Nucleon isovector couplings from $N_f = 2$ lattice QCD

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We compute the axial, scalar, tensor and pseudoscalar isovector couplings of the nucleon as well as the axial tensor and pseudoscalar charges in lattice simulations with $N_f = 2$ mass-degenerate non-perturbatively improved Wilson-Seafermass-Williams-Hohler fermions. The simulations are carried out down to a pion mass of 150 MeV and linear spatial lattice extents of up to 4.6 fm at three different lattice spacings ranging from approximately 0.08 fm to 0.06 fm. Possible excited state contamination is carefully investigated and finite volume effects are studied. The couplings, determined at these lattice spacings, are extrapolated to the physical pion mass. In this limit we find agreement with experimental results, where these exist, with the exception of the magnetic moment. A proper continuum limit could not be performed, due to our limited range of lattice constants, but no significant lattice spacing dependence is detected. Upper limits on discretization effects are estimated and these dominate the error budget.

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I. INTRODUCTION

The electron spectrum measured in nuclear $\beta$-decays led to Pauli’s postulate of an electrically neutral, almost massless particle in his famous letter presented to a meeting of nuclear physicists in 1930 (reprinted and translated in Ref. [1]). The existence of this particle was confirmed in Ref. [2] with the discovery of the electron-antineutrino some 25 years later [3]. The axial coupling (or charge) of the neutron while the vector charge is well known as it quantifies the difference between the zero-induced tensor charge $\tilde{q}_T$ determined (see, e.g., Ref. [4]) and a parameter of fundamental importance in tree-level standard model interactions. However, these observables are carried only by pseudo-vector (PV) processes analogous to the standard model $V-A$ structure. In the isovector channel that we consider here there will be contributions to $g_V = 1$ to second order in the isospin breaking parameter [5], however, we assume isospin symmetry.

\[ \langle p|\bar{u}d|n\rangle \] through the effective field theory description of low energy scattering processes $n + \pi^+ \rightarrow p + \pi^0$, see Ref. [9] and references therein, as well as by the current algebra relations discussed below. The charges $g_T$ and $g_S$ can at present only be determined through lattice simulation.

In this article we compute the isovector nucleon couplings $g_A$, $g_V$, $g_S$, $g_T$, $g_P$ and the induced charges $\tilde{g}_T$ and $\tilde{g}_P$, simulating $N_f = 2$ QCD down to a nearly physical quark mass. For calculations of isovector charges one can rely on standard methods. In particular, quark-line disconnected contributions to correlation functions cancel in the isospin symmetric case which we realize here, i.e., we neglect the mass difference between and the electric charges of up and down quarks.

We extract the couplings from the following form factors at $q^2 = 0$, where — in contrast to the remainder of this article — we employ Minkowski spacetime conventions:

\[ \langle p|\bar{u}d|n\rangle = g_S(q^2)\bar{u}_p(p)u_n(p), \] (1)

\[ \langle p|\bar{u}d|n\rangle = g_P(q^2)\bar{u}_p(p)S_{\nu\mu}(q^2)\gamma^\mu u_n(p), \] (2)

where $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$. Above, we have assumed isospin symmetry [6,7]. The proton and neutron states $|p\rangle$ and $|n\rangle$ carry four-momenta $p_\mu$ and $n_\mu$, respectively. $u_p$ and $u_n$ denote the proton and neutron spinors, $m_N$ the nucleon mass and the momentum transfer is $q_0 = \sqrt{m_N^2 + p_\mu^2}$.
Here we construct the above matrix elements as 
the right hand side of this expression can be extrapolated 
that \( m_{ud} g_P \rangle = m_N g_A(q^2) + \frac{q^2}{4m_N} \bar{g}_P(q^2) \). (7) 
The right hand side of this expression can be extrapolated to \( q^2 = 0 \), giving 
where the second equality is the Goldberger-Treiman relation \( [14] \), \( F_\pi \approx 92 \text{ MeV} \) denotes the pion decay constant and \( g_{\pi NN} \) the pion-nucleon-nucleon coupling. The chiral perturbation theory corrections to this relation due to the non-vanishing pion mass are discussed in Refs. \([13, 15, 17]\). \( m_{ud} g_P \rangle = m_N g_A(q^2) \left[ 1 + O(m_\pi^2) \right] \), (8) 
where the second equality is the Goldberger-Treiman relation \( [14] \), \( F_\pi \approx 92 \text{ MeV} \) denotes the pion decay constant and \( g_{\pi NN} \) the pion-nucleon-nucleon coupling. The chiral perturbation theory corrections to this relation due to the non-vanishing pion mass are discussed in Refs. \([13, 15, 17]\). We will use the first equality in Eq. (7) to determine \( g_P \). 
Equation (7) implies \( \bar{g}_P(q^2) = -4m_N g_A(q^2)/q^2 \) at zero quark mass, which suggests \( \bar{g}_P(q^2) \) is governed by a pion pole at small \( q^2 \) and \( m_\pi \), 
\[
\bar{g}_P(q^2) = \frac{4c_N^2}{m_\pi^2 - q^2} g_A(q^2) + \cdots, 
\] (9)
TABLE I. Details of the ensembles used in this analysis. $N(n)$ indicates the number of configurations $N$ and the number of measurements per configuration $n$. $N_{sm}$ refers to the number of Wuppertal smearing iterations and $t_l$ to the sink-source time differences realized. For small $t_l$-values the numbers of measurements per configuration $n$ were reduced (indicated in brackets after the respective $t_l/a$ entries). Note that the pion and nucleon masses displayed were obtained on the respective ensembles and are not extrapolated to their infinite volume limits. The two errors of $am_\pi$ and $am_N$ are statistical and from varying the fit range, respectively. The error of the pion mass in physical units includes both sources of uncertainty.

| Ensemble | $\beta$ | $a$ [fm] | $\kappa$ | $V$ | $am_\pi$ | $m_\pi$ [GeV] | $am_N$ | $Lm_\pi$ | $N(n)$ | $N_{sm}$ | $t_l/a$ |
|----------|---------|-----------|--------|-----|---------|-------------|--------|----------|---------|----------|---------|
| I        | 5.20    | 0.081     | 0.13596| 32$^4 \times 64$ | 0.11516(73)(11) | 0.2795(18) | 0.4480(31)(06) | 3.69 | 1986(4) | 300 | 13 |
| II       | 5.29    | 0.071     | 0.13620| 24$^3 \times 48$ | 0.15449(69)(26) | 0.4264(20) | 0.4641(53)(05) | 3.71 | 1999(2) | 300 | 15 |
| III      | 0.13620 | 32$^3 \times 64$ | 0.15298(43)(16) | 0.4222(13) | 0.4486(22)(20) | 4.90 | 1998(2) | 300 | 15,17 |
| IV       | 0.13632 | 32$^3 \times 64$ | 0.10675(51)(08) | 0.2946(14) | 0.3855(39)(23) | 3.42 | 2023(2) | 400 | 7(1), 9(1), 11(1), 13,15,17 |
| V        | 40$^3 \times 64$ | 0.10465(37)(08) | 0.2888(11) | 0.3881(32)(12) | 4.19 | 2025(2) | 400 | 15 |
| VI       | 64$^3 \times 64$ | 0.10487(24)(04) | 0.2895(07) | 0.3856(19)(05) | 6.71 | 1232(2) | 400 | 15 |
| VII      | 64$^3 \times 64$ | 0.05786(51)(21) | 0.1597(15) | 0.3486(69)(21) | 2.78 | 3442(2) | 400 | 15 |
| VIII     | 64$^3 \times 64$ | 0.05425(40)(28) | 0.1497(13) | 0.3398(61)(18) | 3.82 | 2177(2) | 600 | 17 |
| IX       | 5.40    | 0.060     | 0.13640 | 32$^3 \times 64$ | 0.15020(53)(06) | 0.2595(09) | 0.3070(26)(43) | 3.82 | 2177(2) | 600 | 17 |
| X        | 1.3647  | 32$^3 \times 64$ | 0.13073(55)(28) | 0.4262(20) | 0.3836(29)(14) | 4.18 | 1999(2) | 450 | 17 |
| XI       | 0.13660 | 48$^3 \times 64$ | 0.07959(25)(09) | 0.2595(09) | 0.3070(26)(43) | 3.82 | 2177(2) | 600 | 17 |

$N_{sm}$ refers to the number of Wuppertal smearing iterations and $t_l$ to the sink-source time differences. The horizontal lines separate different volume ranges.

$\pi \approx 0.08\, fm$
$\pi \approx 0.07\, fm$
$\pi \approx 0.06\, fm$

$am_\pi$ and are not extrapolated to their infinite volume limits. The two errors of $am_\pi$ and $am_N$ are statistical and from varying the fit range, respectively. The error of the pion mass in physical units includes both sources of uncertainty.

To improve the overlap of our nucleon interpolators with the physical ground state, we follow Ref. 58 and employ Wuppertal (Gauss) smearing [59] of the quark fields

$$\phi_x^{(n)} = \frac{1}{1 + 6\delta} \left( \phi_x^{(n-1)} + \delta \sum_{j=\pm 1} U_{x,j} \phi_x^{(n-1)} \right),$$

where we replace the spatial links $U_{x,j}$ by APE-smeared [60] gauge links

$$U_{x,i}^{(n)} = P_{SU(3)} \left( \alpha U_{x,i}^{(n-1)} + \sum_{|j| \neq i} f_{x,j}^{(n-1)} U_{x,j}^{(n-1)} U_{x+a,j}^{(n-1)} \right),$$

with $i \in \{1,2,3\}, j \in \{\pm 1, \pm 2, \pm 3\}$. $P_{SU(3)}$ denotes a projection into the SU(3) group and the sum is over the four spatial “staples”, surrounding $U_{x,i}$. We employ 25 such gauge covariant smearing iterations and use the weight factor $\alpha = 2.5$. Within the Wuppertal smearing we set $\delta = 0.25$ and adjust the number of iterations to optimize the quality of the effective mass plateaus of smeared-smeared nucleon two-point functions.

