The de Finetti structure behind some norm-symmetric multivariate densities with exponential decay

Abstract: We derive a sufficient condition on the symmetric norm $\|\cdot\|$ such that the probability distribution associated with the density function $f(x) \propto \exp(-\lambda \|x\|)$ is conditionally independent and identically distributed in the sense of de Finetti's seminal theorem. The criterion is mild enough to comprise the $\ell_p$-norms as special cases, in which $f$ is shown to correspond to a polynomially tilted stable mixture of products of transformed Gamma densities. In another special case of interest $f$ equals the density of a time-homogeneous load sharing model, popular in reliability theory, whose motivation is a priori unrelated to the concept of conditional independence. The de Finetti structure reveals a surprising link between time-homogeneous load sharing models and the concept of Lévy subordinators.

Keywords: infinite divisibility, Lévy subordinator, de Finetti’s theorem, exchangeability, min-stable multivariate exponential distribution

MSC: 62H05, 60E07

1 Introduction

The present article studies multivariate probability density functions of the form

$$f_d(x) := \frac{\lambda^d}{C_d,\|\cdot\|} e^{-\lambda \|x\|}, \quad x = (x_1, \ldots, x_d) \in (0, \infty)^d,$$

where $C_d,\|\cdot\| > 0$ is a normalizing constant such that $f_d$ integrates to one, and $\|\cdot\|$ is a norm with the property $\|e_i\| = 1$ for all standard unit vectors $e_i, i \in [d] := \{1, \ldots, d\}$. Along a ray $\{t x\}_{t \geq 0}$ with fixed $\|x\| = 1$ the decay of this density is exponential with rate $\lambda$, and the contour lines of the density are precisely the surfaces of $\|\cdot\|$-spheres. We will only consider norms which are invariant with respect to permutations of the components of their argument $x$. This symmetry property implies that the probability distribution associated with $f_d$ is exchangeable. Theorem 3.1 below derives a sufficient criterion on the norm $\|\cdot\|$ such that the probability law associated with $f_d$ is not only exchangeable, but even conditionally iid in the sense of de Finetti’s seminal theorem.

The remainder of the article is organized as follows. Section 2 recalls required mathematical background. Section 3 presents our main result and demonstrates it in case $\|\cdot\| = \|\cdot\|_p$, where $\|\cdot\|_p$ for $p \geq 1$ denotes the usual $\ell_p$-norm defined by

$$\|x\|_p := (|x_1|^p + \ldots + |x_d|^p)^{\frac{1}{p}}, \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d,$$
and \( \| \cdot \|_\infty \) denotes the usual \( \ell_\infty \)-norm defined by

\[
\| x \|_\infty := \max \{ |x_1|, \ldots, |x_d| \}, \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d.
\]

Section 4 applies the main result to the particular case of a norm related to a time-homogeneous load sharing model.

## 2 Required mathematical background

### 2.1 de Finetti’s Theorem

We say that a probability measure \( \mu_{d} \) on \( \mathbb{R}^d \) is \textit{conditionally iid} if there is a random vector \( X = (X_1, \ldots, X_d) \sim \mu_{d} \) on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\) such that conditionally on some sub-\( \sigma \)-algebra \( \mathcal{H} \subset \mathcal{F} \) the components \( X_1, \ldots, X_d \) are iid. In this case, the non-decreasing, right-continuous stochastic process \( \{F_x\}_{x \in \mathbb{R}} \), where \( F_x := \mathbb{P}(X_1 \leq x \mid \mathcal{H}) \), is a random variable taking values in the set of distribution functions, and we may write

\[
\mu_{d}((\infty, x]) = \mathbb{P}(X_1 \leq x_1, \ldots, X_d \leq x_d) = \mathbb{E}\left[ \prod_{i=1}^{d} F_{x_i} \right], \quad x \in \mathbb{R}^d.
\]

We call such mixture representation for a conditional iid probability law \( \mu_{d} \) its \textit{de Finetti structure}, this nomenclature being justified by de Finetti’s theorem, which is recalled in the following. Necessarily, such random vector \( X \) is \textit{exchangeable}, i.e. the probability law of \( X \) is invariant with respect to permutations of components. Conversely, however, not every exchangeable random vector is conditionally iid. The seminal theorem of de Finetti states that the notions “conditionally iid” and “exchangeable” coincide if we let \( d \to \infty \). To wit, if \( X = (X_1, X_2, \ldots) \) is an infinite exchangeable sequence, meaning that its law is invariant with respect to permutations of finitely many, but arbitrarily many, components, then its probability law is necessarily conditionally iid. Conditionally on the tail-\( \sigma \)-algebra \( \mathcal{H} = \cap_{i=1}^\infty \sigma(X_i, X_{i+1}, \ldots) \) of \( X \) the components \( X_1, X_2, \ldots \) form an iid sequence of random variables, see [1] for a textbook account on the topic.

