Triply heavy tetraquark states with the $QQ\bar{q}q$ configuration

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In the framework of the color-magnetic interaction, we systematically investigate the mass splittings of the $QQ\bar{q}q$ tetraquark states and estimated their rough masses in this work. These systems include the explicitly exotic states $c\bar{c}b\bar{q}$ and $b\bar{b}q\bar{q}$ and the hidden exotic states $cc\bar{q}q$, $cb\bar{q}q$, $b\bar{c}q\bar{q}$, and $b\bar{b}q\bar{q}$. If a state around the estimated mass region could be observed, its nature as a genuine tetraquark is favored. The strong decay patterns shown here will be helpful to the experimental search for these exotic states.

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I. INTRODUCTION

In the past decade, many exotic $XYZ$ states were observed in experiments [1–7]. Among them, the charged charmonium-like or bottomonium-like states, the $Z(4430)$ [8–11], the $Z_1(4050)$ [12], the $Z_2(4250)$ [12], the $Z_c(3900)$ [13–16], the $Z_c(3885)$ [17–19], the $Z_c(4020)$ [20, 21], the $Z_{cc}(4255)$ [22, 23], the $Z_{bc}(4200)$ [10], the $Z_b(10610)$ [24] and the $Z_b(10650)$ [24], are considered as possible tetraquark candidates with two heavy quarks. They are also good meson-antimeson molecule candidates.

Besides the hidden flavor case, the exotic charmed mesons also stimulated heated discussions on tetraquark candidates. The possible candidates of the exotic charmed mesons $D_s(2317)$ [25–27], $D_s(2460)$ [25, 26] and $D_s(2632)$ [28] have attracted much attention due to the deviation of their from the quark model expectations [29] and their unexpected decay properties. Cheng and Hou [30] interpreted the $D_s(2317)$ as a $cq\bar{q}\bar{s}$ state to explain its mass and decay behaviors. In Ref. [31], Chen and Li proposed that the $D_s(2317)$ ($D_s(2460)$) is a $DK$ ($D^*K$) molecule while the $D_s(2632)$ is a $c\bar{s}\bar{s}\bar{s}$ state. The decay processes $D_s(2317) \to D^{\mp} \pi^0, D^{*+}_s \gamma$ are studied in a four-quark meson assumption in Ref. [32], which favors the assignment as an iso-triplet. Maiani et al. suggested the $D_s(2632)$ as a $[c\bar{d}\bar{d}\bar{s}]$ state [33] to explain that the $D^0K^+$ mode is suppressed with respect to the $D^+_s\eta$ channel. On the other hand, Liu et al. [34] proposed that the anomalous decay ratio of the $D_s(2632)$ can be understood by assuming a four quark state wave function $\frac{1}{\sqrt{2}}(dsd + sdd + su\bar{u} + us\bar{u} - 2s\bar{s}s)$. The $D_s(2632)$ was not confirmed later. The $D_s(2317)$ and the $D_s(2460)$ mesons are probably conventional charm-strange mesons which are affected largely by the coupled channel effects [35]. The existence of the open flavor tetraquarks remains elusive.

Recently, the DØ Collaboration [36] reported a structure $X(5568)$ in the $B^0\pi^\pm$ invariant mass distribution. This narrow state is about 200 MeV below the $BK$ threshold. Thus, the $X(5568)$ could be a tetraquark state with four different flavors [37–50] rather than a molecule state because the binding is too deep for a weak $BK$ interaction in the isovector case [51–54]. Later, the LHCb Collaboration [55] also investigated the $X(5568)$ state, but found no significant signals, which led some theorists to doubt whether the $X(5568)$ is a genuine resonance [56–66]. The CMS Collaboration did not confirm the state, either [67]. In this work, we move on to the four quark systems with more
heavy quarks to search for more compact tetraquark states, such as \( QQ\bar{Q}\bar{q} \).

In Ref. [68], a structure composed of a charm-anticharm quark pair as well as a light quark-antiquark pair, \( c\bar{c}q\bar{q} \), was proposed. Other exotic mesons with hidden charm including \( c\bar{c}c\bar{c} \) were also discussed. After that, more groups discussed the existence of such exotic states in various methods [69–76]. There are different opinions on the stability of the \( c\bar{c}c\bar{c} \) system. Whether such superheavy tetraquark states exist or not awaits experimental judgment in the future.

Usually, it is difficult for one to distinguish a meson-antimeson molecule from a compact tetraquark. However, for the \( QQ\bar{Q}\bar{q} \) system, the binding force comes from the short-range gluon exchange and a molecule configuration is not favored. If such states do exist, it is also possible that the compact tetraquarks with three heavy quarks and one light quark \( QQ\bar{Q}\bar{q} \) may exist. The binding force is also provided by the short-range gluon exchange. These states look like the excited \( D \) or \( B \) mesons.

Recently, two excited nucleon states \( P_c(4380) \) and \( P_c(4450) \) are observed by the LHCb Collaboration [77–79], where a charm-anticharm pair is observed. Such pentaquarks were predicted in the baryon-meson picture in Refs. [80–83]. It is also possible that a \( c\bar{c} \) or \( b\bar{b} \) pair may exist in the charmed or bottomed mesons. Once such states are really observed, their molecule assignment is not supported while the tetraquark nature is favorable. The experimental and theoretical search for this kind of tetraquark states is missing, as far as we know. In this work, we perform a systematic analysis of the mass spectrum of the \( QQ\bar{Q}\bar{q} \) system, which may provide important information for future experimental research.

Because the interaction strengths between (anti)quarks may be different, it is usually assumed that the diquark substructure exists in multiquark states, which is reflected in the mass spectra. In studying pentaquark states [84], it was argued that the triquark \( (qq\bar{q}) \) substructure results in a lower hadron mass. If one specifies the substructure with a given color-spin state, the two configurations result in different spectra. Here we consider all possible color-spin states and diagonalize the Hamiltonian finally. To understand whether they are equivalent for the compact \( QQ\bar{Q}\bar{q} \) system, we estimate the mass in the diquark-antidiquark \( (QQ)(\bar{Q}\bar{q}) \) picture and the triquark-antiquark \( (QQ\bar{Q})\bar{q} \) picture. One will see that, for the present systems, the two configurations give the same results once the diagonalization is performed.

This paper is organized as follows. In Sec. II, we present the formalism of our calculation. In Sec. III, we show the numerical results for the various \( QQ\bar{Q}\bar{q} \) systems. Finally, we give some discussions and a short summary in the last section.

II. FORMALISM

To describe the chromomagnetic interaction in the \( QQ\bar{Q}\bar{q} \) system, we use the Hamiltonian

\[
H = H_0 + H_{CM} = \sum_i m_i - \sum_{i<j} C_{ij} \lambda_i \cdot \lambda_j \sigma_i \cdot \sigma_j,
\]

(1)

where \( m_i \) is the effective mass of the \( i \)-th quark which includes the constituent quark mass and the binding effect. The \( \sigma_i \) (\( \lambda_i \)) is the Pauli (Gell-Mann) matrix. For antiquarks, \( \lambda_i \) is replaced with \(-\lambda_i^\dagger\). The coupling constant \( C_{ij} = c_0 \delta(r_{ij})/(m_i m_j) \) is determined by the wave function and the constituent quark mass, where \( c_0 \) is related to the interaction constant, \( r_{ij} \) is the distance between the \( i \)-quark and the \( j \)-th quark. In order to calculate the color-spin matrix elements, we adopt the formalism in Ref. [88, 90]. First, we construct the color and spin wave functions by using both the diquark-antidiquark and the triquark-antiquark configurations, and then calculate the color and spin matrix elements with the Hamiltonians \( H_C = -\sum_{i<j} C_{ij} \lambda_i \cdot \lambda_j \) and \( H_S = -\sum_{i<j} C_{ij} \sigma_i \cdot \sigma_j \), respectively. Finally, we get the \( \langle H_{CM} \rangle \) matrices after performing a type of “tensor product” of \( \langle H_C \rangle \) and \( \langle H_S \rangle \).

