Well-posedness for Fractional Dissipative
Benjamin-Ono Equations

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Abstract

This paper is devoted to study the Cauchy problem for the fractional dissipative BO equations

\[ u_t + \mathcal{H}u_{xx} - (D_x^a - D_x^b)u + uu_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \]

\[ u(x, 0) = u_0(x), \]

where \( u = u(x, t) \) is a real valued function, the dissipation parameters satisfy \( 0 < \alpha < \beta \), the operator \( D_x^a \) is defined via the Fourier transform by \( \hat{D}_x^a \varphi(\xi) = |\xi|^\alpha \hat{\varphi}(\xi) \) and \( \mathcal{H} \) denotes the usual Hilbert transform given by

\[ \mathcal{H}\varphi(x) = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{\varphi(y)}{x-y} \, dy = (\text{sgn}(\xi)\hat{\varphi}(\xi))^\vee(x), \quad \text{for } \xi \in \mathbb{R}, \\varphi \in \mathcal{S}(\mathbb{R}). \]

When \( a = \beta \), fDBO corresponds to well-known Benjamin-Ono (BO) equation derived by Benjamin [2] and Ono [18] as a model for long internal gravity waves in deep stratified fluids. The IVP associated to BO equation has been widely studied, see [26, 11, 24, 17, 13, 12, 28, 10, 14, 15] and references therein. Several authors have searched the minimal regularity, measured in the Sobolev scale \( H^s(\mathbb{R}) \), which guarantees that the IVP for BO is locally or globally wellposed (LWP and GWP, resp.). We say that an IVP is LWP when \( s > \max\{3/2 - \beta, -\beta/2\} \) and \( s > -\beta/4 \). For \( \beta \geq 2 \), we show GWP in \( H^s(\mathbb{R}) \), \( s > \max\{3/2 - \beta, -\beta/2\} \). We establish that our results are sharp in the sense that the flow map \( u_0 \mapsto u \) fails to be \( C^2 \) in \( H^s(\mathbb{R}) \), for \( s < -\beta/2 \), and it fails to be \( C^4 \) in \( H^s(\mathbb{R}) \) when \( s < \min\{3/2 - \beta, -\beta/4\} \). When \( 0 < \beta < 1 \), we show ill-posedness in \( H^s(\mathbb{R}) \), \( s \in \mathbb{R} \). Finally, if \( \beta > 3/2 \), we prove GWP in \( H^s(\mathbb{T}) \), \( s > \max\{3/2 - \beta, -\beta/2\} \), and we deduce lack of \( C^2 \) regularity in \( H^s(\mathbb{T}) \) when \( s < -\beta/2 \), in particular we get sharp results when \( \beta \geq 3 \).

Keywords: Benjamin-Ono equation, dissipative-dispersive effects, Locally and Global well-posedness.

1 Introduction and main results

We study the initial value problem (IVP) for the following fractional dissipative Benjamin-Ono (fDBO) equations

\[
\begin{cases}
  u_t + \mathcal{H}u_{xx} - (D_x^a - D_x^b)u + uu_x = 0, & x \in \mathbb{R} \text{ or } x \in \mathbb{T}, \quad t > 0, \\
  u(x, 0) = u_0(x), &
\end{cases}
\]

When \( \alpha = \beta \), fDBO corresponds to well-known Benjamin-Ono (BO) equation derived by Benjamin [2] and Ono [18] as a model for long internal gravity waves in deep stratified fluids. The IVP associated to BO equation has been widely studied, see [26, 11, 24, 17, 13, 12, 28, 10, 14, 15] and references therein. Several authors have searched the minimal regularity, measured in the Sobolev scale \( H^s(\mathbb{R}) \), which guarantees that the IVP for BO is locally or globally wellposed (LWP and GWP, resp.). We say that an IVP is LWP in a functional space \( X \) provided that for every initial data \( u_0 \in X \) there exists \( T = T(\|u_0\|_X) > 0 \) and a unique solution \( u \in X_T \subset C([0, T]; X) \) of the IVP such that the flow-map data solution is locally continuous from \( X \) to \( X_T \). If the above properties are true for any \( T > 0 \), we say that the IVP is GWP. Let us recall some of them for the BO equation: in [26] LWP for \( s > 3 \) was established, in [11] and [24] GWP for \( s \geq 3/2 \), in [28] GWP when \( s \geq 1 \), and finally in [10, 15] GWP when \( s \geq 0 \) was proven. All these results have been obtained by compactness methods. This is a consequence of the results of Molinet, Saut and Tzvetkov in [17] who proved for all \( s \in \mathbb{R} \) that the flow map \( u_0 \mapsto u \) is not of class

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$C^2$ at the origin from $H^s(\mathbb{R})$ to $H^s(\mathbb{R})$. In other words, they showed that one cannot solve the IVP for BO equation by a Picard iterative method implemented on its integral formulation for initial data in the Sobolev space $H^s(\mathbb{R})$, $s \in \mathbb{R}$.

The fDBO equations (1.1) generalize the Chen-Lee (CL) equation

$$
\begin{aligned}
&u_t + \mathcal{H}u_{xx} - (\mathcal{H}u_x + u_x) + uu_x = 0, \quad x \in \mathbb{R} \ (or \ x \in \mathbb{T}), \quad t > 0, \\
u(x,0) = u_0(x),
\end{aligned}
$$

which corresponds to the case $\alpha = 1$ and $\beta = 2$ in fDBO. The CL equation (1.2) was first introduced by Chen and Lee in [6] to describe fluid and plasma turbulence and as a model for internal waves in a two-fluid system. The third and the fourth terms represent the instability and dissipation, respectively. Concerning the initial value problem, it was proved in [22, 20, 21] that (1.2) is GWP in $H^s(\mathbb{R})$ and in $H^s(\mathbb{T})$ when $s > -1/2$. It was also established that the flow map data-solution for (1.2) fails to be $C^3$ at the origin of $H^s(\mathbb{R})$, when $s < -1/2$ and it was proved lack of $C^2$ regularity at the origin of $H^s(\mathbb{T})$ if $s < -1$.

When $\alpha = 1$ and $\beta = 3$, (1.1) becomes the following nonlocal perturbation of the BO (npBO) equation

$$
\begin{aligned}
&u_t + \mathcal{H}u_{xx} - (\mathcal{H}u_x + \mathcal{H}u_{xxx}) + uu_x = 0, \quad x \in \mathbb{R} \ (or \ x \in \mathbb{T}), \quad t > 0, \\
u(x,0) = u_0(x).
\end{aligned}
$$

Recently, it was established in [8] GWP for (1.3) in the Sobolev spaces $H^s(\mathbb{R})$ for any $s > -3/2$ and lack of $C^2$ regularity for the flow map data-solution when $s < -3/2$. It was also studied persistence properties of the flow map in some weighted Sobolev spaces.

Another equation that has never been studied is

$$
\begin{aligned}
&u_t + \mathcal{H}u_{xx} - (\mathcal{H}u_x - u_{xxxx}) + uu_x = 0, \quad x \in \mathbb{R} \ (or \ x \in \mathbb{T}), \quad t > 0, \\
u(x,0) = u_0(x).
\end{aligned}
$$

This equation belongs to the family fDBO and is obtained when $\alpha = 1$ and $\beta = 4$. The models in (1.2), (1.3) and (1.4) have been used in fluids and plasma theory, see (40) in [25].

The form of these types of equations and the dispersive and dissipative effects have motivated us to define the toy model fDBO, in which we study the effects of dissipation for the range $\beta > 0$ with $\alpha < \beta$. We first consider the case $1 < \beta < 2$ with $0 < \alpha < \beta$. Under these conditions, the dispersive effects turn out to be essential to obtain global existence of solutions in the Sobolev spaces $H^s(\mathbb{R})$ of low regularity ($s > -\beta/4$). Therefore to take these effects into account, we will apply a fixed-point argument on some Bourgain type spaces $X^{b,s}$ adapted to the dispersive-dissipative part of fDBO (see (1.8) below).

**Theorem 1.1.** Let $1 < \beta < 2$ with $0 < \alpha < \beta$ fixed and $u_0 \in H^s(\mathbb{R})$ where $s > -\beta/4$. Then for any time $T > 0$ there exists a unique solution $u$ of the integral equation (1.11) in

$$Z_T = C([0,T];H^s(\mathbb{R})) \cap X^{1/2,s}_T.$$

Moreover, the flow map $u_0 \mapsto u(t)$ is smooth from $H^s(\mathbb{R})$ to $Z_T$ and $u$ belongs to $C((0,T],H^\infty(\mathbb{R})).$

The proof of Theorem 1.1 is achieved by deriving a key bilinear estimate in the $X^{b,s}$ spaces. To deduce this result, our approach relays on dyadic decomposition and orthogonality, following the ideas of [29, 27]. More precisely we will apply the dyadic block estimates for the Benjamin-Ono equation [29, Proposition 3.1] to obtain the crucial bilinear estimate.
Next we consider the cases $\beta \geq 2$ with $0 < \alpha < \beta$. Here the fDBO equations behave as a pure dissipative model. Consequently to deduce well-posedness we will apply the methods introduced by Dix [7] (see also, [23, 20, 5]), which rely completely on the dissipation of the equation. This approach consists in a fixed point argument using the integral equation associated to (1.1) on a time weighted $L^2_x$ type spaces (see (1.10) below).

**Theorem 1.2.** Let $\beta \geq 2$ and $0 < \alpha < \beta$ fixed. Consider $u_0 \in H^s(\mathbb{R})$ where $s > \max \{3/2-\beta,-\beta/2\}$. Then for any time $T > 0$ there exists a unique solution $u$ of the integral equation (1.11) in

$$W_T = C([0, T]; H^s(\mathbb{R})) \cap Y^2_T.$$

Moreover, the flow map $u_0 \mapsto u(t)$ is smooth from $H^s(\mathbb{R})$ to $W_T$ and $u$ belongs to $C((0, T], H^\infty(\mathbb{R}))$.

**Remark 1.1.** Theorem 1.2 establishes GWP for the IVP (1.1) in the Sobolev spaces $H^s(\mathbb{R})$, with $s > 3/2-\beta$ if $2 \leq \beta \leq 3$, and $s > -\beta/2$ when $\beta > 3$. Thus gathering the conclusions of Theorem 1.1 and 1.2, we observe that one can solve (1.1) in more singular Sobolev spaces than the Benjamin-Ono equation for which the best known result establishes GWP in $L^2(\mathbb{R})$ (see for instance [10, 15, 9]).

Letting $\beta \geq 1$ and $0 < \alpha < \beta$ fixed in fDBO, we can show that our local results are sharp in the sense that the flow map of the IVP (1.1) fails to be $C^2$ in $H^s(\mathbb{R})$ for $s < -\beta/2$ and it fails to be $C^3$ in $H^s(\mathbb{R})$ when $s < \min \{3/2-\beta,-\beta/4\}$. These statements are equivalent to the fact that we cannot solve the Cauchy problem (1.1) in $H^s(\mathbb{R})$ using a contraction argument on its integral equation by purely using Sobolev spaces either $s < \min \{3/2-\beta,-\beta/4\}$, with $1 \leq \beta < 3$, or for $s < -\beta/2$, when $\beta \geq 3$. Finally, in the case $0 < \beta < 1$ and $\alpha < \beta$ we have that the flow map of the IVP (1.1) fails to be $C^2$ in $H^s(\mathbb{R})$ for all $s \in \mathbb{R}$. We summarize these results in the following theorem.

**Theorem 1.3.** Let $0 < \alpha < \beta$ fixed.

1. Let $\beta \geq 1$ and assume that $s < -\beta/2$. Then there does not exist any time $T > 0$ such that the Cauchy problem (1.1) admits a unique local solution defined on the interval $[0, T]$ and such that the flow-map $u_0 \mapsto u$ is $C^2$ differentiable at zero from $H^s(\mathbb{R})$ to $C([0, T]; H^s(\mathbb{R}))$.

2. Let $\beta \geq 1$ and assume that $s < \min \{3/2-\beta,-\beta/4\}$. Then there does not exist any time $T > 0$ such that the Cauchy problem (1.1) admits a unique local solution defined on the interval $[0, T]$ and such that the flow-map $u_0 \mapsto u$ is $C^3$ differentiable at zero from $H^s(\mathbb{R})$ to $C([0, T]; H^s(\mathbb{R}))$.

3. Let $0 < \beta < 1$ and $s \in \mathbb{R}$. There does not exist $T > 0$ such that the IVP (1.1) admits a unique local solution defined on the interval $[0, T]$ and such that the flow map $u_0 \mapsto u$ is of class $C^2$ in a neighborhood of the origin from $H^s(\mathbb{R})$ to $H^s(\mathbb{R})$.

At the end-point $\beta = 1$, our proof of part (iii) of Theorem 1.3 fails. However, Theorem 1.3 part (ii) provides ill-posedness in $H^s(\mathbb{R})$, for $s < -1/4$. So, it is still not clear what happens to fDBO when $\beta = 1$, $\alpha < \beta$ and $s \geq -1/4$.

Concerning the periodic fDBO equations we can derive GWP and lack of $C^2$ regularity for the IVP associated in $H^s(\mathbb{T})$ employing similar arguments to those used in the $H^s(\mathbb{R})$ case. Indeed, it can be easily seen that the results of Theorem 1.2 hold on the periodic case assuming the constraints $\beta > 3/2$ with $0 < \alpha < \beta$ (see (5.1) below). Regarding the regularity of the flow map data-solution on $H^s(\mathbb{T})$, we have that part (i) of Theorem 1.3 is still valid in this context.

**Theorem 1.4.** Assume that $\beta > 3/2$ with $0 < \alpha < \beta$ fixed. Then the conclusion of Theorem 1.2 and Theorem 1.3 part (i) still hold in the periodic case with $H^s(\mathbb{R})$ replaced by $H^s(\mathbb{T})$ and $Y^2_T$ replaced by $\tilde{Y}^2_T$. In particular these results are sharp for the periodic fDBO equations when $\beta \geq 3$.

**Remark 1.2.** Theorem 1.4 establishes GWP for the IVP (1.1) in $H^s(\mathbb{T})$, with $s > \max \{3/2-\beta,-\beta/2\}$ if $\beta > 3/2$, and for $\beta \geq 1$ the flow map $u_0 \mapsto u$ fails to be $C^2$ in $H^s(\mathbb{T})$ for $s < -\beta/2$. This implies that
the GWP result is sharp for $\beta \geq 3$. It is not clear what happens to the IVP associated to the periodic fDBO equations for either $3/2 < \beta < 3$ and $-\beta/2 \leq s \leq 3/2 - \beta$ or for $0 < \beta \leq 3/2$ and $s \geq -\beta/2$.

When $\alpha = \beta$, we recall that in [14], Molinet proved GWP for the IVP associated to the periodic BO equation in $L^2(\mathbb{T})$.

Now we discuss some consequences of our results for the equations (1.2), (1.3) and (1.4). Here we reprove the results in [22, 20, 21] for the IVP associated to the CL equation (1.2). Thus for this problem we obtain GWP in $H^3(\mathbb{R})$ and in $H^3(\mathbb{T})$ when $s > -1/2$ and we show that the flow map data-solution for (1.2) fails to be $C^3$ at the origin of $H^s(\mathbb{R})$, when $s < -1/2$ and it fails to be $C^2$ at the origin of $H^s(\mathbb{T})$ if $s < -1$.

Regarding the IVP for the npBO equation (1.3) in $H^s(\mathbb{R})$, in this paper we obtain the same GWP in [8] and additionally we extend these results to the periodic setting. Consequently we deduce GWP in $H^s(\mathbb{R})$ and $H^s(\mathbb{T})$ for $s > -3/2$ and sharp results in the sense that the flow map $u_0 \mapsto u$ for npBO fails to be $C^2$ at zero from $H^s(\mathbb{R})$ to $H^s(\mathbb{R})$ or from $H^s(\mathbb{T})$ to $H^s(\mathbb{T})$ when $s < -3/2$.

Finally our results imply that the IVP for (1.4) is GWP in $H^s(\mathbb{R})$ and in $H^s(\mathbb{T})$, when $s > -2$, and its flow map data solution lacks of $C^2$ regularity at the origin of $H^s(\mathbb{R})$ and $H^s(\mathbb{T})$ when $s < -2$.

Remark 1.3. (1) One can extend the results of Theorem 1.2 to the indexes $3/2 < \beta < 2$ with $0 < \alpha < \beta$, obtaining well-posedness for (1.1) in $H^s(\mathbb{R})$ with $s > 3/2 - \beta$. This was already noted for the periodic case in Theorem 1.4. However, Theorem 1.1 leads to better results since its proof is deduced using Bourgain’s spaces containing both dispersive and dissipative effects. Then, when $\beta < 2$, at least with the techniques employed here, we see that the dispersive part plays a key role in the low regularity of the solution.

