A SILTING THEOREM

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Abstract. We give a generalization of the classical tilting theorem. We show that for a 2-term silting complex $P$ in the derived category $D^b(A)$ of a finite dimensional algebra $A$, the algebra $B = \text{End}_{D^b(A)}(P)$ admits a 2-term silting complex $Q$ with the following properties: (i) The endomorphism algebra of $Q$ in the derived category of $B$ is a factor algebra of $A$, and (ii) there are induced torsion pairs in mod $A$ and mod $B$, such that we obtain natural equivalences induced by Hom- and Ext-functors. Moreover, we show how the Auslander-Reiten theory of mod $B$ can be described in terms of the Auslander-Reiten theory of mod $A$.

Introduction

The fundamental idea of tilting theory is to relate the module categories of two algebras by the use of tilting functors. Such functors were introduced by Brenner and Butler, in [BB], who were generalizing the ideas in [BGP] and [APR].

In the seminal paper [HR], Happel and Ringel introduced the concepts of tilting modules and tilted algebras. A tilted algebra is the endomorphism ring of a tilting module over a hereditary finite dimensional algebra. Happel [H] and Cline, Parshall, Scott [CPS] proved that tilting modules induce derived equivalences, and inspired by this Rickard [Ric] introduced the concept of tilting complexes, as a necessary ingredient in developing Morita theory for derived categories.

Over the last 35 years these ideas and concepts have become an essential tool in many branches of mathematics, including algebraic geometry, finite group theory, algebraic group theory and algebraic topology, see [AHK]. More recently, the development of cluster tilting theory, see [K, R], has spurred further interest in the topic and the relation to cluster algebras [FZ].

Let us briefly recall the main ideas from [BB] and [HR]. Let $k$ be a field, let $A$ be a finite dimensional algebra over $k$, and $T$ a tilting module in mod $A$, the category of (finite dimensional) right $A$-modules. That is: $T$ is a module with projective dimension at most 1 (pd $T \leq 1$), with $\text{Ext}_1^A(T, T) = 0$ and such that $|T| = |A|$, where $|X|$ denotes the number of indecomposable direct summand in $X$, up to isomorphism. Let $B = \text{End}_A(T)$. Then $D(T)_B$ is cotilting module over $B$ and $A \cong \text{End}_B(D(T)_B)$. Cotilting modules are defined by replacing pd $T \leq 1$ with id $T \leq 1$ in the definition of tilting modules, where id $T$ is the injective dimension of $T$. Moreover, let $\mathcal{T} = \text{Fac} T$ be the full subcategory of mod $A$ whose objects are generated by $T$, and let $\mathcal{F}$ be the full subcategory of mod $A$ with objects $X$ such that $\text{Hom}_A(T, X) = 0$. Then $(\mathcal{T}, \mathcal{F})$ is a torsion pair in mod $A$. There is also a torsion pair $(X, \mathcal{Y})$ in mod $B$, induced by the cotilting module $D(T)_B$, and Hom- and Ext-functors induce inverse equivalences of $\mathcal{T}$ with $\mathcal{Y}$ and of $\mathcal{F}$ with $X$.

We generalize these results to the following setting. We consider a 2-term silting complex $P$ in the bounded derived category $D^b(A)$. This is just a map between projective modules, considered as a complex, with the property that $\text{Hom}_{D^b(A)}(P, P[1]) = 0$, and such that $P$ generates the bounded homotopy category of projectives $K^b(\text{proj} A)$. Let $B = \text{End}_{D^b(A)}(P)$. It then turns out that mod $A$ and mod $B$ can be compared in a way very similar to the setting with tilting modules.
The concept of silting complexes originated from Keller and Vossieck [KV]. In [HoKM], the relation between 2-term silting complexes and torsion pairs in module categories was first considered. They were mainly dealing with abelian categories with arbitrary coproducts, but we adopt many of their results to our setting.

More recently, there have been several papers, starting with [AiI], often focusing on various (combinatorial) properties on the set of silting complexes. Silting complexes correspond to bounded \( t \)-structures having a heart which is a length category, i.e. there are finitely many simples, and all objects have finite length [KoeY].

The set of 2-term silting complexes has a natural structure of an ordered exchange graph, and as beautifully summarized in [BY], this gives links (expressed as isomorphisms of exchange graphs, see the figure in their introduction) to a plenitude of other structures which have recently been studied. Among these are support \( \tau \)-tilting modules [AdIR] in the module category, and certain bounded \( t \)-structures in the bounded derived category, see [BY, Cor. 4.3]. Starting with a quiver \( Q \), with no loops or oriented 2-cycles, there is a corresponding cluster algebra \( A_Q \), [FZ], and then we obtain also a correspondence with the clusters in \( A_Q \), see [AdIR]. Given \( Q \) as above, and a potential, there is a correspondence with certain bounded \( t \)-structures in the finite-dimensional derived category of a corresponding Ginzburg DG-algebra [BY, KQ].

In this paper and the forthcoming paper [BZ], we consider the endomorphism algebras of 2-term silting complexes, which so far have been less studied. In [BZ] we consider in particular the hereditary case, where such algebras turns out to have a very nice description.

The paper is organized as follows. In the first section, we review some background and notation, and state the main results. In Section 2, we consider links between silting theory, \( t \)-structures and torsion pairs. In Section 3, we prove further properties of 2-term silting complexes, and the main result is proved in Section 4. In Section 5 we apply the main result to obtain some information about the AR-theory of the endomorphism ring of a 2-term silting complex, inspired by similar results in classical tilting, see [ASS].

1. Background and main result

Let \( A \) be a finite dimensional \( k \)-algebra, and \( \text{mod} \, A \) the category of finitely generated right modules. Let \( D^b(A) \) be the bounded derived category, with shift functor \([1]\). Whenever we consider subcategories of \( \text{mod} \, A \) or \( D^b(A) \), they are assumed to be full and closed under isomorphism. For an object \( M \) in an additive category, let \( M \) denote the additive closure, i.e. the full subcategory generated by all direct summands of direct sums of copies of \( M \).

Recall that a torsion pair in \( \text{mod} \, A \), is a pair \((X, \mathcal{Y})\) of subcategories of \( \text{mod} \, A \), with the properties that

- \( \text{Hom}_A(X, Y) = 0 \) if and only if \( Y \) is in \( \mathcal{Y} \), and
- \( \text{Hom}_A(X, \mathcal{Y}) = 0 \) if and only if \( X \) is in \( X \).

If \( M \) is an object in \( \text{mod} \, A \), then there is an exact sequence,

\[
0 \to tM \to M \to M/tM \to 0
\]

called the canonical sequence of \( M \), and with \( tM \) in \( X \) and with \( M/tM \) in \( \mathcal{Y} \). Let \( \text{proj} \, A \) denote the full subcategory of \( \text{mod} \, A \) generated by the projective modules. We consider 2-term complexes \( P \) in \( K^b(\text{proj} \, A) \). These are complexes \( P = \{P_i\} \) with \( P_i = 0 \) for \( i \neq -1, 0 \). Such a complex is called presilting if \( \text{Hom}_{K^b(\text{proj} \, A)}(P, P[1]) = 0 \) and silting if in addition thick \( P = K^b(\text{proj} \, A) \).

Here, for an object \( X \) in \( K^b(\text{proj} \, A) \), we denote by thick \( X \) the smallest triangulated subcategory closed under direct summands containing \( X \). A 2-term silting complex \( P \) is tilting, if in addition \( \text{Hom}_{K^b(\text{proj} \, A)}(P, P[-1]) = 0 \).

Let \( P \) be a 2-term silting complex, and consider the full subcategories of \( \text{mod} \, A \) given by
Theorem 1.1. Let $P$ be a 2-term silting complex in $D^b(A)$, and let $B = \text{End}_{D^b(A)}(P)$.

(a) The pair $(\mathcal{T}(P), \mathcal{F}(P))$ is a torsion pair in $\text{mod} \ A$.

(b) There is a triangle

$$A \to P' \xrightarrow{f} P'' \to A[1]$$

with $P', P''$ in $\text{add} \ A$.

Consider the 2-term complex $Q$ in $D^b(B)$ induced by the map

$$\text{Hom}_{D^b(A)}(P, f): \text{Hom}_{D^b(A)}(P, P') \to \text{Hom}_{D^b(A)}(P, P'').$$

(c) $Q$ is a 2-term silting complex in $D^b(B)$.

(d) There is an algebra epimorphism $\Phi: A \to \overline{A} = \text{End}_{D^b(B)}(Q)$.

(e) $\Phi$ is an isomorphism if and only if $P$ is tilting.

Let $\Phi: \text{proj} \ A \hookrightarrow \text{mod} \ A$ be the induced inclusion functor.

(f) The restriction of the functors $\text{Hom}_{D^b(A)}(P, -)$ and $\Phi^*: \text{Hom}_{D^b(B)}(Q, -[1])$ to $\mathcal{T}(P)$ and $\mathcal{F}(Q)$ is a pair of inverse equivalences.

(g) The restriction of the functors $\text{Hom}_{D^b(A)}(P, -[1])$ and $\Phi^*: \text{Hom}_{D^b(B)}(Q, -)$ to $\mathcal{T}(P)$ and $\mathcal{F}(Q)$ is a pair of inverse equivalences.

