A simplified test of universality in Lattice QCD

David H. Adams
Instituut-Lorentz for Theoretical Physics, Leiden University, Niels Bohrweg 2, NL-2333 CA Leiden, The Netherlands. Email: adams@lorentz.leidenuniv.nl
(16 Dec.'03)

A simplified test of universality in Lattice QCD is performed by analytically evaluating the continuous Euclidean time limits of various lattice fermion determinants, both with and without a Wilson term to lift the fermion doubling on the Euclidean time axis, and comparing them with each other and with zeta-regularised fermion determinant in the continuous time—lattice space setting. The determinant relations expected from universality considerations are found to be violated by a certain gauge field-dependent factor, i.e. we uncover a “universality anomaly”. The physical significance, or lack thereof, of this factor is a delicate question which remains to be settled.

11.15.Ha, 12.38.Gc

The low-lying spectrum of the naive lattice Dirac operator approximates the low-lying spectrum of the continuum Dirac operator but with a 16-fold degeneracy due to fermion doubling [1]. Lattice QCD with a naive fermion is therefore regarded as a regularisation of continuum QCD with 16 degenerate fermion flavours. Lattice QCD with a staggered fermion [2] is regarded as a regularisation of 4 flavour QCD, in accordance with the fact that one naive fermion flavour is equivalent after spin diagonalisation to 4 staggered fermion flavours [3]. Lattice QCD with a Wilson fermion [1], where the Wilson term is added to the naive fermion action to lift the fermion doubling, is regarded as a regularisation of QCD with a single fermion flavour. Implicit in this is a \textit{universality hypothesis}: the LQCD’s with naive fermion, staggered fermion, and Wilson fermion are all in the right universality class to reproduce continuum QCD, the only difference being in the number of continuum fermion flavours the different lattice fermion formulations describe.

It is highly desirable to test this universality hypothesis wherever possible. This is particularly important in view of the fact that LQCD calculations with both dynamical Wilson and staggered fermions are currently being pursued at great effort and expense (see [4] for a recent review with references to the literature). An interesting quantity to consider in this context is the fermion determinant, which appears in the QCD functional integral when the fermions are dynamical. In LQCD with dynamical staggered fermions, the fourth root of the staggered fermion determinant is used as the fermion determinant for a single quark flavour. An important test of the universality hypothesis would therefore be to check whether the fourth power of the Wilson fermion determinant coincides with the staggered fermion determinant in the continuum limit, or, equivalently, whether the 16th power of the Wilson determinant coincides with the naive fermion determinant in this limit. Such a test appears analytically impossible with currently known techniques though. However, a simplified version of this test is feasible: instead of the full continuum limit one can take the continuum limit for one of the spacetime coordinates while keeping the remaining coordinates discrete. The effect of including in the fermion action a Wilson-type term to lift the fermion doubling along the chosen coordinate axis can then be investigated. In this paper we perform such a simplified test of universality by evaluating the continuous Euclidean time limits of the various lattice fermion determinants, both with and without the time part of the Wilson term in the action, and comparing them with each other and with the fermion determinant in the continuous time—lattice space setting. (The fermion determinant in the latter setting is defined via zeta-regularisation.)

On a finite spacetime lattice, with $N_\sigma$ sites along the spacial axes $\sigma = 1, 2, 3$, $N_\beta$ sites along the Euclidean time axis, time lattice spacing=$a$, spacial lattice spacing=$a'$, setting $\beta = aN_\beta=$time length (=inverse temperature in the finite temperature QCD setting), we consider lattice fermion actions of the form $S = a(a')^3 \sum_{(x,\tau)} \bar{\psi}(x,\tau)D^{(r)}(x,\tau)\psi(x,\tau)$ where $(x,\tau)$ runs over the lattice sites and

\begin{equation}
D^{(r)} = \gamma_4 \frac{1}{a} \nabla_4 + \frac{i}{2\beta} \Delta_4 + D_{\text{space}} + m
\end{equation}

\begin{equation}
\nabla_4\psi(x,\tau) = \frac{1}{2}(U_4(x,\tau)\psi(x,\tau + a) - U_4(x,\tau - a)^{-1}\psi(x,\tau - a))
\end{equation}