In the diquark-antidiquark configuration, the allowed base vectors in spin space read

\[
\begin{align*}
\chi_1 &= |(Q_1Q_2)\bar{Q}_3\bar{q}_4\rangle_2, & \chi_2 &= |(Q_1Q_2)\bar{Q}_3\bar{q}_4\rangle_1, \\
\chi_3 &= |(Q_1Q_2)\bar{Q}_3\bar{q}_4\rangle_1, & \chi_4 &= |(Q_1Q_2)\bar{Q}_3\bar{q}_4\rangle_0, \\
\chi_5 &= |(Q_1Q_2)\bar{Q}_3\bar{q}_4\rangle_1, & \chi_6 &= |(Q_1Q_2)\bar{Q}_3\bar{q}_4\rangle_0,
\end{align*}
\]

(2)

where the notation on the right hand side is \( |(Q_1Q_2)_{\text{spin}}(\bar{Q}_3\bar{q}_4)_{\text{spin}}\rangle_{\text{spin}} \). Similarly, the base vectors in spin space in the triquark-antiquark configuration are

\[
\begin{align*}
\zeta_1 &= |(Q_1Q_2)\bar{Q}_3\bar{q}_4\rangle_2, & \zeta_2 &= |(Q_1Q_2)\bar{Q}_3\bar{q}_4\rangle_1, \\
\zeta_3 &= |(Q_1Q_2)\bar{Q}_3\bar{q}_4\rangle_1, & \zeta_4 &= |(Q_1Q_2)\bar{Q}_3\bar{q}_4\rangle_0, \\
\zeta_5 &= |(Q_1Q_2)\bar{Q}_3\bar{q}_4\rangle_1, & \zeta_6 &= |(Q_1Q_2)\bar{Q}_3\bar{q}_4\rangle_0,
\end{align*}
\]

(3)
where the superscripts are color representations. In the triquark-antiquark configuration, there are two triplet representations \(3_{MS}\) and \(3_{MA}\) for the triquark where \(MS (MA)\) means that the first two quarks are symmetric (antisymmetric). The base vectors \(|(Q_1 Q_2 \bar{Q}_3)^{3_{MS}}\rangle\) and \(|(Q_1 Q_2 \bar{Q}_3)^{3_{MA}}\rangle\) seem different from those in the diquark-antidiquark configuration. However, we find that the two configurations give the same results by constructing the explicit color wave functions. That is,

\[
\phi_1 = |(Q_1 Q_2)^{6} (\bar{Q}_3 \bar{Q}_4)^{\bar{6}}\rangle = |(Q_1 Q_2 \bar{Q}_3)^{3_{MS}}\rangle,
\]

\[
\phi_2 = |(Q_1 Q_2)^{3} (\bar{Q}_3 \bar{Q}_4)^{\bar{3}}\rangle = |(Q_1 Q_2 \bar{Q}_3)^{3_{MA}}\rangle,
\]

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\]

For the color \(\otimes\) spin wave functions, the possible Pauli principle restriction has to be considered. In the diquark-antidiquark configuration, we have

\[
\phi_1 \chi_1 = |(Q_1 Q_2)^{6} (\bar{Q}_3 \bar{Q}_4)^{\bar{6}}\rangle_1^{\bar{2}} \delta_{12}, \quad \phi_2 \chi_1 = |(Q_1 Q_2)^{3} (\bar{Q}_3 \bar{Q}_4)^{\bar{3}}\rangle_1^{\bar{2}} \delta_{12},
\]

\[
\phi_1 \chi_2 = |(Q_1 Q_2)^{6} (\bar{Q}_3 \bar{Q}_4)^{\bar{6}}\rangle_1^{\bar{1}} \delta_{12}, \quad \phi_2 \chi_2 = |(Q_1 Q_2)^{3} (\bar{Q}_3 \bar{Q}_4)^{\bar{3}}\rangle_1^{\bar{1}} \delta_{12},
\]

\[
\phi_1 \chi_3 = |(Q_1 Q_2)^{6} (\bar{Q}_3 \bar{Q}_4)^{\bar{6}}\rangle_1^{\bar{0}} \delta_{12}, \quad \phi_2 \chi_3 = |(Q_1 Q_2)^{3} (\bar{Q}_3 \bar{Q}_4)^{\bar{3}}\rangle_1^{\bar{0}} \delta_{12},
\]

\[
\phi_1 \chi_4 = |(Q_1 Q_2)^{6} (\bar{Q}_3 \bar{Q}_4)^{\bar{6}}\rangle_0^{\bar{2}} \delta_{12}, \quad \phi_2 \chi_4 = |(Q_1 Q_2)^{3} (\bar{Q}_3 \bar{Q}_4)^{\bar{3}}\rangle_0^{\bar{2}} \delta_{12},
\]

\[
\phi_1 \chi_5 = |(Q_1 Q_2)^{6} (\bar{Q}_3 \bar{Q}_4)^{\bar{6}}\rangle_0^{\bar{1}} \delta_{12}, \quad \phi_2 \chi_5 = |(Q_1 Q_2)^{3} (\bar{Q}_3 \bar{Q}_4)^{\bar{3}}\rangle_0^{\bar{1}} \delta_{12},
\]

\[
\phi_1 \chi_6 = |(Q_1 Q_2)^{6} (\bar{Q}_3 \bar{Q}_4)^{\bar{6}}\rangle_0^{\bar{0}} \delta_{12}, \quad \phi_2 \chi_6 = |(Q_1 Q_2)^{3} (\bar{Q}_3 \bar{Q}_4)^{\bar{3}}\rangle_0^{\bar{0}} \delta_{12},
\]

where \(\delta_{12} = 0\) if \(Q_1\) and \(Q_2\) are identical quarks, and \(\delta_{12} = 1\) for the other cases. Similarly, the triquark-antiquark base vectors in the color \(\otimes\) spin space are

\[
\phi_1 \zeta_1 = |(Q_1 Q_2)_{1/2} (\bar{Q}_3)^{3_{MS}}\rangle_{1}^{3/2} \delta_{12}, \quad \phi_2 \zeta_1 = |(Q_1 Q_2)_{1/2} (\bar{Q}_3)^{3_{MA}}\rangle_{1}^{3/2} \delta_{12},
\]

\[
\phi_1 \zeta_2 = |(Q_1 Q_2)_{1/2} (\bar{Q}_3)^{3_{MS}}\rangle_{1}^{3/2} \delta_{12}, \quad \phi_2 \zeta_2 = |(Q_1 Q_2)_{1/2} (\bar{Q}_3)^{3_{MA}}\rangle_{1}^{3/2} \delta_{12},
\]

\[
\phi_1 \zeta_3 = |(Q_1 Q_2)_{1/2} (\bar{Q}_3)^{3_{MS}}\rangle_{1}^{3/2} \delta_{12}, \quad \phi_2 \zeta_3 = |(Q_1 Q_2)_{1/2} (\bar{Q}_3)^{3_{MA}}\rangle_{1}^{3/2} \delta_{12},
\]

\[
\phi_1 \zeta_4 = |(Q_1 Q_2)_{0} (\bar{Q}_3)^{3_{MS}}\rangle_{0}^{3/2} \delta_{12}, \quad \phi_2 \zeta_4 = |(Q_1 Q_2)_{0} (\bar{Q}_3)^{3_{MA}}\rangle_{0}^{3/2} \delta_{12},
\]

\[
\phi_1 \zeta_5 = |(Q_1 Q_2)_{0} (\bar{Q}_3)^{3_{MS}}\rangle_{0}^{3/2} \delta_{12}, \quad \phi_2 \zeta_5 = |(Q_1 Q_2)_{0} (\bar{Q}_3)^{3_{MA}}\rangle_{0}^{3/2} \delta_{12},
\]

