Non-perturbative Contributions from Complexified Solutions in $\mathbb{C}P^{N-1}$ Models

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We discuss the non-perturbative contributions from real and complex saddle point solutions in the $\mathbb{C}P^1$ quantum mechanics with fermionic degrees of freedom, using the Lefschetz thimble formalism beyond the gaussian approximation. We find bion solutions, which correspond to (complexified) instanton-antiinstanton configurations stabilized in the presence of the fermionic degrees of freedom. By computing the one-loop determinants in the bion backgrounds, we obtain the leading order contributions from both the real and complex bion solutions. To incorporate quasi zero modes which become nearly massless in a weak coupling limit, we regard the bion solutions as well-separated instanton-antiinstanton configurations and calculate a complexified quasi moduli integral based on the Lefschetz thimble formalism. The non-perturbative contributions from the real and complex bions are shown to cancel out in the supersymmetric case and give an (expected) ambiguity in the non-supersymmetric case, which plays a vital role in the resurgent trans-series. For nearly supersymmetric situation, evaluation of the Lefschetz thimble gives results in precise agreement with those of the direct evaluation of the Schrödinger equation. We also perform the same analysis for the sine-Gordon quantum mechanics and point out some important differences showing that the sine-Gordon quantum mechanics does not correctly describe the 1d limit of the $\mathbb{C}P^{N-1}$ field theory of $\mathbb{R} \times S^1$.

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The application of the resurgence theory and the framework of the complexified path integral to quantum theories has recently been attracting a great deal of attention \[\text{[1–12]}\]. In the resurgence theory, all the perturbative series around nontrivial backgrounds are taken into account, and it is expected that a full semi-classical expansion in perturbative and non-perturbative sectors, which is called a “resurgent” trans-series, leads to unambiguous and self-consistent definition of quantum theories \[\text{[13–45]}\].

While the non-perturbative backgrounds in the resurgent trans-series usually mean exact solutions including instantons, certain quantum theories require taking account of non-solution configurations \[\text{[46–53]}\] as “bions” composed of an instanton and an anti-instanton. Indeed, imaginary ambiguities arising in bion contributions cancel out those arising in non-Borel-summable perturbative series in the double-well and the sine-Gordon quantum mechanics \[\text{[46–50]}\]. From the viewpoint of resurgence theory in ordinary differential equations (ODE) \[\text{[1]}\], the reason why the non-saddle-point configurations play the most relevant role in the resurgent trans-series is unclear.

The resurgence structure in the quantum theories gets clear if we extend the configuration space by complexifying real variables. (Note that complexification of integration contours is a standard technique to perform ordinary zero-dimensional integrals.) In terms of the framework of the complexified path integral, the imaginary ambiguity in the Borel resummation of perturbative series is one of the symptoms of “Stokes phenomena” at \(\arg g^2 = 0\) (\(g^2\) is a perturbation parameter). Here the bion configuration is expected to correspond to the complex saddle point, to which the deformed integration contour is attached for \(\arg g^2 \neq 0\). Such complex contours decomposed from the original integration contour are called Lefschetz thimbles, along which the imaginary part of the action is unchanged \[\text{[54]}\]. Thus, the contribution from each background in the resurgent trans-series can be seen as the integral along the Lefschetz thimble associated with each saddle point. It is notable that Stokes phenomena urge us to complexify the variables when \(\arg g^2 = 0\) is the Stokes line.

Recent investigation on exact solutions of the holomorphic equations of motion for the complexified path integral of quantum mechanics is initiated in Refs. \[\text{[37, 38]}\]. In these papers, it is shown that, in the complexified path integral of double-well and sine-Gordon quantum mechanics with fermionic degrees of freedom, the bion configurations appear as the complexified exact solutions. It indicates that the partition function of quantum theory can be defined by performing integration along the thimbles associated with the exact solutions. However, the integration along
the Lefschetz thimbles has not been performed explicitly, even in the one-loop approximations.

Therefore, what we need to do on this topic are summarized as follows: (i) To calculate the one-loop contributions from the complexified solutions in the quantum mechanical systems with fermionic degrees of freedom. (ii) To show that the results are consistent with the known facts in the supersymmetric cases. (iii) To evaluate the integral along the Lefschetz thimble by considering the complexified quasi moduli integral for the bion configurations.

What we will show in this paper is summarized as follows: We obtain one-loop contributions around real and complex bion (instanton-antiinstanton) solutions in the \( \mathbb{C}P^1 \) and the sine-Gordon quantum mechanics with the parameter \( \epsilon \) corresponding to fermionic degrees of freedom. To consider the path integral along the Lefschetz thimbles associated with the exact solutions, we regard the solutions as well-separated instanton-antiinstanton configurations and calculate the complexified quasi moduli integral along Lefschetz thimbles. It can be viewed as a rigorous version of the Bogomolny–Zinn-Justin prescription. For the supersymmetric parameter set, we show that the one-loop contributions from the real and complex solutions are cancelled out. For nearly supersymmetric case of \( \epsilon \approx 1 \), we find that the result of Lefschetz thimble evaluation correctly reproduces the non-perturbative contribution to the ground state energy, which can be evaluated as an insertion of an operator proportional to \( \epsilon - 1 \), giving the \( \mathcal{O}(\epsilon - 1) \) contribution exactly as a function of \( g \). These non-perturbative contributions from the real and complex bion solutions in the \( \mathbb{C}P^{N-1} \) quantum mechanics are of importance in the resurgent trans-series. We also argue that our result helps to catch the resurgence structure in the 2d sigma model, since the bions in the \( \mathbb{C}P^{N-1} \) sigma model on \( \mathbb{R}^1 \times S^1 \) directly reduces to those in \( \mathbb{C}P^{N-1} \) quantum mechanics by dimensional reduction.

We also perform the same analysis for the sine-Gordon quantum mechanics and find some important differences from the \( \mathbb{C}P^1 \) case, which contradicts with the conjectured relevance of the sine-Gordon quantum mechanics to the two-dimensional \( \mathbb{C}P^{N-1} \) field theory on the compactified space \( \mathbb{R} \times S^1 \).

The organization of this paper is as follows. In Sec.II, we derive \( \mathbb{C}P^{N-1} \) quantum mechanics from two-dimensional \( \mathbb{C}P^{N-1} \) sigma model and discuss (nearly-)supersymmetric cases by dealing with Schrödinger equation exactly. In Sec.III, we show the real and complex bion solutions in complexified \( \mathbb{C}P^{N-1} \) quantum mechanics and calculate their contributions by computing the one-loop determinants in the bion background. In this section, we also manifest the existence of quasi zero modes around the saddle points. In Sec.IV, we perform effective thimble integrals associated with the real and complex bion saddle points, and show that they are in good agreement with the
exact near-supersymmetric results. In Sec.V, we perform the same calculation for the sine-Gordon quantum mechanics, and show its difference from $\mathbb{C}P^{N-1}$ quantum mechanics. Sec.VI is devoted to summary and discussion.

II. $\mathbb{C}P^{N-1}$ QUANTUM MECHANICS FROM 2D FIELD THEORY

In this section, we briefly review the dimensional reduction of the $\mathbb{C}P^{N-1}$ non-linear sigma model on $\mathbb{R} \times S^1$ with a special emphasis on fractional instantons and bion configurations, which are responsible for non-perturbative effects both in the 2d field theory and the quantum mechanics.

A. 2d $\mathbb{C}P^{N-1}$ non-linear sigma model

The action of the $\mathbb{C}P^{N-1}$ model in 2d Euclidean space is given in terms of the inhomogeneous coordinates $\varphi^i(x)$ ($i = 1, \cdots, N - 1$) as

$$S = \frac{1}{g_{2d}^2} \int d^2 x \, G_{ij} \partial_\mu \varphi^i \partial^\mu \bar{\varphi}^j, \quad G_{ij} = \frac{\partial^2}{\partial \varphi^i \partial \bar{\varphi}^j} \log(1 + \varphi^k \bar{\varphi}^k).$$

This model has topologically non-trivial instanton solutions characterized by the topological charge

$$Q = \frac{1}{2\pi} \int dx_1 dx_2 \, i \epsilon^{\mu\nu} G_{ij} \partial_\mu \varphi^i \partial_\nu \bar{\varphi}^j.$$  \hfill (II.2)

To write down instanton configurations, it is convenient to use the homogeneous coordinates $h$ (n-component row vector) which specify a point in $\mathbb{C}P^{N-1}$ as a complex line in $\mathbb{C}^N$, i.e. the physical field corresponds to an equivalence class

$$h(x) \sim V(x) h(x),$$  \hfill (II.3)

where $V(x) \in \mathbb{C}^*$ is an arbitrary nonsingular function. Since $h$ multiplied by any functions should be identified, we can choose one of the non-vanishing component to be unity. For example, if we fix the first component of $h$, the homogeneous and inhomogeneous coordinates are related as

$$h = (1, \varphi^1, \cdots, \varphi^{N-1}).$$  \hfill (II.4)

To discuss application of the resurgence theory to the present theory, it has been extremely useful to compactify one dimension as $x_2 + L \sim x_2$. In the compactified space, one can impose the $\mathbb{Z}_N$-twisted boundary condition

$$\varphi^k(x_1, x_2 + L) = \varphi^k(x_1, x_2)e^{2\pi i k/L}.$$  \hfill (II.5)
In terms of $h(x)$, the $\mathbb{Z}_N$-twisted boundary condition can be rewritten as
\[ h(x_1, x_2 + L) = h(x_1, x_2) \Omega, \quad \Omega = \text{diag} \left[ 1, e^{2\pi i/N}, \ldots, e^{2(N-1)\pi i/N} \right]. \quad (\text{II.6}) \]

In fact, it has been shown that one-loop effective potential for the holonomy in the compactified space favors this boundary condition in various situations \cite{11, 12}. In the presence of the non-trivial holonomy, the continuously degenerated classical vacuum reduces to $N$ discrete points, each of which corresponds to $h(x)$ with only one nonzero component.

The solution of the (anti-)BPS instanton equation $\partial_z \varphi^i = 0$ ($\partial_{\bar{z}} \varphi^i = 0$) is given by an arbitrary (anti-)holomorphic function $h(z)$ ($h(\bar{z})$) with respect to the complex coordinates $z = x_1 + ix_2$. The $\mathbb{Z}_N$-twisted boundary conditions are satisfied if and only if the homogeneous coordinate $h$ takes the form
\[ h = \left( P_1(e^{2\pi z/L}), P_2(e^{2\pi z/L}), \ldots, P_N(e^{2\pi z/L}) \right) \text{diag} \left[ 1, e^{mz}, \ldots, e^{(N-1)mz} \right], \quad (\text{II.7}) \]
where $P_1, P_2, \ldots, P_N$ are polynomial of $\exp\left(\frac{2\pi z}{L}\right)$ and the parameter $m$ is given by
\[ m \equiv \frac{2\pi}{N \cdot L}. \quad (\text{II.8}) \]

The twisted boundary condition allows the fractional instanton (we denote it as $\mathcal{I}$) with topological charge $Q = 1/N$ as the simplest building blocks of BPS solutions \cite{56, 58}. There are $N$ types of fractional instantons, each of which interpolates a pair of adjacent vacua:
\[ h_{\mathcal{I}} = (0, \cdots, 0, 1, a, 0, \cdots, 0) \text{diag} \left[ 1, e^{mz}, \ldots, e^{(N-1)mz} \right]. \quad (\text{II.9}) \]

The fractional instanton has one complex (two real) moduli parameter $a$, which is related to its position $x_1 = \frac{1}{m} \log |a|$ and intrinsic phase $\phi \equiv \arg a$.

If we combine $N$ different types of fractional instantons, we obtain an instanton solution with $Q = 1$
\[ h_{\mathcal{I} \cdots \mathcal{I}} = (1 + a_N e^{Nmz}, a_1, \cdots, a_{N-1}) \text{diag} \left[ 1, e^{mz}, \ldots, e^{(N-1)mz} \right]. \quad (\text{II.10}) \]

These solutions have kink-like behaviors which can be seen from the function $\Sigma(x_1)$ defined by \cite{59}
\[ \Sigma(x_1) = \frac{1}{L} \int_0^L dx_2 \frac{i}{2|h|^2} \left( h \partial_{\bar{z}} h^\dagger - \partial_{\bar{z}} h h^\dagger \right) = \frac{m}{L} \int_0^L dx_2 \frac{\sum_{k=1}^{N-1} k \varphi^k \bar{\varphi}^k}{1 + \sum_{j=1}^{N-1} \varphi^j \bar{\varphi}^j}. \quad (\text{II.11}) \]

Fig. shows the function $\Sigma(x_1)$ for the BPS configurations with fractional and integer topological charges.
If we add a fractional anti-instanton to the left of a fractional instanton, we obtain a bion configuration, which has vanishing topological charge

\[ h_{II} = (0, \cdots, 1, a_+ e^{mx_1} + a_- e^{-mx_1}, \cdots, 0) \text{ diag} \left[ 1, e^{mx_2}, \cdots, e^{(N-1)mx_2} \right]. \]  

(II.12)

The parameters \( a_\pm \) are related to the positions and phases of the constituent fractional instantons \((x_\pm, \phi_\pm)\) as

\[ a_\pm = \exp \left( \mp mx_\pm + i\phi_\pm \right). \]  

(II.13)

Fig. 2 illustrates the bion configuration. The bion configuration becomes a solution of field equation asymptotically for large separation \( x_+ - x_- \), and provides a basis to compute nonperturbative contributions relevant to the resurgence theory.

Fig. 1: Kink-like profiles of (fractional) instanton configurations in \( \mathbb{C}P^3 \) model with \( \mathbb{Z}_4 \)-twisted boundary conditions. Each horizontal dotted line corresponds to the value of \( \Sigma \) in each vacuum. Note that \( \Sigma \) is a periodic quantity with period \( Nm = \frac{2\pi}{L} \) since it is shifted by the transformation (II.3) with non-zero winding number \( w \) \((V(x_2) = e^{2\pi iw\frac{x_2}{L}})\) as \( \Sigma \to \Sigma + wNm \).

\[ B. \mathbb{C}P^{N-1} \text{ quantum mechanics} \]

The two-dimensional field theory on \( \mathbb{R} \times S^1 \) is faithfully reproduced by the Kaluza-Klein decomposition

\[ \varphi^k(x_1, x_2) = \sum_{n \in \mathbb{Z}} \varphi^k_{(n)}(x_1) \exp \left[ i \frac{2\pi}{L} \left( n + \frac{k}{N} \right) x_2 \right], \]  

(II.14)

where we used the mode functions taking into account of the \( \mathbb{Z}_N \)-twisted boundary condition (II.5). Integrating over the compactified coordinate \( x_2 \), the two-dimensional Lagrangian can be rewritten
equivalently as a coupled system of infinitely many quantum mechanical variables $\varphi^k_{(n)}(x_1)$

$$\mathcal{L}_{\text{KK}} = \int_0^L dx_2 \mathcal{L}_{2d}, \quad (\text{II.15})$$

and the coupling constant of the quantum mechanics $g_{1d}$ can be defined as

$$\frac{1}{g_{1d}^2} = \frac{L}{g_{2d}^2}. \quad (\text{II.16})$$

This infinite component quantum mechanics is equivalent to the two-dimensional field theory including the necessity of renormalization, and consequent asymptotic freedom. However, we can retain only finite number of quantum mechanical degrees of freedom and discard the infinite tower of Kaluza-Klein components, as long as we are interested in the semi-classical results from saddle points needed to catch the resurgence structure of the theory at low energies ($E \ll m$). Let us retain only the lowest mode $n = 0$ for each $\varphi^k(x_1, x_2)$ out of an infinite tower of Kaluza-Klein variables to find the $\mathbb{C}P^{N-1}$ quantum mechanics

$$L_{1d} = \frac{1}{g_{1d}^2} G_{kl}^{(0)} \left[ \partial_{x_1} \overline{\varphi^k_{(0)}} \partial_{x_2} \varphi^l_{(0)} + k l m^2 \varphi^k_{(0)} \overline{\varphi^l_{(0)}} \right], \quad G_{kl}^{(0)} = G_{kl}(\varphi^i = \varphi^i_{(0)}). \quad (\text{II.17})$$

We can now recognize that there are only $N$ discrete vacua corresponding to $h = (1, 0, \cdots, 0), \cdots, (0, \cdots, 0, 1)$ because of the $\mathbb{Z}_N$-twisted boundary condition.

