Anderson localization induced by complex potential

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Uncorrelated disorder potential in one-dimensional lattice definitely induces Anderson localization, while quasiperiodic potential can lead to both localized and extended phases, depending on the potential strength. We investigate the Anderson localization in one-dimensional lattice with non-Hermitian complex disorder and quasiperiodic potential. We present a non-Hermitian SSH chain to demonstrate that a nonzero imaginary disorder on-site potential can induce the standard Anderson localization. We also show that the non-Hermitian Aubry-Andre (AA) model exhibits a transition in parameter space, which separates the localization and delocalization phases and is determined by the self-duality of the model. It indicates that a pure imaginary quasiperiodic potential takes the same role as a real quasiperiodic potential in the transition point between localization and delocalization. Remarkably, a system with complex quasiperiodic potential exhibits interference-like pattern on the transition points.

I. INTRODUCTION

The localization phase in quantum systems, which is originally rooted in condensed matter, has recently attracted a lot of theoretical and experimental interest in a variety of fields, including light waves in optical random media, matter waves in optical potentials, sound waves in elastic media, and quantum chaotic systems, since the localization of quantum particles could prevent the transport necessary for equilibration in isolated systems. Anderson localization predicts that single-particle wave functions become localized in the presence of some uncorrelated disorder, leading to a metal-insulator transition caused by the quantum interference in the scattering processes of a particle with random impurities and defects. A conventional Anderson localization is not controllable in one and two-dimensional systems. Nevertheless, localization does not require disorder and fortunately, the Aubry-Andre model, which has quasiperiodic potential, exhibits a transition between a localized and extended phases. In practice, quasiperiodic potential arises naturally in optical experiments using lasers with incommensurate wave vectors. Accordingly, many experiments in such systems have now observed single-particle localization. While any optical system that includes gain or loss is non-Hermitian by nature, it has been shown that the presence of the imaginary potential causes many surprising effects.

Motivated by the recent development of non-Hermitian quantum mechanics, both in theoretical and experimental aspects, we investigate the localization transitions in non-Hermitian regime. Several pioneer works have been devoted to investigate the tight-binding system with non-Hermitian PT-symmetric quasiperiodic potential. The extension from real potential to a complex one raises the question of whether the real and imaginary part of complex potential has correlated effect on the Anderson localization. Especially, since the quasiperiodic potential does possess an intrinsic phase, the phase difference of real and imaginary quasiperiodic potential may influence the Anderson localization transition. This is the main purpose of the present work. We investigate the Anderson localization in one-dimensional lattice with non-Hermitian complex disorder and quasiperiodic potential. We present a non-Hermitian SSH chain to demonstrate that a nonzero imaginary disorder on-site potential can induce the standard Anderson localization. We also show that the non-Hermitian Aubry-Andre (AA) model exhibits a transition in parameter space, which separates the localization and delocalization phases and is determined by the self-duality of the model. It indicates that a pure imaginary quasiperiodic potential takes the same role as a real quasiperiodic potential in the transition point between localization and delocalization. Remarkably, a system with complex quasiperiodic potential exhibits interference-like pattern on the transition point, i.e., the phase difference between real and imaginary quasiperiodic potential determines the boundary of transition. Our approach opens a new way to investigate the interplay of localization and gain/loss in non-Hermitian system.

This paper is organized as follows. In Sec. II we present a non-Hermitian SSH chain with disorder staggered balanced gain and loss. We map this model to an equivalent Hermitian one and show the existence of Anderson localization. In Sec. III we show exactly that a non-Hermitian AA Hamiltonian with imaginary quasiperiodic potential possesses a self-duality and manifests a localization transition at the self-dual point. In Sec. IV we demonstrate that the real and imaginary quasiperiodic potential has interference effect on the Anderson localization transition. Finally, we give a summary in Section V.

