Càdlàg Skorokhod problem driven by a maximal monotone operator

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Abstract

The article deals with existence and uniqueness of the solution of the following differential equation (a càdlàg Skorokhod problem) driven by a maximal monotone operator and with singular input generated by the càdlàg function $m$:

\[
\begin{cases}
  dx_t + A(x_t)(dt) + dk^d_t \ni dm_t, & t \geq 0, \\
x_0 = m_0,
\end{cases}
\]

where $k^d$ is a pure jump function.

The jumps outside of the constrained domain $\text{D}(A)$ are counteracted through the generalized projection $\Pi$, by putting $x_t = \Pi(x_{t-} + \Delta m_t)$, if $x_{t-} + \Delta m_t \not\in \text{D}(A)$.

Approximations of the solution based on discretization and Yosida penalization are considered.

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1 Introduction

Let $\mathcal{H}$ be a separable real Hilbert space and $A : \mathcal{H} \rightharpoonup \mathcal{H}$ be a maximal monotone multivalued operator on $\mathcal{H}$ with the domain $\text{D}(A) = \{z \in \mathcal{H} : A(z) \neq \emptyset\}$ and its graph $\text{Gr}(A) = \{(z, y) \in \mathcal{H} \times \mathcal{H} : z \in \mathcal{H}, y \in A(z)\}$.

Let $\Pi : \mathcal{H} \rightarrow \text{D}(A)$ be a generalized projection on $\text{D}(A)$ (see Definition 8) and $m$ a càdlàg function.

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The main topic of our paper is the existence and uniqueness of a solution $x$ of the following multivalued differential equation with singular input $dm_t$:

$$
\begin{aligned}
\begin{cases}
\frac{dx_t}{dt} + A(x_t)(dt) + dk^d_t \ni dm_t, \quad t \geq 0, \\
x_0 = m_0,
\end{cases}
\end{aligned}
$$

(1)

where $x_t = \Pi(x_{t-} + \Delta m_t)$ if $x_{t-} + \Delta m_t \notin D(A)$ and $k^d$ a pure jump function such that $\Delta k^d_t = \Delta m_t - \Delta x_t$.

The generalized projection $\Pi$ ensures that the jumps from $x_{t-}$ to $x_{t-} + \Delta m_t$ are counteracted whenever these jumps leave the domain.

By a solution of (1) we understand a pair $(x, k)$ of càdlàg functions such that

$$
x_t + k_t = m_t, \quad t \geq 0,
$$

$x_t \in \overline{D(A)}$, for any $t \geq 0$, $k = k^c + k^d$ has locally bounded variation with $k^c$ being its continuous part, $dk^c_t \in A(x_t)(dt)$ (see Definition 48) and

$$
k^d_t = \sum_{0 \leq s \leq t} \Delta k_s = \sum_{0 \leq s \leq t} (I - \Pi) (x_s - \Delta m_s).
$$

In the particular case when $\Pi$ is the orthogonal projection equation (1) is equivalent with the problem

$$
\begin{aligned}
\begin{cases}
x_t + k_t = m_t, \quad t \geq 0, \\
dk_t \in A(x_t)(dt).
\end{cases}
\end{aligned}
$$

(2)

When $m$ has some smoothness properties, the equation (2) was intensely studied in the frame of nonlinear analysis (see, e.g. Barbu [1], Brezis [5] and their references). Strong and generalized solutions were defined. In the case when the interior of the domain of $A$ is nonempty the generalized solution can be reformulated in a similar manner as for the Skorohod problem. In fact, the convex Skorohod problem is a particular case when $A$ is the subdifferential of the indicator function of a closed convex set (see Subsection 1.1 for more details).

Particular cases of the above type equations were considered earlier in many papers. The existence and uniqueness of solution for (2) was proved independently by Răşcanu in [18] (for the infinite dimensional framework) and by Cépa in [7] (the finite dimensional case). Barbu and Răşcanu in [3] studied parabolic variational inequalities with singular inputs (equation formally written as $dx_t + A(x_t)(dt) \ni f_t(dt + dm_t)$ with $A$ of the form $A = A_0 + \partial \varphi$, where $A_0$ is a linear positive defined operator and $\partial \varphi$ is the subdifferential operator associated to a convex lower semicontinuous function $\varphi$ (see Annexes 4.1). It is worth pointing out that the authors of all the mentioned above papers restricted themselves to processes with continuous trajectories.

It is well known that for every nonempty closed convex set $\bar{D} \subset \mathbb{R}^d$ its indicator function $\varphi(x) = I_{\bar{D}}(x) = \begin{cases} 0, & \text{if } x \in \bar{D}, \\ +\infty, & \text{if } x \notin \bar{D}. \end{cases}$ is a convex lower semicontinuous function (see Annexes 4.1). This implies that deterministic equation with reflecting boundary condition in convex domains are special cases of (1). Such equations were introduced by Skorokhod [20] in the one-dimensional case and $D = \mathbb{R}^+$. The multidimensional version of Skorokhod’s equation was
studied in detail by Tanaka in [23] in the case of a convex domain $D$. The reflection problem in non-convex domain was studied by Lions and Sznitman in [13].

Approximations of solutions of the Skorohod problem, considered in the càdlàg case, were studied by Laukajtys and Slomiński in [15]. We also mention that in this paper is involved the classical orthogonal projection on $\overline{D(A)}$, i.e.,

$$\Pi_{\overline{D(A)}}(x) \in \overline{D(A)} \quad \text{and} \quad |\Pi_{\overline{D(A)}}(x) - x| = \inf\{|z - x| : z \in \overline{D(A)}\}.$$  

Skorohod problem with jumps and two time-depending obstacles was treated by Slomiński and Wojciechowski [21].

In the present paper we study existence, uniqueness and approximations of solutions of equation (1), provided that

$$\text{Int} \ (D (A)) \neq \emptyset \quad (3)$$

and $\Pi$ is a generalized projection on the $\overline{D(A)}$. The results concerning the multivalued deterministic equation with jumps, considered in our work are new, even in the case of classical projection $\Pi_{\overline{D(A)}}$. We should mention that assumption (3) is essential for the proof. Although in the finite dimensional case this assumption is not a strong restriction, in the Hilbert space framework this assumption become quite restrictive. But the existence results provided by this research are essential in order to obtain further existence results for the same problem without condition (3) (we recall article [18] where it was considered the continuous version of (2) and where assumption (3) is suppressed). In addition, the infinite dimensional frame considered here allows to consider infinite dimensional systems of type (1).

We outline that the infinite systems of equations (with or without state constraints) describe a various types of real problems which appear in mechanics, in the theory of branching processes, in the theory of neural networks, etc. (see Zautykov [24], Zautykov & Valeev [25] or reference from [16] for a more complete literature scene of applications). Infinite systems of differential equations also appear in partial differential equations which are treated through semidiscretization.

Also very important applications of (1) are in the stochastic reflection problems, in which case $dm_t$ is interpreted as a random noise on the system (see [4, 7, 13, 14, 15, 18, 19, 21]).

The paper is organized as follows. In Section 2 we prove existence and uniqueness of solutions and we give some convergence results in the uniform norm and in the Skorokhod topology $J_1$. We also give examples of projections $\Pi$.

Section 3 is devoted to the Yosida approximations of solutions. If we consider the approximating equation of the form

$$x_t^\varepsilon + \int_0^t A_\varepsilon(x_s^\varepsilon)ds = m_t, \quad t \in \mathbb{R}^+,$$

where $A_\varepsilon(z) = \frac{1}{\varepsilon}(z - J_\varepsilon(z))$ and $J_\varepsilon(z) = (I + \varepsilon A)^{-1}(z)$, then

$$x_t^\varepsilon \to x_t 1_{|\Delta m_t| = 0} + (x_{t-} + \Delta m_t) 1_{|\Delta m_t| > 0}, \quad \forall t \in \mathbb{R}^+, \quad \varepsilon \searrow 0,$$

where $x$ is the solution of (1) associated to the classical projection; also other convergence results are obtained.
When the approximating equation has the form
\[ x^\varepsilon_t + \int_0^t A_\varepsilon(x^\varepsilon_s)ds + \sum_{0 \leq s \leq t} (I - \Pi) (x^\varepsilon_{s-} + \Delta m_s) 1_{\Delta m_s > \varepsilon} = m_t, \quad t \in \mathbb{R}^+, \]
with a generalized projection \( \Pi \), we prove that \( x^\varepsilon \) is uniformly convergent on every compact interval to the solution of (1).

In the paper we use the following notations: \( \mathcal{D}(\mathbb{R}^+, \mathcal{H}) \) is the space of all càdlàg mappings \( y : \mathbb{R}^+ \to \mathcal{H} \) (i.e., \( y \) is right continuous and admits left-hand limits), endowed with the Skorokhod topology \( J_1 \). For \( y \in \mathcal{D}(\mathbb{R}^+, \mathcal{H}) \), \( \delta > 0 \), \( T \in \mathbb{R}^+ \) we denote by \( \gamma_y(\delta, T) \) the modulus of càdlàg continuity of \( y \) (càdlàg modulus of \( y \)) on \([0, T]\), i.e.
\[ \gamma_y(\delta, T) = \inf \{ \max_{i \leq t} O_y([t_{i-1}, t_i)) : 0 = t_0 < \ldots < t_r = T, \inf_{i < r} (t_i - t_{i-1}) \geq \delta \}, \]
where \( O_y(I) := \sup_{s, t \in I} |y_s - y_t| \).

If \( k \in \mathcal{D}(\mathbb{R}^+, \mathcal{H}) \) is a function with locally bounded variation then \( \mathcal{D}_{[t, T]} k_{[0, T]} \) stands for its variation on \((t, T)\) and \( \mathcal{D}_{[t, T]} = \mathcal{D}_{[0, T]} \).
\[ k^c_t := k_t - \sum_{s \leq t} \Delta k_s \quad \text{and} \quad k^d_t := k_t - k^c_t, \quad t, T \in \mathbb{R}^+. \]

Set \( \|x\|_{[s, t]} := \sup_{r \in [s, t]} |x_r| \) and \( \|x\|_{t} := \|x\|_{[0, t]} \).

1.1 Survey on existence results for (2) in the absolutely continuous case

At the end of this section we recall some historical results concerning the existence of a solution for problem (2) without jumps.

Let \( A : \mathcal{H} \Rightarrow \mathcal{H} \) be a maximal monotone operator and (used throughout this section)
\[ m_t := m_0 + \int_0^t f_s ds, \]
where \( m_0 \in \overline{D}(A) \) and \( f \in L^1_{loc}(\mathbb{R}^+; \mathcal{H}) \).

The strong solution of the Cauchy problem
\[ \begin{cases} \dfrac{dx_t}{dt} + A(x_t)(dt) \ni dm_t, \ a.e. \ t > 0, \\ x_0 = m_0, \end{cases} \]
is defined as a continuous function \( x : \mathbb{R}^+ \to \mathcal{H} \) satisfying:
\[ (i) \quad x_t \in D(A), \ a.e. \ t > 0, \]
\[ (ii) \quad \exists h \in L^1_{loc}(\mathbb{R}^+; \mathcal{H}) \text{ such that } h_t \in A(x_t), \ a.e. \ t > 0, \text{ and } \]
\[ x_t + \int_0^t h_s ds = m_t, \ \forall \ t \geq 0, \]
and we will denote \( x = S(A, m_0; f) \).
We introduce the notation

\[ W_{\text{loc}}^1(\mathbb{R}^+;\mathbb{H}) = \left\{ f : \exists a \in \mathbb{H}, \; g \in L_{\text{loc}}^1(\mathbb{R}^+;\mathbb{H}) \text{ such that} \right. \]
\[ f_t = a + \int_0^t g_s ds, \; \forall \; t \geq 0 \} . \]

The following results are recall from Barbu [2, Theorem 1.12] and [1, Chapter IV, Theorem 2.5] respectively. Let us first denote by \( A_\varepsilon \) the Yosida approximation of the operator \( A \), i.e.

\[ A_\varepsilon(x) := \frac{1}{\varepsilon}(x - (I + \varepsilon A)^{-1}(x)) . \]

We mention that \( A_\varepsilon : \mathbb{H} \to \mathbb{H} \) is a single valued maximal monotone operator with \( D(A_\varepsilon) = \mathbb{H} \).

**Proposition 1** Let \( A \) be a maximal monotone operator on \( \mathbb{H} \) and \( m_0 \in D(A) \). If \( f \in W_{\text{loc}}^{1,1}(\mathbb{R}^+;\mathbb{H}) \) then the Cauchy problem (4) has a unique strong solution \( x \in W^{1,\infty}([0,T];\mathbb{H}) \). Moreover if \( x_\varepsilon \) is the solution of the approximate equation

\[ x_\varepsilon^t + \int_0^t A_\varepsilon(x_\varepsilon^s) ds = m_t , \; t \geq 0, \]

then for all \( (u_0,v_0) \in \text{Gr}(A) \) and \( T > 0 \), there exists a constant \( C = C(T,u_0,v_0) > 0 \) such that

1. \( \|x_\varepsilon\|_T \leq C(1 + |m_0| + \|f\|_{L^1(0,T;\mathbb{H})}) \)
2. \( \lim_{\varepsilon \to 0} \|x_\varepsilon - x\|_T = 0. \)

**Proposition 2** Let \( A \) be a maximal monotone operator on \( \mathbb{H} \) such that \( \text{Int}(D(A)) \neq \emptyset \). If \( m_0 \in \overline{D(A)} \) and \( f \in W_{\text{loc}}^{1,1}(\mathbb{R}^+;\mathbb{H}) \), then the Cauchy problem (4) has a unique strong solution \( x \in W_{\text{loc}}^{1,1}(\mathbb{R}^+;\mathbb{H}) \).

By continuity property one can generalize the notion of the solution of equation (4) as follows: \( x \) is a generalized solution of the Cauchy problem (4) with \( m_0 \in \overline{D(A)} \), \( f \in L_{\text{loc}}^1(\mathbb{R}^+;\mathbb{H}) \) (and we shall denote \( x = GS(A,m_0;f) \)) if for all \( T > 0 \),

\[ \diamond \; x \in C([0,T];\mathbb{H}) \]
\[ \diamond \; \text{there exist} \; m_0^n \in D(A), \; f^n \in W^{1,1}([0,T];\mathbb{H}) \; \text{such that} \]
\[ (a) \; m_0^n \to m_0 \; \text{in} \; \mathbb{H}, \]
\[ (b) \; f^n \to f \; \text{in} \; L^1(0,T;\mathbb{H}), \]
\[ (c) \; x^n = S(A,m_0^n;f^n) \to x \; \text{in} \; C([0,T];\mathbb{H}). \]

We have:

**Proposition 3** If \( A \) is maximal monotone operator on \( \mathbb{H} \), \( m_0 \in \overline{D(A)} \) and \( f \in L_{\text{loc}}^1(\mathbb{R}^+;\mathbb{H}) \), then the Cauchy problem (4) has a unique generalized solution \( x : \mathbb{R}^+ \to \mathbb{H} \). If \( x = GS(A,m_0;f) \) and \( y = GS(A,\hat{m}_0;\hat{f}) \) then

\[ |x_t - y_t| \leq |m_0 - \hat{m}_0| + \int_0^t |f_s - \hat{f}_s| ds, \; t \geq 0. \] (5)
Moreover, if \((m_0, \tilde{m}_0) \in \text{Gr}(A), T > 0\), then there exists a constant \(C = C(T, m_0, \tilde{m}_0) > 0\) such that
\[
\|x\|_T \leq C \left( 1 + |m_0| + \|f\|_{L^1(0,T; \mathcal{H})} \right).
\] (6)

In the case when \(\text{Int}(\text{D}(A)) \neq \emptyset\) one can give an equivalent formulation for the generalized solutions, which is analogous to the Skorohod problem.

**Proposition 4** Let \(A : \mathcal{H} \rightrightarrows \mathcal{H}\) be a maximal monotone operator such that \(\text{Int} (\text{D}(A)) \neq \emptyset\), \(m_0 \in \overline{\text{D}(A)}\) and \(f \in L^1_{\text{loc}}(\mathbb{R}^+; \mathcal{H})\). Then:
1. there exists a unique couple of continuous functions \((x, k) : \mathbb{R}^+ \rightarrow \mathcal{H} \times \mathcal{H}\) such that
\[
\begin{align*}
(a) & \quad x(t) \in \overline{\text{D}(A)}, \quad \forall t \geq 0, \\
(b) & \quad k \in \text{BV}_{\text{loc}}(\mathbb{R}^+; \mathcal{H}), \quad k(0) = 0, \\
(c) & \quad x_t + k_t = m_t, \quad \forall t \geq 0, \\
(d) & \quad \int_s^t \langle x_r - \alpha, dk_r - \beta dr \rangle \geq 0, \quad \forall 0 \leq s \leq t, \forall (\alpha, \beta) \in \text{Gr}(A).
\end{align*}
\] (7)
2. \(x = \mathcal{G}\mathcal{S}(A,m_0;f)\) if and only if \(x\) is solution of the problem \((SP)\).
3. the following estimate holds: for all \(T > 0\),
\[
\|x\|_T^2 + \|k\|_T^2 \leq C_T \left( 1 + |m_0|^2 + \|f\|_{L^1(0,T; \mathcal{H})}^2 \right),
\]
where \(C_T\) is a positive constant independent of \(m_0\) and \(f\).

The problem \((SP)\) will be called generalized Skorohod problem associated to the maximal monotone operator \(A\).

**Proposition 5** (see Barbu [1, Chapter III, Section 1]) If \(m_t \equiv m_0 \in \overline{\text{D}(A)}\) then the solution of \((7)\) is given by
\[
x_t = S_A(t)m_0,
\]
where \(\{S_A(t)\}_{t \geq 0}\) is the nonlinear semigroup of contractions generated by \(A\):
\[
S_A(t)y = \lim_{n \to \infty} \left( I + \frac{t}{n}A \right)^{-1} y, \quad y \in \overline{\text{D}(A)}
\]
and the limit is uniform w.r.t. \(t \in [0,T], \forall T > 0\).

