HOW FAR ARE P-ADIC LIE GROUPS FROM ALGEBRAIC GROUPS?

YVES BENOIST AND JEAN-FRANÇOIS QUINT

Abstract. We show that, in a weakly regular $p$-adic Lie group $G$, the subgroup $G_u$ spanned by the one-parameter subgroups of $G$ admits a Levi decomposition. As a consequence, there exists a regular open subgroup of $G$ which contains $G_u$.

Contents

1. Introduction
   1.1. Motivations
   1.2. Main results
   1.3. Plan

2. Preliminary results
   2.1. One-parameter subgroups
   2.2. Weakly regular and regular $p$-adic Lie groups

3. Algebraic unipotent $p$-adic Lie group
   3.1. Definition and closedness
   3.2. Lifting one-parameter morphisms
   3.3. Unipotent subgroups tangent to a nilpotent Lie algebra
   3.4. Largest normal algebraic unipotent subgroup

4. Derivatives of one-parameter morphisms
   4.1. Construction of one-parameter subgroups
   4.2. The group $G_{nc}$ and its Lie algebra $g_{nc}$
   4.3. Derivatives and Levi subalgebras

5. Groups spanned by unipotent subgroups
   5.1. Semisimple regular $p$-adic Lie groups
   5.2. The Levi decomposition of $G_u$
   5.3. Regular semiconnected component
   5.4. Non weakly regular $p$-adic Lie groups

References

2020 Math. subject class. 22E20 ; Secondary 19C09
Key words p-adic Lie group, algebraic group, unipotent subgroup, Levi decomposition, central extension
1. Introduction

1.1. Motivations. When studying the dynamics of the subgroups of a $p$-adic Lie group $G$ on its homogeneous spaces, various assumptions can be made on $G$. For instance, one can ask $G$ to be algebraic as in [10] i.e. to be a subgroup of a linear group defined by polynomial equations.

Another possible assumption on $G$ is regularity. One asks $G$ to satisfy properties that are well-known to hold for Zariski connected algebraic $p$-adic Lie groups: a uniform bound on the cardinality of the finite subgroups and a characterization of the center as the kernel of the adjoint representation (see Definition 2.5). Ratner’s theorems in [13] are written under this regularity assumption.

A more natural and weaker assumption on $G$ in this context is the weak regularity of $G$ i.e. the fact that the one-parameter morphisms of $G$ are uniquely determined by their derivative (see Definition 2.3). Our paper [1] is written under this weak-regularity assumption.

The aim of this text is to clarify the relationships between these three assumptions.

A key ingredient in the proof is Proposition 5.1. It claims
\[ \text{the finiteness of the center of the universal topological central extension of non-discrete simple } p \text{-adic Lie groups}. \]
This is a fact which is due to Prasad and Raghunathan in [12].

1.2. Main results. We will first prove (Proposition 5.5):

\[ \text{In a weakly regular } p \text{-adic Lie group } G, \text{ the subgroup } G_u \text{ spanned by the one-parameter subgroups of } G \text{ is closed and admits a Levi decomposition, } \]
i.e. $G_u$ is a semidirect product $G_u = S_u \ltimes R_u$ of a group $S_u$ by a normal subgroup $R_u$ where $S_u$ is a finite cover of a finite index subgroup of an algebraic semisimple Lie group and where $R_u$ is algebraic unipotent.

As a consequence, we will prove (Theorem 5.12):

\[ \text{In a weakly regular } p \text{-adic Lie group } G \text{ there always exists an open subgroup } G_\Omega \text{ which is regular and contains } G_u. \]
This Theorem 5.12 is useful since it extends the level of generality of Ratner’s theorems in [13] (see [1, Th 5.15]). More precisely, Ratner’s theorems for products of real and $p$-adic Lie groups in [13] are proven under the assumption that these $p$-adic Lie groups are regular. Ratner’s theorems can be extended under the weaker assumption that these $p$-adic Lie groups are weakly regular thanks to our Theorem 5.12.
This Theorem 5.12 has been announced in [1, Prop. 5.8] and has been used in the same paper.

The strategy consists in proving first various statements for weakly regular $p$-adic Lie groups which were proven in [13] under the regularity assumption. To clarify the discussion we will reprove also the results from [13] that we need. But we will take for granted classical results on the structure of simple $p$-adic algebraic groups that can be found in [3], [6], [7] or [11].

1.3. Plan. In the preliminary Chapter 2 we recall a few definitions and examples.

Our main task in Chapters 3 and 4 is to describe, for a weakly regular $p$-adic Lie group $G$, the subset $\mathfrak{g}_G \subset \mathfrak{g}$ of derivatives of one-parameter morphisms of $G$.

In Chapter 3 the results are mainly due to Ratner. We first study the nilpotent $p$-adic Lie subgroups $N$ of $G$ spanned by one-parameter subgroups. We will see that they satisfy $n \subset \mathfrak{g}_G$ (Proposition 3.7). Those $p$-adic Lie groups $N$ are called algebraic unipotent. We will simultaneously compare this set $\mathfrak{g}_G$ with the analogous set $\mathfrak{g}_{G'}$ for a quotient group $G' = G/N$ of $G$ when $N$ is a normal algebraic unipotent subgroup (Lemma 3.8). This will allow us to prove that $G$ contains a largest normal algebraic unipotent subgroup (Proposition 3.8).

In Chapter 4 we will then be able to describe precisely the set $\mathfrak{g}_G$ using a Levi decomposition of $\mathfrak{g}$ (Proposition 4.4). A key ingredient is a technic, borrowed from [1], for constructing one-parameter subgroups in a $p$-adic Lie group $G$ (Lemma 4.1).

In the last Chapter 5, we will prove the main results we have just stated: Proposition 5.5 and Theorem 5.12 using Prasad–Raghunathan finiteness theorem in [12] (see Proposition 5.1). We will end this text by an example showing that, when a $p$-adic Lie group $G$ with $G = G_u$ is not assumed to be weakly regular, it does not always contain a regular open subgroup $H$ for which $H = H_u$ (Example 5.14).

2. Preliminary results

We recall here a few definitions and results from [13].

Let $p$ be a prime number and $\mathbb{Q}_p$ be the field of $p$-adic numbers. When $G$ is a $p$-adic Lie group (see [5]), we will always denote by $\mathfrak{g}$ the Lie algebra of $G$. It is a $\mathbb{Q}_p$-vector space. We will denote by $\text{Ad}_\mathfrak{g}$ or $\text{Ad}$ the adjoint action of $G$ on $\mathfrak{g}$ and by $\text{ad}_\mathfrak{g}$ or $\text{ad}$ the adjoint action of $\mathfrak{g}$ on $\mathfrak{g}$. Any closed subgroup $H$ of $G$ is a $p$-adic Lie subgroup and its Lie algebra $\mathfrak{h}$ is a $\mathbb{Q}_p$-vector subspace of $\mathfrak{g}$ (see [13] Prop. 1.5).

We choose a ultrametric norm $\| \cdot \|$ on $\mathfrak{g}$ with values in $p\mathbb{Z}$. 
2.1. One-parameter subgroups.

**Definition 2.1.** A one-parameter morphism \( \varphi \) of a \( p \)-adic Lie group \( G \) is a continuous morphism \( \varphi : \mathbb{Q}_p \to G; t \mapsto \varphi(t) \). A one-parameter subgroup is the image \( \varphi(\mathbb{Q}_p) \) of an injective one-parameter morphism.

The derivative of a one-parameter morphism is an element \( X \) of \( \mathfrak{g} \) for which \( \text{ad} X \) is nilpotent. This follows from the following Lemma from [13, Corollary 1.2].

**Lemma 2.2.** Let \( \varphi : \mathbb{Q}_p \to \text{GL}(d, \mathbb{Q}_p) \) be a one-parameter morphism. Then there exists a nilpotent matrix \( X \) in \( \mathfrak{gl}(d, \mathbb{Q}_p) \) such that \( \varphi(t) = \exp(tX) \) for all \( t \) in \( \mathbb{Q}_p \).

**Proof.** First, we claim that, if \( K \) is a finite extension of \( \mathbb{Q}_p \), any continuous one-parameter morphism \( \psi : \mathbb{Q}_p \to K^* \) is constant. Indeed, since the modulus of \( \psi \) is a continuous morphism form \( \mathbb{Q}_p \) to a discrete multiplicative subgroup of \( (0, \infty) \), the kernel of \( |\psi| \) contains \( p^k\mathbb{Z}_p \), for some integer \( k \). Since \( \mathbb{Q}_p/p^k\mathbb{Z}_p \) is a torsion group and \( (0, \infty) \) has no torsion, \( \psi \) has constant modulus, that is, \( \psi \) takes values in \( \mathcal{O}^* \), where \( \mathcal{O} \) is the integer ring of \( K \). Now, on one hand, \( \mathcal{O}^* \) is a profinite group, that is, it is a compact totally discontinuous group and hence it admits a basis of neighborhoods of the identity which are finite index subgroups (namely, for example, the subgroups \( 1 + p^k\mathcal{O}, k \geq 0 \)). On the other hand, every closed subgroup of \( \mathbb{Q}_p \) is of the form \( p^k\mathbb{Z}_p \), for some integer \( k \), and hence, has infinite index in \( \mathbb{Q}_p \) and therefore any continuous morphism from \( \mathbb{Q}_p \) to a finite group is trivial. Thus, \( \psi \) is constant, which should be proved.

