Frogs on trees?

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Abstract

We study a system of simple random walks on \( T_{d,n} = (V_{d,n}, E_{d,n}) \), the \( d \)-ary tree of depth \( n \), known as the frog model. Initially there are \( \text{Pois}(\lambda) \) particles at each site, independently, with one additional particle planted at some vertex \( o \). Initially all particles are inactive, except for the ones which are placed at \( o \). Active particles perform (independent) \( t \in \mathbb{N} \cup \{\infty\} \) steps of simple random walk on the tree. When an active particle hits an inactive particle, the latter becomes active. The model is often interpreted as a model for a spread of an epidemic. As such, it is natural to investigate whether the entire population is eventually infected, and if so, how quickly does this happen. Let \( R_t \) be the set of vertices which are visited by the process. Let \( S(T_{d,n}) := \inf\{t : R_t = V_{d,n}\} \). Let \( CT(T_{d,n}) \) be first time in which every vertex was visited at least once, when we take \( t = \infty \). We show that there exist absolute constants, \( c, C > 0 \) such that for all fixed \( \lambda > 0 \), w.h.p. \( c \leq \lambda S(T_{d,n})/n \log n \leq C \) and \( CT(T_{d,n}) \leq 2C\sqrt{\log |V_{d,n}|} \).

Keywords: frog model, simple random walks, cover times, susceptibility, trees.

1 Introduction and results

We study a system of random walks known as the frog model. The frog model on infinite graphs received much attention, e.g. [16, 1, 2, 15, 8, 9]. However, to the best of the author’s knowledge, there are no published results concerning the model on finite graphs. In this paper we study the model in the case that the underlying graph \( G = (V, E) \) is some finite connected simple undirected

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graph. More specifically, we focus mainly on the case that $G$ is a finite $d$-ary tree of depth $n$, denoted by $T_{d,n}$. We employ the convention that the root, which throughout is denoted by $r$ has degree $d$ (while the rest of the non-leaf vertices are of degree $d + 1$). As we soon explain in more details, we study some natural parameters associated with the frog model on finite graphs, which are meaningless in the infinite setup.

The frog model on $G$ with density $\lambda$ can be described as follows. Initially there are Pois($\lambda$) particles at each vertex of $G$, independently. A site of $G$ is singled out and called its origin, denoted by $o$ (when $G$ is a tree, we do not assume that the origin is the root of the tree $r$). An additional particle, denoted by $w_{\text{plant}}$, is planted at $o$ (this is done in order to ensure that the process does not instantly die out). All particles are inactive (sleeping) at time zero, except for those occupying the origin. Each active particle performs a discrete time simple random walk (SRW) of length $\tau$ (for some $\tau \in \mathbb{N} \cup \{\infty\}$) on the vertices of $G$ (i.e. at each step, it moves to a random neighbor of its current position, chosen from the uniform distribution over the neighbor set) after which it cannot become reactivated. We refer to $\tau$ as the particles’ life time. Up to the time a particle dies (i.e. during the $\tau$ steps of its walk), it activates all sleeping particles it hits along its way. From the moment an inactive particle is activated, it performs the same dynamics over its life time $\tau$, independently of everything else. We denote the corresponding probability measure by $P_\lambda$. Notice that there is no interaction between active particles, which means that, once activated, each active particle moves independently of everything else.

We now define two natural parameters for the frog model on a finite graph $G$. Note that (in contrast to the setup in which $G$ is infinite) a.s. there exists a finite minimal lifetime $\tau$ (which is a function of the initial configuration of the particles and the walks they pick) for which every vertex is visited by an active particle before the process “dies out”. We define this lifetime as the susceptibility $S(G)$. Another interesting quantity is the cover time $\text{CT}(G)$, defined as the minimal time by which every vertex is visited by at least one active particle when $\tau = \infty$ (i.e. particles never die). More explicit definitions of the susceptibility and the cover time are given in (2.3).

The name frog model was coined in 1996 by Rick Durrett. It is a particular case of the $A + B \rightarrow 2B$ family of models (see §1.2). Like other models in this family (e.g. [12, 13]), it is often motivated as a model for the spread of a rumor or infection (e.g. [2]). Keeping this interpretation in mind, the
cover time and the susceptibility are indeed natural quantities. The former is (roughly) the minimal time in which “all individuals are infected” and the latter is the minimal lifetime $\tau$ of a virus (more precisely, of an individual infected by a virus), sufficient for “wiping out the entire population”. In this interpretation, the smaller the law of $S(G)$ is (stochastically) the more susceptible the population is.

Most of the literature on the model is focused on the case that the underlying graph on which the particles perform their random walks is $\mathbb{Z}^d$ (for some $d \geq 1$). Beyond the Euclidean setup, there has been much interest in understanding the behavior of the model in the case that the underlying graph is a $d$-ary tree, either finite or infinite (we denote the infinite $d$-ary tree by $T_d$). More specifically, Itai Benjamini asked about the cover time and susceptibility of $T_{d,n}$ (see Problems 1 and 2 below) and about the existence of a phase transition for the model on $T_d$, as the density of walkers varies. Despite several attempts by different groups, until recently there was no progress in neither the finite nor infinite setups, which earned the problem its reputation as a hard problem (I have heard of these attempts from a couple of groups who worked on the problem; see also the paragraph preceding Open Question 5 in [9]). Finally, recently Hoffman et al. [8, 9] showed that the frog model on $T_d$ indeed exhibits a phase transition as the initial state becomes more saturated with particles. Namely, below a critical density of particles it is a.s. transient, and above it is a.s. recurrent (it is still an open problem to determine the existence of such a critical density for all bounded degree non-amenable graphs).

The purpose of this paper is to study the frog model on finite $d$-ary trees. We study two questions presented to us by I. Benjamini [3] (see also [9, Open Question 5]).

**Problem 1.** Show that for every $d \geq 2$ when $\lambda = 1$ there exist some $C_d, \ell > 0$ such that (regardless of the identity of the origin $o$)

$$\lim_{n \to \infty} \mathbb{P}[S(T_{d,n}) > C_d n^\ell] = 0.$$ 

**Problem 2.** Show that for every $d \geq 2$ and $\lambda > 0$, there exists some $f_{d,\lambda} : \mathbb{N} \to \mathbb{N}$ satisfying that $f_{d,\lambda}(n) = o(n^\varepsilon)$ for every $\varepsilon > 0$ such that (regardless of the identity of the origin $o$)

$$\lim_{n \to \infty} \mathbb{P}_\lambda[CT(T_{d,n}) \leq f_{d,\lambda}(|T_{d,n}|)] = 1.$$
There are no published results on the frog model on $T_{d,n}$. To the best of the author’s knowledge, the best known upper bound on $CT(T_{d,n})$ is exponential in $n$, i.e. polynomial in the volume of the tree, $|T_{d,n}|$ (see e.g. the paragraph preceding Open Question 5 in [9]). In light of the aforementioned phase transition it seems plausible that the answer to both Problem 1 and 2 may depend on $\lambda$. Our main result, Theorem 1, resolves Problem 1 by determining for all fixed $\lambda > 0$ the (“typical” w.r.t. $P_\lambda$) value of $S(T_{d,n})$ up to a constant factor. In particular, it asserts that $S(T_{d,n})$ does not exhibit a phase transition w.r.t. $\lambda$. Theorem 2 provides an affirmative answer to Problem 2.

**Theorem 1.** There exist some $C_d, c_d > 0$ such that for all $\lambda > 0$ and $d \geq 2$

$$\lim_{n \to \infty} P_\lambda[c_d \leq \lambda S(T_{d,n})/(n \log n) \leq C_d] = 1.$$

**Theorem 2.** There exists some $C > 0$ such that for all $\lambda > 0$ and $d \geq 2$

$$\lim_{n \to \infty} P_\lambda[CT(T_{d,n}) \leq 2^{C \sqrt{\log |V_{d,n}|}}] = 1.$$

Theorems 1 and 2 both hold uniformly in the identity of the origin $o$. It follows from the assertion of Theorem 1 that $S(T_{d,n})$ does not exhibit a phase transition w.r.t. $\lambda$. This is in contrast with the aforementioned results in [8, 9] concerning $T_d$.

**Problem 3.** Is there a phase transition w.r.t. $\lambda$ in terms of the distribution of $CT(T_{d,n})$ under $P_\lambda$?

We believe that the answer to Problem 3 is positive. The following problem is a more concrete version of Problem 3.

**Problem 4 (3).** Show that for every $d \geq 2$ there exist $\lambda_1(d) \leq \lambda_2(d)$ and some $C_{d,\lambda}, c_{d,\lambda}, \varepsilon_d, \ell > 0$ such that (regardless of the identity of the origin $o$)

$$\forall \lambda > \lambda_2(d), \lim_{n \to \infty} P_\lambda[CT(T_{d,n}) \leq C_{d,\lambda} n^\ell] = 1.$$ $$\forall \lambda < \lambda_1(d), \lim_{n \to \infty} P_\lambda[CT(T_{d,n}) > c_{d,\lambda} 2^{n^{\varepsilon_d}}] = 1.$$

**Remark 1.1.** We believe that if $\lambda < \lambda_1(d)$ and $o$ is taken to be a leaf, then one can replace $CT(T_{d,n})$ above by the activation time of the root (see (2.2)).
Remark 1.2. It is plausible that the frog model on $\mathbb{T}_{d,n}$ exhibits more than one phase transition, in which case it is possible that $\lambda_1(d) < \lambda_2(d)$ above. We believe that $\lambda_1(d) \geq cd$ and that $\lambda_2(d) \leq Cd^2$ (for some absolute constants $c, C > 0$). Perhaps $\lambda_1(d)$ corresponds to the critical value for recurrence for the frog model on $\mathbb{T}_d$, while $\lambda_2(d)$ corresponds to the value below which the occupation measure (for the frog model on $\mathbb{T}_d$) converges to 0 pointwise, while above it, the joint distribution of the number of particles at each site converges to a non-trivial invariant measure. It is not known whether such $\lambda_2(d)$ exists.

We denote the transition kernel of SRW on the underlying graph $G = (V, E)$ (which is clear from context) by $P$. We denote by $p_t(u,v) := P_t(u,v)$ the $t$-steps transition probability from $u$ to $v$.

**Theorem 3.** For every finite regular simple graph $G = (V, E)$ and all $\lambda > 0$

$$\mathbb{P}_\lambda[\lambda S(G) \geq \log |V| - 4 \log \log |V|] \to 1, \quad \text{as } |V| \to \infty. \quad (1.1)$$

Moreover, for all $\lambda > 0$ and $\delta \in [0,1)$ we have that

$$\mathbb{P}_\lambda[S(G) \geq t_{\lambda,\delta}(G)] \to 1, \quad \text{as } |V| \to \infty, \quad (1.2)$$

where $t_{\lambda,\delta}(G) := \min \{ s : \frac{2s\lambda}{\kappa_s} \geq (1 - \delta) \log |V| \}$ and $\kappa_s := \min_{v} \sum_{i=0}^{t} p_i(v, v)$.

**Remark 1.3.** Theorem 3 seems to be especially useful when $G$ is vertex-transitive (Definition 1.1; see Conjecture 1.2 for more on this point). The bound offered by (1.2) is sharp up to a constant factor in the cases that $G$ is either a $d$-dimensional torus ($d \geq 1$) of side length $n$ or a regular expander (this is a topic of a work in progress with I. Benjamini, L.R. Fontes and F.P. Machado; c.f. [4, Theorems 4-5]). Proposition 6.1 gives a certain extension of (1.2) which yields the lower bound on $S(T_{d,n})$ in Theorem 1 (by allowing one to take the minimum in the definition of $\kappa_t$ only w.r.t. the leaf set).

