QUANTIZATION OF SOME MODULI SPACES OF PARABOLIC VECTOR BUNDLES ON $\mathbb{C}P^1$

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Abstract. We address quantization of the natural symplectic structure on a moduli space of parabolic vector bundles of parabolic degree zero over $\mathbb{C}P^1$ with four parabolic points and parabolic weights in $\{0, 1/2\}$. Identifying such parabolic bundles as vector bundles on an elliptic curve, we obtain explicit expressions for the corresponding non-abelian theta functions. These non-abelian theta functions are described in terms of certain naturally defined distributions on the compact group SU(2).

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1. Introduction

Let $X$ be a compact connected Riemann surface, or equivalently a smooth complex projective curve. It is well known that the moduli spaces of vector bundles over $X$ have a canonical symplectic structure [Go], with integral symplectic form. Indeed, being naturally identified with spaces of flat connections on a compact oriented surface, these are important classical phase spaces of Chern-Simons theory. The natural question of their quantization was addressed in many articles [Hi], [AdPW].

The geometric quantization of moduli spaces $\mathcal{N}$ of vector bundles over $X$ in a so-called Kähler polarization leads to what is known as spaces of non-abelian theta functions. More concretely, the Kähler polarized Hilbert spaces, at level $k = 1, 2, \ldots$, are the spaces

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where \( L \) is a determinant line bundle, endowed with a natural Chern connection, whose curvature coincides with the symplectic form. A projectively flat connection was constructed by Hitchin on the space of complex structures on \( N \) [Hi], providing a way of identifying different choices of Kähler polarized Hilbert spaces.

However, an explicit identification between Kähler polarized quantizations and real polarized ones has only been found in a few examples, notably the case when \( X \) is an elliptic curve [AdPW, FMN2], using the relationship between the moduli spaces in this case and a certain class of abelian varieties. In turn, a comparison between real and Kähler quantizations for abelian varieties was obtained using a coherent state transform [FMN2, FMN1, BMN].

In this article, we follow the analogous geometric quantization program for the moduli space \( \mathcal{M}_P(r) \) of parabolic bundles of rank \( r \) over \( \mathbb{C}P^1 \) with four parabolic points and parabolic weights in \( \{0, 1/2\} \) with parabolic degree zero [MS, MY]. It is known that, as in the case of vector bundles, there is a determinant line bundle \( \zeta_P \) over the moduli space of parabolic bundles endowed with a natural Chern connection, whose curvature is the (generally singular) Kähler form.

Let \( X \) be the elliptic curve which has a degree two map to \( \mathbb{C}P^1 \) ramified over the parabolic points. Using the description of the parabolic bundles of above type as holomorphic vector bundles over \( X \) equipped with a lift of the involution corresponding to the degree two covering [Bi1], we see that, for a given choice of parabolic structures on these 4 points, \( \mathcal{M}_P(r) \) has dimension \( d \leq r/2 \) and we have a canonical isomorphism

\[
\mathcal{M}_P(r) \cong X^d/\Gamma_d \cong \mathbb{C}P^d,
\]

where \( \Gamma_d \) is the semi-direct product \((\mathbb{Z}/2\mathbb{Z})^d \rtimes \Sigma_d\) for the natural action on \((\mathbb{Z}/2\mathbb{Z})^d\) of the symmetric group \( \Sigma_d \) for \( d \) elements. Moreover, for the natural polarization line bundle \( L \) on the abelian variety \( X^d \) (associated to a Kähler form of area one on \( X \)), we obtain an isomorphism \( \phi^*\zeta_P \cong L^2 \), where \( \zeta_P \rightarrow \mathcal{M}_P(r) \) is the determinant line bundle and \( \phi : X^d \rightarrow \mathcal{M}_P(r) \) is the natural quotient (see Sections 2 and 3).

This very concrete description allows the expression of the quantization Hilbert space at level \( k \), namely \( H^0(\mathcal{M}_P, \zeta_P^k) \), in terms of the (abelian) theta functions of level \( 2k \) on \( X^d \), and the comparison of real and Kähler polarized Hilbert spaces. For this, we need to apply the framework of [FMN2] for non-abelian theta functions over the moduli space of rank 2 vector bundles, with trivial determinant, over \( X \) (see Section 4). The so-called coherent state transform for Lie groups [Ha] is an analytic tool which, given an invariant Laplacian on a compact Lie group \( K \), associates holomorphic functions on the complexification \( K_C \) to square integrable functions on \( K \). This set up can be extended to appropriate spaces of distributions on \( K \) [FMN1]. Non-abelian theta functions of level \( k \) on \( \mathcal{M}_P \) are then described in terms of \( Ad \)-invariant holomorphic functions on the group \( SL(2, \mathbb{C}) \) with special quasi-periodicity properties. These holomorphic functions are obtained from elements in a vector space of distributions on the compact real form \( SU(2) \) by applying the coherent state transform, for time \( 1/(k + 2) \) (see Theorem 4.7).
2. A moduli space of parabolic vector bundles over $\mathbb{CP}^1$

