A finite temperature version of the Nagaoka–Thouless theorem in the SU(n) Hubbard model

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Abstract
The Aizenman–Lieb theorem for the SU(2) Hubbard model extends the Nagaoka–Thouless theorem for the ground state to finite temperatures, and can be stated simply that the magnetization

\[ m(\beta, b) \]

of the system in a field \( b \) exceeds the pure paramagnetic value

\[ m_0(\beta, b) = \tanh(\beta b). \]

In this paper, we present an extension of the Aizenman–Lieb theorem to the SU(n) Hubbard model. Our proof relies on a random-loop representation of the partition function, which is available when the partition function is presented in terms of path integrals.

1 Introduction

Strongly correlated fermionic many-body systems play an essential role in the study of modern condensed matter physics. In particular, a simplified model describing electrons in crystals, proposed by Kanamori, Gutzwiller, and Hubbard, has been extensively examined theoretically [5, 7, 9] and is currently known as the Hubbard model. Despite its simple mathematical definition, the Hubbard model has been confirmed numerically to describe various phenomena such as metal-insulator transition and magnetic orders. On the other hand, rigorous analysis of the Hubbard model is quite challenging, and mathematical proofs of these numerical predictions, which imply a wide variety of phenomena, are often still missing. In these circumstances, mathematical studies of the magnetic properties of the ground states of the Hubbard model are somewhat of an exception, and several mathematical theorems are known [11, 23]; Nagaoka and Thouless investigated a many-electron system with very high strength of the Coulomb interaction and exactly one hole on the lattice, and proved that the ground state of such a system is ferromagnetic [15, 24]. The Nagaoka–Thouless theorem is the first result concerning the rigorous study of ferromagnetic ground states in the Hubbard model, and has greatly influenced subsequent works on magnetism in the model. Aizenman and Lieb extended the Nagaoka and Thouless result to finite temperatures by applying a method known as the random loop representations to the partition function [1]. Furthermore, the author of this paper extends the Nagaoka–Thouless and Aizenman–Lieb theorems to systems with electron-phonon interaction and electron-quantized radiation field interaction [13, 14].

Recent progress in experimental technologies for ultracold atoms has allowed realizing an extension of the conventional spin-1/2 Hubbard model, the so-called SU(n)Hubbard model with general
SU($n$) symmetry [3, 6, 17, 21, 27]. Theoretical analysis of the SU($n$) model has been developed mainly through numerical calculations, and it has become clear that phenomena that do not appear in systems described by the usual SU(2) Hubbard model occur in systems of the SU($n$) Hubbard model [19, 20, 25, 28]. On the other hand, as in the case of the SU(2) model, rigorous analysis is known to be very challenging. Therefore, the question of how the rigorous results of the SU(2) model can be extended to the SU($n$) model is intriguing from both a physical and a mathematical point of view. As for studies in this direction, we refer to [10, 12, 18, 22, 26]. In particular, in [10], Katsura and Tanaka extend the Nagaoka and Thouless result to the SU($n$) Hubbard model by carefully examining the connectivity condition, which is the key to the proof of the Nagaoka–Thouless theorem. The purpose of this paper is to extend the Aizenman–Lieb theorem, which is a finite temperature version of the Nagaoka–Thouless theorem, to the SU($n$) Hubbard model. The strategy of the proof is to extend the random-loop representation used in the description of the SU(2) model appropriately to the analysis of the partition function of the SU($n$) Hubbard model. Random-loop representations for the partition functions of quantum spin systems are well known as powerful analytical tools in the rigorous study of critical phenomena [2], and the random-loop representation constructed in this paper is expected to have further applications.

The organization of this paper is as follows. In Section 2, we define the SU($n$) Hubbard model and explicitly state the main theorems of this paper. In addition, we make it clear that these main theorems are extensions of the Aizenman–Lieb theorem. Then, in Section 3, we provide necessary preparations for proving the main theorems: we construct the Feynman–Kac–Itô formulas for the heat semigroup generated by the SU($n$) Hubbard model. In Section 4, we first construct a random loop representation for the partition function. Then, using this representation, we give proofs of the main theorems stated in Section 2.

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2 Main results

For $d \geq 2$, let $\Lambda$ be a $d$-dimensional hypercube lattice with one side of length $2\ell$: $\Lambda = (\mathbb{Z} \cap [-\ell, \ell])^d$. The SU($n$) Hubbard Hamiltonian on $\Lambda$ is defined by

$$H^H_\Lambda = \sum_{\sigma=1}^n \sum_{x,y \in \Lambda} t_{x,y} c^*_x,\sigma c_y,\sigma + \sum_{x \in \Lambda} \mu_x n_x + \sum_{x,y \in \Lambda} U_{x,y} n_x n_y. \quad (2.1)$$

Here, $c^*_x,\sigma$ and $c_x,\sigma$ represent the creation and annihilation operators of a fermion with the site $x$ and the flavour $\sigma$, satisfying the usual anti-commutation relations:

$$\{c_x,\sigma, c_y,\tau\} = 0 = \{c^*_x,\sigma, c^*_y,\tau\}, \quad \{c_x,\sigma, c^*_y,\tau\} = \delta_{x,y} \delta_{\sigma,\tau}. \quad (2.2)$$

$n_x$ is the number operator of fermions at the site $x$:

$$n_x = \sum_{\sigma=1}^n c^*_x,\sigma c_x,\sigma. \quad (2.3)$$
\( T = \{ t_{x,y} : x, y \in \Lambda \} \) is a hopping matrix of fermions. In this paper, we assume that \( T \) describes the nearest neighbor hopping:

\[
t_{x,y} = \begin{cases} 
  t & \text{x and y are nearest neighbor} \\
  0 & \text{otherwise.}
\end{cases}
\] (2.4)

We suppose that the Hamiltonian acts on the \( N = |\Lambda| - 1 \) particle space:

\[
\bigwedge^{N} (\ell^2(\Lambda) \otimes \mathbb{C}^n).
\] (2.5)

For the sake of convenience, we assume that the on-site Coulomb interaction is uniform:

\[
U = U_{x,x} \text{ for all } x \in \Lambda.
\] (2.6)

We introduce the operators that will play a fundamental role in this paper by

\[
h_\sigma = N_\sigma - N_{\sigma+1}, \quad \sigma = 1, \ldots, n - 1,
\] (2.7)

where \( N_\sigma \) is the number operator of fermions with the flavour \( \sigma \):

\[
N_\sigma = \sum_{x \in \Lambda} n_{x,\sigma}, \quad n_{x,\sigma} = c^*_{x,\sigma} c_{x,\sigma}.
\] (2.8)

