Avoiding Dark Energy with 1/R Modifications of Gravity

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1 Introduction

The case for alternate gravity is easily made. The best that can be done from observing cosmic motions is to infer the metric \( g_{\mu \nu} \) in some coordinate system. From this one can reconstruct the Einstein tensor and then ask whether or not general relativity predicts it in terms of the observed sources of stress-energy,

\[
(R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R)_{\text{rec}} = 8\pi G (T_{\mu \nu})_{\text{obs}} \quad ? \tag{1}
\]

One way of explaining any disagreement is by positing the existence of an unobserved, “dark” component of the stress-energy tensor,

\[
(T_{\mu \nu})_{\text{dark}} = \frac{1}{8\pi G} (R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R)_{\text{rec}} - (T_{\mu \nu})_{\text{obs}} \tag{2}
\]

This always works, but recent observations make it seem epicyclic.

The theory of nucleosynthesis implies that no more than about 4% of the energy density currently required to make general relativity agree with all observations can consist of any material with which we are presently familiar \[1\] — and only a fraction of this 4% is observed. Just to make general relativity agree with the observed motions of galaxies and galactic clusters we must posit that six times the mass of ordinary matter comes in the form of nonbaryonic, cold dark matter \[2\]. Although there are some plausible candidates for what this might be, no Earth-bound laboratory has yet succeeded in detecting it.

I belong to the minority of physicists who feel that this factor of six already strains credulity. Easing that strain is what led Milgrom to propose MOND \[3\], which can be viewed as a phenomenological modification of gravity in the regime of very small accelerations. There is an impressive amount of observational data in favor of this modification \[4\] — although see \[5\]. Bekenstein has recently constructed a fully relativistic field theory \[6\] which reproduces MOND, and a preliminary analysis of the resulting cosmology works better than many experts thought possible \[7\].
However, the worst problem for conventional gravity comes on the largest scales. To make general relativity agree with the Hubble plots of distant Type Ia supernovae [8, 9, 10], with the power spectrum of anisotropies in the cosmic microwave background [11] and with large scale structure surveys [12], one must accept an additional component of “dark energy” that is about eighteen times larger than that of ordinary matter. This would mean that 96% of the current universe’s energy exists in forms which have so far only been detected gravitationally! Even people who believe passionately in dark matter (and hence accept the factor of six) find this factor of $6+18=24$ difficult to swallow. That is why there has been so much recent interest in modifying gravity to make it predict observed cosmic phenomena without the need for dark energy, and sometimes even without the need for dark matter.

I want to stress that the issue is one of plausibility. There is no problem in inventing field theories which give the required amount of dark energy. The simplest way of doing it is with a minimally coupled scalar [13, 14],

$$\mathcal{L} = -\frac{1}{2} \partial_{\mu} \varphi \partial_{\nu} g^{\mu \nu} \sqrt{-g} - V(\varphi) \sqrt{-g} .$$ (3)

The usual procedure is to begin with a scalar potential $V(\varphi)$ and work out the cosmology, but it is easy to start with whatever cosmological evolution is desired and construct the potential which would support it. I will go through the construction here, both to make the point and so that it can be used later.

On the largest scales the geometry of the universe can be described in terms of a single function of time known as the scale factor $a(t)$,

$$ds^2 = -dt^2 + a^2(t) dx \cdot dx .$$ (4)

The logarithmic time derivative of this quantity gives the Hubble parameter,

$$H(t) \equiv \frac{\dot{a}}{a} .$$ (5)

If we specialize to a solution $\varphi_0(t)$ of the scalar field equations which depends only upon time, the two nontrivial Einstein equations are,

$$3H^2 = 8\pi G \left(\frac{1}{2} \dot{\varphi}_0^2 + V(\varphi_0)\right) ,$$ (6)

$$-2\dot{H} - 3H^2 = 8\pi G \left(\frac{1}{2} \dot{\varphi}_0^2 - V(\varphi_0)\right) .$$ (7)

Let us assume $a(t)$ is known as an explicit function of time, and construct $\varphi_0(t)$ and $V(\varphi)$. By adding (6) and (7) we obtain,

$$-2\dot{H} = 8\pi G\dot{\varphi}_0^2 .$$ (8)

The weak energy condition implies $\dot{H}(t) \leq 0$ so we can take the square root and integrate to solve for $\varphi_0(t)$,
\[ \varphi_0(t) = \varphi_I \pm \int_{t_I}^{t} dt' \sqrt{-\frac{2\dot{H}(t')}{8\pi G}}. \] (9)

One can choose \( \varphi_I \) and the sign freely.

Because the integrand in (9) is always positive, the function \( \varphi_0(t) \) is monotonic. This means we can invert to solve for time as a function of \( \varphi_0 \). Let us call the inverse function \( T(\varphi) \),

\[ \psi = \varphi_0 \left( T(\psi) \right). \] (10)

By subtracting (7) from (6) we obtain a relation for the scalar potential as a function of time,

\[ V = \frac{1}{8\pi G} \left( \dot{H}(t) + 3H^2(t) \right). \] (11)

The potential is determined as a function of the scalar by substituting the inverse function (10),

\[ V(\varphi) = \frac{1}{8\pi G} \left\{ \dot{H} \left( T(\varphi) \right) + 3H^2 \left( T(\varphi) \right) \right\}. \] (12)

This construction gives a scalar which supports any evolution \( a(t) \) (with \( \dot{H}(t) < 0 \)) all by itself. Should you wish to include some other, known component of the stress-energy, simply add the energy density and pressure of this component to the Einstein equations,

\[ 3H^2 = 8\pi G \left( \frac{1}{2} \dot{\varphi}_0^2 + V(\varphi_0) + \rho_{\text{known}} \right), \] (13)

\[ -2\dot{H} - 3H^2 = 8\pi G \left( \frac{1}{2} \dot{\varphi}_0^2 - V(\varphi_0) + p_{\text{known}} \right). \] (14)

Provided \( \rho_{\text{known}} \) and \( p_{\text{known}} \) are known functions of either time or the scale factor, the construction goes through as before.\(^1\)

Using this method one can devise a new field \( \varphi(x) \) which will support any cosmology with \( \dot{H}(t) < 0 \). However, the introduction of such a “quintessence” field raises a number of questions:

1. Where does \( \varphi \) reside in fundamental theory?
2. Why can’t \( \varphi \) couple to fields other than the metric? And if it does couple to other fields, why haven’t we detected its influence in Earth-bound laboratories?
3. Why did \( \varphi \) come to dominate the stress-energy of the universe so recently in cosmological time?
4. Why is the \( \varphi \) field so homogeneous?

\(^1\) This construction seems to be due to Ratra and Peebles [14]. Recent examples of its use include [15] [16] [17].
When a phenomenological fix raises more questions than it answers people are naturally drawn to investigate other fixes. One possibility is that general relativity is not the correct theory of gravity on cosmological scales.

In this talk I shall review gravitational Lagrangians of the form,

$$\mathcal{L} = \frac{1}{16\pi G} \left( R + \Delta R[g] \right) \sqrt{-g},$$  \hspace{1cm} (15)

where $\Delta R[g]$ is some local scalar constructed from the curvature tensor and possibly its covariant derivatives. Examples of such scalars are,

\begin{align*}
\frac{1}{\mu^2} R^{\alpha\beta} R_{\alpha\beta}, & \quad \frac{1}{\mu^2} g^{\mu\nu} R_{\mu\nu}, & \quad \mu^2 \sin \left( \frac{1}{\mu^2} R^{\alpha\beta\rho\sigma} R_{\alpha\beta\rho\sigma} \right). \hspace{1cm} (16)
\end{align*}

I begin by reviewing a powerful no-go theorem which pervades and constrains fundamental theory so completely that most people assume its consequence without thinking. This is the theorem of Ostrogradski [18], who essentially showed why Newton was right to suppose that the laws of physics involve no more than two time derivatives of the fundamental dynamical variables. The key consequence for our purposes is that the only viable form for the functional $\Delta R[g]$ in (15) is an algebraic function of the undifferentiated Ricci scalar,

$$\Delta R[g] = f(R).$$  \hspace{1cm} (17)

I review the Ostrogradski result in section 2, and hopefully immunize you against some common misconceptions about it in section 3. In section 4 I explain why $f(R)$ theories do not contradict Ostrogradski’s result. I also demonstrate that, in the absence of matter, $f(R)$ theories are equivalent to ordinary gravity, with $f(R) = 0$, plus a minimally coupled scalar of the form $\Box \phi$. Then I use the construction given above to show how one can choose $f(R)$ to enforce an arbitrary cosmology. This establishes that an $f(R)$ can be found to support any desired cosmology. In section 5 I discuss problems associated with the particular choice function $f(R) = -\frac{\mu^2}{R}$. Section 6 presents conclusions.

