Massive scalar field near a cosmic string

Devis Iellici

Dipartimento di Fisica, Università di Trento
and Istituto Nazionale di Fisica Nucleare,
Gruppo Collegato di Trento, Italia

Abstract. The ζ function of a massive scalar field near a cosmic string is computed and then employed to find the vacuum fluctuation of the field. The vacuum expectation value of the energy-momentum tensor is also computed using a point-splitting approach. The obtained results could be useful also for the case of self-interacting scalar fields and for the finite-temperature Rindler space theory.

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I. Introduction

In the last few years many authors have considered quantum fields on spacetimes containing conical singularities. These kind of manifolds are relevant for studying fields in presence of real conical singularities, namely existing in the Lorentzian section of the manifold, as in the case of idealized cosmic strings and of 2+1-dimensional gravity. In other cases, conical singularities appear in the finite temperature theory, when adopting the Euclidean path integral with periodic imaginary time formalism. It is the case of spacetimes containing horizons, such as the Rindler and the Schwarzschild spaces, which are much studied in the off-shell approach to the one-loop quantum corrections to the Bekenstein-Hawking entropy. It is worth noticing that, in these latter cases, the conical singularity could be avoided adopting a canonical approach.

As a consequence of this interest in conical spaces, many techniques have been developed to compute relevant quantities in presence of conical singularities (see, among the others, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]). With few exceptions [8, 9, 12, 13, 14, 15, 16], most authors considered massless fields only. Indeed, the introduction of the mass complicates considerably the problem, so that it becomes very difficult to obtain manageable forms for the physical quantities. At this regard, it has to be noted that using the heat-kernel approach [8, 9]

1Electronic address: iellici@science.unitn.it
it seems that all the problems vanish: in fact, for a scalar field of mass \( m \) the integrated heat-kernel is simply related to the massless one as

\[
K^m_t = e^{-tm^2}K^m=0,
\]

and, at least for a simple cone \( R^D-2 \times C_\alpha \), the integrated massless heat-kernel is well known [2, 8, 9]:

\[
K^m=0_t = \sum_{D-2} \left[ \frac{V(C_\alpha)}{4\pi t} + \frac{1}{12} \left( \frac{2\pi}{\alpha} - \frac{\alpha}{2\pi} \right) \right],
\]

where \( 2\pi - \alpha \) is the deficit angle of the cone, \( V(C_\alpha) \) is the volume of the cone and \( \Sigma_{D-2} \) is the volume of the transverse dimensions. On the contrary of the massless case, the Mellin transform of Eq. (1) exists in the massive case, and so it is easy to compute the massive global \( \zeta \) function and from it quantities like the effective action, which have a simple closed form. However, it is well known [11] that this procedure does not yield the correct dependence on the deficit angle of the cone in the massless limit and, moreover, these global quantities are finite, apart for the volume divergences, while the local quantities show a non-integrable singularity near the tip of the cone. Therefore, it is at least dubious that this simple result for the massive case is correct. Besides, in order to compute important quantities such as the vacuum fluctuation of the field and the stress tensor, local rather than global quantities are needed.

The aim of this work is to give a more manageable tool to compute local quantities for a massive scalar field around a cosmic string, and we will accomplish this by computing the local \( \zeta \) function, which will be given as an expansion in powers of \( mr \), where \( r \) is the distance form the string core. Therefore, the obtained expression is useful near the string, where \( mr \ll 1 \). However, since the local quantities diverge at the string core, this is just the interesting region if, for instance, one wants to consider the back-reaction of the energy-momentum tensor on the background metric [17, 18]. The expression we obtain is simple enough to allow us to compute the renormalized vacuum expectation value \( \langle \phi^2(x) \rangle \) and the the energy-momentum tensor, potentially up to arbitrary order in \( (mr)^2 \). Notice that the most common tool used to compute local quantities is the Green’s function, which in the massive case is given in a complicated integral form [12, 13, 15, 8] or as a sum of generalized hypergeometric functions [16].

The capability of treating the massive case is useful also when dealing with self-interacting theories, such as \( \lambda \phi^4 \). In fact, consider an Euclidean path-integral approach to the theory of a self-interacting scalar field: the generating functional of the theory is the given by

\[
Z = \int \mathcal{D}\phi \exp(-S[\phi]),
\]

\[
S[\phi] = \int d^Dx \sqrt{\bar{g}} \left[ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + V(\phi) \right],
\]

where \( V(\phi) \) is a potential. In the usual \( \phi^4 \)-theory, it has the form \( V(\phi) = \frac{1}{4}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4 \), \( \lambda \) being the coupling constant. In the one-loop perturbative approach, the above action is expanded near the minimum \( \hat{\phi} \) of the action up to second order terms

\[
S[\hat{\phi} + \varphi] = S[\hat{\phi}] + \frac{1}{2} \int d^Dx \sqrt{\bar{g}} \varphi [-\Delta + M^2] \varphi,
\]

\[\text{In the cosmic string case, this problem could be overcome renormalizing the action of the string as in [9].}
\]

\[\text{Note that } r = \hbar/mc \text{ is the Compton wavelength of the particle. When the result are translated for the Rindler case, the approximation is valid close to the event horizon.}\]
where \( \Delta \) is the Laplace-Beltrami operator and \( M^2 = V''(\phi) \). In the \( \phi^4 \)-case, we have that \( M^2 = m^2 + \frac{1}{2} \alpha \phi^2 \). Then, the standard step is to assume \( \phi \) is a constant configuration, so that standard passages lead to effective action

\[
W = -\ln Z = S[\phi] + \frac{\hbar}{2} \ln \det[-\Delta + M^2] + \mathcal{O}(\hbar^2),
\]

where, in general, \( S[\phi] \) is proportional to the volume of the spacetime. The determinant in the second term is ultraviolet divergent and it has to be defined by means of some regularization procedure. Convenient ways to regularize the above determinant are, for example, the heat-kernel plus Schwinger’s proper-time regularization or the \( \zeta \)-function regularization.