The $\mathbb{C}P^{N-1}$ quantum mechanics captures restricted configurations in the two-dimensional theory of the form

$$h(x_1, x_2) = h_{(0)}(x_1) \text{ diag.} \left[ 1, e^{i m x_2}, \cdots, e^{i (N-1) m x_2} \right]. \quad (\text{II.18})$$

The function $\Sigma(x_1)$ exhibiting the kink-like profile in Eq. (II.11) reduces to

$$\Sigma(x_1) = \frac{m \sum_{k=1}^{N-1} k \varphi^k_{(0)} \overline{\varphi^k_{(0)}}}{1 + \sum_{j=1}^{N-1} \varphi^j_{(0)} \overline{\varphi^j_{(0)}}}. \quad (\text{II.19})$$
in the $\mathbb{CP}^{N-1}$ quantum mechanics. The fractional instanton in Eq. (II.9) and the bion configuration in Eq. (II.12) take the form of Eq. (II.18). Thus we find that both the fractional instantons and the bion configurations are correctly described in the $\mathbb{CP}^{N-1}$ quantum mechanics. We can show that all other multi-fractional instanton configurations can be correctly described by the $\mathbb{CP}^{N-1}$ quantum mechanics, provided it does not contain multi-fractional-instantons with $|Q| \geq 1$ anywhere locally.

The instanton configuration (II.10) with $Q = 1$ is not reducible to the $\mathbb{CP}^{N-1}$ quantum mechanics, since it does not satisfy (II.18). The action and topological charge densities of the $N$ fractional instanton solution in Eq. (II.10) do exhibit a strong $x_2$ dependence approaching the ordinary single instanton solution when the constituent fractional instantons are compressed in a point. This situation inevitably occurs whenever configurations with $|Q| \geq 1$ are contained. On the other hand, we find that the configurations compatible with the $\mathbb{CP}^{N-1}$ quantum mechanics have action density and topological charge density which are independent of the coordinate $x_2$ of the compact direction. Therefore, configurations with $|Q| < 1$ in the two-dimensional field theory are correctly captured by the $\mathbb{CP}^{N-1}$ quantum mechanics, provided the multi-fractional-instanton configurations with more than unit topological charge is not contained anywhere locally [44].

Once it was conjectured that the $\mathbb{CP}^{N-1}$ model reduces to the sine-Gordon quantum mechanics in the limit of $L \to 0$ (the compactification limit) [11, 12]. However, it has been observed that the relative phase moduli of fractional instanton and anti-instanton is not correctly described by the sine-Gordon quantum mechanics [22, 28, 36]. We discuss the differences between the $\mathbb{CP}^1$ and the sine-Gordon quantum mechanics in Sec. V.

C. $\mathbb{CP}^1$ quantum mechanics with fermion and supersymmetry

To examine bion configurations, it is convenient to introduce a fermionic degree of freedom. Only in this subsection, we use Lorentzian signature instead of Euclidean signature in order to use also Schrödinger equation later. To denote 1d quantities simply, we rewrite without subscript: $-ix_1$ as the Lorentzian time $t$, $\varphi_{(0)}^{k=1} \rightarrow \varphi$, $g_{1d} \rightarrow g$, $G_{11}^{(0)} \rightarrow G$ etc. The Lagrangian of the $\mathbb{CP}^1$ Lorentzian quantum mechanics with a fermion takes the form

$$L = \frac{1}{g^2} G \left[ \partial_t \varphi \partial_t \bar{\varphi} - m^2 \varphi \bar{\varphi} + i \bar{\psi} D_t \psi + \epsilon m (1 + \varphi \partial_\varphi \log G) \bar{\psi} \psi \right], \quad \text{(II.20)}$$

where $G$ is the Fubini-Study metric and $D_t$ is the pullback of the covariant derivative

$$G = \frac{1}{(1 + \varphi \bar{\varphi})^2}, \quad D_t \psi = \left[ \partial_t + \partial_\varphi \partial_\varphi \log G \right] \psi. \quad \text{(II.21)}$$
The parameter $\epsilon$ controls the strength of the interaction between the bosonic and fermionic degrees of freedom. If we set $\epsilon = 1$, this model becomes a supersymmetric system which can be obtained from the 2d $\mathcal{N} = (2,0) \mathbb{C}P^1$ sigma model by an analogous dimensional reduction as the one discussed in the previous subsection.

Since the fermion number $\bar{\psi}\psi$ commutes with the Hamiltonian, we can eliminate $\psi$ by using the conserved fermion number and the associated induced potential. By projecting quantum states onto the subspace of the Hilbert space with a fixed fermion number, we obtain the following purely bosonic Lagrangian (see Appendix A for details)

$$L = \frac{1}{g^2} \frac{\partial \varphi \partial \bar{\varphi}}{(1 + \varphi \bar{\varphi})^2} - V(\varphi \bar{\varphi}), \quad V(\varphi \bar{\varphi}) \equiv \frac{1}{g^2} \frac{m^2 \varphi \bar{\varphi}}{(1 + \varphi \bar{\varphi})^2} - \epsilon m \frac{1 - \varphi \bar{\varphi}}{1 + \varphi \bar{\varphi}}, \quad \text{(II.22)}$$

where we have chosen the fermion number so that the supersymmetric ground state for $\epsilon = 1$ is contained in the subspace of the Hilbert space. The potential $V$ as a function of the latitude $\theta \equiv 2\arctan|\varphi|$ is shown in Fig. 3.

![Fig. 3: The potential $V$ with the contribution of the fermion. The horizontal axis denotes the latitude $\theta \equiv 2\arctan|\varphi|$ on $\mathbb{C}P^1 \cong S^2$.](image)

For $\epsilon = 1$, the ground state wave function $\Psi_0$, which preserves the supersymmetry, is given as the zero energy solution of the Schrödinger equation

$$H \Psi_0 = 0, \quad \text{(II.23)}$$

with the Hamiltonian $H$ of the bosonic theory:

$$H = -g^2 (1 + \varphi \bar{\varphi})^2 \frac{\partial}{\partial \varphi} \frac{\partial}{\partial \bar{\varphi}} + V(\varphi \bar{\varphi}). \quad \text{(II.24)}$$

We find the exact solution of the ground state wave function

$$\Psi_0 = \exp \left( \frac{m}{2g^2} \frac{1 - \varphi \bar{\varphi}}{1 + \varphi \bar{\varphi}} \right). \quad \text{(II.25)}$$
The existence of the supersymmetric state implies that the ground state energy receives no non-perturbative correction for $\epsilon = 1$. For a generic value of $\epsilon$, there can be corrections to the ground state energy. Indeed, we can show that there exist non-perturbative corrections in the near supersymmetric case $\epsilon \approx 1$ by expanding the energy with respect to small $\delta \epsilon \equiv \epsilon - 1$

$$E \approx \frac{\langle 0|\delta H|0\rangle}{\langle 0|0\rangle} = -\delta \epsilon m \left( \frac{1 - \varphi \bar{\varphi}}{1 + \varphi \bar{\varphi}} \right)_{\epsilon=1},$$

(II.26)

where the perturbative Hamiltonian is given by

$$\delta H = H - H_{\epsilon=1} = -\delta \epsilon m \frac{1 - \varphi \bar{\varphi}}{1 + \varphi \bar{\varphi}}.$$

(II.27)

As Eq. (II.26) indicates, we can exactly calculate the leading order coefficients in the small $\delta \epsilon$ expansion of the ground state energy by using the explicit form of the ground state wave function (II.25) as

$$E = \frac{\int dv \delta H|\Psi_0|^2}{\int dv|\Psi_0|^2} + O(\delta \epsilon^2) = \delta \epsilon \left[ g^2 - m \coth \frac{m}{g^2} \right] + O(\delta \epsilon^2),$$

(II.28)

where $dv$ is the standard volume element on $\mathbb{C}P^1$: $dv \equiv d^2\varphi/(1 + \varphi \bar{\varphi})^2$. Note that although we have expanded the energy with respect to $\delta \epsilon$, Eq. (II.28) is non-perturbative as a function of the coupling constant $g$. We can decompose the ground state energy (II.28) into the perturbative and non-perturbative parts

$$E_{\text{pert}} = (g^2 - m) \delta \epsilon + O(\delta \epsilon^2),$$

(II.29)

$$E_{\text{bion}} = -2m e^{-\frac{2m}{s^2}} \delta \epsilon + O \left( e^{-\frac{4m}{s^2}}, \delta \epsilon^2 \right).$$

(II.30)

It is interesting to note that perturbative contributions to this order of $\epsilon - 1$ are terminated at the order $g^2$ without any higher order corrections, and hence there is no ambiguity associated with non-Borel summable asymptotic series. In the following, we will see that contributions of (real and complex) bion configurations correctly reproduce this non-perturbative correction.

### III. BION SADDLE POINTS AND ONE-LOOP APPROXIMATION

In the previous section, we have seen that the ground state energy receives no correction for $\epsilon = 1$ due to the supersymmetry and there exists a non-perturbative correction at least in the near supersymmetric case $\epsilon \approx 1$. In this section, we discuss the non-perturbative correction from the
viewpoint of the saddle point method in the projected model \( \text{(II.22)} \), which we complexify to find a complex bion solution.

In the path integral formalism, the ground state energy can be obtained from the partition function

\[
Z(\beta) = \int \mathcal{D}\varphi \exp(-S_E[\varphi]), \tag{III.1}
\]

where the path integral is performed over configurations satisfying the periodic boundary condition \( \varphi(\tau + \beta) = \varphi(\tau) \). The asymptotic behavior of the partition function in the weak coupling limit \( g \to 0 \) can be obtained by using the saddle point method in the Gaussian approximation:

\[
Z = \sum_{\sigma \in \mathcal{S}} e^{-S_\sigma} \left[ (\det \Delta_\sigma)^{-\frac{1}{2}} + O(g) \right], \tag{III.2}
\]

where \( \mathcal{S} \) denotes a set of saddle points of \( S_E \), \( S_\sigma \) is the value of the action at the saddle point \( \sigma \) and \( \det \Delta_\sigma \) is the one-loop determinant in the saddle point configuration. Let \( Z_0 \) be the contributions from the trivial saddle point (classical vacuum) and \( Z_1 \) be the leading order correction from non-trivial saddle points:

\[
Z = Z_0 + Z_1 + \cdots. \tag{III.3}
\]

Then the non-perturbative correction to the ground state energy can be obtained as

\[
E \approx - \lim_{\beta \to \infty} \frac{1}{\beta} \log Z \approx - \lim_{\beta \to \infty} \frac{1}{\beta} \left[ \log Z_0 + \frac{Z_1}{Z_0} + \cdots \right]. \tag{III.4}
\]

In the following, we compute \( Z_1/Z_0 \) by finding saddle points and calculating one-loop determinants around the saddle points.

### A. Real bion solution

Let us look for saddle points of the Euclidean action which are responsible for the non-perturbative correction by solving the classical equation of motion. Since we are interested in the zero temperature limit, we consider saddle points in the limit \( \beta \to \infty \). In this subsection, as the first step, we show that there is a real bion solution in the model with the potential modified by the fermionic degree of freedom.

To find out classical solutions, it is convenient to use symmetries of the Euclidean action and associated conservation laws. Since the Euclidean action

\[
S_E = \int d\tau \left[ \frac{1}{g^2} \frac{\partial_\tau \varphi \partial_\tau \bar{\varphi}}{(1 + \varphi \bar{\varphi})^2} + V(\varphi \bar{\varphi}) \right], \tag{III.5}
\]
is invariant under the shift of the Euclidean time $\tau \to \tau - \tau_0$, the corresponding “energy” is a conserved quantity

$$E \equiv \frac{1}{g^2} \frac{\partial_\tau \varphi \partial_\tau \bar{\varphi}}{(1 + \varphi \bar{\varphi})^2} - V(\varphi \bar{\varphi}).$$  \hspace{1cm} (III.6)$$

The action is also invariant under the phase rotation $\varphi \to e^{i\phi_0} \varphi$, so that the corresponding angular momentum is a conserved charge

$$l \equiv \frac{i}{g^2} \frac{\partial_\tau \varphi \bar{\varphi} - \partial_\tau \bar{\varphi} \varphi}{(1 + \varphi \bar{\varphi})^2}. \hspace{1cm} (III.7)$$

Since we are interested in saddle point configurations with finite action, we impose the boundary condition so that $\varphi$ is at the minimum of the potential for $\tau \to \pm\infty$:

$$\lim_{\tau \to \pm\infty} \varphi = \lim_{\tau \to \pm\infty} \bar{\varphi} = 0. \hspace{1cm} (III.8)$$

Then it follows that the saddle point configuration cannot have the angular momentum $l$, i.e. the phase of $\varphi$ is a constant of motion. In addition, “the energy conservation law” implies that

$$\frac{1}{g^2} \frac{\partial_\tau \varphi \partial_\tau \bar{\varphi}}{(1 + \varphi \bar{\varphi})^2} - V(\varphi \bar{\varphi}) = \epsilon m = E|_{\varphi=0}. \hspace{1cm} (III.9)$$

We can integrate the energy conservation law to obtain “the real bion solution”.

$$\varphi = e^{i\phi_0} \sqrt{\frac{\omega^2}{\omega^2 - m^2}} \frac{1}{i \sinh \omega (\tau - \tau_0)}, \hspace{1cm} (III.10)$$

where $\omega$ is given by

$$\omega \equiv m \sqrt{1 + \frac{2\epsilon g^2}{m}}. \hspace{1cm} (III.11)$$

The parameters $\tau_0$ and $\phi_0$ are integration constants, i.e. moduli parameters. The orbit of this solution in $\mathbb{C}P^1$ is a great circle starting from the south pole ($\varphi = 0$, minimum of the potential) and passing through the north pole at $\tau = \tau_0$. The phase modulus $\phi_0$ is related to the longitude of the great circle. Thus the moduli space of real bion is a cylinder

$$\mathcal{M}_{\text{bion}} = \mathbb{R} \times S^1. \hspace{1cm} (III.12)$$

These parameters will be eventually integrated to incorporate the contribution of all the bion solutions to the partition function. Precisely speaking, the first factor should be interpreted as an infinitely large $S^1$ with radius $\beta \to \infty$, along which the moduli integration gives a factor of $\beta$ to the single bion contribution $Z_1$. 
The real bion solution can be viewed as a kink-antikink solution with fixed relative position and phase, since it can be rewritten into the kink-antikink form (II.12) in terms of the homogeneous coordinates:

\[ h = \left( 1, a_+ e^{\omega \tau} + a_- e^{-\omega \tau} \right), \quad a_+ = e^{-\omega \tau + i\phi_+}, \quad a_- = e^{\omega \tau - i\phi_-}, \]  

(III.13)

where the positions and phases are given by

\[ \tau_{\pm} = \tau_0 \pm \frac{1}{2\omega} \log \frac{4\omega^2}{\omega^2 - m^2}, \quad \phi_{\pm} = \phi_0 \mp \frac{\pi}{2}. \]  

(III.14)

The Lagrangian for this saddle point configuration takes the form

\[ L = 4m\epsilon \left[ f(\tau - \tau_0) \cosh \omega(\tau - \tau_0) \right]^2 - m\epsilon, \]  

(III.15)

where the function \( f(\tau) \) is given by

\[ f(\tau) \equiv \frac{\omega^2}{\omega^2 + (\omega^2 - m^2) \sinh^2 \omega \tau}. \]  

(III.16)

Neglecting the vacuum value of the Lagrangian, we obtain the action for the real bion as

\[ S_{rb} = 4m\epsilon \int_{-\infty}^{\infty} d\tau \left[ f(\tau - \tau_0) \cosh \omega(\tau - \tau_0) \right]^2 = \frac{2\omega}{g^2} + 2\epsilon \log \frac{\omega + m}{\omega - m}. \]  

(III.17)

This implies that the real bion can give a non-perturbative correction of order \( e^{-S_{rb}} \sim e^{-\frac{2\omega}{g^2}} \). However, as we have seen in the previous section, the ground state energy does not receive any correction for \( \epsilon = 1 \), and hence there should be other saddle point configurations which cancel the contribution of the real bion solution.

