Test for parameter change in the presence of outliers: 
the density power divergence based approach 

Junmo Song\textsuperscript{1} and Jiwon Kang\textsuperscript{2} 

\textsuperscript{1}Department of Statistics, Kyungpook National University
\textsuperscript{2}Department of Computer Science and Statistics, Jeju National University

Abstract

This study considers the problem of testing for parameter change, particularly in the presence of outliers. To lessen the impact of outliers, we propose a robust test based on the density power divergence introduced by Basu et al. (Biometrika, 1998), and then derive its limiting null distribution. Our test procedure can be naturally extended to any parametric model to which MDPDE can be applied. To illustrate this, we apply our test procedure to GARCH models. We demonstrate the validity and robustness of the proposed test through a simulation study. In a real data application to the Hang Seng index, our test locates some change-points that are not detected by the existing tests such as the score test and the residual-based CUSUM test.

Key words and phrases: test for parameter change, robust test, outliers, density power divergence, GARCH models.

1 Introduction

It is often observed, for example, that financial markets fluctuate widely by economic and political events, and it is well known that such events can cause deviating observations in data or structural breaks in underlying models. Over the past decades, most of works have dealt with these phenomena separately. For the former, researchers have developed various robust methods for reducing the impact of outlying observations. For an overview of related theories and methods, see, for example, Marona et al. (2006). The latter has also been extensively studied in the field of change point analysis and vast amount of literature have been devoted to this area. See the recent review papers by Aue and Horvth (2013) and Horvth and Rice (2014). However, there have been relatively few studies addressing the cases that both situations are involved.

This paper is concerned with the problem of testing for parameter change, particularly in the presence of outliers. As is well known, classical estimators such as MLE are very sensitive to outliers. Since various test statistics are constructed based on such estimators, one may naturally
surmise that existing tests are also likely to be affected by outliers. In the literature, Tsay (1988) investigated a procedure for detecting outliers, level shifts, and variance change in a univariate time series and Lee and Na (2005) and Kang and Song (2015) introduced a estimates-based CUSUM test using a robust estimator. Recently, Fearnhead and Rigaill (2018) proposed the penalized cost function to detect the changes in the location parameter and Song (2020) proposed trimmed residual based CUSUM test for diffusion processes. These studies consistently addressed that the previous parameter change tests are also severely damaged by outliers, which obviously indicates that it is not easy to determine whether the testing results are due to genuine changes or not when outlying observations are included in a data set being suspected of having parameter changes.

In this study, we propose a robust test for parameter change using a divergence based method. Divergences are usually taken to evaluate the discrepancy between two probability distributions, but some of them have been popularly used as a way to construct robust estimators. See, for example, Basu et al. (1998), Fujisawa and Eguchi (2008), and Ghosh and Basu (2017) for density power (DP), $\gamma$-, and S-divergence based estimation methods, respectively. In this study, we employ DP divergence (DPD) to construct a robust test. Since Basu et al. (1998) introduced the DPD-based estimation method that yields the so-called minimum DPD estimator (MDPDE), the estimation method has been successfully applied to various parametric models. See, for example, Lee and Song (2009), Kang and Lee (2014), and Song (2017). These studies showed that the corresponding MDPDEs have a strong robust property with little loss in efficiency. Recently, the DPD based method has been extended to testing problems. Basu et al. (2013, 2016) used the objective function of MDPDE to propose Wald-type tests and Ghosh et al. (2016) investigated its properties. Like the MDPDE, the induced tests are found to inherit the robust and efficient properties, and such results motivate us to consider a robust test based on DPD approach. Meanwhile, it is noteworthy that tests based on other divergences such as $\phi$- and S-divergences have also been studied before by several authors. See, for example, Batsidis et al. (2013) and Ghosh et al. (2015). For statistical inference based on divergences, we refer the reader to Pardo (2006).

Our robust test is constructed generalizing the score test for parameter change. More specifically, the test in this paper is obtained by replacing the score function in the score test with the derivatives of the objective function of MDPDE. Since the score function is actually induced from Kullback-Leibler (KL) divergence, our test can be considered as a DP divergence version of the score test. Noting that the DP divergence includes KL divergence, the proposed test is expected to enjoy the merits of the score test as well as robust and efficient properties. For instance, according to the previous studies such as Song and Kang (2018), the score test has a merit in that it produces stable sizes especially when true parameter lies near the boundary of parameter space. Furthermore, just as the score test can be applied to general parametric model, our test procedure is also applicable to any parametric model to which MDPDE can be applied. To demonstrate this, we first introduce the way of constructing a DPD-based test in i.i.d. cases and then apply our test procedure to GARCH models.
This paper is organized as follows. In Section 2, we propose a DPD-based test for parameter change and derive its asymptotic null distribution. In Section 3, we extend our method to GARCH models. We examine our method numerically through Monte Carlo simulations in Section 4. Section 5 illustrates a real data application and Section 6 concludes the paper. The technical proofs are provided in Appendix.

2 DP divergence based test for parameter change

In this section, we review the DP divergence and MDPDE by Basu et al. (1998) and then introduce a robust test statistic based on the divergence.

For two density functions $f$ and $g$, DP divergence is defined by

$$d_\alpha(g, f) := \begin{cases} \int \left\{ f^{1+\alpha}(z) - (1 + \frac{1}{\alpha}) g(z) f^\alpha(z) + \frac{1}{\alpha} g^{1+\alpha}(z) \right\} dz, & \alpha > 0, \\ \int g(z) \left\{ \log g(z) - \log f(z) \right\} dz, & \alpha = 0. \end{cases}$$

As special cases, the divergence includes the KL divergence and $L_2$ distance when $\alpha = 0$ and $\alpha = 1$, respectively. Since $d_\alpha(f, g)$ converges to $d_0(f, g)$ as $\alpha \to 0$, the above divergence with $0 < \alpha < 1$ provides a smooth bridge between KL divergence and the $L_2$ distance.

Let $X_1, \cdots, X_n$ be a random sample from an unknown density $g$. To define an estimator using the divergence, consider a family of parametric densities $\{f_\theta \mid \theta \in \Theta \subset \mathbb{R}^d\}$. Then, the MDPDE with respect to the parametric family $\{f_\theta\}$ is defined as the estimator that minimizes the empirical version of the divergence $d_\alpha(g, f_\theta)$. That is,

$$\hat{\theta}_{\alpha,n} = \arg\min_{\theta \in \Theta} \frac{1}{n} \sum_{i=1}^{n} l_\alpha(X_i; \theta) := \arg\min_{\theta \in \Theta} H_{\alpha,n}(\theta), \quad (1)$$

where

$$l_\alpha(X_i; \theta) = \begin{cases} \int f_\theta^{1+\alpha}(z) dz - \left(1 + \frac{1}{\alpha}\right) f_\theta^\alpha(X_i), & \alpha > 0, \\ -\log f_\theta(X_i), & \alpha = 0. \end{cases}$$

Here, the tuning parameter $\alpha$ plays an important role in controlling the trade-off between robustness and asymptotic efficiency of the estimator. Basu et al. (1998) showed that $\hat{\theta}_{\alpha,n}$ is weakly consistent for $\theta_\alpha := \arg\min_{\theta \in \Theta} d_\alpha(g, f_\theta)$ and asymptotically normal, and demonstrated that the estimators with small $\alpha$ have strong robust properties with little loss in asymptotic efficiency relative to MLE.

In order to focus on the parameter change problem, we assume hereafter that $g$ belongs to the parametric family $\{f_\theta\}$, that is, $g = f_{\theta_0}$ for some $\theta_0 \in \Theta$. In this case, $\theta_\alpha$ becomes equal to the true parameter vector $\theta_0$. From now, for notational convenience, we use $\partial_\theta$ and $\partial^2_{\theta\theta'}$ to denote $\frac{\partial}{\partial \theta}$ and
We now intend to test the following hypotheses in the presence of outliers:

\[ H_0 : X_1, \cdots, X_n \sim i.i.d. f_{\theta_0} \quad v.s. \quad H_1 : \text{not } H_0. \]

For this task, we construct a test statistics using the derivative of the objective function in (1). Our background idea coincides with that of the score test by Horváth and Parzen (1994). Horváth and Parzen (1994) showed that under \( H_0 \),

\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{[ns]} \partial_\theta \log f_{\hat{\theta}_n}(X_i) = -\frac{[ns]}{\sqrt{n}} \partial_\theta H_{\alpha=0,[ns]}(\hat{\theta}_n) \overset{w}{\rightarrow} J^{-1/2} B_d^0(s) \quad \text{in } \mathbb{D}([0, 1], \mathbb{R}^d),
\]

where \( \hat{\theta}_n \) and \( J \) denote the MLE and the Fisher information matrix, respectively, and \( \{ B^0_d(s) | s \geq 0 \} \) is a \( d \)-dimensional standard Brownian bridge, and then used the above to propose the score test for parameter change. In this study, we extend their result to the case of \( \alpha > 0 \).

By using Taylor’s theorem, we have that for each \( s \in [0, 1] \),

\[
\frac{[ns]}{\sqrt{n}} \partial_\theta H_{\alpha,[ns]}(\hat{\theta}_{\alpha,n}) = \frac{[ns]}{\sqrt{n}} \partial_\theta H_{\alpha,[ns]}(\theta_0) + \frac{[ns]}{n} \partial^2_{\theta\theta'} H_{\alpha,[ns]}(\theta^*_{\alpha,n,s}) \sqrt{n}(\hat{\theta}_{\alpha,n} - \theta_0) \quad (2)
\]

where \( \theta_{\alpha,n,s}^* \) is an intermediate point between \( \theta_0 \) and \( \hat{\theta}_{\alpha,n} \). Since \( \partial_\theta H_{\alpha,n}(\hat{\theta}_{\alpha,n}) = 0 \), we have that for \( s = 1 \),

\[
\sqrt{n} \partial_\theta H_{\alpha,n}(\theta_0) + \partial^2_{\theta\theta'} H_{\alpha,n}(\theta^*_{\alpha,n,1}) \sqrt{n}(\hat{\theta}_{\alpha,n} - \theta_0) = 0,
\]

and thus we can express that

\[
\sqrt{n}(\hat{\theta}_{\alpha,n} - \theta_0) = J^{-1}_\alpha \sqrt{n} \partial_\theta H_{\alpha,n}(\theta_0) + J^{-1}_\alpha (B_{\alpha,n} + J_\alpha) \sqrt{n}(\hat{\theta}_{\alpha,n} - \theta_0), \quad (3)
\]

where \( B_{\alpha,n} = \partial^2_{\theta\theta'} H_{\alpha,n}(\theta^*_{\alpha,n,1}) \) and \( J_\alpha \) is the one defined in the assumption \( \text{A6} \) below. Here, putting the above into (2), we obtain

\[
\frac{[ns]}{\sqrt{n}} \partial_\theta H_{\alpha,[ns]}(\hat{\theta}_{\alpha,n}) = \frac{[ns]}{\sqrt{n}} \partial_\theta H_{\alpha,[ns]}(\theta_0) + \frac{[ns]}{n} \partial^2_{\theta\theta'} H_{\alpha,[ns]}(\theta^*_{\alpha,n,s}) J^{-1}_\alpha \sqrt{n} \partial_\theta H_{\alpha,n}(\theta_0) + \frac{[ns]}{n} \partial^2_{\theta\theta'} H_{\alpha,[ns]}(\theta^*_{\alpha,n,s}) J^{-1}_\alpha (B_{\alpha,n} + J_\alpha) \sqrt{n}(\hat{\theta}_{\alpha,n} - \theta_0). \quad (4)
\]

To derive the limiting null distribution of the above, the strong consistency of \( \hat{\theta}_{\alpha,n} \) is required. For this, we assume the following conditions to ensure the strong uniform convergence of the objective function \( H_{\alpha,n}(\theta) \):

\textbf{A1.} The parameter space \( \Theta \) is compact.
A2. The density $f_\theta$ and the integral $\int f_\theta^{1+\alpha}(z)dz$ are continuous in $\theta$.

