Scaling of heat flux and energy spectrum for large and infinite Prandtl number convection

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Under the limit of infinite Prandtl number, we derive analytical expressions for the large-scale quantities, e.g., Péclet number $Pe$, Nusselt number $Nu$, and rms value of the temperature fluctuations $θ_{\text{rms}}$. We complement the analytical work with direct numerical simulations, and show that $Nu \sim Ra^{\gamma}$ with $\gamma \approx (0.29 - 0.33)$, $Pe \sim Ra^\eta$ with $\eta \approx (0.57 - 0.61)$, and $θ_{\text{rms}} \sim \text{const.}$ The Nusselt number is observed to be an intricate function of $Pe$, $θ_{\text{rms}}$, and a correlation function between the vertical velocity and temperature field. The numerically-computed dissipation rates are consistent with Shraiman and Siggia’s exact relations. Using the scaling of large-scale fields, we show that the energy spectrum $E_u(k) \sim k^{-13/3}$, which is in very good agreement with our numerical results. The entropy spectrum $E_\theta(k)$ however has dual branch consisting of $k^{-2}$ and $k^0$ spectra; the $k^{-2}$ branch corresponds to the Fourier modes $\hat{\theta}(0,0,2n)$, which are approximately $-1/(2n\pi)$. We also compute the prefactors of the scaling relations. The scaling relations for Prandtl number beyond 100 match with those for infinite Prandtl number.

1. Introduction

Thermal convection plays a significant role in many engineering applications, as well as in natural phenomena, e.g., mantle convection, atmospheric circulation, stellar convection etc. To simplify the complex nature of convective flow, it is customary to model the flow using a simpler setup called Rayleigh-Bénard convection (RBC), in which a thin horizontal layer of the fluid is heated from below and cooled from the top [Ahlers et al. 2009b; Lohse & Xia 2010]. The two non-dimensional control parameters that characterise RBC flow are the Rayleigh number $Ra$, which is a measure of buoyancy force, and the Prandtl number $Pr$, which is a ratio of kinematic viscosity and thermal diffusivity. The Nusselt number $Nu$ is defined as the ratio of the total heat flux and the conductive heat flux. Experiments and numerical simulations show that the RBC flow depends quite critically on the Prandtl number. In this paper we will describe the scaling of large-scale quantities and the energy spectrum for very large and infinite Prandtl number convection. Physics of large Prandtl number convection is important for understanding convection in Earth’s mantle, viscous oil, etc.

Kraichnan (1962) studied the scaling of Nusselt and Reynolds number theoretically using mixing-length theory, and deduced that $Nu \sim Ra^{1/3}$ for large $Pr$, $Nu \sim (PrRa)^{1/3}$ for small $Pr$, and $Nu \sim 1$ for very small $Pr$. For RBC with very large Rayleigh numbers, called the “ultimate regime”, Kraichnan (1962) argued that that $Nu \propto (Ra/\log(Ra))^{1/2}$. Grossmann & Lohse (2000, 2001, 2002, 2004); Ahlers et al. (2009b); Grossmann & Lohse (2011); Stevens et al. (2013) modelled the scaling of Nusselt and Reynolds numbers using

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Shraiman and Siggia’s exact relations between the dissipation rates and the Nusselt number. For the dissipation rates, they considered the contributions from the bulk and boundary layers separately. They showed that the parameter space \((Ra, Pr)\) is divided into four regimes: I for \(U_{BL}, \theta_{BL}\) dominated regime; II for \(U_{bulk}, \theta_{BL}\) dominated regime; III for \(U_{BL}, \theta_{bulk}\) dominated regime; and IV for \(U_{bulk}, \theta_{bulk}\) dominated regime; here \(U\) and \(\theta\) are the velocity and temperature fields respectively (see figure 1), and \(BL\) stands for the boundary layer. For infinite or very large Prandtl number (kinematic viscosity \(\gg\) thermal diffusivity), most of the flow is dominated by the viscous force. Grossmann & Lohse (2001) denote these regimes as \(I_{\infty}\) and \(III_{\infty}\) depending on the dominance of \(\theta_{BL}\) or \(\theta_{bulk}\) respectively. Throughout the paper we refer to the above scaling as GL scaling. In this paper, we report several numerical simulations for very large and infinite Prandtl number convection; our numerical runs are depicted as filled pink circles in figure 1 and they fall in either \(I_{\infty}\) regime, or at the border of \(I_{\infty}\) and \(I_u\). Also note that there are several mathematically rigorous bounds on the Nusselt number exponent. Whitehead & Doering (2011) and Ingersley et al (2006) showed that \(Nu \leq cRa^{5/12}\) for free-slip boundary condition, while Constantin & Doering (1999) showed that \(Nu \leq cRa^{1/3}(1 + \log(Ra))^{2/3}\) in the limit of infinite Prandtl number.

A large number of experiments (Castaing et al 1989; Cioni et al 1997; Glazier et al 1999; Niemela et al 2000; Ahlers & Xu 2001; Niemela & Sreenivasan 2003; Funfschilling et al 2009; Ahlers et al 2009a) report the Nusselt number exponent to range from 0.26 to 0.32 for \(Ra\) up to \(10^{17}\). However some experiments (He et al 2012; Chavanne et al 1997; Roche et al 2001) exhibit an increase in the exponent from 0.32 to 0.37, thus signalling a signature of ultimate regime. The results from numerical simulations (Kerr 1996; Verzicco & Camussi 1999; Kerr & Herring 2000; Lohse & Toschi 2003; Verzicco & Sreenivasan 2008; Stevens et al 2010; 2011; Verma et al 2012) for small and medium Prandtl numbers and for \(Ra\) up to \(10^{14}\) are consistent with the aforementioned experimental results. However, in this paper we focus on very large and infinite Prandtl number convection. There are only a small number of experiments that investigate large Prandtl number convection. Xia et al (2002) studied heat transport properties of 1-pentanol, tri-
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ethane glycol, dipropylene glycol, and water ($Pr = 4 - 1350$), and reported the Nusselt number exponent to vary from 0.281 to 0.307. Based on these observations, they claimed that $Nu = 0.14Ra^{0.297}Pr^{-0.03}$, thus suggesting a weak dependence of the Nusselt number on the Prandtl number in the large Prandtl number regime. Lam et al. (2002) explored the dependence of Reynolds number ($Re$) on $Ra$ and $Pr$ and showed that $Re = 0.84Ra^{0.40}Pr^{-0.86}$. Since the Péclet number $Pe = RePr$, Lam et al. (2002)'s work indicates a weak dependence of $Pe$ on $Pr$.

Like experiments, numerical simulations of large Prandtl number convection are limited. Silano et al. (2010) numerically analyzed $Nu(Ra, Pr)$ for a wide range of Prandtl numbers ($Pr = 10^{-1}-10^{4}$) and observed Nusselt number exponents ranging from $2/7$ for $Pr = 1$ to 0.31 for $Pr = 10^{3}$. In addition, they reported that for large $Pr$, $Pe \sim Ra^{1/2}$, and a constancy of thermal fluctuations with $Ra$. Roberts (1979) simulated two-dimensional RBC in the limit of $Pr \to \infty$ and $Ra \to \infty$ and reported that $Nu \sim Ra^{1/3}$ for the free-slip runs, and $Nu \sim Ra^{1/5}$ for the no-slip runs. Similar results have been reported by Hansen et al. (1990); Schmalzl et al. (2002), and Breuer et al. (2004) in two- and three-dimensional box simulations with free-slip boundary conditions. The aforementioned researchers also studied the structures of the flow and thickness of the boundary layers.

Another set of important quantities of interest in RBC are the energy spectrum and entropy spectrum ($|\hat{\theta}(k)|^2$). For intermediate Prandtl numbers, L'vov (1991) and L'vov & Falkovich (1992) predict a Bolgiano-Obukhov scaling for small wavenumbers, but Kolmogorov-Obukhov scaling for large wavenumbers. We will show in this paper that the spectra for the large and infinite Prandtl number convection differ from the above. Our arguments are similar to those of Batchelor (1959) for small Prandtl number, except that the role of temperature and velocity is reversed in our work since the Prandtl number is large.

