How NP Got a New Definition: A Survey of Probabilistically Checkable Proofs*

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Abstract

We survey a collective achievement of a group of researchers: the PCP Theorems. They give new definitions of the class NP, and imply that computing approximate solutions to many NP-hard problems is itself NP-hard. Techniques developed to prove them have had many other consequences.

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1. PCP theorems: an informal introduction

Suppose a mathematician circulates a proof of an important result, say Riemann Hypothesis, fitting several thousand pages. To verify it would take you and your doubting colleagues several years. Can you do it faster? Yes, according to the PCP Theorems. He can rewrite his proof so you can verify it by probabilistically selecting (i.e., using a source of random bits) a constant number of bits—as low as 3 bits—to examine in it. Furthermore, this verification has the following properties: (a) A correct proof will never fail to convince you (that is, no choice of the random bits will make you reject a correct proof) and (b) An incorrect proof will convince you with only negligible probability ($2^{-100}$ if you examine 300 bits). In fact, a stronger assertion is true: if the Riemann hypothesis is false, then you are guaranteed to reject any string of letters placed before you with high probability after examining a constant number of bits. (c) This proof rewriting is completely mechanical—a computer could do it—and does not greatly increase its size. (Caveat: Before journal editors rush to adopt this new proof verification, we should mention that it currently requires proofs written in a formal axiomatic system—such as Zermelo Fraenkel set theory—since computers do not understand English.)

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This result has a strong ring of implausibility. A mathematical proof is invalid if it has even a single error somewhere. How can this error spread itself all over the rewritten proof, so as to be apparent after we have probabilistically examined a few bits in the proof? (Note that the simple idea of just making multiple copies of the erroneous line everywhere does not work: the unknown mathematician could hand you a proof in which this does not happen, yet that does not make the proof correct.) The methods used to achieve this level of redundancy are reminiscent of the theory of error-correcting codes, though they are novel and interesting in their own right, and their full implications are still being felt (see Section 3).

1.1. New definition of NP

The PCP Theorems provide interesting new definitions for the complexity class \( \text{NP} \). (Clarification: the singular form “PCP Theorem” will refer to a single result \( \text{NP} = \text{PCP}(\log n, 1) \) proved in \([3, 2]\), and the plural form “PCP Theorems” refers to a large body of results of a similar ilk, some predating the PCP Theorem.) Classically, \( \text{NP} \) is defined as the set of decision problems for which a “Yes” answer has a short certificate verifiable in polynomial time (i.e., if the instance size is \( n \), then the certificate size and the verification time is \( n^c \) for some fixed constant \( c \)). The following are two examples:

3-SAT = satisfiable boolean formulae of the form AND of clauses of size at most 3, e.g., \( (x_1 \lor \neg x_2 \lor x_3) \land (\neg x_1 \lor x_2 \lor x_3) \land (x_4) \). (The certificate for satisfiability is simply an assignment to the variables that makes the formula true.)

\( \text{MATH-THEOREM}_{\text{ZFC}} \) = set of strings of the form \( (T, 1^n) \) where \( T \) is a mathematical statement that is a theorem in Zermelo Fraenkel set theory that has a proof \( n \) bits long. (The “certificate” for theoremhood is just the proof.)

The famous conjecture \( \text{P} \neq \text{NP} \) —now one of seven Millenium Prize problems in math \([19]\)—says that not every \( \text{NP} \) problem is solvable in polynomial time. In other words, though the certificate is easy to check, it is not always easy to find.

The PCP Theorem gives a new definition of \( \text{NP} \): it is the set of decision problems for which a “Yes” answer has a polynomial-size certificate which can be probabilistically checked using \( O(\log n) \) random bits and by examining \( O(1) \) (i.e., constant) number of bits in it.

Our earlier claim about proof verification follows from the PCP Theorem, since \( \text{MATH-THEOREM}_{\text{ZFC}} \) is in \( \text{NP} \), and hence there is a way to certify a YES answer (namely, theoremhood) that satisfies properties (a) and (b). (Property (c) follows from the constructive nature of the proof of the PCP Theorem in \([3, 2]\).)

Motivated by the PCP Theorems, researchers have proved new analogous definitions of other complexity classes such as \( \text{PSPACE} \) \([22]\) and \( \text{PH} \) \([39]\).