Our main purpose is to provide a sufficient criterion on the symmetric norm \( \| \cdot \| \) such that \( f_{d} \) defines the density of a probability law \( \mu_{d} \) which is conditionally iid. Given exchangeability of \( \mu_{d} \) for each \( d \geq 1 \), a prominent sufficient analytical condition is to check whether the measures \( (\mu_{d})_{d \geq 1} \) are consistent in Kolmogorov’s sense:

\[
\mu_n(A \times \mathbb{R}^{n-d}) = \mu_{d}(A), \quad A \subset \mathbb{R}^d \text{ Borel sets,} \quad n > d \geq 1.
\] (1)

In the present situation, this necessarily requires a family of norms \( \| \cdot \| \) that makes sense in arbitrary dimension \( d \), and which is consistent in such a way that \( f_{d} \) is obtained as marginal density of \( f_{n} \), when integrating out \( n - d \) components. It is a priori not obvious which norms satisfy this criterion. Furthermore, even if this analytical condition can be verified for the family \( (f_{d})_{d \geq 1} \), it is a priori unclear how the de Finetti structure looks like, i.e. how the stochastic nature of \( \{F_x\}_{x \in \mathbb{R}} \) can be described. Our solution relies heavily on a non-trivial structural result of [16] on the de Finetti structure associated with survival functions of the form \( x \mapsto \exp(-\|x\|), x \in [0, \infty)^d \). Given a norm \( \| \cdot \| \) that makes sense in arbitrary dimension and is dimension-consistent in the sense that \( \|x\| = \| (x, 0) \| \), we point out that Kolmogorov’s consistency conditions (1) are much easier to verify on the level of survival functions than on densities, since there trivially

\[
\mu_n((x, \infty] \times \mathbb{R}^{n-d}) = e^{-\| (x, 0) \|} = e^{-\|x\|} = \mu_{d}((x, \infty)).
\]

Our main result relies on a transfer of the de Finetti structure of these survival functions, derived in [16], to the algebraically similar density functions \( f_{d} \). In order to formulate this result, required is the mathematical background behind the result of [16], which is recalled in the following.
2.2 Min-stable multivariate exponential laws

While our study of the density \( f \) relies on a rather algebraic motivation, more popular and well-established concepts of multivariate exponential distributions rely on probabilistic motivations, see [5] for an early reference. In particular, if a random vector \( \mathbf{X} = (X_1, \ldots, X_d) \) is such that \( \min\{x_1 X_1, \ldots, x_d X_d\} \) has a univariate exponential distribution for arbitrary \( x_1, \ldots, x_d \geq 0 \) (not all zero), it is said to be \textit{min-stable multivariate exponential}. One reason why this concept has become very popular is that min-stability plays a fundamental role in multivariate extreme-value theory. Concretely, the limiting laws of suitably normalized componentwise minima of iid random vectors necessarily converge to a min-stable multivariate exponential distribution, results in this spirit being pioneered by [3].

It is known at least since [2] that \( \mathbf{X} \) is min-stable multivariate exponential if and only if

\[
P(X_1 > x_1, \ldots, X_d > x_d) = e^{-\|\lambda_1 x_1, \ldots, \lambda_d x_d\|_\lambda}, \quad x \in [0, \infty)^d,
\]

where \( \lambda_i > 0 \) is the exponential rate of the one-dimensional exponentially distributed component \( X_i, i \in [d] \), and the exponent further involves the function

\[
\|\mathbf{x}\|_\lambda := \mathbb{E}\left[ \max_{i=1,\ldots,d} \{|x_i| Y_i\}\right], \quad \mathbf{x} \in \mathbb{R}^d,
\]

which is defined in terms of some generic random vector \( \mathbf{Y} = (Y_1, \ldots, Y_d) \) with non-negative components satisfying \( \mathbb{E}[Y_i] = 1, i \in [d] \). The function \( \mathbf{x} \mapsto \|\mathbf{x}\|_\lambda \) on \([0, \infty)^d\) is called \textit{stable tail dependence function} and is well-known to define an orthant-monotonic norm on \( \mathbb{R}^d \) satisfying \( \|\cdot\|_\infty \leq \|\cdot\|_\lambda \leq \|\cdot\|_1 \). Recall that a norm is called \textit{orthant-monotonic} if it depends on its argument \( \mathbf{x} \) only through \( |\mathbf{x}| = (|x_1|, \ldots, |x_d|) \), see [10]. In dimension \( d = 2 \) the notions “stable tail dependence function” and “orthant monotonic norm” even coincide, but in dimensions \( d \geq 3 \) the norms \( \|\cdot\|_\lambda \) form a proper subfamily of orthant-monotonic norms, see [18]. Loosely speaking, while an orthant-monotonic norm is just non-decreasing in each coordinate (“first order monotonicity”) and convex (“second order monotonicity”), norms of the form \( \|\cdot\|_\lambda \) satisfy monotonicity conditions up to the order \( d \). The probability law of the random vector \( \mathbf{Y} \) is not uniquely determined by the stable tail dependence function \( \mathbf{x} \mapsto \|\mathbf{x}\|_\lambda \) in general. However, under the additional restriction that \( Y/d \) takes values within the unit simplex \( \{y \in [0, 1]^d : \|y\|_1 = 1\} \) it is known to be unique, see [21]. Up to a scale factor, the probability law of \( \mathbf{Y} \) is called \textit{Pickands dependence measure} in this case, the nomenclature dating back to [20].

