A Framework of BSDEs with Stochastic Lipschitz Coefficients through Time Change

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Abstract

In this paper, we suggest an effective technique based on time-change for dealing with a large class of backward stochastic differential equations (BSDEs for short) defined in general space whose drivers have stochastic Lipschitz coefficients. By studying the deep properties of random time change combined with stochastic integral and measure theory, we show the relation between the BSDEs with stochastic Lipschitz coefficients and the ones with deterministic Lipschitz coefficients and stopping terminal time, so they are possible to be exchanged with each other from one type to another. In other words, the stochastic Lipschitz condition is not essential in the context of BSDEs with random terminal time. Next, we derive various results by applying our technique to some types of BSDEs such as Brownian motion BSDE or Markov chain BSDE.

Keywords: Backward Stochastic Differential Equations (BSDEs), time change, stochastic Lipschitz coefficient, random terminal time, Markov chain

MSC: 60H20, 60H15

1. Introduction

Since their first introduction by Bismut [6] in the linear case and the nonlinear extension by Pardoux and Peng [36], Backward stochastic differential equations (BSDEs for short) have been developed rapidly with various types of generalizations in the last decades. BSDEs are closely connected to finance, optimal control and partial differential equation etc.([24, 41, 37, 48]).

Most of BSDEs are concerned with the case of constant time horizon and the uniformly Lipschitz conditions on driver. In many environments, the Lipschitz condition is too restrictive to be assumed, so much effort have been devoted to relax it ([10, 12, 26, 29]).

In this context, El Karoui and Huang [23] studied the BSDEs with stochastic Lipschitz coefficients driven by a general càdlàg martingale and those were developed under weaker conditions in [13]. For the Brownian motion BSDEs, there are some papers going in this direction([2, 8, 5, 46, 40]). Particularly, in [2], Section 3, the existence of the measure solution was stated by the way of examining the weak convergence of a sequence of measures which were constructed using the martingale representation and the Girsanov change of measure. Also, the reflected

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backward stochastic differential equations or backward doubly stochastic differential equations (BDSDEs) with stochastic Lipschitz coefficients were studied in [25, 28, 31, 33, 34, 47].

Recently, the inclusive and generalized BSDEs with jumps were studied in the context of stochastic Lipschtz condition in [35].

Although the details are slightly different, the most techniques for the BSDEs with stochastic Lipschtz conditions are similar to the procedure of BSDEs with Lipschtz conditions.

That is, the techniques consist of using martingale representation theorem, obtaining a priori estimates and finally using the fixed-point arguments.

Other technique was also used in [18], where the Lipschtz approximation to the driver was introduced, some estimates were obtained for the convergence of approximation sequence and finally it was shown that the limit of this sequence is a unique solution.

In this paper, we approach the problem differently by indirect method. The technique is based on time change represented by stochastic Lipschtz coefficients. This time change converts the BSDEs with stochastic Lipschtz condition to the ones with uniformly Lipschtz condition and stopping terminal time on another stochastic basis and these two BSDEs are equivalent in some sense. So, if we know the results of BSDEs with random terminal time and uniformly Lipschtz coefficients, then the results are easily extended to the ones with stochastic Lipschtz coefficients through our framework. In other words, the stochastic Lipschtz condition is not a problem in a setting of BSDEs with random terminal time.

We briefly mention that the opposite argument also holds, that is, the randomness of terminal time do not play an essential role under the stochastic Lipschtz condition. During our discussion, if the integrator of the driver is a general continuous increasing process, it is converted to the typically well-known one, that is, the Lebesgue measure by time change.

Consequently, if we study only the BSDEs to stopping time with standard conditions - the driver satisfies the uniforml Lipschtz continuity, the integrator of the driver is Lebesgue measure, then the research on BSDEs with general conditions - the driver satisfies the stochastic Lipschtz condition, the integrator of the driver is a continuous increasing process is just a corollary of that.

And we apply our technique to the detailed BSDEs and get some improved and new results.

The prototype of BSDEs is of course Wiener-type BSDE, so we first apply our framework to the BSDE driven by Brownian motion. Here, we deal with the stochastic monotonicity condition more generally. It is clear that the better results in the setting of random terminal time we make use of, the better results in the stochastic Lipschtz setting are obtained. On the other hands, the BSDEs with random terminal time were well-studied sufficiently in many papers.

We note that our results include the comparison theorem. In fact, it is a natural question what the behavior of comparison theorem will be like by time change.

Here, we emphasize that the comparison theorem as well as wellposedness for BSDEs are easily extended to the stochastic one by our technique. With respect to the previous results in the setting of stochastic Lipschtz, we guarantee the results under weaker conditions on parameters. Moreover we show some new results in the various settings for BSDEs.

In this paper, we also apply our framework effectively to the Markov chain BSDE. The smart feature is that the discussion on the case of uniformly Lipschtz condition is just inherited to the case of stochastic Lipschtz condition under the same conditions on volumes.

In general, for the wellposedness of BSDEs with stochastic Lipschtz condition, the stronger integrability conditions are required than ones with uniformly Lipschtz condition. The main reason is on the discounting property of the terminal time. This discounting property is contributed to the exponential integrability conditions of volumes and these conditions are influenced by the
Lipschitz coefficients. In fact, discounting property is inherited from the monotonicity of the driver. In our framework, the original BSDE with stochastic Lipschitz condition can be shown as the BSDE to stopping time which is time-changed in reverse and the time-independent discounting rate of this BSDE with constant Lipschitz coefficients is preserved while time change is processed. This means that the stronger integrability conditions are still required if we use the results of BSDEs with random terminal time obtained by using the monotonicity condition as the key tool. But for the Markov chain BSDEs, the results of undiscounted BSDEs to stopping time without assuming the monotonicity which was researched by Samuel N. Cohen[14] make our technique more effective. By passing through the proposed framework, we get a new version of Markov chain BSDEs in the case where the driver has stochastic Lipschitz coefficients for the first time. We also give an example of the real model described as the Markov chain BSDEs with stochastic Lipschitz condition. At the end of the paper, we also show some further uses of time change for the BSDEs.

The rest of this paper is organized as follows. In Section 2, we suggest a general map from the BSDEs with stochastic Lipschitz coefficients to the ones with uniformly Lipschitz coefficients by the technique of time change. We discuss this for BSDEs in general space as in [17]. The applications to the Wiener-type BSDEs are shown in Section 3. We give new results on Markov chain BSDEs in Section 4. In Section 5, we give some concluding remarks.

Let us introduce some useful notations which are used in this paper. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space with a filtration \(\mathcal{F}:=(\mathcal{F}_t)_{t \geq 0}\) satisfying the usual conditions. We shall assume that \(\mathcal{F} = \mathcal{F}_\infty\) and \(\mathcal{F}_0\) is trivial.

- \(\|\cdot\|\) denotes the standard Euclidean norm. If \(z\) is a matrix, \(\|z\| = \text{Trace}(zz^T)\), where \([\cdot]^T\) means the vector transpose.
- \(\mathcal{B}(0, \infty)\) denotes the Borel-\(\sigma\)-field given on \((0, \infty)\).
- \((\Omega, \mathcal{F})\) means the product measurable space. That is \(\Omega := \Omega \times (0, \infty)\) and \(\mathcal{F} := \mathcal{F} \times \mathcal{B}(0, \infty)\).
- \(dQ/d\mu\) denotes the Radon-Nikodym derivative of \(Q\) with respect to \(\mu\), where \(Q\) is absolutely continuous with respect to \(\mu\). If \(\mu\) is Lebesgue measure and \(Q\) is the measure generated by an absolutely continuous function \(f\), then we use \(f^-\) rather than \(dQ/d\mu\).
- \(\mathbb{E}^Q[\cdot]\) means the expectation under measure \(Q\).
- \(L^2(\Omega, \mathcal{F}, \mathbb{P})\) is the space of square-integrable random variables.
- \(\mathcal{L}\) and \(L^c\) are the spaces of local martingales and continuous local martingales, respectively.
- \(\mathcal{H}^2\) is the space of square-integrable martingales.
- \(\mathcal{H}^2_T\) is the space of square-integrable martingales on \([0, T]\).
- \(\mathcal{H}_{loc}^2\) is the space of locally square integrable martingales.
- \(L^2(M) := \{Z \mid Z \text{ is predictable}, \mathbb{E}[\int_0^\infty \|Z_t\|^2 dM_t < \infty] < +\infty\}\) where \(M \in \mathcal{H}^2\).
- \(L^2_c(M) := \{Z \mid Z \cdot 1_{[0,T]} \in L^2(M)\}\), where \(M \in \mathcal{H}_{loc}^2\).
• $L^2_{loc}(M)$ is the space of predictable processes $Z$ for which there exists a localizing sequence $(\tau_n)$ such that

$$\mathbb{E}\left(\int_0^\infty ||Z||^2\,d\langle M \rangle_t \right) = \mathbb{E}\left(\int_0^\tau ||Z||^2\,d\langle M \rangle \right) < +\infty,$$

where $M \in \mathcal{H}^2_{loc}$.

• $L^2_{loc}(M) := \{X \mid X \in L_T^2(M) \text{ for any } T < \infty\}$, where $M \in \mathcal{H}^2_{loc}$.

• $U^2_\theta := \{Y \mid Y$ càdlàg, adapted and $\mathbb{E}\left[\sup_{t \in [0,T]} ||Y_t||^2\right] < +\infty\}$.

• $\mathcal{V}$ is the space of càdlàg, adapted processes which have finite variation on every finite interval.

• $\mathcal{V}^r := \{v \in \mathcal{V} \mid v \text{ is increasing}\}$.

• $\mathcal{A} := \{A \in \mathcal{V} \mid \mathbb{E}[\text{Var}(A(\infty))] < \infty\}$.

• $\mathcal{A}^r_{loc}$ is the space of processes locally belonging to $\mathcal{A}$, that is the space of processes $X$ for which there exists a localizing sequence $(\tau_n)$ such that $X^{\tau_n} \in \mathcal{A}$ for all $n$.

• $\mathcal{A}^r_{loc} := \{X \in \mathcal{A}^r_{loc} \mid X$ is increasing\}.

• $L^2_\theta(0, \tau; \phi) := \{X \mid X$ is progressive, $\mathbb{E}\left[\int_0^\tau \exp(\theta \phi(s))||X(s)||^2 \,ds\right] < \infty\},$

where $\theta \in \mathbb{R}$, $\tau$ is stopping time and $\phi$ is an increasing process.

If $\phi(t) = t$, we write in $L^2_0(0, \tau)$.

• $L^2_\beta(\tau; \phi)$ is the space of random variables $\xi$ such that $\mathbb{E}[\exp(\theta \phi(\tau))\xi^2] < \infty$.

• $L^2_\beta(0, \tau; \phi) := \{Y \mid \beta Y \in L^2_\theta(0, \tau; \phi)\}$.

• $U^2_\beta(0, \tau; \phi) := \{Y \mid Y$ is progressive, $\mathbb{E}[\sup_{s \leq \tau} \exp(\theta \phi(s))||Y(s)||^2] < \infty\}$

If $\phi(t) = t$, we write in $U^2_\beta(0, \tau)$.

