On Buchsbaum bundles on quadric hypersurfaces

E. Ballico  F. Malaspina  P. Valabrega  M. Valenzano

Abstract

Let \( E \) be an indecomposable rank two vector bundle on the projective space \( \mathbb{P}^n \), \( n \geq 3 \), over an algebraically closed field of characteristic zero. It is well known that \( E \) is arithmetically Buchsbaum if and only if \( n = 3 \) and \( E \) is a null-correlation bundle. In the present paper we establish an analogous result for rank two indecomposable arithmetically Buchsbaum vector bundles on the smooth quadric hypersurface \( Q_n \subset \mathbb{P}^{n+1} \), \( n \geq 3 \). We give in fact a full classification and prove that \( n \) must be at most 5. As to \( k \)-Buchsbaum rank two vector bundles on \( Q_3 \), \( k \geq 2 \), we prove two boundedness results.

Keywords: arithmetically Buchsbaum rank two vector bundles, smooth quadric hypersurfaces.

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1 Introduction

Many papers have been written on \( k \)-Buchsbaum indecomposable rank two vector bundles on projective \( n \)-spaces, \( n \geq 3 \), (see for instance [3], [4], [5], [6], [12]). We recall that a vector bundle is called arithmetically Cohen-Macaulay (i.e. 0-Buchsbaum) if it has no intermediate cohomology, while it is called arithmetically Buchsbaum (i.e. 1-Buchsbaum) if it has all the intermediate cohomology modules with trivial structure. Moreover, we say that a bundle is properly arithmetically Buchsbaum if it is so, but it is not arithmetically Cohen-Macaulay. The following result about rank two vector bundles on a projective space is well-known (see [3] and [5]):

**Theorem.** Let \( E \) be an arithmetically Buchsbaum, normalized, rank 2 vector bundle on the projective space \( \mathbb{P}^n \), \( n \geq 3 \). Then \( E \) is one of the following:

1. \( n \geq 3 \): \( E \) is a split bundle;
2. \( n = 3 \): \( E \) is stable with \( c_1 = 0 \), \( c_2 = 1 \), i.e. \( E \) is a null-correlation bundle.

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Therefore, on projective spaces the only properly arithmetically Buchsbaum rank two vector bundles are the null-correlation bundles on $\mathbb{P}^3$.

In the present paper we investigate arithmetically Buchsbaum rank two vector bundles on a smooth quadric hypersurface $Q_n \subset \mathbb{P}^{n+1}$ and can give a full classification, getting the following:

**Theorem 1.** Let $E$ be an arithmetically Buchsbaum, normalized, rank 2 vector bundle on a smooth quadric hypersurface $Q_n$, $n \geq 3$. Then $E$ is one of the following:

i) arithmetically Cohen-Macaulay bundles:
- 1. $n \geq 3$: $E$ is a split bundle;
- 2. $n = 3$: $E$ is stable with $c_1 = -1$, $c_2 = 1$, i.e. $E$ is a spinor bundle;
- 3. $n = 4$: $E$ is stable with $c_1 = -1$, $c_2 = (1,0)$ or $(0,1)$, i.e. $E$ is a spinor bundle;

ii) properly arithmetically Buchsbaum bundles:
- 4. $n = 3$: $E$ is stable with $c_1 = -1$, $c_2 = 2$, i.e. $E$ is associated to two skew lines or to a double line;
- 5. $n = 3$: $E$ is stable with $c_1 = -1$, $c_2 = 3$, and $H^0(Q_3, E(1)) = 0$, i.e. $E$ is associated to a smooth elliptic curve of degree 7 in $Q_3 \subset \mathbb{P}^4$;
- 6. $n = 4$: $E$ is stable with $c_1 = -1$, $c_2 = (1,1)$, i.e. $E$ is the restriction of a Cayley bundle to $Q_4$;
- 7. $n = 5$: $E$ is stable with $c_1 = -1$, $c_2 = 2$, i.e. $E$ is a Cayley bundle;
- 8. $n \geq 6$: no properly arithmetically Buchsbaum bundle exists.

As for $k$-Buchsbaum bundles, $k \geq 2$, on a quadric threefold $Q_3$, we prove two boundedness results for the second Chern class $c_2$, both in the stable and in the non-stable case.

## 2 General facts and notation

We work over an algebraically closed field $k$ of characteristic zero.

### 2.1 Quadric hypersurfaces $Q_n \subset \mathbb{P}^{n+1}$

We denote by $Q_n$ any smooth quadric hypersurface in the projective space $\mathbb{P}^{n+1}$, with $n \geq 2$. We recall some well known facts about quadric hypersurfaces. Firstly, for the first and second Chow groups of $Q_n$ there are natural isomorphisms

$$\text{CH}^i(Q_n) \cong \mathbb{Z} \quad i = 1, 2$$
with the following exceptions:
\[
\text{CH}^1(Q_2) \cong \mathbb{Z} \oplus \mathbb{Z} \quad \text{and} \quad \text{CH}^2(Q_3) \cong \mathbb{Z} \oplus \mathbb{Z},
\]
so an element of these particular Chow groups is identified with a pair \((a, b)\) of integers.

Now, let \(Q_3 \subset \mathbb{P}^4\) be a smooth quadric threefold. Take a general hyperplane section \(Q_2\) and a general conic section \(D\) of \(Q_3\), i.e.
\[
Q_2 = Q_3 \cap K, \quad D = Q_3 \cap K \cap K' = Q_3 \cap K',
\]
where \(K\) and \(K'\) are two general hyperplanes in \(\mathbb{P}^4\). We have the following two inclusion maps \(D \hookrightarrow Q_2 \hookrightarrow Q_3\), so it holds, for the “hyperplane section” line bundles,
\[
i^* \mathcal{O}_{Q_3}(1) = \mathcal{O}_{Q_2}(1,1)
\]
and
\[
j^* \mathcal{O}_{Q_2}(1,1) = \mathcal{O}_D(1) \cong \mathcal{O}_{\mathbb{P}^1}(2),
\]
since the irreducible conic \(D\) is the isomorphic image through the 2-fold Veronese embedding of the projective line \(\mathbb{P}^1\). The above notation \(\mathcal{O}_{Q_2}(1,1)\) has the usual meaning
\[
\mathcal{O}_{Q_2}(1,1) \cong \pi_1^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(1),
\]
where \(\pi_1\) and \(\pi_2\) are the projections onto the two factors in the standard isomorphism \(Q_2 \cong \mathbb{P}^1 \times \mathbb{P}^1\), and moreover, for every integer \(t\),
\[
\mathcal{O}_{Q_2}(t,t) = \mathcal{O}_{Q_2}(1,1)^{\otimes t} \cong \pi_1^* \mathcal{O}_{\mathbb{P}^1}(t) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(t).
\]

### 2.2 Rank 2 vector bundles on \(Q_n\)

Let \(E\) be a rank 2 vector bundle on a smooth quadric hypersurface \(Q_n \subset \mathbb{P}^{n+1}\) with \(n \geq 3\).

**Definition 2.1.** The bundle \(E\) is called **normalized** if it has first Chern class \(c_1 \in \{0, -1\}\). We define the **first relevant level** of \(E\) as the integer
\[
\alpha = \alpha(E) := \min\{t \in \mathbb{Z} \mid h^0(Q_n, E(t)) \neq 0\}.
\]

Let \(H\) be a general hyperplane section of \(Q_n\). We set \(E|_H := E \otimes \mathcal{O}_H\), that is \(E|_H\) is the restriction of the bundle \(E\) to the smooth subvariety \(H \cong Q_{n-1}\). So we define the first relevant level of the restricted bundle \(E|_H\) as the whole number
\[
a = a(E) := \alpha(E|_H) = \min\{t \in \mathbb{Z} \mid h^0(Q_{n-1}, E|_H(t)) \neq 0\},
\]
with the convention that, when \(n = 3\), \(a\) is the first relevant level of \(E|_H\) with respect to the line bundle \(\mathcal{O}_{Q_2}(1,1)\).

It is easy to see that for every vector bundle \(E\) it holds: \(a \leq \alpha\).
Definition 2.2. Let $E$ be a rank 2 vector bundle on $Q_n$ with first Chern class $c_1$ and first relevant level $\alpha$. We say that $E$ is stable if $2\alpha + c_1 > 0$, or equivalently, if $\alpha > 0$ when $E$ is normalized. We say that $E$ is semistable if $2\alpha + c_1 \geq 0$, or equivalently, if $\alpha \geq -c_1$ when $E$ is normalized. Obviously every stable bundle is semistable. Conversely the only semistable bundles which are not stable are those with $c_1 = \alpha = 0$.

We say that $E$ is non-stable if $2\alpha + c_1 \leq 0$, that is if $\alpha \leq 0$ when $E$ is normalized.