Fix a point $p_0 \in \mathbb{CP}^1 \setminus \{0, 1, \infty\}$. Consider the divisor
\begin{equation}
S := \{0, 1, \infty, p_0\} \subset \mathbb{CP}^1.
\end{equation}
Let
\begin{equation}
f : X \rightarrow \mathbb{CP}^1
\end{equation}
be the unique double cover ramified exactly over $D$. Therefore, $X$ is a complex elliptic curve. Let $\text{Pic}^0(X)$ be the moduli space of topologically trivial holomorphic vector bundles on $X$.

**Lemma 2.1.** Any polystable vector bundle $E$ over $X$ of rank $r$ and degree zero is isomorphic to a direct sum $\bigoplus_{i=1}^r L_i$, where $L_i \in \text{Pic}^0(X)$.

The isomorphism classes of line bundles $L_i$, $1 \leq i \leq r$, are uniquely determined by $E$ up to a permutation of $\{1, \cdots, r\}$.

**Proof.** The first statement follows immediately from Atiyah’s classification of holomorphic vector bundles on $X$ (see [At2]). This also follows from the facts that $E$ is given by a representation of the abelian group $\pi_1(X)$ in $U(r)$ [NaSe].

The uniqueness of $L_i$ up to a permutation of $\{1, \cdots, r\}$ follows immediately from [At1, p. 315, Theorem 2(ii)]. \hfill $\Box$

We will consider parabolic vector bundles over $\mathbb{CP}^1$ with $S$ (see (2.1)) as the parabolic divisor. Let $E$ be a holomorphic vector bundle on $\mathbb{CP}^1$. A *quasi-parabolic* structure on $E$ is a filtration of subspaces
\begin{equation}
E_y =: F_{y,1} \supset \cdots \supset F_{y,j} \supset \cdots \supset F_{y,a_y} \supset F_{y,a_y+1} = 0
\end{equation}
over each point $y \in S$. A *parabolic* structure on $E$ is a quasi-parabolic structure as above together with real numbers
\begin{equation}
0 \leq \alpha_{y,1} < \cdots < \alpha_{y,j} < \cdots < \alpha_{y,a_y} < 1
\end{equation}
associated to the quasi-parabolic flags. (See [MS], [MY].) The numbers $\alpha_{y,j}$ in (2.3) are called parabolic weights. The multiplicity of the parabolic weight $\alpha_{y,j}$ is $\dim \mathbb{C} F_{y,j}/F_{y,j+1}$.

For notational convenience, a parabolic vector bundle $(E, \{F_{y,j}\}, \{\alpha_{y,j}\})$ defined as above will also be denoted by $E_*$. The *parabolic degree* is defined to be
\begin{equation}
\text{par-deg}(E_*) := \text{degree}(E) + \sum_{y \in S} \sum_{j=1}^{a_y} \alpha_{y,j} \cdot \dim(F_{y,j}/F_{y,j+1}).
\end{equation}

Fix an integer $r \geq 2$. For each point $y \in S$, fix an integer $m_y \in [0, r]$. Let $\mathcal{M}_P$ be the moduli space of semistable parabolic vector bundles $E_*$ on $\mathbb{CP}^1$ of rank $r$, with $S$ as the parabolic divisor, such that the parabolic weights at a parabolic point $y$ are $1/2$ with multiplicity $m_y$ and $0$ with multiplicity $r - m_y$, and
\begin{equation}
\text{par-deg}(E_*) = 0.
\end{equation}
The moduli spaces of parabolic bundles are irreducible normal complex projective varieties. We will see later that the above moduli space $M_P$ is smooth. Note that $M_P$ is empty if $\sum_{y \in S} m_y$ is an odd integer.

We will assume that $\sum_{y \in S} m_y$ is an even integer.

For any integer $m \geq 1$, we will construct a finite group $\Gamma_m$ equipped with an action of it on the Cartesian product $\text{Pic}^0(X)^m$.

Let $\Sigma_m$ be the group of permutations of $\{1, \cdots, m\}$. This group acts on the Cartesian product $(\mathbb{Z}/2\mathbb{Z})^m$ by permuting the factors. So any permutation $\tau \in \Sigma_m$ sends any $(z_1, \cdots, z_m) \in (\mathbb{Z}/2\mathbb{Z})^m$ to $(z_{\tau^{-1}(1)}, \cdots, z_{\tau^{-1}(m)})$. Let

(2.4) $\Gamma_m := (\mathbb{Z}/2\mathbb{Z})^m \rtimes \Sigma_m$

be the semi-direct product corresponding to this action. So $\Gamma_m$ fits in a short exact sequence

(2.5) $e \longrightarrow (\mathbb{Z}/2\mathbb{Z})^m \longrightarrow \Gamma_m \longrightarrow \Sigma_m \longrightarrow e$

of groups. We will construct a natural action of $\Gamma_m$ on $\text{Pic}^0(X)^m$.

