OBSERVABILITY FOR NON-AUTONOMOUS SYSTEMS

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Abstract. We study non-autonomous observation systems
\[ \dot{x}(t) = A(t)x(t), \quad y(t) = C(t)x(t), \quad x(0) = x_0 \in X, \]
where \((A(t))\) is a strongly measurable family of closed operators on a Banach space \(X\) and \((C(t))\) is a family of bounded observation operators from \(X\) to a Banach space \(Y\). Based on an abstract uncertainty principle and a dissipation estimate, we prove that the observation system satisfies a final-state observability estimate in \(L^r(E; Y)\) for measurable subsets \(E \subseteq [0, T], T > 0\). We present applications of the above result to families \((A(t))\) of uniformly strongly elliptic differential operators as well as non-autonomous Ornstein–Uhlenbeck operators \(P(t)\) on \(L^p(\mathbb{R}^d)\) with observation operators \(C(t)u = u|\Omega(t)\). In the setting of non-autonomous strongly elliptic operators, we derive necessary and sufficient geometric conditions on the family of sets \((\Omega(t))\) such that the corresponding observation system satisfies a final-state observability estimate.

1. Introduction

Let \(X\) and \(Y\) be Banach spaces, \(T > 0\), \((A(t))_{t \in [0,T]}\) a family of operators \(A(t)\): \(D(A(t)) \to X\) on \(X\), and \((C(t))_{t \in [0,T]}\) a family of bounded operators \(C(t): X \to Y\). In this article, we will be concerned with systems of the form
\[ \begin{align*}
\dot{x}(t) &= -A(t)x(t), \quad t \in (0, T], \quad x(0) = x_0, \\
y(t) &= C(t)x(t), \quad t \in [0, T],
\end{align*} \]
where the first equation in (1) describes the evolution of a state function \(x\) which is driven by the operators \(A(t)\) and the second equation describes the observation function \(y\) of the state function \(x\) through the operators \(C(t)\). In particular, we are interested in the following question: for a given measurable subset \(E \subseteq [0, T]\) and \(r \in [1, \infty]\), does there exist a constant \(C_{\text{obs}} \geq 0\) such that, for all initial values \(x_0 \in X\), the observability estimate
\[ \|x(T)\|_X \leq C_{\text{obs}} \left\{ \begin{array}{ll}
& \left( \int_E \|y(t)\|_Y^r \, dt \right)^{1/r}, \quad r \in [1, \infty), \\
& \text{ess sup}_{t \in E} \|y(t)\|_Y, \quad r = \infty,
\end{array} \right. \quad (2) \]
holds? In this case, we say that the system (1) satisfies a final-state observability estimate in \(L^r(E; Y)\). Loosely speaking, final-state observability allows one to retrieve information about the final-state \(x(T)\) by just observing the system through the measurements \(y(t)\) at times \(t \in E\).

\begin{itemize}
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\end{itemize}
In case the families \((A(t))_{t \in [0,T]}\) and \((C(t))_{t \in [0,T]}\) are constant, final-state observability for (1) has been studied thoroughly in the Hilbert space case both in abstract and in concrete situations. Autonomous self-adjoint Schrödinger operators \(A\) in \(L^2\) on bounded domains of \(\mathbb{R}^d\) and a projection \(C u = u|_{\Omega}\) for suitable subsets \(\Omega\) of the domain were considered in [LR95, FL96], as well as in [Mil04a, Mil04b, Gd07, Bar14] for the case of unbounded domains. Moreover, second order elliptic operators \(A\) in \(L^2\) for bounded domains of \(\mathbb{R}^d\) have been studied in [PW13], where also \(E\) in (2) is just a measurable subset of \([0,T]\). Of particular interest in understanding the case of unbounded domains is the specification of necessary and sufficient geometric conditions on \(\Omega\) for observability, which were established in [EV18, WWZZ19] in the case of Hilbert spaces. In the Banach space case, a characterization of observability in terms of geometric conditions was given in [GST20, BGST].

1.1. Abstract Observability and Applications. The main theorem of this article is an abstract observability theorem that combines and generalizes the results from [GST20, BGST] and from [PW13, WZ17]. The main idea is the extension of the Lebeau–Robbiano strategy for deriving observability to the setting of non-autonomous problems (1) and evolution families. Excluding technical details, we summarize the philosophical pillars of this approach as follows:

Given an evolution family generated by the non-autonomous operators \(A(t)\) and an abstract dissipation and uncertainty estimate for the observation operators \(C(t)\), there exists a constant \(C_{\text{obs}}\) such that the final-state observability estimate (2) holds.

All of the above will be made precise in Hypothesis 3.1 and Theorem 3.3. In particular, we will show that our theorem allows us to obtain observability estimates for non-autonomous systems in Banach spaces and to include observation on a measurable set \(E \subseteq [0,T]\) of time. Its proof rests on an interpolation estimate for evolution families presented in Theorem 3.5.

We give two applications for this abstract result: non-autonomous elliptic equations and non-autonomous Ornstein–Uhlenbeck equations. In Section 4, for the first application, we prove the following result, which is also prototypical for our second application:

Consider an observation system (1) consisting of a parabolic equation in \(\mathbb{R}^d\)

\[
\dot{u} = -A(t)u = -\sum_{|\alpha| \leq m} a_\alpha(t) \partial^\alpha u
\]

with time-dependent uniformly elliptic differential operators \(A(t)\) and observation operators \(C(t)\) that are given via the restriction \(u|_{\Omega(t)}\) of functions \(u\) on \(\mathbb{R}^d\) to a time-dependent family of observability sets \(\Omega(t) \subseteq \mathbb{R}^d\). Then, under suitable geometric assumptions on the family \((\Omega(t))_{t \in [0,T]}\), a final-state observability estimate (2) holds.

More precisely, we analyze the connection between final-state observability estimates and the geometry of the observability sets. First and as an extension of the results known for the autonomous setting, in Theorem 4.8, we show that a uniformly thick family of observability sets \(\Omega(t)\) guarantees the existence of a final-state observability
estimate. Then, in Theorem 4.10, we derive a converse implication to Theorem 4.8 which builds upon a weaker notion of thickness.

As our second application, in Section 5, we study non-autonomous Ornstein–Uhlenbeck equations. These equations are well studied in the Hilbert space setting, see, for example, the work [BEP20] and the references contained therein. They are parabolic equations associated with a family of second-order differential operators taking the form

\[ P(t) = \frac{1}{2} \text{tr}(A(t)A(t)^T \nabla_x^2) - \langle B(t)x, \nabla_x \rangle - \frac{1}{2} \text{tr}(B(t)), \]

where \( A, B \in C^\infty(0, T; \mathbb{R}^{d \times d}) \). Note that, in contrast to the elliptic operators considered in Section 4, we have some dependence on the space variable in the first order term. In general, these operators are not elliptic. Therefore, we need to assume a certain Kalman rank condition, which is equivalent to hypoellipticity in the autonomous case, i.e., when the matrices \( A, B \) are independent of \( t \). The main result of this section is Theorem 5.2 which states that for uniformly thick observability sets \( \Omega(t) \) and small final times \( T \), a final-state observability estimate in \( L^p \) holds for the evolution family associated with these Ornstein–Uhlenbeck operators if we assume a Kalman rank condition. In the case of small times, this generalizes an approximate null-controllability estimate from [BEP20].

1.2. Null-controllability. The above question on final-state observability is closely related to a variant of null-controllability properties of the predual system: Given a final time \( T > 0 \), we consider the system

\[ \dot{x}(t) = -A(t)x(t) + B(t)u(t), \quad t \in (0, T], \quad x(0) = x_0, \tag{3} \]

where \( (B(t))_{t \in [0,T]} \) is a family of bounded linear operators \( B(t): U \to X \) for some Banach space \( U \). Problems like (3) for example appear in the field of controllability of parabolic differential equations, where \( (A(t))_{t \in [0,T]} \) is a family of differential operators and \( B(t) = 1_{\Omega(t)} \) is a multiplication operator on \( L^p(\mathbb{R}^d) \) which represents moving control subsets \( (\Omega(t))_{t \in [0,T]} \), cf. [MRR13, CRZ14, LLTT17, BEP20].

Let us introduce two properties which may hold or not hold.

(a) For all measurable subsets \( E \subseteq [0, T] \), there exists \( C > 0 \) such that, for all \( \varepsilon > 0 \) and all \( x_0 \in X \), there exists \( u \in L^r(0, T; U) \) with \( \text{supp} \ u \subseteq E \), \( \|u\|_{L^r(0,T;U)} \leq C\|x_0\|_X \), and \( \|x(T)\|_X < \varepsilon \).

(b) For all measurable subsets \( E \subseteq [0, T] \) and all \( \varepsilon > 0 \), there exists \( C_\varepsilon > 0 \), such that, for all \( x_0 \in X \), there exists \( u \in L^r(0, T; U) \) with \( \text{supp} \ u \subseteq E \), \( \|u\|_{L^r(0,T;U)} \leq C_\varepsilon\|x_0\|_X \), and \( \|x(T)\|_X < \varepsilon \).

Note that the constant \( C \) in (a) is not allowed to depend on \( \varepsilon \), while the constant \( C_\varepsilon \) in (b) may well depend on \( \varepsilon \) and even tend to infinity as \( \varepsilon \) tends to zero. Obviously, (a) \( \Rightarrow \) (b). Both properties are variants of approximate null-controllability. As a consequence of Douglas’ lemma [Dou66] for Hilbert spaces and corresponding results for Banach spaces [DR77, Car88, Vie05, YLC06], the system (3) satisfies property (a) if and only if the dual system, which is of the form (1) with \( A(t) \) replaced by the dual operators \( A(t)' \) and \( C(t) = B(t)' \) for all \( t \in [0,T] \), satisfies a final-state observability estimate in \( L^{r/(r-1)}(0,T;U)' \). Note that, in certain situations, property (a) is equivalent to null-controllability, that is, there exists \( u \in L^r(0, T; U) \) with \( \text{supp} \ u \subseteq E \) such that \( x(T) = 0 \), cf. [Car88, Remark 2.1]. Thus (a) is a natural generalization of
null-controllability from Hilbert spaces to Banach spaces. Likewise, property (b) is equivalent to a so-called weak observability estimate for the dual system. This is spelled out in [Mil10, TWX20, AM] in the setting of Hilbert spaces, and in [EGST] for its generalization to Banach spaces.

1.3. Outline. Let us outline the content of the article. After an introduction to the framework of evolution families for non-autonomous Cauchy problems in Section 2, we will first derive sufficient conditions for observability of abstract non-autonomous systems in Section 3. Building on this abstract result, we will consider the concrete application of non-autonomous elliptic differential operators and relate geometric conditions of the sets of observations to final-state observability in Section 4. Furthermore, we prove an \( L^p \)-observability estimate for evolution families associated with non-autonomous Ornstein–Uhlenbeck equations in Section 5. We collect some properties of the non-autonomous elliptic differential operators as well as the corresponding evolution families in the appendix.

2. Linear Non-autonomous Cauchy Problems and Evolution Families

For Banach spaces \( X \) and \( Y \), let \( \mathcal{L}(X, Y) \) denote the set of bounded linear operators from \( X \) to \( Y \). Similarly, set \( \mathcal{L}(X) := \mathcal{L}(X, X) \). Let \( T > 0 \), and let \( A(t) : D(A(t)) \to X \), \( 0 \leq t \leq T \), be a family of operators in \( X \). Furthermore, let us always assume that the operators \( A(t) \) are closed with common domain \( D(A(t)) = D \) for all \( t \in [0, T] \), where \( D \) densely embeds into \( X \), and that the mapping \( A : [0, T] \to \mathcal{L}(D, X) \) is strongly measurable.

Consider the homogeneous initial value problem
\[
\dot{x}(t) = -A(t)x(t), \quad t \in (0, T], \quad x(0) = x_0,
\]
where \( x_0 \in X \). We will call (NACP) the non-autonomous Cauchy problem for \( A \). Throughout this article, we make use of the following concept of solutions for (NACP); see [Paz83, Chapter 4, Definition 2.8].

**Definition 2.1** (Strong Solution). A function \( x : [0, T] \to X \) is said to be a strong solution of (NACP) if \( x \in W^{1,1}(0, T; X) \cap L^1(0, T; D) \), \( x(0) = x_0 \), and \( \dot{x}(t) = -A(t)x(t) \) for almost all \( t \in (0, T) \).

The following definition provides a natural generalization of operator semigroups to the context of non-autonomous Cauchy problems.

**Definition 2.2** (Evolution Family). Let \( T > 0 \). A two-parameter family of bounded linear operators \( (U(t, s))_{0 \leq s \leq t \leq T} \) on \( X \) is called an evolution family if
(a) \( U(s, s) = \text{Id} \) and \( U(t, s)U(s, r) = U(t, r) \) for \( 0 \leq r \leq s \leq t \leq T \).
If, furthermore,
(b) \( (t, s) \to U(t, s) \) is strongly continuous for \( 0 \leq s \leq t \leq T \),
then we say that the evolution family \( (U(t, s)) \) is strongly continuous.

An evolution family is called exponentially bounded if there exist \( M \geq 1 \) and \( \omega \in \mathbb{R} \) such that \( \|U(t, s)\|_{\mathcal{L}(X)} \leq Me^{\omega(t-s)} \) for all \( 0 \leq s \leq t \leq T \).

An evolution family \( (U(t, s))_{0 \leq s \leq t \leq T} \) is called a (strongly continuous) evolution family for \( A \) if, in addition to conditions (a) (and (b)), the following conditions are satisfied:
(c) For all \( 0 \leq s < T \) and \( x_s \in D \), the function \( x: [s, T] \to X \) defined by \( x(t) = U(t, s)x_s \) is in \( W^{1,1}(s, T; X) \cap L^1(s, T; D) \) and satisfies \( \dot{x}(t) = -A(t)U(t, s)x_s \) for almost all \( t \in (s, T) \).

(d) For all \( 0 < t < T \) and \( x_T \in D \), the function \( x: [t, T] \to X \) defined by \( x(s) = U(T, s)x_T \) is in \( W^{1,1}(t, T; X) \cap L^1(t, T; D) \) and satisfies \( \dot{x}(s) = U(T, s)A(t)x_T \) for almost all \( s \in (t, T) \).

