WELL ORDERED COVERS, SIMPLICIAL BOUQUETS, AND SUBADDITIVITY OF BETTI NUMBERS OF SQUARE-FREE MONOMIAL IDEALS

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Abstract

Well ordered covers of square-free monomial ideals are subsets of the minimal generating set ordered in a certain way that give rise to a Lyubeznik resolution for the ideal, and have guaranteed nonvanishing Betti numbers in certain degrees. This paper is about square-free monomial ideals which have a well ordered cover. We consider the question of subadditivity of syzygies of square-free monomial ideals via complements in the lcm lattice of the ideal, and examine how lattice complementation breaks well ordered covers of the ideal into (well ordered) covers of subideals. We also introduce a family of well ordered covers called strongly disjoint sets of simplicial bouquets (generalizing work of Kimura on graphs), which are relatively easy to identify in simplicial complexes. We examine the subadditivity property via numerical characteristics of these bouquets.

1 Introduction

This paper grew out of investigations into the subadditivity property of syzygies of monomial ideals. For a homogeneous ideal $I$ of a polynomial ring $S$, suppose the maximum degree $j$ such that $\beta_{i,j}(S/I) \neq 0$ is denoted by $t_i$. The subadditivity property is said to hold if we have $t_{a+b} \leq t_a + t_b$ for all positive values of $a$ and $b$.

While the subadditivity property or related inequalities are known to hold in many special cases - certain cases for ideals of codimension $\leq 1$ ([5, Corollary 4.1]); some Koszul rings ([1]); when $I$ monomial ideal and $a = 1$ ([15]); certain homological degrees for Gorenstein algebras ([6]); when $a = 1, 2, 3$ and $I$ monomial ideal generated in degree 2 ([11, 2]); facet ideals of simplicial forests ([9]) - the problem is open for monomial ideals and is known to fail (see Caviglia’s example in [5, Example 4.5]) for general homogeneous ideals.

In the case of monomial ideals, Betti numbers can be calculated as ranks of homology modules of topological objects. In particular, the order complex of the lcm lattice of $I$ (the poset of least common multiples of the minimal monomial generating set of $I$ ordered by division) can be used for

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this purpose. A nonvanishing Betti number, in this context, corresponds by a result of Baclawski \cite{3} to a “complemented” lcm lattice (see Section \cite{4}).

This approach was initiated by the first author in \cite{9}; the existence of complements which have nonvanishing Betti numbers in the “right” homological degrees implies the subadditivity property (Question \cite{4.2}). This approach was further pursued by both authors in \cite{10}, where Question \cite{4.2} was translated into the context of homological cycles in simplicial complexes breaking into smaller ones.

This paper takes yet a different angle. The idea of complements really comes down to the following: take a monomial \( m \) in the lcm lattice of \( I \). Among all the complements of \( m \) in the lcm lattice, can you pick one, say \( m' \), which behaves desirably? If \( m \) is a square-free monomial, this question is even simpler: \( m \) and \( m' \) correspond to subsets \( A \) and \( A' \) of the variables \( \{x_1, \ldots, x_n\} \) such that \( A \cup A' = \{x_1, \ldots, x_n\} \) and the product of the variables in \( A \cap A' \) is not in \( I \). Can we consider the subideals induced on \( A \) and \( A' \) and extract properties from/for them?

The main object of study using this approach will be “well ordered covers” of ideals (Definition \cite{3.2}). The existence of a well ordered cover of size \( i \) is known, via the Lyubeznik resolution, to guarantee a nonvanishing \( i \)th Betti number \cite{8}. In this paper we investigate when complements in the lcm lattice produce subideals with well ordered covers of sizes \( a \) and \( b \) with \( a + b = i \), in order to result in the subadditivity property.

Moreover, we introduce strongly disjoint sets of simplicial bouquets, which we show are always well ordered covers, and we demonstrate how they can be broken up to prove subadditivity in certain homological degrees. An advantage of simplicial bouquets is that they are rather easy to spot in simplicial complexes, as they do not rely on an ordering. Rather, one or more orderings are inherent in the definition (see Theorem \cite{5.4}).

In Section \cite{2} we set up the background on simplicial resolutions. Section \cite{3} introduces the reader to well ordered covers of monomials. Section \cite{4} describes the subadditivity property and contains one of the main results of the paper (Theorem \cite{4.7}) which considers well ordered covers under complementation. In Section \cite{5} we introduce simplicial bouquets and show that certain types of simplicial bouquets are well ordered facet covers. We then apply the results of Section \cite{4} to simplicial complexes that contain strongly disjoint sets of bouquets, and show that the subadditivity property holds in degrees that come from the sizes of the bouquets (Theorem \cite{5.7}). Section \cite{6} offers ways to optimize the order of monomials in a well ordered cover to get the best possible subadditivity results.

With some numerical manipulation, the results of this paper can be adapted to non-square-free monomial ideals via polarization \cite{12}, a method that transforms a monomial ideal into a square-free one which retains many of the algebraic properties of the original ideal, including the minimal free resolution.

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2 Background

2.1 Simplicial complexes and facet ideals

A simplicial complex $\Delta$ on a finite vertex set $V(\Delta)$ is a set of subsets of $V(\Delta)$ such that $\{v\} \in \Delta$ for every $v \in V(\Delta)$ and if $F \in \Delta$, then for every $G \subseteq F$, we have $G \in \Delta$. The elements of $\Delta$ are called faces; the maximal faces with respect to inclusion are called facets, and a simplicial complex contained in $\Delta$ is called a subcomplex of $\Delta$. The set of all facets in $\Delta$ defines $\Delta$ and is denoted by Facets($\Delta$). If Facets($\Delta$) = $\{F_1, \ldots, F_q\}$, then we write $\Delta = \langle F_1, \ldots, F_q \rangle$.

A subcollection of $\Delta$ is a subcomplex of $\Delta$ whose facets are also facets of $\Delta$. If $A \subseteq V(\Delta)$, then the induced subcollection $\Delta[A]$ is the simplicial complex defined as

$$\Delta[A] = \langle F \in \text{Facets}(\Delta) : F \subseteq A \rangle.$$

We say a facet $F$ contains a free vertex $v$ if $v \notin G$ for every facet $G \in \text{Facets}(\Delta) \setminus \{F\}$.

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Given a simplicial complex $\Delta$ on vertices $\{x_1, \ldots, x_n\}$, we can define the facet ideal of $\Delta$ as

$$\mathcal{F}(\Delta) = \left( \prod_{x_i \in F} x_i : F \text{ is a facet of } \Delta \right)$$

which is an ideal of $S = k[x_1, \ldots, x_n]$. Conversely, given a square-free monomial ideal $I \subset S$, the facet complex of $I$ is the simplicial complex

$$\mathcal{F}(I) = \langle F : \prod_{x_i \in F} x_i \text{ is a generator of } I \rangle.$$

Example 2.1. For $I = (xy, yz, zu)$, the simplicial complex $\mathcal{F}(I)$ is below.

```
     y
    /|
   / |
  x---z
    |
    u
```

2.2 Simplicial Resolutions

Any monomial ideal $I$ of $S = k[x_1, \ldots, x_n]$ admits a minimal graded free resolution, which is an exact sequence of free $S$-modules

$$0 \rightarrow \bigoplus_{j \in \mathbb{N}} S(-j)^{\beta_{p,j}} \rightarrow \bigoplus_{j \in \mathbb{N}} S(-j)^{\beta_{p-1,j}} \rightarrow \cdots \rightarrow \bigoplus_{j \in \mathbb{N}} S(-j)^{\beta_{1,j}} \rightarrow S.$$ 

For each $i$ and $j$, the rank $\beta_{i,j}(S/I)$ of the free $S$-modules appearing above are called the graded Betti numbers of the $S$-module $S/I$, and the total Betti number in homological degree $i$ is

$$\beta_i(S/I) = \sum_j \beta_{i,j}(S/I).$$
If $I$ is generated by monomials, the graded Betti numbers can be further refined into sums of multigraded Betti numbers. For a monomial $m$ in $S$, the multigraded Betti number of $S/I$ is of the form $\beta_{i,m}(S/I)$ and we have

$$\beta_{i,j}(S/I) = \sum \beta_{i,m}(S/I)$$  \hspace{1cm} (1)

where the sum is taken over all monomials $m$ of degree $j$ that are least common multiples of subsets of the minimal monomial generating set of $I$.