II. LOCALIZATION IN NON-HERMITIAN SSH CHAIN

We consider a non-Hermitian SSH chain with disorder staggered balanced gain and loss. The simplest tight-
binding model with these features is
\[ H_{\text{SSH}} = (1 + \delta) \sum_{j=1}^{N} a_j^\dagger b_j + (1 - \delta) \sum_{j=1}^{N-1} b_j^\dagger a_{j+1} + \text{H.c.} + i \sum_{j=1}^{N} \gamma_j (a_j^\dagger a_j - b_j^\dagger b_j), \]

where \( \delta \) and \( \gamma_j \), are the distortion factor with unit tunneling constant and the alternating imaginary potential magnitude at dimmer \( j \), respectively. In this work, we focus on the weak limit of imaginary potential \( |\gamma_j| \ll 1 \).

Here \( a_j^\dagger \) and \( b_j^\dagger \) are the creation operator of the particle at the \( l \)th site in \( A \) and \( B \) sub-lattices. The particle can be fermion or boson, depending on their own commutation relations. (A sketch of the lattice has been shown in Fig. [1]). In the following, we will show that a nonzero disorder staggered balanced gain and loss can lead to standard Anderson localization.

\[
\begin{align*}
\alpha_j & = 1 + \frac{\gamma_j}{(1 + \delta)^2 - \gamma_j^2}, \\
\beta_j & = 1 - \frac{\gamma_j}{(1 + \delta)^2 - \gamma_j^2}, \\
\alpha_{j+1} & = 1 + \frac{\gamma_j}{(1 + \delta)^2 - \gamma_j^2}, \\
\beta_{j+1} & = 1 - \frac{\gamma_j}{(1 + \delta)^2 - \gamma_j^2},
\end{align*}
\]

(4)

by introducing particle operators\(^{42,43}\)

\[
\begin{align*}
\alpha_j & = \frac{a_j^\dagger + e^{-i\varphi_j} b_j}{1 + e^{-i\varphi_j}}, \\
\beta_j & = \frac{a_j + e^{i\varphi_j} b_j}{1 + e^{i\varphi_j}},
\end{align*}
\]

with

\[
\tan \varphi_j = \frac{\gamma_j}{\sqrt{(1 + \delta)^2 - \gamma_j^2}}.
\]

Based on the identity

\[
\begin{align*}
(1 + \delta)(a_j^\dagger b_j + b_j^\dagger a_j) + i\gamma_j (a_j^\dagger a_j - b_j^\dagger b_j) \\
= \sqrt{(1 + \delta)^2 - \gamma_j^2} (\bar{\alpha}_j \alpha_j - \bar{\beta}_j \beta_j),
\end{align*}
\]

and the condition

\[
1 + \delta \gg 1 - |\gamma_j| \ll 1,
\]

which lead to \( \cos \varphi_j \approx e^{i\varphi_j} \approx 1 \), one can neglect the transition terms between sites with opposite potential, and get the approximate expression

\[
H_{\text{SSH}} \approx \frac{1}{2} (1 - \delta) \sum_{j=1}^{N-1} (\bar{\alpha}_j \alpha_{j+1} - \bar{\beta}_j \beta_{j+1}) + \text{H.c.} + \sum_{j=1}^{N} V_j (\bar{\alpha}_j \alpha_j - \bar{\beta}_j \beta_j).
\]

(8)

The original system reduces to two independent uniform chains with opposite real on-site random potential \( V_j = \sqrt{(1 + \delta)^2 - \gamma_j^2} \). Hamiltonian \( H_{\text{SSH}} \) is diagonalizable since operators \( (\alpha_j, \beta_j, \bar{\alpha}_j, \bar{\beta}_j) \) satisfy the canonical commutation relations. It indicates that \( H_{\text{SSH}} \) can be regarded as a Hermitian model in the context of biorthogonal inner product. On the other hand, we have shown that if a state

\[ |\psi\rangle = \sum_j (A_j \alpha_j^\dagger |0\rangle + B_j \beta_j^\dagger |0\rangle), \]

with \( A_j B_j = 0 \) for all \( j \), i.e., it has only single-chain component, \( \{ \alpha_j^\dagger |0\rangle \} \) or \( \{ \beta_j^\dagger |0\rangle \} \), the dynamics of \( |\psi\rangle \) is the same as that in a Hermitian chain, exhibiting the standard Anderson localization. Furthermore, the mapping between the Hermitian and non-Hermitian Hamiltonian matrices (i.e., the similar matrix) is a local transformation, which cannot result in localization transition. In conclusion, we provide an example to demonstrate that a nonzero imaginary disorder on-site potential can induce the Anderson localization.