Remark 2.3. (a) Since the early works by Sobolevski˘ı [Sob61] and Tanabe [Tan60], non-autonomous Cauchy problems (NACP) and evolution families have been extensively studied by various authors. For details, we refer to [Tan79, Paz83, AT87, Yag91, Lun95, Nic97, EN00] and the references therein. Most of the aforementioned resources share the algebraic condition (a) in their definition of the evolution family \((U(t, s))\) but rely on other regularity assumptions (b), (c), and (d) and also assume other regularity properties of the operator family \((A(t))\). In the literature, an evolution family is also referred to as an evolution system, evolution operator, evolution process, propagator, or fundamental solution.

(b) The condition in Definition 2.2 (c) states that \( u(t) := U(t, 0)u_0 \) defines a strong solution of (NACP) on \([0, T]\) in the sense of Definition 2.1.

(c) Every strongly continuous evolution family is exponentially bounded.

The following proposition states that the existence of an evolution family for \( A \) already guarantees uniqueness of strong solutions for (NACP).

Proposition 2.4 ([Gal17, Proposition 3.3.4], [GV17, Proposition 4.5]). Let \( T > 0 \) and \((U(t, s))_{0 \leq s \leq t \leq T}\) be an evolution family for \( A \).

(a) If (NACP) has a strong solution \( u \in W^{1,1}(0, T; X) \cap L^1(0, T; D) \), then it satisfies

\[
u(t) = U(t, s)u(s) \quad \text{for} \quad 0 \leq s \leq t \leq T.
\]

In particular, strong solutions are unique.

(b) If \((\tilde{U}(t, s))_{0 \leq s \leq t \leq T}\) is a further evolution family for \( A \), then \( \tilde{U} = U \).

3. Observability for Evolution Families on Measurable Sets in Time

In this section, we prove an abstract observability estimate for evolution families on Banach spaces formulated in Theorem 3.3. Before we state that theorem, we introduce our main hypothesis.

Hypothesis 3.1. Let \( X \) and \( Y \) be Banach spaces, \( T > 0 \), and \((U(t, s))_{0 \leq s \leq t \leq T}\) an exponentially bounded evolution family on \( X \). Let \( C: [0, T] \to \mathcal{L}(X, Y) \) be bounded, and assume that \([0, T] \ni t \mapsto \|C(t)U(t, 0)x_0\|_Y \) is measurable for all \( x_0 \in X \). Let \((P_\lambda)_{\lambda > 0}\) in \( \mathcal{L}(X) \) be bounded. Assume that there exist \( d_0, d_1, \gamma_1 > 0 \) such that

\[
\forall \lambda > 0 \ \forall t \in [0, T] \ \forall x \in X: \quad \|P_\lambda x\|_X \leq d_0 e^{d_1 \lambda t} \|C(t)P_\lambda x\|_Y \quad (4)
\]

and \( d_2 \geq 1 \) and \( d_3, \gamma_2, \gamma_3 > 0 \) with \( \gamma_1 < \gamma_2 \) such that

\[
\forall \lambda > 0 \ \forall 0 \leq s \leq t \leq T \ \forall x_s \in X: \quad \| (\text{Id} - P_\lambda)U(t, s)x_s \|_X \leq d_2 e^{-d_3 \lambda^2 (t-s)^\gamma_3} \|x_s\|_X. \quad (5)
\]

Remark 3.2. The estimate in (4) is an abstract uncertainty principle, while (5) is a dissipation estimate.
Theorem 3.3. Assume Hypothesis 3.1 and let \( E \subseteq [0, T] \) be measurable with positive Lebesgue measure. Then there exists \( C_{\text{obs}} \geq 0 \) such that, for all \( x_0 \in X \) and \( r \in [1, \infty] \), we have
\[
\| U(T, 0)x_0 \|_X \leq C_{\text{obs}} \left( \int_E \| C(t)U(t, 0)x_0 \|_Y^r \, dt \right)^{1/r}, \quad r \in [1, \infty),
\]
and analogously for the observations
\[
G(t) := \| C(t)U(t, 0)x_0 \|_Y, \quad G_\lambda(t) := \| C(t)P_\lambda U(t, 0)x_0 \|_Y, \quad G_\lambda^+(t) := \| C(t)(I - P_\lambda)U(t, 0)x_0 \|_Y.
\]

If \( E = [0, T] \), we give an explicit estimate on \( C_{\text{obs}} \) in Remark 3.7. For the proof of Theorem 3.3, it is convenient to introduce the following shorthand notation: for all \( x_0 \in X \), \( 0 \leq t \leq T \), and \( \lambda > 0 \), we define
\[
F(t) := \| U(t, 0)x_0 \|_X, \quad F_\lambda(t) := \| P_\lambda U(t, 0)x_0 \|_X, \quad F_\lambda^+(t) := \| (I - P_\lambda)U(t, 0)x_0 \|_X.
\]

Lemma 3.4. Let \( F_1, F_2, G, D, C \geq 0 \), \( \theta \in (0, 1) \), and assume that, for all \( \varepsilon \in (0, 1] \), we have
\[
F_2 \leq DF_1 \quad \text{and} \quad F_2 \leq C \left( \varepsilon^{-\frac{\theta}{1-\theta}}G + \varepsilon F_1 \right).
\]
Then we have
\[
F_2 \leq \max \left\{ \frac{C}{\theta^\theta(1-\theta)^{1-\theta}}, D \left( \frac{\theta}{1-\theta} \right)^{1-\theta} F_1 \right\} F_1^\theta G^{1-\theta}.
\]

Proof. If \( F_1 = 0 \) or \( G = 0 \), the statement is obvious. (Note that, in case \( G = 0 \), the second inequality in \( (6) \) yields \( F_2 = 0 \).) Therefore, let \( F_1, G > 0 \). Then the right-hand side of the second inequality in \( (6) \) is minimal for
\[
\varepsilon_0 := \left( \frac{\theta \theta(1-\theta)}{(1-\theta)F_1} \right)^{1-\theta} > 0.
\]
If \( \varepsilon_0 \leq 1 \), we have by assumption
\[
F_2 \leq C \left( \varepsilon_0^{-\frac{\theta}{1-\theta}}G + \varepsilon_0 F_1 \right) = \frac{C}{\theta^\theta(1-\theta)^{1-\theta}} F_1^\theta G^{1-\theta}.
\]
If \( \varepsilon_0 > 1 \), by the first inequality in \( (6) \) and the definition of \( \varepsilon_0 \), we observe
\[
F_2 \leq DF_1^\theta F_1^{-\theta} = DF_1^\theta \left( \frac{\theta \theta(1-\theta)}{(1-\theta)\varepsilon_0^{1/(1-\theta)}} \right)^{1-\theta} < D \left( \frac{\theta}{1-\theta} \right)^{1-\theta} F_1^\theta G^{1-\theta}. \quad \Box
\]

The following theorem is inspired by \cite[Theorem 1.2]{WZ17}, where a similar interpolation estimate is proven in the case of Hilbert spaces and one-parameter semigroups.

Theorem 3.5. Assume Hypothesis 3.1, and let \( \theta \in (0, 1) \). Then there exist \( \tilde{C}_1, \tilde{C}_2, \tilde{C}_3 \geq 0 \) such that, for all \( x_0 \in X \) and \( 0 \leq s < t \leq T \), we have that
\[
\| U(t, 0)x_0 \|_X \leq \tilde{C}_1 \exp \left( \frac{\tilde{C}_2}{(t-s)^{\frac{1-\theta}{2-\theta}}} + \tilde{C}_3(t-s) \right) \| C(t)U(t, 0)x_0 \|_Y^{1-\theta} \| U(s, 0)x_0 \|_X^\theta.
\]
Explicit estimates on $\tilde{C}_1$, $\tilde{C}_2$, and $\tilde{C}_3$ can be inferred from the proof and are summarized in Remark 3.7.

**Proof of Theorem 3.5.** Let $\theta \in (0, 1)$, $x_0 \in X$, and $0 < t \leq T$. Furthermore, let $\lambda > 0$. Using the uncertainty principle (4), we obtain

$$F(t) \leq F_\lambda(t) + F_\lambda^\perp(t) \leq d_0 e^{d_3 \lambda^2} G_\lambda(t) + F_\lambda^\perp(t).$$

Together with the estimate

$$G_\lambda(t) \leq G(t) + G_\lambda^\perp(t) \leq G(t) + \|C(t)\|_{\mathcal{L}(X,Y)} F_\lambda^\perp(t),$$

we get

$$F(t) \leq d_0 e^{d_3 \lambda^2} G(t) + (1 + d_0 \|C(t)\|_{\mathcal{L}(X,Y)}) e^{d_3 \lambda^2} F_\lambda^\perp(t).$$

(7)

Let $0 \leq s < t$. By the algebraic property (a) of evolution families in Definition 2.2 and the dissipation estimate (5), we obtain

$$F_\lambda^\perp(t) = \|(\text{Id} - P_\lambda)U(t,s)U(s,0)x_0\|_X \leq d_2 e^{d_3 (t-s)^2 \gamma^3} F(s).$$

With (7), we have

$$F(t) \leq d_0 e^{d_3 \lambda^2} G(t) + (1 + d_0 \|C(t)\|_{\mathcal{L}(X,Y)}) d_2 e^{d_3 \lambda^2 - d_3 (t-s)^2 \gamma^3} F(s)$$

$$\leq \tilde{c}_1 \left( e^{d_3 \lambda^2} G(t) + e^{d_3 \lambda^2 - d_3 (t-s)^2 \gamma^3} F(s) \right),$$

(8)

where $\tilde{c}_1 := \max\{d_0, (1 + d_0 \|C(\cdot)\|_{\infty})d_2\} > 1$.

Let now $f(\lambda) := d_1 \lambda^2 - \theta d_3 \lambda^2 (t - s)^2 \gamma^3$. Then $f$ attains its maximum at the point

$$\lambda^* := \left( \frac{d_1 \gamma^3_1}{\theta d_3 \gamma_2} \right)^{\frac{1}{2-\gamma_1}} \left( \frac{1}{t-s} \right)^{\frac{2}{2-\gamma_1}}.$$

Thus,

$$f(\lambda) \leq f(\lambda^*) = \left( \frac{d_1 \gamma_1}{\theta d_3 \gamma_2} \right)^{\frac{\gamma_1}{2-\gamma_1}} d_1 \left( 1 - \frac{\gamma_1}{\gamma_2} \right) \left( \frac{1}{t-s} \right)^{\frac{\gamma_1 \gamma_3}{2-\gamma_1}}.$$

This estimate and (8) imply that

$$F(t) \leq \tilde{c}_1 \exp \left( \tilde{c}_2 \left( \frac{1}{t-s} \right)^{\frac{\gamma_1 \gamma_3}{2-\gamma_1}} \right) \left( e^{\theta d_3 \lambda^2 (t-s)^2 \gamma^3} G(t) + e^{-(1-\theta) d_3 \lambda^2 (t-s)^2 \gamma^3} F(s) \right),$$

where

$$\tilde{c}_2 := \left( \frac{d_1 \gamma_1}{\theta d_3 \gamma_2} \right)^{\frac{\gamma_1}{2-\gamma_1}} d_1 \left( 1 - \frac{\gamma_1}{\gamma_2} \right).$$

As $\lambda > 0$ was arbitrary, we conclude that, for all $\varepsilon \in (0, 1]$, we have

$$F(t) \leq \tilde{c}_1 \exp \left( \tilde{c}_2 \left( \frac{1}{t-s} \right)^{\frac{\gamma_1 \gamma_3}{2-\gamma_1}} \right) \left( e^{-(1-\theta) d_3 \lambda^2 (t-s)^2 \gamma^3} G(t) + \varepsilon F(s) \right).$$

(9)

Since $(U(t,s))_{0 \leq s \leq t \leq T}$ is exponentially bounded, there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $F(t) \leq Me^{\omega(t-s)} F(s)$ for all $0 \leq s \leq t \leq T$. Since $\tilde{c}_1 \geq 1$, $M \geq 1$, and
\((\theta^\theta(1-\theta)^{1-\theta})^{-1} \geq (\theta/(1-\theta))^{1-\theta}\) for all \(\theta \in (0,1)\), we have

\[
\max \left\{ \frac{1}{\theta^\theta(1-\theta)^{1-\theta}} \tilde{c}_1 \exp \left( \tilde{c}_2 \left( \frac{1}{t-s} \right)^{\frac{\gamma_1}{\gamma_2}} \right), \left( \frac{\theta}{1-\theta} \right)^{1-\theta} M \omega(t-s) \right\} \leq \frac{1}{\theta^\theta(1-\theta)^{1-\theta}} M \tilde{c}_1 \exp \left( \tilde{c}_2 \left( \frac{1}{t-s} \right)^{\frac{\gamma_1}{\gamma_2}} + \omega_+(t-s) \right),
\]

where \(\omega_+ := \max\{\omega,0\}\). Thus, setting \(\tilde{C}_1 := (\theta^\theta(1-\theta)^{1-\theta})^{-1} M \tilde{c}_1\), \(\tilde{C}_2 := \tilde{c}_2\), and \(\tilde{C}_3 := \omega_+\), the statement of the theorem follows from (9) and Lemma 3.4. □

**Remark 3.6.** For autonomous systems (i.e. \(A(\cdot)\) and \(C(\cdot)\) are time-independent) on Hilbert spaces with self-adjoint generator \(A\) with purely discrete spectrum, the interpolation estimate in Theorem 3.5 is equivalent to the uncertainty principle (4) with spectral projectors \(P_k\), see [PWX17, Theorem 2.1]. Note that, in this case, (5) is trivial. As was pointed out in [PWX17, Remark 2.2], the fact that the system is autonomous is crucial to obtain the equivalence.