The multigraded Betti numbers $\beta_{i,m}(S/I)$ are related to the combinatorics of the ideal $I$. Given a monomial ideal $I$ minimally generated by $m_1, \ldots, m_q$, one can consider a simplicial complex $\Gamma$ on $q$ vertices $\{v_1, \ldots, v_q\}$, where each vertex $v_i$ is labeled with the monomial generator $m_i$, and each face $\tau$ of $\Gamma$ is labeled with the monomial $\text{lcm}(\tau) = \text{lcm}(m_i : m_i \in \tau)$.

We say that $\Gamma$ supports a free resolution of $I$ if the simplicial chain complex of $\Gamma$ can be “homogenized”, using the monomial labels of the faces, to produce a free resolution of $I$. For details of homogenization of a chain complex see [20, Section 55]. The resulting free resolution is called a simplicial resolution.

Taylor’s resolution ([21], see also [20, Construction 26.5]) is an example of a simplicial resolution where the underlying simplicial complex is a full simplex $\Delta(I)$ over the vertex set labeled with the monomial generators $\{m_1, \ldots, m_q\}$ of $I$, called the Taylor complex of $I$. It is known that all simplicial complexes supporting a free resolution of $I$ are subcomplexes of the Taylor complex ([19]), in other words: all simplicial resolutions are contained in the Taylor resolution.

If $\Gamma$ is a simplicial complex supporting a free resolution of $I$, and $m$ is a monomial in $S$, the simplicial subcomplex $\Gamma_{<m}$ is defined as

$$\Gamma_{<m} = \{\tau \in \Gamma : \text{lcm}(\tau) \text{ strictly divides } m\}.$$  

Example 2.2. Consider $I = (xy, yz, zu)$ in $k[x, y, z, u]$. The Taylor complex $\Delta(I)$ and a subcomplex $\Delta(I)_{<xyzu}$ are shown in the following figures.

If $\Gamma$ supports a free resolution of a monomial ideal $I$, then for a fixed integer $i$, the Betti number $\beta_i(S/I)$ is bounded above by the number of $(i - 1)$-faces of $\Gamma$. Therefore, the more we “shrink” the supporting complex $\Gamma$, the better we bound the Betti numbers.
In particular, as stated by Bayer and Sturmfels [4], the multigraded Betti numbers of \( I \) can be determined by the dimensions of reduced homologies of subcomplexes of \( \Delta \). The statement below is from Peeva’s textbook [20].

**Theorem 2.3** ([20], Theorem 57.6). Let \( I \) be a proper monomial ideal of \( S \) which is minimally generated by the monomials \( m_1, \ldots, m_q \), and suppose that \( I \) has a free resolution supported on a simplicial complex \( \Gamma \). For \( i > 0 \) and a monomial \( m \) of positive degree, the multigraded Betti numbers of \( I \) are given by

\[
\beta_{i,m}(S/I) = \begin{cases} 
\dim_k \tilde{H}_{i-2}(\Gamma_{<m}, k) & \text{if } m \mid \text{lcm}(m_1, \ldots, m_q) \\
0 & \text{otherwise.}
\end{cases}
\]

**Example 2.4.** For \( I = (xy, yz, zu) \) in Example 2.2, \( T(I)_{<xyzu} \) is acyclic and hence \( \beta_{i,xyzu}(S/I) = 0 \) for all \( i \).

For a monomial ideal \( I \) of \( S \) which is minimally generated by the set \( G = \{m_1, \ldots, m_q\} \), the lcm lattice of \( I \), denoted by \( \text{LCM}(I) \), is the bounded lattice whose elements are the least common of subsets of \( G \) ordered by divisibility. The top element of \( \text{LCM}(I) \) is \( \hat{1} = \text{lcm}(m_1, \ldots, m_q) \) and the bottom element is \( \hat{0} = 1 \) regarded as the lcm of the empty set. The least common multiple of elements in \( \text{LCM}(I) \) is their join.

**Example 2.5.** The following is the \( \text{LCM}(I) \) for \( I = (xy, yz, zu) \) from Example 2.2.

\[
\begin{array}{c}
xyzu \\
xyz & yzu \\
xy & yz & zu \\
x & y & z & u \\
1
\end{array}
\]

### 3 (Well ordered) covers

Let \( \Delta \) be a simplicial complex. A set \( D \subseteq \text{Facets}(\Delta) \) is called a **facet cover** of \( \Delta \) if every vertex \( v \) of \( \Delta \) belongs to some facet \( F \) in \( D \). A facet cover is called **minimal** if no proper subset of it is a facet cover of \( \Delta \). For instance, the simplicial complex in Example 2.1 has the set \( \{\{x, y\}, \{z, u\}\} \) as a minimal facet cover.

\[
\begin{array}{c}
y \\
\downarrow \\
x \\
\quad \quad \quad \\
\downarrow \\
\text{z} \\
\quad \quad \quad \\
\quad \quad \quad \\
\downarrow \\
\text{u}
\end{array}
\]
We can translate a minimal facet cover of a simplicial complex to its facet ideal. If \( I = (m_1, \ldots, m_q) \) is a square-free monomial ideal and \( \Delta = \mathcal{F}(I) = \langle F_1, \ldots, F_q \rangle \) so that each \( m_i \) is the product of the vertices in \( F_i \), then we say \( m_1, \ldots, m_q \) is a \textbf{(minimal) cover of \( I \) to imply that} \( F_1, \ldots, F_q \) is a (minimal) facet cover of \( \Delta \).

For the sake of simplicity, assume that every variable \( x_i \) appears in at least one of the generators of \( I \). If \( M = \{m_{i_1}, \ldots, m_{i_t}\} \) is a minimal cover of \( I \), then one can see that for every \( j \in \{1, \ldots, t\} \)

\[
\operatorname{lcm}(m_{i_1}, \ldots, \hat{m}_{i_j}, \ldots, m_{i_t}) \neq \operatorname{lcm}(m_{i_1}, \ldots, m_{i_t}) = x_1 \cdots x_n.
\]

This implies that if \( \mathbb{T}(I) \) is the Taylor complex of \( I \), then \( \mathbb{T}(I)_{<x_1^ix_2^j} \) contains the boundary of a hollow \((t-2)\)-cycle, which by Theorem 2.3 means that we could potentially have \( \beta_{t,x_1^ix_2^j}(S/I) \neq 0 \).