If we set

\[
e_{\sigma,\sigma'} = \sum_{x \in \Lambda} c^*_{x,\sigma} c_{x,\sigma'}, \quad \sigma, \sigma' = 1, \ldots, n, \quad \sigma \neq \sigma',
\] (2.9)

the family \( \{ h_\sigma, e_{\sigma,\sigma'} \} \) of operators gives a representation of the \( \mathfrak{su}(n) \) Lie algebra. The family of operators \( \{ h_\sigma \} \) provides a representation of the Cartan subalgebra of \( \mathfrak{su}(n) \) and is known to play an essential role in the representation theory of Lie algebras; see, e.g., [8]. In the case of \( n = 2 \), \( H_\Lambda \) is the conventional Hubbard Hamiltonian, and \( h_1 \) corresponds to the third component \( S_{\text{tot}}(3) \) of the total spin operators. Extending the wording of the \( n = 2 \) case, we call the following Hamiltonian a model describing an interaction between an external magnetic field \( \mathbf{b} = (b_1, \ldots, b_{n-1}) \in \mathbb{R}^{n-1} \) and fermions for convenience:

\[
H^R_\Lambda(\mathbf{b}) = \sum_{\sigma=1}^{n} \sum_{x,y \in \Lambda} t_{x,y} c^*_{x,\sigma} c_{y,\sigma} + \sum_{x \in \Lambda} \mu_x n_x + \sum_{x,y \in \Lambda} U_{x,y} n_x n_y - \sum_{\sigma=1}^{n-1} b_\sigma h_\sigma.
\] (2.10)

In this paper, we examine the system with \( U = \infty \). In order to describe the effective Hamiltonian for such a system, we need some preparations. First, define an orthogonal projection \( Q_\Lambda \) as follows: let \( E_{n_x}(\cdot) \) be the spectral measure of \( n_x \), and let \( Q_{\Lambda,x} = E_{n_x}(\{1\}) \). Define

\[
Q_\Lambda = \prod_{x \in \Lambda} Q_{\Lambda,x}.
\] (2.11)

Then we define the subspace \( \mathfrak{F}_N \) of the \( N \)-particle space by

\[
\mathfrak{F}_N = Q_\Lambda \bigwedge^{N} (\ell^2(\Lambda) \otimes \mathbb{C}^n).
\] (2.12)

---

1Let \( x, y \in \Lambda \). We say that \( x \) and \( y \) are nearest neighbor if \( \|x - y\|_\infty = 1 \), where \( \|x\|_\infty = \max_{j=1}^d |x_j| \).
\( \mathcal{F}_N \) is a Hilbert space of state vectors describing a system with exactly one hole in \( \Lambda \) and a single fermion occupying each site except the hole site. In the case of \( U = \infty \), the energy required for two or more fermions to occupy a single site is infinite. Therefore, the Hilbert space of states of such a system is \( \mathcal{F}_N \), and the effective Hamiltonian is given by

\[
H_\Lambda(b) = Q_\Lambda H^\Lambda_{U=0}(b)Q_\Lambda, \tag{2.13}
\]

where \( H^\Lambda_{U=0}(b) \) denotes the linear operator set to \( U = 0 \) in the definition of \( H^\Lambda_\Lambda(b) \).

The following proposition is a mathematical expression of the intuitive explanation given above:

**Proposition 2.1.** In the limit of \( U \to \infty \), \( H^\Lambda_\Lambda(b) \) converges to \( H_\Lambda(b) \) in the following sense:

\[
\lim_{U \to \infty} \left\| (H^\Lambda_\Lambda(b) - z)^{-1} - (H_\Lambda(b) - z)^{-1} Q_\Lambda \right\| = 0, \quad z \in \mathbb{C} \setminus \mathbb{R}, \tag{2.14}
\]

where \( \cdot \) stands for the operator norm.

Proposition 2.1 can be proved by similar arguments as in the proof of Theorem 2.5 in [13].

In order to state the first main theorem, consider the partition function for \( H_\Lambda(b) \):

\[
Z_\Lambda(\beta; b) = \text{Tr}_{\mathcal{F}_N}[e^{-\beta H_\Lambda(b)}], \quad \beta \geq 0. \tag{2.15}
\]

**Theorem 2.2.** Let \( P_N \) denote the entire partitions of \( N \). \(^a\) For any \( n \in P_N \), there exists a positive number \( D_\beta(n) \) such that

\[
Z_\Lambda(\beta; b) = \sum_{n \in P_N} D_\beta(n) G_\beta(n; b), \tag{2.16}
\]

where, for each \( n = \{n_j\}_{j=1}^k \in P_N \), \( G_\beta(n; b) \) is defined by

\[
G_\beta(n; b) = \prod_{j=1}^k G_\beta(n_j; b), \tag{2.17}
\]

\[
G_\beta(n_j; b) = e^{\beta b_1} + e^{-\beta b_{n-1}} + \sum_{\sigma=2}^{n-1} e^{\beta (b_{\sigma-1} - b_{\sigma})}. \tag{2.18}
\]

**Remark 2.3.** For \( n = 2 \), Theorem 2.2 gives

\[
Z_\Lambda(\beta; b) = \sum_{n \in P_N} D_\beta(n) \prod_{j=1}^k 2 \cosh(\beta b_1 n_j), \tag{2.19}
\]

reproducing the result of Aizenman–Lieb [1] for the ordinary Hubbard model.

To state the next result, we introduce the following symbols:

\[
B_\sigma = \begin{cases} b_1 & \text{if } \sigma = 1 \\ b_\sigma - b_{\sigma-1} & \text{if } 2 \leq \sigma \leq n - 2 \\ -b_{n-1} & \text{if } \sigma = n. \end{cases} \tag{2.20}
\]

\(^a\)Thus, each \( n = \{n_j\}_{j=1}^k \in P_N \) satisfies \( n_j \in \mathbb{N} \) and \( \sum_{j=1}^k n_j = N \).
Theorem 2.4. Let \( n \geq 2 \). Fix \( \sigma \in \{1, 2, \ldots, n-1\} \), arbitrarily. Assume that \( B_\sigma > B_\tau \) for all \( \tau \neq \sigma \). Then one obtains

\[
\langle h_\sigma \rangle_\beta \geq \frac{f_{\beta,\sigma}(b)}{1 + g_{\beta,\sigma}(b)}N, \tag{2.21}
\]

where \( \langle h_\sigma \rangle_\beta \) stands for the thermal expectation of \( h_\sigma \), and the functions \( f_{\beta,\sigma}(b) \) and \( g_{\beta,\sigma}(b) \) are respectively given by

\[
f_{\beta,\sigma}(b) = \frac{1 - e^{-\beta(B_\sigma - B_{\sigma+1})}}{1 + e^{-\beta(B_\sigma - B_{\sigma+1})}}, \tag{2.22}
g_{\beta,\sigma}(b) = \sum_{\tau \neq \sigma, \sigma+1} e^{\beta(B_\tau - B_\sigma)}. \tag{2.23}
\]

For \( n = 2 \), we understand that \( g_{\beta,\sigma}(b) = 0 \).

Remark 2.5. • By using (2.21) and the fact \( g_{\beta,\sigma}(b) \leq n - 2 \), we get

\[
\langle h_\sigma \rangle_\beta \geq \frac{f_{\beta,\sigma}(b)}{n - 1}N. \tag{2.24}
\]

Note that this inequality holds even for \( n = 2 \).

• For \( n = 2 \), if we set \( B_1 = -B_2 = b > 0 \), we can re-derive Aizenman–Lieb’s result \([1]\) from (2.24):

\[
\langle h_1 \rangle_\beta \geq \tanh(\beta b)N. \tag{2.25}
\]

3 Functional integral representations for the semigroup generated by the Hamiltonian

3.1 Preliminaries

3.1.1 Case of a single fermion

We need a functional integral representation for \( e^{-\beta H_\Lambda(b)} \) to prove the main theorems. In this section, for the convenience of the readers, we will outline how to construct the functional integral representation for the SU(\( N \)) Hubbard model.