**Proposition 6** (Brezis [5, Lemma 3.8]) Let \(\alpha^\varepsilon, \alpha \in \mathcal{H}\) such that \(\alpha^\varepsilon \to \alpha\), as \(\varepsilon \to 0\). If \(\alpha \in \overline{\text{D}(A)}\), then for all \(T > 0\),
\[
\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} |S_{A^\varepsilon}(t)\alpha^\varepsilon - S_A(t)\alpha| = 0,
\]
and, if \(\alpha \notin \overline{\text{D}(A)}\), then
\[
\lim_{\varepsilon \to 0} \int_0^T \left| S_{A^\varepsilon}(t)\alpha^\varepsilon - S_A(t)\Pi_{\overline{\text{D}(A)}}(\alpha) \right|^2 dt = 0.
\]
We mention that the classical Skorohod problem corresponds to the case \( A = \partial I_D(x) \) where \( I_D \) is the indicator function of a convex and closed \( D \subset \mathbb{H} \), i.e.
\[
I_D(x) = \begin{cases} 
0, & \text{if } x \in D, \\
+\infty, & \text{if } x \notin D.
\end{cases}
\]

In this case
\[
A = \begin{cases} 
0, & \text{if } x \in D, \\
\mathcal{N}_D(x), & \text{if } x \in \text{Bd} \,(D), \\
\emptyset, & \text{if } x \in \mathbb{R}^d \setminus \bar{D},
\end{cases}
\]
where \( \mathcal{N}_D(x) \) is the outward normal cone to \( D \) at \( x \in \text{Bd} \,(D) \). In this case the term
\[-\partial I_D(x_t)(dt),\]
which is added to the input \( dm_t \), acts, in a minimal way, as an “inward push” that prevents \( x_t \) from exiting the domain \( \bar{D} \).

The following result can be found in Răşcanu [18].

**Theorem 7** If \( A : \mathbb{H} \rightrightarrows \mathbb{H} \) is a maximal monotone operator such that \( \text{Int} \,(D(A)) \neq \emptyset \) and \( m : \mathbb{R}^+ \to \mathbb{H} \) is a continuous function with \( m_0 \in \overline{D(A)} \), then the convex Skorohod problem (7) has a unique solution \((x,k)\).

## 2 The Skorokhod problem with maximal monotone operators

Let \( \mathbb{H} \) be Hilbert space with the inner product \( \langle \cdot, \cdot \rangle \) and the norm induced \( |\cdot| \). Let \( A : \mathbb{H} \rightrightarrows \mathbb{H} \) be a maximal monotone operator (notation \( \rightrightarrows \) means that \( A \) is a multivalued operator).

We formally identify \( A \) with its graph \( \text{Gr} \,(A) = \{(x,y) \in \mathbb{H} \times \mathbb{H} : y \in A(x)\} \). We denote by \( D(A) \), the domain of \( A \)
\[
D(A) = \{x \in \mathbb{H} : Ax \neq \emptyset\}.
\]

**Definition 8** A mapping \( \Pi : \mathbb{H} \to \mathbb{H} \) is a generalized projection on \( \overline{D(A)} \) if

(i) \( \Pi(\mathbb{H}) \subset \overline{D(A)} \) and \( \Pi(x) = x \), for all \( x \in \overline{D(A)} \)

(ii) \( |\Pi(x) - \Pi(y)| \leq |x - y| \), for all \( x, y \in \mathbb{H} \) (i.e. \( \Pi \) is a nonexpansive map).

The (classical) orthogonal projection \( \Pi_{\overline{D(A)}} : \mathbb{H} \to \overline{D(A)} \) is defined by: for all \( x \in \mathbb{H} \)
\[
|x - \Pi_{\overline{D(A)}}(x)| = \inf \{ |x - a| : a \in \overline{D(A)} \}
\]
(8)

(\( \Pi_{\overline{D(A)}}(x) \) is well defined by (8) since \( \overline{D(A)} \) is a closed convex set; see Proposition 36 from Annexes).

It is easily to prove that \( \hat{x} = \Pi_{\overline{D(A)}}(x) \) if and only
\[
\hat{x} \in \overline{D(A)} \text{ and } \langle \hat{x} - x, \hat{x} - a \rangle \leq 0, \text{ for all } a \in \overline{D(A)}.
\]

Hence for all \( x, y \in \mathbb{H} \) we have
\[
\langle x - \Pi_{\overline{D(A)}}(x), \Pi_{\overline{D(A)}}(y) - \Pi_{\overline{D(A)}}(x) \rangle \leq 0 \text{ and } \langle y - \Pi_{\overline{D(A)}}(y), \Pi_{\overline{D(A)}}(x) - \Pi_{\overline{D(A)}}(y) \rangle \leq 0
\]
which yields
\[
|\Pi_{D(A)}^\delta(x) - \Pi_{D(A)}^\delta(y)|^2 \leq (\Pi_{D(A)}^\delta(x) - \Pi_{D(A)}^\delta(y), x - y), \quad \text{for all } x, y \in \mathcal{H}, \quad (10)
\]
If follows that \(\Pi_{D(A)}^\delta\) is a generalized projection on \(\overline{D(A)}\).

There exist important examples of projections on \(\overline{D(A)}\) connected with the so-called
elasticity condition introduced in one-dimensional case in [6] and [21] (see also [19]). Let \(\delta \in [0, 1]\). We will consider the map \(\Pi^\delta : \mathcal{H} \to \mathcal{H}\) defined by
\[
\Pi^\delta(z) = \Pi_{D(A)}^\delta(z) - \delta(z - \Pi_{D(A)}^\delta(z)), \quad z \in \mathcal{H}
\]
and its compositions \(\Pi_n^\delta, n \in \mathbb{N}^*\) of the form
\[
\Pi_n^\delta(z) = \Pi_n \circ \cdots \circ \Pi_1(z), \quad z \in \mathcal{H}, \text{ where } \Pi_1 = \ldots = \Pi_n = \Pi^\delta.
\]

**Proposition 9**

(i) For any \(z \in \mathcal{H}\), \(a \in \text{Int} (D(A))\) and \(B(a, r_0) \subset \overline{D(A)}\) with \(r_0 > 0\):
\[
|\Pi^\delta(z) - a| \leq (1 - \delta^2) |z - \Pi_{D(A)}^\delta(z)|^2 + 2r_0(1 + \delta)|z - \Pi_{D(A)}^\delta(z)| \leq |z - a|.
\]
(ii) The map \(\Pi_n^\delta : \mathcal{H} \to \mathcal{H}\) is a nonexpansive map and \(\Pi_n^\delta(z) = z, \forall z \in \overline{D(A)}\).

(iii) For all \(z \in \mathcal{H}\) there exists the limit \(\lim_{n \to \infty} \Pi_n^\delta(z)\) and \(\Pi_{D(A)}^\delta(z) := \lim_{n \to \infty} \Pi_n^\delta(z)\) is a
generalized projection on \(\overline{D(A)}\).

**Proof.** (i) Clearly,
\[
|\Pi^\delta(z) - a|^2 = |(1 + \delta)(\Pi_{D(A)}^\delta(z) - z) + (z - a)|^2
= (1 + \delta)^2|\Pi_{D(A)}^\delta(z) - z|^2 + |z - a|^2 + 2(1 + \delta)(\Pi_{D(A)}^\delta(z) - z, z - a).
\]
Since \(B(a, r_0) \subset \overline{D(A)}\), from (39) we have for all \(z \in \mathcal{H}\),
\[
(\Pi_{D(A)}^\delta(z) - z, z - a) \leq -r_0 |z - \Pi_{D(A)}^\delta(z)| - |z - \Pi_{D(A)}^\delta(z)|^2.
\]
Consequently,
\[
|\Pi^\delta(z) - a|^2 \leq ((1 + \delta)^2 - 2(1 + \delta)) |\Pi_{D(A)}^\delta(z) - z|^2 + |z - a|^2 - 2(1 + \delta)r_0|\Pi_{D(A)}^\delta(z) - z|,
\]
which implies (i). (ii) By (10) and Lipschitz property of \(\Pi\) we see that, for all \(x, y \in \mathcal{H}\),
\[
|\Pi^\delta(x) - \Pi^\delta(y)|^2
= |(1 + \delta)(\Pi_{D(A)}^\delta(x) - \Pi_{D(A)}^\delta(y)) - \delta(x - y)|^2
= (1 + \delta)^2|\Pi_{D(A)}^\delta(x) - \Pi_{D(A)}^\delta(y)|^2 + \delta^2|x - y|^2
- 2\delta(1 + \delta)(\Pi_{D(A)}^\delta(x) - \Pi_{D(A)}^\delta(y), x - y)
\leq ((1 + \delta)^2 - 2\delta(1 + \delta)) |\Pi_{D(A)}^\delta(x) - \Pi_{D(A)}^\delta(y)|^2 + \delta^2|x - y|^2
\leq |x - y|^2.
\]
Hence for all \( n \in \mathbb{N} \) and \( x, y \in H \)
\[
|\Pi^{\delta,n}(x) - \Pi^{\delta,n}(y)| = |\Pi_n \circ \ldots \circ \Pi_1(x) - \Pi_n \circ \ldots \circ \Pi_1(y)|
\leq |\Pi_{n-1} \circ \ldots \circ \Pi_1(x) - \Pi_{n-1} \circ \ldots \circ \Pi_1(y)|
\leq |x - y|.
\]

(jj) Set \( z_0 = z, z_n = \Pi^{n,\delta}(z), n \in \mathbb{N} \). Clearly, \( z_n = \Pi^\delta(z_{n-1}), n \in \mathbb{N} \). Fix \( a \in \text{Int}(D(A)) \) and \( r_0 > 0 \) such that \( B(a, r_0) \subset D(A) \). By (j) for any \( i \in \mathbb{N} \)
\[
|z_i - a|^2 + 2r_0(1 + \delta)|\Pi_{D(A)}(z_{i-1}) - z_{i-1}| \leq |z_{i-1} - a|^2
\]
and consequently
\[
|z_n - a|^2 + 2r_0(1 + \delta) \sum_{i=0}^{n-1} \left| \Pi_{D(A)}(z_i) - z_i \right| \leq |z_0 - a|^2.
\]

Hence
\[
\sum_{i=0}^{\infty} \left| \Pi_{D(A)}(z_i) - z_i \right| \leq \frac{1}{2r_0(1 + \delta)} |z_0 - a|^2.
\]

By (jj) for any \( n \geq m \)
\[
|z_n - z_m| \leq \sum_{k=m+1}^{n} |z_k - z_{k-1}| = \sum_{k=m+1}^{n} |\Pi^\delta(z_{k-1}) - z_{k-1}|
\leq \sum_{k=m+1}^{n} \left( |\Pi^\delta(z_{k-1}) - \Pi_{D(A)}(z_{k-1})| + |\Pi_{D(A)}(z_{k-1}) - z_{k-1}| \right)
\leq 2 \sum_{k=m+1}^{n} \left| \Pi_{D(A)}(z_{k-1}) - z_{k-1} \right|.
\]

Consequently, \( \{z_n\} \) is a Cauchy sequence. Hence there exists the limit \( z_\infty = \lim_{n \to \infty} z_n \). Since
\[
\lim_{n \to \infty} |\Pi_{D(A)}(z_n) - z_n| = 0,
\]
\( z_\infty \in \overline{D(A)} \) and the proof is complete.

**Example 10** (a) If \( \overline{D(A)} \) satisfies the interior uniform ball condition, then there exists \( n_0 = n_0(z) \in \mathbb{N} \) such that \( \Pi^{\delta,n_0}(z) \in \overline{D(A)} \) and, in this case,
\[
\Pi^{\delta}_{D(A)}(z) = \begin{cases} z, & \text{if } z \in \overline{D(A)}, \\ \Pi_{n_0} \circ \ldots \circ \Pi_1(z), & \text{otherwise}, \end{cases}
\]
where \( \Pi_1 = \ldots = \Pi_{n_0} = \Pi^\delta \) and \( n_0(z) = \min\{k : \Pi_k \circ \ldots \circ \Pi_1(z) \in \overline{D(A)}\} \).

We recall first that the set \( \overline{D(A)} \subset H \) satisfies the Interior Uniform Ball Condition if \( \exists r_0 > 0 \) such that \( \forall z \notin \overline{D(A)}, \)
\[
B(\pi_{D(A)}(z) - r_0u_z, r_0) \subset \overline{D(A)},
\]
where \( u_z := \frac{z - \pi_{D(A)}(z)}{|z - \pi_{D(A)}(z)|} \).

If \( z \in \overline{D(A)} \) then \( n = 0. \) Let now \( z \notin \overline{D(A)} \) and \( u_z := \frac{z - \pi_{D(A)}(z)}{|z - \pi_{D(A)}(z)|} \). Since \( \overline{D(A)} \) satisfies the \( r_0 \)-Interior Uniform Ball Condition,

\[
\pi_{D(A)}(z) - 2r_0 u_z \in \overline{D(A)} - r_0 u_z, r_0 \subset \overline{D(A)}.
\]

Let \( z_0 = z, \ z_1 := \Pi^{\delta} z = \pi_{D(A)}(z) - \delta(z - \pi_{D(A)}(z)). \) Therefore

\[
|z_1 - \pi_{D(A)}(z_1)| \leq |z_1 - (\pi_{D(A)}(z) - 2r_0 u_z)| = |z_1 - \pi_{D(A)}(z)| - 2r_0 \leq \delta|z - \pi_{D(A)}(z)| - 2r_0
\]

and subsequently

\[
|z_2 - \pi_{D(A)}(z_2)| \leq \delta|z_1 - \pi_{D(A)}(z_1)| - 2r_0 \leq \delta^2|z - \pi_{D(A)}(z)| - 2\delta r_0 - 2r_0.
\]

Obviously,

\[
|z_n - \pi_{D(A)}(z_n)| \leq \delta^n|z - \pi_{D(A)}(z)| - 2r_0 (1 + \delta + \cdots + \delta^{n-1}) =: \alpha_n
\]

Now we should consider two cases \( \delta = 1 \) and \( \delta \in (0,1). \) In the both cases it is easy to check that there exists \( n_0 = n_0(z) \in \mathbb{N}^* \) such that

\[
\alpha_{n_0 - 1} > 0 \text{ and } \alpha_{n_0} \leq 0,
\]

which yields \( z_{n_0} = \Pi^{\delta,n_0} z \in \overline{D(A)} \) and consequently \( z_n = z_{n_0} \in \overline{D(A)} , \forall n \geq n_0. \) Therefore \( \Pi^{\delta}_{D(A)} z := z_{n_0}. \)

(b) If \( \overline{D(A)} = \overline{B(a,R)} \), then \( \overline{D(A)} \) satisfies the interior uniform ball condition with \( r_0 = R \) and the same conclusions as in (a) follows.

In the following lemma we collect basic properties of generalized projections \( \Pi : H \to \overline{D(A)} \).

**Lemma 11** Let \( \Pi : H \to \overline{D(A)} \) be a generalized projection on \( \overline{D(A)} \). Then

1. (j) for all \( x, y \in H \)

\[
\langle \Pi(x) - \Pi(y), \Pi(x) - x - \Pi(y) + y \rangle \leq \frac{1}{2} \| \Pi(x) - x - \Pi(y) + y \|^2;
\]

2. (jj) for all \( x \in H \) and \( a \in \overline{D(A)} \)

\[
\langle \Pi(x) - a, \Pi(x) - x \rangle \leq \frac{1}{2} \| \Pi(x) - x \|^2;
\]

3. (jjj) if \( x \in H \) and \( \overline{B(a,r_0)} \subset \overline{D(A)} \), with \( r_0 > 0 \):

\[
r_0 \| \Pi(x) - x \| \leq \langle a - \Pi(x), \Pi(x) - x \rangle + \frac{1}{2} \| \Pi(x) - x \|^2.
\]

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Proof. (j) Indeed for all $x,y \in H$

\[
|\Pi(x) - x - \Pi(y) + y|^2 - 2\langle \Pi(x) - \Pi(y), \Pi(x) - x - \Pi(y) + y \rangle
= |\Pi(x) - \Pi(y)|^2 + |x - y|^2
+ 2\langle \Pi(x) - \Pi(y), y - x \rangle - 2\langle \Pi(x) - \Pi(y), y - x \rangle
= |x - y|^2 - |\Pi(x) - \Pi(y)|^2 \geq 0.
\]

(jj) We put $y = a$ in (j). (jjj) In (jj) we replace $a$ by $a - r_0 \frac{\Pi(x) - x}{|\Pi(x) - x|}$.

The aim of this section is to prove the existence and uniqueness of a solution for a Cauchy problem driven by a maximal monotone operator $A$ and with singular input $dm_t$, formally written as:

\[
\begin{aligned}
    dx_t + A(x_t)(dt) + dk^d_t &\geq dm_t, \quad t \in \mathbb{R}^+, \\
    x_0 = m_0.
\end{aligned}
\]  

(11)

The basic assumptions are:

(i) $A : H \mapsto H$ is a maximal monotone operator,

(ii) $\text{Int} (D(A)) \neq \emptyset$,

(iii) $m \in D(\mathbb{R}^+, H)$, $m_0 = \overline{D(A)}$,

(iv) $\Pi : H \rightarrow H$ is a generalized projection on $D(A)$.

For $y \in D(\mathbb{R}^+, H)$ and $\ell \in D(\mathbb{R}^+, H) \cap BV_{loc}(\mathbb{R}^+, H)$ we denote by $d\ell_t \in A(y_t)(dt)$ if

\[
\int_s^t \langle y_r - \alpha, d\ell_r - \beta dr \rangle \geq 0, \quad \forall 0 \leq s \leq t, \quad \forall (\alpha, \beta) \in \text{Gr}(A).
\]

Definition 12 (Generalized Skorohod problem) We say that function $x \in D(\mathbb{R}^+, H)$ is a solution of equation (11) with its jumps driven by $\Pi$, if there exists $k \in D(\mathbb{R}^+, H)$ such that:

(i) $x_t \in \overline{D(A)}$, $\forall t \in \mathbb{R}^+$

(ii) $k \in BV_{loc}(\mathbb{R}^+, H)$, $k_0 = 0$,

(iii) $x_t + k_t = m_t$, $\forall t \in \mathbb{R}^+$,

(iv) $k = k^c + k^d$, $k_t^d = \sum_{0 \leq s \leq t} \Delta k_s$,

(v) $dk^c_t \in A(x_t)(dt)$

(vi) $x_t = \Pi (x_{t-} + \Delta m_t)$, $\forall t \in \mathbb{R}^+$.

(13)

We note that formulation (13) justifies to call the pair $(x, k)$ solution of a generalized Skorokhod problem associated to $(A, \Pi; m)$ and to denote $(x, k) = \mathcal{SP}(A, \Pi; m)$.