• If we need to show the Euclidean image space $V$, we use $L^2_\theta(0, \tau; \phi, V)$, $L^2_\beta(0, \tau; \phi, V)$ etc.

• $M^2_\beta(0, \tau; \phi; V_1, V_2) := L^2_\beta(\tau; \phi; V_1) \times L^2_\beta(0, \tau; \phi; V_2)$, where $V_1, V_2$ are Euclidean spaces.

1.1. Introducing BSDEs in general space

As in [17], we seem to construct the BSDEs assuming only the usual properties of the filtration and that $L^2(\Omega, \mathcal{F}, \mathbb{P})$ is a separable Hilbert space. Unless otherwise indicated, we should read all equalities(and inequalities) as "up to a measure-zero set" throughout this paper.

Definition 1.1. For $v \in \mathcal{V}^r$, let us define the measure $\mu_v$ on $(\Omega, \mathcal{F})$ as follows.

$$\mu_v(A) := \mathbb{E}\left[\int_0^\infty I_A(\omega, t)\,dv\right], \quad A \in \mathcal{F}$$

where the integral is taken pathwise in a Stieltjes sense.

This measure $\mu_v$ is called the measure induced (or generated) by $v$. 
Note that if $v \in \mathcal{A}_{loc}^+$ then $\mu_v$ gives a $\sigma-$ finite measure on $(\Omega, \mathcal{F})$.

We give a simple version of the well-known Martingale representation theorem below (see [20] or [21]).

**Theorem 1.1** (Martingale representation theorem). Suppose that $L^2(\Omega, \mathcal{F}, \mathbb{P})$ is a separable Hilbert space with an inner product $X \cdot Y = \mathbb{E}[XY].$

Then there exists a sequence of $\mathcal{H}^2-$martingales, $M = (M^1, M^2, ...)$ such that $< M^i, M^j > = 0$ for $i \neq j$ and every $N \in \mathcal{H}^2$ can be represented as

$$N_t = N_0 + \int_0^t Z_u dM_u = N_0 + \sum_{i=1}^{\infty} \int_0^\tau Z_u^i dM^i_u$$

for some sequence of predictable processes, $Z = (Z^1, Z^2, ...)$ satisfying $Z \in L^2(M).$

And the predictable quadratic variation processes of these martingales $< M^i >$ satisfy

$$< M^1 > < M^2 > < M^3 > ...,$$

($>$ denotes absolute continuity of induced measures). If $(N')$ is another such sequence then $< N' > \equiv < M' >$, where $\equiv$ denotes equivalence of induced measures.

**Remark 1.1.** If the space is generated by Brownian motion, the martingale representation theorem holds on infinite interval (see e.g. [21], Theorem 6 or references therein). This also implies the martingale representation theorem on every finite interval.

For a given $k \in \mathbb{N}$, the general type of BSDE is as follows.

$$Y_t = \xi + \int_t^\tau g(\omega, s, Y_s, Z_s) dv_s - \sum_{i=1}^{\infty} \int_t^\tau Z^i_s dv_s,$$

where $\tau$ is an $\mathbb{F}-$stopping time, the terminal value $\xi$ is an $\mathcal{F}_\tau-$measurable random variable with values in $\mathbb{R}$, the driver $g : \Omega \times (0, \infty) \times \mathbb{R}^k \times \mathbb{R}^{k\times\infty} \rightarrow \mathbb{R}^k$ is predictable, $v \in \mathcal{V}$ and the integral of the driver is the Lebesgue-Stieltjes integral with respect to the measures generated by the trajectories of $v$.

A solution of the BSDE (1.3) is a pair of processes $(Y, Z)$ taking values in $\mathbb{R}^k \times \mathbb{R}^{k\times\infty}$, where $Y$ is progressive and $Z$ is predictable.

In this paper, we shall make the following assumption on $v$.

**(A0)** $v$ is a continuous and increasing process.

It follows from (A0) that $v$ is locally bounded and $v \in \mathcal{A}_{loc}^+$.

Noting that the predictable quadratic variation process $< M >$ identifies an induced measure on $\mathcal{F}$ defined by (1.1), suppose that the induced measure $\mu_{< M >}$ has the following Lebesgue decomposition.

$$\mu_{< M >} = \tilde{m}^{(1)} + \tilde{m}^{(2)}, \quad i \in \mathbb{N},$$

where $\tilde{m}^{(1)}$ is absolutely continuous with respect to $\mu_v$ and $\tilde{m}^{(2)}$ is orthogonal to $\mu_v$.

From the generalized Radon-Nikodym Theorem (e.g. see [32], Chapter 3, Proposition 3.49), there exist two processes $m^{(1)}_i, m^{(2)}_i$ such that $\mu_{m^{(1)}} = \tilde{m}^{(1)}$ and $\mu_{m^{(2)}} = \tilde{m}^{(2)}$.

More precisely $m^{(j)}_i = d\pi^j_i dB_j$, $j = 1, 2$, where $\pi^j_i(\omega) := \tilde{m}^{(j)}((0, t] \times B), B \in \mathcal{F}$. Thus

$$< M^i > = m^{(1)}_i + m^{(2)}_i.$$
We can consider (1.5) as the Lebesgue decomposition of $< M' >$.
Let us introduce the stochastic semi-norm $\| \cdot \|_M$, which is defined as
\[
\| z \|_M^2 := \sum_i \| z^i \|^2 \cdot (d\mu^i_1 / d\mu) = \sum_i \| z^i \|^2 \cdot (d\mu^i_2 / d\mu)(\cdot, t),
\]
for every $z = (z^1, z^2, \ldots) \in \mathbb{R}^{k^{\infty}}$.

Now let us consider the finite time BSDE for $T > 0$. We give the following result which is a special case of Theorem 6.1 in [17].

**Lemma 1.2.** Let $T > 0$, $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^k)$ and suppose that $v$ is a deterministic continuous, increasing function which assigns the positive measure to every non-empty interval in $\mathbb{R}^+$. Let $g: \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k^{\infty}} \to \mathbb{R}^k$ be a predictable process such that

1. $\mathbb{E}\left[ \int_0^T \| g(\omega, t, 0, 0) \|^2 dv_t \right] < \infty$
2. For any $y, y' \in \mathbb{R}^k$ and $z, z' \in \mathbb{R}^{k^{\infty}}$, there exists $c > 0$ such that
   \[
   \| g(\omega, t, y, z) - g(\omega, t, y', z') \|^2 \leq c \| y - y' \|^2 + \| z - z' \|_M^2, \quad dv_t \cdot d\mathbb{P} = a.s.
   \]

Then the following BSDE has a unique solution in $U_T^2 \times L_T^2(M)$.
\[
Y_t = \xi + \int_t^T g(\omega, s, Y_s, Z_s) dv_s - \sum_{i=1}^\infty \int_t^T Z^i_s dM^i_s. \tag{1.7}
\]

In the above lemma, the terminal time is constant. We can also consider the BSDE (1.3) with stopping terminal time. Perhaps the Lipschitz condition on driver will be still essential and there will be some further conditions related to stopping terminal time for the existence and uniqueness of (1.3). We will not do research of the existence and uniqueness of such BSDEs with random terminal time in this paper. Our main objective is to show a technique by which the results with respect to stochastic Lipschitz condition are derived from the results with respect to the random terminal time which is considered to be already given.

### 2. Time change and BSDEs

We begin with the definition of time change ([42], Chapter V).

**Definition 2.1.** A time change $C$ is a family $\{ C(s) \mid s > 0 \}$ of stopping times such that the maps $s \to C(s)$ are almost surely increasing and right continuous.

**Definition 2.2.** If $C$ is a time change, a process $X$ is said to be $C$-continuous if $X$ is constant on each interval $[C_s, C_{s+}]$.

We can define the stopped $\sigma$-field $\tilde{\mathcal{F}}_t := \mathcal{F}_{C(t)}$ and get the new stochastic basis $(\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{\mathbb{P}} = (\tilde{\mathcal{F}}_t)_{t \geq 0})$. It can be easily seen that $\tilde{\mathbb{F}}$ also satisfies the usual conditions from the property of stopped $\sigma$-fields. If $X$ is $\mathcal{F}$-progressive then $\tilde{X}_t := X_{C_t}$ is $\tilde{\mathbb{F}}$-adapted and the process $\tilde{X}_t$ is called the time changed process of $X$. We show a typical example of time change below.
Let us consider an increasing and right-continuous adapted process $A$ (so, progressive) with which we associate $C(s) := \inf\{t \mid A(t) > s\}$, \(2.1\)

where $\inf() = +\infty$. This process $C(s)$ is called the inverse of $A(s)$ and we write in $A^{-1}(s)$.

As the stochastic basis satisfies the usual conditions and $A$ is progressive, $A^{-1}(s)$ which is the hitting time of $(s, \infty)$ is a stopping time for every $s > 0$. And obviously it is increasing and right continuous. Thus $C = A^{-1} = \{A^{-1}(s) \mid s > 0\}$ is a time change.

Throughout this section, we suppose that $C$ is almost surely finite and $C_0 = 0$ and for any progressive measurable process $X_t$, $\tilde{X}_t$ means the time changed process of it, unless otherwise indicated. And for the space of processes $V$ with respect to $\mathbb{F}$, $\tilde{V}$ means the corresponding space with respect to $\tilde{\mathbb{F}}$. For example, $\mathbb{L}$ means the space of $\tilde{\mathbb{F}}$–local martingales. We give some main results concerning the property of time change under $C$–continuity below.

**Lemma 2.1.** ([42], Chapter V, Proposition 1.4).

Let $C$ be a time change on $(\Omega, \mathcal{F}, \mathbb{P}, F)$. If $h$ is $\mathcal{F}$–progressive, then $\tilde{h}$ is $\tilde{\mathcal{F}}$–progressive. And if $X$ is a $C$–continuous process of finite variation, then

$$\int_0^C h_u dX_u = \int_0^{\tilde{C}} \tilde{h}_u d\tilde{X}_u.$$

**Lemma 2.2.** ([42], Chapter V, Proposition 1.5)

If $C$ is a time change on $(\Omega, \mathcal{F}, \mathbb{P}, F)$ and $M \in \mathbb{L}^c_s$ satisfies $C$–continuity, then the following hold.

I. $\tilde{M} \in \mathbb{L}^c_s$ and $< \tilde{M} >= < \tilde{M} >$

II. If $h \in L^2_{t, \text{loc}}(M)$, then $\tilde{h} \in L^2_{t, \text{loc}}(\tilde{M})$ and for each $t > 0$

$$\int_0^\infty \tilde{h}_u d\tilde{M}_u = \int_0^{\tilde{C}_t} h_u dM_u.$$

Moreover, if $\xi$ is a non-negative random variable, then

$$\int_0^\xi \tilde{h}_u d\tilde{M}_u = \int_0^{\tilde{C}_t} h_u dM_u \quad \mathbb{P} - \text{a.s.}$$

Now we show the property of time change for general locally square-integrable martingales.

