Aschenbach effect for spinning particles in Kerr spacetime

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I. ABSTRACT

The orbital velocity profile of circular timelike geodesics in the equatorial plane of a Kerr black hole has a non-monotonic radial behavior, provided that the spin parameter $a$ of the black hole is bigger than a certain critical value $a_c \approx 0.9953$. Here the orbital velocity is measured with respect to the Locally Non-Rotating Frame (LNRF), and the non-monotonic behavior, which is known as the Aschenbach effect, occurs only for co-rotating orbits. Using the Mathisson-Papapetrou-Dixon equations for a massive spinning particle, we investigate the Aschenbach effect for test particles with spin. In addition to the black-hole spin, the absolute value of the particles spin and its orientation (parallel or anti-parallel to the black-hole spin) also play an important role for the Aschenbach effect. We determine the critical value $a_c$ of the spin parameter of the Kerr black hole where the Aschenbach effect sets in as a function of the particle spin. We find that for spinning particles in counter-rotating motion with anti-parallel spin around a naked singularity the orbital velocity is increasing on a certain radius interval.

PACS: 04.70.Bw 85.30.Sf

II. INTRODUCTION

As no signal can reach us from inside a black hole, the only way of observing a stationary black hole is by detecting its influence on matter or on light rays that come close to it. In particular, we may observe electromagnetic radiation emitted by matter that orbits a black hole in an accretion disk. As a first approximation, it is reasonable to assume that the particles in an accretion disk move on geodesics; this is true if they have no internal degrees of freedom, if they are not influenced by the interaction with neighboring particles or by external (non-gravitational) fields and if their self-gravity is negligible.

For particles moving on circular geodesics in the equatorial plane of the Kerr spacetime, Aschenbach [1, 2] made an interesting observation. He found that the orbital velocity might become an increasing function of the radius coordinate on some radius interval. This is in contrast to circular geodesic motion in the Schwarzschild metric, and also to circular motion in the Newtonian $1/r$ potential, where the orbital velocity is always a decreasing function of the radius coordinate, see e.g. Shapiro and Teukolsky [3]. More precisely, Aschenbach found that this non-monotonic behavior of the orbital velocity occurs only if the spin parameter, $a$, of the black hole satisfies an inequality $|a| \geq a_c$, where the critical value $a_c \approx 0.9953 M$ is close to the value of an extremal black hole, $|a| = M$, which characterizes the transition to a naked singularity. Moreover, the interval on which the orbital velocity is increasing occurs only for co-rotating, not for counter-rotating, orbits and it is close to but outside of the innermost stable circular orbit. Aschenbach related the non-monotonic behavior of the orbital velocity to the occurrence of certain resonances that could be observed (and, possibly, already have been observed with a few stellar black holes) as peaks
in the power spectrum of the emitted radiation. If the interpretation is correct, the observation of those peaks gives direct information on the spin of the black hole.

The non-monotonic behavior of the orbital velocity, called the Aschenbach effect for short, has also been discussed for (non-geodesic) motion with constant specific angular momentum in the Kerr spacetime [3] and for geodesic motion in the Kerr-(anti-)de Sitter spacetime [3, 6] and in braneworld generalizations of the Kerr spacetime [7].

Here we want to study the Aschenbach effect for particles with spin. To that end, we have to replace the geodesic equation with the Mathisson-Papapetrou-Dixon equations [8–10]. As in the original work by Aschenbach, we restrict to motion in the equatorial plane of the Kerr spacetime. The spinning particle might be a rapidly rotating neutron star or a rapidly rotating hot spot in an accretion disk. In either case, the mass of the particle must be small enough to be negligible in comparison to the mass of the black hole. In the case of a neutron star orbiting a stellar black hole, this puts, of course, some limits on the applicability. It is our main goal to find out how the critical value of the black hole spin and the radius interval in which the Aschenbach effect takes place is influenced by the particle's spin. We leave a discussion of the influence of the spin on the observable resonances for future work.

The paper is organized as follows. In Section III we recall some basic facts about the Mathisson-Papapetrou-Dixon equations. In Section IV we specialize to spinning particles in the equatorial plane of the Kerr metric, calculating in particular the orbital velocity. On the basis of these results, we discuss in Section V the Aschenbach effect for spinning particles.

