Can One See the Number of Colors? * 

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Abstract

We formulate the standard model with an arbitrary number of colors $N_c$. The cancellation of Witten’s global $SU(2)_L$ anomaly requires $N_c$ to be odd, while the cancellation of triangle anomalies determines the consistent $N_c$-dependent values of the quark charges. In this theory, the width of the decay $\pi^0 \to \gamma\gamma$ is not proportional to $N_c^2$. In fact, in the case of a single generation and hence for two quark flavors ($N_f = 2$), $N_c$ does not appear explicitly in the low-energy effective theory of the standard model. Hence, contrary to common lore, it is impossible to see the number of colors in low-energy experiments with just pions and photons. For $N_f \geq 3$, on the other hand, $N_c$ explicitly enters the chiral Lagrangian as the quantized prefactor of the Wess-Zumino-Witten term, but the contribution of this term to photon-pion vertices is completely canceled by the $N_c$-dependent part of a Goldstone-Wilczek term. However, the width of the decay $\eta \to \pi^+\pi^-\gamma$ survives the cancellation and is indeed proportional to $N_c^2$. By detecting the emerging photon, this process thus allows one to literally see $N_c$ for $N_f \geq 3$.

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1 Introduction

In this article we ask if it is possible to deduce the number of colors directly from low-energy experiments with photons, pions, $\eta$-mesons and kaons. For example, can one literally see $N_c$ by detecting the photons emerging from the decay of a neutral pion? Unfortunately, there are several misleading statements in the literature concerning this question. First, one often reads that the standard model is anomalous — and hence inconsistent — for $N_c \neq 3$. Then one reads in almost any textbook on the subject that the width of the decay $\pi^0 \rightarrow \gamma\gamma$ is proportional to $N_c^2$, and that the observed width is consistent only with $N_c = 3$. It was pointed out by Abbas that both of these statements are wrong [1, 2], simply because varying $N_c$ without adjusting the quark charges accordingly is inconsistent. Actually, these first two standard pieces of evidence for $N_c = 3$ do not at all imply that there are three colors. The third standard textbook evidence for three colors is provided by the Drell ratio $R$. When evaluated at high energies, this ratio is sensitive to $N_c$ and its measured value indeed implies $N_c = 3$. Here we ask if there are also low-energy processes which allow one to directly see $N_c$, and which could hence replace the misleading textbook example $\pi^0 \rightarrow \gamma\gamma$.

In order to cancel Witten’s global anomaly [3], the number of colors must be odd in the standard model. The cancellation of triangle anomalies requires the electric charges of the up and down quarks to be

$$Q_u = \frac{1}{2} \left( \frac{1}{N_c} + 1 \right), \quad Q_d = \frac{1}{2} \left( \frac{1}{N_c} - 1 \right). \quad (1.1)$$

For $N_c = 3$ these are the familiar values $Q_u = \frac{2}{3}$ and $Q_d = -\frac{1}{3}$. Keeping these charges fixed while varying $N_c$ is inconsistent, because the anomalies no longer cancel. Of course, in a vector-like theory with only electromagnetic and strong gauge interactions, any electric charge assignment is consistent with any choice of $N_c$. In a chiral gauge theory with electroweak gauge interactions, on the other hand, anomaly cancellation leads to eq.(1.1). If one already knows the quark charges to be $Q_u = \frac{2}{3}$ and $Q_d = -\frac{1}{3}$ (which is what the textbooks implicitly do), both anomaly cancellation and the $\pi^0$ decay indeed imply $N_c = 3$. However, in that case, one could simply say that the observed charges of the proton and the neutron already imply three colors.

In a world with $N_c = 5$ colors, the quark charges are $Q_u = \frac{3}{5}$ and $Q_d = -\frac{2}{5}$. In such a world, baryons consist of five quarks. For example, the proton now contains three up quarks and two down quarks, but still has electric charge $Q_p = 3Q_u + 2Q_d = 1$. The neutron consists of two up quarks and three down quarks and still is electrically neutral. For arbitrary odd $N_c$, the proton is made of $(N_c + 1)/2$ up quarks and $(N_c - 1)/2$ down quarks, while the neutron contains $(N_c - 1)/2$ up quarks
and $(N_c + 1)/2$ down quarks. Hence, as in our world,
\[
Q_p = \frac{N_c + 1}{2}Q_u + \frac{N_c - 1}{2}Q_d = 1,
\]
\[
Q_n = \frac{N_c - 1}{2}Q_u + \frac{N_c + 1}{2}Q_d = 0.
\] (1.2)

There is also still a $\Delta$-isobar with the usual electric charges
\[
Q_{\Delta^{++}} = \frac{N_c + 3}{2}Q_u + \frac{N_c - 3}{2}Q_d = 2,
\]
\[
Q_{\Delta^+} = \frac{N_c + 1}{2}Q_u + \frac{N_c - 1}{2}Q_d = 1,
\]
\[
Q_{\Delta^0} = \frac{N_c - 1}{2}Q_u + \frac{N_c + 1}{2}Q_d = 0,
\]
\[
Q_{\Delta^-} = \frac{N_c - 3}{2}Q_u + \frac{N_c + 3}{2}Q_d = -1.
\] (1.3)

In general, in a world with two flavors and an arbitrary odd number of colors, baryons have equal half-integer valued isospin and spin $1/2 \leq I = S \leq N_c/2$. For example, for $N_c = 5$ there is an additional baryon resonance beyond the $\Delta$-isobar with $I = S = 5/2$. For arbitrary odd $N_c$ the highest of these additional resonances has $I = S = N_c/2$. The member of this multiplet with the largest electric charge consists of $N_c$ up quarks and has $Q = (N_c + 1)/2$ while the member with the most negative charge contains $N_c$ down quarks and has $Q = -(N_c - 1)/2$. Obviously, the high end of the baryon spectrum is sensitive to the number of colors. On the other hand, the familiar baryons $p$, $n$ and $\Delta^{++}$, $\Delta^+$, $\Delta^0$, $\Delta^-$ with their usual electric charges exist for any odd $N_c \geq 3$.

Of course, the interior of a baryon with $N_c \neq 3$ is different from that of a baryon in our world. This definitely has observable consequences at short distances and thus at high energies. For example, the up, down and strange quark contribution to the Drell ratio (with $Q_s = Q_d$) is proportional to
\[
R = \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} \propto N_c(Q_u^2 + 2Q_d^2) = \frac{3}{4}(N_c + \frac{1}{N_c} - \frac{2}{3}N_c^2).
\] (1.4)

Instead, if one uses $Q_u = \frac{2}{3}$ and $Q_d = -\frac{1}{3}$ independent of $N_c$, one obtains the inconsistent textbook result $R \propto \frac{2}{3}N_c$. Still, in either case, $R$ is quite sensitive to the number of colors. However, the above calculation of the Drell ratio is valid only at high energies, above about 1 GeV. Here we ask if and how $N_c$ enters the low-energy electromagnetic physics of pions, $\eta$-mesons and kaons. We will find that, using the consistent charge assignment for the quarks of eq.(1.1), several anomalous processes are $N_c$-independent. This contradicts common lore, which says that, for example, the decay width of $\pi^0 \rightarrow \gamma\gamma$ is proportional to $N_c^2$, thus changing drastically with the number of colors.
The decay $\pi^0 \rightarrow \gamma\gamma$ results from a triangle diagram with an internal quark loop attached to two external electromagnetic $U(1)_{em}$ currents and one external isovector axial current. This diagram is proportional to

$$\text{Tr}(T^3 Q^2) = \frac{N_c}{2} (Q_u^2 - Q_d^2), \quad (1.5)$$

where $T^3 = \frac{1}{2} \sigma^3$ is the diagonal generator of isospin, and $Q = \text{diag}(Q_u, Q_d)$ is the quark charge matrix. Note that the trace in eq.(1.5) implies a sum over color indices.

