APPORXIMATE SOLUTIONS FOR MHD SQUEEZING FLUID FLOW BY
REPRODUCING KERNEL HILBERT SPACE METHOD
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Abstract: In this paper, a steady axisymmetric MHD flow of two dimensional incompressible fluids has been investigated. Reproducing Kernel Hilbert Space Method (RKHSM) is implemented to obtain solution of reduced fourth order nonlinear boundary value problem. Numerical results have been compared with the results that obtained by the Range-Kutta Method (RK-4) and Optimal Homotopy Asymptotic Method (OHAM).

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1 Introduction

Squeezing flows have many applications in food industry, principally in chemical engineering [1]-[4]. Some practical examples of squeezing flow include polymer processing, compression and injection molding. Grimm [5] studied numerically, the thin Newtonian liquids films being squeezed between two plates. Squeezing flow coupled with magnetic field is widely applied to bearing with liquid-metal lubrication [2], [6]-[8]. In this paper, we use RKHSM to study the squeezing MHD fluid flow between two infinite planar plates.

Consider a squeezing flow of an incompressible Newtonian fluid in the presence of a magnetic field of a constant density $\rho$ and viscosity $\mu$ squeezed between two large planar parallel plates, separated by a small distance $2H$ and the plates approaching each other with a low constant velocity $V$, as illustrated in Figure 1 and the flow can be assumed to quasi-steady [1]-[3], [9]. The Navier-Stokes equations [3]-[4] governing such flow in the presence of magnetic field, when inertial terms are retained in the flow, are given as [10]

$$\nabla V.u = 0, \quad (1.1)$$

$$\rho \left[ \frac{\partial u}{\partial t} + (u, \nabla) u \right] = \nabla . T + J \times B + \rho f, \quad (1.2)$$
where \( u \) is the velocity vector, \( \nabla \) denotes the material time derivative, \( T \) is the Cauchy stress tensor,
\[
T = -pI + \mu A_1,
\]
and
\[
A_1 = \nabla u + u^T,
\]
\( J \) is the electric current density, \( B \) is the total magnetic field and
\[
B = B_0 + b,
\]
\( B_0 \) represents the imposed magnetic field and \( b \) denotes the induced magnetic field. In the absence of displacement currents, the modified Ohm’s law and Maxwell’s equations ([11] and the references therein) are given by [10]
\[
J = \sigma \left[ E + u \times B \right], \tag{1.3}
\]
\[
\text{div} \ B = 0, \quad \nabla \times B = \mu_m J, \quad \text{curl} \ E = \frac{\partial B}{\partial t}, \tag{1.4}
\]
in which \( \sigma \) is the electrical conductivity, \( E \) the electric field and \( \mu_m \) the magnetic permeability.

We need the following assumptions [10]:

a) The density \( \rho \), magnetic permeability \( \mu_m \) and electric field conductivity \( \sigma \), are assumed to be constant throughout the flow field region.

b) The electrical conductivity \( \sigma \) of the fluid considers being finite.

c) Total magnetic field \( B \) is perpendicular to the velocity field \( V \) and the induced magnetic field \( b \) is negligible compared with the applied magnetic field \( B_0 \) so that the magnetic Reynolds number is small ([11] and the references therein).

d) We assume that a situation where no energy is added or extracted from the fluid by the electric field, which implies that there is no electric field present in the fluid flow region.
Under these assumptions, the magnetohydrodynamic force involved in Eq. (1.2) can be put into the form

$$J \times B = -\sigma B_0^2 u.$$  \hspace{1cm} (1.5)

We consider an incompressible Newtonian fluid, squeezed between two large planar, parallel smooth plates which is separated by a small distance $2H$ and moving towards each other with velocity $V$. We assume that the plates are nonconducting and the magnetic field is applied along the $z$-axis. For small values of the velocity $V$, as shown in the Figure 1, the gap distance $2H$ between the plates changes slowly with time $t$, so that it can be taken as constant, the flow is steady [2],[9]. An axisymmetric flow in cylindrical coordinates $r, \theta, z$ with $z$-axis perpendicular to plates and $z = \pm H$ at the plates. Since we have axial symmetry, so $u$ is represented by

$$u = (u_r(r, z), 0, u_z(r, z)),$$

when body forces are negligible, Navier-Stokes Eqs. (1.1)-(1.2) in cylindrical coordinates, where there is no tangential velocity ($u_\theta = 0$), are given as [10]

$$\rho \left( u_r \frac{\partial u_r}{\partial r} + u_z \frac{\partial u_r}{\partial z} \right) = -\frac{\partial p}{\partial r} + \left( \frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} + \frac{\partial^2 u_r}{\partial z^2} \right) + \sigma B_0^2 u_r,$$  \hspace{1cm} (1.6)

$$\rho \left( u_z \frac{\partial u_z}{\partial r} + u_z \frac{\partial u_z}{\partial z} \right) = -\frac{\partial p}{\partial r} + \left( \frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r} \frac{\partial u_z}{\partial r} + \frac{\partial^2 u_z}{\partial z^2} \right),$$  \hspace{1cm} (1.7)
where \( p \) is the pressure, and equation of continuity is:

\[
\frac{1}{r} \frac{\partial}{\partial r} (ru_r) + \frac{\partial u_z}{\partial z} = 0. \tag{1.8}
\]

The boundary conditions require

\[
u_r = 0, \quad u_z = -V \quad \text{at} \quad z = H, \tag{1.9}
\]

\[
\frac{\partial u_r}{\partial z} = 0, \quad u_z = 0 \quad \text{at} \quad z = 0.
\]

Introducing the axisymmetric Stokes stream function \( \Psi \) as

\[
u_r = \frac{1}{r} \frac{\partial \Psi}{\partial z}, \quad u_z = -\frac{1}{r} \frac{\partial \Psi}{\partial r}. \tag{1.10}
\]

The continuity equation is satisfied using Eq. (1.10). Substituting Eqs. (1.3)-(1.5) and Eq. (1.10) into the Eqs. (1.7)-(1.8), we obtain

\[
-\rho \frac{1}{r^2} \frac{\partial \Psi}{\partial r} E^2 \Psi = -\frac{\partial p}{\partial r} + \frac{\mu}{r} \frac{\partial E^2 \Psi}{\partial z} - \frac{\sigma B_0^2}{r} \frac{\partial \Psi}{\partial z} \tag{1.11}
\]

and

\[
-\rho \frac{1}{r^2} \frac{\partial \Psi}{\partial z} E^2 \Psi = -\frac{\partial p}{\partial z} + \frac{\mu}{r} \frac{\partial E^2 \Psi}{\partial r}. \tag{1.12}
\]

