Towards a manifestly supersymmetric action
for 11-dimensional supergravity

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Abstract: We investigate the possibility of writing a manifestly supersymmetric action for 11-dimensional supergravity. The construction involves an explicit relation between the fields in the super-vielbein and the super-3-form, and uses non-minimal pure spinors. A simple cubic interaction term for a single scalar superfield is found.
1. Introduction

Eleven-dimensional supergravity \cite{1} is important being the low-energy limit of M-theory, and hence of a strong-coupling limit of string theory. It is the highest-dimensional supersymmetric model including gravity, and gives rise to most lower-dimensional supergravities. With 32 supercharges, it has maximal supersymmetry and a traditional superspace description \cite{2} puts the theory on-shell.

It has been known for some time that pure spinor superfields provide a powerful tool for formulating supersymmetric field and string theories \cite{3,4,5,6,7,8,9,10,11,12,13,14,15,16,17}. This is especially true in models with maximal supersymmetry, where the on-shell closure of the supersymmetry acting on an ordinary superfield is turned into an advantage—the constraint on the ordinary superfield, which enforces the equations of motion, is encoded in a cohomological equation of the type $Q\Psi + \ldots = 0$, which is the equation of motion for the pure spinor superfield. Supermultiplets arise as cohomologies in pure spinor superfields.

It is striking that these cohomologies rely only on the purely algebraic (bosonic) constraint structure of the pure spinor. Not only do the physical fields arise in this way, but also the full set of ghosts and antifields. Pure spinor superfield theory inevitably leads to a Batalin–Vilkovisky (BV) formalism \cite{18,19}.

Much of the work on pure spinors in connection with supersymmetry has been done for strings, and less for supersymmetric field theories. The main difference between the treatments of string theory and field theory is that strings in principle are treated in a first-quantised manner, with interactions represented by vertex operators and the geometry of the world sheet, while theories with fundamental particle excitations are full-fledged field theories. A few maximally supersymmetric field theories have been formulated this way, including $D=10$ super-Yang–Mills (and its dimensional reductions) \cite{7,9} and the Bagger–Lambert–Gustavsson \cite{20,21,22} and Aharony–Bergman–Jafferis–Maldacena \cite{23} conformal models in $D=3$ \cite{16,17}, although none of them have been used for systematic quantum calculations.

So far, no supersymmetric field theory containing gravity has been formulated with manifest supersymmetry beyond the free level. One purpose of the present paper is to examine such formulations. An obvious drawback will be that manifest background invariance is sacrificed, since the form of the BRST operator $Q$ will encode geometric data of the background around which one chooses to expand. On the other hand, all supersymmetry is maintained; our choice is to give this highest priority.

When deciding which supergravity model to try to give a pure spinor superfield formulation, one would of course like the simplest one possible. But on the other hand, maximal supersymmetry is essential for simplicity. For a model with half-maximal or less supersymmetry, the cohomology will give an off-shell multiplet, and one will need yet more constraints.
on the fields to obtain the equations of motion. This happens for example in $D = 6$, $N = 1$ super-Yang–Mills theory, where a second BRST operator effectively sets the auxiliary fields to zero [15]. Similarly, in $D = 10$, $N = 1$ supergravity, the cohomology gives a (partially) off-shell supermultiplet [24]. Therefore we want to begin with a maximally supersymmetric model, and maybe address the question of lower supersymmetry once the maximal case is understood. The only candidates are type IIB supergravity and $D = 11$ supergravity and their dimensional reductions. Type IIB has a self-dual field strength, which complicates the formulation of an action (this is actually reflected in the cohomology, which due to the absence of certain anti-fields does not yield a natural measure [24]). There is no toy model. $D = 11$ supergravity seems to be essentially the only choice.

Pure spinor superfield formulations tend to have some remarkable properties, as an extra bonus in addition to the manifest supersymmetry. The action for $D = 10$ super-Yang–Mills is Chern–Simons-like, and has only a cubic interaction [7]. The conformal models in $D = 3$, whose component actions contain couplings of six scalar fields, simplify enormously in the pure spinor framework, where the matter superfields only have a minimal coupling to the Chern–Simons field [16,17]. Higher order interactions arise when auxiliary fields are eliminated (in both cases the fermionic component of the gauge connection on superspace). One may imagine that these simplifications may turn out to be useful in quantum calculations, where Feynman diagrams will be built with 3-vertices only.

It will be interesting to see to what extent something similar happens for supergravity. There is of course no reason to believe that the action will be polynomial, but there may be simplifications in the series of interactions that makes the theory more tractable. In this sense one may think of the supergravity action as a toy model for closed string field theory [25]. We will not have much to say about this, but plan to investigate the issue in the future.

The linearised cohomology giving $D = 11$ supergravity is known [8,11,26]. There is a fermionic scalar field $\Psi$ of dimension $-3$ and ghost number 3 whose lowest component is the third order ghost for the tensor field. The physical fields of ghost number 0 sit in the field as $\lambda^\alpha \lambda^\beta \lambda^\gamma C_{\alpha \beta \gamma}(x, \theta)$, where $\lambda^\alpha$ is the pure spinor and $C_{\alpha \beta \gamma}$ the lowest-dimensional part of the superspace 3-form $C$. As will be reviewed later, there is a natural measure on the pure spinor space, and it is straightforward to write an action $\int \Psi Q \Psi$ giving the linearised equations of motion [26]. The integrand has ghost number 7 and dimension $-6$. Clearly, since three powers of $\Psi$ already gives ghost number 9, some operators with negative ghost number have to be introduced in the interaction terms. Partial results concerning interactions have been obtained in refs. [26,27,28].

In the superfield treatment of $D = 11$ supergravity [2], the linearised fields can be obtained not only from the 3-form, but also from the super-vielbein. This is reflected in the existence of another pure spinor field $\Phi^a$, where $a$ is a vector index. This field is fermionic, has
ghost number 1 and dimension $-1$ and starts with the diffeomorphism ghost. The physical fields sit in $\Phi^a$ as $\lambda^a E^a(\tau, \theta)$, where $E^a$ is (part of) the linearised lowest-dimensional component superfield of the super-vielbein. One can note that a combination $\sim \lambda^2 \Psi \Phi^2$ has the correct dimension and ghost number to be an interaction term in the action (one can also imagine a term $\sim \lambda^2 \Phi^5$, but it will be ruled out by gauge invariance). The main result of this paper is the reduction of the question of 3-point couplings to the problem of finding the operator $R^a$ relating the two fields as $\Phi^a = R^a \Psi$ and the construction of this operator. When $R^a$ represents an operator cohomology, the interaction is non-trivial and the BV master equation is satisfied to this order in the fields.

