An analogue of Bott’s theorem for Schubert varieties-related to torus semi-stable points

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May 6, 2014

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Abstract

Let $G$ be a simple, simply connected algebraic group over the field $\mathbb{C}$ of complex numbers. Let $B$ be a Borel subgroup of $G$ containing a maximal torus $T$ of $G$. Let $\mathcal{T}_{G/B}$ denote the tangent bundle of the flag variety $G/B$. Let $\tau$ be an element of the Weyl group $W$ and let $X(\tau)$ be the Schubert variety corresponding to $\tau$.

In this paper, we prove the following:

If $G$ is simply laced, then, we have

1. $H^i(X(\tau), \mathcal{L}_{G/B}) = (0)$ for every $i \geq 1$.

2. $H^0(X(\tau), \mathcal{L}_{G/B})$ is the adjoint representation $\mathfrak{g}$ of $G$ if and only if the set of semi-stable points $X(\tau^{-1})^{ss}_T(\mathcal{L}_{\alpha_0})$ with respect to the line bundle associated to the highest root $\alpha_0$ is non-empty.

If $G$ is not simply laced, then, we have

1. $H^i(X(\tau), \mathcal{L}_{G/B}) = (0)$ for every $i \geq 1$.

2. The adjoint representation $\mathfrak{g}$ of $G$ is a $B$-submodule of $H^0(X(\tau), \mathcal{L}_{G/B})$ if and only if the set of semi-stable points $X(\tau^{-1})^{ss}_T(\mathcal{L}_{\alpha_0})$ with respect to the line bundle associated to the highest root $\alpha_0$ is non-empty.

Keywords: Schubert varieties, Highest root, Semi-stable points.
1 Introduction

In [3], Bott proved that for any semisimple algebraic group $G$ over the field of complex numbers, for any Borel sub group $B$ of $G$, all the higher cohomologies $H^i(G/B, \mathcal{T}_{G/B})$ with respect to the tangent bundle $\mathcal{T}_{G/B}$ on the flag variety vanishes. He further showed that the $G$- module of global sections $H^0(G/B, \mathcal{T}_{G/B})$ is the adjoint representation $\mathfrak{g}$ of $G$.

It is a natural question to ask for which Schubert vaeriety $X(\tau)$ in the flag variety $G/B$, $H^i(X(\tau), \mathcal{T}_{G/B})$ with respect to restriction of the tangent bundle $\mathcal{T}_{G/B}$ on the flag variety to $X(\tau)$ vanishes and that the $B$- module of global sections $H^0(X(\tau), \mathcal{T}_{G/B})$ is the adjoint representation $\mathfrak{g}$ of $G$.

The tangent space of $idB$ in $G/B$ as a $T$ module is a direct sum of weight spaces each of which is not dominant except the highest short root and highest long root.

There are interesting and important results have been obtatained for line bundles corresponding to non dominant characters on Schubert varieties. We refer to [1], [4] and [8] for some of the results.

We may also refer to [2] and [9] for recent developments.

However, we do not seem to have a precise answer in the literature for the above mentioned question.

Therefore, this question is of importance in relation to Schubert varieties.

The aim of this paper is to give a necessary and sufficient condition on the Schubert varieties $X(\tau)$ in the simply laced flag variety $G/B$ for which the above question has an affirmative answer.

We now proceed with notation before we describe our result.

The following notation will be maintained throughout this paper except in few places in section 3 where we prove some basic lemmas for algebraic groups over algebraically closed fields of arbitrary characteristic.

Let $\mathbb{C}$ denote the field of complex numbers. Let $G$ a simple, simply connected algebraic group over $\mathbb{C}$. We fix a maximal torus $T$ of $G$ and let $X(T)$ denote the set of characters of $T$. Let $W = N(T)/T$ denote the Weyl group of $G$ with respect to $T$. Let $R$ denote the set of roots of $G$ with respect to $T$.

Let $R^+$ denote the set of positive roots. Let $B^+$ be the Borel sub group of $G$ containing $T$ with respect to $R^+$. Let $S = \{\alpha_1, \ldots, \alpha_l\}$ denote the set of simple roots in $R^+$. Here $l$ is the rank of $G$. Let $B$ be the Borel subgroup of $G$ containing $T$ with respect to the set of negative roots $R^- = -R^+$.

For $\beta \in R^+$ we also use the notation $\beta > 0$. The simple reflection in the Weyl group corresponding to $\alpha_i$ is denoted by $s_{\alpha_i}$.

Let $\mathfrak{g}$ denote the Lie algebra of $G$. Let $\mathfrak{h}$ be the Lie algebra of $T$. Let $\mathfrak{b}$ be the Lie algebra
of $B$.

We have $X(T) \otimes \mathbb{R} = (\mathfrak{h}_\mathbb{R})^*$, the dual of the real form of $\mathfrak{h}$.

The positive definite $W$-invariant form on $(\mathfrak{h}_\mathbb{R})^*$ induced by the Killing form of the Lie algebra $\mathfrak{g}$ of $G$ is denoted by $(\ , \ )$. We use the notation $\langle \ , \ \rangle$ to denote $\langle \nu, \alpha \rangle = \frac{2(\nu, \alpha)}{(\alpha, \alpha)}$.

Let $x_\alpha, y_\alpha, \alpha \in R^+, h_\alpha, \alpha_i \in \mathcal{S}$, denote a Chevalley basis of the Lie algebra of $G$.

We denote by $g_\alpha$ (resp. $g_{-\alpha}$) the one dimensional root subspace of $g$ spanned by $x_\alpha$ (resp. $y_\alpha$).

Let $\leq$ denote the partial order on $X(T)$ given by $\mu \leq \lambda$ if $\lambda - \mu$ is a non negative integral linear combination of simple roots.

We denote by $X(T)^+$ the set of dominant characters of $T$ with respect to $B^+$. Let $\rho$ denote the half sum of all positive roots of $G$ with respect to $T$ and $B^+$.

For any simple root $\alpha$, we denote the fundamental weight corresponding to $\alpha$ by $\omega_\alpha$.

For $w \in W$ let $l(w)$ denote the length of $w$. We define the dot action by $w \cdot \lambda = w(\lambda + \rho) - \rho$.

Let $\alpha_0$ denote the highest root.

We set $R^+(w) := \{ \beta \in R^+ : w(\beta) \in -R^+ \}$.

Let $w_0$ denote the longest element of the Weyl group $W$.

For $w \in W$, let $X(w) := BwB/B$ denote the Schubert variety in $G/B$ corresponding to $w$.

Consider the $T$ action of $G/B$. Schubert varieties $X(w)$ are stable under $T$. Let $\lambda$ be a dominant character of $T$. We denote by $L_\lambda$ denote the line bundle on $G/B$ corresponding to the character $\lambda$ of $B$. We denote the restriction of the line bundle $L_\lambda$ to $X(w)$ as well by $L_\lambda$.

We denote by $X(w)_{ss}^T(L_\lambda)$ the set of all semi-stable points of $X(w)$ with respect to the line bundle $L_\lambda$ for the action of $T$.

So, in particular, we have semi-stable points $X(w)_{ss}^T(L_{\alpha_0})$ with respect to the line bundle $L_{\alpha_0}$ corresponding to the highest root $\alpha_0$.

In this paper, we prove the following theorem for simple, simply connected and simply laced algebraic groups.

**Theorem A**

Let $G$ be a simple, simply connected and simply laced algebraic group over $\mathbb{C}$. Let $\tau \in W$. Then, we have

1. $H^i(X(\tau), \mathcal{T}_{G/B}) = (0)$ for every $i \geq 1$. 

2. $H^0(X(\tau), T_{G/B})$ is the adjoint representation $\mathfrak{g}$ of $G$ if and only if the set of semi-stable points $X(\tau^{-1})^{ss}_T(\mathcal{L}_\alpha)$ is non-empty.

We also prove that

**Theorem B**

Let $G$ be simple, simply connected but not simply laced algebraic group over $\mathbb{C}$. Let $\tau \in W$. Then, we have

1. $H^i(X(\tau), T_{G/B}) = (0)$ for every $i \geq 2$.

2. The adjoint representation $\mathfrak{g}$ is a $B$-submodule of $H^0(X(\tau), T_{G/B})$ if and only if the set of semi-stable points $X(\tau^{-1})^{ss}_T(\mathcal{L}_\alpha)$ is non-empty.

The organisation of the paper is as follows:

Section 2 consists of preliminaries from [5], [6] and [7]. In section 3, we prove theorem A. In section 4, we apply theorem A to certain Schubert varieties related to maximal parabolic subgroups of $G$. For precise statement, see theorem(4.2).

In section 5, we obtain the following theorem on the cohomology modules $H^i(X(c), \mathcal{L}_{c^{-1},0})$ of the line bundle $\mathcal{L}_{c^{-1},0}$ on the Schubert variety $X(c)$ corresponding to a Coxeter element $c$ of $W$. We use theorem A in proving this theorem.

**Theorem C**

1. Let $\tau \in W$. The cohomology module $H^{l(\tau)}(X(\tau), \mathcal{L}_{\tau^{-1},0})$ is the one dimensional trivial representation of $B$.

2. Let $c$ be a Coxeter element of $W$. Then, $H^i(X(c), \mathcal{L}_{c^{-1},0})$ is zero for every $i \neq l(c)$ if and only if both $X(c)^{ss}_T(\mathcal{L}_\alpha)$ and $X(c^{-1})^{ss}_T(\mathcal{L}_\alpha)$ are non-empty.

For a precise detail with notation, see theorem(5.7).

In section 6, we prove theorem B.

## 2 Preliminaries

We denote by $U$ the unipotent radical of $B$. We denote by $P_\alpha$ the minimal parabolic subgroup of $G$ containing $B$ and $s_\alpha$. Let $L_\alpha$ denote the Levi subgroup of $P_\alpha$ containing $T$. We denote by $B_\alpha$ the intersection of $L_\alpha$ and $B$. Then $L_\alpha$ is the product of $T$ and a homomorphic image $G_\alpha$ of $SL(2)$ via a homomorphism $\psi : SL(2) \rightarrow L_\alpha$. (cf. [7, II, 1.1.4]).

We make use of following points in computing cohomologies.
Since $G$ is simply connected, the morphism $\psi : SL(2) \rightarrow G_\alpha$ is an isomorphism, and hence $\psi : SL(2) \rightarrow L_\alpha$ is injective. We denote this copy of $SL(2)$ in $L_\alpha$ by $SL(2, \alpha)$ We denote by $B'_\alpha$ the intersection of $B_\alpha$ and $SL(2, \alpha)$ in $L_\alpha$.

We also note that the morphism $SL(2, \alpha)/B'_\alpha \rightarrow L_\alpha/B_\alpha$ induced by $\psi$ is an isomorphism.

Since $L_\alpha/B_\alpha \rightarrow P_\alpha/B$ is an isomorphism, to compute the cohomology $H^i(P_\alpha/B, V)$ for any $B$-module $V$, we treat $V$ as a $B_\alpha$-module and we compute $H^i(L_\alpha/B_\alpha, V)$.

Given a $w \in W$ the closure in $G/B$ of the orbit of the coset $wB$ is the Schubert variety corresponding to $w$, and is denoted by $X(w)$. We recall some basic facts and results about Schubert varieties. A good reference for all this is the book by Jantzen. (cf [7, II, Chapter 14].)

Let $w = s_{\alpha_1}s_{\alpha_2} \ldots s_{\alpha_n}$ be a reduced expression for $w \in W$. Define

$$Z(w) = \frac{P_{\alpha_1} \times P_{\alpha_2} \times \ldots \times P_{\alpha_n}}{B \times \ldots \times B},$$

where the action of $B \times \ldots \times B$ on $P_{\alpha_1} \times P_{\alpha_2} \times \ldots \times P_{\alpha_n}$ is given by $(p_1, \ldots, p_n)(b_1, \ldots, b_n) = (p_1 \cdot b_1, b_1^{-1} \cdot p_2 \cdot b_2, \ldots, b_{n-1}^{-1} \cdot p_n \cdot b_n)$, $p_j \in P_{\alpha_j}$, $b_j \in B$. We denote by $\phi_w$ the birational surjective morphism $\phi_w : Z(w) \rightarrow X(w)$.

We note that for each reduced expression for $w$, $Z(w)$ is smooth, however, $Z(w)$ may not be independent of a reduced expression.

Let $f_n : Z(w) \rightarrow Z(ws_{\alpha_n})$ denote the map induced by the projection $P_{\alpha_1} \times P_{\alpha_2} \times \ldots \times P_{\alpha_n} \rightarrow P_{\alpha_1} \times P_{\alpha_2} \times \ldots \times P_{\alpha_{n-1}}$. Then we observe that $f_n$ is a $P_{\alpha_n}/B \simeq \mathbb{P}^1$-fibration.

Let $V$ be a $B$-module. Let $L_w(V)$ denote the pull back to $X(w)$ of the homogeneous vector bundle on $G/B$ associated to $V$. By abuse of notation we denote the pull back of $L_w(V)$ to $Z(w)$ also by $L_w(V)$, when there is no cause for confusion. Then, for $i \geq 0$, we have the following isomorphisms of $B$-linearized sheaves

$$R^i f_n^* L_w(V) = L_{ws_{\alpha_n}}(H^i(P_{\alpha_n}/B, L_w(V))).$$

This together with easy applications of Leray spectral sequences is the constantly used tool in what follows. We term this the descending 1-step construction.

We also have the ascending 1-step construction which too is used extensively in what follows sometimes in conjunction with the descending construction. We recall this for the convenience of the reader.

