GERSTENHABER-SCHACK AND HOCHSCHILD COHOMOLOGIES OF CO-FROBENIUS HOPF ALGEBRAS

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 abstract. We show that the Gerstenhaber-Schack cohomology of a co-Frobenius Hopf algebra determines its Hochschild cohomology, and in particular its Gerstenhaber-Schack cohomological dimension bounds its Hochschild cohomological dimension. Together with some general considerations on free Yetter-Drinfeld modules over adjoint Hopf subalgebras and the monoidal invariance of Gerstenhaber-Schack cohomology, this is used to show that Gerstenhaber-Schack cohomological dimension of the quantum symmetry algebra of a finite-dimensional semisimple algebra (including the coordinate algebra of the quantum permutation group) is 3, and bounds its Hochschild cohomological dimension.

1. Introduction

We study homological properties of Hopf algebras by using Yetter-Drinfeld modules and tensor categories techniques. We are especially interested in the following question:

**Question 1.1.** If $A$ and $B$ are Hopf algebras having equivalent tensor categories of comodules, how are their Hochschild cohomologies related? In particular do $A$ and $B$ have the same cohomological dimension?

We have seen in [10] that the Hochschild cohomologies of two such Hopf algebras $A$ and $B$ are indeed closely related, using resolutions of the trivial Yetter-Drinfeld module over $A$ (or over $B$) formed by free Yetter-Drinfeld modules. In the present paper we continue this study along the same line of ideas.

Our first remark in view of Question 1.1 is that there exists at least a cohomology theory for Hopf algebras that is known to be well-behaved with respect to this situation: Gerstenhaber-Schack cohomology [23, 24]. Let $A$ be a Hopf algebra and let $M$ be a Hopf bimodule over $A$: the Gerstenhaber-Schack cohomology $H^*_GS(A, M)$ of $A$ with coefficients in $M$ [24] is defined to be the homology of an explicit bicomplex whose arrows are modeled on the Hochschild complex of the underlying algebra and columns are modeled on the Cartier complex of the underlying coalgebra. When $M = A$ is the trivial Hopf bimodule, then $H^*_GS(A, A) = H^*_b(A)$ is known as the bialgebra cohomology of $A$. These cohomologies, which can also be defined in terms of Yetter-Drinfeld modules, were first introduced in view of applications to deformation theory.

If $A$ and $B$ are Hopf algebras having equivalent tensor categories of comodules, then there exists a tensor equivalence $F : A\mathcal{M}_A \to B\mathcal{M}_B$ between their categories of Hopf bimodules such that for any Hopf bimodule $M$ over $A$, we have $H^*_GS(A, M) \cong H^*_GS(B, F(M))$, and in particular $H^*_b(A) \cong H^*_b(B)$ and $\text{cd}_{GS}(A) = \text{cd}_{GS}(B)$ (where $\text{cd}_{GS}$ denotes the Gerstenhaber-Schack cohomological dimension, defined in the obvious way, see Section 5). We call these properties the monoidal invariance of Gerstenhaber-Schack cohomology.

Going back to Question 1.1 the next question is to study how Hochschild and Gerstenhaber-Schack cohomologies are related. We show that the Gerstenhaber-Schack cohomology of a co-Frobenius Hopf algebra determines its Hochschild cohomology. More precisely, we show that if $A$ is a co-Frobenius Hopf algebra (this means that the category of comodules over $A$ has enough projectives), then there exists a functor $G : A\mathcal{M}_A \to \mathcal{A}\mathcal{M}_A^A$ such that for any $A$-bimodule $M$, we have $H^*(A, M) \cong H^*_GS(A, G(M))$
In particular we have \( \text{cd}(A) \leq \text{cd}_{GS}(A) \) for such a Hopf algebra. Then if \( A \) and \( B \) are co-Frobenius Hopf algebras as in Question 1.1, combining this with the monoidal invariance of Gerstenhaber-Schack cohomology, we get the existence of two functors \( F_1 : \mathcal{AM}_A \to \mathcal{BM}_B \) and \( F_2 : \mathcal{BM}_B \to \mathcal{AM}_A \) such that for any \( A \)-bimodule \( M \) and any \( B \)-bimodule \( N \), we have
\[
H^*(A, M) \simeq H^*_{GS}(B, F_1(M)) \quad \text{and} \quad H^*(B, N) \simeq H^*_{GS}(A, F_2(N))
\]
In particular
\[
\max(\text{cd}(A), \text{cd}(B)) \leq \text{cd}_{GS}(A) = \text{cd}_{GS}(B)
\]
These isomorphisms and this inequality thus provide partial answers to Question 1.1. They lead to the following new question:

**Question 1.2.** Is it true that \( \text{cd}(A) = \text{cd}_{GS}(A) \) for any co-Frobenius Hopf algebra \( A \) over a field of characteristic zero? Is true at least if \( A \) is assumed to be cosemisimple?

A positive answer would give the monoidal invariance of cohomological dimension and fully answer to the last part of Question 1.1 and would also be a natural infinite-dimensional generalization of a famous result by Larson-Radford [33], which states that, in characteristic zero, a finite-dimensional cosemisimple Hopf algebra is semisimple. See the comments at the end of Section 5.

We then apply our general considerations to some concrete classes of Hopf algebras, which were in fact the first motivation for this work.

1. We compute the bialgebra cohomology of \( B(E) \), the universal Hopf algebra associated to the non-degenerate bilinear form corresponding to an invertible matrix \( E \) [20] (including the familiar coordinate algebra \( O(SL_q(2)) \)), removing the cosemisimplicity assumption done in [10].

2. We compute the cohomological dimension of \( B_+(E) \), the adjoint Hopf subalgebra of \( B(E) \), and under a mild assumption on \( E \), its bialgebra cohomology. This is done using some general considerations on free Yetter-Drinfeld modules over adjoint Hopf subalgebras.

3. We compute the bialgebra cohomology of \( A_{\text{aut}}(R, \varphi) \), the quantum symmetry algebra of a semisimple measured algebra \( (R, \varphi) \) of dimension \( \geq 4 \), including in particular the coordinate algebra of the quantum permutation group \( S^+_n \), again under a mild assumption on \( (R, \varphi) \), and we show that \( \text{cd}(A_{\text{aut}}(R, \varphi)) \leq \text{cd}_{GS}(A_{\text{aut}}(R, \varphi)) = 3 \).

As a last comment to further motivate the use of Gerstenhaber-Schack cohomology as an appropriate cohomology theory for Hopf algebras (apart from its use to get information on Hochschild cohomology itself), we would like to point out that, in the examples computed so far, it also has the merit to avoid the “dimension drop” phenomenon usually encountered for quantum algebras (see [27, 28]): the canonical choice of coefficients (the trivial Hopf bimodule) is the good one to get the cohomological dimension. It would be interesting to know if this can be further generalized.

The paper is organized as follows. Section 2 consists of preliminaries. In Section 3 we discuss the cohomological dimension of a Hopf subalgebra and the sub-additivity of the cohomological dimension under extensions. Section 4 is devoted to Yetter-Drinfeld modules: we recall the concept of Free-Yetter Drinfeld module and we introduce the notion of relative projective Yetter-Drinfeld module, which corresponds, via the tensor equivalence between Yetter-Drinfeld modules and Hopf bimodules [44], to the notion of relative projective Hopf bimodule considered in [48]. We show that relative projective Yetter-Drinfeld modules are precisely the direct summands of free Yetter-Drinfeld modules. This section also contains some considerations on free Yetter-Drinfeld over adjoint Hopf subalgebras. In Section 5, after having recalled some basic facts on Gerstenhaber-Schack cohomology, we show, using results from [48], that in the co-Frobenius case, it can be computed by using resolutions by relative projective Yetter-Drinfeld modules. We deduce from this, for any Yetter-Drinfeld module \( V \), an explicit complex that computes the Gerstenhaber-Schack cohomology \( H^*_{GS}(A, V) \) (Theorem 5.4.1). We then show that
Gerstenhaber-Schack cohomology determines Hochschild cohomology (still in the co-Frobenius case). In Section 6 we study the examples mentioned earlier in the introduction. In Section 7 we discuss the Gerstenhaber-Schack cohomological dimension in the setting of Hopf algebras having a projection.

2. Preliminaries

In this preliminary section we fix some notation, we recall some basic definitions and facts on the Hochschild cohomology of a Hopf algebra, and we discuss exact sequences of Hopf algebras.

2.1. Notations and conventions. We work over $\mathbb{C}$ (or over any algebraically closed field of characteristic zero). This assumption does not affect any of the theoretical results in the paper, but is important for the examples we consider. We assume that the reader is familiar with the theory of Hopf algebras and their tensor categories of comodules, as e.g. in \cite{29, 30, 37}. If $A$ is a Hopf algebra, as usual, $\Delta$, $\epsilon$ and $S$ stand respectively for the comultiplication, counit and antipode of $A$. We use Sweedler’s notations in the standard way. The category of right $A$-comodules is denoted $\mathcal{M}_A$, the category of right $A$-modules is denoted $\mathcal{M}_A$, etc... The trivial (right) $A$-module is denoted $\mathbb{C}_\epsilon$. The set of $A$-module morphisms (resp. $A$-comodule morphisms) between two $A$-modules (resp. two $A$-comodules) $V$ and $W$ is denoted $\text{Hom}_A(V,W)$ (resp. $\text{Hom}^A(V,W)$).

2.2. Hochschild cohomology of a Hopf algebra. If $A$ is an algebra and $M$ is an $A$-bimodule, then $H^*(A,M)$ denotes, as usual, the Hochschild cohomology of $A$ with coefficients in $M$. See e.g. \cite{57}.

**Definition 2.1.** The Hochschild cohomological dimension of an algebra $A$ is defined to be

$$\text{cd}(A) = \sup \{n : H^n(A,M) \neq 0 \text{ for some } A \text{- bimodule } M \} \in \mathbb{N} \cup \{\infty\}$$

As noted by several authors (see \cite{22, 29, 25, 13, 17, 10}), the Hochschild cohomology of a Hopf algebra can be described by using a suitable Ext functor on the category of left or right $A$-modules. Indeed, if $A$ is a Hopf algebra and $M$ is an $A$-bimodule, we have

$$H^*(A,M) \simeq \text{Ext}^*_A(\mathbb{C}_\epsilon, M')$$

where the above Ext is in the category of right $A$-modules and $M'$ is $M$ equipped with the right $A$-module structure given by $x \leftarrow a = S(a_{(1)}) \cdot x \cdot a_{(2)}$.

This leads to the following description of the cohomological dimension of a Hopf algebra.