Let now \( \varphi \) be as in the lemma and let \( X \in \mathfrak{gl}(d, \mathbb{Q}_p) \) be the derivative of \( \varphi \). After having simultaneously reduced the commutative family of matrices \( \varphi(t)_{t \in \mathbb{Q}_p} \), the joint eigenvalues give continuous morphisms \( \mathbb{Q}_p \to K^* \) where \( K \) is a finite extension of \( \mathbb{Q}_p \). By the remark above, these morphisms are constant, that is, there exists \( g \) in \( \text{GL}(d, \mathbb{Q}_p) \), such that, for any \( t \) in \( \mathbb{Q}_p \), the matrix \( g\varphi(t)g^{-1} \) is unipotent and upper triangular. We may assume \( g = 1 \). Then \( X \) is nilpotent and upper triangular and it remains to check that the map \( \theta : t \mapsto \varphi(t)\exp(-tX) \) is constant. Since \( \varphi(t) \) commutes with \( X \), this map \( \theta \) is a one-parameter morphism with zero derivative. Hence, the kernel of \( \psi \) is an open subgroup and the matrices \( \theta(t), t \in \mathbb{Q}_p \), have finite order. Since they are unipotent, they equal \( e \), which should be proved. \( \square \)

2.2. Weakly regular and regular \( p \)-adic Lie groups.
Definition 2.3 (Ratner, [13]). A p-adic Lie group $G$ is said to be weakly regular if any two one-parameter morphisms $\mathbb{Q}_p \to G$ with the same derivative are equal.

Note that any closed subgroup of a weakly regular p-adic Lie group is also weakly regular.

Example 2.4 ([13, Cor. 1.3 and Prop. 1.5]). Every closed subgroup of $\text{GL}(d, \mathbb{Q}_p)$ is weakly regular.

Proof. This follows from Lemma 2.2.

Definition 2.5 (Ratner, [13]). A p-adic Lie group $G$ is said to be Ad-regular if the kernel of the adjoint map $\ker(\text{Ad}_g)$ is equal to the center $Z(G)$ of $G$. It is said to be regular if it is Ad-regular and if the finite subgroups of $G$ have uniformly bounded cardinality.

Note that any open subgroup of a regular p-adic Lie group is also regular.

This definition is motivated by the following example.

Example 2.6.  

a) The finite subgroups of a compact p-adic Lie group $K$ have uniformly bounded cardinality.

b) The finite subgroups of a p-adic linear group have uniformly bounded cardinality.

c) The Zariski connected linear algebraic p-adic Lie groups $G$ are regular.

Proof of Example 2.6 (see [13])

a) Since $K$ contains a torsion free open normal subgroup $\Omega$, for every finite subgroup $F$ of $K$, one has the bound $|F| \leq |K/\Omega|$.

b) We want to bound the cardinality $|F|$ of a finite subgroup of a group $G \subset \text{GL}(d, \mathbb{Q}_p)$. This follows from a) since $F$ is included in a conjugate of the compact group $K = \text{GL}(d, \mathbb{Z}_p)$.

c) It remains to check that $G$ is Ad-regular. Let $g$ be an element in the kernel of the adjoint map $\text{Ad}_g$. This means that the centralizer $Z_g$ of $g$ in $G$ is an open subgroup of $G$. This group $Z_g$ is also Zariski closed. Hence it is Zariski open. Since $G$ is Zariski connected, one deduces $Z_g = G$ and $g$ belongs to the center of $G$.

We want to relate the two notions “weakly regular” and “regular”. We first recall the following implication in [13, Cor. 1.3].

Lemma 2.7. Any regular p-adic Lie group is weakly regular.

Proof. Let $\varphi_1 : \mathbb{Q}_p \to G$ and $\varphi_2 : \mathbb{Q}_p \to G$ be one-parameter morphisms of $G$ with the same derivative. We want to prove that $\varphi_1 = \varphi_2$. 

According to Lemma 2.2, the one-parameter morphisms $\text{Ad}_g \varphi_1$ and $\text{Ad}_g \varphi_2$ are equal. Since $G$ is Ad-regular, this implies that, for all $t$ in $\mathbb{Q}_p$, the element $\theta(t) := \varphi_1(t)^{-1}\varphi_2(t)$ is in the center of $G$. Hence $\theta$ is a one-parameter morphism of the center of $G$ with zero derivative. Its image is either trivial or an infinite $p$-torsion group. This second case is excluded by the uniform bound on the finite subgroups of $G$. This proves the equality $\varphi_1 = \varphi_2$. □

The aim of this text is to prove Theorem 5.12 which is a kind of converse to Lemma 2.7.

3. ALGEBRAIC UNIPOTENT $p$-ADIC LIE GROUP

In this chapter, we study the algebraic unipotent subgroups of a weakly regular $p$-adic Lie group. The results in this chapter are mainly due to Ratner.

3.1. Definition and closedness.

We first focus on a special class of $p$-adic Lie groups.

We say that an element $g$ of a $p$-adic Lie group $G$ admits a logarithm if one has $g^{p^n} \longrightarrow e$: indeed, for such a $g$, the map $n \mapsto g^n$ extends as a continuous morphism $\mathbb{Z}_p \to G$ and one can define the logarithm $\log(g) \in \mathfrak{g}$ as being the derivative at 0 of this morphism.

Definition 3.1. A $p$-adic Lie group $N$ is called algebraic unipotent if its Lie algebra is nilpotent, if every element $g$ in $N$ admits a logarithm $\log(g)$, and if the logarithm map $\log : N \to \mathfrak{n}$ is a bijection.

This implies that every non trivial element $g$ of $N$ belongs to a unique one-parameter subgroup of $N$. By definition these groups $N$ are weakly regular. The inverse of the map $\log$ is denoted by $\exp$. These maps $\exp$ and $\log$ are $\text{Aut}(N)$-equivariant.

The following lemma is in [13, Prop. 2.1].

Lemma 3.2. Let $N$ be an algebraic unipotent $p$-adic Lie group. Then the map

$$\mathfrak{n} \times \mathfrak{n} \to \mathfrak{n}; (X,Y) \mapsto \log(\exp X \exp Y)$$

is polynomial and is given by the Baker-Campbell-Hausdorff formula. In other words, a $p$-adic Lie group $N$ is algebraic unipotent if and only if it is isomorphic to the group of $\mathbb{Q}_p$-points of a unipotent algebraic group defined over $\mathbb{Q}_p$.

These groups have been studied in [13, Sect.2] (where they are called quasiconnected).

In particular, we have
Corollary 3.3. Let $N$ be an algebraic unipotent $p$-adic Lie group. Then the exponential and logarithm maps of $N$ are continuous.

Proof of Lemma 3.2. In this proof, we will say that a $p$-adic Lie group is strongly algebraic unipotent if it is isomorphic to the group of $\mathbb{Q}_p$-points of a unipotent algebraic group defined over $\mathbb{Q}_p$. Such a group is always algebraic unipotent.

The aim of this proof is to check the converse. Let $N$ be an algebraic unipotent $p$-adic Lie group. We want to prove that $N$ is strongly algebraic unipotent. Its Lie algebra $\mathfrak{n}$ contains a flag $0 \subset \mathfrak{n}_1 \subset \cdots \subset \mathfrak{n}_r = \mathfrak{n}$ of ideals with $\dim \mathfrak{n}_i = i$. We will prove, by induction on $i \geq 1$, that the set $N_i := \exp(\mathfrak{n}_i)$ is a closed subgroup of $N$ which is strongly algebraic unipotent.

By the induction assumption, the set $N_{i-1}$ is a strongly algebraic unipotent closed subgroup. Since $\mathfrak{n}_{i-1}$ is an ideal, this subgroup $N_{i-1}$ is normal. Let $X_i$ be an element of $\mathfrak{n}_i \setminus \mathfrak{n}_{i-1}$ and $N'_i = \mathbb{Q}_p \rtimes N_{i-1}$ where the action of $t \in \mathbb{Q}_p$ by conjugation on $N_{i-1}$ is given by $t \exp(X) t^{-1} = \exp(e^{\text{ad}X}X)$, for all $X$ in $\mathfrak{n}_{i-1}$. By construction this group $N'_i$ is strongly algebraic unipotent and the map $\psi : N'_i \to N; (t, n) \mapsto \exp(tX_i)n$ is a group morphism. Since $N'_i = \exp(\mathfrak{n}'_i)$, the set $N_i = \exp(\mathfrak{n}_i)$ is equal to the image $\psi(N'_i)$. Hence, by Lemma 3.4 below, the set $N_i$ is a closed subgroup which is isomorphic to $N'_i$ and hence $N_i$ is strongly algebraic unipotent. \[\square\]

The following lemma tells us that an algebraic unipotent Lie subgroup of a $p$-adic Lie group is always closed.

Lemma 3.4. Let $G$ be a totally discontinuous locally compact topological group, $N$ be an algebraic unipotent $p$-adic Lie group and $\varphi : N \to G$ be an injective morphism. Then $\varphi$ is a proper map. In particular, $\varphi(N)$ is a closed subgroup of $G$ and $\varphi$ is an isomorphism of topological groups from $N$ onto $\varphi(N)$.

