Uplink-Downlink Duality for Integer-Forcing

Wenbo He*, Bobak Nazer*, and Shlomo Shamai (Shitz)†

Abstract—Consider a MIMO uplink channel with channel matrix $H$ and a MIMO downlink channel with channel matrix $H^T$. It is well-known that any rate tuple that is achievable on the uplink is also achievable on the downlink under the same total power constraint, i.e., there is an uplink-downlink duality relationship. In this paper, we consider the integer-forcing strategy, in which users steer the channel towards an integer-valued effective channel matrix so that the receiver(s) can decode integer-linear combinations of the transmitted codewords. Recent efforts have demonstrated the benefits of this strategy for uplink, downlink, and interference alignment scenarios. Here, we establish that uplink-downlink duality holds for integer-forcing. Specifically, in the uplink, $L$ transmitters communicate over channel matrix $H$ to an $L$-antenna receiver with target integer matrix $A$. In the downlink, an $L$-antenna transmitter communicates over channel matrix $H^T$ to $L$ single-antenna receivers with target integer matrix $A^T$. We show that any computation rate tuple that is achievable in the uplink is achievable for the same total power in the downlink and vice versa.

I. INTRODUCTION

Interference is a significant challenge to multi-terminal communication. Consider a MIMO uplink channel where $L$ transmitters wish to communicate with an $L$-antenna receiver. While it is well-known that i.i.d. random encoding and joint typicality decoding achieves the capacity region, it is often desirable to find a low-complexity alternative that employs only “single-user” encoding and decoding. For instance, a zero-forcing receiver uses its antennas to null out interference between codewords and then decode them separately, albeit with a rate penalty. Recent efforts have demonstrated the advantages of an integer-forcing receiver, which uses its antennas to create an effective integer-valued channel matrix, decode integer-linear combinations of the codewords, and finally solve for the original messages. As shown in [1], [2], integer-forcing can approach the performance of the optimal joint decoder.

Similarly, consider a MIMO downlink channel where an $L$-antenna transmitter communicates with $L$ single-antenna receivers. As proposed in [3], integer-forcing can be employed as a beamforming strategy: the transmitter steers the channel to an effective integer-valued channel matrix so that the receivers can decode integer-linear combinations of the codewords. It then precodes its messages with the corresponding inverse matrix so that, end-to-end, each receiver gets its desired message. One can also use integer-forcing as a framework for interference alignment, which yields achievable rates beyond what is possible with classical i.i.d. random codes [4].

It is well-known that there is a duality relationship between the capacity regions for MIMO uplink and downlink [5]–[9]. In this paper, we establish that a similar duality relationship holds between uplink and downlink integer-forcing. Owing to space constraints, we limit ourselves to the special case of $L$ single-antenna users, an $L$-antenna basestation, and a full-rank set of desired linear combinations for the receiver(s). Possible applications of this phenomenon include an iterative optimization framework for the integer-forcing interference alignment scheme of [4], and extending the optimality results of [2], [10] to the downlink. Again, due to space considerations, these are deferred to an (upcoming) extended version. The paper is organized as follows. In Sections II and III, we consider the uplink and downlink channels, respectively, and provide a problem statement, coding scheme, and achievable rates. We state our main result in Section IV and end with a brief discussion of iterative optimization in Section V.

We will make use of the following notation. Column vectors will be denoted by boldface lowercase font (e.g., $a \in \mathbb{Z}^L$) and matrices with boldface uppercase font (e.g., $A \in \mathbb{Z}^{L \times L}$). Let $[a]_i$ denote the $i$th coordinate of the vector $a$. We will use $\|a\|$ to represent $\ell_2$-norm of $a$ and $\Tr(A)$ to represent the trace of $A$. We will also use $\diag(a)$ to denote the diagonal matrix formed by using the placing the elements of $a$ along the diagonal. All logarithms are taken to base 2. Subscripts “u” and “d” denote variables associated with the uplink and downlink, respectively. We denote the identity matrix by $I$ and the all-zero column vector of length $k$ by $0_k$. Let $P_\pi$ denote the permutation matrix corresponding to permutation $\pi$.

