THE CANONICAL MODULE OF GT–VARIETIES AND THE NORMAL BUNDLE OF RL–VARIETIES.

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Abstract. In this paper, we study the geometry of GT–varieties $X_d$ with group a finite cyclic group $\Gamma \subset \text{GL}(n+1, \mathbb{K})$ of order $d$. We prove that the homogeneous ideal $I(X_d)$ of $X_d$ is generated by binomials of degree at most 3 and we provide examples reaching this bound. We give a combinatorial description of the canonical module of the homogeneous coordinate ring of $X_d$ and we show that it is generated by monomial invariants of $\Gamma$ of degree $d$ and $2d$. This allows us to characterize the Castelnuovo-Mumford regularity of the homogeneous coordinate ring of $X_d$. Finally, we compute the cohomology table of the normal bundle of the so called RL–varieties. They are projections of the Veronese variety $\nu_d(\mathbb{P}^n) \subset \mathbb{P}^{(\binom{n+d}{n}-1)}$ which naturally arise from level GT–varieties.

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1. Introduction

Through this paper, $\mathbb{K}$ denotes an algebraically closed field of characteristic zero, $R = \mathbb{K}[x_0, \ldots, x_n]$ and $\text{GL}(n+1, \mathbb{K})$ denotes the group of invertible matrices of size $(n+1) \times (n+1)$ with coefficients in $\mathbb{K}$.

In [20], Mezzetti, Miró-Roig and Ottaviani related the existence of homogeneous artinian ideals $I \subset R$ generated by homogeneous forms $F_1, \ldots, F_r$ of degree $d$ failing the weak Lefschetz property in degree $d−1$ to the existence of rational projective varieties of $\mathbb{P}^{(\binom{n+d}{n})−r−1}$

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satisfying a Laplace equation. They called $I$ a Togliatti system. Since then Togliatti systems have been extensively studied as one can see in [1], [6], [9], [18], [19], [21] and [22].

Any Togliatti system $I$ induces a morphism $\varphi_I : \mathbb{P}^n \to \mathbb{P}^{r-1}$ defined as $(F_1, \ldots, F_r)$, its image $\varphi_I(\mathbb{P}^n)$ is called the variety parameterized by $I$. In [19], the authors introduced a new family of Togliatti systems parameterizing varieties with a special geometric property. They called GT-system with cyclic group $\mathbb{Z}/d\mathbb{Z}$ any Togliatti system parameterizing a Galois covering with group $\mathbb{Z}/d\mathbb{Z}$. GT-systems and the varieties parameterized by them have been subsequently studied in [7], [9] and [6], in the latter reference the authors applied invariant theory methods to tackle them.

To be more precise, fix integers $2 \leq n < d$ and $e$ a $d$th primitive root of 1 in $\mathbb{K}$. We denote by $M_{d;\alpha_0,\ldots,\alpha_n}$ the diagonal matrix $\text{diag}(e^{\alpha_0}, \ldots, e^{\alpha_n})$, where $0 \leq \alpha_0 \leq \cdots \leq \alpha_n < d$ are integers such that $\text{GCD}(\alpha_0, \ldots, \alpha_n, d) = 1$ and $\alpha_i < \alpha_j$ for some $i \neq j$. We set $\Gamma := \langle M_{d;\alpha_0,\ldots,\alpha_n} \rangle \subset \text{GL}(n+1, \mathbb{K})$ a finite cyclic group of order $d$ and $R^\Gamma = \{ p \in R \mid g(f) = f, \ \forall g \in \Gamma \}$ the ring of invariants of $\Gamma$. The cyclic extension $\overline{\Gamma} \subset \text{GL}(n+1, \mathbb{K})$ of $\Gamma$ is the finite abelian group of order $d^2$ generated by $M_{d;\alpha_0,\ldots,\alpha_n}$ and $M_{d;1,\ldots,1} = \text{diag}(e, \ldots, e)$. The ring of invariants of $\overline{\Gamma}$ is $R^{\overline{\Gamma}} = \{ p \in R^\Gamma \mid \deg(p) = td, 0 \leq t \}$, often called the $d$th Veronese subalgebra of $R^\Gamma$. In [6] it is proved that $R^{\overline{\Gamma}} = \mathbb{K}[m_1, \ldots, m_{\mu_d}]$, where $m_1, \ldots, m_{\mu_d}$ are all the monomial invariants of $\Gamma$ of degree $d$, and it is shown that the ideal $I_d = (m_1, \ldots, m_{\mu_d})$ is a GT-system. They called GT-variety with group $\Gamma$ the variety $X_d := \varphi_{I_d}(\mathbb{P}^n)$ parameterized by $I_d$ and they started to investigate its geometry. In [13], Gröbner posed the problem of determining whether a monomial projection of the Veronese variety $\nu_d(\mathbb{P}^n) \subset \mathbb{P}^{\binom{n+d}{d}-1}$ parameterized by the set $\mathcal{M}_{n,d} \subset R$ of all monomials of degree $d$ is an arithmetically Cohen Macaulay (shortly aCM) variety. Motivated by this long-standing problem, the authors of [6] proved that any GT-variety $X_d$ is an aCM variety showing that the homogeneous coordinate ring of $X_d$ is isomorphic to $R^{\overline{\Gamma}}$. They also tackled the problem of finding a minimal free resolution of the homogeneous ideal $I(X_d)$ of $X_d$, which they determined for all $GT$-surfaces. Recently in [7], the notions of GT-system and GT-variety have been generalized to any finite group acting on $R$, non necessarily cyclic or even abelian.

In this paper, we address three topics regarding GT-varieties. The first two questions concern explicitly the geometry of any GT-variety $X_d$, in contrast with the last one, which deals with the cohomology of the normal bundle of some smooth rational varieties naturally arising from GT-varieties. First, we find a set of generators of the homogeneous ideal $I(X_d)$ of $X_d$. We prove that $I(X_d)$ is generated by homogeneous binomials of degree at most 3 (Theorem 3.11). We exhibit examples of homogeneous ideals of GT-varieties reaching this bound, which also show that it depends on the group $\Gamma$. Second, we determine the algebraic structure of the canonical module $\omega_{X_d}$ of the homogeneous coordinate ring of $X_d$. 
We identify $\omega_{X_d}$ with the ideal $\text{relint}(I_d) = \langle x_0^{a_0} \cdots x_n^{a_n} \mid 0 \neq a_0 \cdots a_n \rangle$ of $R^\Gamma$ generated by the relative interior of the semigroup ring $K[m_1, \ldots, m_{\mu_d}]$ (Proposition 4.1). We prove that $\text{relint}(I_d)$ is generated by monomials of degree $d$ and $2d$ (Theorem 4.2). This connection allows us to compute the Castelnuovo-Mumford regularity of $R^\Gamma$.

Finally, we introduce a new family of smooth rational monomial projections of the Veronese variety $\nu_d(\mathbb{P}^n) \subset \mathbb{P}^{(n+d)\binom{n+d}{n}-1}$ naturally associated to level $GT$-varieties. A $GT$-variety $X_d$ is called level if the ideal $\text{relint}(I_d)$ is generated only by monomials of degree $d$ and, hence, $R^\Gamma$ is a level ring. An $RL$-variety $X_d$ associated to a level $GT$-variety $X_d$ is a monomial projection of the Veronese variety $\nu_d(\mathbb{P}^n) \subset \mathbb{P}^{(n+d)\binom{n+d}{n}-1}$ parameterized by the set of monomials $M_{n,d} \setminus \{m = x_0^{a_0} \cdots x_n^{a_n} \in \text{relint}(I_d) \mid \deg(m) = d\}$. The name $RL$-variety is conceived to emphasize the relation with the relative interior and the levelness. We give examples of $RL$-varieties in any dimension. Inspired by the recent work of Alzati and Re ([3]), we contribute to the classical open problem of computing the cohomology of the normal bundle of smooth rational varieties (Theorem 5.9). Most results and examples of this topic focus on smooth rational curves and surfaces, see for instance [2], [10] and [25]. We determine the cohomology table of the normal bundle of any $RL$-variety, shading new light on higher dimensions.

Let us see how this work is organized. In Section 2 we gather the basic definitions and results needed in the body of this paper. Section 3 is entirely devoted to find a set of homogeneous binomial generators of the homogeneous ideal $I(X_d)$ of any $GT$-variety $X_d$. We establish that the homogeneous coordinate ring of $X_d$ is isomorphic to $R^\Gamma$ and that $I(X_d)$ is a homogeneous prime binomial ideal. Our main result (Theorem 3.11) proves that $I(X_d)$ is generated by binomials of degree at most 3. We give families of examples of $GT$-varieties whose homogeneous ideals are minimally generated by binomials of degree 2 and 3. In Section 4 we investigate the algebraic structure of the canonical module $\omega_{X_d}$ of the homogeneous coordinate ring of $X_d$. We identify $\omega_{X_d}$ with the ideal $\text{relint}(I_d)$ of $R^\Gamma$, which gives us a combinatorial description of $\omega_{X_d}$. In Theorem 4.2 we show that $\text{relint}(I_d)$ is generated by monomials of degree $d$ and $2d$. We further study $GT$-varieties $X_d$ where $R^\Gamma$ is a level ring, i.e. the canonical module $\text{relint}(I_d)$ of $R^\Gamma$ is generated in only one degree. In particular, we study varieties whose homogeneous coordinate ring $R^\Gamma$ is level and $\text{relint}(I_d)$ is minimally generated in degree $d$. Afterwards in Theorem 4.10 we characterize the Castelnuovo-Mumford regularity of $R^\Gamma$.

Finally in Section 5 we introduce the notions of a level $GT$-variety and its associated $RL$-variety and we give examples of any dimension. Using the new methods of [3], we compute the cohomology table of the normal bundle of any $RL$-variety (see Theorem 5.9).
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2. Preliminaries

In this section, we introduce the main objects and results we use in the body of this paper. First, we define semigroups and normal semigroups, we relate them to invariant theory of finite groups and we see a geometrical interpretation of these objects. For more details the reader can look at [14], [4] and [27]. Finally, we define the weak Lefschetz property, we recall the notions of $GT$–systems and $GT$–varieties and we collect some basic results on this topic.

Semigroup rings and rings of invariants. By a semigroup we mean a finitely generated additive subsemigroup $H = \langle h_1, \ldots, h_t \rangle \subset \mathbb{Z}^{n+1}$. $L(H)$ is the additive subgroup of $\mathbb{Z}^{n+1}$ generated by $H$. We denote by $\mathbb{K}[H] \subseteq R$ the semigroup ring associated to $H$, i.e. the graded $\mathbb{K}$–algebra generated by the monomials $X^{h_j} = x_0^{a_0^j} \cdots x_n^{a_n^j} \in R$ associated to the points $h_j = (a_0^j, \ldots, a_n^j) \in H$, $j = 1, \ldots, t$.

Definition 2.1. A semigroup $H \subset \mathbb{Z}^{n+1}$ is called normal if it satisfies the following condition: if $zh \in H$ for some $h \in L(H)$ and $0 \neq z \in \mathbb{Z}_{\geq 0}$, then $h \in H$.

