Ordinary muon capture as a probe of virtual transitions of $\beta\beta$ decay

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Abstract: A reliable theoretical description of double-beta-decay processes needs a possibility to test the involved virtual transitions against experimental data. Unfortunately, only the lowest virtual transition can be probed by the traditional electron-capture or $\beta^-$-decay experiments. In this article we propose that calculated amplitudes for many virtual transitions can be probed by experiments measuring rates of ordinary muon capture (OMC) to the relevant intermediate states. The first results from such experiments are expected to appear soon. As an example we discuss the $\beta\beta$ decays of $^{76}$Ge and $^{106}$Cd and the corresponding OMC for the $^{76}$Se and $^{106}$Cd nuclei in the framework of the proton-neutron QRPA with realistic interactions. It is found that the OMC observables, just like the $2\nu\beta\beta$-decay amplitudes, strongly depend on the strength of the particle-particle part of the proton-neutron interaction.

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The two-neutrino double beta ($2\nu\beta\beta$) and neutrinoless double beta ($0\nu\beta\beta$) decays proceed via virtual transitions through states of the intermediate double-odd nucleus. In the case of the $2\nu\beta\beta$ decay only the intermediate $1^+$ states are involved, whereas in the case of the $0\nu\beta\beta$ decay all the multipole states $J^\pi$ of the intermediate nucleus are active (see e.g. [1]). Theoretical calculation of these virtual transitions has been a hot subject since two decades and a host of different approaches have been devised to accomplish the task [2]. The most relevant of these models, when only the ground-state-to-ground-state $\beta\beta$ decays are considered, are the nuclear shell model and the proton-neutron random-phase approximation (pnQRPA).

The calculation of the virtual $\beta$-decay type transitions is a formidable task and experimental data on the corresponding electron-capture (EC) or $\beta^-$-decay transitions from the intermediate states to the initial or final ground state can help in fine-tuning the model parameters before calculation of the double-beta-decay rates. Unfortunately, the EC/$\beta^-$-type of measurements can only probe the virtual transition through the lowest intermediate $J^\pi$ state. A further experimental analysis could be done by using the (p,n)
or (n,p) charge-exchange reactions but the extraction of the relevant information is not straightforward.

In this Letter we propose to use the ordinary muon capture (OMC) as a probing tool for the nuclear wave functions involved in the amplitudes of the virtual transitions of the $\beta\beta$ decay. The OMC process is like an EC process, except that the mass of the captured negative muon, $\mu^-$, is about 200 times the electron rest mass. The relevant OMC transitions here are the ones which start from the double-even $0^+$ ground state and end on $J^\pi$ states of the intermediate double-odd nucleus. This means that in the case of the $\beta^-\beta^-$ decays one probes the final leg of a double-beta transition and in the case of the double positron decays one probes the initial leg of the double-beta transition. Due to the large mass energy absorbed by the final nucleus in an OMC transition, one is able to probe intermediate states at high excitation energies, unlike in the EC transitions. A drawback of the OMC method, from the theoretical point of view, is that in addition to the V-A part the muon recoil activates also the induced parts (weak-magnetism and pseudoscalar terms) of the nucleonic current. This means that the theoretical expression for the partial OMC rates (i.e. capture rates to a particular final state) is more complicated than the expressions for the virtual legs of the $\beta\beta$ decay, and the OMC results as such can not be directly interpreted as amplitudes of the virtual transitions in the $\beta\beta$ decay. Instead, the OMC rates can be used to test in a versatile way the many-body wave functions of the $J^\pi$ states of the intermediate nucleus.

In the present work we demonstrate how one can use the partial OMC rates to probe the structure of the $J^\pi$ states of the intermediate nucleus. This is done by making calculations of the OMC rates and the corresponding beta-decay rates in the framework of the proton-neutron quasiparticle random-phase approximation (pnQRPA). As test cases we have chosen the final nucleus $^{76}$Se of the $\beta^-\beta^-$ decay of $^{76}$Ge and the double electron-capture (ECEC) decaying nucleus $^{106}$Cd. In the former case one can test by the OMC on $^{76}$Se the states of $^{76}$As (this is the final virtual leg in the $\beta^-\beta^-$ decay of $^{76}$Ge), the most relevant of which are the $1^+$ states and the $2^-$ states (the ground state of $^{76}$As is a $2^-$ state). In the latter case one can test the $1^+$ states of $^{106}$Ag by the OMC on $^{106}$Cd (this is the initial virtual leg in the ECEC decay of $^{106}$Cd). Also OMC rates to other multipole states can be measured, but the above mentioned states ($1^+$ and $2^-$) have been chosen to serve as demonstration of the method. In addition, some beta-decay data are available for the lowest intermediate excitations in these nuclei so that one can probe to some extent the quality of the calculations for the selected multipoles.

