Modeling 1/f noise

B. Kaulakys$^{1,2}$ and T. Meškauskas$^{1,3}$

$^1$Institute of Theoretical Physics and Astronomy, A. Goštauto 12, 2600 Vilnius, Lithuania
$^2$Department of Physics, Vilnius University, Saulėtekio al. 9, 2040 Vilnius, Lithuania
$^3$Department of Mathematics, Vilnius University, Naugarduko 24, 2006 Vilnius, Lithuania

(Received 29 June 1998)

Abstract

Physical Review E Vol. 58, No. 6, p. 7013-19

The noise of signals or currents consisting from a sequence of pulses, elementary events or moving discrete objects (particles) is analyzed. A simple analytically solvable model is investigated in detail both analytically and numerically. It is shown that 1/f noise may result from the statistics of the pulses transit times with random increments of the time intervals between the pulses. The model also serves as a basis for revealing parameter dependences of 1/f noise and allows one to make some generalizations. As a result the intensity of 1/f noise is expressed through the distribution and characteristic functions of the time intervals between the subsequent transit times of the pulses. The conclusion that 1/f noise may result from the clustering of the signal pulses, elementary events or particles can be drawn from the analysis of the model systems.

PACS number(s): 05.40.+j, 02.50.-r, 72.70.+m
I. INTRODUCTION

The omnipresence of 1/f noise is one of the oldest puzzles of contemporary physics. During more than 70 years since the first observation by Johnson, long-memory processes with long-term correlations have been observed in many types of systems from physics, technology, biology, astrophysics, geophysics and sociology (see [1–4] and references herein). Recently 1/f noise was discovered in human cognition [5], human coordination [6] and even in distribution of prime numbers [7].

Despite the widespread occurrence of fluctuations of signals and variables exhibiting $1/f^\delta$ ($\delta \approx 1$) behavior of the power spectral density $S(f)$ at low frequencies in large diversity of systems no generally recognized explanation of the ubiquity of 1/f noise is still proposed. Physical models of 1/f noise in some physical systems are usually very specialized, complicated (see [1–4] and references herein) and they do not explain the omnipresence of the processes with $1/f^\delta$ spectrum [8–10]. Note also some mathematical analysis [11], models and algorithms of generation of the processes with 1/f noise [12–14]. These models also expose some shortcomings: they are very specific, formal (like "fractional Brownian motion" or half-integral of a white noise signal) or unphysical. They can not, as a rule, be solved analytically and they do not reveal the origin as well as the necessary and sufficient conditions for the appearance of 1/f type fluctuations.

In such a situation the simple analytically solvable model system generating 1/f noise may essentially influence in the reveal of the origin and essence of the effect. Here we present a model which generates 1/f noise in any desirably wide range of frequency. Our model is a result of the search of necessary and sufficient conditions for the appearance of 1/f fluctuations in simple systems affected by the random external perturbations initiated in [15] and originated from the observation of the transition from chaotic to nonchaotic behavior in the ensemble of randomly driven systems [16]. Contrary to the McWhorter model [17] based on the superposition of large number of Lorentzian spectra and requiring a very wide distribution of relaxation times, our model contains only one relaxation rate $\gamma$ and can have an exact 1/f spectrum in any desirably wide range of frequency. The model may be used as a basis for the checking of assumptions made in the derivation of 1/f noise spectrum for different systems. Furthermore, it allows us to make a heuristic presumption for the generalizations of the theory of 1/f noise. Numerical simulations and comparisons with analytical results confirm this supposition.

II. THE MODEL

In many cases, the intensity of some signal or current can be represented by a sequence of random (however, as a rule, mutually correlated) pulses or elementary events $A_k (t - t_k)$. Here the function $A_k (\varphi)$ represents the shape of the $k$ pulse having an influence to the signal $I(t)$ in the region of transit time $t_k$. The signal or intensity of the current of particles in some space cross-section may, therefore, be expressed as

$$I(t) = \sum_k A_k (t - t_k). \quad (1)$$

It is easy to show that the shapes of the pulses influent mainly on the high frequency, $f \geq \Delta t_p$ with $\Delta t_p$ being the characteristic pulse length, power spectral density while fluctuations of
the pulse amplitudes result, as a rule, in the white or Lorentzian but not 1/f noise [18]. Therefore, we restrict our analysis to the noise due to the correlations between the transit times \( t_k \). In such an approach we can replace the function \( A_k(t-t_k) \) by the Dirac delta function \( \delta(t-t_k) \) and the signal express as

\[
I(t) = \sum_k \delta(t-t_k).
\]

(2)

This model also corresponds to the flow of identical objects: electrons, photons, cars and so on. On the other hand, fluctuations of the amplitudes \( A_k \) may result in the additional noise but cannot reduce 1/f noise we are looking for.