## 2 The Theorem of Ostrogradski

Ostrogradski’s result is that there is a linear instability in the Hamiltonians associated with Lagrangians which depend upon more than one time derivative in such a way that the dependence cannot be eliminated by partial integration [18]. The result is so general that I can simplify the discussion by presenting it in the context of a single, one dimensional point particle whose position as a function of time is $q(t)$. First I will review the way the Hamiltonian is constructed for the usual case in which the Lagrangian involves no higher than first time derivatives. Then I present Ostrogradski’s construction for the case in which the Lagrangian involves second time derivatives. And the section closes with the generalization to $N$ time derivatives.
In the usual case of \( L = L(q, \dot{q}) \), the Euler-Lagrange equation is,
\[
\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0. \tag{18}
\]
The assumption that \( \frac{\partial L}{\partial \dot{q}} \) depends upon \( \dot{q} \) is known as nondegeneracy. If the Lagrangian is nondegenerate we can write \( \ddot{q} = F(q, \dot{q}) \) in the form Newton assumed so long ago for the laws of physics,
\[
\ddot{q} = F(q, \dot{q}) \quad \implies \quad q(t) = Q(t, q_0, \dot{q}_0). \tag{19}
\]
From this form it is apparent that solutions depend upon two pieces of initial value data: \( q_0 = q(0) \) and \( \dot{q}_0 = \dot{q}(0) \).

The fact that solutions require two pieces of initial value data means that there must be two canonical coordinates, \( Q \) and \( P \). They are traditionally taken to be,
\[
Q \equiv q \quad \text{and} \quad P \equiv \frac{\partial L}{\partial \dot{q}}. \tag{20}
\]
The assumption of nondegeneracy is that we can invert the phase space transformation (20) to solve for \( \dot{q} \) in terms of \( Q \) and \( P \). That is, there exists a function \( v(Q, P) \) such that,
\[
\frac{\partial L}{\partial \dot{q}} \bigg|_{q=Q} = P. \tag{21}
\]
The canonical Hamiltonian is obtained by Legendre transforming on \( \dot{q} \),
\[
H(Q, P) \equiv P\dot{q} - L, \tag{22}
\]
\[
= P v(Q, P) - L(Q, v(Q, P)). \tag{23}
\]
It is easy to check that the canonical evolution equations reproduce the inverse phase space transformation (21) and the Euler-Lagrange equation (18),
\[
\dot{Q} \equiv \frac{\partial H}{\partial P} = v + P \frac{\partial v}{\partial \dot{P}} - \frac{\partial L}{\partial \dot{q}} \frac{\partial v}{\partial P} = v, \tag{24}
\]
\[
\dot{P} \equiv -\frac{\partial H}{\partial Q} = -P \frac{\partial v}{\partial Q} + \frac{\partial L}{\partial q} + \frac{\partial L}{\partial \dot{q}} \frac{\partial v}{\partial P} = \frac{\partial L}{\partial q}. \tag{25}
\]
This is what we mean by the statement, “the Hamiltonian generates time evolution.” When the Lagrangian has no explicit time dependence, \( H \) is also the associated conserved quantity. Hence it is “the” energy by anyone’s definition, of course up to canonical transformation.

Now consider a system whose Lagrangian \( L(q, \dot{q}, \ddot{q}) \) depends nondegenerately upon \( \ddot{q} \). The Euler-Lagrange equation is,
\[
\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}} = 0. \tag{26}
\]
Non-degeneracy implies that $\frac{\partial L}{\partial \ddot{q}}$ depends upon $\ddot{q}$, in which case we can cast $q^{(4)}$ in a form radically different from Newton’s,

$$q^{(4)} = \mathcal{F}(q, \dot{q}, \ddot{q}, q^{(3)}) \implies q(t) = Q(t, q_0, \dot{q}_0, \ddot{q}_0, q^{(3)}_0).$$

(27)

Because solutions now depend upon four pieces of initial value data there must be four canonical coordinates. Ostrogradski’s choices for these are,

$$Q_1 \equiv q, \quad P_1 \equiv \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}},$$

$$Q_2 \equiv \dot{q}, \quad P_2 \equiv \frac{\partial L}{\partial \ddot{q}}.$$  

(28)  

(29)

The assumption of nondegeneracy is that we can invert the phase space transformation (28-29) to solve for $\ddot{q}$ in terms of $Q_1, Q_2$ and $P_2$. That is, there exists a function $a(Q_1, Q_2, P_2)$ such that,

$$\frac{\partial L}{\partial \ddot{q}} \bigg|_{q=Q_1, \dot{q}=Q_2, \ddot{q}=a} = P_2.$$  

(30)

Note that one only needs the function $a(Q_1, Q_2, P_2)$ to depend upon three canonical coordinates — and not all four — because $L(q, \dot{q}, \ddot{q})$ only depends upon three configuration space coordinates. This simple fact has great consequence.

Ostrogradski’s Hamiltonian is obtained by Legendre transforming, just as in the first derivative case, but now on $\dot{q} = q^{(1)}$ and $\ddot{q} = q^{(2)}$,

$$H(Q_1, Q_2, P_1, P_2) \equiv \sum_{i=1}^{2} P_i q^{(i)} - L,$$

$$= P_1 Q_2 + P_2 a(Q_1, Q_2, P_2) - L(Q_1, Q_2, a(Q_1, Q_2, P_2)).$$

(31)  

(32)

The time evolution equations are just those suggested by the notation,

$$\dot{Q}_i \equiv \frac{\partial H}{\partial P_i} \quad \text{and} \quad \dot{P}_i \equiv -\frac{\partial H}{\partial Q_i}. \quad (33)$$

Let’s check that they generate time evolution. The evolution equation for $Q_1$,

$$\dot{Q}_1 = \frac{\partial H}{\partial P_1} = Q_2,$$

(34)

reproduces the phase space transformation $\dot{q} = Q_2$ in (29). The evolution equation for $Q_2$,

$$\dot{Q}_2 = \frac{\partial H}{\partial P_2} = a + P_2 \frac{\partial a}{\partial P_2} - \frac{\partial L}{\partial \dot{q}} \frac{\partial a}{\partial P_2} = a,$$

(35)
reproduces (30). The evolution equation for \( P_2 \),
\[
\dot{P}_2 = - \frac{\partial H}{\partial Q_2} = -P_1 - P_2 \frac{\partial a}{\partial \dot{q}} + \frac{\partial L}{\partial q} \frac{\partial a}{\partial \dot{q}} + \frac{\partial L}{\partial \ddot{q}} \frac{\partial a}{\partial Q_2} = -P_1 + \frac{\partial L}{\partial \dot{q}} ,
\]  
(36)
reproduces the phase space transformation \( P_1 = \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{q}} \) (28). And the evolution equation for \( P_1 \),
\[
\dot{P}_1 = - \frac{\partial H}{\partial Q_1} = -P_2 \frac{\partial a}{\partial Q_1} + \frac{\partial L}{\partial q} + \frac{\partial L}{\partial \ddot{q}} \frac{\partial a}{\partial Q_1} = \frac{\partial L}{\partial q} ,
\]  
(37)
reproduces the Euler-Lagrange equation (26). So Ostrogradski’s system really does generate time evolution. When the Lagrangian contains no explicit dependence upon time it is also the conserved Noether current. By anyone’s definition, it is therefore “the” energy, again up to canonical transformation.

There is one, overwhelmingly bad thing about Ostrogradski’s Hamiltonian (32): it is \textit{linear} in the canonical momentum \( P_1 \). This means that no system of this form can be stable. In fact, there is not even any barrier to decay. Note also the power and generality of the result. It applies to \textit{every} Lagrangian \( L(q, \dot{q}, \ddot{q}) \) which depends nondegenerately upon \( \ddot{q} \), independent of the details. The only assumption is nondegeneracy, and that simply means one cannot eliminate \( \ddot{q} \) by partial integration. This is why Newton was right to assume the laws of physics take the form (19) when expressed in terms of fundamental dynamical variables.

Adding more higher derivatives just makes the situation worse. Consider a Lagrangian \( L(q, \dot{q}, \ldots, q^{(N)}) \) which depends upon the first \( N \) derivatives of \( q(t) \). If this Lagrangian depends nondegenerately upon \( q^{(N)} \) then the Euler-Lagrange equation,
\[
\sum_{i=0}^{N} \left( -\frac{d}{dt} \right)^i \frac{\partial L}{\partial q^{(i)}} = 0 ,
\]  
(38)
contains \( q^{(2N)} \). Hence the canonical phase space must have \( 2N \) coordinates. Ostrogradski’s choices for them are,
\[
Q_i \equiv q^{(i-1)} \quad \text{and} \quad P_i \equiv \sum_{j=i}^{N} \left( -\frac{d}{dt} \right)^{j-i} \frac{\partial L}{\partial q^{(j)}} .
\]  
(39)
Non-degeneracy means we can solve for \( q^{(N)} \) in terms of \( P_N \) and the \( Q_i \)’s. That is, there exists a function \( A(Q_1, \ldots, Q_N, P_N) \) such that,
\[
\frac{\partial L}{\partial q^{(N)}} \bigg|_{q^{(N)}=A} = P_N .
\]  
(40)
For general \( N \) Ostrogradski’s Hamiltonian takes the form,
\[ H \equiv \sum_{i=1}^{N} P_i q^{(i)} - L, \quad (41) \]
\[ = P_1 Q_2 + P_2 Q_3 + \cdots + P_{N-1} Q_N + P_N A - L\left(Q_1, \ldots, Q_N, A\right). \quad (42) \]

It is simple to check that the evolution equations,
\[ \dot{Q}_i \equiv \frac{\partial H}{\partial P_i} \quad \text{and} \quad \dot{P}_i \equiv -\frac{\partial H}{\partial Q_i}, \quad (43) \]
again reproduce the canonical transformations and the Euler-Lagrange equation. So (42) generates time evolution. Similarly, it is Noether current for the case where the Lagrangian contains no explicit time dependence. So there is little alternative to regarding (42) as “the” energy, again up to canonical transformation.