The organisation of the paper is as follows. In Section 2, we discuss properties of pure spinors in $D = 11$. We introduce non-minimal pure spinors, construct a regularised integration measure and discuss convergence of integrals. There are some principal differences from the “standard” case of 10-dimensional pure spinors. In section 3, we review the known “3-form” and “vielbein” cohomologies in $\Psi$ and $\Phi^a$, respectively. We construct the operator relating the two fields, demanding that it carries cohomology, and investigate some other properties. Section 4 deals with the action. We show that the master equation is satisfied to the relevant order. In Section 5, we end with conclusions and some thoughts about the continuation of the project, in particular higher order interactions.

2. Pure spinors in $D = 11$

2.1. Minimal pure spinors

The anti-commutator of two fermionic derivatives in flat superspace is typically

$$\{D_\alpha, D_\beta\} = -T^c_{\alpha\beta}\partial_c = -2\gamma^c_{\alpha\beta}\partial_c \, .$$

(2.1)

Pure spinors are constrained by

$$\lambda^a \gamma^a = 0$$

(2.2)

in order for the BRST operator

$$q = (\lambda D)$$

(2.3)

to be nilpotent.

The spinors relevant for $D = 11$ supergravity have 32 components. This spinor representation is symplectic. In addition to $\varepsilon_{\alpha\beta}$, also $(\gamma^a)_{\alpha\beta}$ and $(\gamma^{abcd})_{\alpha\beta}$ are antisymmetric in $\alpha\beta$, while $(\gamma^a)_{\alpha\beta}$, $(\gamma^{ab})_{\alpha\beta}$ and $(\gamma^{abcd})_{\alpha\beta}$ are symmetric. The spinor is identical to a chiral spinor in $D = 12$, where $\varepsilon_{\alpha\beta}$ and $\sigma^{(4)}_{\alpha\beta}$ are antisymmetric, while $\sigma^{(2)}_{\alpha\beta}$ and $\sigma^{(5)}_{\alpha\beta}$ are symmetric.
Symplectic spinors, as usual, require special care with conventions to avoid sign errors. We use conventions where indices on gamma matrices are lowered by left and right multiplication with $\varepsilon_{\alpha\beta}$ and raised by its inverse. In this way the sign issues are minimised. We hide spinor indices as much as possible. Some useful relations, both for pure and unrestricted spinors, are listed in Appendix A.

We call the space of $D = 11$ pure spinors $\mathcal{P}$. The dimension of $\mathcal{P}$ is 23. This can be deduced from its decomposition in two $D = 10$ spinors of opposite chirality. If we let $\lambda = (\ell, m)$ the conditions become

$$\ell \sigma^a \ell = (m \tilde{\sigma}^a m), \quad a = 0 \ldots 9,$$

$$\ell m = 0.$$  \hspace{1cm} (2.4)

These equations are solved by $m = iv_a \sigma^a \ell$, where $v_a$ is a unit vector orthogonal to the light-like vector $(\ell \sigma^a \ell)$. Since there is an equivalence under $\delta v^a = (\ell \sigma^a \ell)$, it represents 7 degrees of freedom (a seven-sphere) in addition to the 16 in $\ell$. Clearly, the pure spinor has to be complex, which will also be natural when we consider (Euclidean) integration over pure spinor variables.

In $D = 10$, only one irreducible module remains in the product of any number of pure spinors. If the Dynkin label of the $so(10)$ module of the spinor is (00001), the module of the product of $n$ pure spinors is (0000$n$). In contrast to this, in $D = 11$ the pure spinor bilinear already contains a 2-form and a 5-form, i.e., the modules (01000) and (00002). The number of irreducible modules increases like $n^2$. The irreducible modules occurring in the product of $n$ $\lambda$’s are

$$\bigoplus_{p=0}^{n2} (0, p, 0, 0, n - 2p)).$$  \hspace{1cm} (2.5)

This content (or, rather, the absence of certain modules appearing in the $n$’th symmetric product of a spinor but not in (2.5)) completely determines the zero-mode cohomology of $q = (\lambda D)$.

We will discuss cohomology in the following section, but for the sake of defining integration, we give the table of zero-mode cohomologies (i.e., cohomology of $q$ in a field $\Psi(\lambda, \theta)$) already now. Calculation of the zero-mode cohomology is a purely algebraic problem. It can be done by hand, using the reducibility of the pure spinor constraint as in refs. [7,29,30], or by computer-aided counting of modules as in ref. [11]. We denote modules by Dynkin label, and the cohomology is given in Table 1. We will comment on the cohomology in the following section. For now, we will just use one of its components.

Since there are no singlets in the expansion (2.5) except the constant mode, any function of $\lambda$ must be expressed as a sum of positive powers. All cohomology of a scalar field comes at $\lambda^m \theta^n$ with $m \leq 7$, $n \leq 9$. The “top cohomology” at $\lambda^7 \theta^9$, constructed in the following
section, is a singlet. Picking this cohomology component from a pure spinor superfield has all the correct properties for a measure (ghost number -7, dimension 8, which makes the Lagrangian before integration $\int d^{11}x$ have ghost number 0 and dimension 2) except that it is degenerate. There is of course no non-degenerate “residue” measure if no negative powers are allowed. This is remedied by the non-minimal pure spinors.

There is a special subspace $\mathcal{P}_0$ of (complex) pure spinor space, where $(\lambda \gamma^{ab} \lambda) = 0$, so that one again gets only one irreducible module at each power. Such a “very pure” spinor is a pure spinor in $D = 12$. The dimension of this subspace is 16, which is deduced e.g. from the usual counting in even dimensions using isotropic subspaces. The real dimension of $SO(12)/SU(6)$ is $66 - 35 = 31$, which together with a radius gives 32 real, or 16 complex. When integrating functions of a pure spinor (and its complex conjugate) we need to check for convergence not only at the origin $\lambda = 0$ but also at this codimension 7 subspace. Some operators which we will find in the following section, and which will enter the action, are singular on $\mathcal{P}_0$.

2.2. Non-minimal pure spinors

Non-minimal pure spinors were introduced by Berkovits in ref. [14], with the purpose of formulating a non-degenerate measure for the pure spinors, so that cohomology can be obtained from a v action. They were further elaborated on, and explained in term of Čech and Dolbeault cohomology, by Berkovits and Nekrasov in refs. [31].