We will use the following notation. For any subcategory $\mathcal{T}$ of $\text{mod} \ A$, an $A$-module $X$ in $\mathcal{T}$ is called Ext-projective in $\mathcal{T}$ if $\text{Ext}^1_A(X, Y) = 0$ for all $Y$ in $\mathcal{T}$; dually, $X$ in $\mathcal{T}$ is called Ext-injective in $\mathcal{T}$ if $\text{Ext}^1_A(Y, X) = 0$ for all $Y$ in $\mathcal{T}$. Furthermore, we let $D = \text{Hom}_A(\mathcal{E}, \mathcal{F})$ be the vector-space duality, and let $\nu$ denote the Nakayama functor $\nu = D \text{Hom}_A(\mathcal{E}, \mathcal{F})$, which is an equivalence from $\text{proj} \ A$ to the full subcategory $\text{proj} \ A$ of $\text{mod} \ A$ generated by the injective modules. Then $\nu$ induces an equivalence

$$\nu: K^b(\text{proj} \ A) \to K^b(\text{proj} \ A).$$

It is well known that there is an isomorphism

$$\text{Hom}_{D^b(A)}(X, \nu Y) \cong D \text{Hom}_{D^b(A)}(Y, X)$$

for any $X, Y \in K^b(\text{proj} \ A)$ (see e. g. [H] Chapter 1, Section 4.6).

2. 2-term silting complexes, $t$-structures and torsion pairs

In this section we recall the notion of a $t$-structure [BBD] in a triangulated category, and the interplay between $t$-structures, torsion pairs and 2-term silting complexes.

A pair $(\mathcal{X}, \mathcal{Y})$ of subcategories of $D^b(A)$ is called a $t$-structure if and only if the following conditions hold:

1. $\mathcal{X}[1] \subseteq \mathcal{X}$ and $\mathcal{Y}[-1] \subseteq \mathcal{Y}$;
2. $\text{Hom}_{D^b(A)}(X, Y[-1]) = 0$ for any $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$;
3. for any $C \in D^b(A)$, there is a triangle

$$X \to C \to Y[-1] \to X[1]$$

with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.
Silting complexes give rise to \( t \)-structures in a natural way. For an integer \( m \), consider the pair of subcategories

\[
D^{\geq m}(P) = \{ X \in D^b(A) \mid \text{Hom}_{D^b(A)}(P, X[i]) = 0, \forall i > m \}
\]

and

\[
D^{\leq m}(P) = \{ X \in D^b(A) \mid \text{Hom}_{D^b(A)}(P, X[i]) = 0, \forall i < m \}
\]

in the derived category \( D^b(A) \).

Observe that \( T(P) = D^{\leq 0}(P) \cap \text{mod} \Lambda \) and \( F(P) = D^{\geq 1}(P) \cap \text{mod} \Lambda \). We have the following result. Here, (b) is from \([\text{HoKM}]\) and (a) is from \([\text{KoeY}]\). Note also that a version of (a) was proved in \([\text{HoKM}]\), in the setting of abelian categories with arbitrary coproducts.

**Theorem 2.1.** Let \( P \) be a 2-term silting complex in \( D^b(A) \).

(a) The pair \((D^{\leq 0}(P), D^{\geq 0}(P))\) is a \( t \)-structure in \( D^b(A) \).

(b) The pair \((T(P), F(P))\) is a torsion pair in \( \text{mod} \Lambda \).

The following lemma will be useful for later.

**Lemma 2.2.** For any \( X \in D^b(A) \) and \( i \in \mathbb{Z} \), there is a short exact sequence,

\[
0 \to \text{Hom}_{D^b(A)}(P, H^{i-1}(X)[1]) \to \text{Hom}_{D^b(A)}(P, X[i]) \to \text{Hom}_{D^b(A)}(P, H^i(X)) \to 0.
\]

**Proof.** See \([\text{HoKM}, \text{Lemma 2.5}]\), the proof given there works also in our case. \( \square \)

Let \( C(P) = D^{\leq 0}(P) \cap D^{\geq 0}(P) \) be the heart of the \( t \)-structure \((D^{\leq 0}(P), D^{\geq 0}(P))\). The following summarizes the main features of \( C(P) \).

**Theorem 2.3.** Let \( P \) be a 2-term silting complex in \( D^b(A) \).

(a) \( C(P) \) is an abelian category and the short exact sequences in \( C(P) \) are precisely the triangles in \( D^b(A) \) all of whose vertices are objects in \( C(P) \).

(b) \((T(P)[1], T(P))\) is a torsion pair in \( C(P) \).

(c) For a complex \( X \) in \( D^b(A) \), we have that \( X \) is in \( C(P) \) if and only if \( H^0(X) \) is in \( T(P) \), \( H^{-1}(X) \) is in \( T(P) \) and \( H^0(X) = 0 \) for \( i \neq -1, 0 \).

(d) \( \text{Hom}_{D^b(A)}(P, -) : C(P) \to \text{mod} B \) is an equivalence of (abelian) categories.

**Proof.** Note that (a) is a classical result of \([\text{BBD}]\). Proofs of (b), (c) and (d) can be found in \([\text{HoKM}]\) (although there they proved these in the setting of abelian categories with arbitrary coproducts, but their proofs also work in our case, using Theorem 2.1(a)). We now explain how (b) and (c) can also be seen to follow from \([\text{HRS}]\) Proposition 1.2.1 and Corollary 1.22], which says that for any torsion pair \((T, F)\) in \( \text{mod} A \), we have that the two subcategories

\[
\{ X \in D^b(A) \mid H^i(X) = 0 \text{ for } i > 0 \text{ and } H^0(X) \in T \}
\]

and

\[
\{ X \in D^b(A) \mid H^i(X) = 0 \text{ for } i < 0 \text{ and } H^{-1}(X) \in F \}
\]

form a \( t \)-structure, and that \((T[1], T)\) is a torsion pair in the heart of this \( t \)-structure.

Note first that by Lemma 2.2, we have that

\[
D^{\leq 0}(P) = \{ X \in D^b(A) \mid \text{Hom}_{D^b(A)}(P, H^i(X)) = 0 \text{ for } i > 0 \text{ and } \text{Hom}_{D^b(A)}(P, H^i(X)[1]) = 0, \forall j \geq 0 \}.
\]
Lemma 2.7. For any $A$–module $X$, we have a functorial isomorphism
$$\text{Hom}_{D^b(A)}(P, X) \cong \text{Hom}_A(H^0(P), X)$$
and a monomorphism
$$\text{Hom}_{D^b(A)}(H^0(P), X[1]) \rightarrow \text{Hom}_{D^b(A)}(P, X[1]).$$
Proof. This follows from using \( \text{Hom}_{\mathcal{D}(\mathcal{A})}(\cdot, X) \) on the triangle

\[
H^{-1}(\mathcal{P})[1] \rightarrow \mathcal{P} \rightarrow H^0(\mathcal{P}) \rightarrow H^{-1}(\mathcal{P})[2].
\]

\[\square\]

We next describe some useful properties for the torsion pair corresponding to a 2-term silting complex. A consequence of this is that both \( \mathcal{T}(\mathcal{P}) \) and \( \mathcal{F}(\mathcal{P}) \) are exact categories with enough projectives and injectives. Note that (1) also follows from the proof of \([\text{AdIR}, \text{Theorem 2.7}]\).

**Proposition 2.8.** Let \( \mathcal{P} \) be a 2-term silting complex and \((\mathcal{T}(\mathcal{P}), \mathcal{F}(\mathcal{P}))\) be the torsion pair induced by \( \mathcal{P} \). Then

(1): for any \( X \in \text{mod} \mathcal{A} \), we have that \( X \in \text{add} H^0(\mathcal{P}) \) if and only if \( X \) is Ext-projective in \( \mathcal{T}(\mathcal{P}) \);

(2): for any \( X \in \mathcal{T}(\mathcal{P}) \), there is a short exact sequence

\[
0 \rightarrow L \rightarrow T_0 \rightarrow X \rightarrow 0
\]

with \( T_0 \in \text{add} H^0(\mathcal{P}) \) and \( L \in \mathcal{T}(\mathcal{P}) \);

(3): for any \( X \in \text{mod} \mathcal{A} \), we have that \( X \in \text{add} tvA \) if and only if \( X \) is Ext-injective in \( \mathcal{T}(\mathcal{P}) \);

(4): for any \( X \in \mathcal{T}(\mathcal{P}) \), there is a short exact sequence

\[
0 \rightarrow X \rightarrow T_0 \rightarrow L \rightarrow 0
\]

with \( T_0 \in \text{add} tvA \) and \( L \in \mathcal{T}(\mathcal{P}) \);

(5): for any \( X \in \text{mod} \mathcal{A} \), we have that \( X \in \text{add} H^{-1}(v\mathcal{P}) \) if and only if \( X \) is Ext-injective in \( \mathcal{F}(\mathcal{P}) \);

(6): for any \( X \in \mathcal{F}(\mathcal{P}) \), there is a short exact sequence

\[
0 \rightarrow X \rightarrow F_0 \rightarrow L \rightarrow 0
\]

with \( F_0 \in \text{add} H^{-1}(v\mathcal{P}) \) and \( L \in \mathcal{F}(\mathcal{P}) \);

(7): for any \( X \in \text{mod} \mathcal{A} \), we have that \( X \in \text{add} A/tA \) if and only if \( X \) is Ext-projective in \( \mathcal{F}(\mathcal{P}) \);

(8): for any \( X \in \mathcal{F}(\mathcal{P}) \), there is a short exact sequence

\[
0 \rightarrow L \rightarrow F_0 \rightarrow X \rightarrow 0
\]

with \( F_0 \in \text{add} A/tA \) and \( L \in \mathcal{F}(\mathcal{P}) \).

