A short introduction to local fractional complex analysis

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This paper presents a short introduction to local fractional complex analysis. The generalized local fractional complex integral formulas, Yang-Taylor series and local fractional Laurent’s series of complex functions in complex fractal space, and generalized residue theorems are investigated.

Key words: Local fractional calculus, complex-valued functions, fractal, Yang-Taylor series, local fractional Laurent series, generalized residue theorems

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1 Introduction

Local fractional calculus has played an important role in not only mathematics but also in physics and engineers [1-12]. There are many definitions of local fractional derivatives and local fractional integrals (also called fractal calculus). Hereby we write down local fractional derivative, given by [5-7]

\[ f^{(a)}(x_0) = \frac{d^a f(x)}{dx^a} \bigg|_{x=x_0} = \lim_{\Delta x \to 0} \frac{\Delta^a f(x) - f(x_0)}{(x-x_0)^a} \]  

with \( \Delta^a f(x) = \Gamma(1+\alpha) \Delta f(x) - f(x_0) \), and local fractional integral of \( f(x) \), denoted by [5-6,8]

\[ a I^b_{-b} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t) dt = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta x \to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^a \]  

with \( \Delta t_j = t_{j+1} - t_j \) and \( \Delta t = \max\{\Delta t_1, \Delta t_2, \Delta t_j, \ldots\} \), where for \( j = 0, \ldots, N-1 \), \([t_j, t_{j+1}]\) is a partition of the interval \([a, b]\) and \( t_0 = a, t_N = b \).

More recently, a motivation of local fractional derivative and local fractional integral of complex functions is given [11]. Our attempt, in the present paper, is to continue to study local fractional calculus of complex function. As well, a short outline of local fractional complex analysis will be established.
2 Local fractional calculus of the complex-variable functions

In this section we deduce fundamentals of local fractional calculus of the complex-valued functions. Here we start with local fractional continuity of complex functions.

2.1 Local fractional continuity of complex-variable functions

Definition 1
Given \( z_0 \) and \( |z - z_0| < \delta \), then for any \( z \) we have [11]

\[
|f(z) - f(z_0)| < \varepsilon^\alpha. \tag{2.1}
\]

Here complex function \( f(z) \) is called local fractional continuous at \( z = z_0 \), denoted by

\[
\lim_{z \to z_0} f(z) = f(z_0). \tag{2.2}
\]

A function \( f(z) \) is called local fractional continuous on the region \( \mathfrak{R} \), denoted by

\[
f(z) \in C_\alpha (\mathfrak{R}).
\]

As a direct result, we have the following results:

Suppose that \( \lim_{z \to z_0} f(z) = f(z_0) \) and \( \lim_{z \to z_0} g(z) = g(z_0) \), then we have that

\[
\lim_{z \to z_0} [f(z) \pm g(z)] = f(z_0) \pm g(z_0), \tag{2.3}
\]

\[
\lim_{z \to z_0} [f(z) g(z)] = f(z_0) g(z_0), \tag{2.4}
\]

and

\[
\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f(z_0)}{g(z_0)}, \tag{2.5}
\]

the last only if \( g(z_0) \neq 0 \).

2.2 Local fractional derivatives of complex function

Definition 2
Let the complex function \( f(z) \) be defined in a neighborhood of a point \( z_0 \). The local fractional derivative of \( f(z) \) at \( z_0 \) is defined by the expression [11]

\[
z_0 D_z^\alpha f(z) = \lim_{z \to z_0} \frac{\Gamma(1 + \alpha)[f(z) - f(z_0)]}{(z - z_0)^\alpha}, 0 < \alpha \leq 1. \tag{2.6}
\]

If this limit exists, then the function \( f(z) \) is called to be local fractional analytic at \( z_0 \), denoted by

\[
z_0 D_z^\alpha f(z), \quad \frac{d^\alpha}{dz^\alpha} f(z)\bigg|_{z=z_0} \text{ or } f^{(\alpha)}(z_0).
\]
Remark 1. If the limits exist for all \( z_0 \in \mathbb{R} \), then \( f(z) \) is said to be local fractional analytic in a region \( \mathbb{R} \), denoted by

\[
f(z) \in D(\mathbb{R})
\]

Suppose that \( f(z) \) and \( g(z) \) are local fractional analytic functions, the following rules are valid [11].

\[
\frac{d^\alpha (f(z) \pm g(z))}{dz^\alpha} = \frac{d^\alpha f(z)}{dz^\alpha} \pm \frac{d^\alpha g(z)}{dz^\alpha};
\]

(2.7)

\[
\frac{d^\alpha (f(z)g(z))}{dz^\alpha} = g(z)\frac{d^\alpha f(z)}{dz^\alpha} + f(z)\frac{d^\alpha g(z)}{dz^\alpha};
\]

(2.8)

\[
\frac{d^\alpha \left( \frac{f(z)}{g(z)} \right)}{dz^\alpha} = \frac{g(z)\frac{d^\alpha f(z)}{dz^\alpha} + f(z)\frac{d^\alpha g(z)}{dz^\alpha}}{g(z)^2}
\]

(2.9)

if \( g(z) \neq 0 \);

\[
\frac{d^\alpha (Cf(z))}{dz^\alpha} = C \frac{d^\alpha f(z)}{dz^\alpha},
\]

(2.10)

where \( C \) is a constant;

