Liouville-type theorems for the Navier–Stokes equations

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Abstract. An approach to the study of local regularity of weak solutions of the Navier–Stokes equations is described which is based on the reduction of questions of local smoothness of the original solutions to the proof of Liouville-type theorems for bounded ancient solutions of it. A survey is also given of results on Liouville theorems that are known at present for various classes of ancient solutions of the Navier–Stokes equations.

Bibliography: 55 titles.

Keywords: Navier–Stokes equations, ancient solutions, Liouville theorems.

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This research was carried out with the financial support of the Ministry of Education and Science of the Russian Federation (project no. 14.Z50.31.0037).

AMS 2010 Mathematics Subject Classification. Primary 35B53, 35Q30; Secondary 35D30.

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Introduction

Liouville’s classical theorem says that a harmonic function that is defined and bounded in the whole space must be identically equal to a constant. In this survey we treat the term ‘Liouville-type theorems’ in a somewhat broader sense: we aim at classifying the backwards-in-time solutions of the Navier–Stokes equations that are defined ‘globally’ (in the whole space or a half-space) and have a certain distinguished property. Such a property may be finiteness of this or that norm of the solution, restrictions on the behaviour of the solution at infinity, additional conditions of symmetry type on the solution, and many others.

Apart from being of interest on its own, this problem is important because the modern theory of partial differential equations has methods making it possible to reduce questions of the asymptotic behaviour of solutions of differential equations near singular points to the study of global solutions of the same equations. These methods are universal in the sense they can in fact be applied to arbitrary equations or systems that possess various groups of invariance. The general scheme of such methods is as follows: we look at the original solution in a neighbourhood of a singular point, and according to this solution we use scaling transformations
to construct a sequence of solutions defined in a chain of expanding domains. If we have chosen the parameters of the scaling transformations of the original solution so that the resulting sequence of functions is bounded in a certain norm, then we can often select a subsequence which can be shown to converge to some global solution. It often turns out that this global solution contains some ‘coded’ information about the asymptotic behaviour of the original solution near the singular point. For instance, if the original solution is smooth, then the resulting global solution turns out to be trivial.

This method works in quite different problems in partial differential equations, differential geometry, and mathematical physics. In this paper we discuss whether the above ideas can be used in the study of regularity of weak solutions of the Navier–Stokes equations.

The problem of the global existence of smooth solutions of the Navier–Stokes system (which the Clay Institute included in its list of ‘Millennium Problems’) has attracted the attention of many mathematicians for several decades. Despite numerous attempts to solve it, so far only two types of global results are known for the three-dimensional non-stationary Navier–Stokes system: the existence of weak solutions in the energy class and the existence of smooth solutions for small initial data (in an appropriate norm). Moreover, it still remains an open question whether weak solutions are unique. On the other hand, if smooth solutions do exist, then they are automatically unique in the energy class (weak-strong uniqueness). Therefore, the key question hiding behind the existence problem for smooth solutions is whether the Navier–Stokes system actually gives a deterministic description of the dynamics of viscous fluids for an arbitrary magnitude of the smooth initial data.

An analysis of the regularity properties of weak solutions is one of the possible approaches to answering this question.

One can prove that smooth solutions exist by reducing the Navier–Stokes system to a system of integral equations. In essence, this approach is a variant of the perturbation method: first solve the linearized problem and construct a sequence of iterations, and then ensure the convergence of the iterative procedure by taking a sufficiently small interval of time. This approach enables one to show that smooth (so-called mild) solutions exist locally in time, that is, on a small interval of time whose size depends on the magnitude of the initial data. Whether one can smoothly extend this solution outside the interval is a question requiring further investigation. The question of the behaviour of various norms of the solution when approaching the instant of a possible blowup arises here. Two different behaviours are possible: in the first case one or another scale-invariant norm (that is, a norm invariant under natural scaling transformations of the Navier–Stokes equations) of the solution remains bounded. Such singularities are said to be of type I (the solution has the same type of singularity as a self-similar solution). All the other singularities are put in type II.

In this paper we discuss methods enabling us to reduce questions on the existence of type-I singularities of solutions of the Navier–Stokes equations to questions on whether ancient solutions of the Navier–Stokes system are trivial. We also present a survey of presently known results on Liouville-type theorems for these equations.

The following notation is used in this paper.
• For a Lebesgue-measurable subset $E$ of $\mathbb{R}^n$ let $|E|$ denote its $n$-dimensional Lebesgue measure.

• $\overline{\Omega}$ is the closure of $\Omega$, and $\Omega \Subset \Omega_0$ indicates that $\overline{\Omega}$ is a compact set and $\overline{\Omega} \subset \Omega_0$.

• $\mathbb{R}^+_n := \{ x \in \mathbb{R}^n : x_n > 0 \}$.

• $Q_+ := \mathbb{R}^3 \times ]-\infty, 0[\), $Q_+ := \mathbb{R}^3 \times [0, +\infty[\), $Q^+ := \mathbb{R}_+^3 \times ]-\infty, 0[\).

• $\Omega \subset \mathbb{R}^n$ is an open set, $C_0^\infty(\Omega)$ is the set of smooth functions with compact support in $\Omega$, $\mathcal{D}'(\Omega)$ is the space of distributions on $\Omega$, and $\langle p, \eta \rangle$ is the value of the distribution $p \in \mathcal{D}'(\Omega)$ on a function $\eta \in C_0^\infty(\Omega)$.

• For $a, b \in \mathbb{R}^n$ and $A, B \in \mathbb{R}^{n \times n}$ let

$$a \cdot b := a_i b_i \quad \text{and} \quad A : B := A_{ij} B_{ij}$$

denote the scalar products in $\mathbb{R}^n$ and $\mathbb{R}^{n \times n}$, respectively (we follow the convention of taking the sum from 1 to $n$ over repeating indices), and let

$$a \otimes b := (a_i b_j)$$

be the tensor product in $\mathbb{R}^n$. Denote by $I$ the $n \times n$ identity matrix.

• For a function $\varphi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $\varphi = \varphi(x, t)$, let

$$\varphi_{,k} := \frac{\partial \varphi}{\partial x_k} \quad \text{and} \quad \partial_t \varphi := \frac{\partial \varphi}{\partial t}.$$

• For a vector field $u : \mathbb{R}^3 \to \mathbb{R}^3$ let

$$\nabla u := (u_{i,j}), \quad \text{div} u := u_{i,i}, \quad (u \cdot \nabla) u := (u_j u_{i,j}), \quad \text{curl} u := (u_{3,2} - u_{2,3}, u_{1,3} - u_{3,1}, u_{2,1} - u_{1,2}).$$

• $B(x_0, R) := \{ x \in \mathbb{R}^n : |x - x_0| < R \}$, $B(R) := B(0, R)$, and $B := B(1)$.

• $Q(z_0, R) := B(x_0, R) \times ]t_0 - R^2, t_0[\) for $z_0 = (x_0, t_0)$, $Q(R) := Q(0, R)$, and $Q := Q(1)$.

• $\mathcal{C}(x_0, R) := \{ x \in \mathbb{R}^n : |x - x_0'| < R, |x_n - x_{0n}| < R \}$ is a cylinder in $\mathbb{R}^n$, where $x' := (x_1, \ldots, x_{n-1})$; $\mathcal{C}(R) := \mathcal{C}(0, R)$, and $\mathcal{C} := \mathcal{C}(1)$.

• $\mathscr{D}(z_0, R) := \mathcal{C}(x_0, R) \times ]t_0 - R^2, t_0[\)$ for $z_0 = (x_0, t_0)$, $\mathscr{D}(R) := \mathscr{D}(0, R)$, and $\mathscr{D} := \mathscr{D}(1)$.

• $L^s(\Omega)$ and $W^k_s(\Omega)$, $\dot{W}^k_s(\Omega)$ are the standard Lebesgue and Sobolev spaces.

• $\text{BMO}(\Omega)$ is the set of functions with bounded mean oscillation on $\Omega$, with norm

$$\| f \|_{\text{BMO}(\Omega)} := \sup_{B(x_0, R) \subset \Omega} \frac{1}{|B(R)|} \int_{B(x_0, R)} |f - [f]_{B(x_0, R)}| \, dx,$$

$$[f]_{B(x_0, R)} := \frac{1}{|B(R)|} \int_{B(x_0, R)} f \, dx,$$

and $\text{BMO}^{-1}(\Omega) := \{ \text{div} F \in \mathcal{D}'(\Omega) : F \in \text{BMO}(\Omega) \}$.

• $L^{s,q}(\Omega)$ is the Lorentz space with the norm

$$\| f \|_{L^{s,q}(\Omega)} := \left( s \int_0^\infty \alpha^{q-1} |\{ x \in \Omega : |f(x)| > \alpha \}|^{q/s} \, d\alpha \right)^{1/q},$$
and $L_{s,w}(\Omega) := L^{s,\infty}(\Omega)$ is the weak Lebesgue space with the norm
\[
\|f\|_{L^{s,\infty}(\Omega)} := \sup_{\alpha > 0} \alpha \{x \in \Omega: |f(x)| > \alpha\}^{1/s}.
\]

- $L_s(0,T;X), 1 \leq s < \infty,$ is the space of integrable functions $u: [0,T[ \to X$ taking values in the Banach space $X$,
\[
\|u\|_{L_s(0,T;X)} := \left( \int_0^T \|u(\cdot,t)\|_X^s \, dt \right)^{1/s},
\]
and for $s = \infty$
\[
\|u\|_{L_\infty(0,T;X)} := \text{ess sup}_{t \in [0,T[} \|u(\cdot,t)\|_X.
\]

- For $Q_T := \Omega \times ]0,T[,$ let $L_{s,t}(Q_T) := L_t(0,T;L_s(\Omega))$ denote the anisotropic Lebesgue space with the norm
\[
\|f\|_{L_{s,t}(Q_T)} := \left( \int_0^T \|f(\cdot,t)\|_{L_s(\Omega)}^l \, dt \right)^{1/l},
\]
and for $l = \infty,$
\[
L_{s,\infty}(Q_T) = L_\infty(0,T;L_s(\Omega)) \quad \text{and} \quad \|f\|_{L_{s,\infty}(Q_T)} := \text{ess sup}_{t \in [0,T[} \|f(\cdot,t)\|_{L_s(\Omega)}.
\]

- $W_s^{1,0}(Q) := L_s(-1,0;W_s^1(B)) = \{u \in L_s(Q): \nabla u \in L_s(Q)\},$
\[
\|u\|_{W_s^{1,0}(Q)} := \|u\|_{L_s(Q)} + \|\nabla u\|_{L_s(Q)}.
\]

- $W_s^{2,1}(Q) := \{u \in W_s^{1,0}(Q): \nabla^2 u \in L_s(Q), \partial_t u \in L_s(Q)\},$
\[
\|u\|_{W_s^{2,1}(Q)} := \|u\|_{W_s^{1,0}(Q)} + \|\nabla^2 u\|_{L_s(Q)} + \|\partial_t u\|_{L_s(Q)}.
\]

- For $x \in \mathbb{R}^3$ and $t > 0$
\[
\mathcal{E}(x) := \frac{1}{4\pi|x|} \quad \text{and} \quad \Gamma(x,t) := \frac{1}{(4\pi t)^{3/2}}e^{-|x|^2/(4t)}.
\]

- For $E \subset \mathbb{R}^{n+1},$ $\mathcal{P}^1(E)$ denotes the one-dimensional parabolic Hausdorff measure in $\mathbb{R}^{n+1}$:
\[
\mathcal{P}^1(E) := \lim_{\delta \to 0} \mathcal{P}^1_{\delta}(E),
\]
where
\[
\mathcal{P}^1_{\delta}(E) := \inf \left\{ \sum_k d(U_k): \text{the } U_k \text{ are open sets, } E \subset \bigcup_k U_k, \ d(U_k) < \delta \right\}
\]
and the infimum is over all countable covers of $E$ by open sets $U_k$ with parabolic diameter $d(U_k)$ at most $\delta$; here
\[
d(U) := \sup_{z'=(x',t'), \ z''=(x'',t'') \in U} (|x' - x''| + |t' - t''|^{1/2}).
\]
1. Ancient (backwards-in-time) solutions of the Navier–Stokes equations

Here we describe an approach to the study of local regularity of weak solutions of the Navier–Stokes equations, based on reducing questions on local smoothness of the original solutions to the proof of Liouville-type theorems for bounded ancient solutions corresponding to local singularities of the original problem. In this section we describe different classes of ancient (spatially global, backwards-in-time) solutions of the Navier–Stokes system and list their properties.

1.1. Bounded ancient solutions. Setting $Q_- := \mathbb{R}^3 \times ]-\infty, 0[$, we consider bounded ancient solutions of the Navier–Stokes equations in the whole space:

\[
\begin{aligned}
&\partial_t u - \Delta u + (u \cdot \nabla) u + \nabla p = 0, \\
&\text{div } u = 0
\end{aligned}
\tag{1.1}
\]

which are solutions in the weak sense (of distributions).

**Definition 1.1.** A function $u \in L^\infty(Q_-)$ is called a bounded ancient solution of the Navier–Stokes equations if it satisfies the identity

\[
\int_{Q_-} (u \cdot (\partial_t \eta + \Delta \eta) + u \otimes u : \nabla \eta) \, dx \, dt = 0 \quad \forall \eta \in C_0^\infty(Q_-): \quad \text{div } \eta = 0,
\]

and the divergence-free condition

\[
\int_{Q_-} u \cdot \nabla \varphi \, dx \, dt = 0 \quad \forall \varphi \in C_0^\infty(Q_-).
\]

The pressure can be recovered in the standard way as a distribution: there exists a $p \in \mathcal{D}'(Q_-)$ such that

\[
\int_{Q_-} (u \cdot (\partial_t \eta + \Delta \eta) + u \otimes u : \nabla \eta) \, dx \, dt = \langle p, \text{div } \eta \rangle \quad \forall \eta \in C_0^\infty(Q_-). \tag{1.2}
\]

It is easy to see that each function of the form

\[
u(x, t) = b(t), \quad \text{where } b \in L^\infty(-\infty, 0),
\tag{1.3}
\]
is a bounded ancient solution. We can assign to it a pressure $p(x, t) = -b'(t) \cdot x$, so that the pair $u, p$ satisfies (1.2).

In the case of the linear Stokes system each bounded ancient solution can be shown (see [18]) to have the form (1.3). By contrast, no similar result is known for the non-linear Navier–Stokes system, so there is a problem of classifying bounded ancient solutions. In what follows we consider a special subclass of bounded ancient solutions, the mild bounded ancient solutions.

1.2. The Cauchy problem for the Stokes system and the Green’s function. We let $Q_+ := \mathbb{R}^3 \times ]0, +\infty[$ and consider the Cauchy problem for the linear Stokes system

\[
\begin{aligned}
&\partial_t u - \Delta u + \nabla p = -\text{div } F, \\
&\text{div } u = 0 \\
&u|_{t=0} = a
\end{aligned}
\tag{1.4}
\]
where $F = (F_{ij})$, $F_{ij} : Q_+ \to \mathbb{R}$, is a smooth tensor field and $a : \mathbb{R}^3 \to \mathbb{R}^3$ is a smooth divergence-free vector field. For simplicity let $F$ and $a$ be functions with compact support.

In (1.4) the pressure $p$ is related to the right-hand side $-\text{div } F$ by

$$-\Delta p = \text{div } \text{div } F \quad \text{in } Q_+. \quad \text{(1.5)}$$

This relation does not define $p$ uniquely (for instance, we can add any harmonic function to $p$). Adding the condition

$$\forall t > 0 \quad p(x, t) \to 0 \quad \text{as } |x| \to +\infty$$

to (1.5), we can select in the ‘set of pressures’ a canonical representative $p_F$, which we will call the pressure associated with the right-hand side $F$. For the associated pressure $p_F$ we have

$$p_F(x, t) = -\frac{1}{3} \text{tr } F(x, t) + \lim_{\delta \to 0} \int_{\mathbb{R}^3 \setminus B(x, \delta)} \nabla^2 \mathcal{E}(x - y) : F(y, t) \, dy, \quad \text{(1.6)}$$

where $\mathcal{E}(x) = 1/(4\pi|x|)$. For each $l \in ]1, +\infty[$ the function $p_F$ satisfies the estimate

$$\|p_F(\cdot, t)\|_{L^l(\mathbb{R}^3)} \leq c(l) \|F(\cdot, t)\|_{L^l(\mathbb{R}^3)} \quad \forall t \in ]0, +\infty[.$$ 

Carrying the pressure over to the right-hand side, we can now treat (1.4) as a Cauchy problem for the heat equation:

$$\begin{cases}
\partial_t u - \Delta u = -\text{div } G & \text{in } Q_+, \\
u|_{t=0} = a,
\end{cases}$$

where

$$G(x, t) = (G_{ij}(x, t)) \quad \text{and} \quad G_{ij}(x, t) = F_{ij}(x, t) + p_F(x, t)\delta_{ij}.$$ 

Its solution can be found by the formula

$$u(x, t) = \int_{\mathbb{R}^3} \Gamma(x - y, t) a(y) \, dy - \int_0^t \int_{\mathbb{R}^3} \Gamma(x - y, t - \tau) \text{div } G(y, \tau) \, dy \, d\tau,$$

where $\Gamma(x, t) := \frac{1}{(4\pi t)^{3/2}} e^{-|x|^2/(4t)}$. Since $\text{div } G = 0$, we see from the condition

$$\text{div } a = 0 \quad \text{that } u(\cdot, t) \text{ is divergence free for each } t \in ]0, +\infty[.$$ 

Integrating by parts in the last term and considering that $G(y, t)$ decays sufficiently rapidly as $|y| \to +\infty$, we obtain

$$\int_{\mathbb{R}^3} \Gamma(x - y, t) \frac{\partial G_{ij}}{\partial y_j}(y, t) \, dy = \int_{\mathbb{R}^3} K_{ijk}(x - y, t) F_{jk}(y, t) \, dy,$$

where the tensor $K_{ijk}$ is obtained by differentiating the Oseen tensor $K_{ij}$ (for instance, see [52]):

$$K_{ijk}(x, t) := \frac{\partial K_{ij}}{\partial x_k}(x, t),$$

$$K_{ij}(x, t) := \Gamma(x, t)\delta_{ij} + \frac{\partial^2 \Phi}{\partial x_i \partial x_j}(x, t),$$

$$\Phi(x, t) := \int_{\mathbb{R}^3} \mathcal{E}(x - y) \Gamma(y, t) \, dy.$$
where $\mathcal{E}(x)$ and $\Gamma(x,t)$ were defined above. Thus, the solution $u$ of (1.4) has the representation

$$
  u_i(x,t) = \int_{\mathbb{R}^3} \Gamma(x-y,t)a_i(y)\,dy + \int_0^t \int_{\mathbb{R}^3} K_{ijk}(x-y,t-\tau)F_{jk}(y,\tau)\,dy\,d\tau.
$$

(1.7)

In the next subsection we use (1.7) for the definition of a mild bounded solution of the non-linear Navier–Stokes system. Note that the following estimates proved in [52] hold for the Oseen tensor $K_{ij}$ for any $l, m = 0, 1, 2, \ldots$:

$$
  |\partial_t^l \nabla^m K_{ij}(x,t)| \leq \frac{C_{l,m}}{(|x|^2 + t)^{3/2 + m/2 + l}} ,
$$

yielding for $K_{ijk}$ the estimates

$$
  |\partial_t^l \nabla^m K_{ijk}(x,t)| \leq \frac{C_{l,m}}{(|x|^2 + t)^{2 + m/2 + l}} .
$$

(1.8)

The inequalities (1.8) for the functions $u$ and $p_F$ defined by (1.7) and (1.6) imply the estimates

$$
  \|u\|_{L_{s,1}(Q_T)} \leq C(s, l, s_1, l_1, \sigma)(\|F\|_{L_{s_1,1}(Q_T)} + \|a\|_{L_\sigma(\mathbb{R}^3)})
$$

(1.9)

and

$$
  \|p_F\|_{L_{s_1,1}(Q_T)} \leq C(s_1, l_1)\|F\|_{L_{s_1,1}(Q_T)}
$$

(1.10)

(see [54]), where $Q_T := \mathbb{R}^3 \times [0, T]$ and the exponents

$$
  s \in ]3, +\infty[, \quad s_1 \in \left[\frac{3s}{3+s}, \frac{3s}{3+s} \right], \quad l \in ]2, +\infty[, \quad l_1 \in \left[\frac{2l}{l+2}, \frac{2l}{l+2}\right], \quad \sigma \in ]3, +\infty[,
$$

are connected by the relations

$$
  3\left(\frac{1}{s_1} - \frac{1}{s}\right) + 2\left(\frac{1}{l_1} - \frac{1}{l}\right) = 1, \quad \frac{3}{s} + \frac{2}{l} = \frac{3}{\sigma} .
$$

Since the smooth functions are dense in the corresponding spaces, the inequalities (1.9) and (1.10) also hold for arbitrary divergence-free initial data $a \in L_\sigma(\mathbb{R}^3)$ and right-hand sides $F \in L_{s_1,1}(Q_T)$.

Let us now consider the case when $F \in L_\infty(Q_T)$. The inequality (1.8) implies the estimate

$$
  \|K_{ijk}\|_{L_1(Q_T)} \leq CT^{1/2} .
$$

Thus, each term on the right-hand side of (1.7) is a convolution with kernel in $L_1$, so that we have the following result.

**Theorem 1.1.** Let $a \in L_\infty(\mathbb{R}^3)$, div $a = 0$, and let $F_{ij} \in L_\infty(Q_T)$. Then (1.7) defines a function $u \in L_\infty(Q_T)$ such that

$$
  \|u\|_{L_\infty(Q_T)} \leq C(\|a\|_{L_\infty(\mathbb{R}^3)} + T^{1/2}\|F\|_{L_\infty(Q_T)}) .
$$

(1.11)
For $F \in L_\infty(Q_T)$ the formula (1.6) has no meaning if regarded directly, because the integral on the right-hand side is divergent in general. Nevertheless, for a bounded tensor field $F$ we can also define the associated pressure $p_F$ as a function in $L_\infty(0,T;\text{BMO}(\mathbb{R}^3))$ in a unique way (up to an arbitrary function of $t$).

**Theorem 1.2.** For each tensor field $F \in L_\infty(Q_T)$ there exists a unique function $p_F \in L_\infty(0,T;\text{BMO}(\mathbb{R}^3))$ such that

$$-\Delta p_F = \text{div} \text{div} F \quad \text{in} \mathcal{D}'(Q_T),$$

$$[p_F]_B(t) := \int_B p_F(x,t) \, dx = 0 \quad \text{for almost all } t \in ]0,T[, $$

$$\|p_F\|_{L_\infty(0,T;\text{BMO}(\mathbb{R}^3))} \leq C\|F\|_{L_\infty(Q_T)}.$$

Moreover, if $a \in L_\infty(\mathbb{R}^3)$, div $a = 0$, and the function $u \in L_\infty(Q_T)$ is defined by (1.7), then $u$ and $p_F$ satisfy the Stokes system (1.4) in $Q_T$ in the sense of distributions:

$$\int_{Q_T} u \cdot (\partial_t \eta + \Delta \eta) \, dx \, dt = \int_{Q_T} (F : \nabla \eta + p_F \text{div} \eta) \, dx \, dt \quad \forall \eta \in C_0^\infty(Q_T).$$

(1.12)

In addition, if $\partial_t^k F \in L_\infty(Q_-)$ for $k = 1, \ldots, l$, then $\partial_t^l p_F \in L_\infty(-\infty,0;\text{BMO}(\mathbb{R}^3))$ and the estimate

$$\|\partial_t^l p_F\|_{L_\infty(0,T;\text{BMO}(\mathbb{R}^3))} \leq C_l \sum_{k=0}^l \|\partial_t^k F\|_{L_\infty(Q_T)}$$

(1.13)

holds.