\begin{equation}
\Delta_4\psi(x,\tau) = 2\psi(x,\tau) - U_4(x,\tau)\psi(x,\tau + a) - U_4(x,\tau - a)^{-1}\psi(x,\tau - a)
\end{equation}

\begin{equation}
D^{(r)}_{\text{space}} = \frac{3}{\beta} \nabla_{\text{space}} + \frac{1}{2\beta} \Delta_{\text{space}}
\end{equation}

\begin{equation}
\nabla_{\text{space}}\psi(x,\tau) = \sum_{\sigma=1}^{3} \gamma_\sigma \frac{1}{2}(U_\sigma(x,\tau)\psi(x + \sigma,\tau) - U_\sigma(x - \sigma,\tau)^{-1}\psi(x - \sigma,\tau))
\end{equation}

\begin{equation}
\Delta_{\text{space}}\psi(x,\tau) = 2\psi(x,\tau) - U_\sigma(x,\tau)\psi(x + \sigma,\tau) - U_\sigma(x - \sigma,\tau)^{-1}\psi(x - \sigma,\tau)
\end{equation}
\[ \Delta_{\text{space}} \psi(x, \tau) = \sum_{\sigma=1}^{3} (2\psi(x, \tau) - U_\sigma(x, \tau)\psi(x + \hat{\sigma}, \tau) - U_\sigma(x - \hat{\sigma}, \tau)^{-1}\psi(x - \hat{\sigma}, \tau)) \]  

(6)

For \( r = r' = 0 \) this is the naive fermion action while for \( r = r' \neq 0 \) it is the Wilson action. We are going to evaluate the continuous time limit (\( a \rightarrow 0, N_\beta \rightarrow \infty \) with \( \beta = aN_\beta \) held fixed) of the fermion determinants \( \det D_\alpha^{(r)} \) at \( r = 0 \) and \( r = 1 \) and compare them with each other and with the zeta-regularised fermion determinant \( \det_\zeta D_\alpha \) in the continuous time—lattice space setting. The Dirac operator in the latter setting is

\[ D = \gamma_4(\frac{\partial}{\partial \tau} + A_4) + D_{\text{space}} + m \]  

(7)

with \( A_4(x, \tau) \) being the 4-component of a smooth continuum gauge field such that \( U_4(x, \tau) = Te^{\int_0^1 aA_4(x, \tau + (1-t)a)dt} \) (T=t-ordering) is the lattice transcript. The subscripts “\( \alpha \)” in \( D_\alpha^{(r)} \) and \( D_\alpha \) refer to the operators defined by replacing \( U_4 \rightarrow e^{-\alpha a}U_4 \) in (1)–(6) and \( A_4 \rightarrow A_4 - \alpha \) in (7), respectively. The role of the complex parameter \( \alpha \) is to incorporate the effect of a general boundary condition at the time boundaries: \( D_\alpha^{(r)} \) (resp. \( D_\alpha \)) with periodic time b.c. has the same spectrum and determinant as \( D^{(r)} \) (resp. \( D \)) with time b.c. \( \psi(x, \beta) = e^{a\beta}\psi(x, 0) \). Thus the introduction of \( \alpha \) allows us to always take periodic time b.c. when considering the fermion determinant. It can also be used to incorporate a chemical potential \( \mu \): QCD at finite temperature and density, where the fermion fields satisfy anti-periodic time b.c., corresponds to \( \alpha = \mu + i\pi/\beta \). The gauge fields are required to satisfy periodic time b.c. The spacial b.c.’s for the fermion and gauge fields will not play a role in our considerations; we can simply take them to be periodic.

The term \( \frac{\partial}{\partial \tau} \) in (1) is the “time part” of the usual Wilson term. It lifts the fermion doubling on the Euclidean time axis when \( r \neq 0 \). Therefore, if we think of the continuous time—lattice space setting as the “continuum setting”, then the aforementioned universality hypothesis, which relates the continuum limits of the naive, staggered and Wilson fermion determinants, translates into the following

**Simplified universality hypothesis:**

\[ \lim_{a \rightarrow 0} \det D_\alpha^{(0)} = \left( \lim_{a \rightarrow 0} \det D_\alpha^{(r)} \right)^2 \mod \text{p.i.f.’s} \]  

(8)

where “p.i.f.’s” refers to ‘physically inconsequential factors’. This is now something which can be checked analytically.