(2) It is possible to deduce Theorem 1.2 applying a contraction argument on the $X^{b,s}$ spaces. However, for its simplicity and applicability to periodic equations, we decided to present a different approach following the ideas in [7]. Moreover, since this technique does not have into account the effects of dispersion, the conclusions of Theorem 1.2 and 1.4 hold true for several types of dispersive-dissipative equations like the following model

$$
\begin{cases}
  u_t - \partial_x D_x^a u - (D_x^a - D_x^3)u + uu_x = 0, & x \in \mathbb{R} (or x \in \mathbb{T}), \\
  u(x,0) = u_0(x),
\end{cases}
$$

for all $a \geq 0$.

(3) Our results in Theorems 1.1 and 1.3 are the same as those deduced by Vento [29] for the dissipative Benjamin-Ono (dBO) equations,

$$
  u_t + H u_{xx} + D_x^3 u + uu_x = 0, \quad x \in \mathbb{R}, \quad t > 0.
$$

In the case $0 \leq \beta < 1$, the solution map fails to be $C^2$ in any $H^s(\mathbb{R})$, $s \in \mathbb{R}$. When $1 < \beta \leq 2$, Vento deduced GWP in $H^s(\mathbb{R})$ for all $s > -\beta/4$. His result is sharp in the sense that the flow-map data solution fails to be $C^3$ at the origin of $H^s(\mathbb{R})$, when $s < -\beta/4$.

The organization of the paper is as follows. We begin by introducing some notation and functional spaces to be employed in our arguments. In the following sections we divide our results according to the leading dissipation parameter $\beta > 1$. In section 2, we assume that $1 < \beta < 2$ with $0 < \alpha < \beta$. We first derive some estimates related to the linear fDBO equations on the $X^{b,s}$ spaces. We then proceed to prove the crucial bilinear estimates on these spaces, which leads to the conclusion of Theorem 1.1. In the following section we assume that $\beta \geq 2$ and $0 < \alpha < \beta$. Here we deduce estimates on the $Y_T^p$ spaces to establish Theorem 1.2. In the fourth section we show the ill-posedness results stated in Theorem 1.3. We conclude the paper studying the fDBO equations on the torus in the last section.
1.1 Notation and Preliminaries

The notation we will employ is quite standard. $A \lesssim B$ (for $A$ and $B$ nonnegative) means that there exists $C > 0$ independent of $A$ and $B$ such that $A \leq CB$. Similarly define $A \gtrsim B$ and $A \sim B$. Given $p \in [1, \infty]$, we define its conjugate $p' \in [1, \infty]$ from the relation $1 = \frac{1}{p} + \frac{1}{p'}$. For such values of $p$, we define the Lebesgue spaces $L^p(\mathbb{R})$ by its norm by

$$\|f\|_{L^p} = \left(\int_{\mathbb{R}} |f(x)|^p \, dx\right)^{1/p},$$

with the usual modification when $p = \infty$. We also consider space-time Lebesgue spaces

$$\|f\|_{L^p_t L^q_x} = \|\|f(\cdot, t)\|_{L^q_x}\|_{L^p_t} \quad \text{and} \quad \|f\|_{L^p_x L^q_t} = \|\|f(\cdot, t)\|_{L^p_x}\|_{L^q_t([0,T])}.$$  

The notation we will employ is quite standard. The factor $1/\sqrt{2\pi}$ in the definition of the Fourier transform does not alter our analysis, so will omit it.

Given $s \in \mathbb{R}$, the $L^2$-based Sobolev spaces $H^s(\mathbb{R})$ are defined by

$$H^s(\mathbb{R}) = \{f \in S'(\mathbb{R}) : \|f\|_{H^s} < \infty\},$$

where

$$\|f\|_{H^s} = \left\|\langle \xi \rangle^s \hat{f}(\xi)\right\|_{L^2},$$

and $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$.

Similarly we define $\dot{H}^s(\mathbb{R})$ by its norm $\|f\|_{\dot{H}^s} = \left\|\|\xi|^s \hat{f}(\xi)\|_{L^2}\right\|$. Recall that for $\lambda > 0$,

$$\|f(\lambda \cdot)\|_{H^s} \leq (\lambda^{-1/2} + \lambda^{s-1/2}) \|f\|_{H^s} \quad \text{and} \quad \|f(\lambda \cdot)\|_{\dot{H}^{s}} \sim \lambda^{s-1/2} \|f\|_{H^s}. \quad (1.7)$$

We also consider space-time spaces $H^{b,s}(\mathbb{R}^2)$ endowed with the norm

$$\|f\|_{H^{b,s}} = \left\|\langle \tau \rangle^b \langle \xi \rangle^s \hat{f}(\xi, \tau)\right\|_{L^2_t L^2_x}.$$  

Let $U(\cdot)$ be the unitary group in $H^s(\mathbb{R})$, $s \in \mathbb{R}$ associated to the linear Benjamin-Ono equation, i.e.,

$$\mathcal{F}_x(U(t)\varphi)(\xi) = e^{-i\xi[|\xi| + t]}\hat{\varphi}(\xi),$$

for $t \in \mathbb{R}$ and $\varphi \in H^s(\mathbb{R})$.

We denote by $S(\cdot)$ the $H^s(\mathbb{R})$ semigroup generated by the operator $\mathcal{H}_{xx} - (D_x^a - D_x^b)$, which is equivalently defined via the Fourier transform by

$$\mathcal{F}_x(S(t)\varphi)(\xi) = e^{-i|\xi|t + (|\xi|^{\alpha} - |\xi|^{\beta})t} \hat{\varphi}(\xi), \quad t \geq 0.$$  

We extend $S(\cdot)$ to a linear operator on the whole real axis by setting

$$\mathcal{F}_x(S(t)\varphi)(\xi) = e^{-i|\xi|t + (|\xi|^{\alpha} - |\xi|^{\beta})t} \hat{\varphi}(\xi), \quad t \in \mathbb{R}.$$  

Now when the dissipation terms satisfy $1 < \beta < 2$ with $0 < \alpha < \beta$, we introduce the function space $X^{a,b}$ in the sense of Bourgain [4] and Molinet and Ribaud [16]. Since the linear symbol of (1.1) is $i|\xi| - (|\xi|^{\alpha} - |\xi|^{\beta})$, we define the function space $X^{a,b}$ to be the completion of the Schwartz space $S(\mathbb{R}^2)$ on $\mathbb{R}^2$ endowed with the norm

$$\|u\|_{X^{a,b}} = \left\|\langle \tau + \xi \rangle^{a} (|\xi|^{\alpha} - |\xi|^{\beta})^b \hat{\xi} \hat{\tau} (\xi, \tau)\right\|_{L^2(\mathbb{R}^2)}, \quad (1.8)$$

for $u \in S(\mathbb{R}^2)$. The uniform $X^{a,b}$ norm is defined by $\|u\|_{X^{a,b}} := \sup_{t \geq 0} \|S(t)u\|_{X^{a,b}}$.
or equivalently,

\[ \|u\|_{X_{b,s}^T} = \|\tau \xi| + |\xi|^{\alpha} - |\xi|^{\beta} \|_{L^2(\mathbb{R}^2)}. \]

For \( T > 0 \), we consider the localized spaces \( X_{b,s}^T \) endowed with the norm

\[ \|u\|_{X_{b,s}^T} = \inf \{ \|w\|_{X_{b,s}^T} : w(t) = u(t) \text{ on } [0, T] \}. \]  

(1.9)

Next, we consider the restrictions \( \beta \geq 2 \) with \( 0 < \alpha < \beta \). In this case, we can rely on pure dissipative methods to deduce well-posedness results. Thus, given \( s \in \mathbb{R} \) and \( 0 < t \leq T \leq 1 \) fixed, we denote by

\[ Y_{T}^s = \left\{ u \in C([0, T]; H^s(\mathbb{R})) : \|u\|_{Y_{T}^s} < \infty \right\}, \]

where

\[ \|u\|_{Y_{T}^s} := \sup_{t \in [0, T]} \left( \|u(t)\|_{H^s} + t^{s/\beta} \|u(t)\|_{L^2} \right). \]  

(1.10)

Note that when \( s \geq 0 \), \( Y_{T}^s = C([0, T]; H^s(\mathbb{R})) \) and \( \|u\|_{Y_{T}^s} \sim \|u\|_{L^{\infty}_T H^s}. \)

We mainly work on the integral formulation of (1.1) denoted by

\[ u(t) = S(t)u_0 - \frac{1}{2} \int_0^t S(t-\tau)\partial_x u^2(\tau) \, d\tau, \quad t \geq 0. \]  

(1.11)

valid for any sufficiently regular solution. When the dissipation \( \beta < 2 \) it will be convenient to replace the local-in-time integration (1.11) with a global-in-time truncated equation. Let \( \psi \) be a cutoff function such that

\[ \psi \in C_0^\infty(\mathbb{R}), \quad \text{supp}(\psi) \subset [-2, 2], \quad \psi \equiv 1 \text{ on } [-1, 1], \]

and set \( \psi_T(\cdot) = \psi(\cdot/T) \) for all \( T > 0 \). Thus we can replace (1.11) on time interval \([0, T], T < 1 \) by the equation

\[ u(t) = \psi(t) \left[ S(t)u_0 - \frac{\chi_{R^+}(t)}{2} \int_0^t S(t-\tau)\partial_x (\psi_T^2(\tau)u^2(\tau)) \, d\tau \right], \quad t \geq 0. \]  

(1.12)

\section{Well-Posedness case \( 1 < \beta < 2 \).}

In this section we establish Theorem 1.1 when the dispersion parameters satisfy: \( 1 < \beta < 2 \) with \( 0 < \alpha < \beta \). In this case, we see that the dispersive part of (1.1) has an important role to obtain low regularity solutions. Consequently, we will apply a contraction argument on the integral equation (1.11) on the \( X_{b,s}^T \) spaces, which take into account these effects. The main ingredient to apply this technique is the derivation of a key bilinear estimate (see Proposition 2.1 below).

\subsection{Linear estimates}

In this part we estimate the operator \( \psi(\cdot)S(\cdot) \) as well as the linear operator \( L \) defined by

\[ L : f \mapsto \chi_{R^+}(t)\psi(t) \int_0^t S(t-\tau) f(\tau) \, d\tau. \]

Our approach follows closely the arguments in [16] and [22].

We first study the action of the semigroup \( \{S(t)\}_{t \geq 0} \) on \( H^s(\mathbb{R}), s \in \mathbb{R} \).

\begin{proposition} \label{prop:2.1}
Let \( 0 < \alpha < \beta \) and

\[ \psi_{\alpha,\beta}(t) = \exp \left( \frac{2\alpha}{\beta} \frac{t}{\beta} \right) \left( \frac{\beta - \alpha}{\beta} \right) t, \quad t \in \mathbb{R}. \]

(2.1)
\end{proposition}
Consider $\delta \geq 0$ and $s \in \mathbb{R}$. Then for all $t > 0$ it follows
\[
\|S(t)\phi\|_{H^{s+\lambda}} \lesssim \psi_{\alpha, \beta}(t) \left(1 + t^{-\delta/\beta}\right) \|\phi\|_{H^s},
\] (2.2)
where $\phi \in H^s(\mathbb{R})$ and the implicit constant depends on $\delta$ and $\beta$. Moreover, the map $t \mapsto S(t)\phi$ belongs to $C((0, \infty); H^{s+\delta}(\mathbb{R}))$.

Proof. Noting that for all $0 < \alpha < \beta$ and $t > 0$,
\[
\left\|e^{(|\xi|^{\alpha} - |\xi|^{\beta})t}\right\|_{L^\infty} = \exp \left(\frac{2\alpha}{\beta} \frac{t^{\frac{\beta}{\alpha} - \alpha}}{\beta} \right) = \psi_{\alpha, \beta}(t),
\]
we derive the bound
\[
|e^{(|\xi|^{\alpha} - |\xi|^{\beta})t}| \lesssim \psi_{\alpha, \beta}(t)e^{-|\xi|^{\beta}t/2}.
\] (2.3)
From this and setting $w = t^{1/\beta}\xi$, we see that
\[
\left\|e^{(|\xi|^{\alpha} - |\xi|^{\beta})t}\right\|_{L^\infty} \lesssim \psi_{\alpha, \beta}(t) \left|t^{-1/\beta}w\right|^\delta e^{-|w|^{\beta}t/2}
\]
Since
\[
(1 + t^{-2/\beta}|w|^2)^{\delta/2} \lesssim 1 + t^{-\delta/\beta}|w|^\delta,
\]
we find
\[
\|S(t)\phi\|_{H^{s+\lambda}} \lesssim \left\|\xi\right\|^\delta \left\|e^{(|\xi|^{\alpha} - |\xi|^{\beta})t}\right\|_{L^\infty} \|\phi\|_{H^s}
\]
\[
\lesssim \psi_{\alpha, \beta}(t) \left(1 + t^{-\delta/\beta}\right) \|\phi\|_{H^s}.
\]
This establishes (2.2). The continuity of the map $t \mapsto S(t)\phi$ is deduced arguing as in the proof of Proposition 2.2 in [3].

Lemma 2.1. For all $s \in \mathbb{R}$ and all $\varphi \in H^s(\mathbb{R})$
\[
\|\hat{\psi}(t)\hat{S}(t)\varphi\|_{X^{1/2, s}} \lesssim \|\varphi\|_{H^s}.
\] (2.5)
Proof. By definition of the $\|\cdot\|_{X^{1/2, s}}$-norm, we find the following upper-bound
\[
\|\hat{\psi}(t)\hat{S}(t)\varphi\|_{X^{1/2, s}} \lesssim \left\|\xi\right\|^s \hat{\varphi}(\xi) \left\|\langle\tau\rangle^{1/2} \mathcal{F}_t(g_\xi(t))\right\|_{L^2_1(\mathbb{R})}
\]
\[
+ \left\|\xi\right\|^s \left\|\xi^{\alpha} - |\xi|^{\beta}\right\|^{1/2} \hat{\varphi}(\xi) \|g_\xi(t)\|_{L^2_1(\mathbb{R})}
\] (2.6)
where we have set $g_\xi(t) := \psi(t)e^{(|\xi|^{\alpha} - |\xi|^{\beta})|t|}$. Therefore, in view of (2.6), it is enough to estimate $\|g_\xi\|_{H^b_t}$ for $b \in \{0, 1/2\}$. First assume that $|\xi| \geq 2^{1/\alpha}$, then
\[
\|g_\xi\|_{H^b_t} = \left\|\langle\tau\rangle^{b} \hat{\psi} * \mathcal{F}_t(e^{(|\xi|^{\alpha} - |\xi|^{\beta})|t|})\right\|_{L^2}
\]
\[
\lesssim \left\|\langle\tau\rangle^{b} \hat{\psi}(\tau)\right\|_{L^2} \left\|e^{(|\xi|^{\alpha} - |\xi|^{\beta})|t|}\right\|_{L^2} + \left\|\hat{\psi}(\tau)\right\|_{L^2} \left\|e^{(|\xi|^{\alpha} - |\xi|^{\beta})|t|}\right\|_{H^{b/2}_t},
\] (2.7)
so that (1.7) yields to
\[
\|g_\xi\|_{H^b_t} \lesssim (|\xi|^{\alpha} - |\xi|^{\beta}g^{b-1/2} + |\xi|^{\alpha} - |\xi|^{\beta})^{-1/2}).
\]
(2.8)
On the other hand, notice that when $|\xi| < 2^{1/\alpha}$,
\[
\|g_\xi\|_{H^b} = \left\|\psi(t)e^{(|\xi|^{\alpha} - |\xi|^{\beta})|t|}\right\|_{H^b_t} \lesssim \sum_{n=0}^{\infty} \frac{|\xi|^{\alpha} - |\xi|^{\beta}n}{n!} \|\langle\xi\rangle^n \psi(t)\|_{H^b}. 
\]
Since \( n \geq 1 \), \( \|t|^{\alpha} \psi(t)\|_{H^b} \leq \|t|^{\alpha} \psi(t)\|_{H^1} \lesssim n \), it follows

\[
\|g_{\xi}\|_{H^b} \lesssim 2^{\frac{a}{\alpha}} \sum_{n=1}^{\infty} \frac{2^{a(n-1)}}{(n-1)!} \lesssim 1. \tag{2.9}
\]

Hence combining (2.8) and (2.9) we arrive at

\[
\|g_{\xi}\|_{H^b} \lesssim |\xi|^\alpha - |\xi|^\beta)^{-b},
\]

for \( b \in \{0, 1\} \). The desire estimate now follows substituting the above inequality in (2.6). \qed

Next we deduce some linear estimates dealing with the forcing term.

