Generalized weighted survival and failure entropies and their dynamic versions

Siddhartha Chakraborty and Biswabrata Pradhan

SQC & OR Unit, Indian Statistical Institute, Kolkata, India

ABSTRACT
Generalized weighted survival and failure entropies of order \((h_1, h_2)\) are proposed and their properties are obtained. We further propose the dynamic versions of weighted generalized survival and failure entropies and obtain some properties and bounds. Characterization for Rayleigh and power distributions is done by dynamic weighted generalized entropies. We further consider the empirical versions of generalized weighted survival and failure entropies and, using the difference between theoretical and empirical survival entropies, a test for exponentiality is considered.

1. Introduction

Shannon (1948) introduced the concept of differential entropy and since then it has been playing an important role in the field of information theory, thermodynamics, statistical mechanics and reliability. Let \(X\) be a non-negative absolutely continuous random variable (rv) having cumulative distribution function (cdf) \(F(x)\) and probability density function (pdf) \(f(x)\). Then, Shannon entropy of \(X\) is given by

\[
H(X) = -\int_0^{+\infty} f(x) \log f(x) \, dx,
\]

where \(\log\) is the natural logarithm. For some applications of Shannon’s entropy in various fields one may refer to Bruhn et al. (2001), Camesasca, Kaufman, and Manaszlozcwzer (2006) and Mercurio, Wu, and Xie (2020). There are various generalizations of Shannon entropy considered by many authors. Two most important ones are due to Renyi (1961) and Varma (1966). Renyi’s entropy of \(X\) is given by

\[
H_{\theta_1}(X) = \frac{1}{1 - \theta_1} \log \int_0^{+\infty} f^{\theta_1}(x) \, dx, \quad \theta_1(\neq 1) > 0
\]

and Varma’s entropy of \(X\) is defined as

\[
H_{\theta_1, \theta_2}(X) = \frac{1}{\theta_2 - \theta_1} \log \int_0^{+\infty} f^{\theta_1 + \theta_2 - 1}(x) \, dx, \quad \theta_2 \geq 1, \quad \theta_2 - 1 < \theta_1 < \theta_2.
\]
When \( \theta_1 \to 1 \), \( H_{\theta_1}(X) \to H(X) \). For \( \theta_2 = 1 \), \( H_{\theta_1, \theta_2}(X) \) reduces to \( H_{\theta_1}(X) \) and when both \( \theta_1, \theta_2 \) tends to 1, \( H_{\theta_1, \theta_2}(X) \) tends to \( H(X) \).

If an item has survived up to time \( t \), then in order to incorporate the residual lifetime of the item, Ebrahimi (1996) proposed dynamic entropy as \( H(X; t) = - \int_t^{+\infty} \frac{f(x)}{F(x)} \log \frac{f(x)}{F(x)} \, dx \), where \( F(x) = 1 - F(x) \) is the survival function (sf) of \( X \). Di Crescenzo and Longobardi (2002) proposed the concept of dynamic past entropy measure as \( \tilde{H}(X; t) = - \int_0^t \frac{f(x)}{F(x)} \log f(x) \, dx \).

Recently, Rao et al. (2004) have proposed cumulative residual entropy measure as \( \epsilon(X) = - \int_0^{+\infty} F(x) \log F(x) \, dx \). It may be noted that \( \epsilon(X) \) measures the uncertainty even when cdf exists but pdf does not. Asadi and Zohrevand (2007) proposed the dynamic form of \( \epsilon(X) \) and Di Crescenzo and Longobardi (2009) proposed cumulative entropy \( \tilde{\epsilon}(X) = - \int_0^{+\infty} F(x) \log F(x) \, dx \). Psarrakos and Navarro (2013) introduced generalized cumulative residual entropy as \( \xi_n(X) = \frac{1}{n!} \int_0^{+\infty} F(x)(- \log F(x))^n \, dx \) and Kayal (2016) proposed generalized cumulative entropy as \( \tilde{\xi}_n(X) = \frac{1}{n!} \int_0^{+\infty} F(x)(- \log F(x))^n \, dx \).

Zografos and Nadarajah (2005) proposed survival entropy of order \( \theta_1 \) as \( \xi_{\theta_1}(X) = \frac{1}{1-\theta_1} \log \int_0^{+\infty} F_{\theta_1}(x) \, dx, \ \theta_1(\neq 1) > 0 \) and Abbasnejad et al. (2010) obtained its dynamic version. Abbasnejad (2011) introduced the failure entropy of order \( \theta_1 \) as \( f\xi_{\theta_1}(X) = \frac{1}{1-\theta_1} \log \int_0^{+\infty} F_{\theta_1}(x) \, dx \) and also obtained its dynamic version.

Motivated from Zografos and Nadarajah (2005) and Abbasnejad et al. (2010), Kayal (2015) proposed generalized survival entropy of order \( (\theta_1, \theta_2) \) as

\[
\xi_{\theta_1, \theta_2}(X) = \frac{1}{\theta_2 - \theta_1} \log \int_0^{+\infty} F^{\theta_1 + \theta_2 - 1}(x) \, dx, \ \theta_2 \geq 1, \ \theta_2 - 1 < \theta_1 < \theta_2. \tag{2}
\]

We remark that the integral in Equation (2) is extended to the support of the rv. For a bounded non-negative rv \( X \) with support \([0, l]\), Kayal (2015) proposed generalized failure entropy as

\[
f\xi_{\theta_1, \theta_2}(X) = \frac{1}{\theta_2 - \theta_1} \log \int_0^l F^{\theta_1 + \theta_2 - 1}(x) \, dx, \ \theta_2 \geq 1, \ \theta_2 - 1 < \theta_1 < \theta_2. \tag{3}
\]

They also considered their dynamic versions and obtained characterization results for exponential, Pareto and power distributions. These above measures are shift independent and gives equal weights to the occurrence of events. But in practical situations such as communication theory and Reliability, a shift-dependent measure is often required. To incorporate this issue, Belis and Guiașu (1968) have introduced the concept of weighted entropy as \( H^w(X) = - \int_0^{+\infty} x f(x) \log f(x) \, dx \). Since then, several works have been done on weighted entropies. One may refer to Misagh et al. (2011), Mirali, Baratpour, and Fakoor (2017), Mirali and Baratpour (2017a, 2017b), Rajesh, Abdul-Sathar, and Rohini (2017), Nair, Abdul-Sathar, and Rajesh (2017), Khammar and Jahanshahi (2018), Das (2017), Nourbakhsh, Yari, and Mehrali (2020) and Chakraborty and Pradhan (2021), for details on weighted entropy measures.

In this article, we propose generalized weighted survival and failure entropies of order \( (\theta_1, \theta_2) \) and their dynamic versions. The properties of the proposed entropy measures
are discussed. The rest of the paper is organized as follows. In Section 2, we introduce generalized weighted survival entropy (GWSE) and obtain its properties. The dynamic version of GWSE is discussed in Section 3. Characterization results for Rayleigh distribution are obtained using generalized dynamic weighted survival entropy in Section 4. We propose generalized weighted failure entropy and dynamic failure entropy in Section 5. Characterization results for power distribution are obtained based on generalized dynamic weighted failure entropy. We obtain some inequalities and bounds for the proposed entropy measures in Section 6. The empirical generalized weighted survival and failure entropies are provided in Section 7. A goodness-of-fit test for exponential distribution is discussed in Section 8. Finally, we conclude the paper in Section 9.

2. GWSE of order \((\theta_1, \theta_2)\)

Here we introduce GWSE and obtain some of its properties.

**Definition 2.1.** GWSE of order \((\theta_1, \theta_2)\) is proposed as

\[
\xi_{\theta_1, \theta_2}^w(X) = \frac{1}{\theta_2 - \theta_1} \log \int_0^{+\infty} x\overline{F}_{\theta_1+\theta_2-1}(x)dx, \quad \theta_2 \geq 1, \quad \theta_2 - 1 < \theta_1 < \theta_2. \tag{4}
\]

To illustrate the usefulness of the proposed entropy measure, we consider the following example.

**Example 2.1.** Suppose \(X\) and \(Y\) have pdfs \(f(x) = \frac{1}{b-a}, \quad a < x < b\) and \(g(y) = \frac{1}{b-a}, \quad a + h < y < b + h, \quad h > 0\), respectively. From Equation (2), we have

\[
\xi_{\theta_1, \theta_2}^w(X) = \frac{1}{\theta_2 - \theta_1} \log \frac{b-a}{a \theta_2 + \theta_1 + \theta_2}, \quad \text{From Equation (4) we get,}
\]

\[
\frac{\xi_{\theta_1, \theta_2}^w(X)}{\theta_2 - \theta_1} = \frac{1}{\theta_2 - \theta_1} \log \left[ \frac{(b-a)(a(\theta_1 + \theta_2) + b)}{(\theta_1 + \theta_2)(\theta_1 + \theta_2 + 1)} \right],
\]

\[
\xi_{\theta_1, \theta_2}^w(Y) = \frac{1}{\theta_2 - \theta_1} \log \left[ \frac{(b-a)(a(\theta_1 + \theta_2) + b + h(\theta_1 + \theta_2 + 1))}{(\theta_1 + \theta_2)(\theta_1 + \theta_2 + 1)} \right],
\]

It can be easily shown that \(\xi_{\theta_1, \theta_2}^w(X) < \xi_{\theta_1, \theta_2}^w(Y)\). So we see that \(\xi_{\theta_1, \theta_2}^w(X) = \xi_{\theta_1, \theta_2}^w(Y)\), but GWSE of \(X\) is smaller than GWSE of \(Y\).