Proposition 13 An equivalent definition is: $(x, k) \in D(\mathbb{R}^+, H \times H)$ satisfies (13) with (vi) replaced by

$(vi') \Delta k_s = (I - \Pi)(x_{s-} + \Delta m_s)$
**Proof.** Using the equality  \( x + k = m \) we infer that  \( \Delta k_t = (I - \Pi) (x_t - \Delta m_t) \) is equivalent to  \( \Delta m_t - \Delta x_t = x_t - \Delta m_t - \Pi (x_t - \Delta m_t) \), that is  \( x_t = \Pi (x_t - \Delta m_t) \).

In the case  \( \Pi = \Pi_{\overline{D(A)}} \) one can give a simpler equivalent form of Definition 12.

**Proposition 14** Let  \( m \in \overline{D}(\mathbb{R}^+, \mathcal{H}) \),  \( m_0 \in \overline{D}(A) \). Then  \((x, k) = \mathcal{SP}(A, \Pi_{\overline{D(A)}}; m)\) if and only if

\[
\begin{align*}
(i) & \quad x_t \in \overline{D}(A), \ \forall t \in \mathbb{R}^+ \\
(ii) & \quad k \in BV_{loc}(\mathbb{R}^+; \mathcal{H}), \ k_0 = 0, \\
(iii) & \quad x_t + t = m_t, \ \forall t \in \mathbb{R}^+, \\
(iv) & \quad dk_t \in A(x_t)(dt)
\end{align*}
\]

(14)

**Proof.** Assume that  \((x, k) = \mathcal{SP}(A, \Pi_{\overline{D(A)}}; m)\). Then it holds  \( x_t = m_t - k_t \in \overline{D}(A) \) and  \( k \) is a function with locally bounded variation,  \( k_0 = 0 \) such that

\[
dk_t^c \in A(x_t)(dt).
\]

Since  \( x_r = \Pi_{\overline{D(A)}}(x_{r-} + \Delta m_r) \) and, by Proposition (13),  \( \Delta k_r = (I - \Pi) (x_{r-} + \Delta m_r) \), by characterization 9,

\[
\int_s^t \langle x_r - \alpha, dk_r^d \rangle = \sum_{s < r \leq t} \langle x_r - \alpha, \Delta k_r \rangle \geq 0, \ \forall \alpha \in \overline{D}(A).
\]

Consequently, for all  \((\alpha, \beta) \in \text{Gr}(A)\)

\[
\int_s^t \langle x_r - \alpha, dk_r - \beta dr \rangle = \int_s^t \langle x_r - \alpha, dk_r^c - \beta dr \rangle + \int_s^t \langle x_r - \alpha, dk_r^d \rangle \geq 0.
\]

Conversely, assume that  \((x, k)\) satisfies (14). Set  \( k_t^{(n)} = k_t - \sum_{s \leq t} \Delta k_s 1_{|\Delta k_s| > 1/n}, \ n \in \mathbb{N} \).

Clearly, for all  \( T \geq 0 \),  \( \|k_{r-}^{(n)}\|_T \leq 2 \|k_{r-}^{(n-1)}\|_T \) and  \( \|k^{(n)} - k^{(n-1)}\|_T \to 0 \), as  \( n \to \infty \). In every interval  \((s, t)\) there exist finite number of  \( u_1 < u_2 < ... < u_m \) such that  \( |\Delta k_{u_i}| > 1/n \). Set  \( u_0 = s \) and  \( u_{m+1} = t \) and observe that, for all  \( 0 \leq s < t \) and  \((\alpha, \beta) \in \text{Gr}(A)\)

\[
\int_s^t \langle x_u - \alpha, dk_u^{(n)} - \beta du \rangle = \sum_{i=1}^{m+1} \int_{(u_{i-1}, u_i)} \langle x_u - \alpha, dk_u - \beta du \rangle.
\]

Using next auxiliary result:

**Lemma 15** Let  \( y, z \in \overline{D}(\mathbb{R}^+, \mathcal{H}) \) such that  \( z \in BV_{loc}(\mathbb{R}^+; \mathcal{H}) \). Then the following conditions are equivalent:

\[
\begin{align*}
(i) & \quad \int_{[s,t]} \langle y_r, dz_r \rangle \geq 0, \ \forall 0 \leq s < t, \\
(ii) & \quad \int_{[s,t]} \langle y_r, dz_r \rangle \geq 0, \ \forall 0 \leq s < t, \\
(iii) & \quad \int_{[s,t]} \langle y_r, dz_r \rangle \geq 0, \ \forall 0 \leq s \leq t,
\end{align*}
\]

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we deduce that
\[
\int^t_s \langle x_r - \alpha, dk_r^{(m)} - \beta dr \rangle \geq 0, \ \forall 0 \leq s < t, \ \forall (\alpha, \beta) \in \text{Gr}(A). \tag{15}
\]
and letting \( n \to \infty \) we obtain \( dk_r^{(n)} \in A(x_i)(dt) \). For any \( t \in \mathbb{R}^+ \) we have \( x_t \in \overline{\text{D}(A)} \) and for all \( (\alpha, \beta) \in \text{Gr}(A) \)
\[
0 \leq \int_{\{t\}} \langle x_r - \alpha, dk_r - \beta dr \rangle = \langle x_t - \alpha, \Delta k_t \rangle = \langle x_t - \alpha, x_{t-} + \Delta m_t - x_t \rangle.
\]
Then by (9) it follows \( x_t = \Pi_{\overline{\text{D}(A)}}(x_{t-} + \Delta m_t) \) which completes the proof. \( \blacksquare \)

**Remark 16** Let \((x, k) = \mathcal{SP}(A, \Pi; m)\). We have
\[
|\Delta x_t| \leq |\Delta m_t| \quad \text{and} \quad |\Delta k_t| \leq 2|\Delta m_t|, \ t \in \mathbb{R}^+,
\]
since \( x_{t-} \in \text{D}(A) \) and
\[
\Delta x_t = \Pi(x_{t-} + \Delta m_t) - \Pi(x_{t-}) \quad \text{and} \quad \Delta k_t = \Delta m_t + \Pi(x_{t-}) - \Pi(x_{t-} + \Delta m_t).
\]
Consequently if \( m \) is continuous, then \( x \) and \( k \) are continuous functions and independent of \( \Pi \) \((k^d = 0)\).

**Remark 17** If \((x, k) = \mathcal{SP}(A, \Pi; m)\) and \((\hat{x}, \hat{k}) = \mathcal{SP}(A, \Pi; \hat{m})\) then, taking \( x := x_{t-} + \Delta m_r \) and \( y := \hat{x}_{r-} + \Delta \hat{m}_r \) in Lemma 11, we see that, for any \( a \in \overline{\text{D}(A)} \),
\[
(i) \quad \langle x_r - \hat{x}_r, \Delta k_r - \Delta \hat{k}_r \rangle + \frac{1}{2}|\Delta k_r - \Delta \hat{k}_r|^2 \geq 0
\]
\[
(ii) \quad r_0|\Delta k_r| \leq \langle x_r - a, \Delta k_r \rangle + \frac{1}{2}|\Delta k_r|^2,
\]
where \( r_0 \geq 0 \) is such that \( \overline{B(a, r_0)} \subset \overline{\text{D}(A)} \).

Let \((x, k) = \mathcal{SP}(A, \Pi; m)\). We will use the notation \( x = \mathcal{SP}^{(1)}(A, \Pi; m) \) and \( k = \mathcal{SP}^{(2)}(A, \Pi; m) \). If \( m \) is continuous, since \( x, k \) does not depend on \( \Pi \), we will write \((x, k) = \mathcal{SP}(A; m)\) and \( x = \mathcal{SP}^{(1)}(A; m) \), \( k = \mathcal{SP}^{(2)}(A; m)\).

The following version of the Tanaka’s estimate of the distance between two solutions of the Skorokhod problem will prove useful in what follows.

**Lemma 18** We assume that \( m, \hat{m} \in \mathbb{D}(\mathbb{R}^+; \mathbb{R}) \) with \( m_0, \hat{m}_0 \in \overline{\text{D}(A)} \). If \((x, k) = \mathcal{SP}(A, \Pi; m)\) and \((\hat{x}, \hat{k}) = \mathcal{SP}(A, \Pi; \hat{m})\) then
\[(i) \ \text{for all} \ 0 \leq s < t, \ s, t \in \mathbb{R}^+
\]
\[
\int_s^t \langle x_r - \hat{x}_r, dk_r - d\hat{k}_r \rangle + \frac{1}{2} \sum_{s < r \leq t} |\Delta k_r - \Delta \hat{k}_r|^2 \geq 0; \]
\[(ii) \ \text{for all} \ t \in \mathbb{R}^+
\]
\[
|x_t - \hat{x}_t|^2 \leq |m_t - \hat{m}_t|^2 - 2 \int_0^t \langle m_t - \hat{m}_t - m_s + \hat{m}_s, dk_s - d\hat{k}_s \rangle.
\]

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Proof. (i) Let $0 \leq s < t$ be arbitrary chosen. From Proposition 47 from Annexes, we see that, for all càdlàg functions $(x, k), (\hat{x}, \hat{k})$ satisfying Definition 12, we have

$$\int_s^t \langle x_r - \hat{x}_r, dk^c_r - d\hat{k}^c_r \rangle \geq 0, \quad (16)$$

since $(x, k^c), (\hat{x}, \hat{k}^c) \in A$ (see the Annexes, Proposition 47). On the other hand, using inequality (i) from Remark 17 and also the definition of the Lebesgue-Stieltjes integral (see the Annexes), we deduce that

$$\int_s^t \langle x_r - \hat{x}_r, dk^d_r - d\hat{k}^d_r \rangle + \frac{1}{2} \sum_{s < r \leq t} |\Delta k_r - \Delta\hat{k}_r|^2 \geq 0, \quad 0 \leq s < t, s, t \in \mathbb{R}^+. \quad (17)$$

By combining (16) with (17) we obtain (i). (ii) By Lemma 44

$$|k_t - \hat{k}_t|^2 = \sum_{s \leq t} |\Delta k_s - \Delta\hat{k}_s|^2 + 2 \int_0^t \langle k_s - \hat{k}_s, dk_s - d\hat{k}_s \rangle.$$ 

Hence

$$|x_t - \hat{x}_t|^2 - |m_t - \hat{m}_t|^2 + 2 \int_0^t \langle m_t - \hat{m}_t - m_s + \hat{m}_s, dk_s - d\hat{k}_s \rangle$$

$$= -2 \int_0^t \langle x_s - \hat{x}_s, dk_s - d\hat{k}_s \rangle - \sum_{s \leq t} |\Delta k_s - \Delta\hat{k}_s|^2$$

$$\leq 0$$

and the result follows using step (i). \hfill \blacksquare

**Corollary 19** Under the assumptions (12) the differential equation (11) admits at most one solution $(x, k) = SP (A, \Pi; m)$.

**Proof.** It is a immediate consequence of Lemma 18. \hfill \blacksquare

In Răşcanu [18] (in the context of Hilbert spaces which is considered here) and Cépa [7] (finite dimensional case) it is proved that for any continuous $m$ such that $m_0 \in D(A)$ there exists a unique continuous solution $(x, k) = SP (A; m)$. In particular, for any constant function $\alpha \in D(A)$ there exists a unique solution $SP (A; m)$. In fact, for $m_t \equiv \alpha \in D(A)$ we are in the absolutely continuous case and by Proposition 4 the solution $x_t = S_A (t) \alpha$ where $S_A (t)$ is the nonlinear semigroup generated by $A$.

**Lemma 20** Let assumptions (12) be satisfied and $m \in D (\mathbb{R}^+, \mathbb{H})$ be a step function of the form

$$m_t = \sum_{r \in \pi} m_r 1_{[r, r')} (t),$$

where $\pi \in P_{\mathbb{R}^+}$ and $r'$ is the successor of $r$ in the partition $\pi$. Then there exists a unique solution $(x, k) = SP (A, \Pi; m)$ and the solution is given by

$$(x_t, k_t) = SP (A, \Pi \{x_{t-r} + \Delta m_r\})_{t-r} = S_A (t-r) (\Pi \{x_{t-r} + \Delta m_r\}), \quad (18)$$

for all $t \in [r, r')$, $r \in \pi$. 

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Proof. Clearly 

\[ x_r = \Pi(x_r - \Delta m_r), \quad \forall r \in \pi, \]

and \( x_t = m_t - k_t \in \overline{D(A)}, \quad t \in \mathbb{R}^+ \). Function \( k \) is with locally bounded variation, \( k_0 = 0 \), such that for any \((\alpha, \beta) \in \text{Gr}(A)\)

\[ \int_s^t (x_r - \alpha, dk_r^c - \beta du) \geq 0, \quad 0 \leq s < t, \quad s, t \in \mathbb{R}^+ \]

and the proof is complete.

Lemma 21 Let \((x, k) = \mathcal{SP}(A, \Pi; m)\). Let \( a \in \text{Int}(D(A)) \) and \( r_0 > 0 \) be such that \( B(a, r_0) \subset D(A) \). Let

\[ A_{a, r_0} := \sup \left\{ |\dot{u}| : \dot{u} \in Au, \quad u \in B(a, r_0) \right\} \leq \mu < \infty. \]

Then for every \( 0 \leq s < t \)

\[ r_0 \uparrow k^c_{[s, t]} \leq \int_s^t (x_u - a, dk_u^c) + \frac{1}{2} \sum_{s < u \leq t} |\Delta k_u|^2 + \mu \int_s^t |x_u - a| du + (t - s)r_0 \mu, \]

Proof. From Proposition 49 from Annexes we see that

\[ r_0 \uparrow k^c_{[s, t]} \leq \int_s^t (x_u - a, dk_u^c) + \mu \int_s^t |x_u - a| du + (t - s)r_0 \mu. \]

Using Remark 17-(ii), we deduce that

\[ r_0 \uparrow k^d_{[s, t]} \leq \int_s^t (x_u - a, dk_u^d) + \frac{1}{2} \sum_{s < u \leq t} |\Delta k_u|^2 \]

and the result follows, since \( \uparrow k^c_{[s, t]} \leq \uparrow k^e_{[s, t]} + \uparrow k^d_{[s, t]} \).

Theorem 22 (Convergence results) Assume that:

(i) assumptions (12) are satisfied;

(ii) \( m, m^{(n)} \in \mathbb{D}(\mathbb{R}^+, \mathbb{N}) \) and \((x^{(n)}, k^{(n)}) = \mathcal{SP}(A, \Pi; m^{(n)}), n \in \mathbb{N}^* \).

The following assertions hold true:

(I) If for every \( T > 0, \|m^{(n)} - m\|_T \to 0 \), then

(j) for any \( a \in \text{Int}(D(A)) \), \( r_0 > 0 \) such that \( B(a, r_0) \subset D(A) \), for any \( \mu \geq A_{a, r_0}^\# \), there exist a constant \( C = C(a, r_0, \mu, T, \gamma_m(\cdot, T)) > 0 \) and \( n_0 \in \mathbb{N}^* \) such that for all \( n \geq n_0 \):

\[ \|x^{(n)}\|_T^2 + \uparrow k^{(n)}_{\downarrow T} \leq \|x^{(n)}\|_T^2 + \uparrow k^{(n), c}_{\downarrow T} + \uparrow k^{(n), d}_{\downarrow T} \leq C(1 + \|m\|_T^2); \quad (19) \]

(jj) there exist \( x, k \in \mathbb{D}(\mathbb{R}^+, \mathbb{N}) \) such that

\[ \|x^{(n)} - x\|_T + \|k^{(n)} - k\|_T \to 0, \quad \text{as} \ n \to \infty, \]

and

\[ \|x\|_T^2 + \uparrow k_{\downarrow T} \leq C(1 + \|m\|_T^2); \quad (20) \]
(jjj) \((x, k) = \mathcal{SP}(A, \Pi; m)\).

(II) If \(m^{(n)} \to m\) in \(\mathcal{D}(\mathbb{R}^+, \mathcal{H})\) then there exist \(x, k \in \mathcal{D}(\mathbb{R}^+, \mathcal{H})\) such that

\[
(x^{(n)}, k^{(n)}, m^{(n)}) \to (x, k, m) \quad \text{in} \quad \mathcal{D}(\mathbb{R}^+, \mathcal{H} \times \mathcal{H} \times \mathcal{H}).
\]

and \((x, k) = \mathcal{SP}(A, \Pi; m)\).

(III) Let \(m_0 \in \overline{\mathcal{D}(A)}\) and \(\pi_n \in \mathcal{P}_{\mathbb{R}^+}\) be a partition such that \(\|\pi_n\| \to 0\). If \(m_t^{(n)} = \sum_{r \in \pi_n} m_r 1_{[r, r')} (t)\) denotes the \(\pi_n\)-discretization of \(m\) and \((x^{(n)}, k^{(n)}) = \mathcal{SP}(A, \Pi; m_t^{(n)})\), \(n \in \mathbb{N}\) then there exist \(x, k \in \mathcal{D}(\mathbb{R}^+, \mathcal{H})\) such that

\[
(x^{(n)}, k^{(n)}, m^{(n)}) \to (x, k, m) \quad \text{in} \quad \mathcal{D}(\mathbb{R}^+, \mathcal{H} \times \mathcal{H} \times \mathcal{H})
\]

and \((x, k) = \mathcal{SP}(A, \Pi; m)\).

**Proof.** (I) : (j) Let \(0 \leq t \leq T\). We have

\[
|k_t^{(n)}|^2 = 2 \int_0^t \langle k_r^{(n)}, dk_r^{(n)} \rangle - \sum_{r \leq t} |\Delta k_r^{(n)}|^2
\]

Since \(k_t = m_t - x_t\),

\[
|m_t^{(n)} - x_t^{(n)}|^2 = 2 \int_0^t \langle m_r^{(n)} - x_r^{(n)}, dk_r^{(n)} \rangle - \sum_{r \leq t} |\Delta k_r^{(n)}|^2
\]

and moreover

\[
|m_t^{(n)} - x_t^{(n)}|^2 + 2 \int_0^t \langle x_r^{(n)} - a, dk_r^{(n)} \rangle = 2 \int_0^t \langle m_r^{(n)} - a, dk_r^{(n)} \rangle - \sum_{r \leq t} |\Delta k_r^{(n)}|^2
\]

Using Lemma 21 we obtain

\[
|x_t^{(n)} - m_t^{(n)}|^2 + 2r_0 \sum |k_r^{(n)}|^2 \leq 2\mu t||x_t^{(n)} - a||_t + 2r_0 \mu t + 2 \int_0^t \langle m_r^{(n)} - a, dk_r^{(n)} \rangle \tag{21}
\]

Since \(m \in \mathcal{D}(\mathbb{R}^+, \mathcal{H})\), there exists a partition \(\pi \in \mathcal{P}_{\mathbb{R}^+}\) such that \(\max_{r \in \pi} \mathcal{O}_m ([r, r')] < r_0/2\) (see Remark 39 in the Annexes). We can assume that \(T \in \pi\) (if not, we can replace \(\pi\) by \(\tilde{\pi} = \pi \cup \{T\}\) which also satisfies the above condition). Denote \(N_0 = \text{card} \{r \in \pi : r \leq T\}\).