We label the nucleon source time as $t_l = 0$ and the sink time as $t_s$. The currents are inserted at times $t \in \{0, t_s\}$ and the relevant matrix elements can be extracted from data within the range $t \in [\delta t, t_s - \delta t]$ where $\delta t \geq 2a$, due to the clever term in the action that couples adjacent time slices. Using the sequential source method [61], we employed an inferior quark smearing.

$2$ We also explored stochastic methods [61], see also Refs. 62, 63.
values of $t$ can be realized, essentially without overhead. However, each $t_f$-value requires additional computations of sequential propagators, adding to the cost. On some of our ensembles we vary this distance too, since this may be necessary to parameterize and eliminate excited state contributions. The $t_f$-values used, the numbers of gauge configurations $N$ and measurements per configuration $n$ are also included in Table I. The statistical noise decreases with smaller Euclidean time distances between source and sink, which means we can reduce the number of three-point function measurements in some cases (indicated in brackets after the respective $t_f/a$ entries).

Naively, one would expect the optimal number of smearing iterations $N_{am}$ to somewhat increase with decreasing quark mass and, at a fixed mass, to scale with $1/a^2$, maintaining a smearing radius that is constant in physical units. As can be read off from the table, we approximately follow this rule. In Fig. 2 we compare our effective nucleon masses

$$m_N(t_f + a/2) = a^{-1} \ln \left[ \frac{C_{2pt}(t_f)}{C_{2pt}(t_f + a)} \right]$$

in physical units between ensembles III and X as well as between ensembles I, IV and V, see Fig. 3. These two groups of ensembles correspond to similar pion masses but differ in terms of the lattice spacing. Using our optimized smearing functions in the construction of the nucleon interpolators, we do not detect any significant lattice spacing dependence of the shapes of the resulting effective mass curves. In Fig. 3 the same comparison is made for smeared-smeared pion effective masses. Again, the shapes within each group of ensembles are very similar while obviously in this case we can resolve the small differences between the lower pion masses.

Our nucleon sources were placed at different time slices and spatial positions from configuration to configuration to reduce autocorrelations. Remaining autocorrelations were accounted for by binning subsequent configurations within the jackknife error analysis and varying the bin sizes until they were bigger than four times the respective estimated integrated autocorrelation times.

Recently, many groups investigated the issue of excited state contamination of ground state signals of three-point functions and, indeed, by applying a more careful analysis, varying $t_f$ [21, 24, 27, 28, 32, 36, 37, 39, 40, 57, 65], using a variational approach [25] and/or by optimizing the ground state overlap of the nucleon interpolator [24, 57] significant effects were detected in many matrix elements. Hence, for three of our ensembles, covering the pion masses 150 MeV (VIII), 290 MeV (IV) and 425 MeV (III), we vary the source-sink distance $t_f$ in addition to the position of the current $t$, see Table I. Based on these results and our observation of very similar shapes as a function of time of the effective masses computed from our nucleon two-point functions (see Fig. 2), for the remaining ensembles we fix $t_f \gtrsim 1$ fm.

B. Excited state analysis

The spectral decompositions for two- and three-point functions read

$$C_{2pt}(t_f) = A_0 e^{-m_N t_f} \left( 1 + A_1 e^{-\Delta m_N t_f} + \cdots \right),$$

$$C_{3pt}(t_f, t_s) = A_0 e^{-m_N t_f}$$

$$\times \left[ B_0 + B_01 e^{-\Delta m_N t_f/2} \cosh(\Delta m_N t) + B_1 e^{-\Delta m_N t_f} + \cdots \right],$$

where $\Delta m_N = m_N - m_N$ denotes the mass gap between the nucleon ground state and its first excitation and the ellipses denote contributions from higher excited states. The coefficients $A_0, A_1, B_0, B_01$ and $B_1$ are real if the current is self-adjoint (or anti-self-adjoint) and the same interpolator (i.e. smearing) is used at the source and the sink. Above we assumed the temporal lattice extent to be much bigger than $t_f$ which holds in our case.

For a current $J = \bar{u}d$, a nucleon interpolator $\Phi$, a nucleon state $|N\rangle$ (and first excitation $|N'\rangle$) and a vacuum

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**FIG. 2.** Effective nucleon masses Eq. 13 for five of our ensembles, computed from smeared-smeared two-point functions $C_{2pt}(t_f).$

**FIG. 3.** The same as Fig. 2 for the pion effective mass.
state \( |0\rangle \) the coefficients read: \[ A_0 = \frac{|\langle 0|\Phi|N\rangle|^2}{2m_N}, \quad A_1 = \frac{|\langle 0|\Phi|N\rangle|^2}{2m_N A_0}, \quad (16) \]

\[ B_0 = \frac{\langle N|J|N\rangle}{2m_N}, \quad B_1 = A_1 \frac{\langle N|J|N\rangle}{2m_N A_0}, \quad (17) \]

\[ B_{01} = \frac{2 \text{Re} \left( \langle 0|\Phi|N\rangle \langle N|J|N\rangle \langle N|\Phi|^0 \rangle \right)}{4m_N m_N A_0}. \quad (18) \]

If for instance the transition matrix element \( \langle N|J|N\rangle \) and therefore \( B_{01} \) is small, this does not imply a small coefficient \( B_1 \) and vice versa. Hence it is essential to employ interpolators that minimize overlaps with higher excitations (i.e. \(|\langle 0|\Phi|N\rangle| \ll |\langle 0|\Phi|N\rangle| \) etc.) and to choose \( t_f \) sufficiently large.

For two-point functions excited states are suppressed by factors \( e^{-\Delta m_N t/2} \) while in the three-point functions there exist contributions \( \propto e^{-\Delta m_N t/2} \). If the ratio of the three-point function over the two-point function is constant upon varying \( t \), this indicates a small \( B_{01} e^{-\Delta m_N t/2} \) term, but still terms \((B_1 - A_1)e^{-\Delta m_N t/2}\) may be present that can only be isolated if \( t_f \) is varied as well. Up to such corrections the ratio reads

\[ R(t, t_f) = \frac{C_{3pt}(t, t_f)}{C_{2pt}(t_f)} = \frac{\langle N|J|N\rangle}{2m_N} + \cdots, \quad (19) \]

where \( \langle N|J|N\rangle \) is the matrix element of interest. Fitting this combination to a constant suffers from the obvious caveats described above.

Recently, the summation method \[ 64 \]

\[ \frac{a}{t_f} \sum_{t=0}^{t_f-\delta t} R(t, t_f) = \frac{\langle N|J|N\rangle}{2m_N} + \frac{a}{t_f} + \mathcal{O}(e^{-\Delta m_N t_f}) \quad (20) \]

was advertised \[ 65 \] as a more reliable alternative. In this case corrections \( \propto e^{-\Delta m_N t/2} \) are removed, but a \( c/t_f \) term is introduced, adding a not necessarily small parameter \( c \) to the fit function. We refrain from quoting the corresponding results as direct fits to the known parametrization Eqs. \[ 14 \] and \[ 15 \] are cleaner theoretically and utilize the whole functional dependence of the data on \( t \) and \( t_f \). Since the summation method appears to be very popular, we discuss it in more detail in Sec. [1C] below.

First we discuss \( g_A \). In Fig. 4 we display the ratio Eq. \[ 19 \] of the renormalized (see Sec. [II] below) three-point over the two-point function obtained from ensemble VIII \( (m_s \approx 150 \text{ MeV}, a \approx 0.071 \text{ fm}) \) at \( t_f = 15a \approx 1.07 \text{ fm}, \quad t_t = 12a \quad \text{and} \quad t_t = 9a. \) All three sets are compatible with constants, however, the \( t_t = 9a \approx 0.64 \text{ fm} \) data are significantly lower than the two other sets. This indicates a small \( B_{01} \)-coefficient in Eq. \[ 15 \]. The effect of \( B_1 - A_1 \) (or higher excitations) becomes visible at \( t_t < 1 \text{ fm} \). Whenever \( B_{01} \) could not be resolved, such as in the case shown in the figure, \( g_A \) was obtained from a fit of the plateau to a constant. Otherwise multi-exponential fits Eqs. \[ 14 \] and \[ 15 \] were performed, where \( B_1 \) was set to zero for the ensembles with only one \( t_f \)-value. These multi-exponential fits gave numbers compatible with those obtained by fitting the \( t_f \geq 1 \text{ fm} \) ratios to constants for \( g_A \) as well as for all the other couplings discussed in this article.

In all analyses presented in this article the fit ranges were selected based on the goodness of the correlated \( \chi^2 \)-values and the stability of the results upon reducing the fit range, i.e. increasing the minimal distance between the current and the source-sink \( \delta t \) or reducing the number of \( t_f \)-values entering the fit. A systematic error was then estimated by varying the fit-range, and the parametrization, e.g., allowing for \( B_1 \neq 0 \) in cases where this parameter was consistent with zero.

