A Continuous Transition Between Quantum and Classical Mechanics (II)

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Examples are worked out using a new equation proposed in the previous paper to show that it has new physical predictions for mesoscopic systems.

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I. INTRODUCTION

If one assumes that quantum theory is more fundamental than classical theory, then the emergence of a classical world from a quantum substratum becomes a necessary requirement that quantum theory must satisfy. It is not sufficient to show the emergence of a macroscopic world that is only approximately or partially classical in nature. The accuracy of the prediction of energy loss from binary pulsars by the general theory of relativity is just as impressive as that of quantum electrodynamics for the Lamb shift \[1\]. Unfortunately, nobody has yet succeeded in demonstrating satisfactorily that quantum theory passes this test. The approach that comes closest to doing justice to the problem is that of environment-induced decoherence (EID) \[2\]. Unfortunately, it too falls short of doing full justice because it leaves the notorious measurement problem unsolved. In this paper we will show that a different approach combined with the idea of decoherence is able to bridge the gap between quantum and classical theory smoothly and exactly \[3\] and leads to new predictions.

The most striking feature of quantum theory that distinguishes it from classical theory is the appearance of coherence exhibited by interference phenomena. The idea of EID is based on the observation that the environment surrounding a quantum system can, in effect, monitor some of the system’s observables in such a fashion that the eigenstates of those observables continuously decohere, i.e. the environment not only causes states with macroscopic separations to decohere but also prevents such coherence from reappearing. In technical terms, this is reflected in the fact that the reduced density matrix of the system (obtained by tracing over the environment variables) becomes diagonal in the chosen representation over a time scale that is set by the system parameters, the rate of relaxation and the temperature of the environment (treated as a heat bath). However, this falls short of ensuring that the surviving diagonal elements of the density matrix behave classically. The usual arguments based on properties of the Wigner distribution are deceptive. Although it is true that the time evolution of the Wigner function looks identical to that of the classical Liouville function for all polynomial potentials of order no higher than the second, nevertheless the energy spectra, for example, clearly continue to exhibit the same quantum features as before EID. We will give a number of concrete examples of this and other differences below.

This shortcoming of the EID approach is due to the fact that it is based on the linear and unitary Schrödinger evolution. It is, however, well known that a unitary linear evolution can never lead to a measurement in quantum theory which requires an independent nonlinear, non-unitary process (von Neumann’s “process 1”). This is the crux of the measurement problem that remains unsolved. And unless this is solved or obviated in some way, the gap between quantum theory and classical theory cannot be bridged. Consequently, we would like to explore in detail a different approach proposed in the preceding paper.

The central point of this theory is based on the simple observation that it is possible to give a unified description of classical and quantum theory by introducing a wave function for a classical ensemble of systems. Such a system is described by an action function \(S(x,t)_{cl}\) that satisfies the classical Hamilton-Jacobi equation

\[
\frac{\partial S(x,t)_{cl}}{\partial t} + \frac{1}{2m}(\nabla S(x,t)_{cl})^2 + V(x) = 0
\]  

(1)

and an equation of continuity

\[
\frac{\partial R^2(x,t)_{cl}}{\partial t} + \text{div}(R^2(x,t)_{cl} \frac{p}{m}) = 0
\]  

(2)

for the position distribution function \(R^2(x,t)_{cl}\), where \(p = \nabla S(x,t)_{cl}\) is the momentum. Let us introduce a wave function

\[
\psi(x,t)_{cl} = R(x,t)_{cl} e^{i\sqrt{S(x,t)_{cl}}}
\]  

(3)
For example, consider an ensemble of free particles in one dimension and let

\[ R(x, t)_{cl} = (2\pi \sigma^2)^{-1/4} e^{-\frac{(x-vt)^2}{4\sigma^2}} \]  
\[ S(x, t)_{cl} = px - Et \]  

This describes an ensemble in which the particles have the same velocity \( p/m \) and a position distribution that is Gaussian with a spread \( \sigma \) that does not change with time. In other words, \( \psi(x, t)_{cl} \) is a non-dispersive wave function that behaves classically.

