Report on Progress

Ultracold Fermi gases with emergent SU($N$) symmetry

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Abstract

We review recent experimental and theoretical progress on ultracold alkaline-earth Fermi gases with emergent SU($N$) symmetry. Emphasis is placed on describing the ground-breaking experimental achievements of recent years. The latter include (1) the cooling to below quantum degeneracy of various isotopes of ytterbium and strontium, (2) the demonstration of optical Feshbach resonances and the optical Stern–Gerlach effect, (3) the realization of a Mott insulator of $^{173}_{\text{Yb}}$ atoms, (4) the creation of various kinds of Fermi–Bose mixtures and (5) the observation of many-body physics in optical lattice clocks. On the theory side, we survey the zoo of phases that have been predicted for both gases in a trap and loaded into an optical lattice, focusing on two and three dimensional systems. We also discuss some of the challenges that lie ahead for the realization of such phases such as reaching the temperature scale required to observe magnetic and more exotic quantum orders. The challenge of dealing with collisional relaxation of excited electronic levels is also discussed.

Keywords: ultracold atoms, degenerate fermi gases, strongly correlated systems

(Some figures may appear in colour only in the online journal)

1. Introduction

Prior to the late 20th century, matter was primarily something to be probed, dissected and understood. Now matter is something to be synthesized, organized and exploited for broader purposes, both at the level of basic research and for numerous technological applications. One emerging area of research is to implement Richard Feynman’s pioneering ideas of quantum simulation [1] and quantum information [2]. We want to design artificial, fully controllable quantum systems and use them to model many-body systems to solve otherwise intractable problems in materials physics and other branches of modern quantum science.

Recent advances in the cooling and trapping of alkali atoms have brought us closer to realizing Feynman’s dreams. The simple electronic structure of alkalai atoms, which possess a single valence electron, has allowed us to characterize their hyperfine levels, greatly facilitating the development of effective trapping and quantum control techniques. Using these atoms, experimental physicists have achieved major breakthroughs such as a detailed understanding of the Bose–Einstein Condensate (BEC) to Bardeen–Cooper–Shrieffer (BCS) crossover [3, 4] and the implementation of both the Fermi and Bose Hubbard models [4–8].

Nevertheless, the inherent ‘simplicity’ of alkali atoms introduces major limitations to the phenomena that can be explored with them. For example, the observation of quantum magnetism in Fermi/Bose Hubbard models has been hindered by the low entropy requirements set by the energy scales of their effective
spin–spin interactions. Consequently, systems exhibiting a more complex internal structure could provide an excellent platform for exploring a wider range of many-body phenomena. Complex systems may also lead to the discovery of new states of matter that go beyond the possibilities already offered by conventional condensed-matter systems. There have been exciting advances in this direction during the last few decades as new capabilities have been demonstrated for cooling, trapping and manipulating more complex systems such as trapped ions, magnetic atoms, Rydberg atoms, polar molecules and alkaline-earth atoms (AEAs). In this review, we focus on AEAs.

Strictly speaking, alkaline-earth atoms (AEA) lie in group-II of the periodic table. However, we will also include others with similar electronic structure like Ytterbium (Yb). These atoms have unique atomic properties which make them ideal for the realization of ultra-precise atomic clocks. Lately, as we shall explain below, they are also attracting a great deal of attention for their interesting many-body physics and the possibilities that they offer for the quantum simulation of complex quantum systems.

Nevertheless, before immersing ourselves in the study of their fascinating many-body physics, it is worth recalling that atomic clocks provide one of the most striking illustrations of the unique advantage of AEAs over alkali atoms. State-of-the-art optical clocks use fermionic AEAs, such as Sr or Yb [9]. Those clocks have already surpassed the accuracy of the Cs standard [10]. The most stable of these clocks now operates near the quantum noise limit [11, 12] and just recently, thanks to advances in modern precision laser spectroscopy, is becoming the most precise in the world, even surpassing the accuracy of single ion standards [13]. The stability of the neutral atom optical clocks arises from the extremely long lived singlet, $^1S_0$ and triplet states $^3P_0$, generally referred to as clock states, with intercombination lines both electric and magnetic dipole forbidden and as narrow as a few mHz—nine orders of magnitude lower than a typical dipole-allowed electronic transition (see figure 1). It is impossible to achieve this level of clock stability with conventional alkali atoms, due to the decoherence that arises from the intrinsic sensitivity of the hyperfine ground states to magnetic field fluctuations and/or to intensity and phase noise on the optical fields.

Returning to many-body physics and quantum emulation using AEAs, in this article we attempt to review the experimental and theoretical progress in this area. Given the large amount of recent research, we mainly focus on the consequences of their emergent SU($N$) symmetry of the AEA Fermi gases. Thus, we have tried to capture ‘snapshots’ of the ongoing experimental progress. As far as theory is concerned, we also have attempted to provide a survey of some of the most important and interesting theoretical proposals. Therefore, our selection of topics in the latter regard is rather subjective and the emphasis has been placed on providing a pedagogical introduction to some of the subjects rather than on providing an exhaustive survey of the available literature. As a consequence, some topics have been left out. For instance, the application of AEAs to quantum information processing will not be discussed here and we refer the interested reader to [14]. Another topic that we do not touch in depth is the physics of one-dimensional (1D) systems. This subject has been the focus of theoretical interest in recent years, especially concerning quantum magnetism in 1D lattice systems (see e.g. [15–18] and references therein). For trapped systems on the continuum, we refer the interested reader to the excellent recent review article on this subject by Guan et al [19] and point out that just recently the first experimental exploration of the fascinating role of SU($N$) symmetry in an array of 1D fermionic tubes has been reported in [20].

This article is organized as follows: We begin in section 2 with a review of the work leading to the observation that AEAs posses an emergent SU($N$) symmetry (For a brief review of the group theory relevant to SU($N$), see appendix A.). Although a theoretical prediction, SU($N$) symmetry was based on experimental observations associated with the unique atomic structure of AEAs. The emergent SU($N$) symmetry has not only important consequences in atomic molecular and optical systems and condensed matter physics, but also in other fields in physics as well. In sections 3 and 4, we review the experiments that have been performed in traps and in optical lattices. The review of theoretical results begins in section 5, where the theory of SU($N$) Fermi liquids and their instabilities, including the BCS instability, are surveyed. The discussion in this section mainly applies to gases in a trap, but in section 6 we turn our attention to the quantum phases intrinsic to lattice systems. Focusing on the deep lattice limit, we discuss both the Fermi–Hubbard and Heisenberg models with SU($N$) symmetry. In section 7, we discuss other interesting models that can be engineered using AEAs. We conclude our review in section 8 with a summary and an outlook. Appendix A contains a brief summary of the most important mathematical results about the SU($N$) group; appendices B and C contain some technical details of the topics discussed in section 5.

2. Alkaline-earth Fermi gases: an emergent SU($N$) symmetry

AEAs have a unique atomic structure with fundamental features that make them attractive for the study of many-body phenomena. One appealing property is an emergent SU($N$) symmetry...
symmetry in the nuclear-spin degrees of freedom [21, 22]. Many of the consequences of this symmetry remain to be exploited and understood.

2.1. Background and precedents

To understand how SU(N) symmetry emerges at ultracold temperatures, we begin with the pioneering work by Lee, Huang and Yang [23]. These authors considered the thermodynamic description of interacting gases well below their quantum-degenerate temperature. They argued that, provided the range of the interactions is much shorter than the interparticle distance (i.e. the gas is ‘dilute’), the complicated interatomic potentials are well approximated by the pseudopotential $V(r) = (2\pi\hbar^2a_s/\mu) \delta(r)\delta_0[|r|]$, where $r = |r|$ is the relative separation of the colliding particles, $\mu$ is their reduced mass (equal to half the bare mass for identical particles) and $a_s = -\lim_{k \to 0} \delta_t(k)/k$ is the scattering length ($\delta_t(k)$ is the $s$-wave scattering phase shift). The latter is the only parameter needed to characterize the interactions, since at ultralow temperatures, higher partial waves are suppressed by the centrifugal barrier.

As formulated by Lee, Huang and Yang, the pseudopotential applies to bosons and spin-$\frac{1}{2}$ fermions only. Later, Yip and Ho [24] discovered that for spin-$F$ fermions, this pseudopotential must be generalized to

$$V(r) = \sum_{F=0}^{2F-1} \frac{2\pi\hbar^2a_{Fj}}{\mu} \delta(r)\delta_0[|r|]P_j,$$

where $P_j$ is the projector onto states with total spin equal to $j = 0, 2, \ldots, 2F - 1$. Only the even $F$ values can interact via $s$-wave collisions since (because of) quantum statistics $F$ values are the only ones that have an associated spatial wave function that is antisymmetric. Hence, it follows that $F = 1/2$ scattering lengths are needed to describe the interaction between spin-$F$ fermions. Generally speaking, the differences between the scattering lengths $a_0^F, a_2^F, \ldots, a_{2F-1}^F$ stem from the different configurations that the electronic shell of the colliding atoms can adopt for the possible values of $F$. In the presence of a large magnetic field, $F$ is no longer a good quantum number and the scattering lengths between states with different projections along the quantization direction can also become different [25].

However, (1) can exhibit a much larger symmetry than we naively expect for a higher spin representation of SU(2). As Wu and coworkers [26] noticed for $F = \frac{1}{2}$, the four-component spinor representation of SU(2) is isomorphic to a spinor representation of the SO(5) group without fine tuning. These authors also pointed out that, for the $F = \frac{1}{2}$ members of the AEA family, $^{135}$Ba and $^{137}$Ba, the scattering lengths $a_0^F$ and $a_0^F$ should have similar values because of the completely filled outer electronic shell of barium. These atoms were thus located close to the SU(4) symmetric line in the phase diagram of [26].

Alkali gases with approximate SU(3) symmetries have been considered by a number of authors, beginning with the pioneering work by Modawi and Leggett [27], who studied BCS pairing in a quantum degenerate mixture of the three spin-polarized hyperfine states of $^6$Li. Honerkamp and Hofstetter [28, 29] also considered this system as well as mixtures of $N$ hyperfine states of $^{6}$Li. The three-component $^6$Li system has been recently realized experimentally [30, 31] and evidence of the emergence of a SU(3) symmetry at large magnetic fields (at which the electronic and nuclear-spin degrees of freedom start to become decoupled) has been reported. However, at moderate magnetic fields the SU(3) symmetry breaks down.

2.2. Alkaline-earth and ytterbium atomic gases

For an AEA in the ground state $(^1S_0)$, the electronic degrees of freedom have neither spin nor orbital angular momentum $(J = 0)$. The nuclear spin $(I > 0)$, present only in the fermionic isotopes, is thus decoupled from the electronic state because of the absence of hyperfine interactions. (All bosonic AEAs have $I = 0$ because of their even-even nuclei configuration.) Interestingly, the excited state $^3P_0$ also has, to leading order, vanishing hyperfine interactions and, hence, highly decoupled nuclear and electronic spins [32].

The electronic-nuclear spin decoupling in the fermionic isotopes not only allows for an independent manipulation of their nuclear and electronic degrees of freedom, but also imposes the condition that the scattering parameters involving the $^1S_0$ and $^3P_0$ states should be independent of the nuclear spin, aside from the restrictions imposed by fermionic antisymmetry. Thus, in the clock states, all of the scattering lengths are equal, i.e. $a_0^e = a_0^n$ (for $j = 0, 2, \ldots, 2F - 1$). Under these conditions the interaction and kinetic parts of the Hamiltonian become SU(N) spin-symmetric (where $N = 2I + 1 = 2F + 1$) [21, 22].

