SO(10) COSMIC STRINGS AND BARYON NUMBER VIOLATION

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SO(10) cosmic strings formed during the phase transition Spin(10) → SU(5) × Z₂ are studied. Two types of strings — one effectively Abelian and one non-Abelian — are constructed and the string solutions are calculated numerically. The non-Abelian string can catalyze baryon number violation via the “twisting” of the scalar field which causes mixing of leptons and quarks in the fermion multiplet. The non-Abelian string is also found to have the lower energy possibly for the entire range of the parameters in the theory. Scattering of fermions in the fields of the strings is analyzed, and the baryon number violation cross section is calculated. The role of the self-adjoint parameters is discussed and the values are computed.

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I. INTRODUCTION

Motivated by the Callan-Rubakov effect in the context of magnetic monopoles \[1\], studies have been carried out recently on the possibility that cosmic strings can also catalyze baryon-number violation with strongly enhanced cross sections. It has been shown that the wave function of a fermion scattering off a cosmic string can acquire a large amplification factor near the core of the string, leading to enhancement of the processes that violate baryon number inside the string \[2,3\]. The catalysis processes that have been studied include those mediated by scalar fields and by the grand-unified X and Y gauge bosons in the string core. Although strings, in contrast to monopoles, have no magnetic fields outside, fermions can interact quantum-mechanically with the long-range gauge fields via the Aharonov-Bohm effect. Depending on the flux of the string and the core model used, the enhanced catalysis cross sections (per length) can be of the scale of strong interactions in comparison to the much smaller geometrical cross section \( \sim \Lambda_{\text{GUT}}^{-1} \), where \( \Lambda_{\text{GUT}} \sim 10^{16} \text{ GeV} \). In the early universe when the density of cosmic strings is high, such processes can play important roles, washing out any primordially-generated baryon asymmetry \[4\], or conceivably even generating the baryon to entropy ratio observed today.

Cosmic strings can be produced during certain phase transitions when a gauge group G is broken down to a subgroup H by the vacuum expectation value of some scalar field \( \phi \). The topological criterion for the existence of a string is a nontrivial fundamental homotopy group of the vacuum manifold G/H, denoted by \( \pi_1(G/H) \). For a connected and simply-connected G, the general construction of the scalar field at large distances from the string is given by

\[
\phi(\theta) = g(\theta)\phi_0, \quad g(\theta) = e^{i\tau \theta}.
\]

Here \( \tau \) is some generator of G, \( \theta \) is the azimuthal angle measured around the string, and \( g(0) \) and \( g(2\pi) \) belong to two disconnected pieces of H. In the papers referenced in the previous paragraph, the scalar field responsible for the formation of the string is taken to have the simple form \( \phi(\theta) = e^{i\tau \theta} \phi_0 = e^{i\theta} \phi_0 \). As a result, a non-Abelian string can be modeled by a
U(1) vortex, and the scattering of fermions in the background fields of the string is governed by the Abelian Dirac equation. In general however, for a given $\phi_0$, the generator $\tau$ can be chosen such that $e^{i\tau\theta}\phi_0$ “twists” around the string in more complicated fashion than a phase $e^{i\theta}$ times $\phi_0$. This gives rise to dynamically different strings which are intrinsically non-Abelian \[5\]. One expects the complexity and rich structure of such strings to lead to interesting effects on fermions traveling around them. In particular, we will demonstrate in this paper that for certain $\tau$’s, the twisting of $\phi(\theta)$ can result in mixing of lepton and quark fields, providing a mechanism for baryon number violations distinct from the processes in Abelian strings studied previously.

Since no strings are formed in the minimal SU(5) model, we choose the gauge group SO(10) \[6\] in this paper as an example of grand unified theories in investigating the B-violating process. We will construct string configurations, solve numerically for the undetermined functions, and study the baryon catalysis in the SO(10) theory, although we expect such processes to occur in other non-Abelian theories as well. In SO(10), stable strings can be formed when Spin(10) — the simply-connected covering group of SO(10) — is broken down to SU(5)×$\mathbb{Z}_2$ by the vacuum expectation value of a Higgs field $\phi$ in the 126 representation \[7\]. The generators of SO(10) transform as the adjoint 45, which transforms as $24 + 1 + 10 + \bar{10}$ under SU(5). The 24 and 1 generate the subgroup SU(5)×U(1), where the U(1) includes simultaneous rotations in the 1-2, 3-4, 5-6, 7-8, and 9-10 planes. We are interested in the generators outside SU(5) because to have noncontractible loops at all, $g(\theta)$ in Eq. (1) has to be outside the unbroken H for some $\theta$. We will refer to the U(1) generator as $\tau_{\text{all}}$ and to any of the other 20 basis generators outside SU(5) as $\tau_1$; we name the associated strings as string-$\tau_{\text{all}}$ and string-$\tau_1$, respectively. As we shall see, the scalar field of string-$\tau_1$ causes mixing of leptons and quarks while string-$\tau_{\text{all}}$ is effectively Abelian and no such mixing occurs. Properties of string-$\tau_{\text{all}}$ such as the string mass per unit length \[8\] and its superconducting capability in terms of fermion zero modes \[9\] have been studied. We will compare it with string-$\tau_1$, which will be the main subject of study of this paper.

In Sec. II, we give more detailed discussion of the Higgs 126 and the breaking of Spin(10)
to SU(5) × Z₂, and elaborate on the B-violating mechanism due to the nontrivial winding of the Higgs field. In Sec. III, we write down an ansatz for the field configuration of each string and derive the corresponding equations of motion. The numerical solutions and the energy of the strings are presented in Sec. IV, where we find that τ₁-strings have lower energy than τall-strings, probably for the entire range of the parameters in the theory. Having shown that such strings are energetically favorable, we turn to the scattering problem in Sec. V, where the Dirac equation in the background fields of the strings is solved, and the differential cross section for the B-violating processes in string-τ₁ is calculated. We also comment on the role of the self-adjoint parameters and compute their values using our string solutions.

To establish a common notation and to facilitate reading of this paper, we include in the Appendix a discussion about the relevant aspects of the spinor representation 16 of SO(10), which accommodates a single generation of left-handed fermions.

II. SO(10) STRINGS

There is considerable freedom in the breakings of SO(10) down to the low energy gauge group SU(3) × U(1). Two commonly studied examples include the breaking via an intermediate SU(5), SO(10) → SU(5), and the one via an intermediate Pati-Salam SU(4) × SU(2)_L × SU(2)_R. Details of the symmetry breaking patterns and the Higgs fields inducing the breakings can be found in Ref. 6 and the papers by Slansky and Rajpoot. Kibble, Lazarides and Shafi argued that the strings formed during the phase transition SO(10) → SU(4) × SU(2)_L × SU(2)_R become boundaries of domain walls. Thus in this paper we choose the SU(5) breaking pattern instead for its simplicity. More precisely, we study strings formed when Spin(10) → SU(5) × Z₂ by the vacuum expectation value of a Higgs 126 φ. The nontrivial element of Z₂ corresponds to rotation by 2π in SO(10). The homotopy group π₁(Spin(10)/SU(5) × Z₂) is Z₂; therefore a Z₂ string is formed during this phase transition. The subsequent symmetry breakings can be implemented by the adjoint 45 of SO(10) and the fundamental 10 in the usual fashion:
\( \text{Spin}(10) \xrightarrow{126} \text{SU}(5) \times \mathbb{Z}_2 \)
\( \xrightarrow{45} \text{SU}(3) \times \text{SU}(2) \times \text{U}(1) \times \mathbb{Z}_2 \)
\( \xrightarrow{10} \text{SU}(3) \times \text{U}(1)_{\text{em}} \times \mathbb{Z}_2. \) 

(2)

This \( \mathbb{Z}_2 \) string survives all the symmetry breakings since \( \mathbb{Z}_2 \) is preserved at low energies.

The \textbf{126} representation consists of fifth-rank anti-symmetric tensors satisfying the self-duality condition

\[ \phi_{i_1 \ldots i_5} = \frac{i}{5!} \epsilon_{i_1 \ldots i_{10}} \phi_{i_6 \ldots i_{10}}. \] 

(3)

The component which acquires an expectation value \( \langle \phi \rangle \) transforms as an SU(5) singlet, and to write it down explicitly, we first specify how the SU(5) subgroup is embedded in SO(10). The fundamental representation of SO(10) consists of 10×10 matrices, which can be labeled by indices \( i,i = 1, \ldots, 10 \). The generators of SO(10) in this representation can be written as antisymmetric, purely imaginary matrices. The generators of SU(5) in the fundamental representation are hermitian, traceless 5×5 matrices which can be written as

\[ \tau_{\alpha \beta} = S_{\alpha \beta} + iA_{\alpha \beta}, \] 

(4)

where \( \alpha, \beta = 1, \ldots, 5 \) label the matrix elements, and \( S,A \) are real 5×5 matrices, representing the real and imaginary parts of \( \tau \). Hermiticity and tracelessness of \( \tau \) require \( S_{\alpha \beta} = S_{\beta \alpha}, A_{\alpha \beta} = -A_{\beta \alpha}, \) and \( \text{Tr}S = 0 \). A natural way to embed SU(5) in SO(10) is to treat five-dimensional complex vectors as ten-dimensional real vectors, \textit{i.e.} replace the paired indices \( (\alpha,a) \), where \( \alpha = 1, \ldots, 5 \) label a five-dimensional vector and \( a = 1,2 \) label its real and imaginary parts, by the index \( i, i = 1, \ldots, 10 \). Then, the generators of the subgroup SU(5) of SO(10) can be expressed as

\[ \tau_{\alpha a, \beta b} = i(A_{\alpha \beta} I_{ab} + S_{\alpha \beta} M_{ab}), \] 

(5)

where \( I \) is the 2×2 identity matrix and \( M = i\sigma_2, \sigma_2 \) being the second 2×2 Pauli matrix. One can convince oneself that in this \( (\alpha,a) \) notation, the rank-five antisymmetric Levi-Civita tensor \( \epsilon_{\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5} \) which transforms as an SU(5) singlet in the SU(5) notation becomes
\[ f(a_1...a_5) \epsilon_{a_1a_2a_3a_4a_5}, \]  

where \( f(a_1...a_5) \) is defined to equal the number of \( a_i \) that takes the value 2. It is also straightforward to check that this expression satisfies the self-duality condition (Eq. (3)). Thus \( \langle \phi \rangle \) is written as

\[ \langle \phi_{a_1a_2...a_5} \rangle = \mu \ f(a_1...a_5) \epsilon_{a_1a_2a_3a_4a_5}, \]  

where \( \mu \) is a parameter.

