ON THE GENERALIZED SPRINGER CORRESPONDENCE

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INTRODUCTION

0.1. Let $k$ be an algebraically closed field of characteristic $p \geq 0$. Let $G$ be a connected reductive group over $k$. Let $\mathfrak{A}$ be the set of all pairs $(c, \mathcal{E})$ where $c$ is a unipotent class in $G$ and $\mathcal{E}$ is an irreducible $G$-equivariant local system on $c$ defined up to isomorphism; let $\mathfrak{A}$ be the set of triples $(P, C, S_0)$ (up to $G$-conjugacy) where $P$ is a parabolic subgroup of $G$, $C$ is a unipotent class of the reductive quotient $\bar{P}$ of $P$ and $S_0$ is an irreducible $\bar{P}$-equivariant cuspidal local system on $C$. According to [L1, 6.5], there is a canonical surjective map $\mathfrak{A} \to \mathfrak{A}$ (whose fibres are called blocks) such that the block corresponding to $(P, C, S_0) \in \mathfrak{A}$ is in natural bijection ("generalized Springer correspondence") with the set $\text{Irr}_W$ of irreducible representations (up to isomorphism) of the finite group $W := NL/L$ where $NL$ is the normalizer in $G$ of a Levi subgroup $L$ of $P$. (One can show that $W$ is naturally a Weyl group depending on the block.) The case considered originally by Springer, see [Sp], with some restrictions on $p$, involves the block corresponding to $(B, \{1\}, Q_0)$ where $B$ is a Borel subgroup of $G$. The problem of determining explicitly the generalized Springer correspondence can be reduced to the case where $G$ is almost simple, simply connected. For such $G$ the explicit bijection was determined in [L1], [S2] and the references there, for any block except for

(a) two blocks for $G$ of type $E_6$ with $p \neq 3$, with $W$ of type $G_2$ and

(b) two blocks for $G$ of type $E_8$ with $p = 3$, with $W$ of type $G_2$;

for these blocks the method of [S2] (based mostly on the restriction theorem [L1, 8.3]) had the following gap: it gave the explicit bijection only up to composition with a permutation of $\text{Irr}W$ which interchanges the two 2-dimensional irreducible representations of $W$ and keeps fixed all the other irreducible representations.

0.2. Let $(c, \mathcal{E}) \in \mathfrak{A}$ and let $(c', \mathcal{E}') \in \mathfrak{A}$ be such that $c'$ is contained in the closure $\bar{c}$ of $c$. For an integer $k$ let $m_k^{(c', \mathcal{E}'), (c, \mathcal{E})}$ be the multiplicity of $\mathcal{E}'$ in the local system on $c'$ obtained by restricting to $c'$ the $k$-th cohomology sheaf of the intersection

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cohomology complex $IC(\bar{c}, \mathcal{E})$. Let

$$m_{(c', \mathcal{E}'), (c, \mathcal{E})} = \sum_{k \geq 0} m_{(c', \mathcal{E}'), (c, \mathcal{E})} q^{k/2} \in \mathbb{N}[q^{1/2}]$$

where $q^{1/2}$ is an indeterminate. In [L2, 24.8] it is shown (assuming that $p$ is not a bad prime for $G$) that $m_{(c', \mathcal{E}'), (c, \mathcal{E})} \in \mathbb{N}[q]$, that $m_{(c', \mathcal{E}'), (c, \mathcal{E})} = 0$ if $(c', \mathcal{E}')$, $(c, \mathcal{E})$ are not in the same block and that $m_{(c', \mathcal{E}'), (c, \mathcal{E})}$ is explicitly computable in terms of the generalized Springer correspondence if $(c', \mathcal{E}')$, $(c, \mathcal{E})$ are in the same block.

0.3. In this subsection we consider a block of $G$ as in 0.1(a),(b) where $p$ not a bad prime for $G$; thus we must be in case 0.1(a) and $p \notin \{2, 3\}$. An attempt to close the gap in this case was made in [L2, 24.10], based on computing the polynomials $m_{(c', \mathcal{E}'), (c, \mathcal{E})}$ (see 0.2) for $(c', \mathcal{E}')$, $(c, \mathcal{E})$ in this block. Unfortunately, the attempt in [L2] contained a calculation error. (I thank Frank Lübeck for pointing this out to me.) As a result, the gap remained. In this paper we refine the analysis in [L2, 24.10] and close the gap for the blocks in 0.1(a) with $p \notin \{2, 3\}$ (see Theorem 5.5). We now describe our strategy. Using the algorithm of [L2, 24.10] (based on the two possible scenarios corresponding to the two possible inputs for the generalized Springer correspondence) we compute the polynomials $m_{(c', \mathcal{E}'), (c, \mathcal{E})}$ for $(c', \mathcal{E}')$, $(c, \mathcal{E})$ in the block. We get two sets of polynomials one in each of the two scenarios. In fact the algorithm gives only rational functions in $q$ which, by a miracle, turn out to be in $\mathbb{N}[q]$ in both scenarios. So by this computation one cannot rule out one of the scenarios and a further argument is needed. Let $u \in G$ be a unipotent element of $G$ such that the Springer fibre at $u$ is 3-dimensional (such $u$ is unique up to conjugacy) and let $X_u$ be the “generalized Springer fibre” at $u$ (attached to the block). It is known from [L1] that $\mathcal{W}$ acts naturally on $H^j_c(X_u, ?)$ where ? is a suitable local system on $X_u$ attached to the block. By a known result, from the knowledge of the polynomials $m_{(c', \mathcal{E}'), (c, \mathcal{E})}$ one can extract the trace of a simple reflection of $\mathcal{W}$ on $\sum_j (-1)^j H^j_c(X_u, ?)$. It turns out that this trace is 1 in one scenario and $-1$ in the other scenario. We show that this trace is equal to the Euler characteristic of a certain explicit open subvariety of the Springer fibre at $u$. We then try to compute from first principles this Euler characteristic; the computation occupies most of the paper. The computation does not give the exact value of the Euler characteristic; it only shows that it is one of the numbers 0, 1, 2. Since we already know that it is 1 or $-1$ we deduce that it is 1; moreover, this determines which scenario is real and which one is not.

0.4. Note that Theorem 5.5 completes the explicit determination of the generalized Springer correspondence for any $G$ and any block, assuming that $p$ is not a bad prime for $G$.

In the case where $p$ is a bad prime for $G$, a gap remains, and in 6.2 we state a conjecture about it.

0.5. Notation. All algebraic varieties are assumed to be over $k$ and all algebraic groups are assumed to be affine.
For any connected algebraic group $H$ let $U_H$ be the unipotent radical of $H$ and let $\tilde{H} = H/U_H$, a connected reductive group; let $\pi_H : H \to \tilde{H}$ be the obvious homomorphism. Let $\bar{\mathcal{B}}^H$ be the connected centre of $\tilde{H}$. Let $\mathcal{B}^H$ (resp. $\bar{\mathcal{B}}^H$) be the variety of Borel subgroups of $H$ (resp. $\tilde{H}$). Note that $\beta \mapsto \pi_H^{-1}(\beta)$ is an isomorphism $\mathcal{B}^H \xrightarrow{\sim} \bar{\mathcal{B}}^H$. Now $H$ (resp. $\tilde{H}$) acts by simultaneous conjugation on $\mathcal{B}^H \times \mathcal{B}^H$ (resp. $\bar{\mathcal{B}}^H \times \bar{\mathcal{B}}^H$); the orbits of this action are naturally parametrized by the Weyl group $W^H$ (resp. $\bar{W}^H$) of $H$ (resp. $\tilde{H}$). The identification $\mathcal{B}^H \times \mathcal{B}^H \leftrightarrow \mathcal{B}^H \times \mathcal{B}^H$ via $\pi_H \times \pi_H$, induces an identification $W^H = \bar{W}^H$.

If $H$ is a connected reductive group we denote by $H_{ad}$ the adjoint group of $H$. In this paper $G$ is a fixed connected reductive group. We write $\mathcal{B} = \mathcal{B}^G$, $W = W^G$. Let $\mathcal{O}_w$ be the $G$-orbit on $\mathcal{B} \times \mathcal{B}$ indexed by $w \in W$. The simple reflections in $W$ are denoted as $\{s_i; i \in I\}$ where $I$ is an indexing set. For any $J \subset I$ let $W_J$ be the subgroup of $W$ generated by $\{s_i; i \in J\}$. Let $\epsilon_J : W_J \to \{\pm 1\}$ be the homomorphism given by $\epsilon_J(s_i) = -1$ for all $i \in J$. The set $\mathcal{P}$ of parabolic subgroups of $G$ is naturally partitioned as $\mathcal{P} = \sqcup_{J \subset I} \mathcal{P}_J$, where for $J \subset I$, $\mathcal{P}_J$ consists of parabolic subgroups $P$ with the following property: for $w \in W$, we have $(B, B') \in \mathcal{O}_w$ for some some $B, B'$ in $\mathcal{B}$, $B \subset P$, $B' \subset P$ if and only if $w \in W_J$. Note that $\mathcal{B} = \mathcal{P}_\emptyset$.

Now let $J \subset I$ and let $P \in \mathcal{P}_J$. Then $B^P$ is equal to the closed subvariety $\{B \in \mathcal{B}; B \subset P\}$ of $\mathcal{B}$ and the obvious map $W^P \to W$ identifies $W^P$ with the subgroup $W_J$ of $W$. The set of simple reflections of $W^P = W^P$ becomes the set $\{s_i; i \in J\}$.

For $i \in I$ we write $\mathcal{P}_i$ instead of $\mathcal{P}_{\{i\}}$ and $\mathcal{P}^i$ instead of $\mathcal{P}_{I - \{i\}}$, a class of maximal parabolic subgroups. For any $B \in \mathcal{B}$ there is a unique $P \in \mathcal{P}^i$ such that $B \subset P$; we set $P = B(i)$. A subvariety of $\mathcal{B}$ is said to be an $i$-line if it is of the form $B^P$ for some $P \in \mathcal{P}_i$ (which is necessarily unique). If $P, P'$ are parabolic subgroups of $G$ we write $P \blacklozenge P'$ whenever $P, P'$ contain a common Borel subgroup.

Let $l$ be a fixed prime number such that $l \neq p$. All local systems are assumed to be $\mathbb{Q}_l$-local systems.

## Contents

1. Preliminaries.
2. A trace computation.
3. Computations in certain groups of semisimple rank $\leq 5$.
4. Euler characteristic computations.
5. The main result.
6. Final comments.

### 1. Preliminaries

1.1. We fix a unipotent element $u \in G$; let $\mathcal{B}_u = \{B \in \mathcal{B}; u \in B\}$. This is a nonempty subvariety of $\mathcal{B}$. Let $\mathcal{B}_u$ be the set of irreducible components of $\mathcal{B}_u$. According to Spaltenstein [S1], for $X \in \mathcal{B}_u$, $d_u := \dim X$ depends only on $u$, not
on $X$. For any $X \in \mathcal{B}_u$ let $J_X$ be the set of all $i \in I$ such that $X$ is a union of $i$-lines. We show:

(a) Let $J = J_X$, let $B \in X$ and let $P$ be the unique subgroup in $P_J$ such that $B \subset P$. Then $B^P \subset X$.

Let $B' \in X$. We can find a sequence $B = B_0, B_1, \ldots, B_t = B'$ in $B$ such that for $k = 0, 1, \ldots, t - 1$ we have $(B_k, B_{k+1}) \in \mathcal{O}_{\ast t}$ where $t_k \in J$. We show by induction on $k$ that $B_k \in X$. For $k = 0$ this holds by assumption. Assume now that $k \geq 1$. By the induction hypothesis we have $B_{k-1} \in X$. Since $i_k \in J_X$, the $i_k$-line containing $B_{k-1}$ is contained in $X$. Since $B_k$ is contained in this $i_k$-line we have $B_k \in X$. This completes the induction. We see that $B' \in X$. This proves (a).

We show:

(b) In the setup of (a), assume in addition that $d_u$ is equal to $\nu_J$, the number of reflections in $W_J$. Then $B^P = X$.

Note that $B^P$ is a closed irreducible subvariety of dimension $\nu_J$ of $X$ and $X$ is irreducible of dimension $d_u$. The result follows.

1.2. In this subsection we assume that $G_{ad} = \prod_{i \in I} G_i$ where $G_i \cong PGL_2(k)$ for all $i \in I$. Assume that $B \in \mathcal{B}_u$ is such that for any $i \in I$, the $i$-line through $B$ is contained in $\mathcal{B}_u$. We show that $u = 1$. We write $u = (u_i)$ where $u_i \in G_i$ for all $i$. Let $K = \{i \in I; u_i = 1\}$. Then $\mathcal{B}_u$ is a product of copies of $\mathbb{P}^1$ indexed by $K$. Moreover, $\mathcal{B}_u$ contains an $i$-line if and only if $i \in K$. Our assumption implies that $K = I$ hence $u = 1$, as asserted.

1.3. We shall need the following variant of 1.1(a),(b).

(a) Let $J$ be a subset of $I$ such that $s_j s_{j'} = s_{j'} s_j$ for all $j, j' \in J$. Let $B \in \mathcal{B}_u$ be such that for any $j \in J$ the $j$-line containing $B$ is contained in $\mathcal{B}_u$. Let $P$ be the unique subgroup in $P_J$ such that $B \subset P$. Assume that $\sharp(J) = d_u$. Then $X = B^P$ is an irreducible component of $\mathcal{B}_u$ and $J_X = J$.

We apply 1.2 with $G, u$ replaced by $\tilde{P}, \pi_P(u)$. Note that $\tilde{P}_{ad}$ is a product of copies of $PGL_2(k)$. We see that $\pi_P(u) = 1$ that is, $u \in U_P$. Then any Borel subgroup of $P$ contains $u$. Now (a) follows.

1.4. Let $[\mathcal{B}_u]$ be the $\mathbb{Q}$-vector space with basis $\{X; X \in \mathcal{B}_u\}$. Let $\rho_u$ be the Springer representation of $W$ on the vector space $[\mathcal{B}_u]$. The following property of $\rho_u$ appeared in a letter of the author to Springer (March 1978), see also [Ho]:

(a) Let $i \in I$ and let $X \in \mathcal{B}_u$. Then

\[ s_i X = -X \quad \text{if} \ i \in J_X; \]
\[ s_i X = \sum_{X' \in \mathcal{B}_u; i \in J_X} Z X' \quad \text{if} \ i \notin J_X. \]

For $J \subset I$ let $[\mathcal{B}_u]|_J = \{X \in \mathcal{B}_u; J \subset J_X\}$ and let $[\mathcal{B}_u]|_J$ be the subspace of $[\mathcal{B}_u]$ spanned by $([\mathcal{B}_u]|_J)$. We have $([\mathcal{B}_u]|_J) = \cap_{i \in J} ([\mathcal{B}_u]|_i)$ and $([\mathcal{B}_u]|_J) = \cap_{i \in J} ([\mathcal{B}_u]|_i)$. From (a) we see that for any $i \in J$, we have $([\mathcal{B}_u]|_i) = \{v \in [\mathcal{B}_u]; s_i v = -v\}$.

Hence $([\mathcal{B}_u]|_J) = \cap_{i \in J} \{v \in [\mathcal{B}_u]; s_i v = -v\} = \{v \in [\mathcal{B}_u]; wv = wv = wv \in W_J v\}$. Thus, $\sharp([\mathcal{B}_u]|_J)$ is equal to the multiplicity $(\epsilon_J : \rho_u|_{W_J})$ of $\epsilon_J$ in the $W_J$-module
\( \rho_u|_{W_J}. \) Let \( (\mathcal{B}_u)_J = \{ X \in \mathcal{B}_u; J = J_X \} \) We have \( \mathcal{B}_u|_{J'} = \sum_{J' : J \subseteq J'} \mathcal{B}_u|_{J'} \). Hence \( \mathcal{B}_u|_I = \sum_{J' : J \subseteq J'} (-1)^{|J| - |J'|} \mathcal{B}_u|_{J'} \) so that \( \mathcal{B}_u|_J = \sum_{J' : J \subseteq J'} (-1)^{|J| - |J'|} (\epsilon_{J'} : \rho_u|_{W_J'}). \)

1.5. In this subsection we assume that \( J \subset I \), \( P \in \mathcal{P}_J \) and \( \tilde{u} \in \tilde{P} \) such that \( u \in P \), \( \tilde{u} = \pi_P(u) \) and such that, if \( C \) is the conjugacy class of \( u \) in \( G \) then

(a) \( C \cap \pi_P^{-1}(\tilde{u}) \) is open dense in \( \pi_P^{-1}(\tilde{u}) \).

Then \( C \) is induced (in the sense of [LS]) by the \( \tilde{P} \)-conjugacy class of \( \tilde{u} \).

For any irreducible component \( \xi \) of \( \mathcal{B}_u^\tilde{P} \), the subset \( J_\xi \) of \( J \) is defined as in 1.1 (replacing \( G, I, u \) by \( \tilde{P}, J, \tilde{u} \)). Let \( \tilde{\xi} \) be the image of \( \xi \) under \( \beta \mapsto \pi_P^{-1}(\beta) \); note that \( \tilde{\xi} \) is a closed irreducible subvariety of \( \mathcal{B}_u \) of dimension \( d_u \) hence by (b) is an irreducible component of \( \mathcal{B}_u \). From the definitions we see that

(c) \( J_\xi = J_\tilde{\xi} \) where \( J_\tilde{\xi} \) is defined as in 1.1.

1.6. Let \( i \in I \). Let \( B_{u,i} \) be the set of all \( B \in X \) such that the \( i \)-line through \( B \) is contained in \( \mathcal{B}_u \) or equivalently such that if \( P_i \in \mathcal{P}_i \) is defined by \( B \subset P_i \) then \( u \in U_{P_i} \).

For \( X \in \mathcal{B}_u \) let \( X_i = X \cap B_{u,i} \).

2. A trace computation

2.1. We fix \( J \subset I \) such that \( J \neq I \) and \( P_J \in \mathcal{P}_J \). We write \( \pi_J \) instead of \( \pi_{P_J} \). We fix a unipotent conjugacy class \( C \) of \( \tilde{P}_J \) and let \( S_0 \) be an irreducible cuspidal \( \tilde{P}_J \)-equivariant local system on \( C \). Let \( \mathcal{S} = C \mathcal{Z}_{P_J}^0 \), a locally closed subvariety of \( \tilde{P}_J \); let \( cl(\mathcal{S}) \) be the closure of \( \mathcal{S} \) in \( \tilde{P}_J \). Let \( \mathcal{S} \) be the inverse image of \( S_0 \) under \( \mathcal{S} \to C \) (taking unipotent part). Let \( \mathcal{W} = N_W W_J/W_J \) where \( N_W W_J \) is the normalizer of \( W_J \) in \( W \). For any \( i \in I - J \) let \( N_{W_{Jui}} W_J \) be the normalizer of \( W_J \) in \( W_{Jui} \). From [L1, 9.2] we see that \( N_{W_{Jui}} W_J/W_J \) has order 2 and that \( \mathcal{W} \) is a Coxeter group with simple reflections \( \{ \sigma_i; i \in I - J \} \) where \( \sigma_i \) is the unique nonidentity element of \( N_{W_{Jui}} W_J/W_J \), viewed as an element of \( \mathcal{W} \).

Let \( \bar{X} = \{ (xP_J, g) \in G/P_J \times G; x^{-1}gx \in P_J, \pi_J(x^{-1}gx) \in \mathcal{S} \}. \)

Let \( \bar{Y} \) be the set of all \( g \in G \) such that for some \( xP_J \in G/P_J \) we have \( x^{-1}gx \in P_J, \pi_J(x^{-1}gx) \in cl(\mathcal{S}); \) this is a closed subset of \( G \). Define \( \bar{\pi} : \bar{X} \to \bar{Y} \) by \( \bar{\pi}(xP_J, g) = g \). We define a local system \( \mathcal{S} \) on \( \bar{X} \) by requiring that \( \mathcal{S}(xP_J, g) = S_{\pi_J(x^{-1}gx)} \). Note that \( \mathcal{S} \) is well defined by the \( \tilde{P}_J \)-equivariance of \( S_0 \). Let \( K = \bar{\pi}_! \mathcal{S} \in \mathcal{D}(\bar{Y}) \). From [L1, 4.5, 9.2] we see that \( K \) is an intersection cohomology complex on \( \bar{Y} \) and that we have canonically \( \text{End}(K) = Q_l[\mathcal{W}] \). Let \( g \in \bar{Y} \) and let

\( X_g = \{ xP_J \in G/P_J; x^{-1}gx \in P_J, \pi_J(x^{-1}gx) \in \mathcal{S} \}. \)
Now $X_g$ is a subvariety of $X$ via $xP_J \mapsto (xP_J, g)$. We denote the restriction of $\hat{\mathcal{S}}$ from $X$ to $X_g$ again by $\hat{\mathcal{S}}$. Since $H^j_c(X_g, \hat{\mathcal{S}})$ (for $j \in \mathbb{Z}$) is a stalk of a cohomology sheaf of $K$ at $g$, we see that $H^j_c(X_g, \hat{\mathcal{S}})$ is naturally a $\mathcal{W}$-module.

