MONODROMY OF THE HITCHIN MAP OVER HYPERELLIPTIC CURVES

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1. INTRODUCTION

The purpose of this paper is to study the monodromy of the Hitchin fibration for rank 2 vector bundles over hyperelliptic curves. We reduce the problem to studying a surface braid group generalization of the classical Burau representation [3], and give a combinatorial method for computing this representation.

Let \( M \) be a Riemann surface of genus \( g > 1 \) with canonical bundle \( K \). In [7], Hitchin considered the set of stable Higgs pairs, \((V, \Phi)\), where \( V \) is a rank 2 vector bundle of fixed odd degree and fixed determinant \( \bigwedge^2 V = \xi \), and \( \Phi \in \text{End}(V) \otimes K \). \((V, \Phi)\) is called stable if any \( \Phi \)-invariant line sub-bundle \( L \) of \( V \) has the property that \( \deg(L) < \frac{1}{2} \deg(V) \). If \( \mathcal{G} \) is the group of automorphisms of \( V \) with determinant one, then we study the space \( \mathcal{M} \) which is the quotient of the set of all stable Higgs pairs \((V, \Phi)\) modulo \( \mathcal{G} \). This is a manifold by results of Narasimhan and Ramanan [9].

Hitchin considered the map \( \mathcal{M} \to H^0(M, K^2) \), \((V, \Phi) \mapsto \det(\Phi) \) and showed that the generic fiber is an abelian variety. Specifically, if we let \( H^s \subset H^0(M, K^2) \) be the space of quadratic differentials with simple zeroes, then the restriction \( \mathcal{M}|_{H^s} \) is a fiber bundle of abelian varieties. Monodromy of this bundle is the focus of this paper.

We consider the fibers explicitly. Let \( \omega \in H^s \). There is a spectral cover \( S = S_\omega \to M \) which is a ramified double cover of \( M \) constructed as follows. In the total space, \( TK \), of the canonical bundle, \( K \), take the curve \( S = \sqrt{\omega} \) (Hitchin also shows that this is nonsingular). This is the set of points which map to \( \omega \) under the squaring map taking \( TK \) to the total space of \( K^2 \). Notice that \( c : S \to M \) comes equipped with an involution, \( \tau \), which fixes \( M \) and swaps the sheets of the cover. The Prym variety of \( S \) is the subgroup of the Jacobian of \( S \) on which \( \tau \) acts by \(-1\). In our case, this is just \(-1 \in \mathbb{C}\) acting on \( S \subset TK \to M \). It is known from [7] that if (and only if) the divisor associated to \( \omega \in H^0(M, K^2) \) has simple zeroes,
then the fiber of $M$ over $\omega$ is isomorphic to the Prym variety of $(S, M)$. For a more complete description and reference set for Hitchin’s system, see Section 6 of [7] or Section 8 of [8]. We also describe this in more detail in the proof of Theorem 37.

One may also construct a bundle, $S \to \mathcal{H}^s$ whose fibers are the spectral covers $S_{\omega} \subset TK_M$ and the associated bundle of Jacobians (Section 17), and ask whether this contains $M$ as $\text{Jac}(S)^{\tau=-1}$. In the proof of Theorem 37 we don’t prove this, but we show that it is true up to translation, akin to the statement that $\text{Jac}(S) \cong \text{Pic}^k(S)$ for $k \neq 0$, due to a lack of canonical basepoint. Because of this, these bundles agree homologically, since homology of an abelian variety is translation invariant. Thus to study the monodromy of $M$, we may instead study the monodromy of the Jacobian varieties, or indeed of the $\tau = -1$ part of the homology of $S$.

In Section 20 we show that the $\tau = -1$ part of the homology of $S$ is the restriction of a generalization of the classical Burau representation to (a subgroup of) the surface braid group. We then study this representation.

Throughout this paper, we consider only hyperelliptic curves, $M$. This makes many of the arguments more concrete, but Theorem 11 is the main point where this is used. In Section 23 we show how Theorem 11 is in fact the only obstruction to generalizing.

The main result of this paper may be stated as follows:

**Theorem 1.** To each hyperelliptic $M$ of genus greater than 2, one may associate a graph $\tilde{\Gamma}$ with edge set $E$ and skew bilinear pairing $(e \cdot e')$ on $e, e' \in E$ such that

- The monodromy representation of $\pi_1(H^s)$ is generated by elements $\sigma_e$ labelled by the edges $e \in E$ and the element $\tau$.
- The monodromy representation is a quotient of $\mathbb{Z}E$.
- The action of $\pi_1(H^s)$ on $\mathbb{Z}E$ is given by:

$$\tau e = -e$$

$$\sigma_{e_\alpha} e = e - (e \cdot e_\alpha)e_\alpha.$$

These results are, respectively, Corollary 12 and Corollary 39, Theorem 16, and Theorem 41. The intersection numbers are found in Proposition 24, and Proposition 21 gives the kernel of the quotient.
2. The fundamental group of $H^s$

In this paper, we study the monodromy action of $\pi_1(H^s)$ on this bundle of abelian varieties, $\mathcal{M}|_{H^s}$. We begin by discussing the nature of $\pi_1(H^s)$, and reduce to a study of braid groups. In Proposition 38 we show that, up to $\tau$, the monodromy representation factors through a surface braid group, so that we need only understand the image of $\pi_1(H^s)$ in this group. This image is the result of Theorem 11, for which we begin to lay the groundwork here.

Notice that if $q_1$ and $q_2$ are elements of $H^s$ with the same zeroes, then $q_1/q_2$ is constant, thus $H^s/\mathbb{C}^*$ is the space of effective divisors with simple zeroes linearly equivalent to $K^2$. Call this space $PH^s$. An element of $PH^s$ is determined exactly by its divisor.

**Proposition 2.** The kernel of the map induced from projection $\pi_1(H^s) \to \pi_1(PH^s)$ acts by $\{1, \tau\}$ in the monodromy.

**Proof.** As a quotient by a free $\mathbb{C}^*$ action, the map $H^s \to PH^s$ is a fibration with fiber $\mathbb{C}^*$, so

$$\pi_1(\mathbb{C}^*) \to \pi_1(H^s) \to \pi_1(PH^s)$$

is exact. Thus the kernel is spanned by elements of the form $\gamma(t)\omega$, where $\gamma \in \pi_1(\mathbb{C}^*)$. If $\omega_1 = a^2\omega_2$, then $S_{\omega_1} = aS_{\omega_2}$ in the total space of $K^2$. Thus above $\gamma(t)\omega$ lies the curve $\sqrt{\gamma(t)}S_\omega$, showing that if $\gamma$ is a generator for $\pi_1(\mathbb{C}^*)$, then $\gamma\omega$ acts by the involution $\tau$. \[\square\]

Notice that since the degree of $K$ is $2g-2$ (Gauss-Bonnet, e.g), $K^2$ is of degree $4g-4$. Let $M^{[4g-4]}$ be the configuration space of $4g-4$ distinct, unordered points on $M$. $\pi_1(M^{[4g-4]})$, the surface braid group, has been studied in great detail, see [11, 13, 10]; we also discuss some necessary results in Section 3. Let $\rho : H^s \to M^{[4g-4]}$ be the map which takes a quadratic differential to its zero set. This map factors through $PH^s \hookrightarrow M^{[4g-4]}$, which is an injection by the second paragraph of this section.

**Proposition 3.** $\rho_*\pi_1(H^s) < \ker(\pi_1(M^{[4g-4]}) \to H_1(M, \mathbb{Z}))$.

**Proof.** This proposition will follow very quickly, once we’ve defined the map $\pi_1(M^{[4g-4]}) \to H_1(M, \mathbb{Z})$.

Define the Abel map,

$$A : M^{[4g-4]} \to \text{Jac}(M) \cong \text{Pic}^0(M),$$
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$A(D) = |D| \otimes K^{-2}$. In fact, one may choose any degree $4 - 4g$ basepoint. We chose $K^2$ for the simple reason that $A^{-1}(0) = PH^*$. Therefore,

$$\rho_* : \pi_1(H^*) \to \ker(\pi_1(M^{[4g-4]}) \to \pi_1(\text{Jac}(M))).$$

However, $\text{Jac}(M) = H^0(M, K)^*/H_1(M, \mathbb{Z})$, thus there is a natural isomorphism, $\nu : H_1(\text{Jac}(M)) \to H_1(M, \mathbb{Z})$. Adding Hurewicz, the composition

$$\pi_1(M^{[4g-4]}) \xrightarrow{A_*} \pi_1(\text{Jac}(M)) \xrightarrow{\text{Hurewicz}} H_1(\text{Jac}(M)) \xrightarrow{\nu} H_1(M, \mathbb{Z})$$

takes a braid in $M^{[4g-4]}$, and closes it to a union of homology classes of loops in $M$. The theorem follows from the observation that $A \circ \rho = 0$. □

We will show in Theorem 11 that for $g \geq 3$ this is an isomorphism.

3. Surface braid groups

Here we collect results and definitions concerning the surface braid group. Let $R$ be any (compact) Riemann surface.

Let $R^n$ be the configuration space of distinct ordered subsets of $R$ of cardinality $n$. Denote by $R^{[n]}$ the configuration space of $n$-tuples of distinct unordered points on $R$. $R^{[n]}$ is a quotient of $R^n$ by the symmetric group. Let $R_m = R \setminus X$ for some $X \subset R$ of order $m$. For any choice, $X$, the spaces $R_m$ are homeomorphic, and we intend to study only topological data. Let $R_m^n = (R_m)^n$ and $R_m^{[n]} = (R_m)^{[n]}$. We will also use the notation $R_m^{[m]+[n]}$ to denote the space of all $X,Y \subset R$ which are disjoint and of cardinality $m$ and $n$ respectively.

The following theorem is classical:

**Theorem 4** (Fadell-Neuwirth [5], [3]). The following is a fibration:

$$R_m^n \to R^{n+m} \to R^m.$$

We will use two corollaries.

**Corollary 5.**

$$\pi_1(R_m^{[n]}) \triangleleft \pi_1(R^{[n]+1}).$$

**Corollary 6.**

$$\pi_1(R_m^{[n]} \pi_1(R_n^{[1]}) \to \pi_1(R^{[n]+1}).$$

We will also use part of the structure theory of surface braid groups:

**Theorem 7.** Let $U$ be an open disk in $M$, $m \in M \setminus U$, $X \in U^{[n]} \subset M^{[n]}$. Then
(1) $\pi_1(U^n, X)$ is generated by transpositions $\sigma_1, \ldots, \sigma_{n-1}$ of $x_j$ and $x_{j+1}$ (planar structure theory).

(2) $\pi_1(M \setminus \{m, x_2, x_3, \ldots, x_n\}, x_1)$ is generated by $\sigma'_\alpha$ for $\alpha$ in some basis of $H_1(M \setminus m, \mathbb{Z})$, where $x_1$ traces a path homotopic in $M$ to $\alpha$.

(3) $\pi_1((M \setminus m)^{[n]}, X)$ is generated by $\sigma_1, \ldots, \sigma_{n-1}$ from (1), and $\sigma_\alpha$ from (2).

Here we also give a brief technical definition for transposition. Let $e$ be an interval embedded in $M$ with endpoints $x_1, x_2$. Let $X = \{x_1, \ldots, x_n\} \in M^{[n]}$, and assume also that $x_j \notin e$ for $j > 2$. If $U$ is some contractible neighborhood of $e$, $X \cap U = \{x_1, x_2\}$, then

$$
\pi_1(U^{[2]}, \{x_1, x_2\}) \cong \mathbb{Z}.
$$

We let $s = \{s_1, s_2\} : [0, 1] \to U^{[2]}$ be either generator of this group. Extending by trivial paths $s_j \equiv x_j$, we get an element $\sigma \in \pi_1(M^{[n]})$. We call the elements $\sigma_e$ and $\sigma_e^{-1}$ transpositions associated to $e$.

4. Hyperelliptic Curves

Beginning in this section, we assume that $M$ is hyperelliptic. Notationally, there is some polynomial $f$ of degree $2g + 2$ with distinct roots such that $M$ is the zero set $y^2 = f(x)$. Without loss of generality, $f(0) \neq 0$. This is a two-to-one cover $M \to \mathbb{P}^1$ ramified at the $2g + 2$ roots of $f$. Label (arbitrarily) the points in $M$ above infinity as $\infty^\pm$, and the points above 0 as $0^\pm$. These points will ultimately be the vertices of some graph, $\Gamma$. $M$ is equipped with a hyperelliptic involution $\iota : (x, y) \to (x, -y)$. $\iota$ gives a decomposition of the space of quadratic differentials:

$$
H^0(M, K^2) = H^0(M, K^2)^+ \oplus H^0(M, K^2)^-.
$$

The space of quadratic differentials is generated by $(\frac{dx}{y})^2$, which is holomorphic with divisor $(\frac{dx}{y})^2 = (2g - 2)(\infty^+) + (2g - 2)(\infty^-)$. This element is fixed under the hyperelliptic involution.