The power spectral density of the current (2) is

\[
S(f) = \lim_{T \to \infty} \left\langle \frac{2}{T} \left| \sum_{k=k_{\min}}^{k_{\max}} e^{-i2\pi f t_k} \right|^2 \right\rangle = \lim_{T \to \infty} \left\langle \frac{2}{T} \sum_k \sum_{q=k_{\min}}^{k_{\max}-k} e^{i2\pi f (t_k+q-t_k)} \right\rangle
\]

(3)

where \( T \) is the whole observation time interval, \( k_{\min} \) and \( k_{\max} \) are minimal and maximal values of index \( k \) in the interval of observation and the brackets \( \langle ... \rangle \) denote the averaging over realizations of the process.

In this approach the power spectral density of the signal depends on the statistics and correlations of the transit times \( t_k \) only. It is well known that sequence of random, Poisson, transit times generates white (shot) noise [18]. The sequence of transit times \( t_k \) with random increments, \( t_k = t_{k-1} + \bar{\tau} + \sigma \varepsilon_k \) (where \( \bar{\tau} \) is the average time interval between pulses, \( \{\varepsilon_k\} \) denotes the sequence of uncorrelated normally distributed random variables with zero expectation and unit variance, i.e. the white noise source, and \( \sigma \) is the standard deviation of white noise) results in the Lorentzian spectra [15]. Here we will consider sequences of the transit times with random increments of the time intervals between pulses, \( \tau_k = \tau_{k-1} + \sigma \varepsilon_k \), where \( \tau_k = t_k - t_{k-1} \). It is natural to restrict in some way the infinite Brownian increase or decrease of the intervals \( \tau_k \), e.g. by the introduction of the relaxation to the average period \( \bar{\tau} \) rate \( \gamma \). So, we have the recurrent equations for the transit times

\[
\begin{align*}
    t_k &= t_{k-1} + \tau_k, \\
    \tau_k &= \tau_{k-1} - \gamma (\tau_{k-1} - \bar{\tau}) + \sigma \varepsilon_k.
\end{align*}
\]

(4)

The simplest physical interpretation of the model (4) corresponds to one particle moving in the closed contour with the period of the drift of the particle round the contour fluctuating (due to the external random perturbations) about the average value \( \bar{\tau} \) [19].

III. SOLUTIONS

An advantage of the model (4) is that it may be solved analytically. So, an iterative solution of Eqs. (4) results in an expression for the period

\[
\tau_k = \bar{\tau} + (\tau_0 - \bar{\tau})(1-\gamma)^k + \sigma \sum_{j=1}^{k} (1-\gamma)^{k-j} \varepsilon_j
\]

(5)
where \( \tau_0 \) is the initial period. The dispersion of the period \( \tau_k \) is

\[
\sigma^2_\tau (k) \equiv \langle \tau^2_k \rangle - \langle \tau_k \rangle^2 = \frac{\sigma^2}{2\gamma} \frac{[1 - (1 - \gamma)^{2k}]}{(1 - \gamma/2)} \approx \begin{cases} \sigma^2 k, & 2k \gamma \ll 1 \\ \sigma^2/2\gamma, & 2k \gamma \gg 1. \end{cases}
\] (6)

Therefore, after the characteristic transition to the stationary process time, \( t_{tr} = \bar{\tau}/\gamma \), the dispersion of the period approaches the limiting value \( \sigma^2_\tau = \sigma^2/2\gamma \).

