Nonlinearity

PAPER

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To cite this article: A S Gonchenko and S V Gonchenko 2015 Nonlinearity 28 3403

Manuscript version: Accepted Manuscript

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Lorenz-like attractors in a nonholonomic model of a rattleback.

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Abstract. We study chaotic dynamics in a nonholonomic model of a rattleback stone. We show that, for certain values of parameters characterizing geometrical and physical properties of the stone, a strange Lorenz-like attractor is observed in the model. We study also bifurcation scenarios for appearance and break-down of this attractor.

1. Introduction

Until recently, only hyperbolic and Lorenz-like attractors were considered as genuine strange attractors, i.e., nontrivial attracting closed invariant sets for which the property “every orbit has the positive Lyapunov exponent” is open. However, the situation has changed drastically after the paper by Turaev and Shilnikov [1], in which a class of wild-hyperbolic attractors was introduced. Such attractors are also genuine, however, unlike the hyperbolic and Lorenz ones, they possess homoclinic tangencies and, thus, contain Newhouse wild hyperbolic sets [2]. Note that in the paper [1] there was also constructed an example of four-dimensional flow having a wild spiral attractor that contains a saddle-focus equilibrium. This attractor is structurally unstable, since homoclinic loops of the equilibrium as well as homoclinic tangencies can appear under small perturbations. However, these bifurcations do not lead to the birth of stable periodic orbits due to the fact that the attracting set possesses the pseudo-hyperbolic structure. This means that the flow (or a map) is (i) strongly contracting along certain invariant directions and (ii) it expands volume transverse to these directions. The exact definition see in [1] and in Section 3.1. Properties (i) and (ii) persist at small smooth perturbations [1, 3, 4]. Moreover, the above property (ii) of the extension of volumes insures that every orbit starting near the attractor has a positive Lyapunov exponent.

Another new type of genuine strange attractors, the so-called discrete Lorenz attractors, was found in the paper [5] for a three-dimensional Hénon map. The simplest model of such attractor can be obtained if we consider a three-dimensional flow with the classical Lorenz attractor and add a small time periodic perturbation [3, 6]. The classical Lorenz attractor contains no stable periodic orbits and every its orbit has positive maximal Lyapunov exponent; moreover, these properties are robust, even though the attractor itself is structurally unstable [7, 8, 9]. When a small periodic perturbation is applied the three-dimensional Poincaré map (period map)
Lorenz-like attractors in a nonholonomic model of a rattleback has a discrete Lorenz attractor which possesses the same properties [3]. However, the dynamics of discrete Lorenz attractor is more complicated than the dynamics of the classical one. The time-periodic perturbations leads to the appearance of homoclinic tangencies, saddle tori, periodic saddles with different dimensions of invariant manifolds, and heterodimensional cycles [3]. Thus, the discrete Lorenz attractor is a genuine wild pseudo-hyperbolic attractor.

**Figure 1.** Main steps of the appearance of a homoclinic strange attractor: a discrete Lorenz attractor (below left), a discrete figure-8 attractor (below right).

However, discrete Lorenz attractors found in [5] for three-dimensional Hénon maps have the different origin and nature. In particular, it was shown in [10, 5, 11] that these attractors appear in the maps as result of a local codimension-3 bifurcation of fixed point with multipliers $(-1; -1; +1)$. In this case, for values of parameters close to the bifurcation ones, the flow normal form of the second iteration of the map will coincide, up to small time-periodic terms, with the Shimizu-Morioka system that can possess the Lorenz attractor [12].

Moreover, in [13, 14], there was proposed a simple and universal scenario of emergence of discrete Lorenz attractors in three-dimensional orientable maps. This scenario can be observed even in one parameter families of maps and starts with values of the parameter for which the map has a fixed (periodic) point; then, at varying parameter, this point loses the stability under the soft (supercritical) period doubling bifurcation. Just after this bifurcation, the period-2 attractor is observed in the family.
and the fixed point becomes a saddle of type (2,1), i.e. with two-dimensional stable
and one-dimensional unstable invariant manifolds, and, moreover, its the unstable
multiplier is negative. After further change of parameter, the period-2 attractor loses
stability (no matter what way) and, what is important, homoclinic orbits to the saddle
appear. Moreover, the picture of the new discrete “homoclinic” attractor can be very
similar to the well-known phase portrait of the flow Lorenz attractor (see e.g. Fig. 3)
or, other case, the attractor can have a figure-8 shape, see Remark 1.