Numerical simulation is performed to demonstrate our conclusion. For simplicity, we consider a uniform chain with disorder potential. The Hamiltonian takes the form

\[ H_{\text{And}} = \sum_l a_l^\dagger a_{l+1} + \text{H.c.} + \sum_l \mu a_l^\dagger a_l, \]

(10)
where the potential is simulated by a sequence of random complex variables. According to the theory of Anderson localization, all the eigenstates are localized for any nonzero real random variables. On the other hand, our analysis indicates that imaginary random variables may also induce localized states. We perform the simulation for two cases, (i) \( \mu_1 = \text{ran}(-b,b) e^{i\phi} \), with \( \phi = 0, \pi/4, \) and \( \pi/2 \), (ii) \( \mu_1 = \mu_0 e^{i\text{ran}(-\pi,\pi)} \). Here \( \text{ran}(-b,b) \) denotes a uniform random number within \( (-b,b) \). In case (i) with \( \phi = 0 \), it corresponds to real disorder potential, which is in the framework of Anderson localization. In case (i) with \( \phi = \pi/2 \), it corresponds to imaginary disorder potential, which is the situation of our prediction. In case (i) with \( \phi = \pi/4 \), the real and imaginary disorder potential has the same amplitude. In case (ii), the real and imaginary part of potential is out of phase but keep an identical norm.

To characterize the localization nature, we employ the inverse participation ratio (IPR), as a criterion to distinguish the extended states from the localized ones, which is defined as:

\[
\text{IPR}^{(n)} = \frac{\sum_i |\langle \psi_n | l \rangle|^4}{\left( \sum_i |\langle \psi_n | l \rangle|^2 \right)^2}. \tag{11}
\]

Meanwhile, the average IPR (AIPR) for all energy levels is defined as the form of \( \text{AIPR} = \frac{1}{N} \sum_{n=1}^{N} \text{IPR}^{(n)} \). For spatially extended states, it approaches to zero, whereas it is finite for localized states. For a non-Hermitian system we introduce the quantity, dressed energy

\[
\tilde{\varepsilon}_n = \text{sgn}(|\text{real}(\varepsilon_n)|) |\varepsilon_n| \tag{12}
\]
to specifies the energy in non-Hermitian regime. We plot the result in Fig. 2, which show that the complex random potential of several types can induce Anderson localization as our prediction.

### III. SELF-DUALITY UNDER IMAGINARY POTENTIAL

Consider a uniform chain with imaginary quasiperiodic potential, which has the form

\[
H_{\text{Im}} = \sum_{l=1}^{N} a_l^\dagger a_{l+1} + \text{H.c.} + 2\gamma \sum_{l=1}^{N} \cos(ql) a_l^\dagger a_l, \tag{13}
\]

which is an extension from a standard AA model to a non-Hermitian version by extending \( \gamma \) to \( i\gamma \), and \( q = 2\pi \beta \), \( \beta \) determines the degree of the quasiperiodicity. The simplicity of \( H_{\text{Im}} \) allows for exact theoretical statements in certain cases. For example, the 1D disordered Anderson model \( H_{\text{Ana}} \) with real potential allows for only localized eigenstates at all energies independent of how weak the disorder may be. The standard AA model with the 1D incommensurate potential has either all eigenstates external or localized depending on the strength of the potential. It is due to the self-duality of the Hamiltonian, which concludes that an AA Hamiltonian with irrational \( \gamma \) manifests a localization transition at the self-dual point

\[
|\gamma| = 1, \tag{14}
\]
i.e., all eigenstates are extended (localized) when \( |\gamma| < 1 \) (\( |\gamma| > 1 \)). In the following, it is readily to show that the same thing happens exactly for a non-Hermitian AA model.

