Gauge Symmetry Breaking in Flux Compactification with Wilson-line Scalar Condensate

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Abstract

We discuss the gauge symmetry breaking of six dimensional theories in flux compactification with a magnetic flux background and a constant vacuum expectation value (VEV) for the scalar fields, which are zero modes of extra spatial components of the gauge field. Although the effective potential for the scalar fields are known not to be generated classically and radiatively in a magnetic flux background only, the one-loop effective potential is shown to be generated by the effects of the non-zero constant VEV. As illustrations, we calculate the one-loop effective potential in SU(2) and SU(3) Yang-Mills theories. In both cases, we expect that the potential minimum is located at non-zero VEV and the gauge symmetry breaking takes place.
1 Introduction

Although the Standard Model (SM) has a successful theory, it still has some problems. Many attractive scenarios based on the higher dimensional theory have been proposed as the physics beyond the SM. In particular, flux compactification, which has been studied in string theory \[1,2\], has many attractive aspects: explanation of the generation number of the SM fermions \[3,4\] and computation of Yukawa coupling \[5–8\].

Recently, it has been considered that the quantum corrections to the masses of zero-mode of the scalar field induced from extra components of higher dimensional gauge field (called as Wilson-line (WL) scalar field) are cancelled \[9–13\] and are finite \[14\]. The reason why the quantum corrections are cancelled is that the shift symmetry from translation in extra spaces forbids the mass term of scalar field since the zero-mode of the scalar field can be identified with Nambu-Goldstone (NG) boson of spontaneously broken translational symmetry (or with pseudo-NG boson in \[14\]). This cancellation mechanism may be applied to the hierarchy problem in the SM which is the problem that the quantum corrections to the mass of Higgs field are sensitive to the square of the ultraviolet cutoff scale of the theory. If we regard the Higgs field as the WL scalar field, which is an idea of gauge-Higgs unification \[15–18\], the quantum corrections to the mass of Higgs field is cancelled as mentioned above. In the gauge-Higgs unification, the finite Higgs mass is generated by the quantum corrections \[18–21,23\] controlled by the compactification scale. If the compactification scale is increased by the absence of the new physics discovery, the fine-tuning problem in the Higgs mass parameter is reintroduced. In flux compactification, however, if the translational symmetry in extra spaces is explicitly broken around TeV scale independent of the compactification scale, the light Higgs boson mass is radiatively generated. This logic is also applied to the potential of the WL scalar field.

In this paper, we investigate the gauge symmetry breaking in a higher dimensional theory in flux compactification with a magnetic flux background and a constant WL scalar vacuum expectation value (VEV). First, we consider a six dimensional SU(2) Yang-Mills theory compactified on a torus with a magnetic flux and a constant VEV. Calculating the KK mass spectrum in the presence of both flux background and the constant VEV, we obtain the one-loop effective potential for the WL scalar field. Although the effective potential in the flux background only is not radiatively generated, the effective potential
in both the flux background and the constant VEV is generated at one-loop and the potential minimum at nonvanishing constant VEV is expected. This concludes that gauge symmetry SU(2) is completely broken.

Next, we consider a six dimensional SU(3) Yang-Mills theory compactified on a torus with a magnetic flux and a constant VEV. In this case, we consider two types of configuration where the flux background and the constant VEV are developed. One is that the flux background and the constant VEV are in the eighth and the first components of SU(3). The other is that the flux background and the constant VEV are in the eighth and the sixth components of SU(3). Calculating the one-loop effective potential similarly as is done in SU(2) Yang-Mills theory, we find (expect) that the potential is minimized at nonvanishing constant VEV in the former (latter) case. In the former (latter) case, the gauge symmetry breaking SU(3) → U(1) × U(1)(SU(3) → U(1)) is found, respectively.

This paper is organized as follows. We give a setup of a six-dimensional SU(2) Yang-Mills theory with magnetic flux compactification and introduce the constant VEV in section 2. We furthermore consider a six-dimensional SU(3) Yang-Mills theory with magnetic flux compactification in section 3, where two types of configuration for the flux background and the constant VEV to be taken. In both sections, the one-loop effective potential for the WL scalar field is calculated and the gauge symmetry breaking is discussed. In the last section, we devote our summary. In Appendix A, the calculation of the KK mass spectrum at the second order perturbation is summarized.

2 SU(2) Yang-Mills theory

We consider a six-dimensional SU(2) Yang-Mills theory with two nontrivial backgrounds: a constant magnetic flux background and an ordinary constant vacuum expectation value.

2.1 Set up

Six-dimensional spacetime is $M^4 \times T^2$, where $M^4$ is a Minkowski spacetime and $T^2$ is a two-dimensional square torus. The Lagrangian of SU(2) Yang-Mills theory in six dimensions
is
\[
\mathcal{L}_6 = -\frac{1}{4} F_{MN}^a F^{aMN} \\
= -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2} F_{\mu5}^a F^{a5} - \frac{1}{2} F_{\mu6}^a F^{a6} - \frac{1}{2} F_{56}^a F^{a56},
\]
where the field strength tensor and the covariant derivative are defined by
\[
F_{MN}^a = \partial_M A_N^a - \partial_N A_M^a - ig [A_M, A_N]^a,
\]
\[
D_M A_N^a = D_M A_N^a = (\delta^{ac} \partial_M + g \epsilon^{abc} A_M^b) A_N^c \\
= \partial_M A_N^a - ig [A_M, A_N]^a.
\]
The spacetime indices are \( M, N = 0, 1, 2, 3, 5, 6, \mu, \nu = 0, 1, 2, 3, m, n = 5, 6, \) and the gauge indices are \( a, b, c = 1, 2, 3. \) The metric convention \( \eta_{MN} = \text{diag}(-1, +1, \cdots, +1) \) is employed. \( \epsilon^{abc} \) is a totally anti-symmetric tensor of SU(2).

We discuss how the two backgrounds are introduced in our model. First, the constant magnetic flux, is given by the VEV of the fifth and the sixth component of the gauge fields, \( A_{5,6}^3, \) which must satisfy their classical equation of motion:
\[
D^m \langle F_{mn}^a \rangle = 0.
\]
Second, the ordinary constant background is generated by the quantum correction in the sixth component of the gauge field \( A_{6}^1, \) for simplicity\(^1\) In this section, we choose a solution
\[
\langle A_{5}^1 \rangle = v, \quad \langle A_{5}^2 \rangle = -\frac{1}{2} f x_6, \quad \langle A_{5}^3 \rangle = \frac{1}{2} f x_5 \quad \text{and} \quad \langle A_{6}^{1,2} \rangle = \langle A_{6}^3 \rangle = 0.
\]
\( \langle A_{5,6}^3 \rangle \) introduce a magnetic field parametrized by a constant \( f, \) namely \( \langle F_{56}^3 \rangle = f. \) Note that the flux background spontaneously breaks a translational invariance on the torus. The flux background breaks the gauge symmetry, which is broken to U(1) in this case. The flux is also associated with the degeneracy:
\[
\frac{g}{2\pi} \int_{T^2} dx_5 dx_6 \langle F_{56}^3 \rangle = \frac{g}{2\pi} L^2 f = N \in \mathbb{Z},
\]
where \( L^2 \) is an area of the torus. For simplicity, we set \( L = 1 \) from now on. It is useful to define \( \partial \) and the scalar fields \( \phi^a \) as
\[
\partial \equiv \partial_z = \partial_5 - i \partial_6, \quad z \equiv \frac{1}{2} (x_5 + ix_6), \quad \phi^a = \frac{1}{\sqrt{2}} (A_5^a + i A_6^a).
\]
\(^1\)In general, the constant background can be introduced by the fifth component of the gauge field \( A_{5}^1. \)
In these complex coordinates, the VEVs of \( \phi^1, \phi^3 \) are given by
\[
\langle \phi^1 \rangle = \frac{v}{\sqrt{2}}, \quad \langle \phi^3 \rangle = \frac{f \bar{z}}{\sqrt{2}},
\]
and we expand \( \phi^a \) around the backgrounds:
\[
\phi^a = \langle \phi^a \rangle + \varphi^a,
\]
where \( \varphi^a \) are quantum fluctuations.

The Lagrangian (1) can be rewritten by using the new coordinates (7) as follows:
\[
L_6 = -\frac{1}{4} F_{\mu\nu}^a F^{a \mu \nu} - \partial_\mu \bar{\phi}^a \partial^\mu \phi^a - \frac{1}{2} D A_\mu^a A^{a \mu} \\
- \frac{i}{\sqrt{2}} (\partial_\mu \phi^a A^{a \mu} - \partial_\mu \bar{\phi}^a \partial A^{a \mu}) \\
+ ig \left( \partial_\mu \phi^a [A^a, \bar{\phi}]^a + \partial^\mu \bar{\phi}^a [A^a, \phi]^a \right) \\
- \frac{1}{4} \left( D \bar{\phi}^a + \bar{D} \phi^a + \sqrt{2} g [\phi, \bar{\phi}]^a \right)^2,
\]
where
\[
\begin{cases}
D \Phi^a \equiv (D_5 - i D_6) \Phi^a = \partial \Phi^a - \sqrt{2} g [\phi, \Phi]^a, \\
\bar{D} \Phi^a \equiv (D_5 + i D_6) \Phi^a = \bar{\partial} \Phi^a + \sqrt{2} g [\bar{\phi}, \Phi]^a,
\end{cases}
\]
which express the covariant derivatives with respect to the complex coordinates in the torus. \( \Phi^a \) denote arbitrary fields in the adjoint representation. We can remove the mixing terms between the gauge and the scalar fields in the second line of eq. (10) by introducing the gauge-fixing terms with a gauge parameter \( \xi \):
\[
L_{g-f} = \frac{1}{2\xi} (D_\mu A^{a \mu} + \xi D_m A^{a m})^2
\]
\[
= -\frac{1}{2\xi} D_\mu A^{a \mu} D_\nu A^{a \nu} - \frac{g}{\sqrt{2}} \left( \partial \bar{\phi}^a [A^{a \mu}, A^{a \nu}]^a - \bar{\partial} \phi^a [A^{a \mu}, A^{a \nu}]^a \right)
\]
\[
+ \frac{\xi}{4} (D \phi^a - \bar{D} \bar{\phi}^a)^2 + \frac{i}{\sqrt{2}} (\partial_\mu \phi^a \bar{D} A^{a \mu} - \bar{\partial}_\mu \bar{\phi}^a \partial A^{a \mu}).
\]

The new covariant derivatives \( D, \bar{D} \) are defined by replacing \( \phi^a, \bar{\phi}^a \) in \( D, \bar{D} \) with the VEVs \( \langle \phi^a \rangle, \langle \bar{\phi}^a \rangle \), respectively. Due to the gauge-fixing, we need to introduce the Faddeev-Popov ghost fields and their Lagrangian is
\[
L_{\text{ghost}} = -c^a (D_\mu D^\mu + \xi D_m D^m) c^a.
\]
Then, the total Lagrangian is given by
\[
L_{\text{total}} = -\frac{1}{4} F_{\mu \nu}^{a} F^{a \mu \nu} - \frac{1}{2 \xi} D_{\mu} A^{a \mu} D_{\nu} A^{a \nu} - \partial_{\mu} \tilde{\phi}^{a} \partial^{\mu} \phi^{a} - \frac{1}{2} A^{a \mu} D_{\mu} A^{a \mu}
- \frac{g}{\sqrt{2}} (\partial \phi^{a} [A_{\mu}, A^{\nu}]^{a} - \partial \phi^{a} [A_{\mu}, A^{\nu}]^{a}) + \frac{\xi}{4} (D \phi^{a} - \bar{D} \phi^{a})^2
+ ig \left( \partial_{\mu} \phi^{[a} A^{\mu] a} + \partial^{a} \phi^{[a} [A_{\mu}, A^{\mu}] \right)
- \frac{1}{4} \left( D \phi^{a} + \bar{D} \phi^{a} + \sqrt{2} g \phi \bar{\phi} [A_{\mu}, A^{\mu}] \right)^2
- \bar{c} \left( D_{\mu} D^{\mu} + \xi D_{m} D^{m} \right) c^{a}.
\]
(14)

For simplicity, we choose the Feynman gauge $\xi = 1$ throughout this paper.