Our main technical result in this paper is the following

**Result of calculation:**

\[ \lim_{a \rightarrow 0} \det D_\alpha^{(0)} = \left( \lim_{a \rightarrow 0} \det D_\alpha^{(1)} \right)^2 e^{-\int_0^\beta \text{Tr} \left( \frac{\partial}{\partial \tau} \Delta_{\text{space}}(\tau) \right) d\tau} \mod \text{p.i.f.’s} \]  

(9)

where \( \Delta_{\text{space}}(\tau) \) is defined on the space of lattice spinor fields \( \psi(x) \), living only on the spacial lattice, by replacing \( \psi(x, \tau) \) by \( \psi(x) \) in (6). The p.i.f.’s in (9) are gauge field-independent factors whose only effects are to produce constant (vacuum) shifts in certain physical quantities. They include inverse powers of \( a \) which diverge in the \( a \rightarrow 0 \) limit.

The result (9) reveals a “universality anomaly”: the exponential factor in the right-hand side violates the universality expectation (8). Thus it is important to ascertain the significance, or lack thereof, of this factor. Since it is gauge field-dependent it cannot strictly speaking be regarded as a p.i.f. in the continuous time—lattice space theory. However, since the spacial Wilson term \( \frac{\partial}{\partial \tau} \Delta_{\text{space}} \) formally vanishes in the spacial continuum limit, one could argue that the exponential factor is effectively a p.i.f. when one goes on to take that limit. This is a delicate issue though, since \( \text{Tr} \left( \frac{\partial}{\partial \tau} \Delta_{\text{space}} \right) \) actually diverges in in the \( a' \rightarrow 0 \) limit (note that in the free field case the largest eigenvalue of \( \frac{1}{a'} \Delta_{\text{space}} \) is \( \sim \frac{1}{a'} \)). Further study is required to clarify this issue.

Since the reasoning which leads to the universality expectation (8) is the same as that which leads to the expectation that LQCD with naive, staggered and Wilson fermions are all in the same universality class, (9) is a potential reason for concern about whether the latter universality hypothesis really holds. It would therefore be a significant reassurance if the anomaly factor in (9) can be shown to be physically inconsequential. It should be noted, however, that even if this turns out not to be the case, it would not in itself invalidate the universality hypothesis for LQCD with naive, staggered and Wilson fermions, since the comparison between naive fermion \((r = r' = 0)\) and Wilson fermion \((r = r' \neq 0)\) is not covered by (9) and we could still be lucky in this case. But it would certainly raise a serious concern.

Remarkably, the anomaly in (9) is mirrored by an ambiguity in the zeta-regularised fermion determinant in the continuous time—lattice space setting. The latter can be expressed either as \( \det_\zeta D_\alpha \) or \( \det_\zeta (\gamma_4 D_\alpha) \) (since \( \bar{\psi} = \psi^* \gamma_4 \)). Formally, the determinants of \( D_\alpha \) and \( \gamma_4 D_\alpha \) coincide, but it turns out that the rigorously defined zeta-determinants do not. In fact we find \( \det_\zeta (\gamma_4 D_\alpha) = e^{-\frac{1}{2} \int_0^\beta \text{Tr} \left( \frac{\partial}{\partial \tau} \Delta_{\text{space}}(\tau) \right) d\tau} \det_\zeta D_\alpha \mod \text{p.i.f.’s} \), and
This shows that the lattice regularisations are consistent with zeta-regularisations of the fermion determinant in the continuous time—lattice space setting, and that the requirement that the anomaly factor in (9) be physically inconsequential is also necessary for consistency of continuous time—lattice space QCD when the fermion determinant is defined by zeta-regularisation.

