**L^p ESTIMATES FOR THE BILINEAR HILBERT TRANSFORM FOR 1/2 < p ≤ 2/3: A COUNTEREXAMPLE AND GENERALIZATIONS TO NON-SMOOTH SYMBOLS**

WEI DAI AND GUOZHEN LU

**Abstract.** M. Lacey and C. Thiele proved in [26] (Annals of Math. (1997)) and [27] (Annals of Math. (1999)) that the bilinear Hilbert transform maps $L^{p_1} \times L^{p_2} \rightarrow L^p$ boundedly when $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ with $1 < p_1, p_2 \leq \infty$ and $\frac{2}{3} < p < \infty$. Whether the $L^p$ estimates hold in the range $p \in (1/2, 2/3]$ has remained an open problem since then. In this paper, we prove that the bilinear Hilbert transform does not satisfy any $L^p$ estimates for $p \in \left(\frac{1}{2}, \frac{2}{3}\right)$ (Theorem 1.2). Nevertheless, we can establish $L^p$ estimates for the bilinear Fourier multipliers whose symbols are not identical to but arbitrarily close to that of the bilinear Hilbert transform in the full range $p \in (1/2, \infty)$ (Theorem 1.3).

**Keywords:** Bilinear Hilbert transforms; $L^p$ estimates; random variables; non-smooth symbols; symbols with 1-dimensional singularity; paraproducts.

**2010 MSC** Primary: 42B20; Secondary: 42B15.

1. **Introduction**

The bilinear Hilbert transform is defined by

\[ BHT(f_1, f_2)(x) := p.v. \int_{\mathbb{R}} f_1(x - t)f_2(x + t) \frac{dt}{t}, \]

or equivalently, it can also be written as the bilinear multiplier operator

\[ BHT : (f_1, f_2) \mapsto \int_{\mathbb{R}^2} sgn(\xi_1 - \xi_2)\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)e^{2\pi i x(\xi_1 + \xi_2)}d\xi_1d\xi_2, \]

where $f_1$ and $f_2$ are Schwartz functions on $\mathbb{R}$.

The boundedness problem of the bilinear Hilbert transform was originally posed by A. P. Calderón in connection with the Cauchy integral along Lipschitz curves. He conjectured that the bilinear Hilbert transform was bounded from $L^2 \times L^2$ to $L^1$. Inspired by the two classic proofs of almost everywhere convergence of Fourier series by L. Carleson [3] and C. Fefferman [12], and using delicate orthogonality estimates and combinatorial selection of trees, and deep time-frequency analysis, M. Lacey and C. Thiele proved in [26, 27] the following celebrated $L^p$ estimates for the bilinear Hilbert transform.

**Theorem 1.1.** ([26, 27]) The bilinear operator $BHT$ maps $L^p(\mathbb{R}) \times L^q(\mathbb{R})$ into $L^r(\mathbb{R})$ boundedly for any $1 < p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ and $\frac{2}{3} < r < \infty$.

Research of the first author was partly supported by grants from the NNSF of China, the China Postdoctoral Science Foundation and the research of the second author was partly supported by a US NSF grant.
In general, let \( m(\xi_1, \xi_2) \) be a symbol that is smooth away from the singularity line \( \Gamma := \{(\xi_1, \xi_2) \in \mathbb{R}^2 | \xi_1 = \xi_2 \} \) and satisfies
\[
|\partial^\alpha m(\xi)| \lesssim \frac{1}{\text{dist}(\xi, \Gamma)^{\alpha}}
\]
for every \( \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \Gamma \) and sufficiently many multi-indices \( \alpha \). Throughout this paper, \( A \lesssim B \) means that there exists a universal constant \( C > 0 \) such that \( A \leq CB \). If necessary, we use explicitly \( A \lesssim_{\ast, \ldots, \ast} B \) to indicate that there exists a positive constant \( C_{\ast, \ldots, \ast} \) depending only on the quantities appearing in the subscript continuously such that \( A \leq C_{\ast, \ldots, \ast} B \).

J. Gilbert and A. Nahmod \cite{GilbertNahmod} proved that the \( L^p \) estimates as \( BHT \) are valid for the generalized bilinear multiplier operators \( T_m \) associated with symbol \( m \) in the same range of \( p > \frac{2}{3} \) as that in \cite{Bernicot, Bernicot1}. There has been much work related to the bilinear operators of \( BHT \) type. F. Bernicot \cite{Bernicot} proved a pseudo-differential variant of the multiplier estimates in \cite{Bernicot}. Uniform estimates were obtained by C. Thiele \cite{Thiele}, L. Grafakos and X. Li \cite{GrafakosLi} and X. Li \cite{XuXiangLi}. A two-dimensional bilinear Hilbert transform was studied by C. Demeter and C. Thiele \cite{DemeterThiele}. A maximal variant of Theorem 1.1 was proved by M. Lacey \cite{Lacey} and generalized by C. Demeter, T. Tao and C. Thiele \cite{Demeter}. In C. Muscalu, C. Thiele and T. Tao \cite{MuscaluThieleTao} and J. Jung \cite{Jung}, the authors investigated various trilinear variants of the bilinear Hilbert transform. For more related results involving estimates for multi-linear singular multiplier operators, we refer to the works, e.g., \cite{Bernicot2, Bernicot3, Bernicot4, Bernicot5, Bernicot6, Bernicot7, Bernicot8, Bernicot9, Bernicot10, Bernicot11, Bernicot12, Bernicot13, Bernicot14, Bernicot15, Bernicot16, Bernicot17, Bernicot18, Bernicot19, Bernicot20, Bernicot21, Bernicot22, Bernicot23, Bernicot24, Bernicot25, Bernicot26, Bernicot27, Bernicot28, Bernicot29, Bernicot30} and the references therein.

In multi-parameter cases, there is also a large amount of literature devoted to studying the estimates of multi-parameter and multi-linear operators (see \cite{MuscaluThieleTao, MuscaluThieleTao2, MuscaluThieleTao3, MuscaluThieleTao4, MuscaluThieleTao5, MuscaluThieleTao6, MuscaluThieleTao7, MuscaluThieleTao8, MuscaluThieleTao9, MuscaluThieleTao10, MuscaluThieleTao11, MuscaluThieleTao12, MuscaluThieleTao13, MuscaluThieleTao14, MuscaluThieleTao15, MuscaluThieleTao16, MuscaluThieleTao17, MuscaluThieleTao18, MuscaluThieleTao19, MuscaluThieleTao20, MuscaluThieleTao21, MuscaluThieleTao22, MuscaluThieleTao23, MuscaluThieleTao24, MuscaluThieleTao25, MuscaluThieleTao26, MuscaluThieleTao27, MuscaluThieleTao28, MuscaluThieleTao29, MuscaluThieleTao30} and the references therein). In the bilinear and bi-parameter cases, let \( \Gamma_i \) \((i = 1, 2)\) be subspaces in \( \mathbb{R}^2 \), we consider operators \( T_m \) defined by
\[
T_m(f_1, f_2)(x) := \int_{\mathbb{R}^4} m(\xi, \eta)f_1(\xi_1, \eta_1)f_2(\xi_2, \eta_2)e^{2\pi i x \cdot ((\xi_1, \eta_1) + (\xi_2, \eta_2))}d\xi d\eta,
\]
where the symbol \( m \) satisfies
\[
|\partial^\alpha \xi^{\beta} m(\xi, \eta)| \lesssim \frac{1}{\text{dist}(\xi, \Gamma_1)^{\alpha}} \cdot \frac{1}{\text{dist}(\eta, \Gamma_2)^{\beta}}
\]
for sufficiently many multi-indices \( \alpha = (\alpha_1, \alpha_2) \) and \( \beta = (\beta_1, \beta_2) \). If \( \text{dim} \Gamma_1 = \text{dim} \Gamma_2 = 0 \), C. Muscalu, J. Pipher, T. Tao and C. Thiele proved in \cite{MuscaluPipherTaoThiele1, MuscaluPipherTaoThiele2} that H"older type \( L^p \) estimates are available for \( T_m \); J. Chen and G. Lu established in \cite{ChenLu} such estimates for multipliers with limited smoothness on the symbols. A multi-linear and multi-parameter pseudo-differential operator analogue has also been studied and \( L^p \) estimates have been obtained by W. Dai and G. Lu in \cite{DaiLu}. However, if \( \text{dim} \Gamma_1 = \text{dim} \Gamma_2 = 1 \) with \( \Gamma_1, \Gamma_2 \) non-degenerate in the sense of \cite{DaiLu}, let \( T_m \) be the double bilinear Hilbert transform on polydisks \( BHT \otimes BHT \) defined by
\[
BHT \otimes BHT(f_1, f_2)(x, y) := \text{p. v.} \int_{\mathbb{R}^2} f_1(x - s, y - t)f_2(x + s, y + t)\frac{ds dt}{s t},
\]
then the authors of \cite{DaiLu} also proved that the operator \( BHT \otimes BHT \) does not satisfy any \( L^p \) estimates of H"older type by constructing a counterexample. Nevertheless, under some (slightly better) logarithmic decay assumptions on the symbols, W. Dai and G. Lu proved in \cite{DaiLu} that the bi-parameter operators \( T_m \) defined by \((1.4), (1.5)\) with singularity sets \( \Gamma_1 = \)
\( \Gamma_2 = \{(\xi_1, \xi_2) \in \mathbb{R}^2 | \xi_1 = \xi_2\} \) satisfy the same estimates as \( BHT \). When \( \text{dim} \Gamma_1 = 0 \) and \( \text{dim} \Gamma_2 = 1 \) with \( \Gamma_2 \) non-degenerate in the sense of \cite{31}, P. Silva \cite{37} and the authors of the current paper established in \cite{2} the \( L^p \) estimates of Hölder type for \( T_m \) under some conditions, which addressed the Question 8.2 in \cite{32}.

