Stark units in positive characteristic

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Abstract
We show that the module of Stark units associated to a sign-normalized rank one Drinfeld module can be obtained from Anderson’s equivariant \(A\)-harmonic series. We apply this to obtain a class formula à la Taelman and to prove a several variable log-algebraicity theorem, generalizing Anderson’s log-algebraicity theorem. We also give another proof of Anderson’s log-algebraicity theorem using shtukas and obtain various results concerning the module of Stark units for Drinfeld modules of arbitrary rank.

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Introduction

The power-series \(\sum_{n \geq 1} \frac{z^n}{n}\) is log-algebraic:

\[\sum_{n \geq 1} \frac{z^n}{n} = -\log(1 - z).\]

This identity allows one to obtain the value of a Dirichlet \(L\)-series at \(s = 1\) as an algebraic linear combination of logarithms of circular units. Inspired by examples of Carlitz [1] and Thakur [31], Anderson [1, 2] proved an analogue of this identity for a sign-normalized rank one Drinfeld \(A\)-module, known as Anderson’s log-algebraicity theorem.

When \(A = \mathbb{F}_q[\theta]\) (the genus 0 case), various works [1-3, 7, 8, 10, 13, 17-19, 25-28] have revealed the importance of certain units in the study of special values of the Goss \(L\)-functions at \(s = 1\). To give a simple example, the Carlitz module is considered to play the role of the multiplicative group \(\mathbb{G}_m\) over \(\mathbb{Z}\), and Anderson [1, 2] showed that the images through the Carlitz exponential of some special units give algebraic elements which are the equivalent of the circular units. The special units constructed in such a way are then ‘log-algebraic’. Recently, Taelman [27, 28] introduced the module of units attached to any Drinfeld module and proved a class formula which states that the special value of the Goss \(L\)-function attached to a Drinfeld module at \(s = 1\) is the product of a regulator term arising from the module of units and an algebraic term arising from a certain class module. Also, deformations of Goss \(L\)-series values in Tate algebras are investigated by Pellarin and two of the authors [7, 8, 10, 26]. For higher dimensional versions of Drinfeld modules, we refer the reader to [3, 13, 17-19, 25]. We should
mention that all these works are based on a crucial fact that \( \mathbb{F}_q[\theta] \) is a principal ideal domain, which is no longer true in general.

In the present paper, we develop a new method to deal with higher genus cases. Let \( \phi \) be a Drinfeld \( A \)-module defined over the integral closure \( O_L \) of \( A \) in a finite extension \( L/K \). In [10], Section 2 and in the case \( A = \mathbb{F}_q[\theta] \), two of the authors introduced a sub-module of Taelman’s module of units for \( \phi/O_L \) that is called the module of Stark units attached to \( \phi/O_L \). This terminology is motivated by the fact that when \( L \) is a finite abelian extension of \( K \) and \( \phi \) is the Carlitz module, this module is an analogue of cyclotomic units (see [2, Section 4; 9, Section 7; 10, paragraph 2.6]). We extend the previous work [10] and make a systematic study of these modules of Stark units. For a sign-normalized rank one Drinfeld module, we prove a direct link between the module of Stark units and Anderson’s equivariant \( \phi/A \)-harmonic series, which is an analogue of Stark’s conjectures. It allows us to obtain a class formula à la Taelman and a several variable log-algebraicity theorem in the general context.

Let us give now more precise statements of our results.

Let \( K/\mathbb{F}_q \) be a global function field (\( \mathbb{F}_q \) is algebraically closed in \( K \)), let \( A \) be the ring of elements of \( K \) which are regular outside a fixed place \( \infty \) of \( K \) of degree \( d_\infty \geq 1 \). The completion \( K_\infty \) of \( K \) at the place \( \infty \) has residue field \( \mathbb{F}_q \) and is endowed with the \( \infty \)-adic valuation \( v_\infty : K_\infty \rightarrow \mathbb{Z} \cup \{+\infty\} \). For a \( \in A \), we set: \( \deg a := -d_\infty v_\infty(a) \). We fix an algebraic closure \( \overline{K}_\infty \) of \( K_\infty \), and still denote \( v_\infty : \overline{K}_\infty \rightarrow \mathbb{Q} \cup \{+\infty\} \) the extension of \( v_\infty \) to \( \overline{K}_\infty \). Let \( \tau : \overline{K}_\infty \rightarrow \overline{K}_\infty \) be the \( \mathbb{F}_q \)-algebra homomorphism which sends \( x \) to \( x^q \).

We choose a sign function \( \text{sgn} : \overline{K}_\infty \rightarrow \mathbb{F}_q \), that is, a group homomorphism such that \( \text{sgn} |_{\overline{K}_\infty} = 1 \). Let \( \phi : A \hookrightarrow \overline{K}_\infty \{\tau\} \) be a sign-normalized rank one Drinfeld module (see Section 3.2), that is, there exists an integer \( i(\phi) \in \mathbb{N} \) such that:

\[ \forall a \in A, \quad \phi_a = a + \cdots + \text{sgn}(a)^{i(\phi)} \tau^{\deg a}. \]

Then, the exponential series attached to \( \phi \) is the unique element \( \exp_\phi \in \overline{K}_\infty \{\tau\} \), such that \( \exp_\phi \equiv 1 \pmod{\tau} \), and:

\[ \forall a \in A, \quad \exp_\phi a = \phi_a \exp_\phi. \]

If we write

\[ \exp_\phi = \sum_{i \geq 0} e_i(\phi) \tau^i, \]

with \( e_i(\phi) \in \overline{K}_\infty \), then the field \( H := K(e_i(\phi), i \in \mathbb{N}) \) is a finite abelian extension of \( K \) which is unramified outside \( \infty \) (see Section 3.2). Let \( B \) be the integral closure of \( A \) in \( H \). For all \( a \in A \), we have

\[ \phi_a \in B\{\tau\}. \]

For a non-zero ideal \( I \) of \( A \), we define \( \phi_I \in H\{\tau\} \) to be the monic element in \( H\{\tau\} \) such that

\[ H\{\tau\} \phi_I = \sum_{a \in I} H\{\tau\} \phi_a. \]

In fact, \( \phi_I \in B\{\tau\} \) and we denote its constant term by \( \psi(I) \in B \setminus \{0\} \).

For simplicity, we will work over the abelian extension \( H/K \). We should mention that the results presented below are still valid for any finite abelian extension \( E/K \) such that \( H < E \).

Let \( G = \text{Gal}(H/K) \). For a non-zero ideal \( I \) of \( A \), we denote by \( \sigma_I = (I, H/K) \in G \), where \((\cdot, H/K) \) is the Artin map. Let \( z \) be an indeterminate over \( K_\infty \) and let \( \mathbb{T}_z(K_\infty) \) be the Tate algebra in the variable \( z \) with coefficients in \( K_\infty \). Let’s set

\[ H_\infty = H \otimes_K K_\infty, \]
and
\[ T_z(H_\infty) = H \otimes_K T_z(K_\infty). \]
Let \( \tau : T_z(H_\infty) \to T_z(H_\infty) \) be the continuous \( \mathbb{F}_q[z] \)-algebra homomorphism such that
\[ \forall x \in H_\infty, \quad \tau(x) = x^q. \]
We set
\[ \exp_\phi = \sum_{i \geq 0} e_i(\phi) z^i \tau^i \in H[z]\{\{\tau}\}. \]
Then \( \exp_\phi \) converges on \( T_z(H_\infty) \). Following [10], we introduce the module of \( z \)-units attached to \( \phi/B \):
\[ U(\tilde{\phi}/B[z]) = \left\{ f \in T_z(H_\infty), \exp_\phi(f) \in B[z] \right\}. \]
We denote by \( \text{ev} : T_z(H_\infty) \to H_\infty \) the evaluation at \( z = 1 \). The module of Stark units attached to \( \phi/B \) is defined by (see [10, Section 2])
\[ U_{St}(\phi/B) = \text{ev}(U(\tilde{\phi}/B[z])) \subset H_\infty. \]
Then \( U_{St}(\phi/B) \) is an \( A \)-module in \( H_\infty \) (see Theorem 2.7), that is, \( U_{St}(\phi/B) \) is a \( A \)-module which is discrete and cocompact in \( H_\infty \). In fact, \( U_{St}(\phi/B) \) is contained in the \( A \)-module of the Taelman module of units [27] defined by
\[ H(\phi/B) = \frac{H_\infty}{B + \exp_\phi(H_\infty)}. \]
Following Anderson [1], we introduce the following series (see Section 3.3):
\[ \mathcal{L}(\phi/B; 1; z) = \sum_{\sigma I \in T_z(H_\infty)[G]} \frac{z^{\deg I}}{\psi(I)} \sigma I, \]
where the sum runs through the non-zero ideals \( I \) of \( A \). The equivariant \( A \)-harmonic series attached to \( \phi/B \) is defined by
\[ \mathcal{L}(\phi/B) = \text{ev} (\mathcal{L}(\phi/B; 1; z)) \in H_\infty[G]. \]
One of our main theorems states (see Theorem 3.8) that the module of Stark units \( U_{St}(\phi/B) \) can be obtained from the equivariant \( A \)-harmonic series \( \mathcal{L}(\phi/B) \), which is reminiscent of Stark’s Conjectures [30]:

**Theorem A.** We have
\[ U(\tilde{\phi}/B[z]) = \mathcal{L}(\phi/B; 1; z)B[z]. \]
In particular,
\[ U_{St}(\phi/B) = \mathcal{L}(\phi/B)B. \]
We will present several applications of this theorem.

Firstly, we apply Theorem A to obtain a class formula à la Taelman for \( \phi/B \), by a different method of Taelman’s original one [28]. Roughly speaking, we introduce the Stark regulator
(respectively the regulator defined by Taelman [28]) attached to $\phi/B$ by $[B : U_{St}(\phi/B)]_A \in \mathcal{K}_L^\infty$ (respectively $[B : U(\phi/B)]_A \in \mathcal{K}_L^\infty$) (see Section 2.3). We show (see Theorem 2.7):

**Theorem B.** We have

$$Fitt_A \frac{U(\phi/B)}{U_{St}(\phi/B)} = Fitt_A H(\phi/B),$$

where, for a finite $A$-module $M$, $Fitt_A M$ is the Fitting ideal of $M$.

Observe that $\mathcal{L}(\phi/B)$ induces a $\mathcal{K}_L^\infty$-linear map on $H_L$, and we denote by $\det_{\mathcal{K}_L^\infty} \mathcal{L}(\phi/B)$ its determinant. For a maximal ideal $\mathfrak{P}$ of $B$, by class field theory (see, for example, [21, Section 7.4]), we know that the Fitting ideal $Fitt_A B/\mathfrak{P}$ of $A$ is principal. We denote by $[B/\mathfrak{P}]_A$ the element of $A$ such that $\text{sgn}(B/\mathfrak{P}]_A) = 1$ and $Fitt_A B/\mathfrak{P} = [B/\mathfrak{P}]_A A$. We prove the following formula (see Theorem 3.6):

$$\det_{\mathcal{K}_L^\infty} \mathcal{L}(\phi/B) = \zeta_B(1) := \prod_{\mathfrak{P}} \left(1 - \frac{1}{[B/\mathfrak{P}]_A}\right)^{-1} \in \mathcal{K}_L^\infty,$$

where $\mathfrak{P}$ runs through the maximal ideals of $B$.

Note that $\zeta_B(1)$ is a special value at $s = 1$ of some zeta function $\zeta_B(s)$ introduced by Goss (see [21, Chapter 8]). Therefore, Theorem A and Theorem B imply Taelman’s class formula for $\phi/B$ (see Theorem 3.10 and see Section 1 for the definition of $[H(\phi/B)]_A$):

**Theorem C.** We have

$$\zeta_B(1) = [B : U_{St}(\phi/B)]_A = [B : U(\phi/B)]_A [H(\phi/B)]_A.$$

When the genus of $K$ is zero and $d_\infty = 1$, Taelman’s class formula, its higher dimensional versions, and its arithmetic consequences are now well understood due to the recent works [7, 9, 10, 13, 17–19, 27, 28]. All these works are based on the crucial fact that when $g = 0$ and $d_\infty = 1$, the ring $A$ is a principal ideal domain (when $A$ is not assumed to be principal, the existence of a class formula is still an open problem in general). Using the module of Stark units, we are able to overcome this difficulty, and Theorem C provides a large class of examples of Taelman’s class formula when $A$ is no longer principal. We refer the reader to Section 2.4 for a more detailed discussion.

Secondly, we apply Theorem A to prove a several variable log-algebraicity theorem, generalizing Anderson’s log-algebraicity theorems (Theorem 4.2). We recall that the theorem below is valid for any finite abelian extension $E/K$, $H \subset E$ (see Theorem 4.2 for the precise statement).

**Theorem D.** Let $n \geq 0$ and let $X_1, \ldots, X_n, z$ be $n + 1$ indeterminates over $K$. Let $\tau : K[X_1, \ldots, X_n][[z]] \rightarrow K[X_1, \ldots, X_n][[z]]$ be the continuous $\mathbb{F}_q[[z]]$-algebra homomorphism for the $z$-adic topology such that $\forall x \in K[X_1, \ldots, X_n]$, $\tau(x) = x^q$. Then

$$\forall b \in B, \quad \exp_{\phi} \left( \sum_I \frac{\sigma_I(b)}{\psi(I)} \phi_I(X_1) \cdots \phi_I(X_n) z^{\deg I} \right) \in B[X_1, \ldots, X_n, z],$$

where $I$ runs through the non-zero ideals of $A$. 
For $n \leq 1$ and $d_\infty = 1$, this theorem was due to Anderson [1, Theorem 5.1.1; 2, Theorem 3]:
\[
\forall b \in B, \quad \exp_\phi \left( \sum_I \frac{\sigma_I(b)}{\psi(I)} z^{\deg I} \right) \in B[z],
\]
\[
\forall b \in B, \quad \exp_\phi \left( \sum_I \frac{\sigma_I(b)}{\psi(I)} \phi_I(X) z^{\deg I} \right) \in B[X, z],
\]
where the sum runs through the non-zero ideals of $A$. Again, this result is now well understood when the genus of $K$ is zero (and $d_\infty = 1$) due to the recent works of many people ([7–10, 28, 33] Sections 8.9 and 8.10, and the forthcoming work of Papanikolas [25]). However, to our knowledge, Anderson’s log-algebraicity remains quite mysterious for $g > 0$ until now.

Thirdly, we present an alternative approach to the previous several variable log-algebraicity theorem (Theorem D) via Drinfeld’s correspondence between Drinfeld modules and shtukas. Using the shtuka function attached to $\phi/B$ via Drinfeld’s correspondence, we introduce one variable versions of the previous objects, that is, the modules of $z$-units and Stark units, the equivariant $A$-harmonic series and the $L$-series (see Section 4.2). We prove an analogue of Theorem A in this one variable context (see Theorem 4.9). More generally, we also obtain a several variable log-algebraicity theorem (see Corollary 4.10). In the case $g = 0$ and $d_\infty = 1$, we rediscover the Pellarin’s $L$-series [26] and its several variable variants studied in [5, 7, 8, 10]. We deduce from this another proof of Theorem D (see Section 4.4).

Finally, we prove some results concerning the module of Stark units for Drinfeld modules of arbitrary rank in Section 2. In particular, Theorem B is still valid for any Drinfeld module.

1. Notation

Let $K/\mathbb{F}_q$ be a global function field of genus $g$, where $\mathbb{F}_q$ is a finite field of characteristic $p$, having $q$ elements ($\mathbb{F}_q$ is algebraically closed in $K$). We fix a place $\infty$ of $K$ of degree $d_\infty$, and denote by $A$ the ring of elements of $K$ which are regular outside of $\infty$. The completion $K_\infty$ of $K$ at the place $\infty$ has residue field $\mathbb{F}_q$ and comes with the $\infty$-adic valuation $v_\infty : K_\infty \rightarrow \mathbb{Z} \cup \{+\infty\}$. We fix an algebraic closure $\bar{K}_\infty$ of $K_\infty$ and still denote by $v_\infty : \bar{K}_\infty \rightarrow \bar{\mathbb{Q}} \cup \{+\infty\}$ the extension of $v_\infty$ to the completion $\bar{\mathbb{C}}_\infty$ of $\bar{K}_\infty$.

We will fix a uniformizer $\pi$ of $K_\infty$. Set $\pi_1 = \pi$, and for $n \geq 2$, choose $\pi_n \in \bar{K}_\infty^\times$ such that $\pi_n^\infty = \pi_{n-1}$. If $z \in \bar{\mathbb{Q}}$, $z = \frac{m}{n!}$ for some $m \in \mathbb{Z}$, $n \geq 1$, we set
\[
\pi^z := \pi^m.
\]
Let $\bar{\mathbb{F}}_q$ be the algebraic closure of $\mathbb{F}_q$ in $\bar{K}_\infty$, and let $U_\infty = \{ x \in \bar{K}_\infty, v_\infty(x - 1) > 0 \}$. Then
\[
\bar{K}_\infty^\times = \pi^\infty \times \bar{\mathbb{F}}_q^\times \times U_\infty.
\]
Therefore, if $x \in \bar{K}_\infty^\times$, one can write in a unique way
\[
x = \pi^{v_\infty(x)} \text{sgn}(x)x, \quad \text{sgn}(x) \in \bar{\mathbb{F}}_q^\times, x \in U_\infty.
\]

Let $\mathcal{I}(A)$ be the group of non-zero fractional ideals of $A$. For $I \in \mathcal{I}(A), I \subset A$, we set
\[
\deg I := \dim_{\mathbb{F}_q} A/I.
\]
Then, the function $\deg$ on non-zero ideals of $A$ extends into a group homomorphism
\[
\deg : \mathcal{I}(A) \rightarrow \mathbb{Z}.
\]
Let’s observe that, for $x \in K^\times$, we have
\[
\deg(x) := \deg(xA) = -d_\infty v_\infty(x).
\]
Let \( I \in \mathcal{I}(A) \), then there exists an integer \( h \geq 1 \) such that \( I^h = xA, \ x \in K^\times \). We set
\[
\langle I \rangle := \langle x \rangle^{1/h} \in U_\infty.
\]
Then one shows (see [21, Section 8.2]) that the map \([\cdot] : \mathcal{I}(A) \to K_\infty^\times, I \mapsto \langle I \rangle \pi^{-\deg I/d_\infty} \) is a group homomorphism such that
\[
\forall x \in K^\times, \ [xA] = \frac{x}{\sgn(x)}.
\]
Observe that
\[
\forall I \in \mathcal{I}(A), \ \sgn([I]) = 1.
\]
If \( M \) is a finite \( A \)-module, and \( \text{Fitt}_A(M) \) is the Fitting ideal of \( M \), we set
\[
[M]_A := [\text{Fitt}_A(M)].
\]
Let’s observe that, if \( 0 \to M_1 \to M \to M_2 \to 0 \) is a short exact sequence of finite \( A \)-modules, then
\[
[M]_A = [M_1]_A[M_2]_A.
\]
Let \( E/K \) be a finite extension, and let \( O_E \) be the integral closure of \( A \) in \( E \). Let \( \mathcal{I}(O_E) \) be the group of non-zero fractional ideals of \( O_E \). We denote by \( N_{E/K} : \mathcal{I}(O_E) \to \mathcal{I}(A) \) the group homomorphism such that, if \( \mathfrak{P} \) is a maximal ideal of \( O_E \) and \( P = \mathfrak{P} \cap A \), we have
\[
N_{E/K}(\mathfrak{P}) = P^{[O_E/\mathfrak{P} : A/P]}.
\]
Note that, if \( \mathfrak{P} = xO_E, x \in E^\times \), then
\[
N_{E/K}(\mathfrak{P}) = N_{E/K}(x)A,
\]
where \( N_{E/K} : E \to K \) also denotes the usual norm map.

2. Stark units and \( L \)-series attached to Drinfeld modules

2.1. \( L \)-series attached to Drinfeld modules

Let \( E/K \) be a finite extension, and let \( O_E \) be the integral closure of \( A \) in \( E \). Let \( \tau : E \to E, \ x \mapsto x^q \). Let \( \rho \) be an Drinfeld \( A \)-module (or a Drinfeld module for short) of rank \( r \geq 1 \) defined over \( O_E \), that is, \( \rho : A \to O_E(\tau) \) is an \( \mathbb{F}_q \)-algebra homomorphism such that
\[
\forall a \in A \setminus \{0\}, \quad \rho_a = \rho_{a,0} + \rho_{a,1} \tau + \cdots + \rho_{a,\deg a} \tau^\deg a,
\]
where \( \rho_{a,0}, \ldots, \rho_{a,\deg a} \in O_E, \rho_{a,0} = a, \) and \( \rho_{a,\deg a} \neq 0 \).

Let \( \mathfrak{P} \) be a maximal ideal of \( O_E \), we denote by \( \rho(O_E/\mathfrak{P}) \) the finite dimensional \( \mathbb{F}_q \)-vector space \( O_E/\mathfrak{P} \) equipped with the structure of \( A \)-module induced by \( \rho \).