**Proposition 2.2.** Let $A$ be a Hopf algebra. We have

$$\text{cd}(A) = \sup \{n : \text{Ext}^*_A(\mathbb{C}_\epsilon, M) \neq 0 \text{ for some } A \text{- module } M \}$$

$$= \inf \{n : \text{Ext}^i_A(\mathbb{C}_\epsilon, -) = 0 \text{ for } i > n \}$$

$$= \inf \{n : \mathbb{C}_\epsilon \text{ admits a projective resolution of length } n \}$$

**Proof.** The previous isomorphism ensures that

$$\text{cd}(A) \leq \sup \{n : \text{Ext}^i_A(\mathbb{C}_\epsilon, M) \neq 0 \text{ for some } A \text{- module } M \}$$

If $V$ is a right $A$-module, let $\cdot V$ be the $A$-bimodule whose right structure is that of $V$ and whose left structure is trivial, i.e. given by $\epsilon$. Then $(\cdot V)' = V$, hence the converse inequality holds, and the first equality in the statement is proved, as well as the second one. The last one is shown similarly as in the case of group cohomology, see e.g. \cite{13} Chapter VIII, Lemma 2.1. \hfill $\square$

**Examples 2.3.**

1. If $G$ is a linear algebraic group, with coordinate algebra $\mathcal{O}(G)$, it is well-known that $\text{cd}(\mathcal{O}(G)) = \dim G$.

2. If $\Gamma$ is a (discrete) group, then $\text{cd}(\mathcal{C}\Gamma) = \text{cd}_C(\Gamma)$, the cohomological dimension of $\Gamma$ with coefficients $\mathbb{C}$. We have $\text{cd}(\mathcal{C}\Gamma) = 0$ if and only if $\Gamma$ is finite, and if $\Gamma$ is finitely generated, then $\text{cd}(\mathcal{C}\Gamma) = 1$ if and only if $\Gamma$ contains a free normal subgroup of finite index, see \cite{21}.

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(3) If $A$ is a finite-dimensional Hopf algebra, then either $\text{cd}(A) = 0$ (when $A$ is semisimple) or $\text{cd}(A) = \infty$, a finite-dimensional Hopf algebra being Frobenius and hence self-injective.

2.3. **Exact sequences of Hopf algebras.** A sequence of Hopf algebra maps

$$C \to B \xrightarrow{i} A \xrightarrow{p} L \to C$$

is said to be exact [3] if the following conditions hold:

1. $i$ is injective, $p$ is surjective and $pi = \varepsilon 1$,
2. $\ker p = Ai(B)^+ = i(B)^+ A$, where $i(B)^+ = i(B) \cap \text{Ker}(\varepsilon)$,
3. $i(B) = A^{\text{cop}} = \{a \in A : (\text{id} \otimes p)\Delta(a) = a \otimes 1\} = A^{\text{cop}} = \{a \in A : (p \otimes \text{id})\Delta(a) = 1 \otimes a\}$.

**Proposition 2.4.** Let

$$C \to B \xrightarrow{i} A \xrightarrow{p} L \to C$$

be a sequence Hopf algebra maps $i$ is injective, $p$ is surjective and $pi = \varepsilon 1$. Assume that the antipode of $A$ is bijective. Consider the following three assertions.

1. $A$ is faithfully flat as a right $B$-module and $\text{Ker}(p) = Ai(B)^+ = i(B)^+ A$.
2. $A^{\text{cop}} = A^{\text{cop}} = i(B)$ and $p$ is left or right faithfully coflat ($p$ being automatically faithfully coflat if $L$ is cosemisimple).
3. The sequence is exact.

Then we have $(1) \Rightarrow (3)$ and $(2) \Rightarrow (3)$, and if (3) holds, then we have $(1) \iff (2)$.

An exact sequence satisfying (1) and (2) is called strict [47].

That $(1) \Rightarrow (3)$ holds is well-known (see [3, Proposition 1.2.4], [46, Lemma 1.3], [37, Proposition 3.4.3], or more generally [53, Theorem 1]). Also that $(1) \iff (2)$ if (3) holds is known, see [46, Corollary 1.8]. On the other hand that $(2) \Rightarrow (3)$ holds seems to require more (or different) axioms on $p$ in the references we are aware of (e.g. Proposition 1.2.13 in [3]). As this fact is useful, we develop a proof, which is essentially dual to the proof of $(1) \Rightarrow (3)$.

First recall that a surjective coalgebra map $p : C \to D$ is said to be right faithfully coflat if the functor

$$M_D \to M_C$$

$$V \mapsto V \boxtimes DC$$

is fully exact, i.e. preserves exact sequences and creates them. Here $C$ has the left $D$-comodule structure induced by $p$, and the symbol $\boxtimes D$ stands for the cotensor product of a right $D$-comodule by a left one. One also says that $C$ is (right) $D$-faithfully coflat. There is a similar definition for left faithful coflatness.

**Lemma 2.5.** Let $p : C \to D$ be a (left or right) faithfully coflat coalgebra map. Then the sequence

$$C \boxtimes DC \xrightarrow{\varepsilon \otimes \text{id} - \text{id} \otimes \varepsilon} C \xrightarrow{p} D \to 0$$

is exact.

**Proof.** Put $\phi = \varepsilon \otimes \text{id} - \text{id} \otimes \varepsilon$. The sequence in the statement consists of left and right $D$-colinear maps. First assume right faithful coflatness. By faithful coflatness, it is enough to show that the sequence

$$(C \boxtimes DC) \boxtimes DC \xrightarrow{\varepsilon \otimes \text{id}} C \boxtimes DC \xrightarrow{p \otimes \text{id}} D \boxtimes DC \to 0$$

is exact. It is clear that $p \phi = 0$, and conversely let $\sum x_i \otimes y_i \in \text{Ker}(p \otimes \text{id})$. We have

$$\phi \otimes \text{id}(-\sum x_i \otimes y_{i(1)} \otimes y_{i(2)}) = \sum x_i \otimes y_i - \sum \varepsilon(x_i)y_{i(1)} \otimes y_{i(2)} = \sum x_i \otimes y_i$$

and this gives the result since $\sum x_i \otimes y_{i(1)} \otimes y_{i(2)} \in (C \boxtimes DC) \boxtimes DC$ if $\sum x_i \otimes y_i \in C \boxtimes DC$. A similar proof works with left faithful coflatness.

The next lemma is similar to Remark 1.6 in [47], but we do not assume here that the Hopf ideal $I = \text{Ker}(p)$ is normal.
Lemma 2.6. Let \( p : A \to L \) be a surjective Hopf algebra map with \(^{co}P A = A^{co}P =: B \). Then the map

\[
A \otimes B \longrightarrow A \square_L A \\
x \otimes y \longmapsto x(1) \otimes x(2)y
\]

is a bijection.

Proof. It is a straightforward verification to check that the map

\[
A \square_L A \longrightarrow A \otimes B \\
\sum x_i \otimes y_i \longmapsto \sum x_i(1) \otimes S(x_i(2))y_i
\]

is inverse to the given map. \( \square \)

The proof of (2) \( \Rightarrow \) (3) in Proposition 2.4 is then a direct consequence of the following result.

Proposition 2.7. Let \( p : A \to L \) be a surjective Hopf algebra map such that \(^{co}P A = A^{co}P =: B \) and that \( p \) is left or right faithfully coflat. Then \( \ker(p) = B^+ A = AB^+ \).

Proof. We have a commutative diagram

\[
\begin{array}{ccc}
A \otimes B & \xrightarrow{m \otimes \varepsilon} & A \xrightarrow{q} A/AB^+ \\
\downarrow & & \downarrow \\
A \square_L A & \xrightarrow{\varepsilon \otimes \id \otimes \varepsilon} & A \xrightarrow{p} L
\end{array}
\]

where the left vertical map is the isomorphism in the previous lemma, \( q \) is the canonical map and the vertical map on the right is well-defined since \( AB^+ \subset \ker(p) \). The top sequence is exact, as well as the low one by the first lemma. Thus the right vertical map is an isomorphism and we have \( AB^+ = \ker(p) \). We already can conclude if we assume bijectivity of the antipodes. Otherwise we consider the diagram

\[
\begin{array}{ccc}
B \otimes A & \xrightarrow{\varepsilon \otimes \id \otimes \varepsilon} & A \xrightarrow{q} A/B^+ A \\
\downarrow & & \downarrow \\
A \square_L A & \xrightarrow{\varepsilon \otimes \id \otimes \varepsilon} & A \xrightarrow{p} L
\end{array}
\]

with the left isomorphism

\[
B \otimes A \longrightarrow A \square_L A \\
b \otimes a \longmapsto ba(1) \otimes a(2)
\]

to conclude that \( B^+ A = \ker(p) \). \( \square \)

Remark 2.8. Proposition 2.7 is similar to Proposition 1.2.13 in [3], but without the coadjoint coaction condition. A proof of Proposition 2.7 is provided in the cosemisimple case in [56].

3. Cohomological dimension of a Hopf subalgebra

In this section we discuss the behavior of cohomological dimension when passing to a Hopf subalgebra, which, under mild assumptions, is similar to the group cohomology case.

Proposition 3.1. Let \( B \subset A \) be a Hopf subalgebra. Assume that one of the following conditions holds.

1. \( A \) is projective as a right \( B \)-module.
2. The antipode of \( A \) is bijective and \( A \) is faithfully flat as a right \( B \)-module.
3. \( A \) is cosemisimple.
4. There exists a Hopf algebra map \( \pi : A \to B \) such that \( \pi|_B = \id_B \).
(5) The antipode of $A$ is bijective and $B$ is commutative. Then $\text{cd}(B) \leq \text{cd}(A)$.

**Proof.** If $A$ is projective as a right $B$-module, any projective right $A$-module is projective as a right $B$-module, thus a resolution of length $n$ of $C = \mathcal{M}_A$ yields a resolution of length $n$ in $\mathcal{M}_B$, and thus Proposition 2.2 ensures that $\text{cd}(B) \leq \text{cd}(A)$. Assuming (2), Corollary 1.8 in [46] yields that $A$ is projective as a right $B$-module, and we conclude by (1). If we assume that $A$ is cosemisimple, then its antipode is bijective and by [16] $A$ is faithfully flat as a right $B$-module, and we conclude by (2). If we assume (4), then $A$ is free as a right $B$-module, see [42] (we will come back to this situation in Section 7), thus we conclude by (1). If $B$ is commutative, then $A$ is faithfully flat over $B$ by Proposition 3.12 in [4], and again we conclude by (2). □

The following result is the generalization of the sub-additivity of cohomological dimension under extensions (see e.g. Proposition 2.4 in [14]) with essentially the same proof, using Stefan’s spectral sequence [49] as the natural generalization of the Hochschild-Serre spectral sequence.

**Proposition 3.2.** Let

$$\mathbb{C} \rightarrow B \xrightarrow{i} A \xrightarrow{p} L \rightarrow \mathbb{C}$$

be a strict exact sequence of Hopf algebras, and assume that the antipode of $A$ is bijective. Then we have $\text{cd}(B) \leq \text{cd}(A) \leq \text{cd}(B) + \text{cd}(L)$. If moreover $L$ is semisimple, then $\text{cd}(B) = \text{cd}(A)$.