Consider the action of group $\mathbb{Z}/2\mathbb{Z}$ on $\text{Pic}^0(X)$ defined by the involution $L \mapsto L^*$. Acting coordinate-wise, it produces an action of $(\mathbb{Z}/2\mathbb{Z})^m$ on $\text{Pic}^0(X)^m$. On the other hand, the permutation group $\Sigma_m$ acts on $\text{Pic}^0(X)^m$; as before, the action of any $\tau \in \Sigma_m$ sends any $(z_1, \cdots, z_m) \in \text{Pic}^0(X)^m$ to $(z_{\tau^{-1}(1)}, \cdots, z_{\tau^{-1}(m)})$. These two actions together produce an action of $\Gamma_m$ (constructed in (2.4)) on $\text{Pic}^0(X)^m$. Let

(2.6) $\text{Pic}^0(X)^m \longrightarrow \text{Pic}^0(X)^m / \Gamma_m$

be the quotient for this action. The quotient $\text{Pic}^0(X)^m / (\mathbb{Z}/2\mathbb{Z})^m$ for the subgroup in (2.5) is identified with $(\text{Pic}^0(X)/(\mathbb{Z}/2\mathbb{Z}))^m$. Hence

$$\text{Pic}^0(X)^m / \Gamma_m = \text{Sym}^m(\text{Pic}^0(X)/(\mathbb{Z}/2\mathbb{Z})).$$

Since $\text{Pic}^0(X)/(\mathbb{Z}/2\mathbb{Z}) = \mathbb{CP}^1$, we have

$$\text{Pic}^0(X)^m / \Gamma_m = \text{Sym}^m(\mathbb{CP}^1) = \mathbb{CP}^m.$$

Note that the quotient map in (2.6) factors through the projection

$$\text{Pic}^0(X)^m \longrightarrow \text{Sym}^m(\text{Pic}^0(X)) := \text{Pic}^0(X)^m / \Sigma_m.$$

But the surjective map

$$\text{Sym}^m(\text{Pic}^0(X)) \longrightarrow \text{Pic}^0(X)^m / \Gamma_m$$

in general is not a quotient for a group action because $\Sigma_m$ is not a normal subgroup of $\Gamma_m$.

**Proposition 2.2.** Let $d$ be the (complex) dimension of $\mathcal{M}_P$. Then $d \leq r/2$. If $d > 0$, then the variety $\mathcal{M}_P$ is canonically isomorphic to the quotient $\text{Pic}^0(X)^d / \Gamma_d$ constructed in (2.6).
Proof. Let
\[ \sigma : X \longrightarrow X \]
be the unique nontrivial deck transformation for the covering \( f \) in (2.2).

Let \( E_* \in \mathcal{M}_P \) be a polystable parabolic vector bundle. It corresponds to a unique holomorphic vector bundle \( V \longrightarrow X \) equipped with a lift of the involution \( \sigma \) in (2.7) as an isomorphism of order two
\[ \tilde{\sigma} : V \longrightarrow \sigma^*V \]
of vector bundles [Bi1]; this means that \( \tilde{\sigma} \) is a holomorphic isomorphism of vector bundles, and the composition
\[ V \xrightarrow{\tilde{\sigma}} \sigma^*V \xrightarrow{\sigma^*\tilde{\sigma}} \sigma^*\sigma^*V = V \]
is the identity map. We have
\[ \text{degree}(V) = 0 \]
because \( \text{par-deg}(E_*) = 0 \) [Bi1, p. 318, (3.12)]. The vector bundle \( V \) is polystable because \( E_* \) is polystable [BBN, pp. 350–351, Theorem 4.3]. Therefore, from Lemma 2.1 we know that
\[ V = \bigoplus_{i=1}^{r} L_i, \]
where \( L_i \in \text{Pic}^0(X) \). Recall from Lemma 2.1 that the line bundles \( L_i \) are uniquely determined up to a permutation.

For any line bundle \( L \) on \( X \) of degree zero, the line bundle \( L \otimes \sigma^*L \) descends to \( \mathbb{C}P^1 \), where \( \sigma \) is defined in (2.7). Since \( \text{Pic}^0(\mathbb{C}P^1) = \{O_{\mathbb{C}P^1}\} \), it follows that
\[ \sigma^*L = L^* \]
for all \( L \in \text{Pic}^0(X) \).

