A REMARK ON LOW REGULARITY SOLUTIONS OF THE
CHERN-SIMONS-DIRAC SYSTEM

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Abstract. An alternative proof of low regularity well-posedness for the
Chern-Simons-Dirac system in Coulomb gauge is given which completely
avoids the use of any null structure similarly to a recent result of Bourgain-Candy-Machihara. An unconditional uniqueness result is also given.

1. Introduction and main results
Consider the Chern-Simons-Dirac system in two space dimensions:
\begin{align*}
\frac{1}{2} \epsilon^{\mu\nu\rho} F_{\nu\rho} &= -J^\mu, \\
i \gamma^\mu D^\mu \psi &= m \psi,
\end{align*}
with initial data
\begin{equation}
A_\mu(0) = a_\mu, \quad \psi(0) = \psi_0,
\end{equation}
where we use the convention that repeated upper and lower indices are summed,
Greek indices run over 0,1,2 and Latin indices over 1,2. Here
\begin{align*}
D^\mu &= \partial^\mu - iA^\mu, \\
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu, \\
J^\mu &=\langle \gamma^0 \gamma^\mu \psi, \psi \rangle
\end{align*}
Here \( F_{\mu\nu} : \mathbb{R}^{1+2} \to \mathbb{R} \) denotes the curvature, \( \psi : \mathbb{R}^{1+2} \to \mathbb{C}^2 \), and \( A_\mu : \mathbb{R}^{1+2} \to \mathbb{R} \)
the gauge potentials. We use the notation \( \langle \cdot, \cdot \rangle \) for the inner product in \( \mathbb{C}^2 \),
\( \partial_\mu = \frac{\partial}{\partial x^\mu} \), where we write \((x^0, x^1, ..., x^n) = (t, x^1, ..., x^n)\) and also \( \partial_0 = \partial_t \) and \( \nabla = (\partial_1, \partial_2) \), \( \epsilon^{\mu\nu\rho} \) is the totally skew-symmetric tensor with \( \epsilon^{012} = 1 \), and \( m \geq 0 \).
\( \gamma^\mu \) are the Pauli matrices
\begin{equation*}
\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\end{equation*}
The equations are invariant under the gauge transformations
\begin{align*}
A_\mu &\rightarrow A'_\mu = A_\mu + \partial_\mu \chi, \quad \psi \rightarrow \psi' = e^{i\chi} \phi, \\
D_\mu &\rightarrow D'_\mu = \partial_\mu - iA'_\mu.
\end{align*}
The most common gauges are the Coulomb gauge \( \partial^j A_j = 0 \), the Lorenz gauge \( \partial^\mu A_\mu = 0 \) and the temporal gauge \( A_0 = 0 \).

Our main aim is to give a simple proof of local well-posedness for data \( \psi_0 \in H^s(\mathbb{R}^2) \) for \( s > \frac{1}{4} \), especially we want to show that no null condition is necessary in the case of the Coulomb gauge. Critical with respect to scaling is the case \( s = 0 \). The same result was proven recently in Coulomb as well as Lorenz gauge by M. Okamoto using a null structure of the system. Earlier results were given by Huh

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where

In the sequel we consider this nonlinear Dirac equation with initial condition

\( \psi \in \mathcal{H}_0^+ \), \( a_\mu \in \mathcal{H}_0^+ \) using a null structure, in the Coulomb gauge for \( \psi_0 \in \mathcal{H}_0^{+\times} \), \( a_\mu \in L^2 \), and in temporal gauge for \( \psi_0 \in \mathcal{H}_0^{+\times} \), \( a_\mu \in \mathcal{H}_0^{+\times} + L^2 \), both without using a null structure. Huh-Oh [16] proved local well-posedness in Lorentz gauge for \( \psi_0 \in H^s \), \( a_\mu \in H^s \) for \( s > \frac{1}{2} \) also making use of a null structure. The methods of Okamoto and Huh-Oh are different. Okamoto reduces the problem to a single Dirac equation with cubic nonlinearity for a null structure. The methods of Okamoto and Huh-Oh are different. Okamoto makes use of bilinear estimates in wave-Sobolev spaces given by d’Ancona-Foschi-Selberg [2].

An almost identical result was recently given by Bournaveas-Candy-Machihara [4] who were also able to avoid any use of the null structure of the system. Their proof relies on a bilinear Strichartz estimate given by Klainerman-Tataru [7] whereas we make use of bilinear estimates in wave-Sobolev spaces given by d’Ancona-Foschi-Selberg [2].

