Convergent Homotopy Analysis Method for Solving Linear Systems

H. Nasabzadeh and F. Toutounian

School of Mathematical Sciences, Ferdowsi University of Mashhad, P.O. Box 1159-91775, Mashhad, Iran

Correspondence should be addressed to H. Nasabzadeh; hnasabzadeh@yahoo.com

Received 16 June 2013; Accepted 22 August 2013

Academic Editor: Ting-Zhu Huang

Copyright © 2013 H. Nasabzadeh and F. Toutounian. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

By using homotopy analysis method (HAM), we introduce an iterative method for solving linear systems. This method (HAM) can be used to accelerate the convergence of the basic iterative methods. We also show that by applying HAM to a divergent iterative scheme, it is possible to construct a convergent homotopy-series solution when the iteration matrix \( G \) of the iterative scheme has particular properties such as being symmetric, having real eigenvalues. Numerical experiments are given to show the efficiency of the new method.

1. Introduction

Computational simulation of scientific and engineering problems often depend on solving linear system of equations. Such systems frequently arise from discrete approximation to partial differential equations. Systems of linear equations can be solved either by direct or by iterative methods. Iterative methods are ideally suited for solving large and sparse systems. For the numerical solution of a large nonsingular linear system,

\[
Au = b,
\]

where \( A \in \mathbb{R}^{n \times n} \) is given, \( b \in \mathbb{R}^n \) is known, and \( u \in \mathbb{R}^n \) is unknown, one class of iterative methods is based on a splitting \((M, N)\) of the matrix \( A \), that is,

\[
A = M - N,
\]

where \( M \) is taken to be invertible and cheap to invert, which mean that a linear system with matrix coefficient \( M \) is much more economical to solve than \( (1) \). Based on \((2)\), \((1)\) can be written in the fixed-point form

\[
u^{(k+1)} = Gu^{(k)} + c, \quad k = 0, 1, 2, \ldots
\]

which yields the following iterative scheme for the solution of

\[
u^{(0)} \in \mathbb{R}^n \text{ is arbitrary.}
\]

A sufficient and necessary condition for \((4)\) to converge to the solution of \((1)\) is \( \rho(G) < 1 \), where \( \rho(G) \) denotes spectral radius. Some effective splitting iterative methods and preconditioning methods were presented for solving the linear system of \((1)\), see [1–9]. Recently, Keramati [10], Yusufo˘glu [11], and Liu [12] applied the homotopy perturbation method to obtain the solution of linear systems and deduced the conditions for checking the convergence of homotopy series. In this work, we show how the homotopy analysis method may be regarded as an acceleration procedure based on the iterative method \((4)\). We observe that it is not necessary that the basic method \((4)\) be convergent. When \( \rho(G) > 1 \), it is sufficient that the eigenvalues \( \lambda_i = \text{Re}(\lambda_i) + i \text{Im}(\lambda_i) \), \( i = 1, \ldots, n \), of iteration matrix \( G \) satisfy the relation \( \text{Re}(\lambda_i) < 1 \), \( i = 1, \ldots, n \) (or \( \text{Re}(\lambda_i) > 1 \), \( i = 1, \ldots, n \)). When \( \rho(G) < 1 \), by applying the homotopy analysis method to the basic iterative method \((4)\), we can improve the rate of convergence of the iterative method \((4)\). This paper is organized as follows. In Section 2, we introduce the basic concept of HAM, derive the
conditions for convergence of the homotopy series, and apply
the homotopy analysis method to the Jacobi, Richardson,
SSOR, and SAOR methods. In Section 3, some numerical
examples are presented to show the efficiency of the method.
Finally, we make some concluding remarks in Section 4.

2. Basic Idea of HAM

The homotopy analysis method (HAM) [13, 14] was first
proposed by S. J. Liao in 1992. The HAM was further
developed and improved by S. Liao for nonlinear problem in
[15].