**Proposition 2.2.** For \( w \in S(\mathbb{R}^2) \), consider \( K_{\xi} \) defined on \( \mathbb{R} \) by

\[
K_{\xi}(t) = \psi(t) \int_{\mathbb{R}} \frac{e^{i|\tau|} - e^{(|\xi|^\alpha - |\xi|^\beta)|t|}}{i|\tau| - (|\xi|^\alpha - |\xi|^\beta)\omega(\tau)} \, d\tau. \tag{2.10}
\]

Then, it holds for all \( \xi \in \mathbb{R} \) that

\[
\left\| (i\tau - (|\xi|^\alpha - |\xi|^\beta))^{1/2} F_t(K_{\xi}) \right\|_{L^2(\mathbb{R})} \lesssim \left( \int_{\mathbb{R}} \frac{|\hat{\omega}(\tau)|}{i|\tau| - (|\xi|^\alpha - |\xi|^\beta)} \, d\tau + \left( \frac{\int_{\mathbb{R}} |\hat{\omega}(\tau)|^2}{i|\tau| - (|\xi|^\alpha - |\xi|^\beta)} \, d\tau \right)^{1/2} \right)^{1/2}. \tag{2.11}
\]

**Proof.** We proceed as in [16, Proposition 2.2]. We divide \( K_{\xi} \) in the following manner

\[
K_{\xi}(t) = \psi(t) \int_{|\tau| \leq 1} \frac{e^{i|\tau|} - 1}{i|\tau| - (|\xi|^\alpha - |\xi|^\beta)} \hat{\omega}(\xi) \, d\tau + \psi(t) \int_{|\tau| \geq 1} \frac{1 - e^{(|\xi|^\alpha - |\xi|^\beta)|t|}}{i|\tau| - (|\xi|^\alpha - |\xi|^\beta)} \hat{\omega}(\xi) \, d\tau
\]

\[
+ \psi(t) \int_{|\tau| \geq 1} \frac{e^{i|\tau|}}{i|\tau| - (|\xi|^\alpha - |\xi|^\beta)} \hat{\omega}(\xi) \, d\tau \tag{2.12}
\]

\[
= I + II + III + IV.
\]

We will estimate each contribution of the four terms in (2.12). \hfill \Box

**Contribution of IV.** Noting that \( \langle i\tau - (|\xi|^\alpha - |\xi|^\beta) \rangle \lesssim |i\tau - (|\xi|^\alpha - |\xi|^\beta)| \) when \( \tau \geq 1 \), we deduce

\[
\left\| (i\tau - (|\xi|^\alpha - |\xi|^\beta))^{1/2} F_t(IV) \right\|_{L^2(\mathbb{R})} \lesssim \left\| (i\tau - (|\xi|^\alpha - |\xi|^\beta))^{1/2} \left[ \psi(\tau') \ast \left( \frac{\hat{\omega}(\tau') \chi_{\{|\tau'| \geq 1\}}}{i\tau' - (|\xi|^\alpha - |\xi|^\beta)} \right) \right] \right\|_{L^2(\mathbb{R})} \tag{2.13}
\]

\[
x \int_{\mathbb{R}} \frac{|\hat{\omega}(\tau)|}{i|\tau| - (|\xi|^\alpha - |\xi|^\beta)} \, d\tau.
\]

**Contribution of III.** Clearly,

\[
\left\| (i\tau - (|\xi|^\alpha - |\xi|^\beta))^{1/2} F_t(III) \right\|_{L^2(\mathbb{R})} = \left\| (i\tau - (|\xi|^\alpha - |\xi|^\beta))^{1/2} \left[ \hat{\psi}(\tau') \ast \left( \frac{\hat{\omega}(\tau') \chi_{\{|\tau'| \geq 1\}}}{i\tau' - (|\xi|^\alpha - |\xi|^\beta)} \right) \right] \right\|_{L^2(\mathbb{R})}. \tag{2.14}
\]

Since \( \langle i\tau - (|\xi|^\alpha - |\xi|^\beta) \rangle \leq \langle \tau' \rangle + |i(\tau - \tau') - (|\xi|^\alpha - |\xi|^\beta)| \) for any \( (\tau, \tau') \in \mathbb{R}^2 \), we deduce from Young’s inequality that

\[
\left\| (i\tau - (|\xi|^\alpha - |\xi|^\beta))^{1/2} F_t(III) \right\|_{L^2(\mathbb{R})} \lesssim \left\| \langle \tau' \rangle \hat{\psi}(\tau') \ast \left( \frac{\hat{\omega}(\tau') \chi_{\{|\tau'| \geq 1\}}}{i\tau' - (|\xi|^\alpha - |\xi|^\beta)} \right) \right\|_{L^2(\mathbb{R})} \tag{2.15}
\]

\[
+ \left\| \hat{\psi}(\tau') \ast \left( \frac{\hat{\omega}(\tau') \chi_{\{|\tau'| \geq 1\}}}{i\tau' - (|\xi|^\alpha - |\xi|^\beta)^{1/2}} \right) \right\|_{L^2(\mathbb{R})}
\]

\[
\lesssim \left\| \frac{\hat{\omega}(\tau)}{(i\tau - (|\xi|^\alpha - |\xi|^\beta))^{1/2}} \right\|_{L^2}.
\]
Contribution of $II$. When $\xi \in \{0,1,-1\}$, $II = 0$ and the estimate follows trivially. Thus, we will assume that $\xi \notin \{0,1,-1\}$, i.e., $|\xi|^\alpha - |\xi|^\beta \neq 0$. By Cauchy-Schwarz inequality,

\[
\left\| \langle it - (|\xi|^\alpha - |\xi|^\beta) \rangle^{1/2} F_t(II) \right\|_{L^2(\mathbb{R})} \\
\quad \lesssim \left\| \langle it - (|\xi|^\alpha - |\xi|^\beta) \rangle^{1/2} F_t \left( \psi(t)(1 - e^{i(|\xi|^\alpha - |\xi|^\beta)|t|}) \right) \right\|_{L^2(\mathbb{R})} \\
\quad \times \left( \int_{\mathbb{R}} \frac{|\hat{\psi}(\tau)|^2}{i\tau - (|\xi|^\alpha - |\xi|^\beta)} d\tau \right)^{1/2} \left( \int_{|\tau| \leq 1} \frac{|\langle it - (|\xi|^\alpha - |\xi|^\beta) \rangle|}{|i\tau - (|\xi|^\alpha - |\xi|^\beta)|^2} d\tau \right)^{1/2}
\tag{2.16}
\]

Therefore we obtain

\[
\left\| \langle it - (|\xi|^\alpha - |\xi|^\beta) \rangle^{1/2} F_t \left( \psi(t)(1 - e^{i(|\xi|^\alpha - |\xi|^\beta)|t|}) \right) \right\|_{L^2(\mathbb{R})} \\
\quad \lesssim \left\| \langle it - (|\xi|^\alpha - |\xi|^\beta) \rangle^{1/2} \hat{\psi}(\tau) \right\|_{L^2(\mathbb{R})} \\
\quad + \left\| \langle it - (|\xi|^\alpha - |\xi|^\beta) \rangle^{1/2} F_t \left( \psi e^{i(|\xi|^\alpha - |\xi|^\beta)|t|} \right) \right\|_{L^2(\mathbb{R})} \\
\quad \lesssim (|\xi|^\alpha - |\xi|^\beta)^{1/2},
\tag{2.17}
\]

so that if $|\xi| \geq 2^{\frac{1}{1-n}}$,

\[
\left\| \langle it - (|\xi|^\alpha - |\xi|^\beta) \rangle^{1/2} F_t(II) \right\|_{L^2(\mathbb{R})} \lesssim \left( \int_{\mathbb{R}} \frac{|\hat{\psi}(\tau)|^2}{i\tau - (|\xi|^\alpha - |\xi|^\beta)} d\tau \right)^{1/2}.
\tag{2.18}
\]

Now, if $|\xi| < 2^{\frac{1}{1-n}}$ we deduce

\[
\left\| \langle it - (|\xi|^\alpha - |\xi|^\beta) \rangle^{1/2} F_t \left( \psi(t)(1 - e^{i(|\xi|^\alpha - |\xi|^\beta)|t|}) \right) \right\|_{L^2(\mathbb{R})} \lesssim \left\| \psi(t)(1 - e^{i(|\xi|^\alpha - |\xi|^\beta)|t|}) \right\|_{H^{1/2}} \\
\quad \lesssim \sum_{n=1}^{\infty} \frac{|\xi|^\alpha - |\xi|^\beta|^n}{n!} |||t||^n \psi||_{H^{1/2}}
\tag{2.19}
\]

recalling that $|||t||^n \psi||_{H^{1/2}} \leq |||t||^n \psi||_{H^1} \lesssim n$. Gathering (2.16) and (2.19) we obtain

\[
\left\| \langle it - (|\xi|^\alpha - |\xi|^\beta) \rangle^{1/2} F_t(II) \right\|_{L^2(\mathbb{R})} \lesssim (|\xi|^\alpha - |\xi|^\beta)^{1/2} \left( \int_{\mathbb{R}} \frac{|\hat{\psi}(\tau)|^2}{i\tau - (|\xi|^\alpha - |\xi|^\beta)} d\tau \right)^{1/2}, \quad \forall |\xi| < 2^{\frac{1}{1-n}}.
\tag{2.20}
\]

Hence, estimate $II$ is now a consequence of (2.18) and (2.20).

Estimate $I$. Since $I$ can be rewritten as

\[
I = \psi(t) \int_{|\tau| \leq 1} \sum_{n \geq 1} \frac{(it\tau)^n}{(i\tau - (|\xi|^\beta - |\eta|^\beta) n!} \hat{\psi}(\tau) \psi(t) d\tau,
\]

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Moreover, if \( (f_n) \) is a sequence with \( f_n \to 0 \) in \( X^{-1/2+\delta,s} \), then

\[
\left\| \int_0^t S(t-\tau) f_n(\tau) \, d\tau \right\|_{L^\infty([0,T];H^{s+\beta\delta}([0,T]))} \to 0.
\]
Proof. Without loss of generality, we can set $s = 0$. Since $U(\cdot)$ is a strongly continuous unitary group in $L^2(\mathbb{R})$, it is enough to prove that

$$ t \mapsto U(-t) \int_0^t S(t - \tau) f(\tau) \, d\tau, $$

is continuous from $[0, T]$ to $H^{2\beta}(\mathbb{R})$. Thus, setting $g(x, t) = (U(-t)f(t))(x)$, we have that (2.24) will be proven if we show the continuity of the map

$$ F(\xi, \cdot) : t \in \mathbb{R}^+ \mapsto \langle \xi \rangle^{2\beta} \int_0^t e^{(\langle \xi \rangle^{\alpha} - |\xi|^\beta)t - \tau} F_x(g(\cdot, \tau))(\xi) \, d\tau. \tag{2.26} $$

When $\langle i\tau - (|\xi|^\alpha - |\xi|^\beta) \rangle^{-1/2 + \delta}\tilde{g}(\xi, \tau) \in L^2_{\xi, \tau}(\mathbb{R}^2)$. Applying Fubini’s theorem,

$$ F(t) = \langle \xi \rangle^{2\beta} \int_{\mathbb{R}} e^{it\tau - \langle |\xi|^\alpha - |\xi|^\beta \rangle t} \tilde{g}(\xi, \tau) \, d\tau. \tag{2.27} $$

Thus,

$$ F(t_1) - F(t_2) = \langle \xi \rangle^{2\beta} \int_{\mathbb{R}} \frac{\tilde{g}(\xi, \tau)}{it - (|\xi|^\alpha - |\xi|^\beta)} \left[ (e^{it_1\tau} - e^{it_2\tau}) - (e^{(\langle \xi \rangle^{\alpha} - |\xi|^\beta)t_1} - e^{(\langle \xi \rangle^{\alpha} - |\xi|^\beta)t_2}) \right] \, d\tau. \tag{2.28} $$

We will use Lebesgue Dominated Convergence Theorem to show that $F(t_1) - F(t_2) \to 0$ as $t_1 \to t_2$ on the $L^2$ topology.

We start considering the case $|\xi| \geq 2^{5/\alpha - \frac{1}{2}}$. Then $|\xi|^\alpha - |\xi|^\beta \leq -\frac{|\xi|^\beta}{2}$ and so

$$ \left| e^{(\langle \xi \rangle^{\alpha} - |\xi|^\beta)t_1} - e^{(\langle \xi \rangle^{\alpha} - |\xi|^\beta)t_2} \right| \lesssim 1. $$

Hence, from the above estimate together with the Cauchy-Schwarz inequality and performing the change of variable $\tau = \langle |\xi|^\alpha - |\xi|^\beta \rangle \theta$ we get

$$ |F(t_1) - F(t_2)| \lesssim \langle \xi \rangle^{2\beta} \left( \int_{\mathbb{R}} \frac{|\tilde{g}(\xi, \tau)|^2}{|it - (|\xi|^\alpha - |\xi|^\beta)|^{1 - 2\delta}} \, d\tau \right)^{1/2} \left( \int_{\mathbb{R}} \frac{\langle i\tau - (|\xi|^\alpha - |\xi|^\beta) \rangle^{1 - 2\delta}}{|it - (|\xi|^\alpha - |\xi|^\beta)|^2} \, d\tau \right)^{1/2} \tag{2.29} $$

$$ \lesssim \langle \xi \rangle^{2\beta} \left( \int_{\mathbb{R}} \frac{|\tilde{g}(\xi, \tau)|^2}{|it - (|\xi|^\alpha - |\xi|^\beta)|^{1 - 2\delta}} \, d\tau \right)^{1/2} \left| |\xi|^\alpha - |\xi|^\beta \right|^{-\delta} \left( \int_{\mathbb{R}} \frac{1}{| \theta - 1 - 2\delta |} \, d\theta \right)^{1/2} $$

Assume now that $|\xi| \leq 2^{5/\alpha - \frac{1}{2}}$. We first estimate,

$$ \left| \int \frac{\tilde{g}(\xi, \tau)}{it - (|\xi|^\alpha - |\xi|^\beta)} (e^{it_1\tau} - e^{it_2\tau}) \, d\tau \right| $$

$$ \lesssim \min \{1, |t_1 - t_2|\} \int_{|\tau| \leq 1} \frac{|\tau| |\tilde{g}(\xi, \tau)|}{|it - (|\xi|^\alpha - |\xi|^\beta)|} + 2 \int_{|\tau| \geq 1} \frac{|\tilde{g}(\xi, \tau)|}{|it - (|\xi|^\alpha - |\xi|^\beta)|} \, d\tau $$

$$ \lesssim \left\| (i\tau - (|\xi|^\alpha - |\xi|^\beta)\tilde{g}(\xi, \tau))^{-1/2 + \delta} \right\|_{L^2_{\xi, \tau}} \left[ \left( \int_{|\tau| \leq 1} \langle \tau \rangle^{1 - 2\delta} \right)^{1/2} + \left( \int_{|\tau| \geq 1} \langle \tau \rangle^{-1 - 2\delta} \right)^{1/2} \right] $$

$$ \lesssim \left\| (i\tau - (|\xi|^\alpha - |\xi|^\beta)\tilde{g}(\xi, \tau))^{-1/2 + \delta} \right\|_{L^2_{\xi, \tau}}. \tag{2.30} $$

On the other hand, assume that $t_1, t_2 \leq T$. From the mean-value inequality it easily seen

$$ \left| e^{(\langle \xi \rangle^{\alpha} - |\xi|^\beta)t_1} - e^{(\langle \xi \rangle^{\alpha} - |\xi|^\beta)t_2} \right| \lesssim \psi_{\alpha, \beta}(T) \left| |\xi|^\alpha - |\xi|^\beta \right| |t_1 - t_2|, \tag{2.31} $$