2.2 The mass of the gauge fields $A^{a \mu}_{\mu}$

We will discuss mass eigenstates and eigenvalues of the fields $A^{a \mu}_{\mu}, \phi^{a}, c^{a}$. In this subsection, we find mass eigenvalues and eigenstates of the gauge fields $A^{a \mu}_{\mu}$. The mass term of gauge field corresponds to the background part of $-D A^{a \mu} \bar{D} A^{a \mu}/2$:
\[
L_{AA} = -\frac{1}{2} D A^{a \mu} \bar{D} A^{a \mu} = -\frac{1}{2} A^{a \mu} \left[ -D \bar{D} \right] A^{a \mu}.
\]
(15)

We would like to regard the background covariant derivatives $D, \bar{D}$ as creation and annihilation operators, respectively. In a matrix form, they are expressed as
\[
D = \begin{bmatrix}
\partial & igf \bar{z} & 0 \\
-igf \bar{z} & \partial & igv \\
0 & -igv & \partial
\end{bmatrix}, \quad \bar{D} = \begin{bmatrix}
\bar{\partial} & -igf z & 0 \\
igf z & \bar{\partial} & -igv \\
0 & igv & \bar{\partial}
\end{bmatrix}.
\]
(16)

Diagonalizing them, we obtain
\[
\begin{align*}
D_{\text{diag}} &= \text{diag} \left( \partial, \partial + g \sqrt{f^2 z^2 + v^2}, \partial - g \sqrt{f^2 z^2 + v^2} \right), \\
\bar{D}_{\text{diag}} &= \text{diag} \left( \bar{\partial}, \bar{\partial} - g \sqrt{f^2 z^2 + v^2}, \bar{\partial} + g \sqrt{f^2 z^2 + v^2} \right).
\end{align*}
\]
(17)

Their commutation relations is
\[
[D_{\text{diag}}, \bar{D}_{\text{diag}}]^{ac} = gf^2 \left( \frac{z}{\sqrt{f^2 z^2 + v^2}} + \frac{\bar{z}}{\sqrt{f^2 \bar{z}^2 + v^2}} \right) \left[ \begin{array}{ccc}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{array} \right],
\]
(18)

which depends on extra space coordinates. Therefore, $D, \bar{D}$ cannot be identified with creation and annihilation operators.

Since Kaluza-Klein (KK) mass spectrum cannot be exactly solved by using the creation and annihilation operators, we would like to find them perturbatively by the expansion
in \( v \). In this expansion, \( v L \ll 1 \) or \( v \ll 1 \) in the present case is assumed. In other words, we consider the case where the compactification scale is much larger than the constant VEV \( v \). From (16), we define the unperturbed parts \( \mathcal{D}_3, \overline{\mathcal{D}}_3 \) and the perturbed part \( V \) as

\[
\mathcal{D}_3 \equiv \begin{bmatrix}
\partial & igf \bar{z} & 0 \\
-igf \bar{z} & \partial & 0 \\
0 & 0 & \partial
\end{bmatrix}, \quad \overline{\mathcal{D}}_3 \equiv \begin{bmatrix}
\partial & -igf z & 0 \\
igf z & \partial & 0 \\
0 & 0 & \partial
\end{bmatrix} \tag{19}
\]

and

\[
V \equiv \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & igv \\
0 & -igv & 0
\end{bmatrix}, \tag{20}
\]

respectively. In these notations, the covariant derivatives can be expressed as \( \mathcal{D} = \mathcal{D}_3 + V \) and \( \overline{\mathcal{D}} = \overline{\mathcal{D}}_3 + \overline{V} \). \( \mathcal{D}_3 \) and \( \overline{\mathcal{D}}_3 \) can be identified with creation and annihilation operators, which are diagonalized as follows.

\[
\begin{cases}
\mathcal{D}_{3,\text{diag}} = \text{diag}(\partial - gf \bar{z}, \partial + gf \bar{z}, \partial), \\
\overline{\mathcal{D}}_{3,\text{diag}} = \text{diag}(\partial + gf z, \partial - gf z, \partial).
\end{cases} \tag{21}
\]

Their diagonalizing unitary matrix \( U_3 \), which satisfies \( U_3^{-1} \mathcal{D}_3 U_3 = \mathcal{D}_{3,\text{diag}} \) and \( U_3^{-1} \overline{\mathcal{D}}_3 U_3 = \overline{\mathcal{D}}_{3,\text{diag}} \), is

\[
U_3 = \frac{1}{\sqrt{2}} \begin{bmatrix}
1 & i & 0 \\
i & 1 & 0 \\
0 & 0 & \sqrt{2}
\end{bmatrix}. \tag{22}
\]

The commutation relation between \( \mathcal{D}_{3,\text{diag}} \) and \( \overline{\mathcal{D}}_{3,\text{diag}} \) is

\[
[i \overline{\mathcal{D}}_{3,\text{diag}}, i \mathcal{D}_{3,\text{diag}}]_{ac} = 2gf \begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{bmatrix}. \tag{23}
\]

The creation and annihilation operators are defined as

\[
a = \frac{i}{\sqrt{\alpha_2}} \overline{\mathcal{D}}_{3,\text{diag}}, \quad a^\dagger = \frac{i}{\sqrt{\alpha_2}} \mathcal{D}_{3,\text{diag}}, \tag{24}
\]

where \( \alpha_2 \equiv 2gf \). The components of the creation and annihilation operators are summarized as follows:

\[
\begin{cases}
a_1 \equiv \frac{i}{\sqrt{\alpha_2}} (\partial + gf z), \\
a_2 \equiv \frac{i}{\sqrt{\alpha_2}} (\partial - gf z), \quad \text{and} \quad a_1^\dagger \equiv \frac{i}{\sqrt{\alpha_2}} (\partial - gf \bar{z}), \\
a_3 \equiv \frac{i}{\sqrt{\alpha_2}} \partial, \quad \text{and} \quad a_2^\dagger \equiv \frac{i}{\sqrt{\alpha_2}} (\partial + gf \bar{z}), \tag{25}
\end{cases}
\]
We note that $a_3$ and $a_3^\dagger$ have no flux effects and play no role of creation and annihilation operators. $a_1^\dagger$ and $a_2$ are creation operators and $a_1$ and $a_2^\dagger$ are annihilation operators: the roles of creation and annihilation operators for $a_2$ and $a_2^\dagger$ are inverted due to the commutation relation for the 2-direction $[a_2, a_2^\dagger] = -1$. The ground state mode functions are determined by

$$a_1 \xi_{0,j} = 0, \quad a_2^\dagger \xi_{0,j} = 0,$$

where $\xi_{0,j}$ are the functions of $z$ (in detail, see [5]) and $j$ labels the degeneracy of the ground state: $j = 0, \cdots, |N| - 1$. Higher mode functions $\xi_{n,j}$ are constructed similar to the harmonic oscillator case [7]:

$$\xi_{n_1,j} = \frac{1}{\sqrt{n_1!}}(a_1^\dagger)^{n_1} \xi_{0,j}, \quad \bar{\xi}_{n_2,j} = \frac{1}{\sqrt{n_2!}}(a_2)^{n_2} \bar{\xi}_{0,j} \quad (n_{1,2} \in \mathbb{Z}, n_{1,2} \geq 0).$$

The functions $\xi_{n,j}$ satisfy the orthogonal conditions

$$\int_{T^2} d^2x \bar{\xi}_{n,j} \xi_{n',j'} = \delta_{n,n'} \delta_{j,j'}.$$

To be operated by $a$ or $a^\dagger$, we should define the states with the gauge indices:

$$\psi_{n_1,j}^1 \equiv \begin{bmatrix} \xi_{n_1,j} \\ 0 \\ 0 \end{bmatrix}, \quad \psi_{n_2,j}^2 \equiv \begin{bmatrix} 0 \\ \bar{\xi}_{n_2,j} \\ 0 \end{bmatrix}.$$

Moreover, using the periodic boundary condition of torus, we define the eigenstates for the 3-direction:

$$\psi_{l,m}^3 \equiv \begin{bmatrix} 0 \\ 0 \\ \lambda_{l,m} \end{bmatrix} \quad (l, m \in \mathbb{Z}),$$

where $\lambda_{l,m}$ are the functions of $x_5$ and $x_6$:

$$\lambda_{l,m} \equiv \lambda_{l,m}(x_5, x_6) = \exp[2\pi i(lx_5 + mx_6)].$$

These functions $\lambda_{l,m}$ also satisfy the orthogonal conditions

$$\int_{T^2} d^2x \bar{\lambda}_{l,m} \lambda_{l',m'} = \delta_{l,l'} \delta_{m,m'}.$$
The eigenstates satisfy the following relations

\[
\begin{align*}
    a\psi_{n_1,j}^1 &= \sqrt{n_1} \psi_{n_1-1,j}^1, \\
    a\psi_{n_2,j}^2 &= \sqrt{n_2} \psi_{n_2+1,j}^2, \\
    a\psi_{l,m}^3 &= -\sqrt{\frac{4\pi^2}{\alpha_2}} (l + im) \psi_{l,m}^3,
\end{align*}
\]

and

\[
\begin{align*}
    a^\dagger \psi_{n_1,j}^1 &= \sqrt{n_1 + 1} \psi_{n_1+1,j}^1, \\
    a^\dagger \psi_{n_2,j}^2 &= \sqrt{n_2} \psi_{n_2-1,j}^2, \\
    a^\dagger \psi_{l,m}^3 &= -\sqrt{\frac{4\pi^2}{\alpha_2}} (l - im) \psi_{l,m}^3.
\end{align*}
\]

(33)

For convenience we unify the labels of the eigenstates:

\[
\psi_{\{n_1\}} \equiv \psi_{n_1,j}^1, \quad \psi_{\{n_2\}} \equiv \psi_{n_2,j}^2, \quad \psi_{\{n_3\}} \equiv \psi_{l,m}^3.
\]

(34)

