A transition from a decelerated to an accelerated phase of the universe expansion from the simplest non-trivial polynomial function of $T$ in the $f(R, T)$ formalism

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Abstract In this work we present cosmological solutions from the simplest non-trivial polynomial function of $T$ in $f(R, T)$ theory of gravity, with $R$ and $T$ standing for the Ricci scalar and trace of the energy-momentum tensor, respectively. Although such an approach yields a highly non-linear differential equation for the scale factor, we show that it is possible to obtain analytical solutions for the cosmological parameters. For some values of the free parameters, the model is able to predict a transition from a decelerated to an accelerated expansion of the universe and the values of the deceleration parameter agree with observation.

Keywords Cosmological models · Cosmic acceleration · $f(R, T)$ gravity

1 Introduction

With the modern observational astrophysics, it was discovered that the universe expansion is accelerating. If one considers the universe to be composed mostly by matter, this acceleration is highly non-intuitive since gravity, as an attractive force, should slow down the expansion velocity. The observation of Type Ia Supernovae (Riess et al. 1998; Perlmutter et al. 1999) seems to support a universe mostly ($\sim 70\%$) by an exotic component dubbed dark energy (DE). The DE has an equation of state (EoS) $\omega \sim -1$ which justifies the apparent anti-gravitational aspect of the present universe dynamics. In $\Lambda$CDM cosmology, the DE is mathematically described by a cosmological constant (CC) ($\Lambda$) in the Einstein’s field equations (FEs). However, the CC, coincidence and dark matter problems, missing satellites, hierarchy problem and other shortcomings (see Clifton et al. 2012 and references therein) arising from $\Lambda$CDM model yield the consideration of alternative cosmological models.

We are led, then, to search for some kind of matter fields which could generate the observable outcomes without the $\Lambda$CDM shortcomings. For instance, scalar fields which slowly go down to their potential can produce sufficient negative pressure in order to make the universe expansion to accelerate (Caldwell et al. 1998; Tsujikawa 2013; Moraes and Santos 2014; Khurshudyan et al. 2014; Khurshudyan et al. 2015; Farooq et al. 2011).

Other efficient alternatives to find answers for the cosmological issues mentioned above come from the family of $f(R)$ (de Felice and Tsujikawa 2010; Sotiriou and Faraoni 2010) and $f(R, T)$ (Harko et al. 2011a) theories; the latter focusing not only in generalizing the geometrical terms – those proportional to the Ricci scalar $R$ – in the gravitational part of the action, but also inserting matter terms – those proportional to the trace of the energy-momentum tensor $T$ – on it. According to the $f(R, T)$ gravity authors, the dependence on $T$ comes from the consideration of quantum effects which are neglected in General Relativity (GR) and $f(R)$ theory. Since so far theoretical physicists could not construct a formalism which is able to unify GR with Quantum Mechanics, although attempts to do so can be found in String Theory (Witten 1986; Friedan 1986; Dixon and Harvey 1986; Damour 2003), it is worth considering quantum effects in a theory of gravity.
$f(R, T)$ gravity proved to have a very general aspect in Jamil et al. (2012) since by taking different kinds of energy sources or functional forms for $f(R, T)$, it could retrieve other cosmological models, such as $\Lambda$CDM, Chaplygin gas and quintessence. In this sense it was shown that other alternative models can be seen as $f(R, T)$ special cases.

$f(R, T)$ theory robustness was brought to light, for instance, in the following papers: Moraes and Santos (2016), for which a complete cosmological scenario able to describe all the different dynamical stages of the universe was constructed from the consideration of $f(R, T)$ gravity in the presence of a scalar field (named, $f(R, T, \phi)$) gravity; Moraes (2015), for which a five-dimensional cosmological model was presented and the fifth dimension length scale was shown to shrink with time, departing from other extra-dimensional models where the extra coordinate is imposed to be compactified; Shabani and Farhoudi (2013), for which the evolution of scalar cosmological perturbations was studied.

Another well-behaved cosmological scenarios were constructed from such a theory, as one can check in Moraes and Correa (2016), Moraes (2014, 2016), Rao and Papa Rao (2015), Sahoo and Sivakumar (2015), Shamir (2015), Singh and Singh (2015), Ahmed and Pradhan (2014) and Farajollahi et al. (2012).

