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Asymptotic behavior of varying discrete Jacobi–Sobolev orthogonal polynomials

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Abstract

In this contribution we deal with a varying discrete Sobolev inner product involving the Jacobi weight. Our aim is to study the asymptotic properties of the corresponding orthogonal polynomials and the behavior of their zeros. We are interested in Mehler–Heine type formulae because they describe the essential differences from the point of view of the asymptotic behavior between these Sobolev orthogonal polynomials and the Jacobi ones. Moreover, this asymptotic behavior provides an approximation of the zeros of the Sobolev polynomials in terms of the zeros of other well-known special functions. We generalize some results appeared in the literature very recently.

Keywords:
Sobolev orthogonal polynomials
Jacobi polynomials
Mehler–Heine formulae
Asymptotics
Zeros

1. Introduction

One of the aims of this paper is the study of the asymptotic behavior of sequences of polynomials \( \{Q_{n}(\alpha, \beta, M_{n})\}_{n \geq 0} \) orthogonal with respect to the inner product

\[
(f, g)_{S,n} = \int_{-1}^{1} f(x)g(x)(1-x)^{\alpha}(1+x)^{\beta}dx + M_{n}f^{(j)}(1)g^{(j)}(1),
\]

where \( \alpha > -1, \beta > -1 \), and \( j \geq 0 \).

We assume that \( \{M_{n}\}_{n \geq 0} \) is a sequence of nonnegative real numbers satisfying

\[
\lim_{n \to \infty} M_{n} = M > 0,
\]

where \( M \) is a fixed real number. Notice that this assumption is not very restrictive since the sequence \( \{M_{n}\}_{n \geq 0} \) can behave asymptotically like any real power of the monomial \( n \).

The main motivation to study this type of inner product arises from the papers [1,2]. In [1] the authors work with a measure supported on \([-1, 1]\). However, in [2] the authors deal with measures supported on an unbounded interval. In both cases the authors consider measures with nonzero absolutely continuous part, i.e., they work with the so-called continuous
Sobolev orthogonal polynomials. The main topic in those papers is how to balance the Sobolev inner product to equilibrate the influence of the two measures in the asymptotic behavior of the corresponding orthogonal polynomials. This inspires us to consider the discrete Sobolev inner product

$$\langle f, g \rangle = \int f(x)g(x)(1-x)^\beta(1+x)^\alpha dx,$$

which is a perturbation of a standard inner product. Now, making $M$ dependent on $n$ we can study the influence of the perturbation on the asymptotic behavior of the orthogonal polynomials. The literature on discrete Sobolev (or Sobolev-type) orthogonal polynomials is very wide, so we refer the interested readers on this topic to survey [3] and the references therein.

From here, in [4] the authors found the asymptotic behavior of a family of orthogonal polynomials with respect to a varying Sobolev inner product similar to (1), involving the Laguerre weight $w(x) = x^\alpha e^{-x}$, $\alpha > -1$. We remark that the techniques used in [4] are not useful in this case, and now we need to use more powerful techniques based on those considered in [5]. More recently, in [6] the same authors have even improved these techniques in such a way that they have obtained relevant results for the orthogonal polynomials with respect to a non-varying discrete Sobolev inner product being $\mu_0$, a general measure.

Previously, in [7] J.J. Moreno-Balcázar obtained some results in this direction but only for the case $\alpha = 0$. Again, the method used in that paper does not allow to tackle our problem.

We want to emphasize that our objective is to establish that the size of the sequence $\{M_n\}_{n \in \mathbb{N}}$ has an essential influence on the asymptotic behavior of the orthogonal polynomials with respect to (1), but this influence is only local, that is, around the point where we have introduced the perturbation. In our case, this point is located at $x = 1$. Furthermore, we prove that this influence depends on the size of the sequence $\{M_n\}_{n \in \mathbb{N}}$ and its relation with the parameter $\alpha$ in the Jacobis weight and the order of the derivative in (1). It is important to remark that for a sequence $\{M_n\}_{n \in \mathbb{N}}$, we have a sequence of orthogonal polynomials for each $n$, so we have a square tableau $\{(Q_{\nu}^{\alpha,\beta,n,\mu_0},x)_{\nu \in \mathbb{N}}\}_{n \in \mathbb{N}}$. Here, we deal with the diagonal of this tableau, i.e. $\{Q_0^{\alpha,\beta,n,\mu_0},x\}, Q_1^{\alpha,\beta,n,\mu_0}(x), \ldots, Q_n^{\alpha,\beta,n,\mu_0}(x), \ldots$. At this point, in order to simplify the notation, we will denote $Q_n^{\alpha,\beta,n,\mu_0}(x) = Q_n(x)$.

A second aim of this paper is to establish a simple asymptotic relation between the zeros of the Sobolev polynomials which are orthogonal with respect to (1) and the zeros of combinations of Bessel functions of the first kind. This relation is deduced as an immediate consequence of Mehler–Heine formulæ (Theorem 2) and they have a numerical interest since we can provide an estimate of the zeros of these polynomials.