Our conventions are as follows. The signature of the metric is (+, −, −, −) and we use units where the speed of light is c = 1. We raise and lower indices with the spacetime metric, using Einstein’s summation convention for greek indices running from 0 to 3. Our index conventions for the curvature tensor are such that

\[ R^\tau_{\sigma\mu\nu} = \partial_\mu \Gamma^\tau_{\nu\sigma} - \partial_\nu \Gamma^\tau_{\mu\sigma} + \Gamma^\tau_{\mu\rho} \Gamma^\rho_{\nu\sigma} - \Gamma^\tau_{\nu\rho} \Gamma^\rho_{\mu\sigma}. \quad (1) \]

III. MATHISSON-PAPAPETROU-DIXON EQUATIONS

On a spacetime with curvature tensor \( R^\mu_{\nu\sigma\tau} \), the motion of a spinning particle is determined in the pole-dipole approximation by the Mathisson-Papapetrou-Dixon equations [8–10],

\[ \frac{D}{ds} x^\mu = u^\mu, \quad (2) \]

\[ \frac{D}{ds} p^\mu = -\frac{1}{2} R^\mu_{\nu\sigma\tau} u^\nu S^{\sigma\tau}, \quad (3) \]

\[ \frac{D}{ds} S^{\mu\nu} = p^\mu u^\nu - p^\nu u^\mu. \quad (4) \]

Here \( x^\mu(s) \) is the worldline of a reference point inside the particle, \( u^\mu(s) \) is the corresponding 4-velocity, and \( \frac{D}{ds} \) denotes covariant derivative, in the direction of \( u^\mu(s) \), of tensor fields along the worldline \( x^\mu(s) \). In a given spacetime background, this is a system of first-order ordinary differential equations for the worldline \( x^\mu(s) \), the momentum \( p_\nu(s) \) and the spin tensor \( S^{\mu\nu}(s) = -S^{\nu\mu}(s) \) of the particle. Note that this system of equations is invariant under arbitrary reparametrizations,

\[ \frac{D}{ds} \mapsto k \frac{D}{ds}, \quad u^\mu \mapsto ku^\mu, \quad p_\mu \mapsto p_\mu, \quad S^{\mu\nu} \mapsto S^{\mu\nu}. \quad (5) \]

For our purpose, we find it convenient to choose the proper time parametrization,

\[ g_{\mu\nu} u^\mu u^\nu = 1. \quad (6) \]

If \( u^\mu \) and \( p^\mu \) are timelike and future-oriented, which is usually required for physically reasonable solutions, we may define two real and positive quantities

\[ \mu := \sqrt{g_{\rho\sigma} p^\rho p^\sigma}, \quad m := g_{\mu\nu} u^\mu p^\nu. \quad (7) \]

\( \mu \) is the mass of the particle in the center-of-momentum system whereas \( m \) is the mass in the rest system of an observer comoving along the worldline \( x^\rho(s) \). In general, neither \( m \) nor \( \mu \) is guaranteed to be a constant of motion.
As the system of Mathisson-Papapetrou-Dixon equations is underdetermined, we have to add a supplementary condition

\[ V^\rho S_{\rho\nu} = 0. \]  

(8)

Here \( V^\rho \) is a timelike vector field along the worldline \( x^\mu(s) \) we are free to choose at will. For convenience, we will assume \( V^\rho \) to be normalized according to \( V^\rho V_\rho = 1 \). The most common choices for \( V^\rho \) are \( V^\rho = p^\rho/\mu \) (Tulczyjew-Dixon condition [11, 12]) and \( V^\rho = u^\rho \) (Frenkel-Mathisson-Pirani condition [8, 13, 14]). It is well known that \( \mu \) is a constant of motion if the Tulczyjew-Dixon condition is imposed whereas \( m \) is a constant of motion if the Frenkel-Mathisson-Pirani condition is imposed.

As soon as we have fixed the vector field \( V^\mu \), we can express the spin tensor \( S_{\mu\nu} \) in terms of a spin vector \( S^\rho \),

\[ S_{\mu\nu} = \varepsilon_{\mu\nu\sigma\rho} V^\sigma S^\rho, \quad S^\rho V^\rho = 0, \]  

(9)

where \( \varepsilon_{\mu\nu\sigma\rho} \) is the totally antisymmetric Levi-Civita tensor field (volume form) of the spacetime metric.

The ambiguity in choosing a supplementary condition is understood if we recall that a body with a given spin different from zero must have a minimum size, just to make sure that no parts of the body move at a superluminal speed [15]. Choosing a supplementary condition corresponds to choosing a particular worldline \( x^\mu(s) \) inside the worldtube of such a finite-size body.