Instead of the BRST operator we wrote down in the previous section, $q = (\lambda D)$, one considers

$$Q = q + s = (\lambda D) + (r \frac{\partial}{\partial \bar{\lambda}}) .$$

(2.6)

Here, $\tilde{\lambda}$ is another pure spinor, $(\tilde{\lambda} \gamma^a \tilde{\lambda}) = 0$, and $r$ a fermionic spinor obeying $(\tilde{\lambda} \gamma^a r) = 0$, so that the set of constraints is preserved by $Q$. The last constraint means that also $r$ has 23 independent components. When performing integrals, one considers $\tilde{\lambda}$ to be the complex conjugate of $\lambda$, $\tilde{\lambda}_a = (\lambda^a)^*$. The cohomology of $Q$ is identical to that of $q$ [14], i.e., representatives in cohomology classes can be chosen as independent of $\tilde{\lambda}$ and $r$. It is convenient to assign ghost number $-1$ and dimension $\frac{1}{2}$ to $\tilde{\lambda}$, which leads to ghost number 2 and dimension $\frac{1}{2}$ for $r$.

Due to the reducibility of the modules of products of pure spinors, there are two scalar invariants formed from $\lambda$ and $\tilde{\lambda}$:

$$\xi = (\lambda \tilde{\lambda}) ,$$

$$\eta = (\lambda \gamma^{ab} \lambda)(\tilde{\lambda} \gamma_{ab} \tilde{\lambda}) .$$

(2.7)
The first one, \(\xi\), is formally identical to the one in \(D = 10\). It is positive semidefinite (from now on, we always consider \(\bar{\lambda}\) as the complex conjugate of \(\lambda\)) and vanishes only at the tip of the pure spinor cone, \(\lambda = 0\). The second invariant, \(\eta\), has no counterpart for \(D = 10\) pure spinors. It is negative semidefinite and vanishes only on the codimension 7 subspace \(\mathcal{R}_0\) of \(D = 12\) pure spinors. Fields and operators may contain negative powers of both \(\xi\) and \(\eta\). If we want to separate the behaviour at the origin and at \(\mathcal{R}_0\), it is convenient to consider the projective invariant \(\tilde{\eta} = \xi^{-2}\eta\).

2.3. Integration and singularities

The integration measure for the non-minimal pure spinors is related to the tentative residue considered in the previous subsection. Note that the existence of the singlet at \(\lambda^7\theta^9\) implies that there is (at least) one invariant tensor \(T(\alpha_1...\alpha_7)[\beta_1...\beta_9]\). The number of antisymmetric indices is the same as the number of constraints on a pure spinor (this is a generic feature). We dualise the antisymmetric indices to obtain \(\star T(\alpha_1...\alpha_7)[\beta_1...\beta_23]\). If we follow the same procedure as in \(D = 10\), we would define the integrations over the pure spinor variables as

\[
[d\lambda](\lambda^7)^{\alpha_1...\alpha_7} = \star T(\alpha_1...\alpha_7)[\beta_1...\beta_23]d\lambda^{\beta_1}...d\lambda^{\beta_{23}},
\]

\[
[d\bar{\lambda}](\lambda^7)^{\alpha_1...\alpha_7} = \star T(\alpha_1...\alpha_7)[\beta_1...\beta_{23}]d\bar{\lambda}_{\beta_1}...d\bar{\lambda}_{\beta_{23}},
\]

\[
[dr] = \bar{\lambda}_{\alpha_1}...\bar{\lambda}_{\alpha_7}\star T(\alpha_1...\alpha_7)[\beta_1...\beta_{23}]\frac{\partial}{\partial r_{\beta_1}}...\frac{\partial}{\partial r_{\beta_{23}}}.
\]

Together with full integration \([d\theta] = d^{32}\theta\), these integrations have total dimension 8 and ghost number \(-7\), as desired (these numbers are insensitive to the assignment of dimension and ghost number to \(\bar{\lambda}\)). We will use the notation \(\int [dZ]\) for integration over all coordinates, including \(x\).

These equations are not fully defined as they stand. It turns out that the singlet at \(\lambda^7\theta^9\) is not unique (although the cohomology is). The left hand sides of the first two equations must be projected to contain the same index structure as the right hand sides. This will be the index structure contained in the singlet cohomology, and we let \((\lambda^7)^{\alpha_1...\alpha_7}\) denote this projection.

We now want to find the zero-mode cohomology at \(\lambda^7\theta^9\). When the corresponding problem is addressed in \(D = 10\) one finds one singlet at \(\lambda^3\theta^5\) and none in the “surrounding” positions \(\lambda^2\theta^6\) and \(\lambda^4\theta^4\). One therefore knows that the singlet represents cohomology. In \(D = 11\) we find 3 singlets at \(\lambda^7\theta^9\), and one each in \(\lambda^6\theta^{10}\) and \(\lambda^8\theta^8\). We have to identify the cohomology among the 3 singlets. Refining the analysis, we find that \(\lambda^6\theta^{10}\) is formed through the scalar product of \(\lambda^6\) and \(\theta^{10}\) where both are projected to \((02002)\). We write this as \((\lambda^6 \circ \theta^{10})\). There is a module \((02002)\) already at \(\theta^8\), so we can simplify further to
In a similar fashion, the 3 combinations at $\lambda^7 \theta^9$ can be written $(\lambda^7 \circ_{(02003)} \theta^9)$, $(\lambda^7 \circ_{(03001)} \theta^9)$ and $(\lambda^7 \circ_{(03001)} \theta^9)$, where factors $(\theta \theta)$ have been written out when possible (there are two independent (03001)'s in $\theta^9$). The singlet at $\lambda^8 \theta^8$ is $(\lambda^8 \circ_{(04000)} \theta^8)$. It is clear that $(\lambda^7 \circ_{(02003)} \theta^9)$ is closed, since (00001) $\otimes$ (02003) does not contain (04000). In fact, it is the unique representative for the cohomology, but in order to show that we need to verify that $q \cdot (\lambda^6 \circ_{(02002)} \theta^8) (\theta \theta)$ has no component in $(\lambda^7 \circ_{(02003)} \theta^9)$. We do this by explicit calculation. Concretely, \[
(\lambda^6 \circ_{(02002)} \theta^8)(\theta \theta) = (\lambda \gamma^{ab} \lambda)(\lambda \gamma^{cd} \lambda)(\lambda \gamma^{ijklm} \lambda)(\theta \gamma_{ab p} \theta)(\theta \gamma_{cd q} \theta)(\theta \gamma_{ij}^p \theta)(\theta \gamma_{jl}^q \theta)(\theta \gamma_{kl}^r \theta)(\theta \theta) . \]