Here, we apply the homotopy analysis method (HAM)
to the problem (3) for finding the solution of (1) when
\( \det(A) \neq 0 \). Consider (3), where \( u \) is unknown vector of
(1) and \( G \) is the iteration matrix of an iterative method.
Let \( u_0 \) denote an initial guess of exact solution \( u \), \( h \neq 0 \)
an an initial guess of exact solution \( u \). Therefore,
\[
\nu = u_0 + \sum_{i=1}^{\infty} p^i c_i u_1.
\] (15)

By setting \( p = 1 \), we obtain
\[
\nu = u_0 + \sum_{i=1}^{\infty} G_h^{-1} u_1.
\] (16)

It is obvious that if \( \rho(G_h) < 1 \), then the series, \( \sum_{i=1}^{\infty} G_h^{-1} u_1 \),
converges and we have
\[
\nu = u_0 + (I - G_h)^{-1} u_1
\] (17)
\[
= (I - G)^{-1} c,
\]

which is the exact solution of (3). A series of vectors can be
computed by (14), and our aim is to choose the convergence
control parameter \( h \neq 0 \) so that \( \rho(G_h) < 1 \). For improving
the rate of convergence of iterative method, we present the
following theorem.

**Theorem 1.** Suppose that \( \rho(G) < 1 \), and let \( \lambda_j = \text{Re}(\lambda_j) +
\text{i} \text{Im}(\lambda_j), \) and \( \mu_i, i = 1, 2, \ldots, n \) be the eigenvalues of \( G \)
and \( G_h \), respectively. Let \( \alpha_i = (1 - \text{Re}(\lambda_j))^2 + \text{Im}(\lambda_j)^2 \), \( \beta_i = 2(1 -
\text{Re}(\lambda_j)) \), and let \( g_i(h) = |\mu_i|^2 - \rho(G)^2 = \alpha_i h^2 + \beta_i h + 1 - \rho(G)^2 \),
\( i = 1, 2, \ldots, n \). If \( \alpha_i \neq 0 \) and \( \beta_i - 2 \alpha_i \neq 0, i = 1, 2, \ldots, n, \) then

(i) the quadratic equation \( g_i(h) = 0, i = 1, 2, \ldots, n, \) has
simple real roots \( y_1^{(i)} < y_2^{(i)} < 0 \).

(ii) \( h = -1 \) belongs to the interval \( \max_{i=1}^{n} y_1^{(i)}, \min_{i=1}^{n} y_2^{(i)} \]
and \( \rho(G_{-1}) < \rho(G) < 1, \)

(iii) for each \( h \in [\max_{i=1}^{n} y_1^{(i)}, \min_{i=1}^{n} y_2^{(i)}] \) and \( h \neq -1, \)
the relation \( \rho(G_h) < \rho(G) < 1 \) holds.

**Proof.** (i) We begin by defining two index sets \( N_1 = \{ i \mid |\lambda_i| =
\rho(G) \} \) and \( N_2 = \{ i \mid |\lambda_i| \neq \rho(G) \}. \) Since \( \alpha_i > 0 \) and \( \rho(G) < 1, \)
for \( i \in N_2, \) we have
\[
\text{sign} g_i(-\infty) = 1, \quad g_i(-1) < 0, \quad g_i(0) > 0. \] (18)

So, \( g_i(h), i \in N_2, \) has simple real roots
\[
y_1^{(i)} < -1, \quad -1 < y_2^{(i)} < 0. \] (19)

For \( i \in N_1, \) we have \( \alpha_i - \beta_i < 0 \) and
\[
g_i(-1) = 0, \quad g_i \left( \frac{\alpha_i - \beta_i}{\alpha_i} \right) = 0. \] (20)
in which the iteration matrix \( G \) has the eigenvalues with the desired properties.

**Theorem 3.** Let \( \lambda_i = \text{Re}(\lambda_i) + i\text{Im}(\lambda_i), \) and \( \gamma_i = \text{Re}(\gamma_i) + i\text{Im}(\gamma_i), \) \( i = 1, 2, \ldots, n, \) be the eigenvalues of \( G \) and \( G_0, \) respectively. If \( (1 - \text{Re}(\lambda_i))^2 - (\text{Im}(\lambda_i))^2 > 0 \) (or \( (1 - \text{Re}(\lambda_i))^2 - (\text{Im}(\lambda_i))^2 < 0 \)) for \( i = 1, 2, \ldots, n, \) then \( \text{Re}(\gamma_i) > 1 \) (or \( \text{Re}(\gamma_i) < 1 \)) for \( i = 1, 2, \ldots, n. \)

**Proof.** The proof immediately follows from the fact that \( \text{Re}(\gamma_i) = (1 - \text{Re}(\lambda_i))^2 - (\text{Im}(\lambda_i))^2 + 1. \)

The following corollary shows that by using the modified linear equation (23) and the homotopy analysis method with the corresponding \( G_h = (h + 1)I - hG, \) we can construct a convergent homotopy series for linear system (1).