As discussed above, there are significant development in the understanding of large and infinite Prandtl number convection, especially from the GL scaling (Grossmann & Lohse 2001). Still, many issues remain unresolved in this field. We address some of these issues using analytical approach and direct numerical simulation. Under the limit of infinite Prandtl number, we derive a linear relationship between the velocity and temperature fields, which enables us in deriving interesting exact relations for the Péclet and Nusselt numbers. We also derive energy and entropy spectra analytically using the above relations, as well as the temperature equation. In addition, we have also obtained analytic expressions for the thermal fluctuations, and viscous and entropy dissipation rates. The Nusselt number and entropy dissipation rates are intrinsically dependent on the $Ra$-dependent correlation between the vertical velocity and the temperature fluctuations. These relations provide valuable insights into large $Pr$ convective turbulence. It is important to note that our theoretical work are based on the dimensional and scaling analysis of the equations for the velocity and temperature fields. Our approach differs from GL scaling, which is based on Shraiman & Siggia (1990)'s exact relation, and the scaling of the dissipation rates in the bulk and boundary layers.

To validate our analytical predictions, we perform direct numerical simulations of RBC flows for $Pr = 10^{2}, 10^{3}, \infty$ and Rayleigh numbers varying from $10^{4}$ to $10^{8}$. Using the numerical data we compute the scaling for large-scale quantities like Nusselt and Péclet numbers, rms fluctuations of the temperature field, energy and entropy dissipation rates, and compare them with the analytic predictions. We also compute energy and entropy spectra using the numerical data. We find that our numerical results are in good agreement with the analytical predictions, as well as with GL scaling.

The structure of paper is as follows: In § 2, we present the governing equations and
analytical expressions for various large-scale quantities like Nusselt and Péclet numbers, and energy and entropy dissipation rates. In §3 we discuss the details of our numerical simulations. Scaling relations of $Pe$, $Nu$, and dissipation rates are discussed in §4. In §5 we derive energy and entropy spectra using the dynamical equations in the $Pr = \infty$ limit, and complement them with the numerical results for infinite and large Prandtl number simulations. We conclude in §6.

2. Governing equations and analytic computations

The dynamical equations of RBC under Boussinesq approximations are

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla \sigma + \frac{\alpha g \theta}{\rho_0} \hat{z} + \nu \nabla^2 \mathbf{u}, \quad (2.1)$$

$$\frac{\partial \theta}{\partial t} + (\mathbf{u} \cdot \nabla) \theta = \frac{\Delta}{d} u_z + \nabla^2 \theta, \quad (2.2)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (2.3)$$

where $\mathbf{u} = (u_x, u_y, u_z)$ is the velocity field, $\sigma$ and $\theta$ are respectively the deviations of pressure and temperature from the heat conduction state, $\rho_0$ is the mean density of fluid, $\alpha, \nu$, and $\kappa$ are respectively the thermal expansion coefficient, kinematic viscosity, and thermal diffusivity of fluid, $g$ is the acceleration due to gravity, $\hat{z}$ is the buoyancy direction, and $\Delta$ is the temperature difference between the two plates kept apart by a vertical distance $d$.

For large and infinite Prandtl number convection, it is customary to nondimensionalize the above set of equations using $\sqrt{\frac{\alpha g \Delta d}{\nu \kappa}}$, $d$, and $\Delta$ as the velocity, length, and temperature scales respectively (Silano et al. 2010), which yields

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla \sigma + \frac{Pr \theta}{\sqrt{Ra}} \hat{z} + \frac{Pr}{\sqrt{Ra}} \nabla^2 \mathbf{u}, \quad (2.4)$$

$$\frac{\partial \theta}{\partial t} + (\mathbf{u} \cdot \nabla) \theta = u_z + \frac{\Delta}{d} + \frac{1}{\sqrt{Ra}} \nabla^2 \theta, \quad (2.5)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (2.6)$$

where $Ra = \alpha g \Delta d^3 / \nu \kappa$ and $Pr = \nu / \kappa$. Under the limit of $Pr = \infty$, the momentum equation gets simplified to

$$-\nabla \sigma + \theta \hat{z} + \frac{1}{\sqrt{Ra}} \nabla^2 \mathbf{u} = 0, \quad (2.7)$$

where $\sigma$ is appropriately normalised. The equations for the temperature field and the incompressibility condition remain unchanged. The momentum equation in dimensional form is

$$-\nabla \sigma + \alpha g \theta \hat{z} + \nu \nabla^2 \mathbf{u} = 0. \quad (2.8)$$

The above equation is linear, hence analytically tractable. It is more convenient to analyze the above equation in Fourier space, which is

$$-i k \hat{\sigma}(k) + \alpha g \hat{\theta}(k) \hat{z} - \nu k^2 \hat{u}(k) = 0, \quad (2.9)$$

where $k = (k_x, k_y, k_z)$ is the wavenumber, and $f(k)$ is the Fourier mode of the field $f$. 
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Using the incompressible condition \((\mathbf{k} \cdot \mathbf{u}(\mathbf{k}) = 0)\), we deduce that

\[
\hat{\sigma}(\mathbf{k}) = -\frac{k_z}{k^2} \alpha g \hat{\theta}(\mathbf{k}), \quad (2.10)
\]

\[
\hat{u}_z(\mathbf{k}) = \frac{\alpha g k_z^2}{\nu k^4} \hat{\theta}(\mathbf{k}), \quad (2.11)
\]

\[
\hat{u}_{x,y}(\mathbf{k}) = -\frac{\alpha g}{\nu} k_z k_{x,y} \frac{k^2}{k^6} \hat{\theta}(\mathbf{k}), \quad (2.12)
\]

where \(k^2 \perp = k_x^2 + k_y^2\). The Fourier modes in general do not satisfy the boundary conditions at the plates. Yet, they capture the large-scale modes quite accurately since the energy of the convective flow is dominated by these modes. This feature becomes more significant for infinite Prandtl number since the amplitude of the Fourier modes decreases steeply with the wavenumber \((\sim k^{-2})\), as evident from the aforementioned equations.

Using the above equations, we can derive the total kinetic energy as

\[
E_u = \frac{1}{2} u^2 = \frac{1}{2} \sum_{\mathbf{k}} |\hat{u}(\mathbf{k})|^2 = \left( \frac{\alpha g}{\nu} \right)^2 \sum_{\mathbf{k}} |\hat{\theta}(\mathbf{k})|^2 \frac{k^2}{k^6}. \quad (2.13)
\]

The total energy is dominated by the large-scale flows. Therefore, the Péclet number \(Pe\) is

\[
Pe = \frac{U_L d}{\kappa} = \frac{d}{\kappa} \sqrt{2E_u} = \frac{\alpha g d}{\nu \kappa} \left( \sum_{\mathbf{k}} |\hat{\theta}(\mathbf{k})|^2 \frac{k^2}{k^6} \right)^{1/2}. \quad (2.14)
\]

In terms of nondimensional parameters

\[
Pe = Ra \left( \sum_{\mathbf{k}} |\hat{\theta}(\mathbf{k})|^2 \frac{k^2}{k^6} \frac{1}{d^4 \Delta^2} \right)^{1/2}. \quad (2.15)
\]

One of the generic features of thermal convection in a box is the finite amplitude of \(\hat{\theta}(0,0,2n)\) Fourier modes for small \(n\), e.g., \(n = 1,2,3\). Mishra & Verma (2010) showed using arguments based on energy transfers that

\[
\hat{\theta}(0,0,2n) \approx -\frac{\Delta}{2n\pi}. \quad (2.16)
\]

The \(\hat{\theta}(0,0,2n)\) modes play an important role in determining the vertical profile of temperature. The temperature averaged over horizontal planes drops sharply near the plates (in the boundary layer), and it is approximately constant in the bulk. We will show later in the paper that the temperature drop near the plates gets significant contributions from the \(\hat{\theta}(0,0,2n)\) modes (Verma et al. 2013).