If \( y(z) = (f \circ u)(z) \) where \( u(z) = g(z) \), then

\[
\frac{d^\alpha y(z)}{dz^\alpha} = f^{(\alpha)}(g(z))(g^{(\alpha)}(z))^{\alpha}.
\]

(2.11)

### 2.3 Local fractional Cauchy-Riemann equations

**Definition 3**

If there exists a function

\[
f(z) = u(x,y) + i^\alpha v(x,y),
\]

(2.12)

where \( u \) and \( v \) are real functions of \( x \) and \( y \). The local fractional complex differential equations

\[
\frac{\partial^\alpha u(x,y)}{\partial x^\alpha} - \frac{\partial^\alpha v(x,y)}{\partial y^\alpha} = 0
\]

(2.13)

and

\[
\frac{\partial^\alpha u(x,y)}{\partial y^\alpha} + \frac{\partial^\alpha v(x,y)}{\partial x^\alpha} = 0
\]

(2.14)

are called local fractional Cauchy-Riemann Equations.

**Theorem 1**

Suppose that the function

\[
f(z) = u(x,y) + i^\alpha v(x,y)
\]

(2.15)
is local fractional analytic in a region $\mathcal{R}$. Then we have

$$\frac{\partial^\alpha u(x,y)}{\partial x^\alpha} - \frac{\partial^\alpha v(x,y)}{\partial y^\alpha} = 0$$

(2.16)

and

$$\frac{\partial^\alpha u(x,y)}{\partial y^\alpha} + \frac{\partial^\alpha v(x,y)}{\partial x^\alpha} = 0.$$  

(2.17)

**Proof.** Since $f(z) = u(x,y) + i^\alpha v(x,y)$, we have the following identity

$$f^{(\alpha)}(z_0) = \lim_{z \to z_0} \Gamma(1 + \alpha) \left[ f(z) - f(z_0) \right]/(z - z_0)^\alpha.$$  

(2.18)

Consequently, the formula (2.18) implies that

$$\lim_{\Delta z \to 0} \frac{\Gamma(1 + \alpha) \left[ f(z + \Delta z) - f(z) \right]}{\Delta z^\alpha}$$

$$= \lim_{\Delta x \to 0 \atop \Delta y \to 0} \frac{\Gamma(1 + \alpha) \left[ u(x + \Delta x, y + \Delta y) - u(x, y) + i^\alpha \left( v(x + \Delta x, y + \Delta y) - v(x, y) \right) \right]}{\Delta x^\alpha + i^\alpha \Delta y^\alpha}.$$  

(2.19)

In a similar manner, setting $\Delta y \to 0$ and taking into account the formula (2.19), we have

$$\left( \Delta y \right)^\alpha \to 0$$

such that

$$f^{(\alpha)}(z_0) = \lim_{\Delta y \to 0} \frac{\Gamma(1 + \alpha) \left[ u(x, y + \Delta y) - u(x, y) + i^\alpha \left( v(x, y + \Delta y) - v(x, y) \right) \right]}{i^\alpha \Delta y^\alpha}.$$  

(2.20)

Hence

$$f^{(\alpha)}(z_0) = -i^\alpha \frac{\partial^\alpha u(x,y)}{\partial y^\alpha} + \frac{\partial^\alpha v(x,y)}{\partial x^\alpha}.$$  

(2.21)

If $\Delta x \to 0$, from (2.19) we have $\left( \Delta x \right)^\alpha \to 0$ such that

$$f^{(\alpha)}(z_0) = \lim_{\Delta x \to 0} \frac{\Gamma(1 + \alpha) \left[ u(x + \Delta x, y) - u(x, y) + i^\alpha \left( v(x + \Delta x, y) - v(x, y) \right) \right]}{\Delta x^\alpha}.$$  

(2.22)

Thus we get the identity

$$f^{(\alpha)}(z_0) = \frac{\partial^\alpha u(x,y)}{\partial x^\alpha} + i^\alpha \frac{\partial^\alpha v(x,y)}{\partial x^\alpha}.$$  

(2.24)

Since $f(z) = u(x,y) + i^\alpha v(x,y)$ is local fractional analytic in a region $\mathcal{R}$, we have the following formula

$$f^{(\alpha)}(z_0) = \frac{\partial^\alpha u(x,y)}{\partial x^\alpha} + i^\alpha \frac{\partial^\alpha v(x,y)}{\partial x^\alpha} = -i^\alpha \frac{\partial^\alpha u(x,y)}{\partial y^\alpha} + \frac{\partial^\alpha v(x,y)}{\partial y^\alpha}.$$  

(2.25)

Hence, from (2.25), we arrive at the following identity
and
\[ \frac{\partial^\alpha u(x,y)}{\partial x^\alpha} - \frac{\partial^\alpha v(x,y)}{\partial y^\alpha} = 0 \]  \hspace{1cm} (2.26)

and
\[ \frac{\partial^\alpha u(x,y)}{\partial y^\alpha} + \frac{\partial^\alpha v(x,y)}{\partial x^\alpha} = 0. \]  \hspace{1cm} (2.27)

This completes the proof of Theorem 1.

**Remark 2.** Local fractional C-R equations are sufficient conditions that \( f(z) \) is local fractional analytic in \( \mathbb{R} \).