Proof. We only prove (1) – (4). The proofs of (5) – (8) are similar.

By the second part of Lemma \([2, 7]\), we have that \( H^0(\mathcal{P}) \) is Ext-projective.

Assume \( M \) is Ext-projective in \( \mathcal{T}(\mathcal{P}) = \text{Fac} H^0(\mathcal{P}) \). Then there is an exact sequence

\[
(\natural) \quad 0 \rightarrow L \rightarrow T_0 \xrightarrow{\alpha} M \rightarrow 0
\]

where \( T_0 \xrightarrow{\alpha} M \) is a right add \( H^0(\mathcal{P}) \)-approximation. Since \( \text{Hom}_{\mathcal{A}}(H^0(\mathcal{P}), \alpha) \) is an epimorphism, we have that \( \text{Hom}_{\mathcal{D}(\mathcal{A})}(\mathcal{P}, \alpha) \) is also an epimorphism by Lemma \([2, 7]\). Applying \( \text{Hom}_{\mathcal{D}(\mathcal{A})}(\mathcal{P}, -) \) to (\natural), we have an exact sequence

\[
\text{Hom}_{\mathcal{D}(\mathcal{A})}(\mathcal{P}, T_0) \xrightarrow{\text{Hom}_{\mathcal{D}(\mathcal{A})}(\mathcal{P}, \alpha)} \text{Hom}_{\mathcal{D}(\mathcal{A})}(\mathcal{P}, M) \rightarrow \text{Hom}_{\mathcal{D}(\mathcal{A})}(\mathcal{P}, L[1]) \rightarrow 0.
\]

Then \( \text{Hom}_{\mathcal{D}(\mathcal{A})}(\mathcal{P}, L[1]) = 0 \) which implies that \( L \) is in \( \mathcal{T}(\mathcal{P}) \). Then, by assumption, the sequence (\natural) splits, and hence \( M \) is in add \( H^0(\mathcal{P}) \). This proves (1). Replacing \( M \) with an arbitrary object \( X \) in \( \mathcal{T}(\mathcal{P}) \), we also obtain (2).

For (3) cf. \([\text{Sma}]\) or \([\text{ASS}, \text{Proposition VI.1.11}]\).
We now prove (4). For any $X \in \mathcal{T}(\mathbf{P})$, we have an injective envelope $\alpha: X \to I_0$ with $I_0 \in \add vA$. Considering the canonical exact sequence of $I_0$ in $(\mathcal{T}(\mathbf{P}), \mathcal{F}(\mathbf{P}))$:

$$
\begin{array}{c}
0 \\
\downarrow \\
\downarrow \\
\downarrow \\
0
\end{array}
\xrightarrow{\alpha'} \xrightarrow{\beta} \xrightarrow{\gamma} \xrightarrow{\alpha} I_0 \xrightarrow{\alpha'} I_0/I_0 \xrightarrow{\alpha} 0
$$

we have that $\gamma \alpha = 0$ by $X \in \mathcal{T}(\mathbf{P})$ and $I_0/I_0 \in \mathcal{F}(\mathbf{P})$. So there is a morphism $\alpha': X \to I_0$ such that $\alpha = \beta \alpha'$. Note that $\alpha'$ is injective since $\alpha$ is injective. Let $F_0 = I_0 \in \add vA$ and $L$ be the cokernel of $\alpha'$. Then $L$ is in $\mathcal{T}(\mathbf{P})$, since $\mathcal{T}(\mathbf{P})$ is closed under taking factor modules. □

3. 2-term silting complexes

The first lemma is the analog, for 2-term silting complexes, of the Bongartz completion of classical tilting modules. It can be deduced from [AdIR, Theorem 2.10] and was proven in [DF, IJY, W].

For the reader’s convenience, we give a proof here.

**Lemma 3.1.** Let $\mathbf{P}$ be a 2-term presilting complex in $K^b(\proj A)$. Then there exists a triangle

$$A \to E \to \mathbf{P}'' \to A[1]$$

with $E$ being a 2-term complex in $K^b(\proj A)$ such that $\mathbf{P} \parallel E$ is a 2-term silting complex.

**Proof.** Let $\mathbf{P}'' \to A[1]$ be a right add $\mathbf{P}$–approximation of $A[1]$. Extend it to a triangle

$$(*) \quad A \to E \to \mathbf{P}'' \to A[1].$$

By applying the functors $\Hom_{D^b(A)}(\mathbf{P}, -)$ and $\Hom_{D^b(A)}(-, \mathbf{P})$ to the triangle $(*),$ we have that $\Hom_{D^b(A)}(\mathbf{P}, E[i]) = 0$ for $i > 0$ and $\Hom_{D^b(A)}(E, \mathbf{P}[i]) = 0$ for $i > 0$. Applying $\Hom_{D^b(A)}(-, E)$ yields $\Hom_{D^b(A)}(E, E[i]) = 0$ for $i > 0$. Hence $\mathbf{P} \oplus E$ is a 2-term presilting complex in $K^b(\proj A)$. The triangle $(*)$ shows that $A \in \text{thick}(\mathbf{P} \oplus E)$ and so $\text{thick}(\mathbf{P} \oplus E) = K^b(\proj A)$ which implies that $\mathbf{P} \oplus E$ is a silting complex. □

**Remark 3.2.** Note that if the right add $\mathbf{P}$–approximation in the triangle $(*)$ is minimal, then $E$ does not contain any direct summands whose 0th cohomology is zero since $\Hom_{D^b(A)}(A, A[1]) = 0$. Therefore one can deduce [AdIR, Theorem 2.10] from this proof.

We obtain the following characterization of silting complexes.

**Corollary 3.3.** Let $\mathbf{P}$ be a 2-term presilting complex in $K^b(\proj A)$. Then the following are equivalent:

1. $\mathbf{P}$ is a silting complex in $K^b(\proj A)$;
2. $|\mathbf{P}| = |A|$;
3. there is a triangle $\Delta_{\mathbf{P}}$

$$A \xrightarrow{e} \mathbf{P}' \xrightarrow{f} \mathbf{P}'' \xrightarrow{g} A[1]$$

with $\mathbf{P}', \mathbf{P}'' \in \add \mathbf{P}$.

**Proof.** The equivalence between (1) and (2) is exactly [AdIR, Proposition 3.3], c.f. also [AI, DR, IJY]. The equivalence between (1) and (3) is an immediate corollary of Lemma 3.1, cf. also [W, Theorem 3.5, Proposition 3.9]. □

**Remark 3.4.** Let

$$A \xrightarrow{e} \mathbf{P}' \xrightarrow{f} \mathbf{P}'' \xrightarrow{g} A[1]$$

be an arbitrary triangle with $\mathbf{P}', \mathbf{P}'' \in \add \mathbf{P}$. 
(i) Since \( \text{Hom}_{D^b(A)}(P, P[1]) = 0 \), it follows that \( e \) is a left \( \text{add P} \)-approximation, and that \( g \) is a right \( \text{add P} \)-approximation.

(ii) The map \( e \) is left minimal if and only if \( g \) is right minimal.

(iii) The map \( f: P' \to P'' \) defines a 2-term complex in \( K^b(\text{add } P) \), and is as such, independent of \( e, f \) and \( g \). Here \( K^b(\text{add } P) \) denotes the bounded homotopy category of the additive category \( \text{add P} \).

(iv) Applying \( \text{Hom}_{D^b(A)}(P, -) \) to the map \( f: P' \to P'' \), we obtain a 2-term complex \( Q \) in \( K^b(\text{proj } B) \), which is also independent on \( e, f, g \). This follows from the general fact that \( \text{Hom}_{D^b(A)}(P, -) \) gives an equivalence between \( \text{add } P \) and \( \text{proj } B \).

The following lemmas will be useful later.

**Lemma 3.5.** There is a functorial isomorphism

\[
\text{Hom}_{D^b(A)}(P_0, X) \cong \text{Hom}_B(\text{Hom}_{D^b(A)}(P_0, P), \text{Hom}_{D^b(A)}(P, X))
\]

sending \( f \) to \( \text{Hom}_{D^b(A)}(P, f) \), for any \( P_0 \in \text{add } P \) and \( X \in D^b(A) \).

**Proof.** This follows from the additivity of the functors and from the fact that the defined map is an isomorphism when \( P_0 = P \). \( \square \)

**Lemma 3.6.** For each \( A \)-module \( X \), there are isomorphisms

\[
\text{Hom}_{D^b(A)}(P, X) \cong \text{Hom}_{D^b(A)}(P, tX)
\]

and

\[
\text{Hom}_{D^b(A)}(P, X[1]) \cong \text{Hom}_{D^b(A)}(P, X/tX[1])
\]

as \( B \)-modules. In particular, \( \text{Hom}_{D^b(A)}(P, X) \) is in \( \mathcal{Y}(P) \) and \( \text{Hom}_{D^b(A)}(P, X[1]) \) is in \( \mathcal{X}(P) \) for any \( X \) in \( \text{mod } A \).

