ON THE RELATION BETWEEN ADM AND BONDI
ENERGY-MOMENTA – RADIATIVE SPATIAL INFINITY

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Abstract. In a vacuum spacetime equipped with the Bondi’s radiating metric which is asymptotically flat at spatial infinity including gravitational radiation (Condition D), we establish the relation between the ADM total linear momentum and the Bondi momentum. The relation between the ADM total energy and the Bondi mass in this case was established earlier in [12].

1. Introduction

It is a fundamental problem in gravitational radiation what the relation is between the ADM energy-momentum and the Bondi energy-momentum. Ashtekar and Magnon-Ashtekar demonstrated the mass at spatial infinity is the past limit of the Bondi mass taken as the cut approaches the “point” of spatial infinity in the frame of Penrose conformal compactification [2] (See also the related works [6, 9, 10]). It was proved the standard ADM mass at spatial infinity is the past limit of the Bondi mass for some strongly asymptotically flat, globally hyperbolic vacuum [5], and for vacuum spacetimes equipped with the Bondi’s radiating metric with “strongly” asymptotic flatness at spatial infinity (Condition C) [12]. However, it is presumably believed the assumed asymptotic flatness at spatial infinity in all above works precludes gravitational radiation, at least near spatial infinity.

Under some weaker assumption of asymptotic flatness at spatial infinity (Condition D), the second author also derive a formula relating the ADM total energy to the Bondi mass. In this case, the ADM total energy is no longer the past limit of the Bondi mass and they differ by certain quantity relating to the news of the system. The most importance is that gravitational radiation should be included under this condition. Some physical interpretation of Condition D is provided in [12].

The main difference between Condition C and Condition D is as follows: Under Condition C, the initial data set \( \{ t = t_0 \} \) is standard asymptotically flat in the sense that, in natural coordinates \( \{ x^u \} \), the metric \( g \) and the second fundamental form \( h \) satisfy

\[
\begin{align*}
g_{uv} &= \delta_{uv} + O\left(\frac{1}{r}\right), \\
p_k g_{uv} &= O\left(\frac{1}{r^2}\right), \\
p_l \partial_k g_{uv} &= O\left(\frac{1}{r^3}\right),
\end{align*}
\]

\[
\begin{align*}
h_{uv} &= O\left(\frac{1}{r^2}\right), \\
p_k h_{uv} &= O\left(\frac{1}{r^3}\right).
\end{align*}
\]
However, under **Condition D**, the initial data set \( \{ t = t_0 \} \) is weaker asymptotically flat in the sense that, in natural coordinates \( \{ x^u \} \), the metric \( g \) and the second fundamental form \( h \) satisfy

\[
\begin{align*}
g_{uv} &= \delta_{uv} + O\left( \frac{1}{r} \right), \quad \partial_k g_{uv} = O\left( \frac{1}{r} \right), \quad \partial_l \partial_k g_{uv} = O\left( \frac{1}{r^2} \right), \\
h_{uv} &= O\left( \frac{1}{r} \right), \quad \partial_k h_{uv} = O\left( \frac{1}{r} \right).
\end{align*}
\]

Note that an observer detecting gravitational waves locates always in some finite region of a spacelike hypersurface. Thus the data the observer detects is from the gravitational waves arrived at the spacelike hypersurfaces. Suppose the metric and the second fundamental form of this spacelike hypersurface can be read from the data, then we can calculate the integrands of the ADM total energy-momentum and integrate them in sphere with radius \( r \) included in this region. One may wonder whether gravitational waves can reach spatial infinity. However, as the ADM total energy-momentum is defined as the limit \( r \to \infty \), the data from gravitational waves should reach spatial infinity and **Condition D** should be reasonable when we calculate the ADM total energy-momentum.

In this paper, we will study the relation between the ADM total energy-momentum and the Bondi energy-momentum under **Condition D**. Although the second fundamental form \( h \) of \( t \)-slice falls off as slowly as \( O\left( \frac{1}{r} \right) \), \( h - tr_g(h)g \) falls off surprisingly as \( O\left( \frac{1}{r^2} \right) \) at spatial infinity due to certain mysterious cancelation. Therefore, the ADM total linear momentum is still finite. It is unclear whether the ADM total energy-momentum is a geometric invariant, i.e., independent on the choice of coordinates, under **Condition D**, however, we compare the ADM and Bondi energy-momenta in a fixed coordinate. We believe it should have physical meaning.