A large family of normal semigroups comes from invariant theory, precisely those associated to finite abelian groups acting on $R$. To be more precise, let $\Lambda \simeq \mathbb{Z}/d_1 \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_r \mathbb{Z}$ and choose $d_i$-th primitive roots $e_i$ of $1 \in \mathbb{K}$, $i = 1, \ldots, r$. Therefore $\Lambda$ can be linearly represented in $\text{GL}(n+1, \mathbb{K})$ by means of $r$ diagonal matrices $\text{diag}(e_i^{b_0,i}, \ldots, e_i^{a_{n,i}})$, where $u_{j,i} \in \mathbb{Z}_{\geq 0}$, $0 \leq j \leq n$, $1 \leq i \leq r$. Let $R^\Lambda := \{p \in R \mid \lambda(p) = p \text{ for all } \lambda \in \Lambda\}$ be the ring of invariants of $\Lambda$ acting on $R$. Since $\Lambda$ acts diagonally, each monomial $x_0^{a_0} \cdots x_n^{a_n} \in R$ is mapped into a multiple of itself by every $\lambda \in \Lambda$, and a polynomial $p \in R^\Lambda$ if and only if all its monomials are invariants of $\Lambda$. Thus, by the Noether’s degree bound (see [27 Theorem 2.1.4]), $R^\Lambda$ has a finite basis consisting of monomials of degree at most the order of $\Lambda$. By a basis of $R^\Lambda$ we mean a set of elements $\{\theta_1, \ldots, \theta_t\} \subset R^\Lambda$ which minimally generates $R^\Lambda$ as a $\mathbb{K}$–algebra, i.e. $R^\Lambda = \mathbb{K}[\theta_1, \ldots, \theta_t]$. Let $X^{h_1}, \ldots, X^{h_t}$ be a monomial basis of $R^\Lambda$ and $H = \langle h_1, \ldots, h_t \rangle$. Then $R^\Lambda = \mathbb{K}[H]$. Furthermore, a monomial $x_0^{a_0} \cdots x_n^{a_n} \in R^\Lambda$ if and only if $(a_0, \ldots, a_n)$ satisfies the system of congruences:

\begin{equation}
  a_0u_{0,i} + \cdots + a_nu_{n,i} \equiv 0 \pmod{d_i}, \quad i = 1, \ldots, r.
\end{equation}

Now, if $w \in L(H)$ is such that $zw \in H$ for some $z \in \mathbb{Z}_{\geq 0}$, then $w \in H$, so $H$ is normal. By [14, Theorem 1] or [15, Proposition 13], $\mathbb{K}[H]$ is Cohen Macaulay.

More generally, let $G \subset \text{GL}(n+1, \mathbb{K})$ be a finite group. Geometrically, the ring $R^G$ of invariants can be regarded as the coordinate ring of the quotient of $\mathbb{A}^{n+1}$ by $G$. To be
more precise, set \{f_1, \ldots, f_t\} a basis of \(R^G\), often called a set of fundamental invariants of \(G\), and let \(\mathbb{K}[w_1, \ldots, w_t]\) be the polynomial ring in the new variables \(w_1, \ldots, w_t\). Then the quotient of \(\mathbb{A}^{n+1}\) by \(G\) is given by the morphism \(\pi: \mathbb{A}^{n+1} \to \pi(\mathbb{A}^{n+1}) \subset \mathbb{A}^t\), such that \(\pi(a_0, \ldots, a_n) = (f_1(a_0, \ldots, a_n), \ldots, f_t(a_0, \ldots, a_n))\). Even further, \(\pi\) is a Galois covering of \(\pi(\mathbb{A}^{n+1})\) with group \(G\). For further details on quotients varieties we refer the reader to \[24\]. The ideal \(I(\pi(\mathbb{A}^{n+1}))\) of the quotient variety is called the ideal of syzygies among the invariants \(f_1, \ldots, f_t\); it is the kernel of the homomorphism defined by \(w_i \mapsto f_i, i = 1, \ldots, t\). We denote it by \(\text{syz}(f_1, \ldots, f_t)\). We summarize all these facts in the following proposition.

**Proposition 2.2.** Let \(G \subset \text{GL}(n + 1, \mathbb{K})\) be a finite linear group, \(\{f_1, \ldots, f_t\}\) be a set of fundamental invariants and let \(\pi: \mathbb{A}^{n+1} \to \mathbb{A}^t\) be the induced morphism. Then,

(i) \(\pi(\mathbb{A}^{n+1})\) is the quotient of \(\mathbb{A}^{n+1}\) by \(G\) with affine coordinate ring \(R^G\).

(ii) \(R^G \cong \mathbb{K}[w_1, \ldots, w_t]/\text{syz}(f_1, \ldots, f_t)\).

(iii) \(\pi\) is a Galois covering of \(\pi(\mathbb{A}^{n+1})\) with group \(G\).

**Proof.** See \[26, Section 6\].

The cardinality of a general orbit \(G(a), a \in \mathbb{A}^{n+1}\), is called the degree of the covering. Moreover, if we can find a homogeneous set of fundamental invariants \(\{f_1, \ldots, f_t\}\) of \(G\) such that \(\pi: \mathbb{P}^n \to \mathbb{P}^{t-1}\) is a morphism, then the projective version of Proposition 2.2 is true.

**GT–systems and GT–varieties.** Let \(I \subset R\) be a homogeneous artinian ideal. We say that \(I\) has the weak Lefschetz property (WLP) if there is a linear form \(L \in R_1\) such that, for all integers \(j\), the multiplication map \(\times L: (R/I)^{j-1} \to (R/I)^j\) has maximal rank. In \[20\], Mezzetti, Miró-Roig and Ottaviani proved that the failure of the WLP is related to the existence of varieties satisfying at least one Laplace equation of order greater than 2. More precisely, they proved:

**Theorem 2.3.** Let \(I \subset R\) be an artinian ideal generated by \(r\) forms \(F_1, \ldots, F_r\) of degree \(d\) and let \(I^{-1}\) be its Macaulay inverse system. If \(r \leq \binom{n+d-1}{n-1}\), then the following conditions are equivalent:

(i) \(I\) fails the WLP in degree \(d - 1\);

(ii) \(F_1, \ldots, F_r\) become \(\mathbb{K}\)–linearly dependent on a general hyperplane \(H\) of \(\mathbb{P}^n\);

(iii) the \(n\)-dimensional variety \(Y = \varphi(\mathbb{P}^n), \varphi: \mathbb{P}^n \to \mathbb{P}^{(n+d-r-1)}\) is the rational morphism associated to \((I^{-1})_d\), satisfies at least one Laplace equation of order \(d - 1\).

**Proof.** See \[20, Theorem 3.2\].
Motivated by the above results, Mezzetti, Miró-Roig and Ottaviani introduced the following definitions (see [20] and [18]):

**Definition 2.4.** Let $I \subset R$ be an artinian ideal generated by $r \leq \binom{n+d-1}{n-1}$ forms of degree $d$. We say that:

(i) $I$ is a *Togliatti system* if it fails the WLP in degree $d - 1$.

(ii) $I$ is a *monomial Togliatti system* if, in addition, $I$ can be generated by monomials.

In particular, a Togliatti system is called *smooth* if the variety $Y$ in Theorem 2.3(iii) is smooth. The name is in honour of Togliatti who proved that for $n = 2$ the only smooth Togliatti system of cubics is

$$I = (x_0^3, x_1^3, x_2^3, x_0x_1x_2) \subset \mathbb{K}[x_0, x_1, x_2]$$

([28] and [29]). The systematic study of Togliatti systems was initiated in [20] and it has been continuing in [19], [18], [1], [22] and [21]. Precisely in [18], it was introduced the notion of *GT-system* with group a finite cyclic group. Recently in [8], this notion has been generalized as follows.

**Definition 2.5.** A *GT-system* with a finite group $G$ is an artinian ideal $I_d \subset R$ generated by $r$ forms $F_1, \ldots, F_r$ of degree $d$ such that:

(i) $I_d$ is a Togliatti system.

(ii) The morphism $\varphi_{I_d} : \mathbb{P}^n \to \mathbb{P}^{r-1}$ defined by $(F_1, \ldots, F_r)$ is a Galois covering with group $G$.

If conditions (i) and (ii) holds, we say that $\varphi_{I_d}(\mathbb{P}^n)$ is a *GT-variety with group* $G$.

GT-systems with group a finite cyclic group have been extensively studied in [19], [7] and [6], while in [8], the authors investigate GT-systems with the dihedral group acting on $\mathbb{K}[x_0, x_1, x_2]$. In the last two references, invariant theory techniques have been applied to tackle both objects. Fix $G \subset \text{GL}(n+1, \mathbb{K})$ a finite group of order $d$. Assume that the ring $R^G$ has a basis $\mathcal{B}$ formed by homogeneous invariants of $G$ of degree $d$ and set $I_d$ the ideal generated by $\mathcal{B}$. Keeping this notation, we have the following.

**Proposition 2.6.** If $|\mathcal{B}| \leq \binom{d+n-1}{n-1}$, then $I_d$ is a GT-system with group $G$.

**Proof.** Since $I_d$ contains a homogeneous system of parameters of $R^G$, it is an artinian ideal. By Proposition 2.2, the associated morphism $\varphi_{I_d}$ is a Galois covering with group $G$. By Theorem 2.3, it is enough to prove that $I_d$ fails the WLP in degree $d - 1$, i.e. for any linear form $L \in R_1$, the multiplication map $\times L : (R/I_d)d-1 \to (R/I_d)d$ is not injective. Let $L \in R_1$ and consider $F := \prod_{g \in G} g(L) \in R_d$. We have that $L \cdot F = \prod_{g \in G} g(L) \in R^G$ and, hence, $F \in \ker(\times L)$. \qed
Let us see an illustrative example.

**Example 2.7.** (i) Fix $n = 2$, $d = 5$ and $e$ a 5th primitive root of 1. The finite cyclic group $\mathbb{Z}/5\mathbb{Z}$ can be linearly represented by $\Gamma = \langle \text{diag}(1, e, e^3) \rangle \subset \text{GL}(3, \mathbb{K})$. All the monomial invariants of $\Gamma$ of degree 5 are: $x_0^5, x_0^2x_1^3, x_0^2x_1x_2, x_0x_1x_2^2, x_0^2x_2^3, x_1^2$ (see [6, Example 2.15]). In total we have $r = 5$ monomials. The inequality $r \leq \binom{n+d-1}{n-1}$ is satisfied and the ideal $I_5 \subset R$ generated by them fails the WLP in degree 4. The morphism $\varphi_{I_5} : \mathbb{P}^2 \to \mathbb{P}^4$ is a Galois covering of degree 5 with group $\Gamma$ (see [6, Corollary 3.4] and [19, Theorem 3.4]). Actually, $\varphi_{I_5}(\mathbb{P}^2)$ is the quotient surface of $\mathbb{P}^2$ by $\Gamma$.

(ii) Fix $n = 3$, $d = 4$ and $e$ a 4th primitive root of 1. The diagonal matrix $\text{diag}(1, e, e^2, e^3)$ generates a finite cyclic subgroup $\Gamma \subset \text{GL}(4, \mathbb{K})$ of order 4. There are exactly $r = 10$ monomial invariants of $\Gamma$ of degree 4: $x_0^4, x_0^2x_1x_3, x_0^2x_2, x_0x_2x_3, x_1^4, x_1^2x_3, x_1x_2^2x_3, x_2^4, x_3^4$ (see [9, Example 3.2]). The inequality $r \leq \binom{n+d-1}{n-1}$ is satisfied and the ideal $I_4$ generated by them fails the WLP in degree 3. The associated morphism $\varphi_{I_4} : \mathbb{P}^3 \to \mathbb{P}^9$ is a Galois covering with group $\Gamma$ (see [9, Proposition 3.3] and [6, Corollary 3.4]). Similarly to (i), $\varphi_{I_4}(\mathbb{P}^3)$ is the quotient threefold of $\mathbb{P}^3$ by $\Gamma$.