Definition 2.3. We say that $E$ is an extendable bundle or that it extends to a bundle on $Q_{n+1}$ if there exists a rank 2 vector bundle $F$ on $Q_{n+1}$ such that $E = F|_{Q_n}$, where $Q_n \subset Q_{n+1}$ is a general hyperplane section of $Q_{n+1}$.

Definition 2.4. We say that $E$ is a split bundle if it is (isomorphic to) the direct sum of two line bundles, i.e. $E = O_{Q_n}(a) \oplus O_{Q_n}(b)$ for suitable integers $a$ and $b$. Obviously each split bundle is non-stable.

Definition 2.5. Let $E$ be rank 2 vector bundle on a smooth quadric $Q_n \subset \mathbb{P}^{n+1}$, with $n \geq 3$. We set

$$R = \bigoplus_{t \geq 0} H^0(Q_n, \mathcal{O}_{Q_n}(t)) \quad \text{and} \quad m = \bigoplus_{t \geq 0} H^0(Q_n, \mathcal{O}_{Q_n}(t)),$$

and also

$$H^i_s(Q_n, E) = \bigoplus_{t \in \mathbb{Z}} H^i(Q_n, E(t)) \quad \text{for } i = 0, \ldots, n,$$

which are modules of finite length on the ring $R$.

We say that $E$ is $k$-Buchsbaum, with $k \geq 0$, if for all integers $p, q$ such that $1 \leq p \leq q - 1$ and $3 \leq q \leq n$ it holds

$$m^k \cdot H^p(Q', E|_{Q'}) = 0,$$

where $Q'$ is a general $q$-dimensional linear section of $Q_n$, i.e. $Q'$ is a quadric hypersurface cut out on $Q_n$ by a general linear space $L \subset \mathbb{P}^{n+1}$ of dimension $q + 1$, that is $Q' = Q_n \cap L$.

Obviously $E$ is $k$-Buchsbaum if and only if $E(t)$ is $k$-Buchsbaum for every $t \in \mathbb{Z}$. Moreover, if $E$ is $k$-Buchsbaum, then $E$ is $k'$-Buchsbaum for all $k' \geq k$. So we say that $E$ is properly $k$-Buchsbaum if it is $k$-Buchsbaum but not $(k-1)$-Buchsbaum.

Notice that $E$ is 0-Buchsbaum if and only if $E$ has no intermediate cohomology, i.e. $H^i(Q_n, E(t)) = 0$ for every $t \in \mathbb{Z}$ and $1 \leq i \leq n - 1$. Such a bundle is also called arithmetically Cohen-Macaulay.

Observe also that $E$ is 1-Buchsbaum if and only if $E$ has every intermediate cohomology module with trivial structure over $R$. Such a bundle is also called arithmetically Buchsbaum.

Now we recall some known facts.

Theorem 2.6 (Castelnuovo-Mumford criterion). Let $F$ be a coherent sheaf on $Q_n$ such that $H^i(Q_n, F(-i)) = 0$ for $i > 0$. Then $F$ is generated by global sections and $H^i(Q_n, F(-i+j)) = 0$ for $i > 0, j \geq 0$. 

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Proof. See [8]. □

**Theorem 2.7.** Let $F$ be a vector bundle on $Q_n$, $n \geq 3$. If $F$ is arithmetically Cohen-Macaulay, i.e. it has no intermediate cohomology, then $F$ is a direct sum of line bundles and twisted spinor bundles.

Proof. See [10]. □

**Remark 2.8.** For an account of spinor bundles on quadrics see [13] and [16].

**Definition 2.9.** As introduced in [14], a Cayley bundle $C$ on a smooth quadric hypersurface $Q_5$ is a bundle arising from the following two exact sequences

$$0 \to \mathcal{O}_{Q_n} \to S^* \to G \to 0$$

$$0 \to \mathcal{O}_{Q_n} \to G^*(1) \to C(1) \to 0$$

where $S$ is the spinor bundle on $Q_5$ and $G$ is a stable rank 3 vector bundle with Chern classes $c_1 = c_2 = c_3 = 2$ defined by the generic section of $S^*$, and moreover $C$ is defined by a nowhere vanishing section of $G^*(1)$.

**Theorem 2.10.** Each stable rank 2 vector bundle on $Q_5$ with Chern classes $c_1 = -1$ and $c_2 = 1$ is a Cayley bundle, and these bundles do not extend to $Q_6$. Moreover, the zero locus of a general global section of $C(2)$, $C$ a Cayley bundle, is (isomorphic to) the complete flag threefold $F(0, 1, 2)$ of linear elements of $\mathbb{P}^2$.

Proof. See [14]. □

**Theorem 2.11.** Each stable rank 2 vector bundle $F$ on $Q_4$ with Chern classes $c_1 = -1$ and $c_2 = (1, 1)$ extends in a unique way to a Cayley bundle on $Q_5$. Moreover, such a bundle has a global section of $F(1)$ vanishing on two disjoint 2-planes.

Proof. See [14]. □

### 2.3 Rank 2 vector bundles on $Q_3$

Let $E$ be a rank 2 vector bundle on a smooth quadric threefold $Q_3$. As usual we will use the notation $E(t)$ for the twisted bundle $E \otimes \mathcal{O}_{Q_3}(t)$, for all $t \in \mathbb{Z}$. We identify the Chern classes $c_1$ and $c_2$ of $E$ with integers, and we recall the well known following formulas:

$$c_1(E(t)) = c_1 + 2t$$

$$c_2(E(t)) = c_2 + 2c_1 t + 2t^2.$$  

Moreover, the Hilbert polynomial of the vector bundle $E$ is

$$\chi(E(t)) = \begin{cases} 
(2t + 3)(2t^2 + 6t + 4 - 3c_2)/6 & \text{if } c_1 = 0 \\
(t + 1)(2t^2 + 4t + 3 - 3c_2)/3 & \text{if } c_1 = -1
\end{cases} \quad (1)$$

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(see e.g. [17], Proposition 9 and Corollary 2). In particular we have
\[ \chi(E(-1)) = -\frac{c_2}{2}, \quad \chi(E) = 2 - \frac{3}{2}c_2, \quad \chi(E(1)) = 10 - \frac{5}{2}c_2, \]
when \( c_1 = 0 \), while
\[ \chi(E(-1)) = 0, \quad \chi(E) = 1 - c_2, \quad \chi(E(1)) = 6 - 2c_2. \]
when \( c_1 = -1 \).
If \( E \) is a rank 2 vector bundle on \( Q_3 \), then we define
\[ E|_H := E \otimes \mathcal{O}_H \]
the restriction of the bundle \( E \) to a general hyperplane section \( H \cong Q_2 \) (zero locus of a general global section of \( \mathcal{O}_{Q_3}(1) \)). Moreover we will use the following convention:
\[ E|_H(t) := E|_H \otimes \mathcal{O}_{Q_2}(t, t) \quad \forall t \in \mathbb{Z}. \]
Furthermore, we define the restriction \( E|_D \) of the bundle \( E \), and hence of the bundle \( E|_H \), to a general conic section \( D \) of the threefold \( Q_3 \) as:
\[ E|_D := E \otimes \mathcal{O}_D = E|_H \otimes \mathcal{O}_D. \]
So we have, for each integer \( t \), the following exact sequences
\[ 0 \to E(t-1) \to E(t) \to E|_H(t) \to 0 \] (2)
and
\[ 0 \to E|_H(t-1) \to E|_H(t) \to E|_D(t) \to 0, \] (3)
the so called restriction sequences.
Recall that, by our convention, sequence (2) is in fact
\[ 0 \to E(t-1) \to E(t) \to E|_H \otimes \mathcal{O}_{Q_2}(t, t) \to 0, \]
while sequence (3) is in fact
\[ 0 \to E|_H \otimes \mathcal{O}_{Q_2}(t-1, t-1) \to E|_H \otimes \mathcal{O}_{Q_2}(t, t) \to E|_D(t) \to 0. \]

Let \( E \) be a normalized rank 2 vector bundle on \( Q_3 \) which is \( k \)-Buchsbaum, with \( k \geq 1 \). Let \( H \) be a general hyperplane section of \( Q_3 \) defined by the section \( x \in H^0(Q_3, \mathcal{O}_{Q_3}(1)) \). We denote by \( \phi_t := \phi_{x,t} \) the multiplication map induced by the restriction sequence (2) on the first cohomology groups, i.e.
\[ \phi_t : H^1(Q_3, E(t-1)) \to H^1(Q_3, E(t)) \]
is the multiplication by \( x \). By the long cohomology sequence we have
\[ \ker(\phi_t) \cong \text{coker}(H^0(Q_3, E(t)) \to H^0(Q_2, E|_H(t))). \]
so we have
\[ \dim \ker(\phi_t) \leq h^0(Q_2, E|_H(t)) \quad \forall \, t \in \mathbb{Z}. \tag{4} \]
Taking the composition of \( k \) successive such maps we obtain
\[ \ker(\phi_t \circ \cdots \circ \phi_{t-k+1}) \subseteq H^1(Q_3, E(t-k)); \]
so, if \( E \) is \( k \)-Buchsbaum, we get for each integer \( t \) the following equality
\[ H^1(Q_3, E(t-k)) = \ker(\phi_t \circ \cdots \circ \phi_{t-k+1}) \tag{5} \]
since \( \phi_t \circ \cdots \circ \phi_{t-k+1} \equiv 0 \).