**Proof of Theorem 3.3.** By Hölder’s inequality, it suffices to prove the case \(r = 1\). Let \(x_0 \in X\). Since \((U(t,s))_{0 \leq s \leq t \leq T}\) is exponentially bounded, there exist \(M \geq 1\) and \(\omega \in \mathbb{R}\) such that \(F(\tau) \leq M e^{\omega(\tau-t)} F(t) \leq M e^{\omega_+(\tau-t)} F(t)\) for all \(0 \leq t \leq \tau \leq T\), where \(\omega_+ := \max\{\omega,0\}\). Let \(\theta \in (0,1)\). For all \(0 \leq s < t < \tau \leq T\), we apply Theorem 3.5 and obtain

\[
F(\tau) \leq M \tilde{C}_1 \exp \left( \frac{\tilde{C}_2}{\gamma_2} \frac{\tau}{\gamma_2} + \tilde{C}_3(\tau-s) \right) G(t)^{1-\theta} F(s)^\theta. \tag{10}
\]

For the moment, let us fix \(q \in (0,1)\), which will be specified below. Let \(\ell\) be a Lebesgue point of \(E\). By [PW13, Proposition 2.1], there exists a sequence \((\ell_m)_{m \in \mathbb{N}}\) in \([0,T]\) with \(\ell_m \to \ell\) such that, for all \(m \in \mathbb{N}\), we have

\[
\ell_m > \ell_{m+1}, \quad \ell_{m+1} - \ell_{m+2} = q (\ell_m - \ell_{m+1}), \quad \text{and} \quad |E \cap (\ell_{m+1}, \ell_m)| \geq \frac{\ell_m - \ell_{m+1}}{3}.
\]

We set \(\xi_m := \ell_m + (\ell_m - \ell_{m+1})/6, s = \ell_{m+1}\), and \(\tau = \ell_m\). Then, for \(t \in (\xi_m, \ell_m)\), we have \(t-s \geq \xi_m - \ell_{m+1} = (\ell_m - \ell_{m+1})/6\). Applying (10), we obtain for all \(t \in (\xi_m, \ell_m)\) that

\[
F(\ell_m) \leq \tilde{c}_1 \exp \left( \frac{\tilde{c}_2}{\delta_m^{\gamma_2}} + \tilde{C}_3 \delta_m \right) G(t)^{1-\theta} F(\ell_{m+1})^\theta,
\]

where \(\tilde{c}_1 := M \tilde{C}_1\), \(\tilde{c}_2 := 6^{\gamma_1/\gamma_2} (\ell_{m+1} - \gamma_1) \tilde{C}_2\), and \(\delta_m := \ell_m - \ell_{m+1}\). Let \(\varepsilon > 0\). Then Young’s inequality \(ab \leq \varepsilon a^{1/\theta} + \varepsilon^{-\theta/(1-\theta)}(1-\theta)\theta^\theta/(1-\theta) b^{1/(1-\theta)}\) with \(a = F(\ell_{m+1})^\theta\) yields

\[
F(\ell_m) \leq \varepsilon F(\ell_{m+1}) + \varepsilon^{-\theta/(1-\theta)}(1-\theta)\theta^\theta \tilde{c}_1^{1/\theta} \exp \left( \frac{\tilde{c}_2}{\delta_m^{\gamma_2}} + \tilde{C}_3 \right) G(t).
\]
Taking integral means (with respect to $t$) on $E \cap [\xi_m, \ell_m]$ and using that, by construction, we have $|E \cap [\xi_m, \ell_m]| \geq \delta m/6$, we obtain

$$F(\ell_m) \leq \varepsilon F(\ell_{m+1})$$

$$+ \varepsilon^{-\frac{\theta}{1-\theta}} (1-\theta) \theta^{\frac{\theta}{1-\theta}} \tilde{c}_1^{\frac{1}{\theta}} \exp \left( -\frac{\tilde{c}_2}{(1-\theta)\delta_m^{\gamma_2}} + \frac{\tilde{C}_3}{1-\theta} \right) \frac{6}{\delta_m} \int_{\ell_{m+1}}^\ell 1_E(t) G(t) \, dt$$

and therefore

$$\varepsilon^{\frac{\theta}{1-\theta}} \delta_m \exp \left( -\frac{\tilde{c}_2}{(1-\theta)\delta_m^{\gamma_2}} \right) F(\ell_m) - \varepsilon^{\frac{1}{1-\theta}} \delta_m \exp \left( -\frac{\tilde{c}_2}{(1-\theta)\delta_m^{\gamma_2}} \right) F(\ell_{m+1})$$

$$\leq (1-\theta) \theta^{\frac{\theta}{1-\theta}} \tilde{c}_1^{\frac{1}{\theta}} \exp \left( \frac{\tilde{C}_3}{1-\theta} \right) \int_{\ell_{m+1}}^\ell 1_E(t) G(t) \, dt.$$ 

Setting $\varepsilon := q \exp\left( -(1-\theta)/\delta_m^{\gamma_2/(\gamma_2-\gamma_1)} \right)$ yields

$$\delta_m \exp \left( -\frac{\tilde{c}_2}{(1-\theta)\delta_m^{\gamma_2}} \right) F(\ell_m) - q \delta_m \exp \left( -\frac{\tilde{c}_2}{(1-\theta)\delta_m^{\gamma_2}} \right) F(\ell_{m+1})$$

$$\leq q^{-\frac{\theta}{1-\theta}} (1-\theta) \theta^{\frac{\theta}{1-\theta}} \tilde{c}_1^{\frac{1}{\theta}} \exp \left( \frac{\tilde{C}_3}{1-\theta} \right) \int_{\ell_{m+1}}^\ell 1_E(t) G(t) \, dt.$$ 

Now we set $q := \left( \frac{\tilde{c}_2 + \theta}{\tilde{c}_2 + 1} \right)^{\frac{\gamma_2-\gamma_1}{\gamma_2}}$. With this choice, we have

$$\frac{\tilde{c}_2 + \theta}{\gamma_2} = \frac{\tilde{c}_2 + 1}{\gamma_2} = \frac{\tilde{c}_2 + \theta}{\gamma_2} = \frac{\tilde{c}_2 + 1}{\gamma_2} \frac{\gamma_2}{\gamma_2-\gamma_1} \frac{\gamma_2}{\gamma_2-\gamma_1}$$

which leads us to the estimate

$$\delta_m \exp \left( -\frac{\tilde{c}_2 + \theta}{\gamma_2} \right) F(\ell_m) - \delta_m \exp \left( -\frac{\tilde{c}_2 + \theta}{\gamma_2} \right) F(\ell_{m+1})$$

$$\leq q^{-\frac{\theta}{1-\theta}} (1-\theta) \theta^{\frac{\theta}{1-\theta}} \tilde{c}_1^{\frac{1}{\theta}} \exp \left( \frac{\tilde{C}_3}{1-\theta} \right) \int_{\ell_{m+1}}^\ell 1_E(t) G(t) \, dt.$$ 

Taking the sum over all $m \in \mathbb{N}$, a telescoping sum argument yields

$$\delta_1 \exp \left( -\frac{\tilde{c}_2 + \theta}{\gamma_2} \right) F(\ell_1)$$

$$\leq q^{-\frac{\theta}{1-\theta}} (1-\theta) \theta^{\frac{\theta}{1-\theta}} \tilde{c}_1^{\frac{1}{\theta}} \exp \left( \frac{\tilde{C}_3}{1-\theta} \right) \int_{\ell}^\ell 1_E(t) G(t) \, dt.$$
Hence,

\[
F(\ell_1) \leq q^{-\frac{\alpha}{1-\theta}} \frac{\exp \left( \frac{-\frac{\ell_1}{\ell_1 - \ell_2}}{\theta} \right)}{\ell_1 - \ell_2} \int_E G(t) \, dt.
\]

Now, \( F(T) \leq M e^{\omega (T - \ell_1)} F(\ell_1) \) yields the assertion. \( \square \)

**Remark 3.7.** In this remark, we give explicit estimates on the constants appearing in Theorems 3.5 and 3.3. The proof of Theorem 3.5 shows

\[
\tilde{C}_1 = \frac{1}{\theta^\theta (1 - \theta)^{1 - \theta}} M \max \left\{ d_0, (1 + d_0 \| C(\cdot) \|_\infty) d_2 \right\},
\]

\[
\tilde{C}_2 = \left( \frac{d_1 \gamma_1}{\theta d_3 \gamma_2} \right)^{\frac{\gamma_1}{\gamma_2}} d_1 \left( 1 - \frac{\gamma_1}{\gamma_2} \right),
\]

\[
\tilde{C}_3 = \omega_+.
\]

If \( E = [0, T] \), then the constant \( C_{\text{obs}} \) in Theorem 3.3 can be made explicit as well. Indeed, choosing \( \ell = 0 \) and \( \ell_{m+1} = q^m T \) for \( m \in \mathbb{N}_0 \), we see that \( \ell_1 = T \) and \( \ell_1 - \ell_2 = (1 - q) T \) and the proof of Theorem 3.3 yields the estimate

\[
C_{\text{obs}} \leq \frac{C_1}{T^{1/\theta}} \exp \left( \frac{C_2}{T^{1/\gamma_1}} + C_3 T \right),
\]

where \( T^{1/\infty} := 1 \) and

\[
C_1 = q^{-\frac{\alpha}{1-\theta}} (1 - \theta)^{\frac{\theta}{1-\theta}} M^{\frac{1}{1-\theta}} \tilde{C}_1^{\frac{1}{1-\theta}} 6^{\frac{1}{1-\theta}}, \quad C_2 = \frac{6^{\frac{1}{\gamma_2}} \tilde{C}_2 + \theta}{(1 - q)^{\frac{1}{\gamma_2}}},
\]

\[
C_3 = \frac{\omega_+}{1 - \theta}, \quad q = \left( \frac{6^{\frac{1}{\gamma_2}} \tilde{C}_2 + \theta}{6^{\frac{1}{\gamma_2}} \tilde{C}_2 + 1} \right)^{\frac{\gamma_2}{\gamma_1 \gamma_3}},
\]

with \( \theta \in (0, 1) \). In particular, Theorem 3.3 yields a non-autonomous version of the results in [GST20, Theorem 2.1] and [BGST, Theorem A.1].

4. **Observability for Non-autonomous Elliptic Operators**

In this section, we apply the preceding theory to the evolution family associated with a non-autonomous parabolic equation on the domain \( \mathbb{R}^d \).

4.1. **Non-autonomous Elliptic Operators.** As a preparation, let us introduce the operators and notions used throughout this section. Our aim is to define a family of non-autonomous differential operators and derive \( L^p \)-bounds for the associated evolution family. Let \( T > 0 \).

**Definition 4.1** (Non-autonomous Elliptic Polynomial). Let \( m \in \mathbb{N} \). For \( \alpha \in \mathbb{N}_0^d \) with \( |\alpha| \leq m \), let \( a_\alpha : [0, T] \to \mathbb{C} \). Then we call \( a : [0, T] \times \mathbb{R}^d \to \mathbb{C} \) given by

\[
a(t, \xi) := \sum_{|\alpha| \leq m} a_\alpha(t)(i\xi)^\alpha, \quad t \in [0, T], \, \xi \in \mathbb{R}^d,
\]

non-autonomous polynomial of degree \( m \). The principal symbol of \( a \) is given by

\[
a_m(t, \xi) := \sum_{|\alpha| = m} a_\alpha(t)(i\xi)^\alpha, \quad t \in [0, T], \, \xi \in \mathbb{R}^d.
\]
We call $a$ uniformly strongly elliptic (with respect to $t$) if there exists $c > 0$ such that, for all $t \in [0, T]$ and $\xi \in \mathbb{R}^d$, we have

$$\Re a_m(t, \xi) \geq c|\xi|^m. \quad (11)$$

Note that uniform strong ellipticity implies that $m$ is even such that we will always have $m \geq 2$ for the degree of the non-autonomous polynomial $a$.

We define the Fourier transformation $\mathcal{F}: \mathcal{S}^{'}(\mathbb{R}^d) \rightarrow \mathcal{S}^{'}(\mathbb{R}^d)$ on the Schwartz space by

$$(\mathcal{F}u)(\xi) := \int_{\mathbb{R}^d} e^{-ix\cdot\xi} u(x) \, dx, \quad \xi \in \mathbb{R}^d.$$ 

As usual, we extend $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ to automorphisms of the space of tempered distributions $\mathcal{S}^{'}(\mathbb{R}^d)$.

Using (uniformly strongly elliptic) non-autonomous polynomials as symbols of Fourier multipliers gives rise to a certain class of differential operators.

**Definition 4.2 (Elliptic Operator).** Let $a$ be a non-autonomous polynomial of degree $m \geq 2$. For $t \in [0, T]$, we define $A(t): \mathcal{S}^{'}(\mathbb{R}^d) \rightarrow \mathcal{S}^{'}(\mathbb{R}^d)$ by

$$A(t)u := \mathcal{F}^{-1}(a(t, \cdot)\mathcal{F}u) = \sum_{|\alpha| \leq m} a_{\alpha}(t) \partial^{\alpha} u.$$ 

We call the family $(A(t))_{t \in [0, T]}$ the operator family associated with $a$. If, furthermore, $a$ is uniformly strongly elliptic, we call $(A(t))_{t \in [0, T]}$ elliptic.

Let $a$ be uniformly strongly elliptic and $(A(t))_{t \in [0, T]}$ the associated family of elliptic operators. Note that $A(t)$ leaves $\mathcal{S}(\mathbb{R}^d)$ invariant for all $t \in [0, T]$. Moreover, for $p \in [1, \infty)$ and $t \in [0, T]$, the part $A_p(t)$ of $A(t)$ in $X := L^p(\mathbb{R}^d)$ is a closed and densely defined operator with $D(A_p(t)) = W^{1,p}(\mathbb{R}^d)$ for $p > 1$, while only $W^{1,m}(\mathbb{R}^d) \subseteq D(A_1(t))$, see [Haa06, Chapter 8]. Thus, let us denote $D^p := D(A_p(t))$ for $t \in [0, T]$, noting that $D(A_p(t))$ does not depend on $t$. Furthermore, $\mathcal{S}(\mathbb{R}^d)$ is dense in $D^p$ with respect to the graph norm of $A_p(t)$ for all $t \in [0, T]$. Moreover, in case $p = \infty$, for $t \in [0, T]$, we set $A_\infty(t) := \tilde{A}_1(t)'$, where $(\tilde{A}_1(t))_{t \in [0, T]}$ is the operator family on $L^1(\mathbb{R}^d)$ associated with the uniformly strongly elliptic polynomial $\tilde{a} := a(\cdot, -\cdot)$.

For $p \in [1, \infty)$, we can associate the non-autonomous Cauchy problem

$$\dot{u}(t) = -A_p(t)u(t), \quad t \in (0, T], \quad u(0) = u_0 \in L^p(\mathbb{R}^d)$$

to the operator family $(A_p(t))_{t \in [0, T]}$. Under certain conditions on the coefficients of $a$, we will define a (strongly continuous) evolution family for $(A_p(t))_{t \in [0, T]}$.

