A Note on Community Trees in Networks

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Abstract
We introduce the concept of community trees that summarizes topological structures within a network. A community tree is a tree structure representing clique communities from the clique percolation method (CPM). The community tree also generates a persistent diagram. Community trees and persistent diagrams reveal topological structures of the underlying networks and can be used as visualization tools. We study the stability of community trees and derive a quantity called the total star number (TSN) that presents an upper bound on the change of community trees. Our findings provide a topological interpretation for the stability of communities generated by the CPM.

1 Introduction
The clique percolation method (CPM [13]) is a well-known and powerful algorithm for community detection [16, 11] in networks that has been applied to various complex networks such as social networks [13, 14], biological networks [1, 22], and collaboration networks [15, 14, 9]. Given an order of cliques, say \( k \), the CPM detects communities using overlaps between the \( k \)-cliques. Since the CPM uses cliques to construct the communities, one vertex may be assigned to multiple communities, and this fact can be used to describe the overlaps among the communities [13, 16]. The communities found by the CPM are, thus, called clique communities.

In this paper, we introduce the concept of a community tree of a network. Here, we consider the simplest case where a network is an undirected and unweighted graph. A community tree is a tree structure representing the clique communities discovered by the CPM. A key characteristic of a community tree is: instead of using a fixed order of the cliques \( k \), the construction of a community tree is based on the clique communities for all possible value of \( k \). It uses the fact that the collection of all possible clique communities (regardless of the order of the cliques) forms a tree structure.

A community tree is generalized from the notion of the cluster tree [17, 4, 2, 5, 6, 12] of a function in topological data analysis (TDA [19]). Similar to the fact that a cluster tree summarizes the creation and elimination of connected components of a function, a community tree summarizes the creation and elimination of communities of a network. A community tree naturally generates a persistent diagram [7], a popular analytical tool in TDA. Using a community tree or a persistent diagram, we

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are able to analyze topological structures of a network using the notion of communities. Thus, this paper provides a new direction to bridge TDA and network science.

2 Background

2.1 Clique community and clique percolation method

Let \( G \) be an unweighted, undirected graph (network) with a vertex set \( V(G) \) and an edge set \( E(G) \). A clique refers to a graph \( C \subseteq G \) such that any two distinct vertices \( u, v \in V(C) \) are adjacent (i.e., share an edge). A \( N \)-clique is a clique with \( N \) vertices. Assume there exists \( C_1, \ldots, C_n, n \) different \( k \)-cliques within \( G \). Given an integer level \( k \) where \( k \geq 2 \), the CPM [13] is an algorithm that finds every \( k \)-clique within the network \( G \) (i.e., \( C_1, \ldots, C_n \)) and partitions these \( k \)-cliques based on an adjacency matrix \( A \in \{0, 1\}^{n \times n} \) such that \( A_{ij} = 1 \) if \( C_i \cap C_j \) contains \( k-1 \) vertices and 0 otherwise. Namely, CPM separates \( k \)-cliques into connected components using the adjacency matrix. The subgraph generated by the union of \( k \)-cliques within the same connected component is called a \( k \)-clique community (\( k \)-community for short) [13]. We define a 1-community to be any graph.

If a \( k \)-community \( C = \bigcup_{i=1}^{L} C_i \) is created by \( C_1, \ldots, C_L \), \( L \) \( k \)-cliques, then for any two \( k \)-cliques \( C_i \) and \( C_j \), there exists a sequence of \( k \)-cliques \( C_{i_1} = C_i, C_{i_2}, \ldots, C_{i_m} = C_j \) such that \( C_{i_1} \cap C_{i_{m+1}} \) is a \((k-1)\)-clique. We call such a sequence a \( k \)-clique path \( P \). If a community is formed by \( k \)-cliques, we say that this community has an order \( k \).

2.2 Community trees

An important feature of a \( k \)-community is its nested structure. Since any \( k \)-clique contains \( k \) distinct \((k-1)\)-cliques, any \( k \)-community is a subset (subgraph) of a \((k-1)\)-community. Thus, for any \( k \)-community \( C_k \), there exists a sequences of communities (at different orders) \( C_{k-1}, C_{k-2}, \ldots, C_1 \) such that each \( C_\omega \) is a \( \omega \)-community and

\[
C_k \subset C_{k-1} \subset \cdots \subset C_1.
\]

Note that the 1-community is the original graph \( G \).