**Proof.** By [46, Lemma 1.3], (or more generally [53, Theorem 1], see also [37, Proposition 3.4.3]), the canonical map

$$A \otimes_B A \rightarrow A \otimes L$$

$$a \otimes_B a' \mapsto a a'_{(1)} \otimes p(a'_{(2)})$$

is bijective. Thus $B \subset A$ is an $L$-Galois extension, and $A$ is faithfully flat both as a left and right $B$-module (the antipode of $A$ is bijective). Thus for any $A$-$A$-bimodule $M$ there exists a spectral sequence [49]

$$E_2^{pq} = H^p(L; H^q(B, M)) \Rightarrow H^{p+q}(A, M)$$

The spectral sequence is concentrated in the rectangle $0 \leq p \leq \text{cd}(L), 0 \leq q \leq \text{cd}(B)$, and it follows that for $i > \text{cd}(L) + \text{cd}(B)$, we have $H^i(A, M) = 0$, and this proves the inequality. If $L$ is semisimple, then $\text{cd}(L) = 0$, and hence $\text{cd}(B) = \text{cd}(A)$. □

4. **Yetter-Drinfeld modules**

Let $A$ be a Hopf algebra. Recall that a (right-right) Yetter-Drinfeld module over $A$ is a right $A$-comodule and right $A$-module $V$ satisfying the condition, $\forall v \in V$, $\forall a \in A$,

$$(v \leftarrow a)_{(0)} \otimes (v \leftarrow a)_{(1)} = v_{(0)} \leftarrow a_{(2)} \otimes S(a_{(1)})v_{(1)}a_{(3)}$$

The category of Yetter-Drinfeld modules over $A$ is denoted $\mathcal{YD}^A_1$: the morphisms are the $A$-linear $A$-colinear maps. Endowed with the usual tensor product of modules and comodules, it is a tensor category, with unit the trivial Yetter-Drinfeld module, denoted $\mathbb{C}$.

An important example of Yetter-Drinfeld module is the right coadjoint Yetter-Drinfeld module $A_\text{coad}$: as a right $A$-module $A_\text{coad} = A$ and the right $A$-comodule structure is defined by

$$\text{ad}_r(a) = a_{(2)} \otimes S(a_{(1)})a_{(3)}, \forall a \in A$$

The coadjoint Yetter-Drinfeld module has a natural generalization, discussed in the next subsection.
4.1. Free Yetter-Drinfeld modules. Let $V$ be a right $A$-comodule. The Yetter-Drinfeld module $V \boxtimes A$, defined as follows in \([10]\). As a vector space $V \boxtimes A = V \otimes A$, the right module structure is given by multiplication on the right, and the right coaction $\alpha_{V \boxtimes A}$ is defined by

$$\alpha_{V \boxtimes A}(v \otimes a) = v(0) \otimes a(2) \otimes S(a(1))v(1)a(3)$$

Note that $A_{\text{coid}} = C \boxtimes A$.

A Yetter-Drinfeld module is said to be free if it is isomorphic to $V \boxtimes A$ for some comodule $V$.

The construction of the free Yetter-Drinfeld module on a comodule yields a functor $L = - \boxtimes A : \mathcal{M}^A \rightarrow \mathcal{YD}_A^A$ which is left adjoint to the forgetful functor $R : \mathcal{YD}_A^A \rightarrow \mathcal{M}^A$ (a left-right version of the functor $L$ was first given in \([15]\)). Indeed we have natural isomorphisms

$$\text{Hom}^A(V, R(\alpha)) \rightarrow \text{Hom}_{\mathcal{YD}_A^A}(V \boxtimes A, X)$$

$$f \mapsto \tilde{f}, \quad \tilde{f}(v \otimes a) = f(v) \leftarrow a$$

for any $A$-comodule $V$ and any Yetter-Drinfeld module $X$.

4.2. Relative projective Yetter-Drinfeld modules. We will use the following notion.

**Definition 4.1.** A Yetter-Drinfeld module $P$ over $A$ is said to be relative projective if the functor $\text{Hom}_{\mathcal{YD}_A^A}(P, -)$ transforms exact sequences of Yetter-Drinfeld modules that split as sequences of comodules to exact sequences of vector spaces.

**Proposition 4.2.** Let $P$ be a Yetter-Drinfeld module over $A$. The following assertions are equivalent.

1. $P$ is relative projective.
2. Any surjective morphism of Yetter-Drinfeld module $f : M \rightarrow P$ that admit a section which is a map of comodules admits a section which is a morphism of Yetter-Drinfeld modules.
3. $P$ is a direct summand of a free Yetter-Drinfeld module.

If $A$ is cosemisimple, these conditions are equivalent to $P$ being a projective object of $\mathcal{YD}_A^A$.

**Proof.** The proof of $(1) \Rightarrow (2)$ is similar to the usual one for modules. Assume (2), and consider the surjective Yetter-Drinfeld module morphism $R(P) \boxtimes A \rightarrow P$, $x \otimes a \mapsto x \leftarrow a$. The map $P \rightarrow R(P) \boxtimes A$, $x \mapsto x \otimes 1$ is an $A$-colinear section, so by (2) $P$ is indeed, as a Yetter-Drinfeld module, a direct summand of $R(P) \boxtimes A$.

Assume now that $P$ is free, i.e. $P = V \boxtimes A$ for some comodule $V$, and let

$$0 \rightarrow M \xrightarrow{\eta} N \xrightarrow{\epsilon} Q \rightarrow 0$$

be an exact sequence of Yetter-Drinfeld modules that splits as a sequence of comodules. The sequence

$$0 \rightarrow \text{Hom}_{\mathcal{YD}_A^A}(P, M) \xrightarrow{\iota} \text{Hom}_{\mathcal{YD}_A^A}(P, N) \xrightarrow{\epsilon} \text{Hom}_{\mathcal{YD}_A^A}(P, Q)$$

is exact and we have to show the surjectivity of the map on the right. Let $s : Q \rightarrow N$ be a morphism of comodules such that $s \phi = \text{id}_Q$. Let $\varphi \in \text{Hom}_{\mathcal{YD}_A^A}(V \boxtimes A, Q)$, and let $\varphi_0 : V \rightarrow Q$ be defined by $\varphi_0(v) = \varphi(v \otimes 1)$. $\varphi_0$ is a map of comodules, and so is $s \varphi_0$. Considering now $s \varphi_0 \in \text{Hom}_{\mathcal{YD}_A^A}(V \boxtimes A, N)$, we have $ps \varphi_0 = \varphi$, which gives the expected surjectivity result. Now if $V \boxtimes A \simeq P \oplus M$ as Yetter-Drinfeld modules, then $\text{Hom}_{\mathcal{YD}_A^A}(V \boxtimes A, -) \simeq \text{Hom}_{\mathcal{YD}_A^A}(P, -) \oplus \text{Hom}_{\mathcal{YD}_A^A}(M, -)$, and the usual argument for projective modules work to conclude that $P$ is relative projective.

It is clear that a projective Yetter-Drinfeld module is relative projective, and if $A$ is cosemisimple, a free Yetter-Drinfeld module is a projective object in $\mathcal{YD}_A^A$ (Proposition 3.3 in \([10]\)), hence a direct summand of a free Yetter-Drinfeld module is projective, and so is a relative projective Yetter-Drinfeld module. \(\square\)
4.3. Yetter-Drinfeld modules and Hopf bimodules. In this subsection we briefly recall the category equivalence between Yetter-Drinfeld modules and Hopf bimodules, and check that the notion of relative projective objects for Yetter-Drinfeld modules corresponds to that for Hopf bimodules considered in \[18\].

First recall that a Hopf bimodule over $A$ is an $A$-bimodule and $A$-bicomodule $M$ whose respective left and right coactions $\lambda : M \to A \otimes M$ and $\rho : M \to M \otimes A$ are $A$-bimodule maps. The category of Hopf bimodules over $A$, whose morphisms are the bimodule and bicomodule maps, is denoted $A_M^A$. If $M$ is Hopf bimodule over $A$, then $\text{co}^AM = \{ x \in M \mid \lambda(x) = 1 \otimes x \}$ is a right submodule of $M$, and inherits a right $A$-module structure given by $x \mapsto a = S(a_{(1)})x.a_{(2)}$, making it into a Yetter-Drinfeld module over $A$. This defines a functor

$$A_M^A \to \mathcal{YD}_A^A$$

$$M \mapsto \text{co}^AM$$

Conversely, starting from a Yetter-Drinfeld module $V$, one defines a Hopf bimodule structure on $A \otimes V$ as follows. The bimodule structure is given by

$$a.(b \otimes v).c = abc_{(1)} \otimes (v \leftarrow c_{(2)})$$

and the bicomodule structure is given by the following left and right coactions

$$\lambda : A \otimes V \to A \otimes A \otimes V \quad \rho : A \otimes V \to A \otimes V \otimes A$$

$$a \otimes v \mapsto a_{(1)} \otimes a_{(2)} \otimes v \quad a \otimes v \mapsto a_{(1)} \otimes v_a \otimes a_{(2)} v_{(1)}$$

If $f : V \to W$ is a morphism of Yetter-Drinfeld module, then $\text{id}_A \otimes f : A \otimes V \to A \otimes W$ is a morphism of Hopf bimodules, and hence we get a functor

$$\mathcal{YD}_A^A \to A_M^A$$

$$V \mapsto A \otimes V$$

The two functors just defined are quasi-inverses equivalences.

**Lemma 4.3.** Relative projective objects in $\mathcal{YD}_A^A$ correspond, via the category equivalence $\mathcal{YD}_A^A \simeq A_M^A$, to relative projective objects of $A_M^A$ in the sense of [18].

**Proof.** Let $M$ be a Hopf bimodule over $A$. That $M$ is relatively projective means that the functor $\text{Hom}_{A_M^A}(M, -)$ transforms exact sequences of Hopf bimodules that split as sequences of bicomodules to exact sequences of vector spaces. The proof of the lemma easily reduces to the following statement.

Let $f : V \to W$ be a surjective morphism of Yetter-Drinfeld modules, inducing a surjective morphism of Hopf bimodules $\text{id}_A \otimes f : A \otimes V \to A \otimes W$. Then there exists an $A$-comodule section to $f$ if and only if there exists an $A$-bicomodule section to $\text{id}_A \otimes f$.

Indeed, if $s : W \to V$ is $A$-colinear with $fs = \text{id}_W$, then $\text{id}_A \otimes s : A \otimes W \to A \otimes V$ is $A$-bicolinear and is a section to $\text{id}_A \otimes f$. Conversely starting with an $A$-bicolinear map $T : A \otimes W \to A \otimes V$ with $(\text{id}_A \otimes f)T = \text{id}_A \otimes W$, then the map $s : W \to V$ defined by $s(w) = \varepsilon \otimes \text{id}_V(T(1 \otimes w))$ is $A$-colinear, and satisfies $fs = \text{id}_W$. \[\Box\]

4.4. Adjoint Hopf subalgebras. We now discuss the way to restrict certain free Yetter-Drinfeld to adjoint Hopf subalgebras.

**Proposition 4.4.** Let $B \subset A$ be a Hopf subalgebra. The following assertions are equivalent.

1. For any $a \in A$ and $b \in B$, we have

$$a_{(2)} \otimes S(a_{(1)})ba_{(3)} \in A \otimes B$$

2. For any $B$-comodule $W$, we have $\alpha_{V \otimes B}(W \boxtimes A) \subset (W \boxtimes A) \otimes B$ so that $W \boxtimes A$ is an object of $\mathcal{YD}_B^B$. 