2.3 Min-stable multivariate exponentials and de Finetti

Now consider an arbitrary infinite exchangeable sequence of non-negative random variables \( \mathbf{Y} = (Y_1, Y_2, \ldots) \) with \( \mathbb{E}[Y_1] = 1 \). For arbitrary but fixed \( d \geq 1 \) we may then define a stable tail dependence function \( \|\cdot\|_\lambda \) via (2), see [23]. The associated min-stable multivariate exponential distribution is conditionally iid, its de Finetti structure being revealed in [16]. In order to formulate this result, required is the notion of a stochastic process \( H = \{H_s\}_{s \geq 0} \) that is \textit{strongly infinitely divisible with respect to time} (strong IDT), meaning that for arbitrary \( n \in \mathbb{N} \) the process \( H \) has the same probability distribution as the process \( H^{(1)} + \ldots + H^{(n)} \), where \( H^{(1)}, \ldots, H^{(n)} \) are independent copies of \( H \). Lévy processes form the best-studied, proper subfamily of strong IDT processes, and the corresponding application to the present setting is treated in Section 4 below. Examples for non-Lévy strong IDT processes can be found in [13, 17]. Intuitively, strong IDT processes form a proper subfamily of infinitely divisible processes, in a similar analogy as stable distributions form a subfamily of infinitely divisible distributions, see [13]. In particular, such processes can be described by a drift constant and a Lévy measure on some path space. We always deal with non-decreasing strong IDT processes in our derivations and the Lévy measure in this particular case can be equipped with a probabilistic interpretation in terms of \( \mathbf{Y} \) that is particularly useful for our purpose. To wit, [16] proves that a non-decreasing, right-continuous process \( H = \{H_t\}_{t \geq 0} \) is strong IDT if and only if there exist constants \( b, c \geq 0 \) and an exchangeable sequence \( \mathbf{Y} = (Y_1, Y_2, \ldots) \) of non-negative random variables with \( \mathbb{E}[Y_1] = 1 \) such that

\[
\mathbb{E}\left[ e^{-\sum_{i=1}^d H_i} \right] = e^{-b \|\mathbf{x}\|_1 + c \|\mathbf{x}\|_\lambda}, \quad \mathbf{x} \in [0, \infty)^d, \quad d \in \mathbb{N} \text{ arbitrary}.
\]
Under the normalizing constraint $c = 1 - b$ the involved norm
\[
\|\cdot\|_{b,Y} := b \|\cdot\|_1 + (1 - b) \|\cdot\|_Y
\] (4)
in the exponent takes the value one on unit vectors. This normalizing condition is assumed throughout. With $\varepsilon_1, \ldots, \varepsilon_d$ iid unit scale exponential random variables, independent of $H$, this implies that the random vector $X = (X_1, \ldots, X_d)$ given by
\[
X_i := \inf\{x > 0 : H_x > \varepsilon_i\}, \quad i \in [d],
\]
is min-stable multivariate exponential with associated stable tail dependence function $b \|\cdot\|_1 + (1 - b) \|\cdot\|_Y$. A de Finetti structure for the probability law $\mu_d$ of $X$ is thus revealed to be
\[
\mu_d((-\infty, x]) = \mu_d([0, x]) = \mathbb{E}\left[\prod_{i=1}^d (1 - e^{-H_{x_i}})\right], \quad x \in [0, \infty)^d,
\]
i.e. conditionally on $H$, the random variables $X_1, \ldots, X_d$ are iid with distribution function $x \mapsto 1 - \exp(-H_x), \quad x \geq 0$. We are going to translate this de Finetti structure for survival functions of the form $x \mapsto \exp(-\|x\|_{b,Y})$ into a de Finetti structure for densities of the algebraically similar form $f_d$ with $\|\cdot\| = \|\cdot\|_{b,Y}$.

## 3 The main result and the $\|\cdot\|_p$-case

With a given infinite exchangeable sequence of non-negative random variables $Y = (Y_1, Y_2, \ldots)$ subject to the constraint $\mathbb{E}[Y_1] = 1$, and with some constant $b \in [0, 1]$, we consider the orthant-monotonic norm (4) on $\mathbb{R}^d$. The resulting density $f_d$ describes a probability distribution that is conditionally iid, which is the content of the following theorem, constituting our main finding.

**Theorem 3.1 (de Finetti structure of $f_d$).** *Defining the non-zero and finite constant*
\[
C_{d,\|\cdot\|_{b,Y}} := \int_{(0,\infty)^d} e^{-\|x\|_{b,Y}} \, dx,
\]
*for arbitrary $\lambda > 0$ the function*
\[
f_d(x) = \frac{\lambda^d}{C_{d,\|\cdot\|_{b,Y}}} e^{-\lambda \|x\|_{b,Y}}, \quad x \in (0, \infty)^d,
\]
defines the density of a probability measure $\mu_d$ which is conditionally iid. Concretely, there exists a rightcontinuous, non-decreasing strong IDT process $H = (H_t)_{t \geq 0}$, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that
\[
\mu_d((-\infty, x]) = \mu_d([0, x]) = \mathbb{E}\left[\prod_{i=1}^d \int_0^{x_i} e^{-H_s} \, ds \right],
\]
*where the probability measure $\mathbb{Q}_d$ is equivalent to $\mathbb{P}$ and defined by*
\[
d\mathbb{Q}_d = \left(\frac{\lambda^d}{C_{d,\|\cdot\|_{b,Y}}} e^{-\lambda \|x\|_{b,Y}}\right) \, d\mathbb{P}.
\]
**Proof.** From $\|\cdot\|_{b,Y} \equiv \|\cdot\|_{\infty}$ we obtain
\[
C_{d,\|\cdot\|_{b,Y}} = \int_{(0,\infty)^d} e^{-\|x\|_{b,Y}} \, dx \leq \int_{(0,\infty)^d} e^{-\|x\|_{\infty}} \, dx = d! < \infty,
\]
and
\[
\int_{(0,\infty)^d} e^{-\lambda ||x||_{b,b}} \, dx = \frac{C_{d,||.||_{b,b}}}{\lambda^d}
\]
by a change of variables and the fact that \( \lambda \cdot ||x||_{b,b} = ||\lambda x||_{b,b} \). This shows that \( f_d \) is a density. Now we work on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), on which a non-decreasing strong IDT process \( H = \{H_t\}_{t \geq 0} \) is defined, such that (3) holds. Such a process exists by [16]. We observe