**Proof.** That the function $p_F$ in Theorem 1.2 is unique follows from Liouville’s theorem for harmonic functions, which holds for functions in BMO$(\mathbb{R}^3)$. Now we show that $p_F$ exists. Let $F^m \in C_0^\infty(Q_T)$ be any sequence such that

$$F^m \rightarrow F \quad \text{in} \quad L_\infty(Q_T), \quad \|F^m\|_{L_\infty(Q_T)} \leq \|F\|_{L_\infty(Q_T)}.$$  

(1.14)

Let

$$u_i^m(x,t) = \int_{\mathbb{R}^3} \Gamma(x-y,t) a_i(y) \, dy + \int_0^t \int_{\mathbb{R}^3} K_{ijk}(x-y,\tau) F^m_{jk}(y,\tau) \, dy \, d\tau$$

and

$$p^m(x,t) = -\frac{1}{3} \text{tr} F^m(x,t) + \lim_{\delta \rightarrow 0} \int_{\mathbb{R}^3 \setminus B(x,\delta)} \nabla^2 \mathcal{E}(x-y) : F^m(y,t) \, dy.$$  

(1.15)

Then it is known that $u^m$ and $p^m$ are smooth functions satisfying the relations

$$\begin{cases}
\partial_t u^m - \Delta u^m + \nabla p^m = -\text{div} F^m, \\ \text{div} u^m = 0
\end{cases} \quad \text{in} \ Q_T,$$

(1.16)

and furthermore, in view of (1.14) we have the estimates

$$\|u^m\|_{L_\infty(Q_T)} \leq C\left(\|a\|_{L_\infty(\mathbb{R}^3)} + T^{1/2}\|F\|_{L_\infty(Q_T)}\right)$$
and
\[ \|p^m\|_{L_\infty(0,T;\text{BMO}(\mathbb{R}^3))} \leq C\|F\|_{L_\infty(Q_T)}. \]

Hence we can select subsequences (for which we keep the same notation) such that
\[ u^m \rightharpoonup u \quad \text{in} \quad L_\infty(Q_T), \quad p^m - [p^m]_B \rightharpoonup pF \quad \text{in} \quad L_\infty(0,T;\text{BMO}(\mathbb{R}^3)). \]

It is easy to see that \( pF \) satisfies all the conditions in the theorem. The estimate (1.11) means that (1.7) defines a bounded linear operator from \( L_\infty(Q_T) \) to \( L_\infty(Q_T) \). Consequently, \( u^m \rightharpoonup u \) in \( L_\infty(Q_T) \), and therefore \( \bar{u} = u \) almost everywhere in \( Q_T \). Writing (1.16) in the weak form and taking the limit as \( m \to \infty \) in this integral identity, we obtain (1.12).

To complete the proof it remains to establish (1.13). Note that if \( \partial^k_t F \in L_\infty(Q_-) \) for \( k = 1, \ldots, l \), then we can construct the approximating sequence \( F^m \in C^\infty(Q_-) \) so that, apart from (1.14), we also have
\[ \partial^k_t F^m \rightharpoonup \partial^k_t F \quad \text{in} \quad L_\infty(Q_T) \quad \text{and} \quad \|\partial^k_t F^m\|_{L_\infty(Q_T)} \leq C \sum_{s=0}^{k} \|\partial^s_t F\|_{L_\infty(Q_T)} \quad (1.17) \]

for all \( k = 1, \ldots, l \). Then from (1.15) we obtain
\[ \partial^k_t p^m(x,t) = -\frac{1}{3} \text{tr} \partial^k_t F^m(x,t) + \lim_{\delta \to 0} \int_{\mathbb{R}^3 \setminus B(x,\delta)} \nabla^2 \varepsilon(x-y) : \partial^k_t F^m(y,t) \, dy, \]

and since the singular integral is bounded from \( L_\infty(\mathbb{R}^3) \) to \( \text{BMO}(\mathbb{R}^3) \), it follows that
\[ \|\partial^k_t p^m\|_{L_\infty(0,T;\text{BMO}(\mathbb{R}^3))} \leq C\|\partial^k_t F^m\|_{L_\infty(Q_T)}. \]

Letting \( m \) tend to infinity and taking (1.17) into account, we obtain (1.13). 

**Definition 1.2.** Let \( F \in L_\infty(Q_T) \) and \( a \in L_\infty(\mathbb{R}^3) \), with \( \text{div} \, a = 0 \). Then the function \( u \in L_\infty(Q_T) \) defined by (1.7) is called the mild bounded solution of the problem (1.4), and the function \( pF \in L_\infty(0,T;\text{BMO}(\mathbb{R}^3)) \) defined in Theorem 1.2 is called the pressure associated with this solution.

### 1.3. Mild bounded ancient solutions.

**Definition 1.3.** A function \( u \in L_\infty(Q_-) \) is called a *mild* bounded ancient solution of the Navier–Stokes equations if for all \( t_0 < t < 0 \) it satisfies the identity
\[ u_i(x,t) = \int_{\mathbb{R}^3} \Gamma(x,y,t-t_0)u_i(y,t_0) \, dy \]
\[ + \int_{t_0}^{t} \int_{\mathbb{R}^3} K_{ijk}(x-y,t-\tau)u_j(y,\tau)u_k(y,\tau) \, dy \, d\tau. \quad (1.18) \]

It follows from Theorem 1.2 that each mild bounded ancient solution is a bounded ancient solution in the sense of Definition 1.1. We can also pose the opposite question: when is a bounded ancient solution \( u \) in Definition 1.1 a mild solution in the sense of Definition 1.3? An answer is given by the following equivalent characterization of mild bounded ancient solutions.
Theorem 1.3. A function \( u \in L^\infty(Q_-) \) is a mild bounded ancient solution of the Navier–Stokes system if and only if it is a bounded ancient solution of the Navier–Stokes system in the sense of Definition 1.1 and there exists a function \( p \in L^\infty(-\infty,0;\text{BMO}(\mathbb{R}^3)) \) such that the pair \( u, p \) satisfies the Navier–Stokes equations in \( Q_- \) in the sense of distributions. In that case \( u \) is \( C^\infty \)-smooth in \( \mathbb{R}^3 \times ]-\infty,0] \) and the following estimate holds for all \( l, m = 0, 1, 2, \ldots \):

\[
\| \partial^l_t \nabla^m u \|_{L^\infty(Q_-)} + \| \partial^l_t \nabla^{m+1} p \|_{L^\infty(Q_-)} + \| \partial^l_t p_{u \otimes u} \|_{L^\infty(-\infty,0;\text{BMO}(\mathbb{R}^3))} < +\infty, \quad (1.19)
\]

where \( p_{u \otimes u} \) is the function defined in Theorem 1.2 for the tensor field \( F = u \otimes u \) and connected with \( p \) for almost all \( (x,t) \in Q_- \) by the identity

\[
p_{u \otimes u}(x,t) = p(x,t) - [p(\cdot, t)]_B, \quad \text{where} \quad [p(\cdot, t)]_B := \frac{1}{|B|} \int_B p(x,t) \, dx.
\]

First we explain the general idea of the proof of Theorem 1.3. We can interpret the Navier–Stokes system (1.1) as the heat equation

\[
\partial_t u - \Delta u = f \quad \text{in} \ D'(Q_-) \quad (1.20)
\]

with the right-hand side

\[
f := - \text{div} \, F - \nabla p, \quad F := u \otimes u.
\]

On the other hand, solutions of (1.20) have the representation

\[
u(x,t) = \int_{\mathbb{R}^3} \Gamma(x-y,t-t_0)u(y,t_0) \, dy + \int_{t_0}^t \int_{\mathbb{R}^3} \Gamma(x-y,t-\tau)f(y,\tau) \, dy \, d\tau \quad \forall t_0 < t < 0. \quad (1.21)
\]

Although we can assign a meaning to this representation even when \( f \) is a tempered distribution, we will use only the following result, where the integrals on the right-hand side of (1.21) are treated in the usual sense.

Lemma 1.1. The functions \( u \in L^\infty(Q_-) \) and \( f \in L^\infty(Q_-) \) satisfy (1.20) if and only if the identity (1.21) holds.

We prove Theorem 1.3 in several steps. First, we verify a special case of (1.19):

\[
\nabla u, \nabla^2 u, \partial_t u \in L^\infty(Q_-), \quad (1.22)
\]

\[
\nabla p \in L^\infty(Q_-). \quad (1.23)
\]

The inclusions (1.22) and (1.23) ensure, in particular, that \( f \in L^\infty(Q_-) \) as required in Lemma 1.1. Second, we prove the equality

\[
\int_{t_0}^t \int_{\mathbb{R}^3} \Gamma(x-y,t-\tau)f_i(y,\tau) \, dy \, d\tau = \int_{t_0}^t \int_{\mathbb{R}^3} K_{ijk}(x-y,t-\tau)F_{jk}(y,\tau) \, dy \, d\tau, \quad (1.24)
\]

which can be reduced to the following lemma by integration by parts, provided that the conditions

\[
F \in L^\infty(Q_-), \quad \nabla F \in L^\infty(Q_-),
\]

which are consequences of (1.22), are fulfilled.
Lemma 1.2. Assume that
\[ \nabla^l F \in L_\infty(Q_-), \quad l = 0, 1, 2, \quad \text{and} \quad p \in L_\infty(-\infty, 0; \text{BMO}(\mathbb{R}^3)), \]
and let \( F \) and \( p \) be connected by
\[ -\Delta p = \text{div} \text{ div} F \quad \text{in} \ D'(Q_-). \]
Then \( \nabla p \in L_\infty(Q_-), \) and for all \( t_0 < t < 0 \)
\[ \int_{t_0}^t \int_{\mathbb{R}^3} \nabla_\Gamma(x - y, t - \tau) \frac{\partial p}{\partial y_i}(y, \tau) \, dy \, d\tau \]
\[ = \int_{t_0}^t \int_{\mathbb{R}^3} \frac{\partial^3 \Phi}{\partial x_i \partial x_j \partial x_k}(x - y, t - \tau) F_{jk}(y, \tau) \, dy \, d\tau. \quad (1.25) \]

Proof. Let \( t_0 < t < 0 \) be arbitrary times. We set \( Q_{t_0, t} := \mathbb{R}^3 \times ]t_0, t[ \) and take functions \( F^m \in C_0^\infty(Q_{t_0, t}) \) such that
\[ \nabla^k F^m \rightharpoonup \nabla^k F \quad \text{in} \ L_\infty(Q_{t_0, t}) \quad \text{and} \quad \| F^m \|_{W^k_\infty(Q_{t_0, t})} \leq C \| F \|_{W^k_\infty(Q_{t_0, t})} \quad (1.26) \]
for \( k = 0, 1, 2, \) where we have set
\[ \| F \|_{W^k_\infty(Q_{t_0, t})} := \sum_{s=0}^k \| \nabla^s F \|_{L_\infty(Q_{t_0, t})}. \]

For \( m \in \mathbb{N} \) we construct the functions
\[ p^m(x, t) = -\frac{1}{3} \text{tr} F^m(x, t) + \lim_{\delta \to 0} \int_{\mathbb{R}^3 \setminus B(x, \delta)} \nabla^2 \mathcal{E}(x - y) : F^m(y, t) \, dy. \quad (1.27) \]
Then
\[ -\Delta p^m = \text{div} \text{ div} F^m \quad \text{in} \ Q_{t_0, t}. \quad (1.28) \]
Since the right-hand side of (1.27) defines a bounded linear operator from \( L_\infty(\mathbb{R}^3) \)
to \( \text{BMO}(\mathbb{R}^3) \), taking (1.26) into account we obtain the estimate
\[ \| p^m \|_{L_\infty(t_0, t; \text{BMO}(\mathbb{R}^3))} \leq C \| F \|_{L_\infty(Q_{t_0, t})}. \quad (1.29) \]
Furthermore, for \( x_0 \in \mathbb{R}^3 \) and \( t_0 < t < 0 \) it follows from (1.28) that
\[ \| \nabla p^m(\cdot, t) \|_{L_\infty(B(x_0, 1))} \leq C(\| \nabla^2 F^m(\cdot, t) \|_{L_\infty(B(x_0, 2))} + \| p^m(\cdot, t) \|_{\text{BMO}(B(x_0, 2))}), \]
which in view of (1.26) and (1.29) yields the estimate
\[ \| \nabla p^m \|_{L_\infty(Q_{t_0, t})} \leq C \| F \|_{W^2_\infty(Q_{t_0, t})}. \]
Hence we can select a subsequence such that
\[ \nabla p^m \rightharpoonup \nabla p \quad \text{in} \ L_\infty(Q_{t_0, t}). \]
Let \(\mathcal{P}(\cdot, t) - p(\cdot, t) \in \text{BMO}(\mathbb{R}^3)\)

for almost all \(t \in (t_0, 0]\), it follows from Liouville’s theorem (which holds for harmonic functions in \(\text{BMO}(\mathbb{R}^3)\)) that \(\mathcal{P}(x, t) - p(x, t) = b(t)\) for some \(b \in L_\infty(t_0, t)\). In particular, \(\nabla \mathcal{P} = \nabla p\) in \(Q_{t_0, t}\). By (1.27)

\[
\frac{\partial p^m}{\partial x_i}(x, t) = \int_{\mathbb{R}^3} \mathcal{E}(x - y) \frac{\partial^3 F^m_{jk}}{\partial y_i \partial y_j \partial y_k}(y, t) \, dy,
\]

and therefore for all \(x \in \mathbb{R}^3\) and \(t_0 < t < 0\)

\[
\int_{t_0}^t \int_{\mathbb{R}^3} \Gamma(x - y, t - \tau) \frac{\partial p^m}{\partial y_i}(y, \tau) \, dy \, d\tau = \int_{t_0}^t \int_{\mathbb{R}^3} \Phi(x - y, t - \tau) \frac{\partial^3 F^m_{jk}}{\partial y_i \partial y_j \partial y_k}(y, \tau) \, dy \, d\tau.
\]

Integrating three times by parts on the right-hand side and passing to the limit as \(m \to \infty\), we obtain (1.25) in view of the convergences

\[
\nabla p^m \xrightarrow{\ast} \nabla p \quad \text{in} \quad L_\infty(Q_{t_0, t}) \quad \text{and} \quad F^m \xrightarrow{\ast} F \quad \text{in} \quad L_\infty(Q_{t_0, t}).
\]

Now we verify (1.22). Assuming first that \(u\) is a mild bounded ancient solution of the Navier–Stokes equations in the sense of Definition 1.3, we will prove that the \(\nabla u, \nabla^2 u, \text{ and } \partial_t u \) exist and are bounded in \(Q_-\). To do this we start by establishing several properties of mild solutions of the Cauchy problem for the linear Stokes system. Let \(Q_T := \mathbb{R}^3 \times [0, T[, T > 0\), and consider the Banach space

\[
X_T := \{ v \in L_\infty(Q_T) : t^{1/2} \nabla v, t \nabla^2 v, t \partial_t v \in L_\infty(Q_-), \text{ div } v = 0\}
\]

with the norm

\[
\|v\|_{X_T} := \|v\|_{L_\infty(Q_T)} + \|t^{1/2} \nabla v\|_{L_\infty(Q_T)} + \|t \nabla^2 v\|_{L_\infty(Q_T)} + \|t \partial_t v\|_{L_\infty(Q_T)}.
\]

Let \(a \in L_\infty(\mathbb{R}^3)\) be such that \(\text{div } a = 0\), and define the operator \(A_a(v) := u\), where

\[
u_i(x, t) := \int_{\mathbb{R}^3} \Gamma(x - y, t) a_i(y) \, dy + \int_0^t \int_{\mathbb{R}^3} K_{ijk}(x - y, t - \tau) v_j(y, \tau) v_k(y, \tau) \, dy \, d\tau
\]

(1.30)

The following result holds.

**Proposition 1.1.** For any \(0 < T < +\infty\) and \(a \in L_\infty(\mathbb{R}^3)\) with \(\text{div } a = 0\), the operator \(A_a\) takes \(X_T\) to \(X_T\). Moreover,

\[
\|A_a(v)\|_{X_T} \leq C(\|a\|_{L_\infty(\mathbb{R}^3)} + T^{1/2} \|v\|_{X_T}^2) \quad \forall v \in X_T.
\]

**Proof.** Since it is well known that solutions of the Cauchy problem for the heat equation with initial data in \(L_\infty(\mathbb{R}^3)\) belong to \(X_T\), we verify the statement of the theorem only in the case when \(a = 0\). We follow arguments in [11] with slight modifications. By Theorem 1.1,

\[
\|u\|_{L_\infty(Q_T)} \leq C T^{1/2} \|v\|_{L_\infty(Q_T)}^2
\]
for $u := A_0(v)$. Since the kernel $K_{ijk}(x, t)$ is smooth for $t > 0$, we can differentiate the convolution (1.30) with respect to $x_s$. This gives us that

$$t^{1/2} \frac{\partial u_i}{\partial x_s} = I_1 + I_2,$$

where

$$I_1 := t^{1/2} \int_0^{t/2} \int_{\mathbb{R}^3} \frac{\partial K_{ijk}}{\partial x_s}(x - y, t - \tau)v_j(y, \tau)v_k(y, \tau) \, dy \, d\tau,$$

$$I_2 := t^{1/2} \int_{t/2}^t \int_{\mathbb{R}^3} K_{ijk}(x - y, t - \tau) \frac{\partial}{\partial y_s}(v_j(y, \tau)v_k(y, \tau)) \, dy \, d\tau.$$

From the estimate (1.8) for the kernel and from the relation

$$\int_{\mathbb{R}^3} \frac{dy}{(|x - y|^2 + (t - \tau))^2 + m/2} = \frac{C_m}{(t - \tau)^{1/2 + m/2}}, \quad m = 0, 1, 2, \ldots, \quad (1.32)$$

we obtain for $I_1$ the inequality

$$|I_1| \leq C t^{1/2} \int_0^{t/2} \frac{d\tau}{t - \tau} \|v\|^2_{L_\infty(Q_-)} = C t^{1/2} \|v\|^2_{L_\infty(Q_-)}.$$

To estimate $I_2$ we use the fact that $v \in X_T$:

$$|I_2| \leq C t^{1/2} \int_{t/2}^t \int_{\mathbb{R}^3} \tau^{-1/2}|K_{ijk}(x - y, t - \tau)| \times \|\tau^{1/2}\nabla v(\cdot, \tau)\|_{L_\infty(\mathbb{R}^3)} \|v(\cdot, \tau)\|_{L_\infty(\mathbb{R}^3)} \, dy \, d\tau.$$

Since $(t/\tau)^{1/2} \leq \sqrt{2}$ for $\tau \in [t/2, t]$, using the inequality (1.8) again and the equality (1.32) with $m = 0$, we get that

$$\|t^{1/2}\nabla u\|_{L_\infty(Q_T)} \leq C T^{1/2} \|v\|^2_{X_T}.$$

The estimate

$$\|t\nabla^2 u\|_{L_\infty(Q_T)} \leq C T^{1/2} \|v\|^2_{X_T}$$

is obtained similarly. Now we turn to estimates for the time derivative. Since $v$ is smooth for $t > 0$, we get by differentiating the convolution (1.30) in the standard way that for $t > 0$

$$t \partial_t u = J_1 + J_2 + J_3,$$

where

$$J_1 := t \int_0^{t/2} \int_{\mathbb{R}^3} \frac{\partial K_{ijk}}{\partial t}(x - y, t - \tau)v_j(y, \tau)v_k(y, \tau) \, dy \, d\tau,$$

$$J_2 := t \int_{t/2}^t \int_{\mathbb{R}^3} K_{ijk}(x - y, t - \tau) \frac{\partial}{\partial \tau}(v_j(y, \tau)v_k(y, \tau)) \, dy \, d\tau,$$

$$J_3 := t \int_{\mathbb{R}^3} K_{ijk} \left(x - y, \frac{t}{2}\right) v_j \left(y, \frac{t}{2}\right) v_k \left(y, \frac{t}{2}\right) \, dy.$$
The integrals $J_1$ and $J_2$ are estimated like $I_1$ and $I_2$, respectively, while for $J_3$ we get from (1.8) that

$$|J_3| \leq C\|v\|_{L^2(Q_T)}^2 t \int_{\mathbb{R}^3} \frac{dy}{|x - y|^2 + t/2}^2 \leq Ct^{1/2}\|v\|_{X_T}^2.$$ 

Hence $\|t\partial_t u\|_{L^\infty(Q_T)} \leq CT^{1/2}\|v\|_{X_T}^2$. □

Note that the arguments in the proof of Proposition 1.1 carry over without changes to the case of the operator $A_a : X_T \times X_T \to X_T$ in two variables that is defined by $A_a(v, w) := u$, where

$$u_i(x, t) := \int_{\mathbb{R}^3} \Gamma(x - y, t)a_i(y) dy + \int_0^t \int_{\mathbb{R}^3} K_{ijk}(x - y, -\tau)v_j(y, \tau)w_k(y, \tau) dy d\tau.$$

The operator $A_a(v, w)$ has properties similar to those of $A_a(v)$. In particular, we have the estimate

$$\|A_a(v, w)\|_{X_T} \leq C_0(\|a\|_{L^\infty(\mathbb{R}^3)} + T^{1/2}\|v\|_{X_T}\|w\|_{X_T}) \quad \forall v, w \in X_T$$

(1.34)

with the same constant $C_0 > 0$ as in (1.31).

The next statement was in fact proved in [31] (see also [21]).

**Proposition 1.2.** There exists an absolute constant $\delta > 0$ such that for any $a \in L^\infty(\mathbb{R}^3)$ with $\text{div} a = 0$ and for $T = \delta\|a\|_{L^\infty(\mathbb{R}^3)}^{-2}$ the sequence

$$u^0 = A_a(0), \quad u^{k+1} = A_a(u^k), \quad k \in \mathbb{N},$$

converges in $X_T$ to a fixed point $u \in X_T$ of the operator $A_a$, that is, $u = A_a(u)$. Furthermore, for any $k \in \mathbb{N}$

$$\|u^k\|_{X_T} \leq 2C_0\|a\|_{L^\infty(\mathbb{R}^3)},$$

where $C_0$ is the constant in (1.31). Consequently,

$$\|u\|_{X_T} \leq 2C_0\|a\|_{L^\infty(\mathbb{R}^3)}.$$ 

(1.35)

**Proof.** Let

$$\delta := \min\left\{1, \frac{1}{16C_0^2}\right\}.$$

We start with the estimate $\|u^k\|_{X_T} \leq 2C_0\|a\|_{L^\infty(\mathbb{R}^3)}$ for any $k = 0, 1, 2, \ldots$. The proof goes by induction. For $k = 0$ this estimate is obvious. The induction step $k \to k + 1$ follows from (1.31).

We now show that the $u^k$ form a Cauchy sequence in $X_T$. For any $k$ and $m$ we have

$$u^k - u^m = A_0(u^k + u^m, u^k - u^m),$$

where $A_0(v, w)$ is the operator defined by (1.33) for $a = 0$. Using (1.34), we obtain

$$\|u^{m+1} - u^{k+1}\|_{X_T} \leq C_0T^{1/2}(\|u^k\|_{X_T} + \|u^m\|_{X_T})\|u^k - u^m\|_{X_T}.$$
In view of the inequality \( \|u^k\|_{X_T} \leq 2C_0\|a\|_{L^\infty(\mathbb{R}^3)} \) we have
\[
\|u^{m+1} - u^{k+1}\|_{X_T} \leq 2C_0^2T^{1/2}\|a\|_{L^\infty(\mathbb{R}^3)}\|u^k - u^m\|_{X_T}.
\]
Setting \( T = \delta\|a\|_{L^\infty(\mathbb{R}^3)}^{-2} \) and bearing in mind our choice of \( \delta \), we obtain
\[
\|u^{m+1} - u^{k+1}\|_{X_T} \leq \frac{1}{2}\|u_m - u_k\|_{X_T}.
\]
Hence \( u^k \to u \) in \( X_T \) and \( u = A_a(u) \).