In some publications a dependence of the ratio of the axial three-point over the two-point function on \( t_f \) and on \( t \) is reported that is much stronger than what we observe, see, e.g., Refs. \[ 25 \] \[ 28 \] \[ 30 \] while the results of, e.g., Ref. \[ 59 \] are quite similar to ours. This motivates us to compare two different smearing methods found in the literature on ensemble IX: Jacobi smearing \[ 60 \] and Wuppertal smearing \[ 59 \]. With the optimized root mean

3 In our normalization we assume \( |N\rangle \) to be a one-particle state. However, the precise nature of \( |N\rangle \) does not have any impact on the discussion below nor does it affect any of the arguments or the analysis.
squared smearing radius $r_{\text{RMS}} \approx 0.58$ fm for both methods give similar results, see the comparison between the $N_{\text{sm}} = 225$ Jacobi and the $N_{\text{sm}} = 400$ Wuppertal smearing in Fig. 5. In these cases the parameter $B_{01}$ is statistically compatible with zero. Without realizing additional $t_f$-values we cannot determine $B_1$ but, based on our detailed investigations on ensembles III, IV and VIII, it is reasonable to assume that the effect of this term is statistically insignificant at $t_f = 17a \approx 1.03$ fm.

For the Jacobi algorithm additionally we realize $N_{\text{sm}} = 75$, reducing the smearing radius to $r_{\text{RMS}} \approx 0.37$ fm and $r_{\text{RMS}} \approx 0.34$ fm with and without APE smearing, respectively. This results in some curvature due to the effect of excited states, i.e. the parameter $B_{01}$ now significantly differs from zero. Comparing the two $N_{\text{sm}} = 75$ results illustrates that APE smearing the spatial gauge links is less important than varying the number of smearing iterations. However, APE smearing further increases the overlap with the physical ground state.

For $t_f \to \infty$ and $t \approx t_f/2$ obviously all four data sets must approach the same asymptotic value. However, from the comparison shown in Fig. 5 it is clear that with the two inferior smearing functions $t_f$ needs to be chosen much larger — or at least additional source-sink distances need to be realized, to enable a determination of the parameters $B_1$ and $B_{01}$ and a subsequent extrapolation. Otherwise, in these cases an incorrect result would be obtained: Clearly, the minimal sensible value of $t_f$ does not only depend on the statistical accuracy but also on the quality of the interpolator. For instance, an ideal interpolator $\Phi$ with 100% ground state overlap would, up to issues related to the locality of the action, eliminate the time-dependence altogether.

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4 All three quarks within the interpolator $\Phi_i$, used to create a state with the quantum numbers of the nucleon, are smeared applying the same matrix $A$ to $\delta$-sources. For the case of Wuppertal smearing this matrix $A$ with space and colour indices is iteratively defined in Eq. (11). We compute a gauge invariant smearing function $\psi(r) \geq 0$: $\psi^2(r) = \sum_n \langle (A\delta^c)_n \rangle^2$, where the $\delta$-source has only one non-vanishing entry, at the spatial origin and of colour $a$. The RMS radius is computed in the usual way: $r_{\text{RMS}}^2 = \langle \sum_n r^2\psi(na)\rangle / \langle \sum_n \psi(na) \rangle$, where the sum extends over all (three-dimensional) lattice points and $r^2 = \min[(na)^2, (na - L)^2]$, taking account of the periodic boundary conditions. In principle one could also, by analogy with quantum mechanics, define $r_{\text{RMS}}$ with a weight factor $\psi(r)^2$, rather than $\psi(r)$. Due to the approximately Gaussian profile, this definition will result in a radius that is smaller by a factor of about $\sqrt{2}$ than the numbers we quote.
In Fig. 8 we show data for the renormalized scalar density for the same $m_\pi \approx 150$ MeV ensemble as in Fig. 4. In this case $B_{01}$ significantly differs from zero. We divide the three-point functions by the asymptotic parametrization of the two-point function $A_0 e^{-m_\pi t_f}$, obtained from the combined fit. The curves correspond to the multi-exponential fit Eqs. (14) and (15) with $\delta t = 2a$. $B_1$ is compatible with zero. The figure demonstrates that varying $t_f$ helps to obtain a reliable result. However, it is also clear that within statistical errors the $t_f \approx 15a > 1$ fm data alone would have given the correct value.

Finally, we discuss the tensor charge $g_T$ where the relative errors are — in contrast to $g_S$ — not much bigger than for $g_A$ but excited state contributions are clearly present, as is illustrated in Figs. 7 and 8 for the examples of $m_\pi \approx 150$ MeV and $m_\pi \approx 290$ MeV, respectively. Again, the error bands shown are from multi-exponential fits. In Fig. 9 we compare the different smearing methods for the case of $g_T$. The effect is visible, however, much less dramatic than for $g_A$ (see Fig. 5). In the case of $g_T$ the smearing has only a minor effect on the shape as a function of $t$ but still moves the ratio vertically.

We conclude this section by investigating the lattice spacing dependence of ratios of renormalized three- over two-point functions. This is important as we have only varied $t_f$ on three of our ensembles, albeit at three very different pion masses. From these detailed investigations we concluded that — within the statistics that we have been able to accumulate and with the smearing employed — a single value $t_f \approx 1$ fm was sufficient to obtain the correct ground state results. No lattice spacing effects are visible for effective masses, see Figs. 2 and 3. However, in principle the situation may differ for three-point functions. Therefore, we plot a comparison of the three-point function, normalized with respect to the two-point function, normalized with respect to the two-point function, giving $g_A$ in the limit $0 < t < t_f$ for four of our ensembles.

Similar excited state analyses to those detailed above were carried out for all the couplings on all the different ensembles displayed in Table I also shown in Fig. 11.
FIG. 12. Results on $g^{\overline{MS}}(2 \text{ GeV})$ obtained with the summation method Eq. (21) for different fit ranges $t_f \in [t_{f,\min}, t_{f,\max}]$ and $\delta t/a \in \{2, 3\}$ on ensemble IV ($m_\pi \approx 290 \text{ MeV}, a \approx 0.671 \text{ fm}$). The error band corresponds to the result obtained with the fit method detailed in Sec. III B, including our assignment of systematic errors. All data are normalized with respect to the $\overline{\text{MS}}$ scheme. The error of the renormalization factor is smaller by more than one order of magnitude than any of the statistical errors displayed and can be neglected.

FIG. 13. The same as Fig. 12 for $g^{\overline{MS}}(2 \text{ GeV})$.

C. Comparison with the summation method

The summation method $[61]$ has recently gained in popularity $[63]$. Fitting ratios in $R(t, t_f)$ to a plateau in $t_f$, see Eq. (19), there are corrections of order $\exp(-\Delta m_N t_f)/2$. Instead, the summation method comprises of computing sums

$$S(t_f, \delta t) = \sum_{t_f=\delta t}^{t_f-\delta t} R(t, t_f) = c(\delta t) + \frac{t_f}{2} \left[ \frac{N_{\pi} J_f}{2m_N} + \cdots \right],$$

see Eq. (21), and fitting these linearly in $t_f$ within an interval $t_f \in [t_{f,\min}, t_{f,\max}]$. It is easy to see from Eqs. (14) and (15) that the corrections to the slope, and thereby to the desired matrix element, in this case are only of order $\exp(-\Delta m_N t_f)$. Therefore, for $\delta t$ chosen sufficiently large and $t_{f,\min} \geq t_{f,\max}/2$, the convergence of the slope as a function of $t_{f,\max}$ towards the asymptotic value is faster than the convergence of results of plateau fits as a function of $t_f$ at the price of introducing a second fit parameter $c$. It is not clear why one would compare this procedure to simple plateau fits: In that case, introducing for each $t_f$-value additional fit parameters $c = B_0 \exp(-\Delta m_N t_f/2)$ and $m_N$, the dependence on $\exp(-\Delta m_N t_f/2)$ can be removed too. If more than one $t_f$-value is available, which is a pre-requisite of the summation method, it is also not obvious why one should not attempt the combined fit Eqs. (14) and (15), rather than transforming (and reducing) the available data into sums $S(t_f, \delta t)$.

For $g_A$, with our interpolator, differences between plateau fits, our combined fit and the summation method cannot be resolved statistically as all $R(t, t_f)$ data for different $t_f$ and $t \approx t_f/2$ basically agree within errors. For examples of these ratios, see Figs. 4 and 10 and the $N_{\text{sm}} = 400$ ratio shown in Fig. 5. In Fig. 12 we compare the result of our combined fit (including a systematic error from varying the fit range and parametrization) to results of the summation method Eq. (21) for the example of $g^{\overline{MS}}(2 \text{ GeV})$ on ensemble IV. We employ two different minimal distances $\delta t$ of the summation region in $t$ from the source and sink positions and fit to different intervals $t_f \in [t_{f,\min}, t_{f,\max}]$. Indeed, the summation method converges towards the asymptotic result and the convergence rate improves for larger values of $t_{f,\min}$. The same can be seen in Fig. 13 for the tensor coupling $g^{\overline{MS}}_T(2 \text{ GeV})$.

The form factors $g_T(Q^2)$ and $g_P(Q^2)$ at different virtualities $Q^2$ show a similar behaviour. For the example of the second Mellin moment of the isovector spin-independent structure function $(x)_{a-d}$, a comparison between the methods was presented in Ref. [62]. Also in that case we found agreement between the results of the two methods within the respective $\delta t$- and $t_f$-windows of applicability, however, the combined fits utilize more information than the summation method.