1.2. Optimization, approximation, and PCP theorems

The \( \text{P} \) versus \( \text{NP} \) question is important because of \( \text{NP-completeness} \) (also, \( \text{NP-hardness} \)). Optimization problems in a variety of disciplines are \( \text{NP-hard} \) \([30]\), and so if \( \text{P} \neq \text{NP} \) they cannot be solved in polynomial time. The following is one such optimization problem.
MAX-3SAT: Given a 3-CNF boolean formula \( \phi \), find an assignment to the variables that maximizes the number of satisfied clauses.

Approximation algorithms represent a way to deal with NP-hardness. An algorithm achieves an approximation ratio \( \alpha \) for a maximization problem if, for every instance, it produces a solution of value at least \( \text{OPT} / \alpha \), where \( \text{OPT} \) is the value of the optimal solution. (For a minimization problem, achieving a ratio \( \alpha \) involves finding a solution of cost at most \( \alpha \text{OPT} \).) Note that the approximation ratio is \( \geq 1 \) by definition. For MAX-3SAT we now know a polynomial-time algorithm that achieves an approximation ratio \( 8/7 \) \[40\].

Though approximation algorithms is a well-developed research area (see \[38, 62\]), for many problems no good approximation algorithms have been found. The PCP Theorems suggest a reason: for many NP-hard problems, including MAX-CLIQUE, CHROMATIC NUMBER, MAX-3SAT, and SET-COVER, achieving certain reasonable approximation ratios is no easier than computing optimal solutions. In other words, approximation is NP-hard. For instance, achieving a ratio \( 8/7 - \epsilon \) for MAX-3SAT is NP-hard \[37\].

Why do the PCP Theorems lead to such results? Details appear in the survey \[1\] (and \[Feige 2002\], these proceedings), but we hint at the reason using 3SAT and MAX-3SAT as examples. Cook and Levin \[23, 46\] showed how to reduce any NP problem to 3SAT, by constructing, for any nondeterministic machine, a 3CNF formula whose satisfying assignments represent the transcripts of accepting computations. Thus it is difficult to satisfy all clauses. Yet it is easy to find assignment satisfying \( 1 - o(1) \) fraction of the clauses! The reason is that a computation transcript is a very non-robust object: changing even a bit affects its correctness. Thus the Cook-Levin reduction does not prove the inapproximability of MAX-3SAT. By providing a more robust representation of a computation, the PCP Theorems overcome this difficulty. We note that MAX-3SAT is a central problem in the study of inapproximability: once we have proved its inapproximability, other inapproximability results easily follow (see \[1\]; the observation in a weaker form is originally from work on MAX-SNP \[52\]).

1.3. History and context

PCPs evolved from interactive proofs, which were invented by Goldwasser, Micali, and Rackoff \[34\] and Babai \[5\] as a probabilistic extension of NP and proved useful in cryptography and complexity theory (see Goldreich’s survey \[31\]), including some early versions of PCPs \[24\]. In 1990, Lund, Fortnow, Karloff and Nisan \[48\] and Shamir \[59\] showed IP=PSPACE, thus giving a new probabilistic definition of PSPACE in terms of interactive proofs. They introduced a revolutionary algebraic way of looking at boolean formulae. In retrospect, this algebraization can also be seen as a “robust” representation of computation. The inspiration to use polynomials came from works on program checking \[17\] (see also \[17, 11, 18\]). Babai, Fortnow, and Lund \[7\] used similar methods to give a new probabilistic definition of NEXPTIME, the exponential analogue of NP. To extend this result to NP, Babai, Fortnow, Levin, and Szegedy \[8\] and Feige, Goldwasser, Lovász, Safra, and Szegedy \[20\] studied variants of what we now call probabilistically checkable proof
systems (Babai et al. called their systems holographic proofs).

Feige et al. also proved the first inapproximability result in the PCP area: if any polynomial-time algorithm can achieve a constant approximation ratio for the MAX-CLIQUE problem, then every NP problem is solvable in \( n^{O(\log \log n)} \) time. This important result drew everybody’s attention to the (as yet unnamed) area of probabilistically checkable proofs. A year later, Arora and Safra formalized and named the class PCP and used it to give a new probabilistic definition of NP. (Babai et al. and Feige et al.’s results were precursors of this new definition.) They also showed that approximating MAX-CLIQUE is NP-hard. Soon, Arora, Lund, Motwani, Sudan, and Szegedy proved the PCP Theorem (see below) and showed that MAX-SNP-hard problems do not have a PTAS if \( P \neq NP \). Since the second paper relied heavily on the still-unpublished first paper, the PCP theorem is jointly attributed to \[3,2\]. For surveys of these developments see \[6,31,39,50\].