We can now prove (1.22) for mild bounded ancient solutions of the Navier–Stokes equations.

**Lemma 1.3.** Let \( u \) be a mild bounded ancient solution of the Navier–Stokes equations in \( Q \). Then \( u \) has weak derivatives \( \nabla u, \nabla^2 u, \) and \( \partial_t u \), and they satisfy (1.22).

**Proof.** We follow [23]. Let \( M := \|u\|_{L^\infty(Q)} \) and \( T := \delta M^{-2} \), where \( \delta > 0 \) was defined in Proposition 1.2. Then for each \( t_0 < 0 \) such that \( u(\cdot, t_0) \in L^\infty(\mathbb{R}^3) \) and \( t_0 + T \leq 0 \) the function \( u \) is a mild bounded ancient solution on the interval \([t_0, t_0 + T]\) and corresponds to the initial data \( u(\cdot, t_0) \). Therefore, \( u \) is a smooth function on \( \mathbb{R}^3 \times [t_0, t_0 + T] \), and by (1.35) we have
\[
\sup_{t \in [t_0, t_0 + T]} \sup_{x \in \mathbb{R}^3} \left( |u(x, t)| + (t - t_0)^{1/2}|\nabla u(x, t)| + (t - t_0)|\nabla^2 u(x, t)| + (t - t_0)|\partial_t u(x, t)| \right) \leq 2C_0\|u(\cdot, t_0)\|_{L^\infty(\mathbb{R}^3)} \leq 2C_0M.
\]
Consequently,
\[
\sup_{t_0 + T/2 < t < t_0 + T} \sup_{x \in \mathbb{R}^3} \left( |u(x, t)| + |\nabla u(x, t)| + |\nabla^2 u(x, t)| + |\partial_t u(x, t)| \right) \leq C(T)M,
\]
which yields (1.22). \( \square \)

Next we need the following local smoothness result for solutions of the linear Stokes system (for instance, see [41] or [45]).

**Lemma 1.4.** Let \( s \in ]1, +\infty[ \), and let \( f \in L_s(Q), u \in W^{1,0}_s(Q), \) and \( p \in L_s(Q) \) be functions satisfying the Stokes system in \( Q \) in the sense of distributions:
\[
\begin{cases}
\partial_t u - \Delta u + \nabla p = f, \\
\text{div } u = 0
\end{cases}
in Q.
\]
Then \( u \in W^{2,1}_s(Q(1/2)) \) and \( p \in W^{1,0}_s(Q(1/2)) \), and the following estimate holds:
\[
\|u\|_{W^{2,1}_s(Q(1/2))} + \|\nabla p\|_{L_s(Q(1/2))} \leq C(s)(\|f\|_{L_s(Q)} + \|u\|_{W^{1,0}_s(Q)} + \|p - [p]B\|_{L_s(Q)}).
\]
Here \([p]_B(t) := \frac{1}{|B|} \int_B p(x, t) \, dx\).
We now prove (1.22) and (1.23) in the case when the functions \( u \in L_\infty(Q_-) \) and \( p \in L_\infty(-\infty,0;\text{BMO}(\mathbb{R}^3)) \) satisfy the Navier–Stokes equations in \( Q_- \) in the sense of distributions. In fact, we establish a stronger result.

**Lemma 1.5.** Assume that \( u \in L_\infty(Q_-) \) and \( p \in L_\infty(-\infty,0;\text{BMO}(\mathbb{R}^3)) \) satisfy the Navier–Stokes equations in \( Q_- \) in the sense of distributions. Then for any \( m \in \mathbb{N} \) the derivatives \( \nabla^m u, \nabla^m p, \text{ and } \partial_t \nabla^m u \) exist and are bounded in \( Q_- \). Furthermore,

\[
\|\partial_t \nabla^m u\|_{L_\infty(Q_-)} + \|\nabla^m u\|_{L_\infty(Q_-)} + \|\nabla^m p\|_{L_\infty(Q_-)} \leq C_m,
\]

where the constant \( C_m \) depends on \( \|u\|_{L_\infty(Q_-)} \) and \( \|p\|_{L_\infty(-\infty,0;\text{BMO}(\mathbb{R}^3))} \).

**Proof.** Fix some \( s > 3 \). Since the Navier–Stokes equations can be interpreted as the heat equation \( \partial_t u - \Delta u = -\text{div } F \) in \( \mathcal{D}'(Q_-) \) with the right-hand side \( F = u \otimes u + p \), \( F \in L_{s,\text{loc}}(Q_-) \), we conclude that \( u \in W_{s,\text{loc}}^1(Q_-) \), and for all \( z_0 = (x_0, t_0) \in Q_- \) we have the estimate

\[
\|\nabla u\|_{L_s(Q(z_0,1))} \leq C(\|u \otimes u\|_{L_s(Q(z_0,2))} + \|p - [p]_{B(x_0,2)}\|_{L_s(Q(z_0,2))} + \|u\|_{L_s(Q(z_0,2))}).
\]

By the relations \( u \in L_\infty(Q_-) \) and \( p \in L_\infty(-\infty,0;\text{BMO}(\mathbb{R}^3)) \), this implies that

\[
\sup_{z_0 \in Q_-} \|u\|_{W^{1,0}_s(Q(z_0,1))} \leq C_0,
\]

where the constant \( C_0 \) depends on \( \|u\|_{L_\infty(Q_-)} \) and \( \|p\|_{L_\infty(-\infty,0;\text{BMO}(\mathbb{R}^3))} \). Using Lemma 1.4, we obtain

\[
\|u\|_{W^{2,1}_s(Q(z_0,1/2))} + \|\nabla p\|_{L_s(Q(z_0,1/2))} \\
\leq C(s)(\|\nabla (u \otimes u)\|_{L_s(Q(z_0,1))} + \|u\|_{W^{1,0}_s(Q(z_0,1))} + \|p - [p]_{B(x_0,1)}\|_{L_s(Q(z_0,1))}).
\]

Since \( W^{2,1}_s(Q) \hookrightarrow W^{1,0}_\infty(Q) \), \( s > 3 \), we obtain the estimate

\[
\|\nabla u\|_{L_\infty(Q(z_0,1/2))} \leq C(\|\nabla (u \otimes u)\|_{L_s(Q(z_0,1))} + \|u\|_{W^{1,0}_s(Q(z_0,1))} \\
+ \|p - [p]_{B(x_0,1)}\|_{L_s(Q(z_0,1))})
\]

for any \( z_0 \in Q_- \). Taking (1.38) into account, we arrive at the inequality

\[
\|\nabla u\|_{L_\infty(Q_-)} \leq C_1,
\]

with a constant \( C_1 \) depending on \( \|u\|_{L_\infty(Q_-)} \) and \( \|p\|_{L_\infty(-\infty,0;\text{BMO}(\mathbb{R}^3))} \). On the other hand, (1.39) (where we change the notation for the radius of the ball) gives us that

\[
\sup_{z_0 \in Q_-} \left( \|u\|_{W^{2,1}_s(Q(z_0,1))} + \|\nabla p\|_{L_s(Q(z_0,1))} \right) \leq C'_1,
\]

with a constant \( C'_1 \) also depending on \( \|u\|_{L_\infty(Q_-)} \) and \( \|p\|_{L_\infty(-\infty,0;\text{BMO}(\mathbb{R}^3))} \). Differentiating the Navier–Stokes equations with respect to \( x_k \), repeating the above arguments for \( \tilde{u} := u_{,k} \) and \( \tilde{p} := p_{,k} \), and using (1.41), we now obtain

\[
\|\nabla^2 u\|_{L_\infty(Q_-)} \leq C_2,
\]
with a constant $C_2$ depending on $\|u\|_{L_{\infty}(Q_-)}$ and $\|p\|_{L_{\infty}(-\infty,0;BMO(\mathbb{R}^3))}$. Similarly, if we differentiate the equations again, then for any $m \in \mathbb{N}$ we get the estimate

$$\|\nabla^m u\|_{L_{\infty}(Q_-)} \leq C_m, \tag{1.43}$$

with a constant $C_m$ depending on $\|u\|_{L_{\infty}(Q_-)}$ and $\|p\|_{L_{\infty}(-\infty,0;BMO(\mathbb{R}^3))}$.

Let us now prove that the pressure has bounded derivatives. From the relation

$$- \Delta p = \text{div} \text{div}(u \otimes u) \quad \text{in} \ D'(Q_-) \tag{1.44}$$

we get that for all $x_0 \in \mathbb{R}^3$ and $t < 0$

$$\|p(\cdot,t)\|_{W^2_s(B(x_0,1/2))} \leq C(\|u(\cdot,t) \otimes u(\cdot,t)\|_{W^2_s(B(x_0,1))} + \|p(\cdot,t) - [p]_{B(x_0,1)}(t)\|_{L_s(B(x_0,1))}),$$

which in view of (1.40), (1.42), and the embedding $W^2_s(B) \hookrightarrow W^1_\infty(B)$ gives the estimate

$$\|\nabla p\|_{L_{\infty}(Q_-)} \leq \tilde{C}_1,$$

with a constant $\tilde{C}_1$ depending on $\|u\|_{L_{\infty}(Q_-)}$ and $\|p\|_{L_{\infty}(-\infty,0;BMO(\mathbb{R}^3))}$. Differentiating (1.44) $m$ times with respect to the spatial variables and repeating the above arguments for the derivative, we obtain from (1.43) the estimate

$$\|\nabla^m p\|_{L_{\infty}(Q_-)} \leq \tilde{C}_m. \tag{1.45}$$

Finally, expressing the derivatives $\partial_t \nabla^{m-1} u$ in terms of the spatial derivatives of $u$ and $p$, using the Navier–Stokes equations, and taking (1.43) and (1.45) into account, we complete the proof of (1.37) and thus the lemma. □

Now we can give a proof of the equivalent characterization of mild bounded solutions that was given in Theorem 1.3.

**Lemma 1.6.** A function $u \in L_{\infty}(Q_-)$ is a mild bounded ancient solution of the Navier–Stokes system in the sense of Definition 1.3 if and only if it is a bounded ancient solution of the Navier–Stokes system in the sense of Definition 1.1 and, in addition, there exists a function $p \in L_{\infty}(-\infty,0;BMO(\mathbb{R}^3))$ such that the pair $u, p$ satisfies the Navier–Stokes equations in $Q_-$ in the sense of distributions.

**Proof.** First we assume that $u$ is a mild bounded ancient solution of the Navier–Stokes equations in the sense of Definition 1.3. Then from Lemma 1.3 we conclude that the membership relations (1.22) hold. By Theorem 1.2 the tensor $F := u \otimes u$, $F \in L_{\infty}(Q_-)$, corresponds to a unique function $p_{u \otimes u} \in L_{\infty}(-\infty,0;BMO(\mathbb{R}^3))$ such that for almost all $t < 0$

$$- \Delta p_{u \otimes u}(\cdot,t) = \text{div} \text{div}(u(\cdot,t) \otimes u(\cdot,t)) \quad \text{in} \ D'(\mathbb{R}^3),
\int_B p_{u \otimes u}(x,t) \, dx = 0, \tag{1.46}$$

$$\|p_{u \otimes u}\|_{L_{\infty}(-\infty,0;BMO(\mathbb{R}^3))} \leq C\|u\|^2_{L_{\infty}(Q_-)}.$$

For all $x \in \mathbb{R}^3$ and $t < 0$ the following inequality holds for $p_{u \otimes u}$:

$$\|\nabla p_{u \otimes u}(\cdot,t)\|_{L_{\infty}(B(x,1))} \leq C\left(\|\nabla^2(u \otimes u)(\cdot,t)\|_{L_{\infty}(B(x,2))} + \|p_{u \otimes u}(\cdot,t)\|_{BMO(B(x,2))}\right);$$
because of (1.22) this implies (1.23). Consequently, for
\[ f := -\text{div}(u \otimes u) - \nabla p_{u \otimes u} \]
we have \( f \in L_\infty(Q_-) \). It follows from (1.24) that (1.21) holds for \( u \). By Lemma 1.1 we get that \( \partial_t u - \Delta u = f \) in \( D'(Q_-) \). In view of the definition of \( f \) this means that the Navier–Stokes equations hold in \( Q_- \) in the sense of distributions.

Now we prove the reverse implication in Theorem 1.3. Let \( u \in L_\infty(Q_-) \) and \( p \in L_\infty(-\infty,0;\text{BMO}(\mathbb{R}^3)) \) be functions satisfying the Navier–Stokes equations in \( Q_- \) in the sense of distributions. By Lemma 1.5 the membership relations (1.22) and (1.23) hold, so if \( f := -\text{div}(u \otimes u) - \nabla p \), then \( f \in L_\infty(Q_-) \). Since the Navier–Stokes equations hold in \( Q_- \) in the sense of distributions, \( \partial_t u - \Delta u = f \) in \( D'(Q_-) \). By Lemma 1.1 this means that the identity (1.21) holds. However, then from (1.24) we get that \( u \) is a mild bounded ancient solution in the sense of Definition 1.3. □

In the proof of Lemma 1.6 we showed that each mild bounded ancient solution \( u \in L_\infty(Q_-) \) of the Navier–Stokes equations corresponds to a unique function \( p_{u \otimes u} \in L_\infty(-\infty,0;\text{BMO}(\mathbb{R}^3)) \) defined for the tensor \( F := u \otimes u \) in Theorem 1.2 and satisfying (1.46).

**Definition 1.4.** We call the function \( p_{u \otimes u} \) the pressure associated with the mild bounded ancient solution \( u \).

To complete the proof of Theorem 1.3 it remains to show that mild bounded ancient solutions are infinitely smooth.

**Lemma 1.7.** Under the assumptions of Theorem 1.3 the functions \( u \) and \( p_{u \otimes u} \) are \( C_\infty \)-smooth on \( \mathbb{R}^3 \times ]-\infty,0] \), and the estimate (1.19) holds for all \( l, m = 0, 1, 2, \ldots \).

**Proof.** For brevity let \( p := p_{u \otimes u} \). Theorem 1.2 implies the estimate
\[ \|p\|_{L_\infty(-\infty,0;\text{BMO}(\mathbb{R}^3))} \leq C\|u\|_{L_\infty(Q_-)}^2. \]

In Lemma 1.5 we proved that the derivatives \( \partial_t \nabla^{m-1} u, \nabla^m u, \) and \( \nabla^m p \) exist for any \( m \in \mathbb{N} \) and are bounded in \( Q_- \). Now we show that all the derivatives \( \partial_t^{l} \nabla^m u, l \in \mathbb{N} \), are also bounded. Note that for \( m = 1 \) (1.37) implies that
\[ \|\partial_t u\|_{L_\infty(Q_-)} \leq C_0, \tag{1.47} \]
where the constant \( C_0 \) depends on \( \|u\|_{L_\infty(Q_-)} \).

Since \( p \) satisfies all the conditions in Theorem 1.2 for \( F = u \otimes u \), use of the inequality (1.13) with \( l = 1 \) gives us that
\[ \|\partial_t p\|_{L_\infty(-\infty,0;\text{BMO}(\mathbb{R}^3))} \leq C\left(\|u \otimes u\|_{L_\infty(Q_-)} + \|\partial_t (u \otimes u)\|_{L_\infty(Q_-)}\right). \]

Taking (1.47) into account, we have
\[ \|\partial_t p\|_{L_\infty(-\infty,0;\text{BMO}(\mathbb{R}^3))} \leq C_1 \tag{1.48} \]
for some constant \( C_1 \) depending on \( \|u\|_{L_\infty(Q_-)} \). Differentiating the Navier–Stokes equations with respect to \( t \) and using Lemma 1.5 for the functions \( \partial_t u \) and \( \partial_t p \), we get that for any \( m \in \mathbb{N} \)
\[ \|\partial_t^2 \nabla^{m-1} u\|_{L_\infty(Q_-)} + \|\partial_t \nabla^m u\|_{L_\infty(Q_-)} + \|\partial_t \nabla^m p\|_{L_\infty(Q_-)} \leq C_m, \tag{1.49} \]
where the constant $C_m$ depends on $\|\partial_t u\|_{L^\infty(Q_-)}$ and $\|\partial_t p\|_{L^\infty(-\infty,0;\text{BMO}(\mathbb{R}^3))}$. Taking (1.47) and (1.48) into account, we get an estimate with a constant $\tilde{C}_m$ depending only on $\|u\|_{L^\infty(Q_-)}$.

Using (1.13) with $l = 2$, we obtain

$$
\|\partial_2^2 t p\|_{L^\infty(-\infty,0;\text{BMO}(\mathbb{R}^3))} \leq C_m \left( \|u \otimes u\|_{L^\infty(Q_-)} + \|\partial_t(u \otimes u)\|_{L^\infty(Q_-)} + \|\partial_2^2 t (u \otimes u)\|_{L^\infty(Q_-)} \right).
$$

By (1.37)

$$
\|\partial_2^3 t \nabla^{m-1} u\|_{L^\infty(Q_-)} + \|\partial_2^2 t \nabla^m u\|_{L^\infty(Q_-)} + \|\partial_2^3 t \nabla^m p\|_{L^\infty(Q_-)} \leq C_m
$$

with some constant $C_m$ which by (1.49) and (1.50) depends only on $\|u\|_{L^\infty(Q_-)}$.

Continuing in this way, we first obtain estimates for

$$
\|\partial_l^2 t p\|_{L^\infty(-\infty,0;\text{BMO}(\mathbb{R}^3))}
$$

and then for

$$
\|\partial_l^{l+1} t \nabla^{m-1} u\|_{L^\infty(Q_-)}, \quad \|\partial_l^l t \nabla^m u\|_{L^\infty(Q_-)}, \quad \text{and} \quad \|\partial_l^l t \nabla^m p\|_{L^\infty(Q_-)},
$$

where $l \in \mathbb{N}$ is arbitrary. □

It follows from Theorem 1.3 that a bounded ancient solution $u(x,t) = b(t)$ is mild if and only if $b$ is independent of $t$, so that $b(t) = c$ for some $c \in \mathbb{R}^3$. Therefore, in the class of mild bounded ancient solutions of the linear Stokes system we have Liouville’s theorem: each mild bounded ancient solution of the Stokes system is constant. We are interested in the question of the validity of the analogue of this result for the non-linear Navier–Stokes system:

Is it true that each mild bounded ancient solution of the Navier–Stokes system is a constant?

Thus far this question is open. In the next sections we analyse several cases when the answer is in the affirmative.

1.4. Local energy ancient solutions. Apart from bounded ancient solutions we can also look at other, more general classes of ancient solutions of the Navier–Stokes equations. In particular, we will be interested in the class of so-called local energy ancient solutions. To describe this class, we follow [5] and [32] and start by defining suitable weak solutions of the Navier–Stokes equations.

Definition 1.5. A pair of functions $u$, $p$ is called a suitable weak solution of the Navier–Stokes equations (1.1) in the cylinder $Q = B \times [-1,0[$ if

1) $u \in L^2_{2,\infty}(Q) \cap W^{1,0}_2(Q)$, $p \in L^{3/2}_{3/2}(Q)$, and $\text{div} \, u = 0$ almost everywhere in $Q$,

2) $u$ and $p$ satisfy the Navier–Stokes equations in $Q$ in the sense of distributions,
3) $u$ and $p$ satisfy the local energy inequality in $Q$:

$$
\int_B \zeta |u(x,t)|^2 \, dx + 2 \int_{-1}^t \int_B \zeta |\nabla u|^2 \, dx \, dt' \\
\leq \int_{-1}^t \int_B |u|^2 (\partial_t \zeta + \Delta \zeta) \, dx \, dt + \int_{-1}^t \int_B u \cdot \nabla \zeta (|u|^2 + 2p) \, dx \, dt'
$$

for almost all $t \in ]-1,0[$ and any $\zeta \in C^\infty(Q)$ such that $\zeta \geq 0$ in $Q$ and $\zeta$ vanishes in a neighbourhood of the parabolic boundary of $Q$.

A suitable weak solution in the cylinder $Q(z_0, R)$ is defined similarly.

**Definition 1.6.** Functions $u: Q_- \to \mathbb{R}^3$ and $p: Q_- \to \mathbb{R}$ are called a local energy ancient solution of the Navier–Stokes equations if they are suitable weak solutions of the Navier–Stokes equations in each cylinder $Q(R)$ with $R > 0$.

Local energy ancient solutions arise in a natural way when we consider suitable weak solutions with blowups of type I of the Navier–Stokes equations (see Definition 2.4 below). In contrast to mild bounded ancient solutions, local energy solutions need not be smooth. Nevertheless, they have the property of partial regularity (see Theorem 2.5 below).

## 2. Problem of the unique global solvability of initial-boundary value problems for the Navier–Stokes equations and weak solutions

In this section we recall briefly some known facts on smoothness of solutions of the Navier–Stokes equations.

### 2.1. Leray–Hopf solutions of initial-boundary value problems for the Navier–Stokes equations. Uniqueness.

Let $Q_+ := \mathbb{R}^3 \times [0, +\infty[$, and let $a: \mathbb{R}^3 \to \mathbb{R}^3$ be a function such that $\text{div} \, a = 0$. Consider the Cauchy problem for the Navier–Stokes equations

\[
\begin{aligned}
\partial_t u - \Delta u + (u \cdot \nabla) u + \nabla p &= 0, & \text{in } Q_+, \\
\text{div} \, u &= 0, \\
\left. u \right|_{t=0} &= a.
\end{aligned}
\]

**Definition 2.1.** Let

$$a \in L_2(\mathbb{R}^3), \quad \text{div} \, a = 0 \quad \text{in } \mathbb{R}^3. \quad (2.2)$$

We say that $u: Q_+ \to \mathbb{R}^3$ is a Leray–Hopf solution of the problem (2.1) if the following conditions are satisfied:

1) $u \in L_2,\infty(Q_+): \nabla u \in L_2(Q_+)$ and $\text{div} \, u = 0$ almost everywhere in $Q_+$;
2) for any $t \in [0, +\infty[$ the function $u(\cdot, t)$ belongs to $L_2(\mathbb{R}^3)$, and for any $w \in L_2(\mathbb{R}^3)$ the function

$$t \mapsto \int_{\mathbb{R}^3} u(x, t) w(x) \, dx$$

is continuous on $[0, +\infty[$;
3) the function $u$ satisfies the Navier–Stokes equations in the sense that
\[ \int_{Q_T} (-u \cdot \partial_t \eta + \nabla u : \nabla \eta - u \otimes u : \nabla \eta) \, dx \, dt = 0 \]
for any $\eta \in C_0^\infty(Q_+)$ such that $\text{div} \, \eta = 0$;

4) the function $u$ satisfies the initial condition in the strict $L_2$-sense that
\[ \|u(\cdot, t) - a\|_{L_2(\mathbb{R}^3)} \to 0 \quad \text{as} \quad t \to +0; \]

5) for all $t \in [0, +\infty[$ the function $u$ satisfies the global energy inequality
\[ \frac{1}{2} \int_{\mathbb{R}^3} |u(x, t)|^2 \, dx + \int_0^t \int_{\mathbb{R}^3} |\nabla u(x, \tau)|^2 \, dx \, d\tau \leq \frac{1}{2} \int_{\mathbb{R}^3} |a(x)|^2 \, dx. \]

A global existence theorem is known for Leray–Hopf solutions (see [31]).

**Theorem 2.1.** For any $a$ satisfying (2.2) there exists at least one function $u$ that is a Leray–Hopf solution of the problem (2.1).