We now fix a subset $J'$ of $I$ such that $J \subset J'$. We define $P_{J'} \in \mathcal{P}_{J'}$ by $P_J \subset P_{J'}$. We write $U_{J'}, \pi_{J'}$ instead of $U_{P_{J'}}, \pi_{P_{J'}}$. We shall give an alternative description of the restriction of the $\mathcal{W}$-module $H^j_c(X_g, \hat{\mathcal{S}})$ to the subgroup $\mathcal{W}'$ of $\mathcal{W}$ generated by $\{\sigma_i; i \in J' - J\}.$

We have a commutative diagram of algebraic varieties with cartesian squares

\[
\begin{array}{ccc}
\tilde{M}_{J',g} & \xrightarrow{j} & \tilde{M}_{J'} \\
\pi'' \downarrow & & \pi' \downarrow \\
M_{J',g} & \xrightarrow{j} & M_{J'}
\end{array}
\]

\[
\begin{array}{ccc}
\tilde{M}_{J',g} & \xleftarrow{\omega} & \tilde{E} \times X \\
1 \times \pi' \downarrow & & \pi \downarrow \\
M_{J',g} & \xleftarrow{\omega} & E \times \tilde{Y}
\end{array}
\]

where

\[
X = \{(xP_J, zU_{J'}) \in (P_{J'}/P_J) \times (P_{J'}/U_{J'}); x^{-1}zx \in P_J, \pi_J(x^{-1}zx) \in \mathcal{G}\};
\]

$\tilde{Y}$ is the set of all $zU_{J'} \in P_{J'}/U_{J'}$ such that for some $xP_J \in P_{J'}/P_J$, we have $x^{-1}zx \in P_J, \pi_J(x^{-1}zx) \in cl(\mathcal{G})$;

$\tilde{M}_{J'}$ is the set of all pairs $(xP_J, y(xU_{J'}x^{-1}))$ where $xP_J \in G/P_J$, $y(xU_{J'}x^{-1}) \in (xP_Jx^{-1})/(xU_{J'}x^{-1})$ are such that $\pi_J(x^{-1}yx) \in \mathcal{G}$;

$M_{J'}$ is the set of all pairs $(xP_{J'}, y(xU_{J'}x^{-1}))$ where $xP_{J'} \in G/P_{J'}$, $y(xU_{J'}x^{-1}) \in (xP_{J'}x^{-1})/(xU_{J'}x^{-1})$ are such that for some $v \in P_{J'}$, we have $y \in xvP_{J'}v^{-1}x^{-1}$ and $\pi_{J'}(v^{-1}x^{-1}yxv) \in cl(\mathcal{G})$;

\[
\tilde{M}_{J',g} = \{(xP_J, y(xU_{J'}x^{-1})) \in \tilde{M}_{J'}; g \in xP_Jx^{-1}, g^{-1}y \in xU_{J'}x^{-1}\};
\]

\[
M_{J',g} = \{(xP_{J'}, y(xU_{J'}x^{-1})) \in M_{J'}; g \in xP_{J'}x^{-1}, g^{-1}y \in xU_{J'}x^{-1}\};
\]

\[
E = G/U_{J'};
\]

$j, \tilde{j}$ are the obvious imbeddings; $q, \tilde{q}$ are the obvious projections;

\[
\omega(hU_{J'}, zU_{J'}) = (hP_{J'}, (hz^{-1})(hU_{J'}h^{-1})),
\]

\[
\tilde{\omega}(hU_{J'}, xP_{J'}, zU_{J'}) = (hxP_J, (hz^{-1})(hxU_{J'}x^{-1}h^{-1}));
\]

\[
\pi(xP_J, zU_{J'}) = zU_{J'}, \pi'(xP_J, y(xU_{J'}x^{-1})) = (xP_{J'}, y(xU_{J'}x^{-1})),
\]

\[
\pi''(xP_{J'}, y(xU_{J'}x^{-1})) = (xP_{J'}, y(xU_{J'}x^{-1})).
\]

All maps in the diagram are compatible with the natural actions of $\bar{P}_{J'}$ where the action of $\bar{P}_{J'}$ on the four spaces on the left is trivial. Moreover, $\tilde{\omega}$ and $\omega$ are principal $\bar{P}_{J'}$-bundles. We define a local system $\hat{\mathcal{S}}$ on $X$ by requiring
that $\mathcal{S}_{(xP_J,zU_J)} = \mathcal{S}_{\pi_J(x-1yx)}$. We define a local system $\mathcal{S}$ on $\tilde{M}_{J'}$ by requiring that $\mathcal{S}_{(xP_J,y(xU_J,x^{-1}))} = \mathcal{S}_{\pi_J(x-1yx)}$. Note that $\mathcal{S}, \mathcal{S}$ are well defined by the $P_J$-equivariance of $\mathcal{S}_0$. Note also that $K' := \pi_I \mathcal{S}$ is like $K$ above (with $G$ replaced by $\tilde{P}_{J'}$) hence from [L1, 4.5, 9.2] we see that $K'$ is an intersection cohomology sheaf on $\tilde{Y}$ and we have canonically $\text{End}_{\mathcal{D}(\tilde{Y})}(K') = \text{End}_{\mathcal{D}_{\tilde{P}_{J'}}(\tilde{Y})}(K') = \mathbb{Q}_l[\mathcal{W}']$.

We have an isomorphism $\mathbb{X}_g \xrightarrow{\sim} \tilde{M}_{J',g}$ given by $xP_J \mapsto (xP_J, g(xU_J,x^{-1}))$, under which these two varieties are identified; then the local system $\tilde{j}^{*} \mathcal{S}$ on $\tilde{M}_{J',g}$ becomes $\mathcal{S}$. We have $\tilde{q}^{*} \mathcal{S} = \tilde{\omega}^{*} \mathcal{S}$ hence $q^{*} \pi_I \mathcal{S} = \omega^{*}(\pi_I \mathcal{S})$. The functors

$$\mathcal{D}_{\tilde{P}_{J'}}(\tilde{Y}) \xrightarrow{q^{*}} \mathcal{D}_{\tilde{P}_{J'}}(E \times \tilde{Y}) \xleftarrow{\tilde{\omega}^{*}} \mathcal{D}(M_{J'}) \xrightarrow{j^{*}} \mathcal{D}(M_{J',g})$$

induce algebra homomorphisms

$$\text{End}_{\mathcal{D}_{\tilde{P}_{J'}}(\tilde{Y})}(\pi_I \mathcal{S}) \rightarrow \text{End}_{\mathcal{D}_{\tilde{P}_{J'}}(E \times \tilde{Y})}(q^{*} \pi_I \mathcal{S}) \leftarrow \text{End}_{\mathcal{D}(M_{J'})}(\pi_I \mathcal{S})$$

of which the second one is an isomorphism since $\omega$ is a principal $\tilde{P}_{J'}$-bundle. Taking the composition of the first homomorphism with the inverse of the second one and with the third one and identifying

$$\text{End}_{\mathcal{D}_{\tilde{P}_{J'}}(\tilde{Y})}(\pi_I \mathcal{S}) = \text{End}_{\mathcal{D}(\tilde{Y})}(\pi_I \mathcal{S}) = \mathbb{Q}_l[\mathcal{W}']$$

we obtain an algebra homomorphism

$$\mathbb{Q}_l[\mathcal{W}'] \rightarrow \text{End}_{\mathcal{D}(M_{J',g})}(\pi_I \tilde{j}^{*} \mathcal{S}).$$

It follows that

$$H^j_c(\mathbb{X}_g, \mathcal{S}) = H^j_c(\tilde{M}_{J',g}, \tilde{j}^{*} \mathcal{S}) = H^j_c(M_{J',g}, \pi_I \tilde{j}^{*} \mathcal{S})$$

is naturally a module over the algebra $\text{End}_{\mathcal{D}(M_{J',g})}(\pi_I \tilde{j}^{*} \mathcal{S})$ hence a module over $\mathbb{Q}_l[\mathcal{W}]$. From the definitions we see that this $\mathcal{W}'$-module structure on $H^j_c(\mathbb{X}_g, \mathcal{S})$ coincides with restriction to $\mathcal{W}'$ of the $\mathcal{W}$-module structure on $H^j_c(\mathbb{X}_g, \mathcal{S})$ considered above.

2.2. We now assume that $i \in I - J$ and that $J' = J \cup \{i\}$. Let

$$t_i(g) = \sum_j (-1)^j \text{tr}(\sigma_i, H^j_c(\mathbb{X}_g, \mathcal{S}))$$

where $\sigma_i$ acts by the $\mathcal{W}$-action. By 2.1, we have

$$t_i(g) = \sum_j (-1)^j \text{tr}(\sigma_i, H^j_c(M_{J',g}, \pi_I \tilde{j}^{*} \mathcal{S}))$$
where $\sigma_i$ acts by the $W'$-action. The $\sigma_i$ action on $\pi_1\hat{S}$ induce an $\sigma_i$-action on $\mathcal{H}^j(\pi_1\hat{S})$ and this induces an $\sigma_i$-action on $\mathcal{H}^j(\pi_1\hat{S})$ and an $\sigma_i$-action on

$$H^j_c(M_{J',g}, \mathcal{H}^j(\pi_1\hat{S})).$$

(Here $\mathcal{H}^j()$ denotes the $j'$-th cohomology sheaf). We have a spectral sequence

$$H^j_c(M_{J',g}, \mathcal{H}^j(\pi_1\hat{S})) \Rightarrow H^{j+j'}_c(M_{J',g}, \pi_1\hat{S})$$

which is compatible with the $\sigma_i$-actions. It follows that

$$(a) \quad t_i(g) = \sum_{j,j'} (-1)^j t^j \text{tr}(\sigma_i, H^j_c(M_{J',g}, \mathcal{H}^j(\pi_1\hat{S}))).$$

We now make the further assumption that $g$ is unipotent and that $s_i$ commutes with $W_J$. In this case we have

$$X_g = \{xP_J \in G/P_J; x^{-1}gx \in P_J, \pi_J(x^{-1}gx) \in C\};$$

moreover, the isomorphism $W_{\{i\}} \times W_J \xrightarrow{\sim} W_{J'}$ induced by multiplication corresponds to a direct product decomposition $(\hat{P}_J)_{ad} = H' \times H''$ where $H' \cong PGL_2(k)$, $H'' \cong (P_J)_{ad}$.

Let $\hat{C}$ be the image of $C$ under $\hat{P}_J \to (\hat{P}_J)_{ad} = H''$. Let $C^1$ be the unipotent class in $\hat{P}_J$ whose image in $(\hat{P}_J)_{ad} = H' \times H''$ is $\{1\} \times \hat{C}$. Let $C^r$ be the unipotent class in $\hat{P}_J$ whose image in $(\hat{P}_J)_{ad} = H' \times H''$ is (regular unipotent class in $H'$) $\times \hat{C}$. Then $C^1 \cup C^r$ is exactly the set of unipotent elements in the image of $\pi : X \to \hat{Y}$. Now $\pi : X \to \hat{Y}$ restricts to $\pi^{-1}(C^1) \to C^1$ which is a $\mathbb{P}^1$-bundle and to $\pi^{-1}(C^r) \to C^r$ which is an isomorphism. There is a well defined local system $S_1$ on $C^1$ whose inverse image under $\pi^{-1}(C^1) \to C^1$ is the restriction of $\hat{S}$ to $\pi^{-1}(C^1)$. There is a well defined local system $S_r$ on $C^r$ whose inverse image under $\pi^{-1}(C^r) \to C^r$ is the restriction of $\hat{S}$ to $\pi^{-1}(C^r)$. Let

$$M_{J',g,1} = \{(xP_J, y(xU_1x^{-1})) \in M_{J',g}; \pi_{J'}(x^{-1}gx) \in C^1\},$$

$$M_{J',g,r} = \{(xP_J, y(xU_1x^{-1})) \in M_{J',g}; \pi_{J'}(x^{-1}gx) \in C^r\}.$$ Using the cartesian squares in 2.1 we see that $\pi'' : M_{J',g} \to M_{J',g}$ restricts to $\pi''_1 : \pi''^{-1}(M_{J',g,1}) \to M_{J',g,1}$ which is a $\mathbb{P}^1$-bundle and to $\pi''_r : \pi''^{-1}(M_{J',g,r}) \to M_{J',g,r}$ which is an isomorphism. Moreover there is a well defined local system $S_1$ on $M_{J',g,1}$ such that $\pi''_1^*S_1$ is the restriction $\hat{S}_1$ of $\hat{S}$ to $\pi''^{-1}(M_{J',g,1})$; there is a well defined local system $S_r$ on $M_{J',g,r}$ such that $\pi''_r^*S_r$ is the restriction $\hat{S}_r$ of $\hat{S}$ to $\pi''^{-1}(M_{J',g,r})$. Since $M_{J',g} = M_{J',g,1} \cup M_{J',g,r}$ is a partition and $M_{J',g,1}$ (resp. $M_{J',g,r}$) is closed (resp. open) in $M_{J',g}$ we see that (a) implies

$$t_i(g) = \sum_{j,j'} (-1)^j t^j \text{tr}(\sigma_i, H^j_c(M_{J',g,1}, \mathcal{H}^j(\pi_1\hat{S}_1)))$$

$$+ \sum_{j,j'} (-1)^j t^j \text{tr}(\sigma_i, H^j_c(M_{J',g,r}, \mathcal{H}^j(\pi_1\hat{S}_r))).$$
In the first sum over $j, j'$ we can assume that $j' \in \{0, 2\}$; the $\sigma_t$-action is multiplication by $-1$ if $j' = 2$ and by $1$ if $j' = 0$; in the second sum over $j, j'$ we can assume that $j' = 0$; moreover the $\sigma_t$-action is multiplication by $1$. (Here we use the definition of the $\sigma_i$-action on $K'$.) Thus, we have

$$t_i(g) = - \sum_j (-1)^j \dim H^j_c(M_{j', g, 1}, \mathcal{H}^2(\pi''_{1!}\hat{S}_1)) + \sum_j (-1)^j \dim H^j_c(M_{j', g, 1}, \mathcal{H}^0(\pi''_{1!}\hat{S}_1)) + \sum_j (-1)^j \dim H^j_c(M_{j', g, r, \mathcal{H}^0(\pi''_{r!}\hat{S}_r)}))$$

$$= - \sum_j (-1)^j \dim H^j_c(M_{j', g, 1}, S_1(-2)) + \sum_j (-1)^j \dim H^j_c(M_{j', g, 1}, S_1) + \sum_j (-1)^j \dim H^j_c(M_{j', g, r}, S_r).$$

Note that the Tate twist does not affect the dimension; hence after cancellation we obtain

$$t_i(g) = \sum_j (-1)^j \dim H^j_c(M_{j', g, r}, S_r) = \sum_j (-1)^j \dim H^j_c(\pi''-1(M_{j', g, r}), \hat{S}_r)$$

that is,

$$t_i(g) = \chi(\pi''-1(M_{j', g, r}, \hat{S}_r),$$

where, for an algebraic variety $X$ and a local system $\mathcal{E}$ on $X$ we set $\chi(X, \mathcal{E}) = \sum_j (-1)^j \dim H^j(X, \mathcal{E})$. (We also set $\chi(X) = \chi(X, \mathbb{Q}_l).$)

### 3. Computations in certain groups of semisimple rank $\leq 5$

#### 3.1. Until the end of 3.14 we assume that $G_{ad}$ is of type $D_4$ and that $u \in G$ (see 1.1) is such that $d_u = 3$. Note that $u$ is unique up to conjugation. We can write $I = \{\alpha, \beta, \gamma, \omega\}$ where the numbering is chosen so that each of $s_\alpha s_\omega, s_\beta s_\omega, s_\gamma s_\omega$ has order 3.

The Springer representation $\rho_u$ is a sum of two irreducible representations of $W$: one eight dimensional and one six dimensional. Using 1.4(b) we see that there is a unique $S \in \mathcal{B}_u$ such that $J_S = \{\omega\}$, a unique $\hat{S} \in \mathcal{B}_u$ such that $\hat{S} = \{\alpha, \beta, \gamma\}$, exactly two irreducible components $S_{\beta, \gamma}, S_{\beta, \gamma}$ of $\mathcal{B}_u$ such that $J_{S_{\beta, \gamma}} = J_{S_{\beta, \gamma}} = \{\alpha, \omega\}$, exactly two irreducible components $S_{\alpha, \gamma}, S_{\alpha, \gamma}$ of $\mathcal{B}_u$ such that $J_{S_{\alpha, \gamma}} = J_{S_{\alpha, \gamma}} = \{\beta, \omega\}$ and exactly two irreducible components $S_{\alpha, \beta}, S_{\alpha, \beta}$ of $\mathcal{B}_u$ such that $J_{S_{\alpha, \beta}} = J_{S_{\alpha, \beta}} = \{\gamma, \omega\}.

For any $X \in \mathcal{B}_u$ such that $\omega \in J_X$ and any $i \in \{\alpha, \beta, \gamma\}$ we set

$$X^*_i = \{B \in X; \text{ any } B' \text{ on the same } \omega\text{-line as } B \text{ is in } X_i\}.$$
We will show:

(a) there are well defined parabolic subgroups $P^\alpha \neq \tilde{P}^\alpha$ in $P^\alpha$, $P^\beta \neq \tilde{P}^\beta$ in $P^\beta$, $P^\gamma \neq \tilde{P}^\gamma$ in $P^\gamma$ and $P^\omega$ in $P^\omega$ such that

$$
P^\alpha \cup P^\omega, \tilde{P}^\alpha \cup P^\omega, P^\beta \cup P^\omega, \tilde{P}^\beta \cup P^\omega, P^\gamma \cup P^\omega, \tilde{P}^\gamma \cup P^\omega, \tilde{P}^\gamma \cup P^\omega,
$$

$$
\hat{S} = \{ B \in B; B(\omega) = P^\omega \},
$$

$$
S_{\beta\gamma'} = \{ B \in B; B(\beta) = P^\beta, B(\gamma) = \tilde{P}^\gamma \},
$$

$$
S_{\beta'\gamma} = \{ B \in B; B(\beta) = \tilde{P}^\beta, B(\gamma) = P^\gamma \},
$$

$$
S_{\alpha\gamma} = \{ B \in B; B(\alpha) = P^\alpha, B(\gamma) = P^\gamma \},
$$

$$
S_{\alpha'\gamma'} = \{ B \in B; B(\alpha) = \tilde{P}^\alpha, B(\gamma) = \tilde{P}^\gamma \},
$$

$$
S_{\alpha'\beta'} = \{ B \in B; B(\alpha) = \tilde{P}^\alpha, B(\beta) = \tilde{P}^\beta \},
$$

$$
S^*_\alpha = \{ B \in B; B(\beta) = P^\beta, B(\gamma) = \tilde{P}^\gamma, B(\alpha) \cup P^\omega \},
$$

$$
\sqcup \{ B \in B; B(\beta) = \tilde{P}^\beta, B(\gamma) = P^\gamma, B(\alpha) \cup P^\omega \},
$$

$$
S^*_\beta = \{ B \in B; B(\alpha) = P^\alpha, B(\gamma) = P^\gamma, B(\beta) \cup P^\omega \},
$$

$$
\sqcup \{ B \in B; B(\alpha) = \tilde{P}^\alpha, B(\gamma) \tilde{P}^\gamma, B(\beta) \cup P^\omega \},
$$

$$
S^*_\gamma = \{ B \in B; B(\alpha) = P^\alpha, B(\beta) = P^\beta, B(\gamma) \cup P^\omega \},
$$

$$
\sqcup \{ B \in B; B(\alpha) = \tilde{P}^\alpha, B(\beta) = \tilde{P}^\beta, B(\gamma) \cup P^\omega \}.
$$

(b) Let $Y = \{ B \in S; B(\omega) = P^\omega \}$. Then $Y \subset S_\alpha \cap S_\beta \cap S_\gamma$. Moreover, $Y$ meets any $\omega$-line in $S$ in exactly one point.

From (b) we deduce:

(c) For $i \in \{\alpha, \beta, \gamma\}$ we have $S_i = S^*_i \cup Y$.

The inclusion $S^*_i \cup Y \subset S_i$ is clear. Conversely, let $B \in S_i$ be such that $B \notin Y$. Let $L_B$ be the $i$-line containing $B$; we have $L_B \subset B_u$. Let $L$ be the $\omega$-line containing $B$. We have $L \subset S$ hence $L \subset B_u$. By (b) there exists $B' \in L$ such that $B' \in Y$; in particular we have $B' \in S_i$. Let $L_{B'}$ be the $i$-line containing $B'$; we have $L_{B'} \subset B_u$. We define $\pi \in P^{i,\omega}$ by the condition that $B \subset \pi$. Now $B^{\pi}$ is naturally imbedded in $B$; it is a flag manifold of type $A_2$ since $(s_\omega s_i)^3 = 1$. Note that $\pi$ is a connected reductive group of type $A_2$ and $B^{\pi}$ can be viewed as the flag manifold of $\tilde{\pi}$. Moreover we have $L \subset B^{\pi}, L_B \subset B^{\pi}, L_{B'} \subset B^{\pi}$. Since $u \in B$ we have $u \in \pi$ hence $Ad(u) : B^{\pi} \rightarrow B^{\pi}$ is well defined and its fixed point set contains $L \cup L_B \cup L_{B'}$. 
But a unipotent element in $PGL_3(k)$ whose fixed point set of the flag manifold contains three distinct lines must be the identity element. In particular, for any $B'' \in L$, $\text{Ad}(u)$ acts as identity on the $i$-line through $B''$; thus the $i$-line through $B''$ is contained in $B_u$. In other words we have $B \in S^*_i$. Thus we have $S_i \subset S^*_i \cup Y$. This proves (c) (assuming (b)).

We now deduce from (a),(b) the following statement.

$$Y = S_\alpha \cap S_\beta \cap S_\gamma.$$  

Assume that $B \in S_\alpha \cap S_\beta \cap S_\gamma$ and $B \notin Y$. Using (c) we deduce $B \in (S^*_\alpha \cup Y) \cap (S^*_\beta \cup Y) \cap (S^*_\gamma \cup Y)$. Since $B \notin Y$ we deduce that $B \in S^*_\alpha \cap S^*_\beta \cap S^*_\gamma$. But from (a) we see that

$$S^*_\alpha \cap S^*_\beta \cap S^*_\gamma = \emptyset.$$  

This contradiction shows that $S_\alpha \cap S_\beta \cap S_\gamma \subset Y$. The opposite inclusion is known from (b). This proves (d).

We now show, assuming (a):

(e) $\hat{S} \cap S_{\beta' \gamma}$ is a single $\alpha$-line; $\hat{S} \cap S_{\beta' \gamma}$ is a single $\alpha$-line.