In the remainder of this paper we sketch the derivation of (9)–(10) and give other, more explicit, expressions for the $a \to 0$ limits of $detD_{a}^{(0)}$ and $detD_{a}^{(1)}$. The full details are provided in [5]. It is convenient to regard $\psi(x, \tau)$ as a function $\Psi(\tau)$ living on the lattice sites of the Euclidean time axis and taking values in the vectorspace $W = \{\psi(x)\}$, i.e. the space of lattice spinor fields living on the spacial lattice only. Set $N := \dim W$. Define the linear operator $U_{d}(\tau)$ on $W$ by $(U_{d}(\tau)\psi)(x) = U_{d}(x, \tau)\psi(x)$. The operator $D_{space}(\tau)$ on $W$ is defined similarly by replacing $\psi(x, \tau)$ by $\psi(x)$ in (4)–(6). Since $\Psi(\beta) = \Psi(0)$ we can represent $\Psi$ by the vector $\hat{\Psi} = (\hat{\Psi}(0), \ldots, \hat{\Psi}(N_{\beta} - 1))$ where $\hat{\Psi}(k) = \Psi(ka)$. Then $D_{a}(\tau)$ is represented by

$$\hat{D}(\tau)\hat{\Psi}(k) = \hat{d}_{-1}^{(1)}(k)\hat{\Psi}(k - 1) + \hat{d}_{0}^{(1)}(k)\hat{\Psi}(k) + \hat{d}_{1}^{(1)}(k)\hat{\Psi}(k + 1)$$

where the operators $\hat{d}_{\pm 1}^{(1)}(k) : W \to W$ are given by

$$\hat{d}_{\pm 1}^{(1)}(k) = \frac{1}{2\alpha}((\gamma_{4} - r)\hat{U}_{d}(k), \hat{d}_{0}^{(1)}(1) = \frac{1}{2\alpha}(\gamma_{4} + r)\hat{U}_{d}(k - 1))^{-1}, \hat{d}_{0}^{(1)}(k) + \hat{D}_{space}(k) + m$$

and $\hat{D}_{space}(k) := D_{space}(ka)$. The generalisation of $\hat{D}(\tau)$ to $\hat{D}_{a}(\tau)$, given by $U_{d} \to e^{-\alpha\delta}U_{d}$, is equivalent to $\hat{d}_{\pm 1}^{(1)} \to e^{\mp\alpha\delta}\hat{d}_{\pm 1}^{(1)}$ in (11). After writing $\hat{D}(\tau)$ as an $N_{\beta} \times N_{\beta}$ matrix, its determinant can be straightforwardly evaluated via the method of [6]. The cases $r = 1$ and $r \neq 1$ require separate treatments due to the fact that $\hat{d}_{\pm 1}^{(1)}(k)$ is invertible when $r = 1$ but not when $r = 1$. The details of the calculations are provided in [5]; here we simply quote the results, assuming for convenience that $N_{\beta}$ is even in the $r \neq 1$ case:

$$detD_{a}^{(1)} = \left(\frac{1 - r^{2}}{2\alpha}\right)^{N_{\beta}}e^{-\alpha\beta N}det(\hat{D}(1) - \hat{D}(\tau))^{1/2}$$

$$detD_{a}^{(0)} = \left(\frac{1}{\alpha}\right)^{N_{\beta}}e^{-\alpha\beta N/2}$$

where $\hat{D}(k) := \frac{\hat{d}_{-1}^{(1)}}{d_{0}^{(1)}} \hat{D}(ka) + m$ (i.e. the scalar part of $\hat{D}_{space}(k) + m$). The linear maps $\hat{U}(\tau)(N_{\beta}/2)$ on $W \oplus W$ and $\hat{V}(N_{\beta})$ on $W$ in (12)–(13) are defined as follows. Exploiting the periodicity of the link variables to define $\hat{d}_{\pm 1}^{(1)}(k)$ for all $k \in \mathbb{Z}$, periodic under $k \to k + N_{\beta}$, and thereby define $\hat{D}(\tau)\hat{\Psi}(k)$ for all $k \in \mathbb{Z}$, we consider the equation $\hat{D}(\tau)\hat{\Psi}(k) = 0$ (no periodicity requirement on $\hat{\Psi}(k)$). In the $r \neq 1$ case, since the $\hat{d}_{\pm 1}^{(1)}(k)$’s are invertible, it is clear from (11) that solutions $\hat{\Psi}(k)$ are specified by two initial values. Thus the solution space is isomorphic to $W \oplus W$. Setting $\hat{\Psi}(1) = \hat{\Psi}(2n)$ and $\hat{\Psi}(2) = \hat{\Psi}(2n + 1)$ the solutions are determined from their initial values via an evolution operator:

$$\hat{\Psi}(1) = \hat{\Psi}(2n) = \hat{\Psi}(2n + 1)$$

The operator $\hat{U}(\tau)(N_{\beta}/2)$ appearing in (12) can also be characterised as follows: Due to the $N_{\beta}$–periodicity of the $\hat{d}_{\pm 1}^{(1)}(k)$’s in (11) there is a linear map on the solution space defined by $\hat{\Psi}(k) \to \hat{\Psi}(k + N_{\beta})$, or, equivalently, $(\hat{\Psi}_{1}(1), \hat{\Psi}_{2}(1), \ldots, \hat{\Psi}_{1}(N_{\beta}), \hat{\Psi}_{2}(N_{\beta}))$ $\to (\hat{\Psi}_{1}(1 + N_{\beta}), \hat{\Psi}_{2}(2 + N_{\beta}))$. This map coincides with $\hat{U}(\tau)(N_{\beta}/2)$ when the solution space is identified with $W \oplus W$.

In the $r = 1$ case, solutions to $\hat{D}(\tau)\hat{\Psi}(k) = 0$ are determined by just a single initial value; this is connected with the non-invertibility of $\hat{d}_{\pm 1}^{(1)}(k)$ and can be seen, e.g., from the expression (16) below. Thus the solution space in this case is isomorphic to $W$. The evolution operator $\hat{V}(\tau)$ determines solutions from their initial value through $\hat{\Psi}(k) = \hat{V}(\tau)\hat{\Psi}(0)$. The $\hat{V}(N_{\beta})$ in (13) can be alternatively characterised as the linear map on the solution space which maps $\hat{\Psi}(k) \to \hat{\Psi}(k + N_{\beta})$.

Finite difference approximations to differential operators in one variable and their determinants have been studied in [7,8] and we are going to use a convergence result from there. In the setting of [7,8], specialising to 1st order differential operator, the operator $\hat{L}$ and its finite difference approximation $\hat{L}$ have the forms

$$\hat{L} = L_{1}(\tau)\frac{d}{d\tau} + L_{0}(\tau)$$

$$\hat{L} = \hat{L}_{1}(k)\frac{1}{a}\hat{\partial} + \hat{L}_{0}(k)$$

($\tau \in \mathbb{R}, k \in \mathbb{Z}$) $\partial \in \{\hat{\partial}^{+}, \hat{\partial}^{-}\}$, $\hat{\partial}^{+}\hat{\Psi}(k) = \hat{\Psi}(k+1) - \hat{\Psi}(k)$, $\hat{\partial}^{-}\hat{\Psi}(k) = \hat{\Psi}(k) - \hat{\Psi}(k-1)$, with $L_{1}(\tau), \hat{L}_{j}(k) : W \to W$ being periodic under $\tau \to \tau + \beta$, $k \to k + N_{\beta}$, respectively, and
\[ \hat{L}_j(k) = L_j(ka) + O(a) \quad (j = 0, 1). \] (15)

Then the solutions to \( L\Psi(\tau) = 0 \) and \( \hat{L}\Psi(k) = 0 \) are both determined by a single initial value, so the solution spaces in both cases are isomorphic to \( W \). Solutions \( \hat{\Psi} \) approximate solutions \( \Psi \), i.e. if \( \hat{\Psi}(0) = \Psi(0) \) then \( \hat{\Psi}(k) \approx \Psi(ka) \) for small \( a \). Consequently the evolution operator \( \hat{U}(k) \) for \( \hat{L}\Psi = 0 \) approximates the evolution operator \( U(\tau) \) for \( L\Psi = 0 \). (Explicitly, \( U(\tau) = T e^{-\int_0^\tau L_1(t) \cdots L_0(t) dt} \) \( ) \). In particular one has the following (cf. §3 of [8]):

