Quantum hexaspherical observables for electrons

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A new quantum algebraic description of relativistic electrons, built on a conformal dynamical symmetry $\left(\text{SO} \left(4, 2\right)\right)$, has recently been proposed to treat localization in space-time. It is shown here that localization of an electron may be represented by components of a $\text{SO} \left(4, 2\right)$ vector which are quantum generalizations of the hexaspherical coordinates of classical projective geometry. The shift of this vector under transformations to uniformly accelerated frames is described by $\text{SO} \left(4, 2\right)$ rotations. Hexaspherical observables also allow one to represent the quantum law of free fall under a form explicitly compatible with the same dynamical symmetry.

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I. INTRODUCTION

In quantum field theory as well as in classical field theories such as Maxwell theory or general relativity, fields are represented as functions of coordinate parameters on a classical map of space-time. It is now a common idea that such a classical conception of localization in space-time cannot be considered as satisfactory. In particular, the difficulties met when attempting to quantize the gravitational field suggest that sizeless points in space-time have to be replaced by fuzzy spots with a size at least of the order of Planck length. A lot of work has been devoted to this idea in the domains of non commutative geometry or quantum groups [1–9].

In fact, the insistence on defining positions in space-time as physical observables rather than points on a map dates back at least to Einstein’s introduction of relativistic concepts [10]. This idea was revived in a quantum context by Schrödinger who noticed that positions in space-time should be described as quantum observables if a proper physical meaning is to be attributed to Lorentz transformations [11]. In non relativistic quantum mechanics, this requirement is met for space observables but a time operator is lacking [12]. In relativistic quantum field theory, this unacceptable difference between space and time is cleared up at the expense of abandoning the observable character of space variables as well [13]. The absence of a standard solution to this problem has many implications in present physical theory. It makes the implementation of relativistic symmetries in a quantum framework quite unsatisfactory [14] and plagues the attempts to build up a quantum theory including gravity [15–18].

In the present paper, we vindicate a recently proposed description of localization in space-time which associates quantum observables with positions of an event in space and time. These observables have been first defined for coincidence events between two light rays, in which case they fit Einstein definitions of clock synchronization and space-time localization while obeying the Lorentz transformation laws of classical relativity [19, 20]. They have canonical commutators with momenta and meet the requirements enounced by Schrödinger. This algebraic ‘quantum relativity’ framework is built on the symmetries of electromagnetic field theory. The latter include not only Lorentz transformations of special relativity but also dilatations and conformal transformations to uniformly accelerated frames [21–24].

Invariance under dilatation manifests the insensitivity of light propagation to a conformal metric factor, that is also to a change of space-time scale [25]. Localization observables are defined in terms of Poincaré and dilatation generators. This definition holds for field states containing photons propagating in two different directions which is obviously a preliminary condition for defining a coincidence event. This condition may also be expressed by the property that the mass of the field state differs from zero. Hence, the domain of definition of localization observables does not cover the space of all field states and these observables are not self-adjoint although they are hermitian. This circumvents the common objection against the very possibility of giving a quantum definition of time [26, 27]. Hermitian but not self-adjoint observables are known to allow for a perfectly rigorous treatment which solves the quantum paradoxes of phase and time [28, 29]. Here, localization observables are defined in the enveloping division ring built on symmetry generators through a quantum algebraic calculus well defined as soon as divisions by the mass are carefully dealt with [30].
The shift of mass under conformal transformations to accelerated frames is then found to fit the classical redshift law but written with the quantum positions. It thus reproduces the gravitational potential arising in accelerated frames according to Einstein equivalence principle 23,24. Clearly the extension of these results to massive field theories is impossible as long as mass is treated as a classical constant which breaks conformal symmetry, as is the case in Dirac’s electron theory 35. In modern developments however, electron mass is generated through an interaction with Higgs fields 36 and standard forms of this interaction obey conformal invariance 37. Mass is no longer a classical constant. It is now a quantum operator which changes under frame transformations. Conformal invariance just means that mass unit scales as the inverse of space-time unit so that the Planck constant is preserved 38,39. Using this assumption, it is possible to define localization observables for electrons in the same manner as for 2-photon states. The redshift of mass derived from conformal symmetry is anew found to fit the expectation of Einstein equivalence principle 40.