One can observe that the \( L^p \) estimates for the bilinear operators of \( BHT \) type derived in these previous works are available only for \( p > \frac{2}{3} \). In \cite{24}, by constructing a counterexample, M. Lacey proved that the discrete model operators associated with the bilinear maximal functions cannot be a uniformly bounded bilinear map from \( L^p \times L^q \) into \( L^r \) for \( \frac{1}{2} < r < \frac{2}{3} \). For the endpoint case \( r = \frac{2}{3} \), D. Bilyk and L. Grafakos \cite{2} proved some distributional estimates for \( BHT \) of log type, then F. D. Plinio and C. Thiele \cite{36} improved the distributional estimates by replacing the single logarithmic term with a double logarithmic term.

Since M. Lacey and C. Thiele established the \( L^p \) estimates for \( \frac{2}{3} < p < \infty \) in \cite{26,27}, whether the bilinear operators of \( BHT \) type satisfy \( L^p \) estimates all the way down to \( \frac{1}{2} \) has remained an open problem. Our first main result in this paper is a negative answer (modulo the endpoint case \( p = \frac{2}{3} \)) to this question.

**Theorem 1.2.** The bilinear Hilbert transform does not satisfy any uniform \( L^p \) estimates of Hölder type for any \( p \in (1/2, 2/3) \), that is, the necessary condition for \( L^p \) estimates to be available for the bilinear Hilbert transform is \( \frac{2}{3} \leq p < \infty \).

Although the bilinear Hilbert transform does not satisfy any uniform \( L^p \) estimates for \( p \in (1/2, 2/3) \), by decomposing the bilinear multiplier operator \( T_m \) into a summation of infinitely many bilinear paraproducts without modulation invariance, we can prove that there exists a class of symbols \( m \) (with one-dimensional singularity sets) which also satisfy the symbol estimates of \( BHT \) type operators (see (1.3) investigated in \cite{15}) and are arbitrarily close to the symbols of \( BHT \) type operators (see (1.2) and (1.3)), such that the corresponding bilinear multiplier operators \( T_m \) associated with symbols \( m \) satisfy \( L^p \) estimates all the way down to \( \frac{1}{2} \). Our next result in this paper is the following theorem.

**Theorem 1.3.** For arbitrarily given \( \delta > 0 \), let \( m(\xi_1, \xi_2) \) be a symbol that is smooth away from the singularity line \( \Gamma := \{(\xi_1, \xi_2) \in \mathbb{R}^2 | \xi_1 = \xi_2\} \) and satisfies

\[ |\partial^\alpha m(\xi)| \lesssim \frac{1}{\text{dist}(\xi, \Gamma)^{\frac{1}{3}|\alpha|}} \cdot \exp \left\{ -\delta (1 - \frac{|\alpha|}{3}) \left( \frac{1}{\text{dist}(\xi, \Gamma)} \right) \right\}, \quad 0 \leq |\alpha| \leq 2 \]

and

\[ |\partial^\alpha m(\xi)| \lesssim \frac{1}{\text{dist}(\xi, \Gamma)^{\frac{1}{3}|\alpha|}}, \quad |\alpha| \geq 3 \]

for every \( \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \Gamma \) and sufficiently many multi-indices \( \alpha \), then the bilinear multiplier operator \( T_m \) defined by

\[ T_m(f_1, f_2)(x) := \int_{\mathbb{R}^2} m(\xi_1, \xi_2) \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) e^{2\pi i x(\xi_1 + \xi_2)} \, d\xi_1 d\xi_2 \]

maps \( L^{p_1} \times L^{p_2} \rightarrow L^p \) boundedly for any \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \) with \( 1 < p_1, p_2 \leq \infty \) and \( \frac{1}{2} < p < \infty \). The implicit constants in the bounds depend only on \( p_1, p_2, p \) when \( p > \frac{2}{3} \), also depend on \( \delta \) and tend to infinity as \( \delta \rightarrow 0 \) when \( \frac{1}{2} < p \leq \frac{2}{3} \).
The rest of this paper is organized as follows. In Section 2 we prove Theorem 1.2 by using contraction arguments. Section 3 is devoted to carrying out the proof of Theorem 1.3.

2. Proof of Theorem 1.2

We will prove Theorem 1.2 in this section by using Khinchine’s inequality, Littlewood-Paley inequality and contraction arguments (see e.g., [38]). To this end, we first assume on the contrary that the conclusions in Theorem 1.2 are not true, that is, there exists some \(r_0 \in (\frac{1}{2}, \frac{2}{3})\) and \(1 < p_0, q_0 \leq \infty\) such that \(\frac{1}{r_0} = \frac{1}{p_0} + \frac{1}{q_0}\) and the bilinear Hilbert transform maps \(L^{p_0}(\mathbb{R}) \times L^{q_0}(\mathbb{R})\) into \(L^{r_0}(\mathbb{R})\) boundedly.

**Definition 2.1.** ([30, 35]) For \(J \subseteq \mathbb{R}\) an arbitrary interval, we say that a smooth function \(\Phi_J\) is a bump adapted to \(J\), if and only if the following inequalities hold:

\[
|\Phi_J^{(l)}(x)| \lesssim_{l, \alpha} \frac{1}{|J|^l} \cdot \frac{1}{(1 + \frac{\text{dist}(x, J)}{|J|})^\alpha}
\]

for every integer \(\alpha \in \mathbb{N}\) and for sufficiently many derivatives \(l \in \mathbb{N}\). If \(\Phi_J\) is a bump adapted to \(J\), we say that \(|J|^{-\frac{1}{2}}\Phi_J\) is an \(L^2\)-normalized bump adapted to \(J\).

In order to get a contradiction, let us first consider two \(L^1\)-normalized even Schwartz functions \(\Phi^1(x)\) and \(\Phi^2(x)\) adapted to the interval \([-1, 1]\), such that \(\hat{\Phi}^j \geq 0\) and \(\text{supp} \hat{\Phi}^j \subseteq [-\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}]\) for \(j = 1, 2\). We define Schwartz function \(\Gamma(\xi, \eta) := \hat{\Phi}^1(\xi) \cdot \hat{\Phi}^2(\eta)\) for every \((\xi, \eta) \in \mathbb{R}^2\) such that \(\text{supp} \Gamma \subseteq [-\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}] \times [-\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}]\), then define two \(L^1\)-normalized even Schwartz functions \(\Psi_1(x)\) and \(\Psi_2(x)\) by decomposing the Schwartz function \(\Gamma(\xi, \eta)\) into a product of two \(L^\infty\)-normalized Schwartz functions \(\Gamma(\xi, \eta) =: \hat{\Psi}_1(\xi - \eta) \cdot \hat{\Psi}_2\left(\frac{\xi + \eta}{|\xi - \eta|}\right)\), such that \(\hat{\Psi}_j \geq 0\), \(\text{supp} \hat{\Psi}_j \subseteq [-\frac{1}{2}, \frac{1}{2}]\) and \(\hat{\Psi}_j \geq \frac{1}{2}\) on \([-\frac{1}{4}, \frac{1}{4}]\) for \(j = 1, 2\). In fact, the \(L^\infty\)-normalized functions \(\Psi_2\) may possibly be different as \(\xi - \eta\) varies in \(\text{supp} \hat{\Psi}_1\). However, we only need to let all these functions \(\hat{\Psi}_2\) satisfy the property \(\hat{\Psi}_2 \geq \frac{1}{2}\) on \([-\frac{1}{4}, \frac{1}{4}]\) uniformly for \(\xi - \eta \in [-\frac{1}{4}, \frac{1}{4}] \subseteq \text{supp} \hat{\Psi}_1\), which is enough for our proof (see (2.5) and (2.6)). Therefore, we will ignore the dependence of these functions on the variable \(\xi - \eta\) hereafter and denote them by the same \(L^\infty\)-normalized Schwartz function \(\hat{\Psi}_2\).

Let \(N\) be an arbitrarily fixed large positive integer, we define two sequences of functions

\[
\mathcal{F}_{N,j}(x) := (-1)^j \omega_j \Phi^1(x)e^{2\pi i 2^j x}, \quad \mathcal{G}_{N,k}(x) := (-1)^k \phi_k \Phi^2(x)e^{2\pi i 2^k + N x}
\]

for \(j, k = 1, \cdots, N\), where \(\{\omega_j\}_{j=1}^{N}\) are independent random variables with \(P(\omega_j = j) = \frac{1}{2}\) and \(P(\omega_j = j + \frac{1}{2}) = \frac{1}{2}\) for every \(j = 1, \cdots, N\), and \(\{r_k\}_{k=1}^{N}\) are also independent random variables with \(P(r_k = k) = \frac{1}{2}\) and \(P(r_k = k + \frac{3}{2}) = \frac{1}{2}\) for every \(k = 1, \cdots, N\).
One can observe that

\[ \sum_{j \neq k, j, k = 1}^{N} \text{BHT} \left( \frac{F_{N,j}}{j^{\sigma(r_0)}}, \frac{G_{N,k}}{k^{\sigma(r_0)}} \right)(x) + \sum_{j = k = 1}^{N} \text{BHT} \left( \frac{F_{N,j}(-jN)}{j^{\sigma(r_0)}}, \frac{G_{N,k}(-kN)}{k^{\sigma(r_0)}} \right)(x) \]

\[ = \int_{\mathbb{R}^2} \text{sgn}(\xi - \eta) \left\{ \sum_{j \neq k, j, k = 1}^{N} j^{-\frac{1}{\tau_0}} k^{-\frac{1}{\tau_0}} + \sum_{j = k = 1}^{N} e^{-2\pi i N(j\xi + k\eta)} j^{-\sigma(r_0)} k^{-\sigma(r_0)} \right\} \times \left( -1 \right)^{j\omega_j + k\nu_k} \hat{\Phi}^1(\xi - 2^j) \hat{\Phi}^2(\eta - 2^k + N)e^{2\pi i (\xi + \eta)} d\xi d\eta \]

\[ = \sum_{n = 0}^{N^2 - 1} \varepsilon_n \left\{ \frac{1 - C_{n,N}}{(1 + [\frac{n}{N}])^{\tau_0}(1 + \{\frac{n}{N}\}) N^{\tau_0}} + \frac{C_{n,N} e^{-2\pi i N(1 + [\frac{n}{N}]) N^{\sigma(r_0)}}}{(1 + [\frac{n}{N}])^{2\sigma(r_0)}} \right\} \times \int_{\mathbb{R}^2} \hat{\Phi}^1(\xi - 2^{[\frac{n}{N}]+1}) \hat{\Phi}^2(\eta - 2^{[\frac{n}{N}]+N+1}) e^{2\pi i (\xi + \eta)} d\xi d\eta, \]