A3. There exists a function $B(x)$ such that $|l_\alpha(x;\theta)| \leq B(x)$ for all $x$ and $\theta$ and $E[B(X)] < \infty$.

By the assumption A2, $l_\alpha(x;\theta)$ becomes a continuous function in $\theta$. Hence, it follows that

$$\sup_{\theta \in \Theta} \left| \frac{1}{n} \sum_{i=1}^{n} l_\alpha(X_i;\theta) - E[l_\alpha(X;\theta)] \right| \xrightarrow{a.s.} 0$$

(cf. chapter 16 in Ferguson (1996)). Noting the fact that $E[l_\alpha(X;\theta)] = d_\alpha(f_{\theta_0}, f_\theta) - \frac{1}{2} \int f_\theta(z)dz$, one can see that $E[l_\alpha(X;\theta)]$ has the minimum value at $\theta_0$. Hence, $\hat{\theta}_{\alpha,n}$ converges almost surely to $\theta_0$ by the standard arguments. Assumption A3 is ensured by such condition that $\sup_{x, \theta \in \Theta} f_\theta(x) < \infty$. This condition can usually be obtained by restricting the range of scale parameter. For example, when the normal parametric family $\{N(\mu, \sigma^2)|\mu \in \mathbb{R}, \sigma > 0\}$ is considered, the condition is obtained by considering the parameter space $\Theta = \{ (\mu, \sigma) | \sigma \geq c \}$ for some $c > 0$. Another set of conditions for the strong consistency can be found, for example, in Lee and Na (2005). We introduce further assumptions. Throughout this paper, the symbol $\| \cdot \|$ denotes any norm for matrices and vectors.

A4. The integral $\int f_\theta^{1+\alpha}(z)dz$ is twice differentiable with respect to $\theta$ and the derivative can be taken under the integral sign.

A5. $\partial^2_{\theta\theta}l_\alpha(x;\theta)$ is continuous in $\theta$ and there exists an open neighborhood $N(\theta_0)$ of $\theta_0$ such that $E[\sup_{\theta \in N(\theta_0)} \| \partial^2_{\theta\theta}l_\alpha(X;\theta) \|] < \infty$.

A6. The matrices $K_\alpha$ and $J_\alpha$ defined by

$$K_\alpha = E[\partial_\theta l_\alpha(X;\theta_0)\partial_\theta l_\alpha(X;\theta_0)] \quad \text{and} \quad J_\alpha = -E[\partial^2_{\theta\theta}l_\alpha(X;\theta_0)]$$

exist and are non-singular.

The following is the first main result in this study, which is used as the building block for constructing a robust test.

**Theorem 1.** Suppose that the assumptions A1 - A6 hold. Then, under $H_0$, we have that for $\alpha \geq 0$,

$$K^{-1/2}_\alpha \left[ \frac{\lambda}{n} \partial_\theta H_{\alpha,|n\lambda|}(\hat{\theta}_{\alpha,n}) \right] \xrightarrow{w} B^0_d(s) \quad \text{in} \quad D\left([0,1], \mathbb{R}^d\right),$$

where $\{B^0_d(s)|s \geq 0\}$ is a $d$-dimensional standard Brownian bridge.

Using Theorem 1, one can construct a DPD based test for parameter constancy as follows.

**Theorem 2.** Suppose that the assumptions A1 - A6 hold. Then, under $H_0$, we have that for $\alpha \geq 0$,

$$T^\alpha_n := \max_{1 \leq k \leq n} \frac{k^2}{n} \partial_\theta H_{\alpha,k}(\hat{\theta}_{\alpha,n}) K^{-1}_\alpha \partial_\theta H_{\alpha,k}(\hat{\theta}_{\alpha,n}) \xrightarrow{d} \sup_{0 \leq s \leq 1} \|B^0_d(s)\|_2^2.$$
To implement the test above, it needs to replace $K_\alpha$ with a consistent estimates. As a natural estimator of $K_\alpha$, one can consider to use

$$\hat{K}_\alpha = \frac{1}{n} \sum_{i=1}^{n} \partial_\theta l_\alpha(X_i; \hat{\theta}_{\alpha,n}) \partial_{\theta'} l_\alpha(X_i; \hat{\theta}_{\alpha,n}).$$

Under the condition that $E \sup_{\theta \in \Theta} \| \partial_\theta l_\alpha(X; \theta) \partial_{\theta'} l_\alpha(X; \theta) \| < \infty$ for some neighborhood $\Theta^* \subset \Theta$ of $\theta_0$, it can be shown that $\hat{K}_\alpha$ converges to $K_\alpha$ in probability.

**Remark 1.** Since $-H_{0,n}(\theta)$ is the log likelihood, $T_\alpha^n$ with $\alpha = 0$ becomes the score test presented by Horváth and Parzen (1994)

**Remark 2.** Noting that $\partial_\theta H_{\alpha,n}(\hat{\theta}_{\alpha,n}) = 0$, it can be written that

$$\frac{[ns]}{n} \partial_\theta H_{\alpha,[ns]}(\hat{\theta}_{\alpha,n}) = \frac{1}{\sqrt{n}} \left\{ \sum_{i=1}^{[ns]} \partial_\theta l_\alpha(X_i; \hat{\theta}_{\alpha,n}) - \frac{[ns]}{n} \sum_{i=1}^{n} \partial_\theta l_\alpha(X_i; \hat{\theta}_{\alpha,n}) \right\} = \frac{[ns]}{n} \left( 1 - \frac{[ns]}{n} \right) \sqrt{n} \left( \frac{1}{[ns]} \sum_{i=1}^{[ns]} \partial_\theta l_\alpha(X_i; \hat{\theta}_{\alpha,n}) - \frac{1}{n} \sum_{i=\lceil [ns]+1}^{n} \partial_\theta l_\alpha(X_i; \hat{\theta}_{\alpha,n}) \right).$$

Our test can therefore be regarded as a CUSUM-type test based on $\{\partial_\theta l_\alpha(X_i; \theta)\}$. When $H_0$ is rejected by such CUSUM-type test, the change-point is located as the argument that maximizes the absolute value of the cumulative sum in test statistics. See, for example, Robbins et al. (2011). For the same reason, the change-point estimator of the test above is obtained as

$$\hat{k} := \arg\max_{1 \leq k \leq n} \frac{k^2}{n} \partial_\theta H_{\alpha,k}(\hat{\theta}_{\alpha,n}) \hat{K}_\alpha^{-1} \partial_{\theta'} H_{\alpha}(\hat{\theta}_{\alpha,n}).$$

**Remark 3.** Selection of the optimal $\alpha$ is an important practical issue. Several authors studied decision criteria to choose an optimal $\alpha$. For example, Warwick (2005) proposed a selection rule for $\alpha$ that minimizes the asymptotic mean squared error, Fujisawa and Eguchi (2006) introduced an adaptive method based on the Cramervon Mises divergence, and Durio and Isaia (2011) considered a bootstrap method based on the similarity measure between MDPD estimate and ML estimate. It should, however, be noted that the existing studies dealt with the problem in estimation situation, that is, under the assumption that there exists no parameter change. In testing procedure, the selection of $\alpha$ is more complicated. If $H_0$ is not rejected by the proposed test $T_\alpha^n$ with all $\alpha$ considered, one may employ the existing decision criteria aforementioned. However, in the cases where $H_0$ is rejected, indeed, it seems difficult to establish a decision rule. According to the simulation study below, the empirical power of $T_\alpha^n$ shows a tendency to decrease with an increase in $\alpha$. In particular, $T_\alpha^n$ with small $\alpha$ produces powers almost similar to that of the score test when the data is uncontaminated, while keeping strong robustness. This indicates that small $\alpha$ may be preferred because a large $\alpha$ can lead to a significant loss in powers when the degree of contamination
is not as large as speculated. Based on our simulation results, we recommend to use an \( \alpha \) in \([0.1,0.3]\) when practitioners do not find a proper decision rule.

**Remark 4.** Although, to the best of our knowledge, there are not yet in-depth studies on systematic selection in testing problem, one may consider to choose an \( \alpha \) in terms of forecasting performance. To this end, for each \( \alpha \) under consideration, conduct \( T^\alpha_n \) to detect change-points. Then, using the data from the last change-point, estimate the model and calculate forecasting error measures such as root mean squared errors (cf. Song (2020)). Based on the obtained values, one can select a proper \( \alpha \). In our data analysis, we illustrate the procedure to calculate forecasting errors using the model induced by each \( T^\alpha_n \).

**Remark 5.** The binary segmentation procedure can be used to find multiple changes as do other CUSUM-type tests. That is, first, (i) perform the test \( T^\alpha_n \) on the whole series \( \{X_1, \ldots, X_n\} \). If \( H_0 \) is rejected, split the series into two subseries \( \{X_1, \ldots, X_{\hat{k}}\} \) and \( \{X_{\hat{k}+1}, \ldots, X_n\} \), where \( \hat{k} \) is the one in Remark 2. Then, (ii) repeating the same procedure on each subseries until no change-point is detected, one can locate multiple change-points. For more details on the binary segmentation procedure of CUSUM-type test, see Aue and Horváth (2013) and references therein.

As studied in several papers stated in Introduction, the MDPD estimation procedure can be conveniently applied to various parametric models including time series models and multivariate models. Once such MDPDE is set up, the robust test procedure can be extended to corresponding models. As an illustration, we address a DPD based test for GARCH models using the MDPDE established in Lee and Song (2009). All the remarks mentioned above still hold for the extended cases.

### 3 DP divergence based test for GARCH models

Consider the following GARCH\((p,q)\) model:

\[
X_t = \sigma_t \epsilon_t,
\]

\[
\sigma_t^2 := \sigma_t^2(\theta) = \omega + \sum_{i=1}^{p} \alpha_i X_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2,
\]

where \( \theta = (\omega, \alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q)' \in \Theta \) in \((0, \infty) \times [0, \infty)^{p+q} \) and \( \{\epsilon_t|t \in \mathbb{Z}\} \) is a sequence of i.i.d. random variables with zero mean and unit variance. We assume that the process \( \{X_t|t \in \mathbb{Z}\} \) is strictly stationary and ergodic. The conditions for the existence of stationary and ergodic process can be found, for example, in Bougerol and Picard (1992).

In order to estimate the unknown parameter in the presence of outliers, Lee and Song (2009)
introduced MDPDE for the GARCH model as follows:

\[
\hat{\theta}_{\alpha,n} = \arg\min_{\theta \in \Theta} \frac{1}{n} \sum_{t=1}^{n} \tilde{l}_{\alpha}(X_t; \theta) := \arg\min_{\theta \in \Theta} \tilde{H}_{\alpha,n}(\theta),
\]  

(5)

where

\[
\tilde{l}_{\alpha}(X_t; \theta) = \begin{cases} 
\left( \frac{1}{\sqrt{\tilde{\sigma}_t^2}} \right)^{\alpha} \left\{ \frac{1}{\sqrt{1 + \alpha}} - \left( 1 + \frac{1}{\alpha} \right) \exp \left( -\frac{\alpha}{2} \frac{X_t^2}{\tilde{\sigma}_t^2} \right) \right\}, & \alpha > 0 \\
\frac{X_t^2}{\tilde{\sigma}_t^2} + \log \tilde{\sigma}_t^2, & \alpha = 0
\end{cases}
\]

and \(\{\tilde{\sigma}_t^2|1 \leq t \leq n\}\) is given recursively by

\[
\tilde{\sigma}_t^2 := \tilde{\sigma}_t^2(\theta) = \omega + \sum_{i=1}^{p} \alpha_i X_{t-i}^2 + \sum_{j=1}^{q} \beta_j \tilde{\sigma}_{t-j}^2.
\]  

(6)

Here, the initial values could be any constant values taken to be fixed, neither random nor a function of the parameters. So as to obtain the asymptotic properties of the MDPDE, the following regularity conditions are imposed.

\textbf{A1.} The true parameter vector \(\theta_0 \in \Theta\) and \(\Theta\) is compact.

\textbf{A2.} \(\sup_{\theta \in \Theta} \sum_{j=1}^{q} \beta_j < 1\).

