On Dynkin Games with Unordered Payoff Processes

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Abstract

A Dynkin game is a zero-sum, stochastic stopping game between two players where either player can stop the game at any time for an observable payoff. Typically the payoff process of the max-player is assumed to be smaller than the payoff process of the min-player, while the payoff process for simultaneous stopping is in between the two. In this paper, we study general Dynkin games whose payoff processes are in arbitrary positions. In both discrete and continuous time settings, we provide necessary and sufficient conditions for the existence of pure strategy Nash equilibria and \( \epsilon \)-optimal stopping times in all possible subgames.

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1 Introduction

A Dynkin game, first introduced by Dynkin [2], is a zero-sum, stochastic stopping game between two players where either player can stop the game at any time for an observable payoff. Much research has been done in this field as well as various related problems, for example, [1, 3, 5, 8, 13, 14, 16, 17, 18]. One interesting application of Dynkin games is in two-person game contingent claims. The two-person game contingent claim is defined by Kifer [7], who also proved the existence and uniqueness of its arbitrage price. Further works, such as Hamadène and Zhang [5] and Kallsen and Kühl [6], studied various techniques in its pricing.

Typically the Dynkin game is associated with the payoff processes \( X, Y \) and \( Z \). In particular, the payoff is given by \( X \) if the max-player stops first, \( Y \) if the min-player stops first, and \( Z \) if both players stop at the same time. Standard Dynkin games, commonly studied in literature, refer to cases where the inequality \( X \leq Z \leq Y \) is satisfied. This chapter will present some new results for general Dynkin games, whose payoff processes are in arbitrary positions.

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Sections 2.1 and 3.1 examines the standard Dynkin game in a discrete-time set-up. Well-known results addressing the existence and uniqueness of value as well as optimal stopping times are presented in Propositions 2.5 and 3.3. In Sections 2.2 and 3.2 we establish some original results for the general Dynkin game in both discrete and continuous-time settings. In particular, the main results are Theorems 2.16 and 3.15 which provide sufficient conditions for the existence and uniqueness of value and optimal stopping times. The same conditions are then shown to be necessary for the existence of value in all possible subgames.

The theory of two-person non-zero-sum Dynkin games is not included here. We instead refer the reader to Hamadène and Zhang [5], Hamadène and Hassani [4], Ohtsubo [11, 12], and Shmaya and Solan [15] for some partial results in this area.

2 Discrete-Time Dynkin Games

We first present in Section 2.1 the classic results on discrete-time zero-sum Dynkin games. Subsequently, in Section 2.2, we attempt to provide a complete solution to the problem of existence of a Nash equilibrium for the general zero-sum Dynkin game. It should be stressed that we only deal with stopping games with a finite time horizon; a large body of the existing literature is devoted to stopping games with infinite time horizon and thus also with possibly infinite optimal stopping times.

We will first examine zero-sum stopping games with the random payoff given by

\[ R(\tau, \sigma) = \mathbb{I}_{\{\tau < \sigma\}} X_\tau + \mathbb{I}_{\{\sigma < \tau\}} Y_\sigma + \mathbb{I}_{\{\sigma = \tau\}} Z_\sigma, \]

(1)

where \( X, Y \) and \( Z \) are \( \mathbb{F} \)-adapted and integrable processes. The random times \( \tau \) and \( \sigma \) are chosen from the class \( \mathcal{T}_{[0,T]} \) of \( \mathbb{F} \)-stopping times and they are interpreted as the respective stopping strategies of the two players.

**Remark 2.1.** By assumption, \( \tau, \sigma \leq T \) and thus the values of \( X_T \) and \( Y_T \) are irrelevant in what follows. Therefore, without loss of generality, we adopt the common convention that \( X_T = Z_T = Y_T \).

The following definition deals with the discrete-time case, but its extension to the continuous-time framework is immediate.

**Definition 2.2.** For any fixed date \( t = 0, 1, \ldots, T \), by the Dynkin game \( \text{DG}_t(X,Y,Z) \) started at time \( t \) and associated with the payoff \( R(\tau, \sigma) \), we mean a zero-sum two-person stochastic game in which the goal of the max-player, who controls a stopping time \( \tau_t \in \mathcal{T}_{[t,T]} \), is to maximise the conditional expectation

\[ \mathbb{E}_\mathbb{P} \left( R(\tau_t, \sigma_t) \mid \mathcal{F}_t \right), \]

(2)

while the min-player, controlling a stopping time \( \sigma_t \in \mathcal{T}_{[t,T]} \), wishes to minimise the conditional expectation (2). Also denote by \( \text{DG}(X,Y,Z) \) the family of Dynkin games associated with \( R(\tau, \sigma) \).

For any fixed \( t \) and arbitrary stopping times \( \tau_t \) and \( \sigma_t \) from the class \( \mathcal{T}_{[t,T]} \), formula (1) yields

\[ \mathbb{E}_\mathbb{P} \left( R(\tau_t, \sigma_t) \mid \mathcal{F}_t \right) = \mathbb{E}_\mathbb{P} \left( \sum_{u=t}^{T} \left( \mathbb{I}_{\{u=\tau_t<\sigma_t\}} X_u + \mathbb{I}_{\{u=\sigma_t<\tau_t\}} Y_u + \mathbb{I}_{\{u=\sigma_t=\tau_t\}} Z_u \right) \mid \mathcal{F}_t \right). \]

(3)
We are interested in finding the value process \( V^* \) of \( \text{DG}(X,Y,Z) \), that is, an \( \mathcal{F} \)-adapted process such that, for all \( t = 0,1,\ldots,T \),
\[
V^*_t = \text{ess inf } \text{ess sup } \mathbb{E}_P(R(\tau_1,\sigma_1) \mid \mathcal{F}_t) = \text{ess sup } \text{ess inf } \mathbb{E}_P(R(\tau_t,\sigma_t) \mid \mathcal{F}_t).
\]
In addition, we search for a corresponding Nash (hence also optimal) equilibrium, that is, any pair \((\tau^*_t,\sigma^*_t)\) of optimal stopping times satisfying
\[
V^*_t = \mathbb{E}_P(R(\tau^*_t,\sigma^*_t) \mid \mathcal{F}_t).
\]

### 2.1 Standard Dynkin Game

We first present well-known results for the special class of two-person, zero-sum stopping games in the discrete-time framework (see Neveu [10]).

**Definition 2.3.** By the *standard Dynkin game* \( \text{SDG}(X,Y,Z) \), we mean the stochastic stopping game associated with the payoff \( R \) given by (1) with processes \( X,Y \) and \( Z \) satisfying the following condition: \( X \leq Z \leq Y \).

The following definitions introduce candidates for the value process of the standard zero-sum Dynkin game and the optimal stopping times.

**Definition 2.4.** The process \( V \) is defined by setting \( V_T = Z_T \) and, for any \( t = 0,1,\ldots,T-1 \),
\[
V_t = \min \left\{ Y_t, \max \left\{ X_t, \mathbb{E}_P(V_{t+1} \mid \mathcal{F}_t) \right\} \right\} = \max \left\{ X_t, \min \left\{ Y_t, \mathbb{E}_P(V_{t+1} \mid \mathcal{F}_t) \right\} \right\}.
\]
Furthermore, we set, for any fixed \( t = 0,1,\ldots,T \),
\[
\tau^*_t := \min \left\{ u \in \{ t,t+1,\ldots,T \} \mid V_u = X_u \right\},
\]
\[
\sigma^*_t := \min \left\{ u \in \{ t,t+1,\ldots,T \} \mid V_u = Y_u \right\}.
\]
The assumption that \( X \leq Z \leq Y \) immediately implies that the second equality in (4) holds and, for \( t = 0,1,\ldots,T \),
\[
X_t \leq V_t \leq Y_t,
\]
so that the process \( V \) is bounded below by \( X \) and above by \( Y \). The stopping times \( \tau^*_t \) and \( \sigma^*_t \) capture the first moment \( V \) hits the lower and upper boundaries, respectively, starting from time \( t \). Obviously, if \( V \) is the value process then we also must have, for \( t = 0,1,\ldots,T \),
\[
V_t = \mathbb{E}_P(R(\tau^*_t,\sigma^*_t) \mid \mathcal{F}_t).
\]

The following classic result shows that the process \( V \) given by (4) is indeed equal to the value process \( V^* \) of \( \text{SDG}(X,Y,Z) \). Recall that we work here under the standing assumption that \( X \leq Z \leq Y \); this condition will be relaxed in the foregoing subsection.

**Proposition 2.5.** (i) Let the process \( V \) and the stopping times \( \tau^*_t,\sigma^*_t \) be given by Definition 2.4 Then we have, for arbitrary stopping times \( \tau_1,\tau_2 \in \mathcal{T}_{[t,T]} \),
\[
\mathbb{E}_P(R(\tau^*_t,\sigma_t) \mid \mathcal{F}_t) \geq V_t \geq \mathbb{E}_P(R(\tau_t,\sigma^*_t) \mid \mathcal{F}_t),
\]
and thus also
\[
\mathbb{E}_P(R(\tau^*_t,\sigma_t) \mid \mathcal{F}_t) \geq \mathbb{E}_P(R(\tau^*_t,\sigma^*_t) \mid \mathcal{F}_t) \geq \mathbb{E}_P(R(\tau_1,\sigma^*_t) \mid \mathcal{F}_t).
\]
Hence $(\tau^*_t, \sigma^*_t)$ is a Nash equilibrium of the standard Dynkin game $SDG_t(X, Y, Z)$.

(ii) The process $V_t$ is the value process of the game $SDG(X, Y, Z)$, that is, for every $t = 0, 1, \ldots, T$,

$$V_t = \underset{\tau_t \in T_{[t, T]}^*}{\text{ess inf}} \underset{\sigma_t \in T_{[t, T]}^*}{\text{ess sup}} \mathbb{E}_t \left( R(\tau_t, \sigma_t) \mid F_t \right) = \mathbb{E}_t \left( R(\tau^*_t, \sigma^*_t) \mid F_t \right)$$

and thus $\tau^*_t$ and $\sigma^*_t$ are optimal stopping times as of time $t$. In particular, $V^*_T = Z_T$ and for any $t = 0, 1, \ldots, T - 1$,

$$V^*_t = \min \left\{ V_t, \max \left\{ X_t, \mathbb{E}_t \left( V_{t+1} \mid F_t \right) \right\} \right\}.$$

Proof. (i) We apply the backward induction. The inequalities (8) clearly hold for $t = T$. Assume that (8) holds for some $t$, that is, for arbitrary $\tau_t, \sigma_t \in T_{[t, T]}^*$, \( \mathbb{E}_t \left( R(\tau_t, \sigma_t) \mid F_t \right) \geq V_t \geq \mathbb{E}_t \left( R(\tau^*_t, \sigma^*_t) \mid F_t \right) \). \hfill (10)

We wish to prove that, for arbitrary $\tau_{t-1}, \sigma_{t-1} \in T_{[t-1, T]}^*$,

$$\mathbb{E}_t \left( R(\tau^*_t, \sigma^*_t) \mid F_{t-1} \right) \geq V_{t-1} \geq \mathbb{E}_t \left( R(\tau_{t-1}, \sigma_{t-1}) \mid F_{t-1} \right). \hfill (11)$$

There are essentially two cases, which are dealt with using different arguments.