In [6], [8], and previously in [9], the authors focus on the geometry of $GT$–varieties with group a finite cyclic group or a dihedral group. In [6, Theorem 3.2], it is proved that all $GT$–varieties with group a finite cyclic group are aCM varieties and in [8, Proposition 4.3], it is shown its analogous for $GT$–surfaces with a dihedral group. Furthermore, in [6, Theorem 4.14] and [8, Theorem 4.6], it is determined a minimal free resolution of $GT$–surfaces with group a finite cyclic group and a dihedral group, respectively. Reference [9] is devoted to find a minimal set of homogeneous binomial generators of the homogeneous ideal of certain $GT$–threefolds with group a finite cyclic group. Our goal is to extend the results obtained so far for $GT$–surfaces and $GT$–threefolds to arbitrary $n$-dimensional $GT$–varieties with group a finite cyclic group.

### 3. On the homogeneous ideal of $GT$–varieties

In this section, we look for a system of generators of the homogeneous ideal of $GT$–varieties with group a finite cyclic group. Using combinatorial techniques, we prove that all these ideals are generated by homogeneous binomials of degree 2 and 3. We begin introducing some notations.

**Notation 3.1.** Fix integers $2 \leq n < d$ and $e$ a $d$th primitive root of 1 $\in \mathbb{K}$. We denote by $M_{d; \alpha_0, \ldots, \alpha_n}$ the diagonal matrix $\text{diag}(e^{\alpha_0}, \ldots, e^{\alpha_n})$, where $0 \leq \alpha_0 \leq \cdots \leq \alpha_n < d$ are integers such that $\text{GCD}(d, \alpha_0, \ldots, \alpha_n) = 1$ and $\alpha_i < \alpha_j$ for some $i \neq j$. 
Along this section, we fix a finite cyclic group $\Gamma := \langle M_{d,\alpha_0,\ldots,\alpha_n} \rangle \subset \text{GL}(n + 1, \mathbb{K})$ of order $d$ and we consider the ring $R^\Gamma$ of invariants of $\Gamma$ endowed with the natural grading $R^\Gamma = \bigoplus_{t \geq 1} R_t^\Gamma$, $R_t^\Gamma := R_t \cap R^\Gamma$. The cyclic extension $\Gamma^\Gamma \subset \text{GL}(n + 1, \mathbb{K})$ of $\Gamma$ is the finite abelian group of order $d^2$ generated by $M_{d,\alpha_0,\ldots,\alpha_n}$ and $M_{d,1,\ldots,1}$. We consider the ring $R^\Gamma$ of invariants of $\Gamma^\Gamma$ with the grading $R^\Gamma = \bigoplus_{t \geq 1} R_t^\Gamma$, $R_t^\Gamma := R_{td}^\Gamma \cap R^\Gamma$, often called the $d$th Veronese subalgebra of $R^\Gamma$. We denote by $\mathcal{M}_d := \{m_1, \ldots, m_{\mu_d}\} \subset R$ the set of all monomial invariants of $\Gamma$ of degree $d$, ordered lexicographically. We denote $I_d \subset R$ the monomial artinian ideal generated by $\mathcal{M}_d$. By $\varphi_{I_d} : \mathbb{P}^n \to \mathbb{P}^{\mu_d-1}$, we denote the associated morphism and we set $X_d := \varphi_{I_d}(\mathbb{P}^n) \subset \mathbb{P}^{\mu_d-1}$ its image. In [6], it is established the following.

**Theorem 3.2.** (i) $\mathcal{M}_d$ is a basis of $R^\Gamma$.

(ii) $R^\Gamma$ is the coordinate ring of $X_d$. Hence, $X_d$ is an aCM monomial projection of the Veronese variety $\nu_d(\mathbb{P}^n) \subset \mathbb{P}^{(n+d)}_{n-1}$ from the inverse system $I_d^{-1}$.

(iii) If $\mu_d \leq \binom{d+n-1}{n-1}$, then $I_d$ is a GT-system with group $\Gamma$. In this case, we call $X_d$ a GT-variety with group $\Gamma$. 

**Remark 3.3.** Through the rest of this paper, if the condition $\mu_d \leq \binom{d+n-1}{n-1}$ is satisfied, we refer to $I_d$ as a GT-system and to $X_d$ as a GT-variety.

**Example 3.4.** (i) Fix integers $2 \leq n$ and fix $\Gamma = \langle M_{n+1,0,1,2,\ldots,n} \rangle \subset \text{GL}(n + 1, \mathbb{K})$ (see [6] Example 3.6(iii))). In [7] Theorem 4.8, it is proved that the condition $\mu_{n+1} \leq \binom{2n}{n-1}$ is satisfied, thus $I_d$ is a monomial Togliatti system. By Theorem 3.2(iii), $I_d$ is a GT-system.

(ii) Fix integers $2 = n < d$ and fix $\Gamma = \langle M_{d,0,\alpha_1,\alpha_2} \rangle \subset \text{GL}(3, \mathbb{K})$ (see [19] and [6] Example 3.6(i))). In [19] Theorem 3.4, it is proved that the condition $\mu_d \leq \binom{n-1+d}{n-1}$ is satisfied, thus $I_d$ is a Togliatti system. By Theorem 3.2(iii), $I_d$ is a GT-system.

Let $w_1, \ldots, w_{\mu_d}$ be new variables and set $S := \mathbb{K}[w_1, \ldots, w_{\mu_d}]$. We denote by $I(X_d) \subset S$ the homogeneous ideal of $X_d$. By Proposition 2.2, $I(X_d)$ is the kernel of the morphism $\rho : S \to R$ given by $\rho(w_i) = m_i$. It holds that $I(X_d)$ is the homogeneous binomial prime ideal generated by

$$\mathcal{W}_d = \{w_{i_1} \cdots w_{i_k} - w_{j_1} \cdots w_{j_k} \in S \text{ such that } m_{i_1} \cdots m_{i_k} = m_{j_1} \cdots m_{j_k}, \ k \geq 2\}.$$ 

For any $k \geq 2$, we denote by $\mathcal{W}_{d,k}$ the set of all binomials of $\mathcal{W}_d$ of degree exactly $k$. Our goal is to prove that $I(X_d)$ is generated by binomials of degree 2 and 3, i.e. the ideal $I(X_d) = (\mathcal{W}_{d,2}, \mathcal{W}_{d,3})$. Through families of examples in [3, 8] we observe that this bound is sharp. We start with some definitions.

**Definition 3.5.** Fixed $k \geq 2$, we define a suitable $k$-binomial to be a non-zero binomial $w_\alpha = w_{\alpha_+} - w_{\alpha_-} := \prod_{i=1}^{k} w_{i_1} - \prod_{i=1}^{k} w_{j_1} \in \mathcal{W}_{d,k}$, i.e. $\prod_{i=1}^{k} m_{i_1} = \prod_{i=1}^{k} m_{j_1}$. 
Definition 3.6. Given a suitable $k$–binomial $w^\alpha = w^{\alpha+} - w^{\alpha-} \in \mathcal{W}_{d,k}$, we denote by supp$(w^\alpha)$ (respectively supp$\_$(w$^\alpha$)) the support of the monomial $w^{\alpha+}$ (respectively support of $w^{\alpha-}$). We say that $w^\alpha$ is non trivial if supp$(w^\alpha) \cap$ supp$\_$(w$^\alpha$) $\neq \emptyset$. Otherwise, we say that $w^\alpha$ is trivial.

Definition 3.7. Let $w^\alpha = w^{\alpha+} - w^{\alpha-} \in \mathcal{W}_{d,k}$ be a non trivial suitable $k$–binomial. By an $I(X_d)_k$–sequence from $w^{\alpha+}$ to $w^{\alpha-}$ we mean a finite sequence $(w^1, \ldots, w^t)$ of monomials of $S$ satisfying the following two conditions:

(i) $w^1 = w^{\alpha+}, w^t = w^{\alpha-}$ and 
(ii) For all $1 \leq j < t$, $w^j - w^{j+1}$ is a trivial suitable $k$–binomial.

Example 3.8. Let $\Gamma = \langle M_{6,0,1,2,3} \rangle \subset \text{GL}(4, \mathbb{K})$ be a finite group of order 6. There are $\mu_6 = 16$ monomial invariants of $\Gamma$, we have:

$$
\mathcal{M}_6 = \{x_0^6, x_0^4x_3^2, x_0^3x_1x_2x_3, x_0^3x_3^3, x_0^2x_1^2x_2, x_0^2x_2^2x_3, x_0x_1^4x_2, x_0x_1x_2x_3^3, x_0x_2^3x_3, x_1^3x_3, x_1^2x_2x_3^2, x_1x_2^2x_3, x_2x_3^3\}.
$$

By Theorem 3.2(iii), the ideal $I_6$ generated by them is a $GT$–system and its associated variety $X_6$ is a $GT$–variety. The homogeneous binomials $w_1w_{15} - w_4^2$ and $w_3w_{12}w_{15} - w_6w_9w_{14}$ are non trivial suitable binomials of degree 2 and 3, respectively. Indeed, $\rho(w_1w_{15}) = \rho(w_1)\rho(w_{15}) = (x_0^6)(x_0^6) = (x_0^3x_3^3)^2 = \rho(w_1)^2$ and on the other hand $\rho(w_3)\rho(w_{12})\rho(w_{15}) = (x_0^3x_1x_2x_3)(x_1^3x_3^3)(x_2^6) = (x_0^2x_1^2x_2)(x_0x_1x_2x_3^3)(x_1x_3^3x_3) = \rho(w_6)\rho(w_9)\rho(w_{14})$. Finally, $\{w_3w_{12}w_{15}, w_5w_9w_{15}, w_6w_9w_{14}\}$ is an $I(X_6)_3$–sequence from $w_3w_{12}w_{15}$ to $w_6w_9w_{14}$.

The following result characterizes the inclusion of ideals $\langle \mathcal{W}_{d,k} \rangle \subset \langle \mathcal{W}_{d,k-1} \rangle$, $k \geq 3$. It is key to prove our main results, which we state later.

Proposition 3.9. Fix $k \geq 3$ and let $w^\alpha = w^{\alpha+} - w^{\alpha-} \in \mathcal{W}_{d,k}$ be a suitable $k$–binomial. Then $w^\alpha \in \langle \mathcal{W}_{d,k-1} \rangle$ if and only if there exists an $I(X_d)_k$–sequence from $w^{\alpha+}$ to $w^{\alpha-}$.

Proof. We apply the same arguments as in [9 Proposition 5.4].

We need the following lemma:

Lemma 3.10. Any sequence of $2d-1$ integers in $\{0, \ldots, d-1\}$ contains some subsequence of $d$ integers the sum of which is a multiple of $d$.

Proof. See [11 Theorem].

Theorem 3.11. Let $\Gamma = \langle M_{d,a_0,\ldots,a_n} \rangle \subset \text{GL}(n+1, \mathbb{K})$ be a cyclic group of order $d$ and $X_d \subset \mathbb{P}^{\mu_d-1}$ the variety parameterized by the ideal $I_d = \langle m_1, \ldots, m_{\mu_d} \rangle$ generated by all monomial invariants of $\Gamma$ of degree $d$. Then, the homogeneous ideal $I(X_d)$ of $X_d$ is generated by quadrics and cubics. Precisely, $I(X_d) = \langle \mathcal{W}_{d,2}, \mathcal{W}_{d,3} \rangle$. 