The following lemma shows that \(\xi_{\theta_1, \theta_2}^w(X)\) is a shift-dependent measure.

**Lemma 2.1.** Consider the linear transformation \(Z = aX + b, \) where \(a > 0\) and \(b \geq 0,\) then

\[
\exp\left[ (\theta_2 - \theta_1) \xi_{\theta_1, \theta_2}^w(Z) \right] = a^2 \exp\left[ (\theta_2 - \theta_1) \xi_{\theta_1, \theta_2}^w(X) \right] + ab \exp\left[ (\theta_2 - \theta_1) \xi_{\theta_1, \theta_2}^w(X) \right]. \tag{5}
\]

**Proof.** The results follows using \(\tilde{F}_{aX+b}(x) = F_X\left(\frac{x-b}{a}\right), \quad x \in R.\)

Let \(\tilde{F}_{X_0}(x)\) and \(\tilde{F}(x)\) denote the sfs of the rvs \(X_0\) and \(X\), respectively. \(X_0\) and \(X\) satisfy proportional hazard rate model (see Cox 1972) if \(\tilde{F}_{X_0}(x) = [\tilde{F}(x)]^\theta, \quad \theta > 0.\) The following Lemma compares the GWSE of \(X, X_0\) and \(\theta X.\)
then taking logarithm and dividing by $F(x)$. Let $X$ be a continuous non-negative rv having WMRL $m$. Theorem 2.1.

**Proof.** For $h > 0$, and Pareto distribution with cdf $F(x) = 1 - \left(\frac{x}{a}\right)^b$, $x > b > 0$, $a > 0$, as examples in Table 1 to support Lemma 2.2.

Note that for Pareto distribution when $a_\gamma$, $a_\theta \gamma$ and $\theta_\gamma$ are less than 2, GWSE won’t be finite.

**Definition 2.2** (Misagh et al. 2011). Let $X$ be a continuous non-negative rv with sf $F(x)$, then the weighted mean residual life (WMRL) of $X$ is given by

$$m_F(t) = \int_t^{\infty} x F(x) dx, \quad F(t) > 0.$$  

**Theorem 2.1.** Let $X$ be a continuous non-negative rv having WMRL $m_F(t)$ and GWSE $\xi_{\theta_1, \theta_2}(X)$, then for $\theta_1 + \theta_2 - 1 \geq (\leq) 1$,

$$\xi_{\theta_1, \theta_2}(X) \leq (\geq) \frac{1}{\theta_2 - \theta_1} \log m_F(t).$$

**Proof.** For $\theta_1 + \theta_2 - 1 \geq (\leq) 1$, $x F^{\theta_1 + \theta_2 - 1}(x) \leq (\geq) x F(x)$, taking integral on both sides then taking logarithm and dividing by $(\theta_2 - \theta_1)$ we get the result.

**3. Generalized dynamic weighted survival entropy of order $(\theta_1, \theta_2)$**

Now we define the dynamic version of GWSE to study the uncertainty in the residual life of a component $X$. It is the GWSE of the rv $|X - t| X > t$, $t > 0$.

**Definition 3.1.** Generalized dynamic weighted survival entropy (GDWSE) of order $(\theta_1, \theta_2)$ of a continuous rv $X$ is defined as

$$\zeta_{\theta_1, \theta_2}(X; t) = \frac{1}{\theta_2 - \theta_1} \log \int_t^{\infty} x F^{\theta_1 + \theta_2 - 1}(x) dx, \quad \theta_2 \geq 1, \quad \theta_2 - 1 < \theta_1 < \theta_2.$$
Lemma 3.1. Suppose \( Z = aX + b \), where \( a > 0 \) and \( b \geq 0 \), then

\[
\exp \left[ (\theta_2 - \theta_1) \xi_{\theta_1, \theta_2}^w (Z; t) \right] = a^2 \exp \left[ (\theta_2 - \theta_1) \xi_{\theta_1, \theta_2}^w \left( X; \frac{t - b}{a} \right) \right] + ab \exp \left[ (\theta_2 - \theta_1) \xi_{\theta_1, \theta_2}^w \left( X; \frac{t - b}{a} \right) \right].
\]

Proof. The proof is similar to Lemma 2.1.

In Lemma 3.1 we have considered the generalized dynamic weighted survival entropy measure. Now consider the generalized dynamic survival entropy measure which is defined as

\[
\xi_{\theta_1, \theta_2} (X; t) = \frac{1}{\theta_2 - \theta_1} \log \int_t^{+\infty} \frac{F^{\theta_1 + \theta_2 - 1} (x)}{F^{\theta_1 + \theta_2 - 1} (t)} dx, \quad \theta_2 \geq 1, \quad \theta_2 - 1 < \theta_1 < \theta_2
\]

and it is a shift-independent measure since

\[
\exp \left[ (\theta_2 - \theta_1) \xi_{\theta_1, \theta_2} (Z; t) \right] = a \exp \left[ (\theta_2 - \theta_1) \xi_{\theta_1, \theta_2} \left( X; \frac{t - b}{a} \right) \right].
\]

Remark 3.1. If \( b = 0 \), then from Lemma 3.1, we have

\[
\xi_{\theta_1, \theta_2}^w (Y; t) = \frac{2 \log a}{\theta_2 - \theta_1} + \xi_{\theta_1, \theta_2}^w \left( X; \frac{t}{a} \right). \quad (8)
\]

Now we provide a bound for \( \xi_{\theta_1, \theta_2}^w (X; t) \) in terms of WMRL.

Theorem 3.1. Let \( X \) be a continuous non-negative rv with WMRL \( m_x^e (t) \) and GDWSE \( \xi_{\theta_1, \theta_2}^w (X; t) \), then for \( \theta_1 + \theta_2 - 1 \geq (\leq) 1 \),

\[
\xi_{\theta_1, \theta_2}^w (X; t) \leq (\geq) \frac{1}{\theta_2 - \theta_1} \log m_x^e (t).
\]

Proof. Since \( \frac{F(x)}{F(t)} < 1 \) for \( x > t \), we have \( \frac{F(x)}{F(t)} \theta_1 + \theta_2 - 1 \leq (\geq) \frac{F(x)}{F(t)} \) for \( \theta_1 + \theta_2 - 1 \geq (\leq) 1 \). The result follows using Equation (7).

For illustration of Theorems 2.1 and 3.1, we consider exponential and Pareto distributions. The results are given in Table 2.
Definition 3.2. A non-negative continuous rv $X$ is said to be increasing (decreasing) generalized dynamic weighted survival entropy (IGDWSE (DGDWSE)), if $\xi^w_{\theta_1, \theta_2}(X; t)$ is increasing (decreasing) in $t$ $(\geq 0)$.

Theorem 3.2. A non-negative continuous rv $X$ is IGDWSE (DGDWSE) if and only if $\lambda_F(t) \geq (\leq) \frac{t}{\theta_1 + \theta_2 - 1} \exp\left[-(\theta_2 - \theta_1)\xi^w_{\theta_1, \theta_2}(X; t)\right]$, $\forall t \geq 0$, where $\lambda_F(t) = \frac{f(t)}{F(t)}$ is the hazard function.

Proof. We have

$$\left(\theta_2 - \theta_1\right) \int_0^t x^{\theta_1 + \theta_2 - 1} \right) dx - (\theta_1 + \theta_2 - 1) \log F(t).$$

Differentiating Equation (9) with respect to $t$, we get

$$\left(\theta_2 - \theta_1\right) \int_0^t x^{\theta_1 + \theta_2 - 1} \right) dx - (\theta_1 + \theta_2 - 1) \log F(t).$$

Using Equation (7), we get

$$\left(\theta_2 - \theta_1\right) \int_0^t x^{\theta_1 + \theta_2 - 1} \right) dx - (\theta_1 + \theta_2 - 1) \log F(t).$$

and the result follows from Equation (10).

Definition 3.3 (Shaked and Shanthikumar 2007). Let $X$ and $Y$ be two rvs with sfs $F(x)$ and $G(x)$, respectively. Then $X$ is said to be smaller than $Y$ in the usual stochastic ordering, denoted by $X \leq^s Y$, if $F(x) \leq G(x)$, for all $x$.

Definition 3.4. $X$ is said to be smaller than $Y$ in GWSE ordering, denoted by $X \leq^w Y$, if $\xi^w_{\theta_1, \theta_2}(X) \leq \xi^w_{\theta_1, \theta_2}(Y)$.

Theorem 3.3. Let $X$ and $Y$ be two non-negative continuous rvs with sfs $F(x)$ and $G(x)$, respectively. Then, $X \leq^w Y \Rightarrow X \leq^s Y$.

Proof. Proof easily follows using the definition of GWSE.

Definition 3.5 (Shaked and Shanthikumar 2007). $X$ is said to be smaller than $Y$ in hazard rate ordering, denoted by $X \leq^hr Y$, if $\lambda_F(t) \geq \lambda_G(t)$, $\forall t \geq 0$ or equivalently $\frac{G(t)}{F(t)}$ is increasing in $t$.