The nested structure of clique communities defines a tree structure for the collection of all the communities across various orders. Given any integer \( k \in \mathbb{N} \), let \( C_{k,1}, \ldots, C_{k,J(k)} \) be the \( k \)-communities of \( G \) and

\[
C_k = \{C_{k,j} : j = 1, \ldots, J(k)\}
\]

be the collection of them. Then the collection of all communities

\[
\mathcal{C} = \bigcup_{k \in \mathbb{N}} C_k
\]

has a tree structure. We call the tree generated by \( \mathcal{C} \) the community tree of \( G \). Figure 1 displays an example of a community tree and its corresponding communities.

Every node of a community tree corresponds to a unique community so a community tree informs how different communities are associated to each other. Starting at a leaf of a community tree, when we reduce the order of communities (moving down the tree toward the root), we see how one community morphs into another community. Moreover, communities from different branches of the tree may merge when we follow them down to the root. The order that two communities merge tell us how these two communities overlap. Using a community tree, we can visualize a complex network and its communities.

3 Stability of community trees

3.1 Persistent diagram and tree metrics

To summarize the shape of a community tree, we introduce the concepts of components and persistent diagram (PD) [7]. A component of a community tree is a nested sequence of communities that starts with a community at a leaf of the tree, say a \( T \)-community \( C_T \), and then followed by \( C_{T-1}, C_{T-2}, \ldots, C_1 \) such that

\[
C_T \subset C_{T-1} \subset \cdots \subset C_1,
\]
where $C_\omega$ is a $\omega$-community. Every leaf in a community tree corresponds to a unique component and the entire tree can be reconstructed using all these components. Figure 2 shows the components in the tree of Figure 1.

For each component, we define its birth time (birth order or birth level) to be the highest order of its communities (this corresponds to the order of its leaf community). Two components will merge at certain order/level. Whenever two components merge, we compare their birth time. The one with a higher birth time will stay alive and the other one will be eliminated. And we define the order of this merging to be the death time of the younger component. Note that if two components are at the same age, we arbitrarily pick one of them to stay alive. Moreover, for the component that cannot be assigned a death time using this way, we will set its death time to be 1.

The difference between birth time and death time of a component is called the persistence or life time. A component with a higher persistence is often related to the communities that are more robust against change in the network. Assume that a community tree has $M$ components and each have birth and death time $(d_1, b_1), \cdots, (d_M, b_M)$. A PD of a community tree is the collection of birth time and death time for each component along with the line $x = y$. Namely,

$$\text{PD} = \{ (d_i, b_i) : i = 1, \cdots, M \} \cup \{ (d, b) : d = b \}.$$  

The persistent diagram is a 2D diagram representing topological structures of a community tree. Every community tree admits a unique persistent diagram. The elements in the persistent diagram represents the robustness of each component in the community tree. Figure 3 shows the construction of a persistent diagram using the network presented in Figure 1.

Persistent diagrams provide a way to measure the change in community trees. We define the changes of community tree in terms of the changes of the corresponding persistent diagrams. In particular, we use the bottleneck distance [7] of persistent diagrams to measure the change in community trees.

Figure 1: The construction of a community tree from a network. The top left panel shows the original network (graph). The top right panel shows the resulting community tree. To construct the community tree, we consider $k$-community at various levels. The panels in the bottom row present all the communities at various orders. The alphabetical letter denotes the corresponding position of each community in the cluster tree (top right).
Figure 2: Components of the community tree in Figure 1. There are two components in the community tree since the tree has two leaves. The component 1 starts with the 5-community (the label \( a \) community in Figure 1). The component 2 is the one starts with the 4-community (the label \( c \) community in Figure 1). We see that the two components merge at the order of 3, i.e., their 3-communities (and any community of a smaller order) are the same. Since component 2 has birth time 4, which is lower than component 1 (birth time 5), it is eliminated due to this merging and its death time is 3 (the order of this merging). Note that component 1 never merged into others so its death time is 1.