We may decompose any quadratic differential:

$$
\omega = \omega^+ + \omega^- = p(x)(\frac{dx}{y})^2 + q(x)y(\frac{dx}{y})^2,
$$

such that $\omega^\pm = \pm \omega^\pm$. The degree of $p$ is at most $2g - 2$ and the degree of $q$ is at most $g - 3$. The dimensions of the components of $H^0(M, K^2)$ are $h^0(K^2)^+ = 2g - 1$, and $h^0(K^2)^- = g - 2$. In the genus 2 case, all quadratic differentials are even.
5. GENUS 2

We treat first the special case $g = 2$. This is quite simple and unique because $H^0(M, K^2) = H^0(M, K^2)^+$. Thus a quadratic differential is described by the polynomial $p$ of degree at most 2. Let $D_5^{[2]}$ be the configuration space of pairs of distinct unordered points on the five-punctured disk.

**Theorem 8.** If $M$ is a hyperelliptic curve of genus 2, then $\pi_1(\text{PH}^s)$ is the group, $\pi_1(D_5^{[2]})$, of 2-braids on the 5-punctured disk, and $\pi_1(H^s) \cong \mathbb{Z} \times \pi_1(D_5^{[2]})$.

**Remark 9.** In fact, we show that $\text{PH}^s$ is merely the set of pairs of distinct points on $\mathbb{P}^1 \setminus f^{-1}(0)$, which is $(\mathbb{P}^1)^{[2]}_0 = D_5^{[2]}$.

**Proof.** Notice that $H^0(M, K^2)$ consists of quadratic differentials $\omega = p(x)(\frac{dx}{y})^2$, where $p$ is of degree at most two. Such a polynomial has two zeroes in $\mathbb{P}^1$. These lift to four zeroes in $M$, which will be unique exactly when the two zeroes are distinct and miss the ramification divisor of $M \to \mathbb{P}^1$.

Choosing some $x_0$ in the ramification divisor, $p(x_0) \neq 0$, so $\omega \to (p(x_0), \langle \omega \rangle) \in \mathbb{C}^* \times D_5^{[2]}$ gives a splitting of $H^s \cong \mathbb{C}^* \times D_5^{[2]}$, so this shows $\pi_1(H^s) \cong \mathbb{Z} \times \pi_1(D_5^{[2]})$. \hfill \qedsymbol

6. CELLULAR DECOMPOSITION OF $M$

The objective of the next two sections will be to describe explicitly the structure of $\rho_*\pi_1(H^s)$ as in Theorem 11. We prove this theorem by applying a theorem from [4].

**Theorem 10.** If $M$ is a polyhedron (two-dimensional cell complex) of genus $g$ with $n$ faces such that no face is a neighbor of itself and no two faces share more than one edge, then $B_n^0 = \ker(\pi_1(M^{[n]}) \to H_1(M))$ is generated by the edge set. Specifically, the basepoint of $M^{[n]}$ may be chosen to be a marked point in the interior of each face, and each edge may be viewed as a transposition of the marked points on the faces it separates.

We will give a cellular decomposition of $M$, and then show all generators from Theorem 10 lie in $\rho_*\pi_1(H^s)$, thus proving Theorem 11. This argument relies heavily upon $M$ being hyperelliptic. Also, this is the only place we actually need to use that $M$ is hyperelliptic, so this is the step needed to generalize to any curve (see Section 23).
First of all, notice that a polyhedron of genus two with four faces must necessarily have at least seven edges by Gauss-Bonnet. However, a polyhedron with four faces conforming to the hypotheses of Theorem 1 may have no more than six edges. Thus in the case $g = 2$, we would not be able to apply this theorem.

Now assume $g > 2$. We begin by constructing a triangulation of $M$ by giving a graph $\Gamma_T$ on $M$.

For simplicity we restrict our attention to the case $f(x) = x^{2g+2} - 1$, from which it should be clear how to generalize. In the closing remarks of Section 7, we discuss how one would generalize.

On $\mathbb{P}^1$, $M$ branches over the set $\{x|0 \leq x^{2g+2} \leq 1\}$. That is, the complement of this set in $\mathbb{P}^1$ lifts to a disjoint pair of sets in $M$. Draw the set $\{x| -\infty \leq x^{2g+2} \leq 1\}$. This is a graph $\Gamma_T$ with vertices $0^\pm$ and $\infty^\pm$, as in Figure 1, which gives a triangulation of $M$ into $4g + 4$ faces. The interior of each triangle lies entirely in the upper or lower branch of $M$. Triangles in the upper (e.g.) branch have vertices $(\infty^+, 0^+, 0^-)$. The dual graph $\hat{\Gamma}_T$ to (the fat structure on) $\Gamma_T$ is shown in Figure 2. Its vertices are triangles in the triangulation. In such a way, we get a triangulation for $M$ hyperelliptic of genus $g \geq 3$ (indeed, for $g \geq 1$). Call this the fundamental triangulation of $M$.

$\hat{\Gamma}$ adopts a fat graph structure from $M$, and one sees that the cyclic ordering on the vertices of $\hat{\Gamma}$ is as in Figure 2 for the upper (outer) vertices, and is reversed for the lower (inner) vertices.

Notice that if two triangles are neighbors in a triangulation across an edge, $e$, then removing $e$ from the edge set replaces these two triangles with one quadrilateral. Beginning from the fundamental triangulation which has $4g + 4$ faces, we will erase 8 edges and join 16 triangles to form 8 quadrilaterals. The new polyhedron will have $4g - 4$ faces.
The operation of erasing an edge is easier to envision on \( \hat{\Gamma} \). Erasing an edge collapses two vertices of \( \hat{\Gamma}_T \) along some edge. Such an operation is admissible (with respect to the hypotheses of Theorem 10) if it creates no loops or double edges. In the next section, we will also use the existence of some even quadratic differential \( \omega \) such that the interior of each face contains exactly one zero of \( \omega \). From this data, we may apply Theorem 10.

Figures 3, 4, and 5 illustrate which edges are erased for the genus 3, 5, and 10 surfaces. The eight gray lines will be collapsed.

Explicitly, the edges of \( \Gamma_T \) lying on the upper branch (as well as the lower branch) are parametrized by \( re^{2\pi i (k+1/2)/(2g+2)} \), with \( r > 0 \). Label these edges by \( k \), and likewise for the lower branch. On the upper branch, we remove the edges labelled by \( k = 1, 3, 5, 7 \) and on the lower branch, remove the edges labelled by \( k = 0, 2, 4, 6 \). Let \( \Gamma \) and \( \hat{\Gamma} \) be the new graphs created from \( \Gamma_T \) and \( \hat{\Gamma}_T \) by these deletions. Figures 6, 7, and 8 illustrate \( \hat{\Gamma} \) after applying this operation in \( g = 3, 5, 10 \).

These graphs have \( 4g - 4 \) vertices, no loops and no double edges. Thus this gives an appropriate cellular decomposition for \( M \).
7. Genus 3 and Larger

We now prove the structure theorem for $\pi_1(H^*)$. Recall that in Proposition 3 we showed that $\rho_*\pi_1(H^*)$ is a subgroup of $\ker(\pi_1(M^{[n]}) \to H_1(M))$. In fact for genus greater than 2, this is an equality:

**Theorem 11.** If $M$ is a hyperelliptic curve of genus $g > 2$, then $\rho_*\pi_1(H^*) = \ker(\pi_1(M^{[n]}) \to H_1(M))$.

This theorem, coupled with Corollary 38 allows us to circumvent discussion of generators for $\pi_1(H^*)$ when studying the monodromy. Applying Theorem 10, we get the corollary:

**Corollary 12.** The transpositions associated to the edge set of $\hat{\Gamma}$ generate the image $\rho_*\pi_1(H^*)$.

This observation is in fact the method of proof for the theorem.

**Proof of Theorem 11.** Let $K = \ker(\pi_1(M^{[n]}) \to H_1(M))$. We know by Proposition 3 that $\rho_*\pi_1(PH^*)$ is a subgroup of $K$. By Section 6, we may treat $M$ as a $(4g - 4)$-hedron topologically. We will find a basepoint $\omega \in H^*$ such that each face of $(M, \Gamma)$ contains one zero of $\omega$. Applying Theorem 10 we need to show that all generators of $K$ are in $\rho_*\pi_1(H^*)$. Thus we need to construct the transposition associated to each edge of $M$ (technically one of two possible transpositions).

Recall that $H^0(M, K^2) = H^0(M, K^2)^+ \oplus H^0(M, K^2)^-$, and $\dim(H^0(M, K^2)^-) > 0$ for $g \geq 3$. Let $\zeta = e^{2\pi i / 2g+2}$. Notice that $x \mapsto x^{2g+2} - 2^{2g+2}$ has one root in each triangle of the fundamental triangulation of $M$ (recall Figure 1). Consider the quadratic differential

$$\omega = (x - 2\zeta^2)(x - 2\zeta^4)(x - 2\zeta^6)(x - 2\zeta^8) \prod_{9 \leq j \leq 2g+2} (x - 2\zeta^j)(\frac{dx}{y})^2.$$ 

This has $4g - 4$ simple zeroes, and each face of $M$ contains exactly one zero. Figure 9 shows the zeroes of $\omega$ in the fundamental triangulation of $M$ for $g = 5$. The rays are labelled by $u, l,$ or $u/l$, for whether they represent edges in the upper or lower branch of $M$ or both.

Finally, we must show that transpositions across the edges are contained in $\pi_1(H^*)$. The remainder of this section is a string of lemmas dealing with this issue. □
The reader should be warned that the remainder of this section is quite computational. One might prefer to skip to the last paragraph of this section on first reading.

Let \( \kappa = \sqrt{2g+2} - 1 > 0 \). The roots of \( \omega \) are all of the form \((x, y) = (2\zeta^j, \pm \kappa)\).

For the remainder of this section, we fix \( \omega, \kappa, \zeta \) as above. As our convention, we will assume that the “upper branch” of \( M \) contains the points \((2\zeta^j, (-1)^j \kappa)\).

Lemma 13. Fix \( j, 1 \leq j \leq 2g + 1 \). The pair of points \((x, y) = (2\zeta^j, \pm \kappa)\) are neighbors across some edge \( e_\alpha \). Their transposition is an element of \( \rho \ast \pi_1(H^s) \).

Proof. The two faces containing \( x = \zeta^j \) are parametrized by complex numbers, \( x \), with argument \( \frac{j-1/2}{2g+2} < \theta < \frac{j+1/2}{2g+2} \). Call their union \( D \). One may use \( y \) as a coordinate on the interior of \( D \). In this coordinate, \( D = \{ y/y^2 \not\in \mathbb{R}_{<1} \} \), as in Figure 10.

Let \( \omega(t) = \frac{\omega}{x-2\zeta^j}(x-\alpha(t)) \), where \( x = \alpha(t) \) is an arc in \( D \) connecting \( y = +\kappa \) to \( y = -\kappa \). \( [\omega(t)] \in \rho \ast \pi_1(PH^s) \) is the transposition. As \( \alpha \) changes, the points trace a pair of paths symmetric about \( y = 0 \). One may choose \( \alpha \) explicitly as:

\[
\alpha(t) = 2\zeta^j \left( \frac{1-\varepsilon}{2} + \frac{1+\varepsilon}{2} \cos(t) + i \sin(t) \right)
\]

for any small enough \( \varepsilon \) such that the range of \( \alpha \) is in \( D \). This gives the curve from Figure 10. \( \square \)
Lemma 14.

(1) Let $\lambda = \pm \kappa$. If $a = (2\zeta^j, \lambda)$ and $b = (2\zeta^{j+1}, -\lambda)$ are roots of $\omega$, then the transposition of $a$ and $b$ is in $\rho_*\pi_1(H^s)$.

(2) Let $\lambda = \pm \kappa$. If $a = (2\zeta^j, \lambda)$ and $b = (2\zeta^{j+2}, \lambda)$ are roots of $\omega$ and $(2\zeta^{j+1}, \lambda)$ is not a root, then the transposition of $a$ and $b$ is in $\rho_*\pi_1(H^s)$.

Proof. We prove (1). (2) is identical. The points $a$ and $b$ are neighbors in $M$.

Examine all curves:

$$\omega_\varepsilon(t) = \frac{\omega}{x - 2\zeta^j}(x - 2 + \frac{1}{2} \varepsilon \sin(\pi t) \sin(2\pi t)) + \varepsilon t(1 - t) y \frac{dx}{y}^2,$$

for $\varepsilon \in \mathbb{C}$. For small enough $\varepsilon$, the trajectories of all points other than $2\zeta^j$, and $2\zeta^{j+1}$ are constrained to their faces. If we choose $\varepsilon$ such that

$$\left. \frac{\varepsilon y \frac{dx}{y}^2}{x - 2\zeta^j} \right|_{(x,y) = (2\zeta^{j+1}, -\lambda)}$$

is a small positive multiple of $i\zeta^{j+1}$, then $[\omega(t)]$ is the desired transposition.

□

Lemma 15. Assume that $a = (2\zeta^j, \kappa)$ and $b = (2\zeta^{j+2}, -\kappa)$ are roots of $\omega$ and $(2\zeta^{j+1}, \pm \kappa)$ are not a roots. Furthermore, let $e_\alpha$ be the edge of $\Gamma$ containing the branch point $\zeta^{j+1}$. Then $a$ and $b$ are neighbors across $e_\alpha$, and the transposition associated to $e_\alpha$ is in $\rho_*\pi_1(H^s)$.