B. Complex bion solution

The absence of non-perturbative correction at \( \epsilon = 1 \) implies that there are other saddle points which should be taken into account. However, the configuration (III.10) is the general solution satisfying the boundary condition (III.8). The only way to obtain other saddle point configurations is to extend the configuration space by complexifying the degrees of freedom. Such a procedure is a straightforward generalization of the complexification of integration contour for ordinary finite dimensional integrals, which is a necessary step in the saddle point method [37].

In the case of the complex field \( \varphi \), we independently complexify its real and imaginary parts:

\[ (\varphi, \bar{\varphi}) = (\varphi_R + i\varphi_I, \varphi_R - i\varphi_I) \quad \rightarrow \quad (\varphi_C^R + i\varphi_C^I, \varphi_C^R - i\varphi_C^I). \]  

(III.18)
Consequently, $\bar{\phi}$ becomes an independent complex degree of freedom which is not related to $\phi$ by complex conjugation. In the following, we denote $\tilde{\phi}$ for the complexification of $\bar{\phi}$ to avoid confusion

$$\bar{\phi} \to \tilde{\phi} \neq \text{complex conjugate of } \phi.$$  \hspace{1cm} (III.19)

Then we regard the action $S[\phi, \tilde{\phi}]$ as an analytically continued holomorphic functional of the complexified degrees of freedom

$$S[\phi, \tilde{\phi}] = \int d\tau \left[ \frac{1}{g^2} \frac{\partial_\tau \phi \partial_\tau \tilde{\phi}}{(1 + \phi \tilde{\phi})^2} + V(\phi \tilde{\phi}) \right]. \hspace{1cm} (III.20)$$

We also impose the boundary condition (III.8) with $\tilde{\phi}$ replacing $\bar{\phi}$. By deforming integration contour, the integral can be expressed as a sum of contributions from a set of saddle points of the complexified action $S[\phi, \tilde{\phi}]$.

Since the action is extended as a holomorphic functional, it is invariant under the symmetries of the original action with complexified transformation parameters. Furthermore, the complexified equations of motion take the same forms as those of the original action and hence the configuration (III.10) is still the general solution satisfying the boundary condition (III.8). The important difference in the complexified case is that the integration constants $\tau_0$ and $\phi_0$ are now complex parameters, i.e. the solution $(\phi, \tilde{\phi})$ is a holomorphic function of the moduli parameters $\tau_0$ and $\phi_0$.

The solution (III.10) smoothly varies under small shifts of moduli parameters $\tau_0$ and $\phi_0$. Such solutions are simply related to the real bion solution by the complexified symmetry transformations, and the value of the corresponding action remains the same. Thus the moduli space of bion configurations is also complexified: $\mathcal{M}_{\text{bion}} \to \mathcal{M}^C_{\text{bion}}$. The integration contour of the moduli integral for the partition function can be any middle dimensional contour in the complexified moduli space $\mathcal{M}^C_{\text{bion}}$ as long as it is related to the original real contour $\mathcal{M}_{\text{bion}}$ by a continuous deformation. However, for our purpose, we do not need to consider the deformation of the integration contour and the moduli integral will be performed over $\mathcal{M}_{\text{bion}}$ in the next section. Thus, the bion configurations with the complexified moduli do not give a physically distinct contribution, until the shift in the complexified transformation meets a singularity and produces a jump in the value of the action.

The singular solution can be obtained by a shift, for instance by an amount

$$\tau_0 \to \tilde{\tau}_0 = \tau_0 + \frac{1}{\omega} \frac{\pi i}{2}, \hspace{1cm} (III.21)$$

under which the solution becomes

$$\varphi = e^{i\phi_0} \sqrt{\frac{\omega^2}{\omega^2 - m^2} \frac{1}{\cosh(\omega(\tau - \tau_0))}}, \hspace{0.5cm} \tilde{\phi} = -e^{-i\phi_0} \sqrt{\frac{\omega^2}{\omega^2 - m^2} \frac{1}{\cosh(\omega(\tau - \tau_0))}}. \hspace{1cm} (III.22)$$
(a) $\Sigma(\tau)$ for real bion  \hspace{2cm} (b) $\Sigma(\tau)$ for complex bion

Fig. 4: Kink profiles of for real and complex bions. The complex bion solution has singularities at which $\Sigma(\tau)$ diverges. Note that $\Sigma(\tau)$ can also be complex in the complexified model.

As we will see below, this configuration has singularities at which the action density diverges. Since $\tilde{\varphi}$ is no longer the complex conjugate of $\varphi$, this is a solution of the complexified model and hence we call this configuration “complex bion solution”.

It is worth noting that the shifted solution can also be rewritten into the kink-antikink form

$$
\varphi = \left( e^{\omega(\tau-\tau_+)-i\phi_+} + e^{-\omega(\tau-\tau_-)-i\phi_-} \right)^{-1}, \quad \tilde{\varphi} = \left( e^{\omega(\tau-\tau_+)+i\phi_+} + e^{-\omega(\tau-\tau_-)+i\phi_-} \right)^{-1},
$$

(III.23)

with complexified position parameters $\tau_\pm$:

$$
\tau_\pm = \tau_0 \pm \frac{1}{2\omega} \left( \log \frac{4\omega^2}{\omega^2 - m^2} + \pi i \right), \quad \phi_\pm = \phi_0 - \frac{\pi}{2},
$$

(III.24)

where we have used the fact that the shift $\tau_0 \rightarrow \tau_0 + \frac{1}{\omega} \frac{\pi i}{2}$ can be rewritten as the combination of the shifts $\omega\tau_+ \pm i\phi_+ \rightarrow \omega\tau_+ \pm i\phi_+ + \frac{\pi i}{2} (\text{mod } 2\pi i)$ and $\omega\tau_- \pm i\phi_- \rightarrow \omega\tau_- \pm i\phi_- + \frac{\pi i}{2} (\text{mod } 2\pi i)$.

Therefore, the complex bion solution can also be viewed as a kink-antikink solution with complex relative distance

$$
\tau_r \equiv \tau_+ - \tau_- = \frac{1}{\omega} \left( \log \frac{4\omega^2}{\omega^2 - m^2} + \pi i \right).
$$

(III.25)

Fig. 4 shows the kink-like profiles of the function $\Sigma(\tau)$ in Eq. (II.19), which takes the following form in the complexified theory

$$
\Sigma = m \frac{\varphi \tilde{\varphi}}{1 + \varphi \tilde{\varphi}}.
$$

(III.26)
The value of the Lagrangian in Eq. (III.15) for the shifted solution \((\tau_0 \rightarrow \tilde{\tau}_0)\) is given in terms of the function \(f\) defined in Eq. (III.16) as

\[
L = 4m\epsilon \left[ f(\tau - \tilde{\tau}_0) \cosh \omega(\tau - \tilde{\tau}_0) \right]^2 = -4m\epsilon \left[ \frac{\omega^2 \sinh \omega(\tau - \tau_0)}{\omega^2 - (\omega^2 - m^2) \cosh^2 \omega(\tau - \tau_0)} \right]^2, \tag{III.27}
\]

where we have neglected the vacuum value of the Lagrangian. Since it has second order poles at

\[
\tau_{\pm\text{pole}} = \tau_0 \pm \frac{1}{\omega} \text{arccosh} \sqrt{\frac{\omega^2}{\omega^2 - m^2}}, \tag{III.28}
\]

the shifted solution is a singular solution. To regularize the action, let us turn on small imaginary part of the coupling constant: \(\theta \equiv \text{arg} \ g^2\). Fig. 5 shows the kink profile of the regularized complex bion. Neglecting the vacuum value of the Lagrangian, we obtain the action for the complex bion solution as

\[
S_{cb} = 4m\epsilon \int_{-\infty}^{\infty} d\tau \left[ f(\tau - \tilde{\tau}_0) \cosh \omega(\tau - \tilde{\tau}_0) \right]^2 = 4m\epsilon \int_C d\tau \left[ f(\tau - \tau_0) \cosh \omega(\tau - \tau_0) \right]^2. \tag{III.29}
\]

The integrand in the last expression is the same as that for the real bion whereas the integration contour \(C\) is the line \(\text{Im} \ \tau = -\frac{1}{\omega} \frac{\pi}{2}\) instead of the real axis, and hence the difference of \(S_{rb}\) and \(S_{cb}\) can be calculated by deforming the integration contour from the real axis to \(C\). Although the action is invariant under any smooth deformation of the contour, its value jumps when one of the poles crosses the contour. By evaluating the residue at the pole, we can show that the difference of the action for the real and complex bion is given by

\[
S_{cb} = S_{rb} \pm 2\pi i \epsilon. \tag{III.30}
\]

As shown in Fig. 6, the difference of the action \(S_{cb} - S_{rb}\) is given by the residue at either \(\tau_{\pm\text{pole}}^+\) or \(\tau_{\pm\text{pole}}^-\) depending on the sign of \(\text{arg} \ g^2\). Thus, the contribution of the complex bion has an ambiguity
Fig. 6: The integration contour for $S_{cb} - S_{rb}$ in the complex $\tau$ plane. Depending on the sign of $\arg g^2$, the difference of the action $S_{cb} - S_{rb}$ is given by the residue at either $\tau^+_\text{pole}$ or $\tau^-_\text{pole}$.

for a generic value of $\epsilon$. In the supersymmetric case $\epsilon = 1$, the difference $S_{cb} - S_{rb}$ is $2\pi i$, and hence there is no ambiguity $\exp(-S_{cb}) = \exp(-S_{rb})$.

From the periodicity of the solution and the Lagrangian under the shift of the imaginary part of $\tau_0$, we find that there are only two distinct classes of solutions: real and complex bion solutions as exact solutions of the complexified equation of motion in the $\mathbb{C}P^1$ quantum mechanics. To see that the contributions of the real and complex bions cancel out for $\epsilon = 1$, we have to evaluate the one-loop determinant in the bion background.

C. One-loop determinant in bion backgrounds

In this section, we compute the contributions of the bion configurations by evaluating the one-loop determinant for the fluctuations around the bion backgrounds. For this purpose, we expand the action with respect to the fluctuations $\xi^a$ ($a = 1, 2$) defined by

$$
\begin{pmatrix}
\varphi \\
\tilde{\varphi}
\end{pmatrix} = \begin{pmatrix}
\varphi_{\text{sol}} \\
\tilde{\varphi}_{\text{sol}}
\end{pmatrix} + g \begin{pmatrix}
\epsilon_1 \\
\tilde{\epsilon}_1
\end{pmatrix} \begin{pmatrix}
\xi^1 \\
\xi^2
\end{pmatrix},
$$

where the background bion solution is given by

$$
\varphi_{\text{sol}} = e^{i\varphi_0} \sqrt{\frac{\omega^2}{\omega^2 - m^2 i \sinh(\omega(\tau - \tau_0))}}, \quad \tilde{\varphi}_{\text{sol}} = -e^{-i\varphi_0} \sqrt{\frac{\omega^2}{\omega^2 - m^2 i \sinh(\omega(\tau - \tau_0))}},
$$

with

$$
\text{Im } \tau_0 = \begin{cases}
0 & \text{for real bion} \\
\frac{\pi i}{2\omega} & \text{for complex bion}
\end{cases}
$$
It is convenient to use the following basis of vielbein:
\[
\begin{pmatrix}
e_1 & e_2 \\
\bar{e}_1 & \bar{e}_2
\end{pmatrix} = (1 + |\varphi_{sol}|^2) \begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}}
\end{pmatrix}.
\] (III.34)

Then we obtain the quadratic action of the form
\[
\delta S = \frac{1}{2} \int d\tau \xi^T \Delta \xi,
\]
where \(\Delta\) is a 2-by-2 matrix of second order differential operator around the bion background. The contribution from each bion is given by the determinant of the operator \(\Delta\):
\[
\frac{Z_1}{Z_0} = \sum_{\text{bion}} e^{-S} \int D\xi \exp \left( -\frac{1}{2} \int d\tau \xi^T \Delta \xi \right) = \sum_{\text{bion}} e^{-S} \int d\tau_0 d\phi_0 \sqrt{\det \left( \frac{G}{2\pi} \right)} \sqrt{\det \Delta_0} \sqrt{\det \Delta'},
\] (III.36)

where \(\Delta_0 = -\partial^2_\tau + \omega^2\) is the differential operator around the minimum of the potential and \(\det'\Delta\) denotes the determinant excluding the zero modes. The contribution of zero modes are expressed as the moduli integral over the real moduli space \(M_{\text{bion}} = \mathbb{R} \times S^1\) whose measure is given by the moduli space metric \(G\) (see Appendix C).

