Multiple closed $K$-magnetic geodesics on $S^2$

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Abstract

Let $K$ be a smooth scalar function on the round $2$-sphere. In this paper we face Arnol’d problem [3] about the existence of closed and embedded $K$-magnetic geodesics $\gamma \subset S^2$. We prove the existence of a solution in every sufficiently large energy level and provide a sufficient condition for the existence of two distinct solutions. No assumptions on the sign of $K$ are imposed.

1 Introduction

We deal with the motion $\gamma = \gamma(t)$ of a particle of unit mass and charge in $\mathbb{R}^3$, that experiences the Lorentz force $F$ produced by a magnetostatic field $B$. If the particle is constrained to the standard round sphere $S^2 \subset \mathbb{R}^3$, the motion law reads

$$\gamma'' + |\gamma'|^2 \gamma = K(\gamma) \gamma \wedge \gamma', \quad (1.1)$$

where

$$K(p) := -B(p) \cdot p, \quad p \in S^2.$$  

A trajectory $\gamma(t)$ satisfying \((1.1)\) is called $K$-magnetic geodesic.

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Let us recall the elementary derivation of (1.1). We have $F(\gamma) = \gamma' \wedge B(\gamma)$; due to the constraint $|\gamma| \equiv 1$, the vectors $\gamma$ and $\gamma'$ are orthogonal along the motion. It follows that the projection of $F$ on $T_\gamma S^2 = \langle \gamma \rangle^\perp$ is proportional to $\gamma \wedge \gamma'$, and in fact $F(\gamma) = -\langle B(\gamma) \cdot \gamma \rangle \gamma \wedge \gamma' = K(\gamma) \gamma \wedge \gamma'$. Finally, by differentiating the identity $\gamma \cdot \gamma' \equiv 0$, we see that the tangent component of the acceleration vector is $\gamma'' - (\gamma'' \cdot \gamma) \gamma = \gamma'' + |\gamma'|^2 \gamma$, and thus Newton’s law gives (1.1). Notice that $\gamma'' - (\gamma'' \cdot \gamma) \gamma = \nabla_{S^2}^\gamma \gamma'$, where $\nabla_{S^2}$ is the Levi-Civita connection of $S^2$.

Two remarkable facts immediately follow from (1.1). First, we have $2\gamma' \cdot \gamma' = (|\gamma'|^2)' = 0$. Thus the particle moves with constant scalar speed, say $|\gamma'| \equiv c$, for some $c > 0$. In particular, $\gamma$ is a regular curve. Secondly, we learn from differential geometry that $\gamma$ has geodesic curvature

$$K(\gamma) = \frac{\gamma'' \cdot \gamma \wedge \gamma'}{|\gamma'|^3} = \frac{K(\gamma)}{c}.$$ 

Next, let $c > 0$ and $K : S^2 \to \mathbb{R}$ be given. In [3], see also [4, Problems 1988/30, 1994/14, 1996/18], Arnol’d proposed the following question (actually in a more general setting, where $S^2$ is replaced by an oriented Riemannian surface $(\Sigma, g)$):

Find closed and embedded $K$-magnetic geodesics $\gamma \subset S^2$ with $|\gamma'| \equiv c$. \hfill (P_{K,c})

Problem (P_{K,c}), together with its generalizations, attracted the attention of many authors and has been studied via different mathematical tools, such as variational arguments for multivalued functionals [5] [10] [14] [15], symplectic geometric [3] [6] [7] [8] [12] and index-based topological arguments [11] [13]. In particular, from [13, Theorem 1.1] we have the existence of two distinct solutions to (P_{K,c}), for any positive smooth function $K$ and any $c > 0$. Schneider’s multiplicity result is indeed sharp, that is, Problem (P_{K,c}) might have exactly two distinct solutions, see [13, Theorem 1.3].

As far as we know, all the available results for (P_{K,c}) require that $K$ has constant sign. For example, in [7, Theorem 2.1 (i) and Theorem 2.7], the assumption $K > 0$ guarantees that $\Omega(p) := K(p)d\sigma_p$ (where $d\sigma_p$ is the area form on $S^2$) is a symplectic

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\footnote{We agree that the curves $\gamma_1(t), \gamma_2(t)$ are distinct if $\gamma_1 \neq \gamma_2 \circ g$, for any diffeomorphism $g$.}
form on $\mathbb{S}^2$; it also gives the existence of closed characteristics of the twisted magnetic form $p^*\Omega + cd\lambda$ on the cotangent bundle $T^*\mathbb{S}^2$, being $p: T^*\mathbb{S}^2 \to \mathbb{S}^2$ the projection and $d\lambda$ the canonical symplectic form. In [11] and [13], some crucial a-priori bounds on the length of simple and closed $K$-magnetic geodesics need the assumption $K > 0$ as well.

However, let us notice that the Gauss law for magnetism,

$$\int_{\mathbb{S}^2} K(p) \, d\sigma_p = 0,$$

evidently implies that $K$ cannot have constant sign, see also [3, Problem 1996-17].

In the present paper we admit functions $K \in C^1(\mathbb{S}^2)$ that change sign or vanish somewhere, and we prove the existence of at least a solution to Problem $(P_{K,c})$ for any $c > 0$ sufficiently large. Under the additional assumption (1.2) below, we prove the existence of two distinct solutions. In order to justify this extra assumption, that is clearly not needed if $K$ has constant sign, notice that if $K$ vanishes on some geodesic circle of radius $\pi/2$ about $z \in \mathbb{S}^2$, then $\partial \mathcal{D}_{\pi/2}(z)$ can be parameterized by two $K$-magnetic geodesics that coincide up to orientation.

We are in position to state our main result.

**Theorem 1.1** Let $K \in C^1(\mathbb{S}^2)$ be given. For every $c > 0$ large enough, Problem $(P_{K,c})$ has at least a solution $\gamma$. If in addition $K$ does not vanish on any closed geodesic, or

$$\int_{\partial \mathcal{D}_{\pi/2}(z)} K(q) \, d\sigma_q = \int_{\partial \mathcal{D}_{\pi/2}(-z)} K(q) \, d\sigma_q \quad \text{whenever} \quad K \equiv 0 \quad \text{on} \quad \partial \mathcal{D}_{\pi/2}(z),$$

then for every $c > 0$ large enough, Problem $(P_{K,c})$ has at least two embedded, distinct solutions.

The proof of Theorem 1.1 is based on a Lyapunov-Schmidt finite-dimensional technique, which has been successfully used to face related geometrical problems. In the breakthrough paper [1], Ambrosetti and Badiale showed that the combination of

\[ \text{Recall that changing the orientation of a curve changes the sign of its curvature.} \]
Lyapunov-Schmidt and variational arguments can provide extremely powerful tools. This idea has been applied to tackle quite a large number of variational problems arising from mathematical physics and differential geometry, see the exhaustive list of references in the monograph [2]. We cite also [9], where Arnol’d problem has been studied in case the supporting surface $S^2$ is replaced by the hyperbolic (or the Euclidean) plane.

Problem $(P_{K,c})$ presents completely different features when compared to those studied in [9]. Besides the compactness of the supporting surface $S^2$, the main differences are due to deeper topological issues. In fact, Arnol’d general problem about $K$-magnetic geodesics in an oriented Riemannian manifold $(\Sigma, g)$ admits a (standard) variational formulation if and only if $\Sigma$ is contractible. In particular, a variational formulation for $(P_{K,c})$ through a (non-multivalued) energy functional is not available. To overcome this difficulty, we take advantage of a "local" variational approach which is developed in Section 2. In Section 3 we prove Theorem 1.1 by constructing, for any $c = \varepsilon^{-1} >> 0$, a function $\mathcal{E} : SO(3) \to \mathbb{R}$, whose critical points give rise to solutions to Problem $(P_{K,c})$. The Lusternik-Schnirelmann theorem gives indeed the existence of at least four critical points for $\mathcal{E}$. However, due to the invariances of Problem $(P_{K,c})$, one has that in general only two of them are distinct modulo orientation preserving changes of parameters; the extra assumption in (1.2) guarantees that those two solutions are actually distinct.

Our second main result suggests a way to obtain more and more distinct $K$-magnetic geodesics. It involves the Mel’nikov-type functional

$$F_K(z) = \int_{\mathcal{D}(z)} K(p) \, d\sigma_p, \quad F_K : S^2 \to \mathbb{R}.$$  \hspace{1cm} (1.3)

Roughly speaking, we show that any stable critical point $z_0 \in S^2$ for $F_K$ generates, for any $c > 0$ large enough, a solution to $(P_{K,c})$ that is a perturbation of the closed geodesic about $z_0$. For details, more precise statements and proof see Section 4.

Notation.
The Euclidean space $\mathbb{R}^3$ is endowed with Euclidean norm $|p|$, scalar product $p \cdot q$, and exterior product $p \wedge q$. The canonical basis of $\mathbb{R}^3$ is $\{e_h, \ h = 1, 2, 3\}$. 