From the above outline, it is clear that the one-loop self-interacting case is closely related to the massive case, and, in particular, the regularization of the determinant is the same, provided that \( m^2 \) is replaced by \( M^2 \). Of course, the renormalization of the effective action to remove the ultraviolet divergences is different. In this paper we shall not discuss the self-interacting case and its renormalization explicitly, but we observe that in the cosmic string case, in which \( \langle \phi^2(\tau) \rangle \propto r^{-2} \), it could be more reliable a ‘Hartree-like’ approximation as that employed in \([13]\) and in which in the action \( \lambda \phi^4 \) is replaced by \( \lambda \langle \phi^2 \rangle \phi^2 \).

As we said above, we are mainly concerned in the cosmic-string background. We remind that the space-time around an infinitely long, static and straight cosmic string \([13]\) has the topology \( M^4 = R^2 \times C_\alpha \), where \( C_\alpha \) is the simple two-dimensional cone with deficit angle \( 2\pi - \alpha \), and the metric is

\[
ds^2 = -dt^2 + dz^2 + dr^2 + r^2 d\theta^2, \quad r \in (0, +\infty), \quad \theta \in [0, \alpha], \quad t, z \in (-\infty, +\infty).
\]

The polar angle deficit \( 2\pi - \alpha \) is related to the mass per unit length of the string \( \mu \) by \( 2\pi - \alpha = 8\pi G\mu \). Throughout this paper we shall assume positive deficit angle, so that \( \nu \equiv 2\pi/\alpha > 1 \). For GUT strings \( \nu \sim 10^{-6} \) \([19]\). In the rest of the paper we adopt an Euclidean approach and so we perform a Wick rotation \( \tau = it \): the form of the metric is the same as above with the replacement \( -dt^2 \rightarrow +dr^2 \). Moreover, for technical reasons we will also consider the obvious generalization to the \( D \)-dimensional case, with topology \( R^{D-2} \times C_\alpha \).

We have assumed zero thickness for the string: this is clearly an idealization, since an actual cosmic string should have a small but finite radius, of the order of the Compton wavelength of the Higgs boson involved in the phase transition which gives rise to the cosmic string \([16]\). The internal structure of the string has non-negligible effects even at large distances, as it has been shown by Allen, Kay, and Ottewill \([20]\). However, in the same work it has been shown that for a minimally coupled scalar this dependence on the internal structure is absent, and one can safely use the idealized string. In order to avoid the complications discussed in \([24]\), which are related to the non-uniqueness of the self-adjoint extension of the Laplace-Beltrami operator in the idealized conical space-time, we shall consider the minimally coupled case only. Actually, in Sec. IV we will compute the stress tensor and, for sake of completeness, we will consider arbitrary coupling. Therefore, we have to remember that for \( \xi \neq 0 \) others effects could be present in a realistic cosmic string.

Although the results we will obtain are easily translated to the case of fields at finite temperature in the Rindler space \([1]\) or near the horizon of a Schwarzschild black hole, we
prefer to consider the cosmic string background only. This because the Euclidean Rindler space or Schwarzschild black hole show a conical singularity only when the temperature is different from the Unruh-Hawking one, namely when one considers the ‘off-shell’ theories, but it has been shown that the off-shell quantum states of a field are affected by several pathologies on the event horizon [22] and that these theories do not have a consistent thermodynamics [23].

The rest of this work is organized as follows. In Sec. II we compute the $\zeta$ function of a massive scalar field in the cosmic string background. Then we use this result to compute, in the region $mr \ll 1$, the vacuum fluctuation of the field in Sec. III and the expectation value of the stress tensor in Sec. IV. Section V contains the conclusions. In the rest of the the paper we use units for which $\hbar = c = G = 1$.

II. Computation on the $\zeta$ function

We consider a quasi-free real scalar field in the background given by the (Wick-rotated) metric (2). The action of the theory is given by

$$S[\phi] = \int d^Dx \sqrt{g} \frac{1}{2} A \phi,$$

where $A$ is known as the small fluctuation operator and in our case reads

$$A = -\Delta_D + m^2 + \xi R. \quad (3)$$

Here $\xi$ is a parameter which fixes the coupling of the field to the gravity by means of the scalar curvature $R$. We shall consider the minimally coupled case, $\xi = 0$. Then, by means standard passages one sees that the generating functional of the theory can be expressed in terms of the functional determinant of the small fluctuations operator, which can be conveniently defined by means of the $\zeta$ function regularization [24]:

$$\ln Z_\alpha = -\frac{1}{2} \ln \det \mu^{-2} A = \frac{1}{2} \zeta'(0|A\mu^{-2})$$

where $\zeta(s|A)$ is the global $\zeta$ function related to the operator $A$, the prime indicates the derivative with respect to $s$ and $\mu$ is an arbitrary parameter with the dimensions of a mass needed for dimensional reasons and not to be confused with the mass per unit length of the string. The global $\zeta$ function can be formally written as the integral over the manifold of a local $\zeta$ function $\zeta(s|A)(x)$:

$$\zeta(s|A) = \int_{\mathcal{M}^D} \zeta(s|A)(x) \sqrt{g} d^Dx.$$

Actually, when the manifold is non-compact only the local $\zeta$ function has a precise mathematical meaning, since the integration requires the introduction of cutoffs or smearing functions to avoid divergences.