In other words, the existence of a minimal cover \( M \) of length \( t \) indicates that we might have a “top degree” Betti number \( \beta_{t,n} \). By ordering the generators of \( I \) we can make \( M \) into a “well ordered” cover, which, using a Lyubeznik resolution (a simplicial resolution that is based on ordering the generators of an ideal [18]), guarantees the nonvanishing of \( \beta_{t,n} \) (Theorem 3.4).

\textbf{Definition 3.1} ([8], Definition 3.1). Let \( \Delta \) be a simplicial complex. A sequence of \( F_1, \ldots, F_s \) of facets of \( \Delta \) is called a \textbf{well ordered facet cover} if it is minimal facet cover of \( \Delta \), and for every facet \( F' \notin \{F_1, \ldots, F_s\} \) of \( \Delta \) there exists \( j \leq s-1 \) such that

\[
F_j \subseteq F' \cup F_{j+1} \cup \cdots \cup F_s.
\]

The definition below is an equivalent version of Definition 3.1 stated for monomial ideals.

\textbf{Definition 3.2 (Well ordered cover).} A sequence \( m_1, \ldots, m_s \) of generators of a square-free monomial ideal \( I \) is called a \textbf{well ordered cover} of \( I \) if \( \{m_1, \ldots, m_s\} \) is a minimal cover of \( I \) and for every generator \( m' \notin \{m_1, \ldots, m_s\} \) of \( I \) there exists \( j \leq s-1 \) such that

\[
m_j \mid \operatorname{lcm}(m', m_{j+1}, \ldots, m_s).
\]

\textbf{Example 3.3.} 1. For \( I = (abz, bcz, xyz, axz) \), \( \{abz, bcz, xyz\} \) is a well ordered cover since \( abz \) divides \( \operatorname{lcm}(axz, bcz, xyz) \).

2. For \( I = (xy, yz, zu), \{xy, zu\} \) is a minimal cover that is not a well ordered cover of \( I \) since \( xy \nmid \operatorname{lcm}(yz, zu) \) and \( zu \nmid \operatorname{lcm}(yz, xy) \).
Notice that \( I \) has no well ordered cover since \( \{xy, zu\} \) is the only possible minimal cover and it is not well ordered.

A class of examples of well ordered facet covers is (simplicial) bouquets, which will be discussed in Section 5.

As the following theorem shows, well ordered covers are facets in a Lyubeznik complex, and hence ensure nonvanishing multigraded Betti numbers.

**Theorem 3.4 (8).** Let \( \mathcal{M} = \{m_1, \ldots, m_s\} \) be a well ordered cover of a square-free monomial ideal \( I \). Then there is a total order \( < \) on the set of generators of \( I \) such that \( \mathcal{M} \) is a facet of the Lyubeznik simplicial complex and hence \( \beta_{s, \mathcal{M}}(S/I) \neq 0 \) where \( \mathcal{M} = \text{lcm}(m_1, \ldots, m_s) \).

The converse of the above theorem holds in some cases, for example when \( I \) is the facet of a simplicial forest \( \Delta \), every nonzero Betti number of \( I \) corresponds to a well ordered facet cover of an induced subcollection of \( \Delta \) (8).

4 The subadditivity property

In this section, we will explore how we can use well ordered covers to consider the subadditivity property for the maximal degrees of syzygies of square-free monomial ideals. Let

\[ t_a = \max\{j : \beta_{a,j}(S/I) \neq 0\}. \]

We say that \( I \) satisfies the subadditivity property if for all \( a, b > 0 \) with \( a + b \leq \text{the projective dimension of } S/I, \)

\[ t_{a+b} \leq t_a + t_b. \]

In what follows, we will be moving back and forth between a square-free monomial ideal \( I \) and its facet complex \( \Delta = \mathcal{F}(I) \). For a monomial \( m \in \text{LCM}(I) \) where \( m = x_{i_1} \cdots x_{i_h} \), by \( I_{[m]} \) we mean the facet ideal of the induced subcollection \( \Delta_{[m]} = \Delta_{[x_{i_1}, \ldots, x_{i_h}]} \) or in other words

\[ I_{[m]} = \mathcal{F}(\Delta_{[m]}) = \mathcal{F}(\Delta_{[x_{i_1}, \ldots, x_{i_h}]}). \]

If we set \( \Gamma \) and \( \Gamma' \) to be the Taylor complexes of \( I \) and \( I_{[m]} \), respectively, then Theorem 2.3 indicates that

\[ \beta_{s, m}(S/I) = \beta_{s, m}(S/I_{[m]}) \] (2)

as \( \Gamma_{<m} = \Gamma'_{<m} \). For an integer \( i, \) (1) and (2) show that

\[ \beta_i(S/I) \neq 0 \iff \beta_{i, m}(S/I) \neq 0 \text{ for some } m \in \text{LCM}(I) \iff \beta_{i}(S/I_{[m]}) \neq 0 \text{ for some } m \in \text{LCM}(I). \]
In the rightmost statement in the previous line, the monomial \( m \) is at the “top” of the lcm lattice of \( I_m \). This useful observation tells us that questions about multidegree Betti numbers can be reduced to questions about “top degree” Betti numbers. In the same vein, the subadditivity question can always be rephrased as a “top degree” one. We state this version for square-free monomials below.

**Question 4.1 (Top degree subadditivity).** Suppose \( I = (m_1, \ldots, m_q) \) is a square-free monomial ideal in a polynomial ring \( S \), and \( \text{lcm}(m_1, \ldots, m_q) = x_1 \cdots x_r \). Suppose \( \beta_{i,r}(S/I) \neq 0 \), and \( a, b > 0 \) are such that \( i = a + b \). Then can we show that \( t_a + t_b \geq r = t_{a+b} \)?

In fact, the ambient ring does not really matter, so for the sake of simplicity we can assume in Question 4.1 that \( r = n \) and \( x_1 \cdots x_r = x_1 \cdots x_n \). If \( I \) is a square-free monomial ideal in \( S = k[x_1, \ldots, x_n] \), then since there is only one monomial of degree \( n \) in LCM(I), from (1) we have

\[
\beta_{i,n}(S/I) = \beta_{i,x_1 \cdots x_n}(S/I).
\]

So from now on we will start from this setting.

For a square-free monomial ideal \( I \) in the variables \( x_1, \ldots, x_n \), two monomials \( m, m' \in \text{LCM}(I) \) are called lattice complements if \( \text{lcm}(m, m') = x_1 \cdots x_n \) and \( \text{gcd}(m, m') \notin I \). As a potential way to examine the subadditivity property, the first author raised the following question in [9] (see also [10]):

**Question 4.2 (Betti numbers of lattice complements [9], Question 1.1).** If \( I \) is a square-free monomial ideal in variables \( x_1, \ldots, x_n \), and \( \beta_{i,r}(S/I) \neq 0 \), \( a, b > 0 \) and \( i = a + b \), are there complements \( m \) and \( m' \) in \( \text{LCM}(I) \) with \( \beta_{a,m}(S/I) \neq 0 \) and \( \beta_{b,m'}(S/I) \neq 0 \)?

The existence of lattice complements will establish the subadditivity property of \( I \) simply since \( t_a + t_b \geq \deg(m) + \deg(m') \geq n = t_{a+b} \).

A well ordered cover seems to be a promising place to look for lattice complements as the following example shows.