II. UPLINK CHANNEL

A. Problem Statement

As mentioned above, our uplink problem statement is drawn from [11]. Consider an uplink channel with $L$ single-antenna transmitters and a single $L$-antenna receiver. An uplink integer-forcing code is parametrized by a positive integer $n$ that denotes the block length, a prime number $p$ that denotes the field size, and positive integers $k_{F,\ell} \geq k_{C,\ell}$ for $\ell = 1, \ldots, L$ that determine the number of signal levels available to each user. Let $k_{F,max} = \min_{\ell} k_{F,\ell}$, $k_{C,min} = \min_{\ell} k_{C,\ell}$ and $k = k_{F,max} - k_{C,min}$.

Each transmitter has a message $w_{u,\text{info},\ell} \in \mathbb{Z}_{p}^{k_{F,\ell} - k_{C,\ell}}$ and an encoder $E_{u,\ell} : \mathbb{Z}_{p}^{k_{F,\ell} - k_{C,\ell}} \to \mathbb{R}^n$ that maps its message into a channel input vector $x_{u,\ell,\ell} = E_{u,\ell}(w_{u,\text{info},\ell})$. The rate of the $\ell$th message is $(k_{F,\ell} - k_{C,\ell})/n \log p$. Let $X_u = [x_{u,1} \cdots x_{u,L}]^T$ denote the matrix of transmitted signals. These signals must satisfy a total power constraint $\mathbb{E}[\Tr(X_uX_u^T)] \leq \rho_{u}\nu_{u,\text{total}}$.

* W. He and B. Nazer are with the ECE Department, Boston University, Boston, MA. Emails: whe02@bu.edu, bobak@bu.edu.
† S. Shamai (Shitz) is with the EE Department, Technion, Haifa, Israel. Email: sshlomo@ee.technion.ac.il.

W. He and B. Nazer were supported by NSF grants CCF-1253918 and CCF-1302600. The work of S. Shamai has been supported by the Israel Science Foundation (ISF) and by the European Commission in the framework of the FP7 Network of Excellence in Wireless COMMunications NEWCOM†.
The receiver observes the channel output

\[ Y_u = H_u X_u + Z_u \]  

where \( H_u \) is a fixed \( L \times L \) channel matrix and \( Z_u \) is elementwise i.i.d. Gaussian noise with mean zero and variance one. For a given integer matrix \( A_u = \{a_{u,m,t}\} \in \mathbb{Z}^{L \times L} \), the receiver attempts to decode \( L \) linear combinations of the form

\[ u_{a,m} = \bigoplus_{\ell=1}^{L} [g_{a,m,\ell} w_{u,\ell}] \]  

where \( g_{a,m,\ell} = [a_{u,m,\ell}] \mod p \) and each \( w_{u,\ell} \in \mathbb{Z}_p^k \) satisfies

\[ \tilde{w}_{u,\ell}[i] = \begin{cases} 
1 & 1 \leq i \leq k_{C,\ell} - k_{C_{\min}} \\
0 & k_{C_{\max}} - k_{F,\ell} < i \leq k \\
\tilde{w}_{u,info,\ell}[i] & k_{C,\ell} - k_{C_{\min}} < i \leq k_{F,\max} - k_{F,\ell} 
\end{cases} \]  

where * denotes the coordinate that can take any value in \( \mathbb{Z}_p \) (but is fixed across all linear combinations). The receiver is equipped with a decoder \( D_u : \mathbb{R}^{L \times n} \rightarrow \mathbb{R}_p^{L \times k} \) that outputs an estimate \( \hat{U}_u \) of \( U_u = [u_{1,1} \cdots u_{L,L}]^T \). Owing to space considerations, we will limit ourselves to the special case where \( A_u \) is full rank, which is of interest for MIMO uplink.

For a given channel matrix \( H_u \), integer matrix \( A_u \), and total power constraint \( \rho_u,_{\text{total}} \), we say that the computation rate tuple \( (R_{u,1}, \ldots, R_{u,L}) \) is achievable if, for any \( \epsilon > 0 \) and \( n \) large enough, there exist parameters \( p, k_{C,\ell}, \) and \( k_{F,\ell} \) satisfying \( (k_{F,\ell} - k_{C,\ell})/n \log p \geq R_{u,\ell} \) for \( \ell = 1, \ldots, L \) as well as encoders and a decoder such that \( P(\hat{U}_u \neq U_u) < \epsilon \).

