Critical current of a superconducting wire via gauge/gravity duality

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We describe application of the gauge/gravity duality to study of thin superconducting wires at finite current. The large number $N$ of colors of the gauge theory is identified with the number of filled transverse channels in the wire. On the gravity side, the physics is described by a system of D3 and D5 branes intersecting over a line. We consider the ground state of the system at fixed electric current and find that at zero temperature the normal state is always unstable with respect to appearance of a superconducting component. We discuss relation of our results to recent experiments on statistics of the switching current in nanowires.

Destruction of superconductivity by a current flowing through the superconductor has long been recognized as an important topic both from the viewpoint of fundamental physics and for applications (for a review, see Bardeen [1]). One expects that theoretical treatment should be the simplest for samples that are effectively one-dimensional (1d), i.e., wires in which the superconducting density depends on only one coordinate. Even for this case, however, a complete theoretical treatment of the transition at fixed current has not been forthcoming. The main difficulty lies, somewhat ironically, in the definition of a current-carrying normal state: the equilibrium Fermi distribution used by the conventional mean-field theory is clearly inadequate for the purpose as it carries no current at all.

A related but different problem is a transition caused not by a fixed current (sustained by an external battery) but by an initial winding of the order parameter. Experimentally, this is the condition appropriate for a thin superconducting ring. In this case, the supercurrent is only metastable: there are fluctuations that lower the free energy [2] (provided there is an amount of disorder or a finite temperature [3]); these have become known as phase slips. The relevant question to ask then is whether there is a maximal winding number density beyond which the metastability becomes classical instability.

To define a current-carrying normal state, one needs to include some mechanism that equilibrates the electrons with respect to momenta. One possible approach is to include weak scattering via a kinetic equation. Another, which we adopt here, is to start at the opposite extreme—a theory with strong electron-electron interactions. One may hope that for such a theory there is a complementary (dual) description in terms of weakly coupled collective modes. If these collective modes are the same as seen on the superconducting side, one will have a unified description applicable to both phases, which should make understanding the phase transition easier.

Recently, following the discovery of the gauge/gravity duality [4], carrying out this program has become practical. The gauge/gravity duality allows one to study a strongly coupled $SU(N)$ gauge theory with a large number $N$ of colors by doing calculations in classical gravity, albeit in a higher-dimensional spacetime. Here we use this method to study a strongly-coupled thin superconductor at fixed current. The number $N$ of colors is taken to correspond to the number of populated transverse modes (channels) in the wire. We find that, at zero temperature, the normal-only state is always unstable with respect to appearance of a superconducting component. At sufficiently large currents, $J > J_c$, the instability occurs only for modes with nonzero winding of the order parameter. We interpret this as an indication that at $J > J_c$ at least a part of the total current must be a supercurrent.

The absence of a depairing transition at large currents is surprising. It may have to do with our system being perfectly momentum-conserving. That, for instance, precludes the Landau process—production of quasiparticles with momenta antiparallel to the flow. The situation may change at a finite temperature, due to the presence of thermal quasiparticles. Indeed, as we discuss towards the end, the idea of a near-critical behavior at some finite current is consistent with the results of recent experiments [5–7], at least at not too low temperatures.

We begin by establishing our convention for assembling electron operators into Dirac spinors. Assuming that superconductivity is due to a correlation between oppositely moving electrons in the same transverse mode, we expect it to show in correlation functions of the operators $\psi_R \psi_L$, where the subscripts $R$ and $L$ designate the electron operators with positive and negative momenta, respectively. For this to be a color singlet, $a_R$ should transform as $N = SU(N)$ and $b_L$ as $\bar{N}$ (or vice versa). Thus, we define a 2-component Dirac spinor $\psi$ as follows (omitting the color index $A$):

$$\psi = \left[\frac{\sum_{k>0}(a_{Rk}e^{ikx} + b_{Lk}^\dagger e^{-ikx})}{\sum_{k<0}(a_{Lk}e^{ikx} + b_{Rk}^\dagger e^{-ikx})}\right]. \tag{1}$$