Proof. First, we prove that for all $k \geq 4$, any non trivial suitable $k-$binomial admits an $I(X_d)_k-$sequence. By Proposition 3.9 this implies $\langle W_{d,k} \rangle \subset \langle W_{d,k-1} \rangle$. Fix $k \geq 4$ and let $w^\alpha = w^{\alpha_+} - w^{\alpha_-} = w_{i_1} \cdots w_{i_k} - w_{j_1} \cdots w_{j_k}$ be a non trivial suitable $k-$binomial. For each $w_{i_l}$ (respectively $w_{j_l}$), let $m_{i_l} = x_{i_0}^{a_{i_1}} \cdots x_{i_n}^{a_{i_k}}$ be its associated monomial (respectively $m_{j_l} = x_{j_0}^{b_{j_1}} \cdots x_{j_n}^{b_{j_k}}$, $l = 1, \ldots, k$. We have that

$$\sum_{l=1}^k a_{s}^{i_l} = \sum_{l=1}^k b_{s}^{j_l}, \quad 0 \leq s \leq n.$$ 

We consider the monomials $m_{i_l}, m_{j_l}$ and for each $0 \leq s \leq n$ we define:

$$c_s = \begin{cases} 
0 & \text{if } a_{s}^{i_1} > b_{s}^{j_1} \\
 b_{s}^{i_1} - a_{s}^{j_1} & \text{otherwise.}
\end{cases}$$

This gives rise to a non-zero monomial $m = x_{i_0}^{c_{i_1}} \cdots x_{i_n}^{c_{i_k}} \in R$ of degree strictly smaller than $d$. Clearly, $m$ divides $m_{i_2} \cdots m_{i_k}$ (see (2)). Thus we consider $m' = (m_{i_2} \cdots m_{i_k})/m$, which is a monomial of degree at least $(k-2)d \geq 2d$. We write $m' = x_{j_0}^{a_{j_1}} \cdots x_{j_n}^{a_{j_k}}$. We define the sequence of integers $L = (\alpha_0, \ldots, \alpha_n, \ldots, \alpha_n, \ldots, \alpha_n, \ldots, \alpha_n, \ldots, \alpha_n)$. Since $L$ has length at least $(k-2)d \geq 2d$, by Lemma 3.10 we can find a subsequence $L' = (\alpha_0, \ldots, \alpha_n, \ldots, \alpha_n, \ldots, \alpha_n) \subset L$ of $d$ elements whose sum is a multiple of $d$, which implies that the monomial $x_{i_0}^{c_{i_1}} \cdots x_{i_n}^{c_{i_k}} \in R^\alpha$. By Theorem 3.2(i), we can decompose

$$m_{i_2} \cdots m_{i_k} = m_{i_2} \cdots m_{i_k},$$

where all $m_{i_l} \in R^\alpha$, $2 \leq l \leq k$, are monomials of degree $d$ and in particular:

$$m_{i_k} = x_{j_0}^{a_{j_1}} \cdots x_{j_n}^{a_{j_k}}.$$ 

Notice that we have $m_{i_2} \cdots m_{i_k} = m_{i_2} m_{i_2} \cdots m_{i_k}$. We define $w^2 \in S$ to be the monomial $\rho^{-1}(m_{i_2}) \rho^{-1}(m_{i_2}) \cdots \rho^{-1}(m_{i_k})$. By construction, $w^{\alpha_+} - w^2$ is a trivial suitable $k-$binomial. Observe that $m = x_{i_0}^{c_{i_1}} \cdots x_{i_n}^{c_{i_k}}$ divides $m_{i_2} \cdots m_{i_k}$, thus $m_{j_1}$ divides $m_{i_2} m_{i_2} \cdots m_{i_{k-1}}$. Applying the same argument as before, we factorize

$$m_{i_1} m_{i_2} \cdots m_{i_{k-1}} = m_{i_1} m_{i_2} \cdots m_{i_{k-1}},$$

where $m_{i_1} = m_{j_1}$ and all $m_{i_l} \in R^\alpha$, $2 \leq l \leq k-1$, are monomials of degree $d$. We set

$$w^3 = \rho^{-1}(m_{i_1}) \rho^{-1}(m_{i_2}) \cdots \rho^{-1}(m_{i_k}) \cdot \rho^{-1}(m_{i_1}).$$

Since $m_{i_1} m_{i_2} \cdots m_{i_{k-1}} m_{i_1} = m_{i_1} m_{i_2} \cdots m_{i_k} m_{i_1}$, $w^3 - w^2$ is a suitable trivial $k-$binomial. Furthermore, since $m_{i_1} = m_{j_1}$, also $w^3 - w^{\alpha_-}$ is a trivial suitable $k-$binomial. Therefore $(w_{i_1} \cdots w_{i_n}, w^2, w_{j_1} \cdots w_{j_n})$ is an $I(X_d)_k-$sequence, from which it follows that $\langle W_{d,k} \rangle \subset \langle W_{d,k-1} \rangle$. The argument we have developed only requires that $(k-2)d \geq 2d$, which it is satisfied for all $k \geq 4$. Thus we have proved that for all $k \geq 4$, 

$$\langle W_{d,k} \rangle \subset \langle W_{d,k-1} \rangle \subset \cdots \subset \langle W_{d,3} \rangle,$$

which completes the proof. \qed
Despite \( \mathcal{W}_{d,2} \) always belongs to a minimal set of homogeneous binomial generators of \( I(X_d) \), it is not the case for \( \mathcal{W}_{d,3} \). Even further, this fact depends on the action of the group \( \Gamma = \langle M_{d,0,\alpha_1,\ldots,\alpha_n} \rangle \), as we illustrate in the following examples.

**Example 3.12.** (i) Fix integers \( 2 = n < d \) and fix \( \Gamma = \langle M_{d,0,\alpha_1,\alpha_2} \rangle \subset \text{GL}(3, \mathbb{K}) \), a finite cyclic group of order \( d \) with \( \alpha_1 < \alpha_2 \). By [6 Corollary 4.16], the homogeneous ideal \( I(X_d) \) of the \( \text{GT} \)–surface \( X_d \) is minimally generated by binomials of degree 2 and 3 if \( \text{GCD}(\alpha_1, d) = \text{GCD}(\alpha_2, d) = \text{GCD}(\alpha_2 - 1, d) = 1 \) and \( I(X_d) \) is minimally generated by binomials of degree 2 otherwise.

(ii) Fix integers \( 3 = n < d \) and fix \( \Gamma = \langle M_{d,0,1,2,3} \rangle \subset \text{GL}(4, \mathbb{K}) \). In [3 Corollary 5.7], it is determined a minimal set of homogeneous binomials generators of the homogeneous ideal \( I(X_d) \). Precisely, it is proved that \( I(X_d) \) is minimally generated by binomials of degree 2 if \( d \) is even and \( I(X_d) \) is minimally generated by binomials of degree 2 and 3 if \( d \) is odd.

(iii) Fix integers \( 2 \leq n < d \) and fix \( \Gamma = \langle M_{d,0,\alpha_1,\ldots,\alpha_n} \rangle \subset \text{GL}(n + 1, \mathbb{K}) \). If there exist integers \( 0 < \alpha_i < \alpha_j \) such that \( \text{GCD}(\alpha_i, d) = \text{GCD}(\alpha_j, d) = \text{GCD}(\alpha_j - 1, d) = 1 \), then \( I(X_d) \) is minimally generated by binomials of degree 2 and 3. Indeed, let \( V_d \) be the \( \text{GT} \)–surface associated to the action of the cyclic group \( \Lambda := \langle M_{d,0,\alpha_1,\alpha_j} \rangle \subset \text{GL}(3, \mathbb{K}) \) of order \( d \) acting on \( \mathbb{K}[x_0, x_1, x_j] \). Under a suitable identification of variables, we have that the homogeneous ideal \( I(V_d) = I(X_d) \cap \mathbb{K}[\rho^{-1}(x_0), \rho^{-1}(x_i), \rho^{-1}(x_j)] \). Therefore, if a minimal set of homogeneous binomial generators of \( I(V_d) \) contains a binomial of degree 3, the same holds for \( I(X_d) \).

4. **The canonical module of \( \text{GT} \)–varieties**

The algebraic structure of the canonical module \( \omega_X \) of the homogeneous coordinate ring of an aCM projective variety \( X \) plays a central role in its geometry (see [4, 3, 16]). For example, it can lead us to derive information on the Hilbert function and series, as well as on the Castelnuovo-Mumford regularity, of the homogeneous coordinate ring of \( X \). Let \( \Gamma = \langle M_{d,0,\alpha_1,\ldots,\alpha_n} \rangle \subset \text{GL}(n + 1) \) be a cyclic group of order \( d \) (see Notation 3.1). We denote by \( I_d = (m_1, \ldots, m_{\mu_d}) \subset R \) the ideal generated by all monomial invariants of \( \Gamma \) of degree \( d \), \( \varphi_{I_d} : \mathbb{P}^n \to \mathbb{P}^{\mu_d-1} \) the morphism induced by \( I_d \) and \( X_d = \varphi_{I_d}(\mathbb{P}^n) \subset \mathbb{P}^{\mu_d-1} \) its image. \( X_d \) is an aCM projective variety and its homogeneous coordinate ring \( S/I(X_d) \) is isomorphic to \( R^\Gamma \) (see Theorem 3.2). In this section, we deal with the canonical module \( \omega_{X_d} \) of \( S/I(X_d) \). We identify \( \omega_{X_d} \) with an ideal of \( R^\Gamma \) and we prove that it is generated by monomials of degree \( d \) and \( 2d \). We focus on varieties \( X_d \) the homogeneous coordinate rings \( S/I(X_d) \) of which are level ring, i.e. their canonical modules \( \omega_{X_d} \) are generated in only one degree. Afterwards, we characterize the Castelnuovo-Mumford regularity of \( R^\Gamma \).

Let \( m = x_0^{a_0} \cdots x_n^{a_n} \in R \) be a monomial, we say that \( l_m := (a_0, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^{n+1} \) is the lattice point associated to \( m \). Conversely, given a lattice point \( l = (a_0, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^{n+1} \),
we say that $m_l := x_0^{a_0} \cdots x_n^{a_n}$ is the monomial associated to $l$. We denote $H_d \subset \mathbb{Z}_{\geq 0}^n$ the
semigroup generated by the lattice points $l_1, \ldots, l_{m_d}$ associated to $M_d := \{m_1, \ldots, m_{m_d}\}$. By Theorem 1.1, we
have that $R^\Gamma = \mathbb{K}[H_d]$ and the semigroup $H_d$ coincides with the set of
all solutions $(a_0, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n$ of the systems:
\[
(*)_{t,r} = \begin{cases} 
  y_0 + y_1 + \cdots + y_n &= td \\
  \alpha_0 y_0 + \alpha_1 y_1 + \cdots + \alpha_n y_n &= rd 
\end{cases} 
\]
Therefore $H_d$ is a normal semigroup (see Section 2) with $H_d^+ := \{(a_0, \ldots, a_n) \in H_d \mid a_i > 0, i = 0, \ldots, n\} \neq \emptyset$, for instance $(a, \ldots, d) \in H_d^+$. It holds that $H_d^+$ coincides with the
relative interior of $H_d$. We set $\operatorname{relint}(I_d) := (x_0^{a_0} \cdots x_n^{a_n} \in R^\Gamma \mid 0 \neq a_0 \cdots a_n)$ the ideal of $R^\Gamma$
generated by all monomials associated to $H_d^+$. We have the following.