IV. SPINNING PARTICLE IN THE EQUATORIAL PLANE OF THE KERR SPACETIME

We now specify the background metric to the Kerr metric which reads, in standard Boyer-Lindquist coordinates, [16]

\[ g_{\mu\nu} dx^\mu dx^\nu = \left(1 - \frac{2Mr}{\rho^2}\right) dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - \sin^2 \theta \left(r^2 + a^2 + \frac{2Mra^2 \sin^2 \theta}{\rho^2}\right) d\phi^2 + \frac{4Mar \sin^2 \theta}{\rho^2} dt d\phi \]  

(10)

where

\[ \rho^2 := r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 + a^2 - 2Mr. \]  

(11)

Here \( M \) is the mass parameter and \( a \) is the spin parameter. Both have the dimension of a length. For \( a^2 \leq M^2 \) we have a black hole whereas for \( a^2 > M^2 \) we have a naked singularity.

We want to consider the Mathisson-Papapetrou-Dixon equations with a supplementary condition (8), where for the time being \( V^\mu \) is specified only to be the 4-velocity field of observers in circular motion,

\[ V^\mu \partial_\mu = V^t \partial_t + V^\phi \partial_\phi. \]  

(12)

We are interested in circular motion in the equatorial plane,

\[ \dot{\vartheta} = \pi/2, \quad u^\mu \partial_\mu = u^t \partial_t + u^\phi \partial_\phi, \]  

(13)

with the spin perpendicular to the equatorial plane,

\[ S^\mu \partial_\mu = S^t \partial_t, \quad S^t = -\frac{S}{r}. \]  

(14)

Here \( S \) is a constant of motion,

\[ g_{\mu\nu} S^\mu S^\nu = -S^2, \]  

(15)

that may be positive or negative. We have \( aS > 0 \) if the spin of the particle is parallel to the spin of the black hole and \( aS < 0 \) if it is anti-parallel.

Under these assumptions, evaluating all components of the Mathisson-Papapetrou-Dixon equation (4) yields

\[ p^r = 0, \quad p^\vartheta = 0, \]  

\[ -Su^\varphi V^\varphi + \frac{MS}{r^3} (u^t - au^\varphi) (V^t - aV^\varphi) = (\rho^t - ap^\varphi) u^\varphi - p^\varphi (u^t - au^\varphi). \]  

(16)
Similarly, from (3) we find
\[ \frac{dp^t}{ds} = 0, \quad \frac{dp^\varphi}{ds} = 0, \]
and
\[ M r^2 (p^t - a p^\varphi) (u^t - a u^\varphi) - r^5 p^\varphi u^\varphi = 0. \]
\[ M r^2 (p^t - a p^\varphi) (u^t - a u^\varphi) - r^5 p^\varphi u^\varphi = -3 M S a (u^t - a u^\varphi) (V^t - a V^\varphi) + M S r^2 \left( 2 (u^t - a u^\varphi) V^\varphi + u^\varphi (V^t - a V^\varphi) \right). \] (17)

A. Tulczyjew-Dixon condition

If the Tulczyjew-Dixon supplementary condition \( V^\rho = p^\rho / \mu \) is imposed, \( \mu \) is a constant of motion and it is convenient to characterize the particle’s spin by the dimensionless parameter
\[ s = \frac{S}{M \mu}. \] (18)

Eqs. (16) and (17) specify to
\[ p^t - a p^\varphi = \frac{u^t - a u^\varphi - s M u^\varphi}{u^\varphi - M^2 s}{r^3 (u^t - a u^\varphi)}, \] (19)
\[ p^t - a p^\varphi = \frac{-r^5 M u^\varphi + 2 s M r^2 (u^t - a u^\varphi)}{r^2 (u^t - a u^\varphi) + 3 a s M (u^t - a u^\varphi) - s M r^2 u^\varphi}. \] (20)

If we introduce the angular velocity
\[ \Omega = \frac{u^\varphi}{u^t}, \] (21)
equating the right-hand sides of (19) and (20) yields
\[ \left( 1 + \frac{3 a s M}{r^2} + \frac{2 M^3 s^2}{r^4} \right) (\Omega^{-1} - a)^2 - 3 s M \left( 1 + \frac{a s M}{r^2} \right) (\Omega^{-1} - a) - \frac{r^3}{M} + s^2 M^2 = 0. \] (22)

This is a quadratic equation for \( (\Omega^{-1} - a) \) with solutions
\[ \Omega_{\pm}^{-1} - a = \frac{3 M^2 r^3 s + 3 a M r s^2 \pm \sqrt{M r D}}{2 M r^3 + 6 a M^2 r s + 4 M^4 s^2}. \] (23)

with
\[ D = 4 r^7 + 12 M a r^5 s + 13 M^3 r^4 s^2 + 6 M^4 a r^2 s^3 + (9 a^2 - 8 M r) M^5 s^4. \] (24)

Note that, because of the normalization condition (6), we have
\[ g_{tt} \Omega^{-2} + 2 g_{t\varphi} \Omega^{-1} + g_{\varphi\varphi} = \frac{1}{(u^\varphi)^2}. \] (25)