Proof. If $\varphi$ was not proper, there would exist a sequence $Y_n$ in the Lie algebra $\mathfrak{n}$ of $N$ such that
\[
\lim_{n \to \infty} \varphi(\exp(Y_n)) = e \quad \text{and} \quad \lim_{n \to \infty} \|Y_n\| = \infty.
\]
We write $Y_n = p^{-k_n}X_n$ with integers $k_n$ going to $\infty$ and $\|X_n\| = 1$. Since the group $G$ admits a basis of compact open subgroups, one also has $\lim_{n \to \infty} \varphi(\exp(X_n)) = e$. Let $X$ be a cluster point of the sequence $X_n$. One has simultaneously, $\varphi(\exp(X)) = e$ and $\|X\| = 1$. This contradicts the injectivity of $\varphi$. \[\square\]
3.2. Lifting one-parameter morphisms.

We now explain how to lift one-parameter morphisms.

**Lemma 3.5.** Let $G$ be a $p$-adic Lie group, $N \subset G$ a normal algebraic unipotent closed subgroup, $G' := G/N$, and $\pi : G \to G'$ the projection. Then, for any one-parameter morphism $\varphi'$ of $G'$, there exists a one-parameter morphism $\varphi$ of $G$ which lifts $\varphi'$, i.e. such that $\varphi' = \pi \circ \varphi$.

If $\varphi'$ has zero derivative, one can choose $\varphi$ to have zero derivative.

**Proof.** According to Lemma 3.4 the image $\varphi'(\mathbb{Q}_p)$ is a closed subgroup of $G'$. Hence we can assume that $\varphi'(\mathbb{Q}_p) = G'$.

**First Case**: $N$ is central in $G$. For $k \geq 1$, we introduce the subgroup $Q_k'$ of $G'$ spanned by the element $g_k' := \varphi'(p^{-k})$. Since $Q_k'$ is cyclic and $N$ is central, the group $Q_k := \pi^{-1}(Q_k')$ is abelian. Since the increasing union of these groups $Q_k$ is dense in $G$, the group $G$ is also abelian.

Since $N$ is infinitely $p$-divisible, one can construct, by induction on $k \geq 0$, a sequence $(g_k)_{k \geq 0}$ in $G$ such that $\pi(g_k) = g_k'$ and $p g_k+1 = g_k$.

We claim that $g_0^{p^k} \to e$. Indeed, since $\pi(g_0)^{p^k} \to e$ and since every element $h$ of a $p$-adic Lie group that is close enough to the identity element satisfies $h^{p^k} \to e$, one can find $\ell \geq 0$ and $n$ in $N$ such that $(g_0^{p^\ell} n^{-1})^{p^k} \to e$. As $N$ is algebraic unipotent, we have $n^{p^k} \to e$, and the claim follows, since $N$ is central.

Now, the formulae $\varphi(p^{-k}) = g_k$, for all $k \geq 0$, define a unique one-parameter morphism $\varphi$ of $G$ which lifts $\varphi'$.

Note that when $\varphi'$ has zero derivative, one can assume, after a reparametrization of $\varphi'$, that $\varphi'(\mathbb{Z}_p) = 0$ and choose the sequence $g_k$ so that $g_0 = e$. Then the morphism $\varphi$ has also zero derivative.

**General Case**: The composition of $\varphi'$ with the action by conjugation on the abelianized group $N/[N,N] \simeq \mathbb{Q}_p^d$ is a one-parameter morphism $\psi : \mathbb{Q}_p \to \mathrm{GL}(d, \mathbb{Q}_p)$. According to Lemma 2.2 there exists a nilpotent matrix $X$ such that $\psi(t) = \exp(tX)$ for all $t \in \mathbb{Q}_p$. The image of this matrix $X$ corresponds to an algebraic unipotent subgroup $N_1$ with $[N,N] \subset N_1 \subseteq N$ which is normal in $G$ and such that $N/N_1$ is a central subgroup of the group $G/N_1$. According to the first case, the morphism $\varphi'$ can be lifted as a morphism $\varphi'_1$ of $G/N_1$. By an induction argument on the dimension of $N$, this morphism $\varphi'_1$ can be lifted as a morphism of $G$. \qed
Let $G$ be a $p$-adic Lie group. We recall the notation

(3.1) $\mathfrak{g}_G := \{ X \in \mathfrak{g} \text{ derivative of a one-parameter morphism of } G \}$.

Note that this set $\mathfrak{g}_G$ is invariant under the adjoint action of $G$.

The following lemma tells us various stability properties by extension when the normal subgroup is algebraic unipotent.

**Lemma 3.6.** Let $G$ be a $p$-adic Lie group, $N$ a normal algebraic unipotent subgroup of $G$, and $G' := G/N$.

a) One has the equivalence

$G$ is algebraic unipotent $\iff$ $G'$ is algebraic unipotent.

b) Let $X$ in $\mathfrak{g}$ and $X'$ its image in $\mathfrak{g}' = \mathfrak{g}/\mathfrak{n}$. One has the equivalence

$X \in \mathfrak{g}_G \iff X' \in \mathfrak{g}'_G$.

c) One has the equivalence

$G$ is weakly regular $\iff$ $G'$ is weakly regular.

Later on in Corollary 5.7 we will be able to improve this Lemma.

**Proof.** We denote by $\pi : G \to G/N$ the natural projection.

a) The implication $\Rightarrow$ is well-known. Conversely, we assume that $N$ and $G/N$ are algebraic unipotent and we want to prove that $G$ is algebraic unipotent. Arguing by induction on $\dim G/N = 1$, i.e. that there exists an isomorphism $\varphi' : \mathbb{Q}_p \to G/N$. According to Lemma 3.5 one can find a one-parameter morphism $\varphi$ of $G$ that lifts $\varphi'$. By Lemma 3.4 the image $Q := \varphi(\mathbb{Q}_p)$ is closed and $G$ is the semidirect product $G = Q \ltimes N$. By Lemma 2.2 the one-parameter morphism $t \mapsto \text{Ad}_n \varphi(t)$ is unipotent, and hence the group $G$ is algebraic unipotent.

b) The implication $\Rightarrow$ is easy. Conversely, we assume that $X'$ is the derivative of a one-parameter morphism $\varphi'$ of $G'$. When $X' = 0$, the element $X$ belongs to $\mathfrak{n}$ and, since $N$ is algebraic unipotent, $X$ is the derivative of a one-parameter morphism $\varphi$ of $N$. We assume now that $X' \neq 0$ so that the group $Q' := \varphi'(\mathbb{Q}_p)$ is algebraic unipotent and isomorphic to $\mathbb{Q}_p$. According to point a), the group $H := \pi^{-1}(Q')$ is algebraic unipotent. Since the element $X$ belongs to the Lie algebra $\mathfrak{h}$ of $H$, it is the derivative of a one-parameter morphism $\varphi'$ of $H$.

c) $\Rightarrow$ We assume that $G$ is weakly regular. Let $\varphi'_1$ and $\varphi'_2$ be one-parameter morphisms of $G'$ with the same derivative $X' \in \mathfrak{g}'$. We want to prove that $\varphi'_1 = \varphi'_2$. If this derivative $X'$ is zero, by Lemma 3.5 we can lift both $\varphi'_1$ and $\varphi'_2$ as one-parameter morphisms of $G$ with zero derivative. Since $G$ is weakly regular, both $\varphi_1$ and $\varphi_2$ are trivial and $\varphi'_1 = \varphi'_2$. We assume now that the derivative $X'$ is non-zero. As above, for $i = 1, 2$, the groups $Q'_i := \varphi'_i(\mathbb{Q}_p)$ and $H_i := \pi^{-1}(Q'_i)$ are algebraic unipotent. Since $G$ is weakly regular, and its algebraic
unipotent subgroups $H_1$ and $H_2$ have the same Lie algebra, one gets successively $H_1 = H_2$, $Q'_1 = Q'_2$, and $\varphi'_1 = \varphi'_2$.

$\Leftarrow$ We assume that $G/N$ is weakly regular. Let $\varphi_1$ and $\varphi_2$ be one-parameter morphisms of $G$ with the same derivative $X \in \mathfrak{g}$. We want to prove that $\varphi_1 = \varphi_2$. Since $G/N$ is weakly regular the one-parameter morphisms $\pi \circ \varphi_1$ and $\pi \circ \varphi_2$ are equal and their image $Q'$ is a unipotent algebraic subgroup of $G'$. According to point $a$), the group $H := \pi^{-1}(Q')$ is algebraic unipotent. Since $\varphi_1$ and $\varphi_2$ take their values in $H$, one has $\varphi_1 = \varphi_2$. □

3.3. Unipotent subgroups tangent to a nilpotent Lie algebra.

Proposition 3.7 below describes the nilpotent Lie subgroups of a weakly regular $p$-adic Lie group $G$ which are spanned by one-parameter morphisms.

The following proposition is due to Ratner in [13, Thm. 2.1].

**Proposition 3.7.** Let $G$ be a weakly regular $p$-adic Lie group and $\mathfrak{n} \subset \mathfrak{g}$ be a nilpotent Lie subalgebra. Then the set $\mathfrak{n}_G := \mathfrak{n} \cap \mathfrak{g}_G$ is an ideal of $\mathfrak{n}$ and there exists an algebraic unipotent subgroup $N_G$ of $G$ with Lie algebra $\mathfrak{n}_G$.