For the $^1S_0$ state, Gorshkov et al theoretically determined that the variation of the scattering length for the various nuclear spin components should be smaller $\delta a_s/\bar{a}_s \sim 10^{-9}$ [22]. For the $^3P_0$ electronic state, the decoupling is slightly broken by the admixture with higher-lying $P$ states with $J = 0$. This admixture is very small and the resulting nuclear-spin-dependent variation of the scattering lengths is also expected to be very small, of the order of $10^{-3}$.

The bounds on the variation of the scattering lengths, $\delta a_s/\bar{a}_s$, associated with the various nuclear spin projections, are based on the fact that the scattering length is just a measure of the semiclassical phase, $\Phi$, accumulated by the colliding atoms from the turning point to infinity (computed at zero energy) [22]. The variation of the phase, is thus proportional to $\delta a_s/\bar{a}_s \sim \delta \Phi = \delta V \Delta t/\hbar$, where $\Delta t \sim 1$ ps is the total time in the short-range part of the collision and $\delta V$ is the typical energy difference associated with different nuclear spin projections during this time. For the $^1S_0$ state, the latter can be estimated using second order perturbation theory as $\delta V/\hbar \sim E_{hf}^2/(E_{hf}^0 h) \sim 200$ Hz, where $E_{hf}^0 \sim 300$ MHz is the approximate value for the hyperfine splittings in $^3P_1$ and $E_{hf}^0/\hbar \sim 400$ THz is the optical energy difference between $^1S_0$ and $^3P_1$. This leads to the $10^{-9}$ estimate. For the $^3P_0$ state, the second order formula might be incorrect since the associated molecule states that are separated by the fine structure energy at large distance may come orders of magnitude closer at short range. Thus, assuming $\delta V \sim E_{hf}$ according to first-order
perturbation theory is a more realistic and conservative estimate, which yields $\delta \Phi \sim 10^{-3}$.

### 2.3. Relevance of SU($N$) symmetry

The availability of fermionic systems exhibiting an enlarged SU($N$) symmetry with $N$ as large as 10 can be of interest in other fields of physics beyond research on ultracold gases. For instance, in particle physics, the theory of quantum chromodynamics (QCD), which currently provides the most fundamental description of the atomic nucleus and the nuclear interactions, contains two kinds of SU(3) groups: a global flavor SU(3) symmetry group, whose discovery won the Nobel prize for Gell–Mann and the gauged color SU(3) group. The latter describes the origin of the forces that confine quarks inside baryons and mesons through the exchange of SU(3) gauge bosons known as gluons. In the field of nuclear physics, the SU(6) group has also been considered as a candidate for unifying the description of baryons and mesons into a single group capable of accounting for both the flavor SU(3) and spin SU(2) symmetries [33]. Indeed, the interesting analogies between ultracold gases with enlarged SU($N$) symmetry and cold dense QCD matter have been already noticed by several authors (see, i.e. [21, 34–36] and references therein).

The SU($N$) symmetry can also have remarkable consequences in quantum many-body systems. For example, in a SU(2) antiferromagnet, which is characterized by spin-$\frac{1}{2}$ particles with spin rotation symmetry, every pair of spins minimizes its energy by forming a singlet. The same spin, nevertheless, can participate in only one singlet pair with one of its neighbors. In principle, this constraint can generate geometrical frustration and prevent magnetic ordering. However, spin-$\frac{1}{2}$ particles tend to find a compromise and often become magnetically ordered with decreasing temperature. A typical example of this compromise is found in the SU(2) Heisenberg model on a triangular lattices where, in the ground state, adjacent spins anti-align at 120°.

In contrast, systems with an enlarged number of degrees of freedom and exhibiting SU($N > 2$) spin rotation symmetry, suffer from massive degeneracies. The latter tend to favor the absence of magnetic ordering even classically [37]. Quantum mechanically, massive degeneracies translate into ground states containing massive spin superpositions that give rise to topological order and long-range quantum entanglement [38, 39]. Examples of long-range quantum-entanglement states are fractional quantum-Hall states and the ground state of Kitaev’s toric code [40].

The identification of the SU($N$) symmetry as a unique resource for dealing with unconventional magnetic states has a long history in condensed matter physics [41–46]. A generalization of the symmetry from SU(2) to SU($N$) introduces a perturbative parameter, namely $1/N$. A large $N$ expansion is particularly useful when dealing with problems of quantum magnetism for which there is no other small parameter that allows for a perturbative treatment. The Kondo impurity problem, the Kondo lattice model [41–43] and the Hubbard model [44–46] are some examples of systems where a large $N$ expansion has been shown to be useful. In such approaches, fluctuations about the mean-field solutions appear at order $1/N$. The hope is that even at finite $N$, the $1/N$ corrections can remain irrelevant and thus the mean-field solution is a good approximation. However, in this context, the enlarged SU($N$) symmetry has been often regarded as a mere mathematical construction without a real physical motivation and, in many cases, just as a means for developing approximate solutions for SU(2) systems. The observation that SU($N$) symmetry naturally emerges in the nuclear spin degrees of freedom of fermionic alkaline earth atoms thus raises the exciting potential opportunity of the direct observation of SU($N$) symmetry and its remarkable consequences for the first time in the laboratory.

Finally, enlarged unitary symmetries have been used in other problems in solid state physics such as the quantum-Hall effect in multivalley semiconductors [47, 48]. In such systems, the massive degeneracy of the Landau levels is supplemented by a large degeneracy in spin and valley-spin, which favors ferromagnetic states and complex spin-valley textures [47, 48]. A recent revival of the interest in these systems has been brought about by the discovery of graphene [49], which can be regarded as a two-valley zero-gap semiconductor. Graphene exhibits an SU(4) spin-valley symmetry [49, 50], which, although weakly broken by the long-range part of the Coulomb interaction, plays an important role in determining the properties of the ground state both in the integer [51, 52] and fractional quantum-Hall effect [50, 53]. Finding connections between these phenomena and the many-body physics of AEAs remains an interesting challenge for both experimentalists and theorists.

### 3. Experiments with trapped ultracold gases

Owing to the unique properties of AEAs, substantial experimental efforts have recently been directed at cooling, trapping and manipulating AEAs and many of the capabilities previously demonstrated with alkaline atoms are starting to be reproduced with AEAs. These include (1) laser cooling down to microKelvin temperatures, (2) trapping in optical potentials for several seconds, (3) evaporative cooling to quantum degeneracy, (4) the demonstration of a high degree of control over both internal and external degrees of freedom, (5) imaging and resolving the various hyperfine components using optical Stern–Gerlach, (6) demonstrating control of interaction parameters via optical and magnetic Feshbach resonance and (7) the realization of a Mott Insulator. In this section, we first present a summary of those experimental developments for trapped gases. In the following section, we review the experiments dealing with Fermi gases on optical lattices.

#### 3.1. Experimental determination of the scattering length

A natural manifestation of the SU($N$) symmetry is the conservation of each of the nuclear spin components during a

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4 An exception is the Kondo problem discussed in [41–43], for which $N$ corresponds to the number of possible degenerate magnetic configurations of the impurity atom. See e.g. [182].
collision. This is in stark contrast to the smaller SU(2) symmetry exhibited by alkali atoms that allows for spin-changing processes for $F > \frac{1}{2}$. The spin-changing collisions occur because, as described in section 2.1, in alkali atoms, the scattering lengths depend on the total $F = J + I$ of the colliding atoms and therefore during collisions, the bare nuclear spin degrees of freedom get effectively mixed. Thus, even though the total spin magnetization of the two colliding atoms is always conserved, it is possible to have spin-changing collisions. For example in $^{40}$K, with $F = 9/2$, mode-changing collisions between two atoms, one in $m_F = 5/2$ and the other in $m_F = -5/2$, into states with $m_F = 3/2$ and $m_F = -3/2$ and $m_F = 1/2$ and $m_F = -1/2$ have been reported and controlled [54].

Although the $s$-wave scattering length is the only parameter that fully characterizes the collisional properties of ultracold gases, it is very sensitive to the ground state interatomic potential. Thus naive ab initio calculations in general fail to determine $s$-wave scattering length [55], and consequently, we need to rely on experiments for its actual determination. These experiments include: cross-dimensional rethermalization measurements, measuring the expansion of a condensate after the trapping potential is removed (since the expansion rate depends on the interaction energy of the condensate before its release) and one- and two-color photoassociation (TPA) spectroscopy.

TPA uses two laser beams to measure the binding energy of the weakly bound states of a molecular system [56] (see figure 2). One beam, $L_1$, probes a transition between a pair of colliding ground-state atoms and an excited-molecular bound state. A second beam, $L_2$, probes the bound–bound transition between the excited-molecular bound state and a ground-molecular bound state close to the dissociation threshold. If $L_2$ is close to resonance to the bound–bound transition, it causes the Autler–Townes doublet [57]. When the laser $L_1$ is also on resonance to the free-bound transition, the atomic loss coming from the population of the molecular-excited state is suppressed because of quantum interference (Autler–Townes spectroscopy). In contrast, if both lasers are off-resonant and the frequency difference matches the binding energy of the ground molecular state, the lasers drive a stimulated-Raman transition from the colliding atom pair to the molecular ground state that can be detected as an atom loss (Raman spectroscopy).

For AEAs TPA has become the most reliable and precise way to determine ground-state scattering lengths. This reliability and precision are due to the absence of hyperfine structure in the $^1S_0$ state (with no electronic orbital and spin angular momenta) that gives rise to a simple isotope-independent ground-state molecular potential. The number of bound states in the molecular potential, which can be cleanly extracted from TPA, can then be used as an input parameter in a semiclassical theory [55] which, together with mass scaling, can determine the scattering length of all isotopes with unprecedented precision.

In table 1 we display the measured values of $s$-wave scattering lengths for various fermionic AEAs along with other relevant data such as mass, nuclear spins and emergent SU($N$) symmetries. Note that the scattering length can vary from large negative to large positive values. The magnitude of the $s$-wave scattering length determines the feasibility of reaching quantum degeneracy via evaporative cooling methods for the various isotopes.

Just recently, spectroscopic measurements performed in 1D and 3D lattices, have revealed approximate information about the scattering parameters between the ground ($^2S_0$) and the excited ($^3P_0$) state in $^{173}$Yb [59, 60] and $^{87}$Sr [61] atoms. These measurements have provided further experimental verification of the emergence of SU($N$) symmetry in Yb and Sr ultracold atomic gases.

### 3.2. Towards a quantum degenerate gas

#### 3.2.1. Ytterbium.

The quest to achieve a quantum degenerate gas with Group II atoms started with Yb. Yb has five stable bosonic isotopes $^{168,170,172,174,176}$Yb and two fermionic isotopes, $^{171}$Yb with $I = 1/2$ and $^{173}$Yb with $I = 5/2$.

The first experimental realization of a BEC of $^{172}$Yb was reported in 2003 by the Kyoto group [63]. The lack of hyperfine structure in the ground state of bosonic AEAs prevents both the use of the conventional magnetic trap for BEC production and evaporative cooling by a radio frequency knife. Instead, all-optical trapping and cooling methods are needed. An all-optical formation of a degenerate fermionic $^{173}$Yb gas was achieved by the Kyoto group in 2007 by performing evaporative cooling of a six-nuclear spin-state mixture in an optical dipole trap [64]. Following this achievement, a BEC of $^{170}$Yb [65] and $^{176}$Yb [66] was reported by the same group. The $^{170}$Yb BEC required sympathetic cooling of $^{171}$Yb with $^{174}$Yb, because the $^{176}$Yb has a negative scattering length. A rapid loss of $^{176}$Yb atoms seen after cooling the gas below the transition temperature was consistent with the expected collapse of a $^{176}$Yb condensate due to attractive interactions.