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We isometrically embed the unit sphere $S^2$ into $\mathbb{R}^3$, so that the tangent space $T_zS^2$ at $z \in S^2$ is identified with $\langle z \rangle^1 = \{ p \in \mathbb{R}^3 \mid p \cdot z = 0 \}$. We denote by $\mathcal{D}_\rho(z) \subset S^2$ the geodesic disk of radius $\rho \in (0, \frac{\pi}{2}]$ about $z \in S^2$.

It is convenient to regard at $S^1$ as the unit circle in the complex plane.  

**Function spaces.** Let $m \geq 0$, $n \geq 1$ be integer numbers. We endow $C^m(S^1, \mathbb{R}^n)$ with the standard Banach space structure. If $f \in C^1(S^1, \mathbb{R}^n)$, we identify $f'(x) \equiv f'(x)(ix)$, so that $f' : S^1 \rightarrow \mathbb{R}^n$.

We write $C^m(S^1)$ instead of $C^m(S^1, \mathbb{R})$ and $C^m$ instead of $C^m(S^1, \mathbb{R}^3)$. For $U \subseteq S^2$ we put

$$C^m_U := C^m(S^1, U) = \{ u \in C^m \mid u(x) \in U \text{ for any } x \in S^1 \}.$$  

We identify $U$ with the set of constant functions in $C^2_U$, so that $C^2_U \setminus U = C^2_U \setminus S^2$ contains only nonconstant curves.

The Hilbertian norm in $L^2 = L^2(S^1, \mathbb{R}^3)$ is

$$\| u \|_{L^2}^2 = \int_{S^1} |u(x)|^2 \, dx = \frac{1}{2\pi} \int_{S^1} |u(x)|^2 \, dx,$$

and the orthogonal to $T \subseteq C^0$ with respect to the $L^2$ scalar product is given by

$$T^\perp = \{ \varphi \in C^0 \mid \int_{S^1} u \cdot \varphi \, dx = 0 \text{ for any } u \in T \}.$$  

We regard at $C^2_{S^2}$ as a smooth complete submanifold of $C^2$. If $u \in C^2_{S^2}$, the tangent space to $C^2_{S^2}$ at $u$ is

$$T_uC^2_{S^2} = \{ \varphi \in C^2 \mid u \cdot \varphi \equiv 0 \text{ on } S^1 \}.$$  

If $u$ is regular, that means $u'(x) \neq 0$ for any $x \in S^1$, then

$$T_uC^2_{S^2} = \{ g_1u' + g_2 u \wedge u' \mid g = (g_1, g_2) \in C^2(S^1, \mathbb{R}^2) \}.$$  

**Rotations.** Any complex number $\xi \in S^1$ is identified with the rotation $x \mapsto \xi x$. Recall that $\det(R) = +1$ and $R^{-1} = \underline{t}R$ for any $R \in SO(3)$, where $SO(3)$ is the group of rotations of $\mathbb{R}^3$ and $\underline{t}R$ is the transpose of $R$.

It is well-known that $SO(3)$ is a connected three-dimensional manifold. More precisely, it is a Lie group whose Lie algebra is given by the skew-symmetric matrices, and the tangent
space $T_{\text{Id}}SO(3)$ at the identity matrix is spanned by

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. $$

A simple explanation of this elementary fact follows by introducing the matrices

$$R_1^\xi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \xi_1 & -\xi_2 \\ 0 & \xi_2 & \xi_1 \end{pmatrix}, \quad R_2^\xi = \begin{pmatrix} \xi_1 & 0 & -\xi_2 \\ 0 & 1 & 0 \\ \xi_2 & 0 & \xi_1 \end{pmatrix}, \quad R_3^\xi = \begin{pmatrix} \xi_1 & -\xi_2 & 0 \\ \xi_2 & \xi_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. $$

for $\xi = \xi_1 + i\xi_2 \in S^1$. Clearly $R_h^\xi$ is a rotation about the $\langle e_h \rangle$ axis. By differentiating $R_h^\xi$ with respect to $\xi \in S^1$ at $\xi = 1$ one gets $T_h = dR_h^\xi|_{\xi=1}$, and thus infers that $\{T_h\}$ is a basis for $T_{\text{Id}}SO(3)$. In accordance with the Lie group structure of $SO(3)$, the tangent space to $SO(3)$ at $R \in SO(3)$ is obtained by rotating $T_{\text{Id}}SO(3)$. Hence

$$T_{R}SO(3) = \langle RT_1, RT_2, RT_3 \rangle. $$

Finally, for any $q \in S^2$ we denote by $d_{R}$ the differential of the function $SO(3) \to S^2$, $R \mapsto Rq$, so that $d_{R}(Rq)\tau \in T_{Rq}S^2$ for any $\tau \in T_{R}SO(3)$. We have the formula

$$d_{R}(Rq)(RT_h) = R(e_h \wedge q) = Re_h \wedge Rq. \quad (1.4) $$

## 2 A ”local” variational approach

We put $\varepsilon = c^{-1}$ and study Problem $(P_{K,\varepsilon^{-1}})$ for $\varepsilon$ close to 0. We take advantage of its geometrical interpretation to rewrite it in an equivalent way. Let $\gamma$ be a solution to $(P_{K,\varepsilon^{-1}})$, and let $L_{\gamma}$ be its length. Extend $\gamma$ to an $\varepsilon L_{\gamma}$-periodic function on $\mathbb{R}$ and consider the curve $u \in C^2_{S^2}, u(e^{i\theta}) = \gamma(\frac{\varepsilon L_{\gamma}}{2\pi} \theta)$. Evidently $u$ and $\gamma$ have the same length $L_{\gamma}$ and curvature $\varepsilon K$. Moreover $|u'| \equiv L_{\gamma}/2\pi$ and $u$ solves the system

$$u'' + |u'|^2 u = |u'|\varepsilon K(u) u \wedge u' \quad \text{on } S^1, \quad (2.1) $$

because $\gamma$ solves $(1.1)$. Conversely, any solution $u \in C^2_{S^2} \setminus S^2$ to $(2.1)$ has constant speed $|u'|$, curvature $\varepsilon K(u)$ and gives rise to a solution to $(P_{K,\varepsilon^{-1}})$.
The main goal of the present section is to show that for any point \( p \in \mathbb{S}^2 \), the problem of finding solutions to (2.1) in \( C^2_{S^2 \setminus \{p\}} \), that is an open subset of \( C^2_{S^2} \), can be faced by using variational methods. First, we need to introduce the functional

\[
L(u) = \left( \int_{S^1} |u'|^2 \, dx \right)^{\frac{1}{2}}, \quad L : C^2_{S^2 \setminus \{p\}} \to \mathbb{R}.
\]  

(2.2)

Notice that the Cauchy-Schwarz inequality gives \( L_u \leq 2\pi L(u) \), and equality holds if and only if \( |u'| \) is constant. Moreover, it holds that

\[
L(Ru \circ \xi) = L(u) \quad \text{for any } \xi \in S^1, \ R \in SO(3). 
\]

(2.3)

Finally, we notice that \( L \) is Fréchet differentiable at any \( u \in C^2_{S^2 \setminus S^2} \), with differential

\[
L'(u) \phi = \frac{1}{L(u)} \int_{S^1} u' \cdot \phi' \, dx = \frac{1}{L(u)} \int_{S^1} (-u'' - |u'|^2 u) \cdot \phi \, dx \quad \text{for any } \phi \in T_u C^2_{S^2}. 
\]

(2.4)

In the next lemma we provide a variational reading of the right-hand side of (2.1), see also [10] and [7, Remark 2.2].

**Lemma 2.1** Let \( K \in C^0(S^2) \) and let \( U, V \) be open and contractible subsets of \( S^2 \).

i) There exists a unique \( C^1 \) functional \( A^U_K : C^2_U \to \mathbb{R} \), such that \( A^U_K(u) = 0 \) if \( u \) is constant, and

\[
(A^U_K)'(u) \phi = \int_{S^1} K(u) \phi \cdot u \wedge u' \, dx \quad \text{for any } u \in C^2_U, \ \phi \in T_u C^2_{S^2}; 
\]

(2.5)

ii) If \( R \in SO(3), \ \xi \in S^1 \) and \( u \in C^2_U \), then \( A^{RU}_{K \circ R}(Ru \circ \xi) = A^U_K(u) \);

iii) If \( U \cap V \) is nonempty and contractible, then \( A^U_K(u) = A^V_K(u) \) for any \( u \in C^2_{U \cap V} \);

iv) Let \( u \in C^2_{S^2} \). The function \( p \mapsto A^{S^2 \setminus \{p\}}_K(u) \) is constant on each connected component of \( S^2 \setminus u(S^1) \);

v) Let \( u \in C^2_U \) be a positively oriented parametrization of the boundary of a regular open set \( \Omega_u \subset U \). Then

\[
A^U_K(u) = -\frac{1}{2\pi} \int_{\Omega_u} K(q) \, d\sigma_q.
\]
Proof. Take a 1-form $\beta^U_K$ on $U$, such that
\[
d\beta^U_K = -K(q)\,d\sigma_q, \tag{2.6}\]
where $d\sigma_q$ is the restriction of the volume form on the sphere. We put
\[
A^U_K(u) = \int_{S^1} u^* \beta^U_K = \int_{S^1} \beta^U_K(u)u'\,dx, \quad u \in C^2_U.
\]
It is evident that $A^U_K(u) = 0$ if $u$ is constant. Formula (2.5) can be derived by using Lie differential calculus or local coordinates, like in the proof of [5, Lemma 3]. Elementary arguments and (2.5) give the $C^1$ differentiability of the functional $A^U_K$.