From (a) we have

$$\hat{S} \cap S_{\beta' \gamma} = \{B \in B; B(\omega) = P^\omega, B(\beta) = P^\beta, B(\gamma) = \tilde{P}^\gamma\}$$

$$= \{B \in B; B \subset P^\omega, B \subset P^\beta, B \subset \tilde{P}^\gamma\} = \{B \in B; B \subset P^\omega \cap P^\beta \cap \tilde{P}^\gamma\}.$$

Since $P^\beta \cdot P^\omega, \tilde{P}^\gamma \cdot P^\omega$ and $s_\beta s_\gamma = s_\gamma s_\beta$, the intersection $P^\omega \cap P^\beta \cap \tilde{P}^\gamma$ is a parabolic subgroup in $P^\omega$. This proves the first assertion of (e); the second assertion of (e) is proved in the same way.

### 3.2.

To prove 3.1(a),(b) for $G$ is the same as proving them for $G_{ad}$. Hence we can assume that $G$ is the special orthogonal group associated to an 8-dimensional $k$-vector space $V$ with a given nondegenerate quadratic form $Q : V \to k$ and associate symmetric bilinear form $(,)_i : V \times V \to k$. Until the end of 3.14 we shall adhere to this assumption.

Now, $A := u - 1 : V \to V$ is nilpotent with Jordan blocks of sizes $3, 3, 1, 1$. More precisely, we can find a basis $\{e_i, e'_i; i \in \{0, 1, 2, 3\}\}$ of $V$ such that

- $Q(e_i) = Q(e'_i) = 0$ for $i \in \{0, 1, 2, 3\}$;
- $(e_i, e_j) = (e'_i, e'_j) = 0$ for all $i, j$;
- $(e_i, e'_j) = 1$ if $i \neq j$ are both odd or if $i = j$ are both even;
- $(e_i, e'_j) = 0$ otherwise

and such that

$$Ae_0 = 0, Ae'_0 = 0,$$

$$Ae_1 = e_2 + x e_3, Ae_2 = e_3, Ae_3 = 0, Ae'_1 = -e'_2 + x' e'_3, Ae'_2 = -e'_3, Ae'_3 = 0.$$  

where $x, x' \in k$ satisfy $x + x' = 1$. A subspace $U$ of $V$ is said to be isotropic if $Q|_U = 0$. Let $L_4 = \text{span}(e_3, e'_3, e_0, e_2)$, $L'_4 = \text{span}(e_3, e'_3, e'_0, e'_2)$. $L'_4 = \text{span}(e_3, e'_3, e'_0, e'_2)$.


span(e_3, e'_3, e'_0, e_2), \ L'_4 = \text{span}(e_3, e'_3, e_0, e'_2). These subspaces are isotropic; in 3.4 we will show that they are intrinsic to \( u \); they do not depend on the specific basis used to define \( u \). We identify \( \mathcal{P}^\alpha \) with the variety of all isotropic lines in \( V \); \( \mathcal{P}^\omega \) with the variety of all isotropic planes in \( V \); \( \mathcal{P}^\beta \) with the variety of all isotropic 4-spaces in \( V \) in the \( G \)-orbit of \( \mathcal{L}_4 \) and \( \mathcal{L}'_4 \); \( \mathcal{P}^\gamma \) with the variety of all isotropic 4-spaces in \( V \) in the \( G \)-orbit of \( \mathcal{L}'_4 \) and \( \mathcal{L}''_4 \). (In each case the identification attaches to an isotropic subspace its stabilizer in \( G \)). We identify \( \mathcal{B} \) with the variety of all sequences \( (V_1, V_2, V_4, \tilde{V}_4) \in \mathcal{P}^\alpha \times \mathcal{P}^\omega \times \mathcal{P}^\beta \times \mathcal{P}^\gamma \) such that \( V_1 \subset V_2 \subset V_4, V_2 \subset \tilde{V}_4 \). (Such a sequence is identified with its stabilizer \( \{ g \in G; gV_1k = V_1, gV_2 = V_2, gV_4 = V_4, g\tilde{V}_4 = \tilde{V}_4 \}. \)

We consider the following subspaces of \( V \):

\[
I_2 = A^2 V = \text{span}(e_3, e'_3), \quad I_4 = AV = \text{span}(e_2, e_3, e'_2, e'_3),
\]

\[
K_4 = \ker(A) = \text{span}(e_0, e_3, e'_0, e'_3), \quad K_6 = \ker(A^2) = \text{span}(e_0, e_2, e_3, e'_2, e'_3).
\]

We have \( I_2 \subset I_4 \subset K_6, I_2 \subset K_4 \subset K_6, K_4 \cap I_4 = I_2 \).

3.3. We show:

(a) There are exactly two subspaces \( V_4 \in \mathcal{P}^\beta \) such that \( I_2 \subset V_4 \) and \( \dim(AV_4) \leq 1 \). They are \( \mathcal{L}_4, \mathcal{L}'_4 \).

(b) There are exactly two subspaces \( V_4 \in \mathcal{P}^\gamma \) such that \( I_2 \subset V_4 \) and \( \dim(AV_4) \leq 1 \). They are \( \mathcal{L}'_4, \mathcal{L}''_4 \).

We prove (a). The subspaces \( V_4 \in \mathcal{P}^\beta \) such that \( I_2 \subset V_4 \) are determined by their intersection with \( \mathcal{L}'_4, \mathcal{L}''_4 \); this intersection is of the form \( \text{span}(e_3, e'_3, ae_0 + be'_2) \) where \((a, b) \in k^2 - \{0, 0\} \) hence the required 4-subspaces are of the form \( \text{span}(e_3, e'_3, ae_0 + be'_2, xe_0 + xe'_0 + ze_2 + ze'_2) \) where \( ae_0 + be'_2, xe_0 + xe'_0 + ze_2 + ze'_2 \) are linearly independent and we have \( ax' + bx = 0, xx' = 0 \). Moreover the condition that \( \dim(AV_4) \leq 1 \) is that \( \dim(\text{span}(-be'_3, ze_3 - ze'_3) \leq 1 \), that is \( bz = 0 \). Assume first that \( a \neq 0, b \neq 0 \); then \( z = 0, xx' = 0 \). From \( ax' + bx = 0, xx' = 0 \) we deduce that \( x = x' = 0 \). Since \( ae_0 + be'_2, ze'_2 \) are linearly independent we see that \( x' \neq 0 \) and our \( V_4 \) is \( \text{span}(e_3, e'_3, ae_0 + be'_2, e'_2) = \text{span}(e_3, e'_3, e_0, e'_2) \in \mathcal{P}^\gamma \), a contradiction. Assume now that \( a = 0, b \neq 0 \); then \( z = 0, x = 0 \). Since \( be'_2, xe'_0 + ze'_2 \) are linearly independent we see that \( x' \neq 0 \) and our \( V_4 \) is \( \text{span}(e_3, e'_3, e_2, xe'_0 + ze'_2) = \text{span}(e_3, e'_3, e_0, e'_2) = \mathcal{L}'_4 \). Assume now that \( a \neq 0, b = 0 \); then \( x' = 0 \) and \( zz' = 0 \). Since \( ae_0, xe_0 + ze_2 + ze'_2 \) are linearly independent we have either \( z = 0, z' \neq 0 \) or \( z \neq 0, z' = 0 \). If \( z = 0, z' \neq 0 \) then our \( V_4 \) is \( \text{span}(e_3, e'_3, e_0, xe_0 + z'e'_2) = \text{span}(e_3, e'_3, e_0, e'_2) \in \mathcal{P}^\gamma \), a contradiction. If \( z \neq 0, z' = 0 \) then our \( V_4 \) is \( \text{span}(e_3, e'_3, e_0, xe_0 + ze_2) = \text{span}(e_3, e'_3, e_0, e_2) = \mathcal{L}_4 \). Thus \( V_4 \in \{ \mathcal{L}_4, \mathcal{L}'_4 \} \). Conversely it is clear that if \( V_4 \in \{ \mathcal{L}_4, \mathcal{L}'_4 \} \) then \( V_4 \) satisfies the requirements of (a). This proves (a).

The proof of (b) is entirely similar to that of (a).

3.4. We set \( L_1 = \text{span}(e_3) \in \mathcal{P}^\alpha, \ L'_1 = \text{span}(e'_3) \in \mathcal{P}^\alpha \). We have \( A(\mathcal{L}_4) = \mathcal{L}_1, \ A(\mathcal{L}'_4) = \mathcal{L}'_1 \). In particular, if \( V_4 \) is as in 3.3(a),(b), then \( AV_4 \in \{ L_1, L'_1 \} \) is a one dimensional (isotropic) subspace of \( I_2 \).

From 3.3(a),(b) we see that \( \mathcal{L}_4, \mathcal{L}'_4, \mathcal{L}''_4, \mathcal{L}'''_4 \) are intrinsic to \( u \) and do not depend
Similarly, on the specific basis used to define $u$. It follows also that $L_1, L_1'$ are intrinsic to $u$. Note that $L_1, L_1'$ are the two lines in $I_2$ which are isotropic for the quadratic form $I_2 \to k$ given by $v \mapsto Q(\bar{v})$ where $\bar{v}$ is any vector in $I_4$ such that $A\bar{v} = v$.

3.5. Note that $Q$ induces a nondegenerate quadratic form on $K_4/I_2$ which has exactly two isotropic lines. Hence there are exactly two isotropic 3-spaces contained in $K_4$ and containing $I_2$. They are $\text{span}(e_3, e_3', e_0) = L_4 \cap L_4'$ and $\text{span}(e_3, e_3', e_0) = L_4 \cap L_4'$.

3.6. Let
\[ \tilde{Y} = \{(V_1, V_2, V_4, \tilde{V}_4) \in \mathcal{B}; V_2 = I_2\}. \]

Let $(V_1, V_2, V_4, \tilde{V}_4) \in \tilde{Y}$. We have clearly $AV_1 = 0$. Moreover, since $V_4$ is an isotropic subspace containing $I_2$ we must have $V_4 \subset K_6 = \ker(A^2)$ hence $AV_4 \subset \ker(A) \cap AV = I_4 \cap K_4 = I_2 \subset V_4$. Thus, $AV_4 \subset V_4$. Similarly we have $A\tilde{V}_4 \subset \tilde{V}_4$.

We see that $(V_1, V_2, V_4, \tilde{V}_4) \in \mathcal{B}_u$. Thus, $\tilde{Y} \subset \mathcal{B}_u$. Note that $\tilde{Y}$ is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ hence it is a closed irreducible subvariety of $\mathcal{B}_u$ of dimension 3. Thus $\tilde{Y}$ is an irreducible component of $\mathcal{B}_u$. It satisfies $J_{\tilde{Y}} = \{\alpha, \beta, \gamma\}$. Hence $\tilde{Y} = \tilde{S}$.

3.7. Let
\[ N_1 = \{(V_1, V_2, V_4, \tilde{V}_4) \in \mathcal{B}; V_4 = L_4 \cap L_4', \tilde{V}_4 = L_4'\}, \]
\[ N_2 = \{(V_1, V_2, V_4, \tilde{V}_4) \in \mathcal{B}; V_4 = L_4' \cap L_4', \tilde{V}_4 = L_4'\}. \]

We have $A(L_4 \cap L_4') = 0$, $A(L_4' \cap L_4) = 0$ hence if $(V_1, V_2, V_4, \tilde{V}_4)$ is in $N_1$ or $N_2$ then $AV_2 = 0$. Moreover, each of $L_4, L_4', L_4''$ is $A$-stable. We see that $(V_1, V_2, V_4, \tilde{V}_4) \in \mathcal{B}_u$. Thus, $N_1 \subset \mathcal{B}_u, N_2 \subset \mathcal{B}_u$. Note that $N_1$ and $N_2$ are isomorphic to the flag manifold of $GL_3(k)$ hence they are closed irreducible subvarieties of $\mathcal{B}_u$ of dimension 3. Thus they are (distinct) irreducible components of $\mathcal{B}_u$. They satisfy $J_{N_1} = J_{N_2} = \{\alpha, \omega\}$. Hence $N_1, N_2$ are the same as $S_{\beta'\gamma'}, S_{\beta'\gamma}$ (up to order).

3.8. Let
\[ N_3 = \{(V_1, V_2, V_4, \tilde{V}_4) \in \mathcal{B}; V_1 = L_1, \tilde{V}_4 = L_4\}, \]
\[ N_4 = \{(V_1, V_2, V_4, \tilde{V}_4) \in \mathcal{B}; V_1 = L_1', \tilde{V}_4 = L_4'\}. \]

Let $(V_1, V_2, V_4, \tilde{V}_4) \in N_3$. Let $V_3 = V_4 \cap \tilde{V}_4$. Since $A(\tilde{V}_4) = L_1$ we have $A(V_3) \subset L_1 = V_1 \subset V_3$ that is $uV_3 = V_3$; moreover, $A(V_2) \subset L_1 = V_1$ hence $AV_2 \subset V_2$. Now $V_4$ is the only subspace in its $G$-orbit that contains $V_3$; since $uV_4 \supset uV_3 = V_3$ it follows that $uV_4 = V_4$. The inclusions $AV_1 \subset V_1, A(\tilde{V}_4) \subset \tilde{V}_4$ are obvious. We see that $(V_1, V_2, V_4, \tilde{V}_4) \in \mathcal{B}_u$. Thus, $N_3 \subset \mathcal{B}_u$. Note that $N_3$ is isomorphic to the space of pairs $(V_2, V_3)$ where $V_2 \subset V_3$ are subspaces of $\tilde{V}_4$ with $\dim V_2 = 2, \dim V_3 = 3$; hence $N_3$ is isomorphic to the the flag manifold of $GL_3(k)$. Thus $N_3$ is a closed irreducible subvariety of $\mathcal{B}_u$ of dimension 3. Thus $N_3 \in \mathcal{B}_u$. We have $J_{N_3} = \{\beta, \omega\}$.

Similarly, $N_4 \in \mathcal{B}_u$ and $J_{N_4} = \{\beta, \omega\}$. Hence $N_3, N_4$ are the same as $S_{\alpha\gamma}, S_{\alpha'\gamma'}$ (up to order).
3.9. Let
\[ N_5 = \{(V_1, V_2, V_4, \tilde{V}_4) \in \mathcal{B}; V_1 = L_1, V_4 = \tilde{L}_4\}, \]
\[ N_6 = \{(V_1, V_2, V_4, \tilde{V}_4) \in \mathcal{B}; V_1 = L'_1, V_4 = \tilde{L}'_4\}. \]
A proof completely similar to that in 3.8 shows that \( N_5 \in \mathcal{B}_u, N_6 \in \mathcal{B}_u \). We have \( J_{N_5} = J_{N_6} = \{\gamma, \omega\} \). Hence \( N_5, N_6 \) are the same as \( S_{\alpha\beta}, S_{\alpha'\beta'} \) (up to order).

3.10. Let \( X = \{(V_1, V_2, V_4, \tilde{V}_4) \in \mathcal{B}; V_1 \subset I_2 \subset V_4 \cap \tilde{V}_4, A(V_4 \cap \tilde{V}_4) \subset V_1\} \). We show:
(a) \( X \) is a smooth irreducible projective variety of dimension 3.

The fact that \( X \) is a projective variety is obvious. Now let \( Z \) be the variety of all pairs \( V_1, V_3 \) of isotropic subspaces of \( V \) of dimension 1 and 3 respectively such that \( V_1 \subset I_2 \subset V_3 \) and \( AV_3 \subset V_1 \). Clearly \( \zeta : (V_1, V_2, V_4, \tilde{V}_4) \mapsto (V_1, V_4 \cap \tilde{V}_4) \) makes \( X \) into a \( \mathbf{P}^1 \)-bundle over \( Z \). Hence it is enough to show that \( Z \) is a smooth irreducible surface. Now the space of lines in \( I_2 \) is a projective line and the space of \( V_3 \) containing \( I_2 \) and contained in \( K_6 \) (without the isotropy condition) is a projective 3-space. Hence we can identify \( Z \) with
\[ \{(z_0, z_0', z_2, z_2'), (y, y')) \in \mathbf{P}^3 \times \mathbf{P}^1; z_0 z_0' + z_2 z_2' = 0, z_2 y' + z_2' y = 0\}. \]

The subset of \( Z \) where \( z_2' \neq 0, y' \neq 0 \) is
\[ \{(z_0, z_0', z_2; y) \in k^3 \times k; z_0 z_0' + z_2 = 0, z_2 + y = 0\} \]
\[ = \{(z_0, z_0', z_2) \in k^3; z_0 z_0' + z_2 = 0\} \cong k^2. \]
Similarly the subset of \( Z \) where \( z_2 \neq 0, y \neq 0 \) is \( \cong k^2 \). The subset where \( z_0 \neq 0, y \neq 0 \) is
\[ \{(z_0', z_2, z_2', y') \in k^4; z_0' + z_2 z_2' = 0, z_2 y' + z_2' y = 0\} \cong k^2. \]
The subset where \( z_0 \neq 0, y' \neq 0 \) is \( \cong k^2 \); the subset where \( z_0' \neq 0, y \neq 0 \) is \( \cong k^2 \); the subset where \( z_0' \neq 0, y' \neq 0 \) is \( \cong k^2 \). Thus, \( Z \) is a union of 8 smooth irreducible surfaces (open in \( Z \)) hence \( Z \) is a smooth surface. Since the 8 open subsets above have a nonempty intersection, we see that \( Z \) is irreducible. This proves (a).

Now if \( (V_1, V_2, V_4, \tilde{V}_4) \in X \) then setting \( V_3 = V_4 \cap \tilde{V}_4 \) we have \( AV_1 = 0 \) (since \( V_1 \subset I_2 \)), \( A(V_3) \subset V_1 \subset V_3, AV_2 \subset V_1 \subset V_2 \) hence \( uV_1 \subset V_1, uV_3 \subset V_3, uV_2 \subset V_2 \); since \( V_4 \in \mathcal{P}^\beta, \tilde{V}_4 \in \mathcal{P}^\gamma \) are determined uniquely by the conditions \( V_3 \subset V_4, V_3 \subset \tilde{V}_4 \), we have necessarily \( uV_4 = V_4, u\tilde{V}_4 = \tilde{V}_4 \). Thus, \( X \subset \mathcal{B}_u \). Using this and (a) together with \( \dim \mathcal{B}_u = 3 \) we see that
(b) \( X \) is an irreducible component of \( \mathcal{B}_u \).

It follows that the subsets \( X_\alpha, X_\beta, X_\gamma \) of \( X \) are well defined (see 1.6).

We show:
(c) \( \chi(X) = 12 \).
Here \( \chi() \) is as in 2.2. Since \( X \) is a \( \mathbf{P}^1 \)-bundle over \( Z \), an equivalent statement is:
(d) \( \chi(Z) = 6 \).

The space \( \mathcal{U} \) of isotropic 3-spaces in \( V \) that contain \( I_2 \) hence are contained in \( K_6 \) can be identified with the space of isotropic lines in \( K_6/I_2 \) with its obvious quadratic form hence it is a product \( \mathbb{P}^1 \times \mathbb{P}^1 \). The map \( Z \rightarrow \mathcal{U}, (V_1, V_3) \mapsto V_3 \) is an isomorphism over the complement of two points in \( \mathcal{U} \), namely \( L_4 \cap L_4' \) and \( L_4' \cap L_4' \) and has fibres \( \mathbb{P}^1 \) at each of those two points. It follows that \( \chi(Z) = \chi(\mathbb{P}^1 \times \mathbb{P}^1) - 2 + 2\chi(\mathbb{P}^1) = 4 - 2 + 4 = 6 \), as desired.