**Proposition 2.1.** The following product converges to a principal unit in \( K_\infty^\times \) (that is, an element in \( U_\infty \cap K_\infty^\times \)):
\[
L_A(\rho/O_E) := \prod_{\mathfrak{P}} \left[ \frac{O_E}{\mathfrak{P}} \right]_A^{\rho(O_E/\mathfrak{P})}.
\]
where \( \mathfrak{P} \) runs through the maximal ideals of \( O_E \).
Proof. By [21, Remark 7.1.8.2], we have $H_A \subset E$, where $H_A/K$ is the maximal unramified abelian extension of $K$ such that $\infty$ splits completely in $H_A$. Thus $N_{E/K}(\mathfrak{P})$ is a principal ideal. Observe that

$$Fitt_A \frac{O_E}{\mathfrak{P}} = N_{E/K}(\mathfrak{P}).$$

Thus

$$\left[ \frac{O_E}{\mathfrak{P}} \right]_A = \left[ N_{E/K}(\mathfrak{P}) \right].$$

By [20, Theorem 5.1], there exists a unitary polynomial $P(X) \in A[X]$ of degree $r' \leq r$ such that

$$N_{E/K}(\mathfrak{P}) = P(0)A,$$

$$Fitt_A \rho \left( \frac{O_E}{\mathfrak{P}} \right) = P(1)A,$$

$$v_\infty \left( \frac{(-1)^{r'} P(0)}{P(1)} - 1 \right) \geq \frac{\deg(N_{E/K}(\mathfrak{P}))}{r'd_\infty}.$$

This last assertion comes from the fact that $P(X)$ is a power of the minimal polynomial over $K$ of the Frobenius $F$ of $O_E/\mathfrak{P}$ (see [20, Lemma 3.3]), and that $K(F)/K$ is totally imaginary (that is, there exists a unique place of $K(F)$ over $\infty$). By the properties of [·] (see Section 1), we have

$$\left[ \frac{O_E}{\mathfrak{P}} \right]_A = \frac{(-1)^{r'} P(0)}{P(1)}.$$

The proposition follows. \(\square\)

**Remark 2.2.** The element $L_A(\rho/O_E) \in K_\infty^*$ is called the $L$-series attached to $\rho/O_E$. By the proof of Proposition 2.1, $L_A(\rho/O_E)$ depends on $A, \rho$ and $O_E$, but not on the choice of $\pi$.

Let $F/K$ be a finite extension with $F \subset E$, and such that there exists a unique place of $F$ above $\infty$ (still denoted by $\infty$). Let $A'$ be the integral closure of $A$ in $F$, then $A'$ is the set of elements in $F$ which are regular outside $\infty$. We assume that $\rho$ extends into a Drinfeld $A'$-module: $\rho : A' \rightarrow O_E\{\tau\}$. Let $[\cdot]_{A'} : T(A') \rightarrow K_\infty^*$ be the map constructed as in Section 1 with the help of the choice of a uniformizer $\pi' \in F_\infty^*$. Let $N_{F_\infty/K_\infty} : F_\infty \rightarrow K_\infty$ be the usual norm map.

**Corollary 2.3.** We have

$$N_{F_\infty/K_\infty} (L_{A'}(\rho/O_E)) = L_A(\rho/O_E).$$

**Proof.** Recall that

$$L_{A'}(\rho/O_E) := \prod_{\mathfrak{P}} \frac{\left[ \frac{O_E}{\mathfrak{P}} \right]_{A'}}{\rho(\mathfrak{P}) \left[ \frac{O_E}{\mathfrak{P}} \right]_{A'}},$$
where \( \mathfrak{P} \) runs through the maximal ideals of \( O_E \). Since \( N_{F_{\infty}/K_{\infty}} \) is continuous, we get by the proof of Proposition 2.1:

\[
N_{F_{\infty}/K_{\infty}}(L_{A'}(\rho/O_E)) = \prod_{\mathfrak{P}} N_{F_{\infty}/K_{\infty}}\left( \left[ \frac{O_E}{\mathfrak{P}} \right]_{A'} \right) \cdot \frac{\rho(\mathfrak{P})}{\left[ \rho(\mathfrak{P}) \right]_{A'}}. 
\]

Let \( \mathfrak{P} \) be a maximal ideal of \( O_E \). Since \( \left[ \frac{O_E}{\mathfrak{P}} \right]_{A'} \in F_{\infty}^\times \), we get

\[
N_{F_{\infty}/K_{\infty}}\left( \left[ \frac{O_E}{\mathfrak{P}} \right]_{A'} \right) = N_{F/K}(\left[ \frac{O_E}{\mathfrak{P}} \right]_{A'}). 
\]

But, observe that if \( M \) is a finite \( A' \)-module, we have

\[
N_{F/K}(\text{Fitt}_{A'} M) = \text{Fitt}_{A} M. 
\]

By the proof of Proposition 2.1, \( \left[ \frac{O_E}{\mathfrak{P}} \right]_{A'}/[\rho(O_E/\mathfrak{P})]_{A'} \) is a principal unit in \( F_{\infty}^\times \), and therefore \( N_{F/K}(\left[ \frac{O_E}{\mathfrak{P}} \right]_{A'}/[\rho(O_E/\mathfrak{P})]_{A'}) \) is also a principal unit in \( K_{\infty}^\times \). Again, by the proof of Proposition 2.1, we get

\[
N_{F/K}\left( \left[ \frac{O_E}{\mathfrak{P}} \right]_{A'} \right) = \left[ \frac{O_E}{\mathfrak{P}} \right]_{A}. 
\]

The corollary follows. \( \square \)

2.2. Stark units and the Taelman class module

Let \( E/K \) be a finite extension of degree \( n \), and let \( O_E \) be the integral closure of \( A \) in \( E \). Set

\[
E_{\infty} = E \otimes_K K_{\infty}. 
\]

Let \( M \) be an \( A \)-module, \( M \subset E_{\infty} \), we say that \( M \) is an \( A \)-lattice in \( E_{\infty} \) if \( M \) is discrete and cocompact in \( E_{\infty} \). Observe that if \( M \) is an \( A \)-lattice in \( E_{\infty} \), then there exist \( e_1, \ldots, e_n \in E_{\infty} \) (recall that \( n = [E:K] \)) such that \( E_{\infty} = \oplus_{i=1}^n K_{\infty} e_i, \ N := \oplus_{i=1}^n A e_i \subset M \) and \( \frac{M}{N} \) is a finite \( A \)-module. Note also that \( O_E \) is an \( A \)-lattice in \( E_{\infty} \).

Let \( \tau : E_{\infty} \to E_{\infty}, x \mapsto x^q \). Let \( \rho : A \hookrightarrow O_E\{\tau\} \) be a Drinfeld module of rank \( r \geq 1 \). Then, there exist unique elements \( \exp_{\rho}, \log_{\rho} \in E\{\{\tau\}\} \) such that

\[
\forall a \in A, \quad \exp_{\rho} a = \rho_a \exp_{\rho}, \\
\exp_{\rho} \log_{\rho} = \log_{\rho} \exp_{\rho} = 1. 
\]

The formal series \( \exp_{\rho} \) and \( \log_{\rho} \) are respectively called the exponential series and the logarithm series associated to \( \rho/O_E \). We will write

\[
\exp_{\rho} = \sum_{i \geq 0} e_i(\rho) \tau^i, \\
\log_{\rho} = \sum_{i \geq 0} l_i(\rho) \tau^i, 
\]

with \( e_i(\rho), l_i(\rho) \in E \). Moreover, \( \exp_{\rho} \) converges on \( E_{\infty} \) (see [21, proof of Theorem 4.6.9]).
Observe that \( \exp \) is the module of Stark units associated to \( \rho/O_E \).

Then, as a consequence of [27, Theorem 1], the \( A \)-module \( U(\rho/O_E) \) is an \( A \)-lattice in \( E_\infty \).

**Definition 2.5.** We define the Taelman class module associated to \( \rho/O_E \) by

\[
H(\rho/O_E) = \frac{E_\infty}{O_E + \exp_\rho(E_\infty)}.
\]

Note that \( H(\rho/O_E) \) is an \( A \)-module via \( \rho \), and by [27, Theorem 1], \( H(\rho/O_E) \) is a finite \( A \)-module.

Let \( z \) be an indeterminate over \( K_\infty \), and let \( \mathcal{T}_z(K_\infty) \) be the Tate algebra in the variable \( z \) with coefficients in \( K_\infty \). We set

\[
\mathcal{T}_z(E_\infty) = E \otimes_K \mathcal{T}_z(K_\infty).
\]

Observe that \( E_\infty \subset \mathcal{T}_z(E_\infty) \), and \( \mathcal{T}_z(E_\infty) \) is a free \( \mathcal{T}_z(K_\infty) \)-module of rank \([E : K]\). Let \( \tau : \mathcal{T}_z(E_\infty) \rightarrow \mathcal{T}_z(E_\infty) \) be the continuous \( \mathbb{F}_q[z] \)-algebra homomorphism such that

\[
\forall x \in E_\infty, \quad \tau(x) = x^q.
\]

Let \( \text{ev} : \mathcal{T}_z(E_\infty) \rightarrow E_\infty \) be the surjective \( E_\infty \)-algebra homomorphism given by

\[
\forall f \in \mathcal{T}_z(E_\infty), \quad \text{ev}(f|_{\mathcal{T}_z(E_\infty)}) = f|_{z=1}.
\]

We have \( \ker \text{ev} = (z-1)\mathcal{T}_z(E_\infty) \), and

\[
\forall f \in \mathcal{T}_z(E_\infty), \quad \text{ev}(\tau(f)) = \tau(\text{ev}(f)).
\]

Recall that

\[
\exp_\rho = \sum_{i \geq 0} e_i(\rho) \tau^i, \quad \text{with } e_i(\rho) \in E.
\]

We set

\[
\exp_{\tilde{\rho}} = \sum_{i \geq 0} e_i(\rho) z^i \tau^i \in E[z]\{\{\tau\}\}.
\]

Observe that \( \exp_{\tilde{\rho}} \) converges on \( \mathcal{T}_z(E_\infty) \), and

\[
\forall f \in \mathcal{T}_z(E_\infty), \quad \text{ev}(\exp_{\tilde{\rho}}(f)) = \exp_\rho(\text{ev}(f)).
\]

Let \( \tilde{\rho} : A \rightarrow O_E[z]\{\tau\} \) be the \( \mathbb{F}_q \)-algebra homomorphism given by

\[
\forall a \in A, \quad \tilde{\rho}_a = a + \rho_{a,1} z \tau + \cdots + \rho_{a,r_{\deg a}} z^{r_{\deg a}} \tau^{r_{\deg a}}.
\]

where \( \rho_{a} = a + \rho_{a,1} \tau + \cdots + \rho_{a,r_{\deg a}} \tau^{r_{\deg a}} \). Then

\[
\forall a \in A, \quad \exp_{\tilde{\rho}} a = \tilde{\rho}_a \exp_{\tilde{\rho}}.
\]

**Definition 2.6.** The module of \( z \)-units associated to \( \rho/O_E \) is defined by

\[
U(\tilde{\rho}/O_E[z]) = \{ f \in \mathcal{T}_z(E_\infty), \exp_{\tilde{\rho}}(f) \in O_E[z]\}.
\]

And the module of Stark units associated to \( \rho/O_E \) is defined by

\[
U_{St}(\rho/O_E) := \text{ev}(U(\tilde{\rho}/O_E[z])).
\]

Observe that \( U_{St}(\rho/O_E) \subset U(\rho/O_E) \).
Theorem 2.7. The $A$-module $U_{St}(\rho/O_E)$ is an $A$-lattice in $E_\infty$. Furthermore,

$$\begin{bmatrix} U(\rho/O_E) \\ U_{St}(\rho/O_E) \end{bmatrix}_A = [H(\rho/O_E)]_A.$$ 

Proof. This is a consequence of the proof of [10, Theorem 1]. For the convenience of the reader, we give a sketch of the proof. Let's set

$$H(\tilde{\rho}/O_E[z]) = \frac{T_z(E_\infty)}{O_E[z] + \exp(\tilde{\rho}(T_z(E_\infty)))}.$$ 

Observe that $H(\tilde{\rho}/O_E[z])$ is an $A[z]$-module via $\tilde{\rho}$, and furthermore $H(\tilde{\rho}/O_E[z])$ is a finite $\mathbb{F}_q[z]$-module [10, Proposition 2]. Let's set

$$V = \{x \in H(\tilde{\rho}/O_E[z]) : (z-1)x = 0\}.$$ 

Since $\ker ev = (z-1)T_z(E_\infty)$, the multiplication by $z-1$ on $H(\tilde{\rho}/O_E)$ gives rise to an exact sequence of finite $A$-modules

$$0 \to V \to H(\tilde{\rho}/O_E[z]) \to H(\tilde{\rho}/O_E[z]) \to H(\rho/O_E) \to 0.$$ 

Thus

$$\text{Fitt}_A V = \text{Fitt}_A H(\rho/O_E).$$ 

Now, let's consider the homomorphism of $\mathbb{F}_q[z]$-modules $\alpha : T_z(E_\infty) \to T_z(E_\infty)$ given by

$$\forall x \in T_z(E_\infty), \quad \alpha(x) = \frac{\exp_{\rho}(x) - \exp_{\tilde{\rho}}(x)}{z-1}.$$ 

Observe that

$$(z-1)\alpha(U(\rho/O_E)) \subset O_E + \exp_{\tilde{\rho}}(T_z(E_\infty)),$$

$$\forall a \in A, \forall x \in U(\rho(O_E)), \quad \alpha(ax) - \tilde{\rho}_a(\alpha(x)) \in O_E[z].$$

Thus $\alpha$ induces a homomorphism of $A$-modules:

$$\tilde{\alpha} : U(\rho/O_E) \to V.$$ 

By [10, Proposition 3], this homomorphism is surjective and its kernel is precisely $U_{St}(\rho/O_E)$. The theorem follows. \hfill $\square$

2.3. Co-volumes

Let $V$ be a finite dimensional $K_\infty$-vector space of dimension $n \geq 1$. An $A$-lattice in $V$ is a discrete and cocompact sub-$A$-module of $V$.

Lemma 2.8. Let $M, N$ be two $A$-lattices in $V$. Then there exists an isomorphism of $K_\infty$-vector spaces $\sigma : V \to V$ such that

$$\sigma(M) \subset N.$$ 

Proof. Since $A$ is a Dedekind domain, there exist two non-zero ideals $I, J$ of $A$, and two $K_\infty$-basis $\{e_1, \ldots, e_n\}, \{f_1, \ldots, f_n\}$ of $V$, such that

$$M = \bigoplus_{j=1}^{n-1} Ae_j \oplus Ie_n,$$

$$N = \bigoplus_{j=1}^{n-1} Af_j \oplus Jf_n.$$
Furthermore, \( M \) and \( N \) are isomorphic as \( A \)-modules if and only if \( I \) and \( J \) have the same class in the ideal class group \( \text{Pic}(A) \) of \( A \). Let \( x \in I^{-1}J \setminus \{0\} \). Let \( \sigma : V \to V \) such that

\[
\sigma(e_j) = f_j, \quad j = 1, \ldots, n - 1,
\]

\[
\sigma(e_n) = xf_n.
\]

Then

\[
\sigma(M) \subset N.
\]

Note that if \( M \) and \( N \) are isomorphic \( A \)-modules then we can select \( x \in K^\times \) such that \( I^{-1}J = xA \) and in this case \( \sigma(M) = N \).

\[\tag*{\looseness=-1}
\]

**Lemma 2.9.** Let \( M, N \) be two \( A \)-lattices in \( V \). Let \( \sigma_1, \sigma_2 : V \to V \) be two isomorphisms of \( K_\infty \)-vector spaces such that \( \sigma_i(M) \subset N, i = 1, 2 \). Then

\[
\frac{\text{det}_{K_\infty} \sigma_1}{\text{sgn}(\text{det}_{K_\infty} \sigma_1)} \left[ \frac{N}{\sigma_1(M)} \right]_A^{-1} = \frac{\text{det}_{K_\infty} \sigma_2}{\text{sgn}(\text{det}_{K_\infty} \sigma_2)} \left[ \frac{N}{\sigma_2(M)} \right]_A^{-1}.
\]

**Proof.** Let \( \sigma = \sigma_1 \sigma_2^{-1} \). Since \( \sigma(\sigma_2(M)) = \sigma_1(M) \subset N \), with \( \sigma_2(M) \subset N \), we can find \( a \in A \) with \( \text{sgn} a = 1 \) such that \( a \sigma(N) \subset N \). Set \( U = \frac{1}{a} \sigma_2(M) \cap N \). Then multiplication by \( a \) induces an exact sequence of finite \( A \)-modules:

\[
0 \longrightarrow \frac{U}{\sigma_2(M)} \longrightarrow \frac{N}{\sigma_2(M)} \longrightarrow \frac{N}{a \sigma(N)} \longrightarrow 0
\]

from which we deduce

\[
\left[ \frac{U}{\sigma_2(M)} \right]_A = \left[ \frac{N}{a \sigma(N)} \right]_A = a^n.
\]

And \( a \sigma \) similarly induces an exact sequence of finite \( A \)-modules:

\[
0 \longrightarrow \frac{U}{\sigma_2(M)} \longrightarrow \frac{N}{\sigma_2(M)} \longrightarrow \frac{N}{a \sigma(N)} \longrightarrow 0.
\]

We get

\[
\left[ \frac{N}{\sigma_1(M)} \right]_A = \left[ \frac{N}{\sigma_2(M)} \right]_A^{-1} \left[ \frac{U}{\sigma_2(M)} \right]_A \left[ \frac{N}{a \sigma(N)} \right]_A = \left[ \frac{N}{\sigma_2(M)} \right]_A a^{-n} \frac{\text{det}_{K_\infty}(a \sigma)}{\text{sgn}(\text{det}_{K_\infty}(a \sigma))} \left[ \frac{N}{\sigma_1(M)} \right]_A
\]

\[
= \left[ \frac{N}{\sigma_2(M)} \right]_A \frac{\text{det}_{K_\infty}(\sigma)}{\text{sgn}(\text{det}_{K_\infty}(\sigma))}.
\]

The lemma follows.

Let \( M, N \) be two \( A \)-lattices in \( V \). By Lemma 2.8, there exists an isomorphism of \( K_\infty \)-vector spaces \( \sigma : V \to V \) such that \( \sigma(M) \subset N \), we set

\[
[M : N]_A = \frac{\text{det}_{K_\infty} \sigma}{\text{sgn}(\text{det}_{K_\infty} \sigma)} \left[ \frac{N}{\sigma(M)} \right]_A^{-1}.
\]

By Lemma 2.9, this is well-defined. In particular, if \( M, N \) are two \( A \)-lattices in \( V \) such that \( N \subset M \), then

\[
[M : N]_A = \left[ \frac{M}{N} \right]_A.
\]
If $M, N, U$ are three $A$-lattices in $V$, we get

$$[M : N]_A = [M : U]_A[U : N]_A.$$  

Let $F/K$ be a finite extension such that there exists a unique place of $F$ above $\infty$ (still denoted by $\infty$). Let $A'$ be the integral closure of $A$ in $F$. We assume that $V$ is also an $F_\infty$-vector space. Let $\cdot : A' \to K_\infty$ be the map constructed as in Section 1 with the help of the choice of a uniformizer $\pi' \in F_\infty$. Let $N_{F_\infty/K_\infty} : F_\infty \to K_\infty$ be the usual norm map.

**Lemma 2.10.** Let $M, N$ be two $A'$-lattices in $V$. Then there exists an integer $m \geq 1$ such that $[M : N]_{A'}^m \in F_\infty^\times, [M : N]_A^m \in K_\infty^\times$, and $N_{F_\infty/K_\infty}([M : N]_{A'}^m) = [M : N]_A^m$.

**Proof.** Let $\sigma : V \to V$ be an isomorphism of $F_\infty$-vector spaces such that $\sigma(M) \subset N$, and we set: $I' = \text{Fitt}_{A'}N/\sigma(M)$. Then

$$F_{\text{itt}}A'N/\sigma(M) = N_{F/K}(I').$$

Let $m \geq 1$ be an integer such that $I'^m = xA', \ x \in A' \setminus \{0\}$.

Then

$$[M : N]_{A'}^m = \left(\frac{\det_{F_\infty} \sigma}{\text{sgn}'(\det_{F_\infty} \sigma)}\right)^m \frac{\text{sgn}(x)}{x}.$$  

Furthermore, we have

$$\det_{K_\infty} \sigma = N_{F_\infty/K_\infty}(\det_{F_\infty} \sigma).$$

Thus

$$[M : N]_A^m = \left(\frac{N_{F_\infty/K_\infty}(\det_{F_\infty} \sigma)}{\text{sgn}(N_{F_\infty/K_\infty}(\det_{F_\infty} \sigma))}\right)^m \frac{\text{sgn}(N_{F/K}(x))}{N_{F/K}(x)}.$$  

Therefore

$$N_{F_\infty/K_\infty}([M : N]_{A'}^m) \in [M : N]_A^m F_\infty^\times.$$  

The lemma follows. $\square$

### 2.4. Regulator of Stark units and $L$-series

Let $E/K$ be a finite extension, $E \subset \mathbb{C}_\infty$. Recall that $E_\infty = E \otimes_K K_\infty$. If $M$ is an $A$-lattice in $E_\infty$, then we call $[O_E : M]_A$ the $A$-regulator of $M$.

**Definition 2.11.** Let $\rho : A \to O_E\{\tau\}$ be a Drinfeld module of rank $r \geq 1$. We define the regulator of Stark units associated to $\rho/O_E$ by $[O_E : U_{St}(\rho/O_E)]_A$.

**Proposition 2.12.** Let $\rho : A \to O_E\{\tau\}$ be a Drinfeld module of rank $r \geq 1$. We have

$$[O_E : U_{St}(\rho/O_E)]_A \in U_\infty.$$  

Furthermore, the regulator of Stark units relative to $\rho/O_E$ depends on $\rho, A$ and $O_E$, not on the choice of $\pi$. 