This is in the representation where the Dirac $\gamma$ matrices are given by $\gamma^0 = \sigma_1$, $\gamma^1 = -i\sigma_2$, and $\gamma^5 = \sigma_3$, in terms of the Pauli matrices $\sigma$. The identification of the superconducting channel as $a_R b_L$ implies that $a_{Rk}^\dagger$ creates a $k > 0$ electron, and $b_{Lk}^\dagger$ a $k < 0$ electron (i.e., $a_{Lk}$ creates a $k < 0$ hole). With this convention, the upper and lower components of $\psi$ have opposite electric charges, the su-
perforating channel is \( \tilde{\psi} \psi \) (where \( \tilde{\psi} = \psi^\dagger \gamma^0 \)), and an external electromagnetic potential couples to the axial current \( \tilde{\psi} e^{\mu_5} \gamma^5 \psi \).

To reproduce the physics of such a superconductor on the gravity side, we consider the system of \( N \) coincident D3 branes and a single D5 brane intersecting over a line. As common in applications of the gauge/gravity duality, in the large \( N \), large 't Hooft coupling limit the D3 branes are replaced by their classical geometry while the D5 is considered as a probe, i.e., its effect on the geometry is neglected. The resulting geometry is that of a throat, of coordinate length \( R \), pulled by the D3s out of the flat 10-dimensional (10d) spacetime:

\[
\begin{align*}
 ds^2 &= \frac{1}{\sqrt{f}}(-dt^2 + dx^2) + \sqrt{f} (d\Delta^2 + \Delta^2 d\phi^2) + \\
 &\quad + \sqrt{f} (d\rho^2 + \rho^2 d\Omega_3^2),
\end{align*}
\]

where

\[
f = 1 + \frac{R^4}{(\Delta^2 + \rho^2)^2},
\]

\( t \) and \( x = (x^1, x^2, x^3) \) are coordinates on the D3 worldvolume, \( \Delta \) and \( \phi \) are polar coordinates in the \( (x^8, x^9) \) plane, \( \rho^2 = (x^4)^2 + \ldots + (x^7)^2 \), and \( d\Omega_3^2 \) is metric on a unit 3-sphere. In what follows, we use dimensionless units in which \( R = 1 \). The D3s are located at \( \Delta = \rho = 0 \), where the metric \( \Omega_3^2 \) has a degenerate horizon. This metric is suitable for calculations at zero temperature, the only case for which we present detailed calculations here.

Matter in the fundamental representation of \( SU(N) \) (in our case, the electrons) is described by strings stretching between the D3s and a probe brane \( [8] \). We consider the case when the probe D5 wraps \( x^1, \rho \) and the 3-sphere. Since \( x \equiv x^1 \) is the only spatial direction shared by the D5 and D3s, this brane intersection describes a theory of electrons that live on a \( (1+1) \)-dimensional defect but interact via a \( (3+1) \)-dimensional non-abelian gauge field. Overall the setup is similar to the system of D3 and D7 branes intersecting over a plane, in which the electrons are confined to move in two spatial dimensions \([9,10]\).

Note that in our case there are two directions, \( x^8 \) and \( x^9 \), orthogonal to all branes. In complex notation, the displacement of the D5 relative to the D3s in the \( (x^8, x^9) \) plane is \( \Delta e^{i\phi} \) and forms an order parameter suitable for description of superconductivity. The minimal distance between the D5 and D3s is the quasiparticle gap (in string units).