Since \(\|m^{(n)} - m\|_T \to 0\), there exists \(n_0 \geq N_0\) such that \(\|m^{(n)}\|_T \leq 1 + \|m\|_T\) and \(\max_{r \in \pi} \mathcal{O}_m ([r, r')] < r_0/2\) for all \(n \geq n_0\). Let \(m_{s, n}^{(n), \pi} = \sum_{r \in \pi} m_r^{(n)} 1_{[r, r']} (s), s \geq 0\). We have \(m_{s, n}^{(n), \pi} - m_{s, n}^{(n)} < r_0/2\) for all \(s \geq 0\) and \(n \geq n_0\).

\[
2 \int_0^t \langle m_{s, n}^{(n)} - a, dk_s \rangle = 2 \int_0^t \langle m_{s, n}^{(n)} - m_{s, n}^{(n), \pi}, dk_s \rangle + 2 \int_0^t \langle m_{s, n}^{(n), \pi} - a, dk_{s, n}^{(n)} \rangle \\
\leq r_0 \sum |k_r^{(n)}|^2 + 2 \sum_{r \in \pi, r \leq t} \langle m_r^{(n)} - a, k_r^{(n)} - k_t^{(n)} \rangle + 2 \langle m_t^{(n)} - a, \Delta k_t^{(n)} \rangle \\
\leq r_0 \sum |k_r^{(n)}|^2 + 4N_0 \cdot \|m^{(n)} - a\|_t \cdot \|k_t^{(n)}\|_t \\
\leq r_0 \sum |k_r^{(n)}|^2 + 4N_0 \cdot \|m^{(n)} - a\|_t \cdot \|m^{(n)} - x^{(n)}\|_t
\]
Using this last estimate in (21) we infer for all \( t \in [0, T] \) and \( n \geq n_0 \)

\[
|x_t^{(n)} - m_t^{(n)}|^2 + r_0 \sum_{i=0}^t |\Delta k_i| \leq 2\mu T \left\| (x^{(n)} - m^{(n)}) + (m^{(n)} - a) \right\|_T + 2r_0 \mu T
\]

\[
+ 4N_0 |m^{(n)} - a| T \left\| (x^{(n)} - m^{(n)}) \right\|_T \leq \frac{1}{2} \left\| (x^{(n)} - m^{(n)}) \right\|_T^2 + C(1 + \|m\|_T^2)
\]

with \( C = C(\mu, |a|, T, N_0, r_0) \) a positive constant. Taking, also, in account that

\[ \sum_{r \leq t} |\Delta k_r| \leq \sum_{r \leq t} \langle \Delta k_r \rangle_t \text{ and } \sum_{r \leq t} \langle \Delta k_r \rangle_t = \sum_{r \leq t} (k_r - k_{r-1}) \leq 2 \sum_{r \leq t} k_r \]

the inequality (19) follows. (jj) By Lemma 18 for any \( n, \ell \in \mathbb{N}^+ \), \( n, \ell \geq n_0 \)

\[
\|x^{(n)} - x^{(\ell)}\|_T^2 \leq \|m^{(n)} - m^{(\ell)}\|_T^2 + 4\|m^{(n)} - m^{(\ell)}\|_T \langle \sum_{r \leq T} \langle k^{(n)} \rangle_T + \sum_{r \leq T} \langle k^{(\ell)} \rangle_T \rangle
\]

\[
\leq \left( \frac{1}{n} + \frac{1}{\ell} \right)^2 + 8 \left( \frac{1}{n} + \frac{1}{\ell} \right) C(1 + \|m\|_T^2).
\]

Hence \( x^{(n)} \) and \( k^{(n)} = m^{(n)} - x^{(n)} \) are Cauchy sequences in the spaces of càdlàg functions on \([0, T] \). Thus there exist functions \( x, k \in \mathbb{D}(\mathbb{R}^+, [0, T]) \) such that

\[
\|x^{(n)} - x\|_T + \|k^{(n)} - k\|_T \to 0, \text{ as } n \to \infty.
\]

and by Lemma 45 passing to \( \liminf_{n \to +\infty} \) in (19) we infer (20). (jjj) We prove now \((x, k)\) satisfies conditions (13). Since for all \( t \in [0, T] \) : \( x_t^{(n)} \in \mathbb{D}(\mathbb{R}, x_t^{(n)} + k_t^{(n)} = m_t^{(n)} \), \( x_t^{(n)} = \Pi(x_t^{(n)} + \Delta m_t^{(n)}) \), passing to limit the same conditions are satisfied by \((x, k)\). It remains to show that \( dk_t \in A(x_t) \langle dt \rangle \).

Let \((z, \hat{z}) \in \text{Gr}(A)\) be arbitrary. From Helly-Bray Theorem 46 there exist a subsequence \( n_0 < n_1 < n_2 < \cdots < n_i \to \infty \) and a sequence \( \delta_i \downarrow 0 \) as \( i \to \infty \) such that uniformly with respect to \( s, t \in (0, T] \), \( s \leq t \)

\[
\int_s^t 1_{|\Delta k_t^{(n_i)}| \geq \delta_i} \langle x_r^{(n_i)} - z, dk_r^{(n_i)} \rangle d\hat{r} \to \int_s^t \langle x_r - z, dk_r \rangle d\hat{r}, \text{ as } i \to \infty.
\]

By Remark 17-(ii) we infer

\[
\int_s^t 1_{|\Delta k_t^{(n_i)}| \leq \delta_i} \langle x_r^{(n_i)} - z, dk_r^{(n_i)} \rangle = \sum_{r \in (s, t]} \langle x_r^{(n_i)} - \alpha, \Delta k_r^{(n_i)} \rangle 1_{|\Delta k_r^{(n_i)}| \leq \delta_i}
\]

\[
\geq -\frac{1}{2} \sum_{r \in (s, t]} |\Delta k_r^{(n_i)}|^2 1_{|\Delta k_r^{(n_i)}| \leq \delta_i}
\]

\[
\geq -\frac{1}{2} \delta_i \sum_{r \in (s, t]} \langle k_r^{(n_i)} \rangle_t
\]

\[
\geq -\frac{1}{2} \delta_i C(1 + \|m\|_T^2).
\]

From (13-iv) for \((x^{(n_i)}, k^{(n_i)})\) we have

\[
\int_s^t \langle x_r^{(n_i)} - z, dk_r^{(n_i)} \rangle dr \geq \int_s^t \langle x_r^{(n_i)} - z, dk_r^{(n_i)} \rangle d\hat{r}
\]

(22)
that yields

$$\int_s^t \langle x_r^{(n)} - z, dk_r^{(n)} - \hat{z}dr \rangle \geq \int_s^t 1_{|hk_r^{(n)}| > \Delta} \langle x_r^{(n)} - z, dk_r^{(n)}d \rangle - \frac{1}{2} \delta_1 C(1 + \|m\|_T^2)$$

Passing to lim$_{t \to \infty}$ in this last inequality we deduce using Helly-Bray Theorem 46 and (22):

$$\int_s^t \langle x_r - z, dk_r - \hat{z}dr \rangle \geq \int_s^t \langle x_r - z, dk_r^d \rangle$$

that is

$$\int_s^t \langle x_r - z, dk_r^c - \hat{z}dr \rangle \geq 0$$

for all $(z, \hat{z}) \in Gr(A), \ T > 0$ and $0 \leq s \leq t \leq T$. (II) By the definition of convergence in the Skorokhod topology $J_1$, there exists a sequence of strictly increasing continuous changes of time $\{\lambda^{(n)}\}$ such that $\lambda^{(n)}_0 = 0$, $\lambda^{(n)}_\infty = +\infty$, sup$_{t \geq 0} |\lambda^{(n)}_t - t| \to 0$ and sup$_{t \in [0, T]} |m^{(n)}_{\lambda^{(n)}_t} - m_t| \to 0$, for all $T \in \mathbb{R}^+$. Since $(x^{(n)}_{\lambda^{(n)}_t}, k^{(n)}_{\lambda^{(n)}_t}) = SP(A, \Pi; m^{(n)}_{\lambda^{(n)}_t})$, the first part (I) of the proof yields there exist $x, k \in \mathbb{D}(\mathbb{R}^+, \mathbb{H})$ such that for any $T \in \mathbb{R}^+$,

$$\|x^{(n)}_{\lambda^{(n)}_t} - x\|_T \to 0 \quad \text{and} \quad \|k^{(n)}_{\lambda^{(n)}_t} - k\|_T \to 0,$$

and $(x, k) = SP(A, \Pi; m)$. The proof is complete. (III) Applying Proposition 41 from Annexes, we see that $m^{(n)} \to m$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{H})$ and from (II) the result follows.

We state now the main result of this section:

**Theorem 23** Under the assumptions (12) the differential equation (11) has a unique solution $(x, k) = SP(A, \Pi; m)$.

**Proof.** Uniqueness follows from Corollary 19. Since $m \in \mathbb{D}(\mathbb{R}^+, \mathbb{H})$, there exists a partition $\pi_n \in \mathcal{P}_{\mathbb{R}^+}$ such that max$_{r \in \pi_n} O_m([r, r')) < \frac{1}{n}$, $n \in \mathbb{N}^*$. Let $m^{(n)}_s = \sum_{r \in \pi_n} m_r 1_{[r, r')}(s)$, $s \geq 0$. Let $T > 0$ be arbitrary. Then for all $n \in \mathbb{N}^*$:

$$\|m^{(n)} - m\|_T \leq \frac{1}{n} \quad \text{(23)}$$

By Lemma 20 for the step function $m^{(n)}$, with $m_0^{(n)} = m_0 \in D(A)$, there exists a unique solution $(x^{(n)}, k^{(n)}) = SP(A, \Pi; m^{(n)}), \ n \in \mathbb{N}^*$. Since $\|m^{(n)} - m\|_T \to 0$, by Theorem 22 there exist $x, k \in \mathbb{D}(\mathbb{R}^+, \mathbb{H})$ such that

$$\|x^{(n)} - x\|_T + \|k^{(n)} - k\|_T \to 0, \quad \text{as } n \to \infty,$$

and $(x, k) = SP(A, \Pi; m)$.

**Remark 24** If $m$ is continuous then the solution of the Skorokhod problem $(x, k) = SP(A, \Pi; m)$ is also continuous and is not depending on projections $\Pi$ (see Remark 16). By Theorem 22-(I) for any projection $\Pi$, if $(x^n, k^n) = SP(A, \Pi; m^n), \ n \in \mathbb{N}$ and $\|m^n - m\|_T \to 0, \ T \in \mathbb{R}^+$ then

$$\|x^n - x\|_T \to 0 \quad \text{and} \quad \|k^n - k\|_T \to 0, \ T \in \mathbb{R}^+.$$
Example 25 1. We consider $\mathcal{H}$ a Hilbert space with $\{e_i\}_{i \in \mathbb{N}^*}$ an orthonormal basis and the set of constraints $\mathcal{O} = \overline{\mathcal{O}} \subset \mathcal{H}$ given by

$$\overline{\mathcal{O}} = \{ y \in \mathbb{H} : y^i \geq 0, \ \forall i = 1, N \}.$$ 

We propose the following example of upper bidiagonal infinite dimensional system with constraints and with singular inputs generated by càdlàg functions:

$$\begin{equation}
\begin{aligned}
&\begin{cases}
    dx^1_t + (x^1_t + x^2_t) \ dt + dk^1_t = dm^1_t, \\
    dx^2_t + (x^2_t + x^3_t) \ dt + dk^2_t = dm^2_t, \\
    \cdots \\
    dx^i_t + (x^i_t + x^{i+1}_t) \ dt + dk^i_t = dm^i_t,
    \end{cases}
\end{aligned}
\end{equation}$$

where $m_i = \sum_{i \in \mathbb{N}^*} m_i^i e_i$ with $m_i^i \in \mathbb{D} (\mathbb{R}^+, \mathbb{R})$, $i \in \mathbb{N}^*$.

This system can be written in the abstract form

$$\begin{aligned}
\begin{cases}
    dx_t + A(x_t) \ dt \ni dm_t, \\
    x_0 = m_0,
\end{cases}
\end{aligned}$$

with $A(x) = x + \hat{x} + \partial \Pi_{\mathcal{O}} (x)$, where, for $x = \sum_{i \in \mathbb{N}^*} x^i e_i$, $\hat{x} = \sum_{i \in \mathbb{N}^*} x^{i+1} e_i$. Clearly $\Pi (A) = \overline{\mathcal{O}}$ and $A$ is a maximal monotone operator (as a sum of the linear continuous non-negative operator $L (x) = x + \hat{x}$ and the maximal monotone operator $\partial \Pi_{\mathcal{O}}$).

Applying Theorem 23, we infer that there exists a unique solution (see definition (14)) $(x, k) = \mathcal{SP} (A, \Pi_{\mathcal{O}}, m)$. We outline that solution $(x, k)$ satisfies

$$\begin{aligned}
\begin{cases}
    x_t^i + \int_0^t (x_s^i + x_{s+1}^i) ds + k_t^i = m_t^i, \ \forall i \in \mathbb{N}^*, \\
    \sum_{j=1}^N \int_s^t (x_r^j - a_j \cdot dk_r^j) \leq 0, \ \forall a_j \geq 0, \\
    k_t^i = 0, \ \forall i > N.
\end{cases}
\end{aligned}$$

2. In the above example we can consider a generalized projection $\Pi : \mathcal{H} \to \overline{\mathcal{O}}$,

$$\Pi (x) = \sum_{i \in \mathbb{N}^*} p^i e_i \ \text{ with } \ p^i := (x^i)^+ + [g((x^i)^-) - g(0)]^+,$$

where $g : \mathbb{R} \to \mathbb{R}$ is a 1--Lipschitz function. In this case $(x, k) = \mathcal{SP} (A, \Pi; m)$ and we mention

$$\begin{aligned}
\begin{cases}
    x_t^i + \int_0^t (x_s^i + x_{s+1}^i) ds + k_t^{i,c} + k_t^{i,d} = m_t^i, \ \forall i \in \mathbb{N}^*, \\
    \sum_{j=1}^N \int_s^t (x_r^j - a_j \cdot dk_r^{j,c}) \leq 0 \ \forall a_j \geq 0, \\
    x_t = \Pi (x_t + \Delta m_t), \ t > 0, \\
    k_t^i = 0, \ \forall i > N.
\end{cases}
\end{aligned}$$
3. If we ask the feedback law $dk_t$ to produce, in a minimal way, a boundedness of the form
\[ \sum_{i \in \mathbb{N}^*} (x_i^t)^2 \leq R^2 \]
then we should take $A(x) = x + \dot{x} + \partial I_{\mathbb{O}}(x) + \partial I_{B(0,R)}(x)$ with $\overline{D(A)} = \mathbb{O} \cap B(0,R)$.

4. An interacting particles system with singular input can be modeled by the following infinite dimensional system
\[
\begin{cases}
   dx_t + L(x_t)dt + A(x_t)(dt) \ni dm_t, \\
x_0 = m_0,
\end{cases}
\]
where $L : H \to H$ is a linear continuous non-negative operator and $A = \partial \varphi$ with $\varphi$ a proper convex l.s.c. function given by
\[ \varphi(x) = \sum_{i \neq j} \psi(x_j - x_i) \]
(see also the examples from Sznitman [22] and Cépa [7]).

3 The penalized problem

For all $\varepsilon > 0$ and $z \in H$ let’s define (see the Annexes, subsection 4.1)
\[ J_\varepsilon(z) = (I + \varepsilon A)^{-1}(z), \quad A_\varepsilon(z) = \frac{1}{\varepsilon}(z - J_\varepsilon(z)), \]
where the sequence $\{A_\varepsilon\}$ is the Yosida approximation of the operator $A$. For the properties of $J_\varepsilon$ and $A_\varepsilon$ see Proposition 37 from the Annexes.

3.1 Approximation with unamortized jumps

Let $m^\varepsilon, m \in D(\mathbb{R}^+, H)$ such that, for all $T > 0$, $\|m^\varepsilon - m\|_T \to 0$, as $\varepsilon \to 0$. We will consider equations of the form
\[ x_t^\varepsilon + \int_0^t A_\varepsilon(x_s^\varepsilon)ds = m_t^\varepsilon, \quad t \in \mathbb{R}^+, \quad \varepsilon > 0. \tag{25} \]
Since $A_\varepsilon$ is Lipschitz continuous it is well known that there exists a unique solution $x^\varepsilon$ of (25). On the other hand, $A_\varepsilon : H \to H$ is a maximal monotone operator (is monotone and continuous) with $\overline{D(A_\varepsilon)} = H$. Therefore any generalized projection $\Pi = I = \Pi_{\overline{D(A_\varepsilon)}}$, where $I$ is the identity operator on $H$ and, from Theorem 23 we deduce that there exists a unique solution $(x^\varepsilon, k^\varepsilon) = SP(A_\varepsilon, I; m^\varepsilon) = SP(A_\varepsilon; m^\varepsilon)$, where $k_t^\varepsilon := \int_0^t A_\varepsilon(x_s^\varepsilon)ds$.