This group $N_G$ is unique. It is a closed subgroup of $G$. By construction, it is the largest algebraic unipotent subgroup whose Lie algebra is included in $\mathfrak{n}$.

**Proof.** We argue by induction on $\dim \mathfrak{n}$. We can assume $\mathfrak{n}_G \neq 0$.

**First case :** $\mathfrak{n}$ is abelian. Let $X_1, \ldots, X_r$ be a maximal family of linearly independent elements of $\mathfrak{n}_G$ and, for $i \leq r$, let $\varphi_i$ be the one-parameter morphism with derivative $X_i$. Since $G$ is weakly regular, the group spanned by the images $\varphi_i(Q_p)$ is commutative and the map

$$\psi : Q'_p \to G; (t_1, \ldots, t_r) \mapsto \varphi_1(t_1) \ldots \varphi_r(t_r)$$

is an injective morphism. Its image is a unipotent algebraic subgroup $N_G$ of $G$ whose Lie algebra is $\mathfrak{n}_G$.

**Second case :** $\mathfrak{n}$ is not abelian. Let $\mathfrak{z}$ be the center of $\mathfrak{n}$ and $\mathfrak{z}_2$ the ideal of $\mathfrak{n}$ such that $\mathfrak{z}_2/\mathfrak{z}$ is the center of $\mathfrak{n}/\mathfrak{z}$. If $\mathfrak{n}_G$ is included in the centralizer $\mathfrak{n}'$ of $\mathfrak{z}_2$, we can apply the induction hypothesis to $\mathfrak{n}'$. We assume now that $\mathfrak{n}_G$ is not included in $\mathfrak{n}'$, i.e. there exists

$$X \in \mathfrak{n}_G$$

and $Y \in \mathfrak{z}_2$ such that $[X, Y] \neq 0$.

This element $Z := [X, Y]$ belongs to the center $\mathfrak{z}$. 
We first check that \( Z \) belongs also to \( \mathfrak{g}_G \). Indeed, let \( \mathfrak{m} \) be the 2-dimensional Lie subalgebra of \( \mathfrak{n} \) with basis \( X, Z \). This Lie algebra is normalized by \( Y \). For \( \varepsilon \in \mathbb{Q}_p \) small enough, there exists a group morphism \( \psi : \varepsilon \mathbb{Z}_p \to G \) whose derivative at 0 is \( Y \), and one has \( \text{Ad}(\psi(\varepsilon))Y = e^{e \text{ad}Y}X = X - \varepsilon Z \). Since \( X \) belongs to \( \mathfrak{g}_G \), the element \( X - \varepsilon Z \) also belongs to \( \mathfrak{g}_G \). By the first case applied to the abelian Lie subalgebra \( \mathfrak{m} \), the element \( Z \) belongs to \( \mathfrak{g}_G \).

This means that there exists a one-parameter subgroup \( U \) of \( G \) whose Lie algebra is \( \mathfrak{u} = \mathbb{Q}_p \mathbb{Z} \). Let \( C \) be the centralizer of \( U \) in \( G \). According to Lemma 3.6, the quotient group \( C/U \) is also weakly regular. We apply our induction hypothesis to this group \( C' := C/U \) and the nilpotent Lie algebra \( \mathfrak{n}/\mathfrak{u} \). There exists a largest algebraic unipotent subgroup \( N_C \) in \( C' \) whose Lie algebra is included in \( \mathfrak{n}' \). Hence, using again Lemma 3.6 there exists a largest algebraic unipotent subgroup \( N_C \) of \( C \) whose Lie algebra is included in \( \mathfrak{n} \). Since \( G \) is weakly regular, any one-parameter subgroup of \( G \) tangent to \( \mathfrak{n} \) is included in \( C \) and \( N_C \) is also the largest algebraic unipotent subgroup of \( G \) whose Lie algebra is included in \( \mathfrak{n} \).

3.4. Largest normal algebraic unipotent subgroup.

We prove in this section that a weakly regular \( p \)-adic Lie group contains a largest normal algebraic unipotent subgroup.

Let \( G \) be a \( p \)-adic Lie group. We denote by \( \overline{G}_u \) the closure of the subgroup \( G_u \) of \( G \) generated by all the one-parameter subgroups of \( G \). This group \( \overline{G}_u \) is normal in \( G \). We denote by \( \mathfrak{g}_u \) the Lie algebra of \( \overline{G}_u \). It is an ideal of \( \mathfrak{g} \).

We recall that the radical \( \mathfrak{r} \) of \( \mathfrak{g} \) is the largest solvable ideal of \( \mathfrak{g} \) and that the nilradical \( \mathfrak{n} \) of \( \mathfrak{g} \) is the largest nilpotent ideal of \( \mathfrak{g} \). The nilradical is the set of \( X \) in \( \mathfrak{r} \) such that \( \text{ad}X \) is nilpotent and one has \( [\mathfrak{g}, \mathfrak{r}] \subset \mathfrak{n} \) (see \([5]\)).

When \( G \) is weakly regular, we denote by \( R_u \) the largest algebraic unipotent subgroup of \( G \) whose Lie algebra is included in \( \mathfrak{n} \). It exists by Proposition 3.7.

The following proposition is mainly in \([13\text{, Lem. 2.2}]\).

**Proposition 3.8.** Let \( G \) be a weakly regular \( p \)-adic Lie group.

a) The group \( R_u \) is the largest normal algebraic unipotent subgroup of \( G \).

b) Its Lie algebra \( \mathfrak{r}_u \) is equal to \( \mathfrak{r}_u = \mathfrak{n} \cap \mathfrak{g}_G = \mathfrak{r} \cap \mathfrak{g}_G \).

c) One has the inclusion \( [\mathfrak{g}_u, \mathfrak{r}] \subset \mathfrak{r}_u \).

d) Let \( G' = G/R_u \). Let \( X \) be in \( \mathfrak{g} \) and \( X' \) be its image in \( \mathfrak{g}' = \mathfrak{g}/\mathfrak{r}_u \).
One has the equivalence

\[ X \in g_G \iff X' \in g'_G. \]

**Proof.**

a) We have to prove that any normal algebraic unipotent subgroup \( U \) of \( G \) is included in \( R_u \). Indeed the Lie algebra \( u \) of \( U \) is a nilpotent ideal of \( g \), hence it is included in \( n \) and \( U \) is included in \( R_u \).

b) We already know the equality \( r_u = n \cap g_G \) from Proposition 3.7. It remains to check the inclusion \( r \cap g_G \subset n \). Indeed, let \( X \) be an element in \( r \cap g_G \). Since \( X \) is the derivative of a one-parameter morphism, by Lemma 2.2, the endomorphism \( \text{ad}X \) is nilpotent. Since \( X \) is also in the radical \( r \), \( X \) has to be in the nilradical \( n \).

c) We want to prove that the adjoint action of \( G_u \) on the quotient Lie algebra \( r/r_u \) is trivial. That is, we want to prove that, for all \( X \in g_G \) and \( Y \in r \), one has \( [X,Y] \in r_u \).

By Lemma 2.2, the endomorphism \( \text{ad}X \) is nilpotent. Since \( Y \) is in \( r \), the bracket \( [X,Y] \) belongs to \( n \) and the vector space \( m := \mathbb{Q}_p X \oplus n \) is a nilpotent Lie algebra normalized by \( Y \). Hence, by Proposition 3.7, the set \( m \cap g_G \) is a nilpotent Lie algebra. This set is normalized by \( Y \) since it is invariant by \( e^{\varepsilon \text{ad}Y} \) for \( \varepsilon \in \mathbb{Q}_p \) small enough. In particular the element \( [X,Y] \) belongs to \( g_G \) and hence to \( r_u \).

d) This is a special case of Proposition 3.7. \( \square \)

4. Derivatives of one-parameter morphisms

In this chapter, we describe the set \( g_G \) of derivatives of one-parameter morphisms of a weakly regular \( p \)-adic Lie group \( G \).

4.1. Construction of one-parameter subgroups.

We explain first a construction of one-parameter morphisms of \( G \) borrowed from [1] and [2].

**Lemma 4.1.** Let \( G \) be a \( p \)-adic Lie group and \( g \in G \). Then the vector space \( g_g^+ := \{ v \in g \mid \lim_{n \to \infty} \text{Ad}g^{-n}v = 0 \} \) is included in \( g_G \).

Note that \( g_g^+ \) is a nilpotent Lie subalgebra of \( g \).

The proof relies on the existence of compact open subgroups of \( G \) for which the exponential map satisfies a nice equivariant property. We need some classical definition (see [8]). A \( p \)-adic Lie group \( \Omega \) is said to be a standard group if there exists a \( \mathbb{Q}_p \)-Lie algebra \( I \) and a compact open sub-\( \mathbb{Z}_p \)-algebra \( O \) of \( I \) such that the Baker-Campbell-Hausdorff series converges on \( O \) and \( \Omega \) is isomorphic to the \( p \)-adic Lie group \( O \) equipped with the group law defined by this formula.
In this case, I identifies canonically with the Lie algebra of \( \Omega \), every element of \( \Omega \) admits a logarithm and the logarithm map induces an isomorphism \( \Omega \rightarrow O \). If \( G \) is any \( p \)-adic Lie group, it admits a standard open subgroup (see [8, Theorem 8.29]). If \( \Omega \) is such a subgroup and if \( O \) is the associated compact open sub-\( \mathbb{Z}_p \)-algebra of \( g \), we denote by \( \exp_\Omega : O \rightarrow \Omega \) the inverse diffeomorphism of the logarithm map \( \Omega \rightarrow O \).