Now, it is well known from classical projective geometry that the conformal symmetry in a n-dimensional space is equivalent to rotational symmetry in a (n + 2)-dimensional space 41. In particular, conformal symmetry in Minkowski space-time is equivalent to SO (4, 2) symmetry on a hyperquadric in a 6-dimensional space 42. Dirac and Bhabha have proposed a field description of electrons in such a space 43 and a number of connections between electrons and the SO (4, 2) dynamical symmetry have been studied 44. In the present context where quantum localization observables have been defined, the challenge is raised of finding a representation of these observables explicitly displaying conformal SO (4, 2) symmetry.

A further challenge immediately follows. According to the classical law of inertia, Newton’s equation of motion is not the same in uniformly accelerated frames as in inertial frames, which makes it incompatible with the symmetries of frame transformations. In classical relativity, this difficulty is solved by writing the law of motion as the geodesic equation which transforms covariantly under frame transformations. But this requires the introduction of a space dependent metric tensor representing a classical gravitational field 45. In the quantum algebraic framework, the question is raised of writing the law of motion under a form compatible with conformal dynamical symmetry.

In the present paper we will take up these challenges. We will show that localization of electrons in space-time may be written in terms of quantum hexaspherical observables transformed as components of a SO (4, 2) vector under SO (4, 2) rotations, that is also conformal transformations to accelerated frames. We will exhibit the close connection between hexaspherical variables and mass, thus extending known results of classical projective geometry. We will finally demonstrate that this representation allows one to write a quantum form of the law of free fall which respects conformal symmetry.

The four next sections are mainly devoted to algebraic developments. The physical significance of the results is discussed in the concluding section.

II. CLASSICAL HEXASPHERICAL COORDINATES

Before addressing the localization problem in a quantum context, we recall the definition of hexaspherical coordinates in a classical space-time representation. To this aim, we remind the conformal representation of accelerated frames in classical relativity and we introduce hexaspherical coordinates which constitute a natural extension of space-time coordinates. We also discuss the important role played by the conformal factor.

In classical relativity, uniformly accelerated frames may be identified as flat conformal frames with a metric tensor \( \lambda^2 (x) \eta_{\mu\nu} \) proportional to the Minkowski metric

\[
\eta_{\mu\nu} = \text{diag} (1, -1, -1, -1)
\]

\( \mu, \nu = 0 \ldots 3 \)

The conformal factor \( \lambda (x) \) depends on position in accelerated frames. This dependence is not arbitrary since the metric corresponds to a null curvature. Flat conformal frames are transformed into one another under conformal coordinate transformations generated by Poincaré transformations and inversions or, equivalently, by Poincaré tranformations, dilatations and Bateman-Cunningham transformations

\[
\eta^\mu = \frac{x^\mu - x^2 a^\mu}{1 - 2a_\nu x^\nu + a^2 x^2} \quad (2)
\]

The velocity of light is set to unity. In \( x^\mu \) and \( \eta^\mu \) represent the coordinates of a point in two maps of classical space-time. The transformations (2) form a group which extends the symmetry principles of special relativity to uniform accelerations. In particular, they describe the change of the conformal factor

\[
\lambda (x) = \left( 1 - 2a_\mu x^\mu + a^2 x^2 \right) \lambda (x) \quad (3)
\]

It is always possible to bring the conformal factor \( \lambda \) back to an inertial one, with \( \overline{\lambda} \) independent of \( x \), by applying a well-chosen conformal transformation. Accordingly, geodesic motion in conformal accelerated frames corresponds exactly to the usual relativistic definition of uniformly accelerated motion 46.

Hexaspherical coordinates \( y_\mu \) can be associated with a point \( x \) in classical space-time through

\[
y_+ + y_- = -\lambda \\
y_\mu = \lambda x_\mu \\
y_+ - y_- = \lambda x^2 \quad (4)
\]
Indices in ordinary $4d$ space-time are labelled by Greek letters $(\mu = 0 \ldots 3)$ and manipulated with the Minkowski metric used throughout the paper to raise or lower indices and to evaluate squared vectors

$$x^2 = \eta_{\mu\nu} x^\mu x^\nu \quad (5)$$

Notice that we keep this convention in accelerated frames in contradistinction with the standard covariance convention. Meanwhile, indices in hexaspherical $6d$ space are labelled by Latin letters, with $-$ and $+$ denoting additional dimensions, and they are manipulated with the $6d$ metric

$$\eta_{a b} = \text{diag} (-1, 1, 1, -1, -1, -1) \quad a, b = - , + , 0 \ldots 3 \quad (6)$$

Hexaspherical coordinates $y_a$ associated with points $x_{\mu}$ of ordinary space-time $S$ lie on a quadric $Q$

$$y^2 = \eta_{a b} y^a y^b = 0 \quad (7)$$

Both notations (5) and (7) will be used in the following depending on the context, the first one for points in ordinary space-time and the second one for $6d$ coordinates.