Proof. Let \( \theta \in A \setminus \mathbb{F}_q \), and let \( L = \mathbb{F}_q(\theta), B = \mathbb{F}_q[\theta] \). Let \( [\cdot]_B : \mathcal{I}(B) \to \mathcal{O}_\infty \) be the map as in Section 1 associated to the choice of \( \frac{1}{\theta} \) as a uniformizer of \( \mathcal{O}_\infty \). Then, by Theorem 2.7, we have

\[
[O_E : U_{St}(\rho/O_E)]_B = [O_E : U(\rho/O_E)]_B[H(\rho/O_E)]_B.
\]

Then, by [27, Theorem 2], we get

\[
[O_E : U_{St}(\rho/O_E)]_B \in 1 + \frac{1}{\theta} \mathbb{F}_q \left[ \left. \frac{1}{\theta} \right] \right].
\]

Now, by Lemma 2.10, there exists an integer \( m \geq 1 \) such that

\[
N_{K_\infty/L_\infty} \left( [O_E : U_{St}(\rho/O_E)]_A \right)^m = [O_E : U_{St}(\rho/O_E)]_B^m.
\]

This implies

\[
v_\infty \left( [O_E : U_{St}(\rho/O_E)]_A \right) = 0.
\]

Thus

\[
[O_E : U_{St}(\rho/O_E)]_A \in \mathbb{F}_q^\times \times U_\infty.
\]

But \( \text{sgn}([O_E : U_{St}(\rho/O_E)]_A) = 1 \), thus

\[
[O_E : U_{St}(\rho/O_E)]_A \in U_\infty.
\]

Let \( \pi' \) be another uniformizer of \( K_\infty \), and let \( [\cdot]'_A : \mathcal{I}(A) \to \mathcal{K}_\infty^\times \) be the map as in Section 1 associated to \( \pi' \). Then, by the above discussion, we get

\[
[O_E : U_{St}(\rho/O_E)]_A' \in U_\infty.
\]

Again, by Lemma 2.10, there exists an integer \( m' \geq 1 \) such that

\[
[O_E : U_{St}(\rho/O_E)]_A'^m = [O_E : U_{St}(\rho/O_E)]_A^m.
\]

Since \( [O_E : U_{St}(\rho/O_E)]_A, [O_E : U_{St}(\rho/O_E)]_A \in U_\infty \), we get

\[
[O_E : U_{St}(\rho/O_E)]_A' = [O_E : U_{St}(\rho/O_E)]_A.
\]

This concludes the proof of the proposition. \( \square \)

Let’s set

\[
\alpha_A(\rho/O_E) := \frac{L_A(\rho/O_E)}{[O_E : U_{St}(\rho/O_E)]_A} \in \overline{K}_\infty^\times.
\]

By Proposition 2.12 and Remark 2.2, \( \alpha_A(\rho/O_E) \) depends on \( A, \rho, \) and \( O_E \), not on the choice of \( \pi \). Furthermore,

\[
\alpha_A(\rho/O_E) \in U_\infty.
\]

Let’s also observe that, if \( p^k \) is the exact power of \( p \) dividing \( |\text{Pic}(A)| \), then

\[
\alpha_A(\rho/O_E)^{p^kd_\infty} \in K_\infty^\times.
\]

We have the fundamental result due to Taelman [28, Theorem 1]:

**Theorem 2.13 (Taelman).** Assume that the genus of \( K \) is zero and \( d_\infty = 1 \). Then

\[
\alpha_A(\rho/O_E) = 1.
\]
Proof. Select $\theta \in A \setminus \mathbb{F}_q$ such that $v_\infty(\theta) = 1$. Then $A = \mathbb{F}_q[\theta]$. Let $[\cdot]_A : \mathcal{I}(A) \rightarrow K_\infty^\times$ be the map as in Section 1 associated to the choice of $\frac{1}{\theta}$ as a uniformizer of $K_\infty$. Then, by Proposition 2.12, Theorem 2.7 and [28, Theorem 1]:

$$[O_E : U_{S_l}(\rho/O_E)]_A = [O_E : U(\rho/O_E)]_A [H(\rho/O_E)]_A = L_A(\rho/O_E).$$

This concludes the proof of the theorem. \[\square\]

Corollary 2.14. (1) Let $F/K$ be a finite extension, $F \subset E$, and such that there exists a unique place of $F$ above $\infty$ (still denoted by $\infty$). Let $A'$ be the integral closure of $A$ in $F$. Let $N_{F_\infty/K_\infty} : F_\infty \rightarrow K_\infty$ be the usual norm map. Then, there exists an integer $k \geq 1$ such that $\alpha_{A'}(\rho/O_E)^k \in F_\infty^\times$, $\alpha_A(\rho/O_E)^k \in K_\infty^\times$, and

$$N_{F_\infty/K_\infty}(\alpha_{A'}(\rho/O_E)^k) = \alpha_A(\rho/O_E)^k.$$\[\text{In particular, } \alpha_{A'}(\rho/O_E) = 1 \Rightarrow \alpha_A(\rho/O_E) = 1.\]

(2) If there exists an integer $m \geq 1$ such that $\alpha_A(\rho/O_E)^m \in K_\infty^\times$, then $\alpha_A(\rho/O_E) = 1$. In particular, if $\sigma(\alpha_A(\rho/O_E)) = \alpha_A(\rho^\sigma/\sigma(O_E))$ for all $\sigma \in \text{Aut}_K(C_\infty)$, then $\alpha_A(\rho/O_E) = 1$.

Proof. (1) The first assertion is a consequence of Corollary 2.3 and Lemma 2.10. If $\alpha_{A'}(\rho/O_E) = 1$, then there exists an integer $k \geq 1$ such that $\alpha_A(\rho/O_E)^k = 1$. But, since $\text{sgn}(\alpha_A(\rho/O_E)) = 1$, we get $\alpha_A(\rho/O_E) = 1$.

(2) Let $x = \alpha_{A'}(\rho/O_E)^m \in K_\infty^\times$. Let $P$ be a maximal ideal of $A$, and select an integer $l \geq 1$ such that $P^l$ is a principal ideal. Let $\theta \in A \setminus \mathbb{F}_q$ such that $P^l = \theta A$. Let $L = \mathbb{F}_q(\theta)$ and $B = \mathbb{F}_q[\theta]$. Then, by Taelman’s Theorem (Theorem 2.13), we have

$$\alpha_B(\rho/O_E) = 1.$$\[\text{Therefore, by (1), we have:}\]

$$N_{K/L}(x) \in \mathbb{F}_q^\times.$$\[\text{Since } P \text{ is the only maximal ideal of } A \text{ above } \theta B, \text{ we deduce that } x \text{ is a } P\text{-adic unit. Since this is true for all maximal ideal of } A, \text{ we get:}\]

$$x \in \mathbb{F}_q^\times.$$\[\text{But, } \text{sgn}(\alpha_A(\rho/O_E)) = 1, \text{ thus } \alpha_A(\rho/O_E) = 1.\]

Let’s assume that $\sigma(\alpha_A(\rho/O_E)) = \alpha_A(\rho^\sigma/\sigma(O_E))$ for all $\sigma \in \text{Aut}_K(C_\infty)$. Let $\sigma \in \text{Aut}_K(C_\infty)$. Let $\mathfrak{P}$ be a maximal ideal of $O_E$, then

$$\left[\begin{array}{c}
\sigma(\mathfrak{P}) \\
\sigma(\mathfrak{P})
\end{array}\right]_A = \left[\begin{array}{c}
O_E \\
\mathfrak{P}
\end{array}\right]_A,$$

$$\left[\begin{array}{c}
\rho^\sigma (\sigma(\mathfrak{P})) \\
\sigma(\mathfrak{P})
\end{array}\right]_A = \left[\begin{array}{c}
\rho (O_E) \\
\mathfrak{P}
\end{array}\right]_A.$$\[\text{Thus,}\]

$$L_A(\rho^\sigma/\sigma(O_E)) = L_A(\rho/O_E).$$\[\text{Observe that } \sigma \text{ induces a } K_\infty\text{-algebra isomorphism}\]

$$E_\infty \simeq \sigma(E)_\infty.$$\[\text{Note that } \exp_{\tilde{\rho}} : E[[z]] \rightarrow E[[z]] \text{ is an } \mathbb{F}_q[[z]]\text{-algebra isomorphism. Therefore,}\]

$$U(\tilde{\rho}/O_E[z]) \subset E[[z]].$$
Thus,
\[ U \left( \frac{\rho}{\sigma(O_E)}[z] \right) = \sigma(U \left( \frac{\rho}{O_E}[z] \right)). \]
By the definition of Stark units, we get
\[ U_{St} \left( \frac{\rho}{\sigma(O_E)} \right) = \sigma(U_{St} \left( \frac{\rho}{O_E} \right)). \]
Thus,
\[ [\sigma(O_E) : U_{St} \left( \frac{\rho}{\sigma(O_E)} \right)]_A = [O_E : U_{St} \left( \frac{\rho}{O_E} \right)]_A. \]
Therefore,
\[ \alpha_A \left( \frac{\rho}{\sigma(O_E)} \right) = \alpha_A \left( \frac{\rho}{O_E} \right). \]
We get
\[ \forall \sigma \in \text{Aut}_K(C_\infty), \quad \sigma(\alpha_A(\rho/O_E)) = \alpha_A(\rho/O_E). \]
This implies that \( \alpha_A(\rho/O_E) \) is algebraic over \( K \) and that there exists an integer \( k \geq 0 \) such that
\[ \alpha_A(\rho/O_E)^{\rho_k} \in K^\times. \]
Therefore,
\[ \alpha_A(\rho/O_E) = 1. \]

We do not know whether \( \alpha_A(\rho/O_E) \) is algebraic over \( K \), and it might be too naive to expect that \( \alpha_A(\rho/O_E) = 1 \) in general. However, in the next section, we will prove that, if \( \phi \) is a sign-normalized rank one Drinfeld module and \( E/K \) is a finite abelian extension such that \( H \subset E \), then \( \alpha_A(\phi/O_E) = 1 \) (Theorem 3.10). Recently, Taelman’s class formula [28, Theorem 1] has been generalized by C. Debry to the case where \( A \) is a principal ideal domain [12].

We also prove below that \( \alpha_A(\phi/O_E) \) is invariant under isogeny, which could be considered as an analogue of the isogeny invariance of the Birch and Swinnerton-Dyer conjecture due to Tate [29]:

**Theorem 2.15.** Let \( E/K \) be a finite extension and let \( \rho, \phi : A \to O_E\{\tau\} \) be two Drinfeld \( A \)-modules such that there exists \( u \in O_E\{\tau\} \setminus \{0\} \) with the following property:
\[ \forall a \in A, \quad \rho_a u = u \phi_a, \]
then
\[ \alpha_A(\rho/O_E) = \alpha_A(\phi/O_E). \]

**Proof.** Let \( \mathfrak{P} \) be a maximal ideal of \( O_E \) such that \( u \not\equiv 0 \pmod{\mathfrak{P}} \). Then by [20, Theorem 3.5 and Theorem 5.1], we get
\[ \left[ \rho \left( \frac{O_E}{\mathfrak{P}} \right) \right]_A = \left[ \phi \left( \frac{O_E}{\mathfrak{P}} \right) \right]_A. \]
This implies that there exists an ideal \( I \in \mathcal{I}(A) \) such that
\[ \frac{L_A(\rho/O_E)}{L_A(\phi/O_E)} = [I]. \]
Let \( \zeta \in O_E \setminus \{0\} \) be the constant coefficient of \( u \). Then we have the following equality in \( E\{\{\tau\}\} : \)
\[ \exp_\rho \zeta = u \exp_\phi. \]
Thus, 
\[ \exp_{\widetilde{\rho}} \zeta = \widetilde{u} \exp_{\widetilde{\phi}}, \]
where, if \( u = \sum_{i=0}^{m} u_{i} \tau^{i}, u_{i} \in O_{E}, \widetilde{u} = \sum_{i=0}^{m} u_{i} z^{i} \tau^{i} \). This implies that 
\[ \zeta U(\widetilde{\phi}/O_{E}[z]) \subset U(\widetilde{\rho}/O_{E}[z]). \]
Therefore,
\[ \zeta U_{St}(\phi/O_{E}) \subset U_{St}(\rho/O_{E}). \]
We get
\[ [O_{E} : \zeta U_{St}(\phi/O_{E})]_{A} = [O_{E} : U_{St}(\phi/O_{E})]_{A} \left[ \frac{O_{E}}{\zeta O_{E}} \right]_{A}, \]
and
\[ [O_{E} : \zeta U_{St}(\phi/O_{E})]_{A} = [O_{E} : U_{St}(\rho/O_{E})]_{A} \left[ \frac{U_{St}(\rho/O_{E})}{\zeta U_{St}(\phi/O_{E})} \right]_{A}. \]
Therefore, there exists an element \( J \in I(A) \) such that
\[ \frac{[O_{E} : U_{St}(\rho/O_{E})]_{A}}{[O_{E} : U_{St}(\phi/O_{E})]_{A}} = [J]. \]
Finally, we get
\[ \frac{\alpha_{A}(\rho/O_{E})}{\alpha_{A}(\phi/O_{E})} = [IJ^{-1}]. \]
Let \( x = (\alpha_{A}(\rho/O_{E})/\alpha_{A}(\phi/O_{E}))^{h(q^{d_{\infty}} - 1)} \in K_{\infty}, \) where \( h = |\text{Pic}(A)|. \) Then, by Corollary 2.14, and Theorem 2.13, if \( \theta \in A \setminus F_{q} \), there exists an integer \( k \geq 1 \) such that
\[ N_{K_{\infty}/F_{q}}((\theta)^{x}) = 1. \]
But, by Proposition 2.12, \( x \) is a principal unit in \( K_{\infty} \), thus
\[ N_{K_{\infty}/F_{q}}(\theta)(x) = 1. \]
The above equality being valid for any \( \theta \in A \setminus F_{q} \), by the proof of Corollary 2.14, we deduce that
\[ x = 1. \]
Since \( \text{sgn}(\alpha_{A}(\rho/O_{E})/\alpha_{A}(\phi/O_{E})) = 1 \), we get
\[ \frac{\alpha_{A}(\rho/O_{E})}{\alpha_{A}(\phi/O_{E})} = 1. \]
\[ 3. \quad \text{Stark units associated to sign-normalized rank one Drinfeld modules} \]

3.1. \text{Zeta functions}

In this section, we briefly recall the definition of some zeta functions [21, Chapter 8].

Recall that if \( I \in I(A) \), we have set
\[ [I] = (I)\pi^{-\deg I/d_{\infty}} \in K_{\infty}^{\times}. \]
where \( v_\infty(\langle I \rangle - 1) > 0 \), and

\[
\forall x \in K^\times, \quad \langle xA \rangle = \frac{x}{\text{sgn}(x)} \pi^{-v_\infty(x)}.
\]

Let \( S_\infty = \mathbb{C}_\infty^\times \times \mathbb{Z}_p \) be the Goss 'complex plane'. The group action of \( S_\infty \) is written additively. Let \( I \in \mathcal{I}(A) \) and \( s = (x; y) \in S_\infty \), we set

\[
I^s = \langle I \rangle y^{\deg I} \in \mathbb{C}_\infty^\times.
\]

We have a natural injective group homomorphism: \( \mathbb{Z} \to S_\infty, j \mapsto s_j = (\pi^{-j/d_\infty}, j) \). Observe that

\[
\forall j \in \mathbb{Z}, \forall I \in \mathcal{I}(A), I^{s_j} = [I]_j.
\]

Let \( E/K \) be a finite extension, and let \( O_E \) be the integral closure of \( A \) in \( E \). Let \( \mathfrak{J} \) be a non-zero ideal of \( E \). We have

\[
\forall j \in \mathbb{Z}, N_{E/K}(\mathfrak{J})^{s_j} = \left[ \frac{O_E}{\mathfrak{J}} \right]_A^j.
\]

Letting \( s \in S_\infty \), the following sum converges in \( C_\infty \) [21, Theorem 8.9.2]:

\[
\zeta_{O_E}(s) := \sum_{d \geq 0} \sum_{\mathfrak{J} \subset O_E, \deg(N_{E/K}(\mathfrak{J})) = d} N_{E/K}(\mathfrak{J})^{-s}.
\]

The function \( \zeta_{O_E} : S_\infty \to \mathbb{C}_\infty \) is called the zeta function attached to \( O_E \) and \( [\cdot]_A \). Observe that

\[
\forall j \in \mathbb{Z}, \quad \zeta_{O_E}(j) := \zeta_{O_E}(s_j) = \sum_{d \geq 0} \sum_{\mathfrak{J} \subset O_E, \deg(N_{E/K}(\mathfrak{J})) = d} \left[ \frac{O_E}{\mathfrak{J}} \right]_A^{-j}.
\]

In particular,

\[
\zeta_{O_E}(1) = \prod_{\mathfrak{P}} \left( 1 - \frac{1}{\left[ \frac{O_E}{\mathfrak{P}} \right]_A} \right)^{-1} \in \mathcal{K}_\infty^\times,
\]

where \( \mathfrak{P} \) runs through the maximal ideals of \( O_E \).

**Lemma 3.1.** Let \( H_A \) be the Hilbert class field of \( A \), that is, \( H_A/K \) is the maximal unramified abelian extension of \( A \) in which \( \infty \) splits completely. If \( H_A \subset E \), then the function \( \zeta_{O_E}(\cdot) \) depends only on \( O_E \) and \( \text{sgn} \mid_{K_\infty^\times} \).

**Proof.** Let \( \mathfrak{P} \) be a maximal ideal of \( O_E \). Let \( A' \) be the integral closure of \( A \) in \( H_A \). Let \( P' = \mathfrak{P} \cap A', P = \mathfrak{P} \cap A \). By class field theory, \( \mathfrak{P} \left[ \frac{A'}{A} \right] \) is a principal ideal. Thus

\[
N_{E/K}(\mathfrak{P}) = \theta A,
\]

for some \( \theta \in A \setminus \mathbb{F}_q \). Let \( j \in \mathbb{N}, j \geq 1 \). We have

\[
\left( 1 - \frac{1}{\left[ \frac{O_E}{\mathfrak{P}} \right]_A} \right)^{-1} = \frac{\theta^j}{\text{sgn}(\theta^j)} = \frac{\theta^j}{\text{sgn}(\theta^j)} - 1.
\]
But, observe that
\[ \zeta_{O_E}(j) = \prod_{\mathfrak{p}} \left( 1 - \frac{1}{\frac{O_E}{\mathfrak{p}^j}} \right)^{-1} \in U_\infty \cap K_\infty^\times. \]
The lemma is thus a consequence of [21, Theorem 8.7.1]. □

3.2. Background on sign-normalized rank one Drinfeld modules

For a detailed introduction to the arithmetic of rank one Drinfeld modules, we refer the interested reader to David Hayes’ work [23, 24] and also to [21, Chapter 7].

Let \( \phi : A \to \overline{K}_\infty \{ \tau \} \) be a rank one Drinfeld module such that there exists \( i(\phi) \in \mathbb{N} \) with the following property:
\[ \forall a \in A \setminus \{0\}, \quad \phi_a = a + \cdots + \text{sgn}(a)^{q^{i(\phi)}} \tau^\deg a. \]
Such a Drinfeld module \( \phi \) is said to be sign-normalized. By [21, Theorem 7.2.15], there always exist sign-normalized rank one Drinfeld modules.

From now on, we will fix a sign-normalized rank one Drinfeld module \( \phi : A \to \overline{K}_\infty \{ \tau \} \).