Since \( V \) in (2.9) is isomorphic to \( \sigma^*V \) (see (2.8)), using (2.10),
\[ \bigoplus_{i=1}^{r} L_i = \bigoplus_{i=1}^{r} L_i^* \]
(2.11)
Therefore, all vector bundles on \( X \) corresponding to points of \( \mathcal{M}_P \) are of the form
\[ \bigoplus_{i=1}^{a} (\xi_i \oplus \xi_i^*) \oplus \bigoplus_{j=1}^{r-2a} \eta_j, \]
(2.12)
where \( \eta_j \) are fixed line bundles on \( X \) (these line bundles \( \eta_j \) depend on the numbers \( m_y \) but are independent of the point of the moduli space \( \mathcal{M}_P \)), and the line bundles \( \xi_i, 1 \leq i \leq a \), move over \( \text{Pic}^0(X) \). From (2.11) it follows that
\[ \eta_j = \eta_j^* \]
(2.13)
for all \( j \). We note that any vector bundle as in (2.12) satisfying (2.13) admits a lift of the involution \( \sigma \). Indeed, each \( \eta_j \) has a lift because (2.13) holds. Also, \( \xi_i \oplus \xi_i^* \) has a natural lift of the involution \( \sigma \) because \( \sigma^*\xi_i = \xi_i^* \). Note that the involution of \( \xi_i \oplus \xi_i^* \) interchanges the two direct summands.
Hence we get a surjective morphism
\begin{equation}
\text{Pic}^0(X)^a \longrightarrow M_P
\end{equation}
that sends any \((\xi_1, \cdots, \xi_a)\) to
\[
\bigoplus_{i=1}^a (\xi_i \oplus \xi_i^\ast) \oplus \bigoplus_{j=1}^{r-2a} \eta_j.
\]
This morphism clearly factors through the quotient \(\text{Pic}^0(X)^a/\Gamma_a\) in (2.6).

For a vector bundle
\[
W = \bigoplus_{i=1}^a (\xi_i \oplus \xi_i^\ast),
\]
the unordered pairs \(\{\xi_i, \xi_i^\ast\}\) are uniquely determined by \(W\) up to a permutation of \(\{1, \cdots, a\}\) [At1, p. 315, Theorem 2(ii)]. Using this it follows that the above morphism
\[
\text{Pic}^0(X)^a/\Gamma_a \longrightarrow M_P
\]
is an isomorphism. This completes the proof of the proposition.

3. Determinant line bundle and Kähler form on \(M_P\)

Consider the moduli space \(M_P\) of parabolic vector bundles defined in the previous section. It has a natural (possibly singular) Kähler form; this Kähler form will be denoted by \(\omega_P\). There is a determinant line bundle
\begin{equation}
\zeta \longrightarrow M_P.
\end{equation}
This line bundle \(\zeta\) has a hermitian structure such that the curvature of the corresponding Chern connection coincides with \(\omega_P\). (See [BR], [Bi2], [TZ].)

Consider the dimension \(d\) in Proposition 2.2. Let \(\Gamma_d\) be the group defined in (2.4). The quotient \(\text{Pic}^0(X)^d/\Gamma_d\) is identified with the moduli space \(M_P\) by Proposition 2.2. Let
\begin{equation}
\phi : \text{Pic}^0(X)^d \longrightarrow \text{Pic}^0(X)^d/\Gamma_d = M_P
\end{equation}
be the morphism in (2.14). Since \(\phi\) is the quotient map for the action of \(\Gamma_d\) on \(\text{Pic}^0(X)^d\), the pulled back line bundle \(\phi^*\zeta\) is equipped with a lift of the action of \(\Gamma_d\) on \(\text{Pic}^0(X)^d\), where \(\zeta\) is the determinant line bundle in (3.1).

Let
\begin{equation}
L_0 := \mathcal{O}_{\text{Pic}^0(X)}(\mathcal{O}_X) \longrightarrow \text{Pic}^0(X)
\end{equation}
be the holomorphic line bundle of degree one defined by the point of \(\text{Pic}^0(X)\) corresponding to the trivial line bundle \(\mathcal{O}_X\) on \(X\). For each \(i \in [1, d]\), let
\begin{equation}
q_i : \text{Pic}^0(X)^d \longrightarrow \text{Pic}^0(X)
\end{equation}
be the projection to the \(i\)-th factor. The action of \(\Sigma_d\) on \(\text{Pic}^0(X)^d\) that permutes the factors in the Cartesian product has a natural lift to an action of \(\Sigma_d\) on the line bundle
\[
\bigotimes_{i=1}^d q_i^* L_0 \longrightarrow \text{Pic}^0(X)^d,
\]
where $L_0$ is the line bundle in (3.3).

Recall that the group $(\mathbb{Z}/2\mathbb{Z})^d$ acts on $\text{Pic}^0(X)^d$ using the action of $\mathbb{Z}/2\mathbb{Z}$ on $\text{Pic}^0(X)$ given by the involution $L \mapsto L^*$. Let

$$\sigma_d : (\mathbb{Z}/2\mathbb{Z})^d \rightarrow \text{Aut}(\text{Pic}^0(X)^d)$$

be the corresponding homomorphism.