States of a single fermion are represented by normalized vectors in the Hilbert space:

\[
\ell^2(\Lambda) \otimes \mathbb{C}^n = \ell^2(\Omega), \tag{3.1}
\]

where \( \Omega = \Lambda \times \{1, \ldots, n\} \). The inner product in \( \ell^2(\Omega) \) is given by

\[
\langle f | g \rangle = \sum_{\sigma=1}^n \sum_{x \in \Lambda} f(x, \sigma)^* g(x, \sigma), \quad f, g \in \ell^2(\Omega). \tag{3.2}
\]

Define the free Hamiltonian \( h_0 \) of a single fermion by

\[
(h_0 f)(x, \sigma) = \sum_{\sigma=1}^n \sum_{y \in \Lambda} t_{x,y} \left( f(x, \sigma) - f(y, \sigma) \right), \quad f \in \ell^2(\Omega). \tag{3.3}
\]

\(^1To be precise, \( \langle h_\sigma \rangle_\beta \) is defined by \( \langle h_\sigma \rangle_\beta = \text{Tr}_{\mathbb{C}^n}[h_\sigma e^{-\beta H_\Lambda(b)}]/Z_\Lambda(\beta; b) \).
To represent fundamental physical observables, we introduce the function $\delta_{(x,\sigma)} \left((x, \sigma) \in \Omega\right)$ on $\Omega$ by

$$
\delta_{(x,\sigma)}(y, \tau) = \delta_{x,y} \delta_{\sigma,\tau}, \quad (y, \tau) \in \Omega.
$$

(3.4)

Now, define the function $k_\sigma$ by

$$
k_\sigma = \sum_{x \in \Lambda} \{\delta_{(x,\sigma)} - \delta_{(x,\sigma+1)}\}, \quad \sigma = 1, \ldots, n - 1.
$$

(3.5)

In this paper, for a given function $v$ on $\Omega$, the multiplication operator by $v$ will be denoted by the same symbol:

$$(v f)(x, \sigma) = v(x, \sigma) f(x, \sigma), \quad f \in \ell^2(\Omega).$$

(3.6)

Under this convention, for $b = (b_1, \ldots, b_{n-1}) \in \mathbb{R}^{n-1}$ and a function $\mu : (x, \sigma) \ni \Omega \to \mu_x \in \mathbb{R}$, we define the self-adjoint operator $h_\mu(b)$ on $\ell^2(\Omega)$ by

$$
h_\mu(b) = h_0 + \mu - \sum_{\sigma=1}^{n-1} b_\sigma k_\sigma.
$$

(3.7)

Note that $\mu_x$ does not depend on $\sigma$. The function $\mu$ represents the on-site potential and $b$ represents the external field.

For notational simplicity, set $\mathbb{Z}_+ = \mathbb{Z} \cap [0, \infty)$. Let $(Y_n)_{n \in \mathbb{Z}_+}$ be a discrete time Markov chain with state-space $\Omega$, which is characterized by

$$
P(Y_n = X | Y_{n-1} = Y) = \delta_{x,y} \frac{t_{x,y}}{d(y)}, \quad d(x) = \sum_{y \in \Lambda} t_{x,y}
$$

(3.8)

for $n \in \mathbb{N}$, $X = (x, \sigma) \in \Omega$ and $Y = (y, \tau) \in \Omega$. In the rest of this paper, we work with the fixed probability space $(M, F, P)$. Let $(T_n)_{n \in \mathbb{Z}_+}$ be independent exponentially distributed random variables of parameter 1, independent of $(Y_n)_{n \in \mathbb{Z}_+}$. Then we set

$$
S_n = \frac{T_n}{d(Y_{n-1})}, \quad J_n = S_1 + \cdots + S_n
$$

(3.9)

and

$$
X_t = \sum_{n \in \mathbb{Z}_+} 1_{(J_n \leq t < J_{n+1})} Y_n,
$$

(3.10)

where $1_S$ represents the indicator function of the set $S$. Under this setting, we see that $(X_t)_{t \geq 0}$ is a right continuous process. Furthermore, $J_0 := 0, J_1, J_2, \ldots$ are the jump times of $(X_t)_{t \geq 0}$ and $S_1, S_2, \ldots$ are the holding times of $(X_t)_{t \geq 0}$. Now, set $P_X(\cdot) = P(\cdot | X_0 = X)$ and let $(\mathcal{F}_t)_{t \geq 0}$ be the filtration defined by $\mathcal{F}_t = \sigma(X_s | s \leq t)$. Then we can check that $(M, F, (\mathcal{F}_t)_{t \geq 0}, (P_X)_{X \in \Omega})$ is a strong Markov process, see, e.g., [16, Theorems 2.8.1 and 6.5.4].

The following Feynman–Kac–Itô formula is well-known:

$$
(e^{-th_\mu(b)} f)(X) = \mathbb{E}_X \left[e^{-\int_0^t v(X_t) dt} f(X_t)\right], \quad f \in \ell^2(\Omega), \quad X \in \Omega,
$$

(3.11)

where $\mathbb{E}_X[f]$ is the expected value of $f$ with respect to $P_X$, and the function $v$ on $\Omega$ is defined by

$$
v(X) = \mu(X) - \sum_{\sigma=1}^{n-1} b_\sigma k_\sigma(X).
$$

(3.12)

See, e.g., [4] for a detailed proof. The representation (3.11) is the first step to perform our analysis.
3.1.2 Case of $N$ fermions

Here, we examine an $N$-fermion system. The non-interacting Hamiltonian is given by

$$T = h_\mu(b) \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes h_\mu(b) \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes h_\mu(b).$$  (3.13)

The operator $T$ is a self-adjoint operator acting in $\bigotimes^N \ell^2(\Omega)$, the $N$-fold tensor product of $\ell^2(\Omega)$. In the remainder of this paper, we will freely use the following identification:

$$\bigotimes^N \ell^2(\Omega) = \ell^2(\Omega^N).$$  (3.14)

Note that this identification is implemented by the unitary operator $\iota : \bigotimes^N \ell^2(\Omega) \rightarrow \ell^2(\Omega^N)$ given by

$$\iota(f_1 \otimes \cdots \otimes f_n) = (f_1(X_1) \cdots f_n(X_n))(X_1, \ldots, X_n) \in \Omega^N.$$  (3.15)

Let $\delta^{(j)}_{(x,\sigma)}$ be the multiplication operator on $\bigotimes^N \ell^2(\Omega)$ given by

$$\delta^{(j)}_{(x,\sigma)} = 1 \otimes \cdots \otimes 1 \otimes \delta_{(x,\sigma)} \otimes 1 \otimes \cdots \otimes 1, \quad j = 1, \ldots, N.$$  (3.16)

The Coulomb interaction term is described as the multiplication operator $V$ defined by

$$V = V_0 + V_d,$$  (3.17)

where

$$V_d = U \sum_{x \in X} \sum_{\sigma, \tau = 1}^N \sum_{i,j = 1}^N \delta^{(i)}_{(x,\sigma)} \delta^{(j)}_{(x,\tau)}, \quad V_0 = \sum_{\sigma, \tau = 1}^N \sum_{x \neq y} U_{x,y} \delta^{(i)}_{(x,\sigma)} \delta^{(j)}_{(y,\tau)}.$$  (3.18)