where the variables \( n = (j - 1)N + k - 1 \), the constants

\[ C_{n,N} := \frac{1}{2} \left[ 1 - \text{sgn} \left( N^{(N+1)\{\frac{n}{N+1}\}} - 2 \right) \right] \text{ for } n = 0, \ldots, N^2 - 1, \]

the exponent \( \sigma(r_0) := \frac{2 - r_0}{4r_0} \), \( \{\varepsilon_n\}_{n=0}^{N^2-1} \) are independent and identically distributed random variables with \( P(\varepsilon_n = \pm 1) = \frac{1}{2} \) for every \( n = 0, \ldots, N^2 - 1 \), i.e., \( \{\varepsilon_n\}_{n=0}^{N^2-1} \) is a sequence with Rademacher distribution, and the notation \([x]\) denotes the largest integer which is less than or equal to \( x \), and the remainder part of \( x \) is denoted by \( \{x\} \), that is, \( x = [x] + \{x\} \). By changing the integral variables, we can rewrite the above expression of

\[ \sum_{j \neq k, j, k = 1}^{N} \text{BHT} \left( j^{-\frac{1}{\tau_0}} F_{N,j}, k^{-\frac{1}{\tau_0}} G_{N,k} \right) + \sum_{j = k = 1}^{N} \text{BHT} \left( j^{-\sigma(r_0)} F_{N,j}(-jN), k^{-\sigma(r_0)} G_{N,k}(-kN) \right) \]
in terms of Schwartz functions \( \Psi_1 \) and \( \Psi_2 \) as follows:

\[
\sum_{j\neq k, j, k = 1}^N BHT\left(\frac{F_{N,j}}{j\theta}, \frac{G_{N,k}}{k\theta}\right)(x) + \sum_{j=k=1}^N BHT\left(\frac{F_{N,j}(\cdot - jN)}{j\sigma(\rho_0)}, \frac{G_{N,k}(\cdot - kN)}{k\sigma(\rho_0)}\right)(x)
\]

\[= \sum_{n=0}^{N^2-1} \varepsilon_n \left\{ \frac{1 - C_{n,N}}{(1 + \left[ \frac{n}{N} \right])^2\theta} + \frac{C_{n,N}e^{-2\pi i(1 + \left[ \frac{n}{N} \right])\eta(\xi + \eta)}}{(1 + \left[ \frac{n}{N} \right])2\sigma(\rho_0)} \right\}
\]

\[\times \int_{\mathbb{R}^2} \Gamma(\xi - 2[\frac{\xi}{N}] + 1, \eta - 2[\frac{\eta}{N}] + 1) e^{2\pi i(x/N)(\xi + \eta)} d\xi d\eta\]

\[= \frac{1}{2} \sum_{n=0}^{N^2-1} \varepsilon_n \left\{ (1 - C_{n,N}) \int_{\mathbb{R}^2} \hat{\Psi}_1(v) \hat{\Psi}_2\left(\frac{u}{1 - 2|v|}\right) e^{2\pi i(u\xi + v\eta)} du dv + C_{n,N} \int_{\mathbb{R}^2} \hat{\Psi}_1(v) \hat{\Psi}_2\left(\frac{u}{1 - 2|v|}\right) e^{2\pi i(u\xi + v\eta)} du dv \right\}
\]

\[\times \frac{e^{2\pi i(x-N(1 + [n/N]))(\xi + \eta)}/(1 + \left[ \frac{n}{N} \right]2\sigma(\rho_0))}{(1 + \left[ \frac{n}{N} \right]2\sigma(\rho_0))}
\]

\[= \frac{1}{2} \sum_{n=0}^{N^2-1} \varepsilon_n \left\{ (1 - C_{n,N}) \int_{\mathbb{R}^2} \hat{\Psi}_1(v) |1 - 2|v||\Psi_2(|1 - 2|v||x)dv e^{2\pi i(x-N(1 + [n/N]))(\xi + \eta)}/(1 + \left[ \frac{n}{N} \right]2\sigma(\rho_0)) \right\}
\]

\[+ C_{n,N} \int_{\mathbb{R}^2} \hat{\Psi}_1(v) |1 - 2|v||\Psi_2(|1 - 2|v||x-N(1 + [n/N])) dv \times e^{2\pi i(x-N(1 + [n/N]))(\xi + \eta)}/(1 + \left[ \frac{n}{N} \right]2\sigma(\rho_0)) \right\}
\]

\[= \sum_{n=0}^{N^2-1} \varepsilon_n \Omega_{n,N}(x),
\]

where the variables \( u := \xi + \eta - 2[\frac{\xi}{N}] + 2[\frac{\eta}{N}] + 1 \) and \( v := \xi - \eta - 2[\frac{\xi}{N}] + 2[\frac{\eta}{N}] + 1 \) for arbitrarily large integer \( N \).

Now we consider arbitrary \( x \in [\ell N, \ell N + \frac{1}{2}] \) for \( \ell = 0, 1, \cdots, N \). Since \( \Psi_1(x) \) and \( \Psi_2(x) \) are \( L^1 \)-normalized even Schwartz functions adapted to the interval \([-1, 1]\) and such that \( \hat{\Psi}_j \geq 0, \text{ supp } \hat{\Psi}_j \subseteq [-\frac{1}{2}, \frac{1}{2}] \), \( \hat{\Psi}_j \geq \frac{1}{2} \) on \([-\frac{1}{4}, \frac{1}{4}]\) for \( j = 1, 2 \), one has \( \Psi_2(x - \ell N) \geq 0 \) on \([\ell N - \frac{1}{2}, \ell N + \frac{1}{2}]\), furthermore,

\[
\Psi_2(x - \ell N) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\Psi}_2(\xi) \cos(2\pi x - \ell N) \xi) d\xi \geq \frac{1}{2\pi} \int_{\frac{-1}{2}}^{\frac{1}{2}} \frac{1}{4} \cos \frac{\pi}{4} d\xi \geq \frac{\sqrt{2}}{16\pi}
\]

for every \( x \in [\ell N - \frac{1}{2}, \ell N + \frac{1}{2}] \), and hence we can get an estimate of lower bound for the integral term of \( \Omega_{n,N} \) in the right-hand side of (2.4) on \([\ell N, \ell N + \frac{1}{2}]\):

\[
\left| \int_{\mathbb{R}} \hat{\Psi}_1(v) |1 - 2|v||\Psi_2(|1 - 2|v||x - \ell N)) dv \right| \geq \frac{1}{2} \int_{\frac{-1}{4}}^{\frac{1}{4}} \hat{\Psi}_1(v) \Psi_2(|1 - 2|v||x - \ell N) dv \geq \frac{\sqrt{2}}{128\pi}
\]

for every \( \ell = 0, 1, \cdots, N \).
Therefore, by taking average over all possible choices of i.i.d. random variables \( \{\varepsilon_n\}_{n=0}^{N^2-1} \) and applying Khinchine’s inequality, we can deduce from (2.6) the following estimate for the right hand side of (2.4):

\[
(2.7) \quad \mathbb{E} \left[ \left| \sum_{n=0}^{N^2-1} \varepsilon_n \Omega_{n,N}(x) \right|^{r_0} \right] \sim \left( \sum_{n=0}^{N^2-1} \left| \Omega_{n,N}(x) \right|^2 \right)^{\frac{r_0}{2}} \gtrsim \ell^{-(1-\frac{1}{4p})}
\]

for every \( x \in E_N := \bigcup_{\ell=1}^{N} [\ell N, \ell N + \ell \frac{1}{2}] \) and arbitrary large integer \( N \).

Now we define functions

\[
(2.8) \quad \mathcal{F}_N(x) := \sum_{j=1}^{N} (-1)^{i\omega_j} \mathcal{F}_{N,j}(x) \quad \text{and} \quad \mathcal{G}_N(x) := \sum_{k=1}^{N} (-1)^{kr_k} \mathcal{G}_{N,k}(x)
\]

for arbitrary large integer \( N \). Suppose that the \( L^{r_0} \) estimates of Hölder type hold for bilinear Hilbert transform, one easily observes that \( 1 < p_0 < 2 \) and \( 1 < q_0 < 2 \) because we have assumed \( \frac{1}{2} < r_0 < \frac{2}{3} \), we can deduce from the Minkowski inequality, the Cauchy-Schwartz inequality and the Littlewood-Paley inequality that

\[
(2.9) \quad \left\| \sum_{j \neq k, j,k=1}^{N} BHT \left( \frac{\mathcal{F}_{N,j}}{j^\sigma(r_0)}, \frac{\mathcal{G}_{N,k}}{k^\sigma(r_0)} \right) \right\|_{L^{r_0}}^{r_0} + \left\| \sum_{j=k=1}^{N} BHT \left( \frac{\mathcal{F}_{N,j}}{j^\sigma(r_0)}, \frac{\mathcal{G}_{N,k}}{k^\sigma(r_0)} \right) \right\|_{L^{r_0}}^{r_0} \lesssim \sum_{j=k=1}^{N} \left\| j^{-\sigma(r_0)} \mathcal{F}_{N,j} \right\|_{L^{p_0}}^{r_0} \cdot \left\| k^{-\sigma(r_0)} \mathcal{G}_{N,k} \right\|_{L^{q_0}}^{r_0} \cdot \left( \sum_{k=1}^{N} \frac{1}{j} \cdot \frac{1}{k} \right) \lesssim \left( \sum_{j=1}^{N} \left\| \mathcal{F}_{N,j} \right\|_{L^{p_0}}^{r_0} \right) \cdot \left( \sum_{k=1}^{N} \left\| \mathcal{G}_{N,k} \right\|_{L^{q_0}}^{r_0} \right) \cdot \left( \sum_{m=1}^{N} m^{-2r_0} \right)^{1-r_0} + (\ln N)^2
\]

for arbitrarily large integer \( N \).