To compute the determinant, it is convenient to introduce \(\Xi_0\) defined as
\[
\Xi_0 \equiv \frac{1}{g} \begin{pmatrix}
e_1 & e_2 \\
\bar{e}_1 & \bar{e}_2
\end{pmatrix}^{-1} \begin{pmatrix}
\partial_{\tau_0} \varphi_{sol} & \partial_{\phi_0} \varphi_{sol} \\
\partial_{\tau_0} \bar{\varphi}_{sol} & \partial_{\phi_0} \bar{\varphi}_{sol}
\end{pmatrix}.
\] (III.37)

Its explicit form is
\[
\Xi_0 = \frac{\sqrt{2}}{g} \frac{\sqrt{\omega^2 - m^2}}{\omega} f(\tau - \tau_0) \begin{pmatrix}
\omega \cos \phi_0 \cosh \omega(\tau - \tau_0) & -\sin \phi_0 \sinh \omega(\tau - \tau_0) \\
\omega \sin \phi_0 \cosh \omega(\tau - \tau_0) & \cos \phi_0 \sinh \omega(\tau - \tau_0)
\end{pmatrix}.
\] (III.38)

In terms of \(\Xi_0\), the differential operator \(\Delta\) can be simply written as
\[
\Delta = - \left( \partial_\tau + \partial_\tau \Xi_0 \Xi_0^{-1} \right) \left( \partial_\tau - \partial_\tau \Xi_0 \Xi_0^{-1} \right).
\] (III.39)

It is clear from this form of \(\Delta\) that the column vectors of \(\Xi_0\) are zero modes of the operator \(\Delta\), i.e. \(\Xi_0\) is a basis of the zero modes. As shown in Appendix B, the determinant of operators of the form (III.39) can be obtained by using the formula
\[
\frac{\det'\Delta}{\det \Delta_0} = \frac{\det G}{\det \left( 2MK_+K_- \right)},
\] (III.40)

where the matrices \(G\), \(M\) and \(K_\pm\) are defined by
\[
G = \int_{-\infty}^{\infty} d\tau \Xi_0^T \Xi_0, \quad M = \lim_{\tau \to \pm \infty} \partial_\tau \Xi_0 \Xi_0^{-1}, \quad K_\pm = \lim_{\tau \to \pm \infty} e^{\pm M\tau} \Xi_0.
\] (III.41)
In the present case, \( M = \omega \mathbf{1}_2 \) and \( \mathcal{G} \) is the moduli space metric. The matrices \( K_{\pm} \) can be calculated from the explicit form of \( \Xi_0 \) as

\[
K_{\pm} = \frac{2\sqrt{2}}{g} \frac{\omega}{\sqrt{\omega^2 - m^2}} e^{\pm \omega \tau_0} \left( \begin{array}{c}
\omega \cos \phi_0 & \pm \sin \phi_0 \\
\omega \sin \phi_0 & \pm \cos \phi_0
\end{array} \right). \tag{III.42}
\]

Substituting \( K_{\pm} \) into the formula, we obtain the one-loop determinant

\[
\int \mathcal{D} \xi \exp \left( -\frac{1}{2} \int d\tau \, \xi^T \Delta \xi \right) = \int d\tau_0 d\phi_0 \sqrt{\text{det} \left( \frac{1}{\pi} M K_{+}^\dagger K_{+} \right)} = \int d\tau_0 \frac{16 i e^{-2i \text{Im} \tau_0} \omega^4}{g^2 (\omega^2 - m^2)} , \tag{III.43}
\]

where the overall sign has to be determined by taking into account how the original path integral is deformed. The fact that this determinant is purely imaginary implies that there exists an unstable eigenmode of \( \Delta \) with a negative eigenvalue. We will discuss these issues in the next section.

Combining the real and complex bion contributions and using \( \lim_{\beta \to \infty} \frac{1}{\beta} \int d\tau_0 = 1 \), we obtain

\[
- \lim_{\beta \to \infty} \frac{1}{\beta} \left( \frac{Z_1}{Z_0} \right) = -i(1 - e^{\pm 2\pi i}) \frac{16 \omega^4}{g^2 (\omega^2 - m^2)} \exp \left( \frac{-2 \omega}{g^2} - 2 \epsilon \log \frac{\omega + m}{\omega - m} \right) , \tag{III.44}
\]

where we have used Eq. (III.33) to determine the relative sign of the real and complex bion contributions. This is the leading order non-perturbative correction to the partition function. Note that the saddle point method with the Gaussian approximation gives the asymptotic form for small “Planck constant”, i.e. the inverse of the overall coefficient of the action. Therefore, we should regard the leading order saddle point result (III.44) as the asymptotic form of the partition function in the weak coupling limit \( g \to 0 \) with fixed boson-fermion coupling constant \( \lambda \equiv \epsilon mg^2 \). This result cannot be obtained by conventional methods based on the well-separated kink-antikink ansatz (such as the dilute gas approximation) since the kink-antikink relative distance in the saddle point configurations is not necessarily large for fixed \( \lambda \) as can be seen from Eqs. (III.14) and (III.24).

Although this non-perturbative correction vanishes for \( \epsilon = 1 \) as expected from the discussion of the supersymmetry, it does not reproduce the correct ground state energy for the near supersymmetric case (III.30). This is because the weak coupling limit \( g \to 0 \) with fixed \( \lambda \) is equivalent to the large \( \epsilon \) limit. Therefore it does not agree with the near supersymmetric result with \( \epsilon \approx 1 \).

### D. Normalizable quasi zero modes around bion configurations

In the previous section, we have computed the bion contributions by assuming that fluctuations around the saddle point configurations are small. However, if we take the \( g \to 0 \) limit without fixing \( \lambda = \epsilon mg^2 \), the determinant (III.40) vanishes and hence low frequency (light) modes emerge
in such a limit. Since the truncation at the quadratic order is not valid for nearly massless modes, their contributions are not fully captured in the one-loop determinant. Here, we look for such quasi zero modes which become nearly massless for small $g$ and $\lambda$.

First, note that the exact zero modes given in (III.37) can be obtained by differentiating the background solution with respect to the overall position and phase ($\tau_0, \phi_0$). They can be viewed as superpositions of the position and phase modes localized on the constituent kink and antikink. As can be seen from Eqs. (III.14) and (III.24), the constituent kink and antikink are isolated from each other in the weak coupling limit $\lambda \to 0$:

$$\tau_+ - \tau_- \approx \frac{1}{m} \log \frac{2m^2}{\lambda} \to \infty \quad (\lambda \to 0),$$

(III.45)

and hence any superposition of the position and phase modes becomes massless. This fact implies that the relative modes corresponding to the relative position and phase are nearly massless in the weak coupling regime $\lambda \approx 0$. As with the case of the overall zero modes, such relative modes can also be represented as the derivatives of the background bion solutions with respect to the relative position $\tau_r \equiv \tau_+ - \tau_-$ and phase $\phi_r \equiv \phi_+ - \phi_-:

$$\xi_{\tau_r} = \frac{1}{g} \begin{pmatrix} e_1 & e_2 \\ \bar{e}_1 & \bar{e}_2 \end{pmatrix}^{-1} \begin{pmatrix} \partial_{\tau_r} \varphi_{\text{sol}} \\ \partial_{\tau_r} \tilde{\varphi}_{\text{sol}} \end{pmatrix}, \quad \xi_{\phi_r} = \frac{1}{g} \begin{pmatrix} e_1 & e_2 \\ \bar{e}_1 & \bar{e}_2 \end{pmatrix}^{-1} \begin{pmatrix} \partial_{\phi_r} \varphi_{\text{sol}} \\ \partial_{\phi_r} \tilde{\varphi}_{\text{sol}} \end{pmatrix},$$

(III.46)

where $\varphi_{\text{sol}}$ and $\tilde{\varphi}_{\text{sol}}$ depends on the positions $\tau_{\pm}$ and phases $\phi_{\pm}$ as

$$\varphi_{\text{sol}} = \left( e^{\omega(\tau_+ + i\phi_+)} + e^{-\omega(\tau_- + i\phi_-)} \right)^{-1}, \quad \tilde{\varphi}_{\text{sol}} = \left( e^{\omega(\tau_+ + i\phi_+)} + e^{-\omega(\tau_- + i\phi_-)} \right)^{-1},$$

(III.47)

and $(\tau_{\pm}, \phi_{\pm})$ are set to the values in Eqs. (III.14) and (III.24) after differentiating the solution with respect to the relative parameters $(\tau_r, \phi_r)$. The relative modes $\xi_{\tau_r}$ and $\xi_{\phi_r}$ are approximate normalizable eigenmodes in the weak coupling regime $\lambda \approx 0$, whose eigenvalues are approximately given by

$$\Delta \xi_{\tau_r} \approx \lambda \xi_{\tau_r}, \quad \Delta \xi_{\phi_r} \approx -\lambda \xi_{\phi_r},$$

(III.48)

Note that the negative eigenvalue of $\xi_{\phi_r}$ implies that the bion solution is unstable under a real variation of $\phi_r$.

A numerical analysis implies that the other fluctuation modes seem to have frequencies higher than $\omega$ and hence they are non-normalizable modes with continuous spectra. Their contribution to the path integral can be taken into account by using the one-loop determinant as in the previous subsection.
On the other hand, as with the case of exact zero modes, the contributions of the light normalizable modes should be calculated by means of the quasi moduli integral, i.e. the integral over the quasi moduli parameters \((\tau_r, \phi_r)\):

\[
Z_{\text{bion}} \approx \int d\tau_0 d\phi_0 d\tau_r d\phi_r \det'' \Delta \exp (-V_{\text{eff}}), \tag{III.49}
\]

where \(\det'' \Delta\) denotes the functional determinant excluding both the exact and the quasi zero modes. The effective potential \(V_{\text{eff}}\) is a function of the quasi moduli parameters which can be regarded as a reduced action on “the quasi moduli space”, i.e. the bottom of valley of the action in the configuration space. In the next section, we evaluate the quasi moduli integral \((\text{III.49})\) using an asymptotic form of the effective potential \(V_{\text{eff}}\).

IV. BION CONTRIBUTIONS AND QUASI MODULI INTEGRAL

In the previous section, we have seen that normalizable modes corresponding to the relative position and phase become nearly massless in the weak coupling limit \(g \to 0\). Their contributions may not be fully captured in the one-loop determinant since the action is almost flat along the directions corresponding to the nearly massless modes. In this section, we apply the Lefschetz thimble formalism to our system and compute the contributions of the nearly massless modes by using complexified quasi moduli integrals.

A. Lefschetz thimble formalism

Let us first recapitulate the Lefschetz thimble formalism, which enables us to handle complexified path integrals beyond the level of the Gaussian approximation around saddle points.

In general, a partition function \(Z\) defined as a path integral can be formally rewritten by deforming the integration contour as

\[
Z = \int \mathcal{D}\varphi \exp (-S[\varphi]) = \sum_{\sigma \in \mathcal{S}} n_\sigma Z_\sigma, \tag{IV.1}
\]

where \(\mathcal{S}\) denotes the set of all saddle points of the action \(S\). For simplicity, we assume that all the saddle points are non-degenerated, i.e. there is no flat direction around each saddle point. The contribution from each saddle point \(Z_\sigma\) is given by the path integral over “the Lefschetz thimble \(J_\sigma\)” associated with the saddle point \(\sigma\):

\[
Z_\sigma = \int_{J_\sigma} \mathcal{D}\varphi \exp (-S). \tag{IV.2}
\]
The thimble $\mathcal{J}_\sigma$ is the set of points in the complexified configuration space which can be reached by upward flows from the saddle point $\sigma$:

$$\frac{d\varphi}{dt} = G^{-1} \frac{\delta S}{\delta \varphi}, \quad \lim_{t \to -\infty} \varphi = \varphi_{\text{sol}, \sigma},$$

(IV.3)

where $t$ is a formal flow parameter and $G$ is the target space metric. Note that $\text{Re} S$ is strictly increasing and $\text{Im} S$ is constant along the upward flow

$$\frac{d}{dt} \text{Re} S > 0, \quad \frac{d}{dt} \text{Im} S = 0.$$ (IV.4)

The coefficients $n_\sigma$ indicate how the original integration cycle $C_R$ of the path integral is deformed into the union of $\mathcal{J}_\sigma$:

$$C_R = \sum_{\sigma \in \mathcal{S}} n_\sigma \mathcal{J}_\sigma.$$ (IV.5)

They can also be defined as the intersection numbers between $C_R$ and “the dual thimble $\mathcal{K}_\sigma$” defined as the set of points which flows to the saddle point $\sigma$:

$$\frac{d\varphi}{dt} = G^{-1} \frac{\delta S}{\delta \varphi}, \quad \lim_{t \to \infty} \varphi = \varphi_{\text{sol}, \sigma}.$$ (IV.6)

Since the thimble $\mathcal{J}_\sigma$ and its dual $\mathcal{K}_\sigma$ are defined in terms of the flow, it follows that the real and imaginary parts of the complexified action satisfy

$$\text{Re} S|_{\mathcal{J}_\sigma} \geq \text{Re} S|_{\text{sol}, \sigma} \geq \text{Re} S|_{\mathcal{K}_\sigma}, \quad \text{Im} S|_{\mathcal{J}_\sigma} = \text{Im} S|_{\text{sol}, \sigma} = \text{Im} S|_{\mathcal{K}_\sigma}.$$ (IV.7)

These properties imply that $\mathcal{J}_\sigma$ and $\mathcal{K}_\sigma$ intersect exactly once at the saddle point $\sigma$, and $\mathcal{J}_\sigma$ cannot intersect with $\mathcal{K}_{\sigma'}$ ($\sigma' \neq \sigma$) since $\text{Im} S|_{\mathcal{J}_\sigma} \neq \text{Im} S|_{\mathcal{K}_{\sigma'}}$ for a generic action. Therefore, the intersection pairing of $\mathcal{J}_\sigma$ and $\mathcal{K}_{\sigma'}$, regarded as middle dimensional relative homology cycles, is given by

$$\langle \mathcal{J}_\sigma, \mathcal{K}_{\sigma'} \rangle = \delta_{\sigma \sigma'}. $$ (IV.8)

Using this pairing, we can calculate the coefficients $n_\sigma$ as the intersection number of the original contour $C_R$ and the dual thimble $\mathcal{K}_\sigma$:

$$n_\sigma = \langle C_R, \mathcal{K}_\sigma \rangle.$$ (IV.9)

The perturbative part of the partition function corresponds to $Z_0$ defined as the path integral over the thimble $\mathcal{J}_0$ emanating from the trivial vacuum configuration. Non-perturbative contributions are given by the path integral over thimbles associated with non-trivial saddle points $\sigma$. It is
often the case that the partition function for a real positive coupling constant $g$ is on the Stokes line, i.e. the line on which the thimbles $J_{\sigma}$ and the coefficients $n_{\sigma}$ change discontinuously when we vary the coupling constant in the complex $g$ plane. If $J_{0}$ jumps on the real axis ($\text{Im} \ g = 0$), the perturbative part $Z_{0}$ has an ambiguity depending on how we take the limit $\text{Im} \ g \to \pm 0$. However, the original partition function $Z$ has no ambiguity since it is defined independently of $J_{\sigma}$ and $n_{\sigma}$. Therefore, the ambiguity of $Z_{0}$ has to be canceled by those associated with other non-trivial saddle points. In the case of $\mathbb{C}P^{1}$ quantum mechanics, such saddle points correspond to the bion configurations and their contributions have an ambiguity as can also be seen in the result of the Gaussian approximation (III.44). We will see below that the ambiguity of the bion contribution originates from the discontinuous change of the intersection number $n_{\sigma}$ associated with the bion saddle points.

a. Decomposing degrees of freedom

In the previous section, we have calculated the path integral over the thimbles $J_{\sigma}=rb$ and $J_{\sigma}=cb$ by using the leading order Gaussian approximation. However, such an approximation is not sufficient for small $g$ and $\lambda$, since there emerge nearly massless modes, for which the truncation of the fluctuations at the quadratic order is not valid. In the following, we evaluate the contributions of the such quasi zero modes based on the Lefschetz thimble method.