**Remark 1.4.** The argument in the proof of Theorem 3 is borrowed from Theorems 2 and 4.4 in [4]. It is possible to extend Theorem 3 to the case that $G$ is non-regular. Adapting the argument from Theorem 4.3 in [4] gives

$$\forall \alpha \in (0,1), \exists c_\alpha > 0, \quad \mathbb{P}_\lambda[\lambda S(G) \leq c_\alpha \log |V|] \leq \exp[-c_\alpha^2 r_*^2 |V|^\alpha], \quad (1.3)$$

where $r_* := \min_{u,v \in V} \frac{\deg(u)}{\deg(v)}$. This is meaningful as long as $r_* \geq |V|^{\beta-0.5}$ for some $\beta > 0$. We do not prove (1.3) (the details involved in the translation of
Remark 1.5. We note that for every regular graph $G = (V, E)$ we have that $t_{\lambda,0}(G) \leq C\lambda^{-2}\log^2|V|$ (e.g. [5, Lemma 2.4]), which is tight up to a constant factor as can be seen by considering the $n$-cycle, $C_n$, for which $t_{\lambda,1/2}(C_n) \geq c\lambda^{-2}\log^2|V|$. Indeed, only minor adjustments to the analysis from [4, Theorem 3] are necessary in order to show that there exist $c_1, C_1 > 0$ such that $\lim_{n \to \infty} P_{\lambda}[c_1 \leq \lambda^2 S(C_n)/\log^2 n \leq C_1] = 1$, for all $\lambda > 0$ (see §L.2 for more details on this point). We conjecture that the cycle is in some sense extremal (up to degree dependence).

Proposition 1.1. Let $K_n$ be the complete graph on $n$ vertices. For all $\lambda > 0$

$$\forall \varepsilon \in (0, 1), \quad \lim_{n \to \infty} P_{\lambda}[(1 - \varepsilon) \log n \leq \lambda S(K_n) \leq (1 + \varepsilon) \log n] = 1.$$ 

Remark 1.6. In light of Theorem 3 and Proposition 1.1, it follows that $K_n$ is the regular graph with asymptotically the smallest $S$. This is consistent with the fact that $K_n$ is the regular graph with asymptotically the smallest cover time [6] and also the asymptotically smallest social connectivity time in the random walks social network model [4] (see §L.2).

Remark 1.7. It is not hard to verify that there exists some $C > 1$ such that

$$\forall \lambda > 0, \quad \lim_{n \to \infty} P_{\lambda}[\frac{\lambda CT(K_n)}{C\lambda \log n} \in (1 - \varepsilon, 1 + \varepsilon)] \to 1, \text{ for all } \lambda > 0 \text{ and } \varepsilon \in (0, 1),$$ 

for some constant $C_\lambda$ which perhaps depends on $\lambda$.

1.1 Organization of the paper

In §2 we present an explicit construction of the frog model and give more detailed definitions of $S(G)$ and $CT(G)$. In §3 we prove Theorems 1-2. In §4 we present a collection of auxiliary results concerning SRW on $T_{d,n}$. In §5 we prove Theorem 3. In §6 we prove Proposition 1.1.

1.2 Related models and further questions

The $A + B \to 2B$ family of models (e.g. [12, 13]) are defined by the following role: there are type $A$ and $B$ particles occupying a graph $G$, say with densities
\( \lambda_A, \lambda_B > 0 \). They perform independent SRW with holding probabilities \( p_A, p_B \in [0,1] \) (depending on the type). When a type \( B \) particle collides with a type \( A \) particle, the latter transforms into a type \( B \) particle. The frog model can be considered as a particular case of the above dynamics in which the type \( A \) particles are immobile \( (p_A = 1) \).

In [4], Benjamini and the author study the following model for a social network, called the random walks social network model, or for short, the SN model. Given a graph \( G = (V, E) \), consider Poisson(\(|V|\)) walkers performing independent lazy simple random walks on \( G \) simultaneously, where the initial position of each walker is chosen independently w.p. proportional to the degrees. When two walkers visit the same vertex at the same time they are declared to be acquainted. The social connectivity time, \( SC(G) \), is defined as the first time in which there is a path of acquaintances between every pair of walkers. The main result in [4] is that when the maximal degree of \( G \) is \( d \), then \( c \log |V| \leq SC(G) \leq C_d \log^3 |V| \) w.h.p.. Moreover, \( SC(G) \) is determined up to a constant factor in the cases that \( G \) is a regular expander or a \( d \)-dimensional torus \( (d \geq 1) \) of side length \( n \).

Note that in the SN model all walkers are initially activated and we are not requiring the sequence of times in which some walkers met along a certain path of acquaintances to be non-decreasing. Hence one should expect the SN model to evolve much faster than the frog model, in a sense that (when \( \lambda \) is fixed), for many graphs \( \mathbb{E}[SC(G)] \ll \mathbb{E}[CT(G)] \) (see Remark 1.3). We note that this fails when \( G \) is either the complete graph, or a regular expander. In these cases both terms are of order \( \log |V| \) (the expander case is in fact quite involved).

However, in many examples, even when \( \mathbb{E}[SC(G)] \ll \mathbb{E}[CT(G)] \), it is still the case that \( \mathbb{E}[SC(G)] \) and \( \mathbb{E}[S(G)] \) are of the same order, and several techniques from [4] can be applied successfully to the frog model. Namely, the same technique used in [4] to prove general lower bounds on \( SC(G) \) are used in the proof of Theorem 3. Moreover, the analysis of the two models on expanders and on \( d \)-dimensional tori \( (d \geq 1) \) are similar (in all of these cases \( S(G) \) and \( SC(G) \) are w.h.p. of the same order).

In light of the above discussion the following conjecture is natural.

**Conjecture 1.1** (Benjamini). There exist some \( C_{d,\lambda}, \ell > 0 \), such that for every sequence of finite connected graphs \( G_n = (V_n, E_n) \) with \( |V_n| \to \infty \) we have that \( \lim_{n \to \infty} \mathbb{P}_\lambda[S(G_n) \leq C_{d,\lambda} \log^\ell |V_n|] = 1 \).
We suspect that one can take above \( \ell = 2 \) and \( C_{\lambda,d} = C\lambda^{-2}d^2 \) for some absolute constant \( C > 0 \). Moreover, we suspect that for regular or vertex-transitive \( G \), one can even take above, respectively, \( C_{\lambda,d} = C\lambda^{-2}d^2 \) or \( C_{\lambda,d} = C\lambda^{-2} \) for some absolute constant \( C > 0 \) (c.f. [4, Conjecture 8.3]).

Remark 1.8. In contrast with Conjecture 1.1, \( CT(G) \geq \text{diameter}(G) \).

Definition 1.1. We say that a bijection \( \varphi : V \to V \) is an automorphism of a graph \( G = (V, E) \) if \( \{u, v\} \in E \iff \{\varphi(u), \varphi(v)\} \in E \). A graph \( G \) is said to be vertex-transitive if the action of its automorphisms group, \( \text{Aut}(G) \), on its vertices is transitive (i.e. \( \{\varphi(v) : \varphi \in \text{Aut}(G)\} = V \) for all \( v \)).

Conjecture 1.2. There exists an absolute constants \( C > 0 \) such that for every finite connected vertex-transitive graph \( G = (V, E) \) and all \( \lambda, \delta > 0 \)

\[
\mathbb{P}_\lambda[1 - \delta \leq S(G)/t_\lambda(G) \leq C] \to 1, \quad \text{as} \quad |V| \to \infty,
\]

where \( t_\lambda(G) := \min\{s : 2s/\kappa_s \geq \lambda^{-1} \log |V|\} \) and \( \kappa_t := \min_v \sum_{i=0}^t p^i(v,v) \).

1.3 Notation

We denote \([k] := \{1, 2, \ldots, k\}\) and \([k] := \{0, 1, \ldots, k\}\). We denote the cardinality of a set \( A \) by \(|A|\). By abuse of notation, for a finite tree \( T = (V, E) \) we write \(|T|\) and \( v \in T \) instead of \(|V|\) and \( v \in V \), resp..

We write w.p. as a shorthand for “with probability”. We say that a sequence of events \( A_n \) defined, resp., w.r.t. some probabilistic model on a sequence of graphs \( G_n := (V_n, E_n) \) with \(|V_n| \to \infty\), holds w.h.p. (“with high probability”) if the probability of \( A_n \) tends to 1 as \( n \to \infty \). We write \( o(1) \) for terms which vanish as \( n \to \infty \) (or some other index, which is clear from context). We write \( f_n = o(g_n) \) if \( f_n/g_n = o(1) \).

We shall use \( C, C', C_0, C_1, \ldots \) (resp. \( c, c', c_0, c_1, \ldots \)) to denote positive absolute constants which are sufficiently large (resp. small) to ensure that a certain inequality holds. Similarly, we use \( C_d, c_d \) or \( C_{\lambda,d}, c_{\lambda,d} \) (etc.) to refer to positive constants, whose value may depend on the parameters appearing in subscript (where the same convention applies regarding lower case and upper case letters). Different appearances of the same constant at different places may refer to different numeric values.

For SRW on a graph \( G = (V, E) \), the \textbf{hitting time} of a set \( A \subset V \) is \( T_A := \inf\{t \geq 0 : X_t \in A\} \). Similarly, \( T_A^+ := \inf\{t \geq 1 : X_t \in A\} \). When
\( A = \{ x \} \) is a singleton, we instead write \( T_x \) and \( T^+_x \). Let \( P \) be the transition kernel of SRW on \( G \). We denote by \( p'(u,v) := P'(u,v) \) the \( t \)-steps transition probability from \( u \) to \( v \). We denote by \( P_u \) the law of the entire walk, started from vertex \( u \). The same definitions apply when we consider an arbitrary Markov chain on a finite or countable state space, rather than SRW.

The distance \(\text{dist}(x,y)\) between vertices \( x \) and \( y \) is the minimal amount of edges that one must pass in order to go from \( x \) to \( y \). Vertices are said to be neighbors if they belong to a common edge.

We denote the \( d \)-ary tree of depth \( n \) by \( \mathcal{T}_{d,n} = (\mathcal{V}_{d,n}, \mathcal{E}_{d,n}) \) (when \( d \) and \( n \) are clear from context we write \( (\mathcal{V}, \mathcal{E}) \)). We denote its root by \( r \). Denote by \( L_i \) the \( i \)th level of the tree and the leaf set by \( \mathcal{L} \). For \( x \in \mathcal{V}_{d,n} \) we write \( |x| = \text{dist}(x,r) \) (i.e. \( x \in L_0 \)).