Uniqueness is trivial, because $C^2_U$ is a connected manifold. In particular, for $u \in C^2_U$ the real number $A^U_K(u)$ does not depend on the choice of $\beta^U_K$.

To prove $ii$) take a 1-form $\beta$ in the domain $RU$ such that $d\beta = -(K \circ t_R)\,d\sigma_q$. Clearly $R^*\beta$ is a 1-form in $U$, and $d(R^*\beta) = R^*(d\beta) = -K(q)d\sigma_q$. Thus we can take $\beta^U_K = R^*\beta$ in formula (2.6) and we obtain
\[
A^{RU}_K(Ru) = \int_{S^1} (Ru)^* \beta = \int_{S^1} u^*(R^*\beta) = A^U_K(u)
\]
for any $u \in C^2_U$. The invariance of the area functional with respect to composition with rotations of $S^1$ is immediate.

Now we prove $iii$). If $V \subset U$ and $u \in C^2_V$, then the restriction of $\beta^U_K$ to $V$ can be used to compute $A^V_K(u)$. Thus $A^V_K(u) = A^U_K(u)$. It follows that if two open, connected sets $U, V$ have contractible intersection and $u \in C^2_{U \cap V}$, then $A^{U \cap V}_K(u) = A^U_K(u)$ and $A^{U \cap V}_K(u) = A^V_K(u)$.

Claim $iv$) readily follows from $iii$). In fact, take $p_0 \in S^2 \setminus u(S^1)$ and a small disk $D_\delta(p_0) \subset S^2 \setminus u(S^1)$. For any $p \in D_\delta(p_0)$ we have
\[
A^{S^2 \setminus \{p\}}(u) = A^{S^2 \setminus D_\delta(p_0)}(u) = A^{S^2 \setminus \{p_0\}}(u).
\]
We proved that the function $p \mapsto A^{S^2 \setminus \{p\}}(u)$ is locally constant on $S^2 \setminus u(S^1)$, and hence is constant on each connected component of $S^2 \setminus u(S^1)$.
For the last claim we use Stokes’ theorem to get
\[
2\pi A^U_K(u) = \int_{S^1} u^* \beta^U_K = \int_{\partial \Omega_u} \beta^U_K = \int_{\Omega_u} d\beta^U_K = -\int_{\Omega_u} K(q) d\sigma_q
\]
by (2.6). The lemma is completely proved. \(\square\)

From now on we write
\[
A_K(p; u) = A_K^{S^2\setminus\{p\}}(u), \quad p \in S^2, \quad u \in C^2_{S^2\setminus\{p\}}.
\]

By Lemma 2.1, the functional \(A_K\) enjoys the following properties,

A1) The functional \(A_K(p; \cdot)\) is of class \(C^1\) on \(C^2_{S^2\setminus\{p\}}\), and
\[
A'_K(p; u) \phi = \int_{S^1} K(u) \phi \cdot u \wedge u' \, dx \quad \text{for any } u \in C^2_{S^2\setminus\{p\}}, \quad \phi \in T_u C^2_{S^2}\.
\]

A2) If \(R \in SO(3), \xi \in S^1, \text{ and } u \in C^2_{S^2\setminus\{p\}}, \text{ then } A_K \circ \epsilon(R) \circ \epsilon(\xi)(R) = A_K(p; u).

A3) Let \(u \in C^2_{S^2}\). The function \(p \mapsto A_K(p; u)\) is locally constant on \(S^2 \setminus u(S^1)\).

A4) Let \(u \in C^2_{S^2\setminus\{p\}}\) be a positively oriented parametrization of the boundary of a regular open set \(\Omega_u \subset S^2 \setminus \{p\}\). Then
\[
A_K(p; u) = -\frac{1}{2\pi} \int_{\Omega_u} K(q) \, d\sigma_q.
\]

**Remark 2.2** To find an explicit formula for \(A_K(p; \cdot)\) let \(\Pi_p : S^2 \setminus \{p\} \to \mathbb{R}^2\) be the stereographic projection from the pole \(p\). If \(u \in C^2_{S^2\setminus\{p\}}\), then \(\Pi_p \circ u\) is a curve in \(\mathbb{R}^2\) and \((\Pi_p^{-1})^*(K d\sigma_q) = (K \circ \Pi_p^{-1}) \text{det} J_{\Pi_p^{-1}}(z) dz\) is a 2-form on \(\mathbb{R}^2\). Let \(\tilde{\beta}^p_K\) be a 1-form on \(\mathbb{R}^2\) such that \(d\tilde{\beta}^p_K = (\Pi_p^{-1})^*(K d\sigma_q)\). Then
\[
A_K(p; u) = \int_{S^1} u^*(\Pi_p^* \tilde{\beta}^p_K) = \int_{S^1} (\Pi_p \circ u)^* \tilde{\beta}^p_K.
\]

For instance, if \(K \equiv 1\) is constant one can take
\[
A_1(p; u) = \int_{S^1} \frac{p}{1 - u \cdot p} \cdot u \wedge u' \, dx = 2 \int_{S^1} \frac{p}{|u - p|^2} \cdot u \wedge u' \, dx.
\]
The next lemma provides the predicted "local" variational approach to \([2.1]\).

**Lemma 2.3** Let \(K \in C^0(S^2)\).

i) For any \(p \in S^2\), the functional

\[
E_{\varepsilon K}(p; u) = L(u) + \varepsilon A_K(p; u), \quad E_{\varepsilon K}(p; \cdot) : C^2_{S^2 \setminus \{p\}} \to \mathbb{R}
\]

is of class \(C^1\), with differential

\[
L(u)E'_{\varepsilon K}(p; u)\varphi = \int_{S^1} (-u'' + L(u)\varepsilon K(u)u \wedge u') \cdot \varphi \, dx, \quad \text{for any } \varphi \in T_u C^2_{S^2}. \tag{2.7}
\]

In particular, any critical point \(u \in C^2_{S^2 \setminus \{p\}} \setminus S^2\) for \(E_{\varepsilon K}(p; \cdot)\) solves \([2.1]\).

ii) If \(R \in SO(3), \xi \in S^1\) and \(p \in S^2\), then \(E_{\varepsilon K}^R(Rp; Ru \circ \xi) = E_{\varepsilon K}(p; u)\) for any nonconstant curve \(u \in C^2_{S^2 \setminus \{p\}}\), and thus

\[
E'_{\varepsilon K}(p; u)u' = 0 \quad \text{for any } u \in C^2_{S^2 \setminus \{p\}} \setminus S^2. \tag{2.8}
\]

iii) Let \(u \in C^2_{S^2 \setminus \{p\}} \setminus S^2\). The function \(E_{\varepsilon K}(\cdot; u) : S^2 \setminus u(S^1) \to \mathbb{R}\) is locally constant.

iv) If \(K \in C^1(S^2)\) then the functional \(E_{\varepsilon K}(p; \cdot)\) is of class \(C^2\) on its domain.

**Proof.** Formula \([2.4]\) and the property \(A1)\) of the area functional give the \(C^1\) regularity of \(E_{\varepsilon K}(p; \cdot)\) and \([2.7]\). Let \(u\) be a critical point for \(E_{\varepsilon K}(p; \cdot)\). Take any \(\varphi \in C^2\) and put \(\varphi^\top = \varphi - (\varphi \cdot u) u \in T_u C^2_{S^2}\). We have \(\varphi \cdot u \wedge u' = \varphi^\top \cdot u \wedge u'\) on \(S^1\), and \(u' \cdot (\varphi^\top)' = u' \cdot \varphi' - (\varphi \cdot u)|u'|^2\) because \(u' \cdot u \equiv 0\). Since

\[
0 = L(u)E'_{\varepsilon K}(p; u)\varphi^\top = \int_{S^1} (u' \cdot (\varphi^\top)') + L(u)\varepsilon K(u)\varphi^\top \cdot u \wedge u') \, dx
\]

\[
= \int_{S^1} (u' \cdot \varphi' - (\varphi \cdot u)|u'|^2 + L(u)\varepsilon K(u)\varphi \cdot u \wedge u') \, dx,
\]

and therefore \(u\) solves \(u'' + |u'|^2 u = L(u)\varepsilon K(u)u \wedge u'\) on \(S^1\). Since \(u'' \cdot u' \equiv 0\), we see that \(|u'| \equiv L(u)\) is constant, and thus \(u\) solves \([2.1]\).