## 2. Bondi’s radiating vacuum spacetimes

The Bondi’s radiating vacuum spacetime \( (L^{3,1}, \bar{g}) \) is a vacuum spacetime equipped with the following metric \( \bar{g} = \bar{g}_{ij}dx^i dx^j \)

\[
\bar{g} = \left( \frac{V}{r} e^{2\beta} + r^2 e^{2\gamma} U^2 \cosh 2\delta + r^2 e^{-2\gamma} W^2 \cosh 2\delta \\
+ 2r^2 U W \sinh 2\delta \right) du^2 - 2e^{2\beta} dudr \\
- 2r^2 \left( e^{2\gamma} U \cosh 2\delta + W \sinh 2\delta \right) dud\theta \\
- 2r^2 \left( e^{-2\gamma} W \cosh 2\delta + U \sinh 2\delta \right) \sin \theta dud\phi \\
+ r^2 \left( e^{2\gamma} \cosh 2\delta \theta d\theta^2 + e^{-2\gamma} \cosh 2\delta \sin^2 \theta d\psi^2 \\
+ 2 \sinh 2\delta \sin \theta d\theta d\psi \right),
\]
where $\beta, \gamma, \delta, U, V, W$ are functions of

$$x^0 = u, \ x^1 = r, \ x^2 = \theta, \ x^3 = \psi,$$

$u$ is a retarded coordinate, $r$ is Euclidean distance, $\theta$ and $\psi$ are spherical coordinates, $0 \leq \theta \leq \pi, \ 0 \leq \psi \leq 2\pi$. We assume that $\tilde{g}$ satisfies the outgoing radiation condition.

The metric (2.1) was studied by Bondi, van der Burg, Metzner and Sachs in the theory of gravitational waves in general relativity [3, 8, 11]. They proved that the following asymptotic behavior holds for $r$ sufficiently large if the spacetime satisfies the outgoing radiation condition [11]:

$$\gamma = \frac{c(u, \theta, \psi)}{r} + \frac{C(u, \theta, \psi) - \frac{1}{6} c^3 - \frac{3}{2} cd^2}{r^3} + O\left(\frac{1}{r^4}\right),$$

$$\delta = \frac{d(u, \theta, \psi)}{r} + \frac{H(u, \theta, \psi) + \frac{1}{2} c^2 d - \frac{1}{6} d^3}{r^3} + O\left(\frac{1}{r^4}\right),$$

$$\beta = -\frac{c^2 + d^2}{4r^2} + O\left(\frac{1}{r^4}\right),$$

$$U = -\frac{l(u, \theta, \psi)}{r^2} + \frac{p(u, \theta, \psi)}{r^3} + O\left(\frac{1}{r^4}\right),$$

$$W = -\frac{\bar{l}(u, \theta, \psi)}{r^2} + \frac{\bar{p}(u, \theta, \psi)}{r^3} + O\left(\frac{1}{r^4}\right),$$

$$V = -r + 2M(u, \theta, \psi) + \frac{\bar{M}(u, \theta, \psi)}{r} + O\left(\frac{1}{r^2}\right),$$

where

$$l = c_2 + 2c \cot \theta + d_3 \csc \theta,$$

$$\bar{l} = d_2 + 2d \cot \theta - c_3 \csc \theta,$$

$$p = 2N + 3(cc_2 + dd_2) + 4(c^2 + d^2) \cot \theta - 2(c_3 d - cd_3) \csc \theta,$$

$$\bar{p} = 2P + 2(c_2 d - cd_2) + 3(cc_3 + dd_3) \csc \theta,$$

$$\bar{M} = N_2 + \cot \theta + P_3 \csc \theta - \frac{c^2 + d^2}{2}$$

$$- \left[ (c_2)^2 + (d_2)^2 \right] - 4(cc_2 + dd_2) \cot \theta - 4(c^2 + d^2) \cot^2 \theta - \left[ (c_3)^2 + (d_3)^2 \right] \csc^2 \theta$$

$$+ 4(c_3 d - cd_3) \csc \theta \cot \theta + 2(c_3 d_2 - cd_2) \csc \theta.$$

(We denote $f_i = \frac{\partial f}{\partial x^i}$ for $i = 0, 1, 2, 3$ throughout the paper.) $M$ is the mass aspect and $c_0, d_0$ are the news functions and they satisfy the following equation [11]:

$$M_{0,0} = -\left[ (c_0)^2 + (d_0)^2 \right] + \frac{1}{2} \left( l_{0,2} + l \cot \theta + \bar{l}_{3} \csc \theta \right). \quad (2.2)$$
Let $N_{u_0}$ be a null hypersurface which is given by $u = u_0$ at null infinity. The Bondi energy-momentum of $N_{u_0}$ is defined by [3]:

$$m_{\nu}(u_0) = \frac{1}{4\pi} \int_{S^2} M(u_0, \theta, \psi) n^\nu dS$$

where $\nu = 0, 1, 2, 3$, $S^2$ is the unit sphere,

$$n^0 = 1, \quad n^1 = \sin \theta \cos \psi, \quad n^2 = \sin \theta \sin \psi, \quad n^3 = \cos \theta.$$ 

And $m_0$ is the Bondi mass, $m_i$ is the Bondi momentum.

