A SURVEY ON CONNES’ EMBEDDING CONJECTURE

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1 Introduction

In his famous paper [Co], A. Connes formulated a conjecture which is now one of the most important open problem in Operator Algebras. This importance comes from the works
of many mathematicians (above all Kirchberg [Ki], but also Brown [Br], Collins-Dykema [Co-Dy], Haagerup-Winslow [Ha-Wi1, Ha-Wi2], Radulescu [Ra1, Ra2], Voiculescu [Vo2] and many others) who have found some unexpected equivalent statements showing as this conjecture is transversal to almost all the sub-specialization of Operator Algebras.

In this survey I would like to give a more or less detailed description of all these approaches. In the second chapter I am going to recall, more or less briefly, some preliminary notions (ultrafilters, ultraproducts ...) and give the original formulation of the conjecture. In the third one I am going to describe Radulescu’s algebraic approach via hyperlinear groups. In the forth one I am going to describe Haagerup-Winslow’s topological approach via Effros-Marchal topology. In the fifth one I am going to describe Brown’s theorem which connects Connes’ embedding conjecture with Lance’s weak expectation property. In the sixth one I am going to describe briefly other approaches.

2 Preliminary notions and original formulation of the conjecture

The most important preliminary notions are those of ultrafilter and ultraproduct. We recall the following

Definition 1. Let $X$ be a set and $\mathcal{U}$ a non-empty family of subsets of $X$. We say that $\mathcal{U}$ is an ultrafilter if the following properties are satisfied:

1. $\emptyset \notin \mathcal{U}$
2. $A, B \in \mathcal{U}$ implies $A \cap B \in \mathcal{U}$
3. $A \in \mathcal{U}$ implies $B \in \mathcal{U}$, $\forall B \supseteq A$
4. For each $A \subseteq X$ one has either $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$

Ultrafilters are very useful in topology, since they can be thought as a dual notion of net, allowing to speak about convergence in a very general setting. In this overview we are interested only in the concept of limit along an ultrafilter of a family of real numbers.
**Definition 2.** Let \( \{x_a\}_{a \in A} \) be a family of real numbers and \( U \) an ultrafilter on \( A \). We say that \( \lim_U x_a = x \in \mathbb{R} \) if for any \( \varepsilon > 0 \) one has

\[
\{ a \in A : |x_a - x| < \varepsilon \} \in U
\]

**Remark 3.** In order to understand better this notion of convergence, let us consider a convergent sequence of real numbers \( \{x_n\} \). We want to prove that it is convergent along any ultrafilter \( U \). Let us consider separately two cases: the first one is when \( U \) is principal (i.e. there exists \( B \subseteq \mathbb{N} \) such that \( U \) is the collection of the supersets of \( B \)). In this case, one says that \( B \) is a basis for \( U \); the second one is when \( U \) is not principal. In this last case \( U \) is also called free. We need the following classical

**Lemma 4.** Let \( U \) be an ultrafilter on a set \( X \).

1. If \( U \) is principal, its basis is a singleton.

2. If \( U \) is not principal, it cannot contain finite sets.

**Proof.**

1. Let \( B \) the basis for \( U \). If \( B \) is not a singleton, we can take a non trivial partition of \( B \). One and only one of the sets of this partition must belong into \( U \), contradicting the minimality of \( B \).

2. Assuming the contrary, let \( A \in U \) be a finite set. Take \( a \in A \). Then one set between \( \{a\} \) and \( A \setminus \{a\} \) must belong into \( U \). In the first case \( U \) should be principal with basis \( \{a\} \); in the second one we can repeat the argument until to obtain a singleton.

**Note 5.** Notice that we have used that \( \{x_n\} \) is a sequence just to exclude the case \( \mathbb{N} \setminus A_\varepsilon \in U \). A more refined version of the previous argument however shows that every
bounded net is convergent along a given ultrafilter $\mathcal{U}$. In order to prove it one can follow a Bolzano-Weierstrass argument: let $\{x_a\}_{a \in A} \subseteq [-M, M]$, set $R_1 = [-M, 0], R_2 = (0, M]$ and $F_i = \{a \in A : x_a \in R_i\}$. One and only one between $F_1$ and $F_2$ belongs into the ultrafilter $\mathcal{U}$ (if it is $F_2$, we exchange $R_2$ with $\overline{R_2}$ (we find a subset of $A$ which contains $F_2$ and so it still belongs into $\mathcal{U}$)). By repeating this argument, we find a sequence of closed sets $R_n$, whose diameter halves at each step and containing infinitely many elements of the net. Now $\bigcap R_n$ is a singleton $\{x\}$ ant it easy to prove that $\lim_{\mathcal{U}} x_a = x$.

Now we can introduce the notion of ultraproduct. It depends on the algebraic structure of the objects whose we want to make the product. Thus there are many kinds of ultraproduct. We are interested in just two of them: ultraproduct of metric groups and of type $II_1$ factors. In order to define the ultraproduct of a family of metric groups we firstly recall what metric group means.

**Definition 6.** Let $G$ be a group. A bi-invariant metric on $G$ is a metric on $G$ such that

$$d(gx, gy) = d(x, y) = d(xy, yg) \quad \forall x, y, g \in G$$

The pair $(G, d)$ is called metric group.

Similarly one can define left-invariant or right-invariant metrics, but one can find examples (see [Pe] Ex.2.1) that show as these concepts are not good to define the ultraproduct.

**Notation 7.** Let $\{(G_a, d_a)\}_{a \in A}$ be a family of groups equipped with bi-invariant metrics and $\mathcal{U}$ an ultrafilter on the index set $A$. We set

$$G = \{x \in \prod_{a \in A} G_a : \sup_{a \in A} d_a(x_a, 1_{G_a}) < \infty\}$$

In this way, we assure $\lim_{\mathcal{U}} d_a(x_a, 1_{G_a})$ exists for any $x \in G$. So let

$$N = \{x \in G : \lim_{\mathcal{U}} d_a(x_a, 1_{G_a}) = 0\}$$

We have the following

**Lemma 8.** $N$ is a normal subgroup of $G$.

**Proof.** Of course $1_G = \{1_{G_a}\}_{a \in A} \in N$. Let $x, y \in N$, by using the left invariance and the triangle inequality, one has

$$d_a(x_a y_a, 1_{G_a}) = d_a(y_a, x_a^{-1}) \leq d_a(y_a, 1_{G_a}) + d_a(x_a^{-1}, 1_{G_a}) = d_a(y_a, 1_{G_a}) + d_a(x_a, 1_{G_a}) \to 0$$
Similarly one can prove that if \( x \in \mathbb{N} \), then also \( x^{-1} \in \mathbb{N} \). In order to prove the normality of \( \mathbb{N} \) we need the hypothesis of bi-invariance on \( d \) (see [Pe] Ex. 2.1). Let \( x \in G \) and \( n \in \mathbb{N} \). One has

\[
d_a(x_a n_a x_a^{-1}, 1_{G_a}) = d_a(x_a n_a, x_a) = d_a(n_a, 1_{G_a}) \to 0
\]

Thus \( xnx^{-1} \in \mathbb{N} \).

Thus the quotient \( G/N \) is well-defined as a group and it is easy to verify it is a metric group with respect to the bi-invariant metric

\[
d(xN, yN) = \lim_U d_a(x_a, y_a)
\]

Notice that the metric is well-defined, since \( d_a(x_a, y_a) \leq d_a(x_a, 1_{G_a}) + d_a(y_a, 1_{G_a}) \) and thus the net \( d_a(x_a, y_a) \) is bounded. Consequently it converges along every ultrafilter (see Rem[3]).

**Definition 9.** The metric group \( G/N \) is called ultraproduct of the \( G_a \)'s and it is denoted by \( \prod U G_a \).

We will come back to the ultraproduct of metric groups in the next chapter, when we will describe Radulescu’s algebraic approach to the Conjecture. Now we want to present the construction of the ultraproduct of type \( II_1 \) factors \( M_a \), which is

**Definition 10.** Let \( \{(M_a, tr_a)\}_{a \in A} \) a family of type \( II_1 \) factors equipped with normalized traces \( tr_a \) and \( U \) an ultrafilter on \( A \). Set

\[
M = \{ x \in \prod_{a \in A} M_a : \sup| |x_a|| < \infty \}
\]

and

\[
J = \{ x \in M : \lim_U tr_a(x_a^* x_a)^{1/2} = 0 \}
\]

The quotient \( M/J \) turns out to be a factor of type \( II_1 \) with the trace \( tr(x + J) = \lim_U tr_a(x_a) \) (but it is not easy to prove! see [Pe] pg. 18,19 for a sketch, or the original papers by McDuff (MCD) and Janssen (Ja)). It is called ultraproduct of the \( M_a \)'s. The word ultrapower is referred to the case \( M_a = N \) for every \( a \in A \).

The last preliminary notion is a recall of the type \( II_1 \) hyperfinite factor.
Definition 11. A von Neumann algebra $M$ is called approximately finite dimensional (AFD) if it contains an increasing chain of finite dimensional subalgebras whose union is strongly dense in $M$.

It has been already found out by Murray and von Neumann (Mu-vN) that there is substantially a unique (up to von Neumann algebra isomorphism) AFD factor of type $II_1$, denoted by $R$. It is called hyperfinite factor and it is natural to expect that it is the smallest type $II_1$ factor, in the sense that every type $II_1$ factor contains a copy of $R$. Actually, one has it is the smallest factor of infinite dimension (as Banach space). One can also describe it explicitly. Let us recall the following

Definition 12. Let $G$ be a group and $l^2(G)$ the Hilbert space of all square-summable complex-valued functions on $G$. Each $g \in G$ defines an operator $\lambda_g : l^2(G) \to l^2(G)$ in the following way:

$$\lambda_g(f)(x) = f(g^{-1}x)$$

The group von Neumann algebra of $G$, denoted by $VN(G)$, is the strong operator closure of the subalgebra of $B(l^2(G))$ generated by all the $\lambda_g$'s.

Note 13. A group von Neumann algebra is always finite. A trace is determined by the conditions: $tr(1) = 1$ and $tr(g) = 0, \forall g \neq 1$.

Remark 14. Notice that $\lambda_g^* = \lambda_{g^{-1}}$. Thus $\lambda_g \in U(VN(G))$ and the mapping $G \to U(VN(G))$ defined by $g \to \lambda_g$ embeds $G$ into the unitary group of its group von Neumann algebra.

Note 15. We recall two classical results on the group von Neumann algebra: it has separable predual if and only if the group is discrete; it is a factor if and only if the group is i.c.c., i.e. every conjugacy class except $\{1_G\}$ is infinite.

Note 16. A classical result is that the group von Neumann algebra of $S_{\infty}^{fin}$ (the group of all the permutations of $\mathbb{N}$ which fix all but finitely many elements) is the hyperfinite type $II_1$ factor.

Another way to describe the hyperfinite type $II_1$ factor is the following: $R \cong \bigotimes_{n=1}^{\infty} M_2(\mathbb{C})$. Indeed this von Neumann algebras is a finite factor which contains an increasing family of factors whose union is strongly dense (by using $M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \cong M_4(\mathbb{C})$). From this description it follows that $R \otimes R \cong R$, which we will use later.
Remark 17. The notion of separability for von Neumann algebras cannot be given with respect to the norm topology, since it is trivial. Indeed, if $M$ is an infinite dimensional von Neumann algebras, then it contains a countable family of mutually orthogonal projections, with which (by using the borelian functional calculus) it is easy to construct a copy of $l^\infty$ into $M$. So the unique von Neumann algebras which are norm-separable are the finite-dimensional ones.

The right notion of separability for von Neumann algebras is given by the following classical

Proposition 18. Let $M$ be a von Neumann algebras. The following are equivalents:

1. The predual $M_*$ is norm-separable.
2. $M$ is weakly separable
3. $M$ is faithfully representable into $B(H)$, with $H$ separable.

Definition 19. A von Neumann algebras is called separable if it satisfies one of the previous conditions.

After these preliminary notions we are able to enunciate Connes’ embedding conjecture in its original formulation. In order to simplify notations let us denote $\omega$ a generic free ultrafilter on $\mathbb{N}$ and $R^\omega$ the ultrapower of $R$ with respect to $\omega$.

Conjecture 20. (A. Connes, [Co]) Every separable type $II_1$ factor is embeddable into $R^\omega$.

Remark 21. Assuming Continuum Hypothesis, Ge and Hadwin have proved in [Ge-Ha] that all the ultrapowers of a fixed $II_1$ factor with separable predual with regard to a free ultrafilter on the natural numbers are isomorphic among themselves. More recently, Farah, Hart and Sherman have proved also the converse: for any separable type $II_1$ factor $M$ Continuum Hypothesis is equivalent to the statement that all the tracial ultrapowers of $M$ (with regard to a free ultrafilter on the natural numbers) are isomorphic among themselves (see [Fa-Ha-Sh], Th.3.1). On the other hand, ultrapowers with respect a principal ultrafilter are trivial (being isomorphic to the factor itself!). It follows that Continuum Hypothesis together with Connes’ embedding conjecture implies the existence
of a universal type $II_1$-factor; universal in the sense that it should contain every type $II_1$ factor. Ozawa have proved in [Oz2] that such a universal type $II_1$ factor cannot have separable predual.

Fortunately we don’t have this problem

**Proposition 22.** If $\omega$ is non-principal, then $R^\omega$ is not separable.

**Proof.** We have to prove that $R^\omega$ is not faithfully representable into $B(H)$, with $H$ separable. We recall that if $H$ is separable, then all the (classical) topologies on $B(H)$ are separable, except the norm topology (see [Jo]). Moreover, we recall that the strong topology coincide with Hilbert-Schmidt topology on the bounded sets. So it is enough to prove that $R^\omega$ contains a non-countable family of unitaries $\{u(t)\}$ such that $||u(t) - u(s)||_2 = \sqrt{2}$ for all $t \neq s$.