Another physically interesting example is the harmonic oscillator. Consider the wave packet for which

\[ R(x, t)_{cl} = (\frac{m\omega}{\pi \hbar})^{1/4} e^{-\frac{m\omega}{\hbar}(x-a \cos \omega t)^2} \]  
\[ S(x, t)_{cl} = \frac{1}{4} m\omega a^2 \sin 2\omega t - m\omega xa \sin \omega t \]  

The equation of motion of the particles (obtained from equation (1)) in the standard fashion is

\[ \frac{dp}{dt} = -\nabla V = -m\omega^2 x \]  

whose solution is \( x = a \cos \omega t \). The particles in the ensemble therefore behave exactly classically in spite of the fact that they are described by a wave function that involves \( \hbar \).

The equation of motion that these wave functions must satisfy so that the classical equations (1) and (2) are preserved is most interesting. It is

\[ i\hbar \frac{\partial \psi(x, t)_{cl}}{\partial t} = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(x) - Q_{cl} \right] \psi(x, t)_{cl} \]  

with

\[ Q_{cl} = -\frac{\hbar^2}{2m} \frac{\nabla^2 R(x, t)_{cl}}{R(x, t)_{cl}} \]  

This is a nonlinear equation that reduces to the Schrödinger equation in the limit \( Q_{cl} \) going to zero. \( Q_{cl} \) acts like a potential that destroys linearity and therefore coherence, and makes the system behave exactly classically, in spite of the occurrence of Planck’s constant in the equation. This is a point that is worth emphasizing. The traditional approach has been an attempt to show that quantum mechanics reduces to classical mechanics in the limit \( \hbar \) going to zero. This has not been entirely successful. However, the fact is that \( \hbar \) is not zero in the real world. So, the real problem is to explain how the macroscopic world behaves classically in spite of Plank’s constant not being zero. This has not been nearly sufficiently emphasized in the literature.

We therefore take the point of view advocated in the preceding paper that classical mechanics is a theory in which the Planck constant \( \hbar \) occurs as a fundamental constant through equation (9). The systems nevertheless behave completely classically because of the (hitherto ignored) fundamental interaction \( Q_{cl} \) which is nonlinear and cancels all quantum features exactly. Note that \( Q_{cl} \) remains ‘hidden’ in the usual formulation of classical mechanics because it is not formulated in terms of a wave function. Its form is not \textit{ad hoc}. It is uniquely fixed by the equations (1) and (2), i.e., by the sole requirement of classicality. Quantum theory then emerges naturally and in full glory in the limit that this new fundamental interaction is switched off. This, we believe, is a more reasonable and natural assumption than the usual one that an effective nonlinear term such as \( Q_{cl} \) does somehow arise from a basically linear equation such as the Schrödinger equation. We fully realize that ours will not be an immediately popular view, and most of our readers familiar with the reigning paradigm of physics will be skeptical. We only wish to remind them that our point of view, though perhaps different, is not inconsistent with that of Bohr, the founder of the Copenhagen philosophy. Bohr always emphasized that the classical nature of the measuring apparatus cannot be derived from quantum mechanics. This is in sharp contrast to the prevalent view that it is possible to understand the emergence of the classical world starting from quantum theory. Our experience with the heretical point of view we are pursuing has convinced us that

\[ ^1 \text{We do not wish to speculate on the origin of fundamental constants. We simply postulate them, as in all physical theories so far proposed.} \]
its consequences cannot be dismissed off hand, and need to be seriously explored and confronted with experiments. We believe it is crazy enough that it has a chance of being right.