When acting with $q$ and looking only for components in $(\lambda^7 \circ_{(02003)} \theta^9)$, we can discard everything except the term where $q$ hits $(\theta \theta)$. We then also discard every expression with 3 factors of $(\lambda \gamma^{(2)} \lambda)$, which gives a contribution to the other two singlets (strictly speaking, also $(\lambda^7 \circ_{(02003)} \theta^9)$ contains such terms along with terms $(\lambda \gamma^{(2)} \lambda)^2 (\lambda \gamma^{(5)} \lambda)$ in order to be irreducible, but the latter ones cannot vanish). The relevant part reads
\[
q \cdot (\lambda^6 \circ_{(02002)} \theta^8)(\theta \theta) = \ldots + 2(\lambda \gamma^{ab} \lambda)(\lambda \gamma^{cd} \lambda)(\lambda \gamma^{ijklm} \lambda)(\theta \theta)(\theta \gamma_{jkl}^q \theta)(\theta \gamma_{cd}^r \theta)(\theta \gamma_{ij}^p \theta) . \]

Here we use a relation from Appendix A to bring out indices to the $\lambda$'s, $(\lambda \theta)(\theta \gamma^{klm} \theta) = -2(\lambda \gamma^{[kl]} \theta)(\theta \gamma^{m]r} \theta) - (\lambda \gamma^{klm} \theta)(\theta \theta)$, together with the fact that $(\gamma_m \lambda)_\alpha (\lambda \gamma^{ijklm} \lambda)$ can be thrown away when looking for the coefficient of (02003), to get
\[
\ldots + 4(\lambda \gamma^{ab} \lambda)(\lambda \gamma^{cd} \lambda)(\lambda \gamma^{ijklm} \lambda)(\gamma^{r} \theta)(\theta \gamma_{ab p} \theta)(\theta \gamma_{cd q} \theta)(\theta \gamma_{ij}^p \theta)(\theta \gamma_{kl}^r \theta) , \]

which vanishes for symmetry reasons.

So, the cohomology is uniquely represented by $(\lambda^7 \circ_{(02003)} \theta^9)$ and takes the explicit form
\[
(\lambda^7 \circ_{(02003)} \theta^9) = (\lambda \gamma^{ab} \lambda)(\lambda \gamma^{cd} \lambda)(\lambda \gamma^{ijklm} \lambda)(\gamma^n \theta)(\theta \gamma_{ab p} \theta)(\theta \gamma_{cd q} \theta)(\theta \gamma_{ij}^p \theta)(\theta \gamma_{kl}^r \theta) \quad (2.12)
\]
where $\Lambda^a_{ijklm}$ is in (00003), $\Lambda^a_{ijklm} = \lambda_a (\lambda \gamma^{ijklm} \lambda) - 2(\gamma^{ij} \lambda)\alpha (\lambda \gamma^{lm} \lambda)$. This defines the tensor $T_{a_1 \ldots a_{r} \ldots b_1 \ldots b_s}$ used in defining the integration measure.

The last thing to do is to regularise the integration. The measure alone does not work properly for at least two reasons. The bosonic integration is non-compact, and has to be regularised if the integral of the singlet cohomology at $\lambda^7 \theta^9$ is to be finite. At the same time one needs to get a non-zero result from the full $\theta$-integration, so there must be some factor
saturating the integral with $23\theta$'s. All these demands are reached by the same regularisation, which of course has to be BRST-invariant. We insert a factor $e^{\{Q,\chi\}}$, for some fermion $\chi$. This differs from 1 by a $Q$-exact expression. Therefore, the integration is independent of the choice of $\chi$, when it is well-defined. Choosing $\chi = (\bar{\lambda}\theta)$ gives $\{Q, \chi\} = -\bar{\lambda}\lambda - (\theta r)$.

At the same time as the bosonic integral becomes exponentially convergent at infinity, the final term in the expansion in $r$, $r^{23}\theta^{23}$, saturates the $\theta$ integration with the 23 missing $\theta$'s in the right tensorial structure to pick up the singlet cohomology at $\lambda^7\theta^9$.

There are possibilities of divergences both at the origin and at $\mathcal{P}_0$. Consider first the origin, and let $\rho = \sqrt{(\lambda\bar{\lambda})}$. The radial integration contains $\int d\rho \rho^{45}$. Take an integrand $\sim \lambda^p\bar{\lambda}^p$. The integral is convergent if $p > -23$. Close to $\mathcal{P}_0$, the radial coordinate $\sigma$ is given by $\sigma^2 \sim \eta^2$. The real codimension is 14, so one gets an integration $\int d\sigma \sigma^{13}$. Each factor of $(\lambda\gamma^{(2)}\lambda)$ or $(\bar{\lambda}\gamma^{(2)}\bar{\lambda})$ goes like $\sigma$. The integration measure takes away two factors of $\sigma$. An integrand that behaves like $\sigma^q$ will give a convergent integral if $q > -12$.

3. Supergravity cohomologies and their relation

3.1. The vielbein and 3-form fields

The zero-mode cohomologies in $\Psi$ and $\Phi^a$ were calculated in ref. [11]. We have listed them in Tables 1 and 2 in Appendix B. We note that the linearised supergravity fields are obtained in both fields at ghost number 0. The full cohomology can be understood by noting that if there is cohomology in the next column to the right, these will impose differential constraints on the fields. These antifield cohomologies are in one-to-one correspondence with the equations of motion. It is typical for maximally supersymmetric models that the antifields are present as cohomologies of the same field as the physical fields, so that these will make up on-shell multiplets.

In Table 1, the cohomology of $\Psi$ contains all the ghosts and higher order ghosts relevant for the tensor gauge symmetries and superdiffeomorphisms.

Table 2 gives the cohomology in $\Phi^a$. It is essential that one in addition to the pure spinor constraint consider $\Phi^a$ in the gauge equivalence class $\tilde{\Phi}^a \approx \Phi^a + (\lambda\gamma^a\rho)$ for any spinor $\rho$, otherwise the cohomology would just be the tensor product of the cohomology in $\Psi$ with the vector module. Note that the 3-form potential $C$ only enters this cohomology through its field strength 4-form $H = dC$.

There are some zero-mode cohomology at ghost number zero related to the Weyl connections [33], which have no local degrees of freedom.
3.2. Relating the two fields

We want to relate the field \( \Phi^a \) to the field \( \Psi \) through an operator \( R^a \) of ghost number \(-2\) and dimension 2. This should be possible since they represent the same physical degrees of freedom. It will of course not mean that any cohomology in \( \Psi \) will map to a cohomology in \( \Phi^a \). The \( C \)-field gauge modes, the tensor ghosts and their antifields should of course be annihilated. Neither should it be possible to map something to any cohomology in \( \Phi^a \), there are cohomologies at negative ghost number in \( \Phi^a \) which do not seem to have any physical meaning. It is obvious that \( \Psi \) should be the fundamental field, since it allows a free action (see the following section) and since it encodes the full set of \( \text{BV} \) fields with a symmetry between fields and antifields. Notice for example that it is impossible to get the Chern–Simons term from \( \Phi \) alone.

