x,tX) \right|_{t=0},
\]

which is a linear combination of partial derivatives of \( \mu \) all of them being bounded by some polynomials in \( x \) since \( \mu \) is tempered. In the same way, the derivatives of left-invariant vector fields are linear combinations of higher partial derivatives of compositions of \( \mu \) with itself in the second variable, which are also bounded by some polynomials. Hence the left-invariant vector fields are tempered, and consequently so are the left-invariant derivatives of \( \mu \) and \( \iota \).

Conversely, assume \( \mu \) and \( \iota \) are tempered in the sense of left-invariant vector fields. We will see that the constant vector fields on \( \mathbb{R}^m \) are linear combinations of left-invariant vector fields, the coefficients being tempered functions. Indeed, we have \( X = (\mu_x^{-1})^{-1} \left( \tilde{X}_x^\psi \right) \) and the matrix elements of that inverse matrix are finite sums and products of the matrix elements of the original one, which are tempered, divided by its determinant. Thus all we have to check is that the inverse of the determinant is a tempered function. But \( \frac{1}{\det(\mu_x^{-1})} = \det \left( \mu_x(x) \right)^{-1} \) is tempered since \( \mu \) and \( \iota \) are. \( \square \)

The preceding observation yields us to introduce the following notion:

**Definition 2.12** A Lie group \( G \) is called tempered if there exists a global coordinate system \( \psi : \mathbb{R}^m \to G \) where the multiplication and inverse operations are tempered functions. A smooth function \( f \) on a tempered Lie group is called a tempered function if \( f \circ \psi \) is tempered.

**Remark 2.13** Every tempered Lie group, being diffeomorphic to a Euclidean space, is connected and simply connected. Moreover, by arguments similar to those of Lemma 2.11 a smooth function \( f \) on a tempered Lie group is tempered if and only if for any \( X \in \mathcal{U}(\mathfrak{g}) \), its derivative along \( X \) is bounded by a polynomial function within the global chart \( \mathbb{R}^m \to G \).

**Example 2.14** For any (simply connected) nilpotent Lie group, the exponential coordinates, \( \mathfrak{g} \to G : X \mapsto \exp(X) \), provides \( G \) with a structure of a tempered Lie group. Indeed, in the case of a nilpotent Lie group, the Baker-Campbell-Hausdorff series is finite.

**Remark 2.15** Observe also that we can replace in Lemma 2.11 left-invariant differential operators by right-invariant one. In fact, for a tempered group, left-invariant vector fields are linear combinations of right-invariant one with tempered coefficients and vice versa.

\(^3\)By tempered function, we mean a smooth function \( f \) whose every derivative \( D^\alpha f \) is bounded by a polynomial function \( P_\alpha \). These functions are sometimes called “slowly increasing”. 

Lemma 2.16 Let $G$ be a tempered Lie group. Then the modular weight $\mathfrak{d}$ (cf. Example 2.3) and the modular function $\Delta_G$ are tempered. Moreover, in the transported coordinates, every Haar measure on $G$ is a multiple of a Lebesgue measure on $\mathbb{R}^n$ by a tempered density.

Proof. The conjugate action $\mathbf{C} : G \times G \to G : (g, x) \mapsto gxg^{-1}$ is a tempered map when read in the global coordinate system. Therefore, the evaluation of the restriction of its tangent mapping to the first factor $G \times \{e\}$ on the constant section $0 \oplus X$, $X \in \mathfrak{g}$, of $T(G \times G)$ consists in a tempered mapping:

$$G \to \mathfrak{g} : g \mapsto \mathbf{C}_{\mathfrak{d}(g,e)}(0_g \oplus X).$$

The latter coincides with $g \mapsto \text{Ad}_g(X)$. Varying $X$ in $\mathfrak{g}$ yields the tempered map $\text{Ad} : G \to \text{End}(\mathfrak{g})$. Transporting the group structure of $G$ to $\mathbb{R}^n$ by mean of the global coordinates, it is clear that any Haar measure on $\mathbb{R}^n$ (for the transported group law) is absolutely continuous with respect to the Lebesgue measure. Let $d_G(\xi)$ be a left invariant Haar measure on $G$ transported to $\mathbb{R}^n$. Let also $\rho : \mathbb{R}^n \to \mathbb{R}$ be the Radon-Nikodym derivative of $d_G(\xi)$ with respect to $d\xi$, the Lebesgue measure on $\mathbb{R}^n$. Let $\xi \in \mathbb{R}^n$ be the transported neutral element of $G$. By left-invariance of the Haar measure $d_G(\xi)$, we get

$$\rho(\mu(\xi', \xi)) = \rho(\xi)|\text{Jac}_{L^*_x}(\xi)|, \quad \forall \xi, \xi' \in \mathbb{R}^n,$$

where $\mu(\cdot, \cdot)$ denotes the transported multiplication law on $\mathbb{R}^n$ and $L^*_x$ stands for the associated left translation operator on $\mathbb{R}^n$. Letting $\xi \to \xi_e$, we deduce

$$\rho(\xi) = \rho(\xi_e)|\text{Jac}_{L^*_x}(\xi_e)|, \quad \forall \xi \in \mathbb{R}^n,$$

and we conclude by Lemma 2.11 using the fact that the multiplication law is tempered.

Next, we let $\iota$ the inversion map of $G$ transported to $\mathbb{R}^n$. We have in the transported coordinates:

$$\Delta_G(\xi) = \frac{d_G(\iota(\xi))}{d_G(\xi)} = \frac{d_G(\iota(\xi))}{d\xi} \frac{d\xi}{d_G(\xi)} = \text{Jac}_{\iota}(\xi) \frac{d_G(\iota(\xi))}{d(\iota(\xi))} \frac{d\xi}{d_G(\xi)} = \text{Jac}_{\iota}(\xi) \frac{\rho(\iota(\xi))}{\rho(\xi)},$$

and we conclude using what precedes and the temperedness of the inversion map on $G$.

We now consider the data of a pair $(G, S)$ where $G$ is a connected real Lie group with real Lie algebra $\mathfrak{g}$ and $S$ is a real-valued smooth function on $G$.

Definition 2.17 The pair $(G, S)$ is called tempered (on the left) if the following two properties are satisfied:

(i) The map

$$\phi : G \to \mathfrak{g}^* : x \mapsto \left[ \mathfrak{g} \to \mathbb{R} : X \mapsto \mathbb{d}S_x(\tilde{X}) = (\tilde{X}.S)(x) \right],$$

is a global diffeomorphism.

(ii) The inverse map $\phi^{-1} : \mathfrak{g}^* \simeq \mathbb{R}^n \to G$ endows $G$ with the structure of a tempered Lie group.

Remark 2.18 Within the above situation, the function $S$ is itself automatically tempered. Indeed, in the proof of Lemma 2.11, we have seen that directional derivatives in the coordinate system $\phi$ are expressed as linear combinations of left-invariant vector fields with tempered coefficients. Hence, within a basis $\{X_j\}_{j=1,..,N}$ of $\mathfrak{g}$, denoting $x_j := (\tilde{X}_j.S)(x)$, we have $\partial_x S(x) = \sum_k m^k(x)x_k$ where the $m^k$'s are tempered. This implies that the partial derivatives of every order of $S$ are tempered. In polar coordinates $(r, \theta)$ (associated to the $x_j$'s) one observes that $\partial_r S(r, \theta) = R(\theta)\delta^k_{2k} \partial_{x_k} S$ where $\theta$ belongs to the unit sphere $S^{N-1}$ and where $R(\theta)$ is a (rotation) matrix that smoothly depends on $\theta$. Hence:

$$|S(x)| = \left| C + \int_{r_0}^r \partial_r S(\rho, \theta) d\rho \right| \leq |C| + \int_{r_0}^r |R(\theta)|^k \rho^n d\rho ,$$

which for large $x$ is smaller than a multiple of some positive power of $r$. Therefore the function $S$ as well as every of its derivative has polynomial growth.

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Given a tempered pair \((G,S)\), with \(\mathfrak{g}\) the Lie algebra of \(G\), we now consider a vector space decomposition:

\[
\mathfrak{g} = \bigoplus_{n=0}^{N} V_n ,
\]

and for every \(n = 0, \ldots, N\), an ordered basis \(\{e_j^n\}_{j=1, \ldots, \dim(V_n)}\) of \(V_n\). We get global coordinates on \(G\):

\[
x_j^n := (e_j^n, S)(x), \quad n = 0, \ldots, N, \quad j = 1, \ldots, \dim(V_n).
\]

We choose a scalar product on each \(V_n\) and let \(|\cdot|_n\) be the associated Euclidean norm. Given an element \(A \in \mathcal{U}(\mathfrak{g})\), we let \(\tilde{A}^*\) the formal adjoint of the left-invariant differential operator \(\tilde{A}\), with respect to the inner product of \(L^2(G, d_G)\). We make the obvious observation that \(\tilde{A}^*\) is still left-invariant. Indeed, for \(\psi, \varphi \in C_c^\infty(G)\) and \(g \in G\), we have

\[
\langle L_g^* \tilde{A}^* \psi, \varphi \rangle = \langle \psi, L_g^* \tilde{A} \varphi \rangle = \langle \tilde{A}^* L_g^* \psi, \varphi \rangle.
\]

Moreover, we make the following requirement of compatibility of the adjoint map on \(L^2(G, d_G)\) with respect to the ordered decomposition (17). Namely, denoting for every \(n \in \{0, \ldots, N\}\):

\[
V^{(n)} := \bigoplus_{k=0}^{n} V_k ,
\]

and within the notation (17), we assume:

\[
\forall n = 0, \ldots, N, \quad \forall A \in \mathcal{U}(V_n), \quad \exists B \in \mathcal{U}(V^{(n)}) \text{ such that } \tilde{A}^* = \tilde{B} ,
\]

where the space \(\mathcal{U}(V^{(n)})\) is defined in (17). We now pass to regularity assumptions regarding the function \(S\).

**Definition 2.19** Set

\[
E := \exp\{iS\} .
\]

A tempered pair \((G,S)\) is called (left-) admissible, if there exists a decomposition (17) with associated coordinate system (18), such that for every \(n = 0, \ldots, N\), there exists an element \(X_n \in \mathcal{U}(V_n) \subset \mathcal{U}(\mathfrak{g})\) whose associated multiplier \(\alpha_n\), defined as

\[
\tilde{X}_n E := \alpha_n E ,
\]

satisfies the following properties:

(i) There exist \(C_n > 0\) and \(\rho_n > 0\) such that:

\[
|\alpha_n| \geq C_n (1 + |x_n|^\rho_n) ,
\]

where \(x_n := (x_j^n)_{j=1, \ldots, \dim(V_n)}\).

(ii) For all \(n = 0, \ldots, N\), there exists a tempered function \(0 < \mu_n \in C^\infty(G)\) such that:

(ii.1) For every \(A \in \mathcal{U}(V^{(n)}) \subset \mathcal{U}(\mathfrak{g})\) there exists \(C_A > 0\) such that:

\[
|\tilde{A} \alpha_n| \leq C_A |\alpha_n| \mu_n .
\]

(ii.2) The function \(\mu_n\) is independent of the variables \(\{x_j^r\}_{j=1, \ldots, \dim(V_r)}\), for all \(r \leq n\):

\[
\frac{\partial \mu_n}{\partial x_j^r} = 0, \quad \forall r \leq n, \quad \forall j = 1, \ldots, \dim(V_r) .
\]

**Remark 2.20** Similarly, using right-invariant vector fields one defines the notions of tempered pair on the right as well as right-admissibility. Those notions will be used [4] in a crucial way.
We start with a preliminary result, which gives an upper bound for powers of derivatives of the inverse of a multiplier, in the context of admissible tempered pairs.

**Lemma 2.21** Fix $n = 0, \ldots, N$. Let $\alpha \in C^\infty(G)$ be non-vanishing and $1 \leq \mu \in C^\infty(G)$ such that for every $A \in \mathcal{U}(V^{(n)})$ there exists $C > 0$ with $|A\alpha| \leq C \mu |\alpha|$. Fixing $X \in \mathcal{U}(V^{(n)})$, a monomial of homogeneous degree $M \in \mathbb{N}$, we consider the differential operator

$$D_{X,\alpha} : C^\infty(G) \to C^\infty(G), \quad \Phi \mapsto \tilde{X}(\frac{1}{\alpha} \Phi).$$

Then, for every $r \in \mathbb{N}$, there exist an element $X' \in \mathcal{U}(g)$ of maximal homogeneous degree bounded by $rM$ and a constant $C > 0$ such that for every $\Phi \in C^\infty(G)$ we have:

$$|D_{X,\alpha}^r \Phi| \leq C \frac{\mu^r M}{|\alpha|^r} |\tilde{X}' \Phi|.$$

**Proof.** We start by recalling di Bruno’s formula:

$$\frac{d^r}{dt^r} \left( \frac{1}{f} \right) = \frac{1}{f} \sum_{M} C^r_{M} \prod_{j=1}^{r} \left( \frac{f^{(j)}}{f} \right)^{M_j}, \quad f \in C^\infty(\mathbb{R}),$$

where $M = (M_1, \ldots, M_r)$ runs along partitions of $r$ (i.e. $r = \sum_{j=1}^{r} j M_j$) and where $C^r_{M}$ is some combinatorial coefficient. Within Sweedler’s notations [2], di Bruno formula then yields for $\Phi \in C^\infty(G)$:

$$D_{X,\alpha} \Phi = \sum_{\langle X \rangle} \left( \tilde{X}(1) \frac{1}{\alpha} \right) \left( \tilde{X}(2) \Phi \right) = \sum_{\langle X \rangle} \frac{1}{\alpha} \sum_{\langle X \rangle} \prod \left( \tilde{X}(\frac{1}{\alpha})^{M_j} \left( \tilde{X}(2) \Phi \right) \right),$$

where the second sum and product run over partitions of $M_{(1)} := \deg(X_{(1)}) \leq M$ and where the element $X_j$ is of homogeneous degree $j = 1, \ldots, r$. Of course, we also have that $X_{(1)}$, $X_{(2)}$ and $X_j$ all belong to $\mathcal{U}(V^{(n)})$. Thus, $|\tilde{X}_{j\alpha}| \leq C(X) \mu |\alpha|$, the estimation is satisfied for $r = 1$. For $r = 2$, we observe:

$$D_{X,\alpha}^2 \Phi = \sum_{\langle X \rangle} \left( \tilde{X}(1) \frac{1}{\alpha} \right) \left( \tilde{X}(2) \Phi \right) = \sum_{\langle X \rangle, \langle X \rangle} \left( \tilde{X}(1) \frac{1}{\alpha} \right) \sum_{\langle X \rangle} \left( \tilde{X}(1)(\tilde{X}(1) \frac{1}{\alpha}) \right) \left( \tilde{X}(2) \Phi \right).$$

Di Bruno’s formula for $\frac{1}{\alpha}$ then yields the assertion for $r = 2$. Iterating this procedure, we get that

$$D_{X,\alpha}^r \Phi = \sum_{\langle X \rangle} \prod_{j=1}^{r} \left( \tilde{X}(j) \frac{1}{\alpha} \right) \left( \tilde{X}' \Phi \right), \quad (22)$$

for some elements $X^{(j)}$, $X' \in \mathcal{U}(V^{(n)})$ where the maximal homogeneous degree of $X^{(j)}$ is bounded by $jM$. Therefore, Di Bruno’s formula yields for every $j = 1, \ldots, r$:

$$|\tilde{X}(j) \frac{1}{\alpha}| = \left| \frac{1}{\alpha} \sum_{\langle X \rangle} \prod \left( \tilde{X}(k) \frac{\deg(X^{(j)})}{\alpha} \right) \right| \leq C \frac{\mu^{r \deg(X^{(j)})}}{|\alpha|^r} \leq C \frac{\mu^{r \deg(X^{(j)})}}{|\alpha|^r} \leq C \frac{\mu^{r j M}}{|\alpha|^r}.$$

Therefore since $\frac{r(r+1)}{2} \leq r^2$ and $\mu \geq 1$, we get the (rough) estimation:

$$|D_{X,\alpha}^r \Phi| \leq C \sum_{\langle X \rangle} \prod_{j=1}^{r} \frac{1}{|\alpha|} \mu^{j M} |\tilde{X}' \Phi| \leq C' \sum_{\langle X \rangle} \frac{1}{|\alpha|} \mu^{r j M} |\tilde{X}' \Phi|,$$

which delivers the proof.
We now fix an admissible tempered pair \((G, S)\) and for all \(n = 0, \ldots, N\), we let \(X_n \in \mathcal{U}(V_n)\) as given in Definition \(2.19\) and we let \(\alpha_n, \mu_n \in C^\infty(G)\) be the associated multiplier and tempered function. Accordingly to the previous notations, we introduce the operators:

\[
D_n := D_{X_n, \alpha_n} : C^\infty(G) \to C^\infty(G), \quad \Phi \mapsto \tilde{X}_n^* \left( \frac{1}{\alpha_n} \Phi \right).
\]

Recall that by assumption, there exists \(Y_n \in \mathcal{U}(V^{(n)})\) such that \(\tilde{X}_n^* = \tilde{Y}_n\) and thus, we can apply Lemma \(2.21\) to these operators. For every \(r_n \in \mathbb{N}\), accordingly to the expression \(22\), we write

\[
D^{r_n}_{n+1} \Psi = \sum_{(X_n)} \prod_{j=1}^{r_n} \left( \tilde{X}_n^{(j)} \frac{1}{\alpha_n} \right) \left( \tilde{X}'_{n+1} \Phi \right),
\]

where \(X^{(j)}_n \in \mathcal{U}(V^{(n)})\) and its homogeneous degree is bounded by \(jM_n\), with \(M_n\) the maximal homogeneous degree of \(X_n\) and where the one of \(X'_{n+1}\) is bounded by \(r_nM_n\). Setting

\[
\Psi_n = \prod_{j=1}^{r_n} \left( \tilde{X}_n^{(j)} \frac{1}{\alpha_n} \right),
\]

we then write (abusingly since in fact it is a finite sum of such terms):

\[
D^{r_n}_{n+1} \Psi_n =: \Psi_n \tilde{X}'_{n+1}.
\]

Given a \(N+1\)-tuple of integers \(r' = (r_0, \ldots, r_N) \in \mathbb{N}^{N+1}\), we will be led to consider the operator

\[
D_{r'} := D_{r_0, \ldots, r_N} := D^{r_0}_0 D^{r_1}_1 \cdots D^{r_N}_N,
\]

and according to the previous notations, expressions of the form:

\[
D^{r_n}_{n+1} D^{r_{n+1}}_{n+1} \Phi = \Psi_n \tilde{X}'_{n+1} \left( \Psi_{n+1} \tilde{X}'_{n+1} \Phi \right).
\]

Within Sweedler’s notations, the latter is expressed as

\[
D^{r_n}_{n+1} D^{r_{n+1}}_{n+1} \Phi = \Psi_n \sum_{(X_n)} \left( \left( \tilde{X}_{n+1}^{(1)} \right) \left( X_{n+1}^{(2)} \right) \right) \left( \tilde{X}'_{n+1} \Phi \right).
\]

This leads us to define recursively the following quantities:

\[
\Psi_{n+1, \ldots, n-k} := \left( \tilde{X}'_{n-k} \right)_{(212\ldots)} \Psi_{n+1, \ldots, n-k+1} \in C^\infty(G),
\]

\[
X_{N, \ldots, 0} := \left( X_{N}^{(2)} \right)_{(22)} \left( X_{N-1}^{(2)} \right)_{(22)} \cdots \left( X_{1}^{(2)} \right)_{(22)} X_{N} \in \mathcal{U}(g),
\]

in terms of which we have (with the same abuse as in \(24\) above):

\[
D_{r'} = \Psi_0 \Psi_{1,0} \Psi_{2,1,0} \cdots \Psi_{N,\ldots,0} \tilde{X}'_{N,\ldots,0}.
\]

**Lemma 2.22** Fix \(n = 0, \ldots, N\) and let \(\alpha \in C^\infty(G)\) and \(\mu \in C^\infty(G, \mathbb{R}^*_+)\) satisfying the hypothesis of Lemma \(2.27\). For \(j = 1, \ldots, r\) and \(r \in \mathbb{N}^*\), fix also \(X^{(j)} \in \mathcal{U}(V^{(n)})\) and define

\[
\Psi := \prod_{j=1}^{r} \left( \tilde{X}^{(j)} \frac{1}{\alpha} \right),
\]

where \(\deg(X^{(j)}) \leq jM\), for a given \(M \in \mathbb{N}^*\). Consider a monomial \(Y \in \mathcal{U}(V^{(n)})\), then we have

\[
\tilde{Y} \Psi = \sum_{Y} \prod_{j=1}^{r} \left( \tilde{Y}^{(j)} \frac{1}{\alpha} \right) \quad \text{with} \quad \deg(Y^{(j)}) \leq jM + \deg(Y),
\]

and moreover there exists \(C > 0\) such that

\[
|\tilde{Y} \Psi| \leq C \frac{\mu r^2 M + r \deg(Y)}{|\alpha|^r}.
\]
Proof. The equality is immediate. Regarding the inequality, we first note that by virtue of di Bruno’s formula, we have for a finite sum:

\[
\hat{v}^{(j)} \frac{1}{\alpha} = \frac{1}{\alpha} \sum_{k=1}^{\deg(Y^{(j)})} \frac{\hat{v}^{(j)}}{\alpha}^{\deg(Y^{(j)})}.
\]

Hence

\[
\left| \hat{v}^{(j)} \frac{1}{\alpha} \right| \leq C \frac{\sum_{k=1}^{\deg(Y^{(j)})}}{|\alpha|} \leq C \frac{\mu^{\deg(Y^{(j)})}}{|\alpha|} \leq C \frac{\mu^{M+\deg(Y)}}{|\alpha|}.
\]

We then conclude as in the proof of Lemma 2.21.

From the lemmas above, we deduce an estimate for the ‘coefficient functions’ appearing in the expression of the differential operator \(D_F\) in (20).

Corollary 2.23 Let \((G, S)\) be an admissible tempered pair with decomposition \(g = \bigoplus_{n=0}^{N} V_n\) and accordingly to Definition 2.14, for \(n = 0, \ldots, N\), we let \((X_n, \alpha_n, \mu_n) \in \mathcal{U}(V_n) \times C^\infty(G) \times C^\infty(G)\) be the associated differential operator, multiplier and tempered function. Then, for \(k = 0, \ldots, N\) and \(r_k \in \mathbb{N}^+\), with \(\Psi_{k, \ldots, 0} \in C^\infty(G)\) defined in (23), we have

\[
|\Psi_{k, \ldots, 0}| \leq C_k \frac{r_k^2 M_k + r_k \sum_{j=0}^{k-1} r_j M_j}{|\alpha_k|^r_k},
\]

for some finite non-negative constant \(C_k\) and where \(M_n := \deg(X_n), n = 0, \ldots, N\).

Proof. Observe that

\[
\Psi_{k, \ldots, 0} = \prod_{j=0}^{k-1} (\tilde{X}_j^{(212 \ldots)}) \Psi_k,
\]

where \(\Psi_k\) is defined in (23). Since \((X_j^{(212 \ldots)}) \in \mathcal{U}(V^n)\) with homogeneous degree of is bounded by \(r_j M_j\) for every \(j = 0, \ldots, k - 1\), the estimate we need follows from Lemma 2.22.

We can now state the main technical results of this subsection.

Proposition 2.24 Let \((G, S)\) be an admissible tempered pair and let \(\mu\) be a tempered weight. Then, there exists \(\bar{r} = (r_0, \ldots, r_N) \in \mathbb{N}^{N+1}\) such that for every element \(F \in \mathcal{B}^\mu(G)\), the function \(D_{\bar{r}} F\) belongs to \(L^1(G, d\xi)\). More precisely, there exist a finite constant \(C > 0\) and \(K \in \mathbb{N}\) with \(K \leq \sum_{k=0}^{N} r_k M_k\) and \(M_k = \deg(X_k)\) (with \(X_k \in \mathcal{U}_{M_k}(g)\) as given in Definition 2.17), such that for all \(F \in \mathcal{B}^\mu(G)\), we have:

\[
\|D_{\bar{r}} F\|_1 \leq C \sup_{x \in \mathcal{U}_k(\alpha)} \sup_{\mu(g) |X| K} \left\{ \left| \tilde{X} F(x) \right| \right\} = C \|F\|_{K, \mu, \infty}.
\]

Proof. By Lemma 2.16 in the coordinates (18), the Radon-Nicodym derivative of the left Haar measure on \(G\) with respect to the Lebesgue measure on \(g^*\), is bounded by a polynomial in \(\{x_j^\alpha, j = 1, \ldots, \dim(V_n), n = 0, \ldots, N\}\). By the assumption of temperedness of the weight \(\mu\), the latter is also bounded by a polynomial in the same coordinates. Now, observe from (26), that we have for any \(\bar{r} = (r_1, \ldots, r_N)\) and for \(K = \deg(X_{N, \ldots, 0}) \leq \sum_{k=0}^{N} r_k M_k\):

\[
|D_{\bar{r}} F| \leq |\Psi_0| \cdot |\Psi_{1,0}| \cdot |\Psi_{2,1,0}| \cdots |\Psi_{N,\ldots,0}| \cdot \tilde{X}_{N,\ldots,0} F |
\leq |\Psi_0| \cdot |\Psi_{1,0}| \cdot |\Psi_{2,1,0}| \cdots |\Psi_{N,\ldots,0}| \cdot \tilde{X}_{N,\ldots,0} F |
\leq |\Psi_0| \cdot |\Psi_{1,0}| \cdot |\Psi_{2,1,0}| \cdots |\Psi_{N,\ldots,0}| \cdot |X_{N,\ldots,0} F |
\leq C \frac{\mu}{|K|} \frac{\mu}{|K|} \frac{M_k}{|\alpha_k|^r_k} \frac{M_k}{|\alpha_k|^r_k}.
\]

This will gives the estimate, if we prove that the function in front of \(\|F\|_{K, \mu, \infty}\) in (27) is integrable for a suitable choice of \(\bar{r} \in \mathbb{N}^{N+1}\). We prove a stronger result, namely that given \(\bar{R} = (R_0, \ldots, R_N) \in \mathbb{N}^{N+1}\), there exists \(\bar{r} = (r_0, \ldots, r_N) \in \mathbb{N}^{N+1}\) such that the associated functions \(\Psi_{k, \ldots, 0}\) (which depend on \(\bar{r}\)) satisfy:

\[
|\Psi_0(x)| \cdot |\Psi_{1,0}(x)| \cdot |\Psi_{2,1,0}(x)| \cdots |\Psi_{N,\ldots,0}(x)| \leq C \frac{1}{(1 + |x_0|) R_0 \cdots (1 + |x_N|) R_N}.
\]

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From Corollary 2.23 and writing \( r_k^2 M_k + r_k \sum_{j=0}^{k-1} r_j M_j = r_k \sum_{j=0}^{k} r_j M_j \), we obtain the following estimation:

\[
|\Psi_0| |\Psi_{1,0}| |\Psi_{2,1,0}| \cdots |\Psi_{N,\ldots,0}| \leq C \prod_{k=0}^{N} \frac{\mu_k^{r_k} \sum_{j=0}^{k} r_j M_j}{\alpha_k \mu_k^{r_k}}.
\]

Moreover, by assumption of temperedness, see Definition 2.19 (ii.1), there exist \( \rho_0, \ldots, \rho_N > 0 \) such that

\[
|\Psi_0(x)| |\Psi_{1,0}(x)| |\Psi_{2,1,0}(x)| \cdots |\Psi_{N,\ldots,0}(x)| \leq C \prod_{k=0}^{N} \frac{\mu_k(x)^{r_k} \sum_{j=0}^{k} r_j M_j}{(1 + |x_k|)^{\rho_k r_k}}.
\]

From the hypothesis of Definition 2.19 (ii.2), we deduce that the element \( \mu_N \) is constant. Indicating the variable dependence into parentheses, one also has

\[
\mu_{N-1} = \mu_{N-1}(x_N), \quad \mu_{N-2} = \mu_{N-2}(x_{N-1}, x_N), \quad \ldots \quad \mu_1 = \mu_1(x_2, \ldots, x_N), \quad \mu_0 = \mu_0(x_1, x_2, \ldots, x_N).
\]

Denoting by \( m_n, n = 0, \ldots, N \), the degree of a polynomial function that, in the variables \( \{ \mathbf{r} \} \), dominates the tempered function \( \mu_n \), we obtain the sufficient conditions:

\[
\rho_n r_n - \sum_{k=0}^{n-1} \left( m_k r_k \sum_{j=0}^{k} r_j M_j \right) \geq R_n, \quad n = 0, \ldots, N.
\]

One checks inductively that the latter corresponds to:

\[
r_0 \geq \rho_0^{-1} R_0 \quad \text{and} \quad r_n \geq \rho_n^{-1} R_n + \rho_n^{-1} \sum_{k=0}^{n-1} \left( m_k R_k \sum_{j=0}^{k} R_j M_j \right), \quad n = 1, \ldots, N,
\]

which is always achievable.

Let now \( \mathcal{E} \) be a complex Fréchet space, with topology associated with a countable family of semi-norms \( \{ \| \cdot \|_j \}_j \in \mathbb{N} \). An immediate modification of its proof, lead us to the following version of Proposition 2.24 (the only difference with the former is that now the index \( K \in \mathbb{N} \) may depends on \( j \) via the order of the tempered weight \( \mu_j \)).

**Proposition 2.25** Let \( (G, S) \) be an admissible tempered pair, \( \mathcal{E} \) be a complex Fréchet space and let \( \{ \mu_j \}_j \in \mathbb{N} \) be a family of tempered weights. Then for all \( j \in \mathbb{N} \), there exist \( r_j' \in \mathbb{N}^{N+1}, C_j > 0 \) and \( K_j \in \mathbb{N} \), such that for every element \( F \in \mathcal{B}^{(\mu_j)}(G, \mathcal{E}) \), we have

\[
\int_G \| D_{r_j'} F(g) \|_j \, dG(g) \leq C_j \sup_{X \in U_{K_j}(g)} \sup_{g \in G} \left\{ \frac{\| X F(x) \|_j}{\mu_j^{-1}(g) \| X \|_{K_j}} \right\} =: C_j \| F \|_{j, K_j, \mu_j, \infty}.
\]

### 2.3 An oscillatory integral for admissible tempered pairs

We are now prepared to define our notion of oscillatory integral. The latter follows from Proposition 2.25 above, together with the identity

\[
D^r E = E, \quad \forall r \in \mathbb{N}^{N+1},
\]

with \( E \) the function on \( G \) defined in \( [21] \) and \( D \) the differential operator given in \( [21] \).

**Definition 2.26** Let \( (G, S) \) be an admissible tempered pair, \( \mu \) a tempered weight, \( m \) an element of \( \mathcal{B}^{\mu}(G) \) and \( \{ \mu_j \}_j \in \mathbb{N} \) a family of tempered weights. Let also \( \{ \mu_j' \}_j \in \mathbb{N} \) be another family of tempered weights that dominates the family \( \{ \mu_j \}_j \in \mathbb{N} \) (hence \( \{ \mu_j' \}_j \in \mathbb{N} \) dominates \( \{ \mu_j \}_j \in \mathbb{N} \)). Associated to these weights, let \( r_j' \in \mathbb{N}^{N+1}, j \in \mathbb{N}, \) as given in Proposition 2.25 and let \( D_{r_j'} \) be the differential operators given in [21]. Performing integrations by parts, the Dunford-Petit theorem \([12]\) yields a \( \mathcal{B}^{(\mu_j')}(G, \mathcal{E}) \)-continuous mapping

\[
\mathcal{D}(G, \mathcal{E}) \to \mathcal{E} : F \mapsto \int_G m E F = \int_G E D_{r_j'}(m F).
\]

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Then by Lemma 2.26(v), the latter extends to the following continuous linear mapping:

\[ \int \mathbf{m} \mathbf{E} : \mathcal{B}^{(\mu_j)}(G, \mathcal{E}) \rightarrow \mathcal{E}, \]

that we refer to as an oscillatory integral.

Our next aim is to prove that the oscillatory integral on \( \mathcal{B}^{(\mu_j)}(G, \mathcal{E}) \), does not depend on the choices made. So let \( \mu, \{\mu_j\}_{j \in \mathbb{N}} \) and \( \{\mu'_j\}_{j \in \mathbb{N}} \) as in Definition 2.26. Fix \( F \in \mathcal{B}^{(\mu_j)}(G, \mathcal{E}) \) and chose a sequence \( \{F_n\}_{n \in \mathbb{N}} \) of elements of \( \mathcal{D}(G, \mathcal{E}) \) converging to \( F \) for the topology of \( \mathcal{B}^{(\mu_j)}(G, \mathcal{E}) \). By definition of the oscillatory integral and undoing the integrations by parts at the level of smooth compactly supported \( \mathcal{E} \)-valued functions, we first observe that:

\[ \int_G \mathbf{m} \mathbf{E}(F) = \int_G \mathbf{m} \mathbf{E}(\lim_{n \to \infty} F_n) = \lim_{n \to \infty} \int_G \mathbf{m} \mathbf{E}(F_n) = \lim_{n \to \infty} \int_G \mathbf{m}(g) \mathbf{E}(g) F_n(g) \, d_G(g), \]

where the first limit is in \( \mathcal{B}^{(\mu'_j)}(G, \mathcal{E}) \) and the last two are in \( \mathcal{E} \). Then, the estimate of Proposition 2.25 immediately implies that the limit above is independent of the approximation sequence \( \{F_n\}_{n \in \mathbb{N}} \) chosen. This shows that the oscillatory integral does not depends on the differential operators in \( \mathbf{D}_{\mu'} \) used to define the extension (in the topology of \( \mathcal{B}^{(\mu_j)}(G, \mathcal{E}) \)) of the oscillatory integral from \( \mathcal{D}(G, \mathcal{E}) \) to \( \mathcal{B}^{(\mu_j)}(G, \mathcal{E}) \). Last, to see that the oscillatory integral mapping is also independent of the choice of the family of dominant weights \( \{\mu'_j\}_{j \in \mathbb{N}} \) chosen, it suffices to remark that the approximation sequence constructed in the proof of Lemma 2.16(viii) can be used for any family \( \{\mu'_j\}_{j \in \mathbb{N}} \) such that \( \mu'_j \succ \mu_j \). Of course this will hold provided that we can always find dominant weights. This is certainly the case if there exists a weight dominating the constant weight 1. Thus we have proven:

**Proposition 2.27** Let \( (G, S) \) an admissible tempered pair, \( \mathcal{E} \) a complex Fréchet space, \( \mu, \mu_j \) be tempered weights and \( \mathbf{m} \in \mathcal{B}^\mu(G) \). Assuming that there exists a tempered weight \( \mu_c \) which dominates the constant weight 1, then the oscillatory integral mapping

\[ \int_G \mathbf{m} \mathbf{E} : \mathcal{B}^{(\mu_j)}(G, \mathcal{E}) \rightarrow \mathcal{E}, \]

does not depends on the choice of the integers \( r_j \in \mathbb{N}^{N+1} \) and dominant weights \( \{\mu'_j\} \) given in Definition 2.26. Moreover, given \( F \in \mathcal{B}^{(\mu_j)}(G, \mathcal{E}) \), we have

\[ \int_G \mathbf{m} \mathbf{E}(F) = \lim_{n \to \infty} \int_G \mathbf{m}(g) \mathbf{E}(g) F_n(g) \, d_G(g), \]

where \( \{F_n\}_{n \in \mathbb{N}} \) is an arbitrary sequence in \( \mathcal{D}(G, \mathcal{E}) \), converging to \( F \) in the topology of \( \mathcal{B}^{(\mu_j)}(G, \mathcal{E}) \), for an arbitrary sequence of weights \( \{\mu'_j\} \), which dominates \( \{\mu_j\} \).

**Remark 2.28** Note that Proposition 2.27 does not assert that the oscillatory integral on \( \mathcal{B}^{(\mu_j)}(G, \mathcal{E}) \) is the unique continuous extension of its restriction to \( \mathcal{D}(G, \mathcal{E}) \).

We observe that the of existence of a tempered weight that dominates the constant weight 1 implies that every weight is dominated, which is crucial for the construction of the oscillatory integral. This leads us to introduce the notion of tameness below. We will see that this property holds for negatively curved Kählerian groups, where we can use the modular weight.

**Definition 2.29** A tempered Lie group \( G \), with associated diffeomorphism \( \phi : G \rightarrow g^* \) is called tame if there exist a Euclidean norm \( |.| \) on \( g^* \), a tempered weight \( \mu_\phi \) and two positive constants \( C, \rho \) such that

\[ C (1 + |\phi|^2)^{\frac{\rho}{2}} \leq \mu_\phi. \]
In the constant family case, i.e. for $B^\mu(G,\mathcal{E})$, we can express the oscillatory integral as an absolutely convergent one for each semi-norm $||.||_j$:

$$\int_G \mathfrak{m} \mathcal{E}(F) = \int_G \mathcal{E} D_{\vec{r}}(m F), \quad F \in B^\mu(G,\mathcal{E}),$$

where the label $\vec{r} \in \mathbb{N}^{N+1}$ of the differential operator $D_{\vec{r}}$ is given by Proposition 2.24 and does not depend on the index $j \in \mathbb{N}$ which labels the semi-norms defining the topology of $\mathcal{E}$. However, we cannot have access to such a formula in the case of $B^{(\mu_j)}(G,\mathcal{E})$, since in this case the label $\vec{r} \in \mathbb{N}^{N+1}$ may depend on the index $j \in \mathbb{N}$. But from Proposition 2.24 we have the following weaker statement:

**Proposition 2.30** Let $(G, S)$ an admissible and tame tempered pair, $(\mathcal{E}, \{||.||_j\})$ a complex Fréchet space, $\mu, \mu_j$ be tempered weights and $m \in \mathcal{B}^\mu(G)$. Then for every $j \in \mathbb{N}$, there exists $\vec{r}_j \in \mathbb{N}^{N+1}$, such that for all $F \in B^{(\mu_j)}(G,\mathcal{E})$, we have

$$\int_G \mathfrak{m} \mathcal{E}(F) = \int_G \mathcal{E} D_{\vec{r}_j}(m F),$$

where the right hand side is an absolutely convergent integral for the semi-norm $||.||_j$.

We close this subsection with a natural result on the compatibility of the oscillatory integral with continuous linear maps between Fréchet spaces.

**Lemma 2.31** Let $(G, S)$ be an admissible and tame tempered pair, $(\mathcal{E}, \{||.||_j\})$ and $(\mathcal{F}, \{||.||'_j\})$ two Fréchet spaces and $T : \mathcal{E} \rightarrow \mathcal{F}$ a continuous linear map such that for all $j \in \mathbb{N}$ there exist $l(j) \in \mathbb{N}$ and $C_j > 0$, such that for all $a \in \mathcal{E}$, we have $\|T(a)||'_j \leq C_j \|a||_(l(j))$. Define the map $\hat{T}$ from $C(G,\mathcal{E})$ to $C(G,\mathcal{F})$, by setting

$$(\hat{T}F)(g) := T(F(g)).$$

Then, for any family $\{\mu_j\}$ of tempered weights, $\hat{T}$ is continuous from $B^{(\mu_j)}(G,\mathcal{E})$ to $B^{(\mu_j)}(G,\mathcal{F})$. Moreover for any $m \in \mathcal{B}^\mu(G)$, with $\mu$ another tempered weight, we have

$$T \left( \int_G \mathfrak{m} \mathcal{E}(F) \right) = \int_G \mathfrak{m} \mathcal{E}(\hat{T}F).$$

**Proof.** The continuity of $\hat{T}$ is immediate from our assumptions. Indeed, for $F \in B^{(\mu_j)}(G,\mathcal{E})$ and $j, k \in \mathbb{N}$, we have

$$\|\hat{T}F\|'_{j,k,\mu(j),\infty} = \sup_{X \in U_k(g) \in G} \sup_{\mu_j(\xi)|X|_k} \|\check{X} T(F(g))\|'_j = \sup_{X \in U_k(g) \in G} \sup_{\mu_j(\xi)|X|_k} \|T(\check{X} F(g))\|'_j \leq C_j \sup_{X \in U_k(g) \in G} \sup_{\mu_j(\xi)|X|_k} \|\check{X} F(g)\|'_{l(j)} = C_j \|F\|'_{l(j),k,\mu(j),\infty}.$$

Repeating the arguments for $B^{(\mu_j,\mu_0)}(G,\mathcal{E})$ ($\mu_0$ is the tempered weight associated to tameness) instead of $B^{(\mu_j)}(G,\mathcal{E})$, we see that both sides of (28) define continuous linear maps from $B^{(\mu_j,\mu_0)}(G,\mathcal{E})$ to $\mathcal{F}$. Moreover, it is easy to see that they coincide on $D(G,\mathcal{E})$ and thus they coincide on the closure of $D(G,\mathcal{E})$ inside $B^{(\mu_j,\mu_0)}(G,\mathcal{E})$, which contains $B^{(\mu_j)}(G,\mathcal{E})$ by Lemma 2.8 (vi), as $\{\mu_j\}_{j \in \mathbb{N}}$ dominates $\{\mu_j\}_{j \in \mathbb{N}}$.

### 2.4 A Fubini Theorem for semi-direct products

The aim of this subsection is to prove a Fubini type result for the oscillatory integral on a semi-direct product of tempered pairs. We start with following observation:
Lemma 2.32 Let \((\mathcal{E}, \{\|\cdot\|_j\})\) be a complex Fréchet space, \(G_1, G_2\) be two Lie groups with Lie algebras \(\mathfrak{g}_1, \mathfrak{g}_2\) and \(\mathbf{R} \in \text{Hom}(G_1, \text{Aut}(G_2))\) be an extension homomorphism. Consider \(\{\mu_j\}\) a family of weights on the semi-direct product \(G_1 \ltimes_{\mathbf{R}} G_2\) with sub-multiplicative degrees \(\{L_j, R_j\}\). Set also \(\mu_{1,j}\) and \(\mu_{2,j}\) for the restrictions of the weight \(\mu_j\) to the sub-groups \(G_1, G_2\) and let \(\mathfrak{d}_1\) be the restriction of the modular weight (c.f. Example 2.3) of \(G_1 \ltimes_{\mathbf{R}} G_2\) to \(G_1\). Then the map

\[
F \in C^\infty(G_1 \ltimes_{\mathbf{R}} G_2, \mathcal{E}) \mapsto \tilde{F} := [g_1 \in G_1 \mapsto [g_2 \in G_2 \mapsto F(g_2g_1)] \in C^\infty(G_1, C^\infty(G_2, \mathcal{E})) ,
\]

sends continuously \(B^{(\mu_j)}(G_1 \ltimes_{\mathbf{R}} G_2, \mathcal{E})\) to \(B^{(\mu_{1,j}, \mu_{2,j})}_{j, k \in \mathbb{N}}(G_1, \mathcal{E}^{(\mu_{1,j}, \mu_{2,j})})(G_2, \mathcal{E})\).

Proof. First, observe that for \(g \in G_1 \ltimes_{\mathbf{R}} G_2\) with \(g = g_2g_1, g_1 \in G_1, g_2 \in G_2, F \in C^\infty(G_1 \ltimes_{\mathbf{R}} G_2)\) and \(X^1 \in \mathfrak{g}_1, X^2 \in \mathfrak{g}_2\), we have

\[
\tilde{X}^{1,g_1} \tilde{F}(g_1, g_2) = \tilde{X}^{1,g} F(g) , \quad \tilde{X}^{2,g_2} \tilde{F}(g_1, g_2) = R^{-1}_{g_1} (\tilde{X}^{2,g}) F(g) ,
\]

where we use the same notation for the extension homomorphism and its derivative:

\[
g_2 \rightarrow g_2 , \quad X \mapsto \frac{d}{dt} \bigg|_{t=0} R_{g_1}(e^{tx} X) , \quad g_1 \in G_1 .
\]

From this, it follows that the restriction of a weight on \(G_1 \ltimes_{\mathbf{R}} G_2\) to \(G_1\) or \(G_2\) is still a weight on \(G_1\) or \(G_2\). Indeed, given \(\mu\) a weight on \(G_1 \ltimes_{\mathbf{R}} G_2\), call \(\mu^i, i = 1, 2\), its restriction to the sub-group \(G_i\) and given \(X \in \mathcal{U}(\mathfrak{g}_i)\) call \(X_i\) its image in \(\mathfrak{g}_i \ltimes \mathfrak{g}_2\). Then, Equation (28) yields \(\tilde{X} \mu^i = (\tilde{X}, \mu)^i, i = 1, 2\), which together with \(\mu^i(g) = (\mu^i)(g)\), implies the first condition of Definition 2.1 is satisfied. Sub-multiplicativity at the level of each sub-groups \(G_i, i = 1, 2\), follows from sub-multiplicativity at the level of \(G_1 \ltimes_{\mathbf{R}} G_2\) (with the same sub-multiplicativity degree).

Moreover, (29) also implies that for \(F \in B^{(\mu_j)}(G_1 \ltimes_{\mathbf{R}} G_2, \mathcal{E}), X^1 \in \mathcal{U}_{k_1}(\mathfrak{g}_1), X^2 \in \mathcal{U}_{k_2}(\mathfrak{g}_2)\) and \(k_1, k_2, j \in \mathbb{N}\), we have for \(g_2g_1 \in G_1 \ltimes_{\mathbf{R}} G_2\):

\[
\|\tilde{X}^{1,g_1} \tilde{X}^{2,g_2} \tilde{F}(g_1, g_2)\|_j = \| (\tilde{X}^{1,g_1} R_{g_1}^{-1} (\tilde{X}^{2,g}) F)(g_2g_1)\|_j \\
\leq C(k_1, k_2) \|X^1\|_{k_1} \|X^2\|_{k_2} \| R_{g_1}^{-1} |k_2 \sup_{Y \in \mathcal{U}_{k_1+k_2}(\mathfrak{g}_1 \ltimes \mathfrak{g}_2)} \| Y F(g_2g_1)\|_j \\
\leq C(k_1, k_2) \|X^1\|_{k_1} \|X^2\|_{k_2} \| R_{g_1}^{-1} |k_2 \sup_{\mathcal{U}_{k_1+k_2}(\mathfrak{g}_1 \ltimes \mathfrak{g}_2)} \| Y F(g_2g_1)\|_j \|F\|_{j, k_1+k_2, \mu_j, \infty} \\
\leq C'(k_1, k_2) \|X^1\|_{k_1} \|X^2\|_{k_2} \| R_{g_1}^{-1} (\tilde{X}^{2,g_1}) F\|_{j, k_1+k_2, \mu_j, \infty} \|\tilde{F}\|_{j, k_1+k_2, \mu_j, \infty} ,
\]

by Lemma 2.39 since for \(g_1 \in G_1, R_{g_1}\) coincides with the restriction of \(\mathfrak{d}_1\) to \(g_2\). Thus, labeling by \((j, k_2) \in \mathbb{N}^2\) the semi-norms \(\| \|_{j, k_1, k_2, \mu_j, \infty}\) of \(B^{(\mu_j)}(G_2, \mathcal{E})\), we finally get:

\[
\|\tilde{F}\|_{(j, k_2), k_1, \mu_j, \infty} \leq C'(k_1, k_2) \|\tilde{F}\|_{j, k_1+k_2, \mu_j, \infty} ,
\]

which completes the proof. \(\blacksquare\)

Now, assume that the groups \(G_1\) and \(G_2\) come from admissible tempered pairs \((G_1, S_1)\) and \((G_2, S_2)\). Parametrizing \(g = g_2g_1 \in G_1 \ltimes_{\mathbf{R}} G_2\) with \(g_i \in G_i, i = 1, 2\), we can then set

\[
S : G_1 \ltimes_{\mathbf{R}} G_2 \rightarrow \mathbb{R}, \quad g_2g_1 \mapsto S_1(g_1) + S_2(g_2) ,
\]

and with (21), we set accordingly

\[
E(g_2g_1) := E_1(g_1) E_2(g_2) .
\]

Assume further that \(\mathfrak{d}_1\), the restriction of the modular weight on \(G_1 \ltimes_{\mathbf{R}} G_2\) to \(G_1\) is tempered. Thus for \(\mathbf{m} \in B^{(\mu)}(G_1 \ltimes_{\mathbf{R}} G_2)\) with \(\mu\) a tempered weight, and with \(\tilde{\mathbf{m}}\) the associated function on \(G_1 \times G_2\) as constructed in (29), Lemma 2.32 shows that the map

\[
B^{(\mu_j)}(G_1 \ltimes_{\mathbf{R}} G_2, \mathcal{E}) \rightarrow \mathcal{F} \mapsto \int_{G_2} \int_{G_1} E_1 E_2 \left( \int_{G_1} E_1 (\tilde{\mathbf{m}} F) \right) ,
\]

Assume further that \(\mathfrak{d}_1\), the restriction of the modular weight on \(G_1 \ltimes_{\mathbf{R}} G_2\) to \(G_1\) is tempered. Thus for \(\mathbf{m} \in B^{(\mu)}(G_1 \ltimes_{\mathbf{R}} G_2)\) with \(\mu\) a tempered weight, and with \(\tilde{\mathbf{m}}\) the associated function on \(G_1 \times G_2\) as constructed in (29), Lemma 2.32 shows that the map
is well defined as a continuous linear map. Thus under these circumstances, this map could be used as a definition for the oscillatory integral on the semi-direct product $G_1 \ltimes R G_2$. Moreover, when the pair $(G_1 \ltimes R G_2, S)$ is also tempered and admissible and when the extension homomorphism preserves the Haar measure $d_{G_2}$, then the map above coincides with the oscillatory integral on $G_1 \ltimes R G_2$, as given in Definition 2.20. This is our Fubini-type result in the context of semi-direct product of tempered pairs:

**Proposition 2.33** Within the context of Lemma 2.32 assume further that the groups $G_1$ and $G_2$ come from admissible and tame tempered pairs $(G_1, S_1)$ and $(G_2, S_2)$ and, with $S$ defined in (21), that $(G_1 \ltimes R G_2, S)$ is admissible, tame and tempered too. Assume last that the extension homomorphism $R \in Hom(G_1, Aut(G_2))$ is tempered and preserves the Haar measure $d_{G_2}$. Let also $\mu, \mu_j, j \in N$, be tempered weights on the semi-direct product $G_1 \ltimes R G_2$. Then, for $F \in B^{(\mu)}(G_1 \ltimes R G_2, E)$, $m \in B^{(\mu)}(G_1 \ltimes R G_2)$, with $F$ and $m$ the associated functions on $G_1 \times G_2$, as in (29), we have

$$
\int_{G_1 \ltimes R G_2} \tilde{E}_m(F) = \int_{G_2} \tilde{E}_2 \left( \int_{G_1} \tilde{E}_1 (\tilde{m} \tilde{F}) \right).
$$

**Proof.** Since $R$ is tempered, $d_1$, the restriction of the modular weight on $G_1 \ltimes R G_2$ to $G_1$ is tempered on $G_1$. Thus by Lemma 2.32 the right hand-side of (22) is well defined as a continuous linear map from $B^{(\mu)}(G_1 \ltimes R G_2, E)$ to $E$. Note also that by our assumptions that the pair $(G_1 \ltimes R G_2, S)$ is tempered and admissible, the left hand side of (22) is also well defined as a continuous linear map from $B^{(\mu)}(G_1 \ltimes R G_2, E)$ to $E$, too. Now, take $F \in D(G_1 \ltimes R G_2, E)$ and associate to it $\tilde{F} \in D(G_1, D(G_2, E))$ as in (29). By construction, we have

$$
\int_{G_1 \ltimes R G_2} \tilde{E}_m(F) = \int_{G_1 \ltimes R G_2} E(g) m(g) F(g) d_{G_1 \ltimes R G_2}(g).
$$

Since the extension homomorphism $R$ preserves $d_{G_2}$, we have for $g_1 \in G_1, g_2 \in G_2$:

$$
d_{G_1 \ltimes R G_2}(g_2 g_1) = d_{G_1}(g_1) d_{G_2}(g_2),
$$

which, by the ordinary Fubini Theorem, implies that

$$
\int_{G_1 \ltimes R G_2} \tilde{E}_m(F) = \int_{G_2} E_2(g_2) \left( \int_{G_1} E_1(g_1) m(g_2 g_1) F(g_2 g_1) d_{G_1}(g_1) \right) d_{G_2}(g_2) = \int_{G_2} E_2 \left( \int_{G_1} E_1 (\tilde{m} \tilde{F}) \right).
$$

Thus, both sides of (22) are continuous linear map from $B^{(\mu, \mu)}(G_1 \ltimes R G_2, E)$ ($\mu_0$ is the tempered weight on $G_1 \ltimes R G_2$ associated with tameness) to $E$ and coincide on $D(G_1 \ltimes R G_2, E)$. Therefore, these maps coincide on the closure of $D(G_1 \ltimes R G_2, E)$ inside $B^{(\mu, \mu)}(G_1 \ltimes R G_2, E)$. One concludes using Lemma 2.3 (vi), which shows that the latter closure contains $B^{(\mu)}(G_1 \ltimes R G_2, E)$.

**2.5 A Schwartz space for tempered pairs**

In this subsection, we introduce a Schwartz type functions space, out of an admissible tempered pair $(G, S)$ and prove that it is Fréchet and nuclear. Our notion of Schwartz space is of course closely related, if not in many cases equivalent, to other notions of Schwartz space on Lie groups, but the point here is that it is formulated in terms of the phase function $S$ only. This is the formulation that allows to immediately implement the compatibility with our notion of oscillatory integral.

**Definition 2.34** Let $(G, S)$ be a tempered pair. For all $X \in U(g)$, we let $\alpha_X := E^{-1} \tilde{X} E \in C^\infty(G)$, where $E$ is defined in (21). Then we set

$$
S^S(G) := \{ f \in C^\infty(G) : \forall X, Y \in U(g), \forall n \in N, \sup_{x \in G} |\alpha_X^n(x) (\tilde{Y} f)(x)| < \infty \}.
$$

We first prove that this space is isomorphic to the ordinary Schwartz space of the Euclidean space $g^*$. 

25
Lemma 2.35 Let \( \phi : G \to g^* \) be the diffeomorphism underlying Definition 2.17 associated to an admissible tempered pair \((G, S)\). Fixing a Euclidean structure on \( g^* \), denote by \( S(g^*) \) the ordinary Schwartz space of \( g^* \). Then, \( S^\phi(G) \) coincides with

\[
S^\phi(G) := \left\{ f \in C^\infty(G) : f \circ \phi^{-1} \in S(g^*) \right\}.
\]

In particular, endowed with the transported topology, \( S^\phi(G) \) is a nuclear Fréchet space.

Proof. Recall that \( f \in S^\phi(G) \) if and only if for all \( \alpha, \beta \in \mathbb{N}^{\dim(G)} \), we have

\[
\sup_{\xi \in g^*} \left| \xi^\alpha \partial^\beta (f \circ \phi^{-1})(\xi) \right| < \infty,
\]

while \( f \in S^\phi(G) \) if and only if for all \( X, Y \in \mathcal{U}(g) \) and all \( n \in \mathbb{N} \)

\[
\sup_{x \in G} \left| \alpha^n_X(x) (\hat{Y} f)(x) \right| < \infty.
\]

Fix \( \{X_j\}_{j=1}^{\dim(G)} \) a basis of \( g \) and let \( \{\xi_j\}_{j=1}^{\dim(G)} \) the dual basis on \( g^* \). From the same methods as in Lemma 2.10 one can construct an invertible matrix \( M(\xi) \) which is tempered with tempered inverse and which is such that in the \( \phi \)-coordinates

\[
\tilde{X}_j = \sum_{i=1}^{\dim(G)} M(\xi)_{j,i} \partial_{\xi_i}.
\]

Since by Remark 2.18 \( S \) is tempered, for all \( X \in \mathcal{U}(g) \), the associated multiplier \( \alpha_X \) in \( \phi \)-coordinates is bounded by a polynomial function on \( g^* \). Last, since the pair \((G, S)\) is admissible, associated to the vector space decomposition \( g = \bigoplus_{k=0}^N V_k \), there exist elements \( X_k \in \mathcal{U}(V_k) \) and constants \( \rho_k > 0 \) such that

\[
|\xi| \leq C \left( 1 + \sum_{k=0}^N |\alpha_k(\phi^{-1}(\xi))|^{\rho_k} \right).
\]

Putting these three facts together gives the equality between the two sets of functions on \( G \) and the equivalence of the topologies associated with the semi-norms \( (\| \cdot \|_j)_{j \in \mathbb{N}} \). More generally, when \( E \) is a complex Fréchet space with topology underlying a countable set of semi-norms \( (\| \cdot \|_j)_{j \in \mathbb{N}} \), we define the \( E \)-valued Schwartz space associated to a tempered pair \((G, S)\) as

\[
S^S(G, E) := \left\{ f \in C^\infty(G, E) : \forall X, Y \in \mathcal{U}(g) \text{, } \forall n, j \in \mathbb{N} \text{, } \sup_{x \in G} |\alpha^X_n(x)| \| (\hat{Y} f)(x) \|_j < \infty \right\}.
\]

Note that, when admissible, by nuclearity of \( S^S(G) \), we have \( S^S(G, E) = S^S(G) \hat{\otimes} E \) (for any completed tensor product).

Remark 2.36 In the context of tameness and admissibility, we deduce from Lemma 2.35 that \( S^S(G, E) \) is a Fréchet space for the topology associated with the semi-norms

\[
\| \cdot \|_{j, k, n, \infty} : f \in S^S(G, E) \mapsto \sup_{X \in \mathcal{U}(g)} \sup_{x \in G} \frac{\mu_{\phi}(x)^n \| \tilde{X} f(x) \|_j}{\| X \|_k}, \quad j, k, n \in \mathbb{N},
\]

where \( \mu_{\phi} \) is the weight associated to the tameness of the admissible tempered pair \((G, S)\). This is this set of semi-norms that we are going to use, rather than \( (\| \cdot \|_j)_{j \in \mathbb{N}} \) or \( (\| \cdot \|_j)_{j \in \mathbb{N}} \).

### 2.6 Bilinear mappings from the oscillatory integral

We now present several of results which establish most of the analytical properties we will need to construct our universal deformation formula for actions of Kählerian groups on Fréchet algebras. In all what follows, when considering a Fréchet algebra \( (A, (\| \cdot \|_j)_{j \in \mathbb{N}}) \), we will always assume that the semi-norms are sub-multiplicative, i.e.

\[
\|ab\|_j \leq \|a\|_j \|b\|_j, \quad \forall a, b \in A, \quad \forall j \in \mathbb{N}.
\]

We start with a crucial result. Its proof being very similar to those of Lemma 2.10 we omit it.
Lemma 2.37 Let \((\mathcal{A}, \{\|\cdot\|_j\}_{j \in \mathbb{N}})\) be a Fréchet algebra and let \(\{\mu_j\}\) and \(\{\mu'_j\}\) be two families of weights with sub-multiplicative degrees respectively denoted by \(\{(L_j, R_j)\}\) and \(\{(L'_j, R'_j)\}\). Then the bilinear mapping
\[
\mathcal{R} \otimes \mathcal{R} := \left[ (F, F') \in C^\infty(\mathcal{G}, \mathcal{A}) \times C^\infty(\mathcal{G}, \mathcal{A}) \mapsto \left[ (x, y) \in \mathcal{G} \times \mathcal{G} \mapsto (R_x^* F)(R_y^* F') := \left[ g \in \mathcal{G} \mapsto F(gx)F'(gy) \right] \in \mathcal{A} \right] \right] \in C^\infty(\mathcal{G} \times \mathcal{G}, C^\infty(\mathcal{G}, \mathcal{A})) ,
\]
is jointly continuous from \(\mathcal{B}(\mu_j)(G, \mathcal{A}) \times \mathcal{B}(\mu'_j)(G, \mathcal{A})\) to \(\mathcal{B}(\mu_j \otimes \mu'_j, \phi^k)(G \times \mathcal{G}, \mathcal{B}(L_j, L'_j)(G, \mathcal{A}))\), where \(\phi\) is the modular weight (see Example 2.23) of \(G \times \mathcal{G}\).