**Proof.** Applying \( \text{Hom}_{D^b(A)}(P, -) \) to the canonical exact sequence of \( X \) in the torsion pair \((\mathcal{T}(P), \mathcal{F}(P))\), we have a long exact sequence

\[
0 \to \text{Hom}_{D^b(A)}(P, tX) \to \text{Hom}_{D^b(A)}(P, X) \to \text{Hom}_{D^b(A)}(P, X/tX) \\
\to \text{Hom}_{D^b(A)}(P, tX[1]) \to \text{Hom}_{D^b(A)}(P, X/tX[1]) \to \text{Hom}_{D^b(A)}(P, X/tX[1]) \to 0.
\]

Note that \( \text{Hom}_{D^b(A)}(P, X/tX) = 0 \) by \( X/tX \in \mathcal{T}(P) \) and \( \text{Hom}_{D^b(A)}(P, tX[1]) = 0 \) by \( tX \in \mathcal{F}(P) \). Thus we get the desired isomorphisms. \( \square \)

**Lemma 3.7.** For any 2-term complex \( X : X^{-1} \to X^0 \) in \( K^b(\text{proj } A) \), if \( H^0(X) \cong 0 \cong H^{-1}(X) \), then \( X \cong 0 \).

**Proof.** On the one hand, \( H^0(X) \cong 0 \) implies that \( x \) is an epimorphism, so \( x \) is a retraction. On the other hand, \( H^{-1}(X) \cong 0 \) implies that \( \nu x \) is a monomorphism, so \( \nu x \) is a section. Since \( \nu \) is an equivalence from \( \text{proj } A \) to \( \text{inj } A \), we have that \( x \) is an isomorphism. Hence \( X \cong 0 \). \( \square \)

Recall from Remark 3.4 that \( P \) determines an (up to isomorphism) unique 2-term complex \( Q \) in \( K^b(\text{proj } B) \) given by

\[
\text{Hom}_{D^b(A)}(P, P') \xrightarrow{\text{Hom}_{D^b(A)}(P, f)} \text{Hom}_{D^b(A)}(P, P''),
\]

where \( f \) is the map from the triangle \( \Delta P \).

**Proposition 3.8.** Let \( P \) be a 2-term silting complex. Then the complex \( Q \) defined above is a 2-term silting complex in \( K^b(\text{proj } B) \). Moreover, \( \mathcal{T}(Q) = \mathcal{X}(P) \) and \( \mathcal{F}(Q) = \mathcal{Y}(P) \).
Proof. Let $P_1, \cdots, P_n$ be a complete collection of indecomposable, pairwise non-isomorphic projective $A$-modules. Since the map $e$ from the triangle $\Delta_P$ is a left add $P$-approximation, there are triangles

$$P_i \xrightarrow{e_i} P_i' \xrightarrow{f_i} P_i'' \xrightarrow{g_i} P_i[1]$$

such that the direct sum of these triangles is a direct summand of $\Delta_P$. Let $Q_i$ be the 2-term complex in $K^b(\text{proj } B)$ given by

$$\text{Hom}_{D^b(A)}(P, P_i') \xrightarrow{\text{Hom}_{D^b(A)}(P, f_i)} \text{Hom}_{D^b(A)}(P, P_i''),$$

for each $1 \leq i \leq n$. Then $\oplus_{i=1}^n Q_i$ is isomorphic to a direct summand of $Q$. We claim that $Q_1, \cdots, Q_n$ are nonzero and each two of them have no common direct summands. Indeed, for each $1 \leq i \leq n$,

$$H^0(Q_i) = \ker \text{Hom}_{D^b(A)}(P, f_i) \cong \text{Hom}_{D^b(A)}(P, P_i[1]) \cong \text{Hom}_{D^b(A)}(P, P_i[tP_i][1])$$

and

$$H^{-1}(vQ_i) = \ker v \text{Hom}_{D^b(A)}(P, f_i) \cong \ker \text{Hom}_{D^b(A)}(P, vP_i) \cong \text{Hom}_{D^b(A)}(P, vP_i) \cong \text{Hom}_{D^b(A)}(P, tvP_i)$$

where $(*)$ holds because $\text{Hom}_{D^b(A)}(P, vP) = D \text{Hom}_{D^b(A)}(P, P)$ is an injective generator of mod $B$. If $Q_i \cong 0$ for some $i$, both $H^0(Q_i)$ and $H^{-1}(vQ_i)$ are isomorphic to zero. Then by Corollary 2.5 we have $P_i/tP_i \cong 0 \cong tvP_i$, where the first isomorphism implies that $P_i \in \text{add } P$, and the second implies that $P_i[1] \in \text{add } P$. This is a contradiction. Hence $Q_i \neq 0$. Note that $P_i$ is a projective cover of $P_i/tP_i$ (if $P_i/tP_i \neq 0$) and $vP_i$ is an injective envelope of $tvP_i$ (if $tvP_i \neq 0$). So by Corollary 2.5 for any $i \neq j$, $H^0(Q_i)$ and $H^0(Q_j)$ have no common direct summands, and $H^{-1}(vQ_i)$ and $H^{-1}(vQ_j)$ have no common direct summands. If $Q_i$ and $Q_j$ have a common direct summand $X$, then $H^0(X) \cong 0 \cong X$. By Lemma 3.7 $X \cong 0$. We finish the proof of the claim. Therefore, $|Q_i| \geq |A|$. 

To prove that $Q$ is silting, it is by Corollary 3.3, sufficient to prove that $Q$ is pre-silting. Let $\alpha$ be a morphism in $\text{Hom}_{K^b(\text{proj } B)}(Q, Q[1])$, then it has the following form

$$\text{Hom}_{D^b(A)}(P, P') \xrightarrow{\text{Hom}_{D^b(A)}(P, f)} \text{Hom}_{D^b(A)}(P, P'') \xrightarrow{\text{Hom}_{D^b(A)}(P, f)} \text{Hom}_{D^b(A)}(P, P''')$$

By Lemma 3.5 there is a morphism $h: P' \to P''$ such that $\alpha = \text{Hom}_{D^b(A)}(P, h)$. Since $\text{Hom}_{D^b(A)}(A, A[1]) = 0$, there are morphisms $h_1, h_2$ such that the following commutative diagram is a morphism of triangles:

$$A \xrightarrow{e} P' \xrightarrow{f} P'' \xrightarrow{g} A[1]$$

By Remark 3.4, the morphism $g$ is a right add $P$-approximation of $A[1]$. So there is a morphism $h_3$ such that $h_2 = gh$. Then $g(h - h_3 f) = gh - gh_3 f = gh - h_2 f = 0$. Hence there is a morphism $h_4$ such that $h - h_3 f = fh_4$. 

$$A \xrightarrow{e} P' \xrightarrow{f} P'' \xrightarrow{g} A[1]$$
Applying $\text{Hom}_{D^b(A)}(P, -)$ to $h - h_3f = fh_4$ yields
\[
\alpha = \text{Hom}_{D^b(A)}(P, h_3) \text{Hom}_{D^b(A)}(P, f) + \text{Hom}_{D^b(A)}(P, f) \text{Hom}_{D^b(A)}(P, h_4)
\]
which implies that $\alpha$, regarded as a map in $\text{Hom}_{D^b(A)}(Q, Q[1])$ is null-homotopic. Thus, we have completed the proof of that $Q$ is a silting complex.

Finally we prove that $T(Q) = X(P)$. The proof of $T(Q) = \mathcal{J}(P)$ is similar. We have $H^0(Q) \cong \text{Hom}_{D^b(A)}(P, A[1]) \cong \text{Hom}_{D^b(A)}(P, A/ta[1])$. By Proposition 3.4 it is therefore sufficient to prove that $\text{Fac Hom}_{D^b(A)}(P, A/ta[1]) = X(P)$.

Let $X$ be in $X(P)$. There is then an object $X'$ in $\mathcal{F}(P)$, such that $X = \text{Hom}_{D^b(A)}(P, X'[1])$. By Proposition 2.8, there is a short exact sequence
\[
0 \to L \to F_0 \to X' \to 0
\]
in $\mathcal{F}(P)$, with $F_0$ in $\text{add} A/tA$. Apply now $\text{Hom}_{D^b(A)}(P, [1])$, to obtain a short exact sequence in $\text{mod} B$ showing that $X$ is in $\text{Fac Hom}_{D^b(A)}(P, A/ta[1])$, so we have $X(P) \subseteq \text{Fac Hom}_{D^b(A)}(P, A/tA)$. On the other hand, $A/tA \in \mathcal{F}(P)$ implies $\text{Hom}_{D^b(A)}(P, A/tA[1]) \in X(P)$, hence $\text{Fac Hom}_{D^b(A)}(P, A/ta[1]) \subseteq X(P)$, since $X(P)$ is closed under factor objects. This concludes the proof. □

**Corollary 3.9.** The induced torsion pair $(X(P), \mathcal{J}(P))$ by $P$ in $\text{mod} B$ is functorially finite.

**Proof.** This follows from Proposition 3.8 and [AdIR] Theorem 2.7, Theorem 3.2. □

### 4. A silting theorem

If $P$ is isomorphic to a tilting $A$-module $T$, then $\nu Q[-1]$ is isomorphic to the cotilting $B$-module $D(T)_B = D \text{Hom}_A(T, A)$, and moreover, the endomorphism algebra of this cotilting module is canonically isomorphic to $A$.