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Proof. (1) ⇒ (2) follows from the definition of $\alpha_{V\otimes A}$. Conversely, assuming that (2) holds, take $W = B$ the regular $B$-comodule. Then for any $a \in A$ and $b \in B$, we have
\[
b_{(1)} \otimes a_{(2)} \otimes S(a_{(1)})b_{(2)}a_{(3)} \in A \otimes A \otimes B
\]
and hence
\[
a_{(2)} \otimes S(a_{(1)})ba_{(3)} = a_{(2)} \otimes S(a_{(1)})\varepsilon(b_{(1)})b_{(2)}a_{(3)} \in A \otimes B
\]
Thus (1) holds.

**Definition 4.5.** A Hopf subalgebra $B \subset A$ is said to adjoint if it satisfies the equivalent conditions of Proposition 4.4.

Very often adjoint Hopf subalgebras are obtained in the following way. Recall that a Hopf algebra map $f : A \to L$ is said to be cocentral if $f(a_{(1)}) \otimes a_{(2)} = f(a_{(2)}) \otimes a_{(1)}$ for any $a \in A$.

**Proposition 4.6.** Let $B \subset A$ be a Hopf subalgebra. Assume that there exists a cocentral and surjective Hopf algebra map $p : A \to L$ such that $B = A^\text{cop}$. Then $B \subset A$ is an adjoint Hopf subalgebra. Conversely, if $B \subset A$ is an adjoint Hopf subalgebra, if $A$ and $B$ have bijective antipodes and if $A$ is faithfully flat as a right $B$-module, then there exists a cocentral surjective Hopf algebra map $p : A \to L$ such that $B = A^\text{cop}$.

**Proof.** Let $a \in A$ and $b \in B$. Since $p(b) = \varepsilon(b)1$, we have, using the cocentrality of $p$,
\[
id_A \otimes id_B \otimes p (a_{(2)} \otimes (S(a_{(1)})ba_{(3)}))_{(1)} \otimes (S(a_{(1)})ba_{(3)}))_{(2)}
\]
\[
= id_A \otimes id_B \otimes p (a_{(3)} \otimes S(a_{(2)})b_{(1)}a_{(4)} \otimes S(a_{(1)})b_{(2)}a_{(5)})
\]
\[
= a_{(3)} \otimes S(a_{(2)})b_{(1)}a_{(4)} \otimes ps(a_{(1)})p(b_{(2)})p(a_{(5)})
\]
\[
= a_{(3)} \otimes S(a_{(2)})ba_{(4)} \otimes ps(a_{(1)})p(a_{(5)})
\]
\[
= a_{(2)} \otimes S(a_{(1)})ba_{(3)} \otimes 1
\]
Hence $a_{(2)} \otimes S(a_{(1)})ba_{(3)} \in A \otimes A^\text{cop} = A \otimes B$: this shows that $B \subset A$ is adjoint.

Conversely, assume that $B \subset A$ is adjoint, that $A$ and $B$ have bijective antipodes and that $A$ is faithfully flat as a right $B$-module. Then for any $a \in A$ and $b \in B$, we have
\[
S(a_{(1)})ba_{(3)} = \varepsilon(a_{(2)})\varepsilon(b_{(1)})S(a_{(1)})b_{(2)}a_{(3)} \in B
\]
It is well-known that this implies $B^+A \subset AB^+$, and hence $AB^+ \subset B^+A$ by the bijectivity of the antipodes. It follows that $B^+A$ is a Hopf ideal in $A$, and we denote by $p : A \to A/B^+A = L$ the canonical Hopf algebra surjection. By construction we have $B \subset A^\text{cop}$, and for $b \in B$ we have $p(b) = \varepsilon(b)$. Hence we have for any $a \in A$, $a \otimes 1 = a_{(2)} \otimes p(S(a_{(1)})a_{(3)})$, hence
\[
a_{(2)} \otimes p(a_{(1)}) = (1 \otimes p(a_{(1)}))(a_{(2)} \otimes 1) = (1 \otimes p(a_{(1)})(a_{(3)} \otimes p(S(a_{(2)})a_{(4)})) = a_{(1)} \otimes p(a_{(2)})
\]
and this shows that $p$ is cocommutative. Finally we have $B = A^\text{cop}$ by Corollary 1.8 in [46].

We now discuss a condition that ensures that the restriction of a free Yetter-Drinfeld module to an adjoint Hopf subalgebra as in Proposition 4.4 remains a relative projective Yetter-Drinfeld module.

**Proposition 4.7.** Let $B \subset A$ be a Hopf subalgebra with $B = A^\text{cop}$ for some cocentral and surjective Hopf algebra map $p : A \to L$. Assume that there exists a linear map $\sigma : L \to A$ such that
\begin{enumerate}
  
  \item $p\sigma = id_L$;
  
  \item $\sigma(x)_{(1)} \otimes p(\sigma(x)_{(2)}) = \sigma(x_{(1)}) \otimes x_{(2)}$, for any $x \in L$;
  
  \item $\sigma(x)_{(1)}S(\sigma(x)_{(3)}) \otimes \sigma(x)_{(2)} = 1_B \otimes \sigma(x)$, for any $x \in L$.
\end{enumerate}
Then for any $B$-comodule $W$, the object $W \otimes A \in \mathcal{YD}_B$ is relative projective. Such a map $\sigma$ exists if $A$ is cosemisimple.
Proof. We first claim that for any \(a \in A\), we have
\[
\sigma p(a(1))_1 \otimes S(\sigma p(a(1))_2) a(2) \in A \otimes B
\]
For any \(x \in L\), we have, by (2)
\[
\sigma(x)_{(1)} \otimes \sigma(x)_{(2)} \otimes p(\sigma(x)_{(3)}) = \sigma(x)_{(1)} \otimes \sigma(x)_{(2)} \otimes x_{(2)}
\]
and hence for any \(a \in A\)
\[
\sigma p(a(1))_1 \otimes \sigma p(a(2)) \otimes p(\sigma p(a(3))_{(3)}) = \sigma p(a(1))_1 \otimes \sigma p(a(1))_2 \otimes p(a(2))
\]
We have
\[
(id_A \otimes p \otimes id_A)(id_A \otimes \Delta) (\sigma p(a(1))_1 \otimes S(\sigma p(a(1))_2) a(2))
= \sigma p(a(1))_1 \otimes S(\sigma p(a(1))_3) p(a(2)) \otimes S(\sigma p(a(1))_2) a(3)
= \sigma p(a(1))_1 \otimes \sigma p(a(2)) p(a(3)) \otimes S(\sigma p(a(1))_2) a(4)
= \sigma p(a(1))_1 \otimes 1 \otimes S(\sigma p(a(1))_2) a(2)
\]
and this proves our claim.

We thus get for any \(B\)-comodule \(W\), a linear map
\[
\iota : W \boxtimes A \to (W \boxtimes A) \boxtimes B
\]
\[w \otimes a \mapsto w \otimes \sigma p(a(1))_1 \otimes S(\sigma p(a(1))_2) a(2)\]
that we claim to be a morphism of Yetter-Drinfeld modules over \(B\). That \(\iota\) is a left \(B\)-module map is easily checked. Denoting by \(\beta\) the \(B\)-coaction on \((W \boxtimes A) \boxtimes B\), we have
\[
\beta \iota(w \otimes a) = w(0) \otimes \sigma p(a(1))_2 \otimes S(\sigma p(a(1))_5) a(3) \otimes S(\sigma p(a(1))_6) w(1) a(2) \otimes S(\sigma p(a(1))_3) a(3) \otimes S(\sigma p(a(1))_4) a(4)
= w(0) \otimes \sigma p(a(1))_2 \otimes S(\sigma p(a(1))_3) a(3) \otimes S(\sigma p(a(1))_4) a(2) \otimes S(\sigma p(a(1))_1) w(1) a(4)
= w(0) \otimes \sigma p(a(1))_2 \otimes S(\sigma p(a(1))_3) a(3) \otimes S(\sigma p(a(1))_1) S(\sigma p(a(1))_4) w(1) a(4)
\]
By (3), for \(x \in L\), we have
\[
\sigma(x)_{(2)} \otimes \sigma(x)_{(1)} S(\sigma(x)_{(3)}) = \sigma(x) \otimes 1_B
\]
and hence
\[
\sigma(x)_{(2)} \otimes S(\sigma(x)_{(3)}) \otimes \sigma(x)_{(1)} S(\sigma(x)_{(4)}) = \sigma(x)_{(1)} \otimes S(\sigma(x)_{(2)}) \otimes 1_B
\]
Thus
\[
\beta \iota(w \otimes a) = w(0) \otimes \sigma p(a(1))_1 \otimes S(\sigma p(a(1))_2) a(3) \otimes S(\sigma p(a(1))_2) w(1) a(4)
\]
Now let \(\gamma\) be the \(B\)-coaction on \(W \boxtimes A\). We have
\[
(\iota \otimes id_B) \gamma(w \otimes a) = \iota \otimes id_B(w(0) \otimes a(2) \otimes S(\sigma p(a(1))_2) w(1) a(3))
= w(0) \otimes \sigma p(a(2))_1 \otimes S(\sigma p(a(2))_2) a(3) \otimes S(\sigma p(a(2))_1) w(1) a(4)
= w(0) \otimes \sigma p(a(1))_1 \otimes S(\sigma p(a(1))_2) a(3) \otimes S(\sigma p(a(1))_2) w(1) a(4) = \beta \iota(w \otimes a)
\]
where we have used the cocentrality of \(p\). It follows that \(\iota\) is \(B\)-colinear, and hence is a morphism of Yetter-Drinfeld modules over \(B\). Consider now
\[
\mu : (W \boxtimes A) \boxtimes B \to W \boxtimes A
\]
\[w \otimes a \otimes b \mapsto w \otimes ab\]
It is straightforward to check that \(\mu\) is a morphism of Yetter-Drinfeld modules over \(B\), with \(\mu_\iota = id_{W \boxtimes A}\) and hence we conclude from Proposition 4.2 that \(W \boxtimes A\) is a relative projective Yetter-Drinfeld module over \(B\).
For the last assertion, note that $L$ and $A$ both admit right $B^{\text{cop}} \otimes L$-comodule structures given by

$$L \to L \otimes (B^{\text{cop}} \otimes L), \quad A \to A \otimes (B^{\text{cop}} \otimes L)$$

$$x \mapsto x^{(1)} \otimes 1 \otimes x^{(2)}, \quad a \mapsto a^{(2)} \otimes a^{(1)} S(a^{(3)}) \otimes p(a^{(4)})$$

and that $p$ is $B^{\text{cop}} \otimes L$-colinear. If $A$ is cosemisimple then so is $B$ and so is $L$ (since $p$ is cocentral), hence $B^{\text{cop}} \otimes L$ is cosemisimple. Thus there exists a $B^{\text{cop}} \otimes L$-colinear section to $p$, which satisfies our 3 conditions. \hfill \square

There are also situations where the Hopf algebra in the proposition is not cosemisimple and the map $\sigma$ still exists, see Section 6.