After some algebra we can also obtain an explicit expression for the transit times \( t_k \) \((k \geq 1)\),

\[
t_k = t_0 + k\bar{\tau} + \frac{1-\gamma}{\gamma} \left[1 - (1 - \gamma)^k\right] (\tau_0 - \bar{\tau}) + \frac{\sigma}{\gamma} \sum_{l=1}^{k} \left[1 - (1 - \gamma)^{k+l-1}\right] \varepsilon_l,
\] (7)

where \( t_0 \) is the initial time. The dispersion of the transit time \( t_k \) is

\[
\sigma^2_t (k) \equiv \langle t^2_k \rangle - \langle t_k \rangle^2 = \frac{\sigma^2}{\gamma^2} \left\{ k - 2\frac{1-\gamma}{\gamma} \left[1 - (1 - \gamma)^k\right] + (1 - \gamma)^2 \frac{1 - (1 - \gamma)^{2k}}{1 - (1 - \gamma)^2} \right\}
\]

= \begin{cases} \sigma^2 (k/6 + k^2/2 + k^3/3 + ...), & 2\gamma k \ll 1, \\ (\sigma/\gamma)^2 (k - 3/2\gamma + 5/4 \pm ...), & 2\gamma k \gg 1. \end{cases} (8)

At \( k \gg \gamma^{-1} \) Eq. (7) generates the stationary time series. The difference of the transit times \( t_{k+q} \) and \( t_k \) in Eq. (3) for \( \tau_0 = \bar{\tau} \) or \( 2\gamma k \gg 1 \) is

\[
t_{k+q} - t_k = \bar{\tau} q + \frac{\sigma}{\gamma} \left\{ [1 - (1 - \gamma)^{q}] \sum_{l=1}^{k} (1 - \gamma)^{k+l-1} \varepsilon_l + \sum_{l=k+1}^{k+q} \left[1 - (1 - \gamma)^{k+l-1}\right] \varepsilon_l \right\}, \quad q \geq 0.
\] (9)

The dispersion of this times difference equals

\[
\langle (t_{k+q} - t_k)^2 \rangle - \bar{\tau}^2 q^2 = \frac{\sigma^2}{2} g(q)
\] (10)

where

\[
g(q) = \frac{2}{\gamma^2} \left\{ [1 - (1 - \gamma)^q]^2 \sum_{l=1}^{k} (1 - \gamma)^{2l} + \sum_{l=1}^{q} \left[1 - (1 - \gamma)^l\right]^2 \right\}, \quad q \geq 0.
\] (11)

Summation in Eq. (11) results in

\[
g(q) = \frac{2}{\gamma^2} \left\{ q - \frac{(1 - \gamma)[1 - (1 - \gamma)^q]}{1 - (1 - \gamma)^2} \left\{ 2 + [1 - (1 - \gamma)^q] (1 - \gamma)^{2k} \right\} \right\}.
\] (12)

At \( \gamma q \ll 1 \)

\[
g(q) = \begin{cases} (2k + 1) q^2 + q/3 + 2q^2/3, & 2\gamma k \ll 1, \\ \left(\frac{1}{\gamma} + \frac{1}{2}\right) q^2 + \frac{q}{3} q - \frac{1}{3} q^3, & 2\gamma k \gg 1, \end{cases} \tag{12a}
\]

while for \( 2\gamma k \gg 1 \) we have

\[
g(q) = \frac{2}{\gamma^2} \left\{ q - 2\frac{(1 - \gamma)[1 - (1 - \gamma)^q]}{1 - (1 - \gamma)^2} \right\}.
\] (13)
\[ S(f) \approx \begin{cases} \left( \frac{1}{2} + \frac{1}{2} \right) q^2 + \frac{1}{2} q - \frac{1}{2} q^3, & \gamma q \ll 1, \\ \frac{1}{\gamma} \left( q + \frac{1}{2} \right) - \frac{1}{2} + ..., & q \gg \gamma^{-1} \gg 1. \end{cases} \] 

(13a)

Note that for \( q < 0 \) we should replace in Eq. (9)–(13) \( q \) by \(|q|\) and \( k \) by \(|k|\). Therefore, the function \( g(q) \) at \( k - |q| \gg \gamma^{-1} \) is even, i.e. \( g(-q) = g(q) \).

Substituting Eq. (9) into Eq. (3) and replacing the summations in the exponents by the multiplications of the exponents we have the following expression for the power spectral density

\[
S(f) \approx \lim_{T \to \infty} \left\{ \frac{2}{T} \sum_{k, q=k_{\min} - k}^{k_{\max} - k} e^{i2\pi f \tau_{eq}} \prod_{l=1}^{k} \exp \left\{ i \frac{2\pi f}{\gamma} \left[ 1 - (1 - \gamma)^q \right] (1 - \gamma)^{k+1-l} \epsilon_l \right\} \right\}. 
\]

(3a)