III. $g_V$, $g_A$ AND THE RENORMALIZATION

Following the procedure outlined in Sec. II B we obtain the un-renormalized values $g_A^{\text{lat}}, g_S^{\text{lat}}, g_T^{\text{lat}}$ and $g^{\text{lat}}_F$ listed in Table II. The induced couplings $\tilde{g}_T$ and $\tilde{g}_P$ require an extrapolation of non-forward three-point functions in the virtuality $Q^2$ and will be discussed in detail together with $g_S$ and $g_T$ in Sec. V below. Here we concentrate on $g_V$ and $g_A$. We also list the pion masses and PCAC lattice quark masses, obtained from the axial Ward identity

$$\tilde{m} = \frac{\partial_A (0) |A_4| \pi}{2(0) |P| \pi} \left[ 1 + am(b_A - b_P) \right],$$

(22)
where $|\pi\rangle$ is the physical pion state created by an interpolator of spin/flavour structure $(u\gamma_5d)^\dagger$, $\partial_\mu$ denotes the symmetrized lattice derivative, $P = \bar{u}\gamma_\mu d$ is the local pseudoscalar density and $A_\mu = \bar{u}\gamma_\mu\gamma_5d + ac_\mu d$ $P$ is the non-perturbatively improved axial current ($P$ is automatically order-$a$ improved). $c_A$ was obtained in Ref. [67], the improvement factor $b_4 - b_P$ is explained below and $m$ denotes the lattice vector quark mass defined through

$$m = \frac{1}{2\kappa} \left( \frac{1}{\kappa} - \frac{1}{\kappa_{\text{crit}}} \right),$$

where $\kappa_{\text{crit}}$ is the value of the hopping parameter where the PCAC mass vanishes. The lattice quark masses $m$ can easily be computed from the $\kappa$-values given in Table I and the critical hopping parameter values listed in Table III. The PCAC quark masses $\bar{m}$ (listed in Table II) can be translated into the MS scheme at 2 GeV, upon multiplication with $Z_A/Z_P$ (see below). The pion decay constant is obtained through

$$F_{\pi} = \frac{\langle 0|A_4|\pi\rangle}{\sqrt{2}m_\pi},$$

where we use the normalization that corresponds to the experimental value $F_\pi = Z_A(1 + ambx)F_{\pi}^{\text{lat}} \approx 91$ MeV.

The lattice couplings extracted from the respective matrix elements need to be renormalized too:

$$g_X = Z_X(1 + ambx)g_{X}^{\text{lat}},$$

where $X \in \{S, P, V, A, T\}$. The renormalization factors $Z_X$ and the improvement coefficients $b_X$ depend on the inverse lattice coupling $\beta$. No anomalous dimension is encountered for $g_V$ and $g_T$ due to baryon number conservation and for $g_A$ and $g_T$ due to the PCAC relation. In the other cases we quote the values in the MS scheme at a scale $\mu = 2$ GeV. As detailed in Ref. [68], the renormalization factors are first determined non-perturbatively in the R'MOM scheme, using the Roma-Southampton method [69], and then converted perturbatively at three-loop order to the MS-scheme. The improvement factors $ambx$ were computed in Ref. [70] ($X \in \{S, P, V, A\}$) to one loop and confirmed in Refs. [71, 72], where $b_P$ is given as well. These are very close to unity, due to the smallness of $am$, and can be taken into account perturbatively:

$$b_A = 1 + 0.15219(5)g_2^2, \quad b_V = 1 + 0.15323(5)g_2^2, \quad b_P = 1 + 0.15312(3)g_2^2, \quad b_S = 1 + 0.19245(5)g_2^2, \quad b_T = 1 + 0.1392(1)g_2^2.$$  

In this context we use the “improved” coupling $g_2^2 \equiv -3\ln P = 6/\beta + O(g_4^2)$, where $P$ denotes the average plaquette with the normalization $P = 1$ at $\beta = \infty$. The corresponding chirally extrapolated values of $P$ are displayed in Table III. Note that $b_m = -b_S/2$ as well so the combination $b_A - b_P \approx 0$ were determined non-perturbatively [73] and for $b_S$ we use the interpolating formula of this reference

$$b_S = (1 + 0.19246g_2^2)^2 \frac{1 - 0.3737g_4^{10}}{1 - 0.5181g_4^{14}},$$

instead of the one-loop expression given in Eq. (26).

For convenience we list, in addition to the critical hopping parameter values, the renormalization factors $Z_X$ between the lattice and the MS schemes determined in Ref. [68] (and slightly updated here) in Table III. Note that our $Z_A$-value at $\beta = 5.2$ is by about 2% smaller than that obtained in Ref. [50] from the Schrödinger functional. This is indicative of the $O(a^2)$ difference between cut-off effects of the two methods. This disagreement indeed reduces with increasing $\beta$ [74]. Also note that the ratios $Z = Z_P/(Z_SZ_A)$ are consistent with

| Ensemble | $am_\pi$ | $\bar{m}_\pi$ | $aF_\pi^{\text{lat}}$ | $g_\pi^{\text{lat}}$ | $g_A^{\text{lat}}$ | $g_S^{\text{lat}}$ | $g_T^{\text{lat}}$ |
|----------|---------|-------------|----------------|----------------|----------------|----------------|----------------|
| I        | 0.11516(73)(11) | 0.003670(38)(10) | 0.05056(18)(07) | 1.3714(24)(03) | 1.566(23)(14) | 1.59(17)(05) | 1.239(19)(16) |
| II       | 0.1544(96)(26) | 0.007087(43)(06) | 0.04841(43)(05) | 1.3461(87)(04) | 1.473(31)(04) | 1.15(19)(03) | 1.275(35)(07) |
| III      | 0.15298(43)(16) | 0.007964(32)(10) | 0.04943(28)(03) | 1.3387(17)(01) | 1.550(15)(09) | 1.35(07)(03) | 1.264(14)(11) |
| IV       | 0.10675(51)(08) | 0.003794(27)(06) | 0.04416(30)(05) | 1.3539(57)(05) | 1.491(30)(02) | 1.58(18)(11) | 1.188(30)(11) |
| V        | 0.10465(37)(08) | 0.003734(21)(04) | 0.04499(12)(04) | 1.3473(30)(05) | 1.600(19)(09) | 1.49(14)(03) | 1.267(20)(05) |
| VI       | 0.10487(24)(04) | 0.003749(16)(08) | 0.04490(12)(04) | 1.3445(14)(04) | 1.585(17)(05) | 1.51(09)(02) | 1.221(17)(04) |
| VII      | 0.05786(51)(21) | 0.001129(18)(04) | 0.04048(48)(13) | 1.3395(120)(04) | 1.521(28)(02) | 1.48(38)(05) | 1.196(27)(20) |
| VIII     | 0.05425(40)(28) | 0.000985(17)(08) | 0.04029(30)(03) | 1.3440(110)(17) | 1.540(26)(03) | 1.68(28)(13) | 1.181(17)(07) |
| IX       | 0.15202(53)(06) | 0.009323(21)(13) | 0.04351(33)(03) | 1.3141(15)(02) | 1.489(14)(00) | 1.57(07)(03) | 1.201(22)(10) |
| X        | 0.13073(55)(28) | 0.007005(23)(04) | 0.04152(27)(03) | 1.3190(23)(04) | 1.492(15)(00) | 1.42(10)(01) | 1.249(20)(05) |
| XI       | 0.07959(25)(09) | 0.002633(13)(04) | 0.03651(33)(04) | 1.3233(50)(06) | 1.540(19)(09) | 1.51(15)(02) | 1.179(17)(18) |
TABLE III. The critical hopping parameters $\kappa_{\text{crit}}$, $m = 0$ plaquette values $P$ and renormalization constants [68] of the lattice currents relative to the $\overline{\text{MS}}$-scheme at $\mu = 2$ GeV. The errors given include systematics.

| $\beta$ | $\kappa_{\text{crit}}$ | $P$ | $Z_A$ | $Z_V$ | $Z_S^{\overline{\text{MS}}}(2 \text{ GeV})$ | $Z_P^{\overline{\text{MS}}}(2 \text{ GeV})$ | $Z_T^{\overline{\text{MS}}}(2 \text{ GeV})$ |
|---------|----------------|-----|-------|-------|---------------------------------|---------------------------------|---------------------------------|
| 5.20    | 0.1360546(39)  | 0.53861 | 0.7532(16) | 0.7219(47) | 0.6196(54) | 0.464(12) | 0.8356(15) |
| 5.29    | 0.1364281(12)  | 0.54988 | 0.76487(64) | 0.7365(48) | 0.6153(25) | 0.476(13) | 0.8530(25) |
| 5.40    | 0.1366793(11)  | 0.56250 | 0.77756(33) | 0.7506(43) | 0.6117(19) | 0.498(09) | 0.8715(14) |

In Fig. 14, $g_V/Z_V \equiv g_{V0}^{\text{lat}}(1 + amb_V)$ as a function of $m_\pi^2$ for all ensembles. Symbols are as in Fig. 1. Shown as solid bands are the $1/\zeta$-values determined non-perturbatively [68] (updated in Table III) for the three $\beta$-values.

In Fig. 15, $g_A$ as a function of $m_\pi^2$ for all ensembles. Symbols are as in Fig. 1. The square corresponds to $Lm_\pi \approx 6.7$, circles to $Lm_\pi > 4.1$, stars to $Lm_\pi \in [3.4,4.1]$ and the triangle to $Lm_\pi \approx 2.8$. The line drawn to guide the eye represents the result of a linear fit to the four $m_\pi < 430$ MeV points with $Lm_\pi > 4.1$. We find perfect agreement within errors. The non-perturbative determination of $Z_A$ is very similar to that of $Z_V$. Therefore, based on this independent validation of $g_V = 1$, we would not expect any problems related to the renormalization of $g_A$ either.