One can see from (42) that the Hamiltonian is linear in \( P_1, P_2, \ldots P_{N-1} \).

Only with respect to \( P_N \) might it be bounded from below. Hence the Hamiltonian is necessarily unstable over half the classical phase space for large \( N! \)

3 Common Misconceptions

The no-go theorem I have just reviewed ought to come as no surprise. It explains why Newton was right to expect that physical laws take the form of second order differential equations when expressed in terms of fundamental dynamical variables. Every fundamental system we have discovered since Newton’s day has had this form. The bizarre, dubious thing would be if Newton had blundered upon a tiny subset of possible physical laws, and all our probing over the course of the next three centuries had never revealed the vastly richer possibilities. However — deep sigh — particle theorists don’t like being told something is impossible, and a definitive no-go theorem such as that of Ostrogradski provokes them to tortuous flights of evasion. I ought to know, I get called upon to referee the resulting papers often enough! No one has so far found a way around Ostrogradski’s theorem. I won’t attempt to prove that no one ever will, but let me use this section to run through some of the misconceptions which have been in back of attempted evasions.

To fix ideas it will be convenient to consider a higher derivative generalization of the harmonic oscillator,
\[ L = -\frac{gm}{2\omega^2} q^2 + \frac{m}{2} q^2 - \frac{m\omega^2}{2} q^2. \quad (44) \]

Here \( m \) is the particle mass, \( \omega \) is a frequency and \( g \) is a small positive pure number we can think of as a coupling constant. The Euler-Lagrange equation,
\[-m \left( \frac{g}{\omega^2} q^{(4)} + \ddot{q} + \omega^2 q \right) = 0 \, \]  
(45)

has the general solution,

\[ q(t) = A_+ \cos(k_+t) + B_+ \sin(k_+t) + A_- \cos(k_-t) + B_- \sin(k_-t) \, \]  
(46)

Here the two frequencies are,

\[ k_\pm \equiv \omega \sqrt{\frac{1 \mp \sqrt{1 - 4g}}{2g}} \, \]  
(47)

and the initial value constants are,

\[ A_+ = \frac{k_+^2 q_0 + \dot{q}_0}{k_+^2 - k_+^2} \, , \quad B_+ = \frac{k_+^2 \dot{q}_0 + q_0}{k_+(k_+^2 - k_+^2)} \, , \]  
(48)

\[ A_- = \frac{k_-^2 q_0 + \dot{q}_0}{k_-^2 - k_-^2} \, , \quad B_- = \frac{k_-^2 \dot{q}_0 + q_0}{k_-(k_-^2 - k_-^2)} \, . \]  
(49)

The conjugate momenta are,

\[ P_1 = m \ddot{q} + \frac{gm}{\omega^2} q^{(3)} \quad \Leftrightarrow \quad q^{(3)} = \frac{\omega^2 P_1 - m \omega^2 Q_2}{gm} \, , \]  
(50)

\[ P_2 = -\frac{gm}{\omega^2} \ddot{q} \quad \Leftrightarrow \quad \ddot{q} = -\frac{\omega^2 P_2}{gm} \, . \]  
(51)

The Hamiltonian can be expressed in terms of canonical variables, configuration space variables or initial value constants,

\[ H = P_1 Q_2 - \frac{\omega^2}{2gm} P_2^2 - \frac{m}{2} Q_2^2 + \frac{m \omega^2}{2} Q_1^2 \, , \]  
(52)

\[ = \frac{gm}{\omega^2} q^{(3)} - \frac{gm}{\omega^2} \ddot{q}^2 + \frac{m}{2} \dot{q}^2 + \frac{m \omega^2}{2} q^2 \, , \]  
(53)

\[ = \frac{m}{2} \sqrt{1 - 4g k_+^2} (A_+^2 + B_+^2) - \frac{m}{2} \sqrt{1 - 4g k_-^2} (A_-^2 + B_-^2) \, . \]  
(54)

The last form makes it clear that the “+” modes carry positive energy whereas the “−” modes carry negative energy.

### 3.1 Nature of the Instability

It’s important to understand both how the Ostrogradskian instability manifests and what is physically wrong with a theory which shows this instability. Because the Ostrogradskian Hamiltonian is not bounded below with respect to more than one of its conjugate momenta, one sees that the problem is not reaching arbitrarily negative energies by setting the dynamical variable
to some constant value. Rather it is reaching arbitrarily negative energies by making the dynamical variable have a certain time dependence. People sometimes mistakenly believe they have found a higher derivative system which is stable when all they have checked is that the Hamiltonian is bounded from below for constant field configurations. For example, from expression \( \phi \), we see that our higher derivative oscillator energy is bounded below by zero for \( q(t) = \text{const} \). Negative energies are achieved by making \( \ddot{q} \) large and/or making \( q^{(3)} \) large while keeping \( \dot{q} + gq^{(3)}/\omega^2 \) fixed.

Another crucial point is that the same dynamical variable typically carries both positive and negative energy degrees of freedom in a higher derivative theory. For our higher derivative oscillator this is apparent from expression \( \psi \) which shows that \( q(t) \) involves both the positive energy degrees of freedom, \( A_+ \) and \( B_+ \), and the negative energy ones, \( A_- \) and \( B_- \). And note from expression \( \theta \) that I really mean positive and negative energy, not just positive and negative frequency, which is the usual case in a lower derivative theory.

People sometimes imagine that the energy of a higher derivative theory decays with time. That is not true. Provided one is dealing with a complete system, and provided there is no external time dependence, the energy of a higher derivative system is conserved, just as it would be under those conditions for a lower derivative theory. This conservation is apparent for our higher derivative oscillator from expression \( \phi \).

The physical problem with nondegenerate higher derivative theories is not that their energies decay to lower and lower values. The problem is rather that certain sectors of the theory become arbitrarily highly excited when one is dealing with an interacting, continuum field theory which has nondegenerate higher derivatives. To understand this I must digress to remind you of some familiar facts about the Hydrogen atom.

If you consider Hydrogen in isolation, there is an infinite tower of stationary states. However, if you allow the Hydrogen atom to interact with electromagnetism only the ground state is stationary; all the excited states decay through the emission of a photon. Why is this so? It certainly is not because “the system wants to lower its energy.” The energy of the full system is constant, the binding energy released by the decaying atom being compensated by the energy of the recoil photon. Yet the decay always takes place, and rather quickly. The reason is that decay is terrifically favored by entropy. If we prepare the Hydrogen atom in an excited state, with no photons present, there is one way for the atom to remain excited, whereas there are an infinite number of ways for it to decay because the recoil photon could go off in any direction.

Now consider an interacting, continuum field theory which possesses the Ostrogradskian instability. In particular consider its likely particle spectrum about some “empty” solution in which the field is constant. Because the Hamiltonian is linear in all but one of the conjugate momenta we can increase or decrease the energy by moving different directions in phase space.
Hence there must be both positive energy and negative energy particles — just as there are in our higher derivative oscillator. Just as in that point particle model, the same continuum field must carry the creation and annihilation operators of both the positive and the negative energy particles. If the theory is interacting at all — that is, if its Lagrangian contains a higher than quadratic power of the field — then there will be interactions between positive and negative energy particles. Depending upon the interaction, the empty state can decay into some collection of positive and negative energy particles. The details don’t really matter, all that matters is the counting: there is one way for the system to stay empty versus a continuous infinity of ways for it to decay. This infinity is even worse than for the Hydrogen atom because it includes not only all the directions that recoil particles of fixed energies could go but also the fact that the various energies can be arbitrarily large in magnitude provided they sum to zero. Because of that last freedom the decay is instantaneous. And the system doesn’t just decay once! It is even more entropically favored for there to be two decays, and better yet for three, etc. You can see that such a system instantly evaporates into a maelstrom of positive and negative energy particles. Some of my mathematically minded colleagues would say it isn’t even defined. I prefer to simply observe that no theory of this kind can describe the universe we experience in which all particles have positive energy and empty space remains empty.