**Lemma 2.3.** If $C$ is a time change on $(\Omega, \mathcal{F}, \mathbb{P}, F)$ and $M \in \mathcal{H}_s^2$ is $C$–continuous, then the followings hold.

I. $\tilde{M} \in \tilde{\mathcal{H}}_s^2$ and $< \tilde{M} >= < \tilde{M} >$

II. If $h \in L^2_{t, \text{loc}}(M)$, $\tilde{h} \in L^2_{t, \text{loc}}(\tilde{M})$ and for each $t > 0$

$$\int_0^\infty \tilde{h}_u d\tilde{M}_u = \int_0^{\tilde{C}_t} h_u dM_u.$$

Moreover if $\xi$ is a non-negative random variable then

$$\int_0^\xi \tilde{h}_u d\tilde{M}_u = \int_0^{\tilde{C}_t} h_u dM_u \quad \mathbb{P} - \text{a.s.}$$
Proof. I. For any \( L \in \mathcal{L} \), it is easy to see that \( \tilde{L} \in \mathcal{L} \) from the optional stopping theorem and \( C \)-continuity of \( M \).

As \( M \in \mathcal{H}^2_{loc} \), the predictable quadratic variation \( \langle M \rangle \) is in \( \mathcal{A}^+_{loc} \) and \( M^2 - \langle M \rangle \) is a local martingale from the characterization of \( \mathcal{H}^2_{loc} \) martingale (see e.g. [32], Chapter 3, Proposition 3.64). Therefore \( M^2 - \langle M \rangle = \tilde{M}^2 - \langle M \rangle \) is an \( \tilde{F} \)-local martingale.

Let \( (\tau_n) \) denote the localizing sequence such that \( \langle M \rangle^{\tau_n} \in \mathcal{A}^+ \) for every \( n \).

Then \( \tilde{\tau}_n := C^{-1}_{\tau_n} = \inf \{t : C_t \geq \tau_n \} \) is an \( \tilde{F} \)-stopping time for every \( n \) and \( (\tilde{\tau}_n) \) is a localizing sequence.

Noting that \( M \) is \( C \)-continuous if and only if \( \langle M \rangle \) is \( C \)-continuous (see [42], Chapter IV, Proposition 1.13), \( \langle M \rangle \) is constant on \( [\tau_n, C_{\tau_n}] \).

So \( \mathbb{E}[\langle M \rangle^{\tau_n}(\tilde{\tau}_n)] = \mathbb{E}[\langle M \rangle^{\tau_n}(C_{\tau_n})] = \mathbb{E}[\langle M \rangle(\tau_n)] < \infty \). Hence \( \langle M \rangle \in \tilde{\mathcal{A}}^+_{loc} \).

And \( \langle M \rangle \) is also \( \tilde{F} \)-predictable from the \( C \)-continuity. Accordingly, using again the characterization of \( \mathcal{H}^2_{loc} \) martingale, \( \tilde{M} \in \mathcal{H}^2_{loc} \) and \( \langle \tilde{M} \rangle = \langle M \rangle \).

II. This is a simple consequence of I and Lemma 2.1 together with the relation between stochastic integral and quadratic variation. \( \square \)

**Remark 2.1.** Lemma 2.3 still holds for \( \mathcal{H}^2 \)-martingales under \( C \)-continuity. That is, if \( M \) is \( \mathcal{H}^2 \)-martingale satisfying \( C \)-continuity, then \( \tilde{M} \in \mathcal{H}^2 \). In this case we use the characterization of \( \mathcal{H}^2 \)-martingales (e.g. see [32], Chapter II, Proposition 2.84) and the same procedure is used for the proof.

Now we return to the discussion on BSDE. For the BSDE on which we discuss, the sequence of \( \mathcal{H}^2 \)-martingales \( M^i \) \( (i = 1, 2, \ldots) \) has the martingale representation property on \( (\Omega, \mathcal{F}, \mathbb{P}, \tilde{F}) \).

At this point, the martingale representation on \( (\Omega, \mathcal{F}, \mathbb{P}, \tilde{F}) \) is naturally expected whereas the time changed processes of \( M^i \) \( (i = 1, 2, \ldots) \) are \( \mathcal{H}^2 \)-martingales under \( C \)-continuity by Lemma 2.3.

**Lemma 2.4.** Let \( C \) be a time change and \( \mathcal{H}^2 \)-martingales \( M^i \) \( (i = 1, 2, \ldots) \) be \( C \)-continuous. Then the sequence of \( \tilde{\mathcal{H}}^2 \)-martingales \( \tilde{M}^i \) has the martingale representation property for any \( \mathcal{H}^2 \)-martingale satisfying \( C^{-1} \)-continuity such as in Theorem 1.1.

**Proof.** Let \( \tilde{N} \) be an \( \tilde{\mathcal{H}}^2 \)-martingale satisfying \( C^{-1} \)-continuity. Then \( \tilde{N} = \tilde{N}C^{-1}(C_i) = N_{C_i} \), where \( N_{C_i} := \tilde{N}_{C_i} \). Obviously, \( N \in \mathcal{H}^2 \) by Lemma 2.3. Therefore using Theorem 1.1 and Lemma 2.3,

\[
\tilde{N}_{C_i} = N_{C_i} = N_0 + \sum_{i=1}^{\infty} \int_0^{C_i} \tilde{Z}_t^i \tilde{d}M^i_u = \tilde{N}_0 + \sum_{i=1}^{\infty} \tilde{Z}_t^i \tilde{d}\tilde{M}^i_u,
\]

for some sequence of \( \tilde{F} \)-predictable processes, \( (\tilde{Z}^i) \) satisfying

\[
\mathbb{E} \left[ \sum_{i=1}^{\infty} \int_0^{\infty} (\tilde{Z}_t^i)^2 d < M^i >_u \right] < +\infty.
\]

Using Lemma 2.1 and Lemma 2.3 again,

\[
\mathbb{E} \left[ \sum_{i=1}^{\infty} \int_0^{\infty} (\tilde{Z}_t^i)^2 d < M^i >_u \right] = \mathbb{E} \left[ \sum_{i=1}^{\infty} \int_0^{\infty} (\tilde{Z}_t^i)^2 d < \tilde{M}^i >_u \right].
\]
This leads to
\[ \mathbb{E} \left[ \sum_{i=1}^{\infty} \int_{0}^{\infty} (\tilde{Z}_u^i) d < \tilde{M}^i > u \right] < +\infty. \tag{2.2} \]

Hence for any \( \tilde{N} \in \tilde{H}^2 \), there exists a sequence of \( \tilde{\omega} \)-predictable processes, \( \tilde{Z}(i = 1, 2,...) \) satisfying (2.2) such that
\[ \tilde{N}_t = \tilde{N}_0 + \sum_{i=1}^{\infty} \int_{0}^{t} \tilde{Z}_u^i d\tilde{M}^i_u. \]

Then by using Lemma 2.3, we can easily deduce that the martingales \( \tilde{M}^i, i = 1, 2,... \) are mutually orthogonal. The absolute continuity of the induced measures and the uniqueness of the representation are similarly proved. \( \square \)

If we know the results for the BSDE (1.4) with uniformly Lipschitz condition, it is possible to extend to the case where the driver has the stochastic Lipschitz coefficients. This is the main argument in this section.

Conveniently, we rewrite the BSDE (1.4) omitting the index \( i \) as follows.
\[ Y_t = \xi + \int_{t}^{\tau} g(\omega, s, Y_s, Z_s) d\nu_s - \int_{t}^{\tau} Z_s dM_s, \quad 0 \leq t \leq \tau, \tag{2.3} \]

where \( Z = (Z^1, Z^2, ...) \) and \( M = (M^1, M^2, ...) \).

Assume that the driver of (2.3) satisfies the following stochastic Lipschitz condition.

(A1) There exist predictable processes \( r_t \) and \( u_t \) such that
\[ ||g(\omega, t, y_t, z_t) - g(\omega, t, y'_t, z'_t)|| \leq r_t||y_t - y'_t|| + u_t||z_t - z'_t||M_t, \quad d\mu - a.s. \]

for any \( y_t, y'_t \in \mathbb{R}^{k} \) and \( z_t, z'_t \in \mathbb{R}^{k \times \infty} \), where \( \alpha^2_i := \max(r_i, u_i^2) > \epsilon \) for some \( \epsilon > 0 \) and \( \alpha^2_i \) is pathwise Stieltjes-integrable with respect to \( \nu \) for every finite interval in \( \mathbb{R}^{+} \).

Now we define the following process.
\[ \phi(t) := \int_{0}^{t} \alpha^2_i dv_s. \tag{2.4} \]

The remarkable point is that \( \phi^{-1} \) i.e. the inverse of \( \phi_t \) defined by (2.1) is a time change. We shall make a good use of this process in the view of time change. It is clear that \( \phi^{-1} \) is a.s. finite and \( \phi^{-1}(0) = \phi(0) = 0 \). From now, the symbol \( C \) which has meant time change will be replaced by \( \phi^{-1} \). The focus of this section is on the technique, so we do not have detailed discussion on the space of solutions. The main result in this section is as follows.

**Theorem 2.5.** Let \( \phi(t) \) be a process defined by (2.4) and \( M \) be \( \phi^{-1} \)-continuous. If \( (Y_t, Z_t) \) is a solution of BSDE (2.3) satisfying (A0) and (A1) on \( (\Omega, \mathcal{F}, P, \mathbb{F}) \), then \( (Y_t, z_t) := (Y_{\phi^{-1}(t)}, Z_{\phi^{-1}(t)}) \) is a solution of the following BSDE on \( (\Omega, \mathcal{F}, P, \mathbb{F}) \).
\[ y_t = \xi + \int_{t}^{\tau} g(\omega, s, y_s, z_s) d\nu_s - \int_{t}^{\tau} z_s d\tilde{M}_s, \quad 0 \leq t \leq \tau, \tag{2.5} \]
where
\[ \hat{g}(\omega, s, y, z) := g(\omega, \phi^{-1}(s), y, z)/\alpha^2(\phi^{-1}(s)), \quad \hat{\tau} := \phi(\tau), \quad \hat{M}_t := M_{\hat{\tau}t}. \] (2.6)

The converse is also true, that is if \((y_i, z_i)\) is a solution of the BSDE (2.5), then \((Y_t, Z_t) := (\hat{y}(\cdot), \hat{z}(\cdot))\) is a solution of the BSDE (2.3). Mainly the new driver \(\hat{g}\) of (2.5) satisfies uniform Lipschitz continuity such that for any \(y_i, y'_i \in \mathbb{R}^k\) and \(z_i, z'_i \in \mathbb{R}^{k \times \infty}\),
\[ ||\hat{g}(\omega, t, y_i, z_i) - \hat{g}(\omega, t, y'_i, z'_i)|| \leq ||y_i - y'_i|| + ||z_i - z'_i||_{M_t}, \quad dt \times d\mathbb{P} \text{ a.s.}. \]