To see the thimbles structure in the $\mathbb{C}P^{1}$ quantum mechanics, let us decompose the degrees of freedom as

$$
\begin{pmatrix}
\varphi
\varepsilon
\end{pmatrix} = \begin{pmatrix}
\varphi_{kk}
\varepsilon
\end{pmatrix} + g \begin{pmatrix}
e_1 & e_2
\varepsilon_1 & \varepsilon_2
\end{pmatrix} \begin{pmatrix}
\xi^1
\xi^2
\end{pmatrix}.
$$

The background configuration ($\varphi_{kk}, \varepsilon_{kk}$) is a kink-antikink ansatz with the (quasi)moduli parameters $(\tau_0, \phi_0, \tau_r, \phi_r)$ satisfying the equation of motion up to the quasi zero modes:

$$
\begin{pmatrix}
\frac{\delta S}{\delta \varphi}
\frac{\delta S}{\delta \varepsilon}
\end{pmatrix} = a \begin{pmatrix}
\frac{\partial \varphi_{kk}}{\partial \tau_r}
\frac{\partial \varepsilon_{kk}}{\partial \tau_r}
\end{pmatrix} + b \begin{pmatrix}
\frac{\partial \varphi_{kk}}{\partial \phi_r}
\frac{\partial \varepsilon_{kk}}{\partial \phi_r}
\end{pmatrix}.
$$

The massive fluctuations $\xi^a$ are taken to be orthogonal to the exact and the quasi zero modes. Eq. (IV.11) defines the bottom of the valley of the action parameterized by the quasi moduli parameters $(\tau_r, \phi_r)$ along which the variations of the action in the normal directions vanish.

The massive fluctuations $\xi^a$ are taken to be orthogonal to the exact and the quasi zero modes. Then, Eq. (IV.11) ensures that the expanded action does not have linear term in $\xi$:

$$
S[\varphi, \varepsilon] = V_{\text{eff}}(\tau_r, \phi_r) + \frac{1}{2} \int d\tau \xi^T \Delta_{kk} \xi + O(g),
$$

(IV.12)
where $\Delta_{kk}$ is the differential operator in the background of the kink-antikink configuration. For $g \approx 0$, the flow for each massive eigenmode in $\xi$ can be approximated as a straight line and the integration along such flows gives the one-loop determinant of $\Delta_{kk}$. For the quasi moduli parameters, the gradient flow of the action reduces to that of the effective potential $V_{\text{eff}}$:

$$\frac{d\eta^i}{dt} = g'_{ij} \frac{\partial V_{\text{eff}}}{\partial \eta^j}, \quad (\text{IV.13})$$

where $\eta^1 = \tau_r$, $\eta^2 = \phi_r$ and $g'_{ij}$ is the metric on the quasi moduli space

$$g'_{ij} = \frac{1}{g^2} \int d\tau \frac{1}{(1 + |\varphi_{kk}|^2)^2} \frac{\partial \varphi_{kk}}{\partial \eta^i} \frac{\partial \varphi_{kk}}{\partial \eta^j}. \quad (\text{IV.14})$$

In the weak coupling limit $\lambda \to 0$, the saddle point configurations can be viewed as well-separated kink-antikink pairs, so that the path integral is dominated by the contributions from the well-separated region. In such a case, $\varphi_{kk}$ and $\tilde{\varphi}_{kk}$ can be well approximated by the simple kink-antikink ansatz (III.47). Therefore, the flows around the saddle points can be determined by using the following asymptotic form of the effective potential obtained by substituting the ansatz (III.47) into the action:

$$V_{\text{eff}} \approx \frac{2m}{g^2} - \frac{4m}{g^2} e^{-m\tau_r} \cos \phi_r + 2em\tau_r. \quad (\text{IV.15})$$

Therefore, the bion contributions for small $g$ and $\lambda$ can be obtained by applying the Lefschetz thimble method to the quasi moduli integral

$$\frac{Z_1}{Z_0} \approx \int d\tau_0 d\phi_0 d\tau_r d\phi_r \sqrt{\text{det} \left( \frac{\mathcal{G}}{2\pi} \right) \text{det} \left( \frac{\mathcal{G}'}{2\pi} \right) \text{det}' \Delta_0} \exp (-V_{\text{eff}}). \quad (\text{IV.16})$$

Since the spectrum of $\Delta_{kk}$ for a well-separated kink-antikink configuration can be approximated as two copies of that of a single kink, we find that

$$\sqrt{\frac{\text{det} \Delta_0}{\text{det}' \Delta_{kk}}} = \frac{1}{\sqrt{\mathcal{G}}} \int d\xi \exp \left( -\frac{1}{2} \int d\tau \xi^T \Delta_{kk} \xi \right) \approx \frac{1}{\sqrt{\mathcal{G}'} \Delta_0}. \quad (\text{IV.17})$$

The one-loop determinant in the single kink background $\text{det}' \Delta_k$ can be obtained by using the formula (III.40) as

$$\frac{\text{det} \Delta_0}{\text{det}' \Delta_k} = \frac{1}{\text{det} \mathcal{G}_k} \left( \frac{4m^2}{g^2} \right)^2, \quad (\text{IV.18})$$

where $\mathcal{G}_k$ is the metric on the single kink moduli space. The overall and the relative moduli space metrics also reduce to the two copies of the single kink metric

$$\sqrt{\text{det} \mathcal{G} \text{det} \mathcal{G}'} \approx \text{det} \mathcal{G}_k. \quad (\text{IV.19})$$
In summary, the leading order bion contribution to the ground state energy for small $g$ and $\lambda$ is given by the following quasi moduli integral

$$- \lim_{\beta \to \infty} \frac{1}{\beta} \frac{Z_1}{Z_0} \approx -\frac{8m^4}{\pi g^4} \int d\tau_r d\phi_r \exp (-V_{\text{eff}}). \quad (IV.20)$$

In the following, we apply the Lefschetz thimble method to evaluate the quasi moduli integral with the asymptotic potential (IV.15).

**B. Quasi-moduli Integral in sine-Gordon quantum mechanics**

Before calculating the quasi moduli integral in the $\mathbb{C}P^1$ model, let us briefly discuss the case of the sine-Gordon model, which can be obtained by restricting the $\mathbb{C}P^1$ mechanics to the zero angular momentum ($l = 0$) sector:

$$L = \frac{1}{4g^2} \left( \dot{\varphi}^2 - m^2 \sin^2 \theta \right) + \epsilon m \cos \theta, \quad \varphi = \tan \frac{\theta}{2} e^{i\phi}. \quad (IV.21)$$

This model is not only simpler, but also serves as a useful building block for the $\mathbb{C}P^1$ model. Since the bion solutions in the $\mathbb{C}P^1$ model are in the zero angular momentum sector, the sine-Gordon action (IV.21) also has the corresponding real and complex bion solutions [37, 38]. As shown in [11, 22, 36, 46, 47], the bion contribution with periodic potentials can be expressed in terms of the following quasi zero mode integral with respect to the relative separation $\tau$ (the subscript of $\tau_r$ will be omitted in the following)

$$[I\bar{I}] = \int_{C_R} d\tau e^{-V_{\text{SG}}(\tau)} \quad V_{\text{SG}}(\tau) \equiv -\frac{4m}{g^2} e^{-m\tau} + 2\epsilon m \tau, \quad (IV.22)$$

where $[I\bar{I}]$ denotes the single bion contribution excluding the classical part $e^{-S_{\text{bion}}} = e^{-\frac{2m}{g^2}}$. The original integration cycle is the real axis since the original path integral is over real configurations. Note that the divergence of the integrand (IV.22) in the $\tau \to -\infty$ limit is an artifact of the approximation, since the effective potential is valid only for large $\tau$. In the Bogomolny–Zinn-Justin (BZJ) prescription, with which we can extract physical information form (IV.22), we first regard $g^2$ as a negative number and perform the integral. In the end of the calculation, we analytically continue $g^2$ back to a positive number. Then we end up with a bion contribution $[I\bar{I}]$ with an imaginary ambiguity depending on how the result is analytically continued to real positive $g^2$. It is known that this ambiguity is cancelled out by another imaginary ambiguity emerging from the Borel re-summation of the perturbative series. This is the well-known first step to the resurgence theory. In the reference [38], it has been indicated that the complexified quasi moduli integral is a
more rigorous way to treat the quasi moduli integral which can replace the BZJ prescription. We here extend this idea and clarify the BZJ prescription in terms of the complexification method.

As we have done in Sec. [IIIi], we first introduce a small complex factor as a regulator into the coupling as

$$g^2 \rightarrow g^2 e^{i\theta}. \quad (IV.23)$$

In the end of calculation we take \( \theta \rightarrow \pm 0 \) limits. Now, we complexify the quasi moduli \( \tau \)

$$\tau \in \mathbb{R} \rightarrow \tau = \tau_R + i\tau_I \in \mathbb{C}, \quad (IV.24)$$

with the real part \( \tau_R \) and the imaginary part \( \tau_I \). Let \( \mathcal{S} \) be the set of saddle points in the complex \( \tau \)-plane. Then the original integration cycle can be decomposed as

$$C_R = \sum_{\sigma \in \mathcal{S}} n_\sigma J_\sigma, \quad (IV.25)$$

where \( J_\sigma \) is the thimble associated with the saddle point \( \tau_\sigma \), i.e. the flow line starting from the saddle point \( \tau_\sigma \):

$$\frac{d\tau}{dt} = \frac{1}{2m} \frac{\partial V_{SG}}{\partial \tau}, \quad \lim_{t \to -\infty} \tau = \tau_\sigma, \quad (IV.26)$$

where we have rescaled the flow time parameter \( t \) for notational convenience. Using the decomposition of the integration contour \([IV.25]\), we can rewrite the original integral in \([IV.22]\) as

$$[\bar{I}I] = \sum_{\sigma} n_\sigma Z_\sigma, \quad Z_\sigma = \int_{J_\sigma} d\tau e^{-V_{SG}(\tau)}, \quad (IV.27)$$

The coefficients \( n_\sigma \) is the intersection number between the original integration cycle \( C_R \) and the dual cycle \( K_\sigma \) (dual thimble)

$$n_\sigma = \langle C_R, K_\sigma \rangle \quad (IV.28)$$

a. Thimbles and Dual Thimbles

The effective kink-antikink potential deformed by \( \theta \) is written as

$$V_{SG}(\tau) = \frac{4m}{g^2} e^{-m\tau-i\theta} + 2\epsilon m\tau. \quad (IV.29)$$

What we need to do is just to find the saddle points of the potential, and the thimbles and the dual thimbles associated with them.

The saddle points of the potential \( V_{SG} \) is labeled by an integer \( \sigma \in \mathbb{Z} \):

$$\tau = \tau_\sigma \equiv \frac{1}{m} \left[ \log \frac{2m}{eg^2} + (2\sigma - 1)\pi i - i\theta \right]. \quad (IV.30)$$
The gradient flow equation is given by
\[
\frac{d\tau}{dt} = \frac{1}{2m} \frac{\partial V_{SG}}{\partial \tau} = -\frac{2m}{g^2} e^{m\tau+i\theta} + \epsilon. \tag{IV.31}
\]
This equation can be solved as
\[
\tau(t) = \frac{1}{m} \log \left[ \frac{2m \sin((a - be^{-\epsilon m t}) - \theta)}{m \sigma + a_{\sigma}} \right] - \frac{i}{m} (a - be^{-\epsilon m t}), \tag{IV.32}
\]
where \(a\) and \(b\) are real integration constants. The dual thimbles can be defined as flows reaching the saddle points at \(t \to \infty\). The above solution of the flow equation approach the saddle point \(\tau_{\sigma}\) in Eq. (IV.30), if the integration constant \(a\) is given by
\[
a_{\sigma} = -(2\sigma - 1)\pi + \theta, \quad \sigma \in \mathbb{Z}. \tag{IV.33}
\]
Eliminating \(t\) from the solution (IV.32) and its complex conjugate, we find that the real part \(\tau_R = \text{Re} \tau\) and the imaginary part \(\tau_I = \text{Im} \tau\) are related in the dual thimble \(K_{\sigma}\) as
\[
m\tau_R = \log \left[ \frac{2m \sin(m\tau_I + a_{\sigma})}{m \sigma + a_{\sigma}} \right], \quad (-a_{\sigma} - \pi \leq m\tau_I \leq -a_{\sigma} + \pi). \tag{IV.34}
\]
On the other hand, thimbles are defined by reaching the saddle point at \(t \to -\infty\). Such solutions can be obtained from the general solution (IV.32) by redefining the integration constants as
\[
a = a_{\sigma} + \delta, \quad b = -\delta e^{-\epsilon m t_0}, \tag{IV.35}
\]
and taking the limit \(\delta \to 0:\)
\[
\tau(t) = \frac{1}{m} \log \frac{2m}{eg^2} \left( 1 + e^{m(t-t_0)} \right) - \frac{i}{m} a_{\sigma}. \tag{IV.36}
\]
Therefore, the thimble \(J_{\sigma}\) is the straight line with fixed imaginary part
\[
m\tau_I = -a_{\sigma} = (2\sigma - 1)\pi - \theta. \tag{IV.37}
\]

b. Integral along Lefschetz Thimbles and BZJ prescription

In Fig. 7, the two critical points \(\tau_{\sigma}\), the associated thimbles \(J_{\sigma}\) and the dual cycles \(K_{\sigma}\) are depicted for \(\sigma = 0\) and \(\sigma = 1\). For \(\theta = 0\), the intersection numbers are ill-defined since the dual thimbles never cross the original integration path \(C_R\) although they are asymptotically tangent to \(C_R\). This is one of the reasons why we need to regulate the potential by the imaginary part of the coupling constant.

As shown in Fig. 7, the intersection number jumps when \(\theta\) changes its sign:
\[
(n_0, n_1) = \begin{cases} (0, 1) & \text{for } \theta = +0 \ , \\ (1, 0) & \text{for } \theta = -0 \ , \end{cases} \tag{IV.38}
\]
Fig. 7: Complex integration cycles for bion amplitude in quantum mechanics. The regularization parameter is \( \theta = +0 \) (left) and \( \theta = -0 \) (right).

and hence

\[
\begin{bmatrix}
I_1
\end{bmatrix} = \sum_{\sigma} n_{\sigma} Z_{\sigma} = \begin{cases} 
Z_{\sigma=1} & \text{for } \theta = +0 \\
Z_{\sigma=0} & \text{for } \theta = -0 
\end{cases}
\] (IV.39)

Integrating along each thimble \( \mathcal{J}_\sigma \), we find that

\[
Z_{\sigma} = \int_{\mathcal{J}_\sigma} d\tau \exp(-V_{SG}) = \frac{1}{m} e^{-2\pi i(2\sigma - 1)} \left( \frac{g^2}{4m} e^{i\theta} \right)^{2\epsilon} \Gamma(2\epsilon).
\] (IV.40)

This agrees with the result from the BZJ description. In this calculation of the complex integration, the region where the integrand is divergent is avoided by moving the integration contour \([-\infty, \infty]\) to either of the Lefschetz thimbles \([-\infty \pm i\pi, \infty \pm i\pi]\). This is how one extracts a finite result from the ill-defined integral in the BZJ prescription. Now, it is clear that the ambiguity comes from the sign of regularization parameter \( \theta \) in \( g^2 \rightarrow g^2 e^{i\theta} \).