Based on the above setups, we define the Hamiltonian describing the interacting $N$-fermion system by

$$L(b) = T + V.$$  (3.19)

Since the particles are fermions, the Fermi-Dirac statistics have to be taken into account. For this purpose, we introduce the antisymmetrizer $A_N$ on $\ell^2(\Omega^N)$ by

$$(A_N F)(X) = \sum_{\tau \in \mathfrak{S}_N} \frac{\text{sgn}(\tau)}{N!} F(\tau^{-1} X)$$  (3.20)

for $F \in \ell^2(\Omega^N)$ and $X = (X^{(1)}, \ldots, X^{(N)}) \in \Omega^N$, where $\mathfrak{S}_N$ indicates the permutation group on the set $\{1, \ldots, N\}$, and $\tau X := (X^{(\tau(1))}, \ldots, X^{(\tau(N))})$ for $\tau \in \mathfrak{S}_N$. Note that $A_N$ is the orthogonal projection from $\ell^2(\Omega^N)$ onto $\ell^2_\omega(\Omega^N)$, the space of all antisymmetric functions on $\Omega^N$. Then the Hamiltonian we are interested in, $H^H(b)$, can be expressed as

$$H^H_N(b) = A_N L(b) A_N.$$  (3.21)

Using the representation (3.21), we wish to construct a Feynman–Kac–Itô formula for the semigroup generated by $H^H(b)$. For this purpose, let

$$\Omega^N_\neq = \left\{ X \in \Omega^N \mid X^{(i)} \neq X^{(j)} \text{ for all } i, j \in \{1, \ldots, N\} \text{ with } i \neq j \right\}.$$  (3.22)
Using Eq. (3.14), we have the following identification:

$$
\bigwedge_{1}^{N} \ell^2(\Lambda) \otimes \mathbb{C}^n = \ell^2_{\text{as}}(\Omega^N_{\neq}). 
$$  

(3.23)

Given \( m = (m_1, \ldots, m_N) \in (M)^N \), we set \( X_s^{(j)}(m) = X_s(m_j) \). For each \( m \in (M)^N \), a right-continuous \( \Omega^N \)-valued function

\[
(X_t(m))_{t \geq 0} = (X_t^{(1)}(m), \ldots, X_t^{(N)}(m))
\]

(3.24)
is simply called a path. If we write \( X_t^{(j)}(m) = (x_t^{(j)}(m), \sigma_t^{(j)}(m)) \), then \( \sigma_t^{(j)}(m) \) is called the flavor component of \( X_t^{(j)}(m) \), and \( x_t^{(j)}(m) \) is called the spatial component of \( X_t^{(j)}(m) \), respectively. The collection of the spatial components \((x_t^{(1)}(m), \ldots, x_t^{(N)}(m))\) represents a trajectory of the \( N \)-fermions.

Define the event by

\[
D = D_O \cap D_S,
\]

(3.25)

where

\[
D_O = \left\{ m \in (M)^N \right\} X_s(m) \in \Omega^N_{\neq} \text{ for all } s \in [0, \infty) \right\},
\]

(3.26)

\[
D_S = \left\{ m \in (M)^N \right\} \sigma_s^{(j)}(m) = \sigma_0^{(j)}(m) \text{ for all } j = \{1, \ldots, N\} \text{ and } s \in [0, \infty) \right\}.
\]

(3.27)

Here, in the definition of \( D_S \), \( \sigma_s^{(j)}(m) \) denotes the flavor part of \( X_s^{(j)}(m) \).\(^4\) Note that, for a given \( m \in D \), the path \((X_t(m))_{t \geq 0}\) has characteristics such that \( \sigma_t^{(j)}(m) \) are constant in time, and fermions of the equal flavor never meet each other.

By using the Feynman–Kac–Itô formula for a single fermion (3.11) and Trotter’s product formula, one obtains the following:

**Proposition 3.1.** For every \( X = (X^{(1)}, \ldots, X^{(N)}) \in \Omega^N_{\neq} \) and \( F \in \ell^2_{\text{as}}(\Omega^N) \), we have

\[
\left(e^{-tH_{\Lambda}(b)}F\right)(X) = \mathbb{E}_X \left[ 1_D \exp \left\{ - \int_0^t W(X_s) ds \right\} F(X_t) \right],
\]

(3.28)

where \( \mathbb{E}_X[F] \) represents the expected value of \( F \) associated with the probability measure \( \otimes_{j=1}^N P_{X^{(j)}} \) on \((\Omega^N, M^N)\), and

\[
W(X) = V(X) + \sum_{j=1}^N v(X^{(j)}), \quad X \in \Omega^N_{\neq}.
\]

(3.29)

**3.1.3 The system of U = \infty**

Here, we provide a Feynman–Kac–Itô formula for the semigroup generated by the Hamiltonian \( H_{\Lambda}(b) \) describing the system of \( U = \infty \).

Let

\[
\Omega^N_{\neq, \infty} = \left\{ X \in \Omega^N \left| x^{(i)} \neq x^{(j)} \text{ for all } i, j \in \{1, \ldots, N\} \text{ with } i \neq j \right. \right\}.
\]

(3.30)

\(^4\)To be precise, \( X_s^{(j)}(m) = (x_s^{(j)}(m), \sigma_s^{(j)}(m)) \).
In the above definition, we used the following notations: \( \mathbf{X} = (X^{(1)}, \ldots, X^{(N)}) \) with \( X^{(j)} = (x^{(j)}, \sigma^{(j)}) \). In the remainder of this paper, we freely use the following natural identification:

\[
\mathfrak{F}_N = \ell_2^{\text{as}}(\Omega_{\not\in, \infty}^N),
\]

(3.31)

where \( \mathfrak{F}_N \) is defined by (2.12). Given \( \beta > 0 \), we set \( D_{\infty}(\beta) = D_{O, \infty}(\beta) \cap D \) with

\[
D_{O, \infty}(\beta) = \left\{ \mathbf{m} \in (M)^N \mid X_s(\mathbf{m}) \in \Omega_{\not\in, \infty}^N \text{ for all } s \in [0, \beta] \right\}.
\]

(3.32)

Note that, for each \( \mathbf{m} \in D_{\infty}(\beta) \), there are no fermion encounters in the corresponding path \( (X_t(\mathbf{m}))_{t \in [0, \beta]} \).

Now we are ready to construct a Feynman–Kac–Itô formula for \( e^{-\beta H_\lambda(b)} \).

**Theorem 3.2.** For every \( \mathbf{X} \in \Omega_{\not\in, \infty}^N \) and \( F \in \ell_2^{\text{as}}(\Omega_{\not\in, \infty}^N) \), we have

\[
\left( e^{-\beta H_\lambda(b)} F \right)(\mathbf{X}) = \mathbb{E}_\mathbf{X} \left[ \mathbb{1}_{D_{\infty}(\beta)} \exp \left\{ - \int_0^\beta W_{\infty}(X_s) \, ds \right\} F(\mathbf{X}_t) \right],
\]

(3.33)

where

\[
W_{\infty}(\mathbf{X}) = V_0(\mathbf{X}) + \sum_{j=1}^N v(X^{(j)}), \quad \mathbf{X} \in \Omega_{\not\in, \infty}^N.
\]

(3.34)

Here, recall that \( V_0 \) is given by (3.18).