Definition 3.6. $X$ is said to be smaller than $Y$ in generalized dynamic weighted survival entropy ordering, denoted by $X \leq^dwse Y$, if $\xi^w_{\theta_1, \theta_2}(X; t) \leq \xi^w_{\theta_1, \theta_2}(Y; t)$.

Theorem 3.4. Let $X$ and $Y$ be two non-negative continuous rvs with sfs $F(x)$ and $G(x)$ and hazard rate functions $\lambda_F(t)$ and $\lambda_G(t)$, respectively. If $X \leq^hr Y$ then $X \leq^dwse Y$. 
Proof. Proof follows using the fact that, \( \frac{\tilde{F}(x)}{\tilde{G}(t)} \leq \frac{\tilde{G}(x)}{\tilde{G}(t)} \quad x \geq t. \)

**Theorem 3.5.** Let \( X \) and \( Y \) be two non-negative continuous rvs and \( X \leq (\geq) \) \( Y \). Let \( Z_1 = a_1 X \) and \( Z_2 = a_2 Y \), where \( a_1, a_2 > 0 \). Then \( Z_1 \leq (\geq) Z_2 \), if \( \xi_{\theta_1, \theta_2}^w (X; t) \) is decreasing in \( t > 0 \) and \( a_1 \leq (\geq) a_2 \).

Proof. Suppose \( a_1 \leq a_2 \). Since \( \xi_{\theta_1, \theta_2}^w (X; t) \) is decreasing in \( t \), we have, \( \xi_{\theta_1, \theta_2}^w (X; \frac{t}{a_1}) \leq \xi_{\theta_1, \theta_2}^w (X; \frac{t}{a_2}) \). Again, \( \xi_{\theta_1, \theta_2}^w (Y; \frac{t}{a_2}) \leq \xi_{\theta_1, \theta_2}^w (Y; \frac{t}{a_1}) \) since \( Y \leq Y \). Combining these two inequalities and using Remark 3.1, we have

\[
\xi_{\theta_1, \theta_2}^w (Z_1; t) = \frac{2 \log a_1}{\theta_2 - \theta_1} + \xi_{\theta_1, \theta_2}^w (X; \frac{t}{a_1}) \leq \frac{2 \log a_2}{\theta_2 - \theta_1} + \xi_{\theta_1, \theta_2}^w (Y; \frac{t}{a_2}) = \xi_{\theta_1, \theta_2}^w (Z_2; t).
\]

Hence, the result. Similarly, when \( a_1 \geq a_2 \) and \( X \geq Y \), it can be easily shown that \( Z_1 \geq Z_2 \).

The next theorem shows that \( \xi_{\theta_1, \theta_2}^w (X; t) \) uniquely determines the underlying survival function.

**Theorem 3.6.** Let \( X \) be a non-negative continuous rv having pdf \( f(x) \) and sf \( \tilde{F}(x) \). Assume that \( \xi_{\theta_1, \theta_2}^w (X; t) < +\infty; \ t \geq 0, \ \theta_2 - 1 < \theta_1 < \theta_2, \ \theta_2 \geq 1 \). Then \( \xi_{\theta_1, \theta_2}^w (X; t) \) uniquely determines the sf of \( X \).

Proof. From Equation (10), we have

\[
\lambda_F(t) = \frac{1}{\theta_2 - \theta_1} \left( (\theta_2 - \theta_1) \frac{d}{dt} \xi_{\theta_1, \theta_2}^w (X; t) + t \exp \left[ -(\theta_2 - \theta_1) \xi_{\theta_1, \theta_2}^w (X; t) \right] \right).
\] (11)

Now, let \( X_1 \) and \( X_2 \) be two rvs with sfs \( \tilde{F}_1(t) \) and \( \tilde{F}_2(t) \), GDWSES \( \xi_{\theta_1, \theta_2}^w (X_1; t) \) and \( \xi_{\theta_1, \theta_2}^w (X_2; t) \) and hazard functions \( \lambda_{F_1}(t) \) and \( \lambda_{F_2}(t) \), respectively.

Suppose \( \forall t \geq 0 \),

\[
\xi_{\theta_1, \theta_2}^w (X_1; t) = \xi_{\theta_1, \theta_2}^w (X_2; t)
\]

then from Equation (11) we get \( \lambda_{F_1}(t) = \lambda_{F_2}(t) \). Since hazard function uniquely determines the survival function of the underlying distribution, we conclude that,

\[
\tilde{F}_1(t) = \tilde{F}_2(t), \ \forall t \geq 0.
\]

4. Characterization results based on GWSE and GDWSE

In this section, we obtain some characterization results for Rayleigh distribution based on GDWSE.

**Theorem 4.1.** The rv \( X \) has constant GDWSE if and only if it has a Rayleigh distribution with survival function \( \tilde{F}(x) = e^{-\lambda x^2}; \ x \geq 0, \ \lambda > 0 \).
**Proof.** The if part of the theorem can be easily obtained by using Equation (7). For the only if part, let us assume that \( \tilde{z}_{\theta_1, \theta_2}(X; t) = c \). Differentiating with respect to \( t \) we have

\[
(\theta_1 + \theta_2 - 1)\lambda_F(t) - t \exp\left[-(\theta_2 - \theta_1)\tilde{z}_{\theta_1, \theta_2}(X; t)\right] = 0.
\]

This implies \( \lambda_F(t) = \frac{e^{\theta_1 - \theta_2}}{\theta_1 + \theta_2 - 1}t \), which is the hazard function of a Rayleigh distribution with survival function \( \bar{F}(t) = e^{-\lambda t^2}; \ t \geq 0 \), where \( \lambda = \frac{e^{\theta_1 - \theta_2}}{2(\theta_1 + \theta_2 - 1)} > 0 \) as \( \theta_1 + \theta_2 > 1 \).

**Theorem 4.2.** Let \( X \) be a continuous rv with absolutely continuous survival function \( F \). Then the relation \( (\theta_2 - \theta_1)\tilde{z}_{\theta_1, \theta_2}(X; t) = \log m^*_F(t) - \log (\theta_1 + \theta_2 - 1) \) holds if and only if \( X \) has a Rayleigh distribution.

**Proof.** If part of the theorem is straight-forward. For the only if part, assume \((\theta_2 - \theta_1)\tilde{z}_{\theta_1, \theta_2}(X; t) = \log m^*_F(t) - \log (\theta_1 + \theta_2 - 1)\) holds. Differentiating with respect to \( t \) we get

\[
(\theta_2 - \theta_1) \frac{d}{dt} \tilde{z}_{\theta_1, \theta_2}(X; t) = \frac{d}{dt} m^*_F(t) m_F(t) - m^*_F(t).
\]

Then by using Equation (10) we have

\[
(\theta_1 + \theta_2 - 1)\lambda_F(t) - t \exp\left[-(\theta_2 - \theta_1)\tilde{z}_{\theta_1, \theta_2}(X; t)\right] = \frac{d}{dt} m^*_F(t) m_F(t) - m^*_F(t).
\]

Note that \( m^*_F(t) = \int_{t}^{+\infty} x\frac{F(x)}{F(t)} dx \). This implies

\[
\frac{d}{dt} m^*_F(t) = \lambda_F(t) m^*_F(t) - t.
\]

By putting Equation (13) in Equation (12), after simplification, we get

\[
\lambda_F(t) m^*_F(t) = t,
\]

Combining Equations (13) and (14), we get \( \frac{d}{dt} m^*_F(t) = 0 \). This implies

\[
m^*_F(t) = c,
\]

where \( c \) is a constant. Using Equations (14) and (15), we obtain, \( \lambda_F(t) = \frac{1}{t} \), which is the hazard function of a Rayleigh distribution with survival function \( \bar{F}(t) = e^{\frac{-t^2}{2}} \).

Now we obtain a characterization result of the first order statistics based on GWSE. Let \( X_1, X_2, ..., X_n \) be a random sample of size \( n \) from \( F(x) \). Denote the corresponding order statistics as \( X_{1:n}, X_{2:n}, ..., X_{n:n} \), where \( X_{i:n} \) (\( 1 \leq i \leq n \)) is the \( i \)th-order statistic. The sf of \( X_{1:n} \) is given by \( \bar{F}_{1:n}(x) = \bar{F}^n(x) \) and GWSE of \( X_{1:n} \) is obtained as

\[
\tilde{z}_{\theta_1, \theta_2}(X_{1:n}) = \frac{1}{\theta_2 - \theta_1} \log \int_{0}^{+\infty} x\bar{F}^n(\theta_1 + \theta_2 - 1)(x) dx = \frac{1}{\theta_2 - \theta_1} \log \int_{0}^{1} \frac{\nu^{\theta_1 + \theta_2 - 1}F^{-1}(1 - v)}{f(F^{-1}(1 - v))} dv.
\]

The following lemma from Mirali and Baratpour (2017b) is useful to obtain the characterization result.
Lemma 4.1. If \( \eta \) is a continuous function on \([0, 1]\), such that \( \int_0^1 x^n \eta(x) dx = 0 \), for \( n \geq 0 \), then \( \eta(x) = 0, \ \forall x \in [0, 1] \).

Theorem 4.3. Let \( X \) and \( Y \) be two non-negative continuous rvs having common support \([0, +\infty)\) with cdfs \( F(x) \) and \( G(x) \), respectively. Then \( F(x) = G(x) \) if and only if \( \xi_{\theta_1, \theta_2}(X_{1:n}) = \xi_{\theta_1, \theta_2}(Y_{1:n}), \ \forall n \geq 1 \).