2. Definitions and results

Now we define the class PCP. We will use “language membership” and “decision problem” interchangeably. A \((r(n),q(n))\)-restricted verifier for a language \(L\), where \(r, q\) are integer-valued functions, is a probabilistic turing machine \(M\) that, given an input of size \(n\), checks membership certificates for the input in the following way. The certificate is an array of bits to which the verifier has random-access (that is, it can query individual bits of the certificate).

- The verifier reads the input, and uses \(O(r(n))\) random bits to compute a sequence of \(O(q(n))\) addresses in the certificate.
- The verifier queries the bits at those addresses, and depending upon what they were, outputs “accept” or “reject”.
- \[\forall x \in L \exists \text{ certificate } \Pi \text{ s.t. } Pr[M^{\Pi} \text{ accepts}] = 1, \quad (2.1)\]
- \[\forall x \notin L \forall \text{ certificate } \Pi, \text{ Pr}[M^{\Pi} \text{ accepts}] \leq 1/2 \quad (2.2)\]

(In both cases the probability is over the choice of the verifier’s random string.)

PCP\((r(n),q(n))\) is the complexity class consisting of every language with an \((r(n),q(n))\)-restricted verifier. Since NP is the class of languages whose membership certificates can be checked by a deterministic polynomial-time verifier, NP = \(\cup_{c \geq 0} \text{PCP}(0,n^c)\). The PCP Theorem gives an alternative definition: NP = PCP\((\log n,1)\). Other PCP-like classes have been defined by using variants of the definition above, and shown to equal NP (when the parameters are appropriately chosen). We mention some variants and the best results known for them; these are the “PCP Theorems” alluded to earlier.

1. The probability 1 in condition (2.1) may be allowed to be \(c < 1\). Such a verifier is said to have imperfect completeness \(c\).
2. The probability 1/2 in condition (2.2) may be allowed to be \(s < c\). Such a verifier is said to have soundness \(s\). Using standard results on random walks


on expanders, it can be shown from the PCP theorem that every NP language has verifiers with perfect completeness that use $O(k)$ query bits for soundness $2^{-k}$ (here $k \leq O(\log n)$).

3. The number of query bits, which was $O(q(n))$ above, may be specified more precisely together with the leading constant. The constant is important for many inapproximability results. Building upon past results on PCPs and using Fourier analysis, Håstad recently proved that for each $\epsilon > 0$, every NP language has a verifier with completeness $1 - \epsilon$, soundness $1/2$ and only $3$ query bits. He uses this to show the inapproximability of MAX-3SAT up to a factor $8/7 - \epsilon$.

4. The free bit parameter may be used instead of query bits. This parameter is defined as follows. Suppose the query bit parameter is $q$. After the verifier has picked its random string, and picked a sequence of $q$ addresses, there are $2^q$ possible sequences of bits that could be contained in those addresses. If the verifier accepts for only $t$ of those sequences, then we say that the free bit parameter is $\log t$ (note that this number need not be an integer). Samorodnitsky and Trevisan show how to reduce the soundness to $2^{-k^2/4}$ using $k$ free bits.

5. Amortized free bits may be used. This parameter is $\lim_{s \to 0} f_s / \log(1/s)$, where $f_s$ is the number of free bits needed by the verifier to make soundness $< s$. Håstad shows that for each $\epsilon > 0$, every NP language has a verifier that uses $O(\log n)$ random bits and $\epsilon$ amortized free bits. He uses this to show (using a reduction from [26] and modified by [27]) that MAX-CLIQUE is inapproximable up to a factor $n^{1-\delta}$.

6. The certificate may contain not bits but letters from a larger alphabet $\Sigma$. The verifier’s soundness may then depend upon $\Sigma$. In a $p$ prover 1-round interactive proof system, the certificate consists of $p$ arrays of letters from $\Sigma$. The verifier is only allowed to query 1 letter from each array. Since each letter of $\Sigma$ is represented by $\lceil \log |\Sigma| \rceil$ bits, the number of bits queried may be viewed as $p \cdot \lceil \log |\Sigma| \rceil$. Constructions of such proof systems for NP appeared in [16, 45, 28, 14, 27, 53]. Lund and Yannakakis used these proof systems to prove inapproximability results for SETCOVER and many subgraph maximization problems. The best construction of such proof systems is due to Raz and Safra. They show that for each $k \leq \sqrt{\log n}$, every NP language has a verifier that uses $O(\log n)$ random bits, has $\log |\Sigma| = O(k)$ and soundness $2^{-k}$. The parameter $p$ is $O(1)$.