**Proposition 4.1.** $\operatorname{relint}(I_d)$ is the canonical module of $R^\Gamma$.

**Proof.** See [3] Theorem 6.3.5(b)]

For a complete exposition of the canonical module of normal semigroup rings we refer the reader to [4].

We denote by $C_{d,k} \subset \operatorname{relint}(I_d)$ the set of all monomials of degree $kd$. We have:

**Theorem 4.2.** For any cyclic group $\Gamma = \langle M_{d;0a_0, \ldots, a_n} \rangle \subset \operatorname{GL}(n+1, \mathbb{K})$ of order $d$, $\operatorname{relint}(I_d) = \langle C_{d,1}, C_{d,2} \rangle$.

**Proof.** It is enough to show that for any monomial $m \in C_{d,k}$, $k \geq 3$, there exists a monomial $m' \in C_{d,k-1}$ which divides $m$. This proves that for $k \geq 3$, $\langle C_{d,k} \rangle \subset \langle C_{d,k-1} \rangle \subset \cdots \subset \langle C_{d,2} \rangle$.

We fix an integer $k \geq 3$, a monomial $m = x_0^{a_0} \cdots x_n^{a_n} \in C_{d,k}$ and we set $m_1 = m/(x_0 \cdots x_n) = x_0^{a_0-1} \cdots x_n^{a_n-1}$. Since $d \geq n + 1$ and $k \geq 3$, $m$ is a monomial of degree $kd - (n + 1) \geq 2d$. We define the sequence of integers $L = (a_0, a_0^{-1}, a_0, \ldots, a_n, a_n^{-1}, a_n)$, by [11] Theorem] there exists a subsequence $L' = (a_0, b_0, a_0, \ldots, a_n, b_n, a_n) \subset L$ of $d$ integers the sum of which
is a multiple of $d$. Therefore, $L'$ gives rise a monomial $m_2 := x_0^{b_0} \cdots x_n^{b_n} \in R^\Gamma$ of degree $d$
divides $m_1$. Hence, we can factorize $m = m_2m'$ and by construction $m' \in C_{d,k-1}$ is the
required monomial. \(\Box\)

**Example 4.3.** Let $\Gamma = \langle M_{6;0,1,2,3} \rangle \subset \operatorname{GL}(4, \mathbb{K})$ be a cyclic group of order 6. We have that $C_{6,1} = \{x_0^3 x_1 x_2 x_3, x_0 x_1 x_2 x_3^3\}$ (see Example 3.8) and

$$
C_{6,2} = \{x_0^9 x_1 x_2 x_3, x_0^7 x_1 x_2 x_3^3, x_0^5 x_1 x_2 x_3^5, x_0^3 x_1 x_2 x_3^7, x_0 x_1 x_2 x_3^9, x_0^8 x_1 x_2 x_3^2, x_0^6 x_1 x_2 x_3^4, x_0^4 x_1 x_2 x_3^6, x_0^2 x_1 x_2 x_3^8, x_0 x_1 x_2 x_3^10\}.
$$
Only the following four monomials \( x_0^2x_1^4x_2^3, x_0^2x_1^4x_2^2x_3, x_0^3x_2^3x_3, x_0x_1^6x_2^3x_3, x_0x_1^5x_3^5x_3, x_0x_1x_2x_3^3 \) belong to the ideal \( \langle C_{0,1} \rangle \subset R^\Gamma \). From this observation and Theorem 4.2 we obtain that the canonical module of \( R^\Gamma \) is the ideal:

\[
\text{relint}(I_b) = \langle x_0^3x_1x_2x_3, x_0^2x_1^4x_2^3, x_0^2x_1^4x_2^2x_3, x_0^3x_2^3x_3, x_0x_1^6x_2^3x_3, x_0x_1^5x_3^5x_3, x_0x_1x_2x_3^3 \rangle.
\]

In [9] a minimal free resolution of \( R^\Gamma \) is computed. Set \( S := \mathbb{K}[w_1, \ldots, w_{16}] \) (see Example 3.8), precisely we have:

\[
0 \to S(-14)^4 \oplus S(-15)^2 \to S(-13)^{108} \oplus S(-14)^7 \to S(-12)^{803} \to S(-11)^{2850} \to S(-10)^{6237} \to \\
\to S(-9)^{9064} \to S(-7)^6 \oplus S(-8)^{8811} \to S(-6)^{258} \oplus S(-7)^{5352} \to S(-5)^{844} \oplus S(-6)^{1638} \to \\
S(-4)^{796} \oplus S(-5)^{184} \to S(-3)^{322} \oplus S(-4)^{13} \to S(-2)^5 \to S \to S/I(X_b) \to 0,
\]

verifying as well Theorem 4.2.

We recall that \( R^\Gamma \) is a level ring if its canonical module \( \text{relint}(I_d) \) is generated in only one degree and \( R^\Gamma \) is a Gorenstein ring if it is a level ring and \( \text{relint}(I_d) \) is principal. As a consequence of Theorem 4.2, we have that \( R^\Gamma \) is a level ring if and only if \( \text{relint}(I_d) = \langle C_{d,1} \rangle \) or \( C_{d,1} = \emptyset \). In [9] Corollary 4.13(ii)], it is shown that the homogeneous coordinate ring of any \( GT \)–surface is level. However, the same assertion is not true for \( GT \)–varieties of higher dimensions, see for instance [9]. We investigate further this property, which will play an important role in the last section of this work. Let us first present an interesting family of examples of Gorenstein \( GT \)–varieties.

**Proposition 4.4.** Fix an even integer \( 2 \leq n \leq 100 \) and \( \Gamma = \langle M_{n+1,0,1,2,\ldots,n} \rangle \subset \text{GL}(n+1, \mathbb{K}) \), a cyclic group of order \( n+1 \). Then \( R^\Gamma \) is a Gorenstein ring.

**Proof.** Notice that \( m = x_0 \cdots x_n \in R^\Gamma \). Indeed \( m \) is of degree \( n+1 \) and it is satisfied that \( 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \), which is a multiple of \( n+1 \) since \( n \) is even. It is straightforward to see that \( m \) divides any monomial of \( \text{relint}(I_{n+1}) \). Hence \( \text{relint}(I_{n+1}) = (x_0 \cdots x_n) \) and the proof is complete. \( \square \)

**Proposition 4.5.** Fix integers \( 2 \leq n < d, 1 \leq k \) and assume that \( n \) is even. Let \( \Gamma = \langle M_{d,0,0,\ldots,a_n} \rangle \subset \text{GL}(n+1, \mathbb{K}) \) and \( \Gamma_k = \langle M_{kd,0,0,\ldots,a_n} \rangle \subset \text{GL}(n+1, \mathbb{K}) \) be finite cyclic groups of order \( d \) and \( kd \), respectively. If \( R^\Gamma \) is a Gorenstein ring, then \( R^{\Gamma_k} \) is a level ring.

**Proof.** We denote \( I_d \subset R \) (respectively \( I_{kd} \subset R \)) the ideal generated by all monomials of degree \( d \) (respectively \( kd \)) which are invariants of \( \Gamma \) (respectively \( \Gamma_k \)). We write \( \text{relint}(I_d) = \langle m \rangle \). We want to prove that any monomial \( m' \in C_{d,2kd} \) is divisible by a monomial \( \bar{m} \in C_{d,kd} \). We fix \( m' = x_0^{a_0} \cdots x_n^{a_n} \in C_{d,2kd} \). Notice that \( m' \) is also an invariant of \( \Gamma \), so \( m' \in C_{d,k} \) and
by hypothesis \( m \) divides \( m' \). We define \( m_1 = \frac{m'}{m} \); since \( m_1 \in R^\Gamma \) is a monomial of degree \((2k-1)d\), by Theorem \ref{thm:main}\,(i), there are monomials \( m_2, \ldots, m_{2k} \in R^\Gamma \) of degree \( d \) such that \( m_1 = m_2 \cdots m_{2k} \), and hence \( m' = mm_2 \cdots m_{2k} \).

For each monomial \( m_i, 1 \leq i \leq 2k \), there is a unique integer \( r_i \geq 0 \) such that the lattice point \( l_{m_i} \) is a solution of the system \((*)_{1,r_i}\) induced by \( \Gamma \). By Lemma \ref{lem:subsequence} there is a subsequence \( \langle r_{i_1}, \ldots, r_{i_k} \rangle \subset \langle r_1, \ldots, r_{2k} \rangle \) of \( k \) integers the sum of which is a multiple of \( k \). Therefore, we obtain that \( m_{i_1} \cdots m_{i_k} \in R^{\Gamma_k} \) and \( \overline{m} = m'/(m_{i_1} \cdots m_{i_k}) \in \text{relint}(I_{kd}) \) is the required monomial.

\begin{corollary}
\textbf{Corollary 4.6.} Fix integers \( 2 \leq n, 1 \leq k \) with \( n \) even and fix \( \Gamma_k = \langle M_{k(n+1),0,1,2,\ldots,n} \rangle \subset \text{GL}(n+1, \mathbb{K}) \), a finite cyclic group of order \( k(n+1) \). Then \( R^{\Gamma_k} \) is a level ring.
\end{corollary}

\begin{proof}
It follows directly from Propositions \ref{prop:level} and \ref{prop:level2}.
\end{proof}

The rest of this section concerns the Castelnuovo-Mumford regularity \( \text{reg}(R^\Gamma) \) of \( R^\Gamma \) with \( \Gamma = \langle M_{d,a_0,\ldots,a_n} \rangle \subset \text{GL}(n+1, \mathbb{K}) \) a cyclic group of order \( d \) as in Notation \ref{notation:Gamma}. We characterize \( \text{reg}(R^\Gamma) \) in terms of \( \text{relint}(I_d) \). First we need some preparation.

\begin{definition}
\textbf{Definition 4.7.} A set \( \{y_0, \ldots, y_n\} \subset R^\Gamma \) of \( n \) homogeneous elements is said to be a \textit{homogeneous system of parameters}, shortly h.s.o.p, if \( R^\Gamma \) is a finitely generated \( \mathbb{K}[y_0, \ldots, y_n] \)-module.
\end{definition}

For sake of completeness we prove the following.

\begin{proposition}
\textbf{Proposition 4.8.} \( \{x_0^d, \ldots, x_n^d\} \) is an h.s.o.p of \( R^\Gamma \).
\end{proposition}

\begin{proof}
We consider the graded quotient algebra \( A := R^\Gamma / \langle x_0^d, \ldots, x_n^d \rangle = \bigoplus_{t \geq 1} A_t, \quad A_t = R^\Gamma / \langle x_0^d, \ldots, x_n^d \rangle_{td} = R_{td}^\Gamma / \langle x_0^d, \ldots, x_n^d \rangle_{td} \). Therefore, for \( t \geq n+1 \) we have that \( A_t = \{0\} \) and for \( 1 \leq t \leq n \), a \( \mathbb{K} \)-basis of \( A_t \) is formed by the set of all monomials \( m = x_0^{a_0} \cdots x_n^{a_n} \in R^\Gamma \) of degree \( td \) such that \( a_0 < d, \ldots, a_n < d \). We write \( \theta_1, \ldots, \theta_D \) the set of all such monomials and \( \theta_0 = 1 \). Then, it is clear that \( R^\Gamma = \langle \theta_0, \theta_1, \ldots, \theta_D \rangle \) as a \( \mathbb{K}[x_0^d, \ldots, x_n^d] \)-module.