As we will see later, the triangle diagram gives rise to the effective vertex

$$L_{\pi^0\gamma\gamma} = -i N_c (Q_u^2 - Q_d^2) \frac{e^2}{32\pi^2 F^*_\pi \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}}, \quad (1.6)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (1.7)$$

is the field strength of the electromagnetic vector potential $A_\mu$. Using $Q_u = \frac{2}{3}$ and $Q_d = -\frac{1}{3}$ this leads to the textbook result for the decay width

$$\Gamma(\pi^0 \rightarrow \gamma\gamma) = \left( \frac{N_c}{3} \right)^2 \frac{e^4 M^3_{\pi}}{1024\pi^5 F^2_{\pi}}, \quad (1.8)$$

which is proportional to $N^2_c$. This result compared to the observed width seems to imply $N_c = 3$. However, by fixing the quark charges to $Q_u = \frac{2}{3}$ and $Q_d = -\frac{1}{3}$, one has already implicitly assumed that $N_c = 3$. On the other hand, if one uses the consistent quark charges given in eq.(1.1), one obtains

$$N_c(Q_u^2 - Q_d^2) = \frac{N_c}{4} \left[ (\frac{1}{N_c} + 1)^2 - (\frac{1}{N_c} - 1)^2 \right] = 1, \quad (1.9)$$

and the width turns into

$$\Gamma(\pi^0 \rightarrow \gamma\gamma) = \frac{e^4 M^3_{\pi}}{1024\pi^5 F^2_{\pi}}, \quad (1.10)$$

independent of $N_c$. Of course, the pion mass $M_{\pi}$ and the pion decay constant $F_{\pi}$ also depend on $N_c$ implicitly. However, if one takes $M_{\pi}$ and $F_{\pi}$ from experiment, the implicit dependence is irrelevant and the observed width does not imply $N_c = 3$. On the other hand, if one computes $M_{\pi}$ and $F_{\pi}$ using lattice QCD for different values of $N_c$ — which is certainly a nontrivial task — one would still infer $N_c = 3$. However, using that method, one can deduce the number of colors from any QCD observable, not only from processes like the $\pi^0$ decay.

Since the 1970s, the misleading statement that the $\pi^0 \rightarrow \gamma\gamma$ decay width is proportional to $N^2_c$ has been used to lend support to the correct conclusion that in our world $N_c = 3$. In this context it is sometimes stated that Steinberger, who obtained almost the correct width from a nucleon triangle diagram as early as 1949
from any low-energy effective theory of the standard model even if $N_c \neq 3$. His result is what one obtains for $N_c = 1$, i.e. in a colorless world without quarks. Of course, in 1949 Steinberger did not know about quarks or color, but he was still using a consistent low-energy effective description of our world. Indeed, the standard model with $N_c = 1$, and hence without strong interactions, is also anomaly free. According to eq.(1.1) the “quark” charges are then equal to $Q_u = 1$ and $Q_d = 0$, and the up and down “quarks” are, in fact, just the proton and the neutron. A standard model with the usual Higgs and electroweak sector, but with $N_c = 1$ has a nucleon, but no strong interactions, no spontaneous chiral symmetry breaking, and thus no pions. Still, one can add the pions in the form of a Gell-Mann Levy linear $\sigma$-model \[5\] without spoiling renormalizability or anomaly cancellation. In such a model, which is close to what Steinberger used, one again obtains eq.(1.10) despite the fact that there are no colors at all.

The low-energy physics of the strong interactions is governed by the pseudo-Goldstone bosons of spontaneous chiral symmetry breaking — the pions, $\eta$-mesons and kaons. Chiral perturbation theory offers a systematic approach to describe the Goldstone boson dynamics at low energies \[3\]. In QCD with $N_f$ massless quark flavors, the chiral symmetry is $SU(N_f)_L \otimes SU(N_f)_R$ which breaks spontaneously to $SU(N_f)_L=SU(N_f)_R$. This should be the case for any number of colors, except for $N_c = 2$. The $N_c = 2$ case is special because then quarks and anti-quarks are indistinguishable and the chiral symmetry is $SU(2N_f)$, which is expected to break spontaneously to $Sp(2N_f)$ \[7\]. For any even $N_c$ the baryons are bosons and hence their physics is qualitatively different from that of the real world. Here we are interested in generalisations of the standard model to $N_c \neq 3$ that are at least qualitatively similar to our world. Interestingly, as mentioned before, the cancellation of Witten’s global anomaly, separately for each generation, already limits us to odd $N_c$. In that case, the Goldstone bosons are described by fields in the coset space $SU(N_f)_L \otimes SU(N_f)_R/SU(N_f)_L=SU(N_f)$. In the standard model with $N_f = 2$ and arbitrary odd $N_c$ the electric charge of the up and down quarks is given by $Q = T^3_L + Y = T^3_L + T^3_R + \frac{1}{2}B$. Here $T^3_L$ and $T^3_R$ are the diagonal generators of $SU(2)_L \otimes SU(2)_R$ and $B$ is the baryon number. Since $U(1)_B$ and hence $U(1)_Y$ and $U(1)_{em}$, are not subgroups of $SU(2)_L \otimes SU(2)_R$, it is not straightforward to gauge the electroweak symmetry or even just electromagnetism in the low-energy pion effective theory. It was first realized by Skyrme \[8\] that a baryon current can be constructed from pion fields, although the pions themselves do not carry any baryon number. In particular, there are solitons — the so-called Skyrmions — which indeed represent baryons. While the detailed structure of the Skyrmion (and even the question of its stability against shrinking) are beyond reach of chiral perturbation theory, the fact that pion field configurations with non-zero baryon number exist has profound consequences. In particular, if one wants to
gauge the weak interactions at the level of the effective theory, one must quantize the Skyrmion as a fermion \[9, 10\] in order to cancel Witten’s global anomaly. In order to account for the baryon number contribution \( B \) to the hypercharge \( Y \) or the electric charge \( Q \), one must also include a gauge invariant modification of Skyrme’s baryon number current — the so-called Goldstone-Wilczek current \[11\] — in the chiral Lagrangian. The Goldstone-Wilczek term cancels the triangle anomalies of the lepton sector and is thus necessary to correctly describe the electroweak interactions of pions. In particular, the Goldstone-Wilczek current is responsible for the decay \( \pi^0 \to \gamma \gamma \), which occurs because \( Q_u + Q_d \neq 0 \), i.e. because the quark charge matrix \( Q \) is not a traceless generator of \( SU(2)_{L=R} \). As we will see, for \( N_f = 2 \), the number of colors \( N_c \) does not appear explicitly in the low-energy effective theory of the standard model. Hence, it is then impossible to directly see the number of colors in low-energy experiments with just pions and photons.

In the \( N_f \geq 3 \) case the Wess-Zumino-Witten term \[12, 9\] arises with a quantized prefactor. Hence, besides the more familiar low-energy parameters like \( F_\pi \), the chiral Lagrangian also contains an integer-valued low-energy parameter. As first shown by Witten \[9\], in QCD this parameter is the number of colors \( N_c \). Since all low-energy parameters (like \( F_\pi \)) have some implicit \( N_c \)-dependence, it is perhaps not too surprising that the integer-valued parameter also depends on \( N_c \). However, in contrast to the \( N_f = 2 \) case, this means that \( N_c \) explicitly enters the low-energy Goldstone boson theory for \( N_f \geq 3 \). For example, there is a strong interaction vertex that turns two kaons into three pions. This vertex is directly proportional to \( N_c \). Hence, by scattering pions and kaons at low energies, one can indeed figure out the number of colors. Once one has appreciated the presence of the integer-valued low-energy parameter (with value \( N_c \)), this should not be too surprising. In particular, one is not surprised that other vertices depend, for example, on the low-energy parameter \( F_\pi \), which implicitly depends on \( N_c \). After all, it is natural that strong interaction processes depend implicitly or explicitly on the number of colors. It is perhaps more surprising that electromagnetic probes can see \( N_c \) in low-energy experiments.

When one gauges electromagnetism in a \( N_f = 3 \) theory with quark charges \( Q_u = \frac{2}{3}, Q_d = Q_s = -\frac{1}{3} \) (and hence with \( Q_u + Q_d + Q_s = 0 \)) the Wess-Zumino-Witten term gives a decay width of \( \pi^0 \to \gamma \gamma \) proportional to \( N_c^2 \). However, according to eq.(1.1) this charge assignment implicitly assumes \( N_c = 3 \). For arbitrary odd \( N_c \) one has \( Q_u = (1/N_c + 1)/2, Q_d = Q_s = (1/N_c - 1)/2 \), and hence \( Q_u + Q_d + Q_s = (3/N_c - 1)/2 \neq 0 \). This means that for \( N_c \neq 3 \) the charge matrix is not a traceless generator of \( SU(3)_{L=R} \). Then, as for \( N_f = 2 \), one has to include a Goldstone-Wilczek term even in the \( N_f = 3 \) case. As we will see, this term cancels the contribution of the Wess-Zumino-Witten term to the decay \( \pi^0 \to \gamma \gamma \), and leads to a width that is independent of the number of colors. This is indeed the correct result for the standard model with arbitrary odd \( N_c \). Still, for \( N_f \geq 3 \) there are processes involving photons, pions, \( \eta \)-mesons and kaons that allow one to see the number of
colors. In particular, the width of the decay $\eta \to \pi^+\pi^-\gamma$ is proportional to $N_c^2$. This decay deserves to become the future textbook process that implies that there are indeed three colors in our world.