Note that the scenario proposed in [13] does not require special symmetries for
the map, unlike flows with Lorenz attractors. However, such Lorenzian symmetries
(like $x \to x, y \to -y, z \to -z$) appear here “by themselves” due to the period doubling
bifurcation after which the saddle $O$ has two negative and one positive multipliers.
Therefore, these (Lorenz-like) symmetries exist, in general, only for points of invariant
manifolds of the saddle $O$.

In Fig. 1 this scenario is illustrated for the case when the periodic attractor $O$ is
an asymptotically stable fixed fixed point with real multipliers $\lambda_i, i = 1, 2, 3$. Then,
after the period doubling bifurcation, the point $O$ will have multipliers $\lambda_1 < -1,$
$0 < \lambda_2 < 1, -1 < \lambda_3 < 0$. In the case $|\lambda_3| < \lambda_2$, when the “homoclinic attractor”
is created, the global configuration of $W^u(O)$ can be quite similar to that for the
classical Lorenz attractor, see Fig. 1(below left). Note also that the point $O$ must
have the saddle value $\sigma = |\lambda_1 \lambda_2|$ to be greater than 1. Indeed, in the case $\sigma < 1$,
the condition (ii) of pseudohyperbolicity is evidently violated and the corresponding
homoclinic attractor is a quasiattractor [17], i.e. stable periodic orbits can appear
inside it under arbitrarily small perturbations.

**Remark 1** However, if $|\lambda_3| > \lambda_2$ and $\sigma = \lambda_1 \lambda_3 > 1$, then rather new subject, the so-called “discrete figure-8 attractor”, can arise, see Fig. 1(below right). We know so far
only two examples of systems in which such an attractor was found: a nonholonomic
model of unbalanced ball [15] and a three-dimensional generalized Hénon map [16].
In this case the condition $\sigma > 1$ is again necessary for validity of the condition (ii)
of pseudohyperbolicity. If $\sigma < 1$, the 8-attractor is, in fact, a quasiattractor of Hénon-like
type.

This scenario provides us with examples of truly high-dimensional (i.e. not
two-dimensional) robust chaotic behavior which, as we sure, must occur in various
applications. In the present paper we show that discrete Lorenz attractors can exist
in Poincaré map of a nonholonomic model of a rattleback.

The paper is organized as follows. Section 2 contains necessary facts related to a
nonholonomic model of a rattleback under consideration. Section 3 is the main part
of the paper. In this section we consider the nonholonomic model of rattleback with
parameters (3.1) and show (mostly numerically) that a discrete Lorenz attractor exists
in this model. We study this problem using a research strategy (Section 3.2) based on
fundamental facts from the theory of Lorenz-like attractors (see Section 3.1). Main
results are formulated and discussed in Section 3.3.

### 2. The nonholonomic model of a rattleback.

Recall that, in the rigid body dynamics, the rattleback is a top (usually, of a symmetric
form, see Fig. 2) whose one of the principal inertial axes is vertical and other two axes
are horizontal and they are rotated by some angle with respect to the geometrical
axes.
A nonholonomic model of a rattleback stone is a mathematical model where both the stone and the plane are absolutely rigid and rough, i.e., the stone moves along the plane without slipping and, moreover, the friction force has zero moment. This means that the full energy is conserved which is a certain disadvantage of the model. However, it is well known that the nonholonomic model allows one to explain the main phenomenon of the rattleback dynamics – the nature of reverse, i.e., the rotational asymmetry, which results in the fact that the stone can rotate freely in one direction (e.g., clockwise) but “does not want” to rotate in the opposite direction (counterclockwise). In the latter case it performs several rotations due to inertia, then stops rotating and starts oscillating, then it changes the direction of rotation and finally continues rotating freely (clockwise).

![Figure 2. The sample of a rattleback stone. The main body (of form of elliptic paraboloid) is symmetric with respect to the vertical axis $R_z$, the geometric horizontal axes are $R_x$ and $R_h$. The dynamical asymmetry is achieved due to a massive bar kept on the top of stone, which can be rotated by an angle $\delta$ with respect to the axis $R_x$. If $0 < \delta < \pi/2$, the clockwise rotation of this stone is stable, whereas, the counterclockwise rotation is unstable.](image)