In the massless case, the $\zeta$ function on the cone has been explicitly computed in [11]. In order to compute it in the massive case, the starting point is the well known relation between the local $\zeta$ function and the heat kernel given by the Mellin transform:

$$\zeta^m(s; x) = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} e^{-tm^2} K^D_t = 0(x).$$
The heat kernel of the massive field is related to the massless one by means of the following obvious decomposition:

\[ K_I^m(x) = e^{-tm^2} K_I^{m=0}(x). \]

Now we take into account the following property of the Mellin transform of the product of two functions \(25\)

\[ \int_0^\infty t^{s-1} f(t)g(t) dt = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} F(z)G(s-z)dz, \]

where \( F \) and \( G \) are the Mellin transforms of \( f \) and \( g \) respectively and \( \sigma \) is a real number in the common strip of convergence of the two Mellin transforms. In this way we can write the massive \( \zeta \) function of in terms of the massless one:

\[ \zeta^m(s; x) = \frac{1}{2\pi i\Gamma(s)} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\Gamma(z)\zeta^{m=0}(z; x)m^{2z-2s}\Gamma(s-z)dz}{\Gamma(z)\zeta^{m=0}(z; x)m^{2z-2s}\Gamma(s-z)dz}, \]

where we have used the Mellin transform

\[ \int_0^\infty t^{s-1}e^{-tm^2} dt = m^{-2s}\Gamma(s), \]

which converges for \( \text{Re} > 0 \).

Since we are interested in the \( \zeta \) function of a massive scalar field in the (Euclidean) space \( R^{D-2} \times C_{\alpha} \), we consider the corresponding massless \( \zeta \) function. It has been shown in \(2\) and \(1\) that it is the sum of two parts which converge in separate strips of the complex \( s \)-plane and which are summed only after their analytic continuation:

\[ \zeta^{m=0}(z; x) = \zeta_<(s; x) + \zeta_>(s; x), \]

where

\[ \zeta_<(s; x) = \frac{r^{2s-D}}{(4\pi)^{D/2}\Gamma(s)} \frac{\Gamma(s - \frac{D-1}{2})\Gamma(D - s)}{2\sqrt{\pi}\Gamma(s - \frac{D-2}{2})}, \quad \frac{D-1}{2} < \text{Re} < \frac{D}{2}, \]

\[ \zeta_>(s; x) = \frac{r^{2s-D}}{\alpha(4\pi)^{D/2}\Gamma(s)} \frac{\Gamma(s - \frac{D-1}{2})\Gamma(D - s)}{\sqrt{\pi}\Gamma(s - \frac{D-2}{2})} G_\alpha(s - \frac{D-2}{2}), \quad \frac{D}{2} < \text{Re} < \frac{D}{2} + \nu, \]

and the function \( G_\alpha(s) \) is defined as

\[ G_\alpha(s) = \sum_{n=1}^{\infty} \frac{\Gamma(\nu_n - s + 1)}{\Gamma(\nu_n + s)} \frac{2\pi}{\alpha |n|}, \quad \nu = \frac{2\pi}{\alpha |n|}, \quad \nu \equiv \nu_1 \]

and can be analytically continued in the whole complex plane showing simple poles in \( s = 1 \), with residue \( \alpha/4\pi \), and \( s = \nu + k + 1 \), \( k = 0, 1, 2, \ldots \), \((\alpha \neq 2\pi)\) with obvious residue. Since \( \zeta_< \) and \( \zeta_> \) do not have a common strip of convergence, we must split Eq. \(4\) in two parts: setting \( \text{Re} > D/2 \) we have

\[ \zeta^m_>(s; x) = \frac{1}{2\pi i\Gamma(s)} \frac{r^{D-m-2s}}{(4\pi)^{D/2}\sqrt{\pi}\alpha} \int_{\text{Re} \geq z < \text{Re} s} dz \frac{\Gamma(z - \frac{D-1}{2})G_\alpha(z - \frac{D-2}{2})\Gamma(s - z)(mr)^{2z}}{\Gamma(s - z)(mr)^{2z}} \]

\[ = \frac{r^{2s-D}}{(4\pi)^{D/2}\sqrt{\pi}\alpha\Gamma(s)} \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (mr)^{2n}\Gamma(s + n - \frac{D-1}{2})G_\alpha(s + n - \frac{D-2}{2}) + (mr)^{D-2s} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma(\nu_n + k + 1/2)}{\Gamma(2\nu_n + k + 1)} \Gamma(s - \nu_n - k - \frac{D}{2}) (mr)^{2\nu_n + 2k} \right\}, \]
where we have performed the integral shifting the integration path to the right and picking up the residues of the poles of the integrand. The same procedure can be applied to \( \zeta_m^\kappa(s;x) \), but in this case we have to consider the poles in \( z = s + n \) and \( z = 2 + n \):

\[
\zeta_m^\kappa(s;x) = \frac{1}{2\pi i} \frac{r^{-D}m^{-2s}}{(4\pi)^{D/2} \sqrt{\pi \alpha}} \int_{\text{Re}z > \text{Re}\frac{D}{2}} dz \frac{\Gamma(z-D/2)\Gamma(D/2 - z)}{2\Gamma(z + 1 - D/2)} \Gamma(s - z)(mr)^{2z} = \frac{r^{2s-D}}{(4\pi)^{D/2} \sqrt{\pi \alpha} \Gamma(s)} \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (mr)^{2n} \frac{\Gamma(s + n - D/2)\Gamma(D/2 - s - n)}{2\Gamma(s + n + 1 - D/2)} \right. \\
+ \left. (mr)^{D-2s} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (mr)^{2n} \frac{\Gamma(n + 1/2)\Gamma(s - n - D/2)}{2\Gamma(n + 1)} \right\}.
\]