Some words about our notation. The tensor indices \( i_1, \ldots, i_5 \) of \( \phi_{i_1...i_5} \) will be suppressed for convenience and legibility whenever no ambiguity should arise. In the expressions like \( \tau \phi \) and \( e^{i\tau\theta} \phi \) where \( \tau \) operates on \( \phi \), \( \tau \) is understood to be in the same representation of \( \phi \), \( \text{i.e.} \ \tau \) is the short-hand for

\[ \tau_{i_1...i_5j_1...j_5} = \tau_{i_1j_1}\delta_{i_2j_2}...\delta_{i_5j_5} + \delta_{i_1j_1}\tau_{i_2j_2}...\delta_{i_5j_5} + \ldots \]  

With the symmetry breaking \( \text{Spin}(10) \rightarrow \text{SU}(5) \times \mathbb{Z}_2 \), strings are formed. At spatial infinity, the general form of \( \phi \) is given by Eq. (1). For the energy to be finite, the covariant derivative of \( \phi \), \( D_\mu \phi \equiv \partial_\mu \phi + eA_\mu \phi \), has to vanish at spatial infinity; therefore the gauge field \( A_\mu \) takes the form \( A^\theta = i \frac{1}{e^r} \tau, \ A^r = 0 \), as \( r \rightarrow \infty \). In the core of the string, there is a magnetic flux \( \oint A \cdot d\vec{l} = \frac{2\pi}{e^r} \tau \) pointing in the direction of \( \tau \) in group space. Strings carrying flux pointing in different directions in group space are topologically equivalent since the only nontrivial winding number here is one, but dynamically they can differ. Because the scalar field \( \phi(\theta) \) varies with \( \theta \), the embedding of the unbroken subgroup \( \text{SU}(5) \) in \( \text{SO}(10) \) outside the string also varies with \( \theta \). More precisely, the generators \( \tau_\alpha^a, a = 1, \ldots, 24 \) of the unbroken SU(5) at \( \theta \) are related to the generators \( \tau_\alpha^a \) of the unbroken SU(5) at \( \theta = 0 \) by the similarity transformation

\[ \tau_\theta^a = g(\theta) \tau_0^a g^{-1}(\theta), \ g(\theta) = e^{i\tau\theta}. \]  

Consequently, the fermion fields which transform as 1, 5 and 10 under SU(5) are also rotated as one goes around the string. How the fields mix depends on which direction in group space \( \phi(\theta) \) winds.
The SO(10) generators can be written as 10×10 matrices of the form
\((\tau^{ab})_{ij} = -i(\delta_i^a \delta_j^b - \delta_i^b \delta_j^a)\), where \(a, b\) label the group indices, \(i, j\) label the matrix elements, and \(a, b, i, j\) all run from 1 to 10. In this notation \(\tau_{\text{all}}\) is given by
\[
\tau_{\text{all}} \equiv \frac{1}{5}(\tau^{12} + \tau^{34} + \ldots + \tau^{910}), \tag{10}
\]
where the factor of 1/5 is included for \(\phi(\theta)\) to have a 2\(\pi\) rotational period. It takes a little more effort to write down the \(\tau_i\)'s. Let us first write the SU(5) generators specified by Eq. (5) in terms of \(\tau^{ab}\) given above. The four diagonal generators are trivial. For the other twenty generators, one can group the 10×10 space into 2×2 blocks, and write the 45 \(\tau^{ab}\)'s as \(\tau^{2\alpha-1,2\beta-1}, \tau^{2\alpha-1,2\beta}, \tau^{2\alpha,2\beta-1}\), and \(\tau^{2\alpha,2\beta}\), where \(\alpha, \beta\) both run from 1 to 5. Then it is not hard to see that the twenty linear combinations
\[
\frac{1}{2}(\tau^{2\alpha-1,2\beta} - \tau^{2\alpha,2\beta-1}),
\frac{1}{2}(\tau^{2\alpha-1,2\beta-1} + \tau^{2\alpha,2\beta}), \quad \alpha < \beta. \tag{11}
\]
are all of the form of Eq. (5), and therefore can be chosen to be the twenty off-diagonal generators of SU(5). Note that the superscripts \(\alpha, \beta\) above label the group indices while the subscripts \(\alpha, \beta\) in Eq. (5) label the matrix elements. The twenty \(\tau_i\)'s outside SU(5) then can be expressed by the other twenty linear combinations as
\[
\tau_1 \equiv \frac{1}{2}(\tau^{2\alpha-1,2\beta} + \tau^{2\alpha,2\beta-1}),
\frac{1}{2}(\tau^{2\alpha-1,2\beta-1} - \tau^{2\alpha,2\beta}), \quad \alpha < \beta. \tag{12}
\]

Other than the SU(5) group properties, the linear combinations above can also be classified under the group SO(4), which is locally isomorphic to SU(2)×SU(2). For a given \(\alpha\) and \(\beta\) where \(\alpha < \beta\), the two generators of Eq. (11) plus the diagonal
\[
\frac{1}{2}(\tau^{2\alpha-1,2\beta-1} - \tau^{2\alpha,2\beta}) \tag{13}
\]
can be easily shown to obey the SU(2) algebra. Similarly, the two generators of Eq. (12) plus
\[
\frac{1}{2} (r^{2\alpha-1,2\beta-1} + r^{2\alpha,2\beta}) \quad (14)
\]
generate another SU(2). Thus, for a given \( \alpha \) and \( \beta \) (\( \alpha < \beta \)), the six generators of Eqs. (11-14) generate rotations in the 4-dimensional space spanned by vectors in the \( 2\alpha - 1, 2\alpha, 2\beta - 1, 2\beta \) directions.

### III. FIELD CONFIGURATIONS

The relevant part of the Lagrangian for the SO(10) theory is given by

\[
\mathcal{L} = \frac{1}{4} tr F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^*(D^\mu \phi) - V(\phi) 
\quad (15)
\]

where \( F_{\mu\nu} = -i F^a_{\mu\nu} \tau_a, A_\mu = -i A^a_\mu \tau_a, F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + e[A_\mu, A_\nu], D_\mu = \partial_\mu + eA_\mu ; \)
\( A^a_\mu, a = 1, \ldots, 45, \) are the SO(10) gauge fields and \( \phi \) is the Higgs 126. The most general gauge-invariant and renormalizable potential \( V(\phi) \) contains all the distinct contractions of two and four \( \phi \)'s:

\[
V(\phi) = v_1 \phi_{i_1...i_5} \phi_{i_1...i_5}^* + v_2 (\phi_{i_1...i_5} \phi_{i_1...i_5}^*)^2 \\
+ v_3 \phi_{i_1i_2n_3n_4n_5} \phi_{j_1j_2n_3n_4n_5}^* \phi_{i_3i_4l_3l_4l_5} \phi_{j_3j_4l_3l_4l_5}^* \\
+ v_4 \phi_{i_1i_2n_3n_4n_5} \phi_{j_1j_2n_3n_4n_5}^* \phi_{i_3i_4l_3l_4l_5} \phi_{j_3j_4l_3l_4l_5}^* \\
+ v_5 \phi_{i_1i_2n_3n_4n_5} \phi_{j_1j_2n_3n_4n_5}^* \phi_{i_3i_4l_3l_4l_5} \phi_{j_3j_4l_3l_4l_5}^* \\
+ v_6 \phi_{i_1i_2j_3n_4n_5} \phi_{j_1j_2j_3n_4n_5}^* \phi_{i_3i_4l_3l_4l_5} \phi_{j_3j_4l_3l_4l_5}^*. 
\quad (16)
\]

In writing down the \( v_3 \) through \( v_6 \) terms above, one has to consider two things: (1) the possible ways to contract the indices, and (2) which \( \phi \)'s are to be complex conjugated. One can deal with (1) without the complication of (2) by adopting an equivalent real 252 representation for \( \phi \) because a complex, self-dual 126-dimensional tensor can be thought of as a real, 252-dimensional tensor by dropping the self-duality condition and taking the real parts of the resulting complex, 252-dimensional tensor. One can see there are only four distinct terms and they are terms \( v_3 \) through \( v_6 \) in Eq. (16) above. Then when \( \phi \) is
taken to be complex, two out of the four $\phi$’s have to be complex conjugated to make the potential real. There are three possibilities: $\phi\phi^*\phi\phi^*$, $\phi^*\phi\phi\phi^*$, $\phi\phi\phi\phi^*$, for each of the four contractions $\phi\phi\phi\phi$ when $\phi$ is real. But after the self-duality condition is applied, one can show that only one of the three terms is actually independent.

