Position-dependent-mass: cylindrical coordinates, separability, exact solvability and $\mathcal{PT}$-symmetry

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Abstract
The kinetic energy operator with position-dependent-mass in cylindrical coordinates is obtained. The separability of the corresponding Schrödinger equation is discussed within radial cylindrical mass settings. Azimuthal symmetry is assumed and spectral signatures of various $z$-dependent interaction potentials (Hermitian and non-Hermitian $\mathcal{PT}$-symmetric) are reported.

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1. Introduction

The von Roos Hamiltonian for position-dependent-mass (PDM) quantum particles is known to be associated with an ordering ambiguity problem manifested by the non-unique representation of the kinetic energy operator [1]. In such Hamiltonian

$$H = -\frac{\hbar^2}{4} [m(\vec{r})^\alpha \vec{\nabla} m(\vec{r})^\beta \cdot \vec{\nabla} m(\vec{r})^\gamma + m(\vec{r})^\alpha \vec{\nabla} m(\vec{r})^\beta \cdot \vec{\nabla} m(\vec{r})^\gamma] + V(\vec{r}),$$

(1)

an obvious profile change in the effective potential is introduced when the parametric values of the ambiguity parameters $(\alpha, \beta, \gamma)$ are changed (within the von Roos constraint $\alpha + \beta + \gamma = -1$). Nevertheless, it is known that the continuity conditions at the heterojunction boundaries between two crystals imply $\alpha = \gamma$ (cf, e.g., [2] and the related references cited therein). This would effectively reduce the domain of the acceptable parametric values of the ambiguity parameters. In fact, the PDM Hamiltonian (1) is known to be a descriptive model for many physical problems (such as, but not limited to, many-body problem, electronic properties of semiconductors, etc) [1–30]. It is, moreover, a mathematically challenging and a useful model that enriches the class of exactly solvable quantum mechanical systems.

In the literature, nevertheless, one may find many suggestions on the ambiguity parametric values. For example, Gora and William have suggested that $\beta = \gamma = 0$, $\alpha = -1$, Ben Daniel and Duke $\alpha = \gamma = 0$, $\beta = -1$, Zhu and Kroemer $\alpha = \gamma = -1/2$, $\beta = 0$, Li and Kuhn...
\[ \beta = \gamma = -1/2, \alpha = 0, \] \[ \text{and Mustafa and Mazharimousavi} \quad \alpha = \gamma = -1/4, \beta = -1/2 \quad (\text{cf, e.g., [2, 3] and references therein}). \]

Very recently, we have studied the problem of a singular PDM particle in an infinite potential well and shown that none of the above known parametric ordering sets is admissible within the methodical proposal discussed in [3]. Consequently, the ordering ambiguity conflict does not only depend on the heterojunction boundaries and the Dutra and Almeida’s [4] reliability test (cf, e.g., [3, 4] for more details). The potential and/or the form of the PDM have their say in the process [3]. At the end of the day, however, the consensus is that this ambiguity is mainly attributed to the lack of the Galilean invariance (cf, e.g., [1] on the details of this issue).

In the current methodical proposal, we shall be working with the ambiguity parameters as they are without any discrimination as to which set of ordering is favorable than the other. We discuss the von Roos Hamiltonian (1) using cylindrical coordinates and seek some feasible separability in section 2. Therein, we suggest the PDM to be only radial dependent (i.e. \( m(\mathbf{r}) = m \), \( M(\rho, \varphi, z) = M(\rho, \varphi, z) = M(\rho) = 1/\rho^2 \)) and azimuthal symmetrization is sought through the assumption that

\[ V(\mathbf{r}) = V(\rho, \varphi, z) = \frac{\rho^2}{2} [\tilde{V}(\rho) + \tilde{V}(z)]. \]