However, as of today it is an open question whether weak Leray–Hopf solutions are unique. On the other hand, a uniqueness theorem has been proved in the class of so-called strong solutions. There are several equivalent ways to define strong solutions of the Navier–Stokes equations. Here we present one approach to their definition.

**Definition 2.2.** A Leray–Hopf solution $u$ of the Navier–Stokes equations in a domain $Q_T := \mathbb{R}^3 \times [0, T]$ is called a strong solution if
\[ \nabla u \in L_{2, \infty}(Q_T). \]

One can show that strong solution are a fortiori smooth (see [27]). However, in the general case (without additional assumptions about the structure of the initial data) we know of only two types of results on the existence of strong solutions of (2.1): either local in time existence for initial data of arbitrary magnitude or global existence for small initial data (see [22]). It is important here that strong solutions, if they exist on an interval $]0, T[$, are unique solutions of (2.1) in the class of all Leray–Hopf solutions on this interval. This is the so-called weak-strong uniqueness theorem. Now we state the corresponding results.

**Theorem 2.2.** There exist positive absolute constants $C_0$ and $\varepsilon_0$ such that:

1) for any initial data $a \in W^1_2(\mathbb{R}^3)$ with $\text{div} \, a = 0$ such that $T_* \geq C_0 \|\nabla a\|_{L_2(\mathbb{R}^3)}^{-4}$ there exists a strong solution of (2.1) on the time interval $]0, T_*[$;

2) if the initial data $a \in W^1_2(\mathbb{R}^3)$ with $\text{div} \, a = 0$ satisfy $\|\nabla a\|_{L_2(\mathbb{R}^3)} \leq \varepsilon_0$, then a strong solution of (2.1) exists on the time interval $]0, +\infty[;

3) if $u$ is a strong solution of (2.1) on an interval $]0, T[$ with initial data $a \in W^1_2(\mathbb{R}^3)$ satisfying $\text{div} \, a = 0$, and $v$ is a Leray–Hopf solution of the same problem on the same interval and with the same initial data, then $u = v$ almost everywhere in $\mathbb{R}^3 \times ]0, T[.$
In fact, the uniqueness class of the Navier–Stokes equations is significantly broader than the class of strong solutions. One of the best conditions known to date and ensuring the weak-strong uniqueness of solutions of (2.1) is that one of the solutions belongs to the so-called Ladyzhenskaya–Prodi–Serrin class:

$$\exists s \in [3, +\infty], \exists l \in [2, +\infty]: \frac{3}{s} + \frac{2}{l} = 1, \; u \in L_{s,l}(Q_T), \; Q_T := \mathbb{R}^3 \times ]0, T[. \tag{2.3}$$

**Theorem 2.3.** Let $u$ and $v$ be two Leray–Hopf solutions of the problem (2.1) on an interval $]0, T[$ that correspond to initial data $a \in L^2(\mathbb{R}^3)$ with $\text{div} \, a = 0$, and assume that (2.3) holds for $u$. Then $u = v$ almost everywhere in $Q_T$.

For $s > 3$ Theorem 2.3 was proved in [34] and [51]. The case when $s = 3$ and $l = +\infty$ is different from $s > 3$, and in this case the uniqueness theorem was proved in [24]. In fact, solutions in the Ladyzhenskaya–Prodi–Serrin class are strong. For $3/s + 2/l = 1$ and $s > 3$ this was shown in [25]. That solutions in the class (2.3) are locally smooth was proved in [51] for $3/s + 2/l < 1$ and in [55] for $3/s + 2/l = 1$ and $s > 3$. In the case when $s = 3$ and $l = +\infty$ solutions in the class $L_{3,\infty}$ were shown to be smooth in [12].

The condition (2.3) puts some implicit additional constraints on the initial data $a$. The question of how much we can relax the assumptions about the initial data $a \in L^2(\mathbb{R}^3)$ with $\text{div} \, a = 0$ so that the weak-strong uniqueness still holds for (2.1) has been a subject of extensive study in recent years. Thus far there is some implicit evidence (including results of numerical analysis) that for $a \in L^2(\mathbb{R}^3)$ with $\text{div} \, a = 0$ (initial data in the energy class itself) the uniqueness theorem can fail; see the recent papers [20] and [16] on this issue. We note also that in the class of weak solutions of the Navier–Stokes equations with finite kinetic energy but without the assumption of finite viscous dissipation, the absence of uniqueness for solutions was investigated in [4]; see also references there.

Theorem 2.3 shows that investigating the smoothness properties of weak (Leray–Hopf) solutions of the Navier–Stokes equations is one of the possible approaches to solving one of the fundamental problems in hydrodynamics, the question of whether the Navier–Stokes equations give a deterministic description of flows of viscous fluids. In 2000 the problem of global smooth solvability of the Navier–Stokes equations was named as one of seven ‘millennium problems’ by the Clay Mathematical Institute.\(^1\)

### 2.2. Local smoothness of suitable weak solutions of the Navier–Stokes equations

In this subsection we discuss the question of local regularity of the Navier–Stokes equations. We look at the class of solutions with locally finite energy (note that Leray–Hopf solutions of (2.1) certainly belong to this class). Among such solutions, so-called *suitable weak solutions* (see Definition 1.5 in §1.4) play a special role.

Partial regularity is an important property of suitable weak solutions of the Navier–Stokes equations (see [35], [36], [5]). This means that solutions in this class are sufficiently smooth everywhere with the possible exception of a closed subset $\Sigma$

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\(^1\)See the problem “Navier–Stokes Equation” on their website http://www.claymath.org/millennium-problems, and the paper [28].
of zero one-dimensional parabolic Hausdorff measure. To characterize the singular set more precisely we give the following definition.

**Definition 2.3.** A point \( z_0 \in \Omega \times [0, T] \) is called a regular point of a suitable weak solution \( u, p \) of the Navier–Stokes equations in \( Q_T = \Omega \times [0, T] \) if \( u \) is bounded in a parabolic neighbourhood of \( z_0 \), that is, there exists an \( r > 0 \) such that

\[
Q(z_0, r) \subset Q_T \quad \text{and} \quad u \in L_\infty(Q(z_0, r)).
\]

Otherwise \( z_0 \) is called a singular point of the solution.

It can be shown (for instance, see [41], Chap. 6) that in a neighbourhood of each regular point the velocity field of a suitable weak solution is Hölder-continuous (with any exponent \( \alpha \in [0, 2/3] \) in the parabolic metric) together with all its spatial derivatives. Note that here the exponent \( 2/3 \) is determined by the \textit{a priori} information we have about the pressure (that is, by the assumption that \( p \in L_{3/2}(Q_T) \)). We state the corresponding result.

**Theorem 2.4.** Let \( u, p \) be a suitable weak solution of the Navier–Stokes equations in \( Q_T = \Omega \times [0, T] \), and let \( z_0 \in \Omega \times [0, T] \) be a regular point of this solution. Then there exists an \( r > 0 \) such that \( Q(z_0, r) \subset \Omega \times [0, T] \) and

\[
\nabla^{k-1}u \in C^{\alpha, \alpha/2}(Q(z_0, r)) \quad \forall k \in \mathbb{N}, \quad \forall \alpha \in \left]0, \frac{2}{3}\right[.
\]

It is obvious that the set of regular points of a suitable weak solution is open, so its complement (which we call the singular set of this solution) is relatively closed in \( Q_T \). Now we can state the theorem in [5] on the partial regularity of suitable weak solutions.

**Theorem 2.5.** Let \( u, p \) be a suitable weak solution of the Navier–Stokes equations in \( Q_T = \Omega \times [0, T] \), and let \( \Sigma \) be the singular set of this solution:

\[
\Sigma := \{z_0 \in \Omega \times [0, T] : z_0 \text{ is a singular point of } u, p\}.
\]

Then

\[
\mathcal{P}^1(\Sigma) = 0,
\]

where \( \mathcal{P}^1(\Sigma) \) is the one-dimensional parabolic Hausdorff measure of the set \( \Sigma \).

Theorem 2.5 is deduced using the condition of \( \varepsilon \)-regularity, which was also established in [5].

**Theorem 2.6.** There exists an absolute constant \( \varepsilon_* > 0 \) such that, for each suitable weak solution \( u, p \) of the Navier–Stokes equations in \( Q(z_0, R) \), if

\[
\limsup_{r \to 0} \frac{1}{r} \int_{Q(z_0, r)} |\nabla u|^2 \, dx \, dt < \varepsilon_*,
\]

then \( z_0 \) is a regular point of this solution.
We can also give other conditions for \( \varepsilon \)-regularity of suitable weak solutions in a neighbourhood of a given point. Such conditions usually involve a smallness assumption for a certain functional which is invariant under scaling transformations of the Navier–Stokes equations. Examples of such scale-invariant functionals are given by the scale-invariant energy quantities

\[
A(u, z_0, r) := \frac{1}{r} \operatorname{ess sup}_{t \in (t_0-r^2,t_0)} \int_{B(x_0, r)} |u|^2 \, dx,
\]

(2.4)

\[
E(u, z_0, r) := \frac{1}{r} \int_{Q(z_0, r)} |\nabla u|^2 \, dx \, dt,
\]

(2.5)

and

\[
C(u, z_0, r) := \frac{1}{r^2} \int_{Q(z_0, r)} |u|^3 \, dx \, dt.
\]

(2.6)

For \( z_0 = (0,0) \) we denote the functionals in (2.4), (2.5), and (2.6) by \( A(u, R) \), \( E(u, R) \), and \( C(u, R) \), respectively. The invariance of the functionals in (2.4)–(2.6) under scaling transformations of the Navier–Stokes equations means the following property: if \( F(u, z_0, R) \) is one of these functionals and the functions \( u^\lambda \) and \( p^\lambda \) are defined in \( Q(R/\lambda) \) by

\[
u^\lambda(x, t) := \lambda u(x_0 + \lambda x, t_0 + \lambda^2 t) \quad \text{and} \quad p^\lambda(x, t) := \lambda^2 p(x_0 + \lambda x, t_0 + \lambda^2 t),
\]

(2.7)

where \( z_0 = (x_0, t_0) \) and \( \lambda > 0 \), then

\[
F(u, z_0, R) = F(u^\lambda, R/\lambda).
\]

Now we formulate the condition of \( \varepsilon \)-regularity corresponding to these functionals (see [38]).

\textbf{Theorem 2.7.} There exists a positive absolute constant \( \varepsilon_0 \) such that, for each suitable weak solution \( u, p \) of the Navier–Stokes equations in \( Q(z_0, R) \), if

\[
\min \left\{ \sup_{r < 1} A(u, z_0, r), \sup_{r < 1} C(u, z_0, r), \sup_{r < 1} E(u, z_0, r) \right\} < \varepsilon_0,
\]

(2.8)

then \( z_0 \) is a regular point of this solution.

We can also give other conditions for \( \varepsilon \)-regularity. In particular, conditions were obtained in [17] and [50] for \( \varepsilon \)-regularity in terms of functionals of the form

\[
C_{s,l}(u, z_0, r) := r^{1-3/s-2/l} \left( \int_{t_0-r^2}^{t_0} \int_{B(x_0, r)} |u|^s \, dx \, dt \right)^{l/s} \left( \int_{t_0-r^2}^{t_0} \int_{B(x_0, r)} |u|^l \, dx \, dt \right)^{1/l}, \quad 1 \leq \frac{3}{s} + \frac{2}{l} \leq 2.
\]

(2.9)

Theorem 2.5 allows one to reduce the investigation of local regularity of suitable weak solutions of the Navier–Stokes equations to the investigation of the model problem ‘up to the first singularity’ in a certain canonical domain.
Theorem 2.8. Let $u, p$ be a suitable weak solution in $\Omega \times [0, T[$, and let $z_0 = (x_0, t_0) \in \Omega \times [0, T]$ be a singular point of it. Then there exists an $R > 0$ such that the functions

$$u_R(x, t) = Ru(x_0 + Rx, t_0 + R^2t), \quad p_R(x, t) = R^2p(x_0 + Rx, t_0 + R^2t)$$

are a suitable weak solution of the Navier–Stokes equations in the unit cylinder $Q := B \times ]-1, 0[$ such that $(0, 0)$ is a singular point of it and, in addition,

1) $u_R \in L_\infty(B \times ]-1, t_0[)$ for any $t_0 \in ]-1, 0[,$
2) there exists an $R_1 \in (0, 1)$ such that $u_R \in L_\infty(Q \setminus Q(R_1)).$

The proof of this theorem can be found in [41], Proposition 3.8.

2.3. Scale-invariant functionals and type-I blowups. The following question arises: can the smallness condition for functionals in Theorem 2.7 be replaced by the condition of their finiteness? To date, the answer is not known. So we give the following definition.

Definition 2.4. We say that a suitable weak solution $u, p$ in $Q(z_0, R)$ has a type-I blowup at a point $z_0$ if $z_0$ is a singular point of this solution and, in addition,

$$\min\left\{ \sup_{r<R} A(u, z_0, r), \sup_{r<R} C(u, z_0, r), \sup_{r<R} E(u, z_0, r) \right\} < +\infty. \quad (2.10)$$

It can be shown (see [38]) that if one of the functionals in (2.4)–(2.6) is uniformly bounded, then the other functionals are also bounded.

Theorem 2.9. Let $u, p$ be a suitable weak solution of the Navier–Stokes equations in $Q(z_0, R)$, and suppose that (2.10) holds. Then

$$\max\left\{ \sup_{r<R} A(u, z_0, r), \sup_{r<R} C(u, z_0, r), \sup_{r<R} E(u, z_0, r), \sup_{r<R} D(p, z_0, r) \right\} < +\infty,$$

where the functional $D(p, z_0, r)$ is defined by

$$D(p, z_0, r) = \frac{1}{r^2} \int_{Q(z_0, r)} |p(x, t)|^{3/2} dx \, dt. \quad (2.11)$$

In [50] (see also [49], Theorem 6) a similar result was proved for the functionals $C_{s,l}(u, z_0, R)$ defined by (2.9).

Theorem 2.10. Let $u, p$ be a suitable weak solution of the Navier–Stokes equations in $Q(z_0, R)$, and suppose that for some $s \in ]3, +\infty[$ and $l \in ]2, +\infty[$ such that

$$\max\left\{ 2 - \frac{1}{l}, \frac{3}{2} + \frac{1}{2l} \right\} < \frac{3}{s} + \frac{2}{l} < 2$$

the condition

$$\sup_{r<R} C_{s,l}(u, z_0, r) < +\infty$$

holds. Then (2.10) also holds for $u$. 

The next theorem shows that solutions in $Q = B \times ]-1,0[$ that are singular at $(0,0)$ and satisfy at least one of the pointwise estimates
\begin{equation}
|u(x,t)| \leq \frac{C}{\sqrt{-t}}, \quad (x,t) \in Q,
\end{equation}
and
\begin{equation}
|u(x,t)| \leq \frac{C}{|x|}, \quad (x,t) \in Q,
\end{equation}
are a special case of type-I blowups.

**Theorem 2.11.** Let $u, p$ be a suitable weak solution of the Navier–Stokes equations in $Q$, and assume that at least one of the estimates (2.12) and (2.13) holds for $u$. Then the condition (2.10) also holds for $u$.

In the case of the estimate (2.12) Theorem 2.11 was established in [49], Remark 2. The condition (2.13) can be regarded as a special case of membership of the solution in the Lorentz (also known as the weak Lebesgue) space $L^{3,\infty}$:
\[ u \in L^{\infty}(-1,0; L^{3,\infty}(B)). \]

One can show (for instance, see [1]), that then the scaled energy of the solution is also uniformly bounded (that is, (2.10) holds).

### 2.4. Behaviour of scale-invariant norms of solutions of the Cauchy problem for the Navier–Stokes equations.

Let us consider the Cauchy problem (2.1) for the Navier–Stokes equations. We assume for simplicity that $a \in C_0^\infty(\mathbb{R}^3)$ and $\text{div} \, a = 0$. Then by Theorem 2.2 there is a time interval $[0,T]$ with $T > 0$ depending on $\|\nabla a\|_{L^2(\mathbb{R}^3)}$ on which the problem (2.1) has a unique smooth solution in the class of all Leray–Hopf solutions. For the given initial data $a$ the **blowup time** is understood to be the quantity
\[ T_* := \sup\{T > 0: \text{the problem (2.1) has a smooth solution on }]0, T[\}. \]

In 1934 Leray showed that if $T_* < +\infty$, then for any $s \in ]3, +\infty[$ there exists a positive constant $c_s$ such that
\[ \|u(\cdot, t)\|_{L^s(\mathbb{R}^3)} \geq \frac{c_s}{(T_* - t)^{(s-3)/(2s)}} \quad \forall t \in ]0, T_*[. \]

For $s = 3$ the question arises of the behaviour of the $L_3$-norm of the solution, and also other scale-invariant norms of it, as we approach the blowup time. For the $L_3$-norm an answer was obtained in [39], and we present the corresponding result.

**Theorem 2.12.** Let $u, p$ be a smooth solution of the problem (2.1) with initial data $a \in C_0^\infty(\mathbb{R}^3)$ satisfying $\text{div} \, a = 0$, and let $T_* > 0$ be the blowup time of this solution. Suppose that $T_* < +\infty$. Then
\[ \lim_{t \to T_*-0} \|u(\cdot, t)\|_{L_3(\mathbb{R}^3)} = +\infty. \]

Subsequently, it was shown in [3] that in Theorem 2.12 the $L_3$-norm of the solution can be replaced by a weaker norm in the Lorentz class $L^{3,q}(\mathbb{R}^3)$, with any $q \in [3, +\infty[.$
3. Local smoothness of weak solutions of the Navier–Stokes equations and ancient solutions

In this section we establish connections between the presence of local singularities of weak solutions of the Navier–Stokes equations and a Liouville-type theorem for bounded ancient solutions of the Navier–Stokes system.

3.1. Existence of non-trivial ancient solutions as a necessary and sufficient condition for the existence of singular solutions with finite energy.

One very important property of the Navier–Stokes equations is their invariance under scaling transformations. Let \( u, p \) be a suitable weak solution of the Navier–Stokes equations in the cylinder \( Q(z_0, R) \), \( z_0 = (x_0, t_0) \). Then for any \( \lambda > 0 \) the functions \( u^\lambda, p^\lambda \) defined by (2.7) are also a suitable weak solution of the Navier–Stokes equations in the cylinder \( Q(R/\lambda) \). The scale invariance of the equations enables us to associate with each solution \( u, p \) of the Navier–Stokes equations in \( Q \) a sequence of solutions of the same equations in a chain of expanding domains. Namely, consider a sequence \( \lambda_k \to 0 \) and let \( (x_k, t_k) \in Q \) be some points. Then we define the functions

\[
    u^k(x, t) := \lambda_k u(x_k + \lambda_k x, t_k + \lambda_k^2 t) \quad \text{and} \quad p^k(x, t) := \lambda_k^2 p(x_k + \lambda_k x, t_k + \lambda_k^2 t). \tag{3.1}
\]

If we can construct a sequence \( u^k, p^k \) which is bounded in some norm (for example, one that gives us local control of the energy norm), then by using routine compactness arguments and taking the limit in the equations we find a subsequence \( u^{k_j} \) that converges in this or that sense to an ancient solution \( u^\infty \) of the Navier–Stokes equations.

Thus, we associate a certain ancient solution \( u^\infty \) with the original solution \( u, p \) by selecting the scaling parameters \( x_k, t_k, \) and \( \lambda_k \) in some way or another. The choice of scaling determines the properties of the ancient solution corresponding to the original solution. It turns out that in a certain sense the properties of this ancient solution characterize the asymptotic behaviour of the original solution in a neighbourhood of the singular points. In particular, when the original solution \( u \) has a singularity, we can always organize the scaling procedure so that the corresponding ancient solution is non-trivial. This lets us use the following two steps for investigating the local smoothness of suitable weak solutions of the Navier–Stokes equations:

1) construct an ancient solution \( u^\infty \) corresponding to a given suitable weak solution \( u, p \) and investigate the properties of \( u^\infty \) (such as finiteness of this or that norm, symmetries of the solution, and so on) following from the particular features of the original solution \( u, p \) (but also depending on the choice of the scaling parameters \( x_k, t_k, \) and \( \lambda_k \));

2) investigate whether the ancient solution \( u^\infty \) is trivial (which in our context means identically equal to zero).

In the framework of this approach, establishing Liouville-type theorems for the corresponding class of ancient solutions is the decisive step in the study of local regularity of weak solutions of the Navier–Stokes equations. In addition, as we can see in the case of the linear Stokes system, mild bounded ancient solutions play a prominent role in the class of all bounded ancient solutions. In fact, consider
the bounded ancient solution of the Stokes system defined to be a divergence-free function \( u \in L_\infty(Q_-) \) satisfying the condition

\[
\int_{Q_-} u \cdot (\partial_t \eta + \Delta \eta) \, dx \, dt = 0 \quad \forall \eta \in C_0^\infty(Q_-): \; \text{div} \, \eta = 0.
\]

It is obvious that for the Stokes system there is no Liouville theorem in the class of bounded ancient solutions: there exist solutions of the form \( u(x, t) = b(t) \), where \( b \in L_\infty(-\infty, 0) \) is arbitrary, which are not constants (moreover, bounded ancient solutions are not necessarily smooth). However, if we confine ourselves to mild solutions (which are always smooth), then Liouville’s theorem holds in this class: each mild bounded ancient solution of the Stokes system is identically constant (see [18]).

The paper [47] contains the conjecture that each mild bounded ancient solution of the Navier–Stokes equations also is identically constant. In this generality the conjecture is still open; to date it has been verified only in several special cases, which we list below.

### 3.2. Mild bounded ancient solutions corresponding to local singularities of weak solutions

We describe a scaling procedure that produces mild bounded ancient solutions. It was used for the first time in [23] and [47].