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where $\psi_{\alpha,\beta}(T)$ is defined by (2.1). Then we estimate the remaining term,
\[
\left| \int_{\mathbb{R}} \tau \frac{\hat{g}(\xi, \tau) e^{i\xi t} - e^{i(\xi_1 - \xi_2) t}}{\tau - (|\xi|^\alpha - |\xi|^\beta)^{1-2/\delta}} d\tau \right|
\]
\[
\lesssim \psi_{\alpha,\beta}(T)|t_1 - t_2| \left\| \langle \xi \rangle - (|\xi|^\alpha - |\xi|^\beta)^{-1/2+\delta} \hat{g}(\xi, \cdot) \right\|_{L^2_\xi} \left\| |\xi|^\alpha - |\xi|^\beta \right\| \left( \int_{\mathbb{R}} \frac{\langle \xi \rangle - (|\xi|^\alpha - |\xi|^\beta)^{1-2/\delta}}{\tau - (|\xi|^\alpha - |\xi|^\beta)^{2}} d\tau \right)^{1/2}
\]
\[
\lesssim \psi_{\alpha,\beta}(T)|t_1 - t_2| \left\| \langle \xi \rangle - (|\xi|^\alpha - |\xi|^\beta)^{-1/2+\delta} \hat{g}(\xi, \cdot) \right\|_{L^2_\xi} \left\| |\xi|^\alpha - |\xi|^\beta \right\| \left( |\xi|^\alpha - |\xi|^\beta \right)^{-1} + |\xi|^\alpha - |\xi|^\beta)^{-2/\delta}
\]
\[
\lesssim \psi_{\alpha,\beta}(T)|t_1 - t_2| \left\| \langle \xi \rangle - (|\xi|^\alpha - |\xi|^\beta)^{-1/2+\delta} \hat{g}(\xi, \cdot) \right\|_{L^2_\xi} .
\]

Combining (2.29), (2.30) and (2.32) we deduce
\[
|F(t_1) - F(t_2)| \lesssim \left( \int_{\mathbb{R}} \frac{|\hat{g}(\xi, \tau)|^2}{\tau - (|\xi|^\alpha - |\xi|^\beta)^{1-2/\delta}} d\tau \right)^{1/2} .
\]

Observe that the above estimates imply that the function $\tau \to |\hat{g}(\xi, \tau)|(|\tau - (|\xi|^\alpha - |\xi|^\beta)^{-1} \in L^1_\xi(\mathbb{R})$ for almost every $\xi$, given that $\tau - (|\xi|^\alpha - |\xi|^\beta)^{-1/2+\delta} \in L^2_\xi(\mathbb{R}^2)$. Therefore, since the integrand in (2.28) tends to 0 as $|t_1 - t_2| \to 0$, and it is uniformly bounded on time by the function $\tau \to |\hat{g}(\xi, \tau)|(|\tau - (|\xi|^\alpha - |\xi|^\beta)| \in L^1(\mathbb{R})$ for some fixed constant $c$, it follows from Lebesgue dominated convergence theorem that $|F(t_1) - F(t_2)| \to 0$ as $|t_1 - t_2| \to 0$. Consequently, this convergence, estimate (2.33) and Lebesgue dominated convergence theorem lead to
\[
\lim_{t_1 \to t_2} |F(t_1) - F(t_2)| = 0.
\]

To show (2.25), it suffices to notice
\[
\sup_{t \in [0, T]} \left\| F_n(t) \right\|_{L^2(\mathbb{R})} \lesssim \left\| \langle \xi \rangle - (|\xi|^\alpha - |\xi|^\beta)^{-1/2+\delta} \hat{g}(\xi, \tau) \right\|_{L^2_{\xi, \tau}},
\]
where $F_n$ is defined as $F$ with $g$ replaced by $g_n(\cdot) = F_{x}(U(\cdot) f_n(\cdot))$. \hfill \Box

We conclude this subsection with the following result which will be used in the proof of Theorem 1.1.

**Lemma 2.2.** Let $s \in \mathbb{R}$. For any $\epsilon > 0$ and $T \in (0, 1)$, the following estimate follows
\[
\| \psi_{\tau} u \|_{X^{1/2, s}} \lesssim \tau^{-\epsilon} \| u \|_{X^{1/2, s}} .
\]

**Proof.** By setting $v(x, t) = U(-t) u(x, t)$ we observe
\[
\| \psi_{\tau} u \|_{X^{1/2, s}} = T \left\| \langle \xi \rangle^s \left( \psi_{\tau}^{\delta}(T\cdot) * \hat{u} \right)(\xi, \tau) \right\|_{L^2_{\xi, \tau}} .
\]

Hence using that $|\langle \xi \rangle^s - |\xi|^\beta \rangle \lesssim \langle \tau - \tau_1 \rangle^{1/2} + |\tau_1|^{1/2}, \forall \tau_1 \in \mathbb{R},$

we are reduced to estimate
\[
T \left\| \langle \xi \rangle^s \left( \psi_{\tau}^{\delta}(T\cdot) * (\langle \cdot \rangle^{1/2} \hat{u} \right)(\xi, \tau) \right\|_{L^2_{\xi, \tau}} + T \left\| \langle \xi \rangle^s (| \cdot |^{1/2} \psi_{\tau}^{\delta}(T\cdot) * \hat{u}) \right\|_{L^2_{\xi, \tau}} \]}

\[
=: A_1 + A_2.
\]

By Young’s inequality one finds
\[
A_1 \lesssim T \left\| \langle \xi \rangle^s \right\|_{L^1_{\xi}} \left\| \langle \tau \rangle^{1/2} \hat{u} \right\|_{L^2_{\xi}} \lesssim \| u \|_{X^{1/2, s}} .
\]
In the same manner is deduced

\[ A_2 \lesssim T^{1/2} \left\| \langle \xi \rangle^s \left( \left( 1 \cdot \hat{\varphi}^2 \right)(T\cdot) \right) \right\|_{L_p^{21}} \left\| \hat{\varphi}(\xi, \tau) \right\|_{L^{p'}_{L^2}} , \]  

(2.36)

by Young’s inequality provided that \( p_1 \leq 2, 1 \leq p_2 \) and \( 1/p_1 + 1/p_2 = 3/2 \), or equivalently \( 1/p_2 = 1/2 + 1/p \) with \( 1/p_1 + 1/p = 1 \). By Hölder’s inequality

\[ \left\| \hat{\varphi}(\xi, \tau) \right\|_{L^{p'}_{L^2}} \lesssim \left\| \langle \tau \rangle^{-1/2} \right\|_{L_p} \left\| \langle \tau \rangle^{1/2} \hat{\varphi}(\xi, \tau) \right\|_{L^2} , \]  

(2.37)

for given \( p > 2 \) and almost every \( \xi \in \mathbb{R} \). Gathering (2.36) and (2.37) we arrive at

\[ A_2 \lesssim T^{-(1/2-1/p)} \left\| u \right\|_{X^{1/2,s}} . \]  

(2.38)

From the above estimate we can take any \( p > 2 \) to assure the term \( T^{-\epsilon} \) for arbitrary small \( \epsilon > 0 \). Having into account that \( T \in (0, 1] \) the proof is completed. \( \square \)

### 2.2 Bilinear estimates

To deduce the crucial bilinear result we will apply the dyadic block estimates deduced in [29]. This approach is based on Tao’s \([k; Z]\)-multiplier theory [27].

We first introduce some notation and results to be employed in our arguments (see [27] for a detailed discussion). Let \( Z \) be any abelian additive group with an invariant measure \( d\eta \). For any integer \( k \geq 2 \) we define the hyperplane

\[ \Gamma_k(Z) = \{ (\eta_1, \ldots, \eta_k) \in Z^k : \eta_1 + \cdots + \eta_k = 0 \} , \]

which is endowed with the measure

\[ \int_{\Gamma_k(Z)} f = \int_{Z^{k-1}} f(\eta_1, \ldots, \eta_{k-1}, -(\eta_1 + \cdots + \eta_k)) d\eta_1 \cdots d\eta_k . \]

A \([k; Z]\)-multiplier is defined to be any function \( m : \Gamma_k(Z) \to \mathbb{C} \). The norm of the multiplier \( ||m||_{[k; Z]} \) is defined to be the best constant such that the inequality

\[ \left| \int_{\Gamma_k(Z)} m(\eta) \prod_{j=1}^k f_j(\eta_j) \right| \leq ||m||_{[k; Z]} \prod_{j=1}^k ||f_j||_{L^2(\mathbb{R})} , \]  

(2.39)

holds for all test function \( f_1, \ldots, f_k \) on \( Z \). In other words,

\[ ||m||_{[k; Z]} = \sup_{\|f_j\|_{L^2(\mathbb{R})} \leq 1} \left| \int_{\Gamma_k(Z)} m(\eta) \prod_{j=1}^k f(\eta_j) \right| . \]  

(2.40)

Following the notation in [27], capitalized variables such as \( N_1, L_3, H \) are presumed to be dyadic, i.e. these variables range over numbers of the form \( 2^l \) for \( l \in \mathbb{Z} \). Let \( N_1, N_2, N_3 > 0 \), we define the quantities \( N_{\max} \geq N_{\med} \geq N_{\min} \) to be the maximum, median, and minimum of \( N_1, N_2, N_3 \) respectively. Similarly, we define \( L_{\max} \geq L_{\med} \geq L_{\min} \) whenever \( L_1, L_2, L_3 > 0 \). The quantities \( N_j \) will measure the magnitude of frequencies of our waves, while \( L_j \) measures how closely our waves approximate a free solution.

We adopt the following summation conventions. Any summation of the form \( L_{\max} \sim \ldots \) is a sum over the three dyadic variables \( L_1, L_2, L_3 \geq 1 \), thus for instance

\[ \sum_{L_{\max} \sim H} := \sum_{L_1, L_2, L_3 \geq 1: L_{\max} \sim H} . \]

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Similarly, any summation of the form $N_{\max} \sim \ldots$ sum over the three dyadic variables $N_1, N_2, N_3 > 0$, hence
\[
\sum_{L_{\max} \sim H} := \sum_{L_1, L_2, L_3 \geq 1 : L_{\max} \sim H}
\]
Due to the nonlinear term in (1.1), we will consider $[3; \mathbb{R} \times \mathbb{R}]$-multipliers and the variables will be set as $\eta = (\xi, \tau)$ with the usual Lebesgue measure $d\eta = d\xi d\tau$. We also introduce some notation following \cite{29}. We let
\[
h_0(\theta) = -|\theta|, \quad \lambda_j = \tau_j - h_0(\xi_j), \quad j = 1, 2, 3,
\]
and the resonance function
\[
h(\xi) = h_0(\xi_1) + h_0(\xi_2) + h_0(\xi_3), \quad \xi = (\xi_1, \xi_2, \xi_3).
\]
By a dyadic decomposition of the variables $\xi_j, \lambda_j, h(\xi)$, we will lead to estimate
\[
\|X_{N_1, N_2, N_3, H, L_1, L_2, L_3}\|_{[3; \mathbb{R} \times \mathbb{R}]},
\]
where $X_{N_1, N_2, N_3, H, L_1, L_2, L_3}$ is the multiplier
\[
X_{N_1, N_2, N_3, H, L_1, L_2, L_3} = \chi_{|h(\xi)| \sim H} \prod_{j=1}^3 \chi_{|\xi_j| \sim N_j} \chi_{|\lambda_j| \sim L_j}.
\]
From the identities
\[
\xi_1 + \xi_2 + \xi_3 = 0
\]
and
\[
\lambda_1 + \lambda_2 + \lambda_3 + h(\xi) = 0,
\]
on the support of the multiplier, we see that (2.42) vanish unless
\[
N_{\max} \sim N_{\med}
\]
and
\[
L_{\max} \sim \max(H, L_{\med}).
\]
As a consequence we have the following result.

**Lemma 2.3.** On the support of $X_{N_1, N_2, N_3, H, L_1, L_2, L_3}$, one has
\[
H \sim N_{\max} N_{\min}.
\]

**Proof.** See \cite{29, Lemma 3.1}.

Our arguments mainly depend on the following dyadic blocks estimates for the Benjamin-Ono equation. This result was established in \cite{29, Proposition 3.1}.

**Lemma 2.4.** Let $N_1, N_2, N_3, H, L_1, L_2, L_3 > 0$ satisfying (2.45), (2.46) and (2.47).

(i) In the high modulation case $L_{\max} \sim L_{\med} \gg H$, we have
\[
(2.41) \lesssim L_{\min}^{1/2} N_{\min}^{1/2},
\]
(ii) In the low modulation case $L_{\max} \sim H$,
\[
(\text{a) } ((++) \text{ coherence}) \text{ if } N_{\max} \sim N_{\min} \text{ then }
\]
\[
(2.41) \lesssim L_{\min}^{1/2} L_{\med}^{1/4},
\]

(b) \((\pm\cdot)\) coherence \(\) If \(N_2 \sim N_3 \gg N_1 \) and \(H \sim L_1 \gtrsim L_2, L_3\), then for any \(\gamma > 0\)

\[
(2.41) \lesssim L_{\min}^{1/2} \min(N_{\min}^{-1/2}, N_{\max}^{-1/2-\gamma}, N_{\min}^{1/2-\gamma}) L_{\med}^{1/2-\gamma}.
\]

(2.50)

Similar for any permutations of indexes \(\{1, 2, 3\}\).

(c) In all other cases, the multiplier \((2.41)\) vanishes.

The main goal of this section is to derive the following key bilinear estimate.

**Theorem 2.1.** Let \(1 < \beta \leq 2, 0 < \alpha < \beta \) and \(s > -\beta/4\). For all \(T > 0\), there exists \(\delta, \nu > 0\) such that for all \(u, v \in X^{1/2,s}\) with compact support in \([-T, T]\)

\[
\|\partial_x(uv)\|_{X^{-1/2+\delta,s}} \lesssim T^{\nu} \|u\|_{X^{1/2,s}} \|v\|_{X^{1/2,s}}.
\]

(2.51)

Actually, according to the arguments in [16], we mainly use the following bilinear estimate, which is a direct consequence of Theorem 2.1, together with the triangle inequality

\[
\forall s \geq s^+_c, \ (\xi)(\xi) \leq (\xi)_{s}^{s+c} (\xi)_{s-c}^{s-c} + (\xi)_{s-c}^{s} (\xi - \lambda_{1})_{s-c}^{s}.
\]

(2.52)

**Proposition 2.5.** Given \(s^+_c > -\beta/4\), there exist \(\nu, \delta > 0\) such that for any \(s \geq s^+_c\) and \(u, v \in X^{1/2,s}\) with compact support in \([-T, T]\),

\[
\|\partial_x(uv)\|_{X^{-1/2+\delta,s}} \lesssim T^{\nu} \left(\|u\|_{X^{1/2,s}} \|v\|_{X^{1/2,s}} + \|u\|_{X^{1/2,s}} \|v\|_{X^{1/2,s}}\right).
\]

(2.53)

The next lemma gives the contraction factor \(T^{\nu}\) in our estimates (see [19]).

**Lemma 2.5.** Let \(f \in L^2(\mathbb{R}^2)\) with compact on \([-T, T]\). For any \(\theta > 0\), there exists \(\nu = \nu(\theta)\) such that

\[
\left\| F^{-1} \left(\frac{\mathcal{F}_T f(x, \tau)}{\tau + |\xi|^{\theta}}\right) \right\|_{L^2_{x,\tau}} \lesssim T^{\nu} \|f\|_{L^2_{x,\tau}}.
\]

(2.54)

**Proof of Theorem 2.1.** We follow the arguments in [29]. By duality and Lemma 2.5, it is enough to show

\[
\left\| (\lambda_1 + ||\xi|^\alpha - |\xi|^{\beta})^{1/2} (\lambda_2 + ||\xi|^\alpha - |\xi|^{\beta})^{1/2} (\lambda_3 + ||\xi|^\alpha - |\xi|^{\beta})^{1/2} \right\|_{[3;3;\mathbb{R}^\times \mathbb{R}]} \lesssim 1.
\]

(2.55)

By a dyadic decomposition of the variables \(\xi, \lambda_j\) and \(h(\xi)\), we may assume \(|\xi| \sim N_j, |\lambda_j| \sim L_j\) and \(|h(\xi)| \sim H_j\). By the translation invariance of the \(k; \mathcal{Z}\)-multiplier (see [27, Lemma 3.4]) we may restrict the multiplier to the region \(L_j \gtrsim 1\) and \(N_{\max} \gtrsim 1\). Furthermore, we define

\[
I(N_j) := \inf_{|\xi| \geq N_j} |\xi|^{\alpha} - |\xi|^{\beta},
\]

and analogously we set \(I(N)\). Noting that if \(|\xi| \geq 2^{\delta_1/\omega}\), \(|\xi|^{\alpha} - |\xi|^{\beta}| = |\xi|^{\beta} - |\xi|^{\alpha} \geq |\xi|^{\beta}/2\), we deduce

\[
\begin{cases}
I(N_j) \lesssim \max \left\{N_j^{\omega}, N_j^{\delta_1/\omega}\right\}, & \text{when } N_j \lesssim 1, \\
I(N_j) \sim N_j^{\delta_1}, & \text{when } N_j \gg 1.
\end{cases}
\]

(2.56)

In view of (2.56), we must consider separately the cases \(N \gg 1\) and \(N \sim 1\) (cf. [29, Theorem 3.1]). Gathering the results in [27] (Schur’s test, comparison principle and orthogonality), it is deduced that (2.55) is bounded by one of the following inequalities

\[
\sum_{N_{\max} \sim N_{\med} \sim N} \sum_{L_1, L_2, L_3 \gtrsim 1} \frac{N_{\max}^{\omega} (N_j)^{\delta_1/\omega} (N_{\med})^{\delta_1}}{(L_1 + I(N_j))^{1/2} (L_2 + I(N_{\med}))^{1/2} (L_3 + I(N_j))^{1/2-\delta}} \times \|X_{N_1, N_2, N_3, L_{\max}, L_1, L_2, L_3}\|_{[3;3;\mathbb{R}^\times \mathbb{R}]}.
\]

(2.57)
and
\[ \sum_{N_{\text{max}} \sim N_{\text{med}} \sim N} \sum_{N_{\text{med}} \sim L_{\text{med}}} \sum_{H \leq L_{\text{max}}} \frac{N_3(N_3)^s(N_1)^{-s}(N_2)^{-s}}{(L_1 + I(N_1))^{1/2}(L_2 + I(N_2))^{1/2}(L_2 + I(N_2))^{1/2}} \times \|X_{N_1, N_2, N_3, L_{\text{max}}, L_1, L_2, L_3}\|_{[3; \infty, \infty]} \cdot \] (2.58)

Therefore, we are reduced to bound the above expressions for all \( N \gtrsim 1 \). We will divide our arguments according to Lemma 2.4.

