Fourier and Schur-Weyl transforms applied to XXX Heisenberg magnet

P Jakubczyk¹, T Lulek², D Jakubczyk², B Lulek²

¹University of Rzeszów, Aleja Rejtana 16c, 35-959 Rzeszów, Poland
²Rzeszów University of Technology, ul. Wincentego Pola 2, 35-959 Rzeszów, Poland
E-mail: pjakub@univ.rzeszow.pl

Abstract. Similarities and differences between Fourier and Schur-Weyl transforms have been discussed in the context of a one-dimensional Heisenberg magnetic ring with \(N\) nodes. We demonstrate that main difference between them correspond to another partitioning of the Hilbert space of the magnet. In particular, we point out that application of the quantum Fourier transform corresponds to splitting of the Hilbert space of the model into subspaces associated with the orbits of the cyclic group, whereas, the Schur-Weyl transform corresponds to splitting into subspaces associated with orbits of the symmetric group.

1. Mathematical description of the model
Let us consider a one dimensional Heisenberg magnetic ring of \(N\) nodes, each with a single node spin \(s\) and periodic boundary conditions. Such a model reveals a symmetry under collective unitary rotations \(u \in SU(n)\) in the single-node spaces, and permutations \(\sigma \in \Sigma_N\) of nodes. Mathematically, we have two sets
\[
\hat{N} = \{ j = 1, 2, ..., N \}, \quad \hat{n} = \{ i = 1, 2, ..., n \}, \quad n = 2s + 1
\]
the set of \(N\) nodes of the crystal, and the set of single-node states (or the set of labels of projections \(m_i = s + 1 - i, \ i \in \hat{n}\) of the single-node spin \(s\)), respectively. A state of the whole magnet is given by assigning the spin projection \(i\) to each node \(j\) of the magnet \(f : \hat{N} \rightarrow \hat{n}\), and it can be presented as
\[
|f\rangle = |f(1)f(2)\cdots f(N)\rangle \equiv |i_1, i_2, ..., i_N\rangle, \quad i_j \in \hat{n}, j \in \hat{N}.
\]
The set
\[
\hat{n}^{\hat{N}} = \{ f : \hat{N} \rightarrow \hat{n} \}
\]
of all states of the form (2) constitutes the computational basis \(\langle f | f' \rangle = \delta_{ff'}, \ f, f' \in \hat{n}^{\hat{N}}\) which spans the Hilbert space of the model
\[
\mathcal{H} = l_{\mathbb{C}} \hat{n}^{\hat{N}},
\]
where \(l_{\mathbb{C}} \hat{n}^{\hat{N}}\) denotes linear closure of the set \(\hat{n}^{\hat{N}}\) over the field \(\mathbb{C}\) of complex numbers. The space (4) consist of all linear combinations of magnetic configurations with coefficient being
complex numbers, in particular all eigenvectors of the magnet belong to this space. Moreover it is an arena of action of the Hamiltonian of the model.

In order to exploit permutational symmetry of the model, let us consider the action

$$A : \Sigma_N \times \tilde{n} \tilde{N} \rightarrow \tilde{n} \tilde{N}$$

of the symmetric group $\Sigma_N$ on the basis states (2), as a purely permutational representation

$$A(\sigma)|f\rangle = \left(\begin{array}{c} f \\ f \circ \sigma^{-1} \end{array}\right), \quad f \in \tilde{n} \tilde{N}, \quad \sigma \in \Sigma_N,$$

where $f \circ \sigma^{-1}$ is the composition of mappings $f : \tilde{N} \mapsto \tilde{n}$ and $\sigma^{-1} : \tilde{N} \rightarrow \tilde{N}$, so that $|f \circ \sigma^{-1}| = |\cdots i_{\sigma^{-1}(j)} \cdots|$, and $A(\sigma\sigma') = A(\sigma)A(\sigma')$. This action decomposes the set (3) of all magnetic configurations into orbits $O_\mu$ of the symmetric group

$$O_\mu = \{A(\sigma)|f\rangle \mid \sigma \in \Sigma_N\}$$

labelled by weights $\mu$. The weight is a composition

$$\mu = (\mu_1, \mu_2, \ldots, \mu_n), \quad \sum_{i \in \tilde{n}} \mu_i = N,$$

where a part

$$\mu_i = |\{i_j = i \mid j \in \tilde{N}\}|, \quad i \in \tilde{n}$$

is the occupation number for the single-node state $i \in \tilde{n}$, for $f \in O_\mu$. In other words, the weight $\mu$ characterises distribution of single-node states over nodes of the magnet, in such a way that the part $\mu_i$ is equal to the number of nodes with the single-node state $|i\rangle$.

