Nash Convergence of Mean-Based Learning Algorithms in First Price Auctions

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slides credit to

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Internet Advertising Auctions

- Second Price Auction (SPA): highest bidder wins, pays the 2nd highest bid
- First Price Auction (FPA): highest bidder wins, pays its own bid

\[ v^1 = 800 \]

\[ v^2 = 600 \]
Strategic bidding in FPA

Online learning!

\[ v^1 = \$800 \]
\[ b^1 = \$800 \]
\[ v^2 = \$600 \]
\[ b^2 = \$600 \]
Main Questions:
How will bidders behave in repeated first price auctions if they use online-learning algorithms to learn to bid? (cf., single bidder learning)

Will they converge to a Nash equilibrium?

Our Results
A wide class of online learning algorithms (“mean-based”) converge to a Nash equilibrium in the first price auction (under some assumptions on bidders’ values).
Online Learning in Repeated FPA

Each bidder $i$
- has a **fixed value** $v^i$;
- runs an **online learning algorithm** (mean-based).

**Bidders get feedback from the auction to update algorithms.**

**Infinite horizon:**
Round $t \geq 1$

Bidder $i$ submits a bid $b_t^i$
chosen by its algorithm.

**Bidder who bids the highest (random tie-breaking) wins, pays its own bid.**

**Utility** $u_t^i$ of bidder $i$ at round $t$:
- $v^i - b_t^i$, for the winner;
- 0, for a loser.

**Nash equilibrium of one-shot auction**

Suppose all values and bids are in a **discrete** space normalized to a bounded non-negative integer space $\{0, 1, \ldots, V\}$. 
Online Learning in Repeated FPA

Each bidder $i$
- has a fixed value $v^i$;
- runs an online learning algorithm (mean-based).

Bidder $i$ submits a bid $b_t^i$ chosen by its algorithm.

Bidders get feedback from the auction to update algorithms.

Infinite horizon:
Round $t \geq 1$

FPA single item
- Bidder who bids the highest (random tie-breaking) wins, pays its own bid.
- Utility $u_t^i$ of bidder $i$ at round $t$:
  - $v^i - b_t^i$, for the winner;
  - 0, for a loser.

Nash equilibrium of one-shot auction

Suppose all values and bids are in a discrete space normalized to a bounded non-negative integer space $\{0, 1, \ldots, V\}$. 

[Hon-Snor-Monderer-Sela 98], [Kolumbus-Nisan 22]…
Mean-Based Learning algorithm

[Braverman-Mao-Schneider-Weinberg 2018, Feng-Guruganesh-Liaw-Mehta 2021]

• Let $\alpha_t^i(b)$ be the average utility of bidder $i$ if it bids $b$ in the first $t$ rounds:

$$\alpha_t^i(b) = \frac{1}{t} \sum_{s=1}^{t} u_s^i(b, b_s^{-i})$$

• A learning algorithm is $(\gamma_t)$-mean-based if

$$\alpha_{t-1}^i(b') - \alpha_{t-1}^i(b) > V\gamma_t \Rightarrow Prob( \text{ i bids } b \text{ in round } t ) \leq \gamma_t$$

where $\gamma_t \to 0$

Examples:
• Greedy (Follow the Leader)
• No-regret mean-based learning algorithms
  • $\epsilon$-Greedy
  • Multiplicative Weights Update (MWU)
• Follow the Perturbed Leader
Nash Equilibria of (One-Shot) FPA

• Let $v^1 = v^2 = \ldots = v^M > v^{M+1} = \ldots = v^{M'} > \ldots \geq v^N$.

  - highest-value bidders
  - second-highest-value bidders (if exist)

• Assume each bidder bids strictly smaller than its own value.

• Nash equilibria (omitting corner cases and other bidders):

| $M$       | highest-value bidders                      | second-highest-value bidders |
|-----------|-------------------------------------------|------------------------------|
| $\geq 3$  | $v^1 - 1$                                  | any                          |
| 2         | $v^1 - 1$ or $v^1 - 2$                     | any                          |
| 1         | $v^{M+1} = v^2$                            | $v^{M+1} - 1 = v^2 - 1$     |
Main Results (Informal)

| $M$ | Time-average | Last-iterate |
|-----|--------------|--------------|
| $\geq 3$ | ✓            | ✓            |
| 2    | ✓            | ✗            |
| 1    | ✗            | ✗            |

$M = \#$ bidders with the highest value $v^1$.