A mathematical explanation of this phenomenon seems now simple enough. The fact is that, like most of the well-known nonholonomic mechanical models, the rattleback model is described by a reversible system, i.e., a system that is invariant with respect to the coordinate and time change of the form $X \rightarrow R X$, $t \rightarrow -t$, where $R$ is an involution, i.e. a specific diffeomorphism of the phase space such that $R^2 = Id$. However, in the case of a rattleback stone, this system is, in general, neither conservative nor integrable, although it possesses two independent integrals (see formula (2.4)). Because of this, the system can possess, on the common level set of the integrals, asymptotic stable and completely unstable solutions, stationary (equilibria), periodic (limit cycles) ones and even strange attractors and repellers [18, 19], which are $R$-symmetric with respect to each other. Then, for example, a stable equilibrium $O_s$ corresponds to a stable vertical rotation of the stone, and an unstable equilibrium $O_u = R(O_s)$ corresponds to an unstable rotation in the opposite direction. Such an explanation of the reverse in the rattleback dynamics was given in a series of papers: by I.S. Astapov [20], A.V. Karapetyan [21], A.P. Markeev [22, 23]. We also refer the reader to the papers [24]–[28] in which the rattleback dynamics is
Lorenz-like attractors in a nonholonomic model of a rattleback studied from different points of view.

Nevertheless, the motion of the rattleback stone is still regarded in mechanics as one of the most complicated and poorly studied types of rigid body motion. Moreover, this is one of the few types of motion in which chaotic dynamics was observed [29, 30, 18, 19].

We study the dynamics of a rigid body moving on a plane without slipping. This means that we consider a nonholonomic model of motion in which the contact point of the body has zero velocity, i.e. we have

\[ v + \omega \times r = 0 \quad (2.1) \]

where \( r \) is the radius vector from the center of mass \( C \) to the contact point, \( v \) is the velocity of \( C \) and \( \omega \) is the angular velocity of the body. As usual, the coordinates of all vectors are defined in some coordinate systems that is rigidly attached to the body. Then the equations of motion can be written in the form [32]

\begin{align*}
\dot{M} &= M \times \omega + mr \times (\omega \times r) + mgr \times \gamma, \\
\dot{\gamma} &= \gamma \times \omega, \quad (2.2)
\end{align*}

where

\[ M = I \omega + mr \times (\omega \times r) \quad (2.3) \]

is the angular momentum of the body with respect to the contact point, \( \gamma \) is the unit vertical vector and \( m g \) is the gravity force. The equation (2.2) admits two integrals

\[ H = \frac{1}{2} (M, \omega) - mg(r, \gamma) \quad \text{and} \quad (\gamma, \gamma) = 1, \quad (2.4) \]

the energy integral and the geometric integral, respectively.

We consider the rattleback stone whose surface \( F(r) \) has the shape of the elliptic paraboloid

\[ F(r) = \frac{1}{2} \left( \frac{r_1^2}{a_1} + \frac{r_2^2}{a_2} \right) - (r_3 + h) = 0, \]

where \( a_1 \) and \( a_2 \) are the principal radii of curvature at the paraboloid vertex \((0, 0, -h)\) respectively, and the center of mass is the point \( r_1 = r_2 = r_3 = 0 \). Therefore, the vector \( r \) and \( \gamma \) are related as follows:

\[ r_1 = -a_1 \frac{\gamma_1}{\gamma_3}, \quad r_2 = -a_2 \frac{\gamma_2}{\gamma_3}, \quad r_3 = -h + \frac{a_1 \gamma_1^2 + a_2 \gamma_2^2}{2 \gamma_3^2}. \quad (2.5) \]

It is also assumed that one of the principal axes of inertia is vertical. One of the main features of a rattleback stone is that two other principal axes of inertia are rotated about the geometrical axes by some angle \( \delta \), where \( 0 < \delta < \pi/2 \). Accordingly, the inertia tensor takes the following form [30]:

\[ I = \begin{pmatrix}
I_1 \cos^2 \delta + I_2 \sin^2 \delta & (I_1 - I_2) \cos \delta \sin \delta & 0 \\
(I_1 - I_2) \cos \delta \sin \delta & I_1 \sin^2 \delta + I_2 \cos^2 \delta & 0 \\
0 & 0 & I_3
\end{pmatrix}, \quad (2.6) \]

where \( I_1, I_2 \) and \( I_3 \) are the principal moments of inertia of the stone. We express vectors \( r, \dot{r} \) and \( \omega \) by \( M \) and \( \gamma \) using relations (2.1), (2.3), (2.5) and (2.6). Then the system (2.2) can be represented in the standard form

\[ (M, \dot{\gamma}) = G(M, \gamma, \mu), \quad (2.7) \]

† Recently complex chaotic dynamics has been also found in a model of unbalanced rubber ball (a dynamically asymmetric ball with a displaced center of gravity) moving on the plane, see [31], and in a model of Chaplygin top [15].
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of six-dimensional system with respect to phase variables $M$ and $\gamma$. This system depends also on parameters $\mu$ characterizing the geometrical and physical properties of the stone. Note that on the common level set of the integrals (2.4) the system (2.2) generates the flow on a four-dimension manifold: $M^4 = \{(M, \gamma) : (\gamma, \gamma) = 1, H(M, \gamma) = \text{const}\}$, which is homeomorphic to $S^2 \times S^2$.

