Fano’s inequality is false for a simple Cremona transformation of five-dimensional projective space

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ABSTRACT. A Cremona transformation of five-dimensional projective space is constructed. The degree of the transformation is 7. The inequalities of Fano are not fulfilled for this transformation.

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Introduction

It was shown in [3] and [4] that Fano’s inequalities are false for tree-dimensional projective space and for three-dimensional quadric. Both the birational transformations were of degree 13. Here we construct a simple five-dimensional Cremona transformation of degree 7 and not satisfying the inequalities of Fano .

Let $X$ be a non-singular $n$-dimensional variety such that Pic$(X) = \mathbb{Z}$, and the anticanonical class $(-K_X)$ is ample. If $(-K_X) = rH$ for a generator $H$ of the Picard group, then $X$ is said to be Fano variety of index $r$ ( and of the first kind ). According to the texts mentioned in survey [1], Fano’s inequality is the statement:

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For any birational transformation, 

\[ f: X \dashrightarrow X \]

defined by a linear system of degree \( d > 1 \) (the degree is the number defined by \( f(mH) = dmH \)) there exists an irreducible subvariety \( Y \subset X \) such that 

\[ 0 \leq \dim Y \leq \dim X - 2, \]

\[ \text{mult}_Y(f(|H|)) > \frac{(\dim X - \dim Y - 1) \cdot d}{r}. \]

Five-dimensional projective space \( \mathbb{P}^5 \) has index 6, therefore one can reformulate Fano's inequality:

For any Cremona transformation, 

\[ f: \mathbb{P}^5 \dashrightarrow \mathbb{P}^5 \]

defined by six homogeneous polynomials of the same degree \( d > 1 \) and without a common nonconstant factor,

\[ x'_i = f_i(x_0, x_1, x_2, x_3, x_4, x_5), \quad i = 0, ..., 5, \]

there exists an irreducible subvariety \( Y \subset \mathbb{P}^5 \) such that 

\[ 0 \leq \dim Y \leq 3, \]

\[ \text{mult}_Y(f_i) > \frac{(4 - \dim Y) \cdot d}{6}. \]

for every \( i = 0, ..., 5 \).

The goal of my article is to show that these inequalities do not take place for a certain Cremona transformation of degree 7. That is, I write down the formulas for a Cremona transformation of degree 7 such that for the forms \( f_0, \ldots, f_5 \) defining the transformation, for any surface \( S \subset \mathbb{P}^5 \), for any curve \( C \subset \mathbb{P}^5 \) and for any point \( P \in \mathbb{P}^5 \), one can see that

\[ \min_i(\text{mult}_S(f_i)) \leq 2, \quad \min_i(\text{mult}_C(f_i)) \leq 3, \quad \min_i(\text{mult}_P(f_i)) \leq 4. \]
Construction of the example

Let us consider six homogeneous coordinates for $\mathbb{P}^5$ as the normalized coefficients $x_{00}, x_{01}, x_{11}, x_{02}, x_{12}, x_{22}$ of a ternary quadratic form $F(T_0, T_1, T_2)$,

$$F(T_0, T_1, T_2) = x_{00}T_0^2 + 2x_{01}T_0T_1 + x_{11}T_1^2 + 2x_{02}T_0T_2 + 2x_{12}T_1T_2 + x_{22}T_2^2.$$ 

Let $D = D(x_{00}, x_{01}, x_{11}, x_{02}, x_{12}, x_{22})$ be the discriminant of the ternary quadric,

$$D = \begin{vmatrix} x_{00} & x_{01} & x_{02} \\ x_{01} & x_{11} & x_{12} \\ x_{02} & x_{12} & x_{22} \end{vmatrix}. $$

The set of double points of the cubic discriminant hypersurface consists of the points of the Veronese surface, these points correspond to the ternary quadrics which are perfect squares of linear forms.

We fix parameters $s, t$ and consider six following forms of degree 7.

$$(f_{s,t})_{00} = x_{00}D^2,$$
$$(f_{s,t})_{01} = x_{01}D^2,$$
$$(f_{s,t})_{11} = x_{11}D^2,$$
$$(f_{s,t})_{02} = x_{02}D^2 + x_{01}x_{11}^3D_s + x_{00}^3Dt,$$
$$(f_{s,t})_{12} = x_{12}D^2 + x_{01}x_{11}^3D_t + x_{11}^3Ds,$$
$$(f_{s,t})_{22} = x_{22}D^2 + 2x_{12}x_{11}^3Ds + 2x_{02}x_{00}^3Dt + x_{11}^7s^2 + 2x_{01}x_{11}^3x_{00}^3st + x_{00}^7t^2.$$ 

These six forms define a two-parameter family of rational maps

$$g_{s,t}: \mathbb{P}^5 \dashrightarrow \mathbb{P}^5.$$ 

If $s = t = 0$, all four forms have a common factor $D^2$. After cancelling this, we see that $g_{0,0}$ is the identity transformation. For our example, we need nonzero values of $s, t$. If one of the parameters $s, t$ is not zero, then it is clear that the six forms have no nonconstant common factor. Moreover,

$$D\left((f_{s,t})_{00}, (f_{s,t})_{01}, (f_{s,t})_{11}, (f_{s,t})_{02}, (f_{s,t})_{12}, (f_{s,t})_{22}\right) = D^7.$$ 