Now we analytically continue each term and sum to get the final expression of the local \( \zeta \) function of a massive scalar field:

\[
\zeta_m^\kappa(s;x) = \frac{r^{2s-D} \mu^{2s}}{(4\pi)^{D/2} \sqrt{\pi \alpha} \Gamma(s)} \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (mr)^{2n} I_\alpha(s + n - D/2) \right. \\
+ \left. (mr)^{D-2s} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (mr)^{2n} \frac{\Gamma(n + 1/2)\Gamma(s - n - D/2)}{2\Gamma(n + 1)} \right\},
\]

where the function \( I_\alpha(s) \) is defined as \[11\]

\[
I_\alpha(s) = \frac{\Gamma(s-1/2)}{\sqrt{\pi}} \left[ G_\alpha(s) + \frac{\Gamma(1-s)}{2\Gamma(s)} \right], \\
I_\alpha(0) = \frac{1}{6\nu} \left( \nu^2 - 1 \right), \\
I_\alpha(-1) = \frac{1}{90\nu} \left( \nu^2 - 1 \right) \left( \nu^2 + 11 \right),
\]

(6)

The function \( I_\alpha(s) \) is analytic in the whole complex plane but in \( s = 1 \), where it has a simple pole. Near the pole we have

\[
I_\alpha(s) = \frac{1}{2(s-1)} \left( \nu^{-1} - 1 \right) + \frac{1}{2} \left( \nu^{-1} - 1 \right) (\gamma - 2 \ln 2) - \frac{\ln \nu}{\nu} + O\left( (s - 1)^2 \right),
\]

where \( \gamma \) is the Euler’s constant. In expression (6) we have also reintroduced the arbitrary mass \( \mu \) using the formal relation \( \zeta(s|\mu^{-2}) = \mu^{2s} \zeta(s|Ap^{-2}) \), and which has been omitted in the derivation for simplicity. It must be noted that the above \( \zeta \) function can be obtained in a more direct way, albeit much longer, performing the integrations in the Mellin transform of the spectral representation of the massive heat kernel, and then rearranging the sums and the Euler gamma functions in the generalized hypergeometric functions obtained by the integrations. We prefer the used method since it can be more easily applied to the computation of others quantities, as we will see in the next sections.

Although expression (5) looks awful, it is very simple when \( mr \ll 1 \): considering the
physically interesting case $D = 4$ and terms up to $(mr)^4$

$$
\zeta^m(s;x) = \frac{r^{2s-4} \mu^{2s}}{4\pi \Gamma(s)} \left[ I_\alpha(s-1) - (mr)^2 I_\alpha(s) + \frac{1}{2}(mr)^4 I_\alpha(s+1) + O((mr)^6) \right] \\
+ \frac{m^4 (m/\mu)^{-2s}}{8\pi \alpha \Gamma(s)} \left[ \Gamma(s-2) - \frac{(mr)^2}{2} \Gamma(s-3) \right] \\
+ 2(mr)^2 \frac{\Gamma(\nu + 1/2) \Gamma(s - \nu - 2)}{\sqrt{\pi} \Gamma(2\nu + 1)} + O((mr)^4). \]

In the first term of the first row we recognize the massless $\zeta$ function on $R^2 \times C$ \cite{11}, while the first term in the second row becomes the $\zeta$ function of a massive scalar field in the Minkowski space-time when $\alpha = 2\pi$ \cite{26}. The others terms are clearly corrections due to the presence of the conical singularity. When the singularity is absent, namely when $\alpha = 2\pi$, the first row vanishes, since $I_{2\pi}(s) = 0$, while in the second and third rows only the first term survives, since the others cancel two by two.

At the same order in $mr$, the effective lagrangian density is given by

$$
L(x) = \frac{1}{2} \frac{d}{ds} \zeta^m(s;x)|_{s=0} \\
= \frac{1}{8\pi \alpha \Gamma(s)} \left\{ I_\alpha(-1) - (mr)^2 I_\alpha(0) \\
- \frac{(mr)^4}{2} \left[ \frac{\nu - 1}{\nu} \left( \gamma + \ln \frac{r \mu}{2} \right) + \ln \frac{\nu}{\nu} + \ln \frac{m}{\mu} \right] \right. \\
- \left. \frac{(mr)^6}{24} \left[ 2 G_\alpha(2) + 2 \gamma + 2 \ln \frac{mr}{2} - \frac{11}{6} \right] + 2(mr)^{2\nu+4} \frac{\Gamma(\nu + 1/2) \Gamma(-\nu - 2)}{\sqrt{\pi} \Gamma(2\nu + 1)} \right\}.
$$

where $G_\alpha(n), n \geq 2$, are readily computed being in the region of convergence of the series defining $G_\alpha$. For example

$$
G_\alpha(2) = -\frac{1}{2\nu} \left[ 2\gamma + \psi(\nu^{-1}) + \psi(-\nu^{-1}) \right], \\
G_\alpha(3) = -\frac{1}{6\nu} \left[ 3\gamma - \psi(\nu^{-1}) - 3\psi(-\nu^{-1}) + \psi(-2\nu^{-1}) \right],
$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$. In the limit $\alpha \to 2\pi$ the above effective lagrangian reduces to the usual Coleman-Weinberg potential. It is important to note that the computation of higher orders in the above expansion in powers of $(mr)^2$ does not involve particular complications.