Since Jacobi classical orthogonal polynomials are involved in the varying inner product (1), we recall some of their basic properties. Jacobi polynomials are orthogonal with respect to the standard inner product

$$\langle f, g \rangle = \int f(x)g(x)(1-x)^\beta(1+x)^\alpha dx, \quad \alpha, \beta > -1.$$

In the sequel, we will work with the sequence $\{P_{\nu}^{\alpha,\beta,n,\mu_0}\}_{n \in \mathbb{N}}, \alpha > -1$ and $\beta > -1$, normalized by (see [8, f. (4.1.1)])

$$\langle P_{\nu}^{\alpha,\beta,n,\mu_0}(1) \rangle = \frac{n+\nu}{\nu+1} = \frac{\Gamma(n+\nu+1)}{\Gamma(n+\nu+2)}.$$

The derivatives of Jacobi polynomials satisfy (see, [8, f. (4.21.7)])

$$\langle P_{\nu}^{\alpha,\beta,n,\mu_0}(x)^{(k)} \rangle = \frac{\Gamma(n+\nu+\beta+k+1)}{\Gamma(n+\nu+\beta+1)} \frac{\Gamma(n+\nu+\beta+k+1)}{\Gamma(n+\nu+\beta+1)}, \quad k \geq 0.$$  

Using (3) and (4), we deduce

$$\langle P_{\nu}^{\alpha,\beta,n,\mu_0}(x)^{(k)} \rangle = \frac{\Gamma(n+\nu+\beta+k+1)}{\Gamma(n+\nu+\beta+1)} \frac{\Gamma(n+\nu+\beta+k+1)}{\Gamma(n+\nu+\beta+1)},$$

where $\langle P_{\nu}^{\alpha,\beta,n,\mu_0}(x)^{(k)} \rangle$ denotes the $k$th derivative of $P_{\nu}^{\alpha,\beta,n,\mu_0}$ evaluated at $x = 1$.

We also note that the squared norm of a Jacobi polynomial is (see, [8, f. (4.3.3)])

$$\langle P_{\nu}^{\alpha,\beta,n,\mu_0} \rangle^2 = \frac{2\nu+\beta+1}{\nu+1} \frac{\Gamma(n+\nu+\beta+1)}{\Gamma(n+\nu+\beta+1)} \frac{\Gamma(n+\nu+\beta+1)}{\Gamma(n+\nu+\beta+1)}$$

Finally, we will use the Mehler–Heine formula for classical Jacobi polynomials

**Theorem 1** ([8, Th. 8.1.1]) Let $\alpha, \beta > -1$. Then,

$$\lim_{n \to \infty} n^{-\nu} P_{\nu}^{\alpha,\beta}(x) \cos \left( \frac{\alpha}{2} \right) = \lim_{n \to \infty} \frac{1}{\nu!} P_{\nu}^{\alpha,\beta}(1 - \frac{x^2}{2\nu}) = (x/2)^{-\nu} f_\nu(\alpha),$$




uniformly on compact subsets of \( \mathbb{C} \). Here \( j_n(x) \) denotes the Bessel function of the first kind, i.e.,
\[
J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (n + k + 1)} \left( \frac{x}{2} \right)^{2n+2k}.
\]

We will also use the following limit related to Stirling formula (see, for example, [9, Eq. (11.13)])
\[
\lim_{n \to \infty} \frac{n!}{\Gamma(n+1)} = 1.
\]

We introduce the following notation: If \( a_n \) and \( b_n \) are two sequences of real numbers, then \( a_n \approx b_n \) means that the sequence \( a_n \) converges to \( b_n \).

The paper is organized as follows. In Section 2, we provide some properties of the varying Jacobi–Sobolev orthogonal polynomials which are essential to establish the Mehler–Heine asymptotics for these polynomials in Section 3. Furthermore, as a consequence of this asymptotic formula we deduce the asymptotic behavior of the corresponding zeros. Thus, as we have commented previously, we can see the influence of the parameter \( \gamma \) as a consequence of this asymptotic formulawededucetheasymptoticbehaviorofthecorrespondingzeros. Thus, as we

\section{Varying Jacobi–Sobolev orthogonal polynomials}

It is well known that the classical Jacobi orthogonal polynomials, \( (p_n^{(\alpha, \beta)}(x))_{n=0}^{\infty} \), constitute a basis of the linear space \( P_n[x] \) of polynomials with real coefficients and degree at most \( n \). Therefore, the Jacobi–Sobolev orthogonal polynomial of degree \( n \), \( Q_n(x) \), can be expressed as
\[
Q_n(x) = p_n^{(\alpha, \beta)}(x) + \sum_{j=0}^{\infty} \beta_j p_{n}^{(\alpha, \beta)}(x).
\]

Then, using well-known algebraic tools (see, for example, [10, Sect. 2]) we can deduce
\[
Q_n(x) = p_n^{(\alpha, \beta)}(x) - \frac{M_n}{1 + M_n K_n^{(11)}(1, x)},
\]

with
\[
K_n^{(11)}(x, y) = \sum_{i=0}^{\infty} \frac{(p_n^{(\alpha, \beta)}(x))^i}{|p_n^{(\alpha, \beta)}(y)|^i}.
\]

Next, we give a technical result useful for our purposes, interesting in itself though.