\[
\frac{\lambda^d}{C_{d,||.||_{b,b}}} \int_0^x \cdots \int_0^x e^{-\lambda ||y||_{b,b}} \, dy \cdots dy_1 (3) \frac{\lambda^d}{C_{d,||.||_{b,b}}} \mathbb{E}^P \left[ \prod_{i=1}^d \int_0^{x_i} e^{-H_{s,i}} \, ds \right] = \mathbb{E}^Q_d \left[ \prod_{i=1}^d \int_0^{\infty} e^{-H_{s,i}} \, ds \right],
\]

where the last equality follows from the change of variables \( u = \lambda s \). Further notice that \( Q_d \) is actually a probability measure equivalent to \( \mathbb{P} \), since

\[
\mathbb{E}^P \left[ \left( \int_0^{\infty} e^{-H_s} \, ds \right)^d \right] = \int_{(0,\infty)^d} \mathbb{E}^P \left[ e^{\sum_{i=1}^d H_{s,i}} \right] \, ds = \int_{(0,\infty)^d} e^{-||y||_{b,b}} \, dy = C_{d,||.||_{b,b}}.
\]

The claim is established.

In general, a simulation of the process \( H \) under \( Q_d \) is difficult, since most well-known examples for strong IDT processes rely on path transformations of Lévy processes, see [17], making the law of \( \int_0^{\infty} \exp(-H_s) \, ds \) intractable and hence Theorem 3.1 difficult to apply in concrete cases. The statement has thus rather structural value, since to decide whether a given probability law is conditionally iid or not is rather complicated in general, see [12]. But there is one prominent special case, namely \( ||.||_{b,b} = ||.||_p \) with \( p \in [1, \infty] \), which yields a tractable and interesting example, because the related strong IDT process is very simple. The \( \ell_1 \)-norm corresponds to the trivial strong IDT process \( H_t = t \) (i.e. \( b = 1 \)) and the density \( f_d \) clearly corresponds to independent exponential random variables with rate \( \lambda \), so is well-understood. The case \( p = \infty \) is a special case of [7] and is treated in the following example. The most interesting and (to the best of our knowledge) new case \( p \in (1, \infty) \) is then fleshed out in a separate subsection after the example.

**Example 3.2** (The special case \( ||.||_{b,b} = ||.||_\infty \)). To demonstrate Theorem 3.1, let us show how the traditional result for the case \( ||.||_{b,b} = ||.||_\infty \) is embedded. To this end, we recall that the stable tail dependence function \( ||.||_\infty \) corresponds to \( b = 0 \) and \( Y = (1, 1, \ldots) \), which corresponds to the extreme strong IDT process \( H_t = \infty 1_{\{t > 0\}} \) for \( e \) exponential with unit scale. We observe that

\[
\int_0^{\infty} e^{-H_s} \, ds = e, \quad E[e^d] = d!, \quad \int_0^{\infty} \frac{\lambda x}{e} e^{-H_s} \, ds = \min \left\{ \frac{\lambda x}{e}, 1 \right\}.
\]

In particular, conditionally on \( e \), the last distribution function corresponds to a uniform law on the interval \( [0, e/\lambda] \). It remains to determine the probability law of \( e/\lambda \) under \( Q_d \). To this end, for an arbitrary bounded measurable function \( h : [0, \infty) \to \mathbb{R} \) we observe

\[
\mathbb{E}^Q_d \left[ h \left( \frac{e}{\lambda} \right) \right] = \mathbb{E}^P \left[ \frac{e^d}{d!} h \left( \frac{e}{\lambda} \right) \right] = \int_0^{\infty} \frac{y^d}{d!} h \left( \frac{y}{\lambda} \right) \, dy.
\]
Corollary 3.5. Let \( \lambda > 0 \) and \( p \in [1, \infty) \). Let the random variable \( S \) have the polynomially tilted stable density \( g_{\alpha,1/p}^S \). Conditionally on \( S \), let \( X_1, \ldots, X_d \) be iid with generalized Gamma density \( g_{\alpha,1/p}^S \). Then the random vector \( X = (X_1, \ldots, X_d) \) is conditionally iid and has density

\[
 f_d(x) = \frac{\lambda^d \Gamma(1 + \frac{d}{p})}{d! \Gamma(1 + \frac{1}{p})^d} e^{-\lambda \|x\|_p}, \quad x \in (0, \infty)^d.
\]

Proof. We consider the Fréchet distribution function

\[
 F(x) := e^{-(F(1-1/p)x)^{1-1/p}}, \quad x \geq 0,
\]

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 Consequently, \( e/\lambda \) has an Erlang distribution with scale \( \lambda \) and shape \( d + 1 \) under \( Q_d \), as well-known from [7, Theorem 2]. In that reference, the author derives necessary and sufficient conditions on a function \( g : [0, \infty) \rightarrow [0, \infty) \) such that \( x \mapsto g(\|x\|_\infty) \) equals the density of a probability distribution on \( (0, \infty)^d \) that is conditionally iid. It is shown that this is the case if and only if there is a positive random variable \( M \) such that

\[
 g(x) = \mathbb{E} [1_{\{M > x\}} M^{-d}],
\]

and then a random vector \( X = (X_1, \ldots, X_d) \) with density \( x \mapsto g(\|x\|_\infty) \) may be constructed such that conditionally on \( M \) the components \( X_1, \ldots, X_d \) are iid uniformly distributed on the interval \( [0, M] \). The intersection with the present reference occurs for \( M \) with an Erlang distribution, in which case \( g \) has an exponential form.