**Corollary 4.** If \( G \) has only real eigenvalues, then there exists \( h \neq 0 \) such that the series of vectors generated by

\[
\begin{align*}
 u_1 &= h \left[ (I - G) u_0 - \bar{z} \right], \\
 u_i &= \bar{G}_h u_{i-1}, \quad i = 2, 3, \ldots, \\
&\text{converges to the exact solution of (1).}
\end{align*}
\]

**Proof.** The proof immediately follows from Theorems 2 and 3.

This corollary establishes that the series of vectors generated by (24) always converges if the iteration matrix \( G \) is a symmetric matrix. When \( A \) is symmetric with diagonal elements positive real numbers, (1) can be written as follows:

\[
(D^{-1/2}AD^{-1/2})(D^{1/2}x) = (D^{-1/2}b),
\]

where \( D \) is the diagonal of \( A. \) Denoting again by \( A, x, \) and \( b \) the expressions \( (D^{-1/2}AD^{-1/2}), (D^{1/2}x), \) and \( (D^{-1/2}b), \) respectively, it is obvious the new coefficient \( A \) is still symmetric and can therefore be written in the form \( A = I - L - L^T. \) An immediate consequence of Corollary 4 and the above discussion is the following results.

(i) The series of vectors generated by (24) converges when \( A \) is a symmetric matrix and the iterative method is the Richardson method

\[
 u^{(k+1)} = (I - A) u^{(k)} + b.
\]

(ii) The series of vectors generated by (24) converges when \( A = I - L - U \) is a symmetric matrix and the iterative method is the Jacobi method

\[
 u^{(k+1)} = (L + U) u^{(k)} + b.
\]

(iii) If \( A = I - L - U \) is a symmetric matrix and the iterative method is SAOR method

\[
 u^{(k+1)} = J_{ra} u^{(k)} + c
\]
**Table 2:** The basic method is convergent (Theorem 1).

| Matrix    | Method       | \( \rho(G) \)   | Convergence interval | \( h_{opt} \) | \( \rho(G_{opt}) \) |
|-----------|--------------|-----------------|----------------------|--------------|---------------------|
| pde225    | SOR \((w = r = 1)\) | 0.9776          | \((-1, -0.0743)\)    | -0.7423      | 0.7768              |
| pde225    | AOR \((r = 1, w = 0.8)\) | 0.8053          | \((-1, -0.8096)\)    | -0.9400      | 0.7739              |
| pde225    | SSOR \((w = r = 0.5)\) | 0.8048          | \((-1.4441, -1)\)    | -1.2941      | 0.7474              |
| pde225    | SAOR \((r = 0.5, w = 1)\) | 0.7773          | \((-1, -0.6038)\)    | -0.8920      | 0.6673              |
| cage5     | SOR \((w = r = 1)\) | 0.3388          | \((-1.2814, -1)\)    | -1.1591      | 0.2314              |
| cage5     | AOR \((r = 1.2, w = 1.9)\) | 0.9240          | \((-1, -0.0589)\)    | -0.68        | 0.3355              |
| cage5     | SSOR \((w = r = 0.5)\) | 0.6427          | \((-1.7235, -1)\)    | -1.5235      | 0.4557              |
| cage5     | SAOR \((r = 0.1, w = 0.7)\) | 0.5590          | \((-1.5609, -1)\)    | -1.3756      | 0.3890              |
| pivtol    | SOR \((w = r = 0.1)\) | 0.9487          | \((-3.9226, -1)\)    | -2.7526      | 0.8588              |
| pivtol    | AOR \((r = 0.3, w = 0.6)\) | 0.9958          | \((-1, -0.0128)\)    | -0.7300      | 0.7626              |
| pivtol    | SSOR \((w = r = 0.2)\) | 0.8005          | \((-1.4850, -1)\)    | -1.3450      | 0.7317              |
| pivtol    | SAOR \((r = 0.1, w = 0.4)\) | 0.9199          | \((-1, -0.2198)\)    | -0.83        | 0.6943              |
| bwfb782   | SOR \((w = r = 1)\) | 0.3732          | \((-1.2270, -1)\)    | -1.1470      | 0.3024              |
| bwfb782   | AOR \((r = 1.2, w = 1.9)\) | 0.9942          | \((-1, -0.0045)\)    | -0.65        | 0.3988              |
| bwfb782   | SSOR \((w = r = 0.5)\) | 0.5984          | \((-1.6665, -1)\)    | -1.4704      | 0.4103              |
| bwfb782   | SAOR \((r = 0.1, w = 0.7)\) | 0.5123          | \((-1.5123, -1)\)    | -1.3423      | 0.3453              |
| bwfb782   | Richardson | 0.999998        | \((-8.5201 \times 10^4, -1)\) | -8.0701 \times 10^4 | 0.8952              |