Their products of eigenstates in the different directions are zero.

\[
(\psi_{\{n_a\}}^a)^\dagger \psi_{\{n_c\}}^c = 0 \quad (a \neq c).
\]

(35)

Then we define the unperturbed Hamiltonian \( H_0 \) and the perturbation \( V_1, V_2 \):

\[
-\mathcal{D}\mathcal{D} = U_3 \left[ -\mathcal{D}_{3,\text{diag}} \mathcal{D}_{3,\text{diag}} + \mathcal{D}_{3,\text{diag}} U_3^{-1} V U_3 \right.
\]

\[
- U_3^{-1} V U_3 \mathcal{D}_{3,\text{diag}} + (U_3^{-1} V U_3)^2 \big] U_3^{-1}
\]

\[
\equiv U_3 \left[ H_0 + V_1 + V_1^\dagger + V_2 \right] U_3^{-1}
\]

\[
\equiv U_3 H U_3^{-1}.
\]

(36)

From the previous discussion, \( H_0 \) is expressed as

\[
H_0 = \begin{bmatrix}
\alpha_2 n_1 & 0 & 0 \\
0 & \alpha_2 (n_2 + 1) & 0 \\
0 & 0 & 4\pi^2 (l^2 + m^2)
\end{bmatrix},
\]

(37)

and its eigenstates of the gauge fields are defined by

\[
A_\mu U_3 a^a \equiv \tilde{A}_\mu a \equiv \sum_{\{n_a\}} \tilde{A}_\mu_{\{n_a\}} \psi_{\{n_a\}}^a, \quad (U_3^{-1})^{aa'} A^{a'} \mu \equiv \tilde{A}^{a\mu} \equiv \sum_{\{n_a\}} \tilde{A}^{a\mu}_{\{n_a\}} \psi_{\{n_a\}}^a.
\]

(38)

The perturbation \( V_1 \) and \( V_2 \) are

\[
V_1 = \frac{gv}{\sqrt{2}} \begin{bmatrix}
0 & 0 & \partial - gf \bar{z} \\
0 & 0 & i(\partial + gf \bar{z}) \\
\partial & -i\partial & 0
\end{bmatrix}, \quad V_2 = \frac{g^2 v^2}{2} \begin{bmatrix}
1 & -i & 0 \\
i & 1 & 0 \\
0 & 0 & 2
\end{bmatrix},
\]

(39)

respectively. As was seen from the unperturbed Hamiltonian (37), we find that there are degeneracy: \( \psi_{0,j}^1 \) and \( \psi_{l,m}^3 \), \( \psi_{n+1,j}^1 \) and \( \psi_{n,j}^2 \). We thus should be careful to calculate their energies in perturbation.
For $\psi_{1,j}^1$ and $\psi_{l,m}^3$, the first-order perturbation energy from $V_1 + V_1^\dagger + V_2$, $E_{A,0}^{(1)}$ can be easily obtained

$$E_{A,n=0}^{(1)} = g^2 v^2 / 2, \quad E_{A,3}^{(1)} = g^2 v^2.$$  \hspace{1cm} (40)

Note that we have to solve secular equation for $\psi_{0,j}^1$ and $\psi_{l=0,m=0}^3$ because there exist the degeneracy and the perturbation of $\psi_{l\neq0,m\neq0}^3$ can be obtained by $V_2$. The second-order perturbation energy $E_{A,0}^{(2)}$ for $\psi_{0,j}^1$ is shown in appendix A.

For $\psi_{n+1,j}^1$, $\psi_{n,j}^2$ ($n \geq 0$) and $\psi_{l,m}^3$, the first-order perturbation energy from $V_1 + V_1^\dagger + V_2$, $E_A^{(1)}$ can be easily obtained

$$E_{A,1'}^{(1)} = 0, \quad E_{A,2'}^{(1)} = g^2 v^2,$$

where the mode functions in new direction $1'$ and $2'$ are defined as

$$\psi_{n+1,j}^{1'} \equiv (i \psi_{n+1,j}^1 + \psi_{n,j}^2) / \sqrt{2}, \quad \psi_{n+1,j}^{2'} \equiv (\psi_{n+1,j}^1 + i \psi_{n,j}^2) / \sqrt{2}. \hspace{1cm} (42)$$

The second-order perturbation energy $E_{A,1'}^{(2)}$, $E_{A,2'}$ and $E_{A,3}$ are shown in appendix A.

Thus, we summarize the mass of the gauge fields as

$$m_{A,n=0}^2 = g^2 v^2 / 2 + E_{A,0}^{(2)},$$

$$m_{A,1'}^2 = \alpha_2(n + 1) + E_{A,1'}^{(2)},$$

$$m_{A,2'}^2 = \alpha_2(n + 1) + g^2 v^2 + E_{A,2'}^{(2)},$$

$$m_{A,3}^2 = 4\pi^2(t^2 + m^2) + g^2 v^2 + E_{A,3}^{(2)}, \hspace{1cm} (43)$$

and we find that all of the gauge fields have nonzero mass if $v \neq 0$. Therefore, we conclude that the SU(2) gauge symmetry is completely broken. The fact that non-zero VEV $v$ is realized will be seen in the potential analysis.

### 2.3 The mass of the scalar fields $\varphi^a$

The terms relevant to the scalar mass are

$$\mathcal{L} \supset - \frac{1}{4} \left\{ (\mathcal{D}\varphi + \overline{\mathcal{D}}\varphi)^2 - 2\sqrt{2} g (\partial \langle \varphi^a \rangle + \overline{\partial} \langle \varphi^a \rangle) \left[ \varphi, \overline{\varphi} \right]^a \right\} + \frac{1}{4} (\mathcal{D}\overline{\varphi} - \overline{\mathcal{D}}\varphi)^2$$

$$= -\overline{\varphi} (H + gf \text{ diag}(1, -1, 0)) \varphi,$$  \hspace{1cm} (44)
where

\[
\bar{\varphi}' a' U_3 a' \equiv \bar{\varphi}' \equiv \sum_{\{n_a\}} \bar{\varphi}'_{\{n_a\}} \psi'_{\{n_a\}}, \quad (U_3^{-1}) a' a' \equiv \bar{\varphi}' \equiv \sum_{\{n_a\}} \bar{\varphi}'_{\{n_a\}} \psi'_{\{n_a\}}.
\]

(45)

Since the energy eigenvalues for \(\psi_{n,j}^1\) and \(\psi_{n,j}^2\) are degenerate, we must solve the secular equation. We find the first-order perturbation energy from \(V_1 + V_1^\dagger + V_2\) as

\[
E_{\varphi,1''}^{(1)} = 0, \quad E_{\varphi,2''}^{(1)} = g^2 v^2,
\]

where the mode functions in new direction \(1''\) and \(2''\) are defined as

\[
\psi_{n,j}^{1''} \equiv (i\psi_{n,j}^1 + \psi_{n,j}^2) / \sqrt{2}, \quad \psi_{n,j}^{2''} \equiv (\psi_{n,j}^1 + i\psi_{n,j}^2) / \sqrt{2}.
\]

(47)

The second-order perturbation energy \(E_{\varphi,1''}^{(2)}\), \(E_{\varphi,2''}^{(2)}\) and \(E_{\varphi,3''}^{(2)}\) are shown in appendix A.

Thus, the mass of the scalar fields are obtained as

\[
\begin{align*}
m_{\varphi,1''}^2 &= \alpha_2 \left( n + \frac{1}{2} \right) + E_{\varphi,1''}', \\
m_{\varphi,2''}^2 &= \alpha_2 \left( n + \frac{1}{2} \right) + g^2 v^2 + E_{\varphi,2''}', \\
m_{\varphi,3''}^2 &= 4\pi^2 (l^2 + m^2) + g^2 v^2 + E_{\varphi,3''}.
\end{align*}
\]

(48)

2.4 The mass of the ghost fields \(c^a\)

The terms relevant to the ghost mass is

\[
\mathcal{L} \supset -\bar{c}^a (\mathcal{D}_m \mathcal{D}^m)^{ab} c^b \quad (m = 5, 6),
\]

where

\[
\mathcal{D}_m \mathcal{D}^m = -(-\mathcal{D} \mathcal{D}) - \frac{1}{2} [\mathcal{D}, \mathcal{D}]
\]

\[
= -U_3 \left[ H + \frac{1}{2} \left( [\mathcal{D}_3, \mathcal{D}_3, \mathcal{D}_3, \mathcal{D}_3, \mathcal{D}_3] - V_1 - V_1^\dagger + V_3 + V_3^\dagger \right) \right] U_3^{-1}
\]

\[
= -U_3 \left[ H_0 + V_2 + g f d i a g(1, -1, 0) + V_4 + V_4^\dagger \right] U_3^{-1}.
\]

(50)

\(V_3\) and \(V_4\) are defined as

\[
V_3 \equiv U_3^{-1} V U_3 \mathcal{D}_3, \quad V_4 \equiv \frac{1}{2} (V_1 + V_3).
\]

(51)
Note that the first three terms in the equation (50) are the same as those of the scalar fields. As in previous section, we solve the secular equation and find the first-order perturbation energy from $V_2 + V_4 + V_4^\dagger$ as

$$E_{c,1'}^{(1)} = 0, \quad E_{c,2'}^{(1)} = g^2 v^2. \quad (52)$$

The second-order perturbation energy $E_{c,1''}^{(2)}$, $E_{c,2''}^{(2)}$, and $E_{c,3}^{(2)}$ are summarized in appendix A.

Thus, the mass of the ghost fields are obtained as

$$\begin{align*}
  m_{c,1''}^2 &= \alpha_2 \left( n + \frac{1}{2} \right) + E_{c,1''}^{(2)}, \\
  m_{c,2''}^2 &= \alpha_2 \left( n + \frac{1}{2} \right) + g^2 v^2 + E_{c,2''}^{(2)}, \\
  m_{c,3}^2 &= 4\pi^2(l^2 + m^2) + g^2 v^2 + E_{c,3}^{(2)}. \quad (53)
\end{align*}$$

Note that the mass of ghost fields are the same as that of scalar mass at the first order in Feynman gauge $\xi = 1$. This fact greatly simplifies the potential analysis as will be discussed later.

### 2.5 The analysis of the effective potential

Since the potential of the constant WL scalar fields is not generated at tree level, we have to calculate the one-loop effective potential by use of KK mass spectrum obtained in the previous subsections. We will show that the constant WL scalar VEV can be nonzero and the gauge symmetry is broken.

