Calculation of Band Edge Eigenfunctions and Eigenvalues of Periodic Potentials through the Quantum Hamilton - Jacobi Formalism *

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Abstract

We obtain the band edge eigenfunctions and the eigenvalues of solvable periodic potentials using the quantum Hamilton - Jacobi formalism. The potentials studied here are the Lamé and the associated Lamé which belong to the class of elliptic potentials. The formalism requires an assumption about the singularity structure of the quantum momentum function \( p \), which satisfies the Riccati type quantum Hamilton - Jacobi equation, \( p^2 - i\hbar \frac{d}{dx}p = 2m(E - V(x)) \) in the complex \( x \) plane. Essential use is made of suitable conformal transformations, which leads to the eigenvalues and the eigenfunctions corresponding to the band edges in a simple and straightforward manner. Our study reveals interesting features about the singularity structure of \( p \), responsible in yielding the band edge eigenfunctions and eigenvalues.

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* This paper is dedicated to the memory of Prof. R. A. Leacock.

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I. INTRODUCTION

As is well-known, periodic potentials play a significant role in condensed matter physics. Recently, optical lattices have manifested in Bose-Einstein condensates (BEC)\cite{11}, leading to the analysis of the excitation spectra in such systems. The energy spectrum of periodic potentials is unique due to the existence of energy bands and the solutions of the Schrödinger equation have the Bloch form:

$$\psi(x) = u(x) \exp(ik.x).$$

(1)

Here $u(x)$ has the periodicity of the potential function. These features of the periodic potentials are usually illustrated using the Kronig-Penney model\cite{2}, in which the crystal lattice is approximated by a periodic array of square wells. Imposing the Bloch condition on the solutions of the Kronig-Penney model, one is led to a transcendental equation, which has to be solved in order to obtain the band edge solutions.

Of late, there has been considerable interest in the study\cite{3,4,5,6} and construction\cite{7,8} of new periodic potentials. The periodic lattices in BEC has also renewed interest in this area\cite{9}. The number of exactly solvable (ES) potentials is very limited and in that, exactly solvable periodic potentials constitute a very small number. Quite sometime back it was shown by Scarf that, $V(x) = A/\sin^2 x$, usually treated in a finite domain, can lead to band structure for $-1/4 < A < 0$\cite{10}. In this paper, we study the Lamé and the associated Lamé potentials which belong to the class of elliptic potentials which are usually expressed in terms of the Jacobi elliptic functions $\text{sn}(x,m)$, $\text{cn}(x,m)$ and $\text{dn}(x,m)$. The parameter $m$ is known as the elliptic modulus whose value lies between 0 and 1. By applying supersymmetric quantum mechanics\cite{7,8} and group theoretical techniques to the family of elliptic potentials, many new ES and QES potentials have been constructed\cite{11,12}.

The Lamé potential

$$V(x) = j(j + 1)m \text{sn}^2(x,m),$$

(2)

with $j$ being an integer, has $2j + 1$ bands followed by the continuum. This potential has been extensively studied\cite{13,14}. It is interesting to note that the Schrödinger equation with the Lamé potential

$$\frac{d^2\psi(x)}{dx^2} + \frac{\hbar^2}{2m}(E - j(j + 1)m \text{sn}^2(x,m))\psi(x) = 0,$$

(3)
is a result of separating the Laplace equation in the ellipsoidal coordinates and is referred to
as the Lamé equation\textsuperscript{13,15}. This has unique analytical properties\textsuperscript{13}, which render it useful
in the study of topics ranging from astrophysics\textsuperscript{16,22} to condensed matter physics\textsuperscript{24}. For
example one comes across Lamé equation in problems related to the early epoch of the
universe like the distance red-shift relations\textsuperscript{17} and quantum vacuum fluctuations.\textsuperscript{16} The
Lamé equation occurs in the study of bifurcations in chaotic Hamiltonian systems\textsuperscript{23,25}. It
was shown by Finkel \textit{et.al}\textsuperscript{16} that the matter modes equation, corresponding to the most
general inflation potential in the Minkowski space reduces to the Lamé form.

In comparison, the associated Lamé potential

\begin{equation}
V(x) = pm \text{sn}^2(x, m) + qm \frac{\text{cn}^2(x, m)}{\text{dn}^2(x, m)},
\end{equation}

with \( p = a(a + 1) \) and \( q = b(b + 1) \), did not have a systematic study till now and has a few
scattered references in the mathematical literature. This potential is ES when \( a = b = j \),
with \( j \) being an integer and QES when \( a \neq b \) with \( a, b \) being real. Recently, a systematic
study of the band structure of this potential, for various cases of \( a \) and \( b \), has been done
by Khare and Sukhatme\textsuperscript{26} and the algebraic aspects of the QES associated Lamé potential
have been investigated in\textsuperscript{27,31}.

The band edge eigenfunctions and the eigenvalues of both these potentials have been
found using the conventional techniques of solving the ordinary differential equations\textsuperscript{13,15}.
Here, we propose an alternate method through the quantum Hamilton - Jacobi (QHJ)
formalism, where one can obtain the solutions, without solving the differential equations
explicitly. The QHJ formalism was shown to yield bound state energies and wave functions
for a large number of ES models. The tools used for our earlier studies are sufficient to
derive the results for the periodic potentials also. This method requires making a guess
about the singularity structure of the logarithmic derivative of the wave function \( \psi \) and using
elementary tools from complex variables to fix the QMF completely. Once the form of the
QMF is obtained getting the band edge wave functions and eigenvalues is straightforward.

