LOCAL DIMENSIONS OF MEASURES OF FINITE TYPE ON THE TORUS.

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ABSTRACT. The structure of the set of local dimensions of a self-similar measure has been studied by numerous mathematicians, initially for measures that satisfy the open set condition and, more recently, for measures on \( \mathbb{R} \) that are of finite type.

In this paper, our focus is on finite type measures defined on the torus, the quotient space \( \mathbb{R}/\mathbb{Z} \). We give criteria which ensures that the set of local dimensions of the measure taken over points in special classes generates an interval. We construct a non-trivial example of a measure on the torus that admits an isolated point in its set of local dimensions. We prove that the set of local dimensions for a finite type measure that is the quotient of a self-similar measure satisfying the strict separation condition is an interval. We show that sufficiently many convolutions of Cantor-like measures on the torus never admit an isolated point in their set of local dimensions, in stark contrast to such measures on \( \mathbb{R} \). Further, we give a family of Cantor-like measures on the torus where the set of local dimensions is a strict subset of the set of local dimensions, excluding the isolated point, of the corresponding measures on \( \mathbb{R} \).

1. INTRODUCTION

The local dimension of a probability measure \( \mu \) defined on a metric space, at a point \( x \) in the support of \( \mu \), is the number

\[
\dim_{\text{loc}} \mu(x) = \lim_{r \to 0} \frac{\log \mu(B(x, r))}{\log r}.
\]

It is of interest to determine the local dimensions of a given measure as these numbers quantify the local concentration of the measure. For self-similar measures that satisfy the open set condition (OSC), it is well known that the set of local dimensions is a closed interval whose endpoints are given by a simple formula.

When the OSC condition fails, the situation is more complicated, less understood and can be quite different. Indeed, in [12], Hu and Lau discovered that when \( \mu \) is the three-fold convolution of the classic middle-third Cantor measure on \( \mathbb{R} \), then there is an isolated point in the set of local dimensions, namely at \( x = 0, 3 \). This fact was later established for other ‘overlapping’ Cantor-like measures in [1, 6, 7, 15].

These Cantor-like measures are special examples of equicontractive self-similar measures of finite type on \( \mathbb{R} \), a notion introduced by Ngai and Wang [13] that is weaker than the OSC, but stronger than the weak separation condition [14]. Such measures have a very structured geometry, which makes them more tractable than arbitrary self-similar measures. In [3, 4, 5], Feng conducted a detailed study of equicontractive, self-similar measures of finite type on \( \mathbb{R} \), focussing primarily
on Bernoulli convolutions with contraction factors the inverse of a simple Pisot number. In [9, 11], the authors, together with Ng, showed that if $\mu$ is a measure of finite type on $\mathbb{R}$ that arises from regular probabilities and has full interval support, then the set of local dimensions of $\mu$ at the points in any positive loop class is a closed interval and the local dimensions at periodic points in the loop class are dense within this interval. See Section 2.1 for a definition of regular probabilities. Moreover, there is always a distinguished positive loop class, known as the essential class, which has full measure and is often all but the endpoints of the support of the measure. This is the situation for the Bernoulli convolutions with contraction a simple Pisot inverse and the 3-fold convolution of the middle-third Cantor measure, for instance. If the self-similar measure does not arise from regular probabilities, it is still true that the set of local dimensions at points in the interior of the essential class is a closed interval.

In this paper, we introduce the notion of finite type for measures on the torus $\mathbb{T}$ (the quotient space $\mathbb{R}/\mathbb{Z}$) that are quotients of equicontractive, self-similar measures on $\mathbb{R}$. Examples of such measures include convolutions on the torus of Cantor measures or Bernoulli convolutions with contraction factor the inverse of a Pisot number. These measures are the quotients of the convolutions on $\mathbb{R}$ of the (initial) Cantor measures or Bernoulli convolutions.

We develop a general method for calculating the local dimensions of finite type measures on $\mathbb{T}$ and obtain a simple formula for the local dimensions at the periodic points. With these tools, the same techniques as used for finite type measures on $\mathbb{R}$ show that if the self-similar measure is associated with regular probabilities and the quotient measure has support $\mathbb{T}$, then the set of local dimensions of the quotient measure at the points in any positive loop class is an interval. If we do not assume regular probabilities, under a mild technical assumption it is still true that the set of local dimensions at points in the interior of any positive essential class is an interval. As with finite type measures on $\mathbb{R}$, in either case this interval is the closure of the local dimensions at the periodic points in the class. However, in contrast to the case for finite type measures on $\mathbb{R}$, the essential class for measures of finite type on $\mathbb{T}$ need not be unique or of positive type.

We use these results to prove that if a self-similar measure of finite type on the torus is associated with an IFS that satisfies the strong separation condition, then the set of local dimensions of the quotient measure is not only an interval, but coincides with the set of local dimensions of the original measure. We also give the first (as far as we are aware) non-trivial example of a quotient measure on $\mathbb{T}$ whose set of local dimensions admits an isolated point.

In [1], it was shown that the sets of local dimensions for quotients of $k$-fold convolutions of Cantor measures with contraction factor $1/d$ are intervals whenever $k \geq d$. Although these quotient measures do not have an essential class of positive type, we are able to modify our general approach to give a new proof of this fact. Moreover, we extend this result to what we call complete quotient Cantor-like measures and also prove that the set of local dimensions is the closure of the set of local dimensions of periodic points.

In [7] it was explicitly shown that set of local dimensions of the 3-fold convolution of the Cantor measure with contraction factor $1/3$ on the torus is a strict subset of the set of local dimensions at points in the essential class, $(0, 3)$, of the corresponding measure on $\mathbb{R}$. The authors also comment that a similar proof can be used for all
$d$-fold convolutions of the Cantor measure with contraction factor $1/d$ for $d \geq 3$. In the last section we extend this result to show that for all $k \geq 0$ and all $d$ sufficiently large the $(k + d)$-fold convolution of the Cantor measure with contraction factor $1/d$ shares this property.

2. Finite type quotient measures on $\mathbb{T}$

2.1. Basic definitions and notation. Assume

$$\{S_j(x) = \varrho x + d_j : j = 0, \ldots, k\}$$

is an iterated function system (IFS) of equicontractive similarities on $\mathbb{R}$ and let $p_j$ for $j \in \mathcal{A} = \{0, \ldots, k\}$ denote probabilities, meaning $p_j > 0$ and $\sum p_j = 1$. By the associated self-similar measure $\mu$, we mean the unique measure satisfying the identity

$$\mu = \sum_{j=0}^{k} p_j \mu \circ S_j^{-1}.$$

Its support is the associated self-similar set $K \subseteq \mathbb{R}$, the unique non-empty compact set satisfying $K = \bigcup_{j=0}^{k} S_j(K)$.

There is no loss of generality in assuming $d_0 < d_1 < \cdots < d_k$. If $p_0 = p_k = \min p_j$, then the probabilities are referred to as regular.

We first recall what it means to say such an equicontractive IFS or self-similar measure on $\mathbb{R}$ is of finite type.

Definition 2.1. The iterated function system, $\{S_j(x) = \varrho x + d_j : j = 0, \ldots, k\}$, is said to be of finite type if there is a finite set $F \subseteq \mathbb{R}$ such that for each positive integer $n$ and any two sets of indices $\sigma, \tau \in \mathcal{A}^n$, either

$$\varrho^{-n} |S_\sigma(0) - S_\tau(0)| > \delta \text{ or } \varrho^{-n} (S_\sigma(0) - S_\tau(0)) \in F,$$

where $\delta = (1 - \varrho)^{-1} (\max d_j - \min d_j)$ is the diameter of $K$. If $\{S_j\}$ is of finite type and $\mu$ is an associated self-similar measure, we also say that $\mu$ is of finite type.

Hereafter, we will refer to this notion as ‘finite type on $\mathbb{R}$’ to distinguish it from the notion of ‘finite type on the torus’, which will be the focus of this paper and will be defined shortly.

Measures which satisfy the open set condition are of finite type on $\mathbb{R}$ and measures of finite type on $\mathbb{R}$ satisfy the weak open set condition [14]. The structure of finite type measures and aspects of their multi-fractal analysis is explained in detail in [3 4 5 8 11].

Example 2.2. The self-similar measures associated with the IFS $\{S_0(x) = \varrho x, S_1(x) = \varrho x + (1 - \varrho)\}$ and probabilities $p_0 = p_1 = 1/2$ are known as (uniform) Bernoulli convolutions when $\varrho > 1/2$ and Cantor measures when $\varrho < 1/2$. Cantor measures satisfy the OSC. Bernoulli convolutions do not, but when $\varrho$ is the inverse of a Pisot number the measures are of finite type.

\[1\]A Pisot number is an algebraic integer greater than one, all of whose Galois conjugates are strictly less than one in modulus.
Given two measures \( \mu, \nu \) on \( \mathbb{R} \), by their convolution, \( \mu \ast \nu \), we mean the measure on \( \mathbb{R} \) defined by
\[
\mu \ast \nu(E) = \int_{\mathbb{R}} \mu(E-x) \, d\nu(x) \quad \text{for all Borel sets } E \subseteq \mathbb{R}.
\]

The \( k \)-fold convolutions of the Bernoulli convolutions or Cantor measures are (also) the self-similar measures associated with the IFS \( \{S_j(x) = qx + j(1 - q) : j = 0, 1, ..., k \} \) and probabilities \( p_j = \binom{k}{j}2^{-k} \). These measures do not satisfy the OSC when \( k \geq 1/\varrho \), but are of finite type on \( \mathbb{R} \) whenever \( \varrho \) is the inverse of a Pisot number, such as a positive integer \( \lfloor \).

By the torus, \( T \), we mean the quotient group, \( \mathbb{R} \backslash \mathbb{Z} \). We let \( \pi \) denote the canonical quotient map and denote the usual metric on \( T \) by \( d_\pi \). Sometimes it is convenient to identify the torus as the group \( [0, 1) \) under addition mod 1.

Given a measure \( \mu \) on \( \mathbb{R} \), we let \( \mu_\pi \) be the quotient measure defined by \( \mu_\pi(E) = \mu(\pi^{-1}(E)) \) for any Borel set \( E \subseteq T \). Of course, if \( \mu \) has support \( K \), then \( \mu_\pi \) has support \( \pi(K) \). If \( \mu \) has support contained in \( [0, 1] \), the only difference between the local dimensions of \( \mu \) and \( \mu_\pi \) is that (identifying \( T \) with \( [0, 1) \)) \( \dim_{\text{loc}} \mu_\pi(0) = \min(\dim_{\text{loc}} \mu(0), \dim_{\text{loc}} \mu(1)) \), so this situation is trivial.

**Definition 2.3.** The iterated function system, \( \{S_j(x) = qx + d_j : j = 0, 1, ..., k \} \), defined on \( \mathbb{R} \), is said to be of finite type on the torus \( T \) if there is a finite set \( \Lambda \subseteq \mathbb{R} \) such that for each positive integer \( n \) and any two sets of indices \( \sigma, \tau \in \mathcal{A}^n \), either
\[
d_\pi(\pi(S_\sigma(0)), \pi(S_\tau(0))) > \varrho^n \delta \text{ or } d_\pi(\pi(S_\sigma(0)), \pi(S_\tau(0))) \leq \varrho^n \Lambda,
\]
where \( \delta = (1 - \varrho)^{-1}(\max(d_j - \min(d_j)) \text{ is the diameter of the self-similar set } K \subseteq \mathbb{R} \). If the IFS is of finite type on the torus and \( \mu_\pi \) is the quotient measure of a self-similar measure \( \mu \) associated with the IFS, we also say that \( \mu_\pi \) is of finite type on \( T \).

Later in this section we will show that quotients of the \( k \)-fold convolutions of Bernoulli convolutions or Cantor measures with contraction factors the inverse of a Pisot number are of finite type on \( T \). We will also give an example of a measure which is of finite type on \( \mathbb{R} \), but whose quotient is not of finite type on \( T \).

A translation argument shows there is no loss of generality in assuming \( d_0 = 0 \). We will also suppose that the diameter of the self-similar set \( K \) is an integer \( \delta \). We will leave it to the reader to consider what modifications to our arguments need to be made when the diameter is not an integer.

First, we introduce the important notions of quotient net intervals, neighbours and characteristic vectors. These are motivated by the analogous ideas for measures of finite type on \( \mathbb{R} \).

For each integer \( n \geq 1 \), let \( 0 = h_1 < h_2 < \cdots < h_{s_n - 1} < 1 \) be the collection of elements \( \{\pi(S_\sigma(0)), \pi(S_\sigma(\delta)) : \sigma \in \mathcal{A}^n \} \) in the torus which here it is convenient to identify with \( [0, 1) \). Put \( h_{s_n} = 1 \) and let
\[
\mathcal{F}_{n}^{(\pi)} = \{[h_j, h_{j+1}] : 1 \leq j \leq s_n - 1, \pi^{-1}(h_j, h_{j+1}) \cap K \neq \emptyset \}.
\]
Put \( \mathcal{F}_0^{(\pi)} = \{[0, 1] \} \). Elements in \( \mathcal{F}_n^{(\pi)} \) are called the quotient net intervals of level \( n \). In what follows, we will often omit the adjective ‘quotient’ if this is clear from the context. We note that these are intervals in \( \mathbb{R} \) that are contained in \([0, 1] \). Each
quotient net interval \( \Delta \), of level \( n \geq 1 \), is contained in a unique quotient net interval \( \hat{\Delta} \) of level \( n - 1 \), called its *parent*.

There is a similar notion of net intervals for measures on \( \mathbb{R} \). It is worth observing that an interval in \( \mathcal{F}_n^{(\pi)} \) may not correspond directly to a net interval of level \( n \) in \( \mathbb{R} \) because \( \pi(S_{\sigma}(e_1)) \) and \( \pi(S_{\tau}(e_2)) \) may be adjacent in \( T \) without being adjacent in \( \mathbb{R} \).

Let \( \Delta = [a, b] \in \mathcal{F}_n^{(\pi)} \), \( n \geq 1 \) and for \( l = 0, ..., \delta - 1 \) let \( \Delta^{(l)} = \Delta + l \). As \( \pi^{-1}(a, b) \cap K \) is not empty there must be some \( l \) such that \( \text{int}(\Delta^{(l)}) \cap K = (a + l, b + l) \cap K \) is not empty. Put

\[
\{a_1, \ldots, a_m\} = \{\rho^{-n}(a - S_{\sigma}(0) + l) : \sigma \in A^n, l \in \mathbb{N}, \text{int}(\Delta^{(l)}) \cap S_{\sigma}(K) \neq \emptyset\},
\]

where we assume the real numbers \( \{a_i\} \) satisfy \( a_1 < a_2 < \cdots < a_m \). By the *quotient neighbour set* of \( \Delta \) we mean the tuple

\[
V_n(\pi)(\Delta) = (a_1, \ldots, a_m).
\]

The normalized length of \( \Delta = [a, b] \in \mathcal{F}_n^{(\pi)} \) is denoted

\[
\ell_n(\Delta) = \rho^{-n}|b - a|.
\]

As \( |S_{\sigma}(0) - S_{\sigma}(\delta)| = \rho^n \delta \) when \( \sigma \in A^n \), it follows that for large enough \( n, |b - a| = d_{T}(a, b) \).

Suppose \( \hat{\Delta} \) is the parent of \( \Delta \in \mathcal{F}_n^{(\pi)} \) and \( \Delta_1, ..., \Delta_j \) are the quotient net intervals of level \( n \) that are also children of \( \hat{\Delta} \), listed from left to right, with the same normalized length and quotient neighbour set as \( \Delta \). Define \( r_n(\Delta) \) to be the integer \( r \) such that \( \Delta = \Delta_r \). By the *quotient characteristic vector* of \( \Delta \in \mathcal{F}_n^{(\pi)} \) for \( n \geq 1 \) we mean the triple

\[
C_n(\pi)(\Delta) = (\ell_n(\Delta), V_n(\pi)(\Delta), r_n(\Delta)).
\]

We call \( (\ell_n(\Delta), V_n(\pi)(\Delta)) \) the *reduced quotient characteristic vector* of \( \Delta \). For \( n = 0 \), we define

\[
C_0(\pi)([0, 1]) = (1, (0, 0, 1, 2, \ldots, \delta - 1), 1).
\]

The reason for this choice will be clear later.

We remark that this structure depends only on the similarities \( \{S_j\}_{j=0}^k \) and not on the measure itself.

Here is a very important fact about measures of finite type on \( T \). The same statement (with the appropriate definitions) is known to be true for measures of finite type on \( \mathbb{R} \) [3].

**Lemma 2.4.** *If the IFS is of finite type on \( T \), then there are only finitely many quotient characteristic vectors.*

**Proof.** First, we will check there are only finitely many normalized lengths of quotient net intervals. It is certainly enough to verify this for net intervals of level \( n \) for large enough \( n \), thus we can assume that if \( \Delta = [a, b] \in \mathcal{F}_n^{(\pi)} \), then \( \rho^n \delta < 1/2 \). Since \( |S_{\sigma}(0) - S_{\sigma}(\delta)| \leq \rho^n \delta \) when \( \sigma \in A^n \), it follows that \( |b - a| = d_{T}(\pi(a), \pi(b)) \leq \rho^n \delta \).

Suppose \( a = \pi(S_{\sigma}(0)) \) and \( b = \pi(S_{\tau}(0)) \) for some \( \sigma, \tau \in A^n \) (viewing \( T \) as \( [0, 1] \)). Then

\[
\rho^{-n} |b - a| = \rho^{-n} d_{T}(\pi(S_{\sigma}(0)), \pi(S_{\tau}(0))) \leq \rho^{-n} \delta
\]

and thus by definition \( \rho^{-n} |b - a| \) belongs to the finite set \( \Lambda \). The argument is similar if \( a, b \) are both images of \( \delta \).
Now suppose \( a = \pi(S_\rho(\delta)) \) and \( b = \pi(S_\tau(0)) \). Since \( \pi^{-1}(a, b) \cap K \) is not empty, the definition of a net interval ensures there is some \( \alpha \in \mc{A}^n \) such that \( \pi^{-1}[a, b] \subseteq [S_\alpha(0), S_\alpha(\delta)] \). Of course, \( \rho^{-n}d_\tau(\pi(S_\alpha(0)), \pi(S_\alpha(\delta))) = \delta \). Since
\[
d_\tau(\pi(S_\rho(\delta)), \pi(S_\tau(0))) = d_\tau(\pi(S_\rho(\delta)), \pi(S_\alpha(\delta))) + d_\tau(\pi(S_\rho(0)), \pi(S_\alpha(0)))
\]
we see that \( \rho^{-n}|b - a| = \rho^{-n}d_\tau(\pi(S_\rho(\delta)), \pi(S_\tau(0))) \) belongs to the finite set \( \Lambda \pm \Lambda \pm \Lambda \) (where we assume, without loss of generality, that \( \delta \in \Lambda \)).

Similar, but easier, arguments apply if \( a = \pi(S_\rho(0)), b = \pi(S_\tau(\delta)) \). Consequently, there are only finitely many normalized lengths.

This fact guarantees that each quotient net interval has a bounded number of children. In particular, there can only be finitely many choices for the 3rd component of the characteristic vectors.

