The $n$-dimensional Klein bottle is a real Bott manifold

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Abstract. Recently Davis initiated the study of an $n$-dimensional analogue of the Klein bottle. This generalized Klein bottle occurs as a moduli space of planar polygons for a certain choice of side lengths. Davis determined many topological invariants and manifold-theoretic properties of this space. The main aim of this short note is to show that the $n$-Klein bottle is a real Bott manifold and determine the corresponding Bott matrix. We also determine two other classes of polygon spaces that are total spaces of an iterated $S^1$ bundle. In all these cases we compute the Betti numbers using a formula, due to Suciu and Trevisan.

1. Introduction

The moduli space of planar polygons (or planar polygon space) associated with a length vector $\alpha := (\alpha_1, \ldots, \alpha_{n+3})$, denoted by $\mathcal{M}_\alpha$, is the collection of all closed piecewise linear paths with side lengths $\alpha_1, \alpha_2, \ldots, \alpha_{n+3}$ in the plane viewed up to all isometries. In other words,

$$\mathcal{M}_\alpha := \left\{ (v_1, v_2, \ldots, v_{n+3}) \in (S^1)^{n+3} : \sum_{i=1}^{n+3} \alpha_i v_i = 0 \right\}/O_2,$$

where $S^1$ is the unit circle and the group of isometries $O_2$ acts diagonally. For a generic choice of $\alpha$ (i.e., if the possibility of having lined polygons is excluded) $\mathcal{M}_\alpha$ is a smooth, closed manifold of dimension $n$. Since the diffeomorphism type of a planar polygon space does not depend on the ordering of the side lengths, we assume $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{n+3}$.

Definition 1.1. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n+3})$ be a length vector. A subset $I \subset [n+3]$ is short if

$$\sum_{i \in I} \alpha_i < \sum_{j \notin I} \alpha_j.$$

A subset is long if the complement is short.

In general, the collection of short subsets may be very large. Hence, there is another combinatorial object that efficiently encodes the information about all short subsets (it was introduced by Hausmann [6, Section 1.5]). For a (generic) length vector $\alpha$, consider the following collection of subsets of $[n+3]$:

$$S_{n+3}(\alpha) = \{ J \subset [n+3] : n+3 \in J \text{ and } J \text{ is short} \}.$$

There is a partial order on these subsets given by $I \leq J$ iff $I = \{ i_1, \ldots, i_t \}$ and $\{ j_1, \ldots, j_t \} \subseteq J$ with $i_s \leq j_s$ for $1 \leq s \leq t$.

Definition 1.2. The genetic code of a length vector $\alpha$ is the set of maximal elements of $S_{n+3}(\alpha)$ with respect to the above partial order. The maximal elements are called genes.

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If $A_1, A_2, \ldots, A_k$ are the maximal elements of $S_{n+3}(\alpha)$ with respect to $\leq$ then the genetic code of $\alpha$ is denoted by $(A_1, \ldots, A_k)$. It follows from [6, Lemma 1.2] that the genetic code of a length vector determines the diffeomorphism type.

An $n$-dimensional analogue of the Klein bottle, denoted $K_n$, was introduced by Davis [4] as follows:

\begin{equation}
K_n = \begin{pmatrix} (S^1)^n \\ (z_1, \ldots, z_{n-1}, z_n) \sim (\bar{z}_1, \ldots, \bar{z}_{n-1}, -z_n) \end{pmatrix}.
\end{equation}

The circle $S^1$ is considered as the unit circle in $\mathbb{C}$ and $\bar{z}$ is the complex conjugate. It is easy to see that $K_2$ is the Klein bottle. Davis computed the fundamental group, integral cohomology algebra and the stable homotopy type of $K_n$. Moreover, he obtained an explicit immersion of $K_n$ in $\mathbb{R}^{n+1}$ and an embedding in $\mathbb{R}^{n+2}$.

The following result is an immediate consequence of [6, Proposition 2.1] that justifies the connection with polygon spaces.

**Theorem 1.3.** Let $\alpha$ be a length vector with the genetic code $\langle \{1, 2, \ldots, n-1, n+3\} \rangle$. Then $M_\alpha \cong K_n$.

Another class of manifolds we are interested in are the real Bott manifolds. A real Bott tower is a sequence of $\mathbb{R}P^1$-bundles:

$$M_n \rightarrow M_{n-1} \rightarrow \cdots \rightarrow M_1 \rightarrow \{\ast\},$$

where each $\mathbb{R}P^1$-bundle $M_i \rightarrow M_{i-1}$ is the projectivization of Whitney sum of two real line bundles on $M_{i-1}$, one of them is the trivial line bundle. For each $i$, the manifold $M_i$ is called a real Bott manifold. The homeomorphism type of real Bott manifolds is completely determined by Stiefel–Whitney classes of the line bundles at each step. Hence, an efficient way to encode the homeomorphism type of these manifolds is using a square matrix, called the Bott matrix, containing 0’s and 1’s.

The main aim of this short article is to show that $K_n$ is indeed a real Bott manifold. It is not hard to see that real Bott manifolds are examples of small covers (in the sense of Davis–Januskeiwicz–Scott) with the cube as their quotient polytope. The small cover structure helps us compute the Betti numbers using a result of Suciu and Trevisan. We end the article with a discussion on two more genetic codes whose corresponding polygon spaces have a similar structure.

2. **Realizing $K_n$ as the real Bott manifold and its Betti numbers**

In this section we show that $K_n$ is a real Bott manifold and determine the corresponding Bott matrix. Since real Bott manifolds are also examples of small covers (topological analogues of real toric varieties), we begin the section by defining characteristic functions. Then we define such a function on the $n$-dimensional cube and show that the corresponding small cover is homeomorphic to the $n$-dimensional Klein bottle.

Let $P$ be an $n$-dimensional simple polytope $F = \{F_1, \ldots, F_m\}$ be the set of its facets.

**Definition 2.1.** A function $\chi : F \rightarrow \mathbb{Z}_2^n$ is called characteristic for $P$ if for each vertex $v = F_{i_1} \cap \cdots \cap F_{i_m}$, the $n \times n$ matrix whose columns are $\chi(F_{i_1}), \ldots, \chi(F_{i_m})$ is invertible. Equivalently, we can think of the characteristic function as an $n \times m$-matrix of 0’s and 1’s

$$\chi = \begin{bmatrix} \chi(F_1) & \chi(F_2) & \cdots & \chi(F_n) \end{bmatrix}$$

with the above property satisfied.
An $n$-dimensional small cover $M$ is a closed, smooth manifold with an action of $\mathbb{Z}_2^n$ that is locally isomorphic to the standard action of $\mathbb{Z}_2^n$ on $\mathbb{R}^n$ and such that the orbit space is an $n$-dimensional simple polytope $P$. These manifolds are topological generalizations of real toric varieties. Davis and Januszkiewicz in their seminal work showed how to build a small cover from the quotient polytope (see [5, Section 1.5] for details). Their result states that there is a regular cell structure on the manifold consisting of $2^n$ copies of the quotient polytope as the top-dimensional cells. Here is a brief description. Given a pair $(P, \chi)$ of a small cover and a characteristic function defined on its facets, the corresponding small cover $X(P, \chi)$ is constructed as follows:

$$X(P, \chi) := \{(\mathbb{Z}_2)^n \times P) \setminus \{(t, p) \sim (u, q)\} \quad \text{if } p = q \text{ and } t^{-1}u \in \text{stab}(F_q)$$

where $F_q$ is the unique face of $P$ containing $q$ in its relative interior.