In the next section, we give a brief summary of the QHJ formalism and the steps involved
in obtaining the solutions for any general potential. In sec \textit{III}, we show how one can obtain
the form of the wave functions for the Lamé potential for a general \( j \), followed by sec \textit{IV},
where we take up a special case of \( j = 2 \) and obtain explicit band edge wave functions and
energies. In sec \textit{V}, the form of the wave functions for the general associated Lamé potential
(ES case) is derived. Associated Lamé potential, with \( a = b = 1 \), is worked out in sec VI, followed by the last section containing the concluding remarks.

II. QUANTUM HAMILTON-JACOBI FORMALISM

The QHJ formalism, was initiated by Leacock and Padgett\(^{32,33}\) in 1983 - 84. Using this formalism on a host of ES models, Bhalla \( et.al^{34-37} \) were successful in obtaining the corresponding energy eigenvalues. The method when applied to the QES models, yielded the quasi exactly solvability condition\(^{38} \). With a slight modification of the method, it was shown that, one could obtain the eigenfunctions and the eigenvalues of ES models simultaneously\(^{39} \).

The main aim of this present study is to demonstrate the applicability of the QHJ formalism to the periodic potentials. Since the QHJ formalism and its working are well studied in the literature, we do not go into details here but give only the necessary information. For details, the interested reader is referred to the earlier works\(^{32,39} \) and the references there in.

The main object of interest in the QHJ formalism is the quantum momentum function (QMF) \( p \), which is the logarithmic derivative of the wavefunction \( \psi(x) \)\(^{32,33} \) apart from a factor of \(-i\hbar\)

\[
p = -i\hbar \frac{d}{dx} (\ln \psi(x)). \tag{5}
\]

Hence, by obtaining the expression for the QMF one can obtain the expression for the wavefunction. The QMF satisfies the Riccati equation which is referred to as the QHJ equation,

\[
p^2 - i\hbar p' = 2m(E - V(x)), \tag{6}
\]

which is the special case of the general Riccati equation

\[
A(x)p^2 + B(x)p + C(x) + i\frac{dp}{dx} = 0. \tag{7}
\]

Equation (6) has the most convenient form to work with, in the QHJ formalism. Hence for all the cases studied through this method one tries to bring the equation for the QMF to the form of (6). In general, the solutions of Riccati equation (7) has two types of singularities, the fixed singularities and the moving singularities. The fixed singularities are determined by the singular points of \( A(x) \), \( B(x) \) and \( C(x) \). These appear in every solution and are independent of the initial conditions. Thus for the QHJ equation (6), the fixed singular points originate from the potential. The other singular points \( i.e \), the moving
singular points depend on the initial conditions and need not appear in every solution. It is known that for the Riccati equation only poles can appear as moving singularities. It is known that, the wavefunction for the $n^{th}$ excited state has $n$ zeros. It is then seen from (6) that correspondingly $p$ has $n$ moving poles in between the two classical turning points and the residue at each of this pole is $-i\hbar$. In general there could be other moving poles of $p$ in the finite complex plane but one has little information regarding their location.

In our earlier studies of the ES and QES solvable models it was assumed that the QMF has finite number of moving poles in the finite complex plane, which turned out to be true for all the models studied. Therefore, for the study of the exactly solvable periodic potentials, without losing generality, one can assume that, the QMF has finite number of moving poles in the complex plane. It follows from our assumption that the point at infinity is an isolated singularity.

With this assumption on the singularity structure of the QMF, we proceed to write the QHJ equation (6) as

$$q^2 + q' = V(x) - E,$$  \hfill (8)

with $\hbar = 2m = 1$, where $q = \frac{d}{dx}\psi(x)$. The wavefunction in terms of $q$ will be

$$\psi(x) = \exp(\int q(x)dx).$$  \hfill (9)

One can see from (8) that the residue at the moving poles is unity. In general it has been found useful to change variable $x$ to variable $t = f(x)$ so as to make the coefficients of the Riccati equation rational. After a change of variable, (8) becomes

$$q^2 + F(t)\frac{dq}{dt} + E - \tilde{V}(t) = 0,$$  \hfill (10)

where $F(t) = \frac{df}{dx}$ expressed as a function of $t$ and $\tilde{V}(t)$ is the potential in terms of $t$. One can see that the above equation does not have the convenient form of (8). Hence to get (10) into the form of (8), we perform a transformation from $q(x)$ to $\chi(t)$ as follows,

$$q = F(t)\phi, \quad \phi = \chi - \frac{1}{2}\frac{d}{dt}(\ln F(t)).$$  \hfill (11)

Using the above transformations (10) becomes

$$\chi^2 + \frac{d\chi}{dt} + \frac{E - \tilde{V}(t)}{F^2(t)} - \frac{1}{2}\left(\frac{F''(t)}{F(t)}\right) + \frac{1}{4}\left(\frac{F'(t)}{F(t)}\right)^2 = 0,$$  \hfill (12)
which is in the form of the Riccati equation. The residue at the moving poles is unity. The fixed poles correspond to the zeros of \( F(t) \). One can make use of (12) instead of the original QHJ equation for any general potential. In the next section we show, by taking the explicit example of the general Lamé potential, how one can obtain the form of the band edge wave functions.