More precisely, labeling by \((j, k) \in \mathbb{N}^2\) the semi-norm \(\| \cdot \|_{j, k, \mu_j \otimes \mu'_j, \phi^k, \infty}\) of \(\mathcal{B}(\mu_j \otimes \mu'_j, \phi^k)(G, \mathcal{A})\), for all \((j, k, k')\) in \(\mathbb{N}^3\), there exists \(C > 0\) such that for all \(F \in \mathcal{B}(\mu_j)(G, \mathcal{A})\), \(F' \in \mathcal{B}(\mu'_j)(G, \mathcal{A})\), we have
\[
\| \mathcal{R} \otimes \mathcal{R}(F, F') \|_{(j, k), k', \mu_j \otimes \mu'_j, \phi^k, \infty} \leq C \| F \|_{j, k, \mu_j, \infty} \| F' \|_{j, k, k', \mu'_j, \infty} .
\]

Theorem 2.38 Let \((\mathcal{G} \times \mathcal{G}, \mathcal{S})\) be an admissible and tame tempered pair. Let also \(\mathbf{m} \in \mathcal{B}^\mu(G \times \mathcal{G}, \mathcal{C})\) for some tempered weight \(\mu\) on \(\mathcal{G} \times \mathcal{G}\) and \(\{\mu_j\}\), \(\{\mu'_j\}\) be two families of weights on \(\mathcal{G}\) with sub-multiplicative degrees respectively denoted by \(\{(L_j, R_j)\}\) and \(\{(L'_j, R'_j)\}\), such that the weights \(\mu_j \otimes \mu'_j, j \in \mathbb{N}\), are tempered on \(\mathcal{G} \times \mathcal{G}\). Then, for any Fréchet algebra \((\mathcal{A}, \{\|\cdot\|_j\}_{j \in \mathbb{N}})\), the oscillatory integral
\[
\ast_S := \left[ (F, F') \mapsto \int_{\mathcal{G} \times \mathcal{G}} \mathbf{m}_E \circ \mathcal{R} \otimes \mathcal{R}(F, F') \right] ,
\]
defines a jointly continuous bilinear map from \(\mathcal{B}(\mu_j)(G, \mathcal{A}) \times \mathcal{B}(\mu'_j)(G, \mathcal{A})\) to \(\mathcal{B}(\mu_j \otimes \mu'_j, \phi^k)(G \times \mathcal{G}, \mathcal{A})\). More precisely, for any \((j, k) \in \mathbb{N}^2\) there exist \(C > 0\) and \(l \in \mathbb{N}\) such that for any \(F \in \mathcal{B}(\mu_j)(G, \mathcal{A})\) and \(F' \in \mathcal{B}(\mu'_j)(G, \mathcal{A})\), we have
\[
\| F \ast_S F' \|_{j, k, l, \mu_j \otimes \mu'_j, \phi^k, \infty} \leq C \| F \|_{j, l, \mu_j, \infty} \| F' \|_{j, l, \mu'_j, \infty} .
\]

In particular, one has a continuous bilinear product (not necessarily associative!):
\[
\ast_S : \mathcal{B}(G, \mathcal{A}) \times \mathcal{B}(G, \mathcal{A}) \rightarrow \mathcal{B}(G, \mathcal{A}) .
\]

Proof. By Lemma 2.37 the map
\[
\mathcal{R} \otimes \mathcal{R} : \mathcal{B}(\mu_j)(G, \mathcal{A}) \times \mathcal{B}(\mu'_j)(G, \mathcal{A}) \rightarrow \mathcal{B}(\mu_j \otimes \mu'_j, \phi^k)(G \times \mathcal{G}, \mathcal{A}) ,
\]
is a jointly continuous bilinear mapping. By tameness, for every index \((j, k)\), the tempered weight \(\mu_j \otimes \mu'_j, (R_j \otimes \phi^k, \infty)\) is dominated. Hence the oscillatory integral composed with \(\mathcal{R} \otimes \mathcal{R}\) is well defined as a jointly continuous bilinear mapping. The precise estimate follows by putting together Lemma 2.37 Proposition 2.24 and Proposition 2.31.

We now discuss some issues regarding associativity of the bilinear mapping \(\ast_S\). To this aim, we need to show how to compute the product \(F \ast_S F'\) as the limit of a double sequence of products of smooth compactly supported functions.

Lemma 2.39 Within the context of Theorem 2.38, for \(F \in \mathcal{B}(\mu_j)(G, \mathcal{A})\) and \(F' \in \mathcal{B}(\mu'_j)(G, \mathcal{A})\), we let \(\{F_n\}\), \(\{F'_n\}\) be two sequences in \(\mathcal{D}(G, \mathcal{A})\) converging respectively to \(F\) and \(F'\) for the topologies of \(\mathcal{B}(\mu_j)(G, \mathcal{A})\) and \(\mathcal{B}(\mu'_j)(G, \mathcal{A})\) with \(\{\mu_j\}\) and \(\{\mu'_j\}\), any families of weights on \(G\) which dominate \(\{\mu_j\}\) and \(\{\mu'_j\}\) respectively.

Then we have in \(\mathcal{B}(\mu_j \otimes \mu'_j, \phi^k)(G, \mathcal{A})\):
\[
F \ast_S F' = \lim_{n \rightarrow \infty} \lim_{n' \rightarrow \infty} F_n \ast_S F'_n = \lim_{n' \rightarrow \infty} \lim_{n \rightarrow \infty} F_n \ast_S F'_n .
\]
Proof. Note that the family \( \{ \mu_j^{R_j} \otimes \mu_j^{R_j'} \phi^k \}_{j,k} \) dominates the family \( \{ \mu_j^{R_j} \otimes \mu_j^{R_j'} \phi^k \}_{j,k} \) and consequently, we may view \( R \otimes R(F,F') \) as an element of

\[
B(\mu_j^{R_j} \otimes \mu_j^{R_j'} \phi^k)_{j,k} \in \mathbb{N}
\]

\( G \times G, B(\mu_j^{R_j} \mu_j^{R_j'})_{j,k} \in (G, A) \).

By the estimate of Theorem 2.38, we know that for all \( j, k \in \mathbb{N} \), there exists \( l \in \mathbb{N} \) such that

\[
\| F \ast_S F' \|_{j,k, \mu_j^{R_j}, \mu_j^{R_j'}} \leq C(k, l) \| F \|_{j,k, \mu_j, \mu_j} \| F' \|_{j,k, \mu_j, \mu_j}.
\]

One then concludes by writing \( F \ast_S F' - F_n \ast_S F'_n = (F - F_n) \ast_S F' + F_n \ast_S (F' - F'_n) \) and using that the semi-norm \( \| \cdot \|_{j,k, \mu_j, \mu_j} \) is dominated by \( \| \cdot \|_{j,k, \mu_j} \) and \( \| \cdot \|_{j,k, \mu_j} \) by \( \| \cdot \|_{j,k, \mu_j, \mu_j} \).

\[
\begin{aligned}
\text{Remark 2.40} & \quad \text{In other words, within the setting of Lemma 2.39 we have in } B(\mu_j^{L_j} \mu_j^{L_j'})_{j,k} \text{ in } (G, A): \\
F \ast_S F' &= \lim_{m,n \to \infty} \int_{G \times G} E(x, x') \, m(x, x') \, R^*_m(F_m) \, R^*_n(F'_n) \, d_G(x) \, d_G(x'),
\end{aligned}
\]

for suitable approximation sequences \( \{ F_n \}, \{ F'_n \} \subset D(G, A) \).

**Definition 2.41** Within the context of Theorem 2.38, we say that the product \( \ast_S \), given in (33), is weakly associative when for all \( v_1, v_2, v_3 \in D(G, A) \), one has \( (v_1 \ast_S v_2) \ast_S v_3 = v_1 \ast_S (v_2 \ast_S v_3) \) in \( B(G, A) \).

**Proposition 2.42** Within the context of Theorem 2.38, weak associativity implies strong associativity in the sense that, when weakly associative, for every further family of tempered weights \( \{ \mu_j'' \} \) with sub-multiplicative degrees denoted by \( \{ (L_j^2, R_j^2) \} \) and every element \( (F, F', F'') \in B(\mu_j)(G, A) \times B(\mu_j')(G, A) \times B(\mu_j'')(G, A) \), one has the equality \( (F \ast_S F') \ast_S F'' = F \ast_S (F' \ast_S F'') \) in \( B(\mu_j^2 \mu_j^{L_j^2} \mu_j^{L_j^2})(G, A) \).

**Proof.** Let \( \mu \) a tempered weight on \( G \times G \) which dominates the constant weight 1 (it exists by assumption of tameness.) Consider the element \( \mu \in C^\infty(G) \) defined by \( \nu_\mu : (g, e) \mapsto \mu(\mu, e) \). The latter is then a tempered weight on \( G \) that dominates 1. Hence, all the weights \( \mu_j, \mu_j' \) and \( \mu_j'' \) are dominated e.g. by \( \nu_j : = \nu_\mu \mu_j, \nu_j' : = \nu_\mu \mu_j' \) and \( \nu_j'' : = \nu_\mu \mu_j'' \) respectively.

Let us consider sequences of smooth compactly supported elements \( \{ \Phi_n \}_{n \in \mathbb{N}}, \{ \Phi_n' \}_{n \in \mathbb{N}} \) and \( \{ \Phi_n'' \}_{n' \in \mathbb{N}} \) that converge to the elements \( F, F' \) and \( F'' \) respectively in \( B(\mu_j)(G, A), B(\mu_j')(G, A) \) and \( B(\mu_j'')(G, A) \).

Using separate continuity of \( \ast_S \) and Lemma 2.39 observe the following equality:

\[
\lim_{n \to \infty} \left( \lim_{n' \to \infty} \left( \lim_{n'' \to \infty} \| (\Phi_n \ast_S \Phi_n') \ast_S \Phi_n'' \| \right) \right) = (F \ast_S F') \ast_S F'',
\]

in \( B(\mu_j^2 \mu_j^{L_j^2} \mu_j^{L_j^2})(G, A) \). One then concludes using weak associativity and the commutativity of the limits, as shown in Lemma 2.39.

In subsection 2.20 we have seen how to associate in a canonical way a Schwartz type functions space to a tempered, admissible and tame pair. Hence, starting with such a pair \( (G \times G, S) \), we get a Schwartz space on \( G \times G \). We can also define a one-variable Schwartz space using the continuity of the partial evaluation maps:

**Definition 2.43** Let \( (G \times G, S) \) a tempered admissible and tame pair and \( A \) be a Fréchet algebra. We define the \( A \)-valued Schwartz space on \( G \) associated to \( S \) by

\[
S^S(G, A) := \left\{ g \in G \mapsto f(g, e) \mid f \in S^S(G \times G, A) \right\}.
\]

We endow the latter with the topology induced by the semi-norms:

\[
\| f \|_{j,k,n,\infty} := \sup_{X} \sup_{x \in \mu_\phi} \sup_{e \in G} \frac{\mu_\phi(x)^n \|X f(x)\|_k}{|X|^k}, \quad j, k, n \in \mathbb{N},
\]

with \( \mu_\phi(x) := \mu_\phi(x, e) \) and \( \mu_\phi \) the tempered weight on \( G \times G \) associated with the tameness (Definition 2.29).
The next Lemma shows that the right action on the space of $\mathcal{E}$-valued Schwartz functions, leads us to a $B$-type space for families too.

**Lemma 2.44** Let $(G \times G, S)$ be a tame and admissible tempered pair, $(A, \{\|\cdot\|_j\}_{j \in \mathbb{N}})$ be a Fréchet algebra and $\{\mu_j\}$ be a family of tempered weights with sub-multiplicative degrees denoted by $\{\{(L_j, R_j)\}\}$. Then, for all elements $F \in B(\mu_j(\nu_j)) (G, A)$ and $\varphi \in S^S(G, A)$, the element $(R \otimes R)(F, \varphi)$ (defined in Lemma 2.44) belongs to $B(\nu_j,\kappa,\mu) (G \times G, S^S(G, A))$ with

$$
\mu_{j,k,n} := (\mu_{j}^{n} \otimes \mu_{\phi,1}^{P(n)})^{2k}, \quad j, k, n \in \mathbb{N},
$$

where $\mu_{\phi,1}$ is given in Definition 2.43, $\phi$ is the modular weight of $G \times G$, $P$ is a certain polynomial and $(j, k, n)$ is the labeling of the semi-norms in $S^S(G, A)$.

**Proof.** Using Sweedler’s notation, we have for $X \in \mathcal{U}_k(g)$, $Y_1 \in \mathcal{U}_{k_1}(g)$, $Y_2 \in \mathcal{U}_{k_2}(g)$:

$$
\tilde{X}_g : \left( (\tilde{Y}_1 \otimes \tilde{Y}_2)(x, y), (R_g^* F(g) R_g^* \varphi(g)) \right) = \sum_{(X)} \left( (\Lambda d_{x-1} X_1) \tilde{Y}_1 F(x) \left((\Lambda d_{y-1} X_2) \tilde{Y}_2 \varphi(y)\right) \right),
$$

which yields the following estimation for arbitrary $n \in \mathbb{N}$:

$$
\|\tilde{X}_g : \left( (\tilde{Y}_1 \otimes \tilde{Y}_2)(x, y), (R_g^* F(g) R_g^* \varphi(g)) \right)\|_j \leq \sum_{(X)} |X_{(1)}| |X_{(2)}| |\Lambda d_{x-1} - |\Lambda d_{y-1} - |X_{(1)}|, |X_{(2)}| \sup_{Z_1 \in \mathcal{U}_{k_1}(g)} \|\tilde{Z}_1 F(gx)\|_j \sup_{Z_2 \in \mathcal{U}_{k_2}(g)} \|\tilde{Z}_2 \varphi(gy)\|_j \leq \sum_{(X)} |X_{(1)}| |X_{(2)}| |\Lambda d_{x-1} - |\Lambda d_{y-1} - |X_{(1)}|, |X_{(2)}| \mu_j(gx) \mu_{\phi,1}^{-n}(gy) \|F\|_{j,k+k_1,\mu_1,\infty} \|\varphi\|_{j,k+k_2,\mu_2,\infty},
$$

which by Lemma 2.43 and the estimate 1 is bounded by a constant times

$$
|X|_{j,k+k_1,\mu_1,\infty} \|\varphi\|_{j,k+k_2,\mu_2,\infty}. \quad (37)
$$

Setting $(L, R)$ for the sub-multiplicative degree of $\mu_{\phi}$ and using

$$
\mu_{\phi,1}^{-1}(g) \leq \mu_{\phi,1}^{-1/R}(g) \mu_{\phi,1}^{L/R}(y^{-1}), \quad y, g \in G,
$$

we see that 37 is bounded by

$$
|X|_{j,k+k_1,\mu_1,\infty} \|\varphi\|_{j,k+k_2,\mu_2,\infty}.
$$

So given $N \in \mathbb{N}$, it suffices to chose $n \in \mathbb{N}$ such that $\mu_j^{L_j} \mu_{\phi,1}^{-n/R} \leq \mu_{\phi,1}^{-N}$ and the polynomial $P$ such that $\mu_{\phi,1}^{-nL/R} \leq \mu_{\phi,1}^{P(N)}$. The result follows immediately since $\mu_{\phi,1}$ is a tempered weight by Remark 2.2. \qed

We then deduce the following important consequence of Lemma 2.44.

**Proposition 2.45** Let $(G \times G, S)$ be a tame and admissible tempered pair, $A$ a Fréchet algebra, $\{\mu_j\}$ a family of tempered weights, such that the weights $\mu_j \otimes 1$ are tempered on $G \times G$ and $m \in B^{\mu}(G \times G, \mathbb{C})$ for some tempered weight $\mu$ on $G \times G$. Then the bilinear map $*_{S}$, defined in (35), is jointly continuous on $S^S(G, A)$ and one has the jointly continuous bi-linear map:

$$
*_{S} : B^{(\mu_j)}(G, A) \times S^S(G, A) \rightarrow S^S(G, A) : (F, \varphi) \mapsto L_{*_{S}}(F) : \varphi \mapsto F *_{S} \varphi.
$$

**Remark 2.46** In the context of the proposition above, observe that the restriction to $S^S(G, A) \times S^S(G, A)$ of the bilinear product $*_{S}$ is the (point-wise and semi-norm-wise) absolutely convergent expression:

$$
\varphi_1 *_{S} \varphi_2 = \int_{G \times G} m(x_1, x_2) E(x_1, x_2) R^*_x (\varphi_1) R^*_y (\varphi_2) dG(x_2) dG(x_1).
$$
3 Tempered pairs for Kählerian Lie groups

The aim of this section is to endow each negatively curved Kählerian Lie group with the structure of a tempered, tame and admissible pair. Recall that a Lie group $G$ is called a Kählerian Lie group when it is endowed with an invariant Kähler structure i.e. a left-invariant complex structure $\mathbf{J}$ together with a left-invariant Riemannian metric $\mathbf{g}$ such that the triple $(G, \mathbf{J}, \mathbf{g})$ constitutes a Kähler manifold. Within the present work, we will be concerned with Kählerian Lie groups whose sectional curvature is negative. We call them negatively curved.

3.1 Pyatetskii-Shapiro’s theory

The following definition, due to Pyatetskii-Shapiro [17], describes the infinitesimal structure of negatively curved Kählerian Lie groups.

Definition 3.1 A normal $\mathbf{j}$-algebra is a triple $(\mathfrak{b}, \alpha, \mathbf{j})$ where

(i) $\mathfrak{b}$ is a solvable Lie algebra which is split over the reals, i.e. $\text{ad}_X$ has only real eigenvalues for all $X \in \mathfrak{b}$,

(ii) $\mathbf{j}$ is an endomorphism of $\mathfrak{b}$ such that $\mathbf{j}^2 = -1$ and

\[ [X, Y] + \mathbf{j}[X, Y] + \mathbf{j}[X, \mathbf{j}Y] - [\mathbf{j}X, \mathbf{j}Y] = 0, \quad X, Y \in \mathfrak{b}, \]

(iii) $\alpha$ is a linear form on $\mathfrak{b}$ such that

\[ \alpha([\mathbf{j}X, X]) > 0 \quad \text{if} \quad X \neq 0 \quad \text{and} \quad \alpha([\mathbf{j}X, \mathbf{j}Y]) = \alpha([X, Y]), \quad X, Y \in \mathfrak{b}. \]

We quote the following structure result from [17].

Proposition 3.2 The Lie algebra of a negatively curved Kählerian Lie group always carries a structure of normal $\mathbf{j}$-algebra.

If $\mathfrak{b}'$ is a sub-algebra of $\mathfrak{b}$ which is invariant by $\mathbf{j}$, then $(\mathfrak{b}', \alpha|_{\mathfrak{b}'}, \mathbf{j}|_{\mathfrak{b}'})$ is again a normal $\mathbf{j}$-algebra, called a $\mathbf{j}$-sub-algebra of $(\mathfrak{b}, \alpha, \mathbf{j})$. A $\mathbf{j}$-sub-algebra whose underlying Lie algebra $\mathfrak{b}'$ is an ideal of $\mathfrak{b}$ is called a $\mathbf{j}$-ideal.

Example 3.3 Every Iwasawa factor $AN$ of the simple Lie group $SU(1, n)$ is naturally a negatively curved Kählerian Lie group. Indeed, denoting by $K \simeq U(n)$ a maximal compact sub-group of $SU(1, n)$, one knows that the associated symmetric space $G/K$ is a negatively curved Kählerian $SU(1, n)$-manifold. The associated Iwasawa decomposition $SU(1, n) = ANK$ then yields a global diffeomorphism between $G/K$ and $AN$. Transporting to $AN$ the Kähler structure on $G/K$ under the latter diffeomorphism, then endows $AN$ with a negatively curved Kählerian Lie group structure, called elementary after Pyatetskii-Shapiro.

The infinitesimal structure underlying an elementary normal $\mathbf{j}$-group (cf. the above Example 3.3) may be precisely described as follows. Let $(V, \omega^0)$ be a symplectic vector space of real dimension $2d$. We consider the associated Heisenberg Lie algebra $\mathfrak{h} := V \oplus \mathbb{R}E$. That is, $\mathfrak{h}$ is the central extension of the Abelian Lie algebra $V$, with brackets given by

\[ [v_1, v_2] := \omega^0(v_1, v_2) E, \quad v_1, v_2 \in V, \quad [E, X] := 0, \quad X \in \mathfrak{h}. \]

Definition 3.4 Setting $\mathfrak{a} := \mathbb{R}H$, we consider the split extension of Lie algebras:

\[ 0 \to \mathfrak{h} \to \mathfrak{s} := \mathfrak{a} \times_{\rho_\mathfrak{h}} \mathfrak{h} \to \mathfrak{a} \to 0, \]

with extension homomorphism $\rho_\mathfrak{h} : \mathfrak{a} \to \text{Der}(\mathfrak{h})$ given by

\[ \rho_\mathfrak{h}(H)(v + t E) := [H, v + t E] := v + 2t E, \quad v \in V, \quad t \in \mathbb{R}. \quad (38) \]

The Lie algebra $\mathfrak{s}$ is called elementary normal. Last, we denote by $\mathcal{S}$ the connected simply connected Lie group whose Lie algebra is $\mathfrak{s}$ and we call the later an elementary normal $\mathbf{j}$-group.

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Note that $S$ is a solvable group of real dimension $2d + 2$ and if $V = \{0\}$, $S$ is isomorphic to the affine group of the real line.

It turns out that every negatively curved Kählerian Lie group can be decomposed into elementary pieces: at the infinitesimal level, one has the following result, due to Pyatetskii-Shapiro [17].

**Proposition 3.5** Let $(\mathfrak{b},\alpha,\mathfrak{j})$ be a normal $\mathfrak{j}$-algebra. Then, there exist $\mathfrak{z}$, a one-dimensional ideal of $\mathfrak{b}$ and $V$, a vector subspace of $\mathfrak{b}$, such that setting $\mathfrak{a} := \mathfrak{j} \oplus V \oplus \mathfrak{z}$ underlies an elementary normal $\mathfrak{j}$-ideal of $\mathfrak{b}$. Moreover, the associated extension sequence

$$0 \rightarrow \mathfrak{s} \rightarrow \mathfrak{b} \rightarrow \mathfrak{b}' \rightarrow 0,$$

is split as a sequence of normal $\mathfrak{j}$-algebras and such that:

$$[\mathfrak{b}', \mathfrak{a} \oplus \mathfrak{j}] = 0 \quad \text{and} \quad [\mathfrak{b}', V] \subset V. \quad (39)$$

In particular, every normal $\mathfrak{j}$-algebra $\mathfrak{b}$ admits a decomposition as a sequence of split extensions of elementary normal $\mathfrak{j}$-algebras $\mathfrak{s}_i$, $i = 1, \ldots, N$, of real dimension $2d_i + 2$, $d_i \in \mathbb{N}$:

$$\left(\cdots (\mathfrak{s}_N \ltimes \mathfrak{s}_{N-1}) \ltimes \cdots \ltimes \mathfrak{s}_2) \ltimes \mathfrak{s}_1, \right.$$

such that for all $i = 1, \ldots, N - 1$

$$\left((\mathfrak{s}_N \times \ldots) \ltimes \mathfrak{s}_{i+1}, \mathfrak{a}_i \oplus \mathfrak{j}\right] = 0 \quad \text{and} \quad \left((\mathfrak{s}_N \times \ldots) \ltimes \mathfrak{s}_{i+1}, V_i\right] \subset V_i.$$

**Definition 3.6** A **normal $\mathfrak{j}$-group** $\mathbb{B}$, consists in a connected simply connected Lie group that admits a normal $\mathfrak{j}$-algebra as Lie algebra, i.e. $\mathbb{B} = \exp(\mathfrak{b})$, where $\mathfrak{b}$ is a normal $\mathfrak{j}$-algebra.

At the group level, for $i = 1, \ldots, N - 1$, call $\mathbf{R}^i$ the extension homomorphism at each step:

$$\mathbf{R}^i \in \text{Hom}(\mathfrak{S}_N \times \ldots \times \mathfrak{S}_{i+1}, \text{Aut}(\mathfrak{S}_i)). \quad (40)$$

The conditions given in (39) implies that $\mathbf{R}^i$ takes values in $\text{Sp}(\mathfrak{V}_i, \omega_0^i)$, where $(\mathfrak{V}_i, \omega_0^i)$ denotes the symplectic vector space attached to $\mathfrak{S}_i$.

### 3.2 Geometric structures on elementary normal $\mathfrak{j}$-groups

In this subsection, we review the properties of a symplectic symmetric space structure every elementary normal $\mathfrak{j}$-group is naturally endowed with. The phase function with respect to which an admissible tempered pair will be associated to later on, was defined in [3] in terms of this symplectic symmetric space structure. We start with the definition of a symplectic symmetric space as in [1], which is an adaptation to the symplectic case of the notion of symmetric space as introduced by O. Loos [16].

**Definition 3.7** A **symplectic symmetric space** is a triple $(M, s, \omega)$ where

(i) $M$ is a connected smooth manifold,

(ii) $s$ is a smooth map

$$s : M \times M \rightarrow M, \quad (x,y) \mapsto s_x(y) := s(x, y),$$

such that:

(ii.1) For every $x \in M$, the partial map $s_x : M \rightarrow M$ is an involutive diffeomorphism admitting $x$ as isolated fixed point. The diffeomorphism $s_x$ is called the **symmetry** at point $x$.

(ii.2) For all points $x$ and $y$ in $M$, the following relation holds:

$$s_x \circ s_y \circ s_x = s_{s_x(y)}. \quad (31)$$
(iii) $\omega$ is a non-degenerate differential two-form on $M$ that is invariant under the symmetries:

$$s_x^* \omega = \omega, \quad \forall x \in M.$$ 

A morphism between two symplectic symmetric spaces is defined as a symplectomorphism that intertwines the symmetries.

The following result, attracted from [1], explains why a symplectic symmetric space is indeed a symplectic manifold:

**Proposition 3.8** Let $X$ and $Y$ be smooth tangent vector fields on $M$. Then the formula:

$$\left( \nabla_X Y \right)_x := \frac{1}{2} [X, Y + s_x^* Y], \quad x \in M,$$

defines an affine connection on $M$ called the canonical connection. It is the unique affine connection on $M$ which is invariant under the symmetries. Moreover, $\nabla$ is torsion-free and its Riemann curvature tensor field is parallel. Last, the two-form $\omega$ is parallel as well:

$$\nabla \omega = 0.$$ 

In particular, the two-form $\omega$ is automatically closed, hence symplectic.

Symplectic symmetric spaces always carry a preferred Lie group of transformations [1]:

**Proposition 3.9** The automorphism group of $(M, s, \omega)$:

$$\text{Aut}(M, s, \omega) := \text{Symp}(M, \omega) \cap \text{Aff}(\nabla),$$

constituted by the affine symplectic transformations of $M$ is a Lie group of transformation of $M$ that acts transitively on $M$. Its Lie algebra is called the derivation algebra of $(M, s, \omega)$ and is denoted by $\text{aut}(M, s, \omega)$.

We now pass to the particular case of a given $2d+2$-dimensional elementary normal $j$-group $S$ with associated symplectic form $\omega^S$. Let $a, t \in \mathbb{R}$ and $v \in V \cong \mathbb{R}^{2d}$. The following identification will always be understood:

$$\mathbb{R}^{2d+2} \rightarrow S, \quad x := (a, v, t) \mapsto \exp(aH) \exp(v) \exp(tE).$$

The following result is extracted from [7, 5, 3]:

**Proposition 3.10** Let $S$ be an elementary normal $j$-group.

(i) The map

$$s : (a, v, t) \mapsto \exp(aH) \exp(v) \exp(tE) = \exp(aH) \exp(v) \exp(tE),$$

is a global Darboux chart on $(S, \omega^S)$ in which the symplectic structure reads:

$$\omega^S := 2da \wedge dt + \omega^0.$$ 

(ii) Setting furthermore

$$s_{(a,v,t)}(a', v', t') := (2a - a', 2 \cosh(a - a')v - v', 2 \cosh(2a - 2a')t + \omega^0(v, v') \sinh(a - a') - t'),$$

defines a symplectic symmetric space structure $(S, s, \omega^S)$ on the elementary normal $j$-group $S$.

(iii) The left action $L_x : S \rightarrow S : x' \mapsto x.x'$ defines a injective Lie group homomorphism

$$L : S \rightarrow \text{Aut}(S, s, \omega^S).$$

In the coordinates [11], we have

$$x.x' = (a, v, t)(a', v', t') = (a + a', e^{-a'}v + v', e^{-2a'}t + t' + \frac{1}{2}e^{-a'}\omega^0(v, v')) ,$$

and

$$x^{-1} = (a, v, t)^{-1} = (-a, -e^a v, -e^{2a} t).$$
(iv) The action $R : \text{Sp}(V,\omega^0) \times S \to S$, $(A, (a, v, t)) \mapsto R_A(a, v, t) := (a, Av, t)$ by automorphisms of the normal $j$-group $S$ induces an injective Lie group homomorphism:

$$R : \text{Sp}(V,\omega^0) \to \text{Aut}(S, s, \omega^S), \quad A \mapsto R_A.$$ 

In fact, $\text{Sp}(V,\omega^0) \simeq \text{Aut}(S) \cap \text{Aut}(S, s, \omega^S)$.

Note that in the coordinates (41), the modular function of $S$, $\Delta_S$, reads $e^{(2d+2)a}$.

We now pass to the definition of the three-point phase on $S$. For this we need the notion of “double geodesic triangle” as introduced by A. Weinstein [27] and Z. Qian [28].

**Definition 3.11** Let $(M, s)$ be a symmetric space. A midpoint map on $M$ is a smooth map $M \times M \to M$, $(x, y) \mapsto \text{mid}(x, y)$, such that, for all points $x, y$ in $M$:

$$s_{\text{mid}(x, y)}(x) = y.$$ 

**Remark 3.12** Observe that in the case where the partial maps $s^y : M \to M, x \mapsto s_x(y)$ are global diffeomorphisms of $M$, a midpoint map exists and is given by:

$$\text{mid}(x, y) := (s^x)^{-1}(y).$$ 

Note that in this case, every $\varphi \in \text{Aut}(M, s)$ intertwines the midpoints. Indeed, since for all $x, y \in M$ we have $\varphi(s_y(x)) = s_{\varphi(y)}(\varphi(x))$, we get

$$\varphi(\text{mid}(x, y)) = \text{mid}(\varphi(x), \varphi(y)).$$ 

An immediate computation shows that a midpoint map always exists on the symplectic symmetric space attached to an elementary normal $j$-group:

**Lemma 3.13** For the symmetric space $(\mathbb{S}, s)$ underlying an elementary normal $j$-group, the associated partial maps are global diffeomorphisms. In the coordinates (41), we have:

$$(s^{(a_0, v_0, t_0)})^{-1} : (a, v, t) \mapsto \left(\frac{a + a_0}{2}, \frac{v + v_0}{2 \cosh(\frac{a - a_0}{2})}, \frac{t + t_0}{2 \cosh(a - a_0)} - \omega^0(v, v_0)\frac{\sinh(\frac{a - a_0}{2})}{4 \cosh(a - a_0) \cosh(\frac{a_0}{2})}\right).$$ 

The following statement is proven in [3].

**Proposition 3.14** Let $\mathbb{S}$ be an elementary normal $j$-group.

(i) The affine space $(\mathbb{S}, \nabla)$ is strictly geodesically complete, i.e. two points determine a unique geodesic arc.

(ii) The “double triangle” three-point function

$$\Phi : \mathbb{S}^3 \to \mathbb{S}^3, \quad (x_1, x_2, x_3) \mapsto (\text{mid}(x_1, x_2), \text{mid}(x_2, x_3), \text{mid}(x_3, x_1)),$$

is a $\mathbb{S}$-equivariant (under the left regular action) global diffeomorphism.

Since our space $\mathbb{S}$ has trivial de Rham cohomology in degree two, any three points $(x, y, z) \in \mathbb{S}^3$ define an oriented geodesic triangle $T(x, y, z)$ whose symplectic area is well-defined by integrating the two-form $\omega^\mathbb{S}$ on any surface admitting $T(x, y, z)$ as boundary. With a slight abuse of notation, we set

$$\text{Area}(x, y, z) := \int_{T(x, y, z)} \omega^\mathbb{S}.$$ 

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Definition 3.15 The canonical two-point phase associated to an elementary normal j-group is defined by
\[ S^\text{can}_\text{S}(x_1, x_2) := \text{Area} \left( \Phi^{-1}(e, x_1, x_2) \right) \in C^\infty(S^2, \mathbb{R}) , \]
where \( e := (0, 0, 0) \) denotes the unit element in \( \text{S} \). In the coordinates \( (4.1) \), one has the explicit expression:
\[ S^\text{can}_\text{S}(x_1, x_2) = \sinh(2a_1)t_2 - \sinh(2a_2)t_1 + \cosh(a_1) \cosh(a_2) \omega^0(v_1, v_2) . \]  

The canonical two-point amplitude associated to an elementary normal j-group is defined by
\[ A^\text{can}_\text{S}(x_1, x_2) := \text{Jac}_{\Phi^{-1}}(e, x_1, x_2)^{1/2} \in C^\infty(S^2, \mathbb{R}) . \]

In the coordinates \( (4.1) \), it reads
\[ A^\text{can}_\text{S}(x_1, x_2) = (\cosh(a_1) \cosh(a_2) \cosh(a_1 - a_2))^d \left( \cosh(2a_2) \cosh(2a_2) \cosh(2a_1 - 2a_2) \right)^{1/2} . \]  

3.3 Tempered pair for elementary normal j-groups

The aim of this technical subsection is to prove that the pair \((\text{S} \times \text{S}, S^\text{can}_\text{S})\) is tempered, admissible and tame. We start by splitting the 2d-dimensional symplectic vector space \((V, \omega^0)\) associated to an elementary normal j-group \( \text{S} \) into a direct sum of two Lagrangian subspaces in symplectic duality:
\[ V = \mathfrak{l}^* \oplus \mathfrak{l} , \]
and for every \( v := (x, y) \in \mathfrak{l}^* \oplus \mathfrak{l} \), we set \( xy := \omega^0(x, y) \). The following result establishes temperedness.

Lemma 3.16 The pair \((\text{S} \times \text{S}, S^\text{can}_\text{S})\) is tempered. Moreover, the Jacobian of the map
\[ \phi : \text{S} \times \text{S} \to (\mathfrak{s} \oplus \mathfrak{s})^*, \quad g \rightarrow [X \in \mathfrak{s} \oplus \mathfrak{s} \mapsto \left( \widetilde{X}, S^\text{can}_\text{S}(g) \right)] , \]
is proportional to \((A^\text{can}_\text{S})^2\).

Proof. Let us fix \( \{ f_j \}_{j=1}^d \), a basis of \( \mathfrak{l}^* \) to which we associate \( \{ e_j \}_{j=1}^d \) the symplectic-dual basis of \( \mathfrak{l} \), i.e. it is defined by \( \omega^0(f_i, e_j) = \delta_{i,j} \). We let \( E \) the central element of the Heisenberg Lie algebra \( \mathfrak{h} \subset \mathfrak{s} \) and \( H \) the generator of \( \mathfrak{a} \) in the one dimensional split extension which defines the Lie algebra \( \mathfrak{s} \):
\[ 0 \to \mathfrak{h} \to \mathfrak{s} \to \mathfrak{a} \to 0 , \]

Accordingly, we consider the following basis of \( \mathfrak{s} \oplus \mathfrak{s} \):
\[ \begin{aligned}
H_1 &:= H \oplus \{ 0 \} , & \quad H_2 &:= \{ 0 \} \oplus H , \\
f^1_j &:= f_j \oplus \{ 0 \} , & \quad f^2_j &:= \{ 0 \} \oplus f_j , \\
e^1_j &:= e_j \oplus \{ 0 \} , & \quad e^2_j &:= \{ 0 \} \oplus e_j , \\
E^1_1 &:= E \oplus \{ 0 \} , & \quad E^2_2 &:= \{ 0 \} \oplus E ,
\end{aligned} \]
where the index \( j \) runs from 1 to \( d = \dim(V)/2 \). From Proposition \( \text{3.10 \text{iii}} \) and with the notation \( v := (x, y) \in \mathfrak{l}^* \oplus \mathfrak{l} = V \), we see that the left-invariant vector fields on \( \text{S} \) are given by:
\[ \begin{aligned}
\tilde{H} &= \partial_a - \sum_{j=1}^d (x_j \partial_{x_j} + y_j \partial_{y_j}) - 2t \partial_t , \\
\tilde{f}_j &= \partial_{x_j} - \frac{\partial}{\partial t} \partial_{y_j} , \\
\tilde{e}_j &= \partial_{y_j} + \frac{1}{2} \partial_t , \\
\tilde{E} &= \partial_t .
\end{aligned} \]  

Thus, we find
\[ \begin{aligned}
\tilde{H}_1 S^\text{can}_\text{S} &= 2 \cosh(2a_1)t_2 + 2 \sinh(2a_2)t_1 - e^{-a_1} \cosh(a_2) \omega^0(v_1, v_2) , & \tilde{E}_1 S^\text{can}_\text{S} &= - \sinh(2a_2) , \tag{46} \\
\tilde{H}_2 S^\text{can}_\text{S} &= -2 \cosh(2a_2)t_1 - 2 \sinh(2a_1)t_2 - e^{-a_2} \cosh(a_1) \omega^0(v_1, v_2) , & \tilde{E}_2 S^\text{can}_\text{S} &= \sinh(2a_1) , \\
\tilde{f}_j S^\text{can}_\text{S} &= \cosh(a_1) \cosh(a_2)y_j^2 + \frac{1}{2} \sinh(2a_2)y_j^2 , & \tilde{f}^2_j S^\text{can}_\text{S} &= - \cosh(a_1) \cosh(a_2)y_j^2 - \frac{1}{2} \sinh(2a_1)y_j^2 , \\
\tilde{e}^1_j S^\text{can}_\text{S} &= - \cosh(a_1) \cosh(a_2)x_j^2 - \frac{1}{2} \sinh(2a_2)x_j^2 , & \tilde{e}^2_j S^\text{can}_\text{S} &= \cosh(a_1) \cosh(a_2)x_j^2 + \frac{1}{2} \sinh(2a_1)x_j^2 .
\end{aligned} \]
A computation then shows that the Jacobian of the map \( \phi : S \times S \to (s \oplus s)^* \), underlying Definition 2.17 is given by
\[
2^{2d+2}(\cosh a_1 \cosh a_2 \cosh(a_1 - a_2))^{2d} \cosh 2a_1 \cosh 2a_2 \cosh 2(a_1 - a_2) = 2^{2d+2} A_{\text{can}}(a_1, a_2)^2 \geq 2^{2d+2},
\]
and hence \( \phi \) is a global diffeomorphism. It is also clear from Proposition 8.10 iii), that the multiplication and inversion maps on \( S \times S \) are tempered function in the coordinates (16). Therefore, the pair \((S \times S, S^S_{\text{can}})\) is tempered.

**Remark 3.17** Note that the formal adjoints of the left invariant vector fields (15), with respect to the inner product of \( L^2(S, d\sigma) \) read:
\[
\tilde{H}^* = -\tilde{H} + 2d + 2, \quad f_j^* = -\tilde{f}_j, \quad e_j^* = -\tilde{e}_j, \quad E^* = -\tilde{E},
\]
so that the assumption (20) is trivially satisfied.

We will now prove that the tempered pair \((S \times S, S^S_{\text{can}})\) is admissible and tame. For this, we need a decomposition of the Lie algebra \( s \) and we shall use the following one:
\[
s = \bigoplus_{k=0}^3 V_k \quad \text{where} \quad V_0 := a, \quad V_1 := l^*, \quad V_2 := l \quad \text{and} \quad V_3 := RE.
\]
(47)

Note that both \( V_0 \) and \( V_3 \) are of dimension one, while \( V_1 \) and \( V_2 \) are \( d \)-dimensional. Accordingly, we consider the decompositions of \( s \oplus s \) given by
\[
s \oplus \{0\} = \bigoplus_{k=0}^3 V_{1,k} \quad \text{and} \quad \{0\} \oplus s = \bigoplus_{k=0}^3 V_{2,k},
\]
where the subspaces \( V_{i,k}, i = 1, 2, \) of each factor correspond respectively to the subspaces \( V_k \) of \( s \) within the decomposition (47). We then set:
\[
\mathfrak{U}_k := V_{1,k} \oplus V_{2,k} \quad \text{and} \quad s \oplus s = \bigoplus_{k=0}^3 \mathfrak{U}_k,
\]
(48)
by which we mean that there are four subspaces involved in the ordered decomposition of \( s \oplus s \). We also let
\[
\mathfrak{U}^{(k)} := \bigoplus_{n=0}^k \mathfrak{U}_n, \quad k = 0, 1, 2, 3,
\]
as in (19) and we let \( \mathcal{U}(\mathfrak{U}^{(k)}) \) be the unital sub-algebra of \( \mathcal{U}(s \oplus s) \) generated by \( \mathfrak{U}^{(k)} \) as in (7). Accordingly, we consider the associated tempered coordinates (16):
\[
x_{i,0} := \tilde{H}_i, S^S_{\text{can}}, \quad x_{i,1}^j := \tilde{f}_j^i S^S_{\text{can}}, \quad x_{i,2}^j := \tilde{e}_j^i S^S_{\text{can}}, \quad x_{i,3} := \tilde{E}_i S^S_{\text{can}}, \quad i = 1, 2, \quad j = 1, \ldots, d,
\]
and we use the vector notations:
\[
\tilde{x}_0 := (x_{1,0}, x_{2,0}) \in \mathbb{R}^2, \quad \tilde{x}_1 := (x_{1,1}, x_{2,1}) := ((x_{1,1})^j_{j=1}, (x_{2,1})^j_{j=1}) \in \mathbb{R}^{2d},
\]
\[
\tilde{x}_2 := (x_{1,2}, x_{2,2}) := ((x_{1,2})^j_{j=1}, (x_{2,2})^j_{j=1}) \in \mathbb{R}^{2d}, \quad \tilde{x}_3 := (x_{1,3}, x_{2,3}) \in \mathbb{R}^2.
\]
(49)
According to the notations \((a, v, t) \in \mathbb{R} \times \mathbb{R}^{2d} \times \mathbb{R} \simeq S\) and \( v = (x, y) \in l^* \oplus l = V \), we set
\[
\tilde{a} := (a_1, a_2) \in \mathbb{R}^2, \quad \tilde{x} = (x_1, x_2) \in \mathbb{R}^{2d}, \quad \tilde{y} = (y_1, y_2) \in \mathbb{R}^{2d}, \quad \tilde{t} := (t_1, t_2) \in \mathbb{R}^2.
\]
We consider the functions
\[
s_{12} := \sinh(2a_1)t_2 - \sinh(2a_2)t_1, \quad \Omega_{12} := \omega^0(v_1, v_2), \quad \gamma_{12} := \cosh(a_1) \cosh(a_2),
\]
in term of which we have
\[ S^\mathbb{S}_{\text{can}} = s_{12} + \gamma_{12} \Omega_{12}. \]

Introducing last
\[ A := \begin{pmatrix} \sinh(2a_2) & \cosh(2a_1) \\ -\cosh(2a_2) & -\sinh(2a_1) \end{pmatrix}, \quad B := \begin{pmatrix} -\frac{1}{2} \sinh(2a_2) & -\cosh(a_1) \cosh(a_2) \\ \cosh(a_1) \cosh(a_2) & \frac{1}{2} \sinh(2a_1) \end{pmatrix}, \quad (50) \]

\[ \tilde{\gamma} := (\cosh(a_2)e^{-a_1}, \cosh(a_1)e^{-a_2}), \quad \tilde{\delta} := (-\sinh(2a_2), \sinh(2a_1)), \]

the relations given in (46) can be summarized as:
\[ \tilde{x}_3 = \tilde{\delta}, \quad \tilde{x}_2 = B \tilde{x}, \quad \tilde{x}_1 = -B \tilde{y}, \quad \tilde{x}_0 = 2A \tilde{t} - \Omega_{12} \tilde{\gamma}. \quad (51) \]

We first treat the easiest variables \( \tilde{x}_3 \), which lead to multipliers \( \alpha_3 \) that satisfy property (ii) of Definition 2.19 with constant \( \mu_3 \).

Lemma 3.18 Consider an element \( X \in \mathcal{U}(\mathfrak{g}_3) \) such that the associated multiplier \( \alpha_X \) is invertible. Then, for every \( Y \in \mathcal{U}(\mathfrak{g}^{(3)}) = \mathcal{U}(\mathfrak{s} \oplus \mathfrak{s}) \) there exists a positive constant \( C_Y \) such that
\[ |\tilde{Y} \alpha_X| \leq C_Y |\alpha_X|. \]

Proof. Note first that \( \mathfrak{g}_3 \) turns out to be a two-dimensional Abelian Lie algebra. Note also that \( \alpha_{E_i}, i = 1, 2 \) is independent of the variables \( \tilde{t} \). Thus, given a two-variables polynomial \( P \), we have for \( X = P(E_1, E_2) \in \mathcal{U}(\mathfrak{g}_3) \):
\[ \alpha_X = P(-\sinh(2a_2), \sinh(2a_1)). \]

It also follows from the explicit expression of the left-invariant vector fields given in (46) that \( \tilde{Y} \alpha_X = 0 \) for all \( Y \in \mathcal{U}(\bigoplus_{k=1}^3 \mathfrak{g}_k) \). Hence, it suffices to treat the case of \( Y \in \mathcal{U}(\mathfrak{g}_0) \). Observe that the restriction of \( \tilde{H}_j \) to functions which depend only on \( a_j \), equals \( \partial_{a_j} \). Thus in this case, we see that \( \tilde{Y} \alpha_X \) is a polynomial of the same degree as \( P \), but in the variables \( e^{\pm a_1} \) and \( e^{\pm a_2} \). This is enough to conclude when \( \alpha_X \) is invertible. ■

Next, we treat the variables \( \tilde{x}_2 \) and \( \tilde{x}_1 \). We first observe

Lemma 3.19 There exist finitely many matrices \( B_{(r)} \in M_2(\mathbb{R}[e^{\pm a_1}, e^{\pm a_2}]) \) such that for all integers \( N_1 \) and \( N_2 \), the elements \( \tilde{H}_1^{N_1} \tilde{H}_2^{N_2} B \) consist in a linear combination of the \( B_{(r)} \)'s, where the matrix \( B \) has been defined in (50). The same property holds for the matrix \( A \).

Proof. Set
\[ D := \begin{pmatrix} -\frac{1}{2} \sinh(2a_2) & 0 \\ 0 & \frac{1}{2} \sinh(2a_1) \end{pmatrix}, \quad \Gamma := \gamma_{12} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \]

and observes that
\[ B = D + \Gamma \quad \text{and} \quad \partial_{a_i}^2 D = 4D, \quad \partial_{a_i}^2 \Gamma = \Gamma, \quad i = 1, 2. \]

The derivatives of \( B \) therefore all belong to the space generated by \( D, \Gamma \) and finitely many derivatives. This is enough to conclude since restricted to functions that depend only on the variable \( a \), we have \( H = \partial_a \). The proof for the matrix \( A \) is entirely similar. ■

We can now deduce what we need for the variables \( \tilde{x}_2 \) and \( \tilde{x}_1 \).

Lemma 3.20 There exist finitely many tempered functions \( m_{2,(r)} \) (respectively \( m_{1,(r)} \)) depending on the variables \( \tilde{x}_3 \) only, such that for every element \( X \in \mathcal{U}(\mathfrak{g}^{(3)}) \) (respectively \( X \in \mathcal{U}(\mathfrak{g}^{(1)}) \)), the element \( \tilde{X} \tilde{x}_2 \) (respectively \( \tilde{X} \tilde{x}_1 \)) belongs to the space spanned by \( \{m_{2,(r)}, m_{2,(r)} \tilde{x}_2\} \) (respectively \( \{m_{1,(r)}, m_{1,(r)} \tilde{x}_1\} \)).

Proof. This follows from Lemma 3.19 and the expressions (46) for the invariant vector fields. Indeed, the latter implies that for every \( X \in \mathcal{U}(\bigoplus_{k=1}^3 \mathfrak{g}_k) \) (respectively \( X \in \mathcal{U}(\mathfrak{g}_1) \)) of strictly positive homogeneous degree, \( \tilde{X} \tilde{x}_2 \) (respectively \( \tilde{X} \tilde{x}_1 \)) is either zero or one of the entries of the matrix \( B \). ■
Remark 3.21  Note that in view of the expressions [15] and [30] and by symmetry on $\vec{x}_1$ and $\vec{x}_2$ the assertion in Lemma [3.20] holds for every element $X$ in $\mathcal{U}(s \oplus s)$ for both variables $\vec{x}_1$ and $\vec{x}_2$.

Last, we go to the variables $\vec{x}_0$. The next Lemma is proven using the same type of arguments as in the proof of Lemma [3.19].

Lemma 3.22  There exist finitely many vectors $\gamma(\tau) \in \mathbb{R}^2[e^{\pm a_1}, e^{\pm a_2}]$ such that for all integers $N_1$ and $N_2$, the elements $\tilde{H}_1^{N_1} \tilde{H}_2^{N_2} \gamma$ consist in a linear combination of the $\gamma(\tau)$’s.

Observing that $\tilde{H}_i t_i$ is proportional to $t_i$ and that $\tilde{H}_i \Omega_{12} = -\Omega_{12}$, the Lemmas [3.19] and [3.22] then yield the following result.

Lemma 3.23  There exist finitely many matrices $M(\tau) \in M_2(\mathbb{R}[e^{\pm a_1}, e^{\pm a_2}])$ and finitely many vectors $v(\alpha) \in \mathbb{R}^2[e^{a_1}, a^{a_2}]$ such that for all integers $N_1$ and $N_2$, one has

$$\tilde{H}_1^{N_1} \tilde{H}_2^{N_2} \vec{x}_0 = M_{N_1, N_2} \vec{x}_0 + \Omega_{12} v_{N_1, N_2},$$

with $M_{N_1, N_2} \in \text{span}\{M(\tau)\}$ and $v_{N_1, N_2} \in \text{span}\{v(\alpha)\}$.

The following result is then a direct consequence of Lemmas [3.18], [3.20], and [3.22].

Corollary 3.24  For every $k = 0, \ldots, 3$, there exists a tempered function $0 < m_k$ with $\partial_{x_i} m_k = 0$ for every $j \leq k$ and such that for every $X \in \mathcal{U}(\mathbb{Q}(k))$, there exists $C_X > 0$ with

$$|X \vec{x}_k| \leq C_X m_k (1 + |\vec{x}_k|).$$

Remark 3.25  In fact the function $m_0$ above depends on $\vec{x}_1, \vec{x}_2, \vec{x}_3$ and the functions $m_1$ and $m_2$ depend on $\vec{x}_3$ only (and $m_3$ is constant as it should be).

We are now able to check the admissibility conditions of Definition [2.19] for the tempered pair $(\mathbb{S} \times \mathbb{S}, \mathbb{S}^0_{\text{can}})$.

Proposition 3.26  Define

$$X_0 := 1 - H_1^2 - H_2^2, \quad X_1 := 1 - \frac{d}{\sqrt{2}} \left((f_1^1)^2 + (f_2^1)^2\right), \quad X_2 := 1 - \frac{d}{\sqrt{2}} \left((f_1^2)^2 + (f_2^2)^2\right), \quad X_3 := 1 - E_1^2 - E_2^2.$$

Then the corresponding multipliers $\alpha_k := e^{-i \mathbb{S}^0_{\text{can}}} \tilde{X}_k e^{i \mathbb{S}^0_{\text{can}}}$, $k = 0, \ldots, 3$, satisfy conditions (i) and (ii) of Definition [2.19].

Proof.  We start by observing the following expression of the multiplier:

$$\alpha_k = 1 + |\vec{x}_k|^2 - i \beta_k, \quad k = 0, \ldots, 3,$$

where

$$\beta_k := \tilde{X}_{1,k} x_{1,k} + \tilde{X}_{2,k} x_{2,k},$$

with obvious notations. Then we get

$$\frac{1}{|\alpha_k|^2} = \frac{1}{(1 + |\vec{x}_k|^2)^2 + \beta_k^2} \leq \frac{1}{(1 + |\vec{x}_k|^2)^2},$$

and the first condition of Definition [2.19] is satisfied for $C_k = 1$ and $\rho_k = 2$. Let now $X \in \mathcal{U}(\mathbb{Q}(k))$ of strictly positive order. Then, using Sweedler’s [2], notations we get

$$\tilde{X} \alpha_k = \sum_{(x)} \left( \tilde{X}_1(x) \vec{x}_k \right) \cdot \left( \tilde{X}_2(x) \vec{x}_k \right) - i \tilde{X} \tilde{X}_{1,k} x_{1,k} - i \tilde{X} \tilde{X}_{2,k} x_{2,k}.$$

Since $X_1, X_2, X_{1,k}, X_{2,k} \in \mathcal{U}(\mathbb{Q}(k))$, Corollary [3.24] yields

$$|\tilde{X} \alpha_k| \leq C_1 m_k^2 (1 + |\vec{x}_k|)^2 + C_2 m_k (1 + |\vec{x}_k|).$$

As $1 + |\vec{x}_k|^2 \leq |\alpha_k|$, the second condition of Definition [2.19] is satisfied for $\mu_k = m_k (1 + m_k)$. 

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Last, we prove tameness for the pair \((S \times S, S_{\text{can}}^S)\). We start by a minoration of the modular weight of the group \(S\):

**Lemma 3.27** There exists \(C > 0\) such that for every \(g = (a, v, t) \in S\):

\[
C(\cosh(a) + |v| + |t|) \leq \mathfrak{d}(g).
\]

**Proof.** A quick computation gives:

\[
\text{Ad}_g(a'H + v' + t'E) = a'H + e^a(v' - va') + e^{2a}(t' + 2ta' + \frac{1}{2} \omega^0(v, v'))E,
\]

and

\[
\text{Ad}_{g^{-1}}(a'H + v' + t'E) = a'H + e^{-a}v' + va' + (e^{-2a}t' + 2ta' - \frac{1}{2} e^{-a} \omega^0(v, v'))E,
\]

and the estimate follows immediately. \(\blacksquare\)

**Corollary 3.28** The tempered pair \((S \times S, S_{\text{can}}^S)\) is tame.

**Proof.** Consider any two Lie groups \(G_1, G_2\) with Lie algebras \(g_1, g_2\) respectively. Fix Euclidean structures on \(g_1\) and on \(g_2\) and induce one on \(g_1 \oplus g_2\) by declaring that \(g_1\) is orthogonal to \(g_2\). Then for \(X_1 \oplus X_2 \in g_1 \oplus g_2\) and \((g_1, g_2) \in G_1 \times G_2\), we have:

\[
|\text{Ad}_{(g_1, g_2)}(X_1 \oplus X_2)|^2 = |\text{Ad}_{g_1}X_1 \oplus \text{Ad}_{g_2}X_2|^2 = |\text{Ad}_{g_1}X_1|^2 + |\text{Ad}_{g_2}X_2|^2.
\]

Hence we deduce

\[
|\text{Ad}_{(g_1, g_2)}| \geq \max\left\{ \sup_{|X_1|=1} |\text{Ad}_{g_1}X_1|, \sup_{|X_2|=1} |\text{Ad}_{g_2}X_2| \right\} = \max\left\{ |\text{Ad}_{g_1}|, |\text{Ad}_{g_2}| \right\} \geq \frac{1}{2}(|\text{Ad}_{g_1}| + |\text{Ad}_{g_2}|),
\]

from which we get

\[
\mathfrak{d}_{G_1 \times G_2} \geq \frac{1}{2} \left( \mathfrak{d}_{G_1} \otimes 1 + 1 \otimes \mathfrak{d}_{G_2} \right).
\]

Lemma 3.27 thereof implies that

\[
\mathfrak{d}_{S \times S}(a_1, v_1, t_1; a_2, v_2, t_2) \geq C \left( \cosh(a_1) + \cosh(a_2) + |v_1| + |v_2| + |t_1| + |t_2| \right),
\]

so that with the relations \(51\) in mind, we see that there exists \(C' > 0\) with

\[
(|\mathfrak{f}_0|^2 + |\mathfrak{f}_1|^2 + |\mathfrak{f}_2|^2 + |\mathfrak{f}_3|^2)^{1/2} \leq C' \mathfrak{d}_{S \times S}^4.
\]

According to Definition 2.29 we may set \(\mu_0 = \mathfrak{d}_{S \times S}^4\). Last, temperedness of the modular weight \(\mathfrak{d}\) has been proven in Lemma 2.16. \(\blacksquare\)

We summarize all this by stating the main result of this sub-section:

**Theorem 3.29** Let \(S\) be an elementary normal \(j\)-group and let \(S_{\text{can}}^S\) be the smooth function on \(S \times S\) given in Definition 3.19. Then, the pair \((S \times S, S_{\text{can}}^S)\) is tempered, admissible and tame.

**Remark 3.30** From Remark 3.21 and the above discussion, we observe that setting \(\mathfrak{g}_{12} := \mathfrak{g}_1 \oplus \mathfrak{g}_2\) yields a decomposition into three subspaces: \(s \oplus s = \mathfrak{g}_0 \oplus \mathfrak{g}_{12} \oplus \mathfrak{g}_3\) also underlying admissibility but with associated elements \(X_0, X_3\) and \(X_{12} := X_1 + X_2\). The corresponding multipliers are \(\alpha_0, \alpha_3\) and \(\alpha_{12} = \alpha_1 + \alpha_2\).

### 3.4 Tempered pairs for normal \(j\)-groups

Let \(b\) be a normal \(j\)-algebra, and \(B\) a connected simply connected Lie group with Lie algebra \(b\). We first observe:

**Lemma 3.31** Let \(B = B' \rtimes S_1\) be a Pyatetskii-Shapiro decomposition. Setting \(g_1, h_1 \in S_1, g', h' \in B'\) then we have:

\[
\mathfrak{d}_{S \times B}(g_1g', h_1h') \geq \frac{1}{4} \left( \mathfrak{d}_{S_1}(g_1) + \mathfrak{d}_{S_1}(h_1) + \mathfrak{d}_{B'}(g') + \mathfrak{d}_{B'}(h') \right).
\]
Proof. From the inequality
\[ 2d_{B \times B} \geq d_B \otimes 1 + 1 \otimes d_B, \]
proven in Corollary 4.23, it suffices to show
\[ 2d_B(g_1g') \geq d_{S_1}(g_1) + d_{B'}(g') , \]
for \( g_1 \in S_1, \ g' \in B' \). Set also \( X_1 = (a, v, t) \in s_1 \) and \( X' \in B' \), we observe from Proposition 4.3 that
\[ Ad_{g_1}(X') = X', \]
and
\[ Ad_{g'}(X_1) = (a, A(g') v, t), \]
where \( A(g') \) is the matrix in \( Sp(V_1, \omega_1^0) \) as defined in Proposition 4.3.10 (iv). Fix an Euclidean structure on \( V_j \) and define (inductively) another one on \( b \) by \( \langle (a, v, t), (a', v', t') \rangle = aa' + \langle v, v' \rangle + tt' \) on \( s_1 \) and by declaring that \( s_1 \) and \( b' \) are orthogonal. Hence we get \( |X' + X_1|^2 = |X'|^2 + |X_1|^2 \) for all \( X' \in B' \) and \( X_1 \in s_1 \). Now, let \( g = g_1g' \in B \), with \( g_1 \in S_1 \) and \( g' \in B' \). Then from the observations made above, we deduce:
\[ |Ad_g|^2 \geq \sup_{X' \in B', |X'| = 1} |Ad_{g_1g'}(X')|^2 = \sup_{X' \in B', |X'| = 1} |Ad_{g_1}(X')|^2 + |Ad_{g_1}(Ad_{g'}(X')) - Ad_{g_1}(X')|^2 \]
\[ \geq \sup_{X' \in B', |X'| = 1} |Ad_{g'}(X')|^2 = |Ad_{g'}|_{B'}^2 , \]
while in the other hand, since \( A(g') \in Sp(V_1, \omega_1^0) \)
\[ |Ad_g|^2 \geq \sup_{(a, v, t) \in s_1, (a, v, t) = 1} |Ad_{g_1}(a, v, t)|^2 = \sup_{(a, v, t) \in s_1, (a, v, t) = 1} |Ad_{g_1}(a, A(g') v, t)|^2 \]
\[ = \sup_{(a, v, t) \in s_1, (a, v, t) = 1} |Ad_{g_1}(a, v, t)|^2 = |Ad_{g_1}|_{s_1}^2 . \]
Since moreover \( g'g_1 = R_{g'}(g_1)g' \) and \( |Ad_{R_{g'}(g_1)}|_{s_1} = |Ad_{g_1}|_{s_1} \), we deduce that for all \( g = g_1g' \in B, g_1 \in S_1, \ g' \in B' \), we arrive at
\[ d_B(g) \geq \max \{ d_{S_1}(g_1), d_{B'}(g') \} \geq \frac{1}{2} (d_{S_1}(g_1) + d_{B'}(g') ) , \]
which is all what we needed.