Eliminating the pressure from Eqs. (1.11) and (1.12) by integrability condition we get the compatibility equation as

\[
-\rho \left[ \frac{\partial}{\partial (r, z)} \left( \frac{E^2 \Psi}{r^2} \right) \right] = \frac{\mu}{r} E^2 \Psi - \frac{\sigma B_0^2}{r} \frac{\partial^2 \Psi}{\partial z^2}, \tag{1.13}
\]

where

\[
E^2 = \frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.
\]

The stream function can be expressed as

\[
\Psi(r, z) = r^2 F(z). \tag{1.14}
\]

In view of Eq. (1.14), the compatibility Eq. (1.13) and the boundary conditions (1.9) take the form

\[
F^{(iv)}(z) - \frac{\sigma B_0^2}{r} F''(z) + 2 \frac{\mu}{r} F(z) F'''(z) = 0, \tag{1.15}
\]
subject to

\( F(0) = 0, \quad F''(0) = 0, \)  

\( F(H) = \frac{V}{2}, \quad F'(H) = 0. \)  

(1.16)

Introducing the following non-dimensional parameters

\( F^* = 2F, \quad z^* = \frac{z}{H}, \quad \text{Re} = \frac{\rho HV}{\mu}, \quad m = B_0 H \sqrt{\frac{\sigma}{\mu}} \)

For simplicity omitting the \( \ast \), the boundary value problem (1.15)-(1.16) becomes

\[ F^{(iv)}(z) - m^2 F''(z) + \text{Re} F(z) F'''(z) = 0, \]  

(1.17)

with the boundary conditions

\( F(0) = 0, \quad F''(0) = 0, \)  

\( F(1) = 1, \quad F'(1) = 0, \)  

(1.18)

where Re is the Reynolds number and \( m \) is Hartmann number. This problem has been solved by RKHSM and for comparison it has been compared with the OHAM and numerically with the RK-4 by using maple 16.

The RKHSM which accurately computes the series solution is of great interest to applied sciences. The method provides the solution in a rapidly convergent series with components that can be elegantly computed. The efficiency of the method was used by many authors to investigate several scientific applications. Geng and Cui [13] applied the RKHSM to handle the second-order boundary value problems. Yao and Cui [14] and Wang et al. [15] investigated a class of singular boundary value problems by this method and the obtained results were good. Zhou et al. [16] used the RKHSM effectively to solve second-order boundary value problems. Wang and Chao [18] Li and Cui [19], Zhou and Cui [20] independently employed the RKSHSM to variable-coefficient partial differential equations. Geng and Cui [21], Du and Cui [22] investigated the approximate solution of the forced Duffing equation with integral boundary conditions by combining the homotopy perturbation method and the RKM. Lv and Cui [32] presented a new algorithm to solve linear fifth-order boundary value problems. Cui and Du [25] obtained the representation of the exact solution for the nonlinear Volterra-Fredholm integral equations by using the
reproducing kernel Hilbert space method. Wu and Li [27] applied iterative reproducing kernel Hilbert space method to obtain the analytical approximate solution of a nonlinear oscillator with discontinuities. For more details about RKHSM and the modified forms and its effectiveness, see [12]-[37] and the references therein.

The paper is organized as follows. Section 2 introduces several reproducing kernel spaces and a linear operator. Solution representation in $W^5_2[0,1]$ has been presented in Section 3. It provides the main results, the exact and approximate solution of (1.1) and an iterative method are developed for the kind of problems in the reproducing kernel space. We have proved that the approximate solution converges to the exact solution uniformly. Some numerical experiments are illustrated in Section 4. There are some conclusions in the last section.

2 Preliminaries

2.1. Reproducing Kernel Spaces

In this section, we define some useful reproducing kernel spaces.

**Definition 2.1.** (Reproducing kernel). Let $E$ be a nonempty abstract set. A function $K : E \times E \rightarrow C$ is a reproducing kernel of the Hilbert space $H$ if and only if

$$\forall t \in E, \quad K (., t) \in H,$$

$$\forall t \in E, \forall \varphi \in H, \quad \langle \varphi (.), K (., t) \rangle = \varphi (t).$$  \hspace{1cm} (2.1)

The last condition is called ”the reproducing property”: the value of the function $\varphi$ at the point $t$ is reproduced by the inner product of $\varphi$ with $K (., t)$

**Definition 2.2.** We define the space $W^5_2[0,1]$ by

$$W^5_2[0,1] = \left\{ u \left| u, u', u'', u''', u^{(4)} \right. \text{ are absolutely continuous in } [0,1], \right. \quad \begin{cases} 
\left. u^{(5)} \in L^2[0,1], \quad x \in [0,1], \quad u(0) = u(1) = u'(1) = u''(0) = 0. \right. 
\end{cases}$$
The fifth derivative of $u$ exists almost everywhere since $u^{(4)}$ is absolutely continuous. The inner product and the norm in $W_2^5[0, 1]$ are defined respectively by

$$\langle u, v \rangle_{W_2^5} = \sum_{i=0}^{4} u^{(i)}(0)v^{(i)}(0) + \int_0^1 u^{(5)}(x)v^{(5)}(x)dx, \quad u, v \in W_2^5[0, 1],$$

and

$$\|u\|_{W_2^5} = \sqrt{\langle u, u \rangle_{W_2^5}}, \quad u \in W_2^5[0, 1].$$

The space $W_2^5[0, 1]$ is a reproducing kernel space, i.e., for each fixed $y \in [0, 1]$ and any $u \in W_2^5[0, 1]$, there exists a function $R_y$ such that

$$u = \langle u, R_y \rangle_{W_2^5}.$$

**Definition 2.3.** We define the space $W_2^4[0, 1]$ by

$$W_2^4[0, 1] = \left\{ u \left| u, u', u'', u''' \text{ are absolutely continuous in } [0, 1], \right. \right. \right.$$

$$\left. \left. \left. \left\{ \begin{array}{l} u^{(4)} \in L^2[0, 1], \quad x \in [0, 1]. \end{array} \right. \right. \right\}$$

The fourth derivative of $u$ exists almost everywhere since $u'''$ is absolutely continuous. The inner product and the norm in $W_2^4[0, 1]$ are defined respectively by

$$\langle u, v \rangle_{W_2^4} = \sum_{i=0}^{3} u^{(i)}(0)v^{(i)}(0) + \int_0^1 u^{(4)}(x)v^{(4)}(x)dx, \quad u, v \in W_2^4[0, 1],$$

and

$$\|u\|_{W_2^4} = \sqrt{\langle u, u \rangle_{W_2^4}}, \quad u \in W_2^4[0, 1].$$