In Fig. 15, we show the renormalized axial coupling as a function of the squared pion mass for all ensembles. The different symbols encode the linear lattice extents $Lm_\pi$ and the colours the lattice spacings, see Fig. 1. Finite lattice spacing effects cannot be resolved within our errors. Comparing volumes similar in units of $m_\pi$, $g_A$ increases with decreasing pion mass. It also increases, enlarging the volume at a fixed pion mass: by about 5% increasing $Lm_\pi$ from 3.7 to 4.9 at $m_\pi \approx 425$ MeV and by about 6% going from $Lm_\pi \approx 3.4$ to 4.2 at $m_\pi \approx 290$ MeV. When further pushing $Lm_\pi$ from 4.2 to 6.7, $g_A$ remains constant within a combined error of 1.7%. At the near-physical pion mass the larger volume has an extent $Lm_\pi \approx 3.5$ only, possibly explaining the underestimation of the experimental value by about 7%. Unfortunately, at this pion mass, we do not have a volume with $Lm_\pi > 4.1$ at our disposal which would have required simulating a spatial box of $80^3$ points. There is little effect, however, moving from $Lm_\pi \approx 3.5$ down to $Lm_\pi \approx 2.8$. One should not over-interpret this though as it is conceivable that the volume dependence could be small within some range of volumes, due to other effects competing with $N_\pi$ and $\Delta_\pi$ loop corrections. Naively, one would expect volume effects mediated by pion exchange to be proportional to $m_\pi^2$ when keeping the lattice extent fixed in terms of the pion Compton wave length. Comparing the 290 MeV pion mass points to the 425 MeV points, there is no indication though for the change being larger in the latter case, suggesting a more complex behaviour — at least for $Lm_\pi < 4$. Fitting the $Lm_\pi > 4.1$ values of $g_A(m_\pi^2)$ alone for $m_\pi < 430$ MeV as a linear function of $m_\pi^2$ gives the line drawn in Fig. 15 illustrating the remarks made above. The line suggests consistency with experiment. At the physical point it reads $g_A = 1.242(15)$, two standard deviations below the known value. However, clearly, with few ensembles at small quark masses and $Lm_\pi > 4$, we cannot at present perform such an extrapolation with
FIG. 16. $g_A$ as a function of $m_\pi^2$; our results (RQCD, non-perturbatively improved (NPI) Wilson-clover) in comparison with other results (fermion action used in brackets). $N_f = 2$: QCDSF [26] (NPI Wilson-clover), Mainz [28] (NPI Wilson-clover), ETMC [29] (twisted mass), $N_f = 2 + 1$: LHPC [23] (HEX-smeared Wilson-clover), RBC/UKQCD [27] (domain wall). $N_f = 2 + 1 + 1$: ETMC [35] (twisted mass), PNDME [39] (Wilson-clover on a HISQ staggered sea). Also indicated as a shaded area is the result from extrapolating our $g_A/F_\pi$ data to the physical point, see Sec. IV. 

any confidence, in particular as the slope is expected to change its sign towards very small pion masses, see, e.g., Ref. [25] as well as Sec. IV] below.

Prior to investigating the finite volume behaviour in more detail in the next section, in Fig. 16 we put our $N_f = 2$ results on $g_A$ in perspective, comparing these to recent determinations obtained by other collaborations, namely QCDSF [26], the Mainz group [28] and ETMC [29] for $N_f = 2$, LHPC [23] and RBC/UKQCD [27] for $N_f = 2 + 1$ as well as ETMC [35] and PNDME [39] for $N_f = 2 + 1 + 1$. Most errors displayed are larger than ours, which include the systematics from the renormalization factors, varying fit ranges and parametrizations. This precision is in particular due to our large numbers of measurements and the effort that went into the optimization of the nucleon interpolators. We also indicate in the figure as a shaded area the result of a chiral extrapolation of our data on the ratio $g_A/F_\pi$, which we expect to be less affected by finite volume effects, see Sec. IV.

Note that the recent QCDSF study [26] utilizes a smearing different from ours for $m_\pi > 250$ MeV but has significant overlap in terms of the gauge ensembles and the values of $Z_A$ used. These results also carry quite small errors, however, their $g_A$-values are systematically lower, suggesting in these cases that smearing could be an issue, see Fig. 5. The left-most point of that study, that they associate with $m_\pi \approx 130$ MeV, was obtained using the same smearing that we employ on a sub-set of ensemble VII [$m_\pi(L) \approx 160$ MeV, $Lm_\pi \approx 2.8$, $m_\pi(\infty) \approx 149.5$ MeV]. Their result at this point (left-most circle) is compatible within errors not only with experiment but also with our corresponding high statistics result (second red square from the left).

Within errors all recent determinations (with the exception of $m_\pi > 250$ MeV QCDSF results) are consistent with our data. In particular, differences between including the strange or even the charm quark or ignoring these vacuum polarization effects are not obvious. Moreover, in all studies the $g_A$-values appear to be constant or increasing with decreasing pion mass and, where this could be resolved, correlated with the lattice size. In none of the simulations could any significant lattice spacing effects be detected.

IV. FINITE SIZE EFFECTS AND THE AXIAL CHARGE $g_A$

Above we have seen a noticeable dependence of $g_A$ on the lattice volume for $Lm_\pi < 4.1$. Chiral perturbation theory not only predicts the functional form of the pion mass dependence of hadronic observables but also their finite volume effects, as long as $m_\pi$ is small enough and $\lambda = Lm_\pi$ sufficiently large. To leading non-trivial order [70, 77], the finite size effects on the pion mass read

$$\frac{m_\pi(L) - m_\pi}{m_\pi} = \frac{2}{N_f} h(Lm_\pi, m_\pi), \quad (28)$$

$$h(\lambda, m_\pi) = \frac{m_\pi^2}{16\pi^2 F^2} \sum_{n \neq 0} \frac{K_1(\lambda|n|)}{|\lambda|}, \quad (29)$$

where $F$ is the pion decay constant in the chiral limit, $m_\pi = m_\pi(\infty)$ is the infinite volume pion mass, $\mathbf{n} \in \mathbb{Z}^3$ are integer component vectors and $K_1(x)$ is the modified Bessel function of the second kind.

The only parameter appearing in Eq. (28), apart from $F = 85.8(6)$ MeV [93, 78], is the infinite volume pion mass. Going beyond this order of chiral perturbation theory [79, 80], several low-energy constants (LECs) are encountered, namely $\ell_i, i = 1, 2, 3, 4$ at $O(p^4)$ and $\tilde{\ell}_i(m_\pi), i = 1, 2, \ldots, 6$ at $O(p^6)$ (next-to-next-to-leading order, NNLO). We use the parametrization with NNLO chiral perturbation theory input of Ref. [50] to investigate finite volume effects of the pion mass, setting $F = 86$ MeV and using the FLAG values [73] $\ell_3 = 3.41(41), \ell_4 = 4.62(22)$ for these two LECs. For $\ell_1, \ell_2$ and $\tilde{\ell}_i$ we take the central values given in Ref. [51] that were also used in Ref. [80].

We are now in a position to estimate the infinite volume pion mass. We do this by matching the NNLO finite size formula [50] in each case to the pion mass obtained on the largest available volume. Extrapolating this to infinite volume lowers the central value of the pion mass on ensemble III from 422.2 MeV by half a standard

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5 For each of the ensembles studied by the Mainz group two results are given in their article, obtained from plateau fits and from the summation method. We include the summation results since this appears to be their preferred method.
deviation to 421.5 MeV, that on ensemble VI (289.5 MeV) by 0.02 MeV and that on ensemble VIII from 149.7 MeV by one sixth of a standard deviation to 149.5 MeV. Having eliminated the free parameter by this matching, we can compare the combination \([m_\pi(L) - m_\pi]/m_\pi^3\) to the leading order chiral expectation \(h(\lambda, m_\pi)/m_\pi^2\), see Eqs. (28) and (29), and the NNLO formula of Ref. [80]. This comparison is shown in Fig. 17. Note that we omit the \(m_\pi \approx 150\) MeV data from the figure. In this case \([m_\pi(3.42\text{ fm}) - m_\pi]/m_\pi^3 \approx 3\) GeV\(^{-2}\), well off the scale of the figure, while the leading order prediction Eq. (28) amounts to 0.20 GeV\(^{-2}\) and the NNLO prediction [80] to 0.27 GeV\(^{-2}\). On one hand the expansion seems to break down around \(Lm_\pi \approx 3.5\) where the differences between the leading order and NNLO curves become large. Already the leftmost point shown in the figure appears to deviate from the predictions. On the other hand, in the safe \(Lm_\pi > 4\) region, the exponentially small finite size effects cannot be resolved within the precision of the lattice data.

In Refs. [76, 77] the leading order finite size expression of the pion decay constant is given too:

\[
F_\pi(L) - F_\pi = -2N_c h(Lm_\pi, m_\pi). \tag{30}
\]

The leading order finite volume effect of the axial charge in SU(2) chiral perturbation theory contains the same \(h(\lambda, m_\pi)\) term [82, 83]:

\[
g_A(L) - g_A(\infty) = -4h(Lm_\pi, m_\pi) + D(L, m_\pi, \Delta_0), \tag{31}
\]

where \(g_A^0 = g_A(\infty)\) at \(m_\pi = 0\) and we have suppressed the pion mass dependence of \(g_A(L)\). The correction \(D(L, m_\pi, \Delta_0)\) has been computed taking into account also transitions between the nucleon and the \(\Delta(1232)\) resonance in Ref. [84], using the small scale expansion (SSE) technique [85]. Consequently, it depends on the mass difference \(\Delta_0\) between the nucleon and the real part of the \(\Delta\) pole as well as on the squares of the pion-nucleon-nucleon and pion-nucleon-\(\Delta\) couplings and the ratio of the \(\Delta\) axial charge over the nucleon axial charge \(g_A^0/g_A\).