3.1 The case \( \|\cdot\| = \|\cdot\|_p \)

It is instructive to recall the definitions of two one-dimensional probability laws. On the one hand, the following probability law is introduced in [26].

**Definition 3.3** (Generalized Gamma distribution). A positive random variable is said to have a generalized Gamma distribution with parameters \( \eta > 0 \) (shape), \( \theta > 0 \) (scale), and \( \kappa > 0 \), if its density is given by

\[
 g_{\eta,\theta,\kappa}^G(z) := \frac{\kappa \theta^{-\eta} z^{\eta-1} e^{-(\frac{z}{\theta})\kappa}}{\Gamma(\frac{\eta}{\kappa})}, \quad z > 0.
\]

The traditional Gamma distribution with shape \( \eta > 0 \) and scale \( \theta > 0 \) is obtained for the special case \( \kappa = 1 \).

If \( Z \) has a (traditional) Gamma distribution with density \( g_{\eta,\theta,1}^G \), the random variable \( Z^{1/\kappa} \) has a generalized Gamma distribution with density \( g_{\eta,\theta,\kappa}^G \).

On the other hand, the following probability law is investigated in [4].

**Definition 3.4** (Polynomially tilted stable law). A positive random variable \( S \) is said to be stable with parameter \( \alpha \in (0, 1] \) if its Laplace transform is given by

\[
 \mathbb{E} [e^{-S}] = e^{-\alpha x}, \quad x \geq 0.
\]

We denote the density of \( S \) by \( g_{\alpha}^S \). A random variable \( S \) is said to be polynomially tilted stable with parameters \( \beta \geq 0 \) and \( \alpha \in (0, 1] \), if the density \( g_{\beta,\alpha}^S(x) \) satisfies \( g_{\beta,\alpha}^S(x) \propto x^{-\beta} g_{\alpha}^S(x) \). In particular, \( g_{0,\alpha}^S = g_{\alpha}^S \).

An efficient and exact simulation algorithm for the polynomially tilted stable distribution is derived in [4]. An application of Theorem 3.1 implies the following corollary.

**Corollary 3.5** (de Finetti structure in case of \( \|\cdot\|_p \)). Fix \( \lambda > 0 \) and \( p \in [1, \infty) \). Let the random variable \( S \) have the polynomially tilted stable density \( g_{\alpha,1/p}^S \). Conditionally on \( S \), let \( X_1, \ldots, X_d \) be iid with generalized Gamma density \( g_{\alpha,1/p}^G \). Then the random vector \( X = (X_1, \ldots, X_d) \) is conditionally iid and has density

\[
 f_d(x) = \frac{\lambda^d \Gamma(1 + \frac{d}{p})}{d! \Gamma(1 + \frac{1}{p})^d} e^{-\lambda \|x\|_p}, \quad x \in (0, \infty)^d.
\]

Proof. We consider the Fréchet distribution function

\[
 F(x) := e^{-(F(1-1/p)x)^{1-1/p}}, \quad x \geq 0,
\]
with shape parameter \( p \) and unit mean. Let \( Y = (Y_1, Y_2, \ldots) \) be an iid sequence with \( Y_1 \sim F \). It is well-known and easy to check that \( \|x\|_p = \|x\|_p \). The associated strong IDT process \( H \), defined on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\), is given by \( H_x = S x^p \), \( x \geq 0 \), where the random variable \( S \) has a stable distribution with parameter \( 1/p \), see [16, Example 2]. We observe that

\[
\int_0^\infty & \frac{\lambda x^p}{\Gamma(1/p)} e^{-H_s} \, ds = \int_0^\infty \frac{(\lambda x)^p}{\Gamma(1/p)} y^{\frac{1}{p}-1} e^{-\frac{y}{p}} \, dy = \int_0^\infty \frac{1}{\Gamma(1/p)} \left( \frac{1}{\lambda p} \right)^{1/p} \left( \frac{1}{\lambda p} y \right)^{1/p-1} e^{-\frac{y}{p}} \, dy \\
&= \int_0^\infty g_{1, S^{-1/p}/\lambda, p}(y) \, dy.
\]

Furthermore,

\[
\int_0^\infty e^{-H_s} \, ds = \int_0^\infty e^{-S^{-\frac{1}{p}} \Gamma \left( \frac{1}{p} + 1 \right)} \, ds = \frac{d!}{\Gamma \left( \frac{1}{p} + \frac{d}{p} \right)} E^p[S^{-\frac{1}{p}}],
\]

where the last equality is shown in [27]. For an arbitrary bounded measurable function \( h : [0, \infty) \to \mathbb{R} \) we thus observe

\[
E^{Q_d}[h(S)] = \frac{d!}{\Gamma \left( \frac{1}{p} + \frac{d}{p} \right)} \int_0^\infty h(s) g_{d/p, 1/p}(s) \, ds.
\]

Thus, under the equivalent measure \( Q_d \) the stable random variable \( S \) has the polynomially tilted stable density \( g_{d/p, 1/p} \). In analytical terms, we have shown that

\[
f_d(x) = \frac{A^d \Gamma \left( \frac{1}{p} + \frac{d}{p} \right)}{d! \Gamma \left( \frac{1}{p} + \frac{1}{p} \right)} e^{-\lambda \|x\|_p} = \int_0^\infty \left( \prod_{i=1}^d \frac{1}{\Gamma \left( \frac{1}{p} + \frac{1}{p} \right)} \right) g_{d/p, 1/p}(x_i) \, ds,
\]

and the argument is complete.