Note that we only reach this conclusion if the higher derivative theory possesses both interactions and continuum particles. Our point particle oscillator has no interactions, so its negative energy degree of freedom is harmless. Of course it is also completely unobservable! However, it is conceivable we could couple this higher derivative oscillator to a discrete system without engendering an instability. The feature that drives the instability when continuum particles are present is the vast entropy of phase space. Without that it becomes an open question whether or not there is anything wrong with a higher derivative theory. Of course we live in a continuum universe, and any degree of freedom we can observe must be interacting, so these are very safe assumptions. However, people sometimes delude themselves that there is no problem with continuum, interacting higher derivative models of the universe on the basis of studying higher derivative systems which could never describe the universe because they either lack interactions or else continuum particles.

In this sub-section we have learned:

1. The Ostrogradskian instability does not drive the dynamical variable to a special, constant value but rather to a special kind of time dependence.
2. A dynamical variable which experiences the Ostrogradskian instability will carry both positive and negative energy creation and annihilation operators.
3. If the system interacts then the “empty” state can decay into a collection of positive and negative energy excitations.
4. If the system is a continuum field theory the vast entropy at infinite momentum will make the decay instantaneous.

### 3.2 Perturbation Theory

People sometimes mistakenly believe that the Ostrogradskian instability is avoided if higher derivatives are segregated to appear only in interaction terms. This is not correct if one considers the theory on a fundamental level. One can see from the construction of section 2 that the fact of Ostrogradski’s Hamiltonian being unbounded below depends only upon nondegeneracy, irrespective of how one organizes any approximation technique. However, there is a way of imposing constraints to make the theory agree with its perturbative development. If this is done then there are no more higher derivative degrees of freedom, however, one typically loses unitarity, causality and Lorentz invariance on the nonperturbative level.

I constructed the higher derivative oscillator (44) so that its higher derivatives vanish when \( g = 0 \). If we solve the Euler-Lagrange equation (45) exactly, without employing perturbation theory, there are four linearly independent solutions (46) corresponding to a positive energy oscillator of frequency \( k_+ \) and a negative energy oscillator of frequency \( k_- \). However, we might instead regard the parameter \( g \) as a coupling constant and solve the equations perturbatively. This means substituting the ansatz,

\[
q_{\text{pert}}(t) = \sum_{n=0}^{\infty} g^n x_n(t),
\]

into the Euler-Lagrange equation (45) and segregating terms according to powers of \( g \). The resulting system of equations is,

\[
\ddot{x}_0 + \omega^2 x_0 = 0,
\]

\[
\ddot{x}_0 + \omega^2 x_0 = -\frac{1}{\omega^2} x_0^{(4)},
\]

\[
\ddot{x}_1 + \omega^2 x_1 = -\frac{1}{\omega^2} x_1^{(4)},
\]

and so on. Because the zeroth order equation involves only second derivatives, its solution depends upon only two pieces of initial value data,

\[
x_0(t) = q_0 \cos(\omega t) + \frac{\dot{q}_0}{\omega} \sin(\omega t).
\]

The first correction is,

\[
x_1(t) = -\frac{\omega t}{\omega} q_0 \sin(\omega t) + \frac{t}{2} \dot{q}_0 \cos(\omega t) - \frac{1}{2\omega} \dot{q}_0 \sin(\omega t),
\]

and it is easy to see that the sum of all corrections gives,
\[ q_{\text{pert}}(t) = q_0 \cos(k_+ t) + \frac{q_0}{k_+} \sin(k_+ t) . \]  

(61)

What is the relation of the perturbative solution (61) to the general one (46)? The perturbative solution is what results if we change the theory by imposing the constraints,

\[ \ddot{q}(t) = -k_+^2 q(t) \quad \iff \quad P_2 = \frac{m}{2} \left( 1 - \sqrt{1 - 4g} \right) Q_1 , \]  

(62)

\[ q^{(3)}(t) = -k_+^2 \dot{q}(t) \quad \iff \quad P_1 = \frac{m}{2} \left( 1 + \sqrt{1 - 4g} \right) Q_2 . \]  

(63)

Under these constraints the Hamiltonian becomes,

\[ H_{\text{pert}} = \sqrt{1 - 4g} \left( \frac{m}{2} Q_2^2 + \frac{mk_+^2}{2} Q_1^2 \right) , \]  

(64)

which is indeed that of a single harmonic oscillator. From the full theory, perturbation theory has retained only the solution whose frequency is well behaved for \( g \to 0 \),

\[ k_+ = \omega \left( 1 + \frac{1}{2} g + \frac{7}{8} g^2 + O(g^3) \right) . \]  

(65)

It has discarded the solution whose frequency blows up as \( g \to 0 \),

\[ k_- = \frac{\omega}{\sqrt{g}} \left( 1 - \frac{1}{2} g - \frac{5}{8} g^2 + O(g^3) \right) . \]  

(66)

So what’s wrong with this? In fact there is nothing wrong with the procedure for our model. If the constraints (62-63) are imposed at one instant, they remain valid for all times as a consequence of the full equation of motion. However, that is only because our model is free of interactions. Recall that this same feature means the positive and negative energy degrees of freedom exist in isolation of one another, and there is no decay to arbitrarily high excitation as there would be for an interacting, continuum field theory.

When interactions are present it is more involved but still possible to impose constraints which change the theory so that only the lower derivative, perturbative solutions remain. The procedure was first worked out by Jaén, Llosa and Molina [19], and later, independently, by Eliezer and me [20]. To understand its critical defect suppose we change the “interaction” of our higher derivative oscillator from a quadratic term to a cubic one,

\[ - \frac{gm}{2\omega^3} q^2 \rightarrow - \frac{gm}{6\ell\omega^4} q^3 . \]  

(67)

Here \( \ell \) is some constant with the dimensions of a length. As with the quadratic interaction, the new equation of motion is fourth order,

\[ - m \left[ \frac{d^2}{dt^2} \left( \frac{gq^2}{2\ell\omega^4} \right) + \ddot{q} + \omega^2 q \right] = 0 , \]  

(68)
Its general solution depends upon four pieces of initial value data. However, by isolating the highest derivative term of the free theory,

\[ \ddot{q} = -\omega^2 q - \frac{d^2}{dt^2} \left( \frac{gq^2}{2\ell\omega^4} \right), \]  

and then iteratively substituting (69), we can delay the appearance of higher derivatives on the right hand side to any desired order in the coupling constant \( g \). For example, two iterations frees the right hand side of higher derivatives up to order \( g^2 \),

\[ \ddot{q} = -\omega^2 q - \frac{d^2}{dt^2} \left\{ \frac{g}{2\ell\omega^4} \left[ -\omega^2 q - \frac{d^2}{dt^2} \left( \frac{gq^2}{2\ell\omega^4} \right) \right]^2 \right\}, \]  

\[ = -\omega^2 q + \frac{g}{\ell} \left( \omega^2 q^2 - \dot{q}^2 \right) + \frac{g^2}{2\ell^2\omega^4} q \frac{d^2}{dt^2} \left( \ddot{q}^2 \right) - \frac{g^2}{2\ell^2\omega^4} \frac{d^2}{dt^2} \left[ q \frac{d^2}{dt^2} \left( \ddot{q} \right) \right] - \frac{g^3}{8\ell^3\omega^{12}} \frac{d^2}{dt^2} \left[ \frac{d^2}{dt^2} \left( \ddot{q}^2 \right) \right]. \]  

This obviously becomes complicated fast! However, the lower derivative terms at order \( g^2 \) are simple enough to give if I don’t worry about the higher derivative remainder,

\[ \ddot{q} = -\omega^2 q + \frac{g}{\ell} \left( \omega^2 q^2 - \dot{q}^2 \right) + \frac{g^2}{\ell^2} \left( -6\omega^2 q^3 + 14q\dot{q}^2 \right) + O(g^3). \]  

If we carry this out to infinite order, and drop the infinite derivative remainder, the result is an equation of the traditional form,

\[ \ddot{q} = f(q,\dot{q}). \]  

The canonical version of this equation gives the first of the desired constraints. The second is obtained from the canonical version of its time derivative.

The constrained system we have just described is consistent on the perturbative level, but not beyond. It does not follow from the original, exact equation. That would be no problem if we could define physics using perturbation theory, but we cannot. Perturbation theory does not converge for any known interacting, continuum field theory in 3+1 dimensions! The fact that the constraints are not consistent beyond perturbation theory means there is a nonperturbative amplitude for the system to decay to the arbitrarily high excitation in the manner described in sub-section 3.1. The fact that the constraints treat time derivatives differently than space derivatives also typically leads to a loss of causality and Lorentz invariance beyond perturbation theory.