The $n$-dimensional cube is given by

$$I^n = [-1, 1]^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : -1 \leq x_i \leq 1 \text{ for } 1 \leq i \leq n\}.$$ Consider the following labeling of the facets of $I^n$. For each $1 \leq i \leq n$, we set

$$F_i = I \times \cdots \times \{-1\} \times \cdots I$$

and

$$F_{n+i} = I \times \cdots \times \{1\} \times \cdots I,$$

where $\{-1\}$ and $\{1\}$ is at the $i$th place. Define

$$\chi(F) = \begin{cases} e_i & \text{if } F = F_i \text{ or } F = F_{n+i}, 2 \leq i \leq n, \\ e_1 & \text{if } F = F_1, \\ \sum_{i=1}^n e_i & \text{if } F = F_{n+1}. \end{cases}$$

It is clear that the $n \times 2n$-matrix of $\chi$ is

$$(2) \quad \chi = \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 1 & 0 & \cdots & 1 \end{bmatrix}. $$

**Lemma 2.2.** The function $\chi$ is characteristic for $\mathcal{F}(I^n)$.

**Proof.** Let $v$ be the vertex of $I^n$. Consider the following subcollection of facets of $\mathcal{F}(I^n)$

$$\mathcal{F}(v) = \{F \in \mathcal{F}(I^n) : v \in F\}.$$

Then,

$$\chi(\mathcal{F}(v)) = \begin{cases} \{e_2, \ldots, e_n, \sum_{i=1}^n e_i\}, & \text{if } v \in F_{n+1}, \\ \{e_1, \ldots, e_n\}, & \text{otherwise.} \end{cases}$$

Clearly, in both the cases $\chi(\mathcal{F}(v))$ forms a basis for $\mathbb{Z}_2^n$. Therefore, $\chi$ is a characteristic function on $\mathcal{F}(I^n)$. \hfill \Box

We follow [1] for basics of real Bott manifolds. Given a strictly upper triangular binary matrix (i.e., matrix whose entries are 0 or 1), a real Bott manifold can be constructed as the quotient of the $n$-dimensional torus by a free action of $\mathbb{Z}_2^n$. 3
Definition 2.3. A binary square matrix $A$ is said to be a Bott matrix if there exists a permutation matrix $P$ and a strictly upper triangular binary matrix $B$ such that $A = PBP^{-1}$.

Let $z \in S^1$ and $a \in \{0, 1\}$. Define the notation

$$z(a) := \begin{cases} z & \text{if } a = 0, \\ \bar{z} & \text{if } a = 1. \end{cases}$$

Let $A^i_j$ be the $(i, j)$ entry of a Bott matrix $A$. For $1 \leq i \leq n$ define the involution $a_i$ on $(S^1)^n$ as follows:

$$(3) \quad a_i((z_1, \ldots, z_n)) = (z_1(A^i_1), \ldots, z_{i-1}(A^i_{i-1}), -z_i, z_{i+1}(A^i_{i+1}), \ldots, z_n(A^i_n)).$$

Note that these involutions commute with each other and generate a multiplicative group $\mathbb{Z}_2^n$, which we denote by $G(A)$. Moreover, the action of this group on $(S^1)^n$ is free (see [1, Lemma 2.1]). The real Bott manifold associated with the Bott matrix is defined as the quotient

$$(S^1)^n / G(A).$$

Recall that the $n$-dimensional real Bott manifolds are small covers over $n$-cube. The characteristic function is determined by the Bott matrix. Let $B = [b_{i,j}]$ be the Bott matrix and $F_1, \ldots, F_n, F_{n+1}, \ldots, F_{2n}$ are the facets of $I^n$. Then the corresponding characteristic function is:

$$(4) \quad \chi(F) = \begin{cases} e_i & \text{if } F = F_i \text{ for } 1 \leq i \leq n, \\ e_i + \sum_{k=i+1}^{n} b_{i,k} e_k & \text{if } F = F_{n+i} \text{ for } 1 \leq i \leq n-1, \\ e_n & \text{if } F = F_{2n}. \end{cases}$$

It is easy to see that the matrix of this characteristic function is given by

$$[I_n \mid I_n + B^T],$$

where $I_n$ is the block of $n \times n$ identity matrix.

Now we prove that $K_n$ is indeed a real Bott manifold.

Theorem 2.4. The $n$-dimensional Klein bottle $K_n$ is a real Bott manifold corresponding to the Bott matrix

$$(5) \quad B = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

In particular, $K_n$ is homeomorphic to the small cover $X(I^n, \chi)$, where $\chi$ is defined by the Equation (2).

Proof. By the quotient construction we have

$$M(B) = \frac{(S^1)^n}{G(B)},$$

where $G(B) = < a_1, \ldots, a_n >$ and

$$a_1((z_1, z_2, \ldots, z_n)) = (-z_1, \bar{z}_2, \ldots, \bar{z}_n).$$
\begin{align*}
a_i((z_1, \ldots, z_n)) &= (z_1, \ldots, -z_i, \ldots, z_n),
\end{align*}
for \(2 \leq i \leq n\). Hence,
\begin{align*}
M(B) &\cong \frac{S^1 \times S^{n-1}}{< a_1 > \times < a_2, \ldots, a_n >}.
\end{align*}
Equivalently,
\begin{align*}
M(B) &\cong S^1 \times \mathbb{Z}_2 (\mathbb{R}P^1)^{n-1},
\end{align*}
where the action of \(\mathbb{Z}_2\) is given by \(a_1((z_1, [z_2, \ldots, z_n])) = (-z_1, [\bar{z}_2, \ldots, \bar{z}_n])\). Consequently \(M(B)\) is homeomorphic to \(K_n\).

In case of \(K_n\) the characteristic matrix given by Equation (2) coincides with the characteristic matrix generated by the Bott matrix \(B\). Thus, \(K_n\) is the small cover \(X(I^n, \chi)\).

The mod-2 Betti numbers of small covers were computed by Davis and Januszkiewicz in [5, Theorem 3.1]. In particular the generating function for these Betti numbers is the same as the \(h\) polynomial of the quotient polytope. Ishida [8], gave a formula for rational Betti numbers of real Bott manifolds. Let \(A = [A_j]\) be a Bott matrix and \(M(A)\) be the corresponding real Bott manifold of \(M(A)\).

