The Mean Curvature of First-Order Submanifolds in Geometries with Torsion

Gavin Ball and Jesse Madnick

May 2019

Abstract

We derive formulas for the mean curvature of special Lagrangian 3-folds, associative 3-folds, and coassociative 4-folds in the general case where the ambient space has intrinsic torsion. Consequently, we are able to characterize those SU(3)-structures (resp., G\textsubscript{2}-structures) for which every special Lagrangian 3-fold (resp. associative 3-fold, coassociative 4-fold) is a minimal submanifold.

In the process, we obtain obstructions to the local existence of special Lagrangian 3-folds and coassociative 4-folds in manifolds with torsion.

Contents

1 Introduction 2

1.1 Results on Special Lagrangians .......................... 3
1.2 Results on Associatives and Coassociatives .................. 4
1.3 Organization ........................................ 5

2 Special Lagrangian 3-Folds in SU(3)-Structures 6

2.1 Preliminaries ........................................ 6
2.2 Some SO(3)-Representation Theory ........................ 10
2.3 The Refined Torsion Forms ................................ 14
2.4 Mean Curvature of Special Lagrangian 3-Folds ............... 19

3 Associative 3-Folds and Coassociative 4-Folds in G\textsubscript{2}-Structures 24

3.1 Preliminaries ........................................ 24
3.2 Some SO(4)-Representation Theory ........................ 28
3.3 The Refined Torsion Forms ................................ 33
3.4 Mean Curvature of Associative 3-Folds ....................... 38
3.5 Mean Curvature of Coassociative 4-Folds ..................... 41

References 44
1 Introduction

In their fundamental work on calibrations, Harvey and Lawson [9] defined four new classes of calibrated submanifolds in Riemannian manifolds with special holonomy, summarized in the following table:

| Submanifold       | Ambient Manifold                                    |
|-------------------|-----------------------------------------------------|
| Special Lagrangian n-fold | Riemannian 2n-manifold \((M^{2n}, g)\) with \(\text{Hol}(g) \leq \text{SU}(n)\) |
| Associative 3-fold          | Riemannian 7-manifold \((M', g)\) with \(\text{Hol}(g) \leq \text{G}_2\) |
| Coassociative 4-fold        | Riemannian 7-manifold \((M', g)\) with \(\text{Hol}(g) \leq \text{G}_2\) |
| Cayley 4-fold              | Riemannian 8-manifold \((M^8, g)\) with \(\text{Hol}(g) \leq \text{Spin}(7)\) |

By virtue of being calibrated, each of these submanifolds satisfy a strong area-minimizing property. In particular, they are stable minimal submanifolds. Moreover, by an argument using the Cartan-Kähler Theorem, Harvey and Lawson [9] were able to show that submanifolds of each class exist locally in abundance.

In fact, each of these classes of submanifolds make sense in an even more general class of ambient spaces: namely, that of (Riemannian) manifolds \(M\) equipped with a \(G\)-structure, for \(G = \text{SU}(n)\) or \(\text{G}_2\) or \(\text{Spin}(7)\) as appropriate.

However, in this generalized setting, such submanifolds need not be minimal. This raises the following:

**Minimality Problem:** Let \(M\) be a manifold. Characterize those \(G\)-structures (for \(G = \text{SU}(n)\), \(\text{G}_2\), \(\text{Spin}(7)\)) on \(M\) for which every submanifold in \(M\) of a given class (special Lagrangian, associative, etc.) is a minimal submanifold of \(M\).

We will completely solve the Minimality Problem in the contexts of special Lagrangian 3-folds, associative 3-folds, and coassociative 4-folds by deriving simple formulas for their mean curvature. The case of Cayley 4-folds is being addressed by work in progress, and will be included in an updated version of this report.

Perhaps more fundamentally, in our generalized context the relevant submanifolds need not exist at all, even locally. This raises the natural:

**Local Existence Problem:** Let \(M\) be a manifold. Characterize those \(G\)-structures (with \(G\) as above) on \(M\) for which submanifolds of a given class (special Lagrangian, etc.) exist locally at every point of \(M\).

In this work, we make progress towards the resolution of the Local Existence Problem. More precisely, we obtain the strongest obstructions known to the local existence of special Lagrangian 3-folds and coassociative 4-folds.
1.1 Results on Special Lagrangians

Let \((M^6, \Omega, \Upsilon)\) be a 6-manifold with an SU(3)-structure \((\Omega, \Upsilon) \in \Omega^2(M) \oplus \Omega^3(M; \mathbb{C})\). The first-order local invariants of \((\Omega, \Upsilon)\) are completely encoded in six differential forms, called the torsion forms of the SU(3)-structure, denoted

\[(\tau_0, \hat{\tau}_0, \tau_2, \hat{\tau}_2, \tau_3, \tau_4, \tau_5) \in \Omega^0 \oplus \Omega^0 \oplus \Omega^2 \oplus \Omega^2 \oplus \Omega^3 \oplus \Omega^1 \oplus \Omega^1\]

In order to study special Lagrangian 3-folds in \(M\), we will break the torsion forms into SO(3)-irreducible pieces with respect to a certain splitting of \(TM\). Indeed, in §2.3, we will decompose \(\tau_0, \hat{\tau}_0, \tau_2, \hat{\tau}_2, \tau_3, \tau_4, \tau_5\) into SO(3)-irreducible components, writing

\[
\begin{align*}
\tau_0 &= \tau_0, \quad \tau_2 = (\tau_2)_1 + (\tau_2)_2, \quad \tau_3 = (\tau_3)'_0 + (\tau_3)'_0 + (\tau_3)'_2 + (\tau_3)'_2, \\
\hat{\tau}_0 &= \hat{\tau}_0, \quad \hat{\tau}_2 = (\hat{\tau}_2)_1 + (\hat{\tau}_2)_2,
\end{align*}
\]

and

\[
\begin{align*}
\tau_4 &= (\tau_4)_T + (\tau_4)_N, \\
\tau_5 &= (\tau_5)_T + (\tau_5)_N
\end{align*}
\]

as refined torsion forms (with respect to a certain splitting of \(TM\)). It turns out that the mean curvature of a special Lagrangian can be expressed purely in terms of the refined torsion forms:

**Theorem 2.10:** Let \(M^6\) be a 6-manifold equipped with an SU(3)-structure. The mean curvature vector \(H\) of a phase \(\theta\) special Lagrangian 3-fold immersed in \(M\) is given by

\[
H = -\frac{1}{\sqrt{2}} \cos(3\theta) \left[ (\tau_2)_1 \right]^3 + \frac{1}{\sqrt{2}} \sin(3\theta) \left[ (\hat{\tau}_2)_1 \right]^3 - \sin(\theta) \left[ (\tau_5)_T \right]^5 - \cos(\theta) \left[ J(\tau_5)_N \right]^5
\]

where \(\sharp, \flat\) are certain isometric isomorphisms defined in (2.16) and (2.8), respectively, and where \(J\) is the almost-complex structure of \(M\).

In particular, an SU(3)-structure on \(M\) has the property that every special Lagrangian 3-fold (of every phase) in \(M\) is minimal if and only if \(\tau_2 = \hat{\tau}_2 = \tau_5 = 0\).

This formula can be regarded as a submanifold analogue of the curvature formulas derived by Bedulli and Vezzoni [11] for 6-manifolds with SU(3)-structures. We will also derive an obstruction to the local existence of special Lagrangian 3-folds in the language of refined torsion forms:

**Theorem 2.8:** If a special Lagrangian 3-fold \(\Sigma\) of phase \(\theta\) exists in \(M\), then the following relation holds at points of \(\Sigma\):

\[
\hat{\tau}_0 \sin(3\theta) - \tau_0 \cos(3\theta) = \frac{\sqrt{3}}{6} \left( \sin(\theta) (\tau_3)'_0 + \cos(\theta) (\tau_3)'_0 \right)
\]

where \(\dagger, \ddagger\) are certain isometric isomorphisms defined in (2.19) and (2.20), respectively.

In particular, if \(\tau_3 = 0\), then the phase of every special Lagrangian 3-fold in \(M\) satisfies \(\tan(3\theta) = \tau_0/\hat{\tau}_0\).

**Corollary 2.9:** Fix \(x \in M\) and \(\theta \in [0, 2\pi)\). If every phase \(\theta\) special Lagrangian 3-plane in \(T_x M\) is tangent to a phase \(\theta\) special Lagrangian 3-fold, then \(\tau_3|_x = 0\) and \(\hat{\tau}_0|_x \sin(3\theta) = \tau_0|_x \cos(3\theta)\).
1.2 Results on Associatives and Coassociatives

Let \((M^7, \varphi)\) be a 7-manifold with a \(G_2\)-structure \(\varphi \in \Omega^3(M)\). The first-order local invariants of \(\varphi\) are completely encoded in four differential forms, called the \emph{torsion forms} of the \(G_2\)-structure, denoted \((\tau_0, \tau_1, \tau_2, \tau_3) \in \Omega^0 \oplus \Omega^1 \oplus \Omega^2 \oplus \Omega^3\).

In order to study associative 3-folds and coassociative 4-folds in \(M\), we will break the torsion forms into \(\text{SO}(4)\)-irreducible pieces with respect to a certain splitting of \(TM\). Indeed, in \(\S 3.3\), we will decompose \(\tau_0, \tau_1, \tau_2, \tau_3\) into \(\text{SO}(4)\)-irreducible components, writing

\[
\tau_0 = \tau_0 \\
\tau_1 = (\tau_1)_A + (\tau_1)_C \\
\tau_2 = (\tau_2)_A + (\tau_2)_{1,3} + (\tau_2)_{2,0} \\
\tau_3 = (\tau_3)_{0,0} + (\tau_3)_{0,4} + (\tau_3)_{2,2} + (\tau_3)_{1,3} + (\tau_3)_C
\]

We will refer to the individual pieces

\[
\tau_0, \ (\tau_1)_A, (\tau_1)_C, \ (\tau_2)_A, (\tau_2)_{1,3}, (\tau_2)_{2,0}, \ (\tau_3)_{0,0}, (\tau_3)_{0,4}, (\tau_3)_{2,2}, (\tau_3)_{1,3}, (\tau_3)_C
\]

as \emph{refined torsion forms} (with respect to a certain splitting of \(TM\)). As for special Lagrangians, the mean curvature of associatives and coassociatives can be expressed in terms of the refined torsion:

**Theorem 3.9:** The mean curvature vector \(H\) of an associative 3-fold in \(M\) is given by

\[
H = -3[(\tau_1)_C]^\sharp - \frac{\sqrt{3}}{2}[(\tau_3)_C]^\sharp
\]

where \(
\sharp\) and \(
\sharp\) are certain isometric isomorphisms defined in \((3.5)\) and \((3.18)\), respectively.

In particular, a \(G_2\)-structure on \(M\) has the property that every associative 3-fold in \(M\) is minimal if and only if \(\tau_1 = \tau_3 = 0\). Equivalently, if and only if \(d\varphi = \lambda \ast \varphi\) for some constant \(\lambda \in \mathbb{R}\).

**Theorem 3.12:** The mean curvature vector \(H\) of a coassociative 4-fold in \(M\) is given by

\[
H = -4[(\tau_1)_A]^\sharp + \frac{\sqrt{6}}{3}[(\tau_2)_A]^\sharp
\]

where \(
\sharp\) and \(
\sharp\) are certain isometric isomorphisms defined in \((3.5)\) and \((3.11)\), respectively.

In particular, a \(G_2\)-structure on \(M\) has the property that every coassociative 4-fold in \(M\) is minimal if and only if \(\tau_1 = \tau_2 = 0\). Equivalently, \(d \ast \varphi = 0\).

These formulas can be regarded as a submanifold analogue of the curvature formulas derived by Bryant [3] for 7-manifolds with \(G_2\)-structures. We will also derive an obstruction to the local existence of coassociative 4-folds:

**Theorem 3.10:** If a coassociative 4-fold \(\Sigma\) exists in \(M\), then the following relation holds at points of \(\Sigma\):

\[
\tau_0 = -\frac{\sqrt{42}}{7}[(\tau_3)_{0,0}]^\dagger
\]

where \(
^\dagger\) is an isometric isomorphism defined in \((3.17)\).

In particular, if \(\tau_3 = 0\) and \(\tau_0\) is non-vanishing, then \(M\) admits no coassociative 4-folds (even locally).
Corollary 3.11: Fix $x \in M$. If every coassociative 4-plane in $T_x M$ is tangent to a coassociative 4-fold, then $\tau_0|_x = 0$ and $\tau_3|_x = 0$.

Note that Theorem 3.10 generalizes the well-known fact that nearly-parallel G$_2$-structures (viz., those with $\tau_1 = \tau_2 = \tau_3 = 0$ and $\tau_0$ non-vanishing) cannot admit coassociative 4-folds.

1.3 Organization

In §2, we study special Lagrangian 3-folds in 6-manifolds with SU(3)-structures. In §2.2, we explain how to decompose the relevant SU(3)-modules (e.g., $\Lambda^k(\mathbb{R}^6)$ and $\text{Sym}^2(\mathbb{R}^6)$) that appear in the study of SU(3)-structures into SO(3)-irreducible pieces. We will give explicit descriptions of these submodules, both for our own calculations and in the hopes that our setup will be useful to others.

Then, in §2.3, we use the linear algebra of §2.2 to define the relevant refined torsion forms, and express them in terms of a local SO(3)-frame. Finally, in §2.4, we prove Theorem 2.8, Corollary 2.9, and Theorem 2.10.

The structure of §3 is completely analogous. That is, in §3.2 we decompose the G$_2$-modules appearing in the study of G$_2$-structures into SO(4)-irreducible submodules, providing explicit descriptions as much as possible. In §3.3, we define the corresponding refined torsion forms using the linear algebra of §3.2, and express them in terms of a local SO(4)-frame.

In §3.4, we study associative 3-folds, proving Theorem 3.9. In §3.5, we study coassociative 4-folds, proving Theorem 3.10, Corollary 3.11, and Theorem 3.12.

Acknowledgements: This work has benefited from conversations with Robert Bryant, Thomas Madsen, and Alberto Raffero. The second author would also like to thank McKenzie Wang for his guidance and encouragement.
2 Special Lagrangian 3-Folds in SU(3)-Structures

Our goal in this section is to derive a formula (Theorem 2.10) for the mean curvature of special Lagrangian 3-folds of arbitrary phase in 6-manifolds equipped with an SU(3)-structure. In the process, we observe an obstruction (Theorem 2.8) to their local existence.

These formulas and obstructions will be phrased in terms of refined torsion forms, which we will define in §2.3.1. These refined forms are simply the SO(3)-irreducible pieces of the usual torsion forms $\tau_0, \hat{\tau}_0, \ldots, \tau_5$ of a SU(3)-structure. As such, we will use §2.2 to describe the relevant SO(3)-representation theory needed to decompose $\tau_0, \hat{\tau}_0, \ldots, \tau_5$.

2.1 Preliminaries

In this section, we define both the ambient spaces (in §2.1.2) and submanifolds (in §2.1.3) of interest. We also use this section to fix notation and clarify conventions.

2.1.1 SU(3)-Structures on Vector Spaces

Let $V = \mathbb{R}^6$ equipped with the standard inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $\{e_1, \ldots, e_6\}$ denote the standard (orthonormal) basis of $V$, and let $\{e^1, \ldots, e^6\}$ denote the corresponding dual basis of $V^*$. We will regard $V \cong \mathbb{C}^3$ via the complex structure $J_0$ given by

$$J_0 e_1 = e_4 \quad J_0 e_2 = e_5 \quad J_0 e_3 = e_6.$$  

The standard symplectic form $\Omega_0 = \langle J_0 \cdot, \cdot \rangle$ and complex volume form $\Upsilon_0$ are then given by

$$\Omega_0 = e^{14} + e^{25} + e^{36} \quad \Upsilon_0 = (e^1 + ie^4) \wedge (e^2 + ie^5) \wedge (e^3 + ie^6)$$

Note that $\Upsilon_0$ has real and imaginary parts

$$\text{Re}(\Upsilon_0) = e^{123} - e^{156} + e^{246} - e^{345} \quad \text{Im}(\Upsilon_0) = e^{126} - e^{135} + e^{234} - e^{456}$$

Note also that

$$\text{vol}_0 := \frac{1}{6} \Omega_0^3 = \frac{1}{8} \Upsilon_0 \wedge \overline{\Upsilon}_0 = e^{142536}$$

is a (real) volume form $V$.

For calculations, it will be convenient to express $\Omega_0$ and $\Upsilon_0$ in the form

$$\Omega_0 = \frac{1}{2} \Omega_{ij} e^{ij} \quad \text{Re}(\Upsilon_0) = \frac{1}{6} \epsilon_{ijk} e^{ijk} \quad \text{Im}(\Upsilon_0) = \frac{1}{6} \hat{\epsilon}_{ijk} e^{ijk} \quad (2.1)$$

where the constants $\Omega_{ij}, \epsilon_{ijk}, \hat{\epsilon}_{ijk} \in \{-1, 0, 1\}$ are defined by this formula. For example, $\Omega_{14} = -\Omega_{41} = 1$ and $\epsilon_{123} = -\epsilon_{213} = 1$. Identities involving the $\Omega$- and $\epsilon$-symbols are given in $[\Pi]$.

Remark: The data $\langle \cdot, \cdot \rangle, J_0, \Omega_0, \Upsilon_0$ are not independent of one another, and one can recover $\langle \cdot, \cdot \rangle$ and $J_0$ from the knowledge of $\Omega_0$ and $\Upsilon_0$. Let us be more precise.

In general, suppose $(g, J, \Omega, \Upsilon)$ is a quadruple on $V$ consisting of a positive-definite inner product $g$, a complex structure $J$, a non-degenerate 2-form $\Omega$ defined by $g = \Omega(\cdot, J \cdot)$, and a complex $(3, 0)$-form $\Upsilon \in \Lambda^3(V^*; \mathbb{C})$ for which $\Upsilon \wedge \overline{\Upsilon} \neq 0$. Then $\Upsilon$ is decomposable, satisfies $\Omega \wedge \Upsilon = 0$, the 6-form $\frac{1}{8} \Upsilon \wedge \overline{\Upsilon}$ is a real volume form, and finally

$$g(X, Y) \frac{1}{8} \Upsilon \wedge \overline{\Upsilon} = \frac{1}{2} \iota_X(\Omega) \wedge \iota_Y(\text{Re}(\Upsilon)) \wedge \text{Re}(\Upsilon). \quad (2.2)$$
Conversely, let \((\Omega, \Upsilon) \in \Lambda^2(V^*) \oplus \Lambda^3(V^*; \mathbb{C})\) be a pair consisting of a non-degenerate 2-form \(\Omega\) and a decomposable complex 3-form \(\Upsilon\) satisfying \(\Upsilon \wedge \overline{\Upsilon} \neq 0\) and \(\Omega \wedge \Upsilon = 0\). Then one can recover \((g, J)\) via
\[
\iota_{JX}(\frac{1}{8} \Upsilon \wedge \overline{\Upsilon}) = \frac{1}{2} \iota_X(\text{Re}(\Upsilon)) \wedge \text{Re}(\Upsilon) \quad (2.3a)
\]
\[
g(X, Y) = \Omega(X, JY). \quad (2.3b)
\]
For a proof, see [15]. □

In the sequel, we will always equip \(\Lambda^k(V^*)\) with the usual inner product, also denoted \(\langle \cdot, \cdot \rangle\), given by declaring
\[
\{e^I: I \text{ increasing multi-index}\} \quad (2.4)
\]
to be an orthonormal basis. We let \(\| \cdot \|\) denote the corresponding norm. We will also need both the orthogonal and symplectic Hodge star operators. These are the respective operators \(*, \star: \Lambda^k(V^*) \rightarrow \Lambda^{6-k}(V^*)\) such that every \(\alpha, \beta \in \Lambda^k(V^*)\) satisfy
\[
\alpha \wedge * \beta = \langle \alpha, \beta \rangle \text{vol}_0 \quad \alpha \wedge \star \beta = \Omega_0(\alpha, \beta) \text{vol}_0
\]

We view \(V \simeq \mathbb{R}^6\) as a faithful SU(3)-representation. This SU(3)-representation is irreducible. However, the induced SU(3)-representations on \(\Lambda^k(V^*)\) for \(2 \leq k \leq 5\) are not irreducible. Indeed, \(\Lambda^2(V^*)\) decomposes into irreducible SU(3)-modules as
\[
\Lambda^2(V^*) = \mathbb{R} \Omega_0 \oplus \Lambda^2_6 \oplus \Lambda^2_8,
\]
where
\[
\Lambda^2_6 = \{ \iota_X(\text{Re}(\Upsilon_0)): X \in V\} = \{ * (\alpha \wedge \text{Re}(\Upsilon_0)): \alpha \in \Lambda^1 \}
\]
\[
\Lambda^2_8 = \{ \beta \in \Lambda^2: \beta \wedge \text{Re}(\Upsilon_0) = 0 \text{ and } * \beta = -\beta \wedge \Omega_0 \}
\]

Similarly, \(\Lambda^3(V^*)\) decomposes into irreducible SU(3)-modules as
\[
\Lambda^3(V^*) = \mathbb{R} \text{Re}(\Upsilon_0) \oplus \mathbb{R} \text{Im}(\Upsilon_0) \oplus \Lambda^3_6 \oplus \Lambda^3_{12}
\]
where
\[
\Lambda^3_6 = \{ \alpha \wedge \Omega_0: \alpha \in \Lambda^1 \} = \{ \gamma \in \Lambda^3: * \gamma = \gamma \}
\]
\[
\Lambda^3_{12} = \{ \gamma \in \Lambda^3: \gamma \wedge \Omega_0 = 0 \text{ and } \gamma \wedge \text{Re}(\Upsilon_0) = \gamma \wedge \text{Im}(\Upsilon_0) = 0 \}
\]

In each case, \(\Lambda^k_\ell\) is an irreducible SU(3)-module of dimension \(\ell\). One can obtain similar decompositions of \(\Lambda^4(V^*)\) and \(\Lambda^5(V^*)\) by applying the orthogonal Hodge star operator.

The SU(3)-module \(\text{Sym}^2(V^*)\) is also reducible, splitting as
\[
\text{Sym}^2(V^*) = \mathbb{R} \text{Id} \oplus \text{Sym}^2_+ \oplus \text{Sym}^2_-
\]
where
\[
\text{Sym}^2_+ = \{ h \in \text{Sym}^2(V^*): J_0 h = h, \text{tr}(h) = 0 \}
\]
\[
\text{Sym}^2_- = \{ h \in \text{Sym}^2(V^*): J_0 h = -h \}
\]

In [1], the authors note that the maps
\[
\rho: \text{Sym}^2_+ \rightarrow \Lambda^3_6 \quad \chi: \text{Sym}^2_- \rightarrow \Lambda^3_{12}\]
\[
\rho(h_{ij} e^i e^j) = h_{ik} \Omega_{kj} e^{ij} \quad \chi(h_{ij} e^i e^j) = h_{il} \epsilon_{ljk} e^{ijk}
\]
are SU(3)-module isomorphisms. These isomorphisms will be crucial to our calculations in §2.2.
2.1.2 SU(3)-Structures on 6-Manifolds

**Definition:** Let \( M \) be an oriented 6-manifold. An \( SU(3) \)-structure on \( M \) is a pair \((\Omega, \Upsilon)\) consisting of a non-degenerate 2-form \( \Omega \in \Omega^2(M) \) and a complex 3-form \( \Upsilon \in \Omega^3(M; \mathbb{C}) \) such that at each \( x \in M \), there exists a coframe \( u: T_x M \to \mathbb{R}^6 \) for which \( \Omega|_x = u^*(\Omega_0) \) and \( \Upsilon|_x = u^*(\Upsilon_0) \).