We will demonstrate later in the paper that \(\hat{\theta}(0,0,2n)\) modes dominate the temperature fluctuations for large and infinite Prandtl number convection. Thus,

\[
\theta_L \approx \sqrt{2E_\theta} \approx \Delta. \quad (2.17)
\]

To quantify the contributions from other thermal modes, we define residual temperature fluctuation \(\theta_{res}\), which is defined as

\[
\theta_{res}^2 = \theta^2 - \sum_n |\hat{\theta}(0,0,2n)|^2. \quad (2.18)
\]

It is important to note that \(\hat{u}_z(0,0,n) = 0\) due to the absence of net mass flux across any horizontal cross-section in the box. As a result, the \(\hat{\theta}(0,0,2n)\) modes do not contribute
to the heat flux $H$, which is

$$H \propto \langle u_z \theta \rangle = \sum_k (\hat{u}_z(k)\hat{\theta}(k) + \hat{u}_z^*(k)\hat{\theta}^*(k)). \quad (2.19)$$

Thus, the heat flux gets contributions only from $\theta_{res}$ fluctuations.

The viscous and entropy dissipation rates provide important information about the scaling of large-scale quantities (Shraiman & Siggia 1990; Grossmann & Lohse 2000). Shraiman & Siggia (1990) relate these dissipation rates to the Nusselt number using the following exact relationships:

$$\epsilon_u = \nu \frac{\langle |\nabla \times u|^2 \rangle}{d^2} = \nu \frac{1}{3} (Nu - 1) \frac{Ra}{Pr^2}, \quad (2.20)$$

$$\epsilon_\theta = \kappa \frac{\langle |\nabla \theta|^2 \rangle}{d^2} = \kappa \frac{\Delta^2}{d^2} Nu. \quad (2.21)$$

In viscous regime, we use $\epsilon_u \propto \nu U^3 L / d^2$ and define a normalised viscous dissipation rate as

$$C_{\epsilon_u} = \frac{\epsilon_u}{\nu U_L^2 / d^2} = \frac{(Nu - 1)Ra}{Pe^2}. \quad (2.22)$$

We define a similar normalised dissipation rates for the entropy as

$$C_{\epsilon_{\theta,1}} = \frac{\epsilon_\theta}{\kappa \Delta^2 / d^2} = Nu. \quad (2.23)$$

Note that $\epsilon_u \propto U^3 d$ in turbulent regime, but this formula is not applicable for the large and infinite Prandtl number convection because of the laminar nature of the flow. The temperature equation however is fully nonlinear, hence $\epsilon_\theta \propto U_L \theta_L^2 / d$ (Grossmann & Lohse 2001; Verma et al. 2013b). Therefore we define another normalised entropy dissipation rate as

$$C_{\epsilon_{\theta,2}} = \frac{\epsilon_\theta}{U_L \theta_L^2 / d} = Nu \left( \frac{\Delta}{\theta_L} \right)^2. \quad (2.24)$$

We will compute these dissipation rates using numerical data and use them for validation, as well as for the prediction of Nusselt number scaling.

3. Numerical Method

We solve the nondimensionalized RBC equations (2.4, 2.5, and 2.6) numerically for both the free-slip and no-slip boundary conditions. We simulate free-slip RBC flow in a three-dimensional box of dimension $2\sqrt{2} : 2\sqrt{2} : 1$ using a pseudospectral code Tarang (Verma et al. 2013a). On the top and bottom plates we employ free-slip and isothermal conditions for the velocity and the temperature fields respectively. However, periodic boundary condition is employed on the lateral walls. Fourth-order Runge-Kutta (RK4) method is used for time advancement, and the 2/3 rule for de-aliasing.

To complement the aforementioned free-slip numerical runs, we also simulate RBC flow under no-slip boundary condition in a two-dimensional box of aspect ratio one. We used the spectral element code NEK5000 (Fischer 1997) for this purpose. Runs were performed in a box with $28 \times 28$ spectral elements with 7th-order polynomials within each element, resulting in a 196$^2$ effective grid in the box. For spectrum study however we used 15th-order polynomial, which yields 420$^2$ effective grid points in the box. We will describe below that the two-dimensional no-slip and three-dimensional free-slip runs exhibit similar results.
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We perform simulations for \( Pr = 10^2, 10^3, \infty, \) and \( Ra = 10^4 - 10^8 \). For such large Prandtl numbers, the kinematic viscosity is much larger than the thermal diffusion coefficient, consequently coherent thin plumes are generated in such flows [Schmalzl et al. 2002; Hansen et al. 1990; Breuer et al. 2004; Stevens et al. 2010a]. Figure 2 illustrates the temperature isosurfaces of the flow structures for \( Pr = 10^3, Ra = 6 \times 10^6 \), and for \( Pr = 10^2, Ra = 10^7 \). The figures demonstrate that the plumes become thinner and sharper with an increase of Prandtl number. Earth’s mantle that has very large Prandtl number shows similar structures. Also, \( k_{max}\eta_u \gtrsim 1 \) and \( k_{max}\eta_\theta \gtrsim 1 \) for almost all our simulation runs; here \( \eta_u = (\nu^3/\epsilon_u)^{1/4} \) and \( \eta_\theta = (\kappa^3/\epsilon_u)^{1/4} \) are the Kolmogorov and Batchelor lengths for the velocity and temperature fields respectively. Thus our simulations are numerically well resolved.

We compute various global quantities (e.g., \( \theta_L \), Péclet and Nusselt numbers), and energy and entropy spectra using the numerical data generated by our simulations. These quantities are averaged over 20 eddy turnover time after the flow has reached a steady state. Note that the system takes around a thermal diffusive time to reach a steady state. For better statistical averaging, the Nusselt number is computed by averaging the heat flux over the box volume.

Table 1 exhibits details of our free-slip numerical simulations. In the table, we list the normalised dissipation rates computed using the numerical data and compare them with the ones derived using the exact relations (using the Nusselt number). The estimated values are in very good agreement with the numerically computed ones, thus validating our numerical simulations. The details of our no-slip runs are exhibited in Table 2.

In the next section we will describe the scaling of large-scale quantities derived using the numerical data.
Table 3. Numerically computed values of the Fourier modes \( \hat{\theta}(0, 0, 2) \), \( \hat{\theta}(0, 0, 4) \), \( \hat{\theta}(0, 0, 6) \), and \( \hat{\theta}(0, 0, 8) \) for \( Pr = 10^2, 10^3, \) and \( \infty \). The values obtained from our simulation are in good agreement with \cite{MishraVerma2010}'s theoretical prediction that \( \hat{\theta}(0, 0, 2n) \approx -1/2\pi n \) (the last row of the table).
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Figure 3. Averaged temperature $T(z)$ (dashed blue curve) as a function of $z$ for $Pr = \infty$ and $Ra = 10^8$. The mean temperature remains almost a constant in the bulk, and it displays sharp gradient near the top and bottom plates. The red, sky blue, brown, and black curves (curves with increasing thickness) represent $T_c + \sum_n 2\hat{\theta}(0,0,2n)\sin(2\pi n z)$ for $n = 1, 2, 4, \text{ and } 10$ respectively; here $T_c = 1 - z$ is the conduction profile. The curve for $n = 10$ matches quite well with $T(z)$.

4. Scaling of large-scale quantities

In this section we report average values of the Péclet and Nusselt numbers, as well as that of the temperature fluctuations and dissipation rates, for various Prandtl and Rayleigh numbers. These values are compared with the analytical predictions.

4.1. Temperature fluctuations

For large Prandtl number convection, Silano et al. (2010) showed that the temperature fluctuations are independent of $Ra$ and $Pr$. Here we analyze this issue in more detail. As discussed in the previous section, $\hat{\theta}(0,0,2n)$ modes play an important role in turbulent convection. We compute $\hat{\theta}(0,0,2n)$ modes for small $n$ using the numerical data of our simulation. These values, exhibited in table 3 for some typical parameters, are in good agreement with the predictions of Mishra & Verma (2010) that $\hat{\theta}(0,0,2n) \approx -\Delta / (2n\pi)$. In figure 3 we plot the averaged temperature profile $\bar{T}(z)$, as well as $1 - z + \sum_n 2\hat{\theta}(0,0,2n)\sin(2\pi n z)$ for $n = 1, 2, 4$ and 10 (note that $T_{\text{conduction}} = 1 - z$). The figure demonstrates that the $\bar{T}(z)$ is well approximated by $1 - z + \sum_{n=0}^{10} 2\hat{\theta}(0,0,2n)\sin(2\pi n z)$. Hence we conclude that $\hat{\theta}(0,0,2n)$ modes contribute significantly to $\bar{T}(z)$.