- First, if the game is stopped at time $t - 1$, then the result can be deduced by analysing the relative sizes of processes $X, Y, Z$ and $V$ at time $t$.

- Second, if the game is not stopped at time $t - 1$, then the analysis is reduced to the time $t$ case, which is covered by the induction hypothesis.

Note that since the game is symmetric between the two players, it suffices to establish the upper inequality of (11). The lower inequality can be shown using analogous arguments.

For any $\tau_{t-1}, \sigma_{t-1} \in T_{[t-1, T]}^*$, let us write $\tilde{\tau}_{t-1} := \tau_{t-1} \lor t$, $\tilde{\sigma}_{t-1} := \sigma_{t-1} \lor t$, so that the stopping times $\tilde{\tau}_{t-1}$ and $\tilde{\sigma}_{t-1}$ belong to $T_{[t, T]}$.

We proceed to the proof of the upper inequality in (11), beginning with the case where the game is stopped at time $t - 1$. On the event $\{\tau^*_t = t - 1\}$,

$$\mathbb{E}_t \left( R(\tau^*_t, \sigma_{t-1}) \mid F_{t-1} \right) = \mathbb{I}_{\{\sigma_{t-1} \geq t\}} X_{t-1} + \mathbb{I}_{\{\sigma_{t-1} = t\}} Z_{t-1} \geq X_{t-1} = V_{t-1}.$$ \hfill (12)

On the event $\{\sigma_{t-1} = t - 1 < \tau^*_t\}$, using (7), we obtain

$$\mathbb{E}_t \left( R(\tau^*_t, \sigma_{t-1}) \mid F_{t-1} \right) = Y_{t-1} \geq V_{t-1}. \hfill (13)$$

Now for the case where the game is not stopped at time $t - 1$. On the event $\{\tau^*_t = \tau^*_{t-1} \wedge \sigma_{t-1} \geq t\}$, it follows from Definition 2.3 that $\tau^*_t = \tau^*_{t-1} = \tau^*_t$ and $V_{t-1} > X_{t-1}$, and thus

$$V_{t-1} = \min \left\{ Y_{t-1}, \mathbb{E}_t \left( V_t \mid F_{t-1} \right) \right\}. \hfill (14)$$

Hence

$$\mathbb{E}_t \left( R(\tau^*_t, \sigma_{t-1}) \mid F_{t-1} \right) = \mathbb{E}_t \left( R(\tilde{\tau}^*_{t-1}, \tilde{\sigma}_{t-1}) \mid F_{t-1} \right)$$

$$= \mathbb{E}_t \left( \mathbb{E}_t \left( R(\tilde{\tau}^*_{t-1}, \tilde{\sigma}_{t-1}) \mid F_{t} \right) \mid F_{t-1} \right)$$

$$\geq \mathbb{E}_t \left( V_t \mid F_{t-1} \right) \geq \min \left\{ Y_{t-1}, \mathbb{E}_t \left( V_t \mid F_{t-1} \right) \right\} = V_{t-1}. \hfill (15)$$
Note that inequality (15) follows from the induction hypothesis (10), while equality (16) is an immediate consequence of (14). After combining (12), (13) and (16), we obtain the upper inequality of (11). As mentioned before, the lower inequality can be (11) established by symmetry. The induction is then complete and thus statement (i) is proven.

(ii) We observe that, from (8), the pair $(\tau^*_t, \sigma^*_t)$ is a Nash equilibrium of SDG$_t(X, Y, Z)$. Therefore, the process $V$ satisfies $V_t = E_P(R(\tau^*_t, \sigma^*_t) \mid F_t)$ and thus it is the value process of the standard zero-sum Dynkin game SDG$(X, Y, Z)$. Equality 9 now follows easily.

**Remark 2.6.** It can be easily checked from Definition 2.4 that the stopped process $V_{\tau^*_t \wedge \sigma^*_t}$ is an $F_t$-martingale on the time interval $[t, T]$.

### 2.2 General Dynkin Game

We will now discuss possible generalisations of the standard zero-sum Dynkin game, while still maintaining the zero-sum property of the game. Specifically, we consider the zero-sum Dynkin game associated with the random payoff $R$ given by

$$R(\tau, \sigma) = 1_{\{\tau < \sigma\}} X_\tau + 1_{\{\sigma < \tau\}} Y_\sigma + 1_{\{\sigma = \tau\}} Z_\sigma,$$

where $X, Y$ and $Z$ are $F_t$-adapted, integrable processes. Note that we no longer impose any addition assumptions on their relative sizes (such as $X \leq Z \leq Y$), and thus we deal here with a *general Dynkin game* GDG$(X, Y, Z)$. As in Remark 2.1, without loss of generality, we may and do assume that $X_T = Y_T = Z_T$.

However, since the processes $X, Y$ and $Z$ are now unrestricted, it is easy to construct a Dynkin game without a Nash equilibrium. Our aim in this subsection is to identify necessary and sufficient conditions for the following property:

*For all $t = 0, 1, \ldots, T$, the Dynkin game GDG$_t(X, Y, Z)$ admits a Nash equilibrium.*

The idea is to emulate the progression of the previous subsection, while replacing the inequalities $X \leq Z \leq Y$ by a general set of sufficient conditions. When analysing the existence of a Nash equilibrium, we will employ the backward induction argument, as we did in the proof of Proposition 2.5. The key argument thus boils down to the thorough analysis of the embedded single period game, which starts at time $t$ and is either stopped immediately or it is terminated on the next date.

To motivate the construction of the value process candidate in Definition 2.9 let us temporarily assume there exists a value process $V^*$ for the Dynkin game with the payoff process $R$ given by (17). Also, let $\tau^*_t, \sigma^*_t$ be any pair of optimal stopping times for the game starting at time $t$, so that

$$V^*_t = E_P(R(\tau^*_t, \sigma^*_t) \mid F_t).$$

Let us denote $P_t := E_P(V^*_t \mid F_t)$. The next lemma deals with the single period embedded game.

**Lemma 2.7.** The Nash equilibrium property of a pair $(\tau^*_t, \sigma^*_t)$ of stopping times is equivalent to the following conditions:

- $Y_t \leq V^*_t = Z_t \leq X_t$ on $\{\tau^*_t = t, \sigma^*_t = t\}$,
- $P_t \leq V^*_t = X_t \leq Z_t$ on $\{\tau^*_t = t, \sigma^*_t > t\}$,
- $Z_t \leq V^*_t = Y_t \leq P_t$ on $\{\tau^*_t > t, \sigma^*_t = t\}$,
- $X_t \leq V^*_t = P_t \leq Y_t$ on $\{\tau^*_t > t, \sigma^*_t > t\}$. 

Proof. We note that, when written out in full according to definition (17) of $R$, there are four cases to examine:

\[ V_t^* = Z_t \quad \text{on} \quad \{ \tau_t^* = t, \sigma_t^* = t \}, \]
\[ V_t^* = X_t \quad \text{on} \quad \{ \tau_t^* = t, \sigma_t^* > t \}, \]
\[ V_t^* = Y_t \quad \text{on} \quad \{ \tau_t^* > t, \sigma_t^* = t \}, \]
\[ V_t^* = P_t \quad \text{on} \quad \{ \tau_t^* > t, \sigma_t^* > t \}. \]

The stated conditions now follow easily from the definition of the Nash equilibrium. \qed

Let us write $L_t := Z_t \wedge X_t$ and $U_t := Y_t \vee Z_t$, so that $L_t \leq Z_t \leq U_t$ for $t = 0, 1, \ldots, T$.

**Lemma 2.8.** Assume that $V^*$ is the value process for GDG($X, Y, Z$) and $\tau_t^*, \sigma_t^*$ are optimal stopping times for GDG$_t(X, Y, Z)$. Then:

(i) $L_t \leq V_t^* \leq U_t$;
(ii) $V_t^* = L_t$ on $\{ \tau_t^* = t \}$ and $V_t^* = U_t$ on $\{ \sigma_t^* = t \}$;
(iii) $L_t \leq \mathbb{E}_t(V_{t+1}^* | \mathcal{F}_t) \leq U_t$ on the event $\{ \tau_t^* \wedge \sigma_t^* > t \}$.

Proof. From Lemma 2.7, we deduce easily that $V_t^*$ is always bounded below by $L_t := Z_t \wedge X_t$ and from above by $U_t := Y_t \vee Z_t$, so that part (i) is valid. This makes sense intuitively since $X_t$ and $Z_t$ ($-Y_t$ and $-Z_t$, resp.) are the possible payoffs of the max-player (the min-player, resp.) if he stops at time $t$. Parts (ii) and (iii) also follow easily from Lemma 2.7 \qed

We note that these behaviours of $L, U, V^*, \tau_t^*, \sigma_t^*$ are reminiscent of Definition 2.4 if processes $X$ and $Y$ are replaced by $L$ and $U$, respectively. This observation furnishes a strong motivation for the following definition.

**Definition 2.9.** The process $V$ is defined by setting $V_T = Z_T$ and, for any $t = 0, 1, \ldots, T-1$,

\[ V_t := \min \left\{ U_t, \max \left\{ L_t, \mathbb{E}_t(V_{t+1} | \mathcal{F}_t) \right\} \right\} = \max \left\{ L_t, \min \left\{ U_t, \mathbb{E}_t(V_{t+1} | \mathcal{F}_t) \right\} \right\}, \]

(20)

where $L := X \wedge Z$ and $U := Y \vee Z$. For any fixed $t = 0, 1, \ldots, T$, the stopping times $\tau_t^*$ and $\sigma_t^*$ from $\mathcal{T}_{[t,T]}$ are given by

\[ \tau_t^* := \min \left\{ u \in \{ t, t+1, \ldots, T \} \mid V_u = L_u \right\}, \]
\[ \sigma_t^* := \min \left\{ u \in \{ t, t+1, \ldots, T \} \mid V_u = U_u \right\}. \]

(21) \hspace{1cm} (22)

In the remainder of this section, the process $V$ and stopping times $\tau_t^*, \sigma_t^*$ are as specified by Definition 2.9. To justify Definition 2.9, we will show in Lemma 2.10 that the process $V$ given by (20) is in fact the unique candidate for the value process of the general zero-sum Dynkin game GDG($X, Y, Z$). Of course, the existence of the value process for GDG($X, Y, Z$) is not yet ensured and in fact some additional conditions are needed to achieve this goal (see Assumption 2.11).

Since $L \leq Z \leq U$, it is clear that the second equality in (20) holds and, for $t = 0, 1, \ldots, T$,

\[ L_t \leq V_t \leq U_t. \]

(23)

Let the modified payoff $\tilde{R}$ be given by the following expression

\[ \tilde{R}(\tau, \sigma) := \mathbb{1}_{\{ \tau < \sigma \}} L_\tau + \mathbb{1}_{\{ \sigma < \tau \}} U_\sigma + \mathbb{1}_{\{ \sigma = \tau \}} Z_\sigma. \]

(24)

Then the analysis of the previous section shows that

\[ V_t = \mathbb{E}_t(\tilde{R}(\tau_t^*, \sigma_t^*) | \mathcal{F}_t), \]

(25)
and Proposition 2.5 implies that \((\tau^*_t, \sigma^*_t)\) is a Nash equilibrium of the standard zero-sum Dynkin game \(\text{SDG}(L, U, Z)\) associated with the payoff process \(\tilde{R}\). Obviously, this does not mean that they also provide solution to the general zero-sum Dynkin game \(\text{GDG}(X, Y, Z)\) with the payoff process \(R\). Nevertheless, the following lemma shows that \(V\) is the appropriate candidate of the value process for \(\text{GDG}(X, Y, Z)\).