**Proof.** We split the proof into four steps.

**Step 1** We first show that \(\tilde{\nu}(\cdot, \omega) := \nu(\phi^{-1}(\cdot, \omega), \omega)\) is absolutely continuous for each \(\omega \in \Omega\). As \(\nu\) is increasing and continuous, \(\nu^{-1}\) defined by (2.1) is a time change and \(\nu \equiv \nu^{-1}\)-continuous. Therefore by Lemma 2.1, we can see that
\[ \phi(t) = \int_0^t \alpha^2(s)dv_s = \int_0^{\nu^{-1}(t)} \alpha^2(s)dv_s - \int_t^{\nu^{-1}(t)} \alpha^2(s)dv_s = \int_0^{\nu^{-1}(t)} \alpha^2(s)dv_s = \int_0^{\nu^{-1}(s)} \alpha^2(v_s)ds = \left(\int_s^{\nu^{-1}(s)} \alpha^2(v_s)ds\right)^{-1}(t). \]

Thus, \(\tilde{\nu}_t = [\nu \circ \phi^{-1}](t) = \left[\nu \circ \nu^{-1} \circ \left(\int_0^{\nu^{-1}(s)} \alpha^2(v_s)ds\right)^{-1}\right](t) = \left(\int_0^{\nu^{-1}(s)} \alpha^2(v_s)ds\right)^{-1}(t). \)

Noting that \(\alpha^2(s) > 0\), \(\nu^{-1}\) is strictly increasing and absolutely continuous for each \(\omega \in \Omega\) and so is the reversed process. Hence \(\tilde{\nu}_t\) resp. \(\mu_t\) is absolutely continuous with respect to Lebesgue measure (resp \(dt \times d\mathbb{P}\)) and
\[ d\tilde{\nu}_t/(dt \times d\mathbb{P}) = d\nu_t/dt = 1/\alpha^2 \circ \nu \circ \phi^{-1}(t) = \alpha^{-2}(\phi^{-1}_t). \]

In fact, we can see that \(\tilde{\nu}_t\) resp. \(\mu_t\) is equivalent to Lebesgue measure (resp \(dt \times d\mathbb{P}\)). We also mention that \(\nu \equiv \nu^{-1}\)-continuous.

**Step 2** We derive the Lebesgue decomposition of the measure induced by \(<\tilde{M}>\). First, we show that \(m_1\) is a.s. \(\phi^{-1}\)-continuous. Suppose that \(\nu\) is a constant on \([a, b]\) \((0 \leq a < b)\). Then for any \(c \in [a, b]\) and \(B \in \mathcal{T}\), \(\hat{m}([c, b) \times B) = \mathbb{E}[\int_a^b I_B(\omega) \cdot (d\hat{m}_1/d\mu)(\omega, t)d\nu_t] = 0\). Noting that \(\hat{m}([0, t] \times B) = \int_B m_1^1 d\mathbb{P},\)
\[ 0 = \hat{m}([c, b) \times B) = \hat{m}([0, b) \times B) - \hat{m}([0, c] \times B) = \int_B (m_1^1 - m_1^1) d\mathbb{P}. \]

Hence \(m_1^1\) is a.s. constant on \([a, b]\). Because \(\nu\) is \(\phi^{-1}\)-continuous from **Step 1**, we can see that \(m_1^1\) is a.s. \(\phi^{-1}\)-continuous. Recalling (1.5) and using Lemma 2.3, we obtain (omitting the index \(i\)
\[ <\tilde{M}>_t := <\tilde{M} >_t = m_1^1 + \hat{m}_t^2. \] (2.7)

And the continuity of \(\phi\) which comes from the continuity of \(\nu\) implies \(\phi(\hat{\nu}^{-1}) = t\). Now we can use Lemma 2.1 to show
\[ \mu(\hat{\nu}^i(A)) = \mathbb{E} \left[ \int_0^\infty I_A(\omega, t)d\hat{m}_1^i \right] = \mathbb{E} \left[ \int_0^\infty I_A(\omega, \phi(\omega, t))dm_1^i \right] = \mathbb{E} \left[ \int_0^\infty I_A(\omega, \phi(\omega, t))(d\hat{m}_1^i/d\mu)(\omega, t) \cdot dv_t \right] = \mathbb{E} \left[ \int_0^\infty I_A(\omega, t)(d\hat{m}_1^i/d\mu)(\phi^{-1}(t))d\hat{\nu}_t \right] \]
for any \( A \in \mathcal{F} \), where \([d\overline{m}] / [d\mu^i] (\phi^{-1}) := [d\overline{m}] / [d\mu_i] (\phi, \phi^{-1}(\omega, t))\).

Thus \( \mu^i < dt \) and \( d\mu^i / d\mu = [d\overline{m}] / [d\mu_i] (\phi^{-1}) \).

Noting that \( \mu^i < dt \times d\overline{P} \) by **Step 1**, we can deduce \( \mu^i < dt \times d\overline{P} \) and

\[
d\mu^i / (dt \times d\overline{P}) = [d\overline{m}] / [d\mu_i] (\phi^{-1}) \cdot d\nu_i / dt. \tag{2.8}
\]

Similarly, \( \mu^i \) is orthogonal to \( dt \times d\overline{P} \). This shows that (2.7) is the Lebesgue decomposition of \( < M > \) with respect to \( dt \times d\overline{P} \).

**Step 3.** As \((Y_t, Z_t)\) is the solution of (2.3),

\[
y_t := Y_{\phi^{-1}(t)} = \xi + \int_{\phi^{-1}(t)}^\tau g(\omega, s, y_s, Z_s)dv_s - \int_{\phi^{-1}(t)}^\tau Z_s dM_s, \quad 0 \leq t \leq \tau.
\]

By Lemma 2.1 and **Step 1**,

\[
\int_{\phi^{-1}(t)}^\tau g(\omega, s, y_s, Z_s)dv_s = \int_{\phi^{-1}(t)}^\tau g(\omega, \phi^{-1}(s), Y_{\phi^{-1}(s)} Z_{\phi^{-1}(s)})d\nu_i / ds \cdot ds
\]

\[
= \int_{\phi^{-1}(t)}^\tau g(\omega, \phi^{-1}, Y_{\phi^{-1}(s)} Z_{\phi^{-1}(s)})\alpha^{-2}(\phi^{-1}(s))ds
\]

\[
= \int_{\phi^{-1}(t)}^\tau \overline{g}(\omega, s, y_s, z_s)ds.
\]

By Lemma 2.3 and \( \phi^{-1} \)–continuity of \( M \),

\[
\int_{\phi^{-1}(t)}^\tau Z_s dM_s = \int_{\phi^{-1}(t)}^\tau Z_{\phi^{-1}(s)} dM_{\phi^{-1}(s)} = \int_{\phi^{-1}(t)}^\tau z_s d\overline{M}_s.
\]

So we have

\[
y_t = \xi + \int_{\phi^{-1}(t)}^\tau \overline{g}(\omega, s, y_s, z_s) - \int_{\phi^{-1}(t)}^\tau z_s d\overline{M}_s, \quad 0 \leq t \leq \tau.
\]

As \( Y_t \) is \( \mathcal{F} \)–progressive, \( y_t \) is \( \overline{F} \)–progressive. Due to the fact that all stochastic integrals are indistinguishable from the stochastic integrals of predictable processes, we can consider \( z_t \) is predictable. Accordingly, \((y_t, z_t)\) is a solution of BSDE (2.5) on \((\Omega, \mathcal{F}, \overline{P}, \overline{F})\). Passing back through the above procedure, the converse argument is trivial.

**Step 4.** Finally, we show that \( \overline{g} \) satisfies uniform Lipschitz continuity. It follows from the results in **Step 2** that

\[
\|z_t\|_{\overline{M}_t}^2 = \|z_t\|^2 \cdot [d\mu^i / (dt \times d\overline{P})] = \|z_t\|^2 \cdot [d\overline{m}] / [d\mu_i] (\phi^{-1}) d\nu_i / dt
\]

\[
= \alpha^{-2}(\phi^{-1}) \|z_t\|_{\overline{M}_t}^2 \big|_{\phi^{-1}(t)}.
\]
From the stochastic Lipschitz condition on \( g \),

\[
|\tilde{g}(\omega, t, y, z_0) - \tilde{g}(\omega, t, y', z'_0)| = |\tilde{g}(\omega, \phi^{-1}(t), y, z_0) - \tilde{g}(\omega, \phi^{-1}(t), y', z'_0)|\alpha^{-2}(\phi_t^{-1}) \\
\leq \alpha^{-2}(\phi_t^{-1})[r_{\phi^{-1}(t)}|y_t - y'_t| + u_{\phi^{-1}(t)}(\|z_t - z'_t\|_{\tilde{g}_{\phi^{-1}(t)}})] \\
= \alpha^{-2}(\phi_t^{-1})|r_{\phi^{-1}(t)}|y_t - y'_t| + u_{\phi^{-1}(t)}(\|z_t - z'_t\|_{\tilde{g}_{\phi^{-1}(t)}}) \\
= \frac{r_{\phi^{-1}(t)}}{\max\{r_{\phi^{-1}(t)}, u_{\phi^{-1}(t)}^2\}}|y_t - y'_t| + \frac{u_{\phi^{-1}(t)}}{\sqrt{\max\{r_{\phi^{-1}(t)}, u_{\phi^{-1}(t)}^2\}}}|z_t - z'_t|_{\tilde{g}_{\phi^{-1}(t)}} \\
\leq |y_t - y'_t| + |z_t - z'_t|_{\tilde{g}_{\phi^{-1}(t)}}
\]

for any \( y_t, y'_t \in \mathbb{R}^k \) and \( z_t, z'_t \in \mathbb{R}^{k \times \infty} \). From Step 1, we know that \( \mu_t \) is equivalent to \( dt \times d\tilde{P} \). So Lipschitz property on \( \tilde{g} \) holds \( dt \times d\tilde{P} \)-a.s.

**Remark 2.2.** If the trajectories of \( v \) are strictly increasing, then \( \phi^{-1} \) is strictly increasing and continuous (that is \( \phi^{-1}(\phi(t)) = \phi(\phi^{-1}(t)) = t \)), so we do not have to assume that \( M \) is \( \phi^{-1} \)-continuous. Remark that \( \phi^{-1} \)-continuity of \( M \) is equivalent to \( \phi^{-1} \)-continuity of \( M^2 \).

**Remark 2.3.** In our discussion, the continuity of \( v \) which leads to the continuity of \( \phi \), plays an important role. This guarantees \( v(\phi^{-1}(t)) = \phi(\phi^{-1}(t)) = t \). If \( v \) is a finite variation process possibly with jumps, it may be needed to decompose the Stieltjes measures generated by the trajectories of \( v \) as the continuous part and the discontinuous one. Perhaps it may be non-trivial.

**Remark 2.4.** If we only want to simplify the continuous integrator of driver, it is sufficient to use \( v^{-1} \) as the time change.