The space $W_2^4[0, 1]$ is a reproducing kernel space, i.e., for each fixed $y \in [0, 1]$ and any $u \in W_2^4[0, 1]$, there exists a function $r_y$ such that

$$u = \langle u, r_y \rangle_{W_2^4}.$$
Theorem 2.1. The space $W_2^2[0, 1]$ is a reproducing kernel Hilbert space whose reproducing kernel function is given by

$$R_y(x) = \begin{cases} \sum_{i=1}^{10} c_i(y) x^{i-1}, & x \leq y, \\ \sum_{i=1}^{10} d_i(y) x^{i-1}, & x > y, \end{cases}$$

where, $c_j(y)$ can be deduced easily by using for example MAPLE 16,

$$c_1(y) = 0, \quad c_3(y) = 0,$$

$$c_2(y) = -\frac{1537}{378141715} y^9 + \frac{9374}{378141715} y^8 - \frac{3932}{75628343} y^7 + \frac{608}{54020245} y^6$$
$$+ \frac{8006}{54020245} y^5 + \frac{8006}{10804049} y^4 - \frac{14592}{10804049} y^3 + \frac{5201}{10804049} y,$$

$$c_4(y) = \frac{16061}{13613101740} y^9 + \frac{38}{1134425145} y^8 - \frac{48487}{1134425145} y^7 + \frac{10758}{54020245} y^6$$
$$- \frac{145157}{324121470} y^5 - \frac{145157}{64824294} y^4 + \frac{1509137}{388945764} y^3 - \frac{14592}{10804049} y,$$

$$c_5(y) = \frac{1243}{10890481392} y^9 - \frac{4003}{217809627840} y^8 - \frac{107867}{27226203480} y^7 + \frac{145157}{778915280} y^6$$
$$- \frac{81901}{1944728820} y^5 + \frac{9493633}{6223132224} y^4 - \frac{145157}{64824294} y^3 + \frac{8006}{10804049} y,$$

$$c_6(y) = \frac{1243}{54452406960} y^9 - \frac{4003}{1089048139200} y^8 - \frac{107867}{136131017400} y^7 + \frac{145157}{38894576400} y^6$$
$$- \frac{81901}{9723644100} y^5 + \frac{9493633}{3115661120} y^4 - \frac{145157}{324121470} y^3 + \frac{8006}{54020245} y,$$

$$c_7(y) = -\frac{16061}{1633572208800} y^9 - \frac{19}{68065508700} y^8 + \frac{48487}{136131017400} y^7 - \frac{1793}{1080404900} y^6$$
$$+ \frac{145157}{38894576400} y^5 + \frac{145157}{778915280} y^4 - \frac{1509137}{46673491680} y^3 + \frac{608}{54020245} y,$$
\begin{align*}
c_8(y) &= \frac{3631}{1905834243600} y^9 + \frac{983}{762333697440} y^8 - \frac{18797}{238229280450} y^7 + \frac{48487}{136131017400} y^6 \\
&\quad - \frac{107867}{136131017400} y^5 - \frac{107867}{27226203480} y^4 - \frac{48487}{1134425145} y^3 + \frac{1}{10080} y^2 - \frac{3932}{75628343} y, \\
\end{align*}

\begin{align*}
c_9(y) &= \frac{1537}{15246673948800} y^9 - \frac{4687}{7623336974400} y^8 + \frac{983}{7623336974400} y^7 - \frac{19}{68065508700} y^6 \\
&\quad - \frac{4003}{1089048139200} y^5 - \frac{4003}{217809627840} y^4 + \frac{38}{1134425145} y^3 - \frac{743}{62231322240} y, \\
\end{align*}

\begin{align*}
c_{10}(y) &= -\frac{323}{4288127048100} y^9 + \frac{1537}{15246673948800} y^8 + \frac{3631}{1905834243600} y^7 - \frac{16061}{1633572208800} y^6 \\
&\quad + \frac{1243}{54452406960} y^5 + \frac{1243}{108904813920} y^4 + \frac{16061}{13613101740} y^3 - \frac{1537}{378141715} y + \frac{1}{362880} y, \\
\end{align*}

Similarly, we can obtain,

\begin{align*}
d_1(y) &= \frac{1}{362880} y^9, \quad d_3(y) = \frac{1}{10080} y^7, \\
d_2(y) &= -\frac{1537}{378141715} y^9 - \frac{743}{62231322240} y^8 - \frac{3932}{75628343} y^7 + \frac{608}{54020245} y^6 \\
&\quad + \frac{8006}{54020245} y^5 + \frac{8006}{10804049} y^4 - \frac{14592}{10804049} y^3 + \frac{5201}{10804049} y, \\
\end{align*}

\begin{align*}
d_4(y) &= \frac{16061}{13613101740} y^9 + \frac{38}{1134425145} y^8 - \frac{48487}{1134425145} y^7 - \frac{1509137}{46673491680} y^6 \\
&\quad - \frac{145157}{324121470} y^5 - \frac{145157}{64824294} y^4 + \frac{1509137}{388945764} y^3 - \frac{14592}{10804049} y, \\
\end{align*}

\begin{align*}
d_5(y) &= \frac{1243}{108904813920} y^9 - \frac{4003}{217809627840} y^8 - \frac{107867}{27226203480} y^7 + \frac{145157}{7778915280} y^6 \\
&\quad + \frac{949363}{3115661120} y^5 + \frac{949363}{6223132224} y^4 - \frac{145157}{64824294} y^3 + \frac{8006}{10804049} y, \\
\end{align*}
Proof: Let $u \in W^5_2[0,1]$. By Definition 2.2 we have

$$\langle u, R_y \rangle_{W^5_2} = \sum_{i=0}^{4} u^{(i)}(0)R_y^{(i)}(0) + \int_0^1 u^{(5)}(x)R_y^{(5)}(x)dx. \quad (2.3)$$

Through several integrations by parts for (2.3) we have

$$\langle u, R_y \rangle_{W^5_2} = \sum_{i=0}^{4} u^{(i)}(0) \left[ R_y^{(i)}(0) - (-1)^{(4-i)}R_y^{(9-i)}(0) \right] + \sum_{i=0}^{4} (-1)^{(4-i)}u^{(i)}(1)R_y^{(9-i)}(1) - \int_0^1 u(x)R_y^{(10)}(x)dx. \quad (2.4)$$
Note that property of the reproducing kernel

\[ \langle u, R_y \rangle_{W^2} = u(y). \]