For a general D5 embedding,

\[
\begin{align*}
 \Delta &= \Delta(t, x, \rho, \alpha_i), \\
 \phi &= \phi(t, x, \rho, \alpha_i),
\end{align*}
\]

where \( \alpha_i \) are angles on the 3-sphere. Vibrations of the brane correspond to collective modes of the electron fluid. In the 't Hooft limit \( N \to \infty \) at fixed \( \lambda = g_s N \), where \( g_s \) is the closed string coupling), quantum fluctuations of the brane are suppressed by the large value of the brane tension, so to the leading order the brane can be considered as a classical object. One can then explore various embedding ansatze, which will typically have less coordinate dependence than the most general form \([11-13]\). We assume throughout that \( x^2 = x^3 = 0 \). In addition, all embeddings we consider here are independent of \( \alpha_i \). With this restriction, the Dirac-Born-Infeld (DBI) action of the D5 brane is

\[
S_{DBI} = -2\pi^2 T_5 \int dt dx d\rho^3 f^{3/4} |\det(G_{ab} + F_{ab})|^{1/2},
\]

where \( T_5 \) it the brane tension, \( G_{ab} \) for \( a, b = t, x, \rho \) are the components of the induced metric, and \( F_{ab} = \partial_a A_b - \partial_b A_a \) is a \( U(1) \) gauge field (distinct from the usual electromagnetic field) on the D5 worldvolume. The classical dynamics of the brane is described by the Euler-Lagrange (EL) equations following from the total of \( S_{DBI} \) and a Wess-Zumino term that describes the coupling of the D5 to the background Ramond-Ramond field \([11]\).

A spatially uniform winding of the order parameter can be described by (static) embeddings for which \( \phi(x) = qx \), where \( q \) is a constant, while \( \Delta \) depends on \( \rho \) only. The constant \( \partial_\rho \phi \) sources worldvolume electric field \( F_{t\rho} \) through the Wess-Zumino coupling. This leads to a system of coupled equations for \( \Delta \) and \( F_{t\rho} \). We do not consider this case further here and move on to description of states in which there is initially no winding, i.e., all current is carried by the normal component.

We begin with static \( x \)-independent embeddings \( \Delta = \Delta(\rho) \), \( \phi = 0 \), \( A_t = A_t(\rho) \) with all other components of \( A_\mu \) equal to zero. As explained in [12] (in the context of a different D-brane system), the value \( \mu = A_t(\infty) \) is the chemical potential for the density \( \psi^\dagger \psi \). Turning to \([11]\), we see that the \( k > 0 \) and \( k < 0 \) electrons are oppositely charged with respect to \( \mu \). Thus, in the superconductor, \( \mu \) is conjugate to the electric current. The lines of the radial field \( F_{t\rho} \) have nowhere on the brane to end, so the D5 must extend behind the horizon \([12]\). Hence the boundary condition \( \Delta(\rho = 0) = 0 \): at any nonzero current the superconductor is gapless.

The action \([5]\) for this type of embedding is \( S_{DBI} = -\int dt dF \), where \( F \) is the free energy:

\[
F = 2\pi^2 T_5 \int dx d\rho^3 \sqrt{f}(1 + \Delta^2 - F_{t\rho}^2)^{1/2}.
\]

We use notation \( \Delta_{t\rho} = \partial_{t\rho} \Delta \). The Wess-Zumino term vanishes. The equation of motion for \( A_t \) shows that the current

\[
J = \frac{1}{2\pi^2 T_5} \frac{\delta F}{\delta F_{t\rho}}
\]

is \( \rho \)-independent. Following the usual procedure of Legendre transforming to go from fixed \( \mu \) to fixed \( J \), we
Upon substitution \( \Psi(t, x, \rho) = e^{-i\omega t + ikx} \Delta(\rho) \) for the position of the D5 in the \((x^3, x^9)\) plane, the linearized equation for \( \Delta \) reads

\[
\frac{1}{\sqrt{C}} \partial_\rho \left( \sqrt{C} \Delta, \rho \right) + \omega^2 f_0 \Delta - \frac{k^2 \rho^6 f_0^2}{C} \Delta + \frac{2\Delta}{C} + \frac{4kJ}{\rho C} \Delta = 0,
\]

where \( C = \rho^6 f_0 + J^2 \) and \( f_0 = 1 + 1/\rho^4 \); the last term on the left-hand side is from the WZ action. At \( \rho \to \infty \), this reduces to the spherical wave equation in \((3+1)\) dimensions. At \( \rho \to 0 \), and \( \omega \neq 0 \), the leading asymptotics are \( \Delta(\rho) \sim \rho e^{\pm i\omega/\rho} \). We choose the positive sign, corresponding to waves falling into the horizon.