The relation (4) between points of ordinary space-time $S$ and their hexaspherical representatives is a stereographic projection of $Q$ onto $S$, that is also an inversion. Usually, hexaspherical coordinates $y_a$ are projective coordinates so that the definition of the factor $\lambda$ is not fixed by equation (4). Chosing for this factor the $x$-dependent conformal factor $\lambda(x)$ is however particularly appropriate for different reasons.

First, this choice allows one to write a simple relation between the $6d$ distance $(y - y')^2$ of two points on $Q$ and the metric distance of the two points in $S$

$$(y - y')^2 = \lambda(x) \lambda(x') (x - x')^2 \quad (8)$$

This implies that two points in $S$ with a light-like separation have their hexaspherical representatives on $Q$ also conjugated with respect to $Q$

$$(x - x')^2 = 0 \Rightarrow y^2 = y'^2 = y^a y'_a = 0 \quad (9)$$

Hence, the quadric $Q$ contains straight lines of points conjugated to each other which are hexaspherical images of ordinary light rays in $S$.

Then, conformal coordinate transformations in $S$ are given by mere rotations of hexaspherical coordinates on $Q$. In particular, conformal transformations to accelerated frames (2) correspond to

$$\overline{y}_- + \overline{y}_+ = y_- + y_+ + 2\alpha^\mu y_\mu + \alpha^2 (y_- - y_+)$$
$$\overline{y}_\mu = y_\mu + \alpha_\mu (y_- - y_+)$$
$$\overline{y}_- - \overline{y}_+ = y_- - y_+ \quad (10)$$

The transformation (3) of the conformal factor is just the first line in the preceding equation.

Finally, a light ray remains a light ray under conformal transformations to accelerated frames. The hexaspherical scalar $y^a y'_a$ is preserved by rotations (10) so that, as a consequence of (5), $\lambda(x) \lambda(x') (x - x')^2$ is preserved under conformal frame transformations. This is exactly the property which is needed to demonstrate the conformal invariance of electromagnetic vacuum (17).

At the limit of neighbouring points, the invariance of the hexaspherical scalar (5) is read as a metric property

$$(dy)^2 = \lambda^2 (dx)^2 \quad (11)$$

As a matter of fact, $\sqrt{(dx)^2}$ is the Lorentz interval defined in all frames in terms of the Minkowski tensor $\eta_{\mu\nu}$ and its product by the conformal factor $\lambda$ is the proper time interval. The invariance of this proper time interval under transformations to accelerated frames is here associated with conformal symmetry.

Up to now we have restricted our attention to hexaspherical points lying on the quadric $Q$. Points lying outside $Q$ also have a well known interpretation in classical projective geometry (11). Any point $y_a$ in the $6d$ space indeed defines an hyperplane of points $y'_a$ conjugated to it ($y^a y'_a = 0$) with respect to $Q$. The intersection of this hyperplane with $Q$ is the hexaspherical image of an hyperboloid $H_y$ in ordinary space-time $S$

$$y^2 y'_a = y'^2 = 0 \quad \Leftrightarrow \quad x' \in H_y \quad (12)$$

where $x'$ and $y'$ are related by (4). The characteristic elements of this hyperboloid, namely its center $\omega$ and radius or waist size $\rho$, are related to the hexaspherical coordinates $y^a$

$$x' \in H_y \quad \Leftrightarrow \quad (x' - \omega)^2 + \rho^2 = 0$$
$$y_- + y_+ = -\lambda$$
$$y_\mu = \lambda \omega_\mu$$
$$y_+ - y_- = \lambda (\omega^2 + \rho^2) \quad (13)$$

This relation is such that

$$y^2 = -\lambda^2 \rho^2 \quad (14)$$

The particular case of a null radius $\rho = 0$ corresponds to points $y_a$ which lie on $Q$. In this case the hyperboloid is degenerated into the light cone issued from the point $\omega$ that is also the set of all light rays which intersect this point. In the general case of a non null radius, the hyperboloid may still be built up as a collection of light rays but these light rays no longer intersect the same point.