**High modulation case.** Here \( L_{\text{max}} \sim L_{\text{med}} \gg H \), and so we must show that (2.58) \( \lesssim 1 \). In fact, this result follows under the weaker assumption that \( s > -1/2 \). It easily seen

\[ N_3(N_3)^s(N_1)^{-s}(N_2)^{-s} \lesssim (N_{\text{min}})^{-s}N_{\text{max}}, \]

then (2.48) yields

\[ (2.58) \lesssim \sum_{N_{\text{max}} \sim N_{\text{med}} \sim N} \sum_{N_{\text{med}} \sim L_{\text{med}} \sim 3NN_{\text{min}}} \frac{(N_{\text{min}})^{-s}N_{\text{max}}^{1/2}N_{\text{min}}^{1/2}}{(L_1 + I(N_1))^{1/2}(L_2 + I(N_2))^{1/2}(L_2 + I(N_2))^{1/2}} \] (2.59)

Assuming that \( N \sim 1 \), we have \( N_{\text{max}} \sim N_{\text{min}} \sim 1 \), \( N_{\text{min}} \lesssim 1 \) and \( L_{\text{max}} \sim L_{\text{med}} \gtrsim N_{\text{min}} \). Estimating under these assumptions one gets

\[ \langle L_1 + I(N_1)\rangle^{1/2}\langle L_2 + I(N_2)\rangle^{1/2}\langle L_3 + I(N_3)\rangle^{1/2} \gtrsim L_{\text{min}}^{1/2}L_{\text{max}}^{-1/2}. \] (2.60)

Thus, it follows

\[ (2.58) \lesssim \sum_{N_{\text{min}} \lesssim 1} \sum_{L_{\text{max}} \sim 3NN_{\text{min}}} \frac{(N_{\text{min}})^{-s}L_{\text{min}}^{1/2}L_{\text{max}}^{1/2}}{L_{\text{min}}^{1/2}L_{\text{max}}^{1/2}} \lesssim \sum_{N_{\text{min}} \lesssim 1} N_{\text{min}}^{-s}L_{\text{max}}^{1/2} \] (2.61)

Now suppose that \( N \gg 1 \). In this case, \( N_{\text{max}} \sim N_{\text{med}} \sim N \gg 1 \), \( L_{\text{max}} \sim L_{\text{min}} \gtrsim NN_{\text{min}} \), and so from (2.56) we obtain

\[ \langle L_1 + I(N_1)\rangle^{1/2}\langle L_2 + I(N_2)\rangle^{1/2}\langle L_3 + I(N_3)\rangle^{1/2} \gtrsim L_{\text{min}}^{1/2}L_{\text{max}}^{-1/2}(NN_{\text{min}} + N_{\text{min}})^{1/2-\delta}(NN_{\text{min}})^{1/2-\delta}. \] (2.62)

Then,

\[ (2.58) \lesssim \sum_{N_{\text{max}} \sim N} \sum_{N_{\text{med}} \sim L_{\text{med}} \sim NN_{\text{max}}} \frac{(N_{\text{min}})^{-s}N_{\text{max}}^{1/2}}{L_{\text{min}}^{1/2}L_{\text{max}}^{1/2}(NN_{\text{min}} + N_{\text{min}})^{1/2-\delta}(NN_{\text{min}})^{1/2-\delta}} \] (2.63)

Consequently,

\[ (2.58) \lesssim \sum_{N_{\text{min}} \lesssim 1} \frac{N_{\text{max}}^{1/2}}{N_{\text{min}}^{\beta/2 - \beta\delta}(NN_{\text{min}})^{1/2-\delta}} + \sum_{N_{\text{min}} \gtrsim 1} \frac{N_{\text{min}}^{1/2-\epsilon}}{NN_{\text{min}}^{1/2-\delta} - NN_{\text{min}}^{1/2-\delta}N_{\text{min}}^{1/2}} \]

\[ \lesssim \sum_{N_{\text{min}} \lesssim 1} N_{\text{min}}^{\beta/2 - \beta\delta + (\beta + 1)\delta} + \sum_{N_{\text{min}} \gtrsim 1} \frac{N_{\text{min}}^{1/2 - s + 2\delta + \epsilon}}{N_{\text{min}}^{2\delta - \epsilon(\beta - 1)}} \lesssim 1, \]

which holds when \( \beta > 1 \), \( \delta \ll 1 \), \( \epsilon = 2\delta/(\beta - 1) > 0 \) and \( s > -1/2 \). This completes the estimate (2.58) \( \lesssim 1 \).
Low modulation case: (++) coherence. We show that (2.57) \(\lesssim 1\), assuming contribution (2.49).

In this case \(N_{\min} \sim N_{\text{mid}} \sim N_{\max} \sim N\). If \(N \gg 1\), (2.56) gives

\[
(L_1 + I(N_1))^{1/2}(L_2 + I(N_2))^{1/2}(L_3 + I(N_3))^{1/2-\delta} \gtrsim L_{\min}^{1/2}L_{\max}^{\delta}(L_{\med} + N^{\beta})^{1/2}(L_{\max} + N^{\beta})^{1/2-2\delta}
\]

(2.64)

Consequently, since \(L_{\max} \sim N^2\), we deduce

\[
(2.57) \lesssim \sum_{L_{\max} \sim N^2} \frac{N^{1-s}L_{\min}^{1/2}L_{\med}^{1/4}}{L_{\max}^{1/2}L_{\max}^{\delta}L_{\med}^{1/4}N^{\beta/4}L_{\max}^{1/2-2\delta}} \lesssim \frac{N^{1-s}}{N^{\beta/4}N^{1-4\delta}} \lesssim N^{s-\beta/4+4\delta} \lesssim 1,
\]

(2.65)

when \(s > -\beta/4\) and \(\delta \ll 1\). We consider \(N \sim 1\), so that

\[
(L_1 + I(N_1))^{1/2}(L_2 + I(N_2))^{1/2}(L_3 + I(N_3))^{1/2-\delta} \gtrsim L_{\min}^{1/2}L_{\max}^{\delta}L_{\med}^{1/4}L_{\max}^{1/2-2\delta}.
\]

(2.66)

A similar reasoning as in the previous case leads to

\[
(2.57) \lesssim N^{1-s} \sim 1.
\]

(2.67)

Low modulation case: (+-) coherence. We are reduce to show that (2.57) \(\lesssim 1\) when (2.49) holds. By symmetry it suffices to treat the cases

\[
N_1 \sim N_2 \gg N_3, \text{ and } H \sim L_3 \gtrsim L_1, L_2,
\]

\[
N_2 \sim N_3 \gg N_1, \text{ and } H \sim L_1 \gtrsim L_2, L_3.
\]

(2.68)

In the first case, since \(L_{\max} \sim N_{\max}N_{\min}\), we deduce for \(\gamma = 1\) that

\[
(2.41) \lesssim L_{\min}^{1/2} \min(N_{\min}^{1/2}, N_{\min}^{-1/2}L_{\med}^{1/2}) \lesssim L_{\min}^{1/2}L_{\med}^{1/4}.
\]

(2.69)

Then, when \(N \gg 1\),

\[
(L_1 + I(N_1))^{1/2}(L_2 + I(N_2))^{1/2}(L_3 + I(N_3))^{1/2-\delta} \gtrsim L_{\min}^{1/2}L_{\max}^{\delta}N^{\beta/4}L_{\med}^{1/4}(N_{\max}N_{\min})^{1/2-2\delta},
\]

(2.70)

so that

\[
(2.57) \lesssim \sum_{N_3 > 0} \frac{N_3(N_3)^sN^{-2s}L_{\min}^{1/2}L_{\med}^{1/4}}{L_{\max}^{1/2}L_{\max}^{\delta}N^{\beta/4}L_{\med}^{1/4}(N_3)^{1/2-2\delta}} \lesssim \sum_{N_3 > 0} \frac{N_3(N_3)^sN^{-2s}}{N^{\beta/4+1/2-2\delta}N_3^{1/2-2\delta}} \lesssim \sum_{N_3 \geq 1} N_3^{1/2+2\delta}(N_3)^sN^{-2s-\beta/4+1/2+2\delta}.
\]

(2.71)

Since \(-2s - \beta/4 - 1/2 + 2\delta < 0\) the above inequality allow us to deduce

\[
(2.57) \lesssim \sum_{N_3 \leq 1} N_3^{1/2+2\delta} + \sum_{N_3 \geq 1} N_3^{-s-\beta/4+4\delta} \lesssim 1,
\]

(2.72)
which holds for $\delta \ll 1$ and $s > -\alpha/4$. When $N \sim 1$ one gets

$$
\langle L_1 + I(N_1)\rangle^{1/2} \langle L_2 + I(N_2)\rangle^{1/2} \langle L_3 + I(N_3)\rangle^{1/2} - \delta 
\gtrsim L_{\min}^{1/2} L_{\max}^{1/2} N^{1/2} L_{\med}^{1/2} (N_{\max} N_{\min})^{1/2 - 2\delta},
$$

(2.73)

so since $L_{\min} \gtrsim 1$, $N \sim 1$ and $\delta \ll 1$ we arrive at

$$
\langle L_1 + I(N_1)\rangle^{1/2} \langle L_2 + I(N_2)\rangle^{1/2} \langle L_3 + I(N_3)\rangle^{1/2} - \delta 
\gtrsim L_{\min}^{1/2} L_{\max}^{1/2} (L_{\med} + I(N))^{1/2 - 2\delta}. \quad (2.75)
$$

Thus, (2.50) and (2.75) imply

$$
\sum_{N_1 \leq N} \sum_{L_{\max} \sim N_{N_1}} \frac{N_1^{\delta} \langle N_1 \rangle^{-s} N L_{\min}^{1/2} N_{N_1}^{1/2}}{L_{\min}^{1/2} L_{\max}^{1/2} L_{\med}^{1/2} (L_{\med} + I(N))^{1/2 - 2\delta}} \lesssim \sum_{N_1 \leq N} \frac{N_1^{\delta} \langle N_1 \rangle^{-s} N_{N_1}^{1/2}}{(N_{N_1})^{1/2 - \delta} (N_{N_1}^{-1/2} - 1)^{1/2 + \gamma} + I(N))^{1/2 - 2\delta}. \quad (2.76)
$$

When $N \sim 1$, (2.76) shows

$$
\sum_{N_1 \leq 1} N_1^{\delta} \langle N_1 \rangle^{-s} \lesssim 1. \quad (2.77)
$$

Suppose that $N \gg 1$, so $I(N) \sim N^{\gamma}$. Then when $N \ll 1$, (2.76) implies

$$
\sum_{N_1 \leq 1} N_1^{\delta} N^{(1-\beta)/2 + s/2 + \gamma} \lesssim 1, \quad (2.78)
$$

for $\delta \ll 1$ and $\beta > 1$. If $N \gg 1$, from (2.76) we find

$$
\sum_{N_1 \geq 1} N_1^{\delta - s + 2} N_{N_1}^{1/2 + \delta} \lesssim \sum_{N_1 \geq 1} N_1^{\delta - s - 1/2 + (1+s)(1+\gamma) - 2s/2} N_{N_1}^{-1/2 + (1+\gamma) - 2\delta} \lesssim 1. \quad (2.79)
$$

for $\delta, \gamma \ll 1$, $s > -1/2$ and $\epsilon = (2\delta + 2\gamma(1/2 - \delta))/((\beta - 1) + \gamma) > 0$.

Now, we consider the case $N_{\min}^{1/2} \gtrsim N_{\max}^{1/2} - 1/2 - 2\gamma N_{\min}^{-1/2} L_{\med}^{1/2}$, i.e., $L_{\med} \lesssim N_{N_1}^{-1/2} N_{N_1}^{1/2}$. First we assume that $N \sim 1$. Since the right-hand side of (2.75) is bounded below by $L_{\min}^{1/2} L_{\max}^{1/2}$ for $0 < \delta < 1/2$, one gets

$$
\sum_{N_1 \leq 1} \sum_{L_{\max} \sim N_{N_1}} \frac{N_1^{\delta} \langle N_1 \rangle^{-s} N_{N_1}^{1/2} N_{N_1}^{1/2} \sim N_{N_1}^{-1/2} L_{\med}^{1/2}}{L_{\min}^{1/2} L_{\max}^{1/2}} \lesssim \sum_{N_1 \leq 1} N_{N_1}^{-1/2} \lesssim 1. \quad (2.80)
$$
Now we assume that $N \gg 1$. Inequality (2.75) allow us to deduce

\[
(2.57) \lesssim \sum_{N_1 \geq 1} \sum_{L_{\max} \leq N_1} \frac{\langle N_1 \rangle^{-s} N L_{\min}^{1/2} N^{1/2-1/2\gamma} N_{1}^{-1/2\gamma} L_{\max}^{1/2\gamma}}{(L_{\max} + N^{\beta})^{1/2-\delta}}
\]

\[
(2.81)
\]

So when $N \lesssim 1$, $\delta \ll 1$ and $\beta > 1$, we have

\[
(2.57) \lesssim \sum_{N_1 \leq 1} \sum_{L_{\max} \leq N_1^{1+\gamma}} \frac{N_1^{-1/2\gamma-1/2+\delta} N^{1-1/2\gamma+\delta} N^{\beta/2+\delta} (N^{1-\gamma} N_1^{1+\gamma})^{1/2\gamma}}{(L_{\max} + N^{\beta})^{1/2-\delta}}
\]

\[
(2.82)
\]

When $N_1 \geq 1$,

\[
(2.57) \lesssim \sum_{N_1 \geq 1} \sum_{L_{\max} \leq N_1^{1+\gamma}} \frac{N_1^{-s-1/2-1/2\gamma+\delta} N^{1-1/2\gamma+\delta} (N^{1-\gamma} N_1^{1+\gamma})^{1/2\gamma}}{(L_{\max} + N^{\beta})^{1/2-\delta}}
\]

\[
(2.83)
\]

This concludes the estimates regarding the low modulation case. Thus, the proof of Theorem 2.1 is now completed.

We are in condition to prove Theorem 1.1.

\[\square\]

Proof of Theorem. Following [20] and [21], we divide the proof in three main steps.