Proof. If we let $\omega_\alpha(t) = \frac{\omega}{x - 2\zeta^j}(x - \alpha(t))$, where $\alpha$ is a curve turning clockwise once around $\zeta^{j+1}$, then conjugating the curve from Lemma 14 (2) by $[\omega_\alpha]$, we get the desired result.

□

All edges of $\bar{\Gamma}$ are of the types described in the above lemmas, so we are done.

All of the constructions from this section pass to general hyperelliptic curves as follows:

Let $y^2 = f(x)$ be any hyperelliptic curve of genus $g > 2$. Without loss of generality, $\deg(f) = 2g + 2$, and $f(0) \neq 0$. Let $\omega \in H^s$. Choosing non-intersecting paths from 0 to $\langle \omega \rangle$, and 0 to $\infty$, which alternate in the appropriate way, we may again construct a fundamental triangulation of $M$, and consequently construct $\Gamma$ on $M$. The constructions of the transpositions depended only on curves in $(\mathbb{P}^1)^{2g-2}$ with small deformations proportional to $y \frac{dx}{y}^2$. 
8. Homology of $S$ via $\tilde{\Gamma}$

In this section, we intend to study the cellular homology of $S$ via $\tilde{\Gamma}$. We focus also on the $\tau = -1$ part of the homology, as it is related to the homology of $\text{Prym}(S, M)$. Let $X$ be the ramification divisor of $S \to M$. As such, $X$ is a fixed point set for $\tau$ and each face of $\Gamma$ contains exactly one point of $X$. Choose some realization of $\tilde{\Gamma}$ in $M$ such that the vertices of $\tilde{\Gamma}$ are $X$, each edge of $\tilde{\Gamma}$ crosses $\Gamma$ only once, and this crossing is through its corresponding edge of $\Gamma$.

We continue to use $\omega$ as a basepoint for $H^*$ and let $S = \sqrt{\omega}$. Recall that on $M$ we have the graph $\Gamma$ which makes $M$ a polyhedron. The ramification points of $S \to M$ lie away from $\Gamma$, so that we may lift to $\bar{\Gamma}$, a graph on $S$. Every face, $f$, of $\Gamma$ contains exactly one ramification point. Thus if $f$ is a triangle, then its lift, $\bar{f}$, is a hexagon. Each pair of opposite edges of $\bar{f}$ lies above a single edge of $f$. Likewise quadrilaterals lift to octagons with opposite edges identified under $S \to M$. Opposite vertices are also identified.

Thus $\bar{\Gamma}$ has the same face set as $\Gamma$ with the edge and vertex sets doubled. As a check, the Euler characteristic is $2(4) - 2(6g - 2) + (4g - 4) = 8 - 8g = 2 - 2(4g - 3)$. This agrees with the Riemann-Hurwitz formula, which says that the genus of $S$ is $\bar{g} = 4g - 3$.

We may also study $\tilde{\Gamma}$, the dual to $\bar{\Gamma}$ on $S$, or equivalently, the lift of $\tilde{\Gamma}$. Its 1-skeleton has the same vertex set as $\tilde{\Gamma}$ and all edges doubled. We use the cell structure of $\tilde{\Gamma}$ as a fat graph, or cellular decomposition of $S$, as our cell complex $C_\bullet$ for $S$.

$$C_\bullet = (0 \to \mathbb{Z} \bar{F} \to \mathbb{Z} \bar{E} \to \mathbb{Z} X \to 0).$$

Notice that the differential $\partial$ of $C_\bullet$ is the standard boundary operator. Since $\tau$ is a cellular diffeomorphism, $\partial \tau = \tau \partial$.

9. The Graph $\Gamma$

In this section, we restrict our attention to the case $g = 10$ when drawing pictures, though all arguments go through for any $g > 2$. Herein, we draw the graph $\Gamma$. In Section 13 we will use this to construct $\tilde{\Gamma}$.

Recall that the fat graph structure on $\tilde{\Gamma}_T$ agrees with the planar representation in Figure 2 for the upper vertices, and is opposite for the lower vertices. Once one contracts the appropriate edges, one finds the cyclic orientations for the vertices in Figures 6, 7, and 8 are still clockwise for the upper vertices and counter-clockwise...
for the lower vertices. From this, one can easily draw the boundaries of the faces by following the directed cycles which pass through vertices by entering along one half-edge and leaving along the “next” half edge. There are four such cycles, one for each face of $\hat{\Gamma}$.

The graph $\Gamma$ has four vertices labelled $0^\pm$ and $\infty^\pm$. There are $4g - 4$ edges with multiplicity. To find the neighbors of, for example, $0^+$, one need only draw a small loop $\gamma(t) = \frac{1}{2}e^{it}, t \in [0, \pi]$, around 0 and enumerate the edges crossed by the two curves $c^{-1}\gamma \subset S$. Alternatively, one could find two consecutive edges of $0^+$ in $\hat{\Gamma}$, and find which unique cycle from the previous paragraph follows these edges. Doing this, one represent $\Gamma$ as in Figure 11. In this figure, we’ve restricted to $g = 10$, and labelled each multiple edge of $\Gamma$ by its dual subgraph in $\hat{\Gamma}$. We should point out that in the special case of $g = 3$, the only neighbor of $0^-$ is $0^+$ as in Figure 12.

![Figure 11. $\Gamma$ for $g = 10$.](image)

10. ORIENTATION

Since $\partial(0^+ + 0^- + \infty^+ + \infty^-) = 0$, there exists a unique orientation for each edge such that in $\Gamma$, $-\partial\infty^\pm$ and $\partial0^-$ all are positive sums of edges. Up to a global choice of sign, this orientation is as in Figure 13. Since $\partial^2 = 0$ the consecutive segments $\partial f$ should be joined head to tail.

We orient the edges of $\Gamma$ so that if $e \in E$ corresponds to $e_\Gamma$ as an edge of $\Gamma$, and $e_{\hat{\Gamma}}$ as an edge of $\hat{\Gamma}$ (geometric realizations of $e$), then the intersection pairing on $M$ gives:

$$e_{\hat{\Gamma}} \cdot e_\Gamma = +1.$$
Physically, this means that edges of $\Gamma$ travel from 0 to $\infty$, and “zig-zag” between $0^+$ and $0^-$.

11. The Prym Variety

In this section, we set out to convert the Prym variety into entirely combinatorial data, ultimately proving Theorem 10. A basis for $H_1(S, \mathbb{Z})$ extends to an $\mathbb{R}$-basis for $H^0(S, K)^*$, as it is a full-rank sublattice. Since we are interested in topological data (monodromy), we may ignore holomorphic structure and use the identification

$$\text{Jac}(S) = H_1(S, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}/H_1(S, \mathbb{Z}).$$

Recall that $\hat{\Gamma}$ is embedded in $M$, and is the combinatorial data $(X, E, F)$ with vertices $X$, the ramification divisor. We lift the orientation of the previous section from $\hat{\Gamma}$ to $\tilde{\Gamma}$. Any edge, $e$, of $\hat{\Gamma}$ lifts as a set to a curve $\tilde{e} = c^{-1}(e)$ in $S$ such for
either choice of orientation on $\vec{e}$, $\tau\vec{e} = -\vec{e}$. We know also that $\vec{e}$ is a signed sum of elements of $\bar{\Gamma}$.

Again pick some edge $e \in E$. This lifts as an oriented curve to some $e_1 + e_2 \in \bar{E}$. $\pm (e_1 - e_2)$ are the loops which lie (as sets) over $e$ and on which $\tau$ acts by $-1$.

Let $\psi : \mathbb{R}E \to \mathbb{R}\bar{E}$ be any map such that for $e \in E$, $\psi(e) = \pm (e_1 - e_2)$, where $c(e_1) = c(e_2) = e$, $\tau e_1 = e_2$. There are $2^{\left|G\right|}$ such choices of maps. Any such $\psi$ gives a map $\psi : \mathbb{R}E \to (1 - \tau)\mathbb{R}\bar{E} = (\mathbb{R}\bar{E})^-$. Also choose a map $\Psi : \mathbb{R}F \to \mathbb{R}\bar{E}$ by choosing $f_1, f_2 \in \bar{F}$ above each $f \in F$ with $\tau f_1 = f_2$, and letting $\Psi(f) = \partial(f_1 - f_2)$. We will see that $\Psi \neq \psi \circ \partial$, and that $\Psi$ actually encodes data relevant to the double covering map. Since $\psi$ is an injection, and the image of $\Psi$ lies in the image of $\psi$, we will sometimes use $\psi$ to denote $\Psi^{-1} \psi : \mathbb{R}F \to \mathbb{R}E$.

In the next section, we will construct a specific such pair $(\psi, \Psi)$, but for now we treat general $(\psi, \Psi)$. The following theorem uses the simplicial structure of $S$ to construct the $\tau = -1$ part of $\text{Jac}(S)$. The point of the theorem is that the Jacobian is spanned by $\bar{E}$. Inside this set is $\psi E$ which is generated by elements of the form $e_1 - \tau e_1$. In the quotient $\tau = -1$, The element $\frac{1}{2}(e_1 - \tau e_1) = e_1$ is an honest edge, and the point of the theorem is that these elements span (over $\mathbb{R}$) the Prym variety.

**Theorem 16.** If $E$ and $F$ are the edge and face sets of $\tilde{\Gamma}$, and $(\psi, \Psi)$ is a map as above, then topologically, $(\psi, \Psi)$ induces a homeomorphism:

$$\text{Prym}(S, M) \cong \mathbb{R}E / (\mathbb{R}F + \frac{1}{2}Z\bar{E}).$$

**Remark 17.** One employs the isomorphism as follows. $(\psi, \Psi) : ZE \times ZF \to \bar{E}$, which descends to an isomorphism under the identification

$$\text{Prym} \cong (\mathbb{R}\bar{E}/\mathbb{R}\bar{F} + \mathbb{Z}\bar{E})^-$$

**Proof.** Let $K = \ker(\mathbb{R}\bar{E} \to \mathbb{R}X)$. First we show the identification of Remark 17

$$\text{Prym} \cong \text{Jac}(S)^-$$

$$\cong [H_1(S, \mathbb{Z}) \otimes \mathbb{R} / H_1(S, \mathbb{Z})]^-$$

$$\cong [(\ker(\mathbb{R}\bar{E} \to \mathbb{R}X) / \partial \bar{\mathbb{R}}) / (\ker(\mathbb{Z}\bar{E} \to \mathbb{Z}X) / \partial \mathbb{Z}\bar{F})]^-$$

$$\cong [(K / \partial \mathbb{R}\bar{F}) / (\mathbb{Z}\bar{E} \cap K / \partial \mathbb{Z}\bar{F})]^-$$

$$\cong [K / (\partial \mathbb{R}\bar{F} + \mathbb{Z}\bar{E} \cap K)]^-$$
All of these maps descend from the identity map $\bar{R}\bar{E} \to \bar{R}\bar{E}$, thus are natural. Now assume that $e \in \bar{R}\bar{E}$ such that $e + \tau e \in \partial\bar{R}\bar{F} + \bar{Z}\bar{E}$. Then

$$(1 + \tau)\partial e = \partial(1 + \tau)e \in (\partial^2\bar{R}\bar{F} + \partial\bar{Z}\bar{E})^+ \subset \bar{R}X^+ = 0.$$ 

Thus $\partial e \in (\bar{R}X)^+ = 0$ by Section 8 and $e \in K$. Therefore,

$$[K/(\partial\bar{R}\bar{F} + \bar{Z}\bar{E} \cap K)]^- \cong [\bar{R}E/(\partial\bar{R}\bar{F} + \bar{Z}\bar{E} \cap K)]^-$$

There is a map

$$[\bar{R}\bar{E}/(\partial\bar{R}\bar{F} + \bar{Z}\bar{E} \cap K)]^- \to [\bar{R}\bar{E}/(\partial\bar{R}\bar{F} + \bar{Z}\bar{E})]^-.$$ 

The identification will be complete if we show that this map is injective. Assume that $e \in \partial\bar{R}\bar{F} + \bar{Z}\bar{E}$, and that $e + \tau e \in \partial\bar{R}\bar{F} + \bar{Z}\bar{E} \cap K = (\partial\bar{R}\bar{F} + \bar{Z}\bar{E}) \cap K$. Thus $e + \tau e \in K$, and $e - \tau e \in \bar{R}E^+ \subset K$. So $e \in K$. Thus $e \equiv 0 \mod (\partial\bar{R}\bar{F} + \bar{Z}\bar{E}) \cap K$, and the map is injective.