Denote by $\{\hat{e}^i\}$ the coframe of the standard flat metric $g_0$ on $\mathbb{R}^3$ in polar coordinates,

$$\hat{e}^1 = dr, \quad \hat{e}^2 = r d\theta, \quad \hat{e}^3 = r \sin \theta d\psi.$$ 

Denote by $\{\hat{e}_i\}$ the dual frame $(i = 1, 2, 3)$. The connection 1-form $\{\hat{\omega}_{ij}\}$ is defined by

$$d\hat{e}^i = -\hat{\omega}_{ij} \wedge \hat{e}^j.$$ 

It is easy to find that

$$\hat{\omega}_{12} = -\frac{1}{r} \hat{e}^2, \quad \hat{\omega}_{13} = -\frac{1}{r} \hat{e}^3, \quad \hat{\omega}_{23} = -\frac{\cot \theta}{r} \hat{e}^3.$$ 

The Levi-Civita connection $\hat{\nabla}$ of $g_0$ is given by

$$\hat{\nabla}_i \hat{e}_j = -\hat{\omega}_{ij} \otimes \hat{e}_j.$$ 

We denote $\hat{\nabla}_i \equiv \hat{\nabla}_{\hat{e}_i}$ for $i = 1, 2, 3$ throughout the paper.

Define $C_{(a_1, a_2, a_3)}$ the space of smooth functions in the spacetime which satisfies the following asymptotic behavior at spatial infinity

$$C_{(a_1, a_2, a_3)} = \left\{ f : \lim_{r \to \infty} \lim_{u \to -\infty} r^{a_1} f = O(1), \quad \lim_{r \to \infty} \lim_{u \to -\infty} r^{a_2} \hat{\nabla}_i f = O(1), \quad \lim_{r \to \infty} \lim_{u \to -\infty} r^{a_3} \hat{\nabla}_i \hat{\nabla}_j f = O(1) \right\}. \quad (2.3)$$

In [12], the following four conditions are introduced:

**Condition A:** Each of the six functions $\beta$, $\gamma$, $\delta$, $U$, $V$, $W$ together with its derivatives up to the second orders are equal at $\psi = 0$ and $2\pi$.

**Condition B:** For all $u$,

$$\int_0^{2\pi} c(u, 0, \psi) d\psi = 0, \quad \int_0^{2\pi} c(u, \pi, \psi) d\psi = 0.$$ 

**Condition C:** $\gamma \in C_{(1, 2, 3)}$, $\delta \in C_{(1, 2, 3)}$, $\beta \in C_{(2, 3, 4)}$, $U \in C_{(2, 3, 4)}$, $W \in C_{(2, 3, 4)}$, $V + r \in C_{(0, 1, 2)}$.

**Condition D:** $\gamma \in C_{(1, 1, 1)}$, $\delta \in C_{(1, 1, 1)}$, $\beta \in C_{(2, 2, 2)}$, $U \in C_{(2, 2, 2)}$, $W \in C_{(2, 2, 2)}$, $V + r \in C_{(0, 0, 0)}$. 

Condition A and Condition B ensure that the metric (2.1) is regular, also ensure the following Bondi mass loss formula
\[ \frac{d}{du} m_\nu = -\frac{1}{4\pi} \int_{S^2} [(c,0)^2 + (d,0)^2] n^\nu dS. \]

Condition C ensures the Schoen-Yau’s positive mass theorem at spatial infinity. However, it precludes gravitational radiation, at least near spatial infinity. Condition D should include gravitational radiation. It indicates that, for \( r \) sufficiently large,
\[ \lim_{u \to -\infty} \{ M, c, d, M_0, c_0, d_0, M_A, c_A, d_A \} = O(1) \]
where \( 2 \leq A \leq 3. \)

3. Initial data sets

From now on we assume the “real” time \( t \) is defined as
\[ t = u + r. \]