The experimental possibilities to use the OMC as a probe for the virtual transitions of the $\beta\beta$ decays lie on prospects to use enriched targets of nuclei like $^{48}$Ti, $^{76}$Se, $^{82}$Kr, $^{96}$Mo, $^{100}$Ru, $^{106}$Cd, $^{116}$Sn, $^{130}$Xe, etc. The need for an enriched $(A,N,Z)$ target comes from the fact that even a small admixture of the $(A+1,N+1,Z)$ isotope would be very dangerous, as the probability of the neutron-emission reaction
\[ \mu + (A + 1, N + 1, Z) \to \nu_\mu + (A + 1, N + 2, Z - 1) \to \nu_\mu + (A, N + 1, Z - 1) + \text{neutron} \]

is higher by an order of magnitude, and this channel would produce the same excited states of the final \((A, N + 1, Z - 1)\) nucleus as the OMC in the \((A, N, Z)\) nucleus. If the enrichment is less than 80\% the measurements should be done twice, for the \(A\) and \(A + 1\) isotopes separately, in order to separate the neutron-emission contamination.

Also the analysis of the gamma cascades following the capture to an excited state in the double-odd daughter nucleus has to be performed carefully. This kind of studies have been performed for the 1s-0d shell nuclei in T. P. Gorringe et al. \[4\] by measuring 38 gamma-ray lines and 29 \((\mu, \nu)\) transitions following negative muon capture on \(^{24}\text{Mg}, \, ^{28}\text{Si}, \, ^{31}\text{P}\) and \(^{32}\text{S}\). In the heavier, double-beta-decaying nuclei the density of the final states of the muon capture is larger but there already exists a host of data on the energy levels and their gamma feeding in the intermediate double-odd nuclei. It has to be noted that not all of the gamma rays need to be detected. A set of the lowest ones already yields information on the muon-capture rates to the lowest intermediate states and can thus be used to probe the \(\beta\beta\) virtual transitions to these states. The first experimental proposal concerning the measurement of the partial muon-capture rates in several double-beta-decay candidates, with mass numbers ranging from \(A = 48\) to \(A = 150\), has been submitted to the Paul Scherrer Institut (PSI) in Villigen, Switzerland by the Dubna-Louvain-Orsay-PSI-Jyväskylä collaboration \[5\].

The \(2\nu\beta\beta\) decay proceeds via the \(1^+\) states of the intermediate double-odd nucleus and the corresponding expression for the inverse half-life can be written as

\[ [t_{1/2}^{(2\nu)}(0^+_i \to 0^+_f)]^{-1} = G_{\text{DGT}}^{(2\nu)} \left| M_{\text{DGT}}^{(2\nu)} \right|^2 , \]  

(1)

where \(G_{\text{DGT}}^{(2\nu)}\) is the integral over the phase space of the leptonic variables \[1, 6\]. The nuclear matrix element \(M_{\text{DGT}}^{(2\nu)}\) can be written as

\[ M_{\text{DGT}}^{(2\nu)} = \sum_{m,n}(0^+_f \mid \sum_j \sigma(j) \tau_j^\mp \mid 1^+_m < 1^+_m \mid 1^+_n > (1^+_n \mid \sum_j \sigma(j) \tau_j^\mp \mid 0^+_i)) \left( \frac{1}{2} Q_{\beta\beta} + E_m - M_i \right) / m_e + 1 , \]  

(2)

where the transition operators are the usual Gamow-Teller operators, \(Q_{\beta\beta}\) is the \(2\nu\beta\beta\) \(Q\) value, \(E_m\) is the energy of the \(m\)th intermediate state, \(M_i\) is the mass energy of the initial nucleus, and \(m_e\) is the electron-mass energy. The overlap \(< 1^+_m \mid 1^+_n >\) between the two sets of \(1^+\) states, which are pn-QRPA solutions based on the initial and final ground states, helps in matching the two branches of virtual excitations \[6\].