**Table 3:** The basic method is divergent (Theorem 2).

| Matrix    | Method       | \( \rho(G) \)   | Convergence interval | \( h_{opt} \) | \( \rho(G_{opt}) \) |
|-----------|--------------|-----------------|----------------------|--------------|---------------------|
| pde225    | SOR \((r = \omega = 1.5)\) | 2.4937          | \((-0.4318, 0)\)     | -0.3         | 0.7980              |
| pde225    | AOR \((r = 1, \omega = 1.5)\) | 1.5542          | \((-0.6816, 0)\)     | -0.5         | 0.7745              |
| pde225    | SSOR \((r = 2.01, \omega = 2.01)\) | 1.1112          | \((0, 17.9804)\)    | 12.0540      | 0.9388              |
| pde225    | SAOR \((r = 2, \omega = 3)\) | 4.9844          | \((0, 0.3408)\)     | 0.2510       | 0.7741              |
| cage5     | SOR \((r = \omega = 2)\) | 1.0150          | \((-0.9872, 0)\)     | -0.4         | 0.7425              |
| cage5     | AOR \((r = 1.2, \omega = 3)\) | 2.0370          | \((-0.6584, 0)\)     | -0.43        | 0.3355              |
| cage5     | SSOR \((r = 2.2, \omega = 2.2)\) | 1.8715          | \((0, 2.2948)\)     | 2.0280       | 0.7681              |
| cage5     | SAOR \((r = 2, \omega = 3)\) | 4.0002          | \((0, 0.6666)\)     | 0.458        | 0.3748              |
| pivtol    | SOR \((r = \omega = 1)\) | 4.3002          | \((-0.3773, 0)\)     | -0.33        | 0.7778              |
| pivtol    | AOR \((r = 0.467, \omega = 0.981)\) | 1.5280          | \((-0.7566, 0)\)    | -0.58        | 0.6766              |
| pivtol    | SSOR \((r = \omega = 0.5)\) | 2.1132          | \((-0.6424, 0)\)     | -0.55        | 0.7306              |
| pivtol    | SAOR \((r = 0.1, \omega = 0.7)\) | 2.6670          | \((-0.5454, 0)\)    | -0.47        | 0.7235              |
| bwfb782   | Jacobi | 1.0554          | \((-0.9731, 0)\)     | -0.81305     | 0.6794              |
| bwfb782   | AOR \((r = 1.2, \omega = 3)\) | 2.1398          | \((-0.6352, 0)\)    | -0.4         | 0.3996              |
| bwfb782   | SSOR \((\omega = r = 2.2)\) | 2.1411          | \((0, 1.7528)\)     | 1.5550       | 0.7748              |
| bwfb782   | SAOR \((r = 2, \omega = 3)\) | 4.0000          | \((0, 0.6667)\)     | 0.45         | 0.3570              |