**Example 4.3.** Let \( \Delta = \mathcal{F}(I) = \langle xy, yz, xz, za, ab, bc \rangle \).

According to Macaulay2 [13], \( S/I \) has the following Betti table:

\[
\begin{array}{cccc}
0 & 1 & 2 & 3 & 4 \\
\text{total} : & 1 & 6 & 10 & 7 & 2 \\
0 : & 1 & . & . & . & . \\
1 : & . & 6 & 6 & 1 & . \\
2 : & . & . & 4 & 6 & 2 \\
\end{array}
\]

Here \( \beta_{4,xyzabc}(S/I) \neq 0 \) and hence \( t_4 = 6 \). It is easy to check that \( \mathcal{M} = \{ab, xy, bc, xz\} \) (highlighted above) is a well ordered cover. We consider the following two cases:
1. \( a = 1 \) and \( b = 3 \). Then, using the above Betti table \( t_1 = 2 \) and \( t_3 = 5 \). Here we have \( t_4 < t_1 + t_3 = 7 \). Define the monomials \( m, m' \in \text{LCM}(I) \) as \( \text{lcm} \) of some subsets of \( M \) as follows:

\[
\begin{align*}
\Delta[m] & \quad \Delta[m'] \\
\end{align*}
\]

\[
m = ab = \text{lcm}(ab) \quad \text{and} \quad m' = bcxyz = \text{lcm}(xy, bc, xz).
\]

Then,

\[
\beta_{1, ab}(S/I) = \dim_k \tilde{H}_{-1}(\emptyset, k) \neq 0 \quad \text{and} \quad \beta_{3, xyzc}(S/I) = \dim_k \tilde{H}_1(\langle xy, yz, xz, bc \rangle, k) \neq 0.
\]

Note that \( \text{lcm}(m, m') = xyzabc \) and \( \text{gcd}(m, m') = b \notin I \). As a result, \( m \) and \( m' \) are lattice complements and

\[
t_1 + t_3 = \deg(m) + \deg(m') > 6 = t_4.
\]

2. \( a = b = 2 \). From the Betti table we have \( t_2 = 4 \) and \( t_4 < 2t_2 = 8 \). As in the above case, we take

\[
m = abxy = \text{lcm}(ab, xy) \quad \text{and} \quad m' = bcxz = \text{lcm}(bc, xz).
\]

Here, we also have \( \text{lcm}(m, m') = xyzabc \) and \( \text{gcd}(m, m') = bx \notin I \). So \( m \) and \( m' \) are lattice complements, and we have

\[
\beta_{2, ybab}(S/I) = \dim_k \tilde{H}_0(\langle xy, ab \rangle, k) \neq 0 \quad \text{and} \quad \beta_{2, xzbc}(S/I) = \dim_k \tilde{H}_0(\langle xz, bc \rangle, k) \neq 0.
\]

As a result,

\[
2t_2 = \deg(m) + \deg(m') > 6 = t_4.
\]

**Proposition 4.4.** Let \( M = \{m_1, \ldots, m_s\} \) be a well ordered cover of a square-free monomial ideal \( I \). Define for each \( 1 \leq a \leq s - 1 \) the monomials

\[
m = \text{lcm}(m_1, \ldots, m_a) \quad \text{and} \quad m' = \text{lcm}(m_{a+1}, \ldots, m_s).
\]
Then $m$ and $m'$ are lattice complements in $\text{LCM}(I)$.

**Proof.** Suppose $I = (m_1, \ldots, m_s, n_1, \ldots, n_k)$ and $\text{lcm}(m_1, \ldots, m_s) = \text{lcm}(m, m') = x_1 \cdots x_n$.

Suppose that $m$ and $m'$ are not lattice complements, so $\gcd(m, m') \in I$. In particular, there is a generator $q$ of $I$ such that $q \mid \text{lcm}(m_1, \ldots, m_a)$ and $q \mid \text{lcm}(m_{a+1}, \ldots, m_s)$.

Suppose that $q = m_i$ for some $i = 1, \ldots, s$. If $i \leq a$, then as $m_i \mid \text{lcm}(m_{a+1}, \ldots, m_s)$ we must have $\text{lcm}(m_1, \ldots, \hat{m}_i, \ldots, m_a, m_{a+1}, \ldots, m_s) = \text{lcm}(m_1, \ldots, m_s)$, which contradicts $M$ being a minimal cover of $I$.

The case $i \geq a + 1$ also leads to the same contradiction. Suppose that $q = n_i$ for some $i \in \{1, \ldots, k\}$. Since $M$ is a well ordered cover of $\Delta$, then there exists $\ell \leq s - 1$ such that $m_\ell \mid \text{lcm}(n_i, m_{\ell+1}, \ldots, m_s)$.

If $\ell \leq a$, then $\ell + 1 \leq a + 1$ and as $n_i \mid \text{lcm}(m_{a+1}, \ldots, m_s)$ we have $m_\ell \mid \text{lcm}(m_{\ell+1}, \ldots, m_a, m_{a+1}, \ldots, m_s)$.

Therefore, $\text{lcm}(m_1, \ldots, \hat{m}_\ell, \ldots, m_s) = \text{lcm}(m_1, \ldots, m_s)$.

This also contradicts the minimality of the cover $M$. If $\ell \geq a + 1$, then since $n_i \mid \text{lcm}(m_1, \ldots, m_a)$,

$m_\ell \mid \text{lcm}(n_i, m_{\ell+1}, \ldots, m_s)
\mid \text{lcm}(m_1, \ldots, m_a, m_{\ell+1}, \ldots, m_s)$.

Similarly,

$\text{lcm}(m_1, \ldots, m_a, \hat{m}_\ell, m_{\ell+1}, \ldots, m_s) = \text{lcm}(m_1, \ldots, m_s)$,

a contradiction. Thus, $\gcd(m, m') \notin I$ and hence $m$ and $m'$ are lattice complements in $I$, as required. □

In fact more is true: the second half of the well ordered cover in Proposition 4.4 is itself a well ordered cover.

**Proposition 4.5.** Let $M = \{m_1, \ldots, m_s\}$ be a well ordered cover of a square-free monomial ideal $I$. Define for each $1 \leq a \leq s - 1$ the monomials

$m = \text{lcm}(m_1, \ldots, m_a) \quad \text{and} \quad m' = \text{lcm}(m_{a+1}, \ldots, m_s)$.

Then
1. \{m_1, \ldots, m_a\} is a minimal cover of \(I_{[m]}\).

2. \{m_{a+1}, \ldots, m_s\} is a well ordered cover of \(I_{[m']}\).

3. \(\beta_{s-a,m'}(S/I) \neq 0\).

Proof. Clearly, \(\{m_1, \ldots, m_a\}\) and \(\{m_{a+1}, \ldots, m_s\}\) are covers of \(I_{[m]}\) and \(I_{[m']}\) respectively. Suppose that \(\{m_1, \ldots, m_a\}\) is not minimal, i.e. there exists a proper subset

\[\{m_{i_1}, \ldots, m_{i_k}\} \subset \{m_1, \ldots, m_a\},\]

that is a cover of \(I_{[m]}\). In particular, there exists \(h \in \{1, \ldots, a\} \setminus \{i_1, \ldots, i_k\}\) such that

\[m_h \mid \operatorname{lcm}(m_{i_1}, \ldots, m_{i_k}).\]

Then

\[\operatorname{lcm}(m_1, \ldots, \hat{m}_h, \ldots, m_s) = \operatorname{lcm}(m_1, \ldots, m_s).\]

This contradicts the minimality of the cover \(M\). Thus, \(\{m_1, \ldots, m_a\}\) is a minimal cover of \(I_{[m]}\). Using a similar argument, \(\{m_{a+1}, \ldots, m_s\}\) is also a minimal cover of \(I_{[m']}\).