The next step of the proof is to show that the map $(\psi, \Psi) : \bar{Z}E \times \bar{Z}F \to \bar{Z}\bar{E}$ induces an isomorphism:

$$[\bar{R}\bar{E}/(\partial\bar{R}\bar{F} + \bar{Z}\bar{E})]^- \cong \bar{R}E/(\partial\bar{R}F + \frac{1}{2}\bar{Z}E).$$

First of all, the map is well-defined, since $\partial F + \frac{1}{2}\bar{Z}E \subset \ker(1 + \tau)$. It is injective, because $\bar{R}E \cap (1 + \tau)e(\bar{R}\bar{F} + \bar{Z}\bar{E}) \subset \bar{R}F + \frac{1}{2}\bar{Z}E$. If $e \in \bar{R}E$, then $\frac{1}{2}(1 - \tau)e \in \bar{R}E$, and since $\frac{1}{2}(1 + \tau)e \in \bar{R}E^+$, $e \equiv \frac{1}{2}(1 - \tau)e \mod \bar{R}\bar{F} + \bar{Z}\bar{E}$. Therefore it is surjective as well.

The identification is complete. However, it depends on the choice of $\psi$ (but not $\Psi$). In the next section, we make specific choices for these maps, in order to perform explicit computation.

12. The nature of $S \to M$

In this section we fix $\omega$ as in Section 7

$$\omega = (x - 2\zeta^2)(x - 2\zeta^4)(x - 2\zeta^6)(x - 2\zeta^8) \prod_{9 \leq j \leq 2g+2} (x - 2\zeta^j)(\frac{dx}{y})^2$$

$$= \frac{(x^{2g+2} - 2^{2g+2})}{(x - 2\zeta)(x - 2\zeta^3)(x - 2\zeta^5)(x - 2\zeta^7)}(\frac{dx}{y})^2$$

$$= p(x)(\frac{dx}{y})^2,$$

and study what happens when we lift cycles of $M$ to $S$. Specifically, there are many double covers of $M$ ramified at the points, $X$, which can be distinguished
by which closed loops lift to connected loops. For instance, a double cover of a torus ramified at two points is uniquely determined by a branch cut between the two points. However, the same branch points may give different covers, and there are in fact four distinct double covers of a torus with two given ramification points. Thus we need to do a little work to see which double cover $S \to M$ is given by $\omega$, which is given by Theorem 20.

Lemma 18. If $|x| < 1$, then $p(x) = -r\zeta^{-16}e^{i\theta}$ for $r > 1$ and $-\frac{5\pi}{6} \leq \theta \leq \frac{5\pi}{6}$.

Proof. Clearly, $r > (2^{2g+2} - 1)/2^4 > 1$. Next one writes

$$p(x) = (z^{2g+2} - 2^{2g+2}) \cdot \frac{1}{(x - 2\zeta)} \cdot \frac{1}{(x - 2\zeta^3)} \cdot \frac{1}{(x - 2\zeta^5)}$$

and observes, for example, that the argument of $\frac{1}{(x - 2\zeta)}$ is in the interval $[\pi - \log \zeta - \frac{\pi}{6}, \pi - \log \zeta + \frac{\pi}{6}]$. □

Lemma 19. There exists some $\varepsilon > 0$ such that on $R_\varepsilon = \{(x, y) \in M \mid |x| < 1 + \varepsilon\}$, $\sqrt{p}$ is a two-valued function with image having two components.

Proof. By continuity and Lemma 18 for small $\varepsilon$, $p(R_\varepsilon)$ is a connected set which is never real negative. Thus $\sqrt{p(R_\varepsilon)}$ is a pair of disconnected regions. □

Theorem 20. Let $R = R_\varepsilon$ as above. Let $R_\varepsilon^c = M \setminus R_\varepsilon$. Then $S|_{R_\varepsilon} \to R_\varepsilon$ is a disconnected double cover, and $S$ can be constructed by gluing the unique double cover of the pair of disks $S|_{R_\varepsilon^c} \to R_\varepsilon^c$ appropriately to $S|_{R_\varepsilon}$.

Effectively what this theorem says is that as a four-fold covering $S \to M \to \mathbb{P}^1$, the first map is ramified outside $R_\varepsilon$ and the second is ramified inside $R_\varepsilon$.

Proof. On $R_\varepsilon$, one may choose a well-defined, holomorphic function $\sqrt{p}$. Then $\sqrt{|p|dx dy}$ is a holomorphic one-form on $R_\varepsilon$. The graph of this one-form is one of the components of $S|_{R_\varepsilon}$.

13. The map $(\psi, \Psi)$

We set out in this section to construct the maps $\psi$ and $\Psi$ for general $g > 2$. First we will graph $\Gamma$ as in Figures 11 and 12. Recall the embedding of $\Gamma$ and $\omega^{-1}(0)$ in $M$ from Figure 14. To these markings on $M$, we add the set

$$B_\varepsilon = \{e^{it/2g+2} \mid t \in [0, 2] \cup [4, 6] \cup \bigcup_{4 \leq j \leq g} [2j, 2j + 1]\},$$
as in Figure 14.

Figure 14. $B_c \subset M$ for $g = 10$.

$B_c$ is a set of $2g - 2$ arcs which may be used as branch cuts for the map $c : S \to M$. Thus $c^{-1}(M \setminus B_c)$ is a disjoint union, $S_1 \coprod S_2$, with $\tau S_1 = S_2$ by Theorem 20. Let $0^+, \infty^+ \in S_j$ over $0^\pm, \infty^\pm$. Now let $\Psi(f) = \partial(f_1 - f_2)$ define $\Psi$ on $F$ and extend linearly. Clearly, if $e\bar{e} = e$ and $e$ connects $f$ to $f'$, then $\bar{e}$ connects $f_j$ to $f'_k$ with $j = k$ if $e$ does not cross $B_c$ and $j \neq k$ if $e$ does cross $B_c$. From this observation, we may costruct $\tilde{\Gamma}$ from $\Gamma$. We first highlight in $\tilde{\Gamma}$ the edges that cross $B_c$ in Figure 15.

We observe that $\Psi \neq \psi \circ \partial$ precisely due to the set $B_c$. Working again by example, let $\bar{e}$ be some edge separating $\infty^+_1$ from $0^+_2$. Then $\psi \partial(\infty^+_1 + 0^+_2)$ is supported away from $e$. However, since $e \in B_c$, $0^+_1$ and $\infty^+_1$ are not neighbors, $\Psi(\infty^+_1 0^+_2)$ contains some component, $\pm 2e$.

In our graph of $\Gamma$, we will label exactly one element of each pair $\{e, \tau e\}$, where it is understood that $\tau$ gives a symmetry across the horizontal midline. We also choose orientations on the cycles $\psi(e)$ so that $-\Psi(\infty^\pm)$ are positive sums, and so that the cycles corresponding to edges separating $0^\pm$ appear with positive coefficient in $\Psi(0^-)$. This is as consistent with the conventions of Section 11 as possible.

As an aside, notice that each vertex of $\tilde{\Gamma}$ (ramification point) meets $B_c$ at exactly one edge. This is to be expected for a double cover, and will play an implicit role in Proposition 24.

In our graph of $\tilde{\Gamma}$, which for genus 10 is Figure 16, we will label exactly one element of each pair $\{\bar{e}, \tau \bar{e}\}$, where it is understood that $\tau$ gives a symmetry across the horizontal midline. If for some edge $e \in E$, we have drawn $e_1$ with $ce_1 = e$, then we let $e_2 = \tau e_1$, and define $\psi e = e_1 - e_2$. This extends to give the map $\psi$. Notice that the edges of $B_c$ (Figure 15) are exactly the edges which cross the $\tau$-midline in Figure 16.
We graph the values of $\Psi$ for the four faces in Figures 17-20.

Algebraically, we may label our edges counterclockwise from the real axis in sets upper branch ($u_j$), lower branch ($l_j$) or branch locus ($b_j$), depending on where
their realizations in $\Gamma$ lie. We illustrate this (again in genus 10) in Figure 21, from which the pattern can be deduced. From this we explicitly write $\Psi$ on the faces. Recall that $\Psi(\mathbb{Z}F) \subset \psi(\mathbb{Z}E)$, and $\psi$ is injective, so we will also write $\Psi$ for $\psi^{-1} \circ \Psi : \mathbb{Z}F \to \mathbb{Z}E$.

**Proposition 21.** $\mathbb{Z}E$ is spanned by elements of the form

- $u_j$, $1 \leq j \leq 2g - 2$
- $l_j$, $1 \leq j \leq 2g - 2$
- $b_j$, $1 \leq j \leq 2g + 2$

and we may choose $\psi$ so that a basis for $\mathbb{Z}F \xrightarrow{\psi} \mathbb{Z}E$ is given by:

$$
-\Psi(\infty^-) = \sum_{j=1}^{2g-2} l_j
-\Psi(\infty^+) = \sum_{j=1}^{2g-2} u_j
-\Psi(0^+) = l_1 + l_3 - u_2 - u_4 + \sum_{j=1}^{g+1} (u_{2j-1} - l_{2j}) + \sum_{k=1}^{2g+2} b_k
-\Psi(0^-) = \sum_{j=3}^{g+1} (u_{2j} - l_{2j-1}) + \sum_{k=1}^{2g+2} b_k.
$$

These will represent our standard choice for $(\psi, \Psi)$. We collect this data notationally in the following lemma:

**Lemma 22.** The boundaries of the faces of $\tilde{\Gamma}$ are cycles joined head-to-tail as follows. Brackets have been used to signify “important” subwords, and carriage returns are placed between the square edges and triangle edges in $\partial \Omega^+_1$, as these have different word patterns. It is understood that these are cyclic words. For simplicity, for each edge, $e \in E$, we choose some lift $\tilde{e}$ such that $\psi(e) = \tilde{e} - \tau \tilde{e}$. However, we suppress the bar in our notation.
\[-\partial \infty^-_1 = l_1 l_2 l_3 \cdots l_{2g-2},\]
\[-\partial \infty^+_1 = u_1 u_2 u_3 \cdots u_{2g-2},\]
\[\partial 0^-_i = [b_1 \cdots b_8] [b_9 (\tau l_5) b_{10} u_6] \cdots [b_{2g+1} (\tau l_{2g-3}) b_{2g+2} u_{2g-2}],\]
\[\partial 0^+_i = [(-b_1) (\tau u_1) (-b_2) (\tau l_1)] [(-b_3) u_2 (-b_4) l_2] [(-b_5) (\tau u_3) (-b_6) (\tau l_3)] [(-b_7) u_4 (-b_8) l_4]
\[\cdots [(-b_{2g+1}) (\tau u_{2g-3}) (-b_{2g+2}) l_{2g-2}].\]

14. INTERSECTION PAIRINGS

Theorem \[\text{[41]}\] requires the computation of the intersection pairings on \(S\) of the cycles \(\psi(e)\). For completeness, we include these computations in this section.

**Lemma 23.** Let \(f \in F\) be a face of \(\tilde{\Gamma}\), and let \(e', e\) be oriented edges of \(\tilde{\Gamma}\) such that \(\partial e = x - y\) and \(\partial e' = y - z\). Assume also that \((e', y, e)\) is an interval on \(\partial f\). Then \(e - \tau e\) and \(e' - \tau e'\), viewed as loops in \(S\) have intersection pairing

\[(e - \tau e) \cdot (e' - \tau e') = +1.\]

**Proof.** Notice that \(x \neq y\) and \(z \neq y\) by construction, and that \(e\) and \(e'\) only intersect at \(y \in S\). If we look at a neighborhood of \(y\) in \(S\), We arrive at Figure 22. The dotted lines indicate other faces/edges. From this figure, the result is clear. \[\square\]

Thus, for example, if we let \(f = \infty^+, e = -u_2, e' = -u_1\), so that \((u_2 + \tau u_2) \cdot (-u_1 + \tau u_1) = +1.\)
This lemma in conjunction with Lemma 22 gives the intersection numbers:

**Proposition 24.** Let $u_j$, $l_j$, and $b_j$ denote the cycles in $S$ labelled as in Figure 21. Notice that we are now using the symbols $e$ to denote $\psi(e) = e - \tau e$. Let $e$, $e'$ denote any two of these cycles.

1. $e \cdot e' = 0$ if $e$ and $e'$ do not share a vertex.
2. $u_j \cdot u_{j+1} = -1$
3. $l_j \cdot l_{j+1} = -1$
4. $b_j \cdot b_{j+1} = +1$ for $1 \leq j \leq 7$
5. $u_j \cdot b_{2j} = -u_j \cdot b_{2j+1} - u_{j+1} \cdot b_{2j} = u_{j+1} \cdot b_{2j+1} = +1$, for $j = 1, 3$
6. $u_j \cdot b_{2j} = -u_j \cdot b_{2j+1} - u_{j+1} \cdot b_{2j} = u_{j+1} \cdot b_{2j+1} = -1$, for $j = 2, 4$
7. $l_{j-1} \cdot b_{2j-1} = -l_{j-1} \cdot b_{2j} = -l_j \cdot b_{2j-1} = l_j \cdot b_{2j} = -1$, for $j = 1, 3$
8. $l_{j-1} \cdot b_{2j-1} = -l_{j-1} \cdot b_{2j} = -l_j \cdot b_{2j-1} = l_j \cdot b_{2j} = +1$, for $j = 2, 4$
9. $l_j = b_{j+5} = -l_{j+1} = b_{j+5} = -1$, for $4 \leq j \leq 2g - 3$
10. $u_j = b_{j+5} = -u_{j+1} = b_{j+5} = +1$, for $5 \leq j \leq 2g - 2$

**Proof.** (1) follows from noticing that $\hat{\Gamma}$ may be realized inside $M$, so if $e$ and $e'$ do not intersect on $\hat{\Gamma}$, then they lift to two disjoint sets on $S$.