This paper is organized as follows. In section 2 we derive eq.(1.1) by canceling the anomalies in the standard model with $N_c$ colors. In section 3 we discuss the low-energy pion effective theory for $N_f = 2$ and arbitrary odd $N_c$. Section 4 deals with the $N_f \geq 3$ case. In section 5 we show that in all photon-pion vertices the factor $N_c$ drops out, while it survives in some processes involving photons, pions, $\eta$-mesons and kaons. Finally, section 6 contains our conclusions.

2 A Consistent Standard Model with Arbitrary Odd $N_c$

In this section, following ref.[1], we use gauge anomaly cancellation conditions to determine the electroweak charges of quarks in a standard model with an arbitrary number of colors $N_c$. We leave the Higgs, gauge, and lepton sectors unchanged, and only adjust the quark sector in order to achieve anomaly cancellation. We must pay attention to Witten’s nonperturbative global anomaly, to the perturbative gauge triangle anomalies, as well as to the gravitational anomaly. As in the standard model at $N_c = 3$, we demand anomaly cancellation separately for each generation of fermions, and discuss this in the context of the first generation.

In the lepton sector of the first generation we have an $SU(2)_L$ doublet containing the left-handed neutrino and the left-handed electron, as well as two $SU(2)_L$ singlets: the right-handed neutrino and the right-handed electron. In the quark sector we have $N_c$ $SU(2)_L$ doublets containing the left-handed up and down quarks of different colors, as well as $2N_c$ $SU(2)_L$ singlets containing the right-handed up and down quarks. Hence, the fermion fields of one generation are

$$\begin{pmatrix} \nu_L(x) \\ e_L(x) \end{pmatrix}, \nu_R(x), e_R(x), \begin{pmatrix} u^i_L(x) \\ d^i_L(x) \end{pmatrix}, u^i_R(x), d^i_R(x), i \in \{1, 2, \ldots, N_c\}. \quad (2.1)$$

The leptons are color singlets while the quarks are in the $N_c$-dimensional fundamental representation of $SU(N_c)$. We leave the $U(1)_Y$ hypercharge assignments of the leptons unchanged, i.e.

$$Y_{\nu_L} = Y_{e_L} = -\frac{1}{2}, \quad Y_{\nu_R} = 0, \quad Y_{e_R} = -1. \quad (2.2)$$

Note that in the lepton sector

$$Y = T^3_R - \frac{1}{2} L, \quad (2.3)$$
where $T^3_R$ is the diagonal generator of $SU(2)_R$ and $L$ is lepton number. The corresponding electric charges result from

$$Q_l = T^3_L + Y = T^3_L + T^3_R - \frac{1}{2}L,$$

and are given by $Q_\nu = 0$ and $Q_e = -1$. On the other hand, we leave the quark hypercharges $Y_{u_L} = Y_{d_L} = Y_L$, $Y_{u_R}$, and $Y_{d_R}$ as free parameters to be determined by the anomaly cancellation conditions.

In four space-time dimensions (compactified to the sphere $S^4$) the gauge transformations $L(x) \in SU(2)_L$ fall into two topologically distinct classes characterized by a “winding” number $\text{Sign}[L] = \pm 1 \in \Pi_4[SU(2)_L] = \mathbb{Z}(2)$. Gauge transformations $L(x)$ that can be deformed continuously into the trivial gauge transformation have $\text{Sign}[L] = 1$, while all others have $\text{Sign}[L] = -1$. The fermion determinant of a single $SU(2)_L$ doublet changes sign under a nontrivial $SU(2)_L$ gauge transformation with $\text{Sign}[L] = -1$ and is thus not gauge invariant. In order to obtain a gauge invariant theory one hence needs an even number of $SU(2)_L$ doublets. Since there are one lepton doublet and $N_c$ quark doublets, $N_c$ must be odd in order to cancel Witten’s global anomaly. This implies that the standard model is consistent only if the baryons are fermions.

In the next step we cancel the triangle anomalies which are proportional to

$$A^{abc} = \text{Tr}_L[(T^aT^b + T^bT^a)T^c] - \text{Tr}_R[(T^aT^b + T^bT^a)T^c].$$

Here, the $T^a$ with $a \in \{1, 2, 3\}$ refer to the generators of $SU(2)_L$, $T^4 = Y$, and the $N_c^2 - 1$ remaining $T^a$, with $a - 4 \in \{1, 2, ..., N_c^2 - 1\}$, generate the color gauge group $SU(N_c)$. The traces are over the left- and right-handed fields, respectively. If two indices are color indices and the third index belongs to $U(1)_Y$ (i.e. $c = 4$), the anomaly cancels only if the quark hypercharges satisfy

$$2Y_L - Y_{u_R} - Y_{d_R} = 0. \quad (2.6)$$

When two indices belong to $SU(2)_L$ and the third index belongs to $U(1)_Y$ the anomaly cancellation condition involves both quarks and leptons and takes the form

$$2N_cY_L + Y_{\nu_L} + Y_{e_L} = 0 \Rightarrow Y_L = \frac{1}{2N_c}. \quad (2.7)$$

Finally, when all three indices belong to $U(1)_Y$, the anomaly $A^{444}$ vanishes if

$$N_c(2Y^3_L - Y^3_{u_R} - Y^3_{d_R}) + Y^3_{\nu_L} + Y^3_{e_L} - Y^3_{\nu_R} - Y^3_{e_R} = 0 \Rightarrow 2Y^3_L - Y^3_{u_R} - Y^3_{d_R} = -\frac{3}{4N_c}. \quad (2.8)$$

The cancellation of the gravitational anomaly also yields eq.(2.6) and hence does not imply additional constraints. Combining the anomaly cancellation conditions eqs.(2.4, 2.6, 2.7, 2.8) one finally obtains

$$Y_L = \frac{1}{2N_c}, \quad Y_{u_R} = \frac{1}{2}(\frac{1}{N_c} + 1), \quad Y_{d_R} = \frac{1}{2}(\frac{1}{N_c} - 1). \quad (2.9)$$
In the quark sector we can hence write
\[ Y = T_R^3 + \frac{1}{2} B, \] (2.10)
where \( B = 1/N_c \) is the baryon number of a quark. The quark electric charges are given by
\[ Q_q = T_L^3 + T_R^3 + \frac{1}{2} B, \] (2.11)
which results in the up and down quark charges of eq.(1.1). The general expression for the electric charge, valid for both quarks and leptons, is
\[ Q = T_L^3 + T_R^3 + \frac{1}{2}(B - L). \] (2.12)
Since \( Q \) as well as \( T_L^3 + T_R^3 \) generate symmetries of the standard model, \( B - L \) is a good symmetry as well.

At this point we have constructed an anomaly free generalization of the standard model with an arbitrary odd number of colors. This shows explicitly that the consistency requirement of anomaly cancellation does not imply \( N_c = 3 \). Even the constraint that \( N_c \) must be odd resulted only because we insisted that the anomalies cancel within a single generation. When there is an even number of generations, the global anomaly cancels automatically, and \( N_c \) could then as well be even. In that case, the baryons are bosons which, as a consequence of eq.(1.1), have half-integer charges. For odd \( N_c \), on the other hand, the baryons are fermions with integer electric charges.

It should be noted that after canceling the gauge anomalies, there are still anomalies in some global symmetries. For example, the baryon number current
\[ j_\mu = \frac{1}{N_c} \sum_{i=1}^{N_c} (u_i^L \gamma_\mu u_i^L + \bar{u}_i^R \gamma_\mu \bar{u}_i^R + d_i^L \gamma_\mu d_i^L + \bar{d}_i^R \gamma_\mu \bar{d}_i^R) \] (2.13)
is not conserved due to the 't Hooft anomaly [13]. Its divergence is given by
\[ \partial_\mu j_\mu = -\frac{1}{32\pi^2} \varepsilon_{\mu\nu\rho\sigma} \text{Tr}[W_{\mu\nu}W_{\rho\sigma}] + \frac{1}{32\pi^2} \varepsilon_{\mu\nu\rho\sigma} \text{Tr}[B_{\mu\nu}B_{\rho\sigma}]. \] (2.14)
Here \( W_\mu = igW_\mu^a T^a \) is the \( SU(2)_L \) gauge field with gauge coupling \( g \) and field strength
\[ W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + [W_\mu, W_\nu], \] (2.15)
and \( B_\mu = ig'B_\mu^3 T^3 \) is the \( U(1)_Y \) gauge field with gauge coupling \( g' \) and field strength
\[ B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu. \] (2.16)
The first term in eq.(2.14) results from a triangle diagram with an internal quark loop attached to two external $SU(2)_L$ currents and one external baryon current. This diagram is proportional to

$$\text{Tr}_L[(T^3)^2 B] = N_c \frac{1}{2} \frac{1}{N_c} = \frac{1}{2}. \quad (2.17)$$

The second term in eq.(2.14) comes from a triangle diagram with an internal quark loop attached to two external $U(1)_Y$ currents and one external baryon current. That diagram is proportional to

$$\text{Tr}_L[Y^2 B] - \text{Tr}_R[Y^2 B] = N_c [2Y^2_L - Y^2_{uR} - Y^2_{dR}] \frac{1}{N_c} =$$

$$\frac{1}{4} [2 \frac{1}{N_c} - (\frac{1}{N_c} + 1)^2 - (\frac{1}{N_c} - 1)^2] = -\frac{1}{2}. \quad (2.18)$$

It is interesting that both in eq.(2.17) and in eq.(2.18) the $N_c$-dependence cancels completely. Also, note that an electroweak instanton with topological charge

$$-\frac{1}{32\pi^2} \int d^4x \; \varepsilon_{\mu\nu\rho\sigma} \text{Tr}[W_{\mu\nu}W_{\rho\sigma}] = 1 \quad (2.19)$$

causes violation of baryon number conservation by one unit.