B. Coding Scheme Overview

Lattice Preliminaries: We will employ a version of the asymmetric compute-and-forward scheme of [11]. Recall that a lattice \( \Lambda \) is a discrete additive subgroup of \( \mathbb{R}^n \). Let \( Q_{\chi}(x) \triangleq \arg \min_{x' \in \Lambda} \|x - x'\| \) denote the nearest neighbor quantizer associated to \( \Lambda \) and let \( V \) denote the resulting fundamental Voronoi region, i.e., the set of all points in \( \mathbb{R}^n \) that quantize to 0 (breaking ties systematically). We also define the modulo operation as \([x] \mod \Lambda \triangleq x - Q_{\chi}(x)\). The lattice \( \Lambda_{C,\ell} \) is said to be nested in the lattice \( \Lambda_{C,\ell} \) if \( \Lambda_{C,\ell} \subseteq \Lambda_{C,\ell} \). In this case, \( \Lambda_{C,\ell} \) is called the coarse lattice and \( \Lambda_{F,\ell} \) the fine lattice. A nested lattice codebook \( L = \Lambda_{F,\ell} \cap \mathcal{V}_{\ell,\ell} \) consists of all fine lattice points that fall in the fundamental Voronoi region of the coarse lattice.

The scheme in [11] utilizes \( L \) coarse lattices \( \Lambda_{C,1}, \ldots, \Lambda_{C,L} \) and \( L \) fine lattices \( \Lambda_{F,1}, \ldots, \Lambda_{F,L} \), all of which are nested according to decreasing second moments. From these, it constructs \( L \) nested lattice codebooks, \( \mathcal{L}_\ell = \Lambda_{F,\ell} \cap \mathcal{V}_{\ell,\ell}, \ell = 1, \ldots, L \). Roughly speaking, the coarse lattices are designed to have second moments \( \rho_{u,c,1}, \ldots, \rho_{u,c,L} \), respectively, which in turn implies that if \( s_{u,c} \) is uniformly distributed over \( \mathcal{V}_{\ell,\ell} \), then \( \frac{1}{n} E \| s_{u,c} \|^2 \geq \rho_{u,c} \). The fine lattices are designed to have effective noise tolerances \( \sigma_{u,eff,\ell}^2 \) to be specified later, where \( \delta_\ell \) tends to zero as \( n \rightarrow \infty \). Let \( \Lambda_{F,\max} \) and \( \Lambda_{C,\min} \) denote the finest and coarsest lattice in the collection, respectively. It can be shown that there is an isomorphism \( \Phi \) between \( \Lambda_{F,\max} \cap \mathcal{V}_{\min} \) and \( \mathbb{Z}_p^k \).

Encoding: Each transmitter begins by zero-padding its message to length \( k \), \( w_{u,\ell} = [0_{k_{u,\ell} - k_{C_{\min}}}, w_{u,info,\ell}, 0_{k_{F,\max} - k_{F,\ell}}]^T \), and then mapping it onto its nested lattice codebook \( t_{u,\ell} = \Phi(w_{u,\ell}) \in \mathcal{L}_\ell \). Next, it dithers its lattice codeword \( s_{u,\ell} = [t_{u,\ell} + d_{u,\ell}] \mod \Lambda_{C,\ell} \) where the dither vector \( d_{u,\ell} \) is independently generated according to a uniform distribution over \( \mathcal{V}_{\ell,\ell} \). Finally, it scales by \( \alpha_{u,\ell} \in \mathbb{R} \) to obtain its channel input \( x_{u,\ell} = \alpha_{u,\ell}s_{u,\ell} \).

Decoding: First, we define some useful notation. Let \( D_u \triangleq [d_{u,1} \cdots d_{u,L}]^T \), \( S_u \triangleq [s_{u,1} \cdots s_{u,L}]^T \), \( \alpha_u \triangleq [\alpha_{u,1} \cdots \alpha_{u,L}]^T \), and \( C_u = \text{diag}(\alpha_u) \). Let \( \rho_{\ell,\ell} \triangleq [\rho_{u,1,\ell} \cdots \rho_{u,L,\ell}]^T \) be the uplink coding power vector and \( P_{c,u} \triangleq \text{diag}(\rho_{c,u}) \) the associated diagonal matrix. Also, let \( \tilde{t}_{u,\ell} \triangleq \tilde{q}_{\ell,\ell} - C_{u,\ell} t_{u,\ell} \) and \( \tilde{U}_u = [\tilde{t}_{u,1} \cdots \tilde{t}_{u,L}]^T \).