Let \( I_K \) be the group of id\'eles of \( K \). Let’s consider the following subgroup of the id\'eles of \( K \):
\[ K^\times \left( \prod_{v \neq \infty} O_v^\times \times \ker \text{sgn} \mid_{K_\infty^\times} \right), \]
where for a place \( v \) of \( K \), \( O_v \) denotes the valuation ring of the \( v \)-adic completion of \( K \). By class field theory, there exists a unique finite abelian extension \( H/K \) such that the reciprocity map induces an isomorphism:
\[ \frac{I_K}{K^\times \left( \prod_{v \neq \infty} O_v^\times \times \ker \text{sgn} \mid_{K_\infty^\times} \right)} \simeq \text{Gal}(H/K). \]
The natural surjective homomorphism \( I_K \to \mathcal{I}(A) \) induces an isomorphism given by the Artin map \((\cdot, H/K)\):
\[ \frac{\mathcal{I}(A)}{\mathcal{P}_+(A)} \simeq \text{Gal}(H/K), \]
where \( \mathcal{P}_+(A) = \{ xA, x \in K, \text{sgn}(x) = 1 \} \). Let \( H_A \) be the Hilbert class field of \( A \), that is, \( H_A/K \) corresponds to the following subgroup of the id\'eles of \( K \):
\[ K^\times \left( \prod_{v \neq \infty} O_v^\times \times K_\infty^\times \right). \]
Then \( H/K \) is unramified outside \( \infty \), and \( H/H_A \) is totally ramified at the places of \( H_A \) above \( \infty \). Furthermore,
\[ \text{Gal}(H/H_A) \simeq \frac{\mathbb{F}_q^\times}{\mathbb{F}_q^{\times n}}. \]
If \( w \) is a place of \( H \) above \( \infty \), then the \( w \)-adic completion of \( H \) is isomorphic to
\[ K_\infty \left( -\pi \right)^{(q-1)/(q^d-1)} \].
We denote by \( B \) the integral closure of \( A \) in \( H \) and set \( A' = B \cap H_A \). We observe that \( \mathbb{F}_\infty \subset A' \).
We denote by \(G\) the Galois group \(\text{Gal}(H/K)\). For \(I \in \mathcal{I}(A)\), we set
\[
\sigma_I = (I, H/K) \in G.
\] (3.1)

By [21, Proposition 7.4.2 and Corollary 7.4.9], the subfield of \(C_\infty\) generated by \(K\) and the coefficients of \(\phi_a\) is \(H\). Furthermore [21, Lemma 7.4.5],
\[
\forall a \in A, \quad \phi_a \in B\{\tau\}.
\]
Let \(I\) be a non-zero ideal of \(A\), and let’s define \(\phi_I\) to be the unitary element in \(H\{\tau\}\) such that
\[
H\{\tau\} \phi_I = \sum_{a \in I} H\{\tau\} \phi_a.
\]
We have
\[
\ker \phi_I = \bigcap_{a \in I} \ker \phi_a,
\]
\[
\phi_I \in B\{\tau\},
\]
\[
\deg_{\tau} \phi_I = \deg I.
\]

**Lemma 3.2.** The map \(\psi\) extends uniquely into a map \(\psi : \mathcal{I}(A) \to H^\times\) with the following properties:

1. for all \(I, J \in \mathcal{I}(A), \psi(IJ) = \sigma_J(\psi(I)) \psi(J),\)
2. for all \(I \in \mathcal{I}(A), IB = \psi(I)B,\)
3. for all \(x \in K^\times, \psi(xA) = \frac{x}{\text{sgn}(x)^{q^I(\phi)-q^I(\phi)+\deg I}} \psi(I).\)

In particular, we have
\[
\forall x \in K^\times, \quad \sigma_{xA}(\psi(I)) = \text{sgn}(x)^{q^I(\phi)-q^I(\phi)+\deg I} \psi(I).
\]

**Proof.** Let \(I \in \mathcal{I}(A)\), select \(a \in A, \text{sgn}(a) = 1\), such that \(aI \subset A\). Let’s set
\[
\psi(I) := \frac{\psi(aI)}{a} \in H^\times.
\]

By [21, Theorem 7.4.8 and Theorem 7.6.2], the map \(\psi : \mathcal{I}(A) \to H^\times\) is well defined and satisfies the desired properties. \(\square\)

Note that the map \(\psi\) determines \(H\) and \(H_A\):

**Proposition 3.3.** We have

1. \(H = K(\psi(I), I \in \mathcal{I}(A));\)
2. \(H_A = K(\psi(I), I \in \mathcal{I}(A), \deg I \equiv 0 \pmod{d_\infty}).\)

**Proof.** (1) Let \(\sigma \in \text{Gal}(H/K(\psi(I), I \in \mathcal{I}(A))).\) Let \(J \in \mathcal{I}(A)\) such that \(\sigma = \sigma_J.\) Then
\[
\forall I \in \mathcal{I}(A), \quad \sigma_I(\psi(J)) = \psi(J).
\]
Therefore,
\[
\psi(J) \in K^\times.
\]
Since $JB = \psi(J)B$ (Lemma 3.2), we get that $J = xA$ for some $x \in K^\times$. Thus, for all $I \in \mathcal{I}(A)$, we get
\[ \text{sgn}(x)^{\psi(I) - q(I) + \deg I} = 1. \]
Since $\deg : \mathcal{I}(A) \to \mathbb{Z}$ is a surjective group homomorphism, this implies that $\text{sgn}(x) \in \mathbb{F}_q^\times$ and thus $J \in P_+ (A)$. Therefore $\sigma = 1$.

(2) Set $E = K(\psi(I), I \in \mathcal{I}(A), \deg I \equiv 0 \pmod{d_\infty})$. Observe that
\[ \text{Gal}(H/H_A) = \{\sigma_{xA}, x \in K^\times\}. \]
Thus,
\[ K(\mathbb{F}_\infty) \subset E \subset H_A. \]
We also have
\[ \text{Gal}(H_A/K(\mathbb{F}_\infty)) = \{(I, H_A/K), I \in \mathcal{I}(A), \deg I \equiv 0 \pmod{d_\infty}\}. \]
Let $\sigma \in \text{Gal}(H_A/E)$. Then, there exists $J \in \mathcal{I}(A), \deg J \equiv 0 \pmod{d_\infty}$, such that $\sigma = (J, H_A/K)$. But for all $I \in \mathcal{I}(A), \deg I \equiv 0 \pmod{d_\infty}$, we have
\[ \psi(IJ) = \sigma(\psi(I))\psi(J) = \psi(I)\psi(J), \]
and therefore
\[ (I, H_A/K)(\psi(J)) = \psi(J). \]
This implies
\[ \psi(J) \in K(\mathbb{F}_\infty)^\times. \]
But
\[ JA[\mathbb{F}_\infty] = \psi(J)A[\mathbb{F}_\infty]. \]
Thus $J^{d_\infty}$ is a principal ideal. But
\[ \psi(J^{d_\infty}) = \psi(J)^{d_\infty}. \]
In particular,
\[ \psi(J)^{(d_\infty q^{d_\infty} - 1)/(q-1)} \in K^\times. \]
Thus, if $\delta$ is the Frobenius in $\text{Gal}(K(\mathbb{F}_\infty)/K)$, there exists $\zeta \in \mathbb{F}_\infty^\times$ such that
\[ \delta(\psi(J)) = \zeta \psi(J). \]
Observe that
\[ N_{\mathbb{F}_\infty/K}(\zeta) = 1. \]
Thus,
\[ \zeta = \frac{\mu}{\delta(\mu)}, \]
for some $\mu \in \mathbb{F}_\infty^\times$. This implies that
\[ \psi(J)\mu \in K^\times. \]
Therefore $J$ is a principal ideal and thus $\sigma = 1$. □
We have the following crucial fact:

**Proposition 3.4.** Let $E/K$ be a finite extension such that $H \subset E$. Then

$$L_A(\phi/O_E) = \zeta_{O_E}(1).$$

**Proof.** Let $\mathfrak{P}$ be a maximal ideal of $O_E$. Let $m = \lfloor \frac{O_E}{\mathfrak{P}} \rfloor$. Then

$$N_{E/K}(\mathfrak{P}) = P^m.$$

Since $H \subset O_E$, by class field theory, we get

$$P^m = \theta A, \quad \text{with } \theta \in A, \sigma(\theta) = 1.$$

Since $\phi$ is a rank one Drinfeld module, it implies that

$$\phi \equiv \tau^{m \deg P} \pmod{\mathfrak{P}}.$$

This implies that

$$\left[ \phi \left( \frac{O_E}{\mathfrak{P}} \right) \right]_A = \theta - 1 = \left[ \frac{O_E}{\mathfrak{P}} \right]_A - 1.$$

We get

$$L_A(\phi/O_E) = \prod_{\mathfrak{P}} \left[ \frac{O_E}{\mathfrak{P}} \right]_A^{-1} = \prod_{\mathfrak{P}} \left( 1 - \frac{1}{\left[ \frac{O_E}{\mathfrak{P}} \right]_A} \right)^{-1} = \zeta_{O_E}(1).$$

□

3.3. **Equivariant A-harmonic series: a detailed example**

We keep the notation of Section 3.2. Let $z$ be an indeterminate over $K_\infty$, and recall that $T_z(K_\infty)$ denotes the Tate algebra in the variable $z$ with coefficients in $K_\infty$. Recall that

$$H_\infty = H \otimes_K K_\infty,$$

$$T_z(H_\infty) = H \otimes_K T_z(K_\infty).$$

For $n \in \mathbb{Z}$, we set

$$Z_B(n; z) = \sum_{d \geq 0} \sum_{\mathfrak{P} \in I(B), \mathfrak{P} \subset B, \deg(N_{E/K}(\mathfrak{P})) = d} \left[ \frac{O_E}{\mathfrak{P}} \right]_A^{-n} z^d.$$

Then, by [21, Theorem 8.9.2], for all $n \in \mathbb{Z}$, $Z_B(n; \cdot)$ defines an entire function on $\mathbb{C}_\infty$, and

$$\forall n \in \mathbb{N}, \quad Z_B(-n; z) \in A[z].$$

Observe that

$$\forall n \in \mathbb{Z}, \quad Z_B(n; z) \in T_z(K_\infty),$$

and

$$\forall n \geq 1, \quad Z_B(n; z) = \prod_{\mathfrak{P}} \left( 1 - \frac{z^{\deg(N_{H/K}(\mathfrak{P}))}}{\left[ \frac{O_E}{\mathfrak{P}} \right]_A} \right)^{-1} \in T_z(K_\infty)^\times.$$

Finally, we note that

$$Z_B(1; 1) = \zeta_B(1).$$
Recall that $G = \text{Gal}(H/K)$. Then $G \simeq \text{Gal}(H(z)/K(z))$ acts on $T_z(H_\infty)$. We denote by $T_z(H_\infty)[G]$ the non-commutative group ring where the commutation rule is given by

$$\forall h, h' \in T_z(H_\infty), \forall g, g' \in G, \quad hg.h'g' = hg(h')gg'.$$

Recall that for $I \in \mathcal{I}(A)$, we have set (3.1)

$$\sigma_I = (I, H/K) \in G.$$

**Lemma 3.5.** Let $n \in \mathbb{Z}$. The following infinite sum converges in $T_z(H_\infty)[G]$:

$$\mathcal{L}(\phi/B; n; z) := \sum_{d \geq 0} \sum_{I \in \mathcal{I}(A), I \subset A, \deg I = d} z^{\deg I} \psi(I)^n \sigma_I.$$

Furthermore, for all $n \geq 1$, we have

$$\mathcal{L}(\phi/B; n; z) = \prod_P \left(1 - \frac{z^{\deg P}}{\psi(P)^n \sigma_P}\right)^{-1} \in (T_z(H_\infty)[G])^\times$$

and for all $n \leq 0$

$$\mathcal{L}(\phi/B; n; z) \in B[z][G].$$

**Proof.** Let $n \geq 1$. First let’s observe that for any place $w$ of $H$ above $\infty$:

$$\lim_{I \subset A, \deg I \to +\infty} w(\psi(I)) = +\infty.$$

Let $P$ be a maximal ideal of $A$. Note that

$$\forall k \geq 0, \quad \psi(P^{k+1}) = \sigma_P(\psi(P^k))\psi(P) = \sigma_P(\psi(P))\psi(P^k).$$

Thus,

$$\sum_{m \geq 0} \frac{z^{m \deg P}}{\psi(P^m)^n \sigma_P^m} \in T_z(H_\infty)[G],$$

and we have

$$\left(1 - \frac{z^{\deg P}}{\psi(P)^n \sigma_P}\right) \left(\sum_{m \geq 0} \frac{z^{m \deg P}}{\psi(P^m)^n \sigma_P^m}\right) = \left(\sum_{m \geq 0} \frac{z^{m \deg P}}{\psi(P^m)^n \sigma_P^m}\right) \left(1 - \frac{z^{\deg P}}{\psi(P)^n \sigma_P}\right) = 1.$$

Thus, we have

$$\left(1 - \frac{z^{\deg P}}{\psi(P)^n \sigma_P}\right)^{-1} := \sum_{m \geq 0} \frac{z^{m \deg P}}{\psi(P^m)^n \sigma_P^m} \in (T_z(H_\infty)[G])^\times.$$

Let $P, Q$ be two distinct maximal ideals of $A$. We have

$$\left(1 - \frac{z^{\deg P}}{\psi(P)^n \sigma_P}\right) \left(1 - \frac{z^{\deg Q}}{\psi(Q)^n \sigma_Q}\right) = \left(1 - \frac{z^{\deg Q}}{\psi(Q)^n \sigma_Q}\right) \left(1 - \frac{z^{\deg P}}{\psi(P)^n \sigma_P}\right).$$

Therefore,

$$\mathcal{L}(\phi/B; n; z) = \prod_P \left(1 - \frac{z^{\deg P}}{\psi(P)^n \sigma_P}\right)^{-1} = \sum_{I \in \mathcal{I}(A), I \subset A} \frac{z^{\deg I}}{\psi(I)^n \sigma_I} \in (T_z(H_\infty)[G])^\times.$$
Let \( n \in \mathbb{Z} \). For \( d \in \mathbb{N} \), we set
\[
S_{\psi,d}(B;n) = \sum_{I \in \mathcal{I}(A), I \subseteq A, \deg I = d} \psi(I)^{-n} \sigma_I \in H[G].
\]

Let \( h \) be the order of \( \mathcal{I}(A)/\mathcal{P}_+(A) \). Let \( I_1, \ldots, I_h \in \mathcal{I}(A) \cap A \) be a system of representatives of \( \mathcal{I}(A)/\mathcal{P}_+(A) \). Then
\[
S_{\psi,d}(B;n) = \sum_{j=1}^{h} \psi(I_j)^{-n} \sigma_{I_j} \sum_{a \in K^\times, \sgn(a) = 1, a I_j \subseteq A, \deg(a I_j) = d} a^{-n}.
\]

Now, let’s assume that \( n \leq 0 \). Then, by [4, Lemma 3.2], there exists an integer \( d_0(n, \psi, H) \in \mathbb{N} \) such that, for all \( d \geq d_0(n, \psi, H) \), for all \( j \in \{1, \ldots, h\} \), we have
\[
\sum_{a \in K^\times, \sgn(a) = 1, a I_j \subseteq A, \deg(a I_j) = d} a^{-n} = 0.
\]

Therefore, for \( d \geq d_0(n, \psi, H) \), we have
\[
S_{\psi,d}(B;n) = 0.
\]

Thus,
\[
\forall n \in \mathbb{N}, \quad \mathcal{L}(\phi/B; -n; z) \in B[z][G]. \quad \square
\]

The element \( \mathcal{L}(\phi/B) := \mathcal{L}(\phi/B; 1; 1) \in (H_\infty[G])^\times \) will be called the equivariant A-harmonic series attached to \( \phi/B \).

Note that \( \mathcal{L}(\phi/B; 1; z) \) induces a \( T_z(K_\infty) \)-linear map \( \mathcal{L}(\phi/B; 1; z) : T_z(H_\infty) \to T_z(H_\infty) \). Since \( T_z(H_\infty) \) is a free \( T_z(K_\infty) \)-module of rank \( [H : K] \) (recall that \( T_z(K_\infty) \) is a principal ideal domain), \( \det_{T_z(K_\infty)} \mathcal{L}(\phi/B; 1; z) \) is well defined. We also observe that \( \mathcal{L}(\phi/B) \) induces a \( K_\infty \)-linear map \( \mathcal{L}(\phi/B) : H_\infty \to H_\infty \), and we denote by \( \det_{K_\infty} \mathcal{L}(\phi/B) \) its determinant. Recall that \( \text{ev} : T_z(H_\infty) \to H_\infty \) is the \( H_\infty \)-linear map given by
\[
\forall f \in T_z(H_\infty), \quad \text{ev}(f) = f \mid_{z=1}.
\]

Observe that, if \( \{e_1, \ldots, e_n\} \) is a \( K \)-basis of \( H/K \) (recall that \( n = [H : K] \)), then
\[
H_\infty = \bigoplus_{i=1}^{n} K_\infty e_i, \\
T_z(H_\infty) = \bigoplus_{i=1}^{n} T_z(K_\infty) e_i.
\]

We deduce that
\[
\det_{K_\infty} \mathcal{L}(\phi/B) = \text{ev} \left( \det_{T_z(K_\infty)} \mathcal{L}(\phi/B; 1; z) \right).
\]

**Theorem 3.6.** We have
\[
\det_{T_z(K_\infty)} \mathcal{L}(\phi/B; 1; z) = Z_B(1; z).
\]

In particular,
\[
\det_{K_\infty} \mathcal{L}(\phi/B) = \zeta_B(1).
\]

**Proof.** First, we recall that, by Lemma 3.5, we have the following equality in \( T_z(H_\infty)[G] \):
\[
\prod_{P} \left( 1 - \frac{z^{\deg P}}{\psi(P)^{\deg P}} \sigma_P \right)^{-1} = \mathcal{L}(\phi/B; 1; z),
\]
where $P$ runs through the maximal ideals of $A$, and

$$
\left(1 - \frac{z^{\deg P}}{\psi(P)\sigma_P}\right)^{-1} = \sum_{n \geq 0} \frac{z^{n\deg P}}{\psi(P^n)\sigma_{P^n}}.
$$

By the properties of $\psi$ (Lemma 3.2), we have

$$
\lim_{N \to +\infty} \prod_{\deg P \geq N} \left(1 - \frac{z^{\deg P}}{\psi(P)\sigma_P}\right)^{-1} = 1.
$$

Thus,

$$
\det_{T_z(K_\infty)} \mathcal{L}(\phi/B; 1; z) = \prod_P \det_{T_z(K_\infty)} \left(1 - \frac{z^{\deg P}}{\psi(P)\sigma_P}\right)^{-1}.
$$

Thus, we are led to compute

$$
\det_{T_z(K_\infty)} \left(1 - \frac{z^{\deg P}}{\psi(P)\sigma_P}\right).
$$

But $1 - \frac{z^{\deg P}}{\psi(P)\sigma_P}$ induces a $K[z]$-linear map on $H[z]$. Thus,

$$
\det_{T_z(K_\infty)} \left(1 - \frac{z^{\deg P}}{\psi(P)\sigma_P}\right) = \det_{K[z]} \left(1 - \frac{z^{\deg P}}{\psi(P)\sigma_P}\right) |_{H[z]}.
$$

Let $e \geq 1$ be the order of $P$ in $I(A)/P_+(A)$. Write $(\xi = z^{\deg P}/\psi(P)\sigma_P |_{H[z]}$. We have $\xi^e = (z^e\deg P/\psi(P^e)) \in K[z]$. Since $e$ is the order of $\sigma_P$ in $G$, by Dedekind’s Theorem $\sigma_P, \sigma_{P^2}, \ldots, \sigma_{P^{e-1}}$ are linearly independent over $H(z)$. We deduce that $X^e - \frac{z^{\deg P}}{\psi(P^e)}$ is the minimal polynomial of $\xi$ over $K(z)$ and also over $H^{(\sigma_P)}(z)$, and that

$$
\det_{K[z]} \left(1 - \frac{z^{\deg P}}{\psi(P)\sigma_P}\right) |_{H[z]} = \left(1 - \frac{z^{e\deg P}}{\psi(P^e)}\right)^{[H:K]/e}.
$$

Now, let $\mathfrak{P}$ be a maximal ideal of $B$ above $P$. Then, by class field theory, we have

$$
\left[ \frac{B}{\mathfrak{P}} : \frac{A}{P} \right] = e.
$$

Therefore,

$$
\left[ \frac{B}{\mathfrak{P}} \right]_{\frac{A}{P}} = \psi(P^e).
$$

Thus,

$$
\det_{K[z]} \left(1 - \frac{z^{\deg P}}{\psi(P)\sigma_P}\right) |_{H[z]} = \prod_{\mathfrak{P} | P} \left(1 - \frac{z^{\deg(N_{H/K}(\mathfrak{P}))}}{\left[ \frac{B}{\mathfrak{P}} \right]_{\frac{A}{P}}}\right).
$$

Finally, we get

$$
\det_{T_z(K_\infty)} \mathcal{L}(\phi/B; 1; z) = \prod_{\mathfrak{P}} \left(1 - \frac{z^{\deg(N_{H/K}(\mathfrak{P}))}}{\left[ \frac{B}{\mathfrak{P}} \right]_{\frac{A}{P}}}\right)^{-1},
$$

where $\mathfrak{P}$ runs through the maximal ideals of $B$. Thus,

$$
\det_{T_z(K_\infty)} \mathcal{L}(\phi/B; 1; z) = Z_B(1; z).
$$
Now
\[ \det_{K_\infty} \mathcal{L}(\phi/B) = \text{ev}(\det_{\mathbb{Z}/(K_\infty)} \mathcal{L}(\phi/B; 1; z)) = \text{ev}(Z_B(1; z)) = \zeta_B(1). \]

Although this is not evident, the above theorem reflects a class formula à la Taelman which will be proved in Section 3.5.

3.4. Stark units

We keep the notation of the previous sections. We will need the following basic result:

**Lemma 3.7.** Let \( L/K \) be a finite extension, and let \( O_L \) be the integral closure of \( A \) in \( L \). Let \( \rho: A \to O_L \{ \tau \} \) be a Drinfeld module of rank \( r \geq 1 \). Let \( \exp_\rho, \log_\rho \in 1 + L\{ \tau \} \) be such that
\[
\forall a \in A, \quad \exp_\rho a = \rho_a \exp_\rho,
\]
\[
\exp_\rho \log_\rho = \log_\rho \exp_\rho = 1.
\]
Write
\[
\exp_\rho = \sum_{i \geq 0} e_i(\rho) \tau^i,
\]
\[
\log_\rho = \sum_{i \geq 0} l_i(\rho) \tau^i,
\]
with \( e_i(\rho), l_i(\rho) \in L \).