Let $\{u_n\} \subseteq U(R)$ a sequence of distinct unitaries such that $u_n \neq 1$, for all $n \in \mathbb{N}$ and $\tau(u_n^*u_m) = 0$ for all $n \neq m$.

Let $t \in [\frac{1}{10}, 1)$, for instance $t = 0, 132471...$. Define

$$I_t = \{1, 13, 132, 1324, 13247, 132471, ...\}$$

i.e. $I_t$ is the sequence of the approximations of $t$. Clearly, $\{I_t\}_{t \in [\frac{1}{10}, 1)}$ is uncountable and $I_t \cap I_s$ is finite for all $t \neq s$ (this property forces the choice of $t \geq \frac{1}{10}$).

Now define

$$u_1^{(t)} = 1, u_2^{(t)} = u_2, ..., u_{12}^{(t)} = u_{12}, u_{13}^{(t)} = u_{13}, u_{14}^{(t)} = u_2, ..., u_{131}^{(t)} = u_{131-12}, u_{132}^{(t)} = u_1...$$

i.e. every time we find an element of $I_t$, we start again from $u_1$. Now define $u^{(t)} = \prod_{n \in \mathbb{N}} u_n^{(t)}$. Since $I_t \cap I_s$ is finite (for $t \neq s$), then $u^{(t)}$ and $u^{(s)}$ have only a finite number of common components. Thus we have

$$||u^{(t)} - u^{(s)}||_2^2 = \lim_{n \to \infty} \tau_n((u_n^{(t)} - u_n^{(s)})^* (u_n^{(t)} - u_n^{(s)}))$$

where $\tau_n$ is the normalized trace on the $n$-th copy of $R$. Now we observe that

$$\tau_n((u_n^{(t)} - u_n^{(s)})^* (u_n^{(t)} - u_n^{(s)})) = \begin{cases} 0 & \text{if } u_n^{(t)} = u_n^{(s)} \\ 2 & \text{if } u_n^{(t)} \neq u_n^{(s)} \end{cases}$$
Since \( u_n^{(t)} = u_n^{(s)} \) only on a finite set and since \( \mathcal{U} \) is free (and thus it does not contain finite sets), it follows that
\[
\lim_{\mathcal{U}} \tau_n ((u_n^{(t)} - u_n^{(s)})(u_n^{(t)} - u_n^{(s)})) = 2
\]
and thus \( ||u^{(t)} - u^{(s)}||_2 = \sqrt{2} \).

**Note 23.** Non-separability of \( R^\omega \) has been already proved by several authors ([Fe] and, in greater generality, [Po] Prop. 4.3). We have preferred this proof because it is constructive in the sense that will be more clear in the following section: it allows (together with th. [30]) to produce examples of uncountable groups which embed trace-preserving into \( U(R^\omega) \) and to generalize a theorem by Rădulescu (see also [Ca-Pa]).

### 3 The algebraic approach

The idea of the algebraic approach is attaching the following weaker version of Connes’ embedding conjecture.

**Conjecture 24. (Connes’ embedding conjecture for groups)** For every countable i.c.c. group \( G \), the group von Neumann algebra \( VN(G) \) embeds into a suitable ultrapower \( R^\omega \).

Rădulescu in [Ra1] has worked to find a characterization for those groups which satisfy this weaker version of the Conjecture.

**Definition 25.** Let \( U(n) \) be the unitary group of order \( n \), i.e. the group of \( n \times n \) matrices with complex entries and such that \( u^*u = uu^* = 1 \). The normalized Hilbert-Schmidt distance on \( U(n) \) is
\[
d_{HS}(u, v) = ||u - v||_2 = \sqrt{\frac{1}{n} \sum_{i,j=1}^n |u_{ij} - v_{ij}|} = \frac{1}{\sqrt{n}} \sqrt{tr((u - v)(u - v)^*)}
\]

Bi-invariance of this metric follows from the main property of the trace: \( tr(ab) = tr(ba) \).

**Definition 26.** A group \( G \) is called hyperlinear if it embeds into a suitable ultraproduct of unitary groups of finite rank (equipped with Hilbert-Schmidt distance).
Our purpose is to prove the following characterization theorem

**Theorem 27. (Rădulescu)** The following conditions are equivalent

1. Connes’ embedding conjecture for groups is true.
2. Every countable i.c.c. group is hyperlinear.

This theorem was firstly proved by Rădulescu in the countable case. L. Paunescu and the present author have generalized it to the continuous one. We present our result later (see Cor.37). Now we need some preliminary results.

**Lemma 28.** Let \( M = \prod_{\mathcal{U}} M_a \) be a type \( II_1 \) factor obtained as ultraproduct of type \( II_1 \) factors \( M_a \), equipped with normalized traces \( \text{tr}_a \), with regard to an ultrafilter \( \mathcal{U} \) on the index set. Then

\[
U(M) = \prod_{\mathcal{U}} U(M_a)
\]

i.e. the unitary group of the ultraproduct is the ultraproduct of the unitary groups.

**Proof.** The inclusion \( \supseteq \) is obvious, since the multiplication in the ultraproduct is pointwise. Conversely, let \( v_a \in M_a \) such that \( v = \prod_{\mathcal{U}} v_a \) is unitary, i.e. \( \prod_{\mathcal{U}} v_a^* v_a = 1 \). We have to prove that there exist unitaries \( u_a \) such that \( \prod_{\mathcal{U}} u_a = \prod_{\mathcal{U}} v_a \). Let \( v_a = u_a |v_a| \) the polar decomposition of \( v_a \). Since \( M_a \) is a type \( II_1 \) factor, we can extend the partial isometry \( u_a \) to a unitary operator. So we can assume that \( u_a \) is unitary. Now we can verify that they are just the unitaries which we are looking for. Indeed

\[
\prod_{\mathcal{U}} v_a = \prod_{\mathcal{U}} u_a |v_a| = \prod_{\mathcal{U}} u_a \prod_{\mathcal{U}} |v_a| = \prod_{\mathcal{U}} u_a \prod_{\mathcal{U}} (v_a^* v_a)^{1/2} = \prod_{\mathcal{U}} u_a (\prod_{\mathcal{U}} v_a^* v_a)^{1/2} = \prod_{\mathcal{U}} u_a
\]

\[\blacksquare\]

**Proposition 29. (Elek-Szabó, [El-Sz])** Let \( G \) be a group such that for any finite \( F \subseteq G \) and any \( \varepsilon > 0 \) there exist a natural number \( n \) and a map \( \theta : F \to U(n) \) such that

1. if \( g, h, gh \in F \), then \( \| \theta(gh) - \theta(g)\theta(h) \|_2 < \varepsilon \)
2. if \( 1_G \in F \), then \( \| \theta(1_g) - 1_{U(n)} \|_2 < \varepsilon \)
3. for all distinct \( x, y \in F \), \( \| \theta(x) - \theta(y) \| \geq 1/4 \)
Then $G$ is hyperlinear.

Proof. Choosing $\varepsilon = 1/n$, we have a family of maps $\theta_{F,1/n} : F \to U(F,n)$. We set

$$A = \{(F,1/n), F \subseteq G \text{ finite, } n \geq 1\}$$

partially ordered in a natural way. Let $\mathcal{U}$ be a free ultrafilter on $A$ containing every subset of the form $\{(H,1/m) : H \supseteq F, m \geq n\}$. Now we consider the map

$$\theta : G \ni g \to \prod_{\mathcal{U}} \theta_{F,1/n}(g) \in \prod_{\mathcal{U}} U(F,n)$$

We have to prove that this map is a monomorphism. Let $d_{HS}^{F,n}$ and $d_{HS}$ respectively the Hilbert-Schmidt distance on $U(F,n)$ and on the ultraproduct $\prod_{\mathcal{U}} U(F,n)$. We have

$$d_{HS}(\theta(hg) - \theta(h)\theta(g)) = \lim_{\mathcal{U}} d_{HS}^{F,n}(\theta_{F,1/n}(gh), \theta_{F,1/n}(g)\theta_{F,1/n}(h)) \leq \lim_{\mathcal{U}} 1/n$$

Now we use the particular choice of the ultrafilter in order to conclude that the previous limit must be zero. In a similar way (by using the second property) one can easily prove that $\theta$ is unital. Thus it is an homomorphism. Injectivity follows from the third property applied to a similar argument.

\[\square\]

**Note 30.** Also the converse of the previous proposition is true (see [El-Sz]). Moreover, this proposition shows that the notion of hyperlinearity does not depend on the choice of the ultrafilter.

**Remark 31.** The previous proposition can be viewed in the following way: if one can approximate every finite subset of $G$ with a unitary group of finite rank, then $G$ is hyperlinear. A fundamental application of this fact is the following.

**Corollary 32.** Let $(G,d)$ be a metric group containing an increasing chain of subgroups isomorphic to $U(n), n \in \mathbb{N}$, whose union is dense in $G$ and such that $d_{U(n)} = d_{HS}$. Then a group $H$ is hyperlinear if and only if it embeds into a suitable ultraproduct of $G$.

Proof. If $H$ is hyperlinear, then it embeds into a suitable ultraproduct of unitary groups. This ultraproduct, by hypothesis, embeds into the same ultraproduct of $G$. Conversely,
let $\theta$ be the embedding of $H$ into $\prod_u G$. Let $F = \{f_1, \ldots, f_k\} \subseteq H$ and $\varepsilon > 0$. Let $m$ be a natural number and $(\theta(f_i))_m$ the $m$-th component of $\theta(f_i) \in G^\omega$. Since $F$ is finite and $\theta$ is an embedding, we can choose $m$ such that $(\theta(f_i))_m$ are all distinct (take $m$ in the intersection among the sets on which the $\theta(f_i)_m$’s differ). This intersection cannot be empty, since it is finite intersection of sets belonging into $\omega$). By hypothesis, there exist $u_i \in U(n_i)$ such that $d_{HS}(\theta(f_i)_m - u_i) < \varepsilon$. Let $n = \max\{n_i, i = 1, \ldots, k\}$, we can identify $u_i \in U(n)$. Moreover we can assume that $u_i$ are all distinct. So, we can define $\theta_{F,\varepsilon}(f_i) = u_i$: the first two properties are clearly satisfied; the third one is not obvious and one has to follow a trick known as amplification (see [El-Sz] and [Ra1]).

Lemma 33.

\[ R^\omega \otimes R^\omega \subseteq R^\omega \]

Proof. At first we observe that $(R \otimes R)^\omega \cong R^\omega$, by using the isomorphism $(x_n \otimes y_n)_n \rightarrow (\theta(x_n \otimes y_n))_n$, where $\theta$ is an isomorphism between $R \otimes R$ and $R$. It remains to embed $R^\omega \otimes R^\omega$ into $(R \otimes R)^\omega$. We can do this by using the embedding $(x_n)_n \otimes (y_n)_n \rightarrow (x_n \otimes y_n)_n$.

Lemma 34. Let $G$ be an i.c.c. group and $\theta : G \rightarrow U(M)$ a unitary faithfully representation on a finite von Neumann algebra $M$ with trace $\tau$. Then $|\tau(\theta(g))| < 1, \forall g \in G, g \neq 1_G$.

Proof. Certainly $|\tau(\theta(g))| \leq 1$, since the unitary elements have trace in absolute value $\leq 1$. So we assume $\tau(\theta(g_0)) = \lambda$, with $\lambda$ a complex number with norm one, and we prove that $g_0 = 1_G$. Take $u = \lambda^* \theta(g_0)$. This unitary element has trace one and thus it must be the identity. Indeed

\[ \tau((u - 1)^*(u - 1)) = 2 - 2Re(\tau(u)) = 0 \]

So $\theta(g_0) = \lambda 1$ and thus, setting $h = g_0 g^{-1}$, we have $\theta(h) = \lambda$ and thus $h = g_0$ (by the faithfulness). Therefore $g_0$ is the unique element of its conjugacy class and then $g_0 = 1_G$.

Now we can prove Radulescu’s characterization theorem.
Proof. We have to prove that $VN(G)$ embeds into $R^\omega$ if and only if $G$ is hyperlinear. We start assuming that $VN(G)$ embeds into $R^\omega$. Recalling rem.13 we have that $G$ embeds into $U(VN(G))$ and then into $U(R^\omega)$. It follows that $G$ embeds into $\prod_\omega U(R)$ (by the lemma 28). Now we recall that $R$ contains an increasing family of weakly dense finite dimensional von Neumann factors. Thus $U(R)$ contains an increasing family of subgroups isomorphic to $U(n)$ whose union is dense in $U(R)$ (the density follows from the normality of the trace). Of course we have the restriction property of the distance. So we can use Cor.32 to conclude that $G$ must be hyperlinear. Conversely, by the hypothesis and by Cor. 32 $G$ embeds into $U(R^\omega)$. Let $\theta_1$ be such an embedding and $g \in G$, $g \neq 1$. By Lemma 34 we have $|\tau(\theta_1(g))| < 1$. We define a new embedding $\theta_2 = \theta_1 \otimes \theta_1$. This is still an embedding into $U(R^\omega)$, by Lemma 33. Moreover $\tau(\theta_2(g)) = \tau(\theta_1(g))^2$. By induction we can construct a sequence of embedding $\theta_n = \theta_{n-1} \otimes \theta_1$ and we have $\lim |\tau(\theta_n(g))| = 0$ for each $g \neq 1$. Now, since $G$ is countable, we can write $G = \bigcup_{i=1}^{\infty} A_i$, with $A_1 \subseteq A_2 \subseteq \ldots$ are all finite subsets of $G$. For each $k \in \mathbb{N}$ choose $\theta_{n_k}$ such that $|\tau(\theta_{n_k}(g))| < 2^{-k}$ for all $g \in A_k$, $g \neq 1$. Let us denote $\lambda_k = \theta_{n_k}$. Moreover, if $x \in U(R^\omega)$, $x_n$ denotes the $n$-th component of $x$. Lastly $\tau_n$ denotes the trace on the $n$-th copy of $R$. With these notations, we have

$$\lim_\omega |\tau_n(\lambda_k(g)_n)| = |\tau(\lambda_k(g))| < 2^{-k}$$

Thus, there exists $m_k$ such that $\tau_{m_k}(\lambda_k(g)_{m_k}) < 2^{-k}$. We define $\pi_{m_k} : G \to U(R)$ by setting $\pi_{m_k} = \lambda_k(g)_{m_k}$ and lastly $\pi = \prod_\omega \pi_{m_k} : G \to U(R^\omega)$ by setting $\pi(g) = \prod_\omega \pi_{m_k}(g)$. It is still an embedding and verifies the fundamental property that $\tau(\pi(g)) = 0$ for all $g \neq 1$ and $\tau(\pi(1)) = 1$, indeed for $g \neq 1$

$$|\tau(\pi(g))| = \lim_\omega |\tau_k(\pi_k(g)_k)| = \lim_\omega |\tau_k(\lambda_k(g)_{m_k})| = \lim_\omega |\tau_{m_k}(\lambda_k(g)_{m_k})| < \lim_\omega 2^{-k} = 0$$

So the trace of $\pi(g)$ is exactly the trace of $g$, viewed into the group von Neumann algebra (see Note 13). This means just that we can extend this embedding to $VN(G)$ and find an identification between $VN(G)$ and a subalgebra of $R^\omega$.