The restriction

$$A|_{O_\mu} \equiv R^{\Sigma_N: \Sigma^\mu}$$

of the action $A$ to the orbit $O_\mu$ gives a transitive representation of $\Sigma_N$ with the Young subgroup

$$\Sigma^\mu = \Sigma_{\mu_1} \times \Sigma_{\mu_2} \times \cdots \times \Sigma_{\mu_n}$$

as the stabiliser of an initial magnetic configuration in the orbit $O_\mu$.

2. Quantum Fourier transform on orbits of the cyclic group. The basis of orbits.

It is also the case that the magnet is invariant under the action of the cyclic subgroup $C_N \subset \Sigma_N$, it implies that each $\Sigma_N$-orbit (6) decomposes into $C_N$-orbits, in accordance with the restriction

$$R^{\Sigma_N: \Sigma^\mu} \downarrow C_N = \sum_{\kappa \in K(N)} m(\mu, \kappa) R^{C_N: C_{\kappa}},$$

where $R^{C_N: C_{\kappa}}$ is a transitive representation of $C_N$, with the cyclic group $C_\kappa \triangleleft C_N$ being the stabiliser, so that $\kappa$ is a divisor of $N$, and $K(N)$ is the lattice of all divisors of $N$. The multiplicity $m(\mu, \kappa)$ in equation (11), that is, the multiplicity of occurrence of the transitive representation $R^{C_N: C_{\kappa}}$ of the cyclic group in transitive representation $R^{\Sigma_N: \Sigma^\mu}$ of the symmetric group, or the number of $\kappa$-tuply rarefied $C_N$-orbits in the orbit $O_\mu$ of the symmetric group $\Sigma_N$, is given by a combinatoric formula [1]

$$m(\mu, \kappa) = \begin{cases} \kappa \sum_{\kappa \in K(\gcd(\mu_1, \ldots, \mu_n))} \tilde{\mu}(\kappa') \left(\frac{\mu}{\gcd(\mu_1, \ldots, \mu_n)}\right) \left(\frac{\kappa}{\gcd(\mu_1, \ldots, \mu_n)}\right) & \text{if all } \frac{\mu_i}{\kappa_i}, \ i \in \tilde{n} \text{ are integers} \\ 0 & \text{otherwise.} \end{cases}$$

(12)
Here, gcd(µ₁/κ, ..., µₙ/κ) denotes the greatest common divisor of integers µᵢ/κ, i ∈ ˜n, and ˜µ : Z₊ → {0, ±1} is the standard Möbius function of number theory.

Thus, A ∩ Cₙ (the action A restricted to the subgroup Cₙ) decomposes each Σₙ-orbit (6) into strata labelled by κ ∈ KN, consisting of κ-tuply rarefied Cₙ-orbits. In this way, we achieve new set of labels

$$|µκtk⟩, \ \mu = N, \ \kappa ∈ KN, \ \text{t} ∈ ˜m(µ, Κ) = \{1, 2, ..., m(µ, Κ)\}, \ \ j ∈ ˇκ = \{1, 2, ..., ˇκ\}, \ (13)$$

which classify, in some different way, all basis states of the magnet. Here, t ∈ ˜m(µ, κ) labels orbits of Cₙ on Oₙ in the stratum κ and j ∈ ˇκ labels configurations within Cₙ-orbits, ˇκ = N/κ is the length of the κ-tuply rarefied Cₙ-orbit. We call (13) the basis of orbits in the set (3) of all magnetic configurations [2]. The basis of orbits specifies all positions of the classical counterpart of the Heisenberg magnet with the structure imposed by the cyclic group.

But, sometimes, especially if we work in the field of quantum information [3], it is more convenient to pass from the position representation to the momentum one, and the basis (13) allows us to do it for each Cₙ-orbit separately. In order to present this approach, let us observe, that each Cₙ-orbit spans a subspace in H, which is invariant under Cₙ according to the decomposition of the transitive representation Rₖ∈Cₙ of the cyclic group into irreps

$$R^{C_N:C_κ}_k = \sum_{k ∈ B/κ} Γ_k, \ (14)$$

where Γₖ = exp (2πi κ/N) is the irrep of Cₙ,

$$B/κ = \{k ∈ B \ | \ k/κ ∈ Z\} ⊂ B \ (15)$$

is the κ-tuply rarefied Brillouin zone, and

$$B = \{k = 0, ±1, ±2, ..., \{ ±(N/2 − 1), N/2 \text{ for } N \text{ even} \} \}
\{ ±(N - 1)/2, \text{ for } N \text{ odd} \} \ (16)$$

is the (finite) Brillouin zone of the (one dimensional) magnet ˜N. The equation (14) can be written on the level of bases in the form

$$|µκtk⟩ = \sqrt{\frac{κ}{N}} \sum_{j = 1}^{ˇκ} \exp \left( -\frac{2πi j}{N} \right) |µκj⟩. \ (17)$$

One can see that one dimensional subspaces in Eq. (17) which carry irreps Γₖ, are given by the quantum Fourier transform on orbits of the cyclic group.