In order to show that \(\{m_{a+1}, \ldots, m_s\}\) is a well ordered cover of \(I_{[m']}\), if \(n\) is a minimal generator of \(I_{[m']}\) such that \(n \notin \{m_{a+1}, \ldots, m_s\}\), we need to find \(a + 1 \leq \ell \leq s - 1\) such that

\[m_{\ell} \mid \operatorname{lcm}(n, m_{a+1}, \ldots, m_s).\]

By Proposition 4.4, \(m\) and \(m'\) are lattice complements. As a result, \(n\) cannot be a minimal generator of \(I_{[m]}\); in particular, \(n \notin \{m_1, \ldots, m_a\}\). Since \(M = \{m_1, \ldots, m_s\}\) is a well ordered cover and \(n \notin \{m_1, \ldots, m_s\}\), then there exists \(1 \leq \ell \leq s - 1\) such that

\[m_{\ell} \mid \operatorname{lcm}(n, m_{a+1}, \ldots, m_s).\]

Suppose that \(\ell \leq a\), so \(\ell + 1 \leq a + 1\). Since \(n\) is a minimal generator of \(I_{[m']}\), \(n \mid \operatorname{lcm}(m_{a+1}, \ldots, m_s)\).

Therefore,

\[m_{\ell} \mid \operatorname{lcm}(n, m_{a+1}, \ldots, m_{a+1}, \ldots, m_s)\]

\[\mid \operatorname{lcm}(m_{\ell+1}, \ldots, m_s).\]

Thus,

\[\operatorname{lcm}(m_1, \ldots, \hat{m}_{\ell}, \ldots, m_s) = \operatorname{lcm}(m_1, \ldots, m_s),\]

contradicting the minimality of the cover \(M\). Hence \(a + 1 \leq \ell \leq s - 1\) which makes \(\{m_{a+1}, \ldots, m_s\}\) a well ordered cover of \(I_{[m']}\). Using Theorem 3.4, \(\beta_{s-a,m'}(S/I_{[m']}) \neq 0\), and hence \(\beta_{s-a,m'}(S/I) \neq 0\).

Example 4.6. If \(\Delta = \mathcal{F}(I) = \langle xy, yz, xz, za, ab, bc \rangle\), then \(M = \{ab, xy, bc, xz\}\) is a well ordered cover.
Then, by Proposition 4.5 we have: $\beta_{1,xyz}(S/I) \neq 0$, $\beta_{2,bxyz}(S/I) \neq 0$, $\beta_{3,bcxyz}(S/I) \neq 0$ and $\beta_{4,abcxyz}(S/I) \neq 0$.

Under certain extra conditions, the first part of the well ordered cover is also well ordered, which gives us the subadditivity property in those cases. Note that the subadditivity property in the case $a = 1$ below is also covered by a theorem of Herzog and Srinivasan [15].

**Theorem 4.7 (Well ordered covers and subadditivity).** Let $M = \{m_1, \ldots, m_s\}$ be a well ordered cover of a square-free monomial ideal $I$. Define for each $1 \leq a \leq s - 1$ the monomials

$$m = \text{lcm}(m_1, \ldots, m_a) \quad \text{and} \quad m' = \text{lcm}(m_{a+1}, \ldots, m_s).$$

Supposed one of the following conditions holds:

1. $I_{[m]} = (m_1, \ldots, m_a)$ (e.g. when $a = 1$); or
2. $\gcd(m, m') = 1$.

Then $m$ and $m'$ are lattice complements in $I = \mathcal{F}(\Delta)$ such that $\beta_{a,m}(S/I) \neq 0$. In particular

$$t_s \leq t_a + t_{s-a}.$$  

**Proof.** By Proposition 4.5 we already know that $m_1, \ldots, m_a$ is a minimal cover for $I_{[m]}$. We claim that if either of the two conditions above hold, then it is a well ordered cover. If $I_{[m]} = (m_1, \ldots, m_a)$, then this is trivial, since there is no generator other than $m_1, \ldots, m_a$. Suppose $\gcd(m, m') = 1$, and let $n$ be the minimal monomial generating set of $I_{[m]}$ and $n \not\in \{m_1, \ldots, m_a\}$. Then, $n$ is a minimal generator of $I$ as well. Moreover $m_1, \ldots, m_a$ is a well ordered cover of $I$, and so for some $j \leq s - 1$ we have $m_j | \text{lcm}(n, m_{j+1}, \ldots, m_s)$. On the other hand, since $n | m$ and $\gcd(m, m') = 1$, we must have $j \leq a - 1$ and

$$m_j | \text{lcm}(n, m_{j+1}, \ldots, m_s)$$

which implies that $m_1, \ldots, m_a$ is a well ordered cover of $I_{[m]}$. Therefore, by Theorem 4.4 $\beta_{a,m}(S/I_{[m]}) \neq 0$, and hence $\beta_{a,m}(S/I) \neq 0$.

By Proposition 4.5, $m_{a+1}, \ldots, m_s$ is a well ordered cover for $I_{[m']}$, and $\beta_{s-a,m'}(S/I_{[m]}) \neq 0$, hence $\beta_{s-a,m'}(S/I) \neq 0$.

Now, since $m$ and $m'$ are complements (Proposition 4.4), we have that

$$t_s \leq t_a + t_{s-a},$$

which ends our argument. □
5 Simplicial Bouquets

As proved in [8, Proposition 4.3], an example of a well ordered facet cover for a simple graph is a strongly disjoint set of bouquets (see [22, Definition 1.7] and also [17, Definitions 2.1 and 2.3]). Bouquets are in general much easier to identify in graphs than well ordered edge covers, as one does not need to worry about the order. In this section we define a simplicial counterpart for a strongly disjoint set of bouquets of graphs, and show that they form well ordered facet covers and hence guarantee nonvanishing Betti numbers. Similar to graph bouquets, simplicial bouquets are easy to spot in a picture, and strongly disjoint sets of simplicial bouquets often come with more than one guaranteed order. We then apply the results of the previous section to examine the subadditivity property in the presence of such simplicial bouquets. It must be noted that our definition of a simplicial bouquet is very close to hypergraph bouquets developed in [16, Definition 3.1], and almost the same as the hypergraph bouquets in [7, Definition 2.1].

**Definition 5.1 (Simplicial bouquet).** Let $\Delta$ be a simplicial complex. A simplicial bouquet is a subcollection $B = \langle G_1, \ldots, G_t \rangle$ of $\Delta$ such that each facet $G_i$ has at least one free vertex in $B$, and
\[
\bigcap_{i=1}^{t} G_i \neq \emptyset.
\]

The nonempty intersection of the facets of $B$ is called the root of $B$ and denoted by $\text{Root}(B)$.

A simplicial bouquet in a graph coincides with the usual definition of bouquets in graphs (see [22, Definition 1.7]).