If $e$ and $e'$ are consecutive edges of a word in Lemma 22 then $(e - \tau e) \cdot (e' - \tau e') = +1$ by Lemma 23. This proves the result for all cases except if $e$ and $e'$ are opposite edges on a four-valent vertex. However, if $ee''$ is a consecutive pair of edges on $f_a$ and $(e'')^{-1}e'$ is a pair of edges on $f_b$, then we may apply the argument of Lemma 23 to $f_a \cup_{e''} f_b$, which has consecutive edges $ee'$, to arrive at the result. □

15. **The classical Burau representation**

In this section, we recall the classical Burau representation, and discuss a specialization, which may be helpful later as an analogy. One may find a detailed description in [3].
Throughout this section, if $R$ is any Riemann surface, then let $\sigma \mapsto \bar{\sigma}$ be the Hurewicz homomorphism, $\pi_1(R) \to H_1(R, \mathbb{Z})$. Also use the same notation for the map $\pi_1(R^{[n]}) \to H_1(R, \mathbb{Z})$ defined implicitly in Proposition.

We construct the classical Burau representation. Let $B_n = \pi_1(D^{[n]})$ be the braid group on the disk, $D$, $X \in D^{[n]}$. Let $w$ be the winding number function $w: H_1(D \setminus X, \mathbb{Z}) \to \mathbb{Z}$ such that $w(\partial x) = +1$ for all $x \in X$. $w$ induces a covering space $\tilde{D} \to D \setminus X$ by the map

$$
\pi_1(D \setminus X) \to \mathbb{Z} \\
\sigma \mapsto w(\bar{\sigma}).
$$

This space is equipped with a vertical translation operator $t$. This is the map induced by moving the basepoint “up” along a curve of winding number one, thus increasing the winding number of all points in $\tilde{D}$ by one. Let $\Lambda = \mathbb{Z}[t, t^{-1}]$. $H_1(\tilde{D}, \mathbb{Z})$ is a finite rank $\Lambda$-module, and gives a $\Lambda$-representation $B_n \to \text{Aut}_\Lambda(H_1(\tilde{D}, \mathbb{Z}))$. This is the Burau representation.

Explicitly, $\pi_1(D^{[n]})$ is generated by transpositions $\sigma_1, \ldots, \sigma_{n-1}$, and the Burau representation is an action on $\Lambda X$, given by

$$
\sigma_j x_j = (1 - t)x_j + tx_{j+1} \\
\sigma_j x_{j+1} = x_j \\
\sigma_j x_k = x_k \text{ otherwise}
$$

One finds these relations by letting $x_j$ represent a curve from $\partial D$ to $\partial D$ with only $x_j$ in its interior, lifting to $\tilde{D}$, and rescaling appropriately by some power of $t$.

We consider two specializations, which we will generalize to surfaces. Consider the quotient $t^k = 1$. This is equivalent to constructing the $k$-fold cover $\tilde{D}_k \to D \setminus X$ given by

$$
w \mod k: \pi_1(D \setminus X) \to \mathbb{Z}/k\mathbb{Z}.
$$

Observe that if $k = 2$, and we let $\xi_j = x_j + x_{j+1}$, then $\xi_j$ is homotopy equivalent to a loop in $\tilde{D}_2$ around the points $x_j, x_{j+1} \in X$. Morally these are the cycles $e - \tau e$. On $\tilde{D}_2$, the nontrivial intersection pairings are

$$
\xi_j \cdot \xi_{j+1} = -1.
$$

1That is, defined only for the case $n = 4g - 4$. 
One can compute directly from the definitions above that

\[ \sigma_j(\xi_k) = \xi_k - (\xi_k \cdot \xi_j)\xi_j. \]

Compare this result with Theorems 1 and 41. Indeed, one can apply exactly the argument from Theorem 41 to prove this as well.

Now consider the quotient of the Burau representation by \(1 + t + t^2 + \ldots + t^{k-1} = 0\). To realize this we construct the cover \(\tilde{D}_k \rightarrow D\) which is the \(n\)-point compactification of \(\tilde{D}_k\) by the ramification points, \(X\). This is distinguished from \(\tilde{D}_k\) by the fact that the loop around the point \(x_j \in X\), which is \((1 + t + t^2 + \ldots + t^{k-1})x_j\) is now contractible to 0. We call this the compact Burau representation associated to the quotient \(\Lambda/t^k - 1\), to be consistent with the following section. This has no counterpart for the Burau representation, since

\[ \sum_{i \in \mathbb{Z}} t^i \notin \Lambda. \]

We will mainly be interested in the case \(k = 2\), so the quotients \(t^2 = 1\) and \(t = -1\).

Observe that we can construct these quotients of the Burau representations as monodromy representations of bundles. Let

\[ \tilde{D}_k = \{(X, x, y) \in D^{[n]} \times D \times \mathbb{C}| y^k = \prod_{x_j \in X} (x - x_j)\}, \]

and let \(D_k = \{(X, x, y) \in \tilde{D}| y \neq 0\}\). Then the fibers of these bundles are \(D_k\) and \(\tilde{D}_k\), and the monodromy representations are the quotients discussed above.

16. The surface Burau representation

In this section, we construct a generalization of the classical Burau representation. Later we show that, up to twisting by \(\tau\), the homology of \(\mathcal{M}\) is related to the compact form of this Burau representation via pullback from \(\mathcal{M}^{[4g-4]}\). This will tell us the monodromy representation of \(\pi_1(H^s)\).

There are three obstructions to generalizing the Burau representation to a general Riemann surface \(M\) of positive genus: \(w\) is well-defined only up to \(\mathbb{Z}/n\mathbb{Z}\), there is no \textit{a priori} choice for \(w\) on the complement of \(\ker(H_1(\mathcal{M} \setminus X, \mathbb{Z}) \rightarrow H_1(\mathcal{M}, \mathbb{Z}))\), and monodromy would be determined only up to automorphisms of the cover (as there is no boundary to use to fix the identification).

Let \(M^{[n]+1} = \{(m, X) \in M \times M^{[n]}| m \notin X\}\). This is consistent with the notation of Section 3. For \((m, X) \in M^{[n]+1}\), some choice of winding number, \(w\), and some
positive integer $k|n$, we will construct a $k$-fold covering of $M \setminus X$ with a marked point. Let $(B', m')$ be the based covering space of $(M \setminus X, m)$ associated to the map $\pi_1(M \setminus X) \to \mathbb{Z}/k\mathbb{Z}$ given by $\sigma \mapsto w(\bar{\sigma}) \mod k$. We need the restriction $k|n$ since the “interior” of $\bar{\sigma}$ is not a well-defined notion.\footnote{$w_{\text{int}}(\bar{\sigma}) - w_{\text{ext}}(\bar{\sigma}) = \pm n$.} We may complete this space by adding $X$ as ramification points. Call this new space $B^k_{m_Xw}$. It is topologically a surface of genus

$$\tilde{g} = k(g - 1) + 1 + \frac{(k - 1)n}{2}.$$ 

This space has a vertical translation operator, $t$, so its homology is a module. Let $\tilde{B}^k_{m_Xw} = B^k_{m_Xw} \setminus X$, so that $B^k_{m_Xw}$ is the compact form (compactification by $X$) of $\tilde{B}^k_{m_Xw}$.

**Definition 25.** $\mathcal{W} \to M^{[n]+1}$:

Let $W^k_X$ be the set of winding number functions on $M \setminus X$ modulo $k$:

$$W^k_X = \{ w \in H_1(M \setminus X, \mathbb{Z}/k\mathbb{Z}) | w(\partial x) = +1 \forall x \in X \}.$$ 

This is a set with $k^{2g}$ points. Notice that this is affine, modelled on the lattice $H_1(M, \mathbb{Z}/k\mathbb{Z})^*$. Let $X \in M^{[n]}$, and choose $U \subset M$ contractible open such that $X \subset U$. Then $U^{[n]}$ gives a neighborhood of $X$ in $M^{[n]}$. A winding number function on $M \setminus X$ is defined entirely by its restriction to $H_1(M \setminus U, \mathbb{Z})$. Thus $w$ is well-defined as an element of $W^k_Y$ for all $Y \in U^{[n]}$, giving natural isomorphisms $W^k_Y \cong W^k_X$. We have bundles $\tilde{\mathcal{W}}|_{U^{[n]}} \to U^{[n]}$ for all contractible open $U \subset M$ with fibers $W^k_Y$, and restriction to $U' \subset U$ gives a canonical isomorphism

$$(\tilde{\mathcal{W}}|_U)|_{U'} \cong \tilde{\mathcal{W}}|_{U'},$$

so this gives a well-defined locally constant bundle $\tilde{\mathcal{W}} \to M^{[n]}$. Let

$$\mathcal{W} = \tilde{\mathcal{W}} \times_{M^{[n]}} M^{[n]+1}.$$ 

**Definition 26.** $B \to \mathcal{W}$:

Let $B$ be the set

$$B = \prod B^k_{m_Xw}.$$ 

$B$ has a unique and universal structure as a bundle over $\mathcal{W}$:
Lemma 27. \( \mathcal{B} \) satisfies the following properties:

- \( \mathcal{B} \) is a topological bundle over \( \mathcal{W} \).
- If \( \mathcal{W}' \subset \mathcal{W} \) and \( \mathcal{B}' \to \mathcal{W}' \) is another bundle such that \( \mathcal{B}'|_{(m,X,w)} = B_{mXw}^k \) with section \( \mu \) such that \( \mu(m,X,w) \) lies over \( m \) (a section of marked points), then there exists a unique isomorphism \( \mathcal{B}' \simeq \mathcal{B}|_{\mathcal{W}'} \) preserving the marked point sections.

Proof. Fix \( (m,X,w) \in \mathcal{W} \). Let \( U,V \subset M \) contractible, \( U \cap V = \emptyset, X \subset U, m \in V \). This defines an open neighborhood of \( (m,X,w) \) in \( \mathcal{W} \), since \( \mathcal{W} \) is discrete over \( M[\mathfrak{n}]+1 \). The topology of \( \mathcal{W} \) is generated by such neighborhoods. For any \( m \in V \), \( w \) gives a well-defined unique marked double cover of \( M \setminus U \), and all are canonically isomorphic, as \( V \) is contractible. Now one uses the classical Burau construction to build the double cover on \( U \), and identify the boundary. Specifically, choose a smooth coordinate \( z : U \to \mathbb{C} \) and construct the cover

\[
\{(y,x) \in \mathbb{C} \times U | y^2 = \prod_{x_j \in X} (z - x_j) \} \to U[\mathfrak{n}] \times V.
\]

Since the order of \( X \) is even, the boundary of this cover is a disjoint union of circles on which monodromy of \( \pi_1(U[\mathfrak{n}]) \) acts trivially, so the gluing construction is well-defined.

We now need to show that under restriction in the open cover:

\[
(B|_{(V,U[\mathfrak{n}],w)})|_{(V',(U')[\mathfrak{n}],w)} = B|_{(V',(U')[\mathfrak{n}],w)}
\]

for \( V' \subset V \) and \( U' \subset U \), but this must be true, as on the overlap \( U \setminus U' \), each is just the disjoint union of two annuli: \( U \setminus U' \times \{0,1\} \times (U')[\mathfrak{n}] \times V' \), and elsewhere canonically isomorphic, given that we choose the same coordinate on \( U \) and \( U' \).

The first point is proved.

Now assume that there is some \( \mathcal{B}' \to \mathcal{W}' \), with marked point \( \mu \). There is a canonical map to \( \mathcal{B}|_{\mathcal{W}'} \) by the uniqueness of \( B_{mXw}^n \). We need to show this map is continuous. However, one can recover from \( \mathcal{W}' \) and \( \mu \) the geometry of \( \mathcal{B} \) via the exact construction that gave the first part of the lemma. Then the bundles are easily locally isomorphic (via the global map), so are isomorphic.

We set out to construct a bundle over \( M[\mathfrak{n}] \) whose monodromy generalizes the Burau representation. We have arrived at: \( \mathcal{B} = \prod B_{mXw}^n \to \mathcal{W} \to M[\mathfrak{n}]+1 \to M[\mathfrak{n}] \). This is a far cry from what we want, as it has a legion of fibers telling the extra data...
of winding numbers and marked points. Now we try to reduce this information. First we study the monodromy of $W \to M^{[n]+1}$.

**Proposition 28.** Let $\langle \cdot, \cdot \rangle$ denote the intersection pairing on $H_1(M)$.

- Every element of $\pi_1(M^{[n]+1}, (X, m))$ may be factored as $\sigma \sigma'$, for $\sigma \in \pi_1((M \setminus m)^n, X)$ and $\sigma' \in \pi_1(M \setminus X, m)$
- $\sigma'(w) = w$ for $\sigma' \in \pi_1(M \setminus X)$
- $\sigma(w) = w - \langle \bar{\sigma}, \cdot \rangle$ for $\sigma \in \pi_1((M \setminus m)^n)$

**Proof.** Applying Corollary 6, we see that $\pi_1((M \setminus m)^n) \times \pi_1(M \setminus X) \to \pi_1(M^{[n]+1})$.