III. GENERAL LAMÉ POTENTIAL

The QHJ equation in terms of \( q \), for the Lamé potential is

\[
q^2 + q' + E - j(j + 1)m \text{sn}^2(x, m) = 0. \tag{13}
\]

To proceed further we need to bring the potential to a meromorphic form, for which we do the following the change of variable

\[
t = \text{sn} (x, m), \tag{14}
\]

with \( \text{sn} x = \text{sn} (x + 4K(m)) \). We would like to point out here that, the Lamé equation can be written in five forms namely two algebraic, one trigonometric, one Weierstrassian and one Jacobian, depending on the change of variable. Of all these change of variables we found (14), which gives the Jacobi form of this Lamé equation, to be best suited for our method. It enables one to write the QHJ equation in a form which can be easily analyzed using the QHJ formalism. Another added advantage of this particular choice is that, it maps half of the period parallelogram \( 0 \leq x < 2K(m) \) of the Jacobi elliptic \( \text{sn}(x, m) \) function to the upper half complex plane. One gets the equation for \( \chi \) as

\[
\chi^2 + \frac{d\chi}{dt} + \frac{(mt)^2 + 2m}{4(1 - mt^2)^2} + \frac{t^2 + 2}{4(1 - t^2)^2} + \frac{2E - 2j(j + 1)mt^2 - mt^2}{2(1 - t^2)(1 - mt^2)} = 0, \tag{15}
\]

where

\[
q = \sqrt{(1 - t^2)(1 - mt^2)}\phi, \quad \phi = \chi + \frac{1}{2} \left( \frac{mt}{1 - mt^2} + \frac{t}{1 - t^2} \right). \tag{16}
\]

The following properties of the Jacobi elliptic functions were used.

\[
\frac{d}{dx} \text{sn}(x, m) = cn(x, m)dn(x, m) \tag{17}
\]

and

\[
\text{sn}^2(x, m) + \text{cn}^2(x, m) = 1, \quad \text{sn}^2(x, m) + m\text{dn}^2(x, m) = 1. \tag{18}
\]
For all our further calculations we will regard $\chi$ as the QMF instead of $p$. Using (15), in place of the original QHJ equation (13), one can obtain the expressions for the wave functions by analyzing the singularity structure of $\chi$

**Singularity structure of $\chi$:** Equation (15) shows that $\chi$ has fixed poles at $t = \pm 1$ and $t = \pm 1/\sqrt{m}$ and there are finite number of moving poles in the complex plane. Hence we make an assumption that the point at $\infty$ is an isolated singular point and that there are no singularities of the QMF except for those mentioned above. Therefore one can write $\chi$, separating it into its singular and analytical parts in the following form,

$$
\chi = \frac{b_1}{t - 1} + \frac{b'_1}{t + 1} + \frac{d_1}{t - 1/\sqrt{m}} + \frac{d'_1}{t + 1/\sqrt{m}} + \left( \sum_{k=0}^{n} \frac{1}{t - t_k} \right) + Q(t).
$$

(19)

Here, $Q(t)$ is the analytic part of $\chi$ and the rational terms represent the singular parts. Here, $b'_1, b_1, d'_1$ and $d_1$ are the residues at $t = \pm 1$ and $t = \pm 1/\sqrt{m}$ respectively. From (15), one can see that $\chi$ is bounded at infinity, which makes $Q(t)$ in (19), analytic and bounded at infinity. Hence from Louville’s theorem $Q(t)$ will be a constant $C$. The summation term in (19), represents the singular part coming due to the finite number of moving poles for which the residues are easily found to be one and this term can be written as $\frac{P'_n}{P_n}$, where $P_n \equiv \prod_{k=0}^{n} (t - t_k)$ is an $n^{th}$ degree polynomial. Thus (19) becomes

$$
\chi = \frac{b_1}{t - 1} + \frac{b'_1}{t + 1} + \frac{d_1}{t - 1/\sqrt{m}} + \frac{d'_1}{t + 1/\sqrt{m}} + \frac{P'_n}{P_n} + C.
$$

(20)

The residues $b_1, b'_1, d_1$ and $d'_1$ at the fixed poles can be determined by substituting the Laurent expansion of $\chi$ around these poles. For example, to calculate the residue at $t = 1$, one expands $\chi$

$$
\chi = \frac{b_1}{t - 1} + a_0 + a_1(t - 1) + \ldots \ldots
$$

(21)

Substituting the above equation in (15) and comparing the coefficients of different powers of $(t - 1)$, one gets a quadratic equation in $b_1$ whose roots are

$$
b_1 = \frac{3}{4}, \frac{1}{4}.
$$

(22)

Similarly the two values of the residues at all the other fixed poles $b'_1, d_1$ and $d'_1$ turn out to be

$$
3/4, 1/4.
$$

(23)
Note that, all the values of the residues are independent of the potential parameter $j$. Hence, irrespective of the value of $j$ in the potential, the residues at the fixed poles will take only the above values for any Lamé potential.