### 3 Low-Energy Description of a Single Generation

Let us first restrict ourselves to one generation of fermions. From the point of view of the strong interactions, this is the $N_f = 2$ case of just up and down quark flavors. Due to spontaneous chiral symmetry breaking from $SU(2)_L \otimes SU(2)_R$ to $SU(2)_{L=R}$, the low-energy degrees of freedom are the pseudo-Goldstone pion fields

$$U(x) = \exp(2i\pi^a(x)T_a/F_\pi), \quad (3.1)$$

that live in the coset space $SU(2)_L \otimes SU(2)_R/SU(2)_{L=R} = SU(2)$. Note that we have introduced the generators of $SU(2)$ such that $\text{Tr}(T^aT^b) = \frac{1}{2} \delta_{ab}$. At low energies the pion dynamics is described by chiral perturbation theory. To lowest order, the Euclidean chiral perturbation theory action is given by

$$S[U] = \int d^4x \left\{ \frac{F_\pi^2}{4} \text{Tr}[\partial_\mu U^\dagger \partial_\mu U] - \frac{1}{4} \langle \bar{\Psi}\Psi \rangle \text{Tr}[\mathcal{M}(U + U^\dagger)] \right\}. \quad (3.2)$$

Here $\langle \bar{\Psi}\Psi \rangle$ is the chiral condensate and $\mathcal{M} = \text{diag}(m_u, m_d)$ is the quark mass matrix. For massless quarks the action is invariant under global $SU(2)_L \otimes SU(2)_R$ transformations

$$U'(x) = L^\dagger U(x)R. \quad (3.3)$$
The nontrivial homotopy group $\Pi_3[SU(2)] = \mathbb{Z}$ implies that, at every instant in time, the pion field is characterized by an integer winding number

$$B = \frac{1}{24\pi^2} \int d^3 x \varepsilon_{ijk} \text{Tr}[(U^\dagger \partial_i U)(U^\dagger \partial_j U)(U^\dagger \partial_k U)].$$

(3.4)

Skyrme was first to suggest that $B$ should be identified with baryon number [8]. The baryon current

$$j_\mu = \frac{1}{24\pi^2} \varepsilon_{\mu\nu\rho\sigma} \text{Tr}[(U^\dagger \partial_\nu U)(U^\dagger \partial_\rho U)(U^\dagger \partial_\sigma U)]$$

(3.5)

is topologically conserved, i.e. $\partial_\mu j_\mu = 0$.

The partition function of the pion field theory takes the form

$$Z = \int \mathcal{D}U \exp(-S[U]) \text{Sign}[U].$$

(3.6)

The “winding” number $\text{Sign}[U]$ is an element of the nontrivial homotopy group $\Pi_4[SU(2)] = \Pi_4[S^3] = \mathbb{Z}(2)$. It can be identified as the fermion permutation sign of the Skyrmion soliton. For example, a pion field configuration $U$ in which two Skyrmions interchange their positions as they evolve in time has $\text{Sign}[U] = -1$ [9]. When the Skyrmion is quantized as a fermion, $\text{Sign}[U]$ must be included in the pion path integral in order to correctly implement the Pauli principle for Skyrmions. Similarly, a configuration $U$ in which a single Skyrmion rotates by $2\pi$ during its time evolution also has $\text{Sign}[U] = -1$. The inclusion of $\text{Sign}[U]$ in the path integral therefore automatically ensures that the Skyrmion has half-integer spin. As we have seen in the previous section, the standard model is free of Witten’s global anomaly only for odd $N_c$. Hence, the Skyrmions of the low-energy pion effective theory should be quantized as fermions. As we will see below, one indeed must include $\text{Sign}[U]$ in the path integral in order to cancel the global anomaly also at the level of the effective theory [9, 10].

In the next step we want to gauge both $SU(2)_L$ and $U(1)_Y$ in order to obtain a low-energy effective theory of the full standard model. In particular, we are interested if $N_c$ enters the effective theory explicitly. Of course, as discussed earlier, $N_c$ enters implicitly through $F_\pi$ and $\langle \bar{\Psi}\Psi \rangle$.

Gauging $SU(2)_L$ is straightforward. First, one just replaces ordinary derivatives by covariant derivatives

$$D_\mu U = (\partial_\mu + W_\mu)U.$$  

(3.7)

Second, one replaces the quark mass matrix $\mathcal{M}$ by a coupling to the standard model Higgs field which can be expressed as a matrix

$$\Phi(x) = \begin{pmatrix} \Phi_0^*(x) & \Phi_+(x) \\ -\Phi_+(x) & \Phi_0(x) \end{pmatrix}.$$  

(3.8)
The action then takes the form
\[ S[U, \Phi, W_\mu] = \int d^4x \left\{ \frac{F_{\mu\nu}^2}{4} \text{Tr}[D_\mu U^\dagger D_\mu U] - \frac{1}{4} \langle \bar{\Psi} \Psi \rangle \text{Tr}[\mathcal{F}^\dagger \Phi^\dagger U + U^\dagger \Phi \mathcal{F}^\dagger] \right\}, \tag{3.9} \]
which is invariant under local transformations
\[ U'(x) = L^\dagger(x)U(x), \quad \Phi'(x) = L^\dagger(x)\Phi(x), \quad W'_\mu(x) = L^\dagger(x)(W_\mu(x) + \partial_\mu)L(x). \tag{3.10} \]
In the vacuum the Higgs field has the expectation value \( \Phi(x) = v \mathbb{1} \), and the up and down quark masses are obtained from the matrix of Yukawa couplings \( \mathcal{F} = \text{diag}(f_u, f_d) \) as
\[ M = \mathcal{F} v \Rightarrow m_u = f_u v, \quad m_d = f_d v. \tag{3.11} \]

When one performs an \( SU(2)_L \) gauge transformation \( L(x) \) with \( \text{Sign}[L] = -1 \) in the standard model, the fermion determinant of the leptons changes sign and is thus not gauge invariant. The global anomaly is canceled by an odd number of minus signs due to the \( N_c \) quark \( SU(2)_L \) doublets. At the level of the low-energy effective theory, the leptons are still present but the quarks have been replaced by pion fields. Hence, the question arises how the cancellation of the global anomaly is achieved at the level of the low-energy effective theory. While the pion action \( S[U, \Phi, W_\mu] \) of eq.(3.9) is gauge invariant, the path integral as a whole is not. This is because
\[ \text{Sign}[U'] = \text{Sign}[LU] = \text{Sign}[L] \text{ Sign}[U]. \tag{3.12} \]
Hence, as pointed out by Witten [9] and by D’Hoker and Farhi [10], the \( SU(2)_L \) gauge variation of the fermion permutation sign of the Skyrmions cancels the global anomaly of the leptons. In this way, the effective theory inherits the global anomaly cancellation constraint that \( N_c \) must be odd. The low-energy theory is gauge invariant only if its baryons are quantized as fermions.

At the quark level, we have seen that the hypercharge \( Y = T^3_R + \frac{1}{2}B \) contains the baryon number \( B \). Since \( U(1)_B \) is not a subgroup of \( SU(2)_L \otimes SU(2)_R \), it is not at all straightforward to gauge \( U(1)_Y \). Gauging the \( SU(2)_R \) component of \( Y \) simply amounts to extending the covariant derivative to
\[ D_\mu U = \partial_\mu U + W_\mu U - UB_\mu, \tag{3.13} \]
where \( B_\mu \) is the \( U(1)_Y \) gauge field. However, incorporating the covariant derivatives alone is not sufficient to correctly gauge \( U(1)_Y \). Although the pions themselves do not carry baryon number, it is still important to incorporate the baryon current in the effective theory since \( Y \) contains the baryon number \( B \). In particular, if one would not include the baryon current, the decay \( \pi^0 \rightarrow \gamma\gamma \) would not happen in the effective theory.