If $\sigma' \in \pi_1(M \setminus X)$, then $\sigma'$ acts trivially on $W$, since $W = W' \times M^{[n]} M^{[n]+1}$.

Assume $\sigma \in \pi_1((M \setminus m)^n)$. Let $a_j, b_j$ be a symplectic basis of $H_1(M, \mathbb{Z})$ based at $x_1$ and missing $m$. Since $\pi_1(M \setminus X)$ acts trivially and $\sigma \to \bar{\sigma}$ is a group homomorphism, we may assume by Theorem 7 that $\sigma$ is one of the following:

- a transposition of $x_j$ and $x_k$ across some edge $e$
- $\sigma$ fixes $x_j$, $j > 1$, and $\sigma$ follows $a_j$ or $b_j$.

We study $\sigma(w) - w$:

$$\sigma(w)(\alpha) - w(\alpha) = w(\sigma^{-1} \alpha - \alpha).$$

Assume that $\sigma$ is a transposition around an “edge,” $e$ connecting $x_1$ to $x_k$. Then $\sigma^{-1} \alpha - \alpha$ is supported in a neighborhood of $e$, so

$$\sigma^{-1} \alpha - \alpha = a \partial x - b \partial y.$$ 

However, applying $\sigma$ (an involution) to both sides, we see

$$\alpha - \sigma \sigma^{-1} \alpha = -b \partial x + a \partial y,$$

so $a = b$, and

$$w(\sigma^{-1} \alpha - \alpha) = w(a \partial x - a \partial y) = a - a = 0.$$ 

The proposition follows in this case, since $\bar{\sigma} = e - e = 0$.

Now assume that $\sigma$ fixes $x_j$, for $j > 1$, and $x_1$ traces $a_k$. Again we study $\sigma^{-1} \alpha - \alpha$. This is zero for $\alpha = b_j$ for $j \neq k$, $\alpha = \partial x_j$, $j > 1$, and $\alpha = a_j$, since the supports are disjoint ($\sigma$ may be realized as an automorphism fixing all but a neighborhood of $a_j$). Also,

$$\sigma \partial x_1 = \partial \sigma x_1 = \partial x_1,$$
So $\sigma$ is trivial on all basis elements but $b_k$.

Figure 23 illustrates how $\sigma b_j$ acquires $\partial x_1$ as $x_1$ passes through $b_j$. Replacing $(a_j, b_j)$ by $(b_j, -a_j)$ shows the result for $\bar{\sigma} = b_j$ as well. Thus

$$\sigma w(\alpha) = w(\alpha) - \langle \bar{\sigma}, \alpha \rangle$$

for all generators $\sigma$ of $\pi_1((M \setminus m)^n)$.

Figure 23. $\sigma b_k$.

If $\sigma \in \pi_1(M \setminus X)$, then $\sigma : \mathcal{B}|_{(m, X, w)} \to \mathcal{B}|_{(m, X, w)}$ acts by vertical translation by $t^w(\bar{\sigma})$ (specifically, it preserves the components of $\mathcal{B} \to M^{[n]+1}$ since $\sigma$ moves the basepoint. If $\sigma \in \pi_1((M \setminus m)^n)$, then $\sigma$ acts trivially on $\mathcal{W}$ (preserves the components) only when $\bar{\sigma} \in kH_1(M \setminus X, \mathbb{Z})$. This is because $\sigma(w) = w + \langle \cdot, \bar{\sigma} \rangle$.

For now we fix $k$ and let $K = \{ \sigma \in \pi_1((M \setminus m)^n) | \bar{\sigma} \in \mathbb{Z}\partial m \}$. The group $K$ is the kernel of

$$\pi_1((M \setminus m)^n) \to H_1(M, \mathbb{Z}).$$

Then for any $w \in W_X$, $\sigma(w) = w$, and $\mathcal{B}$ gives a monodromy action of $K$ on $B = B_{m, X}^k$. $H_1(B, \mathbb{Z})$ is a $\Lambda_k$-module, and a $\Lambda_k$-representation of $K$. Furthermore, if $\sigma \in \pi_1(M \setminus X)$, then $\sigma$ acts by $t^w(\bar{\sigma})$ as it changes only the basepoint. Thus in fact, we get a map:

$$\ker(\pi_1(M^{[n]} \to H_1(M, \mathbb{Z})) \to \text{Aut}_{\Lambda_k} H_1(B, \mathbb{Z})/(t \cdot \text{Id}),$$

which we will call the Burau representation. This is not a linear representation, and it depends on the choice of $w$.

Notice that this is $\text{Aut}_{\Lambda_k}$ and not just $\text{Aut}_{\mathbb{Z}}$, since by Corollary \ref{t.Id} $\langle t \cdot \text{Id} \rangle$ is normal in the image of $\pi_1(M^{[n]+1}) \to \text{Aut}_{\mathbb{Z}}(H_1(B, \mathbb{Z}))$. \hfill \square
At this point, we begin to study structure of the Hitchin bundle. Letting \( H_1(\mathcal{M}) \) denote the bundle of integral first homology of the fibers of \( \mathcal{M} \to H^* \) we are studying the action of \( \pi_1(H^*) \) on \( H_1(\mathcal{M}) \). For a construction of \( \mathcal{H}(\cdot) \), see [6]. First, we construct the bundle \( \mathcal{S} \). We use this bundle to model \( H_1(\mathcal{M}) \) by Theorem 37. In Proposition 38, we show that \( H_1(\mathcal{S}) \) is closely akin to the Burau bundle and prove that \( \ker(\pi_1(H^*) \to \pi_1(M[4g-4])) \) acts almost trivially in monodromy (in fact by the group \( \{1, \tau\} \)). We finish by studying the monodromy via the combinatorial techniques of Section 13.

Henceforth, let \( T \) be the total space map, so that \( TK_M \) is the total space of the canonical bundle, and \( TK_M^2 \) the total space of the square of the canonical bundle. The map \( H^* \times M \to TK_M^2, (\omega, m) \mapsto \omega(m) \) is holomorphic. Consider the (holomorphic) map, \( H^* \times TK_M \to TK_M^2 \) defined by \( (\omega, m, q) \mapsto (m, \omega(m) - q^2) \).

We have a variety:

**Definition 29.** The space, \( \mathcal{S} \), is defined as
\[
\mathcal{S} = \{ (\omega, m, q) \in H^* \times TK_M | \omega(m) - q^2 = 0 \}.
\]

Notice that for \( \omega \in H^* \), \( \mathcal{S}|_{\{\omega\}} \cong S_\omega \), naturally. For any \( \omega \in H^* \), there is some neighborhood \( U \subset H^* \) of \( \omega \) such that \( \mathcal{S}|_U \cong U \times S_\omega \) as smooth manifolds. Thus \( \mathcal{S} \) is a smooth bundle \( \mathcal{S} \to H^* \) fibered by double covers of \( M \).

**Definition 30.** Let \( H_1(\mathcal{S}, \mathbb{R}) \to H^* \) be the real bundle defined fiberwise by:
\[
\mathcal{H}_1(\mathcal{S}, \mathbb{R})|_{\{\omega\}} = \mathcal{H}(\mathcal{S})|_\omega \otimes \mathbb{Z} \otimes \mathbb{R}.
\]

Continuous local sections descend from \( \mathcal{O}_R \otimes \mathcal{H}(\mathcal{S}) \). Let \( \text{Jac}(\mathcal{S}) \to H^* \) be the bundle defined fiberwise by:
\[
\text{Jac}(\mathcal{S}) = \mathcal{H}_1(\mathcal{S}, \mathbb{R})/\mathcal{H}(\mathcal{S}).
\]

18. SOME DIFFERENTIAL GEOMETRY

Throughout this section, we use \( H^1_{DR} \) to denote complexified deRham cohomology. We introduce a local coordinate system on \( \text{Jac}(\mathcal{S}) \) via integrals in order to return to a classical definition of the Jacobian as a space of integrals.

**Proposition 31.** If \( dx_1, \ldots, dx_\tilde{g} \in H^1_{DR}(S_\omega) \) with \( dx_1, \ldots, dx_\tilde{g}, d\tilde{x}_1, \ldots, d\tilde{x}_\tilde{g} \) linearly independent, then there exist contractible open \( U \) containing \( \omega \), and \( dz_1, \ldots, dz_\tilde{g} \in H^1_{DR}(S|_U) \), such that \( dz_j|_{S_\omega} = dx_j \) and \( dz_j|_{\eta}, dz_j|_{\eta} \) linearly independent.
Proof. Choose any such $U$ contractible and open. Then $H^1_{DR}(S|_U) \cong H^1_{DR}(S_\omega)$ by restriction, so there exist $dz_j$ such that $dz_j|_{S_\omega} = dx_j$. Let $\omega$ and $\eta$ be elements of $U$. Let $\alpha_\omega \in H_1(S_\omega, \mathbb{Z})$. Then $\alpha$ is homotopic to a curve $\alpha_\eta \in H_1(S_\eta, \mathbb{Z})$, and in fact, $U$ gives a natural isomorphism between the different groups $H_1(S_\eta, \mathbb{Z})$ for $\eta \in U$. Since the $dx_j$ are closed,

$$\int_{\alpha_\omega} dx_j = \int_{\alpha_\eta} dx_j,$$

so that the $dx_j$ give an isomorphism between groups $H^1_{DR}(S_\omega)$ and $H^1_{DR}(S_\eta)$. Specifically, the restrictions of the one-forms remain linearly independent. □

In fact we have the much stronger corollary, which is a family version of Abel-Jacobi (notice that this is not a holomorphic statement):

**Theorem 32.** If $U$ is contractible and open, then there exists $dx = (dx_1, \ldots, dx_3)$, $dx_j \in H^1_{DR}(S|_U)$ such that $\int dx$ gives an isomorphism:

$$\text{Jac}(S)|_U \to U \times \mathbb{C}^3 / \Lambda,$$

where $\Lambda = \{ \int_\alpha dx | \alpha \in H_1(S_\omega, \mathbb{Z}) \}$ for some (any) $\omega \in U$.

Proof. We need only show that if $\alpha \neq 0 \in H_1(S_\eta, \mathbb{Z}) \otimes \mathbb{R}$, then $\int_\alpha dx \neq 0$. However, if $\int_\alpha dx = 0$, then $\int_\alpha d\bar{x} = 0$, and the $dx_j, d\bar{x}_j$ form a basis for complexified deRham cohomology, which gives a contradiction. □

Given this, we can treat degree 0 divisors on $S_\omega$ as elements of $\text{Jac}(S)|_U$. If $D \in \text{Div}^0(S_\omega)$, then

$$\int_D dx = \int_{\alpha_D} dx \mod \Lambda$$

for some $\alpha_D \in H_1(S_\omega, \mathbb{Z}) \otimes \mathbb{R}$. However, since $dx_j, d\bar{x}_j$ span $H^1_{DR}(S_\omega)$, $D \mapsto \alpha_D$ is independent of coordinate $dx$ and is well-defined as a map from $\text{Div}^0(S_\omega)$ to $\text{Jac}(S)|_{U}$.

**Theorem 33.** If $U$ is contractible, $dx$ is as above, and $d_1, \ldots, d_n, e_1, \ldots, e_n$ are smooth sections of $S|_U$, then

$$\sum_{j=1}^n d_j(\omega) - e_j(\omega) \in \text{Div}^0(S_\omega)$$

for all $\omega \in U$, and $\sum_j \int_{e_j(\omega)}^{d_j(\omega)}$ is a smooth section of $\text{Jac}(S)|_U$. 
Proof. By Theorem 32 we need only show that
\[ \omega \rightarrow \sum_{j} \int_{\gamma_j(\omega)} dx \]
is smooth, but this is clear. \qed

19. A canonical section and the isomorphism \( H_1(M) \rightarrow H_1(S)^{-} \).

In this section, we finally prove that \( H_1(M) \) is modelled on \( H_1(S) \) (Theorem 37). We have chosen this differential geometry approach in order to hide the most difficult mathematics in Lemma 35. This uses a deep result about index theory to show that some line bundle has a “meromorphic” section, as one might expect. The casual reader may accept this fact and pass by. One might also choose to consider the isomorphism from a holomorphic point of view, but this leads quickly to a discussion of coherent sheaves.

Let \( \pi_{\omega} : S_{\omega} \rightarrow M \). To each \( \omega \in H^s \), we construct a canonical section \( \sqrt{\omega} \in H^0(S_{\omega}, \pi_{\omega}^* K_M) \). We have the line bundle \( \pi_{\omega}^* K_M \rightarrow S_{\omega} \). A section of this map \( \gamma : S_{\omega} \rightarrow \pi_{\omega}^* K_M \) is an assignment \( \gamma(s) \in K_M|_{\pi(s)} \). Let \( \gamma(s) = s \) be our section. More formally, the identity map \( S_{\omega} \rightarrow TK_M \) factors through \( T(\pi_{\omega}^* K_M) \), and \( S_{\omega} \rightarrow T(\pi_{\omega}^* K_M) \) is a natural, single-valued section of \( \pi_{\omega}^* K_M \). Call this section \( \sqrt{\omega} \).