To sum up, unambiguous definition of moduli integral is obtained by making the coupling constant complex and using the Lefschetz thimble approach. We can regard this method as a more rigorous definition of the BZJ prescription.

C. Bion contributions in \( \mathbb{C}P^1 \) model

a. Quasi-moduli Integral

As discussed above and in [22, 36], the bion contribution in \( \mathbb{C}P^1 \) quantum mechanics can be expressed by the following quasi zero mode integral with respect to the separation \( \tau \) and the relative
phase $\phi$,

$$[\tilde{\cal I}] = \int_{-\pi}^{\pi} d\phi \int_{-\infty}^{\infty} d\tau e^{-V(\tau,\phi)}, \quad V(\tau, \phi) = -\frac{4m}{g^2} \cos \phi e^{-m\tau} + 2\epsilon m \tau. \quad \text{(IV.41)}$$

\(b\). Thimbles and Dual Thimbles

In the integral in question, we first consider the following deformation of the integration contour for the phase $\phi$ \cite{54}. As shown in Fig. 8, we extend the integration path on the $\phi$-plane to make it an integration cycle without boundary in the complex $\phi$ plane. Since the contributions from the paths attached to $\phi = \pm \pi$ (the boundaries of the original path) cancel with each other, the extended contour gives the same value of the bion contribution $[\tilde{\cal I}]$.

We next introduce a small phase for the coupling constant as $g^2 \to g^2 e^{i\theta}$, and complexify both $\tau$ and $\phi$ as

$$\tau = \tau_R + i\tau_I \in \mathbb{C}, \quad \phi = \phi_R + i\phi_I \in \mathbb{C}. \quad \text{(IV.42)}$$

The saddle points are given by the equations,

$$\frac{\partial V}{\partial \tau} = \frac{4m^2 \cos \phi}{g^2} e^{-m\tau-i\theta} + 2\epsilon m = 0 \quad \text{(IV.43)}$$

$$\frac{\partial V}{\partial \phi} = \frac{4m \sin \phi}{g^2} e^{-m\tau-i\theta} = 0. \quad \text{(IV.44)}$$

The solutions are labeled by an integer $\sigma \in \mathbb{Z}$

$$\tau_\sigma = \frac{1}{m} \log \frac{2m}{\epsilon g^2} + \frac{i}{m} (\sigma \pi - \theta), \quad \phi_\sigma = -(\sigma - 1)\pi \pmod{2\pi}. \quad \text{(IV.45)}$$
The physical interpretation of these saddle point will discussed below. To find the thimble $J_\sigma$ and its dual $K_\sigma$, let us redefine the coordinates as
\[ \tau_+ = \tau + \frac{i}{m} \phi, \quad \tau_- = \tau - \frac{i}{m} \phi. \] (IV.46)

Then we can rewrite the effective potential $V$ into two copies of the effective kink-antikink potential in the sine-Gordon model (IV.22):
\[ V = \frac{V_{SG}(\tau_+) + V_{SG}(\tau_-)}{2}, \] (IV.47)

Thus, the solution of the flow equation
\[ \frac{d\tau}{dt} = \frac{1}{2m} \frac{\partial V}{\partial \tau}, \quad \frac{d\phi}{dt} = \frac{m}{2} \frac{\partial V}{\partial \phi}. \] (IV.48)

can be easily obtained as
\[ \tau_i = \frac{1}{m} \log \left[ \frac{2m \sin(a_i - b_i e^{-\epsilon m t} - \theta)}{b_i e^{-\epsilon m t}} \right] - \frac{i}{m} (a_i - b_i e^{-\epsilon m t}), \] (IV.49)

where $i = (+, -)$ and $a_i$ and $b_i$ are integration constants. The flows on the dual thimbles associated with the critical points (IV.45) can be obtained by setting the integration constants as
\[ a_+ = a_{+\sigma} \equiv -\pi + \theta, \quad a_- = a_{-\sigma} \equiv -(2\sigma - 1)\pi + \theta, \] (IV.50)

Eliminating the parameters $b_1$ and $b_2$, we obtain the following equations for the dual thimbles
\[
\begin{align*}
m\tau_R - \phi_I &= \log \left[ \frac{2m \sin(m\tau_I + \phi_R + a_{+\sigma})}{\epsilon g^2 \sin(m\tau_I + \phi_R + a_{+\sigma})} \right], \quad -\pi \leq m\tau_I + \phi_R + a_{+\sigma} \leq \pi, \quad (IV.51) \\
m\tau_R + \phi_I &= \log \left[ \frac{2m \sin(m\tau_I - \phi_R + a_{-\sigma})}{\epsilon g^2 \sin(m\tau_I - \phi_R + a_{-\sigma})} \right], \quad -\pi \leq m\tau_I - \phi_R + a_{-\sigma} \leq \pi. \quad (IV.52)
\end{align*}
\]

The thimbles can also be determined by using the result for the sine-Gordon case (IV.37) as
\[ \text{Im} \tau_+ = -a_{+\sigma} = \pi - \theta, \quad \text{Im} \tau_- = -a_{-\sigma} = (2\sigma - 1)\pi - \theta. \] (IV.53)

Therefore, the thimbles are the planes specified by
\[ m\tau_I = \sigma \pi - \theta, \quad \phi_R = -(\sigma - 1)\pi. \] (IV.54)

The thimbles and dual thimbles projected onto $(\phi_R, \phi_I, \tau_I)$ are shown in Fig.9. They have the same structure as in the sine-Gordon case on the two dimensional slices $\tau_I + \phi_R = -a_{+\sigma}$ and $\tau_I - \phi_R = -a_{-\sigma}$.

As with case of the sine-Gordon quantum mechanics, the dual thimbles $K_1$ and $K'_0$ intersect the original integration contour for $\theta = +0$ while the dual thimbles $K_{-1}$ and $K'_0$ cross the original
Fig. 9: Integration contour, Lefschetz thimbles and dual thimbles for (a) $\theta = -0$ (seen from upper left) and (b) $\theta = +0$ (seen from lower right). The orange lines are the original extended integration contours, while four colored (red, blue, yellow and green) lines and surfaces indicate the thimbles and their duals, respectively. Note that since the integration contour and Lefschetz thimbles are direct products of the $\tau_R$ direction and lines in $(\phi_R, \phi_I, \tau_I)$, their projected images are lines in the three-dimensional space.

contour for $\theta = -0$. Note that $K_0$ and $K'_0$ are the identical thimble related by the shift $\phi_R \rightarrow \phi_R + 2\pi$. By taking into account how the original integration contour is decomposed into the thimbles (see Fig. 10), the sign of the intersection numbers can be determined as

$$\left( n_{-1} , n_0 , n_1 \right) = \begin{cases} \left( -1 , 1 , 0 \right) & \text{for } \theta = -0 \\ \left( 0 , -1 , 1 \right) & \text{for } \theta = +0 \end{cases}.$$ \hspace{1cm} (IV.55)

Therefore, the bion contribution has the ambiguity depending on the sign of $\theta$

$$\left[ \mathcal{I} \mathcal{I} \right] = \begin{cases} Z_{\sigma = 0} - Z_{\sigma = -1} & \text{for } \theta = -0 \\ Z_{\sigma = 1} - Z_{\sigma = 0} & \text{for } \theta = +0 \end{cases}.$$ \hspace{1cm} (IV.56)

c. Integral along Lefschetz Thimbles

Now let us evaluate the integral over the thimbles. Changing the coordinates as

$$\tau \rightarrow \tau' = \tau - \tau_{\sigma} \quad \phi \rightarrow \phi' = \phi - \phi_{\sigma}.$$ \hspace{1cm} (IV.57)

we find that the potential becomes

$$V = 2\epsilon \left( m \tau' + e^{-m \tau'} \cos \phi' + \log \frac{2m}{\epsilon g^2} + \sigma \pi i - i\theta \right).$$ \hspace{1cm} (IV.58)
Fig. 10: Deformation of integration contour. (a) The integration contour can be decomposed into two paths (orange and blue). One of them corresponds to the thimble with \( n = 0 \) and the other can be continuously deformed into the thimble with \( n = 1 \). The shaded regions in the right figure corresponds to the region where \( \text{Re} V < T \ll \text{Re} V_{\text{critical}} \) with some real number \( T \).

The thimble \( \mathcal{J}_\sigma \) corresponds to the two dimensional plane \( \tau' \in \mathbb{R}, \phi' \in i\mathbb{R} \). We can check that the potential satisfies

\[
\text{Re} V \geq 2\epsilon \left( 1 + \log \frac{2m}{\epsilon g^2} \right), \quad \text{Im} V = (\sigma \pi - \theta)\epsilon = \text{const.}, \quad (IV.59)
\]

for \( \tau' \in \mathbb{R}, \phi' \in i\mathbb{R} \). Integrating over the thimbles, we obtain

\[
Z_\sigma = \int_{\mathbb{R}} d\tau' \int_{i\mathbb{R}} d\phi' e^{-V} = \frac{i}{2m} \left( \frac{g^2 e^{i\theta}}{2m} \right)^{2\epsilon} e^{-2\pi i\sigma} \Gamma (\epsilon)^2. \quad (IV.60)
\]

Therefore, the bion contribution is given by

\[
[\mathcal{I}\mathcal{I}] = \frac{1}{m} \left( \frac{g^2 e^{i\theta}}{2m} \right)^{2\epsilon} \sin \epsilon \pi \Gamma (\epsilon)^2 \times \begin{cases} 
e^{\pi i\epsilon} & \text{for } \theta = -0 \\ ne^{-\pi i\epsilon} & \text{for } \theta = +0 \end{cases}. \quad (IV.61)
\]

This result is consistent with the one obtained by applying the Bogomolny–Zinn-Justin prescription for the divergent region \( \tau \to -\infty, |\phi| \leq \pi/2 \). In this calculation of the complex integral, the region where the integrand is divergent is avoided by deforming the integration contour as shown in Fig. 10. This is how one extracts a finite result from the ill-defined integral in the BZJ prescription. Thus, based on the Lefschetz thimble decomposition of the quasi moduli integral together with the complexification of the coupling, we obtain an unambiguous definition of the ill-defined moduli integral.
From Eq. (IV.20) and the result of the complexified quasi moduli integral, we obtain the following non-perturbative correction to the ground state energy:

\[- \lim_{\beta \to \infty} \frac{1}{\beta} \frac{Z_1}{Z_0} \approx - \frac{8m^4}{\pi g^4} [\mathcal{I}] e^{-\frac{2m}{g^2}} \]

\[= -2m \left( \frac{g^2}{2m} \right)^{2(\epsilon-1)} \frac{\sin \epsilon \pi}{\pi} \Gamma(\epsilon) e^{-\frac{2m}{g^2}} \times \begin{cases} e^{\pi i \epsilon} & \text{for } \theta = -0 \\ e^{-\pi i \epsilon} & \text{for } \theta = +0 \end{cases}. \quad (IV.62)\]

For \( \epsilon \approx 1 \), this non-perturbative correction to the ground state energy becomes

\[- \lim_{\beta \to \infty} \frac{1}{\beta} \frac{Z_1}{Z_0} = -2m(\epsilon - 1)e^{-\frac{2m}{g^2}}. \quad (IV.63)\]

This gives correct non-perturbative correction (II.30) in the near supersymmetric case \( \epsilon \approx 1 \).

The result of the Gaussian approximation (III.44) in the weak coupling limit \( g \to 0 \) agrees with Eq. (IV.62) if the gamma function is replaced by its asymptotic form

\[\Gamma(\epsilon) \to \sqrt{\frac{2\pi}{\epsilon}} e^{\epsilon} e^{-\epsilon}. \quad (IV.64)\]

Therefore, these two results agree in the large \( \epsilon \) limit as expected from the discussion in Sec. III.

d. On the saddle points of the original action and the effective bion potential

As we have seen above, the saddle points of the effective bion potential \( V_{\text{eff}} \) are labeled by an integer \( \sigma \). By comparing the values of the quasi moduli at each saddle point and those of the real bion (III.14) and the complex bion (III.24), we find that the saddle points with \( \sigma = 0 \) and \( \sigma = 1 \) correspond to the weak coupling limit (\( g \to 0 \)) of the real and the complex bions, respectively. Actually, the saddle point \( \sigma = -1 \) also corresponds to the complex bion. This is because the imaginary part of the kink positions

\[\tau_\pm = \tau_0 \pm \frac{\tau_r}{2}, \quad (IV.65)\]

is defined modulo \( 2\pi i/\omega \approx 2\pi i/m \) and hence the following shift of the imaginary part does not change the physical configuration.

\[(m\tau_0, m\tau_r) \sim (m\tau_0 - \pi i, m\tau_r + 2\pi i). \quad (IV.66)\]

By using this shift, the value of \( \tau_r \) at the saddle point with \( \sigma = -1 \) can be fixed to that for the complex bion. Nevertheless, the value of the action is different since the overall position \( \tau_0 \) has the imaginary part \( \text{Im} \tau_0 = -\pi i/m \), for which the integration path \( C \) in (III.29) has to be the line \( \text{Im} \tau = +\frac{1}{m} \frac{\pi i}{2} \).
V. COMMENTS ON BION CONTRIBUTIONS IN SINE-GORDON MODEL

As we have in the previous section, the sine-Gordon action [IV.21], which can be obtained by restricting the $\mathbb{C}P^1$ action to the zero angular momentum sector, also has real and complex bion solutions [37, 38]. The crucial difference is that the bions in the sine-Gordon model do not have phase modulus. This is merely one manifestation of the fundamental difference of the topology of the target space: $S^1$ for the sine-Gordon model and $\mathbb{C}P^1 = S^2$ for the $\mathbb{C}P^1$ model. This fact particularly gives a marked difference when we consider quantum theory [63]. Consequently, the non-perturbative contributions to the ground state energy in the sine-Gordon model is different from that in the $\mathbb{C}P^1$ model. In the sine-Gordon model, the Gaussian approximation for the bion contributions which is valid in the limit $g \to 0$ with fixed $\lambda = mg^2$, gives

$$\lim_{\beta \to \infty} \frac{1}{\beta} \frac{Z_1}{Z_0} = 2 \sqrt{\frac{8\omega^5}{\pi g^2(\omega^2 - m^2)}} (1 + e^{\pm 2\pi i\epsilon}) \exp \left[ \frac{2\omega}{g^2} - 2\epsilon \log \frac{\omega + m}{\omega - m} \right], \quad (V.1)$$

while the complexified quasi moduli integral, which is valid in the limit $g \to 0$ with fixed $\epsilon$, gives

$$\lim_{\beta \to \infty} \frac{1}{\beta} \frac{Z_1}{Z_0} = \frac{m}{\pi} (1 + e^{\pm 2\pi i\epsilon}) \Gamma(2\epsilon) \exp \left[ -\frac{2m}{g^2} + (2\epsilon - 1) \log \frac{g^2}{4m} \right], \quad (V.2)$$

corresponding to $\theta = \text{arg} g^2 = -, +$ for upper and lower sign respectively. These results do not agree with the corresponding non-perturbative corrections (III.44) and (IV.62) in the $\mathbb{C}P^1$ model. The mismatch of the ground state energies is due to the difference of the Hamiltonian $H_{\mathbb{C}P^1}$ in Eq. (II.24) for the $\mathbb{C}P^1$ model and that obtained from the Lagrangian [IV.21] for the sine-Gordon model

$$H_{\mathbb{C}P^1}^{\epsilon=0} = -g^2 \left( \partial_{\theta}^2 + \frac{1}{\tan \theta} \partial_{\theta} \right) + \frac{m^2}{4g^2} \sin^2 \theta - \epsilon m \cos \theta = H_{SG} - \frac{g^2}{\tan \theta} \partial_{\theta}. \quad (V.3)$$

The nonperturbative corrections (V.1) and (V.2) vanish in the limit $\epsilon = \frac{1}{2}$. This is in accord with the fact that $\epsilon = \frac{1}{2}$ is the supersymmetric limit of the sine-Gordon model.