Let $a$ be a uniformly strongly elliptic polynomial of degree $m \geq 2$ with coefficients $a_{\alpha} \in L^1(0, T)$ for $|\alpha| \leq m$. Then, as a consequence of the ellipticity estimate (11), for $0 \leq s < t \leq T$, we have

$$e^{-\int_s^t a(\tau, \cdot) \, d\tau} \in \mathcal{S}(\mathbb{R}^d).$$

Thus, for $0 \leq s \leq t \leq T$, we define $U(t, s): \mathcal{S}^{'}(\mathbb{R}^d) \rightarrow \mathcal{S}^{'}(\mathbb{R}^d)$ by

$$U(s, s)u := u, \quad U(t, s)u := \mathcal{F}^{-1}\left(e^{-\int_s^t a(\tau, \cdot) \, d\tau} \mathcal{F}u\right), \quad t > s. \quad (12)$$
It is easy to see that, for $0 \leq s < t \leq T$, the operator $U(t, s)$ is given as a convolution operator with kernel $p_{t,s} \in S(\mathbb{R}^d)$ defined via

$$p_{t,s} := \mathcal{F}^{-1} e^{-\int_s^t a(\tau, \cdot) \, d\tau}.$$  

(13)

The next lemma collects several algebraic properties of $(U(t, s))_{0 \leq s \leq t \leq T}$.

**Lemma 4.3.** Let $(U(t, s))_{0 \leq s \leq t \leq T}$ be the operator family defined in (12).

(a) For $0 \leq r \leq s \leq t \leq T$, we have that

$$U(s, s) = \text{Id} \quad \text{and} \quad U(t, r) = U(t, s)U(s, r).$$

Moreover, $p_{t,r} = p_{t,s} * p_{s,r}$ for all $0 \leq r < s \leq t \leq T$.

(b) For $p \in [1, \infty]$ and $0 \leq s \leq t \leq T$, $U(t, s)$ leaves $L^p(\mathbb{R}^d)$ invariant.

**Proof.** The proof of (a) is straightforward from the definitions of the operator family $(U(t, s))_{0 \leq s \leq t \leq T}$ in (12) and of its kernel in (13). Statement (b) is a consequence of Young’s inequality, see, e.g., [Gra14, Theorem 1.2.10]. □

Let $(U(t, s))_{0 \leq s \leq t \leq T}$ be as in (12) and $p \in [1, \infty]$. For $0 \leq s \leq t \leq T$, we define $U_p(t, s) := U(t, s)|_{L^p(\mathbb{R}^d)}$. By Lemma 4.3, $U_p(t, s)$ is a bounded operator on $L^p(\mathbb{R}^d)$ with $\|U_p(t, s)\|_{\mathcal{L}(L^p(\mathbb{R}^d))} = \|p_{t,s}\|_{L^1(\mathbb{R}^d)}$ for $0 \leq s \leq t \leq T$. Thus, $(U_p(t, s))_{0 \leq s \leq t \leq T}$ is an evolution family on $L^p(\mathbb{R}^d)$ in the sense of Definition 2.2(a).

Under suitable assumptions on the coefficients $a_\alpha$, it is possible to show that the evolution family $(U_p(t, s))_{0 \leq s \leq t \leq T}$ is strongly continuous and exponentially bounded. In fact, the evolution family $(U_p(t, s))_{0 \leq s \leq t \leq T}$ can be seen as the solution operator to the non-autonomous Cauchy problem (NACP) for $(A_\rho(t))_{t \in [0, T]}$. The proof of these facts is postponed to Appendix A. The following theorem summarizes all of these properties.

**Theorem 4.4.** Let $a$ be a uniformly strongly elliptic polynomial of degree $m \geq 2$ with coefficients $a_\alpha \in L^\infty(0, T)$ for $|\alpha| \leq m$. Let $(U(t, s))_{0 \leq s \leq t \leq T}$ be defined as in (12).

(a) Let $p \in [1, \infty]$. Then $(U_p(t, s))_{0 \leq s \leq t \leq T}$ is an exponentially bounded evolution family.

(b) Let $p \in (1, \infty)$. Then $(U_p(t, s))_{0 \leq s \leq t \leq T}$ is the unique evolution family for the family of operators $(A_\rho(t))_{t \in [0, T]}$. 

**Proof.** By Lemma 4.3, $(U_p(t, s))_{0 \leq s \leq t \leq T}$ is an evolution family and Lemma A.2 yields the exponential bound.

Moreover, Proposition A.6 yields that $(U_p(t, s))_{0 \leq s \leq t \leq T}$ is an evolution family for $(A_\rho(t))_{t \in [0, T]}$ in case $p \in (1, \infty)$. Uniqueness follows from Proposition 2.4. □

4.2. **Observability.** In this subsection, we show an observability estimate for the evolution family $(U_p(s, t))_{0 \leq s \leq t \leq T}$ from Subsection 4.1. For this purpose, we introduce the notion of a thick subset $\Omega$ of $\mathbb{R}^d$. Loosely speaking, a thick subset is a set such that the portion of it in a hypercube is bounded away from zero no matter where the hypercube is located. In the following, given a measurable set $\Omega \subseteq \mathbb{R}^d$, let $|\Omega|$ denote its Lebesgue measure.

**Definition 4.5** (Thick Set). Let $L \in (0, \infty)^d$ and $\rho > 0$. 


(a) A set $\Omega \subseteq \mathbb{R}^d$ is called $(L, \rho)$-thick if $\Omega$ is measurable and, for all $x \in \mathbb{R}^d$, we have
\[
\left| \Omega \cap \left( \bigtimes_{i=1}^{d} (0, L_i) + x \right) \right| \geq \rho \prod_{i=1}^{d} L_i.
\]

(b) Let $T > 0$. A family $(\Omega(t))_{t \in [0,T]}$ of sets $\Omega(t) \subseteq \mathbb{R}^d$ is called mean $(L, \rho)$-thick on $[0,T]$ if $\Omega(t)$ is measurable for all $t \in [0,T]$, the mapping $[0,T] \times \mathbb{R}^d \ni (t, x) \mapsto 1_{\Omega(t)}(x)$ is measurable, and, for all $x \in \mathbb{R}^d$, we have
\[
\frac{1}{T} \int_0^T \left| \Omega(t) \cap \left( \bigtimes_{i=1}^{d} (0, L_i) + x \right) \right| \, dt \geq \rho \prod_{i=1}^{d} L_i.
\]

(c) Let $T > 0$. A family $(\Omega(t))_{t \in [0,T]}$ of sets $\Omega(t) \subseteq \mathbb{R}^d$ is called uniformly $(L, \rho)$-thick on $[0,T]$ if $\Omega(t)$ is $(L, \rho)$-thick for all $t \in [0,T]$ and the mapping $[0,T] \times \mathbb{R}^d \ni (t, x) \mapsto 1_{\Omega(t)}(x)$ is measurable.

We call $\Omega \subseteq \mathbb{R}^d$ thick if there exist $L \in (0, \infty)^d$ and $\rho > 0$ such that $\Omega$ is $(L, \rho)$-thick. Likewise, $(\Omega(t))_{t \in [0,T]}$ is called mean/uniformly thick if it is mean/uniformly $(L, \rho)$-thick on $[0,T]$ for some $L \in (0, \infty)^d$ and $\rho > 0$.

Note that equivalent notions of (mean/uniform) thickness are obtained by replacing the hypercubes $\bigtimes_{i=1}^{d} (0, L_i)$ with balls $B(0,R)$ with some radius $R > 0$.

**Example 4.6.** Let $\Omega_1 = [0, \infty)$, $\Omega_2 = (-\infty, 0]$, $T = 2$, and
\[
\Omega(t) := \begin{cases} 
\Omega_1, & t \in [0,1), \\
\Omega_2, & t \in [1,2].
\end{cases}
\]

Then $(\Omega(t))_{t \in [0,T]}$ is mean $(L,1/2)$-thick for all $L > 0$ but not uniformly thick.

**Lemma 4.7.** Let $a$ be a uniformly strongly elliptic polynomial of degree $m \geq 2$ with coefficients $a_\alpha \in L^\infty(0,T)$ for $|\alpha| \leq m$, and let $(U(t,s))_{0 \leq s \leq t \leq T}$ be defined as in (12). For each $t \in [0,T]$, let $\Omega(t) \subseteq \mathbb{R}^d$ be measurable, and assume that $[0,T] \times \mathbb{R}^d \ni (t, x) \mapsto 1_{\Omega(t)}(x)$ is measurable. Let $p \in [1,\infty)$ and $u_0 \in L^p(\mathbb{R}^d)$. Then $[0,T] \ni t \mapsto \|1_{\Omega(t)} U_p(t,0) u_0\|_{L^p(\mathbb{R}^d)}$ is measurable.

**Proof.** By Corollary A.5, $(U_p(t,s))_{0 \leq s \leq t \leq T}$ is strongly continuous for $p \in [1,\infty)$ and strongly continuous w.r.t. the weak*-topology for $p = \infty$. For $p \in [1,\infty)$, this implies directly the measurability of $[0,T] \ni t \mapsto \|1_{\Omega(t)} U_p(t,0) u_0\|_{L^p(\mathbb{R}^d)}$. For $p = \infty$, the measurability follows from the variational description of the $L^{\infty}$-norm via the canonical pairing with $L^1$-elements and the strong continuity of $(U_\infty(t,s))_{0 \leq s \leq t \leq T}$ w.r.t. the topology $\sigma(L^\infty(\mathbb{R}^d),L^1(\mathbb{R}^d))$.

Our first result shows that uniform thickness implies an observability estimate.

**Theorem 4.8.** Let $a$ be a uniformly strongly elliptic polynomial of degree $m \geq 2$ with coefficients $a_\alpha \in L^\infty(0,T)$ for $|\alpha| \leq m$. Let $(U(t,s))_{0 \leq s \leq t \leq T}$ be as in (12). Let $(\Omega(t))_{t \in [0,T]}$ be uniformly thick on $[0,T]$. Let $E \subseteq [0,T]$ be measurable with positive Lebesgue measure and $r \in [1,\infty]$. Then there exists $C_{\text{obs}} \geq 0$ such that, for all $p \in [1,\infty]$ and $u_0 \in L^p(\mathbb{R}^d)$, we have
\[
\|U_p(T,0) u_0\|_{L^p(\mathbb{R}^d)} \leq C_{\text{obs}} \left( \int_E \left( \|U_p(t,0) u_0\|_{L^p(\Omega(t))} \right) \, dt \right)^{1/r}, \quad r \in [1,\infty),
\]
\[
\text{ess sup}_t \|U_p(t,0) u_0\|_{L^p(\Omega(t))}, \quad r = \infty.
\]
Remark 4.9. In the situation of Theorem 4.8, if $E = [0, T]$, then we obtain

$$C_{\text{obs}} \leq \frac{C_1}{T^{1/r}} \exp\left(\frac{C_2}{T^{1/\gamma_1}} + C_3 T\right)$$

for some $C_1, C_2, C_3 \geq 0$, $\gamma_1 = \gamma_3 = 1$, and $\gamma_2 = m$; cf. Remark 3.7.

Proof of Theorem 4.8. This proof consists of two parts. In the first part, we will show a dissipativity estimate, and, in the second part, we will derive an abstract uncertainty estimate. As both estimates do not depend on the value of $p$, it follows from Theorem 3.3 that also the observability constant $C_{\text{obs}}$ can be chosen independently of $p$.

We start by introducing a family of smooth frequency cutoffs. To this end, let $\eta \in C_c^\infty([0, \infty))$ with $0 \leq \eta \leq 1$ such that $\eta(r) = 1$ for $r \in [0, 1/2]$ and $\eta(r) = 0$ for $r \geq 1$. For $\lambda > 0$, we define $\chi_\lambda : \mathbb{R}^d \to \mathbb{R}$ by $\chi_\lambda(\xi) := \eta(|\xi|/\lambda)$. Since $\chi_\lambda \in \mathcal{S}(\mathbb{R}^d)$ for all $\lambda > 0$, we have $\mathcal{F}^{-1} \chi_\lambda \in \mathcal{S}(\mathbb{R}^d)$.

We define $P_\lambda : \mathcal{L}^p(\mathbb{R}^d) \to \mathcal{L}^p(\mathbb{R}^d)$ by $P_\lambda := (\mathcal{F}^{-1} \chi_\lambda) * f$. Then, for all $\lambda > 0$, the operator $P_\lambda$ is a bounded linear operator, the family $(P_\lambda)_{\lambda > 0}$ is uniformly bounded by $\|\mathcal{F}^{-1} \chi_\lambda\|_{\mathcal{L}^1(\mathbb{R}^d)}$, and, for all $f \in \mathcal{S}(\mathbb{R}^d)$, we have $P_\lambda f \in \mathcal{S}(\mathbb{R}^d)$, $\mathcal{F} P_\lambda f = \chi_\lambda \mathcal{F} f \in \mathcal{S}(\mathbb{R}^d)$, and $\text{supp } \mathcal{F} P_\lambda f \subseteq \{ y \in \mathbb{R}^d : |y| \leq \lambda \} \subseteq [-\lambda, \lambda]^d$, see [GST20, Theorem 3.3] for details.

Since $a$ is uniformly strongly elliptic, there exists $c > 0$ such that, for all $t \in [0, T]$ and all $\xi \in \mathbb{R}^d$, we have $\text{Re } a_m(t, \xi) \geq c |\xi|^m$. We define the (autonomous) uniformly strongly elliptic polynomials $b, \tilde{b} : [0, T] \times \mathbb{R}^d \to \mathbb{C}$ by

$$b(t, \xi) := |\xi|^m, \quad \tilde{b}(t, \xi) := \frac{c}{2} |\xi|^m,$$

and set $\tilde{a} := a - \tilde{b}$. Note that $\tilde{a}$ is also uniformly strongly elliptic.

Let $(V(t, s))_{0 \leq s \leq t \leq T}$, $(\tilde{V}(t, s))_{0 \leq s \leq t \leq T}$, and $(\tilde{U}(t, s))_{0 \leq s \leq t \leq T}$ be as in (12) for $b, \tilde{b}$, and $\tilde{a}$, respectively. Note that $\tilde{V}(t, s) = V(\frac{\tau}{2}, \frac{\tau}{2}s)$ for all $0 \leq s \leq t \leq T$.