The local fractional partial equations
\[ \frac{\partial^{2\alpha} u(x,y)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha} u(x,y)}{\partial y^{2\alpha}} = 0 \]  \hspace{1cm} (2.28)

and
\[ \frac{\partial^{2\alpha} v(x,y)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha} v(x,y)}{\partial y^{2\alpha}} = 0 \]  \hspace{1cm} (2.29)

are called local fractional Laplace equations, denoted by
\[ \nabla^{\alpha} u(x,y) = 0 \]  \hspace{1cm} (2.30)

and
\[ \nabla^{\alpha} v(x,y) = 0, \]  \hspace{1cm} (2.31)

where
\[ \nabla^{\alpha} = \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha}}{\partial y^{2\alpha}} \]  \hspace{1cm} (2.32)

is called local fractional Laplace operator.

**Remark 3.** Suppose that \( \nabla^{\alpha} u(x,y) = 0 \) is a local fractional harmonic function in \( \mathbb{R} \).

### 2.4 Local fractional integrals of complex function

**Definition 4**

Let \( f(z) \) be defined, single-valued and local fractional continuous in a region \( \mathbb{R} \). The local fractional integral of \( f(z) \) along the contour \( C \) in \( \mathbb{R} \) from point \( z_p \) to point \( z_q \), is defined as [11]

\[ I_{C^{\alpha}} f(z) = \frac{1}{\Gamma(1+\alpha)} \lim_{\epsilon \to 0} \sum_{i=0}^{n-1} f(z_i)(\Delta z_i)^{\alpha} \]  \hspace{1cm} (2.33)

where for \( i = 0,1,\ldots,n \), \( \Delta z_i = z_i - z_{i-1} \), \( z_0 = z_p \) and \( z_n = z_q \).
For convenience, we assume that
\[ z_0 I_z^{(a)} f(z) = 0 \]  
(2.34)
if \( z = z_0 \).

The rules for complex integration are similar to those for real integrals. Some important results are as follows [11]:

Suppose that \( f(z) \) and \( g(z) \) be local fractional continuous along the contour \( C \) in \( \mathbb{R} \).

\[
\frac{1}{\Gamma(1+\alpha)} \int_C \left[ (f(z) + g(z))(dz)^\alpha \right] = \frac{1}{\Gamma(1+\alpha)} \int_C f(z)(dz)^\alpha + \frac{1}{\Gamma(1+\alpha)} \int_C g(z)(dz)^\alpha;
\]
\[ (2.35) \]

\[
\frac{1}{\Gamma(1+\alpha)} \int_C kf(z)(dz)^\alpha = \frac{k}{\Gamma(1+\alpha)} \int_C f(z)(dz)^\alpha,
\]
\[ (2.36) \]

for a constant \( k \); \n
\[
\frac{1}{\Gamma(1+\alpha)} \int_C f(z)(dz)^\alpha = \frac{1}{\Gamma(1+\alpha)} \int_{z_1} f(z)(dz)^\alpha + \frac{1}{\Gamma(1+\alpha)} \int_{z_2} f(z)(dz)^\alpha,
\]
\[ (2.37) \]

where \( C = C_1 + C_2 \):

\[
\frac{1}{\Gamma(1+\alpha)} \int_{z_1} f(z)(dz)^\alpha = -\frac{1}{\Gamma(1+\alpha)} \int_{z_2} f(z)(dz)^\alpha;
\]
\[ (2.38) \]

\[
\left| \frac{1}{\Gamma(1+\alpha)} \int_C f(z)(dz)^\alpha \right| \leq \frac{1}{\Gamma(1+\alpha)} \int_C \left| f(z) \right|(dz)^\alpha \leq ML,
\]
\[ (2.39) \]

where \( M \) is an upper bound of \( f(z) \) on \( C \) and \( L = \frac{1}{\Gamma(1+\alpha)} \int_C \left| (dz)^\alpha \right| \).

**Theorem 2**

If the contour \( C \) has end points \( z_p \) and \( z_q \) with orientation \( z_p \) to \( z_q \), and if function \( f(z) \) has the primitive \( F(z) \) on \( C \), then we have

\[
\frac{1}{\Gamma(1+\alpha)} \int_C f(z)(dz)^\alpha = F(z_q) - F(z_p).
\]
\[ (2.40) \]

**Remark 4.** Suppose that \( f(z) \in D(\mathbb{R}) \). For \( k = 0,1,...,n \) and \( 0 < \alpha \leq 1 \) there exists a local fractional series

\[
f(z) = \sum_{n=0}^{\infty} \frac{f^{(n\alpha)}(z_0)}{\Gamma(1+n\alpha)} (z-z_0)^{n\alpha}\]
\[ (2.41) \]

with \( f^{(n\alpha)}(z) \in D(\mathbb{R}) \), where \( f^{(n\alpha)}(z) = D_z^{(n\alpha)} \cdots D_z^{(n\alpha)} f(z) \).

This series is called Yang-Taylor series of local fractional analytic function (for real function case, see [12].)
Theorem 3

If \( C \) is a simple closed contour, and if function \( f(z) \) has a primitive on \( C \), then [11]

\[
\frac{1}{\Gamma(1+\alpha)} \oint_C f(z) \, (dz)^\alpha = 0. \tag{2.42}
\]

Corollary 4

If the closed contours \( C_1, C_2 \) is such that \( C_2 \) lies inside \( C_1 \), and if \( f(z) \) is local fractional analytic on \( C_1, C_2 \) and between them, then we have [11]

\[
\frac{1}{\Gamma(1+\alpha)} \int_{C_1} f(z) \, (dz)^\alpha = \frac{1}{\Gamma(1+\alpha)} \int_{C_2} f(z) \, (dz)^\alpha. \tag{4.43}
\]

Theorem 5

Suppose that the closed contours \( C_1, C_2 \) is such that \( C_2 \) lies inside \( C_1 \), and if \( f(z) \) is local fractional analytic on \( C_1, C_2 \) and between them, then we have [11]

\[
\frac{1}{\Gamma(1+\alpha)} \int_{C_1} f(z) \, (dz)^\alpha = \frac{1}{\Gamma(1+\alpha)} \int_{C_2} f(z) \, (dz)^\alpha. \tag{2.44}
\]

3 Generalized local fractional integral formulas of complex functions

In this section we start with generalized local fractional integral formulas of complex functions and deduce some useful results.