In order to calculate the one-loop effective potential as general as possible, we parametrize the KK mass spectrum as follows.

$$m_0^2 \equiv A g^2 v^2, \quad m_n^2 \equiv \alpha(n + x) + B g^2 v^2, \quad m_{l,m}^2 \equiv 4\pi^2(l^2 + m^2) + C g^2 v^2, \quad (54)$$

where $x$ is 0, 1 or 1/2 depending on the fields under consideration. We suppose that $A$, $B$ and $C$ are positive constant. $\alpha$ means $\alpha_2$ or $\alpha_3$, which will be defined in next section.
Then, the typical forms of the one-loop effective potential can be written as

\[ \nu_0(A) \equiv \int \frac{d^4p}{(2\pi)^4} \ln \left[ p^2 + Ag^2v^2 \right], \]  
(55)

\[ \nu_n(\alpha, B; x) \equiv \sum_{n=0}^{\infty} \int \frac{d^4p}{(2\pi)^4} \ln \left[ p^2 + \alpha(n + x) +Bg^2v^2 \right], \]  
(56)

\[ \nu_{i,m}(C) \equiv \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \int \frac{d^4p}{(2\pi)^4} \ln \left[ p^2 + 4\pi^2(l^2 + m^2) + Cg^2v^2 \right]. \]  
(57)

Note that the second order perturbations are neglected in these potentials. For an obvious reason, \( \nu_0(A) \), \( \nu_n(\alpha, B; x) \) and \( \nu_{i,m}(C) \) will be referred as one-loop effective potential of zero-mode type, with-flux type and without-flux type, respectively.

### 2.5.1 The zero-mode type \( \nu_0 \)

First, we consider the effective potential of zero-mode type \( \nu_0 \).

\[ \nu_0(A) = -\int \frac{d^4p}{(2\pi)^4} \int_0^\infty \frac{dt}{t} e^{-(p^2 + Ag^2v^2)t} \]
\[ = -\frac{1}{16\pi^2} \int_0^\infty \frac{dt}{t^3} e^{-Ag^2v^2t}, \]

where Schwinger’s proper time integral is introduced in the first line and the momentum integral is performed in the second line. Obviously, this integral diverges at \( t = 0 \), but we can extract a finite value from it. We will propose the idea later and the regularized effective potential of zero-mode type \( \nu_{\text{reg},0} \) is found as

\[ \nu_{\text{reg},0}(A) = \frac{(Ag^2v^2)^2}{576\pi^2} \left( \frac{9\zeta(3)}{\pi^2} - 1 \right), \]  
(58)

where \( \zeta(3) \) is the Apéry’s constant, \( \zeta(3) = 1.20205 \cdots \) and \( 9\zeta(3)/\pi^2 - 1 = 0.0961381 \cdots \).

### 2.5.2 The with-flux type \( \nu_n \)

Next, we consider the effective potential of with-flux type \( \nu_n \).

\[ \nu_n(\alpha, B; x) = -\sum_n \int \frac{d^4p}{(2\pi)^4} \int_0^\infty \frac{dt}{t} e^{-(p^2 + \alpha(n + x) +Bg^2v^2)t} \]
\[ = -\frac{1}{16\pi^2} \int_0^\infty \frac{dt}{t^3} e^{-\alpha(x + Bg^2v^2/\alpha - 1)t} \]
\[ = -\frac{\alpha^2}{16\pi^2} \int_0^\infty dy e^{-\alpha(x + Bg^2v^2/\alpha - 1)y} \frac{1 - e^{y} - 1}{y^{-2} e^{y} - 1}. \]  
(59)
To calculate this, we try to apply the integral representation of the Hurwitz \( \zeta \) function

\[
\zeta(s,a) = \frac{a^{-s}}{2} + \frac{a^{1-s}}{s-1} + \sum_{k=1}^{n} \frac{B_{2k}}{(2k)!} a^{1-s-2k} (s)_{2k-1} + \frac{1}{\Gamma(s)} \int_{0}^{\infty} dy \frac{e^{-ay}}{y^{1-s}} \left( \frac{1}{e^y - 1} - \frac{1}{y} + \frac{1}{2} - \sum_{k=1}^{n} \frac{B_{2k}}{(2k)!} y^{2k-1} \right),
\]

where \( n \) is a non-negative integer, \( B_n \) is the Bernoulli number and \( (s)_n \) is the Pochhammer symbol. The equation (60) is satisfied with the conditions

\[
\text{Re } s > -(2n+1) \quad (n \in \mathbb{Z}, n \geq 0), \quad s \neq 1, \quad \text{Re } a > 0.
\]

Comparing the integral in (59) with the expression (60), we need to consider a case where \( s = -2 \). Therefore, it is enough to take \( n = 1 \) as follows:

\[
\zeta(s,a) = \frac{a^{-s}}{2} + \frac{a^{1-s}}{s-1} + \frac{sa^{s-1}}{12} + \frac{1}{\Gamma(s)} \int_{0}^{\infty} dy \frac{e^{-ay}}{y^{1-s}} \left( \frac{1}{e^y - 1} - \frac{1}{y} + \frac{1}{2} - \frac{y}{12} \right),
\]

where \( (s)_1 = s \) and \( B_2 = 1/6 \) are used. The integral which includes a term of \( 1/(e^y - 1) \) corresponds to the with-flux type \( \nu_n \). On the other hand, we know the form of \( \zeta(-2,a) \) with elementary functions:

\[
\zeta(-2,a) = -\frac{1}{3} \sum_{k=0}^{3} 3C_k B_{3-k} a^k = \frac{a^{(-2)}}{2} + \frac{a^{1-(-2)}}{-2-1} + \frac{(-2)a^{(-2)-1}}{12},
\]

and we find this to be equivalent to the first line of the equation (62).

When we take \( s = \epsilon - 2 \) (\( \epsilon = (4 - d)/2 \ll 1 \)), the equation (63) can be understood as \( O(\epsilon^0) \) terms of the both sides of the equation (62). Therefore, the integral in the second line of (62) is understood to be \( O(\epsilon) \). Extracting the \( O(\epsilon) \) terms from the equation (62), we obtain

\[
\int_{0}^{\infty} dy \frac{e^{-ay}}{y^3} \left( \frac{1}{e^y - 1} - \frac{1}{y} + \frac{1}{2} - \frac{y}{12} \right) = \frac{1}{2} \left[ \zeta^{(1,0)}(-2,a) - \frac{a}{12} (2 \ln a + 1) + \frac{a^2}{2} \ln a - \frac{a^3}{9} (3 \ln a - 1) \right],
\]

where we used the expansion of \( \zeta(\epsilon - 2, a) \) in \( \epsilon \):

\[
\zeta(\epsilon - 2, a) = \zeta(-2, a) + \epsilon \zeta^{(1,0)}(-2, a) + O(\epsilon^2).
\]
Although each integral of the equation (64) diverges at \( y = 0 \), the right hand side of (64) is finite. In other words, we can interpret that the last three terms of the left hand side in (64) work as the regulators which extract finite quantity from the divergent integral and we are able to evaluate the potential of with-flux type \( \nu_n \). Then, we obtain the regularized one-loop effective potential of with-flux type \( \nu_{\text{reg},n} \)

\[
\nu_{\text{reg},n}(\alpha, B; x) = -\frac{\alpha^2}{32\pi^2} \left[ \zeta(1,0) \left( -2, x + \frac{B g^2 v^2}{\alpha} - 1 \right) \right.
\]

\[
- \frac{1}{12} \left( x + \frac{B g^2 v^2}{\alpha} - 1 \right) \left\{ 2 \ln \left( x + \frac{B g^2 v^2}{\alpha} - 1 \right) + 1 \right\} 
\]

\[
+ \frac{1}{2} \left( x + \frac{B g^2 v^2}{\alpha} - 1 \right)^2 \ln \left( x + \frac{B g^2 v^2}{\alpha} - 1 \right)
\]

\[
- \frac{1}{9} \left( x + \frac{B g^2 v^2}{\alpha} - 1 \right)^3 \left\{ 3 \ln \left( x + \frac{B g^2 v^2}{\alpha} - 1 \right) - 1 \right\} \bigg] . \tag{66}
\]

Note that the quantity \( a = x + B g^2 v^2 / \alpha - 1 \) of the equation (59) is not always positive: if \( a \) is zero or negative, it does not satisfy the condition (61). This situation occur in the case \( x = 0 \) or \( 1/2 \). In that case, we separate the term of \( n = 0 \):

\[
\nu_{\text{reg},n}(\alpha, B; x < 1) = -\int \frac{d^4 p}{(2\pi)^4} \int_0^\infty \frac{dt}{t} e^{-(p^2 + \alpha x + B g^2 v^2)t} 
\]

\[
- \sum_{n=1}^{\infty} \int \frac{d^4 p}{(2\pi)^4} \int_0^\infty \frac{dt}{t} e^{-(p^2 + \alpha(n+x) + B g^2 v^2)t} 
\]

\[
= -\frac{\alpha^2}{16\pi^2} \int_0^\infty \frac{dy}{y^3} e^{-(x + B g^2 v^2 / \alpha) y} - \frac{\alpha^2}{16\pi^2} \int_0^\infty \frac{dy}{y^3} e^{-(x + B g^2 v^2 / \alpha) y} \cdot \tag{67}
\]

Although the first term in (67) is similarly divergent at \( y = 0 \), it can be finite by using (64) as

\[
-\frac{1}{16\pi^2} \int_0^\infty \frac{dt}{t^3} e^{-(\alpha x + B g^2 v^2)t} \to \frac{\alpha^2}{576\pi^2} \left( x + \frac{B g^2 v^2}{\alpha} \right)^2 \left(\frac{9 \zeta(3)}{\pi^2} - 1 \right) .
\]

In this calculation, the terms except for the third term of the left hand side in (64) work as the regulators. Thus, the regularized one-loop effective potential of with-flux type
\( \nu_{\text{reg},n}(\alpha, B; x < 1) \) can be obtained

\[
\nu_{\text{reg},n}(\alpha, B; x < 1) = -\frac{\alpha^2}{32\pi^2} \left[ \zeta^{(1,0)}(-2, x + \frac{Bg^2v^2}{\alpha}) \right.
- \frac{1}{12} \left( x + \frac{Bg^2v^2}{\alpha} \right) \left\{ 2\ln \left( x + \frac{Bg^2v^2}{\alpha} \right) + 1 \right\}
+ \frac{1}{18} \left( x + \frac{Bg^2v^2}{\alpha} \right)^2 \left\{ 9\ln \left( x + \frac{Bg^2v^2}{\alpha} \right) - \frac{9\zeta(3)}{\pi^2} + 1 \right\}
- \frac{1}{9} \left( x + \frac{Bg^2v^2}{\alpha} \right)^3 \left\{ 3\ln \left( x + \frac{Bg^2v^2}{\alpha} \right) - 1 \right\} \right].
\]

(68)

2.5.3 The without-flux type \( \nu_{l,m} \)

Finally, we consider the one-loop effective potential of without-flux type \( \nu_{l,m} \).

\[
\nu_{l,m}(C) = -\sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \int_{0}^{\infty} \frac{dt}{t} \int \frac{d^4p}{(2\pi)^4} e^{-[p^2 + 4\pi^2(l^2+m^2) + Cg^2v^2]t}
= -\frac{1}{16\pi^2} \sum_{l,m} \int_{0}^{\infty} \frac{dt}{t^3} e^{-[4\pi^2(l^2+m^2) + Cg^2v^2]t}
= -\frac{1}{16\pi^2} \int_{0}^{\infty} du e^{-Cg^2v^2/u} \sum_{l,m} e^{-4\pi^2(l^2+m^2)/u}.
\]