| Component 1 | 5-community | 4-community | 3-community | 2-community | 1-community |
|-------------|-------------|-------------|-------------|-------------|-------------|
| ![Component 1](image1) | ![5-community](image2) | ![4-community](image3) | ![3-community](image4) | ![2-community](image5) | ![1-community](image6) |

| Component 2 | 5-community | 4-community | 3-community | 2-community | 1-community |
|-------------|-------------|-------------|-------------|-------------|-------------|
| ![Component 2](image7) | ![5-community](image8) | ![4-community](image9) | ![3-community](image10) | ![2-community](image11) | ![1-community](image12) |

Figure 3: Original network, community tree, and the persistence diagram correspond to Figure 1. By examining the components described in Figure 2, we see that there are two components. Component 1 has birth time \( b_1 = 5 \) and death time \( d_1 = 1 \) and component 2 has a birth time of \( b_2 = 4 \) and death time \( d_2 = 3 \) (the order of merging). The persistent diagram (right panel) is then the two points \((1, 5), (3, 4)\) and the line \( x = y \).
Figure 4: The difference between two networks. We compare one network (top row) to two possible new networks (middle and bottom rows). The first column displays how the networks look like. For the new networks (middle and bottom), the dashed red edges are edges being removed and the solid red edges/vertex are the new edges/vertex being added. The second column presents the community trees. The third column shows the persistent diagrams. The right panel compares persistent diagrams (black: original network; red: new network). The blue line indicates an optimal matching between two persistent diagrams that leads to the bottleneck distance between them. When comparing the first new network (new network -1) to the original network, the RSN concerns the dashed edges (edges being removed), so it equals 2 (the set $V_0$ can be chosen as $V_0 = \{v_3, v_4\}$). The ASN is 1 since the all the added edges are incident to vertex $v_{11}$. Thus, $TSN = 2 + 1 = 3$ gives a conservative bound. In the case of comparing the original network to the second new network (new network - 2), the RSN is 1 and ASN is 0, leading to $TSN = 1$, which agrees with the actual bottleneck distance.

For any two persistent diagrams $PD_1$ and $PD_2$, let $\gamma: PD_1 \mapsto PD_2$ be a bijective (one-to-one and onto) mapping between them. The bottleneck distance is

$$d_\infty(PD_1, PD_2) = \inf_\gamma \sup_{A \in PD_1} \| A - \gamma(A) \|_\infty,$$

where the infimum is taken over all possible bijective mappings. And for a vector $v = (v_1, v_2)$, the norm $\|v\|_\infty = \max\{|v_1|, |v_2|\}$ is the $L_\infty$ norm.

The bottleneck distance has an important relation with components: if a component has persistence $L$, we need a change with at least a bottleneck distance of $L$ to eliminate this component.

Let $T_1$ and $T_2$ denote the two community trees and $PD(T_1), PD(T_2)$ be the corresponding persistent diagrams. We then define the bottleneck distance between the two community trees as

$$d_B(T_1, T_2) = d_\infty(PD(T_1), PD(T_2)).$$

The bottleneck distance measures how the community trees differ in terms of their corresponding persistent diagrams. Figure 4 provides examples of computing bottleneck distance of community trees.

Note that there are other metrics for trees constructed from a function [2, 4, 5, 10, 12]. However, these metrics cannot be applied to community trees because community trees are not constructed from a function.
3.2 Stability theory

For any edge \( e \), let \( \nu(e) \) be the two vertices of \( e \). For two graphs \( G \) and \( G' \), we introduce a quantity called star number that will be useful in deriving the stability theory.

**Definition 1** The removal star number (RSN) of \( G' \) and \( G \) is

\[
\text{RSN}(G', G) = \min\{|V_0| : \nu(e) \cap V_0 \neq \emptyset \ \forall e \in E(G) \setminus E(G')\},
\]

where \( V_0 \) is a collection of vertices and \(|V_0|\) is the number of elements in the set \( V_0 \). The addition star number (ASN) of \( G' \) and \( G \) is

\[
\text{ASN}(G', G) = \min\{|V_0| : \nu(e) \cap V_0 \neq \emptyset \ \forall e \in E(G') \setminus E(G)\}
\]

The total star number (TSN) is the sum of RSN and ASN.

If \( \text{TSN}(G, G') = k \), then we can interpret it as that the change from \( G \) to \( G' \) can be attributed to about \( k \) vertices.