$f(R, T)$ formalism is also able to provide the description of a coupling between matter and geometry, such as other reputed models as those discussed in Harko (2008), Harko and Lobo (2010), Harko et al. (2011b, 2013), Delsate and Steinhoff (2012) and Haghani et al. (2013).

Moreover, in an astrophysical level, the theory also has presented remarkable features, as one can check in Barrientos and Rubilar (2016), Zubair and Noureen (2015), Noureen and Zubair (2015a, 2015b) and Noureen et al. (2015).

We believe the above points among others which we will quote below make reasonable to consider $f(R, T)$ as the underlying theory for a cosmological model. In this sense, that is the aim of the present article.

The cosmological features of the above models are directly related to the functional forms of the $f(R, T)$ functions. In the $f(R)$ theories, the simplest non-trivial polynomial function of the Ricci scalar $R$ was used in a quite seminal work (Starobinsky 2007). A.A. Starobinsky has proposed a class of models for the dependence of the gravitational part of the action on $R$. Those have reproduced $\Lambda$CDM features at recent times and satisfied solar system and laboratory tests. Today, the most popular of these models, known as Starobinsky model (SM), is the one for which the functional form of $f(R)$ is given by $R + \alpha R^2$, with $\alpha$ being a constant.

Our purpose in the present article is to propose a Starobinsky-like model for the dependence on $T$ in the $f(R, T)$ formalism, i.e., we will derive a cosmological scenario from $f(R, T) = f(R) + f(T)$, with $f(T) = \alpha T + \beta T^2$, and $\alpha$ and $\beta$ being constants. This is the simplest non-trivial polynomial function of $T$ for the functionality of $f(R, T)$ as is the SM in $f(R)$ gravity and due to the high non-linearity of the resulting differential equations it has not been used so far in $f(R, T)$ gravity for cosmological or any other purposes.

An $f(R) = R + \alpha R^2$ gravity can generate matter bounce cosmological solutions, as shown in Oikonomou (2015). Observational constraints on the SM parameter can be found in Dev et al. (2008). Furthermore, the structure of neutron stars in SM was discussed in Ganguly et al. (2014), Orellana et al. (2013) and Thongkool et al. (2009) and the collapse of massive stars in Goswami et al. (2014). Studies of gravitational waves in SM can be appreciated in Yang et al. (2011) and Capozziello et al. (2009). SM has presented remarkable features in different areas of Cosmology, Gravitation and Astrophysics. Like a kind of analogy, it might be interesting to check the consequences of linear and quadratic matter terms corrections in Einstein-Hilbert gravitational action. This shall be appreciated below from a cosmological perspective.

The article is organized as follows: in Sect. 2 a brief review of $f(R, T)$ gravity is exposed. We present solutions for the correspondent cosmological parameters, such as scale factor, Hubble parameter and deceleration parameter in Sect. 3. The model is constructed by assuming a matter-dominated universe, i.e., without necessity of assuming the universe dynamics is dominated by a CC or some sort of quintessence. In Sect. 4 we discuss our results and compare them with observation.

2 A brief review of the $f(R, T)$ gravity

The gravitational action in $f(R, T)$ gravity reads

$$S_{grav} = \frac{1}{16\pi} \int f(R, T) \sqrt{-g} d^4x,$$  \hspace{1cm} (1)

with $f(R, T)$ being the general function of $R$ and $T$, and $g$ the determinant of the metric. According to the authors, such a $T$-dependence comes from the consideration of quantum effects. Moreover, throughout this work, we will consider units such that the gravitational constant and the speed of light are equal to 1.

The FE of the $f(R, T)$ gravity are obtained by varying (1) with respect to the metric and read:

$$f_R(R, T) R_{\mu \nu} - \frac{1}{2} f(R, T) g_{\mu \nu} + (g_{\mu \nu} \Box - \nabla_{\mu} \nabla_{\nu}) \Theta_{\mu \nu},$$

$$f_R(R, T) = 8\pi T_{\mu \nu} - f_T(R, T) T_{\mu \nu} - f_T(R, T) \Theta_{\mu \nu},$$  \hspace{1cm} (2)
in which $\Theta_{\mu\nu} = -2T_{\mu\nu} - p g_{\mu\nu}$, $T_{\mu\nu}$ is the Ricci tensor, $T_{\mu\nu} = diag(\rho, -p, -p, -p)$ is the energy-momentum tensor, which we are assuming to be the one of a perfect fluid, with $\rho$ and $p$ representing the matter-energy density and pressure of the universe, respectively. $\Box$ is the D’Alambert operator and $\nabla$ is the covariant derivative. Moreover, $f_R(R, T)$ and $f_T(R, T)$ are the partial derivatives of $f(R, T)$ with respect to $R$ and $T$, respectively.