3. Proof of the PCP theorems

A striking feature of the PCP Theorems is that each builds upon the previous ones. However, a few ideas recur. First, note that it suffices to design verifiers for 3SAT since 3SAT is NP-complete and a verifier for any other language can transform the input to a 3SAT instance as a first step. The verifier then expects a certificate for a “yes” answer to be an encoding of a satisfying assignment; we define this next.
For an alphabet Σ let Σ\(^m\) denote the set of \(m\)-letter words. The \textit{distance} between two words \(x, y \in \Sigma^m\), denoted \(\delta(x, y)\), is the fraction of indices they differ on. For a set \(C \subseteq \Sigma^m\), let the \textit{minimum distance} of \(C\), denoted \(\text{min-dist}(C)\), refer to \(\min_{x, y \in C; x \neq y} \{\delta(x, y)\}\) and let \(\delta(x, C)\) stand for \(\min_{y \in C} \{\delta(x, y)\}\). If \(\text{min-dist}(C) = \gamma\), and \(\delta(x, C) < \gamma/2\), then triangle inequality implies there is a unique \(y \in C\) such that \(\delta(x, y) = \delta(x, C)\). We will be interested in \(C\) such that \(\text{min-dist}(C) \geq 0.5\); such sets are examples of \textit{error-correcting codes} from information theory, where \(C\) is thought of as a map from strings of \(\log |C|\) bits (“messages”) to \(C\). When encoded this way, messages can be recovered even if transmitted over a noisy channel that corrupts up to \(1/4\)th of the letters.

The probabilistically checkable certificate is required to contain the encoding of a satisfying assignment using some such \(C\). When presented with such a string, the verifier needs to check, first, that the string is close to some codeword, and second, that the (unique) closest codeword is the encoding of a satisfying assignment. As one would expect, the set \(C\) is defined using mathematically interesting objects (polynomials, monotone functions, etc.) so the final technique may be seen as a “lifting” of the satisfiability question to some mathematical domain (such as algebra). The important new angle is “local checkability,” namely, the ability to verify global properties by a few random spot-checks. (See below.)

Another important technique introduced in \cite{3} and used in all subsequent papers is \textit{verifier composition}, which composes two verifiers to give a new verifier some of whose parameters are lower than those in either verifier. Verifier composition relies on the notion of a \textit{probabilistically checkable split-encoding}, a notion to which Arora and Safra were led by results in \cite{8}. (Later PCP Theorems use other probabilistically checkable encodings: \textit{linear function codes} \cite{2}, and \textit{long codes} \cite{13, 36, 37}.) One final but crucial ingredient in recent PCP Theorems is \textit{Raz’s parallel repetition theorem} \cite{53}.

### 3.1. Local tests for global properties

The key idea in the PCP Theorems is to design probabilistic local checks that verify global properties of a provided certificate. Designing such local tests involves proving a statement of the following type: if a certain object satisfies some local property “often” (say, in \(90\%\) of the local neighborhoods) then it satisfies a global property. Such statements are reminiscent of theorems in more classical areas of math, e.g., those establishing properties of solutions to PDEs, but the analogy is not exact because we only require the local property to hold in most neighborhoods, and not all.

We illustrate with some examples. (A research area called \textit{Property Testing} \cite{55} now consists of inventing such local tests for different properties.) There is a set \(C \subseteq \Sigma^m\) of interest, with \(\text{min-dist}(C) \geq 0.5\). Presented with \(x \in \Sigma^m\), we wish to read “a few” letters in it to determine whether \(\delta(x, C)\) is small.

1. \textit{Linearity test}. Here \(\Sigma = GF(2)\) and \(m = 2^n\) for some integer \(n\). Thus \(\Sigma^m\) is the set of all functions from \(GF(2)^n\) to \(GF(2)\). Let \(C_1\) be the set of words that correspond to \textit{linear functions}, namely, the set of \(f: GF(2)^n \to GF(2)\) such that \(\exists a_1, \ldots, a_n \in GF(2)\) s.t. \(f(z_1, z_2, \ldots, z_n) = \sum_i a_i z_i\). The
test for linearity involves picking $\overline{z}, \overline{u} \in GF(2)^n$ randomly and accepting if $f(\overline{z}) + f(\overline{u}) = f(\overline{z} + \overline{u})$. Let $\gamma$ be the probability that this test does not accept. Using elementary fourier analysis one can show $\gamma \geq \delta(f, C_1)/2$ (see also earlier weaker results in \cite{18}).