Using the Poisson resummation formula,

\[
\sum_{l=-\infty}^{\infty} \exp \left[ -\frac{4\pi^2(l + v)^2}{u} \right] = \sqrt{\frac{u}{4\pi}} \sum_{r=-\infty}^{\infty} e^{-ur^2/4 + 2\pi ivr}
\]

(69)

with \( v = 0 \), we obtain

\[
\nu_{l,m}(C) = -\frac{1}{64\pi^3} \int_{0}^{\infty} du u^2 e^{-Cg^2v^2/u} \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} e^{-u(r^2+s^2)/4}
= -\frac{1}{64\pi^3} \sum_{r,s \neq 0} \left( \frac{4}{r^2 + s^2} \right)^3 \int_{0}^{\infty} du u^2 e^{-u} \exp \left[ -\frac{Cg^2v^2(r^2 + s^2)}{4u} \right]
+ \frac{1}{64\pi^3} \int_{0}^{\infty} dt e^{-Cg^2v^2t/t^3} \left( -\frac{1}{t} \right),
\]

(70)

where we note that a change of variable \( t = (r^2 + s^2)/u \) is performed in the second line of (70) except for \( r = s = 0 \) mode. For the first term of the equation (70), we consider
applying the modified Bessel function of second kind,

\[ K_\nu(z) = \frac{1}{2} \left( \frac{z}{2} \right)^\nu \int_0^\infty dt t^{-\nu-1} \exp \left( -t - \frac{z^2}{4t} \right) \]

\[ = \frac{1}{2} \left( \frac{z}{2} \right)^{-\nu} \int_0^\infty du u^{\nu-1} \exp \left( -\frac{z^2}{4u} - u \right), \tag{71} \]

where a change of variable \( u = \frac{z^2}{4t} \) is performed in the second line. This is satisfied with the conditions \( \text{Re} \, \nu > -1/2 \), \( |\arg z| < \pi/4 \). Thus, the first term of the equation (70) becomes

\[ -\frac{1}{64\pi^3} \sum_{r,s \neq 0} \left( \frac{4}{r^2 + s^2} \right)^3 \int_0^\infty du u^2 e^{-u} \exp \left[ -\frac{C g^2 v^2 (r^2 + s^2)}{4u} \right] \]

\[ = -\frac{C^{3/2} g^3 v^3}{4\pi^3} \sum_{r,s \neq 0} \left( \frac{1}{r^2 + s^2} \right)^{3/2} K_3 \left( g v \sqrt{C} (r^2 + s^2) \right). \]

The second term of the equation (70) is regularized by using the equation (64):

\[ \frac{1}{64\pi^3} \int_0^\infty dt e^{-C g^2 v^2 t} \left( -\frac{1}{t} \right) \rightarrow \frac{(C g^2 v^2)^3}{4608\pi^3} \left( \frac{9 \zeta(3) \pi^2}{\pi^2} - 1 \right). \]

Therefore, the regularized one-loop effective potential of without-flux type is obtained

\[ \nu_{\text{reg},l,m}(C) = -\frac{(C g^2 v^2)^{3/2}}{4\pi^3} \sum_{r,s \neq 0} \left( \frac{1}{r^2 + s^2} \right)^{3/2} K_3 \left( g v \sqrt{C} (r^2 + s^2) \right) \]

\[ + \frac{(C g^2 v^2)^3}{4608\pi^3} \left( \frac{9 \zeta(3) \pi^2}{\pi^2} - 1 \right). \tag{72} \]

The several types of the one-loop potentials \( \nu_{\text{reg},0}(1), \nu_{\text{reg},n}(\alpha_2, 1; 0), \nu_{\text{reg},n}(\alpha_2, 1; 1) \) and \( \nu_{\text{reg},l,m}(1) \) are shown in Figure 1.

### 2.6 The one-loop effective potential of SU(2) Yang-Mills theory

We now apply the above results obtained in the previous subsections to SU(2) Yang-Mills theory. The effective potentials can be calculated by using the masses of the gauge fields \( A_\mu^a \), the scalar fields \( \varphi^a \) and the ghost fields \( c^a \),

\[ V_A = \frac{3}{2} \frac{1}{(2\pi)^2} \left( \nu_0(1/2) + \nu_{l,m}(1) + \nu_n(\alpha_2, 1; 1) \right), \tag{73} \]

\[ V_\varphi = \frac{1}{2} \frac{1}{(2\pi)^2} \left( \nu_{l,m}(1) + \nu_n(\alpha_3, 1; 1/2) \right), \tag{74} \]

\[ V_c = -\frac{1}{2} \frac{1}{(2\pi)^2} \left( \nu_{l,m}(1) + \nu_n(\alpha_3, 1; 1/2) \right) = -V_\varphi. \tag{75} \]
Since these effective potentials are divergent, we extract the finite value from them. By using (58), (66) and (72), the regularized effective potential is expressed as

\[
V_{\text{reg},A} = \frac{3}{2} \frac{1}{(2\pi)^2} \left( \nu_{\text{reg},0}(1/2) + \nu_{\text{reg},l,m}(1) + \nu_{\text{reg},n}(\alpha_2, 1; 1) \right),
\]

\[
V_{\text{reg},\varphi} = \frac{1}{2} \frac{1}{(2\pi)^2} \left( \nu_{\text{reg},l,m}(1) + \nu_{\text{reg},n}(\alpha_3, 1; 1/2) \right),
\]

\[
V_{\text{reg},c} = -\frac{1}{2} \frac{1}{(2\pi)^2} \left( \nu_{\text{reg},l,m}(1) + \nu_{\text{reg},n}(\alpha_3, 1; 1/2) \right) = -V_{\text{reg},\varphi}.
\]

Since \(V_\varphi\) and \(V_c\) cancel each other because of the same the KK mass spectrum in Feynman gauge, we have only to consider \(V_{\text{reg},A}\) as the total effective potential \(V \equiv V_{\text{reg},A} + V_{\text{reg},\varphi} + \ldots\)

---

**Figure 1:** A picture of \(\nu_{\text{reg},0}(1)\), \(\nu_{\text{reg},n}(\alpha_2, 1; 0)\), \(\nu_{\text{reg},n}(\alpha_2, 1; 1)\) and \(\nu_{\text{reg},l,m}(1)\).
In detail, the total effective potential can be expressed as

$$V = V_{reg,A}$$

$$= \frac{(g^2 v^2)^2}{6144\pi^4} \left( \frac{9\zeta(3)}{\pi} - 1 \right)$$

$$- \frac{3(g^2 v^2)^{3/2}}{32\pi^5} \sum_{r,s\neq 0} \left( \frac{1}{r^2 + s^2} \right)^{3/2} K_3(g v \sqrt{r^2 + s^2}) + \frac{(g^2 v^2)^3}{12288\pi^5} \left( \frac{9\zeta(3)}{\pi} - 1 \right)$$

$$- \frac{3N^2}{16\pi^2} \left[ \zeta^{(1,0)}(-2, \frac{g^2 v^2}{4\pi N}) - \frac{1}{12} \left( \frac{g^2 v^2}{4\pi N} \right)^2 \ln \left( \frac{g^2 v^2}{4\pi N} \right) - \frac{1}{9} \left( \frac{g^2 v^2}{4\pi N} \right)^3 \right]$$

$$+ \frac{1}{2} \left( \frac{g^2 v^2}{4\pi N} \right)^2 \ln \left( \frac{g^2 v^2}{4\pi N} \right) - \frac{1}{9} \left( \frac{g^2 v^2}{4\pi N} \right)^3 \left\{ 3 \ln \left( \frac{g^2 v^2}{4\pi N} \right) - 1 \right\} \right]. \quad (79)$$

![Figure 2: One-loop effective potentials of SU(2) Yang-Mills theory. N = 3 is taken for simplicity. The shaded region represents a range where our perturbation analysis breaks down.](image)

The one-loop effective potentials with $N = 3$ are shown in figure 2. The cancellation of the potentials between the scalar (blue line) and the ghost (green line) loop contributions can be explicitly verified. As can be seen from the total potential (red line) in figure 2, we can conclude that the origin of the potential is at least not a minimum and the minimum of the effective potential is expected to locate at a nonzero constant VEV $v \neq 0$ and the gauge symmetry SU(2) is completely broken.

\footnote{\text{If you do not choose Feynman gauge $\xi = 1$, the cancellation of $V_{reg,c}$ and $V_{reg,\phi}$ does not occur. See [12] for this implications.}}
Here we should note that we cannot determine the location of the minimum since the location is expected to be beyond the range which our perturbation is valid as is shown in figure 2. Even if the value of the minimum is not determined, if we assume that the potential is bounded below, the pattern of gauge symmetry breaking can be confirmed from the gauge boson spectrum (43).

3 SU(3) Yang-Mills theory

In a context of gauge-Higgs unification, SU(3) gauge symmetry is minimal to realize the zero mode of WL scalar as an SU(2) Higgs doublet. In our case, if SU(3) is broken to SU(2) × U(1) by the VEV of the constant magnetic flux, the SU(2) Higgs doublet would appear and the electroweak symmetry breaking can be discussed. As a first step toward the discussion, we now consider a six-dimensional SU(3) Yang-Mills theory. We consider two cases of directions where we introduce the VEVs

\[ \langle \phi^1 \rangle = \frac{v}{\sqrt{2}} \quad \text{and} \quad \langle \phi^8 \rangle = \frac{f e}{\sqrt{2}}, \]  

\[ \langle \phi^6 \rangle = \frac{v}{\sqrt{2}} \quad \text{and} \quad \langle \phi^8 \rangle = \frac{f e}{\sqrt{2}}, \]

where the superscripts of \( \phi \) denote the gauge indices of SU(3) and the structure constant is changed to that of SU(3) accordingly. Note that the flux background breaks the SU(3) gauge symmetry, which is broken SU(2) × U(1). The case (1) is not included in SU(2) theory, because two kinds of VEV are both taken in the components of the unbroken symmetry. Other cases of developing the VEVs are reduced to the above two cases by the gauge rotations.
### 3.1 Case (1)

In this subsection, let us consider the case (1) where the background covariant derivatives \( D, \overline{D} \) are expressed as

\[
D^{ac} = \begin{bmatrix}
\partial & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \partial & igv & 0 & 0 & 0 & 0 & 0 \\
0 & -igv & \partial & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \partial & \frac{\sqrt{3}}{2} igf \bar{z} & 0 & \frac{1}{2} igv & 0 \\
0 & 0 & 0 & -\frac{\sqrt{3}}{2} igf \bar{z} & \partial & -\frac{1}{2} igv & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} igv & \partial & \frac{\sqrt{3}}{2} igf \bar{z} & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{2} igv & 0 & -\frac{\sqrt{3}}{2} igf \bar{z} & \partial & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \partial & \partial
\end{bmatrix}, \tag{82}
\]

\[
\overline{D}^{ac} = \begin{bmatrix}
\bar{\partial} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \bar{\partial} & -igv & 0 & 0 & 0 & 0 & 0 \\
0 & igv & \bar{\partial} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \bar{\partial} & -\frac{\sqrt{3}}{2} igfz & 0 & -\frac{1}{2} igv & 0 \\
0 & 0 & 0 & \frac{\sqrt{3}}{2} igfz & \bar{\partial} & \frac{1}{2} igv & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{2} igv & \bar{\partial} & -\frac{\sqrt{3}}{2} igfz & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} igv & 0 & \frac{\sqrt{3}}{2} igfz & \bar{\partial} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\partial}
\end{bmatrix}. \tag{83}
\]