Now we recall some known results on rank 2 vector bundles on a quadric threefold \( Q_3 \).

**Theorem 2.12** (Splitting Criterion). Let \( E \) be a rank 2 vector bundle on a smooth quadric threefold \( Q_3 \) with first Chern class \( c_1 \) and first relevant level \( \alpha \). If \( E \) is arithmetically Cohen-Macaulay, then \( E \) splits, unless \( 0 < c_1 + 2\alpha < 2 \).

*Proof.* See [11], Theorem 2. \( \square \)

**Corollary 2.13.** The only indecomposable, arithmetically Cohen-Macaulay, normalized, rank 2 vector bundles on a smooth quadric threefold \( Q_3 \) are stable with \( c_1 = -1, c_2 = 1, \) and \( \alpha = 1 \), i.e. they are the spinor bundles.

*Proof.* By the above splitting criterion, if \( E \) is an indecomposable, arithmetically Cohen-Macaulay, normalized, rank 2 vector bundle on \( Q_3 \), then we must have \( c_1 + 2\alpha = 1 \), so the only possibility is \( c_1 = -1 \) and \( \alpha = 1 \), hence \( E \) is stable. Moreover, we have \( 1 - c_2 = \chi(E) = 0 \), that is \( c_2 = 1 \). Therefore \( E \) is a spinor bundle on \( Q_3 \) (see [13]). \( \square \)

**Theorem 2.14** (Restriction Theorem). Let \( E \) be a rank 2 vector bundle on a smooth quadric threefold \( Q_3 \). If \( E \) is stable, then \( E|_H \), the restriction of \( E \) to a general hyperplane section \( H \) of \( Q_3 \), is again stable.

*Proof.* See [4], Theorem 1.6. \( \square \)

**Theorem 2.15**. If \( E \) be a semistable rank 2 vector bundle on a smooth quadric threefold \( Q_3 \) with \( c_1 = 0 \) (respectively \( c_1 = -1 \)), then \( E|_D \), the restriction of \( E \) to a general conic section \( D \) of \( Q_3 \), is isomorphic to \( O_{P^1} \oplus O_{P^1} \) (respectively \( O_{P^1}(-1) \oplus O_{P^1}(-1) \)).

*Proof.* See [4], Corollary 1.5. \( \square \)

### 3 Preliminary results

In the sequel, when we say that a vector bundle is \( k \)-Buchsbaum we always assume that \( k \geq 1 \), hence excluding arithmetically Cohen-Macaulay bundles.
3.1 Stable bundles

**Lemma 3.1.** Let $E$ be a semistable, normalized, rank 2 vector bundle on a smooth quadric threefold $Q_3$ and let $D$ be a general conic section of $Q_3$. Then for every integer $t \geq 0$:

$$h^0(D, E|_D(t)) = 4t + 2c_1 + 2 \quad \text{and} \quad h^1(D, E|_D(t)) = 0.$$

**Proof.** By Theorem 2.15 for every integer $t \geq 0$ for all $t$ smooth quadric threefold $Q_3$. Then

$$h^0(D, E|_D(t)) = 2h^0(P^1, \mathcal{O}_{P^1}(2t + c_1)) = 4t + 2c_1 + 2 \quad \forall t \geq 0,$$

and also $h^0(D, E|_D(t)) = 0$ for each $t < 0$. The vanishing of the first cohomology follows by Serre duality.

**Lemma 3.2.** Let $E$ be a stable, normalized, rank 2 vector bundle on a smooth quadric threefold $Q_3$. Then $c_2 > 0$.

**Proof.** By the stability hypothesis it holds $0 < a \leq \alpha$, where $\alpha = \alpha(E)$ and $a = \lambda(E|_H)$, with $H$ a general hyperplane section of $Q_3$. Therefore $h^0(Q_3, E) = 0$ and $h^3(Q_3, E) = h^0(Q_3, E(-3 - c_1)) = 0$, hence

$$\chi(E) = h^2(Q_3, E) - h^1(Q_3, E) = h^1(Q_3, E(-3 - c_1)) - h^1(Q_3, E) \leq 0,$$

since the multiplication map $\phi_t$ is injective for each $t \leq a - 1$, where $a - 1 \geq 0$.

We have $\chi(E) = 2 - \frac{3}{2}c_2$ if $c_1 = 0$ and $\chi(E) = 1 - c_2$ if $c_1 = -1$, so we get $c_2 \geq 2$ when $c_1 = 0$, and $c_2 \geq 1$ when $c_1 = -1$.

**Remark 3.3.** Observe that the above Lemma is a consequence of Bogomolov’s inequality (see e.g. [10], Theorem 7.3.1), but here we have given a proof based on elementary techniques.

**Lemma 3.4.** Let $E$ be a stable, $k$-Buchsbaum, normalized, rank 2 vector bundle on $Q_3$. Then $h^1(Q_3, E(-1)) \neq 0$ if $c_1 = 0$, and $h^1(Q_3, E) \neq 0$ if $c_1 = -1$.

**Proof.** By Lemma 3.2 it holds $c_2 > 0$, and this implies $\vartheta > 0$, where $\vartheta = \frac{3}{2}c_2 + \frac{1}{2}$ if $c_1 = 0$ and $\vartheta = \frac{3}{2}c_2 - \frac{1}{2}$ if $c_1 = -1$. Moreover $-3/2 + \sqrt{\vartheta} > -1$ for each $c_2 \geq 2$, $c_2$ even, when $c_1 = 0$, and also $-1 + \sqrt{\vartheta} > 0$ for each $c_2 \geq 2$ when $c_1 = -1$. So by [2], Theorem 5.4, we get the claim.

**Lemma 3.5.** Let $E$ be a stable, $k$-Buchsbaum, normalized, rank 2 vector bundle on $Q_3$. Let $H$ be a general hyperplane section of $Q_3$, and let $a$ be the first relevant level of $E|_H$. Then $h^1(Q_3, E(t)) = 0$ for each $t \leq a - k - 1$. 

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Proof. By the stability hypothesis we have $0 < a \leq \alpha$, where $\alpha$ is the first relevant level of $E$. Then we have $h^0(Q_2, E|_H(t)) = 0$ for all $t \leq a - 1$, hence the multiplication map $\phi_t$ is injective for each $t \leq a - 1$. Therefore, the composition of $k$ successive multiplication maps $\phi_t \circ \cdots \circ \phi_{t-k+1}$ is, at the same time, injective and the zero map for each $t \leq a - 1$, so we obtain (see \cite{5}) $h^1(Q_3, E(t)) = 0$ for all $t \leq a - k - 1$. 

**Lemma 3.6.** Let $E$ be a stable, k-Buchsbaum, normalized, rank 2 vector bundle on $Q_3$. Let $H$ be a general hyperplane section of $Q_3$, and let $a$ be the first relevant level of $E|_H$. Then

\[
a \leq \begin{cases} 
  k - 1 & \text{if } c_1 = 0 \\
  k & \text{if } c_1 = -1
\end{cases}
\]

**Proof.** It is enough to use Lemma 3.4 and Lemma 3.5. 

### 3.2 Non-stable bundles

**Lemma 3.7.** Let $E$ be a normalized rank 2 vector bundle on $Q_3$ and $E|_H$ its restriction to a general hyperplane section $H$ of $Q_3$. If $E$ is non-split and non-stable, then

1. $\alpha(E|_H) = \alpha(E) = \alpha$,
2. $h^0(Q_3, E(\alpha)) = h^0(Q_2, E|_H(\alpha)) = 1$,
3. $h^0(Q_3, E(t)) = h^0(Q_3, \mathcal{O}_{Q_3}(t - \alpha))$ for all $t \leq -\alpha - c_1$,
4. $h^0(Q_2, E|_H(t)) = h^0(Q_2, \mathcal{O}_{Q_2}(t - \alpha))$ for all $t \leq -\alpha - c_1$,
5. for each $t \leq -\alpha - c_1$ the following sequence of cohomology groups

\[
0 \to H^0(Q_3, E(t - 1)) \to H^0(Q_3, E(t)) \to H^0(Q_2, E|_H(t)) \to 0
\]

is exact,

6. the multiplication map $\phi_t: H^1(Q_3, E(t - 1)) \to H^1(Q_3, E(t))$ by a general linear form $x \in H^0(Q_3, \mathcal{O}_{Q_3}(1))$ is injective for each $t \leq -\alpha - c_1$.