It is natural to try the comparison theorem under the stochastic Lipschitz condition by means of time change. Suppose that we have two BSDEs satisfying \( (A0), (A1) \) and let \((g, \xi), (\tilde{g}, \tilde{\xi}) \) be the corresponding generators. And let \((Y, Z), (\tilde{Y}, \tilde{Z}) \) be the associated solutions. The following assumption plays an important role to ensure that the comparison theorem holds ([17]).

**(A2)**

1. For every \( j \), there exists \( \tilde{P}_j \) equivalent to \( \tilde{P} \) such that \( f^j \) component of \( X \) as defined by

\[
e^j_t := - \int_0^t e^j_u [g(\omega, u, Y_u, Z_u) - g(\omega, u, \tilde{Y}_u, \tilde{Z}_u)]dv_u + \int_0^t e^j_u [Z_u - \tilde{Z}_u]dM_u
\]

is \( \tilde{P}_j \)-supermartingale.

2. If for all \( r \geq 0 \),

\[
e^j_T - \mathbb{E}^{\tilde{P}}\left[ \int_T^\infty e^j_t g(\omega, u, Y_u, Z_u)dv_u | \mathcal{F}_T \right] \geq e^j_T - \mathbb{E}^{\tilde{P}}\left[ \int_T^\infty e^j_t g(\omega, u, \tilde{Y}_u, \tilde{Z}_u)dv_u | \mathcal{F}_T \right]
\]

for all \( i \), then \( Y_r \geq \tilde{Y}_r \) for all \( r \geq 0 \) componentwise.

The driver satisfying \( (A2) \) is often called balanced. This notation originated from finance, as in some sense, the driver balances the outcomes to hedge. This driver is closely connected to no-arbitrage opportunity and furthermore the condition under which the comparison theorem holds for martingale-type BSDEs possibly with jumps (see [16, 17] or [18], Part IV).
It is obvious that the comparison theorem holds for BSDE (2.3) if and only if the comparison theorem holds for the corresponding BSDE (2.5). Now we shall show that the essential conditions which ensure that the comparison theorem holds are preserved while the time change is processed. We still assume that the BSDE satisfies (A0),(A1) and M is $\phi^{-1}$--continuous.

**Theorem 2.6.** If BSDE (2.3) satisfies (A2), the time changed BSDE (2.5) also satisfies (A2) with respect to filtration $\overline{\mathbb{F}}$.

**Proof.** First by the optional stopping theorem, $e_j^T\overline{X}$ is $\overline{\mathbb{F}}$--supermartingale under $\overline{\mathbb{F}}_j$ for every $j$ using that $e_j^T$ is $\overline{\mathbb{F}}$--supermartingale under $\overline{\mathbb{F}}_j$. Having the similar procedure to Step 3 in the proof of Theorem 2.5, we obtain

$$e_j^T\overline{X}_r = -\int_0^r e_j^T[\overline{g}(\omega, u, \tilde{y}_u, \tilde{z}_u) - \overline{g}(\omega, u, \tilde{y}_u, \tilde{z}_u)]du + \int_0^r e_j^T[\tilde{z}_u - \tilde{z}_u]d\overline{M}_u$$

So the first part of (A2) is satisfied with respect to $\overline{\mathbb{F}}$ for BSDE (2.5). Similarly we can prove that the second part is also satisfied.

We conclude this section with the following statement.

**Interesting remark on terminal time.** When we study the BSDEs with stochastic Lipschtz coefficients, the randomness of terminal time does not play an important role. This is illustrated as follows. Due to the Remark 2.4, we can suppose that the BSDE is given in the following type without loss of generality.

$$Y_t = \xi + \int_t^\tau g(\omega, s, Y_s, Z_s)ds - \int_t^\tau Z_s dM_s.$$  

(2.9)

We use the following process introduced for the quadratic BSDEs in [2].

$$\Phi(\omega, t): = \frac{t}{1 + \tau \land t}, \quad t \geq 0.$$  

After the simple calculation, we get

$$\Phi^{-1}(t) = t/(1 - t), \quad [\Phi^{-1}(t)]' = -(1 - t)^{-2}, \quad 0 \leq t \leq \tau = \Phi(\tau) < 1.$$  

Obviously $\Phi^{-1}$ is time change and we can deduce the following BSDE on $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ equivalent to (2.9) in some sense.

$$y_t = \xi + \int_t^\tau G(\omega, s, y_s, z_s)ds - \int_t^\tau z_s d\tilde{M}_s,$$  

(2.10)

where $G(\omega, s, y, z) := I_{s \leq \tau}g(\omega, \Phi^{-1}(s), y, z)(\Phi^{-1}(s))'$ and $\tilde{M}_s := M_{\Phi^{-1}(s)}$. We mention that the new driver $G$ is stochastic Lipschtz even though the original driver $g$ is uniform Lipschtz. In fact, if we suppose that $g$ has constants $r, u$ as the Lipschtz coefficients, for any $y, y' \in \mathbb{R}^k$ and $z, z' \in \mathbb{R}^{k \times \infty}$,

$$||g(\omega, s, y, z) - g(\omega, s, y', z')|| = I_{s \leq \tau}[||g(\omega, \Phi^{-1}(s))'||r|y - y'| + u||z - z'||_{\tilde{M}_s}||\Phi^{-1}(s)|^{-1/2}]$$

$$\leq r(1 - s)^{-2}[||y - y'|| + u(1 - s)^{-1}||z - z'||_{\tilde{M}_s}]$$

$$\leq r(1 - \tau)^{-2}[||y - y'|| + u(1 - \tau)^{-1}||z - z'||_{\tilde{M}_s}]$$

$$= r(1 + \tau)^{-2}[||y - y'|| + u(1 + \tau)||z - z'||_{\tilde{M}_s},$$
This means that the stopping terminal time of BSDEs can be converted to constant and this operation is adapted to the class of BSDEs with stochastic Lipschtz condition.

3. Wiener-type BSDEs with stochastic monotone coefficients

The well-known and mostly studied type of BSDEs are of course Wiener-type BSDEs. Let $W$ be $d$-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathbb{F} := \{ \mathcal{F}_t \}_{t \geq 0}$ be the natural complete, right continuous filtration generated by $W$. It is worthy to study Wiener-type BSDEs with stochastic Lipschtz conditions. For example, let us consider the pricing problem of a European contingent claim to hedge, $\xi$ is the maturity date. In general, $r_i$ and $u_i$ both will not be bounded and moreover the maturity date will be non-deterministic. In this case the Lipschtz condition does not hold uniformly any more. For the forward-backward BSDEs, when the uncertainty of driver only comes from a solution of forward component, we can give the probabilistic interpretation of a system of semi-linear elliptic PDEs (see [3], Remark 4.6).

We shall have slightly different procedure from Section 2, but this is essentially the same. For the discussion of Martingale-type BSDE, the martingale term is changed into a martingale process, the stochastic Lipschtz condition is adapted to the class of BSDEs with stochastic Lipschtz condition. This means that the stopping terminal time of BSDEs can be converted to constant and this operation is adapted to the class of BSDEs with stochastic Lipschtz condition.

$$Y_t = \xi + \int_t^\tau (r_s Y_s + u_s Z_s) ds + \int_t^\tau Z_s dW_s,$$

where $\xi$ is the contingent claim to hedge, $r_s$ is the interest rate, $u_s$ is the risk premium vector and $T$ is the maturity date. In general, $r_i$ and $u_i$ both will not be bounded and moreover the maturity date will be non-deterministic. In this case the Lipschtz condition does not hold uniformly any more. For the forward-backward BSDEs, when the uncertainty of driver only comes from a solution of forward component, we can give the probabilistic interpretation of a system of semi-linear elliptic PDEs (see [3], Remark 4.6).

We shall have slightly different procedure from Section 2, but this is essentially the same. For the discussion of Martingale-type BSDE, the martingale term is changed into a martingale process. Due to $< W_i > = t$, the stochastic semi-norm defined by (1.6) is obtained as $\| \|_{W_i} = \| \|_{Z}$ for $z \in \mathbb{R}^{k d}$. Let the driver $f$ satisfy the stochastic Lipschtz condition. That is there exist non-negative progressive processes $r_i$ and $u_i$ such that

$$||f(\omega, t, y, z) - f(\omega, t, y', z')|| \leq r_i ||y - y'|| + u_i ||z - z'||, \quad dt \times d\mathbb{P} - a.s. \quad (3.2)$$

for any $y, y' \in \mathbb{R}^k$, $z, z' \in \mathbb{R}^{kd}$. As in Section 2, we assume that there exists $\epsilon > 0$ such that $\alpha^2(t) := \max\{r_i, u_i^2\} > \epsilon$ and that $\alpha^2(t)$ is Lebesgue-integrable on any finite interval in $\mathbb{R}^+$. And we introduce the following strictly increasing, absolutely continuous process:

$$\phi(t) := \int_0^t \alpha^2(s) ds. \quad (3.3)$$

We set $\overline{F} := \mathcal{F}_{\phi^{-1}(t)}$. Now we define stochastic process $\tilde{W}_t$ as follows.

$$\tilde{W}_t := \sum_{s \leq t} [(\phi^{-1}(s))^{-1/2}] dW_{\phi^{-1}(s)}. \quad (3.4)$$
Then \( \bar{W} \) is a continuous \( \bar{F} \)-local martingale and for each \( i, < \bar{W}^i \triangleright_\tau = \int_0^\tau [(\phi^{-1})'(s)]^{-1} dW^i \triangleright_\phi^{-1}(\tau) = \int_0^\tau [(\phi^{-1})'(s)]^{-1} d\phi^{-1}(s) = t \), so it is \( \bar{F} \)-Brownian motion by Lévy’s characterization theorem. If \((Y_t, Z_t)\) is a solution of (3.1), then
\[
y_t := Y_{\phi^{-1}(t)} = \xi + \int_0^\tau f(\omega, s, Y_s, Z_s)ds - \int_0^\tau Z_sdW_s
\]
In fact, noting that
\[
\text{Then } \bar{\phi} \in M \quad \text{if } Y_t, Z_t
\]

We can notice that
\[
\text{Proof.} \quad \text{where }
\]
\[
\text{z}_s := \phi^{-1}(t) \quad \text{and } \quad \bar{f} := \phi(\tau). \quad \text{So, if we set } \bar{f} \quad \text{as}
\]
\[
\bar{f}(\omega, s, y, z) := f(\phi^{-1}(s), Y_{\phi^{-1}(s)}, Z_{\phi^{-1}(s)}) = f(\phi^{-1}(s), (\phi^{-1})'(t)) = f(\phi^{-1}(t))
\]

then \((y_t, z_t) = (Y_{\phi^{-1}(t)}, Z_{\phi^{-1}(t)} \cdot (\phi^{-1})'(t))\) is the solution of the following BSDE on \((\Omega, F, [\bar{P}], \bar{F})\):
\[
y_t = \xi + \int_t^\tau f(\omega, s, Y_s, Z_s)ds - \int_t^\tau z_s d\bar{W}_s, \quad 0 \leq t \leq \bar{\tau}. \quad (3.6)
\]

Conversely, if \((y_t, z_t) = (Y_{\phi^{-1}(t)}, Z_{\phi^{-1}(t)} \cdot (\phi^{-1})'(t))\) is a solution of (3.1) using that \( (\phi^{-1})'(t) = (\phi'(t))^{-1} \). As in Section 2, \( f \) satisfies the uniform Lipschtz continuity. In fact, noting that \( (\phi^{-1})'(t) = \alpha^{-2}(\phi^{-1}(t)) \).
\[
\|\bar{f}(\omega, t, y, z) - \bar{f}(\omega, t, y', z')\| \leq \alpha^{-2}(\phi^{-1})(\|y - y'\| + u_{\phi^{-1}(0)} \cdot (\phi^{-1})'(z - z'))
\]
\[
= \frac{r_{\phi^{-1}(0)}}{\max[r_{\phi^{-1}(0)}, u_{\phi^{-1}(0)}^2]} \|y - y'\| + \frac{u_{\phi^{-1}(0)}}{\sqrt{\max[r_{\phi^{-1}(0)}, u_{\phi^{-1}(0)}^2]}} \|z - z'\|
\]
\[
\leq \|y - y'\| + \|z - z'\|, \quad dt \times d\bar{P} = a.s. \quad (3.7)
\]

Now we are prepared to state some results on Wiener type BSDEs.