The average over realizations of the process coincides with the average over the distribution of the random variables \( \epsilon_l \). Using the fact that random variables \( \epsilon_l \) are independent and mutually uncorrelated we can fulfill the averaging over every random variable \( \epsilon_l \) independently. Therefore, Eq. (3a) may be rewritten in the form

\[
S(f) = \lim_{T \to \infty} \frac{2}{T} \sum_{k, q} e^{i2\pi f \tau_{eq}} \prod_{l=1}^{k} \left\langle \exp \left\{ i \frac{2\pi f}{\gamma} \left[ 1 - (1 - \gamma)^q \right] (1 - \gamma)^{k+1-l} \epsilon_l \right\} \right\rangle. 
\]

(3b)

The result of the averaging of the exponent \( \left\langle \exp \{ ic \epsilon_l \} \right\rangle \) (with \( c \) being a constant) over the normally distributed random variable \( \epsilon_l \) with zero expectation and unit variance is

\[
\left\langle e^{ic \epsilon_l} \right\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\epsilon_l^2/2} d\epsilon_l = e^{-c^2/2}.
\]

Therefore, after the averaging over the normal distribution of the random variables \( \epsilon_l \) Eq. (3b) takes the form

\[
S(f) = \lim_{T \to \infty} \frac{2}{T} \sum_{k, q} e^{i2\pi f \tau_{eq}} \prod_{l=1}^{k} \left\langle \exp \left\{ -\frac{2\pi^2 f^2 \sigma^2}{\gamma^2} \left[ 1 - (1 - \gamma)^q \right] (1 - \gamma)^{2(k+1-l)} \right\} \right\rangle. 
\]

(3c)

Transition in Eq. (3c) from the multiplications of the exponents to the summations in the exponents and transformations in analogy with Eq. (11) of the two sums’ summation indexes \( l \to k + 1 - l \) and \( l \to k + q + 1 - l \), respectively, yield according to Eq. (11) the final expression for the power spectral density

\[
S(f) = \lim_{T \to \infty} \frac{2}{T} \sum_{k, q} e^{i2\pi f \tau_{eq} - \pi^2 f^2 \sigma^2 g(q)}. 
\]

(14)
Since the expansion of the function $g(q)$ in powers of $\gamma |q| \ll 1$ at $2\gamma k \gg 1$ according to Eqs. (12) and (13) is

$$g(q) = \frac{1}{\gamma}q^2 - \frac{1}{3} |q|^3 + \frac{1}{2} q^2 \pm \ldots,$$

(15)

for $f \ll f_\tau = (2\pi\bar{\tau})^{-1}$ and $f < f_2 = 2\sqrt{\gamma}/\pi\sigma$ we can replace the summation in Eq. (14) by the integration

$$S(f) = 2\bar{I} \int_{-\infty}^{+\infty} e^{i2\pi f\bar{\tau}q - \frac{\pi f\sigma^2}{2}\gamma^2} dq.$$  

(16)

where $\bar{I} = \lim_{T \to \infty} (k_{max} - k_{min} + 1)/T = \bar{\tau}^{-1}$ is the averaged current.

Furthermore, at $f \gg f_1 = \gamma^{3/2}/\pi\sigma$ it is sufficient to take into account only the first term of expansion (15), $g(q) = q^2/\gamma$. Integration in Eq. (16) hence yields to 1/f spectrum

$$S(f) = 2\bar{I} \int_{-\infty}^{+\infty} \exp \left[ i2\pi f\bar{\tau}q - \frac{(\pi f\sigma)^2}{\gamma} q^2 \right] dq = \bar{I}^2 \frac{\alpha_H}{f}, \quad f_1 < f < f_2, f_\tau$$

(17)

where $\alpha_H$ is a dimensionless constant (the Hooge parameter)

$$\alpha_H = \frac{2}{\sqrt{\pi}} Ke^{-K^2}, \quad K = \bar{\tau} \sqrt{2\sigma_{\tau}} = \bar{\tau} \sqrt{\gamma}.$$  

(18)

Using an expansion of the function $g(q)$ at $\gamma q \gg 1$ according to expression (13), $g(q) = 2q/\gamma^2$, we obtain from Eq. (16) the Lorentzian power spectrum density for $f < f_1$

$$S(f) = 2\bar{I} \frac{\sigma^2}{\bar{\tau}^2 \gamma^2} \frac{1}{1 + (\pi f\sigma^2/\bar{\tau}^2 \gamma^2)^2} = \bar{I}^2 \frac{4\tau_{rel}}{1 + \tau_{rel}^2 \omega^2}.$$  