Now, if

\[
\begin{aligned}
R_y'(0) + R_y^{(8)}(0) &= 0, \\
R_y^{(3)}(0) + R_y^{(6)}(0) &= 0, \\
R_y^{(4)}(0) - R_y^{(5)}(0) &= 0, \\
R_y^{(5)}(1) &= 0, \\
R_y^{(6)}(1) &= 0, \\
R_y^{(7)}(1) &= 0,
\end{aligned}
\]

then (2.4) implies that,

\[ R_y^{(10)}(x) = -\delta(x - y), \]

when \( x \neq y \), then

\[ R_y^{(10)}(x) = 0, \]

and therefore

\[
R_y(x) = \begin{cases} 
\sum_{i=1}^{10} c_i(y)x^{i-1}, & x \leq y, \\
\sum_{i=1}^{10} d_i(y)x^{i-1}, & x > y.
\end{cases}
\]

Since

\[ R_y^{(10)}(x) = \delta(x - y), \]

we have

\[ \partial^k R_y^+(y) = \partial^k R_y^-(y), \quad k = 0, 1, 2, 3, 4, 5, 6, 7, 8, \]

and
\[ \partial^{9} R_{y+}(y) - \partial^{9} R_{y-}(y) = -1. \]  \hspace{1cm} (2.7)

Since \( R_{y}(x) \in W^{5}_{2}[0, 1] \), it follows that

\[ R_{y}(0) = 0, \ R_{y}(1) = 0, \ R_{y}'(1) = 0, \ R_{y}''(0) = 0. \]  \hspace{1cm} (2.8)

From (2.5)-(2.8), the unknown coefficients \( c_{i}(y) \) and \( d_{i}(y) \) \((i = 1, 2, ..., 12)\) can be obtained. This completes the proof. \( \square \)

**Remark 2.1.** Reproducing kernel function \( r_{y} \) of \( W^{4}_{2}[0, 1] \) is given as

\[
r_{y}(x) = \begin{cases} 
1 + xy + \frac{1}{4} y^{2} x^{2} + \frac{1}{36} y^{3} x^{3} + \frac{1}{144} y^{3} x^{4} - \frac{1}{240} y^{2} x^{5} + \frac{1}{720} y x^{6} - \frac{1}{5040} x^{7}, & x \leq y, \\
1 + yx + \frac{1}{4} y^{2} x^{2} + \frac{1}{36} y^{3} x^{3} + \frac{1}{144} x^{3} y^{4} - \frac{1}{240} x^{2} y^{5} + \frac{1}{720} x y^{6} - \frac{1}{5040} y^{7}, & x > y.
\end{cases}
\]

This can be proved easily as in Theorem 2.1.

### 3 Solution Representation in \( W^{5}_{2}[0, 1] \)

In this section, the solution of equation (1.17) is given in the reproducing kernel space \( W^{5}_{2}[0, 1] \).

On defining the linear operator \( L : W^{5}_{2}[0, 1] \rightarrow W^{4}_{2}[0, 1] \) as

\[
Lu = u^{(4)}(x) + \text{Re} \left( \frac{e^x}{e} \left( x^{3} - 4x^{2} + 4x \right) - m^{2} u''(x) \right) + \text{Re} \left( \frac{e^x}{e} \left( x^{3} + 5x^{2} - 2x - 6 \right) u(x) \right).
\]

Model problem (1.17) changes the following problem:

\[
\begin{align*}
Lu &= M(x, u, u^{(3)}), \quad x \in [0, 1], \\
\quad &\text{subject to } u(0) = 0, \ u(1) = 0, \ u'(1) = 0, \ u''(0) = 0,
\end{align*}
\]  \hspace{1cm} (3.1)

where

\[
M(x, u, u^{(3)}) = -\text{Re} \ u^{(3)}(x) u(x) - \text{Re} \left( \frac{e^x}{e} \right)^{2} \left( x^{3} - 4x^{2} + 4x \right) \left( x^{3} + 5x^{2} - 2x - 6 \right)
\]

\[
- \frac{e^x}{e} \left( x^{3} + 8x^{2} + 8x - 2 \right) + m^{2} \frac{e^x}{e} \left( x^{3} + 2x^{2} - 6x \right).
\]
3.1 The Linear Boundedness of Operator $L$.

**Theorem 3.1.** The operator $L$ defined by (3.1) is a bounded linear operator.

**Proof:** We only need to prove
\[ \|Lu\|_{W^4_2}^2 \leq P \|Lu\|_{W^2_2}^2, \]
where $P$ is a positive constant. By Definition 2.3, we have
\[ \|u\|_{W^2_2}^2 = \langle u, u \rangle_{W^2_2} = \sum_{i=0}^{3} \left[ u^{(i)}(0) \right]^2 + \int_{0}^{1} \left[ u^{(4)}(x) \right]^2 dx, \quad u \in W^4_2[0,1], \]
and
\[ \|Lu\|_{W^4_2}^2 = \langle Lu, Lu \rangle_{W^2_2} = \left[ (Lu)(0) \right]^2 + \left[ (Lu)'(0) \right]^2 + \left[ (Lu)''(0) \right]^2 + \left[ (Lu)^{(3)}(0) \right]^2 + \int_{0}^{1} \left[ (Lu)^{(4)}(x) \right]^2 dx. \]
By reproducing property, we have
\[ u(x) = \langle u, R_x \rangle_{W^5_2}, \]
and
\[ (Lu)(x) = \langle u, (LR_x) \rangle_{W^2_2}, \quad (Lu)'(x) = \langle u, (LR_x)' \rangle_{W^2_2}, \]
\[ (Lu)''(x) = \langle u, (LR_x)'' \rangle_{W^2_2}, \quad (Lu)^{(3)}(x) = \langle u, (LR_x)^{(3)} \rangle_{W^2_2}, \]
\[ (Lu)^{(4)}(x) = \langle u, (LR_x)^{(4)} \rangle_{W^2_2}. \]
Therefore
\[ |(Lu)(x)| \leq \|u\|_{W^2_2} \|LR_x\|_{W^5_2} \leq a_1 \|u\|_{W^5_2}, \text{ (where } a_1 > 0 \text{ is a positive constant)}, \]
\[ |(Lu)'(x)| \leq \|u\|_{W^2_2} \|(LR_x)'\|_{W^5_2} \leq a_2 \|u\|_{W^5_2}, \text{ (where } a_2 > 0 \text{ is a positive constant)}, \]
\[ |(Lu)''(x)| \leq \|u\|_{W^2_2} \|(LR_x)''\|_{W^5_2} \leq a_3 \|u\|_{W^5_2}, \text{ (where } a_3 > 0 \text{ is a positive constant)}, \]
\[ |(Lu)^{(3)}(x)| \leq \|u\|_{W^2_2} \|(LR_x)^{(3)}\|_{W^5_2} \leq a_4 \|u\|_{W^5_2}, \text{ (where } a_4 > 0 \text{ is a positive constant)}, \]