### III. Vacuum fluctuations

Now we use the above $\zeta$ function to compute the (renormalized) value of the vacuum fluctuation of the field \cite{26}:

$$
\langle \phi^2(x) \rangle = \lim_{s \to 1} \zeta(s;x).
$$

Since even in the Minkowski space the vacuum expectation value diverges, we renormalize the vacuum expectation value on the cone subtracting the Minkowski value: in this way we

\footnote{Note that we are interested in the values of the $\zeta$ function at $s = 0, 1$, and we consider positive deficit angles only, so that $\nu > 1$.}
have a remarkable cancellation of the poles, yielding a finite result:

\[
\langle \phi^2(x) \rangle_\alpha - \langle \phi^2(x) \rangle_{2\pi} = \frac{1}{4\pi^2 \alpha^2} \left\{ I_\alpha(0) + \frac{(mr)^2}{\nu} \left[ (\nu - 1) \left( \ln \frac{mr}{\nu} + \gamma - \frac{1}{2} \right) + \ln \nu \right] + \frac{(mr)^4}{8} \left[ 2G_\alpha(2) + 2\gamma - 1 + 2\ln \frac{mr}{2} \right] + \frac{(mr)^{2\nu + 2}\Gamma(\nu + 1/2)\Gamma(-\nu - 1)}{\sqrt{\pi}\Gamma(2\nu + 1)} + O((mr)^6) \right\}. 
\]

(7)

The computation of higher corrections is straightforward. One can verify that Eq. (7) correctly vanish in the limit \( \alpha \to 2\pi \), and that up to order \((mr)^2\) it is identical to the result obtained by Moreira Jnr \[16\]. In particular, we notice the additional logarithmic divergences of the vacuum fluctuations at the conical singularity due to the massive corrections.

IV. Energy-momentum tensor

Another important vacuum average is that of the energy-momentum tensor. While in the massless case it is quite easy to be computed, since its form is fixed from symmetry arguments, in the massive case the computation is much more difficult and, to our knowledge, the massive corrections near the string have never been shown explicitly. Only in \[13\] and \[15\] the explicit form of the energy-momentum tensor has been given for \( mr \gg 1 \), where they found the expected exponential damping factor \( \exp(-2mr) \).

We use a point-splitting approach, in which the vacuum expectation value of the energy momentum tensor is given by the coincidence limit of a non-local differential operator applied to the propagator of the field \[21\]:

\[
\langle T_{\mu\nu}(x) \rangle = i \lim_{x' \to x} D_{\mu\nu}(x, x') G_F(x, x'),
\]

where

\[
D_{\mu\nu}(x, x') \equiv (1 - 2\xi)\nabla_\mu \nabla_\nu' + 2\xi \nabla_\mu \nabla_\nu + (2\xi - 1/2)g_{\mu\nu} \left[ \nabla_\alpha \nabla_\alpha' - m^2 \right],
\]

(8)

and the prime indicates that the derivative has to be taken with respect to \( x' \). Since the propagator can be obtained from the off-diagonal \( \zeta \) function \[21\] as

\[
G_F(x, x') = i \lim_{s \to 1} \zeta(s; x, x'),
\]

we can compute the energy momentum tensor from the \( \zeta \) function as (see also \[24\], \[27\])

\[
\langle T_{\mu\nu}(x) \rangle = -\lim_{s \to 1} \lim_{x' \to x} D_{\mu\nu}(x, x') \zeta(s; x, x').
\]

The partial derivatives of the off-diagonal \( \zeta \) function which appear in the above expression can be computed from the spectral representation in \( D \) dimensions (\( x \equiv (\tau, \vec{z}), \ k = |k| \))

\[
\zeta_D(s; x, x') = \frac{2(4\pi)^{-D/2}}{\alpha \Gamma(D/2)} \int_0^\infty dk \ k^{D-3} \sum_{n=\infty}^\infty \int_0^\infty d\lambda \lambda^{\frac{D}{2} - 1} \left[ \lambda^2 + k^2 + m^2 \right]^{-s} \times J_{\nu_n}(\lambda r')J_{\nu_n}(\lambda r) e^{ik(x-x')+i\frac{2\pi}{\alpha}n(\theta-\theta')},
\]

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and then taking the coincidence limit. In this way one can easily show that

\[ \partial_\theta \partial_\varphi \zeta_D(s; x, x')\big|_{x=x'} = -\partial_\theta^2 \zeta_D(s; x, x')\big|_{x=x'} \]
\[ \partial_r \partial_r \zeta_D(s; x, x')\big|_{x=x'} = -\partial_r^2 \zeta_D(s; x, x')\big|_{x=x'} = 2\pi \zeta_D+2(s; x) \]
\[ \partial_r \partial_r \zeta_D(s; x, x')\big|_{x=x'} = -\partial_r^2 \zeta_D(s; x, x')\big|_{x=x'} = 2\pi \zeta_D+2(s; x) \]
\[ \partial_r \zeta_D(s; x, x')\big|_{x=x'} = \frac{1}{2} \partial_r \zeta_D(s; x). \]