Let $p \in [1, \infty]$. For $f \in \mathcal{L}^p(\mathbb{R}^d)$ and $0 \leq s \leq t \leq T$, we have by definition

$$U_p(t, s)f = \mathcal{F}^{-1}\left(e^{-\int_0^t \tilde{b}(\tau, \cdot) \, d\tau} \mathcal{F} f\right) = \mathcal{F}^{-1}\left(e^{-\int_0^t \tilde{b}(\tau, \cdot) \, d\tau} \mathcal{F} \mathcal{F}^{-1}\left(e^{-\int_0^t \tilde{a}(\tau, \cdot) \, d\tau} \mathcal{F} f\right)\right) = \tilde{V}_p(t, s)\tilde{U}_p(t, s)f.$$ 

By [BGST, Proposition 3.2], we infer that there exists $K_{m,d} \geq 0$, depending only on $m$ and $d$, such that, for all $\lambda > 0$, all $f \in \mathcal{L}^p(\mathbb{R}^d)$, and $0 \leq s \leq t \leq T$, we have

$$\|(\text{Id} - P_\lambda) V_p(t, s)f\|_{\mathcal{L}^p(\mathbb{R}^d)} \leq K_{m,d} e^{-2^{m-3}(t-s)\lambda^m} \|f\|_{\mathcal{L}^p(\mathbb{R}^d)}.$$ 

Thus, we also conclude

$$\|(\text{Id} - P_\lambda) \tilde{V}_p(t, s)f\|_{\mathcal{L}^p(\mathbb{R}^d)} \leq K_{m,d} e^{-2^{m-3}c/2(t-s)\lambda^m} \|f\|_{\mathcal{L}^p(\mathbb{R}^d)}.$$ 

Moreover, by Theorem 4.4 there exist $\tilde{M} \geq 1$ and $\tilde{\omega} \in \mathbb{R}$, depending on $\tilde{a}$ and therefore on $a$, such that $\|\tilde{U}_p(t, s)f\|_{\mathcal{L}^p(\mathbb{R}^d)} \leq M e^{\tilde{\omega}(t-s)}$ for all $0 \leq s \leq t \leq T$. Note that we can choose $\tilde{\omega} = \omega$, where $\omega$ is an exponential growth rate for $(U_p(t, s))_{0 \leq s \leq t \leq T}$ (by choosing the same $c_0$ in Lemma A.1 and inspecting the proof of Lemma A.2). Thus,
for $\lambda > \lambda^* := (2^{m+5} \max\{\omega, 0\}/c)^{1/m}$, $f \in L^p(\mathbb{R}^d)$, and $0 \leq s \leq t \leq T$, we arrive at

$$
\| (\text{Id} - P_\lambda) U_p(t, s) f \|_{L^p(\mathbb{R}^d)} = \| (\text{Id} - P_\lambda) \tilde{V}_p(t, s) \tilde{U}_p(t, s) f \|_{L^p(\mathbb{R}^d)}
\leq K_{m,d} e^{-2^{-m-3}c/2(t-s)} \lambda^m \tilde{M} \omega(t-s) \| f \|_{L^p(\mathbb{R}^d)}
\leq K_{m,d} \tilde{M} e^{-(t-s)2^{-m-5}c\lambda^m} \| f \|_{L^p(\mathbb{R}^d)}.
$$

(14)

Let $L \in (0, \infty)^d$ and $\rho > 0$ such that $(\Omega(t))_{t \in [0,T]}$ is uniformly $(L, \rho)$-thick, $f \in L^p(\mathbb{R}^d)$, and $\lambda > 0$. Since supp $\mathcal{F}P_\lambda f \subseteq [-\lambda, \lambda]^d$, the Logvinenko–Sereda theorem [Kov01, Theorem 3] implies

$$
\| P_\lambda f \|_{L^p(\mathbb{R}^d)} \leq e^{-K d \ln(\rho/K^d)} e^{-2K [\ln(\rho/K^d) + \lambda^m] \| (P_\lambda f)_{|\Omega(t)} \|_{L^p(\Omega(t))}}
$$

(15)

for all $t \in [0, T]$, where $K \geq 0$ is a universal constant. By (14), (15), Theorem 4.4(a), and Lemma 4.7, we conclude that Hypothesis 3.1 is satisfied with $Y = L^p(\mathbb{R}^d)$ and $C(t)$ the restriction operator on $\Omega(t)$ for $t \in [0, T]$. Therefore, Theorem 3.3 yields the assertion.

The following theorem will show a partial converse of Theorem 4.8, namely that a final-state observability estimate implies that the family $(\Omega(t))_{t \in [0,T]}$ is mean thick. For the pure Laplacian on $L^2(\mathbb{R}^d)$ and time-independent set of observability, such a result has first been shown in [EV18, WWZZ19]. In the autonomous case, this has been generalized to strongly elliptic operators in $L^p(\mathbb{R}^d)$ in [GST20, Theorem 3.3]. We also refer to [BEP20, Theorem 5] where a similar result is shown for the non-autonomous Ornstein–Uhlenbeck equation.

**Theorem 4.10.** Let $a$ be a uniformly strongly elliptic polynomial of degree $m \geq 2$ with coefficients $a_\alpha \in L^\infty(0, T)$ for $|\alpha| \leq m$. Let $(\Omega(t))_{t \in [0,T]}$ be such that $\Omega(t) \subseteq \mathbb{R}^d$ is measurable for all $t \in [0, T]$ and $[0, T] \times \mathbb{R}^d \ni (t, x) \mapsto 1_{\Omega(t)}(x)$ is measurable. Let $(U(t,s))_{0 \leq s \leq t \leq T}$ be as in (12). Let $p, r \in [1, \infty)$, and assume there exists $C_{\text{obs}} \geq 0$ such that, for all $u_0 \in L^p(\mathbb{R}^d)$, we have

$$
\| U_p(T, 0) u_0 \|_{L^p(\mathbb{R}^d)} \leq C_{\text{obs}} \left( \int_0^T \| U_p(t, 0) u_0 \|_{L^r(\Omega(t))} \| f \|_{L^r(\Omega(t))} \ dt \right)^{1/r}.
$$

Then the family $(\Omega(t))_{t \in [0,T]}$ is mean thick.

**Proof.** Our proof is inspired by [EV18, WWZZ19, BEP20]. We will show the contrapositive: assume that the family $(\Omega(t))_{t \in [0,T]}$ is not mean thick. Then there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathbb{R}^d$ such that, for all $n \in \mathbb{N}$, we have

$$
\frac{1}{T} \int_0^T |\Omega(t) \cap B(x_n, n)|^{1/p} \ dt < \frac{1}{n}.
$$

(16)

Let $f \in S(\mathbb{R}^d)$, $\| f \|_{L^p(\mathbb{R}^d)} = 1$, and set $f_n := f(\cdot - x_n)$ for $n \in \mathbb{N}$. Let $t \in (0, T)$ and $n \in \mathbb{N}$. Then $U_p(t,0) f_n = p_{t,0} * f_n - p_{t,0} * f(\cdot - x_n)$. Moreover,

$$
\| (U_p(t,0) f_n)_{|\Omega(t)} \|_{L^p(\Omega(t))} = \| 1_{\Omega(t)} U_p(t,0) f_n \|_{L^p(\mathbb{R}^d)}
= \| 1_{\Omega(t)} p_{t,0} * f(\cdot - x_n) \|_{L^p(\mathbb{R}^d)} - \| 1_{\Omega(t) - x_n} p_{t,0} * f \|_{L^p(\mathbb{R}^d)}
+ \| 1_{\Omega(t) - x_n} (1 - 1_{B(0,n)}) p_{t,0} * f \|_{L^p(\mathbb{R}^d)}.
$$

(17)
We first estimate the first summand on the right-hand side of (17). As a consequence of Lemma A.2, there exists $C \geq 0$ such that $\|p_t,0\|_{L^p(\mathbb{R}^d)} \leq C$ for all $t \in (0,T)$, where $\frac{1}{p} + \frac{1}{q} = 1$. By Hölder’s and Young’s inequality, we estimate
\[
\|1_{(\Omega(t) - x_n)} \cap B(0,n) p_t,0 * f\|_{L^p(\mathbb{R}^d)} \leq \| (\Omega(t) - x_n) \cap B(0,n) \|_{L^q(\mathbb{R}^d)} \leq C \| \Omega(t) \cap B(x_n,n) \|.
\]

For the second summand on the right-hand side of (17), we have by Lemma A.2, Hölder’s inequality, and Fubini–Tonelli’s theorem
\[
\|1_{(\Omega(t) - x_n)} \|_{L^p(\mathbb{R}^d)} \leq (1 - 1_{B(0,n)}) p_t,0 * f\|_{L^p(\mathbb{R}^d)} \leq \| p_t,0 \|_{L^p(\mathbb{R}^d)}
\]
\[
\leq \int_{E(0,n)} (\int_{\mathbb{R}^d} C_1 e^{-C_2 |z|^{m/(m-1)}} |f(x-t^{1/m})| \, dz)^p \, dx
\]
\[
= C_1^p \eta^{p/2} \int_{E(0,n)} \left( \int_{\mathbb{R}^d} e^{-C_2 |z|^{m/(m-1)}} \, dz \right)^{p/2} \left( \int_{\mathbb{R}^d} e^{-C_2 |z|^{m/(m-1)}} \, dz \right)^{p/2} \, dx
\]
\[
\leq C_1^p \eta^{p/2} \int_{\mathbb{R}^d} e^{-C_2 |z|^{m/(m-1)}} \, dz \, dx.
\]

Let us focus on estimating the double integral over $\mathbb{R}^d \times \mathbb{C}(0,n)$ in the previous calculation by splitting it up. To this end, let $\varepsilon > 0$. Then there exist $n_0 \in \mathbb{N}$ and $R$ such that
\[
\int_{\mathbb{R}^d} e^{-C_2 |z|^{m/(m-1)}} \, dz \leq \varepsilon.
\]
Consequently, for $n \geq n_0 + T^{1/m} R$, we have $\mathbb{C}(0,n) - t^{1/m} \mathbb{B}(0,R) \subseteq \mathbb{C}(0,n)$ and
\[
\int_{\mathbb{R}^d} e^{-C_2 |z|^{m/(m-1)}} \, dx \, dz
\]
\[
= \int_{\mathbb{R}^d} e^{-C_2 |z|^{m/(m-1)}} \, dx \, dz
\]
\[
\leq \int_{\mathbb{R}^d} e^{-C_2 |z|^{m/(m-1)}} \, dz \, dx \leq \varepsilon \int_{\mathbb{R}^d} e^{-C_2 |z|^{m/(m-1)}} \, dz.
\]

Thus,
\[
\sup_{t \in [0,T]} \| 1_{(\Omega(t) - x_n)} (1 - 1_{B(0,n)}) p_t,0 * f \|_{L^p(\mathbb{R}^d)} \to 0
\]
as $n$ tends to $\infty$. By (16)–(18) we obtain
\[
\int_0^T \| (U_p(t,0) f_n) \|_{L^p(\Omega(t))} \, dt \to 0
\]
as $n$ tends to $\infty$. Since $\| U_p(T,0) f_n \|_{L^p(\mathbb{R}^d)} = \| p_{T,0} * f \|_{L^p(\mathbb{R}^d)} > 0$ for all $n \in \mathbb{N}$, an observability estimate does not hold. \(\square\)

Remark 4.11. (a) Combining Theorem 4.8 and Theorem 4.10, we observe that uniformly thick observability sets allow for a final-state observability estimate, while such an estimate only implies that the observation sets are mean thick. It is an
interesting question whether it is possible to close this gap, either by finding a suitable condition on the observation sets which is equivalent to a final-state observability estimate, or by proving that an observability estimate holds for mean thick sets. Even in the setting of Hilbert spaces and for autonomous problems, i.e. $p = r = 2$ and $A(\cdot)$ time-independent, an answer on this question is still open. However, for a certain class of non-autonomous diffusive evolution equations governed by the Ornstein–Uhlenbeck operator it has recently been proven in [AM] that the corresponding equation is cost-uniform approximate null-controllable, if and only if the family $(\Omega(t))_{t \in [0,T]}$ is mean thick. Here, cost-uniform approximate null-controllable is meant in the sense of property (b) in the introduction, and is thus a weaker property than the observability estimate in the above theorems. In fact, if $r = p = 2$, then cost-uniform approximate null-controllability is equivalent to a so-called weak observability estimate, cf. [AM, Corollary 7.2].

(b) If the family of sets $(\Omega(t))_{t \in [0,T]}$ does not depend on $t$, then uniform thickness is equivalent to mean thickness. In this case, one can prove the statement of Theorem 4.10 for $r = \infty$ as well.

5. Observability for Non-autonomous Ornstein–Uhlenbeck Equations

This section applies the results from Section 3 to evolution families associated with non-autonomous Ornstein–Uhlenbeck equations.

5.1. Non-autonomous Ornstein–Uhlenbeck Operators. Let $p, r \in [1, \infty]$ and $T > 0$. We consider non-autonomous Ornstein–Uhlenbeck equations of the form

$$\dot{u}(t) - P(t)u(t) = 0, \quad t \in (0, T], \quad u(0) = u_0 \in L^p(\mathbb{R}^d)$$

(19)

with the non-autonomous Ornstein–Uhlenbeck operator $P(t)$ given by

$$P(t) := \frac{1}{2} \text{tr}(A(t)A(t)^T \nabla_x^2) - \langle B(t)x, \nabla_x \rangle - \frac{1}{2} \text{tr}(B(t)), \quad t \in [0, T]$$

(20)

where $A, B \in C^\infty((0, T) ; \mathbb{R}^{d \times d})$.