A final comment concerns the limit of small coupling constant, i.e., \( g \to 0 \). One can see from the frequencies (65-66) of our higher derivative oscillator that the negative energy frequency diverges for \( g \to 0 \). Disingenuous purveyors of higher derivative models sometimes appeal to people’s experience with positive
energy modes by arguing that, “the $k_-$ mode approaches infinite frequency for small coupling so it must drop out.” That is false! The argument is quite correct for an infinite frequency positive energy mode in a stable theory. In that case exciting the mode costs an infinite amount of energy which would have to be drawn from de-exciting finite frequency modes. However, a negative energy mode doesn’t decouple as its frequency diverges. Rather it couples more strongly because taking its frequency to infinity opens up more and more ways to balance its negative energy by exciting finite frequency, positive energy modes.

3.3 Quantization

People sometimes imagine that quantization might stabilize a system against the Ostrogradskian instability the same way that it does for the Hydrogen atom coupled to electromagnetism. This is a failure to understand correspondence limits. Conclusions drawn from classical physics survive quantization unless they depend upon the system either being completely excluded from some region of the canonical phase space or else inhabiting only a small region of it. For example, the classical instability of the Hydrogen atom (when coupled to electromagnetism) derives from the fact that the purely Hydrogenic part of the energy,

$$E_{\text{Hyd}} = \frac{\| p \|^2}{2m} - \frac{e^2}{\| x \|},$$

(74)
can be made arbitrarily negative by placing the electron close to the nucleus at fixed momentum. Because this instability depends upon the system being in a very small region of the canonical phase space, one might doubt that it survives quantization, and explicit computation shows that it does not.

In contrast, the Ostrogradskian instability derives from the fact that $P_1 Q_2$ can be made arbitrarily negative by taking $P_1$ either very negative, for positive $Q_2$, or else very positive, for negative $Q_2$. This covers essentially half the classical phase space! Further, the variables $Q_2$ and $P_1$ commute with one another in Ostrogradskian quantum mechanics. So there is no reason to expect that the Ostrogradskian instability is unaffected by quantization.

3.4 Unitarity vs. Instability

Particle physicists who quantize higher derivative theories don’t typically recognize a problem with the stability. They maintain that the problem with higher derivatives is a breakdown of unitarity. In this sub-section I will again have recourse to the higher derivative oscillator (44) to explain the connection between the two apparently unrelated problems.

Let us find the “empty” state wavefunction, $\Omega(Q_1, Q_2)$ that has the minimum excitation in both the positive and negative energy degrees of freedom. The procedure for doing this is simple: first identify the positive and negative energy lowering operators $\alpha_{\pm}$ and then solve the equations,
We can recognize the raising and lowering operators by simply expressing the general solution (46) in terms of exponentials,
\[ q(t) = \frac{1}{2} (A_+ + iB_+) e^{-ik_+ t} + \frac{1}{2} (A_+ - iB_+) e^{ik_+ t} + \frac{1}{2} (A_- + iB_-) e^{-ik_- t} + \frac{1}{2} (A_- - iB_-) e^{ik_- t} . \] (76)

Recall that the \( k_+ \) mode carries positive energy, so its lowering operator must be proportional to the \( e^{-ik_+ t} \) term,
\[ \alpha_+ \sim A_+ + iB_+ , \]
\[ \sim \frac{mk_+}{2} \left( 1 + \sqrt{1-4g} \right) Q_1 + iP_1 - k_+ P_2 - \frac{im}{2} \left( 1 - \sqrt{1-4g} \right) Q_2 . \] (78)

The \( k_- \) mode carries negative energy, so its lowering operator must be proportional to the \( e^{+ik_- t} \) term,
\[ \alpha_- \sim A_- - iB_- , \]
\[ \sim \frac{mk_-}{2} \left( 1 - \sqrt{1-4g} \right) Q_1 - iP_1 - k_- P_2 + \frac{im}{2} \left( 1 + \sqrt{1-4g} \right) Q_2 . \] (80)

Writing \( P_i = -i \frac{\partial}{\partial Q_i} \) we see that the unique solution to (75) has the form,
\[ \Omega(Q_1, Q_2) = N \exp \left[ -\frac{m\sqrt{1-4g}}{2(k_+ + k_-)} \left( k_+ k_- Q_1^2 + Q_2^2 \right) - i\sqrt{g}mQ_1 Q_2 \right] . \] (81)

The empty wave function (81) is obviously normalizable, so it gives a state of the quantum system. We can build a complete set of normalized stationary states by acting arbitrary numbers of + and − raising operators on it,
\[ |N_+, N_-\rangle \equiv \left( \alpha_+^{\dagger} \right)^N \left( \alpha_-^{\dagger} \right)^N |\Omega\rangle . \] (82)

On this space of states the Hamiltonian operator is unbounded below, just as in the classical theory,
\[ H|N_+, N_-\rangle = \left( N_+ k_+ - N_- k_- \right) |N_+, N_-\rangle . \] (83)

This is the correct way to quantize a higher derivative theory. One evidence of this fact is that classical negative energy states correspond to quantum negative energy states as well.

Particle physicists don’t quantize higher derivative theories as we just have. What they do instead is to regard the negative energy lowering operator as
a positive energy raising operator. So they define a “ground state” \( |\Omega\rangle \) which obeys the equations,
\[
\alpha_+ |\Omega\rangle = 0 = \alpha_-^\dagger |\Omega\rangle .
\] (84)

The unique wave function which solves these equations is,
\[
|\Omega\rangle(Q_1, Q_2) = N \exp \left[ -\frac{m\sqrt{1-4g}}{2(k_- - k_+)} \left( k_+ k_- Q_1^2 - Q_2^2 \right) + i\sqrt{g} m Q_1 Q_2 \right] .
\] (85)

This wave function is not normalizable, so it doesn’t correspond to a state of the quantum system. At this stage we should properly call a halt to the analysis because we aren’t doing quantum mechanics anymore. The Schrödinger equation \( H\psi(Q) = E\psi(Q) \) is just a second order differential equation. It has two linearly independent solutions for every energy \( E \): positive, negative, real, imaginary, quaternionic — it doesn’t matter. The thing that puts the “quantum” in quantum mechanics is requiring that the solution be normalizable. Many peculiar things can happen if we abandon allow normalizability [21, 22].

However, my particle theory colleagues ignore this little problem and define a completely formal “space of states” based upon \( |\Omega\rangle \),
\[
|N_+, N_-\rangle \equiv \frac{(\alpha_+^\dagger)^{N_+} (\alpha_-)^{N_-}}{\sqrt{N_+! \sqrt{N_-!}}} |\Omega\rangle .
\] (86)

None of these wavefunctions is any more normalizable than \( |\Omega\rangle(Q_1, Q_2) \), so not one of them corresponds to a state of the quantum system. However, they are all positive energy eigenfunctions,
\[
H|N_+, N_-\rangle = (N_+ k_+ + N_- k_-)|N_+, N_-\rangle .
\] (87)

My particle physics colleagues typically say they define \( |\Omega\rangle \) to have unit norm. Because they have not changed the commutation relations,
\[
[\alpha_+, \alpha_+] = 1 = [\alpha_-, \alpha_-^\dagger] ,
\] (88)

the norm of any state with odd \( N_- \) is negative! The lowest of these is,
\[
\langle 0, 1 | 0, 1 \rangle = \langle \Omega | \alpha_+^\dagger \alpha_- | \Omega \rangle = -\langle \Omega \Omega \rangle .
\] (89)

As I pointed out above, the reason this has happened is that we aren’t doing quantum mechanics any more. We ought to use the normalizable, but indefinite energy eigenstates. What particle physicists do instead is to reason that because the probabilistic interpretation of quantum mechanics requires norms to be positive, the negative norm states must be excised from the space of states. At this stage good particle physicists note that that the resulting model fails to conserve probability [23]. Just as the correctly-quantized, indefinite-energy theory allows processes which mix positive and negative energy particles, so too the indefinite-norm theory allows processes which mix
positive and negative norm particles. It only conserves probability on the space of “states” which includes both kinds of norms. If we excise the negative norm states then probability is no longer conserved.

So good particle physicists reach the correct conclusion — that nondegenerate higher derivative theories can’t describe our universe — by a somewhat illegitimate line of reasoning. But who cares? They got the right answer! Of course bad particle physicists regard the breakdown of unitarity as a challenge for inspired tinkering to avoid the problem. Favorite ploys are the Lee-Wick reformulation of quantum field theory \[24\] and nonperturbative resumptions. The analysis also typically involves the false notion that high frequency ghosts decouple, which I debunked at the end of sub-section 3.2. When the final effort is written up and presented to the world, some long-suffering higher derivative expert gets called away from his research to puzzle out what was done and explain why it isn’t correct. Sigh. The problem is so much clearer in its negative energy incarnation! I could list many examples at this point, but I will confine myself to citing a full-blown paper debunking one of them \[25\]. It is also appropriate to note that Hawking and Hertog have previously called attention to the mistake of quantizing higher derivative theories using nonnormalizable wave functions \[26\].