3.11. Clearly, \( X \) is a union of \( \omega \)-lines. Hence for \( i \in \{ \alpha, \beta, \gamma \} \), \( X_i^* \) is defined as in 3.1. We have

\[
X_\alpha = \{(V_1, V_2, V_4, \tilde{V}_4) \in X; \text{ for any line } V_1' \text{ in } V_2 \text{ we have } AV_1' = 0 \}
\]

hence

\[
X_\alpha = \{(V_1, V_2, V_4, \tilde{V}_4) \in X; V_2 \subset K_4 \}.
\]

We deduce that

\[
X_\alpha^* = \{(V_1, V_2, V_4, \tilde{V}_4) \in X; \text{ for any } V_2' \text{ with } V_1 \subset V_2' \subset V_4 \cap \tilde{V}_4 \text{ we have } V_2' \subset K_4 \}.
\]

Since the various \( V_2' \) such that \( V_1 \subset V_2' \subset V_4 \cap \tilde{V}_4 \) generate \( V_4 \cap \tilde{V}_4 \) we see that

\[
X_\alpha^* = \{(V_1, V_2, V_4, \tilde{V}_4) \in X; I_2 \subset V_4 \cap \tilde{V}_4 \subset K_4 \}
\]

If \( (V_1, V_2, V_4, \tilde{V}_4) \in X_\alpha^* \) then \( A(V_4 \cap \tilde{V}_4) = 0 \) hence \( \dim A(V_4) \leq 1 \), \( \dim A(\tilde{V}_4) \leq 1 \) (since \( V_4 \cap \tilde{V}_4 \) has codimension 1 in \( V_4 \) and in \( \tilde{V}_4 \)). Using 3.3(a),(b) we deduce that \( V_4 \in \{ L_4, L_4' \}, \tilde{V}_4 \in \{ L_4', L_4'' \} \). Thus we have either \( V_4 = L_4, \tilde{V}_4 = L_4' \) or \( V_4 = L_4', \tilde{V}_4 = L_4'' \). We see that

\[
X_\alpha^* = \{(V_1, V_2, V_4, \tilde{V}_4) \in X; V_4 = L_4, \tilde{V}_4 = L_4' \} \\
\cup \{(V_1, V_2, V_4, \tilde{V}_4) \in X; V_4 = L_4', \tilde{V}_4 = L_4'' \}.
\]

(The right hand side is clearly contained in the left hand side.) Since \( A(L_4 \cap L_4') = 0, A(L_4' \cap L_4'') = 0 \), we have

\[
X_\alpha^* = \{(V_1, V_2, V_4, \tilde{V}_4) \in B; V_1 \subset I_2, V_4 = L_4, \tilde{V}_4 = L_4' \} \\
\cup \{(V_1, V_2, V_4, \tilde{V}_4) \in B; V_1 \subset I_2, V_4 = L_4', \tilde{V}_4 = L_4'' \}.
\]

(a)

3.12. We have

\[
X_\beta = \{(V_1, V_2, V_4, \tilde{V}_4) \in X; \text{ for any } V_4' \in \mathcal{P}_\beta \text{ with } V_2 \subset V_4' \text{ we have } u(V_4') = V_4' \}.
\]

Let \( (V_1, V_2, V_4, \tilde{V}_4) \in X_\beta \). Then for any \( V_4' \in \mathcal{P}_\beta \) with \( V_2 \subset V_4' \) we have that \( \tilde{V}_4 \cap V_4' \) is an isotropic 3-space containing \( V_2 \); it is \( u \)-stable (since \( \tilde{V}_4 \) and \( V_4' \) are
Moreover all isotropic 3-spaces containing $V_2$ and contained in $\tilde{V}_4$ are obtained in this way hence are all $u$-stable; it follows that $u$ acts as 1 on $\tilde{V}_4/V_2$ that is, $A(tV_4) \subset V_2$. Thus, $X_\beta \subset \{(V_1, V_2, V_4, \tilde{V}_4) \in X; A(\tilde{V}_4) \subset V_2\}$. A similar argument shows the reverse inclusion. Thus

$$X_\beta = \{(V_1, V_2, V_4, \tilde{V}_4) \in X; A(\tilde{V}_4) \subset V_2\}.$$ 

We deduce that

$$X^*_\beta = \{(V_1, V_2, V_4, \tilde{V}_4) \in X; \text{ for any } V'_2 \text{ with } V_1 \subset V'_2 \subset V_4 \cap \tilde{V}_4 \text{ we have } A(\tilde{V}_4) \subset V'_2\}.$$ 

Since the intersection of all $V'_2$ such that $V_1 \subset V'_2 \subset V_4 \cap \tilde{V}_4$ is $V_1$ we see that

$$X^*_\beta = \{(V_1, V_2, V_4, \tilde{V}_4) \in X; A(\tilde{V}_4) \subset V_1\}$$

that is,

$$X^*_\beta = \{(V_1, V_2, V_4, \tilde{V}_4) \in B; V_1 \subset I_2 \subset V_4 \cap \tilde{V}_4, A(\tilde{V}_4) \subset V_1\}.$$ 

Using 3.3(b), we see that for $\tilde{V}_4$ in the right hand side we have $\tilde{V}_4 \in \{L^1_4, L^1_4\}$ so that $V_1$ is necessarily $L_1$ (if $\tilde{V}_4 = L^1_4$) or $L'_1$ (if $\tilde{V}_4 = L^1_4$). Thus,

$$X^*_\beta = \{(V_1, V_2, V_4, \tilde{V}_4) \in B; I_2 \subset V_4, V_1 = L_1, \tilde{V}_4 = L^1_4\}$$

(a) \quad \sqcup \{(V_1, V_2, V_4, \tilde{V}_4) \in B; I_2 \subset V_4, V_1 = L'_1, \tilde{V}_4 = L^1_4\}.$$ 

An entirely similar argument yields:

$$X_\gamma = \{(V_1, V_2, V_4, \tilde{V}_4) \in S; A(V_4) \subset V_2\},$$

$$X^*_\gamma = \{(V_1, V_2, V_4, \tilde{V}_4) \in B; I_2 \subset \tilde{V}_4, V_1 = L_1, V_4 = L^1_4\}$$

(b) \quad \sqcup \{(V_1, V_2, V_4, \tilde{V}_4) \in B; I_2 \subset \tilde{V}_4, V_1 = L'_1, V_4 = L^1_4\}.$$ 

From 3.11(a) we see that $X^*_\alpha$ has two irreducible components, each one being a $P^1$-bundle over $P^1$ hence is two-dimensional; we see that $X^*_\alpha \neq X$ hence $X$ is not a union of $\alpha$-lines. Similarly from (a),(b) we see that $X^*_\beta \neq X, X^*_\gamma \neq X$ hence $X$ is not a union of $\beta$-lines and $X$ is not a union of $\gamma$-lines. Thus we have $J_X = \{\omega\}$. It follows that

(c) \quad X = S.
3.13. We have
\[ Y = \{(V_1, V_2, V_4, \tilde{V}_4) \in X; V_2 = I_2\} \]
\[ = \{(V_1, V_2, V_4, \tilde{V}_4) \in \mathcal{B}; V_2 = I_2, A(V_4 \cap \tilde{V}_4) \subset V_1\}. \]

Assume that \((V_1, V_2, V_4, \tilde{V}_4) \in Y\). If \(V'_1\) is an isotropic line contained in \(I_2\) then \(AV'_1 = 0\) (since \(AI_2 = 0\)). This shows that \(Y \subset X_\alpha = S_\alpha\). If \(U\) is an isotropic 4-space containing \(I_2\) then \(U \subset K_\delta = \ker(A^2)\) hence \(AU \subset \ker(A) \cap AV = K_4 \cap I_4 = I_2 \subset U\); thus, \(AU \subset U\). This shows that \(Y \subset X_\beta = S_\beta\) and \(Y \subset X_\gamma = S_\gamma\). This proves the first assertion of 3.1(b). Now let \((V_1, V_2, V_4, \tilde{V}_4) \in X\) and let \(L\) be the \(\omega\)-line in \(X\) containing \((V_1, V_2, V_4, \tilde{V}_4)\). By the definition of \(X\) if \(V_2\) is replaced by \(I_2\) the resulting quadruple \((V_1, I_2, V_4, \tilde{V}_4)\) belongs to \(X\) (and even to \(Y\)). This proves the second assertion of 3.1(b).

3.14. From 3.11(a) and 3.12(a),b,c), we see that 3.1(a) holds. Here \(P^\alpha\) (resp. \(\tilde{P}^\alpha\)) is the stabilizer of \(L_1\) (resp. \(L'_1\)); \(P^\beta\) (resp. \(\tilde{P}^\beta\)) is the stabilizer of \(L_4\) (resp. \(L'_4\)); \(P^\gamma\) (resp. \(\tilde{P}^\gamma\)) is the stabilizer of \(L'_4\) (resp. \(L'_4\)).

3.15. Until the end of 3.18 we assume that \(G_{ad}\) is of type \(D_5\). In this case we can write \(I = \{\alpha, \beta, \gamma, \delta, \omega\}\) where the numbering is chosen so that each of \(s_\alpha s_\omega, s_\beta s_\omega, s_\gamma s_\omega, s_\alpha s_\delta\) has order 3. We shall assume that \(u \in G\) (see 1.1) is such that \(d_u = 3\); this determines \(u\) uniquely up to conjugacy. We can find \(P \in \mathcal{P}^\delta\) such that \(u \in P\) and \(\bar{u} := \pi_P(u) \in \bar{P}\) satisfies \(d_{\bar{u}} = 3\) and, if \(C\) is the conjugacy class of \(u\) in \(G\), then \(C \cap \pi_{\bar{P}}^{-1}(\bar{u})\) is open dense in \(\pi_{P}^{-1}(u)\). Since the adjoint group of \(\bar{P}\) is of type \(D_4\), the irreducible component \(S\) of \(\mathcal{B}_u^\bar{P}\) is defined as in 3.1 with \(G, u\) replaced by \(\bar{P}, \bar{u}\). Note that the Weyl group of \(\bar{P}\) is canonically identified with the subgroup of \(W\) generated by \(\{s_\alpha, s_\beta, s_\gamma, s_\omega\}\). The imbedding \(\mathcal{B}^\bar{P} \subset \mathcal{B}\), \(B_1 \mapsto \pi_{\bar{P}}^{-1}(B_1)\) restricts to an imbedding \(\mathcal{B}_u^\bar{P} \subset \mathcal{B}_u\) and to an isomorphism of \(S\) onto an irreducible component \(\tilde{S}\) of \(\mathcal{B}_u\). (See 1.5.) This imbedding carries any \(i\)-line in \(\mathcal{B}^\bar{P}\) (where \(i \in \{\alpha, \beta, \gamma, \omega\}\)) to an \(i\)-line in \(\mathcal{B}\).

Using 1.4(b), we see that there is a unique \(E \in \mathcal{B}_u\) such that \(J_E = \{\beta, \gamma, \delta\}\).

The inverse images under \(\pi_P\) of the parabolic subgroups \(P^\alpha, \tilde{P}^\alpha, P^\beta, \tilde{P}^\beta, P^\gamma, \tilde{P}^\gamma, P^\omega\) as in 3.1(a) (with \(G, u\) replaced by \(\bar{P}, \bar{u}\)) are denoted again by the same letters; thus we have
\(P^\alpha \neq \tilde{P}^\alpha\) in \(\mathcal{P}^\alpha\), \(P^\beta \neq \tilde{P}^\beta\) in \(\mathcal{P}^\beta\), \(P^\gamma \neq \tilde{P}^\gamma\) in \(\mathcal{P}^\gamma\), \(P^\omega \in \mathcal{P}^\omega\),
where \(\mathcal{P}^i\) refers to \(G\).

3.16. We will show that there is a well defined parabolic subgroup \(\hat{P}^\alpha \in \mathcal{P}^\alpha\) such that
(a) \(\hat{P}^\alpha \notin \{P^\alpha, \tilde{P}^\alpha\}\), \(\hat{P}^\alpha \not\subset P^\omega\) and
(b) \(\tilde{S}_\delta = \{B \in \tilde{S}; B(\alpha) = \hat{P}^\alpha\}\),
(c) \(E = \{B \in \mathcal{B}; B(\alpha) = \hat{P}^\alpha, B(\omega) = P^\omega\}\).

To prove this is the same as proving it for \(G_{ad}\). Hence we can assume that \(G\) is the
special orthogonal group associated to a 10-dimensional $k$-vector space $V$ with a given nondegenerate quadratic form $Q : V \to k$ and associate symmetric bilinear form $(, ) : V \times V \to k$. Until the end of 3.18 we shall adhere to this assumption. Now $A := u - 1 : V \to V$ is nilpotent. If $p \neq 2$, $A$ has Jordan blocks of sizes $5, 3, 1, 1$. If $p = 2$, $A$ has Jordan blocks of sizes $4, 4, 1, 1$ and moreover we have $Q(A^2 x) \neq 0$ for some $x \in V$. These conditions describe completely the conjugacy class of $u$.

We identify $P^\alpha$ with the variety of all isotropic lines in $V$; $P^\beta$ with the variety of all isotropic planes in $V$; $P^\gamma$ with the variety of all isotropic 3-spaces in $V$. (In each case the identification attaches to an isotropic subspace its stabilizer in $G$.)

We will describe later $P^\beta$ and $P^\gamma$.

We can find a basis $\{ e_i, e'_i : i \in \{0, 1, 2, 3\} \} \sqcup \{ f, f' \}$ of $V$ such that $P$ is the stabilizer in $G$ of the line $k f$;

$Q(e_i) = Q(e'_i) = 0$ for $i \in \{0, 1, 2, 3\}, Q(f) = Q(f') = 0$;

$(e_i, e_j) = (e'_i, e'_j) = 0$ for all $i, j$ and $(f, e_i) = (f, e'_i) = 0$ for all $i$;

$(e_i, e'_j) = 1$ if $i \neq j$ are both odd or if $i = j$ are both even and $(f, f') = 1$;

$(e_i, e'_j) = 0$ otherwise,

and such that

$Af' = a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 + a'_0 e'_0 + a'_1 e'_1 + a'_2 e'_2 + a'_3 e'_3 + df$

$Ae_0 = b_0 f, Ae'_0 = b'_0 f,$

$Ae_1 = e_2 + xe_3 + b_1 f, Ae_2 = e_3 + b_2 f, Ae_3 = b_3 f, Ae'_1 = -e'_2 + x'e'_3 + b'_1 f,$

$Ae'_2 = -e'_3 + b'_2 f, Ae'_3 = b'_3 f, Af = 0,$

where $x, x' \in k$ satisfy $x + x' = 1$, compare with 3.2. Here

$a_0, a_1, a_2, a_3, a'_0, a'_1, a'_2, a'_3, b_0, b_1, b_2, b_3, b'_0, b'_1, b'_2, b'_3, d$

are elements of $k$ such that

$d + a_0 a'_0 + a_1 a'_3 + a_2 a'_2 + a_3 a'_1 = 0, a'_0 + b_0 = 0, a_0 + b'_0 = 0,$

$a'_3 + a'_2 + xa'_1 + b_1 = 0, a_3 - a_2 + x'a_1 + b'_1 = 0, a'_2 + a'_1 + b_2 = 0, a_2 - a_1 + b'_2 = 0,$

$a'_1 + b_3 = 0, a_1 + b'_3 = 0.$

(These equations express the fact that $u$ is an isometry for $Q$.)

We identify $P^\beta$ with the variety of all isotropic 5-spaces in $V$ in the $G$-orbit of span$(f, e_3, e'_3, e_0, e_2)$; $P^\gamma$ with the variety of all isotropic 5-spaces in $V$ in the $G$-orbit of span$(f, e_3, e'_3, e'_0, e_2)$. (In each case the identification attaches to an isotropic subspace its stabilizer in $G$.) We identify $B$ with the variety of all sequences $(V_1, V_2, V_3, V_5, \tilde{V}_5) \in P^\delta \times P^\alpha \times P^\omega \times P^\beta \times P^\gamma$ such that $V_1 \subset V_2 \subset V_3 \subset V_5, V_3 \subset \tilde{V}_5$. (Such a sequence is identified with its stabilizer $\{ g \in G : gV_1 = V_1, gV_2 = V_2, gV_3 = V_3, gV_5 = V_5, g\tilde{V}_5 = \tilde{V}_5 \}$.)

We now compute the powers $A^k$ for $k = 2, 3, 4$ (here $*$ denotes an element of $k$).

$A^2 f' = a_1 e_2 + (a_2 + xa_1) e_3 - a'_1 e'_2 + (-a'_2 + x'a'_1) e'_3 + * f,$

$A^2 e_0 = 0, A^2 e_1 = e_3 + * f, A^2 e_2 = -a'_1 f, A^2 e_3 = 0,$

$A^2 e'_0 = 0, A^2 e'_1 = e'_3 + * f, A^2 e'_2 = -a_1 f, A^2 e'_3 = 0, A^2 f = 0;$

$A^3 f' = a_1 e_3 + a'_1 e'_3 + * f, A^3 e_0 = 0, A^3 e_1 = -a'_1 f, A^3 e_2 = 0, A^3 e_3 = 0, A^3 e'_0 = 0,$

$A^3 e'_1 = 0, A^3 e'_2 = 0, A^3 e'_3 = 0.
We see that $A^3e'_1 = -a_1f$, $A^3e'_2 = 0$, $A^3e'_3 = 0$, $A^3f = 0$;
$A^4f' = -2a_1a'_1f$, $A^4e_i = A^4e'_i = 0$ for $i = 0, 1, 2, 3$, $A^4f = 0$.

We see that $A^4 = 0$ if $p = 2$ and $A^5 = 0$ without restriction on $p$. Since $A$ has a
Jordan block of size 5 (if $p \neq 2$) and one of size 4 (if $p = 2$) we see that $A^4 \neq 0$ (if
$p \neq 2$) and $A^3 \neq 0$ (if $p = 2$). It follows that $a_1a'_1 \neq 0$ if $p \neq 2$ and $(a_1, a'_1) \neq (0, 0)$
in any case.

Let $\mathcal{I}$ be the image of the map $V \to k$, $v \mapsto Q(A^2v)$. Since
\[ A^2v = \text{span}(f, e_3, e'_3, a_1e_2 - a'_1e'_2), \]
$\mathcal{I}$ is the same as the image of the map $k^4 \to k$,
\[(x, y, x, t) \mapsto Q(xf + ye_3 + ze'_3 + t(a_1e_2 - a'_1e'_2)) = Q(t(a_1e_2 - a'_1e'_2)) = -t^2a_1a'_1. \]

Thus, if $p \neq 2$ we have $\mathcal{I} = k$ since $a_1a'_1 \neq 0$; if $p = 2$ then we already know that
$\mathcal{I} \neq 0$ hence we must have $a_1a'_1 \neq 0$. Thus, without assumption on $p$ we have
$a_1a'_1 \neq 0$ and $\mathcal{I} = k$.

3.17. Let $I_3 = \text{span}(f, e_3, e'_3) \in P^\omega$, $I'_2 = \text{span}(f, a_1e_3 - a'_1e'_3)$, $K_7 = \{v \in
V; (v, V_3) = 0\} = \text{span}(f, e_3, e'_3, e_2, e'_2, e_0, e'_0)$.

Note that $I'_2 = I_3 \cap \ker(A)$; moreover $I'_2, I_3$ are isotropic subspaces. From the
definitions we have
\[ \tilde{S} = \{(V_1, V_2, V_3, V_5, \tilde{V}_5) \in \mathcal{B}; V_1 = kf \subset V_2 \subset I_3 \subset V_5 \cap \tilde{V}_5, A(V_5 \cap \tilde{V}_5) \subset V_2 \}. \]

From the definitions, $\tilde{S}_\delta$ is the set of all $(V_1, V_2, V_3, V_5, \tilde{V}_5) \in \tilde{S}$ such that for any
isotropic line $V'_i$ contained in $V_2$ we have $AV'_1 \subset V'_i$ that is, $AV'_1 = 0$. Since $V_2$
generated by such $V'_i$, we see that
\[ \tilde{S}_\delta = \{(V_1, V_2, V_3, V_5, \tilde{V}_5) \in \tilde{S}; AV_2 = 0\}
= \{(V_1, V_2, V_3, V_5, \tilde{V}_5) \in \tilde{S}; V_2 \subset I_3 \cap \ker(A)\} = \{(V_1, V_2, V_3, V_5, \tilde{V}_5) \in \tilde{S}; V_2 = I'_2\}. \]
This shows that 3.16(b) holds where $\tilde{P}^\alpha$ is the parabolic subgroup in $P^\alpha$ which is the
stabilizer of $I'_2$. From the definitions we see that $P^\alpha$ (resp. $\tilde{P}^\alpha$) is the stabilizer
of $\text{span}(e_3, f)$ (resp. of $\text{span}(e'_3, f)$). Since $\text{span}(e_3, f), \text{span}(e'_3, f), \text{span}(a_1e_3 -
a'_1e'_3, f)$ are distinct (recall that $a_1a'_1 \neq 0$) we see that $\tilde{P}^\alpha \notin \{P^\alpha, \tilde{P}^\alpha\}$. Since $P^\omega$
is the stabilizer of $I_3$ and $I'_2 \subset I_3$ we see that $\tilde{P}^\alpha \leq P^\omega$. Thus, 3.16(a) holds.

3.18. Let
\[ E' = \{(V_1, V_2, V_3, V_5, \tilde{V}_5) \in \mathcal{B}; V_2 = I'_2, V_3 = I_3\}. \]
If $(V_1, V_2, V_3, V_5, \tilde{V}_5) \in E'$ then $AV_1 = AV_2 = AV_3 = 0$; moreover we have $V_5 \subset
K_7$ hence $AV_5 \subset AK_7 \subset I_3 \subset V_5$. Similarly, $AV_5 \subset I_3 \subset V_5$. Thus we have
$(V_1, V_2, V_3, V_5, \tilde{V}_5) \in \mathcal{B}_u$. We see that $E' \subset \mathcal{B}_u$. Now $E'$ is isomorphic to $\mathbf{P}^1 \times
\mathbf{P}^1 \times \mathbf{P}^1$ hence it is a closed irreducible 3-dimensional subvariety of $\mathcal{B}_u$. Thus $E'$
is an irreducible component of $\mathcal{B}_u$. We have $J_{E'} = \{\delta, \beta, \gamma\}$. We see that $E' = E$. From
the description of $E = E'$ given above we see that 3.16(c) holds.
3.19. Until the end of 3.23 we assume that \( G_{ad} \) is of type \( A_5 \) and that \( u \in G \) (see 1.1) is such that \( d_u = 3 \) and the Springer representation \( \rho_u \) is the irreducible representation of \( W \) of dimension five appearing in the third symmetric power of the reflection representation of \( W \). Note that \( u \) is unique up to conjugation. We can write \( I = \{1, 2, 3, 4, 5\} \) where the numbering is chosen so that each of \( s_1 s_2, s_2 s_3, s_3 s_4, s_4 s_5 \) has order 3.

Using 1.4(b) we see that there is a unique \( T \in \mathcal{B}_u \) such that \( J_T = \{3\} \) and a unique \( M \in \mathcal{B}_u \) such that \( J_M = \{1, 3, 5\} \). We shall prove

(a) there are well defined parabolic subgroups \( P^2 \in \mathcal{P}^2, \ P^4 \in \mathcal{P}^4 \) such that \( M = \{B \in \mathcal{B}; B(2) = P^2, B(4) = P^4\} \);

we have \( T_1 = T_5 = \{B \in T; B(2) = P^2, B(4) = P^4\} = M \cap T \) (we denote it by \( T_{15} \));

this is a \( \mathbb{P}^1 \)-bundle over \( \mathbb{P}^1 \) and is a union of 3-lines;

we have \( T_2 = T_4 \) (we denote it by \( T_{24} \)); it intersects any 3-line in \( T \) in exactly one point;

the intersection \( T_{15} \cap T_{24} \) is isomorphic to \( \mathbb{P}^1 \);

there is a unique morphism \( \vartheta : T_{24} \to T_{15} \cap T_{24} \) such that for any \( B \in T_{24} \) we have \( (B, \vartheta(B)) \in \mathcal{O}_w \) for some \( w \in W_{24} \); moreover, \( \vartheta \) is a \( \mathbb{P}^1 \)-bundle.