Now, let also \( b = a \oplus n \) be a decomposition with \( n \) the nilradical of \( b \) and \( a \) its orthogonal complement. It follows then that \( a \) is an abelian sub-algebra, so that \( b = a \times n \) and the group \( B \) may be identified to its Lie algebra \( b \) with product
\[ (a, n) \cdot (a', n') = (a + a', e^{-ad_a n'} \cdot \text{CBH} n'), \]
where \( n \cdot \text{CBH} n' \) denotes the Baker-Campbell-Hausdorff series in the Lie algebra \( n \), which is finite since \( n \) is nilpotent. The following Definition and Lemmas in this subsection are taken from \( \mathbb{H} \).

**Definition 3.32** Let \( \{ H_j \}_{j=1}^n \) and \( \{ N_j \}_{j=1}^m \) be bases of \( a \) and \( n \) respectively. The coordinates system
\[ \mathbb{R}^{n+m} \to a \oplus n, \]
\[ (a_1, \ldots, a_n, n_1, \ldots, n_m) \to \left( \text{arcsinh}(a_1)H_1 + \cdots + \text{arcsinh}(a_n)H_n, n_1N_1 + \cdots + n_mN_m \right), \]
are said to be adapted tempered coordinates for \( B \).

**Lemma 3.33** In any adapted tempered coordinates on \( B \), the multiplication and inverse operations are tempered maps \( \mathbb{R}^{n+m} \times \mathbb{R}^{n+m} \to \mathbb{R}^{n+m} \) and \( \mathbb{R}^{n+m} \to \mathbb{R}^{n+m} \) respectively.

Proof. Let \( a_1, \ldots, a_n, n_1, \ldots, n_m \) be adapted tempered coordinates on \( B \) as in the above definition. Then, since
\[ \sinh(a + a') = \sinh(a) \cosh(a') + \cosh(a) \sinh(a') , \]
the \( \{a_i\} \)-coordinates of the multiplication of \( x, x' \in \mathbb{R}^{n+m} \) read
\[
\sinh(\text{arcsinh}(a_i) + \text{arcsinh}(a'_i)) = a_i \sqrt{1 + a_i^2} + a'_i \sqrt{1 + a'_i^2},
\]
so that they clearly are tempered functions in the \( a_i, a'_i \) variables.

For the \( n \) part, recall that there is a decomposition in real root spaces \( n = \bigoplus_a n_\alpha \) for the adjoint action of \( a \). Now if \( n' \in n_\alpha \), we have
\[
e^{\text{ad}(\text{arcsinh}(a_i)H_1 + \cdots + \text{arcsinh}(a_n)H_n)}_{n'} = e^{\alpha(H_1)} \text{arcsinh}(a_i) + \cdots + \alpha(H_n) \text{arcsinh}(a_n)}_{n'}
\]
which is a tempered function in \( a_1, \ldots, a_n \). As the CBH product in a nilpotent group is polynomial, linearly decomposing \( n_1' N_1 + \cdots n_m' N_m \) along the root space decomposition and using the above computation, we get that the \( n_i \) coordinates of the product of \( x \) and \( x' \) are tempered in all variables.

For the inverse, as \( (a, n)^{-1} = (-a, -e^{-\text{ad}_n}a) \), the above computation also shows the result.

**Lemma 3.34** Let \( b = b' \ltimes s \) be a Pyatetskii-Shapiro decomposition of a normal \( j \)-algebra \( b \), with \( s \) an elementary normal \( j \)-algebra and with corresponding Lie group decomposition \( B = B' \ltimes S \). Denote \( R : B' \to \text{Aut}(S) \) the associated extension homomorphism. Then in any adapted tempered coordinates for \( \mathbb{R}^{n' + m'} \times \mathbb{R}^{1 + m} \to \mathbb{R}^{1 + m} \).

**Proof.** Let \( a_1, \ldots, a_{n'}, a_1', \ldots, a_{n'}', a_{n'+1}, a_{n'+1}', \ldots, a_{n'+m}, a_{n'+m}' \) be adapted tempered coordinates for \( B' \) and \( S \) respectively. The group \( B' \) acts trivially on \( H_{n'+1} \), the generator of \( a \). Moreover, the coordinates \( a_1, \ldots, a_{n'+1}, a_1', \ldots, a_{n'+m} \) are adapted tempered coordinates for \( B \). Indeed, one knows [17, pages 56-57] that the infinitesimal action of \( H_1, \ldots, H_{n'} \) is real semi-simple with spectrum contained in \( \{-\frac{1}{2}, 0, \frac{1}{2}\} \).

Denote \( i' : B' \to B \) and \( i : S \to B \) the inclusions seen through the coordinates. Now by Lemma 3.33, the map
\[
(x', x) \in B' \times S \mapsto i'(x') \cdot i(x) \in B,
\]
is tempered. But the \( n \) part of that product is exactly \( R_{g'}(x) \) and so, this concludes the proof.

We are now prepared to state and prove the main result of this subsection.

**Theorem 3.35** Let \( B \) be a normal \( j \)-group with Pyatetskii-Shapiro decomposition \( B = (S_N \ltimes \cdots \ltimes S_1) \). Parametrizing the elements \( g, g' \in B \) as \( g = g_1 g_2 \cdots g_N \) and \( g' = g'_1 g'_2 \cdots g'_N \) with \( g_i, g'_i \in S_i \), we define
\[
S_{\text{can}}^B : B \times B \to \mathbb{R}, \quad (g, g') \mapsto \sum_{i=1}^{N} S_{\text{can}}^S(g_i, g'_i),
\]
where \( S_{\text{can}}^S \) is the canonical phase of \( S_i \) given in Definition 3.15. Then the pair \((B \times B, S_{\text{can}}^B)\) is tempered admissible and tame.

**Proof.** We will use an induction over \( N \), the number of elementary factors in \( B \). Accordingly, we set \( B = B' \ltimes R S \), with \( B' := (S_N \ltimes \cdots \ltimes S_2) \) and \( S := S_1 \). We then observe that \( \mathbb{R} \times B = (\mathbb{R} \times B') \ltimes R \times R (S \times S) \) and from Lemma 3.34 that the extension homomorphism \( R \times R =: R^2 \) is tempered within adapted coordinates. By Theorem 3.29, the pair \((S \times S, S_{\text{can}}^B)\) is tempered and admissible. By induction hypothesis, the latter also holds for \((B' \times B', S_{\text{can}}^B)\). Moreover, Equations (11) tell that, in the “elementary” case of \( S \times S \), the adapted tempered coordinates and the coordinates associated to the phase function are related to one another through a tempered diffeomorphism. By induction hypothesis, the latter also holds for \( B' \times B' \). Obviously, the extension homomorphism \( R \times R \) is then tempered within the coordinates associated to the phase functions as well. Note that under the parametrization \( g = g_1 g', h = h_1 h' \in B' \), \( g_1, h_1 \in S_1 \), \( g', h' \in B' \in B' \), the multiplication and inverse maps of \( B \) become:
\[
gh = g_1 R_{g'}(h_1) g' h', \quad g^{-1} = R_{g'}(g^{-1}_1) g'^{-1},
\]

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and similarly for $\mathbb{B} \times \mathbb{B}$. From this and the tempereness of the extension homomorphism $\mathbb{R} \times \mathbb{R}$, we see that tempereness of the multiplication and inverion laws in $\mathbb{B} \times \mathbb{B}$ will immediately follow once we will have shown that the map $10$ is a global diffeomorphism from $\mathbb{B} \times \mathbb{B}$ to $(b \otimes b)^*$. We will return to this question while examinating the question of admissibility. To this aim, let us set $G_1 := \mathbb{B}' \times \mathbb{B}'$, $G_2 := S \times S$ and denote respectively by $\mathfrak{g}_1$ and $\mathfrak{g}_2$ their Lie algebras. Let us also set $S_1 := S_{\text{can}}'$, $S_2 := S_{\text{can}}^2$, and let us assume, by induction hypothesis, that the pair $(G_1, S_1)$ is admissible, with associated decomposition $\mathfrak{g}_1 = \oplus_{k=0}^{N_1} \mathfrak{m}_k$. Let us consider an adapted basis of $\mathfrak{g}_1$, $\{w_k\}$, $k = 1, \ldots, \dim(\mathfrak{g}_1)$ with associated coordinates $\{ b \}_{k} := 1_{w_k}.S_1(b)$ on $G_1$. Similarly, let us consider the basis $\{2w_j^r\} | r = 1, 2 ; j = 0, 1, 2, 3\} \text{ of } \mathfrak{g}_2$ adapted to the decomposition $13$, where, for the values 1 and 2, $j$ consists in a multi-index and where $r$ labels the copies of $\mathfrak{S}$ in $\mathfrak{S} \times \mathfrak{S}$. Accordingly, we have the associated coordinate system $14$ on $G_2$ that now reads $2(x)^r_j := 2w_j^r.S_2(x)$. On $G := G_1 \times G_2$, with $S(xb) := S_1(b) + S_2(x)$, $x \in G_2$, $b \in G_1$, we then compute that:

$$1(xb)_k := 1_{w_k}.S(xb) = 1(b)_k \quad \text{and} \quad 2(xb)^r_j := 2w_j^r.S(xb) = R^r_j(2w_j^r).S_2(x).$$

Hence it suffices to look at the properties of the multipliers within $\mathfrak{g}_2$. From Pyatetskii-Shapiro’s theory, we know that the linear operator $\left[ R^r_j(2w_j^r) \right]$ that:

$$R^r_j(2w_j^r) = 2(x)^r_j, \quad \forall j \in \{0, 3\}.$$ 

Hence it suffices to look at the properties of the multipliers within the subspace $V \times V = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. For $j = 1, 2$, however, the action is not trivial but stabilizes component-wise $V \times V$. Accordingly, we set:

$$R^r_j(2w_j^r) =: \sum_{p=1}^{2} \left[R^r_j(2w_j^r) \right] 2w_p^r, \quad \forall j \in \{1, 2\},$$

where, again, $p$ is a multi-index. We therefore have:

$$R^r_j(2w_j^r) = \left[R^r_j(2w_j^r) \right] 2(x)^r_p, \quad \forall j \in \{1, 2\}. \quad (52)$$

From Pyatetskii-Shapiro’s theory, we know that the linear operator $\left[ R^r_j(2w_j^r) \right]$ is of jacobian one. In particular, this implies that the map $10$ is a global diffeomorphism from $G$ to $\mathfrak{g}^*$. (Hence, we have completed the proof of temperedness.) We now consider the ordered decomposition:

$$\mathfrak{g} = \mathfrak{g}_2 \oplus \mathfrak{g}_1 = \left( \oplus_{j=0}^{3} \mathfrak{g}_j \right) \bigoplus \left( \oplus_{k=0}^{N_1} \mathfrak{m}_k \right),$$

where indices occurring on the left ($\mathfrak{g}_2$) are considered as lower than the one on the right ($\mathfrak{g}_1$). Within this setting, we compute that for every element $X \in \mathcal{U}(\mathfrak{g}_2)$:

$$\alpha_X(xb) := e^{-iS(xb)}(\tilde{X}.e^{iS}(xb)) = 2\alpha_{R^r_j(X)}(x);$$

where $2\alpha_X := e^{-iS_x(\tilde{X}.e^{iS})}$ denotes the multiplier on $G_2$. Again, for the extreme values of $j$, we observe that:

$$\alpha_X(xb) = 2\alpha_X(x), \quad \forall X \in \mathcal{U}(\mathfrak{g}_0 \oplus \mathfrak{g}_3),$$

so in these cases the properties underlying admissibility are trivially satisfied. For $j = 1, 2$, we have with the notation $X_j := 1 - \sum_{r=1}^{2} (2w_j^r)^2$ of Proposition $67$,

$$R^2_j(X_j) = 1 - \sum_{r=1}^{2} \left(R^2_j(2w_j^r) \right)^2,$$

which leads to

$$(2)\alpha_{X_j}(xb) = 1 + \sum_{r=1}^{2} \left(2(xb)^r_j \right)^2 - i \sum_{r=1}^{2} \left[R^r_j(2w_j^r) \right] 2w_p^r,2(x)^r_p.$$
where $b = (b_1, b_2) \in B' \times B' = G_1$. This gives the property (i) of Definition 2.19. From the expression (52) and the structure of the elementary case (Lemma 3.20), we then observe that for every homogeneous degree three monomial $A \in U(V \times V)$:

$$\hat{A}_{(2)} \alpha X_j = 0.$$ 

Also, setting $-i \beta X_j (xb) := -i \sum_{r=1}^2 |R_b r_j| \hat{p}_r |w_{pr} \cdot (2(x)r_x)}$, we deduce from the expressions (43) and (46) that, for every $A \in V \times V$: $\beta X_j = 0$. From the expression (52) and setting $2(x)_{12} := (2(x)_1, 2(x)_2)$, we then deduce that for every $A \in U(V \times V)$ of strictly positive degree:

$$\left| \hat{A}_{(2)} \alpha X_1 (xb) + (2) \alpha X_2 (xb) \right| = \left| \hat{A}_{12} (xb)_{12} \right|^2 = \left| \hat{R}_{2}^G (A) \hat{R}_{2}^G (2(x)_{12}) \right|^2 \leq |R_{2}^G | \deg(A) + 2 \mu_{12} (x) \left| (2(x)_{12}) \right|^2,$$

where the last estimate is obtained from Corollary 3.24. Since

$$|(2(x)_{12})|^2 = \left| R_{2}^{G, 1} \hat{R}_{2}^G (2(x)_{12}) \right|^2 \leq \left| R_{2}^{G, 1} \right|^2 \left| (2(x)_{12}) \right|^2,$$

we then get

$$\left| \hat{A}_{(2)} \alpha X_1 (xb) + (2) \alpha X_2 (xb) \right| \leq \left| R_{2}^{G, 1} \right|^2 \left| \deg(A) + 2 \mu_{12} (x) \right| \left| (2(x)_{12}) \right|^2 \leq C \left| \hat{R}_{2}^G \right|^2 \left| \deg(A) + 2 \mu_{12} (x) \right| \left| (2(x)_{12}) \right|^2.$$

But we know that we may assume $\deg(A) \leq 2$, hence

$$\left| \hat{A}_{(2)} \alpha X_1 (xb) + (2) \alpha X_2 (xb) \right| \leq C \left| \hat{R}_{2}^G \right|^2 \left| \mu_{12} (x) \right| \left| (2(x)_{12}) \right|^2 \leq C \left| \hat{R}_{2}^G \right|^2 \left| \mu_{12} (x) \right| \left| (2(x)_{12}) \right|^2.$$

Defining the element $\mu_{12} (xb) := \hat{R}_{2}^G \left( \mu_{12} (x) \right)$ yields admissibility at the level of $V \times V$.

Last, tameness follows from Lemma 3.31 and, within the notations displayed above, from the equality $|2(x)_{12}| = \left| R_{2}^{G, 1} \right| = |2(x)|$.

**Remark 3.36** Last, we observe that Remarks 2.15 and 2.36 and Lemma 3.31 show that the one-variable Schwartz space $S_{\text{smooth}} \left( B \times \mathcal{E} \right)$ associated with the two-variable tempered, admissible and tame pair $(\mathcal{B} \times \mathcal{B}, S_{\text{can}}^G)$ (see Definition 2.43) can be equivalently topologized by the semi-norms

$$f \in S^G (B) \Rightarrow \sup_{X \in \{ X \}} \sup_{x \in B} \left\{ \frac{\hat{R}_{2}^G \left( X \right) \left| (X f(x)) \right|}{\left| X \right|} \right\}, \quad j, k, n \in \mathbb{N},$$

or even with

$$f \in S^G (B) \Rightarrow \sup_{X \in \{ X \}} \sup_{x \in B} \left\{ \frac{\hat{R}_{2}^G \left( X \right) \left| (X f(x)) \right|}{\left| X \right|} \right\}, \quad j, k, n \in \mathbb{N}.$$

### 4 Non-formal star-products

#### 4.1 Star-products on normal j-groups

We consider an elementary normal j-group $\mathcal{S}$ viewed as a symplectic symmetric spaces as in subsection 3.2. We start by recalling the results obtained in [2] and [3].

**Definition 4.1** Set $\tilde{S} := \{(a, v, \xi)\} = \mathbb{R} \times \mathbb{R}^{2d} \times \mathbb{R}$. The **twisting map** is the smooth one-parameter family of diffeomorphisms defined as

$$\phi_\theta : \tilde{S} \rightarrow \tilde{S} : (a, v, \xi) \mapsto \left( a, \cosh \left( \frac{q}{2} \xi \right)^{-1} v, \frac{2}{q} \sinh \left( \frac{q}{2} \xi \right) \right), \quad \theta \in \mathbb{R}^*.$$
Let $\mathcal{S}(\mathbb{S})$ be the Euclidean Schwartz space of $\mathbb{S}$, i.e. the ordinary Schwartz space in the coordinates $(11)$. Accordingly, let $\mathcal{S}(\mathbb{S})'$ be the dual space of tempered distributions. Let us also denote by 

$$(\mathcal{F}u)(a, v, \xi) := \int_{-\infty}^{\infty} e^{-it\xi} u(a, v, t) \, dt ,$$

the partial Fourier transform in the $t$-variable. For $\gamma > 0$, we let $\mathcal{O}_{C, \gamma}(\mathbb{R}^m)$ be the sub-set of smooth functions, the derivatives of which are uniformly polynomially bounded:

$$\mathcal{O}_{C, \gamma}(\mathbb{R}^m) := \{ f \in C^\infty(\mathbb{R}^m) : \exists r > 0 : \forall \alpha \in \mathbb{N}^m : \exists C_\alpha > 0, |\partial^\alpha f(x)| \leq C_\alpha (1 + |x|)^{-\gamma |\alpha|} \} .$$

**Definition 4.2** We denote by $\Theta$, the subspace of $C^\infty(\mathbb{R}, \mathbb{C})$ constituted by the elements $\tau$ such that $\exp \circ \pm \tau$ belong to the space $\mathcal{O}_{C, 1}(\mathbb{R}, \mathbb{C})$.

Let $\tau_0$ be the element of $C^\infty(\mathbb{S})$, given by:

$$\tau_0 := \frac{1}{2} \log \circ \text{Jac}_{\phi_\theta^{-1}} .$$

Viewed as a function of its last variable only, $\tau_0$ belongs to $\Theta$. Indeed, we have:

$$\text{Jac}_{\phi_\theta^{-1}}(a, v, \xi) = 2^{-d} \frac{(1 + \sqrt{1 + \frac{\xi^2}{4}})^d}{\sqrt{1 + \frac{\xi^2}{4}}} .$$

Given an element $\tau \in \Theta$, one defines a linear injection:

$$T_{\theta, \tau} := \mathcal{F}^{-1} \circ \exp(\tau_0 - \tau) \circ (\phi_\theta^{-1})^* \circ \mathcal{F} : \mathcal{S}(\mathbb{S}) \to \mathcal{S}(\mathbb{S})' ,$$

whose adjoint, with respect to the inner product of $L^2(\mathbb{S}, d\xi)$, reads:

$$T_{\theta, \tau}^* := \mathcal{F}^{-1} \circ (\phi_\theta)^* \circ \exp(-\tau_0 - \tau) \circ \mathcal{F} : \mathcal{S}(\mathbb{S}) \to \mathcal{S}(\mathbb{S}) .$$

Note that in particular, the inverse map defines a linear injection from $\mathcal{S}(\mathbb{S})$ to itself:

$$T_{\theta, \tau}^{-1} := \mathcal{F}^{-1} \circ (\phi_\theta)^* \circ \exp(-\tau_0 + \tau) \circ \mathcal{F} : \mathcal{S}(\mathbb{S}) \to \mathcal{S}(\mathbb{S}) .$$

Observe the following immediate fact:

**Lemma 4.3** For $\tau \in \Theta$, the map $T_{\theta, \tau}^{-1} : \mathcal{S}(\mathbb{S}) \to \mathcal{S}(\mathbb{S})$ extends to a unitary operator on $L^2(\mathbb{S}, d\xi)$ if and only if $\tau$ is purely imaginary.

Let $\omega^0$ be the standard symplectic structure of $\mathbb{R}^{2d+2}$ and let $s^0_\theta$ be the Weyl product on $\mathcal{S}(\mathbb{R}^{2d+2})$ given by

$$f_1 s^0_\theta f_2(x) = \frac{1}{(\pi \theta)^{2(d+1)}} \int_{\mathbb{R}^{2d+2} \times \mathbb{R}^{2d+2}} e^{\frac{i}{\theta} S_0(x,y,z)} f_1(y) f_2(z) \, dy \, dz ,$$

where $S_0(x,y,z) := \omega^0(x,y) + \omega^0(y,z) + \omega^0(z,x)$. For $\tau \in \Theta$, denoting by

$$\mathcal{E}_{\theta, \tau}(\mathbb{S}) := T_{\theta, \tau}(\mathcal{S}(\mathbb{S})) ,$$

the range subspace in the tempered distribution space $\mathcal{S}(\mathbb{S})'$, one has the inclusions

$$\mathcal{S}(\mathbb{S}) \subset \mathcal{E}_{\theta, \tau}(\mathbb{S}) \subset C^\infty(\mathbb{S}) .$$

We consider the linear isomorphism:

$$T_{\theta, \tau}^{-1} : \mathcal{E}_{\theta, \tau}(\mathbb{S}) \to \mathcal{S}(\mathbb{S}) .$$
Identifying $\mathbb{S} \simeq \mathbb{R}^{2d+2}$ by mean of the global coordinate system (11), we transport under $T_{\theta, \tau}$ the Weyl’s product on $\mathcal{S}(\mathbb{R}^{2d+2}) \simeq \mathcal{S}(\mathbb{S})$. This yields an associative product:

$$\ast_{\theta, \tau} : \mathcal{E}_{\theta, \tau}(\mathbb{S}) \times \mathcal{E}_{\theta, \tau}(\mathbb{S}) \to \mathcal{E}_{\theta, \tau}(\mathbb{S}) ,$$

given by

$$f_1 \ast_{\theta, \tau} f_2 := T_{\theta, \tau}(T_{\theta, \tau}^{-1}(f_1) \ast_{\theta, \tau}^{S} T_{\theta, \tau}^{-1}(f_2)) , \quad f_1, f_2 \in \mathcal{E}_{\theta, \tau}(\mathbb{S}) .$$

The associative algebra $\mathcal{E}_{\theta, \tau}(\mathbb{S}), \ast_{\theta, \tau}$, endowed with the Fréchet algebra structure transported under $T_{\theta, \tau}$ from $\mathcal{S}(\mathbb{R}^{2d+2})$, satisfies the following properties [2 7]:

**Theorem 4.4** Let $\tau \in \Theta$ and $\theta \neq 0$. Then,

(i) For all compactly supported $u, v \in \mathcal{E}_{\theta, \tau}(\mathbb{S})$, one has the integral representation:

$$u \ast_{\theta, \tau} v = \int_{\mathbb{S} \times \mathbb{S}} K_{\theta, \tau}(x_1, x_2) R^*_{x_1}(u) R^*_{x_2}(v) \, ds(x_1) \, ds(x_2) , \quad (55)$$

where the two-point kernel is given by

$$K_{\theta, \tau}(x_1, x_2) := (\pi \theta)^{-2(d+1)} A_{\theta, \tau}(x_1, x_2) \exp \left\{ \frac{2i}{\theta} S^S_{\text{can}}(x_1, x_2) \right\} , \quad (56)$$

with, in the coordinates \(^4\),

$$A_{\theta, \tau}(x_1, x_2) := A^S_{\text{can}}(x_1, x_2) \exp \left\{ \tau \left( \frac{2}{\theta} \sinh(2a_1) \right) + \tau \left( \frac{2}{\theta} \sinh(-2a_2) \right) - \tau \left( \frac{2}{\theta} \sinh(2a_1 - 2a_2) \right) \right\} ,$$

and with $S^S_{\text{can}}$ and $A^S_{\text{can}}$ defined in [33] and [41].

(ii) The product $\ast_{\theta, \tau}$ is equivariant under the automorphism group of the symplectic symmetric space $(\mathbb{S}, s, \omega^S)$: for all elements $g$ of $\text{Aut}(\mathbb{S}, s, \omega^S)$ and $u, v \in \mathcal{D}(\mathbb{S})$, one has

$$g^* (u \ast_{\theta, \tau} v) = g^* (u \ast_{\theta, \tau} v) .$$

Consider a normal j-group decomposed, following Proposition 3.5 into a semi-direct product $\mathbb{B} = \mathbb{B}' \ltimes \mathbb{S}$ where $\mathbb{S}$ is elementary. One knows from Proposition [33] and [34] that the extension homomorphism $\mathbb{R} : \mathbb{B}' \to \text{Aut}(\mathbb{S})$ underlies a homomorphism from $\mathbb{B}'$ into the isotropy subgroup $\text{Aut}(\mathbb{S}, s, \omega^S)_e$ at the unit element $e$ of $\mathbb{S}$ viewed as a symmetric space:

$$\mathbb{R} : \mathbb{B}' \to \text{Sp}(V, \omega^0) \subset \text{Aut}(\mathbb{S}, s, \omega^S)_e ,$$

where $(V, \omega^0)$ is the symplectic vector space attached to $\mathbb{S}$. In particular, the action of $\mathbb{B}'$ leaves invariant the two-point kernel $K_{\theta, \tau}$ on $\mathbb{S} \times \mathbb{S}$. Iterating the above observation at the level of $\mathbb{B}'$ and translating the “extension Lemma” in [3] within the present framework, we obtain:

**Proposition 4.5** Let $\mathbb{B}$ be a normal j-group with Pyatetskii-Shapiro decomposition $\mathbb{B} = (\mathbb{S}_N \ltimes \ldots \ltimes \mathbb{S}_1)$ and fix $\mathcal{F} := (\tau_1, \ldots, \tau_N) \in \Theta^N$. Parametrizing a group element $g \in \mathbb{B}$ as $g = g_1 \ldots g_N$, with $g_i \in \mathbb{S}_i$, we consider the two-point kernel on $\mathbb{B}$ given by

$$K_{\theta, \mathcal{F}}(g, g') := K_{\theta, \tau_1}(g_1, g'_1) \cdots K_{\theta, \tau_N}(g_N, g'_N) , \quad (57)$$

where $K_{\theta, \tau_i}$ is the two-points kernel on $\mathbb{S}_i \times \mathbb{S}_i$, defined in [33]. Then, the bilinear mapping

$$\ast_{\theta, \mathcal{F}} : [(u, v) \mapsto \int_{\mathbb{B} \times \mathbb{B}} K_{\theta, \mathcal{F}}(g, g') R^*_{g}(u) R^*_{g'}(v) \, db(g) \, db(g') ] ,$$

is associative on

$$\mathcal{E}_{\theta, \mathcal{F}}(\mathbb{B}) := \mathcal{E}_{\theta, \tau_N}(\mathbb{S}_N) \otimes \cdots \otimes \mathcal{E}_{\theta, \tau_1}(\mathbb{S}_1) ,$$

(recall that $\mathcal{E}_{\theta, \tau_1}(\mathbb{S}_1)$ is nuclear). Moreover, at the level of compactly supported functions, the product $\ast_{\theta, \mathcal{F}}$ is equivariant under the left-translations in $\mathbb{B}$.

\(^4\) As usual, we set $x_j = (a_j, v_j, t_j) \in \mathbb{R}^{2d+2}$.
4.2 An oscillatory integral formula for the star-product

In this subsection, we fix $\mathbb{B}$ a normal $j$-group, with Lie algebra $\mathfrak{b}$. We also let $\mathfrak{r} \in \Theta^N$ as above ($N$ is the number of elementary components in $\mathbb{B}$) and form the two-point kernel $K_{\theta, \mathfrak{r}}$ on $\mathbb{B} \times \mathbb{B}$, defined in (57). Proposition 4.5 implies that the deformed product

\[ u \star_{\theta, \mathfrak{r}} v = \int_{\mathbb{B} \times \mathbb{B}} K_{\theta, \mathfrak{r}}(g, g') R^*_g(u) R^*_{g'}(v) \, d_\mathbb{B}(g) \, d_\mathbb{B}(g'), \tag{58} \]

is weakly associative (in the sense of Definition 2.11) and left $\mathbb{B}$-equivariant. The results of section 2 will allow to properly understand the integral in (58) as oscillatory one. As a consequence, we will see that the deformed product extends as a continuous bilinear and associative map on the function space $\mathcal{B}(\mathbb{B}, \mathcal{A})$, for $\mathcal{A}$ a Fréchet algebra. We start with a simple fact:

**Lemma 4.6** Let $\mathbb{B}$ be an elementary normal $j$-group and $\mathfrak{r} \in \Theta^N$. Then the amplitude $A_{\theta, \mathfrak{r}}$, as given in Proposition 4.5, consists in an element of $\mathcal{B}^{\mu_\tau}(\mathbb{B} \times \mathbb{B})$ for a tempered weight $\mu_\tau$.

**Proof.** Consider first the case where $\mathbb{B} = S$ is elementary. Within the notations of subsection 3.3, we have

\[ |\vec{x}_3| = |(x_{1,3}, x_{2,3})| = \left| \left( - \sinh(2a_2), \sinh(2a_1) \right) \right| = \left( \sinh(2a_2)^2 + \sinh(2a_1)^2 \right)^{1/2}, \]

so that the function

\[ \mu_{\text{can}}(x_1, x_2) := \cosh a_1 \cosh a_2, \]

is a tempered weight. As the left invariant vector field $\vec{H}$ on $S$ restricted to functions of depending on the variable $a$ only, coincides with the partial differentiation operator $\partial_a$, we get from the explicit expression

\[ A_{\text{can}}^S(x_1, x_2) = (\cosh a_1 \cosh a_2 \cos(a_1 - a_2))^d \sqrt{\cosh 2a_1 \cosh 2a_2 \cosh 2(a_1 - a_2)}, \]

that there exists $\rho > 0$ such that for any $X \in \mathcal{U}(\mathfrak{g} \oplus \mathfrak{s})$, there exists a constant $C_X > 0$ with

\[ |\vec{X} A_{\text{can}}^S| \leq C_X \rho_{\text{can}}^d. \]

Hence $A_{\text{can}}^S \in \mathcal{B}^{\mu_{\text{can}}}(S \times S)$. Next, since $\mathfrak{r} \in \Theta$, we have $\exp \circ \pm \tau \in \mathcal{O}_{C, 1}(\mathbb{R})$. Thus, there exists $r > 0$ such that the $n$-th derivative of $\exp \circ \pm \tau(x)$ is bounded by $(1 + |x|)^{-n}$. Let us denote by $\deg(\tau)$ such positive number $r$. Since $\exp \circ \pm \tau$ depends on the variable $a$ only, among all elements of $\mathcal{U}(\mathfrak{g} \oplus \mathfrak{s})$, only the powers of $\vec{H}_i$, $i = 1, 2$, give non zero contributions. Therefore, an easy computation shows that for any $X \in \mathcal{U}(\mathfrak{g} \oplus \mathfrak{s})$, there exists a constant $C_X > 0$ with

\[ |\vec{X} \exp \{ \pm \tau \left( \frac{2}{3} \sinh(2a) \right) \} | \leq C_X (1 + |\vec{x}_3|)^{\deg(\tau)}. \]

Hence $A_{\theta, \mathfrak{r}}$ belongs to $\mathcal{B}^{\mu_\tau}(S \times S)$ for $\mu_\tau = \mu_{\text{can}}^{d + 3\deg(\tau)}$.

The general case $\mathbb{B} = \mathbb{B}' \times S$ follows easily by Pyatetskii-Shapiro theory, since only the variables in $V \subset S$ are affected by the action of $\mathbb{B}'$ and that $A_{\theta, \mathfrak{r}}$ is independent of these variables.

We now consider a Fréchet algebra $\mathcal{A}$, with topology underlying a countable family of sub-multiplicative semi-norms $\{ \| \cdot \|_j \}_{j \in \mathbb{N}}$. Combining Lemma 4.6 with Theorem 3.35 leads us to prove that the integral in the expression of the deformed product (55) can be properly understood as an oscillatory one in the sense of section 2. In particular, this allows to define the product $\star_{\theta, \mathfrak{r}}$ on $\mathcal{B}(\mathbb{B}, \mathcal{A})$. This is the main result of this section.

**Theorem 4.7** Let $\mathbb{B}$ be a normal $j$-group. Fix $\mathfrak{r} \in \Theta^N$ and let $\{ \mu_j \}, \{ \mu'_j \}, \{ \mu''_j \}$ be three families of tempered weights on $\mathbb{B}$ of sub-multiplicativity degrees $\{ (L_j, R_j) \}, \{ (L'_j, R'_j) \}, \{ (L''_j, R''_j) \}$. Considering $K_{\theta, \mathfrak{r}}$ the two-point kernel on $\mathbb{B}$ defined in (54), the correspondence

\[ \star_{\theta, \mathfrak{r}} : (F_1, F_2) \in \mathcal{B}^{(\mu_j)}(\mathbb{B}, \mathcal{A}) \times \mathcal{B}^{(\mu'_j)}(\mathbb{B}, \mathcal{A}) \mapsto \int_{\mathbb{B} \times \mathbb{B}} K_{\theta, \mathfrak{r}}(x, x) \star_{\theta, \mathfrak{r}}(F_1) \star_{\theta, \mathfrak{r}}(F_2) \in \mathcal{B}^{(\mu''_j)}(\mathbb{B}, \mathcal{A}), \]
is a jointly continuous bilinear map and is equivariant under the left translations in $\mathbb{B}$ in the sense that for all $g \in \mathbb{B}$, we have

$$L_g^*(F_1 \ast_\theta, \varphi F_2) = (L_g^* F_1) \ast_\theta, \varphi (L_g^* F_2),$$

in $\mathcal{B}^{(\mu_j, R_j \mu_j, R_j ^* \mu_j)}(\mathbb{B}, \mathbb{A})$. Moreover, the map $\ast_{\theta, \varphi}$ is associative in the sense that for every $F \in \mathcal{B}^{(\mu_j)}(\mathbb{B}, \mathbb{A})$, $F' \in \mathcal{B}^{(\mu_j')}(\mathbb{B}, \mathbb{A})$, $F'' \in \mathcal{B}^{(\mu_j'')}(\mathbb{B}, \mathbb{A})$ we have the equality

$$(F \ast_{\theta, \varphi} F') \ast_{\theta, \varphi} F'' = F \ast_{\theta, \varphi} (F' \ast_{\theta, \varphi} F'') \quad \text{in} \quad \mathcal{B}^{(\mu_j, R_j \mu_j, R_j ^* \mu_j, R_j ^* \mu_j, R_j ^* \mu_j)}(\mathbb{B}, \mathbb{A}).$$

In particular, $(\mathcal{B}(\mathbb{B}, \mathbb{A}), \ast_{\theta, \varphi})$ is a Fréchet algebra with jointly continuous product.

**Proof.** That the bilinear map $\ast_{\theta, \varphi}$ (with the domain and image as indicated) is well defined and jointly continuous, follows from Theorem 2.34, Theorem 3.35 and Lemma 1.6. Associativity follows from associativity in $\mathcal{E}_{\theta, \varphi}(\mathbb{B})$, which implies weak associativity in the sense of Definition 2.41 and Proposition 2.42. So, it remains to prove left $\mathbb{B}$-equivariance. We first note that by Lemma 2.38 (ii), the group $\mathbb{B}$ acts continuously from $\mathcal{B}^{(\mu_j)}(\mathbb{B}, \mathbb{A})$ to $\mathcal{B}^{(\mu_j')}(\mathbb{B}, \mathbb{A})$ (for any family of weights $\{\mu_j\}$ of sub-multiplicative degrees $\{(L_j, R_j)\}$) on the left. Also, we have by Lemma 2.39 that $F \ast_{\theta, \varphi} F' = \lim_{n \to \infty} F_n \ast_{\theta, \varphi} F'_n$ in $\mathcal{B}^{(\mu_j', R_j, \mu_j')}(\mathbb{B}, \mathbb{A})$, for any pair of sequences $\{F_n\}$ and $\{F'_n\}$ of smooth compactly supported $\mathbb{A}$-valued functions on $\mathbb{B}$, which converge to $F$ and $F'$, in the topology of $\mathcal{B}^{(\mu_j)}(\mathbb{B}, \mathbb{A})$ and $\mathcal{B}^{(\mu_j')}(\mathbb{B}, \mathbb{A})$ for any sequence of weights $\{\hat{\mu}_j\}$ and $\{\hat{\mu}_j'\}$ dominating $\{\mu_j\}$ and $\{\mu_j'\}$. From continuity of the left regular action (see Lemma 2.38 (ii) and left $\mathbb{B}$-equivariance at the level of $D(\mathbb{B}, \mathbb{A})$, we thus have

$$L_g^*(F \ast_{\theta, \varphi} F') = \lim_{n \to \infty} L_g^*(F_n \ast_{\theta, \varphi} F'_n) = \lim_{n \to \infty} (L_g^* F_n) \ast_{\theta, \varphi} (L_g^* F'_n),$$

where the limits are in $\mathcal{B}^{(\mu_j \mu_j', R_j, R_j', \mu_j')}(\mathbb{B}, \mathbb{A})$. It remains to find specific approximation sequences $\{F_n\}$ and $\{F'_n\}$, such that $\{L_g^* F_n\}$ and $\{L_g^* F'_n\}$ converge to $L_g^* F$ and $L_g^* F'$, in the topology of $\mathcal{B}^{(\mu_j)}(\mathbb{B}, \mathbb{A})$ and $\mathcal{B}^{(\mu_j')}(\mathbb{B}, \mathbb{A})$. For this, we observe that the same construction as in the proof of Lemma 2.38 (viii), does the job. Indeed, recall that there, we have constructed the approximation sequence $\{F_n\}$, by setting

$$F_n := e_n F \in D(\mathbb{B}, \mathbb{A}), \quad e_n := \int_{\mathbb{B}} \psi(g) R_g^*(\chi_{C_n}) \, dt(g) \in D(\mathbb{B}),$$

where $0 \leq \psi \in \mathcal{D}(\mathbb{B})$, $\int_{\mathbb{B}} \psi \, dt = 1$, $\{C_n\}$ is an increasing sequence of relatively compact open sub-sets of $\mathbb{B}$ converging to $\mathbb{B}$ and $\chi_{C_n}$ is the characteristic function of $C_n$. Fixing $g \in \mathbb{B}$ and setting $C_n^g := g C_n$, the sequence $\{C_n^g\}$ is still an increasing sequence of relatively compact open sub-sets of $\mathbb{B}$ converging to $\mathbb{B}$. Also, as

$$e_n^g := L_g^*(e_n) = \int_{\mathbb{B}} \psi(g') R_g^*(\chi_{C_n^g}) \, dt(g') \in D(\mathbb{B}),$$

we deduce that for all $j, k \in \mathbb{N}$:

$$||L_g^*(F_n) - L_g^*(F)||_j, k, \mu_j, \infty = ||(1 - e_n^g) L_g^*(F)||_j, k, \mu_j, \infty,$$

which, by Lemma 2.38 (vi), converges to zero as $L_g^*(F) \in \mathcal{B}^{(\mu_j, R_j)}(\mathbb{B}, \mathbb{A})$ and $\{\mu_j\}$ dominates $\{\mu_j'\}$. 

Let $S_{\text{can}}^{(\mathbb{B}, \mathbb{A})}$ be the one-variable Schwartz space associated to the admissible and tame tempered pair $(\mathbb{B} \times \mathbb{B}, S_{\text{can}}^{(\mathbb{B})})$, constructed in Definition 2.40. The next result follows immediately from Proposition 2.44.

**Proposition 4.8** Let $\mathbb{B}$ be a normal $j$-group and fix $\varphi \in \Theta^N$. Then $\ast_{\theta, \varphi}$ is an associative and jointly continuous product on $S_{\text{can}}^{(\mathbb{B}, \mathbb{A})}$. Moreover, for every family of tempered weights $\{\mu_j\}_{j \in \mathbb{N}}$, the space $\mathcal{B}^{(\mu_j)}(\mathbb{B}, \mathbb{A})$ acts continuously on $S_{\text{can}}^{(\mathbb{B}, \mathbb{A})}$ via

$$L_{\ast_{\theta, \varphi}}(F) : \varphi \mapsto F \ast_{\theta, \varphi} \varphi, \quad F \in \mathcal{B}^{(\mu_j)}(\mathbb{B}, \mathbb{A}), \quad \varphi \in S_{\text{can}}^{(\mathbb{B}, \mathbb{A})}.$$

In particular, $(S_{\text{can}}^{(\mathbb{B}, \mathbb{A})}, \ast_{\theta, \varphi})$ is an ideal of $(\mathcal{B}(\mathbb{B}, \mathbb{A}), \ast_{\theta, \varphi})$. 

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We now see that, as expected, the constant function is an identity for the deformed product.

**Proposition 4.9** Let \( \mathbb{B} \) be a normal \( j \)-group. Fix \( \bar{r} \in \Theta^N \), \( \{ \mu_j \} \) a family of tempered weights of sub-multiplicative degrees \( \{ (L_j R_j) \} \) and \( F \in B^{(\mu_j)}(\mathbb{B}, \mathcal{A}) \). Identifying every element \( a \in \mathcal{A} \) with the function \( [g \in \mathbb{B} \mapsto a \in \mathcal{A}] \in B(\mathbb{B}, \mathcal{A}) \), we have

\[
a \ast_{\theta, \bar{r}} F = a F, \quad F \ast_{\theta, \bar{r}} a = F a,
\]

in \( B^{(\mu_j)}(\mathbb{B}, \mathcal{A}) \). In particular, if \( \mathcal{A} \) is unital, the element \( [g \mapsto 1_\mathcal{A}] \in B(\mathbb{B}, \mathcal{A}) \) is the unit of \( (B(\mathbb{B}, \mathcal{A}), \ast_{\theta, \bar{r}}) \).

**Proof.** Since the constant unit function is a fixed point of the map \( T_{\theta, \bar{r}}^{-1} \), for every \( \varphi \in \mathcal{S}^{\mathcal{B}_{\text{can}}}(\mathbb{B}, \mathcal{A}) \), we have:

\[
\varphi \ast_{\theta, \bar{r}} a = T_{\theta, \bar{r}}(T_{\theta, \bar{r}}^{-1}(\varphi) \ast_0 a),
\]

in \( \mathcal{S}^{\mathcal{B}_{\text{can}}}(\mathbb{B}, \mathcal{A}) \). By Remark 4.38 we see that the transported Schwartz space \( \mathcal{S}^{\mathcal{B}_{\text{can}}}(\mathbb{B}, \mathcal{A}) \) is a (dense) subset of the ordinary Schwartz space \( \mathcal{S}(\mathfrak{b}, \mathcal{A}) \), under the usual identification \( \mathbb{B} \simeq \mathfrak{b} \). Since \( T_{\theta, \bar{r}}^{-1} \) preserves the latter space, we see that \( T_{\theta, \bar{r}}^{-1}(\varphi) \in \mathcal{S}(\mathfrak{b}, \mathcal{A}) \). By \[\text{19}\], we now that the Weyl product admits the constant function as unit element (for the algebra of \( \mathcal{A} \)-valued flat \( \mathfrak{B} \) functions). Thus \( \varphi \ast_{\theta, \bar{r}} a = \varphi a \) and \( a \ast_{\theta, \bar{r}} \varphi = a \varphi \) for all \( \varphi \in \mathcal{S}^{\mathcal{B}_{\text{can}}}(\mathbb{B}, \mathcal{A}) \) and \( a \in \mathcal{A} \). Now, consider the injective homomorphism \( L_{\ast_{\theta, \bar{r}}} \) from \( (B^{(\mu_j)}(\mathbb{B}, \mathcal{A}), \ast_{\theta, \bar{r}}) \) to the algebra of continuous operators acting on \( \mathcal{S}^{\mathcal{B}_{\text{can}}}(\mathbb{B}, \mathcal{A}) \), defined in Proposition 12. From the previous considerations, the associativity of the deformed product and the fact that \( \mathcal{S}^{\mathcal{B}_{\text{can}}}(\mathbb{B}, \mathcal{A}) \) is an ideal of \( B^{(\mu_j)}(\mathbb{B}, \mathcal{A}) \), we get

\[
L_{\ast_{\theta, \bar{r}}}(F \ast_{\theta, \bar{r}} a) = L_{\ast_{\theta, \bar{r}}}(F a), \quad \forall F \in B^{(\mu_j)}(\mathbb{B}, \mathcal{A}),
\]

which entails by injectivity that \( F \ast_{\theta, \bar{r}} a = F a \) in \( B^{(\mu_j^{-1})}(\mathbb{B}, \mathcal{A}) \). As \( F a \in B^{(\mu_j)}(\mathbb{B}, \mathcal{A}) \), we deduce that the equality \( F \ast_{\theta, \bar{r}} a = F a \) holds in fact in \( B^{(\mu_j)}(\mathbb{B}, \mathcal{A}) \). The case of \( a \ast_{\theta, \bar{r}} F \) is entirely similar. \[\blacksquare\]

## 5 Deformation of Fréchet algebras

### 5.1 The deformed product

In this section, we still consider a normal \( j \)-group \( \mathbb{B} \), with Lie algebra \( \mathfrak{b} \) and with \( N \) elementary components in its Pyatetskii-Shapiro decomposition. We also consider a pair \( (\mathcal{A}, \alpha) \), consisting of a Fréchet algebra \( \mathcal{A} \) (with topology determined by a countable set of sub-multiplicative semi-norms \( \{ \| \cdot \|_j \}_{j \in \mathbb{N}} \)), together with a strongly continuous action \( \alpha \) of \( \mathbb{B} \) by automorphisms. We start by more general considerations regarding tempered action:

**Definition 5.1** An action \( \alpha \) of a tempered Lie group \( G \) on a Fréchet algebra \( \mathcal{A} \), is said to be **tempered**, if for all \( j \in \mathbb{N} \) there exists a tempered weight \( \mu^*_j \) such that for all \( a \in \mathcal{A} \) and all \( g \in G \), we have

\[
\| \alpha_g(a) \|_j \leq \mu^*_j(g) \| a \|_j.
\]

**Remark 5.2** Note that for a tempered action and for \( g \in G \) fixed, \( \alpha_g \) acts continuously on \( \mathcal{A} \).

We denote by \( \mathcal{A}^\infty \), be the set of smooth vectors for the action \( \alpha \) of \( \mathbb{B} \) on \( \mathcal{A} \). By strong continuity, \( \mathcal{A}^\infty \) is a dense subspace of \( \mathcal{A} \). On this subset, we consider the infinitesimal form of the action, given for \( X \in \mathfrak{g} \) by:

\[
X^\alpha(a) := \left. \frac{d}{dt} \right|_{t=0} \alpha_{e^{tx}}(a), \quad X \in \mathfrak{g}, \ a \in \mathcal{A}^\infty,
\]

and we extend it to the whole enveloping algebra \( \mathcal{U}(\mathfrak{g}) \), by declaring that the map \( \mathcal{U}(\mathfrak{b}) \to \text{End}(\mathcal{A}^\infty) \), \( X \mapsto X^\alpha \) is an algebra homomorphism. The subspace \( \mathcal{A}^\infty \) carries a finer topology associated with the set of semi-norms:

\[
\| a \|_{j,X} := \| X^\alpha(a) \|_j, \quad a \in \mathcal{A}^\infty, \quad X \in \mathcal{U}(\mathfrak{b}), \ j \in \mathbb{N}.
\]
Considering the PBW basis of $\mathcal{U}(\mathfrak{b})$ associated to an ordered basis of $\mathfrak{b}$ as in [3], one can use only countably many semi-norms to define the topology of $\mathcal{A}^\infty$. The latter are indexed by $(j, k) \in \mathbb{N}^2$, where $j$ refers to the labeling of the initial family of semi-norms $\{ \| \cdot \|_{j,K} \}_{K \in \mathbb{N}}$ of $\mathcal{A}$ and $k$ refers to the labeling of the filtration $\mathcal{U}(\mathfrak{b}) = \cup_{k \in \mathbb{N}} \mathcal{U}_k(\mathfrak{b})$ associated to the chosen PBW basis, as defined in [4]. In turn, $\mathcal{A}^\infty$ becomes a Fréchet space, for the topology associated with the semi-norms

$$
\| \cdot \|_{j,K} : \mathcal{A}^\infty \to [0, \infty), \quad a \mapsto \sup_{x \in \mathcal{U}_k(g)} \frac{\| a \|_{j,K}}{|x_k|} = \sup_{x \in \mathcal{U}_k(g)} \frac{\| X^\alpha(a) \|_j}{|X|^k}, \quad j, k \in \mathbb{N}, \tag{59}
$$

with $\| \cdot \|_K$ the $\ell^1$-norm of $\mathcal{U}_k(\mathfrak{b})$ defined in [3]. As in [9], we have

$$
\| a \|_{j,K} \leq \max_{|\beta| \leq k} \| a \|_{j,K},
$$

with $\{ X^\beta, |\beta| \leq k \}$ the basis [3] of $\mathcal{U}_k(\mathfrak{b})$. Hence the semi-norms [59] are well defined on $\mathcal{A}^\infty$.

In the context of a tempered action on a Fréchet algebra $\mathcal{A}$, we observe that the restriction of the action to $\mathcal{A}^\infty$ is also tempered, but never isometric, even if the action is isometric on $\mathcal{A}$ and unless the group is Abelian.

**Lemma 5.3** Let $(\mathcal{A}, \{ \| \cdot \| \}, \alpha, \{ \mu^\alpha_g \})$ a Fréchet algebra endowed with a tempered action of a tempered Lie group $G$. Then, the restriction of $\alpha$ on $\mathcal{A}^\infty$ is tempered too, with:

$$
\| \alpha_g(a) \|_{j,K} \leq C(k) \delta(g)^k \mu^\alpha_g(a) \| a \|_{j,K}, \quad j, k \in \mathbb{N}, \ g \in G, \ a \in \mathcal{A}^\infty, \tag{59}
$$

where the function $\delta \in C^\infty(G)$ is the modular weight defined in Example 2.3.

**Proof.** First remark

$$
\| \alpha_g(a) \|_{j,K} = \sup_{x \in \mathcal{U}_k(g)} \frac{\| \alpha_g(\mathcal{A}d_g(X))^{\alpha}(a) \|_j}{|X|^k} \leq \mu^\alpha_g(a) \sup_{x \in \mathcal{U}_k(g)} \frac{\| \mathcal{A}d_g(X)^{\alpha}(a) \|_j}{|X|^k}.
$$

As for $X \in \mathcal{U}_k(g)$ and $a \in \mathcal{A}^\infty$, we have

$$
\| X^\alpha(a) \|_j \leq |X|^k \sup_{Y \in \mathcal{U}_k(g)} \frac{\| Y^\alpha(a) \|_j}{|Y|^k} = |X|^k \| a \|_{j,K},
$$

we get, with $|\mathcal{A}d_g|_k$ denoting the operator norm of the adjoint action of $G$ on the normed space $(\mathcal{U}_k(g), \| \cdot \|_k)$:

$$
\| \alpha_g(a) \|_{j,K} \leq \mu^\alpha_g(a) \sup_{x \in \mathcal{U}_k(g)} \frac{|\mathcal{A}d_g(X)|_k}{|X|^k} \| a \|_{j,K} = \mu^\alpha_g(a) |\mathcal{A}d_g|_k \| a \|_{j,K},
$$

and one concludes using Lemma 2.9. \[\square\]

**Example 5.4** Applying the former result to $\alpha = R^\ast$ and $\mathcal{A} = S_{\text{can}}(\mathcal{B})$ (which is its own space of smooth vectors), we see that the right-action of $\mathcal{B}$ on $S_{\text{can}}(\mathcal{B})$ is tempered.

For $a \in \mathcal{A}$, we let $\alpha(a)$ be the $\mathcal{A}$-valued function on $\mathcal{B}$, defined by

$$
\alpha(a) := [g \in \mathcal{B} \mapsto \alpha_g(a) \in \mathcal{A}].
$$

Given $\theta \in \mathbb{R}^\ast$ and $\tau \in \Theta^N$, our goal is to defined a new product $\star^\alpha_{\theta, \tau}$ on $\mathcal{A}^\infty$ by mean of the following formula:

$$
a \star^\alpha_{\theta, \tau} b := (\alpha(a) \star^\alpha_{\theta, \tau} \alpha(b))(e),
$$

and to show that this new algebra structure is compatible with the Fréchet topology of $\mathcal{A}^\infty$.

The following statement is the foundation of our construction:
Lemma 5.5 Let \((\alpha, \{\mu^\alpha_j\})\) be a tempered and strongly continuous action of a Lie group \(G\) on a Fréchet algebra \((A, \{\|\cdot\|_j\})\). Then, we have an equivariant continuous embedding

\[
\alpha : A^\infty \to B^{(u^\alpha_j \circ b)}(G, A^\infty), \quad a \mapsto \alpha(a) = [g \in G \mapsto \alpha_g(a) \in A^\infty].
\]

Proof. Note first that for \(a \in A\) and \(g, g_0 \in G\), we have

\[
\alpha(\alpha_g(a))(g_0) = \alpha_{g_0g}(a) = (R_{g_0}^\ast \alpha(a))(g_0),
\]

and thus \(a \in A \mapsto [g \mapsto \alpha_g(a)] \in C(G, A)\) intertwines the actions \(R^\ast\) and \(\alpha\). Let now \(a \in A^\infty\) and \(X \in U(g)\). By equivariance and strong-differentiability of \(\alpha\) on \(A^\infty\), we get

\[
\tilde{X} \alpha(a) = \alpha(X^\alpha a).
\]

Since for all \(j \in \mathbb{N}\) and all \(a \in A\), we have \(\|\alpha(a)\|_j \leq \mu^\alpha_j\|a\|_j\), we deduce that

\[
\|\alpha(a)\|_{j, k, \mu^\alpha_j \circ b, \infty} = \sup_{X \in U_k(g)} \sup_{g \in G} \|\tilde{X} \alpha(a)\|_j \leq \sup_{X \in U_k(g)} \sup_{g \in G} \|\tilde{X} \alpha(a)\|_j = \|\alpha(a)\|_j.
\]

This analysis shows that \(\alpha : A^\infty \to B^{(u^\alpha_j \circ b)}(G, A)\) is continuous. Now we want to take into account the intrinsic topology of \(A^\infty\) in the target space of the map \(\alpha\). Remark that the topology of \(B^{(u^\alpha_j \circ b)}(G, A)\) is associated with the countable set of semi-norms

\[
\|F\|_{(j, k), \mu^\alpha_j \circ b, \infty} = \sup_{X \in U_k(g)} \sup_{g \in G} \|Y^\alpha \tilde{X} F(g)\|_j.
\]

Since \(\alpha_{g^{-1}} \circ X^\alpha \circ \alpha_g = (\text{Ad}_{g^{-1}} X)^\alpha\) for all \(X \in U(g)\) and \(g \in G\), we get for \(F = \alpha(a)\):

\[
\|\alpha(a)\|_{(j, k), \mu^\alpha_j \circ b, \infty} = \sup_{X \in U_k(g)} \sup_{g \in G} \|Y^\alpha \tilde{X} F(g)\|_j = \sup_{X \in U_k(g)} \sup_{g \in G} \|Y^\alpha \tilde{X} F(g)\|_j.
\]

The next result, although rather obvious, will also play a key role.

Lemma 5.6 Let \(A\) be a Fréchet algebra with topology coming from a family of semi-norms \(\{\|\cdot\|_j\}\) and let \(\{\mu_j\}\) be a family of tempered weights on a Lie group \(G\). Then, the evaluation map at the unit element, \(B^{(u^\mu_j \circ b)}(G, A) \to A, F \mapsto F(e)\), is continuous.

Proof. Fix \(j \in \mathbb{N}\). Assuming that \(\mu_j(e) = 1\), we have for any \(F \in B^{(u^\mu_j \circ b)}(G, A)\):

\[
\|F(e)\|_j = \|F(e)\|_{\mu_j(e)} \leq \sup_{g \in G} \|F(g)\|_{\mu_j(g)} = \|F\|_{j, 0, \mu_j, \infty},
\]

and the result follows immediately.
Last, we need to lift the action $\alpha$ from $\mathcal{A}^\infty$ to $\mathcal{B}^{(\mu_j^\alpha \nu^\theta)}(\mathbb{B}, \mathcal{A}^\infty)$ and to show that this lift acts by automorphisms of the product $\ast_{\theta, \tau}$.

**Lemma 5.7** Let $\{\mu_{j,k}\}_{(j,k)\in \mathbb{N}^2}$ be a family of tempered weights of sub-multiplicative degree $\{(L_{j,k}, R_{j,k})\}$ and $(\alpha, \{\mu_{j,k}^\alpha\})_{j \in \mathbb{N}}$ be a strongly continuous and tempered action of a normal $\mathfrak{j}$-group $\mathbb{B}$ on a Fréchet algebra $(\mathcal{A}, \{\|\cdot\|_j\})_{j \in \mathbb{N}}$. For $g \in \mathbb{B}$, the map

$$\hat{\alpha}_g : F \mapsto \left[ g_0 \in \mathbb{B} \mapsto \alpha_g(F(g_0)) \right],$$

is continuous on $\mathcal{B}^{(\mu_{j,k})}(\mathbb{B}, \mathcal{A}^\infty)$. Moreover, given $(\theta, \tau) \in \mathbb{R}^* \times \Theta^N$, $\hat{\alpha}$ defines an action of $\mathbb{B}$ by automorphisms of the deformed product $\ast_{\theta, \tau}$, in the sense that for all $F \in \mathcal{B}^{(\mu_{j,k})}(\mathbb{B}, \mathcal{A}^\infty)$ and $F' \in \mathcal{B}^{(\mu_{j,k}')}(\mathbb{B}, \mathcal{A}^\infty)$, with $\{\mu_{j,k}'\}$ another family of tempered weights on $\mathbb{B}$ of sub-multiplicative degree $\{(L'_{j,k}, R'_{j,k})\}$, we have

$$\hat{\alpha}_g(F \ast_{\theta, \tau} F') = \hat{\alpha}_g(F) \ast_{\theta, \tau} \hat{\alpha}_g(F'), \quad \forall g \in \mathbb{B},$$

in $\mathcal{B}^{(\mu_{j,k}^\theta \mu_{j,k}')}_{(L_{j,k}^\theta L_{j,k}^\tau)}(\mathbb{B}, \mathcal{A}^\infty)$.