It is easy to check that this does not hold in our setting, that is: in general it does not hold that $\text{End}_{\mathcal{D}^b(B)}(Q)$ is isomorphic to $A$, where $Q$ is the 2-term silting complex in $K^b(\text{proj} A)$, considered in the previous section. However, we prove that $\text{End}_{\mathcal{D}^b(B)}(Q)$ is isomorphic to a factor algebra of $A$. This will then be used to provide mutual equivalences of torsion pairs, as we have in classical tilting theory.

Consider now, as in Remark 3.3 the map $P' \xrightarrow{f} P''$, coming from the triangle $\Delta P$ in Corollary 3.3 as an object $\hat{Q}$ in $K^b(\text{add} P)$, by letting $\hat{Q}' = 0$ for all $i \neq -1, 0$, and recall that $\text{Hom}_{D^b(A)}(P, -)$ induces an algebra isomorphism $\text{End}_{K^b(\text{add} P)}(\hat{Q}) \to \text{End}_{\mathcal{D}^b(B)}(\hat{Q})$.

We will define an algebra-homomorphism $\text{End}_A(A) \to \text{End}_{K^b(\text{add} P)}(\hat{Q})$. For this, represent the object $P'$ by $P^{-1}_\Delta \xrightarrow{p'} P^0_\Delta$, and represent the object $P''$ as the mapping cone of $A \to P'$, that is $P^{-1}_\Delta \oplus A \xrightarrow{(-p', e)} P^0_\Delta$.

Now, let $a$ be an element in $\text{End}_A(A)$. Since $\text{Hom}_{D^b(A)}(P'', P'[1]) = 0$, there is map $b: P' \to P'$ such that $be = ea$. Choose first such a map $b = (b_1, b_2)$. Then, in particular, the following diagram commutes in $\text{proj} A$:

\[
\begin{array}{ccc}
P^{-1}_\Delta & \xrightarrow{p'} & P^0_\Delta \\
| & | & | \\
b_1 & | & b_2 \\
\downarrow & & \downarrow \\
P^{-1}_\Delta & \xrightarrow{p'} & P^0_\Delta
\end{array}
\]
Now since the diagram
\[
\begin{array}{ccc}
A & \xrightarrow{e} & P' \\
\downarrow{a} & & \downarrow{(b_1, b_2)} \\
A & \xrightarrow{e} & P'
\end{array}
\]
commutes in \(K^b(\text{add } P)\), we have a map \(A \xrightarrow{t} P^{-1}_\Delta\), such that \(p't = ea - b_2e\).

Next, consider the endomorphism \(c\) of
\[
P^{-1}_\Delta \oplus A \xrightarrow{(-p' e)} \hat{P}_\Delta^0
\]
given as follows
\[
P^{-1}_\Delta \oplus A \xrightarrow{\begin{pmatrix} b_1 & t \\ 0 & a \end{pmatrix}} \xrightarrow{b_2} P^{-1}_\Delta \oplus A \xrightarrow{(-p' e)} \hat{P}_\Delta^0
\]
It is straightforward to check, that we obtain a morphism of triangles
\[
\begin{array}{ccc}
A & \xrightarrow{e} & P' \\
\downarrow{a} & & \downarrow{b} \\
A & \xrightarrow{e} & P'
\end{array}
\begin{array}{ccc}
P' & \xrightarrow{f} & P'' \\
\downarrow{c} & & \downarrow{g} \\
P' & \xrightarrow{f} & P''
\end{array}
\begin{array}{ccc}
P' & \xrightarrow{g} & A[1] \\
\downarrow{a[1]} & & \downarrow{g} \\
P' & \xrightarrow{g} & A[1]
\end{array}
\]
where \(f\) and \(g\) now denote the maps
\[
P^{-1}_\Delta \xrightarrow{p'} \hat{P}_\Delta^0 \\
P^{-1}_\Delta \oplus A \xrightarrow{(-p' e)} P^0_\Delta
\]

**Proposition 4.1.** The map \(\Phi_P: \text{End}_A(A) \rightarrow \text{End}_{K^b}(\hat{Q})\) given by \(a \mapsto (b, c)\) is a well-defined and surjective algebra morphism with kernel given by
\[
\{v | u \in \text{Hom}_{D^b(A)}(A, P_I), \alpha \in \text{Hom}_{D^b(A)}(P_I, P_{II}[-1]) \text{ and } v \in \text{Hom}_{D^b(A)}(P_{II}[-1], A \text{ with } P_I, P_{II} \in \text{add } P}\}.
\]
Moreover, we have \(\ker \Phi_P = 0\) if and only if \(\text{Hom}_{D^b(A)}(P, P[-1]) = 0\).

**Proof.** The map is clearly additive, so in order to show that \(\Phi_P\) is well-defined, it suffices to show that a map \((b, c)\) in \(\text{End}_{K^b(\text{add } P)}(\hat{Q})\) of the form
\[
P' \xrightarrow{\begin{pmatrix} -1 \\ 0 \end{pmatrix}} P''
\]
where \(p't = -b_2e\), is zero-homotopic in \(K^b(\text{add } P)\).
Consider the map $P'' \xrightarrow{\mu} P'$ defined as follows:

$$
\begin{array}{ccc}
P_{-1}^{-1} \oplus A & \xrightarrow{(-p', e)} & P_{-1}^{0} \\
(-b_{1} -t) & \downarrow & b_{2} \\
\downarrow & & \downarrow \\
P_{-1}^{-1} & \xrightarrow{p'} & P_{-1}^{0}
\end{array}
$$

Then it is easily verified that $\mu f = b$, and that $f \mu = c$ in add $P$. Hence, $\Phi_{P}$ is well-defined, and it is easy to check that it is an algebra homomorphism.

We next show that $\Phi_{P}$ is surjective. Consider an arbitrary map $(b, c)$ in $\text{End}_{\text{add} P}(\hat{Q})$ represented by

$$
\begin{array}{ccc}
P' & \xrightarrow{((b_{1}, b_{2}))(\frac{-1}{0}, 1)} & P'' \\
\downarrow & & \downarrow & & \downarrow \\
P' & \xrightarrow{((b_{1}, b_{2}))(\frac{-1}{0}, 1)} & P''
\end{array}
$$

It is sufficient to show that such map is equivalent to a map of the form

$$
\begin{array}{ccc}
P' & \xrightarrow{((b_{1}, b_{2}))(\frac{-1}{0}, 1)} & P'' \\
\downarrow & & \downarrow & & \downarrow \\
P' & \xrightarrow{((b_{1}, b_{2}))(\frac{-1}{0}, 1)} & P''
\end{array}
$$

for some value of $a$, and for a $u$ satisfying $p'u = ea - b_{2}e$.

Since $cf = fb$, we have that the following maps

$$
\begin{array}{ccc}
P_{-1}^{-1} & \xrightarrow{p'} & P_{-1}^{0} \\
(-c_{1}) & \downarrow & c_{0} \\
\downarrow & & \downarrow \\
P_{-1}^{-1} & \xrightarrow{p'} & P_{-1}^{0}
\end{array}
$$

are homotopic in $K^{b}(\text{proj} A)$. Hence, there exists $(\frac{c_{1}}{c_{3}}) : P_{-1}^{0} \rightarrow P_{-1}^{-1} \oplus A$, such that $(\frac{c_{1}}{c_{3}}) p' = (c_{1} - b_{1})$ and $b_{2} = -c_{0} = (p' e) (\frac{c_{2}}{c_{4}} + ye) = -p'xe + ey$.

It is now straightforward to verify that the map $c = (\frac{c_{1}}{c_{3}}) \oplus (\frac{c_{2}}{c_{4}}) \oplus \frac{c_{0}}{c_{4}} + ye$ is homotopic to the map $\hat{Q}(b_{1}, c_{2} + xe, b_{2})$, and that $u := c_{2} + xe$ satisfies $p'u = ea - b_{2}e$ where $a = c_{4} + ye$. This proves the claim, and hence $\Phi_{P}$ is surjective.

Assume now $a$ is in the kernel in $\Phi_{P}$, so that $(b, c)$ is homotopic to zero. That is, there exists a map

$$
\begin{array}{ccc}
P_{-1}^{-1} \oplus A & \xrightarrow{(-p', e)} & P_{-1}^{0} \\
(d_{1} d_{2}) & \downarrow & w \\
\downarrow & & \downarrow \\
P_{-1}^{-1} & \xrightarrow{p'} & P_{-1}^{0}
\end{array}
$$

such that $(b_{1}, b_{2})$ is homotopic to $(-d_{1}, w) = ((d_{1}, d_{2}), w)((\frac{-1}{0}, 1), 1)$ and such that $((\frac{b_{1}}{b_{2}}), b_{2})$ is homotopic to $((-d_{1}, -d_{2}), w) = ((\frac{-1}{0}, 1), 1)((d_{1}, d_{2}), w)$. 