We consider real \( k \) and complex \( \omega \). Unstable modes of the trivial \( \Delta = 0 \) embedding are eigenmodes of (10) with \( \text{Im} \ \omega > 0 \). They decay exponentially at both \( \rho \to 0 \) and \( \rho \to \infty \). For these boundary conditions, \( \omega^2 \) is purely real; hence \( \omega \) is purely imaginary. Let us return for a moment to the spatially uniform case \( k = 0 \). By numerically solving (10) with Dirichlet boundary conditions, we find that, when \( J < J_c \) but close to it, there is only one unstable mode, and its profile closely matches that of the nontrivial static solution to the nonlinear problem (9). This is evidence that the instability of the trivial embedding develops into one of the nontrivial embeddings we considered earlier. At \( J = J_c \), the frequency crosses zero in a smooth, analytic manner, approximately as \( \text{Im} \ \omega = -0.9(J - J_c) \), and the instability disappears.

On the other hand, for \( k \neq 0 \) and \( J > 0 \), the change of variable to \( z = 1/\rho \) brings the small \( \rho \) (large \( z \)) limit of (10) to the form

\[
-\frac{1}{z^2} \partial_z (z^2 \partial_z \Delta) - \frac{\gamma}{z^2} \Delta = \omega^2 \Delta,
\]

where \( \gamma = 4kJ/(k/J)^2 \). This is a Schrödinger equation with a fall-to-center potential. The potential is supercritical (i.e., the full eq. (10) has an infinite number of bound states) when \( \gamma > 1/4 \) [12]. There is always a band of \( k \) satisfying this condition. We conclude that the normal state remains unstable even at \( J > J_c \), but the instability now occurs only in channels with nonzero winding. We interpret this as an indication that in a stable state at \( J > J_c \), at least part of the total current is a supercurrent.

For calculations at a finite temperature, \( T \), the region near \( \Delta = \rho = 0 \) is replaced by a black hole [4]. This cuts off the large \( z \) region in (11), which for fall-to-center problems typically reduces both the number and the growth rates of the unstable modes. It is of interest to consider the possibility that, at sufficiently high \( T \), there will be a depairing transition associated with a near-critical behavior in the vicinity of some \( J_c(T) \). Indeed, experimentally, superconducting nanowires have been observed to switch to the normal state at currents below the estimated depairing current [3, 7]. Large fluctuations of the switching current found in these experiments have been interpreted [3, 7] as a consequence of phase slips. This interpretation is consistent with a second-order (or nearly so) transition at some \( J = J_c(T) \), as it implies that the
free energy barrier suppressing phase slips can be almost completely removed by bringing the current close to the estimated depairing current.

The following estimate suggests that, if the observed fluctuations of the switching current are indeed a result of a near-critical behavior at $J \approx J_c(T)$, that behavior is controlled by a Gaussian (not necessarily stable) fixed point, at least at temperatures where thermally activated phase slips are thought to be the main effect. The exponential factor in the rate of thermal activation is $\exp(-\delta F/T)$, where $\delta F$ is the free energy barrier and $T$ is the temperature. Near $J = J_c$, $\delta F$ scales as the product of the free energy density and the correlation length, i.e., as $(J_c - J)^{\frac{2}{3} - \alpha - \nu}$, in terms of the conventionally defined critical exponents. For the Gaussian point, $\alpha = 0$ and $\nu = \frac{1}{2}$, so $\delta F \sim (J_c - J)^{3/2}$. Curiously, this is the same scaling as obtained for a Josephson junction [14]. It has been found to provide a good fit to the data in Ref. [6] and for the amorphous samples in Ref. [7] (for the crystalline samples [2], the 5/4 power law has been found to be a better fit). In contrast, for a fixed point obeying hyperscaling, $2 - \alpha - \nu = 0$ and $\delta F$ scales to a constant.

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[1] J. Bardeen, Rev. Mod. Phys. 34, 667 (1962).