The distance between two distinct facets $F, F'$ of $\Delta$, denoted by $\text{dist}_\Delta(F, F')$, is the minimum length $\alpha$ of sequences of facets of $\Delta$ $F = F_0, F_1, \ldots, F_\alpha = F'$ where $F_{i-1} \cap F_i \neq \emptyset$, or $\infty$ if there is no such sequence. We say that $F$ and $F'$ are 3-disjoint in $\Delta$ if $\text{dist}_\Delta(F, F') \geq 3$. A subset $E \subset \text{Facets}(\Delta)$ is said to be pairwise 3-disjoint if every pair of distinct facets $F, F' \in E$ are 3-disjoint in $\Delta$ (see [14, Definitions 2.2 and 6.3]).

**Definition 5.2 ((Strongly disjoint) set of bouquets).** Let $\Delta$ be a simplicial complex. For a set $\mathcal{B} = \{B_1, B_2, \ldots, B_d\}$ of simplicial bouquets of $\Delta$, define
\[
\text{Facets}(\mathcal{B}) = \text{Facets}(B_1) \cup \cdots \cup \text{Facets}(B_d) \quad \text{and} \quad \text{V}(\mathcal{B}) = \text{V}(B_1) \cup \cdots \cup \text{V}(B_d).
\]

Then $\mathcal{B}$ is called strongly disjoint in $\Delta$ if:

1. $\text{V}(B_i) \cap \text{V}(B_j) = \emptyset$ for all $i \neq j$, and
2. we can choose a facet $G_i$ from each $B_i \in \mathcal{B}$ so that the set $\{G_1, \ldots, G_d\}$ is pairwise 3-disjoint in $\Delta$.

We say that $\Delta$ contains a strongly disjoint set of bouquets if there exists a strongly disjoint set of bouquets $\mathcal{B} = \{B_1, \ldots, B_d\}$ of $\Delta$ such that:
1. \( V(\Delta) = V(\mathcal{B}) \), and

2. if \( F \in \text{Facets}(\Delta) \setminus \text{Facets}(\mathcal{B}) \) and \( F \cap G \neq \emptyset \) for some \( G \in \text{Facets}(B_i) \) and \( i \in \{1, \ldots, d\} \), then \( (G \setminus \text{Root}(B_i)) \subseteq F \).

**Example 5.3.** Let \( I = \langle abc, bcd, cdf, def, eg, fg, gh, hi, gi, fi, gx, gy \rangle \) and \( \Delta = \mathcal{F}(I) \). It is easy to check that \( \Delta \) contains a strongly disjoint set of bouquets \( \mathcal{B} = \{B_1, B_2\} \) where the bouquets

\[
B_1 = \langle bcd, abc \rangle \quad \text{and} \quad B_2 = \langle gy, gx, ge, gf, gh, gi \rangle
\]

are highlighted below, and the set of facets \( \{abc, gx\} \) is pairwise 3-disjoint.

![Diagram of \( \Delta \)]

The following statement is a generalization of [8, Proposition 4.3].

**Theorem 5.4 (Strongly disjoint set of bouquets and well ordered facet covers).** Let \( \Delta \) be a simplicial complex which contains a strongly disjoint set of bouquets \( \mathcal{B} = \{B_1, \ldots, B_d\} \). For each \( q \in \{1, \ldots, d\} \) suppose

\[
\text{Facets}(B_q) = \{G_{1}^{q}, G_{2}^{q}, \ldots, G_{b_{q}}^{q}, G_{q}\}
\]

where \( \{G_1, \ldots, G_d\} \) is pairwise 3-disjoint in \( \Delta \). Then for any permutation \( k_1, k_2, \ldots, k_d \) of the integers \( 1, \ldots, d \), the sequence of facets

\[
\underbrace{G_{1}^{k_1}, \ldots, G_{b_{1}}^{k_1}}_{\text{Facets}(B_{k_1})}, \quad \underbrace{G_{1}^{k_2}, \ldots, G_{b_{2}}^{k_2}}_{\text{Facets}(B_{k_2})}, \quad \ldots, \quad \underbrace{G_{1}^{k_d}, \ldots, G_{b_{d}}^{k_d}}_{\text{Facets}(B_{k_d})}
\]

form a well ordered facet cover of \( \Delta \).

**Proof.** Observe that we can identify each monomial generator \( x_1 \ldots x_d \) of \( \mathcal{F}(\Delta) \) with the facet \( \{x_1, \ldots, x_d\} \) of \( \Delta \). The condition that \( V(\Delta) = V(\mathcal{B}) \) and that every facet of \( \mathcal{B} \) has a free vertex guarantees that Facets(\( \mathcal{B} \)) is a minimal facet cover of \( \Delta \). Suppose \( F \in \text{Facets}(\Delta) \setminus \text{Facets}(\mathcal{B}) \). Since the set \( \{G_{k_1}, \ldots, G_{k_d}\} \) is pairwise 3-disjoint in \( \Delta \), \( F \) can intersect at most one of \( \{G_{k_1}, \ldots, G_{k_d}\} \), and so at least one vertex of \( F \) does not belong to \( \bigcup_{j=1}^{d} G_{j} \). Therefore for some \( q \in \{1, \ldots, d\} \) and \( j \in \{1, \ldots, b_{k_d}\} \) we have \( G_{j}^{k_q} \cap F \neq \emptyset \) and so

\[
F \supseteq G_{j}^{k_q} \setminus \text{Root}(B_{k_q}).
\]
Hence
\[ G_j^{k_q} = (G_j^{k_q} \setminus \text{Root}(B_{k_q})) \cup \text{Root}(B_{k_q}) \subseteq F \cup G_{k_q} \subseteq F \cup G_j^{k_q} \cup \cdots \cup G_{k_q}^{k_{q+1}} \cup \cdots \cup G_{k_d} \]
which implies that the sequence above is a well ordered facet cover of \( \Delta \).

Equivalently, strongly disjoint simplicial bouquets produce well ordered covers for facet ideals.

**Example 5.5.** For the ideal in Example 5.3, Theorem 5.4 states that both
\[
\{\text{bcd}, \text{abc}, \text{gy}, \text{ge}, \text{gf}, \text{gh}, \text{gi}, \text{gx}\} \quad \text{and} \quad \{\text{gy}, \text{ge}, \text{gf}, \text{gh}, \text{gi}, \text{gx}, \text{bcd}, \text{abc}\}
\]
are well ordered covers. Note that there are many other options for well ordered covers, obtained by reordering all facets (except for the last one) of each bouquet, or picking another set of pairwise 3-disjoint facets from the set of bouquets.

The following is a generalization of [17, Theorem 3.1]. From here onwards we assume \( S \) is a polynomial ring over a field generated by variables which are vertices of the simplicial complex \( \Delta \).

**Corollary 5.6 (Betti numbers from simplicial bouquets).** Let \( \Delta \) be a simplicial complex, \( W \subseteq V(\Delta) \) and suppose the induced subcollection \( \Delta_{[W]} \) contains strongly disjoint set of bouquets \( \mathcal{B} \) with \( |\text{Facets}(\mathcal{B})| = i \) and \( |W| = |V(\mathcal{B})| = j \). Then \( \beta_{i,j}(S/\mathcal{F}(\Delta)) \neq 0 \).

**Proof.** As discussed in Section 4, \( \beta_{i,j}(S/\mathcal{F}(\Delta)) \neq 0 \) if and only if \( \beta_{i,j}(S/\mathcal{F}(\Delta_{[W]})) \neq 0 \) for some \( W \subseteq V(\Delta) \) with \( |W| = j \). The statement is now a direct consequence of Theorems 3.4 and 5.4.