Our result gives uniqueness in a certain subspace of \( C^0([0,T],H^s) \) of \( X^{s,b} \)-type. Thus it is natural to consider the question whether unconditional uniqueness also holds, namely in \( C^0([0,T],H^s) \). We give a positive answer if \( s > \frac{1}{2} \) using an idea of Zhou [10].

We exclusively study the Coulomb gauge condition \( \partial_\gamma A^\gamma = 0 \). In this case one easily checks using (1) that the potentials \( A_\mu \) satisfy the elliptic equations

\[
A_0 = \Delta^{-1}(\partial_2 J_1 - \partial_1 J_2), \quad A_1 = \Delta^{-1}\partial_2 J_0, \quad A_2 = -\Delta^{-1}\partial_1 J_0.
\]

Inserting this into (2) and defining the matrices \( \alpha_\mu = \gamma^0\gamma^\mu \), \( \beta = \gamma_0 \) we obtain

\[
(i\alpha_\mu \partial_\mu - m\beta)\psi = N(\psi,\psi,\psi),
\]

where

\[
N(\psi_1,\psi_2,\psi_3) = \Delta^{-1}(\partial_2(\alpha_1\psi_1,\psi_2) - \partial_1(\alpha_2\psi_1,\psi_2) + \partial_2(\psi_1,\psi_2)\alpha_1 - \partial_1(\psi_1,\psi_2)\alpha_2)\psi_3.
\]

In the sequel we consider this nonlinear Dirac equation with initial condition

\[
\psi(0) = \psi_0.
\]

Using an idea of d’Ancona - Fosci - Selberg [1] we simplify (5) by considering the projections onto the one-dimensional eigenspaces of the operator \( -i\alpha \cdot \nabla = -i\alpha^2 \partial_\gamma \) belonging to the eigenvalues \( \pm \xi \). These projections are given by \( \Pi_\pm(D) \), where \( D = \sum \) and \( \Pi_\pm(\xi) = \frac{1}{2}(I \pm \frac{\xi}{|\xi|} \cdot \alpha) \). Then \( -i\alpha \cdot \nabla = |D|\Pi_+(D) - |D|\Pi_-(D) \) and \( \Pi_\pm(\xi)\beta = \beta\Pi_\pm(\xi) \). Defining \( \psi_\pm := \Pi_\pm(D)\psi \), the Dirac equation can be rewritten as

\[
(-i\partial_\pm + |D|)\psi_\pm = m\beta\psi_\mp + \Pi_\pm N(\psi_+ + \psi_- + \psi_+ + \psi_-).
\]

The initial condition is transformed into

\[
\psi_\pm(0) = \Pi_\pm \psi_0.
\]

We use the following function spaces and notation. \( H^s_p \) denotes the standard \( L^p \)-based Sobolev space of order \( s \), \( H^s = H^s_2 \), and \( B^{p,q}_s \) the Besov space as defined e.g. in [3]. Let \( \hat{\cdot} \) denote the Fourier transform with respect to space and time. The
standard spaces of Bougain-Klainerman-Machedon type $X_{\pm}^{s,b}$ belonging to the half waves are defined by the completion of $S(\mathbb{R} \times \mathbb{R}^2)$ with respect to

$$\|f\|_{X_{\pm}^{s,b}} = \|\langle \xi \rangle^s (\tau \pm |\xi|)^b \hat{f}(\tau, \xi)\|_{L^2},$$

where $\langle \cdot \rangle := (1 + |\cdot|^2)^{\frac{1}{4}}$. $X_{\pm}^{s,b}[0,T]$ is the space of restrictions to the time interval $[0,T]$. Similarly $H^{s,b}$ denotes the completion of $S(\mathbb{R} \times \mathbb{R}^2)$ with respect to

$$\|f\|_{H^{s,b}} = \|\langle \xi \rangle^s |\tau| - |\xi|)^b \hat{f}(\tau, \xi)\|_{L^2}$$

and $H^{s,b}[0,T]$ its restriction to the time interval $[0,T]$.

We remark the embedding $X_{\pm}^{s,b} \subset H^{s,b}$ for $b \geq 0$.

Finally $a^+$ and $a^−$ denote numbers which are slightly larger and smaller than $a$ respectively, such that $a^− < a < a^+ < a^++$.

We now formulate our results.