The previous proof is quite technical and strongly depending on the hypothesis of countability of $G$. We can simplify and extend it to the uncountable case by using a concept of product between ultrafilters. This notion, together with Prop. 22 also allows to prove that the von Neumann algebra of the free group on a continuous family of generators $VN(F_{N_c})$ is embeddable into $R^\omega$. More applications of the product between ultrafilters
In order to prove it, we recall the classical notion of tensor product between ultrafilters
and we prove that \((R_{\omega})_{\omega'} \cong R_{\omega} \otimes_{\omega'} \).

**Definition 35.** Let \(\omega, \omega'\) be two ultrafilters on \(N\). We define
\[ B \in \omega \otimes \omega' \iff \{ k \in N : \{ n \in N : (k, n) \in B \} \in \omega' \} \in \omega \]

**Theorem 36.**
\[(R_{\omega})_{\omega'} \cong R_{\omega} \otimes_{\omega'} \]

**Proof.** Since operations are component-wise we don’t have algebraic problem. We have only to prove that those factors have the same trace. So we have to prove that
\[ \lim_{k \to \omega} \lim_{n \to \omega'} x^k_n = \lim_{(k, n) \to \omega \times \omega'} x^k_n \]
Let \( x = \lim_{k \to \omega} \lim_{n \to \omega'} x^k_n \). Fixed \( \varepsilon > 0 \), we set
\[ A = \{ k \in N : |\lim_{n \to \omega'} x^k_n - x| < \frac{\varepsilon}{2} \} \in \omega \]
and
\[ A_k = \{ n \in N : |x^k_n - \lim_{n \to \omega'} x^k_n| < \frac{\varepsilon}{2} \} \in \omega' \]
So
\[ B = \{ (k, n) \in N^2 : k \in A, n \in A_k \} \subseteq \{ (k, n) \in N^2 : |x^k_n - x| < \varepsilon \} \]
Since \( B \in \omega \otimes \omega' \) the proof is complete. \( \square \)

In the proof of Radulescu’s theorem we have used the fact that a countable group is
hyperlinear if and only if it embeds into \( U(R_{\omega}) \). The only if part is no longer true for
uncountable groups because they can be too big. So, the right way to extend Radulescu’s
theorem to the uncountable case is the following

**Corollary 37.** For an i.c.c. group \( G \), the following statements are equivalent

1. \( G \) (not necessarily countable) embeds into \( U(R_{\omega}) \).
2. \( VN(G) \) is embeddable into \( R_{\omega} \).

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Proof. Radulescu’s proof of the implication 2. ⇒ 1. does not depend on the countability of $G$. Conversely, we can follow Radulescu’s proof and define $\theta_n$. Then, we define $\theta(g) = \{\theta_n(g)\}_{n \in \mathbb{N}}$. It is an embedding into $U((R^\omega)^\omega)$. By Th.36 one can look at $\theta$ as an embedding into $U(R^{\omega \times \omega})$. Now $\tau(\theta(g)) = \lim_\omega \tau(\theta_n(g)) = 0$, whenever $g \neq 1$.

Here is a nice application of the product between ultrafilters and of the construction of Prop.22. Let $F_{\aleph_c}$ be the free group on a continuous family of generators.

**Corollary 38.** $VN(F_{\aleph_c})$ is embeddable into $R^\omega$.

**Proof.** It is enough to prove that $F_{\aleph_c}$ is embeddable into $U(R^\omega)$ and that such an embedding $\theta$ preserves the trace, i.e. $\tau(\theta(g)) = 0$ if $g \neq 1$ and $\tau(\theta(1)) = 1$. Since $F_{\infty}$ (free group countably generated) is hyperlinear, we have a sequence $\{u_n\} \subseteq U(R^\omega)$ such that

1. $\tau(u_n) = 0$ for all $n \in \mathbb{N}$
2. $\tau(u_n^*u_m) = 0$ for all $n \neq m$
3. $u_n$ have no relations between themselves

This sequence is simply the image of the generators of $F_{\infty}$ into $U(R^\omega)$. Now, we apply the construction of the proof of Prop.22. By using Th.36 we find a copy of $F_{\aleph_c}$ into $U(R^\omega)$ such that the desired property on the trace is satisfied.

**Note 39.** In this last note we want to describe briefly the actual situation of the research around hyperlinear groups. Indeed, in the hope that Connes’ embedding conjecture is true, one can try to prove that any group is hyperlinear. This problem is still open, but there are some positive partial results.

1. We recall the following

**Definition 40.** A group is called residually finite if the intersection of all its normal subgroups of finite index is trivial.

Clearly, finite groups are residually finite. A non-trivial example of residually finite groups is given by the free groups (Sa). Anyway
Every residually finite group is hyperlinear (see [Pe], ex. 4.2)

2. We recall the following

**Definition 41.** A group $G$ is called amenable if for any finite $F \subseteq G$ and $\varepsilon > 0$, there exists a finite $\Phi \subseteq G$ such that for every $g \in F$,

$$|g\Phi \Delta \Phi| < \varepsilon |\Phi|$$

**Folner condition**

where $\Delta$ stands for the symmetric difference: $A \Delta B = (A \cup B) \setminus (A \cap B)$.

For instance, compact groups are amenable (One should use an equivalent definition of amenability, linked to the measure theory. Then amenability of compact groups follows from the finiteness of the Haar measure).

Every amenable group is hyperlinear (see [Pe], ex. 4.4).

Gromov introduced in [Gr] the notion of initially subamenable groups: these are groups for which every finite subset can be multiplicatively embedded into an amenable group. By Prop.29 (and its converse) it follows that hyperlinearity is a local property. Thus, also initially subamenable groups are hyperlinear.

The class of initially subamenable groups is the largest among those we have a result about hyperlinearity (residually finite groups are initially subamenable (see the next proposition)). Thom has been the first who had found an example of hyperlinear group which is not initially subamenable (see [Th]).

Let us conclude this section with the proof that every residually finite group is initially subamenable. This fact is well known but it seems to impossible to find references.

**Definition 42.** A group is called initially subfinite if every finite subset can be multiplicatively embedded into a finite group.

Since finite groups are amenable, it is enough to prove the following

**Proposition 43.** Every residually finite group is initially subfinite.
Proof. Let $G$ be a residually finite group and $F \subseteq G$ be a finite subset. We set $\overline{F} = F \cup \{xy, x, y \in F \cup F^{-1}\}$. So $\overline{F}$ is still finite. Now, for each $x \in \overline{F}$ there exists a normal subgroup $G_x \trianglelefteq G$ with finite index not containing $x$ (up to the case $x = 1$). Let $H = \bigcap_{x \in \overline{F}} G_x$. So $\overline{F} \cap H$ is the empty set or the identity. Moreover $H$ is still a normal subgroup of $G$ with finite index (since it is finite intersection of normal subgroups with finite index). Let $\pi : G \to G/H$ the canonical projection. It remains to observe that $\pi|_{\overline{F}}$ is an embedding which preserves the multiplication from $\overline{F}$ into the finite group $G/H$. 

The converse of the previous proposition is false: in [Th] one can find an example of initially subfinite group which is not residually finite.

4 The topological approach

In this section we want to describe the topological approach by Haagerup and Winslow (see [Ha-Wi1] and [Ha-Wi2]). Let $H$ be a Hilbert space and $vN(H)$ the set of von Neumann algebras acting on $H$, this topological approach is based on the definition of a topology on $vN(H)$, named Effros-Marechal topology. Indeed they were the firsts who have introduced this topology and have studied its properties (see [Ef] and [Ma]); but merely Haagerup and Winslow, thirty years later, have argued the link between this topology and Connes’ embedding conjecture.

There are three different ways to describe the Effros-Marechal topology and one can find in [Ha-Wi1] (th.2.8) the proof that these ways are truly equivalent. Here we shall give only the definitions and we shall describe some interesting properties without giving proofs.

Here is the first definition

**Definition 44.** The Effros-Marechal topology on $vN(H)$ is the weakest topology such that for every $\phi \in B(H)$, the mapping

$$vN(H) \ni M \to ||\phi|_M||$$

is continuous.

The second definition of the Effros-Marechal topology come from a more general definition by Effros (see [Ef2]).
Notation 45. Let $X$ be a compact Hausdorff space, $c(X)$ the set of closed subsets of $X$ and $\omega(x)$ the set of the neighborhoods of a point $x \in X$. Let $\{C_a\} \subseteq c(X)$ and

\[
\lim C_a = \{x \in X : \forall U \in \omega(x), U \cap C_a \neq \emptyset \ \text{eventually}\}
\]

\[
\liminf C_a = \{x \in X : \forall U \in \omega(x), U \cap C_a \neq \emptyset \ \text{frequently}\}
\]

Effros has proved that there is only a topology on $c(X)$, whose convergence is described by the conditions

\[C_a \rightarrow C \ \text{iff} \ \lim C_a = \lim C_a = C\]

Since the unit ball $\text{Ball}(M)$ of a von Neumann algebra $M$ is weakly compact, one can use this notion of convergence in our setting.

Definition 46. Let $\{M_a\} \subseteq vN(H)$ be a net. The Effros-Marechal topology is described by the following notion of convergence:

\[M_a \rightarrow M \ \text{iff} \ \overline{\lim \text{Ball}(M_a)} = \overline{\text{lim \text{Ball}(M_a)}} = \text{Ball}(M)\]

The third definition is by introducing a further notion of convergence in $vN(H)$. First of all we need some definitions.

Notation 47. Let $x \in B(H)$, $so^*(x)$ denotes the set of the neighborhoods of $x$ with respect to the strong* topology.

Definition 48. Let $\{M_a\} \subseteq vN(H)$ be a net. We set

\[\liminf M_a = \{x \in B(H) : \forall U \in so^*(x), U \cap M_a \neq \emptyset \ \text{eventually}\}\]

By th. 2.6 in [Ha-Wi1], $\liminf M_a$ can be thought as the largest element in $vN(H)$ whose unit ball is contained in $\overline{\text{lim Ball}(M_a)}$. This suggests to define $\limsup M_a$ as the smallest element in $vN(H)$ whose unit ball contains $\overline{\lim \text{Ball}(M_a)}$, that is clearly $\overline{\text{lim Ball}(M_a)}''$. So we have quite naturally the following

Definition 49. Let $\{M_a\} \subseteq vN(H)$ be a net. We set

\[\limsup M_a = (\overline{\lim \text{Ball}(M_a)})''\]

Now here is the third description of the Effros-Marechal topology
**Definition 50.** The Effros-Marechal topology on $vN(H)$ is described by the following notion of convergence:

$$M_n \to M \quad \text{iff} \quad \liminf M_n = \limsup M_n = M$$

We recall that in [Ha-Wi1], th. 2.8, they have shown that these three definition of the Effros-Marechal topology are equivalent.

Connes’ embedding conjecture regards the behaviour of separable type $II_1$ factors. So we are interested in the case in which $H$ is separable. In this case it happens that the Effros-Marechal topology is metrizable, second countable and complete (i.e. $vN(H)$ is a Polish space). Moreover, a possible distance is given by the Hausdorff distance between the unit balls:

$$d(M, N) = \max\{\sup_{x \in \text{Ball}(M)} \{\inf_{y \in \text{Ball}(N)} d(x, y)\}, \sup_{x \in \text{Ball}(N)} \{\inf_{y \in \text{Ball}(M)} d(x, y)\}\}$$

where $d$ is a metric on the unit ball of $B(H)$ which induces the weak topology (remember that the weak topology on $\text{Ball}(B(H))$ is metrizable whenever $H$ is separable).

There are many interesting results about the Effros-Marechal topology in the case of separability of $H$. For example, the sets of factors of each of the types $I_n, n \in \mathbb{N}, II_1, II_{\infty}, III_\lambda, \lambda \in [0, 1]$ are Borel subsets of $vN(H)$ without being $G_\delta$-sets, or many others (see [Ha-Wi1] sections 4. and 5.). Anyway it is in the second paper [Ha-Wi2] that Haagerup and Winslow have begun studying density problems relative to the Effros-Marechal topology.

What important subsets of $vN(H)$ are dense? They have found a lot of interesting results and, above all, an equivalent condition to Connes’ embedding conjecture.