The theory described so far is an almost unknown way of describing classical physics. It is interesting but admittedly sterile in the sense that it contains no new physical result. The new physics enters through the postulate that $Q_d$ is the effective potential seen by a classical system that is surrounded by an environment. Equation (9) should then be replaced by the more fundamental equation

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(x) - \lambda(t)Q \right] \psi(x,t)$$

(11)

where the coupling parameter $0 < \lambda(t) \leq 1$ and encapsulates the cumulative effect of random disturbances caused by the environment on the system. This type of cumulative effect of the environment on a system is generic and occurs in a wide class of phenomena. The limit $\lambda(0) \to 0$ corresponds to quantum mechanics (pure Schrödinger equation) provided we also assume that $\psi$ is single-valued, and the limit $\lambda(\infty) = 1$ corresponds to classical physics (equation (9)). The two worlds are smoothly bridged by this equation. However, its specific new predictions really come in the intermediate ranges of $\lambda(t)$ corresponding to mesoscopic systems, and are weakly dependent on the model of the system-environment coupling. But before showing that, we will now demonstrate how this equation makes the diagonal terms of the reduced density matrix of a system behave exactly classically after decoherence.

Before concluding this section, we would like to point out that there are other approaches too to decoherence in which the Schrödinger equation is modified by the addition of ad hoc non-linear, non-unitary terms that cause ‘explicit collapse’. Our approach is totally different in that the primacy of the macroscopic world as opposed to the quantum world makes it unnecessary for us to invoke the concept of collapse. The empirical differences between models of ‘explicit collapse’ and EID (‘false collapse’) have been discussed by Ghose [4].

II. EID: CONVENTIONAL VIEW

Consider a wave function that is initially a superposition of two different position eigenfunctions:

$$\psi(x,0) = \frac{1}{\sqrt{2}} [\psi_1(x,0) + \psi_2(x,0)]$$

(12)

Its density matrix $\rho(x,x',0)$ has off-diagonal terms. In the presence of an environment simulated by a heat bath at a temperature $T$, the time evolution of the density matrix is governed by a master equation of the form [2]

$$\frac{d\rho}{dt} = -\frac{i}{\hbar} [H,\rho] - \gamma(x - x') (\partial_x - \partial_{x'}) \rho + \rho/\tau_D$$

(13)

where $\gamma$ is the relaxation time and $\tau_D = \hbar^2/2m\gamma k_B T(x - x')^2 = \gamma^{-1}(\lambda_T/\Delta x)^2$ where $\lambda_T$ is the thermal de Broglie wavelength. The first term describes the usual quantum evolution, the second term causes dissipation due to friction and the third term is responsible for fluctuations or Brownian motion. It is the last term that is principally responsible for decoherence of states, whose positions are macroscopically distinguishable, over a time scale $\tau_D$ which could be as small as $10^{-23}$ sec for a mass of 1 gm at room temperature ($T = 300$ K) even if the relaxation time $\tau_R = \gamma^{-1}$ is of the order of $10^{17}$ sec, the age of the universe. On the other hand, for microscopic particles like the electron $\tau_D$ could be much larger than $\tau_R$ on atomic and larger time scales. Notice that these last two terms responsible for decoherence according to the prevalent view are both off-diagonal (they vanish when $x = x'$) and have no effect on the first (diagonal) term which remains quantum mechanical.

We will now give two examples of this. Consider a superposition of the harmonic oscillator stationary state $n = 1$ centered around $x = x_1$ and $x = x_2$,

$$\psi_{os}(x,t) = \left( \frac{m\omega \tau}{\pi \hbar} \right)^{1/4} \left[ e^{-\frac{m\omega(x-x_1)^2}{2\tau}} (x-x_1) + e^{-\frac{m\omega(x-x_2)^2}{2\tau}} (x-x_2) \right]$$

(14)

as well as that of a dispersive, free Gaussian quantum mechanical wave packet,

$$\psi_{fg}(x,t) = \frac{1}{\sqrt{2}} \left( \frac{2\pi \sigma^2}{\tau} \right)^{1/4} \left[ e^{ik(x-x_1-ut/2)-(x-x_1-ut)^2/4\sigma^2\tau} + e^{ik(x-x_2-ut/2)-(x-x_2-ut)^2/4\sigma^2\tau} \right]$$