The $^{171}$Yb fermionic isotope has a very small scattering length in its ground state, with $a_s = -0.15$ nm, which prevents...
cooling by direct evaporation. However, in 2010 it was cooled to quantum degeneracy via sympathetic cooling with $^{174}$Yb. This method allowed the realization of the first SU(2) × SU(6) mixture in the presence of $^{177}$Yb, in ultracold gases [67]. Finally, despite the low natural abundance of $^{168}$Yb, of the order of 0.13%, a BEC of this rare atomic species was obtained by direct evaporative cooling in 2011, without the need of an enriched source [68]. Thus, except for $^{172}$Yb, which is unstable to three-body losses because of its large negative scattering length, quantum degenerate gases and/or mixtures of all the stable Yb isotopes have been produced by the Kyoto group. Recently the creation of quantum degenerate gases of Yb has been also reported by Sengstock’s group in Hamburg [69]. The production of quantum degenerate mixtures of fermionic alkali-metal $^6$Li and bosonic Yb [70, 71] and fermionic Yb [71] has also been reported. See [72] for a more detailed review.

### 3.2.2. Calcium
Calcium was the first AEA, truly belonging to the Group-2 elements of the periodic table, that was cooled down to quantum degeneracy. In 2009, the Physikalisches-Technische Bundesanstalt in Germany reported a BEC of $^{40}$Ca [73]. BEC was achieved in spite of the inelastic collisions associated with $^{40}$Ca’s large ground-state s-wave scattering length (18 < $a_s$ < 37 nm) [62] by using a large-volume optical trap for initial cooling. A second $^{40}$Ca BEC was reported in 2012 by Hemmerich’s group in Hamburg [74]. So far, no fermionic isotopes of Ca have have been cooled below the quantum degeneracy.

### 3.2.3. Strontium
Strontium has three relatively abundant isotopes. Two of them are bosonic $^{86}$Sr and $^{88}$Sr with relative abundances ~9.9% and ~82.6%, respectively, and one is fermionic $^{87}$Sr with a relative abundance of ~7.0% and a nuclear spin $I = \frac{9}{2}$.

Initial efforts to reach quantum degeneracy with Sr gases failed because the unfavorable scattering properties of the bosonic isotopes [75, 76]. While the scattering length of $^{86}$Sr is close to zero, the scattering length of the $^{88}$Sr isotope is very large, $a_s = 40$ nm, leading to a large detrimental loss of atoms via three-body recombination. The breakthrough for reaching quantum degeneracy came from the development of an efficient loading scheme that allowed the experimenters to overcome the low natural abundance of $^{84}$Sr (only ~0.56%) and take advantage of $^{88}$Sr’s favorable scattering length, $a_s = 6.5$ nm. A BEC of $^{84}$Sr was almost simultaneously reported by two groups, Schreck’s group (Innsbruck) [77] and Killian’s group (Rice University) [78]. This achievement was followed up by the cooling to quantum degeneracy of a spin-polarized gas of $^{87}$Sr in thermal contact with a BEC of $^{84}$Sr [79] and the corresponding mixture [80] by the same groups respectively. A BEC of $^{86}$Sr was finally created, in despite of its large scattering length, by the Innsbruck group. BEC was achieved by reducing the density in a large-volume optical-dipole trap [81, 82]. In addition, a BEC of $^{88}$Sr was produced by the Rice University group using a sympathetic cooling with $^{87}$Sr [83]. Finally, the quest of developing faster and better pure optical methods for reaching larger and colder samples of AEAs has recently lead to the implementation of a method using laser cooling as the main cooling mechanism [84]. See [85] for a more detailed review.

### 3.3. Control of interactions: optical Feshbach resonances
The ability to tune interactions in ultracold alkali-metal atomic gases using magnetic Feshbach resonances (MFRs) has been a crucial step for the exploration of few- and many-body physics in these systems [25]. MFR, however, cannot be used to tune interactions in the ground state of AEAs because of the lack of magnetic electronic structure.

However, tuning interatomic interactions via optical Feshbach resonances (OFRs) is a feasible route with AEAs. In an OFR, a laser tuned near a photoassociative resonance is used to couple a pair of colliding atoms to a bound molecular level in an excited-electronic level. The coupling induces a Feshbach resonance and modifies the scattering length of the two colliding atoms. Ciurylo et al predicted that OFR could be ideally implemented in AEAs using a transition from the singlet ground state to a metastable triplet level [86]. The possibility of tuning the scattering length with significantly fewer induced losses was based on the long lifetime of the excited-molecular state and a relatively large overlap integral between excited-molecular and ground-collisional wave functions.

There have been a few experimental demonstrations of the use of OFR to modify interaction properties in AEAs, although significant atom loss has always been observed. The modification of the photoassociation spectrum by an OFR in a thermal gas of $^{173}$Yb was reported in [87]. An OFR laser pulse of a 1D optical lattice turned on for several microseconds was used by Yamazaki et al [88] to modulate the mean-field energy in a $^{174}$Yb BEC. Blatt et al [89] used an OFR in a thermal gas of $^{88}$Sr to modify thermalization and loss rates. More recently Yan et al used an OFR [90] to control the collapse and expansion of a $^{88}$Sr BEC via moderate modifications of the scattering length. The use of more deeply bound excited-molecular states to help the suppression of atom-light scattering and to

| Atom species | Mass (u) | Nuclear spin (I) | Symmetry group | Scattering length (nm) |
|-------------|---------|-----------------|----------------|-----------------------|
| $^{171}$Yb | 170.93  | $\frac{1}{2}$  | SU(2)          | −0.15(19) $^{[171]}$Yb, −30.6(3.2) $^{[173]}$Yb (From [58]) |
| $^{173}$Yb | 172.94  | $\frac{5}{2}$  | SU(6)          | 10.55(11) $^{[173]}$Yb, −30.6(3.2) $^{[171]}$Yb (From [58]) |
| $^{87}$Sr  | 86.91   | $\frac{9}{2}$  | SU(10)         | 5.09(10) $^{[87]}$Sr (From [62]) |
reduce the background two-body loss could enhance the utility of OFR in AEAs and efforts in that direction are currently taking place in various labs. One important point to highlight, nevertheless, is that the direct use of OFR to control scattering properties can destroy the SU(4) symmetry since the ground state is directly coupled to an excited state that does possess a hyperfine structure.

3.4. Imaging and detection of nuclear spin components

An important tool for probing AEAs is the ability to spatially resolve the different nuclear spin components (see figure 3). In Group-1 elements (i.e. alkali atoms) hyperfine states can be resolved and imaged by taking advantage of the well-known Stern-Gerlach technique. The latter uses the spin-state-dependent force generated by a magnetic-field gradient to spatially split an expanding atom-cloud into clouds of different hyperfine levels. However, this method cannot be used for AEAs in the clock states for which $J = 0$, because of their small magnetic moment that entirely stems from the nuclear spin. The nuclear magnetic moment is about three orders of magnitude smaller than the Bohr magneton and therefore the separation of the nuclear components would require unaccessible magnetic field gradients. To overcome this difficulty, experiments have successfully taken advantage of the so-called Optical Stern–Gerlach (OSG) effect produced by circularly polarized laser beams. The basic idea is that the spin-dependent light shift generated by circularly polarized beams mimics a fictitious magnetic field that can be used to resolve the nuclear manifold [91]. For the $^{173}$Yb gas [67], one OSG beam was sufficient to separate four of the six nuclear spin states. The remaining two nuclear states could be analyzed by reversing the polarization of the OSG beam. For a $^{87}$Sr gas, the simultaneous application of two OSG beams with opposite circular polarization was required to resolve all the nuclear spin states [92].

An alternative, complementary tool to resolve nuclear spin components uses spectroscopic methods. These methods are ideal for AEAs thanks to their narrow intercombination lines. The first demonstration of this technique was achieved using the ultranarrow $^1S_0^−^3P_0$ transition in an optical lattice clock [32] operated with a cool (at temperature of a few µK), but not quantum degenerate $^{87}$Sr gas. The $^1S_0^−^3P_0$ transition is only allowed because in the excited state, the hyperfine interaction leads to a small admixture of the higher-lying P states [93]. This small admixture strongly affects the magnetic moment, causing the nuclear $g$ factor of the excited state to significantly differ from that of the ground state (i.e. ~50% for strontium).
The differential $g$ factor was used to resolve all ten nuclear spins in a bias magnetic field. The spectroscopy was performed in a deep 1D optical lattice designed to operate at the so-called magic wavelength, at which the light shifts on the clock states are equal and the clock frequency is not perturbed by them [94]. Stellmer et al employed a similar procedure using the $^{1}S_{0}-^{3}P_{1}$ intercombination line to perform nuclear spin-dependent absorption imaging [92].

A fundamental consequence of the SU(N) symmetry is the conservation of the total number of atoms with nuclear spin projection $m_{I} = -I \leq m_{I} \leq I$. This means that an atom with large $I$ such as $^{87}$Sr can reproduce the dynamics of atoms with lower $I$ if one takes an initial state with no population in the extra levels. Stellmer et al tested this fundamental feature by measuring spin relaxation using the nuclear-spin state-detection techniques described above. The spin-relaxation rate constant was found to be less than $5 \times 10^{-15}$ cm$^{3}$s$^{-1}$ [92].

4. Experiments in optical lattices: realization of a SU(6) Mott insulator

Optical lattices provide us with a new way of studying ultracold atomic gases by confining them in periodic arrays that strongly resemble the potential experienced by electrons in crystalline solids. The optical lattice potential is highly controllable and can be used to tune the interatomic interactions, the density, the kinetic energy and even the dimensionality of the system, by tightly confining the atoms along specific directions (see e.g. [4, 95, 96] for a review and references therein).

Alkaline-earth atomic gases trapped in optical lattices realize the SU(N) generalization of the Hubbard model [21, 67]:

$$H = -t_{g} \sum_{\langle i,j \rangle} \left[ c_{i}^{\dagger} c_{j} + h. \text{c.} \right] + \frac{U_{gg}}{2} \sum_{i} n_{i} (n_{i} - 1),$$

(2)

where $\sum_{\langle i,j \rangle}$ stands for summation over nearest-neighbor lattice sites and $c_{i}$ are fermionic annihilation operators of $g$ atoms in nuclear spin $\alpha$ at lattice site $i$. The lattice site index $i$ is associated with the vector $\mathbf{R}_{i} = (R_{xi}, R_{yi}, R_{zi})$, where $R_{xi} = m_{\alpha} a (p = x, y, z)$, $m_{\alpha}$ being positive integers and $a$ is the lattice parameter. The term $n_{i} = c_{i}^{\dagger} c_{i}$ is the operator that measures the total fermion occupation (irrespective of the spin) at the lattice site $i$. The dimensionality of the lattice, $d$, and the lattice spacing, $a$, are determined by the number of counter-propagating laser beams employed to create the lattice potential and the laser wavelength, respectively [4, 95]. Equation (2) describes the dynamics of a dilute ultracold Fermi gas hopping between nearest neighbour lattice sites and interacting only locally. The lattice potential is assumed to be deep enough that only the lowest Bloch band is occupied by the atoms. In this regime, at most $N$ fermions can occupy the same lattice site.