1. Local well-posedness. Let $u_0 \in H^s(\mathbb{R})$, $s > -\beta/4$. We consider the Banach space

   \[ Z = \left\{ u \in X^{1/2, s} : \|u\|_Z = \|u\|_{X^{1/2, s}} + \gamma \|u\|_{X^{1/2, s}} < \infty \right\}, \]

   where \( s^+_c \in (-\beta/4, \min \{0, s\}) \) and \( \gamma \) is defined for all nontrivial \( u_0 \),

   \[ \gamma = \frac{\|u_0\|_{H^s}}{\|u_0\|_{H^s}}. \]

   Given $0 < T \leq 1$ to be chosen later, we define the integral map

   \[ \Psi(u) = \psi(t) \left[ S(t)u_0 - \frac{\chi_{s^+_c + s}(t)}{2} \int_0^t S(t-\tau)\partial_x(\psi(\tau)u(\tau))^2 \ d\tau \right], \]

   \[
   (2.84)
   \]

   for each $u \in Z$. In view of Lemmas 2.1, 2.2, estimate (2.23), Theorem 2.1 and Proposition 2.5 there exist some constants $c, \nu > 0$ such that

   \[ \|\Psi(u)\|_Z \leq c \left( \|u_0\|_{H^{s^+_c}} + \gamma \|u_0\|_{H^s} \right) + c T\nu \|u\|_Z^2, \]

   \[
   (2.85)
   \]

   \[ \|\Psi(u) - \Psi(v)\|_Z \leq c T\nu \|u - v\|_Z \|u + v\|_Z, \]

   \[
   (2.86)
   \]

   for all $u, v \in Z$. Thus, recalling the definition of $\gamma$, we consider $0 < T \leq \min \left\{1, (16c^2 \|u_0\|_{H^{s^+_c}})^{-1/\nu}\right\}$. Then (2.85) and (2.86) imply that $\Psi$ is a contraction on the ball \( \left\{ u \in Z : \|u\|_Z \leq 4c \|u_0\|_{H^{s^+_c}} \right\} \). Consequently the Fixed Point Theorem assures the existence of a solution $u \in X^{1/2, s}$ of (1.11) on the time
interval \([0,T]\), with \(u(0) = u_0\).

The continuity with respect to the initial data follows directly from Proposition 2.4. Moreover, the above contraction argument implies the uniqueness of the solution to the truncated integral equation (1.12). The proof of uniqueness for the integral equation (1.11) can be derived following the same arguments in [16] and [29].

2. Regularity. Now we establish that \(u \in C ([0, T], H^\infty(\mathbb{R})) \cap X_T^{1/2,s}\) and the flow map data solution is smooth. Indeed, in view of Proposition 2.1, \(S(\cdot)u_0 \in C([0, \infty); H^s(\mathbb{R})) \cap C([0, \infty); H^\infty(\mathbb{R}))\). Then it follows from Theorem 2.1, Proposition 2.4 and the local well-posedness that

\[
u \in C ([0, T]; H^s(\mathbb{R})) \cap C ([0, T]; H^{s+\beta}(\mathbb{R})),
\]

where \(T = T(\|u_0\|_{H_x^s})\). Thus, we can deduce by induction, using the uniqueness result and the fact that the time of existence of solutions depends uniquely on the \(H_x^s(\mathbb{R})\)-norm of the initial data that

\[
u \in C ([0, T]; H^s(\mathbb{R})) \cap C ([0, T]; H^\infty(\mathbb{R})).
\]

The smoothness of the flow-map is a consequence of the implicit function theorem (see for instance [1, Remark 3]).

3. Global well-posedness. We define

\[
T^* = \sup \left\{ T > 0 : \exists! \text{ solution of (1.11) in } C ([0, T]; H^s(\mathbb{R})) \cap X_T^{1/2,s} \right\}.
\]

Let \(u \in C ([0, T^*]; H^s(\mathbb{R})) \cap C ([0, T^*]; H^\infty(\mathbb{R}))\) be the local solution of the integral equation associated to (1.1) on the maximal interval \([0, T^*)\). We will prove that \(T^* < \infty\) implies a contradiction. Since \(u\) is smooth by the above step, we have that this function solves (1.1) in a classical sense. Therefore, we can multiply (1.1) by \(u\) and integrating over \(\mathbb{R}\) to obtain

\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2}^2 - \|D^{\alpha/2}u(t)\|_{L^2}^2 + \|D^{\beta/2}u(t)\|_{L^2}^2 = 0,
\]

which leads since \(0 \leq \alpha < \beta\) to

\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2}^2 \leq \|u(t)\|_{L^2}^2.
\]

Let \(t_0 \in (0, T^*)\) fixed. Integrating the above inequality between \(t_0\) and \(t\) and applying Gronwall’s inequality to the resulting expression one gets

\[
\|u(t)\|_{L^2} \leq \|u(t_0)\|_{L^2} e^{(T^* - t_0)} \equiv M, \quad \forall t \in [t_0, T^*).
\]

Recalling that \(s^+_2 \leq 0\), it follows

\[
\|u(t)\|_{H^{s^+_2}} \leq M, \quad \forall t \in [t_0, T^*).
\]

Since the time of existence \(T(\cdot)\) is a nonincreasing function of the \(H^{s^+_2}(\mathbb{R})\)-norm, there exists a time \(\tilde{T} > 0\), such that for all \(\tilde{u}_0 \in H^s(\mathbb{R})\) with \(\|\tilde{u}_0\|_{H^{s^+_2}} \leq M\), there exists a function \(\tilde{u} \in C([0, \tilde{T}]; H^s(\mathbb{R})) \cap X_T^{1/2,s}\) solution of the integral equation (1.11) with \(\tilde{u}(0) = \tilde{u}_0\). Let \(0 < \epsilon < \min \{\tilde{T}, (T^* - t_0)\}\), applying this result to \(\tilde{u}_0 = u(T^* - \epsilon)\), we define

\[
v(t) = \begin{cases}
  u(t), & \text{when } 0 \leq t \leq T^* - \epsilon, \\
  \tilde{u}(t - T^* + \epsilon), & \text{when } T^* - \epsilon \leq t \leq T^* + \tilde{T} - \epsilon.
\end{cases}
\]

Hence, \(v(t)\) is a solution of the integral equation (1.11) on \([0, T^* + \tilde{T} - \epsilon]\) with initial data \(u_0\). Clearly, \(T^* + \tilde{T} - \epsilon > T^*\), which leads to a contradiction to the definition of \(T^*\). This proves the global result.

\[
\square
\]
3 Well-Posedness case $\beta \geq 2$.

This section is devoted to prove local and global well-posedness for the equation (1.1) when the dissipative parameters satisfy, $\beta \geq 2$ and $0 < \alpha < \beta$. Our approach is based on the methods introduced in [7], which rely completely on the dissipation of the equation. Mainly, the strategy to obtain local existence is to construct a contraction mapping from the integral equation (1.11) acting on the Banach spaces $Y^s_\alpha$ defined by (1.10).

We first recall the results in Proposition 2.1 where it was established

$$\|S(t)\phi\|_{H^s} \lesssim \psi_{\alpha,\beta}(t) \|\phi\|_{H^s},$$

(3.1)

for all $\phi \in H^s(\mathbb{R})$, $s \in \mathbb{R}$ and with

$$\psi_{\alpha,\beta}(t) = \exp\left(\frac{2\alpha}{\beta} \frac{\alpha - \alpha}{\beta} t\right), \quad t \in \mathbb{R}. \tag{3.2}$$

It is worth noting that the factor $\psi_{\alpha,\beta}(t)$ will appear frequently in many of our estimates. In order to evaluate the action of the integral equation (1.11) on the $Y^s_\alpha$ spaces, we require the following proposition.

**Proposition 3.1.** Let $\beta > 3/2$, $0 < \alpha < \beta$ fixed and $\psi_{\alpha,\beta}$ given by (2.1).

(i) For all $s \geq 0$ and $t > 0$

$$\left\| \xi^s e^{(\xi^2 - |\xi|^2)t} \right\|_{L^2} \lesssim \psi_{\alpha,\beta}(t) t^{-s/\beta - 1/2}. \tag{3.3}$$

(ii) Let $s \in \mathbb{R}$. Then for any $0 < t \leq 1$ it follows that

$$\left\| \xi (\xi^s e^{(|\xi|^2 - |\xi|^2)t}) \right\|_{L^2} \lesssim \psi_{\alpha,\beta}(t) t^{-r/2} \tag{3.4}$$

for all $r > \max \{(3 + 2s)/\beta, 0\}$.

**Proof.** In view of inequality (2.3) and changing variables by $w = t^{1/\beta} \xi$, we find

$$\left\| \xi^s e^{(|\xi|^2 - |\xi|^2)t} \right\|_{L^2} \lesssim \psi_{\alpha,\beta}(t) \left\| w^s e^{-\xi^2/2} \right\|_{L^2} \lesssim t^{-s/\beta - 1/2}. \tag{3.5}$$

This establish (i). Now, we derive (3.5) when $0 < t \leq 1$. We recall the inequality

$$t^r e^{-|\xi|^2 t} \leq \left(\frac{r}{|\xi|^2}\right)^r e^{-r}, \tag{3.6}$$

which is valid for all $\xi \neq 0$ and $r > 0$. Then, dividing in low and high frequencies, in view of (3.6) we deduce

$$\psi_{\alpha,\beta}(-2t) \left\| \xi (\xi e^{(|\xi|^2 - |\xi|^2)t}) \right\|_{L^2}^2 \lesssim \int_{|\xi| \leq 1} |\xi|^2 (1 + |\xi|^2)^s d\xi + t^{-r} \int_{|\xi| > 1} |\xi|^2 e^{-r} |\xi|^2 d\xi \lesssim t^{-r},$$

where $r > \max \{(3 + 2s)/\beta, 0\}$. This completes the proof of (3.4). \hfill \Box

Now we can estimate the integral equation (1.11) on the spaces $Y^s_\alpha$.

**Proposition 3.2.** Let $0 < \alpha < \beta$, $0 < T \leq 1$ and $\psi_{\alpha,\beta}$ defined by (2.1).

(i) Let $s < 0$ and $\phi \in H^s(\mathbb{R})$. Then

$$\|S(t)\phi\|_{Y^s_\alpha} \lesssim \psi_{\alpha,\beta}(T) \|\phi\|_{H^s}. \tag{3.7}$$

(21)
(ii) Let $\beta > 3/2$ and $s \geq 0$. Then for all $u, v \in C([0, T]; H^s(\mathbb{R}))$ it follows
\[
\left\| \int_0^t S(t - \tau)\partial_x(\tau) d\tau \right\|_{L^\infty T} \leq \psi_{\alpha, \beta}(T) T^{3(2 \beta - 3)} \| u \|_{L^\infty T} \| v \|_{L^\infty T} .
\] (3.7)

(iii) Assume that $\beta > 3/2$ and $\max\{3/2 - \beta, -\beta/2\} < s < 0$. Given $u, v \in Y^s_T$ it follows
\[
\left\| \int_0^t S(t - \tau)\partial_x(\tau) d\tau \right\|_{Y^s_T} \leq \psi_{\alpha, \beta}(T) T^{3(2 \beta - r \beta + 4s)} \| u \|_{Y^s_T} \| v \|_{Y^s_T} ,
\] (3.8)

for some $\max\{(3 + 2s)/\beta, 0\} < r < 2(\beta + 2s)/\beta$.

Proof. Part (i) is a direct consequence of (2.2) (with $\delta = 0$) and using that $0 \leq t \leq T \leq 1$. To deduce (ii), we consider $s > 0$ and the inequality
\[
\left\| (\xi)^s \mathcal{F}(u^2(\tau))(\xi) \right\|_{L^\infty} \leq \| u(\tau) \|^2_{H^s} \leq \| u \|^2_{L^\infty T} H^s .
\]

Thus, in view of the above estimate and (3.3),
\[
\left\| \int_0^t S(t - \tau)\partial_x(\tau) d\tau \right\|_{H^s} \leq \int_0^t \psi_{\alpha, \beta}(\tau) \left\| \xi |(\xi)^s e((\xi)^{s - |\xi|^3})(\xi)(\xi) \right\|_{L^2} \left\| (\xi)^s \mathcal{F}(u^2(\tau))(\xi) \right\|_{L^\infty} d\tau 
\]
\[
\leq \psi_{\alpha, \beta}(T) \left( \int_0^t \tau^{-\frac{2\beta}{3r}} d\tau \right) \| u \|_{L^\infty T} \| v \|_{L^\infty T} .
\] (3.9)

This shows (ii). To deduce (iii), we consider $s < 0$. The definition of the norm on the space $Y^s_T$ yields to the following estimate
\[
\| u(\tau)v(\tau) \|_{L^1} \leq \frac{\| u \|_{Y^s_T} \| v \|_{Y^s_T}}{t^{\frac{2\beta - 2}{3}}},
\] (3.10)

so that in view of (3.4) and performing the change of variables $\sigma = \tau/t$, we deduce
\[
\left\| \int_0^t S(t - \tau)\partial_x(\tau) d\tau \right\|_{H^s} \leq \int_0^t \psi_{\alpha, \beta}(\tau) \left\| \xi |(\xi)^s e((\xi)^{s - |\xi|^3})(\xi)(\xi) \right\|_{L^2} \| u(\tau)v(\tau) \|_{L^1} d\tau 
\]
\[
\leq \psi_{\alpha, \beta}(T) \int_0^t \left\| \xi |(\xi)^s e((\xi)^{s - |\xi|^3})(\xi)(\xi) \right\|_{L^2} d\tau \| u \|_{Y^s_T} \| v \|_{Y^s_T} 
\]
\[
\leq \psi_{\alpha, \beta}(T) T^{\frac{3}{2}(2 \beta - 2 \sigma)} \left( \int_0^1 (1 - \sigma)^{-\frac{2\beta}{3r}} d\sigma \right) \| u \|_{Y^s_T} \| v \|_{Y^s_T} .
\] (3.11)

where $r > \max\{(3 + 2s)/\beta, 0\}$ and $0 \leq t \leq T$. Arguing in a similar manner, we have for all $0 \leq t \leq T$ that
\[
t^\frac{2\beta}{3} \left\| \int_0^t S(t - \tau)\partial_x(\tau)(\tau) d\tau \right\|_{L^2} \leq t^\frac{2\beta}{3} \int_0^t \psi_{\alpha, \beta}(\tau) \left\| \xi |(\xi)^s e((\xi)^{s - |\xi|^3})(\xi)(\xi) \right\|_{L^2} \| u(\tau)v(\tau) \|_{L^1} d\tau 
\]
\[
\leq \psi_{\alpha, \beta}(T) t^\frac{2\beta}{3} \int_0^t \left\| \xi |(\xi)^s e((\xi)^{s - |\xi|^3})(\xi)(\xi) \right\|_{L^2} d\tau \| u \|_{Y^s_T} \| v \|_{Y^s_T} 
\]
\[
\leq \psi_{\alpha, \beta}(T) T^{\frac{3}{2}(2 \beta - 2 \sigma)} \left( \int_0^1 (1 - \sigma)^{-\frac{2\beta}{3r}} d\sigma \right) \| u \|_{Y^s_T} \| v \|_{Y^s_T} .
\] (3.12)

Consequently the right-hand side of inequalities (3.11) and (3.12) impose the conditions
\[
\beta > 3/2, \ \text{and} \ \ s > \max\{3/2 - \beta, -\beta/2\},
\]
with $\max\{(3 + 2s)/\beta, 0\} < r < 2(\beta + 2s)/\beta$. This remark completes the proof of Proposition 3.2. □
Proposition 3.3. Let $\beta > 3/2$, $0 < \alpha < \beta$, $s > \max\{3/2 - \beta, -\beta/2\}$ and $0 < T \leq 1$. Then there exists $\delta = \delta(s, \beta) > 0$ such that the application
\[
t \to \int_0^t S(t - \tau)\partial_x(u^2)(\tau) \, d\tau
\]
is in $C([0, T]; H^{s+\delta}(\mathbb{R}))$, for every $u \in Y_T^\beta$.

Proof. The proof is similar to that of Proposition 4 in [23].

We are in condition to prove Theorem 1.2.

Proof of Theorem 1.2. Since the results in Theorem 1.1 establish well-posedness when $1 < \beta < 2$, we will assume that $\beta \geq 2$ with $0 < \alpha < \beta$. Let $u_0 \in H^s(\mathbb{R})$ with $s > \max\{3/2 - \beta, -\beta/2\}$, we define the integral map
\[
\tilde{\Psi}(u) = S(t)u_0 - \frac{1}{2} \int_0^t S(t - \tau)\partial_x(u^2)(\tau) \, d\tau,
\]
for each $u \in Y_T^\beta$. By Proposition 3.2 there exists a positive constant $c = c(\beta, \alpha)$ such that
\[
\|\tilde{\Psi}(u)\|_{Y_T^\beta} \leq c\left(\|u_0\|_{H^s} + T^{\beta/s(3 + 2\alpha \beta)}\|u\|_{Y_T^\beta}^2\right),
\]
\[
\|\tilde{\Psi}(u) - \tilde{\Psi}(v)\|_{Y_T^\beta} \leq cT^{\beta/s(3 + 2\alpha \beta)}\|u - v\|_{Y_T^\beta} \|u + v\|_{Y_T^\beta},
\]
for all $u, v \in Y_T^\beta$, $0 < T \leq 1$ and $s > \max\{3/2 - \beta, -\beta/2\}$. Here the map $g_\alpha(s)$ is defined according to Proposition 3.2, i.e., $g_\beta(s) = \frac{1}{2\beta}(2\beta - 3)$ for all $s \geq 0$, and $g_\beta(s) = -\frac{1}{2\beta}(2\beta - r\beta + 2s)$, when 
\[
\max\{3/2 - \beta, -\beta/2\} < s < 0, \text{ for some fixed } r \text{ such that } \max\{(3 + 2s)/\beta, 0\} < r < (2\beta + 2s)/\beta.
\]

We consider $R = 2c\|u_0\|_{H^s}$ and $0 < T \leq \min\left\{1, (4cR)^{-\frac{1}{\beta(3 + 2\alpha \beta)}}\right\}$. Then estimates (3.13) and (3.14) imply that $\tilde{\Psi}$ is a contraction on the complete metric space $\left\{u \in Y_T^\beta : \|u\|_{Y_T^\beta} \leq R\right\}$. Therefore, the Fixed Point Theorem implies the existence of a solution $u$ to the integral equation (1.11).