As previously, $y_a$ are projective coordinates of $H_y$ so that the choice of $\lambda$ is not fixed. We now choose $\lambda$ as inversely proportional to the radius $\rho$

$$\lambda^2 = -\frac{k^2}{\rho^2} \quad \Rightarrow \quad y^2 = k^2 \quad (15)$$
The factor $\lambda$ is a conformal factor now associated with $\mathcal{H}_u$ rather than with a point. A given hyperboloid $\mathcal{H}_u$ is transformed into another hyperboloid $\mathcal{H}_{u'}$ under conformal frame transformations and this transformation is still described by the rotation $\text{(10)}$ of hexaspherical coordinates $\text{(13)}$. Since the factor $k$ is preserved under conformal transformations $\text{(14)}$, it may be eliminated from the transformation of the characteristic elements of hyperboloids

$$\frac{1}{\rho} = 1 - 2\alpha^2 \omega^2 + \alpha^2 (\omega^2 + \rho^2)$$

As $\text{(15)}$, these relations show that the radius $\rho$ encodes metric information in projective geometry. It is preserved for Poincaré transformations but changed as a conformal factor for dilatations and transformations to accelerated frames. Equations $\text{(16)}$ thus generalize the laws of differential geometry in a manner which now depends not only on a position $\omega$, but also on a spot size, the radius $\rho$. In the limiting case of an infinitesimal radius $\rho \rightarrow 0$, the conformal factor has just its standard form and the laws of differential geometry are recovered.

We have discussed in some detail these results of classical projective geometry because they announce quantum properties to be obtained in the following where the conformal factor $\lambda$ and the projective constant $k$ will be replaced respectively by the electron mass and the Planck constant.

**III. QUANTUM LOCALIZATION OBSERVABLES**

We come now to the definition of quantum localization observables. This definition will be based upon the algebraic properties obeyed by the generators of the symmetries involved in localization.

We first recall the commutators of Poincaré and dilatation generators

$$(P_\mu, P_\nu) = 0$$

$$(J_{\mu\nu}, P_\rho) = \eta_{\rho\nu}P_\mu - \eta_{\rho\mu}P_\nu$$

$$(J_{\mu\nu}, J_{\rho\sigma}) = \eta_{\rho\sigma}J_{\mu\nu} + \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\mu\rho}J_{\nu\sigma} - \eta_{\nu\sigma}J_{\rho\mu}$$

$$(D, P_\mu) = P_\mu$$

$$(D, J_{\mu\nu}) = 0$$

$$(17)$$

$P_\mu$ and $J_{\mu\nu}$ are the components of energy-momentum vector and angular momentum tensor. $D$ is the generator of dilatations. Algebraic relations $\text{(17)}$ represent at the same time quantum relations between observables and actions of relativistic symmetries on these observables. It is convenient to denote commutators as brackets $(A, B)$ related to the usual quantum notation $[A, B]$.

$$\langle A, B \rangle \equiv \frac{\left[ A, B \right]}{ih} \equiv \frac{AB - BA}{ih}$$

Notice that the Planck constant $\hbar$ is kept as the characteristic scale of quantum effects. Commutators obey the Jacobi identity

$$\langle A, B, C \rangle = \langle A, (B, C) \rangle - \langle B, (A, C) \rangle$$

As discussed in the Introduction, the electron mass should no longer be considered as a classical constant but as a quantum operator. Forthcoming developments will not depend on a particular underlying quantum field theory but only on the hypothesis of conformal symmetry. We will introduce the operator $M$ according to the relativistic definition of mass

$$(P_\mu, M) = (J_{\mu\nu}, M) = 0$$

$$(D, M) = M$$

$$M^2 = P^2$$

$$(19)$$

$M$ is invariant under Poincaré transformations and it has the same conformal weight as energy-momentum.

The definition and properties of localization observables are deduced from conformal algebra. Spin observables are first defined through the Pauli-Lubanski vector and the spin tensor $S_{\mu\nu}$

$$S_{\mu} = -\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} J^{\nu\rho} P^{\sigma}$$

$$(21)$$

$S_{\mu\nu}$ is the completely antisymmetric Lorentz tensor. The square modulus of the Lorentz vector $S^{\mu}$ is a Lorentz scalar $S^2$ with its standard form in terms of a spin number $s$ fixed to the value $\frac{1}{2}$ in the following

$$S^2 = -\hbar^2 s (s + 1) = -\frac{3}{4} \hbar^2$$

Position observables are then defined as

$$X_\mu = \frac{P_\mu}{M^2} \cdot D + \frac{P^\rho}{M^2} \cdot J_{\rho\mu}$$

$$(23)$$

The dot symbol denotes a symmetrized product for non commuting observables

$$A \cdot B \equiv \frac{AB + BA}{2}$$

$$(24)$$

It has to be manipulated with care since it is not associative

$$A \cdot (B \cdot C) - (A \cdot B) \cdot C = \frac{\hbar^2}{4} (B, (A, C))$$

$$(25)$$

We will also use a symmetrized division

$$\frac{A}{B} \equiv A \cdot \frac{1}{B}$$

$$(26)$$
Poincaré and dilatation generators take their usual form in terms of localization observables

\[ J_{\mu\nu} = P_\mu \cdot X_\nu - P_\nu \cdot X_\mu + S_{\mu\nu} \]
\[ D = \bar{P}^\mu \cdot X_\mu \]  