Diagonalizing them, we obtain

\[
\begin{cases}
D_{\text{diag}} = \text{diag} \left( \partial, \partial - gv, \partial + gv, \partial - \frac{1}{2} gv - \frac{\sqrt{3}}{2} gf \bar{z}, \partial - \frac{1}{2} gv + \frac{\sqrt{3}}{2} gf \bar{z}, \partial + \frac{1}{2} gv - \frac{\sqrt{3}}{2} gf \bar{z}, \partial + \frac{1}{2} gv + \frac{\sqrt{3}}{2} gf \bar{z} \right), \\
\overline{D}_{\text{diag}} = \text{diag} \left( \bar{\partial}, \bar{\partial} + gv, \bar{\partial} - gv, \bar{\partial} + \frac{1}{2} gv + \frac{\sqrt{3}}{2} gfz, \bar{\partial} + \frac{1}{2} gv - \frac{\sqrt{3}}{2} gfz, \bar{\partial} - \frac{1}{2} gv + \frac{\sqrt{3}}{2} gfz, \bar{\partial} - \frac{1}{2} gv - \frac{\sqrt{3}}{2} gfz \right).
\end{cases} \tag{84}
\]

Their diagonalizing unitary matrix \( U \) is given by

\[
U = \frac{1}{2} \begin{bmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{2} & \sqrt{2}i & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{2}i & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & i & -1 & i & 0 \\
0 & 0 & 0 & i & 1 & -i & 1 & 0 \\
0 & 0 & 0 & 1 & -i & 1 & i & 0 \\
0 & 0 & 0 & i & -1 & i & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2
\end{bmatrix}. \tag{85}
\]
The commutation relation of $\mathcal{D}$ and $\bar{\mathcal{D}}$ become just a constant matrix:

$$[i\bar{\mathcal{D}}_{\text{diag}}, i\mathcal{D}_{\text{diag}}]^{ac} = \sqrt{3}gf\text{diag}(0, 0, 0, 1, -1, 1, -1, 0),$$

where the creation and annihilation operators can be given as

$$a = \frac{i}{\sqrt{\alpha_3}}\bar{\mathcal{D}}_{\text{diag}}, \quad a^\dagger = \frac{i}{\sqrt{\alpha_3}}\mathcal{D}_{\text{diag}} \ (\alpha_3 \equiv \sqrt{3}gf).$$

Defining further

$$a_{4,6} = \frac{i}{\sqrt{\alpha_3}} \left( \bar{\partial} + \frac{\sqrt{3}}{2} gfz \right), \quad a_{4,6}^\dagger = \frac{i}{\sqrt{\alpha_3}} \left( \partial - \frac{\sqrt{3}}{2} gfz \right),$$

$$a_{5,7} = \frac{i}{\sqrt{\alpha_3}} \left( \partial - \frac{\sqrt{3}}{2} gfz \right), \quad a_{5,7}^\dagger = \frac{i}{\sqrt{\alpha_3}} \left( \partial + \frac{\sqrt{3}}{2} gfz \right)$$

in the matrix form of the creation and annihilation operator, the diagonalized part of $\mathcal{D}_{\text{diag}}$ and $\bar{\mathcal{D}}_{\text{diag}}$ can be expressed as

$$\begin{cases}
(i\bar{\mathcal{D}}_{\text{diag}})^{44} = \sqrt{\alpha_3}a_{4,6} + \frac{1}{2}igv, \\
(i\bar{\mathcal{D}}_{\text{diag}})^{55} = \sqrt{\alpha_3}a_{5,7} + \frac{1}{2}igv, \\
(i\bar{\mathcal{D}}_{\text{diag}})^{66} = \sqrt{\alpha_3}a_{4,6} - \frac{1}{2}igv, \\
(i\bar{\mathcal{D}}_{\text{diag}})^{77} = \sqrt{\alpha_3}a_{5,7} - \frac{1}{2}igv,
\end{cases}
$$

and

$$\begin{cases}
(i\mathcal{D}_{\text{diag}})^{44} = \sqrt{\alpha_3}a_{4,6}^\dagger - \frac{1}{2}igv, \\
(i\mathcal{D}_{\text{diag}})^{55} = \sqrt{\alpha_3}a_{5,7}^\dagger - \frac{1}{2}igv, \\
(i\mathcal{D}_{\text{diag}})^{66} = \sqrt{\alpha_3}a_{4,6}^\dagger + \frac{1}{2}igv, \\
(i\mathcal{D}_{\text{diag}})^{77} = \sqrt{\alpha_3}a_{5,7}^\dagger + \frac{1}{2}igv.
\end{cases}$$

Note that other components are just spatial derivatives and do not play any role of creation and annihilation operators.

### 3.1.1 The mass of the gauge fields $A^a_\mu$

In the same way as the calculations in subsection 2.2 we obtain the gauge mass matrix

$$-\mathcal{D}_{\text{diag}}\bar{\mathcal{D}}_{\text{diag}} \equiv m_A^2,$$

where the components are

$$\begin{cases}
(m_A^2)^{11} = 4\pi^2(l_1^2 + m_1^2), & (m_A^2)^{22} = 4\pi^2 \left( l_2^2 + \left( m_2 - \frac{gv}{2\pi} \right)^2 \right), \\
(m_A^2)^{33} = 4\pi^2 \left( l_3^2 + \left( m_3 + \frac{gv}{2\pi} \right)^2 \right), & (m_A^2)^{44} = \alpha_3n_4 + \frac{1}{4}g^2v^2, \\
(m_A^2)^{55} = \alpha_3(n_5 + 1) + \frac{1}{4}g^2v^2, & (m_A^2)^{66} = \alpha_3n_6 + \frac{1}{4}g^2v^2, \\
(m_A^2)^{77} = \alpha_3(n_7 + 1) + \frac{1}{4}g^2v^2, & (m_A^2)^{88} = 4\pi^2(l_8^2 + m_8^2).
\end{cases}$$
From the massless modes in (92), we find that the SU(2) × U(1) gauge symmetry is broken into U(1) × U(1) if \( v \neq 0 \).

### 3.1.2 The mass of the scalar fields \( \varphi^a \)

The scalar masses are calculated from the terms

\[
2 \left( -D_{\text{diag}} D_{\text{diag}} \right)^{ac} - 4 ig f (U^{-1})^{a'a} f^{8a'c'} U^{c'} \equiv m_{\varphi}^2,
\]

and the resulting KK mass spectrums are found as

\[
\begin{align*}
(m_{\varphi}^2)^{11} &= 4\pi^2(l_1^2 + m_1^2), \\
(m_{\varphi}^2)^{22} &= 4\pi^2 \left( l_2^2 + \left( m_2 - \frac{gv}{2\pi} \right)^2 \right), \\
(m_{\varphi}^2)^{33} &= 4\pi^2 \left( l_3^2 + \left( m_3 + \frac{gv}{2\pi} \right)^2 \right), \\
(m_{\varphi}^2)^{44} &= \alpha_3 \left( n_4 + \frac{1}{2} \right) + \frac{1}{4} g^2 v^2, \\
(m_{\varphi}^2)^{55} &= \alpha_3 \left( n_5 + \frac{1}{2} \right) + \frac{1}{4} g^2 v^2, \\
(m_{\varphi}^2)^{66} &= \alpha_3 \left( n_6 + \frac{1}{2} \right) + \frac{1}{4} g^2 v^2, \\
(m_{\varphi}^2)^{77} &= \alpha_3 \left( n_7 + \frac{1}{2} \right) + \frac{1}{4} g^2 v^2, \\
(m_{\varphi}^2)^{88} &= 4\pi^2 (l_8^2 + m_8^2).
\end{align*}
\]

### 3.1.3 The mass of the ghost fields \( c^a \)

The ghost masses are calculated from the terms

\[
-D_{\text{diag}} D_{\text{diag}} + \frac{\alpha_3}{2} [a, a^\dagger] \equiv m_c^2,
\]

and the resulting KK spectrum are obtained

\[
\begin{align*}
(m_c^2)^{11} &= 4\pi^2(l_1^2 + m_1^2), \\
(m_c^2)^{22} &= 4\pi^2 \left( l_2^2 + \left( m_2 - \frac{gv}{2\pi} \right)^2 \right), \\
(m_c^2)^{33} &= 4\pi^2 \left( l_3^2 + \left( m_3 + \frac{gv}{2\pi} \right)^2 \right), \\
(m_c^2)^{44} &= \alpha_3 \left( n_4 + \frac{1}{2} \right) + \frac{1}{4} g^2 v^2, \\
(m_c^2)^{55} &= \alpha_3 \left( n_5 + \frac{1}{2} \right) + \frac{1}{4} g^2 v^2, \\
(m_c^2)^{66} &= \alpha_3 \left( n_6 + \frac{1}{2} \right) + \frac{1}{4} g^2 v^2, \\
(m_c^2)^{77} &= \alpha_3 \left( n_7 + \frac{1}{2} \right) + \frac{1}{4} g^2 v^2, \\
(m_c^2)^{88} &= 4\pi^2 (l_8^2 + m_8^2).
\end{align*}
\]
3.1.4 The potential calculation

The effective potential can be calculated by using the masses of the gauge fields $A^a_{\mu}$, the scalar fields $\phi^a$, the ghost fields $c^a$ and (56) and (57):

$$V_A = \frac{3}{2} \left( \frac{2}{(2\pi)^2} \right) \left( 2\nu_{t,m}(0) + \nu_+(gv/2\pi) + \nu_-(gv/2\pi) + 2\nu_n(\alpha_3, 1/4; 0) + 2\nu_n(\alpha_3, 1/4; 1) \right),$$

(97)

$$V_\phi = \frac{1}{2} \left( \frac{2}{(2\pi)^2} \right) \left( 2\nu_{t,m}(0) + \nu_+(gv/2\pi) + \nu_-(gv/2\pi) + 4\nu_n(\alpha_3, 1/4; 1/2) \right),$$

(98)

$$V_c = -\frac{1}{2} \left( \frac{2}{(2\pi)^2} \right) \left( 2\nu_{t,m}(0) + \nu_+(gv/2\pi) + \nu_-(gv/2\pi) + 4\nu_n(\alpha_3, 1/4; 1/2) \right) = -V_\phi,$$

(99)

where $\nu_{\pm}(V)$ is defined as

$$\nu_{\pm}(V) \equiv \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \int \frac{d^4p}{(2\pi)^4} \ln \left[ p^2 + 4\pi^2 \{ l^2 + (m \pm V)^2 \} \right].$$

(100)

The terms which do not contain the constant VEV $v$, $\nu_{t,m}(0)$, are irrelevant to determine the potential minimum.