The following inequality holds (see Theorem 22, inequality (19)), there exits $\varepsilon_0 > 0$ such that, $\forall 0 < \varepsilon \leq \varepsilon_0$,
\[ \|x^\varepsilon\|^2_T + \frac{1}{2}k_T^\varepsilon \leq C(1 + \|m\|^2_T), \tag{26} \]
since $|A_\varepsilon(u)| \leq |A^0(u)|$, $\forall u \in D(A)$ and consequently
\[ \sup \left\{ |A_\varepsilon u| : u \in \overline{B(a, r_0)} \right\} \leq A^\#_{a, r_0} \leq \mu. \]
Remark 26 If $m$ is a step function of the form $m_t = \sum_{r \in \pi} m_r 1_{[r, r')} (t)$, where $\pi$ is a partition from $\mathcal{P}_{\mathbb{R}^+}$, then the solution $x^\varepsilon = S\mathcal{P}^{(1)}(A_\varepsilon; m)$ is given by
\[
x^\varepsilon_t = S\mathcal{P}^{(1)}(A_\varepsilon; x^\varepsilon_{t-} + \Delta m_r)_t, \quad \forall t \in [r, r'), \ r \in \pi.
\] (27)

Lemma 27 Let $m, \tilde{m} \in \mathcal{D}(\mathbb{R}^+, \mathbb{H})$. If $x^\varepsilon = S\mathcal{P}^{(1)}(A_\varepsilon; m)$, $\tilde{x}^\varepsilon = S\mathcal{P}^{(1)}(A_\varepsilon; \tilde{m})$ and
\[
k^\varepsilon_s = \int_0^t A_\varepsilon(x^\varepsilon_s) ds, \quad \tilde{k}^\varepsilon_s = \int_0^t A_\varepsilon(\tilde{x}^\varepsilon_s) ds, \ t \in \mathbb{R}^+, \ \varepsilon > 0,
\] then for all $0 \leq s < t$ and $\varepsilon > 0$
\[
|k^\varepsilon_s - \tilde{k}^\varepsilon_s|^2 - |x^\varepsilon_s - \tilde{x}^\varepsilon_s|^2 \leq \left[ |m_t - \tilde{m}_t|^2 - |m_s - \tilde{m}_s|^2 \right]
-2\int_0^t (m_t - \tilde{m}_t - m_r + \tilde{m}_r, \kappa^\varepsilon_s - \tilde{\kappa}^\varepsilon_s) + 2\int_0^s (m_s - \tilde{m}_s - m_r + \tilde{m}_r, \kappa^\varepsilon_s - \tilde{\kappa}^\varepsilon_s).
\]

Proof. Since $A_\varepsilon$ is monotone,
\[
\int_s^t \langle x^\varepsilon_r - \tilde{x}^\varepsilon_r, A_\varepsilon(x^\varepsilon_r) - A_\varepsilon(\tilde{x}^\varepsilon_r) \rangle dr \geq 0.
\] (28)

Consequently, the result follows by the arguments from the proof of Lemma 18-(ii).

We will study the problem of convergence of $\{x^\varepsilon\}$. It is well known that if $m$ is continuous and $m_0 \in \overline{D(A)}$ then, for all $T \geq 0$,
\[
\|x^\varepsilon - x\|_T \rightarrow 0, \ \text{as} \ \varepsilon \rightarrow 0,
\]
where $x = S\mathcal{P}^{(1)}(A; m)$ (see, e.g., Răscanu [18]). In the case where $m$ has jumps the problem is more delicate. We start with the simplest case where $m_t = \alpha \in \mathbb{H}$, $t \in \mathbb{R}^+$.

Lemma 28 Let $\varepsilon > 0$, $\alpha_\varepsilon, \alpha \in \mathbb{H}$ and $x^\varepsilon_t = S\mathcal{P}^{(1)}(A_\varepsilon; \alpha_\varepsilon)_t = S_{A_\varepsilon} (t) \alpha_\varepsilon$ with $\alpha_\varepsilon \rightarrow \alpha$ as $\varepsilon \rightarrow 0$.
(i) If $\alpha \in \overline{D(A)}$ and $x_t = S\mathcal{P}^{(1)}(A; \alpha)_t = S_A (t) \alpha$, then, for all $T \geq 0$,
\[
\|x^\varepsilon - x\|_T \rightarrow 0, \ \text{as} \ \varepsilon \rightarrow 0.
\]
(ii) If $\alpha \notin \overline{D(A)}$ and $x = S\mathcal{P}^{(1)}(A; \Pi_{\overline{D(A)}}(\alpha))$ then for any $0 < \delta \leq T$,
\[
\|x^\varepsilon - x\|_{[\delta, T]} \rightarrow 0, \ \text{as} \ \varepsilon \rightarrow 0.
\]

Proof. (i) We have
\[
|x^\varepsilon_t - x_t| = |S_{A_\varepsilon} (t) \alpha_\varepsilon - S_A (t) \alpha| \\
\leq |\alpha_\varepsilon - \alpha| + |S_{A_\varepsilon} (t) \alpha - S_A (t) \alpha|.
\]
The result follows using Proposition 1. (ii) By Proposition 6 we know that for all $T > 0$, $x^\varepsilon$ tends to $x$ in $L^2([0, T])$, as $\varepsilon \rightarrow 0$. Hence $x^\varepsilon$ tends almost everywhere to $x$ with respect to the Lebesgue measure. Consequently, for any $\delta > 0$ there is $t_0 \in (0, \delta]$ such that $x^\varepsilon_{t_0} \rightarrow x_{t_0}$. Let $\hat{x}^\varepsilon = S\mathcal{P}^{(1)}(A_\varepsilon; \Pi_{\overline{D(A)}}(\alpha))$, $\varepsilon > 0$. By Lemma 27
\[
|x^\varepsilon_t - \hat{x}^\varepsilon_t|^2 \leq |x^\varepsilon_{t_0} - \hat{x}^\varepsilon_{t_0}|^2, \ t \geq t_0, \ \varepsilon > 0.
\]
Finally, by (i), $\|\hat{x}^\varepsilon - x\|_T \rightarrow 0$, as $\varepsilon \rightarrow 0$, which completes the proof.
Theorem 29 Assume that \( m^\varepsilon, m \in \mathcal{D}(\mathbb{R}^+, \mathbb{H}) \), \( m_0 \in \overline{\mathcal{D}(A)} \) and let \( x = \mathcal{S}\mathcal{P}^{(1)}(A, \Pi_{\overline{\mathcal{D}(A)}}; m) \) and \( x^\varepsilon = \mathcal{S}\mathcal{P}^{(1)}(A_\varepsilon; m^\varepsilon) \) for each \( \varepsilon > 0 \).

If for all \( T > 0 \),

\[ \|m^\varepsilon - m\|_T \to 0, \text{ as } \varepsilon \to 0, \]

then

(j) for any \( T > 0 \),

\[ \lim_{\varepsilon \to 0} \left\| x_t - x_t \right\|_T = 0, \text{ where } x_t = x_t |_{\Delta m_t = 0} + (x_t + \Delta m_t) 1_{|\Delta m_t| > 0} \]

and consequently,

\[ \lim_{\varepsilon \to 0} \left( \sup_{t \in [0,T]} |x_t^\varepsilon - x_t| \right) = 0; \]

(jj) for all \( T > 0 \),

\[ \lim_{\varepsilon \to 0} J_\varepsilon(x_t^\varepsilon) = \Pi_{\overline{\mathcal{D}(A)}}(x_t - \Delta m_t) = x_t, \text{ uniformly for } t \in [0,T]; \]

(jjj) for any sequence \( t_\varepsilon \searrow t \) we have \( x_t^\varepsilon \to x_t \) and for \( t_\varepsilon \nearrow t \) we have \( x_t^\varepsilon \to x_t \).

Proof. Since \( m \in \mathcal{D}(\mathbb{R}^+, \mathbb{H}) \), there exists a partition \( \pi_n \in \mathcal{P}_{\mathbb{R}^+} \) such that \( \max_{r \in \pi_n} \mathcal{O}_m([r, r^r)) < 1/n \) (see Remark 39). Let \( m_t^{(n)} = \sum_{r \in \pi_n} m_r 1_{[r, r')} (t) \) and \( m^{(n)}_\varepsilon(t) = \sum_{r \in \pi_n} m^\varepsilon(t) 1_{[r, r')} (t) \). We have

\[ \|m - m^{(n)}\|_T < \frac{1}{n} \text{ and } \|m^{(n)}_\varepsilon - m^{(n)}\|_T \leq \|m^\varepsilon - m\|_T \]

hence

\[ \|m^{(n)}_\varepsilon - m\|_T \leq \|m^\varepsilon - m\|_T + \frac{1}{n} \text{ and } \|m^{(n)}_\varepsilon - m^\varepsilon\|_T \leq 2 \|m^\varepsilon - m\|_T + \frac{1}{n}. \]

Let \( x^{(n)} = \mathcal{S}\mathcal{P}^{(1)}(A_\varepsilon; m^{(n)}_\varepsilon) \) and \( (x^{(n)}, k^{(n)}) = \mathcal{S}\mathcal{P}(A; m^{(n)}) \), \( n \in \mathbb{N}, \varepsilon > 0 \). From Convergence Theorem 22 it follows

\[ \|x^{(n)} - x\|_T + \|k^{(n)} - k\|_T \to 0, \text{ as } n \to \infty, \]

where \((x, k) = \mathcal{S}\mathcal{P}(A; m)\). By Lemma 27 (for \( s = 0 \)) we obtain that

\[ |x_t^\varepsilon - x_t^{(n)}|^2 \leq |m_t^\varepsilon - m_t^{(n)}|^2 + 4|m_t^\varepsilon - m_t^{(n)}|_T + k^{\varepsilon}_{\varepsilon^\varepsilon} + k^{\varepsilon}(n)^{\varepsilon^\varepsilon}, \]

where

\[ k^{\varepsilon}(n) = \int_0^\varepsilon A_\varepsilon(x_s^{\varepsilon}) ds, \quad k^\varepsilon = \int_0^\varepsilon A_\varepsilon(x_s) ds, \quad n \in \mathbb{N}, \varepsilon > 0. \]

Since

\[ \|m^{(n)}_\varepsilon\|_T \leq \frac{1}{n} \quad + \|m^\varepsilon - m\|_T + \|m\|_T, \]

\[ |m_t^{(n)} - m_t^{(n)}| \leq 2\|m^\varepsilon - m\|_T + \frac{2}{n} + |m_t - m_s|, \]

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it is easy to see from the proof of Theorem 22, (I – j) that, there exist \( \varepsilon_0 > 0 \) and \( n_0 \in \mathbb{N}^* \) such that, for all \( 0 < \varepsilon \leq \varepsilon_0 \) and \( n \geq n_0 \),

\[
\|x^{\varepsilon,(n)}\|_T^2 + \|k^{\varepsilon,(n)}\|_T^2 \leq C(1 + \|m\|_T^2).
\]

From this inequality and (26) we deduce that variations \( k^{\varepsilon,i} \), \( k^{\varepsilon,(i)} \), \( \varepsilon \), \( \varepsilon_n \), \( \varepsilon_T \), for any \( T \in \mathbb{R}^+ \), hence from (31) we obtain

\[
|x^{\varepsilon}_{t_i} - x^{\varepsilon,(n)}| \leq C \left( \|m^{\varepsilon} - m\|_T^2 + \|m^{\varepsilon} - m\|_T + \frac{1}{n} \right), \forall t \in [0, T].
\]

Taking into account the definitions we have for all \( r \in \pi_n \),

\[
x^{\varepsilon,(n)}_t = S_{A_t} (t - r) x^{\varepsilon,(n)}_r, \forall t \in (r, r'),
\]

\[
x^{\varepsilon}_r = x^{\varepsilon,(n)}_r + \Delta m^{\varepsilon}_r(n)
\]

and

\[
x^{\varepsilon}_t = S_{A_t} (t - r) x^{\varepsilon}_r, \forall t \in (r, r'),
\]

\[
x^{\varepsilon}_r = \Pi_{\mathcal{D}(A)} (x^{\varepsilon}_r + \Delta m^{\varepsilon}_r(n)).
\]

By Lemma 28 it is clear that for any \( r \in \pi_n \)

\[
\lim_{\epsilon \to 0} x^{\varepsilon,(n)}_t = \begin{cases} x^{(n)}_t, & \text{if } t \in (r, r'), \\ x^{(n)}_r + \Delta m^{(n)}_r, & \text{if } t = r \end{cases} \tag{32}
\]

(and the convergence is uniformly w.r.t. \( t \in [0, T], \forall T \in [0, T] \)). If \( t \in (r, r') \),

\[
|x^{\varepsilon}_t - x_t| \leq |x^{\varepsilon}_t - x^{\varepsilon,(n)}_t| + |x^{\varepsilon,(n)}_t - x^{(n)}_t| + |x^{(n)}_t - x_t|,
\]

then

\[
\limsup_{\epsilon \to 0} \left( \sup_{r \in \pi_n} \max_{t \in (r, r')} |x^{\varepsilon}_t - x_t| \right) \leq C \frac{1}{n} + \|x^{(n)} - x\|_T, \forall n \in \mathbb{N}^*.
\]

If \( t = r \),

\[
|x^{\varepsilon}_t - x_t - \Delta m_t| \leq |x^{\varepsilon}_r - x^{\varepsilon,(n)}_r| + \left|x^{\varepsilon,(n)}_r - x^{(n)}_r - \Delta m^{(n)}_r\right| + \left|x^{(n)}_r + \Delta m^{(n)}_r - x^{(n)}_r - \Delta m^{(n)}_r\right| + \left|x^{(n)}_r + \Delta m^{(n)}_r - x^{(n)}_r - \Delta m^{(n)}_r\right|
\]

then

\[
\limsup_{\epsilon \to 0} \left( \sup_{r \in \pi_n} \max_{t \leq T} |x^{\varepsilon}_{t -} - \Delta m_t| \right) \leq C \frac{1}{n} + \|x^{(n)} - x\|_T, \forall n \in \mathbb{N}^*.
\]

Let \( \bar{x}_t := x_t 1_{|\Delta m| = 0} + (x_t - \Delta m_t) 1_{|\Delta m| > 0} \). Consequently

\[
\sup_{t \in [0,T]} |x^{\varepsilon}_t - \bar{x}_t| \leq \max_{r \in \pi_n} \max_{t \leq T} |x^{\varepsilon}_{t -} - \Delta m_t| + \max_{r \in \pi_n} \max_{t \in (r, r')} |x^{\varepsilon}_{t \wedge T} - x_{t \wedge T}|
\]

and we get

\[
\limsup_{\epsilon \to 0} \|x^{\varepsilon}_T - \bar{x}_T\|_T \leq \delta_n, \forall n \in \mathbb{N}^*,
\]
where $\delta_n \to 0$, as $n \to \infty$. We recall that $x = SP^{(1)} (A, \Pi_{D(A)} m)$ satisfies, for all $t \geq 0$:
\[ |\Delta x_t| \leq |\Delta m_t| \quad \text{and} \quad x_t = \Pi_{D(A)} (x_{t-} + \Delta m_t) \in D(A). \]

(j) Since $\lim_{\epsilon \to 0} x_t^\epsilon = \bar{x}_t$, by Proposition 37-(jj), we have uniformly on every interval $[0, T]$, $J^\epsilon (x_t^\epsilon) \to \Pi_{D(A)} (x_{t-} + \Delta m_t) = x_t$, $t \in [0, T]$.

(jj) is an immediate consequence of the uniform convergence (29).

**Theorem 30** Assume that $m, m^\epsilon \in D (R^+, \mathbb{H})$, $\epsilon > 0$, $m_0 \in \overline{D(A)}$ and let $x^\epsilon = SP^{(1)} (A_\epsilon; m^\epsilon)$.

If $m^\epsilon \longrightarrow m$ in $D (R^+, \mathbb{H})$, as $\epsilon \to 0$, then
(j) for any $t \in R^+$, $x_t^\epsilon \longrightarrow x_t$, provided that $\Delta m_t = 0$ and there exist $C > 0$ and $\epsilon_0 > 0$ such that, $\forall 0 < \epsilon \leq \epsilon_0$,
\[ \int_0^t |A_\epsilon (x_s^\epsilon)| ds \leq C < +\infty. \quad (33) \]

(jj) $(J^\epsilon (x^\epsilon), m^\epsilon) \longrightarrow (x, m)$ in $D (R^+, \mathbb{H} \times \mathbb{H})$.

(jj) if $m$ is continuous and $x = SP^{(1)} (A; m)$ then
\[ \|x^\epsilon - x\|_T \longrightarrow 0, \quad T \in R^+. \]

**Proof.** (j) Since $m^\epsilon \longrightarrow m$ in $D (R^+, \mathbb{H})$, there exist $\lambda^\epsilon : R^+ \to R^+$, a strictly increasing continuous changes of time such that $\lambda_0^\epsilon = 0$, $\lambda^\epsilon_\infty = +\infty$, $\sup_{t \geq 0} |\lambda^\epsilon_t - t| \to 0$ and $|m_{\lambda^\epsilon}^\epsilon - m||_T \to 0$ as $\epsilon \to 0$, for all $T \in R^+$. We have $(x^\epsilon_{\lambda^\epsilon_t}, k^\epsilon_{\lambda^\epsilon_t}) = SP (A_\epsilon; m^\epsilon_{\lambda^\epsilon_t})$, where $k^\epsilon_{\lambda^\epsilon_t} = \int_0^t A_\epsilon (x_s^\epsilon) ds$. Let $T > 0$ be arbitrary fixed and $t \in [0, T]$. Then there exists $\epsilon_0 > 0$ such that $\lambda^\epsilon_{T+1} \geq t$ and, by (26),
\[ \int_0^t |A_\epsilon (x_s^\epsilon)| ds \leq \int_0^{\lambda^\epsilon_{T+1}} |A_\epsilon (x_s^\epsilon)| ds \leq C (1 + ||m||_T+1), \quad \forall 0 < \epsilon \leq \epsilon_0, \]

which is (33). Moreover, by Theorem 29, the convergence follows. (jj) follows from the assumption $m^\epsilon \longrightarrow m$ in $D (R^+, \mathbb{H})$ and from Theorem 29-(jj). (jj) By Remark 40 we deduce $||m^\epsilon - m||_T \to 0$ as $\epsilon \to 0$, which implies $(iii)$, via Theorem 29.

**Remark 31** Note that Theorem 30-(j) implies that $x^\epsilon$ tends to $x$ in the $S$ topology introduced by Jakubowski in [12].