(27)

The shifts of positions under translations, dilatation and rotations also have the classical expressions

\[ (P_\mu, X_\nu) = -\eta_{\mu\nu} \]
\[ (D, X_\mu) = -X_\mu \]
\[ (J_{\mu\nu}, X_\rho) = \eta_{\mu\rho} X_\nu - \eta_{\nu\rho} X_\mu \]  

(28)

Positions in space-time are thus defined as conjugate with respect to momentum observables while properly representing Lorentz symmetry. These results meet the requirements enounced by Schrödinger \[1\] and have to be contrasted with previous studies of the localization problem where only positions in space were introduced \[18, 20\]. Different position components do not commute in the presence of a non vanishing spin \[23\].

\[ (X_\mu, X_\nu) = \frac{S_{\mu\nu}}{M^2} \]  

(29)

This indicates that quantum objects cannot be treated as sizeless points.

Symmetry generators have to be thought of as integrals built on the quantum stress tensor associated with the electron. The squared mass \( M^2 \) is defined in terms of momenta while position observables are obtained in the division ring built on symmetry algebra \[17\]. Hence these observables are highly non linear expressions built on integrals of electron stress tensor. They are hermitian but not self-adjoint observables \[20\]. As recalled in the Introduction, this is not a deficiency but rather a mandatory condition for solving difficulties which are otherwise inescapable.

In a quantum algebraic approach, frame transformations of observables are described as conjugations by group elements. Since such conjugations preserve commutation relations as well as products, any algebraic relation valid in a given frame also holds in any other one. As far as inertial frames are concerned, this property constitutes the very essence of the principle of relativity. Here, this principle is extended to dilatations, that is to say to changes of units which preserve the velocity of light and Planck constant \( \hbar \), and to conformal transformations to accelerated frames. In the following we will focus our attention on the latter which correspond to classical transformations \[4\] and are obtained here by exponentiating infinitesimal generators \( C_\mu \)

\[ \bar{A} = \exp \left( -\frac{\alpha^\mu C_\mu}{i\hbar} \right) A \exp \left( \frac{\alpha^\mu C_\mu}{i\hbar} \right) \]
\[ = A + \alpha^\mu (A, C_\mu) + \frac{\alpha^\mu \alpha^\nu}{2} ((A, C_\mu), C_\nu) + \ldots \]  

(30)

The classical parameters \( \alpha^\mu \) are acceleration components along the 4 space-time directions. Positions and momenta transformed according to these relations preserve the canonical commutators since \( \eta_{\mu\nu} \) is a classical number invariant under conjugations. Quantum algebraic relations are written in all frames in terms of the same Minkowski metric which, as already stated, stands in contradistinction with covariance convention.

The relativistic effects of acceleration are recovered when the results of group conjugations are evaluated. As an important example, the redshifts of an observable under conjugations \[30\] can be obtained from the definition of this observable and from the commutators of the generators \( C_\mu \) with other conformal generators

\[ (D, C_\mu) = -C_\mu \]
\[ (P_\mu, C_\nu) = -2\eta_{\mu\nu} D - 2J_{\mu\nu} \]
\[ (C_\mu, C_\nu) = 0 \]
\[ (J_{\mu\nu}, C_\rho) = \eta_{\rho\nu} C_\mu - \eta_{\nu\rho} C_\mu \]  

(31)

The general problem of evaluating the shifts of observables under transformations to accelerated frames is greatly simplified when the spin number \( s \) is preserved. In this case, closed expressions can be derived for the generators \( C_\mu \) in terms of Poincaré and dilatation generators \[22\]. We assume that this is the case for electrons which have a spin number \( s = \frac{1}{2} \) in all frames and we restrict our attention to the simplest form of the expression of \( C_\mu \)

\[ C_\mu = 2D \cdot X_\mu - P_\mu \cdot \left( X^2 + \frac{3\hbar^2}{4M^2} \right) + 2X^\rho \cdot S_{\rho\mu} \]  

(32)

Electron spin can only take the two values \( \pm \frac{\hbar}{2} \) when measured along any direction transverse to momentum. This property is expressed as the following relation between spin and momentum observables

\[ S_\mu \cdot S_\nu = -\frac{\hbar^2}{4} (\eta_{\mu\nu} - \frac{P_\mu P_\nu}{M^2}) \]  

(33)

Taken with the general results of the present section, these assumptions are sufficient to build up a theory of electrons in uniformly accelerated as well as inertial frames \[10\].