Taking the Fourier transformation

\[
a_l = \frac{1}{\sqrt{N}} \sum_n (-1)^{l} e^{iqn l} a_n, \tag{15}
\]

we can obtain

\[-iH_{\text{Im}} = \gamma \sum_{n} (a_n^\dagger a_{n+1} + \text{H.c.}) + 2\gamma \sum_{n} \cos(qn) a_n^\dagger a_n. \tag{16}\]

Obviously, \( H_{\text{Im}} \) and \(-iH_{\text{Im}} \) has the identical complete set of eigenstates and have self-duality with the self-dual point \( \gamma = 1 \). It indicates that the imaginary quasiperiodic potential takes the same role leading to localization transition. Numerical simulations are given in Fig. 3 and Fig. 4 to demonstrate our conclusion about the self-duality under the imaginary potential model.

### IV. INTERFERENCE EFFECT ON LOCALIZATION TRANSITION

We have seen that a real or imaginary quasiperiodic potential has the same effect on the localization when they show up individually. A natural question is what happens when the quasiperiodic potential is complex. There are many variety of complex potential. In this work, we consider the complex potential with identical frequency

\[
H_{\text{Comp}} = \sum_{j=1}^{N} (a_j^\dagger a_{j+1} + \text{H.c.}) + 2V \sum_{j=1}^{N} \cos(2\pi \beta j) a_j^\dagger a_j \\
+ 2\gamma \sum_{j=1}^{N} \cos(2\pi \beta j + \Phi) a_j^\dagger a_j, \tag{17}
\]

but different phase difference \( \Phi \). In the following, we aim at the effect of \( \Phi \) on the localization transition point. We consider the form of \( \Phi \) in two simple ways. (i) We take the \( \Phi \) in the range of \([0, 2\pi]\) and it is independent of the values of \( V \) and \( \gamma \). (ii) Phase difference \( \Phi \) is taken as a function of \( V \) and \( \gamma \), \( \Phi = m \tan^{-1}(\gamma/V) \), where \( m \) is an integer. We note that the point along \( V \) and \( \gamma \) axes of \( \gamma - V \) plane can be exactly solved as the transition point, i.e., when \( |\gamma| < 1 \) (\(|\gamma| > 1\)) at \( \gamma \) axis, all the eigenstates are extended (or localized), the same as \( \gamma \) axis but for the point \(|V| < 1 \) (\(|V| > 1\)) at \( V \) axis. The numerical simulation is performed in the following procedure. For a given point \((V, \gamma)\), we compute the eigenstates by exact diagonalization. Then the IPR of all the levels and the
average IPR (AIPR) for all energy levels can be obtained. Meanwhile, we confirm that the maximal and minimal IPR has not much deviations from the average, then we can use AIPR to describe the characters of extended (or localized) eigenstates. In Figs. 3 and 4, we find that the localization transition boundary is strongly depends on both the magnitude and the phase of the complex potential. The AIPR profiles exhibit interference-like patterns in the parameter space. We also notice that patterns in both figures are asymmetrically under the switch of $V$ and $\gamma$. It is probably due to the Hermiticity of the hopping term, which breaks the balance between the real and imaginary potential.

V. CONCLUSION

In summary, we have investigated an extension of localization to non-Hermitian system. We have obtained three main results. (i) Based on the equivalence between a disorder non-Hermitian SSH model and a Hermitian uniform chain with real random on-site potential, we have shown exactly that the uncorrelated disorder imaginary on-site potential can induce Anderson localization. (ii) We presented a non-Hermitian AA model, by replacing a real quasiperiodic potential by an imaginary one. It has been shown to have self-duality and thus supports the transition from extension to localization. (iii) Furthermore, we also studied the interactive effect of real and imaginary quasiperiodic potential on the localization transition. Numerical simulations indicates that the phase difference between real and imaginary quasiperiodic potential determines the boundary of transition, exhibiting an interference-like pattern. Our approach opens a new way to investigate the interplay of localization and gain/loss in non-Hermitian system.