**Theorem 3.1.** For any $g \in (\mathbb{Z}/2\mathbb{Z})^d$, there is a canonical isomorphism of holomorphic line bundles

$$\sigma_d(g)^*(\bigotimes_{i=1}^d q_i^*L_0) \sim \bigotimes_{i=1}^d q_i^*L_0,$$

where $\sigma_d$ is the homomorphism in (3.5), and $L_0$ is the line bundle in (3.3).

The line bundle $(\bigotimes_{i=1}^d q_i^*L_0)^{\otimes 2}$ has a canonical lift of the action of $\Gamma_d$ on $\text{Pic}^0(X)^d$.

There is a $\Gamma_d$-equivariant isomorphism of line bundles

$$\bigotimes_{i=1}^d (q_i^*L_0)^{\otimes 2} \sim \phi^*\zeta,$$

where $\phi$ and $\zeta$ are defined in (3.2) and (3.1) respectively.

**Proof.** Consider the automorphism of $\text{Pic}^0(\text{Pic}^0(X))$ induced by the involution of $\text{Pic}^0(X)$ defined by $L \mapsto L^*$. It fixes the line bundle $L_0$ defined in (3.3), because the above involution $L \mapsto L^*$ fixes the point of $\text{Pic}^0(X)$ corresponding to the trivial line bundle $\mathcal{O}_X$ on $X$. Note that $\text{Pic}^0(\text{Pic}^0(X))$ is identified with $\text{Pic}^0(X)$ by sending any $\xi \in \text{Pic}^0(X)$ to $\mathcal{O}_{\text{Pic}^0(X)}(\xi - \mathcal{O}_X)$. This identification commutes with the involutions. Since the point of $\text{Pic}^0(X)$ corresponding to $\mathcal{O}_X$ is fixed by the involution, it follows that

$$\sigma_d(g)^*(\bigotimes_{i=1}^d q_i^*L_0) = \bigotimes_{i=1}^d q_i^*L_0$$

for all $g \in (\mathbb{Z}/2\mathbb{Z})^d$. This proves the first statement of the theorem.

Let $z_0 := (\mathcal{O}_X, \ldots, \mathcal{O}_X) \in \text{Pic}^0(X)^d$ be the point. Note that $\sigma_d(g)(z_0) = z_0$ for all $g \in (\mathbb{Z}/2\mathbb{Z})^d$. In view of (3.6), there is a unique isomorphism

$$\rho : \sigma_d(g)^* \bigotimes_{i=1}^d q_i^*L_0 \rightarrow \bigotimes_{i=1}^d q_i^*L_0$$

which coincides with the identity map of the fiber $(\bigotimes_{i=1}^d q_i^*L_0)_{z_0}$ over the point $z_0$.

Consider the action of $(\mathbb{Z}/2\mathbb{Z})^d$ on $\text{Pic}^0(X)^d$ defined by the homomorphism $\sigma_d$ in (3.5). For any $g \in (\mathbb{Z}/2\mathbb{Z})^d$, there is a canonical lift of the involution $\sigma_d(g)$ of $\text{Pic}^0(X)^d$ to the line bundle

$$(\bigotimes_{i=1}^d q_i^*L_0) \otimes \sigma_d(g)^*(\bigotimes_{i=1}^d q_i^*L_0).$$
Using the isomorphism $\rho$ in (3.7), these lifts of the involutions $\sigma_d(g)$, $g \in (\mathbb{Z}/2\mathbb{Z})^d$, together produce a lift of the action of $(\mathbb{Z}/2\mathbb{Z})^d$ on $\text{Pic}^0(X)^d$ to the line bundle
\[
\left( \bigotimes_{i=1}^d q_i^* L_0 \right)^{\otimes 2} \longrightarrow \text{Pic}^0(X)^d.
\]

We already noted that the action of $\Sigma_d$ on $\text{Pic}^0(X)^d$ has a natural lift to an action of $\Sigma_d$ on the line bundle $\bigotimes_{i=1}^d q_i^* L_0$. This lift to $\bigotimes_{i=1}^d q_i^* L_0$ produces a lift to $(\bigotimes_{i=1}^d q_i^* L_0)^{\otimes 2}$ of the action of $\Sigma_d$ on $\text{Pic}^0(X)^d$. This action of $\Sigma_d$ on $(\bigotimes_{i=1}^d q_i^* L_0)^{\otimes 2}$ and the action of $(\mathbb{Z}/2\mathbb{Z})^d$ on $(\bigotimes_{i=1}^d q_i^* L_0)^{\otimes 2}$ constructed above together produce a lift to $(\bigotimes_{i=1}^d q_i^* L_0)^{\otimes 2}$ of the action of $\Gamma_d$ on $\text{Pic}^0(X)^d$. This proves the second statement of the theorem.