**Proof.** By Proposition 2.1, we have

\[
\lim_{U \to \infty} e^{-\beta H_\lambda(b)} F = e^{-\beta H_\lambda(b)} F.
\]

(3.35)

Denote by \( \mathbb{1}_{D} G_U(\mathbf{X}_t) \) the integrand in the right hand side of (3.28). We split \( \mathbb{E}_\mathbf{X} [\mathbb{1}_{D} G_U(\mathbf{X}_t)] \) into two parts as follows:

\[
\mathbb{E}_\mathbf{X} [\mathbb{1}_{D} G_U(\mathbf{X}_t)] = \mathbb{E}_\mathbf{X} [\mathbb{1}_{D_{\infty}(\beta)} G_{U=0}(\mathbf{X}_t)] + \mathbb{E}_\mathbf{X} [\mathbb{1}_{D \setminus D_{\infty}(\beta)} G_{U=0}(\mathbf{X}_t)].
\]

(3.36)

Because \( \lim_{U \to \infty} G_U(\mathbf{X}_t(\mathbf{m})) = 0 \) for all \( \mathbf{m} \in D \setminus D_{\infty}(\beta) \), we have

\[
\lim_{U \to \infty} \mathbb{E}_\mathbf{X} [\mathbb{1}_D G_U(\mathbf{X}_t)] = \mathbb{E}_\mathbf{X} [\mathbb{1}_{D_{\infty}(\beta)} G_{U=0}(\mathbf{X}_t)]
\]

(3.37)

by the dominated convergence theorem. Combining (3.35) and (3.37), we obtain the desired assertion in Theorem 3.2. \( \Box \)

### 3.1.4 A Feynman–Kac–Itô formula for the partition function

Given \( \beta > 0 \), let

\[
D_{P, \infty}(\beta) = \left\{ \mathbf{m} \in (M)^N \mid \exists \tau \in \mathfrak{S}_N(X_0(\mathbf{m})) \text{ such that } X_\beta(\mathbf{m}) = \tau X_0(\mathbf{m}) \right\},
\]

(3.38)

where \( \mathfrak{S}_N(\mathbf{X}) \) is a subset of \( \mathfrak{S}_N \), and since its definition is a bit complicated, we will provide its precise definition in Definition 3.4 below. We then define the event by

\[
L_\beta = D_{\infty}(\beta) \cap D_{P, \infty}(\beta).
\]

(3.39)

The purpose here is to prove the following theorem:
Theorem 3.3. For every $\beta > 0$, there exists a measure $\mu_\beta$ on $L_\beta$ such that

$$Z_\Lambda(\beta; b) = \int_{L_\beta} d\mu_\beta \prod_{j=1}^N \prod_{\sigma=1}^{n-1} \exp \left\{ \int_0^\beta b_\sigma k_\sigma(X^{(j)}_\sigma) ds \right\},$$

(3.40)

where $k_\sigma$ is given by (3.5).

In order to prove Theorem 3.3, we need some preparations. First, let us construct a complete orthonormal system (CONS) for $\mathfrak{F}_N = \ell^2_N(\Omega_{\neq,\infty})$. Given $X \in \Omega_N$, we set

$$\delta_X = \otimes_{j=1}^N \delta_X^{(j)} \in \ell^2(\Omega_N)$$

(3.41)

and $e_X = A_N \delta_X$, where $A_N$ is the antisymmetrizer defined by (3.26). Then we readily confirm that $\{\delta_X \mid X \in \Omega_N\}$ is a CONS for $\ell^2(\Omega_N)$. To construct a CONS for $\ell^2_N(\Omega_{\neq,\infty})$, we do a little more preparation. Note that $e_\tau X = sgn(\tau) e_X$ holds for all $\tau \in \mathfrak{S}_N$. With this in mind, we introduce an equivalence relation in $\Omega_{\neq,\infty}^N$ as follows: Let $X, Y \in \Omega_{\neq,\infty}^N$. If there exists a $\tau \in \mathfrak{S}_N$ satisfying $Y = \tau X$, then we write $X \equiv Y$. It is easy to see that this binary relation gives an equivalence relation. Let $[X]$ be the equivalence class to which $X$ belongs. We will often abbreviate $[X]$ to $X$ if no confusion occurs. We denote by $[\Omega_{\neq,\infty}^N]$ the quotient set $\Omega_{\neq,\infty}^N / \equiv$. Then we readily confirm that $\{e_X \mid X \in [\Omega_{\neq,\infty}^N]\}$ is a CONS for $\ell^2_N(\Omega_{\neq,\infty})$. This CONS is useful in our analysis below.

Let $Q_\Lambda$ be the orthogonal projection from $\ell^2(\Omega_N)$ to $\mathfrak{F}_N$ given by (2.11). We denote by $T$ the Hamiltonian of the free fermions:

$$T = Q_\Lambda T_{b=0,\mu=0} Q_\Lambda,$$

(3.42)

where $T_{b=0,\mu=0}$ denotes the operator defined as $b = 0$ and $\mu = 0$ in the defining equation of $T$, i.e., (3.13).

We are now ready to state the precise definition of $\mathfrak{S}_N(X)$ that appears in Eq. (3.38):

Definition 3.4. Let $X \in \Omega_{\neq,\infty}^N$. We say that a permutation $\tau \in \mathfrak{S}_N$ is dynamically allowed associated with $X$ if there exists an $n \in \mathbb{Z}_+$ such that

$$\langle \delta_X | T^n \delta_{\tau X} \rangle \neq 0.$$  

(3.43)

We denote by $\mathfrak{S}_N(X)$ the set of all dynamically allowed permutations associated with $X$. Note that if $\tau$ is dynamically allowed, then $\tau$ is always even, i.e., $sgn(\tau) = 1$ [1].

In order to give a characterization of the dynamically allowed permutations, let us introduce some terms. Given $X = (X^{(j)})_{j=1}^N \in \Omega_{\neq,\infty}^N$ and $Y = (Y^{(j)})_{j=1}^N \in \Omega_{\neq,\infty}^N$, define the distance between $X$ and $Y$ by

$$\|X - Y\|_\infty = \max_{j=1,\ldots,N} \|x^{(j)} - y^{(j)}\|_\infty,$$

(3.44)

where $x^{(j)}$ (resp. $y^{(j)}$) is the spatial component of $X^{(j)}$ (resp. $Y^{(j)}$). We say that $X$ and $Y$ are neighbors if $\|X - Y\|_\infty = 1$ and the flavor components of $X^{(j)}$ and $Y^{(j)}$ are equal for all $j = 1, \ldots, n$. A pair $\{X, Y\} \in \Omega_{\neq,\infty}^N \times \Omega_{\neq,\infty}^N$ is called an edge if $X$ and $Y$ are neighbors. A sequence $(X_i)_{i=1}^m \subset \Omega_{\neq,\infty}^N$ is called a path, if $\{X_i, X_{i+1}\}$ is an edge for all $i$. For a given edge $\{X, Y\}$, define the linear operator acting in $\ell^2(\Omega_{\neq,\infty})$ by

$$Q(X, Y) = |\delta_X \rangle \langle \delta_Y|.$$  

(3.45)

This operator is employed to describe the following lemma, which characterizes the dynamically allowed permutations.
Lemma 3.5. Let \( \tau \in \mathcal{G}_N \) and let \( X \in \Omega_{\mathbb{Z}, \infty}^N \). The following (i) and (ii) are mutually equivalent:

(i) \( \tau \) is dynamically allowed associated with \( X \);

(ii) there exists a path \((X_i)_{i=1}^m\) satisfying the following:

\[
X_1 = X \quad \text{and} \quad X_m = \tau X;
\]

\[
\langle \delta X | Q(X_1, X_2)Q(X_2, X_3) \cdots Q(X_{m-1}, X_m) \delta \tau X \rangle > 0.
\]

See [14] for a proof of this lemma.