Theorem 6

Suppose that \( f(z) \) is local fractional analytic within and on a simple closed contour \( C \) and \( z_0 \) is any point interior to \( C \). Then we have

\[
\frac{1}{(2\pi)^a} \frac{1}{i^a} \frac{1}{\Gamma(1+\alpha)} \oint_C \frac{f(z)}{(z-z_0)^a} \, (dz)^\alpha = f(z_0). \tag{3.1}
\]

Proof. From (2.44), we arrive at the formula

\[
\frac{1}{(2\pi)^a} \frac{1}{i^a} \frac{1}{\Gamma(1+\alpha)} \oint_C \frac{f(z)}{(z-z_0)^a} \, (dz)^\alpha = \frac{1}{(2\pi)^a} \frac{1}{i^a} \frac{1}{\Gamma(1+\alpha)} \oint_{C_1} \frac{f(z)}{(z-z_0)^a} \, (dz)^\alpha. \tag{3.2}
\]

where \( C_1 : |(z-z_0)^a| = \epsilon^\alpha \).

Setting \( |(z-z_0)^a| = \epsilon^\alpha \) implies that

\[
z^a - z_0^a = \epsilon^\alpha E_\alpha \left(i^a \theta^a\right) \tag{3.3}
\]
Taking (3.3) and (3.4), it follows from (3.2) that
\[
\frac{1}{(2\pi)^\alpha} \cdot \frac{1}{\Gamma(1+\alpha)} \int_0^{2\pi} \frac{f(z_0 + \varepsilon E(i\theta))}{\varepsilon E_\alpha(i \theta^\alpha)} E_\alpha(i \theta^\alpha) (d\theta)^\alpha \cdot \lim_{\varepsilon \to 0} \frac{1}{(2\pi)^\alpha} \cdot \frac{1}{\Gamma(1+\alpha)} \int_0^{2\pi} f(z_0 + \varepsilon E(i\theta)) (d\theta)^\alpha.
\]
From (3.5), we get
\[
\frac{1}{(2\pi)^\alpha} \cdot \frac{1}{\Gamma(1+\alpha)} \int_0^{2\pi} \left( \lim_{\varepsilon \to 0} f(z_0 + \varepsilon E(i\theta)) \right) (d\theta)^\alpha = \frac{f(z_0)}{(2\pi)^\alpha} \cdot \frac{1}{\Gamma(1+\alpha)} \int_0^{2\pi} (d\theta)^\alpha.
\]
Furthermore
\[
f(z_0) = \frac{f(z_0)}{(2\pi)^\alpha} \cdot \frac{1}{\Gamma(1+\alpha)} \int_0^{2\pi} (d\theta)^\alpha = f(z_0).
\]
Substituting (3.7) into (3.6) and (3.3) implies that
\[
\frac{1}{(2\pi)^\alpha} \cdot \frac{1}{\Gamma(1+\alpha)} \int_0^{2\pi} \left( \lim_{\varepsilon \to 0} f(z_0 + \varepsilon E(i\theta)) \right) (d\theta)^\alpha (dz)^\alpha = f(z_0).
\]
The proof of the theorem is completed.
Likewise, we have the following corollary:

**Corollary 7**

Suppose that \( f(z) \) is local fractional analytic within and on a simple closed contour \( C \) and \( z_0 \) is any point interior to \( C \). Then we have
\[
\frac{1}{(2\pi)^\alpha} \cdot \frac{1}{\Gamma(1+\alpha)} \int_0^{2\pi} \left( \lim_{\varepsilon \to 0} f(z_0 + \varepsilon E(i\theta)) \right) (d\theta)^\alpha (dz)^\alpha = f(z_0).
\]

**Proof.** Taking into account formula (3.1), we arrive at the identity.

**Theorem 8**

Suppose that \( f(z) \) is local fractional analytic within and on a simple closed contour \( C \) and \( z_0 \) is any point interior to \( C \). Then we have
\[
\frac{1}{(2\pi)^\alpha} \cdot \frac{1}{\Gamma(1+\alpha)} \int_0^{2\pi} \left( \lim_{\varepsilon \to 0} f(z_0 + \varepsilon E(i\theta)) \right) (d\theta)^\alpha (dz)^\alpha = f(z_0).
\]

**Proof.** Taking \( f(z) = 1 \), from (3.9) we deduce the result.

**Theorem 9**

Suppose that \( f(z) \) is local fractional analytic within and on a simple closed contour \( C \) and \( z_0 \) is any point interior to \( C \). Then we have
\[
\frac{1}{\Gamma(1+\alpha)} \oint_C \frac{(dz)^\alpha}{(z-z_0)^{\alpha n}} = 0, \text{ for } n > 1.
\] (3.10)

Proof. Taking \( f(z) = 1 \), from (3.9) we deduce the result.

