On the uniqueness of algebraic curves passing through $n$-independent nodes

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October 20, 2015

Abstract

A set of nodes is called $n$-independent if each its node has a fundamental polynomial of degree $n$. We proved in a previous paper [H. Hakopian and S. Toroyan, On the minimal number of nodes determining uniquely algebraic curves, accepted in Proceedings of YSU] that the minimal number of $n$-independent nodes determining uniquely the curve of degree $k \leq n$ equals to $K := (1/2)(k-1)(2n+4-k)+2$. Or, more precisely, for any $n$-independent set of cardinality $K$ there is at most one curve of degree $k \leq n$ passing through its nodes, while there are $n$-independent node sets of cardinality $K-1$ through which pass at least two such curves. In this paper we bring a simple characterization of the latter sets. Namely, we prove that if two curves of degree $k \leq n$ pass through the nodes of an $n$-independent node set $X$ of cardinality $K-1$ then all the nodes of $X$ but one belong to a (maximal) curve of degree $k-1$.

Key words: Algebraic curves, $n$-independent nodes, maximal curves, polynomial interpolation

Mathematics Subject Classification (2010). 14H50, 41A05.

1 Introduction

Denote the space of all bivariate polynomials of total degree $\leq n$ by $\Pi_n$:

$$\Pi_n = \left\{ \sum_{i+j \leq n} a_{ij} x^i y^j \right\}.$$

We have that

$$N := N_n := \dim \Pi_n = (1/2)(n+1)(n+2).$$

Consider a set of $s$ distinct nodes

$$X_s = \{ (x_1, y_1), (x_2, y_2), \ldots, (x_s, y_s) \}.$$
The problem of finding a polynomial $p \in \Pi_n$ which satisfies the conditions

$$p(x_i, y_i) = c_i, \quad i = 1, \ldots, s,$$

(1.1)

is called interpolation problem.

A polynomial $p \in \Pi_n$ is called an $n$-fundamental polynomial for a node $A = (x_k, y_k) \in \mathcal{X}_s$ if

$$p(x_i, y_i) = \delta_{ik}, \quad i = 1, \ldots, s,$$

where $\delta$ is the Kronecker symbol. We denote this fundamental polynomial by $p_k = p^*_A = p^*_A, \mathcal{X}_s$. Sometimes we call fundamental also a polynomial that vanishes at all nodes of $\mathcal{X}_s$ but one, since it is a nonzero constant times a fundamental polynomial.

Next, let us consider an important concept of $n$-independence (see [5], [9]).

**Definition 1.1.** A set of nodes $\mathcal{X}$ is called $n$-independent if all its nodes have $n$-fundamental polynomials. Otherwise, if a node has no $n$-fundamental polynomial, $\mathcal{X}$ is called $n$-dependent.

Fundamental polynomials are linearly independent. Therefore a necessary condition of $n$-independence of $\mathcal{X}_s$ is $s \leq N$.

Suppose a node set $\mathcal{X}_s$ is $n$-independent. Then by the Lagrange formula we obtain a polynomial $p \in \Pi_n$ satisfying the interpolation conditions (1.1):

$$p = \sum_{i=1}^{s} c_i p^*_i.$$

In view of this, we get readily that the node set $\mathcal{X}_s$ is $n$-independent if and only if the interpolating problem (1.1) is solvable, meaning that for any data $(c_1, \ldots, c_s)$ there is a polynomial $p \in \Pi_n$ (not necessarily unique) satisfying the interpolation conditions (1.1).

**Definition 1.2.** The interpolation problem with a set of nodes $\mathcal{X}_s$ and $\Pi_n$ is called $n$-poised if for any data $(c_1, \ldots, c_s)$ there is a unique polynomial $p \in \Pi_n$ satisfying the interpolation conditions (1.1).