### 3.2 Approximation with amortized large jumps

If in the first case we considered an approximation with free jumps, this time we define an approximation absorbing the too large jumps with help of a generalized projection.
Let $m \in D(\mathbb{R}^+, \mathbb{H})$, $m_0 \in \overline{D(A)}$. We will consider for $0 < \varepsilon \leq 1$ the approximating equation of the form

\begin{align*}
(i) & \quad x^\varepsilon \in D(\mathbb{R}^+, \mathbb{H}), \\
(ii) & \quad k^\varepsilon \in D(\mathbb{R}^+, \mathbb{H}) \cap BV_{loc}(\mathbb{R}^+; \mathbb{H}), \quad k_0^\varepsilon = 0, \\
(iii) & \quad x_t^\varepsilon + k_t^\varepsilon = m_t, \quad \forall \ t \geq 0, \\
(iv) & \quad k_t^\varepsilon = k_t^{\varepsilon, c} + k_t^{\varepsilon, d}, \quad k_t^{\varepsilon, d} = \sum_{0 \leq s \leq t} \Delta k_s^{\varepsilon}, \\
(v) & \quad k_t^{\varepsilon, c} := \int_0^t A_\varepsilon(x_s^\varepsilon)ds, \quad \forall \ t \geq 0 \\
(vi) & \quad x_t^\varepsilon = (x_{t-}^\varepsilon + \Delta m_t)1_{|\Delta m_t| \leq \varepsilon} + \Pi(x_{t-}^\varepsilon + \Delta m_t)1_{|\Delta m_t| > \varepsilon}, \quad \forall \ t \geq 0.
\end{align*}

Clearly

$$
\Delta k_t^{\varepsilon} = \Delta k_t^{\varepsilon, d} = \{x_{t-}^\varepsilon + \Delta m_t - \Pi(x_{t-}^\varepsilon + \Delta m_t)\}1_{|\Delta m_t| > \varepsilon}.
$$

We will call $x^\varepsilon$ the solution of the Yosida problem associated with the projection $\Pi$ and we will use the notation $(x^\varepsilon, k^\varepsilon) = \mathcal{YP}(A_\varepsilon, \Pi; m)$, $\varepsilon > 0$. Set $t_0 = 0$, $t_{k+1} = \inf\{t > t_k : |\Delta m_t| > \varepsilon\}$, $k \in \mathbb{N}$ and observe that on every interval $[t_k, t_{k+1})$, $x^\varepsilon$ satisfies the equation

$$
x_t^\varepsilon + \int_{t_k}^t A_\varepsilon(x_s^\varepsilon)ds = \Pi(x_{t_k}^\varepsilon + \Delta m_{t_k}) + m_t - m_{t_k},
$$

which has a solution from Theorem 23 since $A_\varepsilon$ is maximal monotone operator. The solution is

\begin{align*}
x_t^\varepsilon = \begin{cases} S\mathcal{P}^{(1)}(A_\varepsilon; m)_t, & t \in [0, t_1), \\
S\mathcal{P}^{(1)}(A_\varepsilon; \Pi(x_{t_k}^\varepsilon + \Delta m_{t_k}) + m_{t_k} - m_{t_k}), & t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}.
\end{cases}
\end{align*}

We can now state the analogue of Remark 26.

**Lemma 32** If $m$ is a step function of the form $m_t = \sum_{r \in \pi} m_r 1_{[r, r')} (t)$, where $\pi \in \mathcal{P}_{\mathbb{R}^+}$ is a partition and $m_0 \in \overline{D(A)}$, then $x^\varepsilon = \mathcal{YP}(A_\varepsilon, \Pi; m)$ is given by: for all $r \in \pi$ and $t \in [r, r')$,

\begin{align*}
x_t^\varepsilon = \begin{cases} \mathcal{YP}(A_\varepsilon; x_r^\varepsilon + \Delta m_r)_t, & if |\Delta m_r| \leq \varepsilon, \\
\mathcal{YP}(A_\varepsilon; \Pi(x_r^\varepsilon + \Delta m_r))_t, & if |\Delta m_r| > \varepsilon.
\end{cases}
\end{align*}

**Proposition 33** Let $m, \hat{m} \in D(\mathbb{R}^+, \mathbb{H})$ and $m_0, \hat{m}_0 \in \overline{D(A)}$.

If $(x^\varepsilon, k^\varepsilon) = \mathcal{YP}(A_\varepsilon, \Pi; m)$ and $(\hat{x}^\varepsilon, \hat{k}^\varepsilon) = \mathcal{YP}(A_\varepsilon, \Pi; \hat{m})$ then

(i) for any $t \in \mathbb{R}^+$ and $\varepsilon > 0$

\begin{align*}
|x_t^\varepsilon - \hat{x}_t^\varepsilon|^2 \leq |m_t - \hat{m}_t|^2 - 2 \int_0^t \langle m_t - \hat{m}_t - m_s + \hat{m}_s, dk_s^\varepsilon - d\hat{k}_s^\varepsilon \rangle.
\end{align*}

(ii) for any $a \in \text{Int}(D(A))$ and $T \in \mathbb{R}^+$ there exist $\varepsilon_0 > 0$ and $C > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$,

\begin{align*}
||x^\varepsilon||^2_T + \int_T \varepsilon T \leq C(1 + ||m||^2_T).
\end{align*}
Proof. (i) By (28)
\[
\int_0^t \langle x_s^\varepsilon - \hat{x}_s^\varepsilon, dk_s^{\varepsilon,c} - d\hat{k}_s^{\varepsilon,c} \rangle \geq 0, \ t \in \mathbb{R}^+.
\]
Using Remark 17-(i), it follows that
\[
\int_0^t \langle x_s^\varepsilon - \hat{x}_s^\varepsilon, dk_s^{\varepsilon,d} - d\hat{k}_s^{\varepsilon,d} \rangle + \frac{1}{2} \sum_{s \leq t} |\Delta k_s^\varepsilon - \Delta \hat{k}_s^\varepsilon|^2 \geq 0, \ t \in \mathbb{R}^+.
\]
Consequently,
\[
\int_0^t \langle x_s^\varepsilon - \hat{x}_s^\varepsilon, dk_s^{\varepsilon} - d\hat{k}_s^{\varepsilon} \rangle + \frac{1}{2} \sum_{s \leq t} |\Delta k_s^\varepsilon - \Delta \hat{k}_s^\varepsilon|^2 \geq 0, \ t \in \mathbb{R}^+.
\]
Using the same arguments as in the proof of Lemma 18-(ii) we obtain (i). (ii) It is sufficient to use (26), to observe that, as in Remark 17-(ii),
\[
r_0|\Delta k_s^\varepsilon| \leq \langle x_s^\varepsilon - a, \Delta k_s^\varepsilon \rangle + \frac{1}{2}|\Delta k_s^\varepsilon|^2, \ s \in \mathbb{R}^+
\]
and to follow the proof of Theorem 22. \qed

Theorem 34 Assume that $m^\varepsilon \in \mathbb{D}(\mathbb{R}^+, \mathbb{H})$, $m_0^\varepsilon \in \overline{D(A)}$, $(x^\varepsilon, k^\varepsilon) = \mathcal{YP}(A_\varepsilon, \Pi; m^\varepsilon)$, $\varepsilon > 0$ and $(x, k) = \mathcal{SP}(A, \Pi; m)$.

(j) If $||m^\varepsilon - m||_T \to 0$, $T \in \mathbb{R}^+$ then
\[
||x^\varepsilon - x||_T \to 0, \ T \in \mathbb{R}^+.
\]

(jj) If $m^\varepsilon \to m$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{H})$ then
\[
(x^\varepsilon, m^\varepsilon) \to (x, m) \text{ in } \mathbb{D}(\mathbb{R}^+, \mathbb{H} \times \mathbb{H}).
\]

Proof. We use the notation from the proof of Theorem 29. Let $(x^{(n)}, k^{(n)}) = \mathcal{SP}(A, \Pi; m^{(n)})$, $(x^{(n)}_\varepsilon, k^{(n)}_\varepsilon) = \mathcal{YP}(A_\varepsilon, \Pi; m^{(n)}_\varepsilon)$, $n \in \mathbb{N}$, $\varepsilon > 0$. Following the same step as in the proof of Theorem 29 we deduce, for all $T \in \mathbb{R}^+$, $n \in \mathbb{N}$,
\[
\lim_{\varepsilon \to 0} ||x^{(n)}_\varepsilon - x^{(n)}||_T \to 0.
\]
Using the same estimates for the differences $n^{(n)}_\varepsilon - m$ and $m^{(n)}_\varepsilon - m^{(n)}$ and the arguments used in the proof of Theorem 29-(j) the conclusion (j) follows and, in addition, (jj). \qed

Corollary 35 Let $m \in \mathbb{D}(\mathbb{R}^+, \mathbb{H})$, $m_0 \in \overline{D(A)}$ and let $\pi_\varepsilon \in \mathcal{P}_{R^+}$ be a sequence of partitions. If $m^{(\varepsilon)}_t = \sum_{r \in \pi_\varepsilon} m^r \mathbf{1}_{[r, r')} (t)$, for $\varepsilon > 0$, denotes the sequence of discretizations of $m$ and $(x^{(\varepsilon)}, k^{(\varepsilon)}) = \mathcal{YP}(A_\varepsilon, \Pi; m^{(\varepsilon)})$, $\varepsilon > 0$ then
\[
(x^{(\varepsilon)}, m^{(\varepsilon)}) \to (x, m) \text{ in } \mathbb{D}(\mathbb{R}^+, \mathbb{H} \times \mathbb{H}),
\]
where $(x, k) = \mathcal{SP}(A, \Pi; m)$.

Moreover, if $||m^{(\varepsilon)} - m||_T \to 0$, $T \in \mathbb{R}^+$ then
\[
||x^{(\varepsilon)} - x||_T \to 0, \ T \in \mathbb{R}^+.
\]

Proof. It is sufficient to observe that $m^{(\varepsilon)} \to m$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{H})$ and to apply Theorem 34. \qed
4 Annexes

4.1 Maximal monotone operators on Hilbert spaces

Let $H$ be a separable real Hilbert space with the inner product denoted by $\langle \cdot, \cdot \rangle$ and the induced norm $|\cdot|$. A multivalued operator $A : H \rightharpoonup H$ (a point-to-set operator $A : H \to 2^H$) will be seen also as a subset of $H \times H$. In fact we formally identify the multivalued operator $A : H \rightharpoonup H$ with its graph $\text{Gr } (A) = \{(x,y) \in H \times H : y \in A(x)\}$. Denote with $D(A) := \{x \in H : Ax \neq \emptyset\}$, the domain of $A$. Define $A^{-1} : H \rightrightarrows H$ the point-to-set operator defined by: $x \in A^{-1}(y)$ if $y \in A(x)$.

We give some definitions:

• $A : H \rightharpoonup H$ is monotone if $\langle v - y, u - x \rangle \geq 0$, for all $(x,y), (u,v) \in A$.

• $A : H \rightrightarrows H$ is a maximal monotone operator if $A$ is a monotone operator and it is maximal in the set of monotone operators: that is,

$$\langle v - y, u - x \rangle \geq 0, \forall (x,y) \in A \implies (u,v) \in A.$$

We recall:

**Proposition 36** Let $A : H \rightharpoonup H$ be a maximal monotone operator. Then

(a) $\overline{D(A)}$ is a convex subset of $H$.

(b) $Ax$ is a convex closed subset of $H$ for all $x \in D(A)$.

(c) $A$ is locally bounded on $\text{Int } (D(A))$ that is: for every $u_0 \in \text{Int } (D(A))$ there exists $r_0 > 0$ such that

$$\overline{B(u_0, r_0)} := \{u_0 + r_0 v : |v| \leq 1\} \subset D(A)$$

and

$$A_{u_0, r_0}^\# := \sup \{|\hat{u}| : \hat{u} \in A(u_0 + r_0 v), |v| \leq 1\} < \infty.$$

The maximal monotone operator $A$ is uniquely defined by its principal section $A^0 x := \Pi_{Ax}(0)$ in the following sense: if $(x,y) \in \overline{D(A)} \times H$ such that

$$\langle y - A^0u, x - u \rangle \geq 0, \forall u \in D(A)$$

then $(x,y) \in A$.

For each $\varepsilon > 0$ the operators

$$J_\varepsilon x = (I + \varepsilon A)^{-1}(x) \text{ and } A_\varepsilon (x) = \frac{1}{\varepsilon}(x - J_\varepsilon x),$$

from $H$ to $H$ are single-valued. The operator $A_\varepsilon$ is called Yosida’s approximation of the maximal monotone operator $A$. In [1] and [5] we can find the proof of the following properties:

**Proposition 37** Let $A : H \rightrightarrows H$ be a maximal monotone operator. Then:
(j) for all $\varepsilon > 0$ and for all $x, y \in \mathbb{H}$

(i) $(J_\varepsilon x, A_\varepsilon x) \in A,$

(ii) $|J_\varepsilon x - J_\varepsilon y| \leq |x - y|,$

(iii) $|A_\varepsilon x - A_\varepsilon y| \leq \frac{1}{\varepsilon} |x - y|,$

(iv) $A_\varepsilon : \mathbb{H} \to \mathbb{H}$ is a maximal monotone operator.

(jj) if $x_\varepsilon, x \in \mathbb{H}$ and $\lim_{\varepsilon \searrow 0} x_\varepsilon = x,$ then $\lim_{\varepsilon \searrow 0} J_\varepsilon x_\varepsilon = \Pi_{D(A)}(x),$ $\forall x \in \mathbb{H},$ where $\Pi_{D(A)}(x)$ is the orthogonal projection of $x$ on $D(A).

(jj) $\lim_{\varepsilon \searrow 0} A_\varepsilon x = A^0 x \in A x,$ for all $x \in D(A).$

(jv) $|A_\varepsilon x|$ is monotone decreasing in $\varepsilon > 0,$ and when $\varepsilon \searrow 0$

$$|A_\varepsilon (x)| \geq \begin{cases} |A^0 (x)|, & \text{if } x \in D(A), \\ +\infty, & \text{if } x \notin D(A). \end{cases}$$

(v) For all $x, y \in \mathbb{H}$

$$\langle x - y, A_\varepsilon (x) - A_\varepsilon (y) \rangle \geq \varepsilon \left( |A_\varepsilon (x)|^2 + |A_\varepsilon (y)|^2 - 2 \langle A_\varepsilon (x), A_\varepsilon (y) \rangle \right) \geq 0.$$

Proposition 38 Let $A : \mathbb{H} \Rightarrow \mathbb{H}$ be a maximal monotone operator. Let $r_0 \geq 0$ and $B(a, r_0) \subset D(A).$ Then for all $x \in D(A),$ $\hat{x} \in A(x)$ and $\hat{u} \in A(a + r_0 u),$ where $u = 0,$ if $\hat{x} = 0$ and $u = \frac{\hat{x}}{|\hat{x}|}$ if $\hat{x} \neq 0$ the following inequalities hold:

(j) $r_0 |\hat{x}| \leq \langle \hat{x}, x - a \rangle + |\hat{u}| |x - a| + r_0 |\hat{u}|,$

(jj) $r_0 |\hat{x}| \leq \langle \hat{x}, x - a \rangle + A^\#_{x_0, r_0} |x - a| + r_0 A^\#_{x_0, r_0},$

where $A^\#_{x_0, r_0} := \sup \{|\hat{u}| : \hat{u} \in A(a + r_0 u), |v| \leq 1\}.$

Moreover for all $\varepsilon \in (0, 1)$ and $z \in \mathbb{H} :$

(jjj) $r_0 |A_\varepsilon z| \leq \langle A_\varepsilon z, z - a \rangle + |\hat{u}| |z - a| + (|A^0 (a)| + r_0) |\hat{u}|,$

(jv) $r_0 |A_\varepsilon z| \leq \langle A_\varepsilon z, z - a \rangle + A^\#_{x_0, r_0} |z - a| + (|A^0 (a)| + r_0) A^\#_{x_0, r_0}$

(38)

Proof. By monotonicity of $A$ we have

$$r_0 \langle \hat{x}, u \rangle \leq r_0 \langle \hat{x}, u \rangle + \langle \hat{x} - \hat{u}, x - (a + r_0 u) \rangle = \langle \hat{x}, x - a \rangle - \langle \hat{u}, x - a \rangle + r_0 \langle \hat{u}, u \rangle \leq \langle \hat{x}, x - a \rangle + |\hat{u}| |x - a| + r_0 |\hat{u}|,$$

that is (37). Now since $A_\varepsilon (x) \in A(J_\varepsilon (x)),$

$$r_0 |A_\varepsilon x| \leq \langle A_\varepsilon x, J_\varepsilon (x) - a \rangle + |\hat{u}| |J_\varepsilon (x) - a| + r_0 |\hat{u}| \leq \langle A_\varepsilon x, x - a \rangle + |\hat{u}| |J_\varepsilon (x) - a| + |J_\varepsilon (x) - a| + r_0 |\hat{u}| \leq \langle A_\varepsilon x, x - a \rangle + |\hat{u}| |x - x_0| + |\hat{u}| |A^0 (a)| + r_0 |\hat{u}|.$$

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If \( \varphi : \mathbb{H} \to (-\infty, +\infty] \) is a proper convex, lower semicontinuous function then the subdifferential operator \( A = \partial \varphi : \mathbb{H} \rightrightarrows \mathbb{H} \) defined by
\[
\partial \varphi (x) := \{ y \in \mathbb{H} : \langle \dot{x}, z - x \rangle + \varphi (x) \leq \varphi (z), \text{ for all } z \in \mathbb{H} \}
\]
is a maximal monotone operator on \( \mathbb{H} \). In this case \( A_\varepsilon (x) = \nabla \varphi_\varepsilon (x) \) and \( J_\varepsilon (x) = x - \varepsilon \nabla \varphi_\varepsilon (x) \), where \( \varphi_\varepsilon \) is the Moreau-Yosida regularization of \( \varphi \)
\[
\varphi_\varepsilon (x) := \inf \left\{ \frac{|x - z|^2}{2\varepsilon} + \varphi (z) : z \in \mathbb{H} \right\}
\]
If \( \overline{D} \) is a nonempty closed convex subset of \( \mathbb{H} \), then the convexity indicator function \( I_D : \mathbb{H} \to (-\infty, +\infty] \)
\[
I_D (x) := \left\{ \begin{array}{ll}
0, & \text{if } x \in \overline{D}, \\
+\infty, & \text{if } x \in \mathbb{H} \setminus \overline{D},
\end{array} \right.
\]
is a proper convex lower semicontinuous and
\[
\partial I_D (x) = \left\{ \begin{array}{ll}
N_D (x) = \{ \nu : \langle \nu, z - x \rangle \leq 0, \text{ for all } z \in \overline{D} \}, & \text{if } x \in \overline{D}, \\
\emptyset, & \text{if } x \notin \overline{D},
\end{array} \right.
\]
where \( N_D (x) = \{ 0 \} \) if \( x \in \text{Int} (\overline{D}) \) and \( N_D (x) \) is the closed external normal cone to \( \overline{D} \) if \( x \in \text{Bd} (\overline{D}) \). In this case for the maximal operator \( A = \partial I_D \) we have
\[
J_\varepsilon (x) = \Pi_D (x) \quad \text{and} \quad A_\varepsilon (x) = \frac{1}{\varepsilon} (x - \Pi_D (x)) \in \partial I_D (\Pi_D (x)).
\]
Let \( r_0 \geq 0 \) and \( \overline{\{ a, r_0 \}} \subset \overline{D} \), then by Proposition 38 for all \( z \in \mathbb{H} \):
\[
\begin{align*}
r_0 |z - \Pi_D (z)| & \leq \langle z - \Pi_D (z), \Pi_D (z) - a \rangle \\
& = - |z - \Pi_D (z)|^2 + \langle z - \Pi_D (z), z - a \rangle,
\end{align*}
\]
since \( \hat{u} = 0 \in \partial I_D (a + r_0 u) \).