APPENDIX

In this Appendix, we present a detailed derivation for the diagonalization of a non-Hermitian SSH chain with Hamiltonian

$$H_{\text{SSH}} = (1 - \delta) \sum_{j=1}^{N-1} b_j^\dagger a_{j+1} + \text{H.c.} + \sum_{j=1}^{N} h_j. \quad (18)$$

Here each $h_j$ describes a non-Hermitian $\mathcal{PT}$-symmetric dimer

$$h_j = (1 + \delta)(a_j^b b_j + \text{H.c.}) + i\gamma_j(a_j^\alpha a_j - b_j^\beta b_j), \quad (19)$$

at position $j$. It can be diagonalized as the form

$$h_j = \sqrt{(1 + \delta)^2 - \gamma_j^2}(\alpha_j a_j - \beta_j b_j), \quad (20)$$

by introducing canonical particle operators in the aid of biorthogonal inner product. The particle operators

$$\begin{align*}
\alpha_j &= \alpha_j^\alpha + i\gamma_j \beta_j, \\
\beta_j &= \alpha_j^\beta, \\
\overline{\alpha}_j &= \frac{1}{\gamma_j^2 - (1 + \delta)^2} \frac{\alpha_j - \gamma_j \beta_j}{\overline{\beta}_j}, \\
\overline{\beta}_j &= \frac{1}{\gamma_j^2 - (1 + \delta)^2} \frac{\beta_j}{\overline{\alpha}_j},
\end{align*} \quad (21)$$

with

$$\tan \varphi_j = \frac{\gamma_j}{\sqrt{(1 + \delta)^2 - \gamma_j^2}}, \quad (22)$$

or inversely

$$\begin{align*}
\alpha_j &= \frac{\gamma_j}{\sqrt{(1 + \delta)^2 - \gamma_j^2}} \frac{1 + \gamma_j}{\gamma_j^2 - (1 + \delta)^2} \frac{\overline{\beta}_j}{\overline{\alpha}_j}, \\
b_j &= \frac{1 + \gamma_j}{\gamma_j^2 - (1 + \delta)^2} \frac{\alpha_j^\beta}{\alpha_j^\alpha}, \\
\alpha_j^\dagger &= \frac{\gamma_j}{\sqrt{(1 + \delta)^2 - \gamma_j^2}} \frac{1 - \gamma_j}{\gamma_j^2 - (1 + \delta)^2} \frac{\overline{\beta}_j}{\overline{\alpha}_j}, \\
b_j^\dagger &= \frac{1 - \gamma_j}{\gamma_j^2 - (1 + \delta)^2} \frac{\alpha_j^\beta}{\alpha_j^\alpha},
\end{align*} \quad (23)$$

which satisfy the canonical commutation relations

$$[\alpha_j, \overline{\alpha}_j]_\pm = [\beta_j, \overline{\beta}_j]_\pm = \delta_{jj'}, \quad (24)$$

$$[\alpha_j, \alpha_j']_\pm = [\beta_j, \beta_j']_\pm = [\overline{\alpha}_j, \overline{\alpha}_j']_\pm = [\overline{\beta}_j, \overline{\beta}_j']_\pm = 0,$$
for the interchain particles,

\[
[a_j, \overline{a}_{j'}]_\pm = [\overline{a}_j, a_{j'}]_\pm = [a_j, \overline{a}_{j'}]_\pm = [\overline{a}_j, a_{j'}]_\pm = 0,
\]

for the intrachain particles. Straightforward derivations show that

\[
(1 + \delta)(a_j^\dagger b_j + b_j^\dagger a_j) + i \gamma_j (a_j^\dagger a_j - b_j^\dagger b_j)
= \sqrt{(1 + \delta)^2 - \gamma_j^2 (\overline{a}_j a_j - \overline{b}_j b_j)},
\]