✓: Almost surely converge.

✗: May not converge.

• Time-average:
  • (traditional) the empirical distributions of bids approach a Nash equilibrium.
  • (ours) the fraction of rounds where bidders play a Nash equilibrium approaches 1.

• Last-iterate:
  • bidders’ mixed strategy profile approaches a Nash equilibrium.
Main Results (Formal)

• If \( M \geq 3 \), then with probability 1, **both of** the following happen:
  
  \[
  \lim_{t \to \infty} \frac{1}{t} \sum_{s=1}^{t} I[b_s^i = v^1 - 1, \forall i \in M^1] = 1
  \]
  
  \[
  \forall i \in M^1, \lim_{t \to \infty} x_t^i = 1_{v^1-1}
  \]

• If \( M = 2 \), then with probability 1, **one of** the following happen:
  
  \[
  \lim_{t \to \infty} \frac{1}{t} \sum_{s=1}^{t} I[b_s^i = v^1 - 1, \forall i \in M^1] = 1, \text{ and } \forall i \in M^1, \lim_{t \to \infty} x_t^i = 1_{v^1-1}
  \]
  
  \[
  \lim_{t \to \infty} \frac{1}{t} \sum_{s=1}^{t} I[b_s^i = v^1 - 2, \forall i \in M^1] = 1
  \]

• If \( M = 1 \), there exists a mean-based algorithm that does not converge to NE, either in last-iterate or in time-average.
$M = 1$ : Non-Convergence

- Three bidders with $v^1 = 10, v^2 = v^3 = 7$
- Each player uses the **Follow the Leader** algorithm (0-mean based)
- They may generate the following bidding path \((b_t^1, b_t^2, b_t^3)_{t \geq 1}\)
  - \((7, 6, 1), (7, 1, 6), (7, 1, 1), (7, 6, 1), (7, 1, 6), (7, 1, 1), \ldots\)
- \((7, 1, 1)\) happens in 1/3-fraction of rounds but is not a Nash equilibrium
- Do not converge in empirical distribution or last-iterate

- Experiments also show such non-convergence for **no-regret** mean-based algorithms such as MWU
$M \geq 2$: Proof of Convergence

• Intuition:
  • First price auction (with fixed values and $M \geq 2$) can be solved by iterative elimination of dominated strategies. [Hon-Snir-Monderer-Sela 1998]

• Proof Sketch:
  • Example: 3 bidders with the same value $v^1$. NE: all bid $v^1 - 1$.
  • $b \in \{0, 1, \ldots, v^1 - 2, v^1 - 1\}$.
  • $\mathcal{A}_t$ is a $(\gamma_t)$-mean-based algorithm if:
    \[ a_t^i(b') - a_t^{i-1}(b) > V\gamma_t \implies \text{Prob(} i \text{ bids } b \text{ in round } t ) \leq \gamma_t \]

• Challenge: randomness of algorithms.
  • A mean-based algorithm may pick a dominated strategy with a positive probability.

• Technique: time-partitioning and repeated use of Azuma’s inequality. [Feng-Guruganesh-Liaw-Mehta 2021]
Summary &
Open Questions

- Any mean-based learning algorithms converge to the Nash equilibrium in a first price auction with bidders having fixed values, if there are more than one highest-value bidders.
  - **Open question #1**: what’s the convergence rate?
- If there is only one highest-value bidder, not all mean-based learning algorithms converge.
  - **Open question #2**: better algorithms that always converge?
- **Open question #3**: the Bayesian setting of the first price auction.
  - [Feng-Guruganesh-Liaw-Mehta, 2021]: two uniform[0, 1] i.i.d. bidders + mean based algorithms with uniform exploration phase => converge to BNE.

| M  | Time-average | Last-iterate |
|----|--------------|--------------|
| ≥ 3| ✓            | ✓            |
| 2  | ✓            | X            |
| 1  | X            | X            |

Thanks!