In general, the first-order term of these operators has coefficients that are allowed to vary in space. Furthermore, they are not elliptic, so the theory of Section 4 does not apply. However, we will employ the following generalized Kalman rank condition considered in [BEP20] as a substitute for ellipticity. More precisely, for $k \in \mathbb{Z}_+$ and $t \in [0, T]$, define $\widetilde{A}_k(t)$ by induction via the identities

$$\widetilde{A}_0(t) := A(T - t), \quad \widetilde{A}_{k+1}(t) := \frac{d}{dt} \widetilde{A}_k(t) + B(T - t) \widetilde{A}_k(t).$$

We say that the generalized Kalman rank condition holds at time $T$ if

$$\text{span} \left\{ \widetilde{A}_k(T)x : x \in \mathbb{R}^d, k \in \mathbb{Z}_+ \right\} = \mathbb{R}^d.$$

It was shown in [BEP20, Section 6] that, if the generalized Kalman rank condition holds at time $T$, then the problem (19) admits a unique weak solution $u \in C(0, T; L^2(\mathbb{R}^d))$ and gives rise to an evolution family $(U_2(t, s))_{0 \leq s \leq t \leq T}$ on $L^2(\mathbb{R}^d)$ defined by

$$U_2(t, s)f := \mathcal{F}^{-1}\left( e^{-\frac{1}{2} \int_s^t \text{tr} B(\tau) \, d\tau - \frac{1}{2} \int_s^t |A(\tau)^T R(t, \tau) R(t, \tau)^T |^2 \, d\tau} \mathcal{F}(f)(R(t, s)\cdot) \right),$$

(21)
for all $f \in L^2(\mathbb{R}^d)$, where, $(R(t, s))_{0 \leq s \leq t \leq T}$ is the unique family of $d \times d$-matrices such that, for all $s, t \in [0, T]$, we have

$$
\partial_t R(t, s) = B(t)R(t, s), \quad R(s, s) = \text{Id}_{\mathbb{R}^d}.
$$

We recall that, for all $r, s, t \in [0, T]$, it holds that

$$
R(t, s)R(s, r) = R(t, r),
$$

see, e.g., [Cor07, Proposition 15]. For details on the above concept of weak solution, we refer the reader to [BEP20, Section 6] and [BP17, Appendix A.1].

In the following, we will show that the evolution family $(U_2(t, s))_{0 \leq s \leq t \leq T}$, when restricted to $L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, can be extended to an evolution family on $L^p(\mathbb{R}^d)$. As a first step, we rewrite the exponent in (21) through a suitable quadratic form $q_{t,s}$. Let $0 \leq s \leq t \leq T$ and recall that, in [BEP20, Proposition 15, Equation (80)], the authors proved that there exist constants $c, \tilde{c} > 0$ and $m_1 \in \mathbb{N}$ such that, for all $\xi \in \mathbb{R}^d$ and $0 \leq s \leq t \leq \tilde{c}$, it holds that

$$
\int_s^t \left| A(\tau)^T R(t, \tau)^T \xi \right|^2 d\tau \geq c(t - s)^{m_1} |\xi|^2.
$$

Defining the matrix

$$
Q_{t,s} := \int_s^t R(s, \tau)A(\tau)A(\tau)^T R(s, \tau)^T d\tau,
$$

we have

$$
\int_s^t \left| A(\tau)^T R(t, \tau)^T \xi \right|^2 d\tau = \int_s^t \langle R(t, \tau)A(\tau)A(\tau)^T R(t, \tau)^T \xi, \xi \rangle d\tau
= \langle Q_{t,s}R(t, s)^T \xi, R(t, s)^T \xi \rangle =: q_{t,s}(\xi),
$$

where the latest identity is due to the transposed version of (22)

$$
R(s, r)^T R(t, s)^T = R(t, r)^T.
$$

It follows from (23) that $q_{t,s}$ is a positive definite quadratic form for $0 \leq s \leq t \leq \tilde{c}$, and we may rewrite (21) via

$$
U_2(t, s)f := \mathcal{F}^{-1}\left( e^{\int_s^t \frac{1}{2} B(\tau) d\tau} e^{-\frac{q_{t,s}}{2}} (\mathcal{F} f)(R(t, s)^T \cdot) \right), \quad f \in L^2(\mathbb{R}^d).
$$

To prove the desired $L^p$-estimate for the operator $U_2(t, s)$, we employ the following lemma.

**Lemma 5.1.** Let $p \in [1, \infty]$ and $Q, \Lambda \in \mathbb{R}^{d \times d}$ such that $\det(\Lambda) \neq 0$ and the quadratic form

$$
q: \mathbb{R}^d \to \mathbb{R}, \quad \xi \mapsto \langle Q\Lambda^T \xi, \Lambda^T \xi \rangle
$$

is positive definite. Consider the operator $A_{q,\Lambda}: \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ given by

$$
A_{q,\Lambda} f := \mathcal{F}^{-1}\left( e^{-q/2} (\mathcal{F} f)(\Lambda^T \cdot) \right).
$$

Then, for all $p \in [1, \infty]$ and all $f \in \mathcal{S}(\mathbb{R}^d)$, it holds that

$$
\|A_{q,\Lambda} f\|_{L^p(\mathbb{R}^d)} \leq |\det(\Lambda)|^{-1/p'} \|f\|_{L^p(\mathbb{R}^d)},
$$

where $1/p + 1/p' = 1$. 
Proof. Clearly,
\[ A_{q,t} = \mathcal{F}^{-1} e^{-q/2} \mathcal{F} \mathcal{F}^{-1}(\mathcal{F} f)(\Lambda T) = \mathcal{F}^{-1} e^{-q/2} \mathcal{F} \frac{1}{|\det(\Lambda)|} f(\Lambda^{-1}) . \]
Since, by substitution for \( p < \infty \) and directly for \( p = \infty \),
\[ \| \frac{1}{|\det(\Lambda)|} f(\Lambda^{-1}) \|_{L^p(\mathbb{R}^d)} = |\det(\Lambda)|^{1/p-1} \| f \|_{L^p(\mathbb{R}^d)} , \]
we only need to prove that
\[ \| \mathcal{F}^{-1} e^{-q/2} \mathcal{F} f \|_{L^p(\mathbb{R}^d)} \leq \| f \|_{L^p(\mathbb{R}^d)} . \]
By Young’s inequality, it suffices to show that
\[ \| \mathcal{F}^{-1} e^{-q/2} \|_{L^1(\mathbb{R}^d)} \leq 1 . \]
Let \( M \) be the symmetric positive definite matrix such that \( q = \langle M \cdot, \cdot \rangle \) and set \( q' := \langle M^{-1} \cdot, \cdot \rangle \). It is well-known that
\[ \mathcal{F}^{-1} e^{-q/2} = \frac{e^{-\eta/2}}{(2\pi)^{d/2} \det(M)^{1/2}} ; \]
see, for example, [Hör90, Theorem 7.6.1]. Again, by substitution, it follows that
\[ \| \mathcal{F}^{-1} e^{-q/2} \|_{L^1(\mathbb{R}^d)} = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-|y|^2/2} dy = 1 , \]
which yields the assertion. \( \square \)

Note that, from Liouville’s formula
\[ \det(R(t,s)) = e^{\int_t^s \text{tr} B(\tau) \, d\tau} , \] (25)
cf. [Tes12, Lemma 3.11], it follows that \( R(t,s) \) is invertible and, thus, we may rewrite identity (24) using the operator \( A_{q,t,s} R(t,s) \) from Lemma 5.1 as
\[ U_2(t,s) f = e^{\frac{1}{2} \int_s^t \text{tr} B(\tau) \, d\tau} A_{q,t,s} R(t,s) f , \quad f \in S(\mathbb{R}^d) . \] (26)
Furthermore, formula (25) and Lemma 5.1 give the estimate
\[ \| U_2(t,s) f \|_{L^p(\mathbb{R}^d)} \leq e^{\left( \frac{1}{2} - \frac{1}{p} \right) \int_s^t \text{tr} B(\tau) \, d\tau} \| f \|_{L^p(\mathbb{R}^d)} . \]
Therefore, \( U_2(t,s) \) extends to a bounded operator \( U_p(t,s) : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d) \) with norm estimate
\[ \| U_p(t,s) \|_{L(L^p(\mathbb{R}^d))} \leq e^{\left( \frac{1}{2} - \frac{1}{p} \right) \int_s^t \text{tr} B(\tau) \, d\tau} . \] (27)
5.2. Observability. Using a dissipation estimate from [BEP20], we prove the following observability estimate for small final times $T$.

**Theorem 5.2.** Let $p \in (1, \infty)$ and $\tilde{T} > 0$ such that the generalized Kalman rank condition holds at time $\tilde{T}$. Then there exists a constant $\varepsilon \in (0, \tilde{T})$ such that, if $T \in [0, \varepsilon]$, $(\Omega(t))_{t \in [0,T]}$ uniformly thick on $[0,T]$, $E \subseteq [0,T]$ measurable with positive Lebesgue measure, and $r \in [1, \infty]$, then there exists $C_{\text{obs}} \geq 0$ such that, for all $u_0 \in L^p(\mathbb{R}^d)$, we have

$$
\|U_p(T,0)u_0\|_{L^p(\mathbb{R}^d)} \leq C_{\text{obs}} \left\{ \left( \int_0^T \|\left( U_p(t,0)u_0 \right)|_{\Omega(t)} \|_{L^r(\Omega(t))} dt \right)^{1/r}, \quad r \in [1, \infty), \
\right.
\left. \quad \text{ess sup}_{t \in E} \|\left( U_p(t,0)u_0 \right)|_{\Omega(t)} \|_{L^p(\Omega(t))}, \quad r = \infty, \right.
$$

where $(U_p(t,s))_{0 \leq s \leq t \leq T}$ is the evolution family defined in (26).

Note that, in contrast to Theorem 4.8, the constant $C_{\text{obs}}$ may (and will) depend on $p$ (and $\varepsilon$).

**Proof of Theorem 5.2.** We check that Hypothesis 3.1 is satisfied for the choice of $X = Y = L^p(\mathbb{R}^d)$, $C(t)$ the restriction operator to $\Omega(t)$, $(P_\lambda)_{\lambda > 0}$ the family of smooth frequency cutoffs as defined in the proof of Theorem 4.8, and $(U(t,s))_{0 \leq s \leq t \leq \tilde{T}} = (U_p(t,s))_{0 \leq s \leq t \leq T}$ the evolution family on $L^p(\mathbb{R}^d)$ associated with a non-autonomous Ornstein–Uhlenbeck equation as above.

The uncertainty principle follows directly from the Logvinenko–Sereda theorem [Kov01, Theorem 3], so we merely need to check the dissipation estimate. To this end, define the sharp spectral cutoff operator

$$
Q_\lambda : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d), \quad f \mapsto \mathcal{F}^{-1}1_{[-\lambda,\lambda]^d}\mathcal{F}f.
$$

From [BEP20, Proposition 15], it follows immediately that there exist constants $c_0, c_1, \tilde{\varepsilon} > 0$, and $m_1 \in \mathbb{N}$ such that, for all $0 \leq s \leq t \leq T := \tilde{T}$, we have

$$
\|(\text{Id} - P_\lambda)U_2(t,s)\|_{L^1(\mathbb{R}^d)} \leq \|(\text{Id} - Q_{\lambda/(2\sqrt{d})}U_2(t,s))\|_{L^1(\mathbb{R}^d)} \leq c_0e^{-c_1(t-s)^{m_1}\lambda^2},
$$

where $m_1$ is as in (23). On the other hand, it follows from the norm estimate (27) that, for all $0 \leq s \leq t \leq T$, we have

$$
\|(\text{Id} - P_\lambda)U_1(t,s)\|_{L^1(\mathbb{R}^d)}
$$

$$
\leq \|(\text{Id} - P_\lambda)\|_{L^1(\mathbb{R}^d)} \|U_1(t,s)\|_{L^1(\mathbb{R}^d)} \leq Ke^{\frac{1}{2}\int_{s}^{t}B(\tau) d\tau},
$$

where we used uniform boundedness of the family $(P_\lambda)_{\lambda > 0}$ to define the constant

$$
K := 1 + \|\mathcal{F}^{-1}1_{[-\lambda,\lambda]^d}\|_{L^1(\mathbb{R}^d)}.
$$

By the Riesz–Thorin interpolation theorem, for $p \in (1, 2)$, we obtain

$$
\|(\text{Id} - P_\lambda)U_p(t,s)\|_{L^p(\mathbb{R}^d)} \leq (c_0e^{-c_1(t-s)^{m_1}\lambda^2})^{1-\theta}K^{\theta} e^{\frac{\theta}{2}\int_{s}^{t}B(\tau) d\tau},
$$

where

$$
\frac{1}{p} = \frac{1}{1} + \frac{1-\theta}{2}, \quad \text{i.e.} \quad \theta = \frac{1}{p} - \frac{1}{p'} = \frac{1}{2} - 1 \quad \text{and} \quad 1 - \theta = \frac{2}{p'} = 2 - \frac{2}{p},
$$
and thus, setting
\[ M_p := \max_{0 \leq s \leq t \leq T} K^{\frac{2}{p}} e^{\left(\frac{1}{p} - \frac{1}{2}\right) \int_s^t B(\tau) d\tau}, \]
we obtain
\[ \| (\text{Id} - P_\lambda) U_p(t, s) \|_{L^p(\mathbb{R}^d)} \leq M_p (c_0 e^{-c_1 (t-s)^{m_1} \lambda^2})^{2-2/p}. \]
This proves the necessary dissipation estimate in the case that \( p \in (1, 2) \). For the case \( p \in (2, \infty) \), fix \( q \in \mathbb{R} \) such that \( p < q \). If we replace (28) by
\[ \| (\text{Id} - P_\lambda) U_q(t, s) \|_{L^q(\mathbb{R}^d)} \leq K e^{\left(\frac{1}{p} - \frac{1}{2}\right) \int_s^t B(\tau) d\tau} \]
in the above argument, where \( 1/q + 1/q' = 1 \), we end up with
\[ \| (\text{Id} - P_\lambda) U_p(t, s) \|_{L^p(\mathbb{R}^d)} \leq (c_0 e^{-c_1 (t-s)^{m_1} \lambda^2})^{1-\sigma} K^\sigma e^{\left(\frac{1}{p} - \frac{1}{q}\right) \int_s^t B(\tau) d\tau}, \tag{29} \]
where
\[ \frac{1}{p} = \frac{1}{2} - \frac{\sigma}{q}, \quad \text{i.e.} \quad \sigma = \frac{p-2}{p} \cdot \frac{q}{q-2}. \]
Letting \( q \to \infty \) in (29), we obtain
\[ \| (\text{Id} - P_\lambda) U_p(t, s) \|_{L^p(\mathbb{R}^d)} \leq (c_0 e^{-c_1 (t-s)^{m_1} \lambda^2})^{2/p} K^{1-2/p} e^{\left(\frac{1}{p} - \frac{1}{q}\right) \int_s^t B(\tau) d\tau} \]
and therefore, by setting
\[ N_p := \max_{0 \leq s \leq t \leq T} K^{1-2/p} e^{\left(\frac{1}{p} - \frac{1}{q}\right) \int_s^t B(\tau) d\tau}, \]
we obtain the estimate
\[ \| (\text{Id} - P_\lambda) U_p(t, s) \|_{L^p(\mathbb{R}^d)} \leq N_p (c_0 e^{-c_1 (t-s)^{m_1} \lambda^2})^{2/p} \]
for \( p \in (2, \infty) \). In either case, this proves the dissipation estimate. The claim now follows from Theorem 3.3. \( \square \)

We demonstrate the above result by an example concerning the (autonomous) Kolmogorov equation with time-dependent observation sets.