The conditions (1.1) give a system of $s$ linear equations with $N$ unknowns (the coefficients of the polynomial $p$). The poisedness means that this system has a unique solution for arbitrary right side values. Therefore a necessary condition of poisedness is $s = N$. If this condition holds then we obtain from the linear system

**Proposition 1.3.** A set of nodes $\mathcal{X}_N$ is $n$-poised if and only if

$$p \in \Pi_n \text{ and } p \big|_{\mathcal{X}_N} = 0 \implies p = 0.$$

Thus, geometrically, the node set $\mathcal{X}_N$ is $n$-poised if and only if there is no curve of degree $n$ passing through all its nodes.

It is worth mentioning
**Proposition 1.4.** For any set $X_{N-1}$, i.e., set of cardinality $N - 1$, there is a curve of degree $n$ passing through all its nodes.

Indeed, the existence of the curve reduces to a system of $N - 1$ linear homogeneous equations with $N$ unknowns – the coefficients of the polynomial of degree $n$.

It follows from Proposition 1.3 also that a node set of cardinality $N$ is $n$-poised if and only if it is $n$-independent.

Suppose we have an $m$-poised set $X$. From what was said above we can conclude easily that through any $N - 1$ nodes of $X$ there pass a unique curve of degree $n$. Namely the curve given by the fundamental polynomial of the missing node. While through any $N - 2$ nodes of $X$ there pass more than one curve of degree $n$, for example the curves given by the fundamental polynomials of two missing nodes. Thus we have that the minimal number of $n$-independent nodes determining uniquely the curve of degree $n$ equals to $N - 1$.

In [11] we considered this problem in the case of arbitrary degree $k, k \leq n$. We proved that the minimal number of $n$-independent nodes determining uniquely the curve of degree $k \leq n$ equals to $K := (1/2)(k-1)(2n + 4 - k) + 2$. Or, more precisely, for any $n$-independent set of cardinality $K$ there is at most one curve of degree $k \leq n$ passing through its nodes, while there are $n$-independent node sets of cardinality $K - 1$ through which pass at least two such curves. Let us mention that the above described problem in the case $k = n - 1$ was solved in [1].

In this paper we bring a simple characterization of the sets of cardinality $K - 1$ through which pass at least two curves of degree $k$. Namely, we prove that in this case all the nodes of $X$ but one belong to a curve of degree $k - 1$. Moreover, this latter curve is a maximal curve meaning that it passes through maximal possible number of $n$-independent nodes (see Section 3).

At the end let us bring a well-known Berzolari-Radon construction of $n$-poised set (see [2], [12]).

**Definition 1.5.** A set of $N = 1 + \cdots + (n + 1)$ nodes is called Berzolari-Radon set for degree $n$, or briefly $BR_n$ set, if there exist lines $l_1, l_2, \ldots, l_{n+1}$, such that the sets $l_1 \setminus l_1$, $l_3 \setminus (l_1 \cup l_2)$, $\ldots$, $l_{n+1} \setminus (l_1 \cup \cdots \cup l_n)$ contain exactly $(n + 1), n, n - 1, \ldots, 1$ nodes, respectively.

## 2 Some properties of $n$-independent nodes

Let us start with the following simple (see [10], Lemma 2.3)

**Lemma 2.1.** Suppose that a node set $X$ is $n$-independent and a node $A \notin X$ has $n$-fundamental polynomial with respect to the set $X \cup \{A\}$. Then the latter node set is $n$-independent, too.

Indeed, one can get readily the fundamental polynomial of any node $B \in X$ with respect to the set $\mathcal{Y} := X \cup \{A\}$ by using the given fundamental polynomial $p^*_A$ and the fundamental polynomial of $B$ with respect to the set $\mathcal{X}$. 

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Evidently, any subset of $n$-poised set is $n$-independent. According to the next lemma any $n$-independent set is a subset of some $n$-poised set (see, e.g. [7], Lemma 2.1):

**Lemma 2.2.** Any $n$-independent set $X$ with $\#X < N$ can be enlarged to an $n$-poised set.

*Proof.* It suffices to show that there is a node $A$ such that the set $X \cup \{A\}$ is $n$-independent. By Proposition 1.4 there is a nonzero polynomial $q \in \Pi_n$ such that $q|_X = 0$. Now, in view of Lemma 2.1, we may choose a desirable node $A$ by requiring only that $q(A) \neq 0$. Indeed, then $q$ is a fundamental polynomial of $A$ with respect to the set $X \cup \{A\}$.  