2.1. The Andoyer-Deprit variables.

In numerical investigations of the rattleback dynamics we use the so-called Andoyer-Deprit variables $(L, H, G, g, l)$ defined by the formulas [32]

\[
M_1 = \sqrt{G^2 - L^2} \sin l, \quad M_2 = \sqrt{G^2 - L^2} \cos l, \quad M_3 = L,
\]

\[
\begin{align*}
\gamma_1 &= \left( \frac{H}{G} \sqrt{1 - \frac{L^2}{G^2}} + \frac{L}{G} \sqrt{1 - \frac{H^2}{G^2} \cos g} \right) \sin l + \left( 1 - \frac{H^2}{G^2} \sin g \cos l \right), \\
\gamma_2 &= \left( \frac{H}{G} \sqrt{1 - \frac{L^2}{G^2}} + \frac{L}{G} \sqrt{1 - \frac{H^2}{G^2} \cos g} \right) \cos l - \left( 1 - \frac{H^2}{G^2} \sin g \sin l \right), \\
\gamma_3 &= \frac{H L}{G^2} - \sqrt{1 - \frac{L^2}{G^2}} \sqrt{1 - \frac{H^2}{G^2} \cos g}.
\end{align*}
\] (2.8)

By definition (see, e.g., [32]),

\[ H = (M, \gamma) = M_1 \gamma_1 + M_2 \gamma_2 + M_3 \gamma_3. \] (2.9)

On the common level set of two integrals (2.4), the system (2.7) represents the four-dimensional flow $G_E$. Note that the new coordinates $L, H, G, g$ and $l$ are chosen in such a way that the condition $(\gamma, \gamma) = 1$ holds automatically. Thus, the formulas (2.8) specify a one-to-one correspondence between the coordinates $(M, \gamma)$ and $(L, H, G, g, l)$ everywhere except for the planes $L/G = \pm 1$ and $H/G = \pm 1$ (for which the coordinate $l$ and, respectively, $g$ are not defined).

Further, we will investigate the system on the four-dimensional energy levels $H(L, G, H, l, g) = E$. In this case the planes $g = g_0 = \text{const}$ (for appropriate $g_0$) can be considered as cross-sections for orbits of the corresponding four-dimensional flow $G_E$. Thus, we can also study the dynamics of the three-dimensional Poincaré map [29, 30]:

\[ \bar{x} = F_{g_0}(x), \quad x = \left( \frac{L}{G}, \frac{H}{G} \right), \] (2.10)

which is defined in the domain $0 \leq l < 2\pi, -1 < \frac{H}{G} < 1, -1 < \frac{L}{G} < 1$.

2.2. Symmetries in the rattleback model.

The system (2.7) possesses a number of interesting and useful symmetries described by the following lemma.

Lemma 1 [30] In the case under consideration, the system (2.7) is symmetric with respect to the coordinate changes:

(a) \( S_1 : \omega \rightarrow - (\omega_1, - \omega_2, \omega_3), \quad \gamma \rightarrow (\gamma_1, - \gamma_2, \gamma_3) \) \hspace{1cm} (2.11)

and is reversible with respect to the following involutions:

(b) \( \mathcal{I}_1 : \omega \rightarrow - \omega, \quad \gamma \rightarrow \gamma, \quad t \rightarrow - t \)

(c) \( \mathcal{I}_2 : \omega \rightarrow (\omega_1, \omega_2, - \omega_3), \quad \gamma \rightarrow (- \gamma_1, \gamma_2, \gamma_3), \quad t \rightarrow - t \) \hspace{1cm} (2.12)
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Note that these symmetries and involutions are also preserved for the Andoyer-Deprit coordinates. However, they cannot always be linear in this case. But for the Poincaré map (2.10) with the cross-section $g = 0$, which we denote as $F_0$, the symmetries (2.12) remain linear.