Proof. If \( \xi_{\theta_1, \theta_2}(X_{1:n}) = \xi_{\theta_1, \theta_2}(Y_{1:n}) \), then from Equation (16) we have

\[
\int_0^1 v^{\theta_1+\theta_2-1} \left[ \frac{F^{-1}(1-v)}{f(F^{-1}(1-v))} - \frac{G^{-1}(1-v)}{g(G^{-1}(1-v))} \right] dv = 0.
\]

Then from Lemma 3.1, we have \( F^{-1}(1-v) = \frac{G^{-1}(1-v)}{g(G^{-1}(1-v))} \) for all \( v \in (0, 1) \). This reduces to

\[
F^{-1}(w) \frac{d}{dw} F^{-1}(w) = G^{-1}(w) \frac{d}{dw} G^{-1}(w), \text{ where } w = 1 - v \text{ and } \frac{d}{dw} F^{-1}(w) = \frac{1}{f(F^{-1}(w))}. \]

So we have \( F^{-1}(w) = G^{-1}(w), 0 \leq w \leq 1 \). Hence the proof.

5. Generalized weighted failure and dynamic failure entropy of order \((\theta_1, \theta_2)\)

Definition 5.1. For a rv with finite support \([0, l]\), generalized weighted failure entropy (GWFE) of order \((\theta_1, \theta_2)\) is defined as

\[
f_{\xi_{\theta_1, \theta_2}}(X) = \frac{1}{\theta_2 - \theta_1} \log \int_0^l x^{\theta_1+\theta_2-1} \frac{F^{\theta_1}(x)}{F^{\theta_1}(1-x)} dx, \ \theta_2 \geq 1, \ \theta_2 - 1 < \theta_1 < \theta_2.
\] (17)

Example 5.1. Let \( X \) and \( Y \) be two rvs with pdfs \( f(u) = \frac{1}{a}, 0 < u < a \) and \( g(u) = \frac{1}{h}, h < u < a + h, \ h > 0 \), respectively. From Equation (3), we have \( f_{\xi_{\theta_1, \theta_2}}(X) = f_{\xi_{\theta_1, \theta_2}}(Y) = \frac{1}{\theta_2 - \theta_1} \log \frac{a}{\theta_1 + \theta_2} \). From Equation (17), we get

\[
f_{\xi_{\theta_1, \theta_2}}(X) = \frac{1}{\theta_2 - \theta_1} \log \left[ \frac{a^2}{\theta_1 + \theta_2 + 1} \right],
\]

\[
f_{\xi_{\theta_1, \theta_2}}(Y) = \frac{1}{\theta_2 - \theta_1} \log \left[ \frac{a(a(\theta_1 + \theta_2) + h(\theta_1 + \theta_2 + 1))}{(\theta_1 + \theta_2)(\theta_1 + \theta_2 + 1)} \right].
\]

Although \( f_{\xi_{\theta_1, \theta_2}}(X) = f_{\xi_{\theta_1, \theta_2}}(Y) \) but \( f_{\xi_{\theta_1, \theta_2}}(X) \neq f_{\xi_{\theta_1, \theta_2}}(Y) \).

The following Lemma shows the shift-dependency of GWFE.

Lemma 5.1. Suppose \( X \) has finite support \([0, l]\) and \( Z = aX + b \), where \( a > 0 \) and \( b \geq 0 \), then

\[
\exp\left[ (\theta_2 - \theta_1) f_{\xi_{\theta_1, \theta_2}}(Z) \right] = a^2 \exp\left[ (\theta_2 - \theta_1) f_{\xi_{\theta_1, \theta_2}}(X) \right] + ab \exp\left[ (\theta_2 - \theta_1) f_{\xi_{\theta_1, \theta_2}}(X) \right].
\] (18)

Proof. Proof is omitted because it is analogous to the one of Lemma 2.1.

Let \( F_{X_0}(x) \) and \( F(x) \) denote the distribution functions of the rvs \( X_0 \) and \( X \) having finite support, then proportional reversed hazard rate model (see, Gupta, Gupta, and
Table 3. GWFE for uniform and Power distribution, where $\gamma = \theta_1 + \theta_2 - 1$.

| Distribution | $$(\theta_2 - \theta_1)f_{X_{\theta_1,\theta_2}}(X)$$ | $$(\theta_2 - \theta_1)f_{X_{\theta_1,\theta_2}}(x_0)$$ | $$(\theta_2 - \theta_1)f_{X_{\theta_1,\theta_2}}(\theta x)$$ |
|--------------|--------------------------------|--------------------------------|--------------------------------|
| Uniform      | $\log \frac{\theta^2}{\theta_1^2}$ | $\log \frac{\theta^2}{\theta_1^2}$ | $\log \frac{\theta^2}{\theta_1^2}$ |
| Power        | $\log \frac{1}{\theta_1^{\theta_2}}$ | $\log \frac{1}{\theta_1^{\theta_2}}$ | $\log \frac{a^{\theta_2}}{\theta_1^{\theta_2}}$ |

Gupta (1998) is described by the relation $F_X(x) = [F(x)]^\theta$, where $\theta > 0$. The following Lemma compares the GWFE of $X$, $X_0$ and $\theta X$. Proofs are omitted.

**Lemma 5.2.** The following statements hold:

(a) $f_{\tilde{Z}_{\theta_1,\theta_2}}(X_0) = \left(\frac{\theta_2 - \theta_1 - \theta + 1}{\theta_2 - \theta_1}\right)f_{Z_{\theta_1,\theta_2}}(X)$;

(b) $f_{\tilde{Z}_{\theta_1,\theta_2}}(X_0) \leq f_{\tilde{Z}_{\theta_1,\theta_2}}(X) \leq f_{\tilde{Z}_{\theta_1,\theta_2}}(\theta X)$, if $\theta > 1$;

(c) $f_{\tilde{Z}_{\theta_1,\theta_2}}(X_0) \geq f_{\tilde{Z}_{\theta_1,\theta_2}}(X) \geq f_{\tilde{Z}_{\theta_1,\theta_2}}(\theta X)$, if $0 < \theta < 1$.

For illustration, we consider uniform distribution with cdf $F_U(x) = \frac{x}{a}$, $0 < x < a$, $a > 0$ and power distribution with cdf $F_P(x) = x^c$; $0 < x < 1$; $c > 0$. Lemma 5.2 can be easily verified by using the results in Table 3.

**Definition 5.2.** Generalized dynamic weighted failure entropy (GDWFE) of order $(\theta_1, \theta_2)$ of a rv $X$ is the GWFE of the rv $[t - X|X < t]$, $t > 0$. GDWFE of $X$ is defined as

$$f_{\tilde{Z}_{\theta_1,\theta_2}}(X; t) = \frac{1}{\theta_2 - \theta_1} \log \int_0^t x \frac{F_{\theta_1+\theta_2-1}(x)}{F_{\theta_1+\theta_2-1}(t)} dx, \quad \theta_2 \geq 1, \quad \theta_2 - 1 < \theta_1 < \theta_2.$$  

(19)

**Remark 5.1.** Let $X$ be a rv with bounded support $S$ and $l = \sup S < +\infty$. Then, $f_{\tilde{Z}_{\theta_1,\theta_2}}(X; l) = \tilde{Z}_{\theta_1,\theta_2}(X)$. Note that, GDWFE is defined even when the support is unbounded.

**Lemma 5.3.** Suppose $X$ has finite support $[0, l]$ and $Z = aX + b$, where $a > 0$ and $b \geq 0$, then

$$\exp\left[(\theta_2 - \theta_1)f_{\tilde{Z}_{\theta_1,\theta_2}}(Y; t)\right] = a^2 \exp\left[(\theta_2 - \theta_1)f_{\tilde{Z}_{\theta_1,\theta_2}}(X; \frac{t-b}{a})\right] + ab \exp\left[(\theta_2 - \theta_1)f_{\tilde{Z}_{\theta_1,\theta_2}}(X; \frac{t-b}{a})\right].$$

**Remark 5.2.** If $b = 0$, then from Lemma 5.3, we have

$$f_{\tilde{Z}_{\theta_1,\theta_2}}(Y; t) = \frac{2 \log a}{\theta_2 - \theta_1} + f_{\tilde{Z}_{\theta_1,\theta_2}}(X; \frac{t}{a}).$$  

(20)

Now we provide some bounds of GWFE and GDWFE in terms of weighted mean inactivity time (WMIT).
Definition 5.3 (Misagh et al. 2011). The WMIT of a rv $X$ is defined as

$$\mu_F(t) = \int_0^t x \frac{F(x)}{F(t)} \, dx, \quad F(t) > 0. \quad (21)$$

Theorem 5.1. Let $X$ be a non-negative continuous rv having finite support $[0, l]$ with WMIT $\mu_F(t)$, GWFE $f_{\xi_{\theta_1, \theta_2}}(X)$ and GDWFE $f_{\xi_{\theta_1, \theta_2}}(X; t)$. Then for $\theta_1 + \theta_2 - 1 \geq (\leq) 1$, we have

(i) $f_{\xi_{\theta_1, \theta_2}}(X) \leq (\geq) \frac{1}{\theta_2 - \theta_1} \log[\mu_F(l)];$

(ii) $f_{\xi_{\theta_1, \theta_2}}(X; t) \leq (\geq) \frac{1}{\theta_2 - \theta_1} \log[\mu_F(t)].$

Proof. Proofs are similar to that of Theorems 2.1 and 3.1.