Under the condition

\[
1 + \delta \gg 1 - \delta, |\gamma_j| \ll 1,
\]
we have
\[
\cos \varphi_j = \frac{\sqrt{(1 + \delta)^2 - \gamma_j^2}}{1 + \delta} \approx 1 - \frac{\gamma_j^2}{2(1 + \delta)^2} \approx 1, \quad (29)
\]
\[
e^{i\varphi} = \frac{\sqrt{(1 + \delta)^2 - \gamma_j^2} + i\gamma_j}{1 + \delta} \approx 1 + \frac{i\gamma_j}{1 + \delta} \approx 1, (30)
\]
and then
\[
\begin{align*}
 b_j^\dagger a_{j+1} + a_{j+1}^\dagger b_j \\
 & \approx \frac{1}{2}(\overline{\alpha}_j \alpha_{j+1} + \overline{\alpha}_j \beta_{j+1} - \overline{\beta}_j \alpha_{j+1} - \overline{\beta}_j \beta_{j+1}) \\
 & \quad + \frac{1}{2}(\alpha_j^\dagger \overline{\alpha}_j - \alpha_j^\dagger \overline{\beta}_j + \beta_{j+1}^\dagger \overline{\alpha}_j - \beta_{j+1}^\dagger \overline{\beta}_j) \\
 & = \frac{1}{2}\left(\overline{\alpha}_j \alpha_{j+1} + \overline{\beta}_j \beta_{j+1} - \overline{\beta}_j \alpha_{j+1} - \overline{\beta}_j \beta_{j+1}\right) + \text{H.c.}
\end{align*}
\]

Neglecting the transition terms between sites with opposite potential, we have
\[
\begin{align*}
 b_j^\dagger a_{j+1} + a_{j+1}^\dagger b_j & \approx \frac{1}{2}(\overline{\alpha}_j \alpha_{j+1} - \overline{\beta}_j \beta_{j+1}) + \text{H.c.} \\
 & \approx \frac{1}{2}(\alpha_j^\dagger \overline{\alpha}_j - \beta_{j+1}^\dagger \overline{\beta}_j) + \text{H.c.}
\end{align*}
\]

Then we have the approximate expression
\[
H_{SSH} \approx \frac{1}{2}(1 - \delta) \sum_{j=1}^{N-1} (\overline{\alpha}_j \alpha_{j+1} - \overline{\beta}_j \beta_{j+1}) \\
+ \text{H.c.} + \sum_{j=1}^{N} V_j (\overline{\alpha}_j \alpha_j - \overline{\beta}_j \beta_j),
\]
where real potential \( V_j = \sqrt{(1 + \delta)^2 - \gamma_j^2} \). On the other hand, we have
\[
\begin{align*}
[\alpha_j, \alpha_j^\dagger] & = [\beta_j, \beta_j^\dagger] = \delta_j \delta_{jj'} = \frac{\delta_j}{1 - \sin \varphi_j}, \\
[\alpha_j, \beta_j^\dagger] & = [\beta_j, \alpha_j^\dagger] = [\beta_j, \beta_j^\dagger] = [\alpha_j, \beta_j^\dagger] = 0,
\end{align*}
\]
for the interchain particles,
\[
\begin{align*}
[\alpha_j, \beta_j^\dagger] & = i \tan \varphi_j \delta_{jj'}, \\
[\alpha_j, \beta_j^\dagger] & = [\alpha_j^\dagger, \beta_j] = [\beta_j, \beta_j^\dagger] = [\alpha_j^\dagger, \beta_j^\dagger] = 0,
\end{align*}
\]
for the intrachain particles. It indicates that if a state satisfy the relation
\[
|\psi\rangle = \sum_j (A_j \alpha_j^\dagger |0\rangle + B_j \beta_j^\dagger |0\rangle),
\]
with \( A_j B_j = 0 \) for all \( j \), i.e., it has only single-chain component, \( \left\{ \alpha_j^\dagger |0\rangle \right\} \text{ or } \left\{ \beta_j^\dagger |0\rangle \right\} \), the dynamics of \( |\psi\rangle \) is the same as that in a Hermitian chain.

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