Let the notations be as above and write $\tau = s_\gamma w$, with $l(\tau) = l(w) + 1$, for some simple root $\gamma$. Then we have an induced morphism

$$g_1 : Z(\tau) \rightarrow P_\gamma/B \simeq \mathbb{P}^1,$$

with fibres given by $Z(w)$. Again, by an application of the Leray spectral sequences together with the fact that the base is a $\mathbb{P}^1$, we obtain for every $B$-module $V$ the following exact
sequence of $P_\gamma$-modules:

\[
(0) \longrightarrow H^1(P_\gamma/B, R^{i-1}g_{1*}\mathcal{L}_w(V)) \longrightarrow H^i(Z(\tau), \mathcal{L}_\tau(V)) \longrightarrow H^0(P_\gamma/B, R^ig_{1*}\mathcal{L}_w(V)) \longrightarrow (0).
\]

This short exact sequence of $B$-modules will be used frequently in this paper. So, we denote this short exact sequence by $SES$ whenever this is being used.

We also recall the following well-known isomorphisms:

- $\phi_{w*}\mathcal{O}_{Z(w)} = \mathcal{O}_{X(w)}$.
- $R^q\phi_{w*}\mathcal{O}_{Z(w)} = 0$ for $q > 0$.

This together with [7, II. 14.6] implies that we may use the Bott-Samelson schemes $Z(w)$ for the computation and study of all the cohomology modules $H^i(Z(w), \mathcal{L}_w(V))$. Henceforth in this paper we shall use the Bott-Samelson schemes and their cohomology modules in all the computations.

Simplicity of Notation If $V$ is a $B$-module and $\mathcal{L}_w(V)$ is the induced vector bundle on $Z(w)$ we denote the cohomology modules $H^i(Z(w), \mathcal{L}_w(V))$ by $H^i(w, \lambda)$. In particular, if $\lambda$ is a character of $B$ we denote the cohomology modules $H^i(Z(w), \mathcal{L}_\lambda)$ by $H^i(w, \lambda)$.

2.0.1 Some constructions from Demazure’s paper

We recall briefly two exact sequences from [5] that Demazure used in his short proof of the Borel-Weil-Bott theorem (cf. [3]). We use the same notation as in [5]. In the rest of the paper these sequences are referred to as Demazure exact sequences.

Let $\alpha$ be a simple root and let $\lambda \in X(T)$ be a weight such that $\langle \lambda, \alpha \rangle \geq 0$. For such a $\lambda$, we denote by $V_{\lambda,\alpha}$ the module $H^0(P_\alpha/B, \lambda)$. Let $\mathbb{C}_\lambda$ denote the one dimensional $B$-module.

Here, we recall the following lemma due to Demazure on a short exact sequence of $B$-modules: (to obtain the second sequence we need to assume that $\langle \lambda, \alpha \rangle \geq 2$).

**Lemma 2.1.**

\[
(0) \longrightarrow K \longrightarrow V_{\lambda,\alpha} \longrightarrow \mathbb{C}_\lambda \longrightarrow (0).
\]

\[
(0) \longrightarrow \mathbb{C}_{s_\alpha(\lambda)} \longrightarrow K \longrightarrow V_{\lambda-\alpha,\alpha} \longrightarrow (0).
\]

A consequence of the above exact sequences is the following crucial lemma, a proof of which can be found in [5].

**Lemma 2.2.**

1. Let $\tau = ws_\alpha$, $l(\tau) = l(w) + 1$. If $\langle \lambda, \alpha \rangle \geq 0$ then $H^j(\tau, \lambda) = H^j(w, V_{\lambda,\alpha})$ for all $j \geq 0$.

2. Let $\tau = ws_\alpha$, $l(\tau) = l(w) + 1$. If $\langle \lambda, \alpha \rangle \geq 0$, then $H^i(\tau, \lambda) = H^{i+1}(s_\alpha \cdot \lambda)$. Further, if $\langle \lambda, \alpha \rangle \leq -2$, then $H^i(\tau, \lambda) = H^{i-1}(s_\alpha \cdot \lambda)$.  

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3. If $\langle \lambda, \alpha \rangle = -1$, then $H^i(\tau, \lambda)$ vanishes for every $i \geq 0$ (cf. [7], Prop 5.2(b)).

We derive the following easy consequence of the lemma(2.2) which will be used to compute cohomologies in this paper:

**Lemma 2.3.** Let $V$ be an irreducible $L_\alpha$- module. Let $\lambda$ be a character of $B_\alpha$. Then, we have

1. If $\langle \lambda, \alpha \rangle \geq 0$, then $H^0(L_\alpha/B_\alpha, V \otimes \mathbb{C}\lambda)$ is isomorphic to the tensor product of $V$ and $H^0(L_\alpha/B_\alpha, \mathbb{C}\lambda)$, and $H^j(L_\alpha/B_\alpha, V \otimes \mathbb{C}\lambda) = (0)$ for every $j \geq 1$.

2. If $\langle \lambda, \alpha \rangle \leq -2$, $H^0(L_\alpha/B_\alpha, V \otimes \mathbb{C}\lambda) = (0)$, and $H^1(L_\alpha/B_\alpha, V \otimes \mathbb{C}\lambda)$, is isomorphic to the tensor product of $V$ and $H^0(L_\alpha/B_\alpha, \mathbb{C}s_\alpha \cdot \lambda)$.

3. If $\langle \lambda, \alpha \rangle = -1$, then $H^j(L_\alpha/B_\alpha, V \otimes \mathbb{C}\lambda) = (0)$ for every $j \geq 0$.

We recall the following lemma from [2] on indecomposable $B_\alpha$- modules (cf. [2], cor(9.1)).

**Lemma 2.4.** Any finite dimensional indecomposable $B_\alpha$ module $V$ is isomorphic to $V' \otimes \mathbb{C}\lambda$ for some irreducible representation $V'$ of $L_\alpha$, and $\mathbb{C}\lambda$ is an one dimensional representation of $L_\alpha$ given by a character $\lambda$ of $B_\alpha$.

Applying lemma(2.4), we obtain the following lemma.

Let $V$ be a $P_\alpha$ module. Consider the restriction of the module $V$ to $B$. Consider the evaluation map $ev : H^0(P_\alpha/B, V) \rightarrow V$ defined by $ev(s) = s(id_B)$, the value of $s$ at the identity coset $id_B$ of $P_\alpha/B$.

Then, we have

**Lemma 2.5.**

1. The evaluation map $ev : H^0(P_\alpha/B, V) \rightarrow V$ is an isomorphism of $P_\alpha$-modules.

2. $H^i(P_\alpha/B, V) = (0)$ for all $i \geq 1$.

*Proof.* Since the inclusion $L_\alpha/B_\alpha \hookrightarrow P_\alpha/B$ is an isomorphism, by treating the $B$- module as a $B_\alpha$- module, it is sufficient to prove that the evaluation map $ev : H^0(L_\alpha/B_\alpha, V) \rightarrow V$ is an isomorphism and $H^i(L_\alpha/B_\alpha, V) = (0)$ for all $i \geq 1$.

Now, we decompose $V$ into irreducible $L_\alpha$- modules. This is possible since $L_\alpha$ is reductive, and the base field is $\mathbb{C}$.

Since, the cohomologies commute with direct sum, we may assume that $V$ is an irreducible $L_\alpha$- module.

Now, the lemma follows from lemma(2.3(1)), by taking $\lambda = 0$. \qed
We state a combinatorial lemma. For completeness, we give a proof here.

**Lemma 2.6.** Let $G$ be a simple simply laced algebraic group. Let $\alpha \in S$, and $\beta$ be a root different from both $\alpha$ and $-\alpha$. Then, $\langle \beta, \alpha \rangle \in \{-1, 0, 1\}$.

**Proof.** Since $\beta$ and $\alpha$ are not proportional, by using similar arguments in [6] we see that the product $\langle \beta, \alpha \rangle \langle \alpha, \beta \rangle$ is an integer lying in $\{0, 1, 2, 3\}$.

Since $G$ is simply laced, we have $\langle \beta, \alpha \rangle = \langle \alpha, \beta \rangle$.

Since $\langle \beta, \alpha \rangle$ is an integer, $\langle \beta, \alpha \rangle \in \{-1, 0, 1\}$. \hfill $\Box$

Let $\gamma$ be a simple root.

We recall that $sl_{2,\gamma}$ is the simple Lie algebra corresponding to $\gamma$.

We first note that $sl_{2,\gamma}$ is an indecomposable $B_\gamma$-summand of $g$.

The following lemma gives a description of indecomposable $B_\gamma$-summands of $g$.

**Lemma 2.7.** Every indecomposable $B_\gamma$-summand $V$ of $g$ must be one of the following:

1. $V = \mathbb{C} \cdot h$ for some $h \in \mathfrak{h}$ such that $\gamma(h) = 0$.

2. $V = g_\beta \oplus g_{\beta-\gamma}$ for some root $\beta$ such that $\langle \beta, \gamma \rangle = 1$.

3. $V = sl_{2,\gamma}$, the three dimensional irreducible $L_\gamma$-module with highest weight $\gamma$.

**Proof.** Let $V$ be an indecomposable $B_\gamma$-summand of $g$. Let $\lambda$ be a maximal weight of $V$. Then, the direct sum $\bigoplus_{r \in \mathbb{Z}_{\geq 0}} V_{\lambda - r\gamma}$ is a $B_\gamma$-summand of $V$.

Hence, we have $V = \bigoplus_{r \in \mathbb{Z}_{\geq 0}} V_{\lambda - r\gamma}$. By lemma(2.6), the dimension of $V$ must be at most two unless $V = sl_{2,\gamma}$.

Further, if the dimension of $V$ is one, $V = \mathbb{C} \cdot h$ for some $h \in \mathfrak{h}$ such that $\gamma(h) = 0$. Also, if the dimension of $V$ is two, then, we must have $V = g_\beta \oplus g_{\beta-\gamma}$ for some root $\beta$ such that $\langle \beta, \gamma \rangle = 1$.

This completes the proof of the lemma. \hfill $\Box$
3 Proof of theorem A- simply laced Case

In this section, we prove theorem A. Theorem A is stated for only simply laced groups. However, in the first subsection we prove a result for any simple algebraic group over any algebraically closed field of arbitrary characteristic.

3.1 Global sections $H^0(G/B, V)$ for the case when $V$ is a $G$-module

We have the following notation only in this subsection. Let $K$ be an algebraically closed field of arbitrary characteristic. Let $G$ be a simple, simply connected algebraic group over $K$.

In this subsection, we prove that for any $G$-module $V$, the evaluation map

$ev : H^0(G/B, V) \rightarrow V$ given by $ev(s) = s(idB)$ is an isomorphism of $G$-modules (cf lemma(3.2)).

This lemma is a slight generalisation of lemma(2.5(1)). Also, its proof is independent of the characteristic of the base field.

We first prove the following two basic lemmas. For the completeness, we provide a proof here.

Let $H$ be an algebraic group over $K$.

Let $W_1$ and $W_2$ be two finite dimensional rational $H$-modules.

Let $W_2^H$ denote the set of all $H$-invariants of $W_2$. Let $Hom_H(W_1, W_2)$ denote the set of all homomorphism of $H$-modules from $W_1$ into $W_2$.

Consider the linear map $\psi : (W_1^* \otimes W_2)^H \rightarrow Hom_H(W_1, W_2)$ given by

$\psi(f \otimes w)(v) = f(v) \cdot w$.

Then, we have

**Lemma 3.1.** The restriction of $\psi$ to $(W_1^* \otimes W_2)^H$ induces an isomorphism

$\psi : (W_1^* \otimes W_2)^H \rightarrow Hom_H(W_1, W_2)$ of finite dimensional vector spaces over $K$.

**Proof.** Let $\sum_{i=1}^r f_i \otimes e_i \in (W_1^* \otimes W_2)^H$. Let $\psi(\sum_{i=1}^r f_i \otimes e_i) = \phi$.

For any $v \in W_1$ and for any $h \in H$, we have $\phi(h \cdot v) = \sum_{i=1}^r f_i(h \cdot v) \cdot e_i$. Since $\sum_{i=1}^r f_i \otimes e_i$ is $H$-invariant, we have

$\sum_{i=1}^r f_i(h \cdot v) \cdot e_i = \sum_{i=1}^r f_i(v) \cdot (h \cdot e_i) = h \cdot \phi(v)$.

Hence, we have $\phi(h \cdot v) = h \cdot \phi(v)$. Thus, we can see that $\psi(W_1^* \otimes W_2)^H \subset Hom_H(W_1, W_2)$. 9
Proof of \( \text{Hom}_H(W_1, W_2) \subset \psi(W_1^* \otimes W_2)^H \) is similar to that of 
\( \psi(W_1^* \otimes W_2)^H \subset \text{Hom}_H(W_1, W_2) \).

This completes the proof of the lemma.

The following lemma could be well known. For completeness, we give the details of a proof.

We use lemma(3.1) to prove:

**Lemma 3.2.** Let \( V \) be a finite dimensional rational \( G \)-module. Then, the evaluation map 
\( ev : H^0(G/B, V) \longrightarrow V \) is an isomorphism of \( G \)-modules.

**Proof. Step 1**

We first show that the evaluation map 
\( ev : H^0(G/B, V) \longrightarrow V \) is a homomorphism of \( G \)-modules.

Take \( W_1 = H^0(G/B, V) \) and taking \( W_2 = V \).

We first note that \( W_1 \) and \( W_2 \) are both \( G \)-modules. Since \( G/B \) is projective, we see that

**Observation**

the \( B \)-invariants of \( W_1^* \otimes W_2 \) is equal to the \( G \)-invariants of \( W_1^* \otimes W_2 \).

Since the evaluation map 
\( ev : H^0(G/B, V) \longrightarrow V \) is a homomorphism of \( B \)-modules, applying lemma(3.1) to \( H = B \), we can find a vector \( u \in (W_1^* \otimes W_2)^B \) such that \( \psi(u) = ev \).

By the above **Observation**, we have \( u \in (W_1^* \otimes W_2)^G \). We now apply lemma(3.1) to \( H = G \) and conclude that the evaluation map 
\( ev : H^0(G/B, V) \longrightarrow V \) is a homomorphism of \( G \)-modules.