An initial data set \((N_{t_0}, g, h)\) is a spacelike hypersurface in \( L^{3,1} \) which is given by \( \{ t = t_0 \} \). Here \( g \) is the induced metric of \( \tilde{g} \) and \( h \) is the second fundamental form. It is straightforward that
\[
\begin{align*}
\tilde{g} & = \left( 2 + \frac{V}{r} \right) e^{2\beta} + r^2 e^{2\gamma} U^2 \cosh 2\delta \\
& \quad + r^2 e^{-2\gamma} W^2 \cosh 2\delta + 2r^2 UW \sinh 2\delta \right) dr^2 \\
& \quad + r^2 \left( e^{2\gamma} \cosh 2\delta d\theta^2 + e^{-2\gamma} \cosh 2\delta \sin^2 \theta d\psi^2 \\
& \quad + 2 \sinh 2\delta \sin \theta d\theta d\psi \right) \\
& \quad + 2r^2 \left( e^{2\gamma} U \cosh 2\delta + W \sinh 2\delta \right) dr d\theta \\
& \quad + 2r^2 \left( e^{-2\gamma} W \cosh 2\delta + U \sinh 2\delta \right) \sin \theta dr d\psi. \end{align*}
\]

(3.1)
The other components of the metric \( \tilde{g} \) are
\[
\begin{align*}
\tilde{g}_{tt} & = \left( \frac{V}{r} e^{2\beta} + r^2 e^{2\gamma} U^2 \cosh 2\delta \\
& \quad + r^2 e^{-2\gamma} W^2 \cosh 2\delta + 2r^2 UW \sinh 2\delta \right) \\
\tilde{g}_{t1} & = -\left( 1 + \frac{V}{r} \right) e^{2\beta} - r^2 e^{2\gamma} U^2 \cosh 2\delta \\
& \quad - r^2 e^{-2\gamma} W^2 \cosh 2\delta - 2r^2 UW \sinh 2\delta, \\
\tilde{g}_{t2} & = -r^2 \left( e^{2\gamma} U \cosh 2\delta + W \sinh 2\delta \right), \\
\tilde{g}_{t3} & = -r^2 \left( e^{-2\gamma} W \cosh 2\delta + U \sinh 2\delta \right) \sin \theta.
\end{align*}
\]
The inverse $g^{ij}$ of metric tensor $g_{ij}$, the lapse $\mathcal{N} = \left(-\tilde{g}^{tt}\right)^{-\frac{1}{2}}$ and the shift $X_i = \tilde{g}_{ti}$ ($i = 1, 2, 3$) of $N_{t0}$ are derived in [12] as follows:

\[
\begin{align*}
\frac{1}{g_{11}} &= (2 + \frac{V}{r})e^{2\beta}, \\
g_{12} &= -U g_{11}, \\
g_{13} &= -\frac{W}{\sin\theta} g_{11}, \\
g_{22} &= \frac{e^{-2\gamma} \cosh 2\delta}{r^2} + U^2 g_{11}, \\
g_{23} &= \frac{\sinh 2\delta}{r^2 \sin \theta} + \frac{UW}{\sin \theta} g_{11}, \\
g_{33} &= \frac{e^{2\gamma} \cosh 2\delta}{r^2 \sin^2 \theta} + \frac{W^2}{\sin^2 \theta} g_{11}, \\
N^2 &= e^{4\beta} g_{11}, \\
X_1 &= -\left(1 + \frac{V}{r}\right)e^{2\beta} - r^2 e^{2\gamma} U^2 \cosh 2\delta \\
&\quad - r^2 e^{-2\gamma} W^2 \cosh 2\delta - 2r^2 UW \sinh 2\delta, \\
X_2 &= -r^2 \left(e^{2\gamma} U \cosh 2\delta + W \sinh 2\delta\right), \\
X_3 &= -r^2 \left(e^{-2\gamma} W \cosh 2\delta + U \sinh 2\delta\right) \sin \theta.
\end{align*}
\]

The second fundamental form is then given by

\[
[1] h_{ij} = \frac{1}{2N} \left(\nabla_i X_j + \nabla_j X_i - \partial_t \tilde{g}_{ij}\right)_{t=t_0}.
\]

With the help of asymptotic behavior of $\beta, \gamma, \delta, U, V, W$ and Mathematica 5.0, we obtain the asymptotic expansions of $g_{ij}$,