Denote the linear space of polynomials of total degree at most $n$ vanishing on $X$ by

$$P_{n,X} = \{p \in \Pi_n : p|_X = 0\}.$$  

The following is well-known (see e.g. [9])

**Proposition 2.3.** For any node set $X$ we have that

$$\dim P_{n,X} \geq N - \#X.$$  

Moreover, equality takes place here if and only if the set $X$ is $n$-independent.

From here one gets readily (see [10], Corollary 2.4):

**Corollary 2.4.** Let $Y$ be a maximal $n$-independent subset of $X$, i.e., $Y \subset X$ is $n$-independent and $Y \cup \{A\}$ is $n$-dependent for any $A \in X \setminus Y$. Then we have that

$$P_{n,Y} = P_{n,X}. \quad (2.1)$$

*Proof.* We have that $P_{n,X} \subset P_{n,Y}$, since $Y \subset X$. Now, suppose that $p \in \Pi_n$, $p|_Y = 0$ and $A$ is any node of $X$. Then $Y \cup \{A\}$ is dependent and therefore, in view of Lemma 2.1, $p|_A = 0$.  

From (2.1) and Proposition 2.3 (part ”moreover”) we have that

$$\dim P_{n,X} = N - \#Y, \quad (2.2)$$

where $Y$ is any maximal $n$-independent subset of $X$. Thus, all the maximal $n$-independent subsets of $X$ have the same cardinality, which is denoted by $\mathcal{H}_n(X)$ — *the Hilbert $n$-function* of $X$. Hence, according to (2.2), we have that

$$\dim P_{n,X} = N - \mathcal{H}_n(X).$$
3 Maximal curves

An algebraic curve in the plane is the zero set of some bivariate polynomial of degree at least 1. We use the same letter, say $p$, to denote the polynomial $p \in \Pi_k \setminus \Pi_{k-1}$ and the corresponding curve $p$ of degree $k$ defined by equation $p(x, y) = 0$.

According to the following well-known statement there are no more than $n + 1$ n-independent points in any line:

**Proposition 3.1.** Assume that $l$ is a line and $X_{n+1}$ is any subset of $l$ containing $n + 1$ points. Then we have that

$$p \in \Pi_n \quad \text{and} \quad p|_{X_{n+1}} = 0 \implies p = lr,$$

where $r \in \Pi_{n-1}$.

Denote

$$d := d(n, k) := N_n - N_{n-k} = k(2n + 3 - k)/2.$$

The following is a generalization of Proposition 3.1.

**Proposition 3.2 ([13], Prop. 3.1).** Let $q$ be an algebraic curve of degree $k \leq n$ without multiple components. Then the following hold.

i) Any subset of $q$ containing more than $d(n, k)$ nodes is $n$-dependent.

ii) Any subset $X_d$ of $q$ containing exactly $d(n, k)$ nodes is $n$-independent if and only if the following condition holds:

$$p \in \Pi_n \quad \text{and} \quad p|_{X_d} = 0 \implies p = qr,$$

where $r \in \Pi_{n-k}$.

Suppose that $X$ is an $n$-poised set of nodes and $q$ is an algebraic curve of degree $k \leq n$. Then of course any subset of $X$ is $n$-independent too. Therefore, according to Proposition 3.2, at most $d(n, k)$ nodes of $X$ can lie in the curve $q$. Let us mention that a special case of this when $q$ is a set of $k$ lines is proved in [4].

This motivates the following definition (see [13], Def. 3.1).

**Definition 3.3.** Given an $n$-independent set of nodes $X_s$, with $s \geq d(n, k)$. A curve of degree $k \leq n$ passing through $d(n, k)$ points of $X_s$, is called maximal.