We consider here new one-loop effective potentials of without-flux type $\nu_{\pm}$.

$$\nu_{\pm}(V) = -\sum_{l,m} \int_0^\infty \frac{dt}{t} \int \frac{d^4p}{(2\pi)^4} \exp \left( - \left[ p^2 + 4\pi^2 \{ l^2 + (m \pm V) \} \right] t \right)$$

$$= -\frac{1}{16\pi^2} \sum_{l,m} \int_0^\infty \frac{dt}{t^3} \exp \left( -4\pi^2 \{ l^2 + (m \pm V) \} t \right)$$

$$= -\frac{1}{16\pi^2} \int_0^\infty \frac{du}{u} \sum_{l,m} \exp \left( -4\pi^2 \{ l^2 + (m \pm V) \} / u \right).$$

Using the Poisson resummation formula (69), $\nu_{\pm}$ become

$$\nu_{\pm}(V) = -\frac{1}{64\pi^3} \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} e^{\pm 2\pi i vs} \int_0^\infty du u^2 e^{-u(r^2+s^2)/4}.$$

Since the term with $s = 0$ has no dependence on $v$, it may be removed and we obtain the regularized one-loop effective potentials of without-flux type $\nu_{\text{reg}, \pm}$

$$\nu_{\text{reg}, \pm}(V) = -\frac{1}{64\pi^3} \sum_{r=-\infty}^{\infty} \sum_{s \neq 0} e^{\pm 2\pi i vs} \int_0^\infty du u^2 e^{-u(r^2+s^2)/4}$$

$$= -\frac{4}{\pi^3} \sum_{r=-\infty}^{\infty} \sum_{s = 1}^{\infty} \frac{1}{(r^2 + s^2)^3} \cos(2\pi Vs).$$

(101)
Note that this periodic potential often appears in gauge-Higgs unification \cite{22,23}.

By using (66), (68), (72) and (101), each regularized effective potential is expressed as

\[ V_{\text{reg},A} = \frac{3}{2} \frac{1}{(2\pi)^2} \left( 2\nu_{\text{reg},l,m}(0) + 2\nu_{\text{reg},\pm}(gv/2\pi) + 2\nu_{\text{reg},n}(\alpha_3, 1/4; 0) + 2\nu_{\text{reg},n}(\alpha_3, 1/4; 1) \right), \]

\[ V_{\text{reg},\varphi} = \frac{1}{2} \frac{1}{(2\pi)^2} \left( 2\nu_{\text{reg},l,m}(0) + 2\nu_{\text{reg},\pm}(gv/2\pi) + 4\nu_{\text{reg},n}(\alpha_3, 1/4; 1/2) \right), \]

\[ V_{\text{reg},c} = -\frac{1}{2} \frac{1}{(2\pi)^2} \left( 2\nu_{\text{reg},l,m}(0) + 2\nu_{\text{reg},\pm}(gv/2\pi) + 4\nu_{\text{reg},n}(\alpha_3, 1/4; 1/2) \right) = -V_{\text{reg},\varphi}. \]

Because of \( V_{\text{reg},c} = -V_{\text{reg},\varphi} \), we consider \( V_{\text{reg},A} \) as the total effective potential \( V \equiv V_{\text{reg},A} + V_{\text{reg},\varphi} + V_{\text{reg},c} \). In detail, the total one-loop effective potential can be expressed as

\[ V = V_{\text{reg},A} \supset -\frac{3}{\pi^5} \sum_{r=-\infty}^{\infty} \sum_{s=1}^{\infty} \frac{1}{(r^2 + s^2)^3} \cos(gv\delta) \]

\[ -\frac{9N^2}{16\pi^2} \left[ \zeta(1,0) \left( -2, \frac{g^2v^2}{8\sqrt{3}\pi N} \right) - \frac{1}{12} \left( \frac{g^2v^2}{8\sqrt{3}\pi N} \right) \left( 2\ln \frac{g^2v^2}{8\sqrt{3}\pi N} + 1 \right) \right] \]

\[ + \frac{1}{36} \left( \frac{g^2v^2}{8\sqrt{3}\pi N} \right)^2 \left\{ 18\ln \frac{g^2v^2}{8\sqrt{3}\pi N} - \frac{9\zeta(3)}{2\pi^2} + 1 \right\} \]

\[ -\frac{1}{9} \left( \frac{g^2v^2}{8\sqrt{3}\pi N} \right)^3 \left( 3\ln \frac{g^2v^2}{8\sqrt{3}\pi N} - 1 \right). \]

In (105), \( 2\nu_{\text{reg},l,m}(0) \) term was dropped because it is irrelevant to finding the potential minimum.

Figure 3: A picture of the effective potential \( V \) with \( N = 3 \) and its parts \( V_A, V_\varphi \) and \( V_c \). (b) is a closeup of the effective potential (a).
The effective potential \(105\) of SU(3) Yang-Mills theory with \(N = 3\) is shown in Figure 3. Note that \(V_\phi\) and \(V_c\) are cancelled each other and \(V_A\) itself becomes the total potential \(V\). It seems that there is a local minimum at \(gv = 0\), but it is not correct. The behavior of \(V\) around \(gv = 0\) is shown in figure 3(b) and we find that there are two local minima at \(gv \neq 0\). The reason why \(V\) becomes convex upward at \(gv = 0\) is due to the existence of the term \(g^2v^2\ln g^2v^2\) in \(V_A\): it becomes dominant as \(gv\) gets close to zero. Furthermore, we emphasize that the logarithmic term is originated from the potential of with-flux type. If the magnetic flux is absent, we have no such contribution to the potential. Then, the potential is a periodic one as seen in the gauge-Higgs unification, where the origin of the potential becomes convex downward, which implies the origin can be a local minimum. The effects from the potential of with-flux type are very crucial in our analysis of gauge symmetry breaking. Thus, we find that the minimum of the effective potential has a nonzero VEV \(v\neq 0\), and the gauge symmetry SU(2) \(\times\) U(1) is broken into U(1) \(\times\) U(1).

### 3.2 Case (2)

In this section, we consider the case (2).

\[
\langle \phi^6 \rangle = \frac{v}{\sqrt{2}}, \quad \langle \phi^8 \rangle = \frac{f \bar{z}}{\sqrt{2}}.
\]  

(106)

In this case, the components of VEV where the constant WL scalar develops are in the broken generators under SU(3) \(\rightarrow\) SU(2) \(\times\) U(1). This case is similar to section 2. In this situation, the background covariant derivatives \(\mathcal{D}, \bar{\mathcal{D}}\) are expressed as

\[
\mathcal{D}^a = \begin{bmatrix}
\partial & 0 & 0 & 0 & 0 & \frac{1}{2}igv & 0 & 0 & 0 \\
0 & \partial & 0 & \frac{1}{2}igv & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \partial & 0 & 0 & 0 & \frac{1}{2}igv & 0 & 0 \\
0 & -\frac{1}{2}igv & 0 & \partial & \sqrt{3}igf \bar{z} & 0 & 0 & 0 & 0 \\
\frac{1}{2}igv & 0 & 0 & -\sqrt{3}igf \bar{z} & \partial & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \partial & \sqrt{3}igf \bar{z} & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2}igv & 0 & 0 & -\sqrt{3}igf \bar{z} & \partial & \sqrt{3}igv & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{3}igv & \partial & \sqrt{3}igv \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}igv & \partial
\end{bmatrix}.
\]  

(107)
If we diagonalize them, the eigenvalues are the same form as (17). Therefore, we apply the perturbation theory as we have seen in section 2.2. Defining the unperturbed parts to be $D_8$, $\overline{D}_8$ and the perturbation part $V$ such as (19) and (20) respectively, the covariant derivatives can be represented as

$$D = D_8 + V$$

and

$$\overline{D} = \overline{D}_8 + \overline{V}$$. 

Diagonalizing $D_8$, $\overline{D}_8$, we obtain the eigenvalues (84) with $v = 0$. Their diagonalizing unitary matrix is given by

$$U_8 = \frac{1}{\sqrt{2}} \begin{bmatrix}
\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & i & 0 & 0 & 0 \\
0 & 0 & 0 & i & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & i & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & i & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2}
\end{bmatrix},$$

(109)

which is different from $U$ in section 3.1. We can apply the discussion in section 2.2 by replacing $U_3$ in section 2.2 with $U_8$. According to (36), we define $V_1 \equiv D_{8,\text{diag}} U_8^{-1} V U_8$ and $V_2 \equiv (U_8^{-1} V U_8)^2$. In particular, we focus on $V_2$:

$$V_2 = \frac{g^2 v^2}{4} \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & \sqrt{3} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & -2i & 0 \\
0 & 0 & 0 & 0 & 0 & 2i & 2 & 0 \\
0 & 0 & -\sqrt{3} & 0 & 0 & 0 & 0 & 3
\end{bmatrix}. 

(110)

From (92) with $v = 0$ and $V_2$, we find that the pair of $\psi_{l,m}^3$, $\psi_{l,m}^8$ and $\psi_{n+1,j}$, $\psi_{n,j}^7$ are degenerate, and we must solve the secular equations. As a result, the first-order perturbation energy $E^{(1)}$ has

$$E_{3'}^{(1)} = g^2 v^2, \quad E_{8'}^{(1)} = 0, \quad E_{6'}^{(1)} = 0, \quad E_{7'}^{(1)} = g^2 v^2, \quad E_{1'}^{(1)} = g^2 v^2, \quad E_{2'}^{(1)} = g^2 v^2, \quad E_{4'}^{(1)} = g^2 v^2, \quad E_{5'}^{(1)} = g^2 v^2.$$
where the mode functions in new direction $3'$ and $8'$ are defined as

\[
\psi_{l,m}' = (-\psi_{l,m}^3 + \sqrt{3}\psi_{l,m}^8)/2, \quad \psi_{l,m}^8 = (\sqrt{3}\psi_{l,m}^3 + \psi_{l,m}^8)/2,
\]

\[
\psi_{n,j}' = (i\psi_{n+1,j}^6 + \psi_{n,j}^7)/\sqrt{2}, \quad \psi_{n,j}^7 = (\psi_{n+1,j}^6 + i\psi_{n,j}^7)/\sqrt{2}.
\]

Thus, the masses of gauge fields can be obtained as

\[
-\mathcal{D}_{\text{diag}}\overline{\mathcal{D}}_{\text{diag}} \equiv m_A^2,
\]

\[
\begin{align*}
(m_A^2)^{11} &= 4\pi^2(l_1^2 + m_1^2) + \frac{1}{4}g^2v^2, \\
(m_A^2)^{22} &= 4\pi^2(l_2^2 + m_2^2) + \frac{1}{4}g^2v^2, \\
(m_A^2)^{33'} &= 4\pi^2(l_{3'}^2 + m_{3'}^2) + g^2v^2, \\
(m_A^2)^{44} &= \alpha_3n_4 + \frac{1}{4}g^2v^2, \\
(m_A^2)^{55} &= \alpha_3(n_5 + 1) + \frac{1}{4}g^2v^2, \\
(m_A^2)^{66'} &= \frac{1}{2}g^2v^2(n_6 = 0), \quad \alpha_3(n_6' + 1) (n_6' \geq 0), \\
(m_A^2)^{77'} &= \alpha_3(n_7' + 1) + g^2v^2, \\
(m_A^2)^{88'} &= 4\pi^2(l_{3'}^2 + m_{3'}^2).
\end{align*}
\]

where we ignore the second order perturbation energy.