(15)
with \( s_i = \sigma_0 (1 + i \hbar t / 2m \sigma_0^2) \). It is clear from these expressions that although the off-diagonal terms of the corresponding density matrices die out because of decoherence, the diagonal terms remain intact. Although the environment will prevent the off-diagonal terms from reappearing over an effectively infinite time scale, the energy spectra of the remaining diagonal terms will still reveal their initial quantum nature. For example, it is straightforward to see that the harmonic oscillator energy will continue to be \( 3\hbar/2 \) for the \( n = 1 \) stationary state, and

\[
<E> = \frac{p^2}{2m} + \frac{\hbar^2}{8m\sigma_0^2}
\]

for the free Gaussian wave packet. In the latter case it is also clear that the diagonal terms will continue to be dispersive and so maintain their quantum character.

Exactly the same conclusions regarding the energy spectra also follow from the Wigner distributions

\[
W(x,p) = \frac{1}{2\pi \hbar} \int_{-\infty}^{\infty} e^{i px/\hbar} \rho(x - \frac{y}{2}, x + \frac{y}{2}) dy
\]

(for convenience defined here in one space dimension only) in these cases. The fact that the time evolution of the Wigner function happens to coincide with the classical Liouville function,

\[
\frac{dW}{dt} = \partial W/\partial t + \{W,H\}_{PB}
\]

for polynomial potentials of degree less than or equal to two, as already mentioned, is therefore clearly misleading. In fact, one can show that for stationary states of the harmonic oscillator the energy spectrum is given in terms of the Wigner function by the expression

\[
E = \frac{p^2}{2m} + \frac{1}{2} m\omega^2x^2 - \frac{\hbar^2}{8mW} \frac{\partial^2 W}{\partial x^2} - \frac{m\omega^2\hbar^2}{8W} \frac{\partial^2 W}{\partial p^2}
\]

The third and fourth terms are clearly non-classical. It is straightforward to verify this for \( n = 1 \) and \( n = 2 \). In these cases one again obtains the expected quantum mechanical results, \( E = 3\hbar/2, 5\hbar/2 \) for \( n = 1 \) and \( n = 2 \) respectively.

Before concluding this section, let us just note the expression for the rate of change of \( W(x,p) \) that follows from the master equation (13),

\[
\frac{dW}{dt} = -\{W,H\}_{PB} + 2\gamma \frac{\partial (pW)}{\partial p} + D \frac{\partial^2 W}{\partial p^2}
\]

where \( D = 2m\gamma k_B T \). It is the last term in this equation that causes the off-diagonal terms to disappear by a process of diffusion.

### III. EID: HERETICAL VIEW

A consequence of the new equation (11) is that the density matrix \( \rho \) must now satisfy a different evolution equation, namely

\[
\frac{d\rho}{dt} = -\frac{i}{\hbar} [H', \rho]
\]

\[
H' = -\frac{\hbar^2}{2m} \nabla^2 + V(x) - \lambda(t)Q
\]

This clearly shows that the potential \( \lambda(t)Q \) will modify the density matrix and ensure its exact classical behaviour in the limit \( \lambda = 1 \). To see this in terms of the Wigner function, it will be convenient to impose the de Broglie-Bohm guidance condition \( p = mdx/dt = \nabla S \). Then one has

\[
E = -\frac{\partial S}{\partial t} = \frac{p^2}{2m} + V(x) + (1 - \lambda(t))Q
\]

\[
\frac{dp}{dt} = -\nabla [V(x) + (1 - \lambda(t))Q]
\]
The rate of change of the Wigner function can then be written in the form

\[ \frac{dW}{dt} = \frac{\partial W}{\partial t} + \frac{p}{m} \frac{\partial W}{\partial x} - \frac{\partial W}{\partial p} \frac{\partial E}{\partial x} + ... \]