The root induces the following partial order, \( \leq \), on \( \mathcal{V}_{d,n} \). We write \( x \leq y \) if the path from \( y \) to the root goes through \( x \). If \( x \leq y \) we say that \( x \) is the \( \text{dist}(x,y) \)th ancestor of \( y \). If \( x \leq y \) and \( \text{dist}(x,y) = 1 \), we say that \( x \) is the parent of \( y \). We denote the \( i \)th ancestor of a vertex \( x \) by \( \leftarrow^i x \). For \( x, y \in \mathcal{V}_{d,n} \) we denote by \( x \wedge y \) their last common ancestor (the vertex \( z \) such that \( z \leq x \), \( z \leq y \) and \( |z| \) is maximal). The induced tree at \( x \), denoted by \( \mathcal{T}_x \), is the induced tree on the set of \( \{ y : x \leq y \} \). For \( x \in \mathcal{V}_{d,n} \) and \( |\mathcal{T}_x| \leq t \leq |\mathcal{V}_{d,n}| \) we denote by \( \mathcal{T}_x(t) \) the smallest induced tree \( \mathcal{T}_y \) such that \( y \leq x \) and \( |\mathcal{T}_y| \geq t \).

We reserve the term “induced tree” for trees of the form \( \mathcal{T}_x \) as these are the only type of subtrees we shall consider.

Apart from \( \mathcal{T}_{d,n} \) we also consider some of its induced trees and the infinite \( d \)-regular tree \( \mathbb{T}_d := (V(\mathbb{T}_d), E(\mathbb{T}_d)) \). When referring to some tree \( \mathcal{T} \) other than \( \mathcal{T}_{d,n} \), we write \( L_i(\mathcal{T}) \) and \( \mathcal{L}(\mathcal{T}) \) for the \( i \)th level and the leaf set of \( \mathcal{T} \), respectively. We denote the law of SRW on a tree \( \mathcal{T} \) (other than \( \mathcal{T}_{d,n} \)) started from vertex \( v \) by \( P^v_\mathcal{T} \) and similarly, denote the transition probabilities of such a walk by \( P^v_\mathcal{T}(\cdot, \cdot) \).

### 2 A formal construction of the model

The following construction of the model is useful. We denote the set of \( \text{Pois}(\lambda) \) (or \( 1 + \text{Pois}(\lambda) \) for the origin) frogs occupying vertex \( v \) at time 0 by \( \mathcal{W}_v = \{ w^v_1, \ldots, w^v_{|\mathcal{W}_v|} \} \). We can assume that at time 0 there are infinitely many particles occupying each site \( v \), \( \mathcal{J}_v := \{ w^v_i : i \in \mathbb{N} \} \) (where \( w^v_i \) is referred to as the \( i \)-th particle at \( v \)), but only the first \( |\mathcal{W}_v| \) of them are actually involved.
in the dynamics of the model. We may think of each particle \( w_i^u \in \mathcal{J}_u \) as first picking an infinite SRW \((S^{u,i}_t)_{t \in \mathbb{Z}^+}\) according to \( P_u \) and only in the case that \( i \leq |W_u| \) and \( v \) is visited by some active particle, say at time \( s \) (for the first time), does \( w_i^u \) actually performs the first \( \tau \) steps of the SRW it picked (i.e. its position in time \( s + t \) is \( S^{u,i}_t \) for all \( t \in \tau \)).

Let \( G = (V, E) \) be a graph. A walk of length \( k \) in \( G \) is a sequence of \( k+1 \) vertices \((v_0, v_1, \ldots, v_k)\) such that for all \( 0 \leq i < k \) or \( \{v_i, v_{i+1}\} \in E \). Let \( \Gamma_k \) be the collection of all walks of length \( k \) in \( G \). We say that \( w_i^u \in W_u \) picked the path \( \gamma = (\gamma_0, \ldots, \gamma_k) \in \Gamma_k \) if \( S^{i,v}_t = \gamma_t \) for all \( t \in [k] \). For each \( \gamma \in \Gamma_k \) let \( X_\gamma \) denote the number of particles in \( W_\gamma \), other than the particle planted at the origin, which picked the walk \( \gamma \). For a walk \( \gamma = (\gamma_0, \ldots, \gamma_k) \in \Gamma_k \) for some \( k \geq 1 \), we denote \( p(\gamma) := \prod_{i=0}^{k-1} P(\gamma_i, \gamma_{i+1}) \). By Poisson thinning, we have that for every fixed \( k \), the joint distribution of \((X_\gamma)_{\gamma \in \Gamma_k}\) (under \( P_\lambda \)) is that of independent Poisson random variables with \( E[ X_\gamma ] = \lambda p(\gamma) \) for all \( \gamma \in \Gamma_k \).

Consider a collection of homogeneous Poisson processes on \( \mathbb{R}_+ \) with rate 1, \(((M_v(s))_{s \geq 0})_{v \in \mathcal{V}}\) and a collection of simple random walks on \( \mathcal{G} \), \(((S^{v,i}_t)_{t \in \mathbb{Z}^+}; v \in \mathcal{V}, i \in \mathbb{N})\), where for all \( i \) and \( x \), \((S^{v,i}_t)_{t \in \mathbb{Z}^+}\) is the walk picked by the \( i \)th particle in \( \mathcal{J}_u \). We take the walks and the Poisson processes to be jointly independent. We take \( |W^\lambda_u| = M_u(\lambda) + 1_{u=0} \), where \( W^\lambda_u \) denotes the set of particles involved in the dynamics, whose initial position is \( u \), when the density is taken to be \( \lambda \). When clear from context, we omit the superscript \( \lambda \) and write \( W_u \). From this construction it is clear that the laws of \( \mathcal{S}(\mathcal{G}) \) and \( \text{CT}(\mathcal{G}) \) are stochastically decreasing in \( \lambda \).

For \( x, y \in \mathcal{V}(\mathcal{G}) \) such that \( x \neq y \) and \( \tau \in \mathbb{N} \cup \{ \infty \} \) let

\[
\ell_\tau(x, y) := \inf\{j \leq \tau : S^{i,y}_j = y \; \text{for some} \; i \leq |W_x| \}
\]

(2.1)

(where \( \inf \emptyset = \infty \)). The activation time of \( x \) (and also of \( W_x \)) w.r.t. lifetime \( \tau \) is

\[
\text{AT}_\tau(x) := \inf\{\ell_\tau(x_0, x_1) + \cdots + \ell_\tau(x_{m-1}, x_m) \}
\]

where the infimum is over all finite sequences \( o = x_0, x_1, \ldots, x_{m-1}, x_m = x \) where \( x_i \in \mathcal{V}(\mathcal{G}) \). The event \( \text{AT}_\tau(x) = \infty \) is precisely the event that (for lifetime \( \tau \)) site \( x \) is never visited by an active particle, while when finite, \( \text{AT}_\tau(x) \) is the first time in which vertex \( x \) is visited by an active particle (for lifetime \( \tau \)). The cover time and susceptibility of \( \mathcal{G} \) can be defined as

\[
\text{CT}(\mathcal{G}) := \max_{v \in \mathcal{V}} \text{AT}_\infty(v) \quad \text{and} \quad \mathcal{S}(\mathcal{G}) := \inf\{\tau : \max_{v \in \mathcal{V}} \text{AT}_\tau(v) < \infty\}.
\]

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3 Proof of Theorem

In this section we show that there exist some $C_d > 0$ such that for all $\lambda > 0$ and $d \geq 2$

$$\lim_{n \to \infty} \mathbb{P}_{\lambda}[\lambda S(T_{d,n}) \leq C_d n \log n] = 1.$$ 

The proof of the existence of some $c_d > 0$ such that for all $\lambda > 0$ and $d \geq 2$

$$\lim_{n \to \infty} \mathbb{P}_{\lambda}[\lambda S(T_{d,n}) \geq c_d n \log n] = 1,$$

is deferred to the end of §6. The analysis below relies heavily on properties of SRW on $T_{d,n}$ which are for the most part intuitive, but nevertheless are non-trivial. We choose to defer the proofs of all auxiliary results concerning SRW on $T_{d,n}$ to §5. Below we write $V$ instead of $V_{d,n}$.

Proof. As w.h.p. the particle planted at $o$ hits the leaf set by time $8n$ (by adjusting some constants if necessary) we may assume that $o \in \mathcal{L}$. We split the particles into two sets. Let $\mathcal{W}_v$ be the set of particles whose initial position is $v$. For each $v$ we partition $\mathcal{W}_v$ into two disjoint sets $\mathcal{W}_v^1$ and $\mathcal{W}_v^2$ so that $|\mathcal{W}_v^i|_{(v,i) \in V \times \{1,2\}}$ are mutually independent having a Pois($\frac{\lambda}{2}$) distribution, apart from $\mathcal{W}_o^1$, for which $|\mathcal{W}_o^1| - 1 \sim$ Pois($\frac{\lambda}{2}$) (i.e. the planted particle belongs to $\mathcal{W}_o^1$). We call the particles in $\bigcup_{v \in V} \mathcal{W}_v^i$ type $i$ particles ($i = 1, 2$). We first analyze the dynamics only w.r.t. the type 1 particles (as if there are no other type 2 particles). This dynamics is the same as the frog model with density $\lambda/2$.

Recall that $T_o(\tilde{t})$ is the smallest induced tree $T_x$ of size at least $\tilde{t}$, which contains $o$. Denote

$$\tilde{\ell} = \tilde{t}_n := \lceil C_1 d \lambda^{-1} \log |V| \log \log(\max(e^\varepsilon, |V|)) \rceil \quad \text{and} \quad t = t_n := |T_o(\tilde{t})|,$$

where $C_1 > 20$ is as in Corollary 5.2. We take the lifetime of the type 1 particles to be $t$ and of type 2 particles to be $\tau := \lceil td \log d \rceil$. As we soon explain in more details, our goal is to show that $U$, the collection of vertices which are visited by the type 1 dynamics, is w.h.p. sufficiently large so that w.h.p. $V$ is covered by the union of the ranges of the walks performed by the type 2 particles occupying $U$ at time 0, i.e. this union equals $V$. A precise formulation of “sufficiently large” is that the type 1 dynamics “conquers” $T_{d,n}$, where the notion of conquering a tree shall be defined soon. This is similar to a standard technique from percolation theory called Sprinkling.

We now give a few definitions.
Definition 3.1. For every set of vertices $B$, let $\mathcal{W}_B$ denote the collection of type 1 particles whose initial position is in $B$. For a collection of particles $\mathcal{W}'$, let $\mathcal{R}(\mathcal{W}')$ denote the union of the ranges of the first $t$ steps of the walks picked ("picked" in the sense of §2) by the particles in $\mathcal{W}'$. When $\mathcal{W}' = \{w\}$ we write $\mathcal{R}(w)$ instead of $\mathcal{R}(\{w\})$.

Let $\mathcal{H}_i$ be the collection of all type 1 particles such that the length $t$ walk they picked ends at $T^i$ ($i \in [m]$). Let

$$Q_i := \{w \in \mathcal{H}_i : |\mathcal{R}(w) \cap T^i| \geq c't/\log t\},$$

be the collection of all particles in $\mathcal{H}_i$ such that the intersection of the range of their (length $t$) walk with $T^i$ is of size at least $c't/\log t$, where $c' > 0$ is as in part (ii) of Corollary 5.1.