It will be shown later that, the knowledge of the residues is sufficient to obtain the wave function. Hence, at this stage, we should check, which values of the residues, out of the two values, give rise to acceptable wave functions. In the present case, it turns out that both the values, i.e, $3/4$ and $1/4$ for each of the residues, leads to acceptable wave functions.

The only requirement that restricts possible combinations of values is that of parity. The fact that $\psi$ has definite parity implies that $\chi(-t) = -\chi(t)$, which rules out the possibilities $b_1 \neq b'_1$ and $d_1 \neq d'_1$ and hence of all the possible combinations of the values of $b_1$, $b'_1$, $d_1$ and $d'_1$, the only ones which are accepted are those with $b_1 = b'_1$ and $d_1 = d'_1$.

In contrast to the above results, in our earlier study of ES boundstate problems and QES problems, one was lead to a unique choice of the residues at the fixed poles when one demanded that the wave function be square integrable. The other values of the residue did not correspond to any physical solution and were ruled out. In the case of potentials, which exhibit two phases of SUSY, for different ranges of the potential parameters, both values of the residues were useful. One set of residues gave physically acceptable solutions for exact SUSY case and the other set gave for the broken SUSY. In the case of periodic potentials there is no such restriction as square integrability and so there is no way of ruling out one of the values. The reason that only one value of residue is acceptable for the bound state problems in one dimension can be attributed to the fact that the bound state solutions are non-degenerate. Whereas the band edge wavefunctions of the periodic potentials are doubly degenerate, we keep both the values of the residues.

**Behaviour of $\chi$ at infinity**: We have assumed that the point at infinity is an isolated singularity. The form of (15) suggests that $\chi$ is bounded at $\infty$. Therefore, $\chi$ can be expressed as

$$\chi(t) = \lambda_0 + \frac{\lambda_1}{t} + \frac{\lambda_2}{t^2} + \ldots$$

and the coefficients $\lambda_k$'s are fixed using the Riccati equation (15) and one gets

$$\lambda_0 = 0,$$

and two values for $\lambda_1$ as

$$\lambda_1 = j + 1, -j.$$
From (26), one obtains the leading behaviour of $\chi(t)$ at $\infty$ which is dependent on $j$. Equation (20) gives the form of $\chi$ which holds for all finite values of $t$. For large $t$, comparing (20) and (24), one gets

$$2b_1 + 2d_1 + n = \lambda_1.$$  \hspace{1cm} (27)

As the left hand side is positive, it is clear that, only the choice $\lambda_1 = j + 1$ is consistent with the above equation, while the other value $\lambda_1 = -j$ is ruled out. Thus from (27), one gets, the expression for $n$ as

$$n = j + 1 - 2b_1 - 2d_1,$$  \hspace{1cm} (28)

which gives the degree of the polynomial $P_n$.

**Explicit forms of the wavefunctions :** One can obtain the form of the wave functions using (9). Substituting the relations given in (16) in (9), one gets the expression for the wave function in terms of $\chi$ as,

$$\Psi(t) = \exp \int \left( \chi(t) + \frac{1}{2} \left( \frac{mt}{1 - mt^2} + \frac{t}{1 - t^2} \right) \right) dt$$  \hspace{1cm} (29)

$$= \exp \int \left( \frac{(1 - 4b_1)t}{2(1 - t^2)} + \frac{(1 - 4d_1)mt}{2(1 - mt^2)} + \frac{P_n'}{P_n} \right) dt.$$  \hspace{1cm} (30)

which when simplified and written in terms of the original variable $x$, gives

$$\psi(x) = (cn x)^\alpha (dn x)^\beta P_n(sn x),$$  \hspace{1cm} (31)

where, $\alpha = \frac{4b_1 - 1}{2}$ and $\beta = \frac{4d_1 - 1}{2}$. For the sake of simplicity, the elliptic modulus $m$ of the Jacobi elliptic functions will be suppressed here onwards. The four different combinations of the residues (given as sets 1 to 4) give rise to four different forms of the band edge wavefunctions as listed in the table I.

The parity constraint $\chi(-t) = -\chi(t)$, restricts the polynomial $P_n(t)$ to have either only odd or only even powers of $t \equiv sn x$. In the sixth and the seventh columns of table I, the number of linearly independent solutions, for the two cases, $j$ being odd and $j$ being even are given in terms of a positive integer $N$. For odd $j$, $N = (j - 1)/2$ and for even $j$, $N = j/2$. In both the cases the total number of solutions for a particular $j$ is equal to $2j + 1$. It is easy to see that, the four forms and the number of solutions, of a particular form obtained here are in agreement with those already known. For a given set of $b_1$ and $d_1$, $n$ is fixed using (28) and the differential equation for the unknown polynomial $p_n(t)$ can be obtained
by substituting $\chi(t)$ from (20) in (15), which gives $Q = 0$ and

$$P''_n + 4t \left( \frac{b_1}{t^2 - 1} + \frac{md_1}{mt^2 - 1} \right) P'_n + G(t) P_n = 0,$$

(32)

where

$$G(t) = \frac{t^2(4b_1^2 - 2b_1 + 1/4) + 1/2 - 2b_1}{(t^2 - 1)^2} + \frac{(mt)^2(4d_1^2 - 2d_1 + 1/4) + m/2 - 2md_1}{(mt^2 - 1)^2} + \frac{2E - 2j(j + 1)mt^2 + (16mb_1d_1 - 1)mt^2}{2(t^2 - 1)(mt^2 - 1)}.$$

For each set of residues given in table I, this differential equation is equivalent to a system of $n$ linear equations, for the coefficients of different powers of $t$ in $P_n(t)$. The energy eigenvalues are obtained by setting the corresponding determinant equal to zero. We illustrate this process by obtaining the eigenvalues and eigenfunctions explicitly for the case $j = 2$.