As far as \( \partial^2_\varphi \zeta_D(s; x, x')\big|_{x=x'} \) is concerned, in the \( m = 0 \) case it is easy to see that we have

\[ \partial^2_\varphi \zeta_D^{m=0}(s; x, x')\big|_{x=x'} = -\frac{r^{2s-D} \Gamma(s - D-1)}{(4\pi)^{D-2} \sqrt{\pi} \alpha \Gamma(s)} H_\alpha(s - \frac{D-2}{4}), \]

where the function \( H_\alpha(s) \) is defined and studied in the Appendix A. The massive case can then be treated using the off-diagonal version of Eq. (4) with the partial coincidence limit

\[ r = r', z = z' \text{ and } t = t', \]

\[ \partial^2_\varphi \zeta_4(s; \theta, \theta')|_{\theta=\theta'} = \frac{1}{2\pi i \Gamma(s)} \int \Gamma(z) \partial^2_\varphi \zeta_4^{m=0}(s; z, \theta')|_{\theta=\theta'} \Gamma(s - z) m^{2z-2s} dz \]
\[ = \frac{-1}{2\pi i \Gamma(s)} \int_{\text{Re } z > 3} \frac{r^{2s-4}}{4\pi \sqrt{\pi} \alpha \Gamma(s)} \Gamma(z - 3/2) H_\alpha(z - 1) \Gamma(s - z) m^{2z-2s} dz. \]
\[ = -\frac{r^{2s-4} \mu^{2s}}{4\pi \sqrt{\pi} \alpha \Gamma(s)} \left\{ \sum_{n=0}^\infty \frac{(-1)^n}{n!} (mr)^{2n} \Gamma(s + n - 3/2) H_\alpha(s + n - 1) \right\} \]
\[ + (mr)^{4-2s} \sum_{n=1}^\infty \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (mr)^{2n+2k} \sqrt{\nu_n} \frac{\Gamma(s - \nu_n - k - 2) \Gamma(\nu_n + k + 1/2)}{\Gamma(2\nu_n + k + 1)} \].

where, as usual, we have performed the integration shifting the integration contour to the right and picking up the residues at \( z = s + n \) and \( z = \nu_n + k + 2 \) to get an expansion in powers of \( mr \) similar to Eq. (6).

As far as the second derivatives with respect to \( r \) and \( r' \) are concerned, using the following identity, which can be proved using some recursion formulas for the Bessel functions \[ 28],

\[ [\partial_r J_\nu(\lambda r)]^2 = -\frac{\nu^2}{r^2} J_\nu^2(\lambda r) + \frac{1}{2r} \partial_r r \partial_r J_\nu^2(\lambda r) + \lambda^2 J_\nu^3(\lambda r), \]

one can see that

\[ \partial_r \partial_r \zeta_D(s; x, x')\big|_{x=x'} = \frac{1}{2r} \partial_r \partial_r \zeta_D(s; x) + \frac{1}{r^2} \partial^2_\theta \zeta_D(s; x, x')\big|_{x=x'} + \chi D(s; x), \]
\[ \partial^2_\varphi \zeta_D(s; x, x')\big|_{x=x'} = -\frac{1}{2r} \partial_r \zeta_D(s; x) - \frac{1}{r^2} \partial^2_\theta \zeta_D(s; x, x')\big|_{x=x'} - \chi D(s; x), \]

where the function \( \chi D(s; x) \) is defined in Appendix B. If \( D = 4 \) and \( m = 0 \) we simply have

\[ \chi_D^{m=0}(s; x) = 4\pi(s - 2) \zeta_D^{m=0}(s; x), \]

while the massive case is more complicating and is studied in the Appendix.
Now we have all the pieces needed to compute the massive correction to the energy-momentum tensor. For the actual calculation it is convenient to follow \[29\] and \[30\]. We define the renormalized stress tensor as

\[
\langle T_{\mu\nu}(x) \rangle^R_\alpha = \langle T_{\mu\nu}(x) \rangle_\alpha - \langle T_{\mu\nu}(x) \rangle_{2\pi}
\]

which, considering only the first massive correction, turns out to be

\[
\langle T_{\theta\theta}(x) \rangle^R_\alpha = \frac{1}{4\pi\alpha r^2} \left[ 2H_\alpha(0) - (6\xi - 1)I_\alpha(0) + (mr)^2 \left( H_\alpha(1) + \xi \frac{\nu - 1}{\nu} \right) \right],
\]

\[
\langle T_{rr}(x) \rangle^R_\alpha = \frac{1}{4\pi\alpha r^2} \left[ 2H_\alpha(0) - I_\alpha(-1) - (2\xi - 1)I_\alpha(0) + (mr)^2 \left( H_\alpha(1) + \xi \frac{\nu - 1}{\nu} \right) \right],
\]

\[
\langle T_{tt}(x) \rangle^R_\alpha = \frac{1}{8\pi\alpha r^4} \left[ I_\alpha(-1) + 2(4\xi - 1)I_\alpha(0) + (mr)^2 I_\alpha(0) \right],
\]

\[
\langle T_{zz}(x) \rangle^R_\alpha = \langle T_{tt}(x) \rangle^R_\alpha,
\]

The tensor can be written in the more familiar form

\[
\langle T^\nu_{\mu}(x) \rangle^R_\alpha = \frac{-1}{144\pi^2 r^4} \left[ \left( \nu^4 - 1 \right) \text{diag}(1, 1, -3, 1) + 10(6\xi - 1) \left( \nu^2 - 1 \right) \text{diag}(2, -1, 3, 2) 
+ 15(mr)^2(\nu - 1)(12\xi - 1 - \nu)\text{diag}(0, 1, -1, 0) 
- 15(mr)^2 \left( \nu^2 - 1 \right) \text{diag}(1, 0, 0, 1) \right].
\]  