**Lemma 2.10.** For \(t = 0, 1, \ldots, T\), the following properties are valid:
(i) For any fixed \(\tau_t, \sigma_t \in \mathcal{T}_{[t, T]}\), there exist \(\tilde{\tau}_t, \tilde{\sigma}_t \in \mathcal{T}_{[t, T]}\) such that

\[
R(\tilde{\tau}_t, \tilde{\sigma}_t) \geq \tilde{R}(\tau_t, \sigma_t) \geq R(\tau_t, \sigma_t). \tag{26}
\]

(ii) The variable \(V_t\) lies between the minimax and the maximin values of \(\text{GDG}(X, Y, Z)\) so that

\[
\begin{align*}
\inf_{\tau_t \in \mathcal{T}_{[t, T]}} \sup_{\sigma_t \in \mathcal{T}_{[t, T]}} E_p \left( R(\tau_t, \sigma_t) \mid F_t \right) &\geq V_t \geq \inf_{\tau_t \in \mathcal{T}_{[t, T]}} \sup_{\sigma_t \in \mathcal{T}_{[t, T]}} E_p \left( R(\tau_t, \sigma_t) \mid F_t \right). \tag{27}
\end{align*}
\]

(iii) If the \(\text{GDG}(X, Y, Z)\) has a value then it equals to \(V_t\).

**Proof.** (i) We will only prove the upper inequality of (27), as the lower inequalities follows by symmetry. To choose a stopping time \(\tilde{\tau}\) such that \(R(\tilde{\tau}_t, \sigma_t) \geq \tilde{R}(\tau_t, \sigma_t)\), we first compare \(R(\tau_t, \sigma_t)\) and \(\tilde{R}(\tau_t, \sigma_t)\). On the following events, \(R(\tau_t, \sigma_t) \geq \tilde{R}(\tau_t, \sigma_t)\) is automatically satisfied.

\[
\begin{align*}
\{ \tau_t = \sigma_t \}, & \quad R(\tau_t, \sigma_t) = Z_{\tau_t} = \tilde{R}(\tau_t, \sigma_t); \\
\{ \tau_t < \sigma_t \}, & \quad R(\tau_t, \sigma_t) = X_{\tau_t} \geq L_{\tau_t} = \tilde{R}(\tau_t, \sigma_t); \\
\{ \sigma_t < \tau_t, Y_{\sigma_t} \geq Z_{\sigma_t} \}, & \quad R(\tau_t, \sigma_t) = Y_{\sigma_t} = U_{\sigma_t} = \tilde{R}(\tau_t, \sigma_t).
\end{align*}
\]

The problem arises on the event \(\{ \sigma_t < \tau_t, Z_{\sigma_t} > Y_{\sigma_t} \}\), since then

\[
R(\tau_t, \sigma_t) = Y_{\sigma_t} < U_{\sigma_t} = \tilde{R}(\tau_t, \sigma_t).
\]

Let us modify \(\tau\) by setting

\[
\tilde{\tau} = \sigma_t 1_{\{ \sigma_t < \tau_t, Z_{\sigma_t} > Y_{\sigma_t} \}} + \tau_t \left( 1 - 1_{\{ \sigma_t < \tau_t, Z_{\sigma_t} > Y_{\sigma_t} \}} \right).
\]

Then \(\tilde{\tau}\) is indeed an \(F\)-stopping time, since the event \(\{ \sigma_t < \tau_t, Z_{\sigma_t} > Y_{\sigma_t} \}\) belongs to \(\mathcal{F}_{\sigma_t \wedge \tau_t}\). Furthermore, on the event \(\{ \sigma_t < \tau_t, Z_{\sigma_t} > Y_{\sigma_t} \}\), we have that

\[
R(\tilde{\tau}_t, \sigma_t) = R(\tau_t, \sigma_t) = Z_{\tau_t} = U_{\sigma_t} = \tilde{R}(\tau_t, \sigma_t)
\]

and thus for the stopping time \(\tilde{\tau}\) the left-hand side inequality in (27) is satisfied.

(ii) Again, we only show the upper inequality of (27). By Proposition 2.5, \(V_t\) is the value of \(\text{SDG}(L, U, Z)\) associated with \(\tilde{R}\). Let \((\tau^*_t, \sigma^*_t)\) be a Nash equilibrium of \(\text{SDG}(L, U, Z)\). Hence we have, for any \(\sigma_t \in \mathcal{T}_{[t, T]}\),

\[
E_p \left( \tilde{R}(\tau^*_t, \sigma_t) \mid F_t \right) \geq E_p \left( \tilde{R}(\tau^*_t, \sigma^*_t) \mid F_t \right) = V_t.
\]

By part (i), there exists \(\tilde{\tau}_t \in \mathcal{T}_{[t, T]}\) such that \(R(\tilde{\tau}_t, \sigma_t) \geq \tilde{R}(\tau^*_t, \sigma_t)\). Consequently,

\[
\begin{align*}
\inf_{\tau_t \in \mathcal{T}_{[t, T]}} \sup_{\sigma_t \in \mathcal{T}_{[t, T]}} E_p \left( R(\tau_t, \sigma_t) \mid F_t \right) &\geq E_p \left( R(\tilde{\tau}_t, \sigma_t) \mid F_t \right) \geq E_p \left( \tilde{R}(\tau^*_t, \sigma_t) \mid F_t \right) \geq V_t. \tag{28}
\end{align*}
\]

Since (28) holds for all \(\sigma_t \in \mathcal{T}_{[t, T]}\), we must have

\[
\begin{align*}
\inf_{\sigma_t \in \mathcal{T}_{[t, T]}} \sup_{\tau_t \in \mathcal{T}_{[t, T]}} E_p \left( R(\tau_t, \sigma_t) \mid F_t \right) \geq V_t,
\end{align*}
\]

\[\square\]
as required.

(iii) By the definition of the value (see Definition 3.1), if there exists a value $V_t^*$ for $\text{GDG}(X, Y, Z)$, it must satisfy

$$V_t^* = \text{ess inf } \text{ess sup} \mathbb{E}_\mathbb{P}(R(\tau_t, \sigma_t) \mid \mathcal{F}_t) = \text{ess inf } \text{ess sup} \mathbb{E}_\mathbb{P}(R(\tau_t, \sigma_t) \mid \mathcal{F}_t).$$

In view of part (ii), we conclude that necessarily $V_t^* = V_t$.

Even though $V$ is the unique value process candidate for $\text{GDG}(X, Y, Z)$, the existence of the value process has not been established. There are two major obstacles to overcome when attempting to apply the backward induction argument similar to Proposition 2.5 on the payoff process $R$.

First, it is not necessarily true that $V_t = \mathbb{E}_\mathbb{P}(R(\tau_t^*, \sigma_t^*) \mid \mathcal{F}_t)$.

In particular, equality fails to hold if either of the following occurs:

$$Z_{\tau_t^*} = V_{\tau_t^*} < X_{\tau_t^*} \land Y_{\tau_t^*} \quad \text{on the event} \quad \{\tau_t^* < \sigma_t^*\}, \quad (30)$$

$$Z_{\sigma_t^*} = V_{\sigma_t^*} > X_{\sigma_t^*} \lor Y_{\sigma_t^*} \quad \text{on the event} \quad \{\sigma_t^* < \tau_t^*\}. \quad (31)$$

Second, it is possible that $V$ fails to satisfy any of the necessary conditions on $V^*$ established in Lemma 2.7. An exhaustive check shows that the exceptions are:

$$Z_t \leq V_t < X_t \land Y_t, \quad (32)$$

$$Z_t \geq V_t > X_t \lor Y_t. \quad (33)$$

It is crucial to observe that the undesirable scenarios may only occur when $V$ is either greater than $X \lor Y$ or less than $X \land Y$. Therefore, it is natural to introduce the following additional assumption.

**Assumption 2.11.** Let $X, Y$ and $Z$ be $\mathbb{F}$-adapted integrable processes and let the associated process $V$ be given as in Definition 2.6. We postulate that the processes $X, Y$ and $V$ satisfy, for $t = 0, 1, \ldots, T$,

$$X_t \land Y_t \leq V_t \leq X_t \lor Y_t. \quad (34)$$

Assumption 2.11 certainly eliminates the scenarios described in (30)–(33). Since $V$ is defined in terms of $X, Y$ and $Z$, this is really an assumption on $X, Y$ and $Z$, albeit its form is somewhat convoluted, since it also refers to formula (20). In the foregoing example, we provide some more explicit conditions that entail Assumption 2.11.

**Example 2.12.** (i) Let us first consider the conditions from the previous section: $X_T = Z_T = Y_T$ and $X \leq Z \leq Y$. In view of (24), it is clear that Assumption 2.11 is satisfied since $L = X = X \land Y$ and $U = Y = X \lor Y$. This shows that Assumption 2.11 covers the case of the standard zero-sum Dynkin game.

(ii) Suppose $X, Y$ and $Z$ satisfy $X_T = Z_T = Y_T$ and, for all $t \in [0, T]$,

$$X_t \land Y_t > Z_t \quad \Rightarrow \quad X_t \land Y_t \leq \mathbb{E}_\mathbb{P}(X_{t+1} \land U_{t+1} \mid \mathcal{F}_t),$$

$$X_t \lor Y_t < Z_t \quad \Rightarrow \quad X_t \lor Y_t \geq \mathbb{E}_\mathbb{P}(L_{t+1} \lor Y_{t+1} \mid \mathcal{F}_t).$$

One can check that Assumption 2.11 is satisfied.

(iii) It should be acknowledged that various generalisations of the standard Dynkin game were studied in the literature. In particular, Ohtsubo [11] examined the zero-sum Dynkin game with an infinite time horizon under the assumption that

$$X_t \land Y_t \leq Z_t \leq X_t \lor Y_t. \quad (35)$$
Once again, we see that if (35) holds then Assumption 2.11 is satisfied. He established the existence of a Nash equilibrium for the game starting at any date \( t \) under the assumption that \( (X_t) \) and \( (Y_t) \) are mutually independent sequences of i.i.d. random variables (see Corollary 3.2 in [11]).

As a special case, Ohtsubo [11] considered also the game with the payoff

\[ R(\tau, \sigma) := \mathbb{1}_{\{\tau < \sigma\}} X_\tau + \mathbb{1}_{\{\sigma \leq \tau\}} Y_\sigma \]

for arbitrary \( \mathcal{F} \)-adapted, integrable processes \( X \) and \( Y \). Since here \( Z = Y \), so that \( L = X \wedge Y \) and \( U = Y \), it follows easily from (23) that Assumption 2.11 is satisfied. It can be deduced from Proposition 3.1 in [11] that in the finite horizon case the game admits a Nash equilibrium and the value process \( V^* \) satisfies: \( V^*_T = Y_T = Z_T \) and, for \( t = 0, 1, \ldots, T - 1 \),

\[ V^*_t = Y_t \mathbb{1}_{\{Y_t \leq X_t\}} + \min \left\{ Y_t, \max \left\{ X_t, \mathbb{E}_P(V^*_{t+1} | \mathcal{F}_t) \right\} \right\} \mathbb{1}_{\{Y_t > X_t\}}. \tag{36} \]

This result can be seen as a special case of Proposition 2.14 since for \( Y = Z \) equation (20) becomes: \( V_T = Z_T = Z_T \) and

\[ V_t := \min \left\{ Y_t, \max \left\{ X_t \wedge Y_t, \mathbb{E}_P(V_{t+1} | \mathcal{F}_t) \right\} \right\}, \]

which indeed coincides with (36), so that \( V = V^* \) where \( V^* \) is given by (36).