2.1. An example of quantum Fourier transform

For a very simple case, the Heisenberg magnet with N nodes and one spin deviation, we have

$$\mu = \{N − 1, 1\}, \ \kappa = 1, \ \text{t} = 0$$

what means that we have here one orbit of the symmetric group which consists of one stratum, and this stratum consists of just one regular orbit of the cyclic group. Thus from Eq. (17) we have

$$|k⟩ = \frac{1}{√N} \sum_{j = 1}^{N} \exp \left( -\frac{2πi j}{N} \right) |j⟩. \ (18)$$

The transformation (18) is exactly the quantum Fourier transform on the orthonormal basis \{ |j⟩, | j ∈ ˜N \} (c.f. for example [3]).
3. Schur-Weyl transform on orbits of the symmetric group. The basis of Schur-Weyl duality.

Now, we introduce another partitioning of the Hilbert space $\mathcal{H}$ by use of duality relation between the actions of the unitary and symmetric group. Namely, the space $\mathcal{H}$ is the scene of two actions of the symmetric $A : \Sigma_N \times \mathcal{H} \rightarrow \mathcal{H}$ and unitary $B : SU(n) \times \mathcal{H} \rightarrow \mathcal{H}$ groups. These actions can be decomposed into irreps, according to formulas

$$A = \sum_{\lambda \in D_W(N,n)} m(A, \Delta^\lambda) \Delta^\lambda, \quad B = \sum_{\lambda \in D_W(N,n)} m(B, D^\lambda) D^\lambda,$$

(19)

where $\Delta^\lambda$ and $D^\lambda$ are irreducible representations of the symmetric and unitary group, respectively, $D_W(N,n)$ denotes the set of all partitions of the integer $N$ into not more than $n$ parts, and appropriate multiplicities, on the strength of Schur-Weyl duality [4], satisfy relations

$$m(A, \Delta^\lambda) = \dim D^\lambda, \quad m(B, D^\lambda) = \dim \Delta^\lambda.$$

(20)

Relations (20) steam from the quantum-mechanical observation that these two actions mutually commute, that is,

$$[A(\sigma), B(u)] = 0, \quad \sigma \in \Sigma_N, \quad u \in U(n),$$

(21)

despite the fact that both dual groups are, for $N > 2, n > 1$, highly noncommutative. In this way the Hilbert space $\mathcal{H}$ of all quantum states of the composite system decomposes into sectors $\mathcal{H}^\lambda$,

$$\mathcal{H} = \sum_{\lambda \in D_W(N,n)} \oplus \mathcal{H}^\lambda,$$

(22)

such that

$$A|_{\mathcal{H}^\lambda} = (\dim D^\lambda) \Delta^\lambda, \quad B|_{\mathcal{H}^\lambda} = (\dim \Delta^\lambda) D^\lambda.$$

(23)

Thus restriction of the action $A$ to the sector $\mathcal{H}^\lambda$ gives $\dim D^\lambda$ copies of irrep $\Delta^\lambda$ of the symmetric group, whereas restriction of the action $B$ to the same sector $\mathcal{H}^\lambda$ gives $\dim \Delta^\lambda$ copies of irrep $D^\lambda$ of the unitary group. The duality of Schur-Weyl admits therefore the irreducible basis of the form

$$b_{irr} = \{ |\lambda t y \rangle | \lambda \in D_W(N,n), \ t \in \tilde{D}^\lambda, \ y \in \tilde{\Delta}^\lambda \},$$

where $\tilde{D}^\lambda$ and $\tilde{\Delta}^\lambda$ are some standard bases for the irrep $D^\lambda$ and $\Delta^\lambda$, respectively. We call this basis irreducible, because its elements transforms under the action of the symmetric or unitary group according to these irreps. It is convenient to take standard bases in the form

$$\tilde{\Delta}^\lambda = SYT(\lambda), \quad \tilde{D}^\lambda = SSWT(\lambda, \tilde{n}),$$

(24)

where $SYT(\lambda)$ denotes the set of all standard Young tableaux of the shape $\lambda$ in the alphabet $\tilde{N}$ of nodes, and $SSWT(\lambda, \tilde{n})$ is the set of all Weyl tableaux of the shape $\lambda$ in the alphabet $\tilde{n}$ of spins.