After inserting the metric coefficients the condition of \( 1/(u^\varphi)^2 > 0 \) requires that
\[ \left( 1 - \frac{2 M}{r} \right) (\Omega^{-1} - a)^2 + 2 a (\Omega^{-1} - a) - r^2 > 0. \] (26)

This inequality makes sure that the 4-velocity of the particle is timelike, i.e., that the motion is subluminal. If, at a certain radius value \( r \), the discriminant \( D \) defined in (24) is negative, then there is no solution to our motion problem at
this radius value. If $D$ is non-negative, there may be two solutions (typically one co-rotating and the other counter-rotating), one solution or no solution, depending on whether (26) is satisfied for both $\Omega = \Omega_+$ and $\Omega = \Omega_-$, only for one of them, or for neither of them.

For defining the orbital velocity we introduce the Locally Non-Rotating Frame (LNRF)\textsuperscript{[17, 18]}

$$e_0 = \frac{-g_{\varphi\varphi}\partial_t + g_{t\varphi}\partial_\varphi}{\sqrt{-g_{\varphi\varphi}(g_{t\varphi}^2 - g_{\varphi\varphi}g_{tt})}}, \quad e_1 = \frac{\partial_t}{\sqrt{-g_{rr}}}, \quad e_2 = \frac{\partial_\varphi}{\sqrt{-g_{\varphi\varphi}}}, \quad e_3 = \frac{\partial_\theta}{\sqrt{-g_{\varphi\varphi}}}.$$ (27)

This is an orthonormal tetrad if $\Delta > 0$, i.e., everywhere except between the two horizons. Observers with 4-velocity $e_0$ are also known as Zero Angular Momentum Observers (ZAMOs).

For circular motion, the orbital velocity $V$ with respect to the LNRF is determined by

$$u^t \partial_t + u^r \partial_r = N \left( e_0 + V e_3 \right)$$ (28)

where $N$ is a scalar factor. For timelike orbits $V$ takes values between $-1$ and $1$. Comparing coefficients of $\partial_t$ and $\partial_\varphi$ in (28) allows us to express $\Omega = u^\varphi/u^t$ in terms of $V$,

$$\Omega = \frac{g_{t\varphi} + V \sqrt{g_{t\varphi}^2 - g_{\varphi\varphi}g_{tt}}}{-g_{\varphi\varphi}}.$$ (29)

After inserting the metric coefficients and solving for $V$ we find

$$V = \frac{(r^2 + a^2)^2 - a^2\Delta}{r^2 \sqrt{\Delta}} (r^2 + 3a^2 + 3Mr).$$ (30)

With $\Omega = \Omega_\pm$ from (26), this gives us two solutions for the orbital velocity,

$$V_\pm = -\frac{2Ma}{r\sqrt{\Delta}} - \frac{\left(2Mar^3 + 3M^2r(2a^2 + r^2)s + M^3a(4M + 3r)s^2 \pm \sqrt{Mr}\sqrt{D}\right) - 2Ma^2 - ar^3 + 3Mr^3}{\left(2Ma^2r^3 - 2r^6 + 6Mr^2\left(a^2 + r^2\right)s + 2M^3r^3 + 3a^2r^2 + 2Ma^2\right)r\sqrt{\Delta}}.$$ (31)

For the existence of an orbit with velocity $V_+$ (or $V_-$, respectively) at radius value $r$ it is necessary and sufficient that $D$ is non-negative and that $|V_+| < 1$ (or $|V_-| < 1$, respectively).

Far away from the center, (23) and (31) may be approximated as

$$\Omega_\pm = \pm \frac{\sqrt{M}}{\sqrt{r^3}} \left(1 + O((M/r)^{1/2})\right),$$ (32)

$$V_\pm = \pm \frac{\sqrt{M}}{\sqrt{r^3}} \left(1 + O((M/r)^{1/2})\right).$$ (33)

From these equations, we read that, for any choice of $a$ and $s$, there are two circular orbits at all sufficiently large radius values; by (33), the label + refers to an orbit with positive $V$, i.e., a particle moving in the positive $\varphi$ direction with respect to the ZAMOs, whereas the label - refers to an orbit with negative $V$, i.e., a particle moving in the negative $\varphi$ direction with respect to the ZAMOs. This means that for $a > 0$ the + orbit is co-rotating and the - orbit is counter-rotating; for $a < 0$ it is vice versa. Far away from the center, $V_+$ goes monotonically to zero from above and $V_-$ goes monotonically to zero from below. It is the subject of this paper to investigate if and how this monotonic behavior changes closer to the central object. We will see that $V_+$ and $V_-$ may change sign so that it is not always true that for $a > 0$ the + orbit is co-rotating and the - orbit is counter-rotating. Also, $|V_+|$ and $|V_-|$ may become bigger than $1$; in regions where this happens the corresponding orbit does not exist at all.