**Proof.** For $F \in \mathcal{B}^{(\mu_{j,k})}(\mathbb{B}, \mathcal{A}^\infty)$, $X, Y \in \mathcal{U}(\mathfrak{b})$ and $g, g' \in \mathbb{B}$, we have

$$Y^\alpha \left( \hat{\alpha}_g(F)(g') \right) = \alpha_g \left( (\text{Ad}_{g^{-1}} Y)^\alpha (\hat{\mathcal{X}} F(g')) \right).$$

This entails that

$$\|\hat{\alpha}_g(F)\|_{(j,k),k',\mu_{j,k},\infty} = \sup_{X \in \mathcal{U}(\mathfrak{b})} \sup_{g' \in \mathbb{B}} \sup_{Y \in \mathcal{U}(\mathfrak{b})} \frac{\|Y^\alpha \left( \hat{\alpha}_g(F)(g') \right)\|_j}{\mu_{j,k}(g') \|X_k\|_k},$$

$$= \sup_{X \in \mathcal{U}(\mathfrak{b})} \sup_{g' \in \mathbb{B}} \sup_{Y \in \mathcal{U}(\mathfrak{b})} \frac{\|Y^\alpha \left( \hat{\alpha}_g(F)(g') \right)\|_j}{\mu_{j,k'}(g') \|X_{k'}\|_{k'}},$$

$$\leq C(k) \mu_{j,k}^\alpha(g) \partial(g)^k \sup_{X \in \mathcal{U}(\mathfrak{b})} \sup_{g' \in \mathbb{B}} \sup_{Y \in \mathcal{U}(\mathfrak{b})} \frac{\|Y^\alpha \left( \hat{\mathcal{X}} F(g') \right)\|_j}{\mu_{j,k'}(g') \|X_{k'}\|_{k'}},$$

proving the continuity.

Next, consider $F \in \mathcal{B}^{(\mu_{j,k})}(\mathbb{B}, \mathcal{A}^\infty)$ and $F' \in \mathcal{B}^{(\mu_{j,k}')}(\mathbb{B}, \mathcal{A}^\infty)$, together with $\{\tilde{\mu}_{j,k}\}$ and $\{\mu_{j,k}'\}$, two families of tempered weights that dominate respectively $\{\mu_{j,k}\}$ and $\{\mu_{j,k}'\}$. Defining $F_n := F e_n \in \mathcal{D}(\mathbb{B}, \mathcal{A})$ and $F'_n = F' e_n \in \mathcal{D}(\mathbb{B}, \mathcal{A})$ defined in (12), from

$$\hat{\alpha}_g(F_n) = \hat{\alpha}_g(F) e_n, \quad \hat{\alpha}_g(F'_n) = \hat{\alpha}_g(F') e_n,$$

we deduce from Lemma 2.8(viii) that $\{\hat{\alpha}_g(F_n)\}$ and $\{\hat{\alpha}_g(F'_n)\}$ converges to $\{\hat{\alpha}_g(F)\}$ and $\{\hat{\alpha}_g(F')\}$ in the topologies of $\mathcal{B}^{(\mu_{j,k})}(\mathbb{B}, \mathcal{A}^\infty)$ and $\mathcal{B}^{(\mu_{j,k}')}(\mathbb{B}, \mathcal{A}^\infty)$ respectively. Thus, we can use Lemma 2.39 to get the $\hat{\alpha}$-equivariance at the level of smooth compactly supported functions from the commutativity of $\hat{\alpha}$ and $R^*$:

$$\hat{\alpha}_g(F \ast_{\theta, \tau} F') = \hat{\alpha}_g(\lim_{n,n' \to \infty} F_n \ast_{\theta, \tau} F'_n) = \lim_{n,n' \to \infty} \hat{\alpha}_g(F_n \ast_{\theta, \tau} F'_n)$$

$$= \lim_{n,n' \to \infty} \hat{\alpha}_g(F_n) \ast_{\theta, \tau} \hat{\alpha}_g(F'_n) = \hat{\alpha}_g(F) \ast_{\theta, \tau} \hat{\alpha}_g(F'),$$

in $\mathcal{B}^{(\mu_{j,k}^\theta \mu_{j,k}')}_{(L_{j,k}^\theta L_{j,k}^\tau)}(\mathbb{B}, \mathcal{A}^\infty)$, and this concludes the proof.

We are now prepared to state the main result of the first part of this paper:

**Theorem 5.8 (Universal Deformation Formula of Fréchet Algebras)** Let $(\mathcal{A}, \alpha, \mathbb{B})$ be a Fréchet algebra endowed with a tempered and strongly continuous action of a normal $\mathfrak{j}$-group. Let also $\theta \in \mathbb{R}^*$ and $\tau \in \Theta^N$. Then, $(\mathcal{A}^\infty, \ast_{\theta, \tau}^\theta)$ is an associative Frechet algebra with jointly continuous product.
Proof. Let \( \{\mu^j_\alpha\} \) be the family of tempered weights, with sub-multiplicative degrees \( \{(L_j, R_j)\} \), associated with the tempered action \( \alpha \) as in Definition 5.1. Let \( a, b \in \mathcal{A}^\infty \), then by Lemma 5.5, \( \alpha(a), \alpha(b) \in \mathcal{B}(\mu^{\alpha_0} \otimes \mathcal{A}^\infty) \). Then, since \( \mathcal{B} \) is sub-multiplicative of degree \( (1, 1) \), Theorem 5.4 shows that \( \alpha(a) *_{\theta, \overline{\tau}} \alpha(b) \) belongs to \( \mathcal{B}(\mu^{\alpha_0} \otimes \mathcal{A}^\infty) \) and that the map

\[
\mathcal{A}^\infty \times \mathcal{A}^\infty \to \mathcal{B}(\mu^{\alpha_0} \otimes \mathcal{A}^\infty), \quad (a, b) \mapsto \alpha(a) *_{\theta, \overline{\tau}} \alpha(b),
\]

is continuous. Applying Lemma 5.6 for the Fréchet algebra \( \mathcal{A} \) yields that the composition of maps

\[
\mathcal{A}^\infty \times \mathcal{A}^\infty \to \mathcal{B}(\mu^{\alpha_0} \otimes \mathcal{A}^\infty) \to \mathcal{A}^\infty, \quad (a, b) \mapsto \alpha(a) *_{\theta, \overline{\tau}} \alpha(b) \mapsto (\alpha(a) *_{\theta, \overline{\tau}} \alpha(b))(c) =: a *_{\theta, \overline{\tau}} b,
\]

is continuous.

It remains to prove associativity. With \( \hat{\alpha} \) defined in Lemma 5.7 we compute for \( a, b \in \mathcal{A}^\infty \) and \( g \in \mathbb{B} \):

\[
\alpha(a) *_{\theta, \overline{\tau}} b(g) = \alpha_g(a) *_{\theta, \overline{\tau}} b = \alpha_g(\alpha(a) *_{\theta, \overline{\tau}} \alpha(b))(e) = \hat{\alpha}_g(\alpha(a) *_{\theta, \overline{\tau}} \alpha(b))(e).
\]

Using Lemma 5.7 we deduce the equality in \( \mathcal{B}(\mu^{\alpha_0} \otimes \mathcal{A}^\infty) \):

\[
\hat{\alpha}_g(\alpha(a) *_{\theta, \overline{\tau}} \alpha(b)) = \alpha_g(\alpha(a)) *_{\theta, \overline{\tau}} \hat{\alpha}_g(\alpha(b)).
\]

As a short computation shows, for \( a \in \mathcal{A} \) and \( g \in \mathbb{B} \), we have \( \hat{\alpha}_g(\alpha(a)) = L^{-1}_g(\alpha(a)) \). Thus, using the equivariance of the product \( *_{\theta, \overline{\tau}} \) under the left regular action, as stated in Theorem 4.7, we get

\[
\hat{\alpha}_g(\alpha(a)) *_{\theta, \overline{\tau}} \hat{\alpha}_g(\alpha(b)) = L^{-1}_g(\alpha(a)) *_{\theta, \overline{\tau}} L^{-1}_g(\alpha(b)) = L^{-1}_g(\alpha(a) *_{\theta, \overline{\tau}} \alpha(b)),
\]

in \( \mathcal{B}(\mu^{\alpha_0} \otimes \mathcal{A}^\infty) \). Evaluating this equality at the unit element, yields, by Lemma 5.10 that the map

\[
\alpha(\alpha(a) *_{\theta, \overline{\tau}} b)(g) = L^{-1}_g(\alpha(a) *_{\theta, \overline{\tau}} \alpha(b))(e) = (\alpha(a) *_{\theta, \overline{\tau}} \alpha(b))(g).
\]

Hence, we proved that the functions \( \alpha(\alpha *_{\theta, \overline{\tau}} b) \) and \( \alpha(a) *_{\theta, \overline{\tau}} \alpha(b) \) coincide. This implies for \( a, b, c \in \mathcal{A}^\infty \):

\[
a *_{\theta, \overline{\tau}} (b *_{\theta, \overline{\tau}} c) = (\alpha(a) *_{\theta, \overline{\tau}} \alpha(b) *_{\theta, \overline{\tau}} \alpha(c))(e) = (\alpha(a) *_{\theta, \overline{\tau}} \alpha(b) *_{\theta, \overline{\tau}} \alpha(c))(e),
\]

and the associativity of \( *_{\theta, \overline{\tau}} \) on \( \mathcal{A}^\infty \) follows from associativity of \( *_{\theta, \overline{\tau}} \) on the triple Cartesian product of the space \( \mathcal{B}(\mu^{\alpha_0} \otimes \mathcal{A}^\infty) \) as stated in Theorem 4.7.

Remark 5.9 Contrarily to the \( \mathbb{B}^d \)-action case treated in [19], in the non-Abelian situation the original action is no longer an automorphism of the deformed product \( *_{\theta, \overline{\tau}} \) on \( \mathcal{A}^\infty \). This can be understood as the chief reason to introduce the whole oscillatory integrals machinery in section 2 and also to consider the spaces \( \mathcal{B}(\mu^j \otimes \mathcal{A}) \) for families of weights \( \{\mu_j\} \).

To conclude this section, we establish a formula for the deformed product \( *_{\theta, \overline{\tau}} \) on \( \mathcal{A}^\infty \), which in some sense, is more natural. It will also clarify an important point, namely that the universal deformation of algebra \( \mathcal{A} = C_{ru}(\mathbb{B}) \), for the action \( \alpha = R^* \) coincides with \( (\mathcal{B}(\mathbb{B}), *_{\theta, \overline{\tau}}) \).

Proposition 5.10 Let \( \alpha, \{\mu^j_\alpha\} \) be a (strongly continuous) tempered action of a normal \( \mathfrak{j} \)-group \( \mathbb{B} \) on a Fréchet algebra \( \mathcal{A} \). Then, for \( a, b \in \mathcal{A}^\infty \) and \( \theta \in \mathbb{R}, \overline{\tau} \in \Theta^N \), we have

\[
a *_{\theta, \overline{\tau}} b = \int_{\mathbb{B} \times \mathbb{B}} K_{\theta, \overline{\tau}} (\alpha(a) \otimes \alpha(b)),
\]

where we denote

\[
\alpha(a) \otimes \alpha(b) : \mathbb{B} \times \mathbb{B} \to \mathcal{A}^\infty : (x, y) \mapsto \alpha_x(a) \alpha_y(b).
\]

5Remind that \( C_{ru}(\mathbb{B}) \) denotes the \( \mathcal{C}^* \)-algebra of right uniformly continuous and bounded functions on \( \mathbb{B} \).
Example 5.12 Take any Lie group $G$ and define

$$
\mathcal{A} = L^p(G) \cap L^\infty(G), \quad p \in (1, \infty),
$$

normed with $\|\cdot\|_p + \|\cdot\|_\infty$. On this Banach algebra, the right regular action is almost isometric with associated weight given by $\Delta^G_{1/p}$.

Also, to simplify the discussion below, we assume that each weight $\mu_j$ is sub-multiplicative. We start with the simple observation that for any family of tempered weights $\{\mu_j\}$, the extended action (defined in Lemma 5.7) $\hat{\alpha}$ on $\mathcal{B}(\mu_j)(\mathbb{B}, \mathcal{A})$, commutes with the left regular action $L^*$. This leads us to defined the commuting composite action $\beta := \hat{\alpha} \circ L^* = L^* \circ \hat{\alpha}$, explicitly given by:

$$
(\beta_g F)(g_0) := \alpha_g F(g^{-1}g_0), \quad g, g_0 \in \mathbb{B}, \quad F \in \mathcal{B}(\mu_j)(\mathbb{B}, \mathcal{A}).
$$
Note also that by Lemma 2.38 (ii) and Lemma 5.7, for fixed \(g \in \mathbb{B}\), \(\beta_g\) sends continuously \(\mathcal{B}^{(\mu_j)}(\mathbb{B}, \mathcal{A})\) to \(\mathcal{B}^{(\mu_j)}(\mathbb{B}, \mathcal{A})\), if \((L_j, R_j)\) is the sub-multiplicative degree of the weight \(\mu_j\). Thus, in our context of sub-multiplicative weights, \(\beta_g\) is continuous on \(\mathcal{B}^{(\mu_j)}(\mathbb{B}, \mathcal{A})\).

Now observe that for an almost isometric action of a Lie group \(G\) on a Fréchet algebra \(\mathcal{A}\), the map \(\alpha : a \mapsto [g \mapsto \alpha_g(a)]\), is an isometric embedding of \(\mathcal{A}^\infty\) into \(\mathcal{B}^{(\mu_j)}(G, \mathcal{A})\). Indeed for all \(j, k \in \mathbb{N}\), we have

\[
\|\alpha(a)\|_{j,k,\mu_j^{\infty}} = \sup_{X \in \mathcal{U}_k(g)} \sup_{g \in G} \frac{\|X \alpha(a)(g)\|_j}{\mu_j^k(g)} = \sup_{X \in \mathcal{U}_k(g)} \frac{\|X^a a\|_j}{|X_k|} = \|a\|_{j,k}.
\]

(61)

By \(\mathcal{B}^{(\mu_j)}(\mathbb{B}, \mathcal{A}))^\beta\), we denote the closed subspace of \(\mathcal{B}^{(\mu_j)}(\mathbb{B}, \mathcal{A})\) of fixed points for the action \(\beta\). It is then immediate to see that the image of \(\mathcal{A}^\infty\) under \(\alpha\) lies inside \(\mathcal{B}^{(\mu_j)}(\mathbb{B}, \mathcal{A}))^\beta\). Reciprocally, an element \(F\) of \(\mathcal{B}^{(\mu_j)}(\mathbb{B}, \mathcal{A}))^\beta\), satisfies \(F(g) = \alpha_g(F(e))\) for all \(g \in \mathbb{B}\), i.e. \(F = \alpha(a)\) with \(a := F(e) \in \mathcal{A}\). But by our assumption of almost-isometry and (61), we have \(\|a\|_{j,k} = \|F\|_{j,k,\infty}\), for all \(j, k \in \mathbb{N}\), and thus \(a = F(e)\) has to be smooth. This proves that \(\alpha : \mathcal{A}^\infty \to (\mathcal{B}^{(\mu_j)}(\mathbb{B}, \mathcal{A}))^\beta\) is an isomorphism of Fréchet spaces, which is isometric for each semi-norms. Moreover, the map \(\alpha\) is an algebra homomorphism. Indeed, by the arguments given in the proof of Theorem 5.8, applied to the case of an almost-isometric action with sub-multiplicative weights \(\mu_j^a\), for all \(a, b \in \mathcal{A}^\infty\) we have the equality

\[
\alpha(a \ast_{\theta, \overline{\tau}} b) = \alpha(a) \ast_{\theta, \overline{\tau}} \alpha(b) \text{ in } (\mathcal{B}^{(\mu_j)}(\mathbb{B}, \mathcal{A}))^\beta.
\]

In summary, we have proven the following:

**Proposition 5.13** Let \(\theta \in \mathbb{R}^*, \overline{\tau} \in \Theta^N\) and let \((\mathcal{A}, \alpha)\) be a Fréchet algebra endowed with a (strongly continuous) tempered and almost-isometric action with sub-multiplicative weights \(\mu_j^a\), of a normal \(j\)-group \(\mathbb{B}\). Then, we have an isometric isomorphism of Fréchet algebras:

\[
(\mathcal{A}^\infty, \ast_{\theta, \overline{\tau}}^a) \simeq (\mathcal{B}^{(\mu_j)}(\mathbb{B}, \mathcal{A}))^\beta, \ast_{\theta, \overline{\tau}}.
\]

**Remark 5.14** We stress that the assumption of sub-multiplicativity for the family of weights \(\mu_j^a\), associated with the tempered action \(\alpha\), is in fact irrelevant in the previous result. However it is unclear to us whether a similar statement holds without the assumption of almost-isometry.

### 5.3 Functorial properties of the deformed product

To conclude with the deformation theory at the level of Fréchet algebras, we establish some functorial properties. We come back to the general setting of a strongly continuous and tempered action \((\alpha, \{\mu_j^a\})\) of a normal \(j\)-group \(\mathbb{B}\) on a Fréchet algebra \(\mathcal{A}\) (i.e. we no longer assume that the action is almost isometric). We start with the question of algebra homomorphisms.

**Proposition 5.15** Let \((\mathcal{A}, \{\|\cdot\|\}), (\mathcal{F}, \{\|\cdot\|\}), (\mathcal{B}, \{\|\cdot\|\}), (\mathcal{G}, \{\|\cdot\|\}), \alpha, \beta\) two Fréchet algebras endowed with a strongly continuous and tempered actions of a normal \(j\)-group \(\mathbb{B}\) by automorphisms. Let also \(T : \mathcal{A} \to \mathcal{F}\) be a continuous homomorphism such that for all \(j \in \mathbb{N}\) there exist \(k(j) \in \mathbb{N}\) and \(C_j > 0\), such that for all \(a \in \mathcal{A}\), we have

\[
\|T(a)\|_j \leq C_j \|a\|_{k(j)}
\]

and such that \(T\) intertwines the actions \(\alpha\) and \(\beta\). Then for any \(\theta \in \mathbb{R}^*\) and \(\overline{\tau} \in \Theta^N\), the map \(T\) restricts to a homomorphism from \((\mathcal{A}^\infty, \ast_{\theta, \overline{\tau}}^a)\) to \((\mathcal{F}^\infty, \ast_{\theta, \overline{\tau}}^\beta)\).

**Proof.** Since by assumption \(T \circ \alpha = \beta \circ T\), we get for any \(P \in \mathcal{U}(\mathcal{b})\) that \(T \circ P^\alpha = P^\beta \circ T\), which entails that \(T\) restricts to a continuous map from \(\mathcal{A}^\infty\) to \(\mathcal{F}^\infty\). The remaining part of the statement follows then by Lemma 2.31. □
Next, we prove that if a Fréchet algebra is endowed with a continuous involution, then the latter will also define a continuous involution for the deformed product, under the mild condition\(^6\) that \(\bar{\tau}(-a) = \tau(a)\). Indeed, the latter implies that
\[
\overline{K_{\theta,\tau}}(x_1,x_2) = K_{\theta,\tau}(x_2,x_1),
\]
so by Lemma 2.39 we get:

**Proposition 5.16** Let \((A,\alpha)\) be a Fréchet algebras endowed with a (strongly continuous) tempered action of a normal \(\mathfrak{j}\)-group \(B\). Assuming that for \(\theta \in \mathbb{R}^*\) and \(\bar{\tau} \in \Theta^N\), we have \(\tau_j(-a) = \tau_j(a), \ j = 1, \ldots, N\), then any continuous involution of \(A^\infty\) is a continuous involution of \((A^\infty, *_{\theta,\bar{\tau}})\) too.

In a similar way, we deduce from Lemma 2.39 that the deformation is ideal preserving:

**Proposition 5.17** Let \((A,\alpha)\) be a Fréchet algebras endowed with a (strongly continuous) tempered action of a normal \(\mathfrak{j}\)-group \(B\) and \(\theta \in \mathbb{R}^*\), \(\bar{\tau} \in \Theta^N\). If \(I\) is a closed \(\alpha\)-invariant ideal of \(A\), then \(I^\infty\) is a closed ideal of \((A^\infty, *_{\theta,\bar{\tau}})\).

We now examine the consequence of the fact that the constant function is the unit of \(A\).

We recall that a Fréchet algebra \((A,\|\|)\) is a closed \(\mathfrak{j}\)-group.

**Proposition 5.18** Let \((A,\alpha)\) be a Fréchet algebras endowed with a strongly continuous and tempered action of a normal \(\mathfrak{j}\)-group \(B\) and \(\theta \in \mathbb{R}^*\), \(\bar{\tau} \in \Theta^N\). If \(a \in A^\infty\) is fixed by the action \(\alpha\), then for \(b \in A^\infty\), we have
\[
a *_{\theta,\bar{\tau}} b = ab, \quad b *_{\theta,\bar{\tau}} a = ba.
\]

**Proof.** This is a consequence of Proposition 4.9 together with the defining relation of the deformed product:
\[
a *_{\theta,\bar{\tau}} b = (\alpha(a) *_{\theta,\bar{\tau}} \alpha(b))(e) = (a *_{\theta,\bar{\tau}} \alpha(b))(e) = (a \alpha(b))(e) = ab.
\]
The second equality is entirely similar.

Next, we study the question of the existence of a bounded approximate unit for the Fréchet algebra \((A^\infty, *_{\theta,\bar{\tau}})\).

We recall that a Fréchet algebra \((A, \{\|\|_j\})\) admits a bounded approximate unit if there exists a net \(\{e_\lambda\}_{\lambda \in \Lambda}\) of elements of \(A\) such that for any \(a \in A\), there exists \(\{\lambda e_\lambda\}_{\lambda \in \Lambda}\) converges to \(a\) and such that for each \(j \in \mathbb{N}\), there exists \(C_j > 0\) such that for every \(\lambda \in \Lambda\), we have \(\|\lambda e_\lambda\| \leq C_j\).

**Proposition 5.19** Let \((A,\alpha)\) be a Fréchet algebras endowed with a strongly continuous and tempered action of a normal \(\mathfrak{j}\)-group \(B\) and such that \(A\) admits a bounded approximate unit. Then for any \(\theta \in \mathbb{R}^*, \bar{\tau} \in \Theta^N\), the Fréchet algebra \((A^\infty, *_{\theta,\bar{\tau}})\) admits a bounded approximate unit too.

**Proof.** Let \(\{f_\lambda\}\) be a net of bounded approximate units for \(A\), let \(0 \leq \psi \in \mathcal{D}(B)\) of \(L^1\)-norm one and define
\[
e_\lambda := \int_B \psi(g) \alpha_g(f_\lambda) \, d\mu(g).
\]
Observe that even if \(\{f_\lambda\}\) is not smooth, \(\{e_\lambda\}\) is. Indeed, for all \(X \in \mathcal{U}(b)\), we have
\[
X^\alpha e_\lambda = \int_B X \psi(g) \alpha_g(f_\lambda) \, d\mu(g),
\]
and we get for the semi-norms defining the topology of \(A^\infty\), with \(\{\mu_j^\alpha\}\) the family of tempered weights associated to the temperedness of the action \(\alpha\):
\[
\|e_\lambda\|_{j,k} = \sup_{X \in \mathcal{U}(b)} \frac{\|X^\alpha e_\lambda\|_j}{|X|_k} \leq \int_B |X|_k \psi(g) \, d\mu(g) \|\alpha_g(f_\lambda)\|_j \leq \int_B \sup_{X \in \mathcal{U}(b)} \frac{|X|_k \psi(g)}{|X|_k} \mu_j^\alpha(g) \, d\mu(g) \times \|f_\lambda\|_j.
\]
Hence, the net \(\{e_\lambda\}\) belongs to \(A^\infty\) and is semi-norm-wise bounded in \(\lambda \in \Lambda\) as \(\|f_\lambda\|_j\) is. Next, we show that it is indeed an approximate unit for \(A^\infty\): Since \(\int \psi = 1\), we first note that for any \(a \in A\)
\[
e_\lambda a - a = \int_B \psi(g) (\alpha_g(f_\lambda)a - a) \, d\mu(g) = \int_B \psi(g) \alpha_g(f_\lambda \alpha_g(a) - a \alpha_g(a)) \, d\mu(g),
\]
\(^6\)In Lemma 2.39 we will see how to suppress this extra condition.
which gives
\[ \|e_\lambda a - a\|_j \leq \int_{\mathbb{B}} \psi(g) \mu_j^\alpha(g) \|f_{\lambda} \alpha_{g^{-1}}(a) - \alpha_{g^{-1}}(a)\|_j \, d_\mathbb{B}(g), \]
which converges to zero because \( \|f_{\lambda} \alpha_{g^{-1}}(a) - \alpha_{g^{-1}}(a)\|_j \) does by assumptions and because \( \psi \) is compactly supported. The general case is treated recursively exactly as in the proof of Lemma 2.7 (viii). Hence, \( A^\infty \) (with its original algebraic structure) admits a bounded approximate unit too. Now, we will prove that a bounded approximate unit for \( A^\infty \) is also a bounded approximate unit for \( (A^\infty, \ast_{\theta, \tau}) \).

So, let \( \{e_\lambda\} \) be any bounded approximate unit for \( A^\infty \). First observes that if we view the product \( \ast_{\theta, \tau} \) as a bilinear map
\[ \ast_{\theta, \tau} : \mathcal{B}(\mathbb{B}) \times \mathcal{B}(\mu_j^\alpha \sigma_k)(\mathbb{B}, A^\infty) \to \mathcal{B}(\mu_j^\alpha \sigma_k)(\mathbb{B}, A^\infty), \]
a slight adaptation of the arguments of Proposition 4.9 shows that for all \( a \in A^\infty \):
\[ 1 \ast_{\theta, \tau} a = \alpha(a), \]
where 1 denotes the unit element of \( \mathcal{B}(\mathbb{B}) \). Combining this with Proposition 5.10 gives the equality in \( A^\infty \):
\[ e_\lambda \ast_{\theta, \tau} a - a = \int_{\mathbb{B} \times \mathbb{B}} K_{\theta, \tau}(\alpha(e_\lambda) \otimes \alpha(a) - 1 \otimes \alpha(a)) \, d_\mathbb{B}(x) \, d_\mathbb{B}(y), \]
where
\[ \alpha(e_\lambda) \otimes \alpha(a) - 1 \otimes \alpha(a) := [(x, y) \in \mathbb{B} \times \mathbb{B} \mapsto \alpha_x(e_\lambda) \alpha_y(a) - \alpha_y(a)] \in \mathcal{B}(\mu_j^\alpha \sigma_k \otimes \mu_j^\alpha \sigma_k), \]
\( 1, \ast_{\theta, \tau} \mathcal{A}_0 \) is a \( \mathcal{B} \)-sub-group. Moreover, the set of smooth vectors \( \{e_\lambda\}_{\lambda \in \Lambda} \) is bounded in the absolute norm \( \|\cdot\|_{j, k} \) of \( A^\infty \). As, the net \( \{e_\lambda\}_{\lambda \in \Lambda} \) is bounded in the semi-norm \( \|\cdot\|_{j, k} \), we may apply dominated convergence to get
\[ \lim_{\lambda} \|e_\lambda \ast_{\theta, \tau} a - a\|_{j, k} = 0. \]
This concludes the proof as the arguments for \( a \ast_{\theta, \tau} e_\lambda \) are similar.

At last, we show that the deformation associated with a normal \( j \)-group coincides with the iterated deformations of each of its elementary normal \( j \)-sub-groups.

**Proposition 5.20** Let \( \mathbb{B} \) be a normal \( j \)-group with Pyatetskii-Shapiro decomposition \( \mathbb{B} = \mathbb{B}' \ltimes S \), where \( \mathbb{B}' \) is a normal \( j \)-group and \( S \) is an elementary normal \( j \)-group. Let \( \mathcal{A} \) be a Fréchet algebra endowed with a strongly continuous and tempered action \( (\alpha, \{\mu_j^\alpha\}) \) of \( \mathbb{B} \). Denote by \( \alpha^{\mathbb{B}'} \) (respectively by \( \alpha^{\mathbb{B}'} \) respectively by \( \alpha_\mathcal{C} \)) the restriction of \( \alpha \) to \( \mathbb{B}' \) (respectively to \( \mathcal{C} \)). For \( \mathcal{C} \) a sub-space of \( \mathcal{A} \), denote by \( \mathcal{C}_\mathbb{B}^\infty \) (respectively by \( \mathcal{C}_{\mathbb{B}'}^\infty \) the set of smooth vectors in \( \mathcal{C} \) for the action of \( \mathbb{B} \) (respectively of \( \mathbb{B}' \), \( S \)). Then, for \( \theta \in \mathbb{R}^+ \) and \( \tau = (\tau^r, \tau_1) \in \Theta^{N+1} \) \( (N \) is the number of elementary factors in \( \mathbb{B}' \)), we have
\[ (A_{\mathbb{B}'}^\infty, \ast_{\theta, \tau})_{\mathbb{B}'} = (A_{\mathbb{B}'}^\infty, \ast_{\theta, \tau}). \]

**Proof.** Observe that being the restrictions of a strongly continuous and tempered action, the action \( \alpha^{\mathbb{B}'} \) of \( S \) on \( \mathcal{A} \) is also strongly continuous and tempered. But the action \( \alpha^{\mathbb{B}'} \) of \( \mathbb{B}' \) on \( A_{\mathbb{B}'}^\infty \) is also strongly continuous (which is rather obvious) and tempered. To see that, note that for \( g' \in \mathbb{B}' \) and \( a \in A_{\mathbb{B}'}^\infty \), we have
\[ \|\alpha_{g'}(a)\|_{j, k} = \sup_{X \in t_4(s)} \frac{\|X^{\alpha} a_{g'}(a)\|_j}{|X|_k} = \sup_{X \in t_4(s)} \frac{\|\alpha_{g'}((A_{g'}^{-1})X^{\alpha} a)\|_j}{|X|_k} \leq \mu_j^\alpha(g') \sup_{X \in t_4(s)} \frac{\|((A_{g'}^{-1})X)^{\alpha} a\|_j}{|X|_k}. \]
As $\mathbb{B}^\prime$ acts on $\mathbb{S}$ by conjugation, it acts on $\mathcal{U}_k(s)$ and by Lemma 2.9 we deduce that

$$\|\alpha^B_{g'}(a)\|_{j,k} \leq C(k) \mu_j^f(g') \|d(g')^k\|a\|_{j,k},$$

and hence the extension $\alpha^B_{g'}$ of $\mathbb{B}^\prime$ on $A^\infty_\mathbb{S}$ is tempered with associated family of tempered weights given by $\{\mu^f_j\}_{j,k} \in \mathbb{N}^2$. Note also that the subspace of smooth vectors for $\mathbb{B}$ coincides with the subspace of smooth vectors for $\mathbb{B}^\prime$ within the subspace of smooth vectors for $\mathbb{S}$, i.e.

$$A^\infty_\mathbb{S} = (A^\infty_\mathbb{S})^\infty_{g'}.$$

Indeed, the inclusion $A^\infty_\mathbb{S} \subset (A^\infty_\mathbb{S})^\infty_{g'}$ is clear since $a \in (A^\infty_\mathbb{S})^\infty_{g'}$ if and only if for all $X' \in \mathcal{U}(b')$, all $X \in \mathcal{U}(s)$ and all $j \in \mathbb{N}$, we have

$$\|X^{a_{g'}}^X X^{a_{g'}} a\|_j < \infty,$$

and $X'X \in \mathcal{U}(b)$. But this also gives the reversed inclusion since has subgroups, we formulate them automorphisms on the deformed Fréchet algebra $(A^\infty_\mathbb{S}, \alpha^S_{g'})$. First, by Proposition 6.11 and Lemma 2.39 we get with the elements $e_n \in D(\mathbb{S})$ defined in (12), $n \in \mathbb{N}$, and for $a, b \in A^\infty_\mathbb{S}$:

$$a \star_{\theta, \tau}^S b = \int_{\mathbb{S} \times \mathbb{S}} K_{\theta, \tau}(a(a) \star (a)(b)) = \lim_{n,m \to \infty} \int_{\mathbb{S} \times \mathbb{S}} K_{\theta, \tau}(x, y) e_n(x) e_m(y) a_\theta^S(x) e_m(y) a_\theta^S(y) \|b\|_{S}(x) \|b\|_{S}(y).$$

Observe also that (62) shows that for $g' \in \mathbb{B}'$ fixed, the operator $\alpha^S_{g'}$ is continuous on $A^\infty_\mathbb{S}$. From this and the absolute convergence of the integrals in the product $\star_{\theta, \tau}^S$ at the level of compactly supported functions, we deduce that for $a, b \in A^\infty_\mathbb{S}$ and $g' \in \mathbb{B}^\prime$:

$$\alpha^S_{g'}(a \star_{\theta, \tau}^S b) = \lim_{n,m \to \infty} \alpha^S_{g'} \left( \int_{\mathbb{S} \times \mathbb{S}} K_{\theta, \tau}(x, y) e_n(x) a_\theta^S(a) e_m(y) a_\theta^S(b) \|b\|_{S}(x) \|b\|_{S}(y) \right).$$

Remember that

$$R \in \text{Hom}(\mathbb{B}^\prime, \text{Aut}(\mathbb{S}, s, \omega^S) \cap \text{Sp}(V, \omega^0)),$$

where $(V, \omega^0)$ is the symplectic vector space attached to $\mathbb{S}$. Using the invariance of the two-point kernel of the left Haar measure under the action of $\text{Sp}(V, \omega^0) = \text{Aut}(\mathbb{S}) \cap \text{Aut}(s, \omega^S)$, we get:

$$\alpha^S_{g'}(a \star_{\theta, \tau}^S b) = \lim_{n,m \to \infty} \int_{\mathbb{S} \times \mathbb{S}} K_{\theta, \tau}(x, y) e_n(x) a_\theta^S(a) e_m(y) a_\theta^S(b) \|b\|_{S}(x) \|b\|_{S}(y)$$

Thus, both Fréchet algebras $(A^\infty_\mathbb{S}, \star_{\theta, \tau}^S)$ and $(A^\infty_\mathbb{S}, \star_{\theta, \tau}^R)$ are well defined and their underlying sets coincide. It remains to show that their algebraic structures coincide too. But this follows from Proposition 2.33 as the the extension homomorphism $R$ of $\mathbb{B} = \mathbb{B}^\prime \ltimes_R \mathbb{S}$ is tempered.

6 Quantization of polarized symplectic symmetric spaces

We are arrived at the second part of this work. Here we consider the data $(A, \alpha, \mathbb{B})$, consisting in a $C^*$-algebra endowed with a strongly continuous and isometric action of a normal $j$-group. Our aim is to construct a $C^*$-norm on the deformed Fréchet algebra $(A^\infty_\mathbb{S}, \star_{\theta, \tau}^R)$. Our methods rely on the construction of a global pseudo-differential calculus on the symplectic symmetric space underlying an elementary normal $j$-group, in such a way that the associated law of composition of symbols is indeed the star-product $\star_{\theta, \tau}$. Because a good part of our results do not rely on specific properties elementary normal $j$-groups, we formulate them in the more general context of polarized symplectic symmetric spaces.
6.1 Polarized symplectic symmetric spaces

Let \((M, s, \omega)\) be a symplectic symmetric space and let \(\text{Aut}(M, s, \omega)\) its automorphism group and \(\text{aut}(M, s, \omega)\) its derivation algebra (see Definition 5.7 and Proposition 5.9). Choose a base point \(o\) in \(M\). Then the conjugation by the symmetry at \(o\) yields an involutive automorphism of the automorphism group:

\[ \sigma : \text{Aut}(M, s, \omega) \to \text{Aut}(M, s, \omega), \quad g \mapsto s_o \circ g \circ s_o =: \sigma(g) . \]  

We extract the following result from [1]:

**Proposition 6.1** The smallest sub-group \(G(M)\) of \(\text{Aut}(M, s, \omega)\) that is stable under \(\sigma\) and that acts transitively on \(M\) is a Lie sub-group of \(\text{Aut}(M, s, \omega)\). It coincides with group generated by products of an even number of symmetries:

\[ G(M) = \text{gr}\{s_x \circ s_y \mid x, y \in M\} . \]

The group \(G(M)\) is called the **transvection group** of \(M\).

We now come to the notion of polarized symplectic symmetric spaces:

**Definition 6.2** A symplectic symmetric space \((M, \omega, s)\) is said to be **polarizable** if it admits a \(G(M)\)-invariant Lagrangian tangent distribution. A choice of such a transvection-invariant distribution \(W \subset TM\) determines a **polarization** of \(M\), in which case one speaks about a **polarized** symplectic symmetric space.

The infinitesimal version of the notion of symplectic symmetric space is given in the following definition:

**Definition 6.3** A symplectic involutive Lie algebra (shortly a “siLa”) is a triple \((\mathfrak{g}, \sigma, \varpi)\) where \(\mathfrak{g}\) is a finite dimensional real Lie algebra, \(\sigma\) is an involutive automorphism of \(\mathfrak{g}\) and \(\varpi \in \wedge^2 \mathfrak{g}^*\) is a Chevalley two-cocycle on \(\mathfrak{g}\) (valued in the trivial representation on \(\mathbb{R}\)) such that, denoting by

\[ \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} , \]

the \(\pm 1\)-eigenspace decomposition of \(\mathfrak{g}\) associated to the involution \(\sigma =: \text{id}_\mathfrak{k} \oplus (-\text{id}_\mathfrak{p})\), the cocycle \(\varpi\) contains \(\mathfrak{k}\) in its radical and restricts to \(\mathfrak{p} \times \mathfrak{p}\) as a non-degenerate two-form. A **morphism** between two such siLa’s is a Lie algebra homomorphism which intertwines both involutions and two-cocycles.

**Definition 6.4** A transvection symplectic triple is a siLa \((\mathfrak{g}, \sigma, \varpi)\) where

(i) the action of \(\mathfrak{k}\) on \(\mathfrak{p}\) is faithful, and

(ii) \([\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}\) .

Given a symplectic symmetric space \((M, s, \omega)\), we can define a transvection symplectic triple \((\mathfrak{g}, \sigma, \varpi)\) as follows: \(\mathfrak{g}\) is the Lie algebra of the transvection group \(G(M)\), \(\sigma\) is the restriction to \(\mathfrak{g}\) of the differential of the involution \((63)\) and \(\varpi = \pi_* \omega_o\) is the pullback of \(\omega\) at the base point \(o\) by the differential \(\pi_* : \mathfrak{g} \to T_o M\) of the projection \(\pi : G(M) \to M, g \mapsto g.o\). Defining the notion of isomorphism in the obvious way, one has

**Proposition 6.5** The correspondence described above induces a bijection between the isomorphism classes of simply connected symplectic symmetric spaces and the isomorphism classes of transvection symplectic triples.

Generally, the two-cocycle \(\varpi\) is not exact, or equivalently, the symplectic action of the transvection group on the symplectic symmetric space is not Hamiltonian. However, it is always possible to centrally extend the transvection group in such a way that the extended group acts on \(M\) in a Hamiltonian way. The associated moment mapping then consists in a symplectic equivariant covering onto a co-adjoint orbit. The situation we consider in the present article concerns such non-exact transvection triples underlying polarized symplectic symmetric spaces.

**Lemma 6.6** Let \((M, s, \omega)\) be a symplectic symmetric space, polarized by a transvection-invariant Lagrangian distribution \(W \subset TM\). This data corresponds (via the correspondence of Proposition 6.7) to a \(\mathfrak{k}\)-invariant Lagrangian subspace \(W_o\) of \(T_o M\) corresponds to a Lagrangian subspace \(W\) of the symplectic vector space \((\mathfrak{p}, \varpi)\).

Proof. Under the linear isomorphism \(\pi_*|_\mathfrak{p} : \mathfrak{p} \to T_o M\) the subspace \(W_o\) of \(T_o M\) corresponds to a Lagrangian subspace \(W\) of the symplectic vector space \((\mathfrak{p}, \varpi)\). \(\blacksquare\)

\(^7\)Involutions \(\sigma\) are denoted the same way either at the Lie group or Lie algebra level.
According to the previous Lemma, we use the following terminology:

**Definition 6.7** A transvection symplectic triple \((g,\sigma,\varpi)\) is called polarized if it is endowed with \(W\), a \(\mathfrak{t}\)-invariant Lagrangian subspace of \((p,\varpi)\).

Let \((g,\sigma,\varpi)\) be a non-exact transvection symplectic triple polarized by a Lagrangian subspace \(W \subset p\). Let us consider \(\mathcal{D}\), the algebra of \(W\)-preserving symplectic endomorphisms of \(p\). Note that the faithfulness condition (i) of Definition 6.4 implies the inclusion \(\mathfrak{t} \subset \mathcal{D}\). The vector space \(\mathcal{D} \oplus p\) then naturally carries a structure of Lie algebra (containing \(g\)) that underlies a siLa. We centrally extend the latter in order to define a new siLa:

\[
\mathcal{L} := \mathcal{D} \oplus p \oplus \mathbb{R}Z ,
\]

with table given by

\[
[X,Y]_\mathcal{L} := [X,Y] + \varpi(X,Y)Z , \quad [X,Z]_\mathcal{L} := 0 , \quad \forall X,Y \in \mathcal{D} \oplus p ,
\]

where \([.,.]\) denotes the Lie bracket in \(\mathcal{D} \oplus p\) and where we have extended the 2-form \(\varpi\) on \(p\) to a 2-form on the entire Lie algebra \(\mathcal{D} \oplus p\) by zero on \(\mathcal{D}\).

**Lemma 6.8** Let \((g,\sigma,\varpi)\) be a non-exact polarized transvection symplectic triple. Within the notations given above, consider the element \(\xi \in \mathcal{L}^\ast\) defined by

\[
\langle \xi, Z \rangle = 1 , \quad \xi \big|_{\mathcal{D} \oplus p} = 0 .
\]

Define moreover:

\[
\mathcal{D} := \mathcal{D} \oplus \mathbb{R}Z , \quad \sigma_\mathcal{L} := \text{id}_\mathcal{D} \oplus (-\text{id}_p) \quad \text{and} \quad \mathfrak{g} := g \oplus \mathbb{R}Z , \quad \mathfrak{t} = \mathfrak{t} \oplus \mathbb{R}Z , \quad \tilde{\sigma} := \text{id}_\mathfrak{t} \oplus (-\text{id}_p) .
\]

Then, the triples \((\mathcal{L},\sigma_\mathcal{L},\delta\xi)\) and \((\tilde{g},\tilde{\sigma},\delta\xi\big|_{\mathfrak{g}})\) are exact siLa’s.

**Proof.** We give the proof for the first triple \((\mathcal{L},\sigma_\mathcal{L},\delta\xi)\) only, the second case being handled in a similar way. Since for \(X,Y \in p\), we have \([X,Y] \in \mathfrak{t} \subset \mathcal{D}\), we get

\[
\delta\xi(X,Y) = \langle \xi, [X,Y]_\mathcal{L} \rangle = \langle \xi, [X,Y] + \varpi(X,Y)Z \rangle = \varpi(X,Y) ,
\]

which at once proves closedness, non-degeneracy and \(\mathcal{D}\)-invariance of the 2-form \(\delta\xi\) on \(p\).

**Remark 6.9** The exact siLa’s \((\mathcal{L},\sigma_\mathcal{L},\delta\xi)\) and \((\tilde{g},\tilde{\sigma},\delta\xi\big|_{\mathfrak{g}})\) and the non-exact siLa \((g,\sigma,\varpi)\), all represent the same simply connected symplectic symmetric space.

**Definition 6.10** Given an exact siLa \((g,\sigma,\varpi)\) (i.e. \(\varpi = \delta\xi\) for an element \(\xi \in g^\ast\)), by a polarization affiliated to \(\xi\), we mean a \(\sigma\)-stable Lie sub-algebra \(\mathfrak{b}\) of \(g\) containing \(\mathfrak{t}\) and maximal for the property of being isotropic with respect to the two-form \(\varpi\).

Let \((G,\sigma)\) be a connected involutive Lie group associated to the involutive Lie algebra \((g,\sigma)\) underlying an exact siLa with \(\mathfrak{b}\) a polarization as in Definition 6.10. In that context, denote by \(B := \exp\{\mathfrak{b}\}\) the analytic (i.e. connected) Lie sub-group of \(G\) with Lie algebra \(\mathfrak{b}\). We will always assume \(B\) to be closed in \(G\). Since \(B\) is stable under \(\sigma\), the coset space \(G/B\) admits the following natural family of involutions \(\varsigma\):

\[
M \times G/B \to G/B , \quad (gK,g_0B) \mapsto \mathfrak{g}_\sigma K(g_0B) := g\sigma(g^{-1}g_0)B ,
\]

where we recall that under the identification \(M \simeq G/K\) the symmetries read \(s_{gK}(g'K) := g\sigma(g^{-1}g')K\), where \(K\) is the sub-group of \(G\) constituted by \(\sigma\)-invariant elements which are connected to the neutral element.

**Definition 6.11** With the same notations as above, the quadruple \((G,\sigma,\xi,B)\) is called a polarization quadruple. Its infinitesimal version \((g,\sigma,\xi,\mathfrak{b})\) is called the associated infinitesimal polarization quadruple. A morphism between two polarization quadruples \((G_j,\sigma_j,\xi_j,B_j), j = 1,2\), is defined as a Lie group homomorphism

\[
\phi : G_1 \to G_2 ,
\]

that intertwines the involutions, such that \(\phi(B_1) \subset B_2\) and such that, denoting again by \(\phi\) its differential at the unit element, one has \(\phi^*\xi_2 = \xi_1\).
In this subsection, we fix \( \mathfrak{g}, \sigma, \xi, B \) a non-exact transvection symplectic triple polarized by \( W \subset \mathfrak{p} \). Within the context of Lemma 6.8, we set \( \mathfrak{d} := \mathfrak{d} \oplus W \) and \( \mathfrak{b} := \mathfrak{k} \oplus W \). Then the quadruples \( (\mathcal{L}, \sigma, \xi, \mathfrak{B}) \) and \( (\tilde{\mathfrak{g}}, \tilde{\sigma}, \tilde{\xi}, \tilde{\mathfrak{B}}) \) are polarization quadruples.

The latter observation leads us to introduce the following terminology:

**Definition 6.15** Starting from a non-exact polarized transvection triple \( (\mathfrak{g}, \sigma, \varpi) \), the associated polarization quadruple \( (\mathcal{L}, \sigma, \xi, \mathfrak{B}) \) is called the (infinitesimal) full polarization quadruple. The sub-quadruple \( (\tilde{\mathfrak{g}}, \tilde{\sigma}, \tilde{\xi}, \tilde{\mathfrak{B}}) \) is called the associated (infinitesimal) transvection quadruple.

In the sequel, we will denote by \( \mathbb{L} \) the connected, simply connected Lie group with Lie algebra \( \mathcal{L} \) and we will consider the connected Lie sub-group \( \tilde{G} \) of \( \mathbb{L} \) tangent to \( \tilde{\mathfrak{g}} \). We will denote by \( \mathbb{B} \) (respectively \( \mathfrak{B} \)) the connected Lie sub-group of \( \mathbb{L} \) (respectively of \( \tilde{G} \)) associated to \( \mathfrak{B} \) (respectively to \( \mathfrak{b} \)).

### 6.2 Unitary representations of symmetric spaces

In this subsection, we fix \( (\mathfrak{g}, \sigma, \varpi) \) a non-exact polarized transvection triple, to which we associate a transvection quadruple \( (\tilde{G}, \tilde{\sigma}, \tilde{\xi}, \tilde{\mathfrak{B}}) \), according to the construction underlying Definition 6.15. We start with the following pre-quantization condition: in the sequel we will always assume the character \( \xi|_b : \mathfrak{b} \to \mathbb{R} \) exponentiates to \( B \) as a unitary character

\[
\chi : B \to U(1), \quad b \mapsto \chi(b)
\]

such that locally:

\[
\chi(b) := e^{i (\xi|_b, \log(b))}, \quad b \in B.
\]  

(65)

Note that the character is automatically fixed by the restriction to \( B \) of the involution:

\[
\sigma^* \chi = \chi.
\]

Of course, the pre-quantization condition is satisfied when the group \( B \) is exponential, as it will be the case for Pyatetskii-Shapiro’s elementary normal \( j \)-groups.
Lemma 6.16 Let \((\tilde{g}, \tilde{\sigma}, \xi, b)\) be the transvection quadruple of a non-exact transvection triple \((g, \sigma, \varpi)\) such that \(B\) is exponential. Then, the pre-quantization condition is satisfied.

Proof: Since \(B\) is exponential, by the BCH formula, the statement will follow from \(\xi([b, b]_{\tilde{g}}) = 0\). By construction of \(\xi\) (see Definition 6.8), this will follow if the \(Z\)-component of \([b, b]_{\tilde{g}}\) vanishes. But the latter reads \(\varpi(b, b)Z\) which reduces to zero by Definition 6.10 of a polarization quadruple and by Proposition 6.14 which shows that \((\tilde{g}, \tilde{\sigma}, \xi, b)\) is indeed a polarization quadruple.

We then form the line bundle:

\[
E_\chi := \tilde{G} \times_\chi \mathbb{C} \to \tilde{G}/B,
\]

and consider the associated induced representation of \(\tilde{G}\) on the smooth sections \(\Gamma^\infty(E_\chi)\). We will denote the latter representation by \(U_\chi\). Identifying as usual \(\Gamma^\infty(E_\chi)\) with the space of \(B\)-equivariant functions:

\[
\Gamma^\infty(E_\chi) \simeq C^\infty(\tilde{G})^B := \{ \tilde{\varphi} \in C^\infty(\tilde{G}) \mid \tilde{\varphi}(gb) = \varpi(b)\tilde{\varphi}(g), \forall b \in B, \forall g \in \tilde{G} \},
\]

the representation \(U_\chi\) is given by the restriction to \(C^\infty(\tilde{G})^B\) of the left-regular representation:

\[
[U_\chi(g)\tilde{\varphi}]^\chi(g') := \tilde{\varphi}(g^{-1}g'), \quad \forall \tilde{\varphi} \in \Gamma^\infty(E_\chi).
\]

We endow the line bundle \(E_\chi\) with the Hermitean structure, defined in term of the identification (66) by:

\[
h_{gB}(\tilde{\varphi}_1, \tilde{\varphi}_2) := \overline{\tilde{\varphi}_1(g)\tilde{\varphi}_2(g)}, \quad \forall \tilde{\varphi}_1, \tilde{\varphi}_2 \in \Gamma^\infty(E_\chi), \quad gB \in \tilde{G}/B.
\]

We make the assumption that the modular function of \(B\) coincides with the restriction to \(B\) of the modular function of \(\tilde{G}\). By Lemma 6.13 this condition implies the existence of a \(\tilde{G}\)-invariant and \(\sigma_{\tilde{K}}\)-invariant Borelian measure \(d_{\tilde{G}/B}\) on \(\tilde{G}/B\). Here, \(\sigma_{\tilde{K}}\) is the involutive diffeomorphism of \(\tilde{G}/B\) given in (64) for the involutive pair \((\tilde{G}, \tilde{\sigma})\) and subgroup \(B\), underlying the transvection quadruple \((\tilde{G}, \tilde{\sigma}, \xi, B)\). We then let \(\mathcal{H}_\chi\) be the Hilbert space completion of \(\Gamma_c^\infty(E_\chi)\) for the inner product:

\[
\langle \tilde{\varphi}_1, \tilde{\varphi}_2 \rangle := \int_{\tilde{G}/B} h_{gB}(\tilde{\varphi}_1, \tilde{\varphi}_2) d_{\tilde{G}/B}(gB).
\]

Of course, the induced representation \(U_\chi\) of \(\tilde{G}\) then naturally acts on \(\mathcal{H}_\chi\) by unitary operators. Now observe the fact that the character \(\chi\) is invariant under \(\tilde{\sigma}\), implies that the pull back under \(\tilde{\sigma}\) of an equivariant function is again equivariant. Therefore, we get a linear involution:

\[
\Sigma : \mathcal{H}_\chi \to \mathcal{H}_\chi, \quad [\Sigma \tilde{\varphi}]^\chi := \tilde{\sigma}^* \tilde{\varphi}.
\]

Also, the \(\sigma_{\tilde{K}}\)-invariance of the measure \(d_{\tilde{G}/B}\) implies:

\[
\langle \Sigma \tilde{\varphi}_1, \Sigma \tilde{\varphi}_2 \rangle = \int_{\tilde{G}/B} h_{gB}(\Sigma \tilde{\varphi}_1, \Sigma \tilde{\varphi}_2) d_{\tilde{G}/B}(gB) = \int_{\tilde{G}/B} h_{\sigma_{\tilde{K}}(gB)}(\tilde{\varphi}_1, \tilde{\varphi}_2) d_{\tilde{G}/B}(gB) = \langle \tilde{\varphi}_1, \tilde{\varphi}_2 \rangle,
\]

for all \(\tilde{\varphi}_1, \tilde{\varphi}_2 \in \mathcal{H}_\chi\), showing that \(\Sigma\) is not only involutive but also self-adjoint. Thus the element \(\Sigma\) belongs to \(\mathcal{U}_{sa}(\mathcal{H}_\chi)\), the collection of unitary and self-adjoint operators on \(\mathcal{H}_\chi\). When composed with the representation \(U_\chi\) of \(G\), the operator \(\Sigma\) satisfies the following properties, whose proofs consist in direct computations:

**Proposition 6.17** Let \((M, s, \omega)\) be a the simply connected polarized symplectic symmetric space associated to a transvection quadruple \((\tilde{G}, \tilde{\sigma}, \xi, B)\) such that the modular function of \(B\) coincides with the restriction to \(B\) of the modular function of \(\tilde{G}\). Then the map

\[
\tilde{G} \to \mathcal{U}_{sa}(\mathcal{H}_\chi), \quad g \mapsto U_\chi(g) \Sigma U_\chi(g)^*,
\]

is constant on the left lateral classes of \(\tilde{K}\) in \(\tilde{G}\). The corresponding mapping:

\[
\Omega : M = \tilde{G}/\tilde{K} \to \mathcal{U}_{sa}(\mathcal{H}_\chi), \quad \tilde{g}K \mapsto \Omega(\tilde{g}K) := U_\chi(g) \Sigma U_\chi(g)^*,
\]

is linear.
defines a unitary representation of the symmetric space $M = \tilde{G}/\tilde{K}$ in the sense that, for all $x, y$ in $M$ and $g$ in $\tilde{G}$, the following representative properties hold:

$$
\Omega(x)^2 = \text{Id}_{\mathcal{H}}, \quad \Omega(x)^\dagger \Omega(y) \Omega(x) = \Omega(s_x y), \quad U_\chi(g) \Omega(x) U_\chi(g)^* = \Omega(g.x).
$$

**Definition 6.18** The pair $(\mathcal{H}_\chi, \Omega)$ is called the unitary representation of $(M, s)$ induced by the character $\chi$ of $B$.

**Remark 6.19** In a quantum mechanical context, this type of operators are usually referred as phase space symmetry or parity operators, since they reflect the symmetric structure of the classical phase space at the level of the quantum configuration space (see for instance [24, 14] and the references therein).

We are now ready to define our prototype of quantization map on a polarized symplectic symmetric space:

**Definition 6.20** Let $(M, s, \omega)$ be a the polarized symplectic symmetric space. Denote by $L^1(M)$ the space of integrable functions on $M$ with respect to the $G$-invariant (Liouville) measure $d_M$. Denote by $\mathcal{B}(\mathcal{H}_\chi)$ the space of bounded linear operators on the Hilbert space $\mathcal{H}_\chi$. Consider the $G$-equivariant continuous linear map:

$$
\Omega : L^1(M) \to \mathcal{B}(\mathcal{H}_\chi), \quad f \mapsto \Omega(f) := \int_M f(x) \Omega(x) d_M(x).
$$

The latter is called the quantization map of $M$ induced by the transvection quadruple $(\tilde{G}, \tilde{\sigma}, \xi, B)$.

**Remark 6.21** From $||\Omega(x)|| = ||\Sigma|| = 1$ (the norm here is the uniform norm on $\mathcal{B}(\mathcal{H}_\chi)$), we get the obvious estimate $||\Omega(f)|| \leq ||f||_1$, from which the continuity of the quantization map follows. Also, from Proposition [6.17] and from the $G$-invariance of $d_M$, the covariance property at the level of the quantization map reads:

$$
U(g) \Omega(f) U(g)^* = \Omega(\Sigma f), \quad \forall f \in L^1(M), \forall g \in \tilde{G},
$$

where $\Sigma f := [g_0 \tilde{K} \mapsto \tilde{f}(g^{-1} g_0 \tilde{K})]$. The latter extended covariance property is an essential difference between the present construction and the classical coherent-state-quantization approach. Another one is that the quantization defined above is in general not positive, i.e. for $0 \leq f \in L^1(M)$, $\Omega(f)$ is not necessarily a positive operator. But since $\Omega(f)^* = \Omega(\tilde{f})$, it maps real-valued functions to self-adjoint operators.

**Remark 6.22** The quantization map of Definition [6.20] is a generalization of the Weyl quantization, from the point of view of symmetric spaces. Moreover, we will see that for the symmetric space underlying a two-dimensional elementary normal $j$-group (i.e. for the affine group of the real line) this construction coincides with Unterberger’s Fuchs calculus [24].

Our next step is to introduce a functional parameter in the construction of the quantization map. There is several reasons for doing this, among which there is one of a purely analytical nature: obtaining a quantization map which is a unitary operator from the Hilbert space of square integrable symbols, $L^2(M)$, to the Hilbert space of Hilbert-Schmidt operators on $\mathcal{H}_\chi$, denoted by $L^2(\mathcal{H}_\chi)$. This unitarity property will enable us to define a non-formal $*$-product on $L^2(M)$ in a straightforward way.

**Definition 6.23** Identifying a Borelian function $m$ on $\tilde{G}/B$ with the operator on $\mathcal{H}_\chi$ of point-wise multiplication by this function, we let

$$
\Sigma_m := m \circ \Sigma.
$$

When $m$ is locally essentially bounded, the family of operators $U_\chi(g) \Sigma_m U_\chi(g)^*$, $g \in \tilde{G}$, can be defined on the common domain $\Gamma^\infty(E_\chi)$. Note however that the later family of operators is not necessarily constant on the left lateral classes of $\tilde{K}$ in $\tilde{G}$ and unless $\Sigma_\tilde{K} m = m^{-1}$, one loses the involutive property for $\Sigma_m$ (but one always keeps the $\tilde{G}$-equivariance). Also, these operators are bounded on $\mathcal{H}_\chi$, if and only if the function $m$ is essentially bounded. But we will see in Theorem [6.39] that in order to obtain a unitary quantization map we are forced to consider such unbounded $\Sigma_m$’s. We mention a simple self-adjointness criterium, interesting on its own.  

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8 Indeed, the set $\{\Omega(x) : x \in M\}$ viewed as a real Banach sub-manifold of $U_m(\mathcal{H})$, is endowed with a structure of a symmetric space for $S : (\Omega(x), \Omega(y)) \mapsto \Omega(x) \circ \Omega(y) \circ \Omega(x) \circ \Omega(y)$ and the later is (trivially) isomorphic to $(M, s)$. 

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Lemma 6.24 Let $m$ a locally essentially bounded Borelian function on $\tilde{G}/B$ such that $\mu^* \mu = m$. Define

$$\tilde{\Omega}(g) := U(\varphi) \sum m_U(\varphi)^*$$

on the domain

$$B_g := \{ \varphi \in \mathcal{H} : \|m_\varphi\| < \infty \} \quad \text{where} \quad m_\varphi := [g_0B \mapsto m(g^{-1}g_0B)], \quad g \in \tilde{G}.$$ 

Then, $\tilde{\Omega}(g)$ is self-adjoint on $\mathcal{H}$. Moreover, $\Gamma^\infty(E_\chi)$ is a common core for all $\tilde{\Omega}(g)$'s, $g \in \tilde{G}$.

Proof. Note first that the formal adjoint of $\Sigma m$ is $\Sigma \mu^* \mu$. Therefore, when $\mu^* \mu = m$, the operator $\Omega_m(x)$ is symmetric on $\Gamma^\infty(E_\chi)$. Next, we remark that as both $\Sigma$ and $U(\varphi)$, $g \in \tilde{G}$, preserve $\Gamma^\infty(E_\chi)$, we get for $\varphi \in \Gamma^\infty(E_\chi)$:

$$\tilde{\Omega}(g)^2 \varphi = \|m_\varphi\|^2 \varphi,$$

that is, $\tilde{\Omega}(g)$ squares on $\Gamma^\infty(E_\chi)$ to a multiplication operator. Since $\tilde{\Omega}(g)$ is symmetric on the space of smooth compactly supported sections of $E_\chi$, the latter entails that

$$\|\tilde{\Omega}(g)\varphi\| = \|m_\varphi\| \varphi, \quad \forall \varphi \in \Gamma^\infty(E_\chi),$$

and thus $\tilde{\Omega}(g)$ is well defined on $B_g$. Then, the same computation as above, shows that $\tilde{\Omega}(g)$ is also symmetric on its domain $B_g$. Observe that $B_g$ is complete in the graph norm, given by $\|\psi\|^2 + \|m_\varphi\| \varphi^2$, and that $\Gamma^\infty(E_\chi)$ is dense in $B_g$ for this norm. Thus $\tilde{\Omega}(g)$, with domain $B_g$, is a closed operator. Clearly $B_g \subset \text{dom}((\Omega_m(g)^2)$, since $\tilde{\Omega}(g)$ is symmetric on $B_g$.