**Remark 51.** In the case in which $H$ is separable and $\{M_n\} = \{M_n\}$ is a sequence, the definition of the Effros-Marechal topology may be simplified by using the third definition. In particular we have

$$\liminf M_n = \{x \in B(H) : \exists \{x_n\} \in \prod M_n \text{ s.t. } x_n \to s^* x\}$$

**Notation 52.**

$\mathcal{I}_{fin}$ is the set of type I finite factors acting on $H$

$\mathcal{I}$ is the set of type I factors acting on $H$
\( \mathcal{I}_{AFD} \) is the set of approximately finite dimensional factors acting on \( H \)

\( \mathcal{I}_{inj} \) is the set of injective factors acting on \( H \)

We recall that a von Neumann algebra \( M \subseteq B(H) \) is called injective if it is the range of a projection of norm 1. For example every type I von Neumann algebra is injective.

**Theorem 53.** The following statements are equivalent:

1. \( \mathcal{I}_{fin} \) is dense in \( vN(H) \)
2. \( \mathcal{I} \) is dense in \( vN(H) \)
3. \( \mathcal{I}_{AFD} \) is dense in \( vN(H) \)
4. \( \mathcal{I}_{inj} \) is dense in \( vN(H) \)
5. Connes’ embedding conjecture is true

Moreover, a separable type II\(_1\) factor \( M \) is embeddable into \( R^\omega \) if and only if \( M \in \mathcal{I}_{inj} \).

**Proof.** As \( \mathcal{I}_{fin} \subseteq \mathcal{I} \subseteq \mathcal{I}_{AFD} \), the implications 1. \( \Rightarrow \) 2. \( \Rightarrow \) 3. are trivial. The implication 3. \( \Rightarrow \) 1. follows from the fact that AFD factors contain an increasing chain of type \( I_{fin} \) factors, whose union is weakly dense and from the second definition of the Effros-Marechal topology (Def. 46). The equivalence between 3. and 4. is a theorem by A. Connes ([Co]). The equivalence between 4. and 5. is the theorem by Haagerup and Winslow ([Ha-Wi2], Cor. 5.9).

Also the last sentence is proved in [Ha-Wi2] (see Th. 5.8).

Now we want to give a sketch of Haagerup-Winslow’s proof of a theorem by Kirchberg, which gives probably the most unexpected equivalent condition to Connes’ embedding conjecture. Let us recall some concepts on the tensor product of \( C^*\)-algebras. A complete introduction can be found in the forth chapter of the first book by Takesaki ([Ta1]).

**Remark 54.** The algebraic tensor product of two \( C^*\)-algebras is a \(*\)algebra in a natural way, by setting

\[
(x_1 \otimes x_2)(y_1 \otimes y_2) = x_1x_2 \otimes y_1y_2
\]

\[
(x_1 \otimes x_2)^* = x_1^* \otimes x_2^*
\]

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Nevertheless it is not clear how one can define a norm to obtain a $C^*$-algebra (Notice that the product of the norms is not in general a norm on the algebraic tensor product).

**Definition 55.** Let $A_1, A_2$ be two $C^*$-algebras and $A_1 \otimes A_2$ their algebraic tensor product. A norm $\| \cdot \|_\beta$ on $A_1 \otimes A_2$ is called $C^*$-norm if the followings hold

1. $\|xy\|_\beta \leq \|x\|_\beta \|y\|_\beta$, for all $x, y \in A_1 \otimes A_2$
2. $\|x^*x\|_\beta = \|x\|_\beta^2$, for all $x \in A_1 \otimes A_2$

If $\| \cdot \|_\beta$ is a $C^*$-norm on $A_1 \otimes A_2$, $A_1 \otimes_\beta A_2$ stands for the completion of $A_1 \otimes A_2$ with respect to $\| \cdot \|_\beta$. It is a $C^*$-algebra.

Unlucky there is no a unique $C^*$-norm on $A_1 \otimes A_2$ in general, but one can construct by hands at least two of them.

**Definition 56.**

\[ \|x\|_{\max} = \sup\{\|\pi(x)\|, \pi \text{ representation of the } * \text{ algebra } A_1 \otimes A_2\} \]

This norm is called projective or Turumaru’s norm ([Tu]). One can prove that this norm is a $C^*$-norm and the completion of $A_1 \otimes A_2$ with respect to it is denoted by $A_1 \otimes_{\max} A_2$.

The projective norm has the following universal property (see [Ta1], IV.4.7)

**Proposition 57.** Given $C^*$-algebras $A_1, A_2, B$. If $\pi_i : A_i \to B$ are homomorphisms with commuting ranges, then there exists a unique homomorphism $\pi : A_1 \otimes_{\max} A_2 \to B$ such that

1. $\pi(x_1 \otimes x_2) = \pi_1(x_1)\pi_2(x_2)$
2. $\pi(A_1 \otimes_{\max} A_2) = C^*(\pi_1(A_1), \pi_2(A_2))$

**Definition 58.**

\[ \|x\|_{\min} = \sup\{\|(\pi_1 \otimes \pi_2)(x)\|, \pi_1 \text{ representation of } A_1, \pi_2 \text{ representation of } A_2\} \]

This norm is called injective or Guichardet’s norm ([Gu]). One can prove that this norm is a $C^*$-norm and the completion of $A_1 \otimes A_2$ with respect to it is denoted by $A_1 \otimes_{\min} A_2$. 

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Remark 59. Clearly $|| \cdot ||_{\text{min}} \leq || \cdot ||_{\text{max}}$, since representations of the form $\pi_1 \otimes \pi_2$ are particular *representation of the algebraic tensor product $A_1 \otimes A_2$. These norms are different, in general, as Takesaki has shown in [Ta2]. More recently Junge and Pisier have shown, in [Ju-Pi], that $B(l^2) \otimes_{\text{min}} B(l^2) \neq B(l^2) \otimes_{\text{max}} B(l^2)$. Notation $|| \cdot ||_{\text{max}}$ reflects the obvious fact that there are no $C^*$-norm greater than that one. Notation $|| \cdot ||_{\text{min}}$ has the same justification, but it is harder to prove:

Theorem 60. (Takesaki, [Ta2]) $|| \cdot ||_{\text{min}}$ is the smallest $C^*$-norm among those on $A_1 \otimes A_2$.

Definition 61. Let $G$ be a locally compact group. By using the Haar measure, one can consider $L^1(G)$. The universal $C^*$-algebra of $G$ is the enveloping $C^*$-algebra of $L^1(G)$, i.e. the completion of $L^1(G)$ with respect to the norm $||f|| = \sup_\pi ||\pi(f)||$, where $\pi$ runs over all non-degenerate *representation of $L^1(G)$ in a Hilbert space. This norm makes sense by virtue of the classical result: a *homomorphism of an involutive Banach algebra into a $C^*$-algebra is contractive.

Remark 62. Let $\mathbb{F}_\infty$ the free group countably generated. It is a locally compact group with respect to the discrete topology, so we can consider its universal $C^*$-algebra, $C^*(\mathbb{F}_\infty)$. $\mathbb{F}_\infty$ can be canonically embedded into $U(C^*(\mathbb{F}_\infty))$. Unitaries corresponding to such an embedding are called universal.

Here is Kirchberg’s theorem ([Ki]).

Theorem 63. The following statements are equivalent

1. $C^*(\mathbb{F}_\infty) \otimes_{\text{min}} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{\text{max}} C^*(\mathbb{F}_\infty)$

2. Connes’ embedding conjecture is true.

Proof. By using Th.53 we can prove the following implications:

1. If $\mathcal{I}_{\text{jfin}}$ is dense in $vN(H)$, then $C^*(\mathbb{F}_\infty) \otimes_{\text{min}} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{\text{max}} C^*(\mathbb{F}_\infty)$

2. If $C^*(\mathbb{F}_\infty) \otimes_{\text{min}} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{\text{max}} C^*(\mathbb{F}_\infty)$, then $\mathcal{I}_{\text{jfin}}$ is dense in $vN(H)$.
(Proof of 1.)
Let $\pi$ be a representation of the algebraic tensor product $C^*(\mathbb{F}_\infty) \otimes C^*(\mathbb{F}_\infty)$ into $B(H)$. Since $C^*(\mathbb{F}_\infty)$ is separable, we can assume that $H$ is separable. In this way

$$A = \pi(C^*(\mathbb{F}_\infty) \otimes \mathbb{C}1) \quad B = \pi(\mathbb{C}1 \otimes C^*(\mathbb{F}_\infty))$$

belong into $B(H)$, with $H$ separable. Let $\{u_n\}$ be the universal unitaries in $C^*(\mathbb{F}_\infty)$. They are clearly a norm-total sequence. Let $v_n = \pi(u_n \otimes 1) \in A \quad w_n = \pi(1 \otimes u_n) \in B$

Now, let $M = A'' \in \mathcal{N}(H)$. By hypothesis, there exists a sequence $\{F_n\} \subseteq \mathcal{I}_{f_{in}}$ such that $F_n \to M$. So $A \subseteq A'' = \liminf F_n$. Thus we have

$$A \subseteq \liminf F_n$$

Moreover

$$B \subseteq A' = M' = (\limsup F_n)' = \liminf F_n'$$

by the commutant theorem (see [Ha-Wi1] th. 3.5). Now we observe that

$$\{v_n\} \subseteq U(A) \subseteq U(\liminf F_n) = \lim Ball(F_n) \cap U(B(H))$$

where the equality follows from [Ha-Wi1] th.2.6. Let $w(x)$ and $s^*(x)$ respectively the families of weakly and strong* open neighborhoods of an element $x \in B(H)$. We have just proved that for every $n \in \mathbb{N}$ and $W \in w(v_n)$, one has $W \cap Ball(F_n) \cap U(B(H)) \neq \emptyset$ eventually. Now let $S \in s^*(u_n)$. By [Ha-Wi1] Lemma 2.4, there exists $W \in w(v_n)$ such that $W \cap Ball(F_m) \cap U(B(H)) \subseteq S \cap Ball(F_m) \cap U(B(H))$. Now, since the first set must be eventually non empty, also the second one must be the same. This means that we can approximate (in the strong* topology) $v_n$ with elements in $U(F_m)$. So let $\{v_{i,n}\} \subseteq F_n$ such that $v_{i,n} \to s^* v_i$. In a similar way we can find unitaries $w_{i,n}$ in $F'_n$ such that $w_{i,n} \to s^* w_i$. Now let $n$ be fixed, $\pi_{1,n}$ a representation of $C^*(\mathbb{F}_\infty)$ which maps $u_i$ in $v_{i,n}$ and $\pi_{2,n}$ a representation of $C^*(\mathbb{F}_\infty)$ which maps $u_i$ in $w_{1,n}$. We can find these representation because the $u'_i$s have no relations among themselves and because any representation of $G$ extends to a representation of $C^*(G)$. Notice now that the ranges of these representations commute, since $v_n \in A$ and $w_n \in B$, and $A, B$ commute. Moreover, the image of $\pi_{1,n}$ belongs into $C^*(F_n)$ and the image of $\pi_{2,n}$ belongs into $C^*(F'_n)$. So, by the universal
property in Prop. 57, there are unique representations \( \pi_n \) of \( C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty) \) such that
\[
\pi_n(u_i \otimes 1) = v_{i,n} \quad \text{and} \quad \pi_n(1 \otimes u_i) = w_{i,n}
\]
whose image is into \( C^*(F_n, F'_n) \). Now, since \( F_n \) are finite type I factors, one has \( C^*(F_n, F'_n) = F_n \otimes_{\min} F'_n \) and thus \( \pi_n \) splits: \( \pi_n = \sigma_n \otimes \rho_n \), for some \( \sigma_n, \rho_n \) representation of \( C^*(\mathbb{F}_\infty) \) in \( C^*(F_n, F'_n) \). Consequently \( ||\pi_n(x)|| \leq ||x||_{\min} \) for all \( n \in \mathbb{N} \) and \( x \in C^*(\mathbb{F}_\infty) \otimes C^*(\mathbb{F}_\infty) \). On the other hand the sequence \{\( \pi_n \)\} converges to \( \pi \) in a strong* pointwise sense (because \{\( u_n \)\} is total). Therefore
\[
||\pi(x)|| \leq \liminf ||\pi_n(x)|| \leq ||x||_{\min} \quad \forall x \in C^*(\mathbb{F}_\infty) \otimes C^*(\mathbb{F}_\infty)
\]
Since \( \pi \) is arbitrary, it follows that \( ||x||_{\max} \leq ||x||_{\min} \) and the proof of the first implication is complete.

Notice that we had to work with the strong* topology in order to use the inequality \( ||\pi(x)|| \leq \liminf ||\pi_n(x)|| \) which fails in case of weak convergence.

In order to prove 2. we need two preliminary results

Lemma 64. (Haagerup-Winslow, [Ha-Wi2] Lemma 4.3) Let \( A \) be a unital \( C^* \)-algebra and \( \lambda, \rho \) representation of \( A \) in \( B(H) \). Assume \( \rho \) is faithful and satisfies \( \rho \sim \rho \oplus \rho \oplus \ldots \). Then there exists a sequence \( \{u_n\} \subseteq U(B(H)) \) such that
\[
u_n \rho(x) u_n^* \rightarrow^{\ast} \lambda(x) \quad \forall x \in A
\]

Theorem 65. (Choi, [Ch] th.7) Let \( \mathbb{F}_2 \) be the free group with two generators. Then \( C^*(\mathbb{F}_2) \) has a separating family of finite dimensional representations.