The exact form of the operator \( R^a \) is not \emph{a priori} obvious. Finding good operators with negative ghost number is non-trivial. An example of this is the \( b \)-ghost in \( D = 10 \) [14], which contains negative powers of \( \xi = (\bar{\lambda}\lambda) \).

Somewhat surprisingly, the operator \( R^a \) will not contain inverse powers of \( \xi \), but of \( \eta = (\lambda^\gamma\gamma^\lambda)(\bar{\lambda}\gamma_{ij}\bar{\lambda}) \). Its \( r \)-independent part is

\[
R^a_0 = \eta^{-1}(\bar{\lambda}\gamma^{ab}\bar{\lambda})\partial_b .
\]

It is clear that this operator represents non-vanishing cohomology of \( q \). It was not initially clear that this had to be the form of \( R^a_0 \). If one believes that it should capture the relation between some zero-mode cohomology in \( \Phi^a \) to the derivative of a component field in some zero-mode position in \( \Psi \), as \( H \) from \( C \), it seems good. But if one focuses \emph{e.g.} on the behaviour of the lowest component of the entire \( \Phi^a \), the diffeomorphism ghost, which comes at \( \lambda^2\theta^2 \) in \( \Psi \), one might guess an expression containing two antisymmetrised fermionic derivatives (we will nevertheless show below how the correct result is produced). We have shown that such an \( R^a_0 \) (which is not cohomologically equivalent to the form given here) is not possible. Eq. (3.1) was then obtained uniquely from the demand that the cohomology of \( f(\xi, \eta)(\bar{\lambda}\gamma^{ab}\bar{\lambda})\partial_b \) be independent of \( \bar{\lambda} \) (where \( f \) obviously has to be homogeneous of degree \(-2\) in \( \bar{\lambda} \)).

Closedness with respect to \( Q \) means that the \( q \)-cohomology of \( R^a_0 \) is independent of \( \bar{\lambda} \).

We need to find a sequence of operators \( \{ R^a_p \} \) of degree of homogeneity \( p \) in \( r \), such that

\[
[q, R^a_p] = 0 ,
[s, R^a_p] + [q, R^a_{p+1}] = 0 , \quad p = 1, \ldots, P - 1 ,
[s, R^a_P] = 0 . \tag{3.2}
\]
We get
\[ [s, R_0^a] = \partial_\alpha \frac{\partial R_0^a}{\partial \lambda^\alpha} = 2 \eta^{-1}(\tilde{\lambda} \gamma^{ab} r)(\lambda_\gamma^c d r) \lambda_\gamma^c d \lambda \partial_\beta . \] (3.3)

This expression is \( q \)-exact, which can be seen by calculating
\[ [q, (\tilde{\lambda} \gamma^{ab} \lambda)(\tilde{\lambda} \gamma^c d r)(\lambda_\gamma^c d D)] = 2(\tilde{\lambda} \gamma^{ab} \lambda)(\tilde{\lambda} \gamma^c d r)(\lambda_\gamma^c d \lambda) \partial_i \]
\[ + 4 \eta^{-3}(\tilde{\lambda} \gamma^{ab} \lambda)(\tilde{\lambda} \gamma^c d r)(\tilde{\lambda} \gamma^e f r)(\lambda_\gamma^e f \lambda)(\lambda_\gamma^c d D) \] (3.4)

where eq. (A.4) from Appendix A has been used in the last step. We therefore have
\[ R_1^a = \eta^{-2}(\tilde{\lambda} \gamma^{ab} \lambda)(\tilde{\lambda} \gamma^c d r)(\lambda_\gamma^c d D) . \] (3.5)

To next order we have
\[ [s, R_1^a] = 2 \eta^{-2}(\tilde{\lambda} \gamma^{ab} \lambda)(\tilde{\lambda} \gamma^c d r)(\lambda_\gamma^c d D)
+ 4 \eta^{-3}(\tilde{\lambda} \gamma^{ab} \lambda)(\tilde{\lambda} \gamma^c d r)(\tilde{\lambda} \gamma^e f r)(\lambda_\gamma^e f \lambda)(\lambda_\gamma^c d D) . \] (3.6)

In order for this expression to be cancelled by \([q, R_2^a]\), where \( R_2^a \) is gauge invariant, it is necessary to rewrite it in a form where the fermionic derivatives only occur through the combinations \((\lambda D)\) and \((\lambda_\gamma^c d \lambda)\). This is possible, using identities from Appendix A, for the expression \((\lambda_\gamma^e f \lambda)(\lambda_\gamma^c d D)\). Using this combination multiplying \((\tilde{\lambda} \gamma^{ab} \lambda)(\tilde{\lambda} \gamma^c d r)(\tilde{\lambda} \gamma^e f r)\) as in eq. (3.6), the terms \((\lambda_\gamma^e f \lambda)(\lambda_\gamma^c d D)\) in the antisymmetrisation will not contribute, since they are symmetric under the interchange of the pairs \([cd]\) and \([ef]\). With
\[ R_2^a = -16 \eta^{-3}(\tilde{\lambda} \gamma^{ab} \lambda)(\tilde{\lambda} \gamma^c d r)(\tilde{\lambda} \gamma^e f r)(\lambda_\gamma^e f \lambda)(\lambda_\gamma^c d D) \] (3.7)

we get
\[ [q, R_2^a] = -16 \eta^{-3}(\tilde{\lambda} \gamma^{ab} \lambda)(\tilde{\lambda} \gamma^c d r)(\tilde{\lambda} \gamma^e f r)(\lambda_\gamma^e f \lambda)(\lambda_\gamma^c d D)
= -4 \eta^{-3}(\tilde{\lambda} \gamma^{ab} \lambda)(\tilde{\lambda} \gamma^c d r)(\tilde{\lambda} \gamma^e f r) [(\lambda_\gamma^e f \lambda)(\lambda_\gamma^c d D) - (\lambda_\gamma^e f \lambda)(\lambda_\gamma^c d D)] \] (3.8)

\[ = -[s, R_1^a] . \]
It is also straightforward to show that $[s, R^a_2] = 0$. The complete operator
\[ R^a = R^a_0 + R^a_1 + R^a_2 \]
\[ = \eta^{-1}(\bar{\lambda} \gamma^{ab} \lambda) \partial_b + \eta^{-2}(\bar{\lambda} \gamma^{ab} \lambda) (\bar{\lambda} \gamma^{cd} \gamma^r) (\lambda \gamma_{bcd} D) \]
\[ - 16 \eta^{-3}(\bar{\lambda} \gamma^{a[b} \lambda) (\bar{\lambda} \gamma^{cd} \gamma^r) (\lambda \gamma_{fb} \lambda) (\lambda \gamma_{cde} w) . \]

satisfies $[Q, R^a] = 0$.