Definition 5.4. A non-negative rv $X$ having finite support $[0, l]$ is said to be increasing (decreasing) generalized dynamic weighted failure entropy (IGDWFE (DGDWFE)), if $f_{\xi_{\theta_1, \theta_2}}(X; t)$ is increasing (decreasing) in $t$ ($\geq 0$).

Theorem 5.2. A non-negative continuous rv $X$ having finite support $[0, l]$ is IGDWFE (DGDWFE) if and only if

$$r_F(t) \leq (\geq) \frac{t}{\theta_1 + \theta_2 - 1} \exp[-(\theta_2 - \theta_1)f_{\xi_{\theta_1, \theta_2}}(X; t)], \quad \forall t \geq 0,$$

where $r_F(t) = \frac{f(t)}{F(t)}$ is the reversed hazard rate function.

Proof. Differentiating Equation (19), we get

$$(\theta_2 - \theta_1) \frac{d}{dt}f_{\xi_{\theta_1, \theta_2}}(X; t) = t \exp[-(\theta_2 - \theta_1)f_{\xi_{\theta_1, \theta_2}}(X; t)] - (\theta_1 + \theta_2 - 1)r_F(t). \quad (22)$$

The result follows from Equation (22).

Definition 5.5. Let $X$ and $Y$ be two rvs having finite support $[0, l]$. Then, $X$ is said to be smaller than $Y$ in generalized weighted failure entropy ordering, denoted by $X \preceq \text{we} Y$, if $f_{\xi_{\theta_1, \theta_2}}(X) \leq f_{\xi_{\theta_1, \theta_2}}(Y)$.

Definition 5.6 (Shaked and Shanthikumar 2007). $X$ is said to be smaller than $Y$ in reversed hazard rate ordering, denoted by $X \preceq \text{rh} Y$, if $r_F(t) \leq r_G(t)$, $\forall t \geq 0$ or equivalently $\frac{G(t)}{F(t)}$ is increasing in $t$.

Definition 5.7. Let $X$ and $Y$ be two rvs having finite support $[0, l]$. Then, $X$ is said to be smaller than $Y$ in generalized dynamic weighted failure entropy ordering, denoted by $X \preceq \text{dwfe} Y$, if $f_{\xi_{\theta_1, \theta_2}}(X; t) \leq f_{\xi_{\theta_1, \theta_2}}(Y; t)$, for $0 < t < l$. 


Theorem 5.3. Let $X$ and $Y$ be two continuous rv having finite support $[0, l]$ with cdfs $F$ and $G$, respectively. Then, $X \leq Y \Rightarrow X \geq Y$.

Theorem 5.4. Let $X$ and $Y$ be two non-negative continuous rv having finite support $[0, l]$ with cdfs $F$ and $G$ and reversed hazard functions $r_F(t)$ and $r_G(t)$, respectively. Then, $X \leq Y \Rightarrow X \geq Y$.

Proof. Proof follows using that if $X \leq Y$ then $\frac{F(x)}{F(t)} \geq \frac{G(x)}{G(t)}$.

Theorem 5.5. Let $X$ and $Y$ be two non-negative continuous rv having finite support $[0, l]$ and $X \leq (\geq) Y$. Let $Z_1 = a_1X$ and $Z_2 = a_2Y$, where $a_1, a_2 > 0$. Then $Z_1 \leq (\geq) Z_2$, if $f_{\xi_{\theta_1, \theta_2}}(x; t)$ is decreasing in $t > 0$ and $a_1 \leq (\geq) a_2$.

Proof. Proof follows along the same line as Theorem 3.5.

Next theorem shows that GDWFE uniquely determines the distribution function of the underlying distribution.

Theorem 5.6. Let $X$ be a non-negative continuous rv having pdf $f(x)$ and distribution function $F(x)$. Assume that, $f_{\xi_{\theta_1, \theta_2}}(X; t) < +\infty; \forall t \geq 0, \theta_2 - 1 < \theta_1 < \theta_2, \theta_2 \geq 1.$ Then for each $\theta_1$ and $\theta_2$, $f_{\xi_{\theta_1, \theta_2}}(x; t)$ uniquely determines the cdf of $X$.

Proof. From Equation (22), we have

$$r_F(t) = \frac{1}{\theta_1 + \theta_2 - 1} \left( t \exp[-(\theta_2 - \theta_1)f_{\xi_{\theta_1, \theta_2}}(X; t)] - (\theta_2 - \theta_1) \frac{d}{dt}f_{\xi_{\theta_1, \theta_2}}(X; t) \right). \ (23)$$

Let $F_1(t)$ and $F_2(t)$ be two distribution functions with generalized dynamic weighted failure entropies as $f_{\xi_{\theta_1, \theta_2}}(X_1; t)$ and $f_{\xi_{\theta_1, \theta_2}}(X_2; t)$ and the reversed hazard rate functions $r_{F_1}(t)$ and $r_{F_2}(t)$, respectively. Assume that $f_{\xi_{\theta_1, \theta_2}}(X_1; t) = f_{\xi_{\theta_1, \theta_2}}(X_2; t), \forall t \geq 0$ holds. Then from Equation (23), we have $r_{F_1}(t) = r_{F_2}(t).$ Since reversed hazard rate uniquely determines the distribution function of the underlying distribution, we obtain $F_1(t) = F_2(t)$.

Now we provide some characterization results for power distribution based on GDWFE.

Theorem 5.7. Let $X$ be a non-negative rv having support $(0, b)$, with absolutely continuous distribution function $F(x)$ and reversed hazard rate function $r_F(x)$. Then, $X$ has a power distribution with $F(x) = \left(\frac{x}{b}\right)^c, \ 0 < x < b, \ c > 0$ if and only if

$$(\theta_2 - \theta_1)f_{\xi_{\theta_1, \theta_2}}(X; t) = \log k + \log \mu^*_F(t),$$

where $\mu^*_F(t)$ is the WMIT function of $X$, $k(> 0)$ is a constant and $\theta_1 + \theta_2 \neq 2$.

Proof. If part is straight-forward. Suppose the relation $(\theta_2 - \theta_1)f_{\xi_{\theta_1, \theta_2}}(X; t) = \log k + \log \mu^*_F(t)$ holds. Differentiating with respect to $t$ and from Equation (22), we get
\[ t \exp \left[ - (\theta_2 - \theta_1) f_{\xi_{\theta_1, \theta_2}}(X; t) \right] = \frac{d}{dt} \mu_F^* (t) \cdot \frac{d}{dt} \mu_F^* (t). \] (24)

Substituting the value of \((\theta_2 - \theta_1) f_{\xi_{\theta_1, \theta_2}}(X; t)\) in Equation (24) and using the fact that \(d/dt \mu_F^* (t) = t - r_F(t) \mu_F^* (t)\) and after some calculation (24) reduces to

\[ r_F(t) \mu_F^* (t) = \frac{1 - k}{k(\theta_1 + \theta_2 - 2)} t. \] (25)

This implies

\[ \frac{d}{dt} \mu_F^* (t) = \frac{k(\theta_1 + \theta_2 - 1) - 1}{k(\theta_1 + \theta_2 - 2)} t. \]

Integrating with respect to \(t\) and taking \(\mu_F^* (0) = 0\), we get

\[ \mu_F^* (t) = \frac{k(\theta_1 + \theta_2 - 1) - 1 t^2}{k(\theta_1 + \theta_2 - 2)}. \]

From Equation (25), we obtain

\[ r_F(t) = \frac{2(1 - k)}{k(\theta_1 + \theta_2 - 1) - 1} \frac{1}{t} = \frac{c}{t}, \]

where \(c = \frac{2(1-k)}{k(\theta_1 + \theta_2 - 1) - 1} > 0\). Now to determine the appropriate ranges of \(k\), first suppose that \(\theta_1 + \theta_2 - 1 > 1\). Then, for \(c > 0\) we have \(1 > k > \frac{1}{\theta_1 + \theta_2 - 1}\). Again if \(\theta_1 + \theta_2 - 1 < 1\) then we have \(1 < k < \frac{1}{\theta_1 + \theta_2 - 1}\) for \(c > 0\). So we see that \(r_F(t)\) is the reversed hazard rate function of the power distribution with distribution function \(F(x) = \left(\frac{x}{b}\right)^c, 0 < x < b, c > 0\). Hence the result.

**Remark 5.3.** Note that for \(\theta_1 + \theta_2 = 2\), the above relation in Theorem 5.7 becomes an identity. Since \(\theta_1 + \theta_2 = 2 \Rightarrow \theta_1 + \theta_2 - 1 = 1\) and from Equation (19) we get, \((\theta_2 - \theta_1) f_{\xi_{\theta_1, \theta_2}}(X; t) = \log \mu_F^* (t)\).