**Lemma 2** [30] The map $F_0$ is invariant (i.e. it conserves the form) under the following transformations:

\[(a) \quad \tilde{S}_1 : l \rightarrow l + \pi, \quad \frac{L}{G} \rightarrow \frac{L}{G'}, \quad \frac{H}{G} \rightarrow \frac{H}{G'}.
\]

\[(b) \quad \tilde{T}_1 : l \rightarrow l + \pi, \quad \frac{L}{G} \rightarrow -\frac{L}{G'}, \quad \frac{H}{G} \rightarrow -\frac{H}{G'}, \quad F_0 \rightarrow F_0^{-1}\]  

\[(c) \quad \tilde{T}_2 = \tilde{T}_1 \tilde{S}_1 : l \rightarrow l, \quad \frac{L}{G} \rightarrow -\frac{L}{G'}, \quad \frac{H}{G} \rightarrow -\frac{H}{G'}, \quad F_0 \rightarrow F_0^{-1}\]  

**Corollary 1** Let $L^*$ be an orbit of $F_0$. Then $\tilde{S}_1(L^*)$, $\tilde{T}_1(L^*)$ and $\tilde{T}_2(L^*)$ are also orbits of $F_0$. Moreover, the orbits $L^*$ and $\tilde{S}_1(L^*)$, as well as $\tilde{T}_1(L^*)$ and $\tilde{T}_2(L^*)$, are symmetric with respect to each other. The orbits $L^*$ and $\tilde{S}_1(L^*)$ are both in involution with the orbits $\tilde{T}_1(L^*)$ and $\tilde{T}_2(L^*)$.

3. On discrete Lorenz attractors in the rattleback dynamics.

In this section we consider the nonholonomic model of rattleback for the following values of parameters:

\[I_1 = 2, I_2 = 6, I_3 = 7, m = 1, g = 100, a_1 = 9, a_2 = 4, h = 1.\]  

(3.1)

We also take $\delta = 0.485$.

Note that the model with the parameters (3.1) was considered in [33] in which a strange attractor was found, with $E = 770, \delta = 0.405$. This strange attractor is quite similar (by the shape) to the attractor in Fig. 7(f). Since analogous attractors are known to exist in three-dimensional Hénon maps [13, 34] near the boundaries of destruction of discrete Lorenz attractors, the question naturally arises whether a discrete Lorenz attractor exists for close values of the parameters $E$ and $\delta$. The answer is positive and we give below a short review of results obtained.

Before description of main results (Sections 3.2 and 3.3) we make some remarks and recall necessary definitions.

3.1. On pseudo-hyperbolic Lorenz-like attractors.

First of all we give the definition of pseudo-hyperbolicity for diffeomorphisms, which is, in fact, a reformulation of the corresponding definition for flows from [3], see also [6].

Let $f$ be a $C^r$-diffeomorphism, $r \geq 1$ and let $Df$ be its differential. An open bounded domain $D \subset \mathbb{R}^n$ is absorbing for $f$ if $f(D) \subset D$.

**Definition 1** The diffeomorphism $f$ is called pseudo-hyperbolic on $D$ if the following conditions hold.

1) For each point of $D$ there exist two transversal subspaces $N_1$ and $N_2$ continuously depending on the point ($\dim N_1 = k \geq 1, \dim N_2 = n - k$) which are invariant with respect to $Df$:

\[Df(N_1(x)) = N_1(f(x)), \quad Df(N_2(x)) = N_2(f(x)).\]
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and for each trajectory $L : \{x_i \mid x_{i+1} = f(x_i), i = 0, 1, \ldots; x_0 \in \mathcal{D}\}$ the maximal Lyapunov exponent corresponding to the subspace $N_1$ is strictly smaller than the minimal Lyapunov exponent corresponding to the subspace $N_2$, i.e., the following inequality holds:

$$\lim_{n \to \infty} \sup_1 \frac{1}{n} \ln \left( \sup_{\|u\| = 1} \|Df^u(x_0)u\| \right) < \lim_{n \to \infty} \inf_1 \frac{1}{n} \ln \left( \inf_{\|v\| = 1} \|Df^u(x_0)v\| \right).$$

(3.2)

2) The restriction of $f$ to $N_1$ is contracting, i.e., there exist constants $\lambda > 0$ and $C_1 > 0$ such that

$$\|Df^n(N_1)\| \leq C_1 e^{-\lambda n}.$$  

(3.3)

3) The restriction of $f$ to $N_2$ expands volumes exponentially, i.e., there exist such constants $\sigma > 0$ and $C_2 > 0$ such that

$$|\det Df^n(N_2)| \geq C_2 e^{\sigma n}.$$  

(3.4)

The following property immediately follows from this definition:

1° All the orbits in $\mathcal{D}$ are unstable: each orbit has positive maximal Lyapunov exponent

$$\lambda_{\text{max}}(x) = \lim_{n \to \infty} \sup_1 \frac{1}{n} \ln \|Df^n(x)\| > 0$$

Note that the pseudo-hyperbolicity conditions require the expansion of only $(n-k)$-dimensional volumes by the restriction of the diffeomorphism to $N_2$, which makes these conditions different from those for uniform hyperbolicity, where the following condition must hold: $\|Df^{-n}(N_2)\| < C e^{-\sigma n}$, i.e., the uniform expansion should be along all directions in $N_2$. Nevertheless, it is possible to establish the following fact in a standard way [35, 1].