**Theorem 1.1.** Assume $\psi_0 \in H^s(\mathbb{R}^2)$ with $s > 1/4$. Then (5),(6) is locally well-posed in $H^s(\mathbb{R}^2)$. More precisely there are $T > 0$, $b > 1/2$ such that there exists a unique solution $\psi = \psi_+ + \psi_−$ with $\psi_\pm \in X_{\pm}^{s,b}[0,T]$. This solution belongs to $C^0([0,T], H^s(\mathbb{R}^2))$.

The unconditional uniqueness result is the following

**Theorem 1.2.** Assume $\psi_0 \in H^s(\mathbb{R}^2)$ with $s > 1/4$. The solution of (5),(6) is unique in $C^0([0,T], H^s(\mathbb{R}^2))$.

Fundamental for their proof are the following bilinear estimates in wave-Sobolev spaces which were proven by d’Ancona, Foschi and Selberg in the two dimensional case $n = 2$ in [2] in a more general form which include many limit cases which we do not need.

**Theorem 1.3.** Let $n = 2$. The estimate

$$\|uv\|_{H^{-s_0-s_1} \times H^{s_1}} \lesssim \|u\|_{H^{s_1}} \|v\|_{H^{s_2}}$$

holds, provided the following conditions hold:

$$
\begin{align*}
&b_0 + b_1 + b_2 > \frac{1}{2} \\
&b_0 + b_1 > 0 \\
&b_0 + b_2 > 0 \\
&b_1 + b_2 > 0 \\
&s_0 + s_1 + s_2 > \frac{3}{2} - (b_0 + b_1 + b_2) \\
&s_0 + s_1 + s_2 > 1 - (b_0 + b_1) \\
&s_0 + s_1 + s_2 > 1 - (b_0 + b_2) \\
&s_0 + s_1 + s_2 > 1 - (b_1 + b_2) \\
&s_0 + s_1 + s_2 > \frac{1}{2} - b_0 \\
&s_0 + s_1 + s_2 > \frac{1}{2} - b_1 \\
&s_0 + s_1 + s_2 > \frac{1}{2} - b_2 \\
&s_0 + s_1 + s_2 > \frac{3}{4}
\end{align*}
$$
(s_0 + b_0) + 2s_1 + 2s_2 > 1
2s_0 + (s_1 + b_1) + 2s_2 > 1
2s_0 + 2s_1 + (s_2 + b_2) > 1
s_1 + s_2 \geq \max(0, -b_0)
s_0 + s_2 > \max(0, -b_1)
s_0 + s_1 > \max(0, -b_2)

2. Proof of the theorems

Proof of Theorem \[\text{12.1}\] By standard arguments we only have to show

\[
\|N(\psi_1, \psi_2, \psi_3)\|_{X^{s_0} \pm, 1} \lesssim \prod_{i=1}^{3} \|\psi_i\|_{X^{s_i, \pm}}.
\]

where \(\pm\ (i = 1, 2, 3, 4)\) denote independent signs.

By duality this is reduced to the estimates

\[
J := \int (N(\psi_1, \psi_2, \psi_3), \psi_4) dt \, dx \lesssim \prod_{i=1}^{3} \|\psi_i\|_{X^{s_i, \pm}} \|\psi_4\|_{X^{-s_0, \pm}}.
\]

By Fourier-Plancherel we obtain

\[
J = \int q(\xi_1, \ldots, \xi_4) \prod_{j=1}^{4} \hat{\psi}_j(\xi_j, \tau_j) d\xi_1 \ldots d\xi_4 \, d\tau_4,
\]

where * denotes integration over \(\xi_1 - \xi_2 = \xi_4 - \xi_3 =: \xi_0\) and \(\tau_1 - \tau_2 = \tau_4 - \tau_3\) and

\[
q = \frac{1}{|\xi_0|^2} [\xi_0 ((\alpha_1 \hat{\psi}_1, \hat{\psi}_2) (\hat{\psi}_3, \hat{\psi}_4) + (\hat{\psi}_1, \hat{\psi}_2) (\alpha_1 \hat{\psi}_3, \hat{\psi}_4))

- \xi_0 ((\alpha_2 \hat{\psi}_1, \hat{\psi}_2) (\hat{\psi}_3, \hat{\psi}_4) + (\hat{\psi}_1, \hat{\psi}_2) (\alpha_2 \hat{\psi}_3, \hat{\psi}_4))].
\]

The specific structure of this term, namely the form of the matrices \(\alpha_j\) plays no role in the following, thus the null structure is completely ignored.