**Lemma 3.1.** Let the following conditions hold for BSDE (3.1).

1. The stochastic Lipschtz condition (3.2) holds.
2. \( \xi \in L^2(\tau, \Phi) \) and \( f(\omega, s, 0, 0)/\alpha(s) \in L^2(0, \tau, \Phi) \) for some \( \rho > 3 \).

Then BSDE (3.1) has a unique solution \((Y_t, Z_t)\) in \( M^{2, \alpha}(0, \tau; \Phi; \mathbb{R}^k; \mathbb{R}^{k \times d}) \). This solution actually belongs to \( M^{2, \alpha}(0, \tau; \Phi; \mathbb{R}^k; \mathbb{R}^{k \times d}) \) and \( Y \in U^{2, \alpha}_(0, \tau; \Phi) \).

**Proof.** We can notice that \( \Phi(t) \) and \( \bar{f}(\omega, s, 0, 0) \in L^2(0, \tau) \) from
\[
\int_0^\tau \exp(\rho s) \left\| f(\omega, s, 0, 0)/\alpha(s) \right\|^2 ds = \int_0^\tau \exp(\rho s) \left\| f(\omega, \phi^{-1}(s), 0, 0)/\alpha(\phi^{-1}(s)) \right\|^2 d\phi^{-1}(s)
\]
\[
= \int_0^\tau \exp(\rho s) \left\| f(\omega, \phi^{-1}(s), 0, 0) \right\|^2 \left(\left(\phi^{-1})'(s)\right)^2 \right) ds = \int_0^\tau \exp(\rho s) \left\| f(\omega, s, 0, 0) \right\|^2 ds.
\]
Recalling (3.7), the simple application of [19], Theorem 3.4 admits that BSDE (3.6) has a unique solution \((y, z)\) in \(L^2_T(0, \bar{T}; \mathbb{R}^k \times \mathbb{R}^{kd})\) which belongs to \(L^2_T(0, \bar{T}; \mathbb{R}^k \times \mathbb{R}^{kd})\) and \(y \in U^2_{\rho}(0, \bar{T})\).

Noting that \((Y, Z_t) = (\gamma_{\theta(t)}, \zeta_{\theta(t)} \cdot [(\phi'(t))^{1/2}])\), we get the following expressions:

\[
\mathbb{E}\left[\int_0^\bar{T} \exp(\rho s)||y||^2 ds\right] = \mathbb{E}\left[\int_0^\bar{T} \exp(\rho s)||\gamma_{\theta(t)}||^{1/2} \phi'(s) ds\right] \\
= \mathbb{E}\left[\int_0^\bar{T} \exp(\rho s)||\sigma \cdot Y_s||^2 ds\right].
\]

\[
\mathbb{E}\left[\int_0^\bar{T} \exp(\rho s)||z_s||^2 ds\right] = \mathbb{E}\left[\int_0^\bar{T} \exp(\rho s)||\zeta_{\theta(t)}| \phi'(s)|^{1/2}||^2 ds\right] \\
= \mathbb{E}\left[\int_0^\bar{T} \exp(\rho s)||Z_s||^2 ds\right].
\]

\[
\mathbb{E}[\sup_s \exp(\rho s)||y_s||^2 : 0 \leq s \leq \bar{T}] = \mathbb{E}[\sup_t \exp(\rho \phi(t)||Y_t||^2 : 0 \leq s \leq \bar{T}]].
\]

These are sufficient to complete the proof.

In the above lemma, the stochastic Lipschtz condition in \(y\) can be relaxed whereas BSDEs with random terminal time are well adopted under the monotonicity condition. This naturally admits us to give the following main result.

**Theorem 3.2.** Suppose that the following conditions hold for BSDE (3.1).

1. There exist non-negative progressive processes \(u, l\) and progressive process \(r\), satisfying 
   \[\alpha^2(t) := \max[r_{\phi(t)}^{-1}, l_{\phi(t)}^{-1}, u_{\phi(t)}] > \epsilon (r^{-} := \max(\epsilon, 0)) \text{ for some } \epsilon > 0 \text{ such that for any } y, y' \in \mathbb{R}^k \text{ and } z, z' \in \mathbb{R}^{kd};\]

1.1. \(||f(\omega, t, y, z)|| \leq ||f(\omega, t, 0, z)|| + l_{\phi}(||y|| + l'')(\text{where } l'' \in [0, 1])\)

1.2. \((y - y')(f(\omega, t, y, z) - f(\omega, t, y', z')) \leq -r||y - y'||^2\)

1.3. \(||f(\omega, t, y, z) - f(\omega, t, y', z')|| \leq u||z - z'||\)

2. \((\xi + l') \in L^2_T(\tau; \Phi) \text{ and } f(\omega, s, 0, 0)/\alpha(s) \in L^2_0(0, \tau; \Phi) \text{ for some } \rho > 3, \text{ where } \phi(t) := \int_0^t \alpha^2(s) ds.\)

Then the conclusion of Lemma 3.1 holds.

**Proof.** It can be easily seen that \(\tilde{f}\) is Lipschtz continuous in \(z\). The monotonicity and linear growth in \(y\) are shown as follows.

\[
(y - y')(f(\omega, t, y, z) - f(\omega, t, y', z')) \leq \alpha^{-2}(\phi_{\tilde{f}}^{-1}) \cdot (-r_{\tilde{f}}^{-1})||y - y'||^2 \\
\leq \alpha^{-2}(\phi_{\tilde{f}}^{-1}) \cdot r_{\tilde{f}}^{-1}||y - y'||^2 = \frac{r_{\tilde{f}}^{-1}||y - y'||^2}{\max[r_{\tilde{f}}^{-1}, l_{\tilde{f}}^{-1}(u_{\tilde{f}}^{-1})]} \leq ||y - y'||^2.
\]

\[
||f(\omega, t, y, z)|| \leq ||\tilde{f}(\omega, t, 0, z)|| + \alpha^{-2}(\phi_{\tilde{f}}^{-1}) \cdot l_{\tilde{f}}^{-1}||y|| + l''(||y|| + l''').
\]

Now, the result easily follows from [19], Theorem 3.4.
Remark 3.1. Existence and uniqueness results for BSDEs driven by Brownian motion with stochastic Lipschitz coefficients or stochastic monotone coefficients were already given in [3, 5] under stronger assumptions than ours on linear growth coefficient and ρ need to be enough large. For example, ρ is assumed to be larger than 90 in [23] (see the proof of Theorem 6.1 therein).

For the Wiener-type BSDE with random terminal time, the comparison theorem also holds under the conditions for the existence and uniqueness (see [19], Corollary 4.4.2). Thus it is trivial that the comparison theorem holds for the BSDE (3.1). Here we give the stability with respect to perturbations. Comparing to Theorem 3 in [5] we study under weaker assumptions.

Theorem 3.3. Suppose (τ, ξ, f), (τ′, ξ′, f′) are the triples verifying the assumptions of Theorem 3.2 with the same ρ > 3. Let ΔY := Y − Y′, ΔZ := Z − Z′ for (Y, Z) ∈ L^2_{ad}(0, τ; \mathbb{R}^k × \mathbb{R}^{k×d}) and (Y′, Z′) ∈ L^2_{ad}(0, τ'; \mathbb{R}^k × \mathbb{R}^{k×d}) which are solutions of (3.1) corresponding to (τ, ξ, f), (τ′, ξ′, f′), respectively. Then there exist positive numbers β, δ for 3 < β ≤ ρ such that

\[
||ΔY(0)||^2 + βE\int_0^{τ′−}\exp(\theta(τ))/2||ΔZ(s)||^2 ds \leq E||ΔY(0)||^2 + E\int_0^{τ′−}\exp(\theta(τ)/2)||ΔZ(s)||^2 ds + \delta^{-1}E\int_0^{τ′−}\exp(\theta(τ))/2||ΔZ(s)||^2 ds.
\]

Proof. We can adopt the same strategy as the proof of Theorem 3.1 thanks to [19], Theorem 4.4 and omit the proof.

We can consider the case where the driver satisfies stochastic polynomial condition, that is, condition 1 in Theorem 3.2 can be replaced by

\[
||f(ω, t, y, z)|| ≤ ||f(ω, t, 0, z)|| + l_1(||y||^p + l'), \quad p > 1
\]

or more generally

\[
||f(ω, t, y, z)|| ≤ ||f(ω, t, 0, z)|| + l_1(\phi(||y||) + l')
\]

for some continuous, increasing function φ. In this case, we can refer to [10] (or [39]) and there will not be any difficulty. On the other hand, if the stochastic monotone coefficient is always non-negative (that is strictly monotone), we can set r_1 = 0, so it is sufficient to suppose that ρ > 1 in preceding results. If the driver is monotone decreasing, we can get a more useful result by referring to [43].

Theorem 3.4. For BSDE (3.1), we suppose that the conditions 1.2, 1.3 and 2 in Theorem 3.2 and (3.8) hold with l_i = l, r_i = 0, k = 1, l' = 0 for some l, r ∈ \mathbb{R}.

We further assume that \forall t \geq 0, f(ω, t, 0, 0) = 0 and \mathbb{E}[Z] ≤ M for some M ≥ 0. Then there exists a solution (Y, Z) of (3.1) such that |Y| ≤ M and \forall t \geq 0, \mathbb{E}|Z|^2 < ∞.

Proof. We only sketch the proof. We define a process φ(t) := \int_0^t(\alpha^2(s) + 1)ds with which we associate time change. Obviously φ^{-1}(t) ≤ t. Let \tilde{f} denote the driver of time changed BSDE.