Note that if \( \Omega \) and \( \Omega' \) are standard open subgroups of \( G \), the maps \( \exp_\Omega \) and \( \exp_{\Omega'} \) coincide in some neighborhood of 0 in \( g \).

**Lemma 4.2.** Let \( G \) be a \( p \)-adic Lie group, \( \Omega \subset G \) a standard open subgroup and \( \exp_\Omega : O \rightarrow \Omega \) the corresponding exponential map. For every compact subset \( K \subset G \), there exists an open subset \( O_K \subset g \) which is contained in \( O \) and in all the translates \( \text{Ad}^{-1}(O), g \in K \), and such that one has the equivariance property
\[
\exp_\Omega(\text{Ad}^{-1}(O)) = g \exp_\Omega(\text{Ad}^{-1}(O))g^{-1} \text{ for any } v \in O_K, g \in K.
\]

**Proof.** We may assume that \( K \) contains \( e \). The intersection \( \Omega_K := \bigcap_{g \in K} g^{-1}\Omega g \) is an open neighborhood of \( e \) in \( G \). We just choose \( O_K \) to be the open set \( O_K := \log(\Omega_K) \).

**Proof of Lemma 4.1.** This is [1, Lem. 5.4]. For the sake of completeness, we recall the proof. Fix \( g \in G \). Let \( \Omega \) be a standard open subgroup of \( G \) with exponential map \( \exp_\Omega : O \rightarrow \Omega \). By Lemma 4.2, there exists an open additive subgroup \( U \subset O \cap g^+ \) such that \( \text{Ad}^{-1}U \subset O \) and that
\[
\exp_\Omega(u) = g \exp_\Omega(\text{Ad}^{-1}(u))g^{-1} \text{ for any } u \in U.
\]

After eventually replacing \( U \) by \( \bigcap_{k \geq 0} \text{Ad}^{-1}(U) \), we can assume \( \text{Ad}^{-1}(U) \subset U \). Now, for \( k \geq 0 \), let \( U_k := \text{Ad}^{-k}(U) \) and define a continuous map \( \psi_k : U_k \rightarrow G \) by setting
\[
\psi_k(u) = g^k \exp_\Omega(\text{Ad}^{-k}(u))g^{-k} \text{ for any } u \in U_k.
\]

We claim that, for any \( k \), one has \( \psi_k = \psi_{k-1} \) on \( U_{k-1} = \text{Ad}^{-1}(U_k) \). Indeed, let \( u \) be in \( U_k \). As \( u_k := \text{Ad}^{-k}(u) \) belongs to \( U \), we have
\[
\psi_{k+1}(u) = g^k(\exp_\Omega(\text{Ad}^{-1}(u))g^{-1})g^{-k} = g^k \exp_\Omega(u_k)g^{-k} = \psi_k(u).
\]

Therefore, as \( g^+ = \bigcup_{k \geq 0} U_k \), one gets a map \( \psi : g^+ \rightarrow G \) whose restriction to any \( U_k, k \geq 0 \), is \( \psi_k \). For every \( v \) in \( g^+ \), the map \( t \mapsto \psi(tv) \) is a one-parameter morphism of \( G \) whose derivative is equal to \( v \). □
4.2. The group $G_{nc}$ and its Lie algebra $g_{nc}$.

We introduce in this section a normal subgroup $G_{nc}$ of $G$ which contains $G_u$.

Let $G$ be a $p$-adic Lie group and $g_{nc}$ be the smallest ideal of the Lie algebra $g$ such that the Lie algebra $s_c := g / g_{nc}$ is semisimple and such that the adjoint group $\text{Ad}_{s_c}(G)$ is bounded in the group $\text{Aut}(s_c)$ of automorphisms of $s_c$. Let $G_{nc}$ be the kernel of the adjoint action in $s_c$, i.e.

$$G_{nc} := \{ g \in G \mid \text{for all } X \in g, \quad \text{Ad}_g(X) - X \in g_{nc} \}$$

By construction $G_{nc}$ is a closed normal subgroup of $G$ with Lie algebra $g_{nc}$.

**Lemma 4.3.** Let $G$ be a $p$-adic Lie group. Any one-parameter morphism of $G$ takes its values in $G_{nc}$.

In other words, the group $G_u$ is included in $G_{nc}$.

**Proof.** Let $\varphi$ be a one-parameter morphism of $G$. Then $\text{Ad}_{s_c} \circ \varphi$ is a one-parameter morphism of $\text{Aut}(s_c)$ whose image is relatively compact. By Lemma 2.2, this one-parameter morphism is trivial and $\varphi$ takes its values in $G_{nc}$. $\square$

4.3. Derivatives and Levi subalgebras.

We can now describe precisely which elements of $g$ are tangent to one-parameter subgroups of $G$.

We recall that an element $X$ of a semisimple Lie algebra $s$ is said to be nilpotent if the endomorphism $\text{ad}_s X$ is nilpotent. In this case, for any finite dimensional representation $\rho$ of $s$, the endomorphism $\rho(X)$ is also nilpotent.

We recall that a Levi subalgebra $s$ of a Lie algebra $g$ is a maximal semisimple Lie subalgebra, and that one has the Levi decomposition $g = s \oplus r$.

The following proposition is proven in [13, Th. 2.2] under the additional assumption that $G$ is Ad-regular.

**Proposition 4.4.** Let $G$ be a weakly regular $p$-adic Lie group, $r$ be the radical of $g$, $s$ a Levi subalgebra of $g$ and $s_u := s \cap g_{nc}$. One has the equality : $g_G = \{ X \in s_u \oplus r_u \mid \text{ad}X \text{ is nilpotent} \}$.

It will follow from Lemma 1.7 that the Lie algebra $s_u \oplus r_u$ does not depend on the choice of $s$.

The key ingredient in the proof of Proposition 4.4 will be Lemma 4.1. We will begin by three preliminary lemmas. The first two lemmas are classical.
Lemma 4.5. Let $V = \mathbb{Q}_p^d$ and $G$ be a subgroup of $\text{GL}(V)$ such that $V$ is an unbounded and irreducible representation of $G$. Then $G$ contains an element $g$ with at least one eigenvalue of modulus not one.

Proof. Let $A$ be the associative subalgebra of $\text{End}(V)$ spanned by $G$. Since $V$ is irreducible, the associative algebra $A$ is semisimple and the bilinear form $(a,b) \mapsto \text{tr}(ab)$ is non-degenerate on $A$ (see [9, Ch. 17]). If all the eigenvalues of all the elements of $G$ have modulus 1, this bilinear form is bounded on $G \times G$. Since $A$ admits a basis included in $G$, for any $a$ in $A$, the linear forms $b \mapsto \text{tr}(ab)$ on $A$ are bounded on the subset $G$. Hence $G$ is a bounded subset of $A$. □

Lemma 4.6. Let $\mathfrak{s}_0$ be a simple Lie algebra over $\mathbb{Q}_p$ and $S_0 := \text{Aut}(\mathfrak{s}_0)$. All open unbounded subgroups $J$ of $S_0$ have finite index in $S_0$.

Proof. Since $J$ is unbounded, by Lemma 4.5, it contains an element $g_0$ with at least one eigenvalue of modulus not one. Since $J$ is open, the unipotent Lie subgroups $U^+ = \{ g \in S_0 \mid \lim_{n \to \infty} g^{-n} g_0^n = e \}$ and $U^- = \{ g \in S_0 \mid \lim_{n \to \infty} g_0^n g g^{-n} = e \}$ are included in $J$. By [4, 6.2, 6.13], $J$ has finite index in $S_0$. □

The third lemma contains the key ingredient.

Lemma 4.7. Let $G$ be a weakly regular $p$-adic Lie group, $\mathfrak{r}$ the radical of $\mathfrak{g}$, $\mathfrak{s}$ a Levi subalgebra of $\mathfrak{g}$ and $\mathfrak{s}_u := \mathfrak{s} \cap g_{nc}$.

a) One has the inclusion $[\mathfrak{s}_u, \mathfrak{r}] \subset \mathfrak{r}_u$.

b) Every nilpotent element $X$ in $\mathfrak{s}_u$ belongs to $\mathfrak{g}_G$.

Note that Lemma 4.7a does not follow from Proposition 3.8c, since with the definitions of $\mathfrak{g}_u$ and $\mathfrak{s}_u$ that we have given, we do not know yet that $\mathfrak{s}_u = \mathfrak{s} \cap \mathfrak{g}_u$.

Proof. a) By Proposition 3.8 we can assume $\mathfrak{r}_u = 0$. We want to prove that $[\mathfrak{s}_u, \mathfrak{r}] = 0$. Let $\mathfrak{s}_i$, $i = 1, \ldots, \ell$ be the simple ideals of $\mathfrak{s}_u$. Replacing $G$ by a finite index subgroup, we can also assume that the ideals $\mathfrak{s}_i \oplus \mathfrak{r}$ are $G$-invariant. Similarly, let $\mathfrak{r}_j$, $j = 1, \ldots, m$ be the simple subquotients of a Jordan-Hölder sequence of the $G$-module $\mathfrak{r}$.