1. Let \( P \) be a maximal ideal of \( A \). Let \( A_P \) be the \( P \)-adic completion of \( A \). Then
\[
\forall n \geq 0, \quad P q^n e_n(\rho) O_L \subset P O_L \otimes_A A_P,
\]
\[
\forall n \geq 0, \quad P_{q^{\deg(P)}} l_n(\rho) O_L \subset O_L \otimes_A A_P.
\]

2. Let \( \sigma: L \to \overline{K}_\infty \) be a field homomorphism such that \( \sigma|_K = \text{Id}_K \). Then, there exist \( n(\rho, \sigma) \in \mathbb{N}, C(\rho, \sigma) \in ]0; +\infty[ \) such that
\[
\forall n \geq n(\rho, \sigma), \quad v_\infty(\sigma(e_n(\rho))) \geq C(\rho, \sigma) n q^n.
\]

**Proof.** (1) Let \( \theta \in A \setminus \mathbb{F}_q \) such that \( \theta A \rho = PA \rho \). Let \( d = r \deg(\theta) \), and let’s write
\[
\rho_\theta = \sum_{j=0}^{d} \rho_{\theta, j} \tau^j.
\]
From \( \exp_\rho \theta = \rho_\theta \exp_\rho \), we get
\[
\forall n \geq 0, \quad \left( \theta q^n - \theta \right) e_n(\rho) = \sum_{j=1}^{d} \rho_{\theta, j} e_{n-j}(\rho) q^{j-i}
\]
where \( e_i = 0 \) if \( i < 0 \). Since \( e_0(\rho) = 1 \), one proves by induction on \( n \geq 0 \) that
\[
e_n(\rho) \theta q^n \in \theta^{\inf \{q^{-1}; q^n\}} O_L \otimes_A A_P.
\]
Observe that
\[
\forall a \in A, \quad a \log_\rho = \log_\rho a.
\]
Thus, 
\[ \forall a \in A, \forall n \geq 0, \quad \left( a - a^n \right) l_n(\rho) = \sum_{l=1}^{r \deg a} l_{n-l}(\rho) \rho_{a,l}^{n-l}. \]

Thus, if \( n \not\equiv 0 \pmod{\deg P} \), we get 
\[ l_n(\rho)O_E \otimes_A A_P \subset \sum_{l=1}^{n} l_{n-l}(\rho)O_L \otimes_A A_P. \]

If \( n \equiv 0 \pmod{\deg P} \), we have 
\[ \left( \theta - \theta^n \right) l_n(\rho) = \sum_{l=1}^{d} l_{n-l}(\rho) \rho_{\theta,l}^{n-l}. \]

In any case, we get 
\[ \theta \left[ \frac{n}{\deg P} \right] l_n(\rho) \in \sum_{l=1}^{n} \theta \left[ \frac{(n-l)}{\deg P} \right] l_{n-l}(\rho)O_L \otimes_A A_P. \]

Since \( l_0(\rho) = 1 \), we get the desired second assertion by induction on \( n \geq 0 \).

(2) This is a consequence of the proof of [21, Theorem 4.6.9]. We give a proof for the convenience of the reader. We keep the previous notation. In particular, let \( \theta \in A \setminus \mathbb{F}_q \), and write
\[ \rho_{\theta} = \sum_{j=0}^{\deg(\theta)} \rho_{\theta,j} \tau^j, \quad \text{with} \quad \rho_{\theta,j} \in \mathbb{K}_\infty. \]

Recall that \( \rho_{\theta,0} = \theta \). Set \( d = r \deg(\theta) \). Then 
\[ \forall n \geq 0, \quad \left( \theta^n - \theta \right) e_n(\rho) = \sum_{l=1}^{d} \rho_{\theta,l} e_{n-l}(\rho) \tau^l. \]

Set \( u = \frac{\deg(\theta)}{d_{\infty}} = -v_\infty(\theta) \geq 1 \). We get 
\[ v_\infty(e_n(\rho)) / q^n \geq u + \inf \left\{ v_\infty(e_{n-j}(\rho)) / q^{n-j} + v_\infty(\rho_{\theta,j}) / q^n, j = 1, \ldots, d \right\}. \]

Let \( \beta \in [0; u[ \). There exists an integer \( n_0 \) such that 
\[ \forall n \geq n_0, \quad \inf \left\{ \frac{v_\infty(\rho_{\theta,j})}{q^n}, j = 1, \ldots, d \right\} \geq \beta - u. \]

Therefore, 
\[ \forall n \geq n_0, \quad \frac{v_\infty(e_n(\rho))}{q^n} \geq \beta + \inf \left\{ \frac{v_\infty(e_{n-j}(\rho))}{q^{n-j}}, j = 1, \ldots, d \right\}. \]

Thus, for \( n \in [n_0; n_0 + d - 1] \), we get 
\[ \frac{v_\infty(e_n(\rho))}{q^n} \geq \beta + \inf \left\{ \frac{v_\infty(e_{n-j}(\rho))}{q^{n-j}}, j = 1, \ldots, d \right\}. \]

Set 
\[ C = \inf \left\{ \frac{v_\infty(e_{n_0-j}(\rho))}{q^{n_0-j}}, j = 1, \ldots, d \right\}. \]
By induction, we show that if \( n \geq n_0 + md, m \in \mathbb{N} \), then
\[
\frac{v_\infty(e_{n}(\rho))}{q^n} \geq \beta(m + 1) + C.
\]
Therefore there exist \( n_1 \geq n_0, C', C \in \mathbb{Q} \), with \( C' > 0 \), such that
\[
\forall n \geq n_1, \quad v_\infty(e_{n}(\rho)) \geq C'nq^n + C. \quad \Box
\]

Let \( E/K \) be a finite abelian extension \( H \subset E \). Let \( G = \text{Gal}(E/K) \). We denote by \( S_E \) the set of maximal ideals \( P \) of \( A \) which are wildly ramified in \( E/K \) (note that we can have \( S_E = \emptyset \)). Let \( P \) be a maximal ideal of \( A \) such that \( P \notin S_E \). We fix a maximal ideal \( \mathfrak{P} \) of \( O_E \) above \( P \). Let \( D_P \subset G \) be the decomposition group associated to \( P \), that is, \( D_P = \{ g \in G, g(\mathfrak{P}) = \mathfrak{P} \} \). We have a natural surjective homomorphism \( D_P \rightarrow \text{Gal}(\frac{O_E}{\mathfrak{P}}/\mathbb{F}_p) \), \( g \mapsto \bar{g} \). Let \( I_P \) be the inertia group at \( P \), that is, \( I_P = \ker(D_P \rightarrow \text{Gal}(\frac{O_E}{\mathfrak{P}}/\mathbb{F}_p)) \). Then, since \( P \notin S_E \), we have
\[
|I_P| \equiv 0 \pmod{p}.
\]
Let \( \text{Frob}_P \in \text{Gal}(\frac{O_E}{\mathfrak{P}}/\mathbb{F}_p) \) be the Frobenius at \( P \), that is,
\[
\forall x \in \frac{O_E}{\mathfrak{P}}, \quad \text{Frob}_P(x) = x^{q^{\deg P}}.
\]
We set
\[
\sigma_{P,O_E} := \frac{1}{|I_P|} \sum_{\bar{g} \in D_P, \bar{g} = \text{Frob}_P} g \in \mathbb{F}_p[G].
\]
If \( P \in S_E \), we set
\[
\sigma_{P,O_E} = 0.
\]
Note that, if \( L/K \) is a finite abelian extension, \( L \subset E \), and if \( P \) is unramified in \( L \) with \( P \notin S_E \), then
\[
\sigma_{P,O_E} |_{L} = (P, L/K).
\]
If \( I \in \mathcal{I}(A), I \subset A, I = \prod_P P^{m_P} \), we set
\[
\sigma_{I,O_E} = \prod_P \sigma_{P,O_E}^{m_P} \in \mathbb{F}_p[G].
\]
For all \( n \in \mathbb{Z} \), we set
\[
\mathcal{L}(\phi/O_E; n; z) = \sum_{d \geq 0} \sum_{I \in \mathcal{I}(A), I \subset A, \deg I = d} \frac{z^d}{\psi(I)^n} \sigma_{I,O_E} \in H[G[[z]]].
\]
By the proof of Lemma 3.5, we have
\[
\forall n \in \mathbb{Z}, \quad \mathcal{L}(\phi/O_E; n; z) \in \mathbb{T}_z(H_{\infty})[G],
\]
and
\[
\mathcal{L}(\phi/O_E; 1; z) = \prod_P \left( 1 - \frac{z^{\deg P}}{\psi(P)} \sigma_{P,O_E} \right)^{-1} \in (\mathbb{T}_z(H_{\infty})[G])^\times.
\]
Note that, if \( L/K \) is a finite abelian extension, \( H \subset L \subset E \), we have
\[
\mathcal{L}(\phi/O_E; 1; z) \mid_{\mathbb{T}_z(L_{\infty})} = \left( \prod_{P \in S_E \setminus S_L} \left( 1 - \frac{z^{\deg P}}{\psi(P)} \sigma_{P,O_L} \right) \mathcal{L}(\phi/O_L; 1; z) \right) \mid_{\mathbb{T}_z(L_{\infty})}.
\]
We set

\[ I(O_E) = \prod_{P \in SE} P. \]

Recall that

\[ U(\tilde{\phi}/O_E[z]) = \left\{ f \in T(z(E_\infty), \exp_{\tilde{\phi}}(f) \in O_E[z] \right\}. \]

**Theorem 3.8.** We always have

\[ \psi(I(O_E))L(\phi/O_E; 1; z)O_E[z] \subset U(\tilde{\phi}/O_E[z]). \]

Furthermore, if \( S_E = \emptyset \), we have an equality

\[ L(\phi/O_E; 1; z)O_E[z] = U(\tilde{\phi}/O_E[z]). \]

**Proof.** We divide the proof into several steps.

1. We will first work in \( E[[z]] \). Observe that \( \exp_{\tilde{\phi}} : E[[z]] \to \tilde{\phi}(E[[z]]) \) is an isomorphism of \( A \)-modules. In fact, if we write \( \log_{\phi} = \sum_{i \geq 0} l_i(\phi) \tau^i \), then we set

\[ \log_{\tilde{\phi}} = \sum_{i \geq 0} l_i(\phi) z^i \tau^i. \]

Thus, \( \log_{\tilde{\phi}} \) converges on \( E[[z]] \), and \( \log_{\tilde{\phi}} \exp_{\tilde{\phi}} = \exp_{\tilde{\phi}} \log_{\tilde{\phi}} = 1. \)

2. Let \( P \) be a maximal ideal of \( A \). Let \( R_P = S^{-1}O_E \subset E \), where \( S = A \setminus P \). Then

\[ PR_P = \psi(P)R_P. \]

By Lemma 3.7, we have

\[ \exp_{\tilde{\phi}}(PR_P[[z]]) \subset PR_P[[z]], \]

\[ \log_{\tilde{\phi}}(PR_P[[z]]) \subset PR_P[[z]]. \]

Thus,

\[ \exp_{\tilde{\phi}}(PR_P[[z]]) = PR_P[[z]]. \quad (3.2) \]

3. Recall that there exists a sign-normalized rank one Drinfeld module \( \phi := P * \phi : A \rightarrow B(\tau) \) such that

\[ \forall a \in A, \ \ \phi_P \phi_a = \varphi_a \phi_P. \]

Furthermore [21, Theorem 7.4.8],

\[ \forall a \in A, \ \ \varphi_a = \sigma_p \left( \phi_a \right) := \sum_{i=0}^{r-\deg a} \sigma_p(\phi_{a,i}) \tau^i. \]

Thus,

\[ \exp_{\phi} = \sigma_p \left( \exp_{\phi} \right) := \sum_{i \geq 0} \sigma_p(e_i(\phi)) \tau^i, \]

\[ \log_{\phi} = \sigma_p \left( \log_{\phi} \right) := \sum_{i \geq 0} \sigma_p(l_i(\phi)) \tau^i. \]
In particular,
\[
\phi_P \exp_\phi = \sigma_P \left( \exp_\phi \right) \psi(P),
\]
\[
\psi(P) \log_\phi = \sigma_P \left( \log_\phi \right) \phi_P.
\]
The same properties hold for \( \tilde{\phi} \).

(4) Let’s set
\[
U \left( \tilde{\phi}/R_P[[z]] \right) = \left\{ x \in E[[z]]; \exp_\tilde{\phi}(x) \in R_P[[z]] \right\}.
\]

Let’s assume that \( P \notin S_E \). Then, by (1) and (2), \( \exp_\tilde{\phi} \) induces an isomorphism of \( A \)-modules
\[
E[[z]] / PR_P[[z]] \cong \tilde{\phi} \left( E[[z]] / PR_P[[z]] \right).
\]

Therefore, we get an isomorphism of \( A \)-modules
\[
U \left( \tilde{\phi}/R_P[[z]] \right) / PR_P[[z]] \cong \tilde{\phi} \left( R_P[[z]] / PR_P[[z]] \right).
\]

Now observe that
\[
\left( \tilde{\phi}_P - z^{\deg_P \sigma_{P,OE}} \right) \tilde{\phi} \left( R_P[[z]] / PR_P[[z]] \right) = \{0\}.
\]

Furthermore, if \( x \in E[[z]] \setminus R_P[[z]] \), then one can easily verify that
\[
\left( \tilde{\phi}_P - z^{\deg_P \sigma_{P,OE}} \right) (x) \notin PR_P[[z]].
\]

Thus,
\[
\tilde{\phi} \left( R_P[[z]] / PR_P[[z]] \right) = \left\{ x \in \tilde{\phi} \left( E[[z]] / PR_P[[z]] \right); \left( \tilde{\phi}_P - z^{\deg_P \sigma_{P,OE}} \right) (x) = 0 \right\}.
\]

Let \( x \in E[[z]] \), we deduce that
\[
x \in U \left( \tilde{\phi}/R_P[[z]] \right) \iff \left( \tilde{\phi}_P - z^{\deg_P \sigma_{P,OE}} \right) \left( \exp_\tilde{\phi}(x) \right) \in PR_P[[z]].
\]

Observe that, by (3), we have
\[
\sigma_{P,OE} \left( \exp_\tilde{\phi} \right) = \exp_\tilde{\phi},
\]
and also
\[
\tilde{\phi}_P \exp_\tilde{\phi} = \sigma_{P,OE} \left( \exp_\tilde{\phi} \right) \psi(P).
\]

Thus,
\[
x \in U \left( \tilde{\phi}/R_P[[z]] \right) \iff \exp_\tilde{\phi} \left( \psi(P)x - z^{\deg_P \sigma_{P,OE}} \left( \exp_\tilde{\phi}(x) \right) \right) \in PR_P[[z]].
\]

Applying (3.2) for \( \varphi \), we have
\[
x \in U \left( \tilde{\phi}/R_P[[z]] \right) \iff \psi(P)x - z^{\deg_P \sigma_{P,OE}}(x) \in PR_P[[z]].
\]

Thus,
\[
U \left( \tilde{\phi}/R_P[[z]] \right) = \left( 1 - \frac{z^{\deg_P \sigma_{P,OE}}}{\psi(P)} \right)^{-1} R_P[[z]].
\]
(5) Let $P$ be a maximal ideal of $A$. If $P \not\in S_E$, by (4), we have
\[ U \left( \tilde{\phi}/R_P[[z]] \right) = \psi(I(O_E)) \Lambda(\phi/O_E; 1; z) R_P[[z]] \]
\[ = \left( 1 - \frac{z^{\deg P}}{\psi(P)} \sigma_{P,O_E} \right)^{-1} R_P[[z]]. \]

If $P \in S_E$, then
\[ \psi(I(O_E)) \Lambda(\phi/O_E; 1; z) R_P[[z]] = PR_P[[z]] \subset U \left( \tilde{\phi}/R_P[[z]] \right). \]

Since $\psi(I(O_E)) \Lambda(\phi/O_E; 1; z) \in T_z(H_{\infty})[G]$, we get
\[ \psi(I(O_E)) \Lambda(\phi/O_E; 1; z) R[z] \subset T_z(E_{\infty}). \]

Observe that $O_E[[z]] = \bigcap_P R_P[[z]]$. Therefore, we get
\[ \exp_{\tilde{\phi}}(\psi(I(O_E)) \Lambda(\phi/O_E; 1; z) O_E[z]) \subset O_E[[z]] \cap T_z(E_{\infty}) = O_E[z]. \]

Thus, we get the first assertion.

Now, let’s assume that $S_E = \emptyset$. We have
\[ \bigcap_P U \left( \tilde{\phi}/R_P[[z]] \right) = \left\{ x \in E_{\infty}[[z]], \exp_{\tilde{\phi}}(x) \in O_E[[z]] \right\}. \]

By (4), we get
\[ \prod_P \left( 1 - \frac{z^{\deg P}}{\psi(P)} \sigma_{P,O_E} \right) \left\{ x \in E_{\infty}[[z]], \exp_{\tilde{\phi}}(x) \in O_E[[z]] \right\} = O_E[[z]]. \]

Thus,
\[ \left\{ x \in E_{\infty}[[z]], \exp_{\tilde{\phi}}(x) \in O_E[[z]] \right\} = \Lambda(\phi/O_E; 1; z) O_E[[z]]. \]

Hence,
\[ U \left( \tilde{\phi}/R[z] \right) = \Lambda(\phi/O_E; 1; z) O_E[[z]] \cap T_z(E_{\infty}). \]

Since $\Lambda(\phi/O_E; 1; z) \in (T_z(H_{\infty})[G])^\times$, we have
\[ \Lambda(\phi/O_E; 1; z) O_E[[z]] \cap T_z(E_{\infty}) = \Lambda(\phi/O_E; 1; z) O_E[z]. \]

This concludes the proof of the theorem. \qed

3.5. A class formula à la Taelman

Recall that $\psi : T_z(E_{\infty}) \to E_{\infty}$ is the evaluation at $z = 1$.

**Definition 3.9.** We define the **equivariant A-harmonic series** $\Lambda(\phi/O_E)$ attached to $\phi/O_E$ by
\[ \Lambda(\phi/O_E) = \psi(\Lambda(\phi/O_E; 1; z)) \in (H_{\infty}[G])^\times. \]

Note that
\[ \Lambda(\phi/O_E) = \prod_P \left( 1 - \frac{1}{\psi(P)} \sigma_{P,O_E} \right)^{-1} = \sum_{I \in \mathcal{I}(A), I \subset A} \frac{1}{\psi(I)} \sigma_{I,O_E}. \]
Theorem 3.10. We have
\[ \alpha_A(\phi/O_E) = 1, \]
that is,
\[ \zeta_{O_E}(1) = [O_E : U(\phi/O_E)]_A[H(\phi/O_E)]_A. \]
Furthermore,
\[ \psi(I(O_E))\mathcal{L}(\phi/O_E)O_E \subset U_{SL,\phi}(O_E), \]
and
\[ \left[ \frac{U_{SL}(\phi/O_E)}{\psi(I(O_E))\mathcal{L}(\phi/O_E)O_E} \right]_A = \left[ \phi \left( \frac{O_E}{I(O_E)O_E} \right) \right]_A. \]

Proof. (1) Let \( J \subset I(O_E) \) be a finite product of maximal ideals of \( A \). Set
\[ \mathcal{L}_J(\phi/O_E) := \prod_P \left( 1 - \frac{1}{\psi(P)} \sigma_{P,O_E} \right)^{-1} \in (H_\infty[G])^\times, \]
\[ \mathcal{L}_J(\phi/O_E; 1; z) := \prod_P \left( 1 - \frac{z \deg P}{\psi(P)} \sigma_{P,O_E} \right)^{-1} \in (\mathbb{T}_z(H_\infty)[G])^\times, \]
where \( P \) runs through the maximal ideals of \( A \) that do not divide \( J \).