\textbf{Lemma 1.} Let \( (Q_n)_{n=0}^{\infty} \) be the sequence of orthogonal polynomials with respect to (1) and \( 0 \leq k \leq n \), then
\begin{enumerate}[(a)]
  \item \[
  \lim_{n \to \infty} \frac{(Q_n)_{(1)}(1)}{(P_n^{(\alpha, \beta)})_{(1)}(1)} = \begin{cases} \frac{k-j}{x+j+k+1}, & \text{if } \gamma < 2(\alpha + 2j + 1), \\ \theta_{a, b, k}, & \text{if } \gamma = 2(\alpha + 2j + 1), \\ 1, & \text{if } \gamma > 2(\alpha + 2j + 1), \end{cases}
  \]
  \item \[
  \theta_{a, b, k} = \frac{M(k-j)}{(x+j+k+1) \left( M + F^2(\alpha + j + 1)2^{2\alpha+2j+1}(\alpha + 2j + 1) \right)}.
  \]
\end{enumerate}

\textbf{Proof.} Kernel polynomials related to classical families of orthogonal polynomials and their derivatives have been widely studied in the literature. Thus, we can claim that the following limit exists,
\[
\lim_{n \to \infty} K_n^{(11)}(1, 1) \in \mathbb{R}.
\]

It is very easy to check it by using Stolz’s criterion, (5), (6), (7) and the fact that
\[
p^{2\alpha+2j+2k+2} - (n-1)^{2\alpha+2j+2k+2} \approx (2\alpha + 2j + 2k + 2)(n-1)^{2\alpha+2j+2k+1}.
\]
To simplify the computations we introduce the following notation

\[ C_{jk} = \frac{1}{F(a + j + 1)F(a + k + 1)(a + j + k + 1)} \]

We will now prove part (a) of the lemma, by (8)

\[
\lim_{n \to \infty} \frac{Q^{(1, 1)}(n)}{\left(\frac{\alpha}{\beta}\right)^n(1)} = \lim_{n \to \infty} \left[ 1 - \frac{M_n \alpha^{(1, 1), (1, 1)}(n)}{\left(1 + M_n \alpha^{(1, 1)}(1)\right)^{(a + j) + (a + k) + 1}} \frac{(\alpha, \beta)}{1 + M_n^{a + 2b + 2k - 2 - 2y}} \right].
\]

To simplify the computations we introduce the following notation

\[ a_n = M_n \alpha^{(1, 1)}, \quad b_{a,b} = \alpha^{(1, 1)}(1, 1) \]

Then, the above limit becomes

\[
\lim_{n \to \infty} \left[ 1 - \frac{a_n b_{a,b}}{\alpha^{(a+b+2y+2)(1)}(1 + a_n b_{a,b})^{(a+b+2y+2)(1)}} \right] = \frac{1}{F(a + j + 1)F(a + k + 1)} \lim_{n \to \infty} \frac{a_n b_{a,b}}{\alpha^{(a+b+2y+2)(1)}(1 + a_n b_{a,b})^{(a+b+2y+2)(1)}} = \frac{1}{F(a + j + 1)F(a + k + 1)} \lim_{n \to \infty} \frac{1}{a_n b_{a,b} + a_n b_{a,b}}
\]

Therefore, it is necessary to distinguish three cases according to the value of the parameter \( y \). The value of this limit is:

Case \( y > 2(a + 2) + 1 \).

\[
\lim_{n \to \infty} \frac{a_n b_{a,b}}{\alpha^{(a+b+2y+2)(1)}(1 + a_n b_{a,b})^{(a+b+2y+2)(1)}} = 1.
\]

Case \( y < 2(a + 2) + 1 \).

\[
\lim_{n \to \infty} \frac{a_n b_{a,b}}{\alpha^{(a+b+2y+2)(1)}(1 + a_n b_{a,b})^{(a+b+2y+2)(1)}} = \frac{1}{a_n b_{a,b} + a_n b_{a,b}}.
\]
To establish (b) it is enough to prove that we can deduce

\[ \frac{\partial^2}{\partial x^2} \left( \psi_{n_2}^{(2)}(x) \right) = \frac{\partial^2}{\partial x^2} \left( \psi_{n_2}^{(2)}(x) \right) \]

Indeed, from (5) and (6) this limit can be expressed as

\[ \lim_{n \to \infty} \left( \frac{\partial^2}{\partial x^2} \left( \psi_{n_2}^{(2)}(x) \right) \right) = 0. \]

To establish (b) it is enough to prove that

\[ \lim_{n \to \infty} \left( \frac{\partial^2}{\partial x^2} \left( \psi_{n_2}^{(2)}(x) \right) \right) = 0. \]