The Hubbard model is written in a form that is manifestly SU(N) invariant. It is characterized by two energy scales, $t_{g}$ and $U_{gg}$, which correspond to the kinetic and interaction energies respectively and are determined by the depth of the optical lattice potential [4, 67]. $U_{gg}$ is proportional to $a_{s}$, i.e. the $s$-wave scattering length between two atoms in the ground state. Experimentally, the ratio $U_{gg}/t_{g}$ can be tuned by varying the depth of the lattice potential, which, in turn, is controlled by the intensity of the laser beams generating the lattice [4].

Roughly speaking, at absolute temperatures $T \ll t_{g}/k_B$, when the kinetic energy dominates (i.e. $t_{g} \gg U_{gg}$) and away from special values of the lattice filling, $n = \langle n \rangle$, the system is expected to be a Fermi liquid (see section 5). On the other hand, when the lattice filling, $n$, takes integer values $n < N$ and the interaction energy dominates, i.e. $U_{gg} \gg t_{g}$, the hopping of the atoms between lattice sites is strongly suppressed. This suppression occurs because, in order to be able to move around, atoms must pay an energy penalty $= U_{gg}$, which is not available at low temperatures where $T \ll U_{gg}/k_B$. Thus, the system becomes a Mott insulator, in which atom motion is blocked by interactions. This situation is different from the so-called band insulator that happens when $n \equiv N$. In this case, the lowest Bloch band is completely filled and the atom motion is blocked even in the absence of interactions by the Pauli exclusion principle.

The experiment reported in [67], describes the realization of the SU(6) Hubbard model by loading a nuclear spin mixture of $^{173}$Yb atoms in their ground state (g = $^{1}S_{0}$) into a three dimensional (3D) cubic optical lattice with lattice spacing $a = 266 \text{ nm}$. The lattice was generated by $d = 3$ mutually orthogonal pairs of counter-propagating laser beams. In addition to the two terms in (2), in the experiments there is a confining potential generated by the Gaussian curvature of the lattice beams. This potential is described by adding to (2) the term:

$$V_{\text{trap}} = \sum_{i} V_{i} n_{i},$$

(3)

The trapping potential is well approximated by a harmonic trap, i.e. $V_{i} = \frac{1}{2} \sum_{p=x,y,z} (\omega_{p} R_{pi}^{2})$, where $\omega_{p}$ is the trap frequency along the $p = x, y, z$ directions (for example $\omega_{x} = 2 \pi \times 100 \text{ Hz}$ in [67]).

To realize a Mott insulator with SU(N = 6) symmetry, the Kyoto group followed the standard adiabatic-loading procedure used to create Mott insulators in alkali-metal gases [4]. Specifically, an ultracold gas of $^{173}$Yb atoms was first loaded in a 3D dipole trap and then into a deep optical lattice by ramping slowly the lattice depth up to a maximum final value of 13 $E_g$($E_g = h^2 \pi^2 m a^2$ being the recoil energy of the atoms). The loading was checked to be adiabatic by reversing the ramp of the optical lattice and finding that the initial and final temperatures were very close to each other. For the final trapping conditions quoted above and an initial temperature of the gas ($T/T_F \approx 0.2$), the maximum lattice filling was below 2 atoms per site even at the center of the trap. This condition is crucial for probing the Mott insulator phase.

To probe the SU(6) Mott phase and, in particular, to infer its temperature, the Kyoto group used lattice modulation spectroscopy [97–100]. This method applies a small periodic (in time) modulation to the optical lattice depth, heating the gas. When the system enters the Mott phase, the injected energy causes the creation of holes and doublons, i.e. empty sites and...
Theoretical calculations are based on slave particle methods and high-temperature series expansions. Increasing values of the lattice depth (measured in units of the recoil energy, $E_R$), the DPR as a function of frequency exhibits a peak centered at the frequency corresponding to the Mott gap. Reprinted with permission from [101]. © 2012 Nature Publishing Group.

Figure 4. Lattice modulation spectra versus modulation frequency for increasing values of the lattice depth (measured in units of the $^{173}$Yb recoil energy, $E_R$). The series shows the emergence of a peak centered around the frequency corresponding to the Mott gap. Reprinted with permission from [101]. © 2012 Nature Publishing Group.

The doublet production rate (DPR) can be measured by converting the doublons into molecules via photoassociation. The molecules escape rapidly from the trap and thus can be detected as atom loss. For deep lattices, the DPR as a function of frequency exhibits a peak centered at the frequency corresponding to the Mott gap. This technique can be also used to estimate the temperature of the gas in the lattice, which sets the system in the regime $t_d < k_BT < U_{gg}$, see figure 4. Hence, the lattice modulation provides a direct measurement of the Mott gap. This technique can be also used to estimate the temperature of the gas in the lattice, which sets the system in the regime $t_d < k_BT < U_{gg}$ [67]. Theoretical calculations based on slave particle methods and high-temperature series expansions [100] agree with the experimental observations.

By comparing the temperature measurements taken for the Mott insulating phases of SU(6) and SU(2) gases (the latter achieved by optical pumping), it was found that the final temperature for the SU(6) gas was found to be a factor of ~2 or 3 smaller than the one reached for the SU(2) system (see figure 5). The initial $T/T_i$ values, achieved as a result of evaporative cooling, were almost the same for both the SU(2) and SU(6) cases.

These measurements were consistent with the theoretical expectations that systems with $SU(N > 2)$ symmetry, adiabatically loaded on a lattice, can be more efficiently cooled down than SU(2) systems [21, 102, 103]. The cooling can be understood as a direct consequence of the large entropy stored in the spin degrees of freedom in a SU(N) Mott insulator in the $t_d < k_BT < U_{gg}$ regime. See section 6 for a more detailed discussion. Before finishing this section we point out that first efforts towards the observation of a SU(10) Mott insulator in Sr have started [85].

5. Fermi liquids and Their instabilities

5.1 SU(N) Fermi liquid theory and Pomeranchuk instabilities

At temperatures well below the Fermi temperature $T_F$ in a trap or in an optical lattice of weak to moderate trap depth, a gas of AEs is expected to be a Fermi liquid. The latter defines a universality class of interacting fermion systems. As introduced by Landau [104] (see, e.g. [105] for a review), Fermi liquids are characterized by the existence of a gapless Fermi surface (FS) and long-lived low-energy fermionic elementary excitations known as quasiparticles (QPs). The QP states can be put in one-to-one correspondence with the excited states of a non-interacting Fermi gas, implying that the QPs carry the same quantum numbers as noninteracting particles in a Fermi gas.

For a uniform SU(N) symmetric AEA Fermi liquid, the momentum and SU(N) (nuclear) spin are good quantum numbers and therefore a QP distribution function in momentum space, $n_0(p)$, can be defined. At $T = 0$, the ground state of an unpolarized 3D gas of mean density $\rho_0 = N_0/V$ is described by a Fermi distribution at $T = 0$, $n_0(p) = n_0(p) = \theta(p - p_F)$, where $p_F = (4\pi^2 \rho_0 N)^{1/3}$ is the radius of the FS. To describe excitations, it is useful to generalize the QP distribution function to a density matrix, $n_{0\sigma}(p)$, which allows us to describe excited states in which the different orientations of the nuclear spin may be entangled. Following Landau, the grand-canonical free energy (at $T = 0$) of the excited states can be written as [21, 106]:

$$F = F_0 + \sum_p [\epsilon_0(p) - \mu] \delta n_{0\sigma}^2(p) + \frac{1}{2V} \sum_{p, p'} \delta \epsilon(p, p') \delta n_{0\sigma}^2(p) \delta n_{0\sigma}^2(p'),$$

where $\epsilon_0(p)$ is the bare quasi-particle energy, $\mu$ is the zero-temperature chemical potential and $\delta n_{0\sigma}^2(p) = n_{0\sigma}^2(p) - n_0(p) \delta_{\sigma}^2$ is the deviation of the QP distribution with respect to the ground state. For $p = p_F$, $\epsilon_0(p, p') = \mu + (p_F/m^*)^2(p - p_F)$, where $m^*$ is the QP mass and $\mu$ is the zero-temperature chemical potential.

In (4) $\delta \epsilon(p, p')$ is the Landau function describing the (forward-scattering) interactions between the QPs. As explained in appendix B, the Landau function can be parametrized...
in terms of a discrete set of Landau parameters \( \{F_0^F, F_0^p\} \), where \( L = 0, 1, 2, \ldots \) is an integer. The Landau parameters can be obtained to lowest order in the gas parameter \( p_F a_L \) by using the Hartree–Fock approximation. This method, yields \( F_0^F \approx 2(N - 1) p_F a_L / \pi, F_0^p \approx -2p_F a_L / \pi \) and vanishing values for \( L > 0 \) (hence, \( m^* = m \)) [21]. Recently, the Landau parameters to second order in \( p_F a_L \) were obtained by Yip and coworkers [107]. The higher-order corrections are much enhanced at large \( N \). This enhancement means, in particular, that the region at which the Hartree–Fock (HF) results apply rapidly shrinks with \( N \) because the applicability criterion for HF is \( N p_F a_L < 1 \).

For the isotropic Fermi liquid state to be stable, the positivity of the free-energy fluctuations to quadratic order requires that \( F_{L,0}^m > - (2L + 1) \). Otherwise, the system undergoes a Pomeranchuck instability [21, 105, 106] that can result in a permanent deformation of the FS, which may or may not be accompanied by a spontaneous breaking of the SU(\( N \)) symmetry. A notable example of Pomeranchuck instability is Stoner instability, which corresponds to the transition from a spin-unpolarized (i.e., paramagnetic) gas to a polarized gas. For a system in a trap, where the number of particles in each component is fixed, this transition corresponds to the spatial segregation of the different nuclear spin components.

Within Landau Fermi liquid theory, the Stoner instability occurs if \( F_{0}^m < 0 \). Interestingly, to lowest order in \( p_F a_L \), this criterion is \( p_F a_L \approx \frac{\varepsilon}{\pi} \), which is the same for all \( N \) [21]. Yet, the analysis based on Fermi liquid theory of the Stoner instability can be quite misleading [21], as it predicts a continuous phase transition for all values of \( N \). A more careful treatment begins by noticing that the order parameter for \( N > 2 \) is a traceless hermitian matrix \( M \) belonging to the adjoint representation \( N^2 - 1 \) whose components are \( M_{ij} \propto \sum_{k} \langle c_i^{\dagger}(k) c_j^{\dagger}(k) c_j(k) c_i(k) \rangle \). Hence, the change in the Landau free energy at the Stoner transition can be written as:

\[
F - F_0 = a_2 \text{Tr} \, M^2 + a_3 \text{Tr} \, M^3 + a_4 \text{Tr} \, M^4 + \cdots ,
\]  

where \( a_2 \propto (F_0^m + 1) \). For \( N = 2 \), the second term in the right-hand side of (5) vanishes because for a \( 2 \times 2 \) traceless matrix \( \text{Tr} \, M^3 = 0 \). However, this result does not occur for \( N > 2 \), which implies that the Stoner transition is a first-order (i.e. discontinuous) transition at the mean-field level [21]. (For \( N = 2 \) the Stoner transition becomes discontinuous at low temperatures because of fluctuation effects beyond mean-field theory [108, 109].) As a consequence, close to the Stoner transition, the system will exhibit metastability and phase coexistence. Furthermore, at a quantitative level, the Pomeranchuck–Stoner criterion \( F_0^m > -1 \) does not provide a reliable estimate of the transition point [21]. Nevertheless, although qualitatively correct, the above argument assumes that the order parameter, i.e. the matrix elements of \( M \), remain small in the neighborhood of the critical point so that keeping the lowest order terms from an expansion in \( M \) is enough to capture the transition properties. In contrast, a direct numerical minimization of the total free energy shows that the order parameter is not small [110]. Indeed, for an ultracold gas with SU(\( N \) > 2) symmetry, the Stoner transition appears to be strongly first order, although the conclusion obtained from the above free-energy form remains correct only at the qualitative level.