Thus
\[ |(Lu)(0)^2 + [(Lu)'(0)]^2 + [(Lu)''(0)]^2 + [(Lu)^{(3)}(0)]^2 \leq (a_1^2 + a_2^2 + a_3^2 + a_4^2) \|u\|_{W^5_2}^2. \]

Since
\[ (Lu)^{(4)} = \left< u, (LR_x)^{(4)} \right>_{W^2_2}, \]

then
\[ |(Lu)^{(4)}| \leq \|u\|_{W^2_2} \|(LR_x)^{(4)}\|_{W^2_2} = a_5 \|u\|_{W^4_2}, \text{ (where } a_5 > 0 \text{ is a positive constant)}, \]

so, we have
\[ \left[ (Lu)^{(4)} \right]^2 \leq a_5^2 \|u\|_{W^4_2}^2, \]

and
\[ \int_0^1 \left[ (Lu)^{(4)}(x) \right]^2 dx \leq a_5^2 \|u\|_{W^4_2}^2, \]

that is
\[ \|Lu\|_{W^4_2}^2 = [(Lu)(0)^2 + [(Lu)'(0)]^2 + [(Lu)''(0)]^2 + [(Lu)^{(3)}(0)]^2 + \int_0^1 \left[ (Lu)^{(4)}(x) \right]^2 dx \]
\[ \leq (a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2) \|u\|_{W^5_2}^2 = P \|u\|_{W^4_2}^2, \]
where \(P = (a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2) > 0\) is a positive constant. This completes the proof. □

4 The Normal Orthogonal Function System of \(W_2^5[a, b]\)

Let \(\{x_i\}_{i=1}^{\infty}\) be any dense set in \([0, 1]\) and \(\Psi_x(y) = L^* r_x(y)\), where \(L^*\) is adjoint operator of \(L\) and \(r_x(y)\) is given by Remark 2.1. Furthermore, for simplicity let \(\Psi_i(x) = \Psi_{x_i}(x)\), namely,

\[
\Psi_i(x) \overset{def}{=} \Psi_{x_i}(x) = L^* r_{x_i}(x).
\]

Now one can deduce following lemmas.

**Lemma 3.2.** \(\{\Psi_i(x)\}_{i=1}^{\infty}\) is complete system of \(W_2^5[0, 1]\).

**Proof:** For \(u \in W_2^5[0, 1]\), let \(\langle u, \Psi_i \rangle = 0 \ (i = 1, 2, \ldots)\), that is

\[
\langle u, L^* r_{x_i} \rangle = (Lu)(x_i) = 0.
\]

Note that \(\{x_i\}_{i=1}^{\infty}\) is the dense set in \([0, 1]\), therefore \((Lu)(x) = 0\). It follows that \(u(x) = 0\) from the existence of \(L^{-1}\). This completes the proof. □

**Lemma 3.3.** The following formula holds

\[
\Psi_i(x) = (L\eta R_x(\eta)) (x_i),
\]

where the subscript \(\eta\) of operator \(L\eta\) indicates that the operator \(L\) applies to function of \(\eta\).

**Proof:**

\[
\Psi_i(x) = \langle \Psi_i(\xi), R_x(\xi) \rangle_{W_2^5[0,1]} \\
= \langle L^* r_{x_i} (\xi), R_x(\xi) \rangle_{W_2^5[0,1]} \\
= \langle (r_{x_i}) (\xi), (L\eta R_x(\eta)) (\xi) \rangle_{W_2^5[0,1]} \\
= (L\eta R_x(\eta))(x_i).
\]
This completes the proof. \hfill \square

**Remark 3.1.** The orthonormal system \( \{ \Psi_i(x) \}_{i=1}^{\infty} \) of \( W^5_2[0,1] \) can be derived from Gram-Schmidt orthogonalization process of \( \{ \Psi_i(x) \}_{i=1}^{\infty} \),

\[
\Psi_i(x) = \sum_{k=1}^{i} \beta_{ik} \Psi_k(x), \quad (\beta_{ii} > 0, \ i = 1, 2, \ldots) \quad (3.7)
\]

where \( \beta_{ik} \) are orthogonal coefficients.

In the following, we will give the representation of the exact solution of Eq.(1.17) in the reproducing kernel space \( W^5_2[0,1] \).

### 4.1 The Structure of the Solution and the Main Results

**Theorem 3.2.** If \( u \) is the exact solution of (3.1) then

\[
u = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} M(x_k, u(x_k), u^{(3)}(x_k)) \Psi_i(x),
\]

where \( \{ x_i \}_{i=1}^{\infty} \) is a dense set in \([0,1]\).

**Proof:** From the (3.7) and uniqueness of solution of (3.1), we have

\[
u = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle L^* r_{x_k} \rangle_{W^2_2} \Psi_i = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle M(x, u, u^{(3)}), r_{x_k} \rangle_{W^2_2} \Psi_i
\]

\[
= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} M(x_k, u(x_k), u^{(3)}(x_k)) \Psi_i(x).
\]

This completes the proof. \hfill \square

Now the approximate solution \( u_n \) can be obtained by truncating the \( n \)-term of the exact solution \( u \) as

\[
u_n = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} M(x_k, u(x_k), u^{(3)}(x_k)) \Psi_i(x).
\]
Lemma 3.4. Assume \( u \) is the solution of (3.1) and \( r_n \) is the error between the approximate solution \( u_n \) and the exact solution \( u \). Then the error sequence \( r_n \) is monotone decreasing in the sense of \( \|\cdot\|_{W^2} \) and \( \|r_n(x)\|_{W^2} \to 0 \).