In the chiral limit the pion-nucleon-nucleon and pion-nucleon-\(\Delta\) couplings can be re-expressed in terms of \(g_A^0\), see Eq. (8), and the axial transition charge \(e_\Delta^0\) respectively. In the SU(2) quark model \(g_A^0/g_A = 9/5\). Note that, although this may not be obvious immediately, the result of Ref. [82] is identical to the expression of Ref. [84] in terms of the volume-dependence Eq. (31).

In Ref. [26] an approximate cancellation between different contributions to \(D(L, m_\pi, \Delta_0)\) over a large range of \(L\) and \(m_\pi\)-values was observed, which motivated the authors to study the ratio \(g_A/F_\pi\). From Eqs. (30) and (31) we obtain to leading one-loop order (i.e. \(O(\epsilon^2)\) in the SU(2) SSE [85])

\[
g_A(L) = g_A(\infty) \frac{1 - \frac{g_A^0}{g_A(\infty)}[4h(L) - D(L, \Delta_0)]}{1 - 4h(L)} \tag{32}
\]

For \(F_\pi(L)\) also the next-to-leading order and NNLO corrections are known [80], however, to be consistent in terms of the order of the SSE, we do not add these here. We set \(g_A^0 = 1.21\) (see below), \(c_\Lambda^0 = 1.5\) [80], \(g_A^0 = 2.2 \approx (9/5)g_A^0\) and \(\Delta_0 = 272\) MeV [57]. In Fig. 18 we show the resulting curves for the infinite volume pion masses \(m_\pi = 149.5\) MeV, \(m_\pi = 289.5\) MeV and \(m_\pi = 421.5\) MeV as functions of \(Lm_\pi\). The normalization \(g_A(\infty)/F_\pi\) will depend on the pion mass and is adjusted to match the three data sets while the error band is from varying \(g_A^0/g_A(\infty)\) in \([0.9, 1.1]\) within Eq. (32). Indeed, finite volume effects are much reduced, relative to those for \(g_A\) visible in Fig. 17 and these are also broadly consistent with the predicted behaviour.
Finally, in Fig. 19 we show the ratio $g_A / F_\pi$ as a function of the squared pion mass, together with a linear fit to the low mass points, omitting the smallest volume (ensemble VII). Symbols are as in Fig. 1.

![Figure 19](image)

**FIG. 19.** $g_A / F_\pi$ as a function of $m_\pi^2$ for all ensembles, together with a linear fit to the low mass points, omitting the smallest volume (ensemble VII). Symbols are as in Fig. 1.

For $m_\pi < 300$ MeV data, omitting the $Lm_\pi < 3.4$ data point (ensemble VII). This fit, with a reduced $\chi^2 / N_{\text{DF}} = 5.9 / 4$, gives $g_A / F_\pi = 13.88(29)$ GeV$^{-1}$ at $m_\pi = 135$ MeV which compares well with the experimental result $g_A / F_\pi = 13.79(34)$. Using $F_\pi = 92.21(15)$ MeV \[31\] at the physical point as an input, this gives $g_A = 1.280(27)(35)$, where the second error corresponds to the overall uncertainty of assigning physical values to our lattice spacings \[54\] (not shown in the figure).

We remark that towards the chiral limit $g_A$ decreases with decreasing pion mass while the observed increase of the ratio $g_A / F_\pi$ is entirely due to an also decreasing pion decay constant. Towards large pion masses $F_\pi$ will continue to increase while $g_A$ eventually starts decreasing again.

From $F_\pi / F = 1.0744(67)\ [78]$ we obtain the ratio $g_A / g_A^0 = 1.050(14)$, giving $g_A^0 = 1.211(16)$ using $g_A = 1.2723(23)\ [3]$. Using the normalization conventions

$$g_A(m_\pi) = g_A^0 \left(1 + \frac{m_\pi^2}{16\pi^2 F^2} \bar{\ell} + \cdots \right),$$

$$F_\pi = F \left[1 + \frac{m_\pi^2}{16\pi^2 F^2} \bar{\ell}_4 + \cdots \right]$$

for the leading chiral corrections, one obtains

$$g_A(m_\pi) = \frac{g_A^0}{F} + \frac{g_A^0}{16\pi^2 F^3} (\bar{b} - \bar{\ell}_4) m_\pi^2 + \cdots .$$

From our fit we find $\bar{b} - \bar{\ell}_4 = -1.41(36)$ and, using $\ell_4 = 4.62(22)\ [78]$, arrive at the value $\bar{b} = 3.21(42) > 0$ for this LEC: $g_A$ increases with the pion mass (as is also obvious from the ratio $g_A(135\text{ MeV}) / g_A^0 > 1$ above). Note however that $g_A$ is expected to start decreasing towards larger pion masses, due to the effect of the nearby $\Delta(1232)$ resonance \[75\ \[88\]. This is also reflected in the lattice data, see Fig. 15.

We did not detect any lattice spacing effects within our statistical errors and therefore so far have ignored these. Not being able to resolve such differences does not mean they are absent and we will re-address this issue in the summary Sec. VI.

V. THE SCALAR, TENSOR AND PSEUDOSCALAR CHARGES

The scalar and tensor couplings can be obtained directly in the forward limit of Eqs. (1) and (5) while the induced tensor and pseudoscalar charges are extracted from extrapolating the respective form factors Eqs. (3) and (4) to small virtualities. We will also determine the value of the induced pseudoscalar form factor $g_p^S = \tilde{g}_p(Q^2)$ at the virtuality $Q^2 = -q^2 = 0.88 m_\pi^2 \approx 9.82 \cdot 10^{-3}$ GeV$^2$, corresponding to muon capture \[18\].

A. The scalar charge $g_S$

In Fig. 20 we show our results for $g_S$ as a function of $m_\pi^2$. Within their large errors the $m_\pi < 430$ MeV data are consistent with a linear extrapolation and we find no lattice spacing or volume dependence. The result of such an extrapolation to the physical point, fitting the six $m_\pi < 300$ MeV data points with $Lm_\pi > 3.4$ is shown in the figure. We find $g_S^{\text{MS}}(2\text{ GeV}) = 1.02(18)$ for a fit with $\chi^2 / N_{\text{DF}} = 0.48 / 4$.

The charge $g_S$ can, via the conserved vector charge relation, also be obtained as the ratio of the mass splitting of proton and neutron in the absence of electromagnetic interactions over the difference of light quark masses. The determination of this requires either further
FIG. 21. \(g_{S}^{\pi}(2\text{ GeV})\) as a function of \(m_{\pi}^{2}\): our results (RQCD, NPI Wilson-clover) in comparison with other results. \(N_{f} = 2 + 1\): LHPC \(^{32}\) (HEX-smeared Wilson-clover). \(N_{f} = 2 + 1 + 1\): PNDME \(^{39}\) (Wilson-clover on a HISQ staggered sea) and ETMC \(^{31}\) (twisted mass). Also included is the linear extrapolation of our data points.

assumptions or lattice simulations of QCD plus (Q)ED with electrically charged quarks. Recently, such lattice input was used in Ref. \(^{50}\) to give \(g_{S} = 1.02(11)\). However, not all systematic uncertainties were accounted for in the error estimate. The central value agrees with our direct determination.

In Fig. 21 we compare our results on \(g_{S}\) to recent lattice determinations by other groups, namely LHPC \(^{32}\), employing \(N_{f} = 2 + 1\) HEX-smeared Wilson-clover fermions, PNDME \(^{39}\), using clover valence fermions on top of a \(N_{f} = 2 + 1 + 1\) highly improved staggered quark (HISQ) sea and ETMC \(^{31}\left(^{41}\right)\), using \(N_{f} = 2 + 1 + 1\) twisted mass fermions. The errors of LHPC are quite large while there appears to be some tension between our results and those of PNDME. Notwithstanding this, around any single pion mass value all results are compatible with each other as well as with our extrapolation on the level of two standard deviations.

B. The tensor charge \(g_{T}\)

In Fig. 22 we show our results on \(g_{T}\). Again, we cannot detect any lattice spacing or volume effects. Note that for our three \(a \approx 0.071\) fm points at \(m_{\pi} \approx 290\) MeV (\(m_{\pi}^{2} \approx 0.084\) GeV\(^{2}\)), the central value for the largest volume (\(Lm_{\pi} \approx 6.7\)) lies inbetween those for the \(Lm_{\pi} \approx 3.4\) and \(Lm_{\pi} \approx 4.2\) lattices. Again, we show a linear extrapolation to the physical point which gives \(g_{T}^{\pi}(2\text{ GeV}) = 0.85\). Unlike in the case of \(g_{A}\) we regard such an extrapolation of \(g_{T}\) as safe since there are no indications of finite volume effects and our lowest mass point \(m_{\pi} \approx 150\) MeV is already very close to the physical pion mass \(m_{\pi} = 135\) MeV. This conclusion is also supported by Fig. 23 where we compare our results to those of ETMC \(^{31}\) (\(N_{f} = 2 + 1\) domain wall fermions), RBC/UKQCD \(^{30}\) (\(N_{f} = 2 + 1\) HEX-smeared Wilson-clover), LHPC \(^{32}\) (\(N_{f} = 2 + 1 + 1\) sea), and ETMC \(^{31}\) (\(N_{f} = 2 + 1 + 1\) twisted mass). Also included is the linear extrapolation of our data.