Next, we obtain a characterization result of the largest order statistic based on GDWFE. The cdf of \(X_{n:n}\) is given by \(F_{n:n}(x) = F^n (x)\). Suppose \(X\) has finite support \([0, l]\) then the GDWFE of \(X_{n:n}\) is obtained as

\[ f_{\xi_{\theta_1, \theta_2}}(X_{n:n}; t) = \log \int_{0}^{l} \frac{1}{\theta_2 - \theta_1} x F^n(\theta_1 + \theta_2 - 1) (x) dx \]

\[ = \log \int_{0}^{l} \frac{1}{\theta_2 - \theta_1} \frac{1}{\nu^{\theta_1 + \theta_2 - 1} F^{-1}(\nu)} d\nu. \] (26)

**Theorem 5.8.** Let \(X\) and \(Y\) be two non-negative continuous rvs having common finite support \((0, l)\) with cdfs \(F(x)\) and \(G(x)\), respectively. Then \(F(x) = G(x)\) if and only if \(f_{\xi_{\theta_1, \theta_2}}(X_{n:n}; t) = f_{\xi_{\theta_1, \theta_2}}(Y_{n:n}; t)\), \(\forall n \geq 1\).

**Proof.** The only if part is straight forward. For the if part, assume that \(f_{\xi_{\theta_1, \theta_2}}(X_{n:n}; t) = f_{\xi_{\theta_1, \theta_2}}(Y_{n:n}; t)\) holds. Now from Equation (26), we have
\[ \int_0^1 \rho^{(\theta_1 + \theta_2 - 1)} \left[ \frac{F^{-1}(v)}{f(F^{-1}(v))} - \frac{G^{-1}(v)}{g(G^{-1}(v))} \right] dv. \]

Then from Lemma 4.1 we have \( \frac{F^{-1}(v)}{f(F^{-1}(v))} = \frac{G^{-1}(v)}{g(G^{-1}(v))} \) for all \( v \in (0, 1) \). The rest of the proof is similar to the proof of Theorem 4.3.

### 6. Some inequalities and bounds

In this section we provide some upper and lower bounds for generalized weighted survival and failure entropies and their dynamic versions.

**Theorem 6.1.** Let \( X \) be a non-negative continuous random variable with pdf \( f(x) \), cdf \( F(x) \) and sf \( \bar{F}(x) \). The following inequalities hold:

\[
\begin{align*}
(i) & \quad (\theta_2 - \theta_1) \bar{F}^{\theta_1, \theta_2}(X) + (\theta_1 + \theta_2 - 1) \geq H(X) + E(\log X); \\
(ii) & \quad (\theta_2 - \theta_1)f\bar{F}^{\theta_1, \theta_2}(X) + (\theta_1 + \theta_2 - 1) \geq H(X) + E(\log X).
\end{align*}
\]

Inequality (ii) is related to rvs having bounded support.

**Proof.** Using log-sum inequality we have

\[
\int_0^{+\infty} f(x) \log \frac{f(x)}{xF^{\theta_1 + \theta_2 - 1}(x)} \, dx \geq \log \int_0^{+\infty} \frac{f(x) \, dx}{xF^{\theta_1 + \theta_2 - 1}(x)} \int_0^{+\infty} f(x) \, dx
\]

\[
\quad = - \log \int_0^{+\infty} xF^{\theta_1 + \theta_2 - 1}(x) \, dx
\]

\[
\quad = - (\theta_2 - \theta_1) \bar{F}^{\theta_1, \theta_2}(X).
\]

Now the L.H.S. of Equation (27) equals

\[
\int_0^{+\infty} (\log f(x)) f(x) \, dx - \int_0^{+\infty} (\log x) f(x) \, dx - (\theta_1 + \theta_2 - 1) \int_0^{+\infty} \log \bar{F}(x) f(x) \, dx,
\]

which reduces to \(-H(X) - E(\log X) + (\theta_1 + \theta_2 - 1)\). The result follows from Equation (27). The proof of part (ii) is similar to that of part (i).

In the next theorem, we provide lower bound for GDWSE and GDWFE.

**Theorem 6.2.** For a non-negative continuous rv \( X \), the following inequalities hold:

\[
\begin{align*}
(i) & \quad (\theta_2 - \theta_1) \bar{F}^{\theta_1, \theta_2}(X; t) + (\theta_1 + \theta_2 - 1) \geq H(X; t) + \int_t^{+\infty} \frac{f(x)}{\bar{F}(t)} \log (x) \, dx; \\
(ii) & \quad (\theta_2 - \theta_1)f\bar{F}^{\theta_1, \theta_2}(X; t) + (\theta_1 + \theta_2 - 1) \geq H(X; t) + \int_0^t \frac{f(x)}{\bar{F}(t)} \log (x) \, dx.
\end{align*}
\]

**Proof.** (i). From log-sum inequality, we get

\[
\int_t^{+\infty} f(x) \log \frac{f(x)}{x (\bar{F}(x) / \bar{F}(t))^{\theta_1 + \theta_2 - 1}} \, dx \geq \log \int_t^{+\infty} \frac{f(x) \, dx}{x (\bar{F}(x) / \bar{F}(t))^{\theta_1 + \theta_2 - 1}} \int_t^{+\infty} f(x) \, dx
\]

\[
\quad = \bar{F}(t) \left[ \log \bar{F}(t) - (\theta_2 - \theta_1) \bar{F}^{\theta_1, \theta_2}(X; t) \right].
\]
After some simplifications, Equation (28) reduces to
\[
\int_{t}^{+\infty} f(x) \log \frac{f(x)}{x(F(x)/F(t))^{\theta_1+\theta_2-1}} \, dx \geq \int_{t}^{+\infty} (\log f(x)) f(x) \, dx - \int_{t}^{+\infty} (\log x) f(x) \, dx \\
+ (\theta_1 + \theta_2 - 1) \bar{F}(t).
\]

Using the definition of \( H(X; t) \) and after some simplifications, the results follows from Equation (28). Proof of part (ii) follows similarly.

Now we provide an upper bound for GDWSE and GDWFE.

**Theorem 6.3.** For a non-negative continuous rv \( X \) having support \([0, b]\), the following inequality holds:
\[
\frac{\bar{x}^w_{\theta_1, \theta_2}(X; t)}{\theta_2 - \theta_1} \leq \frac{\int_{t}^{b} x \left( \frac{F(x)}{F(t)} \right)^{(\theta_1+\theta_2-1)} \log \left[ x \left( \frac{F(x)}{F(t)} \right)^{(\theta_1+\theta_2-1)} \right] \, dx}{(\theta_2 - \theta_1)} + \frac{\log (b - t)}{\theta_2 - \theta_1}, \quad t < b.
\]

**Proof.** Using log-sum inequality we get,
\[
\int_{t}^{b} x \left( \frac{F(x)}{F(t)} \right)^{(\theta_1+\theta_2-1)} \log \left[ x \left( \frac{F(x)}{F(t)} \right)^{(\theta_1+\theta_2-1)} \right] \, dx \\
\geq \log \frac{\int_{t}^{b} x \left( \frac{F(x)}{F(t)} \right)^{(\theta_1+\theta_2-1)} \, dx}{b - t} \int_{t}^{b} \left( \frac{F(x)}{F(t)} \right)^{(\theta_1+\theta_2-1)} \, dx \\
= \left[ (\theta_2 - \theta_1) \frac{\bar{x}^w_{\theta_1, \theta_2}(X; t)}{\theta_2 - \theta_1} - \log (b - t) \right] \int_{t}^{b} \left( \frac{F(x)}{F(t)} \right)^{(\theta_1+\theta_2-1)} \, dx.
\]
The proof follows from Equation (29).

**Proposition 6.1.** Let \( X \) be a non-negative continuous rv. Then,
\[
f_{\bar{x}^w_{\theta_1, \theta_2}(X; t)} \leq \frac{\int_{t}^{b} x \left( \frac{F(x)}{F(t)} \right)^{(\theta_1+\theta_2-1)} \log \left[ x \left( \frac{F(x)}{F(t)} \right)^{(\theta_1+\theta_2-1)} \right] \, dx}{(\theta_2 - \theta_1)} + \frac{\log (t)}{\theta_2 - \theta_1}.
\]

**Proof.** Proof is similar to Theorem 6.3.

**7. Empirical GWSE and GWFE**

Let \( X_1, X_2, \ldots, X_n \) be a random sample of size \( n \) drawn from a distribution with cdf \( F(x) \) and \( X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n} \) be the corresponding order statistics. Let \( F_n(x) \) be the empirical distribution function (edf) of \( X \) which is defined as
The empirical GWSE is defined as

\[
\hat{w}_{\theta_1, \theta_2}(X) = \frac{1}{\theta_2 - \theta_1} \log \left( \int_0^{+\infty} x \hat{F}_n^{\theta_1 + \theta_2 - 1}(x) \, dx \right)
\]

Substituting \( \hat{F}_n(x) = 1 - \frac{1}{n}, \quad i = 1, 2, \ldots, n - 1 \) in Equation (30), we get

\[
\hat{w}_{\theta_1, \theta_2}(X) = \frac{1}{\theta_2 - \theta_1} \log \left( \sum_{i=0}^{n-1} x \hat{F}_n^{\theta_1 + \theta_2 - 1}(x) \right)
\]

where \( U_{i+1} = \frac{X_{i+1:n}^2 - X_{i:n}^2}{2} \) and \( X_{0:n} = 0 \).

