Scattering in Two Black Hole Moduli Space

Kenji Sakamoto\(^1\) and Kiyoshi Shiraishi\(^2\)

Faculty of Science, Yamaguchi University
Yoshida, Yamaguchi-shi, Yamaguchi 753-8512, Japan

Abstract
In this work, we discuss the quantum mechanics on the moduli space consisting of two maximally charged dilaton black holes. We study the quantum effects resulting from the different structure of the moduli space geometry in the scattering process.

1 Introduction

Recently the study of black hole moduli space has attracted much attention. Quantum black holes have been studied by means of quantum fields or strings interacting with a single black hole. In the past few years the new quantum mechanics of an arbitrary number \(N\) of supersymmetric black holes has been focused. Configurations of \(N\) static black holes parametrize a moduli space. The low-lying quantum states of the system are governed by quantum mechanics on the moduli space. The effective theories of quantum mechanics on a moduli space of Reissner-Nordström multi-black holes were constructed in \([1]\). Recently, (super)conformal quantum mechanics are constructed on a moduli spaces of five-dimensional multi-black holes in the near-horizon limit in \([2]\).

The motivation behind these works is the hope that information of these quantum states will lead to the black hole entropy. The quantum states supported in the near-horizon region can be interpreted as internal states of black holes, and the number of such states is related to the black hole entropy. The other motivation is the expectation that investigation of these moduli spaces will lead to the understanding of \(\text{AdS}_2/\text{CFT}_1\) correspondence and unravel some novel features behind the quantum states in multi-black hole mechanics.

The geometry of black hole moduli spaces was first discussed by Ferrell and Eardley in four dimensions \([3]\). Further the black hole moduli spaces geometry with dilaton coupling in \(N+1\) dimensions was discussed by Shiraishi \([4]\). In these works the structure of the moduli space geometry is different according to dimensions and values of dilaton coupling.

In this work, we discuss quantum mechanics on the moduli space consisting of two maximally charged dilaton black holes. We study the quantum effect resulting from the different structure of the moduli space geometry in the scattering process. In the section 2, we discuss the moduli space structure of two black hole system with dilaton coupling in any dimensions. In the section 3, we study the quantum mechanics on this moduli space and the general view of potential in the (3+1) dimensions. In the section 4, we consider the scattering process on the moduli space. Then we discuss the quantum effects from the different structures of the moduli space geometry. In the section 5, we will give the conclusion and the discussion.

2 The Moduli Space Metric for the System Consisting of Maximally Charged Dilaton Black Holes

The Einstein-Maxwell-dilaton system contains a dilaton field \(\phi\) coupled to a \(U(1)\) gauge field \(A_\mu\) beside the Einstein-Hilbert gravity. In the \(N+1\) dimensions \((N \geq 3)\), the action for the fields with particle sources is

\[
S = \int d^{N+1}x \sqrt{-g} \left[ R - \frac{4}{N-1} (\nabla \phi)^2 - e^{-\left(\frac{4 \alpha}{(N-1)}\right)} \phi F^2 \right]
\]
where $R$ is the scalar curvature and $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. We set the Newton constant $G = 1$. The dilaton coupling constant $\alpha$ can be assumed to be a positive value.

The metric for the $N$-body system of maximally-charged dilaton black holes has been known as [4]

$$ds^2 = -U^{-2}(x)dt^2 + U^{2/(N-2)}(x)dx^2,$$

where

$$U(x) = (F(x))^{(N-2)/(N-2+a^2)},$$

$$F(x) = 1 + \sum_{i=1}^{n} \frac{\mu_i}{(N-1)|x-x_i|^{N-2}}.$$  

Using these expressions, the vector one form and dilaton configuration are written as

$$A = \sqrt{\frac{N-1}{2(N-2+a^2)}(F(x))^{-1} dt},$$

$$e^{-4\alpha \phi/(N-1)} = (F(x))^{2a^2/(N-2+a^2)}.$$  

In this solution, the asymptotic value of $\phi$ is fixed to be zero.

The electric charge $Q_i$ of each black hole are associated with the corresponding mass $m_i$ by

$$m_i = \frac{A_{N-1}(N-1)}{8\pi (N-2+a^2)} \mu_i,$$

$$|Q_i| = \sqrt{\frac{N-1}{2(N-2+a^2)} \mu_i},$$

where $A_{N-1} = 2\pi^{n/2}/\Gamma(\frac{1}{2}N)$.