Of course, when \( SU(2)_L \) is gauged, baryon number conservation is violated by electroweak instantons according to eq.(2.14). When one replaces ordinary derivatives by covariant ones in the baryon current of eq.(3.3), its divergence does not
obey eq. (2.14). Instead, one should consider the Goldstone-Wilczek current [11, 10]

\[ j^\mu_{GW} = \frac{1}{24\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{Tr}[(U^\dagger D^\rho U)(U^\dagger D^\sigma U)] \]

\[ - \frac{1}{16\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{Tr}[W_{\nu\rho}(D^\sigma U U^\dagger)] - \frac{1}{16\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{Tr}[B_{\nu\rho}(U^\dagger D^\sigma U)], \]  

(3.14)

whose divergence is indeed given by

\[ \partial^\mu j^\mu_{GW} = -\frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{Tr}\{W_{\mu\nu}W_{\rho\sigma}\} + \frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{Tr}\{B_{\mu\nu}B_{\rho\sigma}\}. \]  

(3.15)

Since, at the quark level, the \( U(1)_Y \) gauge field couples to baryon number with strength \( g'/2 \), the same must be true in the effective theory. This implies that, in order to gauge \( U(1)_Y \) correctly, one must include the Goldstone-Wilczek current in the low-energy effective action. This is achieved by introducing a Goldstone-Wilczek term

\[ S_{GW}[U, W_\mu, B_\mu] = \frac{g'}{2} \int d^4x \, B^3_\mu j^\mu_{GW}. \]  

(3.16)

It should be noted that this term alone is not gauge invariant. In this respect it is similar to the term \( \text{Sign}[U] \). While \( \text{Sign}[U] \) varies under topologically nontrivial \( SU(2)_L \) gauge transformations, the gauge variance is exactly what one needs to cancel the global anomaly in the lepton sector. Similarly, while \( j^\mu_{GW} \) is both \( SU(2)_L \) and \( U(1)_Y \) gauge invariant, \( S_{GW}[U, W_\mu, B_\mu] \) is only \( SU(2)_L \) gauge invariant, but varies under \( U(1)_Y \) gauge transformations \( B'_\mu = B_\mu + \partial_\mu \varphi \). The violation of gauge invariance is determined by

\[ S_{GW}[U', W_\mu, B'_\mu] - S_{GW}[U, W_\mu, B_\mu] = \int d^4x \, \varphi \partial_\mu j^\mu_{GW} = -\int d^4x \, \varphi \partial_\mu j^\mu_{GW} \]

\[ = \int d^4x \, \varphi \left\{ \frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{Tr}\{W_{\mu\nu}W_{\rho\sigma}\} - \frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{Tr}\{B_{\mu\nu}B_{\rho\sigma}\} \right\}. \]  

(3.17)

This gauge variation is exactly what is needed to cancel the triangle anomalies in the lepton sector and render the whole theory gauge invariant.

The path integral of pions in the background of Higgs and gauge fields finally takes the form

\[ Z[\Phi, W_\mu, B_\mu] = \int \mathcal{D}U \exp(-S[U, \Phi, W_\mu, B_\mu]) \text{Sign}[U] \exp(iS_{GW}[U, W_\mu, B_\mu]). \]  

(3.18)

Nowhere in this expression does \( N_c \) appear as an explicit parameter. It only appears implicitly in \( S[U, \Phi, W_\mu, B_\mu] \) through parameters like \( F_\pi \) and \( \langle \bar{\Psi}\Psi \rangle \). It should be noted that \( S[U, \Phi, W_\mu, B_\mu] \) contains all normal parity contributions to the effective action, not only the leading terms given in eq. (3.2). The Goldstone-Wilczek term, on the other hand, contains the anomalous parity contributions.
Let us also discuss how to gauge just $U(1)_{em}$. According to eq. (2.11), at the quark level the electric charge is given by $Q = T_L^3 + T_R^3 + \frac{1}{2} B$. This implies that the electromagnetic covariant derivative takes the form

$$D_\mu U = \partial_\mu U + ieA_\mu[T^3, U].$$

This is consistent with eq. (3.13) because the photon and $Z^0$ boson fields are related to $W^3_\mu$ and $B^3_\mu$ by

$$W^3_\mu = \frac{g' A_\mu + g Z_\mu}{\sqrt{g^2 + g'^2}}, \quad B^3_\mu = \frac{g A_\mu - g' Z_\mu}{\sqrt{g^2 + g'^2}},$$

and the electric charge is given by

$$e = \frac{gg'}{\sqrt{g^2 + g'^2}}.$$

The Goldstone-Wilczek current now takes the form

$$j^G_W = \frac{1}{24\pi^2} \varepsilon_{\mu
u\rho\sigma} \text{Tr}[(U^\dagger D_\nu U)(U^\dagger D_\rho U)(U^\dagger D_\sigma U)] - \frac{i e}{16\pi^2} \varepsilon_{\mu
u\rho\sigma} F_{\nu\rho} \text{Tr}[T^3 (D_\sigma U^\dagger U + U^\dagger D_\sigma U)].$$

In this case, the theory is vector-like and the baryon current is conserved. The path integral then takes the form

$$Z[A_\mu] = \int \mathcal{D}U \exp(-S[U, A_\mu]) \text{Sign}[U] \exp(iS_{GW}[U, A_\mu]),$$

with

$$S_{GW}[U, A_\mu] = \frac{e}{2} \int d^4x \: A_\mu j^G_W.$$ 

One can now identify the vertex responsible for the decay $\pi^0 \rightarrow \gamma\gamma$. Putting $U \approx 1 + 2i\pi^0 T^3 / F_\pi$, after partial integration the second term in the Goldstone-Wilczek current of eq. (3.22) indeed yields the vertex

$$\mathcal{L}_{\pi^0\gamma\gamma} = -i \frac{e^2}{32\pi^2 F_\pi} \pi^0(\epsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}).$$

By now it should no longer come as a surprise that this vertex is not proportional to $N_c$.

Another vertex that is often claimed to be proportional to $N_c$ is $\mathcal{L}_{\pi^0\pi^0\pi^-\pi^+}$. In the microscopic theory this vertex results from a quark box diagram with three external pion and one external photon line. This diagram is proportional to

$$\text{Tr}(T^3[T^+, T^-]Q) = \frac{N_c}{4}(Q_u + Q_d),$$

(3.26)
where $T^\pm = (T^1 \pm iT^2)/\sqrt{2}$. If one uses $Q_u = \frac{2}{3}$ and $Q_d = -\frac{1}{3}$ independent of $N_c$, this expression is obviously proportional to $N_c$. However, if one uses the consistent quark charges of eq.(1.1) it becomes $N_c$-independent because

$$\frac{N_c}{4}(Q_u + Q_d) = \frac{N_c}{4} \left[ \frac{1}{2} \left( \frac{1}{N_c} + 1 \right) + \frac{1}{2} \left( \frac{1}{N_c} - 1 \right) \right] = \frac{1}{4}. \quad (3.27)$$

Using $U \approx 1 + 2i \pi^T T^a/F_\pi$ as well as $\pi^\pm = (\pi^1 \pm i \pi^2)/\sqrt{2}$, in the anomaly free standard model with consistent quark charges this vertex is

$$\mathcal{L}_{\pi^0 \pi^+ \pi^-} = \frac{e}{4 \pi^2 F_\pi^2} \varepsilon_{\mu \nu \rho \sigma} A_\mu \partial_\nu \pi^0 \partial_\rho \pi^+ \partial_\sigma \pi^-, \quad (3.28)$$

which is again $N_c$-independent. The same is true for any other photon-pion vertex. Hence, contrary to common lore, it is impossible to literally see $N_c$ in experiments with just photons and pions.

\section{Low-Energy Description of $N_f \geq 3$ Flavors}

In a world with $N_f \geq 3$ massless quarks the chiral symmetry is $SU(N_f)_L \otimes SU(N_f)_R$ which is spontaneously broken to $SU(N_f)_{L=R}$. Consequently, the Goldstone boson fields now live in the coset space $SU(N_f)_L \otimes SU(N_f)_R/SU(N_f)_{L=R} = SU(N_f)$. The leading order chiral perturbation theory action takes the same form of eq.(3.2) as in the $N_f = 2$ case. Since $\Pi_3[SU(N_f)] = \mathbb{Z}$ for any $N_f \geq 3$, the Skyrme and Goldstone-Wilczek currents of eqs.(3.3,3.14) also still have the same form.