IV. LAMÉ POTENTIAL WITH $j = 2$

Using the procedure described in the previous section, we obtain the eigenvalues and the eigenfunctions for the supersymmetric potential

$$V_-(x) = 6m \text{sn}^2 x - 2m - 2 + 2\delta,$$

(33)

where $\delta = \sqrt{1 - m + m^2}$. This potential is same as the Lamé potential in (2) with $j = 2$, except for an additive constant $2\delta - 2m - 2$, which has been added to make the lowest energy equal to zero. The equation for $\chi(t)$ is,

$$\chi^2 + \frac{\chi}{dt} + \frac{(mt)^2 + 2m}{4(1 - mt^2)^2} + \frac{t^2 + 2}{4(1 - t^2)^2} + \frac{2E + 4(m + 1 - \delta) - 13mt^2}{2(1 - t^2)(1 - mt^2)} = 0.$$

(34)

From (28), the number of moving poles $n$, for $j = 2$ are

$$n = 3 - 2b_1 - 2d_1,$$

(35)

where the values of $b_1$ and $d_1$ are obtained from (22) and (23). For each set of $b_1$, $d_1$ and $n$, one can write the form and the number of linearly independent solutions by substituting $j = 2$, which gives $N = 1$ in table I. Hence for the case $j = 2$, the form and the number

10
of solutions is as given in table II. To get the unknown polynomial part $P_n(t)$ of the wave function and the band edge energies, one needs to substitute the different sets of $b_1, d_1$ and $n$ from table II, in the differential equation

$$P''_n + 4t \left( \frac{b_1}{t^2 - 1} + \frac{md_1}{mt^2 - 1} \right) P'_n + G(t) P_n = 0,$$

where

$$G(t) = \frac{t^2(4b_1^2 - 2b_1 + 1/4) + 1/2 - 2b_1}{(t^2 - 1)^2} + \frac{(mt)^2(4d_1^2 - 2d_1 + 1/4) + m/2 - 2md_1}{(mt^2 - 1)^2} + \frac{2E + 4(m + 1 - \delta) + (16b_1d_1 - 13)mt^2}{2(t^2 - 1)(mt^2 - 1)}.$$

Thus for various sets of residues one has the band edge energies and the wave functions as:

**Set 1 :** $b_1 = 1/4, d_1 = 1/4, n = 2$

Taking $P_2 = At^2 + Bt + C$, the parity constraint implies

$$B = 0.$$  \hspace{1cm} (37)

Substituting $b_1, d_1$ and $P_2$ in (36), one gets a $2 \times 2$ matrix equation for $A$ and $C$ as follows

$$\begin{pmatrix} E - 2m - 2 - 2\delta & -6m \\ 2 & E + 2m + 2 - 2\delta \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix} = 0.$$  \hspace{1cm} (38)

Equating the determinant of the matrix in (38) to zero, one gets the two values for energy as

$$E_I = 4\delta, \ E_{II} = 0,$$

which in turn give the polynomial as,

$$P_I = m + 1 - \delta - 3mt^2, \ P_{II} = m + 1 + \delta - 3mt^2.$$  \hspace{1cm} (40)

From (31), we see that, the band edge wave functions in terms of $x$ will be

$$\psi_I(x) = m + 1 - \delta - 3m \sin^2 x, \ \psi_{II}(x) = m + 1 + \delta - 3m \sin^2 x.$$  \hspace{1cm} (41)

**Set 2 :** $b_1 = 3/4, d_1 = 1/4, n = 1$

Taking $P_1 = t - t_k$ and substituting in (36), one gets the band edge eigenvalue and $t_k$ as,

$$E = 2\delta - m + 2, \ t_k = 0.$$  \hspace{1cm} (42)
The band edge wavefunction becomes
\[ \psi(x) = cn\,x \, sn\,x. \]  \hspace{1cm} (43)

**Set 3 :** \( b_1 = 1/4, \, d_1 = 3/4, \, n = 1 \)
Proceeding in the same way as in set 2 one gets,
\[ E = 2\delta + 2m - 1, \, \psi(x) = dn\,x \, sn\,x. \]  \hspace{1cm} (44)

**Set 4 :** \( b_1 = 3/4, \, d_1 = 3/4, \, n = 0 \)
In this case \( P_0 \) as a constant. One gets the band edge energy and the wave function to be
\[ E = 2\delta - m - 1, \, \psi(x) = cn\,x \, dn\,x. \]  \hspace{1cm} (45)

Thus one obtains five band edge wave functions and their corresponding energies which agree with those given in 7.