Let \( u, p \) be a suitable weak solution of the Navier–Stokes equations in a cylinder \( Q(z_0, R) \) such that \( z_0 \) is a singular point of this solution. In accordance with Theorems 2.8 and 2.4 we can assume without loss of generality that \( u, p \) is a suitable weak solution in \( Q \) and \( u \) is Hölder continuous on \( B \times [-1, t_0] \) for any \( t_0 \in ]-1,0[ \) and on \( \overline{Q} \setminus Q(R_1) \) for some \( R_1 \in ]0,1[ \), and that \((0,0)\) is a singular point of \( u \). Let

\[
M(t) := \sup_{\tau \in [-1, t]} \|u(\cdot, \tau)\|_{L_\infty(B)}, \quad t \in ]-1,0[.
\]

It follows from the theory of \( \varepsilon \)-regularity that if \((0,0)\) is a singular point of \( u \), then for some \( \varepsilon_0 > 0 \) we have

\[
M(t) \geq \frac{\varepsilon_0}{\sqrt{-t}} \quad \forall t \in ]-1,0[.
\]

Then we can construct sequences \( t_k \rightarrow 0 \) and \( x_k \in B(R_1) \) such that

\[
|u(x_k, t_k)| = M(t_k) \rightarrow +\infty \quad \text{as} \; k \rightarrow \infty. \tag{3.2}
\]

Setting

\[
\lambda_k := \frac{1}{M(t_k)} \quad (\lambda_k \rightarrow 0 \text{ as } k \rightarrow \infty), \tag{3.3}
\]

we define functions \( u^k \) and \( p^k \) on \( B(-\lambda_k^{-1}x_k, \lambda_k^{-1}) \times ]-\lambda_k^2(1 + t_k), -\lambda_k^2 t_k[ \) by the formulae (3.1). We extend \( u^k \) and \( p^k \) by zero to the whole of \( Q_- \). Then for any \( R > 0 \) and sufficiently large \( k \)

\[
Q(R) \subset B(\lambda_k^{-1}x_k, \lambda_k^{-1}) \times ]-\lambda_k^2(1 + t_k), -\lambda_k^2 t_k[,
\]
and the functions $u^k$ are Hölder-continuous on $\overline{Q}(R)$. In addition, for any $R > 0$ and sufficiently large $k$ the functions $u^k$ and $p^k$ have the following properties:

$$u^k \text{ and } p^k \text{ satisfy the Navier–Stokes equations in } Q(R);$$

$$\|u^k\|_{L^\infty(\Omega_\cdot)} = 1; \quad |u^k(0,0)| = 1.$$  

**Theorem 3.1.** Assume that a suitable weak solution $u$, $p$ in the domain $Q$ is Hölder-continuous on $\overline{B} \times [-1,t_0]$ for each $t_0 \in ]-1,0[$ and on $\overline{Q} \setminus Q(R_1)$ for some $R_1 \in ]0,1[,$ and that the point $(0,0)$ is singular for this solution. Let $u^k$ and $p^k$ be the sequences defined by (3.1) for parameters $x_k \in B(R_1)$, $t_k \uparrow 0$, and $\lambda_k \downarrow 0$ satisfying (3.2) and (3.3). Then there exist subsequences $\{u^{kj}\}$ and $\{p^{kj}\}$ such that for any $R > 0$

$$u^{kj} \to u^\infty \text{ uniformly on } Q(R), \quad \nabla u^{kj} \rightharpoonup \nabla u^\infty \text{ in } L_2(Q(R)), \quad p^{kj} \rightharpoonup p^\infty \text{ in } L_{3/2}(Q(R)),$$

where $u^\infty$ is some mild bounded ancient solution of the Navier–Stokes equations and $p^\infty$ is the associated pressure; furthermore,

$$|u^\infty(0,0)| = \|u^\infty\|_{L^\infty(\Omega_\cdot)} = 1.$$  

The reader can find the proof of Theorem 3.1, for instance, in [47], Theorem 2.8 or [41], Proposition 3.10.²

**3.3. Mild ancient solutions in problems with axial symmetry.** Now we look at the case when we know that the original suitable weak solution $u$, $p$ has axial symmetry. Let $(r, \varphi, z)$ be the cylindrical coordinates in $\mathbb{R}^3$ connected with the Cartesian coordinates $(x_1, x_2, x_3)$ by the formulae

$$x_1 = r \cos \varphi, \quad x_2 = r \sin \varphi, \quad x_3 = z,$$

and let $e_r, e_\varphi, e_z$ denote an orthonormal basis for the cylindrical coordinate system. A suitable weak solution $u$, $p$ of the Navier–Stokes equations is said to be axially symmetric if the components of the velocity field in this cylindrical basis and the pressure are independent of the angular variable $\varphi$, so that

$$u(x,t) = u_r(r,z,t)e_r + u_\varphi(r,z,t)e_\varphi + u_z(r,z,t)e_z, \quad p(x,t) = p(r,z,t).$$  

(3.7)

It turns out that in the case of axially symmetric solutions the corresponding mild bounded ancient solutions constructed in Theorem 3.1 are either themselves axially symmetric or are two-dimensional in the sense of being independent of one of the Cartesian variables (after an appropriate rotation of the Cartesian coordinate system). Namely, the following result holds.

²Quite recently the authors learned that D. Albritton and T. Barker proved the following: if there exists a non-trivial mild bounded ancient solution whose scale-invariant energy norms are bounded, then there exists a suitable weak solution of the Navier–Stokes equations in the unit parabolic cylinder $Q$, and it has a type-I blowup at $z = 0$.  

Theorem 3.2. Let \( u, p \) be a suitable weak solution in \( Q \) such that all the conditions of Theorem 3.1 are fulfilled. Also, assume that \( u \) and \( p \) are axially symmetric, so that \( (3.7) \) holds. Let \( u^\infty \) be the mild bounded ancient solution of the Navier–Stokes equations that was constructed in Theorem 3.1 for \( u \) and \( p \), and let \( p^\infty \) be the pressure associated with \( u^\infty \). Then

1) either \( u^\infty \) and \( p^\infty \) are axially symmetric relative to some axis \( y' = y'_0 \) (that is, there exists a \( y_0 \in \mathbb{R}^3 \) such that for any \( t \in ]-\infty, 0[ \) the functions \( y \mapsto u^\infty(y - y_0, t) \) and \( y \mapsto p^\infty(y - y_0, t) \) are axially symmetric);

2) or there exists a direction \( l \in \mathbb{R}^3, |l| = 1 \), such that both \( u^\infty \) and \( p^\infty \) are constant along it, that is,

\[
\frac{\partial u^\infty}{\partial l}(x, t) = 0 \quad \text{and} \quad \frac{\partial p^\infty}{\partial l}(x, t) = 0 \quad \forall (x, t) \in Q_-
\]

Proof. A vector field \( v \) is axially symmetric if and only if 

\[
(\nabla v)e_\varphi = \frac{1}{r}(e_\varphi \otimes e_r - e_r \otimes e_\varphi)v
\]

(this is a direct consequence of the representation of the tensor field \( \nabla v \) in the basis of the cylindrical coordinate system). The last relation is equivalent to

\[
(\nabla v)x'^\perp = \frac{1}{|x'|^2}(x'^\perp \otimes x' - x' \otimes x'^\perp)v,
\]

where \( x' = (x_1, x_2, 0) \) and \( x'^\perp = (-x_2, x_1, 0) \). Now suppose that the vector field \( u \) is axially symmetric and consider the sequence

\[
u^k(y, \tau) := \lambda_k u(x_k + \lambda_k y, t_k + \lambda_k^2 \tau),
\]

where \( \lambda_k \to 0 \). Let

\[
y_k := \frac{1}{\lambda_k}x_k.
\]

Then the condition that \( u \) is axially symmetric,

\[
(\nabla u(x, t))x'^\perp = \frac{1}{|x'|^2}(x'^\perp \otimes x' - x' \otimes x'^\perp)u(x, t),
\]

is equivalent to the identity

\[
(\nabla u^k(y, \tau))(y'_k + y')^\perp
\]

\[
= \frac{1}{|y'_k + y'|^2}((y'_k + y')^\perp \otimes (y'_k + y') - (y'_k + y') \otimes (y'_k + y')^\perp)u^k(y, \tau). \quad (3.8)
\]

In turn, the condition \( p, \varphi = 0 \) in \( \mathcal{D}'(Q_-) \) implies that

\[
\int_{Q_-} p^k(y, t) \frac{(y'_k + y')^\perp}{|y'_k + y'|} \nabla \varphi(y_k + y, t) \, dy \, dt = 0 \quad \forall \varphi \in C_0^\infty(Q_-). \quad (3.9)
\]

Assume that we have the convergences \( (3.4)-(3.6) \). Since \( \lambda_k \to 0 \) and the sequence \( \{x_k\} \) is bounded, at least one of the following two statements holds:
1) there exist a point \( y_0 \in \mathbb{R}^3 \), \( y_0 = y'_0 \), and a subsequence \( \{ y_{k_j} \} \) such that \( y'_{k_j} \rightarrow y'_0 \); 
2) there exists a subsequence \( \{ y_{k_j} \} \) such that \( |y'_{k_j}| \rightarrow +\infty \).

Consider the first case. Let \( y'_{k_j} \rightarrow y'_0 \). Then for this subsequence it follows from (3.8) and the convergences (3.4) and (3.5) that

\[
(\nabla u^\infty(y, \tau))(y'_0 + y')^\perp = \frac{1}{|y'_0 + y'|^2} ((y'_0 + y')^\perp \otimes (y'_0 + y') - (y'_0 + y') \otimes (y'_0 + y')^\perp) u^\infty(y, \tau).
\]

Making the substitutions \( v^\infty(y, \tau) = u^\infty(y - y_0, \tau) \) and \( z = y + y_0 \in \text{this identity, we get that} \)

\[
(\nabla v^\infty(z, \tau))z'^\perp = \frac{1}{|z'|^2} (z'^\perp \otimes z' - z' \otimes z'^\perp) v^\infty(z, \tau) \quad \forall z \in \mathbb{R}^3,
\]

which means that \( v^\infty \) is axially symmetric.

Similarly, taking the limit in (3.9) and recalling the weak convergence (3.6), we obtain

\[
\int_{Q_-} p^\infty(y, t) \frac{(y'_0 + y')^\perp}{|y'_0 + y'|} \nabla \varphi(y_0 + y, t) \, dy \, dt = 0 \quad \forall \varphi \in C_0^\infty(Q_-),
\]

which means that the function \( y \mapsto p^\infty(y - y_0, t) \) is axially symmetric.

Consider now the case when \( |y'_{k_j}| \rightarrow +\infty \) for some subsequence \( \{ y_{k_j} \} \), and let

\[
l_j := \frac{y'_{k_j}}{|y'_{k_j}|} \quad (|l_j| = 1)
\]

and

\[
\alpha_j := \frac{1}{|y_{k_j}|} \quad (\alpha_j \rightarrow 0).
\]

With this notation (3.8) takes the form

\[
(\nabla u^{k_j}(y, \tau))(l_j + \alpha_j y')^\perp = \frac{\alpha_j}{|l_j + \alpha_j y'|^2} ((l_j + \alpha_j y')^\perp \otimes (l_j + \alpha_j y')
- (l_j + \alpha_j y') \otimes (l_j + m_j y')^\perp) u^{k_j}(y, \tau).
\] (3.10)

Similarly, after a suitable replacement of the test function \( \varphi \) the identity (3.10) gives us that

\[
\int_{Q_-} p^k(y, t) \frac{(l_j + \alpha_j y')^\perp}{|l_j + \alpha_j y'|} \nabla \varphi(y, t) \, dy = 0 \quad \forall \varphi \in C_0^\infty(Q_-).
\] (3.11)

Selecting a subsequence again (but keeping the same notation for it), we can assume that \( l_j \rightarrow l \) for some \( l \in \mathbb{R}^3 \), \( |l| = 1 \), \( l = l' \). Then in view of (3.4) and (3.5) we find for this subsequence that the left-hand side of (3.10) converges weakly in \( L_2 \) on each compact subset of \( Q_- \) to a function \( (\nabla u^\infty)l \), while the right-hand side converges
uniformly to zero on each compact subset of $Q_-$. Taking the limit in (3.10), we obtain the identity
\[
\frac{\partial u_\infty}{\partial t} = 0 \quad \text{in } Q_-. 
\]
Similarly, taking the limit in (3.11), we obtain
\[
\int_{Q_-} p_\infty(y, t) \frac{\partial \varphi}{\partial l}(y, t) \, dy \, dt = 0 \quad \forall \varphi \in C_0^\infty(Q_-). 
\]
If the angular component of the axially symmetric velocity field is identically equal to zero, then we can refine Theorem 3.2 as follows.

**Theorem 3.3.** Let $u, p$ be a suitable weak solution in $Q$ such that all the conditions of Theorem 3.2 are fulfilled. Also, assume that the angular component of $u$ vanishes identically, that is,
\[
u(x, t) = u_r(r, z, t)e_r + u_z(r, z, t)e_z.
\]
Let $u_\infty, p_\infty$ be a mild bounded ancient solution as constructed in Theorem 3.2 for $u$ and $p$. Then:
1) if $u_\infty$ and $p_\infty$ are axially symmetric relative to some axis $y' = y'_0$, then $u_\infty$ has the additional property that
\[
u_\infty(y, t) \cdot (y'_0 + y')^\perp = 0 \quad \text{for all } (y, t) \in Q_-
\]
(in other words, the angular component of the function $u \mapsto u_\infty(y - y_0, t)$ vanishes);
2) if there exists a direction $l \in \mathbb{R}^3, |l| = 1$, along which $u_\infty$ and $p_\infty$ are constant, then
\[
u_\infty(y, t) \cdot l = 0 \quad \text{for all } (y, t) \in Q_-. 
\]

Theorem 3.3 is proved like Theorem 3.2.

### 3.4. Other ways to construct ancient solutions. Local energy ancient solutions and a backwards uniqueness theorem.

Now we describe a scaling procedure producing local energy ancient solutions. It was first used in [12] to prove that $L_{3,\infty}$-solutions of the Navier–Stokes equations are locally smooth. In the general case this procedure can be used to analyse suitable weak solutions in a neighbourhood of a singular point at which the solutions have a type-I blowup (see Definition 2.4).

**Theorem 3.4.** Let $u, p$ be a suitable weak solution of the Navier–Stokes equations in $Q$ such that
\[
\min\left\{ \sup_{r < R} A(u, r), \sup_{r < R} C(u, r), \sup_{r < R} E(u, r) \right\} < +\infty.
\]
Let $\lambda_k \to 0$ be some sequence, and let
\[
u^k(x, t) := \lambda_k u(\lambda_k x, \lambda_k^2 t) \quad \text{and} \quad p^k(x, t) := \lambda_k^2 p(\lambda_k x, \lambda_k^2 t).
\]
Then there exist a local energy ancient solution $u_\infty, p_\infty$ of the Navier–Stokes equations and subsequences $\{u^{k_j}\}$ and $\{p^{k_j}\}$ such that:
1) for any $R > 0$

\[ u^{kj} \to u^\infty \text{ in } L_3(Q(R)) \cap C([-R^2, 0]; L_{9/8}(B(R))), \]

\[ p^{kj} \to p^\infty \text{ in } L_{3/2}(Q(R)); \]

2) for $u^\infty$ and $p^\infty$ the scale-invariant functionals (2.4)–(2.6) and (2.11) are uniformly bounded, that is,

\[ \sup_{R > 0} \left( A(u^\infty, R) + E(u^\infty, R) + C(u^\infty, R) + D(p^\infty, R) \right) < +\infty; \]

3) if $(0, 0)$ is a singular point of the solution $u$, $p$ (that is, the original solution has a type-I blowup at $(0, 0)$), then

\[ \|u^\infty\|_{L_3(Q(3/4))} \geq c_0 \]

with some absolute constant $c_0 > 0$, and in particular, $u^\infty \neq 0$.

Theorem 3.4 can be proved using the same method as that used in the proof of a similar result in [12].

Under certain additional assumptions on the original solution $u$, $p$ the local energy ancient solution $u^\infty$ in Theorem 3.4 vanishes identically at a finite moment of time:

\[ u^\infty(\cdot, 0) = 0. \tag{3.12} \]

One instance of such an assumption is the following condition, which can be interpreted as a condition on the ‘blowup profile’ of the original solution at a finite moment of time.

**Theorem 3.5.** Let $u, p$ be a suitable weak solution of the Navier–Stokes equations in $Q$ that satisfies all the conditions of Theorem 3.4, and let $u^\infty$ be the local energy ancient solution constructed in Theorem 3.4 for $u$ and $p$. If

\[ r^{-15/8} \int_{B(r)} |u(x, 0)|^{9/8} \, dx \to 0 \quad \text{as } r \to 0, \tag{3.13} \]

then (3.12) holds for $u^\infty$.

**Proof.** It follows from (3.13) that for any $R > 0$

\[ R^{-5/3}\|u^\infty(\cdot, 0)\|_{L_{9/8}(B(R))} \leq R^{-5/3}\|u^\infty(\cdot, 0) - u^k(\cdot, 0)\|_{L_{9/8}(B(R))} + R^{-5/3}\|u^k(\cdot, 0)\|_{L_{9/8}(B(R))} \]

\[ = R^{-5/3}\|u^\infty(\cdot, 0) - u^k(\cdot, 0)\|_{L_{9/8}(B(R))} + (\lambda_k R)^{-5/3}\|u(\cdot, 0)\|_{L_{9/8}(B(\lambda_k R))}. \]

The first term on the right-hand side tends to zero because $u^k \to u^\infty$ in $C([-R^2, 0]; L_{9/8}(B(R)))$, and the second tends to zero in view of (3.13). □

When (3.12) is satisfied, it would be natural to expect $u^\infty$ to vanish identically in the whole of $Q_-$ (which by implication rules out the possibility of a singularity of the original solution $u, p$ at $(0, 0)$). This is supported by the backwards-in-time uniqueness theorem proved in [12] for the heat equation with lower-order terms.
**Theorem 3.6.** Let $\omega, \nabla \omega, \partial_t \omega, \nabla^2 \omega \in L_{2, \text{loc}}(\mathbb{R}_+^n \times [0,1])$, and suppose that there exist positive constants $c$ and $M$ such that

$$
|\partial_t \omega + \Delta \omega| \leq c(|\omega| + |\nabla \omega|) \quad \text{almost everywhere in } \mathbb{R}_+^n \times ]0,1[,
$$

$$
|\omega(x, t)| \leq e^{M|x|^2} \quad \text{for almost all } (x, t) \in \mathbb{R}_+^n \times ]0,1[.
$$

If

$$
\omega(x, 0) = 0 \quad \text{for almost all } x \in \mathbb{R}_+^n,
$$

then $\omega \equiv 0$ almost everywhere in $\mathbb{R}_+^n \times ]0,1[$.

In the case of the Navier–Stokes equations we can take $\omega$ to be the curl of the velocity field of an ancient solution. We remark that the following result was proved in [12] using Theorems 3.4 and 3.6.

**Theorem 3.7.** Let $u, p$ be a suitable weak solution of the Navier–Stokes equations in $Q$ such that

$$
u \in L_{3, \infty}(Q),
$$

Then $(0,0)$ is a regular point for this solution.

4. Liouville-type theorems for ancient solutions

4.1. Methods for proving Liouville theorems for the Navier–Stokes equations. Distinguishing a ‘governing’ scalar equation in various hydrodynamical problems. In this section we discuss two different approaches enabling one to prove Liouville-type theorems for mild bounded ancient solutions of the Navier–Stokes system.

The first is based on distinguishing a scalar equation connected with the particular features of the problem under consideration and then proving the Liouville theorem for this equation using various methods worked out in the theory of scalar parabolic equations.

The second approach, first used in [40], is based on duality theory. Its central idea consists in replacing a direct proof of the Liouville theorem for the solution of some system by an investigation of the rate of decay of certain norms of the solution of the dual linear system, which involves the original solution as a coefficient.

We start our discussion with the first approach, when Liouville-type theorems are proved by distinguishing an auxiliary scalar equation in the problem. This method can be used in the following contexts: the two-dimensional Navier–Stokes system, the axially symmetric system without the angular component of the velocity, and type-I singularities in an axially symmetric problem. All these problems reduce, in one way or another, to establishing Liouville theorems for the heat equation with a divergence-free drift $b$:

$$
\partial_t \psi - \Delta \psi + b \cdot \nabla \psi = 0 \quad \text{in } Q_-. \tag{4.1}
$$

The particular features of this or that problem are reflected here by the properties of both the drift $b \in L_\infty(Q_-)$ and the solution $\psi \in L_\infty(Q_-)$ itself.

For example, the proof of Liouville’s theorem for the two-dimensional Navier–Stokes system reduces to investigating (4.1). Let us see how equation (4.1) arises in the two-dimensional problem.
Proposition 4.1. Let \( u, p \) be a smooth solution of the Navier–Stokes equations in \( Q_- := \mathbb{R}^2 \times ]-\infty, 0] \), and let \( \omega := \text{curl} \, u = u_{2,1} - u_{1,2} \). Then
\[
\partial_t \omega - \Delta \omega + u \cdot \nabla \omega = 0 \quad \text{in } Q_-.
\]
Furthermore,
\[
\text{if } \ u \in L_\infty(Q_-), \ \text{then } \ \omega \in L_\infty(Q_-).
\]
Thus, under the assumptions of Proposition 4.1 the function \( \omega \) satisfies (4.1) with the scalar function \( \psi = \omega \) and the drift \( b = u \).

Proposition 4.2. Let \( u, p \) be a smooth axially symmetric solution of the Navier–Stokes equations in \( Q_- \), and also assume that the angular component of the vector field \( u \) vanishes identically, that is, the following representation holds in the cylindrical coordinates:
\[
u(x, t) = u_r(r, z, t)e_r + u_z(r, z, t)e_z, \quad p(x, t) = p(r, z, t)
(\text{then \text{curl} } u = \omega_\varphi e_\varphi, \ \text{where } \omega_\varphi := u_{r,z} - u_{z,r}).\]
Let
\[
\eta(x, t) = \frac{\omega_\varphi(r, z, t)}{r}, \quad \text{where } r = |x'| = \sqrt{x_1^2 + x_2^2}, \quad z = x_3.
\]
Then
\[
\partial_t \eta - \Delta \eta + \left( u - 2 \frac{x'}{|x'|^2} \right) \cdot \nabla \eta = 0 \quad \text{in } Q_-.
(4.2)
\]
Moreover,
\[
\text{if } \ u \in L_\infty(Q_-), \ \text{then } \ \omega_\varphi, \eta \in L_\infty(Q_-).
\]
Thus, under the assumptions of Proposition 4.2 the function \( \eta \) satisfies equation (4.1) with \( \psi = \eta \) and the singular drift
\[
b = u - 2 \frac{x'}{|x'|^2}.
\]
Finally, we show how the general case of the axially symmetric problem for the Navier–Stokes equations is also connected with the study of equation (4.1).

Proposition 4.3. Let \( u, p \) be a smooth axially symmetric solution of the Navier–Stokes equations in \( Q_- \), so that it has the following representation in the cylindrical coordinates:
\[
u(x, t) = u_r(r, z, t)e_r + u_\varphi(r, z, t)e_\varphi + u_z(r, z, t)e_z, \quad p(x, t) = p(r, z, t).
\]
Let
\[
\xi(x, t) = ru_\varphi(r, z, t).
\]
Then \( \xi \) satisfies the equation
\[
\partial_t \xi - \Delta \xi + \left( u + 2 \frac{x'}{|x'|^2} \right) \cdot \nabla \xi = 0 \quad \text{in } Q_-.
(4.3)
\]
Thus, under the assumptions of Proposition 4.3 the function $\xi$ satisfies equation (4.1) with $\psi = \xi$ and the singular drift

$$b = u + 2 \frac{x'}{|x'|^2}.$$  

Note that the singular parts of the drift in (4.2) and (4.3) have opposite signs, and this is a significant difference between the two equations. While in the case of (4.2) the singular component of the drift gives a positive addendum to the quadratic form of the elliptic part of the operator, in the case of (4.3) the corresponding addendum is negative. As a result, some methods that work successfully for the equation (4.2) cannot be applied to (4.3).

We note also that when the ancient solution $u$ in Proposition 4.3 is an axially symmetric mild bounded ancient solution obtained in a local axially symmetric problem by means of the scaling procedure described in §3.3, the corresponding function $\xi$ always has the property

$$\xi \in L_\infty(Q_-).$$

This was proved in [47]. Thus, it is reasonable to ask whether a bounded solution $\xi$ of (4.3) must be trivial, because this question is prompted by the analysis of a local hydrodynamical problem.

We should note that in investigating equation (4.1) with a singular drift the question arises as to whether its solution $\psi$ is smooth. However, in most applications to axially symmetric hydrodynamical problems the presence of singular terms in the equation does not reflect the substance of the problem, but is only connected with the use of a singular (cylindrical) change of coordinates. At the same time, when we speak of mild ancient solutions, the velocity field is a smooth function. So in the analysis of (4.1) from the standpoint of applications to the Navier–Stokes equations we can always assume that the scalar function $\psi$ connected with the velocity field of a mild ancient solution of the Navier–Stokes equations is sufficiently smooth (normally we will assume that $\psi \in C^1(\overline{Q_-})$).