Again, to simplify the computations we introduce some notation

\[ a_n = M_n n^2, \quad \text{by (2) we have} \quad \lim_{n \to \infty} a_n = M. \]

\[ b_n = \frac{\beta^2(n + \alpha + \beta + 1)}{\Gamma(n + \alpha + 1)\Gamma(n + \alpha + 2)}, \quad \text{then by (7) \lim}_{n \to \infty} b_n = 1. \]

\[ c_n = \frac{\beta^2(n + \alpha + \beta + 1)}{\Gamma(n + \alpha + 1)\Gamma(n + \alpha + 2)}, \quad \text{then by (7) \lim}_{n \to \infty} c_n = 1. \]

\[ d_n = M_n n^2 \frac{\beta^2(n + \alpha + \beta + 1)}{\Gamma(n + \alpha + 1)\Gamma(n + \alpha + 2)}, \quad \text{then using (2) and (11) we get} \quad \lim_{n \to \infty} d_n = MC_n. \]

\[ E_{n} = \frac{1}{2^n} \frac{1}{\Gamma(n + \alpha + 1)}. \]

In this way, for every \( \gamma \), the above limit is

\[ \lim_{n \to \infty} E_{n} \frac{\beta^2(n + \alpha + \beta + 1)}{\Gamma(n + \alpha + 1)\Gamma(n + \alpha + 2)} = \lim_{n \to \infty} \frac{\beta^2(n + \alpha + \beta + 1)}{\Gamma(n + \alpha + 1)\Gamma(n + \alpha + 2)} = 0. \]

and we have just proved (b).
Proof. Let Lemma 2.

Remark 1. Notice that taking into account (b) in the above lemma, (a) holds true when we consider orthonormal polynomials.

To tackle Mehler–Heine asymptotics we need to expand the Sobolev polynomials \( Q_n \) adequately. The following result gives us this expansion. In a more general framework it has been established in [6, Th. 1]. The idea is that the coefficients \( b_i(n) \) in (12) can be obtained as a solution of a homogeneous linear system of \( j+1 \) equations and \( j+2 \) unknowns. In our concrete case, we can compute explicitly the entries of the corresponding coefficient matrix.

Proposition 1. There exists a family of real numbers \( \{b_i(n)\}_{i=0}^{j+1} \) not identically zero, such that the following connection formula holds

\[
Q_n(x) = \sum_{k=0}^{j+1} b_k(n) (1-x)^{\frac{k}{2}} e^{-i x} P_n^{(\alpha,\beta)}(x), \quad n \geq j + 1. \tag{12}
\]

Lemma 2. Let \( \{b_i(n)\}_{i=0}^{j+1} \) be the coefficients in (12). Then

\[
\lim_{n \to \infty} b_i(n) = b_i \in \mathbb{R}, \quad i \in \{0, 1, \ldots, j+1\}.
\]

Proof. We take the \( i \)th derivative in (12) and evaluate the corresponding expression at \( x = 1 \),

\[
Q_i^{(1)}(1) = \sum_{k=0}^{j+1} b_k(n) \frac{k}{2} \bigg(1 - 1\bigg)^k \bigg( P_n^{(\alpha,\beta)}(1) \bigg)^{(k-i)}.
\]

Then,

\[
Q_i^{(1)}(1) = \sum_{k=0}^{j+1} b_k(n) \frac{k}{2} \bigg(1 - 1\bigg)^k \bigg( P_n^{(\alpha,\beta)}(1) \bigg)^{(k-i)}.
\]

From Lemma 1, \( \lim_{n \to \infty} \frac{\sum_{i=0}^{j+1} b_i(n) x^i}{(\sum_{i=0}^{j+1} (\alpha+i)\Gamma(\alpha+i))^{n+1}} \) exists and its value depends on the value of parameter \( \gamma \) related to the size of the sequence \( \{\mu_i\}_{i=0}^{j+1} \), so

\[
\frac{Q_i^{(1)}(1)}{(P_n^{(\alpha,\beta)}(1))^{k-i}} = \sum_{k=0}^{j+1} b_k(n) \left(1 - 1\right)^k \partial_i^a(k, n)
\]

with \( A(k, n) = \left(\frac{P_n^{(\alpha,\beta)}(1)}{(\sum_{i=0}^{j+1} (\alpha+i)\Gamma(\alpha+i))^{n+1}}\right)^{(k-i)} \).

It only remains to prove that there exists \( \lim_{n \to \infty} A(k, n) \in \mathbb{R} \) and, therefore the coefficients \( \{b_i(n)\}_{i=0}^{j+1} \) are convergent. Indeed

\[
\lim_{n \to \infty} A(k, n) = \lim_{n \to \infty} \frac{P_n^{(\alpha,\beta)}(1)}{(\sum_{i=0}^{j+1} (\alpha+i)\Gamma(\alpha+i))^{n+1}} = A(k, u),
\]

where we denote \( A(k, u) = \frac{P_n^{(\alpha,\beta)}(1)}{(\sum_{i=0}^{j+1} (\alpha+i)\Gamma(\alpha+i))^{n+1}} \).

Remark 2. Let \( b_i = \lim_{n \to \infty} b_i(n) \) with \( i \in \{0, 1, \ldots, j+1\} \). (13) is a recursive algorithm to compute \( b_i \).