Statements ii), iii) follow from \([2.3]\), A2) and A3) (to check \([2.8]\) take the derivative of the identity \(E_{\varepsilon K}(p; u \circ \xi) = E_{\varepsilon K}(p; u)\) with respect to \(\xi \in S^1\) at \(\xi = 1\)). Finally, iv) can be proved via elementary arguments, starting from \([2.7]\). \(\square\)
3 Geodesics

For any rotation \( R \in SO(3) \), the loop
\[
\omega_R(x) = R(x_1, x_2, 0), \quad x = x_1 + ix_2 \in \mathbb{S}^1,
\]
is a parameterization of the boundary of \( D_{\pi/2}(Re_3) \) and solves
\[
\omega_R'' + |\omega_R'|^2 \omega_R = 0, \quad L(\omega_R) = |\omega_R'| = 1. \tag{3.1}
\]
In order to simplify notations, from now on we write
\[
\omega(x) = \omega_{id}(x) = (x_1, x_2, 0), \quad x = x_1 + ix_2 \in \mathbb{S}^1.
\]
The tangent space to the smooth 3-dimensional manifold
\[
\mathcal{S} = \{\omega_R \mid R \in SO(3)\} \subset \mathbb{C}^2 \mathbb{S}^2
\]
at \( \omega_R \in \mathcal{S} \) can be easily computed via formula (1.4). It turns out that
\[
T_{\omega_R} \mathcal{S} = \{q \wedge \omega_R \mid q \in \mathbb{R}^3\} = \langle Re_1 \wedge \omega_R, Re_2 \wedge \omega_R, Re_3 \wedge \omega_R \rangle.
\]
We introduce the function
\[
J_0(u) := -u'' - |u'|^2 u, \quad J_0 : \mathbb{C}^2 \mathbb{S}^2 \setminus \mathbb{S}^2 \to \mathbb{C}^0,
\]
so that \( \mathcal{S} \subset \{J_0 = 0\} \). By (2.4) we have
\[
L(u)L'(u)\varphi = \int_{\mathbb{S}^1} J_0(u) \cdot \varphi dx \quad \text{for any } u \in \mathbb{C}^2 \mathbb{S}^2 \setminus \mathbb{S}^2, \varphi \in T_u \mathbb{C}^2 \mathbb{S}^2. \tag{3.2}
\]
Moreover, for \( u \in \mathbb{C}^2 \mathbb{S}^2 \setminus \mathbb{S}^2, q \in \mathbb{R}^3 \) and \( R \in SO(3) \) it holds that
\[
\int_{\mathbb{S}^1} J_0(u) \cdot q \wedge u dx = 0, \quad J_0(Ru) = RJ_0(u). \tag{3.3}
\]
The first identity readily follows via integration by parts or can be obtained by differentiating the identity \( L(Ru) = L(u) \) with respect to \( R \in SO(3) \). The second one is immediate.
Clearly $J_0$ is of class $C^2$; for $R \in SO(3)$ and $\varphi$ in the tangent space
\[ T_{\omega_R}C^2_{S^2} = \{ \varphi = g_1 \omega'_R + g_2 \omega_R \land \omega'_R \mid g = (g_1, g_2) \in C^2(S^1, \mathbb{R}^2) \}, \quad (3.4) \]
we have
\[ J'_0(\omega_R) \varphi = -\varphi'' - 2(\omega'_R \cdot \varphi') \omega_R - \varphi. \]
Further, the operator $J'_0(\omega_R)$ is self adjoint in $L^2(S^1, \mathbb{R}^3)$, that is,
\[ \int_{S^1} J'_0(\omega_R) \varphi \cdot \tilde{\varphi} \, dx = \int_{S^1} J'_0(\omega_R) \tilde{\varphi} \cdot \varphi \, dx \quad \text{for any} \ \varphi, \tilde{\varphi} \in T_{\omega_R}C^2_{S^2}. \quad (3.5) \]
By differentiating the identity $J_0(\omega_R) = 0$ with respect to $R \in SO(3)$, we see that $T_{\omega_R}S \subseteq \ker J'_0(\omega_R)$. Actually, equality holds, as shown in the next crucial lemma.

**Lemma 3.1 (Nondegeneracy)** Let $R \in SO(3)$. Then

i) $\ker J'_0(\omega_R) = T_{\omega_R}S$;

ii) If $\varphi \in T_{\omega_R}C^2_{S^2}$ and $J'_0(\omega_R) \varphi \in T_{\omega_R}S$, then $\varphi \in T_{\omega_R}S$;

iii) For any $u \in T_{\omega_R}S^\perp$ there exists a unique $\varphi \in T_{\omega_R}C^2_{S^2} \cap T_{\omega_R}S^\perp$ such that $J'_0(\omega_R) \varphi = u$.

**Proof.** One can argue by adapting the computations in [13, Section 5]. We provide here a simpler argument.

Since $J'_0(\omega_R)(R \varphi) = R(J'_0(\omega) \varphi)$ for any $\varphi \in T_{\omega}C^2_{S^2}$, it is not restrictive to assume that $R$ is the identity matrix. By direct computations based on (3.1), one can check that
\[ J'_0(\omega)(\psi \omega') = -\psi'' \omega', \quad J'_0(\omega)(\psi \omega \land \omega') = ( -\psi'' - \psi ) \omega \land \omega' \]
for any $\psi \in C^2(S^1, \mathbb{R})$. Since by (3.4) any function $\varphi \in T_{\omega}C^2_{S^2}$ can be written as
\[ \varphi = (\varphi \cdot \omega') \omega' + (\varphi \cdot \omega \land \omega') \omega \land \omega', \]
we are led to introduce the differential operator $B : C^2(S^1, \mathbb{R}^2) \to C^0(S^1, \mathbb{R}^2)$,
\[ B(g) = -g_1'' e_1 + (-g_2'' - g_2) e_2, \quad g = (g_1, g_2) \in C^2(S^1, \mathbb{R}^2). \]
and the function transform

\[ \Psi \varphi = (\varphi \cdot \omega') e_1 + (\varphi \cdot \omega \land \omega') e_2, \quad \Psi : T_\omega C^2_{S_2} \to C^2(S^1, \mathbb{R}^2), \]

do that

\[ J'_0(\omega) \varphi = \Psi^{-1} B(\Psi \varphi) \quad \text{for any } \varphi \in T_\omega C^2_{S_2}, \quad \Psi(\ker J'_0(\omega)) = \ker B. \quad (3.6) \]

We proved that \( \ker J'_0(\omega) \) and \( T_\omega S \) have both dimension 3, thus they must coincide because \( T_\omega S \subseteq \ker J'_0(\omega) \).

For future convenience we notice that \( \Psi \) is an isometry with respect to the \( L^2 \) norms, and in particular

\[ \int_{S^1} (\Psi \varphi) \cdot (\Psi \tilde{\varphi}) \, dx = \int_{S^1} \varphi \cdot \tilde{\varphi} \, dx \quad \text{for any } \varphi, \tilde{\varphi} \in T_\omega C^2_{S_2}. \quad (3.7) \]

Now we prove \( \text{iii) } \). If \( \tau := J'_0(\omega) \varphi \in T_\omega S \), then \( J'_0(\omega) \tau = 0 \), as \( \ker J'_0(\omega) = T_\omega S \).

But then, using (3.5) we get

\[ \int_{S^1} |J'_0(\omega) \varphi|^2 \, dx = \int_{S^1} J'_0(\omega) \varphi \cdot \tau \, dx = \int_{S^1} J'_0(\omega) \tau \cdot \varphi \, dx = 0. \]

Thus \( J'_0(\omega) \varphi = 0 \), that means \( \varphi \in T_\omega S \).