\[
\begin{align*}
g_{11} &= 1 + \frac{2M}{r} + \frac{1}{r^2} \left(-\frac{c^2}{2} - \frac{d^2}{2} + l^2 + \tilde{l}^2 + \tilde{M}\right) + O\left(\frac{1}{r^3}\right), \\
g_{22} &= r^2 + 2rc + 2c^2 + 2d^2 + O\left(\frac{1}{r}\right), \\
g_{33} &= \left(r^2 - 2rc + 2c^2 + 2d^2\right) \sin^2 \theta + O\left(\frac{1}{r}\right), \\
g_{12} &= -l + \frac{1}{r}\left(-2cl - 2d\tilde{l} + p\right) + O\left(\frac{1}{r^2}\right), \\
g_{13} &= -\tilde{l} \sin \theta + \frac{\sin \theta}{r}\left(2c\tilde{l} - 2d\tilde{l} + \tilde{p}\right) + O\left(\frac{1}{r^2}\right), \\
g_{23} &= 2rd \sin \theta + O\left(\frac{1}{r}\right).
\end{align*}
\]
the asymptotic expansions of $g^{ij}$,

$$
\begin{align*}
g^{11} &= 1 - \frac{2M}{r} + \frac{1}{r^2}\left(\frac{c^2}{2} + \frac{d^2}{2} + 4M^2 - \bar{M}\right) + O\left(\frac{1}{r^3}\right), \\
g^{22} &= \frac{1}{r^2} - \frac{2c}{r^3} + \frac{1}{r^4}\left(2c^2 + 2d^2 + \bar{l}^2\right) + O\left(\frac{1}{r^5}\right), \\
g^{33} &= \csc^2\theta + \frac{2c\csc^2\theta}{r^2} + \frac{\csc^2\theta}{r^4}\left(2c^2 + 2d^2 + \bar{l}^2\right) + O\left(\frac{1}{r^5}\right), \\
g^{12} &= \frac{l}{r^2} - \frac{1}{r^3}\left(2Ml + p\right) + O\left(\frac{1}{r^4}\right), \\
g^{13} &= \frac{l\csc\theta}{r^2} - \frac{\csc\theta}{r^3}\left(2M\bar{l} + \bar{p}\right) + O\left(\frac{1}{r^4}\right), \\
g^{23} &= -\frac{2d\csc\theta}{r^3} + \frac{\bar{l}\csc\theta}{r^4} + O\left(\frac{1}{r^5}\right),
\end{align*}
$$

and the asymptotic expansions of $N$ and $X_i$,

$$
\begin{align*}
N &= 1 - \frac{M}{r} - \frac{1}{r^2}\left(\frac{c^2}{4} + \frac{d^2}{4} - \frac{3M^2}{2} + \frac{\bar{M}}{2}\right) + O\left(\frac{1}{r^3}\right), \\
X_1 &= -\frac{2M}{r} - \frac{1}{r^2}\left(l^2 + \bar{l}^2 + \bar{M}\right) + O\left(\frac{1}{r^3}\right), \\
X_2 &= l + \frac{1}{r}\left(2cl + 2d\bar{l} - p\right) + O\left(\frac{1}{r^2}\right), \\
X_3 &= \bar{l}\sin\theta + \frac{\sin\theta}{r}\left(-2c\bar{l} + 2d\bar{l} - \bar{p}\right) + O\left(\frac{1}{r^2}\right).
\end{align*}
$$

as well as the asymptotic expansions of $h_{ij}$,

$$
\begin{align*}
h_{11} &= \frac{M\theta}{r} + \frac{1}{r^2}\left(2M - MM_0 + \frac{cc_0}{2} + \frac{dd_0}{2} + ll_0 + \frac{M\theta}{2}\right) + O\left(\frac{1}{r^3}\right), \\
h_{22} &= -rc_0 - 2M + Mc_0 - 2cc_0 - 2dd_0 + l_2 + O\left(\frac{1}{r}\right), \\
h_{33} &= \left(rc_0 - 2M - Mc_0 - 2cc_0 - 2dd_0 + l\cot\theta + l\bar{c}\csc\theta\right)\sin^2\theta \\
&\quad + O\left(\frac{1}{r}\right),
\end{align*}
$$

$$
\begin{align*}
h_{12} &= \frac{1}{r}\left(-M_2 - l + c_0l + d_0\bar{l}\right) + O\left(\frac{1}{r^2}\right), \\
h_{13} &= \frac{\sin\theta}{r}\left(-M_3\csc\theta - \bar{l} - c_0\bar{l} + d_0l\right) + O\left(\frac{1}{r^2}\right), \\
h_{23} &= -rd_0 + Md_0 + \frac{1}{2}\left(l_2 - l\cot\theta + l_3\csc\theta\right)\sin\theta + O\left(\frac{1}{r}\right).
\end{align*}
$$

The trace of the second fundamental form is

$$
\begin{align*}
tr_g(h) &= \frac{M\theta}{r} + \frac{1}{r^2}\left(-2M - 3MM_0 + l_2 + l\cot\theta + l\bar{c}\csc\theta \\
&\quad + \frac{cc_0}{2} + \frac{dd_0}{2} + ll_0 + \frac{M\theta}{2}\right) + O\left(\frac{1}{r^3}\right).
\end{align*}
$$
4. ADM and Bondi Total Momenta