Let $\mathcal{N}_X(r)$ denote the moduli space of semistable vector bundles on $X$ of rank $r$ and degree zero. So, $\mathcal{N}_X(r) = \text{Sym}^r(\text{Pic}^0(X)) := \text{Pic}^0(X)^r/\Sigma_r$.

Let
\[
\beta : \text{Pic}^0(X)^d \longrightarrow \mathcal{N}_X(r)
\]
be the morphism defined by
\[
(L_1, \ldots , L_d) \longmapsto \bigoplus_{i=1}^d (L_i \oplus L_i^*) \oplus \bigoplus_{j=1}^{r-2d} \eta_j
\]
(see (2.12)). Let
\[
\gamma : \mathcal{M}_P \longrightarrow \mathcal{N}_X(r)
\]
be the morphism that sends any parabolic vector bundle on $\mathbb{CP}^1$ to the corresponding vector bundle on $X$ (see the proof of Proposition 2.2). Clearly,
\[
\beta := \gamma \circ \phi ,
\]
where $\phi$ is constructed in (3.2).

Let $\zeta_r$ be the determinant line bundle on the moduli space $\mathcal{N}_X(r)$. We will quickly recall the definition/construction of $\zeta_r$. Let
\[
\mathcal{P} \longrightarrow X \times \text{Pic}^0(X)
\]
be a Poincaré line bundle; this means that for each point $\alpha \in \text{Pic}^0(X)$, the restriction $\mathcal{P}_{|_{X \times \{\alpha\}}}$ lies in the isomorphism class of line bundles defined by the point $\alpha$. Let
\[
\pi_2 : X \times \text{Pic}^0(X) \longrightarrow \text{Pic}^0(X)
\]
be the natural projection. Define the line bundle
\[
\mathcal{L} := (\bigwedge^\text{top} R^0 \pi_2^* \mathcal{P})^* \otimes (\bigwedge^\text{top} R^1 \pi_2^* \mathcal{P}) \longrightarrow \text{Pic}^0(X).
\]
It can be shown that $\mathcal{L}$ is independent of the choice of the Poincaré line bundle $\mathcal{P}$. To see this note that any other Poincaré bundle is of the form $\mathcal{P}_1 := \mathcal{P} \otimes \pi_2^* A$, where $A$ is a line bundle on $\text{Pic}^0(X)$. Form the projection formula,
\[
(\bigwedge^\text{top} R^0 \pi_2^* \mathcal{P})^* \otimes (\bigwedge^\text{top} R^1 \pi_2^* \mathcal{P}) = (\bigwedge^\text{top} R^0 \pi_2^* \mathcal{P}_1)^* \otimes (\bigwedge^\text{top} R^1 \pi_2^* \mathcal{P}_1) \otimes A^\otimes \chi ,
\]
where $\chi$ is the Euler characteristic of degree zero line bundles on $X$. Since $\chi = 0$, we have $\mathcal{L} = (\Lambda^\text{top} R^0_\pi \mathcal{P}_1)^* \otimes (\Lambda^\text{top} R^1_\pi \mathcal{P}_1)$.

We will show that $\mathcal{L}$ coincides with $L_0$ defined in (3.3). To prove this, we first note that $\mathcal{O}_X$ is the unique line bundle on $X$ such that $\chi(\mathcal{O}_X) = 0 \neq H^0(X, \mathcal{O}_X)$. From this it follows that the point of $\text{Pic}^0(X)$ defined by $\mathcal{O}_X$ is the canonical theta divisor. This immediately implies that $\mathcal{L}$ is canonically identified with $L_0$.

For each $i \in [1, r]$, let $\mathcal{O}_i$ be the projection of $\text{Pic}^0(X)^r$ to the $i$–th factor. The line bundle
\begin{equation}
\bigotimes_{i=1}^r \mathcal{O}_i \mathcal{L} \to \text{Pic}^0(X)^r
\end{equation}
has a natural action of the group $\Sigma_r$ of permutations of $\{1, \cdots, r\}$. Using this action, the line bundle in (3.13) descends to the quotient $\text{Sym}^r(\text{Pic}^0(X))$ of $\text{Pic}^0(X)^r$. This descended line bundle is the determinant line bundle $\zeta_r$ on $\mathcal{N}_X(r) = \text{Sym}^r(\text{Pic}^0(X))$.

For the map $\gamma$ in (3.9), the pullback $\gamma^* \zeta_r$ coincides with the determinant line bundle $\zeta$ on $\mathcal{M}_P$ [BR], [Bi2]. Therefore, from (3.10) we get an isomorphism
\begin{equation}
\phi^* \zeta \sim (3.14) \beta^* \zeta_r.
\end{equation}
Using the fact that each $\eta_j$ in (3.8) is a fixed line bundle of order two, from the construction of $\zeta_r$ described above it is easy to see that $\beta^* \zeta_r$ has a canonical lift of the action of $\Gamma_d$ on $\text{Pic}^0(X)^d$. As noted earlier, the line bundle $\phi^* \zeta$ is equipped with an action of $\Gamma_d$, where $\phi$ is constructed in (3.2). It is straightforward to check that the isomorphism in (3.14) intertwines the actions of $\Gamma_d$.