There is then a map \( \delta : P^0_\Delta \to P_\Delta^{-1} \) and such that \( p'\delta = b_2 - w \) and \( \delta p' = b_1 + d_1 \), and a map 
\((\epsilon : P^0_\Delta \to P_\Delta^{-1} \oplus A) \) such that 
\[
\begin{pmatrix}
\epsilon \\
\delta
\end{pmatrix}
(\begin{pmatrix}
p' \\
\theta
\end{pmatrix}) = \begin{pmatrix} -ep' & \epsilon \\
-\theta p' & \theta \end{pmatrix} = \begin{pmatrix} b_1 + d_1 & t + d_2 \\
0 & a \end{pmatrix}
\]
and such that \((-ep') \cdot (\epsilon : P^0_\Delta \to P_\Delta^{-1} \oplus A)) = -p'\epsilon + e\theta = b_2 - w.\) Combining these equations we obtain 
\[
p'(\delta + \epsilon) = e\theta \quad (\delta + \epsilon)p' = 0.
\]
Note that in particular we have \( \theta p' = 0 \) and \( \theta e = a \). By this we obtain that the map \( e : A \to A \)
factors as follows

\[
\begin{array}{c}
A \\
\downarrow e \\
P^0_\Delta \\
\downarrow (\delta + \epsilon) \\
P_\Delta^{-1} \oplus A \\
\downarrow (-p') \\
P_\Delta^{-1} \oplus A \\
\downarrow (0 1) \\
A
\end{array}
\]

So we have proved that 
\[
\ker \Phi \subseteq I = \{ \nu \alpha u | u \in \text{Hom}_{D^b(A)}(A, P_I), \alpha \in \text{Hom}_{D^b(A)}(P_I, P_{II}[1]) \text{ and } \nu \in \text{Hom}_{D^b(A)}(P_{II}[1], A) \text{ with } P_I, P_{II} \in \text{add} P \}.
\]

Next, we prove that \( I \subseteq \ker \Phi \). Let \( a \) be an element in \( I \). Since the map \( e : A \to P' \) is a left 
add \( P \)-approximation, and the map \( g : P' \to A[1] \) is a right add \( P \)-approximation, we have that 
\( a = g[-1]|ue \) for some map \( u : P' \to P'[\cdot -1] \).

Now, assume \( u \) is represented by \( P^0_\Delta \xrightarrow{u_1} P_\Delta^{-1} \oplus A \), so we have 
\( \begin{pmatrix} u_1 \\
u_2 \end{pmatrix} = \begin{pmatrix} 0 \\
u_2 \end{pmatrix} \) and \( p'u_1 = eu_2 \) and 
\( a = u_2 e \).

Consider the map in \( \text{End}_{K^b(\text{add} P)}(Q) \) given by
\[
\begin{array}{c}
P' \\
\downarrow f \\
P'' \\
\downarrow f \\
P''
\end{array}
\]

Since \( a = u_2 e \), this map must be homotopic to \( \Phi(a) \). The map \( (0, eu_2) \) is nullhomotopic in 
\( K^b(\text{proj} A) \), since \( u_1 p' = 0 \) and \( eu_2 = p'u_1 \). Moreover, the map \( \begin{pmatrix} 0 & 0 \\
0 & u_2 e \end{pmatrix} \) is also nullhomotopic in 
\( K^b(\text{proj} A) \), since 
\[
\begin{pmatrix} 0 & 0 \\
u_2 & u_2 e \end{pmatrix} = \begin{pmatrix} 0 & 0 \\
0 & 0 \end{pmatrix},
\]
and \( (-p') \cdot \begin{pmatrix} 0 \\
u_2 \end{pmatrix} = eu_2 \). Hence \( a \) is in \( \ker \Phi \).

We are now left with proving that \( \ker \Phi \) is 0 if and only if \( \text{Hom}_{D^b(A)}(P, P[1]) = 0 \). By the first 
part, we have that \( \text{Hom}_{D^b(A)}(P, P[1]) = 0 \) implies that \( \ker \Phi = 0 \). Assume \( \text{Hom}_{D^b(A)}(P, P[1]) \neq 0 \).
0. Then $\text{Hom}_{D^b(A)}(P, P[-1])$ contains a non-zero element $\eta$, which is a chain map:

$$
\begin{array}{ccc}
P^{-1} & \xrightarrow{\eta} & P^0 \\
\downarrow \quad \eta & & \downarrow \\
P^{-1} & \xrightarrow{\eta'} & P^0
\end{array}
$$

So there are $P_i$, $P_j$, indecomposable direct summands of $P^0$, $P^{-1}$ respectively, such that the component of $\eta$ from $P_i$ to $P_j$ is not zero. This induces a non-zero morphism $a_\eta$ in $\text{Hom}_A(A, A)$ which factors through $\eta$. Then $a_\eta$ is in $\text{ker} \Phi_\eta$. This concludes the proof. \qed

The following corollary shows that in the tilting case, our result covers the classical result.

**Corollary 4.2.** Under the same notation as before, $P$ is a tilting complex if and only if $\Phi_\eta$ is an isomorphism. In this case, $Q$ is also tilting.

**Proof.** Clearly $P$ is a tilting complex if and only if $\text{Hom}_{D^b(A)}(P, P[-1]) = 0$. Hence, the equivalence follows directly from the last part of Proposition 4.1. Assume now $\text{Hom}_{D^b(A)}(P, P[-1]) = 0$. It suffices to prove that then also $\text{Hom}_{D^b(B)}(Q, Q[-1]) = 0$. Note that each morphism $\alpha$ in $\text{Hom}_{D^b(B)}(Q, Q[-1])$ has the following form:

$$
\begin{array}{ccc}
\text{Hom}_{D^b(A)}(P, P') & \xrightarrow{\text{Hom}_{D^b(A)}(P_f)} & \text{Hom}_{D^b(A)}(P, P'') \\
\downarrow \quad \alpha & & \downarrow \\
\text{Hom}_{D^b(A)}(P, P') & \xrightarrow{\text{Hom}_{D^b(A)}(P_f)} & \text{Hom}_{D^b(A)}(P, P'')
\end{array}
$$

with $\alpha \text{Hom}_{D^b(A)}(P, f) = 0 = \text{Hom}_{D^b(A)}(P, f)\alpha$. By Lemma 3.5 there exist $h : P'' \to P'$ with $\alpha = \text{Hom}_{D^b(A)}(P, h)$ and $hf = 0 = fh$. Hence there exists the following morphism $h_1$:

$$
\begin{array}{ccc}
A & \xrightarrow{e} & P' & \xrightarrow{f} & P'' & \xrightarrow{g} & A[1] \\
\downarrow h_2 & & \downarrow h_1 & & \downarrow h & & \downarrow &
\\
P''[1] & \xrightarrow{-[1]} & A & \xrightarrow{e} & P' & \xrightarrow{f} & P''
\end{array}
$$

such that $h = eh_1$. Due to $eh_1f = hf = 0$, there exists $h_2$ such that $-g[-1]h_2 = h_1f$. But $h_2 \in \text{Hom}_{D^b(A)}(P', P''[-1]) = 0$, so $h_1$ factors through $g$ and then it is zero since $\text{Hom}_{D^b(A)}(A[1], A) = 0$. Therefore, $h = 0$ which implies that $\alpha = 0$. Thus, $Q$ is tilting. \qed

By now we have proved parts (d) and (e) of Theorem 1.1, we next finish the proofs of (f) and (g). Adopting earlier notation, we let $X(Q) = \text{Hom}_{D^b(B)}(Q, T(Q)[1])$ and $Y(Q) = \text{Hom}_{D^b(B)}(Q, T(Q))$. Now, by Corollary 2.5 we have that $\text{Hom}_{D^b(B)}(Q, \cdot)$ induces equivalences $T(Q) \to Y(Q)$ and $T(Q)[1] \to X(P)$.

**Theorem 4.3.** Let $\Phi_\eta : \text{mod End}_{D^b(B)}(Q) \hookrightarrow \text{mod} A$ be the inclusion functor induced by $\Phi_\eta$. Then $\Phi_\eta(X(Q)) = T(P)$ and $\Phi_\eta(Y(Q)) = T(P)$.