### 2.2.1 Sufficiency of Assumption 2.11

Our goal is to demonstrate that Assumption 2.11 is the necessary and sufficient condition for (18) to hold. We start by examining the sufficiency of Assumption 2.11. To this end, we first prove an auxiliary lemma.

**Lemma 2.13.** Under Assumption 2.11, for each \( t = 0, 1, \ldots, T \), the process \( V \) satisfies:

(i) \( \{V_t > Y_t\} \subseteq \{\tau^*_t = t\} \) and \( \{V_t < X_t\} \subseteq \{\sigma^*_t = t\} \);

(ii) \( V_t = \mathbb{E}_P(R(\tau^*_t, \sigma^*_t) | \mathcal{F}_t) \).

**Proof.** (i) For the first inclusion, let us suppose that \( V_t > Y_t \). Since Assumption 2.11 states that \( V_t \) has to lie in between \( X_t \) and \( Y_t \), we obtain

\[ V_t \leq X_t, \tag{37} \]

From (23) we obtain \( V_t \leq U_t = Y_t \vee Z_t \), and thus we must also have

\[ V_t \leq Z_t. \tag{38} \]

By combining (37) with (38), we obtain \( V_t \leq X_t \wedge Z_t = L_t \). Moreover, by noting that \( V_t \geq L_t \) from (23), we conclude that \( V_t = L_t \) and thus, by (21), the equality \( \tau^*_t = t \) holds, as required.

The second inclusion can be shown using similar arguments.

(ii) It is sufficient to show

\[ V^{\tau^*_t \wedge \sigma^*_t} = R(\tau^*_t, \sigma^*_t) = \mathbb{1}_{\{\tau^*_t < \sigma^*_t\}} X_{\tau^*_t} + \mathbb{1}_{\{\sigma^*_t < \tau^*_t\}} Y_{\sigma^*_t} + \mathbb{1}_{\{\sigma^*_t = \tau^*_t\}} Z_{\sigma^*_t}. \tag{39} \]

On the event \( \{\sigma^*_t < \tau^*_t\} \), we have \( V_{\sigma^*_t} = U_{\sigma^*_t} = L_{\sigma^*_t} = Z_{\sigma^*_t} \) as required. On the event \( \{\sigma^*_t > \tau^*_t\} \), we have \( V_{\tau^*_t} = U_{\tau^*_t} = Z_{\tau^*_t} \) and \( Y_{\tau^*_t} \geq Y_{\sigma^*_t} \). If \( Y_{\tau^*_t} > Y_{\sigma^*_t} \), then from (i), we obtain \( \tau^*_t = \sigma^*_t \); which is a contradiction. We thus conclude that \( V_{\sigma^*_t} = Y_{\sigma^*_t} \), as required.

The case of \( \{\sigma^*_t > \tau^*_t\} \) is similar to the case of \( \{\sigma^*_t < \tau^*_t\} \). This establishes (39). \( \square \)
We are now in a position to show that Assumption 2.11 implies the existence of Nash equilibria for the family of Dynkin games \( \text{GDG}_t(X, Y, Z), t = 0, 1, \ldots, T \).

**Proposition 2.14.** Let the process \( V \) and the stopping times \( \tau^*_t, \sigma^*_t \) be given as in Definition 2.2. If Assumption 2.11 holds then for arbitrary stopping times \( \tau_t, \sigma_t \in T_{[t,T]} \),

\[
\mathbb{E}_p(R(\tau^*_t, \sigma_t) | F_t) \geq \mathbb{E}_p(R(\tau^*_t, \sigma^*_t) | F_t) \geq \mathbb{E}_p(R(\tau_t, \sigma^*_t) | F_t)
\]

and thus \((\tau^*_t, \sigma^*_t)\) is a Nash equilibrium of \( \text{GDG}_t(X, Y, Z) \). Furthermore, the process \( V \) is the value process of \( \text{GDG}(X, Y, Z) \), that is, for every \( t = 0, 1, \ldots, T \),

\[
V_t = \text{ess inf}_{\sigma_t \in T_{[t,T]}} \text{ess sup}_{\tau_t \in T_{[t,T]}} \mathbb{E}_p(R(\tau_t, \sigma_t) | F_t) = \mathbb{E}_p(R(\tau^*_t, \sigma^*_t) | F_t)
\]

and \( \tau^*_t, \sigma^*_t \) are the optimal stopping times as of time \( t \). In particular, \( V^*_T = Z_T \) and for any \( t = 0, 1, \ldots, T - 1 \),

\[
V^*_t = \min \left\{ U_t, \max \left\{ L_t, \mathbb{E}_p(V^*_t | F_t) \right\} \right\}.
\]

**Proof.** The arguments used in this proof will be very similar to the ones from Proposition 2.5 with the help of Lemma 2.13. In view of part (ii) in Lemma 2.13 it is sufficient to show that

\[
\mathbb{E}_p(R(\tau^*_t, \sigma_t) | F_t) \geq V_t \geq \mathbb{E}_p(R(\tau_t, \sigma^*_t) | F_t).
\]  

(40)

To this end, we proceed by backward induction. The inequalities (40) clearly hold for \( t = T \). Assume now that they are true for some \( t < T \). We wish to prove that, for arbitrary \( \tau_{t-1}, \sigma_{t-1} \in T_{[t-1,T]} \),

\[
\mathbb{E}_p(R(\tau^*_{t-1}, \sigma_{t-1}) | F_{t-1}) \geq V_{t-1} \geq \mathbb{E}_p(R(\tau_{t-1}, \sigma^*_{t-1}) | F_{t-1}).
\]  

(41)

We will establish the upper bound of (41), the lower bound follows by the symmetry of the Dynkin game. Again the argument can be split into two main cases: either \( \text{GDG}_{t-1}(X, Y, Z) \) is stopped at time \( t-1 \) or it is continued to time \( t \) and the induction hypothesis becomes relevant. As before, for any \( \tau_{t-1}, \sigma_{t-1} \in T_{[t-1,T]} \), we denote \( \tilde{\tau}_{t-1} := \tau_{t-1} \vee t \) and \( \tilde{\sigma}_{t-1} := \sigma_{t-1} \vee t \), so that \( \tau_{t-1}, \sigma_{t-1} \in T_{[t,T]} \).

Let us examine the case where the game is stopped at time \( t-1 \). On the event \( \{ \tau^*_{t-1} = t-1 \} \), we have

\[
\mathbb{E}_p(R(\tau^*_{t-1}, \sigma_{t-1}) | F_{t-1}) = \mathbb{I}_{\{\sigma_{t-1} = t\}} X_{t-1} + \mathbb{I}_{\{\sigma_{t-1} = t\}} Z_{t-1} \geq L_{t-1} = V_{t-1}.
\]  

(42)

On the event \( \{ \sigma_{t-1} = t-1 < \tau^*_{t-1} \} \), we obtain \( \mathbb{E}_p(R(\tau^*_{t-1}, \sigma_{t-1}) | F_{t-1}) = Y_{t-1} \). If \( V_{t-1} > Y_{t-1} \), then by part (i) in Lemma 2.13 we have that \( \tau^*_{t-1} = t-1 \), which is a contradiction. Hence \( V_{t-1} \leq Y_{t-1} \) and thus

\[
\mathbb{E}_p(R(\tau^*_{t-1}, \sigma_{t-1}) | F_{t-1}) = Y_{t-1} \geq V_{t-1}.
\]  

(43)

Let us now assume that the game is not stopped at time \( t-1 \), that is, we now consider the event \( \{ \tau^*_{t-1} \geq t, \sigma_{t-1} \geq t \} \). We observe that here \( \tau^*_{t-1} = \tilde{\tau}_{t-1} = \tau^*_t \) and \( V_{t-1} > L_{t-1} \), so that (20) yields

\[
V_{t-1} = \min \{ U_{t-1}, \mathbb{E}_p(V_t | F_{t-1}) \}.
\]  

(44)
Consequently,
\[ \mathbb{E}_\mathbb{P}(R(\tau^*_{t-1}, \sigma_{t-1}) | F_{t-1}) = \mathbb{E}_\mathbb{P}(R(\tilde{\tau}^*_{t-1}, \tilde{\sigma}_{t-1}) | F_{t-1}) \]
\[ = \mathbb{E}_\mathbb{P}(\mathbb{E}_\mathbb{P}(R(\tau^*_{t}, \sigma_{t-1}) | F_t) | F_{t-1}) \]
\[ \geq \mathbb{E}_\mathbb{P}(V_t | F_{t-1}) \]
\[ \geq \min \{U_{t-1}, \mathbb{E}_\mathbb{P}(V_t | F_{t-1})\} \]
\[ = V_{t-1}. \]  

Note that (45) follows from the induction hypothesis (40) while (46) follows from (41).

Combining (42), (43) and (46) gives the upper inequality of (41). As already mentioned, the lower inequality of (41) follows by symmetry. Therefore, \((\tau^*_{t}, \sigma^*_{t})\) is a Nash equilibrium of \(GDG_t(X, Y, Z)\) and \(V_t = \mathbb{E}_\mathbb{P}(R(\tau^*_{t}, \sigma^*_{t}) | F_t)\) is the value.

### 2.2.2 Necessity of Assumption 2.11

To prove that Assumption 2.11 is also a necessary condition for property (18) to hold, it suffices to show that if this assumption is violated then there exists \(t\) such that the general Dynkin game \(GDG_t(X, Y, Z)\) does not have a Nash equilibrium. Recall that the process \(V\) in Definition 2.9 was originally chosen to be the value process of the Dynkin game SDG\((L, U, Z)\) associated with the payoff process
\[ \tilde{R}(\tau, \sigma) := \mathbb{I}_{\{\tau < \sigma\}} L_\tau + \mathbb{I}_{\{\sigma < \tau\}} U_\sigma + \mathbb{I}_{\{\sigma = \tau\}} Z_\sigma. \]

Next, in Lemma 2.10 it was shown that if the Dynkin game \(GDG(X, Y, Z)\) associated with the payoff process
\[ R(\tau, \sigma) := \mathbb{I}_{\{\tau < \sigma\}} X_\tau + \mathbb{I}_{\{\sigma < \tau\}} Y_\sigma + \mathbb{I}_{\{\sigma = \tau\}} Z_\sigma, \]

has a value process then it has to be a version of \(V\). Finally, we formulated Assumption 2.11, which was shown to ensure that \(GDG(X, Y, Z)\) has a value process.

**Proposition 2.15.** Suppose that Assumption 2.11 is violated at time \(t \in [0, T]\), that is,
\[ \mathbb{P}(\{V_t < X_t \wedge Y_t\} \cup \{V_t > X_t \vee Y_t\}) > 0. \]  

Then \(GDG_t(X, Y, Z)\) does not have a Nash equilibrium.