However, in contrast to the $N_f = 2$ case, $\Pi_4[SU(N_f)] = \{0\}$ for $N_f \geq 3$. Consequently, any space-time dependent Goldstone boson field $U(x) \in SU(N_f)$ can now be continuously deformed into the trivial field $U(x) = 1$. By introducing a fifth coordinate $x_5 \in [0, 1]$ which plays the role of a deformation parameter, one can extend the 4-dimensional field $U(x)$ to a field $U(x, x_5)$ on the 5-dimensional hemisphere $H^5$ whose boundary $\partial H^5 = S^4$ is (compactified) space-time, such that $U(x, 0) = 1$ and $U(x, 1) = U(x)$. This allows one to write down the Wess-Zumino-Witten term [1] with the action

$$S_{WZW}[U] = \frac{1}{240 \pi^2 i} \int_{H^5} d^5 x \varepsilon_{\mu \nu \rho \sigma \lambda} \text{Tr}[(U^\dagger \partial_\mu U)(U^\dagger \partial_\nu U)(U^\dagger \partial_\rho U)(U^\dagger \partial_\sigma U)(U^\dagger \partial_\lambda U)]. \quad (4.1)$$

Note that the factor $i$ in eq.(4.1) is necessary in order to get a real-valued result. Of course, the 4-dimensional physics should be independent of how the field $U(x, x_5)$ is deformed into the bulk of the fifth dimension. It should only depend on the boundary values $U(x)$, i.e. on the Goldstone boson field in the physical part of space-time. This is possible because the integrand in eq.(4.1) is a total divergence. In particular, $S_{WZW}[U]$ is closely related to the winding number $\Pi_5[SU(N_f)] = \mathbb{Z}$. In fact, if the integration in eq.(4.1) were performed over a sphere $S^5$ instead of the
hemisphere $H^5$, the result would be $2\pi$ times the integer winding number of $U(x, x_5)$. Hence, modulo integers, $S_{WZW}[U]$ gets contributions only from the boundary of $H^5$, i.e. from the 4-dimensional physical space-time $S^4$. Of course, one must still ensure that the integer contribution from the 5-dimensional bulk cancels. This is indeed the case, because $S_{WZW}[U]$ enters the path integral,

$$Z = \int \mathcal{D}U \exp(-S[U]) \exp(iN_c S_{WZW}[U]), \quad (4.2)$$

with a quantized prefactor — the number of colors $N_c$. It should be noted that eq.(4.2) is the natural extension of eq.(3.6) in the $N_f=2$ case. In fact, one can show that for $U(x) \in SU(2)$

$$\exp(iN_c S_{WZW}[U]) = \text{Sign}[U]^{N_c}. \quad (4.3)$$

The argument of the Wess-Zumino-Witten term is a 5-dimensional Goldstone boson field $U(x, x_5) \in SU(N_f)$ which reduces to a 4-dimensional $SU(2)$ field $U(x)$ at the boundary of $H^5$. The argument of the sign factor, on the other hand, is just the 4-dimensional field $U(x) \in SU(2)$. Indeed, in the $N_f \geq 3$ theory, the Wess-Zumino-Witten term $\exp(iN_c S_{WZW}[U])$ plays a similar role as $\text{Sign}[U]$ in the $N_f = 2$ case. In particular, for odd $N_c$ it ensures that the Skyrmion is again quantized as a fermion with half-integer spin \[9\]. It also ensures that the global anomaly is properly canceled when one gauges $SU(2)$.

Unlike in the $N_f = 2$ case, for $N_f \geq 3$ the Wess-Zumino-Witten term also breaks the unwanted intrinsic parity symmetry $P_0$ that is present in the Goldstone boson action $S[U]$, but not in QCD \[\text{\cite{12}}\]. The full parity operation $P$, of course, is a symmetry of QCD, at least for vanishing vacuum angle $\theta = 0$. Parity acts on the pseudo-scalar Goldstone bosons $\pi^a(\vec{x}, t)$ by spatial inversion accompanied by a sign-change, i.e. $P \pi^a(\vec{x}, t) = -\pi^a(-\vec{x}, t)$. For the field $U$ the full parity transformation $P$ takes the form

$$PU(\vec{x}, t) = U^\dagger(-\vec{x}, t), \quad (4.4)$$

while the intrinsic parity $P_0$ leaves out the spatial inversion and thus takes the form

$$P_0 U(\vec{x}, t) = U^\dagger(\vec{x}, t). \quad (4.5)$$

If $P_0$ were a symmetry of QCD, the number of Goldstone bosons would be conserved modulo two, i.e. no strong interaction process could change the number of Goldstone bosons from even to odd. For $N_f = 2$ this is indeed the case, and $P_0$ is, in fact, nothing but $G$-parity \[\text{\cite{11}}\]. For $N_f \geq 3$, on the other hand, intrinsic parity is not a symmetry of QCD. For example, the $\phi$-meson decays both into two kaons and into three pions. However, the Goldstone boson action $S[U]$ is indeed invariant under $P_0$,

$$S[P_0 U] = S[U^\dagger] = S[U], \quad (4.6)$$

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and hence has more symmetry than the underlying QCD action. The Wess-Zumino-Witten action, on the other hand, is odd under $P_0$,

$$S_{ZW}^{[PbU]} = S_{ZW}^{[U^\dagger]} = -S_{ZW}[U],$$

and thus reduces the symmetry of the effective theory to the one of QCD.

It is remarkable that an integer parameter (with value $N_c$) appears explicitly in the low-energy effective theory of QCD with $N_f \geq 3$. In particular, this means that some low-energy processes involving more than two flavors indeed depend directly on how many quarks there are inside a proton. For example, there is a vertex in the Wess-Zumino-Witten term that turns two kaons into three pions. This vertex is directly proportional to $N_c$. When one gauges $SU(2)_L$ and $U(1)_Y$, or even just $U(1)_{em}$, the explicit $N_c$ factor also affects some electroweak processes. For example, by gauging $U(1)_{em}$ in an $N_f = 3$ theory with the quark charge matrix

$$Q' = \text{diag}(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}),$$

Witten has shown that the vertex $L_{\pi^0 \gamma \gamma}$ is proportional to $N_c$ \cite{9}. While this is the correct answer if one varies $N_c$ without adjusting the quark charges, it is not what one gets for the anomaly free standard model with arbitrary odd $N_c$. In that case, the appropriate quark charge matrix is given by

$$Q = \text{diag}(Q_u, Q_d, Q_s) = \text{diag}(\frac{1}{2}(\frac{1}{N_c} + 1), \frac{1}{2}(\frac{1}{N_c} - 1), \frac{1}{2}(\frac{1}{N_c} - 1))$$

and hence for $N_c \neq 3$ it is not a generator of $SU(3)_{L=R}$. This implies that, like in the $N_f = 2$ case, we must also include a Goldstone-Wilczek term in the $N_f = 3$ effective theory. However, in the $N_f = 3$ case, the Goldstone-Wilczek term is a factor of $(1 - N_c/3)$ larger than in the $N_f = 2$ case and we thus obtain

$$Z[A_\mu] = \int \mathcal{D}U \exp(-S[U, A_\mu]) \exp(iN_cS_{ZW}[U, A_\mu]) \exp(i(1 - \frac{N_c}{3})S_{GW}[U, A_\mu]).$$

The $U(1)_{em}$-gauged Wess-Zumino-Witten term takes the form \cite{3, 13, 14, 15}

$$S_{ZW}[U, A_\mu] = S_{ZW}[U] + \frac{e}{48\pi^2} \int d^4x \, \varepsilon_{\mu\nu\rho\sigma} A_\mu$$

$$\times \{ \text{Tr}[Q'(\partial_\nu U \partial_\mu U^\dagger)(\partial_\rho U \partial_\sigma U^\dagger) + (U^\dagger \partial_\nu U)(U^\dagger \partial_\rho U)(U^\dagger \partial_\sigma U)] \}$$

$$- \frac{ie^2}{48\pi^2} \int d^4x \, \varepsilon_{\mu\rho\sigma} A_\mu F_{\nu\rho} \text{Tr}[Q'(\partial_\sigma U \partial_\mu U^\dagger)[Q' + \frac{1}{2} UQ' U^\dagger]$$

$$+ Q'(U^\dagger \partial_\sigma U)[Q' + \frac{1}{2} UQ' U]},$$

(4.11)
In the literature the two terms containing $UQ'U^\dagger$ and $U^\dagger Q'U$ are sometimes replaced by a single term $[9, 15]$. Our analysis agrees with the one in $[16]$. We also write down the Goldstone-Wilczek term for general $N_f$

$$S_{GW}[U, A_\mu] = \frac{e}{48\pi^2} \int d^4x \, \varepsilon_{\mu\rho\sigma\tau} A_\mu \text{Tr}[(U^\dagger \partial_\nu U)(U^\dagger \partial_\rho U)(U^\dagger \partial_\sigma U)]$$

$$- \frac{i e^2}{32\pi^2} \int d^4x \, \varepsilon_{\mu\rho\sigma\tau} A_\mu F_{\nu\rho} \text{Tr}[Q'(\partial_\sigmaUU^\dagger + U^\dagger \partial_\sigma U)].$$  (4.12)

Note that all derivatives in this expression can be replaced by covariant derivatives without changing the result.