\textbf{A3.} If \(q > 0\), \(A_{\theta_0}(z)\) and \(B_{\theta_0}(z)\) have no common root, \(A_{\theta_0}(1) \neq 1\), and \(\alpha_0p + \beta_0q \neq 0\), where \(A_\theta(z) = \sum_{i=1}^{p} \alpha_i z^i\) and \(B_\theta(z) = 1 - \sum_{j=1}^{q} \beta_j z^j\). (Conventionally, \(A_\theta(z) = 0\) if \(p = 0\) and \(B_\theta(z) = 1\) if \(q = 0\).)

\textbf{A4.} \(\theta_0\) is in the interior of \(\Theta\).

The following asymptotics of the MDPDE are established by Lee and Song (2009).

\textbf{Proposition 1.} For each \(\alpha \geq 0\), let \(\{\hat{\theta}_{\alpha,n}\}\) be a sequence of the MDPDEs satisfying (5). Suppose that \(\epsilon_t\)s are i.i.d. random variables from \(N(0,1)\). Then, under the assumptions \textbf{A1-A3}, \(\hat{\theta}_{\alpha,n}\) converges to \(\theta_0\) almost surely. If, in addition, the assumption \textbf{A4} holds, then

\[
\sqrt{n} \left( \hat{\theta}_{\alpha,n} - \theta_0 \right) \overset{d}{\longrightarrow} N \left( 0, \frac{k(\alpha)}{g^2(\alpha)} J_{2,\alpha}^{-1} J_{1,\alpha} J_{2,\alpha}^{-1} \right),
\]

where

\[
k(\alpha) = \frac{(1 + \alpha)^2(1 + 2\alpha^2)}{2(1 + 2\alpha)^{2/5}} - \frac{\alpha^2}{4(1 + \alpha)}, \quad g(\alpha) = \frac{\alpha^2 + 2\alpha + 2}{4(1 + \alpha)^{3/2}},
\]

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Recalling that

\[ J_{1,\alpha} = E \left[ \left( \frac{1}{\sigma_t^2(\theta_0)} \right)^{\alpha + 2} \partial_{\theta} \sigma_t^2(\theta_0) \partial_{\theta'} \sigma_t^2(\theta_0) \right] , \quad J_{2,\alpha} = E \left[ \left( \frac{1}{\sigma_t^2(\theta_0)} \right)^{2+2} \partial_{\theta} \sigma_t^2(\theta_0) \partial_{\theta'} \sigma_t^2(\theta_0) \right] . \]

**Remark 6.** Using (24) in Appendix below, one can see that

\[ k(\alpha)J_{1,\alpha} = E[\partial_{\theta} l_\alpha(X_t; \theta_0) \partial_{\theta'} l_\alpha(X_t; \theta_0)] \quad \text{and} \quad g(\alpha)J_{2,\alpha} = E[\partial_{\theta}^2 \tilde{l}_\alpha(X_t; \theta_0)], \]

where \( l_\alpha(X_t; \theta) \) is the counterpart of \( \tilde{l}_\alpha(X_t; \theta) \) obtained by replacing \( \sigma_t^2 \) with \( \sigma_t^2 \) in (5). The non-singularity of \( J_{2,\alpha} \) is shown in page 337 in Lee and Song (2009), and one can also show the invertibility of \( J_{1,\alpha} \) in a similar fashion.

Now, we construct DPD based test for the following hypotheses:

\[ H_0 : \theta_0 \text{ does not change over } X_1, \ldots, X_n \quad \text{v.s.} \quad H_1 : \text{not } H_0. \]

Using Taylor’s theorem with the same arguments used to obtain (4), we can have that for each \( s \in [0, 1] \),

\[
\frac{[ns]}{\sqrt{n}} \partial_{\theta} \tilde{H}_{\alpha,[ns]}(\hat{\theta}_{\alpha,n}) = \frac{[ns]}{\sqrt{n}} \partial_{\theta} \tilde{H}_{\alpha,[ns]}(\theta_0) + \frac{[ns]}{n} \partial_{\theta\theta} \tilde{H}_{\alpha,[ns]}(\theta_{\alpha,n,s}) J_\alpha^{-1} \sqrt{n} \partial_{\theta} \tilde{H}_{\alpha,n}(\theta_0) + \frac{[ns]}{n} \partial_{\theta\theta} \tilde{H}_{\alpha,[ns]}(\theta_{\alpha,n,s}) J_\alpha^{-1} (B_{\alpha,n} + J_\alpha) \sqrt{n} (\hat{\theta}_{\alpha,n} - \theta_0),
\]

where \( \theta_{\alpha,n,s} \) is an intermediate point between \( \theta_0 \) and \( \hat{\theta}_{\alpha,n} \), \( B_{\alpha,n} = \partial_{\theta\theta} \tilde{H}_{\alpha,n}(\theta_{\alpha,n,1}) \), and \( J_\alpha = -g(\alpha)J_{2,\alpha} \). In Appendix below, we show that the last term in the RHS of the above equation is asymptotically negligible and the first two term converges weakly to Brownian bridge, where Lemmas 4 and 5 play a critical role. The following theorem states that the similar result to Theorem 1 is also obtained in GARCH models.

**Theorem 3.** Suppose that the assumptions A1- A4 hold. Then, under \( H_0 \), we have

\[
\frac{1}{\sqrt{k(\alpha)}} J_{1,\alpha}^{-1/2} \left[ \frac{[ns]}{\sqrt{n}} \partial_{\theta} \tilde{H}_{\alpha,[ns]}(\hat{\theta}_{\alpha,n}) \right] \xrightarrow{w} B_D^{\alpha}(s) \quad \text{in } \mathbb{D}([0, 1], \mathbb{R}^D),
\]

where \( D = p + q + 1 \) and \( \{B_D^{\alpha}(s)\} \) is a \( D \)-dimensional standard Brownian bridge, and thus,

\[
\tilde{T}_n^\alpha := \frac{1}{k(\alpha)} \max_{1 \leq k \leq n} k^2 \frac{[ns]}{n} \partial_{\theta} \tilde{H}_{\alpha,k}(\hat{\theta}_{\alpha,n}) J_{1,\alpha}^{-1} \partial_{\theta} \tilde{H}_{\alpha,k}(\hat{\theta}_{\alpha,n}) \xrightarrow{d} \sup_{0 \leq s \leq 1} \|B_D^{\alpha}(s)\|^2_2.
\]

Recalling that \( k(\alpha)J_{1,\alpha} = E[\partial_{\theta} l_\alpha(X_t; \theta_0) \partial_{\theta'} l_\alpha(X_t; \theta_0)] \), one can estimate \( J_{1,\alpha} \) as follows:

\[
\hat{J}_{1,\alpha} = \frac{1}{k(\alpha)n} \sum_{t=1}^{n} \partial_{\theta} \tilde{l}_\alpha(X_t; \hat{\theta}_{\alpha,n}) \partial_{\theta'} \tilde{l}_\alpha(X_t; \hat{\theta}_{\alpha,n}).
\]
The consistency of $\hat{J}_{1,\alpha}$ is proved in Lemma 6.

**Remark 7.** Berkes et al. (2004) proposed a score test for parameter change in GARCH models. Although their test is constructed using the quasi-MLE of Berkes and Horváth (2004), the test is essentially equal to $\hat{T}_{n}^{\alpha}$ with $\alpha = 0$.

### 4 Simulation results

In the present section, we evaluate the finite sample performance of the proposed test and compare with the score test. All empirical sizes and powers in this section are calculated at 5% significance level based on 2,000 repetitions. The corresponding critical values are obtained via Monte Carlo simulations.

We first consider i.i.d. cases to see the behaviors of the tests in the presence of outliers. For this, we generate contaminated samples $\{X_t\}$ by using the following scheme: $X_t = X_{t,o} + \delta p_t \cdot \text{sign}(X_{t,o})$, where $\{X_{t,o}\}$ is a sequence of i.i.d. random variables from $N(\mu, \sigma^2)$, $\delta$ is a positive constant, and $p_t$s are i.i.d. Bernoulli random variables with success probability $p$. $\{X_{t,o}\}$ and $\{p_t\}$ are assumed to be independent. This setting describes the situation that the original data $\{X_{t,o}\}$ is contaminated by outlier process $\{\delta p_t\}$. Uncontaminated samples are obtained with $p = 0$ or $\delta = 0$. $(\mu, \sigma^2) = (0, 1)$ is considered to evaluate empirical sizes, and we change the parameter $(\mu, \sigma^2)$ at midpoint $t = n/2$ for empirical powers. The empirical sizes and powers are presented in Table 1, where the left sub-table shows the results for uncontaminated case and the right for contaminated case with $p = 1\%$ and $\delta = 10$. In the left sub-table, one can see that $T_{n}^{\alpha}$ yields proper sizes and reasonable powers in all $\alpha$ considered, and the score test $T_{n}^{0}$ shows best performance as expected. It is noteworthy that the power tends to decrease as $\alpha$ increases and that $T_{n}^{\alpha}$ with $\alpha$ close to 0 shows similar performance to $T_{n}^{0}$. In the right sub-table, we can observe the power losses of $T_{n}^{0}$. In particular, $T_{n}^{0}$ is severely compromised in testing for the change in $\sigma^2$, that is, variance change. In contrast, $T_{n}^{\alpha}$ produces

| No outliers | $n$ | $T_{n}^{0}$ | $T_{n}^{\alpha}$ | $p = 1\%, \delta = 10$ | $n$ | $T_{n}^{0}$ | $T_{n}^{\alpha}$ |
|-------------|----|-------------|-----------------|-------------------|----|-------------|-----------------|
| Sizes       | 500 | 0.034       | 0.040 0.044 0.042 0.043 | Sizes            | 500 | 0.026       | 0.047 0.046 0.048 0.046 |
| $(\mu, \sigma^2) = (0, 1)$ | 1000 | 0.041       | 0.046 0.047 0.048 0.052 | $(\mu, \sigma^2) = (0, 1)$ | 1000 | 0.030       | 0.043 0.042 0.046 0.044 |
| $\mu : 0 \rightarrow 0.15$ | 500 | 0.221       | 0.222 0.218 0.214 0.198 | $\mu : 0 \rightarrow 0.15$ | 500 | 0.118       | 0.216 0.207 0.201 0.186 |
| $\quad$ | 1000 | 0.469       | 0.466 0.460 0.436 0.406 | $\quad$ | 1000 | 0.250       | 0.464 0.452 0.436 0.399 |
| $\mu : 0 \rightarrow 0.3$ | 500 | 0.764       | 0.758 0.752 0.735 0.696 | $\mu : 0 \rightarrow 0.3$ | 500 | 0.528       | 0.754 0.740 0.722 0.680 |
| $\quad$ | 1000 | 0.984       | 0.980 0.976 0.972 0.956 | $\quad$ | 1000 | 0.834       | 0.978 0.978 0.972 0.955 |
| $\sigma^2 : 1 \rightarrow 1.25$ | 500 | 0.247       | 0.246 0.232 0.216 0.193 | $\sigma^2 : 1 \rightarrow 1.25$ | 500 | 0.026       | 0.230 0.230 0.222 0.196 |
| $\quad$ | 1000 | 0.503       | 0.503 0.480 0.452 0.387 | $\quad$ | 1000 | 0.038       | 0.466 0.450 0.420 0.376 |
| $\sigma^2 : 1 \rightarrow 1.5$ | 500 | 0.704       | 0.702 0.670 0.638 0.567 | $\sigma^2 : 1 \rightarrow 1.5$ | 500 | 0.040       | 0.698 0.688 0.654 0.574 |
| $\quad$ | 1000 | 0.968       | 0.966 0.958 0.946 0.900 | $\quad$ | 1000 | 0.068       | 0.958 0.950 0.935 0.883 |
empirical powers similar to the powers obtained in the left sub-table, i.e., uncontaminated cases. This indicates that $T_n^\alpha$ is less affected by outliers. Such power losses of the score test and the robustness of the proposed test are clearly shown in Table 2, which present the results for more contaminated cases. In all contaminated cases, size distortions are not observed.