**Proof.** Since \(X_T = Y_T = Z_T = V_T\) then manifestly (47) cannot occur when \(t = T\). Assume, for the sake of contradiction, that (47) holds for some \(t < T\) and there is a Nash equilibrium \((\tau^*_{t}, \sigma^*_{t})\). Then, by part (iii) in Lemma 2.10 \(V_t = V^*_t = \mathbb{E}_\mathbb{P}(R(\tau^*_{t}, \sigma^*_{t}) | F_t)\) is the value of \(GDG_t(X, Y, Z)\).

Assume now that either \(\mathbb{P}(\{V_t < X_t \wedge Y_t\}) > 0\) or \(\mathbb{P}(\{V_t > X_t \vee Y_t\}) > 0\). First, consider the event \(\{V_t < X_t \wedge Y_t\}\). Neither \(\tau^*_t = t < \sigma^*_t\) nor \(\tau^*_t = t < \sigma^*_t\) can occur, as otherwise we would have that either \(V_t = X_t\) or \(V_t = Y_t\), respectively. If \(\tau^*_t = \sigma^*_t = t\) then \(R(t, \sigma^*_t) = Y_t \geq V_t\), contradicting the property of the Nash equilibrium. If \(t < \tau^*_t \wedge \sigma^*_t\) then \(R(t, \sigma^*_t) = X_t > V_t\), which is also a contradiction.

The same argument can be made for the event \(\{V_t > X_t \vee Y_t\}\). We conclude there cannot be a Nash equilibrium for the Dynkin game starting at time \(t\) if condition (47) is valid.

Propositions 2.5 and 2.14 can be combined into the following main result of this section, which explicitly states the condition needed for the existence of a Nash equilibrium for arbitrary payoff processes \(X, Y\) and \(Z\). Theorem 2.16 is thus an essential generalisation of Proposition 2.5 for the standard zero-sum Dynkin game, which only addressed the case of \(X \leq Z \leq Y\).
Theorem 2.16. Let $X, Y$ and $Z$ be $\mathbb{F}$-adapted, integrable processes and let the process $V$ be given by: $V_T := Z_T$ and, for $t = 0, 1, \ldots, T - 1,$

$$V_t := \min \left\{ U_t, \max \{ L_t, \mathbb{E}_\mathbb{P}(V_{t+1} | \mathcal{F}_t) \} \right\}$$

where $L = X \wedge Z$ and $U = Y \vee Z$. The inequality

$$X_t \wedge Y_t \leq V_t \leq X_t \vee Y_t$$

holds for all $t = 0, 1, \ldots, T$ if and only if the Dynkin game $GDG_t(X, Y, Z)$ starting at time $t$ and associated with the payoff

$$R(\tau, \sigma) := 1_{\{\tau < \sigma\}} X_\tau + 1_{\{\sigma < \tau\}} Y_\sigma + 1_{\{\sigma = \tau\}} Z_\sigma.$$

has a Nash equilibrium for all $t = 0, 1, \ldots, T$.

Remark 2.17. Theorem 2.16 answers the question regarding the existence of Nash equilibrium in for the set of Dynkin games starting at all times $t = 0, 1, \ldots, T$. For a Dynkin game starting at a particular value of $t$, the exact result is unclear. Assumption 2.11 certainly provides a sufficient condition, but it is not a necessary condition.

3 Continuous-Time Dynkin Games

In this section, we deal with continuous-time versions of two-person, zero-sum stopping games with a finite time horizon. As previously, we focus on conditions under which the game admits a Nash equilibrium.

3.1 Standard Dynkin Game

In this preliminary subsection, we re-examine the standard zero-sum Dynkin game in continuous-time. We first recall two definitions.

Definition 3.1 (Value). Consider a two-player, zero-sum game $\mathcal{G}$ with strategy spaces $S^1$ and $S^2$ and payoff function $V$. It is said to have a value $V^*$ if

$$V^* = \operatorname{ess inf}_{\sigma \in S^2} \operatorname{ess sup}_{\tau \in S^1} V(\tau, \sigma) = \operatorname{ess sup}_{\tau \in S^1} \operatorname{ess inf}_{\sigma \in S^2} V(\tau, \sigma).$$

Definition 3.2. Suppose that a game $\mathcal{G}$ has a value $V^*$. For $\epsilon \geq 0$, an $\epsilon$-optimal strategy $\tau^\epsilon \in S^1$ for the max-player guarantees the payoff to within $\epsilon$ of the value. In other words

$$\operatorname{ess inf}_{\sigma \in S^2} V(\sigma, \tau^\epsilon) \geq V^* - \epsilon. \quad (48)$$

Similarly, an $\epsilon$-optimal strategy $\sigma^\epsilon \in S^2$ for the min-player satisfies

$$\operatorname{ess sup}_{\tau \in S^1} V(\sigma^\epsilon, \tau) \leq V^* + \epsilon. \quad (49)$$

A strategy profile $(\sigma, \tau)$ is is called an $\epsilon$-equilibrium if it consists of $\epsilon$-optimal strategies for both players.

Note that a Nash equilibrium is a 0-equilibrium. The following result is easy to prove and thus the proof is omitted.
Proposition 3.3. In a two-player, zero-sum game \( \Theta \), the following statements are equivalent.

(i) The game has a value for both players.

(ii) For all \( \epsilon > 0 \), there exist \( \epsilon \)-optimal strategies for both players.

(iii) For all \( \epsilon > 0 \), there exists an \( \epsilon \)-equilibrium.

(iv) For all \( \epsilon > 0 \), there exists a real number \( v^\epsilon \) and a strategy profile \( (\sigma^\epsilon, \tau^\epsilon) \) such that

\[
\text{ess inf}_{\sigma \in S_1} V^1(\sigma, \tau^\epsilon) \geq v^\epsilon \geq \text{ess sup}_{\tau \in S_2} V^1(\sigma^\epsilon, \tau).
\] (50)

Let the time parameter \( t \in [0, T] \) be continuous and the filtration \( \mathbb{F} \) be right-continuous. Let \( X, Y \) and \( Z \) be \( \mathbb{F} \)-adapted, càdlàg processes satisfying the usual integrability condition. Consider the standard Dynkin game SDG\((X, Y, Z)\) starting at \( t \) and with payoff given by

\[
R(\tau, \sigma) = \mathbb{1}_{\{\sigma < \tau\}} X_\tau + \mathbb{1}_{\{\sigma > \tau\}} Y_\sigma + \mathbb{1}_{\{\sigma = \tau\}} Z_\sigma
\] (51)

where \( \tau, \sigma \) are \( \mathbb{F} \)-stopping times and \( X \leq Z \leq Y \). We denote by SDG\((X, Y, Z)\) the family of Dynkin games SDG\((X, Y, Z)\), \( t \in [0, T] \). As in Section 2 without the loss of generality, we set \( X_T = Y_T = Z_T \).

The case of the standard zero-sum continuous-time Dynkin game has been studied by several authors, for example, Lepeltier and Maingueneau [9]. The following result summarises some of these findings.

Theorem 3.4. Consider the standard zero-sum Dynkin game SDG\((X, Y, Z)\) associated with the payoff \( R \) given by formula \( (51) \).

(i) For any \( t \in [0, T] \), the standard zero-sum Dynkin game SDG\((X, Y, Z)\) has a value \( V_t^* \) satisfying

\[
V_t^* = \text{ess inf}_{\sigma_t \in T_{[t, T]}} \text{ess sup}_{\tau_t \in T_{[t, T]}} \mathbb{E}_t (R(\tau_t, \sigma_t) | F_t)
\] (52)

\[
= \text{ess sup}_{\tau_t \in T_{[t, T]}} \text{ess inf}_{\sigma_t \in T_{[t, T]}} \mathbb{E}_t (R(\tau_t, \sigma_t) | F_t).
\]

The value process \( V^* \) of SDG\((X, Y, Z)\) can be chosen to be right-continuous.

(ii) For any \( t \in [0, T] \) and any \( \epsilon > 0 \), the pair of \( \mathbb{F} \)-stopping times \( (\tau_t^\epsilon, \sigma_t^\epsilon) \in T_{[t, T]} \times T_{[t, T]} \) defined by

\[
\sigma_t^\epsilon := \inf \{ u \geq t : Y_u \leq V_u^* + \epsilon\}, \quad \tau_t^\epsilon := \inf \{ u \geq t : X_u \geq V_u^* - \epsilon\}
\] (53)

are \( \epsilon \)-optimal strategies satisfying

\[
\text{ess inf}_{\sigma_t \in T_{[t, T]}} \mathbb{E}_t (R(\tau_t^\epsilon, \sigma_t) | F_t) + \epsilon \geq V_t^* \geq \text{ess sup}_{\tau_t \in T_{[t, T]}} \mathbb{E}_t (R(\tau_t, \sigma_t^\epsilon) | F_t) - \epsilon.
\] (54)

(iii) If we further assume that \( X \) and \( -Y \) are left upper semi-continuous (only have positive jumps), then SDG\((X, Y, Z)\) has a Nash equilibrium \( (\tau_t^*, \sigma_t^*) \in T_{[t, T]} \times T_{[t, T]} \) satisfying

\[
\sigma_t^* := \lim_{\epsilon \to 0} \sigma_t^\epsilon, \quad \tau_t^* := \lim_{\epsilon \to 0} \tau_t^\epsilon
\] (55)

and

\[
\mathbb{E}_t (R(\tau_t^*, \sigma_t) | F_t) \geq \mathbb{E}_t (R(\tau_t^*, \sigma_t^*) | F_t) = V_t^* \geq \mathbb{E}_t (R(\tau_t, \sigma_t^*) | F_t), \quad \forall \tau_t, \sigma_t \in T_{[t, T]}.
\] (56)

Proof. Theorem 3.4 summarises well know results and thus its proof is omitted. \( \square \)
Observe that there may be other \( \epsilon \)-optimal strategy pairs (resp. Nash equilibria) than the ones specified by (53) (resp. (55)). Also \( \sigma^*_t, \tau^*_t \) do not necessarily coincide with stopping times \( \sigma^0_t, \tau^0_t \), which are defined by setting \( \epsilon = 0 \) in (53), that is,

\[
\sigma^0_t := \inf \{ u \geq t : Y_u \leq V^*_u \}, \quad \tau^0_t := \inf \{ u \geq t : X_u \geq V^*_u \}.
\]

In general, we have that \( \sigma^*_t \leq \sigma^0_t \) and \( \tau^*_t \leq \tau^0_t \).

3.1.1 Auxiliary Results

Before moving on to the next subsection, we will first establish several auxiliary properties, which are consequences of Theorem 3.3.

Lemma 3.5. (i) For any \( t \in [0, T] \), we have that \( X_t \leq V^*_t \leq Y_t \).