Lastly, we need to show there are only finitely many neighbours. So suppose \( a_i = \rho^{-n}(a - S_\rho(0) + l) \) where \( \Delta = [a, b] \) and \( (\Delta + l) \cap S_\rho(0, \delta) \) is not empty for \( \sigma \in \mc{A}^n \) and integer \( l \). This guarantees that \( |S_\rho(0) - (a + l)| \leq \rho^n \delta \) and therefore
\[
\rho^{-n}|S_\rho(0) - (a + l)| = \rho^{-n}d_\tau(\pi(S_\rho(0)), \pi(a + l)) \in \Lambda \pm \Lambda \pm \Lambda,
\]
completing the proof. \( \square \)

### 2.2. Examples and Counterexamples

We begin this subsection by exhibiting a family of examples of quotient measures of finite type on the torus.

**Proposition 2.5.** Let \( \beta \) be a Pisot number and \( S_j(x) = \beta^{-1}x + d_j \) for \( d_j \in \Q[\beta] \) and \( j = 0, 1, 2, \ldots, k \). Assume the self-similar set has convex hull the interval \([0, \delta]\) with \( \delta \) an integer. Then the quotient of any associated self-similar measure is of finite type on the torus.

**Lemma 2.6.** Let \( S \subset \Q[\beta] \) be a finite set and \( \Lambda^S(\beta) = \{ \sum_{i=0}^n a_i \beta^i \mid a_i \in S, n \in \N \} \). Then there exists a constant \( c > 0 \) such that if \( y, z \in \Lambda^S(\beta) \), then either \( y = z \) or \( |y - z| > c \).

**Proof.** This is essentially done in [S], but we include it here for completeness.

We first observe that we can assume \( S \subset \Z[\beta] \). To see this we multiply by the least common multiple of the denominators of the \( a_i \). This scales the constant \( c \), but does not alter its existence.

Thus we have that \( \Lambda^S(\beta) \subset \Z[\beta] \). Let \( \sigma_1, \sigma_2, \ldots, \sigma_n \) be the Galois automorphisms on \( \Q[\beta] \). Either \( y = z \) or \( y - z \neq 0 \). In the latter case, we have \( y - z \in \Z[\beta] \), and hence the algebraic norm, \( N(y - z) \), is a non-zero integer. This implies that
\[
|N(y - z)| = \left| \prod_{i} \sigma_i(y - z) \right| \geq 1
\]
and hence if \( y = \sum a_j \beta^j \) and \( z = \sum a_j' \beta^j \), with \( a_j, a_j' \in S \), then
\[
|y - z| \geq \frac{1}{ \prod_{\sigma \neq \text{id}} \sigma_i(y - z) } \geq \frac{1}{ \prod_{\sigma \neq \text{id}} \sum_{j} \sigma_i(a_j - a_j') \sigma_i(\beta) } \]
where \( \text{id} \) is the identity automorphism. Let \( c_\sigma = \max_{a, b \in S} |\sigma(a - b)| > 0 \). We have that
\[
|y - z| \geq \frac{1}{ \prod_{\sigma \neq \text{id}} \sum_{j=0}^\infty c_\sigma |\sigma_i(\beta)| }.
\]
As $\beta$ is Pisot, $|\sigma_i(\beta)| < 1$ for all Galois actions, and hence the right hand side is bounded below, giving the result. \hfill $\square$

**Proof of Proposition 2.7.** Let

$$F = \{ \delta, d_j - \ell : j = 0, \ldots, k; \ell = 0, \ldots, \delta \} \subseteq \mathbb{Q}[\beta]$$

and consider the choices of $\sigma, \tau \in A^\nu$ such that $\beta^n |S_\sigma(0) - S_\tau(0) + \ell|$ for suitable integers $\ell \in \{-\delta, \ldots, \delta\}$ and hence $\beta^n d_\tau(\pi(S_\sigma(0)), \pi(S_\tau(0))) \in \Lambda^{F-F}(\beta)$. According to Lemma 2.6, the absolute values of the non-zero elements of $\Lambda^{F-F}(\beta)$ are bounded away from zero, say $\varepsilon$. Thus there can be at most $(2\delta + 1)/\varepsilon$ elements in $\Lambda^{F-F}(\beta) \cap [-\delta, \delta]$. This proves the finite type property. \hfill $\square$

**Remark 2.7.** If $\mu_\pi, \nu_\pi$ are two measures on $T$, then their convolution (on $T$) is defined by

$$\mu_\pi * \nu_\pi(E) = \int_T \mu_\pi(E - x) \, d\nu_\pi(x)$$

for all Borel sets $E \subseteq T$. (Here the group operation is understood on the torus.) If $\mu_\pi$ and $\nu_\pi$ are the quotients of measures $\mu$ and $\nu$ on $R$ respectively, then the convolution $\mu_\pi * \nu_\pi$ is equal to the quotient of the convolution (on $R$) $\mu * \nu$. In other words, $(\mu * \nu)_\pi = \mu_\pi * \nu_\pi$.

From Proposition 2.5 and Remark 2.7 we immediately deduce the following:

**Corollary 2.8.** Any $k$-fold convolution (taken on $T$) of a Cantor measure or Bernoulli convolution, where the contraction factor is the inverse of a Pisot number, is of finite type on $T$.

As $\beta < 1/2$, the IFS satisfies the OSC and even the strong separation condition, and thus $\mu$ is of finite type on $R$. Let $\sigma_{(n)} = (\sigma_0, \sigma_1, \ldots, \sigma_{n-1})$ and $\tau_{(n)} = (\tau_0, \tau_1, \ldots, \tau_{n-1}) \in A^{n+1}$ be defined by $\sigma_i = 1$ if $i \neq 0, \sigma_i = 0$ otherwise, and $\tau_i = 1 - \sigma_i$. The points $a_n = S_{\sigma_{(n)}}(0)$ and $b_n = S_{\tau_{(n)}}(0)$ are easily seen to be symmetric about 1 with $a_n < 1/2$ and $b_n > 3/2$. Consequently,

$$d_{\tau}(\pi(a_n), \pi(b_n)) = 2 |a - 1/2| = 4(1 - \theta) \sum_{i=n+1}^{\infty} \theta^2 \approx 4(1 - \theta) \theta^{n+1}.$$

Hence there exists a net interval in $F_{n+1}$ with normalized length at most

$$\approx 4\theta^{-(n+1)} \theta^{n+1} (1 - \theta) = 4(1 - \theta) \theta^{2n}.$$

These normalized lengths are not bounded below and hence there cannot be a finite number of characteristic vectors. Therefore $\mu_\pi$ is not of finite type.
2.3. Symbolic representations and the essential class(es). By an admissible path we mean a finite tuple \( \eta = (\gamma_j) \) where each \( \gamma_j \) is the quotient characteristic vector of \( \Delta_j \) and \( \Delta_j \) is the parent of \( \Delta_{j+1} \). Each \( \Delta \in \mathcal{F}_n^{(\pi)} \) can be identified with a unique admissible path \( \eta = \eta(\Delta) = (\gamma_j)_{j=0}^n \) where \( \gamma_0 = \mathcal{A}_0[0,1] \) and \( \gamma_n = \mathcal{C}_n(\Delta) \). We call this the symbolic representation of \( \Delta \).

We will often write \( \Delta_n(x) \) for a net interval of level \( n \) containing \( x \). By the quotient symbolic representation of \( x \in K \) we mean the sequence

\[
[x] = (\mathcal{C}_0(\Delta_0(x)), \ldots, \mathcal{C}_n(\Delta_n(x)), \ldots)
\]

where \( \Delta_n(x) \subseteq \Delta_{n-1}(x) \). If \( x \) is an endpoint of \( \Delta_n(x) \) for some \( n \), then \( x \) is called a boundary point and it can have two symbolic representations. Otherwise, the symbolic representation of \( x \) is unique.

We can also define the notion of quotient loop classes and essential classes in the same manner as was done for measures of finite type on \( \mathbb{R} \). A non-empty subset \( \Omega' \) of quotient characteristic vectors will be called a quotient loop class if whenever \( \alpha, \beta \in \Omega' \), then there are quotient characteristic vectors \( \gamma_j, j = 1, \ldots, n \), such that \( \gamma_1 = \alpha, \gamma_n = \beta \) and \( (\gamma_1, \ldots, \gamma_n) \) is an admissible path with all \( \gamma_j \in \Omega' \). A loop class \( \Omega' \) is called a quotient essential class if, in addition, whenever \( \alpha \in \Omega' \) and \( \beta \in \Omega \) is a child of \( \alpha \), then \( \beta \in \Omega' \). We say that an element \( x \) with symbolic representation \( (\gamma_0, \gamma_1, \ldots) \) is in a loop class (such as in the essential class) if there is some \( J \) such that \( \gamma_j \in \text{loop class} \) for all \( j \geq J \).

The finite type property ensures that every element in \( \pi(K) \) is contained in a quotient loop class if the associated IFS is of finite type on \( \mathbb{T} \).

Feng, in [5, Lemma 6.4], proved that if \( \mu \) is of finite type on \( \mathbb{R} \), then there is exactly one essential class. Surprisingly, this is not true for self-similar measures of finite type on the torus, as the example below demonstrates.

**Example 2.10.** Consider the IFS with maps \( S_j(x) = x/4 + d_j/5 \) for \( j = 0, \ldots, 4 \), \( d_j = 3j \) for \( j = 0, \ldots, 3 \) and \( d_4 = 15 \) and any probabilities \( p_j \). According to Proposition 2.5 this IFS is of finite type on \( \mathbb{T} \). The convex hull of \( K \) is \([0,4]\). Using the computer, we determined that any corresponding quotient measure has 10 reduced quotient characteristic vectors. From the reduced transition diagram, Figure 1, one can see that there are two different essential classes, the first from the reduced characteristic vector labelled 7 and the second consisting of the three reduced characteristic vectors 5, 9, 10.

**Remark 2.11.** In [11, Proposition 3.6] it was shown that, for self-similar measures of finite type on \( \mathbb{R} \), the set of points in the essential class has full \( \mu \)-measure, and full \( H^s \)-measure. Here \( H^s \) is the Hausdorff measure associated to the support of \( \mu \). Despite not necessarily having a unique essential class, a similar proof will show that for \( \mu_\pi \), a self-similar measure of finite type on the torus, the set of points in the union of all essential classes of \( \mu_\pi \) will have full \( \mu_\pi \)-measure and full \( H^s \)-measure where \( H_s \) is the Hausdorff measure associated with the support of \( \mu_\pi \) on the torus.
3. Computing local dimensions

The upper and lower local dimensions at a point $x$ in the support of a probability measure $\mu$ defined on a metric space are defined as

$$\dim_{\text{loc}} \mu(x) = \limsup_{r \to 0} \frac{\log \mu(B(x, r))}{\log r}$$

$$\dim_{\text{loc}} \mu(x) = \liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r}.$$ 

When the two coincide the number is known as the local dimension of $\mu$ at $x$.

As was the case for measures of finite type on $\mathbb{R}$, there is an iterative strategy for computing local dimensions for measures of finite type on $\mathbb{T}$, based upon suitable transition matrices.

**Notation 3.1.** Suppose $\Delta = [a, b] \in \mathcal{F}^{(\pi)}_n$ has parent $\hat{\Delta} = [c, d] \in \mathcal{F}^{(\pi)}_{n-1}$ and assume their quotient neighbour sets are $V^{(\pi)}_n(\Delta) = (a_1, \ldots, a_J)$ and $V^{(\pi)}_{n-1}(\hat{\Delta}) = (c_1, \ldots, c_I)$, respectively. The quotient primitive transition matrix,

$$T(C_{n-1}(\hat{\Delta}), C_n(\Delta)) = (T_{ji})$$

is the $J \times I$ matrix defined by the rule that $T_{ji} = p_{\sigma}$ if there is some $\sigma \in \mathcal{A}^{n-1}$ and integer $l$ with $S_{\sigma}(0) = c - \varrho^{n-l-1} e_j + l$ and $S_{\sigma}(0) = a - \varrho^{n-l-1} e_i + l$. We set $T_{ji} = 0$ otherwise.

Given the net interval $\Delta$ with symbolic representation $\eta = (\gamma_{ij})_{j=0}^n$, we write

$$T(\eta) = T(\gamma_0, \gamma_1) \cdots T(\gamma_{n-1}, \gamma_n).$$
Any such product of primitive transition matrices will be called a *quotient transition matrix*. By an essential primitive transition matrix, we mean a transition matrix \(T(\gamma_{j-1}, \gamma_j)\) where \(\gamma_{j-1}, \gamma_j\) belongs to the essential class.

The definition ensures that each column of a primitive transition matrix contains a non-zero entry. If \(\pi(K) = \mathbb{T}\), then the same statement holds for each row.

We note that as \(S_{\sigma}(0) = q^n a_i + S_{\sigma}(0)\), if \(S_{\sigma}(0) = c - q^{n-1}c_j + l\) and \(S_{\sigma}(0) = a - q^n a_i + l\) for integer \(l\), then

\[
a - q^n a_i + l = q^{n-1}d_s + c - q^{n-1}c_j + l,
\]

so \(s\) is uniquely determined (even if \(\sigma\) and \(l\) are not).

**Proposition 3.2.** Let \(\mu\) be a self-similar measure satisfying (7). Assume the convex hull of the self-similar set is \([0, \delta]\) and that the quotient measure \(\mu_{\pi}\) is of finite type on \(\mathbb{T}\). Let \(\Delta = [a, b]\) be a quotient net interval in \(\mathcal{F}_n(\pi)\) with reduced quotient characteristic vector \((\ell_n(\Delta), (a_1, ..., a_m))\). Then

\[
\mu_{\pi}(\Delta) = \sum_{i=1}^{m} \mu([a_i, a_i + \ell_n(\Delta)]) \sum_{l=0}^{\delta-1} \sum_{\sigma \in \mathcal{A}_n} \sum_{a_i = (S_{\sigma}(0) - a - l)q^{-n}} p_{\sigma}.
\]

**Proof.** Assume \(\pi^{-1}(\Delta) \cap [0, \delta] = \bigcup_{l=0}^{\delta-1} \Delta^{(l)}\) where \(\Delta^{(l)} = \Delta + l = [a + l, b + l]\). Then

\[
\mu_{\pi}(\Delta) = \mu(\pi^{-1}(\Delta)) = \sum_{l=0}^{\delta-1} \mu(\Delta^{(l)}) = \sum_{l=0}^{\delta-1} \sum_{\sigma \in \mathcal{A}_n} p_{\sigma} \mu(S_{\sigma}^{-1}(\Delta^{(l)})).
\]

As \(\mu\) is non-atomic this equals

\[
\mu_{\pi}(\Delta) = \sum_{l=0}^{\delta-1} \sum_{\sigma \in \mathcal{A}_n} p_{\sigma} \mu(S_{\sigma}^{-1}(\Delta^{(l)}))\]  

Note that if \(S_{\sigma}(K) \cap int\Delta^{(l)} \neq \emptyset\), then by definition \((S_{\sigma}(0) - a - l)q^{-n} = a_i\) belongs to the quotient neighbour set of \(\Delta\). Hence

\[
\mu_{\pi}(\Delta) = \sum_{l=0}^{\delta-1} \sum_{i=1}^{m} \sum_{\sigma \in \mathcal{A}_n} p_{\sigma} \mu(S_{\sigma}^{-1}(\Delta^{(l)})).
\]

As \(S_{\sigma}(x) = a - a_i q^n + l + q^n x\), we see that \(S_{\sigma}([a_i, a_i + \ell(\Delta)]) = [a + l, b + l]\). Hence

\[
\mu_{\pi}(\Delta) = \sum_{l=0}^{\delta-1} \sum_{i=1}^{m} \sum_{\sigma \in \mathcal{A}_n} p_{\sigma} \mu([a_i, a_i + \ell(\Delta)])
\]

\[
= \sum_{i=1}^{m} \mu([a_i, a_i + \ell_n(\Delta)]) \sum_{l=0}^{\delta-1} \sum_{\sigma \in \mathcal{A}_n} p_{\sigma}
\]

as claimed. \(\square\)
Notation 3.3. For $n \geq 1$, let
\[ P_n^{(i)} = P_n^{(i)}(\Delta) = \sum_{l=0}^{k-1} \sum_{\sigma \in \mathcal{A}^n} p_{\sigma}, \]
where $a_{i}=(S_{\sigma}(0)-a+l)\ell^{n}$
\[
Q_{n}(\Delta) = (P_{n}^{(1)}, \ldots, P_{n}^{(m)}),
\]
and
\[
Q_{0}([0,1]) = (1, \ldots, 1) \in \mathbb{R}^{d}.
\]
We remark that as $[a_{i}, a_{i} + \ell_{n}(\Delta)] = S_{\sigma}^{-1}([a + l + b + l])$ and $K \cap S_{\sigma}^{-1}(a + l, b + l)$ is not empty, we have $\mu([a_{i}, a_{i} + \ell_{n}(\Delta)]) > 0$. Consequently,
\[
\mu(\Delta) \sim \sum_{i} P_{n}^{(i)}(\Delta) = \|Q_{n}(\Delta)\|
\]
with the constants of comparability independent of $\Delta$ or $n$. (Here the norm of the vector is the sum of the absolute values of the entries.)

Proposition 3.4. For all $n \geq 1$, we have
\[
Q_{n}(\Delta) = Q_{n-1}(\Delta) T(\mathcal{C}_{n-1}(\Delta), \mathcal{C}_{n}(\Delta)).
\]
Proof. First, assume $n \geq 2$. We want to show that for each $i$,
\[
P_{n}^{(i)} = \sum_{j} P_{n-1}^{(j)} t_{ji}
\]
where $T(\mathcal{C}_{n-1}(\Delta), \mathcal{C}_{n}(\Delta)) = (t_{ji})$.

Consider a typical summand in the formula for $P_{n}^{(i)}$, say $p_{\sigma}$ where $\sigma \in \mathcal{A}^n$ and $\ell^{n}(a - S_{\sigma}(0) + l) = a_{i}$ for suitable integer $l$. Put $\sigma = \tilde{\sigma}s$ where $\tilde{\sigma} \in \mathcal{A}^{n-1}$ and $s \in \mathcal{A}$. As $S_{\tilde{\sigma}}(0) = a + l - \ell^{n}a_{i}$, by construction there is some index $j$ such that $S_{\tilde{\sigma}}(0) = c + l - \ell^{n-1}c_{j}$. But then $p_{\rho} = p_{\tilde{\sigma}} p_{s} = p_{\tilde{\sigma}} t_{ji}$ and $p_{\tilde{\sigma}}$ is a summand of $P_{n-1}^{(j)}$.

On the other hand, assume $t_{ji} \neq 0$ and $p_{\tilde{\sigma}}$ is a summand of $P_{n-1}^{(j)}$. Then there must be some $l$ such that $S_{\tilde{\sigma}}(0) = c + l - \ell^{n-1}c_{j}$. Furthermore, there is some integer $k$ and $\tau \in \mathcal{A}^{n}$ such that $S_{\tau}(0) = a + k - \ell^{n}a_{i}$ and $S_{\tau}(0) = c + k - \ell^{n-1}c_{j}$. Assume $\tau = \tilde{\sigma}s$. Then $S_{\tilde{\sigma}}(0) = S_{\tilde{\sigma}}(0) + \ell^{n-1}d_{s} = S_{\tilde{\sigma}}(0) + k - l + \ell^{n-1}d_{s} = S_{\tilde{\sigma}}(0) + k - l$, hence $S_{\tilde{\sigma}}(0) = a + l - \ell^{n}a_{i}$. This observation shows that $p_{\tilde{\sigma}} = p_{\tilde{\sigma}} p_{s}$ is a summand of $P_{n}^{(i)}$ and $t_{ji} = p_{s}$. Together, these observations prove (2), as required.