4 Complex Yang-Taylor’s series and local fractional Laurent’s series

In this section we start with a Yang-Taylor’s expansion formula of complex functions and deduce local fractional Laurent series of complex functions.

4.1 Complex Yang-Taylor’s expansion formula

Definition 5
Let \( f(z) \) be local fractional analytic inside and on a simple closed contour \( C \) having its center at \( z = z_0 \). Then for all points \( z \) in the circle we have the Yang-Taylor series representation of \( f(z) \), given by

\[
f(z) = f(z_0) + \frac{f^{(1)}(z_0)}{\Gamma(1+\alpha)} (z-z_0)^\alpha + \frac{f^{(2\alpha)}(z_0)}{\Gamma(1+2\alpha)} (z-z_0)^{2\alpha} + \ldots + \frac{f^{(k\alpha)}(z_0)}{\Gamma(1+k\alpha)} (z-z_0)^{k\alpha} + \ldots
\] (4.1)

For \( C : |z - z_0| \leq R^\alpha \), we have the complex Yang-Taylor series

\[
f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^{k\alpha}.
\] (4.2)

From (3.44) the above expression implies

\[
a_k = \frac{1}{(2\pi)^i} \frac{1}{\Gamma(1+\alpha)} \oint_C \frac{f(z)}{(z-z_0)^{k+1}\alpha} (dz)^\alpha = \frac{f^{(k\alpha)}(z_0)}{\Gamma(1+k\alpha)},
\] (4.3)

for \( C : |z - z_0| \leq R^\alpha \).

Successively, it follows from (4.3) that

\[
f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^{k\alpha},
\] (4.4)

where

\[
a_k = \frac{1}{(2\pi)^i} \frac{1}{\Gamma(1+\alpha)} \oint_C \frac{f(z)}{(z-z_0)^{k+1}\alpha} (dz)^\alpha = \frac{f^{(k\alpha)}(z_0)}{\Gamma(1+k\alpha)},
\] (4.5)
for \( C : |z - z_0|^\alpha \leq R^\alpha \).

Hence, the above formula implies the relation (4.2).

**Theorem 10**

Suppose that complex function \( f(z) \) is local fractional analytic inside and on a simple closed contour \( C \) having its center at \( z = z_0 \). There exist all points \( z \) in the circle such that we have the Yang-Taylor’s series of \( f(z) \)

\[
f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^{k\alpha},
\]

where

\[
a_k = \frac{1}{(2\pi)^{1+\alpha}} \cdot \frac{1}{\Gamma(1+\alpha)} \oint_C \frac{f(z)}{z - z_0} (dz)^\alpha = \frac{f^{(k\alpha)}(z_0)}{\Gamma(1+k\alpha)},
\]

for \( C : |z - z_0|^\alpha \leq R^\alpha \).

**Proof.** Setting \( C_1 : |z - z_0|^\alpha = R^\alpha \) and using (3.1), we have

\[
f(z) = \frac{1}{(2\pi)^{1+\alpha}} \cdot \frac{1}{\Gamma(1+\alpha)} \oint_C \frac{f(\xi)}{z - \xi} (d\xi)^\alpha.
\]

Taking \( \xi \in C_1 \), we get

\[
\frac{|z - z_0|^{\alpha}}{|\xi - z_0|^{\alpha}} = q^{\alpha} < 1
\]

and

\[
\frac{1}{(\xi - z)^{\alpha}} = \frac{1}{(\xi - z_0)^{\alpha}} \cdot \frac{1}{1 - (z - z_0)^{\alpha}}
\]

\[
= \frac{1}{(\xi - z_0)^{\alpha}} \cdot \frac{1}{1 - \left(\frac{z - z_0}{\xi - z_0}\right)^{\alpha}}
\]

\[
= \sum_{n=1}^{\infty} \frac{1}{(\xi - z_0)^{n+1}\alpha} (z - z_0)^{n\alpha}.
\]

Substituting (4.8) into (4.6) implies that
Taking the Yang-Taylor formula of analytic function into account, we have the following relation

\[
f(z) = \sum_{n=0}^{N-1} \frac{f^{(n\alpha)}(z_0)(z-z_0)^{n\alpha}}{\Gamma(1+n\alpha)} + R_N,
\]

(4.10)

where \( R_N \) is reminder in the form

\[
R_N = \frac{1}{(2\pi)^a i^a} \cdot \frac{1}{\Gamma(1+\alpha)} \sum_{c_1}^\infty \left[ \frac{f(\xi)(z-z_0)^{n\alpha}}{(\xi-z_0)^{(n+1)\alpha}} \right] (d\xi)^a.
\]

(4.11)

There exists a Yang-Taylor series

\[
f(z) = \sum_{n=0}^{\infty} \frac{f^{(n\alpha)}(z_0)(z-z_0)^{n\alpha}}{\Gamma(1+n\alpha)}
\]

(4.12)

where is \( f(z_0) \) is local fractional analytic at \( z = z_0 \).