We also compute the residual temperature fluctuations $\theta_{\text{res}}$ defined using equation (2.18), and observe that

$$\theta_{\text{res}} = a_1 \Delta Ra^{-\delta},$$

as shown in figure 4. We deduce that $\theta_{\text{res}}/\Delta = (0.58 \pm 0.09) Ra^{-0.15 \pm 0.01}$ for $Pr = \infty$, $\theta_{\text{res}}/\Delta = (0.48 \pm 0.03) Ra^{-0.13 \pm 0.01}$ for $Pr = 10^2$, and $\theta_{\text{res}}/\Delta = (0.55 \pm 0.07) Ra^{-0.14 \pm 0.01}$ for $Pr = 10^3$. Thus, the scaling exponents as well as the prefactors of $\theta_{\text{res}}$ for various Prandtl numbers are nearly same for $Pr \geq 100$.

For large $Pr$ convection, we observe that $\theta_{\text{res}} \ll \hat{\theta}(0,0,2)$. Consequently, the rms
fluctuation of $\theta$ is dominated by $\hat{\theta}(0, 0, 2)$ mode, thus yielding

$$\theta_{\text{rms}} \approx \sqrt{2E_0} \approx \theta_L = a_2 \Delta,$$

which is independent of $Ra$ and $Pr$, as depicted in figure 5. The constant $a_2 \approx (0.29 \pm 0.01)$ for free-slip runs with $Pr = \infty, 10^2$ and $10^3$, which is reasonably close the corresponding $\theta_{\text{rms}}$ for intermediate $Pr$ ($Pr \sim 1$) (Verma et al. 2012). For no-slip boundary condition, Silano et al. (2010) report that $a_2 \approx 0.12$. 

Figure 4. Normalised residual temperature fluctuation ($\theta_{\text{res}}/\Delta$) as a function of $Ra$ for free-slip runs. The data for $Pr = \infty$ (red squares), $Pr = 10^2$ (blue down-pointing triangles), and $Pr = 10^3$ (green circles) collapse to a single function $\theta_{\text{res}} \approx a_1 \Delta Ra^{-\delta}$. The prefactors and exponents for the three runs are approximately equal.

Figure 5. Plot of normalised root mean square thermal fluctuations ($\theta_{\text{rms}}/\Delta$) vs. $Ra$: for free-slip runs with $Pr = \infty$ (red squares), $Pr = 10^2$ (blue down-pointing triangles), $Pr = 10^3$ (green circles); for no-slip simulations by Silano et al. (2010) with $Pr = 10^3$ (pink stars) and $Pr = 10^2$ (black hexagons). The figure indicates that $\theta_{\text{rms}}/\Delta$ is independent of $Ra$ and $Pr$ for large-$Pr$. The prefactors for the no-slip data is lower than that for the free-slip data.
Figure 6. Plot of normalised Péclet number ($Pe Ra^{-3/5}$) vs. $Ra$: for the free-slip runs with $Pr = \infty$ (red squares), $Pr = 10^2$ (blue down-pointing triangles), and $Pr = 10^3$ (green circles); and for the no-slip runs with $Pr = 10^2$ (orange triangles) and $Pr = 10^3$ (pink stars; data from Silano et al. (2010)). The prefactors for the no-slip runs are lower than those for the free-slip runs.

4.2. Péclet number scaling

Now we analyse the scaling of the Péclet number for large and infinite Prandtl number convection. Using our numerical data and equation (2.15), we find that

$$\left( \sum_{k} |\hat{\theta}(k)|^2 \frac{k^2}{k^6} \frac{1}{d^4 \Delta^2} \right)^{1/2} \approx Ra^{-\zeta},$$

with $\zeta \approx 0.38$. Hence

$$Pe = a_3 Ra^{1-\zeta} = a_3 Ra^{0.62}.$$  \hspace{1cm} (4.4)

Similar relations are observed for $Pr = 10^2$ and $Pr = 10^3$, as shown in figure 6 in which we plot $Pe Ra^{-3/5}$ vs. $Ra$. We find that $Pe = (0.20 \pm 0.02)Ra^{0.61 \pm 0.01}, (0.27 \pm 0.08)Ra^{0.58 \pm 0.02}, (0.21 \pm 0.04)Ra^{0.59 \pm 0.01}$ for $Pr = \infty, 10^2,$ and $10^3$ respectively. For a no-slip simulation with $Pr = 10^2$, $Pe = (0.05 \pm 0.01)Ra^{0.60 \pm 0.01}$. Note that the prefactors for the no-slip runs are smaller than those for the free-slip runs, which is due to the absence of wall friction for the free-slip boundary condition. These results are in reasonable agreement with the earlier results of Silano et al. (2010) (see figure 6), as well as with the GL scaling that $Pe \approx 0.038 Ra^{2/3}$ (the $Pr = 10^2$ no-slip data set belongs to the $I_2 \leq 1$ regime). Another interesting aspect of the above scaling is its independence from $Pr$, unlike that for moderate $Pr$’s for which $Pe \approx \sqrt{Ra Pr}$ (Grossmann & Lohse 2000; Verma et al. 2012).

Since the Reynolds number $Re = Pe/Pr$, $Re = 0$ for $Pr = \infty$, and $Re$ is small for $Pr \gg 1$. Hence the flow is viscous when the Prandtl number is large or infinite. For very large $Ra$, the Reynolds number tends to become larger than one.

4.3. Nusselt number scaling

Nusselt number is defined as the ratio of total heat flux and the conductive heat flux, i.e.,

$$Nu = \frac{\kappa \Delta/d + \langle u_z T \rangle}{\kappa \Delta/d} = 1 + \langle \frac{u_z d \theta_{res}}{\kappa} \Delta \rangle = 1 + \langle u_z' \theta_{res}' \rangle,$$  \hspace{1cm} (4.5)
where $\theta'_n = \theta_{\text{res}}/\Delta$ is the normalised temperature fluctuation without $\hat{\theta}(0, 0, 2n)$ modes, and $u'_z = u_z d/\kappa$. The absence of $(0, 0, 2n)$ Fourier modes in the above expression is due to the fact that $\hat{u}_z(0, 0, 2n) = 0$ (see §2).

The above expression for $Nu$ can be rewritten as (Verma et al. 2012, 2013)

$$Nu - 1 = \langle u'_z \theta'_{\text{res}} \rangle = C_{u\theta}(Ra) \langle u'^2_x \rangle_V^{1/2} \langle \theta'^2_{\text{res}} \rangle_V^{1/2}$$

where the correlation function between the vertical velocity and temperature fields $C_{u\theta}(Ra)$ is

$$C_{u\theta}(Ra) = \left\langle \frac{\langle u'_z \theta'_{\text{res}} \rangle_V}{\langle u'^2_x \rangle_V^{1/2} \langle \theta'^2_{\text{res}} \rangle_V^{1/2}} \right\rangle_t.$$  

(4.7)

Here, $V$ and $t$ stand for the volume and temporal averages respectively. Our numerical data reveal that $C_{u\theta}(Ra) = a_4 Ra^{-0.15}$, as exhibited in figure 7. We observe that $C_{u\theta} = (3.7 \pm 0.6)Ra^{-0.15 \pm 0.01}$ for $Pr = \infty$, $C_{u\theta} = (3.0 \pm 0.5)Ra^{-0.13 \pm 0.01}$ for $Pr = 10^2$, and $C_{u\theta} = (3.4 \pm 0.2)Ra^{-0.14 \pm 0.01}$ for $Pr = 10^3$. Using above results, and the scaling relations $Pe = a_3 Ra^{1-\zeta}$ and $\theta_{\text{res}} = a_1 Ra^{-\delta}$, we deduce that

$$Nu - 1 \approx Nu \approx C_{u\theta}(Ra) \langle u'^2_x \rangle_V^{1/2} \langle \theta'^2_{\text{res}} \rangle_V^{1/2} \approx a_1 a_3 a_4 Ra^{1-\zeta-0.15} = a_5 Ra^{0.32},$$

(4.8)

with $\zeta \approx 0.38$ and $\delta \approx 0.15$, and $a_5 = a_1 a_3 a_4$. Our arguments show that a subtle variations of $Pe$ and $\theta'$, and the correlation between the vertical velocity and temperature fields yield $Nu \approx a_5 Ra^{0.32}$.