Now suppose $n = 1$. In this case, we need to show that $P_{1}^{(i)} = \sum_{j} t_{ji}$ where $(t_{ji}) = T(\mathcal{C}_{0}, \mathcal{C}_{1}[a, b])$. By definition $t_{ji} = p_{s}$ if $S_{s}(0) = a + l - \ell^{n}a_{i}$ and $0 = l - c_{j}$. Thus the definition of the neighbour set of $[0, 1]$ as $\{0, 1, \ldots, \delta - 1\}$ ensures we have $P_{1}^{(i)} = \sum_{j} t_{ji}$ for each $i$. □

By the matrix norm of matrix $M = (M_{jk})$ we mean $\|M\| = \sum_{jk} |M_{jk}|$. In terms of this notation the previous results combine to yield

Corollary 3.5. There are positive constants $c_{1}, c_{2}$ such that
\[
c_{1} \|T(\eta)\| \leq \mu(\Delta) \leq c_{2} \|T(\eta)\|
\]
whenever $\Delta \in \mathcal{F}_{n}(\pi)$ has symbolic representation $\eta$. 
Given $\Delta_n \in \mathcal{F}_n^{(\sigma)}$, we let $\Delta_n^+ \text{ and } \Delta_n^-$ be the quotient net intervals of level $n$ sharing endpoints with $\Delta_n$, (in the torus sense). If $x$ belongs to the interior of $\Delta_n(x)$ we put

$$M_n(x) = \mu_\pi(\Delta_n(x)) + \mu_\pi(\Delta_n^+(x)) + \mu_\pi(\Delta_n^-(x)),$$

while if $x$ is a boundary point of $\Delta_n(x)$ we put $M_n(x) = \mu_\pi(\Delta_n(x)) + \mu_\pi(\Delta_n'(x))$ where $\Delta_n'(x)$ is the other net interval of level $n$ with $x$ as endpoint. Since all quotient net intervals of level $n$ have lengths comparable to $\rho^n$ one can see in the same manner as in [11] Thm. 2.6 that

$$\dim_{\text{loc}} \mu_\pi(x) = \lim_{n \to \infty} \frac{\log M_n(x)}{n \log \rho},$$

with a similar formula for upper and lower local dimensions. In the special case that the probabilities defining the self-similar measure are regular and $\pi(K) = \mathbb{T}$, the same arguments as given in [9] Lemma 3.5 - Cor. 3.7 show that if $[x] = (\gamma_0, \gamma_1, \ldots)$, then we have the simpler formula,

$$\dim_{\text{loc}} \mu_\pi(x) = \lim_{n \to \infty} \frac{\log \mu_\pi(\Delta_n(x))}{n \log \rho} = \lim_{n \to \infty} \frac{\log \|T(\gamma_0, \ldots, \gamma_n)\|}{n \log \rho}. \tag{3}$$

In the study of finite type measures on $\mathbb{R}$, periodic points played an important role. In a similar fashion, we will say that a point $x \in \pi(K)$ is periodic if it has quotient symbolic representation $[x] = (\gamma_0, \ldots, \gamma_J, \theta^-, \theta^-, \ldots)$ where $\theta$ is an admissible cycle (a path whose first and last letters coincide) and $\theta^-$ is the path with the last letter of $\theta$ deleted. Any point which is a boundary point of some quotient net interval is a periodic point.

As was the case for finite type measures in $\mathbb{R}$, we have the following formula for local dimensions of periodic points. We write $L(\theta^-)$ for the length of the path $\theta^-$.  

**Proposition 3.6.** If $\mu_\pi$ is a measure of finite type on $\mathbb{T}$ and $x$ is a periodic point with period $\theta$, then the local dimension exists and is given by

$$\dim_{\text{loc}} \mu_\pi(x) = \frac{\log \sp(T(\theta))}{L(\theta^-) \log \rho},$$

where if $x$ is a boundary point of a quotient net interval with two different symbolic representations given by periods $\theta$ and $\phi$ of the same length, then $\theta$ is chosen with $\sp(T(\theta)) \geq \sp(T(\phi))$.

The proof is the same as in [11] Proposition 2.7 as that argument only required the formula we have developed for $\mu_\pi(\Delta)$ and the fact that the primitive transition matrices have a non-zero entry in each column.

We will call a quotient loop class positive if there is some path $\eta$ in the loop class for which $T(\eta)$ has strictly positive entries. With the preliminary results we have established, the same arguments as in [9] Section 5 prove the following important result. The details are left to the reader.

**Theorem 3.7.** Suppose $\mu_\pi$ is a quotient measure of finite type on $\mathbb{T}$ associated with regular probabilities and with $\supp \mu_\pi = \mathbb{T}$. Then the set of local dimensions at points in any positive quotient loop class is a closed interval and the local dimensions at the periodic points in the loop class are dense in that interval.

**Remark 3.8.** It was shown in [9] that for measures of finite type on $\mathbb{R}$ the essential class is always a positive loop class. In the next section we will see that the quotient essential class (even if unique) need not be positive.
This theorem can be used to see that, as with measures of finite type on \( \mathbb{R} \),
the sets of local dimensions of measures of finite type on \( T \) may or may not admit isolated points. Here are some examples. The details of these examples can be found in [10].

**Example 3.9.** Consider \( \nu \), the convolution square (on \( \mathbb{R} \)) of the uniform Bernoulli convolution with contraction factor \( \rho \) the inverse of the golden mean. This is the self-similar measure associated with the IFS with similarities \( \rho x \), \( \rho x + 1 - \rho \) and \( \rho x + 2(1 - \rho) \) and probabilities \( 1/4, 1/2 \) and \( 1/4 \) respectively. In [9] Sect. 8.2 it was shown that the set of local dimensions of \( \nu \) admits an isolated point. The quotient measure, \( \nu_\pi \), has one essential class which is positive and hence generates an interval of local dimensions. The quotient essential primitive transition matrices and admissible paths precisely coincide with those of \( \nu \). It follows that the set of local dimensions of \( \nu_\pi \) at points in the quotient essential class coincides with the interval of local dimensions of \( \nu \) at points in its essential class.

There are also two additional maximal quotient loop classes, both of which are simple loops. These generate the same three (periodic boundary) points, namely, \( 0, \rho, 1 - \rho \). The three points have the same local dimension and it can be shown that this value is contained in the interval of local dimensions generated by the quotient essential class. It follows that the set of local dimensions of the quotient measure has no isolated point.

**Example 3.10.** Consider the IFS \( \{ S_j(x) = x/4 + d_j/8 : j = 0, \ldots, 7 \} \) where \( d_j = j \) for \( j = 0, \ldots, 4 \), \( d_5 = 7 \), \( d_6 = 9 \) and \( d_7 = 12 \). This IFS generates the self-similar set \([0, 2]\) and is of finite type on \( T \). There are 10 reduced quotient characteristic vectors and one quotient essential class which consists of one reduced quotient characteristic vector (specifically, 4 non-reduced quotient characteristic vectors). Let \( \mu \) be the self-similar measure arising from the regular probabilities \( p_0 = p_7 = 1/2402, p_1 = p_2 = 1000/2402, p_3 = p_4 = p_5 = p_6 = 100/2402 \). We have verified that the quotient essential class is of positive type when these probabilities are used. Computational work also shows that

\[
[.6283, 1.885] \subseteq \{ \dim \mu_\pi(x) : x \in \text{essential class} \} \subseteq [1.614, 2.053].
\]

There are three other maximal loop classes, all of which are simple loops. Two of these loops correspond to the single point 7/8 whose local dimension is \( \log(100/2402) \approx 2.293 \). One of these two loops corresponds to the path approaching 7/8 from the left, and the other to the path approaching 7/8 from the right. There are countably many points associated with the other maximal loop class, all having local dimension \( \approx 2.286 \) (the spectral radius of the normalized transition matrix of the length one cycle is the root of the polynomial \( x^2 - 100x - 100 \), approximately 100.99). Consequently, the set of local dimensions of \( \mu_\pi \) consists of a closed interval and two isolated points. It is again the case that the interval in the set of local dimensions of \( \mu_\pi \) coincides with the interval component of the set of local dimensions of \( \mu \). The two isolated points are also both isolated points in the set of local dimensions of \( \mu \).

A weaker result can be proven if we do not assume that the probabilities are regular or that the support of the quotient measure is the torus. Note that by an essential quotient transition matrix, we mean a transition matrix \( T(\eta) \) where the path \( \eta \) belongs to the essential class. Recall that if \( \supp \mu_\pi = T \), then there is a non-zero entry on each row of each primitive transition matrix.
Theorem 3.11. Suppose $\mu_\pi$ is a quotient measure of finite type on $\mathbb{T}$. Assume the essential class $E$ is positive and that each essential primitive transition matrix has a non-zero entry in each row. Then the set of local dimensions of $\mu_\pi$ at points in the relative interior of $E$ is a closed interval and the local dimensions at periodic points from $E$ are dense in this interval.

This follows by similar arguments to those given in [11] Sect 3.3]. (The proofs given in Section 4.2 are also similar.)

Next, we will apply this theorem to show that finite-type quotients of self-similar measures on $\mathbb{R}$ satisfying the strong separation condition have the same local dimension as the original measure. Recall that an IFS $\{S_j : j = 1, \ldots, m\}$ with self-similar set $K$ is said to satisfy the strong separation condition if the sets $S_j(K)$ are pairwise disjoint.

Theorem 3.12. Assume the equicontractive IFS $\{S_j = qx + d_j : j = 0, \ldots, k\}$ with $k \geq 1$ satisfies the strong separation condition and is of finite type on $\mathbb{T}$. Let $\{p_j\}_{j=0}^k$ be probabilities and assume $\mu$ is the associated self-similar measure and $\mu_\pi$ is its quotient measure. Then

$$\{\dimloc \mu(x) : x \in K\} = \left\lceil \frac{\log(\max p_i)}{\log q}, \frac{\log(\min p_i)}{\log q} \right\rceil := I$$

$$\pi \in \{\dimloc \mu_\pi(x) : x \in \pi(K)\}.$$

Proof. The first equality is well known (c.f. [2, ch. 11]), so we only need to prove the second.

As usual, denote by $[0, \delta]$ the convex hull of the self-similar set $K$. Since the IFS satisfies the strong separation condition, the Lebesgue measure of $K$ is zero. As $\pi(K)$ can be identified with $\bigcup_{n \in \mathbb{Z}} (K_n - n)$ where $K_n = K \cap [n, n+1)$, it follows that $\pi(K)$ also has Lebesgue measure (on $\mathbb{T}$) equal to zero. Consequently, for $n$ sufficiently large, the Lebesgue measure of $\pi\left(\bigcup_{|\sigma| = n} S_\sigma([0, \delta])\right)$ is $< 1$ and hence there must be intervals $(a_i, a_{i+1}) \subseteq (0, 1)$ such that both $a_i$ and $a_{i+1}$ are in $\pi\left(\bigcup_{|\sigma| = n} S_\sigma([0, \delta])\right)$, while $(a_i, a_{i+1}) \cap \pi\left(\bigcup_{|\sigma| = n} S_\sigma([0, \delta])\right)$ is empty.

As $\pi(K)$ is a perfect set, a $a_{i+1}$ is a limit point and thus there is a quotient net interval of level $n$, say $\Delta_n = [a_{i+1}, a_{i+2}]$, adjacent (in the torus sense) to the empty interval. Moreover, since $(a_i, a_{i+1}) \cap \pi(S_\sigma([0, \delta])) = \emptyset$ for all $|\sigma| = n$, we see that if $[a_{i+1}, a_{i+2}] \subseteq \pi(S_\sigma([0, \delta]))$, then $\pi(S_\sigma(0)) = a_{i+1}$. This implies that the neighbour of $\Delta_n$ is simply the singleton $(0)$.

Let $\Delta_{n+1}$ be the left-most descendant of $\Delta_n$. The same reasoning shows the only neighbour of $\Delta_{n+1}$ is the singleton $(0)$. The same holds more generally for the left-most descendant of each generation.

Choose $\ell$ such that $\delta q^{n+\ell} < |\Delta_n|$ and let $\Delta_{n+\ell}$ be the left-most descendant of $\Delta_n$ at level $n + \ell$. As $\Delta_{n+\ell}$ is the intersection of a set of size $\delta q^{n+\ell}$ with $\Delta_n$, we see that $\Delta_{n+\ell}$ will have normalized length $\delta$. This proves that $\gamma = (\delta, (0))$ is a reduced quotient characteristic vector.

Assume $\pi(S_\sigma([0, \delta])) = \Delta_{n+\ell}$. As the IFS satisfies the strong separation condition, we see that $\pi(S_\sigma([0, \delta]))$ are all disjoint scaled copies of $\Delta_{n+\ell}$, with precisely the same geometry. This shows that the $k + 1$ children of $\gamma$ have again the same reduced characteristic vector $\gamma$. Moreover the probabilities associated to these children, in order from left to right, are $p_0, p_1, \ldots, p_k$, thus the corresponding transition matrices are these $1 \times 1$ positive matrices.
To this point, we have proven that \( \{ \gamma \} \) is one (reduced) essential class of \( \mu_\pi \). Next, we will show that it is the essential class. We prove this by showing \( \gamma \) is a descendent of any quotient characteristic vector.

Let \( \Delta \) be a net interval in \( \mathcal{F}_n(\pi) \). By an argument similar to above we see that there exists a \( j \) such that \( \Delta \) will contain a quotient net interval \( \Delta_0 = [a_{i+1}, a_{i+2}] \subseteq \mathcal{F}_{n+j} \), that is adjacent to a set \( (a_i, a_{i+1}) \) which is disjoint from all \( \pi(S_\pi([0, \delta])) \) for \( |\sigma| = n + j \). We then proceed as before, taking the left-most descendents of \( \Delta_0 \) to find a quotient net subinterval that has characteristic vector \( \gamma \). Hence \( \{ \gamma \} \) is the (reduced) essential class and the essential class consists of the \( k + 1 \) vectors \( \gamma_j = (\delta, (0), j), j = 0, ..., k \).

From the previous theorem it follows that the set of local dimensions at points in the relative interior of the essential class is a closed interval. Assume \( p_\alpha = \min p_i \) and \( p_\beta = \max p_i \). It is easy to see that \( \{ x \} = (\chi_0, \chi_1, ..., \chi_J, \gamma_0, \gamma_k, \gamma_\alpha, \gamma_\alpha, ...,) \) and \( \{ y \} = (\chi_0, \chi_1', ..., \chi'_J, \gamma_0, \gamma_k, \gamma_\beta, \gamma_\beta, ...) \) belong to the interior of the essential class for suitable admissible paths \( \chi_0, \chi_1, ..., \chi_J \) and \( \chi_0, \chi_1', ..., \chi'_J \) and that

\[
\dim_{\text{loc}} \mu_\pi(x) = \frac{\log(\min p_i)}{\log \rho}, \quad \dim_{\text{loc}} \mu_\pi(y) = \frac{\log(\max p_i)}{\log \rho}.
\]

Consequently, the set of local dimensions of \( \mu_\pi \) contains the interval \( I \).

The definition of the quotient measure implies that

\[
\min_{\ell} \left( \dim_{\text{loc}} \mu (x + \ell) \right) \leq \dim_{\text{loc}} \mu_\pi(x) \leq \min_{\ell} \left( \dim_{\text{loc}} \mu (x + \ell) \right)
\]

where the minimum is taken over all integers \( \ell \) such that \( x + \ell \in \text{supp} \mu \). Since the upper and lower local dimensions of \( \mu \) at any point in its support lie in the interval \( I \), it follows that the same is true for \( \dim_{\text{loc}} \mu_\pi(x) \) at any \( x \in \pi(K) \). This completes the proof. \( \square \)

**Remark 3.13.** We remark that Example 2.9 demonstrates that satisfying the strong separation condition and having the convex hull of the support being \([0, 2]\) is not enough to guarantee that the quotient measure is of finite type.

### 4. Quotients of Cantor-like measures of finite type

#### 4.1. Finite-type structure of quotients of Cantor-like measures

In this section, we will investigate the finite type structure and local dimensions of quotients of the Cantor-like measures associated with the IFS

\[
S_j(x) = \frac{1}{d^j} + \frac{j(d-1)}{d} \quad \text{for } j \in \Lambda
\]

and probabilities \( p_j > 0 \), where \( \Lambda \subseteq \{0, 1, ..., k\} \) and \( d \geq 3 \). We will assume the convex hull of \( K \) is the interval \([0, k]\), equivalently, \( 0, k \in \Lambda \). The strong separation condition is satisfied if \( k < d - 1 \), so this case is handled by Theorem 3.12 giving the following.

**Corollary 4.1.** Let \( \mu_\pi \) be the quotient of the self-similar measure \( \mu \) associated with the IFS \( \{S_j\} \) with \( k < d - 1 \) and \( \{0, k\} \subseteq \Lambda \). Then

\[
\{ \dim_{\text{loc}} \mu_\pi(x) : x \in \text{supp} \mu_\pi \} = \left[ -\frac{\log(\max p_i)}{\log d}, -\frac{\log(\min p_i)}{\log d} \right]
\]

\[
= \{ \dim_{\text{loc}} \mu(x) : x \in \text{supp} \mu \}.
\]
Thus we will assume $k \geq d - 1$. If $\Lambda = \{0, 1, \ldots, k\}$, the associated self-similar measures have support equal to the (full) interval $[0, k]$, while the OSC fails if $k > d - 1$.

In the special case that $\Lambda = \{0, 1, \ldots, k\}$ and the probabilities satisfy $p_j = \binom{j}{k} 2^{-k}$, then the associated self-similar measure on $\mathbb{R}$ is the $k$-fold convolution of the uniform Cantor measure with contraction ratio $1/d$. In this case, the corresponding quotient measure is also equal to the $k$-fold convolution (taken on the torus) of the uniform Cantor measure.

Consider the IFS (4) where $\Lambda = \{0, 1, \ldots, k\}$ and as $k \geq d$ and $p_0 < p_j$ for all $j \neq 0, k$, then there is an isolated point in the set of local dimensions.

Of course, the corresponding quotient measures are also of finite type on $\mathbb{T}$. In this section we will prove that for many of these quotient net intervals of level $n$ belong to the set $\{jd^{-n} : j = 0, \ldots, k\}$ and their lengths are at most $kd^{-n}$. Neighbours will always belong to $\{0, \ldots, k\}$.

When $\Lambda = \{0, 1, \ldots, k\}$ more can be said.

**Lemma 4.2.** Consider the IFS (4) where $\Lambda = \{0, 1, \ldots, k\}$ and $k \geq d - 1$.

1. The quotient net intervals of level $n$ are the sets $[jd^{-n}, (j + 1)d^{-n}]$ for $j = 0, \ldots, d^n - 1$.
2. There is only one reduced quotient characteristic vector, namely $(1, (0, 1, \ldots, k - 1))$. It has $d$ (identical) children.

**Proof.** The iterates of 0 at level one are the points $j(d-1)/d$ for $j = 0, \ldots, k$. Taking these mod 1 gives the points $jd/d$ for $j = 0, \ldots, d - 1$. The iterates of $k$ are also in $\mathbb{Z}/d$ and so taken mod 1 give no additional terms.