5. Gerstenhaber-Schack cohomology.

5.1. Generalities. Let $A$ be a Hopf algebra and let $V$ be a Yetter-Drinfeld module over $A$. The Gerstenhaber-Schack cohomology of $A$ with coefficients in $V$, that we denote $H^*_{GS}(A,V)$, was introduced in [23, 24] by using an explicit bicomplex. In fact Gerstenhaber-Schack used Hopf bimodules instead of Yetter-Drinfeld modules to define their cohomology, but in view of the equivalence between Hopf bimodules and Yetter-Drinfeld modules, we shall work with the simpler framework of Yetter-Drinfeld modules (a Yetter-Drinfeld version of the Gerstenhaber-Schack bicomplex is provided in [39]). A special instance of Gerstenhaber-Schack cohomology is bialgebra cohomology, given by $H^*_{\text{GS}}(A) = H^*_{\text{GS}}(A,\mathbb{C})$.

As an example, we have by [40], $H^*_\text{GS}(\Gamma) \simeq H^*(\Gamma,\mathbb{C})$ for any discrete group $\Gamma$. The bialgebra cohomology of $O(G)$ for a connected reductive algebraic group $G$ is also described in [40], Theorem 9.2, and some finite-dimensional examples are computed in [52]. Applications to deformations of pointed Hopf algebras are given in [56].

A key result, due to Taillefer [51], shows that Gerstenhaber-Schack cohomology is in fact an Ext-functor:

$$H^*_\text{GS}(A,V) \simeq \text{Ext}^*_\text{YD}(A,C,V)$$

We will use this description as a definition.

**Definition 5.1.** The Gerstenhaber-Schack cohomological dimension of a Hopf algebra $A$ is defined to be

$$\text{cd}_{\text{GS}}(A) = \sup\{ n : H^n_{\text{GS}}(A,V) \neq 0 \text{ for some } V \in \text{YD}^A \} \in \mathbb{N} \cup \{ \infty \}$$

If $A$ and $B$ are Hopf algebras having equivalent tensor categories of comodules, then the given tensor equivalence $F : \mathcal{M}^A \simeq \otimes \mathcal{M}^B$ induces a tensor equivalence $\hat{F} : \text{YD}^A \simeq \otimes \text{YD}^B$ (see e.g. [11, 10]), this is easy to see thanks to the description of the category of Yetter-Drinfeld modules as the weak center of the category of comodules, see [45]). Hence we get, for any Yetter-Drinfeld module $V$ over $A$, an isomorphism

$$H^*_\text{GS}(A,V) \simeq H^*_\text{GS}(B,\hat{F}(V))$$

and moreover $\text{cd}_{\text{GS}}(A) = \text{cd}_{\text{GS}}(B)$. These properties are what we call the monoidal invariance of Gerstenhaber-Schack cohomology.

5.2. Co-Frobenius Hopf algebras. Recall that a Hopf algebra $A$ is said to be co-Frobenius if there exists a non-zero $A$-colinear map $A \to \mathbb{C}$. By [34], $A$ is co-Frobenius if and only if the category $\mathcal{M}^A$ of right comodules has enough projectives. Finite-dimensional Hopf algebras are co-Frobenius as well as cosemisimple Hopf algebra. See [1] for more examples.

The following result, relying on [48], Proposition 10.5.3, will be a key tool.

**Theorem 5.2.** Let $A$ be a co-Frobenius Hopf algebra and let

$$P = \cdots P_{n+1} \to P_n \to \cdots \to P_1 \to P_0 \to 0$$
be a resolution of \( C \) by relative projective objects of \( YD^A_\cdot \). We have, for any Yetter-Drinfeld module \( V \) over \( A \), an isomorphism

\[
H^*_\text{GS}(A, V) \cong H^*(\text{Hom}_{YD^A}(P, V))
\]

**Proof.** We know, since \( A \) is co-Frobenius, that \( YD^A_\cdot \) has enough projective objects (Corollary 3.4 in [10]). Thus the description of \( H^*_\text{GS}(A, -) \) as an Ext functor \([51]\) yields that if \( Q \) is a a resolution of \( C \) by projective objects of \( YD^A_\cdot \), we have

\[
H^*_\text{GS}(A, V) \cong H^*(\text{Hom}_{YD^A}(Q, V))
\]

for any Yetter-Drinfeld module \( V \). Moreover by \([45]\), Proposition 10.5.3, and Lemma \([43]\) two relative projective resolutions are homotopy equivalent as complexes of Yetter-Drinfeld modules. Thus if we start with an arbitrary resolution \( P \) of \( C \) by relative projective objects, it is homotopy equivalent with an arbitrary resolution \( Q \) of \( C \) by projective objects (such a resolution exists by our assumption), and we get \( H^*(\text{Hom}_{YD^A}(Q, V)) \cong H^*(\text{Hom}_{YD^A}(P, V)) \), which proves the result. \( \square \)

As a first application, we have the following result, whose proof is similar to the one for group cohomology, see \([14]\) Chapter VIII, Lemma 2.1].

**Corollary 5.3.** Let \( A \) be a co-Frobenius Hopf algebra. Then we have

\[
cd_{\text{GS}}(A) = \inf\{n : C \text{ admits a projective resolution of length } n \text{ in } YD^A_\cdot\}
\]

\[
= \inf\{n : C \text{ admits a relative projective resolution of length } n \text{ in } YD^A_\cdot\}
\]

We now use Theorem 5.2 to get an explicit complex to describe Gerstenhaber-Schack cohomology in the co-Frobenius case. Recall \([10]\) that for any \( n \in \mathbb{N} \), the Yetter-Drinfeld module \( A^\otimes n \) is defined as follows:

\[
A^\otimes 0 = C, \quad A^\otimes 1 = C \otimes A = \text{coad}, \quad A^\otimes 2 = A^\otimes 1 \otimes A, \quad \ldots, \quad A^\otimes (n+1) = A^\otimes n \otimes A, \ldots
\]

After the obvious vector space identification of \( A^\otimes n \) with \( A^\otimes n \), the right \( A \)-module structure of \( A^\otimes n \) is given by right multiplication and its comodule structure is given by

\[
\text{ad}^{(n)} : A^\otimes n \longrightarrow A^\otimes n \otimes A
\]

\[
a_1 \otimes \cdots \otimes a_n \longmapsto a_1(2) \otimes \cdots \otimes a_n(2) \otimes S(a_1(1) \cdots a_n(1))a_1(3) \cdots a_n(3)
\]

**Theorem 5.4.** Let \( A \) be a co-Frobenius Hopf algebra and let \( V \) be a Yetter-Drinfeld module over \( A \). The Gerstenhaber-Schack cohomology \( H^*_\text{GS}(A, V) \) is the cohomology of the complex

\[
0 \longrightarrow \text{Hom}^0(A, C, V) \xrightarrow{\partial} \text{Hom}^0(A^\otimes 1, V) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \text{Hom}^0(A^\otimes n, V) \xrightarrow{\partial} \text{Hom}^0(A^\otimes n+1, V) \xrightarrow{\partial} \cdots
\]

where the differential \( \partial : \text{Hom}^0(A^\otimes n, V) \longrightarrow \text{Hom}^0(A^\otimes n+1, V) \) is given by

\[
(\partial f)(a_1 \otimes \cdots \otimes a_{n+1}) = \varepsilon(a_1)f(a_2 \otimes \cdots \otimes a_{n+1}) + \sum_{i=1}^{n} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1})
\]

\[
+ (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) \cdot a_{n+1}
\]

**Proof.** By \([10]\), Proposition 3.6, the standard resolution of \( C \varepsilon \) yields in a fact resolution of \( C \) by free Yetter-Drinfeld modules in the category \( YD^A_\cdot \)

\[
\ldots \longrightarrow A^\otimes n+1 \longrightarrow A^\otimes n \longrightarrow \cdots \longrightarrow A^\otimes 2 \longrightarrow A^\otimes 1 \longrightarrow 0
\]

where each differential is given by

\[
A^\otimes n+1 \longrightarrow A^\otimes n
\]

\[
a_1 \otimes \cdots \otimes a_{n+1} \longmapsto \varepsilon(a_1)a_2 \otimes \cdots \otimes a_{n+1} + \sum_{i=1}^{n} (-1)^i a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}
\]

Since free Yetter-Drinfeld modules are relative projective, we get, after standard identification using the fact that the free functor is left adjoint, the result by Theorem 5.2. \( \square \)
We are now ready to show that Gerstenhaber-Schack cohomology determines Hochschild cohomology in the case of co-Frobenius Hopf algebras.

**Theorem 5.5.** Let $A$ be a co-Frobenius Hopf algebra and let $M$ be an $A$-bimodule. Endow $M \otimes A$ with a Yetter-Drinfeld module structure defined by

$$m \otimes a \mapsto m \otimes a(1) \otimes a(2), \quad (m \otimes a) \leftrightarrow b = S(b(2)) \cdot m \cdot b(3) \otimes S(b(1)) \cdot a \otimes b(4), \quad a, b \in A, \quad m \in M$$

and denote by $\text{cd}(A)$ the resulting Yetter-Drinfeld module. Then we have an isomorphism

$$H^*(A, M) \simeq H^*_{GS}(A, M \# A)$$

In particular we have $\text{cd}(A) \leq \text{cd}_{GS}(A)$.

**Proof.** It is a direct verification to check that $M \# A$ is indeed a Yetter-Drinfeld module over $A$. Recall that since $H^*(A, M) \simeq \text{Ext}^*_{A}(\mathbb{C}, M')$ (see Section 2), the complex to compute $H^*(A, M)$ is

$$0 \rightarrow \text{Hom}(\mathbb{C}, M') \rightarrow \text{Hom}(A, M') \rightarrow \cdots \rightarrow \text{Hom}(A^\otimes n, M') \rightarrow \text{Hom}(A^\otimes n+1, M') \rightarrow \cdots$$

where the differential $\partial : \text{Hom}(A^\otimes n, M') \rightarrow \text{Hom}(A^\otimes n+1, M')$ is given by

$$\partial(f)(a_1 \otimes \cdots \otimes a_{n+1}) = \varepsilon(a_1) f(a_2 \otimes \cdots \otimes a_{n+1}) + \sum_{i=1}^{n} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1})$$

$$+ (-1)^{n+1} S(a_{n+1}(1)) \cdot f(a_1 \otimes \cdots \otimes a_n) \cdot a_{n+1}(2)$$

For all $n \geq 0$, we have linear isomorphisms

$$\text{Hom}^A(A^\otimes n, M \# A) \rightarrow \text{Hom}(A^\otimes n, M')$$

$$f \mapsto (\text{id}_M \otimes \varepsilon) f$$

For $f \in \text{Hom}(A^\otimes n, M \# A)$ and $a_1, \ldots, a_n \in A$, with $f(a_1 \otimes \cdots \otimes a_n) = \sum_i m_i \otimes b_i$, we have

$$\text{id}_M \otimes \varepsilon(f(a_1 \otimes \cdots \otimes a_n)) = a_{n+1}$$

$$= \text{id}_M \otimes \varepsilon \left( \sum_i S(a_{n+1}(2)) \cdot m_i \cdot a_{n+1}(3) \otimes S(a_{n+1}(1)) \cdot b_i \cdot a_{n+1}(4) \right)$$

$$= \sum_i \varepsilon(b_i) S(a_{n+1}(1)) \cdot m_i \cdot a_{n+1}(2)$$

$$= S(a_{n+1}(1)) \cdot ((\text{id}_M \otimes \varepsilon)(f(a_1 \otimes \cdots \otimes a_{n+1})) \cdot a_{n+1}(2)$$

From this computation it follows easily that the previous isomorphisms commute with the differentials (the one for Gerstenhaber-Schack cohomology being given by the complex of Theorem 5.4), and hence the complexes that define both cohomologies are isomorphic. □

We get the results announced in the introduction, providing a partial answer to Question 1.1.1.