To exhaust all possible configurations of the \(QQ\bar{Q}\) system, one replaces each \(Q\) by either \(c\) or \(b\) quark. The cases we need to study are: \(b\bar{b}q\), \(bc\bar{q}\), \(bcq\), \(cc\bar{q}\), \(ccc\bar{q}\), and \(cc\bar{b}q\) \((q = u, d, s)\). They can be divided into two classes: (1) \(b\bar{b}q\), \(bc\bar{q}\), \(cc\bar{q}\) and \(cc\bar{b}q\); and (2) \(bcq\) and \(bc\bar{q}\). Because of the Pauli principle for the first class, \(\delta_{12} = 0\) has to be adopted and thus there are 6 independent color \(\otimes\) spin bases. The Pauli principle and there are twelve independent bases.

### A. The \(b\bar{b}q\), \(bc\bar{q}\), \(cc\bar{q}\) and \(cc\bar{b}q\) systems

The quantum numbers of these systems are \(I(J^P) = \frac{1}{2}(2^+)\), \(\frac{1}{2}(1^+)\), or \(\frac{1}{2}(0^+)\) when \(q = u, d\). The isospin is 0 if \(q = s\). To write the CMI (color-magnetic interaction) matrices in a convenient form, we define the combinations of
the effective couplings: $\mu = 3C_{12} - C_{34}, \nu = C_{12} - 3C_{34}, \alpha = C_{12} + C_{34}, \beta = C_{13} + C_{14}, \gamma = C_{23} + C_{24}, \eta = C_{13} - C_{14}, \tau = C_{23} - C_{24}, \beta' = C_{13} + C_{23}, \gamma' = C_{14} + C_{24}, \eta' = C_{13} - C_{23},$ and $\tau' = C_{14} - C_{24}.$

In the case $J = 2$, there is only one state: $\phi_2\chi_1 = \phi_2\zeta_1$. The average of the CMI is $(H_{CM}) = \frac{8}{3}(\alpha + \beta)$ in both the diquark-antidiquark and triquark-antiquark configurations. For a state with given quark content, one replaces the $C_{ij}$ with appropriate number. For example, for the $b\bar{b}q$ system, $C_{12} = C_{68}, C_{13} = C_{23} = C_{65},$ and so on.

With the diquark-antidiquark base $(\phi_2\chi_3, \phi_1\chi_6)^T$, the CMI matrix in the $J = 0$ case is

$$(H_{CM}) = \begin{pmatrix} \frac{8}{3}(\alpha - 2\beta) & 4\sqrt{3}\beta \\ \frac{4}{3}\alpha & -4\sqrt{3}\beta \end{pmatrix}.$$ (8)

Since $(\phi_2\zeta_4, \phi_1\zeta_6)^T = (\phi_2\chi_3, \phi_1\chi_6)^T$, one also gets this matrix in the triquark-antiquark configuration.

In the $J = 1$ case, one has different CMI matrix elements for the two configurations. The obtained matrix is

$$(H_{CM}) = \begin{pmatrix} \frac{8}{3}(\alpha - \beta) & \frac{8\sqrt{2}}{3}\eta & 8\eta \\ \frac{8\sqrt{2}}{3}\nu & -4\sqrt{2}\beta & \frac{8}{3}\mu \end{pmatrix}.$$ (9)

with the diquark-antidiquark base vector $(\phi_2\chi_2, \phi_2\chi_4, \phi_1\chi_5)^T$. In the triquark-antiquark configuration, one gets

$$(H_{CM}) = \begin{pmatrix} \frac{4}{3}(\mu + 3\nu + 3\beta' - 5\gamma') & 4\sqrt{2}(\mu - 3\nu - 2\gamma') & -\frac{8\sqrt{2}}{3}\gamma' \\ \frac{8}{3}(\mu - 3\beta' + \gamma') & 2\sqrt{2}(3\beta' - \gamma') & \frac{8}{3}\mu \end{pmatrix}.$$ (10)

with the base vector $(\phi_2\zeta_2, \phi_2\zeta_3, \phi_1\zeta_5)^T$. The last matrix elements in the two configurations are the same because $\phi_1\zeta_5 = \phi_1\chi_5$.

**B. The $b\bar{b}q$ and $b\bar{b}q$ systems**

The wave functions are not constrained by the Pauli principle and we have $\delta_{12} = 1$. When the total spin of such systems is $J = 2$, the allowed base states are $\phi_1\chi_1$ and $\phi_2\chi_1$ in the diquark-antidiquark configuration. Then the CMI matrix is

$$(H_{CM}) = \begin{pmatrix} \frac{4}{3}(2\alpha + \beta + \gamma) & -2\sqrt{2}(\eta - \tau) \\ \frac{4}{3}(\alpha + 5\beta + 5\gamma) & -\frac{2}{3}(2\alpha + 3\beta + 5\gamma) \end{pmatrix}.$$ (11)

We use the base vector $(\phi_2\chi_3, \phi_2\chi_6, \phi_1\chi_3, \phi_1\chi_6)^T$ for the CMI matrix in the diquark-antidiquark configuration when the total spin is 0 and get

$$(H_{CM}) = \begin{pmatrix} \frac{8}{3}(\alpha - \beta - \gamma) & -\frac{4}{3}\sqrt{3}(\eta - \tau) \\ -\frac{8}{3}\alpha & 4\sqrt{6}(\beta + \gamma) \\ -4\sqrt{6}(\beta + \gamma) & 0 \\ -\frac{4}{3}(\alpha + 5\beta + 5\gamma) & -\frac{10}{3}\sqrt{3}(\eta - \tau) \end{pmatrix}.$$ (12)

In the case of $J = 1$, we use $(\phi_1\chi_2, \phi_1\chi_4, \phi_1\chi_5, \phi_2\chi_2, \phi_2\chi_4, \phi_2\chi_5)^T$ as the base vector for CMI in the diquark-antidiquark configuration. The obtained CMI matrix reads

$$(H_{CM}) = \begin{pmatrix} -\frac{2}{3}(2\alpha + 5\beta + 5\gamma) & \frac{10}{3}\sqrt{2}(\eta + \tau) & -\frac{10}{3}\sqrt{2}(\beta - \gamma) \\ -\frac{4}{3}\nu & \frac{2}{3}\sqrt{3}(\eta - \tau) & -4(\beta - \gamma) & 0 \\ -4(\beta - \gamma) & 0 & \frac{2}{3}(\eta - \tau) \end{pmatrix}.$$ (13)