From (a) we see that

(b) \( \chi(T_{24}) = 4, \ \chi(T) = 8 \).

where \( \chi() \) is as in 2.2.

3.20. To prove 3.19(a) for \( G \) is the same as proving it for \( G_{ad} \). Hence we can assume that \( G = GL(V) \) where \( V \) is a 6-dimensional \( k \)-vector space. Until the end of 3.23 we shall adhere to this assumption. We can write \( u = A + 1 \) where \( A : V \to V \) is nilpotent with Jordan blocks of sizes 3, 3. Let \( K_2 = \ker A = A^2(V), K_4 = \ker A^2 = A(V) \). We have \( K_2 \subset K_4 \).

For \( i \in I \), we identify \( \mathcal{P}^i \) with the variety of all \( i \)-dimensional subspaces of \( V \). (The identification attaches to a subspace its stabilizer in \( G \).) We identify \( \mathcal{B} \) with the variety of all sequences \( (V_1, V_2, V_3, V_4, V_5) \in \mathcal{P}^1 \times \mathcal{P}^2 \times \mathcal{P}^3 \times \mathcal{P}^4 \times \mathcal{P}^5 \) such that \( V_1 \subset V_2 \subset V_3 \subset V_4 \subset V_5 \). (Such a sequence is identified with its stabilizer \( \{g \in G; gV_1 = V_1, gV_2 = V_2, gV_4 = V_4, gV_5 = V_5\} \).

3.21. Let

\[ M' = \{(V_1, V_2, V_3, V_4, V_5) \in \mathcal{B}; V_2 = K_2, V_4 = K_4 \}. \]

If \( (V_1, V_2, V_3, V_4, V_5) \in M \) then

\[ AV_5 \subset AV = K_4 \subset V_5, AV_3 \subset AK_4 = A\ker(A^2) \subset \ker(A) = K_2 \subset V_3, \]

\[ AV_1 \subset AK_2 = 0, AV_4 = AK_4 = K_2 \subset V_4, AK_2 = 0. \]

We see that \( (V_1, V_2, V_3, V_4, V_5) \in \mathcal{B}_u \). Thus, \( M' \subset \mathcal{B}_u \). Note that \( M' \) is isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \) hence it is a closed irreducible subvariety of \( \mathcal{B}_u \) of dimension 3. Thus it is an irreducible component of \( \mathcal{B}_u \). It satisfies \( J_{M'} = \{1, 3, 5\} \). Hence \( M' = M \).
3.22. Let
\[ T' = \{(V_1, V_2, V_3, V_4, V_5) \in \mathcal{B}; V_1 \subset K_2, K_4 \subset V_5, A^2V_5 = V_1, A\mathcal{V}_4 = V_2, V_2 \subset AV_5 \subset V_4\}, \]
\[ Z = \{(V_1, V_2, V_4, V_5) \in \mathcal{P}^1 \times \mathcal{P}^2 \times \mathcal{P}^4 \times \mathcal{P}^5; V_1 \subset V_2, V_4 \subset V_5, V_1 \subset K_2, K_4 \subset V_5, A^2V_5 = V_1, A\mathcal{V}_4 = V_2, V_2 \subset AV_5 \subset V_4\}. \]

The map \( \zeta : T' \to Z \), \((V_1, V_2, V_3, V_4, V_5) \mapsto (V_1, V_2, V_4, V_5) \) is a \( \mathbb{P}^1 \)-bundle. Let
\[ Z' = \{(V_1, V_5) \in \mathcal{P}^1 \times \mathcal{P}^5; V_1 \subset K_2, K_4 \subset V_5, A^2V_5 = V_1\}. \]
Consider the map \( \zeta' : Z \to Z' \), \((V_1, V_2, V_3, V_4, V_5) \mapsto (V_1, V_5) \). Its fibre at \((V_1, V_5)\) can be identified with the projective line \( \{V_4; A\mathcal{V}_5 \subset V_4 \subset V_5\} \) of the 2-dimensional vector space \( V_5/A\mathcal{V}_5 \) (note that \( V_2 \) is uniquely determined by \( V_4 \) via \( V_2 = A\mathcal{V}_5 \) and it automatically satisfies \( V_1 \subset V_2 \subset AV_5 \) since \( A^2V_5 \subset AV_4 \subset AV_5 \)); thus \( \zeta' \) is a \( \mathbb{P}^1 \)-bundle. Note also that \( Z' \) can be identified with the projective line \( \{V_5; K_4 \subset V_5 \subset V\} \) of the 2-dimensional vector space \( V/K_4 \). We see that \( T' \) is a \( \mathbb{P}^1 \)-bundle over a \( \mathbb{P}^1 \)-bundle over a \( \mathbb{P}^1 \)-bundle. In particular, \( T' \) is a closed smooth irreducible subvariety of dimension 3 of \( \mathcal{B} \). We show that \( T' \subset \mathcal{B}_u \). It is enough to show that if \((V_1, V_2, V_3, V_4, V_5) \in T'\), then \( A\mathcal{V}_i \subset V_i \) for \( i \in I \). For \( i = 5 \) this follows from \( A\mathcal{V}_5 \subset V_4 \subset V_5 \). For \( i = 4 \) this follows from \( A\mathcal{V}_4 = V_2 \subset V_4 \). For \( i = 3 \) this follows from \( A\mathcal{V}_3 \subset AV_4 = V_2 \subset V_3 \). For \( i = 2 \) this follows from \( A\mathcal{V}_2 \subset AV_4 = V_2 \). For \( i = 1 \) this follows from \( A\mathcal{V}_1 \subset AK_2 = 0 \). Since \( d_u = 3 \), we see that \( T' \in \mathcal{B}_u \).

Hence \( T'_i \) is defined for \( i \in I \).

3.23. Note that \( T'_3 = T' \). From the definitions we have
\[ T'_1 = \{(V_1, V_2, V_3, V_4, V_5) \in T'; V_2 = K_2\}, \]
\[ T'_2 = \{(V_1, V_2, V_3, V_4, V_5) \in T'; AV_3 \subset V_1\}, \]
\[ T'_3 = \{(V_1, V_2, V_3, V_4, V_5) \in T'; AV_5 \subset V_3\}, \]
\[ T'_4 = \{(V_1, V_2, V_3, V_4, V_5) \in T'; AV_5 \subset V_4\}, \]
\[ T'_5 = \{(V_1, V_2, V_3, V_4, V_5) \in T'; V_4 = K_4\}. \]

We show:
\( a \)
\[ T'_2 = T'_4, \]
\( b \)
\[ T'_1 = T'_5. \]

Let \((V_1, V_2, V_3, V_4, V_5) \in T'\).
If $AV_5 \subset V_3$ then, since $\dim AV_5 = 3$, we have $AV_5 = V_3$. Hence $AV_5 = A^2V_5 = V_1$. Thus $T'_1 \subset T'_2$.

If $AV_3 \subset V_1$ then $A^2V_3 \subset AV_4 = 0$ hence $V_3 \subset K_4$. Now $A : K_4 \rightarrow K_2$ is surjective and $V_1 \subset K_2$ hence $\{x \in K_4; Ax \in V_1\}$ is a 3-dimensional subspace of $K_4$ containing $V_3$ hence $\{x \in K_4; Ax \in V_1\} = V_3$. Now $AV_5 \subset \{x \in K_4; Ax \in V_1\}$ since $AV_5 \subset K_4$ and $A^2V_5 = V_1$. Hence $AV_5 \subset V_3$ (and even $AV_5 = V_3$). Thus $T'_2 \subset T'_4$.

Assume that $V_4 = K_4$. Now $AV_4 = V_2$, $AK_4 = K_2$ hence $V_2 = K_2$. Thus $T'_3 \subset T'_1$.

Assume that $V_2 = K_2$. We have $K_4 = A^{-1}(K_2)$, $V_4 \subset A^{-1}(V_2) = A^{-1}(K_2)$ so that $V_4 \subset K_4$. We see that $T'_1 \subset T'_5$.

This proves (a),(b). We set $T'_2 = T'_4$, $T'_1 = T'_5$. We have

$$T'_2 = \{(V_1, V_2, V_3, V_4, V_5) \in B; V_1 \subset K_2, K_4 \subset V_5, AV_5 = V_3, AV_3 = V_1, AV_4 = V_2\},$$

$$T'_1 = \{(V_1, V_2, V_3, V_4, V_5) \in B; V_2 = K_2, V_4 = K_4, A^2V_5 = V_1\}.$$

We set

$$3 := T'_2 \cap T'_1 = \{(V_1, V_2, V_3, V_4, V_5) \in B; AV_5 = V_3, AV_3 = V_1, V_2 = K_2, V_4 = K_4\}.$$

Now $\vartheta : T'_2 \rightarrow 3$, $(V_1, V_2, V_3, V_4, V_5) \mapsto (V_1, K_2, V_3, K_4, V_5)$ is a $P^1$-bundle; its fibre at $(V_1, K_2, V_3, K_4, V_5)$ is $\{(V_2, V_4) \in P^2 \times P^4; V_1 \subset V_2 \subset V_3, V_3 \subset V_4 \subset V_5, AV_4 = V_2\}$ which is isomorphic to the projective line $\{V_4; V_3 \subset V_4 \subset V_5\}$ of the 2-dimensional vector space $V_5/V_3$.

Note that 3 is isomorphic to the projective line $\{V_5; K_4 \subset V_5 \subset V\}$ of the 2-dimensional vector space $V/K_4$. Thus $T'_2$ is a $P^1$-bundle over $P^1$ and thus, having dimension 2, is $\ne T'$. Now

$$T'_1 \rightarrow \{(V_1, V_2, V_4, V_5) \in P^1 \times P^2 \times P^4 \times P^5; V_2 = K_2, V_4 = K_4, A^2V_5 = V_1\}$$

is a $P^1$-bundle whose base is isomorphic to the projective line $\{V_5; K_4 \subset V_5 \subset V\}$ of the 2-dimensional vector space $V/K_4$. Thus $T'_1$ is a $P^1$-bundle over $P^1$ and thus, having dimension 2, is $\ne T'$. Also $T'_1$ is a union of 3-lines (the fibres of the map above.)

We now see that $T'_i \ne T'$ for $i \in I - \{3\}$. It follows that $T' = T$, see 3.19.

Now any 3-line in $T$ is the fibre of $\zeta : T' \rightarrow Z$ at some $(V_1, V_2, V_4, V_5) \in Z$. This fibre contains exactly one point in $T'_2$ namely the one defined by $V_3 = AV_5$. This completes the proof of 3.19(a).

**3.24.** In this subsection we assume that $G_{ad}$ is of type $A_2A_2A_1$. We can write $I = \{0, 1, 2, 4, 5\}$ where the notation is chosen so that $s_1s_2, s_4s_5$ have order 3. We shall assume that $u \in G$ (see 1.1) is such that the image of $u$ in $G_{ad}$ has a projection to each of the three factors a subregular element in that factor. We have $d_u = 3$.

Now $B_u$ has four irreducible components: $C_1, C_2, C_3, C_4$ where $J_{C_1} = \{1, 0, 4\}$,
It is clear that there are well defined $Q^i \in \mathcal{P}^i$, $i = 1, 2, 4, 5$, such that $\cap_{i \in \{1, 2, 4, 5\}} Q^i$ contains a Borel subgroup and

\[
\begin{align*}
C_1 &= \{ B \in \mathcal{B}_u; B(2) = Q^2, B(5) = Q^5 \}, \\
C_2 &= \{ B \in \mathcal{B}_u; B(1) = Q^1, B(5) = Q^5 \}, \\
C_3 &= \{ B \in \mathcal{B}_u; B(1) = Q^1, B(4) = Q^4 \}, \\
C_4 &= \{ B \in \mathcal{B}_u; B(2) = Q^2, B(4) = Q^4 \},
\end{align*}
\]

3.25. In this subsection we assume that $G_{ad}$ is of type $A_2A_1A_1$. We can write $I = \{0, 1, 3, 5\}$ where the notation is chosen so that $s_0s_3$ has order 3. We shall assume that $u \in G$ (see 1.1) is such that the image of $u$ in $G_{ad}$ has a projection to each of the three factors a subregular element in that factor. We have $d_u = 3$. Now $\mathcal{B}_u$ has two irreducible components: $C'_1, C'_2$ where $J_{C'_1} = \{1, 3, 5\}$, $J_{C'_2} = \{0, 1, 5\}$. It is clear that there are well defined parabolic subgroups $P^0 \in \mathcal{P}^0, P^3 \in \mathcal{P}^3$ such that $P^0 \cap P^3$ contains a Borel subgroup and

\[
\begin{align*}
C'_1 &= \{ B \in \mathcal{B}_u; B(0) = P^0 \}, \\
C'_2 &= \{ B \in \mathcal{B}_u; B(3) = P^3 \}.
\end{align*}
\]

4. Euler characteristic computations

4.1. In this section we assume that $G_{ad}$ is of type $E_6$. We can write $I = \{0, 1, 2, 3, 4, 5\}$ where the numbering is chosen so that $s_1s_2, s_2s_3, s_3s_4, s_4s_5, s_0s_3$ have order 3. We shall assume that $u \in G$ (see 1.1) is such that $d_u = 3$; this determines $u$ uniquely up to conjugacy. The Springer representation $\rho_u$ of $W$ is a direct sum of two irreducible representations: one of dimension 30 (appearing in the third symmetric power of the reflection representation) and one of dimension 15 (appearing in the fifth symmetric power of the reflection representation). Using 1.4(b) and the knowledge of $\rho_u$ we see that there are exactly

- two irreducible components $X$ of $\mathcal{B}_u$ such that $J_X = \{3\}$ (we call them $S, T$);
- two irreducible components $X$ of $\mathcal{B}_u$ such that $J_X = \{0, 3\}$ (we call them $X(03), X'(03)$);
- one irreducible component $X$ of $\mathcal{B}_u$ such that $J_X = \{0, 1, 4\}$ (we call it $X(014)$);
- one irreducible component $X$ of $\mathcal{B}_u$ such that $J_X = \{0, 2, 4\}$ (we call it $X(024)$);
- one irreducible component $X$ of $\mathcal{B}_u$ such that $J_X = \{0, 2, 5\}$ (we call it $X(025)$);
- one irreducible component $X$ of $\mathcal{B}_u$ such that $J_X = \{0, 1, 5\}$ (we call it $X(015)$);
- one irreducible component $X$ of $\mathcal{B}_u$ such that $J_X = \{1, 3, 5\}$ (we call it $X(135)$).

From 1.1(b) we see that if $K$ is one of $\{0, 1, 4\}, \{0, 2, 4\}, \{0, 2, 5\}, \{0, 1, 5\}, \{1, 3, 5\}$ then there exists a unique $P_K \in \mathcal{P}_K$ such that $X(K) = \{ B \in \mathcal{B}; B \subset P_K \}$.

Now $u$ is induced from a unipotent element $u_1 \in P$ where $P \in \mathcal{P}_{01245}$ and $(u_1, \bar{P})$ is like $(u, G)$ in 3.24; in particular we have $u \in P, u_1 = \pi_P(u)$. The four irreducible components of $\mathcal{B}_{u_1}^P$ (see 3.24) give rise as in 1.5 to four irreducible components of $\mathcal{B}_u$ with the $J_X$ being preserved; these irreducible components of
Indeed the condition that $B$ is a Borel subgroup of $G$. Since the adjoint group of $\bar{Q}$ is a Borel subgroup in $Q$, we see also that $\bar{Q}$ is a parabolic subgroup in $Q$. From the previous argument we see also that $\cap i$ gives rise as in 1.5 to two irreducible components of $\bar{Q}$ being preserved; these irreducible components of $B_u$ must be the same as $X(135), X(015)$. Using 3.25 we see that there exist $\tilde{Q}^i = Q^0 \cap P'$ (relative to $G$) such that

$X(014) = \{B \in B; B(0) = Q^0, B(2) = Q^2, B(4) = Q^4\}$,

$X(024) = \{B \in B; B(1) = Q^1, B(3) = Q^3, B(5) = Q^5\}$,

$X(025) = \{B \in B; B(1) = Q^1, B(3) = Q^3, B(4) = Q^4\}$,

$X(015) = \{B \in B; B(2) = Q^2, B(3) = Q^3, B(4) = Q^4\}$.

(Note that these conditions determine $Q^i$ uniquely and that $Q^3 = P$.) Moreover, $\cap i \in \{1,2,3,4,5\} Q^i$ contains a Borel subgroup. Next we note that $u$ is induced from a unipotent element $u_2 \in \bar{P'}$ where $P' \in \mathcal{P}_{0135}$ and $(u_2, \bar{P'})$ is like $(u, G)$ in 3.25; in particular we have $u \in P', u_2 = \pi_{P'}(u)$. The two irreducible components of $B_{u_2}^P$ (see 3.25) give rise as in 1.5 to two irreducible components of $B_u$ with the $J_X$ being preserved; these irreducible components of $B_u$ must be the same as $X(135), X(015)$. Using 3.25 we see that there exist $\tilde{Q}^i = Q^0 \cap P'$ (relative to $G$) where $i \in \{0,2,3,4\}$ such that

$X(135) = \{B \in B; B(0) = \tilde{Q}^0, B(2) = \tilde{Q}^2, B(4) = \tilde{Q}^4\},$

$X(015) = \{B \in B; B(2) = \tilde{Q}^2, B(3) = \tilde{Q}^3, B(4) = \tilde{Q}^4\}.$

(Note that these conditions determine $\tilde{Q}^i$ uniquely and that $\tilde{Q}^2, \tilde{Q}^4$ contain $P'$.) Moreover, $\cap i \in \{0,2,3,4\} \tilde{Q}^i$ contains a Borel subgroup. Since we have

$X(015) = \{B \in B; B(2) = Q^2, B(3) = Q^3, B(4) = Q^4\}$

$= \{B \in B_u; B(2) = \tilde{Q}^2, B_3 = \tilde{Q}^3, B(4) = \tilde{Q}^4\}$

we see that we have $Q^2 = \tilde{Q}^2, Q^3 = \tilde{Q}^3, Q^4 = \tilde{Q}^4$. We set $Q^0 = \tilde{Q}^0$. Then we have

$X(135) = \{B \in B; B(0) = Q^0, B(2) = Q^2, B(4) = Q^4\}.$

From the previous argument we see also that $Q_0 := Q^1 \cap Q^2 \cap Q^3 \cap Q^4 \cap Q^5$ is a parabolic subgroup in $\mathcal{P}_0$ and $Q_{15} := Q^0 \cap Q^2 \cap Q^3 \cap Q^4$ is a parabolic subgroup in $\mathcal{P}_{15}$. Let $Q_{015} = Q^2 \cap Q^3 \cap Q^4$. We have $Q_{015} \in \mathcal{P}_{015}$ and $Q_0 \subset Q_{015}, Q_{15} \subset Q_{015}$. Since the adjoint group of $Q_{015}$ is of type $A_1 A_1 A_1$, the intersection $Q_0 \cap Q_{15}$ is a Borel subgroup of $Q_{015}$. We see that $Q^0 \cap Q^1 \cap Q^2 \cap Q^3 \cap Q^4 \cap Q^5$ is a Borel subgroup of $G$. We show:

(a) The intersection $X(135) \cap X(024)$ consists of exactly one Borel subgroup $B_0$.

Indeed the condition that $B \in B$ belongs to $X(135) \cap X(024)$ is the same as

$B(0) = Q^0, B(2) = Q^2, B(4) = Q^4, B(1) = Q^1, B(3) = Q^3, B(5) = Q^5$ that is, $B \subset Q^0 \cap Q^1 \cap Q^2 \cap Q^3 \cap Q^4 \cap Q^5$. It remains to use that the last intersection is a Borel subgroup of $G$. 

\[ \Box \]
4.2. Now \( u \) is induced from a unipotent element \( u' \in Q' \) where \( Q' \in \mathcal{P}_{12345} = \mathcal{P}^0 \) and \( (u', Q') \) is like \( (u, G) \) in 3.19; in particular, we have \( u \in Q', u' = \pi_{Q'}(u') \).

The irreducible components \( M \) (resp. \( T \)) of \( B_u^{Q'} \) as in 3.19 (with \( G, u \) replaced by \( Q', u' \)) give rise as in 1.5 to irreducible components of \( B_u \) with the same \( J_X \) which therefore must be equal to \( X(135) \) (resp. to \( X' \in \{S, T\} \); we arrange notation so that \( X' = T \)). From 3.19(a) we see that the following holds:

(a) there are well defined parabolic subgroups \( P'^{2} \in \mathcal{P}^2, P'^{4} \in \mathcal{P}^4 \) such that \( X(135) = \{B \in B; B(0) = Q, B(2) = P'^{2}, B(4) = P'^{4}\} \);

we have \( T \subset \{B \in B; B(0) = Q', B(1) \sqcup P'^{2}, B(5) \sqcup P'^{4}\} \);

we have \( T_1 = T_5 = \{B \in T; B(0) = Q', B(2) = P'^{2}, B(4) = P'^{4}\} = X(135) \cap T \) (we denote it by \( T_{15}\)); this is a \( \mathbb{P}^1 \)-bundle over \( \mathbb{P}^1 \) and is a union of 3-lines;

we have \( T_2 = T_4 \) (we denote it by \( T_{24}\)); it intersects any 3-line in \( T \) in exactly one point;

the intersection \( T_{15} \cap T_{24} \) is isomorphic to \( \mathbb{P}^1 \);

there is a unique morphism \( \vartheta : T_{24} \to T_{15} \cap T_{24} \) such that for any \( B \in T_{24} \) we have \( (B, \vartheta(B)) \in \mathcal{O}_w \) for some \( w \in W_{24} \); moreover, \( \vartheta \) is a \( \mathbb{P}^1 \)-bundle.