(Proof of 2.)
By using Choi’s theorem and the classical embedding of \( \mathbb{F}_\infty \) into \( \mathbb{F}_2 \), we can find a sequence \( \sigma_n \) of finite dimensional representations of \( C^*(\mathbb{F}_\infty) \) such that \( \sigma = \sigma_1 \oplus \sigma_2 \oplus \ldots \) is faithful. Replacing \( \sigma \) with the direct sum infinitely many times of itself, we may assume that \( \sigma \sim \sigma \oplus \sigma \oplus \ldots \). Moreover, by [Ta1] IV.4.9, \( \rho = \sigma \otimes \sigma \) is a faithful representation of \( C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty) \) (because \( \rho \) is factorizable). This representation still satisfies \( \rho \sim \rho \oplus \rho \oplus \ldots \). Furthermore, since it is direct sum of finite dimensional representations, it
is separable and thus we may assume that its image is into $B(H)$, with $H$ separable. Now, given $M \in vN(H)$, let $\{v_n\}, \{w_n\}$ be strong* dense sequences of unitaries in $Ball(M)$ and $Ball(M')$, respectively. Let $\{z_n\}$ be the universal unitaries representing $F_\infty$ in $C^*(F_\infty)$. Now, by hypothesis and by using Prop.57, we have a unique representation $\lambda$ of $C^*(F_\infty) \otimes_{min} C^*(F_\infty)$ such that 

$$
\lambda(z_n \otimes 1) = v_n \quad \text{and} \quad \lambda(1 \otimes z_n) = w_n \quad \forall n \in \mathbb{N}
$$

Let us now observe that $M = \lambda(C^*(F_\infty) \otimes \mathbb{C}1)''$, since $\{v_n\}$ is dense in $Ball(M)$. Now, by lemma 64, we have unitaries $u_n \in U(B(H))$ such that 

$$
\rho(x) u_n^* \rightarrow^* \lambda(x) \quad \forall x \in C^*(F_\infty) \otimes_{min} C^*(F_\infty)
$$

Define 

$$
M_n = u_n \rho(C^*(F_\infty) \otimes \mathbb{C}1)' u_n^*
$$

then 

$$
u_n \rho(\mathbb{C}1 \otimes C^*(F_\infty)) u_n^* \subseteq M_n'
$$

So we have (by using Rem.51) 

$$
\lambda(C^*(F_\infty) \otimes \mathbb{C}1) = \liminf u_n \rho(C^*(F_\infty) \otimes \mathbb{C}1) u_n^* \subseteq \liminf M_n
$$

In a similar way, we obtain 

$$
\lambda(\mathbb{C}1 \otimes C^*(F_\infty)) \subseteq \liminf M_n'
$$

Now, by [Ha-Wi1] th.2.3, $\liminf M_n$ is always a von Neumann algebra, and thus the previous inclusions hold by passing to the strong closure: 

$$
M = \lambda(C^*(F_\infty) \otimes \mathbb{C}1)'' \subseteq \liminf M_n
$$

and 

$$
M' = \lambda(\mathbb{C}1 \otimes C^*(F_\infty))'' \subseteq \liminf M_n'
$$

Now, applying the commutant theorem ($\liminf M_n)' = \limsup M_n'$ (see [Ha-Wi1], th.3.5), we have $M_n \rightarrow M$. Now, we observe that $\rho$ is a type I representation, since it is direct sum of finite dimensional representations, and thus $M_n \in vN_I(H)$. Thus we have just proved that $vN_I(H)$ is dense in $vN(H)$. In particular $vN_{inj}(H)$ is dense in $vN(H)$. Now
it has been already proved by Haagerup and Winslow that $vN_{\text{inj}}$ and $\Im(H)$ (factors into $B(H)$) are $G_\delta$ and $\Im(H)$ is dense. On the other hand, $vN(H)$ is a Polish space and hence a Baire’s space. So, also the intersection $vN_{\text{inj}}(H) \cap \Im(H) = \Im_{\text{inj}}(H)$ must be dense. □

Notice that in the proof of 2. we have used the hypothesis only to apply Prop. We need it to have $\lambda$ and $\rho$ defined on the same $C^*$-algebra and so apply Lemma.

**Note 66.** One can ask what groups $G$ satisfy Kirchberg’s property

$$C^*(G) \otimes_{\min} C^*(G) = C^*(G) \otimes_{\max} C^*(G)$$

or the reduced Kirchberg’s property

$$C^*_r(G) \otimes_{\min} C^*_r(G) = C^*_r(G) \otimes_{\max} C^*_r(G)$$

where $C^*_r(G)$ is the reduced $C^*$-algebra of $G$, i.e. the $C^*$-algebra generated by the image of the left regular representation on $l^2(G)$. Let us denote by $K$ and $K_r$ respectively the classes of group which satisfy Kirchberg’s property and reduced Kirchberg’s property. It follows from a more general result by Conti and Hamhalter (see [Co-Ha]) that

$$K_r \cap \{\text{i.c.c. groups}\} = \{\text{amenable groups}\}$$

What can we say about $K$? Are there any non-amenable examples?

5 Lance’s WEP and QWEP conjecture

Kirchberg’s theorem shows an interesting and unexpected link between von Neumann algebras and $C^*$-algebras. Kirchberg himself, in [Ki] again, has found another interesting link between them; more precisely: Connes’ embedding conjecture is a particular case of a conjecture regarding the structure of $C^*$-algebras: QWEP conjecture. We remind the reader that a $C^*$-algebra is QWEP if it is a quotient of a $C^*$-algebra with Lance’s WEP. So it is natural to ask if there is a direct relation between Connes’ embedding conjecture and WEP. N.P. Brown has found in [Br] that Connes’ embedding conjecture is equivalent to the analogue of Lance’s WEP for separable type $II_1$ factors.

In order to give some details about this let us firstly recall the following
Definition 67. Let $A, B$ be $C^*$-algebras and $\phi : A \to B$ a linear map. For every $n \in \mathbb{N}$ we can define a map $\phi_n : M_n(A) \to M_n(B)$ by setting

$$\phi_n[a_{ij}] = [\phi(a_{ij})].$$

$\phi$ is called completely positive if $\phi_n$ is positive for every $n$.

Note 68. Any *homomorphism between two $C^*$-algebras is automatically c.p. Indeed it is clearly positive. On the other hand $\phi_n$ can be described as $\phi \otimes Id_n$ and thus it is still a *homomorphism, since tensor product of *homomorphisms is still a *homomorphism.

Definition 69. Let $A \subseteq B$ be two $C^*$-algebras. We say that $A$ is weakly cp complemented in $B$ if there exists a unital completely positive map $\phi : B \to A^{**}$ such that $\phi|_A = Id_A$.

Definition 70. We say that a $C^*$-algebra $A$ has the WEP (weak expectation property) if it is weakly cp complemented in $B(H)$ for a faithful representation $A \subseteq B(H)$.

This property is been introduced by Lance in [La], where he proved also that this definition does not depend on the choice of the faithful representation of $A$.

Definition 71. We say that $A$ has QWEP if it is a quotient of a $C^*$-algebra with WEP.

Here is QWEP conjecture, regarding the structure of a $C^*$-algebra.

Conjecture 72. (QWEP conjecture) Every $C^*$-algebra is QWEP.

The unexpected theorem by Kirghberg is

Theorem 73. (Kirchberg) The following statements are equivalent

1. Connes’ embedding conjecture is true.

2. QWEP conjecture is true for separable von Neumann algebras.

A proof of this theorem can be found in the original paper by Kirchberg [Ki] or also in [Oz]. Now we prefer to focus on an easier and equally interesting topic: the von Neumann algebraic analogue of Lance’s WEP and the proof of Brown’s theorem. What follows is just a rewriting of Brown’s paper [Br].
Definition 74. Let $M \subseteq B(H)$ a von Neumann algebra and $A \subseteq M$ a weakly dense $C^*$-subalgebra. We say that $M$ has a weak expectation relative to $A$ if there exists a u.c.p. map $\Phi : B(H) \to M$ such that $\Phi(a) = a$, for all $a \in A$.

Note 75. The notion of injectivity for von Neumann algebras can be given also in the following way: $M \subseteq B(H)$ is injective if there exists a u.c.p. map $\Phi : B(H) \to M$ such that $\Phi(x) = x$, for all $x \in M$. So weak expectation relative property is something less than injectivity. Actually something more precise holds: Brown’s theorem can be read by saying that weak expectation relative property is the limit property of injectivity. We can clarify this interpretation after enunciating the following

Theorem 76. (Brown, [Br] Th.1.2) For a separable type $II_1$ factor $M$ the following conditions are equivalent:

1. $M$ is embeddable into $R^\omega$.
2. $M$ has a weak expectation relative to some weakly dense subalgebra.

We can now clarify the interpretation of the weak expectation relative property as limit of injectivity.

Corollary 77. For a separable type $II_1$ factor the following conditions are equivalent:

1. $M$ has a weak expectation relative property.
2. $M$ is Effros-Marechal limit of injective factors.

Proof. It is an obvious consequence of Th.76 and Th.53.

Our purpose is to present the original proof of Th.76. We need some preliminary result.

For the rest of the chapter let $A$ be a separable $C^*$-algebra. This hypothesis is not necessary, but it is convenient.

Definition 78. A tracial state on $A$ is map $\tau : A_+ \to [0, \infty]$ such that

1. $\tau(x + y) = \tau(x) + \tau(y)$, for all $x, y \in A_+$
2. \( \tau(\lambda x) = \lambda \tau(x) \), for all \( \lambda \geq 0, x \in A_+ \)

3. \( \tau(x^*x) = \tau(xx^*) \) for all \( x \in A \)

4. \( \tau(1) = 1 \)

It clearly extends to a positive functional on the whole \( A \).

**Definition 79.** A tracial state \( \tau \) on \( A \subseteq B(H) \) is called *invariant mean* if there exists a state \( \psi \) on \( B(H) \) such that

1. \( \psi(uTu^*) = \psi(T) \), for all \( u \in U(A) \) and \( T \in B(H) \)

2. \( \psi|_A = \tau \)

**Note 80.** A consequence of Th.84 is that the notion of invariant mean does not depend on the choice of the faithful representation \( A \subseteq B(H) \).

In order to prove Brown’s theorem we need a characterization of invariant means.

We recall the following well-known

**Theorem 81.** (Powers-Størmer inequality, \cite{Po-St}) *Let \( h, k \in L^1(B(H))_+ \). Then*

\[
||h - k||_2^2 \leq ||h^2 - k^2||_1
\]

*where \( || \cdot ||_i \) stands for the \( L^i \) norm on \( L^1(B(H)) \) with respect to the canonical unbounded trace \( Tr \). In particular, if \( u \in U(B(H)) \) and \( h \geq 0 \) has finite rank, then*

\[
||uh^{1/2} - h^{1/2}u||_2 = ||uh^{1/2}u^* - h^{1/2}||_2 \leq ||uhu^* - h||_1^{1/2}
\]

**Lemma 82.** *Let \( H \) be a separable Hilbert space and \( h \in B(H) \) a positive, finite rank operator with rational eigenvalues and \( Tr(h) = 1 \). Then there exists a u.c.p. map \( \Phi : B(H) \to M_q(\mathbb{C}) \) such that*

1. \( tr(\Phi(T)) = Tr(hT) \), for all \( T \in B(H) \)

2. \( |tr(\Phi(uu^*) - \Phi(u)\Phi(u^*))| < 2||uhu^* - h||_1^{1/2} \), for all \( u \in U(B(H)) \)

*Here \( tr \) stands for the normalized trace on \( M_q(\mathbb{C}) \).*
Proof. Let \( v_1, \ldots v_k \in H \) be the eigenvectors of \( H \) and \( \frac{p_i}{q}, \ldots \frac{p_k}{q} \) the corresponding eigenvalues. Thus

1. \( hv_i = \frac{p_i}{q} \)

2. \( \sum_{i=1}^k \frac{p_i}{q} = tr(h) = 1 \). It follows that \( \sum p_i = q \)

Let \( \{w_m\} \) be any orthonormal basis for \( H \). Consider the orthogonal subset of \( H \otimes H \):

\[
B = \{v_1 \otimes w_1, \ldots v_1 \otimes v_{p_1}\} \cup \{v_2 \otimes w_1, \ldots v_2 \otimes w_{p_2}\} \cup \ldots \cup \{v_k \otimes w_1, \ldots v_k \otimes w_{p_k}\}
\]

Let \( V \) be the subspace of \( H \otimes H \) spanned by \( B \) and \( P : H \otimes H \rightarrow V \) the orthogonal projection. Let \( T \in B(H) \), the following formula holds

\[
Tr(P(T \otimes 1)P) = \sum_{i=1}^k p_i < Tv_i, v_i >
\]

Indeed \( P(T \otimes 1)P \) is representable (in the basis \( B \)) by a \( q \times q \) block diagonal matrix whose blocks have dimension \( p_i \) with entries \( ETE \), where \( E : H \rightarrow \text{span}\{v_1, \ldots v_k\} \) is the projection. Now define \( \Phi : B(H) \rightarrow M_q(\mathbb{C}) \) by setting \( \Phi(T) = P(T \otimes 1)P \). We have

\[
tr(\Phi(T)) = \frac{1}{q} Tr(P(T \otimes 1)P) = \sum_{i=1}^k < T\frac{p_i}{q}v_i, v_i > = \sum_{i=1}^k < Thv_i, v_i > = Tr(Th)
\]

Moreover \( \Phi \) is u.c.p. So the first assertion is proved.