As opposed to the $\mathbb{C}P^1$ model, the ambiguity in (V.1) and (V.2) does not vanish in the near supersymmetric regime $\epsilon \approx \frac{1}{2}$. To compare it with the ambiguity in the perturbative part, let us consider the leading order correction to the ground state energy in the near supersymmetric limit:

$$E^{(1)} = E^{(1)}_{\text{pert}} + E^{(1)}_{\text{bion}}, \quad (V.4)$$

where $E^{(1)}$ stands for the leading order coefficient in the small $\delta \epsilon \equiv \epsilon - \frac{1}{2}$ expansion of the ground state energy

$$E^{(1)} \equiv \lim_{\epsilon \to \frac{1}{2}} \partial_{\epsilon} E. \quad (V.5)$$
For $\epsilon = \frac{1}{2}$, the supersymmetric ground state wave function in the sine-Gordon quantum mechanics is given by
\[
\Psi = \exp \left( \frac{m^2 g^2 \cos \theta}{2} \right). \tag{V.6}
\]

By using the standard perturbation theory with respect to small $\epsilon - \frac{1}{2} \equiv \delta \epsilon$, we obtain the correction to the ground state energy in the near supersymmetric case:
\[
E \approx - m \delta \epsilon \frac{I_1(m/g^2)}{I_0(m/g^2)} = - \delta \epsilon g^2 m \frac{\partial}{\partial m} \log I_0(m/g^2), \tag{V.7}
\]
where $I_1(m/g)$ and $I_0(m/g)$ are the modified Bessel function of the first kind. The asymptotic expansion of the Bessel function implies that the perturbative expansion gives the following asymptotic series for the correction to the ground state energy
\[
E^{(1)}_{\text{pert}} = - g^2 m \frac{\partial}{\partial m} \log \left[ \sqrt{\frac{g^2}{2\pi m}} \left( 1 + \cdots + \frac{(2n-1)!!}{n!} \left( \frac{g^2}{8m} \right)^n + \cdots \right) \right], \tag{V.8}
\]
where $K_0(m/g^2)$ is the modified Bessel function of the second kind. Therefore, the non-perturbative part should have the following ambiguity
\[
E^{(1)}_{\text{bion}} = E^{(1)}_{\text{pert}} - \mp 2 i m e^{-\frac{2m}{g^2}} + O(e^{-\frac{4m}{g^2}}), \tag{V.11}
\]
where $K_0(m/g^2)$ is the modified Bessel function of the second kind. Therefore, the non-perturbative part should have the following ambiguity
\[
E^{(1)}_{\text{bion}} = E^{(1)}_{\text{pert}} - \mp 2 i m e^{-\frac{2m}{g^2}} + O(e^{-\frac{4m}{g^2}}). \tag{V.11}
\]

VI. SUMMARY AND DISCUSSION

We have discussed the non-perturbative contributions from the complex saddle points in the $\mathbb{C}P^{N-1}$ and sine-Gordon quantum mechanics with the fermionic degrees of freedom. We obtained non-perturbative contributions from the real and complex bion solutions by using the Gaussian approximation, which is valid in the small coupling limit $g \to 0$ with fixed boson-fermion coupling.
constant $\lambda$ (large $\epsilon$ limit). For small $\epsilon$ including the supersymmetric ($\epsilon = 1$) and the purely bosonic ($\epsilon = 0$) cases, we investigated the integral along the Lefschetz thimbles associated with the saddle points to incorporate the contributions from the quasi zero modes, i.e., the light normalizable mode around the real and complex bion solutions. To evaluate the integrals along thimbles, we treated the bion configurations as well-separated instanton-antiinstanton configurations and calculate the quasi moduli integral with the complexified separation and phase based on the Lefschetz thimble formalism. The final results of the non-perturbative contributions from complexified saddle points in the $\mathbb{C}P^{N-1}$ quantum mechanics are consistent with the known results for the supersymmetric case ($\epsilon = 1$) and the near supersymmetric case ($\epsilon \sim 1$).

Apart from the above main results, we have three more arguments:

(i) As discussed in Ref. [38], we show that the result based on the Bogomolny–Zinn-Justin prescription, in which the sign of coupling $g^2$ is changed and is analytically continued back to the original sign in the final expression, is understood in terms of the complexification of the quasi moduli parameters both in the sine-Gordon quantum mechanics and the $\mathbb{C}P^1$ quantum mechanics. In the complexified quasi moduli integral, we consider Lefschetz thimbles corresponding to decomposed cycles of the deformed integration contour. As we have discussed, these thimbles in the complexified quasi moduli integral are viewed as the approximated versions of the Lefschetz thimbles associated with the saddle points of the original complexified action. To sum up, the quasi moduli integrals for instanton-antiinstanton configuration, which was originally performed based on the BZJ prescription, are nothing but the approximate versions of the Lefschetz thimble integrals associated with the real and complex bion saddle points in the complexified quantum mechanics. This is the reason why the imaginary part of the quasi moduli integral cancels that arising from the perturbative Borel resummation.

(ii) We elucidated the relation of $\mathbb{C}P^{N-1}$ field theory on $\mathbb{R}^1 \times S^1$, and $\mathbb{C}P^{N-1}$ quantum mechanics and sine-Gordon quantum mechanics. We showed that the $\mathbb{C}P^{N-1}$ sigma model on $\mathbb{R}^1 \times S^1$ reduces to $\mathbb{C}P^{N-1}$ quantum mechanics by retaining modes with topological charge less than unity, but not to the sine-Gordon quantum mechanics. We in particular established that bions in the two-dimensional $\mathbb{C}P^{N-1}$ field theory at small radius $L \ll 1$ is correctly described by means of $\mathbb{C}P^{N-1}$ quantum mechanics. These facts indicate that there is a smooth $L \rightarrow 0$ limit of the two-dimensional $\mathbb{C}P^{N-1}$ field theory and the resultant theory is correctly described by the $\mathbb{C}P^{N-1}$ quantum mechanics instead of the sine-Gordon quantum mechanics. After the renormalization procedure of the two-dimensional $\mathbb{C}P^{N-1}$ field theory, we expect that the most important part of the quantum dynamics should result in replacing the two-dimensional coupling by the running
coupling $g_{2d}^2 (1/L)$ with the renormalization scale $\mu = 1/L$

$$\frac{1}{g_{2d}^2 (1/L)} = \frac{L}{g_{2d}^2 (1/L)}$$

(VI.1)

instead of Eq. (II.16). More precise treatment including the quantum effects requires one-loop functional determinant with the appropriate background such as bions at the level of the field theory. This is one of the themes of our future works.

(iii) We note that our real and complex bion solutions are the most general exact solutions of the equations of motion of the complexified $\mathbb{C}P^1$ quantum mechanics (III.20) (deformed by the fermion contributions) with the boundary condition (III.8) on $R^1 (-\infty < \tau < \infty)$. This fact implies that no other exact solutions exist, including multiple bions on $R^1 (-\infty < \tau < \infty)$. On the other hand, the resurgence theory requires that non-perturbative contributions with higher powers of non-perturbative exponential factors should exist, that are expected to come from multiple bion configurations. We anticipate that they should correspond to approximate solutions of the equations of motion which reduces to solutions asymptotically at large separations between bions in the complexified theory. This situation is quite similar to the single bion configuration in the $\mathbb{C}P^1$ quantum mechanics before the complexification. Hence our physical picture is the dilute gas of bions with short-range interactions. A similar picture has been advocated for the case of the sine-Gordon quantum mechanics[38]. Alternatively, we anticipate to obtain exact solutions corresponding to multiple bions if we keep the compactification period of the base space (inverse temperature) as $0 \leq \tau < \beta$. We expect, however, the result should be the same as the above dilute gas picture in the large $\beta$ limit.

(iv) As for discussion, there arises a question whether there exist complexified solutions playing similar roles in Yang-Mills or QCD. One possible connection is to consider $U(N)$ Yang-Mills theory coupled with Higgs fields in the fundamental representation. In the Higgs phase, it allows a non-Abelian vortex whose low-energy dynamics is effectively described by the $\mathbb{C}P^{N-1}$ model localized around the vortex[60,62]. When appropriate fermions are coupled in the original bulk theory, fermion quasi zero modes are also localized around the vortex and the $\mathbb{C}P^{N-1}$ model is coupled to the fermions. If the bulk theory is supersymmetric, the vortex can be BPS and the supersymmetric $\mathbb{C}P^{N-1}$ model is obtained as the vortex theory. Therefore, our complexified solution should be able to be embedded into it either in non-supersymmetric or supersymmetric cases. Complexified bions should be able to be interpreted as those in (complexified) Yang-Mills theory in the bulk as instantons in $\mathbb{C}P^{N-1}$ model correspond to Yang-Mills instantons in the bulk[50]. By taking a decoupling limit of the Higgs phase, we will be able to isolated complexified solutions in Yang-Mills
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Appendix A: Projection of fermionic degree of freedom

In this appendix, we derive the potential induced by projecting the fermionic degree of freedom. By redefining the fermionic degree of freedom as

\[ \chi \equiv \frac{1}{g} \frac{1}{1 + |\varphi|^2} \psi, \quad \bar{\chi} \equiv \frac{1}{g} \frac{1}{1 + |\varphi|^2} \bar{\psi}, \]  
(A.1)

the explicit form of the Lagrangian of the CP\(^1\) quantum mechanics (II.20) becomes

\[ L = \frac{1}{g^2} \frac{|\dot{\varphi}|^2 - m^2|\varphi|^2}{(1 + |\varphi|^2)^2} + i\bar{\chi} \left( \dot{\chi} - \frac{\bar{\varphi} \dot{\varphi} - \varphi \dot{\bar{\varphi}}}{1 + |\varphi|^2} \chi \right) + \epsilon m \frac{1 - |\varphi|^2}{1 + |\varphi|^2} \bar{\chi} \chi. \]  
(A.2)

The corresponding classical Hamiltonian takes the form

\[ H = g^2 (1 + |\varphi|^2)^2 \left[ |p_\varphi|^2 + i(\varphi p_\varphi - \bar{\varphi} \bar{p}_\varphi) \right] + \frac{m^2}{g^2} \frac{|\varphi|^2}{(1 + |\varphi|^2)^2} - \epsilon m \frac{1 - |\varphi|^2}{1 + |\varphi|^2} \bar{\chi} \chi, \]  
(A.3)

where the conjugate momenta are given by

\[ p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = \frac{\dot{\varphi}}{(1 + |\varphi|^2)^2}, \quad p_\bar{\varphi} = \frac{\partial L}{\partial \dot{\bar{\varphi}}} = \frac{\dot{\bar{\varphi}}}{(1 + |\varphi|^2)^2}, \quad -i\bar{\chi} = \frac{\partial L}{\partial \dot{\chi}}. \]  
(A.4)

The canonical commutation relations are

\[ [\varphi, p_\varphi] = [\varphi, \bar{p}_\varphi] = i, \quad \{\chi, \bar{\chi}\} = 1. \]  
(A.5)
Note that for \( \epsilon = 1 \), the Hamiltonian commutes with the supercharges

\[
Q \equiv g(1 + |\varphi|^2) \left[ p_\varphi - \frac{1}{g^2 (1 + |\varphi|^2)^2} \right] \chi, \\
\bar{Q} \equiv g(1 + |\varphi|^2) \left[ p_{\bar{\varphi}} + \frac{1}{g^2 (1 + |\varphi|^2)^2} - \frac{i}{2} \partial_{\bar{\varphi}} \log G \right] \bar{\chi},
\]

which satisfy the supersymmetry algebra

\[
\{Q, \bar{Q}\} = H + mq,
\]

where \( q \) is the conserved angular momentum

\[
q \equiv i(\varphi p_{\bar{\varphi}} - \bar{\varphi} p_\varphi) + \chi \bar{\chi}.
\]

Since the fermion number operator commutes with the Hamiltonian

\[
[H, \chi \bar{\chi}] = 0,
\]

we can consider the projection onto the subspace of the Hilbert space with a fixed fermion number. By projecting onto the subspace with \( \chi \bar{\chi} = 0 \), i.e. the sector annihilated by \( \bar{\chi} \)

\[
\bar{\chi} \langle \Psi \rangle = 0,
\]

the Hamiltonian becomes

\[
H = g^2(1 + |\varphi|^2)^2 p_\varphi p_{\bar{\varphi}} + \frac{m^2}{g^2 (1 + |\varphi|^2)^2} - \epsilon m \frac{1 - |\varphi|^2}{1 + |\varphi|^2}.
\]

Therefore, the potential of the projected model is given by

\[
V = \frac{m^2}{g^2 (1 + |\varphi|^2)^2} - \epsilon m \frac{1 - |\varphi|^2}{1 + |\varphi|^2}.
\]

Appendix B: One-loop determinant

In this appendix, we derive the formula for the functional determinant (III.40). Let \( \Delta \) be the following differential operator acting on \( n \)-component vectors

\[
\Delta = -\partial^2_\tau + V(\tau),
\]

where \( V(\tau) \) is an \( n \)-by-\( n \) matrix such that

\[
\tilde{V} = V, \quad \lim_{\tau \to \pm\infty} V(\tau) = M^2 \quad \text{(diagonal constant matrix)}.
\]
Here and in the following matrices with a bar (such as $\bar{V}$) denote their Hermitian conjugates. We first consider the ratio of the determinants

$$D(\lambda) \equiv \frac{\det(\lambda - \Delta)}{\det(\lambda - \Delta_0)} = \prod_n \frac{\lambda - \lambda_n}{\lambda - \lambda_0^n},$$

where $\Delta_0 = -\partial^2_\tau + M^2$ and $(\lambda_n, \lambda_0^n)$ are eigenvalues of $(\Delta, \Delta_0)$ respectively.