The Euler-Lagrange equations of motion for $\phi$ and $A_\mu$ are given by

$$D_\mu D^\mu \phi = -\frac{\partial V}{\partial \phi^*},$$

$$Tr(\tau^a)^2(\partial_\mu F^a_{\mu\nu} + e f^{abc} A^b_\mu F^c_{\mu\nu})$$

$$= i e \{ (D^\nu \phi)^* (\tau^a \phi) - (\tau^a \phi)^* (D^\nu \phi) \},$$

where $a$ is not summed over, and where a basis has been chosen so that $Tr(\tau^a \tau^b) = \delta^{ab} Tr(\tau^a)^2$.

We construct for string-$\tau_{\text{all}}$ a solution of the following form:

**Ansatz I**:

$$\phi = f(r) e^{i\tau_{\text{all}}^a \phi_0} = f(r) e^{i\theta} \phi_0,$$

$$A^a = i \frac{g(r)}{er} \tau_{\text{all}},$$

$$A^r = 0,$$

where $\phi_0 \equiv \langle \phi \rangle$ as defined in Eq. (7). The boundary conditions on the functions are

$$f(0) = 0, \quad f(r) \xrightarrow{r \to \infty} \mu,$$

$$g(0) = 0, \quad g(r) \xrightarrow{r \to \infty} 1;$$

$V(\phi)$ is minimized at $f = \mu$. Inserting this *ansatz* into the equations of motion and using the relations $\tau_{\text{all}}^a \tau_{\text{all}}^b \phi_0 = \phi_0$ and $(\tau_{\text{all}} \phi_0)^* (\tau_{\text{all}} \phi_0) = \phi_0^* \phi_0 = 3840 \equiv N$, we obtain two coupled differential equations for $f(r)$ and $g(r)$:

$$f'' + \frac{1}{r} f' - \left( \frac{1 - g}{r^2} \right)^2 f = f(v_1 + 2Nv_2 f^2),$$

$$Tr(\tau_{\text{all}}^2) \left( g'' - \frac{1}{r} g' \right) = -2Ne^2(1 - g)f^2,$$  

(21)
where the prime denotes differentiation with respect to $r$, and $Tr(\tau^2_{\text{all}}) = \frac{2}{5}$ from Eq. (10).

An expansion of $f(r)$ and $g(r)$ in powers of $r$ around the origin reveals that $f(r)$ is odd in $r$ with a linear leading term, whereas $g(r)$ is even in $r$ with a quadratic leading term.

Inserting Ansatz I for string-$\tau_{\text{all}}$ into the Lagrangian gives

$$- \mathcal{L}^{\text{all}} = \frac{Tr(\tau^2_{\text{all}})}{2e^2 r^2} g'^2 + N f'^2 + N \frac{(1 - g)^2}{r^2} f^2 + N (v_1 f^2 + N v_2 f^4).$$

(22)

As a consistency check, note that the equations of motion obtained by varying $\mathcal{L}^{\text{all}}$ with respect to the functions $g$ and $f$ are identical to those in Eq. (21).

Note that the parameters $v_3$ through $v_6$ in the potential $V$ are absent from Eq. (21) and $\mathcal{L}^{\text{all}}$ above. This is because whenever one index of a given $\phi$ is contracted with one index of another $\phi$, this index is summed over from 1 through 10, or in the $(\alpha, a)$ notation discussed earlier, from $\alpha = 1$ through 5 and $a = 1, 2$. For a given $\alpha$, the term with $a = 2$ by definition has an extra factor of $i^2 = -1$ compared to the term with $a = 1$. These two terms cancel each other when they are added. Because this is true for every $\alpha$, the third through the sixth terms in $V$ vanish identically for the string-$\tau_{\text{all}}$ ansatz.

To construct an ansatz for string-$\tau_1$, we need to consider separately the two sets of generators in Eq. (12), which will be referred to as

$$\tau_{1+} = \frac{1}{2}(\tau^{2\alpha - 1, 2\beta} + \tau^{2\alpha, 2\beta - 1}),$$

$$\tau_{1-} = \frac{1}{2}(\tau^{2\alpha - 1, 2\beta - 1} - \tau^{2\alpha, 2\beta}), \quad \alpha < \beta.$$  

(23)

As we shall see, it is sufficient to derive the equations of motion for an ansatz based on a generator of the form $\tau_{1+}$. By a simple redefinition, it will then be possible to construct an ansatz based on a generator of the form $\tau_{1-}$. For now, we consider the case when $\tau_1$ has the form $\tau_{1+}$. The simple extension of Ansatz I with $\tau_{\text{all}}$ replaced by $\tau_1$ does not work for string-$\tau_1$. The problem arises from the term $\tau_1 \tau_1 \phi$ on the left-hand side of Eq. (17) in which a new tensor $\phi_0^A$,

$$\tau_1 \tau_1 \phi_0 = \phi_0^A,$$

(24)
is generated, where

\[
\phi_{0i_1...i_5}^A \equiv \begin{cases} 
\phi_{0i_1...i_5} & \text{if two indices take the values} \\
(2\alpha - 1, 2\beta - 1) \text{ or } (2\alpha, 2\beta), \\
0, & \text{otherwise.}
\end{cases}
\]  

(25)

As a result, the differential equations for \(g(r)\) and \(f(r)\) are satisfied only if \(g(r) = 1\) or \(f(r) = 0\) everywhere, which is not consistent with the boundary conditions given by Eq. (20). (Note that the solution \(g = 1\) and \(f = \mu\) is the vacuum field configuration expressed in a singular gauge.)

We construct a nontrivial solution for string-\(\tau_1\) by replacing \(f(r)\phi_0\) and \(\tau_{\text{all}}\) in Ansatz I with \((f_1(r)\phi_0 + f_2(r)\phi_0^A)\) and \(\tau_1\) respectively. Note that \(\phi_0\) is not orthogonal to \(\phi_0^A\) because \(\phi_{0i_1...i_5}^A\phi_{0i_1...i_5}^{*A} \neq 0\). Therefore instead of expanding \(\phi\) in \(\phi_0\) and \(\phi_0^A\), we will use the more convenient basis \(\phi_0^A\) and \(\phi_0^B\) where

\[
\phi_0^B \equiv \phi_0 - \phi_0^A
\]

(26)

and \(\phi_0^B\) is orthogonal to \(\phi_0^A\):

\[
\phi_{0i_1...i_5}^A\phi_{0i_1...i_5}^B = 0.
\]

(27)

From the definition of \(\phi_0^A\) (Eq. (25)) and the properties of \(\phi_0\), one can see that

\[
\phi_{0i_1...i_5}^B = \begin{cases} 
\phi_{0i_1...i_5} & \text{if two indices take the values} \\
(2\alpha - 1, 2\beta) \text{ or } (2\alpha, 2\beta - 1), \\
0, & \text{otherwise}
\end{cases}
\]

(28)

and \(\phi_0^B\) is annihilated by \(\tau_1\):

\[
\tau_1 \phi_0^B = 0.
\]

(29)

The solution constructed for string-\(\tau_1\) is

\textit{Ansatz II}:
\[ \phi = e^{i\tau_1 \theta} \{ f_o(r) \phi^A_0 + f_e(r) \phi^B_0 \} , \]
\[ A^\theta = i \frac{g(r)}{er} \tau_1 , \]
\[ A^r = 0 , \]  

where as will become clear in the next two paragraphs, the functions \( f_o(r) \) and \( f_e(r) \) are named after their odd and even parities in \( r \).

At the origin, we require the fields to be regular. Since \( \phi^B_0 \) is left invariant by \( e^{i\tau_1 \theta} \) (Eq. (29)) but \( \phi^A_0 \) is not, at the origin \( f_e(0) \) can be any constant but \( f_o(0) \) has to vanish. At large \( r \), the scalar field \( \phi \) has to take the form

\[ \phi \xrightarrow{r \to \infty} \mu e^{i\tau_1 \theta} \phi_0 = \mu e^{i\tau_1 \theta} (\phi^A_0 + \phi^B_0) \]  

for the unbroken gauge group to be SU(5), so both \( f_o(r) \) and \( f_e(r) \) approach \( \mu \) at large \( r \).

The boundary conditions on the functions are

\[ f_o(0) = 0 , \quad f_o(r) \xrightarrow{r \to \infty} \mu , \]
\[ f_e(0) = a_0 , \quad f_e(r) \xrightarrow{r \to \infty} \mu , \]
\[ g(0) = 0 , \quad g(r) \xrightarrow{r \to \infty} 1 , \]  

where \( a_0 \) is a constant.