Nevertheless, the experimental values of the gas parameter in a trap (i.e. \( p_F a_L \approx 0.13 \) for \( ^{173}\text{Yb} \)) are far from the critical value corresponding to the Stoner transition. Furthermore, as explained in section 3.3, the enhancement of the scattering length by optical means (i.e. OFRs) breaks SU(\( N \)) symmetry and introduces large atom losses, which may complicate the applicability of the results discussed above. Yet, it may still be possible to observe a transition to a polarized (i.e. spatially segregated) state in a not-too-deep optical lattice, as has been recently suggested by Monte Carlo simulations for two-component mixtures [111]. Such an observation may also occur near half-filling in deep optical lattice, as suggested by a Gutzwiller approximation to the SU(3) Hubbard model [112] and a recent generalization of Nagaoka’s theorem for the SU(\( N \)) Hubbard model [113].

Although the instabilities discussed above may not be accessible under current experimental conditions, the experimental measurement of the Landau parameters is still an interesting open problem and should provide further confirmation that the SU(\( N \)) symmetry survives many-body effects. Indeed,
the lowest $L$ Landau parameters can be obtained from the measurement of the equation of state as was done recently for the two-component Fermi gas [114, 115] or from the measurement of the number fluctuations $(N_{\alpha} - \langle N_{\alpha} \rangle)^2$ of the different spin components [107].

### 5.2. BCS instability and superfluidity

In addition to the Pomeranchuk instabilities, the Fermi liquid state is notoriously unstable against the formation of Cooper pairs, which is known as the BCS instability [116]. Such an instability cannot be described within the framework of Landau-Fermi liquid theory because it involves a scattering channel between Landau QPs that is neglected in Landau’s theory [106, 116]. However, its importance cannot be underestimated since, for arbitrarily weak interactions, the Fermi liquid state is always unstable against Cooper-pair formation [117] at sufficiently low temperatures (whether such temperatures can be experimentally reached is a separate issue). The angular momentum of the Cooper pairs and the transition temperature depend on the details of the QP interaction, with attractive interactions typically leading to pairing in the $s$-wave channel and repulsive interactions requiring higher angular momentum channels [117].

Indeed, multicomponent systems exhibit a richer phase diagram of paired states [27, 29, 34, 35, 36, 118, 119] than two-component systems [3, 4]. In repulsive interactions, for example, pairing in channels other than $s$-wave is possible at very low temperatures [106, 117], currently beyond our experimental reach. Furthermore, $d$-wave pairing is possible below half-filling (i.e. when the number of fermions per site $\leq 1/2$) for repulsive interactions, although a weak coupling analysis [28] shows that the critical temperature rapidly decreases with $N$.

To understand the rich pairing possibilities of multicomponent Fermi gases, it is important to remember that the $s$-wave order parameter of a paired state in a uniform gas is $\Delta^{s}_{\alpha\beta} \propto \sum_{k} (\epsilon^{s}_{\alpha}(k) c^{\dagger}_{\alpha}(k) c^{s}_{\beta}(-k))$. When represented by an $N \times N$ matrix, it corresponds to a skew-symmetric matrix $\Delta = \Delta^{s}_{\alpha\beta}$, where $\Delta$ is a unitary matrix and $\Delta$ is skew-symmetric for which only the matrix entries $\Delta^{s}_{\alpha\beta} \propto \epsilon^{s}_{\alpha\beta}$ of the different spin components may be paired or not, depending on energetic considerations [118]. In general, we can rely on Youla’s decomposition [120, 121] and write $\Delta = U \Delta U^{\dagger}$, where $U$ is a unitary matrix and $\Delta$ is skew-symmetric for which only the matrix entries $\Delta^{s}_{\alpha\beta} \propto \epsilon^{s}_{\alpha\beta}$, where $\Delta^{s}_{\alpha\beta} \propto \epsilon^{s}_{\alpha\beta}$ with $k \leq N$ are non-zero, while the rest vanishes. Physically, this means that it is always possible to find a basis in which component 1 pairs with 2, 3 pairs with 4, etc and the system can be described in terms of $k \leq N$ Cooper-pair condensates. Such pairing states were termed diagonal-pairing states by Cherng and coworkers [118]. For example, for $N = 3$, two components pair, while a third one remains unpaired. In the weak coupling limit, i.e. for $|p_{F}\alpha\beta| \ll 1$, the critical temperature has an exponential form similar to the formula obtained in the two-component mixture case: $T_{c} = (8e_{F} \gamma^{-2/3} n) e^{-2e_{F}/|\epsilon_{\alpha\beta}|}$ [27, 29, 36], where $\gamma = h^{2}/m^{2}$ is the Fermi energy and $\gamma \approx 0.5772$ Euler’s constant. For the entire BCS-to-BCS crossover, $T_{c}$ has been recently obtained by Ozawa and Baym [36], following the method of Nozières and Schmitt–Rink [122], to account for the pairing fluctuations. In the BEC limit where $p_{F}a_{\alpha} \to 0^-$, they obtained $T_{c}/T_{F} \to 0.137$ [36].

Nevertheless, we must emphasize that the existence of $k \leq N$ Cooper pair condensates does not rely upon the SU$(N)$ symmetry and entirely follows from $\Delta$ being a skew-symmetric tensor [118, 119]. On the other hand, SU$(N)$ symmetry is important and leads to a set of Ward-Takahashi identities that are only satisfied by diagonal-pairing states and not by combinations of them [118]. Moreover, the SU$(N)$ symmetry plays a crucial role in determining the number and dispersion of the Nambu–Goldstone (NG) collective modes (akin to the Anderson–Bogoliubov mode in the two-component BCS system). This is illustrated using the SU$(3)$ case in appendix C, where we show that the number of NG modes is not equal to the number of broken-symmetry generators and that for $N$ odd, there are quadratically dispersing NG modes for $N$ odd.

The existence of quadratic gapless modes and, in particular, a gapless-unpaired component for $N$ odd may have important consequences for the superfluidity of the system, according to the Landau criterion [123]. This effect on superfluidity could manifest because the unpaired component and the quadratically dispersing NG modes will cause dissipation when a macroscopic object moves through the fluid. However, Modawi and Leggett [27] have argued for the irrelevance of this criterion when applied to such paired states. As pointed out by these authors, the superfluid fraction at zero temperature for these systems remains finite in spite of the presence of the unpaired component.

It is also worth discussing the effects of population imbalance. This is another aspect in which the behavior of the $N > 2$ systems noticeably deviate from the $N = 2$ case [34, 36, 118]. The reason can be understood using the following group-theoretic argument. As pointed out in the previous section, magnetization can be represented by a hermitian traceless matrix, $M$. As to the pairing function, it is a complex rank-2 tensor, which can be presented by a matrix $\Delta$. Thus, it is possible to construct a scalar invariant that couples pairing and magnetization where $\text{Tr} \Delta^{\dagger} \Delta M = -\Delta^{00} \Delta^{00} \Delta^{00} M^{00}$ (recall that $\Delta^{\alpha\beta} = (\Delta^{00})^{\alpha \beta} = -\Delta^{00}$, see appendix A). Thus, the Ginzburg-Landau free energy reads [36, 118]

$$F - F_{0} = a_{2} \text{Tr} M^{2} + b_{2} \text{Tr} \Delta^{00} + c_{1} \text{Tr} \Delta^{00} M^{00} + a_{3} \text{Tr} M^{3} \cdots,$$

where $b_{2} \alpha (T - T_{c})$, and $T_{c}$ is the critical temperature of the paired state. Hence, for $T < T_{c}$, unless we are dealing with a pairing state such that $\Delta \Delta \propto 1$, the pairing will lead to a finite magnetization (i.e. phase segregation in a trapped system) [36, 118]. Note that this is always the case for $N = 2$, as $\Delta$ is a $2 \times 2$ skew-symmetric matrix, i.e. $\Delta^{00} = \Delta^{00} \epsilon^{00}$ ($\epsilon^{12} = -\epsilon^{21} = 1$) and therefore $\Delta^{\dagger} \Delta = |\Delta|^{2} \propto 1$, that is, proportional to the unit matrix 1. However, this condition is not generally met for $N > 2$ and in particular, never when $N$ is odd. The additional term in the free-energy couplingparing and magnetization is also responsible for turning the transitions
between different diagonal-paired states into first-order transitions [118]. Generic phase diagrams for \( N = 3, 4 \) have been obtained by Cherng, Refael and Demler in [118]. For \( N = 3 \), the phase diagram in the entire BEC-to-BCS crossover has been computed by Ozawa and Baym [36].

Finally, we should mention that attractive interactions in systems with \( N > 2 \) can yield phases involving more complicated bound states like trions, which correspond to the baryons of QCD. This possibility has been studied for three-species gases loaded in an optical lattice [35, 124].

In closing, we remark that the observation of the paired and trionic phases described above relies on the possibility of controlling the sign of the atomic interactions. Whereas this certainly is possible for both alkali atoms, using MFRs [3, 4] and AEAs, using optical resonances (see section 3.3), both methods break the emergent SU(\( N \)) symmetry of the gas. Thus, how much of what has been described in this section remains valid depends on the magnitude of interaction anisotropies, which set the temperature scale above which the SU(\( N \)) symmetry remains a good approximation [118].

6. Alkaline-earth atoms in optical lattices

6.1. Cooling on the lattice

Although Mott-insulating behavior has been experimentally demonstrated [67] (see section 4), it is of great importance for a quantum simulation program to be able to cool down the optical lattice system to a regime where \( k_B T < t_{zz}^2 / U_{gg} \). This is necessary for observing effects of magnetic exchange. Currently the achieved temperature in experiments is still in the range \( t_{zz}^2 / U_{gg} < k_B T < U_{gg} \). This constraint is similar to the issues encountered when studying the SU(2) Hubbard model with cold alkali gases, recent investigations suggest that the large spin degeneracy present in SU(\( N \)) systems can help to reach colder temperatures in fermionic AEAs. In particular, Hazzard et al studied the finite-temperature Mott insulator-to-Fermi-gas crossover, in the regime where \( k_B T > t_f \) by performing a high-temperature series expansion, together with a local density approximation assumption, to deal with the external harmonic potential [102]. They showed that the final temperatures, achievable by the standard experimental protocol of adiabatically ramping up the lattice from a weakly interacting gas in a trap can yield substantially colder Mott insulators. For example, for fixed particle numbers and fixed initial temperatures (relevant to current experiments), they showed that increasing \( N \) from 2 to 10 can lead to Mott insulators more than a factor of five colder. Furthermore, if the initial entropy, instead of the temperature, is what is held fixed, the adiabatic procedure can lead to even better cooling for all \( N \). The latter case seems to be experimentally relevant because the Pauli-blocking effect on evaporative cooling depends on entropy, \( S \propto T / T_F \), with \( T_F \) the Fermi temperature, rather than the bare temperature [67].

The cooling can be understood as a direct consequence of the rapidly increasing entropy in an SU(\( N \)) Mott insulator in the \( t_f < k_B T < U_{gg} \) regime. For the \( n = 1 \) case, the entropy per particle grows as \( S \propto \ln N \), since each of the \( N \) flavors is equally likely to occupy a site. For the experimentally relevant range of \( N \leq 10 \), the logarithm grows faster than the entropy of a quantum-degenerate noninteracting gas in a 3D trap that scales for a fixed initial temperature as \( S \propto N^{1/3} \).