Intuitively, an \( SU(3) \)-structure is a smooth identification of each tangent space \( T_x M \) with \( \mathbb{C}^3 \) in such a way that \((\Omega|_x, \Upsilon|_x)\) is aligned with \((\Omega_0, \Upsilon_0)\). We note that a 6-manifold \( M \) admits an \( SU(3) \)-structure if and only if it is orientable and spin.

Every \( SU(3) \)-structure \((\Omega, \Upsilon)\) on \( M \) induces a Riemannian metric \( g \) and an almost-complex structure \( J \) on \( M \) via the formulas \((2.2)-(2.3)\), reflecting the inclusion \( SU(3) \subseteq SO(6) \cap GL_3(\mathbb{C}) \). We emphasize that, in general, \( J \) need not be integrable, and \( \Omega \) need not be closed. We also caution that the association \((\Omega, \Upsilon) \mapsto g\) is not injective.

The first-order local invariants of an \( SU(3) \)-structure are completely encoded in a certain \( SU(3) \)-equivariant function

\[
T: F_{SU(3)} \to \Lambda^0 \oplus \Lambda^0 \oplus \Lambda^2_0 \oplus \Lambda^2_8 \oplus \Lambda^3_{12} \oplus \Lambda^1 \oplus \Lambda^1 \simeq \mathbb{R}^{42}
\]

called the *intrinsic torsion function*, defined on the total space of the \( SU(3) \)-frame bundle \( F_{SU(3)} \to M \) over \( M \). We think of \( T \) as describing the 1-jet of the \( SU(3) \)-structure.

The intrinsic torsion function is somewhat technical to define — the interested reader can find more information in [8] and [13] — but several equivalent reformulations are available. Most conveniently for our purposes: the intrinsic torsion function of a \( SU(3) \)-structure is equivalent to the data of the 3-form \( d\Omega \) and the complex 4-form \( d\Upsilon \).

In [11], the exterior derivatives of \( \Omega \) and \( \Upsilon \) are shown to take the form

\[
d\Omega = 3\tau_0 \Re(\Upsilon) + 3\tau_0 \Im(\Upsilon) + \tau_3 + \tau_4 \wedge \Omega
\]
\[
d\Re(\Upsilon) = 2\tau_0 \Omega^2 + \tau_5 \wedge \Re(\Upsilon) + \tau_2 \wedge \Omega
\]
\[
d\Im(\Upsilon) = -2\tau_0 \Omega^2 - J\tau_5 \wedge \Re(\Upsilon) + \tau_2 \wedge \Omega
\]

where

\[
(\tau_0, \tilde{\tau}_0, \tau_2, \tilde{\tau}_2, \tau_3, \tau_4, \tau_5) \in \Gamma(\Lambda^0 \oplus \Lambda^0 \oplus \Lambda^2_8 \oplus \Lambda^3_{12} \oplus \Lambda^1 \oplus \Lambda^1)
\]

and we are abbreviating \( \Lambda^k_\ell := \Lambda^k_\ell(T^*M) \), etc. We refer to \( \tau_0, \tilde{\tau}_0, \tau_2, \tilde{\tau}_2, \tau_3, \tau_4, \tau_5 \) as the *torsion forms* of the \( SU(3) \)-structure.

Following standard conventions, we let \( X_0^+, X_0^-, X_2^+, X_2^-, X_3, X_4, X_5 \) denote the vector bundles \( \Lambda^0, \Lambda^0, \Lambda^2_8, \Lambda^3_{12}, \Lambda^1, \Lambda^1 \), respectively. Consider the set \( S \) consisting of the \( 2^7 = 128 \) vector bundles

\[
S = \left\{ 0, \bigoplus_{k=1}^\ell E_k : E_k \in \{X_0^+, X_0^-, X_2^+, X_2^-, X_3, X_4, X_5\}, \ell = 1, \ldots, 7 \right\}.
\]

**Definition:** Let \( E \in S \) be a vector bundle on the list above.

We say that an \( SU(3) \)-structure belongs to the *torsion class* \( E \) iff the torsion forms of the \( SU(3) \)-structure \((\tau_0, \tilde{\tau}_0, \tau_2, \tilde{\tau}_2, \tau_3, \tau_4, \tau_5) \in \Gamma(X_0^+ \oplus \cdots \oplus X_5)\) is valued in \( E \subset X_0^+ \oplus \cdots \oplus X_5 \).

For example, an \( SU(3) \)-structure belongs to the torsion class \( X_0^+ \oplus X_0^- \oplus X_2^+ \oplus X_2^- \) if and only if \( \tau_3 = \tau_4 = \tau_5 = 0 \).
2.1.3 Special Lagrangian 3-Folds

Let $(M^6, \Omega, \Upsilon)$ be a 6-manifold with an SU(3)-structure, and fix a tangent space $(T_x M, \Omega|_x, \Upsilon|_x) \simeq (V, \Omega_0, \Upsilon_0)$. In their work on calibrations, Harvey and Lawson [9] observed that the vector space $(V, \Omega_0, \Upsilon_0)$ possesses an $S^1$-family of distinguished classes of 3-dimensional subspaces — the special Lagrangian 3-planes of a given phase — which we now describe.

For $\theta \in [0, 2\pi)$, consider the complex 3-form $\Upsilon_\theta \in \Lambda^3(V^*; \mathbb{C})$ defined by

$$\Upsilon_\theta := e^{-i\theta} \Upsilon_0.$$ 

We refer to its real part

$$\text{Re}(\Upsilon_\theta) = \text{Re}(e^{-i\theta} \Upsilon_0) \in \Lambda^3(V^*)$$

as the phase $\theta$ special Lagrangian 3-form, following the sign convention of [11] (rather than [9]). Note that $\text{Im}(\Upsilon_\theta) = \text{Re}(\Upsilon_{\theta + \frac{\pi}{2}})$, where $\theta + \frac{\pi}{2}$ is regarded mod $2\pi$. The 3-forms $\Upsilon_\theta$ enjoy the following remarkable property:

**Proposition 2.1** [9]: For each $\theta \in [0, 2\pi)$, the 3-form $\text{Re}(\Upsilon_\theta)$ has co-mass one, meaning that:

$$|\text{Re}(\Upsilon_\theta)(x,y,z)| \leq 1$$

for every orthonormal set $\{x, y, z\} \subset V \simeq \mathbb{R}^6$.

In light of this proposition, it is natural to examine more closely those 3-planes $E \in \text{Gr}_3(V)$ for which $|\text{Re}(\Upsilon_\theta)(E)| = 1$.

**Proposition 2.2** [9]: Let $E \in \text{Gr}_3(V)$ be a 3-plane in $V$. The following are equivalent:

(i) If $\{u, v, w\}$ is an orthonormal basis of $E$, then $\text{Re}(\Upsilon_\theta)(u,v,w) = \pm 1$.

(ii) $E$ is Lagrangian and $\text{Im}(\Upsilon_\theta)|_E = 0$.

If either of these conditions hold, we say that $E$ is special Lagrangian (SL) of phase $\theta$.

Note that every Lagrangian 3-plane is special Lagrangian for some phase $\theta$. Note also that the $S^1$-action on $V = \mathbb{R}^6 \simeq \mathbb{C}^3$ given by

$$e^{i\theta} \cdot (z_1, z_2, z_3) := (e^{i\theta}z_1, e^{i\theta}z_2, e^{i\theta}z_3),$$

induces a “change-of-phase” $S^1$-action on $\text{Lag}(V) = \{E \in \text{Gr}_3(V) : E \text{ Lagrangian}\}$. Explicitly, letting $\{e_1, \ldots, e_6\}$ denote the standard $\mathbb{R}$-basis of $V$, and letting

$$v_1(\theta) = \cos(\theta)e_1 - \sin(\theta)e_4 \quad w_1(\theta) = \sin(\theta)e_1 + \cos(\theta)e_4$$

$$v_2(\theta) = \cos(\theta)e_2 - \sin(\theta)e_5 \quad w_2(\theta) = \sin(\theta)e_2 + \cos(\theta)e_5$$

$$v_3(\theta) = \cos(\theta)e_3 - \sin(\theta)e_6 \quad w_3(\theta) = \sin(\theta)e_3 + \cos(\theta)e_6$$

the set $\{v_1(\theta), v_2(\theta), v_3(\theta)\}$ is an oriented basis for the SL 3-plane $e^{i\theta} \cdot \text{span}(e_1, e_2, e_3)$ of phase $\theta$, and $\{w_1(\theta), w_2(\theta), w_3(\theta)\}$ is an oriented basis for the SL 3-plane $e^{i\theta} \cdot \text{span}(-e_4, -e_5, -e_6)$ of phase $\theta + \frac{\pi}{2}$.

Now, the SU(3)-action on $V$ induces an SU(3)-action on $\text{Gr}_3(V)$. This action on $\text{Gr}_3(V)$ is not transitive: for example, the subset of $\text{Gr}_3(V)$ consisting of special Lagrangian 3-planes of a fixed phase $\theta$ is an SU(3)-orbit. The corresponding stabilizer will play a crucial role in this section:
Proposition 2.3 [9]: Fix $\theta \in [0, 2\pi)$. The Lie group SU(3) acts transitively on the subset of special Lagrangian 3-planes of phase $\theta$:

$$\{ E \in \text{Gr}_3(V) : |\text{Re}(\Upsilon_\theta)(E)| = 1 \} \subset \text{Gr}_3(V)$$

The stabilizer of the SU(3)-action is isomorphic to SO(3).

We may finally define our primary objects of interest:

Definition: Let $(M^6, \Omega, \Upsilon)$ be a 6-manifold equipped with an SU(3)-structure $(\Omega, \Upsilon)$. Identify each tangent space $(T_xM, \Omega|_x, \Upsilon|_x) \cong (V, \Omega_0, \Upsilon_0)$. Fix $\theta \in [0, 2\pi)$.

A special Lagrangian 3-fold of phase $\theta$ in $M$ is a 3-dimensional immersed submanifold $\Sigma \subset M$ for which each tangent space $T_x\Sigma \subset T_xM$ is a special Lagrangian 3-plane of phase $\theta$.

Note that if $d(\text{Re}(\Upsilon)) = 0$, then Re($\Upsilon$) is a calibration whose calibrated 3-planes are the special Lagrangian 3-planes of phase 0. Thus, in this case, the phase 0 special Lagrangian 3-folds are calibrated submanifolds, and hence are minimal submanifolds of $M$.

Similarly, if $d(\text{Im}(\Upsilon)) = 0$, then Im($\Upsilon$) is a calibration, so the phase $\frac{\pi}{2}$ special Lagrangian 3-folds are calibrated submanifolds of $M$.

2.2 Some SO(3)-Representation Theory

Let the group SO(3) act on $\mathbb{R}^3 = \text{span}\{x, y, z\}$ in the usual way. This action extends to an action of SO(3) on the polynomial ring $\mathbb{R}[x, y, z]$. Let $\mathcal{V}_n \subset \mathbb{R}[x, y, z]$ be the SO(3)-submodule of homogeneous polynomials of degree $n$, and let $\mathcal{H}_n \subset \mathcal{V}_n$ denote the SO(3)-submodule of harmonic polynomials of degree $n$, an irreducible SO(3)-module of dimension $2n + 1$. Every finite dimensional irreducible SO(3)-module is isomorphic to $\mathcal{H}_n$ for some $n$.

The Clebsch-Gordan formula gives the decomposition of a tensor product of irreducible SO(3)-modules:

$$\mathcal{H}_a \otimes \mathcal{H}_b \cong \mathcal{H}_{a+b} \oplus \mathcal{H}_{a+b-1} \oplus \cdots \oplus \mathcal{H}_{|a-b|}. \quad (2.7)$$

Only $\mathcal{H}_0$, $\mathcal{H}_1$, and $\mathcal{H}_2$ will play a role in this work.

2.2.1 SO(3) as a subgroup of SU(3)

In our calculations, we shall need a concrete realization of SO(3) as the stabilizer of a special Lagrangian plane. Let SO(3) act on $V \cong \mathbb{R}^6$ via the identification $V \cong \mathcal{H}_1 \oplus \mathcal{H}_1$, and let $e_1, \ldots, e_6$ be an orthonormal basis of $V$ such that:

- $\langle e_1, e_2, e_3 \rangle \cong \mathcal{H}_1$ and $\langle e_4, e_5, e_6 \rangle \cong \mathcal{H}_1$,
- The map $e_i \mapsto e_{i+3}$ is SO(3)-equivariant.

Then the following forms are invariant under the SO(3)-action on $V$:

$$e^{14} + e^{25} + e^{36}, \quad (e^1 + ie^4) \wedge (e^2 + ie^5) \wedge (e^3 + ie^6).$$

Thus, the action of SO(3) on $V$ gives an embedding SO(3) $\subset$ SU(3). The 3-plane $\langle v_1(\theta), v_2(\theta), v_3(\theta) \rangle = \langle \cos(\theta)e_1 - \sin(\theta)e_4, \cos(\theta)e_2 - \sin(\theta)e_5, \cos(\theta)e_3 - \sin(\theta)e_6 \rangle$ is special Lagrangian with phase $\theta$ and is preserved by the action of SO(3).
2.2.2 Decomposition of the 1-forms on $V$

Let $V \cong \mathcal{H}_1 \oplus \mathcal{H}_1$ be as in the previous section. The SO(3)-irreducible decomposition of the 1-forms on $V$ is given by

$$\Lambda^1(V^*) = T \oplus N,$$

where

$$T = \langle e^1, e^2, e^3 \rangle,$$
$$N = \langle e^4, e^5, e^6 \rangle.$$

As abstract SO(3)-modules, we have isomorphisms

$$\mathcal{H}_1 \cong T \cong N \cong \Lambda^2(T) \cong \Lambda^2(N) \quad \mathcal{H}_2 \cong \text{Sym}^2_0(T) \cong \text{Sym}^2_0(N).$$

**Definition:** We let $b : V \to V^*$ via $X^b := \langle X, \cdot \rangle$ denote the usual (index-lowering) musical isomorphism, and let $\sharp : V^* \to V$ denote its inverse. In the sequel, we let $T, N \subset V^*$ denote the images of $T, N \subset V$ under the $\sharp$ isomorphism.

We also let $\tilde{b} : T \to e^{i\theta} \cdot N^2$ denote the map

$$\alpha \mapsto \tilde{\alpha} = \sin(\theta)\alpha^z + \cos(\theta)J_0(\alpha^z). \quad (2.8)$$

Thus, for example, $(e^1)^\sharp = w_1(\theta)$, etc.

2.2.3 Decomposition of the Quadratic Forms on $V^*$

We seek to decompose $\text{Sym}^2(V^*)$ into SO(3)-irreducible submodules. One way to do this is to use $V^* = T \oplus N$ to split

$$\text{Sym}^2(V^*) \cong (\mathbb{R} \text{Id}_T \oplus \text{Sym}^2_0(T)) \oplus (T \oplus N) \oplus (\mathbb{R} \text{Id}_N \oplus \text{Sym}^2_0(N))$$
$$\cong (\mathbb{R} \text{Id}_T \oplus \text{Sym}^2_0(T)) \oplus (\mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2) \oplus (\mathbb{R} \text{Id}_N \oplus \text{Sym}^2_0(N))$$

Alternatively, recall that $\text{Sym}^2(V^*)$ splits into SU(3)-irreducible submodules as

$$\text{Sym}^2(V^*) \cong \mathbb{R} \text{Id} \oplus \text{Sym}^2_+ \oplus \text{Sym}^2_-$$

Explicitly,

$$\text{Sym}^2_+ = \left\{ \begin{bmatrix} h_2 & h_1 \\ -h_1 & h_2 \end{bmatrix} : h_1 \in \text{Skew}(\mathbb{R}^2), \ h_2 \in \text{Sym}^2_0(\mathbb{R}^3) \right\}$$

$$\text{Sym}^2_- = \left\{ \begin{bmatrix} h' + c'Id_3 & h'' + c''Id_3 \\ h'' + c''Id_3 & -h' - c'Id_3 \end{bmatrix} : c', c'' \in \mathbb{R}, \ h', h'' \in \text{Sym}^2_0(\mathbb{R}^2) \right\}$$

where $\text{Skew}(\mathbb{R}^3) \cong \Lambda^2(\mathbb{R}^3)$ denotes the vector space of skew-symmetric $3 \times 3$ matrices. This description makes it plain that

$$\text{Sym}^2_+ = (\text{Sym}^2_+)_1 \oplus (\text{Sym}^2_+)_2 \quad (2.9)$$

where we are defining

$$(\text{Sym}^2_+)_1 := \left\{ \begin{bmatrix} 0 & h_1 \\ -h_1 & 0 \end{bmatrix} : h_1 \in \text{Skew}(\mathbb{R}^3) \right\} = \text{Sym}^2_+ \cap (T \oplus N) \cong \mathcal{H}_1$$

$$(\text{Sym}^2_+)_2 := \left\{ \begin{bmatrix} h_2 & 0 \\ 0 & h_2 \end{bmatrix} : h_2 \in \text{Sym}^2_0(\mathbb{R}^3) \right\} = \text{Sym}^2_+ \cap (\text{Sym}^2_0(T) \oplus \text{Sym}^2_0(T)) \cong \mathcal{H}_2$$
Similarly, we see that
\[ \text{Sym}^2_+ = (\text{Sym}^2)_0 \oplus (\text{Sym}^2)_2 \oplus (\text{Sym}^2)_6 \oplus (\text{Sym}^2)''_2 \]
where we are defining
\[
\begin{align*}
(\text{Sym}^2)_0 &= \left\{ \begin{bmatrix} c' \text{Id}_3 & 0 \\ 0 & -c' \text{Id}_3 \end{bmatrix} : c' \in \mathbb{R} \right\} \\
(\text{Sym}^2)_2 &= \left\{ \begin{bmatrix} h' & 0 \\ 0 & -h' \end{bmatrix} : h' \in \text{Sym}^2_0(\mathbb{R}^3) \right\} \\
(\text{Sym}^2)_6 &= \left\{ \begin{bmatrix} 0 & c'' \text{Id}_3 \\ c'' \text{Id}_3 & 0 \end{bmatrix} : c'' \in \mathbb{R} \right\} \\
(\text{Sym}^2)''_2 &= \left\{ \begin{bmatrix} 0 & h'' \\ h'' & 0 \end{bmatrix} : h'', h'' \in \text{Sym}^2_0(\mathbb{R}^3) \right\}
\end{align*}
\]

### 2.2.4 Decomposition of the 2-forms on \( V^* \)

We now seek to decompose \( \Lambda^2(V^*) \) into \( \text{SO}(3) \)-irreducible submodules. As noted above, \( \Lambda^2(V^*) \) splits into \( \text{SU}(3) \)-irreducible submodules as
\[ \Lambda^2(V^*) \cong \mathbb{R}\Omega_0 \oplus \Lambda_6^2 \oplus \Lambda_8^2 \] (2.10)

On the other hand, using \( V^* = T \oplus \mathbb{N} \), we may also decompose \( \Lambda^2(V^*) \) as
\[ \Lambda^2(V^*) \cong \Lambda^2(T) \oplus (T \otimes \mathbb{N}) \oplus \Lambda^2(\mathbb{N}). \] (2.11)

We will refine both decompositions \([2.10]\) and \([2.11]\) into \( \text{SO}(3) \)-submodules.

To begin, note first that as \( \text{SO}(3) \)-modules, we have that \( \mathbb{R}\Omega_0 \cong \mathcal{H}_0 \) and \( \Lambda^2(T) \cong \mathcal{H}_1 \) and \( \Lambda^2(\mathbb{N}) \cong \mathcal{H}_1 \) are irreducible. Thus, it remains only to decompose \( \Lambda_6^2 \), \( \Lambda_8^2 \), and \( T \otimes \mathbb{N} \).

**Definition:** Recall the isomorphism \( \rho : \text{Sym}^2_+ \rightarrow \Lambda_8^2 \) defined in (2.5). We define
\[
\begin{align*}
(\Lambda_6^2)_T &= \{ \iota_X(\text{Re}(\mathcal{T}_0)) : X \in T^2 \} \\
(\Lambda_6^2)_N &= \{ \iota_X(\text{Re}(\mathcal{T}_0)) : X \in \mathbb{N}^2 \} \\
(\Lambda_8^2)_1 &= \rho((\text{Sym}^2_+)_1) \\
(\Lambda_8^2)_2 &= \rho((\text{Sym}^2_+)_2)
\end{align*}
\]
and
\[ (T \otimes \mathbb{N})_1 = \{ \alpha_1 \wedge \alpha_2 + J_0 \alpha_2 \wedge J_0 \alpha_1 : \alpha_1 \in T, \alpha_2 \in \mathbb{N} \} \].

**Lemma 2.4:** There exist decompositions
\[ \begin{align*}
\Lambda_6^2 &= (\Lambda_6^2)_T \oplus (\Lambda_6^2)_N \\
\Lambda_8^2 &= (\Lambda_8^2)_1 \oplus (\Lambda_8^2)_2 \\
T \otimes \mathbb{N} &= \mathbb{R}\Omega_0 \oplus (T \otimes \mathbb{N})_1 \oplus (\Lambda_8^2)_2
\end{align*} \] (2.12-2.14)
and these consist of \( \text{SO}(3) \)-irreducible submodules.

Thus, the decomposition
\[ \Lambda^2(V^*) = \mathbb{R}\Omega_0 \oplus [(\Lambda_6^2)_T \oplus (\Lambda_6^2)_N] \oplus [(\Lambda_8^2)_1 \oplus (\Lambda_8^2)_2] \]
is \( \text{SO}(3) \)-irreducible and refines \([2.10]\), while
\[ \Lambda^2(V^*) = \Lambda^2(T) \oplus [\mathbb{R}\Omega_0 \oplus (T \otimes \mathbb{N})_1 \oplus (\Lambda_8^2)_2] \oplus \Lambda^2(\mathbb{N}) \]
is \( \text{SO}(3) \)-irreducible and refines \([2.11]\).
We will refine (2.17) into SO(3)-submodules. To begin, note first that
\( R \) are irreducible as SO(3)-modules, while \( \Lambda_2^3 \) are not.

Decomposition (2.13) follows from applying the isomorphism \( \rho: \text{Sym}_2^3 \to \Lambda_8^3 \) to the irreducible decomposition (2.9) of \( \text{Sym}_2^3 \).