On the one hand, by assumption, for all $i \leq \ell$, the group $\text{Ad}_{\mathfrak{s}_i \oplus \mathfrak{r}} (G)$ is unbounded. Hence, by Lemma 4.5 there exists an element $g_i$ in $G$ and $X_i$ in $\mathfrak{g}_i^+ \cap (\mathfrak{s}_i \oplus \mathfrak{r})$ whose image in $\mathfrak{g}/\mathfrak{r}$ is non zero. By Lemma 4.1 there exists a one-parameter morphism $\varphi_i$ of $G$ whose derivative is $X_i$.

On the other hand, since $\mathfrak{r}_u = 0$, by the same Lemma 4.1 for every $g$ in $G$, all the eigenvalues of $\text{Ad}_\mathfrak{r}_u (g)$ have modulus 1. By Lemma 4.5 for all $j \leq m$, the image $\text{Ad}_\mathfrak{r}_u (G)$ of $G$ in any simple subquotient
\( r_j \) is bounded. In particular, the one-parameter morphisms \( \text{Ad}_{r_j} \circ \varphi_i \) are bounded. Hence, by Lemma 2.2, one has \( \text{ad}_{r_j}(X_i) = 0 \). Since \( r_j \) is a simple \( g \)-module, the Lie algebra \( \text{ad}_{r_j}(g) \) is reductive and contains \( \text{ad}_{r_j}(s_i) \) as an ideal. Since \( s_i \) is a simple Lie algebra, this implies \( \text{ad}_{r_j}(s_i) = 0 \). Since the action of \( s_u \) on \( r \) is semisimple, this implies the equality \([s_u, r] = 0\).

b) As in a), we can assume that \( G \) preserves the ideals \( s_i \oplus r \) and that \( r_u = 0 \). According to this point a), the Lie algebras \( s_u \) and \( r \) commute and hence \( s_u \) is the unique Levi subalgebra of \( g_{nc} \) (see [5, §6]). In particular,

\[(4.1) \quad \text{for all } g \in G, \text{ one has } \text{Ad}g(s_u) = s_u.\]

Let \( X \) be a nilpotent element of \( s_u \). We want to prove that \( X \) is the derivative of a one parameter morphism of \( G \). By Jacobson-Morozov theorem, there exists an automorphism \( \psi \) of \( s_u \) such that \( \psi(X) = p^{-1}X \).

Since, for every simple ideal \( s_i \) of \( s_u \), the subgroup \( \text{Ad}_{s_i \oplus r}/G_{nc} \subset \text{Aut}(s_i) \) is unbounded and open, this subgroup has finite index. Hence, remembering also (4.1), there exists \( k \geq 1 \) and \( g \in G_{nc} \) such that \( \text{Ad}g(X) = p^{-k}X \). Then, by Lemma 4.1, \( X \) is the derivative of a one-parameter morphism of \( G \).

\[\square\]

Proof of Proposition 4.4. We just have to gather what we have proved so far. By Proposition 3.8, we can assume \( r_u = 0 \). Let \( X \in g \). We write \( X = X_g + X_r \) with \( X_g \in s \) and \( X_r \in r \).

Proof of the inclusion \( \subset \). Assume that \( X \) is the derivative of a one-parameter morphism \( \varphi \) of \( G \). By Lemma 4.3, \( X \) belongs to \( g_{nc} \) and hence \( X_g \) belongs to \( s_u \). By Lemma 2.2, the endomorphism \( \text{ad}X \) is nilpotent, and hence \( X_g \) is a nilpotent element of the semisimple Lie algebra \( s \). According to Lemma 4.7, \( X_g \) and \( X_r \) commute and \( X_g \) is the derivative of a one-parameter morphism \( \varphi_g \) of \( G \). Then \( X_r \) is also the derivative of a one-parameter morphism \( \varphi_r \) of \( G \), the one given by \( t \mapsto \varphi_g(t)^{-1}\varphi(t) \). Hence \( X_r \) belongs to \( r_u \).

Proof of the inclusion \( \supset \). Assume that \( X_g \) belongs to \( s_u \), \( X_r \) belongs to \( r_u \) and \( \text{ad}X \) is nilpotent. By Lemma 4.7, \( X_g \) and \( X_r \) commute and \( X_g \) is the derivative of a one-parameter morphism \( \varphi_g \) of \( G \). By assumption \( X_r \) is the derivative of a one-parameter morphism \( \varphi_r \) of \( G \). Hence \( X \) is also the derivative of a one-parameter morphism \( \varphi \) of \( G \), the one given by \( t \mapsto \varphi_g(t)\varphi_r(t) \). Hence \( X \) belongs to \( g_G \). \[\square\]
5. Groups spanned by unipotent subgroups

In this chapter, we prove the two main results Proposition 5.5 and Theorem 5.12 that we announced in the introduction.

5.1. Semisimple regular $p$-adic Lie groups.

We recall first a nice result due to Prasad-Raghunathan which is an output from the theory of congruence subgroups.

Let $\mathfrak{s}$ be a semisimple Lie algebra over $\mathbb{Q}_p$, $\text{Aut}(\mathfrak{s})$ the group of automorphisms of $\mathfrak{s}$ and $S_+ := \text{Aut}(\mathfrak{s})_u \subset \text{Aut}(\mathfrak{s})$ the subgroup spanned by the one-parameter subgroups of $\text{Aut}(\mathfrak{s})$. We will say that $\mathfrak{s}$ is totally isotropic if $\mathfrak{s}$ is spanned by nilpotent elements. In this case $S_+$ is an open finite index subgroup of $\text{Aut}(\mathfrak{s})$, see [4, 6.14]. Since $\mathfrak{s} = [\mathfrak{s}, \mathfrak{s}]$, the group $S_+$ is perfect i.e. $S_+ = [S_+, S_+]$. In particular this group admits a universal central topological extension $\tilde{S}_+$, i.e. a group which is universal among the central topological extension of $S_+$. In [12] Prasad-Raghunathan were able to describe this group $\tilde{S}_+$ (special cases were obtained before by Moore, Matsumoto and Deodhar). We will only need here the fact that this group is a finite extension of $S_+$.

**Proposition 5.1. (Prasad–Raghunathan)** Let $\mathfrak{s}$ be a totally isotropic semisimple $p$-adic Lie algebra. Then the group $S_+$ admits a universal topological central extension

$$1 \longrightarrow Z_0 \longrightarrow \tilde{S}_+ \overset{\pi_0}{\longrightarrow} S_+ \longrightarrow 1$$

and its center $Z_0$ is a finite group.

The word universal means that for all topological central extension

$$1 \longrightarrow Z_E \longrightarrow E \overset{\pi}{\longrightarrow} S_+ \longrightarrow 1$$

where $E$ is a locally compact group and $Z_E$ is a closed central subgroup, there exists a unique continuous morphism $\psi : \tilde{S}_+ \to E$ such that $\pi_0 = \pi \circ \psi$.

**Proof.** See [12, Theorem 10.4].

**Remark 5.2.** This result does not hold for real Lie groups: indeed, the center $Z_0$ of the universal cover of $\text{SL}(2, \mathbb{R})$ is isomorphic to $\mathbb{Z}$.

**Corollary 5.3.** Let $\mathfrak{s}$ be a totally isotropic semisimple $p$-adic Lie algebra. For every topological central extension

$$1 \longrightarrow Z_E \longrightarrow E \overset{\pi}{\longrightarrow} S_+ \longrightarrow 1$$

with $E = [E, E]$, the group $Z_E$ is finite.
Proof of Corollary 5.3. Let $\psi : \tilde{S}_+ \to E$ be the morphism given by the universal property. Since, by Proposition 5.1, the group $Z_0$ is finite, the projection $\pi_0 = \pi \circ \psi$ is a proper map, hence $\psi$ is also a proper map and the image $\psi(\tilde{S}_+)$ is a closed subgroup of $E$. Since $E = \psi(\tilde{S}_+)Z_E$, one has the inclusion $[E, E] \subset \psi(\tilde{S}_+)$, and the assumption $E = [E, E]$ implies that the morphism $\psi$ is onto. Hence the group $Z_E = \psi(Z_0)$ is finite. □

Remark 5.4. For real Lie groups, the center $Z_E$ might even be non-discrete. Such an example is given by the quotient $E$ of the product $\mathbb{R} \times \tilde{SL}(2, \mathbb{R})$ by a discrete subgroup of $\mathbb{R} \times Z_0$ whose projection on $\mathbb{R}$ is dense.

5.2. The Levi decomposition of $G_u$.

We prove in this section that in a weakly regular $p$-adic Lie group $G$, the subgroup $G_u$ is closed and admits a Levi decomposition.

Let $G$ be a weakly regular $p$-adic Lie group and $\mathfrak{r}$ be the solvable radical of $\mathfrak{g}$. We recall that $\mathfrak{g}_{nc}$ is the smallest ideal of $\mathfrak{g}$ containing $\mathfrak{r}$ such that the group $\text{Ad}_{\mathfrak{g}/\mathfrak{g}_{nc}}(G)$ is bounded. Let $\mathfrak{s}$ be a Levi subalgebra of $\mathfrak{g}$ and $\mathfrak{s}_u := \mathfrak{s} \cap \mathfrak{g}_{nc}$.