The possibility of reaching colder temperatures in the regime \( t_f < k_B T < U_{gg} \) by storing entropy in the nuclear spin degrees of freedom is encouraging. However, the real motivation is the exploration of exotic SU(\( N \)) magnetism, that requires temperatures colder than \( t_f^2 / U_{gg} < t_f \) for \( U_{gg} \gg t_f \). Whether or not large \( N \) can help us to reach this regime is a crucial, but challenging question. Recently, in [125, 126], Quantum Monte Carlo methods supported by analytic [127] and DMRG (density matrix renormalization group methods) calculations [128] showed that after adiabatically loading a weakly interacting gas into an array of 1D chains, the final temperature decreases with increasing \( N \) even in the regime where \( k_B T < t_f^2 / U_{gg} \). According to these calculations, for current initial conditions, such an adiabatic-loading procedure can allow us to reach temperature scales at which interesting magnetic physics happens, for example the onset of Luttinger-liquid behavior and ground-state algebraic magnetic correlations [128, 129]. The cooling is a consequence of the rapid growth of the entropy with \( N \), in the 1D SU(\( N \)) Heisenberg model (see section 6.3). At low \( T \), the entropy scales as \( S \propto N(N-1) \) [127], even faster than its corresponding entropy in the high-\( T \) limit, where it scales as \( S \propto \ln N \), as discussed above. The quadratic growth can be derived from the exact solution [130] and the fact that there are \( N-1 \) branches of elementary excitations all with the same velocity \( v \propto 2\pi / N \) at small momentum. The quadratic growth of \( S \) with \( N \) brings the temperature of the system down with increasing \( N \) and into the region where ground-state-like correlations start to develop. This was shown in Bonnes et al [125] by computing the relevant spin–spin correlations at finite \( T \) and comparing them to the ones expected for the ground-state from DMRG calculations [128]. Starting from currently achievable temperatures, after adiabatic loading the gas, results in signatures of short-range magnetic ordering that could be seen in the spin structure factor for \( N \gg 4 \) [125, 126]. These calculations suggest that it should be possible to explore features of SU(\( N \)) quantum magnetism in ongoing experiments with AEAs.

6.2. The SU(\( N \)) Hubbard model at weak-to-intermediate coupling

The SU(\( N > 3 \)) Hubbard model is expected to exhibit a phase diagram richer than its SU(2) symmetric counterpart. In the weak-to-intermediate coupling limit, this phase diagram has been explored by Honerkamp and Hofstetter [28] for both the attractive and repulsive cases and by the same authors [29] as well as Rapp et al [35] for \( U < 0 \). Since work on the attractive case has been already reviewed in section 5.2, in this section we focus on the repulsive Hubbard model (\( U_{gg} > 0 \)).

The SU(\( N \)) Hubbard model with repulsive interactions can display various types of ordered phases in addition to the Fermi liquid phase that should be stable for \( U_{gg} / t_f \lesssim 1 \) and lattice fillings well away from incommensurability. Some of those phases break the lattice translation symmetry and may
or may not break the SU(N) symmetry at the same time. In this respect, they are different from the phases discussed in section 5, whose order parameters have no spatial dependence (for a uniform system) because these phases do not spontaneously break translational invariance.

Perhaps the most spectacular example of this phenomenon is a phase that breaks lattice-translation symmetry without breaking SU(N), known as the staggered flux (SF) phase (figure 6(a)). This phase was obtained as a mean-field solution of the Hubbard model shortly after the latter gained relevance as the minimal model for the high-Tc cuprate superconductors \cite{44, 45}. It has been postulated as a candidate to explain the anomalous pseudogap phase of these materials \cite{131}. The mean-field solution obtained by Marston and Affleck \cite{44, 45} is the ground state of the SU(N) Hubbard model for a two dimensional (2D) half-filled lattice (filling fraction \( n = \langle n_i \rangle = N/2 \)) in the \( N \rightarrow + \infty \) limit \cite{44, 45}. Therefore, it is expected \cite{21, 28} that it can be realized using ultracold gases as values of \( N \) can be as large as 10 using \(^{173}\)Yb (table 1). However, as pointed out by Honerkamp and Hofstetter \cite{28}, at values of \( N \leq 6 \), a functional renormalization-group analysis (see also \cite{112}, for a recent variational study) shows that another phase, known as a flavour density wave (FDW) phase (figure 6(b)) is favored over the SF phase. Like the SF phase, the FDW phase also breaks lattice translational symmetry. However, unlike the SF phase, it also breaks SU(N) symmetry. Recently, this FDW state has also been numerically found for \( N = 4 \) using determinant projector Quantum Monte Carlo method \cite{132}. According to \cite{28}, \(^{173}\)Yb is on the borderline for the stabilization of the SF phase, whereas \(^{87}\)Sr is probably a better candidate. Also, the SF phase is characterized by an Ising-order parameter, which is the direction of the angular momentum associated with the fermion current in each plaquette of the 2D square lattice. Thus, in two dimensions, the long-range SF order is stable at finite temperatures, which may facilitate its observation using ultracold atoms.

However, one major challenge for the observation of these phases is not only their relatively low ordering temperatures (compared to \( T_c \)), but also the requirement of a lattice fillings at or near the half-filled lattice (i.e. \( n = \langle n_i \rangle \approx N/2 \)). For large \( N \), this requirement means a relatively tight-confinement trapping potential so that large \( n \) plateaux can form at the center of the trap \cite{21}. However, under such circumstances, it is not clear how stable such a lattice system would be against inelastic losses. For instance, using \(^{173}\)Yb, a half-filled insulating plateau containing \( N/2 = 3 \) atoms per site can be reached at the center of the trap \cite{21}. However, the existence of such a plateau makes the system very susceptible to three-body recombination and the unwanted heating effects associated with such recombination. A precise experimental determination of the lifetime of a high-filling optical lattice for common AEAs is in order. Furthermore, on the theory side, not much is known about how such phases and in particular the SF phase, can be detected in an optical lattice setup.

6.3. Strong coupling limit: the SU(N) Heisenberg model

As discussed in section 4, when AEAs in their ground electronic state are loaded into a deep optical lattice, they provide us with an accurate realization of the SU(N) Hubbard model (see (2)). In the limit of large \( U \) and for integer-filling fractions, the system becomes a Mott insulator. In this regime the motion of the particles only takes place virtually, since adding or removing a particle at a giving lattice site costs energy and the Hamiltonian reduces to an effective spin Hamiltonian. Assuming a translational invariant system for simplicity (setting \( V = 0 \)), the effective model obtained by second-order degenerate perturbation theory is the SU(N) Heisenberg model \cite{22}; i.e.

\[
H = \frac{2t^2}{U_{gg}} \sum_{\langle i,j \rangle} S^\dagger_i \sigma (i) S^\dagger_j (j),
\]

where the spin operators \( S^\dagger_i (i) = e_i^{\dagger} c_i^\dagger \) satisfy the SU(N) algebra \([S^\dagger_i (i), S^\dagger_j (j)] = \delta_{ij} (\delta_{ab} S^\dagger_i (i) - \delta_{ab} S^\dagger_j (j)) \) (see appendix A).

Like the Hubbard model reviewed above, the SU(N) Heisenberg model can also display a rich phase diagram. The phases depend on \( N \), the filling fraction \( n = \langle n_i \rangle \), dimensionality and lattice geometry. The parameter \( k \equiv N n \), chosen to be an integer greater than unity, plays a key role in the analysis of the phase diagram: \( k \) is the minimum number of sites needed to form a SU(N) singlet. The 1D chain with \( n = 1 \) admits an exact solution for all \( N \) \cite{130} and its phase diagram is well understood. Nevertheless, the phase diagram of the 2D model is complex and remains unknown to a great extent. The phase diagram predicted in \cite{37, 133} for a square lattice is shown in
Abelian chiral spin liquid (ACSL), which is a spin-system analog of a fractional quantum-Hall state [38, 39, 138], for \( k \geq 5 \). A VCS is nonmagnetic and breaks lattice symmetries. The ACSL spontaneously breaks parity and time-reversal symmetry, supports excitations with fractional quantum numbers and statistics and has gapless chiral-edge states that carry spin.

Although we have focused our analysis of the phase diagram of the SU(\( N \))-Heisenberg model on a square lattice, which is the simplest to generate in experiments, it is important to mention that it is expected to be even richer in more generic lattice geometries. For example, numerical investigations of the SU(3) Heisenberg model in a triangular lattice predict a perfectly ordered three-sublattice state [139]. On a honeycomb lattice, the SU (3) case will have a dimerized, magnetically-ordered state [140–142] and it has been also predicted that the SU(6) case becomes a ACSL using a large 1/N expansion [143, 144]. Whether or not the ACSL remains the ground state in the experimentally relevant part of the phase diagram, \( k = N \) and \( n = 1 \), is unknown and needs to be validated by experiments.

7. Other exotica: physics beyond the SU(\( N \)) Heisenberg model

In this section we present an overview of some of the recently proposed exciting physics that near-term AEA experiments could explore. Most of these proposals take advantage of the long lifetime of the \( ^3P_0 \) state, in addition to the SU(\( N \)) symmetry in the nuclear spin levels.

7.1. Orbital magnetism

The possibility of independently manipulating the \( ^1S_0 \) and \( ^3P_0 \) states with laser light and thereby to construct identical or different optical lattices for the two states [145] allows for the simulation of two-orbital-SU(\( N \)) Hamiltonians that rely on the interplay of charge, spin and orbital degrees of freedom. The electronic-clock states play the role of the orbital degree of freedom and the corresponding nuclear spins provide the spin degree of freedom. The investigation of orbital physics using alkali-metal atoms has, of course, also been considered. For example, a natural way to add orbital degrees of freedom is to encode the spin in their internal hyperfine states. The ACSL spontaneously breaks parity and time-reversal symmetry, supports excitations with fractional quantum numbers and statistics and has gapless chiral-edge states that carry spin.

Although we have focused our analysis of the phase diagram of the SU(\( N \))-Heisenberg model on a square lattice, which is the simplest to generate in experiments, it is important to mention that it is expected to be even richer in more generic lattice geometries. For example, numerical investigations of the SU(3) Heisenberg model in a triangular lattice predict a perfectly ordered three-sublattice state [139]. On a honeycomb lattice, the SU (3) case will have a dimerized, magnetically-ordered state [140–142] and it has been also predicted that the SU(6) case becomes a ACSL using a large 1/N expansion [143, 144]. Whether or not the ACSL remains the ground state in the experimentally relevant part of the phase diagram, \( k = N \) and \( n = 1 \), is unknown and needs to be validated by experiments.

There \( m \) labels the filling fraction (i.e. \( m = n \)). The dashed–dotted line separates the range of parameters beyond the reach of current experiments (above and to the right of the line) and the range within the reach of the experiments (below and to the left of the line covering the region \( N \leq 10 \) and \( n \leq 5 \)). The predictions for the quantum phases, based on a large-\( N \) expansion and thus valid in the limit \( N \gg 1 \) for \( k \) finite, have been shaded, as well as regions where the ground state is known.