Then it is easy to see that \tilde{f} is uniformly Lipschitz in y with Lipschitz coefficient 1 and monotone decreasing in y. It also satisfies controlled growth condition with coefficient 1. We can easily check that f(ω, t, 0, 0) = 0.
So there exists a solution \((y_t, z_t)\) to the time changed BSDE such that \(|y| < M\) and for any \(t\), \(\int_0^{\tilde{T}} \|z_t\|^2 ds < \infty\) from [43]. Theorem 3.1. Noting that \(Y_t = Y_{\Phi(t)}\) and \(\int_0^{\tilde{T}} \|Z_t\|^2 ds \leq \int_0^{\Phi^{-1}(\tilde{T})} \|Z_t\|^2 ds = \int_0^{\Phi^{-1}(\tilde{T})} \|Z_{\Phi(t)}\|^2 \phi'(s) ds = \int_0^{\tilde{T}} \|z_t\|^2 ds < \infty\), we can complete the proof. 

**Remark 3.2.** We note that the uniqueness and comparison can be also stated under the further conditions using Theorems 3.6 and 3.7 in [43]. In Theorem 3.4, the exponential integrability condition on terminal value and the driver are not made and the same conditions as the case of uniformly Lipschitz were used for the study of the BSDE with stochastic one. This is because the monotone coefficient which makes discounting rate is equal to zero.

### 3.1. Some aspects of further applications

Concluding this section, we shall briefly mention that it is possible to have some further applications to get better or new results. For example, it is not difficult to study \(L^p\)-solution [11], the stability [9, 45] and reflected BSDE [1] in the context of stochastic monotonicity condition through our framework. Although \(L^p\)-solution of BSDE with stochastic Lipschitz condition was already studied in [46], we can make the improved version, due to the preceding results. We can use the results in [38] to study the BSDEs with jumps whose drivers are stochastic Lipschitz. For the martingale-type BSDE with stochastic Lipschitz condition, results of [44] are available. The stochastic partial differential equations (SPDEs) with stochastic Lipschitz terms are connected to the backward doubly SDEs (BDSDEs) with stochastic Lipschitz coefficients and we can refer to [31] concerning the BDSDEs with random terminal time. Perhaps the derived results will be better than the ones in [33, 34] where the constant parameter appeared in integrability condition need to be sufficiently large. The proposed technique can be also applied to the BSDEs on manifolds with geometrical Lipschitz condition studied in [7], which includes the results with respect to the random terminal time (see Section 5 therein), so the geometrical Lipschitz condition can be relaxed from the uniform one to the stochastic one. The wellposedness of Mean-field backward stochastic delay equation with stopping terminal time and Lipschitz driver was stated in [28]. Theorem 3.1, so we can state the counterpart when the Lipschitz continuity is stochastic one. By referring to [27] where the results of second-order BSDEs (2BSDEs) with random terminal time are established, we can study 2BSDEs with stochastic Lipschitz condition. The details are left to the readers and some of them may be non-trivial.

In this section, we used the results obtained under the monotonicity assumption. So, the stronger integrability conditions on the driver and the solutions were still required as in the previous works. We shall apply the proposed technique to the undiscounted BSDEs driven by Markov chains without the monotonicity assumption on driver in the next section.

### 4. Markov chain BSDEs with stochastic Lipschitz coefficients

The BSDEs on Markov chains were first introduced in [15] and have developed in several papers, for example, the comparison theorem in [16] or the case of random terminal time in [14]. We present some preliminaries of the Markov chain BSDEs below.

Consider a continuous time, countable state Markov chain \(X\) on \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})\), where \(\mathbb{F}\) is the natural filtration generated by \(X\). Without loss of generality, we assume that \(X\) takes values from the unit vector \(e_i\) in \(\mathbb{R}^N\), \((N \in \mathbb{N} \cup \{\infty\})\), where \(N\) is the number of states of the chain. We denote by \(\Pi\) the state space. If \(A_{ij}\) denotes the rate matrix of the chain at time \(t\), then \((A_{ij})_{ij} \geq 0, i \neq j\) and
\[ X_t = X_0 + \int_0^t A_u X_u \, du + M_t, \]  
(4.1)

where \( M \) is a pure discontinuous martingale with finite variation. In this section, we further assume the Markov chain has the strong Markov property. Let us consider the following BSDE to stopping time on Markov chain.

\[ Y_t = \xi + \int_t^\tau f(\omega, u, Y_u, Z_u) \, du - \int_t^\tau Z_u \, dM_u, \quad 0 \leq t \leq \tau, \]  
(4.2)

where \( f : \Omega \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \) and \( \int_t^\tau Z_u \, dM_u = \sum_{i=1}^N \int_0^\tau Z_{u_i} \, dM_{u_i} \). Note that (4.2) is contained in the class of BSDEs defined by (1.4) due to [15], Lemma 3.1 where it was shown that the sequence \( (M^i), i = 1, 2, \ldots \) has the martingale representation.

**Definition 4.1.** We define \( \psi \) := \( \text{diag}(A_tX_t) - A_t \text{diag}(X_t) - \text{diag}(X_t)A_t^T \). Then the matrix \( \psi \) is symmetric and positive (semi-)definite and \( f < 0 \) (see [15]).

Due to (1.6), we can set the stochastic semi-norm \( \| \cdot \|_{M_t} \) as follows.

\[ \|Z\|_{M_t}^2 := Z^T_t \psi_t Z_t, \quad Z \in \mathbb{R}^{1 \times N}. \]  
(4.3)

We give further definitions from [14].

**Definition 4.2.** We say that the driver \( f \) is \( \gamma \)-balanced if there exists a random field \( \eta : \Omega \times \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N \), with \( \eta(\cdot, \cdot, z, z') \) predictable and \( \eta(\omega, t, 0, 0) \) Borel-measurable, such that

- \( f(\omega, t, y, z) - f(\omega, t, y, z') = (z - z')^T (\eta(\omega, t, z, z') - AX_t) \)
- for each \( e \in \Pi \), \( (e^T \eta(\omega, t, z, z'))(e^T AX_t) \in [\gamma, \gamma^{-1}] \) for some \( \gamma > 0 \), where \( 0/0 := 1 \)
- \( \eta^T(\omega, t, z_t, z') = 0 \) for \( I \in \mathbb{R}^N \) the vector with all entries \( I \)
- \( \eta(\omega, t, z + \alpha I, z') \) for all \( \alpha \in \mathbb{R} \)

**Remark 4.1.** Note that the above definition of \( \gamma \)-balanced driver is closely related to the notion of balanced driver in Section 2 (see [14], Lemma 3).

**Definition 4.3.** Let \( Q_\gamma \) denote the family of all measures \( Q \) where \( X \) has the compensator \( \eta(t, \omega) \), for \( \eta \) a predictable process with \( \eta^T(\eta(t, \omega)) = 0 \) and \( \frac{\tilde{\gamma}0(\gamma)}{\tilde{\gamma}AX_t} \) for all \( \gamma \in \Pi \), where 0/0 := 1.

That is, \( X_t = X_0 + \int_0^t \eta \, dt + \tilde{Q}-\text{martingale}, Q \in Q_\gamma \).

We give the key result of [14] (see Theorem 3, Remark 4 therein).

**Lemma 4.1.** Suppose that the following conditions are verified for Markov chain BSDE (4.2).

1. \( \xi \) is \( \mathcal{F}_\tau \)-measurable.
2. There exist non-decreasing functions \( K_1, K_2 : \mathbb{R}^+ \to [1, \infty) \) and some constants \( \beta, \tilde{\beta} > 0 \) such that

\[ \mathbb{E}^Q[\xi]_{\mathcal{F}_t} \leq K_1(t), \mathbb{E}^Q[(1 + \tau)^{1/\tilde{\beta}} \xi]_{\mathcal{F}_t} \leq K_2(t), \mathbb{E}^Q[K_1(t)^{1+\tilde{\beta}}]_{\mathcal{F}_t} \leq K_2(t), \]

for all \( \mathbb{P} - \text{a.s. all } Q \in Q_\gamma \) and all \( t \).

\[ \forall j, \Sigma(A_j)_{ij} = 0. \] For the simplicity, we shall assume that \( A \) is uniformly bounded. The Markov chain \( X \) has the following Doob-Meyer decomposition (see [22], Appendix B).
3. \( f : \Omega \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \) is \( \gamma \)-balanced.
4. The discounting terms are uniformly bounded above, that is, there exists a constant \( C_1 \in \mathbb{R} \) such that for any \( y, y', z \) and \( s < t \),
\[
\int_s^t r(\omega, u, y, y', z) du < C_1,
\]
where
\[
r(\omega, u, y, y', z) := \frac{f(\omega, t, y, z) - f(\omega, t, y', z)}{y - y'}.
\]
5. There exists \( C_2 \in \mathbb{R} \), \( \beta \in [0, \beta] \) such that \( |f(\omega, t, 0, 0)| \leq C_2(1 + \hat{\beta}) \).
6. \( f : \Omega \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \) is uniformly Lipschitz in \( y \). That is, there exists a constant \( C \) such that \( |f(\omega, t, y, z) - f(\omega, t, y', z)| \leq C|y - y'| \) for all \( y, y', z \).

Then the BSDE (4.2) has a unique solution such that \( |Y_\tau| \leq (1 + C_2) \exp(C_1)|K_1(\tau)| \).

**Remark 4.2.** Note that the fourth condition is verified if the driver is monotone decreasing.

In Lemma 4.1, the Lipschitz condition in \( y \) is restrictive. We give a simple illustration below with the motion of a particle on a graph. Consider a model for transmission of messages from a node to another node over a network. Let the chain \( X \) describe the motion of a message. Then the probability that the message reaches its target is given as the solution of the following BSDE (see [14], Section 4).
\[
I_{[X_t=\tau]} = Y_t - \int_t^\tau -r_{X_u} Y_u - du + \int_t^\tau Z_u dM_u, \quad 0 \leq t \leq \tau,
\]  \hspace{1cm} (4.4)
where \( r_x \) is the rate by which the node \( x \) loses a message. To suppose that the losing rate at each node is bounded is an assumption rarely satisfied in real world. It depends on the time variable in general and it should be written as \( r(t, X_t) \) which may be unbounded.

The main result of this section is as follows (we shall give the proof later).

**Theorem 4.2.** Suppose that the conditions 1)–5) in Lemma 4.1 are satisfied for BSDE (4.2). Let the driver \( f \) be stochastic Lipschitz in \( y \), that is, there exists a non-negative predictable process \( C(t) \) such that for any \( t, y, z, z' \),
\[
|f(\omega, t, y, z) - f(\omega, t, y', z)| \leq C(t)|y - y'|. \quad (4.5)
\]
Then BSDE (4.2) has a unique solution such that \( |Y_\tau| \leq 2 \exp(C_1)|K_1(\tau)| \).

**Remark 4.3.** Theorem 4.2 admits that the bound of a solution does not depend on \( C_2 \). From this fact, the bound estimate on a solution can be replaced by \( |Y_\tau| \leq (1 + C_2 \wedge 1) \exp(C_1)|K_1(\tau)| \) in Lemma 4.1.

Usually, if one wants to relax the Lipschitz continuity as the stochastic one, it has to be considered that the stronger integrability conditions on terminal value, driver and the solution are required, instead. However, this is not true for undiscounted BSDE. It is because the terminal value and driver of this BSDEs are not needed to be discounted at some rate and one can consider the direct conditions on them respectively.