Let Euclidean coordinates

\[ y^1 = r \sin \theta \cos \psi, \quad y^2 = r \sin \theta \sin \psi, \quad y^3 = r \cos \theta. \]

In polar coordinates, the ADM total energy \( E \) and the ADM total linear momentum \( P_k \) are \([1, 12]\):

\[
E = \frac{1}{16 \pi} \lim_{r \to \infty} \int_{S_r} \left[ \hat{\nabla}^2 g(\hat{e}_i, \hat{e}_j) - \hat{\nabla}_1 \text{tr}_g (g) \right] \hat{e}^2 \wedge \hat{e}^3,
\]

\[
P_k = \frac{1}{8 \pi} \lim_{r \to \infty} \int_{S_r} \left[ h \left( \frac{\partial}{\partial y^k}, \frac{\partial}{\partial r} \right) - g \left( \frac{\partial}{\partial y^k}, \frac{\partial}{\partial r} \right) \text{tr}_g (h) \right] \hat{e}^2 \wedge \hat{e}^3.
\]

Under Condition A, Condition B and Condition D, the second author derived the relation between the ADM total energy and the Bondi mass \([12]\):

\[
E(t_0) = m_0(-\infty) + \frac{1}{2 \pi} \lim_{u \to -\infty} \int_0^\pi \int_0^{2\pi} \left( cc_{0} + dd_{0} \right) \sin \theta \psi d\psi d\theta.
\]

Now we study the relation between the ADM total linear momentum and the Bondi momentum under these conditions and prove the main theorem.

**Theorem 4.1.** Let \( P_k(t_0) \) be the ADM total linear momentum of spacelike hypersurface \( N_{t_0} \) whose metric satisfies \([3, 11]\). Under Condition A, Condition B and Condition D, we have

\[
P_k(t_0) = m_k(-\infty) + \frac{1}{8 \pi} \lim_{u \to -\infty} \int_0^\pi \int_0^{2\pi} P_k d\psi d\theta
\]

for \( k = 1, 2, 3 \), where

\[
P_1 = \left[ (c_{0} + M_{0}) \cos \theta \cos \psi - d_{0} \sin \psi \right] l \sin \theta
\]

\[
+ \left[ (c_{0} - M_{0}) \sin \psi + d_{0} \cos \theta \cos \psi \right] \bar{l} \sin \theta,
\]

\[
P_2 = \left[ (c_{0} + M_{0}) \cos \theta \sin \psi + d_{0} \cos \psi \right] l \sin \theta
\]

\[
- \left[ (c_{0} - M_{0}) \cos \psi - d_{0} \cos \theta \sin \psi \right] \bar{l} \sin \theta,
\]

\[
P_3 = - (c_{0} + M_{0}) l \sin^2 \theta - d_{0} \bar{l} \sin^2 \theta.
\]
Proof: Using the asymptotic expansions of $g_{ij}$ and $h_{ij}$, we obtain

\[
g\left(\frac{\partial}{\partial y^1}, \frac{\partial}{\partial r}\right) = g_{11} n^1 + g_{21} \frac{\cos \theta \cos \psi}{r} - g_{31} \frac{\sin \psi}{r \sin \theta} \\
= \sin \theta \cos \psi + \frac{1}{r} \left(2M \sin \theta \cos \psi \right. \\
- l \cos \theta \cos \psi + \bar{l} \sin \psi \left.) + O\left(\frac{1}{r^2}\right), \right.
\]

\[
g\left(\frac{\partial}{\partial y^2}, \frac{\partial}{\partial r}\right) = g_{11} n^2 + g_{21} \frac{\cos \theta \sin \psi}{r} + g_{31} \frac{\cos \psi}{r \sin \theta} \\
= \sin \theta \sin \psi + \frac{1}{r} \left(2M \sin \theta \sin \psi \right. \\
- l \cos \theta \sin \psi - \bar{l} \cos \psi \left.) + O\left(\frac{1}{r^2}\right), \right.
\]

\[
g\left(\frac{\partial}{\partial y^3}, \frac{\partial}{\partial r}\right) = g_{11} n^3 - g_{21} \frac{\sin \theta}{r} \\
= \cos \theta + \frac{1}{r} \left(2M \cos \theta + l \sin \theta \right) + O\left(\frac{1}{r^2}\right),
\]