The continuity with respect to the initial data is deduced following the same arguments in [21]. To verify uniqueness, we let $u, v \in Y_T^\beta$ solutions of equation (1.11) on the time interval $[0, T]$ with the same initial data $u_0 \in H^s$. The proof of Proposition 3.2 assures the existence of a uniform constant $c = c(\alpha, \beta, s)$ such that for all $0 < T_1 \leq T_2 \leq T$
\[
\|(u - v)\chi_{\{t \geq T_1\}}\|_{Y_T^\beta} \leq cK(T_2 - T_1)^{\beta/s(3 + 2\alpha \beta)}\|u - v\|_{Y_T^\beta},
\]
where $K := \|u\|_{Y_T^\beta} + \|v\|_{Y_T^\beta}$. Thus, taking $T_2 \in (0, (cK)^{-\frac{1}{\beta(3 + 2\alpha \beta)}})$ and $T_1 = 0$, we deduce from (3.15) that $u \equiv v$ on $[0, T_2]$. Therefore, fixing $T_2$, we can iterate this argument until we extend the uniqueness result to the whole interval $[0, T]$.

Following the proof of Theorem 1.1 together with Propositions 2.1 and 3.3, it is easily seen that $u \in C((0, T], H^{\infty}(\mathbb{R}))$ and the flow map data solution is smooth.

Finally, the global well-posedness result is deduced exactly as in [21].

\[\Box\]
4 Ill-posedness result.

In this section we prove Theorem 1.3 arguing similar to [29]. Assume that there exists a time $T > 0$ such that the Cauchy problem (1.1) is locally well-posed in $H^s(\mathbb{R})$ on the interval $[0, T]$ and such that the flow-map data solution

$$\Phi : H^s(\mathbb{R}) \rightarrow C([0, T]; H^s(\mathbb{R})), \quad u_0 \mapsto u(t)$$

is $C^k$ ($k = 2$ and $k = 3$) at the origin. Then, for each $u_0 \in H^s(\mathbb{R})$ we have that $\Phi(\cdot)u_0$ is a solution of the integral equation

$$\Phi(t)u_0 = S(t)u_0 - \frac{1}{2} \int_0^t S(t - \tau) \partial_x (\Phi(\tau)u_0)^2 d\tau.$$  

Since $\Phi(t)(0) = 0$, it follows that

$$u_1(t) := d_0\Phi(t)(u_0) = S(t)u_0,$$
$$u_2(t) := d_0^2\Phi(t)(u_0, u_0) = -\int_0^t S(t - \tau) \partial_x (u_1(\tau)u_1(\tau)) d\tau,$$
$$u_3(t) := d_0^3\Phi(t)(u_0, u_0, u_0) = -3 \int_0^t S(t - \tau) \partial_x (u_1(\tau)u_2(\tau)) d\tau.$$

Assuming that the solution map is of class $C^k$, $k = 2$ and $k = 3$, we must have

$$\|u_k(t)\|_{H^s} \lesssim \|u_0\|_{H^s}^k, \quad \forall u_0 \in H^s(\mathbb{R}). \quad (4.1)$$

In the sequel we will prove that (4.1) does not hold in general when $k = 2$, assuming that $s < -\beta/2$, and when $k = 3$ for $s < \min\{3/2 - \beta, -\beta/4\}$. These results establish Theorem 1.3.

4.1 $C^2$-regularity.

We divide our arguments according to parts (i) and (iii) of Theorem 1.3.

Case dissipation $\beta \geq 1$. Let $0 < \alpha < \beta$ and $s < -\beta/2$ fixed. When $k = 2$, we will show that (4.1) fails for an appropriated function $u_0$. We define $u_0$ by its Fourier transform as follows

$$\widehat{u}_0(\xi) = N^{-s} \omega^{-\frac{1}{2}} (\chi_N(\xi) + \chi_{I_N}(-\xi)),$$  

where $N \gg 1$, $0 < \omega \leq 1$ and $I_N = [N, N + 2\omega]$. A simple calculation shows that $\|u_0\|_{H^s} \sim 1$. Now we proceed to estimate the $H^s$ norm of $u_2(x, t)$. Indeed, taking the Fourier transform in the space variable and changing the order of integration, it follows for all $\xi \in [-\omega/2, \omega/2]$ that

$$\widehat{u}_2(\xi, t) \sim \xi e^{-i(\xi^2 + |\xi|^\alpha - |\xi|^\beta)t} \int_{\mathbb{R}} \widehat{u}_0(\xi - \xi_1) \widehat{u}_0(\xi_1) \frac{e^{i(\xi, \xi_1)t} - 1}{\sigma(\xi, \xi_1)} d\xi_1$$
$$\sim N^{-2s} \omega^{-1} \xi e^{-i(\xi^2 + |\xi|^\alpha - |\xi|^\beta)t} \int_{K_\xi} \frac{e^{i(\xi, \xi_1)t} - 1}{\sigma(\xi, \xi_1)} d\xi_1,$$  

where

$$K_\xi = \{\xi_1 : \xi - \xi_1 \in I_N, \xi_1 \in -I_N\} \cup \{\xi_1 : \xi_1 \in I_N, \xi - \xi_1 \in -I_N\}$$

and $\sigma$ is defined by

$$\sigma(\xi, \xi_1) = i(\xi^2 + |\xi|^\alpha - |\xi|^\beta - |\xi|^\alpha - |\xi|^\beta + |\xi|^\alpha - |\xi|^\beta).$$  

(4.4)
Now, if $\xi \in [-\omega/2, \omega/2]$ and $\xi_1 \in K_{\xi}$, we claim

$$|\sigma(\xi, \xi_1)| \sim N^\beta. \quad (4.5)$$

Indeed, a simple computations shows

$$|\Im \sigma(\xi, \xi_1)| \lesssim \omega N.$$  

On the other hand, when $\xi \in [-\omega/2, \omega/2]$ and $\xi_1 \in K_{\xi}$ we observe

$$\Re \sigma(\xi, \xi_1) \leq |2\omega|^\beta + 2(N + 2\omega)^\alpha - 2N^\beta$$

and

$$\Re \sigma(\xi, \xi_1) \geq -|2\omega|^\alpha + 2N^\alpha - 2(N + 2\omega)^\beta.$$  

Hence, combining the above inequalities the claim (4.5) follows for $N$ large.

Therefore, taking a fixed time $t_N = N^{-\beta-\epsilon} \in (0, T)$, $\epsilon > 0$ small (but arbitrary), it follows from the Taylor expansion of the exponential function and (4.5) that

$$\left| \frac{e^{\sigma(\xi, \xi_1)t_N} - 1}{\sigma(\xi, \xi_1)} \right| = \frac{1}{N^{\beta+\epsilon}} + O\left(N^{-\beta-2\epsilon}\right). \quad (4.6)$$

Then, since $|K_{\xi}| \gtrsim \omega$, (4.6) yields

$$|\hat{u}_2(\xi, t_N)| \chi_{[-\omega/2, \omega/2]} \gtrsim N^{-2s-\beta-\epsilon}|\xi|\chi_{[-\omega/2, \omega/2]}.$$  

Thus, we get a lower bound for the $H^s(\mathbb{R})$-norm of $u_2(x, t_N)$,

$$\|u_2(t_N)\|_{H^s}^2 \gtrsim \int_{-\omega/2}^{\omega/2} |\xi|^{2s} |\xi|N^{-4s-2\beta-2\epsilon} d\xi \gtrsim N^{-4s-2\beta-2\epsilon} \quad (4.7)$$

The above inequality contradicts (4.1) ($k = 2$) for $N$ large enough, since $s < -\beta/2$ and $\|u_0\|_{H^s} \sim 1$.

**Case dissipation** $0 < \beta < 1$. Let $0 < \alpha < \beta$ and $s \in \mathbb{R}$. We define $u_0$ by its Fourier transform

$$\hat{u}_0(\xi) = \omega^{-\frac{1}{2}} \chi_{I_1}(\xi) + \omega^{-\frac{1}{2}} N^{-s} \chi_{I_2}(\xi), \quad (4.8)$$

with $I_1 = [\omega/2, \omega], I_2 = [N, N + \omega]$ and $N \gg 1$, $\omega \ll N$ to be chosen later. Then $\|u_0\|_{H^s} \sim 1$. Computing the Fourier transform of $u_2(t)$ leads to

$$\hat{u}_2(\xi, t) \sim \xi e^{-i(\xi t + (|\xi|^\alpha - |\xi|^\beta)t)} \int_{\mathbb{R}} \hat{u}_0(\xi - \xi_1) \hat{u}_0(\xi_1) \frac{e^{\sigma(\xi, \xi_1)t} - 1}{\sigma(\xi, \xi_1)} d\xi_1,$$

where $\sigma(\xi, \xi_1)$ was defined in (4.4). By support considerations, we have $\|u_2(t)\|_{H^s} \geq \|u_2(t)\|_{H^s}$, where

$$\hat{u}_2(\xi, t) \sim N^{-s} \omega^{-1} \xi e^{-i(\xi t + (|\xi|^\alpha - |\xi|^\beta)t)} \int_{K_{\xi}} \frac{e^{\sigma(\xi, \xi_1)t} - 1}{\sigma(\xi, \xi_1)} d\xi_1$$

and

$$K_{\xi} = \{ \xi_1 : \xi_1 \in I_1, \xi - \xi_1 \in I_2 \} \cup \{ \xi_1 : \xi_1 \in I_2, \xi - \xi_1 \in I_1 \}.$$  

We see that if $\xi_1 \in K_{\xi}$, then $\xi \in [N + \omega/2, N + 2\omega]$ and

$$|\Re \sigma(\xi, \xi_1)| \lesssim N^\beta$$

$$\Im \sigma(\xi, \xi_1) = 2\xi_1(\xi_1 - \xi) \sim \omega N.$$  

We deduce for $\omega = N^{\beta-1} \ll N$ that $|\sigma(\xi, \xi_1)| \sim N^\beta$. Now define $t_N = (N + 2\beta)^{-\beta-\epsilon} \sim N^{-\beta-\epsilon}$. By a Taylor expansion of the exponential function,

$$\frac{e^{\sigma(\xi, \xi_1)t} - 1}{\sigma(\xi, \xi_1)} = t_N + R(t_N, \xi, \xi_1)$$

with

$$R(t_N, \xi, \xi_1) = \frac{e^{\sigma(\xi, \xi_1)t} - 1}{\sigma(\xi, \xi_1)} - t_N.$$
and
\[ |R(t_N, \xi_1)| \lesssim \sum_{k \geq 2} \frac{t_k^k |\sigma(\xi, \xi_1)|^{k-1}}{k!} \lesssim N^{-\beta - 2\epsilon}. \]

Since \( |K_\xi| \sim \omega \), we have that
\[ |\hat{u}_2(t_N)(\xi)| \gtrsim N^{-s+1}\omega^{-1}e^{-(N+2\omega)^{-\epsilon}}\omega N^{-\beta - \epsilon} \chi_{[N+\omega/2,N+2\omega]}(\xi) \gtrsim N^{-s+1-\beta - \epsilon} \chi_{[N+\omega/2,N+2\omega]}(\xi). \]

Then, the lower bound for the \( H^s \)-norm of \( u_2(x, t_N) \)
\[ \|u_2(t_N)\|_{H^s} \gtrsim N^{-s+1-\beta - \epsilon} \left( \int_{N+\omega/2}^{N+2\omega} (1 + |\xi|^{2})^\beta d\xi \right)^{1/2} \sim N^{1-\beta - \epsilon} \omega^{1/2} \sim N^{(1-\beta)/2} - \epsilon. \]

This inequality contradicts (4.1) \((k = 2)\) for \( N \) large enough, \( \epsilon \ll 1 \) and \( \beta < 1 \).

### 4.2 \( C^3 \)-regularity.

Suppose that \( s < \min \{3/2 - \beta, -\beta/4\} \). When \( k = 3 \), we will show that (4.1) fails for an appropriated function \( u_0 \). We consider \( u_0 \) as in (4.2), i.e.,
\[ \hat{u}_0(\xi) = N^{-s} \omega^{-1/2} (\chi_{I_N}(\xi) + \chi_{I_N}(-\xi)), \]

\( N \gg 1 \), \( I_N = [N, N + 2\omega] \) and here \( \omega \ll N \) to be chosen later. Observe that \( \|u_0\|_{H^s} \sim 1 \). Computing the Fourier transform of \( u_3(t) \) for each \( \xi \in \mathbb{R} \) one gets
\[ \hat{u}_3(t)(\xi) \sim \xi \int_{0}^{t} e^{-i\xi(t-\tau)} + (|\xi|^s - |\xi|^3) u_1(\tau) * u_2(\tau)(\xi) d\tau. \]  

(4.9)

Therefore, in view of (4.3) and Fubini’s Theorem it is easily seen
\[ \hat{u}_3(t)(\xi) \sim \xi e^{-i\xi(t-\tau)} (|\xi|^s - |\xi|^3) \int_{\mathbb{R}^2} \hat{u}_0(\xi_1) \hat{u}_0(\xi_2 - \xi_1) \hat{u}_0(\xi - \xi_2) \frac{\xi_2}{\sigma(\xi_2, \xi_1)} \times \left( \frac{e^{\sigma(\xi, \xi_1, \xi_2)t} - 1}{\eta(\xi, \xi_1, \xi_2)} - \frac{e^{\sigma(\xi, \xi_2)t} - 1}{\sigma(\xi, \xi_2)} \right) d\xi_1 d\xi_2, \]  

(4.10)

where \( \sigma(\xi, \xi_1) \) is given by (4.5) and we have set
\[ \eta(\xi, \xi_1, \xi_2) := \sigma(\xi, \xi_2) + \sigma(\xi, \xi_1). \]

By support considerations one observes
\[ \left| \hat{u}_3(t)(\xi) \right| \gtrsim N^{-3s} \omega^{-3/2} \xi e^{\xi(|\xi|^s - |\xi|^3)t} \int_{K_\xi} \frac{\xi_2}{\sigma(\xi_2, \xi_1)} \left( \frac{e^{\sigma(\xi, \xi_1, \xi_2)t} - 1}{\eta(\xi, \xi_1, \xi_2)} - \frac{e^{\sigma(\xi, \xi_2)t} - 1}{\sigma(\xi, \xi_2)} \right) d\xi_1 d\xi_2, \]  

(4.11)

where \( K_\xi = K^1_\xi \cup K^2_\xi \cup K^3_\xi \) and
\[ K^1_\xi = \{(\xi_1, \xi_2) : \xi_1 \in I_N, \xi_2 - \xi_1 \in I_N, \xi - \xi_2 \in -I_N\}, \]
\[ K^2_\xi = \{(\xi_1, \xi_2) : \xi_1 \in I_N, \xi_2 - \xi_1 \in -I_N, \xi - \xi_2 \in I_N\}, \]
\[ K^3_\xi = \{(\xi_1, \xi_2) : \xi_1 \in -I_N, \xi_2 - \xi_1 \in I_N, \xi - \xi_2 \in I_N\}. \]

We will restrict the values of \( \xi \) to the interval \([N + 3\omega, N + 4\omega]\). Then, under this condition and using that \( \omega \ll N \), it follows for \((\xi_1, \xi_2) \in K_\xi \) that
\[ 3\eta(\xi, \xi_1, \xi_2) \sim \omega^2, \text{ and } |\Re \eta(\xi, \xi_1, \xi_2)| \sim N^\beta. \]  

(4.12)
We will divide our arguments in two cases depending on the dissipation parameter $\beta$.