**Proof.** By the definition of $\alpha$ and the assumption that $E$ does not split, $E(\alpha)$ has a global section whose zero locus $Z$ has codimension 2 in $Q_3$, hence, by the Serre correspondence, we have for all $t \in Z$ the short exact sequence

\[
0 \to \mathcal{O}_{Q_3}(t - \alpha) \to E(t) \to \mathcal{I}_Z(t + \alpha + c_1) \to 0
\]

which in cohomology gives

\[
0 \to H^0(Q_3, \mathcal{O}_{Q_3}(t - \alpha)) \to H^0(Q_3, E(t)) \to H^0(Q_3, \mathcal{I}_Z(t + \alpha + c_1)) \to 0
\]

and since $t + \alpha + c_1 \leq 0$ for every $t \leq -\alpha - c_1$ it follows that $H^0(Q_3, E(t)) \cong H^0(Q_3, \mathcal{O}_{Q_3}(t - \alpha))$ for every $t \leq -\alpha - c_1$, in particular we get $h^0(Q_3, E(\alpha)) =$
we obtain in cohomology for all $t$ cohomology sequence. hence we have proved (5). Moreover, (6) follows from (5) through the long sequence obtained by tensoring the previous one with $O_H$

$$0 \to O_{Q_3}(t - \alpha) \to E|_H(t) \to I_{Z \cap H}(t + \alpha + c_1) \to 0$$

from which we get

$$0 \to H^0(Q_2, O_{Q_2}(t - \alpha)) \to H^0(Q_2, E|_H(t)) \to H^0(Q_2, I_{Z \cap H}(t + \alpha + c_1)) \to 0$$

so we have $H^0(Q_2, E|_H(t)) \cong H^0(Q_2, O_{Q_2}(t - \alpha))$ for every $t \leq -\alpha - c_1$ (this is (4)), in particular $h^0(Q_2, E|_H(\alpha)) = h^0(Q_2, O_H) = 1$. Moreover from the exact sequence

$$0 \to O_{Q_3}(-1) \to O_{Q_3} \to O_{Q_2} \to 0$$

we obtain in cohomology for all $t \in \mathbb{Z}$

$$0 \to H^0(Q_3, O_{Q_3}(t - 1)) \to H^0(Q_3, O_{Q_3}(t)) \to H^0(Q_2, O_{Q_2}(t)) \to 0,$$

hence we have proved (5). Moreover, (6) follows from (5) through the long cohomology sequence.

**Lemma 3.8.** Let $E$ be a non-stable, $k$-Buchsbaum, normalized, rank 2 vector bundle on $Q_3$ with first relevant level $\alpha$. Then $h^1(Q_3, E(t)) = 0$ for all $t \leq -\alpha - c_1 - k$.

**Proof.** By Lemma 3.8(6) the multiplication map $\phi_t$ is injective for each $t \leq -\alpha - c_1$. Therefore, the composition $\phi_t \circ \cdots \circ \phi_{t-k+1}$ of $k$ successive multiplication maps is injective and also the zero map for each $t \leq -\alpha - c_1$, so we get the claim.

**Proposition 3.9.** There is no normalized rank 2 vector bundle $E$ on $Q_3$ which is properly $k$-Buchsbaum with $\alpha \leq 1 - k$.

**Proof.** Assume $E$ properly $k$-Buchsbaum with $\alpha \leq 1 - k$ and $c_1 \in \{0, -1\}$. Since $k \geq 1$ we get $\alpha \leq 0$, so $E$ is non-stable. By Lemma 3.8 we have $h^1(Q_3, E(t)) = 0$ for all $t \leq -\alpha - c_1 - k$. On the other hand, the hypothesis $\alpha \leq 1 - k$ implies $-\frac{\alpha + c_1}{2} < -\alpha - c_1 - k \leq -\alpha - c_1 - 1$, so by [2], Theorem 4.3, we must have $h^1(Q_3, E(-\alpha - c_1 - k)) \neq 0$, which is in contradiction with the above vanishing.

**Corollary 3.10.** Let $E$ be a non-stable, normalized, rank 2 vector bundle on $Q_3$. If $E$ is properly $k$-Buchsbaum, then $2 - k \leq \alpha \leq 0$.

**Corollary 3.11.** Every properly arithmetically Buchsbaum rank 2 vector bundle on $Q_3$ is stable.

**Proposition 3.12.** Let $E$ be a non-stable, normalized, rank 2 vector bundle on a smooth quadric threefold $Q_3$ with first relevant level $\alpha$. Then $E|_D$, the restriction of $E$ to a general conic section $D$ of $Q_3$, is isomorphic to $O_D(-\alpha) \oplus O_D(\alpha + c_1)$.
Proof. Let $Z$ be the zero locus of a non-zero global section of $E(\alpha)$, which is a subscheme of $Q_3$ of pure dimension 1, then we have the exact sequence

$$0 \to \mathcal{O}_{Q_3} \to E(\alpha) \to \mathcal{I}_Z(2\alpha + c_1) \to 0.$$  

If we tensor the above sequence by $\mathcal{O}_D(-\alpha)$, where $D$ is a general conic section of $Q_3$, hence not meeting $Z$, we get

$$0 \to \mathcal{O}_D(-\alpha) \to E|_D \to \mathcal{O}_D(\alpha + c_1) \to 0.$$  

By hypothesis $2\alpha + c_1 \leq 0$, therefore we have

$$\text{Ext}^1(\mathcal{O}_D(\alpha + c_1), \mathcal{O}_D(-\alpha)) \cong \text{Ext}^1(\mathcal{O}_D, \mathcal{O}_D(-2\alpha - c_1))$$

$$\cong H^1(D, \mathcal{O}_D(-2\alpha - c_1))$$

$$\cong H^0(D, \mathcal{O}_D(2\alpha + c_1 - 1)) = 0$$

so the above sequence splits, that is $E|_D \cong \mathcal{O}_D(-\alpha) \oplus \mathcal{O}_D(\alpha + c_1)$.

Lemma 3.13. Let $E$ be a non-stable, normalized, rank 2 vector bundle on a smooth quadric threefold $Q_3$ with first relevant level $\alpha$ and let $D$ be a general conic section of $Q_3$. Then for every integer $t \geq 0$:

$$h^0(D, E|_D(t)) = \begin{cases} 0 & \text{for } t \leq \alpha - 1 \\ 2t - 2\alpha + 1 & \text{for } \alpha \leq t \leq -\alpha - c_1 - 1 \\ 4t + 2c_1 + 2 & \text{for } t \geq -\alpha - c_1 \end{cases}$$

Proof. By Proposition 3.12 we have $E|_D \cong \mathcal{O}_D(-\alpha) \oplus \mathcal{O}_D(\alpha + c_1)$, so, taking into account that $\mathcal{O}_D(t) \cong \mathcal{O}_{\mathbb{P}^1}(2t)$ for all $t \in \mathbb{Z}$, with a simple computation we get the claim.

4 Arithmetically Buchsbaum rank 2 bundles

In the present section we give the proof of Theorem 4 but dividing it into two statements: Theorem 4.2 and Theorem 4.3.

First, we analyze what happens on a smooth quadric threefold $Q_3 \subset \mathbb{P}^4$, starting with a technical lemma which we need in the following.

Lemma 4.1. Let $E$ be an arithmetically Buchsbaum rank 2 vector bundle on $Q_3$. If $h^2(Q_3, E(t - 1)) = 0$, then $H^1(Q_3, E(t)) \cong H^1(Q_2, E_H(t))$ through the restriction map and moreover the multiplication map

$$H^1(Q_2, E|_H(t)) \to H^1(Q_2, E|_H(t + 1))$$

is the zero map.
Proof. Consider the following commutative diagram

\[
\begin{array}{ccc}
H^1(Q_3, E(t)) & \xrightarrow{\alpha} & H^1(Q_2, E_H(t)) \\
\downarrow & & \downarrow \\
H^1(Q_3, E(t+1)) & \rightarrow & H^1(Q_2, E_H(t+1))
\end{array}
\]

where the horizontal maps are the restriction maps obtained from restriction sequence (2), while the vertical maps are the multiplication maps by a general linear form not defining the hyperplane section \( H \cong Q_2 \), so by the hypotheses it follows that the left vertical map is the zero map.

Now we are able to give the full classification of arithmetically Buchsbaum rank 2 vector bundles on a quadric threefold \( Q_3 \).

**Theorem 4.2.** Let \( E \) be an arithmetically Buchsbaum, normalized, rank 2 vector bundle on \( Q_3 \). Then \( E \) is one of the following:

1. \( E \) is a split bundle;
2. \( E \) is stable with \( c_1 = -1, c_2 = 1 \), i.e. \( E \) is a spinor bundle;
3. \( E \) is stable with \( c_1 = -1, c_2 = 2 \), i.e. \( E \) is associated to two skew lines or to a double line;
4. \( E \) is stable with \( c_1 = -1, c_2 = 3 \), and \( H^0(Q_3, E(1)) = 0 \), i.e. \( E \) is associated to a smooth elliptic curve of degree 7 in \( Q_3 \subset P^4 \).