Note that maximal line, as a line passing through $n + 1$ nodes, is defined in [3].

We say that a node $A \in \mathcal{X}$ uses a polynomial $q \in \Pi_k$ if the latter divides the fundamental polynomial $p = p_A^*$, i.e., $p = qr$, for some $r \in \Pi_{n-k}$.

Next, we bring a characterization of maximal curves:

**Proposition 3.4 ([13], Prop. 3.3).** Let a node set $\mathcal{X}$ be $n$-poised. Then a polynomial $\mu$ of degree $k$, $k \leq n$, is a maximal curve if and only if it is used by any node in $\mathcal{X} \setminus \mu$.

Note that one side of this statement follows from Proposition 3.2 (ii). In the case of lines this was proved in [3]. For other properties of maximal curves we refer reader to [13].
Proposition 3.5. Assume that $\sigma$ is an algebraic curve of degree $k$, without multiple components, and $\mathcal{X}_s \subset \sigma$ is any $n$-independent node set of cardinality $s$, $s < d(n, k)$. Then the set $\mathcal{X}_s$ can be extended to a maximal $n$-independent set $\mathcal{X}_d \subset \sigma$ of cardinality $d$, where $d = d(n, k)$.

Proof. It suffices to show that there is a point $A \in \sigma \setminus \mathcal{X}_s$ such that the set $\mathcal{X}_s + 1 := \mathcal{X}_s \cup \{A\}$ is $n$-independent. Assume to the contrary that there is no such point, i.e., the set $\mathcal{X}_s + 1 := \mathcal{X}_s \cup \{A\}$ is $n$-dependent for any $A \in \sigma$. Then, in view of Lemma 2.1, $A$ has no fundamental polynomial with respect to the set $\mathcal{X}_s + 1$.

In other words we have

$$p \in \Pi_n \quad \text{and} \quad p|_{\mathcal{X}_s} = 0 \quad \Rightarrow \quad p(A) = 0 \quad \text{for any} \quad A \in \sigma.$$

From here we obtain that

$$\mathcal{P}_{n, \mathcal{X}_s} \subset \mathcal{P}_{n, \sigma} := \{q\sigma : q \in \Pi_{n-k}\}.$$

Now, in view of Proposition 2.3, we get from here

$$N - s = \dim \mathcal{P}_{n, \sigma} \leq \dim \mathcal{P}_{n, \mathcal{X}_s} = N_{n-k}.$$

Therefore $s \geq d(n, k)$, which contradicts the hypothesis of Proposition.

The following lemma follows readily from the fact that the Vandermonde determinant, i.e., the main determinant of the linear system described after Definition 1.2 is a continuous function of the nodes of $\mathcal{X}_N$ (see e.g., [6], Remark 1.14).

Lemma 3.6. Suppose $\mathcal{X}_N = \{(x_i, y_i)\}_{i=1}^N$ is $n$-poised. Then there is a positive number $\epsilon$ such that any set $\mathcal{X}_N' = \{(x'_i, y'_i)\}_{i=1}^N$, for which distance between $(x'_i, y'_i)$ and $(x_i, y_i)$ is less than $\epsilon$, is $n$-poised too.

Finally, let us bring a lemma that follows from a simple Linear Algebra argument (see e.g., [3], Lemma 2.10).

Lemma 3.7. Suppose that two different curves of degree $k$ pass through all the nodes of $\mathcal{X}$. Then for any node $A \notin \mathcal{X}$ there is a curve of degree $k$ passing through all the nodes of $\mathcal{X}$ and $A$.

4 Main result

In a previous paper [11] we determined the minimal number of $n$-independent nodes that uniquely determine the curve of degree $k$, $k \leq n$, passing through them:

Theorem 4.1. Assume that $\mathcal{X}$ is any set of $(d(n, k - 1) + 2)$ $n$-independent nodes lying in a curve of degree $k$ with $k \leq n$. Then the curve is determined uniquely. Moreover, there is a set $\hat{\mathcal{X}}$ of $(d(n, k - 1) + 1)$ $n$-independent nodes such that more than one curves of degree $k$ pass through all its nodes.
Let us mention that this result, in the case \( k = n - 1 \), was established in [1].