3 The $f(R, T) = R + \alpha T + \beta T^2$ cosmology

We are going to assume for $f(R, T)$ the simplest non-trivial polynomial function of $T$ and the simplest dependence on $R$, i.e., $f(R, T) = R + \alpha T + \beta T^2$, in (2), with $\alpha$ and $\beta$ being constants, and $T = \rho - 3p$. Such a functional form benefits from the fact that one can recover GR just by letting $\alpha = \beta = 0$.

Recall that Harko et al. (2011a) have proposed a generalization of $f(R)$ theories, by making the gravitational part of the action to depend generally not only on its geometrical terms, but also on its matter terms. We are dealing here with the same GR geometrical term on the action, but we are generalizing the matter terms (those proportional to $T$) in order to check if such a generalization is able to generate well-behaved cosmological scenarios in the same way SM does, for instance, in Starobinsky (2007), Borowiec et al. (2012) and Odintsov and Oikonomou (2015).

By substituting $f(R, T) = R + \alpha T + \beta T^2$ in (2) yields:

$$G_{\mu\nu} = 8\pi T_{\mu\nu} + \alpha \left[ T_{\mu\nu} + \frac{1}{2} (\rho - p) g_{\mu\nu} \right]$$

$$+ 2\beta (\rho - 3p) \left[ T_{\mu\nu} + \frac{1}{4} (\rho + p) g_{\mu\nu} \right],$$

(3)

for which $G_{\mu\nu}$ stands for the usual Einstein tensor. We have written (3) in such an elegant form in order to explicit the terms proportional to $\alpha$ and $\beta$. We can see that the terms proportional to $\alpha$ carry linear corrections in the matter terms while the terms proportional to $\beta$ carry quadratic corrections.

By developing (3) for a Friedmann-Robertson-Walker metric with null curvature (which is in accordance with recent cosmic microwave background radiation temperature fluctuations, Hinshaw et al. 2013), we obtain

$$3 \left( \frac{\dot{a}}{a} \right)^2 = 8\pi \rho + \frac{1}{2} [\alpha (3\rho - p) + \beta (5\rho + p)(\rho - 3p)],$$

(4)

$$2 \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 = -8\pi p + \frac{1}{2} [\alpha (\rho - 3p) + \beta (\rho - 3p)^2],$$

(5)

with $a$ being the scale factor and dots representing time derivatives.

Let us consider a matter EoS for the universe, i.e., $p = 0$. Such an assumption yields the following differential equation for the scale factor:

$$\frac{\ddot{a}}{a} + 3 \frac{\dot{a}}{10a} + \frac{\lambda_1}{2} - \frac{4}{3\pi} \sqrt{\frac{6}{5\beta}} \frac{a}{\dot{a}} = 0,$$

(6)

where we are assuming that $\alpha = -16\pi/3$.

We can see that (6) has a non-linear character. It might be important to remark that the presence of nonlinearity is not surprising, given that such a behavior is found in a wide range of areas of Physics nowadays de Souza Dutra and Correa (2009, 2010), Correa and de Souza Dutra (2015), Correa et al. (2015a, 2015b) and Correa and Moraes (2016). Because of the nonlinearity, we are led to ask if the problem can be analytically resolved.

We show below that, indeed, it is possible to obtain an analytical solution for the scale factor differential equation above. To do this, let us firstly multiply (6) by the factor $(a/\dot{a})$, obtaining:

$$\frac{\ddot{a}}{a} + 3 \frac{\dot{a}}{10a} + \frac{\lambda_1}{2} - \frac{4}{3\pi} \sqrt{\frac{6}{5\beta}} = 0.$$

(7)

Now, after straightforward manipulations, it is easy to deduce from (7) that the equation for the scale factor can be put in the form

$$\frac{d}{dt} (\ln \dot{a} + \lambda_1 \ln a) = -\lambda_2,$$

(8)

where $\lambda_1 \equiv 3/10$ and $\lambda_2 \equiv 1/2 - 4\pi \sqrt{6/(5\beta)}/3.$

Therefore, by solving (8), we obtain

$$a(t) = a_0 (e^{-\lambda_2 t} + c_0)^{10},$$

(9)

where $a_0$ and $c_0$ are arbitrary constants of integration.