**Corollary 5.6.** Let $A$ and $B$ be co-Frobenius Hopf algebras such that there exists an equivalence of linear tensor categories $M^A \simeq \otimes M^B$. Then there exists two functors

$$F_1 : A^M_A \rightarrow YD_B^B$$

and

$$F_2 : B^M_B \rightarrow YD_A^A$$

such that for any $A$-bimodule over $M$ and any $B$-bimodule $N$, we have

$$H^*(A, M) \simeq H^*_{GS}(B, F_1(M))$$

and

$$H^*(B, N) \simeq H^*_{GS}(A, F_2(N))$$

In particular we have $\max(\text{cd}(A), \text{cd}(B)) \leq \text{cd}_{GS}(A) = \text{cd}_{GS}(B)$.

**Proof.** The construction in the previous theorem clearly yields a functor $A^M_A \rightarrow YD_A^A$, that we compose with the functor $YD_A^A \rightarrow YD_B^B$ from the discussion at the end of subsection 5.1, to get the announced functor $F_1$, and similarly the functor $F_2$. The last claim follows immediately. □
We conclude the section by some considerations on Question 1.2, which asks if \( cd(A) = cd_{GS}(A) \) for any co-Frobenius Hopf algebra \( A \).

Our first remark is that Question 1.2 has indeed a positive answer in the finite-dimensional case: if \( A \) is semisimple, then it is cosemisimple by the Larson-Radford theorem \([33]\), and hence \( \mathcal{YD}_A^A \) is semisimple (since the Drinfeld double \( D(A) \) is then semisimple, see \([33]\)), so we have \( cd(A) = 0 = cd_{GS}(A) \). If \( A \) is not semisimple, then \( cd(A) = \infty = cd_{GS}(A) \). It thus follows that a positive answer to Question 1.2 would provide a natural infinite-dimensional generalization to the above mentioned Larson-Radford theorem.

The second remark is that the characteristic zero assumption is indeed necessary: if \( A \) is a finite-dimensional semisimple non-cosemisimple Hopf algebra, the base field being then necessarily of characteristic \( > 0 \) \([33]\), then \( cd(A) = 0 < cd_{GS}(A) = \infty \).

The last remark is that Question 1.2 has a positive answer in the following (rather trivial) case.

**Example 5.7.** We have \( cd(\mathbb{C}\Gamma) = cd_{GS}(\mathbb{C}\Gamma) \) for any discrete group \( \Gamma \).

**Proof.** Put \( A = \mathbb{C}\Gamma \). The Hopf algebra \( A \) is co-Frobenius since cosemisimple, and by Theorem 5.4 we have \( cd(A) \leq cd_{GS}(A) \). Any right \( A \)-module \( M \), endowed with the trivial coaction, becomes a Yetter-Drinfeld over \( A \) (since \( A \) is cocommutative), denoted \( M_1 \). If \( M = V \otimes A \) is free, then \( (V \otimes A)_1 \simeq V \boxtimes A \) (with \( V \) having the trivial \( A \)-comodule structure) is free as a Yetter-Drinfeld module. Similarly, if \( M \) is a projective \( A \)-module, then \( M_1 \) is projective as a Yetter-Drinfeld module (Proposition 5.2) and if \( f : M \rightarrow N \) is an \( A \)-linear map, then \( f : M_1 \rightarrow N_1 \) is a morphism of Yetter-Drinfeld modules. Hence a resolution of length \( n \) of \( \mathbb{C}\Gamma \) by projective \( A \)-modules yields a resolution of length \( n \) of \( \mathbb{C}\Gamma \) by projective Yetter-Drinfeld modules over \( A \). Hence by Corollary 5.3 we have \( cd_{GS}(A) \leq cd(A) \). \( \square \)

Of course this example is not useful in the situation of Question 1.1 since two group algebras having equivalent tensor categories of comodules are obviously isomorphic. More interesting examples are considered in the next section.

### 6. Application to quantum symmetry algebras

In this section we provide applications of the previous considerations to various quantum symmetry algebras.

**6.1. The universal Hopf algebra of a non-degenerate bilinear form.** Let \( E \in GL_n(\mathbb{C}) \). Recall that the algebra \( B(E) \) \([20]\) is presented by generators \((u_{ij})_{1 \leq i,j \leq n}\) and relations

\[
E^{-1}u^t E u = I_n = uE^{-1}u^t E,
\]

where \( u \) is the matrix \((u_{ij})_{1 \leq i,j \leq n}\). It has a Hopf algebra structure defined by

\[
\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}, \quad \varepsilon(u_{ij}) = \delta_{ij}, \quad S(u) = E^{-1}u^t E
\]

The Hopf algebra \( B(E) \) represents the quantum symmetry group of the bilinear form associated to the matrix \( E \). It can also be constructed as a quotient of the FRT bialgebra associated to Yang-Baxter operators constructed by Gurevich \([26]\). For the matrix

\[
E_q = \begin{pmatrix} 0 & 1 \\ -q^{-1} & 0 \end{pmatrix} \in GL_2(\mathbb{C})
\]

we have \( B(E_q) \cong O(SL_q(2)) \), and thus the Hopf algebras \( B(E) \) are natural generalizations of \( O(SL_q(2)) \). It is shown in \([9]\) that for \( q \in \mathbb{C}^* \) satisfying \( \text{tr}(E^{-1}E^t) = -q - q^{-1} \), the tensor categories of comodules over \( B(E) \) and \( O(SL_q(2)) \) are equivalent.

It was proved in \([10]\) that if \( n \geq 2 \), then \( cd(B(E)) = 3 \) (Theorem 6.1 and Proposition 6.4 in \([10]\), see e.g. \([27]\) for the case \( E = E_q \) and \([17]\) for the case \( E = I_n \)), and the bialgebra cohomology of \( B(E) \) was computed there in the cosemisimple case. We generalize this computation to arbitrary matrices. First we need the following probably well-known result.
Proposition 6.1. The Hopf algebra $B(E)$ is co-Frobenius for any $E \in \text{GL}_n(\mathbb{C})$.

Proof. The assertion at $n = 1$ is obvious, so we assume that $n \geq 2$. Since the co-Frobenius property only depends on the fact that the category of comodules has enough projectives, we just have to prove that $O(\text{SL}_q(2))$ is co-Frobenius. For $q = \pm 1$ or $q$ not a root of unity, it is known that $O(\text{SL}_q(2))$ is cosemisimple hence co-Frobenius, while if $q$ is a root of unity of odd order it is also known that $O(\text{SL}_q(2))$ is co-Frobenius [2]. So assume that $q$ is a root of unity of even order $2N$, with $N \geq 2$. As usual we denote by $a, b, c, d$ the generators of $O(\text{SL}_q(2))$. We denote by $B$ the subalgebra generated by the elements $xy$, with $x, y \in \{a^N, b^N, c^N, d^N\}$. It is a direct verification to check that $B$ is co-Frobenius. But $B$ is a Hopf subalgebra of the Hopf subalgebra of $O(\text{SL}_q(2))$, with $B^+O(\text{SL}_q(2)) = O(\text{SL}_q(2))B^+$. Since $B$ is commutative, we have that $O(\text{SL}_q(2))$ is faithfully flat as a $B$-module by Proposition 3.12 in [4], and hence we get (Proposition 2.4) a strict exact sequence of Hopf algebras

$$\mathbb{C} \rightarrow B \rightarrow O(\text{SL}_q(2)) \rightarrow L \rightarrow \mathbb{C}$$

for some finite-dimensional Hopf algebra $L = O(\text{SL}_q(2))/B^+O(\text{SL}_q(2))$, see [54]. The exact sequence being strict, we know from Theorem 2.10 in [1] that $O(\text{SL}_q(2))$ is co-Frobenius if $B$ is co-Frobenius. But $B$ is a Hopf subalgebra of the Hopf subalgebra of $O(\text{SL}_q(2))$ generated by the elements $a^N, b^N, c^N, d^N$, which is known to be isomorphic to $O(\text{SL}_{\pm 1}(2))$, and hence is cosemisimple. \qed

Theorem 6.2. Let $E \in \text{GL}_n(\mathbb{C})$ with $n \geq 2$. We have

$$H^*_b(B(E)) \simeq \begin{cases} 0 & \text{if } n \neq 0, 3 \\ \mathbb{C} & \text{if } n = 0, 3 \end{cases}$$

Moreover $\text{cd}(B(E)) = 3 = \text{cd}_{\text{GS}}(B(E))$.

Proof. Let $V_E$ be the fundamental $n$-dimensional $B(E)$-comodule with $(u_{ij})$ as matrix of coefficients. Recall ([10], Theorem 5.1) that there exists an exact sequence of Yetter-Drinfeld modules over $B(E)$

$$0 \rightarrow \mathbb{C} \boxtimes B(E) \overset{\phi_1}{\rightarrow} (V^*_E \otimes V_E) \boxtimes B(E) \overset{\phi_2}{\rightarrow} (V^*_E \otimes V_E) \boxtimes B(E) \overset{\phi_3}{\rightarrow} \mathbb{C} \boxtimes B(E) \overset{\epsilon}{\rightarrow} \mathbb{C} \rightarrow 0 \quad \ast$$

which thus yields a resolution of the trivial Yetter-Drinfeld module by free Yetter-Drinfeld modules. Free Yetter-Drinfeld modules are relative projective Yetter-Drinfeld modules, and since $B(E)$ is co-Frobenius, we know from Theorem 5.2 that $\text{cd}_{\text{GS}}(B(E)) \leq 3$, and since by Corollary 5.5 we have $\text{cd}_{\text{GS}}(B(E)) \geq \text{cd}(B(E)) = 3$, we conclude that $\text{cd}(B(E)) = 3 = \text{cd}_{\text{GS}}(B(E))$. It also follows from Theorem 5.2 that $H^*_b(B(E)) = H^*_b(B(E), \mathbb{C})$ is the cohomology of the complex obtained after applying the functor $\text{Hom}_{\mathcal{YD}_{B(E)}}(-, \mathbb{C})$ to the sequence

$$0 \rightarrow \mathbb{C} \boxtimes B(E) \overset{\phi_1}{\rightarrow} (V^*_E \otimes V_E) \boxtimes B(E) \overset{\phi_2}{\rightarrow} (V^*_E \otimes V_E) \boxtimes B(E) \overset{\phi_3}{\rightarrow} \mathbb{C} \boxtimes B(E) \rightarrow 0 \quad \ast$$

We have

$$\text{Hom}_{\mathcal{YD}_{B(E)}}(\mathbb{C} \boxtimes B(E), \mathbb{C}) \simeq \text{Hom}^{B(E)}(\mathbb{C}, \mathbb{C}) \simeq \mathbb{C}$$

and

$$\text{Hom}_{\mathcal{YD}_{B(E)}}(V^*_E \otimes V_E \boxtimes B(E), \mathbb{C}) \simeq \text{Hom}^{B(E)}(V^*_E \otimes V_E, \mathbb{C}) \simeq \mathbb{C}$$

where the last space is generated by the evaluation map. It follows that $H^*_b(B(E))$ is the cohomology of a complex of the form

$$0 \rightarrow \mathbb{C} \rightarrow \mathbb{C} \rightarrow \mathbb{C} \rightarrow \mathbb{C} \rightarrow 0$$

The cohomology of the complex is easily computed using the explicit expressions for the maps $\phi_i$, see Definition 5.2 in [10], and is left to the reader. \qed
6.2. The adjoint Hopf subalgebra $B_+(E)$. As a preliminary step towards the study of quantum symmetry algebras of semisimple algebras, we now study the adjoint subalgebra $B_+(E)$ of $B(E)$.