**Figure 4:** A picture of the effective potential with $N = 3$. The shaded region represents a range where our perturbation analysis breaks down.

In previous discussion, the total one-loop effective potential has been expressed by $V = V_{\text{reg},A}$. From (115), we have

\[
V = \frac{3}{2} \left( \frac{1}{2\pi} \right)^2 \left( 2\nu_{\text{reg},l,m}(1/4) + \nu_{\text{reg},l,m}(1) + \nu_{\text{reg},n}(\alpha_3, 1/4; 0) \\
+ \nu_{\text{reg},n}(\alpha_3, 1/4; 1) + \nu_{\text{reg},n}(\alpha_3, 1; 1) + \nu_{\text{reg},0}(1/2) \right).
\]

(116)
The effective potential of SU(3) Yang-Mills theory with $N = 3$ is shown in Figure 4. The shape of the potential in Figure 4 is similar to that (the red line) in Figure 2. The minimum of the effective potential might have a nonzero constant VEV $v \neq 0$ and the gauge symmetry SU(2) $\times$ U(1) is broken to U(1).

We should comment here again as was done in SU(2) case that we cannot determine the location of the minimum since the location is expected to be beyond the range which our perturbation is valid as is shown in Figure 4. Even if the value of the minimum is not determined, if we assume that the potential is bounded below, the pattern of gauge symmetry breaking can be confirmed from the gauge boson spectrum.

4 Summary

We have studied six dimensional Yang-Mills theories compactified on a torus with a magnetic flux and a constant VEV in this paper. Before constructing realistic models, we have discussed simple models of SU(2) and SU(3) Yang-Mills theories to understand basic structures of the gauge symmetry breaking.

We first gave a setup of SU(2) model and derived the KK masses in terms of perturbation theory in quantum mechanics. By using the KK masses, we calculated the one-loop effective potential. In that computations, we focus on the integral representation of Hurwitz $\zeta$ function and the regularized effective potential could be obtained. From the obtained one-loop effective potential and the mass of the gauge fields, we have seen that the SU(2) gauge symmetry is completely broken because of the flux background and the constant VEV.

Next, we considered an SU(3) model where two types of directions to introduce the flux background and the constant VEV. The extension to SU(3) is necessary for WL scalar fields to be an SU(2) doublet in the SM. In the case of section 3.1, the flux background and the constant VEV were introduced in the 8-th and the 1-st components of SU(3) gauge symmetry, respectively. The one-loop effective potential was calculated and it was found that the potential has a nonzero VEV $v \neq 0$, which implies that the SU(3) gauge symmetry is broken to U(1) $\times$ U(1) by the flux background and the constant VEV. On the other hand, in the case of section 3.2, the flux background and the constant VEV were introduced in the 8-th and the 6-th components of SU(3) gauge symmetry, respectively.
This case corresponds to gauge-Higgs unification in that the constant VEV is taken in one of the component of the broken generators of the original symmetry, SU(3)/(SU(2)×U(1)) in our model. The one-loop effective potential was found to expect a nonzero VEV \( v \neq 0 \) and the SU(3) gauge symmetry is broken to U(1) by the flux background and the constant VEV.

Although the results obtained in this paper are very interesting, it cannot be realistic as it stands. If we identify the WL scalar field with the SM SU(2) Higgs doublet in our SU(3) model, the gauge symmetry breaking pattern SU(3) \( \rightarrow \) SU(2) × U(1) \( \rightarrow \) U(1)(or U(1)×U(1)) is not a correct pattern of the electroweak symmetry breaking SU(2) × U(1) \( \rightarrow \) U(1). In order to realize such an electroweak symmetry breaking, it would be interesting to take into account fermion field contributions to the one-loop effective potential.

In some models discussed in this paper, the location of the potential minimum could not be determined in a parameter region where our perturbation is valid. It would be desirable to obtain the value of the potential minimum in a perturbative region by extending our analysis to the models with fermions. These issues are left for our future work.

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**A The second-order perturbation energy**

In this appendix, we represent the second-order perturbation energy. Since \( V_2 \) has an order of \( \mathcal{O}(g^2v^2) \), the second-order perturbation energy from \( V_2 \) has an order of \( \mathcal{O}((g^2v^2)^2) \) and we neglect it.
A.1 The mass of the gauge field

The second-order perturbation energy from \( V_1 + V_1^\dagger \) for \( \psi_{0,j}^1 \) is

\[
E_{A,0}^{(2)} = - \sum_{(l,m)\neq(0,0)} \left| \int_{T^2} d^2x \left( \psi_{l,m}^3 \right)^\dagger (V_1 + V_1^\dagger) \psi_{0,j}^1 \right|^2 \frac{4\pi^2}{4\pi^2(l^2 + m^2)}
\]

\[
= - \frac{g^2v^2}{2} \left[ \sum_{(l,m)\neq(0,0)} \frac{4\pi^2(l^2 + m^2)}{4\pi^2(l^2 + m^2)} \left| \int_{T^2} d^2x \bar{\lambda}_{l,m} \xi_{0,j} \right|^2 \right]
\]

\[
\equiv - \frac{g^2v^2}{2} \left[ \sum_{(l,m)\neq(0,0)} |C_{l,m,0,j}|^2 \right] .
\]  

We do not describe the integrals \( C_{l,m,0,j} \) in detail.

The second-order perturbation energy from \( V_1 + V_1^\dagger \) for \( \psi_{n+1,j}^{1'}, \psi_{n+1,j}^{2'}, \psi_{l=0,m=0}^3 \) and \( \psi_{l \neq 0, m \neq 0}^3 \) are

\[
E_{A,1'}^{(2)} = - \sum_{l,m} \frac{\left| \int_{T^2} d^2x \left( \psi_{l,m}^3 \right)^\dagger (V_1 + V_1^\dagger) \psi_{n+1,j}^{1'} \right|^2}{4\pi^2(l^2 + m^2) - \alpha_2(n + 1)},
\]

\[
E_{A,2'}^{(2)} = - \sum_{l,m} \frac{\left| \int_{T^2} d^2x \left( \psi_{l,m}^3 \right)^\dagger (V_1 + V_1^\dagger) \psi_{n+1,j}^{2'} \right|^2}{4\pi^2(l^2 + m^2) - \alpha_2(n + 1)},
\]

\[
E_{A,3,l=0,m=0}^{(2)} = \sum_{n,j} \frac{\left| \int_{T^2} d^2x \left( \psi_{l,m}^3 \right)^\dagger (V_1 + V_1^\dagger) \psi_{n+1,j}^{1'} \right|^2}{4\pi^2(l^2 + m^2) - \alpha_2(n + 1)}
\]

\[
+ \sum_{n,j} \frac{\left| \int_{T^2} d^2x \left( \psi_{l,m}^3 \right)^\dagger (V_1 + V_1^\dagger) \psi_{n+1,j}^{2'} \right|^2}{4\pi^2(l^2 + m^2) - \alpha_2(n + 1)}
\]  

and

\[
E_{A,3,l \neq 0, m \neq 0}^{(2)} = \frac{g^2v^2}{2} \sum_{j} |C_{l,m,0,j}|^2 + \sum_{n,j} \frac{\left| \int_{T^2} d^2x \left( \psi_{l,m}^3 \right)^\dagger (V_1 + V_1^\dagger) \psi_{n+1,j}^{1'} \right|^2}{4\pi^2(l^2 + m^2) - \alpha_2(n + 1)}
\]

\[
+ \sum_{n,j} \frac{\left| \int_{T^2} d^2x \left( \psi_{l,m}^3 \right)^\dagger (V_1 + V_1^\dagger) \psi_{n+1,j}^{2'} \right|^2}{4\pi^2(l^2 + m^2) - \alpha_2(n + 1)}.
\]
A.2 The mass of the scalar field

The second-order perturbation energy from $V_1 + V_1^\dagger$, $E^{(2)}_\phi$ are

$$E^{(2)}_{\phi,1''} = -\sum_{l,m} \left| \int_{T^2} d^2x \left( \psi_{l,m}^3 \right)^\dagger \left( V_1 + V_1^\dagger \right) \psi_{n,j}^{1''} \right|^2 \frac{4\pi^2(l^2 + m^2) - \alpha_2(n + 1/2)}{4\pi^2(l^2 + m^2) - \alpha_2(n + 1/2)},$$ (122)

$$E^{(2)}_{\phi,2''} = -\sum_{l,m} \left| \int_{T^2} d^2x \left( \psi_{l,m}^3 \right)^\dagger \left( V_1 + V_1^\dagger \right) \psi_{n,j}^{2''} \right|^2 \frac{4\pi^2(l^2 + m^2) - \alpha_2(n + 1/2)}{4\pi^2(l^2 + m^2) - \alpha_2(n + 1/2)}$$ (123)

and

$$E^{(2)}_{\phi,3} = \sum_{n,j} \left| \int_{T^2} d^2x \left( \psi_{l,m}^3 \right)^\dagger \left( V_1 + V_1^\dagger \right) \psi_{n,j}^{1''} \right|^2 \frac{4\pi^2(l^2 + m^2) - \alpha_2(n + 1/2)}{4\pi^2(l^2 + m^2) - \alpha_2(n + 1/2)}$$ (124)

A.3 The mass of the ghost field

The second-order perturbation energy by $V_4 + V_4^\dagger$, $E^{(2)}_c$ are

$$E^{(2)}_{c,1''} = -\sum_{l,m} \left| \int_{T^2} d^2x \left( \psi_{l,m}^3 \right)^\dagger \left( V_4 + V_4^\dagger \right) \psi_{n,j}^{1''} \right|^2 \frac{4\pi^2(l^2 + m^2) - \alpha_2(n + 1/2)}{4\pi^2(l^2 + m^2) - \alpha_2(n + 1/2)},$$ (125)

$$E^{(2)}_{c,2''} = -\sum_{l,m} \left| \int_{T^2} d^2x \left( \psi_{l,m}^3 \right)^\dagger \left( V_4 + V_4^\dagger \right) \psi_{n,j}^{2''} \right|^2 \frac{4\pi^2(l^2 + m^2) - \alpha_2(n + 1/2)}{4\pi^2(l^2 + m^2) - \alpha_2(n + 1/2)}$$ (126)

and

$$E^{(2)}_{c,3} = \sum_{n,j} \left| \int_{T^2} d^2x \left( \psi_{l,m}^3 \right)^\dagger \left( V_4 + V_4^\dagger \right) \psi_{n,j}^{1''} \right|^2 \frac{4\pi^2(l^2 + m^2) - \alpha_2(n + 1/2)}{4\pi^2(l^2 + m^2) - \alpha_2(n + 1/2)}$$ (127)
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