**Table 4:** The basic method is divergent (Theorem 3).

| Matrix    | Method       | \( \rho(G) \)   | Convergence interval | \( h_{opt} \) | \( \rho(G_{opt}) \) |
|-----------|--------------|-----------------|----------------------|--------------|---------------------|
| Si2       | SOR \((w = r = 1.3)\) | 1.0993          | \((0, 0.8415)\)     | 0.8          | 0.9974              |
| Si2       | AOR \((r = 1.5, w = 0.75)\) | 1.0799          | \((0, 2.7964)\)     | 2.7          | 0.9942              |
| Si2       | SSOR \((w = r = 1.8)\) | 1.24945         | \((0, 2.0007)\)     | 1.9          | 0.9909              |
| Si2       | SAOR \((r = 1.5, w = 0.75)\) | 1.1380          | \((0, 2.0949)\)     | 2            | 0.9916              |
| Si2       | Jacobi | 1.3233          | \((0, 0.3705)\)     | 0.7          | 0.9999              |
| Si2       | Richardson | 40.3813         | \((0, 0.0012)\)     | 0.0011       | 0.9999              |
with
\[ J_{r,w} = (I - rU)^{-1} [(1 - \omega)I + (\omega - r)L + \omega U], \]
\[ c = \omega (I - rU)^{-1} [(2 - \omega)I - (\omega - r)(L + U)] (I - rL)^{-1} b, \]
then the series of vectors generated by (24) converges if
\[ \omega \in (0, 2), \quad \omega + \frac{2 - \omega}{\mu_{\text{min}}} < r < \omega + \frac{2 - \omega}{\mu_{\text{max}}}, \]
where \( \mu_{\text{min}} \) and \( \mu_{\text{max}} \) stand for the minimum and the maximum eigenvalues of \( B = L + U \), respectively.

(iv) If \( A = I - L - U \) is a symmetric matrix and the iterative method is SSOR method, then the series of vectors generated by (24) converges if \( \omega \in (0, 2) \).

3. Numerical Examples

For numerical comparison, we use some matrices from the University of Florida sparse matrix Collection [20]. These matrices with their properties are shown in Table 1. We determined the spectral radii of iteration matrices of the classical SOR, AOR, SSOR, SAOR, Jacobi, and Richardson methods (\( \rho(G) \)) as well as those of the corresponding \( G_h \) after the application of HAM method with the experimentally computed optimal value of \( h \) (\( \rho(G_{h_{\text{opt}}}) \)). In Tables 2–4, we list \( \rho(G), \rho(G_{h_{\text{opt}}}) \), the interval of convergence which introduced in Theorems 1 and 2, and the experimentally computed optimal value of \( h(h_{\text{opt}}) \), and the spectral radius of iteration matrix \( \rho(G_{h_{\text{opt}}}) \) which introduced in Corollary 4.

In Table 2, we consider the convergent classical methods (\( \rho(G) < 1 \)). It is easy to verify that the numerical results are consistent with Theorem 1, and we observe that for the convergent classical methods by choosing suitable convergence control parameter \( h \), the rate of the convergence of the HAM method is faster than that of corresponding classical method.

In Table 3, we consider the divergent classical methods when \( \text{Re}(\lambda_i) > 1 \) (or \( \text{Re}(\lambda_i) < 1 \)) for \( i = 1, 2, \ldots, n \). We can see that the numerical results are consistent with Theorem 2. These results show that by applying the HAM to a divergent scheme which is divergent, it is possible to construct a convergent homotopy-series vectors when the iteration matrix \( G \) has mentioned properties.

In Table 4, we report the results obtained for the symmetric matrix Si2 which has positive diagonal elements. For this example, the classical methods diverge and there exist \( i, j \) such that \( \text{Re}(\lambda_i) < 1 \) and \( \text{Re}(\lambda_j) > 1 \). We observe that the results are consistent with Theorem 3 and Corollary 4. The results show that the HAM method is convergent but the rate of the convergence is slow.

Finally, Tables 3 and 4 show that it is not necessary to choose the \( \rho \) the parameters \( r \) and \( \omega \) in the convergence interval of the classical methods. In the case of divergence, under the assumptions of Theorems 2 and 3, the application of the HAM can generate the convergent homotopy-series vectors for linear system (1).