The receiver begins by projecting its observation by \( B_u \in \mathbb{R}^{L \times L} \) and removing the dithers,

\[ \hat{Y}_u = [B_u Y_u - A_u D_u] \mod \Lambda_{C,\min} \]  

\[ = [A_u (S_u - D_u) + Z_u,_{\text{eff}}] \mod \Lambda_{C,\min} \]  

\[ = [A_u (S_u - D_u) \mod \Lambda_{C,\max} + Z_u,_{\text{eff}}] \mod \Lambda_{C,\min} \]  

\[ = [A_u \tilde{T}_u + Z_u,_{\text{eff}}] \mod \Lambda_{C,\min} \]  

\[ Z_u,_{\text{eff}} \triangleq (B_u H_u C_u - A_u) S_u + B_u Z_u \]  

where the modulo operation is applied to each row separately. Define \( v_{T,\ell,u} = [a_{T,\ell,u}^T \tilde{T}_u]^T \mod \Lambda_{C,\min} \) and note that the \( m \)th row \( \tilde{Y}_{u,m} \) of \( \hat{Y}_u \) can be written as

\[ \tilde{Y}_{u,m} = [v_{T,\ell,u}^T + z_{u,\ell,\ell}^T] \mod \Lambda_{C,\min} \]  

where \( z_{u,\ell,\ell}^T \) is the \( m \)th row of \( Z_{u,\ell,\ell} \).

Since \( s_{u,\ell} \) is uniformly distributed over \( \mathcal{V}_{\ell,\ell} \) (and independent from the other codewords), we have that

\[ \frac{1}{n} E \| Z_{u,\ell,\ell}^T \|^2 \leq \cdots \leq \frac{1}{n} E \| Z_{u,\ell,\ell}^T \|^2. \]  

(Otherwise, reindex the rows of \( A_u \).

The following lemma will enable us to order the users for successive cancellation decoding.

**Lemma 1** ( [12, Lemma 2]). For any integer matrix \( A_u \) for which \( [A_u] \mod p \) is full rank over \( \mathbb{Z}_p \), there exists at least one permutation \( \pi \) and a unit lower triangular matrix \( L_u \in \mathbb{Z}_p^{L \times L} \), such that \( ([L_u A_u^2] \mod p) / \pi \) is upper triangular.

We use this permutation order to set the effective noise tolerances of the fine lattices,

\[ \sigma_{u,\ell,\ell}^2 = \frac{1}{n} E \| Z_{u,\ell,\ell}^T \|^2 \]  

---

1These don’t care entries enable the transmitters to utilize different transmit powers while retaining a connection to \( \mathbb{Z}_p^k \).
The receiver begins by estimating the first linear combination \( \tilde{v}_{u,1} \) from the first row of \( \tilde{Y} \),
\[
\tilde{v}_{u,1} = [Q_{A_{F,n-1}}(\tilde{y}_{u,1})] \mod \Lambda_{C,\min}
\]
\[
\hat{u}_{u,1} = \Phi^{-1}(\tilde{v}_{u,1}).
\]
(10) (11)

It can be shown that this is successful with probability tending to 1 as \( n \) tends to infinity, since \( A_{F,n-1} \) is the finest lattice and is designed to absorb \( z_{u,eff,1} \). Now, assume that for some \( 1 < m < L, v_{u,1}, \ldots, v_{u,m-1} \) have been decoded successfully. The receiver utilizes these integer-linear combinations to cancel out lattice points \( t_{u,\pi^{-1}(m-1)} \) from \( \tilde{y}_{u,m} \),
\[
\tilde{y}_{u,m} = \left[ Q_{A_{F,n-1}(m)}(\tilde{y}_{u,m}) - \sum_{i=1}^{m-1} l_{u,m,i} \tilde{v}_{u,i} \right] \mod \Lambda_{C,\min}
\]
\[
\hat{v}_{u,m} = \Phi^{-1}(\tilde{v}_{u,m})
\]
(12) (13)

where \( l_{u,m,i} \) is the \((m,i)\)th element of \( L_u \). It quantizes the result onto \( \Lambda_{F,n-1}(m) \), which is designed to absorb \( z_{u,eff,m} \), and forms its estimate of the desired linear combination,
\[
\hat{v}_{u,m} = \left[ Q_{A_{F,n-1}(m)}(\tilde{y}_{u,m}) - \sum_{i=1}^{m-1} l_{u,m,i} \tilde{v}_{u,i} \right] \mod \Lambda_{C,\min}
\]
\[
\hat{u}_{u,m} = \Phi^{-1}(\tilde{v}_{u,m})
\]
C. Achievability Rates

The following theorem restates the uplink achievability result from [11] in a form that is useful for establishing a duality relationship.