The perturbed metric and potential can be written in the form

$$ds^2 = -U^{-2}(x)dt^2 + 2N dx dt + U^{2/(N-2)}(x)dx^2,$$

$$A = \sqrt{\frac{N-1}{2(N-2+a^2)}(F(x))^{-1} dt + A_{\mu} dx},$$

where $U(x)$ and $F(x)$ are defined by [3] and [4]. We have only to solve linearized equations with perturbed sources up to $O(v)$ for $N_i$ and $A_i$. (Here $v$ represents the velocity of the black hole as a point source.) We should note that each source plays the role of a maximally charged dilaton black hole.

Solving the Einstein-Maxwell equations and substituting the solutions, the perturbed dilaton field and sources to the action [3] with proper boundary terms, we get the effective Lagrangian up to $O(v^2)$ for $N$-maximally charged dilaton black hole system

$$L = -\sum_{i=1}^{n} m_i + \sum_{i=1}^{n} \frac{1}{2} m_i (v_i)^2 + \frac{(N-1)(N-2+a^2)}{16\pi (N-2+a^2)^2} \int d^N x (F(x))^{2(1-a^2)/(N-2+a^2)} \sum_{i,j} \frac{(n_i \cdot n_j)|v_i - v_j|^2 \mu_i \mu_j}{2|r_i|^{N-1}|r_j|^{N-1}},$$

where $r_i = x - x_i$ and $n_i = r_i/|r_i|$. $F(x)$ is defined by [4]. In general, a naive integration in equation (11) diverges. Therefore, we regularize that divergent terms proportional to $\int d^N x \delta^n(x)/|x|^p (p > 0)$ which
appear when the integrand is expanded must be regularized \[3\). We set them to zero. The prescription is equivalent to carrying out the following replacement in equation (11)

\[
(F(x))^{\frac{2(1-a^2)}{N-2+a^2}} \rightarrow -1 + \left[ 1 + \frac{8\pi(N-2+a^2)}{A_{N-1}(N-2)(N-1)|r_a|^{N-2}} m_a \right]^{\frac{2(1-a^2)}{N-2+a^2}} + \left[ 1 + \frac{8\pi(N-2+a^2)}{A_{N-1}(N-2)(N-1)|r_b|^{N-2}} m_b \right]^{\frac{2(1-a^2)}{N-2+a^2}}.
\]

(12)

After regularization, the effective Lagrangian for two body system (consisting of black hole labeled with \(a\) and \(b\)) can be rewritten. From this rewriting effective Lagrangian, we obtain the metric of the \(N+1\) dimensional moduli space for two-body system as

\[
g_{ab} = \gamma(r)\delta_{ab},
\]

(13)

with

\[
\gamma(r) = 1 - \frac{M}{\mu} - \frac{8\pi(N-a^2)}{A_{N-1}(N-2)(N-1)r_{ab}^{N-2}} M + \frac{M}{m_a} \left( 1 + \frac{8\pi(N-2+a^2)}{A_{N-1}(N-2)(N-1)r_{ab}^{N-2}} m_a \right)^{(N-a^2)/(N-2+a^2)} + \frac{M}{m_b} \left( 1 + \frac{8\pi(N-2+a^2)}{A_{N-1}(N-2)(N-1)r_{ab}^{N-2}} m_b \right)^{(N-a^2)/(N-2+a^2)},
\]

(14)

where \(M = m_a + m_b\), \(\mu = m_a m_b / M\), and \(r_{ab} = |x_a - x_b|\).

### 3 Quantum mechanics in two-black hole moduli space

We consider the quantum mechanics on moduli space. The quantization of moduli parameters has been discussed in \[3\).

Let us introduce a wave function \(\Psi\) on the moduli space, which obeys the Schrödinger equation

\[
i\hbar \frac{d\Psi}{dt} = \left( -\hbar^2 \nabla^2 + \hbar^2 R_{(MS)} \right) \Psi,
\]

(15)

where \(\nabla^2\) is the covariant Laplacian constructed from the moduli space metric and \(R_{(MS)}\) is the scalar curvature of the moduli space. We assume \(\xi = 0\) in this paper though this term may be present in most general case.

To simplify, we fix the case of the \((3+1)\) dimensional case. The partial wave in a stationary state is

\[
\Psi = \psi_m(r)Y_{lm}(\theta, \phi) \exp(-iEt/\hbar),
\]

(16)

where \(Y_{lm}(\theta, \phi)\) is the spherical harmonic function and \(E = \hbar^2 q^2 / (2\mu)\). We redefine the variables as

\[
R = \int \sqrt{\gamma} dr,
\]

\[
\psi = \frac{\chi}{r \sqrt{\gamma}}.
\]

(17)

(18)

The Schrödinger equation (15) is rewritten as

\[
\frac{d^2\chi}{dR^2} + (q^2 - V)\chi = 0,
\]

(19)
where the potential $V$ is

$$V = \frac{\gamma(r\gamma')' - r(\gamma')^2}{2r\gamma'} + \frac{l(l+1)}{r^2\gamma}. \quad (20)$$

Here $'$ stands for $\frac{d}{dr}$.