| $p = 1\%, \delta = 15$ | $n$ | $T_n^0$ | $0.1$ | $0.2$ | $0.3$ | $0.5$ |
|-------------------------|-----|---------|-------|-------|-------|-------|
| Sizes                   | 500 | 0.018   | 0.040 | 0.042 | 0.044 | 0.046 |
| $(\mu, \sigma^2) = (0, 1)$ | 1000| 0.024   | 0.045 | 0.046 | 0.046 | 0.047 |
| $\mu: 0 \rightarrow 0.15$ | 500 | 0.100   | 0.237 | 0.230 | 0.227 | 0.215 |
|                         | 1000| 0.172   | 0.440 | 0.429 | 0.418 | 0.370 |
| $\mu: 0 \rightarrow 0.3$ | 500 | 0.408   | 0.762 | 0.750 | 0.730 | 0.692 |
|                         | 1000| 0.641   | 0.974 | 0.972 | 0.966 | 0.954 |
| $\sigma^2: 1 \rightarrow 1.25$ | 500 | 0.023   | 0.242 | 0.232 | 0.224 | 0.196 |
|                         | 1000| 0.038   | 0.492 | 0.460 | 0.432 | 0.368 |
| $\sigma^2: 1 \rightarrow 1.5$ | 500 | 0.026   | 0.687 | 0.670 | 0.632 | 0.563 |
|                         | 1000| 0.040   | 0.962 | 0.950 | 0.940 | 0.894 |

| $p = 3\%, \delta = 10$ | $n$ | $T_n^0$ | $0.1$ | $0.2$ | $0.3$ | $0.5$ |
|-------------------------|-----|---------|-------|-------|-------|-------|
| Sizes                   | 500 | 0.034   | 0.049 | 0.050 | 0.050 | 0.046 |
| $(\mu, \sigma^2) = (0, 1)$ | 1000| 0.034   | 0.055 | 0.051 | 0.049 | 0.046 |
| $\mu: 0 \rightarrow 0.15$ | 500 | 0.095   | 0.226 | 0.222 | 0.216 | 0.204 |
|                         | 1000| 0.160   | 0.439 | 0.420 | 0.406 | 0.384 |
| $\mu: 0 \rightarrow 0.3$ | 500 | 0.308   | 0.771 | 0.756 | 0.737 | 0.698 |
|                         | 1000| 0.612   | 0.976 | 0.972 | 0.964 | 0.946 |
| $\sigma^2: 1 \rightarrow 1.25$ | 500 | 0.036   | 0.230 | 0.222 | 0.208 | 0.188 |
|                         | 1000| 0.040   | 0.466 | 0.460 | 0.424 | 0.370 |
| $\sigma^2: 1 \rightarrow 1.5$ | 500 | 0.034   | 0.687 | 0.692 | 0.650 | 0.574 |
|                         | 1000| 0.049   | 0.948 | 0.952 | 0.936 | 0.890 |

Next, we examine the performance of $\tilde{T}_n^\alpha$ in the following GARCH(1,1) model:

\[ X_t = \sigma_t \epsilon_t, \]
\[ \sigma_t^2 = w + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2, \]

where \{\epsilon_t\} is a sequence of i.i.d. random variables from $N(0, 1)$. For empirical sizes, we generate samples with $(w, \alpha_1, \beta_1) = (0.5, 0.2, 0.4)$ and $(0.5, 0.15, 0.8)$. The latter parameter value is employed to see the performance in more volatile situation. Two types of outliers, innovation outliers (IO) and additive outliers (AO), are considered. We generate samples with IO by replacing $\epsilon_t$ in the GARCH model above with contaminated error $\tilde{\epsilon}_t = \epsilon_t + |Z_{t,c}|p_t \cdot sign(\epsilon_t)$, where \{Z_{t,c}\} and \{p_t\} are sequences of i.i.d. random variables from $N(0, \sigma_c^2)$ and Bernoulli distribution with parameter $p$, respectively. It is assumed that \{\epsilon_t\}, \{Z_{t,c}\}, and \{p_t\} are all independent. Samples contaminated by AO are obtained by the following model: $X_t = X_{t,o} + |Z_{t,c}|p_t \cdot sign(X_{t,o})$, where \{X_{t,o}\} is the uncontaminated sample from the GARCH(1,1) model above. Simulation results for the cases of $(w, \alpha_1, \beta_1) = (0.5, 0.2, 0.4)$ and $(0.5, 0.15, 0.8)$ are provided in the left and right sub-tables in Tables 3-7 respectively.
Table 3: Empirical sizes and powers of $\tilde{T}^\alpha_n$ and $\tilde{T}^\alpha_n$ in GARCH (1,1) models without outliers.

| n   | $\tilde{T}^\alpha_n$ | $\tilde{T}^\alpha_n$ |
|-----|-----------------------|-----------------------|
|     | 0.1 0.2 0.3 0.5       | 0.1 0.2 0.3 0.5       |
| Sizes |                       |                       |
| $\alpha_1$: 0.2 $\rightarrow$ 0.5 | 1500 0.890 0.898 0.900 0.907 | 1500 0.890 0.898 0.900 0.907 |
| $\beta_1$: 0.4 $\rightarrow$ 0.6 | 1500 0.886 0.886 0.890 0.890 | 1500 0.886 0.886 0.890 0.890 |

Table 4: Empirical sizes and powers of $\tilde{T}^\alpha_n$ and $\tilde{T}^\alpha_n$ in the case of IO contamination with $p = 1\%$ and $\sigma_v^2 = 10$.

| n   | $\tilde{T}^\alpha_n$ | $\tilde{T}^\alpha_n$ |
|-----|-----------------------|-----------------------|
|     | 0.1 0.2 0.3 0.5       | 0.1 0.2 0.3 0.5       |
| Sizes |                       |                       |
| $\alpha_1$: 0.2 $\rightarrow$ 0.5 | 1500 0.214 0.214 0.214 0.214 | 1500 0.214 0.214 0.214 0.214 |
| $\beta_1$: 0.4 $\rightarrow$ 0.6 | 1500 0.210 0.210 0.210 0.210 | 1500 0.210 0.210 0.210 0.210 |

Table 5: Empirical sizes and powers of $\tilde{T}^\alpha_n$ and $\tilde{T}^\alpha_n$ in the case of IO contamination with $p = 3\%$ and $\sigma_v^2 = 10$.

| n   | $\tilde{T}^\alpha_n$ | $\tilde{T}^\alpha_n$ |
|-----|-----------------------|-----------------------|
|     | 0.1 0.2 0.3 0.5       | 0.1 0.2 0.3 0.5       |
| Sizes |                       |                       |
| $\alpha_1$: 0.2 $\rightarrow$ 0.5 | 1500 0.409 0.409 0.409 0.409 | 1500 0.409 0.409 0.409 0.409 |
| $\beta_1$: 0.4 $\rightarrow$ 0.6 | 1500 0.343 0.343 0.343 0.343 | 1500 0.343 0.343 0.343 0.343 |
Table 6: Empirical sizes and powers of $T^0_n$ and $T^\alpha_n$ in the case of AO contamination with $p = 1\%$ and $\sigma^2_v = 10$.

|       | $T^0_n$ |       | $T^\alpha_n$ |
|-------|---------|-------|--------------|
|       | $n$     | 0.1   | 0.2          | 0.3          | 0.5          |
| Sizes | 500     | 0.068 | 0.054        | 0.058        | 0.066        | 0.064        |
| $(w, \alpha_1, \beta_1)$ | 1000 | 0.051 | 0.053        | 0.056        | 0.056        | 0.060        |
| = (0.5, 0.2, 0.4) | 1500 | 0.052 | 0.055        | 0.066        | 0.066        | 0.066        |
| $w: 0.5 \rightarrow 0.8$ | 500 | 0.232 | 0.289        | 0.315        | 0.307        | 0.252        |
|       | 1000 | 0.472 | 0.736        | 0.756        | 0.724        | 0.618        |
|       | 1500 | 0.678 | 0.921        | 0.935        | 0.922        | 0.870        |
| $\alpha_1: 0.2 \rightarrow 0.5$ | 500 | 0.258 | 0.384        | 0.444        | 0.462        | 0.426        |
|       | 1000 | 0.702 | 0.876        | 0.896        | 0.884        | 0.832        |
|       | 1500 | 0.898 | 0.984        | 0.986        | 0.982        | 0.967        |
| $\beta_1: 0.4 \rightarrow 0.6$ | 500 | 0.246 | 0.348        | 0.377        | 0.370        | 0.312        |
|       | 1000 | 0.606 | 0.874        | 0.884        | 0.854        | 0.766        |
|       | 1500 | 0.824 | 0.985        | 0.982        | 0.976        | 0.941        |

Table 7: Empirical sizes and powers of $T^0_n$ and $T^\alpha_n$ in the case of AO contamination with $p = 3\%$ and $\sigma^2_v = 10$.

|       | $T^0_n$ |       | $T^\alpha_n$ |
|-------|---------|-------|--------------|
|       | $n$     | 0.1   | 0.2          | 0.3          | 0.5          |
| Sizes | 500     | 0.157 | 0.096        | 0.092        | 0.094        | 0.092        |
| $(w, \alpha_1, \beta_1)$ | 1000 | 0.139 | 0.088        | 0.085        | 0.086        | 0.092        |
| = (0.5, 0.2, 0.4) | 1500 | 0.121 | 0.090        | 0.092        | 0.090        | 0.087        |
| $w: 0.5 \rightarrow 0.8$ | 500 | 0.190 | 0.246        | 0.322        | 0.338        | 0.302        |
|       | 1000 | 0.289 | 0.647        | 0.754        | 0.748        | 0.685        |
|       | 1500 | 0.411 | 0.874        | 0.923        | 0.924        | 0.887        |
| $\alpha_1: 0.2 \rightarrow 0.5$ | 500 | 0.193 | 0.358        | 0.482        | 0.523        | 0.496        |
|       | 1000 | 0.504 | 0.852        | 0.912        | 0.907        | 0.876        |
|       | 1500 | 0.782 | 0.982        | 0.992        | 0.992        | 0.984        |
| $\beta_1: 0.4 \rightarrow 0.6$ | 500 | 0.182 | 0.318        | 0.432        | 0.466        | 0.432        |
|       | 1000 | 0.368 | 0.807        | 0.876        | 0.874        | 0.818        |
|       | 1500 | 0.566 | 0.968        | 0.982        | 0.984        | 0.968        |

Table 3 reports the results for uncontaminated cases. It can be seen that each $T^\alpha_n$ achieves good sizes in all cases. One can also see that the empirical powers of the tests increase as the sample size increases and the tests yield good powers except for the case where $(w, \alpha_1, \beta_1) = (0.5, 0.15, 0.8)$ changes to $(0.2, 0.15, 0.8)$, say Case*. In this case, the proposed test is observed to be less powerful, so $n = 2000$ is considered only for the case. As in the i.i.d. cases above, $T^\alpha_n$ performs similarly to $T^0_n$ when $\alpha$ is close to 0 and also shows a decreasing trend in powers as $\alpha$ increases. As seen in the
previous Case*, $\tilde{T}_n^{\alpha}$ with $\alpha > 0.3$ can show significant loss in power, thus we recommend not to use too large an $\alpha$.

Results for the IO contaminated cases are presented in Tables 4 and 5. We first note that $\tilde{T}_n^{0}$ exhibits significant power losses whereas $\tilde{T}_n^{\alpha}$ with $\alpha > 0$ maintains good powers also except for Case*. In the case of $(w, \alpha_1, \beta_1) = (0.5, 0.15, 0.8)$, the sizes of $\tilde{T}_n^{0}$ is severely distorted when $p = 3\%$ and $\sigma_v^2 = 10$, but the proposed test shows no distortions. Although $\tilde{T}_n^{0}$ yields higher empirical sizes in this case, substantial power losses are observed in $\tilde{T}_n^{0}$.