For any Poincaré line bundle $\mathcal{P} \to X \times \text{Pic}^0(X)$ (see (3.11)), the pullback
\begin{equation}
(\text{Id}_X \times \sigma_1)^* \mathcal{P}^* \to X \times \text{Pic}^0(X)
\end{equation}
is also a Poincaré line bundle, where $\sigma_1$ is the involution in (3.5) defined by $L \mapsto L^*$. Therefore,
\begin{equation}
(3.15) \bigotimes^\text{top} R^0_\pi \mathcal{P}^* \otimes \bigotimes^\text{top} (R^1_\pi \mathcal{P}^*) = \sigma_1^* \mathcal{L},
\end{equation}
where $\pi_2$ is the projection in (3.12). Since $\mathcal{L} = L_0$,
\begin{equation}
(3.16) \sigma_1^* \mathcal{L} = \sigma_1^* L_0 = L_0;
\end{equation}
the last isomorphism follows from the fact that the point of $\text{Pic}^0(X)$ corresponding to $\mathcal{O}_X$ is fixed by $\sigma_1$ (see also (3.3)). Combining (3.15) and (3.16),
\begin{equation}
(3.17) \bigotimes^\text{top} R^0_\pi \mathcal{P}^* \otimes \bigotimes^\text{top} (R^1_\pi \mathcal{P}^*) = L_0.
\end{equation}
Using (3.17), from the constructions of the line bundle $\zeta_r$ and the morphism $\beta$ in (3.8) it follows that
\begin{equation}
\beta^* \zeta_r = (\bigotimes_{i=1}^d q_i^* L_0)^{\otimes 2}.
\end{equation}
In the second part of the theorem we constructed an action of the group $\Gamma_d$ on the line bundle $(\bigotimes_{i=1}^{d} q_i^* L_0)^{\otimes 2}$. We noted earlier that $\beta^* \zeta_r$ is equipped with a lift of the action of $\Gamma_d$ on $\text{Pic}^0(X)^d$. The above isomorphism of $\beta^* \zeta_r$ with $(\bigotimes_{i=1}^{d} q_i^* L_0)^{\otimes 2}$ is $\Gamma_d$-equivariant.

In view of the fact, noted earlier, that the isomorphism in (3.14) intertwines the actions of $\Gamma_d$, this completes the proof of the theorem. □

There is a unique translation invariant Kähler form $h_0$ on $\text{Pic}^0(X)$ of total volume one. Let

$$\omega := \sum_{i=1}^{d} q_i^* h_0$$

be the Kähler form on $\text{Pic}^0(X)^d$, where $q_i$ is the projection in (3.4). As before, the Kähler form on $\mathcal{M}_P$ will be denoted by $\omega_P$.

**Proposition 3.2.** For the morphism $\phi$ in (3.2),

$$\phi^* \omega_P = 2 \omega.$$  

**Proof.** Let $\omega_r$ be the Kähler form on the moduli space $\mathcal{N}_X(r)$. For the map $\gamma$ in (3.9),

(3.18) $$\gamma^* \omega_r = \omega_P$$

(see [BR]).

It can be shown that

(3.19) $$\beta^* \omega_r = 2 \omega,$$

where $\beta$ is constructed in (3.8). To prove (3.19), we first recall that $\omega_r$ is constructed using the unique unitary flat connection on polystable vector bundles of degree zero over $X$. More precisely, consider the unique unitary flat connection $\nabla$ on a polystable vector bundle

$$E := \bigoplus_{i=1}^{r} L_i \in \mathcal{N}_X(r)$$

(the flat hermitian metric on $E$ is not unique, but the flat hermitian connection is unique). Let $\tilde{\nabla}$ be the flat connection on $\text{End}(E) = E \otimes E^*$ induced by $\nabla$. The tangent space $T_E \mathcal{N}_X(r)$ is identified with

(3.20) $$\bigoplus_{i=1}^{r} H^1(X, \text{End}(L_i)) = H^1(X, \mathcal{O}_X)^{\oplus r} \subset H^1(X, \text{End}(E)).$$

Using the flat unitary structure on $\text{End}(E)$, we can represent elements of $H^1(X, \text{End}(E))$ by the harmonic forms. This yields a $L^2$-metric on $H^1(X, \text{End}(E))$. The restriction of this form to the subspace $H^1(X, \mathcal{O}_X)^{\oplus r}$ in (3.20) coincides with the Kähler form $\omega_r$ on $T_E \mathcal{N}_X(r)$.