The equations of motion for \( \phi \) and \( A_\mu \) are closed when the fields take the form in Ansatz II. We obtain three coupled differential equations for \( f_o(r) \), \( f_e(r) \) and \( g(r) \). The algebra involved in extracting these three equations, however, is considerably more tedious than in the \( \tau_{all} \) case mainly because the forms of \( \phi^A_0, \phi^B_0 \) and \( \tau_1 \) are less symmetric. We will not present the algebra involved and simply quote the results:

\[ f_e'' + \frac{1}{r} f'_e = f_e \left\{ v_1 + Nv_2(f_o^2 + f_e^2) - \frac{N}{25} e^2 \lambda_3 (f_o^2 - f_e^2) \right\} \]
\[ f_o'' + \frac{1}{r} f'_o - \frac{(1 - g)^2}{r^2} f_o \]
\[ = f_o \left\{ v_1 + Nv_2(f_o^2 + f_e^2) + \frac{N}{25} e^2 \lambda_3 (f_o^2 - f_e^2) \right\} \]
\[ Tr(\tau_1^2) \left( g'' - \frac{1}{r} g' \right) = -Ne^2 (1 - g) f_o^2 , \]  

(33)
where \( e^2 \lambda_3 \equiv v_3 + \frac{v_4}{4} + \frac{v_5}{4} + \frac{v_6}{12} \), and \( Tr(\tau_1^2) = 1 \) from Eq. (12). An expansion of \( g, f_o \) and \( f_e \) in powers of \( r \) around the origin gives

\[
\begin{align*}
   f_o(r) &= a_1 r + a_3 r^3 + \ldots, \\
   f_e(r) &= a_0 + a_2 r^2 + \ldots, \\
   g(r) &= b_2 r^2 + b_4 r^4 + \ldots,
\end{align*}
\]

where the coefficients of all the higher terms are related to \( a_0, a_1 \) and \( b_2 \) recursively. The function \( f_o \) is indeed odd and \( f_e \) even in \( r \) as claimed earlier.

Inserting Ansatz II for string-\( \tau_1 \) into the Lagrangian gives

\[ -\mathcal{L}^1 = \frac{Tr(\tau_1^2)}{2e^2 r^2} g'^2 + \frac{N}{2} \left( f'^2 + f_o'^2 \right) + \frac{N (1 - g)^2}{r^2} f_o^2 + V_{ans} \]

where

\[ V_{ans} = \frac{N}{2} \left\{ v_1 (f_o^2 + f_e^2) + \frac{N}{2} v_2 (f_o^2 + f_e^2)^2 \ight. \]
\[ + \frac{N}{50} e^2 \lambda_3 (f_o^2 - f_e^2)^2 \right\}. \]

Here again, note that the equations of motion obtained by varying \( \mathcal{L}^1 \) with respect to the functions \( g, f_o \) and \( f_e \) are identical to those in Eq. (33).

Now let us consider the other case when \( \tau_1 \) has the form of \( \tau_1^- \). One can show that Eq. (24) now is \( \tau_1 \tau_1 \phi_0 = \phi_0^B \), and instead of \( \tau_1 \phi_0^B = 0 \), one has \( \tau_1 \phi_0^A = 0 \). Therefore by switching the definitions of \( \phi_0^A \) and \( \phi_0^B \) in Eqs. (25) and (28), all the equations between (24) and (32) are preserved, and one can show that the equations of motion are unchanged. We conclude that Ansatz II applies to all twenty \( \tau_1 \)'s, where for \( \tau_1^+ \), \( \phi_0^A \) and \( \phi_0^B \) are defined by Eqs. (25) and (28) respectively, but for \( \tau_1^- \), the definitions of the two are reversed. The equations of motion are given by Eq. (33) for all cases.

**IV. NUMERICAL CALCULATIONS**

In this section we present the numerical solutions to the two sets of differential equations (21) and (33) with the appropriate boundary conditions at the origin and some large value
of $r$. We implemented two methods: the “shooting” and the relaxation methods to handle this two-point boundary value problem. In the “shooting” method [12], an initial guess for the free parameters at $r = 0$ was made and then the equations were integrated out to large $r$ where the boundary conditions were specified. As the name of the method suggests, the true solutions were found by adjusting the parameters at $r = 0$ in the beginning of each iteration to reduce the discrepancies from the desired boundary conditions at large $r$ computed in the previous iteration. For string-$\tau_1$, the small-$r$ expansion of the functions in Eq. (34) gives

$g(0) = g'(0) = 0, f_o(0) = f_o''(0) = f'_e(0) = 0$, and $f''_e(0) = 2a_2$, where $a_2$ is related to $a_0, a_1$ and $b_2$, but the values of

\[
\begin{align*}
f_e(0) &= a_0, \\
f'_e(0) &= a_1, \\
g''(0) &= 2b_2,
\end{align*}
\]

were adjusted to match the boundary conditions at large $r$. For string-$\tau_{\text{all}}$, we have shown that $f(r)$ is odd and $g(r)$ is even in $r$, with $f(r) = ar + \ldots$ and $g(r) = br^2 + \ldots$. Thus only the two values $f'(0), g''(0)$ were free parameters. At large $r$, discrepancies from the boundary condition were corrected by the multi-dimensional Newton-Raphson method which computed the corrections to the initial parameters. With an initial guess for the parameters at $r = 0$, this “shooting” process was iterated until the “targets” were met. The fourth-order Runge-Kutta method was used to integrate the equations.

We have also implemented a relaxation scheme for comparison. In this method the first step is to express the string energy as a function of the values of the functions $f$ and $g$ (or $f_e, f_o,$ and $g$) defined on an evenly spaced mesh of points. While a Simpson’s rule approximation worked well for the middle range of parameters, a more sophisticated approximation was used to extend the range of parameters that could be treated. For each interval of two lattice spacings, smooth functions $\tilde{f}$ and $\tilde{g}$ were defined by 2nd order polynomial interpolation from the three mesh points (midpoint and two end points); with the help of a symbolic integration program, the integral defining the energy was carried out exactly for
the interpolated functions. (By this method the energy obtained is a rigorous upper limit on the true ground state string energy.) To avoid divergences caused by the explicit factors of $1/r^2$ in the energy density, the first interval had to be treated more carefully—instead of fitting the functions with a 2nd order polynomial, we fitted the coefficients of the analytically determined power series, such as Eq. (34). Trial functions $f$ and $g$ were chosen, and then the energy was minimized by varying each mesh point one at a time, successively going through the lattice many times. We found it efficient to begin with a coarse mesh which was made successively finer by factors of 2, interpolating the solution at each stage to obtain the first trial solution for the next stage. For the final run in each case we used 2048 points.

We found the results by the two methods to agree to approximately one part in a million or better. In general we were able to explore a wider parameter range with the relaxation method than with the “shooting” method, but the qualitative features given by the “shooting” method remained the same. (The author wishes to thank Alan Guth for implementing the relaxation part of the calculations.)

The dependence of the equations on the parameters in the theory can be simplified if $r, f, f_o$ and $f_e$ are rescaled as $(v_1 < 0)$

$$r \rightarrow \sqrt{-v_1} r,$$
$$\{f, f_o, f_e\} \rightarrow \sqrt{\frac{2Nv_2}{-v_1}} \{f, f_o, f_e\}.$$  \hspace{1cm} (38)

Then only the following combinations of parameters appear in the differential equations:

$$\lambda_2 \equiv \frac{v_2}{c^2},$$
$$\lambda_3 \equiv \frac{1}{c^2} \left( v_3 + \frac{v_4}{4} + \frac{v_5}{4} + \frac{v_6}{12} \right).$$  \hspace{1cm} (39)

The Hamiltonian densities $\mathcal{H}^{\text{all}}$ and $\mathcal{H}^1$ for the two strings are simply $-\mathcal{L}^{\text{all}}$ and $-\mathcal{L}^1$ given by Eqs. (22) and (35) because all fields are assumed to be time-independent. With the same rescaling, one obtains

$$\frac{v_2}{(-v_1)^2} \mathcal{H}^{\text{all}} = \frac{1}{2} \left\{ \frac{2\lambda_2}{5r^2} f'^2 + \frac{1 - g^2}{r^2} f'^2 + \frac{(1 - f^2)^2}{2} \right\}$$  \hspace{1cm} (40)
and
\[
\frac{v_2}{(-v_1)^2} H^1 = \frac{1}{2} \left\{ \frac{\lambda_2}{r^2} g^2 + \frac{f_o^2 + f_e^2}{2} + \frac{(1-g)^2}{2r^2} f_o^2 \\
+ \frac{1}{2} \left( 1 - \frac{f_o^2 + f_e^2}{2} \right)^2 + \frac{\lambda_3}{200\lambda_2} (f_o^2 - f_e^2)^2 \right\}
\] (41)

where the \( \tau_{\text{all}} \) equation depends on \( \lambda_2 \) only but the \( \tau_1 \) equation depends on both \( \lambda_2 \) and \( \lambda_3 \).

Typical solutions for the two strings calculated from the “shooting” method are shown in Figs. 1 and 2, where \( \lambda_2 = 0.132 \) and \( \lambda_3 = 10.25 \). For the same \( \lambda_2 \) and \( \lambda_3 \), the solutions given by the relaxation method appear indistinguishable visually from those in Figs. 1 and 2. For string-\( \tau_{\text{all}} \), we were able to find solutions in the approximate range \( 10^{-2} < \lambda_2 < 10 \) using the “shooting” method and \( 10^{-4} < \lambda_2 < 10^2 \) using the relaxation method. For string-\( \tau_1 \), we explored the range \( 5 \times 10^{-2} < \lambda_2 < 1 \) and \( 0.5 < \lambda_3 < 10^2 \). In general, the functions converged more slowly near the two ends of each range above, and we did not attempt to find solutions beyond these limits. We numerically integrated \( H^{\text{all}} \) and \( H^1 \) for the solutions we computed, and found string-\( \tau_1 \) to have the lower energy for all the parameters we explored. In Fig. 3, the energy density \( 2\pi r H \) of the two solutions shown in Figs. 1 and 2 is plotted, and the energy of string-\( \tau_1 \) is clearly lower. For comparison, we point out that the energy per unit length of string-\( \tau_{\text{all}} \) in the range \( 0.9 < \lambda_2 < 4.0 \) has been calculated by Aryal and Everett [8]. Our values in this range of parameters agree with theirs to within 1%.