**Theorem 2.5** ([8, Lemma 2.1]). Let \(\beta_i(M(A), \mathbb{Q})\) be the \(i\)th rational Betti number of \(M(A)\) and \(A_j\) denotes the \(j\)th column of \(A\). Then
\begin{align*}
\beta_i(M(A), \mathbb{Q}) = \# \{ J \subseteq [n] : |J| = i \text{ and } \sum_{j \in J} A_j = 0 \}.
\end{align*}

The following result is an easy application of above theorem.

**Proposition 2.6.** Let \(\beta_i(K_n, \mathbb{Q})\) be the \(i\)th rational Betti number of \(K_n\). Then
\begin{align*}
\beta_i(K_n, \mathbb{Q}) &= \left\{ \begin{array}{ll}
\binom{n-1}{i} & \text{if } i \text{ is an even integer} \\
\binom{n-1}{i-1} & \text{if } i \text{ is an odd integer}.
\end{array} \right.
\end{align*}

**Proof.** Recall that \(K_n \cong M(B)\) where \(B\) is given by Equation (5). For each \(i \in [n]\) consider the following collection of subsets of \([n]\)
\begin{align*}
\mathcal{S}_B(i) &= \{ J \subseteq [n] : |J| = i \text{ and } \sum_{j \in J} B_j = 0 \}.
\end{align*}
Note that, if \(i\) is an even integer then it is easy to see that
\begin{align*}
\mathcal{S}_B(i) &= \{ J \subseteq [n] : |J| = i \text{ and } 1 \notin J \}
\end{align*}
and if \(i\) is an odd integer then
\begin{align*}
\mathcal{S}_B(i) &= \{ J \subseteq [n] : |J| = i - 1 \text{ and } 1 \in J \}.
\end{align*}
The proposition follows by counting the elements of \(\mathcal{S}_B(i)\).

**Remark 2.7.** We make two observations at this point.

1. If \(n\) is an odd integer then \(\beta_n(K_n) = 1\), i.e., \(K_n\) is orientable.

2. For all values of \(n\),
\begin{align*}
\sum_{i=1}^n \beta_i(K_n, \mathbb{Q}) &= 2^{n-1}.
\end{align*}
We now provide an alternative proof of Proposition 2.6 using a formula given by Suciu and Trevisan [9].

Let $P$ be an $n$-dimensional, simple polytope with $m$ facets and let $K$ be the simplicial complex dual of $\partial(P)$. Let $\chi$ be an $n \times m$ characteristic matrix of $P$ with entries from $\mathbb{Z}_2$. For a subset $T \subseteq [n]$, define

$$\chi_T := \sum_{i \in T} \chi_i,$$

where $\chi_i$ is the $i$th row of $\chi$. Let $K_{\chi,T}$ be the induced subcomplex of $K$ on the vertex set

$$\text{supp}(\chi_T) := \{i \in [m] \mid \text{ith entry of } \chi_T \text{ is nonzero} \}.$$

**Theorem 2.8.** [9] Let $\beta_i$ be the $i$th rational Betti number of a small cover $X(P, \chi)$. Then

$$\beta_i = \sum_{T \subseteq [n]} \tilde{\beta}_{i-1}(K_{\chi,T}, \mathbb{Q}),$$

where $\tilde{\beta}_{i-1}(K_{\chi,T}, \mathbb{Q})$ is the $(i-1)$th reduced rational Betti number of $K_{\chi,T}$.

**Lemma 2.9.** Let $\chi$ be the characteristic function for $I^n$ that is used to build the $n$-Klein bottle $K_n$ and let $T \subseteq [n]$. Then

$$|\text{supp}(\chi_T)| = \begin{cases} 2|T| & \text{if } |T| \text{ is an even integer and } 1 \notin T \\ 2|T| - 1 & \text{if } |T| \text{ is an even integer and } 1 \in T \\ 2|T| & \text{if } |T| \text{ is an odd integer and } 1 \in T \\ 2|T| + 1 & \text{if } |T| \text{ is an odd integer and } 1 \notin T. \end{cases}$$

**Proof.** Let $\chi_i$ be the $i$th row of the characteristic matrix of $\chi$. Note that for $2 \leq i \leq n$, $\chi_i$ contains exactly three 1’s and $\chi_1$ contains exactly two 1’s. Moreover, the $i$th and $(n + i)$th column are same for $2 \leq i \leq n$. For a subset $T \subseteq [n]$, $i \in T \setminus \{1\}$, the entry 1 occurs as the $i$th and $(n + i)$th coordinate of vector $\chi_T$.

Suppose $|T|$ is an odd integer and $1 \notin T$. Then the entry 1 occurs in $\chi_T$ at the $(n + 1)$st position. Note that $T \subseteq [n] \setminus \{1\}$. Therefore, for $i \in T$, the entry 1 is placed at $i$th, $n + i$th and $(n + 1)$st positions. In particular 1 occurs $2|T| + 1$ many times in $\chi_T$.

Now assume that $|T|$ is an odd integer and $1 \in T$. Note that for $i \in T \setminus \{1\}$, the entry 1 already occurred at the $i$th and $(n + i)$th position. So $\chi_T$ contains $2(|T| - 1)$ such 1’s. Two more 1’s are added one of them at the 1st and the other at the $(n + 1)$st position. In particular, the entry 1 occurs $2(|T| - 1) + 2 = 2|T|$ many times in $\chi_T$.

Suppose $|T|$ is an even integer with $1 \notin T$. Then for each $i \in T$, the entry 1 will occur at $i$th and $(n + i)$th position but not at the $(n + 1)$st position. In this case the entry 1 occurs in $\chi_T$ exactly $2|T|$ times.

We now assume that $|T|$ is an even integer and $1 \in T$. Then again as observed above we have, for each $i \in T \setminus \{1\}$, 1 occurs at the $i$th and $(n + i)$th position but won’t occurs at the $(n + 1)$th position of $\chi_T$. So there are $2(|T| - 1)$ such 1’s in $\chi_T$. Since $1 \in T$, one more extra 1 gets added in $\chi_T$. Therefore, there are $2(|T| - 1) + 1 = 2|T| - 1$ many 1’s occurs in $\chi_T$. \hfill \Box

Now we determine the homotopy types of the subcomplexes $K_{\chi,T}$ for any $T \subseteq [n]$. 


Lemma 2.10. Let $K_{X,T}$ be the subcomplex of $K$ defined above. Then,

$$K_{X,T} \cong \begin{cases} S^{|T|}-1 & \text{if } |\text{supp}(\chi_T)| \text{ is an even integer}, \\ \{\star\} & \text{if } |\text{supp}(\chi_T)| \text{ is an odd integer}. \end{cases}$$

Proof. Suppose $|\text{supp}(\chi_T)|$ is an even integer. Then it follows from Lemma 2.9 that, either $|T|$ is an even integer and $1 \notin T$ or $|T|$ is an odd integer and $1 \in T$.