\[ = \frac{\partial W}{\partial t} + \{W, E\}_{PB} + ... \]

\[ = \frac{\partial W}{\partial t} + \{W, E_{cl}\}_{PB} + (1 - \lambda(t))\{W, Q\}_{PB} + ... \] (25)

where the dots represent the conventional EID terms. When \( \lambda \to 0 \), this includes the effect of the quantum potential \( Q \) and is therefore non-classical. However, in the limit \( \lambda = 1 \) this reduces to the familiar classical equation in terms of the Poisson bracket. Since there is complete equivalence between de Broglie-Bohm theory and standard quantum mechanics for Gibbs ensembles, this completes the demonstration that our theory causes complete decoherence by reducing the Wigner function to its genuine classical form. The de Broglie-Bohm theory turns out to be the most convenient and natural framework for establishing this connection between classical and quantum theory.

It is clear from equations (20) and (25) that the conventional theory of EID and our theory lead to different physical predictions. It should therefore be possible, in principle, to test which of these theories corresponds more closely to nature.

**IV. BOHMIAN AND CLASSICAL TRAJECTORIES**

In this section we will give three examples of how Bohmian trajectories smoothly pass over to classical trajectories on complete decoherence. Let us first note that the general equation of motion for the trajectories that follows from equation (11) is

\[ m\frac{d^2x}{dt^2} = -\nabla[V(x) + (1 - \lambda(t))Q] \] (26)

It is very difficult to solve this equation analytically because of the nonlinearity introduced by the potential \( Q \). We will therefore solve it numerically, and for this purpose will make use of an approximation that satisfies the correct boundary conditions. We will replace \( Q \) in this equation by the pure ‘quantum potential’, i.e., by the expression for \( Q \) in the quantum mechanical limit \( \lambda = 0 \) which is easy to calculate for a number of physically interesting cases. Then, equation (24) will still reduce to the classical equation in the limit \( \lambda = 1 \), and to the de Broglie-Bohm equation in the quantum limit \( \lambda = 0 \).

Before we can actually compute the trajectories, we need a model for the system-environment coupling. We shall assume that the environment acts in a completely random manner on the system so that it ‘decays’ exponentially to the classical system (\( \lambda = 1 \)). The, a convenient choice for the parameter \( \lambda(t) \) is \( \lambda(t) = 1 - \exp(-bt) \) with \( b = 1/\tau_D \).

Let us now consider stationary states of the harmonic oscillator for which the action function is \( S = -E_nt \) with \( E_n = (n + 1/2)\hbar\omega \). Then, the momentum \( p = \partial S/\partial x = 0 \), and the particles in the ensemble are all at rest. (This does not, of course, mean that their measured momenta will be zero. For details, see Holland ([6], Chapter 8.) Figure 1(i) shows the trajectories for the states \( n = 0, 1 \) and 2. The quantum potential in this case is easily calculated from the quantum Hamilton-Jacobi equation to be

\[ Q = (n + 1/2)\hbar\omega - \frac{1}{2}m\omega^2x^2 \] (27)

Equation (24) can then be solved numerically for appropriate values of the parameters \( m, \omega \) and \( b \). (We have used the units \( \hbar = c = 1 \) and taken \( m = 1 \) MeV, \( \omega = 1 \), \( t \) in units of \( 10^{21} \) sec and therefore \( b \) in units of \( 10^{-21} \) sec\(^{-1} \). The trajectories are shown in Figures 1(ii), 1(iii) and 1(iv). It is gratifying to see how the trajectories for the stationary states smoothly go over to the oscillating modes in the classical limit (large \( b \)).
Let us next consider the non-dispersive Gaussian wave packet solution of the quantum harmonic oscillator:

\[
 R(x, t) = \left( \frac{m\omega}{\pi \hbar} \right)^{1/4} e^{-\frac{m\omega}{2\hbar}(x-a \cos \omega t)^2}
\]

\[
 S(x, t) = -\frac{1}{2} \hbar \omega t + \frac{1}{4} m\omega a^2 \sin 2\omega t - m\omega xa \sin \omega t
\]