2. Low Degree Test. Here $\Sigma = GF(p)$ for a prime $p$ and $m = p^n$ for some $n$. Thus $\Sigma^m$ is the set of all functions from $GF(p)^n$ to $GF(p)$. Let $C_2$ be the set of words that correspond to polynomials of total degree $d$, namely, the set of $f : GF(p)^n \to GF(p)$ such that there is a $n$-variate polynomial $g$ of degree $d$ and $f(z_1, z_2, \ldots, z_n) = g(z_1, z_2, \ldots, z_n)$. We assume $dn \ll p$ (hence degree is “low”). Testing for closeness to $C_2$ involves picking random lines. A line has the parametric form $\{(a_1 + b_1 t, a_2 + b_2 t, \ldots, a_n + b_n t) : t \in GF(p)\}$ for some $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in GF(p)$. (It is a 1-dimensional affine subspace, hence much smaller than $GF(p)^n$.) Note that if $f$ is described by a degree $d$ polynomial, then its restriction to such a line is described by a univariate degree $d$ polynomial in the line parameter $t$.

- Variant 1: Pick a random line, read its first $d + 1$ points to construct a degree $d$ univariate polynomial, and check if it describes $f$ at a randomly chosen point of the line. This test appears in \cite{56} and is similar to another test in \cite{26}.
- Variant 2: This test uses the fact that in the PCP setting, it is reasonable to ask that the provided certificate should contain additional useful information to facilitate the test. We require, together with $f$, a separate table containing a degree $d$ univariate polynomial for the line. We do the test above, except after picking the random line we read the relevant univariate polynomial from the provided table. This has the crucial benefit that we do not have to read $d + 1$ separate “pieces” of information from the two tables. If $\gamma$ is the probability that the test rejects, then $\gamma \geq \min \{0.1, \delta(f, C_2)/2\}$ (see \cite{2}; which uses \cite{56, 3}).

3. Closeness to a small set of codewords. Above, we wanted to check that $\delta(f, C) < 0.1$, in which case there is a unique word from $C$ in Ball($f, 0.1$). Proofs of recent PCP Theorems relax this and only require for some $\epsilon$ that there is a small set of words $S \subseteq C$ such that each $s \in S$ lies in Ball($f, \epsilon$). (In information theory, such an $S$ is called a list decoding of $f$.) We mention two important such tests.

For degree $d$ polynomials: The test in Variant 2 works with a stronger guarantee: if $\beta$ is the probability that the test accepts, then there are poly$(1/\epsilon)$ polynomials whose distance to $f$ is less than $1 - \epsilon$ provided $p > poly(nd/\beta \epsilon)$ (see \cite{4}, and also \cite{54} for an alternative test).

Long Code test. Here $\Sigma = GF(2)$ and $m = 2^n$ for some integer $n$. Thus $\Sigma^m$ is the set of all functions from $GF(2)^n$ to $GF(2)$. Let $C_3$ be the set of words that correspond to coordinate functions, namely,

$$\{f : GF(2)^n \to GF(2) : \exists i \in \{1, 2, \ldots, n\} \text{ s.t. } f(z_1, z_2, \ldots, z_n) = z_i.\}$$

(This encodes $i \in [1, n]$, i.e., log $n$ bits of information, using a string of length $2^n$, hence the name “Long Code”.) The following test works \cite{37}, though we do
not elaborate on the exact statement, which is technical: Pick \( \pi, w \in GF(2)^n \) and \( \pi \in GF(2)^n \) that is a random vector with 1’s in \( \epsilon \) fraction of the entries. Accept iff \( f(\pi + w) = f(\pi) + f(w + \pi) \). (Note the similarity to the linearity test above.)

3.2. Further applications of PCP techniques

We list some notable applications of PCP techniques. The PCP Theorem is useful in cryptography because many cryptographic primitives involve basic steps that prove Yes/No assertions that are in NP (or even P). The PCP Theorem allows this to be done in a communication-efficient manner. See [12, 51, 10] for some examples. Some stronger forms of the PCP Theorem (specifically, a version involving encoded inputs) have found uses in giving new definitions for polynomial hierarchy [43] and PSPACE [21, 22]. Finally, the properties of polynomials and polynomial-based encodings discovered for use in PCP Theorems have influenced new decoding algorithms for error-correcting codes [35], constructions of pseudorandom graphs called extractors [61, 57] and derandomization techniques in complexity theory (e.g. [60]).

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