**Convergence Theorem:** \( \hat{U}(N_\beta) \to U(\beta) \) for \( a \to 0 \) with \( aN_\beta = \beta \) held fixed.

An obvious variant of this which we will make use of is the following. If \( \pm \) by \( D \) hold when \( \partial \), then the solutions \( \hat{\Psi}(\tau) \) of (14), or its aforementioned variant. We have only been able to do this in the \( r = 0 \) and \( r = 1 \) cases. The problem of evaluating \( \lim_{a \to 0} \det D_\alpha^{(r)} \) in the general \( r \) case therefore remains for future work; new techniques beyond those of [7,8] may be required for this. In the \( r = 1 \) case we specialise to a \( \gamma \)-representation where \( \gamma_4 = (\frac{1}{0} - \frac{0}{-1}) \) and decompose \( W = W_+ \oplus W_- \) so that \( \gamma_4 = \pm 1 \) on \( W_\pm \). Then, in terms of this decomposition,

\[ \hat{D}^{(1)} = \hat{L}_1^{(1)} \frac{1}{a} \left[ \begin{array}{cc} \partial^- & 0 \\ 0 & \partial^+ \end{array} \right] + \hat{L}_0^{(1)} \] (16)

where

\[ \hat{L}_1^{(1)}(k) = \gamma_4 \left( \begin{array}{cc} U_4((k-1)a)^{-1} & 0 \\ 0 & U_4(ka) \end{array} \right) \] (17)

\[ \hat{L}_0^{(1)}(k) = \gamma_4 \left( \frac{1}{a}(1 - U_4((k-1)a)^{-1}) - \frac{1}{a}(U_4(ka) - 1) \right) + D_{\text{space}}(ka) + m \] (18)

Clearly \( \hat{L}_j^{(1)}(k) \) is periodic under \( k \to k + N_\beta \) and

\[ \hat{L}_1^{(1)}(k) = \gamma_4 + O(a), \quad \hat{L}_0^{(1)}(k) = \gamma_4 A_4(ka) + D_{\text{space}}(ka) + m + O(a). \] (19)

The convergence theorem now gives \( \lim_{a \to 0} \hat{U}(N_\beta) = U(\beta) \) where \( U(\tau) \), acting on \( W \), is the evolution operator for \( D\Psi(\tau) = 0 \) with \( D \) being the Dirac operator (7) of the continuous time—lattice space setting. Using this and noting \( \det(1 + aM(k)) = e^{aTr \cdot M(k)} + O(a^2) \) the \( a \to 0 \) limit of (13) is now obtained:

\[ \lim_{a \to 0} \det D_\alpha^{(1)} = \left( \frac{1}{a} \right)^{NN_\beta} e^{-\alpha\beta N/2 + \frac{1}{2} \int_0^\beta Tr \cdot M(\tau) d\tau} \det(1 - e^{\alpha \beta} \Psi(\beta)) \] (20)

The gauge field-independent factor \( 1/a^{NN_\beta} \), which diverges in the \( a \to 0 \) limit, is physically inconsequential; it can at most give rise to an overall constant shift in the calculation of certain physical quantities (such as the energy density in finite temperature QCD).

An application of the zeta-regularised determinant formula for differential operators in one variable, Theorem 1 of [7], leads to an expression for \( \det \zeta D_\alpha \) which coincides with (20) without the \( 1/a^{NN_\beta} \) factor and with \( M(\tau) \) replaced by \( \pm M(\tau) \) (the details of this are given in [5]). The sign \( \pm \) depends on the choice of cut in the complex plane used to define the zeta-determinant. Choosing this so that the sign is “+” we then have \( \lim_{a \to 0} a^{NN_\beta} \det D_\alpha^{(1)} = \det \zeta D_\alpha \) which establishes the first part of (10).

In the \( r = 0 \) case, with \( \Psi \) represented by \( (\hat{\Psi}_1(n), \hat{\Psi}_2(n)) \) as before, we have

\[ \hat{D}^{(0)} = \hat{L}_1^{(0)} \frac{1}{2a} \left[ \begin{array}{cc} \partial^+ & 0 \\ 0 & \partial^- \end{array} \right] + \hat{L}_0^{(0)} \] (21)

where
\[ \hat{L}^{(0)}_1(n) = \begin{pmatrix} 0 & \hat{j}_1(n) \\ \hat{K}_1(n) & 0 \end{pmatrix}, \quad \hat{L}^{(0)}_0(n) = \begin{pmatrix} D_{\text{space}}(2na) + m & \hat{K}_0(n) \\ \hat{j}_0(n) & D_{\text{space}}((2n+1)a) + m \end{pmatrix} \]  