In this section we give a characterization of the case when more than one
curve of degree \( k, k \leq n \), passes through the nodes of an \( n \)-independent set \( \mathcal{X} \)
of cardinality \( d(n, k - 1) + 1 \).

As we will see later this result is a generalization of Theorem 4.1.

**Theorem 4.2.** Given a set of \( n \)-independent nodes \( \mathcal{X} \) with \( \# \mathcal{X} = d(n, k - 1) + 1 \).
Then there are at least \( 2 \) curves of degree \( k \) passing through all nodes of \( \mathcal{X} \) if and only if there exists a maximal curve \( \mu \) of degree \( k - 1 \) passing through \( d(n, k - 1) \) nodes of \( \mathcal{X} \) and the remaining node of \( \mathcal{X} \) is outside of \( \mu \).

**Proof.** Let us start with the inverse implication. Assume that \( d(n, k - 1) \) nodes
of \( \mathcal{X} \) are located on a curve \( \mu \) of degree \( k - 1 \). Therefore, as it is mentioned in
the formulation of Theorem, the curve \( \mu \) is maximal and the remaining node of \( \mathcal{X} \),
which we denote by \( A \), is outside of it: \( A \notin \mu \).

Now, according to Proposition 3.4, we have
\[
\mathcal{P}_{k, \mathcal{X}} = \{ \alpha \mu | \alpha \in \Pi, \alpha(A) = 0 \}.
\]
Therefore we get readily
\[
\dim \mathcal{P}_{k, \mathcal{X}} = \dim \{ \alpha | \alpha \in \Pi, \alpha(A) = 0 \} = 2.
\]

Now let us prove the direct implication. Assume that there are two curves
of degree \( k \) : \( \sigma_1 \) and \( \sigma_2 \) that pass through all the nodes of the \( n \)-independent
set \( \mathcal{X} \), \( \# \mathcal{X} = d(n, k - 1) + 1 \). Next, choose a node \( B \notin \mathcal{X} \) such that the following
three conditions are satisfied:

(i) \( B \) does not belong to any line passing through two nodes of \( \mathcal{X} \),
(ii) \( B \) does not belong to the curves \( \sigma_1 \) and \( \sigma_2 \),
(iii) The set \( \mathcal{X} \cup \{ B \} \) is \( n \)-independent.

Let us verify that one can find a such node. Indeed, in view of Lemma 2.2 we
can start by choosing a node \( B' \) satisfying the condition (iii). Then notice that,
according to Lemma 3.6, for some positive \( \epsilon \) all the nodes in \( \epsilon \) neighborhood of
\( B' \) satisfy the condition (iii). Finally, from this neighborhood we can choose a
node \( B \) satisfying the condition (i) and (ii), too.

In view of Proposition 3.4 there is a curve of degree \( k \) passing through all the
nodes of \( \mathcal{Y} := \mathcal{X} \cup \{ B \} \). Denote a such curve by \( \sigma \). In view of (ii) \( \sigma \) is different
from \( \sigma_1 \) and \( \sigma_2 \).

Next, by using Proposition 3.5 let us extend the set \( \mathcal{Y} \) till a maximal \( n \)-independent set \( \mathcal{Z} \subset \sigma \). Notice that, since \( \# \mathcal{Z} = d(n, k) \), we need to add
\( d(n, k) - (d(n, k - 1) + 2) = n - k \) nodes, denoted by \( C_1, \ldots, C_{n-k} \):
\[
\mathcal{Z} := \mathcal{X} \cup \{ B \} \cup \{ C_i \}_{i=1}^{n-k}.
\]

Thus the curve \( \sigma \) becomes maximal with respect to the set \( \mathcal{Z} \).