Now, by writing down the matrix of \( P(T \otimes 1)P(T^* \otimes 1)P \) in the basis \( B \) we have

\[
Tr(P(T \otimes 1)P(T^* \otimes 1)P) = \sum_{i,j=1}^k |T_{i,j}|^2 min(p_i, p_j)
\]

where \( T_{i,j} = < T v_j, v_i > \). Analogously, by writing down the matrices of \( h^{1/2}T, Th^{1/2} \) and \( h^{1/2}Th^{1/2}T^* \) in any orthonormal basis which begins with \( \{v_1, \ldots v_k\} \) we have

\[
Tr(h^{1/2}Th^{1/2}T^*) = \sum_{i,j=1}^k \frac{1}{q} (p_i p_j)^{1/2} |T_{i,j}|^2
\]

By using these formulas, we can make the following preliminary calculation

\[
|Tr(h^{1/2}Th^{1/2}T^*) - tr(\Phi(T)\Phi(T^*))| =
\]
$$= |\sum_{i,j=1}^{k} \frac{1}{q}(p_ip_j)^{1/2}|T_{i,j}|^2 - \frac{1}{q}Tr(P(T \otimes 1)P(T^* \otimes 1)P)| =$$
$$= |\sum_{i,j=1}^{k} \frac{1}{q}|T_{i,j}|^2((p_ip_j)^{1/2} - \min(p_i,p_j))| \leq$$

by using $\min(p_i,p_j) \leq p_i$

$$\leq \sum_{i,j=1}^{k} \frac{1}{q}|T_{i,j}|^2p_i^{1/2}|p_j^{1/2} - p_i^{1/2}| \leq$$

by using the Holder inequality

$$\leq (\sum_{i,j=1}^{k} \frac{1}{q}|T_{i,j}|^2p_i^{1/2})(\sum_{i,j=1}^{k} \frac{1}{q}|T_{i,j}|^2(p_i^{1/2} - p_j^{1/2}))^{1/2} =$$

$$= ||Th^{1/2}||_2||h^{1/2}T - Th^{1/2}||_2$$

suppose now that $T \in U(B(H))$, so that $||Th^{1/2}||_2 = ||h^{1/2}||_2 = 1$

$$= ||h^{1/2}T - Th^{1/2}||_2 = ||Th^{1/2}T^* - h^{1/2}||_2 \leq$$

by using the Powers-Størmer inequality

$$\leq ||ThT^* - T||_1^{1/2}$$

Now we can prove the second assertion. Indeed we have

$$|Tr(\Phi(TT^*) - \Phi(T)\Phi(T^*))| \leq$$

by using the triangle inequality and the previous calculation

$$|1 - Tr(h^{1/2}T^{1/2}T^*)| + ||ThT^* - h||_1^{1/2} =$$

$$= |Tr(T(T^*) - Tr(h^{1/2}T^{1/2}T^*))| + ||ThT^* - h||_1^{1/2} =$$

$$= |Tr((Th^{1/2} - h^{1/2}T)h^{1/2}T^*)| + ||ThT^* - h||_1^{1/2} \leq$$

by using the Cauchy-Schwarz inequality

$$\leq ||h^{1/2}T^*||_2||h^{1/2}T - Th^{1/2}||_2 + ||ThT^* - h||_1^{1/2}$$

So the assertion follows by using $T \in U(B(H)), Tr(h) = 1$ and by applying the Powers-Størmer inequality once more.
We recall a classical theorem by Choi

**Theorem 83.** (Choi, [Ch2]) Let $A, B$ be two $C^*$-algebras and $\Phi : A \to B$ a u.c.p. map. Then

$$\{a \in A : \Phi(aa^*) = \Phi(a)\Phi(a^*), \Phi(a^*a) = \Phi(a^*)\Phi(a)\} =$$

$$= \{a \in A : \Phi(ab) = \Phi(a)\Phi(b), \Phi(ba) = \Phi(b)\Phi(a), \forall b \in A\}$$

Here is the characterization of invariant means. Other ways to characterize them are in [Br2], Th.3.1, and in [Oz], Th. 6.1.

**Theorem 84.** Let $\tau$ be a tracial state on $A \subseteq B(H)$. Then the followings are equivalent:

1. $\tau$ is an invariant mean.
2. There exists a sequence of u.c.p. maps $\Phi_n : A \to M_{k(n)}(\mathbb{C})$ such that
   
   (a) $||\Phi_n(ab) - \Phi_n(a)\Phi_n(b)||_2 \to 0$ for all $a, b \in A$
   
   (b) $\tau(a) = \lim_{n \to \infty} \text{tr}(\Phi_n(a))$, for all $a \in A$

3. For any faithful representation $\rho : A \to B(H)$ there exists a u.c.p. map $\Phi : B(H) \to \pi_\tau(A)''$ such that $\Phi(\rho(a)) = \pi_\tau(a)$, for all $a \in A$, where $\pi_\tau$ stands for the GNS representation associated to $\tau$.

**Proof.** (1 $\Rightarrow$ 2)

Let $\tau$ be an invariant mean with respect to the faithful representation $\rho : A \to B(H)$. Thus we can find a state $\psi$ on $B(H)$ which extends $\tau$ and such that $\psi(uTu^*) = \psi(u)$, for all $u \in U(A)$ and for all $T \in B(H)$. Since the normal states are dense in $B(H)$ and they are represented in the form $\text{Tr}(h \cdot)$, with $h \in L^1(B(H))$, we can find a net $h_\lambda \in L^1(B(H))$ such that $\text{Tr}(h_\lambda T) \to \psi(T)$, for all $T \in B(H)$. Moreover we remind that $h_\lambda$ is positive and has trace 1. Now, since $\psi(uTu^*) = \psi(u)$, it follows that $\text{Tr}(uh_\lambda u^*T) = \text{Tr}(h_\lambda u^*Tu) \to \psi(u^*Tu) = \psi(T)$ and thus $\text{Tr}(h_\lambda T) - \text{Tr}((uh_\lambda u^*)T) \to 0$, for all $T \in B(H)$, i.e. $h_\lambda - uh_\lambda u^* \to 0$ in the weak topology on $L^1(B(H))$. Now let $\{U_n\}$ be an increasing family of finite sets of unitaries whose union have dense linear span in $A$ and $\varepsilon = \frac{1}{n}$. Let $U_n = \{u_1, \ldots, u_n\}$. Fixed $n$, let us consider the convex hull of the set $\{u_1h_\lambda u_1^* - h_\lambda, \ldots, u_nh_\lambda u_n^* - h_\lambda\}$. Its weak closure contains 0 (because of the previous observation) and coincide with the 1-norm closure, by the Hahn-Banach separation theorem. Thus there exists a convex combination of $h_\lambda$’s, say $h$, such that

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1. $Tr(h) = 1$

2. $||uhu^* - h||_1 < \varepsilon, \forall u \in U_n$

3. $|Tr(uh) - \tau(u)| < \varepsilon, \forall u \in U_n$

Moreover, since finite rank operators are norm dense in $L^1(B(H))$, we can suppose that $h$ is finite rank with rational eigenvalues. Now we can apply Lemma 12 in order to construct a sequence of u.c.p. maps $\Phi_n : B(H) \to M_{k(n)}(\mathbb{C})$ such that

1. $Tr(\Phi_n(u)) \to \tau(u)$

2. $|Tr(\Phi_n(uu^*)) - \Phi_n(u)\Phi_n(u^*)| \to 0$

for every unitary in a countable set whose linear span is dense in $A$. So we have obtained the thesis for unitaries. The second property holds for any $a \in A$, by passing to linear combinations. In order to obtain the first one, we observe that $\Phi_n(uu^*) - \Phi_n(u)\Phi_n(u^*) \geq 0$ and thus the following inequality holds

$$||1 - \Phi_n(u)\Phi_n(u^*)||_2^2 \leq ||1 - \Phi_n(u)\Phi_n(u^*)||tr(\Phi_n(uu^*) - \Phi_n(u)\Phi_n(u^*))$$

and the right hand side tends to zero. Now define $\Phi = \oplus \Phi_n : A \to \Pi M_{k(n)}(\mathbb{C}) \subseteq l^\infty(R)$ and compose with the quotient map $p : l^\infty(R) \to R^\omega$. The previous inequality shows that if $u$ is a unitary such that $||\Phi_n(uu^*) - \Phi_n(u)\Phi_n(u^*)||_2 \to 0$ and $||\Phi_n(u^*u) - \Phi_n(u^*)\Phi_n(u)||_2 \to 0$, then $u$ falls in the multiplicative domain of $p \circ \Phi$. But such unitaries have dense linear span in $A$ and hence the whole $A$ falls in the multiplicative domain of $p \circ \Phi$ (by Choi’s theorem 33). By definition of ultraproduct this just means that $||\Phi_n(ab) - \Phi_n(a)\Phi_n(b)||_2 \to 0$, for all $a \in A$.

(2 $\Rightarrow$ 3)

Let $\Phi_n : A \to M_{k(n)}(\mathbb{C})$ be a sequence of u.c.p. maps with the properties stated in the theorem. Identify each $M_{k(n)}(\mathbb{C})$ with a unital subfactor of $R$ and we can define a u.c.p map $\hat{\Phi} : A \to l^\infty(R)$ by $x \to (\Phi_n(x))_n$. Since the $\Phi'_n$s are asymptotically multiplicative in 2-norm one get a $\tau$-preserving *homomorphism $A \to R^\omega$ by composing with the quotient map $p : l^\infty(R) \to R^\omega$. Note that the weak closure of $p \circ \hat{\Phi}(A)$ into $R^\omega$ is isomorphic to
Thus we are in the following situation

\[ A \xrightarrow{\Phi} l^\infty(R) \xrightarrow{p} R^\omega \subseteq p \circ \Phi(A)'' \cong \pi_\tau(A)'' \]

where \( K \) is a representing Hilbert space for \( l^\infty(R) \) and \( i \) is a natural embedding (\( K \) cannot be separable). Now \( l^\infty(R) \) is injective and let \( E : B(K) \to l^\infty(R) \) a surjective projection of norm 1. Moreover let \( F : R^\omega \to \pi_\tau(A)'' \) a conditional expectation (see [Ta1], Prop.2.36). Thus we are in the following situation

\[ A \xrightarrow{\Phi} l^\infty(R) \xrightarrow{p} R^\omega \xrightarrow{E} \pi_\tau(A)'' \cong p \circ \Phi(A)'' \]

Define \( \Phi : B(H) \to \pi_\tau(A)'' \) by setting \( \Phi = FpEi \). Clearly \( \Phi(\rho(a)) = \pi_\tau(a) \).

(\( 3 \Rightarrow 1 \))
The hypothesis \( \Phi(a) = \pi_\tau(a) \) guarantees that \( \Phi \) is multiplicative on \( A \). By Choi’s theorem \([83]\) it follows that \( \Phi(aTb) = \pi_\tau(a)\Phi(T)\pi_\tau(b) \), for all \( a,b \in A, T \in B(H) \). Let \( \tau'' \) be the vector trace on \( \pi_\tau(A)'' \) and consider \( \tau'' \circ \Phi \). Clearly it extends \( \tau \). Moreover it is invariant under the action of \( U(A) \), indeed

\[ (\tau'' \circ \Phi)(u^*Tu) = \tau''(\pi_\tau(u)^*\Phi(T)\pi_\tau(u)) = \tau''(\Phi(T)) = (\tau'' \circ \Phi)(T) \]

Hence \( \tau \) is an invariant mean.

Another preliminary but very nice result is the following

**Proposition 85.** Let \( M \) be a separable type \( II_1 \) factor. There exists a *-monomorphism \( \rho : C^*(\mathbb{F}_\infty) \to M \) such that \( \rho(C^*(\mathbb{F}_\infty)) \) is weakly dense in \( M \).
Proof. We first observe that $C^*(F_\infty)$ is inductive limit of free products of itself. It can be imagined by partitioning the set of generators in a sequence of countable set (one can do it because $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$). Let $\{X_n\}$ such a sequence. Define $A_n = C^*(X_1,\ldots,X_n)$. Clearly one has $A_n = A_{n-1} \ast C^*(X_n)$, where $\ast$ stands for the free product with amalgamation over the scalar. Moreover $C^*(X_n) \cong C^*(F_\infty)$, and then $A_n \cong A_{n-1} \ast C^*(F_\infty)$. Now let $A$ be the inductive limit of the $A_n$s. Clearly $A = \bigcup A_n = C^*(X_1, X_2, \ldots) \cong C^*(F_\infty)$. Now, by Choi’s theorem we can find a sequence of integers $\{k(n)\}$ and a unital $^*$-monomorphism $\sigma : A \to \Pi M_{k(n)}(\mathbb{C})$. Note that we may naturally identify each $A_i$ with a subalgebra of $A$ and hence, restricting $\sigma$ to this copy, get an injection of $A_i$ into $\Pi M_{k(n)}(\mathbb{C})$. Now we can prove the existence of a sequence of unital $^*$-homomorphism $\rho_i : A_i \to M$ such that

1. Each $\rho_i$ is injective;
2. $\rho_{i+1}|_{A_i} = \rho_i$ where we identify $A_i$ with the ”left side” of $A_i \ast C^*(F_\infty) = A_{i+1}$;
3. The union of $\{\rho_i(A_i)\}$ is weakly dense in $M$.

After finding the $\rho_i$s, it will be enough to define $\rho$ as union of those ones.

We first choose an increasing sequence of projections of $M$ such that $\tau_M(p_i) \to 1$. Then we define the orthogonal projections $q_n = p_n - p_{n-1}$ and consider the type $II_1$ factors $Q_i = q_i M q_i$. Now, by the division property of type $II_1$ factors, we can find a unital embedding $\Pi M_{k(n)} \to Q_i \subseteq M$. By composing with $\sigma$, we get a sequence of embeddings $A \to M$, which will be denoted by $\sigma_i$. Now $p_i M p_i$ is weakly separable and thus there is a countable total family of unitaries. Hence we can find a $^*$-homomorphism $\pi_i : C^*(F_\infty) \to p_i M p_i$ with weakly dense range (take the generators of $F_\infty$ into $C^*(F_\infty)$ and map them into that total family of unitaries). Now we define

$$\rho_1 = \pi_1 \oplus \bigoplus_{j \geq 2} \sigma_j|_{A_1} : A_1 \to p_1 M p_1 \oplus (\Pi_{j \geq 2} Q_j) \subseteq M$$

It is a $^*$-monomorphism, since each $\sigma_i$ is already faithful on the whole $A$. Now define a $^*$-homomorphism $\theta_2 : A_2 = A_1 \ast C^*(F_\infty) \to p_2 M p_2$ as the free product of the $^*$-homomorphism $A_1 \to p_2 M p_2$, $x \to p_2 \rho_1(x) p_2$, and $\pi_2 : C^*(F_\infty) \to p_2 M p_2$. We then put

$$\rho_1 = \theta_2 \oplus \bigoplus_{j \geq 3} \sigma_j|_{A_2} : A_2 \to p_2 M p_2 \oplus (\Pi_{j \geq 3} Q_j) \subseteq M$$

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Clearly $\rho_2|_{A_1} = \rho_1$. In general, we construct a map $\theta_{n+1} : A_n \ast C^*(F_\infty) \to p_{n+1}Mp_{n+1}$ as the free product of the cutdown (by $p_{n+1}$) of $\rho_n$ and $\pi_n$. This map need not be injective and hence we take a direct sum with $\bigoplus_{j \geq n+2}\sigma_j|_{A_{n+1}}$ to remedy this deficiency. These maps have all the required properties and hence the proof is complete (note that the last property follows from the fact that the range of each $\theta_n$ is weakly dense in $p_{n+1}Mp_{n+1}$).