Let $T^1, \ldots, T^m$ ($m = m_{d,n} = \lfloor V_{d,n}((1-o(1)) \right\rfloor$) be the collection of all induced subtrees (in $\mathcal{T}_{d,n}$) of size $t$ (i.e. $T^i = \mathcal{T}_{x_i}$ for some $x_i \in \mathcal{V}$). We label the trees from right to left ($T^1$ being the rightmost). By symmetry, we may assume that $o \in \mathcal{L}(T^1)$. Let $z_0$ be the root of $T^1$, i.e. $T_{z_0} = T^1$. Denote $z_i := \delta_i$ for all $i \in [n - |z_0|]$. We label the children of $z_i$ by $y_{i,0}, y_{i,1}, \ldots, y_{i,t-1}$ so that $z_{i-1} = y_{i,0}$ and $T^t \subset \mathcal{T}_{y_{i,j}}$ for all $i \in [n - |z_0|], j \in [d-1]$ and $jd^{t-1} < \ell \leq (j+1)d^{t-1}$.

We employ a recursive divide and conquer approach. Namely, we analyze the set of activated vertices (by the type 1 dynamics) of $\mathcal{T}_{x_i}$ for each $i \in [n - |z_0|]$ by exploiting the earlier analysis of $\mathcal{T}_{z_{i-1}} = \mathcal{T}_{y_{i,0}}$ and the fact that $\mathcal{T}_{y_{i,j}}$ is isomorphic to $\mathcal{T}_{y_{i,j}}$ for all $j \in [d-1]$. We now define the notion of conquering a tree which is a key concept in our divide and conquer approach.

Definition 3.2. Let $B_i$ be the collection of leaves of $T^i$ which are visited by some type 1 particle before the type 1 dynamics dies out, when the lifetime of type 1 particles is taken to be $t$. We say that the type 1 dynamics with lifetime $t$ conquered $T^i$ if $|B_i| \geq t/4$. We say that it conquered a tree $T_x$, where $x \in \mathcal{V}$ and $|T_x| \geq t$, if it conquered $T^i$ for all $i$ such that $T^i \subset T_x$.

(1) Our ultimate goal: By restricting to the type 1 dynamics, we can reduce the problem of bounding $\mathcal{S}(\mathcal{T}_{d,n})$ to an easier problem: It suffices to show that w.h.p. the type 1 dynamics with lifetime $t$ conquers $\mathcal{T}_{d,n}$, as by part (ii) of Corollary 5.2 this implies that w.h.p. the union of the ranges of (the first $\tau$ steps of) the walks performed by the particles in $\bigcup_{i \in [m]} \bigcup_{w \in B_i} \mathcal{W}_w$ covers $\mathcal{V}$.
Suppose we show that $T_z$ is conquered (for lifetime $t$) w.p. $\geq 1 - p_i(n)$, given that at least $C_2 \lambda^{-1} \log |V|$ of its vertices are activated, and that given that it is conquered, then w.p. at least $(d - 1) p(n)$, the activated type 1 particles whose initial positions is in $T_z$ activate at least $C_2 \lambda^{-1} \log |V|$ vertices of each of $T_{y+1}, \ldots, T_{y+d-1}$. We would then want to argue that by a union bound and our estimate on $T_z = T_{y+1}$, the tree $T_z$ is conquered (for lifetime $t$) w.p. at least $1 - (d - 1) p(n) - dp_i(n)$. This suffices to conclude the proof, as long as $p_0(n)$ and $p(n)$ are both $o(|V|^{-1})$. One difficulty that arises is that possibly the tree $T_z$ was conquered with the assistance of some particles which originally lie outside of $T_z$. It seems possible to overcome this difficulty using Poisson thinning and the FKG inequality (namely, that functions which are non-decreasing w.r.t. the point configuration associated with a Poisson random measure are positively correlated). Instead, we define a refined notion of conquering a tree which avoids this complication. Before giving that definition we need the following auxiliary definition.

**Definition 3.3.** We say that a vertex $v \in T_j$ for some $j \in [m]$ is good for time $t$ if there exist a collection of sites $v_0 = o, v_1, \ldots, v_{\ell+1} = v$ and particles $w_{v_0} \in W_{v_0}^1, w_{v_1} \in W_{v_1}^1, \ldots, w_{v_\ell} \in W_{v_\ell}^1$ such that the following hold.

(i) For all $\ell' \in [\ell + 1]$, $v_{\ell'} \in T^{b_{\ell'}}$ for some $b_{\ell'} \in [m]$ so that $T^{b_{\ell'}} \subset T_{v \wedge o}$.

(ii) For all $i \in [\ell]$, $|v_i \wedge v_{i+1}|$ is non-increasing w.r.t. $i' \in [\ell - i]$ and $|v_i \wedge v_{i-1}|$ is non-increasing w.r.t. $i' \in [\ell - i]$.

(iii) For all $i \in [\ell]$, $v_{i+1} \in R(w_i)$.

(iv) Unless $w_i = w_{\text{plant}}$, where $w_{\text{plant}}$ is the particle planted at the origin, $w_i \in \cup_{\ell : T_j \subset T_{v \wedge o}} \mathcal{H}_\ell$.

**Definition 3.4.** We say that $T_j = T_{x_j}$ ($j \in [m], x_j$ being the root of $T_j$) is internally conquered by the type 1 dynamics with lifetime $t$ if the collection $D^j$ of sites of $T^j$ which are good for time $t$ satisfies

$$|D^j \cap L(T^j)| \geq t/4.$$

We say that $T_z$ is internally conquered (for lifetime $t$), if $T^j$ is internally conquered (for lifetime $t$) for all $j \in [d]$. 

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(2) A sufficient condition for internally conquering some \( T^i \): We now give a simple condition which ensures that the type 1 dynamics conquers a certain \( T^i \) w.p. at least \( 1 - 2|V|^{-4} \): Denote \( k = k_n := \lceil C_2 \lambda^{-1} \log |V| \rceil \), where \( C_2 \) is as in Corollary \( 5.2 \). By part (i) of Corollary \( 5.2 \) for every \( i \in [m] \), every set \( D \subset T^i \) such that \( |D| \geq k \) satisfies that \( |\cal R(\cal W_D \cap \cal H_i) \cap \cal L(T^i)| \geq t/4 \) w.p. at least \( 1 - 2|V|^{-4} \), provided that \( n \) is sufficiently large. We take \( C_1 \) to be sufficiently large so that \( c't/\log t > k \) (where \( c' \) is the constant from the definition of the sets \( \cal Q_1, \ldots, \cal Q_m \)).

We are now in the position to start the recursion. First expose \( \cal R(w_{\text{plant}}) \cap T^i \). It is easy to see that w.h.p., \( |\cal R(w_{\text{plant}}) \cap T^i| > \sqrt{t} \), provided that \( C_1 \) is taken to be sufficiently large. This follows by combining the fact that w.h.p. \( w_{\text{plant}} \) does not leave \( T^i \) by time \( t^{4/5}/c' \) (by Lemma \( 5.1 \) \( c' \) as above), with the fact that w.h.p. it visits at least \( t^{3/4}/\log(t^{3/4}/c') > \sqrt{t} \) different sites by time \( t^{4/5}/c' \) (by decomposing the first \( t^{4/5}/c' \) steps of its walk into blocks of length \( t^{3/4} \), and observing that by Corollary \( 5.1 \) the restriction of the range to each block contains at least \( \sqrt{t} \) vertices with some probability which is bounded from below). We now expose \( \cal R(\cal W_{\cal R(\cal w_{\text{plant}}) \cap T^i} \cap \cal H_1) \). Denote \( U_1 := \cal R(w_{\text{plant}}) \cup \cal R(\cal W_{\cal R(w_{\text{plant}}) \cap T^i} \cap \cal H_1) \). By Corollary \( 5.1 \) w.h.p. \( |U_1| \geq 2k \) and thus either \( \cal R(w_{\text{plant}}) \) or \( \cal R(\cal W_{\cal R(w_{\text{plant}}) \cap T^i} \cap \cal H_1) \) is of size at least \( k \). Let \( F_1 := \cal R(\cal W_{U_1}) \cap \cal L(T^i) \). By part (i) of Corollary \( 5.2 \) we have that

\[
\Pr[|F_1| \geq t/4 \mid |U_1| \geq 2k] \geq 1 - 2|V|^{-4}.
\]

This concludes the analysis of the recursion base. Before proceeding to the recursion step, we make a technical comment. Again by part (i) of Corollary \( 5.2 \) we have that if at time 0 there was a set \( D \subset T^i \) which was activated, then (as above) \( \Pr[|\cal R(\cal W_D) \cap \cal L(T^i)| \geq t/4] \geq 1 - 2|V|^{-4} \). In fact, for the sake of the recursion we would like to be in the situation that we may assume that at time 0 there is such a set \( D \subset T^i \) which is activated rather than conditioning on \( |U_1| \geq 2k \). This can be achieved as follows. We can partition the type 1 particles into two sets of density \( \lambda/4 \) and define \( U_1 \) w.r.t. the first set. As above, \( |U_1| \geq k \) w.h.p.. Conditioned on that, we can continue by restricting the dynamics only to the second set of type 1 particles.

(3) The key for the recursion step: Let \( i \in [n - |z_0|] \) and \( D \subset T_{z_{i-1}} \) be so that \( |D| \geq \frac{1}{4} \{ \ell : T^\ell \subset T_{z_{i-1}} \} \) = \( \frac{1}{4}d^{i-1} \). Let \( j \in [d - 1] \). Let \( \xi_{D,j'} := \cal W_D \cap (\cup_{\ell : T^\ell \subset T_{z_{i-1}}} \cal Q_\ell) \) be the number of particles whose initial position is in \( D \) which picked a (length \( t \)) walk ending at one of the
\( T^\ell \subset T_y^{\ell+1} \), whose range contains at least \( c't / \log t > k \) vertices of \( T^\ell \). By part (ii) of Corollary 5.1 provided that \( C_1 \) is taken to be sufficiently large and that \( n \) is sufficiently large, we have that \( \xi_{D_i,y_1} \) has a Poisson distribution whose mean is at least \( |D| / (pd^{-i}) \geq c_0 \lambda d^{-1} t > 4 \log |V| \) (for some \( p, c_0 > 0 \)), and so we have that \( P(\xi_D,y_j) > 0 \). By part (ii) of Corollary 5.1, provided that \( C_1 \) is taken to be sufficiently large and that \( n \) is sufficiently large, we have that \( \xi_{D_i,y_1} \) has a Poisson distribution whose mean is at least \( |D| \lambda (pd^{-i}) \geq c_0 \lambda d^{-1} t > 4 \log |V| \) (for some \( p, c_0 > 0 \)), and so we have that \( P(\xi_D,y_j) > 0 \).