3.5 Constraints

The only way anyone has ever found to avoid the Ostrogradskian instability on a nonperturbative level is by violating the single assumption needed to make Ostrogradski’s construction: nondegeneracy. Higher derivative theories for which the definition of the highest conjugate momentum \[40\] cannot be inverted to solve for the highest derivative can sometimes be stable. An interesting example of this kind is the rigid, relativistic particle studied by Plyushchay \[27, 28\].

Degeneracy is of great importance because all theories which possess continuous symmetries are degenerate, irrespective of whether or not they possess higher derivatives. A familiar example is the relativistic point particle, whose dynamical variable is \(X^\mu(\tau)\) and whose Lagrangian is,

\[
L = -m\sqrt{-\eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu}.
\]  

(90)

The conjugate momentum is,

\[
P_\mu \equiv \frac{m\dot{X}_\mu}{\sqrt{-X^2}}.
\]

(91)

Because the right hand side of this equation is homogeneous of degree zero one can not solve for \(\dot{X}^\mu\). The associated continuous symmetry is invariance under reparameterizations \(\tau \rightarrow \tau'(\tau)\),

\[
X^\mu(\tau) \rightarrow X'^\mu(\tau) \equiv X^\mu(\tau'^{-1}(\tau)).
\]

(92)
The cure for symmetry-induced degeneracy is simply to fix the symmetry by imposing gauge conditions. Then the gauge-fixed Lagrangian should no longer be degenerate in terms of the remaining variables. For example, we might parameterize so that \( \tau = X^0(\tau) \), in which case the gauge-fixed particle Lagrangian is,

\[
L_{\text{GF}} = -m \sqrt{1 - \dot{X} \cdot \dot{X}}.
\] (93)

In this gauge the relation for the momenta is simple to invert,

\[
P_i \equiv \frac{m \dot{X}_i}{\sqrt{1 - \dot{X} \cdot \dot{X}}} \quad \iff \quad \dot{X}^i = \frac{P^i}{\sqrt{m^2 + P \cdot P}}.
\] (94)

When a continuous symmetry is used to eliminate a dynamical variable, the equation of motion of this variable typically becomes a constraint. For symmetries enforced by means of a compensating field — such as local Lorentz invariance with the antisymmetric components of the vierbein \cite{29} — the associated constraints are tautologies of the form \( 0 = 0 \). Sometimes the constraints are nontrivial, but implied by the equations of motion. An example of this kind is the relativistic particle in our synchronous gauge. The equation of the gauge-fixed zero-component just tells us the Hamiltonian is conserved,

\[
\frac{d}{d\tau} \left( \frac{m \dot{X}_0}{\sqrt{-\eta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu}} \right) = 0 \quad \iff \quad \frac{d}{dt} \left( \sqrt{m^2 + P \cdot P} \right) = 0.
\] (95)

And sometimes the constraints give nontrivial relations between the canonical variables that generate residual, time-independent symmetries. In this case another degree of freedom can be removed (“gauge fixing counts twice,” as van Nieuwenhuizen puts it). An example of this kind of constraint is Gauss’ Law in temporal gauge electrodynamics.

Were it not for constraints of this last type, the analysis of a higher derivative theory with a gauge symmetry would be straightforward. One would simply fix the gauge and then check whether or not the gauge-fixed Lagrangian depends nondegenerately upon higher time derivatives. If it did, the conclusion would be that the theory suffers the Ostrogradskian instability. However, when constraints of the third type are present one must check whether or not they affect the instability. This is highly model dependent but a very simple rule seems to be generally applicable: if the number of gauge constraints is less than the number of unstable directions in the canonical phase space then there is no chance for avoiding the problem. Because the number of constraints for any symmetry is fixed, whereas the number of unstable directions increases with the number of higher derivatives, one consequence is that gauge constraints can at best avoid instability for some fixed number of higher derivatives. For example, the constraints of the second derivative model of Plyushchay are sufficient to stabilize the system \cite{27, 28}, but one would expect it to become unstable if third derivatives were added.
People sometimes make the mistake of believing that the Ostrogradskian instability can be avoided with just a single, global constraint on the Hamiltonian. For example, Boulware, Horowitz and Strominger \[30\] showed the energy is zero for any asymptotically flat solution of the higher derivative field equations derived from the Lagrangian,

\[
\mathcal{L} = \alpha R^2 \sqrt{-g} + \beta R^\mu\nu R_{\mu\nu} \sqrt{-g}.
\] (96)

As I explained in sub-section 3.1, the nature of the Ostrogradskian instability is not that the energy decays but rather that the system evaporates to a very highly excited state of compensating, positive and negative energy degrees of freedom. As long as \(\beta \neq 0\), there are six independent, higher derivative momenta at each space point, whereas there are only four local constants — or five if \(\alpha\) and \(\beta\) are such as to give local conformal invariance. Hence there are two (or one) unconstrained instabilities per space point. There are an infinite number of space points, so the addition of a single, global constraint does not change anything. I should point out that Boulware, Horowitz and Strominger were aware of this, cf. their discussion of the dipole instability.

The case of \(\beta = 0\) is special, and significant for the next section. If \(\alpha\) has the right sign that model has long been known to have positive energy \[31, 32\]. This result in no way contradicts the previous analysis. When \(\beta = 0\) the terms which carry second derivatives are contracted in such way that only a single component of the metric carries higher derivatives. So now the counting is one unstable direction per space point versus four local constraints. Hence the constraints can win, and they do if \(\alpha\) has the right sign.

### 3.6 Nonlocality

I would like to close this section by commenting on the implications of Ostrogradski’s theorem for fully nonlocal theories. In addition to nonlocal quantum field theories \[33, 34, 35\] this is relevant to string field theory \[36, 37, 38\], to noncommutative geometry \[39, 40\], to regularization techniques \[41, 42, 43\] and even to theories of cosmology \[15, 44, 45\]. The issue in each case is whether or not we can think of the fully nonlocal theory as the limit of a sequence of ever higher derivative theories. When such a representation is possible the nonlocal theory must inherit the Ostrogradskian instability.

The higher derivative representation is certainly valid for string field theory because, otherwise, there would be cuts and poles that would interfere with perturbative unitarity. So string field theory suffers from the Ostrogradskian instability \[20\]. The same is true for theories where the nonlocality is of limited extent in time \[46\], although not everyone agrees \[47, 48\]. However, when the nonlocality involves inverse differential operators there need be no problem \[20, 49\]. Indeed, the effective action of any quantum field theory is nonlocal in this way \[49, 50\]! Nor is there necessarily any problem when the nonlocality arises in the form of algebraic functions of local actions \[51\].
4. **$\Delta R[g] = f(R)$ Theories**

From the lengthy argumentation of the previous two sections one might conclude that the only potentially stable, local modification of gravity is a cosmological constant, $\Delta R[g] = -2\Lambda$. However, a close analysis of sub-section reveals that it is also possible to consider algebraic functions of the Ricci scalar. In this section I first explain why such theories can avoid the Ostrogradskian instability. I then demonstrate that they are equivalent to general relativity with a minimally coupled scalar, provided we ignore matter. Finally, I exploit this equivalence, with the construction described in the Introduction, to show how $f(R)$ can be chosen to enforce any evolution $a(t)$.

4.1 Why They Can Be Stable

The alert reader will have noted that the $R + R^2$ model avoids the Ostrogradskian instability. It does this by violating Ostrogradski’s assumption of nondegeneracy: the tensor indices of the second derivative terms in the Ricci scalar are contracted together so that only a single component of the metric carries higher derivatives. This component does acquire a new, higher derivative degree of freedom, and the energy of this degree of freedom is indeed opposite to that of the corresponding lower derivative degree of freedom, just as required by Ostrogradski’s analysis. However, that lower derivative degree of freedom is the Newtonian potential. It carries negative energy, but it is also completely fixed in terms of the other metric and matter fields by the $g_{00}$ constraint. So the only instability associated with it is gravitational collapse. Its higher derivative counterpart has positive energy, at least on the kinetic level; it can still have a bad potential, and the model is indeed only stable for one sign of the $R^2$ term.

None of these features depended especially upon the higher derivative term being $R^2$. Any function for the Ricci scalar would work as well. Note that we cannot allow derivatives of the Ricci scalar, because Ostrogradski’s theorem says the next higher derivative degree of freedom would carry negative energy and there would be no additional constraints to protect it. We also cannot permit more general contractions of the Riemann tensor because then other components of the metric would carry higher derivatives. These components are positive energy in general relativity, so their higher derivative counterparts would be negative, and there would again be no additional constraints to protect the theory against instability.