Consider the first possibility that $|T|$ is an even integer and $1 \notin T$. Let $K$ be the boundary of cross polytope of dimension $n$. Observe that for each $1 \leq i \leq n$ the vertex $i$ of $K$ is antipodal to another vertex $n + i$. Note that $T \subseteq [n] \setminus \{1\}$. Therefore for each $i \in T$, 1 occurs at the $i$th and $(n + i)\text{th}$ position of vector $\chi_T$. Consequently, $K_{X,T}$ can be obtained from $K$ by removing stars of those antipodal vertices which do not belong to $\text{supp}(\chi_T)$. Therefore, the subcomplex $K_{X,T}$ is the boundary of $|T|$-dimensional cross polytope. This gives us $K_{X,T} \cong S^{|T|}-1$.

Now consider the other possibility that $|T|$ is an odd integer and $1 \in T$. Clearly, 1 occurs at the $1$st and $(n + 1)$th position of $\chi_T$. Recall that the vertices in $\text{supp}(\chi_T) \setminus \{n\}$ are antipodal. Therefore, for each $i \in T$, 1 occurs at the $i$th and $(n + i)$th position of vector $\chi_T$. Then it is clear that $K_{X,T}$ is obtained from $K$ by removing stars of antipodal vertices which do not belong to $\text{supp}(\chi_T)$. Therefore, $K_{X,T}$ is the boundary of $|T|$-dimensional cross polytope. This gives $K_{X,T} \cong S^{|T|}-1$.

Now assume that $|\text{supp}(\chi_T)|$ is an odd integer. Then by Lemma 2.9, either $|T|$ is an even integer and $1 \in T$ or $|T|$ is an odd integer and $1 \notin T$. Consider the first possibility that $|T|$ is an even integer and $1 \in T$. Therefore, 1 occurs at the 1st position but not at the $(n + 1)$th position of $\chi_T$. Since the vertices in $\text{supp}(\chi_T) \setminus \{1\}$ are antipodal, it can be easily checked that

$$K_{X,T} \cong S^{|T|}-1 \setminus \text{star}\{n + 1\}.$$  

Clearly, $K_{X,T} \cong \{\star\}$. Similarly in the second possibility, we get that

$$K_{X,T} \cong S^{|T|}-1 \setminus \text{star}\{1\}.$$  

Therefore, $K_{X,T} \cong \{\star\}$. This proves the lemma in second case.  

Since the Lemma 2.10 already determined the homotopy types of subcomplexes $K_{X,T}$, it is easy to compute the rational Betti numbers of $K_n$ using the Suciu-Trevisan formula.

Theorem 2.11. Let $\beta_i$ be the $i$th rational Betti number of $K_n$. Then

$$\beta_i = \begin{cases} \binom{n-1}{i} & \text{if } i \text{ is an even integer} \\ \binom{n}{i-1} & \text{if } i \text{ is an odd integer}. \end{cases}$$

Proof. It follows from the Lemma 2.9 and Lemma 2.10 that the reduced rational homology of $K_{X,T}$ is

$$\tilde{H}_{i-1}(K_{X,T}, \mathbb{Q}) \cong \mathbb{Q}$$

if and only if

1. $|T| = i$ is an even integer and $1 \notin T$.
2. $|T| = i$ is an odd integer and $1 \in T$.

Now we can use Suciu-Trevisan formula to compute the Betti numbers of $K_n$. If $i$ is an even integer then the corresponding Betti number is number of $i$-element subsets $[n]$ not containing 1 and if $i$ is an odd integer then the corresponding Betti number is the number of $i$-element subsets $[n]$ containing 1. This proves the theorem.  

$\square$
Theorem 2.15. The matroid corresponding to a Bott matrix was shown in first part of Proof. in \( \mathbb{Z}_2 \) where the relations are given as follows: \( \alpha \) a cohomology class \( \subseteq \mathbb{C} \) to \( \ast \mathbb{H} \sums \mathbb{B} \) with \( \deg(\alpha) \). Recall that a closed manifold \( M \) of dimension \( 2n \) is cohomologically symplectic if there exists a cohomology class \( \alpha \in H^*(M) \) such that \( \alpha^n \neq 0 \).

Proposition 2.13. Let \( K_n \) be the \( n \)-dimensional Klein bottle. Then

1. \( K_n \) is orientable if and only if \( n \) is an odd integer,
2. for no value of \( n \geq 1 \) the manifold \( K_n \) is cohomologically symplectic.

Proof. It was shown in first part of [1, Lemma 2.2] that the real Bott manifold \( M(A) \) corresponding to a Bott matrix \( A = [A_i^j] \) is orientable if and only if all row sums of \( A \) are zero in \( \mathbb{Z}_2 \). Recall that the Bott matrix \( B \) associated with \( K_n \) is given by Equation (5). All row sums of \( B \) are zero if and only if \( n \) is an odd integer. This proves the first of the lemma.

The second part of [1, Lemma 2.2] says, \( M(A) \) admits a symplectic form if and only if \( |\{k : A_k = A_j\}| \) is even for every \( 1 \leq j \leq n \). Let \( B_i \) is the \( i \)th column of \( B \). For each \( j \in [n] \), consider the collection

\[
B(j) = \{k \in [n] : B_k = B_j\}.
\]

Note that \( |B(1)| = 1 \). Therefore, \( K_n \) never admits a symplectic form. \( \square \)

Remark 2.14. The first part of the above lemma also follows from [4, Proposition 3.1].

Let \( M(A) \) be the real Bott manifold corresponding to a Bott matrix \( A \). The rational cohomology ring \( H^*(M(A), \mathbb{Q}) \) was computed by Choi and Park [2]. They showed that \( H^*(M(A), \mathbb{Q}) \) is completely determined by the binary matroid of \( A \). We refer the reader to [2, Section 4] for more details.

Let \( A \) be a Bott matrix and \( E = \{A_j : 1 \leq j \leq n\} \) be the set of its columns. A subset \( C \subseteq E \) is said to be minimally dependent if every proper subset of \( C \) is linearly independent. Consider the collection

\[
C = \{C : C \subseteq E \text{ is minimally dependent}\}.
\]

The matroid \( T(A) = (E, C) \) is called a binary matroid associated with \( A \) and the elements \( C \in C \) are called circuits.

Theorem 2.15 ([2, Proposition 4.3]). Let \( x_C \) be the formal symbol for the cohomology class corresponding to a circuit \( C \). Then

\[
H^*(M(A), \mathbb{Q}) \cong \frac{\mathbb{Q} < x_C : C \in C >}{\sim},
\]

where the relations are given as follows:

\[
x_{C \cap C'} = \begin{cases} (-1)^{|C||C'|} x_C x_{C'} & \text{if } C \cap C' = \emptyset \\ 0 & \text{if } C \cap C' \neq \emptyset \end{cases},
\]

with \( \text{deg}(x_C) = |C| \).

The binary matroid corresponding to the Bott matrix of \( K_n \) is

\[
C = \{\{1\}, \{i, j\} : 2 \leq i < j \leq n\}.
\]
Let $Y$ be the formal symbol of degree 1 cohomology class corresponding to the singleton set \{1\} and for each \{i, j\} $\in C$, let $X_{ij}$ be the formal symbol of degree 2 cohomology class. Then we have

$$H^*(K_n, \mathbb{Q}) \cong \mathbb{Q}[Y, X_{ij} : 2 \leq i < j \leq n],$$

where the following relations hold for $2 \leq i < j \leq n$ and $2 \leq k < l \leq n$.