The known regions correspond to the well-established \( N = 2 \) and \( n = 1 \) (\( k = 2 \)) case, where anti-ferromagnetic long-range order is favored and the ground state is the so-called Néel state. The generic case \( k = 2 \) shares with SU(2) the crucial property that two adjacent spins can form an SU(N) singlet. This has been studied extensively as a large-\( N \) generalization of SU(2) magnetism [44, 45, 46, 137]. Those studies found that under very general conditions in the large-\( N \) limit, the ground state is nonmagnetic and spontaneously breaks lattice symmetries. It is formed by two-site singlets and is referred to as a valence-bond solid (VBS). Quantum Monte Carlo simulations done for \( N = 3, 4 \), have confirmed that the ground state remains a VBS even at finite \( N \). In contrast, numerical studies of the cases \( N = 3, 4, \) but \( n = 1 \) (or \( k = 3, 4 \)) in a square lattice [134, 135] provide strong evidence of magnetically ordered ground states.

The large-\( N \) expansion predicts two different ground states depending on \( k \): a valence cluster state (VCS) formed by tiling the lattice with multisite singlet clusters for \( k < 5 \) and an

Figure 7. Phase diagram of the SU(\( N \)) Heisenberg model in two dimensions on the square lattice with \( N = m k; m \) is the number of fermions per site and \( k \) is the number of sites needed to form a singlet. Regions where there is substantial evidence for a given ground state, or where the ground state is known, are shaded. The Abelian chiral spin liquid (ACSL) and valence cluster state (VCS) regions on the right are established by large-\( N \) analysis; the boundary between these regions in large \( N \) is shown by a dashed line. For \( k = 2, m = 1 \), the Néel state is the well-known ground state. There is also evidence for magnetic order at \( k = 3, m = 1 \) [134] and \( k = 4, m = 1 \) [135]. Valence-bond solid (VBS) order was found for \( k = 2 \) and \( m = 3, 4 \) [136]. The dashed–dotted line separates the range of parameters beyond the reach of current experiments (above and to the right of the line) and the range within the reach of the experiments (below and to the left of the line). The experimentally relevant part of the phase diagram with the greatest potential for novel ground states, in particular, the ACSL, is indicated with a question mark. Reprinted with permission from [133]. © 2011 American Physical Society.
not have any of the metastability issues of higher bands and allows for nuclear spin-flip processes. Collisional relaxation of the electronic-excited states must, however, be considered [146, 147]. A possible way to deal with the collisional relaxation is to work in the regime where there is only one $^3P_0$ atom per lattice site.

The implementation of the two-band SU(N) Hubbard model with alkaline earth atoms opens untapped opportunities [22, 148, 149], including the implementation of a SU(N) generalization of the SU(2)-Kondo lattice model (one of the canonical models used to study strongly correlated electron systems, such as manganese oxide perovskites [150] and rare-earth and actinide compounds classified as heavy-fermion materials [151]) and a SU(N) generalization of the $N = 2$ Kugel–Khomskii Hamiltonian (used to model the spin-orientational interactions in transition-metal oxides with the perovskite structure [152]). Just recently, Hermele et al and Freedman et al pointed out that a SU(N)-Mott insulator with one ground state atom and one excited-state atom on each site of a square lattice is likely to realize a non-ACSL with quantum statistics sufficient for universal quantum computations [133, 153]. Note that other non-Abelian states such as the fractional quantum-Hall state at the filling fraction $\frac{5}{2}$ [154, 155] or a variety of setups involving Majorana fermions [156] are not rich enough to support universal quantum computation [157]. Recent numerical studies of the phase diagram of the SU(4) Kugel–Khomskii model in a honeycomb lattice predict a quantum spin-orbital liquid ground state [144].

72. Artificial gauge fields

Atoms are neutral particles and as such they do not experience Lorentz forces in the presence of electromagnetic fields. Recently, it has been demonstrated that when a neutral atom is illuminated with properly designed laser fields, its center-of-mass motion can mimic the dynamics of a charged particle. This is the basis of the so-called artificial (synthetic) gauge fields for atoms [158]. Although there have been important advances in implementing these ideas in alkali atoms by coupling their internal or motional states with laser light [159–164] the spontaneous emission of the excited levels always imposes limitations. AEs may be ideal for synthetic gauge-field implementation [165], thanks to the long lifetime of the excited state, its reduced spontaneous emission rate and the possibility of generating antimagic lattice potentials for the clock states, in which the clock states feel exactly the same lattice but with an opposite sign. Antimagic lattice potentials have been shown to facilitate the implementation the so-called optical flux lattices [166, 167]. In addition the large number of decoupled nuclear spin degrees of freedom could facilitate the implementation of SU(N) non-Abelian gauge fields and spin–orbit Hamiltonians exhibiting rich quantum dynamics and connections to high-energy physics [168, 169]. For the use of AEAs for artificial gauge-field implementation however, collisional relaxation of the electronic-excited states could impose important limitations and must be considered [146, 147].

73. Atom-light hybrid systems and many-body physics in optical clocks

Recent advances in modern precision laser spectroscopy, with record levels of stability and residual-laser drift of less than 1 mHz s$^{-1}$ [11–13, 170] are crucial developments that are allowing us to deal with AEA clocks operated under very different conditions than those just few years ago. The level of energy resolution achievable in current atomic clocks is now providing the required capability to systematically spectrally resolve the complex excitation spectrum of an interacting many-body system. This was certainly not the case in prior clock experiments where interaction effects were subdominant. Optical atomic clocks operating with AEAs are thus becoming a new laboratory for the exploration of nonequilibrium many-body phenomena with capabilities not heretofore anticipated [61, 147, 171–177].

Moreover, the combination of this new regime of ultrastable atomic dipoles with optical cavities may become a pathway for the realization of exotic states of matter and light. The idea is to make a leap to using light to mediate interactions between atoms, impose coherence and directly drive dynamics through strong coupling to matter [178, 179]. The long-lived dipoles will allow coherent interaction of many atoms with a single optical-cavity mode over an extraordinarily long time, generating strong correlations. Experiments performed using Raman transitions in alkali vapors to mimic the ultrastable alkaline-earth dipoles, have observed superradiant behavior maintained with less than one photon in the cavity and provide a proof of principle of this outstanding capability [180].

8. Summary and outlook

Much has been achieved since AEA gases were first brought to quantum degeneracy. The creation of BECs rapidly led to the production of quantum degenerate Fermi gases. The latter, as we have discussed above, exhibit an emergent SU(N) symmetry, which makes these gases rather unique many-body systems. Subsequently, the field has evolved rapidly leading to the creation of a SU(6) Mott insulator [67] and, very recently, to the realization of arrays of 1D and quantum-degenerate ultracold $^{173}$Yb gases [20]. These experiments have also demonstrated that, thanks to the large entropy that can be stored in the nuclear spin degree of freedom of the AEA gases, there is much room for improvement in the quest for cooling AEAs to lower and lower temperatures (entropies) using conventional methods such as sympathetic cooling and adiabatic loading into the lattice.

Regardless of all the great progress, what has been experimentally demonstrated so far [20, 67] is just fairly interesting physics related to the ‘charge’ degrees of freedom. The real challenge, i.e. the observation of quantum magnetism and many of the other exotic phases that have been described in previous sections, still needs to be overcome. These phases should become stable to thermal fluctuations well below the hopping-temperature scale $\sim t^2_F/k_B$ and typically at $k_B T \ll t_F^2/U_{gg}$ for the Hubbard model of section 4. As we have emphasized above, we expect that the large spectral
degeneracies introduced by the enlarged SU(N) symmetry will bring about new phenomena that have no counterpart in two-component systems. Some hints of these differences have already shown up in the experiments [20, 67], but there is more to come if we can find a way to remove the entropy from the nuclear degree of freedom. This is a challenge that will require new ideas, perhaps different from those applicable to ultracold gases of alkali atoms.

As we have seen, tuning other parameters like the interactions in AEAAs also requires using different methods like OFRs. Unfortunately, these resonances generally spoil the emergent SU(N) symmetry that makes these gases so special. New ideas are also required in this regard. If an efficient and versatile way is found to tune the interactions while respecting the SU(N) symmetry, the door will open to the exploration of superfluidity and ferromagnetism in these systems. The landscape associated with phases, as we have described in section 5 will be rather rich, exhibiting interesting excitations and topological defects as well as discontinuous phase transitions. These can lead to spectacular phase-segregation patterns (i.e. domains) in a trap. In a different, but complementary direction, although the use of the exquisite precision of optical-lattice clocks to probe AEA many-body physics has already started, there is still lots of room for improvement.

In spite of the limitations of the present, it is important to emphasize that seeds for a bright future of the field have been already planted. We strongly believe that there is much more to come and hopefully many serendipitous discoveries are waiting for us. Some of these discoveries may come in the form of new phases of matter, which do not fit into the relatively narrow framework that we have outlined in this article. Or, they may come by exploring nonequilibrium phenomena with AEA gases. This is a field that, compared to what has been already achieved using alkali gases, remained largely unexplored at the time of writing this article. And as it happens for equilibrium phenomena, we have a new parameter to play with, namely N (or 1/N, depending on the point of view). In conclusion, we hope that this review will become the starting point for many of the bold minds wanting to explore these fascinating new systems.

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Appendix A. Brief digest of SU(N) group theory

In this Appendix we briefly review the most important facts about the special unitary group SU(N). We begin by defining it. To this end, let us first introduce an N-dimensional linear space of complex vectors denoted as $\psi^T = (\psi^1, \ldots, \psi^N)$, where $T$ means transpose and the components $\psi^a$ ($a = 1, \ldots, N$) are complex numbers. In this linear space, we define the scalar product between two vectors $\psi$ and $\chi$ as $\langle \psi | \chi \rangle = \Sigma_a (\psi^a)^* \chi_a$. To lighten the notation, we introduce the dual of the vector $\psi$ defined by $\psi_\alpha = (\psi^\alpha)^*$. This allows us to write $\langle \psi | \chi \rangle = \psi_\alpha \chi^\alpha$, where repeated upper and lower indices are to be summed over, unless stated otherwise. Finally, the norm of $\psi$ can be defined as $\sqrt{\langle \psi | \psi \rangle} = \psi_\alpha \psi^{\alpha*}$. We next use consider the linear transformation

$$\tilde{\psi}^\alpha = U_\alpha^\beta \psi^\beta. \quad (A.1)$$

Hence, the dual $\tilde{\psi}_\alpha = (\tilde{\psi}^\alpha)^* = (\psi^\beta)^* (U^\beta_\alpha) = \psi_\beta (U^\beta_\alpha)^*$, where we have employed that $(U^\beta_\alpha)^\dagger = (U^\beta_\alpha)^*$, where $U^\beta_\alpha$ is the hermitian conjugate of the matrix $U$.