For decomposition (2.14), note that as an SO(3)-module
\[ T \otimes N \cong \mathcal{H}_1 \otimes \mathcal{H}_1 \cong \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2. \]

\[ (2.15) \]

The only trivial SO(3)-module contained in \( \Lambda^2(V^*) \) is \( \mathbb{R}\Omega_0 \), so this must correspond to the trivial component of (2.15). Similarly, the only SO(3)-module isomorphic to \( \mathcal{H}_2 \) contained in \( \Lambda^2(V^*) \) is \( (\Lambda_8^3)_2 \), so this must correspond to the \( \mathcal{H}_2 \) component of (2.15). The inclusion \( (T \otimes N)_1 \subset T \otimes N \) is clear by construction, and since \( (T \otimes N)_1 \cong \mathcal{H}_1 \) we have demonstrated that decomposition (2.14) holds.

\[ \square \]

**Definition:** Recall the isomorphism \( \rho: \text{Sym}_2^3 \to \Lambda_8^3 \) defined in (2.5) and the set \( \{w_1(\theta), w_2(\theta), w_3(\theta)\} \) defined in (2.6). Consider the isomorphisms of SO(3)-modules given by
\[ e^{i\theta} \cdot N^2 \to \text{Skew}(\mathbb{R}^3) \to (\Lambda_8^3)_1 \]
\[ a_p w_p(\theta) \to h = \begin{bmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{bmatrix} \mapsto \frac{1}{\sqrt{2}} \rho \begin{bmatrix} 0 & h \\ h & 0 \end{bmatrix} \]

We will let
\[ \zeta: (\Lambda_8^3)_1 \to e^{i\theta} \cdot N^2 \]
\[ (2.16) \]
denote the inverse of this isomorphism. This map is, in fact, an isometry with respect to the given inner products on \( e^{i\theta} \cdot N^2 \) and \( (\Lambda_8^3)_1 \), due to the factor of \( \frac{1}{\sqrt{2}} \). \[ \square \]

### 2.2.5 Decomposition of the 3-forms on \( V^* \)

We now seek to decompose \( \Lambda^3(V^*) \) into SO(3)-irreducible submodules. As noted above, \( \Lambda^3(V^*) \) splits into SU(3)-irreducible submodules as
\[ \Lambda^2(V^*) \cong \mathbb{R}\text{Re}(\Upsilon_0) \oplus \mathbb{R}\text{Im}(\Upsilon_0) \oplus \Lambda_6^3 \oplus \Lambda_1^3 \]
\[ (2.17) \]

On the other hand, using \( V^* = T \oplus N \), we may also decompose \( \Lambda^2(V^*) \) as
\[ \Lambda^2(V^*) \cong \Lambda^3(T) \oplus (\Lambda^2(T) \otimes N) \oplus (T \otimes \Lambda^2(N)) \oplus \Lambda^3(N). \]
\[ (2.18) \]

We will refine (2.17) into SO(3)-submodules. To begin, note first that \( \mathbb{R}\text{Re}(\Upsilon_0) \cong \mathbb{R}\text{Im}(\Upsilon_0) \cong \mathcal{H}_0 \) are irreducible as SO(3)-modules, while \( \Lambda_6^3 \) and \( \Lambda_1^3 \) are not.

**Definition:** Recall the isomorphism \( \chi: \text{Sym}_2^3 \to \Lambda_{12}^3 \) of (2.5). We define
\[ (\Lambda_6^3)_T = \{ \alpha \wedge \Omega_0: \alpha \in T \} \]
\[ (\Lambda_6^3)_N = \{ \alpha \wedge \Omega_0: \alpha \in N \} \]
\[ (\Lambda_3^3)_T = \chi((\text{Sym}_2^3)_0') \]
\[ (\Lambda_3^3)_N = \chi((\text{Sym}_2^3)_0) \]
\[ (\Lambda_3^3)_0' = \chi((\text{Sym}_2^3)_0') \]
\[ (\Lambda_3^3)_0'' = \chi((\text{Sym}_2^3)_0) \]

**Lemma 2.5:** The decompositions
\[ \Lambda_6^3 = (\Lambda_6^3)_T \oplus (\Lambda_6^3)_N \]
\[ \Lambda_{12}^3 = (\Lambda_{12}^3)_T \oplus (\Lambda_{12}^3)_N \]
\[ (2.19) \]
consist of SO(3)-irreducible submodules.

**Definition:** We define maps \( \uparrow: (\Lambda^3_{12})_0' \rightarrow \mathbb{R} \) and \( \downarrow: (\Lambda^3_{12})_0'' \rightarrow \mathbb{R} \) to be the unique vector space isomorphisms for which

\[
(-\text{Re}(\gamma_0) + 4e^{123}) \uparrow = 2\sqrt{3} \quad (2.19)
\]

\[
(-\text{Im}(\gamma_0) - 4e^{456}) \downarrow = 2\sqrt{3} \quad (2.20)
\]

These maps are isometries (due to the choice of \(2\sqrt{3}\)) with respect to our inner product (2.4).

**Remark:** To refine (2.18) into SO(3)-irreducible submodules, one simply has to decompose \(\Lambda^2(T) \otimes N\) and \(\Lambda^2(N) \otimes T\) into irreducibles. This can be done by, say, tracing through the isomorphisms

\[
\Lambda^2(T) \otimes N \cong T \otimes N \cong T \otimes T \cong \mathbb{R} \oplus \text{Sym}^2(T) \oplus \Lambda^2(T)
\]

and similarly for \(\Lambda^2(N) \otimes T\). Since we will not need such a refinement for this work, we leave the details to the interested reader. \(\square\)

### 2.3 The Refined Torsion Forms

Let \((M^6, \Omega, \Upsilon)\) be a 6-manifold equipped with an SU(3)-structure \((\Omega, \Upsilon)\). Fix a point \(x \in M\), choose an arbitrary phase 0 special Lagrangian 3-plane \(T^2 \subset T_x M\), and let \(N^2 \subset T_x M\) denote its orthogonal 3-plane. Our purpose in this section is to understand how the torsion of the SU(3)-structure decomposes with respect to the splitting

\[
T_x M = T^2 \oplus N^2.
\]

In \$2.3.1\$, we use Lemmas 2.4 and 2.5 to break the torsion forms \(\tau_0, \tau_1, \ldots, \tau_5\) into SO(3)-irreducible pieces called *refined torsion forms*. Separately, in \$2.3.2\$, we set up the SU(3)-coframe bundle \(\pi: F_{SU(3)} \rightarrow M\) following [1], repackaging the original SU(3) torsion forms \(\tau_0, \tau_1, \ldots, \tau_5\) as a pair of functions

\[
T = (T_{ij}): F_{SU(3)} \rightarrow \text{Mat}_{6 \times 6}(\mathbb{R}) \cong \mathbb{R}^{36}
\]

\[
U = (U_i): F_{SU(3)} \rightarrow \mathbb{R}^6
\]

Finally, in \$2.3.3\$, we express the functions \(T_{ij}\) and \(U_i\) in terms of the (pullbacks of the) refined torsion forms.

#### 2.3.1 The Refined Torsion Forms in a Local SO(3)-Frame

Fix \(x \in M\) and split \(T^*_x M = T \oplus N\) as above. All of our calculations in this subsection will be done pointwise, and we will frequently suppress reference to \(x \in M\). By Lemmas 3.5 and 3.6, the torsion forms decompose into SO(3)-irreducible pieces as follows:

\[
\tau_0 = \tau_0 \quad \tau_2 = (\tau_2)_1 + (\tau_2)_2 \quad \tau_3 = (\tau_3)_0' + (\tau_3)_0'' + (\tau_3)_2 + (\tau_3)_2' \quad \tau_4 = (\tau_4)_T + (\tau_4)_N \quad (2.21a)
\]

\[
\tau_5 = (\tau_5)_T + (\tau_5)_N \quad (2.21b)
\]

where here

\[
(\tau_2)_1, (\tau_2)_1' \in (\Lambda^2_5)_1 \quad (\tau_3)_0' \in (\Lambda^3_{12})_0' \quad (\tau_3)_2 \in (\Lambda^3_{12})_2 \quad (\tau_4)_T, (\tau_5)_T \in T
\]

\[
(\tau_2)_2, (\tau_2)_2' \in (\Lambda^2_5)_2 \quad (\tau_3)_0'' \in (\Lambda^3_{12})_0'' \quad (\tau_3)_2' \in (\Lambda^3_{12})_2' \quad (\tau_4)_N, (\tau_5)_N \in N
\]
We refer to $\tau_0, \tilde{\tau}_0, (\tau_2)_1, \ldots, (\tau_5)_N$ are refined torsion forms of the SU(3)-structure at $x$ relative to the splitting $T_x^*M = T \oplus N$.

We seek to express the refined torsion in terms of a local SO(3)-frame. To that end, let \{e_1, \ldots, e_6\} be an orthonormal basis for $T_xM$ for which $T^2 = \text{span}(e_1, e_2, e_3)$ and $N^2 = \text{span}(e_4, e_5, e_6)$. Let \{e^1, \ldots, e^6\} denote the dual basis for $T^*_xM$.

**Index Ranges:** We will employ the following index ranges: $1 \leq p, q \leq 3$ and $4 \leq \alpha, \beta \leq 6$ and $1 \leq i, j, k, \ell, m \leq 6$ and $1 \leq \delta \leq 5$.

**Definition:** Define the 2-forms

\[
\begin{align*}
\Gamma_1 &= -e^{23} - e^{56} \\
\Gamma_2 &= -e^{31} - e^{64} \\
\Gamma_3 &= -e^{12} - e^{45}
\end{align*}
\]

\[
\begin{align*}
\Upsilon_1 &= e^{26} + e^{35} \\
\Upsilon_2 &= e^{16} + e^{34} \\
\Upsilon_3 &= e^{15} + e^{24}
\end{align*}
\]

These 2-forms were obtained by applying $\rho: (\text{Sym}^2_+)_1 \oplus (\text{Sym}^2_+)_2 \to (\Lambda^2_8)_1 \oplus (\Lambda^2_8)_2$ of (2.5) to a suitable basis of $\text{Sym}^2_+$. 

**Lemma 2.6:** We have that:

(a) \{\Gamma_1, \Gamma_2, \Gamma_3\} is a basis of $(\Lambda^2_8)_1$

(b) \{\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4, \Upsilon_5\} is a basis of $(\Lambda^2_8)_2$

**Definition:** Define the 3-forms

\[
\begin{align*}
\Theta_1 &= -e^{245} - e^{364} \\
\Theta_2 &= -e^{145} - e^{356} \\
\Theta_3 &= -e^{164} - e^{256} \\
\Theta_4 &= -e^{156} + e^{264} \\
\Theta_5 &= e^{264} + e^{345}
\end{align*}
\]

\[
\begin{align*}
\Delta_1 &= e^{125} + e^{316} \\
\Delta_2 &= e^{124} + e^{236} \\
\Delta_3 &= e^{314} + e^{235} \\
\Delta_4 &= -e^{315} + e^{234} \\
\Delta_5 &= -e^{126} + e^{315}
\end{align*}
\]

and

\[
\begin{align*}
\Theta_0 &= -\text{Re}(\Upsilon_0) + 4e^{123} \\
\Delta_0 &= -\text{Im}(\Upsilon_0) - 4e^{456}
\end{align*}
\]

These 3-forms were obtained by applying the isomorphism

\[
\chi: (\text{Sym}^2_-)_0 \oplus (\text{Sym}^2_-)_0' \oplus (\text{Sym}^2_-)_2 \oplus (\text{Sym}^2_-)_2' \to (\Lambda^3_1)_0 \oplus (\Lambda^3_2)_0' \oplus (\Lambda^3_1)_0' \oplus (\Lambda^3_2)_0''
\]

of (2.5) to a suitable basis of $\text{Sym}^2_-$. 

**Lemma 2.7:** We have that:

(a) \{\Theta_0\} is a basis of $(\Lambda^3_1)_0$

(b) \{\Delta_0\} is a basis of $(\Lambda^3_1)_0''$

(c) \{\Theta_1, \ldots, \Theta_5\} is a basis of $(\Lambda^3_1)_2$

(d) \{\Delta_1, \ldots, \Delta_5\} is a basis of $(\Lambda^3_1)_2''$

We now express $(\tau_2)_T, (\tilde{\tau}_2)_T, \ldots, (\tau_5)_N$ in terms of the above bases. That is, we define functions
A_p, B_δ, C_p, D_δ and E_δ, E_0, F_δ, F_0 and G_p, J_p, M_p, N_p by:

\[
\begin{align*}
(\tau_2)_1 &= 4A_p \Gamma_p & (\tau_3)_0 &= 4E_0 \Theta_0 & (\tau_4)_\tau &= 12G_p \omega^p \\
(\tau_2)_2 &= 4B_\delta \Upsilon_\delta & (\tau_3)_2 &= 4E_\delta \Theta_\delta & (\tau_4)_N &= 12J_p \omega^{p+3} \\
(\tau_2'_1) &= 4C_p \Gamma_p & (\tau_3'_0) &= 4F_0 \Delta_0 & (\tau_5)_\tau &= 3M_p \omega^p \\
(\tau_2'_2) &= 4D_\delta \Upsilon_\delta & (\tau_3'_2) &= 4F_\delta \Delta_\delta & (\tau_5)_N &= 3N_p \omega^{p+3}
\end{align*}
\]

The various factors of 3, 4, and 12 are included simply for the sake of clearing future denominators.

Note that the bases of Lemmas 2.8 and 2.9 are orthogonal but not orthonormal with respect to the inner product \((2.4)\) on \(\Lambda^k(V^*)\). Indeed, we have:

\[
\begin{align*}
\|\Gamma_p\| &= \sqrt{2} & \|\Theta_\delta\| &= \sqrt{2} & \|\Theta_0\| &= 2\sqrt{3} \\
\|\Upsilon_\delta\| &= \sqrt{2} & \|\Delta_\delta\| &= \sqrt{2} & \|\Delta_0\| &= 2\sqrt{3}
\end{align*}
\]

Thus, in terms of the isometric isomorphisms \((2.8), (2.16), (2.19), (2.20)\) of \(\S 2.2\), we have:

\[
\begin{align*}
[(\tau_3)_0]^\dagger &= 8\sqrt{3} E_0 & [(\tau_2)_1]^2 &= 4\sqrt{2} A_p w_p(\theta) & [(\tau_5)_\tau]^\delta &= 3M_p w_p(\theta) \\
[(\tau_3)_0]^\dagger &= 8\sqrt{3} F_0 & [(\tau_2)_1]^2 &= 4\sqrt{2} C_p w_p(\theta) & [J(\tau_5)_N]^\delta &= 3N_p w_p(\theta)
\end{align*}
\]

We will need these for our calculations in \(\S 2.4\).

### 2.3.2 The Torsion Functions \(T_{ij}\) and \(U_i\)

Let \((M^6, \Omega, \Upsilon)\) be a 6-manifold with an SU(3)-structure \((\Omega, \Upsilon)\), and let \(g\) denote the underlying Riemannian metric. Let \(F_{SO(6)} \to M\) denote the oriented orthonormal coframe bundle of \(g\), and let \(\omega = (\omega^1, \ldots, \omega^6) \in \Omega^1(\Lambda^1; \mathbb{R}^6)\) denote the tautological 1-form. By the Fundamental Lemma of Riemannian Geometry, there exists a unique 1-form \(\psi \in \Omega^1(F_{SO(6)}; \mathfrak{so}(6))\), the Levi-Civita connection form of \(g\), satisfying the first structure equation

\[
d\omega = -\psi \wedge \omega.
\]

Let \(\pi: F_{SU(3)} \to M\) denote the SU(3)-coframe bundle of \(M\). Restricted to \(F_{SU(3)} \subset F_{SO(6)}\), the Levi-Civita 1-form \(\psi\) is no longer a connection 1-form in general. Indeed, according to the splitting \(\mathfrak{so}(6) = \mathfrak{su}(3) \oplus \mathbb{R}^6 \oplus \mathbb{R}\), we have the decomposition

\[
\psi = \gamma + \lambda + \mu,
\]

where \(\gamma = (\gamma_{ij}) \in \Omega^1(\mathfrak{su}(3))\) is a connection 1-form (the so-called natural connection of the SU(3)-structure) and \(\lambda \in \Omega^1(\Lambda^1; \mathbb{R}^6)\) and \(\mu \in \Omega^1(F_{SU(3)}; \mathbb{R})\) are \(\pi\)-semibasic 1-forms. Here, we are viewing

\[
\begin{align*}
\mathbb{R}^6 &\simeq \{ (\epsilon_{ijk}v_k) \in \mathfrak{so}(6) : (v_i) \in \mathbb{R}^6 \} \\
\mathbb{R} &\simeq \{ (a\Omega_{ij}) \in \mathfrak{so}(6) : a \in \mathbb{R} \}
\end{align*}
\]

so that \(\lambda\) and \(\mu\) take the form

\[
\lambda = \begin{bmatrix}
0 & \lambda_3 & -\lambda_2 & 0 & -\lambda_6 & \lambda_5 \\
-\lambda_3 & 0 & \lambda_1 & \lambda_6 & 0 & -\lambda_4 \\
\lambda_2 & -\lambda_1 & 0 & -\lambda_5 & \lambda_4 & 0 \\
0 & -\lambda_6 & \lambda_5 & 0 & -\lambda_3 & \lambda_2 \\
\lambda_6 & 0 & -\lambda_4 & \lambda_3 & 0 & -\lambda_1 \\
-\lambda_5 & \lambda_4 & 0 & -\lambda_2 & \lambda_1 & 0
\end{bmatrix}
\]

\[
\mu = \begin{bmatrix}
0 & 0 & 0 & \mu & 0 & 0 \\
0 & 0 & 0 & 0 & \mu & 0 \\
0 & 0 & 0 & 0 & 0 & \mu \\
\mu & 0 & 0 & 0 & 0 & 0 \\
0 & -\mu & 0 & 0 & 0 & 0 \\
0 & 0 & -\mu & 0 & 0 & 0
\end{bmatrix}
\]
Since \( \lambda \) and \( \mu \) are \( \pi \)-semibasic, we may write

\[
\lambda_i = T_{ij} \omega^j \quad \quad \mu = U_j \omega^j
\]

for some matrix-valued function \( T = (T_{ij}) : F_{SU(3)} \to \text{Mat}_{6 \times 6}(\mathbb{R}) \) and vector-valued function \( U = (U_i) : F_{SU(3)} \to \mathbb{R}^6 \). The 1-forms \( \lambda, \mu \), and hence the functions \( T_{ij} \) and \( U_i \), encode the torsion of the SU(3)-structure. In this notation, the first structure equation reads

\[
d\omega_i = - (\gamma_{ij} + \epsilon_{ijk} \lambda_k + \Omega_{ij} \mu) \wedge \omega_j. \tag{2.25}
\]

**Remark:** The reader may wonder how the functions \( T_{ij}, U_i \) on \( F_{SU(3)} \) are related to the forms \( \tau_0, \tau_1, \tau_2, \tau_3 \) on \( M \). In [1], the authors derive expressions for the pullbacks of the torsion forms in terms of \( T_{ij}, U_i \). That is, they derive

\[
\pi^*(\tau_0) = -\frac{1}{3} \Omega_{ij} T_{ij} \quad \quad \pi^*(\tau_4) = \epsilon_{ijk} T_{ij} \omega^k
\]

\[
\pi^*(\tau_7) = \frac{1}{3} T_{ii} \quad \quad \pi^*(\tau_8) = \epsilon_{ijk} T_{ij} \omega^k + 3 \Omega_{ik} U_i \omega^k
\]

along with similar (more complicated) formulas for \( \pi^*(\tau_2), \pi^*(\tau_5), \pi^*(\tau_7) \). In the next section, we will exhibit a sort of inverse to this, expressing the \( T_{ij}, U_i \) in terms of the refined torsion forms \( \pi^*(\tau_0), \pi^*(\tau_7), \ldots, \pi^*((\tau_5)_r), \pi^*((\tau_5)_n) \). \( \square \)

### 2.3.3 Decomposition of the Torsion Functions

For our computations in §2.4, we will need to express the torsion functions \( T_{ij}, U_i \) in terms of the functions \( A_p, B_q, \ldots, N_p \). To this end, we will continue to work on the total space of the SU(3)-coframe bundle \( \pi : F_{SU(3)} \to M \), pulling back all of the quantities defined on \( M \) to \( F_{SU(3)} \). Following convention, we systematically omit \( \pi^* \) from the notation, so that (for example) \( \pi^*(\tau_0) \) will simply be denoted \( \tau_0 \), etc. Note, however, that \( \pi^*(e^j) = \omega_j \).

To begin, recall that the torsion forms \( \tau_0, \tau_1, \tau_2, \tau_3 \) satisfy

\[
d\Omega = 3 \tau_0 \text{Re}(\Upsilon) + 3 \tau_0 \text{Im}(\Upsilon) + \tau_3 + \tau_4 \wedge \Omega
\]

\[
d\text{Re}(\Upsilon) = 2 \tau_0 \Omega^2 + \tau_5 \wedge \text{Re}(\Upsilon) + \tau_2 \wedge \Omega
\]

\[
d\text{Im}(\Upsilon) = -2 \tau_0 \Omega^2 - J \tau_5 \wedge \text{Re}(\Upsilon) + \tilde{\tau}_2 \wedge \Omega
\]

Into the left-hand sides, we substitute (2.1) and use the first structure equation (2.25) to obtain

\[
\check{e}_{ijk} T_{il} \omega^{jk} = 3 \tau_0 \text{Re}(\Upsilon) + 3 \tau_0 \text{Im}(\Upsilon) + \tau_3 + \tau_4 \wedge \Omega
\]

\[
-\frac{1}{2} (\Omega_{km} \Omega_{lj} - \Omega_{kj} \Omega_{lm}) T_{mi} + \check{e}_{ijk} U_i \omega^{jk} = 2 \tau_0 \Omega^2 + \tau_5 \wedge \text{Re}(\Upsilon) + \tau_2 \wedge \Omega
\]

\[
-\frac{1}{2} (2 \Omega_{kl} T_{ji} - \epsilon_{ijk} U_i) \omega^{jk} = -2 \tau_0 \Omega^2 - J \tau_5 \wedge \text{Re}(\Upsilon) + \tilde{\tau}_2 \wedge \Omega
\]

Into the right-hand sides, we again substitute (2.1), as well as the expansions (2.21) and (2.23).