1. $Y^2 = X_{ij}^2 = 0$,
2. $YX_{ij} = X_{ij}Y$,
3. $X_{ij}X_{kl} = X_{kl}X_{ij}$ if \{i, j\} $\cap \{k, l\} = \emptyset$,
4. $X_{ij}X_{kl} = 0$ if \{i, j\} $\cap \{k, l\} \neq \emptyset$.

3. \textbf{The case of other two long genetic codes}

In this section we define certain characteristic functions on the facets of $P_5 \times I^{n-2}$ and $P_6 \times I^{n-2}$ where $P_n$ an $n$-gon. We also show that the corresponding small covers $X(P_5 \times I^{n-2})$ and $X(P_6 \times I^{n-2})$ are homeomorphic to the planar polygon spaces associated with the genetic codes \{\{1, 2, \ldots, n-2, n, n+3\}\} and \{\{1, 2, \ldots, n-2, n+1, n+3\}\} respectively.

3.1. \textbf{Betti numbers of} $X(P_5 \times I^{n-2}, \chi)$. We refer reader to [7] for the following definition and remark.

\textbf{Definition 3.1.} Let $P$ and $P'$ are two convex polytopes of dimension $d$ and $d'$, both containing the origin. Then their direct sum is a $(d + d'$)-dimensional polytope

$$P \oplus P' = \text{conv}\{(p, 0) \in \mathbb{R}^{d+d'} : p \in P\} \cup \{(0, p') \in \mathbb{R}^{d+d'} : p' \in P'\}.$$

\textbf{Remark 3.2.} Let $P^\Delta$ and $P'^\Delta$ be the dual polytopes of $P$ and $P'$, respectively, containing the origin. Then their direct sum and product is related by the following equation:

$$P \times P' = (P^\Delta \oplus P'^\Delta)^\Delta.$$

In particular, if $P_m$ is the $m$-gon then

$$(P_m \times I^{n-2})^\Delta = P_m \oplus (I^{n-2})^\Delta.$$

To construct the characteristic function over $P_5 \times I^{n-2}$, we give a specific labeling for the facets of $P_5 \times I^{n-2}$ as follows:

- For each $1 \leq i \leq n - 2$,
  $$F_i = P_5 \times I \times \cdots \times \{-1\} \times \cdots \times I,$$
  where \{-1\} is at the $i$th place.
- For each $1 \leq i \leq n - 2$,
  $$F_{n+i} = P_5 \times I \times \cdots \times \{1\} \times \cdots \times I,$$
  where \{1\} is at the $i$th place.
- For $1 \leq i \leq 5$, let $E_i$ is the $i$th side of $P_5$. We set
  $$F_{n-1} = E_1 \times I^{n-2}, F_n = E_2 \times I^{n-2}, F_{2n-1} = E_3 \times I^{n-2},$$
  $$F_{2n} = E_4 \times I^{n-2}, F_{2n+1} = E_5 \times I^{n-2}.$$
Let $\mathcal{F}(P_5 \times I^{n-2})$ be the collection of facets of $P_5 \times I^{n-2}$. We define a function $\chi : \mathcal{F}(P_5 \times I^{n-2}) \to \mathbb{Z}_2^n$ by

$$
\chi(F) = \begin{cases}
  e_i & \text{if } F = F_i \text{ and } F = F_{n+i}, 1 \leq i \leq n \\
  \sum_{i=1}^{n} e_i & \text{if } F = E_5 \times I^{n-2}.
\end{cases}
$$

(6)

**Lemma 3.3.** The function $\chi$ is a characteristic function for $P_5 \times I^{n-2}$.

**Proof.** Observe that

$$
\chi(\mathcal{F}(v)) = \begin{cases}
  \{e_1, \ldots, e_{n-1}, \sum_{i=1}^{n} e_i\} & \text{if } v \in F_{2n+1}, \\
  \{e_1, \ldots, e_n\} & \text{otherwise}.
\end{cases}
$$

Therefore, for any vertex, $\chi(\mathcal{F}(v))$ forms a basis of $\mathbb{Z}_2^n$. Consequently, $\chi$ is the characteristic function.

It is clear that the $n \times (2n+1)$-matrix of $\chi$ is

$$
\chi = \begin{bmatrix}
  1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 \\
  0 & 1 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1 & 1
\end{bmatrix}.
$$

**Theorem 3.4.** We have the following homeomorphism

$$
X(P_5 \times I^{n-2}, \chi) \cong (S^1)^{n-2} \times \Sigma_3 \sim (z_1, \ldots, z_{n-2}, z) \sim (\bar{z}_1, \ldots, \bar{z}_{n-2}, -z),
$$

where $\Sigma_3$ is the orientable surface of genus 3.

**Proof.** Follows from [3, Theorem 4.14, Theorem 5.2] and [6, Proposition 2.1].

**Lemma 3.5.** For any subset $T$ we have:

$$
|\text{supp}(\chi_T)| = \begin{cases}
  2|T| & \text{if } |T| \text{ is an even integer} \\
  2|T| + 1 & \text{if } |T| \text{ is an odd integer}.
\end{cases}
$$

**Proof.** Observe that, each row of the characteristic matrix contains three 1’s and for each $1 \leq i \leq n$, the $i$th and $(n + i)$th column coincides.

It is easy to see that, for each $i \in T$ with $1 \leq i \leq n$, 1 occurs at the $i$th and $(n + i)$th position of vector $\chi_T$. Moreover, if $|T|$ is an odd integer then 1 occurs in $\chi_T$ at the $(2n+1)$th position as well. In particular, 1 occurs $2|T| + 1$ many times in $\chi_S$.

Suppose $|T|$ is an even integer. Then 1 will always occur at $i$th and $(n + i)$th position of $\chi_T$ but won’t occur at the $(2n+1)$th position. Therefore, in this case 1 occurs in $\chi_T$ exactly $2|T|$ times.

**Lemma 3.6.** With the notation as before:

$$
K_{\chi,T} \cong \begin{cases}
  S^{|T|-1} & \text{if } \{n - 1, n\} \subseteq T \text{ and } |T| \text{ is an odd integer} \\
  \{\ast\} & \text{if } \{n - 1, n\} \subseteq T \text{ and } |T| \text{ is an even integer}.
\end{cases}
$$
Proof. Suppose \( \{n - 1, n\} \subseteq T \) and \( |T| \) is an odd integer. Therefore,
\[
\{n - 1, n, 2n - 1, 2n, 2n + 1\} \subseteq \text{supp}(\chi_T).
\]
Clearly, \( P_5 \subseteq K_{\chi,T} \). Therefore, the antipodal vertices which does not belongs to
\[
([n - 2] \cup \{n + i : i \in [n - 2]\}) \cap \text{supp}(\chi_T)
\]
are removed from \( K_{\chi,T} \). Since we have \( K \cong \partial(P_5 \oplus (I^{n-2})\triangle), K_{\chi,T} \cong \partial(P_5 \oplus (I^{|T|}-2)\triangle) \).
Consequently, \( K_{\chi,T} \cong S^{|T|-1} \).