### 4.2. Liouville theorems for scalar parabolic equations.

Here we discuss the validity of Liouville’s theorem for the heat equation with a drift (4.1). This equation often occurs, for instance, in probability theory, and standard methods in general parabolic theory are applicable to it. These methods have been known since the late 1960s (for instance, see [29]) and enable one to prove Liouville’s theorem for this equation (including the case of a drift that is not divergence free) under certain additional assumptions. For example, it is classical to assume that the drift belongs to the Ladyzhenskaya–Prodi–Serrin class

$$b \in L_{s,l}(Q_-), \quad \frac{3}{s} + \frac{2}{l} = 1, \quad s > 3.$$  

However, when equation (4.1) is motivated by problems in hydrodynamics, the drift $b$ usually has the additional property of being divergence-free, and the assumptions on it may be relaxed. Such refinements of Liouville’s theorem for equation (4.1) with divergence-free $b$ were obtained, in particular, in [46] and [33]. Below we describe in greater detail the assumptions on the drift $b$ which were made there.
For all this, the theorems in general parabolic theory listed above cannot be applied to hydrodynamical problems, because in these problems the role of $b$ is played in some way or another by the velocity field of a mild bounded ancient solution of the Navier–Stokes equations, of which we know initially only that it is globally bounded, while in all the above results $b$ must tend to zero at infinity in one sense or another.

The situation is not made any easier by the fact that without additional assumptions Liouville’s theorem fails for smooth globally bounded divergence-free $b$. A corresponding counterexample was given in [46]. Thus, we need additional assumptions to ensure that the solution is trivial in this case.

In this paper we present a survey of known results in general parabolic theory and propose a kind of classification of types of Liouville theorems for the heat equation with a drift. We discuss what properties of solutions and/or drift in (4.1) ensure that Liouville’s theorem holds for this equation. We also discuss which of these properties hold in the case when the corresponding problem was motivated by an investigation of local smoothness for suitable weak solutions of the Navier–Stokes equations.

Unless otherwise stipulated, $b_0$ is a smooth divergence-free vector field in $Q_-$ such that $b_0 \in L_\infty(Q_-)$ throughout what follows. We will consider equation (4.1) in the following three cases:

$$b = b_0, \quad b = b_0 - 2\frac{x'}{|x'|^2}, \quad \text{and} \quad b = b_0 + 2\frac{x'}{|x'|^2}.$$  

In applications to problems in hydrodynamics we can always assume that the function $\psi$ is globally bounded and sufficiently smooth (for instance, in the class $C^1(Q_-)$ and $L_\infty(Q_-)$).

Provisionally, we put all the cases when Liouville’s theorem holds for (4.1) into the following three groups.

1) In the first group we put Liouville theorems where we only want the drift $b$ to be globally bounded, and make additional assumptions on the structure of the solution $\psi$ itself. These results ensure Liouville’s theorem for the Navier–Stokes equations in the two-dimensional case and also Liouville’s theorem for axially symmetric solutions of the Navier–Stokes equations with vanishing angular component of the velocity.

2) In the second group we put Liouville theorems in ‘general' parabolic theory, for equation (4.1) with a smooth divergence-free drift $b = b_0$ and without additional restrictions on the behaviour or structure of the bounded solution $\psi$. In the corresponding theorems additional assumptions are made about the drift itself to ensure the Liouville theorem. The theorems in this group are not directly connected with problems in hydrodynamics, but are a kind of ‘starting point’ for study of the third category of Liouville theorems.

3) In the third group we put Liouville theorems for equation (4.1) with a drift $b = b_0 + 2x'/|x'|^2$. In this case we make additional assumptions on both the drift and the solution itself. The conditions imposed on the regular part $b_0$ of the drift are as in the second category of theorems, and as for the solution $\psi$, we assume that it vanishes on some distinguished axis. In the context of hydrodynamics this result means that Liouville’s theorem holds for axially symmetric solutions of the
Liouville-type theorems

Navier–Stokes equations in the case of type-I blowups (this was originally proved in [23]).

Let us proceed to a thorough description of Liouville theorems in the three groups.

Recall that we put Liouville theorems involving additional assumptions on the solution in the first category. Here are examples of such results.

**Theorem 4.1.** Let \( b = b_0 \) and let \( \psi \in C^1(Q_-) \cap L_\infty(Q_-) \) be a solution of equation (4.1). Also assume that \( \psi \in L_\infty(-\infty,0; L_\infty^{-1}(\mathbb{R}^n)) \), that is,

\[
\exists F \in L_\infty(Q_-; \mathbb{R}^n) : \quad \psi = \text{div} \, F. \tag{4.4}
\]

Then \( \psi \equiv 0 \) in \( Q_- \).

**Theorem 4.2.** Let \( b = b_0 - 2x'/|x'|^2 \), and let \( \psi \in C^1(Q_-) \cap L_\infty(Q_-) \) be a solution of (4.1). Also assume that

\[
|\psi(x,t)| \leq \frac{C}{|x'|} \quad \text{for} \quad |x'| \geq R_0, \quad t \in ]-\infty,0[. \tag{4.5}
\]

Then \( \psi \equiv 0 \) in \( Q_- \).

The proofs of Theorems 4.1 and 4.2 are based on a result first established in [23]. Conceptually, this can be characterized as the ‘stability of the strong maximum principle under small perturbations’. The strong maximum principle itself claims that if a solution of a scalar parabolic equation in a bounded cylindrical domain attains a maximum at a point away from the parabolic boundary of the domain, then this solution is identically equal to a constant. It turns out that if a solution ‘almost attains’ its maximum in some sense at an interior point of a cylindrical domain, then it must be ‘almost constant’. More precisely, the following result holds.

**Lemma 4.1.** Let \( b = b_0 \), let \( \Omega, \Omega_0 \subset \mathbb{R}^n \) be bounded connected domains such that \( \Omega \subset \Omega_0 \), let \( 0 < \tau < T \), and let \( I_0 := ]0,T[ \) and \( I := ]\tau,T[ \). Then for any \( \varepsilon > 0 \) there exists a \( \delta = \delta(\varepsilon, \Omega, \Omega_0, T, \tau) > 0 \) such that, for any \( M > 0 \) and any solution \( \psi \in C^1(\bar{\Omega}_0 \times T_0) \) of equation (4.1) in \( \Omega_0 \times I_0 \), if

\[
\sup_{\Omega_0 \times I_0} |\psi| = M
\]

and

\[
\sup_{x \in \bar{\Omega}} \psi(x,T) > M(1 - \delta),
\]

then

\[
\text{osc}_{\Omega \times I} \psi < \varepsilon M.
\]

In particular, it follows from Lemma 4.1 that for bounded solutions of (4.1) in \( Q_- \) there exist cylinders with arbitrarily large radius on which the solution is close to its maximum value. Namely, we have the following result.
Proposition 4.4. Let $b = b_0$ and let $\psi \in L_\infty(Q_-)$ be a solution of equation (4.1) such that

$$M := \sup_{Q_-} \psi > 0.$$ 

Then for any $R > 0$ there exists a point $z_R = (x_R, t_R) \in Q_-$ such that

$$\psi(x, t) \geq \frac{M}{2} \quad \forall (x, t) \in Q(z_R, R),$$

where $Q(z_R, R) := B(x_R, R) \times ]t_R - R^2, t_R[.$

Proof of Theorem 4.1. We show how Theorem 4.1 follows from Proposition 4.4. Let $\psi \in L_\infty(Q_-)$ be a solution of (4.1) such that (4.4) holds. Let $M := \sup_{Q_-} \psi > 0$. Then by Proposition 4.4, for any $R > 0$ there exists a point $z_R = (x_R, t_R) \in Q_-$ such that

$$\inf_{Q(z_R, R)} \psi \geq \frac{M}{2}.$$ 

From (4.4) we get by integrating by parts that

$$M^2 |Q(R)| \leq \int_{Q(z_R, R)} \psi \, dx \, dt = \int_{Q(z_R, R)} \text{div} \, F \, dx \, dt = \int_{t_R - R^2}^{t_R} \, dt \int_{\partial B(x_R, R)} F \cdot \nu \, ds,$$

where $\nu$ is the outward normal to the ball $B(x_R, R)$. The above relation gives us that

$$M^2 R^{n+2} \leq C \|F\|_{L_\infty(Q_-)} R^{n+1}$$

for some absolute constant $C$. This inequality leads to a contradiction when $R$ is sufficiently large. Hence $M \leq 0$. Replacing $\psi$ by $-\psi$, we get that $-M \leq 0$, that is, $\psi \equiv 0$. $\square$

The proof of Lemma 4.1 is based on the strong maximum principle, which is well known for equation (4.1) with $b = b_0$. However, although it is possible to extend the corresponding statement to $b = b_0 - 2x'/|x'|^2$ (for instance, see [33]), this requires overcoming considerable technical problems related to a singular drift. Below we present a weaker version of Proposition 4.4, in which pointwise estimates of the form (4.6) are replaced by estimates for the measure of the set of points at which the solution is ‘close to maximum’. Using this approach, we can give a simple short proof of Theorem 4.2.

Proposition 4.5. Let $b = b_0 - 2x'/|x'|^2$ and let $\psi \in L_\infty(Q_-)$ be a solution of equation (4.1). Assume that

$$M := \sup_{Q_-} \psi > 0.$$ 

Then for any $R > 0$ there exists a point $z_R = (x_R, t_R) \in Q_-$ such that

$$\left| \left\{ (x, t) \in Q(z_R, R) : \psi(x, t) \geq \frac{M}{2} \right\} \right| \geq \delta |Q(R)|,$$

where $\delta > 0$ depends only on $\|b_0\|_{L_\infty(Q_-)}$ (the value of the constant $\delta$ is found in Proposition 4.7 below).
The advantage of the last estimates is that their proof does not use the technically difficult strong maximum principle, but only uses local estimates for the maxima of solutions, which hold for equation (4.1) with \( b = b_0 \) or \( b = b_0 - 2x'/|x'|^2 \).

**Proposition 4.6.** Let \( b = b_0 - 2x'/|x'|^2 \), and let \( \psi \in C^1(\overline{Q}(R)) \) be a solution of equation (4.1) in \( Q(R) \). Then for any \( k \in \mathbb{R} \)

\[
\| (u-k)_+ \|_{L_\infty(Q(R/2))} \leq C_b \left( \frac{1}{Q(R)} \int_{Q(R)} |(u-k)_+|^2 \, dx \, dt \right)^{1/2}, \tag{4.7}
\]

where \( C_b \) depends only on \( \| b_0 \|_{L_\infty(Q(R))} \).

For bounded solutions of equation (4.1) with \( b = b_0 - 2x'/|x'|^2 \) the estimate (4.7) is proved using the standard Moser iteration technique. In this case the singular component of the drift has constant-sign divergence

\[
\text{div} \frac{x'}{|x'|^2} = -2\pi \delta \Gamma \quad \text{in} \quad \mathcal{D}'(Q(R)),
\]

and the term corresponding to this singular component can be dropped altogether (this is the essential difference between the cases \( b = b_0 - 2x'/|x'|^2 \) and \( b = b_0 + 2x'/|x'|^2 \)).

The following result is an immediate consequence of Proposition 4.6.

**Proposition 4.7.** Let \( b = b_0 - 2x'/|x'|^2 \), and let \( \psi \in C^1(\overline{Q}(R)) \) be a solution of equation (4.1) in \( Q(R) \). Take

\[
\delta := \frac{1}{(4C_b)^2},
\]

where \( C_b \) is the constant in Proposition 4.6, and assume that

\[
M := \sup_{Q(R)} \psi > 0.
\]

Also, suppose that

\[
\left\{ (x,t) \in Q(R) : \psi(x,t) \geq \frac{M}{2} \right\} \leq \delta |Q(R)|.
\]

Then

\[
\sup_{Q(R/2)} \psi \leq \frac{3M}{4}.
\]

To prove Proposition 4.7 it is sufficient to take \( k = M/2 \) in Proposition 4.6 and carry the sign of the supremum outside of the integral sign. Next we deduce Proposition 4.5 from Proposition 4.7.

**Proof of Proposition 4.5.** We argue by contradiction. Assume that there exists an \( R > 0 \) such that for each point \( z_0 = (x_0,t_0) \in Q_- \)

\[
\left\{ (x,t) \in Q(z_0,R) : \psi(x,t) \geq \frac{M}{2} \right\} \leq \delta |Q(R)|.
\]
Consider a sequence \( z_k := (x_k, t_k) \in Q_- \) such that \( \psi(z_k) \to M \) as \( k \to \infty \). Then by Proposition 4.7

\[
\sup_{Q(z_k, R/2)} \psi \leq \frac{3M}{4},
\]

in contradiction to the assumption that \( \psi(z_k) \to M \). \( \Box \)

Now we can sketch the proof of Theorem 4.2.

**Proof of Theorem 4.2.** Assume that (4.5) holds. We proceed by contradiction again. For \( M := \sup_{Q_-} \psi \) we assume that \( M > 0 \). Then by Proposition 4.5, for any \( R > 0 \) there exists a point \( z_R = (x_R, t_R) \in Q_- \) such that

\[
\left\{ (x, t) \in Q(z_R, R) : \psi(x, t) \geq \frac{M}{2} \right\} \geq \delta |Q(R)|.
\]

In this case

\[
\int_{Q(z_R, R)} |\psi| \, dx \, dt \geq \int_{\{ (x, t) \in Q(z_R, R) : \psi(x, t) \geq M/2 \}} \psi \, dx \, dt \geq \frac{M}{2} \delta |Q(R)|.
\]

On the other hand, by the condition (4.5)

\[
\int_{Q(z_R, R)} |\psi| \, dx \, dt \leq \int_{Q(z_R, R)} \frac{C}{|x'|} \, dx \, dt \leq \int_{Q(R)} \frac{C}{|x'|} \, dx \, dt \leq CR^4.
\]

The last two estimates imply that

\[
\frac{M}{2} \delta R^5 \leq CR^4.
\]

This leads to a contradiction if \( R \) is sufficiently small. Consequently, \( M \leq 0 \). Replacing \( \psi \) by \(-\psi\), we obtain \(-M \leq 0\), that is, \( \psi \equiv 0 \). \( \Box \)

We proceed to Liouville theorems in the second group. Recall that it contains Liouville theorems for equation (4.1) with \( b = b_0 \). Thus far we know quite a number of different conditions on a smooth divergence-free drift which ensure that bounded solutions of (4.1) are constant. For example, we present two such conditions. The first was found in [46].

**Theorem 4.3.** Let \( b_0 \) be a smooth divergence-free vector field on \( Q_- \) such that

\[
b_0 \in L_\infty(-\infty, 0; \text{BMO}^{-1}(\mathbb{R}^3)),
\]

so that there exists a skew-symmetric tensor field \( A(x, t) = (A_{ij}(x, t)) \) for which

\[
A_{ij} \in L_\infty(-\infty, 0; \text{BMO}(\mathbb{R}^3)), \quad b_0 = \text{div} A \quad \text{in} \; \mathcal{D}'(Q_-).
\]

Let \( \psi \in C^{\infty}(Q_-) \) be a solution of equation (4.1) with \( b = b_0 \) such that

\[
\psi \in L_\infty(Q_-).
\]

Then \( \psi = \text{const.} \)
Another condition, obtained in [33], requires that the function \( b_0 \) belong to some Morrey class

\[
M^\alpha_{s,l}(Q_-) := \left\{ b \in L_{s,l}(Q_-) := L_t(-\infty,0; L_s(\mathbb{R}^n)) : \sup_{R>0} R^{-\alpha} \| b \|_{L_{s,l}(Q(Q(R))} < +\infty \right\}.
\]

**Theorem 4.4.** Let \( b_0 \) be a smooth divergence-free vector field on \( Q_- \) such that

\[
b_0 \in M^\alpha_{s,l}(Q_-), \quad \text{where} \quad 1 \leq \frac{3}{s} + \frac{2}{l} < 2 \quad \text{and} \quad \alpha = \frac{3}{s} + \frac{2}{l} - 1.
\]

Let \( \psi \in C^\infty(Q_-) \) be a solution of equation (4.1) with \( b = b_0 \) such that \( \psi \in L_\infty(Q_-) \).

Then \( \psi = \text{const} \).

To prove Theorems 4.3 and 4.4, a full parabolic theory (with a version of the strong maximum principle, an estimate for oscillation or a Harnack inequality, and so on) must be developed. Since theorems in the second group have no direct connection with the Navier–Stokes equations, we do not dwell on a discussion of their proofs but refer the reader to the papers mentioned above.

Let us now proceed to the Liouville theorems that we put in the third group. Here is an example of such a result.

**Theorem 4.5.** Let \( b = b_0 + 2x/|x'| \), where \( b_0 \) satisfies the condition

\[
|b_0(x,t)| \leq \frac{A}{|x'|} \quad \forall (x,t) \in Q_-.
\]

Let \( \psi \in C^1(Q_-) \cap L_\infty(Q_-) \) be a solution of (4.1) such that

\[
\psi| \Gamma \times (-\infty,0) = 0, \quad \text{where} \quad \Gamma = \{ x \in \mathbb{R}^3 : x' = 0 \}.
\]

Then \( \psi \equiv 0 \) in \( Q_- \).

Below we give a proof of Theorem 4.5 which is based on ideas in [23]. Another proof, based on an essentially different approach, was found in [33]. It is interesting to note that when we impose the additional assumption that the solution vanishes on the axis, then the presence of the singular term turns out to play perhaps a positive role in the sense that the proof of a Liouville theorem is then simpler (shorter) than for theorems in the second category.

To prove Theorem 4.5 we use Lemma 4.1 again, this time taking \( \Omega \) and \( \Omega_0 \) to be

\[
\Omega(R) := \{ x \in \mathbb{R}^3 : 1 < |x'| < R, \ |x_3| < R \},
\]

\[
\Omega_0(R) := \left\{ x \in \mathbb{R}^3 : \frac{1}{2} < |x'| < 2R, \ |x_3| < 2R \right\}
\]

for some \( R > 1 \), and taking the intervals \( ]\tau, T[ \) and \( ]0, T[ \) to be

\[
I_R := \{ t \in \mathbb{R} : -R^2 < t < 0 \} \quad \text{and} \quad \tilde{I}_R := \{ t \in \mathbb{R} : -4R^2 < t < 0 \}.
\]
Note that \( \Omega(R) \subseteq \Omega_0(R) \) for any \( R > 1 \). Furthermore, for any \( R > 1 \) the drift \( b = b_0 + 2x'/|x'|^2 \) is smooth in the domain \( \Omega_0(R) \times \tilde{I}_R \), and therefore for the equation with this drift all the assumptions of Lemma 4.1 hold in \( \Omega_0(R) \times \tilde{I}_R \). For convenience we present the statement of an adaptation of this lemma to the new notation.

**Lemma 4.2.** Let \( b_0 \) be a smooth vector field on \( Q_- \). Then for any \( \varepsilon > 0 \) and \( R > 2 \) there exists a \( \delta = \delta(\varepsilon, R) > 0 \) such that for each solution \( \psi \in C^1(\Omega_0(R) \times \tilde{I}_R) \) of the equation

\[
\partial_t \psi - \Delta \psi + \left( b_0 + 2 \frac{x'}{|x'|^2} \right) \cdot \nabla \psi = 0 \tag{4.11}
\]

in the domain \( \Omega_0(R) \times \tilde{I}_R \) such that

\[
M := \sup_{\Omega_0(R) \times \tilde{I}_R} \psi > 0
\]

the following implication holds: if

\[
\sup_{\Omega(R) \times \{ t=0 \}} \psi > M(1 - \delta),
\]

then

\[
\text{osc}_{\Omega(R) \times I_R} \psi < \varepsilon M.
\]

Another tool that we need to prove Theorem 4.5 is the following estimate for the maximum of a solution in terms of its oscillation.

**Proposition 4.8.** Let \( b_0 \) be a smooth divergence-free vector field in \( Q_- \) with \( b_0 \in L_\infty(Q_-) \) and satisfying (4.8). Then there exist an absolute constant \( C_0 > 0 \) and a constant \( R_0 > 1 \) depending on \( A \) such that for any \( R \geq R_0 \) and any solution \( \psi \in C^1(\mathcal{D}(R)) \) of equation (4.11) in \( \mathcal{D}(R) \) with

\[
\sup_{\mathcal{D}(R)} \psi > 0 \quad \text{and} \quad \psi|_{\Gamma_R \times I_R} = 0,
\]

where \( \Gamma_R = \{ x \in \mathbb{R}^3 : x' = 0, \ |x_3| < R \} \), the estimate

\[
\sup_{\mathcal{D}(R)} \psi \leq C_0(1 + A) \text{osc}_{\Omega(R) \times I_R} \psi \tag{4.12}
\]

holds.