Of course, this constitutes only one feasible separability of the system (other separability options may also occur), as justified in section 2. In section 3, within the radial cylindrical settings, we consider two examples of fundamental nature. The radial cylindrical “Coulombic” \( \tilde{V}(\rho) = -2/\rho \) and the “harmonic oscillator” \( \tilde{V}(\rho) = a^2 \rho^2/4 \). The spectral signatures of different \( \tilde{V}(z) \) settings on the Coulombic and harmonic oscillator spectra are reported for impenetrable walls at \( z = 0 \) and \( z = L \), for a Morse [31], a non-Hermitian \( PT \)-symmetrized Scarf II [28, 32, 33] and a non-Hermitian \( PT \)-symmetrized Samsonov [28, 34] interaction models where \( P \) denotes parity and \( T \) mimics the time reflection (cf, e.g., [28] and references cited therein on this issue). Our concluding remarks are in section 4.

2. Cylindrical coordinates and separability

Let us consider the kinetic energy operator of the PDM Hamiltonian in (1) and a PDM function of the form \( m(\mathbf{r}) = m, M(\rho, \varphi, z) = M(\rho, \varphi, z) \) (where \( \hbar = m_\odot = 1 \) units are to be used hereinafter). Moreover, we consider the substitutions

\[ \tilde{A} = \alpha M(\rho, \varphi, z)^{\alpha^{-1}} \left[ \tilde{\rho} \tilde{\rho}_\rho + \frac{\tilde{\varphi}}{\rho} \tilde{\rho}_\varphi + \tilde{\zeta} \tilde{\rho}_z \right] M(\rho, \varphi, z), \]

\[ \tilde{B} = \beta M(\rho, \varphi, z)^{\beta^{-1}} \left[ \tilde{\rho} \tilde{\rho}_\rho + \frac{\tilde{\varphi}}{\rho} \tilde{\rho}_\varphi + \tilde{\zeta} \tilde{\rho}_z \right] M(\rho, \varphi, z), \]

\[ \tilde{C} = \gamma M(\rho, \varphi, z)^{\gamma^{-1}} \left[ \tilde{\rho} \tilde{\rho}_\rho + \frac{\tilde{\varphi}}{\rho} \tilde{\rho}_\varphi + \tilde{\zeta} \tilde{\rho}_z \right] M(\rho, \varphi, z) \]

(3)

to imply

\[ \tilde{\nabla} M(\rho, \varphi, z)^{\alpha} = \tilde{\tilde{A}} + M(\rho, \varphi, z)^{\alpha} \tilde{\nabla} \]

\[ \tilde{\nabla} M(\rho, \varphi, z)^{\beta} = \tilde{\tilde{B}} + M(\rho, \varphi, z)^{\beta} \tilde{\nabla} \]

\[ \tilde{\nabla} M(\rho, \varphi, z)^{\gamma} = \tilde{\tilde{C}} + M(\rho, \varphi, z)^{\gamma} \tilde{\nabla} \]

(4)

Using the above identities, one (with \( M(\rho, \varphi, z) \equiv M \) for simplicity of notations) may rewrite

\[ M^\alpha \tilde{\nabla} M^\beta \tilde{\nabla} M^\gamma = M^\alpha (\tilde{\tilde{B}} \cdot \tilde{\tilde{C}}) + M^{\alpha+\beta}[2 \tilde{\tilde{C}} \cdot \tilde{\nabla} + \tilde{\nabla} \cdot \tilde{\tilde{C}}] + M^{-1} \tilde{\nabla}^2 \]

(5)
\[ \nabla^2 \mathbf{A} + \mathbf{A} = \nabla^2 \mathbf{A} = M^\alpha \mathbf{A} + M^\beta \mathbf{B} + M^\gamma \mathbf{A} \cdot \mathbf{B} + M^\delta. \]

(6)

Let us now consider a class of the mass functions defined as

\[ M(\rho, \varphi, z) = g(\rho)f(\varphi)k(z) \Rightarrow \partial_\rho M = M\rho = g_\rho(\rho)f(\varphi)k(z) \Rightarrow \partial_\rho^2 M = M\rho = g_\rho(\rho)f(\varphi)k(z), \]