- Step 1. For \( k = 0 \) we obtain \( b_0 \) in a straightforward way.
- Step 2. For \( k = 1 \) we deduce the value of \( b_1 \) from (13) using step 1. Similarly, for \( k \geq 2 \) we apply (13) in a recursive way.
3. Asymptotics and zeros of varying Jacobi–Sobolev

We focus our attention on the analysis of Mehler–Heine formulas for these discrete Jacobi–Sobolev orthogonal polynomials because we want to know how the discrete part in the inner product \( | \cdot | \) influences the asymptotic behavior of the corresponding orthogonal polynomials. Furthermore, we will prove that this influence is related to the size of the sequence \( |M_{n,\gamma}| \).

**Theorem 2.** For the sequence \( \{Q_n \}_{n\geq 0} \), the following Mehler–Heine formula holds

\[
\lim_{n \to \infty} \frac{Q_n(x/n)}{n^2} = \lim_{n \to \infty} \frac{Q_n \left( 1 - \frac{x}{2n} \right)}{n^2} = \begin{cases} 
\phi_0(n), & \text{if } \gamma > 2(a + j + 1), \\
\phi_{j+1}(n), & \text{if } \gamma < 2(a + j + 1), \\
\end{cases}
\]

uniformly on compact subsets of \( \mathbb{C} \), where

- \( \phi_0(n) = \left( \frac{x}{2} \right)^{-a} \psi_0(x) \),
- \( \psi_{j+1}(n) = \sum_{i=0}^{j+1} b_i \left( \frac{x}{2} \right)^{-a} J_{j+2}(x) \),

with

\[
b_i = (-1)^i \frac{\Gamma(\gamma + i + 1) \sum_{k=0}^{i+1} b(k) \Gamma(i+1) \Gamma(i+\gamma+2) \Gamma(i+\gamma+2)}{\Gamma(2a+i+2) \Gamma(2a+i+1) \Gamma(2a+i+1) \Gamma(2a+i+1)}.
\]

for \( 0 \leq i \leq j + 1 \), and

- \( \psi_{j+1}(n) = \sum_{i=0}^{j+1} b_i \left( \frac{x}{2} \right)^{-a} J_{j+2}(x) \),

where the coefficients \( b_i \) are computed as

\[
b_i = (-1)^i \frac{\Gamma(\gamma + i + 1) \sum_{k=0}^{i+1} b(k) \Gamma(i+1) \Gamma(i+\gamma+2) \Gamma(i+\gamma+2)}{\Gamma(2a+i+2) \Gamma(2a+i+1) \Gamma(2a+i+1) \Gamma(2a+i+1)}, \quad 0 \leq i \leq j + 1.
\]

Notice that in last two cases the coefficient \( b_j \) is computed using the corresponding formula assuming \( \sum_{i=0}^{j-1} b_i = 0 \).

**Proof.** Scaling and taking limits in (12)

\[
\lim_{n \to \infty} \frac{Q_n \left( 1 - \frac{x}{2n} \right)}{n^2} = \lim_{n \to \infty} \frac{1}{n^2} \sum_{i=0}^{j+1} b_i \left( 1 - \frac{x}{2n} \right)^i \binom{\gamma + i + 1}{i} \binom{2a + i + 1}{i+1} \left( 1 - \frac{x}{2n} \right)
\]

\[
= \sum_{i=0}^{j+1} b_i \left( 1 - \frac{x}{2n} \right)^i \binom{\gamma + i + 1}{i} \binom{2a + i + 1}{i+1} \left( 1 - \frac{x}{2n} \right)
\]

\[
= \sum_{i=0}^{j+1} b_i \left( \frac{x}{2} \right)^{-a} J_{j+2}(x) \)

uniformly on compact subsets of \( \mathbb{C} \). Notice that in the last inequality we have used Theorem 1 written in the following way

\[
\lim_{n \to \infty} \binom{\gamma + i + 1}{i} \binom{2a + i + 1}{i+1} \left( 1 - \frac{x}{2n} \right) = z \left( \frac{x}{2} \right)^{-a} J_{j+2}(x),
\]

uniformly on compact subsets of \( \mathbb{C} \), where \( j \) is a fixed nonnegative integer number.
Now, we distinguish three cases according to the value of the parameter $\gamma$.