Proof of Theorem 3.3. We give only a brief outline of the proof. For details, see [14]. We divide the proof into two parts.

**Step 1.** Let \( \mathcal{C}_N^{\infty}(\Omega^N) \) be the set of all symmetric functions on \( \Omega^N \). Let \( F_0, F_1, \ldots, F_{n-1} \) be elements in \( \mathcal{C}_N^{\infty}(\Omega^N) \) that are strictly positive. We set

\[
K_n = F_0 e^{-t_1 H_X(b)} F_1 e^{-t_2 - t_1 H_X(b)} F_2 \cdots F_{n-1} e^{-(\beta - t_{n-1}) H_X(b)}.
\]

(3.46)

Fix \( X \in \Omega_{\mathbb{Z}, \infty}^N \), arbitrarily. We claim that if \( \tau \) is not dynamically allowed associated with \( X \), then one obtains, for all \( n \in \mathbb{N} \) and \( 0 < t_1 < t_2 < \cdots < t_{n-1} < \beta \), that

\[
\langle \delta X | K_n \delta \tau X \rangle = 0.
\]

(3.47)

We shall prove this equation in a step-by-step manner. First, let us consider the case where \( W_{\infty} \equiv 0 \). For simplicity, suppose that \( n = 2 \). Because \( F_0 \) and \( F_1 \) are multiplication operators, it holds that

\[
\langle \delta X | F_0(-T)^{n_1} F_1(-T)^{n_2} \delta \tau X \rangle = 0
\]

(3.48)

for all \( n_1, n_2 \in \mathbb{Z}_+ \). To show this, note that we can express \( T \) as

\[
T = \sum_{(X,Y)} C_{X,Y} Q(X,Y) + \mathcal{D},
\]

(3.49)

where \( \sum_{(X,Y)} \) means sum over all edges, the coefficients \( C_{X,Y} \) satisfy \( C_{X,Y} < 0 \) for each edge \( \{X, Y\} \) and \( \mathcal{D} \) is some multiplication operator. By using the formula (3.49), the equation (3.48) follows from the following property:

\[
\langle \delta X | Q(X_1, X_2)Q(X_2, X_3) \cdots Q(X_{m-1}, X_m) \delta \tau X \rangle = 0
\]

(3.50)

for any path \((X_i)_{i=1}^m\). But this is obvious from Lemma 3.5. Using (3.48), we can prove (3.47) as:

\[
\langle \delta X | K_n \delta \tau X \rangle = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \sum_{n_1}^{n_2} \sum_{n_1}^{n_2} \frac{t_1^{n_1} t_2^{n_2}}{n_1! n_2!} \langle \delta X | F_0(-T)^{n_1} F_1(-T)^{n_2} \delta X \rangle = 0.
\]

(3.51)

Similarly, we can prove (3.47) for general \( n \), when \( W_{\infty} \equiv 0 \).

Next, let us consider the case where \( W_{\infty} \neq 0 \). Again, we will consider the case \( n = 2 \) for simplicity. By using Trotter’s formula, we have

\[
\langle \delta X | K_n \delta \tau X \rangle = \lim_{N_1 \to \infty} \lim_{N_2 \to \infty} \langle \delta X | F_0(e^{-(t_1 T / N_1)} e^{-t_1 W_{\infty} / N_1}) F_1(e^{-(t_2 - t_1) T / N_2} e^{-(t_2 - t_1) W_{\infty} / N_2}) N_1 \delta \tau X \rangle.
\]

(3.52)
By applying the claim for the case where $W_\infty \equiv 0$, we see that the right hand side of (3.52) equals zero. Similarly, we can prove the assertion for general $n$. We are now done with the proof of Eq. (3.47).

**Step 2.** By using Theorem 3.2 and (3.47), we get

$$Z_\Lambda(\beta; b) = \sum_{X \in [\Omega^N_0, \infty]} \langle e_X | e^{-\beta H_\Lambda(b)} e_X \rangle$$

$$= \sum_{X \in [\Omega^N_0, \infty]} \sum_{\tau \in \mathcal{S}_N(X)} \frac{\text{sgn}(\tau)}{N!} \langle \delta_X | e^{-\beta H_\Lambda(b)} \delta_{\tau X} \rangle$$

$$= \sum_{X \in [\Omega^N_0, \infty]} \sum_{\tau \in \mathcal{S}_N(X)} \frac{\text{sgn}(\tau)}{N!} \langle \delta_X | e^{-\beta H_\Lambda(b)} \delta_{\tau X} \rangle$$

$$= \sum_{X \in [\Omega^N_0, \infty]} \sum_{\tau \in \mathcal{S}_N(X)} \frac{\text{sgn}(\tau)}{N!} E_X \left[ 1_{\{X_\beta = \tau X \cap D_{\infty}(\beta) \neq 0\}} e^{-\int_0^\beta W_\infty(X_s)ds} \right]. \tag{3.53}$$

Now let us define the measure on $L_\beta$ by

$$\mu_\beta(B) = \sum_{X \in [\Omega^N_0, \infty]} \sum_{\tau \in \mathcal{S}_N(X)} \frac{1}{N!} E_X \left[ 1_B 1_{\{X_\beta = \tau X \cap D_{\infty}(\beta) \neq 0\}} e^{-\int_0^\beta W_\infty(X_s)ds} \right], \tag{3.54}$$

where $W_{\infty, b=0}(X)$ is defined by setting $b = 0$ in the defining equation of $W_\infty(X)$. Because $\text{sgn}(\tau) = 1$ for all $\tau \in \mathcal{S}_N(X)$, we finally obtain the desired assertion in Theorem 3.3.

\[\square\]

## 4 Proofs of Theorems 2.2 and 2.4

### 4.1 Random loop representations

For a given $m \in L_\beta$, we define the particle world lines of the path $(X_s(m))_{s \in [0,\beta]}$ by

$$x_t(m) = (x^{(1)}_t(m), \ldots, x^{(N)}_t(m)), \tag{4.1}$$

where $x^{(i)}_t(m)$ denotes the spatial component of $X^{(i)}_t(m)$. Since each $x^{(i)}_t(m)$ takes values on $\Lambda$, is piece-wise constant concerning time, and hops to nearest neighbor sites at random times, the particle world lines can be illustrated as a collection of polygonal lines in the space-time picture. Figure 1 depicts typical world lines in a two-dimensional system.