This completes the proof of **Step 1**.

By the description of the global sections of the vector bundle on \( G/B \) associated to the \( B \)-module, \( H^0(G/B, V) \) is the space of all morphisms \( f : G \longrightarrow V \) satisfying \( f(gb) = b^{-1} \cdot f(g) \).

For each \( v \in V \), we associate a morphism \( \phi_v :: G \longrightarrow V \) defined by \( \phi_v(g) = g^{-1} \cdot v \). Clearly, \( \phi_v(gb) = b^{-1} \cdot \phi_v(g) \) for every \( g \in G \) and for every \( b \in B \).

So, we have the map \( \phi : V \longrightarrow H^0(G/B, V) \) given by \( \phi(v) = \phi_v \). Clearly \( \phi \) is injective. Hence, the dimension of \( V \) is at most the dimension of \( H^0(G/B, V) \).

On the other hand, using **Step 1**, we see that the kernel of evaluation map 
\( ev : H^0(G/B, V) \longrightarrow V \) is a \( G \) submodule of \( H^0(G/B, V) \). Now, let \( f \) be in the kernel of \( ev \). Then, \( g^{-1} \cdot f \) is also in the kernel of \( ev \). Hence, we have \( f(g) = 0 \) for every \( g \in G \). Thus, we have \( f = 0 \).

Hence \( ev \) is injective. Since the dimension of \( V \) is at most the dimension of \( H^0(G/B, V) \),
the evaluation map 
\( ev : H^0(G/B, V) \longrightarrow V \) is an isomorphism of \( G \)-modules.
This completes the proof the lemma. □

3.2 Proof of theorem A

In this section, we prove theorem A.

The following notation will be maintained throughout the rest of this section.

Let \( G \) be a simple, simply connected and simply laced algebraic group over \( \mathbb{C} \). Let \( g \) be the Lie algebra of \( G \).

Let \( \tau \in W \). Let \( \gamma \) be a simple root. Let \( V \) be a \( B \)-sub module of \( g \) containing \( b \). We recall the evaluation map \( ev : H^0(\tau, V) \rightarrow V \) by \( ev(s) = s(idB) \), the evaluation of the section at the identity coset \( idB \) as point in \( X(\tau) \).

**Lemma 3.3.** The evaluation map \( ev : H^0(\tau, V) \rightarrow V \) is injective.

**Proof.** Proof is by induction on \( l(\tau) \).

If \( l(\tau) = 0 \), we are done.

So, we may choose a simple root \( \gamma \in S \) such that \( l(\tau) = l(s_\gamma \tau) = l(\tau) - 1 \).

Then, by induction on \( l(\tau) \), we assume that the evaluation map

\( ev : H^0(s_\gamma \tau, V) \rightarrow V \) is injective.

Since \( V \) is a \( B \)-submodule of \( g \), \( H^0(s_\gamma \tau, V) \) is a \( B \)- submodule of \( H^0(s_\gamma \tau, g) \).

Hence, \( H^0(P_\gamma/B, H^0(s_\gamma \tau, V)) \) is a \( B \)-submodule \( H^0(P_\gamma/B, g) \).

On the other hand, since the \( B \)- module \( g \) is a restriction of a \( P_\gamma \) module, and so \( g \) is a \( L_\gamma \) module.

Now, since the inclusion \( L_\gamma/B_\gamma \hookrightarrow P_\gamma/B \) is an isomorphism, by using lemma (2.5(1)) (or lemma(3.2)), the evaluation map \( ev : H^0(P_\gamma/B, g) \rightarrow g \) is an isomorphism of \( B \)- modules. Hence, \( H^0(P_\gamma/B, H^0(s_\gamma \tau, V)) \) is a \( B \)-submodule of \( g \).

Hence, the evaluation map \( ev : H^0(P_\gamma/B, H^0(s_\gamma \tau, V)) \rightarrow H^0(s_\gamma \tau, V) \) is injective. Hence, using the short exact sequence \( SES \) of \( B \)- modules, we see that \( H^0(\tau, V) \) is isomorphic to \( H^0(P_\gamma/B, H^0(s_\gamma \tau, V)) \).

Now, since the evaluation map \( ev : H^0(\tau, V) \rightarrow V \) is the composition of the evaluation maps \( ev : H^0(P_\gamma/B, H^0(s_\gamma \tau, V)) \rightarrow H^0(s_\gamma \tau, V) \), and \( ev : H^0(s_\gamma \tau, V) \rightarrow V \), we conclude that the evaluation map \( ev : H^0(\tau, V) \rightarrow V \) is injective.

This completes the proof of the lemma. □

Let \( \tau \in W \). Let \( \gamma \) be a simple root.
Now, let $V$ be a $B$-sub module of $g$ containing $b$. Then, we have

In view of lemma(2.7) and lemma(3.3), we see that the indecomposable $B_\gamma$- summands of $H^0(\tau, V)$ must be at most 3-dimensional. However, it is not clear what are they precisely. It is important to study them to determine the cohomology modules $H^i(\tau, V)$.

In this context, we prove the following Key lemma.

**Lemma 3.4.** Let $\tau \in W$. Let $\gamma$ be a simple root. Every indecomposable $B_\gamma$- summand $V'$ of $H^0(\tau, V)$ must be one of the following:

1. $V' = C \cdot h$ for some $h \in \mathfrak{h}$ such that $\gamma(h) = 0$.

2. $V' = C \cdot h \bigoplus g_{-\gamma}$ for some $h \in \mathfrak{h}$ such that $\gamma(h) = 1$ and $\nu(h) = 0$ for every simple root $\nu$ different from $\gamma$.

3. $V' = g_\beta$ for some root $\beta$ such that $\langle \beta, \gamma \rangle$ lying in $\{-1, 0, 1\}$.

4. $V' = g_\beta \bigoplus g_{\beta - \gamma}$ for some root $\beta$ such that $\langle \beta, \gamma \rangle = 1$.

5. $V' = sl_{2, \gamma}$, the three dimensional irreducible $L_\gamma$-module with highest weight $\gamma$.

**Proof.** Let $V'$ be an indecomposable $B_\gamma$-summand of $H^0(\tau, V)$. If the weight of the $B_\gamma$-stable line in $V'$ is different from $-\gamma$, then, using lemma(2.7) and lemma(3.3), we see that $V'$ must be one of the types (1), (3) or (4).

Otherwise, $g_{-\gamma}$ is a $B_\gamma$-submodule of $V'$. In this case, we need to show that $g_{-\gamma}$ is a proper subspace of $V'$. That is, either $V' = C \cdot h \bigoplus g_{-\gamma}$ for some $h \in \mathfrak{h}$ such that $\gamma(h) = 1$ and $\nu(h) = 0$ for every simple root $\nu$ different from $\gamma$ or $V' = sl_{2, \gamma}$.

We prove this by induction on $l(\tau)$.

If $l(\tau) = 0$, then, $\tau = id$ and so we are done.

Otherwise, choose a simple root $\alpha$ such that $l(\tau) = 1 + l(s_\alpha \tau)$.

By induction on $l(\tau)$, we assume that for any simple root $\nu$, if $V'$ is an indecomposable $B_\nu$ summand of $H^0(s_\alpha \tau, V)$ containing $g_{-\nu}$, then, either $V' = C \cdot h \bigoplus g_{-\nu}$ for some $h \in \mathfrak{h}$ such that $\nu(h) = 1$ and $\mu(h) = 0$ for every simple root $\mu$ different from $\nu$ or $V' = sl_{2, \nu}$.

We now fix a simple root $\gamma$.

We give the details of proof in 3 different cases as follows.

**Case 1:**
We first assume that $\gamma = \alpha$.

Now, let $V_1$ be an indecomposable $B_\gamma$-summand of $H^0(\tau,V)$ containing $g_{-\gamma}$. Then, using lemm(3.3), we see that there is an indecomposable $B_\gamma$-summand $V'$ of $H^0(s_\gamma,\tau,V)$ containing $g_{-\gamma}$. Since $l(s_\gamma,\tau) = l(\tau) - 1$, by induction on length of $\tau$, we see that $V'$ must be of type (2) or of type (5).

If $V'$ is of type (2), then, $H^0(s_\gamma,V') = (0)$. Hence, $g_{-\gamma}$ can not be a subspace of $H^0(s_\gamma,V')$.

On the other hand, using $SES$, we have $H^0(s_\gamma,H^0(s_\gamma,\tau,V)) = H^0(\tau,V)$. Thus, $g_{-\gamma}$ can not be a subspace of $H^0(\tau,V)$.

This completes the proof for the case when $\alpha = \gamma$.

Case 2:

We assume that $\alpha$ is different from $\gamma$ and $\langle \gamma,\alpha \rangle \neq 0$. By using lemma (2.6), we have $\langle \gamma,\alpha \rangle = -1$.

By a similar argument as in Case 1, we may assume that there is a $B_\gamma$-summand $V'$ of $H^0(s_\gamma,\tau,V)$ containing $g_{-\gamma}$. By induction, $V'$ must be of type (2) or of type (5).

Now, let $U'$ denote the minimal $B_\alpha$-summand of $H^0(\tau,V)$ containing $V'$.

Subcase 1:

If $V'$ is of type (2), then, $V' = C \cdot h \bigoplus g_{-\gamma}$ for some $h \in h$ such that $\gamma(h) = 1$ and $\nu(H) = 0$ for every simple root $\nu$ different from $\gamma$.

Then, an indecomposable $B_\alpha$-summand $V_1$ of $U'$ containing $g_{-\gamma}$ must be of the form $g_{-\gamma} \bigoplus g_{-\gamma-\alpha}$. So, by lemma (2.5), we have $H^0(s_\alpha,V_1) = V_1$.

Hence, $g_{-\gamma}$ must be a sub space of $H^0(s_\gamma,H^0(s_\gamma,\tau,V))$.

Since $\alpha \neq \gamma$, we have $\alpha(h) = 0$. Hence, $C \cdot h$ is a $B_\alpha$-direct summand of $U'$. Hence, $C \cdot h$ must be a $B_\alpha$-submodule of $H^0(s_\alpha,U')$. Hence, $C \cdot h$ must be a subspace of $H^0(s_\gamma,H^0(s_\gamma,\tau,V))$.

Hence, $V' = C \cdot h \bigoplus g_{-\gamma}$ must be a subspace of $H^0(s_\alpha,H^0(s_\alpha,\tau,V))$.

Thus, by using $SES$, we conclude that $V' = C \cdot h \bigoplus g_{-\gamma}$ is a subspace of $H^0(\tau,V)$.

Subcase 2:

Let $V'$ be of type (5). Then, we have $V' = sl_{2,\gamma}$. In this case, $U'$, the minimal $B_\alpha$-summand of $H^0(\tau,V)$ containing $V'$ must contain $H_\gamma = -[X_{-\gamma},X_\gamma]$. Here $[X_{-\gamma},X_\gamma]$ denotes the Lie bracket of $X_{-\gamma}$ and $X_\gamma$ in $g$.

Since $\alpha(H_\gamma) = -1$, we have $[X_{-\alpha},H_\gamma] = X_{-\alpha}$. Hence, $g_{-\alpha}$ must be a subspace of $U'$.

Therefore, by induction applying to the simple root $\nu = \alpha$, $U'$ must either contain $sl_{2,\alpha}$ or the indecomposable $B_\alpha$-module $C \cdot h \bigoplus g_{-\alpha}$ for some $h \in h$ such that $\alpha(h) = 1$ and $\nu(h) = 0$ for every simple root $\nu$ different from $\alpha$. 
Now, if \( g_{\gamma + \alpha} \) is a sub space of \( U' \), then, \( g_{\alpha + \gamma} \oplus g_{\gamma} \) is an indecomposable \( B_\alpha \)-summand of \( U' \). Since \( g_{\alpha + \gamma} + g_{\gamma} \) is a \( L_\gamma \)-module, using lemma(2.5), we see that \( H^0(s_\alpha, g_{\alpha + \gamma} \oplus g_{\gamma}) = g_{\alpha + \gamma} \oplus g_{\gamma} \). Hence, we have

**Observation:**

\[ g_{\alpha + \gamma} \oplus g_{\gamma} \] is a sub space of \( H^0(s_\alpha, U') \).

Since \( U' \) is a \( B_\alpha \)-sub module of \( H^0(s_\alpha \tau, V) \), \( H^0(s_\alpha, U') \) is a \( B_\alpha \)-sub module of \( H^0(s_\alpha, H^0(s_\alpha \tau, V)) \). On the other hand, since \( H^0(s_\alpha, H^0(s_\alpha \tau, V)) \) is a \( B_\gamma \)-module, using **Observation**, we see that the \( B_\gamma \)-span \( sl_{2,\gamma} \) of \( g_{\gamma} \) must be a \( B_\gamma \)-sub module of \( H^0(s_\alpha, H^0(s_\alpha \tau, V)) \).

Using **SES**, we conclude that \( H^0(\tau, V) \) contains \( sl_{2,\gamma} \). This proves that \( H^0(\tau, V) \) contains an indecomposable \( B_\gamma \) summand of type \((5)\).

Now, if \( g_{\gamma + \alpha} \) is not a sub space of \( U' \), then, \( \gamma \) is an indecomposable \( B_\alpha \)-direct summand of \( U' \). Since \( \langle \gamma, \alpha \rangle = -1 \), by lemma, we have \( H^i(s_\alpha, \gamma) = (0) \) for every \( i \in \mathbb{Z}_{\geq 0} \). In particular, \( \gamma \) can not be a subspace of \( H^0(s_\alpha, H^0(s_\alpha \tau, V)) \).

Let \( V' \) be of type \((5)\). Then, we have \( V' = sl_{2,\gamma} \). In this case, \( U' \), the minimal \( B_\alpha \)-summand of \( H^0(\tau, V) \) containing \( V' \) must contain \( B_\gamma \)-span of \( X_\gamma \). In particular, \( H_\gamma \in U' \).