**Theorem 5.7 (Subadditivity from simplicial bouquets).** Let \( \Delta \) be a simplicial complex and \( I = \mathcal{F}(\Delta) \). Suppose \( \Delta \) contains a strongly disjoint set of bouquets \( \mathcal{B} \). Let \( \mathcal{B} = \mathcal{B}' \sqcup \mathcal{B}'' \) be a partition of \( \mathcal{B} \) into disjoint subsets \( \mathcal{B}' \) and \( \mathcal{B}'' \), and let
\[
b' = |\text{Facets}(\mathcal{B}')|, \quad b'' = |\text{Facets}(\mathcal{B}'')|, \quad \text{and} \quad b = b' + b'' = |\text{Facets}(\mathcal{B})|.
\]
Then the monomials
\[
m = \prod_{x \in \text{Facets}(\mathcal{B}')} x \quad \text{and} \quad m' = \prod_{x \in \text{Facets}(\mathcal{B}'')} x
\]
are lattice complements in \( \text{LCM}(I) \) and
\[
\beta_{b',m}(S/I) \neq 0 \quad \text{and} \quad \beta_{b'',m'}(S/I) \neq 0
\]
In particular,
\[
t_b \leq t_{b'} + t_{b''}.
\]

**Proof.** As any bouquet of \( \mathcal{B} \) is either in \( \mathcal{B}' \) or in \( \mathcal{B}'' \), and no two distinct bouquets share any vertices, \( \gcd(m, m') = 1 \). The claim now follows directly from Theorem 4.7 and Theorem 5.4.

**Example 5.8.** Let \( I = \langle ax, ay, bz, bv, bw, cu, cg, yz, az \rangle \) and \( G = \mathcal{F}(I) \). It is easy to check that \( \mathcal{B} = \{ \langle ax, ay \rangle, \langle bz, bv, bw \rangle, \langle cu, cg \rangle \} \) (highlighted below) is a strongly disjoint set of bouquets in \( G \) where \( \{ax, bv, cu\} \subseteq \text{Facets}(\mathcal{B}) \) is a set of pairwise 3-disjoint in \( G \).
According to Corollary 5.6 (or [17, Theorem 3.1] since this is the case of bouquets in graphs) $\beta_{7,10}(S/I) \neq 0$, so that $t_7 = 10$. Using the following order of $\mathcal{B}$

\[
\begin{array}{cccc}
ay, ax & bz, bw, bv & cg, cu \\
\text{Facets((ax,ay))} & \text{Facets((bz,bw,bv))} & \text{Facets((cu,cg))}
\end{array}
\]

and setting $m = axy = \text{lcm}(ay, ax)$ and $m' = bzywcu = \text{lcm}(bz, bw, bv, cg, cu)$, by Theorem 5.7 we have $\beta_{2,m}(S/I) \neq 0$ and $\beta_{5,m'}(S/I) \neq 0$. On the other hand, using the following reordering of $\mathcal{B}$

\[
\begin{array}{cccc}
ay, ax & cg, cu & bz, bw, bv \\
\text{Facets((ax,ay))} & \text{Facets((cu,cg))} & \text{Facets((bz,bw,bv))}
\end{array}
\]

and setting $m = acxyu = \text{lcm}(ay, ax, cg, cu)$ and $m' = bzyw = \text{lcm}(bz, bw, bv)$, by Theorem 5.7 we also get $\beta_{4,m}(S/I) \neq 0$ and $\beta_{3,m'}(S/I) \neq 0$. As a result,

\[t_7 < t_2 + t_5 = 12 \quad \text{and} \quad t_7 < t_3 + t_4 = 13\]

which can be confirmed by using the Betti table for $S/I$.

\[
\begin{array}{cccccccc}
& 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\text{total} & 1 & 9 & 28 & 44 & 40 & 22 & 7 & 1 \\
0 & : & 1 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
1 & : & 9 & 10 & 3 & \ldots & \ldots & \ldots & \ldots \\
2 & : & \ldots & 18 & 33 & 20 & 4 & \ldots & \ldots \\
3 & : & \ldots & \ldots & 8 & 20 & 18 & 7 & 1 \\
\end{array}
\]

**Example 5.9.** For $I$ and $\mathcal{B}$ in Example 5.3, since $|\text{Facets}(B_1)| = 2$ and $|\text{Facets}(B_2)| = 6$, and the two monomials $abcd$ and $efghi$ are complements in $\text{LCM}(I)$, it follows that $\beta_{6,7}(S/I) \neq 0$ and $\beta_{2,4}(S/I) \neq 0$ so that $t_8 \leq t_2 + t_6$.

### 6 Reordering well ordered covers

As we saw in the Section 4, the tail end of every well ordered cover is itself a well ordered cover, and therefore guarantees a nonvanishing Betti number in a certain homological degree. For example, let

\[m_1, \ldots, m_s\] (3)

be a well ordered cover of $I$. As a result of Proposition 4.5, we have that

\[\beta_{i,m'}(S/I) \neq 0 \quad \text{for} \quad 1 \leq i \leq s \text{ and } m' = \text{lcm}(m_{s-i+1}, \ldots, m_s).\] (4)
On the other hand, we also know by Proposition 4.4 that for a fixed $i$, if $m = \text{lcm}(m_1, \ldots, m_{s-i})$ and $m'$ is as in (4), then $m$ and $m'$ are complements. So, if we additionally have

$$\beta_{s-i, m}(S/I) \neq 0$$

then (4) and (5) together would imply that

$$t_s \leq t_i + t_{s-i}.$$ 

In this section we consider extra conditions under which (5) would hold. One particular situation is when the monomials $m_1, \ldots, m_{s-i}$ in the well ordered cover in (3) could be shifted to the end of the ordering, so that $m_{s-i+1}, \ldots, m_s, m_1, \ldots, m_{s-i}$ is also a well ordered cover of $I$. In this case, by Proposition 4.5 we would have (5) automatically, and therefore more cases of the subadditivity inequality (6) could easily follow.

**Example 6.1.** Consider the ideal $I$ and $\Delta = F(I)$ as in Example 5.3 and consider the well ordered cover

$$M : m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8$$

Name $n_1 = cd f$, $n_2 = def$, $n_3 = fi$ and $n_4 = hi$ and for each $k = 1, 2, 3, 4$, set

$$\alpha_k = \max \{ j : m_j | \text{lcm}(n_k, m_{j+1}, \ldots, m_8) \}.$$ 

Then $\alpha_1 = \alpha_2 = 5$, $\alpha_3 = 4$ and $\alpha_4 = 6$. Let $\ell = \min(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = 4$. Using $\ell$, we can modify the order in $M$ as follows:

$$M' : m_4, m_5, m_6, m_7, m_8, m_1, m_2, m_3$$

It is easy to see that $M'$ is also a well ordered cover of $I$.

The example above is a special case of a simple observation, stated in Proposition 6.2, which is often powerful enough to prove the subadditivity inequality (6) for all $i$.