A further property of the operator $R^a$ is that it commutes with the regularisation factor in the measure. It is straightforward to check that $[R^a, (\bar{\lambda} \theta)] = 0$. This means that $R^a$, containing only terms with one derivative, can be partially integrated freely.

3.3. An example: The diffeomorphism ghost

It is generically technically complicated to extract components. We will illustrate the action of the operator we have found on a specific component in the component of the cohomology in $\Psi$, in order to demonstrate that it really gives the correct relation between $\Phi^a$ and $\Psi$. The simplest cohomology that is not expected to be annihilated is that of the diffeomorphism ghost. The zero-mode sits in $\Psi$ at $\lambda^2 \theta^2$. We will choose
\[ \Psi_\xi = (\lambda \gamma_{ij} \lambda)(\theta \gamma^{ijk} \theta) \xi_k(x) + \ldots \]  
(3.10)

(one may equally well choose the structure $(\lambda \gamma_{ijkl} \lambda)(\theta_{ijkl} \theta) \xi_m$; they differ by a $q$-exact term $q \cdot (\lambda^4 \theta)(\theta \theta) \xi_i$). In order to represent cohomology, the diffeomorphism ghost must fulfill $\partial_i(\xi_j) = 0$, so what remains are the ghosts corresponding to isometries (in the flat Minkowski space we start from, translations and Lorentz rotations). Equation (3.10) should be complemented with a term $\lambda^2 \theta^4 \partial \xi$, but it is annihilated by the derivative in $R^a_0$. We now act with $R^a_0$ and obtain
\[ R^a_0 \Psi_\xi = \eta^{-1}(\bar{\lambda} \gamma^{a[m} \lambda)(\lambda \gamma_{ij} \lambda)(\theta \gamma^{ijk} \theta) \partial_m \xi_i . \]  
(3.11)

This expression should represent a cohomology which is independent of $\bar{\lambda}$, and to match the position of the same field in $\Phi^a$ it should be proportional to
\[ \Phi^a_\xi = \xi^a + \frac{1}{4}(\theta \gamma^{aij} \theta) \partial_i \xi_j \]
in cohomology. This form of $\Phi^a_\xi$ is easily checked using the gauge symmetry $\Phi^a \approx \Phi^a + (\lambda^a \rho)$ and antisymmetry of $\partial \xi_j$. To show that $R^a_0 \Psi_\xi$ captures this cohomology, we add exact terms to $R^a_0 \Psi_\xi$. A calculation shows that

$$R^a_0 \Psi_\xi + q \cdot \left( \eta^{-1}(\bar{\lambda} \gamma^{am} \bar{\lambda}) [16(\lambda \gamma_{mi} \theta) \xi^i] - 4(\lambda \gamma^i \theta) (\theta \theta) \partial_i \xi_m + 8(\lambda \gamma_{mij} \theta) (\theta \theta) \partial_i \xi_j + 4(\lambda \gamma_{mk} \theta) (\theta \gamma^{ijk} \theta) \partial_i \xi_j - 4(\lambda \gamma^i \theta) (\theta \gamma^{jk} \theta) \partial_i \xi_j \right)$$

$$= -8 \Phi^a_\xi.$$

The first term acted on by $q$ is the only one contributing to the zero-mode, and its coefficient is fixed by demanding that all $\bar{\lambda}$-dependence disappears. The trick is to demand that after Fierz rearrangements, only terms with $(\bar{\lambda} \gamma^{am} \bar{\lambda})(\lambda \gamma_{mn} \lambda)$ remain, since this gives $-\frac{1}{2} \delta^a_n \eta$ using the gauge invariance of $\Phi^a$. Although it is obvious from the construction of $R^a_0$ that it maps cohomology to cohomology, at least for some fields, it is good to verify this in a concrete case.

4. **The action**

4.1. **The linearised action**

When there is a non-degenerate measure, allowing partial integration by $Q$, the equation of motion $Q \Psi = 0$ is obtained from an action

$$S_0 = \int [dZ] \Psi Q \Psi$$

(4.1)

(here we have suppressed an overall dimensionful constant $G^{-1}$, which is uninteresting at this stage, since all terms in the expansion will carry the same factor).

In a Batalin–Vilkovisky framework, the consistency criterion, generalising $Q^2 = 0$ and encoding invariance as well as gauge algebra, is the master equation,

$$(S, S) = 0.$$
Here, the antibracket is defined as
\[
(A, B) = \int A \frac{\delta}{\delta \Psi(Z)} [dZ] B. \tag{4.3}
\]
It is a fermionic operation, and symmetric under the interchange of bosonic \( A \) and \( B \). It has dimension \(-\text{dim}(\int [dZ]) - 2\text{dim}(\Psi) = D - 2\) and ghost number \(-\text{gh}\#(\int [dZ]) - 2\text{gh}\#(\Psi) = 1\).

We note that the antibracket, which in general contains a symmetrised sum of derivatives with respect to all fields and corresponding antifields, takes an extremely simple form. The master equation is trivially fulfilled by the free action \( S_0 \).

4.2. THE 3-POINT COUPLING

When introducing interactions as deformations of the free action we let \( S = S_0 + S_1 + \ldots \). The master equation to lowest order reads \((S_0, S_1) = 0\). The deformation is non-trivial if \( S_1 \neq (S_0, T_1) \). We will propose a 3-point coupling \( S_1 \).