Taylor expansion with respect to the spin parameter $s$ of (23) and (31) yields

$$\Omega_\pm = \frac{\sqrt{M}}{a\sqrt{M} \pm \sqrt{r^3}} \pm \frac{3\sqrt{M^3}(a \mp \sqrt{Mr})}{2\sqrt{r}\left(a\sqrt{M} \pm \sqrt{r^3}\right)^2} s.$$
\[ \pm \frac{3M^2((M-4r)a\sqrt{Mr} - 9a^3\sqrt{M} \pm (8M-3r)a^2\sqrt{r} \pm 7M\sqrt{r^5})}{8r^2(a\sqrt{M} \pm \sqrt{r^3})^3} s^2 + O(s^3) \]  
\tag{34}

and

\[ V_\pm = \frac{\sqrt{M}(r^2 + a^2 \pm 2a\sqrt{Mr})}{\sqrt{\Delta}(a\sqrt{M} \pm \sqrt{r^3})} \pm \frac{3\sqrt{M^3}(r^3 + 2a^2r + 2M^2a - Mr)}{2\sqrt{r^3}\sqrt{\Delta}(a\sqrt{M} \pm \sqrt{r^3})^2} s \]
\[ \pm \frac{3\sqrt{M^5}(M-4r)a\sqrt{Mr} - 9a^3\sqrt{M} \pm (8M-3r)a^2\sqrt{r} \pm 7M\sqrt{r^5})}{8\sqrt{r^7}\sqrt{\Delta}(a\sqrt{M} \pm \sqrt{r^3})^3} s^2 + O(s^3). \]  
\tag{35}

For vanishing spin, \( s = 0 \), we recover the well-known equations for circular geodesics.

### B. Frenkel-Mathisson-Pirani condition

If the Frenkel-Mathisson-Pirani supplementary condition \( V^\rho = u^\rho \) is imposed, Eq. (16) implies

\[ p^t - ap^\varphi = m(u^t - au^\varphi) + r^2S(u^\varphi)^3 - aS(u^\varphi)^2(u^t - au^\varphi) - \frac{M}{r}Su^\varphi(u^t - au^\varphi)^2 + \frac{aM}{r^3}S(u^t - au^\varphi)^3 \]  
\tag{36}

and

\[ p^\varphi = m u^\varphi + aS(u^\varphi)^3 + \left(1 - \frac{2M}{r}\right)S(u^\varphi)^2(u^t - au^\varphi) - \frac{aMS}{r^3}u^\varphi(u^t - au^\varphi)^2 - \left(1 - \frac{2M}{r}\right)\frac{MS}{r^5}(u^t - au^\varphi)^3. \]  
\tag{37}

In this case \( m \) is a constant of motion, so we characterize the particle's spin by the dimensionless parameter

\[ \tilde{s} = \frac{S}{Mm}. \]  
\tag{38}

Then inserting (37) into (17) and using (25) yields a fourth-order equation for \( \Omega^{-1} - a \),

\[ \left(1 - \frac{2M}{r} + \left(\frac{3a}{r^2} - \frac{5aM}{r^3}\right)M\tilde{s}\right)(\Omega^{-1} - a)^4 + \left(2a + \left(\frac{3M}{r} - 2 + \frac{6a}{r^2}\right)M\tilde{s}\right)(\Omega^{-1} - a)^3 \]
\[ + \left(r^2 - \frac{r^3}{M} - 9aM\tilde{s}\right)(\Omega^{-1} - a)^2 - \left(\frac{2ar^3}{M} + \left(6Mr^2 - r^3\right)\tilde{s}\right)(\Omega^{-1} - a) + \frac{r^5}{M} - a^2\tilde{s} = 0. \]  
\tag{39}