One observes that \( \mathcal{F}_N \in \mathcal{S}(\mathbb{R}) \) satisfy \( \widehat{\mathcal{F}_N} \) is an \( L^\infty \)-normalized Schwartz function and \( \text{supp} \ \widehat{\mathcal{F}_N} \subseteq \bigcup_{j=1}^{N} [2^j - 1, 2^j + 1] \) with \( m(\text{supp} \ \widehat{\mathcal{F}_N}) \simeq N \), and \( \mathcal{G}_N \in \mathcal{S}(\mathbb{R}) \) satisfy \( \widehat{\mathcal{G}_N} \) is an \( L^\infty \)-normalized Schwartz function and \( \text{supp} \ \widehat{\mathcal{G}_N} \subseteq \bigcup_{k=1}^{N} [2^{k+N} - 1, 2^{k+N} + 1] \) with \( m(\text{supp} \ \widehat{\mathcal{G}_N}) \simeq N \). From the generalized Bernstein’s estimates, we can get the following uniform estimates that are independent of all possible choices of \( \{\omega_j\}_{j=1}^{N} \) and \( \{r_k\}_{k=1}^{N} \):

\[
(2.10) \quad \left\| \mathcal{F}_N \right\|_{L^{p_0}} \lesssim N^{1-\frac{1}{p_0}}, \quad \left\| \mathcal{G}_N \right\|_{L^{q_0}} \lesssim N^{1-\frac{1}{q_0}}
\]

for arbitrarily large integer \( N \).
Then we can deduce from (2.9) and (2.10) that the following upper bounds

\[ (2.11) \quad E \left[ \left\| \sum_{j,k=1}^{N} BHT \left( \frac{F_{N,j}}{j^{r_0}}, \frac{G_{N,k}}{k^{r_0}} \right) \right\|_{L^p}^{r_0} \right] \]

\[ \lesssim N^{r_0 (1 - \frac{1}{p_0})} \cdot N^{r_0 (1 - \frac{1}{q_0})} \]

hold true with bounds that are uniform with respect to arbitrary \( N \) large, while the estimates (2.11) and (2.7) yield that

\[ (2.12) \quad E \left[ \left\| \sum_{j,k=1}^{N} BHT \left( \frac{F_{N,j}}{j^{r_0}}, \frac{G_{N,k}}{k^{r_0}} \right) \right\|_{L^p(E_N)}^{r_0} \right] \]

\[ \gtrsim \sum_{\ell=1}^{N} \int_{t_N}^{\ell N + \frac{1}{q}} E \left[ \left\| \sum_{n=0}^{N^2-1} \varepsilon_n \Omega_{n,N}(x) \right\|_{L^{p}}^{r_0} \right] dx \gtrsim \sum_{\ell=1}^{N} \ell^{-1 + \frac{3}{q}} \gtrsim N^{\frac{a}{q}}. \]

Combining the estimates (2.11) and (2.12), we must have

\[ (2.13) \quad N^{\frac{1}{q}} \lesssim N^{1 - \frac{1}{p_0}} \cdot N^{1 - \frac{1}{q_0}} \]

for every large integer \( N \), which implies that \( \frac{1}{2} \leq 1 - \frac{1}{p_0} + 1 - \frac{1}{q_0} \), that is, \( r_0 \geq \frac{2}{3} \). This contradicts with the assumption that \( \frac{1}{2} < r_0 < \frac{2}{3} \) and concludes our proof of Theorem 1.2.

3. Proof of Theorem 1.3

For arbitrarily given \( \delta > 0 \), one can observe that the symbols \( m \) defined by (1.7) and (1.8) also satisfy the estimates

\[ (3.1) \quad |\partial^\alpha m(\xi)| \lesssim \frac{1}{|\xi_1 - \xi_2|^\alpha} \]

for every \( \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \Gamma \) and sufficiently many multi-indices \( \alpha \). Therefore, we deduce from [15] that the bilinear operators \( T_m \) given by (1.9) satisfy \( L^p \) estimates of Hölder type for \( p > \frac{2}{3} \), and the implicit constants in the bounds depend only on \( p_1, p_2, p \). In this section, we will focus on proving \( L^p \) estimates of \( T_m \) for \( p \leq \frac{2}{3} \).

3.1. Decomposition into a summation of infinitely many bilinear multipliers. As we can see from the study of multi-parameter and multi-linear Coifman-Meyer multiplier operators (see e.g. [31, 32, 34, 35]), a standard approach to obtain \( L^p \) estimates of the bilinear operators \( T_m \) is to reduce it into discrete sums of inner products with wave packets (see [34]).

First, we need to decompose the symbol \( m(\xi) \) in a natural way. To this end, we will decompose the region \( \{ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1 \neq \xi_2 \} \) by using Whitney squares with respect to the singularity point \( \{ \xi_1 = \xi_2 = 0 \} \). In order to describe our discretization procedure clearly, let us first recall some standard notation and definitions in [34].

An interval \( I \) on the real line \( \mathbb{R} \) is called dyadic if it is of the form \( I = 2^{-k}[n, n + 1] \) for some \( k, n \in \mathbb{Z} \). An interval is said to be a shifted dyadic interval if it is of the form \( 2^{-k}[j + \alpha, j + 1 + \alpha] \) for any \( k, j \in \mathbb{Z} \) and \( \alpha \in \{0, \frac{1}{3}, -\frac{1}{3}\} \). A shifted dyadic cube is a set of
the form \( Q = Q_1 \times Q_2 \times Q_3 \), where each \( Q_j \) is a shifted dyadic interval and they all have the same length. A **shifted dyadic quasi-cube** is a set \( Q = Q_1 \times Q_2 \times Q_3 \), where \( Q_j \) \((j = 1, 2, 3)\) are shifted dyadic intervals satisfying less restrictive condition \(|Q_1| \simeq |Q_2| \simeq |Q_3|\). One easily observes that for every cube \( Q \subseteq \mathbb{R}^3 \), there exists a shifted dyadic cube \( \tilde{Q} \) such that \( Q \subseteq \frac{7}{10} \tilde{Q} \) (the cube having the same center as \( \tilde{Q} \) but with side length \( \frac{7}{10} \) that of \( \tilde{Q} \)) and \( \text{diam}(Q) \simeq \text{diam}(\tilde{Q}) \).

The same terminology will also be used in the plane \( \mathbb{R}^2 \). The only difference is that the previous cubes now become squares.

For any cube and square \( Q \), we will denote the side length of \( Q \) by \( \ell(Q) \) for short and denote the reflection of \( Q \) with respect to the origin by \( -Q \) hereafter.

By writing the characteristic function \( 1_{\xi_1 \neq \xi_2} \) of the region \( \{ \xi \in \mathbb{R}^2 : \xi_1 \neq \xi_2 \} \) into finite sum of smoothed versions of characteristic functions of the cones \( \{ \xi_2 > |\xi_1| \}, \{ \xi_2 < -|\xi_1| \}, \{ \xi_1 > |\xi_2| \} \) and \( \{ \xi_1 < -|\xi_2| \} \), we can decompose the bilinear operator \( T_n \) into a finite sum of four parts. Since all the operators obtained in this decomposition can be treated in the same way, without loss of generalizations, we will discuss in detail only one of them hereafter, for instance, the bilinear operator \( T_{m,n} \) given by smoothly truncating the symbol \( m \) on the cone \( \mathcal{A} := \{ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_2 > |\xi_1| \} \).

For this purpose, we consider the collection \( \mathcal{Q}_n \) of all shifted dyadic squares \( Q = Q_1 \times Q_2 \) satisfying the property that

\[
(3.2) \quad Q \cap \Gamma = \emptyset \quad \text{and} \quad \text{diam}(Q) \simeq 10^{-4}(\frac{1}{\sqrt{n + 1}} - \frac{1}{\sqrt{n + 2}}) \cdot \text{dist}(Q, (0, 0))
\]

for arbitrary positive integer \( n = 1, 2, \ldots \), where the singularity line \( \Gamma = \{ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1 = \xi_2 \} \). Correspondingly, we decompose the cone \( \mathcal{A} \) into a sequence of cones \( \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n, \ldots \) defined by

\[
(3.3) \quad \mathcal{C}_1 := \{ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : \frac{\sqrt{3}}{2} \leq \frac{\text{dist}(\xi, \Gamma)}{|(\xi_1, \xi_2)|} \leq 1, \xi_2 > |\xi_1| \},
\]

\[
(3.4) \quad \mathcal{C}_2 := \{ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : \frac{\sqrt{3}}{2} \leq \frac{\text{dist}(\xi, \Gamma)}{|(\xi_1, \xi_2)|} \leq \frac{\sqrt{3}}{2}, \xi_2 > |\xi_1| \}
\]

and

\[
(3.5) \quad \mathcal{C}_n := \{ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : \frac{1}{\sqrt{n + 2}} \leq \frac{\text{dist}(\xi, \Gamma)}{|(\xi_1, \xi_2)|} \leq \frac{1}{\sqrt{n + 1}}, \xi_2 > |\xi_1| \}
\]

for every \( n \geq 3 \). The borderlines \( \{ \mathcal{C}_n \}_{n=1}^{\infty} \) of these cones are defined by

\[
(3.6) \quad \mathcal{L}_1 := \{ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_2 = \left( \tan \frac{7}{12}\pi \right) \xi_1 > |\xi_1| \},
\]

\[
(3.7) \quad \mathcal{L}_2 := \{ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_2 = \left( \tan \frac{5}{12}\pi \right) \xi_1 > |\xi_1| \}
\]

and

\[
(3.8) \quad \mathcal{L}_n := \{ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_2 = \left( \tan \left( \frac{\pi}{4} + \arcsin \frac{1}{\sqrt{n + 2}} \right) \right) \xi_1 > |\xi_1| \}
\]
for every $n \geq 3$.