Choose an increasing sequence of relatively compact open sets $\{C_n\}_{n \in \mathbb{N}}$ in $\tilde{G}/B$, converging to $\tilde{G}/B$. For $n \in \mathbb{N}$, let $\chi_n$ be the indicator function of $C_n$. Then of course $\chi_n \varphi \in B_g$ for all $\varphi \in \mathcal{H}$. Note also that for $\varphi \in B_g$, we have by definition of $m_\varphi$ and from the relation $\mu^* \mu = m$,

$$\tilde{\Omega}(g)\varphi = m_\varphi \Omega(x) \varphi = \Omega(x) \mu_\varphi \varphi, \quad x = g \tilde{K} \in M.$$

Thus for $g \in \tilde{G}$, $x = g \tilde{K} \in M$, $\psi \in \text{dom}(\tilde{\Omega}(g)^*)$, $\varphi \in \mathcal{H}$ and using the fact that $\chi_n \varphi$ is essentially bounded (i.e. the associated multiplication operator is bounded), we get

$$\langle \varphi, \chi_n \tilde{\Omega}(g) \varphi \rangle = \langle \tilde{\Omega}(g) \chi_n \varphi, \psi \rangle = \langle \Omega(x) \mu_\varphi \varphi, \psi \rangle = \langle \varphi, \chi_n \varphi \rangle.$$

Using the monotone convergence theorem, we obtain

$$\|\tilde{\Omega}(g)^* \psi\| = \lim_{n \to \infty} \|\chi_n \tilde{\Omega}(g)^* \psi\| = \lim_{n \to \infty} \sup_{\|\varphi\| = 1} |\langle \varphi, \chi_n \tilde{\Omega}(g)^* \psi \rangle| = \lim_{n \to \infty} \sup_{\|\varphi\| = 1} |\langle \varphi, \chi_n \varphi \rangle | = \lim_{n \to \infty} \|m_\varphi \Omega(x) \psi\| = \|m_\varphi \varphi\| = \|\mu_\varphi \varphi\| = \|\mu_\varphi^2 \varphi\|,$$

so that necessarily $\psi \in B_g$. Thus $\text{dom}(\tilde{\Omega}(g)^*) = B_g$, as required.

Note at last that $\Gamma^\infty(E_\chi)$ being dense in each $B_g$ for the graph norm, it is a common core for all the $\tilde{\Omega}(g)$, which are therefore essentially selfadjoint on that domain.

\[ \square \]

Remark 6.25 At this early stage of the construction, it is important to observe that our representation of $\mathcal{H}$ (Proposition 6.17) and the associated quantization map (Definition 6.20) could have been equally defined starting with the full polarization quadruple $(\mathcal{L}, \sigma, \xi, B)$ (see Definition 6.14) of a non-exact polarized transvection triple $(q, \sigma, \tau)$, instead of the transvection quadruple $(\tilde{G}, \tilde{\sigma}, \tilde{\xi}, \tilde{B})$. In particular, all the results of subsections 6.3, 6.4, 6.5 and 6.6 can be thought as arising from the full quadruple. We will make great use of this observation in subsection 6.7.
6.3 Locality and the one-point phase

We next pass to the notion of locality in the context of transvection quadruples, out of which we will be able to give an explicit expression of the operators $\Omega_m(x)$ on $\mathcal{H}_x$.

**Definition 6.26** Within the notations of Definition 6.15, we say that the polarized symplectic symmetric space $(M, s, \omega)$, associated to a non-exact polarized transvection triple $(g, \sigma, \varpi)$, is **local** whenever there exists a sub-group $Q$ of $\tilde{G}$ such that:

(i) The map

$$Q \times B \rightarrow \tilde{G}, \quad (q, b) \mapsto qb.$$  

is a global diffeomorphism. In particular, $Q$ is closed as a sub-group of $\tilde{G}$.

(ii) Denoting by $q$ the Lie algebra of $Q$, one has:

$[q, b \cap p] \subset b$.

(iii) For every $q \in Q$, setting $\tilde{\sigma}q:= (\tilde{\sigma}q)Q = (\tilde{\sigma}q)B$ relatively to the global decomposition $\tilde{G} = Q.B$, one has:

$\chi((\tilde{\sigma}q)B) = 1$.

For a local symplectic symmetric space, the identification $Q \simeq \tilde{G}/B$ allows to transfer the symmetric space structure of the former to the later:

**Lemma 6.27** Let $(M, s, \omega)$ be a local symplectic symmetric space. Then:

(i) The mapping:

$$\mathcal{S}: Q \times Q \rightarrow Q, \quad (q, q') \mapsto \mathcal{S}_q(q') := q (\tilde{\sigma}(q^{-1}q'))^Q,$$

defines a left-invariant structure of symmetric space on the Lie group $Q$.

(ii) Moreover, the global diffeomorphism

$$Q \rightarrow \tilde{G}/B; \quad q \mapsto qB,$$

intertwines the symmetry $\mathcal{S}$ with the involution $\mathcal{S}_\tilde{G}$ defined in [64] for the transvection quadruple $(\tilde{G}, \tilde{\sigma}, \xi, B)$:

$$\mathcal{S}_q q' \mapsto \mathcal{S}_\tilde{G}(q'B), \quad \forall q, q' \in Q.$$

(iii) Under the identification $Q \simeq \tilde{G}/B$ given above, the $(\tilde{G}, \mathcal{S}_\tilde{G})$-invariant measure $d\tilde{G}/B$ on $\tilde{G}/B$ constructed in Lemma 6.13 becomes a $(\tilde{G}, \mathcal{S}_q)$-invariant measure $dQ$ on the Lie group $Q$, which is also a left-invariant Haar measure on $Q$.

(iv) Last, we have an isomorphism of Hilbert spaces $\mathcal{H}_x \simeq L^2(Q, dQ)$ induced by $\tilde{G}$-equivariant isomorphism:

$$C^\infty(\tilde{G})^B \rightarrow C^\infty(Q), \quad \phi \mapsto \varphi := \phi|_Q,$$

under which, we have $\Sigma = \mathcal{S}_q^*$. 

**Proof.** Item (ii) follows from a direct check implying in turn the left-$Q$-equivariance of $\mathcal{S}$ from the left-$\tilde{G}$-equivariance of $\mathcal{S}$. The fact that $\mathcal{S}_q$ fixes $q$ isolately is a consequence of the following observation. Considering the linear epimorphism $p_*: p \rightarrow q = T_qQ \simeq T_B(\tilde{G}/B)$ tangent to the projection $p: M \simeq \tilde{G}/K \rightarrow \tilde{G}/B \simeq Q$, $gK \rightarrow gB$, one observes that for every $X \in p$:

$$\mathcal{S}_K (p_*X) = (p \circ s_K)_* (X) = -p_*X.$$  

Hence $\mathcal{S}_K = -i d_q$ and (i) follows. Last, (iii) and (iv) are immediate consequences of the $Q$-equivariant identification $Q \simeq \tilde{G}/B$.  

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From now on, we will always make the identification $\mathcal{H}_\chi \simeq L^2(Q, d\xi)$, under which we can derive the action of the individual operators $\Omega(x)$, $x \in M$. For this, we need a preliminary result:

**Lemma 6.28** Let $(M, s, \omega)$ be a local symplectic symmetric space and let $\hat{\varphi} \in C^\infty(\tilde{G}, \mathbb{C})^B$ be a $B$-equivariant function. Then, for all $q, q_0 \in Q$ and $b \in B$, one has:

$$L_{qb}^* \circ \hat{\sigma}^* \circ L_{(qb)^{-1}}^* \hat{\varphi}(q_0) = \mathcal{E}(q_0^{-1}q\hat{\sigma}(b^{-1})) \hat{\varphi}(\hat{s}_q q_0),$$

with

$$\mathcal{E}(qb) := \chi(C^1_q(b^{-1}\hat{\sigma}b)),$$

and $C_q(g') := gg'g^{-1}$ denotes the conjugate action of $\tilde{G}$ on itself.

**Proof.** A direct computation yields:

$$L_{qb}^* \circ \hat{\sigma}^* \circ L_{(qb)^{-1}}^* \hat{\varphi}(q_0) = \hat{\varphi}(qb\hat{\sigma}(b^{-1})\hat{\sigma}(q^{-1}q_0)) = \hat{\varphi}(q\hat{\sigma}(q^{-1}q_0)C_{\hat{\sigma}(q_0^{-1}q)}(b\hat{\sigma}(b^{-1}))).$$

Under the assumption of locality (Definition 6.26), we have $[Q, b \cap p] \subset Q[b, b \cap p] \subset b$, hence:

$$C_{\hat{\sigma}(q_0^{-1}q)}(b\hat{\sigma}(b^{-1})) \in B.$$

The $\hat{\sigma}$-invariance of $\chi$ and item (iii) of Definition 6.26 then yield the formula. \[\Box\]

**Remark 6.29** We call the function $\mathcal{E}$ in (68) the one-point phase. Observe that the later is well defined thanks to the second condition $[q, b \cap p] \subset b$ in the assumption of locality (Definition 6.26).

**Corollary 6.30** Let $(M, s, \omega)$ be a local symplectic symmetric space. For $\varphi \in \mathcal{H}_\chi$ and $x = qb\tilde{K} \in M$, $q \in Q$, $b \in B$, we have

$$\Omega(x) \varphi(q_0) = \mathcal{E}(q_0^{-1}q\hat{\sigma}(b^{-1})) \varphi(\hat{s}_q(q_0)),$$

where $\mathcal{E}$ is the phase defined in (68) and $\hat{s}$ is the symmetry of the Lie group $Q$ constructed in Lemma 6.27.

**6.4 Unitarity and midpoints for elementary spaces**

In addition to locality (Definition 6.26), we will assume further conditions on the structure of our polarized symplectic symmetric space $(M, s, \omega)$, which will enable us to give an explicit expression of the three-point kernel associated to a WKB-quantization of $M$ as well as to prove the triviality of the associated Berezin transform (see Definition 6.42 below). Recall that the notion of midpoint map on a symmetric space is given in Definition 6.11.

**Definition 6.31** A local symplectic symmetric space $(M, s, \omega)$ is called elementary when, within the context of the subsection 6.2, the following additional conditions are satisfied:

(i) The symmetric space $(Q, \hat{s})$ is solvable and admits a midpoint map.

(ii) There exists an exponential Lie sub-group $\mathbb{Y}$ of $B$ normalized by $Q$ and such that the semi-direct product $S := Q \ltimes \mathbb{Y} \subset \tilde{G}$, acts simply transitively on $M$.

(iii) Denoting by $\mathfrak{Y}$ the Lie algebra of $\mathbb{Y}$, there exists a global diffeomorphism $\Psi : Q \to \mathfrak{q}$ such that

$$\langle \xi, (\text{Ad}_{q^{-1}} - \text{Ad}_{\hat{s}_q(q^{-1})})y \rangle = \langle \xi, [\Psi(q), y] \rangle, \quad \forall y \in \mathfrak{Y}, \forall q \in Q.$$

(iv) The Lie algebras $\mathfrak{Y}$ and $\mathfrak{q}$ are Lagrangian subspaces in symplectic duality.

\[^9\text{By a result proven in [25], it implies that } Q \text{ is exponential.}\]
Remark 6.32 For \((M, s, \omega)\), an elementary symplectic symmetric space, we always make the \(S\)-equivariant identification:

\[ S = Q \ltimes Y \to M, \quad qb \mapsto qbK. \]

In particular, for \(m\) a locally essentially bounded Borelian function on \(Q\) and \(qb \in S\), the operator \(\Omega_m(qb)\) given in Lemma \(6.24\) will be denoted by \(\Omega_m(qb)\). Note that when \(m = 1\), this notation is coherent with Proposition \(6.17\) since the latter coincide with \(\Omega(qbK)\). We also observe that under the identification \(S \simeq M\), the \(G\)-invariant Liouville measure \(d_M\) on \(M\) is a left Haar measure on \(S\), which under the parametrization \(g = qb, q \in Q, b \in Y\), is proportional to any product of left invariant Haar measures on \(Q\) and on \(Y\). We simply denote the latter by \(db\).

Remark 6.33 Note that as \(Q\) normalizes \(Y\), the restriction to \(S = Q \ltimes Y\) of the representation \(U_\chi\) of \(G\) on \(H_\chi \simeq L^2(Q, dQ)\) given in \((67)\), reads:

\[ U_\chi(qb)\varphi(q_0) = \chi(C_{\tilde{\sigma}^{-1}}(q(b))) \varphi(q^{-1}q_0). \]

In the elementary case, we observe the following relation between the one-point phase \(E\) and the diffeomorphism \(\Psi: Q \to q\):

Lemma 6.34 Let \((M, s, \omega)\) be an elementary symplectic symmetric space. Then, for \(q \in Q\) and \(y \in Y\), we have:

\[ E(q^{-1}e^y) = \exp\{i\langle \xi, [\Psi(q), y]\rangle\}. \]

Proof. By definition, we have for \(q \in Q\) and \(b \in Y\):

\[ E(q^{-1}b) = \chi(C_{\tilde{\sigma}^{-1}}(b^{-1}) \tilde{\sigma}(C_{\tilde{\sigma}^{-1}}(b))). \]

Next, we write

\[ \tilde{\sigma}(q^{-1}) = \tilde{\sigma}(q)^{-1} = (\tilde{\sigma}(q)Q \tilde{\sigma}(q)B)^{-1} = (\tilde{\sigma}(q)B)^{-1}(\tilde{\sigma}(q)Q)^{-1}, \]

to get

\[ C_{\tilde{\sigma}^{-1}}(b^{-1}) \tilde{\sigma}(C_{\tilde{\sigma}^{-1}}(b)) = (C_{\tilde{\sigma}^{-1}}(b^{-1})) (\tilde{\sigma}(\tilde{\sigma}(q)B)^{-1}) \tilde{\sigma}(C_{\tilde{\sigma}^{-1}}(b)) \tilde{\sigma}(\tilde{\sigma}(q)B). \]

Since \(Q\) normalizes \(Y\) and \(B\) is \(\tilde{\sigma}\)-stable, we observe that each of the four factors in the right hand side above, belong to \(B\). Thus, we can split \(E(q^{-1}b)\) accordingly to this decomposition, to get

\[ E(q^{-1}b) = \chi(C_{\tilde{\sigma}^{-1}}(b)) \chi(C_{\tilde{\sigma}^{-1}}(b)). \]

and the result follows from the definition of the diffeomorphism \(\Psi\) and the character \(\chi\). \(\blacksquare\)

The identification \(S \simeq M\), allows us to give a slightly simpler formula for the operators \(\Omega_m(qb)\), \(qb \in S\):

Proposition 6.35 Let \((M, s, \omega)\) be an elementary symplectic symmetric space. Let \(q \in Q, b \in Y\) and let \(m\) be an essentially locally bounded Borelian function on \(Q\). Then the densely defined (on \(B_{qb} = B_q\)–see Definition \(6.23\)) and Lemma \(6.24\) operator \(\Omega_m(qb)\), acts as:

\[ \Omega_m(qb)\varphi(q_0) = m(q^{-1}q_0) E(q_0^{-1}qb) \varphi(q^{-1}q_0), \quad \forall \varphi \in B_q \subset H_\chi, \quad \forall q_0 \in Q. \]

Proof. Using Corollary \(6.30\) Definition \(6.23\) and Lemma \(6.24\) we deduce that for \(\varphi \in B_q\), we have

\[ \Omega_m(qb)\varphi(q_0) = m(q^{-1}q_0) E(q_0^{-1}q\tilde{\sigma}(b^{-1})) \varphi(q^{-1}q_0). \]

Now, write

\[ C_{q_0^{-1}}(\tilde{\sigma}(b)b^{-1}) = C_{q_0^{-1}}(\tilde{\sigma}(b)) C_{q_0^{-1}}(b^{-1}). \]

As \(Q\) normalizes \(Y\) by assumption, we see that the factor on the right of the right hand side above, belongs to \(Y \subseteq B\). Since the left hand side belongs to \(B\) as well (by the second condition of locality–see Definition \(6.20\)), the factor on the left of the right hand side belongs to \(B\) too. Thus, we can split the character \(\chi\) accordingly to this decomposition and the result follows immediately. \(\blacksquare\)
Corollary 6.36 Let \((M, s, \omega)\) be an elementary symplectic symmetric space, \(m\) be an essentially locally bounded Borelian function on \(Q\) and \(f \in \mathcal{D}(S)\). Then the operator \(\Omega_m(f)\) defined by
\[
\Omega_m(f) : \mathcal{D}(Q) \to \mathcal{D}'(Q), \quad \varphi \mapsto \Omega_m(f)\varphi := \left[ \psi \in \mathcal{D}(Q) \mapsto \int_{Q \times S} \psi(q_0) f(qb) \left( \Omega_m(qb)\varphi \right)(q_0) \, d_S(qb) \, d_Q(q_0) \right],
\]
has a distributional kernel given by
\[
\Omega_m(f)[q_0, q] = \text{mid}(e, q_0^{-1}q)^{-1} \left| \text{Jac}_{(\omega^\perp)}^{-1} \right| (q_0^{-1}q) \int_{\mathcal{Y}} f(\text{mid}(q_0, qb)) \, E(\text{mid}(e, q_0^{-1}qb)) \, d_Y(b).
\]

Proof. Observe first that under the decomposition \(S = Q \times Y\), the left Haar measure on \(S\) coincides with the product of left Haar measures on \(Q\) and \(Y\):
\[
d_S(qb) = d_Q(q) \, d_Y(b), \quad \forall q \in Q, \, \forall b \in Y.
\]
For \(f \in \mathcal{D}(S)\) and any Borelian \(m\), it is clear that \(\Omega_m(f)\) defines a continuous operator from \(\mathcal{D}(Q)\) to \(\mathcal{D}'(Q)\) and acts as:
\[
\Omega_m(f)\varphi(q_0) = \int_{Q \times Y} f(qb) \, \text{mid}(e, q_0^{-1}q)^{-1} \left| \text{Jac}_{(\omega^\perp)}^{-1} \right| (q_0^{-1}q) \, \text{mid}(qb) \, E(\text{mid}(e, q_0^{-1}qb)) \, \varphi(q) \, d_Q(q) \, d_Y(b), \quad \varphi \in \mathcal{D}(Q).
\]
For any \(q_0 \in Q\), we set \(q'(q) := \text{mid}(q_0, q)\) and we get from the defining property of the midpoint map that \(q = \text{mid}(q_0, q')\). Moreover, since left-translations are automorphisms of the symmetric space \((Q, \omega)\), we get
\[
\text{mid}(q_0, q') = q_0 \, \text{mid}(e, q_0^{-1}q') = L_{q_0} \circ (\omega^\perp)^{-1} \circ L_{q_0^{-1}}(q'),
\]
the invariance of the Haar measure \(d_Q\) under left translation gives:
\[
\left| \text{Jac}_{\text{mid}(q_0, q)} \right|(q') = \left| \text{Jac}_{(\omega^\perp)^{-1}} \right|(q_0^{-1}q').
\]
Therefore, a direct computation shows that:
\[
\Omega_m(f)\varphi(q_0) = \int_{Q \times Y} f(\text{mid}(q_0, qb)) \, \text{mid}(e, q_0^{-1}q)^{-1} \left| \text{Jac}_{(\omega^\perp)^{-1}} \right|(q_0^{-1}q) \, \text{mid}(qb) \, E(\text{mid}(e, q_0^{-1}qb)) \, \varphi(q) \, d_Q(q) \, d_Y(b),
\]
and the result follows by identification.

Our next aim is to understand the geometrical conditions on the functional parameter \(m\) necessary for the quantization map to extend to a unitary operator from \(L^2(M)\) to the Hilbert space of Hilbert-Schmidt operators on \(\mathcal{H}_\lambda\). For this, we introduce the following specific function on \(Q\):

**Definition 6.37** For \((M, s, \omega)\) an elementary symplectic symmetric space, define the function \(m_0\) on \(Q\) as:
\[
m_0(q) := \left| \text{Jac}_{\omega^\perp(q^{-1})} \text{Jac}_{\psi}(q) \right|^{1/2}.
\]

**Remark 6.38** Observe that both \(\text{Jac}_{\psi}\) and \(\text{Jac}_{\omega^\perp}\) are \(\omega^\perp\)-invariant. Indeed, we have \(\psi \circ \omega^\perp = -\psi\) and \(\omega^\perp \circ \omega^\perp = \omega^\perp \circ \omega^\perp\), hence the claim follows from \(\left| \text{Jac}_{\omega^\perp} \right| = 1\). However, \(m_0\) needs not to be \(\omega^\perp\)-invariant, as \(\left| \text{Jac}_{\omega^\perp} \right|\) needs not to be invariant under the inversion map on \(Q\). Thus, from Lemma 6.24, the quantization map \(\Omega_{m_0}\), needs not to send real functions to self-adjoint operators.

We can now state one of the main results of this section:

**Theorem 6.39** Let \((M, s, \omega)\) be an elementary symplectic symmetric space and \(m\) be a Borelian function on \(Q\) which is (almost everywhere) dominated by \(m_0\) and assume that \(Y\) is Abelian. Then the quantization map:
\[
\Omega_m : f \mapsto \Omega_m(f) := \int_S f(qb) \, \Omega_m(qb) \, d_S(qb),
\]
is a bounded operator from \(L^2(S, d_S)\) to \(L^2(\mathcal{H}_\lambda)\) with
\[
\|\Omega_m\| \leq \left\| \frac{m}{m_0} \right\|_\infty.
\]
Moreover, \(\Omega_m\) is a unitary operator if and only if \(|m| = m_0\).
Proof. Recall that a linear operator $T : \mathcal{D}(Q) \to \mathcal{D}'(Q)$ extends to a Hilbert-Schmidt operator on $L^2(Q, dq)$, if and only its distributional kernel belongs to $L^2(Q \times Q, dq \otimes dq)$. In this case, its Hilbert-Schmidt norm coincides with the $L^2$-norm of its kernel. Thus by Corollary 6.36 we deduce that if $f \in \mathcal{D}(\mathbb{S})$, then the square of the Hilbert-Schmidt norm of $\Omega_m(f)$ reads

$$
\|\Omega_m(f)\|_2^2 = \int |m|^2 \mid \Omega_{\text{me}}(e, q_0^{-1} q) \mid^{-1} \mid \text{Jac}_{\text{me}}\mid^{-1} \mid \omega_{\text{me}}(q_0^{-1} q) \mid f(q_0, q) f(q_0, q) b \times \mathbf{E}(\text{me}(e, q_0^{-1} q) b) \mathbf{E}(\text{me}(e, q_0^{-1} q) b) dq(q) dq(b) dq(q) dq(b) dq(b).
$$

Performing the change of variables $\text{me}(q_0, q) \mapsto q$ (the inverse of the one we performed in the proof of Corollary 6.30) and using the relation between the function $\mathbf{E}$ and the diffeomorphism $\Psi : Q \to q$ given in Lemma 6.34 we get

$$
\|\Omega_m(f)\|_2^2
= \int \left| m \right|^2 \left| q_0^{-1} q_0 \right| \left| \text{Jac}_{\text{me}} \right|^{-1} \left| \omega_{\text{me}}(q_0^{-1} q) \right| \mathcal{F}(q_0, q_0) q_0 \left| \text{Jac}_{\text{me}} \right|^{-1} \left| \omega_{\text{me}}(q_0^{-1} q) \right| f(q_0, q_0) \epsilon(e, q_0, \log(b) - \log(b_0)) \times \mathbf{E}(\text{me}(e, q_0^{-1} q) b) \mathbf{E}(\text{me}(e, q_0^{-1} q) b) dq(q) dq(b) dq(q_0) dq(q_0) dq(b).
$$

where in the last line, we used left-invariance of the Haar measure on $Q$. Setting $w_0 = \Psi(q_0) \in q$ and $dw_0$ the Lebesgue measure on $q$, we get

$$
\|\Omega_m(f)\|_2^2 = \int \left| m \right|^2 \left| \Psi^{-1}(w_0) \right| \mathcal{F}(q_0, q_0) q_0 \left| \text{Jac}_{\text{me}} \right|^{-1} \left| \omega_{\text{me}}(q_0^{-1} q) \right| f(q_0, q_0) \epsilon(e, q_0, \log(b) - \log(b_0)) \times \mathbf{E}(\text{me}(e, q_0^{-1} q) b) \mathbf{E}(\text{me}(e, q_0^{-1} q) b) dq(q) dq(b) dw_0 dq(b).
$$

Next, we use the relation (which follows from the construction of $\xi \in \hat{g}^*$):

$$
\langle \xi, [X, Y] \rangle = \omega(X, Y), \quad \forall X, Y \in \hat{g},
$$

and the fact that $q$ and $\mathfrak{g}$ are Lagrangian subspaces in symplectic duality (see Definition 6.31). Thus, since $\mathfrak{g}$ is Abelian, by elementary Fourier theory, we see that if $|m| \leq C |m_0|$, then one gets:

$$
\|\Omega_m(f)\|_2 \leq C\|f\|_2, \quad \forall f \in \mathcal{D}(\mathbb{S})
$$

By density, we deduce that $\Omega_m(f)$ is Hilbert-Schmidt for all $f \in L^2(\mathbb{S}, ds)$ and with equality of norms if and only if $|m| = m_0$. In this case, a similar computation shows that for any $f_1, f_2 \in L^2(\mathbb{S}, ds)$, we have

$$
\text{Tr} \left[ \Omega_m(f_1)^* \Omega_m(f_2) \right] = \int_\mathbb{S} \mathcal{F}_1(q_0) f_2(q_0) dq_0(q_0),
$$

which terminates the proof.

Remark 6.40 The content of the previous Lemma can be summarized as follow:

$$
\text{Tr} \left[ \Omega_m^2(x) \Omega_m(y) \right] = \delta_2(y) \iff |m| = m_0,
$$

where the trace in the left hand side is understood in the distributional sense on $\mathcal{D}(\mathbb{S} \times \mathbb{S})$. Equivalently, this amount to says that when $|m| = m_0$, the Berezin transform associated to the quantization map $\Omega_m$ (see Definition 6.42 below) is trivial.
6.5 The $\star$-product as the composition law of symbols

**Definition 6.41** Let $(M, s, \omega)$ be an elementary symplectic symmetric space such that $m/m_0 \in L^\infty(Q)$ and such that $\mathcal{Y}$ is Abelian. Then let

$$\sigma_m : L^2(H_X) \to L^2(S, d\theta) ,$$

be the adjoint of the quantization map $\Omega_m$. We call the latter the symbol map.

Recall that the defining property of the symbol map is

$$\langle f, \sigma_m[A] \rangle = \text{Tr} [\Omega_m(f)^* A], \quad \forall A \in L^2(H_X), \quad \forall f \in L^2(S, d\theta) .$$

Hence, the symbol map is formally given by

$$\sigma_m[A](x) = \text{Tr} [A \Omega_m(x)], \quad x \in S . \quad (70)$$

Here again, the trace on the right hand side is understood in the distributional sense on $\mathcal{D}(S)$. Note however that when $m$ is essentially bounded, this expression for the symbol map genuinely holds on $L^1(H_X)$, the ideal of trace-class operators on $H_X$. When $m$ is only locally essentially bounded, it also holds rigorously on the dense subspace of $L^2(H_X)$, consisting in finite linear combinations of rank one operators $|\varphi\rangle\langle\psi|$ with $\psi \in \mathcal{D}(Q)$ and $\varphi \in H_X$ arbitrary.

**Definition 6.42** Let $(M, s, \omega)$ be an elementary symplectic symmetric space. Assuming that $m/m_0 \in L^\infty(Q)$, we then set

$$B_m : L^2(S, d\theta) \to L^2(S, d\theta) , \quad f \mapsto \sigma_m \circ \Omega_m(f) ,$$

and we call this linear operator the Berezin transform of the quantization map $\Omega_m$.

**Remark 6.43** The Berezin transform measures the obstruction for the symbol map to be inverse of the quantization map. Say differently, the unitarity of the quantization map on $L^2(S, d\theta)$ is equivalent to the triviality of the associated Berezin transform.

**Proposition 6.44** Let $(M, s, \omega)$ be an elementary symplectic symmetric space with $\mathcal{Y}$ Abelian and let $m$ a Borelian function on $Q$ such that $m/m_0 \in L^\infty(Q)$ and such that $\mathcal{Y}$ is Abelian. Then

(i) The Berezin transform is a positive bounded operator on $L^2(S, d\theta)$, with

$$\|B_m\| \leq \|m/m_0\|_\infty^2 .$$

(ii) The Berezin transform is a kernel operator with distributional kernel given by

$$B_m[x_1, x_2] = \text{Tr} [\Omega_m(x_1) \Omega_m(x_2)] ,$$

and the latter can be identified with

$$B_m[x_1, x_2] = \delta_{q_1}(q_2) \times \int \frac{|m|^2}{m_0^2} (\Psi^{-1}(w)) e^{i(\xi|w, \log b_1 - \log b_2|)} \, dw , \quad x_j = q_j b_j \in S , \quad j = 1, 2 .$$

**Proof.** The first claim comes from the estimate of Theorem 6.39 and the second comes from the computation done in the proof of this Theorem. \qed

For $m = m_0$, the symbol map $\sigma_m$ is the inverse of the quantization map $\Omega_m$. In particular, the associated Berezin transform is trivial. A G-equivariant associative product $\star_{m_0}$ on $L^2(S, d\theta)$ is then defined:

$$f_1 \star_{m_0} f_2 := \sigma_{m_0} [\Omega_{m_0}(f_1) \Omega_{m_0}(f_2)] , \quad \forall f_1, f_2 \in L^2(S, d\theta) . \quad (71)$$
We deduce from (70) that the product (71) is a three-point kernel product, which distributional kernel given by the operator trace of a product of three $\Omega$:

$$f_1 \ast_{m_0} f_2(x) = \int_{\mathbb{R} \times \mathbb{R}} f_1(y) f_2(z) \text{Tr}[\Omega_{m_0}(x) \Omega_{m_0}(y) \Omega_{m_0}(z)] d\mathfrak{g}(y) d\mathfrak{g}(z).$$

We will return to the explicit form of the three-point kernel and its geometric interpretation in the next subsection.

We now come to an important point. Putting together Remark 6.38 and Theorem 6.39, we see that in the more general context of an arbitrary function $m$, the quantization map $\Omega$ is $\mathbb{S}, \mathcal{S}$-invariant, which implies that the complex conjugation is an involution of the Hilbert algebra $(L^2(\mathbb{S}, d\mathfrak{g}), \ast_{m_0})$:

**Proposition 6.45** Let $(M, s, \omega)$ be an elementary symplectic symmetric space with $\mathcal{Y}$ Abelian. Assuming further that $\mathcal{S} m_0 = m_0$, then for all $f_1, f_2 \in L^2(\mathbb{S}, d\mathfrak{g})$, we have:

$$\overline{f_1 \ast_{m_0} f_2} = \overline{f_2} \ast_{m_0} \overline{f_1}.$$ 

Next, we pass to a possible approach\[10\] to define a $\ast$-product for the quantization map $\Omega_m$ in the more general context of an arbitrary function $m$ (and without the assumption that $\mathcal{Y}$ is Abelian).

**Definition 6.46** Let $(M, s, \omega)$ be a polarized, local and elementary symplectic symmetric space and fix $m$ a locally essentially bounded Borelian function on $Q$. Then we let $L^2_m(\mathbb{S}, d\mathfrak{g})$, be the Hilbert-space of classes of measurable functions on $\mathbb{S}$ for which the norm underlying the following scalar product is finite\[11\]:

$$\langle f_1, f_2 \rangle_m := \int_{Q \times m_0 \mathbb{S}} |m|^{-2} (\Psi^{-1}(w)) \left( \int_{\mathcal{Y}} f_1(qb_1) e^{-i(\xi, [w, \log b_1])} d\gamma(b_1) \right) \left( \int_{\mathcal{Y}} f_2(qb_2) e^{i(\xi, [w, \log b_2])} d\gamma(b_2) \right) dQ(q) dw.$$

**Remark 6.47** Formally, we have

$$\langle f_1, f_2 \rangle_m = \langle f_1, B_m f_2 \rangle = \text{Tr}[\Omega_m(f_1) \ast_m \Omega_m(f_2)],$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of $L^2(\mathbb{S}, d\mathfrak{g})$ and $\text{Tr}$ is the operator trace on $\mathcal{H}_\chi$.

Repeating the computations done in the proof of Theorem 6.39, we deduce following extension of the latter:

**Proposition 6.48** Let $(M, s, \omega)$ be an elementary symplectic symmetric space and let $m$ be a locally essentially bounded Borelian function on $Q$.

(i) The quantization map $\Omega_m$ is a unitary operator from $L^2_m(\mathbb{S}, d\mathfrak{g})$ to $L^2(\mathcal{H}_\chi)$,

(ii) Associated to the quantization map $\Omega_m$, there is a deformed product $\ast_m$ on $L^2_m(\mathbb{S}, d\mathfrak{g})$, which is formally given by:

$$f_1 \ast_m f_2 = B_m^{-1} \circ \sigma_m[\Omega_m(f_1) \Omega_m(f_2)].$$

(iii) $(L^2_m(\mathbb{S}, d\mathfrak{g}), \ast_m)$ is an Hilbert algebra and the complex conjugation is an involution when $s^\ast_m m = m$.

---

\[10\] This is however not the approach we will follow for the symmetric spaces underlying elementary normal $\mathcal{J}$-groups—see Proposition 7.17.

\[11\] We do not exclude the possibility for $L^2_m(\mathbb{S}, d\mathfrak{g})$ to be trivial.
6.6 The three-point kernel

The aim of this subsection is to compute the distributional three-point kernel \( \text{Tr}[\Omega_{m_0}(x)\Omega_{m_0}(y)\Omega_{m_0}(z)] \) of the product \(*_{m_0}\) given in \((71)\). We start with two preliminary results extracted from \([23]\):

**Theorem 6.49** Let \((M, s, \omega)\) be an elementary symplectic symmetric space. Given three points \(q_0, q_1, q_2\) in \(Q\), the equation

\[ s_{q_2} \circ_{m_0} s_{q_0}(q) = q, \]

admits a unique solution \(q \equiv q(q_0, q_1, q_2) \in Q\). Moreover, the double triangle map:

\[ \Phi_Q : Q^3 \to Q^3, \quad (q_0, q_1, q_2) \mapsto (\text{mid}(q_0, q_1), \text{mid}(q_1, q_2), \text{mid}(q_2, q_0)) \]

is a global diffeomorphism.

**Proposition 6.50** Let \((M, s, \omega)\) be an elementary symplectic symmetric space with \(\mathcal{Y}\) Abelian. Assume that \(\text{Jac}_{s^{\omega}}\) is invariant under the inversion map on \(Q\). Let \(q \equiv q(q_0, q_1, q_2)\) denote the unique solution of the equation \(s_{q_2} \circ_{m_0} s_{q_0}(q) = q\) in \(Q\). Set

\[ \mathcal{J}(q_0, q_1, q_2) := |\text{Jac}_{s^{\omega}}|^{-1}(q_0^{-1}q_1) |\text{Jac}_{s^{\omega}}|^{-1}(q_1^{-1}q_2) |\text{Jac}_{s^{\omega}}|^{-1}(q_2^{-1}q_0). \]

Then, the kernel of the product \(*_{m_0}\) \((71)\) is given by:

\[ K_{m_0}(x_0, x_1, x_2) = \mathcal{J}^{1/2}(\Phi_Q^{-1}(q_0, q_1, q_2)) |\text{Jac}_{s^{\omega}}|^{-1}(q_0, q_1, q_2) \times |\text{Jac}_{s^{\omega}}|^{1/2}(q_0^{-1}q) |\text{Jac}_{s^{\omega}}|^{1/2}(q_1^{-1}q) |\text{Jac}_{s^{\omega}}|^{1/2}(q_2^{-1}q) \times E(q^{-1}x_0) E((s_{q_0}q)^{-1}x_1) E((s_{q_0}q)^{-1}x_2). \]

**Proof.** Under the assumption that \(\text{Jac}_{s^{\omega}}\) is invariant, we get from Corollary \((6.30)\) the following expression for the operator kernel of \(\Omega_{m_0}(f)\):

\[ \Omega_{m_0}(f)[q, q_0] = |\text{Jac}_{s^{\omega}}|^{1/2}(\text{mid}(e, q_0^{-1}q)) |\text{Jac}_{s^{\omega}}|^{-1}(q_0^{-1}q_0) \int_{\mathcal{Y}} f(\text{mid}(q_0, q_0)b) E(\text{mid}(e, q_0^{-1}q)b) \text{d}\mathcal{Y}(b). \]

Since for \(f \in L^2(\mathbb{S}, d_3)\), \(\Omega_{m_0}(f)\) is Hilbert-Schmidt, the product of three \(\Omega_{m_0}(f)\)’s is a fortiori trace-class and thus we can employ the formula:

\[ \text{Tr} \left[ \Omega_{m_0}(f_0) \Omega_{m_0}(f_1) \Omega_{m_0}(f_2) \right] = \int_{Q^3} \Omega_{m_0}(f_0)[q_0, q_1] \Omega_{m_0}(f_1)[q_1, q_2] \Omega_{m_0}(f_2)[q_2, q_0] \text{d}Q(q_0) \text{d}Q(q_1) \text{d}Q(q_2). \]

Using **Theorem 6.49** and the formula above for the kernel of \(\Omega_{m_0}(f_j)\), we see that the above trace equals:

\[ \int_{Q^3} |\text{Jac}_{s^{\omega}}|^{1/2}(\text{mid}(e, q_1^{-1}q_0)) |\text{Jac}_{s^{\omega}}|^{1/2}(\text{mid}(e, q_2^{-1}q_1)) |\text{Jac}_{s^{\omega}}|^{1/2}(\text{mid}(e, q_0^{-1}q_2)) \times \mathcal{J}^{1/2}(q_0, q_1, q_2) \times f_0(\text{mid}(q_0, q_1)b_0) f_1(\text{mid}(q_1, q_2)b_1) f_2(\text{mid}(q_2, q_0)b_2) \times E(\text{mid}(e, q_1^{-1}q_0)b_0) E(\text{mid}(e, q_2^{-1}q_1)b_1) E(\text{mid}(e, q_0^{-1}q_2)b_2) \text{d}Q(q_0) \text{d}Q(q_1) \text{d}Q(q_2) \text{d}\mathcal{Y}(b_0) \text{d}\mathcal{Y}(b_1) \text{d}\mathcal{Y}(b_2). \]

Performing the change of variable \((q_0, q_1, q_2) \mapsto \Phi_Q^{-1}(q_0, q_1, q_2)\) and setting \(x_j := q_j b_j \in \mathbb{S}, j = 0, 1, 2\), we get

\[ \int_{\mathbb{S}^3} |\text{Jac}_{s^{\omega}}|^{1/2}(q^{-1}q) |\text{Jac}_{s^{\omega}}|^{1/2}(q_1^{-1}s_{q_0}q) |\text{Jac}_{s^{\omega}}|^{1/2}(q_2^{-1}s_{q_0}q) \mathcal{J}^{1/2}(\Phi_Q^{-1}(q_0, q_1, q_2)) |\text{Jac}_{s^{\omega}}|^{-1}(q_0, q_1, q_2) \times E(q^{-1}x_0) E((s_{q_0}q)^{-1}x_1) E((s_{q_0}q)^{-1}x_2) f_0(x_0) f_1(x_1) f_2(x_2) \text{d}_3(x_0) \text{d}_3(x_1) \text{d}_3(x_2), \]

where \(q \equiv q(q_0, q_1, q_2)\) is the unique solution of the equation \(s_{q_2} \circ_{m_0} s_{q_0}(q) = q\) (see **Lemma 6.49**). The result follows by identification. \(\blacksquare\)
Using Lemma 6.34, we deduce the following expression for the phase in the kernel of the product $*_{m_0}$:

**Corollary 6.51** Write $K_{m_0} = A e^{-i s_{m_0}}$ for the three-point kernel given in Proposition 6.50. Then we have for $x_j = q_j b_j$ in $S$, $q_j$ in $Q$, $b_j$ in $Y$, $j = 0, 1, 2$:

$$S(x_0, x_1, x_2) = \langle q, [\Psi(q_1^{-1} q), \log b_0] \rangle + \langle q, [\Psi(q_1^{-1} 0, q), \log b_1] \rangle + \langle q, [\Psi(q_2^{-1} 0, q), \log b_2] \rangle,$$

where $q \equiv q(q_0, q_1, q_2)$ is the unique solution of the equation $q_0 q_j q_2 = q$ (see Lemma 6.49).

### 6.7 Extensions of polarization quadruples

We first make the observation that, given two polarization quadruples $(G_j, \sigma_j, \xi_j, B_j)$, $j = 1, 2$ (in the general sense of Definition 6.11), a morphism $\phi$ between them yields a $G_1$-equivariant intertwiner:

$$\phi^* : C^\infty(G_2) \to C^\infty(G_1)$$

such that $U_{\chi_1}(g_1) \phi^* \phi_2 = \phi^* (U_{\chi_2}(\phi(g_1)) \phi_2)$. (72)

In the context of the transvection and full polarization quadruples, we observe:

**Lemma 6.52** Let $(M, s, \omega)$ be an elementary symplectic symmetric space, associated to a non-exact polarized transvection triple $(g, \sigma, \omega)$. Consider $(L, \sigma, \xi, B)$ and $(G, \sigma, \xi, B)$ and the full and transvection polarization quadruples as in Definition 6.13. Then, the intertwiner (72) corresponding to the injection $\tilde{G} \to \tilde{L}$ is a linear isomorphism.

**Proof.** The injection $j : \tilde{G} \to \tilde{L}$ induces a global diffeomorphism $\tilde{G}/B \to \tilde{L}/B$, $gB \to gB$. Indeed, the map $\tilde{G}/K \to \tilde{L}/\tilde{D} : \tilde{g}K \to gD$ is an identification. Considering the natural projections $\tilde{G}/K \to \tilde{G}/B$ and $\tilde{L}/\tilde{D} \to \tilde{L}/B$, one observes that the diagram

$$
\begin{array}{ccc}
\tilde{G}/K & \to & \tilde{L}/\tilde{D} \\
\downarrow & & \downarrow \\
\tilde{G}/B & \to & \tilde{L}/B
\end{array}
$$

where $\phi(gB) := gB$, is commutative. In particular, $\phi$ is surjective. Examining its differential proves that is also a submersion. The space $Q = \tilde{G}/B$, being exponential, has trivial fundamental group. The map $\phi$ is therefore a diffeomorphism.

Also the restrictions $C^\infty(G)^B \to C^\infty(Q)$ and $C^\infty(L)^B \to C^\infty(Q)$ are linear isomorphisms and one observes that $j^* \phi|_Q = \tilde{\phi}|_Q$. 

Note that when the modular function of $B$ coincides with the restriction to $B$ of the modular function of $L$, then there exists a $L$-invariant measure on $L/B$. From the isomorphism $L/B \simeq G/B \simeq Q$, we see that the later is a left-Haar measure on $Q$. Hence, under the assumption above, we deduce that the left-Haar measure $d_Q$ is also $L$-invariant. This, together with the above Lemma, yields:

**Lemma 6.53** In the setting of Lemma 6.52 and when the modular function of $B$ coincides with the restriction to $B$ of the modular function of $L$, the injection $\tilde{D} \to \tilde{L}$ induces a unitary representation $\mathcal{R} : \tilde{D} \to U(\mathcal{H}_\chi)$ of the corresponding analytic sub-group $D \subset L$ on the representation space $\mathcal{H}_\chi$ associated to the transvection quadruple $(\tilde{G}, \tilde{\sigma}, \tilde{\xi}, B)$.

Consider now two Lie groups $G_j$, $j = 1, 2$, with unitary representations $(U_j, H_j)$, together with a Lie group homomorphism

$$\rho : G_1 \to G_2,$$

and form the associated semi-direct product $G_1 \ltimes_k G_2$, where

$$R_{g_1}(g_2) := C_{\rho(g_1)}(g_2) = \rho(g_1)g_2\rho(g_1)^{-1}, \quad \forall g_1 \in G_1.$$

(73)

We deduce the representation homomorphism:

$$\mathcal{R} : G_1 \to U(H_2), \quad g_1 \mapsto U_2(\rho(g_1)).$$

Within this setting, we first observe:
Lemma 6.54  Parametrizing an element $g \in G_1 \ltimes_R G_2$ as $g = g_1 . g_2$, the map

$$U : G_1 \ltimes_R G_2 \to U(\mathcal{H}_1 \otimes \mathcal{H}_2), \quad g \mapsto U_1(g_1) \otimes R(g_1) U_2(g_2),$$

defines a unitary representation of $G_1 \ltimes_R G_2$ on the tensor product Hilbert space $\mathcal{H} := \mathcal{H}_1 \otimes \mathcal{H}_2$.

Proof. Let $g_j, g_j' \in G_j, j = 1, 2$. Then, on the first hand:

$$U(g_1 g_2) = U(g_1) \otimes R(g_1) U_2(g_2) \otimes R_2(g_2) \otimes R(g_1) U_2(g_2) R(g_1) U_2(g_2),$$

while on the second hand:

$$U(g_1 g_2) = U(g_1) \otimes R(g_1) U_2(g_2) R(g_1) U_2(g_2),$$

and the proof is complete. \hfill \blacksquare

Consider at last two full polarization quadruples $(L_j, \sigma_j, \xi_j, B_j), j = 1, 2$, associated to two local symplectic symmetric spaces $(M_j, \sigma_j, \xi_j)$. Let also $(\Omega_j, \mathcal{H}_j, \mathcal{H}_j)$ be the unitary representation of $L_j$ and the representation of the symplectic symmetric space $M_j = \tilde{G}_j / \tilde{K}_j = L_j / \tilde{D}_j$ (see Remark 6.25). Finally, let $K$ be a Lie sub-group of $L_1$ that acts transitively on $M_1$, and consider the associated $K$-equivariant identification

$$\varphi : K / (K \ltimes \tilde{D}_1) \to M_1.$$ 

Now, given a Lie group homomorphism $\rho : K \to \tilde{D}_2 \subset L_2$, we can form the semi-direct product $K \ltimes_R L_2$, according to (73). Under these conditions, we have the global identification:

$$(K \ltimes_R L_2) / (K \ltimes \tilde{D}_1) \ltimes_R \tilde{D}_2 \to M_1 \times M_2, \quad (g_1, g_2)(K \ltimes \tilde{D}_1) \ltimes_R \tilde{D}_2 \mapsto (\varphi(g_1 K \ltimes \tilde{D}_1), g_2 \tilde{D}_2).$$  \hfill (74)

Proposition 6.55  Let $U$ be the unitary representation of $K \ltimes_R L_2$ on $\mathcal{H} := \mathcal{H}_1 \otimes \mathcal{H}_2$ constructed in Lemma 6.54. Then under the conditions displayed above,

(i) the map

$$\Omega : K \ltimes R L_2 \to U_{sa}(\mathcal{H}), \quad g \mapsto U(g) \circ (\Sigma_1 \otimes \Sigma_2) \circ U(g)^*,$$

is constant on the left-classes of $(K \ltimes \tilde{D}_1) \ltimes_R \tilde{D}_2$ in $K \ltimes_R L_2$.

(ii) For every $g_1 \in K$ and $g_2 \in L_2$, one has\footnote{Warn the reverse order in the group elements.}

$$\Omega(g_2, g_1) = \Omega_1(g_1) \otimes \Omega_2(g_2).$$

(iii) Under the identification (74), the quotient map

$$\Omega : M_1 \times M_2 \to U_{sa}(\mathcal{H}),$$

is $K \ltimes_R L_2$-equivariant.

Proof. We start by checking item (ii). Observe first that

$$U(g_2 g_1) = U(g_1 \rho_{g_1, g_2}) = U_{\chi_1}(g_1) \otimes R(g_1) U_{\chi_2}(\rho(g_1, g_2)) = U_{\chi_1}(g_1) \otimes U_{\chi_2}(g_2) R(g_1).$$

Now, since $\rho$ is $\tilde{D}_2$-valued, we have for all $g_1 \in K$:

$$R(g_1) \Sigma_2 R(g_1)^* = U_{\chi_2}(\rho(g_1)) \Sigma_2 U_{\chi_2}(\rho(g_1))^* = \Sigma_2,$$

and one has

$$\Omega(g_2 g_1) = (U_{\chi_1}(g_1) \otimes U_{\chi_2}(g_2) R(g_1)) \circ (\Sigma_1 \otimes \Sigma_2) \circ (U_{\chi_1}(g_1)^* \otimes R(g_1)^* U_{\chi_2}(g_2)^*)$$

$$= U_{\chi_1}(g_1) \Sigma_1 U_{\chi_1}(g_1)^* \otimes U_{\chi_2}(g_2) R(g_1) \Sigma_2 R(g_1)^* U_{\chi_2}(g_2)^* = \Omega_1(g_1) \otimes \Omega_2(g_2).$$

This implies (ii) and (i) consequently. Regarding item (iii), one observes at the level of $K \ltimes_R L_2$ that

$$\Omega(g g') = U(g) \Omega(g') U(g)^*,$$

which is enough to conclude. \hfill \blacksquare
Remark 6.56 In the same manner than in Definition 6.28 given a Borelian function $m$ on the product manifold $(\mathbb{L}_1/\mathcal{D}_1) \times (\mathbb{L}_2/\mathcal{D}_2)$, we may define for $g \in K \ltimes \mathbb{L}_2$:

$$\Omega_m(g) := U(g) \circ m \circ (\Sigma_1 \otimes \Sigma_2) \circ U(g)^\ast.$$  

Of course, the above procedure can be iterated, namely one observes:

Proposition 6.57 Let $(\mathbb{L}_j, \sigma_j, \xi_j, \mathcal{B}_j)$, $j = 1, \ldots, N$, be $N$ full polarization quadruples, associated to $N$ elementary symplectic symmetric spaces $(M_j, s_j, \omega_j)$ satisfying the extra conditions of coincidence of the modular function on $\mathcal{B}_j$ with the restriction to $\mathcal{B}_j$ of the modular function of $\mathbb{L}_j$, according to Lemmas 6.52 and 6.53. For every $j = 1, \ldots, N-1$, consider a subgroup $K_j$ that acts transitively on $M_j$ together with a Lie group homomorphism $\rho_j : K_j \to \mathcal{D}_{j+1}$. Set $K_N := \mathbb{L}_N$, assume that for every such $j$, the subgroup $\rho_j(K_j)$ normalizes $K_{j+1}$ in $\mathbb{L}_{j+1}$ and denote by $\mathbb{G}_j$ the corresponding homomorphism from $K_j$ to $\text{Aut}(K_{j+1})$. Then, iterating the procedure described in Proposition 6.55 yields a map

$$\Omega : M_1 \times M_2 \times \cdots \times M_N \to \mathcal{U}_{sa}(\mathcal{H}),$$

into the self-adjoint unitaries on the product Hilbert space $\mathcal{H} := \mathcal{H}_{x_1} \otimes \cdots \otimes \mathcal{H}_{x_N}$ that is equivariant under the natural action of the Lie group $\ldots(((K_1 \ltimes_{\mathbb{L}_1} K_2) \ltimes_{\mathbb{L}_2} K_3) \ltimes_{\mathbb{L}_3} \cdots) \ltimes_{\mathbb{L}_{N-1}} L_N$.  

This 'elementary' tensor product construction for the quantization map on direct products of polarized symplectic symmetric spaces (but with covariance under semi-direct products of sub-groups of the covariance group of each pieces) allows to transfer most of the results of the previous subsections. For notational convenience, we formulate all what follows in the context of two elementary symplectic symmetric spaces, i.e. in the context of Proposition 6.55 rather than in the context of Proposition 6.57.

So in all what follows, we assume we are given two elementary symplectic symmetric spaces $(M_j, s_j, \omega_j)$, $j = 1, 2$ (see Definition 6.31). We also let $S_j = Q_j \ltimes \mathbb{Y}_j$, $j = 1, 2$, the subgroups of $\mathbb{G}_j$ (and thus of $L_j$) that acts simply transitively on $M_j$. We also assume that we are given a homomorphism $\rho : S_1 \to \mathcal{D}_2$ (i.e. the role of $K$ in Proposition 6.55 is played by $S_1$). In this particular context, the identification (7.4) becomes:

$$S_1 \ltimes_{\mathbb{L}_1} S_2 \to M_1 \times M_2, \quad g_1 g_2 \mapsto (g_1 \tilde{D}_1, g_2 \tilde{D}_2).$$

We also let $m_0 := m_1^0 \otimes m_2^0$ be the smooth function on $Q_1 \times Q_2$, where $m_j^0$ is the function on $Q_j$ given in (6.9). Combining Proposition 6.55 with the results of subsections 6.3, 6.4, 6.5 and 6.6 we eventually obtain:

Theorem 6.58 Let $(M_j, s_j, \omega_j)$, $j = 1, 2$, be two elementary symplectic symmetric spaces and consider an homomorphism $\rho : S_1 \to \mathcal{D}_2$. Within the notations given above, we have:

(i) Identifying $\mathcal{H}_{x_1} \otimes \mathcal{H}_{x_2}$ with $L^2(Q_1 \times Q_2, d\mu_1 \otimes d\mu_2)$ and parametrizing an element $g \in S_1 \ltimes_{\mathbb{L}_1} S_2$ as $g = q_2 b_2 q_1 b_1$, $q_j \in Q_j$, $b_j \in \mathbb{Y}_j$, we have for $\varphi \in D(Q_1 \times Q_2)$:

$$\Omega_{m_0}(g) \varphi(q_1, q_2) = m_0^1(q_2 q_1^{-1} q_1) m_0^2(q_2^{-1} q_1 b_1) E_{S_1 \ltimes_{\mathbb{L}_1} S_2}^{S_1}(q_2 b_2) \varphi(q_1, q_2),$$

where

$$E_{S_1 \ltimes_{\mathbb{L}_1} S_2}^{S_1}(q_2 b_2) := E_{S_1}^{S_1}(q_1 b_1) E_{S_2}^{S_2}(q_2 b_2), \quad \forall q_j \in Q_j, \forall b_j \in \mathbb{Y}_j;$$

and $E_{S_1}^{S_1}$, $j = 1, 2$, is the one-point phase attached to each elementary symplectic symmetric space $M_j$ as given in Lemma 6.28.

(ii) Moreover, when $\mathbb{Y}_j$ is Abelian, the map

$$\Omega_{m_0} : L^2(S_1 \ltimes_{\mathbb{L}_1} S_2, d\mu_1 \otimes d\mu_2) \to L^2(\mathcal{H}_{x_1} \otimes \mathcal{H}_{x_2}), \quad f \mapsto \int_{S_1 \ltimes_{\mathbb{L}_1} S_2} f(g) \Omega_{m_0}(g) d\mu_1 \otimes d\mu_2(g),$$

is unitary.
(iii) Denoting by $\sigma_{m_0}$ the adjoint of $\Omega_{m_0}$, the associated deformed product:

$$f_1 \ast_{m_0} f_2 := \sigma_{m_0}[\Omega_{m_0}(f_1) \Omega_{m_0}(f_2)],$$

takes on $\mathcal{D}(\mathcal{S}_1 \ltimes \mathcal{R}_2)$ the expression

$$f_1 \ast_{m_0} f_2(g) = \int_{(\mathcal{S}_1 \ltimes \mathcal{R}_2)^2} K_{m_0}^{S_1 \ltimes R_2}(g, g', g'') f_1(g') f_2(g'') d_{S_1 \ltimes R_2}(g') d_{S_1 \ltimes R_2}(g''),$$

where the three-points kernel $K_{m_0}^{S_1 \ltimes R_2}$ is given, with $g = g_2 g_1$, $g' = g_2' g_1'$ and $g'' = g_2'' g_1''$ by

$$K_{m_0}^{S_1 \ltimes R_2}(g, g', g'') := K_{m_0}^{S_1}(g_1, g_1', g_1'') K_{m_0}^{R_2}(g_2, g_2', g_2''),$$

with $K_{m_0}^{S_j}$, $j = 1, 2$, as given in Proposition 6.56.

7 Quantization of Kählerian Lie groups

7.1 The transvection quadruple of an elementary normal $\mathbf{j}$-group

We start by describing the non-exact transvection siLa $\mathfrak{g}$ as described in subsection 3.2. The transvection quadruple of an elementary normal $\mathbf{j}$-group $\mathcal{S}$ is the connected and simply connected Lie group whose Lie algebra $\mathfrak{g}$ is a one-dimensional split extension of two copies of the Heisenberg algebra:

$$\mathfrak{g} := a \ltimes_{\rho} (\mathfrak{h} \oplus \mathfrak{h}),$$

where, again, $a = \mathbb{R} H$ and the extension homomorphism is given by $\rho := \rho_\mathfrak{h} \oplus (-\rho_\mathfrak{h}) \in \text{Der}(\mathfrak{h} \oplus \mathfrak{h})$, with $\rho_\mathfrak{h}$ defined in (38). The involution $\sigma$ of $\mathfrak{g}$ is given by

$$\sigma(aH + (X \otimes Y)) := (-aH) + (Y \otimes X), \quad \forall a \in \mathbb{R}, \quad \forall X, Y \in \mathfrak{h}.$$  \hfill (76)

One has the associated $(\pm 1)$-eigenspaces decomposition:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathfrak{k} := \mathfrak{h}_+ \quad \text{and} \quad \mathfrak{p} := \mathfrak{a} \oplus \mathfrak{h}_-,$$

where for every subspace $F \subset \mathfrak{h}$, we set $F_{\pm} := \{X \oplus (\pm X), X \in F \} \subset \mathfrak{h} \oplus \mathfrak{h}$ and for every element $X \in \mathfrak{h}$ we let $X_{\pm} := \frac{1}{2}(X \oplus (\pm X)) \in \mathfrak{h} \oplus \mathfrak{h}$. Last, we define $\varpi \in \Lambda^2 \mathfrak{g}$ by

$$\varpi(H, E_{-}) = 2 \quad \text{and} \quad \varpi(v_-, v'_-) = \omega^0(v, v'), \quad \forall v, v' \in V,$$

and by zero everywhere else on $\mathfrak{g} \times \mathfrak{g}$. Note that the latter is $\mathfrak{k}$-invariant and its restriction to $\mathfrak{p}$ is non-degenerate. Also, from $[H, \mathfrak{h}_-] = \mathfrak{h}_+ = \mathfrak{k}$, we deduce that $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$ and clearly the action of $\mathfrak{k}$ on $\mathfrak{p}$ is faithful. Thus, in term of Definition 6.3 we have proven the following:

**Proposition 7.1** The siLa $(\mathfrak{g}, \sigma, \varpi)$ defined by (75), (76) and (77), is a transvection symplectic triple.