Therefore \( E \) is either arithmetically Cohen-Macaulay, cases (1) and (2), or properly arithmetically Buchsbaum, cases (3) and (4), with only one non-zero first cohomology group.

Proof. Let \( E \) be an arithmetically Buchsbaum, normalized, rank 2 vector bundle on \( Q_3 \) and let \( H \) and \( D \) be general hyperplane and conic sections of \( Q_3 \). Since the rank 2 arithmetically Cohen-Macaulay vector bundles on \( Q_3 \) are either split or spinor bundles (see Theorem 2.7), we can assume that \( E \) is properly arithmetically Buchsbaum. We set \( \alpha = \alpha(E) \) and \( a = \alpha(E_H) \). By Corollary 3.11 \( E \) must be stable, and moreover, by Lemma 3.9 and Theorem 2.14 there is no stable properly arithmetically Buchsbaum bundle with \( c_1 = 0 \). Therefore we may assume, always by Lemma 3.6 that \( E \) is stable with \( c_1 = -1 \) and \( a = 1 \). We have \( h^1(Q_3, E(t)) = 0 \) for all \( t \leq -1 \), by Lemma 3.5 and \( h^2(Q_3, E(t)) = 0 \) for all \( t \geq -1 \), by Serre duality. Moreover, by Lemma 3.4 \( h^1(Q_3, E) \neq 0 \). Thanks to the vanishing of 2-cohomology for \( t \geq -1 \), the multiplication map \( H^1(Q_2, E_H(t)) \rightarrow H^1(Q_2, E_H(t+1)) \), by Lemma 4.1 is the zero map for every \( t \geq 0 \). From the restriction sequence (2) we get in cohomology the exact sequence

\[
0 \rightarrow H^1(Q_2, E_H(t)) \rightarrow H^1(D, E_D(t)) = 0
\]
for all \( t \geq 1 \) (taking into account Lemma [3, 1]), so we obtain \( h^1(Q_2, E|_H(t)) = 0 \) for all \( t \geq 1 \). Now from restriction sequence [2] we get in cohomology the exact sequence

\[
0 \to H^1(Q_3, E(t)) \to H^1(Q_2, E|_H(t)) = 0
\]

for every \( t \geq 1 \), hence \( h^1(Q_3, E(t)) = 0 \) for all \( t \geq 1 \). Therefore \( E \) must have \( h^1(Q_3, E(t)) = 0 \) for all \( t \neq 0 \).

Since \( \alpha \geq a = 1 \), we have either \( \alpha = 1 \) or \( \alpha > 1 \); in this last event we have \( h^1(Q_3, E(\alpha - 2)) \neq 0 \), because of [2], Theorem 5.2, and therefore we must have \( \alpha = 2 \). So we have to analyze two possibilities: \( \alpha = 1 \) and \( \alpha = 2 \). In both cases it holds \( 6 - 2c_2 = \chi(E(1)) = h^0(Q_3, E(1)) \geq 0 \), that is \( c_2 \leq 3 \), therefore, being \( E \) properly arithmetically Buchsbaum, we obtain \( c_2 = 2 \) if and only if \( \alpha = 1 \), and \( c_2 = 3 \) if and only if \( \alpha = 2 \).

If \( c_2 = 2 \) and \( \alpha = 1 \), then we get \( h^1(Q_3, E) = 1 \) and \( h^0(Q_3, E(1)) = 2 \). By the Serre correspondence we have an exact sequence on \( Q_3 \) like the following

\[
0 \to \mathcal{O} \to E(1) \to \mathcal{I}_Z(1) \to 0
\]

where \( Z \) is a non-empty locally complete intersection curve of degree 2 with \( \omega_Z \cong \mathcal{O}_Z(-2) \), arithmetic genus \(-1\), and \( h^1(\mathbb{P}^4, \mathcal{I}_{Z, \mathbb{P}^4}(t)) = h^1(Q_3, \mathcal{I}_Z(t)) = 0 \) for all \( t \neq 0 \) and \( h^1(\mathbb{P}^4, \mathcal{I}_Z, \mathbb{P}^4) = h^1(Q_3, \mathcal{I}_Z) = 1 \), so \( E \) is a vector bundle associated to two skew lines or a double line.

If \( c_2 = 3 \) and \( \alpha = 2 \), then we get \( h^1(Q_3, E) = 2 \) and \( h^0(Q_3, E(1)) = 0 \). By the Serre correspondence \( E(2) \) fits into an extension of the following type

\[
0 \to \mathcal{O} \to E(2) \to \mathcal{I}_C(3) \to 0
\]

where \( C \) is a non-empty locally complete intersection curve of degree 7 and arithmetic genus \( 1 \), with \( \omega_C \cong \mathcal{O}_C \) and \( h^1(\mathbb{P}^4, \mathcal{I}_{C, \mathbb{P}^4}(t)) = h^1(Q_3, \mathcal{I}_C(t)) = 0 \) for all \( t \neq 1 \) and \( h^1(\mathbb{P}^4, \mathcal{I}_{C, \mathbb{P}^4}(1)) = h^1(Q_3, \mathcal{I}_C(1)) = 2 \), moreover \( E(2) \) is generated by global sections by Castelnuovo-Mumford criterion (see Theorem [2, 0]), so \( \mathcal{I}_C(3) \) is globally generated too, hence the zero locus of a general section of \( E(2) \) is smooth. Therefore \( E \) is a vector bundle associated to a smooth elliptic curve of degree 7 in \( Q_3 \subset \mathbb{P}^4 \).

Now we can state and prove the classification of arithmetically Buchsbaum rank 2 vector bundles on a quadric hypersurface \( Q_n \subset \mathbb{P}^{n+1} \), with \( n \geq 4 \).

**Theorem 4.3.** The only indecomposable, arithmetically Buchsbaum, normalized, rank 2 vector bundle \( F \) on \( Q_n \), \( n \geq 4 \), are the following:

1. for \( n = 5 \), \( F \) is a Cayley bundle, i.e. \( F \) is a bundle with \( c_1 = -1 \), \( c_2 = 2 \);

2. for \( n = 4 \), \( F \) is a spinor bundle or it has \( c_1 = -1 \), \( c_2 = (1, 1) \), i.e. \( F \) is the restriction of a Cayley bundle to \( Q_4 \).

Moreover, for \( n \geq 6 \) no such bundle exists.
**Proof.** Let $F$ be a rank 2 vector bundle on $Q_n$ as in the statement. Take a general 3-dimensional linear section $Q_3$ of $Q_n$ and set $E := F|_{Q_3}$. By the hypothesis $E$ is an indecomposable, arithmetically Buchsbaum, normalized, rank 2 vector bundle on the quadric threefold $Q_3$, so the only possibilities for $E$ are listed in Theorem 4.2. We analyze each case separately.

(a) If $E$ is a spinor bundle on $Q_3$, then $E$ extends to a spinor bundle on $Q_4$, i.e. a stable bundle with $c_1 = -1$ and $c_2 = (1, 0)$ or $c_2 = (0, 1)$, but does not extend to any further bundle on $Q_n$ for $n \geq 5$ (see [13], Theorem 2.1).

(b) If $E$ is stable with $c_1 = -1$, $c_2 = 2$, then $E$ extends to a vector bundle on $Q_4$ with Chern classes $c_1 = -1$, $c_2 = (1, 1)$, and even to one on $Q_5$ with $c_1 = -1$, $c_2 = 2$ (a Cayley bundle), but to no further bundle on any $Q_n$, $n \geq 6$ (see Definition 2.9 and Theorem 2.10 and 2.11).