**Theorem 1.** For a given channel matrix \( H_d \in \mathbb{R}^{L \times L} \), integer matrix \( A_d \in \mathbb{Z}^{L \times L} \), projection matrix \( B_d \in \mathbb{R}^{L \times L} \), (diagonal) precoding matrix \( C_d \in \mathbb{R}^{L \times L} \), and (diagonal) power matrix \( P_{d,u} \in \mathbb{R}^{L \times L} \), the following computation rate tuple is achievable
\[
R_{u,\ell} = \frac{1}{2} \log^+ \left( \frac{\rho_{d,c,\ell}}{\rho_{d,\text{eff},\pi(\ell)}} \right)\quad \ell = 1, \ldots, L
\]
where \( \pi \) is a valid permutation order for \( A_d \) from Lemma 1.

**III. DOWNLINK CHANNEL**

A. Problem Statement

Our downlink problem statement mirrors that for the uplink channel. It is parameterized by (possibly different values of) a positive integer \( n \), a prime number \( p \), and positive integers \( k_{\ell,F}, k_{\ell,C} \) for \( \ell = 1, \ldots, L \). As before, define \( k_{\ell,F} = \max_{\ell} k_{\ell,F} \), \( k_{\ell,C} = \min_{\ell} k_{\ell,C} \), and \( k = k_{\text{max,F}} - k_{\text{min,C}} \).

The transmitter has \( L \) messages \( w_{d,\text{info},\ell} \in \mathbb{Z}_p \) that maps the messages into a channel input matrix \( X_d = (w_{d,\text{info},1}, \ldots, w_{d,\text{info},L}) \). The rate of the \( \ell \)th message is \( (k_{\ell,F} - k_{\ell,C})/n \log p \). The channel input must satisfy a total power constraint \( E[\text{Tr}(X_d^* X_d)] \leq n \rho_d, \text{total} \).

Let \( y_{d,m} \in \mathbb{R}^n \) denote the observation of receiver \( m \) and let \( Y_d = [y_{d,1} \cdots y_{d,L}]^T \). The receivers’ observations can be expressed succinctly as
\[
Y_d = H_d X_d + Z_d
\]
where \( H_d \) is a fixed \( L \times L \) channel matrix and \( Z_d \) is element-wise i.i.d. Gaussian noise with mean zero and variance one.

B. Coding Scheme Overview

As in the uplink coding scheme, we will employ a collection of nested coarse lattices \( \Lambda_{C,1}, \ldots, \Lambda_{C,L} \) and fine lattices \( \Lambda_{F,1}, \ldots, \Lambda_{F,L} \). From these, we construct \( L \) nested lattice codebooks, \( \Phi_\ell = \Lambda_{F,\ell} \mathcal{V}_{C,\ell}, \ell = 1, \ldots, L \). The coarse lattices are designed to have second moments \( \rho_{d,c,1}, \ldots, \rho_{d,c,L} \), respectively, meaning that if \( s_{d,\ell} \) is uniformly distributed over \( \mathcal{V}_{C,\ell} \), then \( \frac{1}{n}E[\|s_{d,\ell}\|^2] = \rho_{d,c,\ell} \). The fine lattices are designed to have effective noise tolerances \( \sigma_{d,\text{eff},\pi^{-1}(1)} + \delta_n, \ldots, \sigma_{d,\text{eff},\pi^{-1}(L)} + \delta_n \) where \( \delta_n \) tends to zero as \( n \to \infty \). Let \( \Lambda_{\text{max,F}} \) and \( \Lambda_{\text{min,C}} \) denote the finest and coarsest lattice in the collection, respectively.