\(1.005(17)\) with \(\chi^{2}/N_{\text{DF}} = 6.0/4\). Unlike in the case of \(g_{A}\) we regard such an extrapolation of \(g_{T}\) as safe since there are no indications of finite volume effects and our lowest mass point \(m_{\pi} \approx 150\) MeV is already very close to the physical pion mass \(m_{\pi} = 135\) MeV. This conclusion is also supported by Fig. 23 where we compare our results to those of ETMC \(^{31}\) (\(N_{f} = 2 + 1\) domain wall fermions), RBC/UKQCD \(^{30}\) (\(N_{f} = 2 + 1\) domain wall fermions), LHPC \(^{32}\) (\(N_{f} = 2 + 1 + 1\) HEX-smeared Wilson-clover), and ETMC \(^{31}\) (\(N_{f} = 2 + 1 + 1\) twisted mass). No correlation with the sea quark content, volume, lattice action or lattice spacing is obvious. Moreover, all these determinations are statistically consistent with each other as

\[a \approx 0.08\text{ fm}, a \approx 0.07\text{ fm}, a \approx 0.06\text{ fm}\]

\(^{6}\) At \(m_{\pi} = 370\) MeV we show their \(t_{r} = 14a \approx 1.14\) fm result. In this reference also \(N_{f} = 2\) results at \(m_{\pi} \approx 126\) MeV can be found: 1.01(46) at \(t = 12a \approx 1.13\) fm and 1.63(76) at \(t = 14a \approx 1.32\) fm.
well as with our extrapolation.

C. The induced tensor charge $\tilde{g}_T$

The induced tensor coupling $\tilde{g}_T = \kappa_{u-d} \approx \kappa_p - \kappa_n \approx 3.706$ is well-determined experimentally. Computing $\tilde{g}_T$ requires an extrapolation of lattice data obtained at virtualities $Q^2 > 0$ to $Q^2 = 0$. At small $Q^2$ one can expand

$$ g_V(Q^2) = 1 - \frac{r_2^V}{6} Q^2 + O(Q^4), \quad (36) $$

$$ \tilde{g}_T(Q^2) = \tilde{g}_T(0) \left[ 1 - \frac{r_2^T}{6} Q^2 + O(Q^4) \right], \quad (37) $$

where the proton isovector Dirac and Pauli radii $r_1$ and $r_2$ diverge as the pion mass approaches zero. It is well known that the $Q^2$-dependence exhibits a substantial curvature, see, e.g., Refs. 35 38 39 41 42 45 90. This means small $Q^2$-values are required for a controlled extrapolation, in particular at small quark masses where the coefficient $r_2^T$ of the leading $Q^2$-term becomes large. We expect this effect to partially cancel from the ratio

$$ \frac{\tilde{g}_T(Q^2)}{g_V(Q^2)} = \frac{\tilde{g}_T(Q^2)}{g_V(Q^2)} Q^2 \to 0 \tilde{g}_T. \quad (38) $$

Therefore, one of our strategies is to extrapolate this ratio as a linear function of $Q^2$ to $Q^2 = 0$.

Another parametrization that incorporates the curvature is a dipole fit

$$ g_T(Q^2) = \frac{\tilde{g}_T(0)}{(1 + Q^2/m_{\rho}^2)^2}. \quad (39) $$

Note that the electric Sachs form factor reads $G_E(Q^2) = g_V(Q^2) - Q^2/(4m_{\rho}^2)\tilde{g}_T(Q^2)$. Therefore, in the isospin symmetric limit, the squared charge radius is given as $r_2^\rho = r_2^T + \tilde{g}_T/(2m_{\rho}^2)$.

Taylor expanding this expression, the linear approximation Eq. (37) should be valid for $Q^2 < m_{\rho}^2 \approx 12/r_2^\rho$. We show both extrapolations, Eqs. (38) and (39), for our three $m_\pi \approx 290$ MeV volumes (ensembles IV, V and VI, see Fig. 1) in Fig. 24. The $\tilde{g}_T/g_V$ data (shown in the left panel) are compatible with a linear behaviour down to our largest $Q^2 \approx 0.6$ GeV$^2 \approx m_{\rho}^2$ value, however, in this case we restrict ourselves to the range $Q^2 < 0.4$ GeV$^2$ to keep $Q^2 < m_{\rho}^2 \approx m_{\rho}^2$. Note that for $Lm_\pi = 3.4$ only one point lies within this window, so no extrapolation is possible. In the right panel we show the corresponding dipole fits to the $Q^2 < 0.6$ GeV$^2$ data. We see no significant volume dependence between the $Lm_\pi = 3.4, 4.2$ and 6.7 data. Moreover, all five extrapolated values are consistent with each other.

We repeat this procedure for all ensembles and take the
central value from dipole fits, adding in quadrature to the statistical error an uncertainty from taking the difference between using the two extrapolation methods and varying the fit range. The resulting induced tensor charges are shown in Fig. 25 as a function of $m_\pi^2$. Due to the different volumes the numbers of points within the fit ranges vary considerably, thus giving rise to significantly fluctuating error sizes. We extrapolate the $m_\pi < 300$ MeV, $Lm_\pi > 3.4$ data linearly to the physical point, obtaining $\tilde{g}_T = 3.00(8)$, which is significantly smaller than the experimental value 3.706. While there could be a deviation between this value and the one relevant for the isospin symmetric approximation, one would not expect this to exceed eight of our standard deviations. It is interesting that results obtained at larger pion masses are closer to experiment than our lowest mass point, which dominates the extrapolation. Small volumes result in a larger low-momentum cut-off and a significant loss of precision which complicates resolving the volume dependence. In general, the central values increase with the lattice size and this deserves further study.

In Fig. 25 we compare our results on $\tilde{g}_T$ to recent lattice determinations by other groups, namely QCDSF [12], the Mainz group [25, 40] and ETMC [43] for $N_f = 2$, LHPC [45] and RBC/UKQCD [38] for $N_f = 2 + 1$ as well as ETMC [45] and PNDME [39] for $N_f = 2 + 1 + 1$. With the exception of one LHPC point, that carries one
were obtained for this ratio. Here, we find deviations from single pole dominance to increase towards low momenta, thereby ruling out that a dominant part of these violations can be ascribed to lattice spacing effects.

The induced pseudoscalar coupling for muon capture $g_p^\mu$ is defined in Eq. 10. It can be obtained, extrapolating the induced pseudoscalar form factor $(m_{\mu}/m_N)\tilde{g}_P(Q^2)$ to $Q^2 = 9.82\times 10^{-3}$ GeV$^2$. We employ a phenomenological parametrization that incorporates the leading pole:

$$
\frac{m_{\mu}}{m_N} \tilde{g}_P(Q^2) = \frac{c_1}{m_{\pi}^2 + Q^2} + c_2 + c_3 Q^2,
$$

where the parameters $c_1 < 4 m_{\pi}^2 g_A^0$, $c_2$ and $c_3$ are fitted separately for each ensemble. The terms involving $c_2$ and $c_3$ turn out to be necessary to approximate corrections to the pole ansatz, which are regular at positive virtualities.

We display the resulting extrapolations for three pion masses (ensembles III, VI and VIII) in Fig. 28. We are not able to reliably determine the above form factor for $Q^2 > 1$ GeV$^2$ which means results cannot be obtained for the small volume ensembles II, IX and X, where less than four data points are within this range. We show the remaining eight results in Fig. 29 as a function of the squared pion mass. A phenomenological fit of the $m_\pi < 300$ MeV, $L m_\pi > 3.4$ data to the functional form

$$
g_P(m_\pi^2) = \frac{a_1}{m_\pi^2 + a_2},
$$

with parameters $a_1$ and $a_2$, gives $g_P^* = 8.40(40)$ at the physical point with a $\chi^2/N_{\text{DF}} = 6.4/4$. Since our nearly physical $m_\pi \approx 150$ MeV point dominates the extrapolated value, this is robust against changes of the parametrization. The number obtained compares well with the recent experimental determination of the MuCap Collaboration $^{91}$ $g_P = 8.06(55)$ and also with the determinations $g_P = 8.44(23)$ \cite{93} or $g_P = 8.21(9)$ \cite{10} from heavy baryon chiral perturbation theory or $g_P = 8.29^{+24}_{-13}(52)$ \cite{92} from covariant baryon chiral perturbation theory. Previously, the RBC and UKQCD collaborations \cite{85} obtained $g_P = 6.6(1,0)$, extrapolating $N_f = 2 + 1$ domain wall fermion results to the physical point.

The flavour changing coupling constant $g_\pi N N$ between the nucleon and the charged pion is defined as the residue of the pole of the induced pseudoscalar form factor at $Q^2 = -m_\pi^2$:

$$
g_\pi N N \equiv \lim_{Q^2 \to -m_\pi^2} \frac{m_\pi^2 + Q^2}{4 m_N F_\pi} \tilde{g}_P(Q^2).
$$

Implementing the above definition requires an extrapolation of lattice data, which is limited to positive virtualities. Figure 27 demonstrates that corrections to the pole dominance model become significant towards small virtualities. Assuming the parametrization Eq. 41, we obtain $g_{\pi N N} = c_1/(4 m_N F_\pi)$, which then needs to be extrapolated to the physical pion mass. However, it is already obvious from Fig. 28 that a controlled extrapolation of $Q^2 \gtrsim 0.1$ GeV$^2$ data to negative virtualities is hardly possible. Indeed, playing around with different parametrizations of $\tilde{g}_P(Q^2)$ that assume a pole at $Q^2 = -m_\pi^2$, values ranging from $g_{\pi N N} \sim 8$ up to $g_{\pi N N} \sim 14$ can easily be produced from our lattice data.