### 4 Time-homogeneous load sharing models

Let \( E_1, E_2, \ldots, E_d \) be independent, exponential random variables with rate parameters \((d - i + 1) r_i, i \in [d] \). Intuitively, we imagine a system of \( d \) components that is symmetrically exposed to external shocks, which arrive at time points \( T_1, T_2, \ldots, T_d \), where

\[
T_1 := E_1 + \ldots + E_i, \quad i = 1, \ldots, d, \quad T_0 := 0.
\]

We consider the random vector \((X_1, \ldots, X_d) := (T_{P(1)}, \ldots, T_{P(d)})\), where \( P \) is a permutation of \([d] \), independent of \( E_1, \ldots, E_d \), with \( \mathbb{P}(P = \pi) = 1/d! \) for each possible permutation \( \pi \) of \([d] \). That is, \( P \) is uniformly distributed on the set of all possible permutations. At each \( T_i \), one randomly picked component of the system fails, so between \( T_{i-1} \) and \( T_i \) there are \( d - i + 1 \) components left, \( i \in [d] \). Intuitively, between \( T_{i-1} \) and \( T_i \) each single component has the same failure intensity \( r_i \) and the components work independently, so that the (a priori unknown) next component that is hit at time \( T_{i-1} \) has intensity \((d - i + 1) r_i \), since the minimum of \((d - i + 1) \) independent exponentials with rate \( r_i \) has this rate. This explains the explicit occurrence of the constant \((d - i + 1) \) before \( r_i \) in our notation, which is slightly different to the one in [24] for instance. The density of \((X_1, \ldots, X_d)\) is given by

\[
f_d(x) = \left( \prod_{i=1}^d r_i \right) e^{-\sum_{i=1}^d (d - i + 1) r_i (x_i - x_{i-1})}, \quad (5)
\]
with the notation \( x_{[0]} := 0 \) and \( x_{[1]} \leq \ldots \leq x_{[d]} \) denoting an ordered list of \( |x_1|, \ldots, |x_d| \), see [24, Formula (2.5)]. Introducing \( r_{d+1} := 0 \), we observe that the function

\[
\|x\| := \sum_{i=1}^{d} (d-i+1) r_i (x_{[i]} - x_{[i-1]}) = \sum_{i=1}^{d} x_{[i]} ((d-i+1) r_i - (d-i) r_{i+1})
\]

defines an orthant-monotonic norm if the sequence \( i \mapsto (d-i+1) r_i \) is non-increasing. Intuitively, this condition means that the sequence of waiting times \( E_1, \ldots, E_d \) between the arrival of shocks needs to be non-decreasing on average. However, it does not necessarily mean that the sequence \( i \mapsto r_i \) of intensity rates needs to be non-increasing, i.e., the component-idiachronic failure intensities might well increase with observed shocks. In fact, our result in Corollary 4.1 below covers precisely such situations in which \( i \mapsto r_i \) is non-decreasing but \( i \mapsto (d-i+1) r_i \) is non-increasing.

Stochastic models with density \( f_d \) in (5) are considered among others in [24], and are known as **time-homogeneous load sharing models** with reference to the reliability context. See [25] for a recent account on load sharing systems. We point out that, under the condition that \( i \mapsto r_i \) is non-decreasing, they form a subclass of *systems weakened by failure*, as shown in [19]. Such models have also been found useful in financial applications as a model of bankruptcy times for companies, see [8, 9, 28]. In the bivariate case \( d = 2 \) the probability law of \((X_1, X_2)\) is called *exchangeable Freund distribution*, named after [6]. If the failure rates are decreasing, i.e. \( r_2 < r_1 \), the surviving component becomes more robust after the other component is destroyed. The correlation coefficient between \( X_1 \) and \( X_2 \) is negative in this case, hence there is no hope to find a de Finetti representation, since conditionally iid probability laws necessarily exhibit non-negative correlations. However, the more interesting case for applications is the case of non-decreasing failure rates. Corollary 4.1 below shows that if the sequence \( i \mapsto r_i \) satisfies certain monotonicity conditions, then the model is conditionally iid.