**Proposition 6.2 (Reordering well ordered covers).** Let $I = (m_1, \ldots, m_s, n_1, \ldots, n_k)$ be a square-free monomial ideal which has $m_1, \ldots, m_s$ as a well ordered cover. Define for each $1 \leq i \leq k$,

$$\alpha_i = \max \{ j : m_j | \text{lcm}(n_i, m_{j+1}, \ldots, m_s) \}. \quad (7)$$

If $\ell = \min(\alpha_1, \ldots, \alpha_k) > 1$, then

1. for every $i \in \{2, \ldots, \ell\}$ the sequence

$$m_i, m_{i+1}, \ldots, m_s, m_1, \ldots, m_{i-1}$$

is a well ordered cover of $I$;
2. \( t_s \leq t_{s-i} + t_i \) for \( 1 \leq i \leq \ell - 1 \).

**Proof.** Statement 1 follows directly from the definition of well ordered covers. For the second statement, by part 1, for \( 1 \leq i < \ell \), we have

\[
m_{i+1}, \ldots, m_s, m_1, \ldots, m_i
\]

is a well ordered cover for \( I \). By (7)

\[
m_{i+1}, \ldots, m_s
\]

is a well ordered cover of \( I_{[m]} \) where \( m = \text{lcm}(m_{i+1}, \ldots, m_s) \). By Proposition 4.5

\[
m_1, \ldots, m_i
\]

is a well ordered cover of \( I_{[m']} \) where

\[
m' = \text{lcm}(m_1, \ldots, m_i).
\]

By Proposition 4.4 \( m \) and \( m' \) are complements in \( \text{LCM}(I) \), and by Theorem 3.4

\[
\beta_{s-i,m} \neq 0 \quad \text{and} \quad \beta_{i,m'} \neq 0,
\]

which together imply that \( t_s \leq t_{s-i} + t_i \). \( \square \)

**Example 6.3.** The ideal \( I \) from Example 5.3 has the following well ordered cover

\[
M: gy, gx, ge, gf, bcd, gh, gi, abc.
\]

In particular \( \beta_{8,11}(S/I) \neq 0 \) and \( t_8 = 11 \). The reordering in Proposition 6.2 of \( M \) yields the following well ordered cover

\[
M': gf, bcd, gh, gi, abc, gy, gx, ge
\]

where \( \ell = 4 \). By part 2 of Proposition 6.2 and using the new well ordered cover \( M' \), we have:

1. \( t_8 \leq t_7 + t_1 \). Here we take \( m = abcd f gh i xy = \text{lcm}(gf, bcd, gh, gi, abc, gx, ge) \) and \( m' = ge \).
2. \( t_8 \leq t_6 + t_2 \). Here we take \( m = abcd f gh iy = \text{lcm}(gf, bcd, gh, gi, abc, gy) \) and \( m' = gex = \text{lcm}(gx, ge) \). Note that this inequality was also done in Example 5.9 using simplicial bouquets.
3. \( t_8 \leq t_5 + t_3 \). Here we take \( m = abcd f ghi = \text{lcm}(gf, bcd, gh, gi, abc) \) and \( m' = gexy = \text{lcm}(gy, gx, ge) \).

The only remaining case is \( a = b = 4 \). If we take \( m = bcd f gh i = \text{lcm}(gf, bcd, gh, gi) \) and \( m' = abc gexy = \text{lcm}(abc, gy, gx, ge) \), then \( \{abc, gy, gx, ge\} \) is a well ordered cover of \( I_{[m]} \) by Proposition 4.5 and it is easy to check that \( \{gf, bcd, gh, gi\} \) is a well ordered cover of \( I_{[m']} \). Hence \( \beta_{4,m}(S/I) \neq 0 \) and \( \beta_{4,m'}(S/I) \neq 0 \) by Theorem 3.4. Also as \( m \) and \( m' \) are lattice complements (by Proposition 4.4), we get \( t_4 + t_4 \geq \text{deg}(m) + \text{deg}(m') > 11 = t_8 \).
References

[1] L. Avramov, A. Conca and S. Iyengar, Subadditivity of syzygies of Koszul algebras, Math. Ann., 361, no.1-2, 511-534 (2015).

[2] A. Abedelfatah and E. Nevo, On vanishing patterns in j-strands of edge ideals, J. Algebraic Combin. 46, no. 2, 287-295 (2017).

[3] K. Baclawski, Galois connections and the Leray spectral sequence, Advances in Math. 25, no. 3, 191-215 (1977).

[4] D. Bayer and B. Sturmfels, Monomial resolutions, Math. Res. Lett. 5, no. 1-2, 31-46 (1998).

[5] D. Eisenbud, C. Huneke, and B. Ulrich, The regularity of Tor and graded Betti numbers, Amer. J. Math. 128, no. 3, 573-605 (2006).

[6] S. El Khoury and H. Srinivasan, A note on the subadditivity of Syzygies, Journal of Algebra and its Applications, vol.16, no.9, 1750177 (2017).

[7] N. Erey, Bouquets, Vertex Covers and Edge Ideals, Journal of Algebra and its Applications, vol.16, no.5, 1750084 (2017).

[8] N. Erey and S. Faridi, Betti numbers of monomial ideals via facet covers, J. Pure Appl. Algebra 220, no. 5, 1990-2000 (2016).

[9] S. Faridi, Lattice complements and the subadditivity of syzygies of simplicial forests, Journal of Commutative Algebra, Volume 11, Number 4, 535-546 (2019).

[10] S. Faridi and M. Shahada, Breaking up Simplicial Homology and Subadditivity of Syzygies, J. Algebraic Combin., to appear.

[11] O. Fernandez-Ramos and P. Gimenez, Regularity 3 in edge ideals associated to bipartite graphs, J. Algebraic Combin., 39 (2014).

[12] R. Fröberg, On Stanley-Reisner rings, Topics in algebra, Banach Center Publications, 26 Part 2, 57-70 (1990).

[13] D. Grayson and M. Stillman, Macaulay2, a software system for research in algebraic geometry, available at http://www.math.uiuc.edu/Macaulay2/.

[14] H. T. Ha and A. Van Tuyl, Monomial ideals, edge ideals of hypergraphs, and their graded Betti numbers, J. Algebraic Combin., 27, 215-245 (2008).

[15] J. Herzog and H. Srinivasan, On the subadditivity problem for maximal shifts in free resolutions, Commutative Algebra and Noncommutative Algebraic Geometry, II MSRI Publications Volume 68, (2015).
[16] F. Khosh-Ahang and S. Moradi, *Codismantlability and projective dimension of the Stanley-Reisner ring of special hypergraphs*, Proc. Indian Acad. Sci. Math. Sci. 128, no. 1, Paper No. 7, 10 pp (2018).

[17] K. Kimura, *Non-Vanishingness of Betti Numbers of Edge Ideals*, Harmony of Grobner Bases and the Modern Industrial Society, World Scientific, Hackensack 153-168 (2012).

[18] G. Lyubeznik, *A new explicit finite free resolution of ideals generate by monomials in an R-sequence*, J. Pure Appl. Algebra 51, 193-195 (1998).

[19] J. Mermin, *Three simplicial Resolutions*, Progress in commutative algebra 1, 127-141, de Gruyter, Berlin (2012).

[20] I. Peeva, *Graded syzygies*, Algebra and Applications, 14. Springer-Verlag London, Ltd., London (2011).

[21] D. Taylor, *Ideals generated by monomials in an R-sequence*, Thesis, University of Chicago (1966).

[22] X. Zheng, *Resolutions of facet ideals*, Comm. Algebra, 32, 2301-2324 (2004).