\( S \subset H^s \times TK_M \), so let \( \Pi : S \rightarrow H^s \times M \) be the projection on the second factor. Note that \( \Pi|_{\{\omega\}} = \text{id} \times \pi_{\omega} : \{\omega\} \times S_{\omega} \rightarrow \{\omega\} \times M \).

The map \( \text{id} \times (-\text{id}) : H^s \times TK_M \rightarrow H^s \times TK_M \) restricts to the involution \( \tau \) on \( S_{\omega} \), so that \( T = \text{id} \times (-\text{id})|_{S_{\omega}} \) is the involution such that \( T|_{\{\omega\}} = \tau \).

Recall that
\[ M = \{(\Phi, V) \text{ stable pairs}, | \wedge^2 V = \xi \text{ for some fixed } \xi \} \]

\( V \) is holomorphic, so compatible with some operator \( \bar{\partial} \) defining the holomorphic structure.

Lemma 34. For \( \omega \in U \subset H^s \) contractible open, there exists \( U' \subset U \) contractible open containing \( \omega \) and a line bundle \( \mathcal{L} \rightarrow S|_{U'} \) of degree 1 such that \( \mathcal{L}|_{S_{\omega}} \) is holomorphic, and \( \mathcal{L} \) has a section, \( s \) such that \( s|_{S_{\omega}} \) is not the zero section for \( \eta \in U' \).

Proof. Choose any degree one holomorphic line bundle. \qed

In the next lemma, we show that for any family of Riemann surfaces with line bundle, locally there is a family of nontrivial meromorphic sections. This relies on an index theory result about the family of complex structures \( \bar{\partial} \).
Lemma 35. Let $U \subset H^s$ be a contractible open set containing some $\omega \in H^s$ and let $\mathcal{L} \to S|_U$ be a line bundle which is holomorphic on the fibers ($\bar{\partial}_\omega \mathcal{L} = 0$ for all $\omega \in U$). Then there exists some line bundle $\mathcal{L}'$ and sections $s : S|_U \to \mathcal{L} \otimes \mathcal{L}'$ and $s' : S|_U \to \mathcal{L}'$ with $s|_{S_{\eta}} \neq 0$ for $\eta$ in some open subset $U' \subset U$ containing $\omega$.

Effectively, this says that $s/s'$ is a “meromorphic” section of $\mathcal{L}$. We won’t use any properties of $H^s$ in the proof.

Proof. Let $U$, $L_1$, and $s_1$ be given by Lemma 34. Let $d$ be the degree of $\mathcal{L}|_{S_\omega}$ (which is independent of $\omega$). Let $k$ be any positive integer such that $d + k > 2g - 2$. Let $\mathcal{L}' = \mathcal{L} \otimes k$ and $s' = s_1^k$. Then the line bundle $\mathcal{L} \otimes \mathcal{L}'$ is a smooth line bundle, holomorphic on each $\mathcal{L} \otimes \mathcal{L}'|_{S_\omega}$. This is a family of holomorphic line bundles, equipped with a family of elliptic operators, $\bar{\partial}_\omega$. Since the degrees at each $\omega$ are larger than $2g - 2$, these operators are of constant positive rank. Thus there is a family of solutions over $U$ (contractibility), which is the desired section, $s$.\footnote{This nontrivial result may be found as Theorem 9.11 in [2].} We may assume this family is nonzero on some restriction of $U$. \hfill \Box

Proposition 36. Let $\mathcal{L} \to S|_U$ be a line bundle holomorphic and degree zero along the $S_\omega$, $U$ contractible. Then there exists a section $D : U \to \text{Jac}(S)|_U$ such that $\mathcal{L}|_{S_\omega} = [D(\omega)]$.

Proof. Applying the lemma, we get $s, s'$ such that $s/s'$ is a “meromorphic” section of $\mathcal{L}$ over some subset $U' \subset U$. Then the family of divisors associated to $s/s'$

$$D(\omega) = \int_{(s'(\omega))}^{(s(\omega))}$$

is locally the section at every $\omega$. This is well-defined and independent of choices $\mathcal{L}'$, $s$, and $s'$ by the classical theory of Jacobian varieties, thus we can patch these sections together to get a continuous section on all of $U$. \hfill \Box

Theorem 37. $\mathcal{H}_1(M) \cong \mathcal{H}_1(S)^{\tau = -1}$.

Proof. First notice that $\mathcal{H}_1(S) \cong \mathcal{H}_1(\text{Jac}(S))$ naturally, since $\text{Jac}(S)$ is defined locally as $\mathbb{R}^{2g}$ mod the lattice which is $\mathcal{H}_1(S)$. This isomorphism is also $\tau$-equivariant, since the $\tau$ action on $\text{Jac}(S)$ is defined by lifting from $S$. Thus we need only show that $\mathcal{H}_1(M) \cong \mathcal{H}_1(\text{Jac}(S))^{\tau = -1}$.\footnote{This nontrivial result may be found as Theorem 9.11 in [2].}
By Hitchin (we follow [8], Theorem 8.1 closely), for every \( \omega \), there exists some \( L_\omega \) such that \( L \mapsto L \otimes L_\omega \) is an isomorphism:

\[
\mathcal{M}|_{\{\omega\}} \to \text{Pic}(S_\omega, M) = \text{Jac}(S_\omega)^{\tau = -1}.
\]

It follows from this that for any \( L'_\omega \) such that the degree of \( L'_\omega \) equals the degree of \( L_\omega \), the map \( L \mapsto L \otimes L'_\omega \) induces the same map

\[
H_1(\mathcal{M}|_{\{\omega\}}, \mathbb{Z}) \to H_1(\text{Jac}(S_\omega), \mathbb{Z})^{\tau = -1}.
\]

as does \( L \mapsto L \otimes L_\omega \), since homology is translation invariant. Thus there is a well-defined map (as sets):

\[
\mathcal{H}_1(\mathcal{M}) \to \mathcal{H}_1(\text{Jac}(S)).
\]

To any local section of \( \mathcal{M} \to H^* \), we have a map \( \Phi : U \times V \to U \times V \otimes K_M \) on vector bundles over \( U \times M \). The vector bundle \( U \times V \to U \times M \) has a smooth operator \( \bar{\partial}_\omega \) defining the holomorphic structure on the fibers, \( V \). \( \Phi \) has the property that \( \det(\Phi(\omega)) = \omega \). \( \Phi \) pulls back to \( \tilde{\Phi} \) acting on \( \Pi^*(U \times V) \) over \( S_{|U} \). By construction, \( \tilde{\Phi} \) preserves the holomorphic structure on \( \Pi^*(U \times V)|_{S_\omega} \).

\( \tilde{\Phi} : \Pi^*(U \times V) \to \Pi^*(U \times V) \otimes \Pi^*(K_M) \). We also have the canonical section \( \sqrt{\det \Phi} \).

Let \( \mathcal{L}_\Phi = \ker(\tilde{\Phi} - \sqrt{\det \Phi}) \) be the line bundle on \( S_{|U} \), which is holomorphic along the \( S_\omega \), and \( d \) be its degree. The degree is independent of choice of section of \( \mathcal{M} \). Choose a line bundle \( \mathcal{L} \) of degree \(-d\) by Lemma [54]. Then the map \( \mathcal{L}_\Phi \to \mathcal{L}_\Phi \otimes \mathcal{L} \) gives a degree zero line bundle associated to any \( \Phi \) which is holomorphic along the \( S_\omega \). By Proposition [60] we get a map from smooth sections of \( \mathcal{M}|_U \) to smooth sections of \( \text{Jac}(S)|_U \). This shows that the bundles \( \mathcal{M} \) and \( \text{Jac}(S) \) are locally (non-canonically) isomorphic. However, any choice of isomorphism gives the same map:

\[
\mathcal{H}_1(\mathcal{M})|_U \hookrightarrow \mathcal{H}_1(\text{Jac}(S))|_U
\]

defined before, showing that this map is continuous, and thus inducing a global isomorphism to the image, \( \mathcal{H}_1(\text{Jac}(S))^{\tau = -1} \).

□

20. RELATING TO THE BURAU REPRESENTATION

In Section [10] we defined the Burau representation, while in Theorem [57] we showed the homology of the Hitchin bundle \( \mathcal{H}_1(\mathcal{M}) \) was equivalent to \( \mathcal{H}_1(S)^{\tau = -1} \). In this section, we show that \( \mathcal{H}_1(S)^{\tau = -1} \) is indeed a specialization of the Burau bundle. Let \( k = 2 \).
Recall the space $W$ from Section 16, letting $n = 4g - 4$. This is the space of points in $M^{[4g - 4] + 1}$ along with a winding number function. For any $\omega$, we can construct a (mod 2) winding number function $w_\omega$ defined on closed loops $\alpha : [0, 1] \to M \setminus \langle \omega \rangle$ by:

$$w_\omega(\alpha) = \sqrt{\omega(\alpha(1))} \over \sqrt{\omega(\alpha(0))} = \pm 1,$$

which is well-defined, and descends to $H_1(M \setminus \langle \omega \rangle, \mathbb{Z})$. Also, $w_{c\omega} = w_\omega$ for all nonzero constants, $c$. Notice that we computed a similar function explicitly in Theorem 20.

If $\langle \omega \rangle = X$, then for any $m \notin X$, $B|_{(m,X,w_\omega)}$ and $S|_\omega$ are isomorphic, yet there are two choices of isomorphism, depending on the image of the basepoint, $\pm \sqrt{\omega(m)}$.

Recall also that the representation

$$\pi_1(W) \to \text{Aut}(H_1(B|_{(m,X,w_\omega)}, \mathbb{Z}))/\langle t \cdot \text{Id} \rangle$$

is trivial on the fibers of $W \to M^{[4g - 4]}$. Therefore the representation descends to a well-defined representation of $\pi_1(M^{[4g - 4]})$, and likewise for $\pi_1(PH^s) \to \text{Aut}(H_1(S_\omega, \mathbb{Z}))/\langle t \cdot \text{Id} \rangle$. This gives us

**Proposition 38.** Let $\omega \in H^s$. Under $PH^s \to M^{[4g - 4]}$, and either identification, $f : B_{m,X,w_\omega} \to S_\omega$, where we use $\bar{\omega}$ to denote the element of $PH^s$ to avoid confusion, $f$ induces an intertwining isomorphism,

$$\hat{f} : \text{Aut}(H_1(B|_{X}, \mathbb{Z}))/\langle t \cdot \text{Id} \rangle \to \text{Aut}(H_1(S_\omega, \mathbb{Z}))/\langle t \cdot \text{Id} \rangle$$

realized by the commutative diagram:

$$\begin{array}{ccc}
\pi_1(PH^s) & \longrightarrow & \text{Aut}(H_1(S_\omega, \mathbb{Z}))/\langle t \cdot \text{Id} \rangle \\
\downarrow & & \downarrow \hat{f} \\
\pi_1(M^{[4g - 4]}) & \longrightarrow & \text{Aut}(H_1(B|_{X}, \mathbb{Z}))/\langle t \cdot \text{Id} \rangle
\end{array}$$

We will prove this proposition in a moment. However, first realize an important corollary:

**Corollary 39.** In the representation, $\pi_1(H^s) \to \text{Aut}(H_1(S|_\omega, \mathbb{Z}))$, the kernel

$$\ker(\pi_1(H^s) \to \pi_1(M^{[4g - 4]}))$$

acts by $\{1, \tau\}$. 
Proof of Proposition 38. First note that \((\hat{f}\sigma)(\alpha) = (f^{-1})^*(\sigma(f^*\alpha))\) is an isomorphism, as it is invertible. Also, if \(f' = tf\), then \(\hat{f}'(\gamma) = (t\hat{f})\gamma\), which gives a different diagram, however, one commutes if and only if the other commutes, since \(t\hat{f}\) agrees with \(\hat{f}\) on the image of \(\pi_1(H^*)\), since \(t\) commutes with the monodromy action as observed at the end of Section 16.

Let \(U \subset M\) be contractible such that the canonical bundle is trivializable on \(U\) with \(\mathbb{C}^*_\)-equivariant trivialization \(\tilde{\phi} : K_M|_U \to U \times \mathbb{C}\). This induces a trivialization \(\phi : K^*_M|_U \to U \times \mathbb{C}\), respecting tensor product (\(\phi \cong \tilde{\phi} \otimes_U \tilde{\phi}\)). Consider the space

\[
A = \{(z, m, X) \in \mathbb{C}^* \times U \times PH^* | m \notin X\}.
\]

As \(U\) has infinite cardinality (thus greater than \(4g - 4\)), \(A\) has a surjective map \(A \twoheadrightarrow PH^*\). In fact, for every \((z, m, X) \in A\), there exists a unique \(\omega \in H^*\) such that \(\omega|_X = 0\) and \(\phi(\omega(m)) = (m, z^2)\). Thus we get a lift to \(H^*\) which is surjective as well (by \(\mathbb{C}^*_\) equivariance, say).