If \( |T| \) is an even integer then clearly \( 2n + 1 \notin \text{supp}(\chi_T) \). Therefore, \( K_{\chi,T} \cong S^{|T|-1} \setminus \text{star}\{2n+1\}. \)
Clearly, \( K_{\chi,T} \cong \{\ast\}. \)

Lemma 3.7. A closed form formula for the homotopy type in the remaining case is given as follows.
\[
K_{\chi,T} \cong \begin{cases} 
S^{|T|-1} & \text{if } \{n - 1, n\} \notin T \text{ and } |T| \text{ is an even integer} \\
\{\ast\} & \text{if } \{n - 1, n\} \notin T \text{ and } |T| \text{ is an odd integer.}
\end{cases}
\]

Proof. Suppose \( \{n - 1, n\} \notin T \) and \( |T| \) is an even integer. Therefore,
\[
\{n - 1, n, 2n - 1, 2n, 2n + 1\} \notin \text{supp}(\chi_T).
\]
In particular \( P_5 \notin K_{\chi,T} \). It follows from Remark 3.2 that \( K \cong \partial(P_5 \oplus (I^{n-2})\triangle) \). Therefore, it is easy to see that \( K_{\chi,T} \cong \partial((I^{|T|})\triangle) \). Now suppose \( \{n - 1, n\} \notin T \) and \( |T| \) is an odd integer. This gives
\[
\{n - 1, n, 2n - 1, 2n\} \notin \text{supp}(\chi_T)
\]
and \( 2n + 1 \in \text{supp}(\chi_T) \). Note that the vertex \( 2n + 1 \) in \( K \) is adjacent to all the vertices in \([n - 2] \cup \{n + i : i \in [n - 2]\}\).
Therefore, in \( K_{\chi,T} \) the vertex \( 2n + 1 \) is adjacent to
\[
([n - 2] \cup \{n + i : i \in [n - 2]\}) \cap \text{supp}(\chi_T).
\]
This gives \( K_{\chi,T} \) is isomorphic to the cone over \( S^{|T|-1} \) with the apex vertex \( 2n + 1 \) since
\[
K_{\chi,T} \setminus \{2n + 1\} \cong S^{|T|-1}.
\]
This proves the lemma. \( \square \)

Lemma 3.8. If
\[
(1) \text{ either } n - 1 \notin T \text{ and } n \in T, \\
(2) \text{ or } n - 1 \in T \text{ and } n \notin T
\]
then \( K_{\chi,T} \cong S^{|T|-1} \).

Proof. Suppose that \( n - 1 \notin T \) and \( n \in T \) with \( |T| \) is an even integer. This gives
\[
\{n - 1, 2n - 1, 2n + 1\} \notin \text{supp}(\chi_T) \text{ and } \{n, 2n\} \subseteq \text{supp}(\chi_T).
\]
Therefore, \( \text{supp}(\chi_T) \) contains two antipodal vertices from \( P_5 \) and \( 2(|T| - 1) \) vertices from \((I^{n-2})\triangle). \) It is easy to see that \( K_{\chi,T} \cong \partial(I \oplus (I^{|T|-1})\triangle). \) Clearly, \( K_{\chi,T} \cong S^{|T|-1} \). Now suppose \( |T| \) is an odd integer. Then we clearly have
\[
\{n - 1, 2n - 1\} \notin \text{supp}(\chi_T) \text{ and } \{n, 2n, 2n + 1\} \subseteq \text{supp}(\chi_T).
\]
Since \( \{2n, 2n + 1\} \) are adjacent vertices and \( n \) is antipodal to \( 2n \), we can collapse an edge \( \{2n, 2n + 1\} \) to \( 2n \). In particular, we have \( P_5 \cap K_{\chi,T} \cong S^0 \). Therefore, again we have \( K_{\chi,T} \cong \partial(I \oplus (I^{|T|-1})\triangle) \). This proves the lemma in the context of first case. Similar arguments can be used to prove the lemma in second case. \( \square \)
Theorem 3.9. Let $\beta_i$ be the $i$th rational Betti number of $X(P_5 \times I^{n-2}, \chi)$. Then

$$
\beta_i = \begin{cases} 
2 \binom{n-2}{i-1} + \binom{n-2}{i} & \text{if } i \text{ is even and} \\
2 \binom{n-2}{i-1} + \binom{n-2}{i-2} & \text{if } i \text{ is an odd integer}
\end{cases}
$$

Proof. Using Lemma 3.6, Lemma 3.7 and Lemma 3.8 we have $\tilde{H}_{i-1}(K_{\chi,T}, \mathbb{Q}) \cong \mathbb{Q}$ if the following conditions holds:

1. If $|T| = i$ is an odd integer with $\{n-1,n\} \subseteq T$.
2. If $|T| = i$ is an even integer with $\{n-1,n\} \nsubseteq T$.
3. If $|T| = i$ with $n-1 \notin T$ and $n \in T$.
4. If $|T| = i$ with $n-1 \in T$ and $n \notin T$.

We use the Suciu–Trevisan formula to compute the Betti numbers. If $i$ is an even integer then the corresponding rational Betti number is the sum of the cardinalities of $i$-element subsets of $[n]$, type (2), (3) and (4). Similarly, if $i$ is an odd integer then the corresponding Betti number is the sum of the cardinalities of $i$-element subsets of $[n]$, type (1), (3) and (4).

\[ \square \]

Remark 3.10. Note that

$$
\sum_{i=1}^{n} \beta_i(K_n, \mathbb{Q}) = 3 \cdot 2^{n-2}.
$$

Example 3.11. The following table contains first five Betti numbers of $X(P_5 \times I^{n-2}, \chi))$ up to dimension 5.

| $n$ | 0   | 1 | 2 | 3 | 4 | 5 |
|-----|-----|---|---|---|---|---|
| 2   | 1   | 2 | 0 | 0 | 0 | 0 |
| 3   | 1   | 2 | 2 | 1 | 0 | 0 |
| 4   | 1   | 2 | 5 | 4 | 0 | 0 |
| 5   | 1   | 2 | 9 | 9 | 2 | 1 |

Table 1. $\beta_i(X(P_5 \times I^{n-2}, \chi))$.