Recall that \( \mathcal{C}(R) \) and \( \mathcal{D}(R) \) are the sets

\[
\mathcal{C}(R) := \{ x \in \mathbb{R}^3 : |x'| < R, \ |x_3| < R \} \quad \text{and} \quad \mathcal{D}(R) := \mathcal{C}(R) \times ]-R^2, 0[.
\]

**Proof.** We present the proof of Proposition 4.8 in [23] (see also [41]). Let

\[
M := \sup_{\mathcal{D}(R)} \psi.
\]
Integrating by parts, we get from (4.11) and the condition \( \psi \big|_{\Gamma_R \times I_R} = 0 \) that for any \( \eta \in C_0^\infty (\mathcal{Q}(R)) \)
\[
\int_{\mathcal{Q}(R)} (M - \psi) \left( \partial_t \eta + \Delta \eta + \left( b_0 + 2 \frac{x'}{|x'|^2} \right) \cdot \nabla \eta \right) \, dx \, dt = M \int_{\mathcal{Q}(R)} \frac{x'}{|x'|^2} \cdot \nabla \eta \, dx \, dt.
\]

Let \( \eta_0 \) be a function such that
\[
\eta_0(x) = \varphi_1(|x'|) \varphi_2(x_3) \chi(t),
\]
where \( \varphi_1, \varphi_2 \in C_0^\infty (-R, R) \) and \( \chi \in C_0^\infty (-R^2, 0) \) are functions such that:
\[
0 \leq \varphi_1(r) \leq 1; \quad \varphi_1(r) = 1 \text{ for } |r| \leq \frac{R}{2}; \quad |\varphi_1'(r)| \leq \frac{C}{R}; \quad |\varphi_1''(r)| \leq \frac{C}{R^2};
\]
\[
0 \leq \varphi_2(z) \leq 1; \quad \varphi_2(z) = 1 \text{ for } |z| \leq \frac{R}{2}; \quad |\varphi_2'(z)| \leq \frac{C}{R}; \quad |\varphi_2''(z)| \leq \frac{C}{R^2};
\]
\[
0 \leq \chi(t) \leq 1; \quad \chi(t) = 1 \text{ for } t \in \left[ -\frac{3}{4} R^2, -\frac{1}{4} R^2 \right]; \quad |\chi'(t)| \leq \frac{C}{R^2}.
\]

Then \( \eta_0 \in C_0^\infty (\mathcal{Q}(R)) \) satisfies the relations
\[
0 \leq \eta_0(x, t) \leq 1 \quad \text{and} \quad \eta_0(x, t) = 1 \text{ for } (x, t) \in \mathcal{C} \left( \frac{R}{2} \right) \times \left( -\frac{3}{4} R^2, -\frac{1}{4} R^2 \right),
\]
and furthermore,
\[
\| \partial_t \eta_0 \|_{L^\infty(\mathcal{Q}(R))} + \| \nabla^2 \eta_0 \|_{L^\infty(\mathcal{Q}(R))} \leq \frac{C}{R^2} \quad \text{and} \quad \| \nabla \eta_0 \|_{L^\infty(\mathcal{Q}(R))} \leq \frac{C}{R}. \tag{4.14}
\]

In the identity (4.13) we set \( \eta = \eta_0 \) and find an estimate for the right-hand side bearing in mind the relation
\[
\int_{\mathcal{Q}(R)} \frac{x'}{|x'|^2} \cdot \nabla \eta_0 \, dx \, dt = -2\pi \int_{-R^2}^0 dt \int_{-R}^R \tilde{\eta}_0(0, z, t) \, dz
\]
( obtained by integrating by parts) and also the inequality
\[
\int_{-R^2}^0 dt \int_{-R}^R \tilde{\eta}_0(0, z, t) \, dz = \int_{-R}^R \varphi_2(z) \, dz \int_{-R^2}^0 \chi(t) \, dt \geq \frac{R^3}{2}.
\]

This gives us that
\[
M \left| \int_{\mathcal{Q}(R)} \frac{x'}{|x'|^2} \cdot \nabla \eta \, dx \, dt \right| \geq \frac{M}{2} R^3. \tag{4.15}
\]

We partition \( \mathcal{C}(R) \) as follows:
\[
\mathcal{C}(R) = \Omega(R) \cup E_1(R) \cup E_2(R),
\]
where
\[
E_1(R) := \left\{ x \in \mathbb{R}^3 : |x'| < 1, \ |x_3| < \frac{R}{2} \right\},
\]
\[
E_2(R) := \left\{ x \in \mathbb{R}^3 : |x'| < 1, \ \frac{R}{2} \leq |x_3| < R \right\}.
\]
Then the left-hand side of (4.13) can be represented as a sum of three terms $J_0 + J_1 + J_2$, where

$$
J_0 = \int_{\Omega(R) \times I_R} (M - \psi) \left( \partial_t \eta_0 + \Delta \eta_0 + \left( b_0 + 2 \frac{x'}{|x'|^2} \right) \cdot \nabla \eta_0 \right) \, dx \, dt,
$$

$$
J_1 = \int_{E_1(R) \times I_R} (M - \psi) \left( \partial_t \eta_0 + \Delta \eta_0 + \left( b_0 + 2 \frac{x'}{|x'|^2} \right) \cdot \nabla \eta_0 \right) \, dx \, dt,
$$

$$
J_2 = \int_{E_2(R) \times I_R} (M - \psi) \left( \partial_t \eta_0 + \Delta \eta_0 + \left( b_0 + 2 \frac{x'}{|x'|^2} \right) \cdot \nabla \eta_0 \right) \, dx \, dt.
$$

For $J_0$ we have

$$
|J_0| \leq C \frac{\text{osc}}{\Omega(R) \times I_R} \psi \left( \|\partial_t \eta_0\|_{L^\infty(\mathcal{Q}(R))} + \|\nabla^2 \eta_0\|_{L^\infty(\mathcal{Q}(R))} \right) |\mathcal{Q}(R)|
$$

$$
+ C \frac{\text{osc}}{\Omega(R) \times I_R} \psi \|\nabla \eta_0\|_{L^\infty(\mathcal{Q}(R))} \int_{\mathcal{Q}(R)} \left( |b_0| + \frac{1}{|x'|} \right) \, dx \, dt.
$$

We estimate the last term on the right-hand side, taking into account the assumption (4.8) and the identity

$$
\int_{\mathcal{Q}(R)} \frac{1}{|x'|} \, dx \, dt = 4\pi R^4.
$$

As a result, we obtain

$$
|J_0| \leq C(1 + A)R^3 \frac{\text{osc}}{\Omega(R) \times I_R} \psi. \quad (4.16)
$$

Now we estimate $J_1$ and $J_2$. Note that $\nabla \eta_0|_{E_1(R) \times I_R} = 0$. Hence for $J_1$ we get from (4.14) that

$$
|J_1| \leq CM \|\partial_t \eta_0\|_{L^\infty(\mathcal{Q}(R))} |E_1(R) \times I_R| \leq CM. \quad (4.17)
$$

Finally, for $J_2$ we obtain

$$
|J_2| \leq CM \left( \|\partial_t \eta_0\|_{L^\infty(\mathcal{Q}(R))} + \|\nabla^2 \eta_0\|_{L^\infty(\mathcal{Q}(R))} \right) |E_2(R) \times I_R|
$$

$$
+ CM \|\nabla \eta_0\|_{L^\infty(\mathcal{Q}(R))} \int_{E_2(R) \times I_R} \left( |b_0| + \frac{1}{|x'|} \right) \, dx \, dt.
$$

Since

$$
|E_2(R) \times I_R| \leq CR^3, \quad \int_{E_2(R) \times I_R} \left( |b_0| + \frac{1}{|x'|} \right) \, dx \, dt \leq C(1 + A)R^2,
$$

the inequalities (4.14) give us that

$$
|J_2| \leq C(1 + A)MR. \quad (4.18)
$$

Combining the estimates (4.15)–(4.18), we get from (4.13) that

$$
\frac{1}{2} MR^3 \leq C(1 + A)R^3 \frac{\text{osc}}{\Omega(R) \times I_R} \psi + C(1 + A)MR.
$$

Taking $R > R_0$, where $R_0$ satisfies the condition $C(1 + A)R_0^{-2} < 1/4$, we obtain (4.12), and Proposition 4.8 is proved. □
Now we can pass to the proof of Theorem 4.5.

**Proof of Theorem 4.5.** Let $C_0$ and $R_0$ be the constants defined in Proposition 4.8. Let $R = R_0$ and take $\varepsilon > 0$ such that

$$[C_0(1 + A) + 1]\varepsilon < 1. \tag{4.19}$$

For the fixed $\varepsilon$ and $R$, by using Lemma 4.2 we find that $\delta = \delta(\varepsilon, R) > 0$ and let $\delta_0 = \min\{\delta, \varepsilon\}$. Assume that

$$\sup_{Q_-} \psi = M > 0. \tag{4.20}$$

For this $\delta_0 > 0$ we find a point $\widehat{z} = (\widehat{x}, \widehat{t}) \in Q_-$ such that

$$\psi(\widehat{z}) > M(1 - \delta_0).$$

Choose $\lambda > 0$ so that

$$1 < \frac{1}{\lambda}(\widehat{x}_1^2 + \widehat{x}_2^2)^{1/2} < R.$$

We consider the functions

$$\psi^\lambda(x_1, x_2, x_3, t) := \psi(\lambda x_1, \lambda x_2, \widehat{x}_3 + \lambda x_3, \widehat{t} + \lambda^2 t)$$

and

$$b_0^\lambda(x_1, x_2, x_3, t) := \lambda b_0(\lambda x_1, \lambda x_2, \widehat{x}_3 + \lambda x_3, \widehat{t} + \lambda^2 t).$$

Then

$$\sup_{Q_-} \psi^\lambda = M.$$

Let

$$\widehat{x}^\lambda := \left(\frac{\widehat{x}_1}{\lambda}, \frac{\widehat{x}_2}{\lambda}, 0\right).$$

Then $\widehat{x}^\lambda \in \Omega(R)$, where $\Omega(R)$ is the set defined in (4.9), and therefore

$$\psi^\lambda(\widehat{x}^\lambda, 0) = \psi(\widehat{x}, \widehat{t}) \quad \text{and} \quad \sup_{\Omega(R) \times \{t = 0\}} \psi^\lambda > M(1 - \delta_0).$$

The functions $\psi^\lambda$ and $b_0^\lambda$ satisfy the equation

$$\partial_t \psi^\lambda - \Delta \psi^\lambda + \left(b_0^\lambda + \frac{x'}{|x'|^2}\right) \cdot \nabla \psi^\lambda = 0 \quad \text{in} \quad Q_-.$$

In addition, $|b_0^\lambda(x, t)| \leq A/|x'|$. We let

$$M_1 = \|\psi^\lambda\|_{L_\infty(\Omega_0(R) \times \tilde{I}_R)},$$

where $\Omega_0(R)$ and $\tilde{I}_R$ are the sets defined in (4.9) and (4.10), respectively. Then

$$M_1 \geq \psi^\lambda(\widehat{x}^\lambda, 0) > M(1 - \delta_0) > M_1(1 - \delta).$$
Consequently, by Lemma 4.2

\[ \text{osc}_{\Omega(R) \times I_R} \psi^\lambda < \varepsilon M_1 \leq \varepsilon M. \]

On the other hand, it follows from Proposition 4.8 that

\[ \sup_{\Omega(R) \times I_R} \psi^\lambda \leq C_0(1 + A) \quad \text{osc}_{\Omega(R) \times I_R} \psi^\lambda < C_0(1 + A)M\varepsilon. \]

Thus,

\[ M \leq \sup_{\Omega(R) \times I_R} \psi^\lambda + M\delta_0 < [C_0(1 + A) + 1]M\varepsilon. \]

This inequality contradicts (4.19). Hence the assumption (4.20) fails. Replacing \( \psi \) by \( -\psi \), we obtain \( M = 0 \). Theorem 4.5 is proved. \( \square \)

4.3. Applying parabolic theory to problems in hydrodynamics. We will show how the theorems in the previous subsection can be applied to the investigation of mild bounded ancient solutions of the Navier–Stokes equations, starting with the two-dimensional case.

**Theorem 4.6.** Let \( u \) be a mild bounded ancient solution of the Navier–Stokes equations in \( Q_- := \mathbb{R}^2 \times ]-\infty, 0[ \). Then \( u = \text{const} \).

**Proof.** Assume that the hypotheses of Theorem 4.6 are satisfied. Let

\[ \omega := u_{2,1} - u_{1,2}. \]

Then by Proposition 4.1

\[ \omega \in L^\infty(Q_-) \]

and all the assumptions of Theorem 4.1 hold for \( \psi := \omega \) and \( b := u \). Thus, from Theorem 4.1 we get that

\[ \omega = \text{curl} u = 0 \quad \text{in } Q_- \]

In combination with the condition \( \text{div } u = 0 \) in \( Q_- \) this means that \( \Delta u = 0 \) in \( Q_- \). Therefore, in this problem the relation \( u = \text{const} \) follows from the usual Liouville theorem for harmonic functions. \( \square \)

As a direct consequence of Theorem 4.6, in the two-dimensional case suitable weak solutions of the Navier–Stokes equations have no local singularities.

**Theorem 4.7.** Let \( B := \{x \in \mathbb{R}^2: |x| < 1\} \), and let \( u, p \) be a suitable weak solution of the Navier–Stokes equations in \( Q := B \times ]-1, 0[ \) such that

\[ u \in L^\infty(B \times ]-1, t_0[) \quad \text{for each } t_0 \in ]-1, 0[ \]

and

\[ u \in L^\infty(Q \setminus Q(R_1)) \quad \text{for some } R_1 \in ]0, 1[. \]

Then \( u \in L^\infty(Q) \).
Proof. If the solution $u$ had a singularity at a point $x_0 \in B(R_1)$ for $t = 0$, then by the two-dimensional analogue of Theorem 3.1 the ancient solution $u^\infty$ corresponding to that singularity would not be identically zero. On the other hand, we know from Theorem 4.6 that $u^\infty = c_0$ for some $c_0 \in \mathbb{R}^3$. Finally, in the two-dimensional case the energy norm of the solution itself is invariant under scaling transformations of the equation (for instance, see [40]). Hence, in two dimensions the ancient solution $u^\infty$ resulting from the procedure described in §3.2 must be in $L^\infty(-\infty,0;L^2(\mathbb{R}^2))$. This means that $c_0 = 0$, in contradiction to the fact that $u^\infty \not\equiv 0$. \qed

Let us now consider axially symmetric mild bounded ancient solutions of the Navier–Stokes equations whose velocity has no angular component. Liouville’s theorem holds also for this class.

**Theorem 4.8.** Let $u$ be a mild bounded ancient solution of the Navier–Stokes equations in $Q_- := \mathbb{R}^3 \times ]-\infty,0[$, and let $p$ be the associated pressure. Also assume that

$$u(x,t) = u_r(r,z,t)e_r + u_\varphi(r,z,t)e_\varphi + u_z(r,z,t)e_z, \quad \text{and} \quad p(x,t) = p(r,z,t).$$

Then $u = \text{const}$.

**Proof.** Let

$$\omega := \text{curl } u, \quad \omega(x,t) = \omega_\varphi(r,z,t)e_\varphi, \quad \text{and} \quad \eta(x,t) := \frac{\omega_\varphi(r,z,t)}{r}.$$

Then it follows from Proposition 4.2 that $\omega_\varphi, \eta \in L^\infty(Q_-)$ and $|\eta| \leq C/r$, and therefore the functions $\psi := \eta$ and $b := u - 2x'/|x'|^2$ fulfill all the assumptions of Theorem 4.2. Hence from Theorem 4.2 we get that $\eta = 0$ in $Q_-$, so that curl $u = 0$ in $Q_-$. In combination with the condition div $u = 0$ in $Q_-$ this means that $\Delta u = 0$ in $Q_-$. Hence, as in the previous case, the relation $u = \text{const}$ follows again from the standard Liouville theorem for harmonic functions. \qed

It is interesting to note that, in contrast to the two-dimensional case, Theorem 4.8 does not imply directly that axially symmetric suitable weak solutions of the Navier–Stokes equations whose velocity has no angular component cannot have local singularities (the corresponding result can be established using other methods; for instance, see [26] and [30]). As before, using Theorem 3.1 we can construct a non-trivial (not identically zero) ancient solution $u^\infty$ that corresponds to the local singularities in this problem, and Theorem 4.8 ensures that $u^\infty = c_0$, where $c_0 \in \mathbb{R}^3$. However, in contrast to the two-dimensional case, we cannot say that $c_0 = 0$ (which would give a contradiction) because in an axially symmetric problem scaling transformations do not preserve energy norms, so this time $u^\infty$ is not necessarily an energy solution.

Finally, we show how Theorem 4.5 can be applied to problems in hydrodynamics.

**Theorem 4.9.** Let $u$ be a mild bounded ancient solution of the Navier–Stokes equations in $Q_- := \mathbb{R}^3 \times ]-\infty,0[$, and let $p$ be the associated pressure. Also assume that

$$u(x,t) = u_r(r,z,t)e_r + u_\varphi(r,z,t)e_\varphi + u_z(r,z,t)e_z \quad \text{and} \quad p(x,t) = p(r,z,t).$$
Furthermore, suppose that
\[ |u(x, t)| \leq \frac{A}{|x'|}, \quad (x, t) \in Q_- . \tag{4.21} \]
Then \( u \equiv 0 \) in \( Q_- \).

Proof. Let
\[ \xi(x, t) := ru_r \varphi(r, z, t) . \]
Then it follows from (4.21) that \( \xi \in L_\infty(Q_-) \), and it follows from Proposition 4.3 that the functions \( \psi := \xi \) and \( b := u + 2x'/|x'|^2 \) satisfy (4.1). Also, \( \xi|_{\Gamma \times (-\infty, 0)} = 0 \), so all the assumptions of Theorem 4.5 hold for \( \psi = \xi \) and \( b_0 = u \). Therefore, from Theorem 4.5 we get that \( \xi \equiv 0 \) in \( Q_- \), so that \( u_\psi \equiv 0 \). Now Theorem 4.9 follows from Theorem 4.8. \( \square \)

Theorem 4.9 enables us to prove the local smoothness of axially symmetric suitable weak solutions of the Navier–Stokes equations in the case of type-I blowups that are characterized by the condition
\[ |u(x, t)| \leq C |x'|, \quad |u(x, t)| \leq C \sqrt{-t} , \quad (x, t) \in Q. \tag{4.22} \]
This was first proved in [47]. Here we present the corresponding result. Note that if an axially symmetric solution \( u \) has a singular point in \( Q \), then by Theorem 2.5 this point can lie only on the axis \( x' = 0 \), so that by using Theorem 2.8 we can reduce the local regularity question for the axially symmetric solution \( u \) to a study of the model problem ‘up to the first singularity’, which also has axial symmetry.

**Theorem 4.10.** Let \( u, p \) be a suitable weak solution of the Navier–Stokes equations in \( Q \) such that
\[ u \in L_\infty(B \times ]-1, t_0[) \quad \text{for each } t_0 \in ]-1, 0[ \]
and
\[ u \in L_\infty(Q \setminus Q(R_1)) \quad \text{for some } R_1 \in ]0, 1[ . \]
Also, let the functions \( u \) and \( p \) be axially symmetric and let \( u \) satisfy at least one of the estimates in (4.22). Then \( u \in L_\infty(Q) \).

**Proof.** It was shown in [47], Proposition 3.7, that the second condition in (4.22) implies the first, so we can assume without loss of generality that \( u \) satisfies the first estimate in (4.22). Reasoning by contradiction, we let \((0, 0)\) be a singular point of \( u \). Using Theorem 3.1, we construct a non-trivial mild bounded ancient solution \( u^\infty \) corresponding to this singularity. Since \( u \) and \( p \) are axially symmetric, we see from Theorem 3.2 that either \( u^\infty \) and \( p^\infty \) are axially symmetric relative to some axis \( x' = x'_0 \), or there exists an \( l \in \mathbb{R}^3 \) with \( |l| = 1 \) and \( l = (l_1, l_2, 0) \) such that \( \partial u^\infty / \partial l = 0 \) and \( \partial p^\infty / \partial l = 0 \). Furthermore, if the first condition in (4.22) holds, then by the scale invariance of this condition, \( u^\infty \) also has the estimate
\[ |u^\infty(x, t)| \leq \frac{C}{|x'|} , \quad (x, t) \in Q_- . \tag{4.23} \]
If $u^\infty$ and $p^\infty$ are axially symmetric relative to the axis $x' = x'_0$, then by (4.23) the functions

$$v^\infty(x, t) := u^\infty(x - x_0, t) \quad \text{and} \quad q^\infty(x, t) = p^\infty(x - x_0, t)$$

satisfy all the assumptions of Theorem 4.9. Hence $v^\infty \equiv 0$, that is, $u^\infty \equiv 0$, which means that the original solution $u$ cannot have a singularity.

If $\partial u^\infty / \partial l = 0$ and $\partial p^\infty / \partial l = 0$ (we can assume without loss of generality that $u^\infty$ and $p^\infty$ are independent of $x_1$), then the three-dimensional Navier–Stokes system splits into the two-dimensional Navier–Stokes system for the functions $u_2^\infty(x_2, x_3, t)$, $u_3^\infty(x_2, x_3, t)$, and $p^\infty(x_2, x_3, t)$ and a separate parabolic equation for $u_1^\infty(x_2, x_3, t)$:

$$\partial_t u_1^\infty + u_2^\infty u_{1,2}^\infty + u_3^\infty u_{1,3}^\infty - u_{1,22}^\infty - u_{1,33}^\infty = 0.$$  \hfill (4.24)

By Theorem 4.6, $u_2^\infty \equiv 0$ and $u_3^\infty \equiv 0$. From the scalar equation (4.24), use of Theorem 4.1 and (4.22) gives us that $u_1^\infty \equiv 0$. In any case we arrive at a contradiction to the assumption that $u^\infty$ is not identically equal to zero. $\square$

### 4.4. Methods of duality theory.

Duality methods are based on considering, instead of the original system for an ancient (backwards-in-time) solution of the Navier–Stokes equations, the dual (forwards-in-time) linear system involving the original solution as a coefficient. Then the question of whether the original ancient solution is trivial is transformed into the question of the rate of decay in time of various norms of the solution of the Cauchy problem for the dual system. The rate of decay of solutions of the heat equation is optimal in a certain sense. However, adding lower-order terms with various drifts can ‘damage’ this rate. By imposing various conditions on the ancient solution (which is the drift in the dual system) and investigating the asymptotic behaviour in time of solutions of the dual system we can find new classes of ancient solutions in which a Liouville theorem holds.

This scheme enjoyed its first success in [40], in investigations of the two-dimensional problem in a half-space (subsequently, the approach proposed there was developed in [37]). The problem of the Navier–Stokes equations in a half-space is different from the problem in the whole space in the following respect: in passing to the curl form of the equations one ‘loses’ the boundary conditions. As we did in §4.1 for problems in the whole space, in some half-space problems we can also distinguish a ‘governing’ scalar equation. However, the absence of a boundary condition in the half-space case makes the ‘governing’ scalar equation ineffective and the methods described above cannot be used to prove corresponding Liouville theorems.

Thus, the half-space problem does not yield to standard methods of investigation that reduce the problem to a scalar one: new approaches must be developed. One possible approach is based on duality theory. Here we outline its general scheme.

Let $u$, $p$ be a mild bounded ancient solution of the Navier–Stokes equations

$$\begin{align*}
\partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p &= 0, \\
\text{div } u &= 0,
\end{align*} \quad \text{in } Q_-, \quad (4.25)$$
and let $F \in C_0^\infty(Q_+)$ be an arbitrary tensor field, $F = (F_{ij})$, where $Q_+ = \mathbb{R}^3 \times ]0, +\infty[$. Let

$$\tilde{u}(x, t) = -u(x, -t) \quad \text{and} \quad \tilde{F}(x, t) = -F(x, -t), \quad (x, t) \in Q_+,$$

and let $v, q$ be the solution of the Cauchy problem for the linear Stokes system with a drift $\tilde{u}$

$$\begin{cases}
\partial_t v - \Delta v + (\tilde{u} \cdot \nabla)v + \nabla q = \text{div} \tilde{F}, \\
\text{div} v = 0 \\
v|_{t=0} = 0.
\end{cases} \quad \text{in } Q_+, \quad (4.26)$$

Then the functions

$$\tilde{v}(x, t) := v(x, -t) \quad \text{and} \quad \tilde{q}(x, t) = q(x, -t), \quad (x, t) \in Q_-,$$

satisfy the equations

$$\begin{cases}
\partial_t \tilde{v} - \Delta \tilde{v} - (u \cdot \nabla)\tilde{v} + \nabla \tilde{q} = -\text{div} F, \\
\text{div} \tilde{v} = 0 \\
\tilde{v}|_{t=0} = 0.
\end{cases} \quad \text{in } Q_-, \quad (4.27)$$

If the spatial decay of the functions $u, v$ and $p, q$ is such that the integration by parts formulae

$$\int_{\mathbb{R}^3} (-\Delta \tilde{v} - (u \cdot \nabla)\tilde{v}) \cdot u \, dx = \int_{\mathbb{R}^3} (-\Delta u + (u \cdot \nabla)u) \cdot \tilde{v} \, dx,$$

$$\int_{\mathbb{R}^3} \nabla p \cdot \tilde{v} \, dx = 0, \quad \int_{\mathbb{R}^3} \nabla \tilde{q} \cdot u \, dx = 0 \quad (4.28)$$

hold for almost all $t \in ]-\infty, 0[$, then by multiplying (4.25) and (4.27) by $\tilde{v}$ and $u$, respectively, and comparing the left-hand sides we arrive at the equality

$$\frac{d}{dt} \int_{\mathbb{R}^3} u \cdot \tilde{v} \, dx = \int_{\mathbb{R}^3} u \cdot \text{div} F \, dx \quad \text{for almost all } t \in ]-\infty, 0[.$$

Taking $T > 0$ sufficiently large (so that $\text{supp} F \subset \mathbb{R}^3 \times ]-T, 0[$) and integrating the relation obtained with respect to $t \in ]-T, 0[$, we get by the condition $\tilde{v}|_{t=0} = 0$ that

$$\int_{Q_-} u \cdot \text{div} F \, dx = -\int_{\mathbb{R}^3} u(x, -T) \cdot \tilde{v}(x, -T) \, dx = \int_{\mathbb{R}^3} \tilde{u}(x, T) \cdot v(x, T) \, dx. \quad (4.29)$$

If the properties of the coefficient $\tilde{u}$ in (4.26) are such that

$$\int_{\mathbb{R}^3} \tilde{u}(x, T) \cdot v(x, T) \, dx \to 0 \quad \text{as } T \to +\infty, \quad (4.30)$$

then from (4.29) we obtain the identity

$$\int_{Q_-} u \cdot \text{div} F \, dx = 0 \quad \forall F \in C_0^\infty(Q_-).$$
which means that \( u(x, t) = b(t) \) for all \( x \in \mathbb{R}^3 \), where \( b \in L_\infty(-\infty, 0) \) is some function. However, \( u \) is a mild bounded ancient solution of the Navier–Stokes equations, and therefore we must have \( b = \text{const} \).