(7)

which would, in effect, imply that the PDM Schrödinger equation \[ H - E \psi_1(\rho, \varphi, z) = 0 \]

for the Hamiltonian (1) be written as

\[ \partial_\rho^2 + \left( -\frac{M\rho}{\rho} + \frac{M\rho}{\rho^2} \right) \partial_\rho + \frac{1}{\rho^2} \left( \partial_\varphi^2 - \frac{M\varphi}{M} \partial_\varphi + \left( \partial_z^2 - \frac{Mz}{M} \partial_z \right) \right) \psi_1(\rho, \varphi, z) \]

= \{2MV(\rho, \varphi, z) - 2ME - MW(\rho, \varphi, z)\} \psi_1(\rho, \varphi, z),

(8)

where

\[ 2MW(\rho, \varphi, z) = \frac{\zeta}{M^2} \left[ M\rho^2 + M\varphi^2 + Mz^2 \right] - \left( \frac{\beta}{M} + 1 \right) \left[ \frac{M\rho}{\rho} + \frac{M\rho}{\rho^2} + \frac{M\rho}{\rho^3} + \frac{M\rho}{\rho^2} \right] \]

\[ \zeta = \alpha (\alpha - 1) + \gamma (\gamma - 1) - \beta (\beta + 1). \]

(9)

(10)

At this point, one should note that the choice of the mass function in (7) is inspired by the appearance of terms like \( \frac{M\rho}{M} \), \( \frac{M\varphi}{M} \) and \( \frac{Mz}{M} \) as multiplicities of the first-order derivatives in (8). This would, in fact, make the separability of (8) highly feasible and far less complicated. Moreover, following the traditional general wavefunction assumption

\[ \psi_1(\rho, \varphi, z) = R(\rho)\Phi(\varphi)Z(z), \quad \rho \in (0, \infty), \quad \varphi \in (0, 2\pi), \quad z \in (-\infty, \infty), \]

(11)

to ease coordinates separability of (8), we obtain

\[ 0 = 2g(\rho)f(\varphi)k(z) \left[ E - V(\rho, \varphi, z) \right] \]

\[ \Phi'(\varphi) + \frac{\zeta}{2} \left( g(\rho) \right) + \left( \frac{\beta}{M} + 1 \right) \left( \frac{g(\rho)}{\rho^2} + \frac{g''(\rho)}{\rho^2} \right) \]

\[ + \frac{1}{\rho^2} \left( \Phi''(\varphi) + \Phi'(\varphi) f'(\varphi) + \frac{\zeta}{2} \left( f(\varphi) \right) - \left( \frac{\beta}{M} + 1 \right) \frac{f'(\varphi)}{f(\varphi)} \right). \]

(12)

It is obvious that separability is granted through a variety of choices. The simplest of which may be sought in an obviously “manifested-by-equation (12)” general identity of the form

\[ 2MV(\rho, \varphi, z) = 2g(\rho)f(\varphi)k(z) V(\rho, \varphi, z) = \tilde{V}(\rho) + \tilde{V}(z) + \frac{1}{\rho^2} \tilde{V}(\varphi). \]

(13)
In this case, we may avoid any specifications on the forms of \( g(\rho) \), \( f(\phi) \) and \( k(z) \) rather than being mathematically and quantum mechanically “very well” defined. However, the energy term \( 2g(\rho)f(\phi)k(z)E \) in (12) suggests three feasible separabilities for \( f(\phi) = 1 = g(\rho) \) and \( f(\phi) = 1 = g(\rho) \). We focus on one of these cases in what follows.