From $(\phi_1\zeta_1, \phi_2\zeta_1)^T = (\phi_1\chi_1, \phi_2\chi_1)^T$ and $(\phi_1\zeta_4, \phi_1\zeta_6, \phi_2\zeta_4, \phi_2\zeta_6)^T = (\phi_1\chi_3, \phi_1\chi_6, \phi_2\chi_3, \phi_2\chi_6)^T$, it is easy to understand that the CMI matrices in the triquark-antiquark configuration are the same as the above ones for the tensor
We use an estimation \( C \) mainly focus on the results in the second scheme. The meson masses we will use in this work are \[92\]:

\[
\begin{pmatrix}
    -\mu + 15\beta' \\
    -3\mu - 25\gamma' \\
    2\left(5\gamma' - \mu - 15\beta'\right)
\end{pmatrix} \sqrt{2} \begin{pmatrix}
    3\nu - \mu \\
    -10\gamma' \\
    6\mu
\end{pmatrix}
\begin{pmatrix}
    -10\sqrt{6}\tau' \\
    -3\sqrt{2}(3\eta' + 5\tau') \\
    6\sqrt{2}(3\eta' + \tau')
\end{pmatrix}
\begin{pmatrix}
    -12\tau' \\
    3\sqrt{6}(3\beta' - \gamma') \\
    -4\sqrt{6}\gamma'
\end{pmatrix}.
\]

The present study is not a dynamical calculation and we use two schemes to estimate roughly the mass of the \( J = 2 \) and scalar \( J = 0 \) systems, respectively. The difference only occurs in the case of \( J = 1 \). If we choose the base vector \( \left( \phi_1 \zeta_2, \phi_1 \zeta_3, \phi_1 \zeta_5, \phi_2 \zeta_2, \phi_2 \zeta_3, \phi_2 \zeta_5 \right)^T \), the obtained CMI matrix in the triquark-antiquark configuration is

\[
\langle H_{CM} \rangle = \frac{2}{9} \times \begin{pmatrix}
    -\mu + 15\beta' \\
    -3\mu - 25\gamma' \\
    2\left(5\gamma' - \mu - 15\beta'\right)
\end{pmatrix} \sqrt{2} \begin{pmatrix}
    3\nu - \mu \\
    -10\gamma' \\
    6\mu
\end{pmatrix} \begin{pmatrix}
    -10\sqrt{6}\tau' \\
    -3\sqrt{2}(3\eta' + 5\tau') \\
    6\sqrt{2}(3\eta' + \tau')
\end{pmatrix} \begin{pmatrix}
    -12\tau' \\
    3\sqrt{6}(3\beta' - \gamma') \\
    -4\sqrt{6}\gamma'
\end{pmatrix}.
\]

\[14\]

### III. NUMERICAL RESULTS

#### A. Parameters

The parameters \( C_{Qq} \) (\( Q = c, b, q = n, s \) with \( n = u, d \)) can be extracted from the masses of charmed and bottom baryons while the parameters \( C_{QQ} \) \( (C_{Q\bar{q}}) \) are determined from the heavy quarkonium \( (D \text{ and } B) \) mesons. We list the derived parameters in Tab. 1. For \( C_{bc}, C_{bc}, C_{bb} \) and \( C_{cc} \), since the relevant baryons are not observed in experiments, we use an estimation \( C_{bc} = 3.3 \text{ MeV} \) from a quark model calculation \[29\] and use the approximation \( C_{cc} = C_{c\bar{c}} \), \( C_{bc} = C_{b\bar{c}} \), \( C_{cc} = C_{c\bar{c}} \) and \( C_{bb} = C_{b\bar{b}} \).

| TABLE I: The effective coupling parameters in units of MeV. |
|-----------------------------------------------------------|
| **Hadron** | **\langle H_{CM} \rangle** | **Hadron** | **\langle H_{CM} \rangle** | **Parameter** |
|------------|----------------------------|------------|----------------------------|---------------|
| \( N \)    | \(-8C_{nn}\) | \( \Delta \) | \( 8C_{nn} \) | \( C_{nn} = 18.4 \) |
| \( \Sigma \) | \( \frac{8}{3}C_{nn} - \frac{32}{3}C_{ns} \) | \( \Sigma)^* \) | \( \frac{8}{3}C_{nn} + \frac{16}{3}C_{ns} \) | \( C_{ns} = 12.4 \) |
| \( \Xi^0 \) | \( \frac{8}{3}(C_{ss} - 4C_{ns}) \) | \( \Xi^{*0} \) | \( \frac{8}{3}(C_{ss} + C_{ns}) \) |
| \( \Omega \) | \( 8C_{ss} \) | \( \Sigma^{*0} \) | \( \frac{8}{3}(C_{ss} + C_{ns}) \) |
| \( D \) | \(-16C_{cn}\) | \( D^* \) | \( \frac{16}{3}C_{cn} \) | \( C_{cn} = 6.7 \) |
| \( D_s \) | \(-16C_{cs}\) | \( D_s^* \) | \( \frac{16}{3}C_{cs} \) | \( C_{cs} = 6.7 \) |
| \( B \) | \(-16C_{bn}\) | \( B^* \) | \( \frac{16}{3}C_{bn} \) | \( C_{bn} = 2.1 \) |
| \( B_s \) | \(-16C_{bs}\) | \( B_s^* \) | \( \frac{16}{3}C_{bs} \) | \( C_{bs} = 2.3 \) |
| \( \eta_c \) | \(-16C_{cc}\) | \( J/\psi \) | \( \frac{16}{3}C_{cc} \) | \( C_{cc} = 5.3 \) |
| \( \eta_b \) | \(-16C_{bb}\) | \( \Upsilon \) | \( \frac{16}{3}C_{bb} \) | \( C_{bb} = 2.9 \) |
| \( \Sigma_c \) | \( \frac{8}{3}C_{cn} - \frac{32}{3}C_{cn} \) | \( \Sigma_c^* \) | \( \frac{8}{3}C_{nn} + \frac{16}{3}C_{cn} \) | \( C_{en} = 4.0 \) |
| \( \Xi_c \) | \( \frac{8}{3}C_{nn} - \frac{16}{3}C_{cn} - \frac{16}{3}C_{cs} \) | \( \Xi_c^* \) | \( \frac{8}{3}C_{nn} + \frac{16}{3}C_{cn} + \frac{16}{3}C_{cs} \) | \( C_{en} = 4.6 \) |
| \( \Sigma_b \) | \( \frac{8}{3}C_{nn} - \frac{32}{3}C_{bn} \) | \( \Sigma_b^* \) | \( \frac{8}{3}C_{nn} + \frac{16}{3}C_{bn} \) | \( C_{bn} = 1.3 \) |
| \( \Xi_b \) | \( \frac{8}{3}C_{nn} - \frac{16}{3}C_{bn} - \frac{16}{3}C_{bs} \) | \( \Xi_b^* \) | \( \frac{8}{3}C_{nn} + \frac{16}{3}C_{bn} + \frac{16}{3}C_{bs} \) | \( C_{bn} = 1.2 \) |

The present study is not a dynamical calculation and we use two schemes to estimate roughly the mass of the \( QQQ\bar{q} \) systems. In the first scheme, we use the effective quark masses \( m_c = 1724.8 \text{ MeV}, m_b = 5052.9 \text{ MeV}, m_n = 361.8 \text{ MeV} \) and \( m_s = 540.4 \text{ MeV} \) as inputs. These masses are extracted from the known baryons. We have shown in Ref. \[91\] that these values lead to overestimated meson masses and give an upper limit for the ground state tetraquarks. In the second scheme, we determine the tetraquark masses by comparing to the threshold of a two-meson system. We will mainly focus on the results in the second scheme. The meson masses we will use in this work are \[92\]: \( m_T = 9460.3 \text{ MeV}, m_{n_b} = 9398.0 \text{ MeV}, m_{J/\psi} = 3996.9 \text{ MeV}, m_{n_c} = 2983.6 \text{ MeV}, m_D = 1864.8 \text{ MeV}, m_{D^*} = 2007.0 \text{ MeV}, m_{D_s} = 1968.3 \text{ MeV}, m_{D_s^*} = 2112.1 \text{ MeV}, m_B = 5279.4 \text{ MeV}, m_{B^*} = 5325.2 \text{ MeV}, m_{B_s} = 5366.8 \text{ MeV}, m_{B_s^*} = 5415.4 \text{ MeV}, \) and \( m_{B_c} = 6275.6 \text{ MeV} \).
B. The $b\bar{b}q$, $b\bar{b}q$, $c\bar{c}q$ and $c\bar{c}q$ systems in diquark-antidiquark configuration