**Example 5.3.** We consider the evolution family associated with the classical Kolmogorov equation
\[ \partial_t u(t, x, v) - \Delta_x u(t, x, v) + v \cdot \nabla_x u(t, x, v) = 0, \quad (t, x, v) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d, \]
\[ u(0, x, v) = u_0(x, v), \quad u_0 \in L^p(\mathbb{R}^d \times \mathbb{R}^d), \]
which, in the notation of (20), corresponds to the choice of
\[ A(t) = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2} \text{Id}_{\mathbb{R}^d} \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 & \text{Id}_{\mathbb{R}^d} \\ 0 & 0 \end{pmatrix}, \quad t \in [0, T]. \]
It follows that, for \( 0 \leq s \leq t \leq T \) and the form
\[ q_{t,s}(\xi, \eta) = 2(t-s) |\eta|^2 + 2(t-s)^2 \eta \cdot \xi + \frac{2}{3} (t-s)^3 |\xi|^2, \quad (\xi, \eta) \in \mathbb{R}^d \times \mathbb{R}^d, \]
the associated evolution family \( \{U(t,s)\}_{0 \leq s \leq t \leq T} \) is given for \( u \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d) \) by
\[ (FU(t,s)u)(\xi, \eta) = e^{-\frac{1}{2} q_{t,s}(\xi, \eta)} (FU)(\xi, \eta + (t-s)\xi), \quad (\xi, \eta) \in \mathbb{R}^d \times \mathbb{R}^d, \]
which we can again extend to $L^p(\mathbb{R}^d \times \mathbb{R}^d)$ by density for $p \in [1, \infty)$. Note that, for arbitrary choices of $T$, we have for all $0 \leq s \leq t \leq T$

$$q_{t,s}(\xi, \eta) \geq \frac{4 - \sqrt{13}}{3} \min \left\{ T^{-2}, 1 \right\} (t - s)^\frac{1}{2} \left( |\xi|^2 + |\eta|^2 \right).$$

Therefore, estimate (23) holds for $\tilde{T} := T$ in this case. Now, let $(\Omega(t))_{t \in [0,T]}$ be uniformly thick on $[0,T]$, $E \subseteq [0,T]$ measurable with positive Lebesgue measure, and $r \in [1, \infty]$. Then Theorem 5.2 yields for $p \in (1, \infty)$ a final-state observability estimate.

**Remark 5.4.** Theorem 5.2 and Example 5.3 can be regarded as extensions of [BEP20, Corollary 8(i)] and [BEP20, Proposition 9], as we can treat non-autonomous Ornstein–Uhlenbeck equations in $L^p(\mathbb{R}^d)$ for $p \in (1, \infty)$.

**Appendix A. Properties of Non-autonomous Elliptic Operators**

Let $T > 0$, and let $a$ be a uniformly strongly elliptic polynomial of degree $m \geq 2$ with coefficients $a_\alpha \in L^\infty(0,T)$ for $|\alpha| \leq m$. In this appendix, we collect some properties of $(U_p(t,s))_{0 \leq s \leq t \leq T}$, where $(U(t,s))_{0 \leq s \leq t \leq T}$ is as in (12) and $p \in [1, \infty]$.

**A.1. Kernel Estimates and Exponential Boundedness.** We will show that we can find *Gaussian bounds* for the kernel $p_{t,s}$, $0 \leq s < t \leq T$, from (13). We do this in two steps.

**Lemma A.1.** Let $c > 0$ be the uniform ellipticity constant of $a$ from (11). Then, for all $c_0 \in (0,c)$, there exists $\omega \in \mathbb{R}$ such that

$$\text{Re } a(t, \xi) \geq c_0 |\xi|^m - \omega, \quad \text{a.e. } t \in [0,T], \xi \in \mathbb{R}^d.$$  

*Proof.* We have $\text{Re } a_m(t, \xi) \geq c |\xi|^m$ for all $t \in [0,T]$ and $\xi \in \mathbb{R}^d$. Thus, for $c_0 \in (0,c)$, we estimate

$$\text{Re } a(\cdot, \xi) \geq c_0 |\xi|^m + (c - c_0) |\xi|^m + \sum_{|\alpha| < m} \text{Re } (a_\alpha(\cdot) i^\alpha) \xi^\alpha.$$  

Since the coefficient functions are locally essentially bounded, it is easy to see that there exists $\omega \in \mathbb{R}$ such that

$$(c - c_0) |\xi|^m + \sum_{|\alpha| < m} (\text{Re } a_\alpha(t) i^\alpha) \xi^\alpha \geq -\omega, \quad \text{a.e. } t \in [0,T], \xi \in \mathbb{R}^d.$$  

This yields the assertion. $\square$

**Lemma A.2.** There exist $C_1, C_2 \geq 0$, and $\omega \in \mathbb{R}$ such that, for all $0 \leq s < t \leq T$ and all $x \in \mathbb{R}^d$, we have

$$|p_{t,s}(x)| \leq C_1 \frac{1}{(t-s)^{d/m}} e^{\omega(t-s)} e^{-C_2 \left( |x|^m / (t-s) \right)^{1/m-1}}.$$  

In particular, for all $p \in [1, \infty]$ and $0 \leq s < t \leq T$, we have

$$\|U_p(t,s)\|_{L^p(\mathbb{R}^d)} = \|p_{t,s}\|_{L^1(\mathbb{R}^d)} \leq C_1 e^{\omega(t-s)} \int_{\mathbb{R}^d} e^{-C_2 |x|^m / (t-s)} \, dx.$$  

*Proof.* We follow the argument in [TR96, Proposition 2.1]. Note that, although $a(t, \cdot)$ is defined on $\mathbb{R}^d$, since it is a polynomial, we can extend it to $\mathbb{C}^d$ for all $t \in [0,T]$. 

Let \(0 \leq s < t \leq T, x \in \mathbb{R}^d\). Then, for \(\eta \in \mathbb{R}^d\), we obtain via the change of variables formula
\[
p_{t,s}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\xi \cdot \xi - x \cdot \eta} e^{-\int_s^t a(\tau, \xi + i\eta) \, d\tau} \, d\xi.
\]
In view of Lemma A.1, there exist \(c_0, c_1, c_2 > 0\) such that, for almost all \(\tau \in [0, T]\) and all \(\xi, \eta \in \mathbb{R}^d\), we have
\[
\text{Re} \, a(\tau, \xi + i\eta) = \text{Re} \, a(\tau, \xi) + \text{Re} \, a(\tau, i\eta) + \text{Re} \, a_0(\tau)
\]
\[
+ \text{Re} \sum_{|\alpha| \leq m, \alpha \neq 0} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (i\xi)^\beta (\eta)^{\alpha - \beta}
\]
\[
\geq c_0 |\xi|^m - c_1 |\eta|^m - c_2 \left(1 + \sum_{1 \leq k \leq m} \sum_{1 \leq l \leq k} |\xi|^l |\eta|^{k-l}\right).
\]
Note that, in order to estimate \(\text{Re} \, a(\tau, \eta)\), the fact that \(m\) is an even number is crucial as it implies \(a_m(\tau, \eta) = (-1)^m a_m(\tau, \eta)\) which follows directly from Definition 4.1.

Now, by Young’s inequality for products, we can choose \(\omega \geq 0\) such that
\[
c_2 \left(1 + \sum_{1 \leq k \leq m} \sum_{1 \leq l \leq k} |\xi|^l |\eta|^{k-l}\right) \leq \frac{c_0}{2} |\xi|^m + \omega(1 + |\eta|^m).
\]
Thus, we finally arrive at
\[
\text{Re} \, a(\tau, \xi + i\eta) \geq \frac{c_0}{2} |\xi|^m - \sigma |\eta|^m - \omega,
\]
where \(\sigma := c_1 + \omega\). Hence, we can estimate
\[
|p_{t,s}(x)| \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-x \cdot \eta} e^{-\int_s^t \text{Re} \, a(\tau, \xi + i\eta) \, d\tau} \, d\xi
\]
\[
\leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-x \cdot \eta} e^{-(t-s)(\frac{c_0}{2} |\xi|^m - \sigma |\eta|^m - \omega)} \, d\xi
\]
\[
= C_1 \frac{1}{(t-s)^{d/m}} e^{-x \cdot \eta} \omega(t-s) e^{\sigma |\eta|^m},
\]
where \(C_1 := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-\frac{c_0}{2} |\xi|^m} \, d\xi\). Now, for \(\eta := \frac{1}{2} \left(\frac{|x|}{\sigma(t-s)}\right)^{1/(m-1)} \frac{x}{|x|}\), we obtain
\[
|p_{t,s}(x)| \leq C_1 \frac{1}{(t-s)^{d/m}} e^{\omega(t-s)} e^{C_2 \left(|x|^m/(t-s)\right)^{1/m-1}},
\]
where \(C_2 := \frac{2^{m-1} - 1}{2^m - 1}\). Thus, integration yields the assertion for \(\|p_{t,s}\|_{L^1(\mathbb{R}^d)}\). For \(p \in [1, \infty]\), the operator \(U_p(t, s)\) is the convolution operator with kernel \(p_{t,s}\). Thus, we have \(\|U_p(t, s)\|_{L^p(\mathbb{R}^d)} = \|p_{t,s}\|_{L^1(\mathbb{R}^d)}\).

A.2. Strong Continuity. We now show that \((U_p(t, s))_{0 \leq s \leq t \leq T}\) is strongly continuous for \(p \in [1, \infty]\), while \((U_\infty(t, s))_{0 \leq s \leq t \leq T}\) is strongly continuous with respect to the weak* topology.

We start with a subspace of \(S'(\mathbb{R}^d)\) that can be identified with a subset of \(C^\infty(\mathbb{R}^d)\). Let \(O_M(\mathbb{R}^d)\) denote the multiplier space
\[
O_M(\mathbb{R}^d) := \{ f \in C^\infty(\mathbb{R}^d) : \forall g \in S(\mathbb{R}^d), \alpha \in \mathbb{N}^d_0 : \|f\|_{g, \alpha} < \infty\},
\]
where \(\|f\|_{g, \alpha} := \sup_{x \in \mathbb{R}^d} |\partial_x^\alpha f(x)| \cdot |g(x)|\).
Corollary A.4. \( f \) Thus, and induces a locally convex topology on \( \mathcal{O}_M(\mathbb{R}^d) \), cf. [Sch78, Chapter 7, §5, p. 243], [Kru19, Example 5.3]. Note that the multiplication \( \mathcal{O}_M(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \ni (f,g) \mapsto fg \in \mathcal{S}(\mathbb{R}^d) \) is hypocontinuous, in particular separately continuous, see, e.g., [Lar13].

**Proposition A.3.** Let \((f_n)_{n \in \mathbb{N}}\) be in \( \mathcal{C}^\infty(\mathbb{R}^d) \) and \( f \in \mathcal{C}^\infty(\mathbb{R}^d) \) such that, for all \( \alpha \in \mathbb{N}_0^d \), we have \( \sup_{n \in \mathbb{N}} \| \partial^\alpha f_n \|_\infty < \infty \) and \( \partial^\alpha f_n \to \partial^\alpha f \) uniformly on compact sets. Then \((f_n)_n\) is in \( \mathcal{O}_M(\mathbb{R}^d) \), \( f \in \mathcal{O}_M(\mathbb{R}^d) \), and \( f_n \to f \) in \( \mathcal{O}_M(\mathbb{R}^d) \).

Proof. Note that \( \partial^\alpha f \) is bounded for all \( \alpha \in \mathbb{N}_0^d \). Since smooth functions whose derivatives of all orders are bounded clearly belong to \( \mathcal{O}_M(\mathbb{R}^d) \), we obtain that \((f_n)_n \) is in \( \mathcal{O}_M(\mathbb{R}^d) \) and \( f \in \mathcal{O}_M(\mathbb{R}^d) \).

Let \( \alpha \in \mathbb{N}_0^d \) and \( g \in \mathcal{S}(\mathbb{R}^d) \), and let \( \varepsilon > 0 \). Choose a compact subset \( K \subseteq \mathbb{R}^d \) such that

\[
\sup_{x \in K} |g(x)| \leq \frac{\varepsilon}{2 \| g \|_\infty + 1}.
\]

Furthermore, choose \( N \in \mathbb{N} \) such that, for all \( n \geq N \), we have

\[
\sup_{x \in K} |\partial^\alpha(f_n - f)(x)| \leq \frac{\varepsilon}{2 \| g \|_\infty + 1}.
\]

Then we observe for all \( n \geq N \)

\[
\| f_n - f \|_{g,\alpha} = \sup_{x \in \mathbb{R}^d} |g(x) \partial^\alpha (f_n - f)(x)| \\
\leq \sup_{x \in K} |g(x) \partial^\alpha (f_n - f)(x)| + \sup_{x \not\in K} |g(x) \partial^\alpha (f_n - f)(x)| \\
\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Thus, \( f_n \to f \) in \( \mathcal{O}_M(\mathbb{R}^d) \). \( \square \)

**Corollary A.4.** \( (U(t,s))_{0 \leq s \leq t \leq T} \) is strongly continuous on \( \mathcal{S}(\mathbb{R}^d) \).

Proof. Let \( 0 \leq s \leq t \leq T \). Let \(( (t_n, s_n))_{n \in \mathbb{N}} \) be in \( [0, \infty)^2 \), with \( 0 \leq s_n \leq t_n \leq T \) for all \( n \in \mathbb{N} \) and limit \( (t_n, s_n) \to (t, s) \). For \( n \in \mathbb{N} \), set

\[
f_n := e^{-\int_{t_n}^t a(\tau) \, d\tau} \in \mathcal{C}^\infty(\mathbb{R}^d).
\]

Let furthermore \( f := \lim_{n \to \infty} f_n \) denote the pointwise limit. By the uniform strong ellipticity of \( a \), it follows that the convergence of \((f_n)_n\) and its partial derivatives is also uniform on compact subsets of \( \mathbb{R}^d \) (with the limit being the corresponding partial derivative of \( f \) and that, for all \( \alpha \in \mathbb{N}_0^d \), the sequence \( (\partial^\alpha f_n)_n \) is bounded, see also Lemma A.1. By Proposition A.3, we have that \( f_n \to f \) in \( \mathcal{O}_M(\mathbb{R}^d) \).