(22)

\[ \hat{j}_1(n) = \gamma_4 U_4((2n-1)a)^{-1}, \quad \hat{K}_1(n) = \gamma_4 U_4((2n+1)a) \]

(23)

\[ \hat{j}_0(n) = \gamma_4 \frac{1}{2a} \left( U_4(2na) - U_4((2n-1)a)^{-1} \right), \quad \hat{K}_0(n) = \gamma_4 \frac{1}{2a} \left( U_4((2n+1)a) - U_4(2na)^{-1} \right) \]

(24)

Clearly \( \hat{L}^{(0)}_j(n) \) is periodic under \( n \to n + N_\beta/2 \) and

\[ \tilde{L}^{(0)}_1(n) = \begin{pmatrix} 0 & \gamma_4 \\ \gamma_4 & 0 \end{pmatrix} + O(a), \quad \tilde{L}^{(0)}_0(n) = \begin{pmatrix} D_{\text{space}}(2na) + m & \gamma_4 A_4(2na) \\ \gamma_4 A_4(2na) & D_{\text{space}}(2na) + m \end{pmatrix} + O(a). \]

(25)

The convergence theorem then gives \( \lim_{a \to 0} \hat{U}(N_\beta/2) = \mathcal{U}(\beta) \) where \( \mathcal{U}(\beta) \), acting on \( W \oplus W \), is the evolution operator for \( \tilde{D} \left( \frac{q_1}{q_2} \right)(\tau) = 0 \) with

\[ \tilde{D} = \begin{pmatrix} D_{\text{space}}(\tau) + m & \gamma_4 \left( \frac{d}{d\tau} + A_4(\tau) \right) \\ \gamma_4 \left( \frac{d}{d\tau} + A_4(\tau) \right) & D_{\text{space}}(\tau) + m \end{pmatrix}. \]

(26)

Introducing \( \mathcal{O} = \left( \begin{array}{cc} 1 & 1 \\ -1 & 1 \end{array} \right) \) on \( W \oplus W \), we find after a little calculation \( \mathcal{O}^{-1} \tilde{D} \mathcal{O} = \left( \begin{array}{cc} (\gamma_4 \gamma_5)^{-1} D(\gamma_4 \gamma_5) & 0 \\ 0 & 0 \end{array} \right) D \), where \( D \) is the Dirac operator (7). It follows that \( \mathcal{U}(\beta) = \mathcal{O}^{-1} \left( \begin{array}{cc} (\gamma_4 \gamma_5)^{-1} \mathcal{V}(\beta)(\gamma_4 \gamma_5) & 0 \\ 0 & 0 \end{array} \right) \mathcal{O} \). Using this, the \( a \to 0 \) limit of (12) is now obtained:

\[ \lim_{a \to 0} \det D^{(0)}_{\alpha} = \left( \frac{1}{2a} \right)^{NN_\beta} e^{-\alpha \beta N} \det^2(1 - e^{\alpha \beta} \mathcal{V}(\beta)) \]

(27)

Again there is a physically inconsequential, divergent factor, \( (1/2a)^{NN_\beta} \). Comparing (27) with (20), and noting that \( e^\frac{1}{2} \int_0^\beta \text{Tr} M(\tau) d\tau = e^{m_\beta N/2} e^{\frac{1}{2}\int_0^\beta \text{Tr} \left( \frac{\tau^2}{4} \Delta_{\text{space}}(\tau) \right) d\tau} \) where \( e^{m_\beta N/2} \) is a p.i.f., we obtain the claimed result (9). Furthermore, an application of Theorem 1 of [7] gives \( \det \zeta(\gamma_4 D_{\alpha}) = e^{-(\bar{\gamma} \bar{\gamma}) \frac{1}{2} \beta_0 N/2} \det(1 - e^{\alpha \beta} \mathcal{V}(\beta)) \) where the sign \( \bar{\gamma} \) again depends on the choice of cut in the complex plane [5]. It follows that \( \lim_{a \to 0} (2a)^{NN_\beta} \det D^{(0)}_{\alpha} = e^{\pm \beta_0 N} \det \zeta(\gamma_4 D_{\alpha})^2 \).