Then let us consider \( n - k - 1 \) lines \( \ell_1, \ell_2, \ldots, \ell_{n-k-1} \) passing through the
nodes \( C_1, C_2, \ldots, C_{n-k-1} \), respectively. We require that each line passes through
only one of the mentioned nodes and therefore the lines are distinct. We require

also that none of these lines is a component (factor) of \( \sigma \). Finally let us denote by \( \ell \) the line passing through \( B \) and \( C_{n-k} \).

Now notice that the following polynomial of degree \( n \) vanishes at all points of \( \mathcal{Z} \)

\[
\sigma_1 \ell_1 \ell_2 \ldots \ell_{n-k-1}.
\] (4.1)

Consequently, in view of Proposition 3.4, \( \sigma \) divides this polynomial:

\[
\sigma_1 \ell_1 \ell_2 \ldots \ell_{n-k-1} = \sigma q, \quad q \in \Pi_{n-k}.
\] (4.2)

The distinct lines \( \ell_1, \ell_2, \ldots, \ell_{n-k-1} \) do not divide the polynomial \( \sigma \in \Pi_{k} \), therefore all they have to divide \( q \in \Pi_{n-k} \). Thus \( q = \ell_1 \ldots \ell_{n-k-1} \ell' \), where \( \ell' \in \Pi_1 \). Therefore, we get from (4.2):

\[
\sigma_1 \ell = \sigma \ell'.
\] (4.3)

If the lines \( \ell, \ell' \) coincide then the curves \( \sigma_1, \sigma \) coincide, which is impossible. Therefore the line \( \ell \) has to divide \( \sigma \in \Pi_k \):

\[
\sigma_1 = \ell r, \quad r \in \Pi_{k-1}.
\]

Let us study this relation closer. We are going to derive from here that the curve \( r \) passes through all the nodes of the set \( X \) but one. Indeed, \( \sigma \) passes through all the nodes of \( \mathcal{X} \). Therefore these nodes are either in the curve \( r \) or in the line \( \ell \). But this line passes through \( B \), and according to (i), it passes through at most one node of \( \mathcal{X} \). Thus \( r \) passes through at least \( d(n, k-1) \) nodes of \( \mathcal{X} \) and therefore it is a maximal curve of degree \( k-1 \). On the other hand, according to Proposition 3.2, the curve \( r \) of degree \( k-1 \) can pass through at most \( d(n, k-1) \) independent nodes. Thus, we conclude that \( r \) passes through exactly \( d(n, k-1) \) nodes of \( \mathcal{X} \).

\section{5 Two corollaries}

As it was mentioned earlier, our main result – Theorem 4.2 yields the uniqueness result: Theorem 4.1, which states that the minimal number of \( n \)-independent points determining uniquely a curve of degree \( k, k \leq n-1 \) equals to \( d(n, k-1)+2 \) (see [11], Theorem 2.1):

\textbf{Corollary 5.1.} Given a set of \( n \)-independent nodes \( \mathcal{X}, \# \mathcal{X} = d(n, k-1)+2 \). Then there can be at most one curve of degree \( k \) which passes through all its nodes.

\textbf{Proof.} Choose a node \( A \in \mathcal{X} \) and consider the set \( \mathcal{Y} := \mathcal{X} \setminus \{A\} \). If there is at most one curve of degree \( k \) which passes through all nodes of \( \mathcal{Y} \) then we are done. Next suppose, that there are at least two curves of degree \( k \) which pass through all nodes of the set \( \mathcal{Y} \). Then, according to Theorem 4.2, there is a node \( B \in \mathcal{Y} \) and a maximal curve \( \mu_{k-1} \) of degree \( k-1 \) which passes through all the nodes of \( \mathcal{Y} \setminus \{B\} \). Moreover, all the nodes of \( \mathcal{X} \) but \( A \) and \( B \) are located in
the curve $\mu_{k-1}$. Now, in view of Proposition 3.4, any curve of degree $k$ passing through all the nodes of $\mathcal{X}$ has the following form

$$p = \ell\mu_{k-1},$$

where $\ell \in \Pi_1$. Finally notice that the line $\ell$ passes through $A$ and $B$ and therefore is determined in a unique way. Hence $p$ is determined uniquely.