Scaling relations for $Pr = \infty$, $10^2$, and $10^3$ (free-slip runs) computed using our numerical data are $Nu = (0.21 \pm 0.03)Ra^{0.33 \pm 0.01}$, $Nu = (0.34 \pm 0.02)Ra^{0.29 \pm 0.01}$, and $Nu = (0.27 \pm 0.02)Ra^{0.31 \pm 0.01}$ respectively. In addition, for the no-slip run, $Nu = (0.14 \pm 0.03)Ra^{0.29 \pm 0.01}$ for $Pr = 10^2$. The prefactor for the free-slip runs is higher than that for the no-slip runs, which is reasonable since the heat transport is enhanced for the free-slip runs due to lower friction at the top and bottom plates. Also, for $Pr = \infty$ with free-slip run, we observe that $a_1 \approx 0.58$, $a_3 \approx 0.20$, and $a_4 \approx 3.7$, and consequently $a_5 \approx 0.43$, which is in a reasonable agreement with the observed $a_5 \approx 0.21$. Similar
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Figure 8. Normalised Nusselt number \((NuRa^{-0.29})\) as a function of \(Ra\): for free-slip runs with \(Pr = \infty\) (red squares), \(Pr = 10^2\) (blue down-pointing triangles), and \(Pr = 10^3\) (green circles); and for no-slip runs with \(Pr = 10^2\) (orange triangles), \(Pr = 10^3\) (pink stars, data taken from Silano et al. (2010)), and \(Pr = 10^4\) (black hexagons, data taken from Silano et al. (2010)). Experimental data of Xia et al. (2002) for \(Pr = 205\) and \(Pr = 818\) are shown as cyan filled pentagons and burlywood diamonds respectively.

consistency is observed for \(Pr = 10^2\) and \(10^3\) as well. Thus, our scaling results for \(\theta\), \(Pe\) and \(Nu\) are consistent with each other.

The aforementioned scaling results are in good agreement with GL scaling (Grossmann & Lohse 2001), according to which \(Nu = 0.17Ra^{1/3}\) in \(I^\infty\) regime. Our results are also consistent with the experimental results of Xia et al. (2002) for \(Pr = 205\) and \(818\), and the numerical results of Silano et al. (2010) and Roberts (1979) for large \(Pr\) simulations (see figure 8).

4.4. Scaling of dissipation rates

In Section 2, we derived relationships between the normalised dissipation rates and the Nusselt number. In this subsection we compute the normalised dissipation rates \(C_{\epsilon_u}, C_{\epsilon_\theta,1}, C_{\epsilon_\theta,2}\) using numerical data and compare them with the exact results.

From the exact relationship between the viscous dissipation rate and the Nusselt number (equation (2.22)), we obtain

\[
C_{\epsilon_u} = \frac{(Nu - 1)Ra}{Pe^2} = \frac{a_5}{a_3^3}Ra^{\gamma + 2\zeta - 1} = \frac{a_5}{a_3^3}Ra^{0.09},
\]

using \(\gamma = 0.33\) and \(\zeta = 0.38\). For \(Pr = \infty\), \(a_5 \approx 0.21\) and \(a_3 \approx 0.20\) provide the prefactor to be 5.2. From our numerical data (table 1) we find that \(C_{\epsilon_u} = (3.6 \pm 1.4)Ra^{0.16 \pm 0.03}\) for \(Pr = 10^2\), \(C_{\epsilon_u} = (3.3 \pm 1.0)Ra^{0.17 \pm 0.02}\) for \(Pr = 10^3\), and \(C_{\epsilon_u} = (6.4 \pm 1.6)Ra^{0.11 \pm 0.02}\) for \(Pr = \infty\). These computed results are in good agreement with the aforementioned estimates using exact relationships, thus they validate our computations as well show consistency with the other scaling relations.

According to equation (2.23), the normalised entropy dissipation rate is defined as

\[
C_{\epsilon_\theta,1} = \frac{\epsilon_{\theta}}{\kappa \Delta^2/d^2} = Nu = a_5 Ra^\gamma,
\]

has a same scaling as the Nusselt number (see table 1). The scaling of the other normalised
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Figure 9. Plot of normalised viscous dissipation rate $C_\epsilon$ vs. $Ra$ for the free-slip runs with $Pr = \infty$ (red squares), $Pr = 10^2$ (blue down-pointing triangles), and $Pr = 10^3$ (green circles). $C_\epsilon(Ra) \sim Ra^{0.16}$, $Ra^{0.17}$, and $Ra^{0.11}$ for $Pr = 10^2$, $10^3$, and $\infty$ respectively.

Figure 10. Plot of normalised entropy dissipation rate $C_{\theta,2}$ vs. $Ra$ for the free-slip runs with $Pr = \infty$ (red squares), $Pr = 10^2$ (blue down-pointing triangles), and $Pr = 10^3$ (green circles). $C_{\theta,2} \sim Ra^{-0.27}$, $Ra^{-0.25}$, and $Ra^{-0.29}$ for $Pr = 10^2$, $10^3$, and $\infty$ respectively.

Entropy dissipation rate is more complex. Equation (2.24) yields

$$C_{\theta,2} = \frac{\epsilon_\theta}{U_L \theta_L^2/d} = \frac{N_u}{Pe} \left( \frac{\Delta}{\theta_L} \right)^2 = \frac{a_5}{a_3 a_2^2} Ra^{\gamma + \zeta - 1} = \frac{a_5}{a_3 a_2^2} Ra^{-0.29}, \quad (4.11)$$

with $\gamma = 0.33$ and $\zeta = 0.38$. Using the constants $a_i$'s, the prefactor is approximately 13 for $Pr = \infty$. Similar exponents and prefactors are found for other higher $Pr$ simulations. In figure 10 we plot $C_{\theta,2}$ vs. $Ra$, which exhibits $C_{\theta,2} = (12 \pm 1.9) Ra^{-0.27 \pm 0.01}$ for $Pr = 10^2$, $C_{\theta,2} = (10 \pm 2.7) Ra^{-0.25 \pm 0.02}$ for $Pr = 10^3$, and $C_{\theta,2} = (15 \pm 1.0) Ra^{-0.29 \pm 0.01}$ for $Pr = \infty$. These numerical results are in good agreement with the theoretical estimates computed earlier.

The aforementioned scaling of the dissipation rates and their consistency with other
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global quantities like \( Nu \) and \( Pe \) indicate consistency of our arguments. These results are summarized in table 4 for a free-slip simulation with \( Pr = \infty \), and in table 5 for a no-slip simulation with \( Pr = 10^2 \). The exponents and the prefactors are nearly the same for all \( Pr \geq 100 \).

After our discussion on the global quantities, we turn to the computations of energy and entropy spectra, as well as their fluxes.

5. Energy and entropy spectra

Energy and entropy contained in a wavenumber shell of radius \( k \) are called the energy spectrum \( E_u(k) \) and entropy spectrum \( E_\theta(k) \) respectively, i.e.,

\[
E_u(k) = \sum_{k' | k' < k+1} \frac{|\hat{u}(k')|^2}{2},
\]

(5.1)

\[
E_\theta(k) = \sum_{k' | k' < k+1} \frac{|\hat{\theta}(k')|^2}{2}.
\]

(5.2)

Nonlinear interactions lead to a transfer of energy and entropy from smaller wavenumber modes to larger wavenumber modes. These transfers are quantified using energy flux \( \Pi_u(k_0) \) and entropy flux \( \Pi_\theta(k_0) \), which are the fluxes coming out of a wavenumber sphere of radius \( k_0 \) (Verma 2004; Mishra & Verma 2010)

\[
\Pi_u(k_0) = \sum_{k \geq k_0} \sum_{p < k_0} \delta_{k,p+q} \Im \{ (k \cdot \hat{u}(q)) [\hat{u}^*(k) \cdot \hat{u}(p)] \},
\]

(5.3)

\[
\Pi_\theta(k_0) = \sum_{k \geq k_0} \sum_{p < k_0} \delta_{k,p+q} \Im \{ (k \cdot \hat{\theta}(q)) [\hat{\theta}^*(k) \cdot \hat{\theta}(p)] \},
\]

(5.4)

where \( \Im \) is the imaginary part of the argument, and \( k, p, q \) are the wavenumbers of a triad such that \( k = p + q \).