As mentioned in the Introduction, an expression \( \int [dZ] \lambda^2 \Psi \Phi^2 \) has the correct dimension and ghost number. We will now be more specific. In the previous section, we showed that \( \Phi^a = R^a \Psi \), where the operator \( R^a \) of dimension 2 and ghost number \(-2\) is given by eq. (3.9). In order for \( R^a \) to represent cohomology, it was essential that \( \Phi^a \) has the additional gauge invariance \( \Phi^a \approx \Phi^a + (\lambda \gamma^a \rho) \) for an arbitrary \( \rho(Z) \). We should also remember that \( \Phi^a \) is a fermionic field, so any expression like \( \Phi^a \Phi_a \) vanishes; instead we need to contract indices by some antisymmetric matrix. Both these requirements, in addition to those from ghost number and dimension, are met by the insertion of a factor \((\lambda \gamma_{ab} \lambda)\). This realisation was inspired by the similar form of the matter kinetic term in the \( D = 3 \) conformal theories \([16,17]\). The candidate interaction term is
\[
S_1 = \int [dZ] (\lambda \gamma_{ab} \lambda) \Psi R^a \Psi R^b \Psi. \tag{4.4}
\]
It is invariant under gauge transformations of \( \Phi^a \) thanks to the pure spinor Fierz identity \( (\gamma^b \lambda) \alpha (\lambda \gamma_{ab} \lambda) = 0 \). Remember that partial integration of \( R^a \) is allowed. Since \( \Psi^2 = 0 \), partial integration gives back the same expression. The naive calculation is extremely simple. Using the expression for the antibracket, one immediately sees that the condition for the master equation to be fulfilled at this order is that \( R^a \) is \( Q \)-closed, and the condition that the interaction is non-trivial becomes the statement that \( R^a \) is not exact. We have already shown that this is the case.
It should be mentioned that the other candidate deformation of the action matching dimension and ghost number, $\int [dZ](\lambda_{\gamma abcd\epsilon})R^a\Psi R^b\Psi R^c\Psi R^d\Psi$, fails to make sense because it is not gauge invariant.

Before trusting the naive formal calculation, one should check that there are no divergences neglected in the procedure. The most singular term in $R^a$ goes as $\sigma^{-4}$ close to the subspace $\mathcal{P}_0$. For a non-singular $\Psi$, the integrand goes as $\sigma^{-7}$ or slower, so the integral converges. This shows that $S_1$ is a valid 3-point coupling.

4.3. Example of a coupling: The diffeomorphism ghosts

Even though it has been shown that the 3-point coupling constructed is a non-trivial parameter-free deformation of the free action, and thus must represent interacting supergravity, the construction has been made in a rather abstract way. This is indeed the reason that the calculations above are tractable; once physical fields are extracted things tend to become much more complicated. Nevertheless we would like to demonstrate that the expected interactions arise. The example we have chosen is the coupling of diffeomorphism ghosts with their antifields, which would show that the diffeomorphism algebra is deformed from the abelian algebra of the non-interacting theory in the way appropriate for gravity (which is a cohomologically unique deformation [34]). Remember that the gauge algebra is reflected in a coupling $f_{abc}^\lambda c_\lambda^a c_\lambda^b c_\lambda^c$, where $c$ is the ghost and $c^*$ its antifield. An interesting alternative would be to derive the Chern–Simons term $\int C \wedge H \wedge H$, where $H = dC$. This is possible but quite involved, since the zero-modes of $C$ in $\Psi$ and of $H$ in $\Phi^a$ are linear combination of a number of terms, and the projection of products of these on the measure mode at $\lambda^7 \theta^9$ is quite non-trivial.

The relevant terms in the fields are $\Psi \sim \lambda^2(\theta^2 \xi + \theta^4 \partial \xi) + \lambda^3(\theta^7 \xi^* + \theta^9 \partial \xi^*), \Phi \sim \xi + \theta^2 \partial \xi + \lambda^3(\theta^7 \xi^* + \theta^9 \partial \xi^*)$. The coupling term in the Lagrangian is $\lambda^2 \Psi \Phi^2$ which then must be formed to get the singlet at $\lambda^7 \theta^9$. We use the possibility to partially integrate $R^a$ to make the choice to have $\xi^*$ in $\Psi$ and $\xi$ in $\Phi$. Using the symmetry between the zero-mode cohomologies at $\lambda^m \theta^n$ and $\lambda^{7-m} \theta^{9-n}$ we get a term with the structure

$$\xi^*(w_{\gamma jk} w)(D_{\gamma jk} D) \cdot (\lambda_{\gamma ab\lambda}) \xi^c(\theta_{\gamma b_{lm}} \theta) \partial_l \xi m$$

$$\sim \xi^* \xi^j \partial_j \xi^i \quad (4.5)$$

with a non-zero coefficient. We have not included the term with $\partial \xi^*$ and two $\xi$’s, due to the technical difficulty of writing the $\partial \xi^*$ term in $\Psi$, but it has to contribute to the same structure. The notation in eq. (4.5) is a little sloppy, the $w$’s should really be the gauge-covariant derivative that preserve the pure spinor constraint, but acting on products of $\lambda$’s in modules $(0, a, 0, 0, b)$, i.e., in a gauge $(w_{\gamma a} w) \Psi = 0$, they acts as $w$. 
This indicates that the diffeomorphism algebra is obtained in the right way, although only Killing vector ghosts were included here. This deformation is of course accompanied by the appropriate interactions of the physical fields. We hope that this will be convincing evidence that the proposed 3-point coupling indeed gives couplings in $D = 11$ supergravity.

5. Conclusions

We have constructed a very simple 3-point coupling in an action for $D = 11$ supergravity with a scalar superfield displaying manifest supersymmetry. All 3-point couplings between component fields (and ghosts and antifields) are encoded in a single term.

It will be interesting to investigate how this action continues at higher order. A few things can be said more or less directly. When one goes to higher couplings the coincident singularities will need to be regularised. Hopefully this can be achieved using a similar BRST-invariant smearing technique as in ref. [31].

If we for the moment ignore this issue, and consider the term $(S_1, S_1)$ in the master equation, it will give one term $\sim \int (\lambda_{ab}\lambda)(\lambda_{cd}\lambda)R^a\Psi R^b\Psi R^c\Psi R^d\Psi$, which vanishes thanks to the pure spinor constraint. There will also be a term $\sim \int (\lambda_{cd}\lambda)\Psi R^a((\lambda_{ab}\lambda)R^b\Psi)R^c\Psi R^d\Psi$. It is easy to check that even though $[R^a, (\lambda_{ab}\lambda)] \neq 0$ one has $[R^a, (\lambda_{ab}\lambda)]R^b = 0$, so the remainder is $\sim \int (\lambda_{ab}\lambda)(\lambda_{cd}\lambda)\Psi[R^a, R^b]\Psi R^c\Psi R^d\Psi$. The algebraic properties of $R^a$ become important. If the commutator (which is clearly non-zero) is reasonably simple, there may be hope of finding concrete forms for higher order interactions. We hope to be able to continue along this line of pursuit.

There is of course also a number of other questions. The formalism suffers from a lack of background invariance. Can this is some way be remedied? How are the conclusions altered in other backgrounds than flat space? Can U-duality be incorporated in a dimensionally reduced setting? Also, calculation of amplitudes might benefit from having manifest supersymmetry. Path integral calculations of amplitudes requires gauge fixing, which in the BV formalism also includes elimination of antifields. Can this be achieved with a composite “b-ghost” along the same lines as in pure spinor string theory?