**Encoding:** As before, each message is zero-padded to length \( k, w_{d,\ell} = \{0\}_{k_{\ell,C} - k_{\ell,C}} \otimes (w_{d,\text{info},\ell})^T \). The linear combinations are pre-computed and the bottom signal levels are zeroed out to match each receiver’s decoding capability,
\[
\tilde{u}_{d,m}[i] = \{ \mathbb{Q}_{\ell=1}^{d,m} q_{d,m,i} w_{d,\ell}[i] \} \quad 1 \leq i \leq k_{\ell,F} - k_{\ell,C}, \ell = 1, \ldots, L
\]
\[
[\tilde{w}_{d,1} \cdots \tilde{w}_{d,L}]^T = Q_{\ell=1}^{d} [\tilde{u}_{d,1} \cdots \tilde{u}_{d,L}]^T
\]
(15) (16) (17) (18)

2For instance, if \( A_d = I \), we must set \( \theta(m) = m \). Otherwise, some of the bottom signal levels will be lost.
where \( Q_d = \{q_{d,m,t} \} \) and all operations are performed over \( \mathbb{Z}_p \). The results are mapped onto lattice codewords, \( t_{d,t} = \Phi^{-1}(\bar{w}_{d,t}) \), and dithered \( s_{d,t} = [t_{d,t} + \delta_{d,t}] \mod \alpha \gamma_{c,t} \) where \( \alpha_{c,t} \) is independently generated according to a uniform distribution over \( V_{c,t} \). The dithered codewords are collected into a matrix \( S_d = [s_{d,1}, \ldots, s_{d,L}]^T \), multiplied by the beamforming matrix \( B_d \in \mathbb{R}^{L \times L} \), and sent over the channel, \( X_d = B_d S_d \).

**Decoding:** Let \( D_d = [d_{d,1} \ldots d_{d,L}]^T \) be the dither matrix, \( \alpha_d = [\alpha_{d,1} \ldots \alpha_{d,L}]^T \) be the vector of scaling factors to be used at the receivers, \( C_d = \operatorname{diag}(\alpha_d) \), and \( c_{d,m} \) be the \( m \)th row of \( C_d \). Let \( \rho_{d,c} = [\rho_{c,1} \ldots \rho_{c,L}]^T \) be the uplink coding power vector and \( P_{c,d} = \operatorname{diag}(\rho_{d,c}) \) the associated diagonal matrix. Also, let \( \tilde{t}_{d,t} \equiv \tilde{t}_{d,t} - Q_{\gamma_{c,t}}(\tilde{t}_{d,t}) \) and \( \tilde{T}_d = [\tilde{t}_{d,1} \ldots \tilde{t}_{d,L}]^T \).

The \( m \)th receiver scales its observation by \( \alpha_{d,m} \), and removes the dithers. This can be written compactly in matrix form,

\[
\tilde{Y}_d = [C_d Y_d - A_d D_d] \mod \gamma_{c,m} \quad (19)
\]

\[
Z_{d,m} \equiv [A_d \tilde{T}_d + Z_{d,m}] \mod \gamma_{c,m} \quad (20)
\]

Define \( v_{d,m}^T = [a_{d,m}^T \tilde{T}_d] \) mod \( \gamma_{c,m} \) and note that the \( m \)th row \( \tilde{y}_{d,m} \) of \( \tilde{Y}_d \) can be written as

\[
\tilde{y}_{d,m} = [v_{d,m}^T + z_{d,m}^T] \mod \gamma_{c,m} \quad (22)
\]

where \( z_{d,m}^T \) is the \( m \)th row of \( Z_{d,m} \) and satisfies

\[
\frac{1}{n} \mathbb{E} \|z_{d,m}^T\|^2 = \|c_{d,m}^T\|^2 + (c_{d,m}^T B_d - a_{d,m}^T) P_{c,d}(c_{d,m}^T B_d - a_{d,m}^T)^T \quad (23)
\]

In the interest of establishing a duality relationship, we focus on the special case where \( A_d = A_d^T \). Starting from Lemma 1, it can be shown that there exists a lower triangular matrix \( L_d \in \mathbb{Z}_p^{L \times L} \), such that \( L_d P_{\gamma_{c,m}} Q_{\gamma_{c,m}} = \) unit upper triangular, where \( \pi \) is a valid permutation for \( A_d \). It follows that setting \( \theta(m) = \pi(m) \) yields a valid permutation for the downlink with \( \sigma_{d,m}^2 = \frac{1}{n} \mathbb{E} \|z_{d,m}^T\|^2 \). (24)

Decoding is straightforward: each receiver quantizes its observation onto the associated fine lattice and maps back to \( \mathbb{Z}_p^k \).

\[
\hat{v}_{d,m} = [Q_{\gamma_{c,m}}(\tilde{y}_{d,m})] \mod \gamma_{c,m} \quad (25)
\]

\[
\hat{u}_{d,m} = \Phi^{-1}(\hat{v}_{d,m}) \quad (26)
\]

It can be shown that this is successful with probability tending to 1 as \( n \rightarrow \infty \), since \( \Lambda_{\gamma_{c,m}} \) is designed to tolerate \( z_{d,m}^T \).