IV. QUANTUM HEXASPERICAL OBSERVABLES

In classical theory, hexaspherical variables have been built on positions and the conformal factor. We now generalize this definition to the quantum algebraic framework by letting the mass observable play the role of the conformal factor.

To this aim, we consider the shift of mass under transformations \[34\] to accelerated frames. We first obtain the action of \( C_\mu \) on mass

\[
\[ (C_\mu, M) = 2Y_\mu \]
\[ Y_\mu = M \cdot X_\mu \]  \hspace{1cm} (34)

and then iterate this action by making use of (33)

\[ (C_\mu, Y_\nu) = \eta_{\mu\nu} \left( M \cdot X^2 + \frac{3 \, \hbar^2}{4 \, M} \right) \]
\[ \left( C_\mu, M \cdot X^2 + \frac{3 \, \hbar^2}{4 \, M} \right) = 0 \]  \hspace{1cm} (35)

As a consequence, the transformed mass \[ \bar{M} \] is a second-order polynomial of the acceleration parameters. Moreover, quantum hexaspherical observables may be defined which transform as classical hexaspherical coordinates under frame transformations

\[ Y_+ + Y_- = -M \]
\[ Y_\mu = M \cdot X_\mu \]
\[ Y_+ - Y_- = M \cdot X^2 + \frac{3 \, \hbar^2}{4 \, M} \]  \hspace{1cm} (36)

Precisely, these observables have their shifts under finite transformations to accelerated frames \[ \lambda \] read as the classical laws \[ \lambda \]. The shifts are now written in terms of the quantum observables \[ Y_a \] and they have to be dealt with care since they involve operators which do not commute with each other.

With this remark kept in mind, we write the transformation of quantum observables \[ Y_a \] as

\[ \bar{M} = M - 2\alpha \alpha' Y_a + \alpha^2 (Y_+ - Y_-) \]
\[ \bar{Y}_\mu = Y_\mu - \alpha Y_+ - Y_- \]
\[ \bar{Y}_+ - \bar{Y}_- = Y_+ - Y_- \]
\[ \bar{Y}^2 = Y^2 = \hbar^2 \]  \hspace{1cm} (37)

As for classical variables, \( Y^2 \) is evaluated in 6d space whereas the notation \( X^2 \) refers to Minkowski space. Relations \[ (36,37) \] are quantum analogs of the classical expressions \[ (13,14) \] with the classical conformal factor \( \lambda \) identified as the quantum mass and the classical projective constant \( k \) identified as the Planck constant

\[ \rho^2 = s \, (s + 1) \, \frac{\hbar^2}{M^2} = \frac{3 \, \hbar^2}{4 \, M^2} \]
\[ \lambda^2 = -s \, \frac{M^2}{s \, (s + 1)} \]
\[ k^2 = \hbar^2 \]  \hspace{1cm} (38)

The inverse relation of \[ (37) \] is simply obtained by exchanging the roles of the two frames and changing the sign of acceleration parameters \( \alpha_\mu \).

We now write the various commutation relations in a form explicitly displaying rotation symmetry in 6d space. To this aim, the 15 conformal generators are identified as rotation generators \( J_{ab} \) in a 6d space which extend the generators \( J_\mu \) of Lorentz transformations in ordinary space-time

\[ P_\mu = J_{+\mu} + J_{-\mu} \]
\[ D = J_{++} \]
\[ C_\mu = J_{+\mu} - J_{-\mu} \]  \hspace{1cm} (39)

The whole set of conformal commutators \[ (17,31) \] is then collected in a single relation

\[ (J_{ab}, J_{cd}) = \eta_{ac} J_{ad} + \eta_{bd} J_{bc} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} \]  \hspace{1cm} (40)

which is just the definition of \( SO(4,2) \) symmetry. Then the commutators \[ (44,47) \], together with relations \[ (20,28) \], are gathered in a single relation

\[ (J_{ab}, Y_c) = \eta_{bc} Y_a - \eta_{ac} Y_b \]  \hspace{1cm} (41)

which means that the variables \( Y_a \) are transformed as components of a \( SO(4,2) \) vector under \( SO(4,2) \) rotations. In particular, shifts \[ (45) \] under finite transformations to accelerated frames are direct consequences of \[ (11) \].