\[
h\left(\frac{\partial}{\partial y^1}, \frac{\partial}{\partial r}\right) = h_{11} n^1 + h_{21} \frac{\cos \theta \cos \psi}{r} - h_{31} \frac{\sin \psi}{r \sin \theta} \\
= \frac{M_0 \sin \theta \cos \psi}{r} + \frac{1}{r^2} \left[(2M - MM_0 \right. \\
+ \left. \frac{cc_0}{2} + \frac{dd_0}{2} + \bar{l} + \frac{\bar{M}_0}{2}\right) \sin \theta \cos \psi \right. \\
- \left.(M_2 + l - c_0 \bar{l} - \bar{d}_0 \bar{l}) \cos \theta \cos \psi \right. \\
+ \left. (M_3 \csc \theta + \bar{l} + c_0 \bar{l} - \bar{d}_0 \bar{l}) \sin \psi \left.) + O\left(\frac{1}{r^3}\right), \right.
\]

\[
h\left(\frac{\partial}{\partial y^2}, \frac{\partial}{\partial r}\right) = h_{11} n^2 + h_{21} \frac{\cos \theta \sin \psi}{r} + h_{31} \frac{\cos \psi}{r \sin \theta} \\
= \frac{M_0 \sin \theta \sin \psi}{r} + \frac{1}{r^2} \left[(2M - MM_0 \right. \\
+ \left. \frac{cc_0}{2} + \frac{dd_0}{2} + \bar{l} + \frac{\bar{M}_0}{2}\right) \sin \theta \sin \psi \right. \\
- \left.(M_2 + l - c_0 \bar{l} - \bar{d}_0 \bar{l}) \cos \theta \sin \psi \right. \\
- \left. (M_3 \csc \theta + \bar{l} + c_0 \bar{l} - \bar{d}_0 \bar{l}) \cos \psi \left.) + O\left(\frac{1}{r^3}\right), \right.
\]

\[
h\left(\frac{\partial}{\partial y^3}, \frac{\partial}{\partial r}\right) = h_{11} n^3 - h_{21} \frac{\sin \theta}{r} \\
= \frac{M_0 \cos \theta}{r} + \frac{1}{r^2} \left[(2M - MM_0 \right. \\
+ \left. \frac{cc_0}{2} + \frac{dd_0}{2} + \bar{l} + \frac{\bar{M}_0}{2}\right) \cos \theta \right. \\
+ \left. (M_2 + l - c_0 \bar{l} - \bar{d}_0 \bar{l}) \sin \theta \right) + O\left(\frac{1}{r^3}\right).
\]
Denote $K_k = h \left( \frac{\partial}{\partial y^k}, \frac{\partial}{\partial r} \right) - g \left( \frac{\partial}{\partial y^k}, \frac{\partial}{\partial r} \right) \text{tr}_g(h)$. We then obtain

\[ K_1 = \frac{1}{r^2} \left\{ 4M \sin \theta \cos \psi - M_2 \cos \theta \cos \psi + M_3 \csc \theta \sin \psi 
- l_2 \sin \theta \cos \psi - \bar{l}_3 \cos \psi 
+ [(c_0 + M_0 - 2) \cos \theta \cos \psi - d_0 \sin \psi] l 
+ [(c_0 - M_0 + 1) \sin \psi + d_0 \cos \theta \cos \psi] \bar{l} \right\} + O \left( \frac{1}{r^3} \right), \]

\[ K_2 = \frac{1}{r^2} \left\{ 4M \sin \theta \sin \psi - M_2 \cos \theta \sin \psi - M_3 \csc \theta \cos \psi 
- l_2 \sin \theta \sin \psi - \bar{l}_3 \sin \psi 
+ [(c_0 + M_0 - 2) \cos \theta \sin \psi + d_0 \cos \psi] l 
- [(c_0 - M_0 + 1) \cos \psi - d_0 \cos \theta \sin \psi] \bar{l} \right\} + O \left( \frac{1}{r^3} \right), \]

\[ K_3 = \frac{1}{r^2} \left\{ 4M \cos \theta - M_2 \sin \theta - l_2 \cos \theta - \bar{l}_3 \cot \theta 
- [(c_0 + M_0 - 2) \sin \theta + \csc \theta] l - d_0 \bar{l} \sin \theta \right\} + O \left( \frac{1}{r^3} \right). \]

Integrating $K_k$ over $S_r$, and simplifying them by the integration by part, also noting that for fixed $t = t_0$, $r \rightarrow \infty$ is equivalent to $u \rightarrow -\infty$, we finally obtain the proof of the theorem. Q.E.D.