(ii) For \( \epsilon \geq 0 \), let \( \tau^\epsilon_t, \sigma^\epsilon_t \) be as defined in (53). Then

\[
X_{\tau^\epsilon_t} \geq V^*_{\tau^\epsilon_t} - \epsilon, \quad Y_{\sigma^\epsilon_t} \leq V^*_{\sigma^\epsilon_t} + \epsilon.
\]

Proof. (i) The lower bound follows from

\[
V^*_t = \esssup_{\tau \in T_{[t,T]}}, \essinf_{\sigma \in T_{[t,T]}} \mathbb{E}_\mathbb{P}(R(\tau, \sigma) | \mathcal{F}_t) \geq \essinf_{\sigma \in T_{[t,T]}} \mathbb{E}_\mathbb{P}(R(t, \sigma) | \mathcal{F}_t) \geq X_t.
\]

The upper bound can be shown similarly. Part (ii) follows immediately from the right-continuity of \( V^*, X \) and \( Y \).

Lemma 3.6. Let \( G \) and \( H \) be integrable progressively measurable processes. Suppose \( G \) is right lower semicontinuous and \( H \) is right-continuous. If for each \( t \in [0, T] \), \( G_t \leq H_t \) a.s., then for all \( \rho \in T_{[0,T]} \), \( G_\rho \leq H_\rho \) a.s..

Proof. Choose a sequence of decreasing stopping times \( \rho_n \) which takes countably many values and converge to \( \rho \). Then

\[
G_\rho \leq \lim_{n \to \infty} G_{\rho_n} \leq \lim_{n \to \infty} H_{\rho_n} = H_\rho,
\]

as required.

For a fixed \( \sigma \in T_{[0,T]} \), the process \( \hat{R}^\sigma_t := R(t, \sigma) \) is right lower semicontinuous, but not necessarily continuous. So let us define the right-continuous process

\[
\hat{R}^\sigma_t := X_t \mathbb{1}_{\{t < \sigma\}} + Y_\sigma \mathbb{1}_{\{\sigma \leq t\}}.
\]

Since \( Y \geq Z \geq X \), we have that \( \hat{R}^\sigma_t \geq R^\sigma_t \). Consequently, by Lemma 3.6 \( \hat{R}^\rho_t \geq R^\rho_t \) for all \( \rho \in T_{[0,T]} \). On the other hand, since \( \hat{R}^\sigma_t = R(\rho \mathbb{1}_{\{\rho < \sigma^*_t\}} + T \mathbb{1}_{\{\rho \geq \sigma^*_t\}}, \sigma) \), the following Snell envelope of \( R^\sigma_t \) and \( \hat{R}^\sigma_t \)

\[
Q^\sigma_t := \esssup_{\rho \in T_{[0,T]}} \mathbb{E}_\mathbb{P}(R^\rho_t | \mathcal{F}_t) = \esssup_{\rho \in T_{[0,T]}} \mathbb{E}_\mathbb{P}(\hat{R}^\rho_t | \mathcal{F}_t) \tag{58}
\]

is a well-defined right-continuous supermartingale. It follows immediately from Lemma 3.6 that, for all \( \tau \in T_{[0,T]} \),

\[
R(\tau, \sigma) \leq Q^\sigma_\tau. \tag{59}
\]

The process \( Q^\sigma \) can also be used to demonstrate properties of \( V^* \).
Proposition 3.7. Let $V^*$ be as defined in (52) and $Q^\sigma$ be as defined in (53).

(i) For all $\sigma, \tau \in \mathcal{T}_{[0,T]}$,

$$V^*_\sigma \wedge \tau \leq Q^\tau_{\sigma \wedge \tau}. \quad (60)$$

(ii) For $\epsilon \geq 0$, let $\hat{\sigma}_t \in \mathcal{T}_{[0,T]}$ be an arbitrary $\epsilon$-optimal strategy for the min-player in SDG$(X,Y,Z)$ and $\tau_t \in \mathcal{T}_{[0,T]}$ be any $\mathbb{F}$-stopping time. If $P$ is an $\mathcal{F}_T$-measurable random variable satisfying $P \leq Q^\delta_t$, then

$$\mathbb{E}_P(P | \mathcal{F}_t) \leq V^*_t + \epsilon. \quad (61)$$

(iii) For $\epsilon \geq 0$, let $(\hat{\tau}_t, \hat{\sigma}_t) \in \mathcal{T}_{[0,T]} \times \mathcal{T}_{[0,T]}$ be an arbitrary pair of $\epsilon$-optimal strategies of SDG$(X,Y,Z)$. Then for all $\sigma_t, \tau_t \in \mathcal{T}_{[0,T]}$,

$$\mathbb{E}_P(V^*_t \wedge \sigma_t | \mathcal{F}_t) + \epsilon \geq V^*_t \geq \mathbb{E}_P(V^*_t \wedge \tau_t | \mathcal{F}_t) - \epsilon. \quad (62)$$

(iv) If $(\hat{\tau}_t, \hat{\sigma}_t) \in \mathcal{T}_{[0,T]} \times \mathcal{T}_{[0,T]}$ is an arbitrary Nash equilibrium of SDG$(X,Y,Z)$, then $V^*$ is a submartingale on $[t, \hat{\tau}_t]$ and a supermartingale on $[t, \hat{\sigma}_t]$.

(v) For $\epsilon > 0$, if $(\tau^*_t, \sigma^*_t) \in \mathcal{T}_{[0,T]} \times \mathcal{T}_{[0,T]}$ is the pair of $\epsilon$-optimal strategies of SDG$(X,Y,Z)$ defined by (63):

$$\sigma^*_t = \inf \{u \geq t : Y_u \leq V^*_u + \epsilon\}, \quad \tau^*_t = \inf \{u \geq t : X_u \geq V^*_u - \epsilon\},$$

then $V^*$ is a submartingale on $[t, \tau^*_t]$ and a supermartingale on $[t, \sigma^*_t]$.

Proof. (i) Consider the right-continuous process defined by

$$V^*_t := V^*_t \mathbb{1}_{\{t \leq \sigma\}} + Y_t \mathbb{1}_{\{\sigma < t\}}.$$

On the event $\{t < \sigma\}$, we have

$$V^*_t = V^*_t = \text{ess inf}_{\sigma_t \in \mathcal{T}_{[t,T]} } \text{ess sup}_{\tau_t \in \mathcal{T}_{[t,T]} } \mathbb{E}_P(R(\sigma_t, \tau_t) | \mathcal{F}_t) \leq \text{ess sup}_{\tau_t \in \mathcal{T}_{[t,T]} } \mathbb{E}_P(R(\tau_t, \sigma_t) | \mathcal{F}_t) = Q^\tau_{\sigma \wedge \tau}. \quad (64)$$

and on the event $\{\sigma \leq t\}$ we obtain

$$V^*_t = Y_t = R(T, \sigma) \leq \text{ess sup}_{\tau_t \in \mathcal{T}_{[T,T]} } \mathbb{E}_P(R(\tau_t, \sigma) | \mathcal{F}_t) = Q^\tau_{\sigma \wedge \tau}. \quad (65)$$

By combining (64) and (65), we obtain $V^*_t \leq Q^\tau_{\sigma \wedge \tau}$. Applying Lemma 3.6, we have

$$V^*_\sigma \wedge \tau = V^*_\sigma \wedge \tau \leq Q^\tau_{\sigma \wedge \tau}$$

as required.

(ii) By using the optional sampling theorem on $Q$ and the $\epsilon$-strategy property of $\hat{\sigma}_t$,

$$\mathbb{E}_P(P | \mathcal{F}_t) \leq \mathbb{E}_P(Q^\hat{\delta}_t | \mathcal{F}_t) \leq Q^\hat{\delta}_t = \text{ess sup}_{\delta \in \mathcal{T}_{[t,T]} } \mathbb{E}_P(R(\rho, \hat{\sigma}_t) | \mathcal{F}_t) \leq V^*_t + \epsilon,$$

as required.

(iii) The lower bound of (52) follows directly from parts (i) and (ii). The upper bound also follows by the symmetry of the problem.

(iv) To obtain the required result, it suffices to set $\epsilon = 0$ in part (iii).

(v) Again we will only demonstrate the lower bound. By (53), $\sigma^*_t$ is increasing with respect to $\epsilon$. So for any $\delta \in [0, \epsilon]$, we have $\sigma^*_t$ being an $\delta$-optimal strategy with $\sigma^*_t \in \mathcal{T}_{[t, \sigma^*_t]}$. Hence by (iii),

$$\mathbb{E}_P(V^*_t \wedge \sigma^*_t | \mathcal{F}_t) \leq V^*_t + \delta.$$

Since this is true for all choice of $\delta \in [0, \epsilon]$, we must have $\mathbb{E}_P(V^*_t | \mathcal{F}_t) \leq V^*_t$ as required. □
Proposition 3.8. If \((\hat{\tau}, \hat{\sigma})\) is a Nash equilibrium of \(SDG_{t}(X, Y, Z)\) then \((\check{\tau}_{t} \land \tau_{t}^{0}, \check{\sigma}_{t} \land \sigma_{t}^{0})\) is also a Nash equilibrium, where \(\tau_{t}^{0}, \sigma_{t}^{0}\) are defined by (53).

Proof. We will first show that \((\check{\tau}_{t} \land \tau_{t}^{0}, \check{\sigma}_{t} \land \sigma_{t}^{0})\) is a Nash equilibrium. It is sufficient to show that
\[
\text{ess inf}_{\sigma_{t} \in T_{(\tau_{t}, \sigma_{t})}} \mathbb{E}_{\mathbb{P}}(R(\check{\tau}_{t}, \sigma_{t}) | F_{t}) \geq V^{*}_{t} \geq \text{ess sup}_{\tau_{t} \in T_{(\tau_{t}, \sigma_{t})}} \mathbb{E}_{\mathbb{P}}(R(\tau_{t}, \check{\sigma}_{t} \land \sigma_{t}^{0}) | F_{t}).
\] (66)
The upper inequality is clear, since \((\check{\tau}_{t}, \check{\sigma}_{t})\) is a Nash equilibrium. For the lower inequality, the key is to introduce \(Q\), as defined in (58), and then apply Proposition 3.7(ii).

There are two cases to examine:
(a) on the event \(\{\check{\tau}_{t} \land \tau_{t} < \sigma_{t}^{0}\}\), we obtain
\[
R(\tau_{t}, \check{\sigma}_{t} \land \sigma_{t}^{0}) = R(\tau_{t}, \check{\sigma}_{t}) = R(\tau_{t}, \check{\sigma}_{t}, \check{\tau}_{t}) \leq Q_{\check{\tau}_{t} \land \sigma_{t}^{0}},
\] (67) where the last inequality follows from (59);
(b) on the event \(\{\sigma_{t}^{0} \leq \check{\sigma}_{t} \land \tau_{t}\}\), we have that
\[
R(\tau_{t}, \check{\sigma}_{t} \land \sigma_{t}^{0}) = Y_{\sigma_{t}^{0}} \quad \text{or} \quad Z_{\sigma_{t}^{0}} \leq Y_{\sigma_{t}^{0}} = V_{\sigma_{t}^{0}}^{*}
\] (68)
\[
= V_{\check{\tau}_{t} \land \sigma_{t}^{0}}^{*} \leq Q_{\check{\tau}_{t} \land \sigma_{t}^{0}}
\] (69) where the last equality of (68) follows from Lemma 3.5 and the last inequality of (69) follows from (60).