Upon equating coefficients, we obtain a system of 56 = (7) + (7) linear equations relating the 42 = 6^2 + 6 functions \( T_{ij}, U_j \) on the left side to the 42 = dim(H^0(H^3(\text{su}(3)))) functions \( \tau_0, \tau_0, A_p, B_q, \ldots, M_p, N_p \) on the right side. One can then use a computer algebra system (we have used MAPLE) to solve this linear system for the \( T_{ij} \) and \( U_i \).
We now exhibit the result, taking advantage of the SO(3)-irreducible splitting
\[
\text{Mat}_{6 \times 6}(\mathbb{R}) \cong V^* \otimes V^* \cong (T \otimes T) \oplus 2(T \otimes N) \oplus (N \otimes N) \\
\cong (\Lambda^2(T) \oplus \text{Sym}^2_0(T) \oplus \mathbb{R}) \oplus 2(\mathbb{R} \oplus (T \otimes N)_1 \oplus (T \otimes N)_2) \\
\oplus (\Lambda^2(N) \oplus \text{Sym}^2_0(N) \oplus \mathbb{R})
\]
to highlight the structure of the solution. We have
\[
\frac{1}{2} \begin{bmatrix}
0 & T_{12} - T_{21} & T_{13} - T_{31} \\
T_{21} - T_{12} & 0 & T_{23} - T_{32} \\
T_{31} - T_{13} & T_{32} - T_{23} & 0
\end{bmatrix} = \begin{bmatrix}
0 & -C_3 + 3G_3 & C_2 - 3G_2 \\
C_3 - 3G_3 & 0 & -C_1 + 3G_1 \\
-C_2 + 3G_2 & C_1 - 3G_1 & 0
\end{bmatrix}
\]
(2.26a)
\[
\frac{1}{2} \begin{bmatrix}
2T_{11} & T_{12} + T_{21} & T_{13} + T_{31} \\
T_{21} + T_{12} & 2T_{22} & T_{23} + T_{32} \\
T_{31} + T_{13} & T_{32} + T_{23} & 2T_{33}
\end{bmatrix} = \begin{bmatrix}
B_4 - F_4 & B_3 - F_3 & B_2 - F_2 \\
B_3 - F_3 & -B_4 + B_5 + F_4 - F_5 & B_1 - F_1 \\
B_2 - F_2 & B_1 - F_1 & -B_5 + F_5
\end{bmatrix} \\
+ \left(\frac{1}{2} \hat{\tau}_0 - 2F_0\right) \text{Id}_3
\]
(2.26b)
corresponding to $T \otimes T \cong \Lambda^2(T) \oplus \text{Sym}^2_0(T) \oplus \mathbb{R}$ and
\[
\frac{1}{2} \begin{bmatrix}
T_{14} - T_{41} & T_{24} - T_{42} & T_{34} - T_{43} \\
T_{45} - T_{51} & T_{25} - T_{52} & T_{35} - T_{53} \\
T_{16} - T_{61} & T_{26} - T_{62} & T_{36} - T_{63}
\end{bmatrix} = \begin{bmatrix}
D_4 & D_3 - 3J_3 & D_2 + 3J_2 \\
D_3 + 3J_3 & -D_4 + D_5 & D_1 - 3J_1 \\
D_2 - 3J_2 & D_1 + 3J_1 & -D_5
\end{bmatrix} - \frac{1}{2} \hat{\tau}_0 \text{Id}_3
\]
(2.27a)
\[
\frac{1}{2} \begin{bmatrix}
T_{14} + T_{41} & T_{24} + T_{42} & T_{34} + T_{43} \\
T_{45} + T_{51} & T_{25} + T_{52} & T_{35} + T_{53} \\
T_{16} + T_{61} & T_{26} + T_{62} & T_{36} + T_{63}
\end{bmatrix} = \begin{bmatrix}
E_4 & A_3 + E_3 & -A_2 + E_2 \\
-A_3 + E_3 & -E_4 + E_5 & A_1 + E_1 \\
A_2 + E_2 & -A_1 + E_1 & -E_5
\end{bmatrix} + 2E_0 \text{Id}_3
\]
(2.27b)
corresponding to $T \otimes N \cong \mathbb{R} \oplus (T \otimes N)_1 \oplus (T \otimes N)_2$, and
\[
\frac{1}{2} \begin{bmatrix}
2T_{45} & T_{45} - T_{54} & T_{46} - T_{64} \\
T_{54} - T_{45} & 0 & T_{56} - T_{65} \\
T_{64} - T_{46} & T_{65} - T_{56} & 0
\end{bmatrix} = \begin{bmatrix}
0 & -(C_3 + 3G_3) & C_2 + 3G_2 \\
C_3 + 3G_3 & 0 & -(C_1 + 3G_1) \\
-(C_2 + 3G_2) & C_1 + 3G_1 & 0
\end{bmatrix}
\]
(2.28a)
\[
\frac{1}{2} \begin{bmatrix}
2T_{45} & T_{45} + T_{54} & T_{46} + T_{64} \\
T_{54} + T_{45} & 2T_{55} & T_{56} + T_{65} \\
T_{64} + T_{46} & T_{65} & 2T_{66}
\end{bmatrix} = \begin{bmatrix}
B_4 + F_4 & B_3 + F_3 & B_2 + F_2 \\
B_3 + F_3 & -B_4 + B_5 + F_4 + F_5 & B_1 + F_1 \\
B_2 + F_2 & B_1 + F_1 & -B_5 - F_5
\end{bmatrix} \\
+ \left(\frac{1}{2} \hat{\tau}_0 + 2F_0\right) \text{Id}_3
\]
(2.28b)
corresponding to $N \otimes N \cong \Lambda^2(N) \oplus \text{Sym}^2_0(N) \oplus \mathbb{R}$. We also have
\[
\begin{bmatrix}
U_1 \\
U_2 \\
U_3
\end{bmatrix} = \begin{bmatrix}
-4J_1 + N_1 \\
-4J_2 + N_2 \\
-4J_3 + N_3
\end{bmatrix}, \\
\begin{bmatrix}
U_4 \\
U_5 \\
U_6
\end{bmatrix} = \begin{bmatrix}
4G_1 - M_1 \\
4G_2 - M_2 \\
4G_3 - M_3
\end{bmatrix}.
\]
(2.29)
2.4 Mean Curvature of Special Lagrangian 3-Folds

In this section, we derive a formula (Theorem 2.10) for the mean curvature of a special Lagrangian 3-fold in an arbitrary 6-manifold \((M, \Omega, \Upsilon)\) with \(\text{SU}(3)\)-structure \((\Omega, \Upsilon)\). In the process, we observe a necessary condition (Theorem 2.8) for the local existence of special Lagrangian 3-folds.

We continue to let \(\pi: F_{\text{SU}(3)} \to M\) denote the \(\text{SU}(3)\)-coframe bundle of \(M\), and \(\omega = (\omega_T, \omega_N) \in \Omega^1(F_{\text{SU}(3)}; T^\mathbb{C} \oplus \mathbb{N}^2)\) denote the tautological 1-form. As above, \(\gamma = (\gamma_{ij}) \in \Omega^1(F_{\text{SU}(3)}; \mathfrak{su}(3))\) denotes the natural connection 1-form, while \(\lambda = (\lambda_{ij}) \in \Omega^1(F_{\text{SU}(3)}; \mathbb{R}^6)\) and \(\mu \in \Omega^1(F_{\text{SU}(3)}; \mathbb{R})\) are \(\pi\)-semibasic 1-forms encoding the torsion of \((\Omega, \Upsilon)\).

Fix a phase \(\theta \in [0, 2\pi)\) once and for all, and define 1-forms \(\eta, \xi \in \Omega^1(F_{\text{SU}(3)}; \mathbb{R}^6)\) via

\[
\eta = \Re(e^{i\theta}(\omega_T + i\omega_N)) = \cos(\theta)\omega_T - \sin(\theta)\omega_N
\]

\[
\xi = \Im(e^{i\theta}(\omega_T + i\omega_N)) = \sin(\theta)\omega_T + \cos(\theta)\omega_N.
\]

Let \(f: \Sigma \to M^7\) denote an immersion of a phase \(\theta\) special Lagrangian 3-fold into \(M\), and let \(f^*(F_{\text{SU}(3)}) \to \Sigma\) denote the pullback bundle. Let \(B \subset f^*(F_{\text{SU}(3)})\) denote the subbundle of coframes adapted to \(\Sigma\), i.e., the subbundle whose fiber over \(x \in \Sigma\) is

\[
B|_x = \{ u \in f^*(F_{\text{SU}(3)})|_x : u(T_x \Sigma) = e^{i\theta} \cdot T^\mathbb{C}\}
\]

\[
= \{ u \in f^*(F_{\text{SU}(3)})|_x : u(T_x \Sigma) = \text{span}(v_1(\theta), v_2(\theta), v_3(\theta))\}
\]

in the notation of (2.6). We recall (Proposition 2.3) that \(\text{SU}(3)\) acts transitively on the set of special Lagrangian 3-planes with stabilizer \(\text{SO}(3)\), so \(B \to \Sigma\) is a well-defined \(\text{SO}(3)\)-bundle. Note that on \(B\), we have

\[
\xi = 0.
\]

For the rest of §2.4, all of our calculations will be done on the subbundle \(B \subset F_{\text{SU}(3)}\).

We begin by expressing \(\gamma, \lambda, \) and \(\mu\) as block matrices with respect to the splitting \(T_x M \simeq T^\mathbb{C} \oplus \mathbb{N}^2\). The 1-form \(\gamma \in \Omega^1(B; \mathfrak{su}(3))\) takes the block form

\[
\gamma = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} = \begin{bmatrix} 0 & \alpha_{12} & -\alpha_{13} & \beta_{11} & \beta_{12} & \beta_{13} \\ -\alpha_{12} & 0 & \alpha_{23} & \beta_{21} & \beta_{22} & \beta_{23} \\ -\alpha_{13} & -\alpha_{23} & 0 & \beta_{31} & \beta_{32} & \beta_{33} \\ -\beta_{11} & -\beta_{12} & -\beta_{13} & 0 & \alpha_{12} & \alpha_{13} \\ -\beta_{21} & -\beta_{22} & -\beta_{23} & -\alpha_{12} & 0 & \alpha_{23} \\ -\beta_{31} & -\beta_{32} & -\beta_{33} & -\alpha_{13} & -\alpha_{23} & 0 \end{bmatrix}
\]

where \(\alpha_{pq}, \beta_{pq} \in \Omega^1(B)\) are 1-forms with \(\beta_{pq} = \bar{\beta}_{qp}\) and \(\beta_{11} + \beta_{22} + \beta_{33} = 0\). As in §2.3.2, the 1-forms \(\lambda \in \Omega^1(B; \mathbb{R}^6)\) and \(\mu \in \Omega^1(B; \mathbb{R})\) break into blocks as

\[
\lambda = \begin{bmatrix} \lambda_T & \lambda_N \\ \lambda_N & \lambda_T \end{bmatrix} = \begin{bmatrix} 0 & \lambda_3 & -\lambda_2 & 0 & -\lambda_6 & \lambda_5 \\ -\lambda_3 & 0 & \lambda_1 & \lambda_6 & 0 & -\lambda_4 \\ \lambda_2 & -\lambda_1 & 0 & -\lambda_5 & \lambda_4 & 0 \\ 0 & -\lambda_6 & \lambda_5 & 0 & -\lambda_3 & \lambda_2 \\ \lambda_6 & 0 & -\lambda_4 & \lambda_3 & 0 & -\lambda_1 \\ -\lambda_5 & \lambda_4 & 0 & -\lambda_2 & \lambda_1 & 0 \end{bmatrix}
\]

\[
\mu = \begin{bmatrix} 0 & \mu \text{Id}_3 \\ -\mu \text{Id}_3 & 0 \end{bmatrix}.
\]
Next, we adapt our matrix-valued forms to the geometry at hand, which is that of a splitting $T_x M \simeq T_x \Sigma \oplus (T_x \Sigma)^\perp$. To this end, recall that the change-of-phase action on $V \simeq \mathbb{C}^3$ is the $S^1$-action given by $e^{i \theta} \cdot (z_1, z_2, z_3) = (e^{i \theta} z_1, e^{i \theta} z_2, e^{i \theta} z_3)$. Regarding this $S^1$ as a subgroup of $U(3) \leq SO(6)$, we consider the induced $\text{Ad}(S^1)$-action on $\mathfrak{so}(6)$, given explicitly in block form as

$$
\text{Ad}_{\theta} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \cos(\theta) \text{Id}_3 & -\sin(\theta) \text{Id}_3 \\ \sin(\theta) \text{Id}_3 & \cos(\theta) \text{Id}_3 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \cos(\theta) \text{Id}_3 & \sin(\theta) \text{Id}_3 \\ -\sin(\theta) \text{Id}_3 & \cos(\theta) \text{Id}_3 \end{bmatrix}.
$$

Viewing $\mathfrak{so}(6) = \mathfrak{su}(3) \oplus \mathbb{R} \oplus \mathbb{R}^6$, note that our $\text{Ad}(S^1)$-action is trivial on the $\mathfrak{su}(3)$- and $\mathbb{R}$-summands, and thus

$$\text{Ad}_{\theta} \gamma = \gamma \quad \text{Ad}_{\theta} \mu = \mu.$$

However, the $\text{Ad}(S^1)$-action is non-trivial on the $\mathbb{R}^6$-summand. We therefore set $\tilde{\lambda} = \text{Ad}_{\theta} \lambda$, writing

$$
\tilde{\lambda} = \begin{bmatrix} \tilde{\lambda}_T \\ \tilde{\lambda}_N \end{bmatrix} = \begin{bmatrix} \tilde{\lambda}_1 & -\tilde{\lambda}_2 & 0 & -\tilde{\lambda}_6 & -\tilde{\lambda}_5 \\ -\tilde{\lambda}_3 & 0 & \tilde{\lambda}_1 & \tilde{\lambda}_6 & 0 & -\tilde{\lambda}_4 \\ -\tilde{\lambda}_2 & -\tilde{\lambda}_1 & 0 & -\tilde{\lambda}_5 & \tilde{\lambda}_4 & 0 \\ 0 & -\tilde{\lambda}_6 & \tilde{\lambda}_5 & 0 & -\tilde{\lambda}_3 & \tilde{\lambda}_2 \\ \tilde{\lambda}_6 & 0 & -\tilde{\lambda}_4 & \tilde{\lambda}_3 & 0 & -\tilde{\lambda}_1 \\ -\tilde{\lambda}_5 & \tilde{\lambda}_4 & 0 & -\tilde{\lambda}_2 & \tilde{\lambda}_1 & 0 \end{bmatrix}.
$$

Explicitly, we have formulas

$$
\tilde{\lambda}_1 = \cos(2\theta) \lambda_1 + \sin(2\theta) \lambda_4 \quad \tilde{\lambda}_4 = \sin(2\theta) \lambda_1 - \cos(2\theta) \lambda_4
$$

$$
\tilde{\lambda}_2 = \cos(2\theta) \lambda_2 + \sin(2\theta) \lambda_5 \quad \tilde{\lambda}_5 = \sin(2\theta) \lambda_2 - \cos(2\theta) \lambda_5
$$

$$
\tilde{\lambda}_3 = \cos(2\theta) \lambda_3 + \sin(2\theta) \lambda_6 \quad \tilde{\lambda}_6 = \sin(2\theta) \lambda_3 - \cos(2\theta) \lambda_6.
$$

We may now apply these $S^1$-actions to the structure equation (2.25) on $F_{SU(3)}$. Using that $\xi = 0$ on $B \subset F_{SU(3)}$, we deduce the first structure equation on $B$:

$$
d \begin{bmatrix} \eta \\ 0 \end{bmatrix} = -\left( \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} + \begin{bmatrix} \tilde{\lambda}_T \\ \tilde{\lambda}_N \end{bmatrix} + \begin{bmatrix} 0 & \mu \text{Id}_3 \\ -\mu \text{Id}_3 & 0 \end{bmatrix} \right) \wedge \begin{bmatrix} \eta \\ 0 \end{bmatrix}.
$$

In particular, the second line gives

$$
\beta \land \eta = (\tilde{\lambda}_N - \mu \text{Id}_3) \land \eta
$$

or in detail,

$$
\begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{bmatrix} \land \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} -\mu & -\tilde{\lambda}_6 & -\tilde{\lambda}_5 \\ \tilde{\lambda}_6 & -\mu & -\tilde{\lambda}_4 \\ -\tilde{\lambda}_5 & \tilde{\lambda}_4 & -\mu \end{bmatrix} \land \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}
$$

(2.30)

Note that on $B$, the 1-forms $\beta_{pq}$, $\lambda_j$, and $\mu$ are semibasic, and we write

$$
\beta_{pq} = S_{pqr} \eta_r \quad \lambda_i = T_{ij} \omega^j \quad \mu = U_i \omega^i
$$

for some function $S = (S_{pqr}) : B \rightarrow \text{Sym}^2(\mathbb{R}^3) \otimes \mathbb{R}^3$. 

20
Using (2.26)-(2.28), we obtain:

holds at points of $\Sigma$:

Theorem 2.8: Thus from (2.24), we deduce:

In particular, if $\theta = 0$, then the phase of every special Lagrangian 3-fold in $M$ satisfies the relation $\tan(3\theta) = \frac{\tau_0}{\pi_0}$.

Corollary 2.9: Fix $x \in M$ and $\theta \in [0, 2\pi)$. If every phase $\theta$ special Lagrangian 3-plane in $T_xM$ is tangent to a phase $\theta$ special Lagrangian 3-fold, then $\tau_3|_x = 0$ and $\pi_0|_x \sin(3\theta) = \pi_1|_x \cos(3\theta)$.

Proof: The hypotheses imply that equation (2.31) holds for all phase $\theta$ special Lagrangian 3-planes at $x \in M$. Thus, we get an $SU(3)$-invariant linear relation between $\pi_0$, $\pi_0$, and $\tau_3$. This implies that $\tau_3 = 0$ by Schur's Lemma. The statement $\pi_0|_x \sin(3\theta) = \pi_0|_x \cos(3\theta)$ follows immediately. \(\Diamond\)
Second, after using $S_{11r} + S_{22r} + S_{33r} = 0$ for each $r = 1, 2, 3$, we observe that:

$$2(S_{111} + S_{122} + S_{133}) = (T_{26} + T_{62} - T_{35} - T_{53}) \cos(3\theta) + (T_{23} - T_{32} + T_{56} - T_{65}) \sin(3\theta)$$
$$+ (T_{23} - T_{32} - T_{56} + 4U_4) \sin(\theta) + (-T_{26} + T_{62} + T_{35} - T_{53} + 4U_1) \cos(\theta) \quad (2.32a)$$

$$2(S_{121} + S_{222} + S_{233}) = (-T_{16} - T_{61} + T_{34} + T_{43}) \cos(3\theta) + (-T_{13} + T_{31} - T_{46} + T_{64}) \sin(3\theta)$$
$$+ (-T_{13} + T_{31} + T_{46} - 4U_5) \sin(\theta) + (T_{16} - T_{61} - T_{34} + T_{43} + 4U_2) \cos(\theta) \quad (2.32b)$$

$$2(S_{131} + S_{232} + S_{333}) = (T_{15} + T_{51} - T_{24} - T_{42}) \cos(3\theta) + (T_{12} - T_{21} + T_{45} - T_{54}) \sin(3\theta)$$
$$+ (T_{12} - T_{21} + T_{45} + T_{54} - 4U_6) \sin(\theta) + (-T_{15} + T_{51} + T_{24} - T_{42} + 4U_3) \cos(\theta). \quad (2.32c)$$

We are now ready to compute the mean curvature of a phase $\theta$ special Lagrangian 3-fold.

**Theorem 2.10:** Let $\Sigma \subset M$ be a special Lagrangian 3-fold immersed in a 6-manifold $M$ equipped with an SU(3)-structure. Then the mean curvature vector $H$ of $\Sigma$ is given by

$$H = -\frac{1}{\sqrt{2}} \cos(3\theta) \left[[(\tau_2)_1]^2 + \frac{1}{\sqrt{2}} \sin(3\theta) \left[[(\tau_3)_1]^2 - \sin(\theta) \left[[(\tau_5)_1]^2 - \cos(\theta) \left[J(\tau_3)_N\right]^2\right]\right]\right].$$

In particular, the largest torsion class of SU(3)-structures $(\Omega, \mathcal{Y})$ for which every special Lagrangian 3-fold (of every phase) is minimal is $X^+_0 \oplus X^+_0 \oplus X_3 \oplus X_4$.

**Proof:** The mean curvature vector may be computed as follows:

$$\begin{bmatrix}
H_1 \\
H_2 \\
H_3
\end{bmatrix}
\eta^{123} = \begin{bmatrix}
-\beta_{11} & -\beta_{12} & -\beta_{13} \\
-\beta_{21} & -\beta_{22} & -\beta_{23} \\
-\beta_{31} & -\beta_{32} & -\beta_{33}
\end{bmatrix} \wedge \begin{bmatrix}
\eta^{23} \\
\eta^{31} \\
\eta^{12}
\end{bmatrix} = \begin{bmatrix}
-S_{111} + S_{122} + S_{133} \\
S_{121} + S_{222} + S_{233} \\
S_{131} + S_{232} + S_{333}
\end{bmatrix} \eta_{123} \quad (2.33)$$

To evaluate the first term of (2.33), we substitute $\beta_{pq} = S_{pqr}\eta_r$, followed by (2.32), and finally (2.26)-(2.29), to obtain:

$$= 2 \begin{bmatrix}
-A_1 \cos(3\theta) + C_1 \sin(3\theta) + (G_1 - M_1) \sin(\theta) + (J_1 - N_1) \cos(\theta) \\
-A_2 \cos(3\theta) + C_2 \sin(3\theta) + (G_2 - M_2) \sin(\theta) + (J_2 - N_2) \cos(\theta) \\
-A_3 \cos(3\theta) + C_3 \sin(3\theta) + (G_3 - M_3) \sin(\theta) + (J_3 - N_3) \cos(\theta)
\end{bmatrix} \eta_{123}$$

Similarly, to evaluate the second term of (2.33), we substitute $\lambda_i = T_{ij}\omega^j$ and $\mu = U_i\omega^i$ followed by (2.26)-(2.29) to obtain:

$$= 2 \begin{bmatrix}
-\lambda_6 & \lambda_5 \\
\lambda_6 & -\lambda_4 \\
-\lambda_5 & \lambda_4
\end{bmatrix} \begin{bmatrix}
\eta^{23} \\
\eta^{31} \\
\eta^{12}
\end{bmatrix}$$
We conclude that

\[ H_p = -4A_p \cos(3\theta) + 4C_p \sin(3\theta) - 3M_p \sin(\theta) - 3N_p \cos(\theta), \]

and so (2.24) yields

\[ H = -\frac{1}{\sqrt{2}} \cos(3\theta) [(\tau_2)_1]^2 + \frac{1}{\sqrt{2}} \sin(3\theta) [(\tau_2)_1]^2 - \sin(\theta) [(\tau_5)_1]^\parallel - \cos(\theta) [J(\tau_5)_N]^\parallel. \]

Thus, the largest torsion class of SU(3)-structures for which \( H = 0 \) for all phases is the one for which \( \tau_2 = \tilde{\tau}_2 = \tau_5 = 0 \), namely \( X_0^+ \oplus X_0^- \oplus X_3 \oplus X_4 \). ◊

**Remark:** In the following table, we summarize the results above for certain special classes of SU(3)-structures encountered in the literature.

| Name        | Torsion Class | Mean Curvature of Phase \( \theta \) SLags | Necessary Condition for Local Existence of Phase \( \theta \) SLag at a Point |
|-------------|--------------|------------------------------------------|------------------------------------------------|
| CY          | 0            | 0                                        | –                                             |
| NK 1        | \( X_0^+ \)  | 0                                        | \( \tau_0 \cos(3\theta) = 0 \)               |
| NK 2        | \( X_0^- \)  | 0                                        | \( \tilde{\tau}_0 \sin(3\theta) = 0 \)       |
| GCY         | \( X_2^+ \oplus X_2^- \) | \( \frac{1}{\sqrt{2}} \sin(3\theta) [(\tau_2)_1]^2 \) | – |
|             |              | \( - \cos(3\theta) [(\tau_2)_1]^2 \)   |                                           |
| Half-Flat   | \( X_0^+ \oplus X_2^- \oplus X_3 \) | \( \frac{1}{\sqrt{2}} \sin(3\theta) [(\tau_2)_1]^2 \) | \( \frac{\sqrt{2}}{6} (\sin(\theta) [(\tau_3)_0]^\parallel + \cos(\theta) [(\tau_3)_0]^\parallel) \) |
|             |              |                                          | \( = -\tau_0 \cos(3\theta) \)                |
| Symp Half-Flat | \( X_2^- \)  | \( \frac{1}{\sqrt{2}} \sin(3\theta) [(\tau_2)_1]^2 \) | – |
| Balanced    | \( X_3 \)    | 0                                        | \( \sin(\theta) [(\tau_3)_0]^\parallel = -\cos(\theta) [(\tau_3)_0]^\parallel \) |
| Class \( X_4 \) | \( X_4 \)    | 0                                        | – |

Here, we are using the shorthand

- CY = Calabi-Yau
- NK 1 = Nearly-Kähler with convention \( d\Omega = 3\tau_0 \text{Re}(\Upsilon) \) and \( d\text{Im}(\Upsilon) = -2\tau_0 \Omega^2 \)
- NK 2 = Nearly-Kähler with convention \( d\Omega = 3\tilde{\tau}_0 \text{Im}(\Upsilon) \) and \( d\text{Re}(\Upsilon) = 2\tilde{\tau}_0 \Omega^2 \)
- GCY = Generalized Calabi-Yau
- Symp Half-Flat = Symplectic Half-Flat = Special Generalized Calabi-Yau (SGCY)

Both conventions for nearly-Kähler 6-manifolds are found in the literature (contrast, say, [6] with [2]). Generalized Calabi-Yau structures are studied in, for example, [4] and [1].