By Lemma 3.7 and the proof of Theorem 3.8, we have
\[ \left\{ x \in E[[z]], \exp_{\phi}(x) \in \psi(J)O_E[[z]] \right\} = \psi(J)\mathcal{L}_J(\phi/O_E; 1; z)O_E[[z]]. \]

We can conclude as in the proof of Theorem 3.8 that
\[ \psi(J)\mathcal{L}_J(\phi/O_E; 1; z)O_E[[z]] = \left\{ x \in \mathbb{T}_z(E_\infty), \exp_{\phi}(x) \in \psi(J)O_E[[z]] \right\}. \]

Therefore, we have a short exact sequence of \( A \)-modules
\[ 0 \rightarrow \frac{U(\phi/O_E[[z]])}{\psi(J)\mathcal{L}_J(\phi/O_E; 1; z)O_E[[z]]} \rightarrow \tilde{\phi} \left( \frac{O_E[[z]]}{\psi(J)O_E[[z]]} \right) \rightarrow \]
\[ \rightarrow \frac{\mathbb{T}_z(E_\infty)}{\psi(J)O_E[[z]] + \exp_{\phi}(\mathbb{T}_z(E_\infty))} \rightarrow H(\tilde{\phi}/O_E[[z]]) \rightarrow 0. \]

Note that \( \tilde{\phi}(O_E[[z]]/\psi(J)O_E[[z]]) \) is a finitely generated and free \( \mathbb{F}_q[z] \)-module. Let \( \rho \) be the Drinfeld module defined over \( O_E \) such that
\[ \exp_{\rho} = \psi(J)^{-1}\exp_{\phi}\psi(J). \]

Then, the map \( x \mapsto \psi(J)^{-1}x \) induces an isomorphism of \( A \)-modules (the left module is an \( A \)-module via \( \phi \) and the right module is an \( A \)-module via \( \rho \)):
\[ \frac{\mathbb{T}_z(E_\infty)}{\psi(J)O_E[[z]] + \exp_{\phi}(\mathbb{T}_z(E_\infty))} \simeq H(\tilde{\rho}/O_E[[z]]) \]

Observe that \( \ker ev = (z - 1)\mathbb{T}_z(E_\infty) \). Furthermore, since \( O_E[[z]] \cap (z - 1)\mathbb{T}_z(E_\infty) = (z - 1)O_E[[z]] \), we have
\[ U \left( \tilde{\phi}/O_E[[z]] \right) \cap \ker ev = (z - 1)U \left( \tilde{\phi}/O_E[[z]] \right), \]
\[ \psi(J)\mathcal{L}_J(\phi/O_E; 1; z)O_E[[z]] \cap \ker ev = (z - 1)\psi(J)\mathcal{L}_J(\phi/O_E; 1; z)O_E[[z]]. \]
Thus, the evaluation at $z = 1$ induces the following exact sequence of $A$-modules:

\[ 0 \to (z - 1) \frac{U \left( \tilde{\phi}/O_E[z] \right)}{\psi(J) \mathcal{L}_J(\phi/O_E; 1; z)O_E[z]} \to \]

\[ \to \frac{U \left( \tilde{\phi}/O_E[z] \right)}{\psi(J) \mathcal{L}_J(\phi/O_E; 1; z)O_E[z]} \to \frac{U_{S_1}(\phi/O_E)}{\psi(J) \mathcal{L}_J(\phi/O_E)O_E} \to 0. \tag{3.4} \]

Note also that the evaluation at $z = 1$ induces a sequence of $A$-modules:

\[ 0 \to (z - 1) \tilde{\phi} \left( \frac{O_E[z]}{\psi(J)O_E[z]} \right) \to \tilde{\phi} \left( \frac{O_E[z]}{\psi(J)O_E[z]} \right) \to \frac{O_E}{\psi(J)O_E} \to 0. \tag{3.5} \]

For an $\mathbb{F}_q[z]$-module $M$, we denote by $M[z - 1]$ the $(z - 1)$-torsion. By (3.3), (3.4), (3.5) and the Snake Lemma, we get the following exact sequence of finite $A$-modules:

\[ 0 \to H(\tilde{\phi}/O_E[z])[z - 1] \to H(\tilde{\phi}/O_E[z])[z - 1] \to \frac{U_{S_1}(\phi/O_E)}{\psi(J) \mathcal{L}_J(\phi/O_E)O_E} \to \]

\[ \to \phi \left( \frac{O_E}{\psi(J)O_E} \right) \to H(\phi/O_E) \to H(\phi/O_E) \to 0. \]

By the proof of Theorem 2.7, we have

\[ [H(\tilde{\phi}/O_E[z])[z - 1]]_A = [H(\phi/O_E)]_A, \]

\[ [H(\tilde{\phi}/O_E[z])[z - 1]]_A = [H(\phi/O_E)]_A. \]

Thus,

\[ \left[ \frac{U_{S_1}(\phi/O_E)}{\psi(J) \mathcal{L}_J(\phi/O_E)O_E} \right]_A = \left[ \frac{O_E}{\psi(J)O_E} \right]_A. \]

(2) Now, we have

\[ [O_E : \mathcal{L}_J(\phi/O_E)]_A = \frac{\det_{K_{\infty}} \mathcal{L}_J(\phi/O_E)}{\text{sgn}(\det_{K_{\infty}} \mathcal{L}_J(\phi/O_E))}. \]

Thus,

\[ [O_E : \psi(J) \mathcal{L}_J(\phi/O_E)O_E]_A = \left[ \frac{O_E}{\psi(J)O_E} \right]_A \frac{\det_{K_{\infty}} \mathcal{L}_J(\phi/O_E)}{\text{sgn}(\det_{K_{\infty}} \mathcal{L}_J(\phi/O_E))}. \]

And finally, we get

\[ [O_E : U_{S_1}(\phi/O_E)]_A = \left[ \frac{O_E}{\psi(J)O_E} \right]_A \frac{\det_{K_{\infty}} \mathcal{L}_J(\phi/O_E)}{\text{sgn}(\det_{K_{\infty}} \mathcal{L}_J(\phi/O_E))}. \]

Set

\[ L_J = \prod_{\psi \mid J} \left[ \frac{O_E}{\phi(\psi(O_E))} \right]_A. \]

Then

\[ [O_E : U_{S_1}(\phi/O_E)]_A = L_J \frac{\det_{K_{\infty}} \mathcal{L}_J(\phi/O_E)}{\text{sgn}(\det_{K_{\infty}} \mathcal{L}_J(\phi/O_E))}. \]
(3) Let $N \geq 1$, and we define $J_N$ to be the l.c.m. of the product of all maximal ideals of degree $\leq N$ and $I(O_E)$. We have

$$
\lim_{N \to +\infty} L_{J_N} = L_A(\phi/O_E),
$$

$$
\lim_{N \to +\infty} L_{J_N}(\phi/O_E) = 1.
$$

In particular,

$$
\lim_{N \to +\infty} \det_K L_{J_N}(\phi/O_E) = 1.
$$

Thus,

$$
[O_E : U_{St}(\phi/O_E)]_A = L_A(\phi/O_E).
$$

If we apply Theorem 2.7 and Proposition 3.4, we get

$$
\zeta_{O_E}(1) = [O_E : U(\phi/O_E)]_A[H(\phi/O_E)]_A.
$$

4. Log-algebraicity theorem

4.1. A refinement of Anderson’s log-algebraicity theorem

We keep the notation of the previous sections.

**Lemma 4.1.** Let $E/K$ be a finite separable extension, $H \subset E$. Let $P$ be a maximal ideal of $A$ which is unramified in $E$. Let $\lambda_P \in K \setminus \{0\}$ be a root of $\phi_P$. Then

$$
O_E(\lambda_P) = O_E[\lambda_P].
$$

**Proof.** Let $F = E(\lambda_P)$. Recall that $F/E$ is a finite abelian extension unramified outside $P, \infty$, and totally ramified at $P$ [21, Proposition 7.5.18]. We also have

$$
[F : E] = q^{\deg P} - 1.
$$

Write $\phi_P = \sum_{k=0}^{\deg P} \phi_{P,k} \tau^k$, $\phi_{P,k} \in B \subset O_E$. Recall that $\phi_{P,0} = \psi(P)$ and $\phi_{P,\deg P} = 1$. Furthermore, $P$ is unramified in $E/K$ and

$$
\psi(P)O_E = PO_E.
$$

Let

$$
G(X) = \sum_{k=0}^{\deg P} \phi_{P,k} X^{q^k - 1} \in O_E[X].
$$

Then, for any maximal ideal $\mathfrak{P}$ of $O_E$ above $P$

$$
G(X) \equiv X^{\deg P - 1} \pmod{\mathfrak{P}}.
$$

This implies that $G(X)$ is an Eisenstein polynomial at $\mathfrak{P}$ for every maximal ideal $\mathfrak{P}$ of $O_E$ above $P$. Furthermore,

$$
XG'(X) + G(X) = \psi(P).
$$

Therefore,

$$
N_{F/E}(G'(\lambda_P))O_E = P^{\deg P - 2}O_E.
$$

But $P^{\deg P - 2}O_E$ is the discriminant of $O_F/O_E$. Thus $O_F = O_E[\lambda_P]$. \qed
Let $E/K$ be a finite abelian extension, $H \subset E$. Let $n \geq 0$ be an integer, let $X_1, \ldots, X_n$ be $n$ indeterminates over $K$. Let $\tau : E[X_1, \ldots, X_n][[z]] \to E[X_1, \ldots, X_n][[z]]$ be the $\mathbb{F}_q[[z]]$-homomorphism continuous for the $z$-adic topology such that

$$\forall f \in E[X_1, \ldots, X_n], \quad \tau(f) = f^q.$$ 

For a non-zero ideal $I$ of $A$ and for $f = \sum_{i_1, \ldots, i_n \in \mathbb{N}} f_{i_1, \ldots, i_n} X_1^{i_1} \cdots X_n^{i_n} \in E[X_1, \ldots, X_n]$, with $f_{i_1, \ldots, i_n} \in E$, we set:

$$I \ast_E f = \sum_{i_1, \ldots, i_n \in \mathbb{N}} \sigma_{I,O_E}(f_{i_1, \ldots, i_n}) \phi_I(X_1)^{i_1} \cdots \phi_I(X_n)^{i_n},$$

where $\sigma_{I,O_E}$ is defined in Section 3.4. Recall that $I(O_E)$ is the product of maximal ideals of $A$ that are wildly ramified in $E/K$.

**Theorem 4.2.** For all $f \in O_E[X_1, \ldots, X_n]$, we have

$$\exp_{\phi} \left( \psi(I(O_E)) \sum_{I \in \mathcal{I}(A), I \subset A} \frac{I \ast_E f}{\psi(I)} z^{\deg I} \right) \in O_E[X_1, \ldots, X_n, z].$$

In particular, for all $f \in B[X_1, \ldots, X_n]$, we have

$$\exp_{\phi} \left( \sum_{I \in \mathcal{I}(A), I \subset A} \frac{I \ast_H f}{\psi(I)} z^{\deg I} \right) \in B[X_1, \ldots, X_n, z].$$

**Remark 4.3.** This result is a generalization of the Log-Algebraicity Theorems established in [1, 2] (in these papers the theorem is proved for $E = H$, $d_\infty = 1$ and $n \leq 1$). Observe that the original results of Anderson [1, 2] involve the quantity $z^{q^{\deg I}}$ while our results involve $z^{\deg I}$. The use of $z^{\deg I}$ instead of $z^{q^{\deg I}}$ is justified by the fact that equivariant $L$-series appear naturally (see the proof below).

Furthermore, the result in the case $E = H$ can be proved along the same lines as that used to prove [2, Theorem 3].

Following [10, Section 2.6], we will show below how Theorem 3.8 implies the Log-Algebraicity Theorem. Observe also that the case $n = 0$ is a direct consequence of Theorem 3.8.

**Proof.** Let’s write

$$\exp_{\phi} \left( \psi(I(O_E)) \sum_I \frac{I \ast_E f}{\psi(I)} z^{\deg I} \right) = \sum_{m \geq 0} g_m(X_1, \ldots, X_n) z^m,$$

with $g_m(X_1, \ldots, X_n) \in E[X_1, \ldots, X_n]$.

(1) Let $P_1, \ldots, P_n$ be $n$ distinct maximal ideals of $A$ which are unramified in $E$, with $q^{\deg P_i} \geq 3$, $i = 1, \ldots, n$, and for $i = 1, \ldots, n$, let $\lambda_i \neq 0$ be a root of $\phi_{P_i}$. Set

$$F = E(\lambda_1, \ldots, \lambda_n).$$

Then $F/E$ is unramified outside $P_1, \ldots, P_n, \infty$, $F/K$ a finite abelian extension of $K$ which is tamely ramified at $P_1, \ldots, P_n$. Let $O_F$ be the integral closure of $A$ in $F$. Letting $Q$ be any maximal ideal of $A$, if $Q$ is not wildly ramified in $E$, we have [21, Proposition 7.5.4]

$$\sigma_{Q,O_F}(\lambda_i) = \phi_Q(\lambda_i), \quad \text{if} \quad Q \neq P_i,$$

and

$$\sigma_{P_i,O_F}(\lambda_i) = 0.$$
We deduce that
\[ \psi(I(O_E)) \sum_{I} \frac{I \ast_E f}{\psi(I)} z^{\deg I} |_{x_i = \lambda_i} = \psi(I(O_F)) \mathcal{L}(\phi/O_F; 1; z) f(\lambda_1, \ldots, \lambda_n). \]

Therefore, by Theorem 3.8, we get
\[ \forall m \geq 0, \ g_m(\lambda_1, \ldots, \lambda_n) \in O_F. \]

Let \( i \in \{1, \ldots, n\}. \) Then
\[ E(\lambda_i) \cap E(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_n) = E. \]

Furthermore, the discriminant of \( O_{E(\lambda_i)}/O_E \) and \( O_{E(\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_n)}/O_E \) are relatively prime, thus, by Lemma 4.1, we have
\[ O_F = O_E[\lambda_1, \ldots, \lambda_n]. \]

Finally, for \( m \geq 0, \) for \( n \) distinct maximal ideals \( P_1, \ldots, P_n \) of \( A \) that are unramified in \( E/K, \) with \( q^{\deg P_i} \geq 3, i = 1, \ldots, n, \) and for \( i = 1, \ldots, n, \) if \( \lambda_i \neq 0 \) be a root of \( \phi_{P_i}, \) then we have
\[ g_m(\lambda_1, \ldots, \lambda_n) \in O_E[\lambda_1, \ldots, \lambda_n]. \]

This implies
\[ \forall m \geq 0, \ g_m(X_1, \ldots, X_n) \in O_E[X_1, \ldots, X_n]. \]

(2) We fix a \( K \)-embedding of \( \overline{K} \) in \( \mathbb{C}_\infty. \) For \( \sigma \in \text{Gal}(H/K), \) let \( \Lambda(\phi^\sigma) \subset \mathbb{C}_\infty \) be the \( A \)-module of periods of \( \phi^\sigma, \) and let \( \Lambda(\phi^\sigma)_{K_\infty} \) be the \( K_\infty \)-vector space generated by \( \Lambda(\phi^\sigma). \) Then \( \Lambda(\phi^\sigma)_{K_\infty}/\Lambda(\phi^\sigma) \) is compact, thus there exists a constant \( C \in \mathbb{R} \) such that
\[ \forall \sigma \in \text{Gal}(H/K), \forall x \in \Lambda(\phi^\sigma)_{K_\infty}, \ v_\infty(\exp_{\phi^\sigma}(x)) \geq C. \]

Recall that, if \( \sigma \in \text{Gal}(H/K), \) then there exists a non-zero ideal \( J \) of \( A \) such that \( \sigma = (J, H/K) = \sigma_J, \) and we have [21, Theorem 7.4.8]
\[ \phi_J \phi_a = \phi_a^{\sigma_J} \phi_J. \]

Thus,
\[ \exp_{\phi^\sigma} \psi(J) = \phi_J \exp_{\phi}. \]

In particular,
\[ \Lambda(\phi^\sigma) = \psi(J) J^{-1} \Lambda(\phi), \]
\[ \Lambda(\phi^\sigma)_{K_\infty} = \psi(J) \Lambda(\phi)_{K_\infty}. \]

Therefore, there exists a constant \( C' \in \mathbb{R}, \) such that, for all \( \sigma \in \text{Gal}(H/K), x_1, \ldots, x_n \in \Lambda(\phi^\sigma)_{K_\infty}, I \in \mathcal{I}(A), \) we have
\[ v_\infty \left( I \ast_E f^\sigma |_{x_i = \exp_{\phi^\sigma}(x_i)} \right) \geq C', \]
where \( f^\sigma_E \) is the map \( \ast \) attached to \( \phi^\sigma. \) Now, recall that \( \exp_{\phi} = \sum_{j \geq 0} e_j(\phi) \tau^j, \) then there exists a constant \( C'' > 0 \) such that (Lemma 3.7)
\[ \forall \sigma \in \text{Gal}(H/K), \forall \tau \geq 0, \ v_\infty(e_j(\phi^\sigma)) \geq C'' j \tau^j. \]

Note also that there exists \( C''' \in \mathbb{R} \) such that
\[ \forall \sigma \in \text{Gal}(H/K), \forall I \in \mathcal{I}(A), \deg I = m \gg 0, \ v_\infty \left( \frac{1}{\sigma(\psi(I))} \right) \geq \frac{m}{d_\infty} + C'''. \]

This implies that there exists an integer \( m_0 \in \mathbb{N}, \) such that
\[ \forall m \geq m_0, \forall \sigma \in \text{Gal}(E/K), \forall \lambda_1, \ldots, \lambda_n \in \exp_{\phi^\sigma} \Lambda(\phi^\sigma)_{K_\infty}, \ v_\infty(g_m^\sigma(\lambda_1, \ldots, \lambda_n)) > 0. \]
(3) Let $m_0 \in \mathbb{N}$ be as in (2). Let $\lambda_1, \ldots, \lambda_n$ be $n$ torsion points for $\phi$. Let $F = E(\lambda_1, \ldots, \lambda_n)$. Then $F/K$ is a finite abelian extension. Let $w$ be a place of $F$ above $\infty$. Let $i_w : E \to \mathbb{C}_\infty$ be the $K$-embedding of $F$ in $\mathbb{C}_\infty$ corresponding to $w$. Then there exists $\sigma \in \text{Gal}(F/K)$ such that
\[
\forall m \geq 0, \quad i_w(g_m(\lambda_1, \ldots, \lambda_n)) = \sigma(g_m(\lambda_1, \ldots, \lambda_n)) = g_m(\sigma(\lambda_1), \ldots, \sigma(\lambda_n)).
\]
Observe that $\sigma(\lambda_i) \in \exp_{\phi^n}(\Lambda(\phi^n)K_\infty), i = 1, \ldots, n$ [21, Proposition 7.5.16]. Therefore,
\[
\forall m \geq m_0, \quad w(g_m(\lambda_1, \ldots, \lambda_n)) > 0.
\]
Thus, we get that for any place $w$ of $F$ above $\infty$:
\[
\forall m \geq m_0, \quad w(g_m(\lambda_1, \ldots, \lambda_n)) > 0.
\]
But by (1), $\forall m \geq 0, g_m(\lambda_1, \ldots, \lambda_n) \in O_F$. Since $O_F$ is the set of elements of $F$ which are regular outside the places of $F$ above $\infty$, we deduce that
\[
\forall m \geq m_0, \quad g_m(\lambda_1, \ldots, \lambda_n) = 0.
\]
And the above property is true for any $n$ torsion points of $\phi$, thus
\[
\forall m \geq m_0, \quad g_m(X_1, \ldots, X_n) = 0. \quad \square
\]

N. Green and M. Papanikolas obtained explicit formulas for Anderson’s Log-Algebraicity Theorem [1, Theorem 5.1.1] when the genus $g$ of $K$ is one and $d_\infty = 1$ (see [22]).

4.2. Several variable L-series and shtukas

In this section, we present an alternative approach to the several variable Log-Algebraicity Theorem (Theorem 4.2) by using the seminal works of Drinfeld [14–16] (see also [1], [32], and [21], Chapter 6). In the rest of this paper, we will write $\otimes$ instead of $\otimes_{\mathbb{F}_q}$. For example, $X \otimes \mathbb{C}_\infty$ will stand for $X \otimes_{\mathbb{F}_q} \mathbb{C}_\infty$.

We recall some notation for the convenience of the reader. Let $X/\mathbb{F}_q$ be a smooth projective geometrically irreducible curve of genus $g$ whose function field is $K$. We will consider $\infty$ as a closed point of $X$ of degree $d_\infty$. Recall that $K_\infty$ is the completion of $K$ at $\infty$, $\bar{K}_\infty$ is a fixed algebraic closure of $K_\infty$, and $\mathbb{C}_\infty$ is the completion of $\bar{K}_\infty$. Let $\text{sgn} : K_\infty^* \to \mathbb{C}_\infty^*$ be a sign function ($\mathbb{C}_\infty$ is the residue field of $K_\infty$ and $d_\infty = [\mathbb{C}_\infty : \mathbb{F}_q]$), that is, sgn is a group homomorphism such that $\text{sgn} |_{\mathbb{C}_\infty^*} = \text{Id} |_{\mathbb{C}_\infty^*}$. We fix $\pi \in K \cap \text{Ker}(\text{sgn})$ and such that $K_\infty = \mathbb{C}_\infty((\pi))$.

We set $\bar{X} := X \otimes \mathbb{C}_\infty$, and $\bar{A} := A \otimes \mathbb{C}_\infty$. Then $F := \text{Frac}(\bar{A})$ is the function field of $\bar{X}$. We identify $\mathbb{C}_\infty$ with its image $1 \otimes \mathbb{C}_\infty$ in $F$. Note that $\bar{A}$ is the set of elements of $F/\mathbb{C}_\infty$ which are ‘regular outside $\infty$’. We denote by $\tau : F \to F$ the $K$-algebra homomorphism such that
\[
\tau |_{\bar{A}} = \text{Id}_{\bar{A}} \otimes \text{Frob}_{\mathbb{C}_\infty},
\]
where $\forall x \in \mathbb{C}_\infty, \text{Frob}_{\mathbb{C}_\infty}(x) = x^q$. For $m \geq 0$, we also set
\[
\forall x \in F, \quad x^{(m)} = \tau^m(x).
\]
Let $P$ be a point of $\bar{X}(\mathbb{C}_\infty)$. We denote by $P^{(i)}$ the point of $\bar{X}(\mathbb{C}_\infty)$ obtained by applying $\tau^i$ to the coordinates of $P$. If $D \in \text{Div}(\bar{X})$, $D = \sum_{j=1}^n n_{P_j}(P_j)$, $P_j \in \bar{X}(\mathbb{C}_\infty)$, $n_{P_j} \in \mathbb{Z}$, we set
\[
D^{(i)} = \sum_{j=1}^n n_{P_j}(P_j^{(i)}).
\]
If $D = (x)$, $x \in F^\times$, then
\[
D^{(i)} = \left(x^{(i)}\right).
\]
Let $\xi$ be the point of $\tilde{X}(\mathbb{C}_\infty)$ corresponding to the kernel of the map $\tilde{A} \to \mathbb{C}_\infty$, $\sum x_i \otimes a_i \mapsto \sum x_i a_i$. Let $\chi : K \to K \otimes 1$, $x \mapsto x \otimes 1$. Then

$$F = \mathbb{C}_\infty(\chi(K)).$$

Let $\phi$ be a sign-normalized Drinfeld $A$-module of rank one. Drinfeld’s correspondence asserts that there is a bijection between the set of such Drinfeld modules and certain functions in $F$ called shtukas ([32], paragraph 0.3.5, see also [21], Section 7.11), in particular, there exists a function $f \in F^\times$ attached to $\phi$, and let’s first recall some basic properties of $f$. We have

$$V^{(1)} - V + (\xi) - (\infty) = (f),$$

for some point $\infty$ of $X(\mathbb{C}_\infty)$ above $\infty$, and some effective divisor $V$ of $\tilde{X}/\mathbb{C}_\infty$ of degree $g$. The points $\xi$ and $\infty^{(-1)}$ do not belong to the support of $V$ [32, Corollary 0.3.3]. We identify the completion of $F$ at $\infty$ with $\mathbb{C}_\infty(1)$, where $t = \chi(\pi^{(-1)})$. We have a natural sign function $\text{sgn} : \mathbb{C}_\infty(1) \to \mathbb{C}_\infty^{\times}$ attached to $1_t$. When $d_\infty = 1$, we normalize $f$ such that $\text{sgn}(f) = 1$ (such a normalization does not hold when $d_\infty > 1$, see [6]). We set

$$\chi(a) = a \otimes 1 = \sum_{i=0}^{\deg a} \phi_{a,i} f \cdots f^{(i-1)},$$

where $\phi_a = \sum \phi_{a,i} \tau^i$, $\phi_{a,i} \in \mathbb{C}_\infty$, and we recall that

$$\exists n \in \mathbb{N}, \forall a \in A, \quad \phi_{a,\deg a} = sgn(a)q^n,$$

$$\forall a \in A, \quad \phi_{a,0} = a.$$

Let’s write

$$\exp_{\phi} = \sum e_i(\phi)\tau^i, \quad e_i(\phi) \in \mathbb{C}_\infty.$$
We have [32, Proposition 0.3.6]
\[ \forall i \geq 0, \quad e_i(\phi) = \frac{1}{f \cdots f^{(i-1)}} \bigg|_{\xi(i)}. \]

Let \( \mathbb{H} = \text{Frac}(A \otimes B) \subset F \). By Drinfeld’s correspondence (see [21, Chapter 6]), \( f \in \mathbb{H} \). Thus,
\[ f = \sum_{i \geq k} f_i \frac{1}{t_i} \in H \left( \left( \frac{1}{t} \right) \right) \subset C_\infty \left( \left( \frac{1}{t} \right) \right), \]
where \( f_i \in H, \forall i \geq k, f_k \neq 0, k \leq -1 \). We refer the interested reader to [6] for a detailed study of the arithmetic of such a sthuka function \( f \).