- If $\gamma > 2(\alpha + 1)$, we are going to prove that $b_0 = 1$ and $b_i = 0$ if $i \in \{1, 2, \ldots, j + 1\}$.
- If $\gamma = 2(\alpha + 1)$, we can compute $b_0$ from (13). If $k = 0$, then
  \[ Q_{\alpha, 0}\left( k \frac{\pi}{n} \right) = b_0(n)A_0(0, n) \]
  Taking limits when $n \to \infty$, we obtain $b_0 = 1$. If $k = 1$, then according to Lemma 1 we have
  \[ Q_{\alpha, 1}\left( k \frac{\pi}{n} \right) = b_1(n)A_1(1, n) - b_2(n)A_1(1, n) \]
  Taking limits, we have
  \[ 1 = 1 - b_1(1, \omega) \quad \text{then} \quad b_1 = 0. \]
- Applying a recursive procedure we get $b_i = 0$ for $i \in \{1, 2, \ldots, j + 1\}$. To illustrate this procedure we consider the case $\gamma < 2(\alpha + 1)$. Thus, we have $b_0 = 0$ for $i \in \{1, 2, \ldots, j\}$. Then,
  \[ Q_{\alpha, j+1}\left( k \frac{\pi}{n} \right) = b_j(n)A_j(1, n) + \sum_{i=0}^{j} b_i(n)\left( j + 1 \right)^{-1/2}A_i(j + 1, n) \]
  Taking limits, we have
  \[ 1 = 1 + b_{j+1}(-1)^{j+1}(j + 1)A_{j+1}(j + 1, \omega), \]
  then $b_{j+1} = 0$.

We can deduce the coefficients $b_i$ in a recursive way from (13).

For $i \leq 1$, we use Lemma 1 and take limits. Thus, we deduce the coefficients $b_i$ in a recursive way from (13).

Next, we are going to study the zeros of the polynomials $Q_{\alpha, n}\omega$ orthogonal with respect to $\gamma$. The following result was established for the non-varying case within a more general framework by A. G. Meijer in [11, Th. 4.1] (see also [12, Lemma 2]). Actually, that proof can be written in the same way for the varying case, so we omit it.

**Proposition 2.** The polynomial $Q_{\alpha, n}(x)$, $n \geq 1$, has $n$ real and simple zeros and at most one of them is located outside the interval $[-1, 1]$.

We can give more information about the location of the zeros. The case $j = 0$ was considered in [7]. We notice that in that case all the zeros are in the interval $(-1, 1)$. Thus, next we will assume $j > 0$ and we will denote by $y_i \geq y_{i+1} > \cdots > y_{n-k+1} > y_{n}$ the zeros of $Q_{\alpha, n}(x)$.

**Proposition 3.** For $n$ large enough and $j > 0$, we have

- If $\gamma > 2(\alpha + 2j + 1)$, then all zeros of $Q_{\alpha, n}(x)$ are located in $(-1, 1)$.
- If $\gamma < 2(\alpha + 2j + 1)$, then $y_{n+1} > 1$. 
- If $\gamma = 2(\alpha + 2j + 1)$, then $y_{n+1} = 1$ if and only if
  \[ M > \frac{\gamma}{2(\gamma + 2j)(\gamma + 2j + 1)(\gamma + 2j + 1)} \]

**Proof.** We distinguish three cases, but essentially we use Lemma 1 (a) with $k = 0$, and the fact that the leading coefficient of $Q_{\alpha, n}$ is positive. Then,

- If $\gamma > 2(\alpha + 2j + 1)$, then by Lemma 1 $Q_{\alpha, 1}(1) > 0$ for $n$ large enough. Therefore, taking into account Proposition 2, all the zeros are located in $(-1, 1)$.
- If $\gamma < 2(\alpha + 2j + 1)$, then $Q_{\alpha, 1}(1) < 0$ for $n$ large enough, which implies that there is a zero of $Q_{\alpha, n}$ greater than 1 and by Proposition 2 it is the only one.
• If \( y = 2(\alpha + 2) + 1 \), then \( y_{j,1} > 1 \) if and only if \( Q_m(1) < 0 \) for large enough, and this only happens if and only if

\[
M > \frac{2^{\alpha+2} + 2(\alpha + j + 1)(\alpha + 2) + 1}{j} \]

Now we deduce the asymptotic behavior of the zeros of \( Q_m(x) \).

**Proposition 4.** Let \( y_{1,1} > y_{1,2} > \cdots > y_{n,n} = y_{n,n-1} \) be the zeros of \( Q_m(x) \) and \( \varphi_m(x) \), \( \psi_m(x) \), and \( \varphi_{m,j}(x) \) the functions defined in Theorem 2. We assume \( j > 0 \).

1. If \( y > 2(\alpha + 2) + 1 \), then

\[
\lim_{n \to \infty} n^{\alpha + 2}(1 - y_{j,n}) = \mu_{j,n}, \quad i \geq 1,
\]

where \( \mu_{j,n} \) denotes the \( j \)th positive zero of the Bessel function of the first kind.

2. If \( y < 2(\alpha + 2) + 1 \), then

\[
\lim_{n \to \infty} n^{\alpha + 2}(1 - y_{j,n}) = \nu_{j,n}, \quad i \geq 2,
\]

where \( \nu_{j,n} \) denotes the \( j \)th positive zero of the function \( \varphi_{m,j}(x) \).