(c) If $E$ is stable with $c_1 = -1$, $c_2 = 3$, and $a = 2$, then $E$ does not extend to any arithmetically Buchsbaum bundle $F$ on $Q_4$. In fact, assume there exists an extension $F$ of the bundle $E$ on $Q_4$, i.e. $E = F|_{Q_3}$. By the proof of Theorem 4.2 we know that $h^0(Q_3, E(2)) \neq 0$, while $h^1(Q_3, E(t)) = 0$ for all $t \neq 0$ and $h^1(Q_3, E) = 2$. By the assumption, the two vector bundle $F$ and $E$ fit into a restriction sequence like

$$0 \rightarrow F(-1) \rightarrow F \rightarrow E \rightarrow 0,$$

so we get in cohomology the exact sequence

$$0 \rightarrow H^1(Q_4, F(t - 1)) \rightarrow H^1(Q_4, F(t)) \rightarrow 0$$

for all $t \leq -1$ and $t \geq 1$, since $h^0(Q_4, F(t)) = 0$ for all $t \leq 1$. It follows that $h^1(Q_4, F(t)) = h^3(Q_4, F(t)) = 0$ for all $t \in \mathbb{Z}$. Similarly, we get $h^2(Q_4, F(t)) = 0$ for all $t \leq -3$ and $t \geq 0$, and also $h^2(Q_4, F(-1)) = h^2(Q_4, F(-2)) = 2$. Therefore we have $h^1(Q_4, F(1)) = h^2(Q_4, F) = h^3(Q_4, F(-1)) = 0$ and $h^4(Q_4, F(-2)) = h^0(Q_4, F(-1)) = 0$, so by the Castelnuovo-Mumford criterion (see Theorem 2.10) $F$ is 2-regular, hence $F(2)$ is generated by global sections and therefore the zero locus of a general global section of $F(2)$ is a smooth surface $S$ of degree 7 and we have the exact sequence

$$0 \rightarrow \mathcal{O}_{Q_4} \rightarrow F(2) \rightarrow \mathcal{I}_S(3) \rightarrow 0.$$

Since $\det(F(2)) \simeq \mathcal{O}_{Q_4}(3)$ and $\omega_{Q_4} \simeq \mathcal{O}_{Q_4}(-4)$, the adjunction formula gives $\omega_S \simeq \mathcal{O}_{S}(-1)$. Thus $S$ is an anticanonically embedded del Pezzo surface. By the above exact sequence we see that $h^1(\mathbb{P}^4, \mathcal{I}_S(t)) = h^1(Q_4, F(1)) = 0$ for all $t \in \mathbb{Z}$. Using Riemann-Roch on the surface $S$ we obtain $h^0(S, \mathcal{O}_S(1)) = \chi(\mathcal{O}_S(1)) = (\mathcal{O}_S(1) \cdot \mathcal{O}_S(2))/2 + \chi(\mathcal{O}_S) = 7 + 1 = 8$. On the other hand, using the structure sequence $0 \rightarrow \mathcal{I}_S \rightarrow \mathcal{O}_{Q_4} \rightarrow \mathcal{O}_S \rightarrow 0$, we have $h^0(S, \mathcal{O}_S(1)) \leq h^0(\mathbb{P}^4, \mathcal{O}_{Q_4}(1)) = 6$, which is absurd. Therefore, there exists no arithmetically Buchsbaum bundle $F$ on $Q_n$, $n \geq 4$, such that its restriction to a general 3-dimensional linear section $Q_3$ is a stable bundle with $c_1 = -1$, $c_2 = 3$, and $a = 2$. \[\Box\]

**Remark 4.4.** Notice that in Theorem 4.3 the hypothesis that $F$ is an arithmetically Buchsbaum bundle can be weakened to the following one: we can ask
that $F$ is a bundle with 1-Buchsbaum first cohomology, meaning that for every integer $q$ such that $3 \leq q \leq n$ it holds

$$m \cdot H^1_*(Q', F|_{Q'}) = 0,$$

where $Q'$ is a general $q$-dimensional linear section of $Q_n$.

Obviously, by Serre duality, for a rank 2 vector bundle $E$ on a quadric threefold $Q_3$ the two conditions:

a) “$E$ is arithmetically Buchsbaum”

b) “$E$ has 1-Buchsbaum first cohomology”

are equivalent. Instead, on an $n$-dimensional quadric $Q_n$, $n \geq 4$, condition a) implies condition b), but a priori the converse is not true.

5 Boundedness for $c_2$ of $k$-Buchsbaum bundles

In this section we investigate the 0-cohomology of the restriction to a general hyperplane section $E|_H$ of a rank 2 vector bundle $E$ on a quadric threefold $Q_3 \subset P^4$, in order to establish some bounds on the second Chern class $c_2$ of a $k$-Buchsbaum bundle on $Q_3$, both in the stable and in the non-stable case.

We start with the stable case.

Lemma 5.1. Let $E$ be a stable, normalized, rank 2 vector bundle on $Q_3$. Let $H$ be a general hyperplane section of $Q_3$, and let $a$ be the first relevant level of $E|_H$. Then for each $t \geq 0$

$$h^0(Q_2, E|_H(t)) \leq 2t(t + 2 + c_1),$$

Moreover, for $t \geq a$ we have the better bound

$$h^0(Q_2, E|_H(t)) \leq 2t(t + 2 + c_1) - 2(a - 1)(a + 1 + c_1).$$

Proof. Let $D$ be a general conic section of $Q_3$. Then we have the exact sequence

$$0 \to E|_H(i - 1) \to E|_H(i) \to E|_D(i) \to 0$$

for each $i \in \mathbb{Z}$ (where $E|_H(i)$ means $E|_H \otimes \mathcal{O}_{Q_2}(i, i)$ as above), so in cohomology we get the exact sequence

$$0 \to H^0(Q_2, E|_H(i - 1)) \to H^0(Q_2, E|_H(i)) \to H^0(D, E|_D(i))$$

and therefore

$$h^0(Q_2, E|_H(i)) - h^0(Q_2, E|_H(i - 1)) \leq h^0(D, E|_D(i))$$

for every $i \in \mathbb{Z}$. Fix an integer $t \geq a$. Then we obtain, by Lemma 3.1

$$h^0(Q_2, E|_H(t)) = \sum_{i=a}^{t} \left( h^0(Q_2, E|_H(i)) - h^0(Q_2, E|_H(i - 1)) \right) \leq \sum_{i=a}^{t} h^0(D, E|_D(i)) = \sum_{i=a}^{t} (4i + 2c_1 + 2) = 2t(t + 2 + c_1) - 2(a - 1)(a + 1 + c_1).$$
Taking into account that \( a \geq 1 \) and \( h^0(Q_2, E|_H(t)) = 0 \) for \( t < a \), we get the claim. \( \square \)

**Remark 5.2.** Lemma 5.1 holds more generally for any \( O_{Q_2}(1,1) \)-stable, normalized, rank 2 vector bundle \( F \) on \( Q_2 \) with first relevant level, with respect to \( O_{Q_2}(1,1), a = \alpha(F) \).

**Proposition 5.3.** Fix integers \( k \geq 1 \) and \( c_1 \in \{0,-1\} \). Let \( S(k,c_1) \) be the family of all stable, properly \( k \)-Buchsbaum, rank 2 vector bundles \( E \) on \( Q_3 \) with first Chern class \( c_1 \). Set \( C_s(k,c_1) := \{ c_2(E) \in \mathbb{Z} \mid E \in S(k,c_1) \} \). Then \( C_s(k,c_1) \) is finite and \( S(k,c_1) \) is bounded. In particular

\[
C_s(k,0) = \{ c_2 \in 2\mathbb{Z} \mid 2 \leq c_2 \leq 2k(k-1)(2k+5)/3 \},
\]

\[
C_s(k,-1) = \{ c_2 \in \mathbb{Z} \mid 1 \leq c_2 \leq k(k+1)(2k+4)/3 + 1 \}.
\]

**Proof.** Take \( E \in S(k,c_1) \), then \( E \) is a stable, properly \( k \)-Buchsbaum, rank 2 vector bundle on \( Q_3 \) with first Chern class \( c_1 \). The stability of \( E \) implies that \( c_2 = c_2(E) > 0 \) (see Lemma 5.2).

Let \( H \cong Q_2 \) be a general hyperplane section of \( Q_3 \), then \( E|_H \) is stable (with respect to the line bundle \( O_{Q_2}(1,1) \)) by Theorem 2.14.

First assume \( c_1 = 0 \) and set

\[
\eta = \eta(E) := \sum_{i=0}^{k-1} h^0(Q_2, E|_H(i)).
\]

For each \( i \in \mathbb{Z} \) we have by (4)

\[
\dim \ker(\phi_i) \leq h^0(Q_2, E|_H(i)),
\]

where \( \phi_i : H^1(Q_3, E(i-1)) \rightarrow H^1(Q_3, E(i)) \) is the multiplication map by an element \( x \in H^0(Q_3, O_{Q_3}(1)) \) defining the hyperplane section \( H \). By hypothesis \( E \) is a \( k \)-Buchsbaum bundle, so

\[
h^1(Q_3, E(-1)) = \dim \ker(\phi_{k-1} \circ \cdots \circ \phi_0) \leq \eta.
\]

Now

\[
-\frac{c_2}{2} = \chi(E(-1)) = -h^1(Q_3, E(-1)) + h^2(Q_3, E(-1)) \geq -h^1(Q_3, E(-1))
\]

since \( h^0(Q_3, E(-1)) = 0 \) and \( h^3(Q_3, E(-1)) = h^0(Q_3, E(-2)) = 0 \), being \( \alpha > 0 \), so we obtain

\[
c_2 \leq 2h^1(Q_3, E(-1)) \leq 2\eta.
\]