It remains to prove \( \text{iii) } \). Since \( \Psi(T_\omega S) = \ker B \), from (3.6) and (3.7) we have that \( u \in T_\omega S^\perp \) if and only if \( \Psi u \in \ker B^\perp \). In particular, if \( u \in T_\omega S^\perp \), then one can compute the unique solution \( g_u \in \ker B^\perp \) to the system \( B g_u = \Psi u \). The function \( \varphi := \Psi^{-1} g_u \) belongs to \( T_\omega S^\perp \); thanks to (3.6) it solves \( J'_0(\omega) \varphi = u \), and is uniquely determined by \( u \). The lemma is completely proved. \( \square \)

**Remark 3.2** For future convenience we compute

\[ m_{ij} = \int_{S^1} (Re_i \land \omega_R) \cdot (Re_j \land \omega_R) \, dx = \int_{S^1} (e_i \land \omega) \cdot (e_j \land \omega) \, dx = \delta_{ij} - \int_{S^1} \omega_i \omega_j \, dx. \]

We see that the functions \( Re_j \land \omega_R = R(e_j \land \omega) \) provide an orthogonal basis for \( T_\omega S \) endowed with the \( L^2 \) scalar product. More precisely, the matrix \( M \) associated to this scalar product with respect to the basis \( \{Re_j \land \omega_R\} \) is given by

\[
M = \begin{pmatrix}
\frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]
3.1 Finite dimensional reduction and proof of Theorem 1.1

By the remarks at the beginning of Section 2, we are led to study problem (2.1) for \( \varepsilon = c^{-1} \) close to 0. Further, since any solution \( u \) to (2.1) satisfies \( |u'| \equiv L(u) \), we can rewrite (2.1) in the following, equivalent way,

\[
  u'' + |u'|^2 u = L(u)\varepsilon K(u) u \wedge u', \quad u \in C^2_{S^2 \setminus S^2}.
\]  

We will look for solutions to (3.8) by solving \( J_\varepsilon(u) = 0 \), where

\[
  J_\varepsilon(u) = J_0(u) + \varepsilon L(u)K(u) u \wedge u' = -u'' - |u'|^2 u + L(u)\varepsilon K(u) u \wedge u'.
\]  

Thanks to (2.7), we can write

\[
  L(u)E'_K(p; u, \varphi) = \oint_{S^1} J_\varepsilon(u) \cdot \varphi \, dx, \quad \text{for} \ u \in C^2_{S^2 \setminus S^2}, \ p \notin u(S^1), \ \varphi \in T_u C^2_{S^2}. \tag{3.10}
\]  

The regularity assumption on \( K \) implies that \( J_\varepsilon \) is of class \( C^1 \) on its domain. In addition, \( J_\varepsilon(u \circ \xi) = J_\varepsilon(u) \) for any \( \xi \in S^1 \), and integration by parts gives

\[
  \oint_{S^1} J_\varepsilon(u) \cdot u' \, dx = 0 \quad \text{for any} \ u \in C^2_{S^2 \setminus S^2}.
\]  

In general, the identities in (3.3) are not satisfied if \( \varepsilon \neq 0 \), because the perturbation term breaks the invariances of the operator \( J_0 \).

In the next lemma we provide the main tool to prove Theorem 1.1 (see also [13, Section 5.2] for a similar construction in a local framework).

**Lemma 3.3** There exist \( \bar{\varepsilon} > 0 \) and a \( C^1 \) function

\[
  [-\bar{\varepsilon}, \bar{\varepsilon}] \times SO(3) \to C^2_{S^2 \setminus S^2} \quad (\varepsilon, R) \mapsto u_\varepsilon^R
\]

such that \( u_\varepsilon^R \) is an embedded loop, and moreover

i) \( u_\varepsilon^0 = \omega_R \);

ii) \( u_\varepsilon^R \in T_{\omega_R} S^1 \);
iii) $J_\varepsilon(u^R_\varepsilon) \in T_{\omega_R}S$;

iv) The function $[-\varepsilon, \varepsilon] \times SO(3) \to \mathbb{R}$,

$$(\varepsilon, R) \mapsto \mathcal{E}^\varepsilon(R) := E_{\varepsilon K}(-Re_3; u^\varepsilon_R) = L(u^\varepsilon_R) + \varepsilon A_K(-Re_3; u^\varepsilon_R)$$

is well defined, of class $C^1$ on its domain, and $d_R\mathcal{E}^\varepsilon(R)(RT_3) = 0$.

v) $R \in SO(3)$ is critical for $\mathcal{E}^\varepsilon : SO(3) \to \mathbb{R}$ if and only if $J_\varepsilon(u^\varepsilon_R) = 0$.

vi) Put $\mathcal{E}^\varepsilon_0(R) = E_{\varepsilon K}(-Re_3; \omega_R) = 1 + \varepsilon A_K(-Re_3, \omega_R)$. As $\varepsilon \to 0$, we have

$$\mathcal{E}^\varepsilon(R) - \mathcal{E}^\varepsilon_0(R) = o(\varepsilon)$$

(3.11)

uniformly on $SO(3)$, together with the derivatives with respect to $R \in SO(3)$.

**Proof.** Consider the differentiable functions

$$\mathcal{F}_1 : \mathbb{R} \times SO(3) \times (C_{S_2}^2 \setminus S^2) \times \mathbb{R}^3 \to C^0, \quad \mathcal{F}_1(\varepsilon, R, u; \zeta) = J_\varepsilon(u) - \sum_{j=1}^3 \zeta_j (Re_j \wedge \omega_R)$$

$$\mathcal{F}_2 : \mathbb{R} \times SO(3) \times (C_{S_2}^2 \setminus S^2) \times \mathbb{R}^3 \to \mathbb{R}^3, \quad \mathcal{F}_2(\varepsilon, R, u; \zeta) = \sum_{j=1}^3 \left( \int_{S_3} u \cdot Re_j \wedge \omega_R \, dx \right) e_j$$

where $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{R}^3$, and then let

$$\mathcal{F} : \mathbb{R} \times SO(3) \times (C_{S_2}^2 \setminus S^2) \times \mathbb{R}^3 \to C^0 \times \mathbb{R}^3, \quad \mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2).$$

Fix $R \in SO(3)$. Since $J_0(\omega_R) = 0$ by (3.1), then $\mathcal{F}(0, R, \omega_R; 0) = 0$. Our first goal is to solve the equation $\mathcal{F}(\varepsilon, R, u; \zeta) = (0; 0)$ in a neighborhood of $(0, R, \omega_R; 0)$, via the implicit function theorem.

Consider the differentiable function

$$\mathcal{F}(0, R, \cdot ; \cdot) : (u; \zeta) \mapsto \mathcal{F}(0, R, u; \zeta), \quad (C_{S_2}^2 \setminus S^2) \times \mathbb{R}^3 \to C^0 \times \mathbb{R}^3$$

and let

$$\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2) : (T_{\omega_R}C_{S_2}^2) \times \mathbb{R}^3 \to C^0 \times \mathbb{R}^3$$

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be its differential evaluated at \((u; \zeta) = (\omega_R; 0)\). We need to prove that \(L\) is invertible.

Take \(\varphi \in T_{\omega_R}C_{S^2}^2\) and \(p = (p_1, p_2, p_3) \in \mathbb{R}^3\). It is easy to compute

\[
L_1(\varphi; p) = J_0^*(\omega_R)\varphi - \sum_{j=1}^3 p_j (Re_j \wedge \omega_R), \quad L_2(\varphi; p) = \sum_{j=1}^3 \left( \int_{S^1} \varphi \cdot Re_j \wedge \omega_R \, dx \right) e_j.
\]

Next, recall that \(T_{\omega_R}S\) is spanned by the functions \(Re_j \wedge \omega_R\). If \(L_1(\varphi; p) = 0\) then \(J_0^*(\omega_R)\varphi \in T_{\omega_R}S\), hence \(\varphi \in T_{\omega_R}S\) by \(ii)\) in Lemma 3.1, if \(L_2(\varphi; p) = 0\) then \(\varphi \in T_{\omega_R}S^\perp\). Therefore, the operator \(L\) is injective.

Before proving surjectivity we notice that

\[
J_0^*(\omega_R)\varphi \in T_{\omega_R}S^\perp \quad \text{for any } \varphi \in T_{\omega_R}C_{S^2}^2
\]

because of (3.5) and since \(T_{\omega_R}S = \ker J_0^*(\omega_R)\).

Now take arbitrary \(\psi \in C^0\) and \(q = (q_1, q_2, q_3) \in \mathbb{R}^3\). We have to find functions \(\varphi^\top \in T_{\omega_R}S, \varphi^\perp \in T_{\omega_R}S^\perp\) and \(p = (p_1, p_2, p_3) \in \mathbb{R}^3\) such that \(L(\varphi^\top + \varphi^\perp, p) = (\psi, q)\). Since \(T_{\omega_R}S = \ker J_0^*(\omega_R)\) is spanned by the functions \(Re_j \wedge \omega_R\), we only need to solve

\[
\begin{cases}
J_0^*(\omega_R)\varphi^\perp = \psi + \sum_j p_j (Re_j \wedge \omega_R), \quad \varphi^\perp \in T_{\omega_R}S, \quad p \in \mathbb{R}^3 \\
\int_{S^1} \varphi^\top \cdot Re_j \wedge \omega_R \, dx = q_j, \quad \varphi^\top \in T_{\omega_R}S^\perp.
\end{cases}
\]

The tangential component \(\varphi^\top \in T_{\omega_R}S\) is uniquely determined. Thanks to (3.12), we see that the function \(\sum_j p_j (Re_j \wedge \omega_R)\) must coincide with the projection of \(-\psi\) on \(T_{\omega_R}S\). This gives the unknown \(p\). More explicitly, we have

\[
e_h \cdot Mp = \sum_{j=1}^3 p_j \int_{S^1} (Re_h \wedge \omega_R) \cdot (Re_j \wedge \omega_R) \, dx = - \int_{S^1} \psi \cdot Re_h \wedge \omega_R \, dx,
\]

where \(M\) is the invertible matrix in Remark 3.2. Once one knows \(p\), the existence of \(\varphi^\perp\) follows from \(iii)\) in Lemma 3.1 and surjectivity is proved.