From the formula $\log \det = \text{Tr} \log$, it follows that

$$\frac{\partial}{\partial \lambda} \log D(\lambda) = \text{Tr} \left[ \frac{1}{\lambda - \Delta} - \frac{1}{\lambda - \Delta_0} \right] = - \int d\tau \text{Tr} \left[ G(\tau, \tau) - G_0(\tau, \tau) \right],$$

where $G$ and $G_0$ are the Green functions

$$(\Delta - \lambda)G(\tau, \tau') = 1_n \times \delta(\tau - \tau'),$$

$$(\Delta_0 - \lambda)G_0(\tau, \tau') = 1_n \times \delta(\tau - \tau').$$

The Green function can be constructed from exponentially decreasing $n$-by-$n$ matrices $\psi_\lambda^\pm$ satisfying

$$(\Delta - \lambda)\psi_\lambda^\pm = 0, \quad \psi_\lambda^\pm \overset{\tau \to \pm \infty}{\longrightarrow} \exp(\mp \kappa \tau),$$

where $\kappa$ is a diagonal matrix defined by

$$\kappa^2 = M^2 - \lambda,$$

which is positive definite for sufficiently small values of $\lambda$. We can show that the Green function $G(\tau, \tau')$ is given by (see below)

$$G(\tau, \tau') = \psi_\lambda^+ (\tau) \bar{W}_\lambda^{-1} \bar{\psi}_\lambda^- (\tau') \theta(\tau - \tau') + \psi_\lambda^- (\tau) W_\lambda^{-1} \bar{\psi}_\lambda^+ (\tau') \theta(\tau' - \tau),$$

where $W_\lambda$ is the $n$-by-$n$ matrix defined as the Wronskian $W[\psi_\lambda^+, \psi_\lambda^-]$ with $\lambda' = \lambda$:

$$W_\lambda = W[\psi_\lambda^+, \psi_\lambda^-], \quad W[\psi_\lambda^+, \psi_\lambda'^-] \equiv \bar{\psi}_\lambda^+ \frac{\partial}{\partial \tau} \psi_\lambda^- - \frac{\partial}{\partial \tau} \bar{\psi}_\lambda^+ \psi_\lambda'^-. $$

Note that $W_\lambda$ is independent of $\tau$ since $W[\psi_\lambda^+, \psi_\lambda^-]$ satisfies

$$\frac{\partial}{\partial \tau} W[\psi_\lambda^+, \psi_\lambda^-] = (\lambda - \lambda') \bar{\psi}_\lambda^+ \psi_\lambda'^-$$. 

This relation also implies that the trace of the Green function at $\tau' = \tau$ can be rewritten as

$$\text{Tr} G(\tau, \tau) = \lim_{\lambda' \to \lambda} \frac{\partial}{\partial \tau} \text{Tr} \left( W_\lambda^{-1} W[\psi_\lambda^+, \psi_\lambda^-] \right),$$

Similarly, the trace of the Green function of $\Delta_0$ is given by

$$\text{Tr} G_0(\tau, \tau) = \lim_{\lambda' \to \lambda} \frac{\partial}{\partial \tau} \text{Tr} \left( (2\kappa)^{-1} W[e^{-\kappa \tau}, e^{\kappa' \tau}] \right).$$
Using these expression, we can relate the determinant and the asymptotic forms of the Wronskians

\[
\frac{\partial}{\partial \lambda} \log D(\lambda) = - \int_{-\infty}^{\infty} d\tau \text{Tr} \left[ G(\tau, \tau) - G_0(\tau, \tau) \right]
\]

\[
= \lim_{T \to \infty} \lim_{\lambda' \to \lambda} \frac{1}{\lambda - \lambda'} \text{Tr} \left\{ (2\kappa)^{-1} W(e^{-\kappa\tau}, e^{\kappa'\tau}) - W_{\lambda}^{-1} W(\psi^+, \psi^-) \right\}_{\tau = -T}. 
\]

The asymptotic form of \( W[\psi^+, \psi^-] \) can be obtained from those of \( \psi^\pm \). In addition to Eq. (B.7), let us assume the following asymptotic form in the opposite infinity

\[
\psi^\pm_{\lambda} \to \exp(-\kappa\tau) F^\pm(\lambda) + \exp(+\kappa\tau) A^\pm(\lambda) \quad \text{for } x \to -\infty, 
\]

\[
\psi^\pm_{\lambda} \to \exp(+\kappa\tau) F^\pm(\lambda) + \exp(-\kappa\tau) A^\pm(\lambda) \quad \text{for } x \to +\infty, 
\]

where the coefficients \( F^\pm(E) \) and \( A^\pm(E) \) are \( n \)-by-\( n \) matrices. Then the asymptotic forms of the Wronskian can be written as

\[
W[\psi^+, \psi^-] = \begin{cases} 
(\kappa + \kappa') e^{-(\kappa-\kappa') \tau} F^-(\lambda') & \text{for } \tau \to +\infty \\
(\kappa + \kappa') e^{-(\kappa-\kappa') \tau} F^+(\lambda) & \text{for } \tau \to -\infty
\end{cases}. 
\]

In particular, for \( \lambda' = \lambda \)

\[
W_{\lambda} = 2\kappa F^-(\lambda) = 2\kappa F^+(\lambda). 
\]

By using these asymptotic forms and

\[
W[e^{-\kappa\tau}, e^{\kappa'\tau}] = (\kappa + \kappa') e^{-\kappa-\kappa'} \tau,
\]

we find that

\[
\frac{\partial}{\partial \lambda} \log D(\lambda) = \lim_{T \to \infty} \lim_{\lambda' \to \lambda} \text{Tr} \left[ (\kappa + \kappa') e^{-(\kappa-\kappa') \tau} F^-(\lambda) - F^-(\lambda') \right] \left( 2\kappa F^-(\lambda) \right)^{-1}
\]

\[
= \frac{\partial}{\partial \lambda} \log \det F^-(\lambda), 
\]

where we have assumed that \( \lambda < \lambda' \). This implies that \( D(\lambda) \propto \det F^-(\lambda) \). Since \( D(\infty) = 1 \) (by definition) and \( F^-(\infty) = 1_n \) (free wave limit), it follows that

\[
D(\lambda) = \det F^-(\lambda). 
\]

\[e. \, \text{Removing zero modes}\]

Suppose that the operator \( \Delta \) has \( n \) zero modes. Let \( \Xi_0 \) be the basis \( (n \text{-by-} n) \) matrix of the zero modes such that

\[
\Xi_0 \to \exp(\mp M\tau) K^\pm \quad \text{as } \tau \to \pm \infty, 
\]

\[
(\Delta - m^2) K^\pm = 0 
\]

\[
\text{Tr} (G_{\lambda} e^{-\kappa \tau}) = \text{Tr} (G_0 e^{-\kappa \tau}) 
\]

\[
\text{Tr} (G_{\lambda} e^{\kappa \tau}) = \text{Tr} (G_0 e^{\kappa \tau}) 
\]

\[
\text{Tr} (G_{\lambda} e^{\pm \kappa \tau}) = \text{Tr} (G_0 e^{\pm \kappa \tau}) 
\]
where $K^\pm$ are constant $n$-by-$n$ matrices. Note that the operator $\Delta$ can be rewritten as

$$\Delta \Xi_0 = ( - \partial_\tau^2 + V ) \Xi_0 = 0 \quad \iff \quad \Delta = ( - \partial_\tau + \partial_\tau \Xi_0 \Xi_0^{-1} ) ( \partial_\tau - \partial_\tau \Xi_0 \Xi_0^{-1} ). \quad (B.23)$$

The existence of $n$ normalizable zero modes $\Xi_0$ implies that $\psi^\pm_\lambda$ become normalizable at $\lambda = 0$ and hence

$$F^\pm(\lambda = 0) = 0. \quad (B.24)$$

From the definition of $D(\lambda)$ in Eq. (B.3), it follows that the zero mode can be removed by differentiating $D(\lambda)$:

$$\frac{\det' \Delta}{\det \Delta_0} = \frac{(-1)^n \partial^n D}{n! \partial \lambda^n} \bigg|_{\lambda = 0} = \det \left( - \frac{\partial F^-}{\partial \lambda} \right) \bigg|_{\lambda = 0}, \quad (B.25)$$

where $\det'$ stands for the determinant excluding the zero modes. By using the asymptotic behaviors of the Wronskian, the right hand side can be rewritten as

$$\frac{\partial F^-}{\partial \lambda} \bigg|_{\lambda = 0} = \frac{1}{2M} \int_{-\infty}^{\infty} d\tau \frac{\partial}{\partial \tau} \frac{\partial}{\partial \lambda'} W[\psi^+_0, \psi^-_{\lambda'}] \bigg|_{\lambda' = 0}. \quad (B.26)$$

On the other hand, the Wronskian satisfies (B.11) and hence

$$\frac{\partial}{\partial \tau} \frac{\partial}{\partial \lambda'} W[\psi^+_0, \psi^-_{\lambda'}] \bigg|_{\lambda' = 0} = - \bar{\psi}^+_0 \psi^-_{\lambda = 0} = - (\bar{K}^+)^{-1} \Xi_0 (K^-)^{-1}. \quad (B.27)$$

where we have used

$$\lim_{\lambda \to 0} \psi^\pm_\lambda = \Xi_0 (K^\pm)^{-1}. \quad (B.28)$$

From Eqs. (B.26) and (B.27), it follows that

$$\frac{\det' \Delta}{\det \Delta_0} = \det \left( - \frac{\partial F^-}{\partial \lambda} \right) \bigg|_{\lambda = 0} = \frac{\det G}{\det(2M) \det K^+ \det K^-}. \quad (B.29)$$

where $G$ is the matrix of the overlap integrals of zero modes

$$G \equiv \int_{-\infty}^{\infty} d\tau \Xi_0 \bar{\Xi}_0. \quad (B.30)$$

f. Green function

Here we prove the formula for the Green function (B.9). First note that $G(\tau, \tau')$ in Eq. (B.9) satisfies

$$(\Delta - \lambda) G(\tau, \tau') = 0, \quad \text{for } \tau \neq \tau'. \quad (B.31)$$
Therefore, the condition for the Green function (B.3) is satisfied if

\[ 0 = \lim_{\tau' \to \tau + 0} G(\tau, \tau') - \lim_{\tau' \to \tau - 0} G(\tau, \tau'), \tag{B.32} \]

\[ -1_n = \lim_{\tau' \to \tau + 0} \partial_\tau G(\tau, \tau') - \lim_{\tau' \to \tau - 0} \partial_\tau G(\tau, \tau'). \tag{B.33} \]

Define \( \Omega \) by

\[ \Omega = -\partial_\tau \psi_\lambda^+ (\psi_\lambda^+)^{-1} + \partial_\tau \psi_\lambda^- (\psi_\lambda^-)^{-1}. \tag{B.34} \]

Then \( W_\lambda \) can be rewritten as

\[ W_\lambda = \bar{\psi}_\lambda^+ \partial_\tau \psi_\lambda^- - \partial_\tau \bar{\psi}_\lambda^+ \psi_\lambda^- = \bar{\psi}_\lambda^+ [\Omega + \partial_\tau \psi_\lambda^+ (\psi_\lambda^+)^{-1}] \psi_\lambda^- - \partial_\tau \bar{\psi}_\lambda^+ \psi_\lambda^- \\
= \bar{\psi}_\lambda^+ \Omega \psi_\lambda^- + W[\psi_\lambda^+, \psi_\lambda^+] (\psi_\lambda^+)^{-1} \psi_\lambda^- \\
= \bar{\psi}_\lambda^+ \Omega \psi_\lambda^- . \tag{B.35} \]

In the last equality, we have used

\[ W[\psi_\lambda^+, \psi_\lambda^+] = 0 , \quad \therefore \partial_\tau W[\psi_\lambda^+, \psi_\lambda^+] = 0 , \quad W[\psi_\lambda^+, \psi_\lambda^+] \to 0 . \tag{B.36} \]

Then (B.9) can be rewritten as

\[ G(\tau, \tau') = \Omega^{-1} \left[ (\bar{\psi}_\lambda^- (\tau))^{-1} \bar{\psi}_\lambda^- (\tau') \theta (\tau - \tau') + (\bar{\psi}_\lambda^+ (\tau))^{-1} \bar{\psi}_\lambda^+ (\tau') \theta (\tau' - \tau) \right] . \tag{B.37} \]

By using this form of \( G(\tau, \tau') \), we can easily show that

\[ \lim_{\tau' \to \tau + 0} G(\tau, \tau') = \lim_{\tau' \to \tau - 0} G(\tau, \tau') = \Omega^{-1} , \tag{B.38} \]

and

\[ \lim_{\tau' \to \tau + 0} \partial_\tau G(\tau, \tau') - \lim_{\tau' \to \tau - 0} \partial_\tau G(\tau, \tau') = \Omega^{-1} \left[ (\bar{\psi}_\lambda^+ (\tau))^{-1} \partial_\tau \bar{\psi}_\lambda^+ - (\bar{\psi}_\lambda^- (\tau))^{-1} \partial_\tau \bar{\psi}_\lambda^- \right] = -1_n . \tag{B.39} \]

Therefore, \( G(\tau, \tau') \) in Eq. (B.9) satisfies the condition for the Green function Eq. (B.5).

**Appendix C: Measure for the moduli integral**

In this appendix, we derive the measure for the moduli integral. Let us consider fluctuations \( \xi^a \) around a classical background \( \varphi^i_{\text{sol}}(\eta^a) \) as a function of the moduli parameters \( \eta^a \)

\[ \varphi^i = \varphi^i_{\text{sol}}(\eta^a) + \epsilon^i_a \xi^a , \tag{C.1} \]
where $e^i_a$ are vielbein on the target space. The fluctuations are decomposed into zero modes $c^\alpha$ and massive modes $c^I$

$$\xi^\alpha = \sum_{\alpha \in \text{zero modes}} c^\alpha \xi^\alpha_\alpha + \sum_{I \in \text{massive}} c^I \xi^\alpha_I. \quad (C.2)$$

The zero modes are related to the derivative with respect to the moduli parameters $\eta^\alpha$ as

$$\xi^\alpha_\alpha = e^i_a \frac{\partial}{\partial \eta^\alpha} \varphi_i^{\text{sol}}. \quad (C.3)$$

The zero modes and massive modes are normalized as

$$\int d\tau \xi^\alpha_\alpha \xi^\alpha_\beta = G_{\alpha\beta}, \quad \int d\tau \xi^\alpha_I \xi^\alpha_\alpha = 0, \quad \int d\tau \xi^\alpha_I \xi^\alpha_J = \delta_{IJ}, \quad (C.4)$$

where $G$ is the moduli space metric. The naive path integral measure is

$$D\xi = \sqrt{\det \left( \frac{G}{2\pi} \right)} \prod_{\alpha \in \text{zero modes}} dc^\alpha \times \prod_{I \in \text{massive}} dc^I \sqrt{2\pi}. \quad (C.5)$$

To remove the zero mode integral, let us insert the following identity into the path integral:

$$1 = \int d^d\eta^\alpha \det \left( \int dx \xi^a_\alpha e^a_i \partial_\beta \varphi^i \right) \delta^d \left( \int dx \xi^a_\alpha e^a_i \varphi^i \right) \quad (C.6)$$

$$= \int d^d\eta^\alpha \det G \delta^d \left( G_{\alpha\beta} c^\beta \right). \quad (C.7)$$

Integrating over $c^\alpha$, we obtain the measure for the moduli integral

$$\int D\xi (\cdots) = \int d^d\eta^\alpha \sqrt{\det \left( \frac{G}{2\pi} \right)} \prod_{I \in \text{massive}} dc^I \sqrt{2\pi} (\cdots). \quad (C.8)$$

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