We first consider the case \(|\xi_0| \leq 1\). In this case we estimate \(J\) as follows:

\[
\|\langle \nabla\rangle^{-s-1} |\nabla|^s \psi_1, \psi_2\|_{L^2_t L^{\frac{4}{4-s}}} \lesssim \|\langle \nabla\rangle^{-s-1} \alpha_i \psi_1, \psi_2\|_{L^2_t L^{\frac{4}{4}}} \lesssim \|\alpha_i \psi_1, \psi_2\|_{L^2 B^{-s-1}_{\frac{4}{4}}},
\]

\[
\lesssim \|\langle \alpha_i \psi_1, \psi_2\|_{L^2 B^{-s-1}_{\frac{4}{4}, \infty}} \lesssim \|\psi_1\|_{L^2 H^{-s}_{\frac{4}{4}}} \|\psi_2\|_{L^2 H^{-s}_{\frac{4}{4}}},
\]

where we used the embeddings \(B^{-s, -1}_{\frac{4}{4}, \infty} \subset B^{-s, -1}_{\frac{4}{4}, 1} \subset H^{-s, -1}_{\frac{4}{4}}\), which hold by [3], Thm. 6.2.4 and Thm. 6.5.1. The last inequality follows from [9], namely the Lemma in Chapter 4.4.3. The same estimate holds for \(\alpha_i = I\). Similarly we obtain

\[
\|\langle \nabla\rangle^{-s-1} |\nabla|^{-\frac{s}{2}} \alpha_i \psi_3, \psi_4\|_{L^2_t} \lesssim \|\psi_3\|_{L^2 H^s} \||\psi_4\|_{L^2 H^s},
\]

for arbitrary matrices \(\alpha_i\), so that we obtain

\[
J \lesssim \|\psi_1\|_{X^{s_0, \pm}} \|\psi_2\|_{X^{-s_0, \pm}} \|\psi_3\|_{X^{s_1, \pm}} \|\psi_4\|_{X^{-s_1, \pm}},
\]

which is more than enough. From now on we assume \(|\xi_0| \geq 1\). Assume first that \(\frac{3}{4} > s > \frac{1}{2}\). We obtain

\[
|J| \lesssim \sum_{j=1}^{2} (\|\langle \alpha_j \psi_1, \psi_2\|_{H^{-s, -\frac{s}{2}}} \|\langle \psi_3, \psi_4\|_{H^{-s, -\frac{s}{2}}}

+ \|\langle \psi_1, \psi_2\|_{H^{-s, -\frac{s}{2}}} \|\langle \alpha_j \psi_3, \psi_4\|_{H^{-s, -\frac{s}{2}}}).
\]
By Theorem 1.3 with $s_0 = \frac{1}{2}, \ b_0 = -\frac{1}{2}, \ s_1 = s_2 = s, \ b_1 = b_2 = \frac{1}{2} + \epsilon$ for the first factors and $s_0 = \frac{5}{2}, \ b_0 = \frac{5}{2} + \epsilon, \ s_1 = s, \ s_2 = -s, \ b_1 = \frac{5}{2} + \epsilon, \ b_2 = \frac{5}{2} - 2\epsilon$ for the second factors we obtain under the assumption $\frac{3}{4} > s > \frac{1}{4}$:

$$|J| \lesssim \prod_{j=1}^{3} \|\psi_j\|_{H^\frac{3}{2} + \epsilon} \|\psi_s\|_{H^{-\frac{1}{2} - 2\epsilon}}.$$ 

Using the embedding $X^{s,b}_\pm \subset H^{s,b}$ for $s \in \mathbb{R}$ and $b \geq 0$ we obtain the desired estimate.

Next assume $s \geq \frac{3}{4}$. We obtain

$$|J| \lesssim \sum_{j=1}^{2} \left( \|\langle \alpha_j \psi_1, \psi_2 \rangle\|_{H^{\frac{3}{2} + \epsilon}} \|\psi_3, \psi_4\|_{H^{-\frac{1}{2} - \epsilon}} + \|\langle \psi_1, \psi_2 \rangle\|_{H^{\frac{3}{2} + \epsilon}} \|\langle \alpha_j \psi_3, \psi_4 \rangle\|_{H^{-\frac{1}{2} - \epsilon}} \right).$$

By Theorem 1.3 with $s_0 = 1 - s - \epsilon, \ b_0 = -\frac{1}{2} - \epsilon, \ s_1 = s_2 = s, \ b_1 = b_2 = \frac{1}{2} + \epsilon$ for the first factors and $s_0 = s + \epsilon, \ b_0 = \frac{1}{2} + \epsilon, \ s_1 = s, \ s_2 = -s, \ b_1 = \frac{1}{2} + \epsilon, \ b_2 = \frac{1}{2} - 2\epsilon$ for the second factors we obtain the same estimate as before.