Taking into account the relation \( \frac{(z-z_0)^{n\alpha}}{(\xi-z_0)^{(n+1)\alpha}} = q^{\alpha} < 1 \) and \( |f(z)| \leq M \), from (4.11) we get

\[
|R_N| \\
\leq \frac{1}{(2\pi)^a i^a} \cdot \frac{1}{\Gamma(1+\alpha)} \sum_{c_1}^\infty \left[ \frac{f(\xi)(z-z_0)^{n\alpha}}{(\xi-z_0)^{(n+1)\alpha}} \right] (d\xi)^a
\]

(4.13)

\[
\leq \frac{1}{(2\pi)^a \Gamma(1+\alpha)} \sum_{c_1}^\infty \left[ \frac{|M|}{(\xi-z_0)^{\alpha}} \right] (d\xi)^a
\]

\[
\leq \frac{(2\pi)^a R^a}{(2\pi)^a \Gamma(1+\alpha) 1-q^a}
\]

Furthermore

\[
\lim_{N \to \infty} R_N = 0.
\]

From (4.9), we have
\[ f(z) = \sum_{n=1}^{\infty} \left[ \frac{1}{(2\pi)^i} \frac{1}{\Gamma(1+\alpha)} \frac{f(\xi)(d\xi)^\alpha}{\xi^{n+1}} \right] (z-z_0)^{n\alpha}. \]  

(4.14)

Hence

\[ a_n = \frac{1}{(2\pi)^i} \frac{1}{\Gamma(1+\alpha)} \frac{f(\xi)(d\xi)^\alpha}{\xi^{n+1}}. \]  

(4.15)

Hence the proof of the theorem is completed.

4.2 Singular point and poles

Definition 6
A singular point of a function \( f(z) \) is a value of \( z \) at which \( f(z) \) fails to be local fractional analytic. If \( f(z) \) is local fractional analytic everywhere in some region except at an interior point \( z = z_0 \), we call \( f(z) \) an isolated singularity.

If

\[ f(z) = \frac{\phi(z)}{(z-z_0)^{n\alpha}} \]  

(4.16)

and

\[ \phi(z) \neq 0 \]  

(4.17)

where \( \phi(z) \) is local fractional analytic everywhere in a region including \( z = z_0 \), and if \( n \) is a positive integer, then \( f(z) \) has an isolated singularity at \( z = z_0 \), which is called a pole of order \( n \).

If \( n = 1 \), the pole is often called a simple pole;
if \( n = 2 \), it is called a double pole, and so on.

4.3 Local fractional Laurent’s series

Definition 7
If \( f(z) \) has a pole of order \( n \) at \( z = z_0 \) but is local fractional analytic at every other point inside and on a contour \( C \) with center at \( z_0 \), then

\[ \phi(z) = (z-z_0)^{n\alpha} f(z) \]  

(4.18)

is local fractional analytic at all points inside and on \( C \) and has a Yang-Taylor series about \( z = z_0 \) so that
This is called a local fractional Laurent series for \( f(z) \).

More generally, it follows that
\[
f(z) = 
\sum_{k=-\infty}^{\infty} a_k (z - z_0)^{ka},
\]
(4.20)
as a local fractional Laurent series.

For \( C : r^\alpha < |z - z_0|^\alpha < R^\alpha \) we have a local fractional Laurent series
\[
f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^{ka}.
\]
(4.21)

From (3.44), the above expression implies that
\[
a_k = \frac{1}{(2\pi)^{i}\cdot(1+\alpha)} \oint_{C} \frac{f(z)}{(z - z_0)^{(k+1)\alpha}} (dz)^{\alpha},
\]
(4.22)
where \( C : r^\alpha < |z - z_0|^\alpha < R^\alpha \).

Setting \( C_1 : |z - z_0|^\alpha = r^\alpha \) and \( C_2 : |z - z_0|^\alpha = R^\alpha \), from (2.44) we have
\[
f(z) = \frac{1}{(2\pi)^{i}\cdot(1+\alpha)} \oint_{C_1} \frac{f(z)}{(z - z_0)^{(k+1)\alpha}} (dz)^{\alpha} - \frac{1}{(2\pi)^{i}\cdot(1+\alpha)} \oint_{C_2} \frac{f(z)}{(z - z_0)^{(k+1)\alpha}} (dz)^{\alpha}
\]

Successively, it follows from the above that
\[
f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^{ka},
\]
(4.23)
where
\[
a_k = \frac{1}{(2\pi)^{i}\cdot(1+\alpha)} \oint_{C} \frac{f(z)}{(z - z_0)^{(k+1)\alpha}} (dz)^{\alpha},
\]
(4.24)
for \( C : r^\alpha \leq |z - z_0|^\alpha \leq R^\alpha \).

**Theorem 11**

If \( f(z) \) has local fractional analytic at every other point inside a contour \( C \) with center at \( z_0 \),

then \( f(z) \) has a local fractional Laurent series about \( z = z_0 \) so that
\[
f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^{ka}, 0 < \alpha \leq 1,
\]
(4.25)
where for \( C : r^\alpha < |z - z_0|^\alpha < R^\alpha \) we have
\[ a_k = \frac{1}{(2\pi)^a} \cdot \frac{1}{\Gamma(1+\alpha)} \oint_C \frac{f(z)}{(z-z_0)^{\alpha+1}} (dz)^a. \] (4.26)

**Proof.** Setting \( C_1 : |z-z_0|^a = r^a \) and \( C_2 : |z-z_0|^a = R^a \), from (2.44) we have that

\[ f(z) = \frac{1}{(2\pi)^a} \cdot \frac{1}{\Gamma(1+\alpha)} \oint_C \frac{f(\xi)}{(\xi-z_0)^\alpha} (d\xi)^a - \frac{1}{(2\pi)^a} \cdot \frac{1}{\Gamma(1+\alpha)} \oint_C \frac{f(\xi)}{(\xi-z_0)^\alpha} (d\xi)^a. \] (4.27)