2° The pseudo-hyperbolicity conditions are not violated under small $C^r$-perturbations of the system. Moreover, the spaces $N_1$ and $N_2$ change continuously.

These two conditions imply that if the diffeomorphism $f$ has an attractor in $\mathcal{D}$, then this attractor is strange§ and does not contain stable periodic orbits, which also do not appear under small perturbations. In other words, pseudo-hyperbolic attractors are genuine strange attractors. The discrete Lorenz attractors form a certain subclass of the class of pseudo-hyperbolic attractors.

Note that the dynamical properties of the geometric Lorenz model [8] under small time-periodic perturbations were investigated in [3]. It was also shown that the properties of pseudo-hyperbolicity and chain transitivity|| of a non perturbed Lorenz attractor hold for a periodically perturbed attractor as well. Thus, the Poincaré

§ Note that the expansion of volumes prevents the attractor to be a finite union of smooth submanifolds (at least for all close systems).

|| i.e. when any two points in an invariant set can be joined by an $\varepsilon$-orbit belonging to the set, for any sufficiently small $\varepsilon$. 

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map also possesses here a pseudo-hyperbolic attractor \( A \), which appears to be a basic example of a discrete Lorenz attractor.

The same conclusions can also be drawn without assuming that the map under consideration is a Poincaré map of a system periodic in time and close to an autonomous one. For this general case the corresponding definition of a discrete Lorenz attractor was given in [34]. Formally, this definition requires to satisfy main conditions which hold for the Poincaré map of periodically perturbed Lorenz attractor, including the geometrical analogy\(^1\) and conditions for the spectrum of Lyapunov exponents due to the Definition 1. Thus, for numerical study of a discrete Lorenz attractor, we need to check the required geometrical analogy and the fact that the numerically obtained Lyapunov exponents \( \lambda_1, \lambda_2, \lambda_3 \) satisfy conditions \( \lambda_3 < \min\{\lambda_1, \lambda_2\} \), \( \lambda_3 < 0 \) and \( \lambda_1 + \lambda_2 > 0 \) (some analogs of conditions (3.2), (3.3) and (3.4), respectively).

In this paper we study the dynamics of the Poincaré map (2.10). For more definiteness, we consider the model with the physical parameters (3.1) and \( \delta = 0.485 \), we take the cross-section \( g_0 = 0 \) and then consider a one parameter family \( T_E \) of maps (2.10) with governing parameter \( E \) – the value of energy. In Section 3.2 we describe the corresponding research strategy. The obtained results are collected in Section 3.3.

3.2. A strategy of qualitative and numerical study of discrete Lorenz attractors on example of the family \( T_E \).

When studying dynamics and bifurcations in the family \( T_E \) for appropriate values of \( E \), we will act by employing the following strategy which, in fact, is justified by Definition 1 and its corollaries 1* and 2*.

1) Verify the geometrical similarity of our attractor \( A_{E^*} \) found in the rattleback model and the classical Lorenz attractor. Here the strange attractor which was found for \( E = E^* = 752 \) is examined.

In particular, this similarity manifests itself in the fact that our three-dimensional map \( F_{0E} \) possesses the following features: (i) it has a fixed saddle point \( O^* \) belonging to the attractor \( A_{E^*} \) with the multipliers of \( \lambda_1, \lambda_2, \gamma \) such that \( |\lambda_2| < |\lambda_1| < 1 < |\gamma| \), \( \lambda_1 > 0, \lambda_2 < 0, \gamma < -1 \) and \( |\lambda_1\gamma| > 1 \); (ii) the manifolds \( W^s(O) \) and \( W^s(O) \) have nonempty intersection; (iii) the phase portraits look “similar”, see Fig. 3, and, moreover, the main bifurcations leading to appearance of the attractor are quite analogous to those in the Lorenz system, see item 4) below.

We mention that negative values of the multipliers \( \lambda_2 \) and \( \gamma \) imply the Lorenz symmetry \( (x \rightarrow x, y \rightarrow -y, z \rightarrow -z) \) of the homoclinic structure. Moreover, for the values of the parameter \( E \) close to \( E^* \), the manifold \( W^u \) will intersect \( W^s \) strictly from one side of the strong stable invariant manifold \( W^{ss}(O) \), which is tangent to the eigendirection corresponding to the multiplier \( \lambda_2 \) of \( O^* \), which implies the homoclinic configuration of the figure-eight-butterfly similar to the Lorenz attractor.