We are in position to apply the implicit function theorem for any fixed \(R \in SO(3)\). Actually, by a compactness argument, we have that there exist \(\varepsilon' > 0\) and uniquely determined differentiable functions

\[
\begin{align*}
u : (-\varepsilon', \varepsilon') \times SO(3) & \to C_{S^2}^2 \setminus S^2, \quad u : (\varepsilon, R) \mapsto u_R^\varepsilon, \\
\zeta : (-\varepsilon', \varepsilon') \times SO(3) & \to \mathbb{R}^3, \quad \zeta : (\varepsilon, R) \mapsto \zeta^\varepsilon(R) = (\zeta_1^\varepsilon(R), \zeta_2^\varepsilon(R), \zeta_3^\varepsilon(R))
\end{align*}
\]
such that
\[ \mathcal{F}(\varepsilon, R, u_R^\varepsilon; \zeta^\varepsilon(R)) = 0, \quad u_R^0 = \omega_R, \quad \zeta^0(R) = 0. \]
Clearly the function \((\varepsilon, R) \mapsto u_R^\varepsilon\) is differentiable. Since \(\omega_R\) is embedded, then \(u_R^\varepsilon\) is embedded as well, provided that \(\varepsilon'\) is small enough.

Condition \(i\) in the Lemma is fulfilled; \(ii\) follows from \(\mathcal{F}_2(\varepsilon, R, u_R^\varepsilon; \zeta^\varepsilon(R)) = 0\) while \(\mathcal{F}_1(\varepsilon, R, u_R^\varepsilon; \zeta^\varepsilon(R)) = 0\) gives \(iii\).

Now we prove that \(iv\) holds for any \(\tau \in (0, \varepsilon')\), provided that \(\varepsilon'\) is small enough. Since \(|\omega + e_3| \geq 1\) and \(u_R^\varepsilon \to \omega_R\) uniformly on \(S^1\) as \(\varepsilon \to 0\), we can assume that
\[ |u_R^\varepsilon(x) + Re_3| \geq \frac{1}{2} \quad \text{for any } x \in S^1, (\varepsilon, R) \in (-\varepsilon', \varepsilon') \times SO(3). \]
In particular, Lemma 2.3 guarantees that the function \(E^\varepsilon(R) = E_{\varepsilon K}(-Re_3; u_R^\varepsilon)\) is well defined and of class \(C^1\) on \(SO(3)\), for any \(\varepsilon \in (-\varepsilon', \varepsilon')\). By \(iii\) in Lemma 2.3, we have that the derivative of \(p \mapsto E_{\varepsilon K}(p; u_R^\varepsilon)\) vanishes for \(p \in S^2 \setminus u_R^\varepsilon(S^1)\), and we can compute
\[ d_R E^\varepsilon(R)(RT_h) = E'_{\varepsilon K}(-Re_3; u_R^\varepsilon)(d_R u_R^\varepsilon(RT_h)) \quad \text{for } h \in \{1, 2, 3\}, \tag{3.13} \]
where \(E'_{\varepsilon K}(-Re_3; \cdot)\) is the differential of the energy with respect to curves running in \(C^2_{\varepsilon, \omega} \setminus \{-Re_3\}\). The \(C^1\) dependence of \(E^\varepsilon(R)\) on \(\varepsilon\) and thus on the pair \((\varepsilon, R)\) is evident.

Next, notice that \(R_3^\varepsilon \omega = \omega \circ \xi\) for any rotation \(\xi \in S^1\) (recall that \(R_3^\varepsilon\) rotates \(S^2\) about the \(e_3\) axis). Hence \(RR_3^\varepsilon \omega = \omega \circ \xi\) and \(T_{RR_3^\varepsilon \omega} S = \{\tau \circ \xi \mid \tau \in T_{\omega \circ \xi} S\}\) for any \(R \in SO(3)\). Taking also \(ii\), \(iii\) into account, we have that
\[ u_R^\varepsilon \circ \xi \in (T_{RR_3^\varepsilon \omega} S)_{\xi}, \quad J_\xi(u_R^\varepsilon \circ \xi) = J_\xi(u_R^\varepsilon) \circ \xi \in T_{RR_3^\varepsilon \omega} S. \]
Since in addition \(u_R^\varepsilon \circ \xi\) is close to \(\omega_R \circ \xi = RR_3^\varepsilon \omega\) in the \(C^2\)-norm by \(i\), we see that
\[ u_{RR_3^\varepsilon} = u_R^\varepsilon \circ \xi \tag{3.14} \]
by the uniqueness of the function \(\varepsilon \mapsto u_R^\varepsilon\) given by the implicit function theorem. By differentiating \((3.14)\) with respect to \(\xi\) at \(\xi = 1\) we obtain \(d_R u_R^\varepsilon(RT_3) = (u_R^\varepsilon)'\), that compared with \((2.8)\) gives \(E'_{\varepsilon K}(-Re_3; u_R^\varepsilon)(d_R u_R^\varepsilon(RT_3)) = E'_{\varepsilon K}(-Re_3; u_R^\varepsilon)(u_R^\varepsilon)' = 0\). Thus \(d_R E^\varepsilon(R)(RT_3) = 0\) by \((3.13)\), and \(iv\) is proved.
To prove that $v$ holds for $\varepsilon$ small enough, first take $R \in SO(3)$, $h \in \{1, 2, 3\}$ and notice that the condition $u^\varepsilon_R \in T_\omega S^\bot$ trivially gives

$$d_R \left( \int_{S^1} u^\varepsilon_R \cdot R(e_j \wedge \omega) \, dx \right)(RT_h) = 0.$$  

We compute $d_R R(e_j \wedge \omega)(RT_h) = Re_h \wedge (R(e_j \wedge \omega)) = R(e_h \wedge (e_j \wedge \omega))$. Since in addition $u^\varepsilon_R \cdot R(e_h \wedge (e_j \wedge \omega)) = -(Re_h \wedge u^\varepsilon_R) \cdot (Re_j \wedge \omega_R)$ we obtain

$$m^\varepsilon_{hj}(R) := \int_{S^1} d_R u^\varepsilon_R(RT_h) \cdot Re_j \wedge \omega_R \, dx = \int_{S^1} (Re_h \wedge u^\varepsilon_R) \cdot (Re_j \wedge \omega_R) \, dx. \quad (3.15)$$

Since $u^\varepsilon_R \to \omega_R$ uniformly for $R \in SO(3)$, from (3.15) we obtain

$$m^\varepsilon_{hj}(R) = \int_{S^1} (Re_h \wedge \omega_R) \cdot (Re_j \wedge \omega_R) \, dx + o(1) = m_{hj} + o(1),$$

where $m_{hj}$ are the entries of the invertible matrix $M$ in Remark 3.2. It follows that the $3 \times 3$ matrix $M^\varepsilon_R = (m^\varepsilon_{hj}(R))_{j,h=1,2,3}$ is invertible for any $R \in SO(3)$, if $\varepsilon$ is small enough.

We are in position to conclude the proof of $v$). We know that there exists a differentiable function $(\varepsilon, R) \mapsto \zeta^\varepsilon(R) \in \mathbb{R}^3$ such that

$$J_\varepsilon(u^\varepsilon_R) = \sum_{j=1}^3 \zeta^\varepsilon_j(R) (Re_j \wedge \omega_R). \quad (3.16)$$

On the other hand, (3.13) and (3.10) give

$$L(u^\varepsilon_R)d_R E^\varepsilon(R)(RT_h) = \int_{S^1} J_\varepsilon(u^\varepsilon_R) \cdot d_R u^\varepsilon_R(RT_h) \, dx, \quad (3.17)$$

by (3.16) and recalling (3.15) we obtain

$$L(u^\varepsilon_R)d_R E^\varepsilon(R)(RT_h) = \sum_{j=1}^3 m^\varepsilon_{hj}(R) \zeta^\varepsilon_j(R) = e_h \cdot M^\varepsilon_R(\zeta^\varepsilon(R)).$$

If $\varepsilon \approx 0$ so that the matrix $M^\varepsilon_R$ is invertible, then $R$ is a critical matrix for $E^\varepsilon$ if and only if $\zeta^\varepsilon(R) = 0$, which is equivalent to say that $J_\varepsilon(u^\varepsilon_R) = 0$.  