Half-flat structures have been used by Hitchin [10] to construct \( G_2 \)-manifolds via evolution equations. Symplectic half-flat structures are studied in [16] and [1], the latter work referring to them as “special generalized Calabi-Yau” structures.

Balanced SU(3)-structures on connected sums of copies of \( S^3 \times S^3 \) are constructed in [7]. Hypersurfaces in 6-manifolds with balanced SU(3)-structures are studied in [5]. Nilmanifolds with SU(3)-structures of class \( X_4 \) are constructed in §4.6 of [14]. ◊
3 Associative 3-Folds and Coassociative 4-Folds in $G_2$-Structures

Our goal in this section is to derive formulas (Theorems 3.9 and 3.12) for the mean curvature of associative 3-folds and coassociative 4-folds in 7-manifolds equipped with a $G_2$-structure. In the process, we observe an obstruction (Theorem 3.10) to the local existence of coassociative 4-folds.

These formulas and obstructions will be phrased in terms of refined torsion forms, which we will define in §3.3.1. These refined forms are simply the SO(4)-irreducible pieces of the usual torsion forms $\tau_0, \tau_1, \tau_2, \tau_3$ of a $G_2$-structure. As such, we will use §3.2 to describe the relevant SO(4)-representation theory needed to decompose $\tau_0, \tau_1, \tau_2, \tau_3$.

3.1 Preliminaries

In this section, we define both the ambient spaces (in §3.1.2) and submanifolds (in §3.1.3) of interest. We also use this section to fix notation and clarify conventions.

3.1.1 $G_2$-Structures on Vector Spaces

Let $V = \mathbb{R}^7$ equipped with the standard inner product $\langle \cdot, \cdot \rangle$, norm $\| \cdot \|$, and an orientation. Let $\{e_1, \ldots, e_7\}$ denote the standard (orthonormal) basis of $V$, and let $\{e^1, \ldots, e^7\}$ denote the corresponding dual basis of $V^*$. The associative $3$-form is the alternating 3-form $\phi_0 \in \Lambda^3(V^*)$ defined by

$$\phi_0 = e^{123} + e^1 \wedge (e^{45} + e^{67}) + e^2 \wedge (e^{46} - e^{57}) + e^3 \wedge (-e^{47} - e^{56})$$

The coassociative $4$-form is the alternating 4-form $\ast \phi_0 \in \Lambda^4(V^*)$ given by the Hodge dual $\ast$ of $\phi_0$. Explicitly:

$$\ast \phi_0 = e^{4567} + e^{23} \wedge (e^{45} + e^{67}) + e^{13} \wedge (-e^{46} + e^{57}) + e^{12} \wedge (-e^{47} - e^{56}).$$

For calculations, it will be convenient to express $\phi_0$ and $\ast \phi_0$ in the form

$$\phi_0 = \frac{1}{6} \epsilon_{ijk} e^{ijk} \quad \ast \phi_0 = \frac{1}{21} \epsilon_{ijkl} e^{ijkl}$$

where the constants $\epsilon_{ijk}, \epsilon_{ijkl} \in \{-1, 0, 1\}$ are defined by this formula. For example, $\epsilon_{123} = \epsilon_{145} = 1$ and $\epsilon_{347} = \epsilon_{356} = -1$. Identities involving the $\epsilon$-symbols are given in [3].

Remark: The associative and coassociative forms admit simple descriptions via the algebra of the octonions $\mathbb{O}$.

Equip $\mathbb{O} \simeq \mathbb{R}^8$ with the standard (euclidean) inner product and split $\mathbb{O} = \text{Re}(\mathbb{O}) \oplus \text{Im}(\mathbb{O}) \simeq \mathbb{R} \oplus \mathbb{R}^7$, where $\text{Re}(\mathbb{O}) := \text{span}_\mathbb{R}(1)$ is the real line and $\text{Im}(\mathbb{O}) := \text{Re}(\mathbb{O})^\perp$ is its orthogonal complement. Under the identification $V \simeq \text{Im}(\mathbb{O})$, the associative and coassociative forms are given by

$$\phi_0(x, y, z) = \langle x, y \times z \rangle$$

$$\ast \phi_0(x, y, z, w) = \frac{1}{2} \langle x, [y, z, w] \rangle,$$

for $x, y, z \in V$, where $y \times z := \text{Im}(\bar{y}z) = \frac{1}{2}(\bar{y}z - \bar{z}y)$ is the octonionic cross product, and $[y, z, w] := (yz)w - y(zw)$ is the associator, measuring the failure of associativity of multiplication in $\text{Im}(\mathbb{O})$. See [4] for a proof.

Consider the $\text{GL}(V)$-action on $\Lambda^3(V^*)$ given by pullback: $A \cdot \gamma := A^* \gamma$ for $A \in \text{GL}(V)$ and $\gamma \in \Lambda^3(V^*)$. It is a classical result of Schouten (see [Bryant 87] for a proof) that the stabilizer of $\phi_0 \in \Lambda^3(V^*)$ is the compact Lie group $G_2$, i.e.:

$$G_2 \cong \{ A \in \text{GL}(V) : A^* \phi_0 = \phi_0 \}.$$
We let \( \Lambda^3_+ (V^*) \) denote the orbit of \( \phi_0 \in \Lambda^3 (V^*) \) under this \( G_2 \)-action, i.e.:

\[
\Lambda^3_+ (V^*) := \{ A^* \phi_0 : A \in \text{GL}(V) \} \cong \frac{\text{GL}(V)}{G_2}.
\]

In [3], it is noted that \( \Lambda^3_+ (V^*) \subset \Lambda^3 (V^*) \simeq \mathbb{R}^{35} \) is an open subset with two connected components, each diffeomorphic to \( \mathbb{R}^{37} \times \mathbb{R}^{28} \).

The isomorphism \( G_2 \cong \{ A \in \text{GL}(V) : A^* \phi_0 = \phi_0 \} \) lets us regard \( G_2 \) as a subgroup of \( \text{GL}(V) \), which in turn lets us view \( V \simeq \mathbb{R}^7 \) as a faithful \( G_2 \)-representation. It can be shown (see [Bryant 87]) that this \( G_2 \)-representation is irreducible.

However, the induced \( G_2 \)-representations on \( \Lambda^k (V^*) \) for \( 2 \leq k \leq 5 \) are not irreducible. Indeed, \( \Lambda^2 (V^*) \) decomposes into irreducible \( G_2 \)-modules as

\[
\Lambda^2 (V^*) = \Lambda^2_7 \oplus \Lambda^2_{14},
\]

where

\[
\Lambda^2_7 = \{ \beta \in \Lambda^2 (V^*) : * (\phi_0 \wedge \beta) = 2 \beta \}
\]

\[
\Lambda^2_{14} = \{ \beta \in \Lambda^2 (V^*) : * (\phi_0 \wedge \beta) = -\beta \}
\]

Similarly, \( \Lambda^3 (V^*) \) decomposes into irreducible \( G_2 \)-modules as

\[
\Lambda^3 (V^*) = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}
\]

where

\[
\Lambda^3_1 = \mathbb{R} \phi_0
\]

\[
\Lambda^3_7 = \{ *(\alpha \wedge \phi_0) : \alpha \in \Lambda^1 \}
\]

\[
\Lambda^3_{27} = \{ \gamma \in \Lambda^3 : \gamma \wedge \phi_0 = 0 \text{ and } \gamma \wedge * \phi_0 = 0 \}
\]

In each case, \( \Lambda^k \) is an irreducible \( G_2 \)-module of dimension \( \ell \). Via the Hodge star \( * : \Lambda^k (V^*) \to \Lambda^{7-k} (V^*) \), one can obtain similar decompositions of \( \Lambda^4 (V^*) \) and \( \Lambda^5 (V^*) \).

In the sequel, we will always equip \( \Lambda^k (V^*) \) with the usual inner product, also denoted \( \langle \cdot , \cdot \rangle \), given by declaring

\[
\{ e^I : I \text{ increasing multi-index} \}
\]

(3.1)

to be an orthonormal basis. We let \( \| \cdot \| \) denote the corresponding norm.

For our calculations in §3.2, we will need the \( G_2 \)-equivariant map \( i \), defined on decomposable elements of \( \text{Sym}^2_0 (V^*) \) as follows:

\[
i : \text{Sym}^2_0 (V^*) \to \Lambda^3 (V^*)
\]

\[
i (\alpha \circ \beta) = \alpha \wedge * (\beta \wedge * \phi_0) + \beta \wedge * (\alpha \wedge * \phi_0).
\]

It is shown in [3] that the image of \( i \) is \( \Lambda^3_{27} \), so that the map with restricted image \( i : \text{Sym}^2_0 (V^*) \to \Lambda^3_{27} \) is an isomorphism of \( G_2 \)-modules. It is also remarked that with respect to the orthonormal basis \( \{ e^1, \ldots, e^7 \} \) of \( V^* \), one has

\[
i (h_{ij} e^i \circ e^j) = e_{ik} = h_{ij} e^{jkl}.  
\]

25
To invert \( i \), one can use the map

\[
j: \Lambda^2_{27}(V^*) \to \text{Sym}^2_0(V^*)
\]

\[
j(\gamma)(v, w) = *(\iota_v \phi_0 \wedge \iota_w \phi_0 \wedge \gamma)
\]

which satisfies \( j \circ i = 8 \text{Id}_{\text{Sym}^2_0(V^*)} \).

Finally, we remark that from the associative 3-form \( \phi_0 \), one can recover the inner product \( \langle \cdot, \cdot \rangle \) and volume form \( \text{vol} = e^{1-7} \) via

\[
\langle X, Y \rangle \text{vol} = \frac{1}{6} (\iota_X \phi_0) \wedge (\iota_Y \phi_0) \wedge \phi_0 \quad (3.3a)
\]

\[
\text{vol} = \phi_0 \wedge *\phi_0. \quad (3.3b)
\]

From these identities, one can show that, in fact, \( G_2 \) preserves both \( \langle \cdot, \cdot \rangle \) and the orientation on \( V \), so we may regard \( G_2 \leq \text{SO}(V, \langle \cdot, \cdot \rangle) \simeq \text{SO}(7) \).

### 3.1.2 \( G_2 \)-Structures on 7-Manifolds

**Definition:** Let \( M \) be an oriented 7-manifold. A \( G_2 \)-structure on \( M \) is a differential 3-form \( \phi \in \Omega^3(M) \) such that \( \phi|_x \in \Lambda^3_x(T^*_x M) \) at each \( x \in M \). That is, at each \( x \in M \), there exists a coframe \( u: T_x M \to \mathbb{R}^7 \) for which \( \phi|_x = u^*(\phi_0) \).

Intuitively, a \( G_2 \)-structure is a smooth identification of each tangent space \( T_x M \) with \( \text{Im}(\mathbb{O}) \) in such a way that \( \phi|_x \) and \( \phi_0 \) are aligned: \( (T_x M, \phi|_x) \simeq (\text{Im}(\mathbb{O}), \phi_0) \). We remark that a 7-manifold \( M \) admits a \( G_2 \)-structure if and only if it is orientable and spin: see [3] for a proof.

Every \( G_2 \)-structure \( \phi \) on \( M \) induces a Riemannian metric \( g_\phi \) and an orientation form \( \text{vol}_\phi \) on \( M \) via the formulas \( (3.3) \), reflecting the inclusion \( G_2 \leq \text{SO}(7) \). We caution, however, that the association \( \phi \mapsto g_\phi \) is not injective: different \( G_2 \)-structures may induce the same Riemannian metric. For a discussion of this point, see [3].

The first-order local invariants of a \( G_2 \)-structure are completely encoded in a certain \( G_2 \)-equivariant function

\[ T: F_{G_2} \to \Lambda^0 \oplus \Lambda^1 \oplus \Lambda^2_{14} \oplus \Lambda^3_{27} \simeq \mathbb{R}^{49} \]

called the **intrinsic torsion function**, defined on the total space of the \( G_2 \)-frame bundle \( F_{G_2} \to M \) over \( M \). We think of \( T \) as describing the 1-jet of the \( G_2 \)-structure.

The intrinsic torsion function is somewhat technical to define — the interested reader can find more information in [8] and [13] — but several equivalent reformulations are available. Most conveniently for our purposes: the intrinsic torsion function of a \( G_2 \)-structure is equivalent to the data of the 4-form \( d\phi \) and the 5-form \( d*\phi \).

In [3], the exterior derivatives of \( \phi \) and \( *\phi \) are shown to take the form

\[
d\phi = \tau_0 *\phi + 3 \tau_1 \wedge \phi + *\tau_3 \quad (3.4a)
\]

\[
d*\phi = 4 \tau_1 \wedge *\phi + \tau_2 \wedge \phi. \quad (3.4b)
\]

where

\[
(\tau_0, \tau_1, \tau_2, \tau_3) \in \Gamma(\Lambda^0(T^*M) \oplus \Lambda^1(T^*M) \oplus \Lambda^2_{14}(T^*M) \oplus \Lambda^3_{27}(T^*M))
\]

We refer to \( \tau_0, \tau_1, \tau_2, \tau_3 \) as the **torsion forms** of the \( G_2 \)-structure.

Following standard conventions, we let \( W_1, W_7, W_{14}, W_{27} \) denote the vector bundles \( \Lambda^0(T^*M) \),
\[ \Lambda^1(T^*M), \Lambda^2_{14}(T^*M), \Lambda^3_{27}(T^*M), \text{respectively.} \]

Consider the set \( S \) consisting of the \( 2^4 = 16 \) vector bundles
\[ S = \{ 0, W_i, W_i \oplus W_j, W_i \oplus W_j \oplus W_k, W_1 \oplus W_7 \oplus W_{14} \oplus W_{27} : i, j, k \in \{ 1, 7, 14, 27 \} \}. \]

**Definition:** Let \( E \in S \) be a vector bundle on the list above.

We say that a \( G_2 \)-structure belongs to the torsion class \( E \) iff the torsion forms of the \( G_2 \)-structure \( (\tau_0, \tau_1, \tau_2, \tau_3) \in \Gamma(W_1 \oplus W_7 \oplus W_{14} \oplus W_{27}) \) is valued in \( E \subset W_1 \oplus W_7 \oplus W_{14} \oplus W_{27} \).

For example, a \( G_2 \)-structure belongs to the torsion class \( W_7 \oplus W_{27} \) if and only if \( \tau_0 = \tau_2 = 0 \).

### 3.1.3 Associative 3-Folds and Coassociative 4-Folds

Let \( (M^7, \varphi) \) be a 7-manifold with a \( G_2 \)-structure, and consider a tangent space \( (T_x M, \varphi|_x) \cong (V, \phi_0) \). The vector space \( (V, \phi_0) \) possesses two distinguished classes of subspaces — associative 3-planes and coassociative 4-planes (to be defined shortly) — first studied by Harvey and Lawson [9] in their work on calibrations. Indeed, they observed that \( \phi_0 \) and \( \ast \phi_0 \) enjoy the following remarkable property:

**Proposition 3.1 [9]:** The associative 3-form \( \phi_0 \) and coassociative 4-form \( \ast \phi_0 \) have co-mass one, meaning that
\[
|\phi_0(x, y, z)| \leq 1 \quad \text{and} \quad |\ast \phi_0(x, y, z, w)| \leq 1
\]
for every orthonormal set \( \{ x, y, z, w \} \) in \( V \cong \mathbb{R}^7 \).

In light of this proposition, it is natural to examine more closely those 3-planes \( A \in \text{Gr}_3(V) \) (respectively, 4-planes \( C \in \text{Gr}_4(V) \)) for which \( |\phi_0(A)| = 1 \) (resp., \( |\ast \phi_0(C)| = 1 \)).

**Proposition 3.2 [9]:** Let \( A \in \text{Gr}_3(V) \) be a 3-plane in \( V \). The following are equivalent:

(i) If \( \{ u, v, w \} \) orthonormal basis of \( A \), then \( \phi_0(u, v, w) = \pm 1 \).

(ii) For all \( u, v, w \in A \), we have \( [u, v, w] = 0 \).

(iii) \( A = \text{span}\{u, v, u \times v\} \) for some linearly independent set \( \{u, v\} \).

If any of these conditions hold, we say that \( A \) is an **associative 3-plane**.

**Proposition 3.3 [9]:** Let \( C \in \text{Gr}_4(V) \) be a 4-plane in \( V \). The following are equivalent:

(i) If \( \{ x, y, z, w \} \) is an orthonormal basis of \( C \), then \( \ast \phi_0(x, y, z, w) = \pm 1 \).

(ii) \( C^\perp \) is associative

(iii) \( \phi_0|_C = 0 \).

If any of these conditions hold, we say that \( C \) is a **coassociative 4-plane**.

For proofs of the above propositions, we refer the reader to [9] and [12].

The \( G_2 \)-action on \( V \) induces \( G_2 \)-actions on the Grassmannians \( \text{Gr}_k(V) \) of \( k \)-planes in \( V \). While these actions are transitive for \( k = 1, 2, 5, 6 \), they are not transitive for \( k = 3, 4 \): indeed, the (proper) subsets consisting of associative 3-planes and coassociative 4-planes are \( G_2 \)-orbits. The corresponding stabilizer, recorded in the following proposition, will play a crucial role in this work:
Proposition 3.4 [9]: The Lie group $G_2$ acts transitively on the subset of associative 3-planes and on the subset of coassociative 4-planes:

$$\{ E \in \text{Gr}_3(V) : |\phi_0(E)| = 1 \} \subset \text{Gr}_3(V)$$
$$\{ E \in \text{Gr}_4(V) : |\ast \phi_0(E)| = 1 \} \subset \text{Gr}_4(V).$$

In both cases, the stabilizer of the $G_2$-action is isomorphic to SO(4).

We may finally define our primary objects of interest:

Definition: Let $(M^7, \varphi)$ be a 7-manifold equipped with a $G_2$-structure $\varphi$. Identify each tangent space $(T_x M, \varphi|_{T_x M}) \cong (V, \phi_0)$.

An associative 3-fold in $M$ is a 3-dimensional immersed submanifold $\Sigma \subset M$ for which each tangent space $T_x \Sigma \subset T_x M$ is an associative 3-plane.

Similarly, a coassociative 4-fold in $M$ is a 4-dimensional immersed submanifold $\Sigma \subset M$ for which each tangent space $T_x \Sigma \subset T_x M$ is a coassociative 4-plane.

Note that if $d\varphi = 0$, then $\varphi$ is a calibration whose calibrated 3-planes are the associative 3-planes in $T_x M$. Thus, in this case, an associative 3-fold is a calibrated submanifold, and hence a minimal submanifold of $M$.

Similarly, if $d\ast \varphi = 0$, then $\ast \varphi$ is a calibration whose calibrated 4-planes are the coassociative 4-planes in $T_x M$. Thus, in this case, a coassociative 4-fold is a calibrated submanifold, and hence a minimal submanifold of $M$.

3.2 Some SO(4)-Representation Theory

The Lie group SO(4) is double-covered by the simply-connected group SU(2) × SU(2). The complex irreducible representations of SU(2) × SU(2) are exactly the tensor products $V_p \otimes V_q$ of irreducible SU(2)-representations for each factor. The complex irreducible representations of SU(2) are well known to be the spaces of homogeneous polynomials in two variables of fixed degree, $V_p = \text{Sym}^p (\mathbb{C}[x,y])$.

Let $V_{p,q}^\mathbb{R}$ denote $V_p \otimes \mathbb{R}$. We think of $V_{p,q}^\mathbb{R}$ as the space of homogeneous polynomials in $(x, y; w, z)$ of bidegree $(p, q)$. When $p$ and $q$ have the same parity the representation $V_{p,q}^\mathbb{R}$ descends to a representation of SO(4), and each of these representations has a real structure induced by the map $(x, y, w, z) \mapsto (y, -x, z, -w)$. This yields a complete description of the real representations of SO(4).

We work with real representations, letting $V_{p,q}$ denote the real representation underlying $V_{p,q}^\mathbb{R}$.

In this language, the standard 4-dimensional representation of SO(4) is $V_{1,1}$, while the adjoint representation $\mathfrak{so}(4)$ is $V_{2,0} \oplus V_{0,2}$. The ordering of the subscripts is chosen so that the representation $\Lambda^2(R^4)$ of SO(4) on the self-dual 2-forms is $V_{0,2}$.

The Clebsch-Gordan formula applied to each SU(2) representation gives the irreducible decomposition of a tensor product of SO(4)-modules:

$$V_{p_1,q_1} \otimes V_{p_2,q_2} \cong \bigoplus_{i=0}^{[p_1-p_2]} \bigoplus_{j=0}^{[q_1-q_2]} V_{p_1+p_2-2i, q_1+q_2-2j}.$$  

3.2.1 SO(4) as a subgroup of $G_2$

In our calculations we shall need a concrete realization of SO(4) as the stabilizer of an associative or coassociative plane. Let SO(4) act on $V \cong \mathbb{R}^7$ via the identification $V \cong V_{0,2} \oplus V_{1,1}$, and let
(e₁, ..., e₇) be an orthonormal basis of V such that

- \langle e₁, e₂, e₃ \rangle \cong V_{0,2} \text{ and } \langle e₄, e₅, e₆, e₇ \rangle \cong V_{1,1},
- The map

\[ e₁ \mapsto e₄ + e₆, \quad e₂ \mapsto e₅ - e₇, \quad e₃ \mapsto -e₄ - e₅ \]

is SO(4)-equivariant.

Then the 3-form

\[ e^{123} + e^1 \wedge (e^{45} + e^{67}) + e^2 \wedge (e^{46} - e^{57}) + e^3 \wedge (-e^{47} - e^{56}) \]

is SO(4)-invariant, and thus the action of SO(4) on V gives an embedding SO(4) ⊂ G₂. The 3-plane \langle e₁, e₂, e₃ \rangle is associative and preserved by the action of SO(4), while the 4-plane \langle e₄, e₅, e₆, e₇ \rangle is coassociative and preserved by the action of SO(4).

### 3.2.2 Decomposition of 1-Forms on V*

Let V be as in the previous section. We have

\[ Λ¹(V^*) = A ⊕ C, \]

where

\[ A \cong \langle e₁, e₂, e₃ \rangle, \quad C \cong \langle e₄, e₅, e₆, e₇ \rangle. \]

As abstract SO(4)-modules, we have isomorphisms A \cong V_{0,2} and C \cong V_{1,1}.

**Notation:** We let \( \♭ : V \rightarrow V^* \) via \( X^\♭ := \langle X, \cdot \rangle \) denote the usual musical (index-lowering) isomorphism induced by the inner product \( \langle \cdot, \cdot \rangle \) on V, and let

\[ \#: V^* \rightarrow V \] \hspace{1cm} (3.5)

denote its inverse. In the sequel, we let \( A^\sharp, C^\sharp \subset V \) denote the images of \( A, C \subset V^* \) under the \( \# \) isomorphism.