This equation can be analytically solved for \( \Omega^{-1} - a \), using a standard method for solving a fourth-order equation, but the resulting expressions are quite awkward and will not be given here. If we Taylor expand with respect to \( \tilde{s} \), we find that the four solutions (\( \Omega_+, \Omega_-, \Omega_+, \Omega_- \)) are

\[ \Omega_\pm = \frac{\sqrt{M}}{a\sqrt{M} \pm \sqrt{r^3}} \pm \frac{3\sqrt{M^3}(a \mp \sqrt{Mr})}{2\sqrt{r}(a\sqrt{M} \pm \sqrt{r^3})^2} \tilde{s} + \frac{3\sqrt{M^5}(6r^2(a \mp \sqrt{Mr})^2 - (a\sqrt{M} \pm \sqrt{r^3})K_\pm)}{8r^3(a\sqrt{M} \pm \sqrt{r^3})^3} \tilde{s}^2 + O(\tilde{s}^3) \]  
\tag{40}

where

\[ K_\pm = \frac{(9r - 43M)a^2r + (3M - r)Mr^2 + 2(9a^2 + 11Mr - 4r^2)a\sqrt{Mr}}{2a\sqrt{M} \pm \sqrt{r(r - 3M)}} \]  
\tag{41}

and

\[ \hat{\Omega}_\pm = \pm \frac{2M - r}{2aM \pm r\sqrt{\Delta}} + \frac{M((r^2 - 5Mr + 6M^2)r - 2aM(a \pm \sqrt{\Delta}))}{2r(2aM \pm r\sqrt{\Delta})^2} \tilde{s} + \frac{M^3\left(\sqrt{\Delta}Q \pm 2MP\right)}{8Mr^3\sqrt{\Delta}(2aM \pm r\sqrt{\Delta})^3} \tilde{s}^2 + O(\tilde{s}^3), \]  
\tag{42}
where
\[ Q = 8a^4M(5M + 3r) - 8a^2Mr(9M^2 + 6M - 5r^2) + r^3(66M^3 - 69M^2r + 20Mr^2 - r^3) \] (43)
and
\[ P = 4a^5(5M + 3r) - 2a^3r(28M^2 + 13Mr - 13r^2) + 3ar^2(10M^3 + 11M^2r - 18Mr^2 + 5r^3). \] (44)

The corresponding orbital velocities read
\[
\mathcal{V}_\pm = \frac{\sqrt{M}(r^2 + a^2 \mp 2a\sqrt{Mr})}{\sqrt{\Delta}(\pm a\sqrt{M} \pm \sqrt{r^3})} \pm \frac{3\sqrt{M^3}(r^3 + a^2r + 2Ma^2)(a \mp \sqrt{\Delta})}{2\sqrt{\Delta}(a\sqrt{M} \pm \sqrt{r^3})^2} \hat{s}
\]
\[ + \frac{3\sqrt{M}r(a \mp \sqrt{Mr})(3(a \mp \sqrt{Mr})^2 - (a\sqrt{M} \pm \sqrt{r^3})K_\pm)}{4r^2(a\sqrt{M} \pm \sqrt{r^3})^2\Delta} \hat{s}^2 + O(\hat{s}^3) \] (45)
and
\[
\hat{\mathcal{V}}_\pm = \mp 1 + \frac{M(r^3 + a^2r + 2Ma^2)(r^3 - 5Mr^2 + 6M^2r - 2aM(a \pm \sqrt{\Delta}))}{2r^2\sqrt{\Delta}(2Ma \pm r\sqrt{\Delta})^2} \hat{s}
\]
\[ \pm \frac{M^2(r^3 + a^2r + 2Ma^2)(2MP \pm \sqrt{\Delta}Q)}{8r^4\Delta(2Ma \pm r\sqrt{\Delta})^2} \hat{s}^2 + O(\hat{s}^3). \] (46)

We see that the hatted solutions are unphysical because $|\mathcal{V}_\pm|$ becomes the velocity of light for $\hat{s} \to 0$, both for the $+$ and the $-$ branch. The orbital velocity is even superluminal for small spin values of one sign. This observation was made already in a paper by Costa et al. [19] where the four exact solutions are worked out for the Schwarzschild case $a = 0$. If we discard the hatted solutions, we are left with two solutions, given in [44] and [45], that have, indeed, the correct geodesic limit for $\hat{s} \to 0$. They coincide not only to zeroth but also to first order with the solutions from the Tulczyjew-Dixon condition, see [34] and [35], where $s$ in the Tulczyjew-Dixon case has to be replaced by $\hat{s}$ in the Frenkel-Mathisson-Pirani case. The second and higher-order terms, however, are different. This confirms the known result that different supplementary conditions give different results unless one linearizes with respect to the particle’s spin.

V. ASCHENBACH EFFECT IN THE EQUATORIAL PLANE OF KERR SPACETIME

We consider orbits of spinning particles with the Tulczyjew-Dixon condition, i.e., with orbital velocity given by [31]. To within linear approximations with respect to the spin, the results are valid also for other supplementary conditions. Without loss of generality, we assume $a > 0$. Then $s = S/(M\mu)$ is positive if the particle’s spin is parallel to the spin of the black hole and negative if it is anti-parallel.