Let us consider the PDM function to be only an explicit function of \( \rho \). Namely, we choose \( f(\phi) = 1 = k(z) \) and \( g(\rho) = \rho^{-2} \) so that \( M(\rho, \phi, z) = M(\rho) = \rho^{-2} \). Under these settings, equation (12) collapses into a simple separable form

\[
0 = \left[ \frac{\Phi''(\phi)}{\Phi(\phi)} + 2E - \tilde{V}(\phi) + 2(\xi - \beta - 1) \right] + \rho^2 \left[ \frac{R''(\rho)}{R(\rho)} + \frac{3}{\rho} \frac{R'(\rho)}{R(\rho)} - \tilde{V}(\rho) + \frac{Z''(z)}{Z(z)} - \tilde{V}(z) \right].
\]

Equation (14) with azimuthal symmetry (i.e. \( \tilde{V}(\phi) = 0 \)) would immediately imply that

\[
\left[ \frac{\Phi''(\phi)}{\Phi(\phi)} + 2E + 2(\xi - \beta - 1) \right] = K_{\phi}^2
\]

and

\[
\left[ \frac{R''(\rho)}{R(\rho)} + \frac{3}{\rho} \frac{R'(\rho)}{R(\rho)} + \frac{K_{\rho}^2}{\rho^2} - \tilde{V}(\rho) \right] + \left[ \frac{Z''(z)}{Z(z)} - \tilde{V}(z) \right] = 0.
\]

In due course, the solution of (15) reads \( \Phi(\phi) = \exp(i m \phi) \) where \( m = 0, \pm 1, \pm 2, \ldots \) is the magnetic quantum number and \( \Phi(\phi) \) satisfies the single valued condition \( \Phi(\phi) = \Phi(\phi + 2\pi) \). Moreover, we obtain

\[
K_{\phi}^2 = 2E + 2(\xi - \beta - 1) - m^2.
\]

Consequently, one may cast

\[
\frac{Z'(z)}{Z(z)} - \tilde{V}(z) = -K_z^2
\]

and

\[
\frac{R''(\rho)}{R(\rho)} + \frac{3}{\rho} \frac{R'(\rho)}{R(\rho)} + \frac{K_{\rho}^2}{\rho^2} - \tilde{V}(\rho) = K_z^2.
\]

In the following section, we consider \( \tilde{V}(\rho) \) to represent a “Coulombic” and a “harmonic oscillator” and find the spectral signatures of different \( \tilde{V}(z) \) potentials of (18) on the overall spectra.

3. Two examples: the radial cylindrical Coulombic and the harmonic oscillator

A priori, we remove the first-order derivative in the radial cylindrical part of (19) and redefine \( R(\rho) = \rho^{-3/2}U(\rho) \)

(20)
to obtain

\[
-U''(\rho) + \left[ \frac{3/4 - K_{\rho}^2}{\rho^2} + \tilde{V}(\rho) \right] U(\rho) = -K_z^2 U(\rho).
\]

In fact, this 1D radial cylindrical Schrödinger equation provides an effective tool to study the effect of different \( \tilde{V}(z) \) settings of (18) on the spectra of two interesting models of fundamental nature: the Coulombic and the harmonic oscillator [30]. Of course such effects could be tested for other models.
Let us take a Coulombic radial cylindrical model \( \tilde{V}(\rho) = -2/\rho \). In this case, equation (21) would read

\[
-U''(\rho) + \left[ \frac{\ell^2 - 1/4}{\rho^2} - \frac{2}{\rho} \right] U(\rho) = -K^2 U(\rho),
\]

where \( \ell = (1 - K^2)/2 \), \( K_z = (n_\rho + \ell + 1)^{-1} \) and \( n_\rho = 0, 1, 2, \ldots \) is the radial quantum number. Hence, \( K_z = 1/(n_\rho + \sqrt{1 - K^2}) \) and

\[
E = \left( \frac{m^2 + 3}{2} \right) - (\xi - \beta) - \frac{1}{2} \left( \frac{1}{K_z} - n_\rho - 1 \right)^2, \quad (23)
\]

where \( K_z \) is to be determined through the solution of (18) under different \( \tilde{V}(z) \) settings.