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Appendix A: Spinor and pure spinor identities in $D = 11$

We will list some identities that have been useful for calculations.

Fierz rearrangements are always made between spinors at the right and left of two spinor products. The general Fierz identity reads

$$ (AB)(CD) = \sum_{p=0}^{5} \frac{1}{32^p} (B\gamma^{a_1\ldots a_p} C)(A\gamma_{a_p\ldots a_1} D) $$

(with an overall minus sign if $A$ and one of $B$ and $C$ are fermionic). For bilinears in a pure spinor $\lambda$ this reduces to

$$ (A\lambda)(\lambda B) = -\frac{1}{2^4} (\lambda\gamma^{ab}\lambda)(A\gamma_{ab}B) + \frac{1}{32^4} (\lambda\gamma^{abcdef}\lambda)(A\gamma_{abcdef}B) .$$

From the constraint on the spinor $r$, $(\bar{\lambda}\gamma^a r) = 0$, one derives

$$ (\bar{\lambda}\gamma^{[ij}\lambda)(\bar{\lambda}\gamma^{kl]}r) = 0 .$$

The gauge invariance $\Phi^a \approx \Phi^a + (\lambda\gamma^a\rho)$ implies that

$$ M^{ai}(\lambda\gamma_{bi}\lambda) = \frac{1}{2} e^a M^{ij}(\lambda\gamma_{ij}\lambda) ,$$

where $a$ is the index carried by $\Phi^a$.

Various useful identities for a pure spinor $\lambda$ include

$$ (\gamma_j \lambda)_{\alpha}(\lambda\gamma^{ij}\lambda) = 0 ,$$
$$ (\gamma_i \lambda)_{\alpha}(\lambda\gamma^{abcd\lambda}) = 6(\gamma^{[ab}\lambda)_{\alpha}(\lambda\gamma^{cd]}\lambda) ,$$
$$ (\gamma_{ij} \lambda)_{\alpha}(\lambda\gamma^{abcdef}\lambda) = -18(\gamma^{[ab}\lambda)_{\alpha}(\lambda\gamma^{cd]}\lambda) ,$$
$$ (\gamma_{ijk} \lambda)_{\alpha}(\lambda\gamma^{abijk}\lambda) = -42\lambda_{\alpha}(\lambda\gamma^{ab}\lambda) ,$$
$$ (\gamma_{ij} \lambda)_{\alpha}(\lambda\gamma^{abcdij}\lambda) = -24(\gamma^{[ab}\lambda)_{\alpha}(\lambda\gamma^{cd]}\lambda) ,$$
$$ (\gamma_i \lambda)_{\alpha}(\lambda\gamma^{abcd\lambda}) = \lambda_{\alpha}(\lambda\gamma^{abcdef}\lambda) - 10(\gamma^{[ab}\lambda)_{\alpha}(\lambda\gamma^{cd]}\lambda) ,$$

(A.5)
and for a fermionic spinor $\theta$:

$$
\theta_\alpha (\theta \gamma^{abcd} \theta) = -2(\gamma^{[ab\cdot j]} \theta)_{\alpha} (\theta \gamma^{cd]} t \theta) - (\gamma^{abcd} \theta)_{\alpha} (\theta \theta) ,
$$

$$
(\gamma^{ij} \theta)_{\alpha} (\theta \gamma^{a[ij]} \theta) = 6(\gamma^{a} \theta)_{\alpha} (\theta \theta) .
$$

(A.6)
Appendix B: Tables of cohomologies

The horizontal direction is the expansion in $\lambda$, i.e., in decreasing ghost number of the component fields, and the vertical is the expansion of the superfields in terms of $\theta$ (downward). The columns have been shifted in order to place fields of same dimension on the same row.

| gh#  |  3 |  2 |  1 |  0 | -1 | -2 | -3 | -4 | -5 |
|------|----|----|----|----|----|----|----|----|----|
| dim  | -3 |    |    |    |    |    |    |    |    |
| $-\frac{5}{2}$ | . | . |    |    |    |    |    |    |    |
| -2   | . | . | (10000) | . |    |    |    |    |    |
| $-\frac{3}{2}$ | . | . | . | . | . |    |    |    |    |
| -1   | . | . | (01000) | (10000) | . | . |    |    |    |
| $-\frac{1}{2}$ | . | . | (00001) | . | . | . | . |    |    |
| 0    | . | . | . | . | . | (00000) | (00100) | (20000) | . | . | . |
| $\frac{1}{2}$ | . | . | . | . | . | (00001) | (10001) | . | . | . | . |
| 1    | . | . | . | . | . | . | . | . | . | . | . | . |
| $\frac{3}{2}$ | . | . | . | . | . | (00001) | (10001) | . | . | . | . |
| 2    | . | . | . | . | . | (00000) | (00100) | (20000) | . | . | . | . |
| $\frac{5}{2}$ | . | . | . | . | . | . | . | (00001) | . | . | . | . |
| 3    | . | . | . | . | . | (01000) | (10000) | . | . | . | . |
| $\frac{7}{2}$ | . | . | . | . | . | . | . | . | . | . | . | . |
| 4    | . | . | . | . | . | . | . | (10000) | . | . | . | . |
| $\frac{9}{2}$ | . | . | . | . | . | . | . | . | . | . | . | . |
| 5    | . | . | . | . | . | . | . | . | (00000) | . | . | . |

*Table 1. The cohomology in $\Psi$.  

[21]
| gh# | 1 | 0 | -1 | -2 | -3 | -4 | -5 |
|-----|---|---|----|----|----|----|----|
| dim = -1 | (10000) |
| -\frac{1}{2} | (00001) | • |
| 0 | • | (20000) | • |
| \frac{1}{2} | • | (00001) | (10001) | • | • |
| 1 | • | (00010) | (10000) | • | • | • |
| \frac{3}{2} | • | • | (00001) | (10001) | • | • | • |
| 2 | • | • | (00000)(00002) | (00100)(01000) | (10000)(20000) | • | • | • |
| \frac{5}{2} | • | • | • | • | • | • | • | • |
| 3 | • | • | • | (00000)(00002) | (00100)(01000) | (10000)(20000) | • | • | • |
| \frac{7}{2} | • | • | • | (00001) | (10001) | • | • | • |
| 4 | • | • | • | • | (00010) | (10000) | • | • |
| \frac{9}{2} | • | • | • | • | (00001) | (10001) | • | • |
| 5 | • | • | • | • | | | (20000) | • | • |
| \frac{11}{2} | • | • | • | • | • | (00001) |
| 6 | • | • | • | • | • | (10000) |
| \frac{13}{2} | • | • | • | • | • | • |

*Table 2. The cohomology in $\Phi^\theta$.***