Our first task is to find out for which values of the relevant parameters the Aschenbach effect occurs, i.e., for which values of $a$ and $s$ there is an interval of the radius coordinate on which the orbital velocity is increasing with the radius coordinate. We first consider the black-hole case, $a \leq M$, and we restrict to the domain of outer communication, i.e., to the region outside of the outer horizon. By inspection, we find that $|\mathcal{V}_-|$ is always monotonically decreasing with $r$ on this domain, so we only have to discuss $\mathcal{V}_+$. On the considered domain, $\mathcal{V}_+$ is always positive, i.e., it describes co-rotating orbits.

Eliminating from the two equations
\[
d\mathcal{V}_+/dr = 0, \quad d^2\mathcal{V}_+/dr^2 = 0 \tag{47}
\]
the radius coordinate $r$ and solving for $a$ gives us the critical value $a = a_c(s)$ of the black-hole spin where the Aschenbach effect sets in. This can be done only numerically. Figure [4] shows the result. In this figure the region where $\mathcal{V}_+$ increases with $r$ is shown shaded (in orange). The lower boundary curve of this region gives the critical
black-hole spin, \( a_c(s) \), as a function of \( s \). For each value of \( s \), the non-monotonic behavior of \( V_+ \) as a function of \( r \) is present for all values \( a > a_c(s) \); at \( a_c(s) \) this function has a saddle-point. We have cut off the shaded (orange) region at \( a = M \) because at the moment we are only considering the black-hole case, not the naked-singularity case.

We see that for spinless particles \((s = 0)\) the Aschenbach effect sets in at \( a_c(0) \approx 0.9953M \), which is the result that was found in the original work by Aschenbach [1, 2]. If \( s \) is negative (i.e., if the spin of the particle is anti-parallel to the spin of the black hole), \( a_c(s) \) is bigger than \( a_c(0) \). If \( s \) is positive (i.e., if the spin of the particle is parallel to the spin of the black hole), \( a_c(s) \) can be smaller than \( a_c(0) \). The minimum value of \( a \) where the Aschenbach effect could set in is at \( a_c(s) \approx 0.9810M \) which happens for a particle spin of \( s \approx 0.47 \). So we see that for spinning particles the critical value of \( a \) may be reduced only by about one percent in comparison to the case of spinless particles.

An interesting result which we can also read from Figure 1 is that for large (positive or negative) spins there is no Aschenbach effect around black holes: We have to restrict to

\[-0.11 < s < 1.05,\]  

otherwise a non-monotonic behavior of \( V_+ \) occurs only for naked singularities \((a > M)\) but not for black holes.

For small values of the spin parameter \( s \) we may restrict to a Taylor approximation of \( a_c(s) \),

\[a_c(s) = \left( 0.9953 - 0.0517s - 0.0164s^2 + O(s^3) \right)M, \]

which again was found numerically. The radius coordinate where the saddle-point occurs is at

\[r_c(s) = \left( 1.5363 + 1.2155s - 3.9655s^2 + O(s^3) \right)M.\]
FIG. 2: Region outside of a Kerr black hole where \( dV_+ / dr > 0 \) for a spinless particle, \( s = 0 \).

It is also instructive to view the domain where \( V_+ \) is increasing with \( r \) in an \( r - a \) diagram. This is shown in Figures 2, 3, and 4 for \( s = 0, s > 0 \) and \( s < 0 \), respectively. In all three pictures the region between the horizons is shown in black. The parameter \( a \) is restricted to values \( a < M \) and only the domain of outer communication is considered. We have already said that in this domain \( V_+ \) is always positive. In the pictures the region where \( dV_+ / dr > 0 \) is shown cross-hatched (in orange). For \( s > 0 \) this region may be bigger than for the spinless case, whereas for \( s < 0 \) it is always smaller.

FIG. 3: Region outside of a Kerr black hole where \( dV_+ / dr > 0 \) for a particle with spin parallel to the spin of the black hole, \( s = 0.05 \).
FIG. 4: Region outside of a Kerr black hole where $dV_+/dr > 0$ for a particle with spin anti-parallel to the spin of the black hole, $s = -0.05$.

In Figure 5 we show the non-monotonic behavior of $V_+$ for parallel and anti-parallel particle spin in comparison to the case of a spinless particle.

FIG. 5: $V_+$ as a function of $r$, for $a = 0.997 M$ and $s = 0$ (solid, red), $s = 0.05$ (dashed, blue) and $s = -0.05$ (dotted, black).

Having clarified what happens for black holes in the domain of outer communication, we now briefly discuss the Aschenbach effect in the entire parameter space, i.e., we allow $a$ to take values bigger than $M$ and we also consider, in the black-hole case, the domain inside the inner horizon. (Between the horizons no timelike circular orbits can exist.) As in the equatorial plane the passage through $r = 0$ is blocked by the ring singularity, we do not consider the domain where $r < 0$.