Now let us define the following disjoint collections of shifted dyadic squares

$$(3.9) \quad Q_n := \{Q = Q_1 \times Q_2 : Q \in Q_n, \ Q \cap C_n \neq \emptyset \ \text{and} \ Q \cap L_n = \emptyset\}$$

for $n = 1, 2, \cdots$. Since the set of squares $\{\frac{1}{10}Q \mid Q \in \bigcup_{n=1}^{\infty} Q_n\}$ also forms a finitely overlapping cover of the cone $A := \bigcup_{n=1}^{\infty} C_n = \{\xi = (\xi_1, \xi_2) : |\xi_1| < \xi_2\}$, by a standard partition of unity, we can write the smoothed characteristic function $\tilde{\chi}_{\{\xi_1 < \xi_2\}}$ of the cone $A$ as

$$(3.10) \quad \tilde{\chi}_{\{\xi_1 < \xi_2\}} = \sum_{n=1}^{\infty} \sum_{Q \in Q_n} \phi_Q(\xi_1, \xi_2),$$

where each $\phi_Q$ is a smooth bump function adapted to $Q$ and supported in $\frac{8}{10}Q$. One observes that all the collections $Q_n$ ($n = 1, 2, \cdots$) can be decomposed further into at most $10^8 n^\frac{3}{2}$ (modulo some fixed constant $C_0$ that is independent of $n = 1, 2, \cdots$) disjoint sub-collections $Q_n^k$ ($1 \leq k \leq C_0 10^8 n^\frac{3}{2}$) which contains only one unique shifted dyadic square $Q$ with the fixed scale $\ell(Q) = 2^{l_0}$ for some arbitrarily given integer $l_0 \in \mathbb{Z}$, that is,

$$(3.11) \quad Q_n = \bigcup_{k=1}^{\sim 10^8 n^\frac{3}{2}} Q_n^k$$

for every $n = 1, 2, \cdots$.

In order to prove $L^p$ estimates for $T_m$ (Theorem 1.3), it’s enough for us to investigate the bilinear operator $T_{m,A}$ given by

$$(3.12) \quad T_{m,A}(f_1, f_2)(x) := \int_{\mathbb{R}^2} m(\xi_1, \xi_2) \tilde{\chi}_{\{\xi_1 < \xi_2\}}(\xi_1, \xi_2) \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) e^{2\pi i x (\xi_1 + \xi_2)} d\xi_1 d\xi_2.$$

By using (3.10) and (3.11), we can decompose $T_{m,A}$ into a summation of bilinear multiplier operators:

$$(3.13) \quad T_{m,A} = \sum_{n=1}^{N} \sum_{k=1}^{\sim 10^8 n^\frac{3}{2}} T_{m,A,n}^k,$$

where the bilinear operators $T_{m,A,n}^k$ are given by

$$(3.14) \quad T_{m,A,n}^k(f_1, f_2)(x) := \sum_{Q \in Q_n^k} \int_{\mathbb{R}^2} m(\xi_1, \xi_2) \phi_Q(\xi_1, \xi_2) \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) e^{2\pi i x (\xi_1 + \xi_2)} d\xi_1 d\xi_2$$

for every $n = 1, 2, \cdots$ and $1 \leq k \lesssim 10^8 n^\frac{3}{2}$. Therefore, the proof of Theorem 1.3 can be reduced to proving $L^p$ estimates for each single bilinear multipliers $T_{m,A,n}^k$ with the constants in the bounds being independent of $k$ and having enough decay which is acceptable for summation with respect to $n$, that is, the following proposition.

**Proposition 3.1.** For every $\delta > 0$, $n = 1, 2, \cdots$ and $1 \leq k \lesssim 10^8 n^\frac{3}{2}$, let $T_{m,A,n}^k$ be the bilinear multiplier operator defined by (3.14) with the symbol $m$ satisfies the differential estimates (1.7) and (1.8), then we have

$$(3.15) \quad \|T_{m,A,n}^k(f_1, f_2)\|_{L^p(\mathbb{R})} \lesssim_{n,p,p_1,p_2,\delta} \|f_1\|_{L^{p_1}(\mathbb{R})} \cdot \|f_2\|_{L^{p_2}(\mathbb{R})},$$
provided that \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \) with \( 1 < p_1, p_2 \leq \infty \) and \( \frac{1}{2} < p < \infty \). Moreover, the implicit constants in the bounds satisfy

\[
C_{n,p,p_1,p_2} \lesssim p^{-\sqrt{\pi} + n^{-3}}.
\]

3.2. Reduce each single bilinear multiplier to a discrete model operator. Now we consider arbitrarily fixed \( n = 1, 2, \ldots \) and \( 1 \leq k \lesssim 10^8 n^{\frac{1}{2}} \). For each shifted dyadic square \( Q \in \mathcal{Q}_n^k \), one observes that there exist bump functions \( \phi_{Q,i} \) (\( i = 1, 2 \)) adapted to the shifted dyadic interval \( Q \) such that \( \text{supp} \phi_{Q,i} \subseteq \frac{1}{10} Q \) and \( \phi_{Q,i} \equiv 1 \) on \( \frac{1}{10} Q \) (\( i = 1, 2 \)) respectively. Notice that \( \text{supp} \phi_Q \subseteq \frac{3}{10} Q \); thus one has \( \phi_{Q,1} \cdot \phi_{Q,2} \equiv 1 \) on \( \text{supp} \phi_Q \). Since \( \xi_1 \in \text{supp} \phi_{Q,1} \subseteq \frac{1}{10} Q_1 \) and \( \xi_2 \in \text{supp} \phi_{Q,2} \subseteq \frac{1}{10} Q_2 \), it follows that \( -\xi_1 - \xi_2 \in \frac{1}{10} Q_1 - \frac{1}{10} Q_2 \), and as a consequence, one can find a shifted dyadic interval \( Q_3 \) with the property that \( -\frac{1}{10} Q_1 - \frac{1}{10} Q_2 \subseteq \frac{1}{10} Q_3 \) and satisfying \( |Q_1| = |Q_2| \approx |Q_3| \). In particular, there exists bump function \( \phi_{Q,3} \) adapted to \( Q_3 \) and supported in \( \frac{1}{10} Q_3 \) such that \( \phi_{Q,3} \equiv 1 \) on \( -\frac{1}{10} Q_1 - \frac{1}{10} Q_2 \).

We denote by \( \mathcal{Q}_n^k \) the collection of all shifted dyadic quasi-cubes \( Q := Q_1 \times Q_2 \times Q_3 \) with \( Q_1 \times Q_2 \in \mathcal{Q}_n^k \) and \( Q_3 \) be defined as above. Assuming this we then observe that, for any \( Q \) in such a collection \( \mathcal{Q}_n^k \), there exists a unique shifted dyadic cube \( \tilde{Q} \) in \( \mathbb{R}^3 \) such that \( Q \subseteq \frac{1}{10} \tilde{Q} \) and with property that \( \text{diam}(Q) \approx \text{diam}(\tilde{Q}) \). This allows us in particular to assume further that \( \mathcal{Q}_n^k \) is a collection of shifted dyadic cubes (that is, \( |Q_1| = |Q_2| = |Q_3| = \ell(Q) \)).

Now consider the trilinear form \( \Lambda_{m.A,n}^k(f_1, f_2, f_3) \) associated to \( T_{m.A,n}^k(f_1, f_2) \), which can be written as

\[
\Lambda_{m.A,n}^k(f_1, f_2, f_3) := \int_{\mathbb{R}^2} T_{m.A,n}^k(f_1, f_2)(x)f_3(x)dx = \sum_{Q \in \mathcal{Q}_n^k} \int_{\xi_1 + \xi_2 + \xi_3 = 0} m_Q(\xi_1, \xi_2, \xi_3)(f_1 \ast \phi_{Q,1})(\xi_1)(f_2 \ast \phi_{Q,2})(\xi_2)(f_3 \ast \phi_{Q,3})(\xi_3)d\xi,
\]

where \( \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \), while

\[
m_Q(\xi_1, \xi_2, \xi_3) := m(\xi_1, \xi_2) \cdot \phi_{Q_1 \times Q_2}(\xi_1, \xi_2) \cdot \tilde{\phi}_{Q,3}(\xi_3),
\]

where the function \( \phi_{Q_1 \times Q_2}(\xi_1, \xi_2) \) is one term of the partition of unity defined in (3.10). \( \tilde{\phi}_{Q,3} \) is an appropriate smooth function of variable \( \xi_3 \) supported on a slightly larger interval (with a constant magnification independent of \( \ell(Q) \)) than \( \text{supp} \phi_{Q,3} \), which equals 1 on \( \text{supp} \phi_{Q,3} \). We can decompose \( m_Q(\xi_1, \xi_2, \xi_3) \) as a Fourier series:

\[
m_Q(\xi_1, \xi_2, \xi_3) = \sum_{n_1, n_2, n_3} C_{n_1, n_2, n_3}^Q e^{2\pi i (n_1 x_1 + n_2 x_2 + n_3 x_3) / \ell(Q)},
\]

where the Fourier coefficients \( C_{n_1, n_2, n_3}^Q \) are given by

\[
C_{n_1, n_2, n_3}^Q = \int_{\mathbb{R}^3} m_Q(\ell(Q)\xi_1, \ell(Q)\xi_2, \ell(Q)\xi_3)e^{-2\pi i (n_1 \xi_1 + n_2 \xi_2 + n_3 \xi_3)}d\xi_1 d\xi_2 d\xi_3
\]
for every shifted dyadic cube $Q \in \mathbb{Q}_n^k$. Then, by a straightforward calculation, we can rewrite (3.17) as
\[
\Lambda_{m,A,n}^k(f_1, f_2, f_3) = \sum_{Q \in \mathbb{Q}_n^k} \sum_{n_1, n_2, n_3 \in \mathbb{Z}} C_{n_1, n_2, n_3}^{\delta, Q} \int_{\mathbb{R}} (f_1 \ast \hat{\phi}_{Q,1})(x - \frac{n_1}{\ell(Q)}) \times (f_2 \ast \hat{\phi}_{Q,2})(x - \frac{n_2}{\ell(Q)}) (f_3 \ast \hat{\phi}_{Q,3})(x - \frac{n_3}{\ell(Q)}) dx.
\]
(3.21)

**Definition 3.2.** (34 [40]) An arbitrary dyadic rectangle of area 1 in the phase-space plane is called a *Heisenberg box* or *tile*. Let $P := I_P \times \omega_P$ be a tile. An $L^2$-normalized wave packet on $P$ is a function $\Phi_P$ which has Fourier support $\text{supp} \hat{\Phi}_P \subseteq \frac{n}{10} \omega_P$ and obeys the estimates
\[
|\Phi_P(x)| \lesssim |I_P|^{-\frac{1}{2}} \left(1 + \frac{\text{dist}(x, I_P)}{|I_P|}\right)^{-M}
\]
for all $M > 0$, where the implicit constant depends on $M$.