The expressions for the \(0\nu\beta\beta\) amplitudes are more involved and are given e.g. in \[1, 6, 7\]. In this case all the intermediate multipoles are involved in the calculation of the half-life and the transition operators in the virtual amplitudes contain spherical harmonics coming from the multipole expansion of the neutrino-exchange potential.
The formalism needed for the calculation of the OMC rate is developed in Ref. [13]. Here we cite the main results of Ref. [13], according to which we can write an explicit formula for the capture rate as

$$W = 4P(\alpha Z m'_\mu)^3 \frac{2J_f + 1}{2J_i + 1} \left(1 - \frac{q}{m_\mu + AM}\right) q^2,$$

with $A$ being the mass number of the initial and final nuclei, $Z$ the charge number of the initial nucleus, and $m'_\mu$ the reduced muon mass. Furthermore, $\alpha$ denotes the fine-structure constant, $M$ the average nucleon mass, $m_\mu$ the muon mass, and $q$ the magnitude of the exchanged momentum between the captured muon and the nucleus. The term $P$ in equation (3) can be written as

$$P = \sum_{\kappa u} |g_N M[0\bar{u}u]S_{0u}(\kappa)\delta_{\bar{u}u}$$

$$- g_A M[1\bar{u}u]S_{1u}(\kappa) - \frac{g_N^A}{M} M[1\bar{u}p]\delta_{\bar{u}u},$$

$$+ \sqrt{3}(g_N Q/2M) \left(\sqrt{(\bar{l} + 1)/(2\bar{l} + 1)} M[0\bar{l} + 1u +] \delta_{\bar{l}+1,u}ight. + \sqrt{\bar{l}/(2\bar{l} - 1)} M[0\bar{l} - 1u -] \delta_{\bar{l}-1,u} \left. \right) S_{1u}(\kappa)$$

$$+ \sqrt{\frac{3}{2}} (g_N Q/M)(1 + \mu_p - \mu_u) \left(\sqrt{\bar{l} + 1} W(11u\bar{u}, 1\bar{l} + 1) M[1\bar{l} + 1u +] \right. + \sqrt{\bar{l} W(11u\bar{u}, 1, \bar{l} - 1)} M[1\bar{l} - 1u -] \left. \right) S'_{1u}(\kappa)$$

$$+ (g_A/M) M[0\bar{l}p] S'_{0u}(\kappa) \delta_{\bar{l}u} + \frac{1}{2} (g_A - g_F) (Q/2M)$$

$$\times \left(\sqrt{(\bar{l} + 1)/(2\bar{l} + 1)} M[1\bar{l} + 1u +] + \sqrt{\bar{l}/(2\bar{l} + 1)} M[1\bar{l} - 1u -] \right)$$

$$\times S'_{0u}(\kappa) \delta_{\bar{l}u}]^2,$$

where the $W()$ symbols are the usual Racah coefficients, and the definition for $\bar{l}$, the matrix elements $M[k\bar{w}u(\pm p)]$ and the geometrical factors $S_{ku}(\kappa)$ and $S'_{ku}(\kappa)$ can be found in Refs. [13, 14].

Finally, we note that a rather direct bridge between the $\beta\beta$ and the OMC processes can be established in the limit $q, Z \to 0$, since from the definition of the OMC matrix elements in Ref. [13, 14] it follows that

$$[101] \to \hat{J}_f \frac{\sqrt{3}}{4\pi} M_{GT},$$

(5)
where $M_{GT}$ is the reduced matrix element

$$M_{GT} = (1^+_s|| \sum_j \sigma(j) \tau^+_j||0^+)$$

(6)

of Eq. (4). Here the $0^+$ state denotes either the final (for $\beta^-\beta^-$ decay) or initial (for $\beta^+/EC\beta^+/EC$ decay) $0^+$ state of the $\beta\beta$-decay process.

In the numerical computations we have used the proton model space 1p-0f-2s-1d-0g-0h $\frac{1}{2}^+$ both for the $A = 76$ isobars and the $A = 106$ isobars. On the neutron side the same model space was used for the $A = 76$ isobars, but for the $A = 106$ isobars the extended neutron space 1p-0f-2s-1d-0g-2p-1f-0h was adopted. The corresponding single-particle energies were obtained by using the Woods-Saxon (WS) well with the parametrization of Ref. 8. This basis was used recently in Ref. 9 for calculation of $0\nu\beta\beta$-decay rates in the $A = 76$ system. For the $A = 106$ system, the neutron and proton WS single-particle energies were adjusted near the corresponding Fermi surfaces according to data on neighboring proton-odd and neutron-odd nuclei. This is the same basis which was used in Ref. 10 for a successful description of the beta and double-beta data on the $A = 106$ isobars.