We view \( H \) as a function field over \( \chi(K) = K \otimes 1 \). Let \( K = \text{Frac}(A \otimes A) \). Let \( \infty \) be the unique place of \( K/\chi(K) \) which is above the place \( \infty \) of \( K/F_q \). Then the completion of \( K \) above \( \infty \) is
\[ K_\infty = \chi(K)(F_\infty)((1 \otimes \pi)). \]
Observe that the set of elements of \( K/\chi(K) \) which are regular outside \( \infty \) is
\[ A := A[\chi(K)] = K \otimes A. \]

We set \( B := B[\chi(K)] = K \otimes B \), then \( B \) is the integral closure of \( A \) in \( \mathbb{H} \). Let \( G = \text{Gal}(H/K) \simeq \text{Gal}(\mathbb{H}/\mathbb{K}) \). Let \( \varphi : A \to \mathbb{H}\{\tau\} \) be the \( \chi(K) \)-algebra homomorphism such that
\[ \forall a \in A, \quad \varphi_a = \sum_{i=0}^{\deg a} \phi_{a,i} f \cdots f^{(i-1)} \tau^i \in \mathbb{H}\{\tau\}. \]
Let \( \exp_\varphi \in \mathbb{H}\{\tau\} \) be the following element:
\[ \exp_\varphi = \sum_{i \geq 0} f \cdots f^{(i-1)} e_i(\phi) \tau^i = \sum_{i \geq 0} \frac{f \cdots f^{(i-1)}}{f \cdots f^{(i-1)} |_{\xi(i)}} \tau^i. \]
Then
\[ \forall a \in A, \quad \exp_\varphi a = \varphi_a \exp_\varphi. \]

We set \( \mathbb{H}_\infty = \mathbb{H} \otimes_{\mathbb{K}} K_\infty \). We still denote by \( \tau : \mathbb{H}_\infty \to \mathbb{H}_\infty \) the continuous morphism of \( \chi(K) \)-vector spaces that extends \( \tau : \mathbb{H} \to \mathbb{H} \). Let’s observe that \( \exp_\varphi \) converges on \( \mathbb{H}_\infty \).

Let \( \mathfrak{p} \) be a maximal ideal of \( B \). Then \( \mathfrak{p} \mathbb{B} \) is a maximal ideal of \( \mathbb{B} \). Let \( v_\infty : \mathbb{H} \to \mathbb{Z} \cup \{+\infty\} \) be the valuation on \( \mathbb{H} \) attached to \( \mathfrak{p} \mathbb{B} \). Since for all \( a \in A, \chi(a) = \sum_{j=0}^{\deg a} \phi_{a,j} f \cdots f^{(j-1)} \), we deduce that
\[ \forall i \geq 0, \quad v_\infty \left( f^{(i)} \right) = q^i v_\infty(f) = 0. \]
However, we warn the reader that, if \( g > 0 \), we have
\[ f \notin \mathfrak{p}. \]

We set
\[ W(B) = \oplus_{i \geq 0} B f \cdots f^{(i-1)}. \]

**Lemma 4.4.** (1) \( W(B) \) is a \( A \otimes B \)-module containing \( A \otimes B \), furthermore \( W(B) \) is a \( A \otimes A \)-module via \( \varphi \).
(2) Let \( W(B)\mathcal{B} \) be the \( \mathcal{B} \)-module generated by \( W(B) \). Let \( \mathfrak{P} \) be a maximal ideal of \( B \). The inclusion \( \mathcal{B} \subset W(B)\mathcal{B} \) induces an equality:

\[
\frac{\mathcal{B}}{\mathfrak{P}\mathcal{B}} = \frac{W(B)\mathcal{B}}{\mathfrak{P}W(B)\mathcal{B}}.
\]

(3) \( W(B)\mathcal{B} \) is a fractional ideal of \( \mathcal{B} \). In particular, it is discrete in \( \mathbb{H}_\infty \).

**Proof.** We have

\[
\forall i \geq 0, \forall a \in A, \quad \chi(a)f \cdots f(i-1) = \sum_{j=0}^{\deg a} \phi_{a,j} f \cdots f(i+j-1) \in W(B).
\]

Observe that \( \forall i, j \geq 0, f \cdots f(j-1) \tau^{j} (f \cdots f(i-1)) = f \cdots f(i+j-1) \).

The assertion 1) follows.

We set \( O_{\mathcal{P}} = \{ x \in \mathbb{H}, v_\mathcal{P}(x) \geq 0 \} \). Since \( \frac{O_{\mathcal{P}}}{\mathcal{P}} \simeq \frac{\mathcal{B}}{\mathfrak{P}} \) and \( \mathcal{B} \subset W(B)\mathcal{B} \subset O_{\mathcal{P}} \), the assertion 2) holds.

Let’s prove the assertion 3). Note that \( A \otimes H \) is the set of elements of \( \mathbb{H} \) which are regular outside \( \bar{\infty} \). By the expression (4.1) of the divisor of \( f \cdots f(i-1), i \geq 0 \), there exists \( a \in A \otimes B \setminus \{0\} \) such that \( \forall i \geq 0, \quad af \cdots f(i-1) \in A \otimes H \).

Since for every maximal ideal \( \mathfrak{P} \) of \( B \), and for all \( i \geq 0, v_\mathfrak{P}(f \cdots f(i-1)) = 0 \), we deduce that \( \forall i \geq 0, \quad af \cdots f(i-1) \in A \otimes B \).

Thus, there exists \( a \in \mathcal{B} \setminus \{0\} \) such that \( aW(B) \subset \mathcal{B} \). Since \( \mathcal{B} \) is discrete in \( \mathbb{H}_\infty \), we get the desired result. \( \square \)

Let’s observe that, by Lemma 4.4, \( W(B)\mathcal{B} \) is an \( A \)-module via \( \varphi \). Let \( \mathfrak{P} \) be a maximal ideal of \( B \), then, again by Lemma 4.4, \( \frac{B}{\mathfrak{P}B} \) is an \( A \)-module via \( \varphi \), and we denote this latter \( A \)-module by \( \varphi(\frac{B}{\mathfrak{P}B}) \).

**Lemma 4.5.** Let \( \mathfrak{P} \) be a maximal ideal of \( B \). Then

\[
\text{Fitt}_{A\varphi} \left( \frac{\mathcal{B}}{\mathfrak{P}\mathcal{B}} \right) = \left( \left[ \frac{B}{\mathfrak{P}B} \right]_{A} - \chi \left( \left[ \frac{B}{\mathfrak{P}B} \right]_{A} \right) \right)_{A}.
\]

**Proof.** Recall that

\[
\left[ \frac{B}{\mathfrak{P}B} \right]_{A} = \psi(P^e),
\]

where \( e = \dim_{A/P} B/\mathfrak{P} \). Set \( aA = P^e \) where \( \text{sgn}(a) = 1 \). Then

\[
\chi(a) = \sum \phi_{a,i} f \cdots f(i-1).
\]

Therefore,

\[
\forall x \in \frac{\mathcal{B}}{\mathfrak{P}\mathcal{B}}, \quad \varphi_{a-\chi(a)}(x) = 0.
\]

Thus, by similar arguments to those of [7, Lemma 5.8], we have an \( A \)-module isomorphism:

\[
\varphi \left( \frac{\mathcal{B}}{\mathfrak{P}\mathcal{B}} \right) \simeq \frac{A}{(a-\chi(a))A}.
\]

\( \square \)
Let $\mathcal{M}$ be an $\mathcal{A}$-module such that $\mathcal{M}$ is a finite dimensional $\chi(K)$-vector space and its Fitting ideal is principal, $\text{Fitt}_\mathcal{A}(\mathcal{M}) = x\mathcal{A}$, then we set

$$[\mathcal{M}]_\mathcal{A} = \frac{x}{\text{sgn}(x)},$$

where $\text{sgn} : \mathcal{K}_\infty^\times \to \chi(K)(\mathcal{F}_\infty)^\times$ is defined by

$$\text{sgn} \left( \sum_{i \geq 0} \beta_i(1 \otimes \pi)^i \right) = \beta_{i_0} \quad \text{with} \quad \beta_i \in \chi(K)(\mathcal{F}_\infty) \quad \text{and} \quad \beta_{i_0} \neq 0.$$

By the above Lemma, we can form the $L$-series attached to $\varphi/\mathcal{W}(\mathcal{B})$:

$$L(\varphi/\mathcal{W}(\mathcal{B})) = \prod_{K} \frac{[\frac{\varphi}{\mathcal{W}}]}{[\varphi(\frac{\mathcal{B}}{\mathcal{W}})_\mathcal{A}]} = \prod_{K} \left( 1 - \frac{\chi(\frac{\varphi}{\mathcal{W}}, [\mathcal{B}])}{\chi([\varphi(\frac{\mathcal{B}}{\mathcal{W}})])} \right)^{-1} \in \mathcal{K}_\infty^\times.$$

Note that $L(\varphi/\mathcal{W}(\mathcal{B}))$ is in fact an element in the $\infty$-adic completion of $\mathcal{K}_\infty[\chi(A)] = A \otimes \mathcal{K}_\infty$ which is an affinoid algebra over $\mathcal{K}_\infty$, and $L(\varphi/\mathcal{W}(\mathcal{B}))$ is a special value of a twisted zeta function (see [4, Section 5.2]).

Let $z$ be an indeterminate. The map $\tau : \mathbb{H}_\infty \to \mathbb{H}_\infty$ extends uniquely into a continuous homomorphism (for the $z$-adic topology) of $\mathcal{F}_\mathcal{g}[[z]]$-algebras $\tau : \mathbb{H}_\infty[[z]] \to \mathbb{H}_\infty[[z]]$. Let $T_z(\mathbb{H}_\infty) \subset \mathbb{H}_\infty[[z]]$ be the $\infty$-adic completion of $\mathbb{H}_\infty[[z]]$, that is, an element $g \in T_z(\mathbb{H}_\infty)$ can be uniquely written $g = \sum_{i \geq 0} g_i z^i, g_i \in \mathbb{H}_\infty$, such that $\lim_{i \to +\infty} g_i = 0$. We also denote by $T_z(\mathcal{K}_\infty)$ the $\infty$-adic completion of $\mathcal{K}_\infty[[z]]$. Note that $T_z(\mathbb{H}_\infty)$ is a free $T_z(\mathcal{K}_\infty)$-module of rank $[H : K]$, and if $(e_1, \ldots, e_n)$ is a $K$-basis of $H$ ($n = [H : K]$), then

$$T_z(\mathbb{H}_\infty) = \oplus_{i=1}^n e_i T_z(\mathcal{K}_\infty).$$

Observe also that $G$ acts on $T_z(\mathbb{H}_\infty)$ and $T_z(\mathbb{H}_\infty)$ is a free $T_z(\mathcal{K}_\infty)[G]$-module of rank one by the normal basis Theorem. We denote by $T_z(\mathbb{H}_\infty)[G]$ the ring

$$T_z(\mathbb{H}_\infty)[G] := \oplus_{\sigma \in G} T_z(\mathbb{H}_\infty)\sigma,$$

where the product rule is given by

$$\forall \sigma_1, \sigma_2 \in G, \forall g_1, g_2 \in T_z(\mathbb{H}_\infty), \quad (g_1 \sigma_1)(g_2 \sigma_2) = g_1 \sigma_1(g_2) \sigma_1 \sigma_2.$$

Let’s set

$$\exp_\varphi = \sum_{i \geq 0} \frac{f \cdots f^{(i-1)}}{f \cdots f^{(i-1)}} \xi^i \tau^i \in \mathbb{H}[z] \{\{\tau\}\}.$$

Let $I$ be a non-zero ideal of $\mathcal{A}$. We set

$$u_I = \sum_{i=0}^{\deg I} \phi_{I,i} f \cdots f^{(i-1)} \in \mathcal{W}(\mathcal{B}),$$

where $\phi_I = \sum_{i=0}^{\deg I} \phi_{I,i} \tau^i, \phi_{I,i} \in \mathcal{B}$. Note that if $I = aA$, we have

$$u_I = \frac{\chi(a)}{\text{sgn}(a)^n}.$$
Furthermore, we prove (see [1], Section 3.7 for the case $d_\infty = 1$):

**Lemma 4.6.** Let $I, J$ be two non-zero ideals of $A$. We have
\[
 u_I \mid \xi = \psi(I), \\
 \sigma_I(f)u_I = f\tau(u_I), \\
 u_{I,J} = \sigma_I(u_J)u_I.
\]

**Proof.** We only give a sketch of the proof, the interested reader is referred to [6] for a detailed proof. The fact that $u_I \mid \xi = \psi(I)$ comes from the definition of $u_I$. Note that we have a natural isomorphism of $B$-modules
\[
 \gamma_\phi : W(B) \simeq B\{\tau\}, \quad f \cdots f^{(i-1)} \mapsto \tau^i.
\]
In particular,
\[
 \forall x \in W(B), \quad \gamma_\phi(fx) = \tau^{\gamma_\phi(x)}, \\
 \forall x \in W(B), \forall a \in A, \quad \gamma_\phi(\chi(a)x) = \gamma_\phi(x)\phi_a.
\]

By explicit reciprocity law (see [21, Theorem 7.4.8]), we have
\[
 \forall a \in A, \quad \phi_I\phi_a = \sigma_I(\phi)_a\phi_I.
\]

By direct calculations, we deduce from this
\[
 \sigma_I(f)u_I = f\tau(u_I),
\]
Now, let $J$ be a non-zero ideal of $A$. We have
\[
 \gamma_\phi(u_{I,J}) = \phi_{I,J} = \sigma_I(\phi)_J\phi_I.
\]
But, since $\forall i \geq 0, \sigma_I(f \cdots f^{(i-1)})u_I = f \cdots f^{(i-1)}u_I^{(i)}$, we have
\[
 \gamma_\phi(\sigma_I(u_J)u_I) = \sigma_I(\phi)_J\phi_I.
\]
Thus,
\[
 u_{I,J} = \sigma_I(u_J)u_I.
\]

We deduce that if $P, Q$ are maximal ideals of $A$:
\[
 \left(1 - \frac{u_P}{\psi(P)}z^{\deg P}\sigma_P\right)\left(1 - \frac{u_Q}{\psi(Q)}z^{\deg Q}\sigma_Q\right) = \left(1 - \frac{u_Q}{\psi(Q)}z^{\deg Q}\sigma_Q\right)\left(1 - \frac{u_P}{\psi(P)}z^{\deg P}\sigma_P\right).
\]

For every integer $n \geq 1$, we set
\[
 \left(1 - \frac{u_P}{\psi(P)}z^{\deg P}\sigma_P\right)^{-1} := \sum_{k \geq 0} \frac{u_P}{\psi(P)^k}z^{k\deg P}\sigma_{P^k} \in \mathbb{T}_z(\mathbb{H}_\infty)[G].
\]

We define
\[
 \forall n \geq 1, \quad \mathcal{L}(\varphi; n; z) = \prod_P \left(1 - \frac{u_P}{\psi(P)^n}z^{\deg P}\sigma_P\right)^{-1} \in \left(\mathbb{T}_z(\mathbb{H}_\infty)[G]\right)^\times,
\]

where $P$ runs through the maximal ideals of $A$. Note that, for any $n \geq 1$, $\mathcal{L}(\varphi; n; z)$ induces a $\mathbb{T}_z(\mathbb{K}_\infty)$-linear endomorphism of $\mathbb{T}_z(\mathbb{H}_\infty)$, and we denote by $\det_{\mathbb{T}_z(\mathbb{K}_\infty)} \mathcal{L}(\varphi; n; z)$ its determinant. Let’s set
\[
 W(B[z]) = \oplus_{i \geq 0} B[z]f \cdots f^{(i-1)} \subset \mathbb{H}[z].
\]


**Proposition 4.7.** We have

\[ \forall n \geq 1, \quad \det_{\mathbb{T}_z(K_{\infty})} \mathcal{L}(\varphi; n; z) = \prod_{\mathfrak{P}} \left( 1 - \frac{\chi(\left[ \frac{B}{\mathfrak{P}} \right]_A) z^{\deg N_{H/K}(\mathfrak{P})}}{\left[ \frac{B}{\mathfrak{P}} \right]_A} \right)^{-1} \in \mathbb{T}_z(K_{\infty})^\times, \]

where \( \mathfrak{P} \) runs through the maximal ideals of \( B \).

**Proof.** The proof is similar to that of Theorem 3.6. We give a sketch of the proof for the convenience of the reader.

Let \( n \geq 1 \). We have

\[ \det_{\mathbb{T}_z(K_{\infty})} \mathcal{L}(\varphi; n; z) = \prod_{\mathfrak{P}} \det_{\mathbb{K}[z]} \left( 1 - \frac{u_\mathfrak{P}}{\psi(\mathfrak{P})^n z^{\deg P} \sigma_\mathfrak{P}} |_{\mathbb{H}(z)} \right)^{-1}. \]

Let \( P \) be a maximal ideal of \( A \). Let \( e \geq 1 \) be the order of \( P \) in \( \text{Pic}(A) \). Then \( 1, \sigma_\mathfrak{P}, \ldots, \sigma_\mathfrak{P}^{e-1} \) are linearly independent over \( \mathbb{H}(z) \). We have

\[ \left( \frac{u_\mathfrak{P}}{\psi(\mathfrak{P})^n z^{\deg P} \sigma_\mathfrak{P}} \right)^e = \frac{\chi(\psi(\mathfrak{P}^e)^\sigma_\mathfrak{P}) z^{e \deg P}}{\psi(\mathfrak{P}^e)^n} \in \mathbb{K}[z]. \]

Thus the minimal polynomial of \( \frac{u_\mathfrak{P}}{\psi(\mathfrak{P})^n z^{\deg P} \sigma_\mathfrak{P}} |_{\mathbb{H}(z)} \) over \( K(z) \) (and also over \( \mathbb{H}(\sigma_\mathfrak{P}^e)(z) \)) is equal to

\[ X^e - \frac{\chi(\psi(\mathfrak{P}^e)^\sigma_\mathfrak{P}) z^{e \deg P}}{\psi(\mathfrak{P}^e)^n} \in \mathbb{K}[z][X]. \]

Therefore the characteristic polynomial of \( \frac{u_\mathfrak{P}}{\psi(\mathfrak{P})^n z^{\deg P} \sigma_\mathfrak{P}} |_{\mathbb{H}(z)} \) over \( K(z) \) is equal to

\[ \left( X^e - \frac{\chi(\psi(\mathfrak{P}^e)^\sigma_\mathfrak{P}) z^{e \deg P}}{\psi(\mathfrak{P}^e)^n} \right)^{[H:K]/e}. \]

One obtains the desired result by the same arguments as that used in the proof of Theorem 3.6. \( \square \)

**Remark 4.8.** Let \( L = \chi(K)(\mathbb{F}_\infty)((t^{d_\infty-1} - \sqrt{-1})) \), and let \( \tau : L \to L \) be the continuous morphism of \( \chi(K) \)-algebras such that \( \forall x \in \mathbb{F}_\infty((t^{d_\infty-1} - \sqrt{-1})) \), \( \tau(x) = x^g \). Then there exists an element \( \omega \in L^\times \) (unique up to the multiplication of an element in \( \chi(K)^\times \)) such that

\[ \tau(\omega) = f \omega. \]

This element is a generalization of the special function introduced by Anderson and Thakur in [3]. The existence of this element (combined with the log-algebraicity theorem) gives new arithmetic informations on special values of \( L \)-series. We refer the interested reader to [22] for the case \( g = d_\infty = 1 \), and to [6] for the general case.

### 4.3. Stark units and several variable log-algebraicity theorem

We set

\[ U(\varphi/W(B[z])) = \left\{ x \in \mathbb{T}_z(\mathbb{H}_\infty), \exp_\varphi(x) \in W(B[z]) \right\}. \]

The following result is a twisted (by the shtuka function \( f \)) version of [1, Theorem 5.1.1].