By substituting the parameters into the CMI matrices in the previous section and diagonalizing the matrices, we obtain the eigenvalues of the CMI and the tetraquark masses with $M = \sum_i m_i + \langle H_{CM} \rangle$. In the second scheme, we use the formula $M = M_{ref} - \langle H_{CM} \rangle_{ref} + \langle H_{CM} \rangle$.

| $J^P$ | $\langle H_{CM} \rangle$ | Eigenvalue | Eigenvector | Mass | $(J/\psi D)$ |
|-------|------------------|------------|-------------|------|-------------|
| 2$^+$ | 56.8             | 56.8       | 1           | 5593.0 | 5097.4     |
| 1$^+$ | (−7.2 −5.3 −11.2) | −72.8     | {−0.17, −0.77, −0.61} | 5463.4 | 4967.8     |
|       | (−5.3 −17.9 −67.9) | 69.4      | {−0.07, −0.61, 0.79} | 5605.6 | 5110.0     |
|       | (−11.2 −67.9 15.9) | −5.7      | {0.98, −0.18, −0.04} | 5530.5 | 5034.9     |
| 0$^+$ | (−39.2 117.6) | −124.6    | {−0.81, 0.59} | 5411.6 | 4916.0     |
|       | (117.6 37.2) | 122.6     | {−0.59, −0.81} | 5658.8 | 5163.3     |

| $J^P$ | $\langle H_{CM} \rangle$ | Eigenvalue | Eigenvector | Mass | $(J/\psi D_s)$ |
|-------|------------------|------------|-------------|------|-------------|
| 2$^+$ | 58.4             | 58.4       | 1           | 5773.2 | 5202.5     |
| 1$^+$ | (−5.6 −5.3 −11.2) | −76.0     | {−0.15, −0.78, −0.60} | 5638.8 | 5068.2     |
|       | (−5.3 −22.7 −67.9) | 67.1      | {−0.08, −0.60, 0.80} | 5781.9 | 5211.2     |
|       | (−11.2 −67.9 15.1) | −4.3      | {0.98, −0.17, −0.03} | 5710.5 | 5139.8     |
| 0$^+$ | (−37.6 117.6) | 124.7     | {0.59, 0.81} | 5839.5 | 5268.9     |
|       | (117.6 39.6) | −122.7    | {−0.81, 0.59} | 5592.1 | 5021.4     |

We present the CMI matrices, eigenvalues, eigenvectors and the estimated masses for the $c\bar{c}n$ ($n = u, d$) and the $cc\bar{s}$ systems in Table II. If these states really exist, probably the masses are slightly above the values in the last column. The reason is that the mass estimated with a color-spin interaction seems underestimated and a correction from the additional kinetic energy is probably needed [91, 93]. The mass splitting between the $c\bar{c}n$ tetraquarks with different spins is at most 250 MeV. The maximum splitting for the $cc\bar{s}$ tetraquarks is similar. We plot the relative positions of these states in Fig. 1, where the masses in the threshold approach are adopted.

![FIG. 1: Proposed $c\bar{c}q$ (left) and $b\bar{b}q$ (right) tetraquark states. The solid (dashed) line corresponds to the case $q = u, d (q = s)$. The dotted line indicates various meson-meson thresholds. When a number in the subscript of a meson-meson state is equal to the spin of an initial state, the decay for the initial state into that meson-meson channel through $S$- or $D$-wave is allowed. The masses are given in units of MeV.](image)

The quantum numbers of these $c\bar{c}n$ ($cc\bar{s}$) tetraquarks are the same as those of $D^*_0$, $D_1$, and $D^*_2$ ($D^*_0$, $D^*_1$, and $D^*_2$). The orbital or radial excitation cannot induce a state 2500 MeV higher than the ground state. Therefore, once the predicted states could be observed, it is easy to identify them as $D$- or $D^*_2$-like mesons with an excited charm-anticharm pair. The argument is similar to the work in predicting the hidden charm pentaquarks [80]. Compared
with the pentaquark case, the binding force for the present system is dominantly provided by the gluon exchange interaction. The contribution from the meson exchange is highly suppressed and the interaction between a charmonium and a heavy-light meson is not strong. As a result, a molecule or cusp interpretation is not favored once the resonance structure is observed. In this sense, one may observe a genuine tetraquark.

TABLE III: Results for the $bbb\bar{q}$ systems in units of MeV. The masses in the fifth column are calculated with the effective quark masses and are theoretical upper limits. The last column lists masses estimated from the $\Upsilon B$ ($\Upsilon B_s$) threshold.

| $J^P$ | $(H_{\text{CM}})$ | Eigenvalue | Eigenvector | Mass | $(\Upsilon B)$ |
|-------|--------------------|------------|-------------|------|----------------|
| 2+    | 24.5               | 24.5       | 1           | 15545.0 | 14782.4 |
|       | (−2.1 3.0 6.4)     | (32.9)     | {0.09, −0.62, 0.78} | 15553.4 | 14790.7 |
| 1+    | (3.0 −2.7 −28.3)   | (−27.0)    | {0.25, −0.75, −0.62} | 15493.5 | 14730.8 |
|       | (6.4 −28.3 9.9)    | (−0.8)     | {0.96, 0.25, 0.09}  | 15519.7 | 14757.1 |
| 0+    | (−15.5 49.0)       | (52.2)     | {0.59, 0.81}      | 15572.7 | 14810.1 |
|       | (49.0 16.8)        | (−50.9)    | {−0.81, 0.59}     | 15469.6 | 14706.9 |

| $J^P$ | $(H_{\text{CM}})$ | Eigenvalue | Eigenvector | Mass | $(\Upsilon B_s)$ |
|-------|--------------------|------------|-------------|------|-----------------|
| 2+    | 24.8               | 24.8       | 1           | 15723.9 | 14873.2 |
|       | (−2.9 2.3 4.8)     | (34.2)     | {0.06, −0.63, 0.78} | 15733.3 | 14882.6 |
| 1+    | (2.3 −1.9 −29.4)   | (−26.9)    | {0.20, −0.75, −0.63} | 15672.2 | 14821.5 |
|       | (4.8 −29.4 10.0)   | (−2.1)     | {0.98, 0.19, 0.08} | 15697.0 | 14846.3 |
| 0+    | (−16.8 50.9)       | (−53.8)    | {−0.81, 0.59}  | 15645.3 | 14794.6 |
|       | (50.9 16.4)        | (−53.4)    | {−0.59, −0.81} | 15752.5 | 14901.8 |

TABLE IV: Results for the $ccb\bar{q}$ systems in units of MeV. The masses in the fifth column are calculated with the effective quark masses and are theoretical upper limits. The last column lists masses estimated from the $B_c D$ ($B_c D_s$) threshold.