**Theorem 2** Let \( G \) and \( G' \) be two graphs. Then their community trees satisfy

\[
d_B(T(G), T(G')) \leq \text{TSN}(G', G).
\]

The proof of Theorem 2 is long so we defer it to the appendix. Theorem 2 provides a powerful result: the difference between two community trees is bounded by their TSN. This implies that it is possible that the community tree does not change much even the network has been substantially changed. For instance, if a network has a vertex of high degree, removing all edges attached to this vertex only contributes to a TSN = 1 effect on the community tree. So the community tree may remain unchanged or only slightly changed. In a sense, the TSN describes an upper bound to the effective change to the community tree. Figure 4 provides the TSN for comparing different networks. We also include the actual bottleneck distance as a reference.

TSN can be computed without constructing community trees and persistent diagrams. Thus, to roughly compare the community tree difference between two networks, we do not need to actually build the community trees and persistent diagrams but just compute their TSN.

However, computing the TSN could be very difficult as described in the following theorem.

**Theorem 3** Let \( G \) and \( G' \) be two graphs. Computing the RSN\((G', G)\) or ASN\((G', G)\) is an NP-complete problem.

**Proof.** We only prove the case of RSN since the case of ASN is just swap the role of \( G \) and \( G' \). Let \( V(G) \) denotes the vertex in \( G \) and \( E^\Delta = E(G) \setminus E(G') \). We define a new graph \( G^\Delta = (V(G), E^\Delta) \). This graph is the graph where the edges representing the edge difference between \( G \) and \( G' \).

Thus, the number \( \text{RSN}(G', G) \) is to find the minimum number of vertices in \( V(G) \) such that every edge in \( E^\Delta = E(G) \setminus E(G') \) is incident to at least one element in the subset of vertices. Namely, \( \text{RSN}(G', G) \) is equivalent to the size of minimum vertex cover of the graph \( G^\Delta \). Since finding the minimum vertex cover is an NP-complete problem [20, 8], computing RSN\((G', G)\) is also NP-complete.

\( \square \)

4 Examples

We apply community trees to real network datasets in Figure 5. The left panel displays the community tree and persistent diagram of a dolphin social network\(^6\). This dataset describes the social network of 62 dolphins (with 159 edges) in a community of Doubtful Sound, New Zealand [3]. The middle panel shows the result of the Zachary’s karate club network\(^7\) [21]. This network contains 34 vertices and

\(^6\)http://vlado.fmf.uni-lj.si/pub/networks/data/ucidata.htm#zachary

\(^7\)http://www-personal.umich.edu/~mejn/netdata/
Figure 5: Examples of community trees and persistent diagrams in real networks. We consider three famous networks: the dolphin network, Zachary’s karate club network, and Zebra network. The top row shows their community trees and the bottom row shows their persistent diagrams. The number attached to each node of community trees indicates the number of vertices belong to that node (community). The number attached to each point in the persistent diagram indicates number of components with the corresponding birth and death time.

78 edges. The right panel presents the community tree and persistent diagram of a zebra network\(^7\) [18] that involves 27 zebras (27 vertices) and 111 interactions (111 edges).

We also attach a number to each node of a community tree to describe the size of that node/community (size: number of vertices belonging to that node). In persistent diagram, we attach a number to a point when there are multiple components with the same birth and death time.

5 Discussion and Future Work

In this paper, we introduce the concept of community trees and persistent diagrams of networks. To study the stability of community trees, we use the bottleneck distance of the corresponding persistent diagrams. We prove that the bottleneck distance is upper bounded by a quantity called \(TSN\) that can be evaluated without constructing community trees. All these concepts are related to TDA, and, thus, this paper presents a new way to apply TDA to network science.

Here, we comment on some possible future directions based on the current work.

- **Practical algorithm for bounding the \(TSN\).** The \(TSN\) is a powerful tool to control the change in a community tree without constructing the entire tree (and the corresponding persistent diagram). But as Theorem 3 has proved, computing the \(TSN\) is an NP-complete problem. To use the \(TSN\) to bound the change of networks, we need a fast algorithm that provides a useful bound of the \(TSN\). Finding such an algorithm will be left for future work.

- **Visualization tool using community tree.** Community trees provide a nice and easy illustration of the network. Thus, it can be used as a visualization tool for a complex network. We plan to design visualization methods using community trees and investigate the information loss during the visualization process.