Suppose that we showed that given that \( |U_1| \geq 2k \), for all \( i \in \{j\} \) we have that \( T_z \) is conquered internally w.p. at least 1 - \( 3|V|^{-4} \sum_{\ell=0}^{d} d^\ell \). Denote

\[
G_i := \bigcup_{\ell : T^\ell \subset T_z} D_\ell \cap L(T^\ell) 
\]

(where \( D_\ell \) is as in Definition 3.4). Note that \( G_j \) is a random set which is measurable w.r.t. the \( \sigma \)-algebra generated by \( R(w_{\text{plant}}) \) and \( R(W_{\mathcal{U}_t} \cap T^\ell \cap (\mathcal{U}_{t : T'' \subset T_{z_j}} \mathcal{H}_{\ell})) \). In particular, \( G_j \) is independent of the walks picked by the particles in \( W_{\mathcal{U}_t} \cap T_{z_j} \), and of the walks picked by particles in \( W_{\mathcal{U}_t} \cap T_{z_j} \) whose length \( t \) walk ends outside \( T_{z_j} \) (more precisely, independent of the number of particles who picked each such path). Thus conditioned on the identity of \( G_j \) being a certain (fixed) set of size at least \( \frac{1}{4} |\{ \ell : T^\ell \subset T_z \}| \) (as implied by \( T_z \) getting conquered internally for lifetime \( t \)), by (3) and Poisson thinning, it follows that w.p. at least 1 - \( 3|V|^{-4} \sum_{\ell=0}^{d} d^\ell \), for every \( \ell \in [d-1] \) there is some \( T^{ae} \subset T_{y_{j+1}} \) such that \( W_{G_j} \cap Q_{ae} \) is non-empty. By the definition of the \( Q_i \)'s and the choice of constants, if this occurs, for each \( T_{y_{j+1}} \) (\( \ell \in [d-1] \)) there is some \( T^{ae} \subset T_{y_{j+1}} \), which contains at least \( k \) sites which are good for lifetime \( t \). This is precisely what is needed in order to apply the recursion. Applying the recursion on each of the \( d-1 \) trees \( T_{y_{j+1}} \) (\( \ell \in [d-1] \)) we get that \( T_{z_{j+1}} \) is conquered internally w.p. at least

\[
1 - (d-1)|V|^{-4} + 3|V|^{-4} \sum_{\ell=0}^{d} d^\ell \geq 1 - 3|V|^{-4} \sum_{\ell=0}^{j+1} d^\ell
\]

(using a union bound). This concludes the proof, using (1). \( \square \)

4 Proof of Theorem 2

We employ some of the definitions and notation from the proof of Theorem 1. We again look only on the type 1 dynamics. As opposed to the proof of Theorem 1 here we consider walks of increasing lengths.

Proof. Similarly to the proof of Theorem 1 it suffices to analyze the time it takes the type 1 dynamics to conquer \( T_{d,n} \). Let \( t, k, m, T^1, \ldots, T^m \) and
\(z_0, \ldots, z_{n-|z_0|} = \mathbf{r}, (y^j_i)_{i \in [n-|z_0|], j \in [d-1]}\) be as in the proof of Theorem 1. Again, we may assume that \(o \in \mathcal{L}(T^1)\). We consider in the proof below a modified dynamics. As in the proof of Theorem 1 we may assume that at time 0 there is a set \(D \subset T^1\) of size at least \(k\) which is activated.

Denote \(s_0 := t\) and \(t_0 := 2t\) (\(s\) for “size” and \(t\) for “time”). We define recursively, \(t_\ell := t_0 3^\ell\) and \(s_{\ell+1} := \min(s_\ell(\frac{\lambda t}{M \log |V|})^{1/2}, |V|)\), for some constant \(M > 0\) to be determined later. Observe that there exists some \(c'_0 > 0\) such that (provided that \(n\) is sufficiently large) for all \(\ell\) we have,

\[s_\ell \geq 3^{c'_0} \sum_{i=1}^\ell i \geq 3^{c_0} \ell^2.\]

Denote, \(\ell_* := \min\{\ell : s_\ell = |V|\}\). Note that \(t_{\ell_*} \leq 2^{C \sqrt{\log |V|}}\). We consider the dynamics (of the type 1 particles) in which, for all \(i \in [\ell_* - 1]\), once an activated particle leaves \(T_o(s_i)\) before time \(t_i\) (where time is measured here w.r.t. the entire process, not the walk of that particular particle), it is frozen up to time \(t_i\), at which time it continues its walk. We may think of the (infinite) walk picked by each particle in the aforementioned frozen model as the same one it picks in the original model, but in the frozen model, the walk is delayed in the case the particle leaves \(T_o(s_i)\) before time \(t_i\) for some \(i\). Note that the cover time for the frozen frog model is at least as large as the cover time of the original model.

Assuming that at time 0 there is a set \(D \subset T^1\) of size at least \(k\) which is activated, we will show that for all \(0 \leq i \leq \ell_*\), the frozen frog model satisfies that w.p. at least \(1 - C' |V|^{-4} \sum_{j=0}^i d^j\), by time \(t_i\), for every \(T^j \subset T_o(s_i)\), the set of leaves of \(T^j\) which were activated is of size at least \(t/4\) and the collection of type 1 particles whose initial position belongs to this set is of size at least \(\lambda t/16\). Denote the last event by \(A_i\). Using (1) from the proof of Theorem 1 together with the estimate \(t_{\ell_*} \leq 2^{C \sqrt{\log |V|}}\), this implies the assertion of Theorem 2.

The case \(i = 0\) follows from the analysis in the proof of Theorem 1. Here we use the fact that for each single particle, the additional requirement of not escaping \(T_o(s_0)\) by time \(t_0\) (or doing so only after a certain requirement is already satisfied by its walk) changes all relevant probabilities regarding that particle only by a constant factor. We later use this also for \(i > 0\).

Suppose now that the aforementioned claim holds for all \(i \in [j]\) for some \(j < \ell_*\). On the event \(A_j\), at time \(t_j\) there are at least \(c'_0 \lambda |T_o(s_j)|\) activated
(type 1) particles which are either inside $\mathcal{T}_o(s_j)$, or at the parent of the root of $\mathcal{T}_o(s_j)$ (these are the particles which are unfrozen at time $t_j$). Consider the collection of $\approx (\frac{M}{M \log |V|})^{1/2}$ induced subtrees of $\mathcal{T}_o(s_j)$ of size $|\mathcal{T}_o(s_j)|$. Denote them by $\mathcal{T}_{j+1}^0 = \mathcal{T}_o(s_j), \mathcal{T}_{j+1}^1, \ldots, \mathcal{T}_{j+1}^{r_{j+1}}$. Using the definition of $s_{j+1}$ and part (ii) of Corollary 5.1 for each $\ell \in [r_{j+1}]$, the probability of each of the $(\geq c_0' \lambda |\mathcal{T}_o(s_j)|)$ activated particles at time $t_j$ of visiting at least $k$ distinct vertices of at least one $\mathcal{T}^\ell \subset \mathcal{T}_{j+1}^{r_{j+1}}$ by time $2t_j$ (without escaping $\mathcal{T}_o(s_{j+1})$ before doing so) is at least $c_0t_j|\mathcal{T}_o(s_j)| |\mathcal{T}_o(s_{j+1})|^{-2} \geq c_0 M^{\lambda - 1} \log |V| / |\mathcal{T}_o(s_j)|$. By Chernoff bound, it follows that if $M$ is taken to be sufficiently large, then on the event $A_j$, for each $\ell \in [r_{j+1}]$, the probability that there is at least one activated particle at time $t_j$ that visits at least $k$ distinct vertices of at least one $\mathcal{T}^\ell \subset \mathcal{T}_{j+1}^{r_{j+1}}$ by time $2t_j$ is at least $1 - |V|^{-4}$. As there are $t_j$ time units between time $2t_j$ and $t_{j+1} = 3t_j$ this allows one to perform the recursion step. The proof is concluded similarly to the proof of Theorem 1 by a union bound over all $\mathcal{T}_{j+1}^1, \ldots, \mathcal{T}_{j+1}^{r_{j+1}}$, using the aforementioned estimate obtained from the application of Chernoff bound and using the recursion for $i = j$ on each of these $r_{j+1}$ trees.

\[ \square \]

5 Auxiliary results on SRW on finite $d$-ary trees

In this section we develop some auxiliary results concerning SRW on $\mathcal{T}_{d,n}$. We start by giving some results concerning the distribution of the hitting time of the root (which can be used to study the hitting time distribution of an arbitrary $y \in \mathcal{V}_{d,n}$ starting from any vertex in $\mathcal{T}_o$). We later give some estimates on $p^j_t(x,y)$ and then use them to study the range of SRW on $\mathcal{T}_{d,n}$.

We start by introducing some notation which shall be used throughout the section. Fix some $n, d$. Consider SRW on $\mathcal{T}_{d,n} = (\mathcal{V}_{d,n}, \mathcal{E}_{d,n})$. Denote its root be $r$. Denote by $\mathcal{L}_i$ the $i$th level of the tree and the leaf set by $\mathcal{L}$. Let $o$ be a leaf. Let $(X_s)_{s=0}^\infty$ be SRW on $\mathcal{T}_{d,n}$. Note that $Y_s := |X_s|$ is a birth and death chain on $[n] := \{0, 1, \ldots, n\}$ with transition matrix $Q(0,1) = 1 = Q(n,n-1)$ and $Q(i,i-1) = (d+1)^{-1} = 1 - Q(i,i+1)$ for $i \in [n - 1]$. Denote its law starting from state $i$ by $\mathbb{P}_i$ and its stationary distribution by $\pi$, where $\pi_0 := \frac{d}{2|\mathcal{E}_{d,n}|} = \frac{d}{2(d^n-1)}$, $\pi_n := \frac{|\mathcal{L}|}{2|\mathcal{E}_{d,n}|} = \frac{(d-1)d^n}{2(d^n-1)}$ and for $1 \leq j < n$, $\pi_j := \frac{|\mathcal{E}_j|(d+1)}{2|\mathcal{E}_{d,n}|} = \frac{(d^2-1)d^j}{2(d^n-1)}$. Elementary considerations involving effective resistance
(e.g. [14, Example 9.9]) show that for all \(i \in [n-1]\),
\[
q_i := P^i_n[T_0 < T_n] = \frac{d^{-i} - d^{-n}}{1 - d^{-n}} \quad \text{and} \quad P_r[T_r^+ < T_L] = q_i = \frac{d^{-1} - d^{-n}}{1 - d^{-n}}. \quad (5.1)
\]
Using the Doob’s h-transform (e.g. [14, Section 17.6.1]), we get that for every \(j \in [n], \ i \in [n-1]\) and \(s \geq 1\)
\[
P^j_n[Y_{s+1} = i - 1 \mid Y_s = i, s \leq T_0^+ < T_n^+] = \frac{q_{i-1}}{(d+1)q_i}
\]
\[
= \frac{d(1 - d^{-i-1})}{(d + 1)(1 - d^{-n})} \geq \frac{d}{d + 1}. \quad (5.2)
\]
Hence \(\mathbb{E}_r[T_r^+ \mid T_r^+ < T_L] \leq \mathbb{E}_{d,n}^n[T_n^+] = \frac{2d}{d-1}\) from which it is not hard to show
that \(\mathbb{E}_r[T_L \mid T_r^+ > T_L] = \frac{d+1}{d+1} n - \frac{d}{d-1}\) (the value of the constant \(\frac{4d}{d-1}\) in the
r.h.s. shall play no role in what comes). Consequently,
\[
\frac{2(d^n - 1)}{d - 1} = 2|\mathcal{E}_{d,n}|/d = \mathbb{E}_r[T_r^+] = P_r[T_r^+ < T_L]\mathbb{E}_r[T_r^+ \mid T_r^+ < T_L]
\]
\[
+P_r[T_r^+ > T_L](|\mathbb{E}_r[T_L \mid T_r^+ > T_L] + \mathbb{E}_o[T_r]) = \frac{d-1}{d} \mathbb{E}_o[T_r] + \frac{d+1}{d} n - O(1),
\]
and thus
\[
\max_{v \in \mathcal{V}_{d,n}} \mathbb{E}_v[T_r] = \mathbb{E}_o[T_r] = \frac{2d(d^n - 1)}{(d - 1)^2} - \frac{d + 1}{d - 1} n - O(1). \quad (5.3)
\]
**Lemma 5.1.** For every \(v \in \mathcal{V}_{d,n}\) and \(i \in \mathbb{N}\),
\[
P_v[T_r \geq 16id^{n-1}] \leq 2^{-i}. \quad (5.4)
\]
Conversely, there exists some \(c > 0\) such that such that
\[
\forall 16n \leq t \leq d^{n-1}, \quad ctd^{-(n-1)} \leq \min_{v \in \mathcal{V}_{d,n}} P_v[T_r \leq t] \leq td^{-(n-1)}. \quad (5.5)
\]
\[
\forall t \geq 16n, \quad \min_{v \in \mathcal{V}_{d,n}} P_v[T_r \leq t] \geq e^{-ctd^{-(n-1)}}. \quad (5.6)
\]
Moreover, there exist some \(c_1, c_2, C_1, C_2 > 0\) such that for every \(v \in \mathcal{V}_{d,n}\)
\[
\forall C_1n \leq t \leq c_1d^{n-1}, \quad c_2d^{-(n-1)} \leq P_v[T_r = t] \leq C_2d^{-(n-1)}. \quad (5.7)
\]
Proof. By (5.3) \( \max_{v \in V_{d,n}} E_v[T_v] \leq 8d^{n-1} \). Hence by Markov inequality \( \max_{v \in V_{d,n}} P_v[T_v \geq 16d^{n-1}] \leq 1/2 \). Averaging over \( (X_{16jd^n-1})_{j=1}^i \) and applying the Markov property \( i \) times yields

\[
\max_{v \in V_{d,n}} P_v[T_v \geq 16id^{n-1}] \leq \left( \max_{v \in V_{d,n}} P_v[T_v \geq 16d^{n-1}] \right)^i \leq 2^{-i}
\]