5 Numerical Results

In this section, comparisons of results have been made through different Reynolds numbers \( Re \) and magnetic field effect \( m \). All computations are performed by Maple 16. The RKM does not require discretization of the variables, i.e., time and space, it is not effected by computation round off errors and one is not faced with necessity of large computer memory and time. The accuracy of the RKM for the MHD squeezing fluid flow is controllable and absolute errors are small with present choice of \( x \) (see Tables 1-12). The numerical results we obtained justify the advantage of this methodology.

| \( x \) | Numerical Solution \((RK - 4)\) | Approximate Solution | Absolute Error \(10^{-10}\) | Relative Error \(10^{-9}\) | Time (s) |
|------|-----------------|---------------------|------------------------|-------------------|--------|
| 0.1  | 0.150294        | 0.15029400074386619072 | 7.4386 \times 10^{-10} | 4.9494071002 \times 10^{-9} | 2.948  |
| 0.2  | 0.297481        | 0.29748099943286204844 | 5.6713 \times 10^{-10} | 1.9064678132 \times 10^{-9} | 2.980  |
| 0.3  | 0.438467        | 0.43846699936146542481 | 6.3853 \times 10^{-10} | 1.4562887861 \times 10^{-9} | 2.870  |
| 0.4  | 0.570189        | 0.57018899983086605298 | 1.6913 \times 10^{-10} | 2.9662786728 \times 10^{-10} | 2.792  |
| 0.5  | 0.689624        | 0.68962399932753349664 | 6.7246 \times 10^{-10} | 9.7512050531 \times 10^{-10} | 2.824  |
| 0.6  | 0.793796        | 0.79379600052975674440 | 5.2975 \times 10^{-10} | 6.6737139569 \times 10^{-10} | 2.902  |
| 0.7  | 0.879779        | 0.87977900034152532706 | 3.4152 \times 10^{-10} | 3.8819445231 \times 10^{-10} | 2.964  |
| 0.8  | 0.944696        | 0.9446960002147585921  | 2.1478 \times 10^{-10} | 2.2735976357 \times 10^{-10} | 2.808  |
| 0.9  | 0.985707        | 0.98570699945336089741 | 5.46639 \times 10^{-10} | 5.5456550738 \times 10^{-10} | 2.761  |
| 1.0  | 1.0             | 1.0                 | 0.0                     | 0.0               | 2.902  |

Table 4.1. Numerical results at \( m = 1 \) and \( Re = 1 \).
| $x$ | Numerical Solution ($RK - 4$) | OHAM       | RKHSM |
|-----|-------------------------------|------------|-------|
| 0.0 | 0.0                           | 0.0        | 0.0   |
| 0.1 | 0.150294                      | 0.150265   | 0.15029400074386619072 |
| 0.2 | 0.297481                      | 0.297424   | 0.29748099943286204844 |
| 0.3 | 0.438467                      | 0.438387   | 0.43846699936146542481 |
| 0.4 | 0.570189                      | 0.570093   | 0.57018899983086605298 |
| 0.5 | 0.689624                      | 0.68952    | 0.68962399932753349664 |
| 0.6 | 0.793796                      | 0.793695   | 0.79379600052975674440 |
| 0.7 | 0.879779                      | 0.879695   | 0.879779000034152532706 |
| 0.8 | 0.944696                      | 0.944641   | 0.94469600021478585921 |
| 0.9 | 0.985707                      | 0.985687   | 0.98570699945336089741 |
| 1.0 | 1.0                           | 1.0        | 1.0   |

Table 4.2. Comparison between RK-4, OHAM and RKHSM solutions at $m = 1$ and $Re = 1$. 
| $x$  | Numerical Solution $(RK - 4)$ | Approximate Solution | Absolute Error | Relative Error | Time (s) |
|------|-------------------------------|----------------------|---------------|---------------|----------|
| 0.1  | 0.137044                      | 0.13704399924397146430 | 7.560285 × 10^{-10} | 5.51668468 × 10^{-9} | 3.261    |
| 0.2  | 0.272494                      | 0.27249400041809657591 | 4.180965 × 10^{-10} | 1.53433314 × 10^{-9} | 3.542    |
| 0.3  | 0.404637                      | 0.40463699937791012358 | 6.220898 × 10^{-10} | 1.53740235 × 10^{-9} | 2.949    |
| 0.4  | 0.531508                      | 0.53150799980699743080 | 1.930025 × 10^{-10} | 3.63122604 × 10^{-10} | 3.541    |
| 0.5  | 0.650756                      | 0.65075599905912100256 | 9.408789 × 10^{-10} | 1.44582454 × 10^{-9} | 3.089    |
| 0.6  | 0.759478                      | 0.75947799979255971384 | 2.074402 × 10^{-10} | 2.73135345 × 10^{-10} | 2.996    |
| 0.7  | 0.854035                      | 0.85403499924057783299 | 7.594221 × 10^{-10} | 8.89216679 × 10^{-10} | 3.026    |
| 0.8  | 0.929817                      | 0.92981700082221438640 | 8.222143 × 10^{-10} | 8.84275493 × 10^{-10} | 7.582    |
| 0.9  | 0.980963                      | 0.98096299961587653980 | 3.841234 × 10^{-10} | 3.91577929 × 10^{-10} | 3.291    |
| 1.0  | 1.0                           | 1.0                  | 0.0            | 0.0            | 2.902    |

Table 4.3. Numerical results at $m = 3$ and $Re = 1$. 

| $x$  | Numerical Solution $(RK - 4)$ | OHAM       | RKHSN |
|------|-------------------------------|------------|-------|
| 0.0  | 0.0                           | 0.0        | 0.0   |
| 0.1  | 0.137044                      | 0.13709    | 0.13704399924397146430 |
| 0.2  | 0.272494                      | 0.272583   | 0.27249400041809657591 |
| 0.3  | 0.404637                      | 0.404759   | 0.40463699937791012358 |
| 0.4  | 0.531508                      | 0.531649   | 0.53150799980699743080 |
| 0.5  | 0.650756                      | 0.650894   | 0.65075599905912100256 |
| 0.6  | 0.759478                      | 0.759591   | 0.75947799979255971384 |
| 0.7  | 0.854035                      | 0.854106   | 0.85403499924057783299 |
| 0.8  | 0.929817                      | 0.929845   | 0.92981700082221438640 |
| 0.9  | 0.980963                      | 0.980966   | 0.98096299961587653980 |
| 1.0  | 1.0                           | 1.0        | 1.0   |


Table 4.4. Comparison between RK-4, OHAM and RKHSM solutions at $m = 3$ and $Re = 1$.