- **Effects from stochastic updates on community trees.** In dynamic networks, networks change over time. We may model the change of networks by a stochastic model where edges and nodes may be created or eliminated with certain probabilities. How the community trees will change under such stochastic model will be an interesting research topic. Studies on

\(^7\)http://moreno.ss.uci.edu/data.html#zebra
this problem will allow us to understand the stability of communities generated by the CPM when the network is dynamic.

- Connections to overlapping communities. The community tree presents a new way to characterize overlapping communities. The CPM was used to detect overlapping communities by the fact that the same vertices may appear in different communities [13, 16]. Using a community tree, we can define the overlap between two communities by the merging between their corresponding components. We will study how different notions of overlaps are related to each other.

A Proof of Theorem 2

Before we prove Theorem 2, we first recall a property for two networks that differ only by one vertex:

**Lemma 4** Let \( G_1, G_2 \) be two networks such that \( G_2 = G_1 \setminus \{v\} \), where \( v \in V(G_1) \). Then

\[
d_B(T(G_1), T(G_2)) \leq 1.
\]

**Proof of Theorem 2.** Let \( G \) and \( G' \) be the two graphs being considered. We define \( G_1 \) to be the graph with vertex \( V(G_1) = V(G) \cap V(G') \) and edges \( E(G_1) = E(G) \cap E(G') \).

By triangle inequality,

\[
d_B(T(G), T(G')) \leq d_B(T(G), T(G_1)) + d_B(T(G_1), T(G')).
\]

We will derive bounds on both quantities.

**Part I: Bounding** \( d_B(T(G), T(G_1)) \). Let \( V_{RSN} \) denotes a possible candidate \( V_0 \) in Definition 1 that leads to \( RSN(G, G') \). Namely,

\[
\forall e \in E(G) \setminus E(G'), \nu(e) \cap V_{RSN} \neq \emptyset
\]

and \( \|V_{RSN}\| = RSN(G, G') \), where \( \|V\| \) denotes the cardinality of a vertex set \( V \). Note that it is easy to see that \( RSN(G, G') = RSN(G, G_1) \).

Define \( G_2 = G \setminus V_{RSN} \) be the induced subgraph of \( G \) by removing every vertex in \( V_{RSN} \). Because \( RSN(G, G') = RSN(G, G_1), G_2 \subset G_1 \subset G \). This further implies

\[
d_B(T(G), T(G_1)) \leq d_B(T(G), T(G_2)).
\]

Let \( V_{RSN} = \{v_1, \cdots, v_{RSN(G,G')}\} \). We define a sequence of subgraphs \( G^*_0, \cdots, G^*_{RSN(G,G')} \) such that \( G^*_0 = G \) and

\[
G^*_t = G^*_{t-1} \setminus \{v_t\},
\]

for \( t = 1, \cdots, RSN(G, G') \). Note that \( G^*_{RSN(G,G')} = G_2 \). Namely, the sequence \( G^*_0, \cdots, G^*_{RSN(G,G')} \) is a sequence of graphs from \( G \) to \( G_2 \) that differ by only one vertex. Because \( G_t \) and \( G^*_{t+1} \) differ by only one vertex, by Lemma 4 and triangle inequalities

\[
d_B(T(G), T(G_2)) \leq \sum_{t=1}^{RSN(G,G')} d_B(T(G^*_{t-1}), G^*_t) \leq \sum_{t=1}^{RSN(G,G')} 1 = RSN(G, G').
\]

Combining the above inequality and equation (2), we conclude

\[
d_B(T(G), T(G_1)) \leq RSN(G, G').
\]

**Part II: Bounding** \( d_B(T(G_1), T(G')) \). By Definition 1, \( RSN(G_a, G_b) = ASN(G_b, G_a) \) for any two graphs \( G_a \) and \( G_b \). Thus, \( ASN(G, G') = RSN(G', G) \).

Replacing \( G \) by \( G' \) in **Part I**, we conclude

\[
d_B(T(G'), T(G_1)) \leq RSN(G', G) = ASN(G, G').
\]

Using the above inequality and equations (3) and (1), we obtain

\[
d_B(T(G), T(G')) \leq d_B(T(G), T(G_1)) + d_B(T(G_1), T(G'))
\]

\[
\leq RSN(G, G') + ASN(G, G')
\]

\[
= TSN(G, G').
\]

\( \square \)
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