| $J^P$ | $(H_{\text{CM}})$ | Eigenvalue | Eigenvector | Mass | $(B, D)$ |
|-------|--------------------|------------|-------------|------|---------|
| 2+    | 44.3               | 44.3       | 1           | 8908.6 | 8344.7 |
|       | (−9.1 −12.8 −27.2) | (70.7)     | {−0.16, −0.62, 0.77} | 8935.0 | 8371.1 |
| 1+    | −12.8 3.7 −56.6    | −60.6      | {−0.48, −0.63, −0.61} | 8803.7 | 8239.8 |
|       | −27.2 −56.6 19.5   | 4.0        | {0.86, −0.47, −0.19} | 8868.3 | 8304.4 |
| 0+    | −35.7 98.0         | −107.5     | {−0.81, 0.59}  | 8756.8 | 8192.9 |
|       | 98.0 26.4          | 98.1       | {−0.59, −0.81} | 8962.4 | 8398.5 |

| $J^P$ | $(H_{\text{CM}})$ | Eigenvalue | Eigenvector | Mass | $(B, D_s)$ |
|-------|--------------------|------------|-------------|------|---------|
| 2+    | 44.0               | 44.0       | 1           | 9086.9 | 8447.9 |
|       | (−9.3 −12.8 −27.2) | (71.1)     | {−0.16, −0.62, 0.77} | 9114.0 | 8475.0 |
| 1+    | −12.8 4.5 −56.6    | −60.3      | {−0.48, −0.63, −0.61} | 8982.6 | 8343.6 |
|       | −27.2 −56.6 19.6   | 4.0        | {0.86, −0.47, −0.20} | 9046.9 | 8407.9 |
| 0+    | −36.0 98.0         | −107.8     | {−0.81, 0.59}  | 8935.1 | 8296.1 |
|       | 98.0 26.0          | 97.8       | {−0.59, −0.81} | 9140.7 | 8501.7 |

Let us take a look at the possible S-wave strong decay channels of these compact tetraquark states from Fig. 1. For the scalar states, the dominant channels are possibly $\eta D, \eta D_s, J/\psi D^*$, and $J/\psi D_s^*$. For the axial vector states, possible decay channels are $J/\psi D, J/\psi D^*, \eta D_s, J/\psi D_s, J/\psi D_s^*$. For the tensor states, the possible decay channels are $J/\psi D^*$ and $J/\psi D_s^*$. Whether relevant channels are open or not depends on the tetraquark mass and flavor conservation. For example, the tensor meson with $M = 5097$ (5203) is around the threshold of $J/\psi D^*$ ($J/\psi D_s^*$) and the decay is marginal. However, it is still possible to find a resonance slightly above the threshold if one considers the limitation of the present estimation method. More channels are possible if the $D$-wave decays are considered. In
Fig. 1, we present several numbers in the subscript of the meson-meson states. When a number is equal to the spin of an initial state, the decay for the initial state into that meson-meson channel through $S$- or $D$-wave is allowed.

By replacing the charm quark with the bottom quark, we get the results for the $b\bar{b}b\bar{b}$ and $b\bar{b}b\bar{s}$ systems in Table III. The rough position is also given in Fig. 1 and similar analysis for the possible decay channels is straightforward.

Now we move on to the $c\bar{c}b\bar{q}$ and $b\bar{b}c\bar{q}$ systems. Such states are explicitly exotic. Up to now, the doubly charmed baryon $\Xi_{cc}$ has not been confirmed since the first announcement by the SELEX Collaboration \cite{94}. There is no experimental result on the search for the proposed $T_{cc}$ ($cc\bar{q}q$) tetraquark. But their existence is not excluded. If one replaces a light antiquark in the $T_{cc}$ with an anti-bottom quark, one gets a currently discussed tetraquark system $c\bar{c}b\bar{q}$. The exchange of bottom and charm results in another exotic tetraquark system $b\bar{b}c\bar{q}$. We present the results for the $c\bar{c}b\bar{q}$ and $b\bar{b}c\bar{q}$ systems in Tabs. IV and V, respectively. In Fig. 2, we display the rough positions of these possible tetraquarks, where the masses in the threshold approach are adopted. It is easy to judge their possible decays from the figure.

TABLE V: Results for the $b\bar{b}c\bar{q}$ systems in units of MeV. The masses in the fifth column are calculated with the effective quark masses and are theoretical upper limits. The last column lists masses estimated from the $B_cB$ ($B_cB_s$) threshold.

| $J^P$ | $(H_{CM})$ | Eigenvalue | Eigenvector | Mass | $(B_cD)$ |
|-------|------------|------------|-------------|------|-----------|
| $b\bar{b}c\bar{q}$ system | | | | |
| $2^+$ | 34.9 | 34.9 | 1 | 12405.9 | 11766.9 |
| $1^+$ | 5.1 3.8 8.0 | -48.9 | {0.14, -0.85, -0.51} | 12322.1 | 11683.1 |
|       | 8.0 -31.7 5.5 | 25.5 | {0.24, -0.47, 0.85} | 12396.5 | 11757.5 |
|       | -9.9 54.9 | 48.4 | {0.57, 0.82} | 12439.4 | 11800.4 |

FIG. 2: Proposed $c\bar{c}b\bar{q}$ (left) and $b\bar{b}c\bar{q}$ (right) tetraquark states. The solid (dashed) line corresponds to the case $q = u, d$ ($q = s$). The dotted line indicates various meson-meson thresholds. When a number in the subscript of a meson-meson state is equal to the spin of an initial state, the decay for the initial state into that meson-meson channel through $S$- or $D$-wave is allowed. The masses are given in units of MeV.
TABLE VI: Results for the \( bc\bar{c}q \) systems in units of MeV. The masses in the fifth column are calculated with the effective quark masses and are theoretical upper limits. The last two columns list masses estimated from the TD (TD_{s}) and BB (B_{s}B_{c}) thresholds, respectively.

| \( J^P \) | \( (H_{CM}) \) | Eigenvalue | Eigenvector | Mass | (TD) | (BB_{s}) |
|--------|---------|------------|-------------|------|-------|----------|
| 2^{+}  | \( (32.3 \ -11.9) \) | 51.3 | \([-0.53, 0.85]\) | 12243.7 | 11468.1 | 11692.7 |
|        | \([-11.9 \ 43.9]\) | 24.8 | \([-0.85, -0.53]\) | 12217.2 | 11441.7 | 11662.2 |
|        | \(-56.1 -12.3\) | 23.6 | 11.9 | 20.0 | -10.4 | 12096.8 | 11321.3 | 11545.8 |
|        | 12.3 | 14.0 | 20.0 | 0.0 | 42.4 | 12246.1 | 11470.5 | 11695.1 |
| 1^{+}  | 23.6 | 14.0 | 11.5 | -10.4 | -42.4 | 0.0 | 12414.9 | 11369.4 | 11593.9 |
|        | 11.9 | 20.0 | -10.4 | -7.7 | 4.9 | 9.4 | 12223.4 | 11447.8 | 11672.4 |
|        | 20.0 | 0.0 | 0.0 | -42.4 | -4.9 | 1.6 | 5.6 | 12176.8 | 11401.3 | 11625.8 |
|        | \(-10.4 -42.4\) | 0.0 | 9.4 | 5.6 | -22.9 | \(-2.2\) | \([0.02, -0.03, -0.04, 0.89, -0.23, 0.40]\) | 12190.2 | 11414.7 | 11639.2 |
| 0^{+}  | \(-27.7 -9.7\) | 23.8 | 73.5 | 106.1 | -24.2 | \(-70.5\) | \([-0.74, 0.31, -0.24, 0.55]\) | 12121.9 | 11346.4 | 11570.9 |
|        | \(-9.7 -36.8\) | 73.5 | \([0.18, 0.82, 0.55, 0.02]\) | 12202.7 | 11427.1 | 11651.7 |