Let \( g \in \mathcal{S}(\mathbb{R}^d) \). Then \( \mathcal{F}(g) \in \mathcal{S}(\mathbb{R}^d) \) and therefore \( f_n \mathcal{F}(g) \to f \mathcal{F}(g) \) in \( \mathcal{S}(\mathbb{R}^d) \). Thus,

\[
U(t_n, s_n)g = \mathcal{F}^{-1}(f_n \mathcal{F}(g)) \to \mathcal{F}^{-1}(f \mathcal{F}(g)) = U(t, s)g
\]

by continuity of \( \mathcal{F}^{-1} \). \( \square \)
Corollary A.5. Let $p \in [1, \infty]$. Then $(U_p(t,s))_{0 \leq s \leq t \leq T}$ is strongly continuous. Moreover, $(U_{\infty}(t,s))_{0 \leq s \leq t \leq T}$ is strongly continuous with respect to the weak*-topology.

Proof. Note that Lemma A.2 yields uniform boundedness of $(U_p(t,s))_{0 \leq s \leq t \leq T}$ for all $p \in [1, \infty]$.

Since $\mathcal{S}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$ is dense for $p \in [1, \infty)$, Corollary A.4 yields that the family $(U_p(t,s))_{0 \leq s \leq t \leq T}$ is strongly continuous for $p \in [1, \infty)$. For $0 \leq s \leq t \leq T$, we have $U_{\infty}(t,s) = V_1(t,s)'$, where $(V_1(t,s))_{0 \leq s \leq t \leq T}$ is the evolution family on $L^1(\mathbb{R}^d)$ associated with the non-autonomous polynomial

$$a(\cdot, -\cdot): (t, \xi) \mapsto a(t, -\xi) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} a_\alpha(t) \xi^\alpha$$

which is also uniformly strongly elliptic since $m$ is necessarily even. Thus, the second assertion follows.

A.3. $(U_p(t,s))_{0 \leq s \leq t \leq T}$ as an Evolution Family for $(A_p(t))_{t \in [0,T]}$. Now, we establish the relation between the evolution family $(U_p(t,s))_{0 \leq s \leq t \leq T}$ and the family of differential operators $(A_p(t))_{t \in [0,T]}$.

Proposition A.6. Let $p \in (1, \infty)$ and $u \in D^p$.

(a) Let $0 \leq s < T$. Then $U_p(\cdot, s)u \in W^{1,1}(s,T;L^p(\mathbb{R}^d)) \cap L^1(s,T;D^p)$ and, for almost all $t \in (s,T)$, we have $\partial_t(U_p(t,s)u) = -A_p(t)U_p(t,s)u$.

(b) Let $0 < t \leq T$. Then $U_p(t,\cdot)u \in W^{1,1}(t,T;L^p(\mathbb{R}^d)) \cap L^1(t,T;D^p)$ and, for almost all $s \in (t,T)$, we have $\partial_s(U_p(T,s)u) = U_p(T,s)A_p(s)u$.

Proof. (a) By Young’s inequality and the $L^1$-bound of the kernel in Lemma A.2, we observe $U_p(\cdot, s)u \in L^1(s,T;L^p(\mathbb{R}^d))$. Note that $(s,T) \ni t \mapsto p_{t,s}$ is weakly differentiable and $\partial_t p_{t,s} = -A_p(t)p_{t,s}$ for all $t \in (s,T)$. For the weak derivative of $U_p(\cdot, s)u$, we have

$$\partial_t(U_p(t,s)u) = \partial_t(p_{t,s} * u) = (\partial_t p_{t,s})*u = (A_p(t)p_{t,s})*u = p_{t,s}*(-A_p(t)u) = -A_p(t)(p_{t,s} * u) = -A_p(t)U_p(t,s)u$$

for almost all $t \in (s,T)$. In particular, the closedness of $A_p(t)$ implies $U_p(t,s)u \in D^p$ for almost all $t \in (s,T)$.

Note that $D^p = W^{p,m}(\mathbb{R}^d)$ and the Sobolev norm and the graph norms $\| \cdot \|_{A_p(t)}$ are equivalent, i.e. there exists $C > 0$ such that

$$\frac{1}{C}\|v\|_{W^{p,m}(\mathbb{R}^d)} \leq \|v\|_{A_p(t)} \leq C\|v\|_{W^{p,m}(\mathbb{R}^d)}$$

for all $t \in [s,T]$ and $v \in D^p$ which follows from uniform strong ellipticity of $a$ and the boundedness of the coefficients $a_\alpha$. In particular, $\|A_p(t)u\|_{L^p(\mathbb{R}^d)} \leq C\|u\|_{W^{p,m}(\mathbb{R}^d)}$ for all $t \in [s,T]$ and $u \in D^p$, and therefore

$$\int_s^T \|A_p(t)u\|_{L^p(\mathbb{R}^d)} dt < \infty,$$

so, by Young’s inequality and the kernel bound from Lemma A.2, we observe that $\partial_t U_p(\cdot, s)u \in L^1(s,T;L^p(\mathbb{R}^d))$ as well as $U_p(\cdot, s)u \in L^1(s,T;D^p)$.

The proof of (b) follows the same lines as the proof of (a).
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REFERENCES

[AM] P. Alphonse and J. Martin. Approximate null-controllability with uniform cost for the hypoelliptic Ornstein-Uhlenbeck equations. arXiv:2201.01516v2 [math.AP].

[AT87] P. Acquistapace and B. Terreni. A unified approach to abstract linear nonautonomous parabolic equations. *Rend. Semin. Mat. Univ. Padova*, 78:47–107, 1987.

[Bar14] V. Barbu. Exact null internal controllability for the heat equation on unbounded convex domains. *ESAIM Control Optim. Calc. Var.*, 20(1):222–235, 2014.

[BEP20] K. Beauchard, M. Egidi, and K. Pravda-Starov. Geometric conditions for the null-controllability of hypoelliptic quadratic parabolic equations with moving control supports. *C. R. Math. Acad. Sci. Paris*, 358(6):651–700, 2020.

[BGST] C. Bombach, D. Gallaun, C. Seifert, and M. Tautenhahn. Observability and null-controllability for parabolic equations in $L_p$-spaces. arXiv:2005.14503v2 [math.FA].

[BP17] K. Beauchard and K. Pravda-Starov. Null-controllability of non-autonomous Ornstein–Uhlenbeck equations. *J. Math. Anal. Appl.*, 456(1):496–524, 2017.

[Car88] O. Carja. On constraint controllability of linear systems in Banach spaces. *J. Optim. Theory Appl.*, 56(2):215–225, 1988.

[Cor07] J.-M. Coron. *Control and Nonlinearity*, volume 136 of *Mathematical Surveys and Monographs*. Amer. Math. Soc., Providence, RI, 2007.

[CRZ14] F. W. Chaves-Silva, L. Rosier, and E. Zuazua. Null-controllability of a system of viscoelasticity with moving control. *J. Math. Pures Appl.*, 101(2):198–222, 2014.

[Dou66] R. G. Douglas. On majorization, factorization, and range inclusion of operators on Hilbert space. *Proc. Amer. Math. Soc.*, 17(2):413–415, 1966.

[DR77] S. Dolecki and D. L. Russell. A general theory of observation and control. *SIAM J. Control Optim.*, 15(2):185–220, 1977.

[EGST] M. Egidi, D. Gallaun, C. Seifert, and M. Tautenhahn. Sufficient criteria for stabilization properties in Banach spaces. arXiv:2108.09028v1 [math.OC].

[EN00] K.-J. Engel and R. Nagel. *One-Parameter Semigroups for Linear Evolution Equations*, volume 194 of *Graduate Texts in Mathematics*. Springer, New York, 2000.

[EV18] M. Egidi and I. Veselić. Sharp geometric condition for null-controllability of the heat equation on $\mathbb{R}^d$ and consistent estimates on the control cost. *Arch. Math.*, 111(1):85–99, 2018.

[FI96] A. V. Fursikov and O. Y. Imanuvilov. *Controllability of Evolution Equations*, volume 34 of *Sukh kangařok*. Seoul National University, Seoul, 1996.

[Gal17] C. Gallarati. *Maximal Regularity for Parabolic Equations with Measurable Dependence on Time and Applications*. PhD thesis, Delft University of Technology, 2017.

[Gd07] M. González-Burgos and L. de Teresa. Some results on controllability for linear and nonlinear heat equations in unbounded domains. *Adv. Differential Equations*, 12(11):1201–1240, 2007.

[Gra14] L. Grafakos. *Classical Fourier Analysis*. Graduate Texts in Mathematics. Springer, New York, 3rd edition, 2014.

[GST20] D. Gallaun, C. Seifert, and M. Tautenhahn. Sufficient criteria and sharp geometric conditions for observability in Banach spaces. *SIAM J. Control Optim.*, 58(4):2639–2657, 2020.

[GV17] C. Gallarati and M. Veraar. Maximal regularity for non-autonomous equations with measurable dependence on time. *Potential Anal.*, 46(3):527–567, 2017.

[Haa06] M. Haase. *The Functional Calculus for Sectorial Operators*, volume 169 of *Operator Theory: Advances and Applications*. Birkhäuser, Basel, 2006.
L. Hörmander. *The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis*. Classics in Mathematics. Springer, 2nd edition, 1990. Reprint, 2003.

O. Kovrijkine. Some results related to the Logvinenko–Sereda theorem. *Proc. Amer. Math. Soc.*, 129(10):3037–3047, 2001.

K. Kruse. The approximation property for weighted spaces of differentiable functions. *Banach Center Publ.*, 119:233–258, 2019.

J. Larcher. Multiplications and convolutions in L. Schwartz’ spaces of test functions and distributions and their continuity. *Analysis (Berlin)*, 33(4):319–332, 2013.

J. Le Rousseau and G. Lebeau. On Carleman estimates for elliptic and parabolic operators. Applications to unique continuation and control of parabolic equations. *ESAIM Contr. Optim. Calc. Var.*, 18(3):712–747, 2012.

J. Le Rousseau, G. Lebeau, P. Terpolilli, and E. Trélat. Geometric control condition for the wave equation with a time-dependent observation domain. *Anal. PDE*, 10(4):983–1015, 2017.

G. Lebeau and L. Robbiano. Contrôle exact de l’équation de la chaleur. *Comm. Partial Differential Equations*, 20(1–2):335–356, 1995.

A. Lunardi. *Analytic semigroups and optimal regularity in parabolic problems*. Modern Birkhäuser Classics. Birkhäuser, Basel, 1995.

L. Miller. Geometric bounds on the growth rate of null-controllability cost for the heat equation in small time. *J. Differential Equations*, 204(1):202–226, 2004.

L. Miller. How violent are fast controls for Schrödinger and plate vibrations? *Arch. Ration. Mech. Anal.*, 172(3):429–456, 2004.

L. Miller. A direct Lebeau-Robbiano strategy for the observability of heat-like semigroups. *Discrete Contin. Dyn. Syst. Ser. B*, 14(4):1465–1485, 2010.

P. Martin, L. Rosier, and P. Rouchon. Null controllability of the structurally damped wave equation with moving control. *SIAM J. Control Optim.*, 51(1):660–684, 2013.

G. Nickel. Evolution semigroups for nonautonomous Cauchy problems. *Abstr. Appl. Anal.*, 2(1–2):73–95, 1997.

A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*, volume 44 of *Applied Mathematical Sciences*. Springer, New York, 1983.

K. D. Phung and G. Wang. An observability estimate for parabolic equations from a measurable set in time and its applications. *J. Eur. Math. Soc. (JEMS)*, 15(2):681–703, 2013.

K. D. Phung, G. Wang, and Y. Xu. Impulse output rapid stabilization for heat equations. *J. Differential Equations*, 263(8):5012–5041, 2017.

L. Robbiano. Fonction de coût et contrôle des solutions des équations hyperboliques. *Asymptot. Anal.*, 10(2):95–115, 1995.

L. Schwartz. *Théorie des distributions*. Hermann, Paris, 1978.

P. E. Sobolevski˘ı. *Equations of parabolic type in a Banach space* (russian). *Trudy Moskov. Mat. Obšč.*, 10:297–350, 1961.

H. Tanabe. On the equations of evolution in a Banach space. *Osaka J. Math.*, 12(2):363–376, 1960.

H. Tanabe. *Equations of evolution*, volume 6 of *Monographs and Studies in Mathematics*. Pitman, London, 1979.

G. Teschl. *Ordinary Differential Equations and Dynamical Systems*, volume 140 of *Graduate Studies in Mathematics*. Amer. Math. Soc, Providence, RI, 2012.

A. F. M. Ter Elst and D. W. Robinson. Elliptic operators on Lie groups. *Acta Appl. Math.*, 44(1–2):133–150, 1996.

E. Trélat, G. Wang, and Y. Xu. Characterization by observability inequalities of controllability and stabilization properties. *Pure Appl. Anal.*, 2(1):93–122, 2020.

A. Vieru. On null controllability of linear systems in Banach spaces. *Systems Control Lett.*, 54(4):331–337, 2005.

G. Wang, M. Wang, C. Zhang, and Y. Zhang. Observable set, observability, interpolation inequality and spectral inequality for the heat equation in $\mathbb{R}^n$. *J. Math. Pures Appl. (9)*, 126:144–194, 2019.
[WZ17] G. Wang and C. Zhang. Observability inequalities from measurable sets for some abstract evolution equations. *SIAM J. Control Optim.*, 55(3):1862–1886, 2017.

[Yag91] A. Yagi. Abstract quasilinear evolution equations of parabolic type in Banach spaces. *Boll. Unione Mat. Ital.*, 5(7):341–368, 1991.

[YLCO6] X. Yu, K. Liu, and P. Chen. On null controllability of linear systems via bounded control functions. In *2006 American Control Conference*, pages 1458–1461, Piscataway, 2006. IEEE.

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