We recall that $G_u$ is the subgroup of $G$ spanned by all the one-parameter subgroups of $G$, that $R_u$ is the subgroup of $G$ spanned by all the one-parameter subgroups of $G$ tangent to $\mathfrak{r}$, and we define $S_u$ as the subgroup of $G$ spanned by all the one-parameter subgroups of $G$ tangent to $\mathfrak{s}$. Note that we don’t know yet, but we will see it in the next proposition, that $S_u$ is indeed a closed subgroup with Lie algebra equal to $\mathfrak{s}_u$.

Proposition 5.5. Let $G$ be a weakly regular $p$-adic Lie group.

a) The group $R_u$ is closed. It is the largest normal algebraic unipotent subgroup of $G$.

b) The group $S_u$ is closed. Its Lie algebra is $\mathfrak{s}_u$, and the morphism $\text{Ad}_{\mathfrak{g}_{nc}} : S_u \to (\text{Aut} \mathfrak{s}_u)_u$ is onto and has finite kernel.

c) The group $G_u$ is closed. One has $G_u = S_uR_u$ and $S_u \cap R_u = \{e\}$.

Remark 5.6. In a real Lie group, the group tangent to a Levi subalgebra is not necessary closed, as for example, if $G = (S \times \mathbb{T})/Z$ where $S$ is the universal cover of $\text{SL}(2, \mathbb{R})$, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $Z$ is the cyclic subgroup spanned by $(z_0, \alpha_0)$ with $z_0$ a generator of the center of $S$ and $\alpha_0$ an irrational element of $\mathbb{T}$.

Proof of Proposition 5.5. a) This follows from Proposition 3.8.
b) Let $E := \overline{S_u}$ be the closure of $S_u$ and $S_+ := (\text{Aut } s_u)_u$. Note that this group $E$ normalizes $s_u$. We want to apply Corollary 5.3 to the morphism

$$E \xrightarrow{\pi} S_+$$

where $\pi$ is the adjoint action $\pi := \text{Ad}_{s_u}$.

We first check that the assumptions of Corollary 5.3 are satisfied. Since $E$ is weakly regular the kernel $Z_E$ of this morphism $\pi$ commutes with all the one-parameter subgroups tangent to $s_u$. Hence $Z_E$ is equal to the center of $E$. Since $S_+$ is spanned by one-parameter subgroups, and since by Proposition 4.4 any nilpotent element $X$ of $s_u$ is tangent to a one parameter subgroup $\varphi$ of $E$, this morphism $\pi$ is surjective.

Now, by Jacobson Morozov Theorem, for any nilpotent element $X$ of $s_u$, there exists an element $H$ in $s_u$ such that $[H, X] = X$. Let $\varphi : \mathbb{Q}_p \to G$ be the one-parameter subgroup tangent to $X$, which exists by Proposition 4.4. Since $G$ is weakly regular, one has, for $t$ in $\mathbb{Q}_p$ and $g_\varepsilon := \exp(\varepsilon H)$ with $\varepsilon$ small,

$$g_\varepsilon \varphi(t)g_\varepsilon^{-1}\varphi(t)^{-1} = \varphi(e^\varepsilon t)\varphi(-t) = \varphi((e^\varepsilon - 1)t).$$

This proves that $\varphi(\mathbb{Q}_p)$ is included in the derived subgroup $[E, E]$. In particular one has $E = [E, E]$.

According to Corollary 5.3 the kernel $Z_E$ is finite. In particular, one has $\dim E = \dim s_u$. Since $s_u$ is totally isotropic, one can find a basis of $s_u$ all of whose elements are nilpotent. By Lemma 1.7 all these elements are in $\mathfrak{g}_G$. Hence, by the implicit function theorem, the group $S_u$ is open in $E$. Therefore, $S_u$ is also closed and $S_u = E$.

c) Since the adjoint action of $R_u$ on $\mathfrak{g}_{nc}/\mathfrak{c}$ is trivial, the intersection $S_u \cap R_u$ is included in the kernel $Z_E$ of the adjoint map $\text{Ad}_{s_u}$. Since, by $b$), this kernel is finite, and since the algebraic unipotent group $R_u$ does not contain finite subgroups, one gets $S_u \cap R_u = \{e\}$.

It remains to check that $S_u R_u$ is closed and that $G_u = S_u R_u$. Thanks to Propositions 3.8 and 4.4 we can assume that $R_u = \{e\}$. In this case, we know from point $b$) that $S_u$ is closed and from Proposition 4.4 that $G_u = S_u$.

Here are a few corollaries. The first corollary is an improvement of Lemma 3.6.

**Corollary 5.7.** Let $G$ be a $p$-adic Lie group, $H$ a normal weakly regular closed subgroup of $G$ such that $H = H_u$, and $G' := G/H$.

a) One has the equality $G'_u = G_u/H$.

b) Let $X$ in $\mathfrak{g}$ and $X'$ its image in $\mathfrak{g}' = \mathfrak{g}/\mathfrak{h}$. One has the equivalence

$$X \in \mathfrak{g}_G \iff X' \in \mathfrak{g}'_{G'}.$$
c) One has the equivalence
\[ G \text{ is weakly regular} \iff G' \text{ is weakly regular}. \]

**Remark 5.8.** The assumption that \( H = H_u \) is important. For instance the group \( G = \mathbb{Q}_p \) and its normal subgroup \( H = \mathbb{Z}_p \) are weakly regular while the quotient \( G/H \) is not weakly regular.

**Proof.** We prove these three statements simultaneously. Since \( H = H_u \), according to Proposition 5.5, the group \( H \) admits a Levi decomposition \( H = S R \) where \( R \) is a normal algebraic unipotent Lie subgroup and \( S \) is a Lie subgroup with finite center \( Z \) whose Lie algebra is semisimple, totally isotropic, and such that the adjoint map \( \text{Ad}_{\mathfrak{s}} : S \to \text{Aut}(\mathfrak{s})_u \) is surjective. Note that \( R \) is also a normal subgroup of \( G \).

Let \( C \) be the centralizer of \( H = S \) in \( G \). Since \( H \) is normal in \( G \) and \( H = H_u \), \( H \) is weakly regular, \( C \) is also the kernel of the adjoint action of \( G \) on \( h = \mathfrak{s} u \). Therefore, by Proposition 5.5, the image of the group morphism \( H \times C \to G; (h, c) \mapsto hc \) has finite index in \( G \). Its kernel is isomorphic to \( H \cap C = Z \) and hence is finite. When this morphism \( H \times C \to G \) is an isomorphism, our three statements are clear. The general case reduces to this one thanks to Lemma 5.9 below. \( \square \)

**Lemma 5.9.** Let \( G \) be a locally compact topological group, \( Z \) be a finite central subgroup of \( G \) and \( \varphi : \mathbb{Q}_p \to G/Z \) be a continuous morphism. Then \( \varphi \) may be lifted as a continuous morphism \( \tilde{\varphi} : \mathbb{Q}_p \to G \).

**Proof.** Let \( H \) be the inverse image of \( \varphi(\mathbb{Z}_p) \) in \( G \). Then \( H \) is totally discontinuous. In particular it contains an open compact subgroup \( U \) such that \( U \cap Z = \{e\} \), so that \( U \) maps injectively in \( G/Z \). Let \( \ell \) be an integer such that \( \varphi(p^\ell \mathbb{Z}_p) \subset U Z/Z \). After rescaling, we can assume that \( \ell = 0 \). We let \( g_0 \) be the unique element of \( U \) such that \( \varphi(1) = g_0 Z \).

Since \( U \) maps injectively in \( G/Z \), we have \( g_0^{p^k} \xrightarrow{k \to \infty} e \).

Let \( X \) be the group of elements of \( p \)-torsion in \( Z \) and \( Y \) be the group of elements whose torsion is prime to \( p \). For any \( k \geq 0 \) pick some \( g_k \) in \( G \) such that \( \varphi(p^{\ell-k} \mathbb{Z}_p) = g_k Z \) and let \( x_k \) and \( y_k \) be the elements of \( X \) and \( Y \) such that \( g_k^{p^k} = g_0 x_k y_k \). We let \( z_k \) be the unique element of \( Y \) such that \( z_k^{p^k} = y_k \). Replacing \( y_k \) by \( g_k z_k^{-1} \), we can assume that \( z_k = e \).

Since \( x_k \) only takes finitely many values, we can find a \( x \) in \( X \) and an increasing sequence \( (k_n) \) such that, for any \( n, x_{k_n} = x \). Now, since \( x \) is a central \( p \)-torsion element, one has \( (g_0 x)^{p^k} \xrightarrow{k \to \infty} e \). Since, for any \( n \),
\(g_{k_n}^n = g_0 x\), there exists a unique morphism \(\tilde{\varphi} : \mathbb{Q}_p \to G\) such that, for any \(n\), \(\tilde{\varphi}(p^{-k_n}) = g_{k_n}\) and \(\tilde{\varphi}\) clearly lifts \(\varphi\). □

The second corollary is an improvement of Proposition 4.4.

**Corollary 5.10.** Let \(G\) be a weakly regular \(p\)-adic Lie group. One has the equality: \(\mathfrak{g}_G = \{X \in \mathfrak{g}_u \mid \text{ad}X \text{ is nilpotent}\}\), where \(\mathfrak{g}_u\) is the Lie algebra of \(G_u\).