We call \( \{\theta_0, \theta_1, \ldots, \theta_D\} \) a set of \textit{secondary invariants} of \( \Gamma \). For \( 1 \leq i \leq D \), we denote \( \delta_i = \text{deg}(\theta_i) \). We set \( e_0 = 1 \) and we define \( e_j, j = 1, \ldots, n \), the multiplicities of the sequence of degrees \( (\delta_1, \delta_1, \ldots, \delta_D) \). Notice that \( e_1 = \mu_d - (n + 1) \). Moreover, we have the following.

\begin{proposition}
\textbf{Proposition 4.9.} (i) The number of secondary invariants of \( R^\Gamma \) is \( D + 1 = d^{n-1} \).

(ii) The Hilbert series \( \text{HS}(R^\Gamma, z) \) of the ring \( R^\Gamma \) is

\begin{equation}
\text{HS}(R^\Gamma, z) = \frac{1 + e_1 z + \cdots + e_n z^n}{(1 - z)^{n+1}}.
\end{equation}

In particular, the degree of \( X_d \) is \( d^{n-1} \).
\end{proposition}

\begin{proof}
See \cite[Proposition 2.3.6]{27}.
\end{proof}
Theorem 4.10. With the above notation,

\[ n \leq \text{reg}(R^\Gamma) \leq n + 1. \]

The equality \( \text{reg}(R^\Gamma) = n + 1 \) holds if and only if \( C_{d,1} \neq \emptyset \).

Proof. The right inequality follows immediately from [3]. We set \( m = x_0^d \cdots x_n^d \) and \( m' = x_0^{d-1} \cdots x_n^{d-1} \). Lemma 3.10 assures the existence of a monomial of degree \((n - 1)d\) in \( R^\Gamma \) dividing \( m' \), and hence it assures the existence of secondary invariants of degrees smaller or equal to \((n - 1)d\), so \( e_{n-1} > 0 \) which gives us the left inequality. Now, \( \text{reg}(R^\Gamma) = n + 1 \) if and only if \( e_n > 0 \). If \( e_n > 0 \), then there exists a secondary invariant \( \theta \) of degree \( nd \) and we obtain \( m/\theta \in C_{d,1} \). Conversely, let \( p = x_0^{a_0} \cdots x_n^{a_n} \in C_{1,d} \). Notice that necessarily \( a_i < d \), \( i = 0, \ldots, n \), thus \( m/p \) is a secondary invariant of degree \( nd \).

To end this section, we present some examples illustrating the last results. They also bring to light how the Hilbert series and regularity of \( R^\Gamma \) can be deduced by just looking at the set of invariants of \( \Gamma \) of degree smaller or equal to \( nd \), and vice versa.

Example 4.11. (i) Fix integers \( 2 = n < d \) and fix \( \Gamma = \langle M_{d,0,\alpha_1,\alpha_2} \rangle \subset \text{GL}(3, \mathbb{K}) \) with \( 0 < \alpha_1 < \alpha_2 \). Let \( \lambda, 0 < \mu \leq \frac{d}{\text{GCD}(\alpha_1, d)} \) be the uniquely determined integers satisfying \( \alpha_2 = \frac{\lambda \alpha_1}{\text{GCD}(\alpha_1, d)} + \frac{\mu}{\text{GCD}(\alpha_1, d)} \). In [6] Proposition 4.12, it is proved that

\[ \text{HS}(R^\Gamma, z) = \frac{d - \theta(\alpha_1, \alpha_2, d) + 2 z^2 + d - \theta(\alpha_1, \alpha_2, d) - 4 z}{(1 - z)^3}, \]

where \( \theta(\alpha_1, \alpha_2, d) = \text{GCD}(\alpha_1, d) + \text{GCD}(\lambda, d') + \text{GCD}(\lambda - 1, d') \). By Proposition 4.9 there are exactly \( \frac{d - \theta(\alpha_1, \alpha_2, d) - 4}{2} \) secondary invariants of degree \( d \) and \( \frac{d - \theta(\alpha_1, \alpha_2, d) + 2}{2} \) secondary invariants of degree \( 2d \). From this it can be easily deduced that \( \text{reg}(R^\Gamma) = 3 \).

(ii) Fix \( n = 3, d = 4 \) and fix \( \Gamma = \langle M_{4,0,1,2,3} \rangle \subset \text{GL}(4, \mathbb{K}) \). We denote \( I_4 \) its associated GT–system. The ideal \( I_4 \) is generated by the following 10 monomials

\[ x_0^4, x_0^2 x_2, x_0^2 x_1 x_3, x_0 x_1^2 x_2, x_0 x_1 x_3^2, x_1^4, x_1^2 x_2^2, x_1 x_2 x_3, x_2^4, x_3^4. \]

Therefore \( \text{relint}(I_4) = \langle C_{4,2} \rangle \) with \( C_{4,2} = \{ x_0^4 x_1 x_2^2 x_3, x_0^4 x_1^2 x_2 x_3, x_0^3 x_1 x_2^3, x_0^3 x_1 x_3^2, x_0^2 x_2^2 x_3^2, x_0^2 x_1 x_2 x_3^2, x_0 x_1^2 x_2 x_3^2, x_0 x_1 x_2^2 x_3^2, x_0 x_1 x_2 x_3 x_3^3 \} \). There are 6 and 9 secondary invariants of degree \( d \) and \( 2d \), respectively, and by Theorem 4.10 \( \text{reg}(R^\Gamma) = 3 \). By Proposition 4.9 the Hilbert series of \( R^\Gamma \) is the following

\[ \text{HS}(R^\Gamma, z) = \frac{9 z^2 + 6 z + 1}{(1 - z)^4}. \]

(iii) Let \( \Gamma = \langle M_{6,0,1,2,3} \rangle \subset \text{GL}(4, \mathbb{K}) \) be a cyclic group of order 6. The ideal \( I_6 \) is generated by 16 monomial invariants of \( \Gamma \) of degree 6 and the ideal \( \text{relint}(I_6) = \langle x_0^3 x_1 x_2 x_3, x_0^2 x_1^4 x_2^2 x_3^2, x_0^2 x_1 x_2^6 x_3, x_0 x_1^2 x_2^3 x_3^2, x_0 x_1 x_2^3 x_3^3 \rangle \) (see Examples 3.8 and 1.3). Therefore, \( R^\Gamma \) has 12
secondary invariants of degree $d$, it has 2 secondary invariants of degree $3d$ and by Theorem 4.10 or Example 4.3, $\text{reg}(R^\Gamma) = 4$. Hence from Proposition 4.9, we can deduce immediately that the Hilbert series of the ring $R^\Gamma$ is the following

$$HS(R^\Gamma, z) = \frac{2z^3 + 21z^2 + 12z + 1}{(1 - z)^3}.$$ 

(iv) Fix integers $2 \leq n$ and $d = n + 1$ with $n$ even, and fix $\Gamma = \langle M_{(n+1);0,1,2,...,n} \rangle \subset \text{GL}(n + 1, \mathbb{K})$. By Corollary 4.4, $R^\Gamma$ is Gorenstein and by Theorem 4.10, we obtain that $\text{reg}(\Gamma) = n + 1$.

5. Cohomology of normal bundles of RL–varieties

In this section, we introduce a new family of smooth rational monomial projections of the Veronese variety $\nu_d(\mathbb{P}^n) \subset \mathbb{P}^{\binom{n+d}{n} - 1}$ which naturally arises from level GT–varieties (see Definition 5.1) and we study their canonical modules. We called them RL–varieties to stress the link with the notions of the relative interior and levelness. We devote the rest of this work to determine the cohomology of the normal bundle of any RL–variety (see Theorem 5.9). Both, the coordinate ring and the canonical module of GT–varieties, play an important role on our computations and the proof of Theorem 5.9 is inspired by [3].

In Section 4 we have seen that the canonical module $\omega_{X_d}$ of a GT–variety $X_d$ with group a finite cyclic group $\Gamma = \langle M_{d;\alpha_0,...,\alpha_n} \rangle \subset \text{GL}(n + 1, \mathbb{K})$ of order $d$ is identified with the ideal $\text{relint}(I_d) = (x_0^{\alpha_0} \cdots x_n^{\alpha_n} \in R^\Gamma \mid 0 \neq a_0 \cdots a_n)$ and we have proved that $\text{relint}(I_d)$ is generated by monomials of degree $d$ and $2d$. We begin with the following definition.

**Definition 5.1.** Let $X_d$ be a GT–variety with group a finite cyclic group $\Gamma = \langle M_{d;\alpha_0,...,\alpha_n} \rangle \subset \text{GL}(n + 1, \mathbb{K})$ of order $d$ and associated GT–system $I_d$. We say that $X_d$ is a level GT–variety if $R^\Gamma$ is a level ring and, in addition, $\text{reg}(R^\Gamma) = n + 1$. Equivalently, if its canonical module $\text{relint}(I_d) = \langle C_{d,1} \rangle$. (See Proposition 4.1 and Theorem 4.2).

Let us see some examples of level GT–varieties of any dimension $2 \leq n$, and next a necessary condition on the matrices $M_{d;\alpha_0,...,\alpha_n}$ for $X_d$ be level.

**Example 5.2.** (i) All GT–surfaces are level (see [3] Corollary 4.13).

(ii) Fix integers $4 \leq n, 1 \leq k$ with $n$ even. For $d := k(n + 1)$ and for the finite cyclic group $\Gamma = \langle M_{d;0,1,2,...,n} \rangle \subset \text{GL}(n + 1, \mathbb{K})$ of order $d$, the associated GT–variety $X_d$ is level (see Corollary 4.6).

(iii) Fix integers $3 \leq n, 1 \leq t$ with $n$ odd and fix a finite cyclic group $\Gamma_t = \langle M_{t(n+1);0,1,...,1,2} \rangle$ of order $d = t(n + 1)$. Then $X_{t(n+1)}$ is a level GT–variety.
We consider new variables $a$ setup. is not satisfied. Many notions and results of this section remain true in this more general systems:

$$\begin{align*}
\mathcal{Y}_0 + \mathcal{Y}_1 + \cdots + \mathcal{Y}_{n-1} + \mathcal{Y}_n &= d \\
\mathcal{Y}_1 + \cdots + \mathcal{Y}_{n-1} + 2\mathcal{Y}_n &= rd,
\end{align*}$$

$r = 0, 1, 2$. We fix hypothesis. Let $\Gamma = \{\alpha, \beta, \gamma\}$. Fix integers $\mathcal{Y}_0, \mathcal{Y}_1, \ldots, \mathcal{Y}_{n-1}, \mathcal{Y}_n$. For any solution $(\alpha, \beta, \gamma)$, it holds that $\text{GCD}(\alpha, \beta, \gamma) = 1$. Therefore, for any monomial $\mathcal{Y}_0, \mathcal{Y}_1, \ldots, \mathcal{Y}_n$ of order $\mathcal{Y}_0$ of one of the systems $(\ast)$, it holds that $\mathcal{Y}_0, \mathcal{Y}_1, \ldots, \mathcal{Y}_n$ are $\{\gamma, d-2\gamma, \gamma \mid \gamma = 0, \ldots, \frac{d}{2}\}$. Therefore we have

$$\mu_d = 2 + \sum_{\gamma=0}^{\frac{d}{2}} \left(\frac{d - 2\gamma + n - 2}{n - 2}\right).$$

Since $x_0 \cdots x_n$ is an invariant of $\Gamma_0 \subset \text{GL}(n + 1, \mathbb{K})$, as in the proof of Proposition 4.3 it follows that $R^\Gamma$ is Gorenstein; by Proposition 4.5 we only have to check that $\mu_d \leq \left(\frac{n - 1 + d}{n - 1}\right)$. For $n \geq 3$, it holds that

$$\mu_d = 2 + \sum_{\gamma=0}^{\frac{d-n}{2}} \left(\frac{\gamma}{n - 2}\right) \leq 2 + \sum_{\gamma=0}^{\frac{d-n}{2}} \left(\frac{\gamma}{n - 2}\right) - (n - 1) \leq \sum_{\gamma=0}^{\frac{d-n}{2}} \left(\frac{\gamma}{n - 2}\right) = \left(\frac{n - 1 + d}{n - 1}\right).$$

\[ \square \]

**Proposition 5.3.** Fix integers $2 \leq n < d$, a finite cyclic group $\Gamma = \{\mathcal{Y}_0, \mathcal{Y}_1, \ldots, \mathcal{Y}_n\} \subset \text{GL}(n + 1, \mathbb{K})$ of order $d$ and set $I_d$ the ideal generated by all monomial invariants of $\Gamma$ of degree $d$. If $\mathcal{C}_{d,1} \neq \emptyset$, then there are at least three indices two by two distinct.