The algebra $B_+(E)$ is, by definition, the subalgebra of $B(E)$ generated by the elements $u_{ij}u_{kl}$, $1 \leq i, j, k, l \leq n$. It is easily seen to be a Hopf subalgebra. Also it is easily seen that $B_+(E) = B(E)^{op}$, where $p$ is the cocentral Hopf algebra map $B(E) \to \mathbb{C}Z_2$, $u_{ij} \mapsto \delta_{ij}g$, where $g$ stands for the generator of $Z_2$, the cyclic group of order 2.

**Lemma 6.3.** Assume that $\text{tr}(E^{-1}E^t) \neq 0$. Then there exists a linear map $\sigma : \mathbb{C}Z_2 \to B(E)$ satisfying the conditions of Proposition 4.7.

**Proof.** Consider the matrix $F = E(E^t)^{-1} = (\alpha_{ij})$. We have $\text{tr}(F) = \text{tr}(E^{-1}E^t) = t \neq 0$. Consider the element $x = t^{-1}\sum_{ij} \alpha_{ij}u_{ij} \in B(E)$ and let $\sigma : \mathbb{C}Z_2 \to B(E)$ be the unique linear map such that $\sigma(1) = 1$ and $\sigma(g) = x$. It is straightforward to check that $\sigma$ indeed satisfies the conditions of Proposition 4.7.

**Theorem 6.4.** Let $E \in \text{GL}_n(\mathbb{C})$ with $n \geq 2$. Then we have $\text{cd}(B_+(E)) = 3$, and if moreover $\text{tr}(E^{-1}E^t) \neq 0$, then $\text{cd}_{\text{GS}}(B_+(E)) = 3$.

**Proof.** We have, by Proposition 4.4, a strict exact sequence of Hopf algebras

$$\mathbb{C} \to B_+(E) \to B(E) \to \mathbb{C}Z_2 \to \mathbb{C}$$

so it follows from Proposition 3.2 that $\text{cd}(B_+(E)) = \text{cd}(B(E)) = 3$. Moreover $B_+(E)$ is co-Frobenius by Theorem 2.13 in [1], since $B(E)$ is. Hence by Theorem 6.4 we have $\text{cd}_{\text{GS}}(B_+(E)) \geq \text{cd}(B_+(E)) = 3$. Consider now the exact sequence of free Yetter-Drinfeld modules over $B(E)$ from [10]:

$$0 \to \mathbb{C} \otimes B(E) \xrightarrow{\phi_1} (V_E^* \otimes V_E) \otimes B(E) \xrightarrow{\phi_2} (V_E^* \otimes V_E) \otimes B(E) \xrightarrow{\phi_3} \mathbb{C} \otimes B(E) \xrightarrow{\epsilon} \mathbb{C} \to 0$$

All the $B(E)$-comodules involved in the left terms are in fact comodules over $B_+(E)$, so we have, by Proposition 4.2, an exact sequence of Yetter-Drinfeld modules over $B_+(E)$. Assume now that $\text{tr}(E^{-1}E^t) \neq 0$. The previous lemma ensures that we are in the situation of Proposition 4.7, so all the terms in the sequence (except the last one of course) are relative projective Yetter-Drinfeld modules over $B_+(E)$. We conclude from Theorem 5.2 that $\text{cd}_{\text{GS}}(B_+(E)) \leq 3$, and hence that $\text{cd}_{\text{GS}}(B_+(E)) = 3$.

To compute the bialgebra cohomology of $B_+(E)$, we need some preliminaries. We specialize at $E_q = \begin{pmatrix} 0 & \cdots & 1 \\ -q^{-1} & \cdots & 0 \end{pmatrix}$ and we put $A = B(E_q) = O(\text{SL}_n(2))$ (with its standard generators $a, b, c, d$) and $B = B_+(E_q)$. The assumption $\text{tr}(E^{-1}E^t) \neq 0$ is then $q + q^{-1} \neq 0$. Recall from Subsection 4.4 that if $W$ is a $B$-comodule, then $W \boxtimes A$ is a Yetter-Drinfeld module over $B$.

**Lemma 6.5.** We have, for any $B$-comodule $W$, a vector space isomorphism

$$\text{Hom}_{YD_B^p}(W \boxtimes A, \mathbb{C}) \to \text{Hom}^B(W, \mathbb{C}) \oplus \text{Hom}^B(W, \mathbb{C})$$

$$\psi \mapsto (\psi(- \otimes 1), \psi(- \otimes \chi))$$

where $\chi = q^{-1}a + qd$.

**Proof.** Let $\psi \in \text{Hom}_{YD_B^p}(W \boxtimes A, \mathbb{C})$. That both $\psi(- \otimes 1)$ and $\psi(- \otimes \chi)$ are $B$-comodule maps follow from the fact that 1 and $\chi$ are coinvariant for the co-adjoint action of $A$. We have, for any $w \in W$, using the $B$-linearity

$$\psi(w \otimes b) = \psi(w \otimes b(ad - q^{-1}bc)) = \psi(w \otimes bad) = q\psi(w \otimes abd) = 0$$

and similarly $\psi(w \otimes c) = 0$. We also have

$$\psi(w \otimes d) = \psi(w \otimes d(ad - q^{-1}bc))) = \psi(w \otimes dad) = \psi(w \otimes \sigma d^2) = \psi(w \otimes a)$$

These identities, together with the fact that $A = B \oplus B'$, where $B' = XB$ and $X = \{a, b, c, d\}$, show that the map in the statement of the lemma is injective.
For $(\psi_1, \psi_2) \in \text{Hom}^B(W, \mathbb{C}) \oplus \text{Hom}^B(W, \mathbb{C})$, we define a linear map $\psi : W \otimes A \to \mathbb{C}$ by

$$\psi(w \otimes (y + y')) = \psi_1(w)\varepsilon(y) + (q + q^{-1})^{-1}\psi_2(w)\varepsilon(y'), \quad y \in B, \; y' \in B'$$

It is clear that $\psi$ is $A$-linear and a direct verification to check that $\psi$ is a map of $B$-comodules, for the co-action of $W \otimes A$. Hence we have $\psi \in \text{Hom}_{YD_B}(W \otimes A, \mathbb{C})$, and clearly $\psi(- \otimes 1) = \psi_1$ and $\psi(- \otimes \chi) = \psi_2$. Therefore our map is surjective, and we are done.

**Theorem 6.6.** Let $E \in \text{GL}_n(\mathbb{C})$ with $n \geq 2$ and $\text{tr}(E^{-1}E') \neq 0$. Then

$$H^n_0(B_+(E)) \simeq \begin{cases} 0 & \text{if } n \neq 0, 3 \\ \mathbb{C} & \text{if } n = 0, 3 \end{cases}$$

**Proof.** The monoidal invariance of bialgebra cohomology enables us to assume that $E = E_q$ as in the previous discussion, of which we keep the notations. We denote by $V$ the fundamental $A$-comodule of dimension 2, of which we fix a basis $e_1, e_2$. We have an exact sequence of Yetter-Drinfeld over $A$ (and over $B$)

$$0 \to C \otimes A \xrightarrow{\phi_1} (V^* \otimes V) \otimes A \xrightarrow{\phi_2} (V^* \otimes V) \otimes A \xrightarrow{\phi_3} C \otimes A \xrightarrow{\epsilon} C \to 0$$

with for any $x \in A$ (see the proof of Lemma 5.6 in [10])

$$\phi_1(x) = e_1^* \otimes e_1 \otimes ((-q^{-1} + qd)x) + e_1^* \otimes e_2 \otimes (-cx) + e_2^* \otimes e_1 \otimes (-bx) + e_2^* \otimes e_2 \otimes ((-q + q^{-1})a)x$$

$$\phi_2(e_1^* \otimes e_1 \otimes x) = e_2^* \otimes e_1 \otimes (1 - q^{-1})a$$

$$\phi_3(e_1^* \otimes e_1 \otimes x) = (a - 1)x, \quad \phi_3(e_1^* \otimes e_2 \otimes x) = bx$$

and by Lemma 6.3, Proposition 4.7 and Theorem 5.2, the bialgebra cohomology of $B$ is the cohomology of the complex

$$0 \to \text{Hom}_{YD_B}(C \otimes A, \mathbb{C}) \xrightarrow{\phi_1} \text{Hom}_{YD_B}(V^* \otimes V) \otimes A, \mathbb{C} \xrightarrow{\phi_2} \text{Hom}_{YD_B}(V^* \otimes V) \otimes A, \mathbb{C} \xrightarrow{\phi_3} \text{Hom}_{YD_B}(C \otimes A, \mathbb{C}) \to 0$$

We have, by the previous lemma, $\text{Hom}_{YD_B}(C \otimes A, \mathbb{C}) \simeq \mathbb{C}^2$, and

$$\text{Hom}_{YD_B}(V^* \otimes V) \otimes A, \mathbb{C} \simeq \text{Hom}^B(V^* \otimes V, \mathbb{C}) \oplus \text{Hom}^B(V^* \otimes V, \mathbb{C}) \simeq \mathbb{C}^2$$

Therefore the previous complex is isomorphic to a complex of the form

$$0 \longrightarrow \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \longrightarrow 0$$

The reader will easily write down explicitly this complex and compute its cohomology, yielding the announced result for the bialgebra cohomology of $B$.