**Remark:** The potentials are completely determined by $\psi$ and $\langle \bullet \rangle$. We have $A_\mu \sim |\nabla|^{-\langle \psi, \psi \rangle}$, so that for $s < 1$:

$$\|A_\mu\|_{H^{2s}} \lesssim \|\langle \psi, \psi \rangle\|_{H^{2s-1}} \lesssim \|\langle \psi, \psi \rangle\|_{L^{\frac{1}{2}}} \lesssim \|\psi\|_{L^{\frac{1}{2}}}^2 \lesssim \|\psi\|_{H^s}^2, \quad \text{for } \psi, \psi, \psi,$$

and

$$\|A_\mu\|_{H^s} \lesssim \|\langle \psi, \psi \rangle\|_{H^{s-1}} \lesssim \|\psi\|_{H^s}^2, \quad \text{for } \psi, \psi, \psi.$$

Thus we obtain for $0 < \epsilon \ll 1$ and $s < 1$:

$$A_\mu \in C^{0}(0, T, \tilde{H}^{2s} \cap \tilde{H}^s).$$

**Proof of Theorem 1.2** We first show $\psi_\pm \in X^{\frac{3}{4},1}_\pm [0, T]$. We have to prove

$$\|N(\psi_1, \psi_2, \psi_3)\|_{L^2([0, T], L^2_\pm)} \lesssim \prod_{j=1}^{3} \|\psi_j\|_{L^\infty([0, T], H^{\frac{3}{2}}_\pm)},$$

where the implicit constant may depend on $T$. This follows from the estimate

$$\|\langle \nabla \rangle^{-1} \langle \alpha_1 \psi_j, \psi_k \rangle \psi_3\|_{L^2_\pm} \lesssim \|\langle \nabla \rangle^{-1} \langle \alpha_1 \psi_j, \psi_k \rangle \psi_3\|_{L^2_\pm} \lesssim \|\langle \alpha \psi_j, \psi_k \rangle\|_{L^\infty} \|\psi_3\|_{L^2_\pm} \lesssim \|\psi_j\|_{L^\infty} \|\psi_k\|_{L^\infty} \|\psi_3\|_{L^2_\pm},$$

and a similar estimate for the term $\|\langle \nabla \rangle^{-1} \langle \psi_j, \psi_k \rangle \alpha \psi_3\|_{L^2_\pm}$.

Assume now $\psi \in C^0([0, T], \tilde{H}^{\frac{3}{2}+\epsilon})$, $\epsilon > 0$. Then we have shown that $\psi_\pm \in X^{\frac{3}{4}+\epsilon,0}_\pm [0, T] \cap X^{\frac{3}{4},1}_\pm [0, T]$. By interpolation we get $\psi_\pm \in X^{\frac{3}{4}+\epsilon,\frac{3}{4}+\epsilon}_\pm [0, T]$ for $\epsilon \ll 1$. Assume now that $\psi, \psi' \in C^0([0, T], \tilde{H}^{\frac{3}{2}+\epsilon})$ are two solutions of (3), (4). Then we have

$$\sum_{\pm} \|\psi_\pm - \psi'_\pm\|_{X^{\frac{3}{4}+\epsilon}_\pm [0, T]} \lesssim T^{0+} \sum_{\pm} \|N(\psi, \psi, \psi) - N(\psi', \psi', \psi')\|_{X^{\frac{3}{4}+\epsilon}[0, T]} \quad (9)$$

$$\lesssim T^{0+} \sum_{\pm} \left( \|N(\psi_\pm - \psi'_\pm, \psi_\pm, \psi_\pm)\|_{X^{\frac{3}{4}+\epsilon}[0, T]} + \|N(\psi_\pm, \psi_\pm - \psi'_\pm, \psi_\pm)\|_{X^{\frac{3}{4}+\epsilon}[0, T]} + \|N(\psi_\pm, \psi_\pm, \psi_\pm - \psi'_\pm)\|_{X^{\frac{3}{4}+\epsilon}[0, T]} \right)$$
Here $\pm, \pm_j$ ($j = 1, 2, 3$) denote independent signs. We want to show that for the first term the following estimate holds:

$$J := \int (N(\psi_{\pm 1} - \psi_{\pm 1}', \psi_{\pm 2}, \psi_{\pm 3}), \psi_4) dx dt$$

$$\lesssim \|\psi_{\pm 1} - \psi_{\pm 1}'\|_{X^{\frac{1}{2}, \frac{1}{2}+} H^{0, \frac{1}{2}+}[0,T]} + \|\psi_{\pm 2}\|_{X^{\frac{1}{2}, \frac{1}{2}+} H^{0, \frac{1}{2}+}[0,T]} + \|\psi_{\pm 3}\|_{X^{\frac{1}{2}, \frac{1}{2}+} H^{0, \frac{1}{2}+}[0,T]} + \|\psi_4\|_{X^{0, \frac{1}{2}+}}. \quad (10)$$