**Proof.** We prove that $\Phi_\eta(Y(Q)) = T(P)$. The proof of $\Phi_\eta(X(Q)) = T(P)$ is similar. By Proposition 3.3 we have that $T(Q) = X(P)$, so we obtain that

$$
Y(Q) = \text{Hom}_{D^b(B)}(Q, T(Q)) = \text{Hom}_{D^b(B)}(Q, X(P)) = \text{Hom}_{D^b(B)}(Q, \text{Hom}_{D^b(A)}(P, T(P)[1])).
$$

Then to complete the proof, we only need to prove that for any $Y \in T(P)$, there is an isomorphism of $A$-modules $Y \cong \text{Hom}_{D^b(B)}(Q, \text{Hom}_{D^b(A)}(P, Y[1]))$. Note first that $\text{Hom}_{D^b(B)}(Q, \text{Hom}_{D^b(A)}(P, Y[1]))$
is the kernel of the map
\[ \text{Hom}_B(\text{Hom}_{D^A}(P, f), \text{Hom}_{D^A}(P, Y[1])) : \text{Hom}_B(\text{Hom}_{D^A}(P, P'), \text{Hom}_{D^A}(P, Y[1])) \rightarrow \text{Hom}_B(\text{Hom}_{D^A}(P, P'), \text{Hom}_{D^A}(P, Y[1])). \]

By Lemma 3.5, this is isomorphic to the kernel of
\[ \text{Hom}_{D^A}(f, Y[1]) : \text{Hom}_{D^A}(P', Y[1]) \rightarrow \text{Hom}_{D^A}(P', Y[1]). \]

Applying \( \text{Hom}_{D^A}(\_ , Y[1]) \) to the triangle \( \Delta_P \), and using that \( \text{Hom}_{D^A}(P', Y) = 0 \), since \( Y \) is in \( \mathcal{F}(P) \), we obtain that ker \( \text{Hom}_{D^A}(f, Y[1]) \) \( \cong \text{Hom}_A(A, Y) \).

Using the construction of \( \Phi_P \), we have the commutative diagram

\[
\begin{array}{ccc}
A & \rightarrow & P' \\
| & & | \\
\downarrow & & \downarrow c \\
A & \rightarrow & P'' \\
& & A[1]
\end{array}
\]

where both of the first row and the second row are \( \Delta_P \) and \( \Phi_P(a) = (b, c) \), it is straightforward to check that the above isomorphisms are \( A \)-module maps. Thus, by \( Y \cong \text{Hom}_A(A, Y) \), we get the desired isomorphism.

\[ \square \]

5. Auslander-Reiten theory

As an application of Theorem 1.1 we show how the AR-theory of \( B = \text{End}_{D^A}(P) \) can be understood in terms of the AR-theory of \( A \). In the case where \( A \) is hereditary, we obtain particularly strong results. These will turn out to be essential for studying the so-called silted algebras obtained as \( \text{End}_{D^A}(P) \), for a 2-term silting complex \( P \) over a hereditary algebra \( A \). Such algebras are investigated and characterized in [BZ].

5.1. Connecting sequences. In this section we describe almost split sequences in \( \text{mod} \ B \). Similarly as in classical tilting theory, we call an almost split sequence in \( \text{mod} \ B \) whose left term lies in \( \mathcal{Y}(P) \) and whose right term lies in \( \mathcal{X}(P) \) a connecting sequence. We denote the AR-translation in a module category by \( \tau \).

**Lemma 5.1.** If \( 0 \rightarrow Y \rightarrow E \rightarrow X \rightarrow 0 \) is a connecting sequence, then there exists an indecomposable projective \( A \)-module \( P_i \) such that \( Y \cong \text{Hom}_{D^A}(P, vP_i) \).

**Proof.** Since \( Y \in \mathcal{Y}(P) \) and \( X = \tau^{-1}Y \in \mathcal{X}(P) \), by [Sma] Lemma 0.1, \( Y \) is Ext-injective in \( \mathcal{Y}(P) \). Then by Proposition 2.3(3), there is an indecomposable \( A \)-module \( Y' \in \text{add} vA \) such that \( Y \cong \text{Hom}_{D^A}(P, Y') \). Note that for each indecomposable projective \( A \)-module \( P_i \), if \( vP_i \neq 0 \), then it is indecomposable since \( vP_i \) is its injective envelope. So \( Y \cong \text{Hom}_{D^A}(P, vP_i) \) for some indecomposable projective \( A \)-module \( P_i \). Hence \( Y \cong \text{Hom}_{D^A}(P, vP_i) \) by Lemma 3.6 \( \square \)

Note that \( \text{Hom}_{D^A}(P, vP_i) = 0 \) if and only if \( vP_i \in \mathcal{F}(P) \) if and only if \( vP_i \in \text{add} H^{-1}(vP) \) if and only if \( P_i[1] \in \text{add} P \). The following lemma is a generalization of the connecting lemma in tilting theory.

**Lemma 5.2.** Let \( P_i \) be an indecomposable projective \( A \)-module with \( P_i[1] \notin \text{add} P \). Then
\[ \tau^{-1} \text{Hom}_{D^A}(P, vP_i) \cong \text{Hom}_{D^A}(P, P_i[1]). \]

In particular, \( \text{Hom}_{D^A}(P, vP_i) \) is an injective \( B \)-module if and only if \( P_i \in \text{add} P \).
Proof. By Corollary 3.3 there is a triangle

\[ P_i \rightarrow P'_i \rightarrow P''_i \rightarrow P_i[1] \]

with \( P'_i, P''_i \in \text{add } P \). Applying \( \text{Hom}_{D^b(A)}(\cdot, P) \) to the triangle, we have an exact sequence

\[
\text{Hom}_{D^b(A)}(P''_i, P) \xrightarrow{\text{Hom}_{D^b(A)}(f_i, P)} \text{Hom}_{D^b(A)}(P'_i, P) \rightarrow \text{Hom}_{D^b(A)}(P_i, P) \rightarrow 0
\]

which is a projective presentation of \( \text{Hom}_{D^b(A)}(P_i, P) \) as a left \( B \)-module. For \( P_0 \in \text{add } P \), we have a functorial isomorphism

\[
\text{Hom}_{D^b(A)}(P, P_0) \cong \text{Hom}_B(\text{Hom}_{D^b(A)}(P_0, P), \text{Hom}_{D^b(A)}(P, P))
\]

g \mapsto (h \mapsto hg)

as right \( B \)-modules. Hence we have the following commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_{D^b(A)}(P, P'_i) & \cong & \text{Hom}_B(\text{Hom}_{D^b(A)}(P_0, P), B) \\
\downarrow & & \downarrow \\
\text{Hom}_{D^b(A)}(P, P''_i) & \cong & \text{Hom}_B(\text{Hom}_{D^b(A)}(P'_i, P), B)
\end{array}
\]

Then we have that

\[ \text{Tr} \text{Hom}_{D^b(A)}(P_i, P) \cong \text{coker} \text{Hom}_B(\text{Hom}_{D^b(A)}(f_i, P), B) \cong \text{coker} \text{Hom}_{D^b(A)}(P, f_i) \cong \text{Hom}_{D^b(A)}(P, P_i[1]).\]

Therefore by the definition of Auslander-Reiten translation, we have

\[ \tau^{-1} \text{Hom}_{D^b(A)}(P, \nu P_i) \cong \text{Tr} \text{D} \text{Hom}_{D^b(A)}(P, \nu P_i) \cong \text{Tr} \text{Hom}_{D^b(A)}(P_i, P) \cong \text{Hom}_{D^b(A)}(P, P_i[1]). \]

Note that we have \( \text{Hom}_{D^b(A)}(P, P_i[1]) = 0 \) if and only if \( P_i \in \mathcal{T}(P) \) if and only if \( P_i \in \text{add } H^0(P) \) if and only if \( P_i \in \text{add } P \). Then the last statement of this lemma follows from this, combined with the fact that if \( P_i \in \text{add } P \), then \( P_i[1] \not\in \text{add } P \).

Hence, we have shown that the connecting sequences are of the form

\[ 0 \rightarrow \text{Hom}_{D^b(A)}(P, \nu P_i) \rightarrow E \rightarrow \text{Hom}_{D^b(A)}(P, P_i[1]) \rightarrow 0.\]

It remains to describe the middle term \( E \).

**Corollary 5.3.** Let \( P_i \) be an indecomposable projective \( A \)-module with \( P_i \notin \text{add } P \) and \( P_i[1] \notin \text{add } P \) and \( E \) be the middle term of the almost split sequence starting at \( \text{Hom}_{D^b(A)}(P, \nu P_i) \). Then the canonical sequence of \( E \) in the torsion pair \( (\mathcal{X}(P), \mathcal{Y}(P)) \) is

\[ 0 \rightarrow \text{Hom}_{D^b(A)}(P, \nu P_i) \rightarrow E \rightarrow \text{Hom}_{D^b(A)}(P, \nu P_i/S_i) \rightarrow 0 \]

where \( \text{rad } P_i \) denotes the radical of \( P_i \) and \( S_i \) is the simple module \( P_i/\text{rad } P_i \).