In order to formulate our result, we recall that a Lévy subordinator \( L = \{L_t\}_{t \geq 0} \) is a non-decreasing, right-continuous stochastic process, which has independent and stationary increments. A textbook account on the topic is [22]. Lévy subordinators form a subfamily of non-decreasing strong IDT processes. While an appropriate analytical description of an arbitrary non-decreasing strong IDT process may be given via (3) in terms of the constants \( b, c \geq 0 \) and a norm \( \| \cdot \|_Y \), in the special case of a Lévy subordinator \( L \) it is more common (and traditional) to describe its probability law in terms of a *drift constant* \( b_L \geq 0 \) and a *Lévy measure* \( \nu_L \) on \((0, \infty)\), which is an arbitrary measure on \((0, \infty)\) satisfying \( \int_{[0,\infty]} \min\{1, x\} \nu_L(du) < \infty \). Concretely, the law of \( L \) is fully determined by the law of \( L_1 \) due to the stationary and independent increments, and the Laplace transformation of \( L_1 \) equals

\[
\mathbb{E}\left[ e^{-x L_1} \right] = \exp \left\{ -b_L x - \int_{[0,\infty]} (1 - e^{-x u}) \nu_L(du) \right\}, \quad x \geq 0.
\]

**Corollary 4.1** (de Finetti structure of some load sharing models). Assume there exists a constant \( \lambda > 0 \) and a random variable \( V \) taking values on \([0, 1]\), defined on some generic probability space \((\Omega, \mathcal{F}, \mathbb{P})\), such that

\[
r_i := \frac{\lambda}{d-i+1} \sum_{j=0}^{d-i} \mathbb{E} [V^j], \quad i \in [d].
\]

Then the time-homogeneous load sharing model \( X \) has density given by

\[
f_d(x) = \left( \prod_{i=1}^{d} r_i \right) e^{-\lambda \sum_{i=1}^{d} \mathbb{E}[V^{d-i} x_{[i]}]}, \quad x \in (0, \infty)^d,
\]

and the associated probability law \( \mu_d \) is conditionally iid. Concretely, let \( L = \{L_t\}_{t \geq 0} \) be a Lévy subordinator, defined on \((\Omega, \mathcal{F}, \mathbb{P})\), with drift \( b_L := -\lambda \mathbb{P}(V = 1) \) and Lévy measure

\[
\nu_L(dx) = \frac{\lambda}{1 - e^{-x}} \mathbb{P}(-\log(V) \in dx), \quad x \in (0, \infty).
\]
Then
\[
\mu_d((-\infty, \mathbf{x})) = \mu_d([0, \mathbf{x}]) = \mathbb{E}^\mathbb{Q}_d \left[ \prod_{i=1}^d \int_0^{x_i} e^{-L_i s} \, ds \right],
\]
where the probability measure \( \mathbb{Q}_d \) is equivalent to \( \mathbb{P} \) and defined by
\[
d\mathbb{Q}_d = \left( \prod_{i=1}^d r_i \right) \left( \int_0^\infty e^{-L_i s} \, ds \right)^d \, d\mathbb{P}.
\]

\textbf{Proof.} The statement is a special case of Theorem 3.1 applied to the norm \( \| \cdot \|_{b_j, \lambda, Y} \) associated with the strong IDT process \( H_t = L_{t/\lambda} \). The associated law of the sequence \( \mathbf{Y} \) is not required for the present proof, but the interested reader can find it explained in [16, Example 1]. The bijection between random variables \( V \) on \([0, 1]\) and Lévy subordinators \( L \) is a result due to [15, Lemma 4.1.3, p. 91], and for the respective link between \( L \) and the multivariate survival function
\[
\mathbf{x} \mapsto e^{-\lambda \sum_{i=1}^d E[V^{d-i}] x_i], \quad \mathbf{x} \in [0, \infty)^d,
\]
we refer to [15, Remark 3.55, p. 82]. These results have also been published in [14]. The proof then becomes a direct corollary to Theorem 3.1. In particular, \( C_{d, \| \cdot \|_{b_j, \lambda, Y}} \) is given by
\[
d! \int_0^\infty \cdots \int_0^\infty \int_0^\infty \int_0^\infty e^{-E[V^d]} \, dy_d \cdots \, dy_1
\]
\[
= \frac{d!}{\prod_{i=1}^{d-1} \sum_{j=1}^i \mathbb{E}[V]^j} = \frac{\lambda^d}{\prod_{i=1}^d r_i}.
\]

To decide whether a given sequence \( r_1, \ldots, r_d \) of failure rates satisfies the criterion (6) of Corollary 4.1 is equivalent to the so-called truncated Hausdorff moment problem, i.e. to decide whether a sequence \( (a_0, \ldots, a_{d-1}) \) can possibly be the moment sequence of some random variable \( V \) on \([0, 1]\), i.e. \( a_n = \mathbb{E}[V^n], n = 0, \ldots, d - 1 \). This problem is satisfactorily solved at least since [11], so the criterion of the corollary can efficiently be verified in concrete cases.

\textbf{Remark 4.2} (Non-necessity of the theorem). \textit{If \( r_i \) satisfy the criterion (6) of Corollary 4.1, then \( (d - i + 1) r_i = \lambda \mathbb{E}[1 + V + V^2 + \cdots + V^{d-i}] \) is non-increasing in \( i \). In contrast, \( r_i \) is non-decreasing in \( i \), since \( r_i \) is the arithmetic average of the first \( d - i \) moments of \( V \), and this moment sequence is non-increasing. Intuitively, the exponential rates of the random variables \( E_1, \ldots, E_d \) are non-increasing. Thus, even though the rate of each individual component increases after a shock, the waiting times between observed shocks to the system increase on average. It is intuitive to believe, however, that increasing exponential rates of the \( E_i \) can also be obtained with a probability measure that is conditionally iid, as the extreme limiting case \( r_1 < r_2 = \cdots = r_d = \infty \), which is not covered by Corollary 4.1, shows (we obtain the upper Fréchet-Hoeffding bound then, which is conditionally iid). This implies that the criterion of Corollary 4.1 is only sufficient for the existence of a de Finetti structure but cannot be necessary, as will be demonstrated further in Example 4.3 below. A generalization of Corollary 4.1 to a sufficient and necessary condition on the rates \( r_i \) appears to be an interesting open problem for future research.}