3.2. Betti numbers of $X(P_6 \times I^{n-2}, \chi)$. To construct the characteristic function over $P_6 \times I^{n-2}$, we give a specific labeling for its facets:

- For each $1 \leq i \leq n-2$, we set
  $$F_i = P_6 \times I \times \cdots \times \{i\} \times \cdots \times I$$
  where $\{i\}$ is at the $i$th place.
- For each $1 \leq i \leq n-2$, we set
  $$F_{n+1+i} = P_6 \times I \times \cdots \times \{i\} \times \cdots \times I$$
  where $\{i\}$ is at the $i$th place.
- For $1 \leq i \leq 6$, let $E_i$ is the $i$th side of $P_6$. Then we set
  $$F_{n-1} = E_1 \times I^{n-2}, \quad F_n = E_2 \times I^{n-2}, \quad F_{2n-1} = E_3 \times I^{n-2},$$
  $$F_{2n} = E_4 \times I^{n-2}, \quad F_{2n+1} = E_5 \times I^{n-2}, \quad F_{2n+1} = E_6 \times I^{n-2}.$$
Let $\mathcal{F}(P_6 \times I^{n-2})$ be the collection of facets of $P_6 \times I^{n-2}$. We define a function
\[
\chi : \mathcal{F}(P_6 \times I^{n-2}) \rightarrow \mathbb{Z}_2^n
\]
as
\[
\chi(F) = \begin{cases}
e_i & \text{if } F = F_i \text{ and } F = F_{n+1+i}, 1 \leq i \leq n \\
\sum_{i=1}^{n} e_i & \text{if } F = F_{n+1} \text{ and } F = F_{2n+2}.
\end{cases}
\]

**Lemma 3.12.** The function $\chi$ is a characteristic function for $P_6 \times I^{n-2}$.

**Proof.** Note that
\[
\chi(\mathcal{F}(v)) = \begin{cases}
\{e_1, \ldots, e_{n-1}, \sum_{i=1}^{n} e_i\} & \text{if either } v \in F_{n+1} \text{ or } v \in F_{2n+2}, \\
\{e_1, \ldots, e_n\} & \text{otherwise}.
\end{cases}
\]

It is clear that the $(n \times 2n)$-matrix of $\chi$ is
\[
\chi = \begin{bmatrix}
1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 \\
0 & 1 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 1 & 0 & \cdots & 1 & 1
\end{bmatrix}.
\]

**Theorem 3.13.** There is the following homeomorphism
\[
X(P_6 \times I^{n-2}, \chi) \cong (S^1)^{n-2} \times \Sigma_4 \cong (\bar{z}_1, \ldots, \bar{z}_{n-2}, z) \sim (\bar{z}_1, \ldots, \bar{z}_{n-2}, -z),
\]
where $\Sigma_4$ is the orientable surface of genus 4.

**Proof.** The proof of this theorem is the same as that of Theorem 3.4. We just have to replace the genetic code by $\langle \{1, 2, \ldots, n-1, n+2, n+3\} \rangle$. \[\square\]

**Lemma 3.14.** With the notation as before we have
\[
|\text{supp}(\chi_T)| = \begin{cases}
2|T|, & \text{if } |T| \text{ is an even integer}, \\
2|T| + 2, & \text{if } |T| \text{ is an odd integer}.
\end{cases}
\]

**Proof.** Observe that, each row of the characteristic matrix contains four 1’s and for each $1 \leq i \leq n+1$, the $i$th and $(n+1+i)$th columns coincide.

For each $i \in T$ with $1 \leq i \leq n$, the entry 1 occurs at the $i$th and $(n+1+i)$th position. Moreover, if $|T|$ is an odd integer then the entry 1 occurs in $\chi_T$ at the $(n+1)$th and $(2n+2)$nd position. In particular, 1 occurs 2$|T|$ + 2 many times in $\chi_S$.

Suppose $|T|$ is an even integer. Then the entry 1 will always occur at $i$th and $(n+1+i)$th position but it won’t occur at the $(n+1)$st and $(2n+2)$nd position. Therefore, in this case the entry 1 occurs exactly 2$|T|$ times. \[\square\]

Now we determine the homotopy type of the subcomplexes. As before the computations are in two parts depending on the conditions on $T$.

**Lemma 3.15.**
\[
K_{\chi,T} \cong \begin{cases}
S^{|T|-1} & \text{if } \{n-1, n\} \subseteq T \text{ and } |T| \text{ is an odd integer} \\
S^{|T|-2} & \text{if } \{n-1, n\} \subseteq T \text{ and } |T| \text{ is an even integer}.
\end{cases}
\]
Proof. Suppose \( \{n - 1, n\} \subseteq T \) and \( |T| \) is an odd integer. Note that for each \( i \in T \) with \( 1 \leq i \leq n \), \( 1 \) occurs at the \( i \)th and \( (n + 1 + i) \)th position of \( \chi_T \). Since \( |T| \) is an odd integer, \( 1 \) occurs at \( (n + 1) \)st and \( (2n + 2) \)nd position of \( \chi_T \) as well. Therefore, 
\[
\{n - 1, n, n + 1, 2n, 2n + 1, 2n + 2\} \subseteq \text{supp}(\chi_T).
\]

Since the above set forms a vertex set of \( P_6 \), we have \( P_6 \subseteq K_{\chi,T} \). The remaining vertices of \( K_{\chi,T} \) are given by 
\[
\{i : i \in T\} \cup \{n + 1 + i : i \in T\}.
\]

Note that \( K \cong P_6 \oplus (I^{n-2})^\Delta \). Observe that the above vertices are from the \( (I^{n-2})^\Delta \) factor of \( K \). Therefore, \( K_{\chi,T} \cong \partial(P_6 \oplus \bigoplus_{i \in T \cap [n-2]} I_i) \), where \( I_i = I \). Now it is clear that 
\[
K_{\chi,T} \cong \partial(P_6 \oplus (I^{|T|-2})^\Delta) \cong S^{|T|-1}.
\]

Now assume that \( \{n - 1, n\} \not\subseteq T \) and \( |T| \) is an even integer. Therefore, \( 1 \) does not occur at the \( (n + 1) \)st and \( (2n + 2) \)nd position of \( \chi_T \). This gives 
\[
\{n + 1, 2n + 2\} \not\subseteq \text{supp}(\chi_T) \quad \text{and} \quad \{n - 1, n, 2n, 2n + 1\} \not\subseteq \text{supp}(\chi_T)
\]

since \( \{n - 1, n\} \not\subseteq T \). Clearly, we have \( P_6 \cap K_{\chi,T} \cong S^0 \). Now it is easy to see that \( K_{\chi,T} \cong \partial(I \oplus \bigoplus_{i \in T} I_i) \), where \( I_i = I \) for all \( i \). Therefore, \( K_{\chi,T} \cong \partial((I^{|T|-1})^\Delta) \cong S^{|T|-2} \). This proves the lemma. \( \square \)

**Lemma 3.16.**
\[
K_{\chi,T} \cong \begin{cases} S^{|T|} & \text{if } \{n - 1, n\} \not\subseteq T \text{ and } |T| \text{ is an odd integer,} \\ S^{|T|-1} & \text{if } \{n - 1, n\} \not\subseteq T \text{ and } |T| \text{ is an even integer.} \end{cases}
\]

**Proof.** Suppose \( \{n - 1, n\} \not\subseteq T \) and \( |T| \) is an even integer. Therefore, \( 1 \) does not occur at the \( (n + 1) \)st and \( (2n + 2) \)nd position of \( \chi_T \). This gives 
\[
\{n + 1, 2n + 2\} \not\subseteq \text{supp}(\chi_T) \quad \text{and} \quad \{n - 1, n, 2n, 2n + 1\} \not\subseteq \text{supp}(\chi_T).
\]

Therefore, \( P_6 \not\subseteq K_{\chi,T} \). Since \( T \subseteq [n - 2] \), \( K_{\chi,T} \cong \partial(\bigoplus_{i \in T} I_i) \), where \( I_i = I \) for all \( i \). Therefore, \( K_{\chi,T} \cong \partial((I^{|T|})^\Delta) \cong S^{|T|-1} \).