4.2 Equivalent Scalar Representation

The general Lagrangian we wish to consider takes the form,

$$\mathcal{L} = \frac{1}{16\pi G} \left( R + f(R) \right) \sqrt{-g} .$$

(97)
If we ignore the coupling to matter the modified gravitational field equation consists of the vanishing of the following tensor,
\[ 16\pi G \frac{\delta S}{\sqrt{-g}} \delta g^{\mu\nu} = \left[ 1 + f'(R) \right] R_{\mu\nu} - \frac{1}{2} [R + f(R)] g_{\mu\nu} + g_{\mu\nu} [f'(R)]^\rho_\rho - [f'(R)]_{;\mu\nu} . \] (98)

There is an old procedure for reformulating this as general relativity with a minimally coupled scalar. I don’t know whom to credit, but I will give the construction.

The first step is to define an “equivalent” theory with an auxiliary field \( \phi \) which is defined by the relation,
\[ \phi \equiv 1 + f'(R) \iff R = R(\phi) . \] (99)

Inverting the relation determines the Ricci scalar as an algebraic function of \( \phi \).

We can then define an auxiliary potential for \( \phi \) by Legendre transformation,
\[ U(\phi) \equiv (\phi - 1) R(\phi) - f\left( R(\phi) \right) \implies U'(\phi) = R(\phi) . \] (100)

Now consider the equivalent scalar-tensor theory whose Lagrangian is,
\[ L_E \equiv \frac{1}{16\pi G} \left( \phi R - U(\phi) \right) \sqrt{-g} . \] (101)

Its field equations are,
\[ 16\pi G \frac{\delta S_E}{\sqrt{-g}} \frac{\delta}{\delta \phi} = R - U'(\phi) = 0 , \] (102)
\[ 16\pi G \frac{\delta S_E}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} = \phi R_{\mu\nu} - \frac{1}{2} \left( \phi R - U(\phi) \right) g_{\mu\nu} + g_{\mu\nu} \phi^\rho_\rho - \phi_{\mu\nu} = 0 . \] (103)

The scalar equation (102) implies \( \phi = 1 + f'(R) \), whereupon the tensor equations (103) reproduce the original modified gravity equations (98).

The final step is to define a new metric \( \tilde{g}_{\mu\nu} \) and a new scalar \( \varphi \) by the change of variables,
\[ \tilde{g}_{\mu\nu} \equiv \phi g_{\mu\nu} \iff g_{\mu\nu} = \exp\left[ -\sqrt{\frac{4\pi G}{3}} \varphi \right] \tilde{g}_{\mu\nu} , \] (104)
\[ \varphi \equiv \sqrt{\frac{3}{4\pi G}} \ln(\phi) \iff \phi = \exp\left[ \sqrt{\frac{4\pi G}{3}} \varphi \right] . \] (105)

In terms of these variables the equivalent Lagrangian takes the form,
\[ L_E = \frac{1}{16\pi G} \tilde{R} \sqrt{-\tilde{g}} - \frac{1}{2} \partial_\mu \varphi \partial_\nu \varphi \tilde{g}^{\mu\nu} \sqrt{-\tilde{g}} - V(\varphi) \sqrt{-\tilde{g}} , \] (106)

where the scalar potential is,
\[ V(\varphi) \equiv \frac{1}{16\pi G} U\left( \exp\left[ \sqrt{\frac{4\pi G}{3}} \varphi \right] \right) \exp\left[ -\sqrt{\frac{16\pi G}{3}} \varphi \right] . \] (107)

This is general relativity with a minimally coupled scalar, as claimed.
4.3 Reconstructing \( f(R) \) from Cosmology

I want to show how to choose \( f(R) \) to support an arbitrary \( a(t) \).\(^3\) Recall from the Introduction that one can choose the potential of a quintessence model such as (106) to support any homogeneous and isotropic cosmology for its metric \( \tilde{g}_{\mu\nu} \). However, we cannot immediately exploit this construction because it is the metric \( g_{\mu\nu} \) which is assumed known, not \( \tilde{g}_{\mu\nu} \). We must explain how to infer the one from the other without knowing \( f(R) \).

Because the relation (104) between \( g_{\mu\nu} \) and \( \tilde{g}_{\mu\nu} \) is a conformal transformation, it makes sense to work in a coordinate system in which each metric is conformal to flat space. This is accomplished by changing from co-moving time \( t \) to conformal time \( \eta \) through the relation, \( d\eta = dt/a(t) \),

\[
ds^2 = -dt^2 + a^2(t)dx \cdot dx = a^2\left(-d\eta^2 + dx \cdot dx\right). \tag{108}\]

The \( g_{\mu\nu} \) element takes the same form in conformal coordinates, but note that its different scale factor implies a different co-moving time,

\[
ds^2 = \tilde{a}^2\left(-d\tilde{\eta}^2 + d\tilde{x} \cdot d\tilde{x}\right) = -d\tilde{t}^2 + \tilde{a}^2(\tilde{t})dx \cdot dx. \tag{109}\]

From relation (104) we infer,

\[
a(t) = \tilde{a}(\tilde{t}) \exp\left[-\sqrt{\frac{\pi G}{3}}\phi_0(\tilde{t})\right]. \tag{110}\]

We denote differentiation with respect to \( \eta \) by a prime, and one should note the relation between derivatives with respect to the various times,

\[
\frac{\partial}{\partial \eta} = a \frac{\partial}{\partial t} = a \frac{\partial}{\partial \tilde{t}}. \tag{111}\]

Differentiating the logarithm of (104) with respect to \( \eta \) and using the relation (8) between \( \tilde{a} \) and \( \phi_0 \) gives,

\[
\frac{a'}{a} = \frac{\tilde{a}'}{\tilde{a}} - \sqrt{\frac{\pi G}{3}}\phi_0' = \frac{\tilde{a}'}{\tilde{a}} - \sqrt{\frac{1}{12}}\tilde{a}'. \tag{112}\]

This is a nonlinear but first order differential equation for the variable \( \tilde{a} \) in terms of the known function, \( a(t(\eta)) \). At the worst it can be solved numerically.

Once we have \( \tilde{a} \) the potential \( V(\varphi) \) can be constructed using the procedure explained in the Introduction. We then compute the auxiliary potential,

\[
U(\phi) = 16\pi G\phi^2V\left(\sqrt{\frac{3}{4\pi G}}\ln(\phi)\right). \tag{113}\]

\(^3\) For a somewhat different construction which achieves the same end, see [17, 52].
The auxiliary field can be expressed in terms of the Ricci scalar from the algebraic relation,

\[ U'(\phi) = R \iff \phi = \Phi(R). \] (114)

And we finally recover the function \( f(R) \) by Legendre transformation,

\[ f(R) = \left( \Phi(R) - 1 \right) R - U(\Phi(R)). \] (115)

5 Problems with \( f(R) = -\mu^4/R \)

In view of the construction of sub-section it is not surprising but rather inevitable that an \( f(R) \) can be found to support late time acceleration, or indeed, any other evolution. However, the method is not guaranteed to produce a simple model, so the discovery that \( f(R) = -\mu^4/R \) works is quite noteworthy \[53, 54\]. It may also be significant that models of this type seem to follow from fundamental theory \[56\].

To derive acceleration in this model consider its field equations,

\[ (1 + \mu^4/R^2) R_{\mu\nu} - \frac{1}{2} (1 - \mu^4/R^2) R g_{\mu\nu} + \left( g_{\mu\nu} \Box - D_{\mu} D_{\nu} \right) \frac{\mu^4}{R^2} = 8\pi G T_{\mu\nu}. \] (116)

Setting \( T_{\mu\nu} = 0 \) and searching for constant Ricci scalar solutions gives,

\[ (1 + \mu^4/R^2) R_{\mu\nu} - \frac{1}{2} (1 - \mu^4/R^2) R g_{\mu\nu} = 0 \iff R_{\mu\nu} = \pm \frac{\sqrt{3}}{4} \mu^2 g_{\mu\nu}. \] (117)

The plus sign corresponds to acceleration.

In addition to proposing the model, Carroll, Duvvuri, Trodden and Turner \[53\] also showed that it suffers from a very weak tachyonic instability in the absence of matter. Because the only new higher derivative degree of freedom resides in the Ricci scalar, we may as well derive an equation for it alone from the trace of (116),

\[ - R + \frac{3\mu^4}{R} + \Box \left( \frac{3\mu^4}{R^2} \right) = 0. \] (118)

Now perturb about the accelerated solution,

\[ R = +\sqrt{3}\mu^2 + \delta R \implies -2\delta R - \frac{2}{\sqrt{3}\mu^2} \delta R + O(\delta R^2) = 0. \] (119)

By comparing the linearized equation for \( \delta R \) with that of a positive mass-squared scalar,

\[ (\Box - m^2)\varphi = 0, \] (120)

4 Although extensions involving \( R^\mu\nu R_{\mu\nu} \) and \( R^\rho\sigma\mu\nu R_{\rho\sigma\mu\nu} \) have also been studied \[55\], they must be ruled out on account of the Ostrogradskian instability.
we see that $\delta R$ behaves like a tachyon with $m^2 = -\sqrt{3}\mu^2$. However, because explaining the current phase of acceleration requires $\mu \sim 10^{-33}$ eV, the resulting instability is not very serious. I should note that the existence of a tachyonic instability in no way contradicts the Ostrogradskian analysis that this model’s higher derivative degree of freedom carries positive kinetic energy.