Consider now $(\mathfrak{g}, \hat{\sigma}, \hat{\delta})$ the exact siLa constructed out of the non-exact siLa $(\mathfrak{g}, \sigma, \varpi)$ as in Lemma 6.8. Recall that $\mathfrak{g}$ is the one-dimensional central extension of $\mathfrak{g}$ with generator $Z$ and table

$$[X, Y]_{\mathfrak{g}} = [X, Y]_{\mathfrak{g}} + \varpi(X, Y) Z, \quad \forall X, Y \in \mathfrak{g}.$$  \hfill (78)

The involution $\hat{\sigma}$ equals $\text{id}_{\mathfrak{g}} \oplus (-\text{id}_{\mathfrak{p}})$, where $\mathfrak{k} = \mathfrak{k} \oplus \mathbb{R} Z$ and $\xi \in \mathfrak{g}^*$ is defined by $\langle \xi, Z \rangle = 1$ and $\xi|_{\mathfrak{g}} = 0$. Accordingly, we set $\hat{\mathcal{G}} = \exp(\hat{\mathfrak{g}})$ and $\hat{K} = \exp(\hat{\mathfrak{k}})$. We identify $\hat{\mathfrak{g}}$ with $\hat{\mathcal{G}}$ via the global chart:

$$aH + v_1 \oplus v_2 + t_1 E \oplus t_2 E + \ell Z \mapsto \exp(aH) \exp(v_1 \oplus v_2 + t_1 E \oplus t_2 E + \ell Z),$$

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where \(a,t_1,t_2,\ell \in \mathbb{R}\) and \(v_1,v_2 \in V\). The group law of \(\tilde{G}\) in these coordinates then reads:

\[
(a,v_1,v_2,t_1,t_2,\ell)(a',v_1',v_2',t_1',t_2',\ell') = \left( a + a', e^{-a'} v_1 + v_1', e^{a'} v_2 + v_2', e^{-2a'} t_1 + t_1' + \frac{1}{2} e^{a'} \omega^0(v_1,v_1'), e^{2a'} t_2 + t_2' + \frac{1}{2} e^{a'} \omega^0(v_2,v_2'), \ell + \ell' + (e^{-2a'} - 1)t_1 + (e^{2a'} - 1)t_2 + \frac{1}{2} \omega^0(e^{-a'}v_1 - e^{a'}v_2, v_1' - v_2') \right),
\]

and the inversion map is given by:

\[
(a,v_1,v_2,t_1,t_2,\ell)^{-1} = (-a, -e^{a}v_1, -e^{a}v_2, -e^{2a}t_1, -e^{-2a}t_2, -\ell - (e^{2a} - 1)t_1 - (e^{-2a} - 1)t_2).
\]

Under the parametrization of \(\tilde{G}\) given above, we consider the following global coordinates system on \(\tilde{G}/\tilde{K}\):

\[
\tilde{G}/\tilde{K} \to \mathbb{R}^{2d+2}, \quad (a,v_1,v_2,t_1,t_2,\ell) \tilde{K} \mapsto (a,v_1-v_2, t_1-t_2 - \frac{1}{2} \omega^0(v_1,v_2)).
\]

From the formula \(s_{\tilde{g}}(g'\tilde{K}) = g\tilde{\sigma}(g'^{-1}g')\tilde{K}\) for the symmetry on \(\tilde{G}/\tilde{K}\), we deduce the following isomorphism of symplectic symmetric spaces:

**Proposition 7.2** Under the identifications \(\mathbb{S} \simeq \mathbb{R}^{2d+2} \simeq \tilde{G}/\tilde{K}\) associated with the charts \([11]\) and \([30]\), the symplectic symmetric space \(\tilde{G}/\tilde{K}\) underlying the exact siLa \((\tilde{g}, \tilde{\sigma}, \delta\xi)\) defined above, is isomorphic to the symplectic symmetric space \((\mathbb{S}, s, \omega^g)\) underlying an elementary normal \(j\)-group \(\mathbb{S}\), as given in subsection \([7.1]\).

Next, we need to endow \((\mathbb{S}, s, \omega^g)\) with a structure of polarized symplectic symmetric space. From Lemma \([6.1]\), it suffices to specify \(W\), a \(t\)-invariant Lagrangian subspace of \(\mathfrak{g}\). To this aim, we again consider the splitting of the \(2d\)-dimensional symplectic vector space \((V, \omega^0)\) into a direct sum of two Lagrangian subspaces in symplectic duality:

\[ V = \mathfrak{t}^* \oplus \mathfrak{t}. \]

Relatively to this decomposition, we define:

\[ W := \mathfrak{l}_- \oplus \mathbb{R} E_- \subset \mathfrak{g}. \]

Following then Proposition \([6.14]\), we let

\[ \mathfrak{b} := \mathfrak{k} \oplus W = \mathfrak{k} \oplus \mathbb{R} Z \oplus W, \]

be the polarization algebra. Accordingly with the terminology introduced in Definition \([6.15]\), we call \((\tilde{\mathfrak{g}}, \tilde{\sigma}, \xi, \mathfrak{b})\) the transverse quadruple of the symplectic symmetric space \((\mathbb{S}, s, \omega^g)\).

**Remark 7.3** Note that \(\tilde{G}, \tilde{K}\) and \(\mathfrak{B} := \exp(\mathfrak{b})\) are all unimodular. In particular, this implies that \(\tilde{G}\)-invariant measures on the homogeneous spaces \(\tilde{G}/\tilde{K}\) and \(\tilde{G}/\mathfrak{B}\) do exist.

Last, we need to specify the local and elementary structures underlying the polarized symplectic symmetric space \((\mathbb{S}, s, \omega^g)\), as introduced in Definition \([6.26]\) and Definition \([6.31]\) respectively. We first note:

**Lemma 7.4** Let

\[ \mathfrak{q} := \mathfrak{a} \oplus (\mathfrak{t}^* \oplus 0) \quad \text{and} \quad \mathfrak{q} :=(0 \oplus 0) \oplus \mathbb{R}((E \oplus 0) + Z). \]

Then \(\mathfrak{q}\) is a Lie sub-algebra of \(\tilde{\mathfrak{g}}\) supplementary to \(\mathfrak{b}\) and \(\mathfrak{q}\) is an Abelian sub-algebra of \(\mathfrak{b}\) which is normalized by \(\mathfrak{q}\). Moreover, the associated semi-direct product \(\mathfrak{q} \ltimes \mathfrak{q}\) is naturally isomorphic to the Lie algebra \(\mathfrak{s}\) and induces the vector space decomposition \(\tilde{\mathfrak{g}} = \mathfrak{s} \oplus \mathfrak{k}\).

**Proof.** First observe that for all \(X \in \mathfrak{h}\), one has \(X \oplus 0 = X_- + X_+ \in \mathfrak{h} \oplus \mathfrak{h}\) and therefore

\[ \varpi(H, E \oplus 0) = \varpi(H, E_- + E_+) = \varpi(H, E_-) = 2, \]

\[ \varpi(v \oplus 0, v' \oplus 0) = \varpi(v_- + v_+, v'_- + v'_+) = \varpi(v_-, v'_-) = \omega^0(v, v'), \quad \forall v, v' \in V. \]

The fact that \(\mathfrak{q}\) is a Lie sub-algebra follows from

\[ [H, \mathfrak{t}^* \oplus 0]_{\tilde{\mathfrak{g}}} = [H, \mathfrak{t}^* \oplus 0] + \varpi(H, \mathfrak{t}^* \oplus 0)Z = \mathfrak{t}^* \oplus 0, \]
Lemma 7.6

Similarly, for all $y \in \Gamma^*$, one has:

$$[y \oplus 0, x \oplus 0 + t(E \oplus 0 + Z)]_{\hat{\sigma}} = [y \oplus 0, x \oplus 0] + \omega^0(y, x)Z = \omega^0(y, x)(E \oplus 0 + Z).$$

The rest of the statement is immediate.\hfill \blacksquare

Remark 7.5 Neither $q$ nor $\bar{\mathfrak{g}}$ are $\hat{\sigma}$-stable. However, since $\hat{\sigma}(q) = a \oplus (0 \oplus \Gamma^*)$ and $[0 \oplus \Gamma^*, \bar{\mathfrak{g}}] = 0$, one sees that $\hat{\sigma}(q)$ normalizes $\bar{\mathfrak{g}}$ as well.

Lemma 7.6 Equipped with the sub-group $Q = \exp\{q\}$ of $\hat{G}$, the polarized symplectic symmetric space $(\mathcal{S}, \sigma, \omega^0)$ is local in the sense of Definition 6.26.

Proof. Note first that

$$\mathfrak{b} = \mathfrak{k} \oplus W \oplus \mathbb{R}Z = \mathfrak{h}_+ \oplus \mathfrak{l}_- \oplus \mathfrak{RE}_- \oplus \mathbb{R}Z = \Gamma^*_+ \oplus (1 \oplus \Gamma) \oplus (\mathbb{R}E \oplus \mathbb{R}E) \oplus \mathbb{R}Z.$$

Thus, under the parametrization of $\hat{G}$, we have

$$B = \{(0, n \oplus m_1, n \oplus m_2, t_1, t_2, \ell) : m_1, m_2 \in \Gamma, n \in \Gamma^*, t_1, t_2, \ell \in \mathbb{R}\},$$

and

$$Q = \{(a, n, 0, 0, 0, 0) : n \in \Gamma^*, a \in \mathbb{R}\}.$$ 

Thus for $q = (a, n, 0, 0, 0, 0)$ and $b = (0, n' \oplus m_1', n' \oplus m_2', t_1', t_2', \ell') \in B$, we have using (80):

$$q.b = (a, n + n') \oplus m_1', n' \oplus m_2', t_1' + \frac{1}{\sqrt{2}} \omega^0(n, m_1'), t_2' + \frac{1}{\sqrt{2}} \omega^0(n, m_1' - m_2')),$$

from which we deduce that the map

$$Q \times B \rightarrow \hat{G}, \quad (q, b) \mapsto q.b,$$

is a global diffeomorphism (i.e. the first condition of Definition 6.26 is satisfied). For the second condition, observe that as $\mathfrak{b} \cap \mathfrak{p} = \mathfrak{l}_- \oplus \mathbb{R}E_-$, we get

$$[\mathfrak{a}, \mathfrak{b} \cap \mathfrak{p}] = [\mathfrak{a}, \mathfrak{l}_-] \oplus [\mathfrak{a}, \mathbb{R}E_-] = \Gamma^*_+ \oplus \mathbb{R}E_+ \subset \mathfrak{h}_+ = \mathfrak{k} \subset \mathfrak{b}.$$

To check the last condition, consider $q = (a, n, 0, 0, 0, 0) \in Q$, with $a \in \mathbb{R}$, $n \in \Gamma^*$. We then have

$$\hat{\sigma}q = (-a, -n, 0, 0, 0, 0)(0, n, 0, 0, 0) = (\hat{\sigma}q)^Q (\hat{\sigma}q)^B,$$

since $(0, n, 0, 0, 0) \in \Gamma^*_+ \subset \mathfrak{h}_+ \subset \mathfrak{b}$. Thus, $\chi((\hat{\sigma}q)^B) = 1$ since for $b = (0, n \oplus m_1, n \oplus m_2, t_1, t_2, \ell) \in B$, we have $\chi(g) = e^{it}$.\hfill \blacksquare
Next, we come to the symmetric space structure of the group $Q$:

**Lemma 7.7** In the global chart:

$$q \simeq a \oplus \mathfrak{t} \to Q, \quad (a, n) \mapsto \exp(aH)\exp(n \oplus 0),$$

the left invariant symmetric space structure $\mathfrak{s}$ on $Q$ described in Lemma 6.27 reads:

$$\mathfrak{s}_{(a, n)}(a', n') = (2a - a', 2\cosh(a - a')n - n').$$

Moreover, the symmetric space $(Q, \mathfrak{s})$ admits a midpoint map, which in the coordinates above, is given by:

$$\text{mid}((a, n), (a_0, n_0)) = \left(\frac{a + a_0}{2}, \frac{n + n_0}{2\cosh(\frac{a - a_0}{2})}\right).$$

**Proof.** By definition we have $\mathfrak{s}_q q' = q\tilde{\sigma}(q^{-1}q')^Q$ and the formula for the symmetry follows easily from \([49]\) and \([52]\). The formula for the midpoint map comes from a direct computation of the inverse diffeomorphism of the partial map $\mathfrak{s}_q := [Q \ni q' \mapsto q, q' \in Q].$ ■

**Remark 7.8** Setting $A := \exp(a)$ and $N := \exp(\mathfrak{t} \oplus 0)$ we have the global decomposition $Q = AN$ and for $q = an \in Q$, the symmetry at the neutral element reads $\mathfrak{s}_q a^{-1}n^{-1}$. Also, the global chart

$$\tilde{G}/B \to \mathbb{R}^{d+1}, \quad (a, n_1 + m_1, n_2 + m_2, t_1, t_2, \ell)B \mapsto (a, n_1 - n_2),$$

identifies $\tilde{G}/B$ with $Q$ via the coordinate system \([88]\).

**Lemma 7.9** The Abelian sub-group $\mathbb{Y} := \exp(\mathfrak{y})$ of $\tilde{G}$, endows the local symplectic symmetric space $(\mathbb{S}, \omega^\mathbb{S})$ with an elementary structure in the sense of Definition 6.31.

**Proof.** We already know by Lemma 7.4 that the left invariant symmetric space $(Q, \mathfrak{s})$ admits a midpoint map. We also know by Lemma 7.4 that $\mathbb{Y}$ is normalized by $Q$ and that $\mathbb{S}$ is isomorphic to $Q \times \mathbb{Y}$. But we need to know that $Q \times \mathbb{Y}$ acts simply transitively on the symmetric space $\tilde{G}/\tilde{K}$. For this, let $g = (a, v, 0, t, 0, t) \in Q \times \mathbb{Y}$ and $g' = (a', v', 0, t', 0, t') \in \tilde{G}$. Then we get

$$gg' = (a + a', e^{-a'}v + v_1, v_2', e^{-2a'} + t + t_1, t + t_2 + e^{-2a'} - 1) + \frac{1}{2}e^{0^\mathbb{S}(e^{-a'}v, v_1 - v_2))},$$

and thus in the chart \([50]\) of $\tilde{G}/\tilde{K}$, we get:

$$gg' \tilde{K} \mapsto (a + a', e^{-a'}v + v_1' - v_2, e^{-2a'}t + t_1' - t_2 - \frac{1}{2}2\omega(0(e^{-a'}v, v_1', v_2'))).$$

This means that under the identification $\mathbb{S} \simeq \tilde{G}/\tilde{K}$, $Q \times \mathbb{Y} \simeq \mathbb{S}$ acts by left translations and the second condition of Definition 6.31 is verified. For the third condition, note that under the parametrization \([78]\) of $\tilde{G}$, we have

$$\mathbb{Y} = \{(0, m, 0, 0, 0, t) : m \in \mathbb{I}, t \in \mathbb{R}\}.$$ 

Take $q = (a, n, 0, 0, 0, 0) \in Q$, $a \in \mathbb{R}$, $n \in \mathbb{I}$ and $b = e^y = (0, m, 0, t, 0, t) \in \mathbb{Y}$, $m \in \mathbb{I}$, $t \in \mathbb{R}$. A computation then shows that

$$\langle \xi, (\text{Ad}_q^{-1} - \text{Ad}_{(\mathfrak{s}_q)^{-1}})y \rangle = 2\sinh(2a)t + 2\cosh(a)\omega^0(n, m),$$

which entails that

$$\Psi(a, n) = (2\sinh(2a), 2\cosh(a)n).$$

The last condition follows from \([79]\). ■
Remark 7.10 Parametrizing $\tilde{G}$ as in $[78]$, $S \cong \tilde{G}/\tilde{K}$ as in $[80]$ and $Q \cong \tilde{G}/B$ as in $[85]$, we have the following expression for the action of $\tilde{G}$ on $S$:

$$(a, v_1, v_2, t_1, t_2, \ell)(a', v', t') = (a + a', e^{-a'}v_1 - e^{a'}v_2 + v', e^{-2a'}t_1 - e^{2a'}t_2 + t' - \frac{1}{2}a'v_2),$$

and on $Q$:

$$(a, n_1 \oplus m_1, n_2 \oplus m_2, t_1, t_2, \ell)(a', n') = (a + a', e^{-a'}n_1 - e^{a'}n_2 + n').$$

Remark 7.11 From similar methods than those leading to Lemma 3.27, we deduce that for $q = (a, n) \in Q$, we have:

$$C(\cosh(a) + |n|) \leq d_Q(q).$$

From the Remark above and in analogy with Remark 3.36, we define the Fréchet valued Schwartz space of $Q$ $\mathcal{S}(Q, \mathcal{E})$, as the set of smooth functions such that all left (or right) derivatives decrease faster than any power of $d_Q$. The latter space is Fréchet for the semi-norms:

$$f \in \mathcal{S}(Q, \mathcal{E}) \mapsto \sup_{X \in U_{k}(q)} \sup_{x \in Q} \left\{ \frac{\partial^{k} \|Xf\|_{\mathcal{E}}}{\|X\|_{k}} \right\}, \quad j, k, n \in \mathbb{N},$$

or even for

$$f \in \mathcal{S}(Q, \mathcal{E}) \mapsto \sup_{X \in U_{k}(q)} \sup_{x \in Q} \left\{ \frac{\partial^{k} \|Xf\|_{\mathcal{E}}}{\|X\|_{k}} \right\}, \quad j, k, n \in \mathbb{N}.$$ .

7.2 Quantization of elementary normal $j$-groups

In this subsection, we specialize the different ingredients of our quantization map in the case of the elementary symplectic symmetric space $(S, s, \omega^S)$ underlying an elementary normal $j$-group. We also (re)introduce a real parameter $\theta$ in the definition of the character $[83]$

$$\chi_{\theta}(b) := \exp\left\{ \frac{\theta}{2} \langle \xi, \log(b) \rangle \right\}, \quad b \in B, \quad \theta \in \mathbb{R}^*,$$

which is globally defined as $B$ is exponential. By Lemma 6.13 and Remark 7.3, the Haar measure $d_S$ on $S$ (respectively $d_Q$ on $Q$) is invariant under both $s^*$ (respectively $s_0^*$) and $\tilde{G}$. Observe that under the parametrization $[S1]$ of the group $B$, we have $\chi_{\theta}(b) = \exp\left\{ \frac{1}{\theta} \ell \right\}$. Note that within the chart $[S33]$, any left-invariant Haar measure $d_Q$ on $Q$ is a multiple of the Lebesgue measure on $q$ (these facts are transparent in Equations [12], [S31] and in Remark 7.10). Also, within the chart $[11]$, any left invariant Haar measure $d_S$ on $S$ is a multiple of the Lebesgue measure on $q \times \mathbb{Y}$. By Remark 6.33, the restriction to $S = Q \times \mathbb{Y}$ of the induced representation $U_{\chi_{\theta}}$ (that we denote by $U_{\theta}$ from now on) of $\tilde{G}$ on $L^2(Q, d_Q)$ reads within the charts $[11]$ on $S$ and $[S33]$ on $Q$:

$$U_{\theta}(a, v, t)\psi(a_0, n_0) = \exp\left\{ \frac{\theta}{2} \left( e^{2(a-a_0)t} + \omega^S \left( \frac{1}{2} e^{a-a_0}n - n_0, e^{a-a_0}m \right) \right) \right\} \psi(a_0 - a, n_0 - e^{a-a_0}n),$$

where $(a, v, t) \in S$ with $a, t \in \mathbb{R}$ and $v = n \oplus m \in \mathbb{Y} \oplus \mathbb{Y} = V$ and $(a_0, n_0) \in Q$ with $a_0 \in \mathbb{R}$ and $n_0 \in \mathbb{Y}$.

Remark 7.12 In accordance with the notations of earlier sections, from now, we make explicit the dependence in the parameter $\theta \in \mathbb{R}^*$ in all the objects we are considering. For instance, we now set $\Omega_{\theta, m}$ instead $\Omega_m$ for the quantization map, $*_{\theta, m}$ instead of $*_m$ for the associated composition product, $K_{\theta, m}$ instead of $K_m$ for its three-points kernel, $E_{\theta}$ instead of $E$ for the one-point phase etc.

Regarding the one-point phase $E_{\theta}$ of Lemma 6.28, we get from Lemma 6.34 and $[80]$

$$E_{\theta}(a, v, t) = \exp\left\{ \frac{\theta}{2} \left( \sinh(2a)t + \omega^S(\cosh(a)n, \cosh(a)m) \right) \right\}.$$

Moreover from $[S4]$ and $[80]$, we observe:

$$|\text{Jac}_{\omega^S}(a, n)| = 2^{d+1} \cosh(a)^d, \quad |\text{Jac}_{\psi}(a, n)| = 2^{d+1} \cosh(2a) \cosh(a)^d,$$

so that the element $m_0$ given in 69 reads:

$$m_0(a, n) = 2^{d+1} \cosh(2a)^{1/2} \cosh(a)^d.$$
Proposition 7.13 Parametrizing $S$ as in \cite{[11]} and $Q$ as in \cite{[55]}, we have the following expression for the action on $L^2(Q,d_Q)$ of the unitary quantizer $\Omega_{\theta,m_0}(x)$, $x \in S$, associated with the polarized symplectic symmetric space underlying an elementary normal $j$-group $S$:

$$\Omega_{\theta,m_0}(a,v,t)\psi(a_0,n_0) = 2^{d+1} \exp \left\{ \frac{2i}{\theta} \left( \sinh(2a - 2a_0)t + \omega^0(\cosh(a - a_0)n - n_0, \cosh(a - a_0)m) \right) \right\} \times \cosh(2a - 2a_0)^{1/2} \cosh(a - a_0)^d \psi(2a - a_0, 2 \cosh(a - a_0)n - n_0).$$

Remark 7.14 Observe that $|\text{Jac}_e|$ is $s_i$-invariant, as it is an even function of the variable $a$ only. Thus, by Lemma 6.24 and Remark 6.38, we deduce that the unitary quantization map $\Omega_{\theta,m_0}$ is also compatible with the natural involutions of its source and range spaces (the complex conjugation on $L^2(S,d_S)$ and the adjoint on $L^2(L^2(Q,d_Q))$. In particular, it sends real-valued functions to self-adjoint operators.

Our next result is one of the key step of this section: it renders transparent the link between sections \cite{[4]} and \cite{[5]} and sections \cite{[4]} and \cite{[5]}. For this, we need the explicit expression of the tri-kernel $K_{\theta,m_0}$ of the product \cite{[7]} for an elementary normal $j$-group $S$. First, observe that the unique solution of the equation $s_{g_2} \circ s_{g_1} \circ s_{g_0}(q) = q$, $q, q_0, q_1, q_2 \in Q$, as given in Lemma 6.40 reads

$$q = (a_0 - a_1 + a_2, 2 \cosh(a_0 - a_1)n_2 - 2 \cosh(a_2 - a_0)n_1 + 2 \cosh(a_1 - a_2)n_0).$$

From \cite{[53]}, we extract

Lemma 7.15 Within the notations of Proposition 6.54, we have

$$\mathcal{J} = |\text{Jac}_Q|.$$ Then, Proposition 6.50 and a straightforward computation gives $K_{\theta,m_0} = A_{m_0} e^{\frac{2i}{\theta}s}$ with:

$$A_{m_0}(x_1,x_2,x_3) = m_0(a_1 - a_2)m_0(a_2 - a_3)m_0(a_3 - a_1) = 2^{3d+3} \cosh(2a_1 - 2a_2)^{1/2} \cosh(a_1 - a_2)^d \times \cosh(2a_2 - 2a_3)^{1/2} \cosh(a_2 - a_3)^d \cosh(2a_3 - 2a_1)^{1/2} \cosh(a_3 - a_1)^d,$$

and

$$S(x_1,x_2,x_3) = \sinh(2a_1 - 2a_2)t_3 + \sinh(2a_3 - 2a_1)t_2 + \sinh(2a_2 - 2a_3)t_1 + \cosh(a_1 - a_2) \cosh(a_2 - a_3) \omega^0(v_1,v_3) + \cosh(a_1 - a_2) \cosh(a_2 - a_3) \omega^0(v_1,v_1) + \cosh(a_3 - a_1) \cosh(a_1 - a_2) \omega^0(v_3,v_2).$$

By identification, we thereof obtain:

Proposition 7.16 For $g_1,g_2,g_3 \in S$ and $\theta \in \mathbb{R}^+$, we have

$$K_{\theta,m_0}(g_1,g_2,g_3) = K_{\theta,0}(g_1^{-1}g_2,g_1^{-1}g_2),$$

where the three-point kernel in the left hand side of the above equality is given in Proposition 6.54 and the two-point kernel in the right hand side is given in Theorem 7.4 for $\tau = 0$. In particular, the products $\star_{\theta,m_0}$ and $\star_{\theta,0}$ coincide on $L^2(S,d_S)$.

Before giving the link between the generic kernels $K_{\theta,m}$ and $K_{\theta,\tau}$, hence a fortiori between the generic products $\star_{\theta,m}$ and $\star_{\theta,\tau}$, we will give the relation between our quantization map with Weyl’s one. Denote by $\Omega^0$, the Weyl quantization map of $S$ in the Darboux chart \cite{[11]}. For a function $f$ on $S$, $\Omega^0(f)$ is an operator on $L^2(\mathbb{R}^{d+1}) \simeq L^2(Q,d_Q)$ given (up to a normalization constant) by

$$\Omega^0(f)\psi(q_0,n_0) := C(\theta,d) \int e^{-\frac{\theta}{2}(2(a_0-a)t + \omega^0(n_0-n,m))} f(\frac{a+n}{2}, \frac{a+n}{2}, m,t) \psi(a,n) \, da \, dn \, dt.$$

Then, recall that for $\tau \in \Theta$ (see Definition 4.2), the inverse $T_{\theta,\tau}^{-1}$ of the map \cite{[53]} is continuous on on the ‘flat’ Schwartz space $S(S)$. As the Weyl quantization maps continuously Schwartz functions to trace-class operators, we deduce that $\Omega^0 \circ T_{\theta,\tau}^{-1}$ is well defined and continuous from $S(S)$ to $L^1(L^2(Q,d_Q))$. From this, we get

\footnote{By this we mean the ordinary Schwartz space in the global chart \cite{[11]}.}
Proposition 7.17 Let $\tau \in \Theta$ (see Definition 4.2). Then, as continuous operators from $S(\mathbb{S})$ to the trace ideal $L^1(L^2(Q, dQ))$, we have

$$\Omega^0 \circ T_{\theta, \tau}^{-1} = \Omega_{\theta, m}, \quad \text{where} \quad m(a, n) = m_0(q) \exp \left\{ \tau \left( \frac{2}{\pi} \sinh(2a) \right) \right\}. \quad (87)$$

**Proof.** By density, it suffices to show that for $f \in S(\mathbb{S})$, $\Omega_{\theta, m}(f)$ and $\Omega^0(T_{\theta, \tau}^{-1}(f))$ coincide on $D(Q)$. Note then that for $f \in S(\mathbb{S})$, we have

$$T_{\theta, \tau}^{-1}(f)(a, v, t) = 2\pi \int \cosh \left( \frac{\theta}{2} \right) \frac{1/2}{\cosh \left( \frac{\theta}{2} \right)} e^{\tau \left( \frac{d}{2} \sinh(\theta) \right)} f(a, \cosh \left( \frac{\theta}{2} \right)^{-1} v, t') e^{i\xi t - \frac{2}{\pi} \sinh \left( \frac{\theta}{2} \right) t'} d\xi dt'. \quad \text{Hence, for} \quad \psi \in D(Q), \quad \text{we get} \quad \Omega^0(T_{\theta, \tau}^{-1}(f)) \psi(a_0, n_0) = C_2(d) \int \cosh \left( \frac{\theta}{2} \right) \frac{1/2}{\cosh \left( \frac{\theta}{2} \right)} e^{\tau \left( \frac{d}{2} \sinh(\theta) \right)} f(a, m, n, t') e^{i\xi t - \frac{2}{\pi} \sinh \left( \frac{\theta}{2} \right) t'} \times f(a - a_0, 2\cosh(\theta/2)n - n_0) \psi(a, n) d\xi dt' da dm dt'. \quad \text{Performing the change of variables} \quad a \mapsto 2a - a_0, \quad n \mapsto 2\cosh(\theta/2)n - n_0 \quad \text{and} \quad m \mapsto \cosh(\theta/2)m, \quad \text{we get} \quad \Omega^0(T_{\theta, \tau}^{-1}(f)) \psi(a_0, n_0) = C_3(d) \int \cosh \left( 2(a - a_0) \right) \frac{1/2}{\cosh \left( 2(a - a_0) \right)} e^{\tau \left( \frac{d}{2} \sinh(2a) \right)} f(a, n, m, t') e^{i\xi t - \frac{2}{2} \sinh \left( \frac{\theta}{2} \right) t'} \times f(2a - a_0, 2\cosh(\theta/2)n - n_0) \psi(2a - a_0, 2\cosh(\theta/2)n - n_0) d\xi dt' da dm dt', \quad \text{Integrating out the} \quad t \quad \text{variable yields a factor} \quad \delta(\xi - \frac{d}{2}(a_0 - a)) \quad \text{and thus we get} \quad \Omega^0(T_{\theta, \tau}^{-1}(f)) \psi(a_0, n_0) = C_3(d) \int \cosh \left( 2(a - a_0) \right) \frac{1/2}{\cosh \left( 2(a - a_0) \right)} e^{\tau \left( \frac{d}{2} \sinh(2a) \right)} f(a, n, m, t') e^{i\xi t - \frac{2}{2} \sinh \left( \frac{\theta}{2} \right) t'} \times f(2a - a_0, 2\cosh(\theta/2)n - n_0) \psi(2a - a_0, 2\cosh(\theta/2)n - n_0) da dm dt', \quad \text{which by Proposition 7.13 coincides with} \quad \Omega_{\theta, m}(f). \quad \blacksquare$$

From the above result and the defining relation $[\mathbf{S}_4]$ for the product $\star_{\theta, \tau}$ for a generic element $\tau \in \Theta$, we then deduce:

**Proposition 7.18** To every $\tau \in \Theta$, associate a right-$N$-invariant function $m$ on $Q$ as in $[\mathbf{S}_7]$. Then, the three point kernel $K_{\theta, m}$ of the product $\star_{\theta, m}$ defined in Proposition 6.48 (ii), is related to the two-point kernel $K_{\theta, \tau}$ given in Theorem 4.4 via:

$$K_{\theta, m}(g_1, g_2, g_3) = K_{\theta, \tau}(g_1^{-1} g_2, g_1^{-1} g_2), \quad \forall g_1, g_2, g_3 \in \mathbb{S}. \quad \text{In particular, the product} \quad \star_{\theta, m} \quad \text{is well defined on} \quad \mathbb{B}(\mathbb{S}) \quad \text{and coincide with} \quad \star_{\theta, \tau}. \quad \text{Remark 7.19 From now on, to indicate that a right-$N$-invariant borelian function $m$ on $Q$ is associated to an element $\tau \in \Theta$, as in $[\mathbf{S}_7]$, we just write $m \in \Theta(\mathbb{S})$.} \quad \text{Remark 7.20 Observe that, considering the ‘double triangle’ three-point function $\Phi_3$ given in Proposition 6.48 (ii), and writing $K_{\theta, m} = A_m e^{\Phi_3}$, for $m$ a right-$N$-invariant function on $Q$, we have:}$$

$$A_m(x_1, x_2, x_3) = m(a_1 - a_2) \frac{m_0^2(a_2 - a_3)}{m(a_2 - a_3)} m(a_3 - a_1)$$

$$= \left| \text{Jac}_{q_3^{-1}}(x_1, x_2, x_3) \right|^{1/2} \frac{m(a_1 - a_2)}{m_0(a_1 - a_2)} \frac{m_0(a_2 - a_3)}{m(a_2 - a_3)} \frac{m(a_3 - a_1)}{m_0(a_1 - a_2)},$$

$^{14}$Observe that $S^{\text{S-con}}(\mathbb{S}) \subset \mathbb{S}$.}
By standard Fourier-analysis arguments, we deduce:

\[
B_{\theta,m}[x_1,x_2] = \delta(a_1 - a_2) \delta(n_1 - n_2) \delta(m_1 - m_2) \int \left| m \right|^2 m_0 \left( \frac{1}{2} \arcsinh \left( \frac{t_0}{m_0} \right) \right) e^{\frac{2}{t_0} (t_1 - t_2)} \, dt_0.
\]

By standard Fourier-analysis arguments, we deduce:

\[
B_{\theta,m} = \left| m \right|^2 m_0 \left( \frac{1}{2} \arcsinh \left( \frac{t}{m} \right) \right).
\]

Finally, let us discuss the question of the involution for the generic product \( \star_{\theta,m} \). Since in general, the formal adjoint of \( \Omega_{\theta,m}(x) \) on \( L^2(Q, dq) \) is \( \Omega_{\theta,m}^\dagger(x) \), we deduce

\[
\int_1 \star_{\theta,m} \int_2 = \int_1 \star_{\theta,m} \int_2.
\]

Hence, we obtain that the natural involution for the product \( \star_{\theta,m} \) is

\[
\star_{\theta,m} : f \mapsto \frac{m}{\Omega_{\theta,m}} \left( \frac{1}{2} \arcsinh \left( \frac{t}{m} \right) \right) f.
\]

**Remark 7.21** From Theorem 6.39 and the previous expression for the involution, we observe that the element \( m_0 \) defined in (69) is uniquely determined by the requirement that the associated quantization map is both unitary and preserving.

### 7.3 Quantization of normal j-groups

Consider now a normal j-group \( B = (\mathbb{S}_N \times \mathbb{R}^{N-1} \ldots) \times \mathbb{R}_1, \mathbb{S}_1 \) with associated extension morphisms

\[
\mathbb{R}^j \in \text{Hom}( (\mathbb{S}_N \times \ldots) \times \mathbb{S}_{j+1} \times \text{Sp}(V_j, \omega_j^0)), \quad j = 1, \ldots, N - 1
\]

as in (40). We wish to apply Proposition 5.57 to this situation. For this, recall that for \( \mathbb{S} \) an elementary normal j-group viewed as an elementary symplectic symmetric space, \( \mathcal{O} \) denotes the Lie algebra of \( W \)-preserving symplectic endomorphisms of \( \mathfrak{p} \) where the Lagrangian subspace \( W \) has been chosen to be \( \mathfrak{l}_0 \oplus \mathbb{R} E_\perp \cong \mathfrak{l}_0 \oplus \mathbb{R} E \).

**Proposition 7.22** Denote by \( \mathcal{O}_0 \) the stabilizer Lie sub-algebra in \( \mathfrak{sp}(V, \omega^0) \) of the Lagrangian subspace \( l \).

(i) Let \( \text{Sym}(l) \) be the space of endomorphisms of \( l \) that are symmetric with respect to a given Euclidean scalar product on \( l \). Let also

\[
\eta : \text{End}(l) \times \text{Sym}(l) \to \text{Sym}(l), \quad (T, S) \mapsto T \circ S + S \circ T^t.
\]

Then, endowing \( \text{Sym}(l) \) with the structure of an Abelian Lie algebra, one has the isomorphism:

\[
\mathcal{O}_0 \cong \text{End}(l) \ltimes_{\eta} \text{Sym}(l).
\]

(ii) The Lie algebra \( \mathcal{O}_0 \) contains an Iwasawa component of \( \mathfrak{sp}(V, \omega^0) \).

(iii) Letting \( \mathcal{O}_0 \) trivially act on the central element \( E \) of \( \mathfrak{h} \) induces an isomorphism:

\[
\mathcal{O} \cong \mathcal{O}_0 \ltimes \mathfrak{h}.
\]

**Proof.** Item (i) is immediate from an investigation at the matrix form level. Item (iii) follows from the fact that the derivation algebra of the non-exact polarized transvection symplectic triple \((\mathfrak{g}, \sigma, \omega)\) underlying an elementary normal j-group \( \mathbb{S} \) admits the symplectic Lie algebra \( \mathfrak{sp}(V, \omega^0) \) as Levi-factor \( \mathfrak{g} \). Item (ii) follows from a dimensional argument combined with Borel’s conjugacy Theorem of maximal solvable sub-groups in complex simple Lie groups. Indeed, on the first hand, the dimension of the Iwasawa factor of \( \mathfrak{sp}(V, \omega^0) \) equals \( \dim \mathfrak{sp}(V, \omega^0) - \dim \mathfrak{u}(d) \) that is \( 2d + \frac{2(d-1)}{2} - d^2 = d(d+1) \) with \( 2d := \dim V = 2 \dim l \). On the other hand, the dimension of the Borel factor in \( \text{End}(l) \) equals \( \frac{d(d-1)}{2} \) which equals \( \dim \text{Sym}(l) \). Hence \( \mathcal{O}_0 \) contains a maximal solvable Lie sub-algebra of dimension \( 2(d + \frac{d(d-1)}{2}) = d(d+1) \). Borel’s Theorem then yields the assertion since \( \mathfrak{sp}(V, \omega^0) \) is totally split.
From \[3\], we observe that the full polarization quadruple of $\mathbb{S}$ underlies the Lie group $L = \text{Sp}(V, \omega_0) \times \hat{G}$. Hence:

**Corollary 7.23** Let $\mathbb{S}$ be an elementary normal $j$-group viewed as an elementary symplectic symmetric space (see Definition 6.31) and let $(L, \sigma, \xi, B)$ the associated full polarization quadruple (see Definition 6.12). Then we have the global decomposition $L = QB$ and moreover $\Delta_1|_B = \Delta_B$.

We can now prove the conditions needed to apply Proposition 6.57.

**Proposition 7.24** Let $B = (S_1 \ltimes \mathbb{R}^{N-1} \ldots) \ltimes \mathbb{R}$; $S_1$ be a normal $j$-group, to which one associates the full polarization quadruples $(L_j, \sigma_j, \xi_j, B_j), j = 1, \ldots, N,$ of the $S_j$’s. Then, there exists an homomorphism $\rho_j : S_j \to \hat{D}_{j-1}$ such that its image $\rho_j(S_j)$ normalizes $S_{j-1}$ in $L_{j-1}$ and such that the extension homomorphism $R_j$ constructed in (73), coincides with $R_j \in \text{Hom}(S_j, \text{Aut}(S_{j-1}))$ in (89).

**Proof.** Firstly, by Pyatetskii-Shapiro’s theory \[17\], one knows that the action of $S_j$ on $S_{j-1}$ factors through a solvable subgroup of $\text{Sp}(V_{j-1}, \omega_{j-1}^0)$, setting $(AN)_{j-1}$ the Iwasawa factor of $\text{Sp}(V_{j-1}, \omega_{j-1}^0)$, we thus get an homomorphism:

$$\tilde{\rho}_j : S_j \to (AN)_{j-1}.$$ 

But Proposition 7.22 (ii) asserts that $(AN)_{j-1}$ is a subgroup of $\exp\{D_{0,j-1}\}$, where $D_{0,j-1}$ is the stabilizer Lie sub-algebra in $\mathfrak{sp}(V_{j-1}, \omega_{j-1}^0)$ of the Lagrangian subspace $L_{j-1}$. Combining this with the isomorphism of Proposition 7.24 (iii), yields another homomorphism:

$$\hat{\rho}_j : (AN)_{j-1} \to D_{j-1} \subset \hat{D}_{j-1}.$$ 

Hence $\rho_j := \hat{\rho}_j \circ \hat{\rho}_j$ is the desired homomorphism. Now, observe that by \[3\] Proposition 2.2 item (i), the group $S_{j-1}$ viewed as a subgroup of $L_{j-1}$, is normalized by $\text{Sp}(V_{j-1}, \omega_{j-1}^0)$ for the action given in Proposition 6.10 (iv) and that this action is precisely the one associated with the extension homomorphism $R_j$ in the decomposition (89). Thus, all what remains to do is to prove that the extension homomorphisms $R_j$ and $R_j$ coincide. Here, $R_j(g) := C_{\rho_j(g)} \in \text{Aut}(S_{j-1}), g \in S_j$ is the extension homomorphism constructed in (73). But that $R_j = R_j$ follows from a very general fact about homogeneous spaces. Namely, observe that if $M = G/K$, then action of the isotropy $K \times M \to M, (k, gK) \to kgK$, lifts to $G$ as the restriction to $K$ of the conjugacy action $K \times G \to G, (k, g) \to kgK$ (indeed: $kgK = kgK$).

From this, we deduce that Proposition 6.57 and Theorem 6.58 are valid in the case of a normal $j$-group. Moreover, we also deduce that the associated product $*_{\theta, m}$ coincides with $*_{\theta, \varphi}$ of Proposition 4.3.

**Proposition 7.25** Let $B$ be a normal $j$-group. To every $\varphi \in \Theta$, we associate a function $m$ on $Q_N \times \ldots Q_1$ by $m = m_N \otimes \ldots \otimes m_1$ where $m_j$ is related to $\tau_j$ as in (87). Then, the three-point kernel $K_{\theta, m}$ of the product $*_{\theta, m}$ defined in Theorem 6.35 (iii), is related with the two-point kernel $K_{\theta, \varphi}$ given in Proposition 4.3 (77) via:

$$K_{\theta, m}(g_1, g_2, g_3) = K_{\theta, \varphi}(g_1^{-1}g_2, g_1^{-1}g_2). \quad \forall g_1, g_2, g_3 \in \mathbb{B}.$$ 

In particular, the product $*_{\theta, m}$ is well defined on $\mathcal{B}(\mathbb{B})$ and coincide with $*_{\theta, \varphi}$.

**Remark 7.26** To indicate that a function $m = m_N \otimes \ldots \otimes m_1$ on $Q_N \times \ldots Q_1$ is related to elements $\tau_j \in \Theta$ as in (87), we just write $m \in \Theta(\mathbb{B})$.

We also quote the following extension of Proposition 7.17.

**Proposition 7.27** Let $B$ be a normal $j$-group. For any $m \in \Theta(\mathbb{B})$, the quantization map $\Omega_{\theta, m}$ is a continuous operator from $S(\mathbb{S}) := S(S_N) \otimes \ldots \otimes S(S_1)$ to $L^1(L^2(Q_N, dQ_N) \otimes \ldots \otimes L^2(Q_1, dQ_1)).$

\[15\]Observe that $S^\infty_{\text{can}}(\mathbb{B}) \subset S(\mathbb{B})$. 

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8 Deformation of $C^*$-algebras

Throughout this section, we consider a $C^*$-algebra $A$, endowed with an isometric and strongly continuous action $\alpha$ of a normal $j$-group $B$. In section 7 we have seen how to deform the Fréchet algebra $A^\infty$, consisting of smooth vectors for the action $\alpha$. Our goal here is to construct a $C^*$-norm on $(A^\infty, *_{BM})$, in order to get, after completion, a deformation theory at the $C^*$-level. We stress that from now on, the isometricity assumption of the action is fundamental. The way we will define this $C^*$-norm is based on the pseudo-differential calculus introduced in the previous two sections.

The basic ideas of the construction can be summarized as follow. Consider $\mathcal{H}$ an Hilbert space carrying a faithful representation of $A$. We will thereof identify $A$ with its image in $B(\mathcal{H})$. Let $S^S_0(B, A)$ be the $A$-valued one-point Schwartz space associated to the tempered pair $(B \times B, S^B_0)$ as given in Definition 2.31. Since this space is a sub-set of the flat $A$-valued Schwartz space of $B$, Proposition 7.27 shows that for every $m \in \Theta(B)$, the map

$$f \in S^S_0(B, A) \mapsto \Omega_{g,m}^B(f) := \int_B f(x) \otimes \Omega_{g,m}^B(x) \, d_B(x)$$

is well defined and takes values in $A \otimes K(\mathcal{H}_x)$, where $\mathcal{H}_x := L^2(Q_1, d_{Q_1}) \otimes \cdots \otimes L^2(Q_N, d_{Q_N})$. Then, the main step is to extend the map $[\Omega]_B$, from $S^S_0(B, A)$ to $B(B, A)$. As for $a \in A^\infty$, the $A$-valued function $\alpha(a) := [g \in B \mapsto \alpha_g(a) \in A]$ belongs to $B(B, A)$, we will define a new norm on $A^\infty$ by setting

$$\|a\|_{g,m} := \|\Omega_{g,m}^B(\alpha(a))\|,$$

where the norm on the r.h.s. denotes the $C^*$-norm of $B(\mathcal{H} \otimes \mathcal{H}_x)$. This will eventually be achieved by proving a non-Abelian $C^*$-valued version of the Calderón-Vaillancourt Theorem in the context of the present pseudo-differential calculus. This theorem will be proven using wavelet analysis and oscillatory integrals methods.

8.1 Wavelet analysis

Let $B$ be a normal $j$-group, with Pyatetskii-Shapiro decomposition

$$B = (S_N \times \ldots) \ltimes S_1,$$

where the $S_j$’s, $j = 1, \ldots, N$, are elementary normal $j$-groups. Recall that our choice of parametrization is:

$$S_N \times \cdots \times S_1 \to B, \quad (g_N, \ldots, g_1) \mapsto g_1 \cdots g_N.$$

Remark 8.1 Observe that the extension homomorphism at each step, $R^j$, being valued in $Sp(V_j, \omega_j)$, it preserves any left invariant Haar measure $d_{B_j}$ on $S_j$:

$$(R^j_{g'})^* d_{B_j} = d_{B_j}, \quad \forall g' \in (S_N \times \ldots) \ltimes S_{j-1}. \quad (91)$$

This implies that the product of left invariant Haar measures $d_{B_1} \otimes \cdots \otimes d_{B_N}$ defines a left invariant Haar measure on $B$ under both parametrization $g = g_1 \ldots g_N$ or $g = g_N \ldots g_1$ of $g \in B$.

The aim of this subsection is to construct a weak resolution of the identity on the tensor product Hilbert space $L^2(Q_N, d_{Q_N}) \otimes \cdots \otimes L^2(Q_1, d_{Q_1}) := \mathcal{H}_x$ from a suitable family of coherent states for $B$, that we now introduce.

Definition 8.2 Let $B$ be a normal $j$-group. Given a mother wavelet $\eta := \eta_N \otimes \cdots \otimes \eta_1 \in D(Q_N) \otimes \cdots \otimes D(Q_1)$, let $\{\eta_x\}_{x \in B}$ be the family of coherent states defined by

$$\eta_x := U_\theta(x)\eta, \quad x \in B,$$

where $U_\theta$ is the unitary representation of $B$ on $\mathcal{H}_x$ constructed in Lemma 6.54 for the morphism underlying Proposition 7.24.\footnote{Given an Hilbert space $\mathcal{H}$, $K(\mathcal{H})$ denotes the $C^*$-algebra of compact operators.}
Observe that in the elementary case, we have:
\[
\eta_x(q_0) = E^0_q(q^{-1} q_0 b) \eta(q^{-1} q_0), \quad x = qb \in S, \quad q_0 \in Q,
\]  
where the phase \( E^0_q \) is defined by
\[
E^0_q(x) := \chi_\rho(C^{-1} q^{-1} b), \quad x = qb \in S.
\]
In the generic case, setting \( B = B' \times_{\mathbb{R} S_1} B' \) with \( B' \) a normal \( j \)-group and \( S_1 \) an elementary normal \( j \)-group, we have for \( \eta = \eta' \otimes \eta^1, \eta' \in D(Q_N \times \cdots \times Q_2), \eta^1 \in D(Q_1) \) and parametrizing \( g \in B' \) as \( g = g' g_1, g' \in B' \), \( g_1 \in S_1 \):
\[
\eta_g = \eta_{g'} \otimes \eta^1_{\rho(g') g_1},
\]
where \( \rho : B' \to B \) is the homomorphism underlying Proposition 7.24.

**Proposition 8.3** Let \( B \) be a normal \( j \)-group, \( E \) a complex Fréchet space, with topology defined by a countable family of semi-norms \( \{ || \cdot ||_j \}_{j \in \mathbb{N}} \) and let \( \eta := \eta_N \otimes \cdots \otimes \eta_1 \in D(Q_N) \otimes \cdots \otimes D(Q_1) \). Then, the map
\[
F^q : L^\infty(Q_N \times \cdots \times Q_1, \mathbb{C}) \to L^\infty(B, E),
\]
\[
f \mapsto \left[ x \in B \mapsto \int f(q_N, \ldots, q_1) \eta_x(q_N, \ldots, q_1) dQ_n(q_N) \ldots dQ_1(q_1) \right] \subset \mathcal{F}^q,
\]
induces a continuous map \( \mathcal{F}^q : S(Q_N \times \cdots \times Q_1, \mathbb{C}) \to L^1(B, E) \).

**Proof.** Assume first that \( B = S \) is an elementary normal \( j \)-group. We denote by \( E^0_q \) the element of \( C^\infty(S) \) given in 7.24, so that with \( x = qb \in S, q \in Q, b \in \mathbb{V} \), we have for every \( f \in S(Q, E) \):
\[
(F^q f)(x) = \int f(q_0) E^0_q(q^{-1} q_0 b) \eta(q^{-1} q_0) dQ(q_0) = \int Q f(q_0) E^0_q(q_0 b) \eta(q_0) dQ(q_0).
\]
Decomposing as usual \( q = a \oplus n \), the left invariant vector fields associated with \( H \) the generator of \( a \) and \( \{ f_j \}_{j=1}^d \) a basis of \( n \), read in the coordinates 7.24:
\[
\tilde{H} = \partial_a - \sum_{j=1}^d n_j \partial_{n_j}, \quad \tilde{f}_j = \partial_{n_j}, \quad j = 1, \ldots, d.
\]
Moreover, in the chart 11 of \( S \), with \( x = (a, n \oplus m, t) \), the function \( E^0_q \) takes the following form:
\[
E^0_q(x) = \exp \left\{ \frac{i}{\hbar} \left( e^{-2a t} - e^{-a} \omega^0(n, m) \right) \right\}.
\]
Hence, defining
\[
i \tilde{H} E^0_q =: \alpha E^0_q \quad \text{and} \quad -\sum_{j=1}^d \tilde{f}_j^2 E^0_q =: \beta E^0_q,
\]
a simple computation gives
\[
\alpha(x) = -\frac{2}{\hbar} (e^{-2a} - e^{-a} \omega^0(n, m)), \quad \beta(x) = \theta^{-2} e^{-2a} |m|^2,
\]
where \( |m|^2 = \sum_{j=1}^d \omega^0(f_j, m)^2 \). Moreover, it is easy to see that both \( \alpha \) and \( \beta \) are eigenvectors of \( \tilde{H} \) with eigenvalue \(-2\) and that \( \tilde{f}_j \beta = 0 \). Hence setting \( \tilde{P} := 1 - \sum_{j=1}^d \tilde{f}_j^2 \), we get by integration by part on the \( q_0 \)-variables and with \( k \in \mathbb{N} \) arbitrary:
\[
(F^q f)(x) = \int E^0_q(q_0 b) (1 - \tilde{H}^2_{q_0}) \left\{ 1 + \frac{1}{(1 + \alpha(q_0 b)^2 - 2i \alpha(q_0 b))(1 + \beta(q_0 b)^2) \tilde{P}_{q_0}^k [f(q_0) \eta(q_0)]} \right\} dQ(q_0).
\]
This easily entails that
\[
\|(F^n f)(x)\|_j \leq C(k) \int \frac{\| (1 - \tilde{H} q_0) \tilde{P}^k [f(q_0) \eta(q_0)] \|_j}{(1 + \alpha(q_0 b)^2)(1 + \beta(q_0 b))} \, d\varrho(q_0).
\]

By left invariance of $\tilde{H}$ and $X_j$, we get up to a redefinition of $f \in \mathcal{S}(Q, \mathcal{E})$ and of $\eta \in \mathcal{D}(Q)$:
\[
\|(F^n f)(x)\|_j \leq C(k) \int \frac{\| f(q_0) \|_j |\eta(q_0)|}{(1 + \alpha(q_0 b)^2)(1 + \beta(q_0 b))} \, d\varrho(q_0).
\]

In coordinates, the $L^1$-norm of $\|(F^n f)(x)\|_j$ is bounded by:
\[
\int \frac{\| f(a, n_0, e^{-a_0} n + n_0) \|_j |\eta(a_0, n_0)|}{(1 + 4 \pi^2 e^{-2n_0 t} - e^{-a_0}(n, m)^2)(1 + \theta^{-2} \cosh(a)^2 \|m\|^2)} \, d\theta_0 \, dn_0 \, d\eta_0 \, dm \, dt
\leq \int \frac{\| f(a, n) \|_j |\eta(a_0, n_0)| e^{a_0}}{(1 + 4 \pi^2 t^2)(1 + \theta^{-2} \|m\|^2)} \, d\theta_0 \, dn_0 \, d\eta_0 \, dm \, dt,
\]
which is finite for $k = [d/2] + 1$ since $f$ is integrable in any semi-norm $\|\cdot\|_j$ and $\eta$ is compactly supported.

Clearly, the arguments above survive when considering direct products of elementary $j$-groups. Now, we make the simple observation that the quantity
\[
\|F^n f\|_{1,j} := \int_{B} \|(F^n f)(g)\|_j \, d\varrho(g),
\]
for the semi-direct product $B = (S \times \ldots \times S_1)$ equals the same quantity for the direct product $S \times \ldots \times S_1$. Indeed, set $B = B' \times \times \times S_1$ and $g = g', g_1 \in B$, $g' \in B'$, $g_1 \in S_1$, we have from [44], the observation made in Remark 8.1 and left-invariance of the Haar measure $d\varrho_1$:
\[
\int_{B' \times S_1} \|(F^n f)(g)\|_j \, d\varrho' \times S_1(g) = \int_{B' \times S_1} \|(F^n f)(g', \rho(g')g_1)\|_j \, d\varrho'(g') \, d\varrho_1(g_1)
= \int_{B' \times S_1} \|(F^n f)(g', g_1)\|_j \, d\varrho'(g') \, d\varrho_1(g_1) = \int_{B' \times S_1} \|(F^n f)(g)\|_j \, d\varrho' \times S_1(g).
\]

This concludes the proof.  

We then deduce the following consequence:

**Corollary 8.4** Let $B$ be a normal $j$-group and $\eta := \eta N \otimes \cdots \otimes \eta 1 \in \mathcal{D}(Q_N) \otimes \cdots \otimes \mathcal{D}(Q_1)$. Then, the maps $[B \ni x \mapsto \langle \eta, \eta_x \rangle]$ and $[B \ni x \mapsto \langle \eta, \eta_{x-1} \rangle]$ belong to $L^1(B, d\varrho)$.  

**Proof.** This follows from Proposition 8.3 with $\mathcal{E} = \mathbb{C}$ since $\langle \eta, \eta_x \rangle = \mathcal{F}^N(\eta)(x)$ and $\langle \eta, \eta_{x-1} \rangle = \mathcal{F}^1(\eta)(x)$.  

The next result is probably well known but since we are unable to locate a proof in the literature and since we use it several times, we deliver a proof.

**Lemma 8.5** Let $(X, \mu)$ be a $\sigma$-finite measure space and $\mathcal{H}$ a separable Hilbert space. Consider an element
\[
K \in L^\infty(X \times X, \mu \otimes \mu; B(\mathcal{H})�,
\]
such that
\[
c_1^2 := \sup_{x \in X} \int_X \|K(x, y)\|_{B(\mathcal{H})} \, d\mu(y) < \infty \quad \text{and} \quad c_2^2 := \sup_{y \in X} \int_X \|K(x, y)\|_{B(\mathcal{H})} \, d\mu(x) < \infty.
\]
Then, the associated kernel operator is bounded on $L^2(X, \mu; \mathcal{H})$ with operator norm not exceeding $c_1 c_2$.  

\[\text{[17]}\text{It can be viewed as a Banach space valued version of Shur’s test Lemma.}\]
Proof. Let $T_K$ the operator associated with the kernel $K$. For $\Phi, \Psi \in L^2(X, \mu; \mathcal{H}) \cap L^1(X, \mu; \mathcal{H})$, we have $|\langle \Phi, T_K \Psi \rangle| < \infty$. Moreover, the Cauchy-Schwarz inequality gives
\[
|\langle \Phi, T_K \Psi \rangle| = \left| \int_{X \times X} \langle \Phi(x), K(x, y) \Psi(y) \rangle d\mu(y) d\mu(x) \right|
\leq \int_{X \times X} \|\Phi(x)\|_\mathcal{H} \|K(x, y)\|_{\mathcal{B}(\mathcal{H})} \|\Psi(y)\|_\mathcal{H} d\mu(y) d\mu(x)
\leq \left( \int_{X \times X} \|\Phi(x)\|_\mathcal{H}^2 \|K(x, y)\|_{\mathcal{B}(\mathcal{H})} d\mu(y) d\mu(x) \right)^{1/2} \left( \int_{X \times X} \|K(x, y)\|_{\mathcal{B}(\mathcal{H})} \|\Psi(y)\|_\mathcal{H}^2 d\mu(y) d\mu(x) \right)^{1/2}
\leq c_1 c_2 \|\Phi\|_{L^2(X, \mu; \mathcal{H})} \|\Psi\|_{L^2(X, \mu; \mathcal{H})},
\]
and the claim follows immediately.

We are now able to prove that the family of coherent states $\{\eta_x\}_{x \in \mathbb{B}}$ provides a weak resolution of the identity.

**Proposition 8.6** Let $\mathbb{B}$ a normal $j$-group, $\mathcal{H}$ an Hilbert space and $\eta := \eta_N \otimes \cdots \otimes \eta_1 \in \mathcal{D}(Q_N) \otimes \cdots \otimes \mathcal{D}(Q_1) \setminus \{0\}$. Then, for all $\Phi, \Psi \in \mathcal{H} \otimes \mathcal{H}_X$ the following relation holds:
\[
\langle \Psi, \Phi \rangle_{\mathcal{H} \otimes \mathcal{H}_X} = C(\eta)^{-1} \int_{\mathbb{B}} \langle \langle \eta_x, \Psi \rangle_{\mathcal{H}_X}, \langle \eta_x, \Phi \rangle_{\mathcal{H}_X} \rangle_{\mathcal{H}} d\mathbb{B}(x),
\]
where $\langle \eta_x, \Psi \rangle_{\mathcal{H}_X}$ is the vector in $\mathcal{H}$ is defined by:
\[
\langle \varphi, \langle \eta_x, \Psi \rangle_{\mathcal{H}_X} \rangle_{\mathcal{H}} := \langle \varphi \otimes \eta_x, \Psi \rangle_{\mathcal{H} \otimes \mathcal{H}_X}, \forall \varphi \in \mathcal{H},
\]
and the constant $C(\eta)$ equals $(2\pi \delta^{\dim(\mathbb{B})}/2)\|\Delta_{Q_j} \eta_j\|_2^2 \cdots \|\Delta_{Q_N} \eta_N\|_2^2$, with $\Delta_{Q_j}$ the modular function of $Q_j$.