3. If \( y = 2(\alpha + 2) + 1 \), we have two cases:

   a) If \( M \leq 2^{\alpha+2} + 2(\alpha + 1)(\alpha + 2) + 1 \), then \( y_{n,1} \leq 1 \), for large enough, and

\[
\lim_{n \to \infty} n^{\alpha + 2}(1 - y_{j,n}) = l_{j,n}, \quad i \geq 1,
\]

where \( l_{j,n} \) denotes the \( j \)th positive zero of the function \( \varphi_{m,j}(x) \).

b) If \( M > 2^{\alpha+2} + 2(\alpha + 1)(\alpha + 2) + 1 \), then

\[
\lim_{n \to \infty} n^{\alpha + 2}(1 - y_{j,n}) = l_{j,n}, \quad i \geq 2,
\]

where \( l_{j,n} \) denotes the \( j \)th positive zero of the function \( \varphi_{m,j}(x) \).

**Proof.** It follows from Theorem 2, Proposition 3, and Hurwitz’s Theorem (see [8, Th. 1.91.3]).

To illustrate Theorem 2 we are going to recover the case \( j = 0 \) obtained in [7]. In that paper the author uses monic polynomials, and here we are considering a different normalization, i.e. the leading coefficient of \( Q_m \) is

\[
\frac{\Gamma(2n + \alpha + \beta + 1)}{\Gamma(2n + \alpha + 1)}.
\]

Therefore, it is necessary to do some easy computations. We use the relations (see, [9, f 10.6.1], [11, 6.1.18])

\[
\ell_{m}(x) = \frac{2(\alpha + 1)}{x} \ell_{m+1}(x) = -\ell_{m-2}(x), \quad (15)
\]

as well as

\[
\Gamma(2n) = \Gamma(n + \frac{1}{2})\sqrt{\pi}, \quad (16)
\]

First, using (7) and (16) we get

\[
\frac{\Gamma'(2n + \alpha + \beta + 1)}{\Gamma(2n + \alpha + 1)\Gamma(n + \alpha + \beta + 1)} = \frac{\sqrt{\pi}}{2^{\alpha+\beta}n}\Gamma(n + \frac{1}{2})\Gamma\left(n + \frac{1}{2} + \frac{1}{4}\right)\Gamma\left(n + \frac{1}{2} + \frac{1}{4} + \frac{1}{4}\right)\Gamma\left(n + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}\right)
\]

In [7] it was obtained

\[
\lim_{x \to \infty} \frac{2^{\alpha+\beta+1} \cos(x/n)}{n^2 + x^2} = \begin{cases} 
-2^{\alpha+\beta+1} \sqrt{n^2 x^2 + 1}, & \text{if } y < 2\alpha + 2,
-2^{\alpha+\beta+1} \sqrt{n^2 x^2 + a_{\alpha,\beta} x^2 + 1}, & \text{if } y = 2\alpha + 2,
2^{\alpha+\beta} \sqrt{n^2 x^2 + 1}, & \text{if } y > 2\alpha + 2,
\end{cases}
\]

where

\[
a_{\alpha,\beta} = \frac{-2M(\alpha + 1)}{M + 2^{\alpha+\beta+1}\Gamma(\alpha + 2)\Gamma(\alpha + 1)}.
\]
such as it is described in Proposition 4. We have computed the four largest zeros of the polynomials $Q_4(x)$ illustrated in Tables 1 and 2.

We can observe that

$$\lim_{n \to \infty} \frac{2^{n+1+2}p_n(x,\beta,M)}{n^{1/4} \sqrt{\pi}} = \begin{cases} -2^n x^2 \beta_{n+2}(x), & \text{if } \gamma < 2a + 2, \\ -2^n (\beta_{n+2}(x) + a_n x \beta_{n+1}(x)), & \text{if } \gamma = 2a + 2, \\ 2^n \beta_{n+2}(x), & \text{if } \gamma > 2a + 2. \end{cases} \quad (17)$$

Therefore, it only remains to compare the limit functions in (14) and (17). The case $\gamma > 2a + 2$ is trivial. We pay attention to the other two cases.

- $\gamma < 2a + 2$.
  In this case $b_0 = 0$ and $b_1 = -1/2$. Thus we have

  $$\psi_0(x) = -x^{\gamma/2} J_{\gamma+2}(x) = -2^\gamma x^{\gamma/2} \beta_{\gamma+2}(x),$$

- $\gamma = 2a + 2$.
  In this case,

  $$b_0 = -\frac{F^2(a + 1)^2x^{2\gamma+1}(a + 1)}{M + F^2(a + 1)2^{2\gamma+1}(a + 1)},$$

  $$b_1 = \frac{2(M + F^2(a + 1)2^{2\gamma+1}(a + 1))^{-1}}{M}.$$

By using (15) we deduce

$$\psi_0(x) = b_0 \left(\frac{x}{2}\right)^{-\gamma} J_\gamma(x) + 2b_1 \left(\frac{x}{2}\right)^{-\gamma} J_{\gamma+2}(x),$$

$$= \frac{F^2(a + 1)^22^{2\gamma+1}x^{2\gamma+1}(a + 1)}{M + F^2(a + 1)2^{2\gamma+1}(a + 1)} \left(\frac{x}{2}\right)^{-\gamma} J_\gamma(x),$$

$$+ \frac{2(M + F^2(a + 1)2^{2\gamma+1}(a + 1))^{-1}}{M} \left(\frac{x}{2}\right)^{-\gamma} J_{\gamma+2}(x),$$