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To prove the last claim of the lemma we take $R \in SO(3)$ and compute the Taylor expansion formula of the function

$$f_R(\varepsilon) = \mathcal{E}^\varepsilon(R) - \mathcal{E}_0^\varepsilon(R) = L(u_R^\varepsilon) - 1 + \varepsilon (A_K(-Re_3; u_R^\varepsilon) - A_K(-Re_3; \omega_R))$$

at $\varepsilon = 0$. Clearly $f_R(0) = 0$. Now we recall that $L'(\omega_R) = 0$ because $\omega_R$ is a geodesic, and we write

$$f'_R(\varepsilon) = (L'(u_R^\varepsilon) - L'(\omega_R))(\partial_\varepsilon u_R^\varepsilon) + \varepsilon A'_K(-Re_3; u_R^\varepsilon)(\partial_\varepsilon u_R^\varepsilon)$$

$$+ (A_K(-Re_3; u_R^\varepsilon) - A_K(-Re_3; \omega_R)).$$

To take the limit as $\varepsilon \to 0$, we notice that $\partial_\varepsilon u_R^\varepsilon$ is uniformly bounded in $C^2_0$ because the function $(\varepsilon, R) \mapsto u_R^\varepsilon$ is of class $C^1$. Further, $L'(u_R^\varepsilon) \to L'(\omega_R)$ in the norm operator, $A'_K(-Re_3; u_R^\varepsilon)(\partial_\varepsilon u_R^\varepsilon)$ remains bounded and $A_K(-Re_3; u_R^\varepsilon) \to A_K(-Re_3; \omega_R)$. In conclusion, we have that $f'_R(0) = 0$, and therefore $f_R(\varepsilon) = o(\varepsilon)$ as $\varepsilon \to 0$, uniformly on $SO(3)$. That is, (3.11) holds true "at the zero order".

To conclude the proof we have to handle the derivatives of $\mathcal{E}^\varepsilon(R) - \mathcal{E}_0^\varepsilon(R)$ with respect to $R$, along any direction $RT_h \in T_RSO(3)$. We use (3.16), the second equality in (3.15) and then (3.16) again to obtain

$$\begin{align*}
\lim_{h \to 0} \int_{S^1} J_0(u_R^\varepsilon) \cdot (d_R u_R^\varepsilon(RT_h)) \, dx &= \sum_{j=1}^3 \zeta_j^\varepsilon(R) \int_{S^1} (d_R u_R^\varepsilon(RT_h)) \cdot (Re_j \land \omega_R) \, dx \\
&= \sum_{j=1}^3 \zeta_j^\varepsilon(R) \int_{S^1} (Re_h \land u_R^\varepsilon) \cdot (Re_j \land \omega_R) \, dx = \int_{S^1} J_\varepsilon(u_R^\varepsilon) \cdot (Re_h \land u_R^\varepsilon) \, dx.
\end{align*}$$

By (3.9), the last integral can be written as

$$\begin{align*}
\int_{S^1} J_0(u_R^\varepsilon) \cdot (Re_h \land u_R^\varepsilon) \, dx &= \varepsilon L(u_R^\varepsilon)A'_K(-Re_3; u_R^\varepsilon)(Re_h \land u_R^\varepsilon) \\
&= \varepsilon L(u_R^\varepsilon)A'_K(-Re_3; u_R^\varepsilon)(Re_h \land u_R^\varepsilon)
\end{align*}$$

because of (3.3). Thus (3.17) leads to the new formula

$$d_R \mathcal{E}^\varepsilon(R)(RT_h) = \varepsilon A'_K(-Re_3; u_R^\varepsilon)(Re_h \land u_R^\varepsilon).$$
On the other hand, it is easy to see that

\[ d_R \mathcal{E}_0^\varepsilon(R)(RT_h) = \varepsilon A_K'(-Re_3; \omega_R)(d_R(\omega_R)(RT_h)) = \varepsilon A_K'(-Re_3; \omega_R)(Re_h \wedge \omega_R), \]

because \( A_K(\cdot; \omega_R) \) is locally constant, and we can conclude that

\[ d_R(\mathcal{E}^\varepsilon(R) - \mathcal{E}_0^\varepsilon(R))(RT_h) \]

\[ = \varepsilon \left( A_K'(-Re_3; u_R^\varepsilon)(Re_h \wedge u_R^\varepsilon) - A_K'(-Re_3; u_R^\varepsilon)(Re_h \wedge \omega_R) \right) = o(\varepsilon), \]

because \( u_R^\varepsilon \to \omega_R \). The lemma is completely proved. □

**Proof of Theorem 1.1** Let \( \varepsilon \) be given by Lemma 3.3. For any \( c > \varepsilon^{-1} \), let \( \varepsilon := c^{-1} < \varepsilon \) and \((\varepsilon, R) \mapsto u_R^\varepsilon, (\varepsilon, R) \mapsto \mathcal{E}^\varepsilon(R)\) be the functions in Lemma 3.3. To every critical point \( R^\varepsilon \) for \( \mathcal{E}^\varepsilon \) corresponds a curve \( u_R^\varepsilon \) that solves \( J^\varepsilon(u_R^\varepsilon) = 0 \). Hence \( u_R^\varepsilon \) solves \( \mathcal{E} \) and, as explained at the beginning of Section 2, yields a solution to \((P_{K,\varepsilon^{-1}}) = (P_{K,c})\).

Now, if \( \mathcal{E}^\varepsilon \) is constant, then \( u_R^\varepsilon \) solves \( \mathcal{E} \) for every \( R \in SO(3) \) and the conclusions in Theorem 1.1 hold. Otherwise, take \( R^\varepsilon, \bar{R}^\varepsilon \in SO(3) \) achieving the minimum and the maximum value of \( \mathcal{E}^\varepsilon \), respectively. Then \( u^\varepsilon := u_R^\varepsilon \) and \( \bar{u}^\varepsilon := u_{\bar{R}}^\varepsilon \) solve \( \mathcal{E} \) and this concludes the proof of the existence part.

Next, assume that \( \mathcal{E}^\varepsilon \) is not constant, and that \( u^\varepsilon = \bar{u}^\varepsilon \circ g \) for a diffeomorphism \( g \) of \( S^1 \). To conclude the proof we have to show that \( \mathcal{E} \) can not hold.

We have \( E_{\varepsilon K}(z^\varepsilon, u^\varepsilon) < E_{\varepsilon K}(\bar{z}^\varepsilon, \bar{u}^\varepsilon) \), that is,

\[ L(u^\varepsilon) + \varepsilon A_K(z^\varepsilon, u^\varepsilon) < L(\bar{u}^\varepsilon) + \varepsilon A_K(\bar{z}^\varepsilon, \bar{u}^\varepsilon) \tag{3.18} \]

where \( z^\varepsilon = -R^\varepsilon e_3, \bar{z}^\varepsilon = -\bar{R}^\varepsilon e_3 \). Since \(|(u^\varepsilon)'|, |(\bar{u}^\varepsilon)'|\) are constant, then \(|g'|\) is constant as well. Thus \(|g'| = 1\) and \( L(u^\varepsilon) = L(\bar{u}^\varepsilon) \). Therefore, (3.18) implies

\[ A_K(z^\varepsilon, u^\varepsilon) \neq A_K(\bar{z}^\varepsilon, \bar{u}^\varepsilon) \tag{3.19} \]

for any \( \varepsilon \neq 0 \). In particular, \( g \) can not be a positive rotation of the circle by the property \( A2) \) of the area functional. Thus \( g \) is a counterclockwise rotation of \( S^1 \). Recall that \( u^\varepsilon \) has curvature \( \varepsilon K(u^\varepsilon) \) and \( \bar{u}^\varepsilon \) has curvature \( \varepsilon K(\bar{u}^\varepsilon) \). Since changing the orientation of a curve changes the sign of its curvature, we have that at any point
\( p \in \Gamma := \overline{u^\varepsilon(S^1)} = \overline{w^\varepsilon(S^1)} \) we have \( K(p) = -K(p) \). It follows that \( K \equiv 0 \) on \( \Gamma \), and hence \( \Gamma \) is the boundary of a half-sphere \( D_{\frac{\pi}{2}}(w^\varepsilon) \). We can assume that \( u^\varepsilon \) is a positive parameterization of \( \partial D_{\frac{\pi}{2}}(w^\varepsilon) \). Then \( z^\varepsilon \notin D_{\frac{\pi}{2}}(w^\varepsilon) \) because \( u^\varepsilon \approx \omega_{w^\varepsilon} \), see i) in Lemma 3.3 Next, since \( u^\varepsilon \) parameterizes the same geodesic with opposite direction, then \( \overline{u^\varepsilon} \) a positive parameterization of \( \partial D_{\frac{\pi}{2}}(-w^\varepsilon) \) and \( \overline{u^\varepsilon} \notin D_{\frac{\pi}{2}}(-w^\varepsilon) \). In particular, from the properties \( A3 \) and \( A4 \) of the area functional we infer

\[
A_K(z^\varepsilon, u^\varepsilon) = A_K(-w^\varepsilon, u^\varepsilon) = -\frac{1}{2\pi} \int_{D_{\frac{\pi}{2}}(w^\varepsilon)} K(q) \, d\sigma_q
\]

\[
A_K(z^\varepsilon, \overline{u^\varepsilon}) = A_K(w^\varepsilon, \overline{u^\varepsilon}) = -\frac{1}{2\pi} \int_{D_{\frac{\pi}{2}}(-w^\varepsilon)} K(q) \, d\sigma_q,
\]

that compared with (3.19) shows that (1.2) is violated.