Now we define $\varphi_{n_i}^{\nu_i}(\xi_i) := e^{2\pi i n_i \xi_i/|Q_i|} \cdot \hat{\phi}_{Q_i,i}(\xi_i)$ for $i = 1, 2, 3$. By the construction of the collection $\mathbb{Q}_n^k$ of shifted dyadic cubes, there exists only one unique cube $Q \in \mathbb{Q}_n^k$ such that $\ell(Q) = |Q_1| = |Q_2| = |Q_3| = 2^l$ for every $l \in \mathbb{Z}$. By splitting the real line $\mathbb{R}$ into disjoint union of unit intervals, performing the $L^2$-normalization procedure and simple calculations, we can rewrite (3.21) as
\[
\Lambda_{m,A,n}^k(f_1, f_2, f_3) = \sum_{n_1, n_2, n_3 \in \mathbb{Z}} \sum_{Q \in \mathbb{Q}_n^k} \int_{0}^{1} \sum_{P \in \mathbb{P}_n^k} \left| P \right|^2 \sum_{l} C_{n_1, n_2, n_3}^{\delta, Q} \left( f_1, \Phi_{P_1}^{1, n_1, \nu_1} \right) \left( f_2, \Phi_{P_2}^{2, n_2, \nu_2} \right) \left( f_3, \Phi_{P_3}^{3, n_3, \nu_3} \right) d\nu
\]
where the notation $\left( \cdot, \cdot \right)$ denotes the complex scalar $L^2$ inner product, the Fourier coefficients $C_{n_1, n_2, n_3}^{\delta, Q} := C_{n_1, n_2, n_3}^{\delta, Q}$, the tri-tiles $P := (P_1, P_2, P_3)$, the tiles $P_j := I_{P} \times \omega_{P_j}$ with time intervals $I_{P_j} := I = 2^{-m}[m, m + 1] =: I_P$ and the frequency intervals $\omega_{P_j} := Q_j$ for $j = 1, 2, 3$, the frequency cubes $Q_P := Q = \omega_{P_1} \times \omega_{P_2} \times \omega_{P_3} \in \mathbb{Q}_n^k$. $\mathbb{P}_n^k$ denotes the collection of such tri-tiles $P$ with frequency cubes $Q_P \in \mathbb{Q}_n^k$, while the $L^2$-normalized wave packets $\Phi_{P_i}^{i, n_i, \nu_i}$ associated with the Heisenberg boxes $P_i$ are defined by $\Phi_{P_i}^{i, n_i, \nu_i}(x) := \hat{\phi}_{Q_i,i}(x) := 2^{-\frac{1}{2}|\nu_i|} e^{-2^{\frac{1}{2}}|\nu_i|} (2^{-1}(m + \nu_i) - x)$ for $i = 1, 2, 3$.

By taking advantage of the differential estimates (1.7) and (1.8) for symbol $m(\xi_1, \xi_2)$, one deduces from the expression of Fourier coefficients (3.20) and integrating by parts sufficiently many times that
\[
|C_{n_1, n_2, n_3}^{\delta, Q} | \lesssim \prod_{j=1}^{3} \frac{1}{(1 + |n_j|)^{1000}} \cdot \left( e^{-\delta \sqrt{n + 1}} + \frac{e^{-\frac{2\delta}{3} \sqrt{n + 1}}}{n + 1} + \frac{e^{-\frac{\delta}{3} \sqrt{n + 1}}}{(n+1)^2} + \sum_{i=3}^{3000} \frac{1}{(n+1)^i} \right)
\]
(3.23)
for any tri-tiles $P \in \mathbb{P}_n^k$.

Observe that the rapid decay with respect to the parameters $n_1, n_2, n_3 \in \mathbb{Z}$ in (3.23) is acceptable for summation, all the functions $\Phi_{P_j}^{i,j,n_1,n_2,n_3} (j = 1, 2, 3)$ are $L^2$-normalized and are wave packets associated with the Heisenberg boxes $P_j$ uniformly with respect to the parameters $n_j$, therefore we only need to consider from now on the part of the trilinear form $\Lambda_{m,A,n}^k (f_1, f_2, f_3)$ defined in (3.22) corresponding to $n_1 = n_2 = n_3 = 0$:

$$
\Lambda_{m,A,n}^k (f_1, f_2, f_3) := \sum_{P \subseteq \mathbb{P}_n^k} \sum_{|P|} \sum_{|P|} \sum_{|P|} C_{Q,P}^\delta \langle f_1, \Phi_{P_1}^{1,\nu} \rangle \langle f_2, \Phi_{P_2}^{2,\nu} \rangle \langle f_3, \Phi_{P_3}^{3,\nu} \rangle d\nu,
$$

where $C_{Q,P}^\delta := C_{0,0,0}^{\delta} \Phi_{P_1}^{i,\nu} := \Phi_{P_1}^{i,0,\nu}$ for $\nu \in [0, 1]$ and $i = 1, 2, 3$.

The bilinear operator corresponding to the trilinear form $\Lambda_{m,A,n}^k (f_1, f_2, f_3)$ can be written as

$$
\hat{\Pi}_{\mathbb{P}_n^k}^\delta (f_1, f_2) (x) := \sum_{P \subseteq \mathbb{P}_n^k} \sum_{|P|} \sum_{|P|} \sum_{|P|} C_{Q,P}^\delta \langle f_1, \Phi_{P_1}^{1,\nu} \rangle \langle f_2, \Phi_{P_2}^{2,\nu} \rangle \Phi_{P_3}^{3,\nu} (x) d\nu.
$$

Since $\hat{\Pi}_{\mathbb{P}_n^k}^\delta (f_1, f_2)$ is an average of some discrete bilinear model operators depending on the parameters $\nu \in [0, 1]$, it is enough to prove the $L^p$ estimates of Hölder-type for each of them, uniformly with respect to the parameter $\nu$. From now on, we will do this in the particular case when the parameter $\nu = 0$, but the same argument works in general. By Fatou’s lemma, we can also restrict the summation in the definition (3.25) of $\hat{\Pi}_{\mathbb{P}_n^k}^\delta (f_1, f_2)$ on arbitrary finite sub-collections $\mathbb{P}$ of $\mathbb{P}_n^k$, and prove the estimates are uniform with respect to different choices of the set $\mathbb{P}$.

Therefore, one can reduce the bilinear operator $\hat{\Pi}_{\mathbb{P}_n^k}^\delta$ further to the discrete bilinear model operator $\Pi_{\mathbb{P}_n^k}^\delta$ defined by

$$
\Pi_{\mathbb{P}_n^k}^\delta (f_1, f_2) (x) := \sum_{P \subseteq \mathbb{P}_n^k} \sum_{|P|} \sum_{|P|} \sum_{|P|} C_{Q,P}^\delta \langle f_1, \Phi_{P_1}^{1,\nu} \rangle \langle f_2, \Phi_{P_2}^{2,\nu} \rangle \Phi_{P_3}^{3,\nu} (x),
$$

where $\Phi_{P_j}^{i,j} := \Phi_{P_j}^{i,0}$ for $j = 1, 2, 3$ respectively, $P \subseteq \mathbb{P}_n^k$ is arbitrary finite sub-collection of tri-tiles contained in $\mathbb{P}_n^k$. By (3.23), one has the following estimates for the Fourier coefficients $C_{Q,P}^\delta$:

$$
|C_{Q,P}^\delta| \lesssim e^{-\delta \sqrt{n}} + n^{-3},
$$

therefore, we can normalize the model operator $\Pi_{\mathbb{P}_n^k}^\delta$ by changing the coefficients $C_{Q,P}^\delta$ into 1 and only need to prove $L^p$ estimates for the normalized model operator with the constants in the bounds depending only on $p$, $p_1$, $p_2$ and independent of $\delta$, $n$ and $k$.

As have discussed above, we now reach a conclusion that the proof of Proposition 3.1 can be reduced to proving the following $L^p$ estimates for an arbitrary single model operator $\Pi_{\mathbb{P}_n^k}^\delta$. 
Proposition 3.3. For every \( n = 1, 2, \ldots \) and \( 1 \leq k \lesssim 10^8 n^2 \), if the finite collection \( \mathbb{P} \subseteq \mathbb{P}^k_n \) is chosen arbitrarily as above, then the discrete model operator \( \Pi^{n,k}_p \) defined by

\[
\Pi^{n,k}_p(f_1, f_2)(x) := \sum_{P \in \mathbb{P}^k \subseteq \mathbb{P}^k_n} \frac{1}{|I_P|^{\frac{1}{2}}} \langle f_1, \Phi^1_{P_1} \rangle \langle f_2, \Phi^2_{P_2} \rangle \Phi^3_{P_3}(x)
\]

maps \( L^{p_1}({\mathbb{R}}) \times L^{p_2}({\mathbb{R}}) \to L^p({\mathbb{R}}) \) boundedly for any \( 1 < p_1, p_2 \leq \infty \) and \( \frac{1}{2} < p < \infty \) satisfying \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \). Moreover, the implicit constants in the bounds depend only on \( p_1, p_2, p \) and are independent of \( n, k \) and the particular choice of finite sub-collection \( \mathbb{P} \).

3.3. Estimates of each single bilinear discrete model operator. In this section, we prove Proposition 3.1 by carrying out the proof of Proposition 3.3 for bilinear discrete model operator \( \Pi^{n,k}_p \) defined by (3.28).