Next we consider the radial cylindrical harmonic oscillator model \( \tilde{V}(\rho) = a^2 \rho^2 / 4 \) to obtain

\[
E = \left( \frac{m^2 + 3}{2} \right) - (\xi - \beta) - \frac{1}{2} \left( \frac{K^2}{a} + 2n_\rho + 1 \right)^2, \quad (24)
\]

where, again, \( K_z \) is to be determined through the solution of (18) under different \( \tilde{V}(z) \) settings in the following subsections. Nevertheless, it is obvious that the PDM spectral signature is documented through the ambiguity parameters’ appearance in the constant shift (i.e. \[-(\xi - \beta - 1)\] as evident from (15) and (17) and included in the energy eigenvalues of (23) and (24)).

3.1. Spectral signature of impenetrable walls at \( z = 0 \) and \( z = L \)

Let us now consider that the above-mentioned PDM particle is trapped to move between two impenetrable walls at \( z = 0 \) and \( z = L \). We may then take

\[
\tilde{V}(z) = \begin{cases} 
0, & 0 < z < L \\
\infty, & \text{elsewhere.}
\end{cases} \quad (25)
\]

Consequently, equation (18) reads

\[
Z''(z) + K^2_z Z(z) = 0, \quad (26)
\]

where \( Z(z) \) satisfies the boundary conditions \( Z(z = 0) = 0 = Z(z = L) \) and implies

\[
Z(z) = \sin K_z z, \quad K_z = n_\xi \pi / L, \quad n_\xi = 1, 2, 3, \ldots. \quad (27)
\]

Hence, \( K_z^2 = n_\xi^2 \pi^2 / L^2 \) and the quantum PDM particle here is quasi-free in the \( z \)-direction (i.e. \( \tilde{V}(z) = 0 \)) but constrained to move between the two impenetrable walls at \( z = 0 \) and \( z = L \). The spectral signature of such \( z \)-dependent potential settings is therefore clear. That is, a quantum particle endowed with a PDM \( M(\rho, \phi, z) = M(\rho) = \rho^{-2} \) and subjected to an interaction potential of the form

\[
V(\rho, \phi, z) = -2 + \rho^2 \tilde{V}(z), \quad (28)
\]

with \( \tilde{V}(z) \) defined in (25), would admit exact energy eigenvalues given by

\[
E_{n_\rho, n_\xi, n_\phi} = \left( \frac{m^2 + 3}{2} \right) - (\xi - \beta) - \frac{1}{2} \left( \frac{L}{n_\xi \pi} - n_\rho - 1 \right)^2. \quad (29)
\]

On the other hand, a quantum particle endowed with a PDM \( M(\rho, \phi, z) = M(\rho) = \rho^{-2} \) subjected to an interaction potential of the form

\[
V(\rho, \phi, z) = a^2 \rho^4 / 4 + a^2 \tilde{V}(z), \quad (30)
\]
with $\tilde{V}(z)$ defined in (25), would be accompanied by exact energy eigenvalues of the form

$$E_{n_{\rho},m,n_\beta} = \left(\frac{m^2 + 3}{2}\right) - (\xi - \beta) - \frac{1}{2} \left[\frac{n_J^2 \pi^2}{a L^2} + 2n_{\rho} + 1\right]^2. \quad (31)$$

### 3.2. Spectral signatures of a $\tilde{V}(z)$ Morse model

Consider a Morse-type interaction $\tilde{V}(z) = D(e^{-2z} - 2e^{-z})$, $D > 0$, in (18). We may then closely follow the methodical proposal of Chen [31] to obtain

$$K_z^2 = \left(\sqrt{\frac{D}{\epsilon}} - \frac{1}{2} \frac{n_\beta}{2} \right), \quad n_\beta = 0, 1, 2, 3, \ldots. \quad (32)$$

where one should consider $2m = h = 1$, $a \to \epsilon$, $E \to K_z^2$ and $x \to z$ of Chen [31] to match our settings in (18). Therefore, a PDM quantum particle endowed with $M(\rho, \phi, z) = M(\rho) = \rho - \frac{2}{2}$ and subjected to an interaction potential of the form

$$V(\rho, \phi, z) = -2\rho + D\rho^2(e^{-2z} - 2e^{-z}), \quad D > 0, \quad (33)$$

would admit exact energy eigenvalues given by

$$E_{n_{\rho},m,n_\beta} = \left(\frac{m^2 + 3}{2}\right) - (\xi - \beta) - \frac{1}{2} \left[\sqrt{\frac{D}{\epsilon}} - \frac{n_\beta}{2} - n_{\rho} - 1\right]^2. \quad (34)$$

Obviously, the condition $(\sqrt{D/\epsilon} - \frac{1}{2} \frac{n_\beta}{2}) > 0$ is manifested here and ought to be enforced, otherwise complex pairs of energy eigenvalues are obtained in the process.