One of the most important properties of the two strings we investigate in this paper is whether string-\( \tau_1 \) has lower energy than string-\( \tau_{\text{all}} \). We just showed that this is true for some range of the parameters. To systematically explore a wider parameter range, however, it is very laborious and time-consuming to calculate the \( \tau_1 \) solutions for different \( \lambda_2 \) and \( \lambda_3 \) first and then compute the corresponding energy. Instead, we employ an upper-bound argument to reduce the two-dimensional parameter space \( (\lambda_2, \lambda_3) \) to one. We set \( f_o = f_e \equiv f_1 \) in the Lagrangian and take \( g(r), f_1(r) \) as trial functions for string-\( \tau_1 \). The advantage in using \( f_o = f_e \) is that the last term in Eq. (41) vanishes, and the equations no longer depend on \( \lambda_3 \). Moreover, Eqs. (40) and (41) then have the same functional form, differing only in the coefficients of the first and the third terms, and one can solve the equations for string-\( \tau_1 \) the
same way as for string-$\tau_{\text{all}}$ using different values of $\lambda_2$. The corresponding energy, denoted by $E_1(f_o = f_e)$, gives an upper bound on the true energy of string-$\tau_1$ by the variational principle. If $E_1(f_o = f_e) < E_{\text{all}}$ for a given $\lambda_2$, then one can conclude that string-$\tau_1$ has the lower energy for that value of $\lambda_2$ and all values of $\lambda_3$. (Note that in the limit of $\lambda_3 \to \infty$, the trial functions approach the true string solution because for the energy to be finite, the last term in Eq. (41) requires $f_o \to f_e$.) Our result is presented in Fig. 4, where the ratio $E_1(f_o = f_e)/E_{\text{all}}$ is plotted as a function of $\log \lambda_2$ for $10^{-4} < \lambda_2 < 2.5 \times 10^3$. Note that $E_1(f_o = f_e)/E_{\text{all}} < 1$ for all 7 decades of $\lambda_2$, and is approaching an asymptote of 1 (or possibly less than 1) as $\lambda_2 \to 0$. For large $\lambda_2$, we find the individual curves of $E_{\text{all}}$ vs. $\log \lambda_2$ and $E_1$ vs. $\log \lambda_2$ approach straight lines, suggesting that the ratio $E_1(f_o = f_e)/E_{\text{all}}$ levels off at a constant for large $\lambda_2$. We conclude that string-$\tau_1$ has lower energy than string-$\tau_{\text{all}}$ for $10^{-4} < \lambda_2 < 2.5 \times 10^3$ and all $\lambda_3$, and probably is the ground state for the entire range of the parameters in the theory.

V. SCATTERING SOLUTIONS

To study the scattering of fermions by an SO(10) cosmic string, one first needs to understand the 16-dimensional spinor representation of SO(10) to which the left-handed fermions are assigned. Spinor representations certainly have been discussed in the literature, but to establish a common notation, we discuss in the Appendix the construction of the generators, the sixteen states and the identification of states with fermions that are relevant to this paper.

Now we proceed to study the Dirac equation

$$(i \not \! \partial - e A^a \tau^a - m)\psi = 0$$  \hspace{1cm} (42)$$

in the background fields of string-$\tau_{\text{all}}$ and $\tau_1$: $A^a_\mu \tau^a = A^a_\mu \tau_{\text{all}}$ and $A^1_\mu \tau_1$. As shown in the Appendix, the fermion fields can be written as a 16-dimensional column vector where each component is identified with a fermion given by Eq. (A.16). The generators $\tau_{\text{all}}$ and $\tau_1$ can
be written as $16 \times 16$ hermitian matrices, where $\tau_{\text{all}}$ is diagonal with one diagonal entry equal to $\frac{1}{2}$, ten entries equal to $\frac{1}{10}$ and five entries equal to $-\frac{3}{10}$. For $\tau_1$, we choose $\tau_1 = \frac{1}{2}(\tau^{58} + \tau^{67})$ for illustration. We find that $\tau_1$ takes the block-diagonal form

$$-\tau_1 = \frac{1}{2} \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}, \quad (43)$$

where

$$B = \begin{pmatrix} 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 \end{pmatrix}, \quad (44)$$

and $I$ is the $2 \times 2$ identity matrix. For string-$\tau_{\text{all}}$, since $\tau_{\text{all}}$ is diagonal, Eq. (42) decouples into sixteen equations, one for each component of the wave function, and there is no mixing of leptons and quarks due to twisting of the Higgs. However, since the sixteen eigenvalues of $\tau_{\text{all}}$ are all fractional, all sixteen fermions scatter nontrivially off the string via the Aharonov-Bohm effect. As pointed out by previous studies, the wave functions of these fermions can be strongly enhanced near the core of the string, leading to strong B-violating processes inside the string.

In the case of string-$\tau_1$, upon diagonalizing $\tau_1$ by a unitary matrix $U$ and simultaneously rotating the fermion basis $\psi$ in Eq. (A.16) to $\tilde{\psi} \equiv U\psi$, we can write $\tilde{\psi}$ as

$$\tilde{\psi} = (e^-, u^c_1, e^-, u_1^c, e^c - u^c_1, \nu^c - d_1, u^c_2, u_2^c, d_3, d_2, \nu_c + d_1^c, u_3 + d_2^c, u_3 - d_2^c, u_2 + d_3^c, u_2 - d_3^c, u_1, \nu, e^+, d_1^c)_L$$

$$u_3 + d_2^c, u_3 - d_2^c, u_2 + d_3^c, u_2 - d_3^c, u_1, \nu, e^+, d_1^c)_L$$

$$\quad (45)$$

and Eq. (42) again decouples into sixteen equations of the form

$$(i \partial + e\lambda_i A^1 - m)\tilde{\psi}_i = 0,$$  

$$\quad (46)$$

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where each $\tilde{\psi}_i$ interacts with the gauge field with coupling strength $e \lambda_i$; $\lambda_i$ are the eigenvalues of $-\tau_1$. The eigenvalues are $\lambda_i = \frac{1}{2}$ for $e + u^c_1, \nu^c + d_1, u_3 + d_2^c, u_2 + d_3^c$, $\lambda_i = -\frac{1}{2}$ for $e - u^c_1, \nu^c - d_1, u_3 - d_2^c, u_2 - d_3^c$, and $\lambda_i = 0$ for all others. Since the $e + u^c$ and $e - u^c$ components have opposite eigenvalues, we expect a pure $e$ or $u^c$ to turn into a mixture of $e$ and $u^c$ as it propagates around the string, producing baryon-number violation.

Before calculating the scattering amplitude, we first comment on the choice of gauge in this problem. The fields in Ansatz II (See Eq. (30)) for string-$\tau_1$ were constructed in a gauge where the scalar field $\phi$ winds with $\theta$ and the gauge field falls off as $r^{-1}$ at large $r$. The particle content, however, is probably most transparent in a different gauge where $\phi$ does not wind with $\theta$ and $A_\mu \to 0$ at large $r$ everywhere except on a sheet of singularities at $\theta = 0$. We will refer to the former as the $1/r$-gauge and the latter as the “sheet” gauge, in analogy with the “string” gauge of a magnetic monopole. Continuing to work in the diagonalized basis, the fermion fields in the “sheet” gauge, $\tilde{\psi}_0$, are related to those in the $1/r$-gauge, $\tilde{\psi}$, by the gauge transformation

$$\tilde{\psi}_0 = e^{-i\tau_1(\pi - \theta)} \tilde{\psi}.$$  \hspace{1cm} (47)

We will solve the Dirac equation and calculate the scattering amplitude in the $1/r$-gauge, and then write down the baryon-number violating cross section in the “sheet” gauge.