The nuclear hamiltonian of the two isobaric regions was obtained from the Bonn one-boson-exchange potential 11 with empirical renormalization by using the phenomenological pairing gaps, giant Gamow-Teller resonances, and spectroscopic data on nuclei close to the relevant isobars. For more details the reader is referred to the articles 9, 10. As in very many calculations in the recent past 1 also the results of the present pnQRPA calculation in the $1^+$ channel for the first intermediate $1^+$ state depend strongly on the strength parameter $g_{pp}$ of the particle-particle part of the proton-neutron interaction. This strong dependence on $g_{pp}$ also strongly influences the $2\nu\beta\beta$-decay rates leading to a strong suppression of the $2\nu\beta\beta$ matrix element, as first discussed in Ref. 12. For that reason, to demonstrate the corresponding effects on the OMC observables, we present our analysis as a function of $g_{pp}$ for the $A = 76$ isobars. For the $A = 106$ isobars we adopt the value $g_{pp} = 0.8$ coming from the exhaustive spectroscopy analysis of Ref. 10.

Partial muon-capture rates $W$ were calculated for transitions $^{76}\text{Se}(0^+_{g.s.}) \rightarrow 76\text{As}(1^+, 2^-)$ and $^{106}\text{Cd}(0^+_{g.s.}) \rightarrow 106\text{Ag}(1^+)$. In these calculations we have used the Goldberger-Treiman (PCAC) value $g_P/g_A = 7$ for the ratio of the coupling strengths of the induced pseudoscalar and axial-vector parts of the charged weak current. More discussion of this matter can be found in Ref. 14. Furthermore, the nuclear-matter renormalized value of $g_A = 1.0$ was adopted for the axial-vector strength. We have plotted the corresponding results in Figs. 2-3 as a function of $g_{pp}$. Here one can clearly see that the capture rates to the $1^+_1$ and $2^-_1$ states are strongly dependent on the value of the particle-particle strength $g_{pp}$ of the proton-neutron force. This dependence for the higher-lying states is smaller, although a notable exception is the OMC to the $1^+_2$ state in $^{106}\text{Ag}$, as seen in Fig. 3. In Fig. 4 one can see that the states $1^+_2$ and $1^+_3$ cross at about $g_{pp} = 0.85$, and that for $g_{pp}$ values
beyond 1.1 one obtains very small values for the OMC rate to the first 1+ state. Beyond this point the recoil matrix element [011p] grows unphysically large and cancels against the contributions coming from the other nuclear matrix elements. The unphysically large value of the recoil matrix element stems from the excessive growth of the ground-state correlations, leading to the same order of magnitude for the forward- and backward-going amplitudes in the pnQRPA approach. This happens close to the breaking point of the pnQRPA, which in this case is around the value \( g_{pp} \approx 1.2 \).

Partial capture rates as well as the corresponding log \( ft \) values for \( \beta^- \) transitions, for specific values of \( g_{pp} \), can be found in Table 1. The value \( g_{pp} = 0.9 \) for the 1+ channel in \( A = 76 \) was chosen because it would yield a sensible log \( ft \) value for the \( \beta^- \) decay of the 1\(_1\)\(^+\) state (for \( g_{pp} = 1.0 \) the log \( ft \) is far too large). The log \( ft \) value for the unique first-forbidden \( \beta^- \) transition from the 2\(_2\)\(^-\) state is practically independent of the value of \( g_{pp} \) and thus the unrenormalized value of \( g_{pp} = 1.0 \) was chosen. For the \( A = 106 \) isobars, the value \( g_{pp} = 0.8 \) is based on spectroscopy [10], as discussed earlier. The obtained log \( ft \) values can be compared with the available data, namely log \( ft(76\text{As}(2^-) \rightarrow 76\text{Se}(0^+_g,s)) = 9.7 \) and log \( ft(106\text{Ag}(1^+) \rightarrow 106\text{Cd}(0^+_g,s)) > 4.2 \) [15]. Thus, a reasonable agreement with available data is obtained, and the predicted OMC rates can be used for estimating the possibility of experimental determination of the discussed partial OMC rates. In addition to the quoted partial OMC rates, we list in Table 2 the total muon-capture rates for each calculated \( J^\pi \) multipole. The evaluation of the total rates has been done for the above discussed values of \( g_{pp} \). The total OMC rates, like the partial ones, are dependent on the value of \( g_{pp} \).