The theorem is completely proved. \( \square \)

4 Multiplicity

Let \( K \in C^1(S^2) \) be given. The Mel’nikov function \( F_K : S^2 \to \mathbb{R} \) in (1.3) is clearly of class \( C^1 \). We adopt and recall here the definition of stable critical point proposed in [2, Chapter 2], see also [9].

**Definition 4.1** Let \( \Omega \subset S^2 \) be open. We say that \( F_K \) has a stable critical point in \( \Omega \) if there exists \( r > 0 \) such that any function \( G \in C^1(\overline{\Omega}) \) satisfying \( \|G - F_K\|_{C^1(\overline{\Omega})} < r \) has a critical point in \( \Omega \).

If \( F_K \) is not constant, then it has at least two distinct stable critical points, namely, its minimum and its maximum. Different sufficient conditions to have the existence of (possible multiple) stable critical points \( z \in \Omega \) for \( F_K \) are easily given via elementary calculus. For instance, one can assume that one of the following conditions holds:

i) \( \nabla F_K(z) \neq 0 \) for any \( z \in \partial \Omega \), and \( \deg(\nabla F_K, \Omega, 0) \neq 0 \), where ”deg” is Browder’s topological degree;
\[ \min_{\partial \Omega} F_K > \min_{\Omega} F_K \quad \text{or} \quad \max_{\partial \Omega} F_K < \max_{\Omega} F_K; \]

\[ F_K \text{ is of class } C^2 \text{ on } \Omega, \text{ it has a critical point } z_0 \in \Omega, \text{ and the Hessian matrix of } F_K \text{ at } z_0 \text{ is invertible.} \]

In this section we show that any stable critical point \( z_0 \) for \( F_K \) gives rise, for any \( c > 0 \) large enough, to a solution \( \gamma^c \) to Problem \( (P_{K,c}) \) which is a perturbation of the closed geodesic about \( z_0 \). Taking advantage of the remarks at the beginning of Section 2, we only need to show that for any stable critical point \( z_0 \) for \( F_K \) and for any \( \varepsilon = c^{-1} \approx 0^+ \), there exists a solution \( u^\varepsilon \) to (3.8), such that \( u^\varepsilon \) is close to the closed geodesic about \( z_0 \). In fact, the next theorem holds.

**Theorem 4.2** Let \( K \in C^1(S^2) \) be given. Assume that \( F_K \) has a stable critical point in an open set \( \Omega \subset S^2 \), such that \( \overline{\Omega} \subset S^2 \).

Then for every \( \varepsilon \in \mathbb{R} \) close enough to 0, there exists a point \( z_\varepsilon \in \Omega \), an embedding \( \omega^\varepsilon : S^1 \to S^2 \) parameterizing the boundary of a circle of geodesic radius \( \pi/2 \) about \( z_\varepsilon \), and a solution \( u^\varepsilon \) to (3.8), such that \( \|u^\varepsilon - \omega^\varepsilon\|_{C^2} = O(\varepsilon) \).

**Proof.** We can assume \( -e_3 \notin \overline{\Omega} \). Otherwise, take any rotation \( R \in SO(3) \) such that \( -e_3 \notin R\overline{\Omega} \), and look for a solution \( \tilde{u}^\varepsilon \) to

\[
u'' + |u'|^2 u = L(u) \varepsilon (K \circ t^R)(u) u \wedge u' \quad \text{on } S^1,
\]

in a \( C^2 \)-neighborhood of a geodesic circle about some point \( \tilde{z}^\varepsilon \in R\Omega \). Conclude by noticing that \( u^\varepsilon := t^R \tilde{u}^\varepsilon \) solves (3.8) and approaches the geodesic circle about \( R^1 \tilde{z}^\varepsilon \in \Omega \).

Next, for \( z \in S^2 \setminus \{-e_3\} \) consider the rotation

\[
N(z) = \begin{pmatrix}
1 - \frac{z_1^2}{1 + z_3} & -\frac{z_1 z_2}{1 + z_3} & z_1 \\
-\frac{z_1 z_2}{1 + z_3} & 1 - \frac{z_2^2}{1 + z_3} & z_2 \\
-z_1 & -z_2 & z_3
\end{pmatrix},
\]

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that maps $e_3$ to $z$. Clearly the function $N : \mathbb{S}^2 \setminus \{ -e_3 \} \to SO(3)$ is differentiable; its differential $dN(z)$ at any $z \in \mathbb{S}^2 \setminus \{ -e_3 \}$ is a linear map $T_z\mathbb{S}^2 \to T_{N(z)}SO(3)$. We have

\begin{align}
T_z\mathbb{S}^2 &= \langle N(z)e_1, N(z)e_2 \rangle \tag{4.1} \\
T_{N(z)}SO(3) &= \langle dN(z)(N(z)e_1), dN(z)(N(z)e_2) \rangle \oplus \langle N(z)T_3 \rangle. \tag{4.2}
\end{align}

Equality (4.1) and the inclusion $\supseteq$ in (4.2) are trivial. To conclude the proof of (4.2) we need to show that the matrices

\begin{align*}
dN(z)(N(z)e_1), & \quad dN(z)(N(z)e_2), \quad N(z)T_3
\end{align*}

are linearly independent. By differentiating the identity $N(z)e_3 = z$ one gets

\[
dN(z)\tau \cdot e_3 = \tau, \quad \tau \in T_z\mathbb{S}^2.
\]

By choosing $\tau = N(z)e_h$, $h = 1, 2$ we infer that the third columns of the matrices $dN(z)(N(z)e_1), dN(z)(N(z)e_2)$ are linearly independent. Thus the matrices $dN(z)(N(z)e_1), dN(z)(N(z)e_2)$ are linearly independent as well. On the other hand, the third column on $N(z)T_3$ is identically zero, that concludes the proof of (4.2).

Now, take the differentiable functions $(\varepsilon, R) \mapsto u^\varepsilon_R \in C^2_{S^2}, (\varepsilon, R) \mapsto \mathcal{E}^\varepsilon(R) \in \mathbb{R}$ given by Lemma 3.3. To simplify notations, for $z \in \mathbb{S}^2 \setminus \{ -e_3 \}$ we write

\[
\tilde{\mathcal{E}}^\varepsilon(z) = \mathcal{E}^\varepsilon(N(z)) = E_\varepsilon K(-z; U^\varepsilon_{N(z)}), \quad \tilde{\mathcal{E}}^\varepsilon_0(z) = \mathcal{E}^\varepsilon_0(N(z)) = E_\varepsilon K(-z; N(z)\omega).
\]

Notice that $N(z)\omega$ parameterizes $\partial D_{\pi/2}(z)$. Therefore, using $ii)$ in Lemma 2.3 property A4) and elementary computations we get

\begin{align}
\tilde{\mathcal{E}}^\varepsilon_0(z) &= L(N(z)\omega) + \varepsilon A_K(-z; N(z)\omega) \\
&= L(\omega) - \frac{\varepsilon}{2\pi} \int_{D_{\pi/2}(z)} K(q)d\sigma_q = L(\omega) - \frac{\varepsilon}{2\pi} F_K(z). \tag{4.3}
\end{align}

Next, for any small $\varepsilon \neq 0$ consider the function

\[
G^\varepsilon(z) = \frac{2\pi}{\varepsilon} (\tilde{\mathcal{E}}^\varepsilon(z) - L(\omega))
\]
and use (4.3) together with iv) in Lemma 3.3 to get
\[
\|G^\varepsilon + F_K\|_{C^1(\Omega)} = \frac{2\pi}{|\varepsilon|} \left\|E_{\varepsilon K}(-z; u^*_K(z) - E_{\varepsilon K}(z))\right\|_{C^1(\Omega)} = o(1)
\]
as \(\varepsilon \to 0\). We see that for \(\varepsilon\) small enough the function \(G^\varepsilon\) has a critical point \(z^\varepsilon \in \Omega\). Thus, for any \(\tau \in T_{z^\varepsilon}S^2\) we have
\[
0 = d_{z^\varepsilon} \check{E}^\varepsilon(z^\varepsilon) \tau = d_R \check{E}^\varepsilon(N(z^\varepsilon)) (d_{z^\varepsilon} N(z^\varepsilon) \tau).
\]
Taking (4.2) and iv) in Lemma 3.3 into account, we infer that the matrix \(N(z^\varepsilon)\) is critical for \(\check{E}^\varepsilon\). Thus, by arguing as for Theorem 1.1 we have that the curve \(u^\varepsilon := u^*_K(z^\varepsilon)\) is a solution to \((PK,\varepsilon^{-1})\).

\[
\square
\]

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