**Case dissipation $1 \leq \beta < 2$.** In view of (4.12) we are lead to choose $\omega = N^{\beta/2} \ll N$ with $N \gg 1$. Hence $|\eta(\xi, \xi_1, \xi_2)| \sim N^\beta$ and since $1 \leq \beta < 2$,

\[
\frac{\xi_2}{\sigma(\xi_2, \xi_1)} \sim N^{-1}.
\]  

(4.13)

Next, we consider

\[
t_N := N^{-\beta-\epsilon},
\]  

(4.14)

with $0 < \epsilon \ll 1$. We divide the estimate of $|\tilde{u}_3(t_N)|(\xi)$ on $[N + 3\omega, N + 4\omega]$ as follows

\[
|\tilde{u}_3(t_N)(\xi)|\chi_{[N+3\omega,N+4\omega]}(\xi) \gtrsim N^{-3+1}N^{-3/2} \int_{K_\xi} \frac{\xi_2}{\sigma(\xi_2, \xi_1)} \left( \frac{e^{\eta(\xi, \xi_1, \xi_2)t_N} - 1}{\eta(\xi, \xi_1, \xi_2)} \right) d\xi_1 d\xi_2 \chi_{[N+3\omega,N+4\omega]}(\xi)
\]

\[
- N^{-3+1}N^{-3/2} \int_{K_\xi} \frac{\xi_2}{\sigma(\xi_2, \xi_1)} \left( \frac{e^{\sigma(\xi, \xi_2)t_N} - 1}{\sigma(\xi, \xi_2)} \right) d\xi_1 d\xi_2 \chi_{[N+3\omega,N+4\omega]}(\xi)
\]

\[
= B_1 - B_2.
\]

To estimate $B_1$, we observe

\[
\frac{e^{\eta(\xi, \xi_1, \xi_2)t_N} - 1}{\eta(\xi, \xi_1, \xi_2)} = t_N + O(N^{-\beta-2\epsilon}).
\]

This yields in view of (4.13) and $|K_\xi| \sim \omega^2$ to

\[
B_1 \gtrsim N^{-3+1}N^{-3/2}N^{-1}N^{-\beta-\epsilon} \chi_{[N+3\omega,N+4\omega]}(\xi)
\]

\[
\sim N^{-3+3\beta/4-\epsilon} \chi_{[N+3\omega,N+4\omega]}(\xi).
\]

(4.15)

To deal with $B_2$, we observe $|\sigma(\xi, \xi_2)| \gtrsim \omega N$ for $(\xi, \xi_2) \in K_\xi$ and $\xi \in [N + 3\omega, N + 4\omega]$. Thus, since $|\sigma(\xi, \xi_2)t_N| \leq 1$, we get

\[
B_2 \lesssim N^{-3+1}N^{-3/2}N^{-2\omega} \chi_{[N+3\omega,N+4\omega]}(\xi)
\]

\[
\sim N^{-3+1-\beta/4} \chi_{[N+3\omega,N+4\omega]}(\xi).
\]

(4.16)

Since $-3s - \beta/4 - 1 < -3s - 3\beta/4 - \epsilon$ given that $\beta < 2$, we conclude

\[
B_1 - B_2 \gtrsim N^{-3+3\beta/4-\epsilon} \chi_{[N+3\omega,N+4\omega]}(\xi),
\]

for $N \gg 1$. Thus, from this fact we derive the following lower bound for the $H^s$-norm of $u_3(x, t_N)$,

\[
\|u_3(t_N)\|_{H^s} \gtrsim N^{-3+1-\beta/4}N^{1/2}N^s \sim N^{-2s-\beta/2-\epsilon}.
\]

The above inequality contradicts (4.1) ($k = 3$) for $N$ large given that $\|u_0\|_{H^s} \sim 1$, $s < -\beta/4$ and $0 < \epsilon \ll 1$.

**Case dissipation $\beta \geq 2$.** In this case the contributions of $B_1$ and $B_2$ are equivalent. Thus, we must proceed estimating the whole right-hand side of (4.11). To do so, we let $\omega = \epsilon_1 N$ with $0 < \epsilon_1 \ll 1$ to be chosen later. We first observe that (4.12) shows that $\eta(\xi, \xi_1, \xi_2) \sim N^\beta$. Moreover, we claim

\[
|\sigma(\xi, \xi_2)| \sim N^\beta,
\]

(4.17)

\[
|\sigma(\xi_2, \xi_1)| \sim N^\beta,
\]

(4.18)

for $\epsilon_1 > 0$ small enough, $N$ sufficiently large and $(\xi_1, \xi_2) \in K_\xi$ with $\xi \in [N + 3\omega, N + 4\omega]$. For the sake of brevity, we only give a proof to (4.17), since (4.18) follows in a similar manner. On these terms, since the imaginary part of $\sigma(\xi, \xi_2)$ is of order $O(N^2)$, we are reduced to show that $|\Re(\sigma(\xi, \xi_1))| \sim N^\beta$. 

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Suppose that \((\xi_1, \xi_2) \in K_\xi^1\). So, since \(2N \leq \xi_2 \leq 2N + 4\omega\), we deduce
\[
\Re \sigma(\xi, \xi_2) \leq N^\alpha((2 + 4\epsilon_1)^\alpha + (1 + 2\epsilon_1)^\alpha - (1 + 3\epsilon_1)^\alpha) - N^\beta((2 + 3\epsilon_1)^\beta + 1 - (1 + 4\epsilon_1)^\beta),
\]
and
\[
\Re \sigma(\xi, \xi_2) \geq N^\alpha((2 + 3\epsilon_1)^\alpha + 1 - (1 + 4\epsilon_1)^\alpha) - N^\beta((2 + 4\epsilon_1)^\beta + (1 + 2\epsilon_1)^\beta - (1 + 3\epsilon_1)^\beta),
\]
which clearly leads to \(|\Re \sigma(\xi, \xi_1)| \sim N^\beta\) for \(\epsilon_1\) small enough.

On the other hand, when \((\xi_1, \xi_2) \in K_\xi^2 \cup K_\xi^3\), we have \(\omega \leq \xi_2 \leq 2\omega\) and it follows
\[
\Re \sigma(\xi, \xi_2) \leq -N^\alpha((1 + 3\epsilon_1)^\alpha - (1 + 2\epsilon_1)^\alpha - (2\epsilon_1)^\alpha) + N^\beta((1 + 4\epsilon_1)^\beta - 1 - \epsilon_1^\beta)
\]
and
\[
\Re \sigma(\xi, \xi_2) \geq -N^\alpha((1 + 4\epsilon_1)^\alpha - 1 - \epsilon_1^\beta) + N^\beta((1 + 3\epsilon_1)^\beta - (1 + 2\epsilon_1)^\beta - (2\epsilon_1)^\beta).
\]
Note that in this case the constants with factor \(N^\beta\) tend to zero as \(\epsilon_1 \to 0\) and they are always positive if \(\epsilon_1 > 0\). To see this, the mean value inequality yields
\[
(1 + 4\epsilon_1)^\beta - 1 - \epsilon_1^\beta \geq (1 + 3\epsilon_1)^\beta - (1 + 2\epsilon_1)^\beta - (2\epsilon_1)^\beta
\]
and
\[
\geq \beta \epsilon_1 \left(1 + 2\epsilon_1\right)^{\beta - 1} - \frac{2}{\beta} (2\epsilon_1)^{\beta - 1} > 0,
\]
given that \(\beta \geq 2\) and \(\epsilon_1 > 0\).

Therefore, gathering the above estimates, we can choose \(\epsilon_1 > 0\) small to fix the sign of the constants involving \(N^\beta\). Consequently, we take \(N\) large to absorb the terms with \(N^\alpha\). At the end, we will find that \(|\Re \sigma(\xi, \xi_2)| \sim N^\beta\) as claimed.

Next, we consider \(t_N\) as in (4.14) with \(0 < \epsilon \ll 1\). By the Taylor expansion of the exponential function we find
\[
\frac{1}{\sigma(\xi_2, \xi_1)} \left( \frac{e^{\eta(\xi_1, \xi_2) t_N} - 1}{\eta(\xi_1, \xi_2)} - \frac{e^{\sigma(\xi_2, \xi_1) t_N} - 1}{\sigma(\xi_2, \xi_1)} \right) = \frac{1}{N^{2\beta + 2\epsilon}} + R(\xi, \xi_1, \xi_2),
\]
where
\[
|R(\xi, \xi_1, \xi_2)| \leq \sum_{k=3}^\infty \left| \eta(\xi_1, \xi_2) k^{-1} - \sigma(\xi_1, \xi_2) k^{-1} \right| \frac{t_N^k}{k!} \leq O \left( \frac{1}{N^{2\beta + 3\epsilon}} \right).
\]

In view of the above inequality the main contribution of (4.20) is given by \(N^{-2\beta - 2\epsilon}\). Thus, for \(N\) large,
\[
\Re \left( \frac{1}{\sigma(\xi_2, \xi_1)} \left( \frac{e^{\eta(\xi_1, \xi_2) t_N} - 1}{\eta(\xi_1, \xi_2)} - \frac{e^{\sigma(\xi_2, \xi_1) t_N} - 1}{\sigma(\xi_2, \xi_1)} \right) \right) \gtrsim N^{-2\beta - 2\epsilon}.
\]
Since \(\xi_2 \sim N\) and \(|K_\xi| \sim \omega^2\), we get from (4.21) that
\[
\left| u_3(t_N)(\xi) \right| \gtrsim N^{-3s} \omega^{-3/2} \xi |e^{-((|\xi_1| + |\xi_2|) t_N)}| \int_{K_\xi} \frac{\xi_2}{\sigma(\xi_2, \xi_1)} \left( \frac{e^{\eta(\xi_1, \xi_2) t_N} - 1}{\eta(\xi_1, \xi_2)} - \frac{e^{\sigma(\xi_2, \xi_1) t_N} - 1}{\sigma(\xi_1, \xi_2)} \right) d\xi_1 d\xi_2
\]
\[
\gtrsim N^{-3s} \omega^{-3/2} N \int_{K_\xi} \frac{\xi_2}{\sigma(\xi_2, \xi_1)} \left( \frac{e^{\eta(\xi_1, \xi_2) t_N} - 1}{\eta(\xi_1, \xi_2)} - \frac{e^{\sigma(\xi_2, \xi_1) t_N} - 1}{\sigma(\xi_1, \xi_2)} \right) d\xi_1 d\xi_2
\]
\[
\gtrsim N^{-3s + 2} \omega^{-3/2} N^{-2\beta - 2\epsilon} \omega^2.
\]
The above inequality gives the following lower bound for the $H^s$-norm of $u_3$,
\[
\|u_3(t_N)\|_{H^s}^2 \geq \int_{\mathbb{R}} \langle \xi \rangle^{2s} |u_3(t_N)(\xi)|^2 \chi_{[N+3\omega,N+4\omega]}(\xi) \, d\xi
\]
\[
\gtrsim \omega^2 N^{-4s+4-4\beta-4\epsilon} \sim N^{-4s+6-4\beta-4\epsilon},
\]
which in turn contradicts (4.1) for $N$ large given that $\|u_0\|_{H^s} \sim 1$, $s < 3/2 - \beta$ and $\epsilon > 0$ is arbitrary.

## 5 fDBO on the Torus

In this section we briefly indicate the modifications needed to prove Theorem 1.4. We first introduce some notation. The periodic Sobolev spaces $H^s(T)$ are endowed with the norm
\[
\|\phi\|_{H^s}^2 = \sum_{k \in \mathbb{Z}} \langle k \rangle^{2s} |\hat{\phi}(k)|^2.
\]

Let $s \in \mathbb{R}$ and $0 < t \leq T \leq 1$ fixed. We consider the spaces
\[
\mathcal{L}_T^s = \left\{ u \in C([0,T]; H^s(T)) : \|u\|_{L^\infty_T H^s_x} < \infty \right\},
\]
where
\[
\|u\|_{L^\infty_T H^s_x} := \sup_{t \in [0,T]} \left( \|u(t)\|_{H^s_x(T)} + \|u(t)\|_{L^2_x(T)} \right).
\]

Note that when $s \geq 0$, $\mathcal{L}_T^s = C([0,T]; H^s_x(T))$ and $\|u\|_{\mathcal{L}_T^s} \sim \|u\|_{L^\infty_T H^s_x}$, since $0 < T \leq 1$.

Comparing with the real line case, here the semigroup $\{S(t)\}_{t \geq 0}$ on $H^s(T)$ is contractive. This follows since $|k|^\alpha - |k|^\beta \leq 0$ for all integer $k$, consequently the multiplier associated satisfies, $e^{(|k|^\alpha - |k|^\beta)t} \leq 1$ for all $t \geq 0$.

On the other hand, since the proof of Proposition 2.1 and 3.2 depends on some change of variables, we must proceed with a bit more care.

**Proposition 5.1.** The results of Proposition 2.1 and those of Proposition 3.1 with $0 < t \leq 1$ still hold in the periodic case. Moreover, the factor $\psi_{\alpha,\beta}(t)$ defined by (2.1) in these estimates can be considered as $1$.

**Proof.** Let $m \geq 0$ and $s \in \mathbb{R}$. Since $|k|^\alpha - |k|^\beta \leq |k|^\beta/2$ when $|k| \geq 2^{1/\alpha}$, and $|k|^\alpha - |k|^\beta \leq 0$ for all integer $k$, we find
\[
\left\| |k|^m (k)^s e^{(|k|^\alpha - |k|^\beta)t} \right\|^2_{\ell^2(\mathbb{Z})} \lesssim \sum_{0 \leq k \leq 2^{1/\alpha}} |k|^{2m} \langle k \rangle^{2s} + \sum_{k > 2^{1/\alpha}} |k|^{2m} \langle k \rangle^{2s} e^{-|k|^\beta t}
\]
\[
\lesssim \sum_{0 \leq k \leq 2^{1/\alpha}} |k|^{2m} \langle k \rangle^{2s} + \sum_{k > 2^{1/\alpha}} \int_{k-1}^k |\xi + 1|^{2m} \langle \xi \rangle^{2s} e^{-|\xi|^\beta t} \, d\xi
\]
\[
\lesssim 1 + \left\| (\xi)^s e^{-|\xi|^\beta t} \right\|^2_{L^2(\mathbb{R})} + \left\| (\xi)^m (\xi)^s e^{-|\xi|^\beta t} \right\|^2_{L^2(\mathbb{R})}.
\]

Therefore, inequality (5.2) allow us to argue exactly as in the proof of Propositions 2.1 and 3.1 to derive analogous time decay estimates. Furthermore one can see that the factor $\psi_{\alpha,\beta}(t)$ is not needed to bound the exponential term $e^{-|\xi|^\beta t}$ in each of these estimates. Finally, since $0 < t \leq 1$, the constant in the right-hand side of (5.2) can be bounded by $t^{-r}$, for any $r > 0$.\qed
Gathering the above results we deduce that Proposition 3.2 is valid in the periodic setting. Thus, for the range \( \beta > \frac{3}{2} \) with \( 0 < \alpha < \beta \), Theorem 1.2 still holds with the same line of arguments for its proof, changing \( H^s(\mathbb{R}) \) and \( Y^s_T \), respectively by \( H^s(\mathbb{T}) \) and \( \tilde{Y}^s_T \). This concludes the existence part of Theorem 1.4.

Finally, we show that part (i) of Theorem 1.3 applies to the periodic case. Essentially, one can argue as in the deduction of Theorem 1.3. Here, we define the function \( u_0 \) via its Fourier series by

\[
\hat{u}_0(k) = \begin{cases} 
N^{-s}, & \text{if } k = N \text{ or } k = 1 - N, \\
0, & \text{otherwise,}
\end{cases}
\]

for \( N \gg 1 \). Noting that \( \sigma(1, N) = \sigma(1, 1 - N) \), it is easily seen

\[
\hat{u}_2(1, t) = N^{-2s}e^{-it}e^{\sigma(1, N)t} - \frac{1}{\sigma(1, N)}.
\]

Therefore since \( |\sigma(1, N)| \sim N^3 \) for \( N \) large, we can follow the ideas behind (4.6) with \( t_N = N^{-\beta - \epsilon} \) to obtain

\[
\|u_2(t_N)\|_{H^s} \gtrsim |\hat{u}_2(1, t_N)| \gtrsim N^{-2s-\beta-\epsilon}.
\]

Thus, (5.5) contradicts (4.1) \((k = 2)\) given that \( \|u_0\|_{H^s} \sim 1 \) and \( s < -\beta/2 \) with \( 0 < \epsilon \ll 1 \).

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