Moreover, by Lemma 5.1 we have

\[
\eta = \sum_{i=0}^{k-1} h^0(Q_2, E|_H(i)) \leq \sum_{i=0}^{k-1} 2i(i+2) = \frac{1}{3}k(k-1)(2k+5).
\]
Therefore we obtain
\[ 2 \leq c_2 \leq 2k(k - 1)(2k + 5)/3. \]

Now assume \( c_1 = -1 \) and set
\[ \eta = \eta(E) := \sum_{i=1}^{k} h^0(Q_2, E|_H(i)). \]

In this case we have, under the hypothesis that \( E \) is \( k \)-Buchsbaum,
\[ h^1(Q_3, E) = \dim \ker(\phi_k \circ \cdots \circ \phi_1) \leq \eta. \]

It holds
\[ 1 - c_2 = \chi(E) = -h^1(Q_3, E) + h^2(Q_3, E) \geq -h^1(Q_3, E) \]

since \( h^0(Q_3, E) = 0 \) and \( h^3(Q_3, E) = h^0(Q_3, E(-2)) = 0 \), being \( \alpha > 0 \), so we obtain
\[ c_2 \leq h^1(Q_3, E) + 1 \leq \eta + 1. \]

Moreover, by Lemma 5.1 we have
\[ \eta = \sum_{i=1}^{k} h^0(Q_2, E|_H(i)) \leq \sum_{i=1}^{k} 2i(i + 1) = \frac{1}{3}k(k + 1)(2k + 4). \]

Therefore we obtain
\[ 1 \leq c_2 \leq k(k + 1)(2k + 4)/3 + 1. \]

Finally, the boundedness of \( S(k, c_1) \) follows from the finiteness of \( C_s(k, c_1) \) and the boundedness of the family of all stable vector bundles with fixed rank and Chern classes (see [10], Theorem 3.3.7).

**Remark 5.4.** Notice that the above result can be improved if we take into account the following facts: the multiplication maps \( \phi_i \) are injective for all \( i \leq a - 1 \), and also, by Lemma 3.6 it holds \( a \leq k - 1 \) if \( c_1 = 0 \) and \( a \leq k \) if \( c_1 = -1 \), so we can set
\[ \eta = \sum_{i=a}^{k-1} h^0(Q_2, E|_H(i)) \] if \( c_1 = 0 \), and
\[ \eta = \sum_{i=a}^{k} h^0(Q_2, E|_H(i)) \] if \( c_1 = -1 \),

hence using the bound for \( h^0(Q_2, E|_H(i)) \) depending on \( a \) of Lemma 5.1 we obtain the following better upper bounds:
\[ c_2 \leq \begin{cases} 
2(2k + 4a + 1)(k - a)(k - a + 1)/3 & \text{if } c_1 = 0, \\
2(k + 2a)(k - a + 1)(k - a + 2)/3 + 1 & \text{if } c_1 = -1,
\end{cases} \]

that are dependent on \( k \) and on \( a = \alpha(E|_H) \) (which is not so easy to compute).
Remark 5.5. Applying Proposition 5.3 in the case of 1-Buchsbaum rank 2 vector bundles on $Q_3$ we obtain:

$$C_s(1, 0) = \emptyset \quad \text{and} \quad C_s(1, -1) = \{c_2 \in \mathbb{Z} \mid 1 \leq c_2 \leq 5\},$$

so we find again the fact that there exists no stable properly 1-Buchsbaum bundle with first Chern class $c_1 = 0$.

Moreover, in the case of 2-Buchsbaum bundles, we obtain:

$$C_s(2, 0) = \{c_2 \in 2\mathbb{Z} \mid 2 \leq c_2 \leq 12\} \quad \text{and} \quad C_s(2, -1) = \{c_2 \in \mathbb{Z} \mid 1 \leq c_2 \leq 17\}.$$

Observe that, thanks to Theorem 4.2, we know that it holds $C_s(1, -1) = \{1, 2, 3\}$, therefore the above upper bound on $c_2$ is not sharp.

Now we consider the non-stable case.

Remark 5.6. Observe, that, by Corollary 3.11 there exist no rank 2 vector bundle on $Q_3$ which is non-stable and properly 1-Buchsbaum.

Lemma 5.7. Let $E$ be a non-stable, normalized, rank 2 vector bundle on $Q_3$ with first relevant level $\alpha$ and let $H$ be a general hyperplane section of $Q_3$. Then for each $t \geq -\alpha - c_1 + 1$

$$h^0(Q_2, E|_H(t)) \leq 2t(t + 2 + c_1) + 2\alpha^2 + 2c_1\alpha + 1 + c_1.$$

Proof. Let $D$ be a general conic section of $Q_3$. Then we have, see the proof of Lemma 5.1

$$h^0(Q_2, E|_H(i)) - h^0(Q_2, E|_H(i - 1)) \leq h^0(D, E|_D(i))$$

for every $i \in \mathbb{Z}$. Fix an integer $t \geq -\alpha - c_1 + 1$. Then, using Lemma 3.13 we obtain

$$h^0(Q_2, E|_H(t)) \leq h^0(Q_2, E|_H(-\alpha - c_1)) + \sum_{i=-\alpha-c_1+1}^{t} h^0(D, E|_D(i)) =$$

$$= (-2\alpha - c_1 + 1)^2 + \sum_{i=-\alpha-c_1+1}^{t} (4i + 2c_1 + 2) =$$

$$= 2t(t + 2 + c_1) + 2\alpha^2 + 2c_1\alpha + 1 + c_1. \quad \square$$

Proposition 5.8. Fix integers $k \geq 2$ and $c_1 \in \{0, -1\}$. Let $\mathcal{N}(k, c_1)$ be the family of all non-stable, properly $k$-Buchsbaum, rank 2 vector bundles $E$ on $Q_3$ with first Chern class $c_1$. Set $C_{ns}(k, c_1) := \{c_2(E) \in \mathbb{Z} \mid E \in \mathcal{N}(k, c_1)\}$. Then $C_{ns}(k, c_1)$ is finite and $\mathcal{N}(k, c_1)$ is bounded. In particular

$$C_{ns}(k, 0) = \{c_2 \in 2\mathbb{Z} \mid -2(k - 2)^2 < c_2 \leq 2(k - 1)(k + 1)(2k + 3)/3\},$$

$$C_{ns}(k, -1) = \{c_2 \in \mathbb{Z} \mid -2(k - 1)(k - 2) < c_2 \leq 2(k - 1)(k^2 + 4k + 6)/3 + 1\}.$$
Proof. Let $\mathcal{N}_\alpha(k, c_1)$ be the subset of $\mathcal{N}(k, c_1)$ containing all non-stable, properly $k$-Buchsbaum, rank 2 vector bundles $E$ with $c_1(E) = c_1$ and $\alpha(E) = \alpha$. The minimal curve corresponding to any non-zero global section of $E(\alpha)$ has degree $\delta = c_2 + 2c_1\alpha + 2\alpha^2 > 0$, so we have $c_2 > -2c_1\alpha - 2\alpha^2$, which is a lower bound for $c_2$ of any bundle $E \in \mathcal{N}_\alpha(k, c_1)$. Let $H$ be a general hyperplane section of $Q_3$, then $\alpha(E|_H) = \alpha$ by Lemma 5.7(1). First assume $c_1 = 0$. With the same reasoning as in the proof of Proposition 5.3 we have

$$h^1(Q_3, E(-1)) = \dim \ker(\phi_{k-1} \circ \cdots \circ \phi_0) \leq \eta,$$

where

$$\eta := \sum_{i=\alpha+1}^{k-1} h^0(Q_2, E|_H(i)),$$

since the multiplication maps $\phi_t$ are injective for every $t \leq -\alpha$ (see Lemma 3.7(6)), and also $-\alpha + 1 \leq k - 1$ by Corollary 3.10. Moreover

$$-\frac{c_2}{2} = \chi(E(-1)) = h^0(Q_3, E(-1)) - h^1(Q_3, E(-1)) + h^2(Q_3, E(-1)) - h^3(Q_3, E(-1)) \geq h^1(Q_3, E(-1))$$

since $h^0(Q_3, E(-1)) - h^3(Q_3, E(-1)) = h^0(Q_3, E(-1)) - h^0(Q_3, E(-2)) \geq 0$, so we obtain

$$c_2 \leq 2h^1(Q_3, E(-1)) \leq 2\eta.$$

By Lemma 5.7 we have

$$\eta = \sum_{i=-\alpha+1}^{k-1} h^0(Q_2, E|_H(i)) \leq \sum_{i=-\alpha+1}^{k-1} \left(2i(i+2)+2\alpha^2+1\right) = \frac{1}{3}k(k-1)(2k+5) + \frac{1}{3}\alpha(\alpha-1)(2\alpha-7) + (k-1+\alpha)(2\alpha^2+1) = \frac{1}{3}(k-1+\alpha)\left[(k+1)(2k+3) + \alpha(8\alpha-7-2k)\right].$$