Moreover, a quantum particle with $M(\rho, \phi, z) = M(\rho) = \rho - \frac{2}{2}$ subjected to an interaction potential

$$V(\rho, \phi, z) = a^2 \rho^4/4 + D\rho^2(e^{-2z} - 2e^{-z}), \quad D > 0, \quad (35)$$

would indulge the exact energy eigenvalues

$$E_{n_{\rho},m,n_\beta} = \left(\frac{m^2 + 3}{2}\right) - (\xi - \beta) - \frac{1}{2} \left[\frac{1}{a} + \frac{\sqrt{D}}{\epsilon} - \frac{n_\beta}{2} + n_{\rho} + 1\right]^2. \quad (36)$$

### 3.3. $\mathcal{PT}$-symmetrized $\tilde{V}(z)$ spectral signatures

We may now consider a $\mathcal{PT}$-symmetrized $\tilde{V}(z)$ Scarf II in (18) so that

$$\tilde{V}(z) = -\frac{3}{4} + \frac{A^2}{4 \cosh^2 z} + i \frac{A \sinh z}{\cosh^2 z}, \quad \text{with} \quad A = 2 \left(\frac{n_z}{2}\right), \quad n_z = 0, 1, 2, 3, \ldots. \quad (37)$$

where the corresponding Hamiltonian is known to be a non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian that admits exact eigenvalues (cf, e.g., [28, 32, 33]) of the form

$$K_z^2 = \begin{cases} \left(\frac{n_z + 1 - A}{2}\right)^2, & n_z = 0, 1, 2, 3, \ldots < \frac{A - 1}{2} \quad \text{for} \quad A \geq 2, \\ \frac{1}{4}, & \text{for} \quad A < 2. \end{cases} \quad \text{for} \quad A \geq 2. \quad (38)$$

Hence, a quantum particle with $M(\rho, \phi, z) = M(\rho) = \rho - \frac{2}{2}$ moving in

$$V(\rho, \phi, z) = -2\rho - \rho^2 \left[\frac{3 + A^2}{4 \cosh^2 z} + \frac{1}{4} \frac{A \sinh z}{\cosh^2 z}\right] \quad (39)$$
would encounter complex pairs of energy eigenvalues since \( K_z = i (n_z + \frac{1-A}{4}) \) (i.e. \( E_{n_{\rho},m,n_z} \in \mathbb{C} \)), whereas, when the same PDM-particle is moving in

\[
V(\rho, \varphi, z) = a^2 \rho^4/4 - \rho^2 \left[ \frac{3 + A^2}{4 \cosh^2 \zeta} + i \frac{A \sinh \zeta}{\cosh^2 \zeta} \right],
\]

it would admit exact and real energy eigenvalues as

\[
E = \left( \frac{m^2 + 3}{2} \right) - (\zeta - \beta) - \frac{1}{2} \left( \frac{K_z^2}{a} + 2n_{\rho} + 1 \right)^2,
\]

with \( K_z^2 \) defined in (38). Of course, this should never be attributed to \( \mathcal{PT} \)-symmetricity or non-\( \mathcal{PT} \)-symmetricity of the original Hamiltonian (1) with the attendant complex non-Hermitian settings. It is very much related to the nature of separability we followed in this methodical proposal.