**Theorem 4.9.** We have

\[ U(\varphi/W(B[z])) = \mathcal{L}(\varphi; 1; z)W(B[z]). \]
In particular,
\[ \exp_{\bar{\varphi}}(\mathcal{L}(\varphi; 1; z)W(B[z])) \subset W(B[z]), \]

**Proof.** The proof is similar to that of Theorem 3.8. We give a sketch of the proof for the convenience of the reader.

Observe that \( \exp_{\bar{\varphi}} : \mathbb{H}[[z]] \to \mathbb{H}[[z]] \) is an isomorphism of \( A[[z]] \)-modules. Furthermore, if we set
\[ W(H[[z]]) = \oplus_{i \geq 0} H[[z]] f \cdots f(i-1), \]
we get
\[ \exp_{\bar{\varphi}}(W(H[[z]])) = W(H[[z]]). \]

Let
\[ W(B[[z]]) = \oplus_{i \geq 0} B[[z]] f \cdots f(i-1) \subset \mathbb{H}[[z]]. \]
Let \( P \) be a maximal ideal of \( A \). Let \( WP = S^{-1}W(B[[z]]) \), where \( S = A \setminus P \). Then
\[ PW_P = \psi(P)WP. \]
By Lemma 3.7, we have
\[ \exp_{\bar{\varphi}}(PW_P) = PW_P. \]
If
\[ \bar{\phi}_P = \sum_{i=0}^{\deg P} \phi_{P,i} \tau^i, \]
we set
\[ \tilde{\bar{\varphi}}_P = \sum_{i=0}^{\deg P} \bar{\phi}_{P,i} f \cdots f(i-1) z^i \tau^i. \]
We have
\[ \tilde{\bar{\varphi}}_P \exp_{\bar{\varphi}} = \exp_{\sigma_P} \psi(P), \]
where
\[ \exp_{\sigma_P} = \sum_{i \geq 0} \sigma_P(e_i(\bar{\phi})) f \cdots f(i-1) z^i \tau^i. \]
Let’s set
\[ U(\tilde{\bar{\varphi}}/WP) = \left\{ x \in \mathbb{H}[[z]], \exp_{\tilde{\bar{\varphi}}}(x) \in WP \right\} \subset W(H[[z]]). \]
We have an isomorphism of \( A[[z]] \)-modules induced by \( \exp_{\bar{\varphi}} \):
\[ \frac{U(\tilde{\bar{\varphi}}/WP)}{PW_P} \simeq \tilde{\bar{\varphi}} \left( \frac{WP}{PW_P} \right). \]
Note that
\[ \forall i \geq 0, \quad \sigma_P(f \cdots f(i-1)) u_P = f \cdots f(i-1) \tau^i(u_P) \in W(B). \]
Therefore,
\[ \left( \tilde{\bar{\varphi}}_P - z^{\deg P} u_P \sigma_P \right) \tilde{\bar{\varphi}} \left( \frac{WP}{PW_P} \right) = \{0\}. \]
Since \( u_P \) is a ‘\( P \)-unit’, for \( x \in W(H[[z]]) \setminus W_P \), \((\tilde{\varphi}_P - z^{\deg P} u_P \sigma_P)(x) = 0 \) is not \( P \)-integral as an element of \( H[[z]] \). Thus,
\[
\tilde{\varphi} \left( \frac{W_P}{PW_P[[z]]} \right) = \left\{ x \in \tilde{\varphi} \left( \frac{W(H[[z]])}{PW_P} \right), (\tilde{\varphi}_P - z^{\deg P} u_P \sigma_P)(x) = 0 \right\}.
\]
Let \( x \in W(H[[z]]) \), we deduce that
\[
x \in U(\tilde{\varphi}/W_P) \iff (\tilde{\varphi}_P - z^{\deg P} u_P \sigma_P) \left( \exp_{\tilde{\varphi}}(x) \right) \in PW_P.
\]
Thus,
\[
x \in U(\tilde{\varphi}/W_P) \iff \exp_{\sigma_P} \left( \psi(P)x - z^{\deg P} u_P \sigma_P(x) \right) \in PW_P.
\]
Lemma 3.7 implies
\[
x \in U(\tilde{\varphi}/W_P) \iff \psi(P)x - z^{\deg P} u_P \sigma_P(x) \in PW_P.
\]
Thus,
\[
U(\tilde{\varphi}/W_P) = \left( 1 - \frac{z^{\deg P} u_P \sigma_P}{\psi(P)} \right)^{-1} W_P.
\]
Observe that \( W(B[[z]]) = \bigcap_P W_P \). We conclude that
\[
W(B[[z]]) = \exp_{\tilde{\varphi}}(\mathcal{L}(\varphi; 1; z)W(B[[z]])).
\]
By Lemma 4.4, we get
\[
\exp_{\tilde{\varphi}}(\mathcal{L}(\varphi; 1; z)W(B[z])) \subset T_z(\mathbb{H}_\infty) \cap W(B[[z]]) = W(B[z]).
\]
Recall that
\[
U(\tilde{\varphi}/W(B[z])) = \left\{ x \in T_z(\mathbb{H}_\infty), \exp_{\tilde{\varphi}}(x) \in W(B[z]) \right\}.
\]
Then
\[
U(\tilde{\varphi}/W(B[[z]])) = \mathcal{L}(\varphi; 1; z)W(B[[z]]) \cap T_z(\mathbb{H}_\infty).
\]
But recall that
\[
\mathcal{L}(\varphi; 1; z) \in (T_z(\mathbb{H}_\infty)[G])^\times.
\]
Thus,
\[
U(\tilde{\varphi}/W(B[z])) = \mathcal{L}(\varphi; 1; z)W(B[z]). \quad \square
\]
Let \( ev : T_z(\mathbb{H}_\infty) \to \mathbb{H}_\infty \) be the evaluation map at \( z = 1 \). Then by Proposition 4.7, we get
\[
L(\varphi/W(B)) = \det_{\mathbb{K}_\infty} \left( ev(\mathcal{L}(\varphi; 1; z)) \right),
\]
where
\[
\exp_{\tilde{\varphi}}(ev(\mathcal{L}(\varphi; 1; z))W(B)) \subset W(B).
\]
And also
\[
\exp_{\tilde{\varphi}}(ev(\mathcal{L}(\varphi; 1; z))W(B)\mathcal{B}) \subset W(B)\mathcal{B}.
\]
If we define the regulator of Stark units \( ev(\mathcal{L}(\varphi; 1; z))W(B)\mathcal{B} \) as follows:
\[
[W(B)\mathcal{B} : ev(\mathcal{L}(\varphi; 1; z))W(B)\mathcal{B}]_A := \det_{\mathbb{K}_\infty} \left( ev(\mathcal{L}(\varphi; 1; z)) \right),
\]
then
\[ L(\varphi/W(B)) = [W(B) \mathbb{B} : \text{ev}(\mathcal{L}(\varphi; 1, z))W(B) \mathbb{B}]_h. \]

We now briefly discuss the several variable version of Theorem 4.9. Let \( s \geq 0 \) be an integer. Let
\[ K_s = \text{Frac} \left( A^{\otimes s} \right), \]
where
\[ A^{\otimes s} = A \otimes \mathbb{F}_q \cdots \otimes \mathbb{F}_q A. \]

If \( s = 0 \), then \( K_0 = \mathbb{F}_q \). Let
\[ H_s = \text{Frac} \left( A^{\otimes s} \otimes \mathbb{F}_q B \right), \]
\[ K_s = \text{Frac} \left( A^{\otimes s} \otimes \mathbb{F}_q A \right). \]

For \( i = 1, \ldots, s \), let
\[ \chi_i : A \to H_s, \quad a \mapsto (1 \otimes \cdots \otimes 1 \otimes a \otimes \cdots \otimes 1) \otimes 1, \]
where \( a \) appears at the \( i \)th position. We still denote by \( \chi_i : H_s \to H_s \) the homomorphism of \( H \)-algebras such that
\[ \forall a \in A, \quad \chi_i(a) = \chi_i(a). \]

We view \( H_s \) and \( K_s \) as functions fields over \( K_s \otimes 1 \). Let \( \infty \) be the unique place of \( K_s/K_s \otimes 1 \) above the place \( \infty \) of \( K/F_q \). Then
\[ K_{s, \infty} = (K_s \otimes 1)(\mathbb{F}_\infty)((1 \otimes s \otimes \pi)), \]
and we set
\[ H_{s, \infty} = H_s \otimes_{K_s} K_{s, \infty}. \]

Let \( T_z(H_{s, \infty}) \) be the Tate algebra in the variable \( z \) with coefficients in \( H_{s, \infty} \). Let \( \tau : T_z(H_{s, \infty}) \to T_z(H_{s, \infty}) \) be the continuous homomorphism of \( (K_s \otimes 1)[z] \)-algebras such that
\[ \forall x \in H_\infty, \quad \tau(x) = x^q. \]

Let’s set
\[ W_s(B[z]) = \bigoplus_{i_1, \ldots, i_s \geq 0} B[z] \prod_{j=1}^s \chi_j(f) \cdots \tau^{(i_j-1)}(\chi_j(f)) \subset H_s[z]. \]

In particular \( W_0(B[z]) = B[z] \). By similar arguments as those of the proof of Lemma 4.4, we show that \( W_s(B[z]) \) is discrete in \( T_z(H_{s, \infty}) \). For \( n \geq 1 \), we set
\[ \mathcal{L}(\varphi_s; n; z) := \prod_P \left( 1 - \frac{\prod_{j=1}^s \chi_j(u_P)}{\psi(P)^n} z^{\deg P \sigma_P} \right)^{-1} \in (T_z(H_{s, \infty})[G])^\times, \]
where \( P \) runs through the maximal ideals of \( A \). Then, by the same proof as that of Proposition 4.7, for all \( n \geq 1 \), we get
\[ \det_{T_z(K_{s, \infty})} \mathcal{L}(\varphi_s; n; z) = \prod_{\mathfrak{p}} \left( 1 - \left( \prod_{j=1}^s \chi_j \left( \left[ \frac{B}{\mathfrak{p}B} \right]_A \right) \right) z^{\deg N_{H/K}(\mathfrak{p})} \right)^{-1} \in T_z(K_{s, \infty})^\times, \]
where \( \mathfrak{p} \) runs through the maximal ideals of \( B \).
We define
\[ \exp_{\tilde{\phi}} = \sum_{i \geq 0} e_i(\phi) \left( \prod_{j=1}^{s} \chi_j(f) \cdots \tau^{i-1}(\chi_j(f)) \right) z^i \tau^i \in \mathbb{F}_q \{ \{ \tau \} \}. \]

Then \( \exp_{\tilde{\phi}} \) converges on \( T_s(\mathbb{H}_{s,\infty}) \), and we set
\[ U(\tilde{\phi}/W_s(B[z])) = \{ x \in T_s(\mathbb{H}_{s,\infty}), \exp_{\tilde{\phi}}(x) \in W_s(B[z]) \}. \]

By similar arguments as those of the proof of Theorem 4.9, we get

**Corollary 4.10.** We have
\[ U(\tilde{\phi}/W_s(B[z])) = L(\phi_{s};1;z)W_s(B[z]). \]

**Example 4.11.** We consider the Carlitz example, where \( g = 0 \) and \( d_{\infty} = 1 \). Observe that there exists \( \theta \in K \) such that \( \text{sgn}(\theta) = 1 \), and \( A = \mathbb{F}_q[\theta] \). Thus, \( K = \mathbb{F}_q(\theta) \), and \( K_{\infty} = \mathbb{F}_q((1/\theta)) \).

Let \( \phi : A \to \overline{K}_{\infty}\{\tau\} \) be the Carlitz module defined by
\[ \phi_{\theta} = \theta + \tau. \]

Then the Carlitz exponential is given by
\[ \exp_{\phi} = \sum_{i \geq 0} \frac{1}{D_i} \tau^i, \]
where for \( i \geq 0 \), \( D_i = \prod_{k=0}^{i-1} (\theta^k - \theta^k) \).

The Hilbert class field \( H \) of \( K \) is \( K \), and then \( B = A \). Then, the shtuka function \( f \in K \otimes H \) associated to the Carlitz module via the Drinfeld correspondence is given by
\[ f = \theta \otimes 1 - 1 \otimes \theta. \]

Let \( s \geq 0 \) be an integer. For \( i = 1, \ldots, s \), let \( t_i = \chi_i(\theta) \). We have
\[ \chi_i(f) = t_i - \theta, \]
\[ H_s = K_s = \mathbb{F}_q(t_1, \ldots, t_s, \theta), \]
\[ H_{s,\infty} = K_{s,\infty} = \mathbb{F}_q(t_1, \ldots, t_s) \left( \left( \frac{1}{\theta} \right) \right). \]

For \( i \geq 0, j = 1, \cdots s \), set
\[ b_i(t_j) = \prod_{k=0}^{i-1} \left( t_j - \theta^k \right). \]

We get
\[ W_s(B[z]) = A[t_1, \ldots, t_s][z]. \]

Observe that
\[ \exp_{\tilde{\phi}} = \sum_{i \geq 0} \frac{\prod_{j=1}^{s} b_i(t_j)}{D_i} z^i \tau^i. \]

We have
\[ L(\phi_{s};1;z) = \sum_{a \in A_+} \frac{a(t_1) \cdots a(t_s)}{a} z^{\text{deg}_a a}, \]
where $A_+$ denotes the set of monic polynomials in $A = \mathbb{F}_q[\theta]$. In particular, for $s = 1$, we recover the zeta function introduced by Pellarin [26].

Corollary 4.10 implies

\[ \exp_{\varphi_z} (\mathcal{L}(\varphi_z; 1) A[t_1, \ldots, t_s, z]) \subset A[t_1, \ldots, t_s, z]. \]

We refer the interested reader to [5, 7, 8, 10], for arithmetic applications of this latter result.

4.4. Another proof of Anderson’s log-algebraicity theorem

**Corollary 4.12.** Let $n \geq 0$ and let $X_1, \ldots, X_n, z$ be $n+1$ indeterminates over $K$. Let $\tau : K[X_1, \ldots, X_n][[z]] \to K[X_1, \ldots, X_n][[z]]$ be the continuous $\mathbb{F}_q[[z]]$-algebra homomorphism for the $z$-adic topology such that $\forall x \in K[X_1, \ldots, X_n], \tau(x) = x^q$. Then

\[ \forall b \in B, \quad \exp_{\varphi_z} \left( \sum_I \delta I(b) \phi_I(X_1) \cdots \phi_I(X_n) z^{\deg I} \right) \in B[X_1, \ldots, X_n, z], \]

where $I$ runs through the non-zero ideals of $A$, and

\[ \exp_{\varphi_z} = \sum_{i \geq 0} e_i(\phi) z^i \tau^i. \]

**Proof.** We first treat the case $n = 0$. Let $b \in B$. By Theorem 4.9, we get

\[ \forall k \geq 0, \quad \sum_{\deg I + i = k} e_i(\phi) f \cdots f^{(i-1)} \frac{\tau^i(u I \sigma_I(b))}{\psi(I)^q} \in W(B), \]

and

\[ \forall k \gg 0, \quad \sum_{\deg I + i = k} e_i(\phi) f \cdots f^{(i-1)} \frac{\tau^i(u I \sigma_I(b))}{\psi(I)^q} = 0. \]

The coefficient of $f \cdots f^{(k-1)}$ in $\sum_{\deg I + i = k} e_i(\phi) f \cdots f^{(i-1)} \frac{\tau^i(u I \sigma_I(b))}{\psi(I)^q}$ is

\[ \sum_{\deg I + i = k} e_i(\phi) \frac{\sigma_I(b)^q}{\psi(I)^q} . \]

Therefore,

\[ \forall k \geq 0, \quad \sum_{\deg I + i = k} e_i(\phi) \frac{\sigma_I(b)^q}{\psi(I)^q} \in B . \]

\[ \forall k \gg 0, \quad \sum_{\deg I + i = k} e_i(\phi) \frac{\sigma_I(b)^q}{\psi(I)^q} = 0 . \]

Thus,

\[ \exp_{\varphi_z} \left( \sum_I \delta I(b) \phi_I(X_1) \cdots \phi_I(X_n) z^{\deg I} \right) \in B[z] . \]

We now assume that $n \geq 1$. We have an isomorphism of $B[z]$-modules

\[ \gamma : W(B[z]) \to \bigoplus_{i_1, \ldots, i_n \geq 0} B[z] X_1^{i_1} \cdots X_n^{i_n} \]
such that
\[ \forall i_1, \ldots, i_n \in \mathbb{N}, \quad \gamma \left( \prod_{j=1}^{n} \chi_j \left( f \cdots f^{(i_j-1)} \right) \right) = \prod_{j=1}^{n} X_j^{q_{ij}}. \]

Observe that
\[ \gamma \circ n \prod_{j=1}^{n} \chi_j \left( f \right) = \tau \circ \gamma. \]

Furthermore,
\[ \gamma \left( \left( \prod_{j=1}^{n} \chi_j \left( u_I \right) \right) \right) = \phi_I \left( X_1 \right) \cdots \phi_I \left( X_n \right). \]

Thus, we get by Corollary 4.10:
\[ \exp_{\tilde{\phi}} \left( \mathcal{L}(\varphi_n; 1; z)b \right) \in W_n(B[z]), \]

and thus
\[ \exp_{\tilde{\phi}} \left( \sum_{I} \sigma_I(b) \phi_I \left( X_1 \right) \cdots \phi_I \left( X_n \right) z^{\deg I} \right) \in \bigoplus_{i_1, \ldots, i_n \geq 0} B[z] X_1^{q_{i_1}} \cdots X_n^{q_{in}}. \]  

**Remark 4.13.** Let \( s \geq 1 \) be an integer and let \( B\{\tau_1, \ldots, \tau_s\} \) be the non-commutative polynomial ring in the variables \( \tau_1, \ldots, \tau_s \), such that
\[ \tau_i \tau_j = \tau_j \tau_i, \]
\[ \forall b \in B, \forall n \geq 0, \quad \tau_i^n b = b^n \tau_i. \]

For \( i = 1, \ldots, s \), we set
\[ \forall a \in A, \quad \varphi^a_{i,a} = \sigma_a \tau_i^j \in B\{\tau_1, \ldots, \tau_s\}, \]

and
\[ \forall a \in A, \quad \varphi^a = \sigma_a \tau_i^j \in B\{\tau_1, \ldots, \tau_s\}, \]

where \( \tau = \tau_1 \cdots \tau_s \).

Let \( W_s(B) = \bigoplus_{i_1, \ldots, i_s} B \prod_{j=1}^{s} \chi_j \left( f \right) \cdots \tau_i^{j-1} \left( \chi_j \left( f \right) \right) \subset \mathbb{R}_s \). Then \( W_s(B) \) is an \( A^{\otimes s} \otimes B \)-module. Let \( j \in \{1, \ldots, s\} \). Let \( a \in A \), we have a natural \( B \)-module homomorphism:
\[ \tilde{\chi}_j(a) : B\{\tau_1, \ldots, \tau_s\} \rightarrow B\{\tau_1, \ldots, \tau_s\}, \]

such that
\[ \forall i_1, \ldots, i_s \in \mathbb{N}, \quad \tilde{\chi}_j(a) \left( \tau_1^{i_1} \cdots \tau_s^{i_s} \right) = \left( \tau_j^{i_j} \varphi_j, a \right) \prod_{k=1}^{s} \tau_k^{i_k}. \]

Observe that
\[ \forall i, j \in \{1, \ldots, s\}, \forall a, b \in A, \quad \tilde{\chi}_j(a) \circ \tilde{\chi}_i(b) = \tilde{\chi}_i(b) \circ \tilde{\chi}_j(a). \]
Thus $B\{\tau_1, \ldots, \tau_s\}$ becomes an $A^{\otimes s} \otimes B$-module via

$$\forall x \in B\{\tau_1, \ldots, \tau_s\}, \quad \left( \sum_i b_i \prod_j \chi_j(a_{i,j}) \right) \cdot x = \sum_i b_i \left( \prod_j \tilde{\chi}_j(a_{i,j}) \right)(x).$$

Then, by the proof of Corollary 4.12, we have an $A^{\otimes s} \otimes B$-module isomorphism:

$$B\{\tau_1, \ldots, \tau_s\} \cong W_s(B).$$

In particular, $B\{\tau_1, \ldots, \tau_s\}$ is a finitely generated $A^{\otimes s} \otimes B$-module of rank one. The case $s = 1$ was already observed by G. Anderson ([21], p. 230, line 21 - there is a misprint in line 24, since in general $f \not\in A \otimes C_{\infty}$). If $I$ is a non-zero ideal of $A$, we define $I \ast : B\{\tau_1, \ldots, \tau_s\} \rightarrow B\{\tau_1, \ldots, \tau_s\}$ to be the $B$-module homomorphism such that

$$I \ast (\tau_1^{i_1} \cdots \tau_s^{i_s}) = \sum_{j_1, \ldots, j_s \in \{0, \ldots, \deg I\}} \phi_{I,j_1}^{i_1} \cdots \phi_{I,j_s}^{i_s} \tau_1^{i_1 + j_1} \cdots \tau_s^{i_s + j_s},$$

where $\phi_I = \sum_{k=0}^{\deg I} \phi_{I,k} \tau^k$.

Let $\mathcal{L} : B\{\tau_1, \ldots, \tau_s\} \rightarrow H\{\{\tau_1, \ldots, \tau_s\}\}$ be defined as follows:

$$\mathcal{L} \left( \sum_{i_1, \ldots, i_s} b_{i_1, \ldots, i_s} \tau_1^{i_1} \cdots \tau_s^{i_s} \right) = \sum_{i_1, \ldots, i_s} \sum_I \frac{\sigma_I(b_{i_1, \ldots, i_s})}{\psi(I)} I \ast (\tau_1^{i_1} \cdots \tau_s^{i_s}).$$

Then by Corollary 4.10, we get that the multiplication by $\exp_\phi \in H\{\{\tau\}\}$ on $H\{\{\tau_1, \ldots, \tau_s\}\}$ yields to the following property:

$$\forall x \in B\{\tau_1, \ldots, \tau_s\}, \quad \exp_\phi(\mathcal{L}(x)) \in B\{\tau_1, \ldots, \tau_s\}.$$

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