Therefore we obtain

$$-2\alpha^2 < c_2(E) \leq 2(k-1+\alpha)\left[(k+1)(2k+3) + \alpha(8\alpha-7-2k)\right]/3$$

for each bundle $E \in \mathcal{N}_\alpha(k, 0)$. Now, by Corollary 3.10 it holds

$$\mathcal{N}(k, 0) = \bigcup_{2-k \leq \alpha \leq 0} \mathcal{N}_\alpha(k, 0),$$

and also

$$\frac{1}{3}k(k-1)(2k+5) + \frac{1}{3}\alpha(\alpha-1)(2\alpha-7) + (k-1+\alpha)(2\alpha^2+1) \leq \frac{1}{3}k(k-1)(2k+5) + (k-1)(2(k-2)^2+1) = \frac{1}{3}(k-1)(8k^2-19k+27)$$

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since
\[ \frac{1}{3} \alpha (\alpha - 1)(2\alpha - 7) + \alpha (2\alpha^2 + 1) \leq 0 \quad \text{and} \quad \alpha^2 \leq (k - 2)^2 \]
for \(2 - k \leq \alpha \leq 0\); so we can say that every vector bundle in \(\mathcal{N}(k, 0)\) has second Chern class \(c_2\) such that:
\[ c_2 \in 2\mathbb{Z} \quad \text{and} \quad -2(k - 2)^2 < c_2 \leq 2(k - 1)(8k^2 - 19k + 27)/3. \]

Now assume \(c_1 = -1\). In this case we have
\[ h^1(Q_3, E) = \dim \ker(\phi_k \circ \cdots \circ \phi_1) \leq \eta, \]
where
\[ \eta := \sum_{i=-\alpha+2}^{k} h^0(Q_2, E|_H(i)), \]

since the multiplication maps \(\phi_t\) are injective for every \(t \leq -\alpha + 1\), and also \(-\alpha + 2 \leq k\) by Corollary 3.10. Moreover
\[ 1 - c_2 = \chi(E) = h^0(Q_3, E) - h^1(Q_3, E) + h^2(Q_3, E) - h^3(Q_3, E) \geq -h^1(Q_3, E) \]
since \(h^0(Q_3, E) - h^3(Q_3, E) = h^0(Q_3, E) - h^0(Q_3, E(-2)) \geq 0\), so we obtain
\[ c_2 \leq h^1(Q_3, E) + 1 \leq \eta + 1. \]

By Lemma 5.7 we have
\[ \eta = \sum_{i=-\alpha+2}^{k} h^0(Q_2, E|_H(i)) \leq \sum_{i=-\alpha+2}^{k} \left( 2i(i + 1) + 2\alpha^2 - 2\alpha \right) = \]
\[ = \frac{2}{3} k(k+1)(k+2) + \frac{2}{3} (\alpha - 1)(\alpha - 2)(\alpha - 3) + (k-1+\alpha)(2\alpha^2 - 2\alpha) \]
\[ = \frac{2}{3} (k-1+\alpha) \left[ k^2 + 4k + 6 + \alpha(4\alpha - 8 - k) \right]. \]
Therefore we obtain
\[ 2\alpha - 2\alpha^2 < c_2(E) \leq 2(k - 1 + \alpha) \left[ k^2 + 4k + 6 + \alpha(4\alpha - 8 - k) \right] / 3 + 1 \]
for each bundle \(E \in \mathcal{N}_\alpha(k, -1)\). Moreover, we have
\[ \frac{2}{3} k(k+1)(k+2) + \frac{2}{3} (\alpha - 1)(\alpha - 2)(\alpha - 3) + (k-1+\alpha)(2\alpha^2 - 2\alpha) < \]
\[ < \frac{2}{3} k(k+1)(k+2) + 2(k-1)(k-2) = \]
\[ = \frac{2}{3} (4k^2 - 9k + 17) - 4 \]

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since
\[ \frac{2}{3}(\alpha - 1)(\alpha - 2)(\alpha - 3) + \alpha(2\alpha^2 - 2\alpha) < 0 \quad \text{and} \quad \alpha^2 - 2\alpha \leq 2(k - 1)(k - 2) \]
for \(2 - k \leq \alpha \leq 0\); so we can say that every vector bundle in \(\mathcal{N}(k, -1)\) has second Chern class \(c_2\) such that:
\[ -2(k - 1)(k - 2) < c_2 \leq \frac{2}{3}k(4k^2 - 9k + 17) - 4. \]

Finally, let \(\mathcal{F}(c_1, c_2, \alpha)\) be the family of all non-stable, rank 2 vector bundles \(E\) on \(Q_3\) with \(c_1(E) = c_1, c_2(E) = c_2\), and \(\alpha(E) = \alpha\). By Corollary 3.10 it holds
\[ \mathcal{N}(k, c_1) \subseteq \bigcup_{2 - k \leq \alpha \leq 0} \mathcal{F}(c_1, c_2, \alpha). \]

The boundedness of each family \(\mathcal{F}(c_1, c_2, \alpha)\) follows from [10], Theorem 1.7.8 (in the Kleiman criterion choose all the constants equal to two). Therefore, the finiteness of \(C_{ns}(k, c_1)\) implies the boundedness of \(\mathcal{N}(k, c_1)\). \(\square\)

**References**

[1] E. Arrondo, I. Sols, *Classification of smooth congruence of low degree*, J. Reine Angew. Math. **393** (1989), 199–219

[2] E. Ballico, P. Valabrega, M. Valenzano, *Non-vanishing Theorems for rank 2 vector bundles on threefolds*, arXiv:1005.2080v1 [math.AG], to appear on Rend. Istit. Mat. Univ. Trieste

[3] M. Chang, *Characterization of arithmetically Buchsbaum subschemes of codimension 2 in \(\mathbb{P}^n\)*, J. Differential Geom. **31** (1990), 323–341

[4] L. Ein, I. Sols, *Stable vector bundles on quadric hypersurfaces*, Nagoya Math. J. **96** (1984), 11–22

[5] Ph. Ellia, M. Fiorentini, *Quelques remarques sur les courbes arithmétique Buchsbaum de l’espace projectif*, Ann. Univ. Ferrara Sez. VII Sc. Mat. **33** (1987), 89–111

[6] Ph. Ellia, A. Sarti, *On codimension two \(k\)-Buchsbaum subvarieties of \(\mathbb{P}^n\)*, in: Commutative Algebra and Algebraic Geometry (Ferrara), Lecture Notes in Pure and Appl. Math. 206, Dekker, New York, 1999, pp. 81–92

[7] R. Hartshorne, Algebraic Geometry, GTM 52, Springer, Berlin, 1977

[8] R. Hernandez, I. Sols, *On a family of rank-3 bundles on \(Gr(1,3)\)*, J. Reine Angew. Math. **360** (1985), 124–135
[9] G. Horrocks, *Vector bundles on the punctured spectrum of a ring*, Proc. London Math. Soc. **14** (3) (1964), 689–713

[10] D. Huybrechts, M. Lehn, *The Geometry of Moduli Spaces of Sheaves*, Aspects of Mathematics E 31, Max-Planck-Institut für Mathematik, Bonn, 1997

[11] C. Madonna, *A Splitting Criterion for rank 2 vector bundles on hypersurfaces in \( \mathbb{P}^4 \)*, Rend. Sem. Mat. Univ. Polit. Torino **56** (1998), no. 2, 43–54

[12] N. Mohan Kumar, A.P. Rao, *Buchsbaum bundles on \( \mathbb{P}^n \)*, J. Pure Appl. Algebra **152** (2000), no. 1–3, 195–199

[13] G. Ottaviani, *Spinor bundles on Quadrics*, Trans. Amer. Math. Soc. **307** (1988), no 1, 301–316

[14] G. Ottaviani, *On Cayley bundles on the five-dimensional quadric*, Bollettino U.M.I. (7) **4-A** (1990), 87–100

[15] G. Ottaviani, M. Szurek, *On moduli of stable 2-bundles with small Chern classes on \( \mathbb{Q}_3 \)*, Ann. Mat. Pura Appl. **167** (1994), 191–241

[16] I. Sols, *On spinor bundles*, J. Pure Appl. Algebra **35** (1985), 85–94

[17] M. Valenzano, *Rank 2 reflexive sheaves on a smooth threefold*, Rend. Sem. Mat. Univ. Polit. Torino **62** (2004), no. 3, 235–254

BALLICO EDOARDO
Dipartimento di Matematica, Università di Trento, 38123 Povo (TN), Italy
e-mail: ballico@science.unitn.it

MALASPINA FRANCESCO
Dipartimento di Matematica, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy
e-mail: francesco.malaspina@polito.it

VALABREGA PAOLO
Dipartimento di Matematica, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy
e-mail: paolo.valabrega@polito.it

VALENZANO MARIO
Dipartimento di Matematica, Università di Torino, via Carlo Alberto 10, 10123 Torino, Italy
e-mail: mario.valenzano@unito.it