**Proof.** By contradiction, we assume $M_{d,0,0,\ldots,0} = M_{d,0,0,\ldots,0}$ with $0 < a < d$ such that $\text{GCD}(a, d) = 1$. Therefore, for any monomial $m \in R^\Gamma$ of degree $d$ it holds that $\text{supp}(m) \in \{x_0, \ldots, x_1\}$ or $\text{supp}(m) \in \{x_{l+1}, \ldots, x_n\}$. In other words, $\mathcal{C}_{d,1} = \emptyset$ which contradicts our hypothesis. \[ \square \]

**Remark 5.4.** Let $\Gamma = \{\mathcal{Y}_0, \mathcal{Y}_1, \ldots, \mathcal{Y}_n\} \subset \text{GL}(n + 1, \mathbb{K})$ be a finite cyclic group of order $d$ and $I_d$ the monomial artinian ideal generated by the set of all monomials $\{m_1, \ldots, m_{\mu_d}\} \subset R^\Gamma$ of degree $d$. Definition 5.1 can be extended to any variety $X_d = \varphi_{I_d}^\mathbb{P} \subset \mathbb{P}^{\mu_d-1}$ even if it is not a GT–variety, or equivalently, $I_d$ is not a Togliatti system, i.e. the condition $\mu_d \leq \left(\frac{n - 1 + d}{n - 1}\right)$ is not satisfied. Many notions and results of this section remain true in this more general setup.
From now onwards, we fix a level $GT$--variety $X_d$ with group a finite cyclic group $\Gamma = \langle M_{d_0, \ldots, d_n} \rangle \subset \text{GL}(n+1, \mathbb{K})$ of order $d$ and we denote $I_d$ its associated $GT$--system. We denote $\eta_d = |C_d|$; i.e. the number of monomials of degree $d$ in $\text{relint}(I_d)$ and we set $N_d := \binom{n+d}{d} - \eta_d - 1$. By $f_d : \mathbb{P}^n \to \mathbb{P}^{N_d}$ we denote the morphism induced by the inverse system $\text{relint}(I_d)^{-1}$. We denote $\mathcal{X}_d = f_d(\mathbb{P}^n) \subset \mathbb{P}^{N}$ and we call $\mathcal{X}_d$ the $RL$--variety associated to $X_d$. We have the following.

**Proposition 5.5.** $\mathcal{X}_d$ is a smooth rational variety and $f_d$ is an embedding.

**Proof.** $\mathcal{X}_d$ is a toric variety parametrized by all monomials of degree $d$ in $\text{relint}(I_d)^{-1}$. It is straightforward to check that $\mathcal{X}_d$ satisfies the smoothness criterion for toric varieties \cite[Chapter 5 - Corollary 3.2]{[3]}. In particular, $\text{relint}(I_d)^{-1}$ contains all monomials $x_i^{d-1} x_j$ for all $i, j \in \{0, \ldots, n\}$, which is a sufficient condition for $f_d$ to be an embedding. \hfill $\square$

In \cite{[3]}, the authors develop a new method to compute the cohomology of the normal bundle of smooth rational projections of the Veronese variety $\nu_{n,d}(\mathbb{P}^n) \subset \mathbb{P}(n+d)_n^{-1}$ for which the parametrization is an embedding. Let $\mathcal{X}_d$ be an $RL$--variety associated to a level $GT$--variety $X_d$ with group $\Gamma$. By Proposition 5.5, any $RL$--variety $\mathcal{X}_d$ is of this kind. Furthermore, the relation between $\mathcal{X}_d$ and $\text{relint}(I_d) \subset R^P$ allows us to apply this approach to any $RL$--variety $\mathcal{X}_d$. In this setting, we have the following presentation of the normal bundle $N_{\mathcal{X}_d}$ of the $RL$--variety $\mathcal{X}_d \subset \mathbb{P}^{N_d}$ (see \cite[(3.3)]{[3]}):

$$(4) \quad 0 \to \mathcal{O}_{\mathbb{P}^n}^{n+1}(1) \to \mathcal{O}_{\mathbb{P}^n}^{N_d+1}(d) \to N_{\mathcal{X}_d} \to 0.$$ 

Taking the long sequence of cohomology for (4), we determine the cohomology $\mathbb{K}$--vector spaces $H^i(\mathcal{X}_d, N_{\mathcal{X}_d}(-k))$ in most cases, as the following result shows.

**Proposition 5.6.** Let $\mathcal{X}_d \subset \mathbb{P}^{N_d}$ be an $RL$--variety of dimension $n \geq 2$, we have:

(i) for all $0 < i < n - 1$ and $k \in \mathbb{Z}$, $H^i(\mathcal{X}_d, N_{\mathcal{X}_d}(-k)) = 0$.

(ii) $h^0(\mathcal{X}_d, N_{\mathcal{X}_d}(-k)) = \begin{cases} (N_d + 1)\binom{n+d-k}{n} - (n+1)\binom{n+1-k}{n} & k \leq 1 \\ (N_d + 1)\binom{n+d-k}{n} & 1 < k \leq d \\ 0 & \text{otherwise}. \end{cases}$

**Proof.** We fix $k \in \mathbb{Z}$. We twist (4) by $\mathcal{O}_{\mathbb{P}^n}(-k)$ and then we consider the long exact sequence of cohomology. For any $i$ and $k$ we obtain

$$(5) \quad \to H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^{N_d+1}(d-k)) \to H^i(\mathcal{X}_d, N_{\mathcal{X}_d}(-k)) \to H^{i+1}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^{n+1}(1-k)) \to$$

From the additivity of the cohomology, it follows the vanishing $H^i(\mathcal{X}_d, N_{\mathcal{X}_d}(-k)) = 0$ for all $0 < i < n - 1$. In addition, we obtain the presentation

$$(6) \quad 0 \to H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^{N_d+1}(1-k)) \to H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^{N_d+1}(d-k)) \to H^0(\mathcal{X}_d, N_{\mathcal{X}_d}(-k)) \to 0.$$
The result follows from the Bott formulas for the cohomology of \( \mathbb{P}^n \) (see [23]).

Thus far, we have determined the dimension of \( H^i(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) \) for any \( i \) and \( k \) except:

\[ H^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) \text{ and } H^n(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)). \]

To compute them, we apply Proposition 5.6(i) to the long exact sequence of cohomology (5).

For any \( k \) we obtain the exact sequence

\[ 0 \to H^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) \to H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^{n+1}(1-k)) \to H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^{N_d+1}(d-k)) \to H^n(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) \to 0. \]

As an immediate result, we get that for all \( k < d + n + 1 \):

\[ H^n(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) = 0 \quad \text{and} \quad H^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) \cong H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^{n+1}(1-k)). \]

Thus,

\[ h^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) = \begin{cases} (n+1)(k-2) & n + 2 \leq k \leq d + n + 1 \\ 0 & k \leq n + 1 \end{cases} \]

We focus on computing \( H^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) \) for \( k \geq d + n + 1 \). We need some preparation.

By \( \partial_{x_0}, \ldots, \partial_{x_n} \) we denote the linear operators acting on \( R \) as partial derivatives. Let \( m \in R_l \) be a monomial and we write \( m = x_0^{a_0} \cdots x_n^{a_n} \). We denote \( \partial_m \) the composition of linear operators \( \partial_{x_0} a_0 \cdots \partial_{x_n} a_n \).

**Lemma 5.7.** Let \( m \) be a monomial of degree \( k - d - n - 1 \), let \( q \) and \( q' \) be monomials of degree \( k - d - n - 1 \) such that \( m \) divides both \( q \) and \( q' \), and let \( 0 \leq i \neq j \leq n \) be integers. Then \( x_i \partial_m q \) and \( x_j \partial_m q' \) are linearly independent if and only if for any monomial \( m' \neq m \) of degree \( k - d - n - 1 \) which divides \( q \) and \( q' \), \( x_i \partial_m q \) and \( x_j \partial_m q' \) are linearly independent.

**Proof.** We write \( m' = x_0^{b_0} \cdots x_n^{b_n}, m = x_0^{a_0} \cdots x_n^{a_n}, q = x_0^{a_0} \cdots x_n^{a_n}, q' = x_0^{a_0} \cdots x_n^{a_n} \). Assume that \( x_i \partial_m q \) and \( x_j \partial_m q' \) are linearly independent and there is \( m' \neq m \), which divides \( q \) and \( q' \), such that \( x_i \partial_m q \) and \( x_j \partial_m q' \) are linearly dependent. Therefore we have the equality

\[ x_0^{a_0-b_0} \cdots x_i^{a_i-b_i+1} \cdots x_n^{a_n-b_n} = x_0^{a_0-b_0} \cdots x_j^{a_j-b_j+1} \cdots x_n^{a_n-b_n}, \]

which implies \( a_i = a_i' \), \( 0 \leq l \neq i, j \leq n, a_i = a_i' - 1 \) and \( a_j = a_j' + 1 \). Then we obtain

\[ x_i \partial_m q = A x_0^{a_0-c_0} \cdots x_j^{a_j-c_j} \cdots x_i^{a_i-c_i+1} \cdots x_n^{a_n-c_n} \]

\[ x_j \partial_m q' = B x_0^{a_0-c_0} \cdots x_j^{a_j-1-c_j+1} \cdots x_i^{a_i-c_i+1} \cdots x_n^{a_n-c_n} \]

for some \( A, B \in \mathbb{K} \setminus \{0\} \), which is a contradiction. \( \Box \)

An RL-variety \( \mathcal{X}_d \subset \mathbb{P}^{N_d} \) of dimension \( n \geq 2 \) is a smooth rational variety embedded in \( \mathbb{P}^{N_d} \). In [3], the authors introduce a new method to compute the cohomology of the normal bundle of varieties of this kind. With the notation of [3], we write the embedding
$f_d : \mathbb{P}(U) \to \mathbb{P}^{N_d}$ with $U = R_d^\vee$. The $RL$-variety $X_d = f_d(\mathbb{P}(U))$ is the projection in $\mathbb{P}^{N_d}$ of the Veronese variety $\nu_d(\mathbb{P}(U)) \subset \mathbb{P}^{\binom{n+d}{n}-1}$ from the projective space $\mathbb{P}(T)$ of dimension $\binom{n+d}{n} - N_d$ where $T^\vee$ is identified with the $\mathbb{K}$-vector subspace of $R_d$ generated by all the monomials of degree $d$ in $\text{relint}(I_d) = (a_0^{a_0} \cdots x_n^{a_n} \in R^\vee_1 \mid 0 \neq a_0 \cdots a_n)$. Let $0 \leq i \neq j \leq n$, $l \geq 1$ and $t \geq 1$ be integers. We denote $D_{i,j} : S^tU \otimes S^tU \to S^{t-1}U \otimes S^{t-1}U$ the linear map $\partial_{x_i} \otimes \partial_{x_j} - \partial_{x_j} \otimes \partial_{x_i}$.