### C. Achievable Computation Rates

Overall, the downlink scheme yields the following achievable rates.

**Theorem 2.** For a given channel matrix \( H_d \in \mathbb{R}^{L \times L} \), integer matrix \( A_d = A_d^T \), beamforming matrix \( B_d \in \mathbb{R}^{L \times L} \), (diagonal) projection matrix \( C_d \in \mathbb{R}^{L \times L} \), and (diagonal) power matrix \( P_{c,d} \in \mathbb{R}^{L \times L} \), the following computation rate tuple is achievable

\[
R_{d,t} = \frac{1}{2} \log^+ \left( \frac{\rho_{c,t} \sigma_{d,m}^2}{\sigma_{d,m}^2 \rho_{c,m}^2} \right) \quad \ell = 1, \ldots, L \quad (27)
\]

where \( \pi \) is a valid permutation order for \( A_d \) from Lemma 1.

### IV. Uplink-Downlink Duality

**Theorem 3.** For a given uplink channel matrix \( H_u \), integer matrix \( A_u \), and (diagonal) power matrix \( P_{c,u} \) that meets the total power constraint \( \rho_{d,c,u} \gamma_{c,u} = \rho_{u,t} \), total, let \( R_{u,1}, \ldots, R_{u,L} \) be a computation rate tuple that is achievable under Theorem 1 using projection matrix \( B_u \), precoding matrix \( C_u = \gamma_{c,u} \), and permutation \( \pi \). Then, for the downlink channel matrix \( H_u = H_u^T \), integer matrix \( A_u = A_u^T \), there exists a (diagonal) power matrix \( P_{c,d} \) with total power usage \( \rho_{d,c,u} \gamma_{c,u} = \rho_{d,c} \), such that the computation rate tuple \( R_{d,1} = \gamma_{u,1}, \ldots, R_{d,L} = \gamma_{u,L} \) is achievable under Theorem 2 using (diagonal) projection matrix \( C_u = C_u^T \) and precoding matrix \( B_d = B_d^T \). The same relationship can be established starting from an achievable rate tuple for the downlink and going to the uplink.

We will need some basic results on non-negative matrices for the proof. A vector or a matrix is non-negative (i.e., \( F \geq 0 \)) if all its entries are non-negative. A vector or a matrix is positive (i.e., \( F > 0 \)) if all its entries are positive. A square matrix \( F \) is a Z-matrix if all its off-diagonal elements are non-positive. An M-matrix is a Z-matrix with eigenvalues whose real parts are positive.

**Lemma 2.** Let \( F \) be a square Z-matrix. The following statements are equivalent:

i) \( F \) is a non-singular M-matrix with a non-negative inverse. That is, \( F^{-1} \) exists and \( F^{-1} \geq 0 \)

ii) There exists \( x \geq 0 \) satisfying \( Fx > 0 \)

iii) Every real eigenvalue of \( F \) is positive.

See [13] for a proof.

**Proof of Theorem 3:** Our proof is inspired by the approach of [7]. Let \( \gamma_{c,t} = \gamma_{c,t} \gamma_{c,m} \) and \( \beta_{c,t} = \beta_{c,t} \beta_{c,m} \) denote the SINRs for the \( \ell \)th uplink and downlink users, respectively. Define \( \gamma = (\gamma_1, \ldots, \gamma_L)^T \) and \( \beta = (\beta_1, \ldots, \beta_L)^T \).

Recall \( \rho_{c,u} \) and \( \rho_{c,d} \) are the power vectors corresponding to the coarse lattices for the uplink and downlink, respectively.