V. ASSOCIATED LAMÉ POTENTIAL

The associated Lamé potential is ES when \( a = b = j \). In this case the potential expression becomes
\[ V(x) = j(j+1)m \left( sn^2x + \frac{cn^2x}{dn^2x} \right). \]  \hspace{1cm} (46)
The QHJ equation with the associated Lamé potential is
\[ q^2 + q' + E - mj(j+1) \left( sn^2x + \frac{cn^2x}{dn^2x} \right) = 0. \]  \hspace{1cm} (47)
Doing the change of variable \( t = sn\,x \) and proceeding in the same way as in the case of general Lamé potential described in sec II, one gets the equation for \( \chi(t) \) as,
\[ \chi^2 + \frac{d\chi}{dt} + \frac{(mt)^2 + 2m(1 - 2j(j+1))}{4(1 - mt^2)^2} + \frac{2 + t^2}{4(1 - t^2)^2} + \frac{2E - mt^2(1 + 2j(j+1))}{2(1 - t^2)(1 - mt^2)} = 0. \]  \hspace{1cm} (48)

**Singularity structure of** \( \chi(t) \): Similar to the Lamé potential, \( \chi(t) \) has fixed poles at \( t = \pm 1 \) and \( t = \pm 1/\sqrt{m} \), along with \( n \) moving poles in the entire complex \( t \) plane and further we assume that, the point at infinity is an isolated singularity. \( \chi \) has no other singular points, except at \( t = \pm 1 \) and \( t = \pm 1/\sqrt{m} \). Following the same procedure, used in sec II, one obtains the residues at \( t = \pm 1 \) as
\[ b_1 = \frac{3}{4}, \, b_1' = \frac{3}{4}. \]  \hspace{1cm} (49)
which is independent of $j$. The residues at $t = \pm 1/\sqrt{m}$ turn out to be

$$d_1 = \frac{3 + 2j}{4}, \frac{1 - 2j}{4}, \quad d'_1 = \frac{3 + 2j}{4}, \frac{1 - 2j}{4},$$

(50)

which are $j$ dependent. Knowing the singularity structure of $\chi$, one can write it as

$$\chi = \frac{b_1}{t - 1} + \frac{b'_1}{t + 1} + \frac{d_1}{t - 1/\sqrt{m}} + \frac{d'_1}{t + 1/\sqrt{m}} + \frac{P'_n}{P_n},$$

(51)

valid for all $t$ similar to the Lamé potential.

**Behavior of $\chi$ at infinity:** The behavior of $\chi$ for large $t$, as fixed from the QHJ equation (48) is

$$\chi = \lambda_0 + \frac{\lambda_1}{t} + \frac{\lambda_2}{t^2} + ....$$

(52)

and one gets the two values for $\lambda_1$ as

$$\lambda_1 = j + 1, -j.$$  

(53)

This result should agree with the large $t$ behavior of $\chi$ given by (51), which is

$$\chi = b_1 + b'_1 + d_1 + d'_1 + n.$$  

(54)

As before, parity requirement implies $b_1 = b'_1$ and $d_1 = d'_1$. Comparing the leading terms for large $t$, in (51) and (54) we get,

$$2b_1 + 2d_1 + n = \lambda_1.$$  

(55)

In this case both the values of $\lambda_1$ are allowed because $d_1$ and $d'_1$ can take negative values unlike the case of Lamé potential where $\lambda_1 = -j$ was ruled out. Thus one has two cases $\lambda_1 = j + 1$ and $\lambda_1 = -j$.

**Case 1 : $\lambda = j + 1$**

The values of $n$, which describe the number of zeros, for different combinations of $b_1$ and $d_1$ values are given in table III. The sets 3 and 4 give $n = -1$ and $n = -2$ respectively and will not be considered, as $n$ should be greater than or equal to 0. The sets 1 and 2 will give positive values of $n$ only if $j$ is positive. Hence, this table is used when $j$ is positive.

**Case 2 : $\lambda_1 = -j$**

The $n$ values for various sets of $b_1$ and $d_1$ with $\lambda_1 = -j$ are listed in table IV. As in the previous case of $\lambda_1 = j + 1$, we will not consider the first two sets in the above table.
Note that the potential is invariant under the transformation $j \rightarrow -j - 1$. From the two tables III and IV, we see that two different values of $j$, one positive ($j = j' > 0$) and another negative ($j = -j' - 1 < 0$) leads to the same expression for the potential described by (41). These two values also lead to the same answers for the wave functions and eigenvalues. Hence it is sufficient to restrict $j$ to positive values alone. Thus using the two sets of combinations given in table III and the expressions for the wave functions given in (31) one gets the following explicit forms and the number of solutions given in table V.

In the next section, we obtain explicit expressions for the wavefunctions and energies for $j = 1$.

VI. ASSOCIATED LAMÉ POTENTIAL WITH $j = 1$

We perform the calculation with the supersymmetric potential

$$V_-(x) = 2m \text{sn}^2 x + 2m \frac{\text{cn}^2 x}{\text{dn}^2 x} - 2 - m + 2\sqrt{1 - m}, \quad (56)$$

corresponding to $j = 1$. Proceeding in the same way as in the previous sections, the equation for $\chi(t)$ is found to be

$$\chi^2 + \frac{d\chi}{dt} + \frac{(mt)^2 - 6m}{4(mt^2 - 1)^2} + \frac{t^2 + 2}{4(t^2 - 1)^2} + \frac{2E + 4 + 2m - 4\sqrt{1 - m} - 5mt^2}{2(1 - mt^2)(1 - t^2)} = 0 \quad (57)$$

and the form for $\chi(t)$ is

$$\chi = \frac{b_1}{t - 1} + \frac{b'_1}{t + 1} + \frac{d_1}{t - 1/\sqrt{m}} + \frac{d'_1}{t + 1/\sqrt{m}} + \frac{P_n'}{P_n}. \quad (58)$$