| \( J^P \) | \( (H_{CM}) \) | Eigenvalue | Eigenvector | Mass | (TD_{s}) | (BB_{s}) |
|--------|---------|------------|-------------|------|----------|----------|
| 2^{+}  | \( (32.3 \ -11.3) \) | 51.4 | \([-0.51, 0.86]\) | 12242.4 | 11571.7 | 11783.4 |
|        | \([-11.3 \ 44.7]\) | 25.6 | \([-0.86, -0.51]\) | 12396.6 | 11545.9 | 11757.6 |
|        | \(-56.7 -13.2\) | 22.6 | 11.3 | 19.2 | -11.2 | 12275.3 | 11424.6 | 11636.3 |
|        | \(-13.2 0.4\) | 13.3 | 19.2 | 0.0 | -43.0 | 54.9 | \([-0.66, 0.36, -0.34, -0.27, -0.32, 0.37]\) | 12425.0 | 11574.4 | 11786.0 |
| 1^{+}  | 22.6 | 13.3 | 11.6 | -11.2 | -43.0 | 0.0 | 12324.0 | 11473.3 | 11685.0 |
|        | 19.2 | 0.0 | 0.0 | -43.0 | -5.3 | 17.3 | \([-0.03, 0.71, -0.32, 0.28, -0.48]\) | 12353.7 | 11563.1 | 11714.7 |
|        | \(-11.2 -43.0\) | 0.0 | 9.1 | 5.3 | -23.2 | \(-2.5\) | \([-0.02, -0.02, -0.08, 0.88, -0.27, 0.37]\) | 12368.5 | 11517.8 | 11729.5 |
| 0^{+}  | \(-28.5 -9.2\) | 22.6 | 74.5 | 143.7 | \(-165.9\) | \([-0.28, -0.48, 0.80, 0.21]\) | 12205.1 | 11354.4 | 11566.1 |
|        | \(-9.2 -36.0\) | 74.5 | \([0.17, 0.82, 0.54, 0.03]\) | 12382.3 | 11531.6 | 11743.3 |

C. The \( bc\bar{c}q \) and \( bc\bar{c}\bar{q} \) systems in the diquark-antidiquark configuration

The \( bc\bar{c}q \) and \( bc\bar{c}\bar{q} \) are also hidden-bottom and hidden-charm systems respectively. Their features are different from the states in the last subsection. The former case corresponds to the excited \( D \) and \( D_{s} \) mesons with much higher masses than the \( cc\bar{c}q \). The latter case corresponds to the excited \( B \) and \( B_{s} \) mesons with lower masses than the \( bb\bar{b}q \). Now the first two heavy quarks are different in the flavor space and there is no constraint from the Pauli principle. Therefore, the number of allowed states is doubled. There are two types of meson-meson threshold one may compare to, \( (\bar{b}\bar{d})(b\bar{q}) \) \( (\bar{b}c)(c\bar{q}) \) and \( (cc')(\bar{b}\bar{q}) \) for the \( bc\bar{c}q \) \( bc\bar{c}q \) case. We use both of them in estimating the tetraquark masses. The number of the possible strong decay channels is also bigger than in the previous cases. We show the numerical results in Tabs. VI and VII for the \( bc\bar{c}q \) and \( bc\bar{c}\bar{q} \) systems, respectively. At present, we cannot determine the accurate values of the masses without solving the bound state problem. One needs further study to answer which set of masses is more physical. The rough positions for these states are given in Fig. 3, where we use the masses estimated with the \( BB_{c}, B_{s}B_{c} \), \( BJ/\psi \), and \( B_{s}/\psi \) thresholds.

When discussing the decay patterns, we do not include possible final states containing \( B_{s}^{*} \). First, we focus on the \( bc\bar{c}q \) case. For the states with \( J^P = 0^{+} \), possible S-wave channels are \( \eta B, \eta D, \chi D^{*}, \chi B_{c}, \eta B_{s}, \eta D_{s}, \chi D_{s}^{*}, \chi B_{c}^{*}, \eta B_{s}^{*}, \eta D_{s}^{*}, \chi D_{s}^{*}, \) and \( B_{s}B_{c} \). For the states with \( J^P = 1^{+} \), possible channels are \( \chi D, \eta B, \eta D^{*}, \eta B_{s}, \eta D_{s}, \eta B_{s}^{*}, \eta D_{s}^{*} \), and \( B_{s}B_{c} \). For the \( J^P = 2^{+} \) states, possible channels are just \( \chi D^{*} \) and \( \chi B_{c}^{*} \). Secondly, we take a look at the \( bc\bar{c}q \) case. The possible S-wave channels for \( J^P = 0^{+} \) are \( \eta B, \eta D, \chi D^{*}, \) \( BB_{c}, \eta B_{s}, \eta D_{s}, \chi D_{s}^{*}, \) and \( B_{s}B_{c} \). For \( J^P = 1^{+} \), the channels are \( J/\psi B, \eta B_{s}, \eta B^{*}, \chi D^{*}, \chi B_{c}^{*} \), and \( B_{s}B_{c} \). For \( J^P = 2^{+} \), the channels are \( J/\psi B^{*}, \chi D^{*}, \chi B_{c}^{*} \), and \( B_{s}B_{c}^{*} \). More channels are possible if the \( D \)-wave decay is considered. Whether the channels are open or not is easy to judge from Fig. 3.

D. Numerical results in triquark-\( q \) configuration

In section II, we have found that the diquark-antidiquark configuration and the triquark-antiquark configuration give identical results for the cases \( J = 2 \) and \( J = 0 \) because the flavor-color-spin wave functions are the same. For
TABLE VII: Results for the $b c \bar{c}q$ systems in units of MeV. The masses in the fifth column are calculated with the effective quark masses and are theoretical upper limits. The last two columns list masses estimated from the $B, D$ ($B, D_s$) and $B J/\psi$ ($B, J/\psi$) thresholds, respectively.

### $b c \bar{c}n$ system

| $J^P$ | $<H_{CM}>$ | Eigenvalue | Eigenvector | Mass | $(B, D, B J/\psi)$ |
|-------|-------------|------------|-------------|------|-------------------|
| 2$^+$ | 42.7, -7.4  | 53.3       | -0.57, 0.82 | 8917.6, 8353.7, 8435.0 |
|       | -7.4, 48.3  | 37.6       | -0.82, -0.57 | 8901.9, 8380.8, 8419.2 |
| 1$^+$ | -67.7, -0.9 | 83.11, 7.4 | 26.4, -0.8 | 8566 | 8566 |
|       | 11.9, 8.7   | 36.5       | -0.14, 0.58, 0.04, 0.08, 0.69 | 8800.8, 8236.9, 8318.2 |
| 0$^+$ | 31.1, 8.7   | 56.7       | 0.06, 0.70, 0.42, 0.22, -0.26, -0.45 | 8921.0, 8357.1, 8438.3 |
|       | 7.4, 26.4   | 41.0       | 0.07, -0.39, 0.71, -0.15, -0.50, 0.28 | 8905.3, 8341.4, 8422.6 |
|       | 26.4, 0.0   | -25.0      | 0.64, -0.01, 0.37, -0.16, 0.65, -0.04 | 8839.3, 8275.4, 8356.7 |
|       | -0.8, -49.2 | 4.9        | -0.13, -0.02, -0.05, -0.86, -0.08, -0.47 | 8866.9, 8305.3, 8386.5 |