Suppose \( X \) is a rv with bounded support \( S \) and \( \sup S = l \). Let \( X_1, X_2, \ldots, X_n \) be the corresponding order statistics and \( X_{(n+1):n} = l \). Then the empirical GWFE can be obtained as

\[
\hat{f}_{\theta_1, \theta_2}(X) = \frac{1}{\theta_2 - \theta_1} \log \left( \sum_{i=1}^{n} U_{i+1} \left( 1 - \frac{i}{n} \right)^{\theta_1 + \theta_2 - 1} \right)
\]

8. Application

In this section we consider the difference between \( \hat{w}_{\theta_1, \theta_2}(X) \) and its empirical version \( \hat{w}_{\theta_1, \theta_2}(X) \) as a test statistic for testing exponentiality. Let \( X_1, X_2, \ldots, X_n \) be iid rvs from a non-negative absolutely continuous cdf \( F \). Let \( F_0(x, \lambda) = 1 - e^{-\lambda x}, \ x > 0, \ \lambda > 0, \) denote the cdf of an exponential distribution with parameter \( \lambda \). We want to test the hypothesis

\[
H_0 : F(x) = F_0(x, \lambda) \ vs. \ H_1 : F(x) \neq F_0(x, \lambda).
\]

Now consider the absolute difference between \( \hat{w}_{\theta_1, \theta_2}(X) \) and \( \hat{w}_{\theta_1, \theta_2}(X) \) as \( D = |\hat{w}_{\theta_1, \theta_2}(X) - \hat{w}_{\theta_1, \theta_2}(X)| \). If \( X \sim \exp(\lambda) \), then \( \hat{w}_{\theta_1, \theta_2}(X) = \frac{2}{\theta_1 - \theta_2} \log(\hat{\lambda}(\theta_1 + \theta_2 - 1)) \) and \( D \) reduces to \( D = |\hat{w}_{\theta_1, \theta_2}(X) - \frac{2}{\theta_1 - \theta_2} \log(\hat{\lambda}(\theta_1 + \theta_2 - 1))| \), where \( \hat{\lambda} = 1/\hat{X} \) is the maximum likelihood estimator (mle) of \( \lambda \). \( D \) measures the distance between GWSE and empirical GWSE and large values of \( D \) indicate that the sample is from a non-exponential family. Note that, the statistic \( D \) is scale invariant. Now consider the monotone transformation \( T = \exp(-D) \), we have \( 0 < T < 1 \). Under the null hypothesis, \( D \overset{D}{\rightarrow} 0 \) and hence
Critical values of $T$

| $n$ | $T_{0.01,n}$ | $T_{0.05,n}$ | $T_{0.10,n}$ | $n$ | $T_{0.01,n}$ | $T_{0.05,n}$ | $T_{0.10,n}$ |
|-----|--------------|--------------|--------------|-----|--------------|--------------|--------------|
| 4   | 0.15452      | 0.18056      | 0.19797      | 22  | 0.30519      | 0.34955      | 0.38022      |
| 5   | 0.17084      | 0.19897      | 0.22030      | 23  | 0.30818      | 0.35743      | 0.39111      |
| 6   | 0.18293      | 0.21504      | 0.23581      | 24  | 0.31159      | 0.36074      | 0.39332      |
| 7   | 0.19529      | 0.23172      | 0.25352      | 25  | 0.32099      | 0.36921      | 0.40241      |
| 8   | 0.20218      | 0.24136      | 0.26676      | 26  | 0.32446      | 0.37260      | 0.40443      |
| 9   | 0.21664      | 0.25408      | 0.28284      | 27  | 0.33040      | 0.37802      | 0.41150      |
| 10  | 0.22490      | 0.26454      | 0.29226      | 28  | 0.33125      | 0.38068      | 0.41445      |
| 11  | 0.23571      | 0.27731      | 0.30546      | 29  | 0.33722      | 0.38670      | 0.41889      |
| 12  | 0.24354      | 0.28555      | 0.31272      | 30  | 0.33753      | 0.38741      | 0.42157      |
| 13  | 0.24885      | 0.29078      | 0.32062      | 35  | 0.35176      | 0.40709      | 0.44064      |
| 14  | 0.25726      | 0.30001      | 0.33029      | 40  | 0.37078      | 0.42298      | 0.45628      |
| 15  | 0.26489      | 0.31010      | 0.34048      | 45  | 0.38136      | 0.43473      | 0.47075      |
| 16  | 0.27563      | 0.31945      | 0.34892      | 50  | 0.39466      | 0.44945      | 0.48360      |
| 17  | 0.27848      | 0.32157      | 0.35377      | 60  | 0.41155      | 0.47032      | 0.50530      |
| 18  | 0.28509      | 0.33113      | 0.36187      | 70  | 0.43122      | 0.48648      | 0.52112      |
| 19  | 0.29347      | 0.33772      | 0.36784      | 80  | 0.45398      | 0.50520      | 0.54058      |
| 20  | 0.29454      | 0.34134      | 0.37401      | 90  | 0.46436      | 0.51746      | 0.55323      |
| 21  | 0.29890      | 0.34726      | 0.38058      | 100 | 0.47475      | 0.52805      | 0.56257      |

$T \overset{P}{\rightarrow} 1$. So we reject $H_0$ at the significance level $\alpha$ if $T < T_{x,n}$, where $T_{x,n}$ is the lower $\alpha$-quantile of the edf of $T$.

The sampling distribution of $T$ under $H_0$ is intractable. So to obtain the critical points $T_{x,n}$ by simulations we generate 10000 samples of size $n$ from a standard exponential distribution for $n = 4(1)30$, $30(5)50$ and $50(10)100$. For each $n$, the lower $\alpha$-quantile of the edf of $T$ is used to determine $T_{x,n}$. The critical points vary for different choices of $(\theta_1, \theta_2)$. The critical points of 90%, 95% and 99% are presented in Table 4 for $\theta_1 = 0.26$ and $\theta_2 = 1.25$.

8.1. Power comparison

We compare the power of the test with two other exponentiality tests based on entropy. Ebrahimi, Habibullah, and Soofi (1992) proposed a goodness-of-fit test for exponential distribution based on the Kullback–Leibler divergence and Baratpour and Rad (2012) proposed an exponentiality test based on cumulative residual entropy.

We compare the performance of $T$ with the test statistics proposed by Baratpour and Rad

$$T^* = \frac{\sum_{i=1}^{n-1} \frac{n-i}{n} \log \left( \frac{n-i}{n} \right) (X_{(i+1):n} - X_{i:n}) + \sum_{i=1}^{n} \frac{X_i^2}{2} \sum_{i=1}^{n} X_i}{2 \sum_{i=1}^{n} X_i},$$

and with the test statistics provided by Ebrahimi et al.

$$KL_{mn} = \exp \left( H_{mn} - \log \bar{x} - 1 \right),$$

where $H_{mn} = \frac{1}{n} \sum_{i=1}^{n} \log \left( \frac{n}{2m} (X_{(i+m):n} - X_{(i-m):n}) \right)$ is the Vasicek’s entropy estimate. The window size $m$ is a positive integer less that $\frac{n}{2}$, $X_{i:n} = X_{1:n}$ if $i < 1$ and $X_{i:n} = X_{n:n}$ if $i > n$. We reject the null hypothesis for large values of $T^*$ and for small values of $KL_{mn}$. 

We calculated the powers of the tests based on 10000 samples of size $n = 10(5)25$. We obtained the powers for significance level $\alpha = 0.01$ and $\alpha = 0.05$. For power computation we consider two alternative distributions Weibull $(p,1)$ with pdf $f_{W}(x) = px^{p-1}e^{-x^p}$, $x, p > 0$ and Gamma $(q,1)$ with pdf $f_{GA}(x) = e^{-x^q}/\Gamma(q)$, $x, q > 0$. The powers for Weibull and gamma alternatives are proposed in Tables 5 and 6, respectively. It is observed that the powers of the test $T$ is similar to that of $T^*$ but lower than that of $KL_{mn}$ for small sample size $n = 10$. However, for moderate to large sample sizes the proposed test $T$ behaves similar to $KL_{mn}$ and $T^*$.

### 8.2. Example

Consider the dataset given in Grubbs (1971) that provides the mileages for 19 military personnel carriers that failed in service. The mileages are 162, 200, 271, 320, 393, 508, 539, 629, 706, 778, 884, 1003, 1101, 1182, 1463, 1603, 1984, 2355, and 2880.

Here, $T = 0.41271$ and for $n = 19$ and $\alpha = 0.01$, from Table 4, we can obtain the critical point as $T_{0.01, 19} = 0.29347$. So we cannot reject the null hypothesis that the failure time follows exponential distribution.

### 9. Conclusion

We proposed generalized weighted survival and failure entropy and their dynamic versions and obtained some properties and bounds. We showed that generalized dynamic weighted survival and failure entropies uniquely determine the underlying distribution. Also characterization results based on smallest and largest order statistics are obtained.
We also provide the empirical versions of the entropy measures and using the difference between GWSE and its empirical version we perform test of exponentiality. The test depends on the choice of $h_1$ and $h_2$. Optimal choice of $h_1$ and $h_2$ is an important issue. One can choose $(h_1, h_2)$ in such a way that the asymptotic variances of the empirical generalized weighted survival and failure entropies are minimum. More work will be needed in this direction.

**Acknowledgments**

The authors express their sincere thanks to the reviewer for valuable comments which lead to an improvement over the earlier version of the manuscript.