Following [1], we can associate arbitrary particle world lines with a collection of loops in the space-time picture by the following procedure:

- we begin by plotting the loops from the position of each of the particles at time $t = 0$;
- we trace the time evolution of a particle’s position in space-time; note that because the strength of the Coulomb interaction is infinite, the particles never encounter each other;
- when the trace line reaches the time at $t = \beta$, the trace line reappears in the same location at time $t = 0$ by regarding time as periodic;
- we continue the above procedures until the trace line is closed.
Each path is completely characterized by the following three conditions: The initial configuration of the particles: \(x_0\), the particle world lines, and the flavor assigned along the world lines. Therefore, we can specify each path by assigning a flavor to each loop. Figure 1 depicts typical loops consisting of particle world lines in a two-dimensional system.

Now, suppose we are given the set of loops \(\{\ell_1, \ldots, \ell_k\}\) corresponding to a path \((X_s(m))_{s \in [0, \beta]}\). In general, a pair of a loop \(\ell\) and a flavor \(\sigma, \gamma = (\ell, \sigma)\), is called a \textit{flavored loop}. Then the path \((X_s(m))_{s \in [0, \beta]}\) can be identified with the collection of the flavored loops \(\Gamma(m) = \{\gamma_1, \ldots, \gamma_k\}\), where \(\gamma_j = (\ell_j, \sigma_j)\). The set \(\Gamma(m)\) is called the \textit{flavored random loop}.

By using Theorem 3.3, we obtain the following:

**Theorem 4.1.** The partition function has the following random loop representation:

\[
Z_{\Lambda}(\beta; b) = \int_{L_\beta} d\mu_\beta \prod_{\gamma \in \Gamma} \exp\{\beta w_{\gamma} f(\sigma_{\gamma})\},
\]

(4.2)

where, for each \(\gamma = (\ell_{\gamma}, \sigma_{\gamma}) \in \Gamma\), \(w_{\gamma}\) denotes the absolute value of the winding number of the loop \(\ell_{\gamma}\), and the function \(f\) on \(\{1, \ldots, n\}\) is given by

\[
f(\sigma) = \begin{cases} 
   b_1 & (\sigma = 1) \\
   -b_{\sigma-1} + b_\sigma & (2 \leq \sigma \leq n-1) \\
   -b_{n-1} & (\sigma = n).
\end{cases}
\]

(4.3)

**Proof.** Given \(m \in L_\beta\), consider the flavored loops \(\Gamma(m) = \{\gamma_1, \ldots, \gamma_k\}\) corresponding to the path \((X_s(m))_s\). When expressed as \(\gamma_j = (\ell_j, \sigma_j)\), each loop \(\ell_j\) can be represented in terms of the particle world lines: \(\ell_j = \bigcup_{s \in [0, \beta]} \bigcup_{i \in I_j} x_s^{(i)}(m)\), where we denote by \(I_j\) the set of particle labels that
Note that for a given /f_lavored random loops 
four.osf/ /two.osf
Proof of Theorem /two.osf/ /two.osf
comprise each \( \ell \),. Using these symbols, we get
\[
\prod_{j=1}^{N} \prod_{\sigma=1}^{n-1} \exp \left\{ \int_0^\beta b_\sigma k_\sigma (X^{(j)}_s(m)) ds \right\} = \prod_{j=1}^{k} \prod_{\sigma=1}^{n-1} \exp \left\{ \int_0^\beta b_\sigma k_\sigma (X^{(j)}_s(m)) ds \right\}
\]
\[
= \prod_{j=1}^{k} C(\gamma_j). \tag{4.4}
\]

Recalling (3.5), we obtain
\[
\sum_{i \in I_j} \delta_x (X^{(i)}_m) = \sum_{i \in I_j} \sum_{x \in \Lambda} \sum_{\sigma=1}^{n-1} b_\sigma \{ \delta_x (X^{(i)}_m) - \delta_x (X^{(i)}_m) \}
\]
\[
= \sum_{i \in I_j} \sum_{x \in \Lambda} \delta_x (X^{(i)}_m) f(\sigma_j) = |I_j| f(\sigma_j). \tag{4.5}
\]
Combining this with the fact \(|I_j| = w(\gamma_j)\), we conclude that \( C(\gamma) = e^{\beta w(\gamma)} f(\gamma). \)

4.2 Proof of Theorem 2.2

Note that for a given flavored random loops \( \Gamma(m) = \{ \gamma_1, \ldots, \gamma_k \} \), the collection of corresponding winding numbers \( w_{\Gamma(m)} = (w_{\gamma_1}, \ldots, w_{\gamma_k}) \) is a partition of \( N \):
\[
\sum_{j=1}^{k} w_{\gamma_j} = N. \tag{4.6}
\]
Thus, as \( m \) runs in \( L_\beta \), \( w_{\Gamma(m)} \) runs through various partitions of \( N \).

To state a technical lemma, we prepare some symbols. Let \( L \) and \( F \) be the collections of the loops and the flavors associated with \( \Gamma(m) \), respectively:
\[
L(\Gamma(m)) = (\ell_\gamma : \gamma \in \Gamma(m)), \quad F(\Gamma(m)) = (\sigma_\gamma : \gamma \in \Gamma(m)). \tag{4.7}
\]
Note that, if \( \#\Gamma(m) = k \), then \( F(\Gamma(m)) \) belongs to \( \mathcal{F}_k := \{1, \ldots, n\}^k \). Given \( F \in \{1, \ldots, n\}^k \), we set
\[
S(k; F) = \{ m \in L_\beta : \#\Gamma(m) = k, \quad F(\Gamma(m)) = F \}. \tag{4.8}
\]
Next, we divide the set of partitions of \( N \), \( P_N \), as follows:
\[
P_N = \bigcup_{k=1}^{N} P_N(k), \tag{4.9}
\]
where we set \( P_N(k) = \{ n \in P_N : \#n = k \} \). For each \( F \in \mathcal{F}_k \) and \( n \in P_N(k) \), define
\[
S(k; F; n) = \{ m \in S(k; F) : w_F = n \}, \tag{4.10}
\]
where \( w_F \) is the collection of the winding numbers of the loops associated with \( \Gamma \). It is essential in the proof of Theorem 2.2 that \( S(k; F) \) can be partitioned as follows:
\[
S(k; F) = \bigsqcup_{n \in P_N(k)} S(k; F; n). \tag{4.11}
\]
Lemma 4.2. Fix \( k \in \{1, \ldots, n\} \), arbitrarily. We also fix \( n \in P_N(k) \). Then \( \mu_\beta(S(k; F; n)) \) is independent of \( F \) and constant on \( \mathcal{F}_k \). Setting
\[
D_\beta(n) = \mu_\beta(S(k; F; n)),
\]
we obtain the following identity:
\[
\sum_{F \in \mathcal{F}_k} \int_{S(k; F; n)} \sum_{\gamma \in \Gamma} d\mu_\beta \prod_{\gamma \in \Gamma} \exp\{\beta w_\gamma f(\sigma_\gamma)\} = D_\beta(n) \mathcal{G}_\beta(n; b),
\]
where \( \mathcal{G}_\beta(n; b) \) is given by (2.17).