**Proposition 5.8.** Let $X_d \subset \mathbb{P}^{N_d}$ be an $RL$-variety of dimension $n \geq 2$ associated to a level GT-variety $X_d$ with group $\Gamma = \langle M_{d,a_0,\ldots,a_n} \rangle \subset \text{GL}(n+1, \mathbb{K})$. Then,

$$h^{n-1}(X_d, \mathcal{N}_{X_d} \linebreak \langle -k \rangle) = \begin{cases} \eta_d + \frac{n(d-1)}{d} \binom{n+d-1}{n} & k = d + n + 1 \\ (n+1)\eta_d & k = d + n + 2 \\ 0 & k \geq d + n + 3 \end{cases}$$

Proof. By [3 Theorem 2], we obtain $h^{n-1}(X_d, \mathcal{N}_{X_d}(-d - n - 1)) = \dim(\mu^{-1}(T))$, where $\mu : U \otimes S^{d-1}U \to S^dU$ is the multiplication map, and for all $k \geq d + n + 2$:

$$H^{n-1}(X_d, \mathcal{N}_{X_d}(-k)) = (S^k(-d-n-1)U \otimes T) \cap (\bigcap_{0 \leq i,j,r,s \leq n} (\ker(D_{i,j} \circ D_{r,s}))).$$

In particular, for $k = d + n + 1, d + n + 2$ we can conclude that

$$h^{n-1}(X_d, \mathcal{N}_{X_d}(-d - n - 1)) = \eta_d + \frac{n(d-1)}{d} \binom{n+d-1}{n}$$

and $H^{n-1}(X_d, \mathcal{N}_{X_d}(-d - n - 2)) = U \otimes T$. Moreover, for $k \geq d + n + 3$ we have

$$H^{n-1}(X_d, \mathcal{N}_{X_d}(-k)) \cong \{ x_0 \otimes q_0 + \cdots + x_n \otimes q_n \in R_1 \otimes R_{k-n-2} \mid x_0 \partial_m(q_0) + \cdots + x_n \partial_m(q_n) \in \text{relint}(I_d) \text{ for all monomial } m \in R_{k-d-n-1} \},$$

where $\text{relint}(I_d)$ denotes the $\mathbb{K}$-vector subspace of $R_d$ generated by the set $C_{d-1} \subset \text{relint}(I_d)$ of monomials of degree $d$. We want to prove that $H^{n-1}(X_d, \mathcal{N}_{X_d}(-k)) = 0$ for all $k \geq d + n + 3$. Assume that there exist $q_0, \ldots, q_n \in R_{k-n-2}$ and a monomial $m \in R_{k-d-n-1}$ such that $0 \neq u_m := x_0 \partial_m(q_0) + \cdots + x_n \partial_m(q_n) \in \text{relint}(I_d)$. Therefore, any monomial that appears in $u_m$ belongs to $\text{relint}(I_d) \subset \Gamma$. Let $q \in R_{k-n-2}$ be a monomial such that $0 \neq x_i \partial_m q$ is a monomial that occurs in $u_m$. By Lemma 3.7, we have that if $x_0 \otimes q_0 + \cdots + x_n \otimes q_n \in H^{n-1}(X_d, \mathcal{N}_{X_d}(-k))$, then for any monomial $m' \in R_{k-d-n-1}$, $x_i \partial_{m'} q \in \text{relint}(I_d) \subset R_1^\vee = R_1$. We will show that there always exists a monomial $m' \in R_{k-d-n-1}$ dividing $q$ such that $x_i \partial_{m'} q \notin R_1$. Thus, it concludes $H^{n-1}(X_d, \mathcal{N}_{X_d}(-k)) = 0$ for all $k \geq d + n + 3$. Notice that, by Proposition 3.3, $M_{d,a_0,\ldots,a_n}$ has three indices $a_i, a_j, a_t$ two by two distinct. We consider monomials $q = x_0^{a_0} \cdots x_n^{a_n} \in R_{k-n-2}$ and $m = x_0^{b_0} \cdots x_n^{b_n} \in R_{k-d-n-1}$ such that $m$ divides $q$ and $x_i \partial_m q \in \text{relint}(I_d)$. In particular, we have that $b_j < a_j$ for all $0 \leq j \neq i \leq n$ and $b_i \leq a_i - 1$. 

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By assumption, \( x_i \partial_m q := x_0^{c_0} \cdots x_n^{c_n} = x_0^{a_0-b_0} \cdots x_i^{a_i-b_i+1} \cdots x_n^{a_n-b_n} \in \mathbb{R}_1^T \). We distinguish two cases.

**Case 1:** \( 0 < b_i \). If \( \alpha_i = 0 \) or \( \alpha_i > 0 \) and \( 2 \alpha_i - \alpha_j \neq 0 \mod d \), we define the monomial \( m' = x_0^{b_0} \cdots x_i^{b_i+1} \cdots x_h^{b_h} \cdots x_n^{b_n} \). Then \( x_i \partial_m q = x_0^{a_0+c_0} \cdots x_i^{a_i+c_i-1} \cdots x_h^{a_h+c_h+1} \cdots x_n^{a_n+c_n} \). Otherwise, \( 2 \alpha_i - \alpha_i \neq 0 \mod d \) and we define \( m' = x_0^{b_0} \cdots x_i^{b_i+1} \cdots x_h^{b_h} \cdots x_n^{b_n} \). Then \( x_i \partial_m q = x_0^{c_0} \cdots x_i^{c_i+1} \cdots x_n^{c_n} \) and its associated point does not verify the linear congruence equation \( \alpha_0 y_0 + \cdots + \alpha_n y_n \equiv 0 \mod d \).

**Case 2:** \( b_i = 0 \). We take \( 0 < b_i \), and we can assume that \( \alpha_i, \alpha_j \) are different pair-wise. We define \( m' = x_0^{b_0} \cdots x_i^{b_i+1} \cdots x_h^{b_h} \cdots x_n^{b_n} \). Then \( x_i \partial_m q = x_0^{c_0} \cdots x_i^{c_i-1} \cdots x_h^{c_h+1} \cdots x_n^{c_n} \) and its associated point does not verify the linear congruence equation \( \alpha_0 y_0 + \cdots + \alpha_n y_n \equiv 0 \mod d \).

In any case, we have constructed a monomial \( m' \in \mathbb{R}_{k-d-n-1} \) dividing \( q \) such that \( x_i \partial_m q \notin \mathbb{R}_1^T \) and the proposition follows.

Directly from (5.3) and Proposition 5.8, we obtain \( h^n(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) \) for all \( k \geq d + n + 1 \). We summarize all the computations and establish the main result of this section.

**Theorem 5.9.** Fix a level \( GT \)-variety \( X_d \) with group a finite cyclic group \( \Gamma = \langle M_{d, \alpha_0, \ldots, \alpha_n} \rangle \subset GL(n+1, \mathbb{K}) \) of order \( d \) and associated \( GT \)-system \( I_d \). Set \( \eta_d := |C_{d,1}| \) the number of monomials of relint \((I_d)\) of degree \( d \) and \( N_d := \binom{n+d}{d} - \eta_d - 1 \). Let \( \mathcal{X}_d \subset \mathbb{P}^{N_d} \) be the RL-variety of dimension \( n \geq 2 \) associated to \( X_d \). It holds:

(i) for \( 0 < i < n - 1 \) and for all \( k \in \mathbb{Z} \), \( h^i(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) = 0 \).

(ii)

\[
\begin{align*}
\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) = \begin{cases} 
(N_d + 1) \binom{n+d-k}{n} - (n+1) \binom{n+1-k}{n} & k \leq 1 \\
(N_d + 1) \binom{n+d-k}{n} & 1 < k \leq d \\
0 & \text{otherwise}.
\end{cases}
\end{align*}
\]

(iii)

\[
\begin{align*}
\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) = \begin{cases} 
(n+1) \binom{k-2}{n} & n + 2 \leq k < d + n + 1 \\
\eta_d + \frac{n(d-1)}{d} \binom{n+d-1}{n} & k = d + n + 1 \\
(n+1) \eta_d & k = d + n + 2 \\
0 & k \leq n + 1 \text{ or } k \geq d + n + 3.
\end{cases}
\end{align*}
\]

(iv)

\[
\begin{align*}
\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d}(-k)) = \begin{cases} 
(N_d + 1) \binom{k-d-1}{n} - (n+1) \binom{k-2}{n} & k \geq d + n + 3 \\
0 & \text{otherwise}.
\end{cases}
\end{align*}
\]

We end this work by showing two examples pointing out Theorem 5.9. All the computations have been made with the software Macaulay2 ([17]).
Example 5.10. (i) We fix $d = 5$ and $\Gamma = \langle M_{5,0,1,2} \rangle \subset \text{GL}(3, \mathbb{K})$ a cyclic group of order 5. The ideal $I_5 = (x_5^5, x_0^2 x_1 x_2^2, x_0 x_1^3 x_2, x_1^5, x_2^5) \subset K[x_0, x_1, x_2]$ is the $GT-$system generated by all monomial invariants of $\Gamma$ of degree 5. Its associated $GT-$variety $X_5$ is level with $\text{relint}(I_5) = (x_2 x_0 x_1 x_2^2, x_0 x_1^3 x_2)$ and $\eta_5 = 2$. We present the cohomology table from degree $-10$ to 0 of the normal bundle $N_{X_5}$ of the smooth rational variety $X_5$ parametrized by the inverse system $\text{relint}(I_5)^{-1}$.

\[
\begin{array}{cccccccccc}
-10 & -9 & -8 & -7 & -6 & -5 & -4 & -3 & -2 & -1 & 0 \\
2 & 150 & 82 & 30 & . & . & . & . & . & . & . \\
1 & . & . & 6 & 26 & 30 & 18 & 9 & 3 & . & . \\
0 & . & . & . & . & . & 19 & 57 & 114 & 190 & 282 \\
\end{array}
\]

(ii) We fix $d = 4$ and $\Gamma = \langle M_{4,0,1,1,2} \rangle \subset \text{GL}(4, \mathbb{K})$ a cyclic group of order 4. The associated $GT-$system $I_4 = (x_4^4, x_0^2 x_1 x_2 x_3, x_0 x_1 x_2 x_3, x_0 x_1^2 x_3, x_1^4, x_1^3 x_2, x_1^2 x_2^2, x_1 x_2^3, x_2^4, x_3^4) \subset K[x_0, x_1, x_2, x_3]$ and its associated $GT-$threefold $X_4$ is level with $\text{relint}(I_4) = (x_0 x_1 x_2 x_3)$ (see Example 5.2(iii)). We present the cohomology table from degree $-9$ to 0 of the normal bundle $N_{X_4}$ of the smooth rational variety $X_4$ parametrized by the inverse system $\text{relint}(I_4)^{-1}$.

\[
\begin{array}{cccccccccc}
-9 & -8 & -7 & -6 & -5 & -4 & -3 & -2 & -1 & 0 \\
3 & 710 & 344 & 116 & . & . & . & . & . & . \\
2 & . & . & 4 & 46 & 40 & 16 & 4 & . & . \\
1 & . & . & . & . & . & . & . & . & . \\
0 & . & . & . & . & 34 & 136 & 340 & 676 & 1174 \\
\end{array}
\]

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