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In fact, this identity expresses the invariance of the discriminant under triangular transformation of variables $T_0, T_1, T_2$. Using the latter identity, it is not hard to see that
\[
(f_{-s,-t})_{ij}((f_{s,t})_{00}, (f_{s,t})_{01}, (f_{s,t})_{11}, (f_{s,t})_{02}, (f_{s,t})_{12}, (f_{s,t})_{22}) = x_{ij}D^{16},
\]
that is,
\[
g_{-s,-t} \circ g_{s,t} = \text{the identity transformation}.
\]
Thus $g_{s,t}$ is rationally invertible and is a Cremona transformation. More generally,
\[
g_{s,t} \circ g_{p,q} = g_{s+p,t+q},
\]
and we get a two-parameter group of Cremona transformations. These transformations induce biregular automorphisms of an affine open subset of the five-dimensional projective space, the complement of the discriminant cubic hypersurface $D = 0$. Indeed, the above formula of the discriminant transformation proves this (moreover, one can see below the exact calculation of the fundamental points of such a transformation).

**Remark.** The formulas for $g_{s,t}$ can be generalized for obtaining an infinite dimensional family of automorphisms of the complement to the discriminant hypersurface. The construction of general formulas resembles the trick used on page 8 of the Max-Planck Institute preprint [2]. If $U(x, y, z)$ is a ternary cubic form, $\phi_m(u, d), \psi_m(u, d)$ are binary forms of degree $m$, 
\[
\phi = \phi_m(U(x_{00}, x_{01}, x_{11}), D),
\]
\[
\psi = \psi_m(U(x_{00}, x_{01}, x_{11}), D),
\]
then following transformation
\[
x'_{00} = x_{00}D^{2m},
\]
\[
x'_{01} = x_{01}D^{2m},
\]
\[
x'_{11} = x_{11}D^{2m},
\]
\[
x'_{02} = x_{02}D^{2m} + x_{01}D^m\psi + x_{00}D^m\phi,
\]
\[
x'_{12} = x_{12}D^{2m} + x_{01}D^m\phi + x_{11}D^m\phi,
\]
\[
x'_{22} = x_{22}D^{2m} + 2x_{12}D^m\phi + 2x_{02}D^m\psi + x_{11}\phi^2 + 2x_{01}\phi\psi + x_{00}\psi^2.
\]
is a Cremona transformation inducing a biregular automorphism of the mentioned affine subset.
Let us return to our two-parameter family. We fix nonzero values of the parameters $s, t$, for example, $s = t = 1$, and consider the corresponding Cremona transformation

\[
\begin{align*}
x'_{00} &= x_{00}D^2, \\
x'_{01} &= x_{01}D^2, \\
x'_{11} &= x_{11}D^2, \\
x'_{02} &= x_{02}D^2 + (x_{01}x_{11}^3 + x_{10}^4)D, \\
x'_{12} &= x_{12}D^2 + (x_{01}x_{00}^3 + x_{11}^4)D, \\
x'_{22} &= x_{22}D^2 + 2(x_{12}x_{11}^3 + x_{02}x_{00}^3)D + (x_{11}^7 + 2x_{01}x_{11}x_{00}^3 + x_{00}^7).
\end{align*}
\]

First of all, we find the points $P \in \mathbb{P}^5$ where each form on the right hand side has positive multiplicity (that is, the set of all common zeros of these right hand sides, or, in other words, the fundamental points of the transformation).

The first right hand side vanishes if

- either $x_{00} = 0$,
- or $D = 0$,
- or simultaneously $x_{00} = D = 0$.

If $x_{00} = 0$ but $D \neq 0$ then using the other five formulas, one sees that for other five coordinates of a fundamental point, the equalities $x_{01} = x_{11} = x_{02} = x_{12} = x_{22} = 0$ take place. This case is not a point of $\mathbb{P}^5$.

The case $D = 0$ but $x_{00} \neq 0$ is realizable. The points satisfying

\[x_{00} \neq 0 \quad D = 0, \quad x_{11}^7 + 2x_{01}x_{11}^3x_{00}^3 + x_{00}^7 = 0\]

are fundamental, but they are of multiplicity 1 for one of three forms $x'_{02}, x'_{12}, x'_{22}$ at least. Indeed, if such a point is double on the discriminant hypersurface, then it is non-singular on the hypersurface of seventh degree. If both the expressions $(x_{01}x_{11}^3 + x_{00}^4), \ (x_{01}x_{00}^3 + x_{11}^4)$ in $x'_{02}, x'_{12}$ vanish , and the point is fundamental , then the multiplicity of $x'_{22}$ at the point is equal to 1.

If $x_{00} = D = 0$ and the point is fundamental, then also $x_{11} = 0$ and either $x_{01} = 0$ or $(2x_{02}x_{12} - x_{01}x_{22}) = 0$, or both the expressions vanish . If
the point is simple on the discriminant cubic and has multiplicity more than 2 on every $x_{ij}'$, then all the homogeneous coordinates vanish. If the point is double on $D = 0$, then $x_{01} = x_{02} = x_{12} = 0$, that is the only non-zero coordinate of the point is $x_{22}$. The multiplicity of $x_{22}'$ at the point is 4.

References

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