In Fig. 4 we present histograms of all calculated partial capture rates with the corresponding excitation energy in the final nucleus. The total capture rates of Table 2 have been obtained by integrating these distributions over the energy. As can be seen, the OMC rate distribution for 1\(_1\)\(^+\) capture is very different for the \( A = 76 \) and the \( A = 106 \) isobars, the distribution peaking strongly between 10 MeV and 20 MeV for the \( A = 106 \) system and below 4 MeV for the \( A = 76 \) system. The 2\(_1\)\(^-\) distribution concentrates on low energies, the 2\(_2\)\(^-\) contribution being notable.

In conclusion, in the present work we have studied the OMC rates as tests of the nuclear wave functions involved in the amplitudes of the virtual transitions of the two-neutrino and neutrinoless double beta decay. It was found that the OMC observables (partial and total capture rates to 1\(_1\)\(^+\) and 2\(_2\)\(^-\) states) depend on the strength \( (g_{pp}) \) of the particle-particle part of the proton-neutron interaction. The same was observed for the 2\(_2\)\(^-\)\(\beta\)-decay amplitudes already more than a decade ago. The strongest dependence is associated with the decays via the first 1\(_1\)\(^+\) state, whereas the dependence of the 2\(_2\)\(^-\) states on \( g_{pp} \) is much weaker than for the 1\(_1\)\(^+\) states. On the experimental side, the first results from the measurements of the partial muon-capture rates to the double-odd intermediate nuclei involved in the virtual transitions of the double beta decay will be available soon. The first experimental proposal has been submitted to the PSI, including the \( A = 76 \) and
$A = 106$ systems discussed in this article.

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Table 1: log $ft$ values of $\beta^-$ decays of three lowest states of the $1^+$ and $2^-$ multipolarities, together with the corresponding muon capture rates $W$. For $A = 76$ the values $g_{pp} = 0.9$ ($J^\pi = 1^+$) and $g_{pp} = 1.0$ ($J^\pi = 2^-$) were used. For $A = 106$ the value $g_{pp} = 0.8$ was adopted. The ratio $g_P/g_A = 7$ was assumed in the calculations.

| Quantity | $A = 76$ | $A = 106$ |
|----------|----------|----------|
| log $ft(J^\pi_i \to 0^+_{g.s.})$ | 5.77 | 4.30 |
| $W$ [$10^3$ 1/s] | 31.17 | 325.52 |

Table 2: Total muon capture rates with $g_P/g_A = 7$ for the $1^+$ and $2^-$ multipoles.

| Captures | $W_{tot}$ [$10^3$ 1/s] | $g_{pp}$ |
|----------|-------------------------|----------|
| $^{76}$Se($0^+_{g.s.}$) $\to$ $^{76}$As($1^+$) | 359 | 0.9 |
| $^{76}$Se($0^+_{g.s.}$) $\to$ $^{76}$As($2^-$) | 2020 | 1.0 |
| $^{100}$Cd($0^+_{g.s.}$) $\to$ $^{100}$Ag($1^+$) | 5385 | 0.8 |
Figure 1: Partial capture rates $^{76}\text{Se}(0^+_{\text{g.s.}}) \rightarrow ^{76}\text{As}(1^+_1)$ as functions of $g_{pp}$ with $g_P/g_A = 7$. 
Figure 2: Partial capture rates $^{76}\text{Se}(0^{+}_{g.s.}) \rightarrow ^{76}\text{As}(2^{-}_{i})$ as functions of $g_{pp}$ with $g_{P}/g_{A} = 7$.

Figure 3: Partial capture rates $^{106}\text{Cd}(0^{+}_{g.s.}) \rightarrow ^{106}\text{Ag}(1^{+}_{i})$ as functions of $g_{pp}$ with $g_{P}/g_{A} = 7$. 

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Figure 4: Partial capture rates as functions of the excitation energy in the final nucleus.

- $^1\gamma$ capture rates in $^{39}\text{Se}$.
  - $g_{\nu\gamma} = 0.9$, $g_{\nu}/g_{\alpha} = 7$.

- $^2\gamma$ capture rates in $^{78}\text{Se}$.
  - $g_{\nu\gamma} = 1$, $g_{\nu}/g_{\alpha} = 7$.

- $^1\gamma$ capture rates in $^{100}\text{Cd}$.
  - $g_{\nu\gamma} = 0.8$, $g_{\nu}/g_{\alpha} = 7$. 