Taking the right side of (4.27) into account implies that for \( \xi \in C_2 \)

\[ \left| \frac{(\xi-z_0)^\alpha}{(z-z_0)^\alpha} \right| = \frac{|\xi-z_0|^a}{R^a} = q^a < 1 \] (4.28)

and

\[ |f(\xi)| \leq M. \] (4.29)

By using (4.29) it follows from (4.27) that

\[ \frac{1}{(2\pi)^a} \cdot \frac{1}{\Gamma(1+\alpha)} \oint_C \frac{f(\xi)}{(\xi-z_0)^\alpha} (d\xi)^a = \frac{1}{(2\pi)^a} \cdot \frac{1}{\Gamma(1+\alpha)} \sum_{n=0}^{\infty} \oint_C \frac{f(\xi)}{(\xi-z_0)^{\alpha+n+1}} (d\xi)^a (z-z_0)^{-na}. \] (4.30)

From (4.27) we get

\[ \frac{1}{(2\pi)^a} \cdot \frac{1}{\Gamma(1+\alpha)} \oint_C \frac{f(\xi)}{(\xi-z_0)^\alpha} (d\xi)^a = \frac{1}{(2\pi)^a} \cdot \frac{1}{\Gamma(1+\alpha)} \sum_{n=0}^{\infty} \oint_C \frac{f(\xi)}{(\xi-z_0)^{\alpha+n+1}} (d\xi)^a (z-z_0)^{-na} + R_N \] (4.31)

where

\[ \lim_{N \to \infty} R_N = \lim_{N \to \infty} \frac{1}{(2\pi)^a} \cdot \frac{1}{\Gamma(1+\alpha)} \sum_{n=0}^{\infty} \oint_C \frac{f(\xi)}{(\xi-z_0)^{\alpha+n+1}} (d\xi)^a (z-z_0)^{-na} \]

is reminder.

Since \( |f(\xi)| \leq M \), taking \( \left| \frac{\xi-z_0}{z-z_0} \right|^a = q^a < 1 \), we have

\[ |R_N| \leq \frac{1}{(2\pi)^a} \cdot \frac{1}{\Gamma(1+\alpha)} \sum_{n=0}^{\infty} \left| \oint_C \frac{f(\xi)}{(\xi-z_0)^{\alpha+n+1}} (d\xi)^a \right| \left| \frac{\xi-z_0}{z-z_0} \right|^a. \]
\[ \frac{1}{(2\pi)^\alpha} \frac{1}{\Gamma(1+\alpha)} \sum_{n=-\infty}^{\infty} \left[ \frac{1}{c_i} \left| \frac{1}{(\xi - z_0)^\alpha} \right| \frac{1}{z - z_0} \right] (d\xi)^\alpha \]

Furthermore

\[ \lim_{N \to \infty} R_N = 0. \]

Hence

\[ -\frac{1}{(2\pi)^\alpha} \frac{1}{i^\alpha} \frac{1}{\Gamma(1+\alpha)} \sum_{n=-\infty}^{\infty} \left[ \frac{1}{c_i} \left| \frac{1}{(\xi - z_0)^\alpha} \right| \frac{1}{z - z_0} \right] (d\xi)^\alpha \]

\[ = \frac{1}{(2\pi)^\alpha} \frac{1}{i^\alpha} \frac{1}{\Gamma(1+\alpha)} \sum_{n=0}^{\infty} \left[ \frac{1}{c_i} \left| \frac{1}{(\xi - z_0)^\alpha} \right| \frac{1}{z - z_0} \right] (d\xi)^\alpha \]

Combing the formulas (4.30) and (4.33), we have the result.

Hence, the proof of the theorem is finished.

5 Generalized residue theorems

In this section we start with a local fractional Laurent series and study generalized residue theorems.

Definition 8
Suppose that \( z_0 \) is an isolated singular point of \( f(z) \). Then there is a local fractional Laurent series

\[ f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^{\alpha k} \]

valid for \( |z-z_0|^\alpha \leq R_\alpha \). The coefficient \( a_{-\alpha} \) of \( (z-z_0)^{-\alpha} \) is called the generalized residue of \( f(z) \) at \( z = z_0 \), and is frequently written as

\[ \text{Res}_{z=z_0} f(z). \]

One of the coefficients for the Yang-Taylor series corresponding to
\[ \phi(z) = (z-z_0)^{n \alpha} f(z), \quad (5.3) \]

the coefficient \( a_{-1} \) is the residue of \( f(z) \) at the pole \( z = z_0 \). It can be found from the formula

\[ \text{Res} f(z) = a_{-1} = \lim_{z \to z_0} \frac{1}{(1+n\alpha)} \frac{d^{(n-1)\alpha}}{dz^{(n-1)\alpha}} \left( (z-z_0)^{n \alpha} f(z) \right) \quad (5.4) \]

where \( n \) is the order of the pole.

Setting \( f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^{k \alpha} \), the expression (5.3) yields

\[ \phi(z) = (z-z_0)^{n \alpha} \sum_{k=-\infty}^{\infty} a_k (z-z_0)^{k \alpha} = a_{-\alpha} + a_{-\alpha+1} (z-z_0)^{\alpha} + a_{-1} (z-z_0)^{(n-1)\alpha} + \ldots \quad (5.5) \]

We know that this is

\[ a_{-1} = \frac{\phi^{(n-1)\alpha}(z_0)}{\Gamma(1+n\alpha)}, \quad (5.6) \]

which is the coefficient of \((z-z_0)^{(n-1)\alpha}\).