5. Axi-Symmetric spacetimes

Gravitational waves in axi-symmetric spacetimes were first studied by Bondi, van der Burg and Metzner [3]. In this case the metric of the spacetime is

\[ \tilde{g} = \left( \frac{V}{r} e^{2\beta} + r^2 e^{2\gamma} U^2 \right) du^2 - 2e^{2\beta} du dr 
- 2r^2 e^{2\gamma} U du d\theta + r^2 \left( e^{2\gamma} d\theta^2 + e^{-2\gamma} \sin^2 \theta d\psi^2 \right) \]

where $\beta$, $\gamma$, $U$ and $V$ are functions of $u$, $r$ and $\theta$. This implies that

\[ c = c(u, \theta), \quad d = 0, \quad l = c_2 + 2c \cot \theta, \quad \bar{l} = 0. \]

**Corollary 5.1.** Let $E(t_0)$, $P_k(t_0)$ be the ADM total energy and the ADM total linear momentum of spacelike hypersurface $N_{t_0}$ in axi-symmetric vacuum radiating spacetimes \([\Sigma_t]\). Under Condition A, Condition B and
Condition D, we have

\[
\begin{align*}
E(t_0) &= m_0(-\infty) + \lim_{u \to -\infty} \int_0^\pi c c_0 \sin \theta d\theta, \\
P_1(t_0) &= m_1(-\infty), \\
P_2(t_0) &= m_2(-\infty), \\
P_3(t_0) &= m_3(-\infty) - \frac{1}{4} \lim_{u \to -\infty} \int_0^\pi \left[ -(c_0)^2 + \frac{c_{220}}{2} \\
&+ c_{20} \cot \theta \right] (c_2 + 2c \cot \theta) \sin^2 \theta d\theta.
\end{align*}
\]

6. Appendix

In this appendix, we provide an addendum to [12]. There are two sign differences between the Bondi radiating metric (2.1) in [12] and the metric used in current paper as well as the original paper [3]. They result some sign changes in certain formulas but the main theorems in [12] still holds true without any change. We give as follows the corresponding formulas for [12] with respect to the metric used in current paper.

1) page 263, (2.1) in [12] changes to (2.1) in current paper.

2) page 267, (2.5) in [12] changes to

\[
\bar{g} = \left( \frac{V}{r} e^{2\beta} + r^2 e^{2\gamma} U^2 \cosh 2\delta \\
+ r^2 e^{-2\gamma} W^2 \cosh 2\delta + 2r^2 U W \sinh 2\delta \right) dt^2 \\
- 2 \left( (1 + \frac{V}{r}) e^{2\beta} + r^2 e^{2\gamma} U^2 \cosh 2\delta \\
+ r^2 e^{-2\gamma} W^2 \cosh 2\delta + 2r^2 U W \sinh 2\delta \right) dt dr \\
- 2r^2 \left( e^{2\gamma} U \cosh 2\delta + W \sinh 2\delta \right) d\theta d\theta \\
- 2r^2 \left( e^{-2\gamma} W \cosh 2\delta + U \sinh 2\delta \right) \sin \theta d\theta d\psi \\
+ \left( (2 + \frac{V}{r}) e^{2\beta} + r^2 e^{2\gamma} U^2 \cosh 2\delta \\
+ r^2 e^{-2\gamma} W^2 \cosh 2\delta + 2r^2 U W \sinh 2\delta \right) d\psi^2 \\
+ 2 \sinh 2\delta \sin \theta d\theta d\psi \\
+ 2r^2 \left( e^{2\gamma} U \cosh 2\delta + W \sinh 2\delta \right) d\psi^2 \\
+ 2r^2 \left( e^{-2\gamma} W \cosh 2\delta + U \sinh 2\delta \right) \sin \theta d\psi.
\]
3) page 268, (3.1) in [12] change to (3.1) in current paper.

4) page 269, line -9 to line -8 in [12] changes to

$$g(\vec{e}_1, \vec{e}_1) = \left[ (2 + \frac{V}{r}) e^{2\beta} + r^2 e^{2\gamma} U^2 \cosh 2\delta + r^2 e^{-2\gamma} W^2 \cosh 2\delta + 2r^2 U W \sinh 2\delta \right]_{t=t_0}.$$

5) page 271, line 2 to line 3 in [12] change to

$$\frac{1}{g^{11}} = (2 + \frac{V}{r}) e^{2\beta},$$

6) page 271, line -7 to line -6 in [12] change to

$$X_1 = -\left( 1 + \frac{V}{r} \right) e^{2\beta} - r^2 e^{2\gamma} U^2 \cosh 2\delta - r^2 e^{-2\gamma} W^2 \cosh 2\delta - 2r^2 U W \sinh 2\delta$$

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