(19)

Here $\omega = 2\pi f$ and $\tau_{rel} = D_t = \sigma^2/2\bar{\tau}^2 \gamma^2$ is the “diffusion” coefficient of the time $t_k$ according to Eqs. (7) and (8). The model is, therefore, free from the unphysical divergence of the spectrum at $f \to 0$; for $f \ll f_0 = \bar{\tau}^2 \gamma^2/\pi\sigma^2 = 1/2\pi\tau_{rel}$ we have from Eq. (19) the white noise

$$S(f) = \bar{I}^2 \left(2\sigma^2/\bar{\tau}^2 \gamma^2\right).$$  

(20)

Therefore, the model containing only one relaxation rate $\gamma$ for sufficiently small parameter $\gamma$ can produce an exact 1/f-like spectrum in any desirably wide range of frequency, $f_1 < f < f_2, f_\tau$. Furthermore, due to the contribution to the transit times $t_k$ of the large number of the very separated in time random variables, $\varepsilon_l$ ($l = 1, 2, \ldots k$), our model represents a 'long-memory' random process.

IV. GENERALIZATIONS AND NUMERICAL ANALYSIS

Eqs. (16)–(20) describe quite well the power spectrum of the random process (4). As an illustrative example in Fig. 1 the numerically calculated power spectral density averaged over five realizations of the process (4) is compared with the analytical calculations according to Eqs. (16)–(20). The analytical results are in good agreement with the numerical simulations.
Note, that analytical results predict not only the slope and intensity of 1/f noise but the frequency range \( f_1 \div f_2 \), of 1/f noise and intensity of the very low frequency, \( f \ll f_0 \), white noise (20) as well.

This model may also be generalized for the non-Gaussian and for the continuous perturbations of the systems’ parameters resulting in the fluctuations of the period \( \tau \). So, for perturbations by the non-Gaussian sequence of random impacts \( \{\varepsilon_k\} \) with zero expectations Eqs. (1)–(13) remain valid. Only the result (14) of the averaging over realizations of the process in the case of the non-Gaussian perturbations may have different form. Consider now such a situation in more detail.

The power spectral density (3) may be rewritten in the form

\[
S(f) = 2\bar{I} \langle \sum_q e^{i2\pi f\tau_k(q)q} \rangle
\]  
(21)

where the transit times \( t_{k+q} \) and \( t_k \) difference is expressed as

\[
t_{k+q} - t_k = \sum_{l=k+1}^{k+q} \tau_k(q) q, \quad q \geq 0
\]  
(22)

and the brackets denote the averaging over the time (index \( k \)) and over the realizations of the process. Here \( \tau_k(q) \equiv (t_{k+q} - t_k) / q \) is the averaged time interval between the subsequent transit times in the time interval \( t_k \div t_{k+q} \). Note that for the slow (diffusive-like) fluctuations of the averaged interval \( \tau_k(q) \) with the change of the index \( k \) Eq. (22) is valid also when \( q < 0 \), i.e. \( t_{k+q} - t_k = \tau_{k-q}(q) q \approx \tau_k(q) q, \quad q < 0 \). At \( 2\pi f \tau_k(q) \ll 1 \) we may replace the summation in Eq. (21) by the integration and do not take into account the dependence of \( \tau_k(q) \) on \( q \). In such a case Eq. (21) yields

\[
S(f) = 2\bar{I} \int_{-\infty}^{+\infty} e^{i2\pi f\tau_k(q)} dq = 2\bar{I} \int_{-\infty}^{+\infty} \langle e^{i2\pi f\tau_k(q)} \rangle dq.
\]  
(23)

Here the averaging over \( k \) and over the realizations of the process coincides with the averaging over the distribution of the periods \( \tau_k \), i.e.

\[
\langle e^{i2\pi f\tau_k} \rangle = \int_{-\infty}^{+\infty} e^{i2\pi f\tau} \psi(\tau) d\tau = \chi_\tau(2\pi f q)
\]  
(24)

where \( \psi(\tau) \) is the periods \( \tau_k \) distribution density and \( \chi_\tau(\vartheta) \) is the characteristic function of the distribution \( \psi(\tau) \). Taking into account the property of the characteristic function

\[
\int_{-\infty}^{+\infty} \chi_\tau(\vartheta) d\vartheta = 2\pi \psi(0)
\]

we have from Eqs. (23) and (24) the final expression for the power spectral density

\[
S(f) = 2\bar{I} \psi(0) / f.
\]  
(25)
Substituting into Eq. (25) the value \( \psi(0) = \exp\left(-\frac{\bar{\tau}^2}{2\sigma^2}\right) / \sqrt{2\pi\sigma}\tau \) for the Gaussian distribution of the periods \( \tau_k \) we recover the result (17)–(18).