At step $n$ the iterates of 0 are the points $\sum_{i=1}^{n} j_i d^{-i}(d - 1)$, where $0 \leq j_i \leq k$ and as $k \geq d - 1$, taking these mod 1 again this gives all $jd^{-n}$. The iterates of $k$ add no new terms. Thus the net intervals at each level are as claimed.

To determine the neighbours, consider the net interval $[j/d, (j + 1)/d]$ of level one. The neighbours are the integers of the form $d(j/d + l - S_J(0))$ where $l$ is an integer, $J \in \{0, 1, \ldots, k\}$ and

$$\left[\frac{j}{d} + l, \frac{j + 1}{d} + l\right] \subseteq [S_J(0), S_J(k)] = \left[\frac{J(d-1)}{d}, \frac{J(d-1)+k}{d}\right].$$

The inequalities implied by (5) ensure that such integers are contained in $\{0, 1, \ldots, k - 1\}$. Easy computations show that if $i \in \{0, \ldots, j\}$, $J = d(j-i)$ and $l = d - 1 - (j-i)$, then $i = d(j/d + l - S_J(0))$ and all the requirements are satisfied. Similarly, for $i \in \{j+1, \ldots, k-1\}$, we see that $d(j/d + l - S_J(0)) = i$ when $J = i - j = l$. Further, all the additional requirements are met. This proves the neighbours are precisely the set $\{0, 1, \ldots, k - 1\}$. In particular, there is are $d$ children at level one of the parent interval $[0, 1]$, but only one reduced characteristic vector, $(1, (0, 1, \ldots, k - 1))$.

The same statement holds for the higher levels because of self-similarity.
Remark 4.3. We remark that even when \( \Lambda \) is a proper subset of \( \{0, 1, \ldots, k\} \), the conclusions of this lemma are often true. For instance, one can check that this is the case if \( d = 4, k = 7 \) and \( \Lambda \) consists of all but one integer \( j \) chosen from \( \{1, \ldots, 6\} \).

From here on we will restrict our attention to IFS and quotients of their self-similar measures for which the conclusion of the lemma holds. We will refer to these as complete quotient Cantor-like measures. Note that the support of such quotient measures is the full torus and they have \( d \) primitive transition matrices that we label as \( T(\ell) \) for \( \ell = 0, \ldots, d - 1 \), where \( \ell \) denotes the \( \ell \)th child from left to right. These matrices are computed in the next lemma.

Lemma 4.4. The primitive transition matrices \( T(\ell) \) for \( \ell \in \{0, \ldots, d - 1\} \) for a complete quotient Cantor-like measure satisfying (4) are the \( k \times k \) matrices with \( j, i \) entry equal to

\[
T(\ell)_{ji} = p_s \text{ if } \frac{\ell - (i - 1) + (j - 1)d}{d - 1} = s \text{ with } s \in \Lambda
\]

and 0 otherwise.

Example 4.5. Consider the case when \( d = 4, k = 7 \) and \( \Lambda = \{0, 1, \ldots, 7\} \). The unique reduced characteristic vector is \( (1, (0, 1, \ldots, 6)) \) and there are four transition matrices

\[
T(0) = \begin{bmatrix}
p_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & p_1 & 0 & 0 & p_0 & 0 & 0 \\
0 & 0 & p_2 & 0 & 0 & p_1 & 0 \\
p_4 & 0 & 0 & p_3 & 0 & 0 & p_2 \\
0 & p_5 & 0 & 0 & p_4 & 0 & 0 \\
0 & 0 & p_6 & 0 & 0 & p_5 & 0 \\
0 & 0 & 0 & p_7 & 0 & 0 & p_6 \\
\end{bmatrix}
\]

\[
T(1) = \begin{bmatrix}
0 & p_0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & p_1 & 0 & 0 & p_0 & 0 \\
p_3 & 0 & 0 & p_2 & 0 & 0 & p_1 \\
0 & p_4 & 0 & 0 & p_3 & 0 & 0 \\
0 & 0 & p_5 & 0 & 0 & p_4 & 0 \\
p_7 & 0 & 0 & p_6 & 0 & 0 & p_5 \\
0 & 0 & 0 & 0 & 0 & p_7 & 0 \\
\end{bmatrix}
\]

\[
T(2) = \begin{bmatrix}
0 & 0 & p_0 & 0 & 0 & 0 & 0 \\
p_2 & 0 & 0 & p_1 & 0 & 0 & p_0 \\
0 & p_3 & 0 & 0 & p_2 & 0 & 0 \\
0 & 0 & p_4 & 0 & 0 & p_3 & 0 \\
p_6 & 0 & 0 & p_5 & 0 & 0 & p_4 \\
0 & p_7 & 0 & 0 & p_6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & p_7 & 0 \\
\end{bmatrix}
\]

\[
T(3) = \begin{bmatrix}
p_1 & 0 & 0 & p_0 & 0 & 0 & 0 \\
p_2 & 0 & 0 & p_1 & 0 & 0 & p_0 \\
0 & p_3 & 0 & 0 & p_2 & 0 & 0 \\
0 & 0 & p_4 & 0 & 0 & p_3 & 0 \\
p_5 & 0 & 0 & p_4 & 0 & 0 & p_3 \\
p_6 & 0 & 0 & p_5 & 0 & 0 & p_4 \\
0 & p_7 & 0 & 0 & p_6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & p_7 & 0 \\
\end{bmatrix}
\]

We remark that these are also the primitive transition matrices in the case that \( \Lambda \) omits only one integer \( j \) other than 0 or 7, with the understanding that the entries denoted \( p_j \) have value zero.

Proof of Lemma 4.4. The \( \ell \)th child of parent \( [0, 1] \) is the net interval \( \left[\frac{\ell}{d}, \frac{\ell + 1}{d}\right] \). The \( j \)th neighbour of \( [0, 1] \) is \( j - 1 \) and the \( i \)th neighbour of \( \left[\frac{\ell}{d}, \frac{\ell + 1}{d}\right] \) is \( i - 1 \). The entry \( T(\ell)_{ji} \) will be \( p_s \) where \( s \in \{0, \ldots, k\} \) is such that

\[
0 - (j - 1) + \frac{s(d - 1)}{d} = \frac{i}{d} - \frac{(i - 1)}{d}.
\]

Solving for \( s \) yields the desired result. □
The transition matrices of complete quotient Cantor-like measures have not only a non-zero entry in each column, but also a non-zero entry in each row as the support of the quotient measure is full.

We next permute the rows and columns of these transition matrices to produce matrices $\tilde{T}(\ell)$ with a natural block structure, where block $(i, j)$ for $i, j \in \{1, \ldots, d-1\}$ has size $\left(\left\lfloor \frac{i-1}{\ell} \right\rfloor + 1\right) \times \left(\left\lfloor \frac{j-1}{\ell} \right\rfloor + 1\right)$. The $(i', j')$ entry of block $(i, j)$ of $\tilde{T}(\ell)$ will have as its entry the $(i+i'(d-1), j+j'(d-1))$ entry of $T(\ell)$, $i'$ and $j'$ being indexed beginning at 0. Doing this permutation is not necessary, but it makes the algebraic manipulations simpler in what follows.

Given such a block matrix $B$, we will write $B(i, j)$ for the $(i, j)$ block. We will say that block matrix $B$ (with this structure) is of type $r$ if $B(i, j) \neq 0$ only if $j - i \equiv r \pmod{d-1}$.

It is easy to see that the permuted transition matrix $\tilde{T}(\ell)$ is type $\ell$ (mod $(d-1)$). Furthermore, $\tilde{T}(\ell)$ has the special property that if $j - i \equiv \ell \pmod{d-1}$, then the matrix $\tilde{T}(\ell)(i, j)$ has at least one non-zero entry in each row and column.

**Example 4.6.** Consider a complete quotient Cantor-like measure with $d = 4$ and $k = 7$. Permuting the rows and columns yields:

$$
\tilde{T}(0) = \begin{bmatrix}
p_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p_4 & p_3 & p_2 & 0 & 0 & 0 & 0 \\
0 & p_7 & p_6 & p_0 & 0 & 0 & 0 \\
0 & 0 & 0 & p_1 & p_0 & 0 & 0 \\
0 & 0 & 0 & 0 & p_5 & p_4 & 0 \\
0 & 0 & 0 & 0 & 0 & p_2 & p_1 \\
0 & 0 & 0 & 0 & 0 & 0 & p_6 & p_5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & p_0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad \tilde{T}(1) = \begin{bmatrix}
p_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p_4 & p_3 & p_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & p_0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & p_4 & p_3 \\
0 & 0 & 0 & 0 & 0 & 0 & p_7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p_1 & 0 & 0 & 0 & 0 & 0 \\
p_5 & p_4 & p_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad \tilde{T}(2) = \begin{bmatrix}
p_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p_4 & p_3 & p_2 & 0 & 0 & 0 & 0 \\
0 & p_7 & p_6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & p_3 & p_2 & 0 & 0 \\
0 & 0 & 0 & 0 & p_7 & p_6 & 0 \\
0 & 0 & 0 & 0 & 0 & p_3 & p_2 \\
0 & 0 & 0 & 0 & 0 & 0 & p_7 & p_6 \\
p_1 & p_0 & 0 & 0 & 0 & 0 & 0 \\
p_5 & p_4 & p_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

Hereafter, when we speak of a block matrix we will mean a matrix $B$ with this block structure of type $\ell$ for some $\ell$ and having the property that $B(i, j)$ has a non-zero entry in each row and column if $j - i \equiv \ell \pmod{d-1}$. The block matrices of type 0 will be called block diagonal. By a block positive matrix we will mean a block matrix of type $\ell$ with all entries of $B(i, j)$ strictly positive for $j - i \equiv \ell \pmod{d-1}$. It is worth noting that a block matrix of type $\ell$ is also of type $\ell'$ for all $\ell' \equiv \ell \pmod{d-1}$.

We will say that a periodic point with period $\theta$ is a block diagonal (positive) periodic point if the block matrix $\bar{T}(\theta)$ is block diagonal (and block positive).

We record here some routine facts about block matrices (with this definition) that follow from block multiplication and the fact that each non-zero block has a non-zero entry in each row and column.

**Lemma 4.7.** Suppose $A, B, C$ are block matrices.

(i) If $A, B$ are types $a, b$ respectively, then $AB$ is a block matrix of type $(a + b) \pmod{d-1}$.
(ii) If $B$ is block positive, then $AB$ and $BA$ are block positive.

(iii) There are positive constants $c_1, c_1'$, depending on $B$, such that
\[ c_1 \|A\| \leq \min(\|AB\|, \|BA\|) \leq \max(\|AB\|, \|BA\|) \leq c_1' \|A\| . \]

(iv) If $B$ is block positive, then there is a positive constant $c_0$, depending on $B$, such that
\[ \|ABC\| \geq c_0 \|A\| \|C\| . \]

Being a product of block matrices, every permuted transition matrix of complete quotient Cantor-like measures is a block matrix.

Block positive matrices have further good properties.

**Lemma 4.8.** Suppose $B$ is a block positive matrix.

(i) If $AB$ is block diagonal, then there is a constant $c = c(B)$ such that
\[ \text{sp}(AB) \leq \|AB\| \leq c \cdot \text{sp}(AB). \]

(ii) If $B$ is block diagonal, then there is a constant $c_2 = c_2(B)$ such that
\[ \|B^n\| \leq c_2 \text{sp}(B^n) \text{ for all } n. \]

Further, there exists an index $j$ such that $\text{sp}(B^n) = \text{sp}(B(j,j))$ for all $n$.

**Proof.** (i) Assume $A$ is type $a$, so $B$ is type $-a$. Then $B(a+j, j)$ is a positive matrix for each $j$ (here $a + j$ should be understood mod $(d - 1)$) and hence by [9] Lemma 3.15(2) there are constants $C_j$ such that
\[ \|A(j, a+j)B(a+j, j)\| \leq C_j \text{sp}(A(j, a+j)B(a+j, j)) \]
\[ \leq C_j \text{sp}(AB(j, j)). \]

Since $AB$ is diagonal, this is dominated by $C_j \text{sp}(AB)$. But
\[ \|AB\| = \sum_{j=1}^{d-1} \|AB(j, j)\| = \sum_j \|A(j, a+j)B(a+j, j)\|, \]
so we can take $c = \max_j C_j(d - 1)$.

(ii) As each square matrix $B(j, j)$ is positive, [9] Lemma 3.15(3)] implies that for each $j$ there is a constant $C'_j$ such that \[ \|B(j, j)^n\| \leq C'_j \text{sp}(B(j, j))^n \leq C'_j \text{sp}(B^n) \]
for each $n$. Hence
\[ \|B^n\| = \sum_j \|B(j, j)^n\| \leq \max_j C'_j(d - 1) \text{sp}(B^n) \leq c_2 \text{sp}(B^n). \]

The final comment follows because $B$ is block diagonal. \hfill \square

**Lemma 4.9.** Suppose $\Lambda = \{0, ..., k\}$. The permuted transition matrix $(\tilde{T}(0) \tilde{T}(d-1))^k$ is block diagonal and positive.

**Proof.** It is easiest to see this if we look at $(T(0)T(d-1))^k$. We must show that if $j \equiv i \mod (d - 1)$, then \[ \left((T(0)T(d-1))^k\right)_{ij} > 0. \]
Let $\ell_0 = i$ and $\ell_2k = j$. Note that
\[ \left((T(0)T(d-1))^k\right)_{ij} = \sum_{\ell_1} \cdots \sum_{\ell_{2k-1}=1} T(0)_{\ell_{i_1}, \ell_{1+\epsilon_1}} \cdots T(0)_{\ell_{2(k-2)+\epsilon_{2(k-2)-1}}, \ell_{2(k-1)+\epsilon_{2(k-1)-1}}} T(d-1)_{\ell_{2k-1}, j} \]
\[ = \sum_{\ell_1} \cdots \sum_{\ell_{2k-1}=1} 2k \prod_{r=1}^{2k} p_{\ell_{r-1} - 1 - \tau_{\ell_{r-1}}} \tau(\epsilon) \]
where \( \tau(r) = 0 \) if \( r \) is odd and \( \tau(r) = 1 \) if \( r \) is even. It suffices to show that there exists \( \ell_1, \ell_2, \ldots, \ell_{2k-1} \) such that

\[
\prod_{r=1}^{2k} p_{\ell_{r-1} - 1 - \frac{\ell_r - \ell_{r-1}}{d-1} + \tau(r)} > 0.
\]

We will define the \( \ell_r \) inductively. We set \( \ell_0 = i \). We define

\[
\ell_r = \begin{cases} 
\ell_{r-1} + (d-1) & \text{if } \ell_{r-1} + (d-1) \leq j \text{ and } \ell_{r-1} + \tau(r) \geq 2 \\
\ell_{r-1} - (d-1) & \text{if } \ell_{r-1} - (d-1) \geq j \text{ and } \ell_{r-1} + \tau(r) \leq k, \\
\ell_{r-1} & \text{otherwise}
\end{cases}
\]

Recall that \( 1 \leq i, j \leq k \).

- If \( \ell_r = \ell_{r-1} + (d-1) \), then \( \ell_r + \tau(r) \geq 2 \) and \( \ell_r < j \) so \( \ell_r + \tau(r) \leq j \leq k \). Thus \( \ell_r - 1 - \frac{\ell_r - \ell_{r-1}}{d-1} + \tau(r) = \ell_{r-1} - 1 - 1 + \tau(r) \in \{0, 1, \ldots, k-1\} \), hence the associated probability is non-zero.

- If \( \ell_r = \ell_{r-1} - (d-1) \), then \( \ell_r + \tau(r) \leq k \) and \( \ell_r > j \), so \( \ell_r - 1 - \frac{\ell_r - \ell_{r-1}}{d-1} + \tau(r) = \ell_{r-1} - 1 + 1 + \tau(r) \in \{1, \ldots, k\} \) and hence the associated probability is non-zero.

- If \( \ell_r = \ell_{r-1} \) then \( \ell_{r-1} - 1 - \frac{\ell_r - \ell_{r-1}}{d-1} + \tau(r) = \ell_{r-1} - 1 + \tau(r) \in \{0, 1, \ldots, k\} \) and again the associated probability is non-zero.

We note that for \( r \) even we always have \( \ell_r - 1 + \tau(r) \geq 2 \) and for \( r \) odd we always have \( \ell_{r-1} + \tau(r) \leq k \). If \( i = \ell_0 < j \), say \( j = i + m(d-1) \) for \( m \geq 1 \), then \( \ell_t = j \) for \( t \geq 2m \) and therefore \( \ell_{2k} = j \). A similar statement holds if \( i = \ell_0 > j \) and, of course, if \( i = j \), then \( \ell_r = i = j \) for all \( r \).

Hence equation (6) is satisfied which proves the result. \( \square \)

When \( \Lambda \) is a proper subset of \( \{0, \ldots, k\} \), it is still possible for this transition matrix to be positive. For example, this can easily be seen to be true in the case that \( d = 4, k = 7, p_j = 0 \) for one of \( j = 1, \ldots, 6 \).

Another simple, but useful, fact is the following. By an interior periodic point, we mean a periodic point that is not a boundary point.

**Lemma 4.10.** Assume the complete quotient Cantor-like measure admits a permuted transition matrix that is block positive. Then there is a finite set \( \mathcal{F} \) of block (permuted) transition matrices such that for each block matrix \( A \) there is some \( B \in \mathcal{F} \) so that \( AB \) and \( BA \) are block positive and diagonal and, furthermore, any periodic point with period \( \theta \) satisfying \( \overline{T}(\theta) = AB \) or \( BA \) is an interior periodic point.

**Proof.** Let \( F \) denote any block diagonal and positive permuted transition matrix. This is guaranteed to exist since the product of a block positive matrix with any block matrix is still block positive. For each \( r = 1, \ldots, d-1 \), let \( B_r \) be a block matrix of type \( r \). Put \( \mathcal{F} = \{FB_1B_{r-1} : r = 0, 1, \ldots, d-2\} \). The matrices \( FB_1B_{r-1} \) are block positive and type \( r \). If \( A \) is any block matrix, it has type \( -r \) for some \( r = 1, \ldots, d-1 \) and hence \( FB_1B_{r-1}A \) and \( AFB_1B_{r-1} \) are block diagonal, positive matrices. The choice of \( B_1B_{r-1} \) ensures that they give rise to interior periodic points. \( \square \)
4.2. Local dimensions for complete quotient Cantor-like measures. In this section we prove that the set of local dimensions for all complete quotient Cantor-like measures which admit a block positive, transition matrix is an interval. We will also prove that the local dimensions at the interior, positive periodic points are dense.

We begin by proving that the local dimensions at periodic points are dense in the set of all local dimensions.

**Theorem 4.11.** Assume \( \mu_\pi \) is a complete quotient Cantor-like measure which admits a block positive, transition matrix. Then the set of local dimensions at block diagonal, positive, interior periodic points is dense in the set of all local (upper or lower) dimensions of \( \mu_\pi \).

**Proof.** Fix \( x \in \mathbb{T} \). We will see how to approximate the lower local dimension of \( \mu_\pi \) at \( x \) by the local dimensions of block diagonal, positive, interior periodic points. The other cases are similar.