The Goldberger-Treiman relation $g_{\pi N N} \approx m_N g_A/F_\pi$ does not require such an extrapolation, however, it is subject to $O(m_\pi^2)$ corrections. The relative difference between $g_{\pi N N}$ defined in Eq. 43 and this approximation is known as the Goldberger-Treiman discrepancy

$$
\Delta_{\pi N} = \frac{1}{g_{\pi N N}} \left[ g_{\pi N N} - m_N \frac{g_A}{F_\pi} \right] \bigg|_{m_\pi = 135 \text{ MeV}}.
$$
Using the experimental values of \( m_N, g_A \) and \( F_\pi \), the Goldberger-Treiman relation amounts to \( g_{\pi NN} \approx 12.96(3) \) while determinations of \( g_{\pi NN} \) from \( N\pi \) scattering data result in values \( g_{\pi NN} = 14.11(20) \) \cite{62}, \( g_{\pi NN} = 13.76(8) \) \cite{93} or \( g_{\pi NN} = 13.69(19) \) \cite{95}. We remark that obtaining these values also involves extrapolating in \( Q^2 \). Combining the last number quoted above with the Goldberger-Treiman relation translates into \( \Delta_{\pi NN} = 0.053(13) \). Experimental data, both from nucleon-nucleon scattering and pionic atoms, have been analysed systematically in the framework of covariant baryon chiral perturbation theory in Ref. \cite{96} (see also references therein), with the central values obtained for \( g_{\pi NN} \) ranging from 13.0 to 14.1, depending on the experimental input and the method used (with or without including the \( \Delta \) resonance).

In Fig. 30 we plot the combination
\[
\frac{m_N g_A}{F_\pi} = m_N \frac{g_{A}^{\text{lat}}}{F_\pi^{\text{lat}}} = g_{\pi NN} \left[ 1 + O \left( \frac{m_{\pi}^2}{m_N^2} \right) \right] \tag{45}
\]

\( \text{versus } m_{\pi}^2 \), see Eq. \( \text{[8]} \). As demonstrated in Sec. \text{[IV]} finite volume effects between \( g_A \) and \( F_\pi \) partially cancel, however, the nucleon mass adds a new source of volume dependence. Extrapolating the combination Eq. \( \text{(45)} \) to the physical pion mass corresponds to the Goldberger-Treiman approximation while extrapolating it to \( m_\pi = 0 \) gives the pion-nucleon-nucleon coupling in the chiral limit. A linear fit to the \( Lm_\pi > 4.1 \) data (indicated as a line) results in \( g_{\pi NN}(m_\pi = 0) = 13.62(32) \). This is broadly consistent with the phenomenological values \cite{62,95} that can differ by \( O(m_{\pi}^2) \) terms. Note, however, that this fit overestimates the known value \( m_N g_A/F_\pi \approx 12.96 \) at the physical pion mass by two standard deviations. We conclude that while our results are consistent with expectations, predicting \( g_{\pi NN} \) at \( m_\pi > 0 \) or determining the Goldberger-Treiman discrepancy \( \Delta_{\pi NN} \) requires different methods, not least due to the significant violations of single pole dominance illustrated in Fig. \text{[27]}.

Finally, in Fig. 31 we show the pseudoscalar charge, obtained from the first equality in Eq. \( \text{(5)} \).

\[
g_{P}^{\text{MS}}(2 \text{ GeV}) = Z_P \frac{m_N}{m_{\text{ud}}} g_{A}^{\text{lat}} (1 + amb_A). \tag{46}
\]

Note that order-\( a \) improvement is already incorporated into our definition Eq. \( \text{(22)} \) of the lattice PCAC mass \( m_\pi \), which is why the coefficient \( b_1 \) rather than \( b_P \) appears above. \( Z_P, \kappa_{\text{crit}} \), and \( P \) [needed to compute \( amb_A \), see Eqs. \( \text{(23)} \) and Eq. \( \text{(26)} \)] can be found in Table \text{[III]} and \( g_{A}^{\text{lat}} \), the nucleon and lattice PCAC masses in Table \text{[II]} We expect \( g_P \) to diverge like \( 1/m_{ud} \) and thus, using the Gell-Mann-Oakes-Renner relation, to be proportional to \( 1/m_{ud}^2 \). Such a curve is drawn to guide the eye. Using the \( N_f = 2 \) value \( m_{\text{ud}}^{\text{MS}}(2 \text{ GeV}) = 3.6(2) \text{ MeV} \) of the FLAG Working group \cite{78}, from Eq. \( \text{(8)} \) we expect \( g_{P}^{\text{MS}}(2 \text{ GeV}) = 332(19) \) at the physical point. Our data are broadly consistent with this value: obviously our quark mass, extrapolated to \( m_\pi = 135 \text{ MeV} \), is consistent with the FLAG average.

\section{VI. SUMMARY}

We have computed all nucleon charges that may be relevant for non-standard model (and standard model) transitions \cite{0,8} between the neutron and the proton in lattice simulations with \( N_f = 2 \) mass-degenerate flavours of sea quarks. These isovector couplings are by definition valence quark quantities. Therefore, we do not expect significant effects from including strange (or charm) sea quarks. This claim is substantiated by comparison
TABLE IV. Summary of results, extrapolated to the physical point. The first errors contain statistics and systematics. The second errors are estimates of lattice spacing effects. $g_A$ was obtained, dividing by $F_\pi$ and therefore a scale setting error is included in the first error, that is not subject to further lattice spacing effects. To determine $g_A^0$ in the chiral limit, the experimental $g_A$-value was used as an input. The experimental $g_A$ and $\tilde{g}_\pi$ values were obtained as follows. To leading order in $\delta a$, the induced coupling $g_A$ and, by implication, the pseudoscalar and induced pseudoscalar form factors. These could be much reduced, considering ratios over the pion decay constant $F_\pi$, which shares a similar finite volume behaviour. Consistency checks were made, regarding the renormalization. The known results for $g_A$ and $g_A$ were reproduc.

The charges, extrapolated to the physical point, as well as $g_A$ in the chiral limit are summarized in Table IV. The first errors displayed contain our statistical and systematical uncertainties related to fit ranges and parametrizations used. The second errors are estimates of the maximally possible discretization effects. These were obtained as follows. To leading order in $a$, assuming $O(a^n)$ discretization effects, we can write $g(a) = g(0) + \delta a^n \Delta a^n = g(0) + \Delta a g$, where $g(0)$ denotes the continuum limit, $g(a)$ the result for this coupling determined at a fixed lattice spacing and the dimensionless constant $\delta_a$ is unknown. We varied the lattice constant from $a \approx 0.081$ fm down to $a \approx 0.060$ fm. The non-detection of any discretization effect means that our error on a coupling $g$ is bigger than the associated variation: $\Delta g > (0.081^{+0.060})|\delta_a|$. Our extrapolated results are dominated by points at $a = 0.071$ fm, meaning that we cannot exclude lattice corrections $\Delta a g = 0.071^0|\delta_a| < 0.071^n \Delta g/(0.081^{+0.060}) \approx 1.7 \Delta g (n = 2)$. Therefore, we multiply our errors by this factor. For the induced couplings $g^*$ and $\tilde{g}_\pi$ the leading discretization effects are linear in $a$ which is why in these cases we allow for discretization errors of $3.7 \Delta g$.

The errors not related to the lattice spacing vary significantly between different couplings. Therefore, our estimates of lattice spacing effects — if obtained as detailed above — become large for some of the channels. However, there is no obvious reason why some couplings should carry much larger discretization effects than others. This means in some cases, in particular for $g_S$ and $g_P^*$, our discretization error assignment may be overly conservative. However, in the absence of a real continuum limit extrapolation, we do not see any way of reliably estimating this remaining uncertainty.

In addition to the results displayed in Table IV we find values for the pion-nucleon-nucleon coupling $g_{\pi N N}$, defined in the chiral limit, consistent with experimental estimates, which may not be too surprising, given that $g_A$ comes out correctly. However, violations of the pole dominance model are found to be large, see Fig. 27. We also quote $g_P^{\overline{MS}}(2 \text{ GeV}) = 332(19)$, which is no independent determination as it relies on the FLAG Working Group quark mass average [78]. Moreover, we determined the low energy constant $\overline{b} = 3.21(42)$, defined in Eq. [33], that encodes the leading order chiral correction to $g_A^0$.

The disagreement between the anomalous magnetic moment $\tilde{g}_\pi = \tilde{g}_\pi(0)$ and experiment (see Table IV) is puzzling and deserves further attention. The determination of the induced couplings is less direct than computing $g_V$, $g_A$, $g_S$ and $\tilde{g}_\pi$ since it requires extrapolating form factors to vanishing virtuality, where the momentum resolution on a finite volume becomes an issue. The error of this extrapolation to the forward limit reduces with the minimal momentum available $\pi/L$ while finite volume effects are dominantly functions of the combination $L m_\pi$. Therefore, $L m_\pi \approx 3.5$ results at $m_\pi \approx 290 \text{ MeV}$ carry much larger errors than at $m_\pi \approx 150 \text{ MeV}$, which may hide finite volume effects. Moreover, we find excited state contributions to increase with $Q^2$. This behaviour, while under control at each single value of $Q^2$, may become amplified in the slope of the form factor and its extrapolation. We will discuss form factors in detail, including $\tilde{g}_\pi(Q^2)$, in a forthcoming publication.

While lattice calculations of baryon structure have not yet reached the level of precision of computations of quantities related to meson properties, it is now possible to obtain predictions, e.g., for the isovector scalar and tensor charges, with uncertainties that have an impact on beyond-the-standard-model phenomenology and in other cases, e.g., for $g_P^*$, to reduce errors to a level that is competitive with experimental determinations. The next obvious step is to significantly vary the lattice spacing, thus enabling a controlled continuum limit extrapolation, further reducing the remaining uncertainties.
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