2) Verify numerically the strangeness and pseudo-hyperbolicity of the attractor \( A_{E^*} \).

At this stage we investigate the spectrum \( \lambda_1, \lambda_2, \lambda_3 \) of the Lyapunov exponents of the map \( T_{E^*} \) on the attractor \( A^* \) and show that this spectrum, where \( \lambda_1 > \lambda_2 > \lambda_3 \), satisfies the following conditions: (1) \( \lambda_1 > 0 \); (2) \( \lambda_1 + \lambda_2 + \lambda_3 < 0 \); (3) \( \lambda_1 + \lambda_2 > 0 \). The conditions (1) and (2) imply that the attractor \( A^* \) is strange and the condition

\( ^1 \) between the discrete Lorenz attractor and the attractor in the Poincaré map of periodically perturbed Lorenz attractor
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(3) holds when the attractor is pseudo-hyperbolic (the map expands two-dimensional areas transversal to the strong contraction direction related to the exponent $\Lambda_3 < 0$).

3) Plot numerically the dependence of the maximal exponent $\Lambda_1$ for some ranges of the parameter $E$ containing this value, $E = E^*$, for which the attractor $A_{E^*}$ exists, see Fig. 4.

At this stage we verify (only numerically) that our attractor is not a quasi-attractor, i.e., it does not contain stable periodic orbits of large periods, which do not appear under perturbations either. As is seen from Fig. 4, the graph is in the domain $\Lambda_1 > 0$ and looks like a continuous function, whereas, if $A_{E^*}$ were a quasi-attractor, the “holes” would be observed on the plot containing the ranges of $\Lambda_1 < 0$ corresponding to “stability windows”.

4) Investigate (mostly numerically) the main bifurcations starting at the stable fixed point and leading to the appearance of discrete Lorenz attractors (including $A_{E^*}$).

We also trace the main stages of destruction of strange attractor.

In principle, this item may seem unnecessary but we suppose that it could be the most interesting since here one can follow a certain “genetic” connection between the phenomena observed in flows with the Lorenz attractors (Lorenz model, Shimizu-Morioka model etc.) and those observed in the rattleback model. Moreover, as the calculations show, our Poincaré map $T_E$ behaves like a “small perturbation of the time shift of the flow in the geometric Lorenz model” for corresponding values of $E$. Formally, this circumstance can be caused by the interesting fact that the middle Lyapunov exponent $\Lambda_2$ is very close to zero (for the flow case it is simply equal to zero): during calculations it demonstrates small oscillations in the range between 0.00007 and 0.00015, see [5] for a discussion of this topic. But what is really interesting is that the bifurcations leading to the appearance of strange attractor are here almost identical to those in the Lorenz model [36], see. Fig. 5(b)–(f).

Figure 3. (a) a Lorenz-like attractor for $E = E^* = 752$ in the rattleback model (about 10000 iterations of some initial point are shown); b) the projection of the Lorenz attractor from the Lorenz model onto the $(x, z)$ plane.
3.3. Results of the numerical study.

Below we show the results of numerical investigations performed according to the items 1)–4) of the strategy.

1) Fig. 3 shows (a) iterations of a single point of the attractor $A_{E^*}$ of the map $T_E$ for $E = E^* = 752$ (for an appropriate angle of projection) and (b) the projection of orbits of the classical Lorenz attractor from the Lorenz model for $r = 28$, $\sigma = 10$, and $b = 8/3$ onto the $(x, z)$ plane.

The fixed saddle point $O$ with coordinates of $l = 3.650; L/G = 0.669; H/G = -0.384$ on the attractor $A_{E^*}$ has the multipliers $\lambda_1 = 0.996; \lambda_2 = -0.664; \gamma = -1.312$. If one draws its unstable manifolds ("separatrices"), then, as expected, they will have "loops" (due to the existence of the homoclinic intersection), see Fig. 6(a), in contrast with the unstable separatrices of the Lorenz attractor in flows which appear to be sufficiently monotonous spirals.

2) For the attractor $A_{E^*}$ at $E = E^* = 752$ the spectrum of the Lyapunov exponents was obtained as follows: $\lambda_1 = 0.0248; \lambda_2 = -0.2445; \lambda_3 = 0.00007 < \lambda_2 < 0.00015$.

Evidently, the conditions $\lambda_1 > 0, \lambda_1 + \lambda_2 + \lambda_3 < 0$ and $\lambda_1 + \lambda_2 > 0$ hold here.

3) On the graph of Fig. 4 the dependence of the maximal Lyapunov exponent $\lambda_1 = \lambda_1(E)$ on $E$ is shown for the ranges (a) $[752; 752.01]$ and (b) $[752; 753]$ of the parameter $E$.