In the presence of an infinitely-thin $\tau_1$-string along the $z$-axis, the gauge field $A^1_\mu$ takes the form $A^{1r} = A^{1z} = 0, A^{1\theta} = \frac{1}{er}$, where $(r, \theta)$ denote the usual polar coordinates with $\theta$ running counter-clockwise from the positive $x$-axis. Owing to the symmetry along the $z$-axis, the matrix $\gamma_3$ in Eq. (46) drops out, and with the choice for the $\gamma$-matrices

$$\gamma_0 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix},$$
$$\gamma_2 = \begin{pmatrix} -i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$  \hspace{1cm} (48)

Eq. (46) decouples into two independent equations for the upper and lower 2-component
spinors of $\tilde{\psi}_i$, where the two equations differ by the sign of the mass term. Writing the upper spinor of $\tilde{\psi}_i$ as
\[
\begin{pmatrix}
\chi_1(r) \\
\chi_2(r)e^{i\theta}
\end{pmatrix} e^{in\theta - iEt},
\] (49)
one can show
\[
\begin{pmatrix}
m - E & -i \left( \partial_r + \frac{n + \lambda_i + 1}{r} \right) \\
-i \left( \partial_r - \frac{n + \lambda_i}{r} \right) & -m - E
\end{pmatrix}
\begin{pmatrix}
\chi_1 \\
\chi_2
\end{pmatrix} = 0,
\] (50)
and the solutions are Bessel functions of order $(n + \lambda_i)$ and $-(n + \lambda_i)$:
\[
\begin{pmatrix}
\chi_1 \\
\chi_2
\end{pmatrix} = \begin{pmatrix}
J_{\pm(n + \lambda_i)}(kr) \\
\pm \frac{ik}{E + m} J_{\pm(n + \lambda_i + 1)}(kr)
\end{pmatrix}, \quad k = \sqrt{E^2 - m^2}.
\] (51)
The appropriate boundary conditions to impose, as pointed out in Ref. 14, are the square-integrability of the wave functions near the origin and a self-adjoint Hamiltonian. The usual requirement that wave functions be regular at the origin is sometimes too strong and has to be relaxed. Since $J_{\nu}(r) \sim r^\nu/(2^{\nu/2}\nu!)$ for small $r$, one can see that the solutions above are square-integrable if the + sign is chosen for the modes $n + \lambda_i > 0$, and the − sign for $n + \lambda_i < -1$. For the mode $-1 < n + \lambda_i < 0$, however, both choices are square-integrable albeit neither is regular at the origin, and the solution takes the form
\[
\begin{pmatrix}
\chi_1 \\
\chi_2
\end{pmatrix} = \begin{pmatrix}
sin \mu J_{n + \lambda_i} + \cos \mu J_{-(n + \lambda_i)} \\
\pm \frac{ik}{E + m} (\sin \mu J_{n + \lambda_i + 1} - \cos \mu J_{-(n + \lambda_i + 1)})
\end{pmatrix},
\] (52)
where $\mu$ is the self-adjoint parameter.

The scattering amplitude $f^{\lambda_i}(\theta)$ for the $i$th fermion in $\tilde{\psi}$ appears in the asymptotic wave function written as the sum of the incident plane wave and the scattered part:
\[
\tilde{\psi}_i \sim u_E e^{-i\lambda_i(\pi - \theta)} e^{i(kx - Et)} + \sqrt{\frac{i}{r}} v_E e^{-i\lambda_i(\pi - \theta)} f^{\lambda_i}(\theta) e^{i(kr - Et)},
\] (53)
where $u_E$ and $v_E$ are given by
\[ u_E = \begin{pmatrix} 1 \\ \frac{k}{E+m} \end{pmatrix}, \quad v_E = \begin{pmatrix} 1 \\ \frac{k}{E+m} e^{i\theta} \end{pmatrix}. \] (54)

Expanding \( e^{ikx} = e^{ikr \cos \theta} \) and \( e^{ikr} \) in Bessel functions using

\[ e^{ikr \cos \theta} = \sum_{n=-\infty}^{\infty} i^n J_n(kr) e^{in\theta}, \] (55)

and with

\[ f^{\lambda_i}(\theta) = \sum_{n=-\infty}^{\infty} f_n^{\lambda_i} e^{in\theta}, \] (56)

Eq. (53) can be matched to the solutions in Eq. (51) mode by mode at large \( r \). Then the scattering amplitude can be calculated:

\[ f^{\lambda_i}(\theta) = \frac{i}{\sqrt{2\pi k}} e^{-i[\lambda_i] \theta} \left( \frac{\sin \left( \frac{\theta}{2} - \frac{\pi \lambda_i}{2} \right)}{\sin \frac{\theta}{2}} - e^{2i\delta} \right), \] (57)

where \([\lambda_i]\) denotes the largest integer less than or equal to \( \lambda_i \), and \( \delta \) is related to \( \lambda_i \) and the self-adjoint parameter \( \tan \mu \) by

\[ \tan \delta = \frac{1 - \tan \mu \tan \frac{\lambda_i \pi}{2}}{1 + \tan \mu}. \] (58)

With the gauge transformation Eq. (47), one can easily see that \( (\tilde{\psi}_0)_i \) in the “sheet” gauge is given by Eq. (53) without the phase \( e^{-i\lambda_i(\pi-\theta)} \).

To illustrate the processes that violate the baryon number, we consider an incident beam of electrons propagating in the fields of the string. We will study the \((e, u^c)\)-subspace and ignore other fermions since \( e \) in \( \psi \) is mixed with \( u^c \) only. In the “sheet” gauge, the eigenstates of \( \tau_1 \) can be written as

\[ e + u^c = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e - u^c = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \] (59)

and the electron is simply given by

\[ e = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}. \] (60)
An incident wave of electrons can be written as

\[ \tilde{\psi}_0^{\text{inc}} = u_E \left( \frac{1}{2} \right) e^{i(kx-Et)} , \]  

which scatters into

\[ \tilde{\psi}_0^{\text{sca}} = \sqrt{\frac{i}{r}} v_E \left\{ f^\frac{1}{2}(\theta) \left( \begin{array}{c} \frac{1}{2} \\ 0 \end{array} \right) + f^{-\frac{1}{2}}(\theta) \left( \begin{array}{c} 0 \\ \frac{1}{2} \end{array} \right) \right\} e^{i(kr-Et)} . \]  

Note that the suppressed index on the 2-component spinors \( u_E \) and \( v_E \) should not be confused with the index associated with the 2-component eigenvectors used here to label the \( e + u_c \) and \( e - u_c \) components of the Dirac field. Rewriting \( \tilde{\psi}_0^{\text{sca}} \) above as

\[ \tilde{\psi}_0^{\text{sca}} = \sqrt{\frac{i}{r}} v_E \left\{ f^\frac{1}{2}(\theta) \left( \begin{array}{c} \frac{1}{2} \\ 0 \end{array} \right) - f^{-\frac{1}{2}}(\theta) \left( \begin{array}{c} 0 \\ \frac{1}{2} \end{array} \right) \right\} e^{i(kr-Et)} , \]  

one finds that the scattered wave consists of a mixture of electrons and \( u_c \)-quarks.

The differential cross section per unit length for the production of \( u \)-quark is defined by

\[ \frac{d\sigma}{d\theta} = \lim_{r \to \infty} \frac{\vec{J}^{u}_{\text{sca}} \cdot \vec{r}}{\vec{J}^{u}_{\text{inc}}} \]  

where \( J^i = \bar{\psi} \gamma^i \psi \). Substituting \( \tilde{\psi}_0^{\text{inc}} \) and \( \tilde{\psi}_0^{\text{sca}} \) into the currents, one obtains

\[ \frac{d\sigma}{d\theta} = \frac{1}{4} \left| f^\frac{1}{2}(\theta) - f^{-\frac{1}{2}}(\theta) \right|^2 , \]  

which can be written out using Eq. (57) as

\[ \frac{d\sigma}{d\theta} = \frac{1}{2\pi k} \left\{ \cos^4 \frac{\theta}{2} \sin^2 \left( \frac{\theta - 2\delta}{2} \right) \right\} . \]  

The calculation above was done in the limit of zero string width. Now let us examine the string core. The structure of the string core is “encoded” in the self-adjoint parameter \( \delta \) (or \( \mu \), related to \( \delta \) by Eq. (58)), which appears in the differential cross section above. In general the self-adjoint parameter is determined either from physical properties at the
origin or sometimes by symmetry arguments. Since the string solutions have already been
obtained in the previous section, we can find \( \mu \) by solving Eq. (50) numerically for the mode
\(-1 < n + \lambda_i < 0\), using the realistic form \( g(r)/r \) for the gauge field computed earlier in
place of the \( 1/r \) in Eq. (50). As we have shown, \( \lambda_i = \pm \frac{1}{2} \) for the fermions that scatter
nontrivially off the \( \tau_1 \)-string. Thus the special mode satisfying \(-1 < n + \lambda_i < 0\) takes the
value \( n + \lambda_i = -\frac{1}{2} \), where \( n = -1 \) for \( \lambda_i = \frac{1}{2} \) and \( n = 0 \) for \( \lambda_i = -\frac{1}{2} \). Recall that in the
calculation of \( g(r) \), the radial distance \( r \) was rescaled to the dimensionless \( \sqrt{-v_1 r} \) \((v_1 < 0)\),
where \( v_1 \) is the quadratic coupling in the Higgs potential in Eq. (16). Rescaling \( \chi_2 \) and \( r \) by
\[
\chi_2 \rightarrow i \frac{E + m}{k} \chi_2, \\
r \rightarrow \sqrt{-v_1} r,
\]
and replacing \( \lambda_i \) in Eq. (50) by \( \lambda_i g(r) \), Eq. (50) can be rewritten as
\[
\partial_r \chi_1 = \frac{g(r) - 2}{2r} \chi_1 + \beta \chi_2 \\
\partial_r \chi_2 = -\frac{g(r)}{2r} \chi_2 - \beta \chi_1
\]
for \( \lambda_i = \frac{1}{2}, n = -1 \), and
\[
\partial_r \bar{\chi}_1 = -\frac{g(r)}{2r} \bar{\chi}_1 + \beta \bar{\chi}_2 \\
\partial_r \bar{\chi}_2 = \frac{g(r) - 2}{2r} \bar{\chi}_2 - \beta \bar{\chi}_1
\]
for \( \lambda_i = -\frac{1}{2}, n = 0 \). The parameter \( \beta \) is defined by
\[
\beta \equiv k/\sqrt{-v_1},
\]
and the bars over \( \chi_1, \chi_2 \) are used to distinguish the solutions of \( \lambda_i = -\frac{1}{2} \) from those of
\( \lambda_i = \frac{1}{2} \). Upon closer inspection of the two sets of equations above, one finds that Eq. (69)
is in fact identical to Eq. (68) if \( \bar{\chi}_1 \) is identified with \( \chi_2 \) and \( \bar{\chi}_2 \) with \(-\chi_1 \). What about the
boundary conditions at the origin? In Eq. (49), for \( n = -1 \), the upper component depends
on \( \theta \) but the lower component does not, and vice versa for \( n = 0 \). Therefore \( \chi_1 \) and \( \bar{\chi}_2 \)
must vanish at the origin for the solution to be continuous, but \( \chi_2 \) and \( \bar{\chi}_1 \) can be nonzero.
at \( r = 0 \). One thus has \( \bar{\chi}_1 = \chi_2 \) and \( \bar{\chi}_2 = -\chi_1 \). Since Eq. (68) is linear, the value of \( \chi_2(0) \) can be chosen arbitrarily when integrating the differential equations.