(10)

In the limit \(m \to 0\) the result is in agreement with that obtained by others authors \[3, 31\]. The components \(\langle T_{rr}(x) \rangle^R_\alpha\) and \(\langle T_{\theta\theta}(x) \rangle^R_\alpha\) satisfy the equation

\[
\frac{d}{dr}(rT^r_r) = T^\theta_\theta,
\]

which follows from the conservation law \(\nabla_\mu T^\mu_\nu = 0\). It is interesting to note that in the corrections to the stress tensor of order \((mr)^2\) are not present the logarithmic divergences which are instead present in the vacuum fluctuations at the same order (see Eq. \[6\]). The logarithmic terms appear in higher order corrections, which can also be computed with the method we have developed. Actually only the logarithms in the term of order \((mr)^4\) give rise to divergences, since at higher orders they are multiplied by positive powers of \(r\). Indeed, the correction of order \((mr)^4\) to \(\langle T_{\theta\theta}(x) \rangle^R_\alpha\) is

\[
\frac{1}{4\pi\alpha r^2} \left\{ \frac{(mr)^4}{8} \left[ 2\ln \frac{\nu + 1}{\nu} + (2\xi - 1)A(r) + 4\xi - 1 \right] 
+ (mr)^2 + 2\nu^2 \left[ 2\xi + (4\xi - 1)\nu \right] \frac{\Gamma(-\nu - 1)\Gamma(\nu + 1/2)}{\sqrt{\pi}\Gamma(2\nu + 1)} \right\} ,
\]

from which one can also compute the correction to \(\langle T_{rr}(x) \rangle^R_\alpha\) by means of the conservation law. The correction to \(\langle T_{tt}(x) \rangle^R_\alpha\) is

\[
\frac{1}{8\pi\alpha r^4} \left\{ \frac{(mr)^4}{4} \left[ 2B(r) - (4\xi - 1)(A(r) + 1) - \frac{1}{\nu} \right] 
+ (mr)^2 + 2\nu^2 \left[ 2\xi + (4\xi - 1)\nu^2 \frac{\Gamma(-\nu - 1)\Gamma(\nu + 1/2)}{\sqrt{\pi}\Gamma(2\nu + 1)} \right] \right\} .
\]
In the above equations we have set
\[
A(r) = 2G_{\alpha}(2) + 2\gamma + 2 \ln \frac{mr}{2},
\]
\[
B(r) = \frac{\nu - 1}{\nu} \left( \ln \frac{mr}{2} + \gamma - \frac{1}{2} \right) + \ln \nu.
\]

An interesting point to be discussed is the dependence on the parameter \(\xi\) which fixes the coupling of the scalar field with the gravity. In the introduction we said that we would consider the minimally coupled case only, \(\xi = 0\). This because it has been shown in [20] that the idealized conical space is a good model of the space-time of a cosmic string only for the minimally coupled case. For nonminimally coupled fields quantities like \(\langle \phi^2(x) \rangle\) will depend on the details of the metric in the core of the string even very far from the string. However, one could be interested, e.g., in finite-temperature fields in the Rindler space, where there is a true conical singularity and not just an idealization of a non-singular metric. In these cases it is interesting to consider also nonminimally coupled fields.

From the mathematical point of view, it is not clear which is the meaning of the field equation \([-\Delta + m^2 + \xi R] \phi = 0\) when the curvature \(R\) has Dirac’s delta singularities, and our choice of \(\xi = 0\) allowed us to avoid the problem. A possible way to define the problem is to smooth out the singularity, as done in [3, 31, 10]: as a result, one finds that when the regularization of the singularity is removed to recover the conical space also the dependence of the Green’s function on the parameter \(\xi\) vanishes [31]. Then we can argue that also the \(\zeta\) function is independent of \(\xi\), since the modes used to construct the \(\zeta\) function are essentially the same as those for the Green’s function.

The dependence on the parameter \(\xi\) comes back into play when considering the stress tensor. It is worth while noticing that even when the manifold if flat everywhere (in our case there is a Dirac’s delta singularity in the curvature at the core of the string) the parameter \(\xi\) remains in the theory as a relic of the fact that \(T_{\mu\nu}\) is obtained by varying the metric \(g_{\mu\nu}\) in the field Lagrangian [21]. One can then see that, if \(R = 0\), the global conserved quantities as total energy should not depend on the value of \(\xi\). This is because the contributions to those quantities due to \(\xi\) are discarded into boundary surface integrals which generally vanish. However, this is not the case dealing with the cosmic string because such integrals diverge at the origin. Notice that similar problems appear working in subregions of the Minkowski space in presence of boundary conditions [21]. In the point-splitting approach, the stress tensor is obtained applying the \(\xi\)-dependent operator \(D_{\mu\nu}(x, x')\) (see Eq. (8)) to the Green’s function or to the \(\zeta\) function. Therefore, if the above argument that the Green’s function (or the \(\zeta\) function) is independent of \(\xi\) holds, it is not contradictory to compute the energy-momentum tensor applying \(D_{\mu\nu}(x, x')\) with \(\xi \neq 0\) to the the Green’s function computed setting \(\xi = 0\), as we have done above and as done by most authors.