Thus, the question of the validity of Liouville’s theorem for a mild bounded ancient solution \( u \) of the system (4.25) reduces to the question of whether (4.30) holds for solutions of the system (4.26). Note that in passing we used the assumption that solutions of (4.26) tend to zero appropriately with respect to the space variables (so that we can integrate by parts as in the formulae (4.28)). In practice it turns out that for a broad class of ancient solutions (that is, the rate of the drift \( \bar{u} \) in the problem (4.26)) it is fairly easy to verify (4.28), and the main difficulty lies with proving the convergence (4.30), that is, in the question of the rate of decay as \( t \to +\infty \) of various norms of the solution \( v \) of (4.26) for a given drift \( \bar{u} \).

5. Problem in a half-space

We pointed out already that the main difficulty in dealing with the Navier–Stokes equations in a half-space (or in any domain), in contrast to the problem in the whole space, is that in passing to the equations in the curl form we ‘lose’ the boundary conditions. As a result, many of the methods developed in previous sections for analysing ancient solutions in the whole space cannot be applied to ancient solutions in a half-space.

5.1. The Stokes problem in a half-space. Let \( Q^+_\pm := \mathbb{R}^3_+ \times ]-\infty, 0[ \). In this subsection we present results in [18] and [19] on bounded ancient solutions of the linear Stokes system in a half-space

\[
\begin{aligned}
\partial_t u - \Delta u + \nabla p &= 0, \\
\text{div } u &= 0, \\
|u|_{x_3=0} &= 0.
\end{aligned}
\tag{5.1}
\]

We start by defining a bounded ancient solution of this problem.

**Definition 5.1.** A function \( u \in L_\infty(Q^+_\pm) \) is called a bounded ancient solution of the problem (5.1) if it satisfies the identities

\[
\int_{Q^+_\pm} u \cdot (\partial_t \eta + \Delta \eta) \, dx \, dt = 0 \quad \forall \eta \in C_0^\infty(Q^-) : \quad \text{div } \eta = 0, \quad \eta|_{x_3=0} = 0
\]

and

\[
\int_{\mathbb{R}^3_+} u(x, t) \cdot \nabla \varphi(x) \, dx = 0 \quad \forall \varphi \in C_0^\infty(\mathbb{R}^3) \quad \text{for almost all } t < 0.
\]

An exhaustive description of this class of ancient solutions was given in [18] and [19].

**Theorem 5.1.** Each bounded ancient solution \( u \) of the problem (5.1) has the form

\[
u(x, t) = (b_1(x_3, t), b_2(x_3, t), 0)
\]

for some functions \( b_1, b_2 \in L_\infty(\mathbb{R}^3_+ \times ]-\infty, 0[) \).
5.2. Ancient solutions in a half-space. By analogy with the linear Stokes problem we define bounded ancient solutions of the Navier–Stokes equations as follows.

**Definition 5.2.** A function \( u \in L_\infty(Q_+^-) \) is called a bounded ancient solution of the Navier–Stokes equations in \( Q_+^- \) if it satisfies the identities

\[
\int_{Q_+^-} \left( u \cdot (\partial_t \eta + \Delta \eta) + u \otimes u : \nabla \eta \right) dx dt = 0
\]

\( \forall \eta \in C_0^\infty(Q_-) : \ \text{div} \ \eta = 0, \ \eta|_{x_3=0} = 0, \)

and

\[
\int_{\mathbb{R}_+^3} u(x, t) \cdot \nabla \varphi(x) \ dx = 0
\]

\( \forall \varphi \in C_0^\infty(\mathbb{R}^3) \) for almost all \( t < 0. \)

As in the case of the whole space, from the standpoint of the validity of Liouville-type theorems, the class of bounded ancient solutions in a half-space is too broad: it certainly contains solutions that are not identically constant. Therefore, we introduce a narrower class of ancient solutions in a half-space, mild bounded ancient solutions.

**Definition 5.3.** A function \( u \in L_\infty(Q_+^-) \) is called a mild bounded ancient solution of the Navier–Stokes equations in \( Q_+^- \) if for all \( t \) with \( t_0 < t < 0 \) it satisfies the identity

\[
u_i(x, t) = \int_{\mathbb{R}^3} G_{ij}(x, y, t - t_0) u_j(y, t_0) dy + \int_{t_0}^t \int_{\mathbb{R}^3} K_{ijk}(x, y, t - \tau) u_j(y, \tau) u_k(y, \tau) dy d\tau, \quad (5.2)
\]

where \((G_{ij})\) is the tensor defined by the formulae

\[
G(x, y, t) = G^1(x, y, t) + G^2(x, y, t),
\]

\[
G^1_{ij}(x, y, t) = \delta_{ij} (\Gamma(x - y, t) - \Gamma(x - y^*, t)),
\]

\[
G^2_{ij}(x, y, t) = 4 \frac{\partial}{\partial x_\beta} (x - z) \frac{\partial \varphi}{\partial x_i}(x - z) \Gamma(z - y^*, t) dz, \quad \beta = 1, 2,
\]

\[
y^* = (y', -y_3), \quad \text{and} \quad K = (K_{ijk}) \text{ is the tensor defined by}
\]

\[
K_{ijk}(x, y, t) = \frac{\partial^3 \Phi_{ij}}{\partial y_\beta \partial y_\gamma \partial y_k}(x, y, t) - \frac{\partial^3 \Phi_{is}}{\partial y_\gamma \partial y_j \partial y_k}(x, y, t),
\]
where the components of the tensor \( \Phi = (\Phi_{ij}) \) are by definition the solutions of the following boundary-value problems:

\[
\begin{align*}
\Delta_y \Phi_{i\beta}(x, y, t) &= G_{i\beta}(x, y, t) \quad \text{in } Q_+^+, \\
\frac{\partial \Phi_{i\beta}}{\partial y_3} \bigg|_{y_3=0} &= 0, \quad \beta = 1, 2; \\
\Phi_{i\beta}(x, y, t) &\to 0 \quad \text{as } |y| \to +\infty,
\end{align*}
\]

\[
\begin{align*}
\Delta_y \Phi_{i3}(x, y, t) &= G_{i3}(x, y, t) \quad \text{in } Q_-^+, \\
\Phi_{i3}(x, y, t) &\to 0 \quad \text{as } |y| \to +\infty.
\end{align*}
\]

Note that (5.2) is based on a representation in [53] for solutions of the Stokes problem in a half-space (see also [54]).

The class of mild bounded ancient solutions in a half-space was characterized in [2]; this characterization plays the same role for the problem in a half-space that Theorem 1.3 plays for the problem in the whole space.

**Theorem 5.2.** A function \( u \in L_\infty(Q_-^+) \) is a mild bounded ancient solution of the Navier–Stokes system if and only if it is a bounded ancient solution of the Navier–Stokes system in \( Q_-^+ \) in the sense of Definition 5.2, there exists a function \( p = p^1 + p^2 \) such that the pair \( u, p \) satisfies

\[
\int_{Q_-^+} (u \cdot (\partial_t \eta + \Delta \eta) + u \otimes u : \nabla \eta + p \text{div} \eta) \, dx \, dt = 0
\]

\[\forall \eta \in C_0^\infty(Q_-) : \text{div} \eta = 0, \quad \eta|_{x_3=0} = 0,\]

and the following conditions hold:

1) \( p^1 = p_{u \otimes u} \), where for almost all \( t \in ]-\infty, 0[ \) the function \( p_{u \otimes u} \in L_\infty(-\infty, 0; \text{BMO}(\mathbb{R}_3^+)) \) satisfies

\[
\int_{\mathbb{R}_3^+} p_{u \otimes u} \Delta \varphi \, dx = - \int_{\mathbb{R}_3^+} u \otimes u : \nabla^2 \varphi \, dx \quad \forall \varphi \in C_0^\infty(\mathbb{R}^3) : \quad \frac{\partial \varphi}{\partial x_3} \bigg|_{x_3=0} = 0; \quad (5.3)
\]

2) for almost all \( t \in ]-\infty, 0[ \) the function \( p^2(\cdot, t) \) is harmonic in \( \mathbb{R}_3^+ \) and

\[
|\nabla p^2(x, t)| \leq c \log \left(2 + \frac{1}{x_3}\right) \quad \forall (x, t) \in Q_-^+,
\]

and moreover,

\[\forall t \in ]-\infty, 0[ \sup_{x' \in \mathbb{R}^2} |\nabla p^2(x, t)| \to 0 \quad \text{as } x_3 \to +\infty.\]

In addition, \( u \) is \( C^\infty \)-smooth in \( \mathbb{R}_3^+ \times ]-\infty, 0[ \), and for any \( l, m = 0, 1, 2, \ldots \)

\[
\|\partial_t^l \nabla^m u\|_{L_\infty(Q_-^+)} + \|\partial_t^l \nabla^{m+1} p\|_{L_\infty(Q_-^+)} + \|\partial_t^l p_{u \otimes u}\|_{L_\infty(-\infty, 0; \text{BMO}(\mathbb{R}_3^+))} < +\infty.
\]
Thus, as in the case of the spatial problem, it is characteristic for bounded ancient solutions in a half-space that belong to the class of mild bounded ancient solutions that we can associate with them a pressure in a certain class. For \( u \in L^\infty(Q^+) \) the relation (5.3) determines a unique function \( p_{u \otimes u} \) in \( L^\infty(-\infty, 0; \text{BMO}(\mathbb{R}^3_+)) \) with the normalization

\[
\int_{B^+} p_{u \otimes u}(x, t) \, dx = 0 \quad \text{for almost all } t < 0,
\]

and this function satisfies the estimate

\[
\|p_{u \otimes u}\|_{L^\infty(-\infty, 0; \text{BMO}(\mathbb{R}^3_+))} \leq C\|u\|^2_{L^\infty(Q^+)}
\]

for some absolute positive constant \( C \). By analogy with the problem in the whole space we call \( p_{u \otimes u} \) the pressure associated with the mild bounded ancient solution \( u \) in a half-space.

### 5.3. Connection between boundary regularity and ancient solutions in a half-space.

The paper [48] contains an analogue of Theorem 3.1 for a half-space. It turns out that, depending on the asymptotic behaviour of the solution in a neighbourhood of a singular point, there are two types of ancient solutions that can correspond to singularities of suitable weak solutions of the Navier–Stokes equations: either the mild bounded ancient solutions in the whole space (which we have already considered earlier), or the mild bounded ancient solutions in a half-space described in this section. We explain this in greater detail.

Let \( u, p \) be a solution of the initial-boundary value problem

\[
\begin{aligned}
\partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p &= 0, & & \text{in } \mathbb{R}^3_+ \times ]0, T[, \\
\text{div } u &= 0, & & \text{in } \mathbb{R}^3_+ \times ]0, T[, \\
u_{x_3=0} &= 0, & & u_{t=0} = a,
\end{aligned}
\]

and for simplicity suppose that \( a \in C^\infty(\mathbb{R}^3_+) \) and \( \text{div } a = 0 \). Assume that \( u \) and \( p \) are smooth functions on \( \mathbb{R}^3_+ \times [0, t_0] \) for each \( t_0 < T \), where \( T < +\infty \) is the blowup time for this solution:

\[
\lim_{t \to T^-} \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^3_+)} = +\infty.
\]

Setting

\[
M(t) := \sup_{\tau \in [0, t]} \|u(\cdot, \tau)\|_{L^\infty(\mathbb{R}^3_+)}, \quad t \in ]0, T[,
\]

we construct sequences \( t_k \nearrow T \) and \( x_k \in \mathbb{R}_+^3 \) such that

\[
\|u(x_k, t_k)\| = M(t_k) \to +\infty \quad \text{as } k \to \infty.
\]

Note that, as follows from [5], the sequence \( \{x_k\} \) is bounded. Setting

\[
\lambda_k := \frac{1}{M(t_k)}, \quad \lambda_k \to 0 \quad \text{as } k \to \infty,
\]

for simplicity suppose that \( a \in C^\infty(\mathbb{R}^3_+) \) and \( \text{div } a = 0 \). Assume that \( u \) and \( p \) are smooth functions on \( \mathbb{R}^3_+ \times [0, t_0] \) for each \( t_0 < T \), where \( T < +\infty \) is the blowup time for this solution:

\[
\lim_{t \to T^-} \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^3_+)} = +\infty.
\]

Setting

\[
M(t) := \sup_{\tau \in [0, t]} \|u(\cdot, \tau)\|_{L^\infty(\mathbb{R}^3_+)}, \quad t \in ]0, T[,
\]

we construct sequences \( t_k \nearrow T \) and \( x_k \in \mathbb{R}_+^3 \) such that

\[
\|u(x_k, t_k)\| = M(t_k) \to +\infty \quad \text{as } k \to \infty.
\]

Note that, as follows from [5], the sequence \( \{x_k\} \) is bounded. Setting

\[
\lambda_k := \frac{1}{M(t_k)}, \quad \lambda_k \to 0 \quad \text{as } k \to \infty,
\]
we define the functions
\[ u^k(x, t) := \lambda_k u(x_k + \lambda_k x, t_k + \lambda_k^2 t) \quad \text{and} \quad p^k(x, t) := \lambda_k^2 p(x_k + \lambda_k x, t_k + \lambda_k^2 t) \] (5.7)
on the set \( \mathbb{R}^3_+ \times ]-\lambda_k^2(1 + t_k), -\lambda_k^2 t_k[ \). We extend \( u^k \) and \( p^k \) to the whole of \( Q^+_\) by zero. For any \( R > 0 \), if \( R \) is sufficiently large, then
\[ Q^+(R) := B^+(R) \times ]-R^2, 0[ \subset \mathbb{R}^3_+ \times ]-\lambda_k^2(1 + t_k), -\lambda_k^2 t_k[ , \]
where the functions \( u^k \) are Hölder continuous on \( \overline{Q}^+(R) \). Furthermore, for any \( R > 0 \), when \( k \) is sufficiently large the functions \( u^k \) and \( p^k \) have the following properties:

\( u^k \) and \( p^k \) satisfy the Navier–Stokes equations in \( \mathbb{R}^3_+ \times ]-R^2, 0[ , \)
\[ \|u^k\|_{L_\infty(Q_-)} = 1, \quad |u^k(0, 0)| = 1. \]

The following result was proved in [48].

**Theorem 5.3.** Assume that the initial-boundary value problem (5.4) has a smooth solution \( u, p \) in the domain \( \mathbb{R}^3_+ \times ]0, T[ , \) and let \( T < +\infty \) be the blowup time for this solution. Let \( u^k \) and \( p^k \) be sequences defined by (5.7) with parameters
\[ x_k = (x^k_1, x^k_2, x^k_3) \in \mathbb{R}^3_+, \quad t_k \nearrow T, \] and \( \lambda_k \searrow 0 \)
that satisfy (5.5) and (5.6). Then:

1) if \( \lambda_k^{-1} x^k_3 \to +\infty \), then there exists a subsequence \( \{u^{k_j}\} \) converging uniformly on compact subsets of \( \mathbb{R}^3 \times ]-\infty, 0[ \) to a mild bounded ancient solution \( u^\infty \) of the Navier–Stokes equations in \( Q_- \) such that
\[ |u^\infty(0, 0)| = \|u^\infty\|_{L_\infty(Q_-)} = 1; \]

2) if \( \lambda_k^{-1} x^k_3 \to x^0_3 \in [0, +\infty[ , \) then there exists a subsequence \( \{u^{k_j}\} \) converging uniformly on compact subsets of \( \mathbb{R}^3_+ \times ]-\infty, 0[ \) to a mild bounded ancient solution \( u^\infty \) of the Navier–Stokes equations in \( Q^-_\) such that
\[ |u^\infty(0, x^0_3, 0)| = \|u^\infty\|_{L_\infty(Q^-_)} = 1. \]

**5.4. Liouville theorems for a half-space.** At the time of writing this survey the authors knew of only two works on Liouville theorems for the non-stationary Navier–Stokes equations in a half-space. In both only the two-dimensional case was considered. Therefore, in this subsection we assume that
\[ \mathbb{R}^2_+ := \{(x_1, x_2) \in \mathbb{R}^2, \ x_2 > 0\}, \quad Q^+_\ := \mathbb{R}^2_+ \times ]-\infty, 0[ , \]
and the functions \( u: Q^-_\to \mathbb{R}^2 \) and \( p: Q^+_\to \mathbb{R} \) satisfy the two-dimensional Navier–Stokes system in \( Q^+_\)
\[ \begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = 0, & \text{in } Q^+_\, , \\ \text{div } u = 0 \\ u|_{x_2=0} = 0. \end{cases} \]

In [40] the duality method (see §4.4 here) was used to establish the Liouville theorem. We present the result obtained there.
Theorem 5.4. Let \( u \) be a mild bounded ancient solution of the Navier–Stokes equations in \( Q^+ \). Also, assume that
\[
\|u\|_{L^\infty(-\infty, 0; L^2(\mathbb{R}^2_+))} < \infty.
\]
Then \( u \equiv 0 \).

The result in [14] is also related to the two-dimensional case. We give its statement, adapting it to our definition of mild bounded ancient solutions.

Theorem 5.5. Let \( u \) be a mild bounded ancient solution of the Navier–Stokes equations in \( Q^+ \). Also, assume that there exists a constant \( C > 0 \) such that
\[
\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^2_+)} \leq \frac{C}{\sqrt{-t}} \quad \forall t \in ]-\infty, 0[.
\]
and moreover,
\[
\omega(x, t) > 0 \quad \forall (x, t) \in Q^+_+, \text{ where } \omega := u_{2,1} - u_{1,2}.
\]
Then \( u \equiv 0 \).

We see that both Theorems 5.4 and 5.5 are based on the assumption of finiteness of this or that scale-invariant norm of a solution (the assumptions (5.8) and (5.9), respectively). Moreover, [14] also uses the assumption that the curl of the solution has constant sign.

6. Liouville theorems for stationary Navier–Stokes equations

In contrast to Liouville theorems for ancient solutions of the Navier–Stokes equations, the corresponding theorems for these equations in the stationary case are not directly connected with the study of local singularities of weak solutions. Moreover, in the case of ancient solutions the assumption that a global solution is bounded is a natural consequence of the procedure described above of scaling transformations of local weak solutions with specially chosen scaling parameters. On the other hand, in the stationary case such an assumption has no similar motivation and is simply postulated. Nevertheless, Liouville theorems for the Navier–Stokes equations in the stationary case can be of independent theoretical interest, and below we briefly list some of the presently known results in this direction.

6.1. Various statements of Liouville-type problems for stationary Navier–Stokes equations.

Consider in \( \mathbb{R}^3 \) the stationary Navier–Stokes system
\[
\begin{align*}
-\Delta u + (u \cdot \nabla)u + \nabla p &= 0, \\
\text{div } u &= 0
\end{align*}
\]
in \( \mathbb{R}^3 \). (6.1)

It is known (for instance, see [27]) that if a weak solution \( u, p \) of this system has locally finite energy, then it is \( C^\infty \)-smooth. There are different ways to pose the question of whether globally bounded solutions of this system are trivial. We present the two most popular forms of such a question (see [13], § X.9).
**Question 1.** Let $u, p$ be a smooth solution of (6.1), and assume that

$$u \in L_\infty(\mathbb{R}^3).$$

Is it true that $u = \text{const}$?

**Question 2.** Let $u, p$ be a smooth solution of (6.1), and assume that

$$\int_{\mathbb{R}^3} |\nabla u|^2 \, dx < +\infty,$$

$$u(x) \to 0 \quad \text{as} \quad |x| \to +\infty.$$

Is it true that $u \equiv 0$?

At the time this survey was written both questions were open. The following result (see [13], Theorem X.9.5) will be a starting point for us.

**Theorem 6.1.** Let $u, p$ be a smooth solution of (6.1), and assume that

$$u \in L_9/2(\mathbb{R}^3).$$

Then $u \equiv 0$.

Recently the result in Theorem 6.1 was improved in [8] by involving a logarithmic factor.

**6.2. Stationary linear systems with a drift.** In this subsection we present the known (as of today) results on Liouville theorems for the stationary Stokes system with a smooth divergence-free drift $b$:

$$
\begin{align*}
-\Delta u + (b \cdot \nabla)u + \nabla p &= 0, \\
\text{div } u &= 0
\end{align*}
$$

in $\mathbb{R}^3$. (6.2)

We are guided by the results established in [46] and [33] for a scalar elliptic equation. We combine these into the following theorem.

**Theorem 6.2.** Let $b: \mathbb{R}^n \to \mathbb{R}^n$ be a smooth divergence-free vector field that satisfies at least one of the two conditions

$$b \in \text{BMO}^{-1}(\mathbb{R}^n)$$

or

$$\exists q \in \left[ \frac{n}{2}, n \right]: \sup_{1 < R < +\infty} R^{1-n/q} \|b\|_{L_q(B(R))} < +\infty,$$

and let $w: \mathbb{R}^n \to \mathbb{R}$ be a smooth scalar function such that

$$-\Delta w + b \cdot \nabla w = 0 \quad \text{in } \mathbb{R}^n.$$

If $w \in L_\infty(\mathbb{R}^n)$, then $w \equiv \text{const}$.

For (6.2) the following result was proved in [43] and [42].
Theorem 6.3. Let \( b : \mathbb{R}^n \to \mathbb{R}^n \) be a smooth divergence-free vector field that satisfies at least one of the two conditions

\[ b \in \text{BMO}^{-1}(\mathbb{R}^3) \]

or

\[ \exists q \in \left[ \frac{3}{2}, 3 \right] : \sup_{1 < R < +\infty} R^{1-3/q} \| b \|_{L_q(w(B(R))} < +\infty. \]

Let \( u : \mathbb{R}^3 \to \mathbb{R}^3 \) and \( p : \mathbb{R}^3 \to \mathbb{R} \) be smooth functions satisfying (6.2), and assume that

\[ \exists s \in [2, 6] : \sup_{1 < R < +\infty} R^{1/2-3/s} \| u \|_{L_s(B(R))} < +\infty. \]  

(6.3)

If \( u \in L_\infty(\mathbb{R}^3) \), then \( u \equiv \text{const} \).

We see that, in comparison with the scalar case of Theorem 6.2 the additional assumption (6.3) is imposed on the solution in Theorem 6.3.

6.3. Liouville-type theorems for the stationary Navier–Stokes equations.

In this subsection we state results on Liouville theorems for the stationary Navier–Stokes system (6.1). It is known (for instance, see [27]) that each weak solution of this system is smooth. So throughout the subsection \( u \) and \( p \) will be smooth functions in \( \mathbb{R}^3 \).

The following result was proved in [44], but we do not present it in the most general form, but only in the special cases of greatest interest to us.

Theorem 6.4. Let \( u \) and \( p \) be smooth functions satisfying the system (6.1), and assume at least one of the conditions

\[ u \in \text{BMO}^{-1}(\mathbb{R}^3) \]

or

\[ \exists q \in \left[ \frac{3}{2}, 3 \right] : \sup_{1 < R < +\infty} R^{1-3/q} \| u \|_{L_q(B(R))} < +\infty. \]

If \( u \in L_\infty(\mathbb{R}^3) \), then \( u \equiv \text{const} \).

Interestingly, Theorem 6.4 is not a direct consequence of Theorem 6.3 with \( b = u \), because the condition (6.3) is not formally satisfied in that case. This is an indication that the non-linear system (6.1) is in some sense ‘nicer’ than the linear system (6.2) with a ‘passive drift’: in the case of (6.1) the ‘drift’ and the solution are explicitly linked, and information about the solution obtained by investigating the equations gives us additional information about the properties of the ‘drift’.

The reader can find other results on Liouville theorems for the stationary Navier–Stokes equations in [13], [15], [46], [6], [10], [8], [7], and [9] and in references in these papers.

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