Since \( \alpha(H_\gamma) = -1 \), we have \( [X_{-\alpha}, H_\gamma] = X_{-\alpha} \). Hence, \( g_{-\alpha} \) must be a sub space of \( U' \).

Therefore, by induction applying to the simple root \( \alpha \), \( U' \) must either contain \( sl_{2,\alpha} \) or it must contain the indecomposable \( B_\alpha \)-module \( \mathbb{C} \cdot h \oplus g_{-\alpha} \) for some \( h \in h \) such that \( \alpha(h) = 1 \) and \( \nu(h) = 0 \) for every simple root \( \nu \) different from \( \alpha \). In either cases, \( U' \) contains a vector \( h' \) of \( h \) which is linearly independent to \( H_\gamma \) and \( \alpha(h') = 1 \).

Hence, we can find a vector \( h \) in the vector subspace spanned by \( h' \) and \( H_\gamma \) such that \( \gamma(h) = 1 \) and \( \nu(h) = 0 \) for every simple root different from \( \gamma \). Therefore, \( \mathbb{C} \cdot h \) is a \( B_\alpha \)-direct summand of \( U' \). Hence, we see that \( H^0(s_\alpha, \mathbb{C} \cdot h) = \mathbb{C} \cdot h \) is a subspace of \( H^0(s_\alpha, U') \).

Thus, the \( B_\gamma \)-span \( \mathbb{C} \cdot h \oplus g_{-\gamma} \) of \( \mathbb{C} \cdot h \) is a subspace of \( H^0(s_\alpha, U') \). Using **SES**, we conclude that \( \mathbb{C} \cdot h \oplus g_{-\gamma} \) is a \( B_\gamma \)-direct summand of \( H^0(\tau, V) \).

**Case 3:**

We assume that \( \langle \gamma, \alpha \rangle = 0 \).

Proof in this case is similar but actually simpler than that of **Case 2**. 

\[ \square \]

Let \( G \) be simply laced.

Let \( V \) be a \( B \)-sub module of \( g \) containing \( h \).

**Lemma 3.5.** Let \( \tau \in W \). Then, we have \( H^i(\tau, V) = (0) \) for every \( i \geq 1 \).

**Proof.** Proof is by induction on \( l(\tau) \).

If \( l(\tau) = 0 \), we are done.
Otherwise, we choose a simple root $\gamma \in S$ be such that $l(s_\gamma \tau) = l(\tau) - 1$.

By lemma(3.4), every indecomposable $B_\gamma$-summand $V'$ of $H^0(s_\gamma \tau, V)$ must be one of the 5 types given in lemma(3.4).

Hence, using lemma(2.3), we conclude that $H^i(P_\gamma / B, V')$ is zero for every indecomposable $B_\gamma$-summand $V'$ of $H^0(s_\gamma \tau, V)$ and for every $i \geq 1$.

Thus, we have shown that

\begin{equation*}
\text{Observation :}
\end{equation*}

\begin{equation*}
H^i(P_\gamma / B, H^0(s_\gamma \tau, V)) = (0) \text{ for all } i \geq 1.
\end{equation*}

By induction on $l(\tau)$, we have $H^i(s_\gamma \tau, V)$ is zero for all $i \geq 1$. Now, using Observation and using the short exact sequence SES of $B$ modules, we conclude that $H^i(\tau, V)$ is zero for all $i \geq 1$.

This completes the proof of lemma.

\[\square\]

Let $V_1$ be a $B$-sub module of $g$ containing $b$. Let $V_2$ be a $B$-sub module of $V_1$ containing $b$.

Let $\tau \in W$. The natural projection $\Pi : V_1 \longrightarrow V_1/V_2$ of $B$-modules induces a homomorphism of $B$-modules $\Pi_\tau : H^0(\tau, V_1) \longrightarrow H^0(\tau, V_1/V_2)$ of $B$-modules.

We now deduce the following lemma as a consequence of the lemma(3.5).

**Lemma 3.6.**

1. $H^i(\tau, V_1/V_2)$ is zero for all $i \geq 1$.

2. $\Pi_\tau : H^0(\tau, V_1) \longrightarrow H^0(\tau, V_1/V_2)$ is a surjective homomorphism of $B$-modules whose kernel is $H^0(\tau, V_2)$.

**Proof.** Proof of (1):

We have the short exact sequence $0 \longrightarrow V_2 \longrightarrow V_1 \longrightarrow V_1/V_2 \longrightarrow 0$ of $B$-modules.

Applying $H^i(\tau, -)$ to this short exact sequence of $B$-modules, we obtain the following long exact sequence of $B$-modules:

\begin{equation*}
\cdots H^i(\tau, V_2) \longrightarrow H^i(\tau, V_1) \longrightarrow H^i(\tau, V_1/V_2) \longrightarrow H^{i+1}(\tau, V_2) \cdots
\end{equation*}

By lemma(3.5), $H^i(\tau, V_2)$, $H^i(\tau, V_1)$ and $H^{i+1}(\tau, V_2)$ are all zero for every $i \geq 1$. Thus, we conclude that $H^i(\tau, V_1/V_2) = (0)$ for every $i \geq 1$.

This proves (1).
Proof of (2):

Taking $i = 0$ in Observation and using $H^1(\tau, V_2) = (0)$, we obtain the following short exact sequence

$$(0) \longrightarrow H^0(\tau, V_2) \longrightarrow H^0(\tau, V_1) \longrightarrow H^0(\tau, V_1/V_2) \longrightarrow (0).$$

This proves (2).

We have

Corollary 3.7. Let $\tau \in W$. Let $\alpha$ be a positive root. Then, $H^i(\tau, \alpha) = (0)$ for every $i \geq 1$.

Proof. Let $V_1 := \bigoplus_{\mu \leq \alpha} g_\mu$ denote the direct sum of the weight spaces of $g$ of weights $\mu$ satisfying $\mu \leq \alpha$.

Let $V_2 := \bigoplus_{\mu < \alpha} g_\mu$ denote the direct sum of the weight spaces of $g$ of weights $\mu$ satisfying $\mu < \alpha$.

It is clear that $V_2$ is a $B$-sub module of $g$ containing $b$ and $V_1$ is a $B$-sub module of $g$ containing $V_2$.

Since $g_\alpha$ is one dimensional and is isomorphic to the quotient $V_1/V_2$, we have

$H^i(\tau, \alpha) = H^i(\tau, g_\alpha) = H^i(\tau, V_1/V_2)$ for every $i \geq 1$. Hence, by lemma(3.6), $H^i(\tau, \alpha) = (0)$ for every $i \geq 1$.

This completes the proof of corollary.

We now prove the following theorem.

Let $\tau \in W$. Let $\alpha_0$ denote the highest root.

Then, we have

Theorem 3.8. Let $G$ be simple, simply connected and simply laced algebraic group over $\mathbb{C}$. Let $\tau \in W$.

1. $H^i(X(\tau), \mathcal{T}_{G/B}) = (0)$ for every $i \geq 1$.

2. $H^0(X(\tau), \mathcal{T}_{G/B})$ is the adjoint representation $g$ of $G$ if and only if the set of semi-stable points $X(\tau^{-1})_{ss}(\mathcal{L}_{\alpha_0})$ is non-empty.

Proof. Since the tangent space of $G/B$ at the point $idB$ is $g/b$, the tangent bundle $\mathcal{T}_{G/B}$ is the homogeneous vector bundle $\mathcal{L}(g/b)$ on $G/B$ associated to the $B$-module $g/b$.

Hence, it is sufficient to prove the following:

1. $H^i(\tau, g/b) = (0)$ for every $i \geq 1$. 

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2. $H^0(\tau, g/b)$ is the adjoint representation $g$ of $G$ if and only if the set of semi stable points $X(\tau^{-1})^{ss}_T(\mathcal{L}_{\alpha_0})$ is non-empty.

We prove this now.

Let $V_1 := g$ and let $V_2 := b$. The natural projection $\Pi : g \rightarrow g/b$ of $B$-modules induces a homomorphism $\Pi_T : H^0(\tau, g) \rightarrow H^0(\tau, g/b)$ of $B$-modules.

Proof of (1) follows from lemma(3.6(1)).

Since the evaluation map $ev : H^0(\tau, g) \rightarrow g$ is an isomorphism, in order to prove (2), it is sufficient to prove that the kernel of the linear map $\Pi_T : H^0(\tau, g) \rightarrow H^0(\tau, g/b)$ is zero if and only if $X(\tau^{-1})^{ss}_T(\mathcal{L}_{\alpha_0})$ is non-empty.

By ([10], lemma(2.1)), $X(\tau^{-1})^{ss}_T(\mathcal{L}_{\alpha_0})$ is non empty if and only if $-\alpha_0 \in \tau(R^+)$. Hence, we have

Observation 1: $X(\tau^{-1})^{ss}_T(\mathcal{L}_{\alpha_0})$ is non-empty if and only if $-\alpha_0 \in \tau(R^+)$. On the other hand from lemma(3.6(2)), we have $Ker(\Pi_T) = H^0(\tau, b)$. Hence, using lemma(3.3), we see that $Ker(\Pi_T)$ is a $B$-submodule of $b$.

Since there is a unique $B$-stable line in $b$ and that is of weight $-\alpha_0$, we conclude that $Ker(\Pi_T)$ is a non-zero $B$-submodule of $b$ if and only if the $-\alpha_0$-weight space of $H^0(\tau, b)$ is non zero.

Hence, $Ker(\Pi_T)$ is non-zero if and only if $-\alpha_0 \in \tau(R^-)$. Reformulating this statement, we have:

$Ker(\Pi_T)$ is zero if and only if $-\alpha_0 \in \tau(R^+)$.

Using Observation 1, we see that $Ker(\Pi_T)$ is zero if and only if the set of semi-stable points $X(\tau^{-1})^{ss}_T(\mathcal{L}_{\alpha_0})$ is non-empty.

This completes the proof of (2).

Let $\tau \in W$. Let $\alpha_0$ denote the highest root.

Let $h^0(\tau, \alpha)$ denote the character of the $T$-module $H^0(\tau, \alpha)$.

Corollary 3.9. $\sum_{\alpha \in R^+} h^0(\tau, \alpha) = \text{Char}(g)$ if and only if the set of semi-stable points $X(\tau^{-1})^{ss}_T(\mathcal{L}_{\alpha_0})$ is non-empty.

Proof. Let $U^+$ denote the unipotent radical of $B^+$. Let $u^+$ denote the Lie algebra of $U^+$.

Since the natural map $u^+ \rightarrow g/b$ is an isomorphism, there is a total ordering $\{\beta_1, \beta_2, \ldots, \beta_N\}$ of positive roots $R^+$ such that the $B$-module $g/b$ has a filtration of sub modules $V_0 := g/b \supset$
$V_1 \supset V_2 \supset \cdots \supset V_{N-1} \supset V_N = (0)$ with each successive quotients $V_i/V_{i+1}$ is one dimensional and is isomorphic to $\mathfrak{g}_{\beta_i}$. Hence, we have $H^j(\tau, V_i/V_{i+1}) = H^j(\tau, \beta_i)$.

Using corollary (3.4), we have $H^j(\tau, V_i/V_{i+1}) = 0$ for every $j \geq 1$ and for every $i = 1, 2, \cdots, N - 1$.

Hence, $H^0(\tau, \mathfrak{g}/\mathfrak{b})$ has a filtration of $B$-sub modules $H^0(\tau, \mathfrak{g}/\mathfrak{b}) \supset H^0(\tau, V_1) \supset H^0(\tau, V_2) \supset \cdots \supset H^0(\tau, V_{N-1}) \supset (0)$ such that each successive quotient $H^0(\tau, V_i)/H^0(\tau, V_{i+1})$ is isomorphic to $H^0(\tau, \beta_i)$.

Hence, we have

$\textbf{Observation 2}$ \hspace{1cm} $\text{Char}(H^0(\tau, \mathfrak{g}/\mathfrak{b})) = \sum_{i=1}^N \text{Char}(H^0(\tau, \beta_i)) = \sum_{\alpha \in R^+} \text{Char}(H^0(\tau, \alpha))$.

Using lemma (2.5(1)), we have $H^0(\tau, \mathfrak{g}) = \mathfrak{g}$ when ever $l(\tau) = 1$. Now, using induction on $l(\tau)$, and again lemma (2.5(1)) successively, we conclude that $H^0(\tau, \mathfrak{g}) = \mathfrak{g}$ for every $\tau \in W$.

Therefore, we have $\text{Char}(H^0(\tau, \mathfrak{g})) = \text{Char}(\mathfrak{g})$. Hence, using Step 1, we see that $\text{Char}(\mathfrak{g}) = \text{Char}(H^0(\tau, \mathfrak{g}/\mathfrak{b}))$ if and only if $X(\tau^{-1})^{ss}_T(\mathcal{L}_\alpha)$ is non-empty.

Proof of theorem follows using Observation 2 in the above statement.

\[ \square \]

4 Schubert varieties related to maximal parabolic subgroups:

In this section, we apply the main theorem to certain Schubert varieties related to maximal parabolic subgroups of $G$. For a precise statement, see theorem (4.2).

**Lemma 4.1.** Let $w \in W$, and let $\beta$ be a positive root. Let $\gamma$ be a simple root $\gamma$ such that $l(ws_\gamma) = l(w) - 1$ and $\langle \beta, \gamma \rangle = -1$. Then, we have $H^i(w, \beta) = 0$ for every $i \geq 0$.

**Proof.** Proof of this lemma follows from lemma (2.2(3)).

\[ \square \]

Let $\alpha \in S$. Let $Q_\alpha$ denote the maximal parabolic subgroup of $G$ containing $B$ all $s_\beta$, where $\beta$ running over all simple roots different from $\alpha$.