According to eq.(4.10), for $N_c = 3$ no Goldstone-Wilczek term arises, and the contribution from the Wess-Zumino-Witten term alone gives the full answer. In that case, the quark charge matrix of eq.(4.8) that was used by Witten, is indeed the one of the consistent standard model. The Wess-Zumino-Witten term alone contributes a vertex $L_{\pi^0\gamma\gamma}$ that is proportional to

$$N_c(Q_u^2 - Q_d^2) = \frac{N_c}{3},$$  (4.13)

and hence a factor $N_c/3$ stronger than the correct vertex in the consistent standard model for arbitrary odd $N_c$. However, for $N_c \neq 3$, there is also the Goldstone-Wilczek term, which contributes $(1 - N_c/3)$ times the correct vertex that was already obtained in the $N_f = 2$ case. The $N_c$-dependent part of the Goldstone-Wilczek term completely cancels the contribution of the Wess-Zumino-Witten term, and hence the $N_c$-independent part alone indeed gives the correct strength

$$\frac{N_c}{3} + (1 - \frac{N_c}{3}) = 1.$$  (4.14)

The above consideration can be trivially extended to a general number $N_f$ of light flavors. Let us consider the consistent standard model with arbitrary odd $N_c$ and with several generations of fermions. We assume that there are $N_u \geq 1$ light up-type quarks (up, charm, top) with charge $Q_u = (1/N_c + 1)/2$ and $N_d \geq 1$ light down-type quarks (down, strange, bottom) with charge $Q_d = (1/N_c - 1)/2$. In this case, $N_f = N_u + N_d$ and the charge matrix of the light quarks takes the form

$$Q = Q' + (1 - N_c \frac{N_d - N_u}{N_f}) \frac{1}{2} B,$$  (4.15)

where $Q'$ is a traceless diagonal generator of $SU(N_f)_{L=R}$. In the corresponding low-energy effective theory,

$$Z[A_\mu] = \int DU \exp(-S[U, A_\mu]) \exp(i N_c S_{WZW}[U, A_\mu])$$

$$\times \exp(i(1 - N_c \frac{N_d - N_u}{N_f}) S_{GW}[U, A_\mu]),$$  (4.16)

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there is a Goldstone-Wilczek current which contributes \((1 - N_c(N_d - N_u)/N_f)\) times the correct vertex \(L_{\pi^0\gamma\gamma}\). The Wess-Zumino-Witten term, on the other hand, is now coupled to the photon field by using the remaining \(SU(N_f)_{L=R}\) generator \(Q'\), which has diagonal elements \(Q'_u\) and \(Q'_d\) with

\[
Q'_u = Q_u - \frac{1}{2} \left( \frac{N_d - N_u}{N_f} \right) = \frac{N_d}{N_f}, \quad Q'_d = Q_d - \frac{1}{2} \left( \frac{N_d - N_u}{N_f} \right) = -\frac{N_u}{N_f}. \tag{4.17}
\]

The strength of the vertex \(L_{\pi^0\gamma\gamma}\) that results from the Wess-Zumino-Witten term is given by

\[
N_c(Q^2_u - Q^2_d) = N_c \frac{N_d^2 - N_u^2}{N_f^2} = N_c \frac{N_d - N_u}{N_f}. \tag{4.18}
\]

As before, the \(N_c\)-dependent part of the Goldstone-Wilczek term completely cancels the contribution from the Wess-Zumino-Witten term, and the \(N_c\)-independent part of the Goldstone-Wilczek term gives the correct strength

\[
N_c \frac{N_d - N_u}{N_f} + (1 - N_c \frac{N_d - N_u}{N_f}) = 1. \tag{4.19}
\]

\section{How Can One See \(N_c\) for \(N_f \geq 3\) ?}

Since the decay \(\pi^0 \to \gamma\gamma\) does not allow one to see \(N_c\), we now ask if other processes do. First, we consider only photons and pions (but no kaons or \(\eta\)-mesons) by embedding a pion \(SU(2)\) sub-matrix \(\tilde{U}(x)\) into \(SU(N_f)\)

\[
U(x) = \begin{pmatrix} \tilde{U}(x) & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \tag{5.1}
\]

where \(\mathbb{1}\) is the \((N_f - 2) \times (N_f - 2)\) unit matrix. Then only a 2 \(\times\) 2 sub-matrix \(\tilde{Q}'\) of the full \(N_f \times N_f\) matrix \(Q'\) enters the calculation. Using

\[
\tilde{Q}' = \begin{pmatrix} Q'_u & 0 \\ 0 & Q'_d \end{pmatrix} = \frac{N_d - N_u}{2N_f} + T^3, \quad \tilde{Q}'^2 = \begin{pmatrix} Q'^2_u & 0 \\ 0 & Q'^2_d \end{pmatrix} = \frac{N_d^2 + N_u^2}{2N_f^2} + \frac{N_d - N_u}{N_f} T^3, \tag{5.2}
\]

it is straightforward to show that

\[
N_c(S_{WZW}[U, A_\mu] - S_{WZW}[\tilde{U}]) + (1 - N_c \frac{N_d - N_u}{N_f})S_{GW}[U, A_\mu] = S_{GW}[\tilde{U}, A_\mu], \tag{5.3}
\]

where \(S_{GW}[\tilde{U}, A_\mu]\) is the \(N_c\)-independent \(N_f = 2\) result of eq.(3.24). This shows that the \(N_f \geq 3\) result is fully consistent with the \(N_f = 2\) calculation. In particular, all photon-pion vertices contained in the Wess-Zumino-term are completely canceled by the \(N_c\)-dependent piece of the Goldstone-Wilczek term. Hence, as we concluded before, one cannot see \(N_c\) directly in low-energy experiments of photons and pions alone.
Next, we consider processes involving just photons and $\eta$-mesons in a theory with $N_u = 1$ and $N_d = 2$ (and hence $N_f = N_u + N_d = 3$). In that case, we write

$$U(x) = \exp(2i\eta^8 T^8/F_\pi),$$

with $T^8 = \frac{1}{2}\lambda^8$, and we use $Q’ = T^3 + T^8/\sqrt{3}$. Then it is again straightforward to show that

$$N_c(S_{WZW}[U, A_\mu] - S_{WZW}[U]) + (1 - N_c\frac{N_d - N_u}{N_f}) S_{GW}[U, A_\mu]$$

$$= N_c(S_{WZW}[U, A_\mu] - S_{WZW}[U]) + (1 - \frac{N_c}{3}) S_{GW}[U, A_\mu]$$

$$= \frac{e^2}{32\sqrt{3}\pi^2 F_\pi} \int d^4 x \eta^8 \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma},$$

(5.5)

which again does not contain $N_c$ explicitly. In the microscopic theory this vertex results from a quark triangle diagram that is proportional to

$$\text{Tr}(T^8 Q^2) = \frac{N_c}{2\sqrt{3}} (Q^2_u - Q^2_d) = \frac{1}{2\sqrt{3}},$$

(5.6)

It should be noted that the physical $\eta$-meson is a mixture of the flavor octet $\eta^8$ and a flavor singlet $\eta^1$. In order to properly discuss the mixing, one must hence also include the $\eta^1$ field in the chiral Lagrangian. The combination of $\eta^1$ and $\eta^8$ orthogonal to the physical $\eta$-meson, is the $\eta’$-meson. Indeed, in the large $N_c$ limit the $\eta’$-meson also becomes a Goldstone boson and should be included in the chiral Lagrangian [18, 19, 20]. Anomalous decays of Goldstone bosons and, in particular, the issue of meson mixing are discussed in [21]. The quark triangle diagram describing the decay of the flavor singlet $\eta^1 \rightarrow \gamma\gamma$ in the microscopic theory is proportional to

$$\text{Tr}(\frac{1}{\sqrt{6}} Q^2) = \frac{N_c}{\sqrt{6}} (Q^2_u + 2 Q^2_d) = \frac{3}{4\sqrt{6}} (N_c + \frac{1}{N_c} - \frac{2}{3}),$$

(5.7)

which is $N_c$-dependent. Hence, due to mixing, the width of the decay $\eta \rightarrow \gamma\gamma$ indeed depends explicitly on $N_c$. However, this dependence is not so simple, because it is influenced by the amount of mixing which itself implicitly depends on $N_c$. Therefore, this decay is not too well suited for replacing the misleading textbook example $\pi^0 \rightarrow \gamma\gamma$ for providing experimental support for three colors.