We first consider the cases \( n \geq 2 \). It’s well known that a standard approach to prove \( L^p \) estimates for one-parameter \( n \)-linear operators with singular symbols (e.g., Coifman-Meyer multiplier, BHT and one-parameter paraproducts) is the generic estimates of the corresponding \( (n+1) \)-linear forms consisting of estimates for different \( L^p \), \( L^q \), \( L^r \) families of \( L^2 \)-normalized bump functions adapted to dyadic intervals \( I_P \) and \( \{ \Phi^i_{P_i} \} \) \( (i = 2, 3) \) also have the integral zero property that \( \int_{Q} \Phi^i_{P_i}(x)dx = 0 \), thus the maximal operator \( M(f_1) \) and the square operators \( S(f_2) \), \( S(f_3) \) are bounded on every \( L^p \) space for \( 1 < p < \infty \), and the implicit constants in the bounds will depend only on \( p \). The desired \( L^p \) estimates can be easily deduced from Hölder estimates in the particular cases \( 1 < p_1, p_2 < \infty \). Indeed, let \( f_1 \in L^{p_1} \), \( f_2 \in L^{p_2} \) and \( f_3 \in L^{p_3} \) with \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \), \( 1 < p_1, p_2 < \infty \) and \( \| f_3 \|_{L^{p_3}} = 1 \), then one can derive that

\[
\left| \int_{\mathbb{R}} \Pi^{n,k}_p(f_1, f_2)(x)f_3(x)dx \right| \lesssim \left| \int_{\mathbb{R}} \sum_{P \in \mathbb{P}^k} \frac{|\langle f_1, \Phi^1_{P_1} \rangle| \langle f_2, \Phi^2_{P_2} \rangle| \langle f_3, \Phi^3_{P_3} \rangle|}{|I_P|^{\frac{1}{2}}} \chi_{I_P}(x)dx \right|
\]

\[
\lesssim \int_{\mathbb{R}} \left| M(f_1)(x)S(f_2)(x)S(f_3)(x)dx \right| \lesssim \| M(f_1) \|_{L^{p_1}} \| S(f_2) \|_{L^{p_2}} \| S(f_3) \|_{L^{p_3}} \lesssim \| f_1 \|_{L^{p_1}} \| f_2 \|_{L^{p_2}},
\]

and hence

\[
\| \Pi^{n,k}_p(f_1, f_2) \|_{L^p({\mathbb{R}})} \lesssim_{p, p_1, p_2} \| f_1 \|_{L^{p_1}({\mathbb{R}})} \cdot \| f_2 \|_{L^{p_2}({\mathbb{R}})}.
\]

By the multi-linear interpolations (see [17, 19, 30, 31]) and the symmetry of operators \( \Pi^{n,k}_p \), in order to prove \( L^p \) estimates in the general cases \( 1 < p_1, p_2 \leq \infty \) and \( \frac{1}{2} < p < \infty \), we only need to prove that the bilinear model operators \( \Pi^{n,k}_p \) satisfy the \( L^{\frac{1}{2}+\epsilon, \infty} \) estimates for \( \epsilon > 0 \) arbitrarily small. To do this, by using the duality lemma for \( L^{\infty} \) (see [30, 31]) and scaling invariance, we fix \( p_1, p_2 \) two numbers larger than 1 and arbitrarily close to 1,
let functions $f_1$, $f_2$ such that $\|f_1\|_{L^p} = \|f_2\|_{L^p} = 1$ and a measurable set $E \subseteq \mathbb{R}$ satisfying $|E| = 1$, our goal is to find a dominative subset $E' \subseteq E$ with comparable measure $|E'| \simeq 1$ such that the corresponding trilinear forms $\Lambda_{P, n}^k$ satisfy the estimates:

$$
(3.29) \quad |\Lambda_{P, n}^k(f_1, f_2, f_3)| = \left| \sum_{p \in \mathbb{P} \subseteq \mathbb{P}_n} \frac{1}{|I_p|^{1/2}} \langle f_1, \Phi_{P_1}^{1} \rangle \langle f_2, \Phi_{P_2}^{2} \rangle \langle f_3, \Phi_{P_3}^{3} \rangle \right| \lesssim 1,
$$

where $f_3 := \chi_{E'}$.

Since one can essentially regard the operators $\Pi_{\mathbb{P}}^{n, k}$ given by (3.28) as one-parameter para-products, we can use the stopping-time decomposition arguments based on square operators $S$ and maximal operators $M$ developed by C. Muscalu, J. Pipher, T. Tao and C. Thiele in [32, 33, 35] to prove (3.29). We will give a brief proof for (3.29) here.

By using the generic decomposition lemma (Lemma 3.1 in [35]), one can estimate (3.29) by

$$
(3.30) \quad |\Lambda_{P, n}^k(f_1, f_2, f_3)| \lesssim \sum_{\ell \in \mathbb{N}} 2^{-1000\ell} \sum_{p \in \mathbb{P} \subseteq \mathbb{P}_n} \frac{1}{|I_p|^{1/2}} |\langle f_1, \Phi_{P_1}^{1} \rangle| |\langle f_2, \Phi_{P_2}^{2} \rangle| |\langle f_3, \Phi_{P_3}^{3, \ell} \rangle|,
$$

where the new bump functions $\Phi_{P_3}^{3, \ell}$ also satisfy the integral zero property as $\Phi_{P_3}^{3}$ but have the additional property that $\text{supp}(\Phi_{P_3}^{3, \ell}) \subseteq 2^\ell I_P$, and hence the square operators $S^\ell$ (which are defined in terms of $\Phi_{P_3}^{3, \ell}$ instead of $\Phi_{P_3}^{3}$) are bounded on any $L^p$ ($1 < p < \infty$) space as well with the constants in the bounds depending only on $p$. For every $\ell \in \mathbb{N}$, we define the sets as follows:

$$
(3.31) \quad \Omega_{-10\ell} := \{ x \in \mathbb{R} : M(f_1)(x) > C2^{10\ell} \} \cup \{ x \in \mathbb{R} : S(f_2)(x) > C2^{10\ell} \},
$$

$$
(3.32) \quad \tilde{\Omega}_{-10\ell} := \{ x \in \mathbb{R} : M(\chi_{\Omega_{-10\ell}})(x) > \frac{1}{100} \},
$$

and

$$
(3.33) \quad \tilde{\tilde{\Omega}}_{-10\ell} := \{ x \in \mathbb{R} : M(\chi_{\tilde{\Omega}_{-10\ell}})(x) > 2^{-\ell} \}.
$$

Finally, we define the exceptional set

$$
(3.34) \quad U := \bigcup_{\ell \in \mathbb{N}} \tilde{\tilde{\Omega}}_{-10\ell}.
$$

It is clear that $|U| < \frac{1}{10}$ if $C$ is a large enough constant, which we fix from now on. Then, we define the dominative subset $E' := E \setminus U$ and observe that $|E'| \simeq 1$.

Now we fix $\ell \in \mathbb{N}$ and consider the corresponding inner sum in (3.30). One easily observes that it’s enough for us to consider the sub-collection $\mathbb{P}_\ell \subseteq \mathbb{P} \subseteq \mathbb{P}_n$ consisting of all tri-tiles $P \in \mathbb{P}$ with time intervals $I_P$ satisfying $I_P \cap \tilde{\Omega}_{-10\ell}^c \neq \emptyset$ in the inner sum of (3.30) (other parts are equal to 0), it follows that $|I_P \cap \Omega_{-10\ell}| \leq \frac{1}{100} |I_P|$ for such tri-tiles $P \in \mathbb{P}_\ell$.

Now we define three different decomposition procedures for functions $f_1$, $f_2$ and $f_3$ respectively. First, define

$$
\Omega_{n_1}^1 := \{ x \in \mathbb{R} | M(f_1)(x) > C2^{-n_1} \}, \quad \Omega_{n_2}^2 := \{ x \in \mathbb{R} | S(f_2)(x) > C2^{-n_2} \}
$$
and
\[
P^i_{n_i, \ell} := \left\{ P \in \mathbb{P} \left| |P| \left| \Omega_{n_i}^i \right| > \frac{|P|}{100}, |P| \left| \Omega_{n_i-1}^i \right| \leq \frac{|P|}{100} \right\}
\]
for \(i = 1, 2\) and every \(n_1, n_2 \geq -10\ell\), which produces the sets \(\{\Omega_{n_i}^i\}_{n_i \geq -10\ell}, \{\Omega_{n_2}^2\}_{n_2 \geq -10\ell},\) \(\{P^1_{n_1, \ell}\}_{n_1 \geq -10\ell}\) and \(\{P^2_{n_2, \ell}\}_{n_2 \geq -10\ell}\) with \(P_{-10\ell, \ell} = P_{-10\ell, \ell} = \emptyset\), and one has \(P_\ell = \bigcup_{n_i \geq -10\ell} P^i_{n_i, \ell}\) for \(i = 1, 2\). Then, we choose \(N > 0\) large enough such that for every \(P \in \mathbb{P}\), one has \(|P \cap \Omega_{-N}^3| \leq \frac{|P|}{100}\), where \(\Omega_{-N}^3 := \{x \in \mathbb{R} | S^\ell(f_3)(x) > C2^N\}\). For every \(n_3 \geq -N\), we define sets
\[
\Omega_{n_3}^3 := \{x \in \mathbb{R} | S^\ell(f_3)(x) > C2^{-n_3}\}
\]
and
\[
P^3_{n_3, \ell} := \left\{ P \in \mathbb{P} \left| |P| \left| \Omega_{n_3}^3 \right| > \frac{|P|}{100}, |P| \left| \Omega_{n_3-1}^3 \right| \leq \frac{|P|}{100} \right\},
\]
which produces the sets \(\{\Omega_{n_3}^3\}_{n_3 \geq -N}\) and \(\{P^3_{n_3, \ell}\}_{n_3 \geq -N}\) with \(P_{-N, \ell} = \emptyset\), and one has \(P_\ell = \bigcup_{n_3 \geq -N} P^3_{n_3, \ell}\).