To study the scattering process, we find the general view of potential on the moduli space. The potential for dilaton coupling $a^2 = 0$, $1/3$ and $1$ are plotted in Fig.1.

![Potential plots for different $a^2$ values](image)

Figure 1: The potential $V$ as the function of $r$ and angular momentum $l$, A for $a^2 = 0$, B for $a^2 = 1/3$ and C for $a^2 = 1$.

In the case of $a^2 = 0$ for all values of $l$ and $a^2 = 1/3$ for $l = 0$, the potential has a maximal value. Then the incoming particle into two black hole moduli space are scattered or coalesced in the scattering process. On the other hand, in the case of $a^2 = 1/3$ for all values of $l \neq 1$ and $a^2 = 1$ all values of $l$, the incoming particle are always scattered away. We notice that the value of $a^2 = 1/3$ is the critical point of moduli structures [4]. For simplicity, we will study the scattering-away processes in the case of $a^2 = 1/3$ and $a^2 = 1$.

4 Scattering on two-black hole moduli space

We consider the case of $a^2 = 1/3$ and $a^2 = 1$. As the first study of quantum effects, we discuss the scattering process in the WKB approximation. For the WKB approximation, the phase shift in the scattering process are

$$\delta_l = -qR_0 + \int_{R_0}^{\infty} dR \left( \sqrt{q^2 - V(R) - \frac{1}{4R^2}} - q \right) + \frac{2l+1}{4} - \pi, \quad (21)$$

where $R_0$ is the solution of $V(R_0) = q^2 - \frac{1}{4R_0^2}$.

The partial cross section are

$$\sigma_l = \frac{4\pi}{q^2} (2l+1) \sin^2 \delta_l. \quad (22)$$

The deflection angle are

$$\Theta = \pi + \int_{R_0}^{\infty} dR \frac{\partial}{\partial l} \sqrt{q^2 - V(R) - \frac{1}{4R^2}}. \quad (23)$$

The phase shift $\delta_l$ and the deflection angle $\Theta$ for $q = 0.1$ and $0.2$ are plotted in Fig.2. The solid line in Fig.2 A2, B2 represents the classical deflection angle which are obtained from the impact parameter. For $a^2 = 1$, the effects for the different value of $q$ are not found, and the deflection angle are almost correspond to the classical deflection angle. Although, for $a^2 = 1/3$, we can find that the behaviours of the phase shift and the deflection angle depend on the incoming particle energy $q$. Then the deflection angle are different from the classical deflection angle at the small value of $l/q$. These effects are interpreted as the quantum effects arisen by the difference of the moduli space structures in the scattering process. Although we used the WKB semi-classical approximation in this analysis, the quantum effects can be obtained sufficiently.
5 Conclusion

In this work, we studied the quantum effects on the different moduli structure. We obtained the black hole moduli space metrics in any dimensions. In the (3+1) dimensions, we considered the quantum mechanics on the moduli space, and investigated the potential for $a^2 = 0, 1/3$ and 1. For the process in which the incoming particle are always scattered away, $a^2 = 1/3$ and 1, we obtained the phase shift and the deflection angle in the WKB approximation. Then we revealed the quantum effects in scattering process for $a^2 = 1/3$. We considered that these effects due to the difference in the moduli space structure. So, the moduli space structure affect to the quantum scattering process.

In the further study, we will consider the behavior in the different dimensions and for the other values of dilaton coupling $a^2$. The moduli space structures are different in the other dimensions and at the other values of dilaton coupling $a^2$. In (4+1) dimensions for $a^2 = 1$, the moduli space structure are resemble the structure in (3+1) dimensions for $a^2 = 1/3$. Then we expect that the similar quantum effects are also arisen. In this work, we consider the quantum effects in the WKB approximation. So, we will more precisely study in quantum mechanically analysis in further study.

References

[1] G. W. Gibbons and P. J. Ruback, Phys. Rev. Lett. 57, 1492 (1986).
[2] J. Michelson, and A. Strominger, JHEP 09, 005 (1999).
[3] F. Ferrell, and D. Eardley, Phys. Rev. Lett. 59, 1617 (1987).
[4] K. Shiraishi, Nucl. Phys. B402, 399 (1993).
[5] L. Infeld and J. Plebański, Motion and Relativity (Pergamon Press, 1960).
[6] J. Traschen and R. Ferrell, Phys. Rev. D45, 2628 (1992).