For \( Pr = \infty \), the momentum equation (2.8) yields

\[
\alpha g(\theta_{res}) \approx \frac{\nu u_l}{l^2}.
\]

(5.5)

We assume that \( (\theta_{res})_l = Ra^{-\delta} \theta_l \) with \( \delta \) defined in equation (4.1). We also assume a constant entropy flux, which yields

\[
\epsilon_\theta = \frac{\theta^2 U_L}{d} C_{\epsilon\theta,2} = \frac{\theta_l^2 u_l}{l} C_{\epsilon\theta,2}.
\]

(5.6)

We derive expressions of \( \theta_L \) and \( U_L \) using equations (4.2) and (4.4). After substitutions of these expressions in the above equations we obtain

\[
u_l = (a_2 a_3)^{1/3} \frac{K}{d} Ra^{(3-2\delta-\zeta)/3} \left( \frac{l}{d} \right)^{5/3},
\]

(5.7)

\[
\theta_l = (a_2 a_3)^{1/3} \Delta Ra^{(\delta-\zeta)/3} \left( \frac{l}{d} \right)^{-1/3},
\]

(5.8)

and therefore the energy and entropy spectra are

\[
E_u(k) = \frac{\nu^2}{k} = (a_2 a_3)^{2/3} d \left( \frac{K}{d} \right)^2 Ra^{2(3-2\delta-\zeta)/3} (kd)^{-13/3},
\]

(5.9)

\[
E_\theta(k) = \frac{\theta^2}{k} = (a_2 a_3)^{2/3} d \Delta^2 Ra^{2(\delta-\zeta)/3} (kd)^{-1/3}.
\]

(5.10)
We define normalised spectra $\tilde{E}_u(k)$ and $\tilde{E}_\theta(k)$ as

$$\tilde{E}_u(k) = \frac{E_u(k)}{d^2 \left( \frac{\delta}{2} \right)^2 Ra^{2(3-2\delta-\zeta)/3}} = \left( a_2^2 a_3 \right)^{2/3}(kd)^{-13/3}, \quad (5.11)$$

$$\tilde{E}_\theta(k) = \frac{E_\theta(k)}{d^\Delta R_d^{2(8-\zeta)/3}} = \left( a_2^2 a_3 \right)^{2/3}(kd)^{-1/3}. \quad (5.12)$$

For $Pr = \infty$, the prefactor $(a_2^2 a_3)^{2/3}$ computed using $a_i$’s is approximately $6.5 \times 10^{-2}$.

In figure 11 we plot kinetic spectrum $E_u(k)$ and normalised kinetic spectrum $\tilde{E}_u(k)k^{13/3}$ for $Pr = \infty$ and $Ra = 10^8$. Normalised kinetic spectrum appears nearly constant for more than one decade of wavenumber with $\tilde{E}_u(k) \approx (0.06 \pm 0.02)k^{-13/3}$. This result is in a very good agreement with the predictions based on scaling arguments. Note
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Figure 13. For no-slip box with $Pr = 10^2$ and $Ra = 10^7$, the kinetic energy spectrum $\tilde{E}_u(k)$ (thick red curve), and the compensated kinetic energy spectrum $\tilde{E}_u(k) k^{13/3}$ (thin blue curve). The scaling is similar to that for the free-slip runs, except for a smaller prefactor.

Figure 14. Plot of entropy spectrum $\tilde{E}_\theta(k)$ vs. $k$ for $Pr = \infty$ and $Ra = 10^8$ (free-slip run). The upper branch of the spectrum represents $\tilde{\theta}(0,0,2n)$ modes that exhibits $k^{-2}$ scaling (red dashed curve). The lower branch is somewhat flat.

that the $E_u(k)$ follows neither the Bolgiano-Obukhov nor the Kolmogorov-Obukhov scaling [L'vov & Falkovich 1992; Mishra & Verma 2010] since the velocity field is viscous. The above scaling law for the kinetic energy spectrum also holds for $Pr = 10^2$ and $10^3$ for free-slip boundary condition, and for $Pr = 10^2$ for no-slip boundary condition, as exhibited in figures 12 and 13 respectively. However the prefactors for the free-slip runs are larger than that for the no-slip run due to lower frictional forces for the free-slip simulations.

The entropy spectrum, however, is more complex. In figure 14 we plot the entropy spectrum $\tilde{E}_\theta(k)$ for $Pr = \infty$ and $Ra = 10^8$, which exhibits a dual branch. The upper branch, which corresponds to the $\tilde{\theta}(0,0,2n)$ Fourier modes, follows $k^{-2}$ energy spectrum since $\tilde{\theta}(0,0,2n) \approx -1/2n\pi$ (see §2 and Mishra & Verma 2010). The lower branch is
Figure 15. Plot of entropy spectrum \( \tilde{E}_\theta(k) \) for \( Pr = 10^2 \) and \( Ra = 10^7 \) (free-slip run), which exhibits similar behaviour as \( Pr = \infty \).

Figure 16. Plot of entropy spectrum \( \tilde{E}_\theta(k) \) for \( Pr = 10^2 \) and \( Ra = 10^7 \) (no-slip run), which exhibits similar scaling as large-\( Pr \) free-slip runs.

the energy spectrum of the Fourier modes other than \( \hat{\theta}(0,0,2n) \), and it follows nearly a flat spectrum. The nature of the entropy spectrum is very different from our phenomenological predictions that \( E_\theta(k) \propto k^{-1/3} \). This discrepancy is due to the boundary condition (the conducting plates) as described in §2 and Mishra & Verma (2010). Similar behaviour is observed for \( Pr = 10^2 \) with the free-slip boundary condition, and for \( Pr = 10^2 \) with no-slip boundary condition, as exhibited in figures 15 and 16 respectively.

We also compute the entropy flux \( \Pi_\theta(k_0) \) using the numerical data. In Figure 17 we plot \( \Pi_\theta(k) \) for \( Pr = \infty \) and \( Ra = 10^8 \), and for \( Pr = 10^2 \) and \( Ra = 10^7 \). The plots indicate a nearly constant entropy flux in the inertial regime. The consistency of \( \Pi_\theta(k) \) is due to the dominance of nonlinear term in the temperature equation. The kinetic energy flux \( \Pi_u(k) \) however is zero for \( Pr = \infty \) due to the absence of the nonlinearity in the velocity equation. For large \( Pr \) runs, \( \Pi_u(k) \) is very small due to weak nonlinearity.

Table 4 summarises the scaling results for \( Pr = \infty \) simulation under free-slip boundary
RBC is very similar to that for \( Pr = 0 \). They are somewhat limited at present.

The estimated values using analytical arguments agree quite well with the numerically computed ones. The formulas for the scaling functions using \( \theta \) are taken from Grossmann & Lohse (2001)'s theoretical work (the \( \Gamma \) regime). The computed \( \theta \) is taken from Silano et al. (2010). The estimated \( PC \) and \( Nu \) scaling are from Grossmann & Lohse (2001)'s theoretical work (the \( \Gamma \) regime).

The computed \( \theta \) is taken from Silano et al. (2010). The estimated \( PC \) and \( Nu \) scaling are from Grossmann & Lohse (2001)'s theoretical work (the \( \Gamma \) regime).