4) Fig. 5 illustrates the main stages of evolution of a discrete Lorenz attractor in the map $T_E$ for the parameter $E$ growing from $E = 747$ to $E = E^* = 750$.

Initially the attractor is a stable fixed point $O$, Fig. 5(a). Then, at $E = E_1 = 747.61$, it undergoes a period-doubling bifurcation, and the stable cycle $P = (p_1, p_2)$ of period two becomes an attractor, Fig. 5(b). At $E = E_2 = 748.4395$ the "homoclinic figure-eight-butterfly" of the unstable manifolds (separatrices) of the saddle $O$ is created, Fig. 5(c), which then gives rise to a saddle-type closed invariant curve $L = (L_1, L_2)$ of period two (where $T_E(L_1) = L_2, T_E(L_2) = L_1$), the curves $L_1$ and $L_2$ surround the point $p_1$ and $p_2$, respectively. For the same values of parameters, the unstable separatrices of $O$ are changing their behavior and now, for $E_2 < E < E_3$, the plane...
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Figure 5. The main stages of evolution of the Lorenz-like attractor in the map \( T_E \). Figs. (a) and (f) show iterations of some starting point, and Figs. (b)–(e) show unstable manifolds of the fixed point \( O \).

Moreover, together with the closed period-2 invariant curve \( \mathcal{L} \), an invariant limit set \( \Omega \) is born here, [36], which is not attracting yet. As the numerical calculations show, for \( E = E_3 \approx 748.97 \) the separatrices “lie” on the stable manifold of the curve \( \mathcal{L} \) and then leave it. Almost immediately after that, at \( E = E_4 \approx 748.98 \), the period-2 cycle \( P \) sharply loses stability under a subcritical torus-birth bifurcation: the closed invariant curve \( \mathcal{L} \) merges with the cycle \( P \), after that the cycle becomes a saddle and the curve disappears. The value of \( E = E_4 \) is the bifurcation moment of the creation of strange attractor – the invariant set \( \Omega \) becomes attracting. Even for the parameter values close to \( E = E_3 \) (and \( E > E_3 \)) the separatrices start to unwind, see Fig. 5(e) and their configuration becomes similar to the Lorenzian one, which also applies to the phase portrait, see Fig. 5(f).

Fig. 6 shows the behavior of (a) manifolds \( W^u(O^*) \) and (b) iterations of the points on the attractor \( A^* \) of map \( T_E \). (here \( E = E^* = 752 \)). This attractor is studied in items 1)–3) above.

Fig. 7 shows some stages of destruction of the discrete Lorenz attractor, which is related to the appearance of resonant stable invariant curves , (b), (d) and (e), and the chaotic regimes (torus-chaos), (c) and (f). Note that for \( E > 790 \) nothing remains of the discrete Lorenz attractor and the representative points run away from its neighborhood, going to a new stable regime – the spiral attractor, observed in [18, 33].
4. Conclusion.

The Lorenz attractors for flows play a special role in the theory of dynamical chaos. Until recently, these and hyperbolic attractors were the only ones which were classified as “genuine” strange attractors, which, in particular, do not allow the appearance of stable periodic orbits under small perturbations. After publication of the paper [1]
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by Turaev and Shilnikov the situation changed drastically. They not only provided an example of a wild hyperbolic spiral attractor that must be regarded as a genuine strange attractor but also introduced a new class of \textit{pseudo-hyperbolic attractors}. Thus, a new trend related to the study of such strange attractors appeared in the theory of dynamical chaos.

Discrete Lorenz attractors should be considered as very interesting examples of such genuine attractors. However, unlike the flow Lorenz attractors, their mathematical theory has not been constructed yet. Although, the basic elements of this theory already exist, [1, 3, 5, 34], and are applied.

In particular, the example of a discrete Lorenz attractor found in this paper for the rattleback model is, as we know, the first one for models of real systems. However, we are sure that discrete Lorenz attractors exist in other models, one needs to make a more detailed search. Especially that such attractors are not exotic, and they can appear in dynamical systems (e.g. in three-dimensional maps) as a result of simple and universal bifurcation scenarios [13, 14].

\textbf{Acknowledgements.} The authors are grateful to M. Malkin for fruitful discussions and useful comments. This work has been partially supported by the Russian Scientific Foundation Grant 14-41-00044 and the RFBR grants 13-01-05589 and 14-01-00344. Section 3 is carried out by the RScIF-grant (project No.14-12-00811). A. Gonchenko is supported by basic funding of Russian Ministry of Education and Science.

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