Proof. Let \( \mathcal{G}_n \) be the permutation group on the set \( \{1, 2, \ldots, n\} \). For any \( \varepsilon = (\varepsilon_j)_{j=1}^N \in \mathcal{G}_n \), we define the unitary operator \( U_\varepsilon \) on \( \mathcal{S}_n \) by
\[
(U_\varepsilon F)(X) = F(X_\varepsilon), \quad F \in \mathcal{S}_n,
\]
where \( X_\varepsilon = (X_\varepsilon^{(1)}, \ldots, X_\varepsilon^{(n)}) \) is defined by \( X_\varepsilon^{(j)} = (x^{(j)}, \varepsilon_j(\sigma^{(j)})) \). Because
\[
U_\varepsilon H_\Lambda(0)U_\varepsilon^{-1} = H_\Lambda(0)
\]
holds for any \( \varepsilon \in \mathcal{G}_n \), we readily confirm that \( \mu_\beta(S(k; F; n)) \) is independent of \( F \) and constant on \( \mathcal{F}_k \).

Since the function \( \prod_{\gamma \in \Gamma} \exp\{\beta w_\gamma f(\sigma_\gamma)\} \) is constant on \( S(k; F; n) \), we see that
\[
\int_{S(k; F; n)} \sum_{F \in \mathcal{F}_k} \sum_{\gamma \in \Gamma} d\mu_\beta \prod_{\gamma \in \Gamma} \exp\{\beta w_\gamma f(\sigma_\gamma)\} = \sum_{F \in \mathcal{F}_k} \sum_{\gamma \in \Gamma} \prod_{\gamma \in \Gamma} \exp\{\beta w_\gamma f(\sigma_\gamma)\}
\]
where we set \( F = (\sigma_1, \ldots, \sigma_k) \) and \( n = (n_1, \ldots, n_k) \). Therefore, by combining the first half of the statement with this fact, we get the following:
\[
\sum_{F \in \mathcal{F}_k} \int_{S(k; F; n)} \sum_{\gamma \in \Gamma} d\mu_\beta \prod_{\gamma \in \Gamma} \exp\{\beta w_\gamma f(\sigma_\gamma)\} = \sum_{F \in \mathcal{F}_k} D_\beta(n) \prod_{j=1}^k \exp\{\beta w_j f(\sigma_j)\}
\]
\[
= D_\beta(n) \prod_{j=1}^k \sum_{\gamma \in \Gamma} \exp\{\beta w_j f(\sigma_j)\}
\]
\[
= D_\beta(n) \mathcal{G}_\beta(n; b).
\]
We are now done with the proof of Lemma 4.2.

Completion of the proof of Theorem 2.2. Using the fact that \( L_\beta \) can be divided as
\[
L_\beta = \bigcup_{k=1}^N \bigcup_{F \in \mathcal{F}} S(k; F) = \bigcup_{k=1}^N \bigcup_{F \in \mathcal{F}} \bigcup_{n \in P_N(k)} S(k; F; n),
\]
the partition function can be expressed as follows:
\[
Z_\Lambda(\beta; b) = \sum_{k=1}^N \sum_{n \in P_N(k)} \sum_{F \in \mathcal{F}_k} \int_{S(k; F; n)} \sum_{\gamma \in \Gamma} d\mu_\beta \prod_{\gamma \in \Gamma} \exp\{\beta w_\gamma f(\sigma_\gamma)\}
\]
\[
= \sum_{k=1}^N \sum_{n \in P_N(k)} D_\beta(n) \mathcal{G}_\beta(n; b)
\]
\[
= \text{the RHS of (2.16)}.
\]
This completes the proof of Theorem 2.2.
4.3 Proof of Theorem 2.4

First, note that the expected value \( \langle h_\sigma \rangle \) can be expressed as follows

\[
\beta \langle h_\sigma \rangle = \frac{\partial}{\partial b_\sigma} \log Z_\Lambda(\beta; b) = Z_\Lambda(\beta; b)^{-1} \sum_{n \in P_N} \sum_{j=1}^k D_\beta(n) G_\beta(n; b) \frac{\partial}{\partial b_\sigma} G_\beta(n_j; b) \frac{\partial}{\partial b_\sigma} G_\beta(n_j; b).
\]  (4.20)

In the following, we estimate \( \frac{\partial}{\partial b_\sigma} G_\beta(m; b) / G_\beta(m; b) \) from below. To this end, we set

\[
C_\sigma(m) = e^{\beta m B_\sigma} + e^{\beta m B_{\sigma+1}},
\]  (4.21)

\[
S_\sigma(m) = e^{\beta m B_\sigma} - e^{\beta m B_{\sigma+1}}.
\]  (4.22)

Then \( G_\beta(m; b) \) can be expressed as

\[
G_\beta(m; b) = \sum_{\sigma=1}^{n-1} e^{\beta m B_\sigma} = C_\sigma(m) + H_\sigma(m),
\]  (4.23)

where

\[
H_\sigma = \sum_{\tau \neq \sigma, \sigma+1} e^{\beta m B_\tau}.
\]  (4.24)

From this representation, the equality

\[
\frac{\partial}{\partial b_\sigma} G_\beta(m; b) = \beta m S_\sigma(m)
\]  (4.25)

follows immediately. Hence, we obtain

\[
\frac{\partial}{\partial b_\sigma} G_\beta(m; b) / G_\beta(m; b) = \beta m \frac{S_\sigma / C_\sigma}{1 + H_\sigma / C_\sigma}.
\]  (4.26)

If we define the function \( f \) by \( f(x) = \frac{1-e^{-x}}{1+e^{-x}} \), then since \( f \) is monotonically increasing, we have the following inequality:

\[
\frac{S_\sigma(m)}{C_\sigma(m)} = f(\beta m (B_\sigma - B_{\sigma+1})) \geq f(\beta (B_\sigma - B_{\sigma+1})) = f_{\beta, \sigma}(b),
\]  (4.27)

where we use the fact that \( B_\sigma > B_{\sigma+1} \) and \( f_{\beta, \sigma}(b) \) is defined by (2.22). A quick examination, on the other hand, also reveals the following inequality:

\[
\frac{H_\sigma(m)}{C_\sigma(m)} = \sum_{\tau \neq \sigma, \sigma+1} e^{\beta m (B_\tau - B_\sigma)} \leq \sum_{\tau \neq \sigma, \sigma+1} e^{\beta m (B_\tau - B_\sigma)} \leq g_{\beta, \sigma}(b),
\]  (4.28)

where \( g_{\beta, \sigma}(b) \) is given by (2.23). Here, we used the assumption that \( B_\sigma > B_\tau (\tau \neq \sigma) \) in deriving the second inequality. Putting the above inequalities together, we get

\[
\frac{\partial}{\partial b_\sigma} G_\beta(m; b) / G_\beta(m; b) \geq \beta m \frac{f_{\beta, \sigma}(b)}{1 + g_{\beta, \sigma}(b)},
\]  (4.29)

which implies that

\[
\text{the RHS of (4.20)} \geq \beta \frac{f_{\beta, \sigma}(b)}{1 + g_{\beta, \sigma}(b)} N,
\]  (4.30)

where we use the fact \( \sum_{j=1}^k n_j = N \). We are now done with the proof of Theorem 2.4. \( \square \)
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