**6.3. Bialgebra cohomology and cohomological dimensions of $A_{\text{aut}}(R, \varphi)$.** Let $(R, \varphi)$ be a finite-dimensional measured algebra: this means that $R$ is a finite-dimensional algebra and $\varphi : R \to \mathbb{C}$ is a linear map (a measure on $R$) such that the associated bilinear map $R \times R \to \mathbb{C}$, $(x, y) \mapsto \varphi(xy)$ is non-degenerate. Thus a a finite-dimensional measured algebra is a Frobenius algebra together with a fixed measure. A coaction of a Hopf algebra $A$ on a finite-dimensional measured algebra $(R, \varphi)$ is an $A$-comodule structure on $R$ making it into an $A$-comodule algebra and such that $\varphi : R \to \mathbb{C}$ is $A$-colinear. It is well-known that there exists a universal Hopf algebra coacting on $(R, \varphi)$ (see [55] in the compact case with $R$ semisimple and [8] in general), that we denote $A_{\text{aut}}(R, \varphi)$ and call the quantum symmetry algebra of $(R, \varphi)$. The following particular cases are of special interest.
(1) For $R = \mathbb{C}^n$ and $\varphi = \varphi_n$ the canonical integration map (with $\varphi_n(e_i) = 1$ for $e_1, \ldots, e_n$ the canonical basis of $\mathbb{C}^n$), we have $A_{\text{aut}}(\mathbb{C}^n, \varphi_n) =: A_{s}(n)$, the coordinate algebra on the quantum permutation group $S_n$, presented by generators $x_{ij}$, $1 \leq i, j \leq n$, submitted to the relations

$$\sum_{i=1}^{n} x_{ii} = 1 = \sum_{i=1}^{n} x_{ik} x_{ij} = \delta_{kj}^i x_{ij}, \quad x_{ki} x_{ji} = \delta_{kj} x_{ij}, \quad 1 \leq i, j, k \leq n$$

Its Hopf algebra structure is defined by

$$\Delta(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj}, \quad \varepsilon(x_{ij}) = \delta_{ij}, \quad S(x_{ij}) = x_{ji}$$

The Hopf algebra $A_{s}(n)$ is infinite-dimensional if $n \geq 4$ [55].

(2) For $R = M_2(\mathbb{C})$ and $q \in \mathbb{C}^*$, let $\text{tr}_q : M_2(\mathbb{C}) \to \mathbb{C}$ be the $q$-trace, i.e. $\text{tr}_q(g) = q g_{11} + q^{-1} g_{22}$ for $g = (g_{ij}) \in M_2(\mathbb{C})$. Then we have $A_{\text{aut}}(M_2(\mathbb{C}), \text{tr}_q) \cong \mathcal{O}(\text{PSL}_q(2))$, the latter algebra being $B_+(E_q)$ in the notation of the previous subsection (it is often denoted $\mathcal{O}(\text{SO}_{q^2/2}(3))$, see e.g. [39]). The above isomorphism $A_{\text{aut}}(M_2(\mathbb{C}), \text{tr}_q) \to \mathcal{O}(\text{PSL}_q(2))$ is constructed using the universal property of $A_{\text{aut}}(M_2(\mathbb{C}), \text{tr}_q)$, and the verification that it is indeed injective is a long and tedious computation, as in [19].

Let $(R, \varphi)$ be a finite-dimensional measured algebra. Since $\varphi \circ m$ is non-degenerate, where $m$ is the multiplication of $R$, there exists a linear map $\delta : \mathbb{C} \to R \otimes R$ such that $(R, \varphi \circ m, \delta)$ is a left dual for $R$, i.e.

$$((\varphi \circ m) \otimes \text{id}_R) \circ (\text{id}_R \otimes \delta) = \text{id}_R = (\text{id}_R \otimes (\varphi \circ m)) \circ (\delta \otimes \text{id}_R)$$

Following [43], we put

$$\tilde{\varphi} = \varphi \circ m \circ (m \otimes \text{id}_R) \circ (\text{id}_R \otimes \delta) : R \to \mathbb{C}$$

and we say that $(R, \varphi)$ (or $\varphi$) is normalizable if $\varphi(1) \neq 0$ and if there exists $\lambda \in \mathbb{C}^*$ such that $\tilde{\varphi} = \lambda \varphi$.

It is shown in [43] (Corollary 4.9), generalizing earlier results from [5, 6, 18], that if $(R, \varphi)$ is a finite-dimensional semisimple measured algebra with $\dim(R) \geq 4$ and $\varphi$ normalizable, then there exists $q \in \mathbb{C}^*$ with $q + q^{-1} \neq 0$ such that

$$\mathcal{M}^{A_{\text{aut}}(R, \varphi)} \cong \otimes \mathcal{O}(\text{PSL}_q(2))$$

The parameter $q$ is determined as follows. First consider $\lambda \in \mathbb{C}^*$ such that $\tilde{\varphi} = \lambda \varphi$ and choose $\mu \in \mathbb{C}^*$ such that $\mu^2 = \lambda \varphi(1)$. Then $q$ is any solution of the equation $q + q^{-1} = \mu$ (recall that $\mathcal{O}(\text{PSL}_q(2)) = \mathcal{O}(\text{PSL}_{-q}(2))$, so the choice of $\mu$ does not play any role).

As an example, for $R = (\mathbb{C}^n, \varphi_n)$ as above (and $n \geq 4$), $\varphi_n$ is normalizable with the corresponding $\lambda$ equal to 1, and $q$ is any solution of the equation $q + q^{-1} = \sqrt{n}$.

**Theorem 6.7.** Let $(R, \varphi)$ be a finite-dimensional semisimple measured algebra with $\dim(R) \geq 4$ and $\varphi$ normalizable. Then we have

$$H^0_{\text{cd}}(A_{\text{aut}}(R, \varphi)) \cong \begin{cases} 0 & \text{if } n \neq 0, 3 \\ \mathbb{C} & \text{if } n = 0, 3 \end{cases}$$

and $\text{cd}(A_{\text{aut}}(R, \varphi)) \leq \text{cd}_{\text{GS}}(A_{\text{aut}}(R, \varphi)) = 3$

**Proof.** The proof follows immediately from the combination of the above monoidal equivalence, the monoidal invariance of Gerstenhaber-Schack cohomology, Theorem 6.4, Theorem 6.6 and Theorem 5.3. □

Note that the length 3 resolution of the trivial Yetter-Drinfeld module over $\mathcal{O}(\text{PSL}_q(2))$ by relative projective Yetter-Drinfeld modules considered in the previous subsection (see the proof of Theorem 6.6) transports to a length 3 resolution of the trivial Yetter-Drinfeld module over $A_{\text{aut}}(R, \varphi)$ by relative projective Yetter-Drinfeld modules (see Theorem 4.1 in [10]), and in particular this yields a length 3 projective resolution of the trivial module over $A_{\text{aut}}(R, \varphi)$. 18
We have not been able to write down this resolution explicitly enough to compute Hochschild cohomology groups and show that \(cd(A_{\text{aut}}(R, \varphi)) = 3\). We believe that this is true however.

**Remark 6.8.** Of course we get in particular that \(cd(A_s(n)) \leq 3 = cd_{GS}(A_s(n))\) for \(n \geq 4\), and hence the \(L^2\)-Betti numbers \(31\) \(\beta_k^2(A_s(n))\) vanish for \(k \geq 4\), and we have as well \(\beta_0^2(A_s(n)) = 0\) by \(32\).

7. Hopf algebras with a projection

It is natural to ask whether similar results to those of Section 2 hold for Gerstenhaber-Schack cohomological dimension. A positive answer to Question 1.2 would of course provide an affirmative answer for co-Frobenius Hopf algebras. So far, our only positive result in this direction is the following one, in the setting of Hopf algebras with a projection 12 35.

**Proposition 7.1.** Let \(B \subset A\) be a Hopf subalgebra. Assume that there exists a Hopf algebra map \(\pi : A \to B\) such that \(\pi|_B = \text{id}_B\) and that \(A\) and \(B\) are co-Frobenius. Then we have \(cd_{GS}(B) \leq cd_{GS}(A)\).

**Proof.** The inclusion \(B \subset A\) together with the Hopf algebra map \(\pi : A \to B\) induce a vector space preserving linear exact tensor functor

\[
F : \mathcal{YD}_A \longrightarrow \mathcal{YD}_B
\]

where if \(V\) is Yetter-Drinfeld module over \(A\), then \(F(V) = V\) as a vector space, the \(B\)-module structure is the restriction of that of \(A\), and the \(B\)-comodule structure is given by \((\text{id}_V \otimes \pi)\alpha\), where \(\alpha\) is the original co-action of \(A\). We claim that it is enough to show that \(F\) sends relative projective Yetter-Drinfeld modules over \(A\) to relative projective Yetter-Drinfeld modules over \(B\). Indeed, if we have a length \(n\) resolution of the trivial Yetter-Drinfeld module over \(A\) by relative projectives, the functor \(F\) will transform it into a \(A\) a length \(n\) resolution of the trivial Yetter-Drinfeld module over \(B\) by relative projectives, and hence by Corollary 5.3 we have \(cd_{GS}(B) \leq cd_{GS}(A)\).

As usual, put \(R = \text{co}^\pi A = \{a \in A \mid \pi(a_{(1)}) \otimes a_{(2)} = 1 \otimes a\}\). This is a subalgebra of \(A\) and we have \((\text{id} \otimes \pi)\Delta(R) \subset R \otimes B\), which endows \(R\) with a right \(B\)-comodule structure. For any \(a \in A\), we have \(a_{(2)}\pi S^{-1}(a_{(1)}) \in R\) (since \(A\) is co-Frobenius, its antipode is bijective 11), and thus we have a linear isomorphism 12 35

\[
A \longrightarrow R \otimes B
\]

\[
a \longrightarrow a_{(3)}\pi S^{-1}(a_{(2)}) \otimes \pi(a_{(1)})
\]

whose inverse is the restriction of the multiplication of \(A\). Let \(V\) be a right \(A\)-comodule: it also has a right \(B\)-comodule structure obtained using the projection \(\pi : A \to B\), that we denote \(V_\pi\). Consider now the map

\[
F(V \boxtimes A) \longrightarrow (V_\pi \otimes R) \boxtimes B
\]

\[
v \otimes a \longrightarrow v \otimes a_{(3)}\pi S^{-1}(a_{(2)}) \otimes \pi(a_{(1)})
\]

This is an isomorphism by the previous considerations, and it is a direct verification to check that it is a morphism of Yetter-Drinfeld modules over \(B\). Hence the functor \(F\) sends free Yetter-Drinfeld modules over \(A\) to free Yetter-Drinfeld modules over \(B\), and since it is additive, it sends, by Proposition 4.2 relative projective Yetter-Drinfeld modules over \(A\) to relative projective Yetter-Drinfeld modules over \(B\). This concludes the proof. □

As an illustration, consider the hyperoctahedral Hopf algebra \(A_h(n)\) 7. This is the algebra presented by generators \(a_{ij}\), \(1 \leq i, j \leq n\), submitted to the relations

\[
\sum_{l=1}^n a_{il}^2 = 1 = \sum_{l=1}^n a_{il}^2, \quad a_{ik}a_{ij} = 0 = a_{ji}a_{kj} \quad \text{if} \quad j \neq k, \quad 1 \leq i, j, k \leq n
\]
Its Hopf algebra structure is given by the same formulas as those for $A_s(n)$. There exist Hopf algebra maps $i : A_s(n) \to A_h(n)$, $x_{ij} \mapsto a_{ij}^2$, $\pi : A_h(n) \to A_s(n)$, $a_{ij} \mapsto x_{ij}$, such that $\pi i = \text{id}$. Hence we deduce from the previous proposition that $\text{cd}_{GS}(A_h(n)) \geq \text{cd}_{GS}(A_s(n))$, and hence if $n \geq 4$, that $\text{cd}_{GS}(A_h(n)) \geq 3$.

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