Now assume that \( \{n - 1, n\} \not\subseteq T \) and \( |T| \) is an odd integer. Therefore, \( 1 \) occurs at the \( (n + 1) \)st and \( (2n + 2) \)nd position of \( \chi_T \). Therefore, 
\[
\{n + 1, 2n + 2\} \subseteq \text{supp}(\chi_T) \quad \text{and} \quad \{n - 1, n, 2n, 2n + 1\} \not\subseteq \text{supp}(\chi_T).
\]

Since \( T \subseteq [n - 2] \), \( K_{\chi,T} \cong \partial(I \oplus \bigoplus_{i \in T} I_i) \), where \( I_i = I \) for all \( i \). Note that the first factor in the previous direct sum is corresponding to \( \{n + 1, 2n + 2\} \). Therefore, 
\[
K_{\chi,T} \cong \partial((I^{|T|} + 1)^\Delta) \cong S^{|T|}.
\]

This proves the lemma. \( \square \)

**Lemma 3.17.** If either

1. \( n - 1 \notin T \) and \( n \in T \) or,
2. \( n - 1 \in T \) and \( n \notin T \)

then \( K_{\chi,T} \cong S^{|T|-1} \).

**Proof.** Suppose \( n - 1 \notin T \) and \( n \in T \) with \( |T| \) is an odd integer. Therefore, the entry \( 1 \) occurs at the \( n \)th, \( (2n + 1) \)st, \( (n + 1) \)st and \( (2n + 2) \)nd position of \( \chi_T \) but it doesn’t occur at the \( (n - 1) \)st and the \( (2n) \)th position. This clearly gives 
\[
\{n, n + 1, 2n + 1, 2n + 2\} \subseteq \text{supp}(\chi_T) \quad \text{and} \quad \{n - 1, 2n\} \not\subseteq \text{supp}(\chi_T).
\]
Since $T \setminus \{n\} \subseteq [n - 2]$, $K_{X,T} \cong \partial(I \oplus [I_{i=1}^{[T]-1} I_i])$, where $I_i = I$ for all $i$. Note that the first factor in the above direct sum is corresponding to $\{n, 2n + 1\}$. Therefore,

$$K_{X,T} \cong \partial((I^{[T]}\triangle)) \cong S^{[T]-1}.$$  

Now suppose $n - 1 \notin T$ and $n \in T$ with $|T|$ is an even integer. Therefore, 1 does not occur at the $n - 1$th, $(2n)$th, $(n + 1)$th, $(2n + 2)$th position of vector $\chi_T$ but occurs at the $n$th and $(2n + 1)$th position. In particular, we have

$$\{n - 1, 2n, n + 1, 2n + 2\} \not\subseteq \text{supp}(\chi_T) \text{ and } \{n, 2n + 1\} \subseteq \text{supp}(\chi_T).$$

Since $T \setminus \{n\} \subseteq [n - 2]$, $K_{X,T} \cong \partial(I \oplus [I_{i=1}^{[T]-1} I_i])$, where $I_i = I$ for all $i$. Note that the first factor in the above direct sum is corresponding to $\{n, 2n + 1\}$. Therefore,

$$K_{X,T} \cong \partial((I^{[T]}\triangle)) \cong S^{[T]-1}.$$  

This proves the lemma in the first case. Similar steps can be followed to prove the second case.

\textbf{Theorem 3.18.} Let $\beta_i$ be the $i$th rational Betti number of $X(P_6 \times I^{n-2}, \chi)$. Then

$$\beta_i = \begin{cases} 3\left(\binom{n-2}{i-1}\right) + \left(\binom{n-2}{i}\right) & \text{if } i \text{ is an even integer}, \\ 3\left(\binom{n-2}{i-1}\right) + \left(\binom{n-2}{i+2}\right) & \text{if } i \text{ is an odd integer}. \end{cases}$$

\textbf{Proof.} Let $i$ be an odd integer. Then from Lemma 3.15, Lemma 3.16 and Lemma 3.17 we have $\hat{H}_{i-1}(K_{X,T}, \mathbb{Q}) \cong \mathbb{Q}$ if the following conditions holds:

(1) If $|T| = i$ with $\{n - 1, n\} \subseteq T$.
(2) If $|T| = i + 1$ with $\{n - 1, n\} \subseteq T$.
(3) If $|T| = i$ with $n - 1 \notin T$ and $n \in T$.
(4) If $|T| = i$ with $n - 1 \in T$ and $n \notin T$.

Note that the cardinality of type (1) sets is $\binom{n-2}{i-2}$ and the cardinalities of type (2), type (3) and type (4) sets are same and it is equal to $\binom{n-2}{i-1}$ in each case. Now theorem follows by adding these cardinalities.

Now suppose $i$ is an even integer. Then again we can use Lemma 3.15, Lemma 3.16 and Lemma 3.17 to get the $(i-1)$th reduced rational homology of $K_{X,T}$. We have $\hat{H}_{i-1}(K_{X,T}, \mathbb{Q}) \cong \mathbb{Q}$ if the following conditions holds:

(1) If $|T| = i$ with $\{n - 1, n\} \not\subseteq T$.
(2) If $|T| = i - 1$ with $\{n - 1, n\} \not\subseteq T$.
(3) If $|T| = i$ with $n - 1 \notin T$ and $n \in T$.
(4) If $|T| = i$ with $n - 1 \in T$ and $n \notin T$.

Note that the cardinality of type (1) sets is $\binom{n-2}{i}$ and the cardinalities of type (2), type (3), type (4) sets are same and it is equal to $\binom{n-2}{i-1}$ in each case. This proves the theorem.

\textbf{Remark 3.19.} Note that

$$\sum_{i=1}^{n} \beta_i(K_n, \mathbb{Q}) = 4 \cdot 2^{n-2}.$$  

\textbf{Example 3.20.} The following table contains first five Betti numbers of $X(P_6 \times I^{n-2}, \chi)$ up to the dimension 5.
Call the total space of an iterated $S^1$-bundle starting with a closed (non-orientable) surfaces as a Bott-type manifold. It is easy to see that these manifolds are small covers over $P_i \times I^{n-2}$ where $P_i$ is an $i$-gon for an appropriate $i > 2$. One could explore the topological and combinatorial aspects of Bott-type manifolds. More precisely, one can ask:

Given $i > 2$, how many Bott-type manifolds exist, up to diffeomorphism, over $P_i \times I^{n-2}$? and How to characterize Bott-type manifolds, up to diffeomorphism, in terms of some combinatorial data associated with the fibration?

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