Tables 6 and 7 summarize the results for the cases of the AO contamination. We can also see the power losses of $\tilde{T}_n^{0}$, but not as large as the IO contaminated cases. This indicates that the score test is more affected by IO than by AO. Interestingly, $\tilde{T}_n^{0}$ is almost insensitive to AO in the case of $(w, \alpha_1, \beta_1) = (0.5, 0.15, 0.8)$. But, even in this cases, $\tilde{T}_n^{\alpha}$ outperforms $\tilde{T}_n^{0}$.

Overall, our simulation results strongly support the validity of the proposed test. In this section, we can see that our test is sufficiently robust against outliers and the test with $\alpha$ close to 0 is as powerful as the score test when data is not contaminated. However, we also observe that $\tilde{T}_n^{\alpha}$ can suffer from power losses when a large $\alpha$ is employed. So, one should be careful not to use $\alpha$ that is too large. As mentioned in Remark 3, we recommend to use an $\alpha$ in $[0.1,0.3]$ based on the simulation results.

5 Real data analysis

In this section, we illustrate a real data application to the Hang Seng index in Hong Kong stock market. Our data consists of daily closing prices from Jan 2, 1986 to April 30, 1990. The index series \{X_t\} and its return series \{r_t\}, where $X_t$ is the index value at time $t$ and $r_t = 100 \log(X_t/X_{t-1})$, are displayed in Figures 1 and 2 respectively. In Figure 2 we can see some deviating observations which may interfere with correct statistical inferences. This data was previously analyzed by Lee and Song (2009), where they fitted GARCH(1,1) model to the data and estimated the model using MDPD estimator. In the present analysis, we also fit GARCH(1,1) model to the index data from 1986 to 1989 and examine whether there were parameter changes during the period. And then, based on each testing result, we calculate one-step-ahead out-of-sample forecasts for the conditional variance. A proper $\alpha$ for the data is chosen based on the forecasting results.

ML estimates of GARCH(1,1) model for the period from 1986 to 1989 are obtained as follows: $\hat{w} = 0.112$, $\hat{\alpha}_1 = 0.282$, and $\hat{\beta}_1 = 0.740$. Here, it should be noted that $\hat{\alpha}_1 + \hat{\beta}_1$ is greater than one, indicating that the parameters are estimated out of the stationary region of parameter space. One can guess that outlying observations unduly affected the ML estimation. As addressed in the studies such as Hillebrand (2005), parameter changes can also result in spuriously high estimates of $\alpha_1 + \beta_1$. Taking into account both possible cases, we perform the proposed test $\tilde{T}_n^{\alpha}$ for $\alpha \in \{0,0.1,0.2,\cdots,1.0\}$, where $\tilde{T}_n^{0}$ is the score test. For comparison, we additionally conduct the
Figure 1: Time series plot of the Hang Seng index from Jan 2, 1986 to April 30, 1990

Figure 2: Return series of the Hang Seng index from Jan 2, 1986 to April 30, 1990

following residual-based CUSUM test for parameter change:

\[
T_n^R := \frac{1}{\sqrt{n \hat{\tau}_n}} \max_{1 \leq k \leq n} \left| \sum_{t=1}^{k} \hat{\epsilon}_t^2 - \frac{k}{n} \sum_{t=1}^{n} \hat{\epsilon}_t^2 \right|
\]

where \( \hat{\epsilon}_t \) denotes the residual in GARCH(1,1) model and \( \hat{\tau}_n^2 = \frac{1}{n} \sum_{t=1}^{n} \hat{\epsilon}_t^4 - \left( \frac{1}{n} \sum_{t=1}^{n} \hat{\epsilon}_t^2 \right)^2 \). Under \( H_0 \), \( T_n^R \) converges in distribution to \( \sup_{0 \leq s \leq 1} |B^\alpha(s)| \), where \( \{B^\alpha(s) | s \geq 0\} \) is the standard Brownian bridge (cf. Kulperger and Yu (2005) and Song and Kang (2018)). \( T_n^R \) and its p-value are obtained to be 0.935 and 0.653, respectively. Hence, \( T_n^R \) do not reject \( H_0 \).

Table 8 provides the test statistics, p-values, and estimated change-points. We first note that
Table 8: Test statistics [p-value] and estimated change point of the score test $\tilde{T}_n^0$ and $\tilde{T}_n^\alpha$

| $\alpha$ | 0   | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $\tilde{T}_n^\alpha$ | 0.681 | 3.069 | 3.755 | 4.051 | 4.256 | 4.369 | 4.288 | 4.102 | 3.887 | 3.679 | 3.491 |
| [0.964] | [0.046] | [0.015] | [0.008] | [0.005] | [0.005] | [0.008] | [0.011] | [0.017] | [0.023] |
| chg.pt   | 575 | 575 | 575 | 568 | 568 | 568 | 568 | 568 | 568 | 568 |

$\tilde{T}_n^\alpha$ with $\alpha > 0$ produce p-values less than 0.05 whereas $\tilde{T}_n^0$ yields the p-value of 0.964. That is, for all $\alpha$ considered, the proposed tests reject $H_0$ but the score test do not reject. Recalling the simulation results that $\tilde{T}_n^0$ suffer from power losses in the presence of outliers, we could presume that $\tilde{T}_n^0$ misses a significant parameter change. $\tilde{T}_n^\alpha$ with $\alpha \leq 0.3$ and $\alpha \geq 0.4$ locate 575 (May 4, 1988) and 568 (April 25, 1988), respectively, as the change-points. To detect further changes, we conduct $\tilde{T}_n^\alpha$ for each $\alpha > 0$ using the binary segmentation method in Remark 5, and one more change-point (Aug 17, 1987) is detected by $\tilde{T}_n^\alpha$ with $\alpha \leq 0.3$. The sub-period obtained by each $\tilde{T}_n^\alpha$ and estimation results are presented in Table 9. From the table, one can see that all values of $\hat{\alpha}_1 + \hat{\beta}_1$ are less than one in all sub-periods. In particular, for every $\alpha$, $\hat{\alpha}_1 + \hat{\beta}_1$ in the last sub-period is estimated to be smaller than in the previous sub-periods and $\hat{w}$ is obtained to be larger than before.

Table 9: ML and MDPD estimates for each sub-period obtained by $\tilde{T}_n^\alpha$

| Period                          | 1/2/86 - 12/29/89 |
|---------------------------------|-------------------|
|                                 | $\hat{w}$ | $\hat{\alpha}_1$ | $\hat{\beta}_1$ |
| MLE                             | 0.112 | 0.282 | 0.740 |
| Sub-Period                      | $\hat{w}$ | $\hat{\alpha}_1$ | $\hat{\beta}_1$ |
|                                 | $\hat{w}$ | $\hat{\alpha}_1$ | $\hat{\beta}_1$ |
|                                 | 0.1   | 0.049 | 0.063 | 0.900 |
|                                 | 0.2   | 0.050 | 0.058 | 0.903 |
|                                 | 0.3   | 0.051 | 0.053 | 0.906 |
|                                 | 0.4   | 0.062 | 0.020 | 0.931 |
|                                 | 0.5   | 0.063 | 0.021 | 0.929 |
|                                 | 0.6   | 0.065 | 0.022 | 0.927 |
|                                 | 0.7   | 0.067 | 0.023 | 0.925 |
|                                 | 0.8   | 0.069 | 0.025 | 0.922 |
|                                 | 0.9   | 0.071 | 0.026 | 0.919 |
|                                 | 1.0   | 0.072 | 0.028 | 0.916 |
|                                 | 0.136 | 0.051 | 0.764 |
|                                 | 0.118 | 0.042 | 0.793 |
|                                 | 0.110 | 0.040 | 0.802 |
|                                 | 0.108 | 0.040 | 0.802 |
|                                 | 0.108 | 0.042 | 0.799 |
|                                 | 0.108 | 0.043 | 0.794 |
|                                 | 0.109 | 0.045 | 0.789 |

Now, we calculate one-step-ahead out-of-sample forecasts of the conditional variance and compare forecasting performance of the models without and with parameter changes. Forecasting using the model with changes means that predicted values are obtained using the data after the
Table 10: One-step-ahead forecasting errors for the models without and with parameter changes

| Estimator | MLE | MDPDE with $\alpha$ |  
|-----------|-----|----------------------|
| $\alpha$  | 0   | 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0 |
| Model     | without change | with change |
| RMSE      | 2.336 | 2.125 2.127 2.131 2.138 2.143 2.145 2.147 2.147 2.147 2.148 |

last change-point. That is, letting $t_c$ be the last change-point, prediction of $\sigma^2_{t+1}$ is conducted using the observation $\{r_{t_c+1}, \ldots, r_t\}$. For $\alpha \leq 0.3$ and $\alpha \geq 0.4$, $t_c$ is 575 and 568, respectively. In the case of no change, $t_c$ is 0. For the purpose of comparison, we estimate the model without change using MLE. This situation describes that the data is analyzed without any robust methods. For the period from Jan 1990 to April 1990, total 80 observations, one-step-ahead predicted value, $\hat{\sigma}^2_{t,t+1}$, of the conditional variance is calculated as follows:

$$\hat{\sigma}^2_{t,t+1} = \hat{\omega}_t + \hat{\alpha}_t r_t^2 + \hat{\beta}_t \hat{\sigma}^2_t(\hat{\omega}_t, \hat{\alpha}_t, \hat{\beta}_t),$$

where $\hat{\omega}_t, \hat{\alpha}_t,$ and $\hat{\beta}_t$ are the estimates obtained using the data $\{r_{t_c+1}, \ldots, r_t\}$ and $\hat{\sigma}^2_t(\hat{\omega}_t, \hat{\alpha}_t, \hat{\beta}_t)$ is the one recursively calculated as in [6]. Since the true conditional variances are unobservable, we use $r_{t+1}^2$ as a proxy of $\sigma^2_{t+1}$. The following root mean squared error (RMSE) is considered to evaluate forecasting performance:

$$\sqrt{\frac{1}{80} \sum_{t=987}^{1066} (r_{t+1}^2 - \hat{\sigma}^2_{t,t+1})^2},$$

where $r_{988}$ and $r_{1067}$ are the return values at Jan 2, 1990 and April 30, 1990, respectively. Table 10 present the forecasting errors. One can see that the model with parameter change produces the smaller RMSE. In terms of forecasting performance, a proper $\alpha$ can be selected as 0.1, which produce smallest RMSE. Based on the estimation and the forecasting results, we can therefore conclude that the model with parameter change is better fitted to the data.

Our empirical findings support the usefulness of our proposed test. The proposed test can detect the parameter changes in the presence of deviating observations, whereas the score test and the residual-based CUSUM test miss the significant changes. The parameters are estimated comparatively differently in each sub-period divided by the proposed test, and the models accommodating parameter change show better forecasting performances. In such situation that seemingly outliers are included in a data set being suspected of having parameter changes, we expect that our test can be a promising tool for detecting parameter change.
6 Concluding remark

In this study, we proposed a robust test for parameter change using the DP divergence. Since the DP divergence includes KL divergence, our test can be viewed as a generalized version of the score test. Under regularity conditions, the limiting null distribution of the proposed test is established. Our simulation results demonstrated that the proposed test is robust to outliers whereas the score test is damaged by outliers. In particular, like the estimators induced from DP divergence, our test with small $\alpha$ is also observed to maintain strong robustness with little loss in power relative to the score test. In the real data analysis, the usefulness of the proposed test is demonstrated by locating some change-points that are not detected by the score test and the residual-based CUSUM test.

The MDPD estimation procedure can be conveniently applied to various parametric models including time series models and multivariate models. As seen in Section 3, once such MDPDE is set up, our test procedure can be readily extended to these models. We leave these extensions as a possible topic of future study.

7 Appendix

In the present section, we provide the proofs of Theorems 1 and 3 for the case of $\alpha > 0$. The following lemma is very helpful in proving Theorem 1.

Lemma 1. Suppose that the assumptions A1-A6 in Theorem 7 hold. Then, under $H_0$, we have

$$\max_{1 \leq k \leq n} \frac{k}{n} \left\| \partial_{\theta}^2 l_\alpha(X; \theta_k) + J_\alpha \right\| = o(1) \quad a.s.,$$

where $\{\hat{\theta}_{\alpha,n,k} \mid 1 \leq k \leq n, n \geq 1\}$ is any double array of $\Theta$-valued random vectors with $\|\hat{\theta}_{\alpha,n,k} - \theta_0\| \leq \|\hat{\theta}_{\alpha,n} - \theta_0\|$.