Since we have also \( X(135) = \{B \in B; B(0) = Q^0, B(2) = Q^2, B(4) = Q^4\} \) we see that we must have \( Q' = Q^0, P'^{2} = Q^2, P'^{4} = Q^4 \). Hence we have \( T \subset \{B \in B; B(0) = Q^0, B(1) \sqcup Q^2, B(5) \sqcup Q^4\} \).

4.3. Now \( u \) is induced from a unipotent element \( u'' \in \tilde{Q}'' \) where \( Q'' \in \mathcal{P}_{0234} \) and \( (u'', \tilde{Q}'') \) is like \( (u, G) \) in 3.1 with \( \alpha = 0, \beta = 2, \gamma = 4, \omega = 3 \); in particular we have \( u \in Q'', u'' = \pi_{Q''}(u) \). The irreducible components \( \tilde{S}, S_{\beta', \gamma}, S_{\beta', \gamma} \) (resp. \( S \)) of \( B_u^{Q''} \) defined as in 3.1 (with \( G, u \) replaced by \( \tilde{Q}'', u'' \)) give rise as in 1.5 to irreducible components of \( B_u \) with the same \( J_X \) which therefore must be equal to \( X(024), X(03), X'(03) \) (resp. \( X'' \in \{S, T\} \)); since \( \chi(S) = \chi(T) = 12 \) and \( \chi(S) = \chi(T) = 8 \) (see 3.10(c), 3.19(b)), we must have \( X'' = S \). We set \( Q'' = P^1 \cap P^5 \) where \( P^1 \in \mathcal{P}^1, P^5 \in \mathcal{P}^5 \). Let \( P^0 \neq \tilde{P}^0 \) in \( \mathcal{P}^0 \), \( P^2 \neq \tilde{P}^2 \) in \( \mathcal{P}^2 \), \( P^4 \neq \tilde{P}^4 \) in \( \mathcal{P}^4 \), \( P^3 \in \mathcal{P}^3 \) (with \( \mathcal{P}^i \) referring to \( G \)) be the inverse images under \( \pi_{Q''} \) of the parabolic subgroups with the same names in \( Q'' \).

From 3.1(a) we see that the following holds:

(a) We have

\[
X(024) = \{B \in B; B(1) = P^1, B(3) = P^3, B(5) = P^5\}
\]

\[
X(03) = \{B \in B; B(1) = P^1, B(2) = P^2, B(4) = \tilde{P}^1, B(5) = P^5\},
\]

\[
X'(03) = \{B \in B; B(1) = P^1, B(2) = \tilde{P}^2, B(4) = P^4, B(5) = P^5\},
\]

\[S \subset \{B \in B; B(1) = P^1, B(5) = P^5\}.\]

Since

\[
X(024) = \{B \in B; B(1) = Q^1, B(3) = Q^3, B(5) = Q^5\},
\]

we see that \( P^1 = Q^1, P^3 = Q^3, P^5 = Q^5 \). Thus,

\[
X(03) = \{B \in B; B(1) = Q^1, B(2) = P^2, B(4) = \tilde{P}^1, B(5) = Q^5\},
\]
4.4. Now $u$ is induced from a unipotent element $'u' \in 'Q$ where $'Q \in P_{01234} = \mathcal{P}^5$ and $(u,G)$ is like $(u,G)$ in 3.15 with $\delta = 1, \alpha = 2, \beta = 0, \gamma = 4, \omega = 3$; in particular we have $u \in 'Q', u = \pi_u Q(u)$. The irreducible components $S, E$ of $B_u$ defined in 3.15 (with $G, u$ replaced by $(u,G)$) give rise as in 1.5 to the irreducible components $S$ and $X(014)$ of $B_u$. From 3.16(a),(b),(c) we deduce that there is a well defined $\hat{P}^2 \in \mathcal{P}$ such that

$$X(014) = \{ B \in B; B(2) = \hat{P}^2, B(3) = Q^3, B(5) = 'Q' \},$$

$$\hat{P}^2 \notin \{ P^2, \tilde{P}^2 \},$$

$$S_1 = \{ B \in S; B(2) = \hat{P}^2 \}.$$

Since we have also

$$X(014) = \{ B \in B; B(2) = Q^2, B(3) = Q^3, B(5) = Q^5 \},$$

we see that $\hat{P}^2 = Q^2$, $'Q = Q^5$. Thus we have

$$Q^4 \notin \{ P^4, \hat{P}^4 \},$$

$$S_1 = \{ B \in S; B(2) = Q^2 \}.$$

4.5. Now $u$ is induced from a unipotent element $''u'' \in ''Q$ where $''Q \in P_{02345} = \mathcal{P}^1$ and $(''u,'''Q)$ is like $(u,G)$ in 3.15 with $\delta = 5, \alpha = 4, \omega = 3, \beta = 2, \gamma = 0$; in particular we have $u \in ''Q', u = \pi'_u Q(u)$. The irreducible components $S, E$ of $B_{''u}$ defined in 3.15 (with $G, u$ replaced by $''Q,'''u)$ give rise as in 1.5 to the irreducible components $S$ and $X(025)$ of $B_{''u}$. From 3.16(a),(b),(c) we deduce as in 4.4 that:

$$Q^4 \notin \{ P^4, \hat{P}^4 \},$$

$$S_5 = \{ B \in S; B(4) = Q^4 \}.$$

4.6. We show:

(a) $X(135) \cap X(03) = \emptyset, X(135) \cap X'(03) = \emptyset.$

Recall that

$$X(135) = \{ B \in B; B(0) = Q^0, B(2) = Q^2, B(4) = Q^4 \},$$

$$X(03) = \{ B \in B; B(1) = Q^1, B(2) = P^2, B(4) = \hat{P}^4, B(5) = Q^5 \},$$

$$X'(03) = \{ B \in B; B(1) = Q^1, B(2) = \tilde{P}^2, B(4) = P^4, B(5) = Q^5 \}.$$

If $B \in X(135) \cap X(03)$ then $B(2) = Q^2 = P^2$. This contradicts $Q^2 \neq P^2$. Similarly if $B \in X(135) \cap X'(03)$ then $B(2) = Q^2 = \tilde{P}^2$. This contradicts $Q^2 \neq \tilde{P}^2$. This proves (a).

Using $T_{15} = T \cap X(135)$ we see that (a) implies:

(b) $T_{15} \cap X(03) = \emptyset, T_{15} \cap X'(03) = \emptyset.$
4.7. We show:
(a) *The intersection of a 0-line in \( B \) with \( T \) is either a point or is empty.*
Assume that this intersection contains two distinct points \( B', B'' \). We have
\((B', B'') \in \mathcal{O}_{s_0}\).
Since \( T \subset Q_0^0 \), we have \( B' \in Q_0^0 \), \( B'' \in Q_0^0 \). Since
\( Q_0^0 \in \mathcal{P}_{12345} \), we have \((B', B'') \in \mathcal{O}_w\) where \( w \in W_{12345} \). This contradicts \( w = s_0 \); (a) is proved.
We show:
(b) *The intersection \( T \cap X(024) \cap X(03) \) is either a point or is empty.*
The intersection \( T \cap X(024) \cap X'(03) \) is either a point or is empty.
This follows from (a) since \( X(024) \cap X(03) \) is a 0-line and \( X(024) \cap X'(03) \) is a
0-line, see 4.3(b).

4.8. We write \( X(135) \cap X(024) = \{B_0\} \) as in 4.1(a). We show:
(a) \( T_{15} \cap T_{24} \cap T_0 \subset \{B_0\} \).
Let \( B \in T_{24} \cap T_0 \). Let \( P \in \mathcal{P}_{024}^0 \) be such that \( B \subset P \). Since \( B \in T_0 \cap T_2 \cap T_4 \),
we see from 1.3(a) that \( X = B^{\mathbb{P}} \) is an irreducible component of \( B_u \) and that
\( J_X = \{0, 2, 4\} \). Thus we have \( X = X(024) \). We see that \( B \in X(024) \). Thus
\( T_{24} \cap T_0 \subset X(024) \). Recall that \( T_{15} = X(135) \cap T \). Thus \( T_{15} \subset X(135) \)
and \( T_{15} \cap T_{24} \cap T_0 \subset X(024) \cap X(135) = \{B_0\} \); now (a) follows.

4.9. We show:
(a) *The morphism \( \vartheta : T_{24} \to T_{24} \cap T_{15} \), see 3.19, restricts to a morphism
\( \vartheta_0 : T_{24} \cap T_0 \to T_{24} \cap T_{15} \cap T_0 \). Moreover, \( \vartheta_0 \) is a \( \mathbb{P}^1 \)-bundle.*
Let \( B \in T_{24} \cap T_0 \) and let \( B' = \vartheta(B) \in T_{24} \cap T_{15} \). As in the proof of 4.8(a) we
have \( B \in X(024) \). Recall that \( (B, B') \in \mathcal{O}_w \) for some \( w \in W_{24} \). Since \( B \in X(024) \)
it follows that \( B' \in X(024) \). Thus \( B' \in T_{24} \cap T_{15} \cap T_0 \). Thus, \( \vartheta_0 \) is well
defined. Conversely, let \( B' \in T_{24} \cap T_{15} \cap T_0 \) and let \( B \in \vartheta^{-1}(B') \). Again we have
\( B' \in X(024) \) and \( (B, B') \in \mathcal{O}_w \) for some \( w \in W_{24} \). It follows that \( B \in X(024) \)
so that \( B \in T_{24} \cap T_0 \). We see that \( \vartheta_0 \) is a \( \mathbb{P}^1 \)-bundle. This proves (a).

We show:
(b) *If \( B_0 \in T \) (hence \( B_0 \in T_{24} \cap T_{15} \cap T_0 \)) then \( T_{24} \cap T_0 \) is a projective
line whose intersection with any 3-line in \( T \) has exactly one point or is empty.*  If
\( B_0 \not\in T \), then \( T_{24} \cap T_0 = \emptyset \).
Using (a), we see that it is enough to show the following statement:

*If \( B, B' \) are Borel subgroups in \( T_{24} \cap T_0 \) such that \( B, B' \) are on the same 3-line
in \( B \) then \( B = B' \).*
Since \( \vartheta(B) = \vartheta(B') = B_0 \), we have \( (B, B_0) \in \mathcal{O}_w \), \( (B', B_0) \in \mathcal{O}_{w'} \), for some \( w, w' \)
in \( W_{24} \). It follows that \( (B, B') \in \mathcal{O}_y \) for some \( y \in W_{24} \). Since \( B, B' \) are on the
same 3-line we have also \( (B, B') \in \mathcal{O}_z \) for some \( z \in W_3 \). This forces \( y = z = 1 \).
This proves (b).

4.10. We identify \( Y \subset S \) with a subvariety of \( S \). We show:
(a) *If \( B_0 \not\in T \) then \( S \cap T = \emptyset \). If \( B_0 \in T \) then \( S \cap T \) is the union of all 3-lines
which intersect \( \{B \in T_{24} \cap T_0 ; \vartheta(B) = B_0\} \cap Y \).*
For $K \subset I$ we denote by $w(K)$ an element of $W_K$. Let $B \in S \cap T$. The 3-line through $B$ must intersect $Y$ in a unique point $B_1$ and it must intersect $T_{24}$ in a unique point $B_2$. Let $B_3 = \vartheta(B_2) \in T_{24} \cap T_{15} \subset X(135)$. We have $(B_3, B_0) \in \mathcal{O}_{w(135)}$. We have $Y \subset X(024)$, hence $(B_1, B_0) \in \mathcal{O}_{w(024)}$. We have $(B, B_1) \in \mathcal{O}_{w(3)}$, hence $(B, B_0) \in \mathcal{O}_{w(3)w(024)}$.

We have $(B, B_2) \in \mathcal{O}_{w'(3)}$, $(B_2, B_3) \in \mathcal{O}_{w(24)}$, $(B_3, B_0) \in \mathcal{O}_{w(135)}$. If $w(24) \neq 1$ it follows that $(B, B_0) \in \mathcal{O}_{w'(3)w(24)w(135)}$ hence $w'(3)w(24)w(135) = w(3)w(024)$ so that $w(135) - 1, w'(3) = w(3), w(024) = w(24)$ and $B_3 = B_0, B_1 = B_2 \in T_{24} \cap Y, (B_1, B_0) \in \mathcal{O}_{w(24)}$. Since $B_3 \in T$ and $B_3 = B_0$ we have also $B_0 \in T$ (hence $B_0 \in T_{24} \cap T_{15} \cap T_0$) and $B_1 \in \{B' \in T_{24} \cap T_0; \vartheta(B) = B_0\} \cap Y$.

If $w_{24} = 1$ we must have $B_2 = B_3$, $(B_3, B_0) \in \mathcal{O}_{w_1(3)}$. Since $B_3 \in T$ and $T$ is a union of 3-lines it follows that $B_0 \in T$. From $(B, B_2) \in \mathcal{O}_{w'(3)}$, $(B_2, B_0) \in \mathcal{O}_{w_1(3)}$ we obtain $(B, B_0) \in \mathcal{O}_{w_2(3)}$.

Now (a) follows (we have used 4.9(b)).

**4.11.** We show:

(a) Assume that $B_0 \in T$. Let $U = \{B \in T_{24} \cap T_0; \vartheta(B) = B_0\}$ (a projective line). Then either $U \cap Y \cong \mathbb{P}^1$ or $U \cap Y$ consists of two points or $U \cap Y$ consists of one point.

Recall that $Y$ can be viewed as the variety

$$\{(z_0, z'_0, z_2, z'_2, y, y') \in \mathbb{P}^3 \times \mathbb{P}^1; z_0z'_0 + z_2z'_2 = 0, z_2y' + z'_2y = 0\}$$

or setting $z_0 = ac, z'_0 = bd, z_2 = -ad, z'_2 = bc$, as the variety

$$\{((a, b), (c, d), (y, y')) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1; -ady' + bcy = 0\}.$$

Now $U$ can be viewed as a subvariety of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of the form

$$\{((a, b), (c, d), (y, y')) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1; c = m_1a + m_2b, d = m'_1a + m'_2b, y = y_0, y' = y_0\}$$

where $(y_0, y'_0) \in \mathbb{P}^1$ is fixed and $(m_1, m_2, m'_1, m'_2) \in k^4$ is fixed such that $m_1m'_2 - m_2m'_1 \neq 0$. Then $U \cap Y$ becomes

$$\{((a, b), (c, d), (y, y')) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1; -a(m'_a + m'_2b)y'_0 + b(m_1a + m_2b)y_0 = 0, y = y_0, y' = y_0\}$$

$$\cong \{((a, b) \in \mathbb{P}^1; -a(m'_a + m'_2b)y'_0 + b(m_1a + m_2b)y_0 = 0\}$$

$$= \{((a, b) \in \mathbb{P}^1; -m'_y'0a^2 + (-m'_2y'_0 + m_1y_0)ab + m_2y_0b^2 = 0\}.$$

It follows that, if each of $-m'_1y'_0, -m'_2y'_0 + m_1y_0, m_2y_0$ is zero then $U \cap Y \cong \mathbb{P}^1$; otherwise, $U \cap Y$ consists of one or two points. This proves (a).

**4.12.** We set $S^o = S - (S_0 \cup S_1 \cup S_2 \cup S_4 \cup S_5)$. This is an open dense subset of $S$. We have the following result.
Proposition 4.13. We have \( \chi(S^\circ) = 1 \).

Under the isomorphism \( S \xrightarrow{\sim} S, S_5 \) (resp. \( S_1 \)) corresponds to a closed subsets \( R_5 \) (resp. \( R_1 \)) of \( S \) and \( S^\circ \) corresponds to the open dense subset \( S^\circ := S - (S_0 \cup R_1 \cup S_2 \cup S_4 \cup R_5) \) of \( S \). It is enough to prove that

\[
\chi(S^\circ) = 1.
\]

The proof is given in 4.14-4.16.

4.14. To prove that \( \chi(S^\circ) = 1 \) we can identify our \( S \) with \( X \subset B \) as in 3.12(c) in such a way that \( S_0 = X_\alpha, S_2 = X_\beta, S_4 = X_\gamma \) and, if

\[
A : V \rightarrow V, L_1, L_1', L_4, L_4', L_4', I_2, K_6
\]

are as in 3.2, 3.4, then \( R_5 \) is identified with

\[
\{(V_1, V_2, V_4, \tilde{V}_4) \in X; V_4 = \tilde{V}_4^1\},
\]

\( R_1 \) is identified with

\[
\{(V_1, V_2, V_4, \tilde{V}_4) \in X; \tilde{V}_4 = \tilde{V}_4^1\},
\]

where \( \tilde{V}_4^1 \) is an isotropic 4-space in \( V \) in the same family as \( L_4^1, L_4'^1 \) but distinct from each of them and \( \tilde{V}_4^1 \) is an isotropic 4-space in \( V \) in the same family as \( L_4^1, L_4'^1 \) but distinct from each of them; moreover we have \( I_2 \subset \tilde{V}_4, I_2 \subset \tilde{V}_4^1 \).

Recall that \( X_\alpha = X_\alpha^* \cup Y, X_\beta = X_\beta^* \cup Y, X_\gamma = X_\gamma^* \cup Y \), where

\[
X_\alpha^* = \{(V_1, V_2, V_4, \tilde{V}_4) \in \mathcal{B}; V_1 \subset I_2, V_4 = L_4^1, \tilde{V}_4 = L_4'^1\}
\]

\[
\sqcup \{(V_1, V_2, V_4, \tilde{V}_4) \in \mathcal{B}; V_1 \subset I_2, V_4 = L_4^1, \tilde{V}_4 = L_4\},
\]

\[
X_\beta^* = \{(V_1, V_2, V_4, \tilde{V}_4) \in \mathcal{B}; I_2 \subset V_4, V_1 = L_1, \tilde{V}_4 = L_4\}
\]

\[
\sqcup \{(V_1, V_2, V_4, \tilde{V}_4) \in \mathcal{B}; I_2 \subset V_4, V_1 = L_1, \tilde{V}_4 = L_4', \tilde{V}_4 = L_4'^1\},
\]

\[
X_\gamma^* = \{(V_1, V_2, V_4, \tilde{V}_4) \in \mathcal{B}; I_2 \subset \tilde{V}_4, V_1 = L_1, V_4 = L_4\}
\]

\[
\sqcup \{(V_1, V_2, V_4, \tilde{V}_4) \in \mathcal{B}; I_2 \subset \tilde{V}_4, V_1 = L_1, V_4 = L_4', \tilde{V}_4 = L_4'^1\},
\]

\[
Y = \{(V_1, V_2, V_4, \tilde{V}_4) \in X; V_2 = I_2\}.
\]

4.15. As in 3.10 let \( Z \) be the variety of all pairs \( V_1, V_3 \) of isotropic subspaces of \( V \) of dimension 1 and 3 respectively such that \( V_1 \subset I_2 \subset V_3 \) and \( AV_3 \subset V_1 \) and let \( \mathcal{U} \) be the space of isotropic 3-spaces in \( V \) that contain \( I_2 \) hence are contained in \( K_6 \); we have \( \mathcal{U} \cong \mathbb{P}^1 \times \mathbb{P}^1 \). From 3.10(d) we have \( \chi(Z) = 6 \).