(alternatively, this could be proved by induction). We now prove (5.5). First, observe that it suffices to consider the case that \( v = o \in \mathcal{L} \) (by symmetry, any leaf achieves the minimum). Consider the aforementioned birth and death chain \( (Y_s) \) started from state \( n \). The excursions from \( n \) to itself can be partitioned into ones during which the chain does not visit \( 0 \) and ones in which it does. Let \( \xi_i \) be the length of the \( i \)th excursion of the former type and let \( Z := T_0 - \sup\{s < T_0 : Y_s = n\} \) be the length of the journey from \( n \) to \( 0 \) in the first excursion of the latter type. Let \( U := \{0 < s < T_0 : Y_s = n\} \).

Denote \( \ell := \lceil t/8 \rceil \). By (5.2), \( P_n^{|n|}[Z > 4n] \leq P_0^{|n|}[T_n > 4n] \leq e^{-c_3n} \). Similarly, \( P_n^{|n|}[\sum_{i=1}^\ell \xi_i > t - 4n] \leq e^{-c_4t} \) (using \( t \geq 16n \), \( E_n^{|n|}[\xi_i] \leq \frac{4d}{t-1} \leq 4 \) and the fact that \( E_n^{|n|}[e^{\xi_i/4}] < \infty \)). By (5.1) and the inclusion-exclusion formula, \( P_n^{|n|}[U \leq \ell] \geq \ell q_{n-1} - \left(\frac{\ell}{2}\right) q_{n-1} \geq 0.5\ell q_{n-1} \geq c_5 t d^{-(n-1)} \), where we have used (5.1) and the fact that \( t \leq d^{n-1} \). Finally,

\[
P_o[T_\ell \leq t] \geq P_n^{|n|}[U \leq \ell, \sum_{i=1}^\ell \xi_i \leq t - 4n, Z \leq 4n]
\]

\[
= P_n^{|n|}[U \leq \ell] P_n^{|n|}[\sum_{i=1}^\ell \xi_i \leq t - 4n] P_n^{|n|}[Z \leq 4n] \geq c t d^{-(n-1)}.
\]

Conversely, as \( \xi_i \geq 2 \), \( P_o[T_\ell \leq t] \leq P_n^{|n|}[U \leq \lfloor \ell \rfloor q_{n-1}] \leq t d^{n-1} \).

The proof of (5.6) is similar to that of (5.5), using the fact that the law of \( U \) under \( P_n^{|n|} \) is Geometric with parameter \( q_{n-1} \). It is left as an exercise.

We now prove (5.7). First observe that (by adjusting \( c_1, C_1 \) if necessary) it suffices to consider the case that \( v \in \mathcal{L} \). In this case, the law of \( T_v \) under \( P_v \) is log-concave and thus (5.7) follows from (5.5) (c.f. [7] Section 6.5). \( \square \)

**Lemma 5.2.** For every \( t \leq d^{n-1} \) and \( u, v \in V_{d,n} \),

\[
p^t(u, v) \leq C_1 \deg(v)/t.
\]

Conversely, for every \( u \in V_{d,n}, v \in \mathcal{L}, \) and \( d^{n-1-|u \wedge v|} \leq t \leq d^{n-1} \),

\[
p^t(u, v) \geq c_1/dt, \quad \text{if } t - (n - |u|) \text{ is even.}
\]
Proof. We first need some auxiliary calculations. Let \( \tau_0 := T_{\ell_0} \). Let \( M_t := \max \{ i : \tau_i \leq t \} \) be the maximal distance of the walk from \( \mathcal{L} \) by time \( t \). Then

1. For every \( t \in \mathbb{N} \) and \( x, y \in \mathcal{V}_{d,n} \),
   \[
   p^i(y, x) = \sum_{i \geq n-|x \wedge y|} P_y[M_t = i]P_y[X_t \in \mathcal{L}_{|x|} \mid M_t = i]d^{-(i+|x|-n)}. \tag{5.10}
   \]

2. For all \( 0 \leq i \leq j \leq n \), \( t > 0 \) and \( y \in \mathcal{L}_i \) such that \( t - (j - i) \) is even
   \[
   P_y[X_t \in \mathcal{L}_j] \leq P_y[T_{\ell} > t] + 4\pi_j P_y[T_{\ell} \leq t]. \tag{5.12}
   \]

3. Denote \( f(t) := t^{-1/2}[2d/(d + 1)^2]^t \). Let \( y \in \mathcal{V}_{d,n} \) then
   \[
   \max_u P^{2t}(y, u) \leq C f(t) + (d + 1)d^{-(n-|y|)} \tag{5.13}
   \]

Consider the walk started from \( y \). Recall that \( \bar{y}_n \) is the \( n \)th ancestor of \( y \). Let \( i \geq n - |y| \). Denote \( z_i := \frac{y_n}{y_i} \). If \( M_t = i \geq n - |y| \), then \( T_{z_i} \leq t \) and \( \{ X_s : s \leq t \} \subset \mathcal{T}_{z_i} \). If in addition \( X_t \in \mathcal{L}_j \) for some \( j \geq i \), then by symmetry \( X_t \) is equally likely to be at each of the \( d^{j-i} \) vertices of \( \mathcal{L}_{j-i}(\mathcal{T}_{z_i}) \). This clearly implies \((5.10)\). For \( 5.11 \) observe that similar reasoning yields that

\[
P_x[X_{2t} = x \mid |X_{2t}| = |x|, M_t = j] = \sum_{\ell \geq j} d^{-\ell-|x|+n} P_x[M_{2t} = \ell \mid |X_{2t}| = |x|, M_t = j] \leq d^{-j-|x|+n}.
\]

We now prove \( 2 \). Recall that \( \pi \) denotes the stationary distribution of \( Y_s = |X_s| \). Since the \( L_{\infty} \) distance from \( \pi \) is non-decreasing in time

\[
\max_{0 \leq \ell \leq n} \sup_{s \geq 0} P[|X_s| = \ell \mid |X_0| = n]/\pi_\ell = 1/\pi_n \leq 4.
\]

Using the Markov property, we get that

\[
P_y[X_t \in \mathcal{L}_j] - P_y[T_{\ell} > t] \leq P_y[T_{\ell} \leq t] \sup_{s \geq 0} P[|X_s| = j \mid |X_0| = k] \\
\leq P_y[T_{\ell} \leq t] \pi_j \left[ \max_{0 \leq \ell \leq n} \sup_{s \geq 0} P[|X_s| = \ell \mid |X_0| = n]/\pi_\ell \right] \leq 4\pi_j P_y[T_{\ell} \leq t].
\]

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We now prove (3).

\[ P^{2t}(y, u) = P_y[X_{2t} = u, T_L > 2t] + P_y[X_{2t} = u, T_L \leq 2t] \]

The first term on the r.h.s. can be bounded from above by

\[ \max_{z,u} P^T_z[Z_{2t} = u] \leq \frac{d + 1}{d} \max_{z} P^T_z[Z_{2t} = z] \leq C f(t) \]

(the first inequality is obtained by comparison with the infinite \(d\)-regular tree, \(\mathbb{T}_d^{\text{reg}}\) which is vertex transitive and hence \(\max_{u,v} P^T_{2t}(u, v) = P^T_{2t}(v, v)\)).

Finally, given \(T_L \leq 2t\) and that \(M_{T_L} = j \leq |y|\), we have that \(X_{T_L}\) has the uniform distribution over \(\mathcal{L}(\mathbb{T}_{(y-1)j})\) and is independent of \(T_L\). Using the fact that the \(L_\infty\) distance from the stationary is non-increasing in time and the triangle inequality, by averaging over \(T_L\) and \(M_{T_L}\) we get that

\[ \max_{u \in V} P_y[X_{2t} = u, T_L \leq 2t] \leq (d + 1)/|\mathcal{L}(T_y)| = (d + 1)d^{-|y|}. \]

This concludes the proof of (5.13).

We now prove (5.8). It is standard that \(\max_{x,y} p^{2t}(x,y) = \max_{x} p^{2t}(x,x)\), for all \(t \geq 0\). Since \(\max_{x,y} p^{t}(x,y)\) is non-increasing in \(t\), it suffices to consider even \(t\) and the case \(u = v = x\). By (5.13) we only need to treat the case that \(t \in \{32d^{-1}, 32d^i\}\) for some \(i \geq n - |x|\). Denote \(z_j := T_{[x]+i-j-n+1}\) and let \(o \in \mathcal{L}(T_{z_j})\). Then by (5.3) and (5.4) (applied to \(\mathbb{T}_{z_j}\) instead of \(\mathbb{T}_{d,n}\)) for every \(1 \leq j \leq i - (n - |x|)\),

\[ P_x[M_{t/2} < i - j] = P_x[T_{z_j} > t/2] \leq P_x[T_{z_j} > 16d^i \mathbb{E}_o[T_{z_j}]] \leq 2^{-d^i}. \]

Using similar reasoning as in part (2), it is not hard to show that

\[ P_x[X_t \in \mathcal{L}_{|x|} | M_{t/2} = j] \leq P_{x}^{[n]}[T_n > t/2] + 4\pi_{|x|} P_{n-j}^{[n]}[T_n \leq t/2]. \]

This together with (5.14), (5.11) and our choice of \(t\) yield (5.8) (we leave the details as an exercise).