Using (9) we are able to find the Hubble parameter $H = \dot{a}/a$ and the deceleration parameter $q = -\ddot{a}/(\dot{a}H)$, for which negative values of the latter indicate an accelerated expansion of the universe.

From (9) we derive

$$H = -\frac{10}{13c_0 e^{\lambda_2 t} + 1},$$

(10)

$$q = -\left( \frac{13}{10}c_0 e^{\lambda_2 t} + 1 \right).$$

(11)

Below we are going to depict the evolution of $H$ and $q$ through time. The discussion of our results will be given in the next section. For now, by invoking the initial condition $a(0) = 0$, we obtain $c_0 = -1$.  

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4 Discussion

The CC problem is one of the greatest mysteries of Cosmology today. Although $\Lambda$CDM model can provide a great matching between theoretical predictions and observations, it still lacks a convincing explanation for the DE physical interpretation. There is a huge discrepancy between the observed value of the CC density (Hinshaw et al. 2013) and its value obtained from Particle Physics theoretical predictions (Weinberg 1989). Since the CC is considered to be the responsible for the cosmic acceleration, it is worth trying to describe such a dynamical phenomenon without invoking it.

One possibility for it would be the $f(R)$ theories. However, classical tests of GR for the Solar System (weak field) regime seem to rule out most of the $f(R)$ models which have been proposed so far (Erickcek et al. 2006; Chiba et al. 2007; Capozziello et al. 2007; Olmo 2007). Quintessence models can also be a possibility. They propose DE is in the form of a time varying scalar field $\phi$ which slowly rolls down toward its potential minimum. However, the potential energy of the scalar field needs a fine-tuning to describe such a dynamical phenomenon without invoking it.

Equation (6) is the differential equation for the scale factor one obtains when considering the Starobinsky-like model for the $T$-dependence and a standard $R$-dependence in $f(R, T)$ gravity. In such an equation we have taken $p = 0$, i.e., we are assuming the universe dynamics is dominated by matter. Moreover we have taken $\alpha = -16\pi/3$ for the sake of its construction.

We have presented (9) as (6) solution. From it, we were able to derive (10)–(11) as the Hubble and deceleration parameters, respectively.

The time evolution of these cosmological parameters is depicted in Figs. 1 and 2. We have shown the behavior of $H$ and $q$ for different values of $\beta$. The constant $c_0$ was set to $-1$ because of the initial condition $a(0) = 0$. Moreover, when plotting $H$ and $q$ there was no necessity of assuming any value for $a_0$ in (9).

In Fig. 1 we can see that for different values of $\beta$, the predicted Hubble parameters are well-behaved. Firstly they are all restricted to positive values, which is expected in an expanding universe. Also, in standard cosmology the Hubble parameter is proportional to the inverse of the Hubble time $t_H$, as $H \propto 1/t_H$. Such a feature is also being respected in Fig. 1.

Figure 2 depicts the evolution of the deceleration parameter for different values of $\beta$. The deceleration parameter is
defined as $-\ddot{a}/a^2$, so that negative values of it describe an acceleration of the universe expansion. We can see that for $\beta = \pi$ and $\beta = 5\pi$ the universe expansion has previously slowed down its velocity, which is fundamental during its evolution, in order to allow large scale structure formation. Then, it reached an epoch in which its expansion speeded up. Such a scenario still prevails and is known as DE era in standard cosmology. On the other hand, from the (red) solid line curve in Fig. 2, we realize that for $\beta = 30\pi$ it is not possible to describe an accelerating expansion, since $q$ is restricted to increasing positive values.

A more sensitive analysis of Fig. 2 reveals that the time evolution of $q$ for $\beta = \pi$ and $\beta = 5\pi$ is in agreement with cosmological observations, as we will argue below. Firstly, for low values of time our results agree with those obtained in Daly and Djorgovski (2004) and Giostri et al. (2012), which have used type Ia supernovae data, baryon acoustic oscillations and cosmic microwave background observation to constrain cosmological parameters, the former assuming the $\Lambda$CDM cosmological model while the latter a kinematical approach.