Proof. By A5, we have

$$E \sup_{\theta \in N(\theta_0)} \left\| \partial_{\theta}^2 l_\alpha(X; \theta) - \partial_{\theta}^2 l_\alpha(X; \theta_0) \right\| < \infty. \quad (8)$$

Then, for any $\epsilon > 0$, we can take a neighborhood $N_\epsilon(\theta_0)$ such that

$$E \sup_{\theta \in N_\epsilon(\theta_0)} \left\| \partial_{\theta}^2 l_\alpha(X; \theta) - \partial_{\theta}^2 l_\alpha(X; \theta_0) \right\| < \epsilon \quad (9)$$

by decreasing the neighborhood in (8) to the singleton $\theta_0$. Since $\hat{\theta}_{\alpha,n}$ converges almost surely to
Due to (9), we can see that

\[ \theta_0, \text{ we have that for sufficiently large } n, \]

\[ \max_{1 \leq k \leq n} \frac{k}{n} \left\| \partial_{\theta \theta}^2 H_{\alpha,k}(\hat{\theta}_{\alpha,n,k}) + J_\alpha \right\| \]

\[ \leq \max_{1 \leq k \leq n} \frac{k}{n} \left\| \partial_{\theta \theta}^2 H_{\alpha,k}(\hat{\theta}_{\alpha,n,k}) - \partial_{\theta \theta}^2 H_{\alpha,k}(\theta_0) \right\| + \max_{1 \leq k \leq n} \frac{k}{n} \left\| \partial_{\theta \theta}^2 H_{\alpha,k}(\theta_0) + J_\alpha \right\| \]

\[ \leq \frac{1}{n} \sum_{i=1}^{n} \sup_{\theta \in N_i(\theta_0)} \left\| \partial_{\theta \theta}^2 l_\alpha(X_i; \theta) - \partial_{\theta \theta}^2 l_\alpha(X_i; \theta_0) \right\| + \max_{1 \leq k \leq n} \frac{k}{n} \left\| \partial_{\theta \theta}^2 H_{\alpha,k}(\theta_0) + J_\alpha \right\| \]

\[ := I_n + II_n \quad a.s. \]

Due to (9), we can see that

\[ \lim_{n \to \infty} I_n = E \sup_{\theta \in N_i(\theta_0)} \left\| \partial_{\theta \theta}^2 l_\alpha(X; \theta) - \partial_{\theta \theta}^2 l_\alpha(X; \theta_0) \right\| < \epsilon \quad a.s. \]

Also, using the fact that \( \left\| \partial_{\theta \theta}^2 H_{\alpha,n}(\theta_0) + J_\alpha \right\| \) converges to zero almost surely, we have

\[ \max_{1 \leq k \leq \sqrt{n}} \frac{k}{n} \left\| \partial_{\theta \theta}^2 H_{\alpha,k}(\theta_0) + J_\alpha \right\| \leq \frac{1}{\sqrt{n}} \sup_{\theta \in n} \left\| \partial_{\theta \theta}^2 H_{\alpha,n}(\theta_0) + J_\alpha \right\| = o(1) \quad a.s. \quad (10) \]

and

\[ \max_{\sqrt{n} < k \leq n} \left\| \partial_{\theta \theta}^2 H_{\alpha,k}(\theta_0) + J_\alpha \right\| = o(1) \quad a.s., \quad (11) \]

which subsequently yield \( II_n = o(1) \) a.s. The lemma is therefore obtained.

\[ \square \]

**Proof of Theorem 1**

Recall in (4) that

\[ \left[ \frac{n s}{\sqrt{n}} \partial_{\theta} H_{\alpha,[ns]}(\hat{\theta}_{\alpha,n}) \right] = \left[ \frac{n s}{\sqrt{n}} \partial_{\theta} H_{\alpha,[ns]}(\theta_0) + \frac{n s}{n} \partial_{\theta \theta}^2 H_{\alpha,[ns]}(\theta^*_n) J^{-1}_\alpha \sqrt{n} \partial_{\theta} H_{\alpha,n}(\theta_0) \right] \]

\[ + \left[ \frac{n s}{n} \partial_{\theta \theta}^2 H_{\alpha,[ns]}(\theta^*_n,n) J^{-1}_\alpha (B_{\alpha,n} + J_\alpha) \sqrt{n} (\hat{\theta}_{\alpha,n} - \theta_0) \right]. \]

Since \( E[\partial_{\theta} l_\alpha(X; \theta_0)] = 0 \) and \( \{\partial_{\theta} l_\alpha(X_i; \theta_0)\} \) is a sequence of i.i.d. random vectors, it follows from the functional central limit theorem that

\[ \left[ \frac{n s}{\sqrt{n}} \partial_{\theta} H_{\alpha,[ns]}(\theta_0) \right] \overset{w}{\to} K^{1/2}_\alpha B_d(s) \quad \text{in } \mathbb{D}([0,1], \mathbb{R}^d), \]

where \( \{B_d(s)|0 \leq s \leq 1\} \) is a \( d \)-dimensional standard Brownian motion. Thus, by the continuous
where \( \theta \) holds for Lemmas 4 and 5. The first lemma states that the functional central limit theorem also defined in Remark 6.

Using Lemma 1 and the fact that \( \sqrt{n} \partial_{\theta} H_{\alpha,n}(\theta_0) \) is \( OP(1) \), we can see that

\[
\sup_{0 \leq s \leq 1} \| I_{\alpha,n}(s) - I_{\alpha,n}^{\circ}(s) \|
= \sup_{0 \leq s \leq 1} \left[ \frac{[ns]}{n} \partial_{\theta}^2 H_{\alpha,[ns]}(\theta_0) \right] J^{-1} \sqrt{n} \partial_{\theta} H_{\alpha,n}(\theta_0) \left[ \frac{[ns]}{n} \sqrt{n} \partial_{\theta} H_{\alpha,n}(\theta_0) \right] \\
\leq \left\| J^{-1} \sqrt{n} \partial_{\theta} H_{\alpha,n}(\theta_0) \right\| \max_{1 \leq k \leq n} \frac{k}{n} \left\| \partial_{\theta}^2 H_{\alpha,k}(\theta_{\alpha,n}^*, k) + J_{\alpha} \right\| = o_P(1),
\]

where \( \theta_{\alpha,n,k}^* \) denotes the one corresponding to \( \theta_{\alpha,n,s}^* \) when \( [ns] = k \), which together with \( (12) \) yields

\[
I_{\alpha,n}(s) \overset{w}{\rightarrow} K_{\alpha}^{1/2} B_d^\circ(s) \quad \text{in} \quad \mathbb{D}([0,1], \mathbb{R}^d).
\]

Next, note that by Lemma 1

\[
\sup_{0 \leq s \leq 1} \left( \frac{[ns]}{n} \right) \left\| \partial_{\theta}^2 H_{\alpha,[ns]}(\theta_{\alpha,n,s}^*) \right\| \leq \max_{1 \leq k \leq n} \frac{k}{n} \left\| \partial_{\theta}^2 H_{\alpha,k}(\theta_{\alpha,n,k}^*) + J_{\alpha} \right\| = O(1) \quad \text{a.s.}
\]

and

\[
\left\| J^{-1}_{\alpha}(B_{\alpha,n} + J_{\alpha}) \right\| \leq \left\| J^{-1}_{\alpha} \right\| \max_{1 \leq k \leq n} \frac{k}{n} \left\| \partial_{\theta}^2 H_{\alpha,k}(\theta_{\alpha,n,k}^*) + J_{\alpha} \right\| = o(1) \quad \text{a.s.}
\]

Further, using \( \sqrt{n} \partial_{\theta} H_{\alpha,n}(\theta_0) = O_P(1) \) again and \( B_{\alpha,n} + J_{\alpha} = o_P(1) \) obtained in \( (15) \), one can see from \( (3) \) that \( \sqrt{n}(\theta_{\alpha,n} - \theta_0) = O_P(1) \). Therefore, combing this, \( (14) \), and \( (15) \), we have that

\[
\sup_{0 \leq s \leq 1} \| H_{\alpha,n}(s) \| = o_P(1).
\]

This completes the proof. \( \square \)

The following part is provided for Theorem 3 in Section 3. Without confusion, \( H_{\alpha,k}(\theta) \) is hereafter used to denote the one obtained by replacing \( \tilde{l}_\alpha(X_t; \theta) \) with \( l_\alpha(X_t; \theta) \) in \( (5) \). \( l_\alpha(X_t; \theta) \) is defined in Remark 6.

Theorem 3 is shown by similar arguments to that of Theorem 1. Fundamental lemmas for the proof are Lemmas 4 and 5. The first lemma states that the functional central limit theorem also holds for \( \{ \partial_{\theta} \tilde{l}_\alpha(X_t; \theta_0) \} \) and the latter one provides the result corresponding to Lemma 1, which is also usefully used in proving Theorem 3. Lemmas 2 and 3 are given to verify the aforementioned two lemmas, where we employ the technical results in Francq and Zakoian (2004) and Lee and Song (2009) to prove the lemmas.
Lemma 2. Suppose that the assumptions A1-A4 in Theorem hold. Then, under $H_0$, we have

$$\frac{1}{\sqrt{n}} \sum_{t=1}^{n} \| \partial_\theta l_\alpha(X_t; \theta_0) - \partial_\theta \tilde{l}_\alpha(X_t; \theta_0) \| = o(1) \quad \text{a.s.,} \quad (16)$$

and, for some neighborhood $V_1(\theta_0)$ of $\theta_0$,

$$\frac{1}{n} \sum_{t=1}^{n} \sup_{\theta \in V_1(\theta_0)} \| \partial_\theta^2 l_\alpha(X_t; \theta) - \partial_\theta^2 \tilde{l}_\alpha(X_t; \theta) \| = o(1) \quad \text{a.s.,} \quad (17)$$

$$\frac{1}{n} \sum_{t=1}^{n} \sup_{\theta \in V_1(\theta_0)} \| \partial_\theta l_\alpha(X_t; \theta) \partial_\theta^2 l_\alpha(X_t; \theta) - \partial_\theta \tilde{l}_\alpha(X_t; \theta) \partial_\theta \tilde{l}_\alpha(X_t; \theta) \| = o(1) \quad \text{a.s.} \quad (18)$$

Proof. The proof of the part (iv) in Lee and Song (2009), page 337, implies the first two results in the lemma, and thus we omit the proofs of (16) and (17). To verify (18), we introduce the following technical results in Francq and Zakoian (2004):

$$\sup_{\theta \in \Theta} \{ |\sigma_t^2 - \tilde{\sigma}_t^2| \lor \| \partial_\theta \tilde{\sigma}_t^2 - \partial_\theta \sigma_t^2 \| \} \leq K \rho^t \quad \text{a.s.} \quad \text{for all } t \geq 1,$$ \quad (19)

$$E \sup_{\theta \in \Theta^*} \left| \frac{1}{\sigma_t} \partial_\theta \sigma_t^2 \right|^d < \infty, \quad E \sup_{\theta \in \Theta^*} \left| \frac{1}{\sigma_t^2} \partial_\theta \theta_j \sigma_t^2 \right|^d < \infty \quad \text{for all } d \in \mathbb{N},$$ \quad (20)

where $K > 0$ and $\rho \in (0, 1)$ denote universal constants, which can take different values from line to line, and $\Theta^*$ is a compact set such that $\theta_0 \in \Theta^* \subset \Theta^o$. The following moment result can be found in Lemma 1 in Lee and Song (2009): for all $d \in \mathbb{N}$,

$$E \sup_{\theta \in V_1(\theta_0)} \frac{X_{2d}^d}{\sigma_t^d} < \infty,$$ \quad (21)

where $V_1(\theta_0)$ is a neighborhood of $\theta_0$. We shall also use the results in page 340 in Lee and Song (2009). That is,

$$|h_\alpha(\tilde{\sigma}_t^2)| \leq \left(1 + \frac{X_t^2}{\sigma_t^2}\right), \quad |h_\alpha(\sigma_t^2) - h_\alpha(\tilde{\sigma}_t^2)| \leq K \left(1 + \frac{X_t^2}{\sigma_t^2} + \frac{X_t^4}{\sigma_t^4}\right) \rho^t,$$ \quad (22)

$$\left| \left( \frac{1}{\sigma_t^2} \right)^{\frac{\alpha}{2}+1} - \left( \frac{1}{\tilde{\sigma}_t^2} \right)^{\frac{\alpha}{2}+1} \right| \leq \frac{K \rho^t}{\sigma_t^2},$$ \quad (23)

where

$$h_\alpha(x) = -\frac{\alpha}{2\sqrt{1 + \alpha}} + \frac{1 + \alpha}{2} \left(1 - \frac{X_t^2}{x}\right) \exp \left(-\frac{\alpha X_t^2}{2x}\right).$$
Here, we note that
\[
\partial_{\theta^i} l_\alpha(X_t; \theta) = \frac{h^\alpha (\sigma^2_t)}{\sigma^2_t} \partial_{\theta^i} \sigma^2_t.
\] (24)