Define \( G_u \equiv \operatorname{diag}(\alpha_{d,1}^2, \ldots, \alpha_{d,L}^2) \) and \( J_u \equiv \operatorname{diag}((\|b_{u,\pi(m)}\|_2)^2, \|b_{u,\pi(L)}\|_2^2) \) and \( M_u \) as the \( L \times L \) matrix with \( (m, \ell) \)th entry \( (b_{u,\pi(m)}^T h_{u,\pi(m)} - a_{u,\pi(m)})^2 \), where \( b_{d,m} \) is the \( \ell \)th column of \( H_d \). The uplink SINR conditions can be expressed as

\[
(I - \operatorname{diag}(\gamma) M_u) \rho_{c,u} \geq 0 \quad (28)
\]

Similarly, define \( G_d \equiv \operatorname{diag}(\alpha_{d,1}^2, \ldots, \alpha_{d,L}^2) \) and \( J_d \) to be the \( L \times L \) matrix with \( (m, \ell) \)th entry \( b_{d,m}^2 \), where \( b_{d,m} \) is the \( (m, \ell) \)th entry of \( B_d \), and \( M_d \) to be the \( L \times L \) matrix \((m, \ell)\)th entry \( \alpha_{d,\pi(m)}^2 b_{d,m}^2 \) where \( b_{d,m} \) is...
Finally, it can be shown that the total power in the downlink equals the uplink SIRs monotonically increase with each iteration, and, since the total power is bounded, this algorithm will converge to a local optimum. Our simulations (which are deferred to an extended version) verify this behavior.

VI. CONCLUSIONS

We have demonstrated that uplink-downlink duality holds for integer-forcing. Our future efforts will concentrate on generalizing this result to include non-square channel and integer coefficient matrices as well as investigating iterative algorithms to optimize beamforming and projection matrices.

REFERENCES

[1] Z. Jhan, B. Nazer, U. Erez, and M. Gastpar, “Integer-forcing linear receivers,” in IEEE Trans. IT, accepted. Available online: http://arxiv.org/abs/1003.5966.

[2] O. Ordentlich and U. Erez, “Precoded integer-forcing equalization universally achieves the MIMO capacity up to a constant gap,” Submitted, 2013. Available online: http://arxiv.org/abs/1301.6393.

[3] S.-N. Hong and G. Caire, “Compute-and-Forward Strategies for Cooperative Distributed Antenna Systems,” in IEEE Trans. IT, vol. 59, pp. 5227–5243, Sep. 2013.

[4] V. Ntanos, V. Cadambe, B. Nazer, and G. Caire, “Integer-forcing interference alignment,” in ISIT 2013, Istanbul, Turkey, July 2013.

[5] G. Caire and S. Shamai, “On the achievable throughput of a multiantenna Gaussian broadcast channel,” IEEE Trans. IT, vol. 49, pp. 1691–1706, Jul. 2003.

[6] S. Vishwanath, N. Jindal, and A. Goldsmith, “Duality, achievable rates, and sum-rate capacity of Gaussian MIMO broadcast channels,” IEEE Trans. IT, vol. 49, pp. 2658–2668, Oct. 2003.

[7] P. Viswanath and D. Tse, “Sum capacity of the vector Gaussian broadcast channel and uplink-downlink duality,” IEEE Trans. IT, vol. 49, pp. 1912–1921, Aug. 2003.

[8] W. Yu and J. M. Cioffi, “Sum capacity of Gaussian vector broadcast channels,” IEEE Trans. IT, vol. 50, pp. 1875–1892, Sep. 2004.

[9] H. Weingarten, Y. Steinberg, and S. Shamai, “The capacity region of the Gaussian multiple-input multiple-output broadcast channel,” IEEE Trans. IT, vol. 52, pp. 3936–3964, Sep. 2006.

[10] O. Ordentlich, U. Erez, and B. Nazer, “Successive integer-forcing and its sum-rate optimality,” in 51st Allerton Conference, (Monticello, IL), Oct. 2013.

[11] V. Ntanos, V. Cadambe, B. Nazer, and G. Caire, “Asymmetric compute-and-forward,” in 51st Allerton Conference, (Monticello, IL), Oct. 2013.

[12] O. Ordentlich, U. Erez, and B. Nazer, “The approximate sum capacity of the symmetric K-user Gaussian interference channel,” IEEE Trans. IT, to appear. Available online: http://dx.doi.org/10.1109/TIT.2014.2316136.

[13] R. Plemons, “M-matrix characterizations. I-nonsingular M-matrices,” Linear Algebra and its Applications, vol. 18, no. 2, pp. 175–188, 1977.