We are now ready to define the SU(N) group as the set of linear transformations that preserve the norm of vectors. Mathematically, $\langle \psi | \psi \rangle = \langle \tilde{\psi} | \tilde{\psi} \rangle$. Hence, using (A.1) leads to:

$$\langle \psi | \tilde{\psi} \rangle = \psi_\alpha (U^\beta_\alpha)^\dagger \psi^\beta = \psi_\alpha \psi^{\alpha*} = \langle \tilde{\psi} | \psi \rangle, \quad (A.2)$$

which, for arbitrary $\psi$, is only possible provided $(U^\beta_\alpha)^\dagger U^\beta_\alpha = \delta^\alpha_\beta$, that is, in matrix notation:

$$U^\dagger U = U U^\dagger = 1, \quad (A.3)$$

where $1$ is the unit matrix. Hence, $U^{-1} = U^\dagger$, or, in other words, $U$ is a unitary matrix. Furthermore, it also follows that $\det(U^\dagger) = \det(U) = 1$, which implies that $\det(U^\dagger) = 1$. If $U$ belongs to SU(N), then $\det(U) = \epsilon_1 \epsilon_2 \cdots \epsilon_N = 1$, where $\epsilon_1 \epsilon_2 \cdots \epsilon_N$ is the fully anti-symmetric Levi-Civita symbol; e.g. for $N = 3$, $\epsilon_{123} = - \epsilon_{213} = - \epsilon_{312} = 1$, etc. The vector and its dual define the two fundamental irreducible representations of SU(N), which are denoted as $N$ and $\overline{N}$, respectively. We can consider tensors with upper and lower indices that transform as products of these two fundamental representations. For instance, $\phi^{\alpha \beta}$ belongs to the $N \otimes \overline{N}$ representation transforming as $\psi^a \chi^b$. Note that the transformation properties of the tensors respect the permutation symmetries of their indices. Thus, for $\phi^{\alpha \beta} = \phi^{\beta \alpha}$, $\phi^{\alpha \beta} = \phi^{\beta \alpha} = - \phi^{\alpha \beta}$ (antisymmetric tensor, the transformed tensor $\tilde{\phi}^{\alpha \beta} = U^\alpha_\gamma U^\beta_\delta \phi^{\gamma \delta}$ is also (anti) symmetric). Hence, since the tensor $\tilde{\phi}^{\alpha \beta} = \phi^{\alpha \beta} + \phi^{\beta \alpha}$, where $[ \ldots ]$ stands for anti-symmetrization of the indices and $(\ldots)$ for symmetrization, we see that the representation $N \otimes \overline{N}$ is reduced to $N(N-1)/2 \oplus N(N+1)/2$. Furthermore, an SU(N) transformation respects the trace of a tensor (the latter being understood as the result of contracting an upper and a lower index). Hence, for instance, $\phi^{\alpha \beta} \rightarrow \frac{1}{N} \phi^{\alpha \beta} + (\sqrt{\phi^{\alpha \beta} - 1/Nq_{\alpha \beta}})$ that is, $N \otimes \overline{N} = 1 \oplus N^2 - 1$

Finally, let us consider the linear transformations in the neighborhood of the unit element of the group (i.e. $1$). For $N = 2$, SU(2) $\simeq O (3)$, the rotation group. For this group we know that any finite rotation can be obtained as the product of an infinite set of infinitesimal rotations. The latter differ from unity $1$ by an infinitesimal amount, i.e. $U = 1 + i \epsilon T$, where $\epsilon \ll 1$ is a real parameter and $T$ is a matrix whose properties
we determine in what follows. From (A.3) it follows that \(U^T U = (1 - i e T^r) (1 + i e T^r) = 1 - i e(T^{r-T}) + O(e^2) = 1\), that is,

\[
T^r = T^r.
\]

Moreover, the unit determinant condition requires that \(1 = \det(1 + i e T) = \exp[i \text{tr}(1 + i e T)] = 1 + i e \text{tr} T\), where we have employed the identity \(\det A = \exp[\text{tr} A]\). Therefore, \(\text{tr} T = 0\). Thus, the \(N \times N\) matrices \(T\) are hermitian (see (A.4)) and traceless. When expressed in terms of the matrix components, \((A.4)\) reads \((T_{xy}^r) = T_{yx}^r\). In other words, the diagonal elements of \(T\) are real and the \(N(N - 1)/2\) upper and lower diagonal are the complex conjugate to each other. Hence, it follows that \(T\) depends only on \(2 \times N(N - 1)/2 + N = N^2\) real parameters. The traceless condition imposes a further constraint, which yields \(N^2 - 1\) for the number of independent \(T\) matrices, which are denoted as \(T^a\), with \(a = 1, ..., N^2 - 1\). Thus, a general infinitesimal \(N(N)\) transformation can be written as \(U = 1 + \delta\varepsilon^a T^a\), where \(\varepsilon^a \ll 1\) are \(N^2 - 1\) real numbers. For \(N = 2\), there are \(2^2 - 1 = 3\) matrices proportional to the Pauli matrices \(T^a = \frac{\varepsilon^a}{2}, \varepsilon = x, y, z\); \(2\) the latter satisfy the angular momentum algebra \([T^a, T^b]\) = \(i\varepsilon^{abc} T^c\), where \(\varepsilon^{abc}\) is the fully antisymmetric Levi–Civita symbol. This is an example of a Lie algebra. For \(N(N) > 2\), the Lie algebra is characterized by a set of structure constants \(f_{ab}^c\) different from \(\varepsilon^{abc}\):

\[
[T^a, T^b] = i f_{ab}^c T^c. \tag{A.5}
\]

The \(N^2 - 1, N \times N\) traceless hermitian matrices, \(T^a\), are the generators of the Lie algebra. Furthermore, they provide a basis for the linear space of \(N \times N\) traceless hermitian matrices. Among them, we can distinguish \(N - 1\) that are diagonal (like \(T^0 = \sigma^0/2\) for \(SU(2)\)), which form a set known as the Cartan basis. A representation for Cartan basis operators is \(T_1 = \frac{1}{2}\text{diag}(1, -1, 0, ..., 0), T_{12} = \frac{1}{\sqrt{2}}\text{diag}(1, 1, -2, 0, ..., 0), \ldots, T_{r-1} = \frac{1}{\sqrt{2(2-r)}}\text{diag}(1, 1, 1, ..., -r, ..., 0)\), for \(r = 2, ..., N\). The other matrices are chosen hermitian and nondiagonal and contain a single nonvanishing element equal to either \(1/\sqrt{2}\) or \(-1/\sqrt{2}\). This basis is conveniently normalized so that \(\text{tr} T^a T^b = \text{traceless} 12\varepsilon^{ab}\). Another convenient basis for \(U(3) = U(1) \times SU(3)\) is provided by the projection operators \(X^a = \{\alpha\}_{b}\), where \(a, b = 1, ..., N\). In this basis, the Lie algebra takes a very simple form:

\[
\left[X^a, X^b\right] = \delta^b_a X^a - \delta^a_b X^b. \tag{A.6}
\]

Furthermore, \(n = X^g\) commutes with all the generators \(X^g\) and corresponds to the generator of the \(U(1)\) subgroup in \(U(3) = U(1) \times SU(3)\). Note that the nondiagonal generators \((\alpha \neq \beta)\) are not hermitian, whereas the diagonal ones are traceless. However, this basis has the advantage that it can be readily represented in a second quantization: Let \(c_a\) transform according to the \(N\) irrep and \(c_a^\dagger\) transform according to \(\mathcal{N}\), then \(X^g = c_a^\dagger c^a\), provided the constraint \(n = c_a^\dagger c^a = 1\) is enforced.

### Appendix B. Fermi liquid parameters

We can exploit the \(SU(N)\) symmetry and write the Landau QP occupation and the Landau function as [21]:

\[
\delta n^g_L(p) = \frac{1}{N} \delta n_L(p) \delta^g_L + \sum_{a=1}^{N-1} m_a(p) (T^a)^g_L, \tag{B.1}
\]

\[
f^{gs}(p, p') = f^0(p, p') - \sum_{a=1}^{N-1} (T^a)^g_L (T^a)^0_L, \tag{B.2}
\]

where we have exploited the fact that \(\delta n^g_L(p)\) is a \(N \times N\) (hermitian) density matrix that can be split into a trace \([\delta n_L(p) = \delta n^g_L(p)]\), which describes fluctuations in the total QP number, and a traceless part. The latter can be conveniently expanded in terms of the generators of the \(SU(N)\) algebra (see appendix A) and describes the nuclear spin fluctuations. In group theoretic language, \(\delta n^g_L(p)\) is a rank-2 tensor in the (reducible) \(N \otimes \mathcal{N} = \mathcal{N} \otimes \mathcal{N} - 1\) representation (see appendix A for definitions). Likewise, the fourth rank tensor of Landau functions belongs to the (reducible) representation \(N \otimes \mathcal{N} \otimes N \otimes \mathcal{N} = \mathcal{N} \otimes \mathcal{N} \otimes (N - 1)\) + non-scalar representations and therefore it can be parametrized in terms of two scalar functions as in (B.2). Because the QP are only well-defined excitations in the neighborhood of the FS (otherwise they scatter each other), for \(|p| = |p'| = p_F\), rotational invariance requires that the Landau functions depend only on \(\cos \theta = p p' / p_F^2\). Thus, it is conventional [105] to express the Landau functions \(f^0(\cos \theta)\) and \(f^m(\cos \theta)\) in terms of the dimensionless Landau parameters \(F^L_{m,m}\):
modes is not equal to the number of broken symmetry generators (see, e.g. [181] and references therein). Qualitatively, this can be understood by writing down an effective Lagrangian for the order parameter spinor field $\Phi(r)$. Besides the $U(3)$ symmetry, the latter is constrained by space rotation invariance, which leads to

$$L = i\bar{\Phi}^\dagger(r,t) \partial_0 \Phi(r,t) - \frac{K_1}{2} \nabla^2 \Phi^\dagger(r,t) \nabla \Phi(r,t) - V(\Phi^\dagger \Phi) + \ldots,$$

(C.1)

where $K_1$ is a constant, the potential $V(\Phi^\dagger \Phi)$ has a minimum for $\Phi^\dagger \Phi = \phi_0^2$, i.e. $V = \frac{1}{2} (\Phi^\dagger \Phi - \phi_0^2)^2$ (we take $\phi_0$ real without loss of generality) and the dots stand for higher-order gradient terms. Note that in a relativistic (i.e. Lorentz-invariant) or in a particle-hole symmetric theory, the first term in the right-hand side of (C.1) would be forbidden and should be replaced by $\sim \partial_0 \Phi^\dagger \partial_0 \Phi$. For such theories, the number of NG modes equals the number of broken-symmetry generators [181]. However, in the nonrelativistic case, as we shall see next, this term is responsible for a dramatic change in the number and long-wave length dispersion of the NG modes. If we parameterize the small fluctuations about the minimum as $\Phi(r,t) = (\phi_1(r,t), \phi_2(r,t), [\phi_0 + \delta \phi_1(r,t)] e^{i\omega(r,t)})$, we see that the linearized equations of motion for $\delta \phi_1(r,t)$ and $\delta \phi_2(r,t)$ imply that the phase ($\theta$) NG mode has linear dispersion, i.e. $\omega \propto \theta$ as $k \to 0$. However, the $\phi_{1,2}$ NG modes disperse quadratically, i.e. $\omega \propto \theta^2$ as $k \to 0$. Furthermore, the number of NG modes is three, which is different from the number of broken-symmetry generators ($=5$) because, upon quantization, the fields $\phi_1^\dagger, \phi_2^\dagger$ and $\phi_1, \phi_2$ cannot be treated as independent degrees of freedom as they correspond to the creation and annihilation of the same eigenmode [34, 119]. Another lesson from this example is that quadratic modes correspond to fluctuations in the pairing function of the two paired components with unpaired one, i.e. $\phi_1 = \Delta_23$ and $\phi_2 = \Delta_{13}$. This is because, in the linearized equations that follow from (C.1), $\phi_1, \phi_2$ are not coupled to each other and to $\phi_3$. However, $\delta \phi_3 = \phi_3 - \phi_0 \sim \delta \rho_3 e^{i\omega} + \delta \rho_3^* e^{-i\omega}$ are coupled, which leads to the linear dispersion for $\theta$. This observation also generalizes to larger odd values of $N = 3, 5, 7, \ldots$, implying that for $N = 5$, there are four quadratic NG modes, etc. The quadratic coupling of the NG modes involving the unpaired component can also be explained using gauge invariance arguments [119]. For even values of $N$, the NG modes disperse linearly at small $q$ [119].

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