### 4.2 Skorohod space

Let \((\mathbb{H}, \langle , \rangle)\) be a separable real Hilbert space with the induced norm \(| . | \).

A function \( x : \mathbb{R}^+ \to \mathbb{H} \) is a càdlàg function if for every \( t \in \mathbb{R}^+ \) the left limit \( x_{t-} := \lim_{s \uparrow t} x_s \) and the right limit \( x_{t+} := \lim_{s \downarrow t} x_s \) exist in \( \mathbb{H} \) and \( x_{t+} = x_t \) for all \( t \geq 0 \); by convention \( x_{0-} = x_0 \).

Denote by \( \mathbb{D} (\mathbb{R}^+, \mathbb{H}) \) the set of càdlàg functions \( x : \mathbb{R}^+ \to \mathbb{H} \) and \( \mathbb{D} ([0, T], \mathbb{H}) \subset \mathbb{D} (\mathbb{R}^+, \mathbb{H}) \) the subspace of paths \( x \) that stop at the instant \( T \) that is \( x \in \mathbb{D} ([0, T], \mathbb{H}) \) if \( x \in \mathbb{D} (\mathbb{R}^+, \mathbb{H}) \) and \( x_t = x^T_t := x_{t\wedge T} \) for all \( t \geq 0 \). The spaces of continuous functions will be denoted by \( \mathcal{C} (\mathbb{R}^+, \mathbb{H}) \) and \( \mathcal{C} ([0, T], \mathbb{H}) \), respectively.

We say that \( \pi = \{ t_0, t_1, t_2, \ldots \} \) is a partition of \( \mathbb{R}^+ \) if
\[
0 = t_0 < t_1 < t_2 < \ldots \quad \text{and} \quad t_n \to +\infty.
\]
Let $\pi$ be a partition and $r \in \pi$. We denote by $r'$ the successor of $r$ in the partition $\pi$, i.e.

$$
\text{if } r = t_i \text{ then } r' := t_{i+1}.
$$

(40)

Write

$$
\|\pi\| := \sup \{ r' - r : r \in \pi \}
$$

and

$$
\text{mesh} (\pi) := \inf \{ r' - r : r \in \pi \}.
$$

The set of all partitions of $\mathbb{R}^+$ will be denoted $\mathcal{P}_{\mathbb{R}^+}$. Given a function $x : \mathbb{R}^+ \to \mathbb{H}$ we define

- the norm sup by: $\|x\|_T = \sup_{t \in [0,T]} |x_t|$ and $\|x\|_\infty = \sup_{t \geq 0} |x_t|$.
- the oscillation of $x$ the on a set $F \subset \mathbb{R}^+$ by: $\mathcal{O}_x(F) = \omega_x(F) = \sup_{t,s \in F} |x_t - x_s|$.
- the modulus of continuity $\mu_x : \mathbb{R}^+ \to \mathbb{R}^+$ by: $\mu_x(\varepsilon) = \mu(\varepsilon;x) = \sup_{t \in \mathbb{R}^+} \mathcal{O}_x([t,t+\varepsilon])$.
- the càdlàg modulus by: $\gamma_x(\varepsilon) = \omega_x'((\varepsilon)) := \inf_{\pi} \{ \max_{r \in \pi} \mathcal{O}_x([r,r')) : \text{mesh}(\pi) > \varepsilon \}$, where $\pi \in \mathcal{P}_{\mathbb{R}^+}$ is a partition an $\gamma_x(\varepsilon,T) = \omega_x'(\varepsilon,T) := \gamma_{x,T}(\varepsilon)$.

**Remark 39** We remark that

1. function $x : \mathbb{R}^+ \to \mathbb{H}$ is continuous on $[0,T]$ if and only if $\lim_{\varepsilon \to 0} \mu_x(\varepsilon;x^T) = 0$.
2. function $x$ is càdlàg on $[0,T]$ if and only if $\lim_{t \to 0} \gamma_x(\varepsilon,T) = 0$.
3. $\gamma_x(\varepsilon) \leq \mu_x(2\varepsilon)$.
4. if $x \in \mathcal{D}([0,T],\mathbb{H})$ then for each $\delta > 0$ there exists a partition $\pi \in \mathcal{P}_{\mathbb{R}^+}$ such that $\max_{r \in \pi} \mathcal{O}_x([r,r')) < \delta$ and consequently:

   (a) there exists a sequence of of partitions $\pi_n \in \mathcal{P}_{\mathbb{R}^+}$ such that $\max_{r \in \pi_n} \mathcal{O}_x([r,r')) < \frac{1}{n}$.

Therefore $x$ can be uniformly approximate by simple functions constant on intervals:

$$
x_t^n = \sum_{r \in \pi_n} x_r 1_{[r,r')} (t), \quad t \geq 0,
$$

and $\|x^n - x\|_\infty \to 0$ as $\|\pi_n\| \to 0$.

(b) for each $\delta > 0$ there exist a finite number of points $t \in [0,T]$ such that

$$
|x_t - x_{t-}| \geq \delta.
$$

(c) $\|x\|_T < \infty$ and the closure of $x([0,T])$ is compact.

Let $\Lambda$ the collection of the strictly increasing functions $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$, $\lambda(0) = 0$, $\lambda(+\infty) = +\infty$, (the space of time scale transformations). The Skorohod topology on $\mathcal{D}([0,T],\mathbb{H})$ it the topology defined by one of the two topologically equivalent metrics: for all $x,y \in \mathcal{D}([0,T],\mathbb{H})$

$$
d^0(x,y) = \inf_{\lambda \in \Lambda} (\|x - y \circ \lambda\|_T + \|\lambda - I\|_\infty), \quad \text{and}
$$

$$
d^1(x,y) = \inf_{\lambda \in \Lambda} \left( \|x - y \circ \lambda\|_T + \ln \left( \sup_{\varepsilon > 0} \mu_{\lambda}(\varepsilon) \right) \right),
$$

where $a \lor b := \max \{a,b\}$ and $I : \mathbb{R}^+ \to \mathbb{R}^+$ is the identity function.
Remark 40. We have:
1. $d^3(x, y) \leq \|x - y\|_T$ for all $x, y \in D([0, T], \mathbb{H})$.
2. $D([0, T], \mathbb{H}), d^3$ is a separable metric space, but not complete.
3. $(D([0, T], \mathbb{H}), d^1)$ is separable complete metric space (Polish space).
4. if $x^n \to x$ in $D([0, T], \mathbb{H})$ if and only if there exists $\lambda_n \in \Lambda$ such that
   $$\|\lambda_n - I\|_{\infty} + \|x^n \circ \lambda_n - x\|_T \to 0, \text{ as } n \to \infty,$$
5. if $x^n \to x$ in $D([0, T], \mathbb{H})$, then $x_n(t) \to x(t)$ holds for continuity points $t$ of $x$; moreover
   if $x$ is continuous, then $\|x^n - x\|_T \to 0$.
6. if $x^n \to x$ in $D([0, T], \mathbb{H})$, then
   $$\sup_{n \in \mathbb{N}} \|x^n\|_T < \infty \quad \text{and} \quad \limsup_{\varepsilon \to 0} \gamma_{x_n}(\varepsilon, T) < \infty.$$
7. $D(\mathbb{R}^+, \mathbb{H})$ is a Polish space under the metric
   $$d(x, y) = \sum_{T \in \mathbb{N}} 2^{-T} d^1(x^T, y^T), \quad x, y \in D(\mathbb{R}^+, \mathbb{H}).$$

The topology $\tau$ generated by $d$ on $D(\mathbb{R}^+, \mathbb{H})$ is called the Skorohod topology (is denoted by $J_1$)
and coincides on $C(\mathbb{R}^+, \mathbb{H})$ with the topology of uniform convergence on bounded intervals.
Also the Skorohod topology coincides on $D([0, T], \mathbb{H}) \subset D(\mathbb{R}^+, \mathbb{H})$ with the topology generated
of $d^0$ or $d^1$.

Using [8, Chapter 3, Proposition 6.5] it is easy to see that:

Proposition 41. Let $m \in D(\mathbb{R}^+, \mathbb{H})$ and $\pi_n \in \mathcal{P}_{\mathbb{R}^+}, n \in \mathbb{N}$, be a partition such that $\|\pi_n\| \to 0$.
If
   $$m^n_t := \sum_{r \in \pi_n} m_r 1_{[r, r')] (t),$$
then
   $$m^n \to m \text{ in } D(\mathbb{R}^+, \mathbb{H}).$$

4.3 Bounded variation functions

Let $[a, b]$ be a closed interval from $\mathbb{R}$ and $\mathcal{P}_{[a,b]}$ be the set of the partitions
$$\pi = \{a = t_0 < t_1 < \cdots < t_n = b\}, \quad n \in \mathbb{N}^*.$$
Denote by $||\pi|| = \sup\{t_{i+1} - t_i : 0 \leq i \leq n - 1\}$.

We define the variation of a function $k : [a, b] \to \mathbb{H}$ corresponding to the partition $\pi \in \mathcal{P}_{[a,b]}$
$$V_\pi(k) := \sum_{i=0}^{n-1} |k_{t_{i+1}} - k_{t_i}|$$
and the total variation of $k$ on $[a, b]$ by
$$\|k\|_{[a, b]} = \sup_{\pi \in \mathcal{P}_{[a,b]}} V_\pi(k) = \sup \left\{ \sum_{i=0}^{n-1} |k_{t_{i+1}} - k_{t_i}| : \pi \in \mathcal{P}_{[a,b]} \right\}.$$
If $[a, b] = [0, T]$ then $\|k\|_{[0, T]} = \|k\|_{[0, T]}$. 


Remark that the above series is well defined since \( k \) is a pure jump function and \( k \geq t \) be arbitrary and passing to the limit for \( N \to \infty \). Thus

\[ V_{\pi_N}(k) \xrightarrow{N \to \infty} k_T^\uparrow \]

Proof. Clearly \( V_{\pi_N}(k) \) is increasing with respect to \( N \) and \( V_{\pi_N}(k) \leq k_T^\uparrow \). Let \( \pi \in \mathcal{P}_{[0,T]} \) be arbitrary \( \pi = \{0 = t_0 < t_1 < \cdots < t_n = T\} \) and let \( j_{i,N} \) be the integer smallest integer grater or equal to \( \frac{t_i}{2^N} \). Then \( j_{i,N} \frac{T}{2^N} \geq t_i \) and \( \lim_{N \to \infty} (j_{i,N} \frac{T}{2^N}) = t_i \). We have

\[
V_\pi(k) = \sum_{i=0}^{n-1} |k_{t_{i+1}} - k_{t_i}| \\
\leq \sum_{i=0}^{n-1} \left| k_{t_{i+1}} - k_{j_{i+1,N} \frac{T}{2^N}} \right| + \left| k_{j_{i+1,N} \frac{T}{2^N}} - k_{j_{i,N} \frac{T}{2^N}} \right| + \left| k_{j_{i,N} \frac{T}{2^N}} - k_{t_i} \right| \\
\leq 2 \sum_{i=0}^{n-1} \left| k_{j_{i,N} \frac{T}{2^N}} - k_{t_i} \right| + V_{\pi_N}(k)
\]

and passing to the limit for \( N \to \infty \) we obtain

\[
V_\pi(k) \leq \lim_{N \to \infty} V_{\pi_N}(k) \leq k_T^\uparrow, \quad \forall \pi \in \mathcal{P}_{[a,b]}. 
\]

Hence \( \lim_{N \to \infty} V_{\pi_N}(k) = k_T^\uparrow \).

Definition 43 A function \( k : [a, b] \to \mathbb{H} \) has bounded variation on \([a, b]\) if \( k_T^\uparrow < \infty \). The space of bounded variation functions on \([a, b]\) will be denoted by \( BV([a, b]; \mathbb{H}) \). By \( BV_{loc}(\mathbb{R}^+; \mathbb{H}) \) we denote the space of the functions \( k : \mathbb{R}^+ \to \mathbb{H} \) such that \( k_T^\uparrow < \infty \) for all \( T > 0 \).

Let \( k \in \mathcal{D}(\mathbb{R}^+; \mathbb{H}) \). The function \( k \) has the decomposition \( k_t = k_t^c + k_t^d \) where

\[
k_t^d = \sum_{0 \leq s \leq t} \Delta k_s, \quad \text{with} \quad \Delta k_s = k_s - k_{s-}
\]

is a pure jump function and \( k_t^c = k_t - k_t^d \) is a continuous function. The series which define \( k_t^d \) is convergent since \( \sum_{0 \leq s \leq t} |\Delta k_s| \leq k_T^\uparrow < \infty \).

If \( x \in \mathcal{D}(\mathbb{R}^+; \mathbb{H}) \) and \( k \in \mathcal{D}(\mathbb{R}^+; \mathbb{H}) \), we define

\[
[x, k]_t := \sum_{0 \leq s \leq t} \langle \Delta x_s, \Delta k_s \rangle
\]

Remark that the above series is well defined since

\[
[|x, k|]_t = \left| \sum_{0 \leq s \leq t} \langle \Delta x_s, \Delta k_s \rangle \right| \leq \sum_{0 \leq s \leq t} |\Delta x_s| |\Delta k_s| \leq 2 \left( \sup_{0 \leq s \leq t} |x_s| \right) k_T^\uparrow.
\]
We recall now some results due to [9]. If \( k \in \mathcal{D}(\mathbb{R}^+, \mathbb{H}) \cap BV_{loc}(\mathbb{R}^+; \mathbb{H}) \) then there exists a unique \( \mathbb{H} \)-valued, \( \sigma \)-finite measure \( \mu_k : \mathcal{B}_{\mathbb{R}^+} \to \mathbb{H} \) such that

\[
\mu_k ((s, t]) = k_t - k_s, \quad \text{for all } 0 \leq s < t.
\]

and the total variation measure is uniquely defined by

\[
|\mu_k| ((s, t]) = \uparrow k^+_t - \downarrow k^-_s.
\]

The Lebesgue-Stieltjes integral on \((s, t]\)

\[
\int_s^t \langle x_r, dk_r \rangle := \int_{(s, t]} \langle x_r, \mu_k (dr) \rangle = \int_{(s, t]} \langle x_r, \mu_{k^+} (dr) \rangle + \int_{(s, t]} \langle x_r, \mu_{k^+} (dr) \rangle
\]

is defined for all Borel measurable function such that \( \int_s^t |x_r| d\uparrow k^+_r := \int_{[s, t]} |x_r| |\mu_k| (dr) < \infty \), and in this case

\[
\left| \int_s^t \langle x_r, dk_r \rangle \right| \leq \int_s^t |x_r| d\uparrow k^+_r \leq \left( \sup_{s < r \leq t} |x_r| \right) (\uparrow k^+_t - \downarrow k^-_s).
\]

We add that the Lebesgue-Stieltjes integral on \([t]\)

\[
\int_{[t]} \langle x_r, dk_r \rangle := \langle x_t, \Delta k_t \rangle.
\]

Let \( \pi_n \in \mathcal{P}_{\mathbb{R}^+} \) be a sequence of partitions such that \( ||\pi_n|| \to 0 \) as \( n \to \infty \). Denote

\[
[r]_n = [r]_{\pi_n} := \max \left\{ s \in \pi_n : s < r \right\} \quad \text{and} \quad [r]_n = [r]_{\pi_n} := \min \left\{ s \in \pi_n : s \geq r \right\}.
\]

Then for all \( x \in \mathcal{D}(\mathbb{R}^+, \mathbb{H}) \) and \( r \geq 0 \) we have \( x_{[r]_n} \to x_{r-} \) and \( x_{[r]_n} \to x_r \). By the Lebesgue dominated convergence theorem as \( n \to \infty \)

\[
\sum_{s \in \pi_n} \langle x_{s \wedge t}, k_{s'-\wedge t} - k_{s\wedge t} \rangle = \int_0^t \langle x_{[r]_n}, dk_r \rangle \to \int_0^t \langle x_{r-}, dk_r \rangle,
\]

and

\[
\int_0^t \langle x_r, dk_r \rangle = \int_0^t \langle x_{r-}, dk_r \rangle + [x, k]_t
\]

(41)
Remark that if $k \in D(\mathbb{R}^+,\mathcal{H}) \cap BV_{\text{loc}}(\mathbb{R}^+;\mathcal{H})$ then

$$|k_t|^2 - |k_0|^2 = \sum_{s \in \pi_n} \langle k_s' \wedge t, k_s' \wedge t - k_s \wedge t \rangle + \sum_{s \in \pi_n} \langle k_s \wedge t, k_s' \wedge t - k_s \wedge t \rangle$$

$$\rightarrow \int_0^t \langle x_r, dk_r \rangle + \int_0^t \langle x_{r-}, dk_r \rangle = 2 \int_0^t \langle k_r, dk_r \rangle - [k,k]_t.$$ 

Hence

Lemma 44 If $k \in D(\mathbb{R}^+,\mathcal{H}) \cap BV_{\text{loc}}(\mathbb{R}^+;\mathcal{H})$ then

$$\int_0^t \langle k_r, dk_r \rangle = \frac{1}{2} |k_t|^2 - \frac{1}{2} |k_0|^2 + \frac{1}{2} [k,k]_t.$$ 

Lemma 45 Let $k: \mathbb{R}^+ \to \mathcal{H}$ and $k^n \in D(\mathbb{R}^+,\mathcal{H}) \cap BV_{\text{loc}}(\mathbb{R}^+;\mathcal{H})$. If for all $T \geq 0$,

$$\|k^n - k\|_T \to 0, \quad \text{and} \quad \sup_{n \in \mathbb{N}^+} \|k^n\|_T = M < +\infty,$$

then $k \in D(\mathbb{R}^+,\mathcal{H}) \cap BV([0,T];D(\mathbb{R}^+;\mathcal{H}))$ and $\|k\|_T \leq M$.