We consider the case $|s_0| \leq 1$. Similarly as in the proof of Theorem 1.1 we obtain

$$|J| \lesssim \|\psi_{\pm 1} - \psi_{\pm 1}'\|_{X^{\frac{1}{2}, \frac{1}{2}+} H^{0, \frac{1}{2}+}[0,T]} + \|\psi_{\pm 2}\|_{X^{\frac{1}{2}, \frac{1}{2}+} H^{0, \frac{1}{2}+}[0,T]} + \|\psi_{\pm 3}\|_{X^{\frac{1}{2}, \frac{1}{2}+} H^{0, \frac{1}{2}+}[0,T]} + \|\psi_4\|_{X^{0, \frac{1}{2}+}},$$

which is more than sufficient. For $|s_0| \geq 1$ we obtain

$$|J| \lesssim \sum_{j=1}^2 \left( \|\langle \alpha_j (\psi_{\pm 1} - \psi_{\pm 1}'), \psi_{\pm 2}\rangle\|_{H^{0, \frac{1}{2}+}} \|\psi_{\pm 3}, \psi_4\|_{H^{-\frac{1}{2}, 0}} \right)$$

$$+ \|\langle \psi_{\pm 1} - \psi_{\pm 1}', \psi_{\pm 2}\rangle\|_{H^{0, \frac{1}{2}+}} \|\psi_{\pm 3}, \psi_4\|_{H^{-\frac{1}{2}, 0}}$$

$$\lesssim \|\psi_{\pm 1} - \psi_{\pm 1}'\|_{X^{\frac{1}{2}, \frac{1}{2}+} H^{0, \frac{1}{2}+}[0,T]} + \|\psi_{\pm 2}\|_{X^{\frac{1}{2}, \frac{1}{2}+} H^{0, \frac{1}{2}+}[0,T]} + \|\psi_{\pm 3}\|_{X^{\frac{1}{2}, \frac{1}{2}+} H^{0, \frac{1}{2}+}[0,T]} + \|\psi_4\|_{X^{0, \frac{1}{2}+}},$$

where we used Theorem 1.3 for the first factor with the choice $s_0 = \frac{1}{2}$, $b_0 = 0$, $s_1 = 0$, $b_1 = \frac{1}{2} + \epsilon$, $s_2 = \frac{1}{4} + \frac{\epsilon}{2}$, $b_2 = \frac{1}{4} + \epsilon$ and for the second factor with $s_0 = \frac{1}{2}$, $b_0 = 0$, $s_1 = \frac{1}{4} + \frac{\epsilon}{2}$, $b_1 = \frac{1}{4} + \epsilon$, $s_2 = 0$, $b_2 = \frac{1}{4} - \epsilon$. The embedding $X^{s,b}_{\pm} \subset H^{s,b}$ for $b \geq 0$ gives (10). The other terms in (9) are treated similarly. We obtain

$$\sum_{\pm} \|\psi_{\pm} - \psi_{\pm}'\|_{X_{\pm}^{\frac{1}{2}, \frac{1}{2}+}[0,T]} \lesssim T^{4+1} \sum_{j=1}^2 \left( \|\psi_{\pm j}\|_{X_{\pm j}^{\frac{1}{2}, \frac{1}{2}+}[0,T]} + \|\psi_{\pm j}'\|_{X_{\pm j}^{\frac{1}{2}, \frac{1}{2}+}[0,T]} \right) \sum_{\pm} \|\psi_{\pm} - \psi_{\pm}'\|_{X_{\pm}^{0, \frac{1}{2}+}[0,T]}.$$

For sufficiently small $T$ this implies $\|\psi_{\pm} - \psi_{\pm}'\|_{X_{\pm}^{0, \frac{1}{2}+}[0,T]} = 0$, thus local uniqueness. By iteration $T$ can be chosen arbitrarily. \qed

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