The self-adjoint parameters \( \mu \) for \( \lambda_i = \frac{1}{2} \) and \( \bar{\mu} \) for \( \lambda_i = -\frac{1}{2} \) are determined by matching the solutions of Eq. (68) to the asymptotic expression in Eq. (52) at some radius \( r \). For \( n + \lambda_i = \frac{1}{2} \), the Bessel functions in Eq. (52) are simply \( J_{\pm \frac{1}{2}} \), which have the analytic forms

\[
J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x, \quad J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x.
\]

Then Eq. (52) leads to the simple expression for \( \mu \) and \( \bar{\mu} \):

\[
\frac{\chi_1}{\chi_2} = \tan(\mu + \beta r),
\]

\[
\frac{\bar{\chi}_1}{\bar{\chi}_2} = \tan(\bar{\mu} + \beta r),
\]

which can be inverted to give \( \mu \) and \( \bar{\mu} \) at a given \( r \), using \( \chi_1 \) and \( \chi_2 \) computed from Eq. (68).

Using Eq. (72) and trigonometric identities, one finds

\[
\bar{\mu} = \mu + \frac{\pi}{2}.
\]

Note that the solutions depend on \( \beta \) which appears in Eq. (68), and the quartic couplings \( \lambda_2, \lambda_3 \) in the Higgs potential. The parameter \( \beta \) defined in Eq. (70) measures the ratio of the incident fermion momentum \( k \) to the Higgs mass parameter \( \sqrt{-v_1} \), which is of the order of GUT energy scale. To put it another way, \( \beta \) measures the string width relative to the wavelength of the incident fermion. In Fig. 5, we set \( \beta = 1 \) and plot \( \mu \) computed from Eq. (72) at a given \( r \) for three sets of \( \lambda_2 \) and \( \lambda_3 \). The true value of \( \mu \) is given by the limit \( r \to \infty \). In Fig. 6, we choose the same set of parameter as in Figs. 1-3: \( \lambda_2 = 0.132 \) and \( \lambda_3 = 10.25 \); \( \mu \) is shown for five values of \( \beta \) ranging from 0.1 to 2.0. One can see that as \( \beta \) decreases, \( i.e. \) when the wavelength of the fermion becomes large compared to the string width, \( \mu \) decreases.
VI. CONCLUSIONS

We constructed two types of strings, string-\(\tau_{\text{all}}\) and string-\(\tau_1\), in the SO(10) grand unified theory. They are topologically equivalent but dynamically different strings, produced during the phase transition \(\text{Spin}(10) \rightarrow \text{SU}(5) \times \mathbb{Z}_2\) in the early universe. String-\(\tau_{\text{all}}\) is effectively Abelian, and can catalyze baryon number violation with a strong cross section via grand-unified processes inside the string. It has been the subject of study in several recent papers. The richer Higgs structure of string-\(\tau_1\), on the other hand, has been shown in this paper to induce baryon catalysis by mixing components in the fermion multiplet, turning leptons into quarks as they travel around the string. The underlying B-violating mechanism is the “twisting” of the scalar field, which leads to different unbroken SU(5) subgroups around the string. This mechanism is distinct from the grand-unified processes which can only occur inside the string core where the GUT symmetry is restored.

The corresponding string solutions have been calculated numerically with both the “shooting” and the relaxation methods. The energy of both strings was computed. With an additional upper bound argument, we found string-\(\tau_1\) to have lower energy than string-\(\tau_{\text{all}}\) in a wide range of parameters: \(10^{-4} < \lambda_2 < 2.5 \times 10^3\) and all \(\lambda_3\). The ratio of the upper bound on \(\tau_1\) energy to the \(\tau_{\text{all}}\) energy increases as \(\lambda_2\) decreases, and possibly approaches one from below as \(\lambda_2 \rightarrow 0\). Scattering of fermions in the fields of string-\(\tau_1\) has also been analyzed, and the B-violating cross section is given by Eq. (66). We conclude that string-\(\tau_1\) is more stable than string-\(\tau_{\text{all}}\), and can catalyze baryon decay with strong cross sections via the interesting mechanism of Higgs field twisting.

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**APPENDIX**

The generators of SO(2n) in the spinor representation can be constructed from a set of $2^n \times 2^n$ hermitian matrices $\Gamma_a^{(n)}$, $a = 1, \ldots, 2n$, which satisfy the Clifford algebra

$$\{\Gamma_a^{(n)}, \Gamma_b^{(n)}\} = 2\delta_{ab}.$$  \hspace{1cm} (A.1)

Starting with the two Pauli matrices for $n = 1$

$$\Gamma^{(1)}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma^{(1)}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$  \hspace{1cm} (A.2)

one can iteratively build the higher-dimensional $\Gamma_a^{(n+1)}$ from the $\Gamma_a^{(n)}$ by

$$\Gamma_a^{(n+1)} = \begin{pmatrix} \Gamma_a^{(n)} & 0 \\ 0 & -\Gamma_a^{(n)} \end{pmatrix}, \quad a = 1, \ldots, 2n$$

$$\Gamma_{2n+1}^{(n+1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\Gamma_{2n+2}^{(n+1)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. $$  \hspace{1cm} (A.3)

One can check that these $\Gamma$ matrices satisfy the Clifford algebra. The $\frac{2n(2n-1)}{2}$ generators of SO(2n) are constructed by

$$M_{ab} = \frac{1}{4i}[\Gamma_a, \Gamma_b], \quad a, b = 1, \ldots, 2n $$  \hspace{1cm} (A.4)

where $M_{ab}$ satisfy the SO(2n) commutation relations

$$[M_{ab}, M_{cd}] = -i(\delta_{bc}M_{ad} + \delta_{ad}M_{bc} - \delta_{ac}M_{bd} - \delta_{bd}M_{ac}).$$  \hspace{1cm} (A.5)

Thus far, we have used the explicit matrix notation to construct $\Gamma$ and $M$. For convenience, however, we will use an alternative notation in which each of the $2^n \times 2^n$ matrices
is written as a tensor product of \( n \) independent Pauli matrices, each acting on a different two-dimensional space. We choose the convention that the first matrix from the right in the tensor product acts on the largest \( 2 \times 2 \) block in the matrix notation, while the second from the right acts on the next, and so on, with the matrix on the left acting on the smallest \( 2 \times 2 \) block. In this notation, the 10 \( \Gamma \)'s of \( \text{SO}(10) \) given by Eq. (A.3) become

\[
\begin{align*}
\Gamma_1 &= \sigma_1 \sigma_3 \sigma_3 \sigma_3, \\
\Gamma_2 &= \sigma_2 \sigma_3 \sigma_3 \sigma_3, \\
\Gamma_3 &= \mathbb{I} \sigma_1 \sigma_3 \sigma_3, \\
\Gamma_4 &= \mathbb{I} \sigma_2 \sigma_3 \sigma_3, \\
\Gamma_5 &= \mathbb{I} \mathbb{I} \sigma_1 \sigma_3, \\
\Gamma_6 &= \mathbb{I} \mathbb{I} \sigma_2 \sigma_3, \\
\Gamma_7 &= \mathbb{I} \mathbb{I} \mathbb{I} \sigma_1, \\
\Gamma_8 &= \mathbb{I} \mathbb{I} \mathbb{I} \sigma_2, \\
\Gamma_9 &= \mathbb{I} \mathbb{I} \mathbb{I} \mathbb{I} \sigma_1, \\
\Gamma_{10} &= \mathbb{I} \mathbb{I} \mathbb{I} \mathbb{I} \sigma_2,
\end{align*}
\]

(A.6)

and the 45 generators \( M \) can be found accordingly. Furthermore, one can write down the five diagonal \( M \)'s that generate the Cartan sub-algebra:

\[
\begin{align*}
M_{12} &= \frac{1}{2} \sigma_3 \mathbb{I} \mathbb{I} \mathbb{I}, \\
M_{34} &= \frac{1}{2} \mathbb{I} \sigma_3 \mathbb{I} \mathbb{I}, \\
M_{56} &= \frac{1}{2} \mathbb{I} \mathbb{I} \sigma_3 \mathbb{I}, \\
M_{78} &= \frac{1}{2} \mathbb{I} \mathbb{I} \mathbb{I} \sigma_3, \\
M_{910} &= \frac{1}{2} \mathbb{I} \mathbb{I} \mathbb{I} \mathbb{I} \sigma_3.
\end{align*}
\]

(A.7)

The eigenvalues of the five generators above can be used to label the states in the spinor representation. Let \( \frac{1}{2} \xi_1, \ldots, \frac{1}{2} \xi_5 \) be the eigenvalues of \( M_{12}, \ldots, M_{910} \) respectively with \( \xi_i = +1 \) or \( -1 \), and denote the states by

\[
| \xi_1 \xi_2 \xi_3 \xi_4 \xi_5 \rangle.
\]

(A.8)

This 32-dimensional representation is reducible to two 16-dimensional irreducible representations because there exists a chirality operator

\[
\chi \equiv (-i)^5 \Gamma_1 \Gamma_2 \cdots \Gamma_{10} \\
= \sigma_3 \sigma_3 \sigma_3 \sigma_3 \sigma_3,
\]

(A.9)
which satisfies the commutation relations

\[ \{ \chi, \Gamma_i \} = 0, \quad [\chi, M_{ab}] = 0. \tag{A.10} \]

Moreover,

\[ \chi |\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5\rangle = \prod_i \epsilon_i |\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5\rangle, \tag{A.11} \]

where the eigenvalue \( \prod_i \epsilon_i \) is +1 or -1 depending on whether the number of spins that are down (\( \epsilon_i = -1 \)) is even or odd.