Let \( Z_1 = \{(V_1, V_3) \in Z; V_3 \subset L_4\}, Z_5 = \{(V_1, V_3) \in Z; V_3 \subset \tilde{V}_4\} \). Let \( \mathcal{U}_1 = \{V_3 \in \mathfrak{U}; V_3 \subset L_4\}, \mathcal{U}_5 = \{V_3 \in \mathfrak{U}; V_3 \subset \tilde{V}_4\} \). Note that \( \mathcal{U}_1 \cong \mathbb{P}^1, \mathcal{U}_5 \cong \mathbb{P}^1 \). By
the argument in the proof of 3.10(d), the map \( Z \to \mathcal{U} \), \((V_1, V_3) \to V_3\), restricts to an isomorphism \( Z_1 \cong \mathcal{U}_1 \) and to an isomorphism \( Z_2 \cong \mathcal{U}_2 \) (note that \( \mathcal{U}_1, \mathcal{U}_2 \) do not contain the exceptional points \( \to L_4 \cap L_4^! \) and \( \to L_4' \cap L_4' \)). We see that \( Z_1 \cong \mathbb{P}^1 \), \( Z_5 \cong \mathbb{P}^1 \). It follows that
\[
\chi(Z_1) = \chi(Z_5) = 2.
\]

We have \( Z_1 \cap Z_5 = \{(V_1, V_3) \in Z; V_1 \subset I_2, V_3 = \to L_4 \cap L_4^! \} \cup \{(V_1, V_3) \in Z; V_1 \subset I_2, V_3 = \to L_4' \cap L_4' \}, \]
\[
\to Z_\alpha^* = \{(V_1, V_3) \in Z; V_1 = L_1, V_3 \subset L_4^! \} \cup \{(V_1, V_3) \in Z; V_1 = \to L_4', V_3 \subset L_4' \}, \]
\[
\to Z_\beta^* = \{(V_1, V_3) \in Z; V_1 = \to L_4, V_3 \subset \to L_4^! \} \cup \{(V_1, V_3) \in Z; V_1 = \to L_4', V_3 \subset \to L_4' \}, \]
\[
\to Z_\gamma^* = \{(V_1, V_3) \in Z; V_1 = \to L_4, V_3 \subset \to L_4 \} \cup \{(V_1, V_3) \in Z; V_1 = \to L_4', V_3 \subset \to L_4 \}, \]

We have
\[
\chi(Z_\alpha^* \cap Z_\beta^*) = \chi(Z_\alpha^* \cap Z_\gamma^*) = \chi(Z_\beta^* \cap Z_\gamma^*) = 2.
\]

Thus, each of \( Z_\alpha^*; Z_\beta^*; Z_\gamma^* \) is a disjoint union of two copies of \( \mathbb{P}^1 \); each of \( Z_\alpha^* \cap Z_\beta^* \), \( Z_\alpha^* \cap Z_\gamma^* \), \( Z_\beta^* \cap Z_\gamma^* \) consists of two points. We see that
\[
\chi(Z_\alpha^*) = \chi(Z_\beta^*) = \chi(Z_\gamma^*) = 4,
\]
\[
\chi(Z_\alpha^* \cap Z_\beta^*) = \chi(Z_\alpha^* \cap Z_\gamma^*) = \chi(Z_\beta^* \cap Z_\gamma^*) = 2
\]
\[
\chi(Z_\alpha^* \cap Z_\beta^* \cap Z_\gamma^*) = 0.
\]

We have
\[
\chi(Z_\alpha^* \cup Z_\beta^* \cup Z_\gamma^*) = \chi(Z_\alpha^*) + \chi(Z_\beta^*) + \chi(Z_\gamma^*)
\]
\[
- \chi(Z_\alpha^* \cap Z_\beta^*) - \chi(Z_\alpha^* \cap Z_\gamma^*) - \chi(Z_\beta^* \cap Z_\gamma^*) + \chi(Z_\alpha^* \cap Z_\beta^* \cap Z_\gamma^*)
\]
\[
= 4 + 4 + 4 - 2 - 2 - 2 + 0 = 6.
\]
We have

\[ Z^*_\alpha \cap Z_1 = \emptyset, Z^*_\gamma \cap Z_1 = \emptyset, \]
\[ Z^*_\beta \cap Z_1 = \{(L'_1, \hat{V}_4 \cap L'_4)\} \cup \{(L_1, \hat{V}_4 \cap L_4)\}, \]
\[ Z^*_\alpha \cap Z_5 = \emptyset, Z^*_\beta \cap Z_5 = \emptyset, \]
\[ Z^*_\gamma = \{(L_1, \hat{V}_4 \cap L'_4)\} \cup \{(L'_1, \hat{V}_4 \cap L'_4)\}. \]

Thus

\[ (Z^*_\alpha \cup Z^*_\beta \cup Z^*_\gamma) \cap Z_1 = \{(L'_1, \hat{V}_4 \cap L'_4)\} \cup \{(L_1, \hat{V}_4 \cap L_4)\}, \]
\[ (Z^*_\alpha \cup Z^*_\beta \cup Z^*_\gamma) \cap Z_5 = \{(L_1, \hat{V}_4 \cap L'_4)\} \cup \{(L'_1, \hat{V}_4 \cap L'_4)\}. \]

Hence

\[ (Z^*_\alpha \cup Z^*_\beta \cup Z^*_\gamma) \cap (Z_1 \cap Z_5) = \emptyset. \]

We have

\[ \chi((Z^*_\alpha \cup Z^*_\beta \cup Z^*_\gamma) \cap Z_1) = \chi((Z^*_\alpha \cup Z^*_\beta \cup Z^*_\gamma) \cap Z_5) = 2, \]
\[ \chi(Z^*_\alpha \cup Z^*_\beta \cup Z^*_\gamma) \cap (Z_1 \cap Z_5)) = 0 \]

hence

\[ \chi((Z^*_\alpha \cup Z^*_\beta \cup Z^*_\gamma) \cap (Z_1 \cup Z_5)) \]
\[ = \chi((Z^*_\alpha \cup Z^*_\beta \cup Z^*_\gamma) \cap Z_1) + \chi((Z^*_\alpha \cup Z^*_\beta \cup Z^*_\gamma) \cap Z_5) \]
\[ - \chi((Z^*_\alpha \cup Z^*_\beta \cup Z^*_\gamma) \cap (Z_1 \cap Z_5)) = 2 + 2 - 0 = 4. \]

We have

\[ \chi((Z^*_\alpha \cup Z^*_\beta \cup Z^*_\gamma) \cup (Z_1 \cup Z_5)) \]
\[ = \chi(Z^*_\alpha \cup Z^*_\beta \cup Z^*_\gamma) + \chi(Z_1 \cup Z_5) \]
\[ - \chi((Z^*_\alpha \cup Z^*_\beta \cup Z^*_\gamma) \cap (Z_1 \cup Z_5)) = 6 + 3 - 4 = 5. \]

4.16. We have a partition \( Z = Z' \cup Z'' \) where \( Z' = Z^*_\alpha \cup Z^*_\beta \cup Z^*_\gamma \cup Z_1 \cup Z_5 \) and \( Z'' = Z - Z' \). Recall that \( \zeta : (V_1, V_2, V_4, \hat{V}_4) \mapsto (V_1, V_4 \cap \hat{V}_4) \) makes \( X \) into a \( \mathbb{P}^1 \)-bundle over \( Z \) and that

\[ Y = \{(V_1, V_2, V_4, \hat{V}_4) \in X; V_2 = I_2\}. \]

Recall that \( \zeta \) restricts to an isomorphism \( Y \xrightarrow{\sim} Z \). Hence setting \( Y' = \zeta^{-1}(Z') \cap Y \), \( Y'' = \zeta^{-1}(Z'') \cap Y \), we have a partition \( Y = Y' \cup Y'' \) and \( \zeta \) restricts to isomorphisms \( Y' \xrightarrow{\sim} Z', Y'' \xrightarrow{\sim} Z'' \). We have

\[ \chi(Y'') = \chi(Z'') = \chi(Z) - \chi(Z') = 6 - 5 = 1. \]
We have
\[ X_\alpha^* = \zeta^{-1}(Z_\alpha^*), \quad X_\beta^* = \zeta^{-1}(Z_\beta^*), \quad X_\gamma^* = \zeta^{-1}(Z_\gamma^*), \]
\[ R_1 = \zeta^{-1}(Z_1), \quad R_5 = \zeta^{-1}(Z_5). \]
Hence
\[ X_\alpha^* \cup X_\beta^* \cup X_\gamma^* \cup R_1 \cup R_5 = \zeta^{-1}(Z'), \]
so that
\[ \chi(X_\alpha^* \cup X_\beta^* \cup X_\gamma^* \cup R_1 \cup R_5) = \chi(\zeta^{-1}(Z')) = 2\chi(Z') = 10. \]
Recall from 3.1(c) that
\[ X_\alpha \cup X_\beta \cup X_\gamma \cup R_1 \cup R_5 = X_\alpha^* \cup X_\beta^* \cup X_\gamma^* \cup R_1 \cup R_5 \cup Y \]
hence we have a partition
\[ X_\alpha \cup X_\beta \cup X_\gamma \cup R_1 \cup R_5 = (X_\alpha^* \cup X_\beta^* \cup X_\gamma^* \cup R_1 \cup R_5) \cup Y'', \]
so that
\[ \chi(X_\alpha \cup X_\beta \cup X_\gamma \cup R_1 \cup R_5) = \chi(X_\alpha^* \cup X_\beta^* \cup X_\gamma^* \cup R_1 \cup R_5) + \chi(Y'') = 10 + 1 = 11. \]
We have \( \chi(X) = 2\chi(Z) = 12 \) (see 3.10(c),(d)) hence
\[ \chi(X - (X_\alpha \cup X_\beta \cup X_\gamma \cup R_1 \cup R_5)) = \chi(X) - \chi(X_\alpha \cup X_\beta \cup X_\gamma \cup R_1 \cup R_5) = 12 - 11 = 1. \]
This proves 4.13(a). Proposition 4.13 is proved.

4.17. We show:
(a) Let \( \mathcal{L} \) be a 3-line in \( T \). Then either \( \mathcal{L} \subset T_0 \) or \( \mathcal{L} \cap T_0 \) is exactly one point.
Let \( B \in \mathcal{L} \). Let \( P \in \mathcal{P}_{03} \) be such that \( B \subset P \). We have \( u \in P \). Let \( u' = \pi_P(u) \).
Let \( \tilde{\mathcal{L}} = \{ \pi_P(B'); B' \in \mathcal{L} \} \). Note that \( \tilde{\mathcal{L}} \) is a line contained in \( B_u^P \). It follows that \( u' \) is not regular unipotent in \( P \). Since \( P_{ad} \) is of type \( A_2 \), \( u' \) must be subregular or 1. If \( u' = 1 \) then \( B_{u'}^P = B^P \) so that for any \( B' \in \mathcal{L} \) the 0-line through \( B' \) is contained in \( B^P \) hence is contained in \( B_u^P \); thus \( \mathcal{L} \subset T_0 \). If \( u' \) is subregular then \( B_{u'}^P \) is the union of two projective lines which intersect in a single point. We see that there is a unique \( B' \in \mathcal{L} \) such that the 0-line through \( B' \) is contained in \( B_u^P \); thus \( B' \in T_0 \). This proves (a).

4.18. We set \( T^\circ = T - (T_0 \cup T_1 \cup T_2 \cup T_4 \cup T_5) \). This is an open dense subset of \( T \). We have the following result.
Proposition 4.19. If $B_0 \notin \mathbf{T}$ then $\chi(T^{\circ}) = 0$. If $B_0 \in \mathbf{T}$ then $\chi(T^{\circ}) = 1$.

Let $\mathfrak{V}$ be the variety of all 3-lines in $\mathbf{T}$. We have a partition $\mathfrak{V} = \mathfrak{V}_{15} \sqcup \mathfrak{W} \sqcup \mathfrak{V} \sqcup \mathfrak{V}''$ where

- $\mathfrak{V}_{15}$ is the variety of all 3-lines $\mathcal{L}$ such that $\mathcal{L} \subset T_{15}$;
- $\mathfrak{W}$ is the variety of all 3-lines $\mathcal{L}$ such that $\mathcal{L} \subset \mathbf{T} - T_{15}$, $\mathcal{L} \subset T_0$;
- $\mathfrak{V}'$ is the variety of all 3-lines $\mathcal{L}$ such that $\mathcal{L} \subset \mathbf{T} - T_{15}$, $\mathcal{L} \cap T_0$ is exactly one point and that point is in $T_{24}$;
- $\mathfrak{V}''$ is the variety of all 3-lines $\mathcal{L}$ such that $\mathcal{L} \subset \mathbf{T} - T_{15}$, $\mathcal{L} \cap T_0$ is exactly one point and that point is not in $T_{24}$.

The fact that this partition is well defined follows from 3.19(a) and the fact that $T_{15}$ is a union of 3-lines. Let $\xi : T \to \mathfrak{V}$ be the map which associates to $B \in \mathbf{T}$ the 3-line containing $B$.

The inverse images of $\mathfrak{V}_{15}, \mathfrak{W}, \mathfrak{V}', \mathfrak{V}$ under $\xi$ are denoted by $T_{15}, T', T''$. (This agrees with our earlier definition of $T_{15}$.) We have $T = T_{15} \sqcup T \sqcup T' \sqcup T''$. We set $T^{\circ} = T' \cap T^\circ$, $T'' = T'' \cap T^{\circ}$. Clearly, we have $T^{\circ} = T^{\circ} \sqcup T^{\circ}$. Let $T'' \to \mathfrak{V}''$ be the restriction of $\xi$. This is a fibration with each fibre being a projective line from which two points have been removed: one in $T_{24}$ and one in $T_0$. It follows that $\chi(T''^{\circ}) = 0$. Let $T'' \to \mathfrak{V}'$ be the restriction of $\xi$. This is a fibration with each fibre being a projective line from which one point has been removed: the one in $T_{24}$ (which is also in $T_0$). It follows that $\chi(T'') = \chi(\mathfrak{V}')$. We have

$$\chi(T^{\circ}) = \chi(T''^{\circ}) + \chi(T''^{\circ}) = 0 + \chi(\mathfrak{V}') = \chi(\mathfrak{V}')$$

If $B_0 \notin \mathbf{T}$ then $T_{24} \cap T_0 = \emptyset$ (see 4.9(b)) hence $\mathfrak{V}' = \emptyset$ and $\chi(T^{\circ}) = \chi(\mathfrak{V}') = 0$.

Now assume that $B_0 \in \mathbf{T}$. Then by 4.9(b), $\mathfrak{V}'$ is isomorphic to $T_{24} \cap T_0 - T_{24} \cap T_{15} \cap T_0$ and this is a projective line minus a point. Thus, in this case, $\chi(T^{\circ}) = \chi(\mathfrak{V}') = 1$.

4.20. We show:

(a) If $B_0 \notin \mathbf{T}$ then $S \cap T = \emptyset$; hence $S^{\circ} \cap T^{\circ} = \emptyset$ and $\chi(S^{\circ} \cap T^{\circ}) = 0$.

(b) If $B_0 \in \mathbf{T}$ then $\chi(S \cap T^{\circ})$ is 1 or 0.

(c) If $B_0 \in \mathbf{T}, B_0 \notin S$ then $\chi(S^{\circ} \cap T^{\circ})$ is 2 or 1.

Now (a) follows from 4.10(a). Next we assume that $B \in \mathbf{T}$. As in 4.11(a), let $U = \{B \in T_{24} \cap T_0; \vartheta(B) = B_0\}$. We have a fibration $S^{\circ} \cap T^{\circ} \to (U \cap Y) - \{B_0\}$. This associates to $B$ the intersection of the 3-line through $B$ with $T_{24}$, see 4.2(a); each fibre is a projective line with a point removed. It follows that $\chi(S^{\circ} \cap T^{\circ}) = \chi((U \cap Y) - \{B_0\})$. This equals $\chi(U \cap Y)$ if $B_0 \notin Y$ (that is if $B_0 \notin S$) and it equals $\chi(U \cap Y) - 1$ if $B_0 \in Y$ (that is if $B_0 \in S$). It remains to use 4.11(a).

4.21. We show:

(a) If $B_0 \notin \mathbf{T}$ then $\chi(S^{\circ} \cup T^{\circ}) = 1$.

(b) If $B_0 \in \mathbf{T}$ then $\chi(S^{\circ} \cup T^{\circ}) \in \{0, 1, 2\}$.

We have $\chi(S^{\circ} \cup T^{\circ}) = \chi(S^{\circ}) + \chi(T^{\circ}) - \chi(S^{\circ} \cap T^{\circ})$. It remains to use 4.13, 4.19 and 4.20(a),(b),(c).
4.22. Let $\mathcal{B}_u^c = \mathcal{B}_u - (\mathcal{B}_{u,0} \cup \mathcal{B}_{u,1} \cup \mathcal{B}_{u,2} \cup \mathcal{B}_{u,4} \cup \mathcal{B}_{u,5})$. We show:

(a) $\mathcal{B}_u^c = \mathcal{S}^c \cup \mathcal{T}^c$.

The inclusion $\mathcal{S}^c \cup \mathcal{T}^c \subset \mathcal{B}_u^c$ is obvious. Conversely, let $B \in \mathcal{B}_u^c$. It is enough to show that $B \in \mathcal{S} \cup \mathcal{T}$. We have $B \in X$ for some $X \in \mathcal{E}$. If $X$ is $\mathcal{S}$ or $\mathcal{T}$, we are done. Thus we can assume that $X \notin \{\mathcal{S}, \mathcal{T}\}$. We have $J_X \neq \{3\}$ and since $J_X \neq \emptyset$ we have $i \in J_X$ for some $i \in I - \{3\}$ and, in particular, $X \subset \mathcal{B}_{u,0} \cup \mathcal{B}_{u,1} \cup \mathcal{B}_{u,2} \cup \mathcal{B}_{u,4} \cup \mathcal{B}_{u,5}$. Since $B \in X$, we have $B \in \mathcal{B}_{u,0} \cup \mathcal{B}_{u,1} \cup \mathcal{B}_{u,2} \cup \mathcal{B}_{u,4} \cup \mathcal{B}_{u,5}$. This contradicts $B \in \mathcal{B}_u^c$. This proves (a).

From (a) and 4.21(a),(b) we deduce

(b) We have $\chi(\mathcal{B}_u^c) \in \{0, 1, 2\}$.

4.23. Let $\mathcal{P}_u^{reg}$ be the set of all $P \in \mathcal{P}_{1245}$ such that $u \in P$ and $\pi_P(u)$ is a regular unipotent element of $\bar{P}$. We define a map $\phi : \mathcal{P}_u^{reg} \to \mathcal{B}_u$ by $P \mapsto B$ where $B$ is the unique Borel subgroup such that $u \in B \subset \bar{P}$. Let $\mathcal{B}_u^{reg} = \mathcal{B}_u - (\mathcal{B}_{u,1} \cup \mathcal{B}_{u,2} \cup \mathcal{B}_{u,4} \cup \mathcal{B}_{u,5})$. We show:

(a) $\phi$ defines an isomorphism $\phi_0 : \mathcal{P}_u^{reg} \sim \mathcal{B}_u^{reg}$.

If $P, P' \in \mathcal{P}_u^{reg}$ and $\phi(P) = \phi(P') = B$ then $P, P'$ are parabolic subgroups in $\mathcal{P}_{1245}$ containing $B$, hence $P = P'$. Assume now that $P \in \mathcal{P}_u^{reg}$ and let $B = \phi(P)$. For $i \in \{1, 2, 4, 5\}$ let $P_i$ be the parabolic subgroup in $\mathcal{P}_i$ such that $B \subset P_i$. We have $P_i \subset P$ and $u \in U_B$. Since $\pi_P(u)$ is regular unipotent in $\bar{P}$ we have $\pi_P(u) \notin \pi_P(U_{P_i})$ that is, $u \notin U_{P_i}$; but this is the same as saying that $B \neq \mathcal{B}_{u,i}$. Thus, $\phi(P) \in \mathcal{B}_u^{reg}$. We see that $\phi$ restricts to an injective map $\phi_0 : \mathcal{P}_u^{reg} \to \mathcal{B}_u^{reg}$.

Now let $B \in \mathcal{B}_u^{reg}$. Let $P$ be the unique parabolic subgroup in $\mathcal{P}_{1245}$ such that $B \subset P$. We have $u \in P$. Again, for $i \in \{1, 2, 4, 5\}$, let $P_i$ be the parabolic subgroup in $\mathcal{P}_i$ such that $B \subset P_i$. We have $P_i \subset P$ and $u \in U_B$. Since $B \notin \mathcal{B}_{u,i}$, we have $u \notin U_{P_i}$ hence $\pi_P(u) \notin \pi_P(U_{P_i})$. It follows that $\pi_P(u)$ is regular unipotent in $\bar{P}$. Thus $\phi_0$ is surjective hence a bijection. We omit the proof of the fact that $\phi_0$ is an isomorphism.

4.24. We show:

(a) Let $\mathcal{L}$ be a 0-line in $\mathcal{B}$ such that $\mathcal{L} \cap \mathcal{B}_u^{reg} \neq \emptyset$. If $\mathcal{L} \subset \mathcal{B}_u$, then $\mathcal{L} \subset \mathcal{B}_u^{reg}$. If $\mathcal{L} \notin \mathcal{B}_u$, then $\mathcal{L} \cap \mathcal{B}_u = \mathcal{L} \cap \mathcal{B}_u^{reg}$ is a single point.

Assume first that $\mathcal{L} \subset \mathcal{B}_u$. Let $B \in \mathcal{L} \cap \mathcal{B}_u^{reg}$. If $\mathcal{L}$ contains some $B' \neq B$ such that $B' \in \mathcal{B}_{u,i}$ for some $i \in \{1, 2, 4, 5\}$ then, applying 1.1(a) with $B$ replaced by $B'$ and $J = \{0, i\}$ we see that $\mathcal{B}^{P} \subset \mathcal{B}_u$ where $P \in \mathcal{P}_{0,i}$ contains $B'$. We have $B \in \mathcal{B}^{P}$ and the line of type $i$ through $B$ is contained in $\mathcal{B}^{P}$ hence in $\mathcal{B}_u$, so that $B \in \mathcal{B}_{u,i}$; this contradicts $B \in \mathcal{B}_u^{reg}$. We see that $\mathcal{L} \subset \mathcal{B}_u^{reg}$.

Assume next that $\mathcal{L} \notin \mathcal{B}_u$. Clearly, the intersection of any 0-line with $\mathcal{B}_u$ is either empty, or $\mathcal{L}$, or a point. In our case $\mathcal{L} \cap \mathcal{B}_u$ is not $\mathcal{L}$ and is not empty hence it is a point. Hence $\mathcal{L} \subset \mathcal{B}_u^{reg}$ is either empty or a point; by assumption it is nonempty hence it is a point. This proves (a).

4.25. From 4.24(a) we see that we have a partition $\mathcal{B}_u^{reg} = \mathcal{B}_u^{reg} \cup \mathcal{B}_u''^{reg}$ where $\mathcal{B}_u''^{reg}$ is the union of the 0-lines contained in $\mathcal{B}_u^{reg}$ and $\mathcal{B}_u^{reg}$ is the set of all
Let $B \in \mathcal{B}_u^{reg}$ such that the 0-line through $B$ intersects $\mathcal{B}_u$ in exactly one point, $B$. Note that $'\mathcal{B}_u^{reg} = \mathcal{B}_u^{r}$. Hence, using 4.22(b), we have

(a) $\chi('\mathcal{B}_u^{reg}) \in \{0, 1, 2\}$.

For any $P \in \mathcal{P}_{1245}$ we denote by $Q_P$ the unique parabolic subgroup in $\mathcal{P}_{01245}$ such that $P \subset Q_P$; note that $(Q_P)_{ad} = \bar{P}_{ad} \times H_P$ (canonically) where $H_P \cong \operatorname{PGL}_2(k)$ (we use that $s_0$ commutes with $W_{1245}$). Let $'\mathcal{P}_u^{reg}$ (resp. $'\mathcal{P}_u^{reg}$) be the set of all $P \in \mathcal{B}_u^{reg}$ such that the image of $u \in Q_P$ under the obvious composition $Q_P \to (Q_P)_{ad} \to H_P$ is $1 \in H_P$ (resp. a regular unipotent element of $H_P$).

From the definitions we see that under the isomorphism $\phi_0 : \mathcal{P}_u^{reg} \xrightarrow{\sim} \mathcal{B}_u^{reg}$ in 4.23(a), the subset $'\mathcal{P}_u^{reg}$ of $\mathcal{P}_u^{reg}$ corresponds to the subset $'\mathcal{B}_u^{reg}$ of $\mathcal{B}_u^{reg}$; hence $'\mathcal{P}_u^{reg} = \mathcal{P}_u^{reg} - ''\mathcal{P}_u^{reg}$ corresponds under $\phi_0$ to $'\mathcal{B}_u^{reg} = \mathcal{B}_u^{reg} - ''\mathcal{B}_u^{reg}$. Using this and (a) we deduce

(b) $\chi('\mathcal{P}_u^{reg}) \in \{0, 1, 2\}$.