Now we define sub-collection \(P^\ell_{n_1, n_2, n_3} := P^1_{n_1, \ell} \cap P^2_{n_2, \ell} \cap P^3_{n_3, \ell}\) and \(\Omega_{n_1, n_2, n_3}^\ell := \bigcup_{P \in P^\ell_{n_1, n_2, n_3}} I_P\). The inner sum in the right-hand side of (3.30) can be estimated by
\[
\sum_{n_1, n_2 > -10\ell, n_3 > -N} |f_1, \Phi^1_{P_1}| \left| f_2, \Phi^2_{P_2} \right| \left| f_3, \Phi^3_{P_3} \right| |I_P \cap \bigcup_{i=1}^{3} \Omega_{n_i-1}^i| \lesssim \int_{(\bigcup_{i=1}^{3} \Omega_{n_i-1}^i) \cap \Omega_{n_1, n_2, n_3}^\ell} M(f_1)(x)S(f_2)(x)S^\ell(f_3)(x) dx
\]
\[
\lesssim \sum_{n_1, n_2 > -10\ell, n_3 > -N} 2^{-(n_1+n_2+n_3)} |\Omega_{n_1, n_2, n_3}^\ell|.
\]
By the boundedness of the operators \(M, S, S^\ell\), one easily get the estimates
\[
|\Omega_{n_1, n_2, n_3}^\ell| \lesssim 2^{n_1 p_1} 2^{n_2 p_2} 2^{n_3 \mu}
\]
for any \(\mu \geq 1\), as a consequence, combining this with the estimates (3.30) and (3.35), we derive the following lower estimates for the trilinear form:
\[
|A^k_{n, \ell}(f_1, f_2, f_3)| \lesssim \sum_{\ell \in \mathbb{N}} 2^{-100\ell} \left\{ \sum_{n_1, n_2 > -10\ell, n_3 > 0} 2^{-(1-p_1\theta_1)n_1} 2^{-(1-p_2\theta_2)n_2} 2^{-(1-\mu\theta_3)n_3} + \sum_{n_1, n_2 > -10\ell, -N < n_3 < 0} 2^{-(1-p_1\theta_1')n_1} 2^{-(1-p_2\theta_2')n_2} 2^{-(1-\mu\theta_3')n_3} \right\}
\]
\[
\lesssim \sum_{\ell \in \mathbb{N}} 2^{-100\ell} \lesssim 1,
\]
where the parameter \(\theta_i\) and \(\theta_i'\) are chosen appropriately, which implies the desired estimate (3.29).

As to the cases \(n = 1\) and \(1 \leq k \lesssim 10^8\), we will perform the discretized square operators \(S\) on the functions \(f_1\) and \(f_2\) and the maximal operator \(M\) on function \(f_3\), and estimates the
corresponding trilinear form $\Lambda_{k,n}^p(f_1, f_2, f_3)$ by $S(f_1)$, $S(f_2)$ and $M(f_3)$, the rest of the proof are completely similar to the $n \geq 2$ cases.

This completes the proof of Proposition 3.3 and also concludes the proof of Proposition 3.1 at the same time.

3.4. Conclusions. We can deduce from Proposition 3.1 and (3.13) the following $L^p$ estimates for the bilinear operator $T_{mA}$ (associated with symbol $m$ smoothly truncated on the cone $A$):

$$\|T_{mA}(f_1, f_2)\|_{L^p(\mathbb{R})} \lesssim \sum_{n=1}^{\infty} \sum_{k=1}^{\sim 10^8 n^2} \|T_{mA,n}^k(f_1, f_2)\|_{L^p(\mathbb{R})}$$

$$\lesssim_{p, p_1, p_2} \sum_{n=1}^{\infty} \sum_{k=1}^{\sim 10^8 n^2} \{e^{-\delta \sqrt{n} + n^{-3}}\} \|f_1\|_{L^{p_1}(\mathbb{R})} \|f_2\|_{L^{p_2}(\mathbb{R})} \lesssim_{p, p_1, p_2, \delta} \|f_1\|_{L^{p_1}(\mathbb{R})} \cdot \|f_2\|_{L^{p_2}(\mathbb{R})}$$

for any $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ with $1 < p_1, p_2 \leq \infty$ and $\frac{1}{2} < p < \infty$, where the implicit constants in the bounds depend only on $p_1, p_2, p, \delta$ and tend to infinity as $\delta \to 0$. One can observe that (3.38) also implies the $L^p$ estimates of the original bilinear operator $T_m$ for $p > \frac{1}{2}$.

This concludes the proof of Theorem 1.3.

References

[1] F. Bernicot, Local estimates and global continuities in Lebesgue spaces for bilinear operators, Anal. PDE, 1(2008), 1-27.
[2] D. Bilyk and L. Grafakos, Distributional estimates for the bilinear Hilbert transform, J. Geom. Anal., 16(2006), no. 4, 563-584.
[3] L. Carleson, On convergence and growth of partial sums of Fourier series, Acta Math. 116 (1966), 135-157.
[4] J. Chen and G. Lu, Hörmander type theorems for multi-linear and multi-parameter Fourier multiplier operators with limited smoothness, Nonlinear Analysis, 101(2014), 98-112.
[5] M. Christ and J. Journé, Polynomial growth estimates for multilinear singular integrals, Acta Math., 159(1987), 51-80.
[6] R. Coifman and Y. Meyer, Au delà des opérateurs pseudo-différentiels, Astérisque, 57(1978).
[7] R. Coifman and Y. Meyer, Wavelets, Calderon–Zygmund Operators and Multilinear Operators, translated from the 1990 and 1991 French originals by D. Salinger, Cambridge Studies in Advanced Mathematics, vol. 48, Cambridge University Press, Cambridge, 1997.
[8] W. Dai and G. Lu, $L^p$ estimates for multi-linear and multi-parameter pseudo-differential operators, to appear in Bull. Soc. Math. Fr., [arXiv:1308.4062].
[9] W. Dai and G. Lu, $L^p$ estimates for bilinear and multi-parameter Hilbert transforms, preprint, [arXiv:1403.0624].
[10] C. Demeter, T. Tao and C. Thiele, Maximal multi-linear operators, Trans. Amer. Math. Soc., 360(2008), no. 9, 4989-5042.
[11] C. Demeter and C. Thiele, On the two dimensional bilinear Hilbert transform, Amer. J. Math., 132(2010), no. 1, 201-256.
[12] C. Fefferman, Pointwise convergence of Fourier series, Ann. of Math. 98 (1973), 551-571.
[13] R. Fefferman and E. M. Stein, Singular integrals on product spaces, Adv. Math., 45(1982), no. 2, 117-143.
[14] L. Grafakos and X. Li, Uniform estimates for the bilinear Hilbert transform I, Ann. Math., 159(2004), 889-933.
[15] J. Gilbert and A. Nahmod, *Bilinear operators with non-smooth symbols I*, J. Fourier Anal. Appl., 7 (2001), no. 5, 435-467.

[16] L. Grafakos and R. H. Torres, *Multilinear Calderón Zygmund theory*, Adv. Math., 165 (2002), 124-164.

[17] L. Grafakos and T. Tao, *Multilinear interpolation between adjoint operators*, J. Funct. Anal., 199 (2003), 379-385.

[18] Q. Hong and G. Lu, *Symbolic calculus and boundedness of multi-parameter and multi-linear pseudo-differential operators*, Advanced Nonlinear Studies, 14 (4) (2014).

[19] S. Janson, *On interpolation of multilinear operators*, Lecture Notes in Mathematics, vol. 1302, Springer, 290-302.

[20] J. Journé, *Calderón-Zygmund operators on product spaces*, Rev. Mat. Iberoamericana, 1 (1985), no. 3, 55-91.

[21] J. Jung, *Iterated trilinear Fourier integrals with arbitrary symbols*, preprint, arXiv:1311.1573

[22] R. Kesler, *Mixed estimates for degenerate multilinear oscillatory integrals and their tensor product generalizations*, preprint, arXiv:1311.2322

[23] C. Kenig and E. M. Stein, *Multilinear estimates and fractional integration*, Math. Res. Lett., 6 (1999), 1-15.

[24] M. T. Lacey, *The bilinear maximal functions map into $L^p$ for $2/3 < p \leq 1$*, Ann. Math., 155 (2000), 35-57.

[25] M. T. Lacey and J. Metcalfe, *Paraproducts in one and several parameters*, Forum Math. 19 (2007), no. 2, 325-351.

[26] M. T. Lacey and C. Thiele, *$L^p$ estimates for the bilinear Hilbert transform*, Ann. Math., 146 (1997), 693-724.

[27] M. T. Lacey and C. Thiele, *On Calderón’s conjecture*, Ann. Math., 150 (1999), 475-496.

[28] X. Li, *Uniform estimates for the bilinear Hilbert transform II*, Revista Mat. Iberoamericana, 22 (2006), 1069-1126.

[29] P. Luthy, *Bi-Parameter Maximal Multilinear Operators*, 2013.

[30] C. Muscalu and W. Schlag, *Classical and Multilinear Harmonic Analysis, II*, Cambridge Studies in Advanced Mathematics, vol. 138, Cambridge University Press, Cambridge, 2013.

[31] C. Muscalu, T. Tao and C. Thiele, *Multilinear multipliers given by singular symbols*, J. Amer. Math. Soc., 15 (2002), 469-496.

[32] C. Muscalu, J. Pipher, T. Tao and C. Thiele, *Bi-parameter paraproducts*, Acta Math., 193 (2004), no. 2, 269-296.

[33] C. Muscalu, J. Pipher, T. Tao and C. Thiele, *A short proof of the Coifman-Meyer multilinear theorem*, 2004.

[34] C. Muscalu, T. Tao and C. Thiele, *$L^p$ estimates for the biest II. The Fourier case*, Math. Ann., 329 (2004), 427-461.

[35] C. Muscalu, J. Pipher, T. Tao and C. Thiele, *Multi-parameter paraproducts*, Revista Mat. Iberoamericana, 22 (2006), 963-976.

[36] F. D. Plinio and C. Thiele, *Endpoint bounds for the bilinear Hilbert transform*, preprint, arXiv:1403.5978

[37] P. Silva, *Vector valued inequalities for families of bilinear Hilbert transforms and applications to bi-parameter problems*, preprint, arXiv:1203.3251

[38] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, 43. Monographs in Harmonic Analysis, III. Princeton University Press, Princeton, NJ, 1993. xiv+695 pp.

[39] C. Thiele, *A uniform estimate*, Ann. Math., 156 (2002), 519-563.

[40] C. Thiele, *Wave Packet Analysis*, CBMS Conference Series in Mathematics, vol. 105, American Mathematical Society, Providence, RI, 2006.

School of Mathematical Sciences, Beijing Normal University, Beijing 100875, P. R. China
E-mail address: daiwei@bnu.edu.cn

Department of Mathematics, Wayne State University, Detroit, MI 48202, U. S. A.
E-mail address: gzlu@wayne.edu