Let $w_\alpha$ denote the unique minimal representative of the longest element $w_0$ of $W$ with respect to the maximal parabolic subgroup $Q_\alpha$.

Recall that $R^+(\tau) := \{ \beta \in R^+ : \tau(\beta) \in -R^+ \}$.

Let $\tau \in W$ be such that $\tau \geq w_\alpha$. The following theorem describe the character of $\mathfrak{g}$ in terms of the sum of characters $h^\alpha(\beta)$ of $H^0(\tau, \beta)$, $\beta$ running over all elements of $R^+(\tau)$.

**Theorem 4.2.** For any $\tau \geq w_\alpha$, we have $\sum_{\beta \in R^+(\tau)} h^\alpha(\tau, \beta)) = \text{Char}(\mathfrak{g})$.
Proof. Step 1:

We first show that \( \sum_{\beta \in R^+} \text{Char}(H^0(\tau, \beta)) = \text{Char}(\mathfrak{g}) \).

Since \( w_\alpha(\omega_\alpha) = w_0(\omega_\alpha) \), it must be a non trivial negative dominant character of \( T \).

Since \( \alpha_0 \geq \nu \) for every simple root \( \nu \), \( \langle w_\alpha(\omega_\alpha), \alpha_0 \rangle \leq -1 \). Since the form \( \langle , \rangle \) is \( W \)-invariant, \( \langle \omega_\alpha, w_\alpha^{-1}(\alpha_0) \rangle \leq -1 \). Hence, \( w_\alpha^{-1}(\alpha_0) \) must be a negative root.

Using ([10], lemma(2.1)), we conclude that \( X(w_\alpha^{-1})^s \tau_s(\mathcal{L}_{\alpha_0}) \) is non empty.

By theorem (3.8), we conclude that \( \sum_{\beta \in R^+(\tau)} \text{Char}(H^0(\tau, \beta)) = \text{Char}(\mathfrak{g}) \).

This proves Step 1.

Step 2:

We now show that \( H^0(\tau, \beta) = (0) \) for every \( \beta \notin R^+(\tau) \).

We first note that a \( \beta \in R^+ \) belongs to \( R^+(w_\alpha) \) if and only if \( \alpha \leq \beta \).

So, the highest root \( \alpha_0 \) lies in the set \( R^+(w_\alpha) \).

Proof of Step 2 is by descending induction on \( l(\tau) \).

Since \( \tau \geq w_\alpha \) and since \( R^+(w_\alpha) = \{ \nu \in R^+ : \nu \geq \alpha \} \), we have \( R^+(w_\alpha) \subset R^+(\tau) \).

Now, since \( \beta \notin R^+(\tau) \), we have \( \beta \notin \alpha \) So, \( \beta \) must be different from \( \alpha_0 \). Hence, there is a simple root \( \gamma \) such that \( \langle \beta, \gamma \rangle = -1 \).

If \( l(\tau s_\gamma) = l(\tau) - 1 \), we have \( \langle \beta, \gamma \rangle = -1 \). By using lemma(4.1), we see that \( H^0(\tau, \beta) = (0) \).

Otherwise, we have \( l(\tau s_\gamma) = l(\tau) + 1 \). So, we have \( \tau s_\gamma \geq w_\alpha \) and \( l(w_\alpha) - l(\tau s_\gamma) = l(w_\alpha) - l(\tau) - 1 \). Now, since \( s_\gamma(\beta) \notin R^+(\tau s_\gamma) \), using induction on \( l(w_\alpha) - l(\tau) \), we have

**Observation:**

\( H^0(\tau s_\gamma, s_\gamma(\beta)) = (0) \).

We now consider the following short exact sequence of \( B \)-modules:

\[
(0) \rightarrow \mathbb{C}_\beta \rightarrow H^0(s_\gamma, s_\gamma(\beta)) \rightarrow \mathbb{C}_{s_\gamma(\beta)} \rightarrow (0).
\]

Applying \( H^0(\tau, \cdot) \) to this short exact sequence and using corollary (3.4), we obtain the following short exact sequence of \( B \)-modules:

\[
(0) \rightarrow H^0(\tau, \beta) \rightarrow H^0(\tau, H^0(s_\gamma, s_\gamma(\beta))) \rightarrow H^0(\tau, s_\gamma(\beta)) \rightarrow (0).
\]

Since \( H^0(\tau, H^0(s_\gamma, s_\gamma(\beta))) = H^0(\tau s_\gamma, s_\gamma(\beta)) \), the above short exact sequence can be written as:

\[
(0) \rightarrow H^0(\tau, \beta) \rightarrow H^0(\tau s_\gamma, s_\gamma(\beta)) \rightarrow H^0(\tau, s_\gamma(\beta)) \rightarrow (0).
\]
Now, from Observation, we have $H^0(\tau s_\gamma, s_\gamma(\beta)) = (0)$. Using this in the short exact sequence, we conclude that $H^0(\tau, \beta) = (0)$.

This proves Step 2.

Proof of theorem follows from Step 1 and Step 2.

Let $G$ be a simple, simply connected and simply laced algebraic group over $\mathbb{C}$. Let $\alpha$ be a simple root.

Let $Q_\alpha$ be the maximal parabolic sub group of $G$ containing $B$ and all $s_\beta$, $\beta$ running over all simple roots different from $\alpha$.

We derive the following corollary as an application of theorem(4.2).

Corollary 4.3. Let $\mathcal{T}_{G/Q_\alpha}$ denote the tangent bundle of $G/Q_\alpha$. Then, we have

1. $H^i(G/Q_\alpha, \mathcal{T}_{G/Q_\alpha}) = (0)$ for every $i \geq 1$.
2. $H^0(G/Q_\alpha, \mathcal{T}_{G/Q_\alpha})$ is the adjoint representation $\mathfrak{g}$ of $G$.

Proof. Let $w_\alpha$ denote the unique minimal representative of the longest element $w_0$ of $W$ with respect to the maximal parabolic subgroup $Q_\alpha$.

Let $\phi$ denote the birational morphism from $X(w_\alpha)$ onto $G/Q_\alpha$ given by the composition of the natural projection $p : G/B \rightarrow G/Q_\alpha$ and the inclusion $X(w_\alpha) \hookrightarrow G/B$.

Since the direct image $\phi_*(\mathcal{O}_{X(w_\alpha)})$ of the structure sheaf $\mathcal{O}_{X(w_\alpha)}$ of $X(w_\alpha)$ is the structure sheaf $\mathcal{O}_{G/Q_\alpha}$ of $G/Q_\alpha$ and all higher direct images are zero, we see that $\phi^* : H^i(G/Q_\alpha, \mathcal{T}_{G/Q_\alpha}) \rightarrow H^i(X(w_\alpha), \phi^*(\mathcal{T}_{G/Q_\alpha}))$ is an isomorphism for every $i \geq 0$.

Let $\mathfrak{q}_\alpha$ denote the Lie algebra of $Q_\alpha$. Then, $p^*(\mathcal{T}_{G/Q_\alpha})$ is actually the homogeneous vector bundle on $G/B$ associated to the $B$-module $\mathfrak{g}/\mathfrak{q}_\alpha$.

Let $Q_\alpha^+$ be the parabolic subgroup of $G$ opposite to $Q_\alpha$ containing $B^+$ and all $s_\beta$, $\beta$ running over all simple roots different from $\alpha$.

Since the Lie algebra of unipotent radical of $Q_\alpha^+$ is $\bigoplus_{\beta \in R^+(w_\alpha)} \mathfrak{g}_\beta$, we can use arguments similar to proof of corollary(3.9) to obtain the following:

There is a total ordering $\{\beta_1, \beta_2, \cdots \beta_m\}$ of positive roots in $R^+(w_\alpha)$ such that the $B$-module $\mathfrak{g}/\mathfrak{q}_\alpha$ has a filtration of sub modules $V_0 := \mathfrak{g}/\mathfrak{q}_\alpha \supset V_1 \supset V_2 \supset \cdots V_{m-1} \supset V_m = (0)$ with each successive quotients $V_i/V_{i+1}$ is one dimensional and is isomorphic to $\mathfrak{g}_{\beta_i}$.

Hence, we have $H^j(\tau, V_i/V_{i+1}) = H^j(\tau, \beta_i)$.

Using corollary(3.4), we have $H^j(\tau, V_i/V_{i+1}) = (0)$ for every $j \geq 1$ and for every $i = 1, 2, \cdots N - 1$.

This completes proof of (1).
Proof of (2) follows from theorem(4.2).

5 Top cohomology module $H^l(\tau)(\tau, \tau^{-1} \cdot 0)$

Throughout this section, we assume that $G$ is a simple, simply connected and simply laced algebraic group over $\mathbb{C}$.

In this section, we show that for any $\tau \in W$ the top cohomology $H^l(\tau)(\tau, \tau^{-1} \cdot 0)$ is the one dimensional trivial representation of $B$. We also prove that for a given Coxeter element $c$ of $W$, $H^i(c, c^{-1} \cdot 0)$ is zero for every $i \neq l(c)$ if and only if both $X(c)^{ss}_T(L_{\alpha_0})$ and $X(c^{-1})^{ss}_T(L_{\alpha_0})$ are non-empty.

We first obtain some application of theorem(4.2) in the following subsection.

5.1 Yang-Zelevinsky’s proposition on Coxeter elements

In this subsection, we obtain a corollary on Schubert varieties $X(c^j)$ corresponding to some power $c^j$ of any given Coxeter element $c$ as an application of theorem(4.2). In the proof of this corollary, we use a Proposition about Coxeter elements by Yang and Zelevinsky. See [12, Proposition(1.3)].

We first recall that an element $c$ of $W$ is said to be a Coxeter element if it has a reduced expression of the form $c = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_l}$, where $i_j \neq i_k$ whenever $j \neq k$ and $l$ is the rank of $G$.

We now state the following proposition from [12].

**Proposition 5.1** (Yang-Zelevinsky). Let $c$ be a Coxeter element. Let $\alpha$ be a simple root. Then, there is a $j \in \mathbb{N}$ such that $c^j(\omega_\alpha) = w_0(\omega_\alpha)$.

We now use this proposition and theorem(4.2) to obtain the following corollary.

Let $c$ be a Coxeter element.

**Corollary 5.2.** Then, there is a $j \in \mathbb{N}$ such that $\sum_{\beta \in R^+(c^j)} h^0(c^j, \beta) = \text{Char}(g)$.

**Proof.** We first fix a simple root $\alpha$. By proposition(5.1), there is a $j \in \mathbb{N}$ such that $c^j(\omega_\alpha) = w_0(\omega_\alpha)$. For this choice of $j$, we have $c^j \geq w_\alpha$. Proof of corollary follows from theorem(4.2) by taking $\tau = c^j$. 

1
We found some interesting facts about Coxeter elements in the study of torus quotients. For instance, see [[10], theorem(4.2)] and see [[11], theorem(3.3)].

We now obtain the following corollary from [[10], theorem(4.2)] in the context of the character of \( g \).

Let \( c \) be a Coxeter element. Let \( h \) denote the order of the Coxeter element \( c \). Let \( C \) denote the cyclic subgroup of \( W \) generated by \( c \). Let \( C' \) denote the complement subset of the singleton set \( \{ id \} \) in \( C \). That is, let \( C' := \{ c^j : j = 1, 2, \ldots h - 1 \} \).

Then, we have

**Corollary 5.3.** \( \sum_{\tau \in C'} \sum_{\beta \in R^+(\tau)} h^0(\tau, \beta) = (h - 1)\text{Char}(g) \) if and only if both \( X(c)^{ss}_T(L_{\alpha_0}) \) and \( X(c^{-1})^{ss}_T(L_{\alpha_0}) \) are non-empty.

**Proof.** Of Necessary condition:

We first prove that if \( \sum_{\tau \in C'} \sum_{\beta \in R^+(\tau)} h^0(\tau, \beta) = (h - 1)\text{Char}(g) \), then, both \( X(c)^{ss}_T(L_{\alpha_0}) \) and \( X(c^{-1})^{ss}_T(L_{\alpha_0}) \) are non-empty.

By lemma(3.6), \( H^0(\tau, g/b) \) is a quotient of \( g \). So, the character of \( H^0(\tau, g/b) \) must be less than or equal to the character of \( g \). Further, by theorem(3.8), if \( \sum_{\beta \in R^+(\tau)} h^0(\tau, \beta) = \text{Char}(g) \) then, \( X(\tau^{-1})^{ss}_T(L_{\alpha_0}) \) is non-empty.

Hence, if \( \sum_{\tau \in C'} \sum_{\beta \in R^+(\tau)} h^0(\tau, \beta) = (h-1)\text{Char}(g) \), then, we must have \( \sum_{\beta \in R^+(c^{-1})} h^0(c^{-1}, \beta) = \text{Char}(g) \) and \( \sum_{\beta \in R^+(c)} h^0(c, \beta) = \text{Char}(g) \).

Hence, by using above arguments, we conclude that both \( X(c)^{ss}_T(L_{\alpha_0}) \) and \( X(c^{-1})^{ss}_T(L_{\alpha_0}) \) are non-empty.

This proves **Necessary condition**. **Proof of Sufficient condition:**

We now prove that if both \( X(c)^{ss}_T(L_{\alpha_0}) \) and \( X(c^{-1})^{ss}_T(L_{\alpha_0}) \) are non-empty, then, \( \sum_{\tau \in C'} \sum_{\beta \in R^+(\tau)} h^0(\tau, \beta) = (h - 1)\text{Char}(g) \).

Now, if both \( X(c)^{ss}_T(L_{\alpha_0}) \) and \( X(c^{-1})^{ss}_T(L_{\alpha_0}) \) are non-empty, then by [10, theorem (4.2)], we have \( G \) must be of type \( A_n \) and either \( c = s_{a_n} s_{a_{n-1}} \cdots s_{a_1} \) or \( c^{-1} = s_{a_n} s_{a_{n-1}} \cdots s_{a_1} \).