**Proof.** This follows from Proposition 4.4 since, by Proposition 5.5, one has the equality \(\mathfrak{g}_u = \mathfrak{s}_u \oplus \mathfrak{r}_u\). □

The last corollary tells us that a weakly regular \(p\)-adic Lie group \(G\) with \(G = G_u\) is “almost” an algebraic Lie group.

**Corollary 5.11.** Let \(G\) be a weakly regular \(p\)-adic Lie group such that \(G = G_u\). Then there exists a Lie group morphism \(\psi : G \to H\) with finite kernel and cokernel where \(H\) is the group of \(\mathbb{Q}_p\)-points of a linear algebraic group defined over \(\mathbb{Q}_p\).

**Proof.** According to Proposition 4.4, \(G = G_u\) is a semidirect product \(G_u = S_u \ltimes R_u\). We choose \(H\) to be the semi direct product \(H := S' \ltimes R_u\) where \(S'\) is the Zariski closure of the group \(\text{Ad}(S_u)\) in \(\text{Aut}(\mathfrak{g}_u)\). Note that, since \(G\) is weakly regular, any automorphism of \(\mathfrak{g}_u\) induces an automorphism of \(R_u\). We define the morphism \(\psi : G_u \to H\) by \(\psi(g) = (\text{Ad}(s), r)\) for \(g = sr\) with \(s \in S_u, r \in R_u\). Proposition 4.4 tells us also that this morphism \(\psi\) has finite kernel and cokernel. □

### 5.3. Regular semiconnected component.

We are now ready to prove the following theorem which was the main motivation of our paper.

**Theorem 5.12.** Let \(G\) be a weakly regular \(p\)-adic Lie group. Then, there exists an open regular subgroup \(G_\Omega\) of \(G\) which contains all the one-parameter subgroups of \(G\).

**Remark 5.13.** Let \(\Omega\) be a standard open subgroup of \(G\). We define the \(\Omega\)-semiconnected component of \(G\) as its open subgroup \(G_\Omega := \Omega G_u\) (see [13]). In this language, Theorem 5.12 states that, the \(\Omega\)-semiconnected component of a weakly regular \(p\)-adic Lie group is regular, if the standard subgroup \(\Omega\) is small enough.

**Proof of Theorem 5.12.** We will need some notations. Let \(\mathfrak{s}\) be a Levi subalgebra of \(\mathfrak{g}\), \(\mathfrak{s}_u := \mathfrak{s} \cap \mathfrak{g}_{nc}\), \(\mathfrak{s}'\) the centralizer of \(\mathfrak{s}_u\) in \(\mathfrak{s}\), \(\mathfrak{r}\) the radical of \(\mathfrak{g}\), and \(\mathfrak{r}'\) the centralizer of \(\mathfrak{s}_u\) in \(\mathfrak{r}\).

We can choose a standard subgroup \(\Omega'_S\) of \(G\) with Lie algebra \(\mathfrak{s}'\), and a standard subgroup \(\Omega'_R\) of \(G\) with Lie algebra \(\mathfrak{r}'\) such that \(\Omega'_S\)
normalizes $\Omega_R'$ and the semidirect product $\Omega' := \Omega'_S \Omega'_R$ is a standard subgroup of $G$ with Lie algebra $\mathfrak{s}' \oplus \mathfrak{r}'$. Since $G$ is weakly regular, by shrinking $\Omega'$, we can assume that it commutes with $S_u$ and normalizes $R_u$.

We claim that, if $\Omega'$ is small enough, the group $$G_\Omega := \Omega' G_u$$ is an open regular subgroup of $G$.

**First step**: Openness. One has the equalities $\mathfrak{s} = \mathfrak{s}' \oplus \mathfrak{s}_u$ and, according to Proposition 3.8, $\mathfrak{r} = \mathfrak{r}' + \mathfrak{r}_u$. Hence $G_\Omega$ is open in $G$.

**Second step**: Ad-regularity. Let $J \subset G_\Omega$ be the kernel of $\text{Ad}\, g$. We want to prove that $J$ is the center of $G_\Omega$. Since $J$ acts trivially on the quotient $\mathfrak{g}/\mathfrak{r} \simeq \mathfrak{s}' \oplus \mathfrak{s}_u$, by Proposition 5.5 one has the inclusion $J \subset J' := Z_E \Omega'_R R_u$ where $Z_E$ is a finite subgroup of $S_u$. One has $J_u = J \cap R_u$ and this group is algebraic unipotent. Hence, by Lemma 3.6 the quotient $R_u/J_u$ is also algebraic unipotent. By Lemma 3.4, the group $R_u/J_u$ is closed in the group $J'/J$. This means that $R_u J$ is closed in $J'$, hence that $J/J_u$ is closed in the compact group $J'/R_u$. In particular, $J/J_u$ is compact. Therefore, there exists a compact set $K \subset J$ such that

$$J = K J_u.$$ 

We can now prove that if $\Omega'$ is small enough the group $J$ commutes with $G_\Omega$. This is a consequence of the following three facts.

(i) Since $J$ is the kernel of $\text{Ad}\, g$ and $G$ is weakly regular, $J$ commutes with $G_u$.

(ii) Since $J_u$ is a subgroup of $R_u$ whose Lie algebra $\mathfrak{j}_u$ is included in the center of $\mathfrak{g}$, if we choose $\Omega'$ small enough, one has $\text{Ad}_{J_u}(\Omega') = \{e\}$, and the group $J_u$ commutes with $\Omega'$.

(iii) Since $K$ is compact and $\text{Ad}_{\mathfrak{g}}(K) = \{e\}$, by Lemma 4.2 if we choose $\Omega'$ small enough, the group $K$ commutes with $\Omega'$.

**Third step**: Size of finite subgroups. We want a uniform upper bound on the cardinality of the finite subgroups of $G_\Omega$. This follows from the inclusions $R_u \subset G_u \subset G_\Omega$ of normal subgroups and from the following three facts.

(i) Since the group $R_u$ is algebraic unipotent, it does not contain finite groups.

(ii) Since, by Proposition 5.3 the group $G_u/R_u$ is a finite extension of a linear group, by Example 2.6, its finite subgroups have bounded cardinality.
(iii) Since the group $G_{\Omega}/G_u$ is a compact $p$-adic Lie group, by Example 2.6, its finite subgroups have bounded cardinality. $\square$

5.4. **Non weakly regular $p$-adic Lie groups.**

Not every $p$-adic Lie group is weakly regular. Here is a surprising example.

Example 5.14. There exists a $p$-adic Lie group $G$ with $G = G_u$ which does not contain any open weakly regular subgroup $H$ with $H = H_u$.

We will give the construction of such a group $G$ with Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{Q}_p)$, but we will leave the verifications to the reader.

We recall that the group $G_0 := \text{SL}(2, \mathbb{Q}_p)$ is an amalgamated product $G_0 = K \ast I_0 K$ where $K := \text{SL}(2, \mathbb{Z}_p)$ and $I_0 := \{k \in K \mid k_{21} \equiv 0 \text{ mod } p\}$ is an Iwahori subgroup of $K$. We define $G$ as the amalgamated product $G = K \ast I K$ where $I \subset I_0$ is an open subgroup such that $I \neq I_0$.

The morphism $G \to G_0$ is a non central extension. Using the construction in Lemma 4.1 one can check that $G$ is spanned by one-parameter subgroups. However, one can check that the universal central extension $\tilde{G}_0 \to G_0$ can not be lifted as a morphism $\tilde{G}_0 \to G$. 
References

[1] Y. Benoist, J.-F. Quint, Stationary measures and invariant subsets of homogeneous spaces (II), *Jour. Am. Math. Soc.* 26 (2013) 659-734.

[2] Y. Benoist, J.-F. Quint, Lattices in $S$-adic Lie groups, *Journal of Lie Theory* 24 (2014) 179-197.

[3] A. Borel, J. Tits, Groupes réductifs, *Publ. Math. IHES* 27 (1965) 55-150.

[4] A. Borel, J. Tits, Homomorphismes “abstraits” de groupes algébriques simples, *Ann. of Math.* 97 (1973) 499-571.

[5] N. Bourbaki, Groupes et Algèbres de Lie, chapitre 1, *CCLS* (1971).

[6] F. Bruhat, J. Tits *Groupes réductifs sur un corps local I*, Publ. IHES 41 (1972) p.5-252.

[7] F. Bruhat, J. Tits *Groupes réductifs sur un corps local II*, Publ. IHES 60 (1984) p.5-184.

[8] J. Dixon, M. duSautoy, A. Mann, D. Segal, *Analytic Pro-p Groups*, CUP (1991).

[9] S. Lang, *Algebra*, Addison-Wesley (1964).

[10] G. Margulis, G. Tomanov, Invariant measures for actions of unipotent groups over local fields on homogeneous spaces, *Invent. Math.* 116 (1994) 347-392.

[11] V. Platonov, A. Rapinchuk, *Algebraic groups and number theory*. Academic Press (1994).

[12] G. Prasad, M. Raghunathan, Topological central extensions of semi-simple groups over local fields, *Ann. of Math.* 119 (1984) 143-268.

[13] M. Ratner, Raghunathan’s conjectures for cartesian products of real and p-adic Lie groups, *Duke Math. J.* 72 (1995) 275-382.

[14] J.P. Serre, *Arbres, amalgames, SL2*, *Asterisque* 46 (1972).

[15] J. Tits, Reductive groups over local fields, *Pr. Sym. P. Math.* 33 (1979) 22-69.