Since different processes result in the Gaussian distribution it is likely that perturbation by the non-Gaussian impacts \( \{\epsilon_k\} \) in Eq. (4) yields nevertheless the Gaussian distribution of the periods \( \tau_k \). For the demonstration of such statement and validity of the approach (21)–(25) we have performed numerical analysis of the model (1)–(4) for different distributions of the perturbations \( \{\epsilon_k\} \). Figures (2) and (3) represent the calculated power spectral densities for the rectangular (uniform) and asymmetric \( \chi^2 \) distributions of the sequence \( \{\epsilon_k\} \) with zero expectations and the same variances and other parameters like those for Fig. 1. We can notice only the slight dependence of the spectra on the distribution function of the perturbing impacts \( \{\epsilon_k\} \) with the same expectations and variances. These results confirm also the presumptions made in the derivation (21)–(24) of the 1/f noise intensity (25).

V. CONCLUDING REMARKS

Analysis of the exactly solvable model of 1/f noise display main features of the noise and serves as a basis for revealing of the origin and parameter dependences of the flicker noise. This allows us to make generalizations of the model resulting in the expression for the 1/f noise intensity through the integral of the characteristic function of the distribution of the time intervals between the subsequent transition times of the elementary events, pulses or particles.

It should be noticed, however, that Eq. (25) represent an idealized 1/f noise. The real systems have finite relaxation time and, therefore, expression of the noise intensity in the form (23) is valid only for \( f > (2\pi\tau_{rel})^{-1} \) with \( \tau_{rel} \) being the relaxation time of the period’s \( \tau_k \) fluctuations. On the other hand, due to the deviations from the approximation \( t_{k+q} - t_k = \tau_k q \) at large \( q \), for sufficiently low frequency we can obtain the finite intensity of \( 1/f^\delta \) (\( \delta \simeq 1 \)) noise even in the case \( \psi(0) = 0 \) but for the signals with fluctuations resulting in the dense concentrations of the transit times \( t_k \). Generalizations of the approach (21)–(25) and analysis of the deviations from the idealized 1/f noise expression (25) are subjects of separate investigations.

ACKNOWLEDGMENTS

Stimulating discussions with Dr. A. Bastys and support from the Alexander von Humboldt Foundation and Lithuanian State Science and Studies Foundation are acknowledged.
REFERENCES

[1] F. N. Hooge, T. G. M. Kleinpenning, and L. K. J. Vadamme, Rep. Prog. Phys. 44, 479 (1981); P. Dutta and P. M. Horn, Rev. Mod. Phys. 53, 497 (1981); Sh. M. Kogan, Usp. Fiz. Nauk 145, 285 (1985) [Sov. Phys. Usp. 28, 170 (1985)]; M. B. Weissman, Rev. Mod. Phys. 60, 537 (1988); M. J. Kirton and M. J. Uren, Adv. Phys. 38, 367 (1989); V. Palenskis, Lit. Fiz. Sb. 30, 107 (1990) [Lithuanian Phys. J. 30(2), 1 (1990)]; F. N. Hooge, IEEE Trans. Electron Devices 41, 1926 (1994); G. P. Zhigal’skii, Usp. Fiz. Nauk 167, 623 (1997) [Phys.-Usp. 40, 599 (1997)].

[2] T. Musha and Higuchi, Jap. J. Appl. Phys. 15, 1271 (1976); X. Zhang and G. Hu, Phys. Rev. E 52, 4664 (1995); M. Y. Choi and H. Y. Lee, Phys. Rev. E 52, 5979, (1995).

[3] W. H. Press, Comments Astrophys. 7, 103 (1978); A. Lawrence, M. G. Watson, K. A. Pounds, and M. Elvis, Nature (London) 325, 694 (1987); M. Gartner, Sci. Am. 238, 16 (1978); S. Maslov, M. Paczuski, and P. Bak, Phys. Rev. Lett. 73, 2162 (1994); M. Usher and M. Stenmler, ibid 74, 326 (1995); Y. Shi, Fractals 4, 547 (1996); V. B. Ryabov, A. V. Stepanov, P. V. Usik, D. M. Vavrin, V. V. Vinogradov, and Yu. F. Yurovsky, Astron. Astrophys. 324, 750 (1997).