Proof. Let $0 < \varepsilon \leq 1$. Then

$$|k_{t+\varepsilon} - k_t| \leq 2 \|k_s - k^n_s\|_{t+\varepsilon} + |k^n_{t+\varepsilon} - k^n_t|$$ 

and therefore

$$\limsup_{\varepsilon \to 0} |k_{t+\varepsilon} - k_t| \leq 2 \|k_s - k^n_s\|_{t+1}, \quad \text{for all } n \in \mathbb{N}^*.$$ 

Hence $k_{t+} = k_t$. Let now a sequence $\pi_N \in \mathcal{P}_{[0,T]}$ such that $V_{\pi_N} \to \uparrow k_T^\uparrow_T$ as $N \to \infty$.

From the definition of $\uparrow k_T^\uparrow_T$ we have $V_{\pi_N} \leq \uparrow k^n_T^\uparrow_T \leq M$. Since $k^n_t \to k_t$ for all $t \in [0,T]$, $V_{\pi_N} \to V_{\pi_N} (k)$. Hence $V_{\pi_N} (k) \leq M$, for all $N \in \mathbb{N}^*$, and passing to the limit as $N \to \infty$ we obtain that $\|k\|_T \leq M$.

Theorem 46 (Helly-Bray) Let $n \in \mathbb{N}^+$, $x^n, x, k \in D(\mathbb{R}^+,\mathcal{H})$ and $k^n \in D(\mathbb{R}^+,\mathcal{H}) \cap BV_{\text{loc}}(\mathbb{R}^+;\mathcal{H})$.

such that for all $T$ :

(i) \quad $\|x^n_s - x_s\|_T \to 0$, as $n \to \infty$,

(ii) \quad $\|k^n_s - k_s\|_T \to 0$, as $n \to \infty$, \quad and

(iii) \quad $\sup_{n \in \mathbb{N}^+} \|k^n\|_T = M < +\infty$.

Then $k \in D(\mathbb{R}^+,\mathcal{H}) \cap BV([0,T];\mathbb{R}^d)$, $\uparrow k_T^\uparrow_T \leq M$, and uniformly with respect to $s,t \in [0,T]$, $s \leq t$ :

$$\int_s^t \langle x^n_r, dk^n_r \rangle \to \int_s^t \langle x_r, dk_r \rangle, \quad \text{as } n \to \infty. \quad (42)$$

Moreover

$$\int_s^t |x_r| d\uparrow k^n_t \leq \liminf_{n \to +\infty} \int_s^t |x^n_r| d\uparrow k^n_t \quad \text{for all } 0 \leq s \leq t \leq T \quad (43)$$

and there exist a a subsequence $n_i$ $\to \infty$ and a sequence $\delta_i \to 0$ as $i \to \infty$ such that uniformly with respect to $s,t \in [0,T]$, $s \leq t$ :

$$\int_s^t 1_{|\Delta k^n_r| > \delta} \langle x^n_r, dk^n_r \rangle \to \int_s^t \langle x_r, dk_r \rangle, \quad \text{as } i \to \infty. \quad (44)$$

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Proof. From Lemma 45 we have $k \in D(\mathbb{R}^+, H) \cap BV ([0, T]; \mathbb{R}^d)$ and $\int_{[0, T]} k_r \, dt \leq M$. Step 1. Let $\varepsilon > 0$ and $\pi_\varepsilon \in \mathcal{P}_{\mathcal{R}^+}$ be such that $\max_{r \in \pi_\varepsilon} \mathcal{O}_x (r, r') < \varepsilon$. Denote $x_u^\varepsilon = \sum_{r \in \pi_\varepsilon} x_r 1_{[r, r')} (u), \quad u \geq 0$. We have $|x_u - x_u^\varepsilon| < \varepsilon$ for all $u \geq 0$ and consequently

$$\left| \int_s^t \langle x_r^n, dk_r \rangle - \int_s^t \langle x_r, dk_r \rangle \right| \leq \int_s^t \left| \langle x_r - x_r^\varepsilon, dk_r - dk_r \rangle \right| \leq 2 \varepsilon \|x\| \|k\|_T \text{ card } \{ r \in \pi_\varepsilon : r \leq T \}$$

because

$$\int_0^t \langle x_r^\varepsilon, dk_r \rangle = \sum_{r \in \pi_\varepsilon} \langle x_r \wedge_t, k_r \wedge_t - k_r \wedge_t \rangle.$$ 

Hence

$$\lim_{n \to \infty} \left[ \sup_{0 < s \leq t \leq T} \left| \int_s^t \langle x_r^n, dk_r^n \rangle - \int_s^t \langle x_r, dk_r \rangle \right| \right] \leq 2 \varepsilon, \quad \forall \varepsilon > 0.$$ 

that yields (42) \textbf{Step 2.} Let $\alpha \in C ([0, T]; \mathbb{R}^d), \|\alpha\|_T \leq 1$. Then

$$\int_s^t |x_r| \langle \alpha_r, dk_r \rangle = \lim_{n \to \infty} \int_s^t |x_r^n| \langle \alpha_r, dk_r^n \rangle \leq \liminf_{n \to +\infty} \int_s^t |x_r^n| \int_{[0, T]} k_r \, dt = \frac{1}{2t},$$ 

and passing to $\sup_{\|\alpha\|_T \leq 1}$ we get (43). \textbf{Step 3.} Recall that the set $\{ |\Delta k_r| : r \in [0, T] \}$ is at most countable. Let $\delta \notin \{ |\Delta k_r| : r \in [0, T] \}$. Since

$$\left| \int_s^t 1_{|\Delta k_r| > \delta} \langle x_r, dk_r^d \rangle - \int_s^t \langle x_r, dk_r^d \rangle \right| \leq \int_0^T 1_{|\Delta k_r| \leq \delta} |x_r| \int_{[0, T]} k_r \, dt,$$

by Lebesgue dominated convergence theorem there exists a sequence $\delta_i \searrow 0, \delta_i \notin \{ |\Delta k_r| : r \in [0, T] \}$ such that

$$\sup_{0 < s \leq t \leq T} \left| \int_s^t 1_{|\Delta k_r| > \delta_i} \langle x_r, dk_r^d \rangle - \int_s^t \langle x_r, dk_r^d \rangle \right| < \frac{1}{2t}, \quad \text{for all } i \in \mathbb{N}.$$ 

Note that for each $i$ fixed the set $Q_i = \{ r \geq 0 : |\Delta k_r| > \delta_i \}$ is finite and consequently, using the uniform convergence of $k^n$, there exists $\ell_i \in \mathbb{N}$ such that $|\Delta k_r^n| > \delta_i$ for all $n \geq \ell_i$ and for all $r \in Q_i$. We have

$$\left| \int_s^t 1_{|\Delta k_r^n| > \delta_i} \langle x_r^n, dk_r^d \rangle - \int_s^t \langle x_r, dk_r^d \rangle \right| \leq D_{i,n} + E_{i,n} + F_i$$

with

$$D_{i,n} = \left| \int_s^t \left( 1_{|\Delta k_r^n| > \delta_i} - 1_{|\Delta k_r| > \delta_i} \right) \langle x_r^n, dk_r^d \rangle \right|,$$

$$E_{i,n} = \left| \int_s^t 1_{|\Delta k_r| > \delta_i} \langle x_r^n, dk_r^d \rangle - \int_s^t 1_{|\Delta k_r| > \delta_i} \langle x_r, dk_r^d \rangle \right|,$$

and

$$F_i = \left| \int_s^t 1_{|\Delta k_r| > \delta_i} \langle x_r, dk_r^d \rangle - \int_s^t \langle x_r, dk_r^d \rangle \right| \leq \frac{1}{2t}.$$ 

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Since
\[ |1_{|\Delta k^p| > \delta_i} - 1_{|\Delta k^r| > \delta_i}| \leq 1_{|\Delta k^p - \Delta k^r| > \delta_i - \delta_{i+1}} + 1_{|\Delta k^r| > \delta_{i+1}} |1_{|\Delta k^p| > \delta_i} - 1_{|\Delta k^r| > \delta_i}| \leq 1_2\|k^n - k\|_T \geq \delta_i - \delta_{i+1} + 1_{|\Delta k^r| > \delta_{i+1}} |1_{|\Delta k^p| > \delta_i} - 1_{|\Delta k^r| > \delta_i}|, \]
we deduce
\[ D_{i,n} \leq \|x^n\|_T \|k^{n+}_T - k^n_T T_1 2_2\|k^n - k\|_T \geq \delta_i - \delta_{i+1} + \sum_{r \in Q_{i+1}} |1_{|\Delta k^p| > \delta_i} - 1_{|\Delta k^r| > \delta_i}\|x^n_k, \Delta k^n_r\rangle \]
\[ \leq \left[ 2_2\|k^n - k\|_T \geq \delta_i - \delta_{i+1} + \sum_{r \in Q_{i+1}} |1_{|\Delta k^p| > \delta_i} - 1_{|\Delta k^r| > \delta_i}\right] \|x^n\|_T \|k^n - k\|_T \text{card}(Q_i). \]

It follows there exists \( n_i \geq \ell_i \) such that
\[ D_{i,n} = 0 \text{ and } E_{i,n} < \frac{1}{2i} \text{ for all } n \geq n_i. \]

Hence
\[ \left| \int_s^t 1_{|\Delta k^p| > \delta_i} \langle x^{ni}_r, dk^{d,ni}_r \rangle - \int_s^t \langle x_r, dk^d_r \rangle \right| < \frac{1}{i}, \text{ for all } i \in \mathbb{N}^*. \]

We give, also, other auxiliary results used in the paper

**Proposition 47** Let \( A : H \rightrightarrows H \) be a maximal monotone operator and \( A : D (\mathbb{R}^+, H) \rightrightarrows D (\mathbb{R}^+, H) \cap \text{BV}([0, T]; D (\mathbb{R}^+, H)) \) be defined by: \( (x, k) \in A \) if \( x \in D (\mathbb{R}^+, H) \), \( k \in D (\mathbb{R}^+, H) \cap \text{BV}([0, T]; D (\mathbb{R}^+, H)) \) and
\[ \int_s^t \langle x_r - z, dk_r - d\zeta dr \rangle \geq 0, \text{ } \forall (z, \zeta) \in A, \forall 0 \leq s \leq t. \] (45)

Then the relation (45) is equivalent to: for all \( u, \hat{u} \in D (\mathbb{R}^+, H) \) such that \( (u_r, \hat{u}_r) \in A, \forall r \geq 0 \)
\[ \int_s^t \langle x_r - u_r, dk_r - \hat{u}_r dr \rangle \geq 0, \text{ } \forall 0 \leq s \leq t, \] (46)
and \( A \) is a monotone operator, that is, for all \( (x, k), (y, \ell) \in A \)
\[ \int_s^t \langle x_r - y_r, dk_r - d\ell_r \rangle \geq 0, \text{ } \forall 0 \leq s \leq t. \]

Moreover \( A \) is a maximal monotone operator.
Proof. (45)$\implies$(46): Let $\forall u, \hat{u} \in D(R^+, H)$ be such that $(u_r, \hat{u}_r) \in A, \forall r \geq 0$. Let $\pi_n$ the partition $0 < \frac{1}{n} < \frac{2}{n} < \cdots$ and
\[
[r]_n = \min \{j \in \mathbb{N} : r \leq j\}. \]
Then
\[
\int_{t}^{t} \langle x_r - u_r, dk_r - \hat{u}_rdr \rangle = \lim_{n \rightarrow \infty} \int_{t}^{t} \langle x_r - u_{[r]}_n, dk_r - \hat{u}_{[r]}dr \rangle \geq 0. \]

(46)$\implies$(45): The implication is obtained for $u_r = z$ and $\hat{u}_r = \hat{z}$. Let $(x, k), (y, \ell) \in A$ be arbitrary. Then for all $u, \hat{u} \in D(R^+, H)$ such that $(u_r, \hat{u}_r) \in A, \forall r \geq 0$ we have for all $0 \leq s \leq t$,
\[
\int_{s}^{t} \langle y_r - u_r, d\ell_r - \hat{u}_rdr \rangle \geq 0 \quad \text{and} \quad \int_{s}^{t} \langle x_r - u_r, dk_r - \hat{u}_rdr \rangle \geq 0. \]
We put here
\[
\langle u_r = J_\varepsilon \left(\frac{x_r + y_r}{2}\right) = \frac{x_r + y_r}{2} - \varepsilon A_\varepsilon \left(\frac{x_r + y_r}{2}\right) \rangle \quad \text{and} \quad \hat{u}_r = A_\varepsilon \left(\frac{x_r + y_r}{2}\right). \]
Since $A$ is a maximal operator on $H$, $D(A)$ is convex and $\lim_{\varepsilon \rightarrow 0} \varepsilon A_\varepsilon (u) \rightarrow 0, \forall u \in \overline{D(A)}$. Also for all $\varepsilon A_\varepsilon (u) = \varepsilon |A_\varepsilon (u) - A_\varepsilon (a)| + \varepsilon |A_\varepsilon (a)| \leq |u - a| + \varepsilon |A^0 (a)|$.

Adding member by member the inequalities we obtain:
\[
0 \leq \frac{1}{2} \int_{s}^{t} \langle y_r - x_r, d\ell_r - dk_r \rangle + \varepsilon \int_{s}^{t} \langle A_\varepsilon \left(\frac{x_r + y_r}{2}\right), d\ell_r + dk_r \rangle. \]
Passing to $\lim_{\varepsilon \rightarrow 0}$ we obtain $\int_{s}^{t} \langle y_r - x_r, d\ell_r - dk_r \rangle \geq 0$. $A$ is a maximal monotone operator since if $(y, \ell) \in D(R^+, H) \times [D(R^+, H) \cap BV_{loc}(R^+; H)]$ satisfies
\[
\int_{s}^{t} \langle y_r - x_r, d\ell_r - dk_r \rangle \geq 0, \quad \forall (x, k) \in A, \]
then this last inequality is satisfied for all $(x, k)$ of the form $(x_t, k_t) = (z, \hat{z})$, where $(z, \hat{z}) \in A$, and consequently (from the definition of $A$) $(y, \ell) \in A$. The proof is complete.

Definition 48 We write $dk_t \in A (x_t) (dt)$ if
\[
(a_1) \quad x \in D(R^+, H) \quad \text{and} \quad x_t \in \overline{D(A)} \quad \text{for all} \quad t \geq 0 \]
\[
(a_2) \quad k \in D(R^+, H) \cap BV_{loc}(R^+; H), \quad k_0 = 0, \]
\[
(a_3) \quad \langle x_t - u, dk_t - \hat{u}dt \rangle \geq 0, \quad \text{on} \quad R^+, \quad \forall (u, \hat{u}) \in A. \]

Proposition 49 Let $A \subset H \times H$ be a maximal subset and $A$ be the realization of $A$ on $D(R^+, H) \times [D(R^+, H) \cap BV_{loc}(R^+; H)]$ defined by (45). Assume that $\text{Int} (D(A)) \neq \emptyset$. Let $a \in \text{Int} (D(A))$ and $r_0 > 0$ be such that $B(a, r_0) = \{u \in H : |u - a| \leq r_0\} \subset D(A)$. Then
\[
A^\#_{a, r_0} := \sup \left\{ |\hat{u}| : \hat{u} \in Au, \ u \in B(a, r_0) \right\} < \infty. \]
and for all \((x, k) \in A:\)

\[
    r_0 d \mathbb{D}^r_s t \leq \langle x_t - a, dk_t \rangle + A_{a, r_0}^\# (|x_t - a| + r_0) dt \quad (47)
\]

as signed measure on \(\mathbb{R}^+\). Moreover for all \(0 \leq s \leq t, y \in D (\mathbb{R}^+, \mathbb{R})\) and \(0 < \varepsilon \leq 1: \)

\[
    r_0 \int_s^t |A_x y_r| dr \leq \int_s^t \langle y_r - a, A_x y_r \rangle dr + A_{a, r_0}^\# \int_s^t [|y_r - a| + |A^0 (a)| + r_0] dr \quad (48)
\]

**Proof.** Since \(A\) is local bounded on \(\text{Int} (D (A))\), for \(a \in \text{Int} (D (A))\), there exist \(r_0 > 0\) such that \(a + r_0 v \in \text{Int} (D (A))\) for all \(|v| \leq 1\) and

\[
    A_{a, r_0}^\# := \sup \{|\hat{z}| : \hat{z} \in A_z, z \in B (a, r_0)\} < \infty.
\]

Let \(0 \leq s = t_0 < t_1 < \ldots < t_n = t \leq T, \max_i (t_{i+1} - t_i) = \delta_n \rightarrow 0\). We put in (45) \(z = a + r_0 v\). Then

\[
    \int_{t_i}^{t_{i+1}} \langle x_r - (a + r_0 v), dk_r - \hat{z} dr \rangle \geq 0, \quad \forall |v| \leq 1, \forall 0 \leq s \leq t \leq T,
\]

and we get

\[
    r_0 \langle k_{t_{i+1}} - k_{t_i}, v \rangle \leq \int_{t_i}^{t_{i+1}} \langle x_r - a, dk_r \rangle + A_{a, r_0}^\# \int_{t_i}^{t_{i+1}} |x_r - a| dr + r_0 A_{a, r_0}^\# (t_{i+1} - t_i),
\]

for all \(|v| \leq 1\). Hence

\[
    r_0 |k_{t_{i+1}} - k_{t_i}| \leq \int_{t_i}^{t_{i+1}} \langle x_r - a, dk_r \rangle + A_{a, r_0}^\# \int_{t_i}^{t_{i+1}} |x_r - a| dr + r_0 A_{a, r_0}^\# (t_{i+1} - t_i)
\]

and adding member by member for \(i = 0\) to \(i = n - 1\) the inequality

\[
    r_0 \sum_{i=0}^{n-1} |k_{t_{i+1}} - k_{t_i}| \leq \int_s^t \langle x_r - a, dk_t \rangle + A_{a, r_0}^\# \int_s^t |x_r - a| dr + (t - s) r_0 A_{a, r_0}^\#
\]

holds and clearly (47) follows. Since \((J_{\varepsilon} y, \int_0^t A_{\varepsilon} y, dr) \in A, \) from (47) we easily obtain (48). \(\blacksquare\)

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