Substituting (58) in (57), one is left with the differential equation

$$P''_n + 4t \left( \frac{b_1}{t^2 - 1} + \frac{md_1}{mt^2 - 1} \right) P'_n + G(t) P_n = 0, \quad (59)$$

where

$$G(t) = \frac{t^2(4b_1^2 - 2b_1 + 1/4) + 1/4 - 2b_1}{(t^2 - 1)^2} + \frac{(mt)^2(4d_1^2 - 2d_1 + 1/4) - 3m/2 - 2md_1}{(mt^2 - 1)^2} + \frac{2E + 4 + 2m - 4\sqrt{1 - m} + (16b_1d_1 - 5)mt^2}{2(t^2 - 1)(mt^2 - 1)}.$$

Substituting $j = 1$ in table III, one gets the band edge eigenfunctions and eigenvalues from set 1 and set 2 as follows.
**Set 1**: \( b_1 = 3/4 , \ d_1 = -1/4 , \ n = 1 \)

This combination gives only one solution with

\[
E = 2 - m + 2\sqrt{1-m} , \ \psi(x) = \frac{\text{cn} \ x \ \text{sn} \ x}{\text{dn} \ x}.
\] (60)

**Set 2**: \( b_1 = 1/4 , \ d_1 = -1/4 , \ n = 2 \)

This combination gives two solutions. The band edge energies are

\[
E_I = 0 , \ E_{II} = 4\sqrt{1-m}
\] (61)

and the corresponding wave functions are

\[
\psi_I(x) = \frac{1}{m} \left( \text{dn} \ x + \frac{\sqrt{1-m}}{\text{dn} \ x} \right) , \ \psi_{II}(x) = \frac{1}{m} \left( \text{dn} \ x - \frac{\sqrt{1-m}}{\text{dn} \ x} \right) ,
\] (62)

which match with the wave functions in [4].

**VII. CONCLUSIONS**

In this paper, we have studied the ES periodic potentials of the elliptic class namely the Lamé and the associated Lamé potentials. Using the QHJ formalism we have obtained here the general form of the band edge wave functions for these potentials, with the potential parameters taking only integer values. Also, the band edge eigenfunctions and the energy eigenvalues have been obtained explicitly for \( j = 2 \), for the Lamé potential and \( p = q = 2 \), for the associated Lamé potential. In the process, we have studied \( p \), the logarithmic derivative of the wave function for these elliptic potentials and have found an interesting singularity structure exhibited by \( p \) in the complex domain.

The QHJ formalism has been successful in obtaining the energy eigenvalues and the wave functions for a large variety of ES models and periodic potentials. The most important steps in obtaining the solutions using QHJ formalism have been, the choice of the change of variable and the ability to guess the right singularity structure of the QMF. It should be noted here that, our approach has been to assume a singularity structure of the QMF as simple as possible. Once a correct choice has been found, one could obtain the solutions in a most straightforward fashion even if the resulting equations were not of a simple form. From our earlier study of ES\textsuperscript{39}, QES\textsuperscript{38} and the present study of periodic potentials, some interesting features of these models appear to be correct. All the models whether ES, QES
or periodic potentials, which we have studied so far, share a common property that ‘the QMF’, $\chi$, becomes a rational function after a suitable change of variable. Therefore, in the finite complex plane it has finite number of moving poles, described by the parameter $n$. Our assumption that the point at infinity is an isolated singular point is equivalent to the condition that the QMF has finite number of moving poles.

There are interesting differences in details of the singularity structure for different potential models. For the ES models, each value of $n$ corresponds to an energy level and an eigenfunction with only $n$ real zeros. These zeros correspond to the $n$ moving poles of the QMF and are confined to the classical region only. For the QES models, $n$ appears as a parameter in the potential and of the infinite possible states only $n$ states can be obtained analytically. The QMF for all these states will have the same number of moving poles which are both real and complex. Correspondingly the wavefunctions for each state will have the same number ($\equiv n$) of zeroes, of which the number of real zeros are in accordance with the oscillation theorem. The number of real zeros increases as one goes from the lowest state to the highest state possible.

On the other hand, the exactly solvable periodic potential models show quite a different kind of distribution of the moving poles in the complex plane. Like QES models, the moving poles of the QMF consist of both real and complex poles. But unlike the QES models, where the number of poles of QMF for all the known states is same, for ES periodic potential models one finds groups of solutions, with number of moving poles remaining same within a group but varying from one group to another group of solutions. This point becomes clear from the form of the solutions listed in table I for the Lamé potential. Different groups of solutions are precisely the different sets of the solutions listed in table I. In each set, the degree of the polynomial $P_n(sn x)$ is same and the total number of zeros of the band edge wave functions in the interval $0 \leq x < 2K(m)$ are same. For example, for the $N+1$ linear independent solutions belonging to set 1 the total number of zeros of the wavefunctions will be $j$. However for the solutions belonging to set 2 there will be $j-1$ zeros of $P_n(sn x)$ and one zero corresponding to $cn x$, thus a total of $j$ zeros. For all the sets the total number of zeros are given in the last column of table I. However the number of real zeros in a group increases with increasing energy and correspondingly the number of complex zeros will decrease. For all values of $j$, we will have utmost four such groups. A similar pattern is observed for ES associated Lamé potential.
When one computes the residues at the fixed pole one gets two solutions due to the quadratic nature of the QHJ equation. For the bound state problems, both ES and QES considered in the earlier papers, the square integrability of the wave function had to be insisted. This allowed us to pick the right residue, out of the two values, which gave the physically acceptable wave function, but in the present case of periodic potential models, there is no such restriction and, both values of the residues lead to acceptable solutions. The only restriction on the possible combination of the residues comes from the requirements of parity.