| $x$ | Numerical Solution $(RK - 4)$ | Approximate Solution | Absolute Error | Relative Error | Time (s) |
|-----|-------------------------------|----------------------|----------------|---------------|----------|
| 0.1 | 0.114976                      | 0.11497599095960418967 | $9.040395 \times 10^{-9}$ | $7.86285469 \times 10^{-8}$ | 4.290 |
| 0.2 | 0.229882                      | 0.22988199268533318687 | $7.314666 \times 10^{-9}$ | $3.18192238 \times 10^{-8}$ | 4.134 |
| 0.3 | 0.344604                      | 0.34460400584434350472 | $5.844343 \times 10^{-9}$ | $1.69595927 \times 10^{-8}$ | 4.477 |
| 0.4 | 0.458904                      | 0.45890399132822355411 | $8.671776 \times 10^{-9}$ | $1.88967113 \times 10^{-8}$ | 4.275 |
| 0.5 | 0.572276                      | 0.5722759999680104400 | $3.198956 \times 10^{-11}$ | $5.5898832 \times 10^{-11}$ | 3.931 |
| 0.6 | 0.683628                      | 0.68362799155831029523 | $8.441689 \times 10^{-9}$ | $1.23483673 \times 10^{-8}$ | 4.556 |
| 0.7 | 0.790607                      | 0.79060700783664672119 | $7.836646 \times 10^{-9}$ | $9.9121899 \times 10^{-9}$ | 4.461 |
| 0.8 | 0.888173                      | 0.88817300466724146312 | $4.667241 \times 10^{-9}$ | $5.25487879 \times 10^{-9}$ | 3.885 |
| 0.9 | 0.965578                      | 0.96557800220185786369 | $2.201857 \times 10^{-9}$ | $2.28035214 \times 10^{-9}$ | 5.007 |
| 1.0 | 1.0                           | 1.0                  | 0.0            | 0.0            | 2.902   |

Table 4.5. Numerical results at $m = 8$ and $Re = 1$. 
| $x$ | Numerical Solution $(RK - 4)$ | OHAM | RKHSM |
|-----|-------------------------------|------|-------|
| 0.0 | 0.0                           | 0.0  | 0.0   |
| 0.1 | 0.114976                      | 0.11507 | 0.11497599095960418967 |
| 0.2 | 0.229882                      | 0.230068 | 0.22988199268533318687 |
| 0.3 | 0.344604                      | 0.344866 | 0.34460400584434350472 |
| 0.4 | 0.458904                      | 0.459205 | 0.45890399132822355411 |
| 0.5 | 0.572276                      | 0.572545 | 0.5722759999680104400 |
| 0.6 | 0.683628                      | 0.683769 | 0.68362799155831029523 |
| 0.7 | 0.790607                      | 0.790543 | 0.79060700783664672119 |
| 0.8 | 0.888173                      | 0.887936 | 0.88817300466724146312 |
| 0.9 | 0.965578                      | 0.965381 | 0.96557800220185786369 |
| 1.0 | 1.0                           | 1.0  | 1.0   |

Table 4.6. Comparison between RK-4, OHAM and RKHSM solutions at $m = 8$ and $Re = 1$. 
| $x$ | Numerical Solution ($RK−4$) | Approximate Solution | Absolute Error | Relative Error | Time (s) |
|-----|----------------------------|----------------------|----------------|---------------|----------|
| 0.1 | 0.105391                   | 0.10539098947593257979 | $1.0524067 \times 10^{-8}$ | $9.985736372 \times 10^{-8}$ | 4.134    |
| 0.2 | 0.210782                   | 0.2107819933190829   | $6.6809171 \times 10^{-9}$ | $3.16958616 \times 10^{-8}$ | 5.101    |
| 0.3 | 0.316173                   | 0.3161729190893567630 | $8.0910643 \times 10^{-8}$ | $2.559062387 \times 10^{-7}$ | 3.010    |
| 0.4 | 0.421563                   | 0.4215629919618786430 | $8.0381213 \times 10^{-9}$ | $1.906742611 \times 10^{-8}$ | 3.198    |
| 0.5 | 0.526952                   | 0.5269519479728988   | $5.2027101 \times 10^{-8}$ | $9.873214486 \times 10^{-8}$ | 3.042    |
| 0.6 | 0.632324                   | 0.632323981769674315 | $1.8230325 \times 10^{-8}$ | $2.883067175 \times 10^{-8}$ | 3.074    |
| 0.7 | 0.737586                   | 0.7375860570172070642 | $5.7017207 \times 10^{-8}$ | $7.730245295 \times 10^{-8}$ | 3.089    |
| 0.8 | 0.842051                   | 0.84205103495023398982 | $3.4950233 \times 10^{-8}$ | $4.150607741 \times 10^{-8}$ | 3.073    |
| 0.9 | 0.940861                   | 0.94086101815219431313 | $1.8152194 \times 10^{-8}$ | $1.929317328 \times 10^{-8}$ | 3.135    |
| 1.0 | 1.0                        | 1.0                   | 0.0            | 0.0            | 2.902    |

Table 4.7. Numerical results at $m = 20$ and $Re = 1$. 

| $x$ | Numerical Solution ($RK−4$) | OHAM | RKHSN |
|-----|----------------------------|------|-------|
| 0.0 | 0.0                        | 0.0  | 0.0   |
| 0.1 | 0.105391                   | 0.105312 | 0.10539098947593257979 |
| 0.2 | 0.210782                   | 0.210625 | 0.2107819933190829 |
| 0.3 | 0.316173                   | 0.315938 | 0.3161729190893567630 |
| 0.4 | 0.421563                   | 0.421249 | 0.4215629919618786430 |
| 0.5 | 0.526952                   | 0.526551 | 0.5269519479728988 |
| 0.6 | 0.632324                   | 0.631824 | 0.632323981769674315 |
| 0.7 | 0.737586                   | 0.736971 | 0.7375860570172070642 |
| 0.8 | 0.842051                   | 0.841352 | 0.84205103495023398982 |
| 0.9 | 0.940861                   | 0.94035  | 0.94086101815219431313 |
| 1.0 | 1.0                        | 1.0    | 1.0   |
Table 4.8. Comparison between RK-4, OHAM and RKHSM solutions at $m = 20$ and $Re = 1$.