The generalized residue is thus

\[ \text{Res} f(z) = a_{-1} = \frac{\phi^{(n-1)\alpha}(z_0)}{\Gamma(1+n\alpha)}, \quad (5.7) \]

where \( \phi(z) = (z-z_0)^{n \alpha} f(z) \).

**Corollary 12**

If \( f(z) \) is local fractional analytic within and on the boundary \( C \) of a region \( \mathcal{R} \) except at a number of poles \( a \) within \( \mathcal{R} \), having a residue \( a_{-1} \), then

\[ \frac{1}{(2\pi)^\alpha \Gamma(1+\alpha)} \oint_C f(z)(dz)^\alpha = \text{Res} f(z) \quad (5.8) \]

**Proof.** Taking into account the definitions of local fractional analytic function and the pole we have local fractional Laurent’s series

\[ f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^{k \alpha} \quad (5.9) \]

and therefore

\[ f(z) = \cdots + a_{-\alpha} (z-z_0)^{-n \alpha} + \cdots + a_{-1} (z-z_0)^{-\alpha} + a_0 + \cdots + a_{\alpha} (z-z_0)^{n \alpha} + \cdots. \quad (5.10) \]

Hence we have the following relation

\[ \frac{1}{\Gamma(1+\alpha)} \oint_C f(z)(dz)^\alpha = \frac{1}{\Gamma(1+\alpha)} \oint_C \left( \sum_{k=-\infty}^{\infty} a_k (z-z_0)^{k \alpha} \right)(dz)^\alpha. \quad (5.11) \]
furthermore

\[ \frac{1}{\Gamma(1+\alpha)} \oint_C f(z)(dz)^{\alpha} = \frac{1}{\Gamma(1+\alpha)} \oint_C \frac{a_1}{(z-z_0)^{\alpha}}(dz)^{\alpha} . \]  

(5.12)

From (3.9), it is shown that

\[ \frac{1}{(2\pi)^{\alpha}} \cdot \frac{1}{\Gamma(1+\alpha)} \oint_C f(z)(dz)^{\alpha} = \frac{1}{(2\pi)^{\alpha}} \cdot \frac{1}{\Gamma(1+\alpha)} \oint_C \frac{a_1}{(z-z_0)^{\alpha}}(dz)^{\alpha} = a_1 . \]  

(5.13)

Hence we have the formula

\[ \frac{1}{\Gamma(1+\alpha)} \oint_C f(z)(dz)^{\alpha} = (2\pi)^{\alpha} i^{\alpha} a_1 . \]  

(5.14)

Taking into account the definition of generalized residue, we have the result.

This proof of the theorem is completed.

From (5.8), we deduce the following corollary:

**Corollary 13**

If \( f(z) \) is local fractional analytic within and on the boundary \( C \) of a region \( \mathcal{R}^\alpha \) except at a finite number of poles \( z_0, z_1, z_2, \ldots \), within \( \mathcal{R}^\alpha \), having residues \( a_{-1}, b_{-1}, c_{-1}, \ldots \) respectively, then

\[ \frac{1}{(2\pi)^{\alpha}} i^{\alpha} \Gamma(1+\alpha) \oint_C f(z)(dz)^{\alpha} = \sum_{i=0}^{\infty} \text{Res} f(z) = a_{-1} + b_{-1} + c_{-1} + \ldots . \]  

(5.15)

It says that the local fractional integral of \( f(z) \) is simply \( (2\pi)^{\alpha} i^{\alpha} \) times the sum of the residues at the singular points enclosed by the contour \( C \).

**6 Applications: Gauss formula of complex function**

**Theorem 14**

Suppose that \( f(z) \) is local fractional analytic and \( \omega \) is any point, then for the circle

\[ |z - \omega|^\alpha = |R^\alpha E_\alpha (i^{\alpha} \theta^\alpha)| \]

we have

\[ f(\omega) = \frac{1}{(2\pi)^{\alpha}} \cdot \frac{1}{\Gamma(1+\alpha)} \int_0^{2\pi} f(\omega + RE(i\theta))(d\theta)^{\alpha} . \]  

(6.1)

**Proof.** By using (3.1) there exists a simple closed contour \( C \) and \( z_0 \) is any point interior to \( C \) such that

\[ f(\omega) = \frac{1}{(2\pi)^{\alpha}} i^{\alpha} \Gamma(1+\alpha) \oint_C f(z)(dz)^{\alpha} . \]  

(6.2)

When \( C \) can been taken to be \( \omega^\alpha + R^\alpha E_\alpha (i^{\alpha} \theta^\alpha) \) for \( \theta \in [0, 2\pi] \), substituting the relations

\[ (z - \omega)^\alpha = R^\alpha E_\alpha (i^{\alpha} \theta^\alpha) \]  

(6.3)
and
\[ (dz)^\alpha = i^\alpha R^\alpha E_\alpha \left( i^{\alpha \theta} \right) (d\theta)^\alpha, \quad (6.4) \]
in (6.2) implies that
\[ f(\omega) = \frac{1}{(2\pi)^{\alpha} i^{\alpha}} \frac{1}{\Gamma(1+\alpha)} \int_{C} \frac{f(\omega + RE(i\theta)) i^\alpha R^\alpha E_\alpha \left( i^{\alpha \theta} \right) (d\theta)^\alpha}{R^\alpha E_\alpha \left( i^{\alpha \theta} \right)} \quad (6.5) \]
and some cancelling gives the result.

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