We have now written the quantum algebraic description of electrons in terms of relations quite analogous to classical projective geometry. But this description is no longer classical and, in particular, quantum hexaspherical observables do not commute. Their commutators are deduced from previously written results

\[ (Y_\mu, M) = P_\mu \]
\[ (Y_\mu, Y_\nu) = J_{\mu\nu} \]
\[ (Y_+ - Y_-, M) = 2D \]
\[ (Y_+ - Y_-, Y_\mu) = C_\mu \]  \hspace{1cm} (42)

and they may be collected in a single \( SO(4,2) \) expression

\[ (Y_a, Y_b) = J_{ab} \]  \hspace{1cm} (43)

\section{V. THE LAW OF FREE FALL}

As already emphasized, the mass observable takes the place of the conformal factor in the quantum algebraic framework. We will now show that quantum mass effectively allows one to write the law of free fall in a constant gravity field. To this aim, we will consider an inertial frame with generators \( J_{ab} \) and hexaspherical observables \( Y_a \) as well as a second frame, with generators \( J_{ab} \) and hexaspherical observables \( Y_a \), which is accelerated with respect to the inertial one. The trajectories defined in inertial in the inertial frame do appear as accelerated in the accelerated frame. In other words, they are the geodesic trajectories in the constant gravity field associated with this uniform acceleration.

We first remark that the concept of motion may be defined in the quantum algebraic framework as the action of a commutator with the inertial mass observable \( \bar{M} \)

\[ F' = (F, \bar{M}) \]  \hspace{1cm} (44)
As a consequence of Jacobi identity, the Leibniz rule is obeyed by this differentiation operator
\[(FG)' = F'G + FG'\] (45)
This would be true for the commutator with any observable but the choice of inertial mass \(\bar{M}\) as the generator of motion leads to conservation of Poincaré generators in the inertial frame
\[\mathcal{T}_\mu = \mathcal{T}_{\mu\nu} = 0\] (46)
The laws of inertial motion may also be written
\[\mathcal{Y}_\mu = \mathcal{M} \cdot \mathcal{X}_\mu = \mathcal{T}_\mu\]
\[\mathcal{Y}_\mu'' = \mathcal{M} \cdot \mathcal{X}_\mu'' = 0\] (47)
The choice of \(\bar{M}\) for generating motion fixes the definition of inertial frames but motion can as well be written in accelerated frames. The inertial mass \(\bar{M}\) may indeed be expressed in terms of the mass \(M\) evaluated in the accelerated frame and of a position dependent conformal factor \(\Lambda\)
\[\bar{M} = \frac{M}{\Lambda}\]
\[\frac{1}{\Lambda} = 1 - 2\alpha' X_\mu + \alpha^2 \left( X^2 + \frac{3}{4} \frac{\hbar^2}{M^2} \right)\] (48)
The latter is now a quantum operator which depends on quantum localization observables \(X_\mu\) and \(M\). The position dependence has nearly the same form as in the classical case except for the last term which is proportional to the squared spin. The motion of any observable evaluated in the accelerated frame, say the position \(X_\mu\), is then obtained as its commutator with \(\frac{\bar{M}}{\Lambda}\). The expressions obtained in this manner are quantum extensions of the laws of geodesic motion of classical relativity. They contain classically looking terms arising from the canonical commutators between momenta and positions and purely quantum terms depending on spin.

At this point, it is worth emphasizing that these spin terms are direct consequences of symmetry considerations. Quantum hexaspherical observables do not commute and their commutators are equal to the rotation generators. For ordinary space-time indices in particular, the commutator \((Y_\mu, Y_\nu)\) is just equal to the ordinary angular momentum \(J_{\mu\nu}\). It contains an orbital part which corresponds to the canonical commutators between momenta and positions. It also involves a spin part which fits the commutator between different position components. Hence, the fact that position components do not commute and have spin components as their commutators is directly connected with conformal dynamical symmetry. In the present quantum algebraic approach, the equivalence principle is nothing but another expression for this dynamical symmetry and the spin terms appearing in the equations of geodesic motion are consequences of this principle.