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Table 4. Summary of the scaling functions for various quantities for free-slip runs with \( Pr = \infty \). The estimated values using analytical arguments agree quite well with the numerically computed ones. The formulas for the scaling functions using \( \theta \)’s are listed in the second column.

| Quantity | Formula          | Estimated          | Computed                                      |
|----------|------------------|--------------------|-----------------------------------------------|
| \( \theta_{rms}/\Delta \) | \( a_1 Ra^{-5} \) | –                  | \( (0.58 \pm 0.09) Ra^{-0.15\pm0.01} \)          |
| \( \theta_{rms}/\Delta \) | \( a_2 \)          | –                  | \( (0.29 \pm 0.01) \)                           |
| \( PC \) | \( a_3 Ra^{-1} \) | \( 0.16 Ra^{0.62} \) | \( (0.20 \pm 0.02) Ra^{0.61\pm0.01} \)          |
| \( C_{u,\theta} \) | \( a_4 Ra^{-0.15} \) | –                  | \( (3.7 \pm 0.6) Ra^{-0.15\pm0.01} \)           |
| \( Nu \) | \( a_5 Ra^{-7} \) | \( 0.43 Ra^{0.32} \) | \( (0.21 \pm 0.03) Ra^{0.33\pm0.01} \)          |
| \( C_{e} \) | \( \frac{a_6}{Ra^{0.5}} Ra^{\tau+2} \) | \( 5.2 Ra^{0.09} \) | \( (6.4 \pm 1.6) Ra^{-0.11\pm0.02} \)          |
| \( \tilde{E}_c(k) \) | \( (a_7 a_8)^{2/3} k^{-13/3} \) | \( 0.065 k^{-13/3} \) | \( (0.06 \pm 0.02) k^{-13/3} \)              |
| \( \tilde{E}_\theta(k) \) | \( (a_9 a_8)^{2/3} k^{-13/3} \) | \( 0.065 k^{-1/3} \) | Dual branches                                |

Table 5. Summary of the scaling functions for various quantities for no-slip runs with \( Pr = 10^2 \). The computed \( \theta_{rms} \) is taken from Silano et al. (2010). The estimated \( PC \) and \( Nu \) scaling are taken from Grossmann & Lohse (2001)'s theoretical work (the \( \Gamma \) regime).

The estimated values using analytical arguments agree quite well with the numerically computed ones. The formulas for the scaling functions using \( \theta \) are taken from Grossmann & Lohse (2001)'s theoretical work (the \( \Gamma \) regime). The computed \( \theta \) is taken from Silano et al. (2010). The estimated \( PC \) and \( Nu \) scaling are from Grossmann & Lohse (2001)'s theoretical work (the \( \Gamma \) regime).

condition, and table \[5\] for \( Pr = 10^2 \) simulation under no-slip boundary condition. Here we list the theoretically-estimated as well as the numerically-computed values. The two sets of values are in good agreement with each other. The scaling for large \( Pr \) \( (Pr \geq 100) \) RBC is very similar to that for \( Pr = \infty \) RBC. Our data for no-slip boundary condition are somewhat limited at present.
6. Conclusions and discussions

In this paper we derive scaling properties of the large-scale quantities (e.g., Péclet and Nusselt numbers), as well as that of energy and entropy spectra for infinite Prandtl number convection. In the above derivation we exploit the linear equation for the velocity field. We also verify these scaling relations with numerical results for the infinite and large Prandtl numbers ($Pr \geq 100$). We observe that the scaling of flows with large Prandtl number ($Pr \geq 100$) is same as that for the infinite Prandtl number. These relations are independent of the Prandtl number. Our results are consistent with the earlier theoretical predictions of Grossmann & Lohse (2001), experimental results of Xia et al. (2002), and the numerical results of Silano et al. (2010). Note however that the analytical work of Grossmann & Lohse (2001) is based on Shraiman & Siggia (1990)'s exact relations and modelling of the dissipation rates at the bulk and boundary layers. While our theoretical work is based on the dimensional and scaling analysis of the dynamical equation of the velocity and temperature fields, as well as several inputs from the numerical simulations.

A summary of our results is as follows.

(a) The temperature field is dominated by the Fourier modes $\hat{\theta}(0, 0, 2n)$, which are approximately $-1/(2n\pi)$ for small $n$ in accordance with the predictions by Mishra & Verma (2010). The modes other than $\hat{\theta}(0, 0, 2n)$ are termed as “residual modes” whose rms fluctuations scale as $Ra^{-\delta}$ with $\delta \approx 0.15$. Due to the dominance of $\hat{\theta}(0, 0, 2n)$ modes, the large-scale temperature field is approximately $\sqrt{\sum |\hat{\theta}(0, 0, 2n)|^2} \sim \Delta$, where $\Delta$ is the temperature difference between the hot and cold plates. These results are in good agreement with those of Silano et al. (2010). We show that the residual modes play a very important role in the scaling of Nusselt number, energy spectrum, etc.

(b) The Péclet number, which is proportional to the large-scale velocity, scales as $Ra^{1-\zeta}$ with $\zeta \approx 0.38$. Note that the Reynolds number in the large $Pr$ limit is small, i.e., $Re \ll 1$. These results are consistent with the theoretical predictions of Grossmann & Lohse (2001) and the numerical results of Silano et al. (2010).

(c) The Nusselt number scales as $Ra^\gamma$ with the exponent lying in the range from 0.29 to 0.33, which is consistent with the results of Grossmann & Lohse (2001), Silano et al. (2010), Roberts (1979), Xia et al. (2002), and Constantin & Doering (1999). This scaling arises due to a complex interplay between the residual modes, Péclet number, and the velocity-temperature correlation.

(d) The normalised viscous and entropy dissipation rates are functions of $Ra$. We observe that

$$C_{\epsilon_u} = \frac{\epsilon_u}{\nu U_L^2/d^2} \approx Ra^{b_1},$$

$$C_{\epsilon_\theta,2} = \frac{\epsilon_{\theta}}{U_L \theta_L^2/d} \approx Ra^{-b_2},$$

with $b_1 \approx 0.11$ and $b_2 \approx 0.29$. These relations are consistent with the $Nu$ scaling derived using the exact relations of Shraiman & Siggia (1990). Here we derive an explicit $Ra$-dependent normalised dissipation rates for large $Pr$ for the first time.

(e) Using analytical arguments, we derive that the energy spectrum $E_u(k) \sim k^{-13/3}$. Our simulations verify this power law in the inertial range for both the free-slip and no-slip boundary conditions.

(f) We predict that the entropy spectrum $E_{\theta}(k) \sim k^{-1/3}$. Unfortunately this power law is not observed in the numerical simulations. Instead, we find a dual branch consisting of an upper branch with $k^{-2}$ spectrum corresponding to $\hat{\theta}(0, 0, 2n) \approx -1/(2n\pi)$,
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and a nearly flat lower branch. The dual branch is due to the presence of boundary layers [Mishra & Verma (2010)].

Our numerical simulations show that the free-slip and no-slip boundary conditions provide similar scaling relations for the global quantities, as well as for the energy and entropy spectra. However, the prefactors of the Péclet and Nusselt numbers, and that of energy spectrum are smaller for the no-slip condition than that for free-slip condition. This discrepancy is due to smaller frictional force experienced by the flow for the free-slip boundary condition. The similarities of the scaling functions between the free-slip and no-slip convection are due to the dominance of the large-scale flows, which have similar structures for both the free-slip and no-slip boundary conditions. Note that viscous boundary layers pervade the whole box for $Pr = \infty$, hence they determine the properties of the bulk flow.

In summary, we derived scaling relations for large-scale quantities, and energy and entropy spectra for large and infinite Prandtl number convection. The scaling properties are independent of the Prandtl number in this regime. Our analytical and numerical results are consistent with earlier results of Grossmann & Lohse (2001), Silano et al. (2010), and Xia et al. (2002).

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REFERENCES

Ahlers, G., Funfschilling, D., & Bodenschatz, E. 2009a Transitions in heat transport by turbulent convection at Rayleigh numbers up to $10^{15}$. New J. Phys. 11, 123001.

Ahlers, G., Grossmann, S., & Lohse, D. 2009b Heat transfer and large scale dynamics in turbulent Rayleigh-Bénard convection. Rev. Mod. Phys. 81, 503–537.