We first consider the $+$ branch of solutions. Figures 6, 7 and 8 show the entire domain where the Aschenbach effect takes place for $s = 0$, $s > 0$ and $s < 0$, respectively. This domain is characterized by the properties that the discriminant $D$ is positive, $V_+$ lies between $-1$ and $1$, and $|V_+|$ increases with $r$. We see that this is the union of
two domains: On the first one, shown cross-hatched (in orange), $\mathcal{V}_+^+$ is positive, i.e., the orbits are co-rotating. On the second one, shown hatched (in blue), $\mathcal{V}_+$ is negative, i.e., the orbits are counter-rotating. Only the cross-hatched (orange) region has an intersection with the domain of outer communication of black holes; this intersection was shown, enlarged, in Figures 2, 3 and 4. We see that the Aschenbach effect is largely taking place in the naked-singularity domain. In Figure 9 we show the non-monotonic behavior of $\mathcal{V}_+$ in a naked-singularity spacetime for parallel and anti-parallel particle spin in comparison to the case of a spinless particle.

![Diagram](image1)

**FIG. 6:** Entire domain where $|\mathcal{V}_+|^{+}$ is increasing with $r$, for $s = 0$

![Diagram](image2)

**FIG. 7:** Entire domain where $|\mathcal{V}_+|$ is increasing with $r$, for $s = 0.05$.
Whereas in the domain of outer communication of a black hole the velocity $|V_-|$ is always decreasing with $r$, this is no longer true if we consider the entire parameter space. For any value of $a > 1$ and negative spin values in a certain interval that depends on $a$, the velocity $|V_-|$ is monotonically increasing on a certain $r$ interval. This interval is bounded on the lower side by a radius value where $V_-$ is zero which means that the particles are hovering at rest with respect to the ZAMOs, and on the upper side by a limiting radius where $V_- = -1$ which corresponds to a counter-rotating orbit at the speed of light. There is no minimum-maximum structure. This region is shown in Figure 10 cross-hatched (in orange). For the picture we have chosen the rather big value of $s = -0.8$ because for smaller values the region would be so narrow that it could hardly be seen.

FIG. 8: Entire domain where $|V_+|$ is increasing with $r$, for $s = -0.05$.

FIG. 9: $V_+$ versus $r$, for $a = 1.001 M$ and $s = 0$ (solid, red), $s = 0.05$ (dashed, blue) and $s = -0.05$ (dotted, black).
VI. CONCLUSIONS

Up to now, we have much better information on the masses than the spins of black-hole candidates. In our view, the astrophysical relevance of the Aschenbach effect is in the fact that it provides a method of determining the spins of (some) black holes because its occurrence is associated with a certain parametric resonance of vertical and radial epicyclic oscillations. The latter are observable as peaks in the power spectrum emitted from matter orbiting the black hole.

The main motivation of the present paper is in the fact that we wanted to investigate if and how Aschenbach’s results are modified if the radiating source is spinning. If we think of a hot spot, orbiting the black hole in an accretion disk, this modification might be non-negligible, in particular if we want to rely on the value of $a_c$ up to several digits after the decimal point. The results obtained in this paper could, of course, also be applied to a neutron star orbiting a black hole. To be sure, as we worked with the Mathisson-Papapetrou-Dixon equations throughout, in any case one has to be aware of the fact that we restricted to situations where the test-particle approximation is valid.

Our analysis was based on the exact (i.e., fully analytical) solutions for the orbital velocity in the Locally Non-Rotating Frame (LNRF), for a spinning test particle. Thereupon, we have numerically determined the critical value of the black-hole spin parameter, $a_c$, where the Aschenbach effect sets in, in dependence of the spin parameter $s$ of the test particle. This is only a first, but crucially important, step towards our goal. The second step would be to investigate the influence of the particle’s spin on the parametric resonances. We are planning to do this in a follow-up paper.

We have investigated in this paper not only the case of black holes but also of naked singularities. The latter are, of course, much more speculative than black holes. However, we believe that it should be kept in mind that the Aschenbach effect occurs also for naked singularities, and even in a much wider parameter range than for black holes, and that, for the discussion of parametric resonances, the case of a naked singularity should not be completely ignored.

Acknowledgment

VP is grateful to Oldřich Semerák for helpful discussions on the motion of spinning particles in general relativity. JK wishes to thank ZARM, Bremen, for hospitality where part of this work was done. Moreover, VP gratefully
acknowledges support from Deutsche Forschungsgemeinschaft within the Research Training Group 1620 “Models of Gravity”.

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