Its quantum potential is given by

\[
 Q = -\frac{1}{2} m\omega^2 (x - a \cos \omega t)^2 + \frac{1}{2} \hbar \omega
\]

Equation (28) can be numerically integrated in this case also (in the same units as before with \(\sigma_0 \approx 10^{-10} \text{ MeV}^{-1}\)). The quantum orbits (corresponding to \(\lambda = 0\)) are shown in Figure 2(i). Over the ensemble (obtained by taking different initial positions) the particles oscillate over different centre points and cross the classical amplitude \(a\). Note that their trajectories do not cross. The trajectories after decoherence are shown in Figures 2(ii), 2(iii), and 2(iv). They remain within the classical amplitude \(a\) but cross one another, as expected.
Finally, let us consider the one dimensional free wave packet corresponding to eqn. (5). Then,

\[ R(x,t) = (2\pi\sigma^2)^{-1/4} e^{-(x-ut)^2/4\sigma^2} \]

\[ S(x,t) = \frac{\hbar}{2} \tan^{-1} \left( \frac{\hbar t}{2m\sigma_0^2} \right) + mu(x - \frac{1}{2}ut) + (x - ut)^2 \frac{\hbar^2t^2}{8m\sigma_0^2\sigma^2} \]

with \( \sigma^2 = \sigma_0^2[1 + \frac{\hbar^2t^2}{4m^2\sigma_0^4}] \). Such a wave packet spreads in time. Its quantum potential is given by

\[ Q = \frac{\hbar^2}{4m\sigma^2} [1 - (x - ut)^2/2\sigma^2] \]

Again it is possible to solve equation (20) numerically in this case. The quantum mechanical trajectories for a packet at rest (\( u = 0 \)) are shown in Figure 3(i). They are hyperbolic trajectories that do not cross. The packet contracts for \( t < 0 \) and expands for \( t > 0 \). The 'meso' and classical trajectories are shown in Figures 3(ii), 3(iii), and 3(iv).

Figure 4 is a plot of the velocities as functions of time. It is clear from these plots that the accelerations cease and the velocities become constant in the \( t > 0 \) region: the trajectories become parallel and the packet does not spread any more.
V. CONCLUSIONS

The smooth transition between quantum (i.e., Bohmian) and classical trajectories exhibited in the previous section are new results that have interesting physical consequences. The mesoscopic systems implied by the theory proposed in the previous paper and elaborated in this paper are a new prediction in the sense that they are conceptually different from "mesoscopic" states described by quantum mechanics. According to our theory these new systems are neither fully quantum nor fully classical but provide a smooth link between the two. If the new formulation is correct, it should be possible to observe, for example, oscillations of tiny cantilevers and study how these oscillations change as their coupling to the environment is varied, and compare them with the predicted curves (like those in Fig. 2). It should be possible to construct such cantilevers [7]. Other physical mesoscopic systems are under study at present using the new formulation.

If this formulation turns out to be correct, it would imply that the de Broglie-Bohm theory is, in a sense, more fundamental than standard quantum theory. Although these two theories are physically completely equivalent when Gibbs ensembles are used, nevertheless the physical interpretations are radically different. Furthermore, the conventional de Broglie-Bohm theory can be non-ergodic in certain multi-particle cases whereas the corresponding standard quantum mechanical systems are ergodic [8]. This implies that the time average of at least one observable in such systems is different from its space average in the de Broglie-Bohm theory, leading to a direct conflict with standard quantum theory in which the two averages are the same. A realistic two-photon experiment is under preparation at Torino to distinguish between standard quantum mechanics and the de Broglie-Bohm theory. The same experiment would also give a clear verdict on the truth or falsity of the new formulation.

VI. ACKNOWLEDGEMENT

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