Recall that \( \Theta \) is also a compact subset in \((0, \infty) \times [0, \infty)^{p+q} \). Then, we have
\[
\frac{1}{\sigma^2_t} \vee \frac{1}{\sigma^2_t} \leq \sup_{\theta \in \Theta} \frac{1}{w} \leq K.
\]
Using this, (19), (22), and (23), we have
\[
\begin{align*}
|\partial_{\theta^i} l_\alpha(X_t; \theta) - \partial_{\theta^i} \tilde{l}_\alpha(X_t; \theta)| \\
= \left| \left( h^\alpha (\sigma^2_t) - h^\alpha (\tilde{\sigma}^2_t) \right) \left( \frac{1}{\sigma^2_t} \right)^{\frac{3}{2}+1} \partial_{\theta^i} \sigma^2_t + h^\alpha (\tilde{\sigma}^2_t) \left\{ \left( \frac{1}{\sigma^2_t} \right)^{\frac{3}{2}+1} - \left( \frac{1}{\tilde{\sigma}^2_t} \right)^{\frac{3}{2}+1} \right\} \partial_{\theta^i} \sigma^2_t \\
+ h^\alpha (\tilde{\sigma}^2_t) \left( \frac{1}{\tilde{\sigma}^2_t} \right)^{\frac{3}{2}+1} (\partial_{\theta^i} \sigma^2_t - \partial_{\theta^i} \tilde{\sigma}^2_t) \right| \\
\leq K \left( 1 + \frac{X^2_t}{\sigma^2_t} + \frac{X^4_t}{\sigma^2_t} \right) \frac{1}{\sigma^2_t} \partial_{\theta^i} \sigma^2_t |\rho^t| + K \left( 1 + \frac{X^2_t}{\tilde{\sigma}^2_t} \right) \frac{1}{\sigma^2_t} \partial_{\theta^i} \sigma^2_t |\rho^t| + K \left( 1 + \frac{X^2_t}{\sigma^2_t} \right) \rho^t \\
\leq K \left( 1 + \frac{X^2_t}{\sigma^2_t} + \frac{X^4_t}{\sigma^2_t} \right) \left( 1 + \left| \frac{1}{\sigma^2_t} \partial_{\theta^i} \sigma^2_t \right| \right) |\rho^t| := K P_{t,i}(\theta) |\rho^t|.
\end{align*}
\]
Since \(|\partial_{\theta^i} l_\alpha(X_t; \theta)| \leq K P_{t,i}(\theta)\), we can also see from the above that
\[
|\partial_{\theta^i} \tilde{l}_\alpha(X_t; \theta)| \leq |\partial_{\theta^i} l_\alpha(X_t; \theta)| + K P_{t,i}(\theta) \leq K P_{t,i}(\theta),
\]
and thus, we have
\[
|\partial_{\theta^i} l_\alpha(X_t; \theta) \partial_{\theta^j} l_\alpha(X_t; \theta) - \partial_{\theta^i} \tilde{l}_\alpha(X_t; \theta) \partial_{\theta^j} \tilde{l}_\alpha(X_t; \theta)|
\]
\[
\leq |\partial_{\theta^i} l_\alpha(X_t; \theta)||\partial_{\theta^j} l_\alpha(X_t; \theta) - \partial_{\theta^j} \tilde{l}_\alpha(X_t; \theta)| + |\partial_{\theta^i} \tilde{l}_\alpha(X_t; \theta)||\partial_{\theta^j} l_\alpha(X_t; \theta) - \partial_{\theta^j} \tilde{l}_\alpha(X_t; \theta)|
\]
\[
\leq K P_{t,i}(\theta) P_{t,j}(\theta) |\rho^t|.
\]
Using the Cauchy-Schwarz inequality with the first result in (20) and (21), we can see that for all \( d \in \mathbb{N} \),
\[
E \sup_{\theta \in \Theta} P_{t,i}^d(\theta) < \infty.
\] (25)
Therefore, since
\[
\sum_{t=1}^{\infty} P \left( \rho^t \sup_{\theta \in \Theta} P_{t,i}(\theta) P_{t,j}(\theta) > \epsilon \right) \leq \frac{1}{\epsilon} \sum_{t=1}^{\infty} \rho^t \sqrt{E \sup_{\theta \in \Theta} P_{t,i}^2(\theta)} \sqrt{E \sup_{\theta \in \Theta} P_{t,j}^2(\theta)} < \infty,
\]
\( \rho^l \sup_{\theta \in V_1(\theta_0)} P_{t,i}(\theta) P_{t,j}(\theta) \) converges to zero with probability one. Hence, (18) is asserted from the Cesàro lemma.

**Lemma 3.** Suppose that the assumptions A1-A4 in Theorem 3 hold. Then, under \( H_0 \), we have that for some neighborhood \( V_2(\theta_0) \) of \( \theta_0 \),

\[
E \sup_{\theta \in V_2(\theta_0)} \left\| \partial_\theta l_\alpha(X_t; \theta) \partial_\theta l_\alpha(X_t; \theta) \right\| < \infty \quad \text{and} \quad E \sup_{\theta \in V_2(\theta_0)} \left\| \partial_{\theta \theta}^2 l_\alpha(X_t; \theta) \right\| < \infty.
\]

**Proof.** Since \( \left| \partial_\theta l_\alpha(X_t; \theta) \partial_\theta l_\alpha(X_t; \theta) \right| \leq KP_{t,i}(\theta) P_{t,j}(\theta) \), the first result is obtained by (25) and the Cauchy-Schwarz inequality. A straightforward calculation shows that

\[
\partial_{\theta \theta}^2 l_\alpha(X_t; \theta) = h_\alpha(\sigma_t^2) \left( \frac{1}{\sigma_t^2} \right)^{\frac{3}{2} + 1} \partial_{\theta \theta} \partial_\theta \sigma_t^2 + m_\alpha(\sigma_t^2) \left( \frac{1}{\sigma_t^2} \right)^{\frac{3}{2} + 2} \partial_\theta \sigma_t^2 \partial_\theta \sigma_t^2.
\]

where

\[
m_\alpha(x) = \frac{\alpha(2 + \alpha)}{4\sqrt{1 + \alpha}} - \frac{1 + \alpha}{2} \left\{ 1 + \frac{\alpha}{2} - (2 + \alpha) \frac{X_t^2}{x} + \frac{\alpha X_t^4}{2 x^2} \right\} \exp \left( -\frac{\alpha X_t^2}{2 x} \right).
\]

The second one can also be readily shown by using (20) and (21), so we omit the proof for brevity.

**Lemma 4.** Suppose that the assumptions A1-A4 in Theorem 3 hold. Then, under \( H_0 \), we have

\[
\frac{1}{\sqrt{k(\alpha)}} J_{1,\alpha}^{-1/2} \frac{[ns]}{\sqrt{n}} \partial_\theta \tilde{H}_{\alpha,\lfloor ns \rfloor}(\theta_0) \xrightarrow{w} B_D(s) \quad \text{in} \quad \mathbb{D}([0,1], \mathbb{R}^D),
\]

where \( B_D \) is a standard \( D \)-dimensional Brownian motion and \( D = p + q + 1 \).

**Proof.** By simple algebra, we have \( E[h_\alpha(\sigma_t^2)] = 0 \), and hence \( \{\partial_\theta l_\alpha(X_t; \theta_0)\} \) becomes a martingale difference process. Since \( \{\partial_\theta l_\alpha(X_t; \theta_0)\} \) is also strictly stationary and ergodic, we can obtain that by the functional central limit theorem for martingale difference,

\[
\frac{1}{\sqrt{k(\alpha)}} J_{1,\alpha}^{-1/2} \frac{[ns]}{\sqrt{n}} \partial_\theta H_{\alpha,\lfloor ns \rfloor}(\theta_0) \xrightarrow{w} B_D(s) \quad \text{in} \quad \mathbb{D}([0,1], \mathbb{R}^D).
\]  

(26)

Due to (16), we also have

\[
\sup_{0 \leq s \leq 1} \frac{[ns]}{\sqrt{n}} \left\| \partial_\theta H_{\alpha,\lfloor ns \rfloor}(\theta_0) - \partial_\theta \tilde{H}_{\alpha,\lfloor ns \rfloor}(\theta_0) \right\| \leq \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \left\| \partial_\theta l_\alpha(X_t; \theta_0) - \partial_\theta \tilde{l}_\alpha(X_t; \theta_0) \right\| = o(1) \quad \text{a.s.,}
\]

which together with (26) ensures the lemma.
Lemma 5. Suppose that the assumptions A1-A4 in Theorem 3 hold. Then, under $H_0$, we have

$$\max_{1 \leq k \leq n} \frac{k}{n} \left\| \partial_{\theta^2} \hat{H}_{\alpha,k}(\theta^*_{\alpha,n,k}) + J_\alpha \right\| = o_P(1).$$

where $\{\theta^*_{\alpha,n,k} | 1 \leq k \leq n, n \geq 1\}$ is any double array of $\Theta$-valued random vectors with $\|\theta^*_{\alpha,n,k} - \theta_0\| \leq \|\hat{\theta}_{\alpha,n} - \theta_0\|$.