### $b c \bar{s}$ system

| $J^P$ | $<H_{CM}>$ | Eigenvalue | Eigenvector | Mass | $(B, D, B J/\psi)$ |
|-------|-------------|------------|-------------|------|-------------------|
| 2$^+$ | 44.5, -6.8  | 53.4       | -0.61, 0.79 | 8700.3, 8457.3, 8525.6 |
|       | -6.8, 48.1  | 39.3       | -0.79, -0.61 | 8902.2, 8443.2, 8511.5 |
| 1$^+$ | -69.2, -1.9 | 30.2, 8.0  | 25.6, -1.6  | 8285 | 8285 |
|       | 19.0, 8.0   | 57.7       | 0.05, 0.75, 0.34, 0.22, -0.20, -0.49 | 8981.5, 8342.5, 8410.8 |
|       | 30.2, 8.0   | 40.7       | 0.08, -0.31, 0.76, -0.13, -0.51, 0.22 | 8908.6, 8444.6, 8512.9 |
| 0$^+$ | 25.6, 0.0   | -28.8      | 0.66, -0.01, 0.35, -0.10, 0.65, -0.03 | 9014.1, 8375.1, 8443.4 |
|       | -1.6, -49.8 | 5.5        | -0.10, -0.03, -0.04, -0.88, -0.04, -0.46 | 9048.4, 8409.4, 8469.7 |

![FIG. 3: Proposed $b c \bar{c}q$ (left) and $b c \bar{s}$ (right) tetraquark states. The solid (dashed) line indicates various meson-meson thresholds. When a number in the subscript of a meson-meson state is equal to the spin of an initial state, the decay for the initial state into that meson-meson channel through S- or D-wave is allowed. The masses are given in units of MeV.](image-url)

The $J = 1$ case, their explicit spin wave functions are different, which results from the coupling order in the spin space. The resulting color-spin bases are different and the CMI matrices are not equal if the color-spin mixing is not considered. After the diagonalization for the matrix ($H_{CM}$) is performed, one finds that the results in the two configurations are also equal. We show in Tab. VIII the diagonal of the matrix and its eigenvalues in the triquark-antiquark configuration for the case $J = 1$. It is obvious that the tetraquark spectra in these two configurations are
the same from the comparison with previous results. By comparing the numbers in the two columns of Tab. VIII, one understands the importance of the mixing effect in the triquark-antiquark configuration.

| System | diag(⟨H_CM⟩) | Eigenvalues |
|--------|---------------|-------------|
| ccďq   | (−19.3, −5.8, 15.9) | (−72.8, 69.4, −5.7) |
| ccďš   | (−22.0, −6.3, 15.1) | (−76.0, 67.1, −4.3) |
| bbďq   | (0.4, −5.2, 9.9) | (32.9, −27.0, −0.8) |
| bbďš   | (−0.1, −4.7, 10.0) | (34.2, −26.9, −2.1) |
| ccďš   | (−12.6, 7.3, 19.5) | (70.7, −60.6, 4.0) |
| ccďš   | (−12.2, 7.4, 19.6) | (71.1, −60.3, 4.0) |
| bbďš   | (−10.6, −9.7, 6.3) | (−44.8, 26.7, 4.1) |
| bbďš   | (−14.1, −9.9, 5.5) | (−48.9, 25.5, 4.8) |
| bbďš   | (−29.7, −25.6, 11.5, −8.3, −1.1, −22.9) | (−95.6, 53.7, −47.5, 31.0, −15.6, −2.2) |
| bbďš   | (−31.1, −25.2, 11.6, −8.3, −0.8, −23.2) | (−95.7, 54.0, −47.0, 31.5, −17.3, −2.5) |
| bbďš   | (−15.7, −40.4, 7.9, −17.1, −9.9, −15.7) | (−105.0, −63.5, 56.7, 41.0, −25.0, 4.9) |
| bbďš   | (−15.5, −39.7, 7.1, −20.2, −10.2, −14.1) | (−106.4, −61.4, 57.7, 40.7, −28.8, 5.5) |

## IV. DISCUSSIONS

In this work, we have calculated the mass splittings for the $QQ\bar{Q}\bar{q}$ type tetraquark systems with the color-magnetic interaction, where $q = u, d, s$ and $Q = c$ or $b$. We have also estimated roughly the mass positions of these states. We have considered both the diquark-antidiquark $[\langle QQ\rangle\langle \bar{Q}\bar{q}\rangle]$ and triquark-antiquark $[\langle QQ\rangle\langle \bar{Q}\bar{q}\rangle]$ configurations and obtained the same numerical results. We notice that one does not need to distinguish the configurations for a compact $QQ\bar{Q}\bar{q}$ system once the mixing between different color-spin states is considered.

The role of the color-spin mixing is different for the $J = 2$, $J = 1$, and $J = 0$ cases in a specific system. For all the discussed systems, the $J^P = 0^+$ states get the largest mass gap from the mixing effect and all the highest and lowest states are scalar. In our calculation, the largest mass gap for the $ccďš(ccďš)$, $bbďš(bbďš)$, $cbďš(ccďš)$, $bbďš(bbďš)$, $bbďš(bbďš)$ and $ccďš(ccďš)$ systems are 247(248) MeV, 103(107) MeV, 206(206) MeV, 112(116) MeV, 239(240) MeV and 279(285) MeV, respectively.

We have used two parameter schemes to estimate the masses of the compact $QQ\bar{Q}\bar{q}$ tetraquarks. In the effective quark mass scheme, our results are only theoretical upper limits. In the reference threshold method, the rough masses probably are close to the physical ones. We collect the rough masses for the systems in Table IX, where the tetraquark states without constraint from the Pauli principle and those with exotic flavor are labeled. The results are preliminary since the calculation does not involve dynamics. More studies are needed to clarify the mass spectrum of the $QQ\bar{Q}\bar{q}$ systems. Whether there exist possible stable $QQ\bar{Q}\bar{q}$ tetraquarks also needs dynamical investigations. It is easy to find the possible rearrangement decay patterns from Figs. 1, 2, and 3.

To summarize, we have calculated the mass splittings and estimated the rough masses of the $QQ\bar{Q}\bar{q}$ tetraquarks which are explicitly exotic (ccďš and bbďš) or hidden exotic (excited $D (D_s)$ and $B (B_s)$). We list their decay patterns as shown in the figures which may be helpful to experimental search. If such a high mass $q\bar{q}$-like state could be observed, its tetraquark nature is easy to be identified. Moreover, such a state should be accompanied by many partner states. We want to emphasize that the derived mass splittings should be reliable, although the present simple chromomagnetic interaction model cannot give accurate predictions of the tetraquark masses. However, one can get the masses of its partner states using the mass splittings derived in the present work once a $QQ\bar{Q}\bar{q}$ tetraquark is observed in the future. Hopefully these intriguing states can be searched for at LHC. The ccďš may also be produced at BELLE2.

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TABLE IX: Comparison for the masses of different meson systems with only one light antiquark. The symbol (*) means that the system is not constrained by the Pauli principle and the number of mesons is doubled comparing to the states constrained by the Pauli principle. The symbol ($) indicates explicitly exotic tetraquark states.

| System Mass (GeV) | System Mass (GeV) |
|------------------|-------------------|
| $cc\bar{q}$      | $bbb\bar{q}$      |
| $\sim 11.5$      | $\sim 14.7$       |
| $cc\bar{b}$      | $bb\bar{c}\bar{q}$|
| $\sim 8.2$       | $\sim 11.6$       |
| $cc\bar{c}$      | $*bc\bar{c}q$     |
| $\sim 5.0$       | $\sim 8.2$        |
| $c\bar{q}$       | $b\bar{q}$        |
| $\sim 2.0$       | $\sim 5.3$        |

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