Ahlers, G. & Xu, X. 2001 Prandtl-number dependence of heat transport in turbulent Rayleigh-Bénard convection. Phys. Rev. Lett. 86, 3320–3323.

Batchelor, G. K. 1959 Small scale variation of convected quantities like temperature in a turbulent fluid. J. Fluid Mech. 5, 113.

Breuer, M., Wessling, S., Schmalzl, J., & Hansen, U. 2004 Effect of inertia in Rayleigh-Bénard convection. Phys. Rev. E 69, 026302.

Castaing, B., Gunaratne, G., Kadanoff, L., Libchaber, A., & Heslot, F. 1989 Scaling of hard thermal turbulence in Rayleigh-Bénard convection. J. Fluid Mech. 204, 1.

Chavanis, X., Chilla, F., Castaing, B., Hebral, B., Chabaud, B. & Chauzy, J. 1997 Observation of the ultimate regime in Rayleigh-Bénard convection. Phys. Rev. Lett. 79, 3648–3651.

Cioni, S., Ciliberto, S., & Sommeria, J. 1997 Strongly turbulent Rayleigh-Bénard convection in mercury: comparison with results at moderate Prandtl number. J. Fluid Mech. 335, 111.

Constantin, P. & Doering, C. R. 1999 Infinite Prandtl number convection. J. Stat. Phys. 94, 159–172.

Fischer, P. F. 1997 An overlapping Schwarz method for spectral element solution of the incompressible Navier-Stokes equations. J. Comp. Phys. 133 (1), 84–101.

Funfschilling, D., Bodenschatz, E. & Ahlers, G. 2009 Search for the “ultimate state” in turbulent Rayleigh-Bénard convection. Phys. Rev. Lett. 103, 014503.

Glazier, J., Segawa, T., Naert, A. & Sano, M. 1999 Evidence against ’ultrahard’ thermal turbulence at very high Rayleigh numbers. Nature 398, 307–310.

Grossmann, S. & Lohse, D. 2000 Scaling in thermal convection: a unifying theory. J. Fluid Mech. 407, 27.

Grossmann, S. & Lohse, D. 2001 Thermal convection for large Prandtl numbers. Phys. Rev. Lett. 86, 3316.
Grossmann, S. & Lohse, D. 2002 Prandtl and Rayleigh number dependence of the Reynolds number in turbulent thermal convection. Phys. Rev. E 66, 016305.

Grossmann, S. & Lohse, D. 2004 Fluctuations in turbulent Rayleigh-Bénard convection: The role of plumes. Phys. Fluids 16, 4462.

Grossmann, S. & Lohse, D. 2011 Multiple scaling in the ultimate regime of thermal convection. Phys. Fluids 23, 045108.

Hansen, U., Yuen, D. A. & Kroening, S. E. 1990 Transition to hard turbulence in thermal convection at infinite Prandtl number. Phys. Fluids 2, 2157.

Clinical, G. R., Kerswell, R. R. & Plasting, S. C. 2006 Infinite-Prandtl-number convection: part 2. a singular limit of upper bound theory. J. Fluid Mech. 560, 159–227.

Kerr, R. 1996 Rayleigh number scaling in numerical convection. J. Fluid Mech. 310, 139–179.

Kerr, R. & Herring, J. 2000 Prandtl number dependence of Nusselt number in direct numerical simulations. J. Fluid Mech. 419, 325–344.

Kraichnan, R. 1962 Turbulent thermal convection at arbitrary Prandtl number. Phys. Fluids 5, 1374.

Lam, S., Shang, X.-D., Zhou, S.-Q. & Xia, K.-Q. 2002 Prandtl number dependence of the viscous boundary layer and the Reynolds numbers in Rayleigh-Bénard convection. Phys. Rev. E 65, 066306.

Lohse, D. & Toschi, F. 2003 Ultimate state of thermal convection. Phys. Rev. Lett. 90.

Lohse, D. & Xia, K. Q. 2010 Small-scale properties of turbulent Rayleigh-Bénard convection. Ann. Rev. Fluid Mech. 42, 335–364.

L’vov, V. S. 1991 Spectra of velocity and temperature fluctuations with constant entropy flux of fully developed free-convective turbulence. Phys. Rev. Lett. 67, 687.

L’vov, V. S. & Falkovich, G. 1992 Conservation laws and two-flux spectra of hydrodynamic convective turbulence. Physica D 57, 85.

Mishra, P. K. & Verma, M. K. 2010 Energy spectra and fluxes for Rayleigh-Bénard convection. Phys. Rev. E 81, 056316.

Niemela, J. J., Skrbek, L., Sreenivasan, K. R. & Donnelly, R. J. 2000 Turbulent convection at very high Rayleigh numbers. Nature 404, 837–840.

Niemela, J. J. & Sreenivasan, K. R. 2003 Confined turbulent convection. J. Fluid Mech. 481, 355–384.

Robert, G. O. 1979 Fast viscous Bénard convection. Geophys. Astrophys. Fluid Dyn. 12, 235–272.

Roche, P. -E., Castaing, B., Chabaud, B. & Hébral, B. 2001 Observation of the 1/2 power law in Rayleigh-Bénard convection. Phys. Rev. E 63, 045303(R).

Schmalzl, J., Breuer, M. & Hansen, U. 2002 The influence of the Prandtl number on the style of vigorous thermal convection. Geophys. Astrophys. Fluid Dyn. 96, 381–403.

Shraiman, B. I. & Siggia, E. 1990 Heat transport in high-Rayleigh-number convection. Phys. Rev. A 42, 3650–3653.

Silano, G., Sreenivasan, K. R. & Verzicco, R. 2010 Numerical simulations of Rayleigh–Bénard convection for Prandtl numbers between $10^{-3}$ and $10^9$ and Rayleigh numbers between $10^5$ and $10^9$. J. Fluid Mech. 662, 409–446.

Stevens, R., Clercx, H. & Lohse, D. 2010a Optimal Prandtl number for heat transfer in rotating Rayleigh-Bénard convection. New J. Phys. 12, 075005.

Stevens, R., Lohse, D. & Verzicco, R. 2011 Prandtl and Rayleigh number dependence of heat transport in high Rayleigh number thermal convection. J. Fluid Mech. 688.

Stevens, R., Poel, E. P., Grossmann, S. & Lohse, D. 2013 The unifying theory of scaling in thermal convection: The updated prefactors. arXiv:1301.7096.

Stevens, R., Verzicco, R. & Lohse, D. 2010b Radial boundary layer structure and Nusselt number in Rayleigh-Bénard convection. J. Fluid Mech. 643.

Verma, M. K. 2004 Statistical theory of magnetohydrodynamic turbulence: Recent results. Phys. Rep. 401, 229–380.

Verma, M. K., Chatterjee, A. G., Reddy, K. S., Yadav, R. K., Paul, S., Chandra, M. & Samtanay, R. 2013a Benchmarking and scaling studies of a pseudospectral code tarang for turbulence simulations. Submitted to Pramana.
Verma, M. K., Mishra, P. K., Pandey, A. & Paul, S. 2012 Scalings of field correlations and heat transport in turbulent convection. *Phys. Rev. E* **85**, 016310.

Verma, M. K., Pandey, A., Mishra, P. K. & Chandra, M. 2013b Role of bulk flow in turbulent convection. *arXiv:1301.1240*.

Verzicco, R. & Camussi, R. 1999 Prandtl number effects in convective turbulence. *J. Fluid Mech.* **383**, 55–73.

Verzicco, R. & Sreenivasan, K. 2008 A comparison of turbulent thermal convection between conditions of constant temperature and constant heat flux. *J. Fluid Mech.* **595**, 203–219.

Whitehead, J. P. & Doering, C. R. 2011 Ultimate state of two-dimensional Rayleigh-Bénard convection between free-slip fixed-temperature boundaries. *Phys. Rev. Lett.* **106**, 244501–1–4.

Xia, K.-Q., Lam, S. & Zhou, S.-Q. 2002 Heat-flux measurement in high-Prandtl-number turbulent Rayleigh-Bénard convection. *Phys. Rev. Lett.* **88**, 064501.