Quantum geodesic equations may be laid down in a much simpler manner by using hexaspherical observables. As the observables \(Y_a\) are linear superpositions of \(Y_a\) (see [35]), the quantum laws of free fall are obtained as
\[Y''_\mu = 2\alpha_\mu \bar{M}\]
\[M'' = 2\alpha^2 \bar{M}\]
\[Y''_\mu - Y'' = 2\bar{M}\] (49)
The first equation describes a force \(Y''_\mu\) proportional to the constant gravity field \(2\alpha_\mu\) and to the mass \(\bar{M}\). The mass entering this law is the inertial mass, that is also the generator of motion \(\bar{M}\). This inertial mass is a constant of motion whilst the mass \(M\) evaluated in the accelerated frame varies according to the second equation in (49).

VI. DISCUSSION

In the present paper, we have defined quantum observables \(Y_a\) which correspond to the hexaspherical coordinates of classical projective geometry. These observables involve not only space-time position observables but also the mass observable. The latter describes metric properties in the quantum algebraic framework, playing the same role as the conformal factor in classical relativity.

Localization observables \(Y_a\) are associated with an electron localized in space and time. Transformations between various uniformly accelerated frames correspond to SO(4, 2) rotations of these observables. In summary, quantum as well as relativistic properties of electrons are described by a ‘non commutative conformal geometry’ which is essentially determined by the conformally invariant commutators ([10],[11],[13]). These results clearly indicate that the conceptions of space-time inherited from classical relativity have to be revised for quantum objects. In particular localization of electrons can no longer be thought of in terms of sizeless points. The best classical picture for localization of electrons obtained in this paper corresponds to the center of an hyperboloid having a waist size or a radius proportional to spin and inversely proportional to mass. Accordingly, the best classical picture of relativistic transformations of electrons is given by the projective geometry of hyperboloids rather than by the geometry of points. Furthermore, the geometrical elements of the hyperboloids, its center and waist size parameter, have to be considered as non commutative operators. In this context sizeless classical points appear as unobservable entities and this certainly raises questions about the pertinence of classical representations of space-time and infinitesimal geometry when applied to quantum problems.

Problems with classical representations of space-time are usually expected to arise at a typical size of the order of Planck length, in connection with the difficulties of quantum gravity. Here in contrast, electrons appear
as fuzzy spots with a typical size $\frac{\varnothing}{M}$, where $S$ is a spin component and $M$ the mass, of the order of Compton wavelength. We have seen that position components do not commute and have spin components as their commutators, as a direct consequence of conformal dynamical symmetry. Then, dispersions in position have to obey an Heisenberg inequality with a typical length just of the order of Compton wavelength.

This typical size might appear as astonishing when contrasted with the fact that quantum field theory is certainly still efficient at smaller length scales. At this point, it is worth recalling that an equivalent set of observables may be defined for the positions of an electron in space-time [10]. In that representation, position observables commute with each other and, hence, may be considered as quantum algebraic extensions of the position variables of standard Dirac theory. There is however a price to be paid for this simplification. Commuting position components are no longer hermitian and their non hermitian part is related to spin. This means that quantum field theory manages to deal with the non commutativity of localization observables at the prize of representing it in terms of internal spin variables. This has certainly permitted impressive achievements with however the drawback of renouncing to the principles of conformal dynamical symmetry which are shown here to lie at the root of the theory of electrons.

We have seen that mass plays the role of a conformal factor, thus determining the space-time scale. At the same time, it allows to represent the law of free fall by a uniform acceleration with respect to inertial frames.

constant gravity field may be considered as arising from a uniform acceleration with respect to inertial frames. No reference to any classical field is needed for writing it. This means that the choice of specific frames as defining inertia cannot be justified from purely algebraic properties. Accelerated frames being included in conformal symmetry, there is no longer any privilege for the case of a null acceleration.

The specific law of free fall corresponding to a constant gravity field is obtained through a contraction of the conformally invariant expression (50).

Expression (50) has exactly the same form in any conformal frames including uniformly accelerated as well as inertial frames. No reference to any classical field is needed for writing it. This means that the choice of specific frames as defining inertia cannot be justified from purely algebraic properties. Accelerated frames being included in conformal symmetry, there is no longer any privilege for the case of a null acceleration.

The quantum algebraic framework has the ability of describing not only localization in space-time and relativistic symmetries associated with frame transformations, but it may also accomodate the description of motion. Up to now, this description has been restricted to constant gravity fields, that is also to flat conformal frames but, even with this restriction, it has extended the symmetry principles of special relativity to include the equivalence principle.

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