To conclude we note that the new QHJ equation, obtained after a change of variable, involves rational functions in the independent variable. In order to make progress, we made crucial assumption that there are finite number of moving poles implying that the point at infinity is an isolated singular point. This important assumption has been used for all the models studied and its consistency with other equations has given useful restrictions on the acceptable solutions. Our present study completes this series of investigation of ES models from the QHJ approach. We now have a good starting point to take up models which are not ES or QES. For these models, making a proper guess about the singularity structure of the QMF does not appear to be possible. A study of the distribution of the moving poles in the complex plane may have a relation to the classical properties of the system. Such a study is interesting in its own right and can be best done numerically. We hope that here also, the established results in the complex variable theory will provide a useful scheme to obtain numerical, possibly approximate analytical solutions for the wave functions and energies.

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TABLE I: Form of the wave functions $\psi(x)$, number of linear independent solutions and the total number of zeros of $\psi(x)$ for the general Lamé potential $V(x) = j(j+1)m\,sn^2x$.

| set | $b_1$ | $d_1$ | $n$ | $\psi$ in terms of x | Number of solutions | Total zeros of $\psi(x)$ |
|-----|-------|-------|-----|-----------------------|---------------------|--------------------------|
| 1   | 1/4   | 1/4   | $j$ | $P_j(sn\,x)$          | $N + 1$             | $N + 1$                  |
| 2   | 3/4   | 1/4   | $j-1$ | $cn\,xP_{j-1}(sn\,x)$ | $N + 1$             | $N$                      |
| 3   | 1/4   | 3/4   | $j-1$ | $dn\,xP_{j-1}(sn\,x)$ | $N + 1$             | $j-1$                    |
| 4   | 3/4   | 3/4   | $j-2$ | $cn\,xdn\,xP_{j-2}(sn\,x)$ | $N$             | $j-1$                    |

TABLE II: The form of the wave functions and the number of linear independent solutions for the Lamé potential with $j = 2$, $V_\pm = 6msn^2x - 2m - 2 + 2\delta$ where $\delta = \sqrt{1 - m + m^2}$.

| set | $b_1$ | $d_1$ | $n$ | $\psi$ in terms of x | Number of LI solutions |
|-----|-------|-------|-----|-----------------------|------------------------|
| 1   | 1/4   | 1/4   | 2   | $P_2(sn\,x)$          | 2                      |
| 2   | 3/4   | 1/4   | 1   | $cn\,xP_1(sn\,x)$    | 1                      |
| 3   | 1/4   | 3/4   | 1   | $dn\,xP_1(sn\,x)$    | 1                      |
| 4   | 3/4   | 3/4   | 0   | $cn\,xdn\,xP_0(sn\,x)$ | 1                      |

TABLE III: Values of $n$ for different combinations of $b_1$ and $d_1$ for $\lambda_1 = j + 1$ for the associated Lamé potential $V(x) = j(j+1)m(sn^2x + cn^2x/dn^2x)$.

| set | $b_1$ | $d_1$ | $n$ |
|-----|-------|-------|-----|
| 1   | 1/4   | $(1 - 2j)/4$ | $2j$ |
| 2   | 3/4   | $(1 - 2j)/4$ | $2j - 1$ |
| 3   | 1/4   | $(3 + 2j)/4$ | $-1$ |
| 4   | 3/4   | $(3 + 2j)/4$ | $-2$ |
TABLE IV: Various sets of $b_1$ and $d_1$ for $\lambda_1 = -j$ for associated Lamé potential $V(x) = j(j + 1)m(sn^2x + cn^2x/dn^2x)$.

| set | $b_1$ | $d_1$ | $n$ |
|-----|-------|-------|-----|
| 1   | $1/4$ | $(1 - 2j)/4$ | $-1$ |
| 2   | $3/4$ | $(1 - 2j)/4$ | $-2$ |
| 3   | $1/4$ | $(3 + 2j)/4$ | $-2j - 2$ |
| 4   | $3/4$ | $(3 + 2j)/4$ | $-2j - 3$ |

TABLE V: Form of the wavefunctions for the associated Lamé potential $V(x) = j(j + 1)m(sn^2x + cn^2x/dn^2x)$ for positive values of $j$.

| set | form of $\psi$ | No of solutions |
|-----|----------------|-----------------|
|     | $j = 2N$       | $j = 2N + 1$    |
| 1   | $\frac{cnxP_{2j-1}(snx)}{(dnx)^j}$ | $2N$ | $2N + 1$ |
| 2   | $\frac{P_{2j}(snx)}{(dnx)^j}$ | $2N + 1$ | $2N + 2$ |