We now prove (5.9). Fix some \(u \in \mathcal{V}_{d,n}, v \in \mathcal{L}\) and \(d^{n-1-|u \wedge v|} \leq t \leq d^{n-1}\). Denote \(m_t := \lfloor \log_d t \rfloor\). Using (5.6) it is not hard to show that there exists some \(\delta > 0\) such that \(P_u[M_{t/2} = m_t = M_t] \geq \delta\). Using the fact that (by a standard coupling argument) for all \(\ell \in \mathbb{N}\) the conditional distribution...
of \((Y_s)_{s=0}^t\) started from \(n - m_t\), conditioned on \(T_{n-m_{t-1}} > \ell\), stochastically dominates its unconditional distribution, by averaging over \(T_{L_{n-m_{t}}}\) we have

\[
P_u(|X_t| = n \mid M_{t/2} = m_t = M_t) \geq \inf_{\ell \geq t/2} P_{n-m_{t}}[Y_{\ell} = n] \geq c_2.
\]

Finally, we get that

\[
P^t(u, v) \geq P_u[M_{t/2} = m_t = M_t]P_u(|X_{2t}| = n \mid M_{t/2} = m_t = M_t)d^{-m_t} \geq \frac{c_1}{dt}.
\]

□

Recall that \(T_x(t)\) is the smallest induced tree containing \(x\) of size at least \(t\).

**Corollary 5.1.** Let \(R_t := \{X_i : 0 \leq i \leq t\}\). Denote \(g(t) = t/\log(t + 1)\). There exist \(c, c', M > 0\) such that the following hold for all \(d \geq 2\) and \(n \geq M\).

(i) For every \(\delta \in (0, 1), y \in \mathcal{V}_{d,n}, d^{n-|y|-1} + 8(n - |y| + 1) \leq t \leq d^{n-1}\) and \(A \subset \mathcal{L}(T_y(t))\) of size \(|A| \geq \delta|\mathcal{L}(T_y(t))|\),

\[
\forall z \in T_y, \quad P_z[|R_t \cap A| > c'\delta g(t) \text{ and } X_t \in T_y(t)] \geq c\delta^2. \tag{5.15}
\]

(ii) For all \(\log_d(32n) < i < n\) and \(x, y \in \mathcal{L}_{n-i}\) \((x \neq y)\) the following holds. Let \(u \in \mathcal{L}(T_y)\). Let \(T_{y,s}^{\ell_{s}}, \ldots, T_{y,s}^{\ell_1}\) be the collection of all induced subtrees of \(T_y\) of size \(|T_y(s)|\) (where \(s \leq d^i\)). For all \(z \in T_x\) and \(32i \leq s \leq t \leq d^i\),

\[
P_z[\max_{\ell \in [\ell_i]} |R_t \cap T_{y,s}^{\ell_{s}}| > c'g(s) \text{ and } X_t \in T_y] \geq ctd^{-(n+i-2|x\wedge y|)}. \tag{5.16}
\]

**Proof.** We first argue that there exist \(C_1, c_1 > 0\) such that for every \(y \in \mathcal{V}_{d,n}\)

\[
\forall 8(n - |y|) < t \leq d^{n-1}, \quad c_1g(t) \leq \mathbb{E}_y[|R_t \cap \mathcal{L}|] \leq C_1g(t). \tag{5.17}
\]

Clearly, it suffices to consider the case that \(y \in \mathcal{L}\) and \(t \in [d^{n-1}]\). Let \(y \in \mathcal{L}\). For every \(v \in \mathcal{L}\) denote \(e_v(s) := \sum_{i=0}^{s} p^i(y, v), a(s) := \sum_{i=0}^{s} p^i(y, y)\) and \(p_v(s) := P_y[T_v \leq s]\). As \(e_v(r) = \sum_{i=0}^{r} P_y[T_v = i]a(r - i)\), for all \(v \in \mathcal{L}\) and \(r \in \mathbb{N}\), by Lemma 5.2

\[
ce_v(t)/\log t \leq e_v(t)/a(t) \leq p_v(t) \leq e_v(2t)/a(t) \leq Ce_v(2t)/\log t,
\]

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Using the fact that $\sum_{v \in L} e_v(s) = E_y[\{i \leq s : X_i \in L\}] = \Theta(s)$, summing over $v \in L$ concludes the proof of (5.17).

We now prove (5.15). Again, it suffices to consider the case that $z \in L(T_y)$. Let $z \in L(T_y)$. First note that

$$\forall s_1, s_2 \geq 1, \quad P_z[R_t \cap L \geq s_1 + s_2] \leq P_z[R_t \cap L \geq s_1]P_z[R_t \cap L \geq s_2]$$

(if $\ell(j) := \inf\{i : |R_i \cap L| = j\}$, then given that $\ell(s_1) \leq t$, in order that also $\ell(s_1 + s_2) \leq t$, the walk, which at time $\ell(s_1)$ is at some leaf, has to visit $s_2$ new leaves in $t - \ell(s_1)$ steps). Thus $P_z[|R_t \cap L| \geq j]\{4E_z[|R_t \cap L|]\} \leq 4^{-j}$ for all $j \geq 1$. In particular,

$$E_z[|R_t \cap A|^2] \leq C_2(E_z[|R_t|])^2 \leq C_3g^2(t).$$

We argue that

$$E_z[|R_t \cap A|1_{X_t \in T_y(0)}] > c_t\delta g(t). \tag{5.18}$$

This implies (5.15) by the Paley-Zygmund inequality (e.g. [11, Lemma 4.1]).

Observe that since $A \subset L(T_y(t))$ for all $a \in A$,

$$P_z[X_t \in T_y(t) | a \in R_t] \geq P_z[X_t \in T_y(t)] \geq c_5 > 0$$

(by conditioning on $T_a$ and using obvious monotonicity to deduce that the fact that at time $X_{T_a} \in L(T_y(t))$ and $T_a \leq t$ can only improve the chance of being at $T_y(t)$ at time $t$). Hence, in order to prove (5.15) it suffices to show that $E_z[|R_t \cap A|] > c_4\delta g(t)$. By Lemma 5.1 with probability at least $c_6 > 0$ the walk reaches the root of $T_y(t)$ and then by time $t/2$ returns to $L(T_y(t))$ before escaping $T_y(t)$. On this event, the expected number of visits to $L(T_y(t))$ between time $t/2$ and time $t$ is at least $c_7t$. By symmetry, on the aforementioned event, every $v \in L(T_y(t))$ has the same contribution to the aforementioned mean. Using the same reasoning as in the proof of (5.17), it follows that $P_z[T_v \leq t] \geq c_8/\log t$ for every $v \in L(T_y(t))$. This concludes the proof of (5.15). We now prove (5.16).

In the setup of part (ii), by Lemma 5.1, for all $z \in T_x$ and $32i \leq s \leq t \leq d^i$.

$$P_z[T_{L(T_y)} < t/2] \geq c_9d^{-(n+i-2)|x\wedge y|}.$$  

This together with (5.15) and the Markov property imply (5.16). \qed

**Corollary 5.2.** There exist absolute constants $C_1', C_2 > 1$ such that the following hold for all $d \geq 2$, $C_1 \geq C_1'$ and $\lambda > 0$. Let $o \in L$. Denote $s = s_n := \lceil C_1\lambda^{-1}\log |V_{d,n}| \log \log |V_{d,n}| \rceil$ and $t = t_n := |T_o(s)|$.  

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Proof. We first prove part (i). Denote the collection of particles occupying \( B \) at time 0 (in the setup of part (i)) whose position at time \( T \) be at \( W \) by \( C \). Hence if \( m \) is large, then the probability that \(|W|\geq c\) is at most \( 2\log|V_d,n|\) for all \( n \geq C_3 \), (where \( C_3 > 0 \) may depend on \( C_1 \)).

(ii) Let \( T_1, \ldots, T_m \) (\( m = \frac{|V_d,n|}{t(1-o(1))} \)) be the collection of all induced subtrees (in \( T_d,n \)) of size \( t \) (i.e. \( T^i = \mathcal{T}_x^i \) for some \( x \in V_d,n \)), where \( t \) is as in part (i). For all \( i \in [m] \) let \( A_i \subset \mathcal{L}(T^i) \) be of size \( |A_i| \geq t/4 \). Denote \( A = \bigcup_{i \in [m]} A_i \). If at each \( a \in A \) there are initially \( \text{Pois}(\lambda/2) \) (independently for different vertices) particles performing simultaneously independent SRWs of length \( r := [dt\log d] \) on \( T_d,n \), then the probability that there is some vertex of \( V_d,n \) which is not visited by any of the walks is at most \( |V_d,n|^{-4} \).

Proof. We first prove part (i). Denote the collection of particles occupying \( B \) at time 0 (in the setup of part (i)) whose position at time \( t \) is in \( T_o(s) \) by \( W_B := \{w_1, \ldots, w_{|W'_B|}\} \). Since each particle has chance at least \( c_0 > 0 \) to be at \( T_o(s) \) at time \( t \), by Poisson thinning, if \( C_2 \) is taken to be sufficiently large, then the probability that \( |W'_B| \leq \frac{C_2 c_0}{4} \log|V_d,n| \) is at most \( |V_d,n|^{-4} \). Let \( (w_i(j))_{j=0}^t \) be the walk performed by \( w_i \). Denote \( J_i := \{w_i(j) : j \in ]t[\} \cap \mathcal{L}(T_o(s)) \) and \( F_i := \bigcup_{j : j \in [i]} J_i \).

Let \( U_i \) be the event that \( |F_i| \geq t/4 \) or \( |F_i \setminus F_{i-1}| \geq ct/\log t \), for some \( c > 0 \) to be determined later (where \( F_0 := \emptyset \)). Denote \( Z_i := 1_{U_i} 1_{|W'_B| \geq i} \). It follows from \( (5,13) \) that we can pick some \( c, c' > 0 \) such that

\[
\mathbb{E}[Z_i | F_{i-1}, |W'_B| \geq c'] \geq c' 1_{|W'_B| \geq i}
\]

a.s., for all \( i \in \mathbb{N} \). It follows that, conditioned on \( |W'_B| = \ell > \frac{C_2 c_0}{4} \log|V_d,n| \), the distribution of \( \sum_{i \in [\ell]} Z_i \) stochastically dominates the \( \text{Bin}(\ell, c') \) distribution. Hence if \( C_2 \geq 60/(c_0 c') \), and \( n \) is sufficiently large so that \( t \leq 5\log|V_d,n| \) then

\[
P\left[ \sum_{i \in [\ell]} Z_i \leq \frac{\log t}{4c} \mid |W'_B| = \ell \right] < |V_d,n|^{-4}, \quad \text{for all} \quad \ell > \frac{C_2 c_0}{4} \log|V_d,n|.
\]
The proof of part (i) is concluded by noting that on the event \( \sum_{i \in [|W_B|]} Z_i > \frac{\log t}{4c} \), it must be the case that \( |F|_{W_B} | \geq t/4 \).

We now prove part (ii). Denote by \( Y_u \) the number of particles from \( A \) which visit vertex \( u \) by time \( r \). Let \( z \in A \). Then if \( C_1 \) is taken to be sufficiently large, by Poisson thinning \( Y_u \) has a Poisson distribution of mean

\[
\mu_u := \frac{\lambda}{2} \sum_{a \in A} P_a[T_u \leq r] \geq \frac{\lambda}{2(d+1)} \sum_{a \in A} P_u[T_a \leq r] \geq \frac{\lambda \sum_{a \in A} \sum_{i \in [r]} P^i(u, a)}{2(d+1) \sum_{a \in A} P^i(z, z)} \geq \frac{c_1 \lambda \mathbb{E}_u[|\{i \in [r] : X_i \in A\}|]}{2(d+1) \log r} \geq \frac{c_2 \lambda r}{2(d+1) \log r} \geq 5 \log |\mathcal{V}_{d,n}| \]

(assuming \( C_1 \) is sufficiently large). The proof is concluded by applying a union bound over \( \mathcal{V}_{d,n} \) \( (e^{-5 \log |\mathcal{V}_{d,n}|} |\mathcal{V}_{d,n}| = |\mathcal{V}_{d,n}|^{-4}) \).