$$= -\left(\frac{x}{2}\right)^{-\gamma} J_\gamma(x) + \frac{2M + 2^{2\gamma+1}F^2(a + 1)(a + 1)}{M} \left(\frac{x}{2}\right)^{-\gamma} J_{\gamma+2}(x),$$

$$= -2^n (\beta_{n+2}(x) + a_n x \beta_{n+1}(x)).$$

4. Numerical experiments

In this section we illustrate the previous results on the zeros of the polynomials $Q_n(x)$ with some numerical experiments where we have taken $j = 3$ for all of them. Thus, we are dealing with the varying Sobolev inner product

$$\langle f, g \rangle_s = \int_1^1 f(x)g(x)(1 - x^2)^{1/2} (1 + x^2)^{1/2} dx + M_3 f^{(3)}(1) g^{(3)}(1).$$

We have used the mathematical software Mathematica® 8.0 for the computations. In all the numerical experiments we have computed the four largest zeros of the polynomials $Q_n(x)$ and the corresponding scaled zeros for several values of $n$. We only show one example for each possible case. In the tables about the scaled zeros we show their asymptotic behavior such as it is described in Proposition 4.

- Case $\gamma > 2(a + 2 + 1)$.
  We choose the following values:

  $$a = 3, \quad \beta = 1, \quad \gamma = 25, \quad \text{and } M_3 = \frac{3e^\beta}{(6e^\beta + 4)^{3/2}}.$$
and we note by $V$ the quantity which appears in Proposition 4, i.e.

$$V = 2^{a+iβ} - 2^{-iβ} (a+j+1)(a+j+2)j^2 (a+j+1)$$

Thus, with this data

$$V = 2^{1/1} \frac{15128}{75} \frac{31}{10} = 1119.0037947.$$
According to Proposition 4 we have two possible choices of $M$ which determine two different asymptotic behaviors of the zeros. In Tables 5 and 6 we show the case $M < n$ where $M = 5$. We can see that the largest zero of $Q_n$ is always less than 1. However, when $M > n$ then $x_{n+1} > 1$ for $n$ large enough and this is illustrated in Table 7 for $M = 10^8$. In Table 8 the asymptotic behavior of the scaled zeros is shown.

Finally, we illustrate Theorem 2 plotting the curves corresponding to the limit functions and to the scaled polynomials $Q_n \left(1 - \frac{x}{2M}\right)$ with $n = 150$ and $n = 500$. In all the figures we have used the same values for the parameters as those ones taken previously in the numerical experiments about the zeros (see Figs. 1–4).

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References

[1] M. Almansa, A. Martínez-Finkelshtein, M.L. Rebolet, Asymptotics properties of balanced extremal Sobolev polynomials: common case, J. Approx. Theory 100 (2000) 44-59.
[2] M. Almansa, J.J. Moreno-Balzózar, A. Peña, M.L. Rebolet, Sobolev orthogonal polynomials: Balance and asymptotics, Trans. Amer. Math. Soc. 361 (2009) 547-590.
[3] F. Marcellán, V. Xu, On Sobolev orthogonal polynomials, Expo. Math. 31 (2013) 308-352.
[4] J.J. Moreno-Balzózar, F. Marcellán, J.J. Moreno-Balzózar, Varying discrete Laguerre–Sobolev orthogonal polynomials: asymptotic behavior and zeros, Appl. Math. Comput. 227 (2013) 81-94.
[5] A. Peña, M.L. Rebolet, Discrete-Laguerre–Sobolev expansions: A CFTP type inequality, J. Math. Anal. Appl. 385 (2012) 254-263.
[6] A. Peña, M.L. Rebolet, Correction formulas for general discrete Sobolev polynomials. Mehler–Fejer asymptotics, Appl. Math. Comput. 261 (2015) 272-285.
[7] J.J. Moreno-Balzózar, Varying Jacobi–Koornwinder orthogonal polynomials: local asymptotic behavior and zeros, Ramanujan J. 28 (2012) 79-88.
[8] G. Szegő, Orthogonal Polynomials, fourth ed., in: Amer. Math. Soc. Colloq. Publ., vol. 23. Amer. Math. Soc., Providence, RI, 1975.
[9] R.A. Askey, R. Roy, Basic hypergeometric functions, in: NIST Handbook of Mathematical Functions, Cambridge University Press, Cambridge, UK, 2010, pp. 135-147.
[10] F. Marcellán, A. Ronveaux, On a class of orthogonal polynomials with respect to a discrete Sobolev inner product, Indag. Math. (N.S.) 11 (4) (1999) 497-508.
[11] T.T. Eby, Zero distribution of orthogonal polynomials in a certain discrete Sobolev space, J. Math. Anal. Appl. 372 (2010) 529-532.
[12] M. Almansa, G. López, M.L. Rebolet, Some properties of zeros of Sobolev-type orthogonal polynomials, J. Comput. Appl. Math. 69 (1996) 171-179.
[13] M. Abramowitz, I. Stegun, Handbook of Mathematical Functions, Dover, New York, 1972.