Fix a subsequence \((n_\ell)\) such that

\[
\dim_{loc, \mu_\pi}(x) = \lim_{\ell} \frac{\log M_{n_\ell}(x)}{n_\ell \log(1/d)}.
\]

For each \( n_\ell \), let \( \Delta'_{n_\ell} \) denote a choice of \( \Delta_{n_\ell}(x) \), \( \Delta^+_{n_\ell}(x) \), \( \Delta^-_{n_\ell}(x) \) such that \( \mu_\pi(\Delta'_{n_\ell}) \leq M_{n_\ell}(x) \leq 3\mu_\pi(\Delta^-_{n_\ell}) \) and let \( (\gamma_0(\ell), \ldots, \gamma_{n_\ell}(\ell)) \) be the symbolic representation of \( \Delta'_{n_\ell} \). Then

\[
\dim_{loc, \mu_\pi}(x) = \lim_{\ell} \frac{\log \left\| \tilde{T}(\gamma_0(\ell), \ldots, \gamma_{n_\ell}(\ell)) \right\|}{n_\ell \log(1/d)}.
\]

For notational ease, let \( A_\ell = \tilde{T}(\gamma_0(\ell), \ldots, \gamma_{n_\ell}(\ell)) \) and choose a block positive matrix \( B_\ell \) from the finite set \( \mathcal{F} \) of Lemma 4.10 so that \( A_\ell B_\ell \) is block diagonal and if the periodic point \( y_\ell \) with period \( \theta_\ell \) satisfies \( \tilde{T}(\theta_\ell) = A_\ell B_\ell \), then \( y_\ell \) is a block diagonal, positive, interior periodic point.

By Lemma 4.8 there is a constant \( c \), which depends only on the finite set \( \mathcal{F} \), such that

\[
sp(A_\ell B_\ell) \leq \|A_\ell B_\ell\| \leq c \cdot sp(A_\ell B_\ell).
\]

Thus for each \( \ell \) there is a constant \( c_\ell \) so that \( c_\ell \|A_\ell B_\ell\| = sp(A_\ell B_\ell) \), where \( c_\ell \) is bounded above and below from zero. Similarly, by Lemma 4.7 there is a constant \( c_\ell' \), bounded above and below, such that \( \|A_\ell B_\ell\| = c_\ell' \|A_\ell\| \). Moreover,

\[
\dim \mu_\pi(y_\ell) = \frac{\log sp(A_\ell B_\ell)}{L(\theta_\ell) \log(1/d)} = \frac{\log (c_\ell c_\ell') \|A_\ell\|}{L(\theta_\ell) \log(1/d)}.
\]

But \( L(\theta^-_\ell) = n_\ell + L(B_\ell) \) (where by \( L(B_\ell) \) we mean the number of permuted primitive transition matrices whose product is \( B_\ell \) and \( L(B_\ell) \) is bounded because \( B_\ell \) is chosen from a finite set. Thus an easy calculation shows that

\[
\left| \dim \mu_\pi(y_\ell) - \frac{\log \|A_\ell\|}{n_\ell \log(1/d)} \right| \to 0 \text{ as } \ell \to \infty
\]

and therefore \( \dim \mu_\pi(y_\ell) \to \dim_{loc, \mu_\pi}(x) \).

**Theorem 4.12.** Assume \( \mu_\pi \) is a complete quotient Cantor-like measure which admits a block positive transition matrix. Suppose \( y, z \) are block diagonal, positive, interior periodic points. Given any real number \( R \), with \( \dim_{loc, \mu_\pi}(y) < R < \)
dim_{loc} \mu_\pi(z), and \varepsilon > 0 there is a periodic point x with
\[ |R - \dim_{loc} \mu_\pi(x)| < \varepsilon. \]

Proof. Assume y has period \phi and z period \theta where \( A = \tilde{T}(\phi) \) and \( B = \tilde{T}(\theta) \) are block diagonal and positive. Given \( R \) as above, choose \( 0 < t < 1 \) such that
\[ R = t \dim_{loc} \mu_\pi(y) + (1 - t) \dim_{loc} \mu_\pi(z). \]
Choose integers \( n_\ell, m_\ell \to \infty \) such that
\[ L(\phi^-) m_\ell \to t. \]
Let \( \alpha = sp(A) \) and \( \beta = sp(B) \) and assume \( \alpha = sp(A(j, j)) \) and \( \beta = sp(B(i, i)) \).

With this notation,
\[ R = \frac{t \log \alpha}{L(\phi^-) \log(1/d)} + \frac{(1 - t) \log \beta}{L(\theta^-) \log(1/d)}. \]

Choose block positive matrices \( M, N \) from the finite set \( \mathcal{F} \) of Lemma 4.10 so that \( M \) is type \( j - i \) and \( N \) is type \( i - j \).

By Lemmas 4.7 and 4.8 and the block structure, there are positive constants \( c, c' \), independent of \( \ell \), which may vary from one occurrence to another, such that
\[ sp(B^{m_\ell} MA^{n_\ell} N) \geq c \| B^{m_\ell} MA^{n_\ell} \| \geq c \| B^{m_\ell} MA^{n_\ell} (i, j) \| \]
\[ = c \| B^{m_\ell} (i, i) M(i, j) A^{n_\ell} (j, j) \| \]
\[ \geq c \| B^{m_\ell} (i, i) \| \| A^{n_\ell} (j, j) \| \geq c \beta^{m_\ell} \alpha^{n_\ell}. \]

On the other hand, as \( A, B \) are both block diagonal and positive, we also have
\[ sp(B^{m_\ell} MA^{n_\ell} N) \leq c' \| B^{m_\ell} \| \| A^{n_\ell} \| \leq c' \beta^{m_\ell} \alpha^{n_\ell}. \]

It follows that if we let \( x_\ell \) be the interior periodic point with period \( B^{m_\ell} MA^{n_\ell} N \), then
\[ \lim_{\ell} \dim_{loc} \mu_\pi(x_\ell) = \lim_{\ell} \frac{\log sp(B^{m_\ell} MA^{n_\ell} N)}{m_\ell \log \beta + n_\ell \log \alpha} = R. \]

\[ \square \]

**Theorem 4.13.** Assume \( \mu_\pi \) is a complete quotient Cantor-like measure which admits a block positive transition matrix. Suppose \( x_n \) are block diagonal, positive, interior periodic points. Then there is some \( x \in \tilde{T} \) such that
\[ \dim_{loc} \mu_\pi(x) = \lim_{n} \dim_{loc} \mu_\pi(x_n) \]
\[ \overline{\dim}_{loc} \mu_\pi(x) = \lim_{n} \sup \dim_{loc} \mu_\pi(x_n) \]
\[ \underline{\dim}_{loc} \mu_\pi(x) = \lim_{n} \inf \dim_{loc} \mu_\pi(x_n). \]

Proof. Assume \( x_n \) has period \( \theta_n \) with \( \tilde{T}(\theta_n) \) block diagonal and positive. Let
\[ S = \lim \frac{\log(sp \tilde{T}(\theta_n))}{L(\theta_n) \log(1/d)} \quad \text{and} \quad I = \lim \frac{\log(sp \tilde{T}(\theta_n))}{L(\theta_n^2) \log(1/d)}. \]

By passing to a subsequence, we can assume
\[ \dim_{loc} \mu_\pi(x_2n) = \frac{\log(sp \tilde{T}(\theta_{2n}))}{L(\theta_{2n}^2) \log(1/d)} \to S. \]
and
\[ \dim_{loc} \mu(x_{2n+1}) = \frac{\log(s_p T(\theta_{2n+1}))}{L(\theta_{2n+1}) \log(1/d)} \to I. \]

We will define a rapidly increasing sequence \((i_n)\) and then put \(x\) to be
\[ x = (\gamma_0, \theta_1, \theta_1, \theta_2, \theta_2, \theta_3, \theta_3, \ldots). \]

The sequence \((i_n)\) will be inductively defined, with \(i_n\) depending on \(i_j, \theta_j\) for \(j = 1, \ldots, n - 1, \theta_n\) and \(\theta_{n+1}\). The choice will be clear from the arguments that follow.

The proof that \(x\) has the required properties is similar to that of [9, Theorem 5.5] and [11, Theorem 3.13]. We only sketch the main ideas here.

Put \(N_n = 1 + \sum_{j=1}^{n} i_j L(\theta_j^-)\) so that \(\Delta_{N_n}(x) = (\gamma_0, \theta_1, \theta_1, \theta_2, \theta_2, \theta_3, \theta_3, \ldots, \theta_n, \ldots, \theta_n)\).

Since \(x_n\) is an interior point, \(\Delta_{N_n}(x), \Delta_{N_n}^+(x), \Delta_{N_n}^-(x)\) have a common ancestor \((\gamma_0, \theta_1, \theta_1, \theta_2, \theta_2, \theta_3, \theta_3, \ldots, \theta_n, \ldots, \theta_n)\) = \(\Delta_{N_n-L(n)}(x)\). Applying Corollary 3.5 and Lemma 4.7 there are constants \(a_n, b_n, c_n\) depending on \(i_j, \theta_j, j = 1, \ldots, n - 1\) and \(\theta_n\), so that
\[ M_{N_n}(x) = a_n \mu(\Delta_{N_n-L(n)}(x)) \]
\[ = b_n \left\| \tilde{T}(\gamma_0, \theta_1, \theta_1, \theta_2, \theta_2, \theta_3, \theta_3, \ldots, \theta_n, \ldots, \theta_n) \right\| = c_n \left\| \tilde{T}(\theta_n)^{i_n-1} \right\|. \]

By choosing \(i_n\) large enough we can ensure that
\[ \frac{\left\| \tilde{T}(\theta_n)^{i_n-1} \right\|}{L(\theta_n)(i_n - 1)} = \frac{\log(s_p T(\theta_n))}{L(\theta_n)} \to 0, \]
\[ \frac{|a_n| + |b_n| + |c_n|}{i_n} \to 0 \]

and
\[ \frac{L(\theta_n)(i_n - 1)}{N_n} \to 1. \]

Consequently,
\[ \frac{\log M_{N_n}(x)}{N_n \log(1/d)} - \frac{\log(s_p T(\theta_n))}{L(\theta_n) \log(1/d)} \to 0 \]
as \(n \to \infty\). It follows that \(\frac{\log M_{N_n}(x)}{N_n \log(1/d)}\) tends to \(S\) along the even and odd subsequence. This shows that \(\dim_{loc} \mu(x) \geq S\) and \(\dim_{loc} \mu(x) \leq I\).

Now consider arbitrary
\[ N = 1 + \sum_{j=1}^{n} i_j L(\theta_j^-) + q_{n+1} L(\theta_{n+1}^-) + r \]
where $0 \leq q_{n+1} < i_n + 1$, $0 \leq r < L(\theta_{n+1})$ and $r > 0$ if $q_{n+1} = 0$. Then $\Delta_N(x)$, $\Delta_N^+(x)$, $\Delta_N^-(x)$ have common ancestor $\Delta_{J_N}(x)$ where

$$J_N = 1 + \sum_{j=1}^n i_j L(\theta_j^-) + (q_{n+1} - 1)L(\theta_n^-)$$

if $q_{n+1} > 1$ and $J_N = 1 + \sum_{j=1}^{n-1} i_j L(\theta_j^-) + (i_n - 1) L(\theta_n^-)$ otherwise. Similar arguments to above show that we should study the limiting behaviour of

$$\frac{\log \mu_x(\Delta_{J_N}(x))}{N \log(1/d)},$$

hence it suffices to study the limiting behaviour of

$$E_n = \frac{\log \|T(\gamma_0, \theta_1, ..., \theta_n)\|}{N \log(1/d)}$$

(omit suitable modifications if $q_{n+1} = 0, 1$). Applying Lemma 2.7(iv), with the positive matrix $\tilde{T}(\theta_n)$ as the matrix $B$, there is a constant $c_0(n)$, depending on $i_j, \theta_j$, $j = 1, ..., n - 1$ and $\theta_n$ such that $E_n$ dominates

$$\frac{\log c_0(n) + \log \|T(\theta_n)^{i_n}\|}{N \log(1/d)} + \frac{\log \|T(\theta_{n+1})^{q_{n+1} - 1}\|}{N \log(1/d)} \geq \frac{\log c_0(n)}{N \log(1/d)} + \frac{\log \text{sp}(\tilde{T}(\theta_n)^{i_n - 1})}{N \log(1/d)} + \frac{\log \text{sp}(\tilde{T}(\theta_{n+1})^{q_{n+1} - 1})}{N \log(1/d)}.$$

Choose a sequence $(\varepsilon_n)$ tending to 0. For $i_n$ sufficiently large, $\frac{\log c_0(n)}{N \log(1/d)} < \varepsilon_n$. Furthermore, with possibly larger $i_n$ we have

$$\frac{\log \text{sp}(\tilde{T}(\theta_{n+1})^{q_{n+1} - 1})}{N \log(1/d)} = \frac{\log \text{sp}(\tilde{T}(\theta_{n+1})^{q_{n+1} - 1})}{(q_{n+1} - 1)L(\theta_{n+1}) \log(1/d)} \frac{(q_{n+1} - 1)L(\theta_{n+1})}{N} \geq \frac{\log \text{sp}(\tilde{T}(\theta_{n+1}))}{L(\theta_{n+1}) \log(1/d)} t_n - \varepsilon_n$$

for

$$t_n = \frac{q_{n+1} L(\theta_{n+1}) + r}{N}.$$

As $1 - t_n = 1 + \sum_{j=1}^n i_j L(\theta_j^-)$, we similarly have

$$\frac{\log \text{sp}(\tilde{T}(\theta_n))^{i_n - 1}}{N \log(1/d)} \geq \frac{\log \text{sp}(\tilde{T}(\theta_n))}{L(\theta_n) \log(1/d)} (1 - t_n) - \varepsilon_n.$$

Thus

$$E_n \geq \frac{\log \text{sp}(\tilde{T}(\theta_n))}{L(\theta_n) \log(1/d)} (1 - t_n) + \frac{\log \text{sp}(\tilde{T}(\theta_{n+1}))}{L(\theta_{n+1}) \log(1/d)} t_n - 3 \varepsilon_n.$$
Appealing to Lemma 4.8(ii), we similarly deduce that for large enough $i_n$

$$E_n \leq \frac{\log \|T(\theta_n)^{i_n}\| + \log \|T(\theta_{n+1})^{q_{n+1}-1}\|}{N \log(1/d)} + \varepsilon_n$$

$$\leq \frac{\log \sp(T(\theta_n)^{i_n})}{N \log(1/d)} + \frac{\log \sp(T(\theta_{n+1})^{q_{n+1}-1})}{N \log(1/d)} + 2\varepsilon_n$$

$$\leq \frac{\log \sp(T(\theta_n))}{L(\theta_n) \log(1/d)}(1 - t_n) + \frac{\log \sp(T(\theta_{n+1}))}{L(\theta_{n+1}) \log(1/d)}t_n + 3\varepsilon_n.$$

Together these estimates show that $E_n$ lies within $3\varepsilon_n$ of the same convex combinations of $\log \sp(T(\theta_n))$ and $\log \sp(T(\theta_{n+1}))$. This proves that $\lim \inf E_n$ and $\lim \sup E_n$ lie in the interval $[I, S]$ and hence the same is true for $\dim_{loc} \mu_\pi(x)$ and $\dim_{loc} \mu_{\pi}(x)$. \qed

Combining these results we immediately deduce the following.

**Corollary 4.14.** The set of (upper, lower) local dimensions of any complete quotient Cantor-like measure that admits a block positive transition matrix is the closed interval which is the closure of the set of local dimensions at block diagonal, positive, interior periodic points. In particular, if $\mu_\pi$ is the quotient of the self-similar measure $\mu$ associated with the IFS $(\mathbf{2})$ with $k \geq d - 1$ and $\Lambda = \{0, 1, \ldots, k\}$, then the set of local dimensions is a closed interval.

### 4.3. Bounds on the local dimensions of Cantor-like measures.

In [7] it was shown that the set of local dimensions of the quotient of the 3-fold convolution of the middle third Cantor measure $\nu$, is a proper subset of the interval component of the set of local dimensions of $\nu$. In this subsection, we will see that this is true, more generally, for quotients of $(d + k)$-fold convolutions of uniform Cantor measures with contraction factor $1/d$, provided $d$ is sufficiently large.

For matrix $M$, let $\|M\|_{min} = \min_j \sum_i |M_{ij}|$ denote the minimum column sum. Of course, $\|M\| \geq \|M\|_{min}$ and in [11] it is observed that $\|M_1M_2\|_{min} \geq \|M_1\|_{min}\|M_2\|_{min}$.

There is a refinement of this for block matrices. Let $T$ be a block matrix with $D$ non-zero blocks and recall that by $T(i, j)$ we mean the block $(i, j)$ submatrix of $T$. The arithmetic/geometric mean inequality implies that

$$\|T\| \geq \sum_{\text{non-zero blocks}} \|T(i, j)\|_{min}$$

$$\geq D \left( \prod_{\text{non-zero blocks}} \|T(i, j)\|_{min} \right)^{1/D}$$

$$\geq \left( \prod_{\text{non-zero blocks}} \|T(i, j)\|_{min} \right)^{1/D}.$$
More generally, for block matrices $T_k$,

$$\|T_1 T_2 \cdots T_t\| \geq \left( \prod_{(i,j) \text{ non-zero blocks of } T_1 T_2 \cdots T_t} \|T_1 T_2 \cdots T_t(i,j)\|_{\min} \right)^{1/D}$$

$$= \left( \prod_{(i,j) \text{ non-zero blocks of } T_1 T_2 \cdots T_t} \prod_{k=1}^t \|T_k(i_k, j_k)\|_{\min} \right)^{1/D}$$

$$= \left( \prod_{k=1}^t \prod_{(i,j) \text{ non-zero blocks of } T_k} \|T_k(i,j)\|_{\min} \right)^{1/D}$$

where $T_1 T_2 \cdots T_t(i,j) = \prod_{k=1}^t T_k(i_k, j_k)$.

Combined with (3), this observation directly yields the following upper bound on local dimensions.

**Proposition 4.15.** Suppose $\mu_\pi$ is a complete quotient Cantor-like measure with contraction factor $1/d$ and regular probabilities. If

$$\left( \prod_{(i,j) \text{ non-zero blocks}} \|\tilde{T}(\ell)(i,j)\|_{\min} \right)^{1/(d-1)} \geq \theta$$

for each primitive transition matrix $\tilde{T}(\ell)$, then

$$\sup_x \dim_{\text{loc}} \mu_\pi(x) \leq \frac{\log \theta}{\log 1/d}.$$

**Proposition 4.16.** Let $d, k$ be non-negative integers with $d \geq 3$ and let $\nu = \nu(d, k)$ be the $(d + k)$-fold convolution of the uniform Cantor measure with contraction factor $1/d$ and $\nu_\pi$ the associated measure on the torus. For any fixed $k$,

$$\{\dim_{\text{loc}} \nu_\pi(x) : x \in \text{supp } \nu_\pi\} \subseteq \{\dim_{\text{loc}} \nu(x) : x \in \text{supp } \nu, x \neq 0, d + k\}$$

provided $d$ is sufficiently large.