4. Conclusion

In this paper, we proposed to apply the homotopy analysis method to the classical iterative methods for solving the linear system of equations. The theoretical results show the HAM can be used to accelerate the convergence of the basic iterative methods. In addition, we show that by applying the HAM to a divergent iterative scheme, it is possible to construct a convergent homotopy-series solution when the iteration matrix \( G \) of the iterative scheme has particular properties. The numerical experiments confirm the theoretical results and show the efficiency of the new method.

References

[1] R. S. Varga, Matrix Iterative Analysis, Prentice-Hall, Englewood Cliffs, NJ, USA, 1962.

[2] D. M. Young, Iterative Solution of Large Linear Systems, Academic Press, New York, NY, USA, 1971.

[3] L. A. Hageman and D. M. Young, Applied Iterative Methods, Academic Press, New York, NY, USA, 1981, Computer Science and Applied Mathematics.

[4] Y.-T. Li, C.-X. Li, and S.-L. Wu, “Improvements of preconditioned AOR iterative method for \( L \)-matrices,” Journal of Computational and Applied Mathematics, vol. 206, no. 2, pp. 656–665, 2007.

[5] H. Wang and Y.-T. Li, “A new preconditioned AOR iterative method for \( L \)-matrices,” Journal of Computational and Applied Mathematics, vol. 229, no. 1, pp. 47–53, 2009.

[6] L. Wang and Y. Song, “Preconditioned AOR iterative methods for \( M \)-matrices,” Journal of Computational and Applied Mathematics, vol. 226, no. 1, pp. 114–124, 2009.

[7] J. H. Yun, “A note on preconditioned AOR method for \( L \)-matrices,” Journal of Computational and Applied Mathematics, vol. 220, no. 1-2, pp. 13–16, 2008.

[8] X. Zhang and T.-Z. Huang, “Modified iterative methods for nonnegative matrices and \( M \)-matrices linear systems,” Computers & Mathematics with Applications, vol. 50, no. 10–12, pp. 1587–1602, 2005.

[9] T.-Z. Huang, X.-Z. Wang, and Y.-D. Fu, “Improving Jacobi methods for nonnegative \( H \)-matrices linear systems,” Applied Mathematics and Computation, vol. 186, no. 2, pp. 1542–1550, 2007.

[10] B. Keramati, “An approach to the solution of linear system of equations by He’s homotopy perturbation method,” Chaos, Solitons & Fractals, vol. 41, no. 1, pp. 152–156, 2009.

[11] E. Yusufoglu, “An improvement to homotopy perturbation method for solving system of linear equations,” Computers & Mathematics with Applications, vol. 58, no. 11-12, pp. 2231–2235, 2009.
[12] H.-K. Liu, "Application of homotopy perturbation methods for solving systems of linear equations," Applied Mathematics and Computation, vol. 217, no. 12, pp. 5259–5264, 2011.

[13] S. Liao, Beyond Perturbation: Introduction to the Homotopy Analysis Method, vol. 2 of Modern Mechanics and Mathematics, Chapman & Hall/CRC Press, Boca Raton, Fla, USA, 2003.

[14] S. J. Liao, The proposed homotopy analysis technique for the solution of nonlinear problem [Ph.D. thesis], Shanghai Jiao Tong University, Shanghai, China, 1992.

[15] S. Liao, “On the homotopy analysis method for nonlinear problems,” Applied Mathematics and Computation, vol. 147, no. 2, pp. 499–513, 2004.

[16] J.-H. He, “Homotopy perturbation technique,” Computer Methods in Applied Mechanics and Engineering, vol. 178, no. 3-4, pp. 257–262, 1999.

[17] J.-H. He, “A coupling method of a homotopy technique and a perturbation technique for non-linear problems,” International Journal of Non-Linear Mechanics, vol. 35, no. 1, pp. 37–43, 2000.

[18] J.-H. He, “Homotopy perturbation method: a new nonlinear analytical technique,” Applied Mathematics and Computation, vol. 135, no. 1, pp. 73–79, 2003.

[19] A. Hadjidimos and A. Yeyios, "Symmetric accelerated over-relaxation (SAOR) method," Mathematics and Computers in Simulation, vol. 24, no. 1, pp. 72–76, 1982.

[20] T. A. Davis and Y. Hu, "The University of Florida sparse matrix collection," Association for Computing Machinery, vol. 38, no. 1, article 1, 2011.