Combining (67) and (69), we conclude that in both cases
\[
R(\tau_{t}, \check{\sigma}_{t} \land \sigma_{t}^{0}) \leq Q_{\check{\tau}_{t} \land \sigma_{t}^{0}}.
\]
Now apply Proposition 3.7(ii), setting \(P = R(\tau_{t}, \check{\sigma}_{t} \land \sigma_{t}^{0})\) and \(\epsilon = 0,\)
\[
\mathbb{E}_{\mathbb{P}}(R(\tau_{t}, \check{\sigma}_{t} \land \sigma_{t}^{0}) | F_{t}) \leq V^{*}_{t}.
\]
This establishes (66) and thus \((\check{\tau}_{t}, \check{\sigma}_{t} \land \sigma_{t}^{0})\) is a Nash equilibrium. Finally, using similar arguments to replace \(\hat{\tau}_{t}\) by \(\check{\tau}_{t} \land \tau_{t}^{0}\), we obtain the required result.

3.2 General Dynkin Game

The goal of this section is to study the general Dynkin game \(GDG_{t}(X, Y, Z)\) with the payoff
\[
R(\tau, \sigma) = 1_{\{\tau < \sigma\}} X_{\tau} + 1_{\{\sigma < \tau\}} Y_{\sigma} + 1_{\{\sigma = \tau\}} Z_{\sigma}.
\] (70)
Hence we no longer postulate that \(X \leq Z \leq Y\). Similarly as in Section 2.2 our goal here is to find the necessary and sufficient conditions for the following property:

For all \(t \in [0, T]\) and \(\epsilon > 0\), the Dynkin game \(GDG_{t}(X, Y, Z)\) has \(\epsilon\)-optimal strategies. (71)

Furthermore, we would also like to explore the necessary and sufficient conditions for the following property:

For all \(t \in [0, T]\), the Dynkin game \(GDG_{t}(X, Y, Z)\) has a Nash equilibrium. (72)

Motivated by the discrete-time case examined in Subsection 2.2 we begin by defining \(L := Z \land X\) and \(U := Z \lor Y\). It is clear that \(L\) and \(U\) are càdlàg processes satisfying the usual integrability condition and \(L \leq Z \leq U\). Again it makes sense to consider the Dynkin game \(SDG(L, U, Z)\) associated with the payoff
\[
\hat{R}(\tau, \sigma) = 1_{\{\tau < \sigma\}} L_{\tau} + 1_{\{\sigma < \tau\}} U_{\sigma} + 1_{\{\sigma = \tau\}} Z_{\sigma}.
\] (73)
In light of Theorem 3.3, we introduce the following notation.

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Let us modify 

(iii) If the 

game GDG

(ii) The value

(i) For any fixed 

We begin by observing that an analogue of Lemma 2.10 can be readily applied to the continuous-time case.

Lemma 3.10. For \( t \in [0, T] \), the following properties are valid.

(i) For any fixed \( \tau_1, \sigma_1 \in T_{[t,T]} \), there exist \( \tilde{\tau}_1, \tilde{\sigma}_1 \in T_{[t,T]} \) such that

\[
R(\tilde{\tau}_1, \tilde{\sigma}_1) \geq \tilde{R}(\tau_1, \sigma_1) \geq R(\tau_1, \sigma_1). \tag{76}
\]

(ii) The value \( V_t \) of SDG\(_t\)(\( L, U, Z \)) lies between the minimax and the maximin values of the game GDG\(_t\)(\( X, Y, Z \)). In other words,

\[
\text{ess inf} \, \text{ess sup} \, \mathbb{E}_\mathbb{P}(R(\tau_1, \sigma_1) | \mathcal{F}_t) \geq V_t \geq \text{ess sup} \, \text{ess inf} \, \mathbb{E}_\mathbb{P}(R(\tau_1, \sigma_1) | \mathcal{F}_t). \tag{77}
\]

(iii) If the GDG\(_t\)(\( X, Y, Z \)) has a value then it equals to \( V_t \).

Proof. (i) The proof is identical to the proof of Lemma 2.10. We will only prove the upper inequality in (76), as the lower inequalities follows by symmetry. To choose a stopping time \( \tilde{\tau} \) such that \( R(\tilde{\tau}_1, \tilde{\sigma}_1) \geq \tilde{R}(\tau_1, \sigma_1) \), we first compare \( R(\tau_1, \sigma_1) \) and \( \tilde{R}(\tau_1, \sigma_1) \). On the following events, \( R(\tau_1, \sigma_1) \geq \tilde{R}(\tau_1, \sigma_1) \) is automatically satisfied

\[
\begin{align*}
\{ \tau_1 = \sigma_1 \}, & \quad R(\tau_1, \sigma_1) = Z_{\tau_1} = \tilde{R}(\tau_1, \sigma_1), \\
\{ \tau_1 < \sigma_1 \}, & \quad R(\tau_1, \sigma_1) = X_{\tau_1} \geq L_{\tau_1} = \tilde{R}(\tau_1, \sigma_1), \\
\{ \sigma_1 < \tau_1, Y_{\sigma_1} \geq Z_{\sigma_1} \}, & \quad R(\tau_1, \sigma_1) = Y_{\sigma_1} = U_{\sigma_1} = \tilde{R}(\tau_1, \sigma_1).
\end{align*}
\]

The problem arises on the event \( \{ \sigma_1 < \tau_1, Z_{\sigma_1} > Y_{\sigma_1} \} \), since then

\[
R(\tau_1, \sigma_1) = Y_{\sigma_1} < U_{\sigma_1} = \tilde{R}(\tau_1, \sigma_1).
\]

Let us modify \( \tau \) by setting

\[
\tilde{\tau} = \sigma_t \mathbb{I}_{\{ \sigma_t < \tau_1, Z_{\sigma_t} > Y_{\sigma_t} \}} + \tau_t (1 - \mathbb{I}_{\{ \sigma_t < \tau_1, Z_{\sigma_t} > Y_{\sigma_t} \}}) \tag{78}
\]

Then \( \tilde{\tau} \) is indeed an \( \mathbb{F} \)-stopping time, since the event \( \{ \sigma_t < \tau_1, Z_{\sigma_t} > Y_{\sigma_t} \} \) belongs to \( \mathcal{F}_{\sigma_t \wedge \tau_1} \). Furthermore, on the event \( \{ \sigma_t < \tau_1, Z_{\sigma_t} > Y_{\sigma_t} \} \) we have that

\[
R(\tilde{\tau}_t, \sigma_t) = R(\sigma_1, \sigma_t) = Z_{\sigma_t} = U_{\sigma_t} = \tilde{R}(\tau_t, \sigma_t)
\]
and thus for the stopping time $\hat{\tau}$ the left-hand side inequality in (76) is satisfied.

(ii) Again, we only show the upper inequality of (77). By Theorem 3.3, $V_t$ is the value of the game $SDG_t(L, U, Z)$. For any $\epsilon > 0$, $(\tau^*_t, \sigma^*_t)$ (see Definition 3.9(ii)) is a pair of $\epsilon$-optimal strategy for $SDG_t(L, U, Z)$. Hence we have, for any $\sigma_t \in \mathcal{T}_{[t,T]}$,

$$\mathbb{E}_P(\tilde{R}(\tau^*_t, \sigma_t) \mid \mathcal{F}_t) \geq V_t - \epsilon.$$ (80)

By part (i), there exists $\hat{\tau}_t \in \mathcal{T}_{[t,T]}$ such that $R(\hat{\tau}_t, \sigma_t) \geq \tilde{R}(\tau^*_t, \sigma_t)$. Consequently,

$$\text{ess sup}_{\tau_t \in \mathcal{T}_{[t,T]}} \mathbb{E}_P(R(\tau_t, \sigma_t) \mid \mathcal{F}_t) \geq \mathbb{E}_P(\tilde{R}(\tau^*_t, \sigma_t) \mid \mathcal{F}_t) \geq \mathbb{E}_P(\hat{R}(\tau^*_t, \sigma_t) \mid \mathcal{F}_t) \geq V_t - \epsilon.$$ (79)

Since (79) holds for all $\sigma_t \in \mathcal{T}_{[t,T]}$ and $\epsilon > 0$, we must have

$$\text{ess inf}_{\sigma_t \in \mathcal{T}_{[t,T]}} \text{ess sup}_{\tau_t \in \mathcal{T}_{[t,T]}} \mathbb{E}_P(R(\tau_t, \sigma_t) \mid \mathcal{F}_t) \geq V_t,$$

as required.

(iii) The proof is the same as in Lemma 3.10. By the definition of the value (see Definition 3.1), if there exists a value $V^*_t$ for the game $GDG_t(X, Y, Z)$, then it must satisfy

$$V^*_t = \text{ess inf}_{\sigma_t \in \mathcal{T}_{[t,T]}} \text{ess sup}_{\tau_t \in \mathcal{T}_{[t,T]}} \mathbb{E}_P(R(\tau_t, \sigma_t) \mid \mathcal{F}_t) = \text{ess sup}_{\tau_t \in \mathcal{T}_{[t,T]}} \text{ess inf}_{\sigma_t \in \mathcal{T}_{[t,T]}} \mathbb{E}_P(R(\tau_t, \sigma_t) \mid \mathcal{F}_t).$$ (80)

In view of part (ii), we conclude that the equality $V^*_t = V_t$ necessarily holds. \[ \square \]

Based on the intuition of the discrete case (see Subsection 2.2), we begin with the following condition, with the aim of achieving (71) and (72).

**Assumption 3.11.** Let $X, Y$ and $Z$ be $\mathcal{F}$-adapted integrable, càdlàg processes and let the associated process $V$ be given as in Definition 3.9(i). We postulate that the processes $X, Y$ and $V$ satisfy, for all $t \in [0,T]$,

$$X_t \land Y_t \leq V_t \leq X_t \lor Y_t.$$ (81)

### 3.2.1 Sufficiency of Assumption 3.11

**Proposition 3.12.** For all $t \in [0,T]$, $\epsilon \geq 0$, let $V_t, \sigma^*_t$ and $\tau^*_t$ be defined as in Definition 3.9. Under Assumption 3.11, we have the following:

(i) For some $\epsilon \geq 0$, if $\sigma_t \in \mathcal{T}_{[t,T]}$ satisfies $\sigma_t \leq \sigma^*_t$, then for all $\tau_t \in \mathcal{T}_{[t,T]}$,

$$R(\tau_t, \sigma_t) \leq Q^{\sigma^*_t}_u.$$ (82)

where $Q^{\sigma^*_t}$ is defined by

$$Q^{\sigma^*_t}_u := \text{ess sup}_{\rho \in \mathcal{T}_{[u,T]}} \mathbb{E}_P(\tilde{R}(\rho, \sigma_t) \mid \mathcal{F}_u), \quad u \in [t,T].$$

(ii) The process $V$ is the value process of $GDG(X, Y, Z)$. For any $\epsilon > 0$, the stopping times $\sigma^*_t, \tau^*_t$ are $\epsilon$-optimal strategies of $GDG_t(X, Y, Z)$, satisfying

$$\text{ess inf}_{\sigma_t \in \mathcal{T}_{[t,T]}} \mathbb{E}_P(R(\tau^*_t, \sigma_t) \mid \mathcal{F}_t) + \epsilon \geq V_t \geq \text{ess sup}_{\tau_t \in \mathcal{T}_{[t,T]}} \mathbb{E}_P(R(\tau_t, \sigma^*_t) \mid \mathcal{F}_t) - \epsilon.$$ (83)

(iii) If $(\tau^*_t, \sigma^*_t)$ is an arbitrary Nash equilibrium of $SDG_t(L, U, Z)$, then $(\tau^*_t \land \tau^*_t, \sigma^*_t \land \sigma^*_t)$ is a Nash equilibrium of $GDG_t(X, Y, Z)$. 