**Proof.** We first demonstrate that for $\Phi \in \mathcal{H} \otimes \mathcal{H}_X$, the map $x \mapsto \langle \eta_x, \Phi \rangle_{\mathcal{H}_X} \in \mathcal{H}$ belongs to $L^2(\mathbb{B}, \mathcal{H})$. To see this, let $\{\mathbb{B}_j\}_{j \in \mathbb{N}}$ be an increasing sequence of relatively compact subsets of $\mathbb{B}$, which converges to $\mathbb{B}$. For each $j \in \mathbb{N}$, we define the operator
\[
T^0_j : L^2(\mathbb{B}_j, \mathcal{H}) \rightarrow \mathcal{H} \otimes \mathcal{H}_X, \quad F \mapsto \int_{\mathbb{B}_j} F(x) \otimes \eta_x d\mathbb{B}(x).
\]
Clearly, each $T^0_j$ is bounded:
\[
\|T^0_j F\|_{\mathcal{H} \otimes \mathcal{H}_X} \leq \int_{\mathbb{B}_j} \|F(x)\|_\mathcal{H} \|\eta_x\|_{\mathcal{H}_X} d\mathbb{B}(x)
\leq \|\eta\|_{\mathcal{H}_X} \text{meas}(\mathbb{B}_j)^{1/2} \left( \int_{\mathbb{B}_j} \|F(x)\|_\mathcal{H}^2 d\mathbb{B}(x) \right)^{1/2} = \|\eta\|_{\mathcal{H}_X} \text{meas}(\mathbb{B}_j)^{1/2} \|F\|_{L^2(\mathbb{B}_j, \mathcal{H})}.
\]
To see that the family $\{T^0_j\}_{j \in \mathbb{N}}$ is in fact uniformly bounded, note that the adjoint of $T^0_j$ reads:
\[
T^{0*}_j : \mathcal{H} \otimes \mathcal{H}_X \rightarrow L^2(\mathbb{B}_j, \mathcal{H}), \quad \Phi \mapsto \left[ x \in \mathbb{B}_j \mapsto \langle \eta_x, \Phi \rangle_{\mathcal{H}_X} \in \mathcal{H} \right].
\]
Hence for $F \in L^2(\mathbb{B}_j, \mathcal{H})$ we get
\[
|T^{0*}_j F(x)|^2 = \int_{\mathbb{B}_j} F(y) \langle \eta_x, \eta_y \rangle d\mathbb{B}(y),
\]
that is $|T^0_j|^2 = S^0_j \otimes \text{Id}_{\mathcal{H}_X}$, where $S^0_j \in \mathcal{B}(L^2(\mathbb{B}_j))$ is a kernel operator with kernel $K^{0*}_j(x, y) = \langle \eta_x, \eta_y \rangle$. Applying Lemma 5.4, a simple change of variable gives $\|S^0_j\| \leq \|\{x \mapsto \langle \eta, \eta_x \rangle\}\|_1 := C$ which is finite by Lemma 5.3 and of course, is uniform in $j \in \mathbb{N}$. Finally, since
\[
\int_{\mathbb{B}_j} \|\langle \eta_x, \Phi \rangle_{\mathcal{H}_X}\|_{\mathcal{H}}^2 d\mathbb{B}(x) = \|T^{0*}_j \Phi\|_{L^2(\mathbb{B}_j, \mathcal{H})}^2 \leq C \|\Phi\|_{\mathcal{H} \otimes \mathcal{H}_X}^2,
\]
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taking the limit $j \to \infty$ gives
\[
\int_{S} \|\langle \eta, \Phi \rangle_{\mathcal{H}} \|^2_{\mathcal{H}} \, d_{\mathcal{H}}(x) \leq C ||\Phi||^2_{\mathcal{H}}
\]
as needed. The rest of the proof is computational. Assume first that $\mathbb{B} = S$ is elementary. In this case, for $\Phi, \Psi \in \mathcal{H} \otimes \mathcal{H}$ and $\eta \in D(Q)$, we have in chart 41 and from (92):
\[
\int_{S} \langle \langle \eta, \Phi \rangle_{\mathcal{H}}, \langle \eta, \Phi \rangle_{\mathcal{H}} \rangle_{\mathcal{H}} \, d_{\mathcal{H}}(x) = \int \langle \Psi(a_0, n_0), \Phi(a_1, n_1) \rangle_{\mathcal{H}} \eta(a_0 - a, n_0 - e^{a - a_0} n) \eta(a_1 - a, n_1 - e^{a - a_1} n)
\]
\[
\times \exp \left\{ -i \left( (e^{-2a_0} - e^{-2a_1}) \epsilon + \omega(n, m) \right) - e^{a_0}(n_0 e^{-a_0} - n_1 e^{-a_1}, m) \right\} \, da \, dn \, dt \, da_0 \, dn_0 \, da_1 \, dn_1.
\]
Integrating out the $t$-variable yields a factor $2\pi \theta e^{-2a_0 + 2a_1} \delta(a_0 - a_1)$, the former expression then becomes
\[
2\pi \theta \int \langle \Psi(a_0, n_0), \Phi(a_0, n_0) \rangle_{\mathcal{H}} \eta(a_0 - a, n_0 - e^{a - a_0} n) \eta(a_0 - a, n_1 - e^{a - a_0} n)
\]
\[
\times e^{-2a_0 + 2a_1} \exp \left\{ -i \left( e^{a - a_0} \omega_0(n_0 - n_1, m) \right) \right\} \, da \, dn \, da_0 \, dn_0 \, dn_1.
\]
Integrating the $m$-variables, yields a factor $(2\pi \theta)^d e^{-d(a_0 + da_0) \delta(n_0 - n_1)}$ and we get, up to a constant:
\[
\int \langle \Psi(a_0, n_0), \Phi(a_0, n_1) \rangle_{\mathcal{H}} \eta(a_0 - a, n_0 - e^{a - a_0} n) \eta(a_0 - a, n_0 - e^{a - a_0} n) \, e^{-(d+2)(a - a_0)} \, da \, dn \, da_0 \, dn_0,
\]
which after an affine change of variable gives
\[
(2\pi \theta)^{d+1} \langle \Psi, \Phi \rangle_{\mathcal{H} \otimes \mathcal{H}} \int |\eta(a, n)|^2 e^{2(d+2)a} \, da,
\]
which is all we needed.

The case of a generic normal $j$-group $\mathbb{B}$, can be treated with the same reduction method than the one used in Proposition 8.3. Observe first that the arguments above survive when considering direct products of elementary normal $j$-groups. Now, for $\mathbb{B} = \mathbb{B}' \ltimes \mathbb{S}_1$, we get with the |bra⟩|ket| notation and with $g = g'g_1 \in \mathbb{B}$, $g' \in \mathbb{B}'$, $g_1 \in \mathbb{S}_1$:
\[
\int_{\mathbb{B}' \times \mathbb{S}_1} |\eta_{g'} \rangle \langle \eta_{g'} | \, d_{\mathbb{B}' \times \mathbb{S}_1}(g) = \int_{\mathbb{B}' \times \mathbb{S}_1} |\eta_{g'g_1} \rangle \langle \eta_{g'g_1} | \, d_{\mathbb{B}' \times \mathbb{S}_1}(g_1)
\]
\[
= \int_{\mathbb{B}' \times \mathbb{S}_1} |\eta_{g'g_1} \rangle \langle \eta_{g'g_1} | \, d_{\mathbb{B}' \times \mathbb{S}_1}(g_1) = \int_{\mathbb{B}' \times \mathbb{S}_1} |\eta_g \rangle \langle \eta_g | \, d_{\mathbb{B}' \times \mathbb{S}_1}(g),
\]
and the claim for semi-direct products of elementary normal $j$-groups follows from the claim for direct products of elementary normal $j$-groups. \hfill \qed

Remark 8.7 In the following, we will absorb the constant $C(\eta)^{1/2}$ of Proposition 8.3 in a redifinition of the mother wavelet $\eta = \eta_N \otimes \cdots \otimes \eta_1 \in D(Q_N) \otimes \cdots \otimes D(Q_1)$.

Remark 8.8 Other types of weak resolution of the identity can be constructed in this setting. For instance, setting $\tilde{\eta}_x := \Omega^B_{m_0}(x)\eta$, where $\eta$ is arbitrary in $\mathcal{H}_x$, we have from the unitarity of the quantization map $\Omega^B_{m_0}$:
\[
\langle \psi, \phi \rangle = ||\eta||^{-2} \int_{\mathbb{B}} \langle \psi, \tilde{\eta}_x \rangle \langle \tilde{\eta}_x, \phi \rangle \, d_{\mathcal{H}}(x),
\]
for all $\phi, \psi \in \mathcal{H}_x$. Similarly, let $W^\eta_{y,z}$ be the Wigner function on $\mathbb{B}$, associated to a pair of wavelets $\eta_x, \eta_y$:
\[
W^\eta_{x,y}(z) := \sigma_m \langle \eta_x | \eta_y \rangle (z) = \langle \eta_y | \Omega^B_{m_0}(z) | \eta_x \rangle, \quad x, y, z \in \mathbb{B}.
\]
Then, these Wigner functions may be used to construct a weak resolution of the identity on $L^2(\mathbb{B}, d_{\mathcal{H}})$: For all $f_1, f_2 \in L^2(\mathbb{B}, d_{\mathcal{H}})$, we have
\[
\langle f_1, f_2 \rangle = ||\eta||^{-4} \int_{\mathbb{B} \times \mathbb{B}} \langle f_1, W^\eta_{x,y}(z) \rangle \langle W^\eta_{x,y} f_2 \rangle \, d_{\mathcal{H}}(x) \, d_{\mathcal{H}}(y).
\]

\[^{18}\]In what follows, $|\varphi\rangle|\psi\rangle$ is the rank one operator given by $\xi \mapsto \langle \psi, \xi \rangle \varphi$.  

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We then deduce:

\[ \text{coordinates system on } S \]

From the expressions (45) of the left invariant vector fields of \( S \), consider the following decomposition of the Lie algebra \( \mathfrak{h} \times \mathfrak{h} \) by mean of the quadratic form

\[ H \times H \to \mathbb{C}, \quad (\phi, \psi) \mapsto \langle \phi \otimes \eta_x, A \psi \otimes \eta_y \rangle_{\mathfrak{h} \otimes \mathfrak{h}}. \]

Assuming further that

\[ \sup_{y \in \mathbb{B}} \int_{\mathbb{B}} \| (\eta_x, A \eta_y)_{\mathfrak{h}} \|_{\mathfrak{g}(\mathfrak{h})} \, d_{\mathbb{B}}(x) < \infty, \quad \sup_{x \in \mathbb{B}} \int_{\mathbb{B}} \| (\eta_x, A \eta_y)_{\mathfrak{h}} \|_{\mathfrak{g}(\mathfrak{h})} \, d_{\mathbb{B}}(y) < \infty, \]

then \( A \) extends to a bounded operator on \( \mathfrak{h} \otimes \mathfrak{h} \).

**Proof.** Since \( \eta_x \) is smooth and compactly supported, our assumption about the domain of \( A \) ensures that \( \langle \eta_x, A \eta_y \rangle_{\mathfrak{h}} \) is well defined as an element of \( \mathcal{B}(\mathfrak{h}) \). Thus, Lemma 8.5 applied to \( (X, \mu) = (\mathbb{B}, d_{\mathbb{B}}) \), yields that the operator \( \tilde{A} \) on \( L^2(\mathbb{B}, \mathfrak{h}) \) given by

\[ \tilde{A} F(x) := \int_{\mathbb{B}} \langle \eta_x, A \eta_y \rangle_{\mathfrak{h}} F(y) \, d\mu(y), \]

is bounded with

\[ \|
\tilde{A}
\| \leq \left( \sup_{y \in \mathbb{B}} \int_{\mathbb{B}} \| (\eta_x, A \eta_y)_{\mathfrak{h}} \|_{\mathfrak{g}(\mathfrak{h})} \, d_{\mathbb{B}}(x) \right)^{1/2} \left( \sup_{x \in \mathbb{B}} \int_{\mathbb{B}} \| (\eta_x, A \eta_y)_{\mathfrak{h}} \|_{\mathfrak{g}(\mathfrak{h})} \, d_{\mathbb{B}}(y) \right)^{1/2}. \]

For \( \Phi \in \mathfrak{h} \otimes \mathfrak{h} \), define the \( \mathfrak{h} \)-valued function on \( \mathbb{B} \): \( \tilde{\Phi} := [x \in \mathbb{B} \mapsto (\eta_x, \Phi)_{\mathfrak{h}} \in \mathfrak{h}] \). By Proposition 8.6 we know that \( \tilde{\Phi} \) belongs to \( L^2(\mathbb{B}, \mathfrak{h}) \) with \( \|	ilde{\Phi}\|_{\mathfrak{h} \otimes \mathfrak{h}} = \|	ilde{\Phi}\|_{L^2(\mathbb{B}, \mathfrak{h})} \). Take now \( \Phi, \Psi \in \text{dom} \ A \). In this case, we can use twice the resolution of the identity to get

\[ \langle \Phi, A \Psi \rangle_{\mathfrak{h} \otimes \mathfrak{h}} = \int \langle \langle \eta_x, \Phi \rangle_{\mathfrak{h}}, \langle \eta_x, A \eta_y \rangle_{\mathfrak{h}} \langle \eta_y, \Psi \rangle_{\mathfrak{h}} \rangle_{\mathfrak{h}} \, d_{\mathbb{B}}(x) \, d_{\mathbb{B}}(y) = \langle \tilde{\Phi}, \tilde{A} \tilde{\Psi} \rangle_{L^2(\mathbb{B}, \mathfrak{h})}. \]

Therefore, we conclude that

\[ \| \langle \Phi, A \Psi \rangle_{\mathfrak{h} \otimes \mathfrak{h}} \| \leq \| \tilde{\Phi} \|_{L^2(\mathbb{B}, \mathfrak{h})} \| \tilde{\Psi} \|_{L^2(\mathbb{B}, \mathfrak{h})} \| \tilde{A} \| = \| \tilde{\Phi} \|_{\mathfrak{h} \otimes \mathfrak{h}} \| \tilde{\Psi} \|_{\mathfrak{h} \otimes \mathfrak{h}} \| \tilde{A} \| < \infty \]

and the result follows immediately.

### 8.2 A tempered pair from the one-point phase

Let \( \mathbb{S} \) be an elementary normal \( j \)-group. Consider the one-point phase \( E_\theta \) defined in (68) and given by

\[ E_\theta(qb) = \chi_\theta \left( C_q(b^{-1}sb) \right) = e^{\frac{1}{2} S(qb)}, \]

(95)

The aim of this section it to prove that the pair \( (\mathbb{S}, \mathbb{S}) \), is tempered, admissible and tame. For this, we consider the following decomposition of the Lie algebra \( s \) (i.e. the one we used in Equation (47)):

\[ s = \bigoplus_{k=0}^{3} V_k \quad \text{where} \quad V_0 := a, \quad V_1 := \mathfrak{l}^*, \quad V_2 := \mathfrak{l} \quad \text{and} \quad V_3 := \mathbb{R}E. \]

As usual, we fix \( \{ f_j \}_{j=1}^d \), a basis of \( \mathfrak{l}^* \) to which we associate \( \{ e_j \}_{j=1}^d \) the symplectic-dual basis of \( \mathfrak{l} \), defined by \( \omega^0(f_i, e_j) = \delta_{i,j} \). Associated to the decomposition \( v = n + m \in \mathfrak{l}^* \oplus \mathfrak{l} = V \), we get coordinates

\[ n_j := \omega^0(n, e_j), \quad m_j := \omega^0(f_j, m), \quad j = 1, \ldots, d. \]

From the expressions (45) of the left invariant vector fields of \( \mathbb{S} \), in the chart (11), we get the following coordinates system on \( \mathbb{S} \):

\[ x_0 := \tilde{H} S = 2e^{-2at} - (1 + e^{-2a}) \omega^0(n, m), \quad x_1 := \tilde{f}_j S = (1 + e^{-2a}) m_j, \]

\[ x_2 := \tilde{e}_j S = (1 + e^{2a}) m_j, \quad x_3 := \tilde{E} S = \sinh(2a). \]

We then deduce:
Lemma 8.10 The pair $(S, S)$ is tempered in the sense of Definition 2.17. Moreover, the Jacobian of the map
\[ \phi: \mathbb{S} \to s^*, \ g \mapsto [s \mapsto \mathbb{R}, \ X \in s \mapsto (X S)(g)] , \]
is proportional to $m_0 \times \Delta_S^{-1/(d+1)}$.

The following Lemma is actually all what we need to prove admissibility (in the sense of Definition 2.19) of the tempered pair $(S, S)$:

Lemma 8.11 For every $k \in \{0, 1, 2, 3\}$, there exists a tempered function $m_k > 0$ with $\partial x_j m_k = 0$ for every $j \leq k$ and such that for every $X \in U(V(k))$, there exists $C_X > 0$ with
\[ |X x_k| \leq C_X m_k (1 + |x_k|) . \]

Proof. From the computations, for $k \in \mathbb{N}^*$ and $i, j = 1, \ldots, d$:
\[
\tilde{H}^k x_0 = (-1)^k (2^{2k+1} e^{-2a} t - 2^k (1 + 2^k e^{-2a}) \omega_0(n, m)) , \\
\tilde{H}^k x_1 = (-1)^k (1 + 3^k e^{-2a}) m_j , \quad \tilde{f}_i x_1 = 0 , \\
\tilde{H}^k x_2 = ((-1)^k + e^{2a}) n_j , \quad \tilde{f}_i x_2 = (1 + e^{2a}) \delta_i^j , \quad \tilde{e}_i x_2 = 0 , \\
\tilde{H}^k x_3 = 2^{k+1} \begin{cases} 
\cosh(2a), & k \text{ even} , \\
\sinh(2a), & k \text{ odd} , 
\end{cases} \tilde{f}_i x_3 = 0 , \quad \tilde{e}_i x_3 = 0 , \quad \tilde{E} x_3 = 0 ,
\]
and elementary estimates, we obtain:
\[
|\tilde{X} x_0| \leq C_X (1 + |x_0|)|1 + |x_1||x_2|) , \quad \forall X \in U(V_0) , \\
|\tilde{X} x_1| \leq C_X (1 + |x_1|) , \quad \forall X \in U(V(1)) , \\
|\tilde{X} x_2| \leq C_X (1 + |x_2|)(1 + |x_3|) , \quad \forall X \in U(V(2)) , \\
|\tilde{X} x_3| \leq C_X (1 + |x_3|) , \quad \forall X \in U(s) ,
\]
and the claim follows with $m_0(x) = (1 + |x_1||x_2|)$, $m_1(x) = 1$, $m_2(x) = (1 + |x_3|)$, $m_3(x) = 1$.

Repeating the arguments of the proof of Proposition 3.26 we deduce admissibility for the tempered pair $(S, S)$.

Lemma 8.12 Define
\[ X_0 := 1 - H^2 \in U(V_0) , \quad X_1 := 1 - \sum_{j=1}^d f_j^2 \in U(V_1) , \quad X_2 := 1 - \sum_{j=1}^d c_j^2 \in U(V_2) , \quad X_3 := 1 - E^2 \in U(V_3) . \]

Then the corresponding multipliers $\alpha_k := E^{-1} \tilde{X}_k E$ satisfy conditions (i) and (ii) of Definition 2.19 with $\rho_k = 2$ and the $\mu_k$’s are given by the $m_k$’s of Lemma 8.11.

At last, we observe that tameness (see Definition 2.20) follows from Lemma 8.27 and arguments very similar to those of Corollary 8.28. We then summarize all this by stating the main result of this section:

Theorem 8.13 Let $S$ be an elementary normal $j$-group and let $S \in C\infty(S)$ as given in (05). Then the pair $(S, S)$ is tempered, admissible and tame.

Remark 8.14 For $B$ a generic normal $j$-group $B$, we could also define a one-point tempered pair, by setting
\[ E_B^B := \exp \left( \frac{2i}{B^S} \right) : B \to \mathbb{U}(1), \quad S^B : B \to \mathbb{R}, \quad g \mapsto \sum_{j=1}^N S_{0j}(g_j) , \]
where $S^B_j$ is the one-point phase of each elementary factor of $B$ in the parametrization $g = g_1 \ldots g_N \in B$, relative to a Pyatetskii-Shapiro decomposition. Then temperedness and admissibility will follow from arguments very similar than those of Theorem 8.35.

Remark 8.15 The one-point Schwartz space $S^{\text{can}}(S)$ associated with the two-point pair $(S \times S, S_{\text{can}})$ coincides with the one-point Schwartz space $S^S(S)$ associated with the one-point pair $(S, S)$.
8.3 Extension of the oscillatory integral for elementary normal \( j \)-groups

Associated to a tempered, admissible and tame pair \((G, S)\), we have constructed in subection \[2\] a continuous linear map for any Fréchet space \( E \) and any element \( m \in \mathcal{B}^\mu(G) \) (with \( \mu \) a tempered weight on \( G \)):

\[
\int EM : \mathcal{B}^{(\mu_j)}(G, E) \to E,
\]

which extends the ordinary integral on \( \mathcal{D}(G, E) \) and that we called the oscillatory integral. The aim of the present subsection is to explain how for the tempered pair \((S, S)\) of Theorem \[8.13\] one can enlarge the domain of definition of the oscillatory integral. For this let \( E \) be a Fréchet space with topology underlying a countable family of semi-norms \( \{ \| \cdot \|_j \}_{j \in \mathbb{N}} \), \( \{ \mu_j \}_{j \in \mathbb{N}} \) a family of tempered weights on \( S \) and \( \nu \) be a fixed tempered and \( Y \)-right-invariant weight \[\overline{\mu}\] on \( S \). Let us then consider the following subspace of \( C^\infty(S, E) \):

\[
\mathcal{B}^{(\mu_j)}(S, E) := \left\{ F \in C^\infty(S, E) : \forall (j, X, Y) \in \mathbb{N} \times \mathcal{U}(q) \times \mathcal{U}(\mathbb{Z}), \exists C : \| \widetilde{X} \widetilde{Y} F(qb) \|_j \leq C \nu(q)^{\text{deg}(X)} \mu_j(qb) \right\}.
\]

This space may be understood as a variant of the symbol space \( \mathcal{B}^{(\mu_j)}(S, E) \), where a specific dependance of the family of weights \( \{ \mu_j \}_{j \in \mathbb{N}} \) in the degree of the derivative is allowed. We endow the latter space with the following set of semi-norms:

\[
\| F \|_{j, k_1, k_2, \mu, \nu, \infty} := \sup_{X \in \mathcal{U}_l(q)} \sup_{\nu \in \mathcal{U}_2(\mathbb{Z})} \sup_{qb \in S} \left\{ \| \widetilde{X} \widetilde{Y} F(qb) \|_j \frac{\| X \|_{k_1} \| Y \|_{k_2}}{\nu(q)^{k_1} \mu(q)^{k_2}} \right\}, \quad j, k_1, k_2 \in \mathbb{N}, \quad (77)
\]

where \( \cup q \in \mathcal{U}_l(q) \), \( \cup q \in \mathcal{U}_2(\mathbb{Z}) \) are the filtrations of \( \mathcal{U}(q) \) and \( \mathcal{U}(\mathbb{Z}) \) associated to the choice of PBW basis as explained in \[8\]. As expected, the space \( \mathcal{B}^{(\mu_j)}(S, E) \) is Fréchet for the topology induced by the semi-norms \[77\] and most the properties of Lemma \[2.8\] remain true.

**Lemma 8.16** Let \((S, E, \{ \| . \|_j \}_{j \in \mathbb{N}})\) as above, let \( \{ \mu_j \}_{j \in \mathbb{N}}, \{ \mu'_j \}_{j \in \mathbb{N}} \) be two families of weights on \( S \) and let \( \nu, \nu' \) be two right-\( Y \)-invariant weights on \( S \).

(i) The space \( \mathcal{B}^{(\mu_j)}(S, E) \) is Fréchet.

(ii) The bilinear map:

\[
\mathcal{B}^{(\mu_j)}(S) \times \mathcal{B}^{(\mu'_j)}(S, E) \to \mathcal{B}^{(\mu_j \mu'_j)}(S, E), \quad (u, F) \mapsto [g \in S \mapsto u(g) F(g) \in E],
\]

is jointly continuous.

(iii) If there exists \( C > 0 \) such that \( \nu' \leq C \nu \) and if for every \( j \in \mathbb{N} \), there exists \( C_j > 0 \) such that \( \mu'_j \leq C_j \mu_j \),

then \( \mathcal{B}^{(\mu_j)}(S, E) \subset \mathcal{B}^{(\mu_j \mu'_j)}(S, E) \) continuously.

(iv) Assume that \( \nu \succ \nu' \) and that \( \mu_j \succ \mu'_j \) for every \( j \in \mathbb{N} \). Then, the closure of \( \mathcal{D}(S, E) \) in \( \mathcal{B}^{(\mu_j)}(S, E) \)

contains \( \mathcal{B}^{(\mu_j)}(S, E) \). In particular, \( \mathcal{D}(S, E) \) is a dense sub-set of \( \mathcal{B}^{(\mu_j \mu'_j)}(S, E) \) for the induced topology of \( \mathcal{B}^{(\mu_j \mu'_j)}(S, E) \).

**Proof.** The first assertion follows from the fact that a countable projective limit of Fréchet spaces is Fréchet and that \( \mathcal{B}^{(\mu_j)}(S, E) \) can be realized as the countable projective limit of the family of Banach spaces underlying the norms \( \sum_{i=0}^j \sum_{k_1=0}^{k_2} \sum_{l_2=0}^{l_2} \| i, j, k_1, \mu, \nu, \infty \). The proof of all the other statements are identical to their counter-parts in Lemma \[2.8\].

\[19\]We may view \( \nu \) as a function on \( Q \).
We are now able to prove our extension result for the oscillatory integral associated to the admissible, tempered and tame pair \((\mathbb{S}, \mathbb{S})\):

**Theorem 8.17** Let \(\{\mu_j\}_{j \in \mathbb{N}}\) be family of tempered weights on \(\mathbb{S}\), \(\nu\) a \(\mathbb{Y}\)-right-invariant tempered weight on \(\mathbb{S}\) and \(\mathbf{m}\) an element of \(\mathcal{B}^\nu(\mathbb{S})\) for another tempered weight \(\mu\) on \(\mathbb{S}\). Let also \(\mathbf{D}_F\), \(\vec{r} \in \mathbb{N}^4\), be the differential operator constructed in (24). Then for all \(j \in \mathbb{N}\), there exist \(j_j \in \mathbb{N}^4\), \(C_j > 0\) and \(K_j, L_j \in \mathbb{N}\), such that for every element \(F \in \mathcal{B}^{\{\mu_j\}}(\mathbb{S}, \mathcal{E})\), we have

\[
\int_{\mathbb{S}} \|\mathbf{D}_F \mathbf{m} F(g)\|_j \, d\mathbb{S}(g) \leq C_j \|F\|_{j, K_j, L_j, \mu_j, \nu, \infty} .
\]

Consequently, the oscillatory integral constructed in Definition 2.24 for the tame and admissible tempered pair \((\mathbb{S}, \mathbb{S})\), originally defined in \(\mathcal{B}^{\{\mu\}}(\mathbb{S}, \mathcal{E})\), extends as a continuous map:

\[
\int_{\mathbb{S}} \mathbf{m} E : \mathcal{B}^{\{\mu_j\}}(\mathbb{S}, \mathcal{E}) \to \mathcal{E} ,
\]

**Proof.** The proof is very similar to those of Proposition 2.24 so we focus on the differences due to the particular behavior at infinity of an element of \(\mathcal{B}^{\{\mu_j\}}(\mathbb{S}, \mathcal{E})\).

By Lemma 8.10, the Radon-Nicodym derivative of the left Haar measure on \(\mathbb{S}\) with respect to the Lebesgue measure on \(\mathbb{R}^d\), is bounded by a polynomial of order \(2d + 4\) in the coordinate \(x_3\). For each \(j \in \mathbb{N}\), the weight \(\mu_j\) is also bounded by a polynomial in \(x_0, x_1, x_2, x_3\). Now, observe that by construction of the operator \(\mathbf{D}_F\) in (24), we have for any \(\vec{r} = (r_0, r_1, r_2, r_3) \in \mathbb{N}^4\), with \(K_1 = 2r_0 + 2r_1\), \(K_2 = 2r_2 + 2r_3\) and with the notations given in (26):

\[
|\mathbf{D}_F F| \leq |\Psi_0| |\Psi_{1,0}| |\Psi_{2,1,0}| |\Psi_{3,2,1,0}| |\vec{X}_{3,2,1,0} F| \\
\leq C |\Psi_0| |\Psi_{1,0}| |\Psi_{2,1,0}| |\Psi_{3,2,1,0}| |\mu_j| \nu^{2r_0 + 2r_1} \|F\|_{j, K_1, K_2, \mu_j, \nu, \infty} .
\]

(98)

This will give the estimate we need, if we prove that the function in front of \(\|F\|_{j, K_1, K_2, \mu_j, \nu, \infty}\) in (98) is integrable for a suitable choice of \(\vec{r} \in \mathbb{N}^4\). We prove a stronger result, namely that given \(\vec{H} \in \mathbb{N}^4\), there exists \(\vec{r} \in \mathbb{N}^4\) such that

\[
|\Psi_0| |\Psi_{1,0}| |\Psi_{2,1,0}| |\Psi_{3,2,1,0}| |\nu^{2r_0 + 2r_1} \leq C \frac{1}{(1 + |x_0|)^{R_0}(1 + |x_1|)^{R_1}(1 + |x_2|)^{R_2}(1 + |x_3|)^{R_3}} .
\]

From Corollary 2.23 and Lemma 8.12 we obtain the following estimation:

\[
|\Psi_0| |\Psi_{1,0}| |\Psi_{2,1,0}| |\Psi_{3,2,1,0}| \leq C \frac{1 + |x_1| |x_2|^{2r_0}}{(1 + |x_0|)^{2r_0}(1 + |x_1|)^{2r_1}(1 + |x_3|)^{2r_2}(1 + |x_3|)^{2r_3}} .
\]

At last (this is the main difference with the proof of Proposition 2.24) note that \(\nu\), the tempered function on \(Q\), can be bounded by \(|x_2|^{p_2}|x_3|^{p_3}\) for some integers \(p_2, p_3\). Hence \(|\Psi_0| |\Psi_{1,0}| |\Psi_{2,1,0}| |\Psi_{3,2,1,0}| \nu^{2r_0 + 2r_1}\) is smaller than

\[
C (1 + |x_0|)^{-2r_0}(1 + |x_1|)^{-2r_1 + 2r_2}(1 + |x_2|)^{-2r_2 + 2r_3 + 2p_2(\rho_0 + r_1)}(1 + |x_3|)^{-2r_3 + 2r_1(\rho_0 + r_1) + 2p_3(\rho_0 + r_1)} ,
\]

and the claim follows.

\[\square\]

**8.4 A Calderón-Vaillancourt type estimate**

For \(j = 1, \ldots, N\), fix \(\mathbf{m}\), a \(\mathbb{Y}\)-right-invariant tempered weight on \(S_j\) (that we identify in a natural manner as a function on \(Q_j\)), in the sense of Definition 2.12 for the tempered pair \((S_j, S^{\nu_j})\) underlying Theorem 8.13. Our aim here is to prove that for \(F \in \mathcal{B}(\mathbb{E}, A)\), the operator \(\Omega_{\delta, \mathbf{m}}(F)\), defined via a suitable quadratic form on \(\mathcal{H} \otimes \mathcal{H}_\nu\), is bounded.\(^{20}\)

We start by proceeding formally, in order to explain our global strategy. Also,\(^{20}\)

\[\text{Observe that this property holds for } F \in S^{\infty}_{\text{can}}(\mathbb{E}, A), \text{ by Proposition 8.27 as } S^{\infty}_{\text{can}}(\mathbb{E}, A) \subset S(\mathbb{E}, A).\]

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Lemma 8.18

Next, we exchange the integrals over $S$ and $Q$ and expand the scalar product of $H_x$ to obtain:

$$
\langle \eta_x, \Omega_{\theta,m}(F) \eta_x \rangle_{H_x} = \int_{S \times Q} F(q_0 q b) \hat{m}(q_0) m(q^{-1}) E(q b) \eta(q_0 \mathfrak{S} e q) d_Q(q_0) d_S(q b).
$$

Given $F \in \mathcal{B}(S, A)$, this suggest to define

$$
F^\eta(q b, q_0) := F(q_0 q b) \eta(q_0 \mathfrak{S} e q),
$$

so that with $\hat{m}(q b) := m(q^{-1})$ and with the notations of Proposition 8.3, we will have

$$
\langle \eta_x, \Omega_{\theta,m}(F) \eta_x \rangle_{H_x} = \mathcal{F}_\pi \left( \int_S E(y) \hat{m}(y) F^\eta(y, y) d_S(y) \right)(x).
$$

Consequently, we obtain

$$
\langle \Phi, \Omega_{\theta,m}(F) \Psi \rangle_{H \otimes H_x} = \int_{S \times S} \langle \eta_x, \Psi \rangle_{H_x} \mathcal{F}_\pi \left( \int_S E(z) \hat{m}(z) (L_{y^{-1}_y}^* F)^\eta(z) d_Q(z) \right)(y^{-1} x), \Psi \rangle_{H_x} d_S(x) d_S(y).
$$

Surprisingly, this is the right hand side of the (formal) equality above which gives rise to a well defined and bounded quadratic form on $H \otimes H_x$, once the integral sign in the middle is replaced by an oscillatory one in the sense of Theorem 8.17 for the tempered pair $(S, S)$.

Coming back to the case of a generic normal $j$-group $B$, the most important step is to understand the properties of the corresponding map $F \mapsto F^\eta$ given in (99).

Lemma 8.18 Let $A$ be a C*-algebra, $B$ be a normal $j$-group with Pyatetskii-Shapiro decomposition $B = (S_N \times \ldots) \times S_1$ and $\eta \in \mathcal{D}(Q_N \times \ldots \times Q_1)$. Then the map

$$
F \mapsto F^\eta := \left[ q_N b_N \in S_N \mapsto \left[ q_{N-1} b_{N-1} \in S_{N-1} \mapsto \ldots \right], q_1 b_1 \in S_1 \mapsto \left[ (q'_N, \ldots, q'_1) \in Q_N \times \ldots \times Q_1 \mapsto F(q'_1 q_1 b_1 \ldots q_{N-1} b_{N-1} q_N b_N) \eta(q'_N q_N, \ldots, q'_1 q_1) \in A \right] \ldots \right],
$$

is continuous from $\mathcal{B}(B, A)$ to

$$
\mathcal{B}^{(S_N \times \ldots \times S_1)} \left( [q'_N \in Q_N] \left( S_N, B^{(S_N \times \ldots \times S_1)} \left( S_N, \ldots \rightarrow B^{(S_1)} \left( S_1, \mathcal{S}(Q_N \times \ldots \times Q_1, A) \right) \right) \right) \right).
$$

where $\mathcal{V}_Q$ (respectively $\mathcal{V}_S$) denotes the modular weight of $Q_j$ (respectively of $S_j$).

Proof. For notational convenience, we assume that $B$ contains only two elementary factors, i.e. $B = S_2 \times S_1$ with $S_1, S_2$ elementary normal $j$-groups. This is enough to understand the global mechanism and the proof for a generic normal $j$-group with an arbitrary number of elementary factors will then follow by induction, without essential difficulties. In this simplified case, we have to prove that the map

$$
F \mapsto F^\eta := \left[ q_2 b_2 \in S_2 \mapsto \left[ q_1 b_1 \in S_1 \mapsto \left[ (q'_2, q'_1) \in Q_2 \times Q_1 \mapsto F(q'_1 q_1 b_1 q_2 b_2) \eta(q'_2 q_2, q'_1 q_1) \in A \right] \right] \ldots \right],
$$

is continuous from $\mathcal{B}(B, A)$ to

$$
\mathcal{B}^{(S_2 \times S_1)} \left( S_2, B^{(S_1)} \left( S_1, \mathcal{S}(Q_2 \times Q_1, A) \right) \right).
$$

(100)
By the discussion following Remark 7.11 it is clear that one may regard \(S(Q_2 \times Q_1, A)\) as a Fréchet space for the topology induced by the following countable set of semi-norms:

\[
\|f\|_{k,j} := \sup_{X \in \mathcal{U}_k(q_2 \oplus q_1)} \sup_{(q_2, q_1) \in Q_2 \times Q_1} \left\{ \frac{\partial_{Q_2} (q_2)^j \partial_{Q_1} (q_1)^j \|X f(q_1, q_2)\|}{|X|^k} \right\}, \quad (j, k) \in \mathbb{N}^2.
\]

Note also that the natural Fréchet topology of the space \(\mathcal{U}(A)\) is associated with the following countable family of semi-norms (indexed by \((k_2, l_2, k_1, l_1, k, j) \in \mathbb{N}^6\)):

\[
\Phi \mapsto \sup_{X \in \mathcal{U}_{k_2}(q_2)} \sup_{X' \in \mathcal{U}_{l_2}(q_2)} \sup_{Y' \in \mathcal{U}_{l_1}(q_1)} \sup_{q_2 \in Q_2} \sup_{q_1 \in Q_1} \frac{\partial_{Q_2} (q_2)^j \partial_{Q_1} (q_1)^j}{\partial_{S_2} (q_2 b_2)^{n_2(k_2, l_2, k, j)} \partial_{S_1} (q_1 b_1)^{n_1(k_2, l_2, k, j)} \partial_{Q_1} (q_1)^{2k_1} \partial_{S_1} (q_1)^{2k_1} \|X|^{k_2} \|X'|^{l_2} \|Y'|^{l_1} \|Z|^{k}}.
\]

We denote the latter semi-norm by \(\|\cdot\|_{k_2, l_2, k_1, l_1, k, j}\). Then, for \((X, X', Y, Y', Z^2, Z^1) \in \mathcal{U}(q_2) \times \mathcal{U}(q_2) \times \mathcal{U}(q_1) \times \mathcal{U}(q_1) \times \mathcal{U}(q_2) \times \mathcal{U}(q_1)\), we get within Sweedler’s notation:

\[
Z_{1}^{q_1} Z_{1}^{q_1} \hat{Y}_{q_1 b_1} \hat{Y}_{q_1 b_1} \hat{X}_{q_2 b_2} \hat{X}_{q_2 b_2} F^0(q_1 b_1; q_2 b_2; q_2, q_1) = \sum_{(X)} \sum_{(Y)} \sum_{(X')} \sum_{(Y')} \sum_{Z^1} \sum_{Z^2} \left( Z_{1}^{q_1} Z_{1}^{q_1} \hat{Y}_{q_1 b_1} \hat{Y}_{q_1 b_1} \hat{X}_{q_2 b_2} \hat{X}_{q_2 b_2} F(q_1 b_1 q_2 q_2 q_2 b_2) \right) \left( Z_{1}^{q_1} Z_{1}^{q_1} \hat{Y}_{q_1 b_1} \hat{Y}_{q_1 b_1} \hat{X}_{q_2 b_2} \hat{X}_{q_2 b_2} \eta(q_2^e q_2, q_2^e q_1) \right).
\]

From the same reasoning as those in the proof of Lemma 2.6 (v), we deduce that

\[
\|F^0\|_{k_2, l_2, k_1, l_1, k, j} \leq C \sup_{X \in \mathcal{U}_{k_2}(q_2)} \sup_{X' \in \mathcal{U}_{l_2}(q_2)} \sup_{Y' \in \mathcal{U}_{l_1}(q_1)} \sup_{q_2 \in Q_2} \sup_{q_1 \in Q_1} \frac{\partial_{Q_2} (q_2)^j \partial_{Q_1} (q_1)^j}{\partial_{S_2} (q_2 b_2)^{n_2(k_2, l_2, k, j)} \partial_{S_1} (q_1 b_1)^{n_1(k_2, l_2, k, j)} \partial_{Q_1} (q_1)^{2k_1} \partial_{S_1} (q_1)^{2k_1} \|X|^{k_2} \|X'|^{l_2} \|Y'|^{l_1} \|Z^2|^{k_2} \|Z^1|^{l_1}},
\]

where the first supremum is over:

\((q_2 b_2, q_1 b_1, q_2, q_1) \in S_2 \times S_1 \times Q_2 \times Q_1\),

the second over:

\((X, X', Y, Y', Z^2, Z^1) \in \mathcal{U}_{k_2}(q_2) \times \mathcal{U}_{l_2}(q_2) \times \mathcal{U}_{k_1}(q_1) \times \mathcal{U}_{l_1}(q_1) \times \mathcal{U}_{k_2}(q_2) \times \mathcal{U}_{k_1}(q_1)\),

and the third over:

\((X, Y, Z^2, Z^1) \in \mathcal{U}_{k_2}(q_2) \times \mathcal{U}_{k_1}(q_1) \times \mathcal{U}_{k_2}(q_2) \times \mathcal{U}_{k_1}(q_1)\).

Next, we observe:

\[
Z_{1}^{q_1} Z_{1}^{q_1} \hat{Y}_{q_1 b_1} \hat{Y}_{q_1 b_1} \hat{X}_{q_2 b_2} \hat{X}_{q_2 b_2} F(q_1 b_1 q_2 b_2)
\]

\[
= (Z_{1}^{q_1} A d_{q_2 b_2} - 1)(Z_{2}^{q_1} A d_{q_2 b_2} - 1)(Y Y') \hat{X} \hat{X} ' F(q_1 b_1 q_2 b_2).
\]

Then, note that if we expand a right-invariant differential operator \(X\) in a PBW-basis of left-invariant one, the coefficient functions will be bounded by \(d_{\deg(X)}\) and thus for any \(F \in B(S, A)\) and \(X \in \mathcal{U}(s)\), the element \(X F\) will belong to \(B^{\deg(X)}(S, A)\). This observation together with Lemma 2.9 entails that the \(F\)-dependent supremum in (101) is bounded by:

\[
C \|F\|_{2k+k_1+l_1+k_2+l_2} \sup_{X \in \mathcal{U}_{k_2}(q_2 b_2)} \sup_{q_2 \in Q_2} \partial_{S_2} (q_2 b_2)^{2k+k_1+l_1} \partial_{S_1} (q_1 b_1)^{k}.
\]

For \(\eta\)-dependent term in (101), we first note:

\[
Z_{1}^{q_1} Z_{1}^{q_1} \hat{Y}_{q_1 b_1} \hat{X}_{q_2 b_2} \eta(q_2^e q_2, q_2^e q_1) = \hat{Y}_{q_1 b_1} \hat{X}_{q_2 b_2} (Z_{1}^{q_1} Z_{1}^{q_1} \eta) (q_2^e q_2, q_2^e q_1) ,
\]

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so that up to a redefinition of $\eta$, we can ignore the right-invariant vector fields. Next, we observe that with $q = (a, n)$, $q' = (a', n')$ in the coordinates $\xi$, we have:

$$ q' \xi^e (q) = \left( 2a + a', e^{-2a} n' + 2 \cosh(a) n \right) . $$

With $H$ the generator of $a$ and $\{ f_j \}_{j=1}^d$ a basis of $\Gamma^*$, the associated left-invariant vector fields on $Q$ read:

$$ \tilde{H} = \partial_a - \sum_{j=1}^d n_j \partial_{n_j} , \quad \tilde{f}_j = \partial_{n_j} . $$

Choosing $\eta = \eta_2 \otimes \eta_1$ with $\eta_j \in D(Q_j)$, it is enough to treat each variables separately. So just assume that $\eta \in D(Q)$. Now, for $N = (N_1, \ldots, N_d) \in \mathbb{N}^d$ with $|N| = k$, we have with $\tilde{f}^N := \tilde{f}_{N_1}^1 \ldots \tilde{f}_{N_d}^d$, $\partial^N := \partial_{n_1}^{N_1} \ldots \partial_{n_d}^{N_d}$ and setting $q_0 := q' \xi^e (q) \in Q$:

$$ \tilde{f}^N \eta (q_0) = 2^k \cosh(a)^k (\partial^N \eta)(q_0) . $$

Since $\cosh(a) \leq 2 \cosh(a'/2) \cosh(a_0/2)$, the latter and Remark 7.11 entail that

$$ |\tilde{f}^N \eta (q_0)| \leq C \cosh(a'/2)^k \cosh(a_0/2)^k (\partial^N \eta)(q_0) \leq C \Delta_Q (q')^k \cosh(a_0/2)^k (\partial^N \eta)(q_0) . $$

On the other hand, we have

$$ \tilde{H}_q \eta (q_0) = 2 (\partial_a \eta)(q_0) - 2 \sum_{j=1}^d (n_j e^{-a} + n_j' e^{-2a}) (\partial_{n_j} \eta)(q_0) . $$

Since $w_j := n_j e^{-a} + n_j' e^{-2a}$ is an eigenvector of $\tilde{H}_q$ with eigenvalue $-2$, we deduce that for $k \in \mathbb{N}$, $\tilde{H}_q^k \eta (q_0)$ is a linear combinations of the ordinary derivatives of $\eta$, with coefficients given in the ring $\mathbb{C}[w_j]$ of order at most $k$. Moreover, the rough estimate:

$$ |w| = |(w_1, \ldots, w_d)| \leq 4 \cosh(a_0) \cosh(a') (|n_0| + |n'|) , $$

gives by Remark 7.11

$$ \left| \tilde{H}_q^k \eta (q_0) \right| \leq C \Delta_Q (q')^k \cosh(a_0)^k |n_0|^k \left| P(\partial_a, \partial_{n_j}) \eta \right| (q_0) , $$

for a suitable polynomial $P$. This implies that the $\eta$-dependent term in (101) is bounded by:

$$ \left| \tilde{Z}_k^\eta \tilde{Z}_j^\eta \tilde{X}_q \tilde{X}_{q_2} (q' \xi^e q_2, q' \xi^e q_1) \right| \leq C \sup_{(q_2 b_2, q_1 b_1, q' q_1') \in S_2 \times S_1 \times Q_2 \times Q_1} \frac{\Delta_Q (q_2)^{j+k} \Delta_Q (q_1)^{j++k} \Delta_{S_2} (q_2 b_2)^{2k+j+k} \Delta_{S_2} (q_2 b_2)^{2k+i} \Delta_{S_1} (q_1 b_1)^{2k}}{\Delta_{S_2} (q_2 b_2)^{n_2(k_1, k_1, k_1)} \Delta_{Q_2} (q_2)^{2k} \Delta_{S_1} (q_1 b_1)^{n_1(k_1, k_1)} \Delta_{Q_1} (q_1)^{2k}} \left| \tilde{\eta} \right| (q' \xi^e q_2, q' \xi^e q_1) , $$

By the sub-multiplicativity and the invariance under the inversion map of the modular weights, we deduce that the quantity in the line above is smaller than:

$$ \frac{\Delta_Q (q_2)^{2k+j+k} \Delta_{S_2} (q_2 b_2)^{2k+i} \Delta_{S_1} (q_1 b_1)^{k} \Delta_{Q_1} (q_1)^{2k} \Delta_{Q_2} (q_2)^{2k} \Delta_{S_1} (q_1 b_1)^{n_1(k_1, k_1)} \Delta_{Q_1} (q_1)^{2k}}{\Delta_{S_2} (q_2 b_2)^{n_2(k_1, k_1, k_1)} \Delta_{Q_2} (q_2)^{2k} \Delta_{S_1} (q_1 b_1)^{n_1(k_1, k_1)} \Delta_{Q_1} (q_1)^{2k}} \left| \tilde{\eta} \right| (q' \xi^e q_2, q' \xi^e q_1) . $$

Because $\tilde{\eta}$ is compactly supported, the quantity in the last line above is smaller than a constant. Also, since $\xi^e (a, n) = (2a, 2 \cosh(a) n)$, we deduce that $\Delta_Q (\xi^e q) \leq C \Delta_Q (q)^2$. Thus, the expression above is bounded by

$$ C(\eta) \frac{\Delta_Q (q_2)^{2k+2j} \Delta_{S_2} (q_2 b_2)^{2k+i} \Delta_{S_1} (q_1 b_1)^{2k+2j} \Delta_{Q_1} (q_1)^{2k+2j} \Delta_{S_1} (q_1 b_1)^{k} \Delta_{S_2} (q_2 b_2)^{n_2(k_1, k_1, k_1)} \Delta_{S_1} (q_1 b_1)^{n_1(k_1, k_1)} \Delta_{Q_1} (q_1)^{2k} \Delta_{Q_2} (q_2)^{2k} \Delta_{S_1} (q_1 b_1)^{n_1(k_1, k_1)} \Delta_{Q_1} (q_1)^{2k}}{\Delta_{S_2} (q_2 b_2)^{n_2(k_1, k_1, k_1)} \Delta_{S_1} (q_1 b_1)^{n_1(k_1, k_1)} \Delta_{Q_1} (q_1)^{2k}} , $$

and one concludes by suitably choosing $n_1(k, j)$ and $n_2(k_1, l_1, k, j)$. 

We are now ready to prove a non-Abelian (curved) and $C^*$-valued version of the Calderón-Vaillancourt estimate, the main result of this section.

**Theorem 8.19** Let $\mathcal{B}$ be a normal $j$-group, $A$ a $C^*$-algebra, $F \in \mathcal{B}(\mathcal{B}, A)$, $\eta = \eta_N \otimes \cdots \otimes \eta_1 \in \mathcal{D}(Q_N) \otimes \cdots \otimes \mathcal{D}(Q_1)$ and $m \in \Theta(\mathcal{B})$. Define for $x, y \in \mathcal{B}$, the element of $A$ given by
\[
\langle \eta_x, \Omega_{\eta, m}(F) \eta_y \rangle_{\mathcal{H}_x} := \mathcal{F} \left( \mathcal{F}^{-1} \left( E_{\eta_1}^{S_1} \hat{m}_1 \cdots \left( \mathcal{F}^{-1} \left( E_{\eta_N}^{S_N} \hat{m}_N \left[ g_N \in S_N \mapsto \cdots \left[ g_1 \in S_1 \mapsto \left( L_{y^{-1}}^* F \right)^{\eta}(g_N, \ldots, g_1; \cdot) \right] \right) \right) \right) \right) (y^{-1} x),
\]
where $\hat{m}(q) = m(q^{-1})$ and where $E_{\eta_i}^{S_i}$ is the one-point phase of $S_i$ as defined in (95). Then we have:
\[
\sup_{x \in \mathcal{B}} \int_{\mathcal{B}} \| \langle \eta_x, \Omega_{\eta, m}(F) \eta_y \rangle_{\mathcal{H}_x} \| \, d_{\mathcal{B}}(y) < \infty, \quad \sup_{y \in \mathcal{B}} \int_{\mathcal{B}} \| \langle \eta_x, \Omega_{\eta, m}(F) \eta_y \rangle_{\mathcal{H}_x} \| \, d_{\mathcal{B}}(x) < \infty.
\]
Consequently (see Proposition 8.9), the operator $\Omega_{\eta, m}(F)$ on $\mathcal{H} \otimes \mathcal{H}_x$ defined by mean of the quadratic form
\[
\Psi, \Phi \in \mathcal{H} \otimes \mathcal{H}_x \mapsto \langle \eta_x, \Psi \rangle_{\mathcal{H}_x} \langle \eta_y, \Phi \rangle_{\mathcal{H}_x},
\]
is bounded. Moreover, there exists $k \in \mathbb{N}$ (depending only on $\dim \mathcal{B}$ and on the order of the polynomial in $\partial_{Q_N} \otimes \cdots \otimes \partial_{Q_1}$ that majorizes $|m|$) and $C > 0$, such that for all $F \in \mathcal{B}(\mathcal{B}, A)$ we have
\[
\| \Omega_{\eta, m}(F) \| \leq C \| F \|_{k, \infty} = C \sup_{X \in \mathcal{U}(\mathcal{B})} \sup_{x \in \mathcal{B}} \| \overline{X} F(x) \|.
\]

**Proof.** To simplify the notation for the matrix element given in (102), we write
\[
\langle \eta_x, \Omega_{\eta, m}(F) \eta_y \rangle_{\mathcal{H}_x} = \mathcal{F} \left( \mathcal{F}^{-1} \left( E_{\eta}^{\mathcal{B}} \hat{m} \left[ z \mapsto \left( L_{y^{-1}}^* F \right)^{\eta}(\cdot, z) \right] \right) \right) (y^{-1} x),
\]
where $E_{\eta}^{\mathcal{B}}$ is given in (95). Observe that this notation is coherent with our Fubini type Theorem 6.3. Thus,
\[
\sup_{y \in \mathcal{B}} \int_{\mathcal{B}} \| \langle \eta_x, \Omega_{\eta, m}(F) \eta_y \rangle_{\mathcal{H}_x} \| \, d_{\mathcal{B}}(y) = \sup_{y \in \mathcal{B}} \int_{\mathcal{B}} \| \mathcal{F} \left( \mathcal{F}^{-1} \left( E_{\eta}^{\mathcal{B}} \hat{m} \left[ z \mapsto \left( L_{y^{-1}}^* F \right)^{\eta}(\cdot, z) \right] \right) \right) (y^{-1} x) \| \, d_{\mathcal{B}}(x) = \sup_{y \in \mathcal{B}} \left\| \mathcal{F} \left( \mathcal{F}^{-1} \left( E_{\eta}^{\mathcal{B}} \hat{m} \left[ z \mapsto \left( L_{y^{-1}}^* F \right)^{\eta}(\cdot, z) \right] \right) \right) (y^{-1} x) \right\| \, d_{\mathcal{B}}(x).
\]
The fact that this quantity is finite follows then by combining Proposition 8.3 with Theorem 8.17 and Lemma 8.18 and the fact that $L_{y^{-1}}^*$ maps $\mathcal{B}(\mathcal{B}, A)$ to itself isometrically. The second case is similar since
\[
\langle \eta_x, \Omega_{\eta, m}(F) \eta_y \rangle_{\mathcal{H}_x}^* = \langle \eta_y, \Omega_{\eta, m}(F^*) \eta_x \rangle_{\mathcal{H}_x}.
\]
The final estimation we give is a consequence of Proposition 8.3 together with the estimates underlying Proposition 8.3, Theorem 8.17 and Lemma 8.18.

**8.5 The deformed $C^*$-norm**

Now, we assume that our $C^*$-algebra $A$ is equipped with a strongly continuous and isometric action $\alpha$ of a normal $j$-group $\mathcal{B}$. We stress that the results of this section cannot hold true in the more general context of tempered actions. This is the main difference between the deformation theory at the level of Fréchet and $C^*$-algebras. Given an element $a \in A$, we construct as usual the $A$-valued function $\alpha(a)$ on $\mathcal{B}$:
\[
\alpha(a) := [g \in \mathcal{B} \mapsto \alpha_g(a) \in A].
\]
Thus, from Theorem 5.8 we can deform the Fréchet algebra structure on the set of smooth vectors \( A^\infty \) by mean of the deformed product
\[
a \ast_{\theta, m}^\alpha b := (\alpha(a) \ast_{\theta, m} \alpha(b))(e), \quad a, b \in A^\infty.
\]
We have seen in [SS] how to modify the original involution at the level of \( B(\mathbb{B}, A) \). At the level of the Fréchet algebra \( A^\infty \), an obvious observation leads to:

**Lemma 8.20** Let \( B \) be a normal \( j \)-group. For \( m \in \Theta(\mathbb{B}) \), the following defines a continuous involution of the Fréchet algebra \( (A^\infty, \ast_{\theta, m}^\alpha) \):
\[
\ast_{\theta, m} : A^\infty \to A^\infty, \quad a \mapsto m_N \left( \frac{1}{2} \arcsinh(\frac{1}{2} E_N) \right) \circ \cdots \circ m_1 \left( \frac{1}{2} \arcsinh(\frac{1}{2} E_1) \right) a^*,
\]
where \( E_N, \ldots, E_1 \) are the central elements of the Heisenberg Lie algebras attached to each elementary factors of \( \mathbb{B} \).

**Remark 8.21** Note that when \( \frac{1}{2} m_j = m_j \), \( j = 1, \ldots, N \), there is no modification of the involution.

The construction of a pre-\( C^* \)-structure on \( (A^\infty, \ast_{\theta, m}^\alpha) \) follows then from Theorem 8.19 and from the following immediate result (compare with Lemma 5.5):

**Lemma 8.22** Let \( (A, \alpha, \mathbb{B}) \) be a \( C^* \)-algebra endowed with a strongly continuous and isometric action of a normal \( j \)-group. Then, we have an isometric equivariant embedding \( \alpha : A^\infty \to \mathcal{B}(\mathbb{B}, A) \).

**Proof.** The equivariance property of \( \alpha \) is obvious and implies (with the fact that \( \alpha \) is an isometry) that for any \( k \in \mathbb{N} \):
\[
\|\alpha(a)\|_{k, \infty} = \sup_{g \in \mathbb{B}} \sup_{X \in \mathcal{U}_k(b)} \frac{\|X g \alpha g(a)\|}{\|X\|_k} = \sup_{g \in \mathbb{B}} \sup_{X \in \mathcal{U}_k(b)} \frac{\|\alpha g(X^\alpha a)\|}{\|X\|_k} = \sup_{X \in \mathcal{U}_k(b)} \frac{\|X^\alpha a\|}{\|X\|_k} = \|a\|_k,
\]
and the proof follows.

Combining the previous lemma with Theorem 8.19 we deduce the following inequality:

**Corollary 8.23** Let \( (A, \alpha, \mathbb{B}) \) be a \( C^* \)-algebra endowed with a strongly continuous and isometric action of a normal \( j \)-group and \( m \in \Theta(\mathbb{B}) \). Then, there exists \( k \in \mathbb{N} \) and \( C > 0 \) such that for any \( a \in A^\infty \), we have:
\[
\|\Omega_{\theta, m}(\alpha(a))\| \leq C \|a\|_k := C \sup_{X \in \mathcal{U}_k(b)} \left\{ \frac{\|X^\alpha a\|}{\|X\|_k} \right\}.
\]

**Proposition 8.24** Let \( m \in \Theta(\mathbb{B}) \). Then the following defines a \( C^* \)-norm on the involutive deformed Fréchet algebra \( (A^\infty, \ast_{\theta, m}^\alpha, \ast_{\theta, m}^\alpha) \):
\[
a \in A^\infty \mapsto \|a\|_{\theta, m} := \|\Omega_{\theta, m}(\alpha(a))\|,
\]
where the norm in the r.h.s. is the spatial \( C^* \)-norm on \( A \otimes B(H_\chi) \), and the operator \( \Omega_{\theta, m}(\alpha(a)) \) is defined in Theorem 8.19. Accordingly, we let \( A_{\theta, m} \) the \( C^* \)-completion of \( A^\infty \) that we abusively call the \( C^* \)-deformation of \( A \).

**Proof.** By construction we have for all \( a, b \in A^\infty \)
\[
\Omega_{\theta, m}(\alpha(a \ast_{\theta, m}^\alpha b)) = \Omega_{\theta, m}(\alpha(a))^\ast \Omega_{\theta, m}(\alpha(b)),
\]
and the claim follows immediately.

**Remark 8.25** We already know that at the level of the deformed pre-\( C^* \)-algebra \( (A^\infty, \ast_{\theta, m}^\alpha) \), the action of the group \( \mathbb{B} \) is no longer by automorphism. But at the level of the deformed \( C^* \)-algebra there is no action of \( \mathbb{B} \) at all.
In a way very analogous to Proposition \[5.20\] we can show that the $C^*$-deformation associated with a normal $j$-group coincides with the iterated $C^*$-deformations of each of its elementary normal sub-groups. To see this, fix $\mathbb{B}$ be a normal $j$-group with Pyatetskii-Shapiro decomposition $\mathbb{B} = B' \times S$ and $A$ a $C^*$-algebra endowed with a strongly continuous and isometric action $\alpha$ of $\mathbb{B}$. Of course, $\alpha^\beta$, the restriction of $\alpha$ to the subgroup $S$, is strongly continuous on $A$. Let us fix also $m = m' \otimes m \in \Theta(\mathbb{B})$, with $m' \in \Theta(B')$ and $m \in \Theta(S)$. Then, we can perform the $C^*$-deformation of $A$ by mean of the action of $\mathbb{S}$. We call this deformed $C^*$-algebra $A^\beta_{\theta,m}$. Then $B'$ acts strongly continuously by $\ast$-homomorphisms on $A^\beta_{\theta,m}$. Indeed, it has been shown in the proof of Proposition \[5.20\] that the subspace of smooth vectors for $\mathbb{B}$ coincides with the subspace of smooth vectors for $B'$ within the subspace of smooth vectors for $S$. In turns, $A^\infty$, the set of smooth vectors for $\mathbb{B}$ on $A$, is dense in $A^\beta_{\theta,m}$. As the action of $B'$ is (obviously) strongly continuous and by $\ast$-homomorphisms (as shown in the proof of Proposition \[5.20\] too) on $A^\infty$, a density argument yields the result. Thus, we can perform the $C^*$-deformation of $A^\beta_{\theta,m}$ by mean of the action of $B'$. We call this deformed $C^*$-algebra $(A^\beta_{\theta,m})_{\beta,m'}$. But we could also perform the $C^*$-deformation of $A$ by mean of the action of $B$ directly. We call this deformed $C^*$-algebra $A_{\theta,m}^\beta$. Now, the precise result of Proposition \[5.20\] is that at the level of the (common) dense subspace $A^\infty$, both constructions coincide. Thus it suffices to show that the $C^*$-norms of $(A^\beta_{\theta,m})_{\beta,m'}$ and $A_{\theta,m}^\beta$ coincide on $A^\infty$. But this easily follows from our construction. Indeed, the $C^*$-norm of $(A^\beta_{\theta,m})_{\beta,m'}$ at the level of $A^\infty$, is by definition the map $$a \mapsto \|\Omega_{\theta,m}(\{z' \in B' \mapsto \Omega_{\theta,m}(\{z \in S \mapsto \alpha_{z'z}(a))\})\|.$$ But by the construction of section \[103\] we precisely have $$\Omega_{\theta,m}(\{z' \in B' \mapsto \Omega_{\theta,m}(\{z \in S \mapsto \alpha_{z'z}(a))\}) = \Omega_{\theta,m}^\beta(\alpha(a)).$$ Thus, we have proven the following:

**Proposition 8.26** Let $\mathbb{B}$ be a normal $j$-group with Pyatetskii-Shapiro decomposition $\mathbb{B} = B' \times S$, where $B'$ is a normal $j$-group and $S$ is an elementary normal $j$-group. Let $A$ be a $C^*$-algebra endowed with a strongly continuous isometric action $\alpha$ of $\mathbb{B}$. Within the notations displayed above, we have: $$A^\beta_{\theta,m} = (A^\beta_{\theta,m})_{\beta,m'}.$$ In the remaining part of this section we prove that the deformed $C^*$-norm constructed above coincide with the $C^*$-norm of bounded and adjointable operators on a $C^*$-module. This will make clearer the analogies with the construction of Rieffel in \[19\] for the Abelian case and it also explains the choice of the spatial tensor product in Theorem \[8.19\].