For $q_0$, the present value of the deceleration parameter, the references above predict $-0.35 \pm 0.15$ and $-0.31 \pm 0.11$, respectively. For both $\beta = \pi$ and $5\pi$ the deceleration parameter eventually assumes the above predicted values. We also note that within this regime of values for $\lambda$, the higher its value, the later the universe transits to an accelerated expansion era. In other words, as one increases the $\lambda$ values, the universe gets more time to establish the large scale structures. In future works, the values of $\beta$ can be highly constrained from this perspective.

Furthermore, from Fig. 2, we note that our solutions characterize a de Sitter solution as $t \rightarrow \infty$ ($q = -1$). Such a tendency can also be seen in the Chaplygin gas (Bento et al. 2002; Dev et al. 2003) and Dvali-Gabadadze-Porrati (Dvali et al. 2000) models.

By analysing the cosmological features above from the perspective of the $\lambda_2$ parameter values, we can see that for $\beta < 128\pi^2/15$, $\lambda_2 < 0$. From (11), we see that in order to be able to predict an accelerating universe ($q < 0$ after a certain period of time), $\lambda_2$ must be negative. On the contrary, $q$ will always assume increasing positive values (recall that $c_0 = -1$).

It is worth remarking that departing from a number of cosmological models obtained from different metrics or functional forms in $f(R, T)$ gravity, such as Adhav (2012), Reddy et al. (2012a, 2012b, 2013a, 2013b, 2014), Reddy and Kumar (2013), Rao and Neelima (2013), Samanta (2013a, 2013b), Ram and Priyanka (2013), Sharif and Zubair (2014), Sahoo et al. (2014), Singh and Sharma (2014), Sharma and Singh (2014), Moraes (2014), Mishra and Sahoo (2014) and Yadav (2014), among many others, our model was able to predict a deceleration parameter which evolves with time and we could describe the transition from a decelerated era to an accelerated stage. Such an argumentation motivates the search for new kinds of $f(R, T)$ cosmological scenarios.

Furthermore, the Starobinsky-like functional form for $f(R, T)$ should be tested in another levels. We have shown that it is able to describe a transition from a decelerated to an accelerated expansion of the universe as a milestone in the theory. However, naturally, it still lacks astrophysical investigations, such as stellar equilibrium configurations of compact stars through Tolman-Oppenheimer-Volkoff equation (Tolman 1939; Oppenheimer and Volkoff 1939), gravitational wave polarization (Eardley et al. 1973; Newman and Penrose 1962a, 1962b) and gravitational lensing (Alhamzawi and Alhamzawi 2016), for instance. Those points might be raised in future works in order to check the reliability of the Starobinsky-like model.

To finish, we would like to remark that the material correction terms proportional to linear and quadratic functions of $\rho$ and $p$ (check (4)–(5), for instance) are, indeed, the responsible for the cosmic acceleration in the present model. We will argue about this statement in the next paragraph.

Consider (5) for a matter-dominated universe with $\alpha = \beta = 0$. The solution for the scale factor in this case is $a(t) = c_1(3t - 2c_2)^{2/3}$. Such a scale factor yields $q = 1/2$ independently of the values of the constants $c_1$ and $c_2$, which is the value obtained from standard Friedmann equations in a matter-dominated universe ($p = 0$). Apart from the fact that such a deceleration parameter is constant, it is positive, i.e., it is not in agreement with an accelerated expansion. Therefore, we have shown that linear and quadratic material corrections in the gravitational action can generate a well-behaved cosmological model, with a varying deceleration parameter, which through its evolution predicts a transition from a decelerated to an accelerated phase of the universe expansion.

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References

Adhav, K.S.: Astrophys. Space Sci. 339, 365 (2012)
Ahmed, N., Pradhan, A.: Int. J. Theor. Phys. 53, 289 (2014)
Alhamzawi, A., Alhamzawi, R.: Int. J. Mod. Phys. D 25, 1650020 (2016)
Alvarenga, F.G., et al.: Phys. Rev. D 87, 103526 (2013)
Barrientos, O.J., Rubilar, G.F.: Phys. Rev. D 93, 024021 (2016)
Bento, M.C., et al.: Phys. Rev. D 66, 043507 (2002)
Borowiec, A., et al.: J. Cosmol. Astropart. Phys. 2, 27 (2012)
Caldwell, R.R., et al.: Phys. Rev. Lett. 80, 1582 (1998)
Capozziello, S., et al.: Phys. Rev. D 76, 104019 (2007)
Capozziello, S., et al.: Gen. Relativ. Gravit. 41, 2313 (2009)