4.26. We now assume that $G$ is simply connected and $p \neq 3$. Let $J = \{1, 2, 4, 5\} \subset I$. We fix $P_j \in \mathcal{P}_j$. Let $C$ be the regular unipotent conjugacy class of $\bar{P}_j$. Let $S_0$ be an irreducible cuspidal $\bar{P}_j$-equivariant local system on $C$. Up to isomorphism there are two such local systems, one for each nontrivial character of the group (cyclic of order 3) of connected components of the centralizer in $\bar{P}_j$ of an element in $C$. We shall use the notation and results of 2.1, 2.2 for this $G, J, P_j, S_0$.

In our case $\mathcal{W}$ in 2.1 is a Weyl group of type $G_2$ with simple reflections $\sigma_0, \sigma_3$. Now $X_u$ and the local system $\mathcal{S}$ on it are defined as in 2.1 (with $g = u$). Recall from 2.1 that $H_c^j(X_u, \mathcal{S})$ is naturally a $\mathcal{W}$-module for $j \in \mathbb{Z}$. We have the following result.

(a) $\sum_j (-1)^j \operatorname{tr}(\sigma_0, H^j_c(X_u, \mathcal{S})) \in \{0, 1, 2\}$.

Let $J' = J \cup \{0\}$ and let $M_{J', g, r}, \pi'', \dot{S}_r$ be as in 2.1, 2.2 with $g = u, i = 0$. We set $\pi''^{-1}(M_{J', u, r}) = M_r$. From 2.2(b) we see that the left hand side of (a) is equal to $\chi(M_r, \dot{S}_r)$. From the definitions we see that there exists an unramified principal covering $\psi : \dot{M}_r \to M_r$ with group $\mathbb{Z}/3$ (with generator $\kappa$) and $\theta \in \check{Q}_l - \{1\}$ with $\theta^3 = 1$ such that the stalk of $\dot{S}_r$ at any $x \in M_r$ is equal to the vector space of functions $f : \psi^{-1}(x) \to \check{Q}_l$ such that $f(\kappa \tilde{x}) = \theta f(\tilde{x})$ for any $\tilde{x} \in \psi^{-1}(x)$. It follows that

$$\chi(M_r, \dot{S}_r) = \sum_{h=0}^2 \sum_j (-1)^j \operatorname{tr}((\kappa^h)^*, H^j_c(\dot{M}_r, \check{Q}_l))\theta^h/3;$$

$$\chi(M_r) = \sum_{h=0}^2 \sum_j (-1)^j \operatorname{tr}((\kappa^h)^*, H^j_c(M_r, \check{Q}_l))/3,$$

where $(\kappa^h)^*$ is induced by the action of $\kappa^h$ on $\dot{M}_r$. If $h \in \{1, 2\}$ then $\kappa^h : \dot{M}_r \to \dot{M}_r$ has no fixed points and it has order 3. Since $p \neq 3$ we can apply the fixed point
formula in [DL, 3.2] to see that \( \sum_j (-1)^j \text{tr}(\langle \kappa^h \rangle^*, H_j^\sigma(\mathcal{M}_r, \mathcal{Q}_r)) = 0 \) for \( h \in \{1, 2\} \). It follows that

\[
\chi(\mathcal{M}_r, \mathcal{S}_r) = \sum_j (-1)^j \dim H_j^\sigma(\mathcal{M}_r, \mathcal{Q}_r))/3,
\]

\[
\chi(\mathcal{M}_r) = \sum_j (-1)^j \dim H_j^\sigma(\mathcal{M}_r, \mathcal{Q}_r))/3,
\]

so that

\[
\chi(\mathcal{M}_r, \mathcal{S}_r) = \chi(\mathcal{M}_r).
\]

Thus, to prove (a), it is enough to show that

\[
\chi(\mathcal{M}_r) \in \{0, 1, 2\}.
\]

From the definitions we see that the map \( \mathcal{M}_r \to \mathcal{P}_{u, \sigma}^\text{reg}, (xP_J, y(xU_J; x^{-1})) \mapsto xP_Jx^{-1} \), is a isomorphism. Hence it is enough to show that \( \chi(\mathcal{P}_{u, \sigma}^\text{reg}) \in \{0, 1, 2\} \). But this is known from 4.25(b). This completes the proof of (a).

5. The main result

5.1. In this section we assume that \( G \) is simply connected of type \( E_6 \) and that \( p \neq 3 \). We write \( I = \{0, 1, 2, 3, 4, 5\} \) as in 4.1. Let \( J = \{1, 2, 4, 5\} \subset I \) and let \( P_J \in \mathcal{P}_J \). Let \( C \) be the regular unipotent class in \( \hat{P}_J \). There are exactly two irreducible cuspidal \( \hat{P}_J \)-equivariant local systems on \( C \); we fix one of them, say \( \mathcal{S}_0 \). Let \( \mathcal{W} = N_{W_J}W_J/W_J \) where \( N_{W_J}W_J \) is the normalizer of \( W_J \) in \( W \). This is a finite Coxeter group of type \( G_2 \) with simple reflections \( \sigma_0, \sigma_3 \) defined as in 4.1. Now the block defined by \( (P_J, C, \mathcal{S}_0) \) consists of six pairs \( (C_h, \mathcal{E}_h) \), \( h \in H := \{0, 1, 3, 4, 9, 12\} \), where \( C_h \) is a unipotent class of \( G \) such that, for \( u_h \in C_h \) we have \( d_{u_h} = h \) and \( \mathcal{E}_h \) is an irreducible \( G \)-equivariant local system on \( C_h \), uniquely determined up to isomorphism by \( h \). In [S2], \( C_0, C_1, C_3, C_4, C_9, C_{12} \) are denoted by \( E_6, E_6(a_1), A_5A_1, A_5, 2A_2A_1, 2A_2 \) respectively. We shall assume that \( u \in G \) (see 1.1) belongs to \( C_3 \).

Let \( \text{Irr}\mathcal{W} \) be the set of irreducible representations (up to isomorphism) of \( \mathcal{W} \). We have \( \text{Irr}\mathcal{W} = \{1, s, \epsilon, \epsilon', \rho, \rho'\} \) where \( 1, s, \epsilon, \epsilon' \) are one-dimensional, \( \rho, \rho' \) are two-dimensional, 1 is the unit representation, and we have \( \rho \otimes \rho = \rho' \otimes \rho' = 1 + \rho' + s, \rho \otimes \rho' = \epsilon + \epsilon' + \rho \); these properties distinguish \( \rho \) from \( \rho' \) and \( s \) from \( 1, \epsilon, \epsilon' \) but not \( \epsilon \) from \( \epsilon' \). We can distinguish \( \epsilon \) from \( \epsilon' \) by the following requirements: \( \sigma_0 \) acts as \(-1\) on \( \epsilon \) and as \( 1 \) on \( \epsilon' \); \( \sigma_3 \) acts as \(-1\) on \( \epsilon' \) and as \( 1 \) on \( \epsilon \). We have also \( \epsilon \otimes \epsilon = \epsilon' \otimes \epsilon' = s \otimes s = 1, \epsilon \otimes \epsilon = s, \epsilon \otimes s = \epsilon', \epsilon' \otimes s = \epsilon, \rho \otimes s = \rho, \rho' \otimes s = \rho', \rho \otimes \epsilon = \rho \otimes \epsilon' = \rho', \rho' \otimes \epsilon = \rho' \otimes \epsilon' = \rho \).

The generalized Springer correspondence for our block provides a bijection \( \text{Irr}\mathcal{W} \leftrightarrow \{(C_h, \mathcal{E}_h); h \in H\} \). For \( h \in H \) let \( E_h \in \text{Irr}\mathcal{W} \) be corresponding to \( (C_h, \mathcal{E}_h) \) under this bijection. According to [S2], either (a) or (b) below holds:

(a) \( E_0 = 1, E_1 = \epsilon, E_3 = \rho, E_4 = \rho', E_9 = \epsilon', E_{12} = s \);
(b) \( E_0 = 1, E_1 = \epsilon, E_3 = \rho', E_4 = \rho, E_9 = \epsilon', E_{12} = s \).
5.2. Let $q$ be an indeterminate. For any representation $E$ of $W$ we set

$$\Gamma_E = (1/12) \sum_{w \in W} (q^2 - 1)(q^6 - 1)\text{tr}(w, E)/\det(q - w) \in \mathbb{N}[q].$$

We have

$$\Gamma_1 = q^6, \Gamma_{\epsilon} = \Gamma_{e'} = q^3, \Gamma_{\rho} = q + q^5, \Gamma_{\rho'} = q^2 + q^4, \Gamma_s = 1.$$ 

It follows that the matrix $(\Gamma_{E_h \otimes E_{h'}})_{(h, h') \in H \times H}$ is equal to

\[
\begin{pmatrix}
q^6 & q^3 & q + q^5 & q^2 + q^4 & q^3 & 1 \\
q^3 & q^6 & q^2 + q^4 & q + q^5 & q^3 & 1 \\
q^2 + q^4 & q + q^5 & 1 + q^2 + q^4 + q^6 & q + 2q^3 + q^5 & q^2 + q^4 & q + q^5 \\
q + q^3 & q^2 + q^4 & q + 2q^3 + q^5 & 1 + q^2 + q^4 + q^6 & q + q^5 & q^2 + q^4 \\
q^3 & 1 & q^2 + q^4 & q + q^5 & q^3 & q^6 \\
1 & q^3 & q + q^5 & q^2 + q^4 & q^3 & q^6
\end{pmatrix}
\]

if 5.1(a) holds and to

\[
\begin{pmatrix}
q^6 & q^2 & q^4 & q + q^5 & q^3 & 1 \\
q^2 & q^4 & q^2 + q^4 & q + q^5 & q^3 & 1 \\
q^2 + q^4 & q^2 + q^4 & q^2 + q^4 & q + q^5 & q^2 + q^4 \\
q + q^3 & q + q^5 & q + 2q^3 + q^5 & 1 + q^2 + q^4 + q^6 & q + q^5 \\
q^3 & 1 & q + q^5 & q^2 + q^4 & q^3 & q^6 \\
1 & q^3 & q + q^5 & q^2 + q^4 & q^3 & q^6
\end{pmatrix}
\]

if 5.1(b) holds. It follows that the matrix $M := (q^{-h-h'}\Gamma_{E_h \otimes E_{h'}})_{(h, h') \in H \times H}$ is equal to

\[
\begin{pmatrix}
q^6 & q^2 & q^4 & q^2 + q^4 & q^3 & 1 \\
q^2 & q^4 & q^2 + q^4 & q + q^5 & q^3 & 1 \\
q^2 + q^4 & q^2 + q^4 & q^2 + q^4 & q + q^5 & q^2 + q^4 \\
q + q^3 & q + q^5 & q + 2q^3 + q^5 & 1 + q^2 + q^4 + q^6 & q + q^5 \\
q^3 & 1 & q + q^5 & q^2 + q^4 & q^3 & q^6 \\
1 & q^3 & q + q^5 & q^2 + q^4 & q^3 & q^6
\end{pmatrix}
\]

if 5.1(a) holds and to

\[
\begin{pmatrix}
q^6 & q^2 & q^4 & q^2 + q^4 & q^3 & 1 \\
q^2 & q^4 & q^2 + q^4 & q + q^5 & q^3 & 1 \\
q^2 + q^4 & q^2 + q^4 & q^2 + q^4 & q + q^5 & q^2 + q^4 \\
q + q^3 & q + q^5 & q + 2q^3 + q^5 & 1 + q^2 + q^4 + q^6 & q + q^5 \\
q^3 & 1 & q + q^5 & q^2 + q^4 & q^3 & q^6 \\
1 & q^3 & q + q^5 & q^2 + q^4 & q^3 & q^6
\end{pmatrix}
\]

if 5.1(b) holds.
5.3. We can write uniquely $M = ^t\Pi A \Pi$ where $\Pi, A$ are matrices indexed by $H \times H$ such that $A$ is diagonal and $^t\Pi$ is upper triangular with 1 on diagonal. More precisely $A$ is equal to

\[
\begin{array}{cccccc}
q^{-2}(q^2 - 1)(q^6 - 1) & 0 & 0 & 0 & 0 & 0 \\
0 & q^{-4}(q^2 - 1)(q^6 - 1) & 0 & 0 & 0 & 0 \\
0 & 0 & q^{-8}(q^2 - 1)(q^6 - 1) & 0 & 0 & 0 \\
0 & 0 & 0 & q^{-8}(q^6 - 1) & 0 & 0 \\
0 & 0 & 0 & 0 & q^{-18}(q^6 - 1) & 0 \\
0 & 0 & 0 & 0 & 0 & q^{-18}
\end{array}
\]

both in case 5.1(a) and in case 5.1(b); moreover $^t\Pi$ is equal to

\[
\begin{array}{cccccc}
1 & 0 & q^2 & q^2 & q^6 & q^6 \\
0 & 1 & 0 & q^2 & 0 & q^8 \\
0 & 0 & 1 & 1 & q^4 & q^4 + q^8 \\
0 & 0 & 0 & 1 & q^4 & q^4 + q^6 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{array}
\]

if 5.1(a) holds and to

\[
\begin{array}{cccccc}
1 & 0 & 0 & q^3 & q^6 & q^6 \\
0 & 1 & q & q & 0 & q^8 \\
0 & 0 & 1 & 1 & q^5 & q^5 + q^7 \\
0 & 0 & 0 & 1 & q^3 & q^3 + q^7 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{array}
\]

if 5.1(b) holds.

Let $h \leq h'$ in $H$; let $\Pi_{h',h}$ be the $(h',h)$-entry of $\Pi$ or the $(h,h')$-entry of $^t\Pi$. We write $\Pi_{h',h} = \sum_{k \in \mathbb{N}} \Pi_{h',h}^k \in \mathbb{N}[q]$ where $\Pi_{h',h}^k \in \mathbb{N}$.

Note that $C_{h'}$ is contained in the closure $\bar{C}_h$ of $C_h$. For $k' \in \mathbb{Z}$ let

$$\mathcal{H}^{k'} IC(\bar{C}_h, \mathcal{E}_h)|_{C_{h'}}$$

be the restriction to $C_{h'}$ of the $k'$-th cohomology sheaf of the intersection cohomology complex $IC(\bar{C}_{h'}, \mathcal{E}_{h'})$; this is a $G$-equivariant local system on $C_h$.

In the remainder of this subsection we assume that $p \notin \{2, 3\}$. From [L2, 24.8] we have that

$$\mathcal{H}^{k'} IC(\bar{C}_h, \mathcal{E}_h)|_{C_{h'}} = 0$$

for $k'$ odd and that for $k \in \mathbb{Z}$, $\mathcal{H}^{2k} IC(\bar{C}_h, \mathcal{E}_h)|_{C_{h'}}$ is isomorphic to a direct sum of $\Pi_{h',h}^k$ copies of $\mathcal{E}_{h'}$. 
5.4. In this subsection we assume that $p \notin \{2, 3\}$.

Recall from 2.1, 2.2 that

$$X_u = \{xP_j \in G/P_j; x^{-1}gx \in P_j, \pi_j(x^{-1}ux) \in C\}$$

carries a local system $\hat{S}$ induced from $S_0$ and that for any $j \in \mathbb{Z}$, $W$ acts naturally on $H_c^j(X_u, \hat{S})$. (Note that $H_c^j(X_u, \hat{S}) = 0$ for odd $j$.) Let $m_{j,h}$ be the number of times the irreducible $W$-module $E_h$ appears in the $W$-module $H_c^j(X_u, \hat{S})$. Using [L1, 6.5], we see that for $h \leq 3$ we have $\sum_j m_{j,h}q^j = \Pi_{3,h}q^h \in \mathbb{Z}[q]$. Using the tables in 5.3 we deduce that, as a $W$-module, $H_c^j(X_u, \hat{S})$ is $\rho$ if 5.1(a) holds and is $\rho'$ if 5.1(b) holds; $H_c^j(X_u, \hat{S})$ is 1 if 5.1(a) holds and is $\epsilon$ if 5.1(b) holds; $H_c^j(X_u, \hat{S})$ is 0 if $j \neq \{4, 6\}$. It follows that

(a) $\sum_j (-1)^j \text{tr}(\sigma_0, H_c^j(X_u, \hat{S}))$ is equal to $\text{tr}(\sigma_0, \rho) + 1 = 1$ if 5.1(a) holds and to $\text{tr}(\sigma_0, \rho') - 1 = -1$ if 5.1(b) holds.

From 4.26 we see that $\sum_j (-1)^j \text{tr}(\sigma_0, H_c^j(X_u, \hat{S})) = \chi('\mathcal{P}_u^\text{reg} where '\mathcal{P}_u^\text{reg}$ is as in 4.25. Thus, $\chi('\mathcal{P}_u^\text{reg}$ is equal to 0 if 5.1(a) holds and to 1 if 5.1(b) holds. By 4.25(b) we have $\chi('\mathcal{P}_u^\text{reg}$ is in $\{0, 1, 2\}$. We deduce the following result in which we assume that $p \notin \{2, 3\}$.

**Theorem 5.5.** (a) Statement 5.1(a) holds.

(b) We have $\chi('\mathcal{P}_u^\text{reg}$ = 1.

5.6. One can show that the statement of 5.5 remains true when $p = 2$. In this case the references to 4.26 and 4.25(b) can still be used. Although the statements at the end of 5.3 are not known in this case, the statement 5.4(a) remains valid in this case.

6. Final Comments

6.1. In this and the next subsection we assume that $G$ is simply connected/adjoint of type $E_8$ and that $p = 3$. We write $I = \{0, 1, 2, 3, 4, 5, 6, 7\}$ where the numbering is chosen so that $s_1s_2, s_2s_3, s_3s_4, s_4s_5, s_5s_6, s_6s_7, s_0s_3$ have order 3. Let $J = \{0, 1, 2, 3, 4, 5\} \subset I$ and let $P_j \in \mathcal{P}_j$. Let $C$ be the regular unipotent class in $\tilde{P}_j$. There are exactly two irreducible $\tilde{P}_j$-equivariant local systems on $C$; we fix one of them, say $S_0$. Let $W = N_W/W_J$ where $N_W/W_J$ is the normalizer of $W_J$ in $W$. This is a finite Coxeter group of type $G_2$ with simple reflections $\sigma_6, \sigma_7$ defined as in 4.1. Now the block defined by $(P_j, C, S_0)$ consists of six pairs $(C_h, E_h), h \in H := \{0, 1, 3, 4, 9, 12\}$, where $C_h$ is a unipotent class of $G$ such that, for $u_h \in C_h$ we have $d_{u_h} = h$ and $E_h$ is an irreducible $G$-equivariant local system on $C_h$, uniquely determined up to isomorphism by $h$. In [S2], $C_0, C_1, C_3, C_4, C_9, C_{12}$ are denoted by $E_8, E_8(a_1), E_7A_1, E_7, E_6A_1, E_6$ respectively. We shall assume that $u \in G$ (see 1.1) belongs to $C_3$. We identify the present $W$ with the group denoted by $W$ in 5.1 by requiring that $\sigma_6, \sigma_7$ correspond respectively to $\sigma_3, \sigma_0$ in 5.1. Note that $s_7$ commutes with $W_J$.  


The generalized Springer correspondence for our block provides a bijection
\( \text{Irr} \mathcal{W} \leftrightarrow \{(C_h, \mathcal{E}_h); h \in H\} \). For \( h \in H \) let \( E_h \in \text{Irr} \mathcal{W} \) be corresponding to \( (C_h, \mathcal{E}_h) \) under this bijection. According to [S2], either (a) or (b) below holds:

(a) \( E_0 = 1, E_1 = \epsilon, E_3 = \rho, E_4 = \rho', E_9 = \epsilon', E_{12} = s \);
(b) \( E_0 = 1, E_1 = \epsilon, E_3 = \rho', E_4 = \rho, E_9 = \epsilon', E_{12} = s \).

(Compare with 5.1(a), (b).)

**Conjecture 6.2.** Statement 6.1(a) holds.

Recall from 2.1, 2.2 that
\[ X_u = \{xP_j \in G/P_j; x^{-1}gx \in P_j, \pi_j(x^{-1}ux) \in C\} \]
carries a local system \( \hat{S} \) induced from \( S_0 \) and that for any \( j \in \mathbb{Z} \), \( \mathcal{W} \) acts naturally on \( H^j_c(X_u, \hat{S}) \). From [L2] one can deduce, using computations like those in 5.2, that

(a) \( \sum_j (-1)^j \text{tr}(\sigma_7, H^j_c(X_u, \hat{S})) \) is equal to \( \text{tr}(\sigma_7, \rho) + 1 = 1 \) if 6.1(a) holds and to \( \text{tr}(\sigma_7, \rho') - 1 = -1 \) if 6.1(b) holds.

Hence to prove the conjecture it is enough to show that

(b) \( \sum_j (-1)^j \text{tr}(\sigma_7, H^j_c(X_u, \hat{S})) \in \mathbb{N} \).

By arguments similar to those in 4.26 we see that the sum in (b) is equal to the Euler characteristic of
\[ B_u - (B_{u,0} \cup B_{u,1} \cup B_{u,2} \cup B_{u,3} \cup B_{u,4} \cup B_{u,5} \cup B_{u,7}) \]
with coefficient in a local system of rank 1 defined by \( S_0 \). Hence it is enough to show that this Euler characteristic is in \( \mathbb{N} \). It is not clear how to do this. One difficulty is that the argument in 4.26 (based on [DL]) is not applicable hence in the Euler characteristic above the local system cannot be replaced by \( \mathcal{Q}_l \).

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