V. Conclusions

In this paper we have studied the \(\zeta\) function of a massive scalar field in a cosmic string background, and we have obtained an expression, Eq. (3), which is useful in the region near the core of the string, \(mr \ll 1\). By means of this expression we have computed the massive corrections to the vacuum fluctuations, Eq. (7), and to the energy-momentum tensor, Eq. (10), up to order \((mr)^4\), going beyond the known results. Higher corrections are also computable.
Possible extensions of this paper are the inclusion of a magnetic flux carried by the string, which gives Aharonov-Bohm effects, and the case of spin-1/2 fields. Also the limit $mr \gg 1$ is worth studying, for example rewriting the $\zeta$ function in terms of the hypergeometric function $1_F2[a; b, c; z]$ and then using its asymptotic behaviour for large $z$. However, the vacuum fluctuations and the energy-momentum tensor in this limit have been already obtained by others authors with different methods [13, 15].

As a possible application of the results of this paper we point out the study of the back-reaction of the energy momentum tensor of a massive scalar field on the background metric of the cosmic string, as done by Hiscock [17] and Guimarães [18]. The computation seems feasible since the first massive correction to the energy momentum tensor does not involve logarithmic terms. We hope to cover this topic in a future paper.

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Appendix A

In this Appendix we study the function $H_\alpha(s)$, which is defined as the analytic continuation of following series:

$$H_\alpha(s) = \sum_{n=1}^{\infty} \nu_n^2 \frac{\Gamma(\nu_n - s + 1)}{\Gamma(\nu_n + s)}, \quad \nu_n = \frac{2\pi}{\alpha |n|}.\$$

It can be studied and analytically continued proceeding exactly as for the function $G_\alpha(s)$ studied in the Appendix of [11]. Then one sees that the series converges for $\text{Re}s > 2$ and that the analytic continuation for $\lfloor n/2 \rfloor < \text{Re}s < 2$ (here $\lfloor n/2 \rfloor$ represents the integer part of $n/2$) is given by

$$\sum_{j=0}^{\lfloor n/2 \rfloor + 1} c_j(s) \nu_j^{3-2s-2j} \zeta_R(2s + 2j - 3) + \sum_{k=1}^{\infty} \nu_k^2 f_n(\nu_k, s),$$

where $\zeta_R$ is the Riemann $\zeta$ function and the function $f_n(\nu, s)$ is generally unknown, but vanishes for $s = 1/2, 0, -1/2, -1, \ldots$. The coefficients $c_j(s)$ vanish for $s = -n/2, (n = -1, 0, 1, 2, \ldots) for all j > (n + 1)/2, and the first ones have been given in [11].

The function $H_\alpha(s)$ has then a simple pole in $s = 2$ and near this pole we have

$$H_\alpha(s) = \frac{1}{2\nu(s-2)} + \frac{1}{\nu} (\gamma - \ln \nu) + \mathcal{O}((\nu - 2)^2).$$

Moreover, it has simple poles at $s = 1 + \nu_n + k, k = 1, 2, \ldots$, due to the gamma function in the numerator of the terms of the series, with obvious residue. Finally, it is possible to compute the value of the function $H_\alpha(s)$ for some useful value of $s$:

$$H_\alpha(0) = \frac{1}{120\nu}(\nu^4 - 1),$$

$$H_\alpha(1) = -\frac{1}{12\nu}(\nu^2 - 1).$$
Appendix B

In this Appendix we study the function $\chi_D(s; x)$, which is defined by the analytic continuation of the following spectral representation

$$\chi_D(s; x) = \frac{2(4\pi)^{-\frac{D-2}{2}}}{\alpha \Gamma((D-2)/2)} \int_0^\infty dk \, k^{D-3} \sum_{n=-\infty}^\infty \int_0^\infty d\lambda \, \lambda^3 [\lambda^2 + k^2 + m^2]^{-s} J_{\nu}(\lambda r),$$  \hspace{1cm} (11)

which is the same as $\zeta_D(s; x)$ with $d\lambda \lambda \rightarrow d\lambda \lambda^3$. Note that the massless case is trivial, since in that case the function $\chi$ is simply related to the $\zeta$ function, see Eq. (9). In the massive case, we can proceed in analogy to the massive $\zeta$ function, namely employing Eq. (4): considering $D = 4$ and using Eq. (9)

$$\chi_{m=4}^m(s; x) = \frac{1}{2\pi i \Gamma(s)} \int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(z) \chi_{m=4}^m(z; x) m^{2z-2s} \Gamma(s-z) \, dz = \frac{4\pi}{2\pi i \Gamma(s)} \int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(z) (z-s) \zeta_{m=6}^m(z; x) m^{2z-2s} \Gamma(s-z) \, dz.$$

Then we go on as usual splitting $\zeta_{m=6}^m(z; x)$ as $\zeta = \zeta_\langle + \zeta_\rangle$ and performing the integrals shifting the integration contour to the left and picking up the residues of the poles. The final result is an expansion in powers of $mr$:

$$\chi_{D=4}^m(s; x) = \frac{r^{2s-6}}{4\pi \alpha \Gamma(s)} \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (s+n-2)(mr)^{2n} I_n(s+n-2) \right. \\
+ \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (n+1)(mr)^{n+2} \frac{\Gamma(n+1/2) \Gamma(s-n-3)}{2\sqrt{\pi} \Gamma(n+1)} \right. \\
+ (mr)^{6-2s} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (\nu_n + k+1)(mr)^{2\nu_n+2k} \frac{\Gamma(\nu_n + k + 1/2) \Gamma(s-\nu_k - k - 3)}{2\sqrt{\pi} \Gamma(2\nu_n + k + 1)} \right\}. $$  \hspace{1cm} (12)

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