With out loss of generality, we may assume that \( c = s_{a_n} s_{a_{n-1}} \cdots s_{a_1} \). Now, a simple calculation shows that \( c^r = w_{a_r} \) for every \( r = 1, 2, \cdots n \).

Hence, we have \( C' = \{ w_{a_r} : r = 1, 2, \cdots n \} \).

Now, proof of **Sufficient condition** follows by using theorem(4.2).

This completes the proof of corollary.

\(\square\)

Let \( c \) be a Coxeter element of \( W \). We choose an ordering \( \{\alpha_1, \alpha_2, \cdots \alpha_n\} \) of simple roots such that \( c = s_{a_n} s_{a_{n-1}} \cdots s_{a_2} s_{a_1} \) is a reduced expression for \( c \).
By proposition (5.1), for every \( j \in \{1, 2, \cdots n\} \), there is a positive integer \( m_j \) such that \( c^{m_j}(\omega_{\alpha_j}) = w_0(\alpha_j) \). For this choice of \( m_j \), we have

\[
1 = \langle \omega_{\alpha_j}, \alpha_j \rangle = \langle c^{m_j}(\omega_{\alpha_j}), c^{m_j}(\alpha_j) \rangle = \langle w_0(\omega_{\alpha_j}), c^{m_j}(\alpha_j) \rangle.
\]

Hence, we see that \( c^{m_j}(\alpha_j) \) is a negative root.

To prove theorem C, we now proceed as follows:

Let \( J' \) denote the set of all integers \( j \) in \( \{1, 2, \cdots n\} \) for which there is a positive integer \( a_j \) such that \( c^{a_j}(\alpha_j) \) is a simple root for every \( i = 0, 1, 2, \cdots a_j - 1 \) and \( c^{a_j}(\alpha_j) \) is a negative root. Let \( J \) denote the set of all elements \( j \) in \( J' \) such that \( c^{-1}(\alpha_j) \) is not a simple root.

The following three lemmas describe some properties of the set \( J \) which will be used in the proof of theorem (5.7).

**Lemma 5.4.** Let \( i \) and \( j \) be two distinct elements of \( \{1, 2, \cdots n\} \). Then, \( c(\alpha_i) = \alpha_j \) if and only if the following holds:

1. \( j \) is the unique element in \( \{1, 2, \cdots i - 1\} \) such that \( \langle \alpha_j, \alpha_i \rangle \neq 0 \).

2. \( i \) is the unique element in \( \{j + 1, j + 2, \cdots n\} \) such that \( \langle \alpha_j, \alpha_i \rangle \neq 0 \).

**Proof.** Proof follows by our chosen ordering \( \{\alpha_1, \alpha_2, \cdots \alpha_n\} \) of simple roots such that \( c = s_{\alpha_n}s_{\alpha_{n-1}} \cdots s_{\alpha_2}s_{\alpha_1} \) is a reduced expression for \( c \). \( \square \)

We also have

**Lemma 5.5.** Let \( i \) and \( j \) be two distinct elements of \( J \). Then, we have \( \langle c^p(\alpha_j), c^q(\alpha_k) \rangle = 0 \) for any two distinct elements \( j \) and \( k \) of \( J \) and for every \( p = 0, 1, 2, \cdots a_j - 1 \) and for every \( q = 0, 1, 2, \cdots a_k - 1 \).

**Proof.** By the definition of \( J \), we can see that

**Observation 1:**

\( c^m(\alpha_j) \neq \alpha_k \) for any \( m = 0, 1, \cdots a_j - 1 \) and \( c^m(\alpha_k) \neq \alpha_j \) for any \( m = 0, 1, \cdots a_k - 1 \).

For instance, we can prove **Observation** as follows:

Since \( k \in J \), we see that \( c^t(\alpha_k) \) is a simple root for every \( t = 0, 1, \leq m \). On the other hand, since \( j \in J \), \( c^{-1}(\alpha_j) \) is not simple. Thus, we have \( p = q \). This is a contradiction as \( j \neq k \).

We now proceed to prove the lemma.

With out loss of generality, we may assume that \( p \leq q \).
Hence, we have
\[ \langle c^p(\alpha_j), c^q(\alpha_k) \rangle = \langle \alpha_j, c^{q-p}(\alpha_k) \rangle. \]

If \( \langle \alpha_j, c^{q-p}(\alpha_k) \rangle \) is non-zero, either we have \( \alpha_j = c^{q-p}(\alpha_k) \) or \( \langle \alpha_j, c^{q-p}(\alpha_k) \rangle = -1 \).
\( \alpha_j = c^{q-p}(\alpha_k) \) is not possible by Observation 1.

So, we may assume that \( \langle \alpha_j, c^{q-p}(\alpha_k) \rangle = -1 \). Let \( c^{q-p}(\alpha_k) = \alpha_t \) for some \( t \in \{1, 2, \cdots n\} \).
With out loss of generality, we may assume that \( j < t \). Since \( \langle \alpha_j, \alpha_t \rangle \) is non-zero, \( c(\alpha_t) \) must be positive. Since \( q-p \leq a_k - 1 \), \( c(\alpha_t) \) must be a simple root.
Since \( \langle \alpha_j, \alpha_t \rangle \) is non-zero, we have \( c(\alpha_t) = \alpha_j \). Hence, we have \( c^{q+1-p}(\alpha_k) = \alpha_j \). This is not possible by Observation 1.

This completes proof of the lemma.

For any \( j \in J \), we take \( \phi_j = s_{\alpha_j} s_{c(\alpha_j)} \cdots s_{c^{a_j-1}(\alpha_j)} \).
Then, we have

**Lemma 5.6.**

1. \( \phi_j \) commutes with \( \phi_k \) for any \( j \) and \( k \) in \( J \).

2. Let \( \phi \) denote the product \( \prod_{j \in J} \phi_j \). Then, we can write \( c = \tau \phi \) such that \( l(c) = l(\phi) + l(\tau) \).

3. Let \( r \in \{1, 2, \cdots n\} \) be such that \( s_{\alpha_r} \leq \tau \). Then, we have \( \text{height}(c(\alpha_r)) \geq \text{height}(\phi(\alpha_r)) \).

**Proof.**

Proof of (1):
Proof of (1) follows from lemma(5.5).

Proof of (2):
Proof of (2) follows from the fact that \( R^+(\phi) \subset R^+(c) \).

Proof of (3):
Since \( s_{\alpha_r} \leq \tau \), we have \( s_{\alpha_r} \leq c \) and \( s_{\alpha_r} \notin \phi \). Hence, \( \phi(\alpha_r) \) is a positive root.
Since \( s_{\alpha_r} \notin \phi \), \( c(\alpha_r) \neq c^j(\alpha_j) \) for any \( j \in J \) and for any \( t = 0, 1, \cdots a_j - 1 \).
If \( \phi(\alpha_r) = \alpha_r \), then, we have
\[
\text{height}(c(\alpha_r)) = 1 = \text{height}(\alpha_r) = \text{height}(\phi(\alpha_r)).
\]
Further, if \( c(\alpha_r) \) is a simple root, then, \( \phi(\alpha_r) = \alpha_r \). Proof is similar to the above case.

So, we may assume that \( c(\alpha_r) \) is not a simple root and \( \phi(\alpha_r) \neq \alpha_r \). Hence, we can use lemma(5.4) to conclude that either there are two distinct elements \( j \) and \( k \) in \( \{1, 2, \cdots r-1\} \).
such that both $\langle \alpha_j, \alpha_r \rangle$ and $\langle \alpha_k, \alpha_r \rangle$ are non-zero or there is a positive integer $j < r$ such that $s_{\alpha_j} \leq \phi$ and a positive integer $k > j$, $k \neq r$ such that both $\langle \alpha_j, \alpha_r \rangle$ and $\langle \alpha_k, \alpha_j \rangle$ are non-zero.

If there are two distinct elements $j$ and $k$ in $\{1, 2, \cdots r - 1\}$ such that both $\langle \alpha_j, \alpha_r \rangle$ and $\langle \alpha_k, \alpha_r \rangle$ are non-zero, then, we must have $c(\alpha_r) \geq \phi(\alpha_r)$. Hence, we have \( \text{height}(c(\alpha_r)) \geq \text{height}(\phi(\alpha_r)) \).

Otherwise, we use lemma(5.4) to conclude that $c(\alpha_k) \neq \alpha_j$. Hence, we have $s_{\alpha_k} \not\in \phi$. Thus, we have $c(\alpha_r) + \alpha_r \geq \phi(\alpha_r) + \alpha_k$.

Hence, we have $\text{height}(c(\alpha_r)) \geq \text{height}(\phi(\alpha_r))$.

This completes the proof of (3).

\[\square\]

### 5.2 Proof of theorem C

In this subsection, we prove theorem C as follows:

**Theorem 5.7.** 1. Let $\tau \in W$. The cohomology module $H^{l(\tau)}(\tau, \tau^{-1} \cdot 0)$ is the one dimensional trivial representation of $B$.

2. Let $c$ be a Coxeter element of $W$. Then, $H^i(c, c^{-1} \cdot 0)$ is zero for every $i \neq l(c)$ if and only if both $X(c)^{ss}{\tau}(\mathcal{L}_{\alpha_0})$ and $X(c^{-1})^{ss}{\tau}(\mathcal{L}_{\alpha_0})$ are non-empty.

**Proof.** Proof of (1):

Since $\tau \cdot \tau^{-1} \cdot 0 = 0$, by the Borel-Weil-Bott’s theorem, $H^{l(\tau)}(w_0, \tau^{-1} \cdot 0)$ is the one dimensional trivial representation of $G$. On the other hand, by [9, Proposition (4.2)], $H^{l(\tau)}(\tau, \tau^{-1} \cdot 0)$ is non-zero.

By [9, corollary (4.3)], the restriction map $H^{l(\tau)}(w_0, \tau^{-1} \cdot 0) \rightarrow H^{l(\tau)}(\tau, \tau^{-1} \cdot 0)$ is surjective. Thus, $H^{l(\tau)}(\tau, \tau^{-1} \cdot 0)$ is the one dimensional trivial representation of $G$. This proves (1).

**Proof of (2):**

**Proof of sufficient condition:**

We first prove that if both $X(c)^{ss}{\tau}(\mathcal{L}_{\alpha_0})$ and $X(c^{-1})^{ss}{\tau}(\mathcal{L}_{\alpha_0})$ are non-empty, then, $H^i(c, c^{-1} \cdot 0)$ is zero for every $i \neq l(c)$.

We now assume that both $X(c)^{ss}{\tau}(\mathcal{L}_{\alpha_0})$ and $X(c^{-1})^{ss}{\tau}(\mathcal{L}_{\alpha_0})$ are non-empty.

Then by [10, theorem (4.2)], we have $G$ must be of type $A_n$ and either $c = s_{\alpha_n} s_{\alpha_{n-1}} \cdots s_{\alpha_1}$ or $c^{-1} = s_{\alpha_n} s_{\alpha_{n-1}} \cdots s_{\alpha_1}$.

With out loss of generality, we may assume that $c = s_{\alpha_n} s_{\alpha_{n-1}} \cdots s_{\alpha_1}$.
Step 1} We show that $c^r = w_{\alpha_r}$ for every $r = 1, 2, \cdots n$.

Using a simple computation, we see that $c(\alpha_1) = -(\sum_{t=1}^{n} \alpha_t)$ and that $c(\alpha_j) = \alpha_{j-1}$ for every $j = 2, 3, \cdots n$.

Now, let $\overline{m}$ denote the remainder when $m$ is divided by $n + 1$.

Using recursion on $r$, we can show that $c^r(\alpha_r) = -(\sum_{t=1}^{n} \alpha_t)$ and $c^r(\alpha_j) = \alpha_{n+1+j-r}$ for every $j \neq r$.

Hence, we have $R^+(c^r) = \{ \beta \in R^+ : \beta \geq \alpha_r \}$ for every $r = 1, 2, \cdots n$. On the otherhand, we have $R^+(w_{\alpha_r}) = \{ \beta \in R^+ : \beta \geq \alpha_r \}$. Thus, we have $c^r = w_{\alpha_r}$ for every $r = 1, 2, \cdots n$.

This proves Step 1.

We consider the natural projection $\pi_r : G/B \rightarrow G/Q_{\alpha_r}$ given by $\pi_r(xB) = xQ_{\alpha_r}$.

Since $R^+(w_{\alpha_r}) = \{ \beta \in R^+ : \beta \geq \alpha_r \}$, $w_{\alpha_r}^{-1} \cdot 0$ is equal to $\sum_{\beta \geq \alpha_r} -\beta$ of all negative roots $-\beta$ such that $\beta \geq \alpha_r$.

On the otherhand, we have $\sum_{\beta \geq \alpha_r} -\beta$ is equal to the multiple $(n + 1)\omega_{\alpha_r}$ of the fundamental weight $\omega_{\alpha_r}$ corresponding to the simple root $\alpha_r$.

Since $L_{-(n+1)\omega_{\alpha_r}}$ is the canonical line bundle on $G/Q_{\alpha_r}$, $H^i(G/Q_{\alpha_r}, L_{-(n+1)\omega_{\alpha_r}})$ vanishes for every $i \neq dim(G/Q_{\alpha_r})$. Since $dim(G/Q_{\alpha_r}) = l(w_{\alpha_r})$, $H^i(G/Q_{\alpha_r}, L_{-(n+1)\omega_{\alpha_r}})$ vanishes for every $i \neq l(w_{\alpha_r})$.

Thus, we have

Observation 1: $H^i(G/Q_{\alpha_r}, L_{w_{\alpha_r}^{-1}})$ vanishes for every $i \neq l(w_{\alpha_r})$.