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Proof. (i) We will make use of (59) and (60), that is,
\[ \bar{R}(\tau_t, \sigma_t) \lor V_{\tau_t, \sigma_t} \leq Q_{\tau_t}^{\sigma_t} \]
There are a few cases to check:
(a) On the event \{\sigma_t = \tau_t\},
\[ R(\tau_t, \sigma_t) = Z_{\sigma_t} = \bar{R}(\tau_t, \sigma_t) \leq Q_{\tau_t}^{\sigma_t}. \]  
(b) On the event \{\sigma_t < \tau_t\},
\[ R(\tau_t, \sigma_t) = Y_{\sigma_t} \leq U_{\sigma_t} = \bar{R}(\tau_t, \sigma_t) \leq Q_{\tau_t}^{\sigma_t}. \]  
(c) On the event \{\tau_t < \sigma_t\}, certainly \tau_t < \sigma_t \leq \sigma_t^\epsilon. From the definition of \sigma_t^\epsilon in Definition (ii), we must have
\[ V_{\tau_t} < U_{\tau_t}. \]  
We now consider the following subcases:
(c.1) If \( Y_{\tau_t} \geq Z_{\tau_t} \), then by (56) \( Y_{\tau_t} = U_{\tau_t} > V_{\tau_t} \). Since Assumption 3.11 requires \( V \) to lie between \( X \) and \( Y \), we must have
\[ R(\tau_t, \sigma_t) = X_{\tau_t} \leq V_{\tau_t} \leq Q_{\tau_t}^{\sigma_t}. \]  
(c.2) If \( Y_{\tau_t} < Z_{\tau_t} \), then by (56) \( Z_{\tau_t} = U_{\tau_t} > V_{\tau_t} \). Now by Lemma 3.10(i), \( V_{\tau_t} \geq L_{\tau_t} = Z_{\tau_t} \land X_{\tau_t} \). Hence we must have
\[ R(\tau_t, \sigma_t) = X_{\tau_t} = L_{\tau_t} \leq V_{\tau_t} \leq Q_{\tau_t}^{\sigma_t}. \]  
In view of (54), (55), (57) and (58), we conclude that \( R(\tau_t, \sigma_t) \leq Q_{\tau_t}^{\sigma_t} \) for all cases, establishing (52).
(ii) By Proposition 3.3 and Lemma 3.10(iii), it is sufficient to establish (58), or
\[ \text{ess inf}_{\sigma_t \in [T_{\tau_t}, \tau_t]} \mathbb{E}_P(R(\tau_t, \sigma_t) \mid F_{\tau_t}) + \epsilon \geq V_{\tau_t} \geq \text{ess sup}_{\tau_t \in [T_{\tau_t}, \tau_t]} \mathbb{E}_P(R(\tau_t, \sigma_t) \mid F_{\tau_t}) - \epsilon. \]
We will only establish the lower bound, since the upper bound follows by symmetry. From part (i), we know that \( R(\tau_t, \sigma_t^\epsilon) \leq Q_{\tau_t}^{\sigma_t} \) for all \( \tau_t \in [T_{\tau_t}, \tau_t] \). Since \sigma_t^\epsilon is an \( \epsilon \)-optimal strategy, we can apply Proposition 3.7(i). By setting \( P = R(\tau_t, \sigma_t^\epsilon) \), we have, for all \( \tau_t \in [T_{\tau_t}, \tau_t] \),
\[ \mathbb{E}_P(R(\tau_t, \sigma_t^\epsilon) \mid F_{\tau_t}) \leq V_{\tau_t}^* + \epsilon. \]
Hence
\[ \text{ess sup}_{\tau_t \in [T_{\tau_t}, \tau_t]} \mathbb{E}_P(R(\tau_t, \sigma_t^\epsilon) \mid F_{\tau_t}) \leq V_{\tau_t}^* + \epsilon, \]
as required.
(iii) By Proposition 3.8 \( (\tau_t^* \land \tau_t^0, \sigma_t^\epsilon \land \sigma_t^0) \) is also a Nash equilibrium of GDG_t(L, U, Z), satisfying \( \tau_t^\epsilon \land \tau_t^0 \leq \tau_t^0 \) and \( \sigma_t^\epsilon \land \sigma_t^0 \leq \sigma_t^0 \). Since a Nash equilibrium is also a pair of \( 0 \)-optimal strategies, we can simply use the same argument as before, but with \( \epsilon = 0 \).

In general, not all \( \epsilon \)-optimal strategies (resp. Nash equilibria) of GDG_t(L, U, Z) are necessarily \( \epsilon \)-optimal strategies (resp. Nash equilibria) of GDG_t(X, Y, Z). Proposition 3.12 only applies to \( \epsilon \)-optimal strategies (resp. Nash equilibria) stopping no later than \( \tau_t^\epsilon \) and \( \sigma_t^\epsilon \) (resp. \( \tau_t^0 \) and \( \sigma_t^0 \)).
3.2.2 Necessity of Assumption 3.11

Proposition 3.13. Suppose that Assumption 3.14 is violated at time $t \in [0, T]$, that is, almost surely

$$\{V_t < X_t \land Y_t\} \cup \{V_t > X_t \lor Y_t\} \neq \emptyset.$$  \hspace{1cm} (89)

Then there exists $\epsilon > 0$ such that the Dynkin game GDG$_t(X, Y, Z)$ does not have $\epsilon$-optimal strategies. In particular, GDG$_t(X, Y, Z)$ has no Nash equilibrium.

Proof. Since $X_T = Y_T = Z_T = V_T$ then manifestly (89) cannot occur when $t = T$. Assume, for the sake of contradiction, that (89) holds for some $t < T$ and there exists a pair of $\epsilon$-optimal strategies $(\tau_0, \sigma_0)$ for all $\epsilon > 0$. Then, by Proposition 3.3 and Lemma 3.10(iii), $V_t$ must be the value of GDG$_t(X, Y, Z)$.

Assume now that either $\mathbb{P}(V_t < X_t \land Y_t) > 0$ or $\mathbb{P}(V_t > X_t \lor Y_t) > 0$. First, consider the event $\{V_t < X_t \land Y_t\}$. Then there exists $\epsilon > 0$ such that $\mathbb{P}(V_t + \epsilon < X_t \lor Y_t) > 0$. On that event, let us consider

$$\tau_0^t = t \mathbb{1}_{\{\sigma_0^t = t\}} + T \mathbb{1}_{\{\sigma_0^t = t\}}.$$

Then $R(\tau_0^t, \sigma_0^t)$ is either $X_t$ or $Y_t$. But then

$$\underset{\tau \in \mathcal{T}_[t,T]}{\text{ess sup}} \mathbb{E}_t \left( R(\tau, \sigma_0^t) \mid \mathcal{F}_t \right) \geq \mathbb{E}_t \left( R(\tau_0^t, \sigma_0^t) \mid \mathcal{F}_t \right) \geq X_t \land Y_t > V_t + \epsilon,$$

contradicting the $\epsilon$-optimal property of $\sigma_0^t$. The same argument can be applied to the event $\{V_t > X_t \lor Y_t\}$. Hence there exists $\epsilon > 0$ such that GDG$_t(X, Y, Z)$ does not have $\epsilon$-optimal strategies.

Proposition 3.14. Under Assumption 3.11, if $(\tau_0^*, \sigma_0^*)$ is an arbitrary Nash equilibrium of GDG$_t(X, Y, Z)$, then it is also a Nash equilibrium of SDG$_t(L, U, Z)$.

Proof. We want to prove that, for all $\sigma_t, \tau_t \in \mathcal{T}_[t,T],$

$$\tilde{R}(\tau_0^*, \sigma_t) \geq \tilde{R}(\tau_0^*, \sigma_0^*) \geq \tilde{R}(\tau_t, \sigma_0^*).$$  \hspace{1cm} (90)

By Lemma 3.10(i), there exists $\tilde{t}_0 \in \mathcal{T}_[t,T]$ such that

$$R(\tilde{t}_0, \sigma_0^*) \geq \tilde{R}(\tau_t, \sigma_0^*).$$

Since $(\tau_0^*, \sigma_0^*)$ is a Nash equilibrium of GDG$_t(X, Y, Z),$

$$V_t = R(\tau_0^*, \sigma_0^*) \geq R(\tilde{t}_0, \sigma_0^*) \geq \tilde{R}(\tau_t, \sigma_0^*).$$

Hence the lower bound of (90) is established. The upper bound can be proven similarly. Therefore, $(\tau_0^*, \sigma_0^*)$ is a Nash equilibrium of SDG$_t(L, U, Z)$.

To summarise the necessity and sufficiency results of this section, we now combine Theorem 3.4 with Propositions 3.12, 3.13 and 3.14.

Theorem 3.15. Suppose $X, Y, Z$ are integrable càdlàg progressive processes satisfying $X_T = Y_T = Z_T$ and let $L = X \land Z$ and $U = Y \lor Z$. Consider the family of Dynkin games GDG$_t(X, Y, Z)$ associated with the payoff

$$R(\tau, \sigma) = \mathbb{1}_{\{\tau < \sigma\}} X_\tau + \mathbb{1}_{\{\sigma < \tau\}} Y_{\sigma} + \mathbb{1}_{\{\sigma = \tau\}} Z_{\sigma}.$$
(i) The Dynkin game $GDG_t(X, Y, Z)$ has a value and a pair of $\epsilon$-optimal strategies for all $t \in [0, T]$ and $\epsilon > 0$ if and only if Assumption 3.11 holds. In particular, the unique value process $V^*$ is given by

$$V^*_t = \text{ess inf}_{\sigma \in T} \text{ess sup}_{\tau \in T} \mathbb{E}_\tau (R(\tau, \sigma) | F_t) = \text{ess sup}_{\tau \in T} \text{ess inf}_{\sigma \in T} \mathbb{E}_\sigma (R(\tau, \sigma) | F_t)$$

and a pair of $\epsilon$-optimal strategies $(\tau^*_t, \sigma^*_t)$ is given by

$$\sigma^*_t := \inf \{u \geq t : U_u \leq V_u + \epsilon \}, \quad \tau^*_t := \inf \{u \geq t : L_u \geq V_u - \epsilon \}.$$

(ii) The Dynkin game $GDG_t(X, Y, Z)$ has a Nash equilibrium for all $t \in [0, T]$ if and only if Assumption 3.11 holds and the Dynkin game $SDG_t(L, U, Z)$ has a Nash equilibrium for all $t \in [0, T]$. If we further assume that $L$ and $-U$ only have positive jumps, then $GDG_t(X, Y, Z)$ has a Nash equilibrium $(\tau^*_t, \sigma^*_t)$ given by

$$\sigma^*_t = \lim_{\epsilon \to 0} \sigma^*_t, \quad \tau^*_t = \lim_{\epsilon \to 0} \tau^*_t.$$

(iii) Fix $t \in [0, T]$. If Assumption 3.11 holds, then $GDG_t(X, Y, Z)$ has a Nash equilibrium if and only if $SDG_t(L, U, Z)$ has a Nash equilibrium.

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