5.1 Inside Matter

Dolgov and Kawasaki [57] showed that a radically different result emerges when this model is considered inside a static distribution of matter,\

$$T_{\mu\nu} = \rho \delta_{\mu}^0 \delta_{\nu}^0 \quad \text{with} \quad 8\pi G\rho \equiv M^2 \gg \mu^2 .$$

(121)

In that case the trace of (116) gives,

$$-R + \frac{3\mu^4}{R} + \Box \left( \frac{3\mu^4}{R^2} \right) = -M^2 .$$

(122)

As might be expected, the static Ricci scalar solution in this case is dominated by $M$ rather than $\mu$,

$$R_0 = \frac{1}{2} \left( M^2 + \sqrt{M^4 + 12\mu^4} \right) \simeq M^2 .$$

(123)

Perturbing about this solution gives,

$$R = R_0 + \delta R \quad \implies \quad -\delta R - \frac{3\mu^4}{R_0^2} \delta R - \frac{6\mu^4}{R_0^3} \Box \delta R + O(\delta R^2) = 0 .$$

(124)

Comparing with the reference scalar (120) now reveals an enormous tachyonic mass,

$$m^2 = -\frac{R_0}{2} \frac{R_3}{6\mu^2} \simeq -\frac{M^6}{6\mu^4} !$$

(125)

Plugging in the numbers for the density of water ($\rho \sim 10^3$ kg/m$^3$) gives $M \sim 10^{-18}$ eV, implying a tachyonic mass of magnitude $|m| \sim 10^{12}$ eV = 10$^3$ GeV!

As disastrous as this problem might seem, Dick [58] and Nojiri and Odintsov [59] have shown that it can be avoided by changing the model slightly,

$$f(R) = -\frac{\mu^4}{R} + \frac{\alpha}{2\mu^2} R^2 \quad \implies \quad -R + \frac{3\mu^4}{R} + 3\Box \left( \frac{\mu^4}{R^2} + \frac{\alpha}{\mu^2} R \right) = 0 .$$

(126)

Because an $R^2$ term has global conformal invariance, it makes no contribution to the trace for constant $R$. Hence the cosmological solution of $R = +\sqrt{3}\mu^2$ is not affected, nor is the static solution inside the matter distribution (121).

However, the equation for linearized perturbations inside matter changes to,
\[ -\delta R - \frac{3\mu^4}{R_0^3} \delta R + 3 \left( -\frac{2\mu^4}{R_0^3} + \frac{\alpha}{\mu^2} \right) \Box \delta R = 0 \]  

(127)

The instability of Dolgov and Kawasaki was driven by the smallness of \(2\mu^4/R_0^3\). By simply taking \(\alpha\) positive and of order one the tachyon becomes a positive mass-squared particle of \(m^2 \sim \mu^2/\alpha\).

5.2 Outside Matter

Marc Soussa and I analyzed force of gravity outside a matter distribution [60]. Although our analysis was for the original \(f(R) = -\mu^4/R\) model, there would be only slight differences for the extended model (126). So our result seems to foreclose this possibility, but see [61].

The tachyonic instability could be studied using the perturbed Ricci scalar, but the gravitational force requires use of the metric. We perturbed about the de Sitter solution with Hubble constant \(H = \mu/(48)^{\frac{1}{4}}\) in co-moving coordinates,

\[ ds^2 = -(1-h_{00})dt^2 + 2a(t)h_{0i}dt dx^i + a^2(t)(\delta_{ij} + h_{ij})dx^i dx^j \quad \text{with} \quad a(t) = e^{Ht}. \]

(128)

In the gauge,

\[ h_{\mu\nu} - \frac{1}{2} h_{\mu} + 3 h_{\nu}[\ln(a)]_{,\nu} = 0, \]

(129)

with \(h \equiv -h_{00} + h_{ii}\), the perturbed Ricci scalar takes the form,

\[ \delta R = -\frac{1}{2} \partial^2 h + 2H \partial h. \]

(130)

Our strategy was first to solve the de Sitter invariant equation for the perturbed Ricci scalar, then reconstruct the gauge-fixed metric.

We assumed a matter density of the form,

\[ \rho(t, x) = \frac{3M}{4\pi R_0^3} \theta\left( R_y - a(t)|x| \right). \]

(131)

The exterior field equation has a simple expression in terms of the coordinate \(y \equiv a(t)H|x|\),

\[ \left[(1-y^2) \frac{d^2}{dy^2} + \frac{2}{y} \left(1-2y^2\right) \frac{d}{dy} + 12\right] \delta R = 0. \]

(132)

The solution takes the form,

\[ \delta R = \beta_1 f_0(y) + \beta_2 f_{-1}(y), \]

(133)

where \(f_0\) and \(f_{-1}\) are hypergeometric functions whose series expansions are,
\[ f_0(y) = 1 - 2y^2 + \frac{1}{5}y^4 + \ldots , \quad (134) \]
\[ f_{-1}(y) = \frac{1}{y} \left( 1 - 7y^2 + \frac{14}{3}y^4 + \ldots \right) . \quad (135) \]

We only need the behavior for small \( y \) because \( y = 1 \) is the Hubble radius! Matching to the source at \( y = HR_g \) determines the combination coefficients to be,
\[ \beta_1 \simeq \frac{3GM}{R_g^3} , \quad \beta_2 \simeq -12GMH . \quad (136) \]

This last step might seem bogus because we needed to regard the mass density as a small perturbation on the cosmological energy density \( \mu^4 \), whereas the opposite would be the case for galaxies or clusters of galaxies. However, this will only make changes of order one in the \( \beta_i \)'s. In particular, the asymptotic solution must still take the form \( (133) \).

The next step is solving for the trace of the perturbed metric. It turns out that relation \( (130) \) can also be expressed very simply using the variable \( y \),
\[ \left[ \left( y^2 - 1 \right) \frac{d}{dy} + \frac{1}{y} \left( 5y^2 - 2 \right) \right] h'(y) = \frac{2}{H^2} \delta R . \quad (137) \]

We only need to solve for the derivative of \( h \) because that is what gives the gravitational force in the geodesic equation. The solution is,
\[ h'(y) = -\frac{2GM}{H^2 R_g^3} y + O(y^2) . \quad (138) \]

This should be compared to the general relativistic prediction,
\[ h'_{\text{GR}}(y) = -\frac{4GMH}{y^2} + O(1) \quad \Rightarrow \quad \frac{h'}{h'_{\text{GR}}} = \frac{1}{2} \left( \frac{\|x\|}{R_g} \right)^3 . \quad (139) \]

One consequence is that the force between the Milky Way and Andromeda galaxies would be about a million times larger than predicted by general relativity!

6 Conclusions

The potential of a quintessence scalar can be chosen to support any cosmology, but the epicyclic nature of this construction suggests we consider modifications of gravity. Ostrogradski's theorem \([18]\) limits local modifications of gravity to just algebraic functions of the Ricci scalar. Models of this form can give a late phase of cosmic acceleration such as we are currently experiencing. However, they can be tuned to give anything else as well. They seem every
bit as epicyclic as scalar quintessence. Further, the $f(r) = -\mu^4/R$ model is problematic, both inside and outside matter sources.\(^5\)

An interesting and largely overlooked possibility for modifying gravity is the fully nonlocal effective action that results from quantum gravitational corrections. In weak field perturbation theory it has long been known that the most cosmologically significant one loop corrections are not of the $R^2$ form usually studied but rather of the form $R \ln(R) R$. More potentially interesting is the possibility of very strong infrared effects from the epoch of primordial inflation\[^{63}\].

It can be shown that quantum gravitational corrections to the inflationary expansion rate grow with time like powers of $\ln(a)$. Although suppressed by very small coupling constants, the exponential growth in $a(t)$ during inflation must eventually cause the effect to become nonperturbatively strong\[^{60, 67}\]. Similar secular growth occurs as well for minimally coupled scalar field theories\[^{68, 69}\], in which context Starobinskii has developed a technique for summing the leading powers of $\ln(a)$ at each loop order\[^{70, 71}\]. If Starobinskii’s technique can be generalized to quantum gravity\[^{72, 73}\], it might result in a nonlocal effective gravity theory for late time cosmology in which a large, bare cosmological constant is almost completely screened by a nonperturbative quantum gravitational effect. In such a formalism the current phase of acceleration might result from a very slight mismatch between the bare cosmological constant and the quantum effect which screens it. It is even conceivable that one could reproduce the phenomenological successes of MOND\[^{3, 4}\] with such a nonlocal metric theory, although it would have to unstable against decay into galaxy-scale gravitational waves\[^{74}\].

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\(^5\) Observations also rule out the somewhat different version of this model that results from regarding the connection and the metric as independent, fundamental variables in the Palatini formalism\[^{62}\].
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