In the next step we consider the interactions of photons, pions, $\eta$-mesons and kaons, again using $N_u = 1$, $N_d = 2$ and $N_f = 3$. In this case, it is easier to literally see $N_c$. In particular, the vertex

$$\mathcal{L}_{\eta^8 \pi^+ \pi^-} = \frac{eN_c}{4\sqrt{3}\pi^2 F^3_\pi} \epsilon_{\mu\nu\rho\sigma} A_\mu \partial_\nu \eta^8 \partial_\rho \pi^+ \partial_\sigma \pi^-,$$

(5.8)

is proportional to $N_c$. It is interesting to note that only the Wess-Zumino-Witten term contributes to this process. Hence, in this case, there is no cancellation with
the $N_c$-dependent part of the Goldstone-Wilczek term. In the microscopic theory this process results from a quark box diagram that is proportional to

$$\text{Tr}(T^8[T^+, T^-]Q) = \frac{N_c}{4\sqrt{3}} (Q_u - Q_d) = \frac{N_c}{4\sqrt{3}}. \quad (5.9)$$

Because of mixing, one must also consider the corresponding vertex for the flavor singlet $\eta^1$. The interactions of photons with three Goldstone bosons and the effect of meson mixing on the decay $\eta \to \pi^+ \pi^- \gamma$ have been discussed in [21, 22]. The quark box diagram describing the decay of the flavor singlet $\eta^1 \to \pi^+ \pi^- \gamma$ is proportional to

$$\text{Tr}(\frac{\mathbb{1}}{\sqrt{6}}[T^+, T^-]Q) = \frac{N_c}{2\sqrt{6}} (Q_u - Q_d) = \frac{N_c}{2\sqrt{6}}. \quad (5.10)$$

Since both vertices $\mathcal{L}_{\eta^8 \pi^+ \pi^- \gamma}$ and $\mathcal{L}_{\eta^1 \pi^+ \pi^- \gamma}$ are proportional to $N_c$, the vertex $\mathcal{L}_{\eta \pi^+ \pi^- \gamma}$ involving the physical $\eta$-meson is also proportional to $N_c$. Hence, the width of the decay $\eta \to \pi^+ \pi^- \gamma$ is proportional to $N_c^2$ and the observed width indeed implies $N_c = 3$. This decay should hence replace the textbook example $\pi^0 \to \gamma \gamma$ for lending experimental support to the fact that there are three colors in our world.

Also the vertices

$$\mathcal{L}_{\eta^0 K^0 \overline{K^0}} = \frac{e(N_c - 1)}{8\pi^2 F_\pi^3} \varepsilon_{\mu \nu \rho \sigma} A_\mu \partial_\nu \pi^0 \partial_\rho \partial_\sigma \overline{K^0},$$

$$\mathcal{L}_{\eta^0 K^+ K^-} = \frac{e(N_c + 1)}{8\pi^2 F_\pi^3} \varepsilon_{\mu \nu \rho \sigma} A_\mu \partial_\nu \pi^0 \partial_\rho K^+ \partial_\sigma K^-,\quad \text{Tr}(T^8[U^+, U^-]Q) = \frac{N_c}{4\sqrt{3}}(Q_u - Q_d) = \frac{N_c}{4\sqrt{3}}. \quad (5.11)$$

are explicitly $N_c$-dependent. However, for kinematic reasons these processes do not contribute to single particle decays and are hence more difficult to observe experimentally. In the microscopic theory the vertices of eq.(5.11) result from quark box diagrams that are proportional to

$$\text{Tr}(T^3[U^+, U^-]Q) = -\frac{N_c}{4}Q_d = \frac{1}{8}(N_c - 1),$$

$$\text{Tr}(T^3[V^+, V^-]Q) = \frac{N_c}{4}Q_u = \frac{1}{8}(N_c + 1),$$

$$\text{Tr}(T^8[U^+, U^-]Q) = \frac{N_c}{4\sqrt{3}}3Q_d = \frac{\sqrt{3}}{8}(1 - N_c),$$

$$\text{Tr}(T^8[V^+, V^-]Q) = \frac{N_c}{4\sqrt{3}}(Q_u + 2Q_d) = \frac{\sqrt{3}}{8}(1 - \frac{N_c}{3}), \quad (5.12)$$

respectively. Here we have used

$$V^\pm = \frac{1}{\sqrt{2}}(T^4 \pm iT^5), \quad U^\pm = \frac{1}{\sqrt{2}}(T^6 \pm iT^7). \quad (5.13)$$
6 Summary and Conclusions

We have considered a consistent standard model with arbitrary odd $N_c$ and with $N_u$ light up-type quarks as well as $N_d$ light down-type quarks. The partition function of the corresponding low-energy effective theory for the Goldstone bosons of the strong interactions in the background of an electromagnetic gauge field then takes the form

$$Z[A_{\mu}] = \int D U \exp(-S[U,A_{\mu}]) \exp(iN_c S_{WZW}[U,A_{\mu}]) \times \exp(i(1 - N_c \frac{N_d - N_u}{N_f})S_{GW}[U,A_{\mu}]).$$ (6.1)

In the two flavor case ($N_f = 2$), the Wess-Zumino-Witten term reduces to $\text{Sign}[U] = \pm 1 \in \Pi_2[SU(2)] = Z(2)$ (for odd $N_c$) and thus becomes $N_c$-independent. With $N_u = N_d = 1$ the Goldstone-Wilczek term also becomes $N_c$-independent, and hence $N_c$ does not appear explicitly in the low-energy effective theory of the standard model. In particular, the width of the decay $\pi^0 \to \gamma\gamma$, which is entirely due to the Goldstone-Wilczek term, is not proportional to $N_c^2$. Also other photon-pion vertices, like $\mathcal{L}_{\pi^0\pi^+\pi^-\gamma}$, do not depend on $N_c$ explicitly. Of course, the low-energy effective theory still depends implicitly on the number of colors, because quantities like $F_\pi$ are $N_c$-dependent. Hence, if one computes $F_\pi$ in a nontrivial lattice QCD calculation with $N_c$ colors, and then compares, for example, with the observed $\pi^0 \to \gamma\gamma$ decay width, one will correctly conclude that $N_c = 3$ in our world. However, if one takes the value of $F_\pi$ from experiment, it is impossible to literally see the number of colors by detecting the photons emerging from the decay of the neutral pion.

In the three flavor case with $N_u = 1$, $N_d = 2$ ($N_u + N_d = N_f = 3$) and $N_c = 3$ the Goldstone-Wilczek term vanishes and the contribution to the $\pi^0$ decay seems to be entirely due to the Wess-Zumino-Witten term. However, for general $N_c$, the $N_c$-dependences of both terms cancel and the resulting width for the decay $\pi^0 \to \gamma\gamma$ stems from the $N_c$-independent part of the Goldstone-Wilczek term only. The cancellations of the $N_c$-dependent terms are not limited to the vertex $\mathcal{L}_{\pi^0\gamma\gamma}$, but appear for all electromagnetic processes involving only pions and photons. Still, for $N_f \geq 3$ there are indeed some processes that allow one to literally see the number of colors. For example, the width of the decay $\eta \to \pi^+\pi^-\gamma$ is proportional to $N_c^2$ and the observed width indeed implies that there are three colors in our world. This decay should hence replace the textbook process $\pi^0 \to \gamma\gamma$ lending experimental support to $N_c = 3$.

It should be noted that our discussion does not apply to very large values of $N_c$. In that case, the $\eta'$-meson becomes light and should be included in the chiral Lagrangian [18, 19, 20]. It would be interesting to repeat our arguments in that situation, especially in order to further address the issue of meson mixing in the decay $\eta \to \pi^+\pi^-\gamma$. It might also be worthwhile to reconsider those calculations that hold the quark charges fixed while taking the large $N_c$ limit. When one uses
the consistent quark charges of eq. (1.1), one would expect to obtain results for electroweak processes with a more well-behaved dependence on the number of colors.

As one would have expected, at the end of this paper we still conclude that in our world \( N_c = 3 \). However, we have sometimes been taught to believe this fact for the wrong reasons. We conclude this paper by expressing our hope that in the future some textbooks will reflect the results of the discussion presented here.

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