**Corollary 5.2.** Let $\mathcal{X}$ be an $n$-poised set of nodes and $\ell$ be a used line which passes through exactly 3 nodes. Then it is used either by exactly one or by exactly three nodes from $\mathcal{X}$. Moreover, if it is used by exactly three nodes, then they are noncollinear.

**Proof.** Assume that $\ell \cap \mathcal{X} = \{A, B, C\}$. Assume also that there are two nodes $P, Q \in \mathcal{X}$ using the line $\ell$:

$$p_P^* = \ell q_1, \quad p_Q^* = \ell q_2,$$

where $q_1, q_2 \in \Pi_{n-1}$.

Both the polynomials $q_1, q_2$ vanish at $N - 5$ nodes of the set $\mathcal{Y} := \mathcal{X} \setminus \{A, B, C, P, Q\}$. Hence these $N - 5 = d(n, n - 2) + 1$ nodes do not uniquely determine curve of degree $n - 1$ passing through them. By the Proposition 4.2 there exists a maximal curve $\mu_{n-2}$ of degree $n - 2$ passing through $N - 6$ nodes of $\mathcal{Y}$ and the remaining node denoted by $R$ is outside. Now, according to Proposition 3.4 $\mu_{n-2}$ divides the fundamental polynomial of the node $R$:

$$p_R^* = \mu_{n-2} q,$$

where $q \in \Pi_2$. This quadratic polynomial $q$ has to vanish at the three nodes $A, B, C \in \ell$. Hence $q = \ell\ell'$ with $\ell' \in \Pi_1$. Therefore, in view of Proposition 3.4 with $n = 2$, the node $R$ uses the line $\ell$:

$$p_R^* = \mu_{n-2} \ell\ell' \quad \ell' \in \Pi_1.$$

Hence if two nodes $P, Q \in \mathcal{X}$ use the line $\ell$ then there exists a third node $R \in \mathcal{X}$ using it and all the nodes $\mathcal{Y} := \mathcal{X} \setminus \{A, B, C, P, Q, R\}$ are located in a maximal curve $\mu_{n-2}$ of degree $n - 2$:

$$\mathcal{Y} \subset \mu_{n-2}.$$  

Next, let us show that there is no fourth node using $\ell$. We will prove this by the way of contradiction. Assume that, except of the nodes $P, Q, R$, there is a fourth node $S$ that uses $\ell$. Of course we have that $S \in \mathcal{Y}$.

Then $P$ and $S$ are using $\ell$ therefore, as was proved above, there exists a third node $T \in \mathcal{X}$ (which may coincide or not with $Q$ or $R$) using it and all the nodes $\mathcal{Y} := \mathcal{X} \setminus \{A, B, C, P, Q, S, T\}$ are located in a maximal curve $\tilde{\mu}_{n-2}$ of degree $n - 2$. We have also that

$$p_S^* = \tilde{\mu}_{n-2} \ell\ell'' \quad \ell'' \in \Pi_1.$$
Now, notice that both $\mu_{n-2}$ and $\tilde{\mu}_{n-2}$ pass through all the nodes of the set

$$Z := X \setminus \{A, B, C, P, Q, R, S, T\} \text{ with } \#Z \geq N - 8.$$  

Now, according to the Corollary [5.1] with $k = n - 2$, $N - 8 = d(n, n - 3) + 2$ nodes determine the curve of degree $n - 2$ passing through them uniquely. Thus $\mu_{n-2}$ and $\tilde{\mu}_{n-2}$ coincide.

Therefore, in view of (5.3) and (5.4), $p_S^*$ vanishes at all the nodes of $Y$, which is a contradiction since $S \in \mathcal{Y}$. \hfill \Box

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