Finally, on the strength of the Schur-Weyl duality (20), and after choosing basis elements like in Eq. (24), we obtain

$$R^{\Sigma_N : \Sigma_\mu} \cong \sum_{\lambda \in \mu} K_{\lambda \mu} \Delta^\lambda,$$

(25)

here, the sum runs over all partitions $\lambda$ greater or equal $\mu$ in dominance order, and $K_{\lambda \mu}$ denotes the Kostka number [5]. Combinatorially, $K_{\lambda \mu}$ is equal to the number of all semstandard Weyl
tableaux of the shape $\lambda$ and weight $\mu$. Decomposition (25) can be written at the level of bases in a form

$$|\mu \lambda ty\rangle = \sum_{f \in O_\mu} \langle \mu f | \lambda ty \rangle |\mu f\rangle,$$

(26)

where coefficients $\langle \mu f | \lambda ty \rangle$ form a unitary matrix, which transforms the initial basis $O_\mu$ of magnetic configurations to the irreducible one of the Schur-Weyl duality. We refer to it as to the Kostka matrix at the level of bases. A method of calculation of such transformation matrix relies on representation theory technique called pattern calculus [6] and was developed in the work [7].

One can see that states given by Eq. (26) are certain wave packets of magnetic configurations which belongs to the orbit $O_\mu$ of the symmetric group. These states have strictly defined symmetry described by Weyl $t$ and Young $y$ tableaux, what means that they transform under the action of the symmetric and unitary groups according to these irreps.

3.1. An example of Kostka matrix at the level of bases.

For a very simple case, the Heisenberg magnet with $N$ nodes and one spin deviation, we have

$$\mu = \{N - 1, 1\},$$

so that the Schur-Weyl basis has the form

$$b_{irr} = \{ |\lambda y\rangle | \lambda \in \{\{N\}, \{N - 1, 1\}\}, y \in SYT(\lambda) \}.$$  

The Hilbert space $\mathcal{H}$ carries the transitive representation $R_{\Sigma^N:\Sigma^{N-1}}$, which can be decomposed into irrep of the symmetric group

$$R_{\Sigma^N:\Sigma^{N-1}} = \Delta^{\{N\}} + \Delta^{\{N-1, 1\}}$$

(27)

where $\Delta^{\{N\}}$ is one dimensional irrep spanned on Young tableau $\ydiagram 1 \cdots N$, whereas $\Delta^{\{N-1, 1\}}$ is the $(N - 1)$-dimensional irrep spanned on Young tableaux of the form $y_{j'} = \ydiagram 2 \cdots j', \ 2 \leq j' \leq N$.

Eq. (27) can be written at the level of bases

$$|\lambda y\rangle = \sum_{j \in \mathbb{N}} \langle j | \lambda y \rangle |j\rangle.$$  

(28)

where $|j\rangle$ denotes the magnetic configurations (a states with spin deviation at the node $j$) and the coefficient $\langle j | \lambda y \rangle$ can be computed from the formula

$$\langle j | \lambda y \rangle = \begin{cases} \frac{1}{\sqrt{N}} & \text{for } \lambda = \{N\}, \ y = \ydiagram 1 2 \cdots N, \\ -\frac{1}{\sqrt{(j' - 1)j'}} & \text{for } \lambda = \{N - 1, 1\}, \ 1 \leq j' < j, \\ \sqrt{\frac{j'-1}{j}} & \text{for } \ j = j', \\ 0 & \text{for } j' > j. \end{cases}$$

(29)
4. Final remarks and conclusions
Two transformations, quantum Fourier and Schur-Weyl, are very similar. The first one is strictly connected with translation symmetry of the magnet and corresponds to splitting of the Hilbert space into subspaces associated with the orbits of the cyclic group. The second, however, arises from permutational symmetry and correspond to splitting of the Hilbert space into subspaces spanned on the orbits of the symmetric group.

These two decompositions of the Hilbert space are very important, because subspaces they create are invariant under the action of the Hamiltonian so that diminishes the size of the eigenproblem by the factor $N$ (in the case of Fourier transform) or $\dim \lambda$ (in the case of Schur-Weyl transform).

References
[1] Lulek B 1992 Acta Phys. Pol. B22 371
[2] Milewski J Ambrozko E 2008 J. Phys. Conf. Ser. 104 012040
[3] Nielsen M A Chuang I L 2000 Quantum Computation and Quantum Information (Cambridge: Cambridge Univ. Press.)
[4] Weyl H 1931 Gruppentheorie und Quantenmechanik (Leipzig: Hirzel), (English translation: 1950 The Theory of Groups and Quantum Mechanics New York, Dover)
[5] Foulkes H O 1974 A survey of some combinatorial aspects of symmetric functions (Paris: Gauthier-Villars)
[6] Louck J D 2009 Unitary symmetry and combinatorics (Singapore: World Scientific)
[7] Lulek T Jakubczyk P Jakubczyk D 2004 Mol. Phys. 102 1279