One may wish to consider the \( \mathcal{PT} \)-symmetric Samsonov [28, 34] model

\[ \tilde{V}(z) = \frac{1}{\cos z + 2i \sin z}, \quad z \in [-\pi, \pi], \] (42)

in (18). In this case \( Z(-\pi) = Z(\pi) = 0 \) and

\[ K_z^2 = n_z^2/4, \quad n_z = 1, 3, 4, \ldots, \] (43)

with a missing state \( n_z = 2 \) (the reader may refer to [34] for more details on this missing state). Hence, for a PDM quantum particle endowed with \( M(\rho, \varphi, z) = M(\rho) = 1/\rho^2 \) and subjected to an interaction potential of the form

\[
\tilde{V}(\rho, \varphi, z) = -2\rho - \frac{1}{\cos z + 2i \sin z}, \quad z \in [-\pi, \pi],
\]

the exact energy eigenvalues would read

\[
E_{n_{\rho},m,n_z} = \left( \frac{m^2 + 3}{2} \right) - (\zeta - \beta) - \frac{1}{2} \left( \frac{2}{n_z} - n_{\rho} - 1 \right)^2,
\]

whereas, for

\[
V(\rho, \varphi, z) = a^2 \rho^4/4 - \frac{1}{\cos z + 2i \sin z}, \quad z \in [-\pi, \pi],
\]

the exact energy eigenvalues would read

\[
E_{n_{\rho},m,n_z} = \left( \frac{m^2 + 3}{2} \right) - (\zeta - \beta) - \frac{1}{2} \left( \frac{n_z^2}{4a} + 2n_{\rho} + 1 \right)^2,
\]

where \( n_z = 1, 3, 4, \ldots \).

4. Concluding remarks

The kinetic energy operator in the PDM Hamiltonian (1) is a problem with many aspects that are yet to be explored. In the current work, we tried to study this problem within the context of cylindrical coordinates \((\rho, \varphi, z)\). In due course, the essentials related with the kinetic energy operator in (1) are reported. The separability of the Schrödinger equation is sought through a radial cylindrical position-dependent-mass \( M(\rho, \varphi, z) = M(\rho) = 1/\rho^2 \) accompanied by an azimuthally symmetrized interaction potential \( V(\rho, \varphi, z) = \rho^2[\tilde{V}(\rho) + \tilde{V}(z)]/2 \), where \( \tilde{V}(\varphi) = 0 \). Such a combination is not a unique one and some other separability settings could be sought. However, we have chosen to stick with the above-mentioned combination
for it leads to a handy though rather constructive separable system of the one-dimensional Schrödinger equations (15), (18) and (19).

Assuming azimuthal symmetrization of the problem at hand and within the radial settings, we consider two examples of fundamental nature. The radial cylindrical Coulombic \( \tilde{V}(\rho) = -2/\rho \) and the radial cylindrical harmonic oscillator \( \tilde{V}(\rho) = a^2 \rho^2/4 \). They are indeed exactly solvable within the settings of (21) and admit exact energy eigenvalues documented in (23) and (24), respectively. Nevertheless, the appearance of \( K_z \) and \( K_z^2 \) in (23) and (24), respectively, offered an opportunity to study their spectral signatures mandated by different \( \tilde{V}(z) \) interaction models. Namely, the spectral signatures of \( \tilde{V}(z) \) for impenetrable walls at \( z = 0 \) and \( z = L \) (27), for a Morse (32), for a non-Hermitian \( \mathcal{P}\mathcal{T} \)-symmetrized Scarf II (38) and for a non-Hermitian \( \mathcal{P}\mathcal{T} \)-symmetrized Samsonov \([28, 34]\) (43) are reported.

To summarize, we have assumed azimuthal symmetry and used the radial cylindrical Coulomb and harmonic oscillator to obtain exact eigenvalues for a new set of interaction potentials (represented in their general form in (2) and detailed in (28), (30), (33), (35), (39), (40), (44) and (46)). In fact, under such azimuthal symmetrization and \( \tilde{V}(z) \) setting, this set of exactly solvable models may grow up as long as one can find exactly solvable radially cylindrical models (hereby, exact solvability may even include numerically exactly solvable models). The recipe as how to collect the energy eigenvalues is clear in the above methodical proposal.

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