**Proof.** Since \( (\mathcal{T}(P), \mathcal{F}(P)) \) is a torsion pair, \( S_i \) is either in \( \mathcal{T}(P) \) or in \( \mathcal{F}(P) \). We refer to the proof of [ASS, Corollary VI.4.10] where the first part (i.e. the case \( S_i \in \mathcal{T}(P) \)) works in our case by a small suitable modification. However, the second part does not work in our case, instead, one need to use the dual proof of the first part. So for the convenience of readers, we give a proof for the case \( S_i \in \mathcal{F}(P) \). Applying \( \text{Hom}_{D^b(A)}(P, \nu) \) to the short exact sequence \( 0 \rightarrow \text{rad } P_i \rightarrow P_i \rightarrow S_i \rightarrow 0 \) yields a short exact sequence

\[ 0 \rightarrow \text{Hom}_{D^b(A)}(P, \text{rad } P_i[1]) \rightarrow \text{Hom}_{D^b(A)}(P, P_i[1]) \rightarrow \delta \rightarrow \text{Hom}_{D^b(A)}(P, S_i[1]) \rightarrow 0.\]
Consider the short exact sequences

\[
\begin{array}{ccccccccc}
0 & \rightarrow & S_i & \xrightarrow{\alpha} & vP_i & \xrightarrow{\beta} & vP_i/S_i & \rightarrow & 0 \\
& & & \downarrow{\gamma} & & & & \downarrow{\gamma} & \\
& & & vP_i/tyP_i & & & & 0 & \\
\end{array}
\]

Since \( P_i[1] \not\in \mathbf{P} \), we have that \( \gamma \colon vP_i \rightarrow vP_i/tyP_i \) is not an isomorphism. Then the composition \( \gamma \alpha = 0 \). So \( \gamma \) factors through \( \beta \). Because \( \text{Hom}_{D^b(A)}(P, \gamma[1]) \) is an isomorphism by Lemma 3.6, the map \( \text{Hom}_{D^b(A)}(P, \beta[1]) \) is a monomorphism. Hence we have a short exact sequence

\[
0 \rightarrow \text{Hom}_{D^b(A)}(P, vP_i) \xrightarrow{\theta} \text{Hom}_{D^b(A)}(P, vP_i/S_i) \rightarrow \text{Hom}_{D^b(A)}(P, S_i[1]) \rightarrow 0
\]

where \( \theta = \text{Hom}_{D^b(A)}(P, \beta) \). Since \( \text{Hom}_{D^b(A)}(P, S_i[1]) \in \mathbf{X}(P) \) and \( \text{Hom}_{D^b(A)}(P, vP_i/S_i) \in \mathbf{Y}(P) \) by Lemma 3.6 and \( \text{Hom}_{D^b(A)}(P, S_i[1]) \neq 0 \) by \( S_i \in \mathcal{F}(P) \), the sequence is not split. In particular, \( \theta \) is not a section, so there is a commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \text{Hom}_{D^b(A)}(P, vP_i) & \xrightarrow{\theta} & \text{Hom}_{D^b(A)}(P, vP_i/S_i) & \rightarrow & \text{Hom}_{D^b(A)}(P, S_i[1]) & \rightarrow & 0 \\
& & \downarrow{\delta} & & \downarrow{\delta} & & \downarrow{h} & & \\
0 & \rightarrow & \text{Hom}_{D^b(A)}(P, vP_i) & \xrightarrow{\theta} & \text{Hom}_{D^b(A)}(P, vP_i/S_i) & \rightarrow & \text{Hom}_{D^b(A)}(P, S_i[1]) & \rightarrow & 0
\end{array}
\]

where the upper sequence is the AR-sequence starting at \( \text{Hom}_{D^b(A)}(P, vP_i) \). Note that \( h \neq 0 \), since otherwise the upper sequence would be split exact.

Since \( \text{Hom}_{D^b(A)}(P, P_i[1]) \cong \text{Hom}_{D^b(A)}(P, P_i/tP_i[1]) \) by Lemma 3.6 and \( \text{Hom}_{D^b(A)}(P, -[1]) \) is an equivalence from \( \mathcal{F}(P) \) to \( \mathbf{X}(P) \) by Corollary 2.5, we have that

\[
\text{Hom}_{B}(\text{Hom}_{D^b(A)}(P, P_i[1]), \text{Hom}_{D^b(A)}(P, S_i[1])) \cong \text{Hom}_{A}(P_i/tP_i, S_i).
\]

Using that \( S_i \) is in \( \mathcal{F}(P) \), by assumption, we have \( \text{Hom}_{A}(P_i/tP_i, S_i) \cong \text{Hom}_{A}(P_i, S_i) \), which is a one dimensional space. Therefore, since \( h \neq 0 \), it equals \( k \delta \), for an element \( k \in \mathbf{k} \). Hence, \( \ker h \cong \text{Hom}_{D^b(A)}(P, \text{rad} P_i[1]) \). Using the snake lemma, we obtain the following commutative
5.2. Separating and splitting silting complexes. Recall that a torsion pair \((\mathcal{X}, \mathcal{Y})\) in \(\text{mod}\ A\) is called split (or sometimes splitting) if each indecomposable \(A\)–module lies either in \(\mathcal{X}\) or in \(\mathcal{Y}\), see [ASS]. In other words, \((\mathcal{X}, \mathcal{Y})\) is split if and only if \(\text{Ext}_A^1(Y, X) = 0\) for all \(X \in \mathcal{X}\) and \(Y \in \mathcal{Y}\).

**Definition 5.4.** Let \(A\) be a finite dimensional algebra, let \(P\) be a 2-term silting complex in \(K^b(\text{proj} \ A)\) and \(B = \text{End}_{D^b(A)}(P)\). Then

1. \(P\) is called separating if the induced torsion pair \((\mathcal{T}(P), \mathcal{F}(P))\) in \(\text{mod}\ A\) is split, and
2. \(P\) is called splitting if the induced torsion pair \((\mathcal{X}(P), \mathcal{Y}(P))\) in \(\text{mod}\ B\) is split.

**Lemma 5.5.** A 2-term silting complex \(P\) is splitting if and only if \(\text{Ext}_A^2(\mathcal{T}(P), \mathcal{F}(P)) = 0\).

**Proof.** This follows from the second isomorphism in Corollary 2.6. \(\square\)

Note that in particular Lemma 5.5 implies that if \(A\) is hereditary, then all 2-term silting complexes are splitting. In a forthcoming paper, [BZ], we study endomorphism rings of 2-term silting complexes over hereditary algebras. We now state a result which is of particular importance for describing the AR-theory of silted algebras.

**Proposition 5.6.** If a silting complex \(P\) is splitting, then any almost split sequence in \(\text{mod}\ B\) lies entirely in either \(\mathcal{X}(P)\) or \(\mathcal{Y}(P)\), or else it is of the form

\[
0 \to \text{Hom}_{D^b(A)}(P, vP_i) \to \text{Hom}_{D^b(A)}(P, \text{rad} \ P_i[1]) \oplus \text{Hom}_{D^b(A)}(P, vP_i/S_i) \to \text{Hom}_{D^b(A)}(P, P_i[1]) \to 0,
\]

where \(P_i\) is an indecomposable projective \(A\)–module with \(P_i \notin \text{add} \ P\) and \(P_i[1] \notin \text{add} \ P\). Moreover, almost split sequences in \(\mathcal{T}(P)\) and \(\mathcal{F}(P)\) are by \(\text{Hom}_{D^b(A)}(P, -)\) and \(\text{Hom}_{D^b(A)}(P, [-1])\) mapped to almost split sequences in \(\mathcal{Y}(P)\) and \(\mathcal{X}(P)\), respectively.

**Proof.** The first statement follows from Lemma 5.2 and Corollary 5.3.

For the second statement, we only prove the statement for \(\mathcal{T}(P)\), since the proof for \(\mathcal{F}(P)\) is similar. Let

\[
0 \to X_1 \xrightarrow{a} X_2 \xrightarrow{\beta} X_3 \to 0
\]

be an almost split sequence in \(\mathcal{T}(P)\). Then by Corollary 2.5, we have a short exact sequence in \(\mathcal{Y}(P)\):

\[
0 \to \text{Hom}_{D^b(A)}(P, X_1) \xrightarrow{\text{Hom}_{D^b(A)}(P, a)} \text{Hom}_{D^b(A)}(P, X_2) \xrightarrow{\text{Hom}_{D^b(A)}(P, \beta)} \text{Hom}_{D^b(A)}(P, X_3) \to 0
\]
where $\text{Hom}_{D(A)}(P, X_1)$ and $\text{Hom}_{D(A)}(P, X_3)$ are indecomposable. Let $Y$ be an indecomposable $B$-module, then $Y \in \mathcal{X}(P)$ or $Y \in \mathcal{Y}(P)$. To complete the proof, by e.g. [ASS, Theorem IV.1.13], it is sufficient to prove the following claim: each homomorphism from $Y$ to $\text{Hom}_{D(A)}(P, X_3)$ which is not a split epimorphism factors through $\text{Hom}_{D(A)}(P, X_3)$ is an equivalence from $\mathcal{T}(P)$ to $\mathcal{Y}(P)$. Now we assume that $Y \in \mathcal{X}(P)$. Then $\text{Hom}_B(Y, \text{Hom}_{D(A)}(P, X_3)) = 0$, so there is nothing left to prove.

**Proposition 5.7.** Each separating 2-term silting complex $P$ is a tilting complex.

**Proof.** By Corollary 4.2, it is sufficient to prove that $\Phi_P$ is an isomorphism. This is equivalent to prove that the induced functor $\Phi_\ast: \text{mod } \text{End}_{D(B)}(Q) \leftrightarrow \text{mod } A$ is an equivalence. Since $\Phi_\ast$ is always fully faithful, we only need to prove that $\Phi_\ast$ is surjective. Since $P$ is separating, each indecomposable $A$-module $M$ is either in $\mathcal{T}(P)$ or in $\mathcal{F}(P)$. Then by Theorem 4.3 there is an $N \in \text{mod } \text{End}_{D(B)}(Q)$ such that $\Phi_\ast(N) = M$. Thus, we complete the proof.

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