Since the restriction of $\pi_r$ to $X(w_{\alpha_r})$ is a birational morphism, the pull back map $\pi^*_r : H^1(w_{\alpha_r}, w_{\alpha_r}^{-1} \cdot 0) \rightarrow H^1(G/Q_{\alpha_r}, L_{w_{\alpha_r}^{-1}})$ is an isomorphism for every $i$.

Proof of sufficient condition follows from Observation 1.

Proof of necessary condition:

We now prove the necessary condition.

Let $c$ be a Coxeter element of $W$ such that $H^i(c, c^{-1} \cdot 0)$ is zero for every $i \neq l(c)$. Firstly, we can find an ordering $\{\alpha_1, \alpha_2, \cdots \alpha_n\}$ of simple roots such that $c = s_{\alpha_n}s_{\alpha_{n-1}} \cdots s_{\alpha_2}s_{\alpha_1}$ is a reduced expression for $c$.

Let $J$ be as in subsection 5.1. For any $j \in J$, we take $\phi_j = s_{\alpha_j}c(\alpha_j) \cdots s_{\alpha_j^{-1}(\alpha_j)}$ as in subsection 5.1. By lemma(5.6), we see that $\phi_j$ commutes with $\phi_k$ for any $j$ and $k$ in $J$.

As in lemma(5.6), we denote the product $\Pi_{j \in J} \phi_j$ of the $\phi_j$’s by $\phi$. as in lemma(5.6), we can write $c$ as a product $c = \tau \phi$ with $l(c) = l(\phi) + l(\tau)$.

Claim:

We first show that $c^{-1} \cdot 0 - \phi_j^{-1} \cdot 0$- weight space of $H^{l(\phi_j)}(\phi_j, c^{-1} \cdot 0)$ is non-zero.

We first note that $-\phi_j^{-1} \cdot 0$ is equal to the sum of all positive roots $\beta$ in $R^+(\phi_j)$. The set
$R^+ (\phi_j)$ consists of precisely the roots of the form $\sum_{i=r}^{a_j-1} c^i (\alpha_j)$, $r$-running over all integers from 0, 1, $\cdots$, $a_j - 1$.

Since $c^i (\alpha_j)$ is a simple root for every $i = 0, 1, 2, \cdots a_j - 1$ and $c^{a_j} (\alpha_j)$ is a negative root, we have $c^{a_j} (\alpha_j) \leq - \sum_{i=0}^{a_j-1} c^i (\alpha_j)$. On the other hand, we have $c^{-1} \cdot 0 = c^{-1}(\rho) - \rho$.

Hence, we have

$$\langle c^{-1} \cdot 0, c^{a_j-1} (\alpha_j) \rangle = \langle \rho, c(c^{a_j-1}(\alpha_j)) \rangle - \langle \rho, c^{a_j-1}(\alpha_j) \rangle \leq \langle \rho, \sum_{i=0}^{a_j-1} c^i (\alpha_j) \rangle - 1 = -a_j - 1.$$

For simplicity of notation, we let $\gamma_i = c^i (\alpha_j)$ for every $i = 0, 1, 2, \cdots a_j - 1$.

By lemma(2.3), $c^{-1} \cdot 0 + a_j \gamma_{a_j-1}$-weight space of $H^1(s_{\gamma_{a_j-1}}, c^{-1} \cdot 0)$ is non-zero.

Further, we have,

$$\langle c^{-1} \cdot 0 + a_j \gamma_{a_j-1}, \gamma_{a_j-2} \rangle = \langle \rho, c(\gamma_{a_j-2}) \rangle - \langle \rho, \gamma_{a_j-2} \rangle - a_j = -a_j.$$

Since each $\gamma_{a_j-2}$-string of weights of $H^1(s_{\gamma_{a_j-1}}, c^{-1} \cdot 0)$ is of length one, each indecomposable $B_{\gamma_{a_j-2}}$-module is one dimensional.

In particular, the one dimensional $\mathbb{C}_{c^{-1} \cdot 0 + a_j \gamma_{a_j-1}}$ is $B_{\gamma_{a_j-2}}$-direct summand of $H^1(s_{\gamma_{a_j-1}}, c^{-1} \cdot 0)$.

Using the same argument again, we see that the one dimensional space $\mathbb{C}_{c^{-1} \cdot 0 + a_j \gamma_{a_j-1} + (a_j - 1) \gamma_{a_j-2}}$ is $B_{\gamma_{a_j-3}}$-direct summand of $H^2(s_{\gamma_{a_j-2}} s_{\gamma_{a_j-1}}, c^{-1} \cdot 0)$.

Proceeding recursively, we can show that $c^{-1} \cdot 0 - \sum_{i=0}^{a_j-1} (i + 1) \gamma_i$- weight space of $H^l(\phi_j)(\phi_j, c^{-1} \cdot 0)$ is non-zero.

Since $\sum_{i=0}^{a_j-1} (i+1) \gamma_i = \phi_j^{-1} \cdot 0$, it follows that $c^{-1} \cdot 0 - \phi_j^{-1} \cdot 0$- weight space of $H^l(\phi_j)(\phi_j, c^{-1} \cdot 0)$ is non-zero.

Using the same process, we can show that $c^{-1} \cdot 0 - \phi^{-1} \cdot 0$- weight space of $H^l(\phi)(\phi, c^{-1} \cdot 0)$ is non-zero.

This proves the Claim.

We now prove that the $c^{-1} \cdot 0 - \phi^{-1} \cdot 0$- weight space of $H^l(\phi)(c, c^{-1} \cdot 0)$ is non-zero.

Let $r \in \{1, 2, \cdots n\}$ be such that $s_{\alpha_r} \leq \tau$. That is, $s_{\alpha_r} \leq c$ but $s_{\alpha_r} \not\geq \phi$.

$$\langle c^{-1} \cdot 0 - \phi^{-1} \cdot 0, \gamma_r \rangle = \langle \rho, c(\alpha_r) \rangle - \langle \rho, \phi(\alpha_r) \rangle = \text{height}(c(\alpha_r)) - \text{height}(\phi(\alpha_r)).$$

Hence, using lemma(5.6), we see that $\text{height}(c(\alpha_r)) \geq \text{height}(\phi(\alpha_r))$.

Thus, we conclude that the line bundle $\mathcal{L}_{c^{-1} \cdot 0 - \phi^{-1} \cdot 0}$ corresponding to the one dimensional $B_{\alpha_r}$-module $\mathbb{C}_{c^{-1} \cdot 0 - \phi^{-1} \cdot 0}$ is an effective line bundle on $P_{\alpha_r}/B$.
Hence, \( C_{c^{-1} \cdot 0 - \phi^{-1} \cdot 0} \) is a \( B_{\alpha_t} \)-direct summand of \( H^0(s_{\alpha_t}, H^l(\phi, c^{-1} \cdot 0)) \) for every \( t \neq r \) such that \( s_{\alpha_t} \leq \tau \).

Using the same argument recursively, we conclude that the \( c^{-1} \cdot 0 - \phi^{-1} \cdot 0 \)-weight space of \( H^0(\tau, H^l(\phi, c^{-1} \cdot 0)) \) is non-zero.

Using \( SES \), we see that \( c^{-1} \cdot 0 - \phi^{-1} \cdot 0 \)-weight space of \( H^l(\phi, c^{-1} \cdot 0) \) is non-zero.

On the other hand, by hypothesis, we have \( H^i(c, c^{-1} \cdot 0) = 0 \) for every \( i \neq l(c) \). Hence, \( J \) can have only point \( \{ j \} \) as the \( \phi_i \)'s commute with each other.

Further, by our chosen reduced expression \( c = s_{\alpha_n} s_{\alpha_{n-1}} \cdots s_{\alpha_2} s_{\alpha_1} \) for \( c, j \) must be \( n \) and \( c^k(\alpha_n) = \alpha_n - k \) for every \( k = 0, 1, \ldots n - 1 \).

Hence \( G \) must be of type \( A_n \) and the ordering of the simple roots is simply the ordering in its Dynkin diagram.

This proves the necessary condition.

This completes the proof of theorem.

\[ \square \]

Let \( c \) be a Coxeter element of \( W \). Let \( C \) denote the cyclic subgroup of \( W \) generated by \( c \). Then, the order of \( C \) is the Coxeter number and we denote it by \( h \). The sum \( \sum_{\tau \in \mathcal{C}} \sum_{i=0}^{l(\tau)} h^i(\tau, \tau^{-1} \cdot 0) \) of the characters \( h^i(\tau, \tau^{-1} \cdot 0) \) of the cohomology modules \( H^j(\tau, \tau^{-1} \cdot 0) \), \( \tau \) running over all elements of \( C \) and \( j \) running over all integers from \( \{0, 1, 2, \ldots l(\tau)\} \) is an element of the representation ring \( \mathbb{Z}[X(T)] \) of \( T \).

The following corollary is another application of our main theorem.

**Corollary 5.8.** \( \sum_{\tau \in \mathcal{C}} \sum_{i=0}^{l(\tau)} h^i(\tau, \tau^{-1} \cdot 0) \) is equal to the Coxeter number \( h \) if and only if both \( X(c)_T^s(L_{\alpha_0}) \) and \( X(c^{-1})_T^s(L_{\alpha_0}) \) are non-empty.

**Proof.** By theorem(5.5(1)), \( H^l(\tau, \tau^{-1} \cdot 0) \) is the one dimensional trivial representation of \( B \) for every element \( \tau \) in \( W \). Therefore, the sum \( \sum_{\tau \in \mathcal{C}} \sum_{i=0}^{l(\tau)} h^i(\tau, \tau^{-1} \cdot 0) \) is equal to \( h \).

Again using theorem (5.5(2)), we see that \( \sum_{\tau \in \mathcal{C}} \sum_{i=0}^{l(\tau)-1} h^i(\tau, \tau^{-1} \cdot 0) \) is zero if and only both \( X(c)_T^s(L_{\alpha_0}) \) and \( X(c^{-1})_T^s(L_{\alpha_0}) \) are non-empty.

This completes proof of corollary.

\[ \square \]

### 6 Proof of theorem B

Throughout this section, we assume that \( G \) is simple, simply connected algebraic group over \( \mathbb{C} \) which is not simply laced.
We first prove that when $G$ is not simply laced, then there is a positive root $\beta$ and a simple root $\alpha$ such that $s_\alpha \cdot \beta$ is the highest short root.

**Lemma 6.1.** Let $G$ be a simple algebraic group which is not simply laced. Then, there is a positive root $\beta$ and a simple root $\alpha$ such that $s_\alpha \cdot \beta$ is the highest short root.

**Proof.** If $G$ is of type $G_2$, then, the simple roots $\alpha_1$ and $\alpha_2$ satisfy the following:

$$\langle \alpha_1, \alpha_2 \rangle = -1 \text{ and } \langle \alpha_2, \alpha_1 \rangle = -3.$$  

Here, we follow the convention in [6].

In this case, we take $\beta = \alpha_2$ and $\alpha = \alpha_1$. Hence, $s_\alpha \cdot \beta = \alpha_2 + 2\alpha_1$ is the highest short root.

Hence, we may assume that $G$ is a simple algebraic group of type $B_n$, $C_n$ or $F_4$.

Let $\nu$ be the highest short root. We now show that there is a simple root $\alpha$ such that $\nu + \alpha$ is a root and $\langle \nu, \alpha \rangle = 0$.

To show that $\nu + \alpha$ is a root, it is sufficient to show that the weight space $\mathfrak{g}_{\nu + \alpha}$ is non-zero.

On the other hand, $\mathfrak{g}_{\nu + \alpha}$ is non-zero is a statement independent of the characteristic. So, we may assume that $k = \mathbb{C}$. Hence, $\mathfrak{g}$ is an irreducible $G$-module.

Hence, $\mathfrak{g}_{\alpha_0}$ is the unique $B^+$-stable line in $\mathfrak{g}$. Hence, there is a simple root $\alpha$ such that $ad_{\mathfrak{h}_\alpha}(g_{\nu})$ is non-zero. Thus, $\nu + \alpha$ is a root.

Since $\nu$ is dominant, we have

**Observation 1**

$$\langle \nu, \alpha \rangle \geq 0.$$  

On the other hand, since $G$ is not of type $G_2$, $\langle \nu + \alpha, \alpha \rangle \leq 2$. Hence, we have $\langle \nu, \alpha \rangle \leq 0$.

By Observation, we have $\langle \nu, \alpha \rangle = 0$.

Proof of the lemma follows by taking $\beta = s_\alpha \cdot \nu$.

Let $\alpha$ and $\beta$ be as in lemma(6.1).

Let $\tau \in W$ be such that $s_\alpha \leq \tau$.

Let $V = \bigoplus_{\mu \leq \beta} \mathfrak{g}_\mu$ be the direct sum of all $T$-weight spaces of weights $\mu$ satisfying $\mu \leq \beta$. Clearly $V$ is a $B$-sub module of $\mathfrak{g}$ containing $\mathfrak{b}$.

Then, we have

**Lemma 6.2.** The $s_\alpha \cdot \beta$- weight space of $H^1(\tau, V)$ is a non-zero.

**Proof.** Since $s_\alpha \cdot \beta$ is the highest short root, $s_\alpha \cdot \beta$ is dominant character of $T$. Hence, by the Borel-Weil-Bott’s theorem, $H^1(w_0, s_\alpha \cdot \beta)$ is an irreducible representation of $G$ with highest weight $s_\alpha \cdot \beta$.  

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On the other hand, by [9, Proposition (4.2)], \( H^1(s_\alpha, s_\alpha \cdot \beta) \) is non-zero.

By [9, corollary (4.3)], the restriction map 
\[ H^1(w_0, s_\alpha \cdot \beta) \to H^1(s_\alpha, s_\alpha \cdot \beta) \] is surjective.

Thus, the restriction map 
\[ H^1(\tau, s_\alpha \cdot \beta) \to H^1(s_\alpha, s_\alpha \cdot \beta) \] is also surjective.