Now we can prove Brown’s theorem

**Theorem 86. (Brown)** Let $M$ be a separable type $II_1$ factor. The followings are equivalent:

1. $M$ is embeddable into $R^\omega$.

2. $M$ has the weak expectation property relative to some weakly dense subalgebra.

**Proof.** ($1 \Rightarrow 2$)

Let $M$ be embeddable into $R^\omega$. By Prop.85 we may identify $C^*(F_\infty)$ with a weakly dense subalgebra $A$ of $M$. We want to prove that $M$ has the weak expectation property relative to $A$. Let $\tau$ the unique normalized trace on $M$, more precisely we will prove that $\pi_\tau(M)$ has the weak expectation property relative to $\pi_\tau(A)$. Indeed $\tau$ is faithful and w-continuous and hence $\pi_\tau(M)$ and $\pi_\tau(A)$ are respectively copies of $M$ and $A$ and $\pi_\tau(A)$ is still weak dense in $\pi_\tau(M)$. We first prove that $\tau|_A$ is an invariant mean. Take $\{u_n\}$ universal generators of $F_\infty$ into $A$. Let $n$ be fixed, since $u_n \in R^\omega$ it is $|| \cdot ||_2$-limit of unitaries in $R$; on the other hand, the unitary matrices are weakly dense in $U(R)$ and hence they are $|| \cdot ||_2$-dense in $U(R)$ (since w-closed convex subsets coincide with the $|| \cdot ||_2$-closed convex ones (see [Jo])). Thus we can find a sequence of unitary matrices which converges to $u_n$ in norm $|| \cdot ||_2$. Let $\sigma$ be the mapping which sends each $u_n$ to such a sequence. Since the $u_n$’s have no relations, we can extend $\sigma$ to a *homomorphism $\sigma : C^*(F_\infty) \to \Pi M_k(C) \subseteq l^\infty(R)$. Let $p : l^\infty(R) \to R^\omega$ be the quotient mapping. By the 2-norm convergence we have $(p \circ \sigma)(x) = x$ for all $x \in C^*(F_\infty)$. Let $p_n : \Pi M_k(C) \to M_n(C)$ be the projection, by the definition of trace in $R^\omega$, we have

$$\tau(x) = lim_{n \to \omega} tr_n(p_n(\sigma(x)))$$

where $tr_n$ is the normalized trace on $M_n(C)$. Now we can apply [S4.2] by setting $\phi_n = p_n \circ \sigma$ (they are u.c.p. since they are *homomorphisms) and conclude that $\tau|_A$ is an invariant
mean. Now consider \( \pi_\tau(M) \subseteq B(H) \) and \( \pi_\tau(A) = \pi_\tau\vert_A(A) \subseteq B(H) \). By Th.84 there exists a u.c.p. map \( \Phi : B(H) \to \pi_\tau(A)' = \pi_\tau(M) \) such that \( \Phi(a) = \pi_\tau(a) \). Thus \( M \) has the weak expectation property relative to \( C^*(F_\infty) \).

\[(2 \Rightarrow 1)\]

Let \( A \subseteq M \subseteq B(H) \) be weakly dense and \( \Phi : B(H) \to M \) a u.c.p. map which fixes \( A \). Let \( \tau \) be the unique normalized trace on \( M \). After identifying \( A \) with \( \pi_\tau(A) \), we are under the hypothesis of 84.3) and thus \( \tau\vert_A \) is an invariant mean. By Th.84 it follows that there exists a sequence \( \phi_n : A \to M_k(n)(\mathbb{C}) \) such that

1. \( ||\phi_n(ab) - \phi_n(a)\phi_n(b)||_2 \to 0 \) for all \( a, b \in A \)
2. \( \tau(a) = \lim_{n \to \infty} tr_n(\phi_n(a)) \), for all \( a \in A \)

Let \( p : l^\infty(R) \to R^\omega \) be the quotient mapping. The previous properties guarantee that the u.c.p. mapping \( A \to R^\omega, \Phi : x \to p(\{\phi_n(x)\}) \) is a \( * \)homomorphism which preserves \( \tau\vert_A \). It follows it mapping is injective too, since \( \Phi(x) = 0 \Rightarrow \Phi(x^*x) = 0 \Rightarrow \tau(x^*x) = 0 \Rightarrow x^*x = 0 \Rightarrow x = 0 \). Observe now that the weak closure of \( A \) into \( R^\omega \) is isomorphic to \( M \) (they are algebraically isomorphic and have the same trace) and hence \( M \) embeds into \( R^\omega \).

6 A few words about other approaches

6.1 Relation with Hilbert’s 17th problem

The original version of Hilbert’s 17th problem is very simple. Let us recall that \( \mathbb{R}[x_1,...,x_n] \) denotes the ring of polynomials with \( n \) indeterminates and real coefficients and \( \mathbb{R}(x_1,...,x_n) \) denotes the quotient field of \( \mathbb{R}[x_1,...,x_n] \).

\textbf{Problem 87. (Hilbert’s 17th)} Given a polynomial \( f \in \mathbb{R}[x_1,...,x_n] \) which is non-negative for all substitutions \( (x_1,...,x_n) \in \mathbb{R}^n \). Is it possible to write \( f \) as sum of squares of elements in \( \mathbb{R}(x_1,...,x_n) \)?

The affirmative answer was given by Emil Artin, in 1927 (see [Ar]). He gave a very abstract solution. Actually, now we have also an algorithm to construct such a decomposition. It has been recently found by Delzell (see [De]).
More recently, many mathematicians have looked for generalizations of the problem. The first and most intuitive one is the following

**Problem 88.** Are the matrices with entries in $\mathbb{R}[x_1, \ldots, x_n]$ which are always positive semidefinite (i.e. for all substitutions $(x_1, \ldots, x_n) \in \mathbb{R}^n$) sum of squares of symmetric matrices with entries in $\mathbb{R}(x_1, \ldots, x_n)$?

Also in this case an affirmative answer was given independently by Gondard-Ribenoim (see [Go-Ri]) and Procesi-Schacher (see [Pr-Sc]). Also in this case, for a constructive solution one had to wait for thirty years: it has been just found by Hillar and Nie in 2006 (see [Hi-Ni]).

Other generalizations come from Geometry and Operator Algebras. Let us recall the following

**Definition 89.** An $n$-manifold $M$ is called irreducible if for any embedding of $S^{n-1}$ into $M$ there exists an embedding of $B^n$ into $M$ such that the image of the boundary $\partial B^n$ coincides with the image of $S^{n-1}$.

**Problem 90. (Geometric version)** Let $M$ be a paracompact irreducible analytic manifold and $f : M \to \mathbb{R}$ a non-negative analytic function. Can $f$ be written as a sum of squares of meromorphic functions?

We recall that meromorphic functions are analytic functions in the whole domain except a set of isolates points which are poles. So, rational functions are meromorphic and one can recognize a generalization of Hilbert’s 17th problem.

This problem was solved by Ruiz (see [Ru]) in the case of compact manifold. In the general case there are lots of approaches in course, but a complete solution is known only for $n = 2$ (see [Ca]).

Now we want to describe briefly the formulation of the problem in terms of Operator Algebras. It is due to Rădulescu, who proved in [Ra2] the equivalence between it and Connes’ embedding conjecture.

The basic idea is to generalize analytic functions with formal series. Let $Y_1, \ldots, Y_n$ be $n$ indeterminates. We set

$$\mathcal{I}_n = \{(i_1, \ldots, i_p); p \in \mathbb{N}, i_1, \ldots, i_p \in \{1, \ldots, n\}\}$$
If \( I = \{i_1, \ldots, i_p\} \in \mathcal{I}_n \), we set \( Y_I = Y_{i_1} \cdots Y_{i_p} \). Let

\[
V = \left\{ \sum_{I \in \mathcal{I}_n} a_I Y_I, a_I \in \mathbb{C}, \| \sum_{I} a_I Y_I \|_R = \sum |a_I| R^{\|I\|} < \infty, \forall R > 0 \right\}
\]

So, for any \( R > 0 \), we have a norm on \( V \). Rădulescu proved, in [Ra2] Prop.2.1, that the norms \( \| \cdot \|_R \) define a structure of Frechet space on \( V \), i.e. locally convex space, metrizable (with a metric invariant by translations) and complete. In this case \( V^* \) is separating for \( V \) and thus we can consider the \( \sigma(V, V^*) \)-topology on \( V \).

Now we want to generalize the notion of "square" and "sum of squares". Starting from the classical theory, in which the squares are elements of the form \( a^*a \), the first step is to define an adjoint operation on \( V \).

**Definition 91.** We set \((Y_{i_1} \cdots Y_{i_p})^* = Y_{i_p} \cdots Y_{i_1}\), and \( a^* = \overline{a} \) for the coefficients. We can extend this mapping by linearity to an adjoint operation on \( V \).

Now, observing that series are too general to obtain in a finite number of steps, we have quite naturally the following

**Definition 92.** We say that \( q \in V \) is sum of squares if it is in the weak closure of the set of the elements of the form \( \sum p^*p \), \( p \in V \).

Now we observe that the formulation of Hilbert’s 17th problem with matrices regards matrices whose entries are REAL polynomial, the geometric formulation regards analytic functions with REAL values. So, recalling that REAL in operator algebras becomes SELF-ADJOINT, we have that our natural setting to generalize Hilbert’s 17th problem is not \( V \), but \( V_{sa} = \{ v \in V : v^* = v \} \).

It remains only to generalize the notion of positivity.

**Definition 93.** Let \( p \in V_{sa} \). We say that \( p \) is positive semidefinite if for every \( N \in \mathbb{N} \) and for every \( X_1, \ldots, X_n \in M_N(\mathbb{C}) \), one has

\[
\text{tr}(p(X_1, \ldots, X_n)) \geq 0
\]

\( V^+_{sa} \) will denote the set of positive semidefinite elements of \( V_{sa} \).

In order to arrive to the generalization of Hilbert’s 17th problem we have to do a last
Remark 94. In the case of polynomials in $\mathbb{R}[Y_1, ..., Y_n]$, we have $Y_I - Y_{\tilde{I}} = 0$, for any permutation $\tilde{I}$ of $I$. In our non-commutative case, we cannot have this equality and thus we have to identify elements which differ by permutation. A way to do this identification is given by the following

Definition 95. Two elements $p, q \in V_{sa}^+$ are called cyclic equivalent if $p - q$ is weak limit of sums of scalars multiples of monomials of the form $Y_I - Y_{\tilde{I}}$, where $\tilde{I}$ is a cyclic permutation of $I$.

In this way, we have the following

Problem 96. (Operator Algebra version) Is every element in $V_{sa}^+$ cyclic equivalent to a sum of squares?

Here is the beautiful and unexpected theorem by Rădulescu.

Theorem 97. (Rădulescu, [Ra2] Cor.1.2) The following statements are equivalent

1. Connes’ embedding conjecture is true.
2. Operator algebra version of Hilbert’s 17th problem has affirmative answer.

Following Radulescu some authors have began an approach to Connes’ embedding problem via sums of hermitian squares. In this last page we want to describe briefly the main result of Klep-Schweighofer’s work (see [Kl-Sc] and also [Ju-Po] for a development).

Let $K$ be the real or the complex field and $V = K[Y_1, ..., Y_n]$. So, the first difference between this approach and Radulescu’s one is that Klep and Schweighofer work with polynomial and Radulescu works with formal series. Other differences are given by the choice of the adjoint operation and the cyclic equivalence. More precisely, they take the identity operation (on the inderminates) as adjoint operation and the following as equivalence

Definition 98. $p, q \in V$ are called equivalent if $p - q$ is sum of commutators. We write $p \sim q$.  

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Once again this equivalence relation is clearly trivial in the commutative case.

On the other hand, the notion of positivity introduced by Klep and Schweighofer is a little less strong

**Definition 99.** \( f \in V \) is called positive semidefinite if for any \( s \in \mathbb{N} \) and for any contractions \( A_1, \ldots, A_n \in M_s(\mathbb{R}) \) one has

\[
\text{tr}(f(A_1, \ldots, A_n)) \geq 0
\]

The set of positive semidefinite element will be denoted by \( V^+ \).

Now we give the definition of quadratic module, which is the major difference with Radulescu’s formulation.

**Definition 100.** A subset \( M \subseteq V_{sa} \) is called quadratic module if the followings hold

1. \( 1 \in M \)

2. \( M + M \subseteq M \)

3. \( p^*Mp \subseteq M \), for all \( p \in V \).

The quadratic module generated in \( V \) by the elements \( 1 - X_1^2, \ldots, 1 - X_n^2 \) will be denoted by \( Q \).