This gives rise to a commutative diagram:

\[
\begin{array}{ccc}
\mathcal{H}_1(S) & \xleftarrow{a^*} & \mathcal{H}_1(S) \\
\downarrow & & \downarrow \\
\mathcal{H}_1(B) & \xrightarrow{b^*} & \mathcal{H}_1(B) \\
\downarrow & & \downarrow \\
H^* & \xrightarrow{a} & A \\
\downarrow & & \downarrow \\
PH^* & \xrightarrow{b} & W \\
\downarrow & & \downarrow \\
M^{[4g-4]} & \xrightarrow{a} & M^{[4g-4]} \\
\end{array}
\]

We intend to show that \(a^*\mathcal{H}_1(S)\) and \(b^*\mathcal{H}_1(B)\) are isomorphic. Once we have this, clearly

\[
a^*\mathcal{H}_1(S)/(t \cdot \text{Id}) \cong b^*\mathcal{H}_1(B)/(t \cdot \text{Id})
\]

are trivial on fibers of \(A \to M^{[4g-4]}\), so the result will follow.

We in fact will show something stronger, that \(a^*S\) and \(b^*B\) are isomorphic. This isomorphism boils down to a universality statement, Lemma 27, that there is only one double cover with given data \((X, w, m)\) of ramification divisor, winding number and marked point in the cover.

Let \(W' = bA\). Then \(b^*\mathcal{H}_1(B) \cong \mathcal{H}_1(B)|_W \times \mathbb{C}^*\). Let

\[
Q = \{((z, m, X), b) \in \mathcal{H}_1(S)|z = 1\}.
\]

Then \(a^*\mathcal{H}_1(S) \cong Q \times \mathbb{C}^*\) by the map

\[
((z, m, X), b) \mapsto ((1, m, X), \frac{b}{z}, z).
\]
This map is well-defined since if \( \omega(\pi_\omega(b)) = b^2 \), then \( b^2 \in S_\omega \). It is also clearly continuous with continuous inverse. Now applying the Lemma 27 to \( Q \to W' \) and \( H_1(B)|_{W'} \), we get
\[
Q \cong H_1(B)|_{W'} \cong b^*H_1(B).
\]

### 21. Combinatorics of \( H_1(\mathcal{M}) \)

In this section, we use the results of Section 11 to turn \( H_1(\mathcal{M}) \) into combinatorial data. First notice
\[
H_1(\mathcal{M}|_\{\omega\}) \cong H_1(\text{Prym}(S_\omega, M), \mathbb{Z})
\cong \mathbb{R}E/(\mathbb{R}F + \frac{1}{2}Z\mathbb{E}).
\]
Also recall that the monodromy of \( H_1(\mathcal{M}) \to H^* \) is modelled on \( H_1(S) \), so that studying the effect on \( \pi_1(H^*) \) on \( \frac{1}{2}Z\mathbb{E} \) gives us this by
\[
H_1(S)|_{\{\omega\}} \cong \frac{1}{2}Z\mathbb{E}/(\mathbb{R}F \cap \frac{1}{2}Z\mathbb{E}).
\]
Given Theorem 11, then, our only remaining task is to compute for edges \( e_\alpha, e_\beta \in E \), the values
\[
\sigma_{e_\alpha}\frac{1}{2}\phi(e_\beta) \in \frac{1}{2}Z\phi(E).
\]
First, however, we discuss a result making the introduction of the Burau representation more salient. Let \( B_{\Lambda_2} \) be the Burau module. This is a free module over \( \Lambda_2 = \mathbb{Z}[t]/(t^2 - 1) \), so
\[
B_\mathbb{Z}^{-} = B_{\Lambda_2}/(t = -1)
\]
is an honest \( \mathbb{Z} \)-module with a (\( \mathbb{Z} \)-projective) representation
\[
K = \ker(\pi_1(M^{[n]}) \to H_1(M)) \to \text{Aut}(B_\mathbb{Z}^-)/(\pm 1).
\]
We choose this notation based on the observation that
\[
B_\mathbb{Z} = B_{\Lambda_1} = B_{\Lambda_2}/(t = +1).
\]
Notice that in \( H_1(\text{Prym}(S_\omega, M), \mathbb{Z}) \), \( \tau \) acts by \(-1\), so
\[
\pi_1(H^*) \to \text{Aut}(H_1(\text{Prym}(S_\omega, M), \mathbb{Z})
\]
descends to a projective representation
\[
\pi_1(\text{PH}^*) \to \text{Aut}(H_1(\text{Prym}(S_\omega, M), \mathbb{Z})/(\pm 1).
Theorem 40. The standard homomorphism \( H_1(\text{Prym}(S_\omega, M), \mathbb{Z}) \to H^1(S_\omega, \mathbb{Z}) \to B^-_Z \) induces a commutative diagram and \( \pi_1(\text{PH}^*) \)-module isomorphism:

\[
\begin{array}{ccc}
\pi_1(\text{PH}^*) & \longrightarrow & \text{Aut}(H_1(\text{Prym}(S_\omega, M), \mathbb{Z}))/\langle \pm 1 \rangle \\
\mathbb{K} & \longrightarrow & \text{Aut}(B^-_Z)/\langle \pm 1 \rangle \\
\end{array}
\]

\( \cong \)

Proof. In fact, commutativity of this diagram follows immediately from Proposition 38. The crux of the statement is that these modules are isomorphic. However, this also follows from Proposition 38, once one observes that \( \hat{f} \) is an intertwining isomorphism and is \( \Lambda_2 \)-equivariant. \( \Box \)

22. Monodromy

Now we are ready to study the monodromy action. We know this action is generated by the transpositions, so we let \( \bar{\sigma} \in \rho_* \pi_1(H^*) \) realize one of these transpositions. By Corollary 39 we know that if we choose any lift \( \sigma \in \pi_1(H^*) \), then any other lift acts in the manner of \( \sigma \) or \( \tau \sigma \). The degree of freedom comes from the ambiguity of identifying \( S_\omega \) with \( S_{\sigma \omega} \), as we have discussed.

In this section, we study the effect of transpositions on \( M \). Recall that \( \Gamma \) makes \( M \) a polyhedron, \( c : S \to M \) is a double cover ramified at marked points, one per face of \( M \). We would like to study the action of \( \rho_* \pi_1(H^*) \) on the edge set of \( \hat{\Gamma} \). The faces of \( M \) are labelled by vertices of \( \hat{\Gamma} \). Let \( X_j \) be a face of \( \Gamma \) and let \( \{ x_j \} = X_j \cap X \) be the corresponding vertex of \( \hat{\Gamma} \). Assume \( x_1 \) and \( x_2 \) are neighbors across \( e \alpha \), with associated transposition \( \sigma_\alpha \). Assume, once and for all, that every transposition is realized by each point \( x_j \) moving counterclockwise around \( e \).

Theorem 41. If \( e_\alpha \in E \) and \( \overline{\sigma}_\alpha \in \rho_* \pi_1(H^*) \), \( \overline{\sigma}_\alpha \) has a lift \( \sigma_\alpha \in \pi_1(H^*) \) such that for any \( e \in \mathbb{Z}E \),

\[
\sigma_\alpha e = e - (e \cdot e_\alpha)e_\alpha.
\]

Proof. We choose the lift \( \sigma_\alpha \) of \( \overline{\sigma}_\alpha \) which acts homotopically trivially on \( S \setminus c^{-1}(X_1 \cup X_2) \). Edges away from \( X_1 \cup X_2 \) are unaffected by this transposition, however any edge from \( x_1 \) or \( x_2 \) may be altered. \( X_1 \cup X_2 \) is simply connected by construction, and has two ramification points. Recall that we are studying the effect of \( \sigma \) on \( E \subset \hat{E} \). Notice that the loop \( \phi_{e_\alpha} \) is in fact homotopic to one component of \( \partial c^{-1}(X_1 \cup X_2) \), thus in fact to a loop lying outside \( c^{-1}(X_1 \cup X_2) \). This implies that \( \sigma_\alpha \) acts trivially
Figure 24. Monodromy action of $\sigma_\alpha$ on $e$.

on $e_\alpha$. In fact, we have seen that $\sigma_\alpha$ acts trivially on any $e \in E$ such that $e \cdot e_\alpha = 0$, where $\cdot$ is the intersection pairing on $S$.

Notice that if $e_1 \cdot e_\alpha = e_2 \cdot e_\alpha = +1$, then $e_1 - e_2$ is homotopy equivalent to a curve supported away from $c^{-1}(X_1 \cap X_2)$, so $\sigma e_1 - e_1 = \sigma e_2 - e_2$. By linearity, then,

$$\sigma_\alpha e = e - (e \cdot e_\alpha)e_q$$

for some as yet undetermined $e_q$.

The set $c^{-1}(X_1 \cup X_2)$ may be realized topologically as the union of two annuli joined along their inner circles (preserving orientation). The effect of $\sigma$ on each annulus is a positive half-twist of these circles. Whe rejoined, we see that $\sigma$ is the operation of a Dehn twist at $e_\alpha$. Thus

$$\sigma_\alpha e = e \pm (e \cdot e_\alpha)e_\alpha.$$ 

Notice that

$$e \cdot (\sigma_\alpha e - e) = \pm (e \cdot e_\alpha)^2,$$

and it is enough to show that there is some $e$ such that that this quantity is negative.

In some small neighborhood of $e_\alpha$, the monodromy is computed as in the Figures 24 and 25. $e$ is a curve which crosses the branch cut $e_\alpha$, so the dark curves lie on the upper branch and the light curves lie on the lower branch. Applying the monodromy, we get Figure 24.

One then graphs $e$ and $\sigma_\alpha e - e$ as in Figures 25 and 26, and finds that their intersection is $-1$ as desired. Of course the sign was originally chosen by the choice of transposition $\sigma_\alpha^\pm$. 

□
23. A NOTE REGARDING NON-HYPERELLIPTIC CURVES

In this section, we show how, given Theorem 11 for general $M$, that the rest of the work in this paper still applies. Indeed, we need only that there is a compatible cellular decomposition:

**Theorem 42.** Let $N$ be any curve of genus $g \geq 3$ with $\eta \in H^0(N, K_N^2)^*$, and $M$ hyperelliptic with $\omega \in H^0(M, K_M^2)^*$ as in Section 2. Then $S_{\eta}$ and $S_{\omega}$ are homeomorphic as double covers.

**Proof.** Let $X = \langle \eta \rangle$ and choose $x_0 \in N \setminus X$. Any closed curve through $x_0$ missing $X$ is homotopic in $N$ to another such curve with winding number zero (by dragging the original curve past some subset of $X$). Thus there is a set of closed curves $a_j, b_j$ of winding number 0 such that $[a_j], [b_j]$ is a symplectic basis for $H_1(N, \mathbb{Z})$. Excising these curves, we find a contractible open set $U_N \subset N$ containing all of the ramification points, $X$. By construction, this set lifts to a two-component set in $S_{\eta}$. Also, $U$ is the interior of a polygon, from which one can form $N$ by identifying sides in a standard way, and $S_{\eta}$ is defined uniquely by this polygon.

We used no data about $N$ in constructing this decomposition, so we could do the same for $M$ and $\omega$, finding $U_M$. However, $U_M$ and $U_N$ are homeomorphic as
punctured open disks, and indeed, there is a homeomorphism preserving the gluing relation. Thus we get two choices for lifts to homeomorphisms of \( S_\eta \) and \( S_\omega \).

Now that these are homeomorphic, we may transport the cellular decomposition of \((S_\omega, M)\) to \((S_\eta, N)\). This is the only other data that relies on hyperellipticity, so we see that, indeed, if Theorem 11 holds in general, then all other arguments hold as well.

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References

[1] P. Bellingeri, On Presentations of Surface Braid Groups J. Algebra 274 (2004) no. 2, 5543-563.
[2] N. Berline, E. Getzler, M. Vergne, Heat Kernels and Dirac Operators (2004) Springer, Berlin; New York.
[3] J. Birman, Braids, Links, and Mapping Class Groups, (1974) Princeton University Press, Princeton NJ.
[4] D. J. Copeland, A Special Subgroup of the Surface Braid Group (2004) \texttt{arXiv:math.GR/0409461}.
[5] E. Fadell, L. Neuwirth, Configuration Spaces, Math.Scand. 10 (1962) 111-118.
[6] P. Griffiths, J. Harris, Principles of Algebraic Geometry, (1994) John Wiley and Sons, Inc., New York.
[7] N. Hitchin, Stable bundles and integrable systems, Duke Mathematics Journal (1987) 54, No. 1, 91-114.
[8] N. Hitchin, The Self-duality Equations on a Riemann Surface Proc. London Math. Soc (3) (1987) 55 no. 1, 59-126.
[9] M. S. Narasimhan, S. Ramanan, Deformations of the Moduli Space of vector bundles over an algebraic curve, Ann. Math. (1975), 1
[10] G.P Scott, Braid Groups and the Group of Homeomorphisms of a Surface, Proc. Camb. Phil. Soc. (1970), 60, 605-617.