We assign the sixteen left-handed fermions to the states of positive chirality, i.e. states with even number of \( \epsilon_i = -1 \). The explicit identification of states to fermions can be achieved by first breaking the \( \text{SO}(10) \) \( 10 \times 10 \) representation into an upper \( 6 \times 6 \) and a lower \( 4 \times 4 \) blocks for the subgroups \( \text{SO}(6) \) and \( \text{SO}(4) \), and then embedding \( \text{SU}(3) \) in \( \text{SO}(6) \) and \( \text{SU}(2) \) in \( \text{SO}(4) \). The generators for \( \text{SO}(4) \) are \( M_{ab}, a, b = 7, 8, 9, 10 \), and with the choice \[ \tau_i = \frac{1}{2} \epsilon_{ijk} M_{jk} - M_{i10}, \quad i, j, k = 7, 8, 9 \tag{A.12} \]

for the generators of \( \text{SU}(2) \), one can easily verify that the last two spins in \( |\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5\rangle \) label the \( \text{SU}(2) \) states with \( |++\rangle, |--\rangle \) labeling the doublets and \( |++\rangle, |--\rangle \) the singlets. Similarly, the first three spins in \( |\epsilon_1\epsilon_2\epsilon_3\epsilon_4\epsilon_5\rangle \) label the \( \text{SU}(3) \) states with \( |+++\rangle, |---\rangle \) labeling the singlets, and \( |+++\rangle, |---\rangle \) with their permutations labeling the \( \text{SU}(3) \) triplets. One also needs the charge operator \( Q \) to make the assignment unique. In \( \text{SU}(5) \), \( Q = \text{diag}(1/3, 1/3, 1/3, 0, -1) \), which takes the form

\[ Q = \frac{1}{3} (M_{12} + M_{34} + M_{56}) - M_{910}. \tag{A.13} \]

In the \( \text{SO}(10) \) spinor representation,

\[ Q|\epsilon_1...\epsilon_5\rangle = \left\{ \frac{1}{6} (\epsilon_1 + \epsilon_2 + \epsilon_3) - \frac{\epsilon_5}{2} \right\} |\epsilon_1...\epsilon_5\rangle. \tag{A.14} \]

Putting all the above together one obtains
\begin{align}
|+++-+\rangle &= \nu^e, \quad |+++--\rangle = e^+ \\
|--+++angle &= u_c^e, \quad |--++-angle = d_1^c \\
|--+++angle &= u_c^e, \quad |-+---\rangle = d_2^c \\
|+-+-+\rangle &= u_c^e, \quad |+-++-\rangle = d_3^c \\
|--++-\rangle &= \nu, \quad |--+++\rangle = e^- \\
|+++-+\rangle &= u_1, \quad |+++--\rangle = d_1 \\
|++-++\rangle &= u_2, \quad |+-++-\rangle = d_2 \\
|--++-\rangle &= u_3, \quad |--++-\rangle = d_3.
\end{align}

Since we already know how to express the generators $M_{ab}$ as matrices, we can write the states as a single 32-dimensional column vector which is projected into two 16-dimensional vectors of positive and negative chirality by the operator $P_{\pm} \equiv \frac{1}{2}(1 \pm \chi)$. We find

$$\psi = (\nu^e \ u_1^c \ u_2^c \ u_3^c \ d_1^c \ d_2^c \ d_1^e \ u_2 \ u_1 \ \nu \ e^+ \ d_1^c \ d_2^c \ d_3^c)_{L}.$$  \hfill (A.16)

In this paper, we studied two types of strings: string-$\tau_{\text{all}}$, where $\tau_{\text{all}}$ is given by Eq. (10), and string-$\tau_1$, where $\tau_1$ can be any of the generators in Eq. (12). It is easy to see that in terms of $M_{ab}$, $\tau_{\text{all}}$ is written as

$$\tau_{\text{all}} = \frac{1}{5}(M_{12} + M_{34} + M_{56} + M_{78} + M_{910}),$$  \hfill (A.17)

and $|\epsilon_1 \ldots \epsilon_5\rangle$ is an eigenstate of $\tau_{\text{all}}$ with eigenvalue $\frac{1}{10}\sum_i \epsilon_i$. For the left-handed fermions above, $\frac{1}{10}\sum_i \epsilon_i = \frac{1}{2}$ for $\nu^e$, $\frac{1}{10}$ for $e^+$, $u, d, u^c$, and $-\frac{3}{10}$ for $\nu, e^-, d^c$.

To study how $\tau_1$ act on the fermions, we write $\tau_{1+}$ and $\tau_{1-}$ defined in Eq. (23) as a product of five Pauli matrices using Eqs. (A.4) and (A.6), and then replace the matrices $\sigma_1$ and $\sigma_2$ by the usual raising and lowering operators $\sigma_{\pm} = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$. One obtains

$$\begin{align}
\tau_{1+} &= \frac{1}{2}(\tau_{2\alpha-1,2\beta} + \tau_{2\alpha,2\beta-1}) \\
&= I \ldots I \sigma_+ \sigma_3 \ldots \sigma_3 \sigma_+ I \ldots I \\
&\quad + I \ldots I \sigma_- \sigma_3 \ldots \sigma_3 \sigma_- I \ldots I
\end{align}$$  \hfill (A.18)
and

\[
\tau_{1-} = \frac{1}{2}(\tau^{2a-1,2\beta-1} - \tau^{2a,2\beta})
\]

\[
= I\ldots I\sigma_+\sigma_3\ldots\sigma_3\sigma_-I\ldots I
\]

\[
-I\ldots I\sigma_-\sigma_3\ldots\sigma_3\sigma_+I\ldots I
\]

(\ref{eq:19})

where \(\alpha, \beta = 1, \ldots, 5, \alpha < \beta\), and the two \(\sigma_{\pm}\) matrices in each term occupy the \(\alpha\)th and \(\beta\)th positions from the left. Now one can read off from the list of fermions above which particles are mixed by a given \(\tau_1\). For generators of the form \(\tau_{1+}\), one immediately finds that except for the case \(\alpha = 4, \beta = 5\), all mix leptons with quarks; when \(\alpha = 4, \beta = 5\), the generator mixes \((e^+, \nu^c)\), \((u_1^c, d_1^c)\), \((u_2^c, d_2^c)\), and \((u_3^c, d_3^c)\). For generators of the form \(\tau_{1-}\), leptons are mixed with quarks when \(\alpha = 1, 2, \) or 3 and \(\beta = 4 \) or 5.
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FIGURES

FIG. 1. The solution of string-$\tau_1$, $g(r)$, $f_o(r)$, $f_e(r)$, as a function of dimensionless $r$ for the case $\lambda_2 = 0.132, \lambda_3 = 10.25$. The function $g(r)$ represents the spatial dependence of the gauge field, and $f_o(r), f_e(r)$ represent that of the Higgs field.

FIG. 2. The solution of string-$\tau_{all}$, $g(r), f(r)$, as a function of dimensionless $r$ for the same case as in Fig. 1. Here $g(r)$ represents the spatial dependence of the gauge field and $f(r)$ that of the Higgs field.

FIG. 3. The radial energy density $2\pi r \mathcal{H}(r)$ (in units of $v_1^2/v_2$) of string-$\tau_1$ and $\tau_{all}$, computed from the solutions in Figs. 1 and 2.

FIG. 4. The ratio of the upper bound on $\tau_1$ energy, $E_1(f_o = f_e)$, over the $\tau_{all}$ energy, $E_{all}$, as a function of $\lambda_2$. $E_1(f_o = f_e)$ is calculated by setting $f_o = f_e$ in the Lagrangian.

FIG. 5. The self-adjoint parameter $\mu$ computed from Eq. (72) at a given $r$, for three sets of $(\lambda_2, \lambda_3)$: (0.132, 10.25), (0.264, 20.50) and (0.528, 41.0), where $\beta = 1$. The true value of $\mu$ is given in the limit $r \to \infty$.

FIG. 6. The self-adjoint parameter $\mu$ computed from Eq. (72) at a given $r$ for different ratios of $\beta$, where $\lambda_2 = 0.132, \lambda_3 = 10.25$. 