In Theorem 4.2, the conditions on stopping time seem to be unfamiliar and it is required to afford an example when they are satisfied. In this context, S. N. Cohen [14] showed that the direct conditions on stopping time are satisfied when the stopping time is a hitting time of a subset of \( \Pi \)
under the uniform ergodicity of the chain by the way of examining the exponential ergodicity of the chain under the perturbations of rate matrix. One can observe that the above hitting time only depends on the character of the chain. In Theorem 4.2, the driver is stochastic Lipschitz only in $y$ and the $\gamma$–balanced condition related to $z$ is still required. On the other hand, it was shown in two uniform and stochastic Lipschitz settings that the conditions on stopping time and terminal value for the wellposedness of BSDE (4.2) coincide. These lead to the following result (see [14], Lemma 6).

**Lemma 4.3.** Suppose that rate matrix is time-homogeneous under the measure $\mathbb{P}$ and the chain is uniformly ergodic. Let $\tau$ be the first hitting time of a set $\Xi \subseteq \Pi$ and $\xi$ be a random variable of the form $\xi = g(\tau, X_\tau)$ for some function $g(t, x) \leq k(1 + \beta^t)$ for some $k, \beta > 0$. Then there exist functions $K_1, K_2$ satisfying the requirements of Theorem 4.2.

When the terminal time and terminal value have the forms like in Lemma 4.3 and the driver is Markovian, that is, $f(\omega, t, y, z) = f(X_{\tau(t)}, t, y, z)$ for some $f$, we can give the ODE system with boundary condition which describes the solution of BSDE in the context of stochastic Lipschitz assumption (see [14], Theorems 6 or 7). Now we seem to prove Theorem 4.2 by means of time change described in Section 2.

**Proof of Theorem 4.2** We define the process $\phi$ as follows (this is based on the same idea as in Section 2).

$$\phi(t) := \int_0^t \alpha^2(s) ds, \quad \alpha^2(s) := \max\{C(s), C_2, 1\}. \quad (4.6)$$

Then it follows that $t \geq \phi^{-1}(t)$ from $\phi(t) \geq \int_0^t 1 ds = t$. We set $\tilde{F}_t := \mathcal{F}_{\phi^{-1}(t)}, \tilde{\mathbb{P}} := \{\mathcal{F}_t\}_{t \geq 0}$ as in Section 2. As $X$ is a strong Markov chain, $\tilde{X} := X_{\phi^{-1}}(t)$ is also a strong Markov chain with respect to $\tilde{\mathbb{P}}$ (e.g. see [4], Chapter 22, Section 3). Using the expression (4.1),

$$\tilde{X}_t = X_0 + \int_0^{\phi^{-1}(t)} A_u X_u du + M_{\phi^{-1}(t)} = \tilde{X}_0 + \int_0^{\phi^{-1}(t)} \tilde{A}_u \tilde{X}_u du + \tilde{M}_t, \quad (4.7)$$

where $\tilde{A}_u := A_{\phi^{-1}(u)} \cdot (\phi^{-1})'(u)$ and $\tilde{M}_t := M_{\phi^{-1}}(t)$. We recall that $\tilde{M} = (\tilde{M}_i, i = 1, 2, ..., N)$ is a sequence of orthogonal martingales which has martingale representation which $\tilde{\mathbb{P}}$ (see Lemma 2.4). Therefore, (4.7) is the (unique) Doob-Meyer decomposition of $\tilde{X}$. If we denote by $R_u$ the rate matrix of $\tilde{X}$, then

$$R_u \tilde{X}_u = \tilde{A}_u \tilde{X}_u, \quad dt \times d\tilde{\mathbb{P}} - a.s. \quad (4.8)$$

It follows that $R_u$ is uniformly bounded from $(\phi^{-1})'(t) = \frac{1}{\phi'(\phi^{-1}(t))} = \alpha^{-2}(\phi^{-1}(t)) \leq 1$, so the $\tilde{\mathcal{F}}$–chain $\tilde{X}$ is also regular. We can consider that the random rate matrix $\tilde{A}_u$ plays the role of transition rate matrix of $\tilde{X}$. Next, we shall show that $\tilde{f}(\omega, s, y, z) := f(\omega, \phi^{-1}(s), y, z) \cdot (\phi^{-1})'(s)$ is $\gamma$–balanced with respect to $\tilde{\mathbb{P}}$. We define $\tilde{\eta}(\omega, t, z, z') := \eta(\omega, \phi^{-1}(t), z, z') \cdot (\phi^{-1})'(t)$. Then by
the definition and (4.8), we have
\[
\tilde{f}(\omega, t, y, z) - \tilde{f}(\omega, t, y', z') = (f(\omega, \phi^{-1}(t), y, z) - f(\omega, \phi^{-1}(t), y', z'))(\phi^{-1})'(t)
\]
\[
= (z - z')(\Phi(\omega, \phi^{-1}(t), y', z') - A_{\phi^{-1}(t)}X_{\phi^{-1}(t)}(\phi^{-1})'(t)
\]
\[
= (z - z')'(\Phi(\omega, t, z, z') - A_tX_t) = (z - z')'(\Phi(\omega, t, z, z') - R_t\tilde{X}_t),
\]
\[
(e_t^T\Phi(\omega, t, z, z'))/(e_t^TA_tX_t) = (e_t^T\Phi(\omega, t, z, z'))/(e_t^TA_tX_t).
\]
\[
\int_t^T\eta(\omega, t, z, z') = \int_t^T\eta(\omega, t, z, z')(\phi^{-1})'(t) = 0,
\]
\[
y_t = \xi + \int_t^Tf(\omega, s, y_s, z_s)ds - \int_t^Tz_s\tilde{M}_s. \quad (4.9)
\]

We have already seen that \(\tilde{f}\) is \(\gamma\)-balanced.

Let us define the non-decreasing functions \(\tilde{K}_1(t) := K_1(\phi^{-1}(t))\) and \(\tilde{K}_2(t) := K_2(\phi^{-1}(t))\). Then \(\forall Q \in Q_f, \mathbb{E}^Q[\xi(\tilde{F}_t)] \leq K_1(\phi^{-1}(t)) = \tilde{K}_1(t)\) and \(\mathbb{E}^Q[\tilde{K}_1(\tau)^{1/2}\tilde{F}_t] = \mathbb{E}^Q[\tilde{K}_1(\tau)^{1/2}\tilde{F}_{\phi^{-1}(t)}] \leq K_2(\phi^{-1}(t)) = \tilde{K}_2(t)\).

Using the assumptions on \(f\), we can get the following expressions on \(\tilde{f}\).

\[
|\tilde{f}(\omega, t, 0, 0)| = \alpha^2(\phi^{-1}(t))|f(\omega, \phi^{-1}(t), 0, 0)|
\]
\[
\leq (1 + \phi^{-1}(t))\hat{\beta}\cdot C_2|\mathbb{E}(\phi^{-1}(t)) + C_2 + 1| \leq 1 + \phi^{-1}(t)\hat{\beta} \leq 1 + \hat{\beta},
\]
\[
\int_0^T(\tilde{f}(\omega, t, y, z) - \tilde{f}(\omega, t, y', z'))(y - y')du = \int_0^Tr(\omega, \phi^{-1}(t), y, y', z)du
\]
\[
\leq \int_0^{\phi^{-1}(t)}r(\omega, u, y, y', z)du \leq C_1,
\]
\[
|\tilde{f}(\omega, t, y, z) - \tilde{f}(\omega, t, y', z')| = |\tilde{f}(\omega, \phi^{-1}(t), y, z) - \tilde{f}(\omega, \phi^{-1}(t), y', z)| \cdot \alpha^{-2}(\phi^{-1}(t))
\]
\[
\leq C(\phi^{-1}(t))[y - y']/(\max(C(\phi^{-1}(t)), C_2, 1)) \leq |y - y'|.
\]

So BSDE (4.9) has a unique solution satisfying \(|y_t| \leq 2 \exp(C_1)|\tilde{K}_1(t)|\) by Lemma 4.1. If we set \((Y_t, Z_t) := (y_{\phi(t)}, z_{\phi(t)})\), Theorem 2.5 shows that it is a solution of BSDE (4.2). Because the solution \(y_t\) of (4.9) is unique up to indistinguishability, \(Y_t\) is also unique up to indistinguishability. And \(|Y_t| = |y_{\phi(t)}| \leq 2 \exp(C_1)|\tilde{K}_1(\phi(t))| = 2 \exp(C_1)|\tilde{K}_1(t)|.\]

**Remark 4.4.** We note that the comparison theorem for BSDE (4.2) holds under the stochastic Lipschitz condition from the corresponding comparison theorem for BSDE (4.9) (see [14], Theorem 5).
4.1. Additional use of time change

In Theorem 4.2, the $5^{th}$ condition is not strictly necessary. In fact, it is sufficient to suppose that $f(\omega, s, 0, 0)/(1 + \beta^2)$ is integrable on every finite interval in $\mathbb{R}^+$. For any $m > 1$, consider the process $\tilde{\phi}(t) := m \int_0^t |f(\omega, s, 0, 0)/(1 + \beta^2)| + 1|ds$ with which we associate time change. Then

$$|f(\omega, t, 0, 0)| = \frac{|f(\omega, \tilde{\phi}^{-1}(t), 0, 0)|}{1 + |f(\omega, \tilde{\phi}^{-1}(t), 0, 0)|/(1 + \tilde{\phi}^{-1}(t)\beta^2)} \cdot \frac{1}{m} \leq \frac{1 + \tilde{\phi}^{-1}(t)\beta}{m} \leq \frac{1 + \beta^2}{m}$$

Also we have the bound on solution such that $|Y_t| \leq (1 + 1/m) \exp(C_1)|K_1(t)|$, for all $m > 1$. Taking $m \to \infty$, we get $|Y_t| \leq \exp(C_1)|K_1(t)|$. So we can show the improvement of Theorem 4.2.

**Theorem 4.4.** Suppose that conditions 1–4, 6 in Lemma 4.1 and the stochastic Lipschitz condition (4.5) hold. We further assume that $f(\omega, s, 0, 0)/(1 + \beta^2)$ is integrable on every finite interval in $\mathbb{R}^+$. Then there exists a unique solution to BSDE (4.2) such that $|Y_t| \leq \exp(C_1)|K_1(t)|$. If the driver is monotone decreasing, then $|Y_t| \leq |K_1(t)|$.

In this subsection, we made the use of time change away from the discussion on Lipschitz continuity. Perhaps, there will be other problems to which we can apply time change effectively in the range of stochastic calculus.

5. Conclusion

In this paper, we showed that the technique for dealing with the BSDEs with stochastic Lipschitz coefficients by time change. The technique says that when we study the BSDE to stopping time, the Lipschitz condition can be given as the stochastic one. Also roughly speaking, most of the results of BSDEs obtained under the Lipschitz continuity may be extended to the case of stochastic Lipschitz continuity. Of course, this is only possible when we are aware of the results with respect to random terminal time and this admits the importance on the study of them.

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