### 3.2.3 Decomposition of 2-Forms on V*

We now seek to decompose \( Λ²(V^*) \) into SO(4)-irreducible submodules. As noted in §3.1 above, \( Λ²(V^*) \) splits into \( G₂ \)-irreducible submodules as

\[ Λ²(V^*) \cong Λ²⁺ \oplus Λ²⁻ \] \hspace{1cm} (3.6)

On the other hand, using \( V^* = A \oplus C \), we may also decompose \( Λ²(V^*) \) as

\[ Λ²(V^*) \cong Λ²(A) \oplus (A \otimes C) \oplus Λ²⁺(C) \oplus Λ²⁻(C). \] \hspace{1cm} (3.7)

We will refine both decompositions (3.6) and (3.7) into SO(4)-submodules.

To begin, note first that as SO(4)-modules, we have that \( Λ²(A) \cong Λ²⁺(C) \cong V_{0,2} \) and \( Λ²⁻(C) \cong V_{1,1} \).
\( V_{2,0} \) are irreducible. Thus, it remains only to decompose \( \Lambda^2_7, \Lambda^2_{14} \), and \( A \otimes C \).

**Definition:** We define

\[
(\Lambda^2_7)_A := \Lambda^2_7 \cap (\Lambda^2(A) \oplus \Lambda^2(C)) = \{ \iota_X \phi_0 : X \in A^2 \}
\]
\[
(\Lambda^2_7)_C := \Lambda^2_7 \cap (A \otimes C) = \{ \iota_X \phi_0 : X \in C^2 \}
\]
\[
(\Lambda^2_{14})_A := \Lambda^2_{14} \cap (\Lambda^2(A) \oplus \Lambda^2_{+}(C))
\]
\[
(\Lambda^2_{14})_{1,3} := \Lambda^2_{14} \cap (A \otimes C)
\]
\[
(\Lambda^2_{14})_{2,0} := \Lambda^2_{14} \cap \Lambda^2_7(C).
\]

The reader can check that, in fact, \( \Lambda^2_7(C) \subset \Lambda^2_{14} \), so that \( (\Lambda^2_{14})_{2,0} = \Lambda^2_{14} \cap \Lambda^2_7(C) = \Lambda^2_7(C) \).

Consider the SO(4)-module isomorphism

\[
L : A \rightarrow (\Lambda^2_7)_A
\]
\[
L(\alpha) = \iota_{\alpha^2} \phi_0 = *(\alpha \wedge * \phi_0).
\]

For \( \beta \in \Lambda^2(V^*) \), write \( \beta = \beta|_{\Lambda^2_7(A)} + \beta|_{A \otimes C} + \beta|_{\Lambda^2_7(C)} \), where \( \beta|_E \in E \) for \( E \in \{ \Lambda^2(A), A \otimes C, \Lambda^2_7(C) \} \).

Define SO(4)-equivariant maps

\[
L_A : A \rightarrow \Lambda^2(A)
\]
\[
L_A(\alpha) = L(\alpha)|_{\Lambda^2(A)}
\]
\[
L_C : A \rightarrow \Lambda^2_7(C)
\]
\[
L_C(\alpha) = L(\alpha)|_{\Lambda^2_7(C)}
\]

It is straightforward to check that \( L_A \) and \( L_C \) are well-defined SO(4)-module isomorphisms, and that \( L_A = *_A \) coincides with the usual Hodge star operator on \( A \). Finally, we define the map

\[
W : A \rightarrow (\Lambda^2_{14})_A
\]
\[
W(\alpha) = 2L_A(\alpha) - L_C(\alpha)
\]

Again, the reader can check that \( W \) is a well-defined SO(4)-module isomorphism. We caution that the maps \( L, L_C, \) and \( W \) are not isometries.

**Lemma 3.5:** The decompositions

\[
\Lambda^2_7 = (\Lambda^2_7)_A \oplus (\Lambda^2_7)_C \tag{3.8}
\]
\[
\Lambda^2_{14} = (\Lambda^2_{14})_A \oplus (\Lambda^2_{14})_{1,3} \oplus (\Lambda^2_{14})_{2,0} \tag{3.9}
\]
\[
A \otimes C = (\Lambda^2_7)_C \oplus (\Lambda^2_{14})_{1,3} \tag{3.10}
\]

consist of SO(4)-irreducible submodules.

Thus, the decomposition

\[
\Lambda^2(V) = [(\Lambda^2_7)_A \oplus (\Lambda^2_7)_C] \oplus [(\Lambda^2_{14})_A \oplus (\Lambda^2_{14})_{1,3} \oplus (\Lambda^2_{14})_{2,0}]
\]

is SO(4)-irreducible and refines [3.6], while

\[
\Lambda^2(V) = \Lambda^2(A) \oplus [(\Lambda^2_7)_C \oplus (\Lambda^2_{14})_{1,3}] \oplus \Lambda^2_7(C) \oplus \Lambda^2_7(C)
\]

30
is SO(4)-irreducible and refines \((3.7)\).

**Proof:** The decomposition \((3.8)\) follows from the isomorphism \(V \to \Lambda^2_7, X \mapsto \iota_X(\varphi_0)\) and the irreducible decomposition \(V \cong A \oplus C\).

By a dimension count, the SO(4)-invariant subspace \(\Lambda^2_{14}\) of \(\Lambda^2(V^*)\) must be isomorphic to the SO(4)-module \(V_{0,2} \oplus V_{2,0} \oplus V_{1,3}\), while, by the Clebsch-Gordan formula, the space \(A \otimes C\) is isomorphic to \(V_{1,3} \oplus V_{1,1}\). It follows from Schur’s lemma that the space \((\Lambda^2_{14})_{1,3}\) is isomorphic to the SO(4)-module \(V_{1,3}\). The space \((\Lambda^2_{14})_A\) is the image of \(A\) under the isomorphism \(W\) defined above, so it is an irreducible SO(4)-module, while the space \((\Lambda^2_{14})_{2,0}\) is isomorphic to \(\Lambda^2(C) \cong V_{2,0}\) so it is an irreducible SO(4)-module. Thus, the decomposition \((3.9)\) consists of irreducible SO(4)-modules.

To see that \((3.10)\) is an irreducible decomposition, note that we have already shown that both \((\Lambda^2_7)_C\) and \((\Lambda^2_{14})_{1,3}\) are irreducible SO(4)-modules. \(\Box\)

**Definition:** We define the map

\[
\beta : (\Lambda^2_{14})_A \to A^2
\]

\[
\beta \mapsto \beta^2 = \sqrt{6}(W^{-1}(\beta))^2
\]

The map \(\beta\) is an SO(4)-module isomorphism, and (because of the factor of \(\sqrt{6}\)) an isometry with respect to the inner product \((3.1)\) on \(\Lambda^2(V^*)\). \(\square\)

### 3.2.4 Decomposition of the Quadratic Forms on \(V^*\)

Before turning to the decomposition of \(\Lambda^2(V^*)\), we take a moment to decompose \(\text{Sym}^2_0(V^*)\) into SO(4)-irreducible pieces. To this end, we first use \(V^* = A \oplus C\) to split

\[
\text{Sym}^2(V^*) \cong \mathbb{R} \text{Id}_A \oplus \text{Sym}^2_0(A) \oplus (A \otimes C) \oplus \mathbb{R} \text{Id}_C \oplus \text{Sym}^2_0(C).
\]

Each of these summands is SO(4)-irreducible, with the exception of \(A \otimes C\), which splits into irreducible summands as

\[
A \otimes C \cong (A \otimes C)_{1,3} \oplus (A \otimes C)_C
\]

where \((A \otimes C)_{1,3}\) and \((A \otimes C)_C\) are submodules isomorphic to \(V_{1,3}\) and \(C\), respectively.

Here we must confess to employing a slight abuse of notation. In §3.2.1, we used \(A \otimes C\) to denote a submodule of \(\Lambda^2(V^*)\), whereas here in §3.2.2, we are using the same symbol \(A \otimes C\) to denote a submodule of \(\text{Sym}^2_0(V^*)\). Abstractly, these two SO(4)-modules are isomorphic, as are their irreducible summands. By Schur’s Lemma, there is a one-dimensional family of SO(4)-module isomorphisms \((\Lambda^2_7)_C \cong (A \otimes C)_C\) and \((\Lambda^2_{14})_{1,3} \cong (A \otimes C)_{1,3}\).

For computations, we will make use of the particular SO(4)-module isomorphism

\[
s : (\Lambda^2_7)_C \to (A \otimes C)_C
\]

defined as follows. In terms of a basis \(\{e^1, \ldots, e^6\}\) of \(V^*\) with \(A = \text{span}(e^1, e^2, e^3)\) and \(C = \text{span}(e^4, e^5, e^6)\), the map \(s\) will formally replace \(\wedge\) symbols with \(\circ\) symbols in each \(e^i \wedge e^j\) term with \(i < j\). So, for example,

\[
s(-e^{15} - e^{26} + e^{37}) = -e^1 \circ e^5 - e^2 \circ e^6 + e^3 \circ e^7.
\]

Finally, we remark that \(\text{Sym}^2_0(V^*)\) decomposes into irreducible SO(4)-modules as

\[
\text{Sym}^2_0(V^*) = \text{Sym}^2_0(A) \oplus (A \otimes C)_{1,3} \oplus (A \otimes C)_C \oplus \text{Sym}^2_0(C) \oplus \mathbb{R} E_0,
\]

where

\[
E_0 = \text{diag}(4, 4, 4, -3, -3, -3) \in \text{Sym}^2_0(V^*).
\]
3.2.5 Decomposition of 3-Forms on $V^*$

We now turn to $\Lambda^3(V^*)$. As noted in §3.1, $\Lambda^3(V^*)$ splits into $G_2$-irreducible submodules as

$$
\Lambda^3(V^*) \cong \Lambda^3_1 \oplus \Lambda^3_2 \oplus \Lambda^3_{27}.
$$

(3.15)

The summand $\Lambda^3_1 \cong \mathbb{R}$ is SO(4)-irreducible, but the summands $\Lambda^3_2$ and $\Lambda^3_{27}$ are not.

On the other hand, using $V^* \cong A \oplus C$, we also have the decomposition:

$$
\Lambda^3(V^*) \cong \Lambda^3(A) \oplus (\Lambda^2(A) \otimes C) \oplus (A \otimes \Lambda^2_+(C)) \oplus (A \otimes \Lambda^2_-(C)) \oplus \Lambda^3(C).
$$

(3.16)

Three of these summands are SO(4)-irreducible, namely $\Lambda^3(A) \cong V_{0,0}$ and $A \otimes \Lambda^2_+(C) \cong V_{2,2}$ and $\Lambda^3(C) \cong V_{1,1}$. Meanwhile, the second and third summands $\Lambda^2(A) \otimes C$ and $A \otimes \Lambda^2_-(C)$ are not.

As in §3.2.1 above, we will refine both (3.15) and (3.16) into SO(4)-irreducible submodules, though only the refinement of (3.15) will be used in this work. We begin with (3.15).

**Definition:** Recall the isomorphism $i: \text{Sym}^3_0(V^*) \to \Lambda^3_{27}$ of §3.2 and recall the SO(4)-irreducible splitting of $\text{Sym}^3_0(V^*)$ given in (3.13). We define

$$(\Lambda^3)^A \triangleq \{ (\alpha \land \phi_0) : \alpha \in A \} \quad \text{(3.14)}$$

$$(\Lambda^3)^C \triangleq \{ (\alpha \land \phi_0) : \alpha \in C \}$$

$$\Lambda^3_0 = (\Lambda^3)^A \oplus (\Lambda^3)^C$$

$$\Lambda^3_{27} = (\Lambda^3_{27})_0 \oplus (\Lambda^3_{27})_4 \oplus (\Lambda^3_{27})_{2,2} \oplus (\Lambda^3_{27})_{1,3} \oplus (\Lambda^3_{27})_c$$

Lemma 3.6: The decompositions

$$
\Lambda^3 = (\Lambda^3)^A \oplus (\Lambda^3)^C
$$

$$
\Lambda^3_{27} = (\Lambda^3_{27})_{0,0} \oplus (\Lambda^3_{27})_{0,4} \oplus (\Lambda^3_{27})_{2,2} \oplus (\Lambda^3_{27})_{1,3} \oplus (\Lambda^3_{27})_c
$$

consist of SO(4)-irreducible submodules.

**Definition:** Recall the isomorphisms $i: \text{Sym}^3_0(V^*) \to \Lambda^3_{27}$ of §3.2 and $s: (\Lambda^3)^C \to (A \otimes C)_C$ of (3.12). We define $\dagger: (\Lambda^3_{27})_{0,0} \to \mathbb{R}$ to be the unique vector space isomorphism for which

$$
\dagger(E_0) = 4\sqrt{42}
$$

(3.17)

where $E_0$ is as in (3.14). The map $\dagger$ is an isometry (due to the choice of $4\sqrt{42}$) with respect to the inner products (3.1).

We will also need the composition of SO(4)-module isomorphisms

$$
C^\sharp \to (\Lambda^3)^C \to (\Lambda^3_{27})_C
$$

$$
X \mapsto \iota_X \psi_0 \mapsto \frac{1}{2\sqrt{3}} (i \circ s)(\iota_X \psi_0).
$$

This map is an isometry due to the factor of $\frac{1}{2\sqrt{3}}$. We denote the inverse of this isometric isomorphism by

$$
\dagger: (\Lambda^3_{27})_C \to C^\sharp
$$

(3.18)

□

**Remark:** Extend the isomorphism $L_C: A \to \Lambda^3_+(C)$ to an isomorphism $L_C: A \otimes A \to A \otimes \Lambda^3_+(C)$ by the identity on the first $A$-factor, and split $A \otimes A = \mathbb{R} \oplus \text{Sym}^3_0(A) \oplus \Lambda^3(A)$. Extend the Hodge
star operator \(*_A: A \to \Lambda^2(A)\) to an isomorphism \(*_A: A \otimes C \to \Lambda^2(A) \otimes C\) by the identity on the C-factor, and recall the decomposition \(A \otimes C = (\Lambda^2_A)_C \oplus (\Lambda^2_{14})_{1,3}\).

Defining
\[
(\Lambda^2(A) \otimes C)_C := *_A[(\Lambda^2_A)_C] \\
(\Lambda^2(A) \otimes C)_{1,3} := *_A[(\Lambda^2_{14})_{1,3}]
\]
we obtain decompositions
\[
\Lambda^2(A) \otimes C = (\Lambda^2(A) \otimes C)_C \oplus (\Lambda^2(A) \otimes C)_{1,3} \\
A \otimes \Lambda^2_A(C) = (A \otimes \Lambda^2_A(C))_{0,0} \oplus (A \otimes \Lambda^2_A(C))_{0,4} \oplus (A \otimes \Lambda^2_A(C))_A
\]
consisting of SO(4)-irreducible submodules. □

**Remark:** The reader can check that some of the above submodules of \(\Lambda^3(V^*)\) are, in fact, equal to one another. Namely, we have the equalities
\[
A \otimes \Lambda^3_A(C) = (\Lambda^3_{27})_{2,2} \\
(\Lambda^2(A) \otimes C)_{1,3} = (\Lambda^3_{27})_{1,3} \\
(\Lambda^2(A) \otimes C)_C := (\Lambda^3_{27})_{0,0} \\
(\Lambda^2_A(C))_0 := (\Lambda^3_{27})_{0,4} \\
(\Lambda^2_A(C))_A := (\Lambda^3_{27})_A
\]
□

### 3.3 The Refined Torsion Forms

Let \((M^7, \varphi)\) be a 7-manifold equipped with a \(G_2\)-structure \(\varphi\). Fix a point \(x \in M\), choose an arbitrary associative 3-plane \(A^3 \subset T_x M\), and let \(C^2 \subset T_x M\) denote its orthogonal coassociative 4-plane. Our purpose in this section is to understand how the torsion of the \(G_2\)-structure decomposes with respect to the splitting
\[
T_x M = A^3 \oplus C^2.
\]

In §3.3.1, we use the decompositions of Lemmas 3.5 and 3.6 to break the torsion forms \(\tau_0, \tau_1, \tau_2, \tau_3\) into SO(4)-irreducible pieces called *refined torsion forms*. Separately, in §3.3.2, we set up the \(G_2\)-coframe bundle \(\pi: F_{G_2} \to M\) following [3], repackaging the original \(G_2\) torsion forms \(\tau_0, \tau_1, \tau_2, \tau_3\) as a matrix-valued function
\[
T = (T_{ij}): F_{G_2} \to \text{Mat}_{7 \times 7}(\mathbb{R}) \simeq \mathbb{R}^{49}.
\]
Finally, in §3.3.3, we express the functions \(T_{ij}\) in terms of the (pullbacks of the) refined torsion forms.

#### 3.3.1 The Refined Torsion Forms in a Local SO(4)-Frame

Fix \(x \in M\) and split \(T^*_x M = A \oplus C\) as above. All of our calculations in this subsection will be done pointwise, and we will suppress reference to \(x \in M\). By the Lemmas 3.5 and 3.6, the torsion forms \(\tau_0, \tau_1, \tau_2, \tau_3\) decompose into SO(4)-irreducible pieces as follows:
\[
\tau_0 = \tau_0 \tag{3.19a} \\
\tau_1 = (\tau_1)_A + (\tau_1)_C \tag{3.19b} \\
\tau_2 = (\tau_2)_A + (\tau_2)_{1,3} + (\tau_2)_{2,0} \tag{3.19c} \\
\tau_3 = (\tau_3)_{0,0} + (\tau_3)_{0,4} + (\tau_3)_{2,2} + (\tau_3)_{1,3} + (\tau_3)_c \tag{3.19d}
\]
Lemma 3.7: (no summation).

Define the 3-forms

\[ \phi_A = e^{123} \]
\[ \phi_C = \sum e^p \wedge \tau_p \]

(\text{no summation}).

We refer to \( \tau_0, (\tau_1)_A, (\tau_1)_C, \ldots, (\tau_3)_C \) as the \textit{refined torsion forms} of the G2-structure at \( x \) relative to the splitting \( T^*_x M = A \oplus C \).

We seek to express the refined torsion forms in terms of a local SO(4)-frame. To that end, let \( \{e_1, \ldots, e_7\} \) be an orthonormal basis for \( T_x M \) for which we have \( A^2 = \text{span}(e_1, e_2, e_3) \) and \( C^2 = \text{span}(e_4, e_5, e_6, e_7) \). Let \( \{e^1, \ldots, e^7\} \) denote the dual basis for \( T^*_x M \).

\textbf{Index Ranges:} We will employ the following index ranges: \( 1 \leq p, q \leq 3 \) and \( 4 \leq \alpha, \beta \leq 7 \) and \( 1 \leq i, j, k, \ell, m \leq 7 \) and \( 1 \leq \delta \leq 8 \) and \( 1 \leq a \leq 5 \).

\textbf{Definition:} Define the 2-forms

\[
\begin{align*}
\Upsilon_1 &= e^{45} + e^{67} \\
\Upsilon_2 &= e^{46} - e^{57} \\
\Upsilon_3 &= -e^{47} - e^{56} \\
\Omega_1 &= e^{45} - e^{67} \\
\Omega_2 &= e^{46} + e^{57} \\
\Omega_3 &= e^{47} - e^{56} \\
\Delta_1 &= e^{17} + e^{24} \\
\Delta_2 &= e^{16} + e^{25} \\
\Delta_3 &= -e^{15} + e^{26} \\
\Delta_4 &= -e^{14} + e^{36} \\
\Delta_5 &= e^{16} + e^{34} \\
\Delta_6 &= -e^{17} + e^{35} \\
\Delta_7 &= -e^{14} + e^{36} \\
\Delta_8 &= e^{15} + e^{37}
\end{align*}
\]

We also define

\[ \Gamma_p = 2 \ast_A e^p - \Upsilon_p \]

(\text{no summation}).

\textbf{Lemma 3.7:} We have that:

(a) \{\Gamma_1, \Gamma_2, \Gamma_3\} is a basis of \( (\Lambda^2_{14})_A \).

(b) \{\Delta_1, \ldots, \Delta_8\} is a basis of \( (\Lambda^3_{14})_{1,3} \).

(c) \{\Omega_1, \Omega_2, \Omega_3\} is a basis of \( (\Lambda^3_{14})_{2,0} \).

\textbf{Definition:} Define the 3-forms

\[
\begin{align*}
\phi_A &= e^{123} \\
\phi_C &= \sum e^p \wedge \tau_p \\
\lambda_{pq} &= e^p \wedge \Omega_q \\
\nu_{\alpha} &= (i \circ s)(\iota_{e_{\alpha}} \phi_0)
\end{align*}
\]

\[
\begin{align*}
\kappa_1 &= e^1 \wedge \Upsilon_2 - e^2 \wedge \Upsilon_1 \\
\kappa_2 &= e^1 \wedge \Upsilon_3 + e^3 \wedge \Upsilon_1 \\
\kappa_3 &= e^2 \wedge \Upsilon_3 + e^3 \wedge \Upsilon_2 \\
\kappa_4 &= e^1 \wedge \Upsilon_1 - e^2 \wedge \Upsilon_2 \\
\kappa_5 &= e^2 \wedge \Upsilon_2 - e^3 \wedge \Upsilon_3.
\end{align*}
\]

\[
\begin{align*}
\mu_1 &= e^{237} + e^{314} \\
\mu_2 &= e^{236} + e^{315} \\
\mu_3 &= -e^{235} + e^{316} \\
\mu_4 &= -e^{234} + e^{317} \\
\mu_5 &= e^{236} + e^{124} \\
\mu_6 &= -e^{237} + e^{125} \\
\mu_7 &= -e^{234} + e^{126} \\
\mu_8 &= e^{235} + e^{127}
\end{align*}
\]

34
Note that $\varphi = \phi_A + \phi_C$.

**Lemma 3.8:** We have that:

(a) $\{6\phi_A - \phi_C\}$ is a basis of $(A^3_{27})_{0,0}$
(b) $\{\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5\}$ is a basis of $(A^3_{27})_{0,4}$
(c) $\{\lambda_{pq} : 1 \leq p, q \leq 3\}$ is a basis of $(A^3_{27})_{2,2}$
(d) $\{\mu_1, \ldots, \mu_8\}$ is basis of $(A^3_{27})_{1,3}$
(e) $\{\nu_4, \nu_5, \nu_6, \nu_7\}$ is a basis of $(A^3_{27})_C$

We now express $(\tau_1)_A, (\tau_1)_C,$ etc., in terms of the above bases. That is, we define functions $A_p, B_\alpha$ and $C_p, D_\delta, E_\rho$ and $F, G_\delta, J_{pq}, L_\delta, M_\alpha$ by:

$$
(\tau_1)_A = 6A_p e^p \quad (\tau_2)_A = 12C_p \Gamma_p \quad (\tau_3)_{0,0} = 12F (6\phi_A - \phi_C) \quad (3.20a)
$$

$$
(\tau_1)_C = 6B_\alpha e^\alpha \quad (\tau_2)_{1,3} = 12D_\delta \Delta_\delta \quad (\tau_3)_{0,4} = 6G_\alpha \kappa_\alpha \quad (3.20b)
$$

$$
(\tau_2)_{2,0} = 12E_\rho \Omega_\rho \quad (\tau_3)_{2,2} = 12J_{pq} \lambda_{pq} \quad (3.20c)
$$

$$
(\tau_3)_{1,3} = 12L_\delta \mu_\delta \quad (3.20d)
$$

$$
(\tau_3)_C = 6M_\alpha \nu_\alpha \quad (3.20e)
$$

The various factors of 6 and 12 are included simply for the sake of clearing future denominators.