**Proof.** It is easy to see that

$$\{\dim_{\text{loc}} \nu_\pi(x) : x \in \text{supp } \nu_\pi\} \subseteq \{\dim_{\text{loc}} \nu(x) : x \in \text{supp } \nu, x \neq 0, d + k\}$$

by noting that

$$\min_{\ell} \left( \dim_{\text{loc}} \mu(x + \ell) \right) \leq \dim_{\text{loc}} \mu_\pi(x) \leq \min_{\ell} \left( \dim_{\text{loc}} \mu(x + \ell) \right).$$

Hence we need only show the strict inclusion. In fact, we will show that $\sup_x \dim_{\text{loc}} \nu_\pi(x) < \sup_{x \neq 0, d+k} \dim_{\text{loc}} \nu(x)$.

There is no loss in assuming $k \leq d - 2$. In this case, [1] Thm. 6.1] gives the formula

$$\sup_{x \neq 0, d+k} \dim_{\text{loc}} \nu(x) = \frac{\log \beta}{\log 1/d}.$$
for
\[
\beta = \frac{p_{r+d+1} + p_r + \sqrt{(p_{r+d+1} - p_r)^2 + 4p_{r+1}p_{r+d}}}{2},
\]
where \( r = \left\lfloor \frac{k}{2} \right\rfloor \) and
\[
p_j = \binom{d+k}{j} 2^{-(d+k)}.
\]
Using the bound \( \binom{d+k}{j} \leq (d+k)^j \) for \( j \leq (d+k)/2 \) one can easily verify that
\[
\beta \leq C_0(d+k)^{k/2+1} 2^{-(d+k)}
\]
for a constant \( C_0 \) independent of \( d \). Thus
\[
\forall x \neq 0, d+k \quad \dim_{\text{loc}} \nu(x) \geq \frac{\log 2^{d+k} - \log C_0(d+k)^{k/2+1}}{\log d}.
\]

Next, we will apply Proposition 4.15 to find an upper bound on \( \dim_{\text{loc}} \nu_\tau(x) \).

For this, we will need to obtain lower bounds on \( \|\tilde{T}(\ell)(i,j)\|_\min \) for the primitive transition matrices \( \tilde{T}(\ell) \), \( \ell = 0, \ldots, d-1 \). Each of these block matrices has \( d-1 \) non-zero blocks \( (i,j) \) where \( j - i \equiv \ell \mod (d-1) \). We recall that the \( (i,j) \) block is of size \( \left( \frac{d+k}{d-1} \right) + 1 \times \left( \frac{d+k}{d-1} \right) + 1 \) and so has either one or two rows, the latter if and only if \( i \leq k+1 \), and similarly for the columns. A calculation shows that the \( (i',j') \) entry of block \( (i,j) \) is equal to \( p_{i-1-j'+\ell} \) if \( j \geq i \) and \( p_{i-1-j'+\ell} \) if \( j < i \).

First, suppose block \( (i,j) \) has one row (hence \( i > k+1 \)). Then the \( j' \) column sum is either \( p_{i-1-j'} \) (when \( j \geq i \)) or \( p_{i-j'} \) (when \( j < i \)). If there is also only one column (equivalently, \( j > k+1 \)), then
\[
\|\tilde{T}(\ell)(i,j)\|_\min \geq \begin{cases} p_{i-1} & \text{if } k+1 < i \leq (d+k)/2 \\ p_i & \text{if } i > (d+k)/2 \end{cases}.
\]

If, instead, the block has two columns (when \( j \leq k+1 \) and therefore \( j < i \)), we obtain the same conclusion for the minimum column sum.

Otherwise, block \( (i,j) \) has two rows, (i.e., \( i \leq k+1 \leq (d+k)/2 \)). Similar reasoning shows that then
\[
\|\tilde{T}(\ell)(i,j)\|_\min \geq \min(p_{i-2}, 2^{-(d+k)}).
\]

Let \( t = k+2 \). For large enough \( d \), \( p_k \leq p_{k+1} \) and \( \binom{d+k}{s} \geq \binom{d+k}{t} \) for \( t \leq s \leq \left\lfloor \frac{d+k}{2} \right\rfloor \) and for \( s > \left\lfloor \frac{d+k}{2} \right\rfloor \), \( p_s = p_{d+k-s} \). Applying these facts, it follows that
\[
\prod_{(i,j) \text{ non-zero blocks}} \|\tilde{T}(\ell)(i,j)\|_\min \geq 2^{-(d+k)(d-1)} \prod_{i=k+3}^{\left\lfloor \frac{d+k}{2} \right\rfloor} \left( \frac{d+k}{i} \right) \prod_{i > \left\lfloor \frac{d+k}{2} \right\rfloor} \left( \frac{d+k}{i} \right) \geq 2^{-(d+k)(d-1)(d+k)^{d-6-k}}.
\]
Thus for sufficiently large $d$,

\[
\left( \prod_{(i,j)} \left\| \mathcal{T}(\ell)(i, j) \right\|_{\min} \right)^{1/(d-1)} \geq 2^{-(d+k)}(d+k)^{3(k+2)/4}.
\]

It follows from Proposition 4.15 and (7) that for large enough $d$,

\[
\sup_x \dim_{\text{loc}} \nu(x) \leq \frac{\log 2^{-(d+k)}(d+k)^{3(k+2)/4}}{\log 1/d} = \frac{\log 2^{d+k} - \log(d+k)^{3(k+2)/4}}{\log d} < \sup_{x \neq 0, d+k} \dim_{\text{loc}} \nu(x).
\]

\[\square\]

**Remark 4.17.** We conjecture that when $3 \leq d \leq m$, the set of local dimensions of the quotient of the $m$-fold convolution of a uniform Cantor measure with contraction factor $1/d$ is a proper subset of the set of local dimensions at essential points of the pre-quotient measure. We have checked this numerically for all $3 \leq d \leq m \leq 10$. See Table S1 of [10]. Numerical evidence suggests this is also true for $m = d - 1$ for $d \geq 4$. It is known by Theorem 3.12 that the two sets of local dimensions will be equal for $m < d - 1$.

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[15] P. Shmerkin, *A modified multi-fractal formalism for a class of self-similar measures with overlap*, Asian J. Math 9(2005), 323-348.
Consider the measure $\mu$ given by the maps $S_i(x) = x/4 + d_i$ with $d_0 = 0$, $d_1 = 3/5$, $d_2 = 6/5$, $d_3 = 9/5$, and $d_4 = 3$. The convex hull of this IFS is $[0, 4]$. Let $\mu_\pi$ be the quotient measure. The reduced transition diagram has 10 reduced characteristic vectors. The reduced characteristic vectors are:

- Reduced characteristic vector 1: $(1, (0, 1, 2, 3))$
- Reduced characteristic vector 2: $(4/5, (0, 4/5, 8/5, 16/5))$
- Reduced characteristic vector 3: $(8/5, (0, 4/5, 8/5, 12/5))$
- Reduced characteristic vector 4: $(4/5, (0, 8/5, 12/5, 16/5))$
- Reduced characteristic vector 5: $(4/5, (0, 4/5, 12/5, 16/5))$
- Reduced characteristic vector 6: $(4/5, (0, 4/5, 8/5, 12/5))$
- Reduced characteristic vector 7: $(4/5, (0, 4/5, 8/5, 12/5, 16/5))$
- Reduced characteristic vector 8: $(4/5, (4/5, 8/5, 12/5, 16/5))$
- Reduced characteristic vector 9: $(4/5, (4/5, 8/5, 16/5))$
- Reduced characteristic vector 10: $(4/5, (0, 8/5, 12/5))$

The maps are:

- RCV 1 $\rightarrow$ [2, 3, 4, 5]
- RCV 2 $\rightarrow$ [2, 6, 7, 7]
- RCV 3 $\rightarrow$ [7, 7, 7, 7, 8, 4, 5, 2]
- RCV 4 $\rightarrow$ [6, 7, 8, 4]
- RCV 5 $\rightarrow$ [5, 9, 10, 5]
- RCV 6 $\rightarrow$ [7, 7, 7, 7]
- RCV 7 $\rightarrow$ [7, 7, 7, 7]
- RCV 8 $\rightarrow$ [8, 4, 5, 2]
- RCV 9 $\rightarrow$ [9, 10, 5, 9]
- RCV 10 $\rightarrow$ [10, 5, 9, 10]

See Figure S1 for the transition diagram. This has (normalized) transition matrices:

$$T(1, 2) = \begin{bmatrix} p_1 & 0 & 0 & 0 \\ 0 & 0 & p_2 & 0 \\ 0 & p_4 & 0 & p_3 \\ p_5 & 0 & 0 & 0 \end{bmatrix} \quad T(1, 3) = \begin{bmatrix} 0 & p_1 & 0 & 0 \\ p_3 & 0 & 0 & p_2 \\ 0 & 0 & p_4 & 0 \\ 0 & p_5 & 0 & 0 \end{bmatrix}$$
Figure S1. $4x - 1$ with $d_i \in [0, 3/5, 6/5, 9/5, 3]$, Full set + Essential class

$T(1, 4) = \begin{bmatrix} p_2 & 0 & p_1 & 0 \\ 0 & p_3 & 0 & 0 \\ 0 & 0 & 0 & p_4 \\ 0 & 0 & p_5 & 0 \end{bmatrix}$ $\quad$ $T(1, 5) = \begin{bmatrix} 0 & p_2 & 0 & p_1 \\ p_4 & 0 & p_3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p_5 \end{bmatrix}$

$T(2, 2) = \begin{bmatrix} p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & p_1 \\ 0 & 0 & p_3 & 0 \\ 0 & p_5 & 0 & 0 \end{bmatrix}$ $\quad$ $T(2, 6) = \begin{bmatrix} 0 & p_1 & 0 & 0 \\ 0 & 0 & p_2 & 0 \\ p_4 & 0 & 0 & p_3 \\ 0 & 0 & p_5 & 0 \end{bmatrix}$

$T(2, 7) = \begin{bmatrix} p_3 & 0 & 0 & p_2 \\ 0 & p_4 & 0 & 0 \\ 0 & 0 & 0 & p_3 \\ 0 & 0 & 0 & p_5 \end{bmatrix}$ $\quad$ $T(2, 7) = \begin{bmatrix} p_2 & 0 & 0 & p_1 \\ 0 & p_3 & 0 & 0 \\ 0 & 0 & p_4 & 0 \\ 0 & 0 & 0 & p_5 \end{bmatrix}$

$T(3, 7) = \begin{bmatrix} p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & p_3 & 0 \\ 0 & 0 & 0 & p_4 \end{bmatrix}$ $\quad$ $T(3, 7) = \begin{bmatrix} 0 & p_1 & 0 & 0 \\ 0 & 0 & p_2 & 0 \\ p_4 & 0 & 0 & p_3 \\ 0 & 0 & 0 & p_4 \end{bmatrix}$
\[
T(3, 7) = \begin{bmatrix}
0 & 0 & p_1 & 0 & 0 \\
p_3 & 0 & 0 & p_2 & 0 \\
0 & p_4 & 0 & 0 & p_3 \\
0 & 0 & 0 & 0 & 0 \\
p_5 & 0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad
T(3, 7) = \begin{bmatrix}
p_2 & 0 & 0 & p_1 & 0 \\
0 & p_3 & 0 & 0 & p_2 \\
0 & 0 & p_4 & 0 & 0 \\
p_5 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
T(3, 8) = \begin{bmatrix}
p_2 & 0 & 0 & p_1 \\
0 & p_3 & 0 & 0 \\
0 & 0 & p_4 & 0 \\
p_5 & 0 & 0 & 0 \\
\end{bmatrix}, \quad
T(3, 4) = \begin{bmatrix}
0 & p_2 & 0 & 0 \\
p_4 & 0 & p_3 & 0 \\
0 & 0 & 0 & p_4 \\
p_5 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
T(3, 5) = \begin{bmatrix}
p_3 & 0 & p_2 & 0 \\
0 & p_4 & 0 & p_3 \\
0 & 0 & 0 & 0 \\
0 & 0 & p_5 & 0 \\
\end{bmatrix}, \quad
T(3, 2) = \begin{bmatrix}
0 & p_3 & 0 & p_2 \\
0 & 0 & p_4 & 0 \\
p_5 & 0 & 0 & 0 \\
0 & 0 & 0 & p_5 \\
\end{bmatrix}
\]

\[
T(4, 6) = \begin{bmatrix}
p_1 & 0 & 0 & 0 \\
0 & p_3 & 0 & 0 \\
0 & 0 & p_4 & 0 \\
p_5 & 0 & 0 & 0 \\
\end{bmatrix}, \quad
T(4, 7) = \begin{bmatrix}
0 & p_1 & 0 & 0 \\
p_4 & 0 & 0 & p_3 \\
0 & 0 & 0 & p_4 \\
p_5 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
T(4, 8) = \begin{bmatrix}
p_4 & 0 & 0 & p_3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & p_5 \\
\end{bmatrix}, \quad
T(4, 4) = \begin{bmatrix}
p_2 & 0 & p_1 & 0 \\
0 & p_4 & 0 & 0 \\
p_5 & 0 & 0 & 0 \\
0 & 0 & 0 & p_5 \\
\end{bmatrix}
\]

\[
T(5, 5) = \begin{bmatrix}
p_1 & 0 & 0 & 0 \\
0 & p_2 & 0 & p_1 \\
0 & 0 & p_4 & 0 \\
p_5 & 0 & 0 & 0 \\
\end{bmatrix}, \quad
T(5, 9) = \begin{bmatrix}
0 & p_1 & 0 & 0 \\
0 & p_2 & 0 & 0 \\
0 & 0 & p_4 & 0 \\
p_5 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
T(5, 10) = \begin{bmatrix}
p_3 & 0 & p_2 \\
0 & 0 & 0 \\
0 & 0 & p_5 \\
\end{bmatrix}, \quad
T(5, 5) = \begin{bmatrix}
p_2 & 0 & p_1 & 0 \\
0 & p_3 & 0 & p_2 \\
p_5 & 0 & 0 & 0 \\
0 & 0 & 0 & p_5 \\
\end{bmatrix}
\]

\[
T(6, 7) = \begin{bmatrix}
p_1 & 0 & 0 & 0 & 0 \\
0 & p_2 & 0 & 0 & p_1 \\
0 & 0 & p_3 & 0 & 0 \\
0 & 0 & 0 & p_4 & 0 \\
0 & 0 & p_1 & 0 & 0 \\
\end{bmatrix}, \quad
T(6, 7) = \begin{bmatrix}
0 & p_1 & 0 & 0 & 0 \\
0 & 0 & p_2 & 0 & 0 \\
p_4 & 0 & 0 & p_3 & 0 \\
0 & 0 & 0 & 0 & p_4 \\
p_2 & 0 & 0 & p_1 & 0 \\
\end{bmatrix}
\]

\[
T(6, 7) = \begin{bmatrix}
p_3 & 0 & 0 & p_2 & 0 \\
p_4 & 0 & 0 & p_3 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad
T(6, 7) = \begin{bmatrix}
p_5 & 0 & 0 & 0 & 0 \\
0 & p_3 & 0 & 0 & p_2 \\
0 & 0 & p_4 & 0 & 0 \\
p_5 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
There are two essential classes in this case. The first is given by the reduced characteristic vector 7 and the second by the reduced characteristic vectors 5, 9 and 10.

There are also two non-essential maximal loop classes, given by the reduced characteristic vectors 4 and 8 and by the reduced characteristic vector 2.
S2. Details for Example 3.9 – Part 1

Consider \( \varrho \), the root of \( x^2 + x - 1 \) and the maps \( S_i(x) = \varrho x + d_i \) with \( d_0 = 0 \), \( d_1 = 1 - \varrho \), and \( d_2 = 2 - 2\varrho \). This measure is of full support on \( [0, 2] \). The probabilities are given by \( p_0 = 1/4 \), \( p_1 = 1/2 \), and \( p_2 = 1/4 \). We first give the information for this measure on \( \mathbb{R} \). The reduced transition diagram has 40 reduced characteristic vectors. The reduced characteristic vectors are:

- Reduced characteristic vector 1: \( (1, 0) \)
- Reduced characteristic vector 2: \( (1/2\varrho, 0) \)
- Reduced characteristic vector 3: \( (1/2\varrho, (0, 1/2\varrho)) \)
- Reduced characteristic vector 4: \( (1 - \varrho, (0, 1/2\varrho, \varrho)) \)
- Reduced characteristic vector 5: \( (1/2\varrho, (1 - \varrho, 1 - 1/2\varrho)) \)
- Reduced characteristic vector 6: \( (1/2\varrho, (1 - 1/2\varrho)) \)
- Reduced characteristic vector 7: \( (1/2 - 1/2\varrho, (0, 1/2\varrho)) \)
- Reduced characteristic vector 8: \( (-1/2 + \varrho, (0, 1/2 - 1/2\varrho, 1/2)) \)
- Reduced characteristic vector 9: \( (1/2 - 1/2\varrho, (0, -1/2 + \varrho, 1/2\varrho, \varrho)) \)
- Reduced characteristic vector 10: \( (1/2 - 1/2\varrho, (0, 1/2 - 1/2\varrho, 1/2\varrho, 1/2, 1/2 + 1/2\varrho)) \)
- Reduced characteristic vector 11: \( (-1/2 + \varrho, (0, 1/2 - 1/2\varrho, 1 - \varrho, 1/2, 1 - 1/2\varrho)) \)
- Reduced characteristic vector 12: \( (1/2 - 1/2\varrho, (0, -1/2 + \varrho, 1/2\varrho, 1/2, \varrho, 1/2 + 1/2\varrho)) \)
- Reduced characteristic vector 13: \( (1/2 - 1/2\varrho, (0, 1/2 - 1/2\varrho, 1/2\varrho, 1/2, 1 - 1/2\varrho, 1/2 + 1/2\varrho)) \)
- Reduced characteristic vector 14: \( (-1/2 + \varrho, (1/2 - 1/2\varrho, 1 - \varrho, 1/2, 1 - 1/2\varrho, 3/2 - \varrho)) \)
- Reduced characteristic vector 15: \( (1/2 - 1/2\varrho, (0, 1/2\varrho, 1/2, 1/2 + 1/2\varrho, 1/2\varrho, 1/2 + 1/2\varrho)) \)
- Reduced characteristic vector 16: \( (1/2 - 1/2\varrho, (1/2 - 1/2\varrho, 1/2\varrho, 1/2, 1 - 1/2\varrho, 1/2 + 1/2\varrho)) \)
- Reduced characteristic vector 17: \( (-1/2 + \varrho, (1 - \varrho, 1 - 1/2\varrho, 3/2 - \varrho)) \)
- Reduced characteristic vector 18: \( (1/2 - 1/2\varrho, (1/2, 1/2 + 1/2\varrho, 1/2\varrho, 1/2 - 1/2\varrho, 3/2 - \varrho)) \)
- Reduced characteristic vector 19: \( (-1/2 + \varrho, (0, 1/2 - 1/2\varrho, 1/2\varrho, 1/2, 1 - 1/2\varrho, 1/2 + 1/2\varrho)) \)
- Reduced characteristic vector 20: \( (1 - 3/2\varrho, (0, -1/2 + \varrho, 1/2\varrho, -1/2 + 3/2\varrho, \varrho, 1/2 + 1/2\varrho, 3/2\varrho)) \)
- Reduced characteristic vector 21: \( (-1/2 + \varrho, (1 - 3/2\varrho, 1/2 - 1/2\varrho, 1 - \varrho, 1/2, 1 - 1/2\varrho, 3/2 - \varrho)) \)
- Reduced characteristic vector 22: \( (1/2 - 1/2\varrho, (0, 1/2 - 1/2\varrho, 1/2\varrho, 1/2, \varrho, 1/2 + 1/2\varrho)) \)
- Reduced characteristic vector 23: \( (-1/2 + \varrho, (0, 1/2 - 1/2\varrho, 1 - \varrho, 1/2, 1 - 1/2\varrho, 1/2 + 1/2\varrho)) \)
- Reduced characteristic vector 24: \( (1 - 3/2\varrho, (0, -1/2 + \varrho, 1/2\varrho, 1/2, \varrho, 1/2 + 1/2\varrho, 3/2\varrho)) \)
- Reduced characteristic vector 25: \( (-1/2 + \varrho, (1 - 3/2\varrho, 1/2 - 1/2\varrho, 1 - \varrho, 3/2 - 3/2\varrho, 1 - 1/2\varrho, 3/2 - \varrho)) \)
- Reduced characteristic vector 26: \( (-1/2 + \varrho, (0, 1 - 3/2\varrho, 1/2 - 1/2\varrho, 1 - \varrho, 1/2, 1 - 1/2\varrho, 3/2 - \varrho)) \)
- Reduced characteristic vector 27: \( (1/2 - 1/2\varrho, (0, -1/2 + \varrho, 1/2 - 1/2\varrho, 1/2\varrho, 1/2, \varrho, 1/2 + 1/2\varrho)) \)
• Reduced characteristic vector 28: \((-1/2 + \varrho, 0, 1/2 - 1/2\varrho, 1/2\varrho, 1 - \varrho, 1/2, 1 - 1/2\varrho, 1/2 + 1/2\varrho))
• Reduced characteristic vector 29: \((-1/2 + \varrho, 0, -1/2 + \varrho, 1/2\varrho, -1/2 + 3/2\varrho, 1/2\varrho, 1/2 + 1/2\varrho, 3/2\varrho))
• Reduced characteristic vector 30: \((-1/2 + \varrho, (1 - 3/2\varrho, 1/2 - 1/2\varrho, 1 - \varrho, 1/2, 3/2 - 3/2\varrho, 1 - 1/2\varrho, 3/2 - \varrho))
• Reduced characteristic vector 31: \((1/2 - 1/2\varrho, 0, 1/2 - 1/2\varrho, 1/2\varrho, 1/2, \varrho, 1 - 1/2\varrho, 1/2 + 1/2\varrho))
• Reduced characteristic vector 32: \((-1/2 + \varrho, 0, 1/2 - 1/2\varrho, 1 - \varrho, 1/2, 1 - 1/2\varrho, 1/2 + 1/2\varrho, 3/2 - \varrho))
• Reduced characteristic vector 33: \((-1/2 + \varrho, 0, 1/2 - 1/2\varrho, 1/2\varrho, 1/2, \varrho, 1 - 1/2\varrho, 1/2 + 1/2\varrho))
• Reduced characteristic vector 34: \((-1/2 + \varrho, (1 - 3/2\varrho, 1/2 - 1/2\varrho, 1 - \varrho, 1/2, 1 - 1/2\varrho, 1/2 + 1/2\varrho, 3/2 - \varrho))
• Reduced characteristic vector 35: \((1/2 + \varrho, 0, 1 - 3/2\varrho, 1/2 - 1/2\varrho, 1 - \varrho, 1/2, 3/2 - 1/2\varrho, 3/2 - \varrho))
• Reduced characteristic vector 36: \((-1/2 + \varrho, (0, 1 - 3/2\varrho, 1/2 - 1/2\varrho, 1 - \varrho, 1/2, 3/2 - 1/2\varrho, 3/2 - \varrho))
• Reduced characteristic vector 37: \((-1/2 + \varrho, (0, 1 - 3/2\varrho, 1/2 - 1/2\varrho, 1 - \varrho, 1/2, 3/2 - 3/2\varrho, 1 - 1/2\varrho, 3/2 - \varrho))
• Reduced characteristic vector 38: \((1/2 - 1/2\varrho, (0, -1/2 + \varrho, 1/2 - 1/2\varrho, 1/2\varrho, 1/2, \varrho, 1 - 1/2\varrho, 1/2 + 1/2\varrho))
• Reduced characteristic vector 39: \((-1/2 + \varrho, (0, 1 - 1/2\varrho, 1/2\varrho, 1 - \varrho, 1/2, 1 - 1/2\varrho, 1/2 + 1/2\varrho, 3/2 - \varrho))

The maps are:
• RCV 1 \rightarrow [2, 3, 4, 5, 6]
• RCV 2 \rightarrow [2, 7]
• RCV 3 \rightarrow [8, 9, 10]
• RCV 4 \rightarrow [11, 12, 13, 14]
• RCV 5 \rightarrow [15, 16, 17]
• RCV 6 \rightarrow [18, 6]
• RCV 7 \rightarrow [8, 9]
• RCV 8 \rightarrow [10]
• RCV 9 \rightarrow [11, 12]
• RCV 10 \rightarrow [19, 20, 21]
• RCV 11 \rightarrow [22]
• RCV 12 \rightarrow [23, 24, 25]
• RCV 13 \rightarrow [19, 20, 21]
• RCV 14 \rightarrow [22]
• RCV 15 \rightarrow [23, 24, 25]
• RCV 16 \rightarrow [13, 14]
• RCV 17 \rightarrow [15]
• RCV 18 \rightarrow [16, 17]
• RCV 19 \rightarrow [19, 20]
• RCV 20 \rightarrow [26]
• RCV 21 \rightarrow [27]
See Figure S2 for the transition diagram. This has (normalized) transition matrices:

\[
T(1, 2) = \begin{bmatrix} 1 \end{bmatrix} \quad T(1, 3) = \begin{bmatrix} 2 & 1 \end{bmatrix} \\
T(1, 4) = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \quad T(1, 5) = \begin{bmatrix} 1 & 2 \end{bmatrix} \\
T(1, 6) = \begin{bmatrix} 1 \end{bmatrix} \quad T(2, 2) = \begin{bmatrix} 1 \end{bmatrix} \\
T(2, 7) = \begin{bmatrix} 2 & 1 \end{bmatrix} \quad T(3, 8) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix} \\
T(3, 9) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \end{bmatrix} \quad T(3, 10) = \begin{bmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 \end{bmatrix} \\
T(4, 11) = \begin{bmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad T(4, 12) = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 \end{bmatrix} \\
T(4, 13) = \begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 \end{bmatrix} \quad T(4, 14) = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
T(5, 15) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad T(5, 16) = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\
T(5, 17) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad T(6, 18) = \begin{bmatrix} 1 & 2 \end{bmatrix} \\
T(6, 6) = \begin{bmatrix} 1 \end{bmatrix} \quad T(7, 8) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix} 
\]
Figure S2. $x^2 + x - 1$ with $d_i \in [0, 1 - \varphi, 2 - 2\varphi]$, Full set + Essential class
\[ T(7, 9) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 \end{bmatrix} \]
\[ T(8, 10) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 1 \end{bmatrix} \]
\[ T(9, 11) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 \end{bmatrix} \]
\[ T(9, 12) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 \end{bmatrix} \]
\[ T(10, 19) = \begin{bmatrix} 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \]
\[ T(10, 20) = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \]
\[ T(10, 21) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \]
\[ T(10, 22) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \]
\[ T(12, 23) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \]
\[ T(12, 24) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \]
\[ T(12, 25) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \]
\[
\begin{align*}
T(13, 19) &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix} \\
T(13, 20) &= \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 \\
1 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix} \\
T(13, 21) &= \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 \\
1 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix} \\
T(14, 22) &= \begin{bmatrix}
2 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 1 & 0 \\
0 & 1 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix} \\
T(15, 23) &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 2 & 0 & 1 \\
0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix} \\
T(15, 24) &= \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix} \\
T(15, 25) &= \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix} \\
T(16, 13) &= \begin{bmatrix}
2 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix} \\
T(16, 14) &= \begin{bmatrix}
2 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 2 & 0 \\
0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix} \\
T(17, 15) &= \begin{bmatrix}
1 & 2 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix} \\
T(18, 16) &= \begin{bmatrix}
1 & 2 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\end{bmatrix}
\end{align*}

\[
T(18, 17) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad T(19, 19) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}
\]

\[
T(19, 20) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}
\]

\[
T(20, 26) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}
\]

\[
T(21, 27) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\]

\[
T(22, 28) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}
\]
\[ T(22,29) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix} \]

\[ T(22,30) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} \]

\[ T(23,31) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 2 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix} \]

\[ T(24,32) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix} \]

\[ T(25,24) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix} \]
\[
T(25, 25) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
T(26, 27) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
T(27, 28) = \begin{bmatrix}
0 & 2 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 2 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

\[
T(27, 29) = \begin{bmatrix}
1 & 0 & 2 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 2 \\
0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

\[
T(27, 30) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 \\
1 & 0 & 2 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]
\[
T(28, 33) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 2 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

\[
T(28, 34) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

\[
T(29, 35) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

\[
T(30, 36) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

\[
T(30, 37) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]
\[
T(31, 28) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

\[
T(31, 29) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

\[
T(31, 30) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

\[
T(32, 31) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 2 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

\[
T(33, 28) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]
\[ T(33, 29) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \\
\end{bmatrix} \]

\[ T(34, 38) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix} \]

\[ T(35, 39) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 2 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix} \]

\[ T(36, 40) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix} \]
\[ T(37, 29) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 
\end{bmatrix} \]

\[ T(37, 30) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 
\end{bmatrix} \]

\[ T(38, 36) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 2 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 
\end{bmatrix} \]

\[ T(38, 37) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 2 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 
\end{bmatrix} \]
\[
T(39, 28) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

\[
T(39, 29) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

\[
T(39, 30) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

\[
T(40, 33) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}
\]
The essential class is: [28, 29, 30, 33, 34, 35, 36, 37, 38, 39, 40]. The essential class is of positive type. An example is the path [28, 34, 38, 36, 40, 34, 38, 37]. The essential class is not a simple loop. This spectral range will include the interval [2.469158042, 2.481194304]. The minimum comes from the loop [28, 33, 28]. The maximum comes from the loop [28, 33, 28]. These points will include points of local dimension [.992399434, 1.002504754]. The Spectral Range is contained in the range [2.038390910, 2.701372314]. The minimum comes from the column sub-norm on the subset 3, 4 of length 20. The maximum comes from the total column sup-norm of length 10. These points will have local dimension contained in [.815720713, 1.400908289].

There are 4 additional maximal loops.

Maximal Loop Class: [25]. The maximal loop class is a simple loop. It’s spectral radius is an isolated points of 2.481194304. These points have local dimension .992399434.

Maximal Loop Class: [19]. The maximal loop class is a simple loop. It’s spectral radius is an isolated points of 2.481194304. These points have local dimension .992399434.

Maximal Loop Class: [6]. The maximal loop class is a simple loop. It’s spectral radius is an isolated points of 1. These points have local dimension 2.880840181.

Maximal Loop Class: [2]. The maximal loop class is a simple loop. It’s spectral radius is an isolated points of 1. These points have local dimension 2.880840181.

The set of local dimensions will contain
\[[.9924, 1.003] \cup \{2.881\}\]

This has 2 components.

The set of local dimensions is contained in
\[[.8157, 1.401] \cup \{2.881\}\]

This has 2 components.

S3. Details for Example 3.9 – Part 2

Consider \(\varrho\), the root of \(x^2 + x - 1\) and the maps \(S_i(x) = \varrho x + d_i\) with \(d_0 = 0, d_1 = 1 - \varrho,\) and \(d_2 = 2 - 2\varrho\). This measure is of full support on [0, 2]. The probabilities are given by \(p_0 = 1/4, p_1 = 1/2,\) and \(p_2 = 1/4\). We now give detailed information for the quotient measure. The reduced transition diagram has 38 reduced characteristic vectors. The reduced characteristic vectors are:

- Reduced characteristic vector 1: \((1, (0, 1))\)
• Reduced characteristic vector 2: \((1 - \varrho, (0, 1 - \varrho, 1, \varrho + 1))\)
• Reduced characteristic vector 3: \((-1 + 2\varrho, (1 - \varrho, 2 - 2\varrho, 2 - \varrho))\)
• Reduced characteristic vector 4: \((1 - \varrho, (0, \varrho, 1, \varrho + 1))\)
• Reduced characteristic vector 5: \((-1 + 2\varrho, (1 - \varrho, 1, 2 - \varrho))\)
• Reduced characteristic vector 6: \((1 - \varrho, (0, \varrho, 2\varrho, \varrho + 1))\)
• Reduced characteristic vector 7: \((1 - \varrho, (0, 1 - \varrho, \varrho, 1, 2 - \varrho, \varrho + 1))\)
• Reduced characteristic vector 8: \((-1 + 2\varrho, (1 - \varrho, 2 - 2\varrho, 1, 2 - \varrho, 3 - 2\varrho))\)
• Reduced characteristic vector 9: \((1 - \varrho, (0, \varrho, 1, 2\varrho, \varrho + 1))\)
• Reduced characteristic vector 10: \((-1 + 2\varrho, (0, 1 - \varrho, 1, 2 - \varrho, \varrho + 1))\)
• Reduced characteristic vector 11: \((-3\varrho + 2, (0, -1 + 2\varrho, \varrho, 2\varrho, \varrho + 1, 3\varrho))\)
• Reduced characteristic vector 12: \((-1 + 2\varrho, (-3\varrho + 2, 1 - \varrho, 2 - 2\varrho, 2 - \varrho, 3 - 2\varrho))\)
• Reduced characteristic vector 13: \((1 - \varrho, (0, 1 - \varrho, \varrho, 1, 2\varrho, \varrho + 1))\)
• Reduced characteristic vector 14: \((-1 + 2\varrho, (0, 1 - \varrho, 2 - 2\varrho, 1, 2 - \varrho, \varrho + 1))\)
• Reduced characteristic vector 15: \((1 - \varrho, (0, -1 + 2\varrho, \varrho, 1, 2\varrho, \varrho + 1))\)
• Reduced characteristic vector 16: \((-1 + 2\varrho, (0, 1 - \varrho, \varrho, 1, 2 - \varrho, \varrho + 1))\)
• Reduced characteristic vector 17: \((-3\varrho + 2, (0, -1 + 2\varrho, \varrho, -1 + 3\varrho, 2\varrho, \varrho + 1, 3\varrho))\)
• Reduced characteristic vector 18: \((-1 + 2\varrho, (-3\varrho + 2, 1 - \varrho, 2 - 2\varrho, 1, 2 - \varrho, 3 - 2\varrho))\)
• Reduced characteristic vector 19: \((1 - \varrho, (0, 1 - \varrho, \varrho, 1, 2\varrho, \varrho + 1))\)
• Reduced characteristic vector 20: \((-1 + 2\varrho, (0, 1 - \varrho, 2 - 2\varrho, 1, 2 - \varrho, \varrho + 1))\)
• Reduced characteristic vector 21: \((-3\varrho + 2, (0, -1 + 2\varrho, \varrho, 1, 2\varrho, \varrho + 1, 3\varrho))\)
• Reduced characteristic vector 22: \((-1 + 2\varrho, (-3\varrho + 2, 1 - \varrho, 2 - 2\varrho, 3 - 3\varrho, 2 - \varrho, 3 - 2\varrho))\)
• Reduced characteristic vector 23: \((-1 + 2\varrho, (0, 1 - \varrho, 2 - 2\varrho, 1, 2 - \varrho, 3 - 2\varrho))\)
• Reduced characteristic vector 24: \((-1 + 2\varrho, (0, -3\varrho + 2, 1 - \varrho, 2 - 2\varrho, 1, 2 - \varrho, 3 - 2\varrho))\)
• Reduced characteristic vector 25: \((1 - \varrho, (0, -1 + 2\varrho, 1 - \varrho, \varrho, 1, 2\varrho, \varrho + 1))\)
• Reduced characteristic vector 26: \((-1 + 2\varrho, (0, 1 - \varrho, 2 - 2\varrho, 1, 2 - \varrho, \varrho + 1))\)
• Reduced characteristic vector 27: \((-3\varrho + 2, (0, -1 + 2\varrho, \varrho, -1 + 3\varrho, 1, 2\varrho, \varrho + 1, 3\varrho))\)
• Reduced characteristic vector 28: \((-1 + 2\varrho, (-3\varrho + 2, 1 - \varrho, 2 - 2\varrho, 1, 3 - 3\varrho, 2 - \varrho, 3 - 2\varrho))\)
• Reduced characteristic vector 29: \((1 - \varrho, (0, 1 - \varrho, \varrho, 1, 2\varrho, 2 - \varrho, \varrho + 1))\)
• Reduced characteristic vector 30: \((-1 + 2\varrho, (0, 1 - \varrho, 2 - 2\varrho, 1, 2 - \varrho, \varrho + 1, 3 - 2\varrho))\)
• Reduced characteristic vector 31: \((-1 + 2\varrho, (0, 1 - \varrho, \varrho, 1, 2\varrho, 2 - \varrho, \varrho + 1))\)
• Reduced characteristic vector 32: \((-3\varrho + 2, (0, -1 + 2\varrho, \varrho, -1 + 3\varrho, 2\varrho, -1 + 4\varrho, \varrho + 1, 3\varrho))\)
• Reduced characteristic vector 33: \((-1 + 2\varrho, (0, -3\varrho + 2, 1 - \varrho, 2 - 2\varrho, 1, 2 - \varrho, \varrho + 1, 3 - 2\varrho))\)
• Reduced characteristic vector 34: \((-3\varrho + 2, (0, -1 + 2\varrho, 1 - \varrho, \varrho, 1, 2\varrho, \varrho + 1, 3\varrho))\)
• Reduced characteristic vector 35: \((-1 + 2\varrho, (-3\varrho + 2, 1 - \varrho, 3 - 4\varrho, 2 - 2\varrho, 3 - 3\varrho, 2 - \varrho, 3 - 2\varrho))\)
• Reduced characteristic vector 36: \((-1 + 2\varrho, (0, -3\varrho + 2, 1 - \varrho, 2 - 2\varrho, 1, 3 - 3\varrho, 2 - \varrho, 3 - 2\varrho))\)
• Reduced characteristic vector 37: $(1 - \varrho, (0, -1+2\varrho, 1-\varrho, \varrho, 1, 2\varrho, 2-\varrho, \varrho+1))$
• Reduced characteristic vector 38: $(-1 + 2\varrho, (0, 1 - \varrho, \varrho, 2 - 2\varrho, 1, 2 - \varrho, \varrho + 1, 3 - 2\varrho))$

The maps are:

• RCV 1 $\rightarrow [2, 3, 4, 5, 6]$
• RCV 2 $\rightarrow [7, 8]$
• RCV 3 $\rightarrow [9]$
• RCV 4 $\rightarrow [10, 11, 12]$
• RCV 5 $\rightarrow [13]$
• RCV 6 $\rightarrow [14, 15]$
• RCV 7 $\rightarrow [16, 17, 18]$
• RCV 8 $\rightarrow [19]$
• RCV 9 $\rightarrow [20, 21, 22]$
• RCV 10 $\rightarrow [7]$
• RCV 11 $\rightarrow [23]$
• RCV 12 $\rightarrow [15]$
• RCV 13 $\rightarrow [16, 17, 18]$
• RCV 14 $\rightarrow [19]$
• RCV 15 $\rightarrow [20, 21, 22]$
• RCV 16 $\rightarrow [16, 17]$
• RCV 17 $\rightarrow [24]$
• RCV 18 $\rightarrow [25]$
• RCV 19 $\rightarrow [26, 27, 28]$
• RCV 20 $\rightarrow [29]$
• RCV 21 $\rightarrow [30]$
• RCV 22 $\rightarrow [21, 22]$
• RCV 23 $\rightarrow [19]$
• RCV 24 $\rightarrow [25]$
• RCV 25 $\rightarrow [26, 27, 28]$
• RCV 26 $\rightarrow [31, 32]$
• RCV 27 $\rightarrow [33]$
• RCV 28 $\rightarrow [34, 35]$
• RCV 29 $\rightarrow [26, 27, 28]$
• RCV 30 $\rightarrow [29]$
• RCV 31 $\rightarrow [26, 27]$
• RCV 32 $\rightarrow [36]$
• RCV 33 $\rightarrow [37]$
• RCV 34 $\rightarrow [38]$
• RCV 35 $\rightarrow [27, 28]$
• RCV 36 $\rightarrow [34, 35]$
• RCV 37 $\rightarrow [26, 27, 28]$
• RCV 38 $\rightarrow [31, 32]$

See Figure S3 for the transition diagram. This has (normalized) transition matrices:

$$T(1, 2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix} \quad T(1, 3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$
Figure S3. $x^2 + x - 1$ with $d_i \in [0, 1 - \varrho, 2 - 2\varrho]$, Full set + Essential class.
| $T(1,4)$ | $T(1,5)$ |
|---------|---------|
| \[
\begin{bmatrix}
2 & 1 & 0 & 0 \\
0 & 0 & 1 & 2
\end{bmatrix}
\] | \[
\begin{bmatrix}
2 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\] |
| $T(1,6)$ | $T(2,7)$ |
|---------|---------|
| \[
\begin{bmatrix}
1 & 2 & 1 & 0 \\
0 & 0 & 1 
\end{bmatrix}
\] | \[
\begin{bmatrix}
2 & 1 & 0 & 0 \\
0 & 1 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}
\] |
| $T(2,8)$ | $T(3,9)$ |
|---------|---------|
| \[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\] | \[
\begin{bmatrix}
1 & 2 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 2
\end{bmatrix}
\] |
| $T(4,10)$ | $T(4,11)$ |
|---------|---------|
| \[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 2 & 1 & 0 & 0 \\
0 & 1 & 2 & 0 & 1 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}
\] | \[
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 2 & 1 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\] |
| $T(4,12)$ | $T(5,13)$ |
|---------|---------|
| \[
\begin{bmatrix}
1 & 0 & 2 & 1 & 0 \\
0 & 0 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\] | \[
\begin{bmatrix}
2 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 2 & 1 \\
0 & 0 & 0 & 1 & 2
\end{bmatrix}
\] |
| $T(6,14)$ | $T(6,15)$ |
|---------|---------|
| \[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\] | \[
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\] |
| $T(7,16)$ | $T(7,17)$ |
|---------|---------|
| \[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\] | \[
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 \\
1 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\] |
\[
T(7, 18) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 \\
1 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 1 & 0
\end{bmatrix}
\]

\[
T(8, 19) = \begin{bmatrix}
0 & 1 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

\[
T(9, 20) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

\[
T(9, 21) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

\[
T(9, 22) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

\[
T(10, 7) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

\[
T(11, 23) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

\[
T(12, 15) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 \\
2 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
\]
\[
\begin{align*}
T(13, 16) &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} & T(13, 17) &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \\
T(13, 18) &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} & T(14, 19) &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \\
T(15, 20) &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \\
T(15, 21) &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
T(15, 22) &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
T(16, 16) &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\end{align*}
\]
\[ T(16, 17) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix} \]

\[ T(17, 24) = \begin{bmatrix}
0 & 1 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix} \]

\[ T(18, 25) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix} \]

\[ T(19, 26) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix} \]

\[ T(19, 27) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix} \]
$T(19, 28) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$T(20, 29) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$T(21, 30) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$T(22, 21) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$T(22, 22) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$
$$T(23,19) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 1 & 0 \\
0 & 1 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

$$T(24,25) = \begin{bmatrix}
1 & 0 & 0 & 2 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}$$

$$T(25,26) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1
\end{bmatrix}$$

$$T(25,27) = \begin{bmatrix}
1 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1
\end{bmatrix}$$

$$T(25,28) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$
\[
T(26,31) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

\[
T(26,32) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

\[
T(27,33) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
T(28,34) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\end{bmatrix}
\]

\[
T(28,35) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]
\[
T(29,26) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]
\[
T(29,27) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]
\[
T(29,28) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]
\[
T(30,29) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 2 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]
\[
T(31,26) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]
\[
T(31, 27) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

\[
T(32, 36) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 2 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
T(33, 37) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 2 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

\[
T(34, 38) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]
\[
T(35, 27) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
T(35, 28) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
T(36, 34) = \begin{bmatrix}
1 & 0 & 0 & 2 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 2 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

\[
T(36, 35) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 2 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]
\[ T(37,26) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ T(37,27) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \]

\[ T(37,28) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ T(38,31) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]
The essential class is: [26, 27, 28, 31, 32, 33, 34, 35, 36, 37, 38]. The essential class is of positive type. An example is the path [26, 32, 36, 34, 38, 32, 36, 35]. The essential class is not a simple loop. This spectral range will include the interval [2.469158042, 2.481194304]. The minimum comes from the loop [27, 33, 37, 27]. The maximin comes from the loop [26, 31, 26]. These points will include points of local dimension [.992399434, 1.002504754]. The Spectral Range is contained in the range [2.038390910, 2.701372314]. The minimum comes from the column sub-norm on the subset 3, 4 of length 20. The maximum comes from the total column sup-norm of length 10. These points will have local dimension contained in [.815720713, 1.400908289].

There are 2 additional maximal loops.

Maximal Loop Class: [22]. The maximal loop class is a simple loop. It’s spectral radius is an isolated points of 2.481194304. These points have local dimension .992399434.

Maximal Loop Class: [16]. The maximal loop class is a simple loop. It’s spectral radius is an isolated points of 2.481194304. These points have local dimension .992399434.

The set of local dimensions will contain

[.9924, 1.003]

This has 1 components.

The set of local dimensions is contained in

[.8157, 1.401]

This has 1 components.

S4. Details for Example 3.10

Consider the measure given by the maps $S_i(x) = x/4 + d_i$ with $d_0 = 0, d_1 = 1/8, d_2 = 1/4, d_3 = 3/8, d_4 = 1/2, d_5 = 7/8, d_6 = 9/8,$ and $d_7 = 3/2$. The probabilities are given by $p_0 = 1/2402, p_1 = 500/1201, p_2 = 500/1201, p_3 = 50/1201, p_4 = 50/1201, p_5 = 50/1201, p_6 = 50/1201$, and $p_7 = 1/2402$. We consider the quotient measure. The reduced transition diagram has 10 reduced characteristic vectors. The reduced characteristic vectors are:

- Reduced characteristic vector 1: (2, 0, 2))
- Reduced characteristic vector 2: (1, 0, 1))
- Reduced characteristic vector 3: (1, 0, 1, 2))
- Reduced characteristic vector 4: (1, 0, 1, 2, 3))
Figure S4. $4x - 1$ with $d_i \in \left[0, 1/8, 1/4, 3/8, 1/2, 7/8, 9/8, 3/2\right]$, Full set + Essential class

- Reduced characteristic vector 5: $(1, (1, 2, 3))$
- Reduced characteristic vector 6: $(1, (2, 3))$
- Reduced characteristic vector 7: $(1, (0, 3))$
- Reduced characteristic vector 8: $(1, (0, 2, 3))$
- Reduced characteristic vector 9: $(1, (0, 1, 3))$
- Reduced characteristic vector 10: $(2, (0, 1, 2))$

The maps are:

- $RCV_1 \rightarrow [2, 3, 4, 4, 4, 5, 6, 7]$
- $RCV_2 \rightarrow [4, 4, 4, 4]$
- $RCV_3 \rightarrow [4, 4, 4, 4]$
- $RCV_4 \rightarrow [4, 4, 4, 4]$
- $RCV_5 \rightarrow [4, 4, 5, 8]$
- $RCV_6 \rightarrow [9, 10, 6]$
- $RCV_7 \rightarrow [7, 2, 3, 4]$
- $RCV_8 \rightarrow [9, 3, 4, 4]$
- $RCV_9 \rightarrow [4, 4, 4, 4]$
- $RCV_{10} \rightarrow [4, 4, 4, 4, 4, 5, 8]$

See Figure S4 for the transition diagram. This has (normalized) transition matrices:

\[
T(1, 2) = \begin{bmatrix} 1 & 0 \\ 0 & 100 \end{bmatrix} \quad T(1, 3) = \begin{bmatrix} 1000 & 1 & 0 \\ 100 & 0 & 100 \end{bmatrix}
\]
\[
\begin{array}{ll}
T(1, 4) &= \begin{bmatrix}
1000 & 1000 & 1 & 0 \\
0 & 100 & 0 & 100 \\
\end{bmatrix} & T(1, 4) &= \begin{bmatrix}
1000 & 1000 & 1 \\
0 & 0 & 100 & 0 \\
\end{bmatrix} \\
T(1, 4) &= \begin{bmatrix}
100 & 100 & 1000 & 1000 \\
1 & 0 & 0 & 100 \\
\end{bmatrix} \\
T(1, 5) &= \begin{bmatrix}
100 & 1000 & 1000 \\
1 & 0 & 0 \\
\end{bmatrix} \\
T(1, 6) &= \begin{bmatrix}
100 & 100 \\
1 & 0 \\
\end{bmatrix} \\
T(1, 7) &= \begin{bmatrix}
100 & 100 \\
0 & 1 \\
\end{bmatrix} \\
T(2, 4) &= \begin{bmatrix}
1 & 0 & 0 & 0 \\
100 & 100 & 1000 & 1000 \\
\end{bmatrix} \\
T(2, 4) &= \begin{bmatrix}
1000 & 100 & 1000 \\
0 & 100 & 0 \\
\end{bmatrix} \\
T(2, 4) &= \begin{bmatrix}
100 & 100 & 1000 \\
0 & 100 & 0 \\
\end{bmatrix} \\
T(3, 4) &= \begin{bmatrix}
1000 & 1000 & 1000 \\
0 & 100 & 0 \\
\end{bmatrix} \\
T(3, 4) &= \begin{bmatrix}
1000 & 1000 & 1000 \\
0 & 100 & 0 \\
\end{bmatrix} \\
T(3, 4) &= \begin{bmatrix}
1000 & 1000 & 1 \\
0 & 100 & 0 \\
\end{bmatrix} \\
T(3, 4) &= \begin{bmatrix}
1000 & 1000 & 1 \\
0 & 100 & 0 \\
\end{bmatrix} \\
T(4, 4) &= \begin{bmatrix}
1000 & 1000 & 1000 \\
0 & 100 & 0 \\
\end{bmatrix} \\
T(4, 4) &= \begin{bmatrix}
1000 & 1000 & 1000 \\
0 & 100 & 0 \\
\end{bmatrix} \\
T(4, 4) &= \begin{bmatrix}
1000 & 1000 & 1000 \\
0 & 100 & 0 \\
\end{bmatrix} \\
T(4, 4) &= \begin{bmatrix}
1000 & 1000 & 1000 \\
0 & 100 & 0 \\
\end{bmatrix} \\
T(5, 4) &= \begin{bmatrix}
1000 & 1000 & 1000 \\
0 & 100 & 0 \\
\end{bmatrix} \\
T(5, 4) &= \begin{bmatrix}
1000 & 1000 & 1000 \\
0 & 100 & 0 \\
\end{bmatrix} \\
T(5, 4) &= \begin{bmatrix}
1000 & 1000 & 1000 \\
0 & 100 & 0 \\
\end{bmatrix} \\
T(5, 5) &= \begin{bmatrix}
100 & 0 & 100 \\
0 & 1 & 0 \\
\end{bmatrix} \\
T(5, 5) &= \begin{bmatrix}
100 & 0 & 100 \\
0 & 1 & 0 \\
\end{bmatrix} \\
T(5, 8) &= \begin{bmatrix}
100 & 0 & 100 \\
0 & 0 & 1 \\
\end{bmatrix} \\
T(6, 9) &= \begin{bmatrix}
0 & 100 & 0 \\
1 & 0 & 100 \\
\end{bmatrix} \\
T(6, 9) &= \begin{bmatrix}
100 & 100 \\
0 & 1 \\
\end{bmatrix} \\
T(6, 6) &= \begin{bmatrix}
100 & 0 \\
0 & 1 \\
\end{bmatrix} \\
T(7, 7) &= \begin{bmatrix}
1 & 0 \\
1 & 0 \\
\end{bmatrix} \\
\end{array}
\]
The essential class is: [4]. The essential class is of positive type. An example is the path [4, 4, 4, 4]. The essential class is not a simple loop. This spectral range will include the interval [175.9930871, 1005.242481]. The minimum comes from the loop [4, 4, 4, 4]. The maximum comes from the loop [4, 4, 4, 4, 4]. These points will include points of local dimension [.628346304, 1.885322743]. The Spectral Range is contained in the range [139.5519097, 1025.016616]. The minimum comes from the total row sub-norm of length 5. The maximum comes from the total row sup-norm of length 5. These points will have local dimension contained in [.614294428, 2.052681190].
There are 3 additional maximal loops.

Maximal Loop Class: [7]. The maximal loop class is a simple loop. It’s spectral radius is an isolated points of 100.00. These points have local dimension 2.293082124.

Maximal Loop Class: [6]. The maximal loop class is a simple loop. It’s spectral radius is an isolated points of 100.00. These points have local dimension 2.293082124.

Maximal Loop Class: [5]. The maximal loop class is a simple loop. It’s spectral radius is an isolated points of 100.9901951. These points have local dimension 2.285974508.

The set of local dimensions will contain
\[ \left[ 0.6283, 1.885 \right] \cup \{ 2.286 \} \cup \{ 2.293 \} \]
This has 3 components.

The set of local dimensions is contained in
\[ \left[ 0.6143, 2.053 \right] \cup \{ 2.286 \} \cup \{ 2.293 \} \]
This has 3 components.

S5. Details for Remark 4.17

Below we have provided details to support our conjecture in Remark 4.17. Let \( \nu \) be the \( m \)-fold convolution of the Cantor measure with ratio of contraction 1/d. We know that:

\[ \{ \dim_{\text{loc}} \nu_\pi(x) : x \in \text{supp } \nu_\pi \} \subseteq \{ \dim_{\text{loc}} \nu(x) : x \in \text{supp } \nu, x \not= 0, d + k \} \]

We wish to show that \( \sup_\pi \dim_{\text{loc}} \nu_\pi(x) < \sup_{x \not= 0, m} \dim_{\text{loc}} \nu(x) \). We do this by showing that the upper bound for \( \sup_{x \not= 0, d+k} \dim_{\text{loc}} \nu(x) \) is bounded below by \( b \) (dependent on \( m \) and \( d \)), \( \sup_{x \not= 0, d+k} \dim_{\text{loc}} \nu(x) \) is bounded above by \( a \) (dependent on \( m \) and \( d \)), and then observe that \( a < b \) in every case except \( m = 2, d = 3 \).
| $m$ | $d$ | Upper bound of $\dim \nu(x)$ bounded below by | Upper bound of $\dim \nu_{\pi}(x)$ bounded above by | Prop 4.16 holds | Depth of Norm Searched |
|-----|-----|---------------------------------------------|---------------------------------------------|----------------|-----------------|
| 2   | 3   | 1.261859507                                | 1.261859507                                | false          | 10              |
| 3   | 3   | 1.133544891                                | 1.077324384                                | true           | 1               |
| 4   | 3   | 1.058745493                                | 1.049820435                                | true           | 2               |
| 5   | 3   | 1.027566600                                | 1.025209036                                | true           | 3               |
| 6   | 3   | 1.014334772                                | 1.011259593                                | true           | 2               |
| 7   | 3   | 1.00657727                                 | 1.005988453                                | true           | 3               |
| 8   | 3   | 1.003425327                                | 1.002515549                                | true           | 3               |
| 9   | 3   | 1.001330543                                | 1.001153367                                | true           | 4               |
| 10  | 3   | 1.000793712                                | 1.000649489                                | true           | 3               |
| 3   | 4   | 1.500000000                                | 1.235839584                                | true           | 1               |
| 4   | 4   | 1.321490682                                | 1.166666667                                | true           | 1               |
| 5   | 4   | 1.207518750                                | 1.084691151                                | true           | 1               |
| 6   | 4   | 1.132742274                                | 1.075965367                                | true           | 2               |
| 7   | 4   | 1.096322539                                | 1.057813726                                | true           | 2               |
| 8   | 4   | 1.060623161                                | 1.035146820                                | true           | 2               |
| 9   | 4   | 1.046554702                                | 1.031111632                                | true           | 2               |
| 10  | 4   | 1.028706359                                | 1.027211789                                | true           | 2               |
| 4   | 5   | 1.722706233                                | 1.292096752                                | true           | 1               |
| 5   | 5   | 1.515580565                                | 1.188044511                                | true           | 1               |
| 6   | 5   | 1.374997393                                | 1.138225274                                | true           | 1               |
| 7   | 5   | 1.273287000                                | 1.162457996                                | true           | 1               |
| 8   | 5   | 1.218846960                                | 1.147466087                                | true           | 1               |
| 9   | 5   | 1.160225830                                | 1.139336787                                | true           | 1               |
| 10  | 5   | 1.134263284                                | 1.089576553                                | true           | 1               |
| 5   | 6   | 1.934264036                                | 1.317946834                                | true           | 1               |
| 6   | 6   | 1.707969651                                | 1.239189644                                | true           | 1               |
| 7   | 6   | 1.547411229                                | 1.167218645                                | true           | 1               |
| 8   | 6   | 1.426500033                                | 1.176448870                                | true           | 1               |
| 9   | 6   | 1.357136478                                | 1.171217752                                | true           | 1               |
| 10  | 6   | 1.281621364                                | 1.179936574                                | true           | 1               |
| 6   | 7   | 2.137243123                                | 1.366428213                                | true           | 1               |
| 7   | 7   | 1.896901635                                | 1.274709522                                | true           | 1               |
| 8   | 7   | 1.720507429                                | 1.212928829                                | true           | 1               |
| 9   | 7   | 1.584821135                                | 1.193520824                                | true           | 1               |
| 10  | 7   | 1.502709990                                | 1.190060439                                | true           | 1               |
| 7   | 8   | 2.333333333                                | 1.403398681                                | true           | 1               |
| 8   | 8   | 2.082091702                                | 1.322408586                                | true           | 1               |
| 9   | 8   | 1.892690635                                | 1.250208526                                | true           | 1               |
| 10  | 8   | 1.745082630                                | 1.215234300                                | true           | 1               |
| 8   | 9   | 2.523719013                                | 1.449978323                                | true           | 1               |
| 9   | 9   | 2.263677476                                | 1.362744592                                | true           | 1               |
| 10  | 9   | 2.06319598                                | 1.295581060                                | true           | 1               |
| 9   | 10  | 2.709269961                                | 1.490378946                                | true           | 1               |
| 10  | 10  | 2.441914915                                | 1.409160390                                | true           | 1               |

Table S1. Table caption here