**Theorem 101. (Klep-Schweighofer)** The following statements are equivalent

1. Connes’ embedding conjecture is true.

2. The following Radulescu’s type implication holds

\[
f \in V^+ \quad \Rightarrow \quad \forall \varepsilon > 0, \exists q \in Q \text{ s. t. } f + \varepsilon \sim q
\]

6.2 Voiculescu’s entropy

In order to show the relation between Connes’ embedding conjecture and Voiculescu’s free entropy, we have to recall briefly Voiculescu’s definition. References for this part are the preliminary sections of the papers by Voiculescu [Vo1] and [Vo2]. A motivation for these definitions can be found in [Vo3].
Note 102. We recall a construction of the entropy of a random variable which outcomes the set \( \{1, \ldots, n\} \) with probabilities \( p_1, \ldots, p_n \). The microstates are the set
\[
\{1, \ldots, n\}^N = \{ f : \{1, \ldots, N\} \to \{1, \ldots, n\} \}
\]
The set of microstates which \( \varepsilon \)-approximate the discrete distribution \( p_1, \ldots, p_n \) is
\[
\Gamma(p_1, \ldots, p_n, N, \varepsilon) = \{ f \in \{1, \ldots, n\}^N : \left| \frac{|f^{-1}(i)|}{N} - p_i \right| < \varepsilon \ \forall i = 1, \ldots, n \}
\]
where \( |f^{-1}(i)| \) is the number of elements in the counter-image. Now, one takes the limit of
\[
N^{-1} \log |\Gamma(p_1, \ldots, p_n, N, \varepsilon)|
\]
as \( N \to \infty \) and then lets \( \varepsilon \) go to zero. Thus we obtain the classical formula \( \sum p_i \log p_i \) for the entropy.

Voiculescu has generalized this construction to the non-commutative setting of von Neumann algebras.

Notation 103. (Lebesgue measure instead of the discrete one) Let \( k \) be a positive integer and \( (M_k(\mathbb{C})_{\text{sa}})^n \) the set of \( n \)-tuples of self-adjoint \( k \times k \) complex matrices. Let \( \lambda \) be the Lebesgue measure on \( (M_k(\mathbb{C})_{\text{sa}})^n \) corresponding to the Euclidean norm
\[
\|\|(A_1, \ldots, A_n)\|\|_{HS}^2 = Tr(A_1^2 + \ldots + A_n^2)
\]
where \( Tr \) is the non-normalized trace on \( M_k(\mathbb{C}) \).

Notation 104. (Microstates are matrices) Fixed \( \varepsilon, R > 0 \) and \( m, k \in \mathbb{N} \). Let \( X_1, \ldots, X_n \) free random variables on a finite factor \( M \). We set
\[
\Gamma_R(X_1, \ldots, X_n; m, k, \varepsilon) = \{ (A_1, \ldots, A_n) \in M_k(\mathbb{C})_{\text{sa}}^n \text{ s.t.} \}
\]
\[
\begin{align*}
\{ & ||A_j|| \leq R \\
& |tr(A_{i_1} \cdots A_{i_p}) - \tau(X_{i_1} \cdots X_{i_p})| < \varepsilon \ \forall (i_1, \ldots, i_p) \in \{1, \ldots, n\}^p, 1 \leq p \leq n \}
\end{align*}
\]
Definition 105. (Generalization of the process of limit)
\[
\chi_R(X_1, \ldots, X_n; m, k, \varepsilon) = \log \lambda(\Gamma_R(X_1, \ldots, X_n; m, k, \varepsilon))
\]
\[
\chi_R(X_1, \ldots, X_n; m, \varepsilon) = \limsup_{k \to \infty}(k^{-2}\chi_R(X_1, \ldots, X_n; m, k, \varepsilon) + 2^{-1}n \log(k))
\]
\[ \chi_R(X_1, \ldots, X_n) = \inf \{ \chi_R(X_1, \ldots, X_n; m, \varepsilon), m \in \mathbb{N}, \varepsilon > 0 \} \]

\[ \chi(X_1, \ldots, X_n) = \sup \{ \chi_R(X_1, \ldots, X_n), R > 0 \} \]

\( \chi(X_1, \ldots, X_n) \) is called free entropy of the variables \( X_1, \ldots, X_n \).

**Note 106.** The factor \( k^{-2} \) instead of \( k^{-1} \) comes from the normalization. The addend \( 2^{-1} \cdot \log(k) \) is necessary, since otherwise \( \chi_R(X_1, \ldots, X_n; m, \varepsilon) \) should be always equal to \(-\infty\).

By definition it follows that the free entropy can be equal to \(-\infty\). Voiculescu himself has found some examples.

**Proposition 107.** ([Vo2], Prop. 3.6,c)) If \( X_1, \ldots, X_n \) are linearly dependent, then \( \chi(X_1, \ldots, X_n) = -\infty \).

In order to have \( \chi(X_1, \ldots, X_n) > -\infty \) we need at least that \( \Gamma_R(X_1, \ldots, X_n, m, k, \varepsilon) \) is not empty for some \( k \), i.e. the finite subset \( X = \{X_1, \ldots, X_n\} \) of \( M_{sa} \) has microstates. This requirement is equivalent to Connes’ embedding conjecture:

**Theorem 108.** Let \( M \) be a type \( II_1 \) factor. The following conditions are equivalent

1. Every finite subsets \( X \subseteq M_{sa} \) has microstates.

2. \( M \) is embeddable into \( R^\omega \).

**Proof.** (1. \( \Rightarrow \) 2.)

Let \( Y = \{x_1, x_2, \ldots\} \) a norm-bounded generating set for \( M \). Fix \( m \in \mathbb{N} \) and \( \varepsilon = \frac{1}{m} \). By hypothesis, there exist a natural number \( k \) and \( A_1^{(m)}, \ldots, A_m^{(m)} \in M_k(\mathbb{C}) \) which are microstates for \( x_1, \ldots, x_m \). It has been proved by Voiculescu (see [Vo2]) that one can choose \( ||A_i^{(m)}|| \leq ||x_i|| \). Let \( \pi_k : M_k(\mathbb{C}) \rightarrow R \) any unital *monomorphism. Define \( a_i^m = \pi_k(A_i^{(m)}) \) and \( b_i = \{a_i^m\}_{m=1}^\infty \in l^\infty(R) \), where \( a_i^m = 0 \), if \( i > m \). Let \( z_i \) be the image of \( b_i \) into \( R^\omega \). The mapping \( x_i \rightarrow z_i \) extends to an embedding \( M \hookrightarrow R^\omega \).

(2. \( \Rightarrow \) 1.)

Let \( X = \{x_1, \ldots, x_n\} \subseteq R^\omega \). These elements are 2-norm limit of element of \( R \) and thus we can find \( a_1, \ldots, a_n \in R \) whose mixed moments approximate those of the \( x_i \)'s (indeed \( \tau_{R^\omega}(x_{i_1} \cdots x_{i_p}) = \lim \tau_R(x_{i_1}^{(n)} \cdots x_{i_p}^{(n)}) \)). Thus the implication follows by noting that every finite subsets of \( R \) is 2-norm approximately contained in some copy of \( M_k(\mathbb{R}) \), for some \( k \) sufficiently large. \( \square \)

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6.3 Collins and Dykema’s approach via eigenvalues

Connes’ embedding problem regards the approximation of the operators in a separable type $II_1$ factor via matrices. The basic idea of the approach by Collins and Dykema is that such an approximation must reflect on the eigenvalues: the eigenvalues of an operator in a separable type $II_1$ factor should be approximated by the eigenvalues of the matrices. This is just the basic idea, but there are some problems:

1. What does eigenvalue mean for an operator in a separable type $II_1$ factor?
2. In which sense those eigenvalues are approximated by the eigenvalues of the matrices?

We start by answering to the first question.

Let $M$ be a separable type $II_1$ factor and $\tau$ its unique faithful normalized trace. For any $a \in M_{sa}$ we can define the distribution of $a$ as the Borel measure $\mu_a$, supported on the spectrum of $a$, such that

$$\tau(a^n) = \int_{\mathbb{R}} t^n d\mu_a(t) \quad n \geq 1$$

**Definition 109.** Let $a \in M_{sa}$. The eigenvalue function of $a$ is the function $\lambda_a : [0, 1) \to \mathbb{R}$ defined by

$$\lambda_a(t) = \sup \{ x \in \mathbb{R} : \mu_a((x, +\infty)) > t \}$$

This definition generalizes what happens in $M_N(\mathbb{C})$ as follows:

Let $a \in M_N(\mathbb{C})_{sa}$ and let $\alpha = (\alpha_1, ..., \alpha_N)$ be its eigenvalue sequence, i.e. $\alpha_1, ..., \alpha_N$ are the eigenvalues of $a$ listed in non-increasing order and according to their multiplicity. In this case one has

$$\lambda_a(t) = \alpha_j$$

where $j$ is defined by the property $\frac{j-1}{N} \leq t < \frac{j}{N}$.

Now we pass to the second question. First of all we need a topology with respect to we can consider the approximations. We denote

$$\mathcal{F} = \{ f : [0, 1) \to \mathbb{R} \text{ right – continuous, non – increasing and bounded} \}$$
Clearly any eigenvalue function belongs into \( \mathcal{F} \). Conversely, given \( f \in \mathcal{F} \) and \( M \in \mathcal{I}_{II} \), there exists \( a \in M_{sa} \) such that \( \lambda a = f \). In this way we are able to identify \( \mathcal{F} \) with the set of eigenvalue functions. On the other hand

\[
\mathcal{F} = \{ \text{compactly supported Borel measures on } \mathbb{R} \} \subseteq C(\mathbb{R})^\ast
\]

Therefore a natural topology on \( \mathcal{F} \) (and thus on the set of eigenvalue functions too) is the weak* topology on \( C(\mathbb{R})^\ast \).

Above we said that the notion of eigenvalue function for operators generalizes that for matrices. Now we need to give a little formalization of this fact. Let \( \mathbb{R}^N \leq \mathbb{R}^N \subseteq \mathbb{R}^N \leq \mathbb{R}^N \) be the set of \( N \)-tuples of real numbers listed in non-increasing order. The correspondence

\[
\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N \rightarrow \lambda_\alpha(t) = \alpha_j \quad \text{where} \quad \frac{j - 1}{N} \leq t < \frac{j}{N}
\]

gives an embedding \( \mathbb{R}^N \leq \mathbb{R}^N \subseteq \mathcal{F} \). This embedding is very good, since it preserves the affine structure (the affine structure on \( \mathcal{F} \) is defined by taking the usual scalar multiplication and sum of functions; the affine structure on \( \mathbb{R}^N \leq \mathbb{R}^N \) comes from \( \mathbb{R}^N \)).

Now the idea is that Connes’ embedding conjecture should be equivalent in something like the density of \( \mathbb{R}^N \leq \mathbb{R}^N \leq \mathcal{F} \). Actually it happens something more precise and elegant. In two words: Connes’ embedding conjecture is equivalent to the possibility of approximating the eigenvalue function of operators of the form

\[
a_1 \otimes x_1 + a_2 \otimes x_2 \quad a_i \in M_N(\mathbb{C}), x_i \in M \quad (M \in \mathcal{I}_{II})
\]

with the eigenvalue function of operators of the form

\[
a_1 \otimes y_1 + a_2 \otimes y_2 \quad a_i, y_i \in M_N(\mathbb{C})
\]

where the eigenvalues functions of the \( y_i \)'s are the same of those of the \( x_i \)'s (after the embedding \( \mathbb{R}^N \leq \mathcal{F} \)).

We give some details in order to arrive to the correct enunciation of Collins-Dykema’s theorem. Let \( \alpha, \beta \in \mathbb{R}^N \leq \mathbb{R}^N \leq \mathbb{R}^N \leq \mathbb{R}^N \), \( d \in \mathbb{N}, a_1, a_2 \in M_{Nd}(\mathbb{C})_{sa}, M \in \mathcal{I}_{II} \). We denote

\[
R_{a_1, a_2}^{\alpha_1, \alpha_2} = \{ \lambda C, C = a_1 \otimes U(diag(\alpha) \otimes Id_d)U^* + a_2 \otimes V(diag(\beta) \otimes Id_d)V^*, U, V \in U(M_{Nd}(\mathbb{C})) \}
\]

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\[ K_{\alpha,\beta,\infty}^{a_1,a_2} = \bigcup_{d \in \mathbb{N}} K_{\alpha,\beta,d}^{a_1,a_2} \]

where the closure is respect to the weak* topology on \( F \).

\[ L_{\alpha,\beta,M}^{a_1,a_2} = \{ \lambda_C, C = a_1 \otimes x_1 + a_2 \otimes x_2 \} \]

where \( x_1, x_2 \in M \) whose eigenvalue functions agree with those of the matrices \( \text{diag}(\alpha) \) and \( \text{diag}(\beta) \).

At last we denote

\[ L_{\alpha,\beta}^{a_1,a_2} = \bigcup_{M \in \exists \mathbb{I}_1} L_{\alpha,\beta,M}^{a_1,a_2} \]

Here is Collins-Dykema’s theorem

**Theorem 110.** (Collins-Dykema, [Co-Dy], Th. 4.6) The following statements are equivalent

1. Connes’ embedding conjecture is true.
2. \( L_{\alpha,\beta}^{a_1,a_2} = K_{\alpha,\beta,\infty}^{a_1,a_2} \)

**Proof.** (Sketch). If Connes’ embedding conjecture is true, then \( L_{\alpha,\beta}^{a_1,a_2} = L_{\alpha,\beta,R^\omega}^{a_1,a_2} \). On the other hand \( L_{\alpha,\beta,R^\omega}^{a_1,a_2} = K_{\alpha,\beta,\infty}^{a_1,a_2} \). Hence the first implication easily follows. Conversely, one can suppose that \( M \) is generated by two self-adjoint elements \( x_1, x_2 \). Approximating \( x_1, x_2 \) we can assume their eigenvalue function belong into \( \mathbb{R}^N_\leq \), for some \( N \). By adding constants we may also assume that \( x_1, x_2 \) are positive and invertible. Now a theorem by Collins and Dykema (see [Co-Dy], 3.6) shows that \( x_1, x_2 \) have microstates and thus the thesis follows from Th. 108.

\( \square \)

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All comments are very much welcome! I will be happy to correct inaccuracies or omissions in future versions.