**ORCID**

Siddhartha Chakraborty [http://orcid.org/0000-0002-2437-8396](http://orcid.org/0000-0002-2437-8396)
Biswaabrata Pradhan [http://orcid.org/0000-0001-6440-6876](http://orcid.org/0000-0001-6440-6876)

**References**

Abbasnejad, M. 2011. Some characterization results based on dynamic survival and failure entropies. *Communications for Statistical Applications and Methods* 18 (6):787–98. doi:[10.5351/CKSS.2011.18.6.787](https://doi.org/10.5351/CKSS.2011.18.6.787).

Abbasnejad, M., N. R. Arghami, S. Morgenthaler, and G. M. Borzadaran. 2010. On the dynamic survival entropy. *Statistics & Probability Letters* 80 (23/24):1962–71. doi:[10.1016/j.spl.2010.08.026](https://doi.org/10.1016/j.spl.2010.08.026).

---

**Table 6. Power comparison for the tests $T$, $T^*$ and $KL_{mn}$ when the alternative is the gamma distribution.**

| $n$ | $q$ | $x$ | $KL_{mn}$ | $T^*$ | $T$ |
|-----|-----|-----|-----------|-------|-----|
| 10  | 5   | 0.01| 0.7418    | 0.5652| 0.5334|
|     |     | 0.05| 0.9393    | 0.8306| 0.8160|
|     | 6   | 0.01| 0.8500    | 0.6876| 0.6567|
|     |     | 0.05| 0.9749    | 0.9155| 0.8962|
|     | 7   | 0.01| 0.9180    | 0.7856| 0.7430|
|     |     | 0.05| 0.9898    | 0.9578| 0.9369|
| 15  | 5   | 0.01| 0.9344    | 0.7858| 0.7636|
|     |     | 0.05| 1         | 0.9458| 0.9316|
|     | 6   | 0.01| 0.9786    | 0.8987| 0.8687|
|     |     | 0.05| 1         | 0.9823| 0.9725|
|     | 7   | 0.01| 0.9933    | 0.9498| 0.9221|
|     |     | 0.05| 1         | 0.9950| 0.9861|
| 20  | 5   | 0.01| 1         | 0.9006| 0.8752|
|     |     | 0.05| 1         | 0.9818| 0.9669|
|     | 6   | 0.01| 1         | 0.9677| 0.9433|
|     |     | 0.05| 1         | 0.9970| 0.9898|
|     | 7   | 0.01| 1         | 0.9908| 0.9773|
|     |     | 0.05| 1         | 0.9993| 0.9953|
| 25  | 5   | 0.01| 1         | 1     | 0.9403|
|     |     | 0.05| 1         | 1     | 0.9873|
|     | 6   | 0.01| 1         | 1     | 0.9793|
|     |     | 0.05| 1         | 1     | 0.9966|
|     | 7   | 0.01| 1         | 1     | 0.9923|
|     |     | 0.05| 1         | 1     | 0.9992|
Asadi, M., and Y. Zohrevand. 2007. On the dynamic cumulative residual entropy. *Journal of Statistical Planning and Inference* 137 (6):1931–41. doi:10.1016/j.jspi.2006.06.035.

Baratpour, S., and A. H. Rad. 2012. Testing goodness-of-fit for exponential distribution based on cumulative residual entropy. *Communications in Statistics – Theory and Methods* 41 (8):1387–96. doi:10.1080/03610926.2010.542857.

Belis, M., and S. Guiasu. 1968. A quantitative-qualitative measure of information in cybernetic systems (Corresp.). *IEEE Transactions on Information Theory* 14 (4):593–94. doi:10.1109/TIT.1968.1054185.

Bruhn, J., L. E. Lehmann, H. Röcke, T. W. Bouillon, and A. Hoeft. 2001. Shannon entropy applied to the measurement of the electroencephalographic effects of desflurane. *The Journal of the American Society of Anesthesiologists* 95 (1):30–35.

Camesasca, M., M. Kaufman, and I. Manas-Zloczower. 2006. Quantifying fluid mixing with the Shannon entropy. *Macromolecular Theory and Simulations* 15 (8):595–607. doi:10.1002/mats.200600037.

Chakraborty, S., and B. Pradhan. 2021. On weighted cumulative Tsallis residual and past entropy measures. *Communications in Statistics – Simulation and Computation*. Advance online publication. doi:10.1080/03610918.2021.1897623.

Cox, D. R. 1972. Regression models and life-tables. *Journal of the Royal Statistical Society: Series B (Methodological)* 34 (2):187–202.

Das, S. 2017. On weighted generalized entropy. *Communications in Statistics – Theory and Methods* 46 (12):5707–27. doi:10.1080/03610926.2014.960583.

Di Crescenzo, A., and M. Longobardi. 2002. Entropy-based measure of uncertainty in past lifetime distributions. *Journal of Applied Probability* 39 (02):434–40. doi:10.1017/S002190020002266X.

Di Crescenzo, A., and M. Longobardi. 2009. On cumulative entropies. *Journal of Statistical Planning and Inference* 139 (12):4072–87. doi:10.1016/j.jspi.2009.05.038.

Ebrahimi, N. 1996. How to measure uncertainty in the residual life time distribution. *Sankhya: The Indian Journal of Statistics, Series A* 58:48–56.

Ebrahimi, N., M. Habibullah, and E. S. Soofi. 1992. Testing exponentiality based on Kullback–Leibler Information. *Journal of the Royal Statistical Society: Series B (Methodological)* 54 (3):739–48.

Grubbs, F. E. 1971. Approximate fiducial bounds on reliability for the two parameter negative exponential distribution. *Technometrics* 13 (4):873–76. doi:10.1080/00401706.1971.10488858.

Gupta, R. C., P. L. Gupta, and R. D. Gupta. 1998. Modeling failure time data by Lehman alternatives. *Communications in Statistics – Theory and Methods* 27 (4):887–904. doi:10.1080/03610929808832134.

Kayal, S. 2015. On generalized dynamic survival and failure entropies of order $(\alpha, \beta)$. *Statistics & Probability Letters* 96:123–32. doi:10.1016/j.spl.2014.09.017.

Kayal, S. 2016. On generalized cumulative entropies. *Probability in the Engineering and Informational Sciences* 30 (4):640–62. doi:10.1017/S0269964816000218.

Khammar, A. H., and S. M. A. Jahanshahi. 2018. On weighted cumulative residual Tsallis entropy and its dynamic version. *Physica A: Statistical Mechanics and Its Applications* 491:678–92. doi:10.1016/j.physa.2017.09.079.

Mercurio, P. J., Y. Wu, and H. Xie. 2020. An entropy-based approach to portfolio optimization. *Entropy* 22 (3):332. doi:10.3390/e22030332.

Mirali, M., and S. Baratpour. 2017a. Dynamic version of weighted cumulative residual entropy. *Communications in Statistics – Theory and Methods* 46 (22):11047–59. doi:10.1080/03610926.2016.1257711.

Mirali, M., and S. Baratpour. 2017b. Some results on weighted cumulative entropy. *Journal of the Iranian Statistical Society* 16 (2):21–32.

Mirali, M., S. Baratpour, and V. Fakoor. 2017. On weighted cumulative residual entropy. *Communications in Statistics – Theory and Methods* 46 (6):2857–69. doi:10.1080/03610926.2015.1053932.
Misagh, F., Y. Panahi, G. H. Yari, and R. Shahi. 2011. Weighted cumulative entropy and its estimation. In 2011 IEEE International Conference on Quality and Reliability, 477–80. Bangkok: IEEE.

Nair, R. S., E. I. Abdul-Sathar, and G. Rajesh. 2017. A study on dynamic weighted failure entropy of order \( a \). American Journal of Mathematical and Management Sciences 36 (2):137–49. doi:10.1080/01966324.2017.1298063.

Nourbakhsh, M., G. Yari, and Y. Mehrali. 2020. Weighted entropies and their estimations. Communications in Statistics – Simulation and Computation 49 (5):1142–58. doi:10.1080/03610918.2016.1140776.

Psarrakos, G., and J. Navarro. 2013. Generalized cumulative residual entropy and record values. Metrika 76 (5):623–40. doi:10.1007/s00184-012-0408-6.

Rajesh, G., E. I. Abdul-Sathar, and S. N. Rohini. 2017. On dynamic weighted survival entropy of order \( a \). Communications in Statistics – Theory and Methods 46 (5):2139–50. doi:10.1080/03610926.2015.1033552.

Rao, M., Y. Chen, B. C. Vemuri, and F. Wang. 2004. Cumulative residual entropy: A new measure of information. IEEE Transactions on Information Theory 50 (6):1220–28. doi:10.1109/TIT.2004.828057.

Rényi, A. 1961. On measures of entropy and information. In Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics. Oakland, CA: The Regents of the University of California.

Shaked, M., and J. G. Shanthikumar. 2007. Stochastic orders. New York: Springer Science & Business Media.

Shannon, C. E. 1948. A mathematical theory of communication. Bell System Technical Journal 27 (3):379–423. doi:10.1002/j.1538-7305.1948.tb01338.x.

Varma, R. S. 1966. Generalizations of Renyi’s entropy of order \( a \). Journal of Mathematical Sciences 1:34–48.

Zografos, K., and S. Nadarajah. 2005. Survival exponential entropies. IEEE Transactions on Information Theory 51 (3):1239–46. doi:10.1109/TIT.2004.842772.