Proof. For any $\epsilon > 0$, using a similar argument in (9) together with Lemma 3, one can take a neighborhood $N_\epsilon(\theta_0)$ of $\theta_0$ such that

$$E \sup_{\theta \in N_\epsilon(\theta_0)} \left\| \partial_{\theta^2} \tilde{l}_\alpha(X_t; \theta) - \partial_{\theta^2} l_\alpha(X_t; \theta) \right\| < \epsilon. \quad (27)$$

Let $V_\epsilon(\theta_0) = V_1(\theta_0) \cap N_\epsilon(\theta_0)$, where $V_1(\theta_0)$ is the one given in Lemma 2. Since $\hat{\theta}_{\alpha,n}$ converges almost surely to $\theta_0$, we have that for sufficiently large $n$,

$$\max_{1 \leq k \leq n} \frac{k}{n} \left\| \partial_{\theta^2} \tilde{H}_{\alpha,k}(\theta^*_{\alpha,n,k}) + J_\alpha \right\| 
\leq \max_{1 \leq k \leq n} \frac{k}{n} \left\| \partial_{\theta^2} \tilde{H}_{\alpha,k}(\theta^*_{\alpha,n,k}) - \partial_{\theta^2} H_{\alpha,k}(\theta^*_{\alpha,n,k}) \right\| + \max_{1 \leq k \leq n} \frac{k}{n} \left\| \partial_{\theta^2} H_{\alpha,k}(\theta^*_{\alpha,n,k}) - \partial_{\theta^2} H_{\alpha,k}(\theta_0) \right\| 
+ \max_{1 \leq k \leq n} \frac{k}{n} \left\| \partial_{\theta^2} H_{\alpha,k}(\theta_0) + J_\alpha \right\| 
\leq \frac{1}{n} \sum_{t=1}^{n} \sup_{\theta \in V_\epsilon(\theta_0)} \left\| \partial_{\theta^2} \tilde{l}_\alpha(X_t; \theta) - \partial_{\theta^2} l_\alpha(X_t; \theta) \right\| + \frac{1}{n} \sum_{t=1}^{n} \sup_{\theta \in V_\epsilon(\theta_0)} \left\| \partial_{\theta^2} l_\alpha(X_t; \theta) - \partial_{\theta^2} l_\alpha(X_t; \theta_0) \right\| 
+ \max_{1 \leq k \leq n} \frac{k}{n} \left\| \partial_{\theta^2} H_{\alpha,k}(\theta_0) + J_\alpha \right\| 
:= I_n + II_n + III_n \text{ a.s.}$$

First, one can see that $I_n = o(1)$ a.s. by (17). Using (27) and the ergodicity of $\{l_\alpha(X_t; \theta)\}$, we also have

$$\lim_{n \to \infty} II_n = E \sup_{\theta \in V_\epsilon(\theta_0)} \left\| \partial_{\theta^2} l_\alpha(X_t; \theta) - \partial_{\theta^2} l_\alpha(X_t; \theta_0) \right\| < \epsilon \text{ a.s.}$$

Finally, observe that $\|\partial_{\theta^2} H_{\alpha,n}(\theta_0) + J_\alpha\|$ converges to zero almost surely by the ergodic theorem. In the same way as in (10) and (11), it can be shown that $III_n = o(1)$ a.s., which completes the proof.
Proof of Theorem 3
We first show in [7] that
\[
\tilde{I}_{a,n}(s) := \frac{[ns]}{\sqrt{n}} \frac{1}{n} \frac{\partial^2}{\partial \theta^2} \tilde{H}_{\alpha,\lfloor ns \rfloor}(\theta) \left[ \theta^* \right] \left[ \tilde{H}_{\alpha,\lfloor ns \rfloor}^{-1} \right] \sqrt{n} \frac{1}{n} \frac{\partial^2}{\partial \theta^2} \tilde{H}_{\alpha,\lfloor ns \rfloor}(\theta)
\]
\[
\quad \overset{w}{\rightarrow} \sqrt{k(\alpha)} \tilde{J}_{1,\alpha}^{1/2} B_d^{1/2}(s) \quad \text{in} \quad \mathbb{D}([0,1], \mathbb{R}^D).
\]
Due to Lemma 4, we have
\[
\tilde{I}_{a,n}(s) := \frac{[ns]}{\sqrt{n}} \frac{1}{n} \frac{\partial^2}{\partial \theta^2} \tilde{H}_{\alpha,\lfloor ns \rfloor}(\theta) - \frac{[ns]}{\sqrt{n}} \frac{1}{n} \frac{\partial^2}{\partial \theta^2} \tilde{H}_{\alpha,\lfloor ns \rfloor}(\theta) \overset{w}{\rightarrow} \sqrt{k(\alpha)} \tilde{J}_{1,\alpha}^{1/2} B_d^{1/2}(s) \quad \text{in} \quad \mathbb{D}([0,1], \mathbb{R}^D).
\]
Observe that \(\sqrt{n} \partial \theta \tilde{H}_{\alpha,\lfloor ns \rfloor}(\theta) = O_P(1)\) by Lemma 4 with \(s = 1\). Then, it follows from Lemma 5 that
\[
\sup_{0 \leq s \leq 1} \left\| \tilde{I}_{a,n}(s) - \tilde{I}_{a,n}(s) \right\| = \sup_{0 \leq s \leq 1} \left\| \frac{[ns]}{\sqrt{n}} \frac{1}{n} \frac{\partial^2}{\partial \theta^2} \tilde{H}_{\alpha,\lfloor ns \rfloor}(\theta^*) \right\| \leq \left\| J_{\alpha}^{-1/2} \sqrt{n} \partial \theta \tilde{H}_{\alpha,\lfloor ns \rfloor}(\theta) \right\| \leq o(1) \quad \text{a.s.},
\]
where \(\theta^*_{\alpha,n,k}\) denotes \(\theta^*_{\alpha,n,k,n}\), and thus (28) is obtained.

Next, note from Proposition 1 that \(\sqrt{n}(\tilde{\theta}_{\alpha,n} - \theta_0) = O_P(1)\). By Lemma 5, we have
\[
\sup_{0 \leq s \leq 1} \left\| \frac{[ns]}{\sqrt{n}} \frac{1}{n} \frac{\partial^2}{\partial \theta^2} \tilde{H}_{\alpha,\lfloor ns \rfloor}(\theta^*) \right\| \leq \max_{1 \leq k \leq n} \left\| \frac{1}{n} \frac{\partial^2}{\partial \theta^2} \tilde{H}_{\alpha,\lfloor ns \rfloor}(\theta^*) \right\| \leq O_P(1),
\]
and
\[
\left\| J_{\alpha}^{-1/2} \sqrt{n} \partial \theta \tilde{H}_{\alpha,\lfloor ns \rfloor}(\theta) \right\| \leq \left\| J_{\alpha}^{-1} \max_{1 \leq k \leq n} \left\| \frac{1}{n} \frac{\partial^2}{\partial \theta^2} \tilde{H}_{\alpha,\lfloor ns \rfloor}(\theta^*) \right\| \right\| \leq o_P(1),
\]
and consequently,
\[
\sup_{0 \leq s \leq 1} \left\| \frac{[ns]}{\sqrt{n}} \frac{1}{n} \frac{\partial^2}{\partial \theta^2} \tilde{H}_{\alpha,\lfloor ns \rfloor}(\tilde{\theta}_{\alpha,n}) - \tilde{I}_{a,n}(s) \right\| = o_P(1),
\]
which together (28) establishes the theorem.

Lemma 6. Suppose that the assumptions A1-A4 in Theorem 3 hold. Then, under \(H_0\), we have
\[
\frac{1}{n} \sum_{l=1}^{n} \partial \phi_l(X_t; \tilde{\theta}_a) \partial \phi \tilde{L}_a(X_t; \tilde{\theta}_a) \overset{P}{\rightarrow} E \left[ \partial \phi_l(X; \theta_0) \partial \phi \tilde{L}_a(X; \theta_0) \right].
\]
Proof. Using the first result in Lemma 3, we can also take a neighborhood $N_ϵ(θ_0)$ such that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} \sup_{θ \in N_ϵ(θ_0)} \left\| \partial_θ l_α(X_t; θ) \partial_θ l_α(X_t; θ) - \partial_θ l_α(X_t; θ_0) \partial_θ l_α(X_t; θ_0) \right\| = E \sup_{θ \in N_ϵ(θ_0)} \left\| \partial_θ l_α(X_t; θ) \partial_θ l_α(X_t; θ) - \partial_θ l_α(X_t; θ_0) \partial_θ l_α(X_t; θ_0) \right\| < ϵ \text{ a.s.}$$

(29)

Since $\hat{θ}_n$ converges to $θ_0$ almost surely, the lemma follows from (18), (29) and the ergodic theorem.

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References
Aue, A. and Horváth, L. (2013). Structural breaks in time series. Journal of Time Series Analysis 34, 116.

Basu, A., Harris, I. R., Hjort, N. L. and Jones, M. C. (1998). Robust and efficient estimation by minimizing a density power divergence. Biometrika 85, 549-559.

Basu, A., Mandal, A., Martin, N. and Pardo, L. (2013). Testing statistical hypotheses based on the density power divergence. Annals of the Institute of Statistical Mathematics 65, 319-348.

Basu, A., Mandal, A., Martin, N. and Pardo, L. (2016). Generalized Wald-type tests based on minimum density power divergence estimators. Statistics 50, 1-26.

Batsidis, A., Horváth, L., Martin, N., Pardo, L. and Zografos, K. (2013). Change-point detection in multinomial data using phi-divergence test statistics. Journal of Multivariate Analysis 118, 53-66.

Berkes, I. and Horváth, L. (2004). The efficiency of the estimators of the parameters in GARCH processes. The Annals of Statistics 32, 633-665.

Berkes, I., Horváth, L. and Kokoszka, P. (2004). Testing for parameter constancy in GARCH(p,q) models. Statistics & Probability Letters 4, 263-273.

Bougerol, P. and Picard, N. (1992). Stationarity of GARCH processes and of some nonnegative time series. Journal of Econometrics 52, 115127.

Durio, A. and Isaia, E.D. (2011). The minimum density power divergence approach in building robust regression models. Informatica 22, 43–56.
Fearnhead, P. and Rigail, G. (2018). Changepoint detection in the presence of outliers. *Journal of the American Statistical Association*, 1-15.

Ferguson, T.S. (1996). *A Course in Large Sample Theory*. Chapman & Hall/CRC, New York.

Francq, C. and Zakoïan, J.-M. (2004). Maximum likelihood estimation of pure GARCH and ARMA-GARCH processes. *Bernoulli* 10, 605-637.

Fujisawa, H. and Eguchi, S. (2006). Robust estimation in the normal mixture model. *Journal of Statistical Planning and Inference* 136, 3989-4011.

Fujisawa, H., and Eguchi, S. (2008). Robust parameter estimation with a small bias against heavy contamination. *Journal of Multivariate Analysis* 99, 2053-2081.

Ghosh, A. and Basu, A. (2017). The minimum S-divergence estimator under continuous models: the Basu-Lindsay approach. *Statistical Papers* 58, 341-372.

Ghosh, A., Basu, A. and Pardo, L. (2015). On the robustness of a divergence based test of simple statistical hypotheses. *Journal of Statistical Planning and Inference* 161, 91-108.

Ghosh, A., Mandal, A., Martín, N. and Pardo, L. (2016). Influence analysis of robust Wald-type tests. *Journal of Multivariate Analysis* 147, 102-126.

Hillebrand, E. (2005). Neglecting parameter changes in GARCH models. *Journal of Econometrics* 129, 121-138.

Horváth, L. and Parzen, E. (1994). Limit theorems for Fisher-score change processes. *Changepoint problems, IMS Lecture Notes-Monograph Series* 23, 157-169.

Horváth, L. and Rice, G. (2014). Extensions of some classical methods in change point analysis. *TEST* 23, 219-255.

Kang, J. and Lee, S. (2014). Minimum density power divergence estimator for Poisson AR models. *Computational Statistics & Data Analysis* 80, 44-56.

Kang, J. and Song, J. (2015). Robust parameter change test for Poisson autoregressive models. *Statistics & Probability Letters* 104, 14-21.

Kulperger, R. and Yu, H. (2005). High moment partial sum processes of residuals in GARCH models and their applications. *The Annals of Statistics* 33, 2395-2422.

Lee, S. and Na, O. (2005). Test for parameter change based on the estimator minimizing density-power divergence measures. *Annals of the Institute of Statistical Mathematics* 57, 553-573.

Lee, S. and Song, J. (2009). Minimum density power divergence estimator for GARCH models. *TEST* 18, 316-341.
Maronna, R. A., Martin, R. D. and Yohai, V. J. (2006). *Robust Statistics: Theory and Methods*. Wiley.

Pardo, L. (2006). *Statistical Inference Based on Divergence Measures*, Chapman and Hall/CRC.

Robbins, M., Gallagher, C., Lund, R. and Aue, A. (2011). Mean shift testing in correlated data. *Journal of Time Series Analysis* 32, 498-511.

Song, J. (2017). Robust estimation of dispersion parameter in discretely observed diffusion processes. *Statistica Sinica* 27, 373-388.

Song, J. (2020). Robust test for dispersion parameter change in discretely observed diffusion processes. *Computational Statistics & Data Analysis* 142, 106832

Song, J. and Kang, J. (2018). Parameter change tests for ARMAGARCH models. *Computational Statistics & Data Analysis* 121, 41-56.

Tsay, R. S. (1988). Outliers, level shifts, and variance changes in time series. *Journal of Forecasting* 7, 1-20.

Warwick, J., (2005). A data-based method for selecting tuning parameters in minimum distance estimators. *Computational Statistics & Data Analysis* 48, 571–585.