[4] M. Kobayashi and T. Musha, IEEE Trans. Biomed. Eng. BME-29, 456 (1982); R. Voss, Phys. Rev. Lett. 68, 3805 (1992); 76, 1978 (1996).

[5] D. L. Gilden, T. Thornton, and M. W. Mallon, Science 267, 1837 (1995).

[6] Y. Chen, M. Ding, and J. A. S. Kelso, Phys. Rev. Lett. 79, 4501 (1997).

[7] M. Wolf, Physica A 241, 493 (1997).

[8] Proc. 13th Int. Conf. on Noise in Physical Systems and 1/f Fluctuations, Palanga, Lithuania, 29 May-3 June 1995. Eds. V. Bareikis and R. Katilius (World Scientific, Singapore, 1995).

[9] Proc. 14th Int. Conf. on Noise in Physical Systems and 1/f Fluctuations, Leuven, Belgium, 14-18 July 1997. Eds. C. Claeyts and E. Simoen (World Scientific, Singapore, 1997).

[10] Proc. 1th Int. Conf. on Unsolved Problems of Noise, Szeged, Hungary, 3-7 Sept. 1996. Eds. Ch. Doering, L. B. Kiss, and M. F. Shlesinger (World Scientific, Singapore, 1997).

[11] B. B. Mandelbrot and J. W. Van Ness, SIAM Rev. 10, 422 (1968); E. Masry, IEEE Trans. Inform. Theory 37, 1173 (1991); B. Ninness, ibid 44, 32 (1998).

[12] H. J. Jensen, Physica Scripta 43, 593 (1991); H. F. Quyang, Z. Q. Huang, and E. J. Ding, Phys. Rev. E 50, 2491 (1994); E. Milotti, ibid 51, 3087 (1995).

[13] J. Kumičak, Ann. Physik 3, 207, (1994); J. Kumičak, in Ref [4], p. 93; H. Akbane and M. Agu, in Ref [4], p. 601 ; J. Timmer and M. König, Astron. Astrophys. 300, 707 (1995); M. König and J. Timmer, Astron. Astrophys. Suppl. Ser. 124, 589 (1997).

[14] S. Sinha, Phys. Rev. E 53, 4509 (1996); T. Ikeguchi and K. Aihara, ibid 55, 2530 (1997).

[15] B. Kaulakys and G. Vektaris, in Ref [8], p. 677; B. Kaulakys and T. Meškauskas, in Ref [4], p. 126; xxx.lanl.gov/abs/chaodyn/9504009.

[16] B. Kaulakys and G. Vektaris, Phys. Rev. E 52, 2091 (1995); xxx.lanl.gov/abs/chaodyn/9504009.

[17] J. Bernamont, Ann. Physik 7, 71 (1937); M. Surdin, J. Phys. Radium (Serie 7) 10, 188 (1939); F. K. du Pré, Phys. Rev. 78, 615 (1950); A. Van der Ziel, Physica 16, 359 (1950); A. L. McWhorter, in: Semiconductor surface physics, edited by R. H. Kingston (Univ. Penn., Philadelphia, 1957), p. 207; F. N. Hooge, in Ref [4], p. 3.
Captions to the figures of the paper

FIG. 1. Power spectral density vs frequency of the current generated by Eqs. (2)–(4) with the Gaussian distribution of the random increments \( \{ \varepsilon_k \} \) for different parameters \( \bar{\tau}, \sigma, \) and \( \gamma \). The sinuous fine curves represent the averaged over five realizations results of numerical simulations, the heavy lines correspond to the numerical integration of Eq. (16) with \( g(q) \) from Eq. (13), and the thin straight lines represent the analytical spectra according to Eqs. (17) and (18).

FIG. 2. Same as in Fig. 1 but for the uniform distribution of the random increments \( \{ \varepsilon_k \} \). Note that for the non-Gaussian distributions of the random perturbations we have no explicit expression analogous to Eq. (16) for the integral representation of the noise power spectral density.

FIG. 3. Same as in Fig. 1 and Fig. 2 but for the asymmetric \( \chi^2 \) distribution of the random increments \( \{ \varepsilon_k \} \).
Fig. 1
Fig. 2
Fig. 3