Since \( e^{\pm \beta_0 N} \) is a p.i.f. this establishes the second part of (10).

In the free fermion case, \( Z = \det D^{(r)}_{\alpha} \) with \( \alpha = \mu + i\beta \) is the partition function of a free lattice Dirac fermion gas at temperature \( T = 1/\beta \) and chemical potential \( \mu \). After noting \( \det(1 - e^{\alpha \beta} \mathcal{V}(\beta)) = \det(1 - e^{\beta_0} e^{-\beta H}) \) where \( H = \gamma_4 (D_{\text{space}} + m) \) it is easy to check that physical quantities such as the mean energy \( \langle E \rangle = -\frac{\mu}{\beta} \log Z \) (with \( \beta \mu \) held constant) calculated from the expressions in this paper are consistent with the standard results described, e.g., in [9]. One subtlety is that in the \( r = 1 \) case \( \lim_{a \to 0} \langle E \rangle \) has an additional shift of \( -2 \sum_p \left( \frac{1}{2} \Delta_{\text{space}}(p) + m \right) \) due to the exponential factor in (20), while such a shift does not occur in the \( r = 0 \) case (cf. (27)). This shift can also be seen from the standard calculation in §18.11 of [9]: There the computation is reduced to a weighted sum of the residues of a certain function \( q(z)/z \). In addition to the poles at \( z = z_{\pm \delta} \) discussed there, there is also a pole at \( z = 0 \) and its residue is readily seen to give the aforementioned shift in \( \lim_{a \to 0} \langle E \rangle \). In the \( r = 0 \) case there is no pole at \( z = 0 \) and hence no corresponding shift in that case.

We have considered only the simplest lattice fermion formulations in this paper, namely the naive and Wilson fermions (with the connection to staggered fermions due to the relation between the naive and staggered fermion determinants). More recent formulations of major current interest are the domain wall [10] and overlap [11] fermions, as well as other chirally improved approaches to lattice fermions [13]. It appears feasible to evaluate the continuous time limit of the domain wall fermion determinant (with finite, discretised fifth dimension) along the same lines as here; this is currently under investigation. Doing this in the case of overlap fermions seems more technically difficult though. We remark that the overlap fermion determinant has previously been successfully tested (in both the chiral and vector cases) against known continuum expressions in 2 dimensions [12].

I thank Pierre van Baal and Claude Bernard for useful discussions and feedback. The author is supported by the European Commission, contract HPMF-CT-2002-01716.

---

[1] K.G. Wilson, Phys. Rev. D 10, (1974) 2445.
[2] J.B. Kogut, Phys. Rev. D 11, (1975) 395; L. Susskind, Phys. Rev. D 16 (1977) 3031.
[3] N. Kawamoto and J. Smit, Nucl. Phys. B 192 (1981) 100; H. Kluberg-Stern et al., Nucl. Phys. B 220 (1983) 447.
[4] K. Jansen, hep-lat/0311039.
[5] D.H. Adams, *On the QCD fermion determinant at finite temperature and chemical potential*. In preparation.
[6] P. Gibbs, Phys. Lett. B 172 (1986) 53.
[7] D. Burghelea, L. Friedlander and T. Kappeler, Commun. Math. Phys. 138 (1991) 1.
[8] R. Forman, Commun. Math. Phys. 147 (1992) 485.
[9] H.J. Rothe, *Lattice gauge theories: An introduction*. (2nd edition) World Scientific, Singapore, 1997.
[10] D.B. Kaplan, Phys. Lett. B 288 (1992) 342; Y. Shamir, Nucl. Phys. B 406 (1993) 90.
[11] H. Neuberger, Phys. Lett. B 417 (1998) 141.
[12] R. Narayanan and H. Neuberger, Phys. Lett. B 348 (1995) 549 (see also C.D. Fosco and S. Randjbar-Daemi, Phys. Lett. B 354 (1995) 383); T.-W. Chiu, Phys. Rev. D 58:074511 (1998).
[13] BGR Collaboration, hep-lat/0307013; W. Bietenholz, Nucl. Phys. B 644 (2002) 223.