Note that the bases of Lemmas 3.8 and 3.9 are orthogonal but not orthonormal with respect to the inner product $(3.1)$ on $\Lambda^k(V^*)$. Indeed, we have:

$$
\|\Gamma_p\| = \sqrt{6} \quad \|\Omega_\rho\| = \sqrt{2} \quad \|\mu_\delta\| = \sqrt{2} \quad \|\kappa_\alpha\| = 2
$$

$$
\|\Delta_\delta\| = \sqrt{2} \quad \|6\phi_A - \phi_C\| = \sqrt{42} \quad \|\nu_\alpha\| = 2\sqrt{3} \quad \|\lambda_{pq}\| = \sqrt{2}
$$

Thus, in terms of the isometric isomorphisms $(3.5), (3.11), (3.17), (3.18)$ of §3.2, we have:

$$
[(\tau_1)_A]^2 = 6A_p e_p 
[(\tau_2)_A]^2 = 12\sqrt{6} C_p e_p 
[(\tau_3)_{0,0}]^2 = 12\sqrt{3} F 
(3.21a)
$$

$$
[(\tau_1)_C]^2 = 6B_\alpha e_\alpha 
[(\tau_3)_C]^2 = 12\sqrt{3} M_\alpha e_\alpha 
(3.21b)
$$

We will need these for our calculations in §§3.4 and 3.5. □

### 3.3.2 The Torsion Functions $T_{ij}$

Let $(M^7, \varphi)$ be a 7-manifold with a $G_2$-structure $\varphi$, and let $g_\varphi$ denote the underlying Riemannian metric. Let $F_{SO(7)} \rightarrow M$ denote the oriented orthonormal coframe bundle of $g_\varphi$, and let $\omega = (\omega^1, \ldots, \omega^7) \in \Omega^1(F_{SO(7)}; \mathbb{R}^7)$ denote the tautological 1-form. By the Fundamental Lemma of Riemannian Geometry, there exists a unique 1-form $\psi \in \Omega^1(F_{SO(7)}; \mathfrak{so}(7))$, the Levi-Civita connection form of $g_\varphi$, satisfying the First Structure Equation

$$
d\omega = -\psi \wedge \omega.
$$

Let $\pi : F_{G_2} \rightarrow M$ denote the $G_2$-coframe bundle of $M$. Restricted to $F_{G_2} \subset F_{SO(7)}$, the Levi-Civita 1-form $\psi$ is no longer a connection 1-form in general. Indeed, according to the splitting $\mathfrak{so}(7) = \mathfrak{g}_2 \oplus \mathbb{R}^7$, we have the decomposition

$$
\psi = \theta + 2\gamma,
$$

35
where $\theta = (\theta_{ij}) \in \Omega^1(\mathbb{F}_{G_2}; \mathfrak{g}_2)$ is a connection 1-form (the so-called natural connection of the $G_2$-structure $\varphi$) and $\gamma \in \Omega^1(\mathbb{F}_{G_2}; \mathbb{R}^7)$ is a $\pi$-semi-basic 1-form. Here, we are viewing $\mathbb{R}^7 = \{(\epsilon_{ijk}v_k) \in \mathfrak{so}(7): (v_1, \ldots, v_7) \in \mathbb{R}^7\}$, so that $\gamma$ takes the form

$$\gamma = \begin{bmatrix}
0 & \gamma_3 & -\gamma_2 & \gamma_5 & -\gamma_4 & \gamma_7 & -\gamma_6 \\
-\gamma_3 & 0 & \gamma_1 & \gamma_6 & -\gamma_7 & -\gamma_4 & \gamma_5 \\
\gamma_2 & -\gamma_1 & 0 & -\gamma_7 & -\gamma_6 & \gamma_5 & \gamma_4 \\
-\gamma_5 & -\gamma_6 & \gamma_7 & 0 & \gamma_1 & \gamma_2 & -\gamma_3 \\
\gamma_4 & \gamma_7 & -\gamma_6 & -\gamma_1 & 0 & -\gamma_3 & -\gamma_2 \\
-\gamma_7 & \gamma_4 & -\gamma_5 & -\gamma_2 & \gamma_3 & 0 & \gamma_1 \\
\gamma_6 & -\gamma_5 & -\gamma_4 & \gamma_3 & \gamma_2 & -\gamma_1 & 0 
\end{bmatrix}.$$ 

Since $\gamma$ is $\pi$-semi-basic, we may write

$$\gamma_i = T_{ij}\omega^j$$

for some matrix-valued function $T = (T_{ij}): \mathbb{F}_{G_2} \to \text{Mat}_{7 \times 7} (\mathbb{R})$. The 1-form $\gamma$, and hence the functions $T_{ij}$, encodes the torsion of the $G_2$-structure. In this notation, the first structure equation reads

$$d\omega_i = - (\theta_{ij} + 2\epsilon_{ijk}\gamma_k) \wedge \omega_j \quad (3.22)$$

**Remark:** The reader may wonder how the functions $T_{ij}$ are related to the forms $\tau_0, \tau_1, \tau_2, \tau_3$. In [2], Bryant expresses the torsion forms $\tau_0, \tau_1, \tau_2, \tau_3$ in terms of $T_{ij}$ as:

$$\pi^*(\tau_0) = \frac{24}{7}T_{ii},$$

$$\pi^*(\tau_1) = \epsilon_{ijk}T_{ij}\omega_k,$$

$$\pi^*(\tau_2) = 4T_{ij}\omega_i \wedge \omega_j - \epsilon_{ijk\ell}T_{ij}\omega_k \wedge \omega_\ell,$$

$$\pi^*(\tau_3) = -\frac{3}{2}\epsilon_{ijk\ell}(T_{ij} + T_{ji})\omega_{jkl} + \frac{18}{7}T_{ii}\sigma.$$ 

In the next section, we will exhibit a sort of inverse to this, expressing the $T_{ij}$ in terms of the refined torsion forms $\pi^*(\tau_0), \pi^*((\tau_1)A), \ldots, \pi^*((\tau_3)C)$. \hfill \Box

### 3.3.3 Decomposition of the Torsion Functions

For our computations in §3.4 and §3.5, we will need to express the torsion functions $T_{ij}$ in terms of the functions $A_p, B_\alpha, \ldots, L_\delta, M_\alpha$. To this end, we will continue to work on the total space of the $G_2$-coframe bundle $\pi: \mathbb{F}_{G_2} \to M$, pulling back all of the quantities defined on $M$ to $\mathbb{F}_{G_2}$. Following common convention, we systematically omit $\pi^*$ from the notation, so that (for example) $\pi^*(\tau_0)$ will simply be denoted $\tau_0$, etc. Note, however, that $\pi^*(\epsilon^j) = \omega_j$.

To begin, recall that the torsion forms $\tau_0, \tau_1, \tau_2, \tau_3$ satisfy

$$d\varphi = \tau_0 \ast \varphi + 3\tau_1 \wedge \varphi + \ast \tau_3$$

$$d \ast \varphi = 4\tau_1 \ast \varphi + \tau_2 \wedge \varphi.$$ 

Into the left-hand sides, we substitute $\varphi = \frac{1}{6}\epsilon_{ijk\ell} \omega^{ijk\ell}$ and $\ast \varphi = \frac{1}{24}\epsilon_{ijk\ell} \omega^{ijk\ell}$ and use the first structure equation (3.22) to obtain

$$\epsilon_{ijk\ell} T_{lm} \omega^{mjk\ell} = \tau_0 \ast \varphi + 3\tau_1 \wedge \varphi + \ast \tau_3$$

$$-\epsilon_{ijk} T_{lm} \omega^{mijk} = 4\tau_1 \ast \varphi + \tau_2 \wedge \varphi.$$
Into the right-hand sides, we again substitute \( \varphi = \frac{1}{6} \epsilon_{ijk} \omega^{ijk} \) and \( \ast \varphi = \frac{1}{24} \epsilon_{ijkl} \omega^{ijkl} \), as well as the expansions (3.19) and (3.20).

Upon equating coefficients, we obtain a system of 56 = \( \binom{7}{3} + \binom{7}{1} \) linear equations relating the 49 = \( 7^2 \) functions \( T_{ij} \) on the left side to the 49 = \( \dim(H^{0,2}(g_2)) \) functions \( \tau_0, A_p, B_0, \ldots, L_0, M_a \) on the right side. One can then use a computer algebra system (we have used MAPLE) to solve this linear system for the \( T_{ij} \).

We now exhibit the result, taking advantage of the SO(4)-irreducible splitting

\[
\text{Mat}_{7 \times 7}(\mathbb{R}) \cong V^* \otimes V^* \cong (A \otimes A) \oplus 2(A \otimes C) \oplus (C \otimes C) \\
\cong (\Lambda^2(A) \oplus \text{Sym}^2_0(A) \oplus \mathbb{R}) \oplus 2((A \otimes C)_{1,3} \oplus (A \otimes C)_C) \\
\oplus (\Lambda^2_+(C) \oplus \Lambda^2_-(C) \oplus \mathbb{R} \oplus \text{Sym}^2_0(C))
\]

to highlight the structure of the solution.

We have

\[
\frac{1}{2} \begin{bmatrix}
0 & T_{12} - T_{21} & T_{13} - T_{31} \\
T_{21} - T_{12} & 0 & T_{23} - T_{32} \\
T_{31} - T_{13} & T_{32} - T_{23} & 0
\end{bmatrix} = \begin{bmatrix}
0 & A_2 + 2C_3 & -(A_2 + 2C_2) \\
-(A_3 + 2C_3) & 0 & A_1 + 2C_1 \\
A_2 + 2C_2 & -(A_1 + 2C_1) & 0
\end{bmatrix}
\]

\[
\frac{1}{2} \begin{bmatrix}
2T_{11} & T_{12} + T_{21} & T_{13} + T_{31} \\
T_{21} + T_{12} & 2T_{22} & T_{23} + T_{32} \\
T_{31} + T_{13} & T_{32} + T_{23} & 2T_{33}
\end{bmatrix} = -\begin{bmatrix}
G_4 & G_1 & G_2 \\
G_1 & G_5 - G_4 & G_3 \\
G_2 & G_3 & G_5
\end{bmatrix} + (-4F + \frac{1}{24} \tau_0) \text{Id}_3
\]

corresponding to \( A \otimes A \cong \Lambda^2(A) \oplus \text{Sym}^2_0(A) \oplus \mathbb{R} \) and

\[
\frac{1}{2} \begin{bmatrix}
T_{41} + T_{14} & T_{42} + T_{24} & T_{43} + T_{34} \\
T_{51} + T_{15} & T_{52} + T_{25} & T_{53} + T_{35} \\
T_{61} + T_{16} & T_{62} + T_{26} & T_{63} + T_{36} \\
T_{71} + T_{17} & T_{72} + T_{27} & T_{73} + T_{37}
\end{bmatrix} = \begin{bmatrix}
L_4 + L_7 & -L_1 & -L_5 \\
L_3 - L_8 & -L_2 & -L_6 \\
-L_2 & -L_5 & -L_3 & -L_7 \\
-L_1 & L_6 & -L_4 & -L_8
\end{bmatrix} + \begin{bmatrix}
-M_5 & -M_6 & M_7 \\
M_4 & M_7 & M_6 \\
-M_7 & M_4 & -M_5 \\
M_6 & -M_5 & -M_4
\end{bmatrix}
\]

and

\[
\frac{1}{2} \begin{bmatrix}
T_{41} - T_{14} & T_{42} - T_{24} & T_{43} - T_{34} \\
T_{51} - T_{15} & T_{52} - T_{25} & T_{53} - T_{35} \\
T_{61} - T_{16} & T_{62} - T_{26} & T_{63} - T_{36} \\
T_{71} - T_{17} & T_{72} - T_{27} & T_{73} - T_{37}
\end{bmatrix} = \begin{bmatrix}
D_4 + D_7 & -D_1 & -D_5 \\
D_3 - D_8 & -D_2 & -D_6 \\
-D_2 & -D_5 & -D_3 & -D_7 \\
-D_1 & D_6 & -D_4 & -D_8
\end{bmatrix} + \begin{bmatrix}
-B_5 & -B_6 & B_7 \\
B_4 & B_7 & B_6 \\
-B_7 & B_4 & -B_5 \\
B_6 & -B_5 & -B_4
\end{bmatrix}
\]

corresponding to \( A \otimes C \cong (A \otimes C)_{1,3} \oplus (A \otimes C)_C \), and

\[
\frac{1}{2} \begin{bmatrix}
0 & T_{45} - T_{54} & T_{46} - T_{64} & T_{47} - T_{74} \\
T_{54} - T_{45} & 0 & T_{56} - T_{65} & T_{57} - T_{75} \\
T_{64} - T_{46} & T_{65} - T_{56} & 0 & T_{67} - T_{76} \\
T_{74} - T_{47} & T_{75} - T_{57} & T_{76} - T_{67} & 0
\end{bmatrix} = \begin{bmatrix}
0 & A_1 - C_1 & A_2 - C_2 & -A_3 + C_3 \\
-A_1 + C_1 & 0 & -A_3 + C_3 & -A_2 + C_2 \\
-A_2 - C_2 & A_3 - C_3 & 0 & A_1 - C_1 \\
A_3 - C_3 & A_2 - C_2 & -A_1 + C_1 & 0
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
0 & E_1 & E_2 & E_3 \\
-E_1 & 0 & -E_3 & E_2 \\
-E_2 & E_3 & 0 & -E_1 \\
-E_3 & -E_2 & E_1 & 0
\end{bmatrix}
\]

37
and
\[
\frac{1}{2} \begin{bmatrix}
2T_{44} & T_{45} + T_{54} & T_{46} + T_{64} & T_{47} + T_{74} \\
T_{54} + T_{45} & 2T_{55} & T_{56} + T_{65} & T_{57} + T_{75} \\
T_{64} + T_{46} & T_{65} & 2T_{66} & T_{67} + T_{76} \\
T_{74} + T_{47} & T_{75} + T_{57} & T_{76} + T_{67} & 2T_{77}
\end{bmatrix} =
\begin{bmatrix}
-J_{11} - J_{22} + J_{33} & J_{23} + J_{32} & -J_{13} - J_{31} & J_{12} - J_{21} \\
J_{23} + J_{32} & -J_{11} + J_{22} - J_{33} & -J_{21} - J_{12} & -J_{13} + J_{31} \\
-J_{12} - J_{31} & -J_{12} - J_{21} & J_{11} - J_{22} - J_{33} & -J_{23} + J_{32} \\
J_{12} - J_{21} & -J_{13} + J_{31} & -J_{23} + J_{32} & J_{11} + J_{22} + J_{33}
\end{bmatrix} + \left(3F + \frac{1}{24}\right) \text{Id}_4
\]
corresponding to \( C \otimes C \cong \Lambda^2(C) \oplus \Lambda^2(C) \oplus R \oplus \text{Sym}^2(C) \).

The above relations are more than we need for this work. In fact, we will only make use of the following relations, which can be read off from the above:
\[
\epsilon_{\alpha\beta\rho} T_{\beta\rho} = -3(B_\alpha + M_\alpha)
\]
and
\[
T_{44} + T_{55} + T_{66} + T_{77} = 3F + \frac{1}{24} T_0
\]
and
\[
-(T_{45} - T_{54}) - (T_{67} - T_{76}) = -4(A_1 - C_1)
\]
\[
(T_{57} - T_{75}) - (T_{46} - T_{64}) = -4(A_2 - C_2)
\]
\[
(T_{47} - T_{74}) + (T_{56} - T_{65}) = -4(A_3 - C_3).
\]

### 3.4 Mean Curvature of Associative 3-Folds

In this section, we derive a formula (Theorem 3.9) for the mean curvature of an associative 3-fold in an arbitrary 7-manifold \((M, \varphi)\) with \(G_2\)-structure \(\varphi\).

We continue with the notation of §3.3, letting \(\pi: F_{G_2} \to M\) denote the \(G_2\)-coframe bundle of \(M\), and \(\omega = (\omega_A, \omega_C) \in \Omega^1(F_{G_2}; \Lambda^A \oplus \Lambda^C)\) denoting the tautological 1-form. We remind the reader that \(\theta = (\theta_{ij}) \in \Omega^1(F_{G_2}; \mathfrak{g}_2)\) is the natural connection 1-form, and that \(\gamma = (\gamma_{ij}) \in \Omega^1(F_{G_2}; \mathbb{R})\) is a \(\pi\)-semibasic 1-form encoding the torsion of \(\varphi\). We will continue to write \(\gamma_{ij} = \epsilon_{ijk} \gamma_k\) and \(\gamma_i = T_{ij} \omega^j\) for \(T = (T_{ij}): F_{G_2} \to \text{Mat}_{7 \times 7}(\mathbb{R})\).

Let \(f: \Sigma^3 \to M^7\) denote an immersion of an associative 3-fold into \(M\), and let \(f^*(F_{G_2}) \to \Sigma\) denote the pullback bundle. Let \(B \subset f^*(F_{G_2})\) denote the subbundle of coframes adapted to \(\Sigma\), i.e., the subbundle whose fiber over \(x \in \Sigma\) is
\[
B_x = \{ u \in f^*(F_{G_2})_x : u(T_x \Sigma) = \Lambda^2 \oplus 0 \}
\]
We recall (Proposition 3.4) that \(G_2\) acts transitively on the set of associative 3-planes with stabilizer \(\text{SO}(4)\), so \(B \to \Sigma\) is a well-defined \(\text{SO}(4)\)-bundle. Note that on \(B\), we have
\[
\omega_C = 0.
\]
For the rest of §3.4, all of our calculations will be done on the subbundle \(B \subset F_{G_2}\).
We now exploit splitting $T_p M = T_p \Sigma \oplus (T_p \Sigma)^\perp \simeq A^8 \oplus C^2$ to decompose $\theta$ and $\gamma$ into $SO(4)$-irreducible pieces. To decompose the connection 1-form $\theta \in \Omega^1(B; \mathfrak{g}_2)$, we split
\[
\mathfrak{g}_2 \cong [\mathfrak{g}_2 \cap (\Lambda^2(A) \oplus \Lambda^2_+(C))] \oplus [\mathfrak{g}_2 \cap (A \otimes C)] \oplus [\mathfrak{g}_2 \cap \Lambda^2_-(C)],
\]
so that $\theta$ takes the block form
\[
\theta = \begin{bmatrix} \rho(\zeta) & -\sigma^T \\ \sigma & \zeta + \xi \end{bmatrix} = \begin{pmatrix}
0 & 2\zeta_3 & -2\zeta_2 \\
-2\zeta_3 & 0 & 2\zeta_1 \\
2\zeta_2 & -2\zeta_1 & 0 \\
\sigma_4 + \sigma_7 & -\sigma_1 & -\sigma_5 \\
\sigma_3 - \sigma_8 & -\sigma_2 & -\sigma_6 \\
-\sigma_2 - \sigma_5 & -\sigma_3 & -\sigma_7 \\
-\sigma_1 + \sigma_6 & -\sigma_4 & -\sigma_8 \\
\sigma_4 - \sigma_7 & -\sigma_3 + \sigma_8 & \sigma_2 + \sigma_5 & \sigma_1 - \sigma_6 \\
\sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \\
\sigma_5 & \sigma_6 & \sigma_7 & \sigma_8 \\
\zeta_1 + \xi_1 & 0 & \zeta_3 - \xi_3 & \zeta_2 + \xi_2 \\
\zeta_2 - \xi_2 & -\zeta_3 + \xi_3 & 0 & -\zeta_1 + \xi_1 \\
\zeta_3 - \xi_3 & -\zeta_2 - \xi_2 & \zeta_1 - \xi_1 & 0
\end{pmatrix}.
\]

Similarly, the 1-form $\gamma \in \Omega^1(B; \mathbb{R}^7)$ breaks into block form as:
\[
\gamma = \begin{bmatrix} \gamma_A & -(\gamma_C)^T \\ \gamma_C & (\gamma_A)_+ \end{bmatrix} = \begin{pmatrix}
0 & \gamma_3 & -\gamma_2 & \gamma_5 & -\gamma_4 & \gamma_7 & -\gamma_6 \\
-\gamma_3 & 0 & \gamma_1 & \gamma_6 & -\gamma_7 & \gamma_5 & -\gamma_4 \\
\gamma_2 & -\gamma_1 & 0 & -\gamma_7 & \gamma_6 & \gamma_5 & -\gamma_4 \\
-\gamma_5 & -\gamma_6 & \gamma_7 & 0 & \gamma_1 & \gamma_2 & -\gamma_3 \\
\gamma_4 & \gamma_7 & \gamma_6 & -\gamma_1 & 0 & -\gamma_3 & -\gamma_2 \\
-\gamma_7 & \gamma_4 & -\gamma_5 & -\gamma_2 & \gamma_3 & 0 & \gamma_1 \\
\gamma_6 & -\gamma_5 & -\gamma_4 & \gamma_3 & \gamma_2 & -\gamma_1 & 0
\end{pmatrix}.
\]

In this notation, the first structure equation (3.22) on $B$ reads:
\[
d\left(\omega_A\right) = -\left(\begin{pmatrix} \rho(\zeta) & -\sigma^T \\ \sigma & \zeta + \xi \end{pmatrix} + 2 \begin{bmatrix} \gamma_A & -(\gamma_C)^T \\ \gamma_C & (\gamma_A)_+ \end{bmatrix}\right) \wedge \left(\omega_A\right).
\]

In particular, the second line gives
\[
0 = -(\sigma + 2\gamma^C) \wedge \omega_A
\]
or in detail,
\[
\begin{pmatrix}
\sigma_4 + \sigma_7 & -\sigma_1 & -\sigma_5 \\
\sigma_3 - \sigma_8 & -\sigma_2 & -\sigma_6 \\
-\sigma_2 - \sigma_5 & -\sigma_3 & -\sigma_7 \\
-\sigma_1 + \sigma_6 & -\sigma_4 & -\sigma_8 \\
\end{pmatrix} \wedge \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = -2 \begin{pmatrix}
-\gamma_5 & -\gamma_6 & \gamma_7 \\
\gamma_4 & \gamma_7 & \gamma_6 \\
-\gamma_7 & \gamma_4 & -\gamma_5 \\
\gamma_6 & -\gamma_5 & -\gamma_4
\end{pmatrix} \wedge \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \tag{3.26}
\]

Note that on $B$, the 1-forms $\sigma_\delta$ and $\gamma_\alpha$ are semibasic, and we write
\[
\sigma_\delta = S_{\delta p} \omega^p \quad \quad \quad \quad \quad \gamma_\alpha = T_{\alpha p} \omega^p
\]
for some function $S = (S_{\delta p}) : B \to V_{1,3} \otimes A$, recalling our index ranges $1 \leq p \leq 3$ and $4 \leq \alpha \leq 7$ and $1 \leq \delta \leq 8$.