6 Lower bounds on the susceptibility

The approach taken here is borrowed from [4]. Fix some regular graph \( G = (V, E) \). Consider the frog model on \( G \) with parameter \( \lambda \). Let \( A \subset V \). Denote

\[
e_{u,v}(s) := \sum_{i=0}^{s} p^i(u, v), \quad m_A(s) := \min_{u \in A} e_{u,u}(s).
\]

Let \( Y_a(t) \) be the number of particles not occupying \( a \) at time 0, other than the planted particle \( w_{\text{plant}} \), which visit vertex \( a \) by time \( t \), if at time 0 all of the vertices are activated. In the notation of §2, if the index of the planted walker at \( o \) is 1, then

\[
Y_a(t) := |\{(i, v) \in \mathbb{N} \times (V \setminus \{a\}) : i \leq |W_v|, a \in \{S^m_j : j \in [t]\}, (i, v) \neq (1, o)\}|.
\]

Observe that

\[
\mathcal{S}(G) > \max\{t : \sum_{v \in V \setminus \{o\}} 1_{Y_v(t) = 0} > t\}
\]

\( (w_{\text{plant}} \text{ can activate at most } t \text{ sites by time } t) \). By Poisson thinning

\[
Y_a(t) \sim \text{Pois}(\mu_a(t)), \quad \mu_a(t) := \lambda \sum_{v \in V \setminus \{a\}} P_v[T_a \leq t], \quad (6.1)
\]

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for all \(a \in V\) and \(t\). Note that by reversibility
\[
\mu_a(t) = \lambda \sum_{v \in V \setminus \{a\}} P_v[T_v \leq t] \leq \lambda \sum_{v \in V \setminus \{a\}} e_{a,v}(t) = \lambda \left( t + 1 - e_{a,a}(t) \right) \leq \lambda t.
\]
Also, \(P_v[T_a \leq t] \leq \frac{e_{v,a}(2t)}{e_{a,a}(t)} \leq \frac{e_{a,v}(2t)}{m_A(t)}\), for all \(a, v \in V\) and \(t > 0\). Whence
\[
\mu_a(t) \leq 2\lambda t/m_A(t). \tag{6.2}
\]

**Proposition 6.1.** In the above notation, let \(Z_a(t) := 1_{Y_a(t) = 0}\) and \(Z_A(t) := \sum_{a \in A \setminus \{\emptyset\}} Z_a(t)\). Let \(\alpha_t(A) := \min(1, \frac{2}{m_A(t)})\) and \(p_A(t) := \max_{a,b \in A} P_a[T_b < t]\).

1. For every \(a \neq b \in A\) we have that \(E_\lambda[Z_a(t)] \geq e^{-\lambda \alpha_t(A)}\) and
\[
\frac{E_\lambda[Z_a(t)Z_b(t)]}{E_\lambda[Z_a(t)]E_\lambda[Z_b(t)]} = e^{E_\lambda[Y_{a,b}(t)]} \leq e^{2\lambda p_A(t)}, \text{ where} \tag{6.3}
\]
\(Y_{a,b}(t) := |\{(i, v) \in N \times (V \setminus \{a, b\}) : i \leq |W_v|, a, b \in \{S^{v,i}_j : j \in [t]\}, (i, v) \neq (1, \emptyset)\}|.

2. Consequently,
\[
\text{Var} (Z_A(t)) - E_\lambda[Z_A(t)] = \sum_{a \neq b \in A \setminus \{\emptyset\}} \text{Cov}(Z_a(t), Z_b(t)) \leq (E_\lambda[Z_A(t)])^2 (e^{2\lambda p_A(t)} - 1)
\]
\[
\text{Pr}_\lambda[Z_A(t) \leq \frac{1}{2} E_\lambda[Z_A(t)]] \leq \frac{4 \text{Var}(Z_A(t))}{(E_\lambda[Z_A(t)])^2} \leq \frac{4}{E_\lambda[Z_A(t)]} + 4(e^{2\lambda p_A(t)} - 1).
\]

3. For every \(A \subset V\) and \(s, t > 0\), there exists \(B_s \subset A\), such that \(|B_s| \geq \frac{|A|}{1 + st^2}\) and \(p_{B_s}(t) < \frac{1}{st}\). In particular, \(E_\lambda[Z_{B_s}(t)] \geq \frac{|A|}{4st^2 e^{\lambda \alpha_t(A)}}\) and
\[
\text{Pr}_\lambda[Z_{B_s}(t) \leq \frac{1}{2} E_\lambda[Z_{B_s}(t)]] < \frac{4}{E_\lambda[Z_{B_s}(t)]} + 4(e^{2\lambda/s} - 1).
\]

**Proof.** For \(E_\lambda[Z_a(t)] \geq e^{-\lambda \alpha_t(A)}\) use \((6.1)-(6.2)\). For \((6.3)\) see \[4, Proposition 4.2\]. Part (2) follows from part (1) using Chebyshev’s inequality. Part (3) follows from the fact that for every \(a \in A\), \(\sum_{b \in A \setminus \{a\}} P_a[T_b \leq t] \leq t\) and so \(|\{b \in A \setminus \{a\} : P_a[T_b \leq t] \geq \frac{1}{st}\}| \leq st^2\). Hence, by systematically deleting some vertices from \(A\) in a naive manner, we obtain a set \(B_s\) satisfying the desired properties (at each stage \(r\), if the current set is \(A_r\) and there is some \(a \in A_r\) such that \(|\{b \in A_r \setminus \{a\} : P_a[T_b \leq t] \geq \frac{1}{st}\}| > 0\), then we set \(A_{r+1} := A_r \setminus \{b \in A_r \setminus \{a\} : P_a[T_b \leq t] \geq \frac{1}{st}\}\)). \(\square\)
Proof of Theorem 3. We first prove (1.1). Set \( k := \lfloor \lambda^{-1}(\log|V| - 4\log \log|V|) \rfloor \). Apply Proposition 6.1 with \( A = V, t = k \) and \( s = \frac{1}{8}\lambda^3 k \). Then \( \mathbb{E}_\lambda[Z_{B,t}(t)] \geq \frac{|V|}{4st^2s'n} > 2k \). Thus

\[
\mathbb{P}_\lambda[Z_{B,t}(t) \leq k] < \frac{1}{2k + 4(e^{2\lambda/s} - 1)},
\]

which tends to 0 as \( |V| \to \infty \). We now prove (1.2). Let \( \delta \in (0, 1) \). Apply Proposition 6.1 with \( A = V, t = t_{\lambda, \delta}(G) - 1 \) and \( s = \lfloor \frac{|V|^{3/2}}{16} \rfloor \). By Remark 1.5 we have that \( s \to \infty \) as \( |V| \to \infty \). Note that with this choice of parameters \( m_A(t) := \kappa_t \) and so \( 2\lambda t/m_V(t) \leq (1 - \delta) \log|V| \). Thus \( \mathbb{E}_\lambda[Z_{B,t}(t)] \geq \frac{|V|}{4st^2e^{-4\lambda t/m_V(t)}} > 4|V|^{3/2} > 2t \), provided that \( |V| \) is sufficiently large. Hence (provided that \( |V| \) is sufficiently large)

\[
\mathbb{P}_\lambda[Z_{B,t}(t) \leq t] < |V|^{-\delta/2} + 4(e^{2\lambda/s} - 1),
\]

which tends to 0 as \( |V| \to \infty \). \( \square \)

Proof of the lower bound on \( \mathcal{S}(\mathcal{T}_{d,n}) \). In the notation of Proposition 6.1, by (5.9) we have that \( m_\mathcal{L}(\ell) \geq c' \ell \log \ell \) for all \( \ell \in \mathbb{N} \). Then for some \( t = [c\lambda^{-1} \log |\mathcal{T}_{d,n}| \log \log |\mathcal{T}_{d,n}|] \) we have that \( 2\lambda t/m_\mathcal{L}(t) \leq \frac{1}{4} \log n \). The proof is concluded by applying Proposition 6.1 with \( A = \mathcal{L} \), \( t \) as above and \( s = |\mathcal{L}|^{1/2} \).

\( \square \)

7 Proof for Proposition 1.1

Fix \( \lambda > 0 \) and \( \varepsilon \in (0, 1/10) \). Let \( \lambda \) be the density of walkers. Consider the case that the lifetime of the particles is \( s_n := \lfloor \lambda^{-1}(1 + \varepsilon) \log n \rfloor \). By Theorem 3 it suffices to show that \( \mathbb{P}_\lambda[\mathcal{S}(K_n) > s_n] = o(1) \). Label the vertex set by \([n]\).

We may assume that \( o = 1 \). Let \( W_i \) be the particles occupying site \( i \) at time 0. Stage 1: First expose (the first \( s_n \) steps of) the walk of each \( w \in W_1 \). Let \( D_1 \) be the union of the ranges of those walks. Set \( U_1 = \{1\} \) and \( A_1 = D_1 \setminus U_1 \).

Stage 2: Let \( K := \lambda^{-1}K_\varepsilon \) for some \( K_\varepsilon > 0 \) to be determined later. Expose the first \( K \) steps of each \( w \in W_{\min A_1} \). Denote the union of the ranges of these walks by \( D_2 \). Set \( U_2 := \{1, \min A_1\} \), \( A_2 := (A_1 \cup D_2) \setminus U_2 \).

Stage i: As long as \( A_i \) is non-empty continue as in stage 2: Expose the first \( K \) steps of each \( w \in W_{\min A_i} \). Denote the union of the ranges of these walks by \( D_{i+1} \). Set \( U_{i+1} := \{1, \min A_1, \ldots, \min A_i\} \), \( A_{i+1} := (A_i \cup D_{i+1}) \setminus U_{i+1} \).

Let \( A = A_{i_*} \), where \( i_* = \min\{i : A_i = \emptyset\} \). Note that \( |A_1| > \lambda^{-1}(1 + \varepsilon/2) \log n \) w.h.p., and so \( |W_{A_1}| \geq \log n \) w.h.p., where \( W_B := \cup_{b \in B} W_b \), for
every $B \subset [n]$. Using the concentration of the Poisson distribution around its mean, and standard arguments from the analysis of the giant component of a super-critical Erdős and Rényi random graph, it is not hard to show that if $K_\varepsilon$ is taken to be sufficiently large then w.h.p. $|A| \geq (1-\varepsilon/3)n$ and $|W_A| > \lambda(1-2\varepsilon/3)n$, where $W_A := \cup_{a \in A} W_a$. After exposing $A$ and the first $K$ (resp. $s_n$) steps of the walks performed by the particles in $W_A \setminus W_1$ (resp. $W_1$), we now expose the remaining $s_n-K$ steps of the walks of the particles in $W_A \setminus W_1$. For sufficiently large $n$ we have that $(s_n-K)(\lambda(1-2\varepsilon/3)n) \geq (1+\varepsilon/6)n \log n$ (as $\varepsilon < 1/10$). The proof is concluded by comparison with the coupon collector problem (e.g. [14, Proposition 2.4]).

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