\textbf{Example 4.3} (The general case for \( d = 2 \)). Consider the case \( d = 2 \), i.e. the density
\[
f_2(x_1, x_2) = r_1 r_2 e^{-(2 r_1 - r_2) x_{[1]} - r_2 x_{[2]}}, \quad x_1, x_2 > 0.
\]

The function
\[
\| (x_1, x_2) \| := (2 r_1 - r_2) x_{[1]} + r_2 x_{[2]}, \quad x_1, x_2 \geq 0,
\]
equals the restriction to $[0, \infty)^2$ of some proper norm on $\mathbb{R}^2$ for all $r_2 \geq r_1 > 0$. But only if the additional requirement $r_1 \geq r_2/2$ is in place, this function equals the restriction of an orthant-monotonic norm, in which case the definition (7) is valid for arbitrary $x_1, x_2 \in \mathbb{R}$ (and not only in the positive orthant). The density $f_2$ is conditionally iid if and only if $r_2 \geq r_1 > 0$ (without the further restriction $r_1 \geq r_2/2$ as required in Corollary 4.1). To see this, we first consider two special cases. For $r_2/2 = r_1$ we obtain the well-known special case $\|\cdot\| = \|\cdot\|_\infty$, which is conditionally iid. For $r_2 = r_1$, we obtain the well-known special case $\|\cdot\| = \|\cdot\|_1$ of independent components. Corollary 4.1 covers all cases in between these two special cases, since $\|\cdot\|_\infty \leq \|\cdot\|_{b,Y} \leq \|\cdot\|_1$. However, the case when $r_1 < r_2/2$ is not covered. Instead, if we postulate only $r_2 > r_1$ and $r_1 \neq r_2/2$ (i.e. exclude the two simple special cases above), we let $B$ have a $\beta(p, q)$-distribution with parameters $p = \min(2r_1, 2(r_2 - r_1))/2r_1 - r_2$ and $q = 3$. Conditionally on $U := -\log(B)/2r_1 - r_2$ the components $X_1, X_2$ of $(X_1, X_2) \sim f_2$ are iid with density equal to
\[
x \mapsto \frac{|2r_1 - r_2| e^{-(2r_1 - r_2)x} x}{1 - e^{-(2r_1 - r_2) U}}, \quad x \in (0, U),
\]
as can readily be verified. It is observed that this density is non-increasing if and only if $r_1 \geq r_2/2$, and only in this case falls into the scope of Corollary 4.1.

**Example 4.4** (A simple special case). Consider the Lévy subordinator $L_t = \lambda \alpha t + \infty \cdot \mathbf{1}_{\{t \geq r_1\}}$, where $E_{\lambda(1-a)}$ is an exponential random variable with rate $\lambda(1-a)$ for some parameters $a \in [0, 1)$ and $\lambda > 0$, on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The associated random variable $V$ has a Bernoulli distribution with success probability $a$, and $r_1 = \lambda (1 + (d - i) a)/(d - i + 1), i \in [d]$. We observe that the conditional density under $\mathbb{P}$ is given by
\[
x \mapsto \frac{\lambda a e^{-\lambda a x}}{1 - e^{-\lambda a E_{\lambda(1-a)}}}, \quad x \in (0, E_{\lambda(1-a)}).
\]

It remains to compute the density of $E_{\lambda(1-a)}$ under $Q_d$. To this end, let $h : (0, \infty) \to \mathbb{R}$ continuous and bounded, and observe
\[
\mathbb{E}^{Q_d}[h(E_{\lambda(1-a)})] = \mathbb{E}^{\mathbb{P}} \left[ \frac{\lambda^d}{d!} \left( \prod_{i=1}^{d} (1 + (i - 1) a) \right)^d \left( 1 - e^{-\lambda a E_{\lambda(1-a)}} \right)^d h(E_{\lambda a}) \right]
\]
\[
= \int_0^\infty \frac{1}{d!} \left( \prod_{i=1}^{d} (1 + (i - 1) a) \right) \lambda (1 - a) e^{-\lambda (1-a) u} \left( 1 - e^{-\lambda a u} \right)^d h(u) du
\]
\[
= \int_0^\infty \frac{\Gamma(d + 1 + (1 - a)/a)}{\Gamma((d - a)/a) \Gamma(d + 1)} \lambda e^{-\lambda (1-a) u} \left( 1 - e^{-\lambda a u} \right)^d h(u) du.
\]

We observe from this that under $Q_d$ the random variable $E_{\lambda(1-a)}$ has the same distribution as $U := -\log(B)/(\lambda a)$, where $B$ has a $\beta(p, q)$-distribution with parameters $p = (1 - a)/a$ and $q = d + 1$. Intuitively, conditionally on $U$, the random variables $X_1, \ldots, X_d$ are iid and follow an exponential distribution with rate $\lambda a$ conditioned to be smaller than $U$. For $d = 2$ this example coincides precisely with the de Finetti representation of Example 4.3 in the case $r_1 > r_2/2$, and thus provides a generalization to $d = 2$.

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