Now, the 24 functions $S_{\delta p}$ and the 12 functions $T_{\alpha p}$ are not independent: the equation (3.26) amounts to $12 = 4 \binom{3}{2}$ linear relations among them. Explicitly:
\[
\begin{pmatrix}
S_{13} - S_{52} & S_{43} + S_{73} + S_{51} - S_{42} - S_{72} - S_{11} \\
S_{23} - S_{62} & S_{33} + S_{83} + S_{61} - S_{32} - S_{82} - S_{21} \\
S_{33} - S_{72} & -S_{23} - S_{53} + S_{71} S_{22} + S_{52} - S_{31} \\
S_{43} - S_{82} & -S_{13} + S_{63} + S_{81} S_{12} - S_{62} - S_{41}
\end{pmatrix} = -2 \begin{pmatrix}
T_{63} + T_{72} & -T_{53} - T_{71} & T_{52} - T_{61} \\
T_{62} - T_{73} & T_{43} - T_{61} & -T_{42} + T_{71} \\
-T_{43} - T_{52} & T_{51} - T_{73} & T_{11} + T_{72} \\
-T_{12} + T_{53} & T_{41} + T_{63} & -T_{51} - T_{62}
\end{pmatrix}
\]

39
In particular, these relations imply:

\[
\begin{align*}
S_{41} + S_{71} - S_{12} - S_{53} &= -4\epsilon_{4op}T_{op} \\
S_{31} - S_{81} - S_{22} - S_{63} &= -4\epsilon_{5op}T_{op} \\
- S_{21} - S_{51} - S_{32} - S_{73} &= -4\epsilon_{6op}T_{op} \\
- S_{11} + S_{61} - S_{12} - S_{83} &= -4\epsilon_{7op}T_{op}
\end{align*}
\]

With these calculations in place, we may finally compute the mean curvature of an associative 3-fold:

**Theorem 3.9:** Let \( \Sigma \subset M \) be an associative 3-fold immersed in a 7-manifold \( M \) equipped with a G\(_2\)-structure. Then the mean curvature vector \( H \) of \( \Sigma \) is given by

\[
H = -3[(\tau_1)\omega]^2 - \frac{\sqrt{3}}{2}[(\tau_3)\omega]^3.
\]

In particular, the largest torsion class of G\(_2\)-structures \( \varphi \) for which every associative 3-fold is minimal is \( W_1 \oplus W_{14} = W_1 \cup W_{14} \), i.e., the class for which \( d\varphi = \lambda \ast \varphi \) for some \( \lambda \in \mathbb{R} \).

**Proof:** The mean curvature vector may be computed as follows:

\[
\begin{bmatrix}
H_4 \\
H_5 \\
H_6 \\
H_7
\end{bmatrix}
\omega^{123} = 
\begin{bmatrix}
\psi_{41} & \psi_{42} & \psi_{43} \\
\psi_{51} & \psi_{52} & \psi_{53} \\
\psi_{61} & \psi_{62} & \psi_{63} \\
\psi_{71} & \psi_{72} & \psi_{73}
\end{bmatrix}
\wedge
\begin{bmatrix}
\omega^{23} \\
\omega^{31} \\
\omega^{12}
\end{bmatrix}
\wedge
\begin{bmatrix}
\sigma_4 + \sigma_7 & -\sigma_1 & -\sigma_5 \\
\sigma_3 - \sigma_8 & -\sigma_2 & -\sigma_6 \\
-\sigma_2 - \sigma_5 & -\sigma_3 & -\sigma_7 \\
-\sigma_1 + \sigma_6 & -\sigma_4 & -\sigma_8
\end{bmatrix}
\wedge
\begin{bmatrix}
\omega^{23} \\
\omega^{31} \\
\omega^{12}
\end{bmatrix}
\]

To evaluate the first term in (3.28), we substitute \( \sigma_3 = S_{\delta p}\omega^p \), followed by (3.27), and finally (3.23), to obtain:

\[
\begin{bmatrix}
\sigma_4 + \sigma_7 & -\sigma_1 & -\sigma_5 \\
\sigma_3 - \sigma_8 & -\sigma_2 & -\sigma_6 \\
-\sigma_2 - \sigma_5 & -\sigma_3 & -\sigma_7 \\
-\sigma_1 + \sigma_6 & -\sigma_4 & -\sigma_8
\end{bmatrix}
\wedge
\begin{bmatrix}
\omega^{23} \\
\omega^{31} \\
\omega^{12}
\end{bmatrix}
= 
\begin{bmatrix}
S_{41} + S_{71} - S_{12} - S_{53} \\
S_{31} - S_{81} - S_{22} - S_{63} \\
- S_{21} - S_{51} - S_{32} - S_{73} \\
- S_{11} + S_{61} - S_{12} - S_{83}
\end{bmatrix}
\omega^{123}
\]

\[
= -4
\begin{bmatrix}
\epsilon_{4op}T_{op} \\
\epsilon_{5op}T_{op} \\
\epsilon_{6op}T_{op} \\
\epsilon_{7op}T_{op}
\end{bmatrix}
\omega^{123} = -12
\begin{bmatrix}
B_4 + M_4 \\
B_5 + M_5 \\
B_6 + M_6 \\
B_7 + M_7
\end{bmatrix}
\omega^{123}
\]

Similarly, to evaluate the second term in (3.28), we substitute \( \gamma_3 = T_{op}\omega^p \) followed by (3.28) to obtain:

\[
\begin{bmatrix}
-\gamma_5 & -\gamma_6 & \gamma_7 \\
\gamma_4 & \gamma_7 & \gamma_6 \\
-\gamma_7 & \gamma_4 & -\gamma_5 \\
\gamma_6 & -\gamma_5 & -\gamma_4
\end{bmatrix}
\wedge
\begin{bmatrix}
\omega^{23} \\
\omega^{31} \\
\omega^{12}
\end{bmatrix}
= -2
\begin{bmatrix}
\epsilon_{4op}T_{op} \\
\epsilon_{5op}T_{op} \\
\epsilon_{6op}T_{op} \\
\epsilon_{7op}T_{op}
\end{bmatrix}
\omega^{123} = -6
\begin{bmatrix}
B_4 + M_4 \\
B_5 + M_5 \\
B_6 + M_6 \\
B_7 + M_7
\end{bmatrix}
\omega^{123}
\]

We conclude that

\[
H_\alpha = -18B_\alpha - 18M_\alpha,
\]

40
and so (3.21) yields
\[ H = -3[(\tau_1)c]^2 - \frac{\sqrt{3}}{2}[(\tau_3)c]^2. \]
In particular, the largest torsion class for which \( H = 0 \) for all associatives is the one for which \( \tau_1 = \tau_3 = 0 \), which is \( W_1 \oplus W_{14} = W_1 \cup W_{14}. \)

3.5 Mean Curvature of Coassociative 4-Folds

In this section, we derive a formula (Theorem 3.12) for the mean curvature of a coassociative 4-fold in an arbitrary 7-manifold \((M, \varphi)\) with \(G_2\)-structure \(\varphi\). In the process, we observe a necessary condition (Theorem 3.10) for the local existence of coassociative 4-folds. We continue with the notation of §3.3.

Let \( f: \Sigma^4 \to M^7 \) denote an immersion of a coassociative 4-fold into \( M \), and let \( f^*(F_{G_2}) \to \Sigma \) denote the pullback bundle. Let \( B \subset f^*(F_{G_2}) \) denote the subbundle of coframes adapted to \( \Sigma \), i.e., the subbundle whose fiber over \( x \in \Sigma \) is
\[ B|_x = \{ u \in f^*(F_{G_2})|_x : u(T_x \Sigma) = 0 \oplus C^2 \}. \]
We recall (Proposition 3.4) that \( G_2 \) acts transitively on the set of coassociative 4-planes with stabilizer \( SO(4) \), so \( B \to \Sigma \) is a well-defined \( SO(4) \)-bundle. Note that on \( B \), we have
\[ \omega_A = 0. \]
For the rest of §3.5, all of our calculations will be done on the subbundle \( B \subset F_{G_2} \).

As in §3.4, we use the splitting \( T_x M = (T_x \Sigma)^1 \oplus T_x \Sigma \simeq A^2 \oplus C^2 \) to decompose \( \theta \) and \( \gamma \) into \( SO(4) \)-irreducible pieces. The result is the identical: the connection 1-form \( \theta \in \Omega^1(B; g_2) \) takes the block form
\[
\theta = \begin{bmatrix}
\rho(\zeta) & -\sigma^T \\
\sigma & \zeta + \xi
\end{bmatrix}
= \begin{bmatrix}
0 & 2\zeta_3 & -2\zeta_2 & -\sigma_4 + \sigma_7 & -\sigma_3 + \sigma_8 & \sigma_2 + \sigma_5 & \sigma_1 - \sigma_6 \\
-2\zeta_3 & 0 & 2\zeta_1 & 0 & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \\
2\zeta_2 & 0 & 0 & \sigma_5 & \sigma_6 & \sigma_7 & \sigma_8 \\
-\sigma_1 + \sigma_7 & 0 & -\sigma_2 - \sigma_5 & -\sigma_3 - \sigma_7 & \sigma_1 + \sigma_6 & -\sigma_4 - \sigma_8 & -\zeta_3 - \zeta_2 - \zeta_1 & 0 & 0
\end{bmatrix}.
\]
and the 1-form \( \gamma \in \Omega^1(B; \mathbb{R}^7) \) takes the block form
\[
\gamma = \begin{bmatrix}
\gamma_A & -(\gamma_C)^T \\
\gamma_C & (\gamma_A)_+
\end{bmatrix}
= \begin{bmatrix}
0 & \gamma_3 & -\gamma_2 & 75 & -74 & 77 & -74 & 76 \\
-\gamma_3 & 0 & \gamma_1 & 76 & -77 & -74 & 75 & 74 \\
\gamma_2 & -\gamma_1 & 0 & -77 & -76 & 75 & 74 & 73 \\
-\gamma_5 & -\gamma_6 & \gamma_7 & 0 & \gamma_1 & \gamma_2 & -\gamma_3 & 74 \\
\gamma_4 & \gamma_7 & \gamma_6 & -\gamma_1 & 0 & -\gamma_3 & -\gamma_2 & 73 \\
-\gamma_7 & \gamma_4 & -\gamma_5 & -\gamma_2 & \gamma_3 & 0 & \gamma_1 & 72 \\
\gamma_6 & -\gamma_5 & -\gamma_4 & \gamma_3 & \gamma_2 & -\gamma_1 & 0
\end{bmatrix}.
\]
In this notation, the first structure equation (3.22) on \( B \) reads:
\[
d\left(\begin{array}{c}
0 \\
\omega_C
\end{array}\right) = -\left(\begin{array}{c}
\rho(\zeta) & -\sigma^T \\
\sigma & \zeta + \xi
\end{array}\right) + 2\left(\begin{array}{c}
\gamma_A & -(\gamma_C)^T \\
\gamma_C & (\gamma_A)_+
\end{array}\right) \wedge \left(\begin{array}{c}
0 \\
\omega_C
\end{array}\right).
\]
In particular, the first line gives
\[ 0 = (\sigma^T + 2(\gamma_C)^T) \wedge \omega_C \]
or in full detail,
\[
\begin{bmatrix}
-\sigma_4 - \sigma_7 & -\sigma_3 + \sigma_8 & \sigma_2 + \sigma_5 & \sigma_1 - \sigma_6 \\
\sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \\
\sigma_5 & \sigma_6 & \sigma_7 & \sigma_8 
\end{bmatrix} \wedge \begin{bmatrix}
\eta_1^1 \\
\eta_2^1 \\
\eta_3^1 \\
\eta_4^1 
\end{bmatrix} = -2 \begin{bmatrix}
\gamma_5 & -\gamma_4 & \gamma_7 & -\gamma_6 \\
\gamma_6 & -\gamma_7 & -\gamma_4 & \gamma_5 \\
-\gamma_7 & -\gamma_6 & \gamma_5 & \gamma_4 
\end{bmatrix} \wedge \begin{bmatrix}
\eta_1^2 \\
\eta_2^2 \\
\eta_3^2 \\
\eta_4^2 
\end{bmatrix} \tag{3.29}
\]

Note that on \( B \), the 1-forms \( \sigma_\delta \) and \( \gamma_\alpha \) are semibasic, so we can write
\[
\sigma_\delta = S_{\delta \alpha} \omega^\alpha \quad \quad \gamma_\beta = T_{\beta \alpha} \omega^\alpha
\]
for some function \( S = (S_{\delta \alpha}) : B \to V_{1,3} \otimes C \), recalling our index ranges \( 1 \leq p \leq 3 \) and \( 4 \leq \alpha, \beta \leq 7 \) and \( 1 \leq \delta \leq 8 \).

Note that the 32 functions \( S_{\delta \alpha} \) and the 16 functions \( T_{\beta \alpha} \) are not independent: the equation \[3.29\] shows that they satisfy \( 3(\frac{1}{2}) = 18 \) linear relations. Explicitly:
\[
\begin{bmatrix}
S_{15} - S_{24} & S_{55} - S_{64} & S_{84} + S_{45} + S_{75} - S_{34} \\
S_{16} - S_{34} & S_{56} - S_{74} & S_{54} + S_{24} + S_{46} + S_{76} \\
S_{26} - S_{35} & S_{66} - S_{75} & S_{14} - S_{64} + S_{47} + S_{77} \\
S_{17} - S_{44} & S_{57} - S_{84} & S_{55} - S_{66} + S_{25} + S_{36} \\
S_{27} - S_{45} & S_{67} - S_{85} & -S_{65} + S_{15} + S_{37} - S_{37} \\
S_{37} - S_{46} & S_{77} - S_{86} & -S_{66} - S_{57} + S_{16} - S_{27} 
\end{bmatrix} = 2 \begin{bmatrix}
-T_{74} - T_{65} & -T_{64} + T_{75} & T_{44} + T_{55} \\
-T_{44} - T_{56} & T_{54} + T_{76} & -T_{74} + T_{65} \\
-T_{45} + T_{76} & T_{55} + T_{66} & T_{64} + T_{57} \\
-T_{54} - T_{67} & T_{44} + T_{77} & -T_{75} - T_{46} \\
T_{55} + T_{77} & T_{45} + T_{67} & T_{65} + T_{47} \\
T_{56} + T_{47} & T_{46} - T_{57} & T_{66} + T_{77} 
\end{bmatrix}
\]

We make two observations on this system of equations. First, we observe the relation
\[(T_{44} + T_{55}) + (T_{55} + T_{66}) + (T_{55} + T_{77}) + (T_{66} + T_{77}) + (T_{44} + T_{77}) + (T_{44} + T_{66}) = 0.
\]

Substituting \[3.24\], this equation simplifies to
\[3F + \frac{1}{24} \tau_0 = 0.
\]

Substituting \[3.21\], we have proved:

**Theorem 3.10:** If a coassociative 4-fold \( \Sigma \) exists in \( M \), then the following relation holds at points of \( \Sigma \):
\[
\tau_0 = -\frac{\sqrt{42}}{7} \left| (\tau_3)_{0,0} \right| \tag{3.30}
\]
In particular, if \( \tau_3 = 0 \) and \( \tau_0 \) is non-vanishing (so the torsion takes values in \( (W_1 \oplus W_7 \oplus W_{14}) - (W_7 \oplus W_{14}) \)), then \( M \) admits no coassociative 4-folds (even locally).

**Corollary 3.11:** Fix \( x \in M \). If every coassociative 4-plane in \( T_x M \) is tangent to a coassociative 4-fold, then \( \tau_0|_x = 0 \) and \( \tau_3|_x = 0 \).

**Proof:** The hypotheses imply that equation \[3.30\] holds for all coassociative 4-planes at \( x \in M \). Thus, we have a \( G_2 \)-invariant linear relation between \( \tau_0|_x \) and \( \tau_3|_x \). This implies that \( \tau_0|_x = 0 \) and \( \tau_3|_x = 0 \) by Schur’s Lemma. ◇
Second, we observe the three relations
\[
(S_{17} - S_{67}) + (S_{26} + S_{56}) + (-S_{35} + S_{85}) - (S_{44} + S_{74}) = -4(T_{45} - T_{54}) - 4(T_{67} - T_{76}) \quad (3.31a)
\]
\[
S_{14} + S_{25} + S_{36} + S_{47} = 4(T_{57} - T_{75}) - 4(T_{16} - T_{64}) \quad (3.31b)
\]
\[
S_{54} + S_{65} + S_{76} + S_{87} = 4(T_{47} - T_{74}) + 4(T_{56} - T_{65}). \quad (3.31c)
\]

We may now compute the mean curvature of a coassociative 4-fold:

**Theorem 3.12:** Let \( \Sigma \subset M \) be a coassociative 4-fold immersed in a 7-manifold \( M \) equipped with a G\(_2\)-structure. Then the mean curvature vector \( H \) of \( \Sigma \) is given by
\[
H = -4[(\tau_1)_{\Sigma}]^2 + \frac{\sqrt{6}}{3}[(\tau_2)_{\Sigma}]^2.
\]

In particular, the largest torsion class of G\(_2\)-structures \( \varphi \) for which every coassociative 4-fold is minimal is \( W_1 \oplus W_{27} \), i.e., the class for which \( d \ast \varphi = 0 \).

**Proof:** Let \( \beta_\alpha := \ast_{\Sigma}(\omega^\alpha) \in \Omega^3(B) \) and \( \text{vol}_\Sigma = \omega^{4567} \). The mean curvature vector may be computed as follows:
\[
\begin{bmatrix}
H^1 \\
H^2 \\
H^3
\end{bmatrix} \text{vol}_\Sigma = \begin{bmatrix}
\psi_{14} & \psi_{15} & \psi_{16} & \psi_{17} \\
\psi_{24} & \psi_{25} & \psi_{26} & \psi_{27} \\
\psi_{34} & \psi_{35} & \psi_{36} & \psi_{37}
\end{bmatrix} \wedge \begin{bmatrix}
\beta_4 \\
\beta_5 \\
\beta_6 \\
\beta_7
\end{bmatrix}
\]
\[
= \begin{bmatrix}
-\sigma_4 - \sigma_7 & -\sigma_3 + \sigma_8 & \sigma_2 + \sigma_5 & \sigma_1 - \sigma_6 \\
\sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \\
\sigma_5 & \sigma_6 & \sigma_7 & \sigma_8
\end{bmatrix} \wedge \begin{bmatrix}
\beta_4 \\
\beta_5 \\
\beta_6 \\
\beta_7
\end{bmatrix} + 2 \begin{bmatrix}
\gamma_5 & -\gamma_4 & \gamma_7 & -\gamma_6 \\
\gamma_6 & -\gamma_7 & -\gamma_4 & \gamma_5 \\
-\gamma_7 & -\gamma_6 & \gamma_5 & \gamma_4
\end{bmatrix} \wedge \begin{bmatrix}
\beta_4 \\
\beta_5 \\
\beta_6 \\
\beta_7
\end{bmatrix}
\]

(3.32)

To evaluate the first term in (3.32), we substitute \( \sigma_\delta = S_{\delta \mu \omega^\nu} \), followed by (3.31), and finally (3.25), to obtain:
\[
\begin{bmatrix}
-\sigma_4 - \sigma_7 & -\sigma_3 + \sigma_8 & \sigma_2 + \sigma_5 & \sigma_1 - \sigma_6 \\
\sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \\
\sigma_5 & \sigma_6 & \sigma_7 & \sigma_8
\end{bmatrix} \wedge \begin{bmatrix}
\beta_4 \\
\beta_5 \\
\beta_6 \\
\beta_7
\end{bmatrix} = 4 \begin{bmatrix}
-(T_{45} - T_{54}) - (T_{67} - T_{76}) \\
(T_{57} - T_{75}) - (T_{46} - T_{64}) \\
(T_{47} - T_{74}) + (T_{56} - T_{65})
\end{bmatrix} \text{vol}_\Sigma
\]
\[
= 16 \begin{bmatrix}
-A_1 + C_1 \\
-A_2 + C_2 \\
-A_3 + C_3
\end{bmatrix} \text{vol}_\Sigma
\]

Similarly, to evaluate the second term in (3.32), we substitute \( \gamma_\alpha = T_{\alpha \mu \omega^\nu} \) followed by (3.25) to obtain:
\[
2 \begin{bmatrix}
\gamma_5 & -\gamma_4 & \gamma_7 & -\gamma_6 \\
\gamma_6 & -\gamma_7 & -\gamma_4 & \gamma_5 \\
-\gamma_7 & -\gamma_6 & \gamma_5 & \gamma_4
\end{bmatrix} \wedge \begin{bmatrix}
\beta_4 \\
\beta_5 \\
\beta_6 \\
\beta_7
\end{bmatrix} = 2 \begin{bmatrix}
-(T_{45} - T_{54}) - (T_{67} - T_{76}) \\
(T_{57} - T_{75}) - (T_{46} - T_{64}) \\
(T_{47} - T_{74}) + (T_{56} - T_{65})
\end{bmatrix} \text{vol}_\Sigma
\]
\[
= 8 \begin{bmatrix}
-A_1 + C_1 \\
-A_2 + C_2 \\
-A_3 + C_3
\end{bmatrix} \text{vol}_\Sigma
\]

43
We conclude that

\[ H_p = -24A_p + 24C_p \]

and so (3.21) yields

\[ H = -4[\langle \tau_1 \rangle \rangle^2 + \frac{\sqrt{6}}{3} [\langle \tau_2 \rangle \rangle^2. \]

In particular, the largest torsion class for which \( H = 0 \) for all coassociatives is the one for which \( \tau_1 = \tau_2 = 0 \), which is \( W_1 \oplus W_{27} \).

References

[1] Lucio Bedulli and Luigi Vezzoni. The Ricci tensor of SU(3)-manifolds. *J. Geom. Phys.*, 57(4):1125–1146, 2007.
[2] Robert L Bryant. On the geometry of almost complex 6-manifolds. *Asian Journal of Mathematics*, 10(3):561–605, 2006.
[3] Robert L. Bryant. Some remarks on \( G_2 \)-structures. *Proceedings of Gökova Geometry-Topology Conference 2005*, 2006.
[4] Paolo De Bartolomeis and Adriano Tomassini. On the maslov index of lagrangian submanifolds of generalized calabi–yau manifolds. *International Journal of Mathematics*, 17(08):921–947, 2006.
[5] Marisa Fernández, Adriano Tomassini, Luis Ugarte, and Raquel Villacampa. Balanced hermitian metrics from su (2)-structures. *Journal of Mathematical Physics*, 50(3):033507, 2009.
[6] Lorenzo Foscolo and Mark Haskins. New \( G_2 \)-holonomy cones and exotic nearly Kähler structures on \( S^6 \) and \( S^3 \times S^3 \). *Ann. of Math. (2)*, 185(1):59–130, 2017.
[7] Jixiang Fu, Jun Li, Shing-Tung Yau, et al. Balanced metrics on non-kähler calabi-yau threefolds. *Journal of Differential Geometry*, 90(1):81–129, 2012.
[8] Robert B. Gardner. *The method of equivalence and its applications*, volume 58 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1989.
[9] Reese Harvey and H Blaine Lawson. Calibrated geometries. *Acta Mathematica*, 148(1):47–157, 1982.
[10] Nigel Hitchin. Generalized calabi–yau manifolds. *Quarterly Journal of Mathematics*, 54(3):281–308, 2003.
[11] Dominic D Joyce. *Riemannian holonomy groups and calibrated geometry*, volume 12. Oxford University Press, 2007.
[12] Dietmar A Salamon and Thomas Walpuski. Notes on the octonions. In *Proceedings of the Gökova Geometry-Topology Conference 2016*, 2017. https://arxiv.org/abs/1005.2820.
[13] Simon Salamon. *Riemannian geometry and holonomy groups*, volume 201 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1989.
[14] Nils Schoemann. Almost hermitian structures with parallel torsion. *Journal of Geometry and Physics*, 57(11):2187–2212, 2007.

[15] Fabian Schulte-Hengesbach. *Half-flat structures on Lie groups*. PhD thesis, Universität Hamburg, 2010.

[16] Adriano Tomassini and Luigi Vezzoni. On symplectic half-flat manifolds. *manuscripta mathematica*, 125(4):515–530, 2008.

Duke University  
Department of Mathematics  
Durham, NC, USA, 27708  
E-mail address: gavin.ball@duke.edu

McMaster University  
Department of Mathematics & Statistics  
Hamilton, ON, Canada, L8S 4K1  
E-mail address: madnickj@mcmaster.ca