This completes proof the lemma.

\[ \square \]

Let \( G \) be a simple, simply connected algebraic group over \( \mathbb{C} \) which is not simply laced.

Let \( V_1 \) be a \( B \)-sub module of \( g \) containing \( b \). Let \( V_2 \) be a \( B \)-sub module of \( V_1 \) containing \( b \). Let \( \tau \in W \).

The natural projection \( \Pi : V_1 \to V_1/V_2 \) of \( B \)-modules induces a homomorphism of \( B \)-modules \( \Pi_\tau : H^0(\tau, V_1) \to H^0(\tau, V_1/V_2) \) of \( B \)-modules.

We now deduce the following lemma as a consequence of the above lemma.

Let \( \tau \in W \). Let \( \gamma \) be a simple root. Let \( V \) be a \( B \)-sub module of \( g \) containing \( b \). Then, we have the following lemma on indecomposable sub modules of \( H^1(\tau, V) \) similar to lemma(3.4) except that (2) and (5) are possible only if \( \gamma \) is a short root.

**Lemma 6.3.** Every indecomposable \( B_\gamma \)- summand \( V' \) of \( H^1(\tau, V) \) must be one of the following:

1. \( V' = C \cdot H \) for some \( H \in h \) such that \( \gamma(H) = 0 \).

2. \( V' = C \cdot H \bigoplus g_{-\gamma} \) for some \( H \in h \) such that \( \gamma(H) = 1 \) and \( \nu(H) = 0 \) for every simple root \( \nu \) different from \( \gamma \).

3. \( V' = g_\beta \) for some short root \( \beta \) different from \( \gamma \).

4. \( V' = g_\beta \bigoplus g_{\beta-\gamma} \) for some short root \( \beta \).

5. \( V' = sl_{2,\gamma} \), the three dimensional irreducible \( L_\gamma \)-module with highest weight \( \gamma \).

**Proof.** Proof is by induction on \( l(\tau) \).

If \( l(\tau) = 0 \), then, \( \tau = id \) and so we are done.

Otherwise, choose a simple root \( \alpha \) such that \( l(\tau) = 1 + l(s_\alpha \tau) \).
By induction, every indecomposable $B_\gamma$-summand $V'$ of $H^1(s_\alpha \tau, V)$ must be one of the above mentioned 5 types.

We first assume that $\gamma = \alpha$.

Now, using lemma(2.3(1)), we see that if $V'$ is one of the types (1), (4), and (5), then, $H^0(s_\gamma, V')$ must be of the same type in $H^0(s_\gamma, H^1(s_\gamma \tau, V))$. In $V'$ is of type (2), using lemma(2.3(3)), we see that $H^0(s_\gamma, V') = (0)$.

In type (3), using lemma(2.3), we see that $H^0(s_\gamma, V')$ is either zero or is one of the types (3) or (4).

We may therefore assume that $\alpha \neq \gamma$.

In this case, we use lemma(2.3) to see that if $V'$ is of type different from type (5), then $H^0(s_\alpha, V')$ must be of the same type in $H^0(s_\alpha, H^1(s_\alpha \tau, V))$.

If $V'$ is of type (5), we again use lemma (2.3) to conclude that $H^0(s_\alpha, V')$ must be of type (2) in $H^0(s_\alpha, H^1(s_\alpha \tau, V))$.

(Here, we use the induction hypothesis that $V'$ can be of type(5) only if $\gamma$ is a short root. Hence, we have $\langle \gamma, \alpha \rangle = -1$)

We recall SES:

$$(0) \rightarrow H^1(s_\gamma, H^0(s_\gamma \tau, V)) \rightarrow H^1(\tau, V) \rightarrow H^0(s_\gamma, H^1(s_\gamma \tau, V)) \rightarrow (0).$$

This completes the proof for the case when $\alpha = \gamma$.

We may therefore assume that $\alpha \neq \gamma$.

In this case, we use lemma(2.3) to see that if $V'$ is of type different from type (5), then $H^0(s_\alpha, V')$ must be of the same type in $H^0(s_\alpha, H^1(s_\alpha \tau, V))$.

If $V'$ is of type (5), we again use lemma (2.3) to conclude that $H^0(s_\alpha, V')$ must be of type (2) in $H^0(s_\alpha, H^1(s_\alpha \tau, V))$.

(This is because $\langle \gamma, \alpha \rangle = -1$)

Proof the lemma is completed using SES.

\[\square\]

For any $B$ module $V$, and for any $\tau \in W$, we recall the evaluation map $ev : H^0(\tau, V) \rightarrow V$ by $ev(s) = s(idB)$, the evaluation of the section at the identity coset $idB$ as point in $X(\tau)$.

Then, we have

**Lemma 6.4.** Let $V$ be a $B$-sub module of $\mathfrak{g}$ containing $\mathfrak{b}$. Then, we have

1. The evaluation map $ev : H^0(\tau, V) \rightarrow V$ is injective.
2. \( H^i(\tau, V) \) is zero for all \( i \geq 2 \).

Proof. Proof of (1) is similar to that of lemma(3.3).

Proof of (2):
Proof is by induction on \( l(\tau) \).
If \( l(\tau) = 0 \), we are done.
Otherwise, choose a simple root \( \gamma \in S \) be such that \( l(s_\gamma \tau) = l(\tau) - 1 \).
By lemma(6.3), every indecomposable \( B_\gamma \)-summand \( V' \) of \( H^1(s_\gamma \tau, V) \) must be one of the 5 types given in lemma(6.3).
Hence, using lemma(2.3), we conclude that \( H^i(P_\gamma/B, V') \) is zero for every indecomposable \( B_\gamma \)-summand \( V' \) of \( H^1(s_\gamma \tau, V) \) and for every \( i \geq 1 \).

Hence, we have \( H^i(P_\gamma/B, H^1(s_\gamma \tau, V)) = (0) \) and for every \( i \geq 1 \).
Since \( \dim(P_\gamma/B) = 1 \), \( H^i(P_\gamma/B, H^0(s_\gamma \tau, V)) = (0) \) for every \( i \geq 2 \).
Thus, we have shown that

Observation 1

1. \( H^i(P_\gamma/B, H^1(s_\gamma \tau, V)) = (0) \) for all \( i \geq 1 \).
2. \( H^i(P_\gamma/B, H^0(s_\gamma \tau, V)) = (0) \) for every \( i \geq 2 \).

By induction on \( l(\tau) \), we have

Observation 2

\( H^i(s_\gamma \tau, V) \) is zero for all \( i \geq 2 \).

Now, using Observation 1, we conclude that \( H^i(\tau, V) \) is zero for all \( i \geq 2 \).

We recall SES: For every \( i \geq 1 \), we have:

\[
(0) \rightarrow H^1(s_\gamma, H^{i-1}(s_\gamma \tau, V)) \rightarrow H^i(\tau, V) \rightarrow H^0(s_\gamma, H^i(s_\gamma \tau, V)) \rightarrow (0).
\]

Using Observation 1 and Observation 2 in the above short exact sequence, we conclude that \( H^i(\tau, V) = (0) \) for every \( i \geq 2 \).

This completes the proof of (2).

We have the following corollary as an application of the lemma.

Corollary 6.5. Let \( \tau \in W \). Let \( \alpha \) be a positive root. Then, \( H^i(\tau, \alpha) = (0) \) for every \( i \geq 2 \).
Proof. Let $V_1 := \bigoplus_{\mu \leq \alpha} g_{\mu}$ denote the direct sum of the weight spaces of $g$ of weights $\mu$ satisfying $\mu \leq \alpha$.

Let $V_2 := \bigoplus_{\mu < \alpha} g_{\mu}$ denote the direct sum of the weight spaces of $g$ of weights $\mu$ satisfying $\mu < \alpha$.

It is clear that $V_2$ is a $B$-sub module of $g$ containing $b$ and $V_1$ is a $B$-sub module of $g$ containing $V_2$.

Since $g_{\alpha}$ is one dimensional and is isomorphic to the quotient $V_1/V_2$, we have

Observation 1

$H^i(\tau, \alpha) = H^i(\tau, g_{\alpha}) = H^i(\tau, V_1/V_2)$ for every $i \geq 2$.

We have the short exact sequence $(0) \rightarrow V_2 \rightarrow V_1 \rightarrow V_1/V_2 \rightarrow (0)$ of $B$-modules.

Applying $H^i(\tau, -)$ to this short exact sequence of $B$-modules, we obtain the following

Observation 2

$$\cdots \rightarrow H^i(\tau, V_2) \rightarrow H^i(\tau, V_1) \rightarrow H^i(\tau, V_1/V_2) \rightarrow H^{i+1}(\tau, V_2) \rightarrow \cdots$$

By lemma(6.4), $H^i(\tau, V_1)$ and $H^{i+1}(\tau, V_2)$ are all zero for every $i \geq 2$. Using Observation 2, we conclude that $H^i(\tau, V_1/V_2) = (0)$ for every $i \geq 2$.

Proof of corollary follows by applying Observation 1.

We now prove theorem B.

**Theorem 6.6.** Let $G$ be a simple, simply connected but not simply laced algebraic group over $\mathbb{C}$. Let $\tau \in W$. Then, we have

1. $H^i(X(\tau), T_{G/B}) = (0)$ for every $i \geq 1$.

2. The adjoint representation $g$ is a $B$-submodule of $H^0(X(\tau), T_{G/B})$ if and only if the set of semi-stable points $X(\tau^{-1})^{ss}(L_{\alpha})$ is non-empty.

Proof. Proof is similar to that of theorem A.

We provide a proof here for completeness.

Since the tangent space of $G/B$ at the point $idB$ is $g/b$, the tangent bundle $T_{G/B}$ is the homogeneous vector bundle $L(g/b)$ on $G/B$ associated to the $B$-module $g/b$.

Hence, it is sufficient to prove the following:

1. $H^i(\tau, g/b) = (0)$ for every $i \geq 1$. 

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2. The adjoint representation $g$ of $G$ is a $B$-sub module of $H^0(\tau, g/b)$ if and only if the set of semi stable points $X(\tau^{-1})_{ss}^T(\mathcal{L}_{\alpha_0})$ is non-empty.

We prove this now.

Let $V_1 : g$ and let $V_2 := b$. The natural projection $\Pi : g \rightarrow g/b$ of $B$-modules induces a homomorphism $\Pi^\tau : H^0(\tau, g) \rightarrow H^0(\tau, g/b)$ of $B$-modules.

Proof of (1):

We have the short exact sequence $(0) \rightarrow b \rightarrow g \rightarrow g/b \rightarrow (0)$ of $B$-modules.

Applying $H^i(\tau, -)$ to this short exact sequence of $B$-modules, we obtain the following long exact sequence of $B$-modules:

Observation 1

$$\cdots H^i(\tau, b) \rightarrow H^i(\tau, g) \rightarrow H^i(\tau, g/b) \rightarrow H^{i+1}(\tau, b) \cdots$$

On the other hand, by lemma(2.5(2)), we have $H^i(\tau, g) = (0)$ for every $i \geq 1$. Further, by lemma(6.4), we have $H^{i+1}(\tau, b) = (0)$ for every $i \geq 1$. Applying this in the above long exact sequence of $B$-modules, we conclude that $H^i(\tau, g/b) = (0)$ for every $i \geq 1$.

This proves (1).

Proof of 2:

Since $H^0(\tau, g) = g$, in order to prove (2), it is sufficient to prove that the kernel of the linear map $\Pi^\tau : H^0(\tau, g) \rightarrow H^0(\tau, g/b)$ is zero if and only if $X(\tau^{-1})_{ss}^T(\mathcal{L}_{\alpha_0})$ is non-empty.

We now show that the kernel of the linear map $\Pi^\tau : H^0(\tau, g) \rightarrow H^0(\tau, g/b)$ is zero if and only if $X(\tau^{-1})_{ss}^T(\mathcal{L}_{\alpha_0})$ is non-empty.

By [10, lemma(2.1)], $X(\tau^{-1})_{ss}^T(\mathcal{L}_{\alpha_0})$ is non empty if and only if $\tau^{-1}(-\alpha_0) \in R^+$. Hence, we have

Observation 2 $X(\tau^{-1})_{ss}^T(\mathcal{L}_{\alpha_0})$ is non empty if and only if $-\alpha_0 \in \tau(R^+)$. It is easy to see that $Ker(\Pi^\tau) = H^0(\tau, b)$. Hence, using lemma(6.4(1)), we see that $Ker(\Pi^\tau)$ is a $B$-submodule of $b$.

Since there is a unique $B$- stable line in $b$ and that is of weight $-\alpha_0$, we conclude that $Ker(\Pi^\tau)$ is a non-zero $B$-submodule of $b$ if and only if the $-\alpha_0$-weight space of $H^0(\tau, b)$ is non zero.

Hence, $Ker(\Pi^\tau)$ is non-zero if and only if $-\alpha_0 \in \tau(R^-)$.

Reformulating this statement, we have:

$Ker(\Pi^\tau)$ is zero if and only if $-\alpha_0 \in \tau(R^+)$. 

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Using Observation 2, we see that $\text{Ker}(P_{\tau})$ is zero if and only if the set of semi-stable points $X(\tau^{-1})_{s}^{T}(L_{\alpha_0})$ is non-empty.

This completes the proof of (2).

**Remark:** The second statement of theorem A does not hold for an arbitrary $\tau$ in case of $G$ is not simply laced.

*Reason:*

For instance, let $G$ be of type $B_2$. Let $\alpha_1$ and $\alpha_2$ be two simple roots such that $\langle \alpha_1, \alpha_2 \rangle = -2$ and $\langle \alpha_2, \alpha_1 \rangle = -1$.

We can take $\tau = s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}$. We see that $H^1(\tau, b)$ is one dimensional representation $\mathbb{C}_{-(\alpha_1+\alpha_2)}$ of $B$.

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