| $x$  | Numerical Solution ($RK - 4$) | Approximate Solution | Absolute Error | Relative Error | Time (s) |
|------|-------------------------------|----------------------|----------------|----------------|----------|
| 0.1  | 0.158104                      | 0.15810400012535311729 | $1.25353117 \times 10^{-10}$ | $7.928522826 \times 10^{-10}$ | 5.304    |
| 0.2  | 0.311962                      | 0.31196200057873017887 | $5.78730178 \times 10^{-10}$ | $1.855130364 \times 10^{-9}$ | 7.332    |
| 0.3  | 0.457539                      | 0.45753900003164153289 | $3.16415328 \times 10^{-11}$ | $6.915592526 \times 10^{-11}$ | 5.913    |
| 0.4  | 0.591193                      | 0.59119300033029000468 | $3.30290004 \times 10^{-10}$ | $5.586838894 \times 10^{-10}$ | 6.272    |
| 0.5  | 0.709771                      | 0.70977100026331200670 | $2.63312006 \times 10^{-10}$ | $3.709816359 \times 10^{-10}$ | 5.757    |
| 0.6  | 0.810642                      | 0.81064200064720692438 | $6.47206924 \times 10^{-10}$ | $7.983880978 \times 10^{-10}$ | 6.256    |
| 0.7  | 0.891666                      | 0.89166599939606220359 | $6.03937796 \times 10^{-10}$ | $6.039377964 \times 10^{-10}$ | 6.396    |
| 0.8  | 0.95112                       | 0.95112000044608660232 | $4.46086602 \times 10^{-10}$ | $4.690119035 \times 10^{-10}$ | 5.101    |
| 0.9  | 0.987612                      | 0.98761199979328069240 | $2.06719307 \times 10^{-10}$ | $2.093122679 \times 10^{-10}$ | 5.616    |
| 1.0  | 1.0                           | 1.0                  | 0.0            | 0.0            | 2.902    |

Table 4.9. Numerical results at $m = 1$ and $Re = 4$. 
| $x$ | Numerical Solution $(RK - 4)$ | OHAM | RKHSM |
|-----|-----------------------------|------|-------|
| 0.0 | 0.0                         | 0.0  | 0.0   |
| 0.1 | 0.158104                    | 0.156218 | 0.15810400012535311729 |
| 0.2 | 0.311962                    | 0.308363 | 0.31196200057873017887 |
| 0.3 | 0.457539                    | 0.452557 | 0.4575390003164153289 |
| 0.4 | 0.591193                    | 0.585287 | 0.59119300033029000468 |
| 0.5 | 0.709771                    | 0.703518 | 0.70977100026331200670 |
| 0.6 | 0.810642                    | 0.804726 | 0.81064200064720692438 |
| 0.7 | 0.891666                    | 0.886838 | 0.89166599939606220359 |
| 0.8 | 0.95112                     | 0.948051 | 0.95112000044608660232 |
| 0.9 | 0.987612                    | 0.986529 | 0.98761199979328069240 |
| 1.0 | 1.0                         | 1.0   | 1.0   |

Table 4.10. Comparison between RK-4, OHAM and RKHSM solutions at $m = 1$ and $Re = 4$. 
| $x$ | Numerical Solution ($RK - 4$) | Approximate Solution | Absolute Error | Relative Error | Time (s) |
|-----|-------------------------------|----------------------|----------------|---------------|----------|
| 0.1 | 0.167616                      | 0.1676160001397322991| $1.39732299 \times 10^{-10}$ | $8.3364535 \times 10^{-10}$ | 5.569 |
| 0.2 | 0.329031                      | 0.32903100221406728329 | $2.21406728 \times 10^{-9}$ | $6.7290537 \times 10^{-9}$ | 6.365 |
| 0.3 | 0.478907                      | 0.47890699791462877619 | $2.08537122 \times 10^{-9}$ | $4.3544388 \times 10^{-9}$ | 7.378 |
| 0.4 | 0.613252                      | 0.6132519950552162812 | $4.49447837 \times 10^{-9}$ | $7.3289257 \times 10^{-9}$ | 7.254 |
| 0.5 | 0.729428                      | 0.7294279984508679063 | $1.5449132 \times 10^{-9}$ | $2.117979 \times 10^{-9}$ | 6.271 |
| 0.6 | 0.825843                      | 0.82584300690485584332 | $6.9048558 \times 10^{-9}$ | $8.3609788 \times 10^{-9}$ | 7.425 |
| 0.7 | 0.901576                      | 0.9015760084025340903 | $8.4042534 \times 10^{-9}$ | $9.3217359 \times 10^{-9}$ | 6.162 |
| 0.8 | 0.901576                      | 0.9015751838249656701 | $8.16175 \times 10^{-7}$ | $9.052759 \times 10^{-7}$ | 7.410 |
| 0.9 | 0.988978                      | 0.98897799997420425356 | $2.579574 \times 10^{-11}$ | $2.6083235 \times 10^{-11}$ | 7.910 |
| 1.0 | 1.0                           | 1.0                  | 0.0            | 0.0            | 2.902 |

Table 4.1. Numerical results at $m = 1$ and $Re = 10$.  

| $x$ | Numerical Solution ($RK - 4$) | OHAM | RKHSM |
|-----|-------------------------------|------|-------|
| 0.0 | 0.0                           | 0.0  | 0.0   |
| 0.1 | 0.167616                      | 0.175911 | 0.1676160001397322991 |
| 0.2 | 0.329031                      | 0.344336 | 0.32903100221406728329 |
| 0.3 | 0.478907                      | 0.498671 | 0.47890699791462877619 |
| 0.4 | 0.613252                      | 0.633941 | 0.61325199550552162812 |
| 0.5 | 0.729428                      | 0.747277 | 0.7294279984508679063 |
| 0.6 | 0.825843                      | 0.838004 | 0.82584300690485584332 |
| 0.7 | 0.901576                      | 0.907244 | 0.90157600840425340903 |
| 0.8 | 0.901576                      | 0.956954 | 0.9015751838249656701 |
| 0.9 | 0.988978                      | 0.988387 | 0.98897799997420425356 |
| 1.0 | 1.0                           | 1.0   | 1.0   |
Table 4.12. Comparison between RK-4, OHAM and RKHSM solutions at $m = 1$ and $Re = 10$.

6 Conclusion

In this paper, we introduced an algorithm for solving the MHD squeezing fluid flow with boundary conditions. The method gives more realistic series solutions that converge very rapidly in physical problems. The approximate solution obtained by the present method is uniformly convergent.

Clearly, the series solution methodology can be applied to much more complicated nonlinear differential equations and boundary value problems. However, if the problem becomes nonlinear, then the RKM does not require discretization or perturbation and it does not make closure approximation. Results of numerical examples show that the present method is an accurate and reliable analytical method for the MHD squeezing fluid flow problem with boundary conditions.

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