**Definition 8.27** Let $m \in \Theta(\mathbb{B})$. Then, for $f_1, f_2 \in S_{\text{can}}(\mathbb{B}, A)$, we define the $A$-valued paring:

$$\langle f_1, f_2 \rangle_{\theta,m} := \int \langle \eta_x, \Omega_{\theta,m}(f_1 \ast \theta,m f_2) \eta_x \rangle d_{\mathbb{B}}(x),$$

where where $\{\eta_x\}_{x \in \mathbb{B}} \subset \mathcal{H}_X$ is the family of coherent states given in Definition \[8.2\] and the involution $\ast_{\theta,m}$ on $S_{\text{can}}(\mathbb{B}, A)$ is defined by:

$$\ast_{\theta,m} : f \mapsto \frac{m_1}{m_N} \cdots \frac{m_1}{m_1} \arcsinh\left(\frac{1}{n}E_N\right) \circ \cdots \circ \arcsinh\left(\frac{1}{n}E_1\right)f^*.$$ 

In the last formula, $E_N, \ldots, E_1$ denote the central elements in each Heisenberg Lie algebra attached to each elementary components in $\mathbb{B}$ and $f^* := [x \in \mathbb{B} \mapsto f(x)^*] \in S_{\text{can}}(\mathbb{B}, A)$.

**Proposition 8.28** Endowed with the paring \[103\] and action $$A^\infty \times S_{\text{can}}(\mathbb{B}, A) \to S_{\text{can}}(\mathbb{B}, A), \quad (a, f) \mapsto [g \in \mathbb{B} \mapsto f(g)a],$$
the space $S_{\text{can}}(\mathbb{B}, A)$ becomes a pre-$C^*$-module for the $C^*$-algebra $A$. 

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Proof. Note first that for any \( f \in S_{S^*}(\mathcal{B}, A) \), we have by Proposition [8.6]

\[
\langle \eta_x, \Omega_{\theta, m}(f) \eta_x \rangle_{\mathcal{B}}(x) = \int \langle \eta_x, \eta_y \rangle \langle \eta_y, \Omega_{\theta, m}(f) \eta_x \rangle_{\mathcal{B}}(x) \, d\mathcal{B}(y) = \int \langle \eta, \eta_{x^{-1}y} \rangle \langle \eta_y, \Omega_{\theta, m}(f) \eta_x \rangle_{\mathcal{B}}(x) \, d\mathcal{B}(y).
\]

Thus,

\[
\int \|\langle \eta_x, \Omega_{\theta, m}(f) \eta_x \rangle\|_{\mathcal{B}}(x) \leq \sup_{x \in \mathbb{B}} \int \|\langle \eta, \eta_{x^{-1}y} \rangle\|_{\mathcal{B}}(y) \times \sup_{y \in \mathbb{B}} \int \|\langle \eta_y, \Omega_{\theta, m}(f) \eta_x \rangle\|_{\mathcal{B}}(x)
\]

\[
= \int \|\langle \eta, \eta_y \rangle\|_{\mathcal{B}}(y) \times \sup_{y \in \mathbb{B}} \int \|\langle \eta_y, \Omega_{\theta, m}(f) \eta_x \rangle\|_{\mathcal{B}}(x) < \infty,
\]

by Corollary [8.4] and Theorem [8.13] (since \( S_{S^*}(\mathcal{B}, A) \subset B(\mathcal{B}, A) \)). Thus, we conclude from the stability of \( S_{S^*}(\mathcal{B}, A) \) under \( \ast_{\theta, m} \) (see Proposition [4.8]), that the paring \( \langle \ldots \rangle_{\theta, m} \) is well defined. Moreover, testing this paring on the dense subset \( \text{span}\{a\varphi, a \in A, \varphi \in S_{S^*}(\mathcal{B})\} \) of \( S_{S^*}(\mathcal{B}, A) \), we see that \( \langle S_{S^*}(\mathcal{B}, A), S_{S^*}(\mathcal{B}, A) \rangle_{\theta, m} = AA \), which is dense in \( A \). Next, we observe that the paring can be rewritten as

\[
\langle f_1, f_2 \rangle_{\theta, m} = \int \langle \eta_x, \Omega_{\theta, m}(f_1)^* \Omega_{\theta, m}(f_2) \eta_x \rangle_{\mathcal{B}}(x).
\]

This shows that \( \langle f_1, f_2 \rangle_{\theta, m}^\ast = \langle f_2, f_1 \rangle_{\theta, m} \) and proves positivity and non-degeneracy. Last, it is clear that \( \langle f_1, f_2 \rangle_{\theta, m}a = \langle f_1, f_2a \rangle_{\theta, m} \) for all \( a \in A \) and all \( S_{S^*}(\mathcal{B}, A) \).

\[\Box\]

Remark 8.29 It can be shown that the paring can be rewritten as:

\[
\langle f_1, f_2 \rangle_{\theta, m} = \int_{\mathbb{B}} f_1^\ast \ast_{\theta, m} f_2 \, d\mathcal{B}(g) = \text{Tr}(\Omega_{\theta, m}(f_1)^* \Omega_{\theta, m}(f_2)).
\]

However, this is by far less convenient expressions, as shown in the proof of Theorem [8.31] below.

Definition 8.30 Let \( m \in \Theta(\mathcal{B}) \). For \( F \in B(\mathcal{B}, A) \), let \( L_{\theta, m}(F) \) be the operator on \( S_{S^*}(\mathcal{B}, A) \) given by

\[
L_{\theta, m}(F) \, f = F \ast_{\theta, m} f.
\]

By Proposition [4.8], the operator \( L_{\theta, m}(F) \), \( F \in B(\mathcal{B}, A) \), acts continuously on \( S_{S^*}(\mathcal{B}, A) \). Moreover, \( L_{\theta, m}(F) \) is adjointable, with adjoint given by \( L_{\theta, m}(F^\ast) \). Indeed, for all \( f_1, f_2 \in S_{S^*}(\mathcal{B}, A) \) and \( F \in B(\mathcal{B}, A) \), we have

\[
\langle f_1, L_{\theta, m}(F) f_2 \rangle_{\theta, m} = \int \langle \eta_x, \Omega_{\theta, m}(f_1^\ast \ast_{\theta, m} F \ast_{\theta, m} f_2) \eta_x \rangle_{\mathcal{B}}(x) = \int \langle \eta_x, \Omega_{\theta, m}((F^\ast \ast_{\theta, m} f_1)^\ast \ast_{\theta, m} f_2) \eta_x \rangle_{\mathcal{B}}(x) = \langle L_{\theta, m}(F^\ast) f_1, f_2 \rangle_{\theta, m}.
\]

Note also that the operators \( L_{\theta, m}(F) \) all commutes with the right-action of \( A \). But we have more, since in fact \( L_{\theta, m}(F) \), for \( F \in B(\mathcal{B}, A) \), belongs to the \( C^* \)-algebra of \( A \)-linear adjointable endomorphisms of the pre-\( C^* \)-module \( S_{S^*}(\mathcal{B}, A) \). Indeed, from the operator inequality on \( A \otimes \mathcal{H} \)

\[
\Omega_{\theta, m}(f^\ast \ast_{\theta, m} F^\ast \ast_{\theta, m} F \ast_{\theta, m} f) = \Omega_{\theta, m}(f)^* \Omega_{\theta, m}(F)^2 \Omega_{\theta, m}(f) \leq \||\Omega_{\theta, m}(F)||^2 \Omega_{\theta, m}(f^\ast \ast_{\theta, m} \ast_{\theta, m} f)
\]

we deduce for \( F \in B(\mathcal{B}, A) \) and \( f \in S_{S^*}(\mathcal{B}, A) \), the operator inequality on \( A \):

\[
\langle L_{\theta, m}(F) f, L_{\theta, m}(F) f \rangle_{\theta, m} \leq \||\Omega_{\theta, m}(F)||^2 \langle f, f \rangle_{\theta, m}.
\]

Hence we get

\[
\|L_{\theta, m}(F)\| \leq \||\Omega_{\theta, m}(F)||,
\]

(104)
where the norm on the left hand side denotes the norm of the $C^*$-algebra of $A$-linear adjointable endomorphisms of the pre-$C^*$-module $S^{S_{\text{cas}}}(B, A)$. Now, observe the dense embedding of the algebraic tensor product
\[ A \otimes_{\text{alg}} B(B) \to B(B, A), \quad \sum_i a_i \otimes \phi_i \mapsto [g \in B \mapsto \sum_i \phi_i(g) a_i \in A]. \]

Via this embedding, the norm in the right hand side of (104) is by construction the restriction to $A \otimes_{\text{alg}} B(B)$ of the minimal (spatial) $C^*$-norm on $A \otimes_{\text{alg}} B$, where $B$ is the $C^*$-completion of $\{\Omega_{\theta, m}(F), F \in B(B)\}$ in $B(H_\chi)$. Hence, we deduce that
\[ \|L^{\theta, m}(F)\| \geq \|\Omega_{\theta, m}(F)\|, \quad \forall F \in A \otimes_{\text{alg}} B(B), \]

which by density implies that
\[ \|L^{\theta, m}(F)\| \geq \|\Omega_{\theta, m}(F)\|, \quad \forall F \in B(B, A). \]

Thus we have proven the following:

**Theorem 8.31** Let $B$ be a normal $\ast$-group, $A$ a $C^*$-algebra and $m \in \Theta(B)$. Then
\[ \|L^{\theta, m}(F)\| = \|\Omega_{\theta, m}(F)\|, \quad \forall F \in B(B, A), \]

where the norm in the left hand side is the one of the $C^*$-algebra of $A$-linear adjointable endomorphisms of the pre-$C^*$-module $S^{S_{\text{cas}}}(B, A)$ and the norm in the right hand side is the spatial $C^*$-norm of $A \otimes B(H_\chi)$.

Back to the case where $A$ carries a strongly continuous action $\alpha$ by $*$-morphisms, we deduce:

**Proposition 8.32** Let $(A, \alpha)$ be a $C^*$-algebra endowed with a strongly continuous and isometric action of a normal $\ast$-group $B$ and let $m \in \Theta(B)$. Then, the $C^*$-norm on the involutive Fréchet algebra $(A^\infty, *_{\theta, m}^\alpha, *_{\theta, m})$ given by
\[ a \in A^\infty \mapsto \|L^{\theta, m}(\alpha(a))\|, \]

coincides with the deformed norm $\|\cdot\|_{\theta, m}$ of Proposition 8.23.

### 8.6 Functorial properties of the deformation

In this subsection, we collect the main functorial properties of the deformation. We still consider a $C^*$-algebra $A$, endowed with a strongly continuous and isometric action $\alpha$ of a normal $\ast$-group $B$. Given an element $m \in \Theta(B)$, we form $A_{\theta, m}$ the $C^*$-deformation of $A$. We let $B_{\theta, m}(B, A)$ (respectively $\overline{\Omega_{\theta, m}(B, A)}$) the $C^*$-completion of the pre-$C^*$-algebra $(B(B, A), *_{\theta, m}, *_{\theta, m})$ (respectively $(S^{S_{\text{cas}}}(B, A), *_{\theta, m}, *_{\theta, m})$) for the $C^*$-norm
\[ F \mapsto \|\Omega_{\theta, m}(F)\|. \]

Firstly, we observe from Proposition 8.21 the following isomorphism:

**Lemma 8.33** Let $A$ be a $C^*$-algebra and $m \in \Theta(B)$. Then we have:
\[ \overline{\Omega_{\theta, m}(B, A)} \simeq \mathcal{K}(H_\chi) \otimes A. \]

Now, we come to the question of bounded approximate units for the deformed $C^*$-algebra $A_{\theta, m}$. From the existence of a bounded approximate unit for the undeformed $C^*$-algebra $A$, Proposition 5.19 shows that the pre-$C^*$-algebra $(A^\infty, *_{\theta, m}^\alpha, *_{\theta, m})$ possesses a bounded approximate unit as well. Thus, we deduce from Corollary 8.23

**Proposition 8.34** Let $(A, \alpha)$ be a $C^*$-algebra endowed with a strongly continuous action of a normal $\ast$-group $B$ and $m \in \Theta(B)$. Then $A_{\theta, m}$ possesses a bounded approximate unit $\{e_\lambda\}_{\lambda \in \Lambda}$ consisting of elements of $A^\infty$.

Next, we observe that the two-sided ideal $(S^{S_{\text{cas}}}(B, A), *_{\theta, m}, *_{\theta, m})$ is essential in $(B(B, A), *_{\theta, m}, *_{\theta, m})$: 99
Proposition 8.35 Let $A$ be a $C^*$-algebra and $m \in \Theta(\mathbb{B})$. Then we have for all $F \in \mathcal{B}(\mathbb{B}, A)$:

$$
\|\Omega_{\theta,m}(F)\| = \sup \{ \|\Omega_{\theta,m}(F \ast_{\theta,m} f)\| : f \in \mathcal{S}^{S_{\text{fin}}}(\mathbb{B}, A) \}, \quad \|\Omega_{\theta,m}(f)\| \leq 1
$$

Proof: This is verbatim the arguments of [19, Proposition 4.11], combined with the equality $\|\Omega_{\theta,m}(F)\| = \|L_{\theta,m}^{-\infty}(F)\|$ of Proposition 5.32 for all $F \in \mathcal{B}(\mathbb{B}, A)$ (thus for $f \in \mathcal{S}^{S_{\text{fin}}}(\mathbb{B}, A)$ too) and with the existence of bounded approximate units of the pre-$C^*$-algebra $\mathcal{S}^{S_{\text{fin}}}(\mathbb{B}, A)$ as shown in Proposition 5.19.

The proof of the next two results is word for word the one of the corresponding results in the flat situation, given in [19, Proposition 4.12 and Proposition 4.15].

Proposition 8.36 Let $A$ be a $C^*$-algebra, $I$ an essential ideal of $A$ and $m \in \Theta(\mathbb{B})$. Then the $C^*$-norm on $(\mathcal{B}(\mathbb{B}, A), \ast_{\theta,m})$ given in Theorem 5.4 is the same than the $C^*$-norm of Proposition 5.32 for the restriction of the action of $\mathcal{B}(\mathbb{B}, A)$ on $\mathcal{S}^{S_{\text{fin}}}(\mathbb{B}, I)$.

Proposition 8.37 Let $A$ be a $C^*$-algebra and $m \in \Theta(\mathbb{B})$. The $C^*$-algebra $\mathcal{B}_{\Theta,m}(\mathbb{B}, A)$ is isomorphic to the $C^*$-deformation of the algebra of $A$-valued right uniformly continuous and bounded functions on $\mathbb{B}$, $C_{ru}(\mathbb{B}, A)$, for the right regular action of $\mathbb{B}$.

The following two results treat the question of morphisms and ideals. They can be proven exactly as [19, Theorem 5.7, Proposition 5.8 and Proposition 5.9], by using Proposition 5.12 and Proposition 5.17.

Proposition 8.38 Fix $m \in \Theta(\mathbb{B})$ and let $(A, \alpha)$ and $(B, \beta)$ be two $C^*$-algebras endowed with a strongly continuous actions of $\mathbb{B}$. Then, if $T : A \to B$ is a continuous homomorphism which intertwines the actions $\alpha$ and $\beta$, its restriction $T^\infty : A^\infty \to B^\infty$ extends to a continuous homomorphism $T_{\theta,m} : A_{\theta,m} \to B_{\theta,m}$. If moreover $T$ is injective (respectively surjective) then $T_{\theta,m}$ is injective (respectively surjective) too.

Proposition 8.39 Fix $m \in \Theta(\mathbb{B})$ and let $(A, \alpha)$ be a $C^*$-algebra endowed with a strongly continuous action of $\mathbb{B}$ and let also $I$ be an $\alpha$-invariant (essential) ideal of $A$. Then $I_{\theta,m}$ is an (essential) ideal of $A_{\theta,m}$.

8.7 Invariance of the $K$-theory

In this final subsection, we show that the $K$-theory is an invariant of our $C^*$-deformation, exactly as in the Abelian case [20]. We still consider a $C^*$-algebra $A$, endowed with a strongly continuous and isometric action $\alpha$ of a normal $\mathfrak{j}$-group $\mathbb{B}$. Given an element $m \in \Theta(\mathbb{B})$, we form $A_{\theta,m}$, the $C^*$-deformation of $A$. We endow $\mathcal{B}_{\theta,m}(\mathbb{B}, A)$ with the action $\hat{\alpha}$ of $\mathbb{B}$ defined on $\mathcal{B}(\mathbb{B}, A)$ by

$$
(\hat{\alpha}_g F)(x) := \alpha_g(F(g^{-1}x)) \quad (105)
$$

We will show that $\hat{\alpha}$ is a proper action of $\mathbb{B}$ on the sub-$C^*$-algebra $\mathcal{S}_{\theta,m}(\mathbb{B}, A)$, and that $A_{\theta,m}$ is the generalized fixed point algebra for this action, in the sense of [18]. We also let $\hat{\alpha}$ to denote the extension of $\alpha$ from $A$ to $\mathcal{B}(\mathbb{B}, A)$, given by

$$
(\hat{\alpha}_g F)(x) = \alpha_g(F(x))
$$

so that

$$
\hat{\alpha}_g = \hat{\alpha}_g \circ L_g^* = L_g^* \circ \hat{\alpha}_g, \quad \forall g \in \mathbb{B}.
$$

This also shows that the infinitesimal form of $\hat{\alpha}$ on $\mathcal{B}(\mathbb{B}, A^\infty)$ is related with the infinitesimal form of $\alpha$ on $A^\infty$ by

$$
(X^\alpha f)(x) = X^{\alpha}(f(x)), \quad \forall X \in \mathcal{U}(\mathfrak{b}), \quad \forall f \in \mathcal{B}(\mathbb{B}, A)
$$

We start with some preliminary results:

Lemma 8.40 The action $\hat{\alpha}$ is isometric on $\mathcal{B}_{\theta,m}(\mathbb{B}, A)$ and its restriction to $\mathcal{S}_{\theta,m}(\mathbb{B}, A)$ is strongly continuous.

Proof. The isometry follows from the covariance of the pseudo-differential calculus: for all $g \in \mathbb{B}$, we have

$$
\Omega_{\theta,m}(\hat{\alpha}_g F) = \alpha_g \otimes \text{Id}_{\mathcal{S}(\mathfrak{b})}(\Omega_{\theta,m}(L_g^* F)) = \alpha_g \otimes \text{Id}_{\mathcal{S}(\mathfrak{b})}(U_{\theta}(g) \circ \Omega_{\theta,m}(F) \circ U_{\theta}(g)^*) \circ \forall F \in \mathcal{B}(\mathbb{B}, A).
$$

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For the strong continuity of $\hat{\alpha}$ on $S_{0,m}(\mathbb{B}, A)$, it suffices (by density) to prove it on $S_{S,\gamma}(\mathbb{B}, A)$. Since by Theorem 8.19 the operator norm is weaker than the Fréchet topology of $\mathcal{B}(\mathbb{B}, A)$, it suffices to prove strong continuity on $S_{S,\gamma}(\mathbb{B}, A)$ for the induced Fréchet topology of $\mathcal{B}(\mathbb{B}, A)$. As $S_{S,\gamma}(\mathbb{B}, A)$ is a subset of $\mathcal{B}(\mathbb{B}, A)$ stable under the inversion map of $\mathbb{B}$ and that the right regular action $R^*$ is strongly continuous on $\mathcal{B}(\mathbb{B}, A)$, we deduce that the left regular action $L^*$ is strongly continuous on $S_{S,\gamma}(\mathbb{B}, A)$. Since $\hat{\alpha} = \hat{\alpha} \circ L^* = L^* \circ \hat{\alpha}$, it then remains to prove strong continuity of $\hat{\alpha}$ on $S_{S,\gamma}(\mathbb{B}, A)$ for the induced Fréchet topology of $\mathcal{B}(\mathbb{B}, A)$. As $\hat{\alpha}$ commutes with the right regular action $R^*$, it commutes with the left-invariant differential operators. By isometry of $\alpha$, strong continuity everywhere will follow from strong continuity at the neutral element $e_B$. This also implies that for $f \in S_{S,\gamma}(\mathbb{B}, A)$, we have in the semi-norms defining the topology of $\mathcal{B}(\mathbb{B}, A)$:

$$
\| \hat{\alpha}_g(f) - f\|_{k,\infty} = \sup_{X \in \mathcal{U}_k(b)} \frac{\|X(\hat{\alpha}_g f) - Xf\|_{\infty}}{|X|_k} = \sup_{X \in \mathcal{U}_k(b)} \frac{\|\hat{\alpha}_g(X f) - X f\|_{\infty}}{|X|_k}, \quad \forall g \in \mathbb{B}.
$$

Hence it suffices to consider the case $k = 0$. By a compactness argument, we see that the strong continuity of $\alpha$ on $A$ implies strong continuity of $\hat{\alpha}$ on $\mathcal{D}(\mathbb{B}, A)$ for the uniform norm, namely

$$
\| \hat{\alpha}_g(f) - f\|_{\infty} \to 0, \quad g \to e_B,
$$

and one concludes by density of $\mathcal{D}(\mathbb{B}, A)$ in $S_{S,\gamma}(\mathbb{B}, A)$.

We now come to the question of the properness of the action $\hat{\alpha}$ on $S_{0,m}(\mathbb{B}, A)$, in the sense of [13]. For this property to hold, we need to find a dense $\hat{\alpha}$-invariant $*$-subalgebra $B_0$ of $S_{0,m}(\mathbb{B}, A)$, such that for all $f_1, f_2 \in B_0$, the maps

$$
[g \in \mathbb{B} \mapsto \Delta_{\mathbb{B}}(g)^{-1/2} f_1 \ast_{\theta, m} \hat{\alpha}_g(f_2^{*_{\theta, m}})] \quad \text{and} \quad [g \in \mathbb{B} \mapsto f_1 \ast_{\theta, m} \hat{\alpha}_g(f_2^{*_{\theta, m}})],
$$

belong to $L^1(\mathbb{B}, S_{0,m}(\mathbb{B}, A))$ and denoting by $M(B_0)^{\hat{\alpha}}$ the subalgebra of the multiplier algebra of $S_{0,m}(\mathbb{B}, A)$ of elements which preserve $B_0$ and which are invariant under the extension of $\hat{\alpha}$ to $M(S_{0,m}(\mathbb{B}, A))$ we have

$$
\left[f \in B_0 \mapsto \int_{\mathbb{B}} f \ast_{\theta, m} \hat{\alpha}_g(f_1^{*_{\theta, m}} \ast_{\theta, m} f_2) d\mathbb{B}(g) \right] \in M(B_0)^{\hat{\alpha}}.
$$

Our candidate for $B_0$ is $S_{S,\gamma}(\mathbb{B}, A^\infty)$. Note first:

**Lemma 8.41** Let $m \in \Theta(\mathbb{B})$. Then $(S_{S,\gamma}(\mathbb{B}, A^\infty), \ast_{\theta, m})$ is a dense $\hat{\alpha}$-invariant $*$-subalgebra of $S_{0,m}(\mathbb{B}, A)$. 

**Proof.** That $S_{S,\gamma}(\mathbb{B}, A^\infty)$ is closed under the involution $\ast_{\theta, m}$ and under the action $\hat{\alpha}$ of $\mathbb{B}$ is clear. Since $S_{S,\gamma}(\mathbb{B}, A^\infty)$ is dense in $S_{S,\gamma}(\mathbb{B}, A)$ for the Fréchet topology of the later, since $S_{S,\gamma}(\mathbb{B}, A)$ is dense on $S_{0,m}(\mathbb{B}, A)$ for the $C^*$-topology of the latter and since the $C^*$-topology is weaker than the Fréchet topology on $S_{S,\gamma}(\mathbb{B}, A)$, the density statement follows. Last, that $(S_{S,\gamma}(\mathbb{B}, A^\infty), \ast_{\theta, m})$ is a subalgebra of $S_{0,m}(\mathbb{B}, A)$ follows from Proposition 8.3.

The two conditions underlying properness relies on an important preliminary result:

**Lemma 8.42** Let $f_1, f_2 \in S_{S,\gamma}(\mathbb{B}, A^\infty)$. Then the map

$$
\left[(x_1, x_2) \in \mathbb{B} \times \mathbb{B} \mapsto \left[ y \in \mathbb{B} \mapsto R_{x_2}(f_1) R_{x_2}^{*_{\theta}}(\hat{\alpha}_g(f_2)) = [z \in \mathbb{B} \mapsto f_1(z x_1) \alpha_g(f_2(y^{-1} z x_2)) \in A]\right]\right],
$$

belongs to $\mathcal{B}^{(\nu_{_{\mathbb{B}}})}(\mathbb{B} \times \mathbb{B}, S_{S,\gamma}(\mathbb{B}, S_{S,\gamma}(\mathbb{B}, A^\infty)))$, with $\delta$ the modular weight on $\mathbb{B} \times \mathbb{B}$.

**Proof.** Recall that the Fréchet topology of $S_{S,\gamma}(\mathbb{B}, A^\infty)$ can be associated with the set of semi-norms

$$
\|f\|_{(j,k),l} := \sup_{X \in \mathcal{U}_k(b)} \sup_{Y \in \mathcal{U}_l(b)} \sup_{x \in \mathbb{B}} \frac{\|X^n Y^m f(x)\|}{|X|_k |Y|_l},
$$

and the one of $\mathcal{B}^{(\mu_{_{\mathbb{B}}})}(\mathbb{B} \times \mathbb{B}, S_{S,\gamma}(\mathbb{B}, S_{S,\gamma}(\mathbb{B}, A^\infty)))$ can be associated with the set of semi-norms

$$
\|F\|_{(j_1,k_1,j_2,k_2),m} := \sup_{X_1 \in \mathcal{U}_k(b)} \sup_{X_2 \in \mathcal{U}_l(b)} \sup_{Y_1 \in \mathcal{U}_j(b)} \sup_{Y_2 \in \mathcal{U}_j(b)} \sup_{Z \in \mathcal{U}_m(b)} \sup_{x_1, x_2, y, z \in \mathbb{B}} \frac{\|X_1^n Y_1^m Z^n f(x_1, x_2; y, z)\|}{|X_1|_{k_1} |Y_1|_{j_1} |Y_2|_{j_2} |Z|_m}.
$$

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Applying this to the four-point function constructed in \[106\], we get using Swedler’s notation:

\[
Z^n \tilde{Y}_{2,z} \tilde{Y}_{1,y} \tilde{X}_{(x_1,x_2)} \left[ f_1(zx_1) \alpha_y (f_2(y^{-1}zx_2)) \right] \\
= \sum_{(X)} Z^n \tilde{Y}_{2,z} \tilde{Y}_{1,y} \left[ (\tilde{X}_1 f_1)(zx_1) \alpha_y \left( (\tilde{X}_2 f_2)(y^{-1}zx_2) \right) \right] \\
= \sum_{(X)} \sum_{(Y)} Z^n \tilde{Y}_{2,z} \left[ (\tilde{X}_1 f_1)(zx_1) \alpha_y \left( (Y^\alpha_{1,1} Y^\alpha_{1,2} \tilde{X}_2 f_2)(y^{-1}zx_2) \right) \right] \\
= \sum_{(X)} \sum_{(Y)} \sum_{(Z)} Z^n \left[ (\tilde{A}_{x_1} Y^\alpha_{1,1} \tilde{X}_1 f_1)(zx_1) \alpha_y \left( (\tilde{A}_{x_2} Y^\alpha_{1,2} \tilde{X}_2 f_2)(y^{-1}zx_2) \right) \right] \\
= \sum_{(X)} \sum_{(Y)} \sum_{(Z)} \left( Z^n \tilde{A}_{x_1} Y^\alpha_{1,1} \tilde{X}_1 f_1)(zx_1) \alpha_y \left( (\tilde{A}_{x_2} Y^\alpha_{1,2} \tilde{X}_2 f_2)(y^{-1}zx_2) \right) \right).
\]

Hence we get

\[
\sup_X \sup_Y \sup_Z \left\| Z^n \tilde{Y}_{2,z} \tilde{Y}_{1,y} \tilde{X}_{(x_1,x_2)} \left[ f_1(zx_1) \alpha_y (f_2(y^{-1}zx_2)) \right] \right\| \\
\leq C(l, k_1, k_2, m) \left\| \tilde{B}(x_1, x_2)^{k_2} \tilde{B}(y)^m \left\| f_1 \right\|_{(N,l+k_2),m} \left\| f_2 \right\|_{(M,l+k_1+k_2),k_1+m} \right. \\
\right. 
\]

As \( M, N \) are arbitrary, using the sub-multiplicativity of the modular weight, we can find \( n(j_1, k_1, j_2, k_2, m) \) such that the fraction above is uniformly bounded. This achieves the proof.

**Proposition 8.43** For \( m \in \Theta(\mathbb{B}) \) and \( f_1, f_2 \in S^{\text{sc}}(\mathbb{B}, A^\infty) \), the maps

\[
[g \in \mathbb{B} \mapsto \Delta_{\mathbb{B}}(g)^{-1/2} f_1 \ast_{\theta, m} \hat{\alpha}_g(f_2^{*m})] \quad \text{and} \quad [g \in \mathbb{B} \mapsto f_1 \ast_{\theta, m} \hat{\alpha}_g(f_2^{*m})],
\]

belong to \( L^1(\mathbb{B}, \hat{S}_{\theta,m}(\mathbb{B}, A)) \). Moreover, the map

\[
f \in S(\mathbb{B}, A^\infty) \mapsto \int_{\mathbb{B}} f \ast_{\theta, m} \hat{\alpha}_g(f_1^{*m} \ast_{\theta, m} f_2) \, d\mathbb{B}(g),
\]

belongs to the multiplier algebra \( M(S^{\text{sc}}(\mathbb{B}, A^\infty)) \). Consequently, the action \( \hat{\alpha} \) of \( \mathbb{B} \) on \( \hat{S}_{\theta,m}(\mathbb{B}, A) \) is proper in the sense of \([19]\).

**Proof.** The first part of the claim follows from the definition of the product \( \ast_{\theta, m} \) in term of an oscillatory integral:

\[
f_1 \ast_{\theta, m} \hat{\alpha}_g(f_2^{*m}) = \int_{\mathbb{B} \times \mathbb{B}} K_{\theta,m}(x_1, x_2) R_{x_1}^*(f_1) R_{x_2}^*(\hat{\alpha}_g(f_2^{*m})) \, d\mathbb{B}(x_1) \, d\mathbb{B}(x_2),
\]

combined with Lemma 8.42 and with the inclusion \( S^{\text{sc}}(\mathbb{B}, S^{\text{sc}}(\mathbb{B}, A^\infty)) \subset L^1(\mathbb{B}, \hat{S}_{\theta,m}(\mathbb{B}, A)) \), together with the fact that the multiplication by \( \Delta_{\mathbb{B}}^{-1/2} \) is continuous on \( S^{\text{sc}}(\mathbb{B}, \hat{S}_{\theta,m}(\mathbb{B}, A)) \) as it is a tempered function. For the second part, we again use Lemma 8.42 which shows since \( S^{\text{sc}}(\mathbb{B}, A^\infty) \) is an algebra for the product \( \ast_{\theta, m} \), that the map

\[
f \mapsto \int_{\mathbb{B}} f \ast_{\theta, m} \hat{\alpha}_g(f_1^{*m} \ast_{\theta, m} f_2) \, d\mathbb{B}(g),
\]
sends continuously $S^{S_{\text{can}}}(B, A^\infty)$ to itself. Since the $\hat{\alpha}$-invariance of the map above is rather clear, it remains to show that the latter extends from $S_{\theta,m}(B, A)$ to itself, i.e. that it is indeed an element of the multiplier algebra of $S_{\theta,m}(B, A)$. For this we need to find a convenient expression for

$$\int_B f_1 \ast_{\theta,m} \hat{\alpha}_g(f_2) \, d_{S}(g), \quad \forall f_1, f_2 \in S(B, A^\infty), \quad \forall y \in B.$$  

By Proposition [103], we can express the oscillatory integral underlying the product $\ast_{\theta,m}$ in term of absolutely convergent integral. More precisely, writing $K_{\theta,m} = e^{2\pi S_{\text{can}}} A_{\theta,m}$, we have

$$\int_B f_1 \ast_{\theta,m} \hat{\alpha}_g(f_2) \, d_{S}(g) = \int B \, e^{2\pi S_{\text{can}}(x_1,x_2)} D_{x_1,x_2} \left[ A_{\theta,m}(x_1,x_2) f_1(yx_1) \alpha_g(f_2(g^{-1}yx_2)) \right] \, d_{S}(g) \, d_{S}(x_1) \, d_{S}(x_2),$$

where one can choose the operator $D$ such that the coefficients decay (in a tempered way) as fast as one wishes, so that taking into account the decay of $f_1, f_2$, the triple-integral above is absolutely convergent. As $D$ commutes with left translations, we get after the change of variable $g \mapsto yx_2g$:

$$\int_B f_1 \ast_{\theta,m} \hat{\alpha}_g(f_2) \, d_{S}(g) = \int B \, e^{2\pi S_{\text{can}}(x_1,x_2)} D_{x_1,x_2} \left[ A_{\theta,m}(x_1,x_2) f_1(yx_1) \alpha_{yx_2g}(f_2(g^{-1})) \right] \, d_{S}(g) \, d_{S}(x_1) \, d_{S}(x_2).$$

By Fubini Theorem, the triple integral above becomes:

$$\int e^{2\pi S_{\text{can}}(x_1,x_2)} D_{x_1,x_2} \left[ A_{\theta,m}(x_1,x_2) f_1(yx_1) \alpha_{yx_2}(f_2(g^{-1})) \right] \, d_{S}(x_1) \, d_{S}(x_2),$$

which means that as expected, we can write

$$\int_B f_1 \ast_{\theta,m} \hat{\alpha}_g(f_2) \, d_{S}(g) = f_1 \ast_{\theta,m} \alpha \left( \int B \, \alpha_g(f_2(g^{-1})) \, d_{S}(g) \right).$$

Now, we observe that the map

$$f \mapsto \int_B \alpha_g(f(g^{-1})) \, d_{S}(g),$$

sends continuously $S^{S_{\text{can}}}(B, A^\infty)$ to $A^\infty$. Indeed, for $X \in \mathcal{U}(b)$, we have

$$X^\alpha \int_B \alpha_g(f(g^{-1})) \, d_{S}(g) = \int_B \alpha_g((\text{Ad}_{g^{-1}}X)^\alpha(f(g^{-1}))) \, d_{S}(g),$$

which entails that

$$\|X^\alpha \int_B \alpha_g(f(g^{-1})) \, d_{S}(g)\| \leq \int_B \|\text{Ad}(g)^{\deg(X)}X^\alpha(f(g^{-1}))\| \, d_{S}(g),$$

which is finite since the group inverse map is continuous on $S^{S_{\text{can}}}(B, A)$. This is enough to conclude in view of Lemma [102].

The sub-$C^*$-algebra of $M(S_{\theta,m}(B, A))$ generated by the operations described in the previous Proposition is called the generalized fixed point algebra of $S_{\theta,m}(B, A)$ and is denoted symbolically by $S_{\theta,m}(B, A)^\alpha$. Also, it is proven in [103] that the linear span of

$$\left\{ \left[ g \in B \mapsto \Delta(g)^{-1/2} f_1 \ast_{\theta,m} \hat{\alpha}_g(f_2^{\theta,m}) \in S^{S_{\text{can}}}(B, A^\infty) \right] : f_1, f_2 \in S^{S_{\text{can}}}(B, A^\infty) \right\},$$

forms a sub-algebra of the crossed product $B \ltimes_{\alpha} S_{\theta,m}(B, A)$, whose closure is strongly Morita equivalent to the generalized fixed point algebra, with equivalence bi-module given by $S_{\theta,m}(B, A)$ itself. Our final task is to show that the on the one hand $S_{\theta,m}(B, A)^\alpha \simeq A_{\theta,m}$ and on the other that the action is saturated, meaning that the algebra generated by the set of functions $f \mapsto \Delta(g)^{-1/2} f_1 \ast_{\theta,m} \hat{\alpha}_g(f_2^{\theta,m})$ is dense in the crossed product $B \ltimes_{\alpha} S_{\theta,m}(B, A)$. Note that here, there is no distinction here between the reduced and full crossed product algebras, since $B$ is solvable and thus amenable.
Lemma 8.44  With the notations as above, we have $\overline{S_{\theta,m}(\mathbb{B}, A)}^\mathbf{\mathcal{A}} \simeq A_{\theta,m}$.

Proof. Call $P : S^{S_{\text{can}}(\mathbb{B}, A^\infty)} \to A^\infty$ the map given in (107). As observed earlier, $P$ sends $S^{S_{\text{can}}(\mathbb{B}, A^\infty)}$ to $A^\infty \subset A_{\theta,m}$. As $\overline{S_{\theta,m}(\mathbb{B}, A)}^\mathbf{\mathcal{A}}$ is the closure of the image of $P(S^{S_{\text{can}}(\mathbb{B}, A^\infty)})$ under the map $\alpha : A^\infty \to B(\mathbb{B}, A)$, in $M(\overline{S_{\theta,m}(\mathbb{B}, A)}^\mathbf{\mathcal{A}})$ and that $B(\mathbb{B}, A) \subset M(\overline{S_{\theta,m}(\mathbb{B}, A)}^\mathbf{\mathcal{A}})$ isometrically, we get that $\overline{S_{\theta,m}(\mathbb{B}, A)}^\mathbf{\mathcal{A}} \subset \overline{A_{\theta,m}(\mathbb{B}, A)}$.

Now, let $\varphi_j \in S^{S_{\text{can}}(\mathbb{B})}$ and $a_j \in A^\infty$, $j = 1, 2$, so that $\varphi_j \otimes a_j \in S^{S_{\text{can}}(\mathbb{B}, A^\infty)}$ and

$$P(\varphi_1 \otimes a_1 *_{\theta,m} \varphi_2 \otimes a_2) = \int_{\mathbb{B}} \alpha_y(a_1 a_2) \varphi_1 \varphi_2(g^{-1}) \, d_\mathbb{B}(g) .$$

By Proposition 5.19 we can let $\varphi_1$ ranges over an approximate unit for the product $*_{\theta,m}$. Thus, $P(\varphi_1 \otimes a_1 *_{\theta,m} \varphi_2 \otimes a_2)$ will converges to

$$\int_{\mathbb{B}} \alpha_y(a_1 a_2) \varphi_2(g^{-1}) \, d_\mathbb{B}(g) ,$$

for the Fréchet topology of $A^\infty$. Let then $\varphi_2$ ranging over approximate $\delta$-functions supported at the neutral element. The latter integral will converges to $a_1 a_2$ for the Fréchet topology of $A^\infty$, and thus for the norm topology of $A_{\theta,m}$. We conclude by the density of $A^\infty. A^\infty$ in $A_{\theta,m}. A_{\theta,m}$ and by the density of products in a $C^*$-algebra.

To prove that the action $\hat{\alpha}$ of $B$ on $\overline{S_{\theta,m}(\mathbb{B}, A)}^\mathbf{\mathcal{A}}$ is saturated, we need an inversion formula for the product $*_{\theta,m}$. This result (for an elementary normal $j$-group) is extracted from [1] but we provide the detailed arguments for the sake of completeness.

Proposition 8.45 Let $m \in \Theta(\mathbb{B})$, $\mathbb{B}$ a normal $j$-group and $f_1, f_2 \in S^{S_{\text{can}}(\mathbb{B}, A)}$. Then for all $z \in \mathbb{B}$, we have the absolutely convergent representation:

$$\int K^\mathbb{B}_{-\theta \frac{m^2}{m}}(x_1, x_2) \left( R^{*}_{x_1} f_1 \right) \left( R^{*}_{x_2} f_2 \right)(z) \, d_\mathbb{B}(x_1) \, d_\mathbb{B}(x_2) = f_1(z) \cdot f_2(z) .$$

Proof. Fix $z \in \mathbb{B}$ and consider the continuous map

$$S^{S_{\text{can}}(\mathbb{B}, A)} \to S^{S_{\text{can}}(\mathbb{B}, A)}, \quad f \mapsto \left[ x \in \mathbb{B} \mapsto (R^{*}_{x} f)(z) = f(zx^{-1}) \in A \right] .$$

Then, using arguments very similar to those leading to Lemmas 2.37 and 2.44, we can deduce that the map

$$(f_1, f_2) \mapsto \left[ (y_1, y_2) \in \mathbb{B} \times \mathbb{B} \mapsto \left[ (x_1, x_2) \mapsto (R^{*}_{y_1} f_1)(z) (R^{*}_{y_2} f_2)(z) \in A \right] \right] ,$$

is continuous from $S^{S_{\text{can}}(\mathbb{B}, A)} \times S^{S_{\text{can}}(\mathbb{B}, A)}$ to $\mathcal{L}(\{\mu\} (\mathbb{B} \times \mathbb{B}, S^{S_{\text{can}}(\mathbb{B} \times \mathbb{B}, A)}))$ for a suitable family $\{\mu_j\}_{j \in \mathbb{N}}$ of tempered weights on $\mathbb{B} \times \mathbb{B}$. Hence, the map

$$(f_1, f_2) \mapsto \left[ (x_1, x_2) \in \mathbb{B} \times \mathbb{B} \mapsto (R^{*}_{x_1} f_1) \left( R^{*}_{x_2} f_2 \right)(z) \right] ,$$

is continuous from $S^{S_{\text{can}}(\mathbb{B}, A)} \times S^{S_{\text{can}}(\mathbb{B}, A)}$ to $S^{S_{\text{can}}(\mathbb{B} \times \mathbb{B}, A)}$, which in turn entails that the map

$$(f_1, f_2) \mapsto \left[ (x_1, x_2) \in \mathbb{B} \times \mathbb{B} \mapsto K^\mathbb{B}_{-\theta \frac{m^2}{m}}(x_1, x_2) \left( R^{*}_{x_1} f_1 \right) \left( R^{*}_{x_2} f_2 \right)(z) \right] ,$$

is continuous from $S^{S_{\text{can}}(\mathbb{B}, A)} \times S^{S_{\text{can}}(\mathbb{B}, A)}$ to $S^{S_{\text{can}}(\mathbb{B} \times \mathbb{B}, A)}$, since the kernel $K^\mathbb{B}_{-\theta \frac{m^2}{m}}$ is tempered. In summary, the quantity

$$\int K^\mathbb{B}_{-\theta \frac{m^2}{m}}(x_1, x_2) \left( R^{*}_{x_1} f_1 \right) \left( R^{*}_{x_2} f_2 \right)(z) \, d_\mathbb{B}(x_1) \, d_\mathbb{B}(x_2) ,$$

is well defined as an absolutely convergent integral for all $z \in \mathbb{B}$. Next, from Remark 2.40 for the restriction of $*_{\theta,m}$ to $S^{S_{\text{can}}(\mathbb{B}, A)} \times S^{S_{\text{can}}(\mathbb{B}, A)}$, we have the (point-wise) absolutely convergent expression:

$$f_1 *_{\theta,m} f_2 = \int_{\mathbb{B} \times \mathbb{B}} K^\mathbb{B}_{\theta,m}(x_1', x_2') R^{*}_{x_1'} (f_1) R^{*}_{x_2'} (f_2) \, d_\mathbb{B}(x_1') \, d_\mathbb{B}(x_2') .$$

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Also, by construction, we have 

$$B_{x,t}(f_1) \ast_B m_{x,t} R^*_{x,t-1}(f_2) \, ds(x_1) \, ds(x_2) =$$

$$K^B_{-\theta, m_{x,t}}(x_1, x_2) \left( \int K^B_{\theta, m}(x_1', x_2') R^*_{x_1', x_2'}(f_1) R^*_{x_2, x_2'}(f_2) \, ds(x_1') \, ds(x_2') \right) \, ds(x_1) \, ds(x_2) =$$

$$K^B_{-\theta, m_{x,t}}(x_1, x_2) \left( \int K^B_{\theta, m}(x_1', x_2') R^*_{x_1', x_2'}(f_1) R^*_{x_2', x_2}(f_2) \, ds(x_1') \, ds(x_2') \right) \Delta^-_{B}(x_1) \Delta^-_{B}(x_2) \, ds(x_1) \, ds(x_2) .$$

The rest of the proof is computational: we are going to show (in the distributional sense and up to numerical pre-factors) that

$$T^B_{x_1, x_2} := \int K^B_{-\theta, m_{x,t}}(x_1, x_2) K^B_{\theta, m}(x_1' x_2', x_2) \Delta^-_{B}(x_1) \Delta^-_{B}(x_2) \, ds(x_1) \, ds(x_2) = \delta^B_{c}(x_1') \delta^B_{c}(x_2') ,$$

where $\delta^B_{c}$ is the Dirac measure on $B$ supported at $g$. We first prove that without loss of generality, we may assume that $B$ is an elementary normal $J$-group. Indeed, let $B' \times_{R} S_1$ be the Pyatetskii-Shapiro decomposition of $B$ and assume that $T^B = \delta^B_{c} \otimes \delta^B_{c}$ and $T^B = \delta^S_{1} \otimes \delta^S_{1}$. As usual, an element $g \in B$ is parametrized as $g = g_1 g'$, where $g' \in B'$, $g_1 \in S_1$. Firstly, we observe that under this decomposition, the modular function of $B$ is the product of modular functions of $B'$ and $S_1$. This follows from the decomposition of left invariant Haar measures $ds = ds' \otimes ds_1$, from the relation (91) in Remark 8.1 and the definition of the modular function:

$$\Delta_{B}(g) = \frac{ds(g^{-1})}{ds(g)} = \frac{ds((R g')^{-1} g^{-1})}{ds(g')} = \frac{ds((R g')^{-1} g^{-1}) ds(g')}{ds_1(g_1) ds_1(g')} = (R g')^{-1} ds_1(g_1) ds_1(g') \Delta_{B}(g') .$$

Also, by construction, we have

$$K^B_{\theta, m}(x, y) = K^B_{\theta, m}(x', y') K^{S_1}_{\theta, m_1}(x_1, y_1) ,$$

and thus

$$K^B_{\theta, m}(x y, s t) = K^B_{\theta, m_1}(x' y', s' t') K^{S_1}_{\theta, m_1}(x_1 R_{x'}(y_1), s_1 R_{s'}(t_1)) .$$

Hence, for $x = x_1 x_2$, $s = s_1 s_2$, $y = y_1 y'$, $t = t_1 t'$, we get from our induction hypothesis:

$$T^B_{x, s} = \int K^B_{-\theta, m_{x,t}}(y', t') K^{S_1}_{\theta, m_1}(y_1, t_1) K^B_{\theta, m}(x' y', s' t') K^{S_1}_{\theta, m_1}(x_1 R_{x'}(y_1), s_1 R_{s'}(t_1)) \times \Delta^-_{B}(y') \Delta^-_{S_1}(y_1) \Delta^-_{B}(t') \Delta^-_{S_1}(t_1) \, ds_1(y_1) \, ds_1(t_1)$$

$$= T^B_{x_2', s_2'} \int K^{S_1}_{-\theta, m_{x_2,t}}(y_1, t_1) K^{S_1}_{\theta, m_1}(x_1 R_{x'}(y_1), s_1 R_{s'}(t_1)) \times \Delta^-_{S_1}(y_1) \Delta^-_{S_1}(t_1) \, ds_1(y_1) \, ds_1(t_1)$$

$$= \delta^B_{c}(x') \delta^B_{c}(t') \int K^{S_1}_{-\theta, m_{x_2,t}}(y_1, t_1) K^{S_1}_{\theta, m_1}(x_1 y_1, s_1 t_1) \times \Delta^-_{S_1}(y_1) \Delta^-_{S_1}(t_1) \, ds_1(y_1) \, ds_1(t_1)$$

$$= \delta^B_{c}(x') \delta^B_{c}(t') T^{S_1}_{x_1, s_1} = \delta^B_{c}(x') \delta^B_{c}(t') \delta^B_{c}(s_1) \delta^B_{c}(s_1) = \delta^B_{c}(x) \delta^B_{c}(t) .$$
Now, for $B = S$ an elementary normal $j$-group, using the explicit expression for the two-point kernels, we find in the coordinates $^{(1)}$ and up to numerical pre-factors:

\[ T^S_{x_1', x_2'} = \int \frac{m(a_1 - a_2)m(-a_1 - a_1')m(a_2 + a_2') m^2(a_1)m^2(a_2)m^2(a_1 + a_1' - a_2 - a_2')} {m(-a_1)m(a_2)m(a_1 + a_1' - a_2 - a_2')} \Delta_5(a_1)\Delta_5(a_2) \times \exp \left\{ \frac{2i}{v} \left( t_1 \left( \sinh(2a_2) - \sinh(2a_2 + 2a_2') \right) - t_2 \left( \sinh(2a_1) - \sinh(2a_1 + 2a_1') \right) \right) \right\} \times \exp \left\{ \frac{2i}{v} \left( \sinh(2a_1 + 2a_1')e^{-2a_1}t_2' - \sinh(2a_2 + 2a_2')e^{-2a_2}t_1' \right) \right\} \times \exp \left\{ \frac{2i}{v} \left( \cosh a_1 \cosh a_2 - \cosh(a_1 + a_1') \cosh(a_2 + a_2') \right) \omega^0(v_1, v_2) \right\} \times \exp \left\{ \frac{i}{v} \left( \sinh(2a_2 + 2a_2')e^{-a_2} \omega^0(v_1, v_1') + 2 \cosh(a_1 + a_1') \cosh(a_2 + a_2')e^{-a_2} \omega^0(v_1, v_2') \right) \right\} \times \exp \left\{ - \frac{2i}{v} \left( \cosh(a_1 + a_1') \cosh(a_2 + a_2')e^{-a_1} \omega^0(v_1', v_2') \right) \right\}.

Integrating out the variables $t_1, t_2$, yields a factor $\cosh(2a_1)^{-1} \cosh(2a_2)^{-1} \delta(a_1') \delta(a_2')$, and thus we get:

\[ T^S_{x_1, x_2} = \delta(a_1') \delta(a_2') \int \frac{m^2(a_1)m^2(a_2)m^2(a_1 + a_2)} {\cosh(2a_1)\cosh(2a_2)\Delta_5(a_1)\Delta_5(a_2)} \exp \left\{ \frac{2i}{v} \left( \sinh(2a_1)e^{-2a_1}t_2' - \sinh(2a_2)e^{-2a_1}t_1' \right) \right\} \times \exp \left\{ \frac{i}{v} \left( \sinh(2a_2)e^{-a_2}v_1' + 2 \cosh(a_1) \cosh(a_2)e^{-a_2}v_2' \right) \right\} \times \exp \left\{ - \frac{2i}{v} \left( \sinh(2a_2)e^{-a_2}v_1 + 2 \cosh(a_2) \cosh(a_1)e^{-a_2}v_2 \right) \right\} \times \exp \left\{ - \frac{2i}{v} \left( \cosh(a_1) \cosh(a_2)e^{-a_1}\omega^0(v_1', v_2') \right) \right\} da_1 da_2 dv_1 dv_2.

Integrating out the variables $v_1, v_2$, yields a factor:

\[ \delta \left( \sinh(2a_2)e^{-a_2}v_1' + 2 \cosh(a_1) \cosh(a_2)e^{-a_2}v_2' \right) \delta \left( \sinh(2a_1)e^{-a_2}v_1' + 2 \cosh(a_1) \cosh(a_2)e^{-a_2}v_2' \right) \delta \left( \sinh(2a_1)e^{-a_2}v_1' + 2 \cosh(a_1) \cosh(a_2)e^{-a_2}v_2' \right) \delta \left( \sinh(2a_1)e^{-a_2}v_1' + 2 \cosh(a_1) \cosh(a_2)e^{-a_2}v_2' \right) \delta \left( \sinh(2a_1)e^{-a_2}v_1' + 2 \cosh(a_1) \cosh(a_2)e^{-a_2}v_2' \right) \delta \left( \sinh(2a_1)e^{-a_2}v_1' + 2 \cosh(a_1) \cosh(a_2)e^{-a_2}v_2' \right) \delta \left( \sinh(2a_1)e^{-a_2}v_1' + 2 \cosh(a_1) \cosh(a_2)e^{-a_2}v_2' \right) \delta \left( \sinh(2a_1)e^{-a_2}v_1' + 2 \cosh(a_1) \cosh(a_2)e^{-a_2}v_2' \right).

Observe that the Jacobian of the map

\[ V \times V \rightarrow V \times V, \]

\[ (v_1', v_2') \mapsto \left( \sinh(2a_2)e^{-a_2}v_1' + 2 \cosh(a_1) \cosh(a_2)e^{-a_2}v_2', \sinh(2a_1)e^{-a_2}v_1 + 2 \cosh(a_1) \cosh(a_2)e^{-a_2}v_2 \right), \]

is proportional to

\[ e^{-2d(a_1 + a_2)} \cosh(a_1)^{2d} \cosh(a_2)^{2d} \cosh(a_1 - a_2)^{2d}. \]

Thus, the former $\delta$ function is proportional to:

\[ e^{2d(a_1 + a_2)} \cosh(a_1)^{-2d} \cosh(a_2)^{-2d} \cosh(a_1 - a_2)^{-2d} \delta(v_1') \delta(v_2'), \]

and consequently,

\[ T^S_{x_1', x_2'} = \delta(a_1') \delta(a_2') \delta(v_1') \delta(v_2') \times \int e^{-2(a_1 + a_2)} \cosh(2a_1 - 2a_2) \exp \left\{ \frac{2i}{v} \left( \sinh(2a_1)e^{-2a_1}t_2' - \sinh(2a_2)e^{-2a_1}t_1' \right) \right\} da_1 da_2.

But the pre-factor $e^{-2(a_1 + a_2)} \cosh(2a_1 - 2a_2)$ in the expression above is exactly the Jacobian of the map:

\[ \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad (a_1, a_2) \mapsto \left( \sinh(2a_1)e^{-2a_2}, -\sinh(2a_2)e^{-2a_1} \right), \]

so that

\[ T^S_{x_1', x_2'} = \delta(a_1') \delta(a_2') \delta(v_1') \delta(v_2') \int \exp \left\{ \frac{2i}{v} \left( u_1 t_2' + u_2 t_1' \right) \right\} du_1 du_2 = \delta(a_1') \delta(a_2') \delta(v_1') \delta(v_2') \delta(t_1) \delta(t_2) = \delta^S(x_1) \delta^S(x_2).

This concludes the proof.\[\square\]
Lemma 8.46 The action $\hat{\alpha}$ of $B$ on $\overline{S_{\theta,m}(B,\Lambda)}$ is saturated, that is, the convolution algebra generated by the elements

$$[g \in B \mapsto \Delta(g)^{-1/2} f_1 *_{\theta,m} \hat{\alpha}_g(f_2 *_{\theta,m} B) \in S_{\text{Scan}}(B, A^\infty) ], \quad f_1, f_2 \in S_{\text{Scan}}(B, A^\infty), \quad (109)$$
is dense in the crossed product $C^*$-algebra $B \ltimes \hat{\alpha} \overline{S_{\theta,m}(B,\Lambda)}$.

Proof. Call $E_0$ the sub-algebra of $B \ltimes \hat{\alpha} \overline{S_{\theta,m}(B,\Lambda)}$ generated by the elements given in (109) and $E$ its closure for the $C^*$-norm of the crossed product algebra. We need to show that $E = B \ltimes \hat{\alpha} \overline{S_{\theta,m}(B,\Lambda)}$. By Lemma 8.42 we know that

$$E_0 \subset S_{\text{Scan}}(B, S_{\text{Scan}}(B, A^\infty)) \subset L^1(B, \overline{S_{\theta,m}(B,\Lambda)}),$$

so that $E \subset B \ltimes \hat{\alpha} \overline{S_{\theta,m}(B,\Lambda)}$. Since $E$ is a two-sided ideal of $B \ltimes \hat{\alpha} \overline{S_{\theta,m}(B,\Lambda)}$ [15] Theorem 1.5], the converse inclusion will clearly follow if we prove the existence of a bounded approximate unit of $B \ltimes \hat{\alpha} \overline{S_{\theta,m}(B,\Lambda)}$ consisting of elements of $E$. But this follows from the following two arguments.

The inversion formula of Proposition 8.45 gives in this context:

$$\Delta(g)^{-1/2} \int K_{\theta,m}^{-1/2} (u, v) R_u^{*_{\theta,m}} (f_1 *_{\theta,m} g) R_v^{*_{\theta,m}} (\hat{\alpha}_g(f_2)) d\mu(u) d\mu(v) = \Delta(g)^{-1/2} f_1 *_{\theta,m} \hat{\alpha}_g(f_2),$$

point-wise for any $f_1, f_2 \in S_{\text{Scan}}(B, A^\infty)$. Since the right regular action $R^*$ preserves $S_{\text{Scan}}(B, A^\infty)$ and commutes with the action $\hat{\alpha}$ given in (109), approximating the Riemann integral above by Riemann sums, shows that the maps

$$[g \in B \mapsto \Delta(g)^{-1/2} f_1 *_{\theta,m} \hat{\alpha}_g(f_2) \in S(B, A^\infty) ], \quad \forall f_1, f_2 \in S(B, A^\infty),$$
belong to $E$ as well. But with such maps, it is easy to construct an approximate unit in $B \ltimes \hat{\alpha} \overline{S_{\theta,m}(B,\Lambda)}$. Indeed, as

$$\int R_u^* (\delta^\theta) L_v^* R_v^* (\delta^\theta) d\mu(u) = \delta^\theta (g) \otimes 1,$$

where 1 above is the constant unit function of $B(\Theta)$, it suffices to consider the net of elements $\Delta(g)^{-1/2} \varphi_\lambda \otimes a_\lambda \hat{\alpha}_g (\varphi_\lambda \otimes a_\lambda)$, where $\varphi_\lambda$ are in $S_{\text{Scan}}(B)$ and approximate the Dirac measure supported at the neutral element, and where $a_\lambda$ is a bounded approximate unit for $A^\infty$.

Corollary 8.47 The deformed $C^*$-algebra $A_{\theta,m}$ is strongly Morita equivalent to the reduced crossed product $B \ltimes \hat{\alpha} \overline{S_{\theta,m}(B,\Lambda)}$.

Proof. By [15] Theorem 1.5], $E$ (as described above) is strongly Morita equivalent to the generalized fixed point algebra $\overline{S_{\theta,m}(B,\Lambda)}$ for the action $\hat{\alpha}$. Then one concludes using Lemmas 8.44 and 8.46.

We are now able to prove the main result of this subsection:

Theorem 8.48 For all $m \in \Theta(B)$ and $\theta \in \mathbb{R}^*$, we have $K_\ast(A_{\theta,m}) \simeq K_\ast(A), \ast = 0, 1$.

Proof. From Corollary 8.47 we have $K_\ast(A_{\theta,m}) \simeq K_\ast(B \ltimes \hat{\alpha} \overline{S_{\theta,m}(B,\Lambda)})$. As $B$ is solvable, we can use the Thom-Connes isomorphism to deduce that $K_\ast(A_{\theta,m}) \simeq K_{\ast+\dim(B)}(\overline{S_{\theta,m}(B,\Lambda)})$, but as $B$ is even dimensional, we get $K_\ast(A_{\theta,m}) \simeq K_\ast(\overline{S_{\theta,m}(B,\Lambda)})$. Last, from Lemma 8.33 we know that $\overline{S_{\theta,m}(B,\Lambda)} \simeq K(\mathcal{H}_\lambda) \otimes A$ and thus $K_\ast(A_{\theta,m}) \simeq K_\ast(A)$ as needed.
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