The equality in (3.19) follows from the above description of $\omega_r$. Note that the factor 2 in (3.19) appears because the map $\beta$ constructed in (3.8) involves both $L_i$ and $L_i^*$, and the involution of $\text{Pic}^0(X)$ defined by $L \mapsto L^*$ preserves the translation invariant Kähler form $h_0$ on $\text{Pic}^0(X)$.

The proposition follows from (3.18), (3.19) and (3.10). □
There is a unique additive complex Lie group structure on $X$ with $f^{-1}(0)$ as the identity element, where $f$ is the map in (2.2). We fix this Lie group structure on $X$. The identity element $f^{-1}(0)$ will be denoted by $e$.

There is a natural complex group homomorphism

$$X \rightarrow \text{Pic}^0(X)$$

defined by $x \mapsto \mathcal{O}_X(x - e)$. This isomorphism will be useful here.

4. Non-abelian theta functions

In [FMN2], non-abelian theta functions on the moduli space of trivial determinant vector bundles of rank of $n$ on the elliptic curve $X$ were studied in terms of Weyl anti-invariant distributions in $SU(n)$. Let us recall briefly that construction, paying particular attention to the case $n = 2$, which will be especially relevant for the description of the Hilbert space associated to the quantization of the moduli space of parabolic bundles $\mathcal{M}_P$.

4.1. $SL_n(\mathbb{C})$ non-abelian theta functions on an elliptic curve. We start by writing the elliptic curve $X$ in the form:

$$X = \mathbb{C}/(\mathbb{Z} \oplus \tau \mathbb{Z}), \text{ for some } \tau \in \mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}.$$  

Let $\mathfrak{h}$ be the Cartan subalgebra of $sl_n(\mathbb{C})$ consisting of diagonal matrices of trace zero, and let $\hat{\Lambda}$ denote its coroot lattice. To be concrete, we identify $sl_n(\mathbb{C})$ with the space of traceless $n \times n$ complex matrices and $\mathfrak{h}$ with the space of diagonal matrices of trace zero.

Let $\mathcal{M}_X(n)$ be the moduli space semistable vector bundles $E$ over $X$ of rank $n$ with $\bigwedge^n E = \mathcal{O}_X$.

Consider the abelian variety

$$M = X \otimes \hat{\Lambda} \cong \mathfrak{h}/(\hat{\Lambda} \oplus \tau \hat{\Lambda}).$$

The Weyl group $W$ of $sl_n(\mathbb{C})$, given by the permutations of $\{1, \cdots, n\}$, acts naturally on $M$, via its natural action on $\mathfrak{h}$. As shown in [Lo, La], the moduli space $\mathcal{M}_X(n)$ can be naturally identified with the quotient under this action

$$\mathcal{M}_X(n) = M/W \cong \mathbb{C}P^{n-1}.$$

To consider the quantization of $\mathcal{M}_X(n)$, we use the symplectic form $\omega$ induced from the symplectic structure on $X$ and the determinant line bundle $L \rightarrow \mathcal{M}_X(n)$ whose curvature form coincides with $\omega$ [Qu].

Let

$$p : \mathbb{C}/\mathbb{Z} \cong \mathbb{C}^* \rightarrow \text{Pic}^0(X) \cong X$$

be the projection defined by $z + \mathbb{Z} \mapsto z + \mathbb{Z} + \tau \mathbb{Z}$, where $z \in \mathbb{C}$. The maximal torus of diagonal matrices in $SL_n(\mathbb{C})$ will be denoted by $T_\mathbb{C}$ and is canonically identified with $\mathfrak{h}/\hat{\Lambda}$. Let

$$q : SL_n(\mathbb{C}) \rightarrow SL_n(\mathbb{C})/SL_n(\mathbb{C}) \cong T_\mathbb{C}/W$$
be the quotient map for the conjugation action of $\text{SL}_n(\mathbb{C})$ on itself. We have the following commutative diagram:

$$
\begin{align*}
\text{SL}_n(\mathbb{C}) & \xrightarrow{q} T_C / W & & \leftarrow T_C = \mathfrak{h} / \tilde{\Lambda} \\
\mathcal{M}_X(n) & \leftarrow M & & \leftarrow \mathfrak{h},
\end{align*}
$$

(4.3)

where the composition $\text{SL}_n(\mathbb{C}) \to T_C / W \to \mathcal{M}_X(n)$ corresponds to the Schottky map described in [FMN2].

Note that the Verlinde numbers are:

$$
\dim H^0(\mathcal{M}_X(n), L^k) = \binom{n - 1 + k}{k}.
$$

(4.4)

Let $H(V)$ be the space of all holomorphic functions on a complex manifold $V$. Given a level $k$, the subspace of $H(\mathfrak{h})$ consisting of functions $\theta$ satisfying the identity

$$
\theta(v + \tilde{\alpha} + \tau \tilde{\beta}) = \left(e^{-2\pi i \beta(v) - \pi i r(\beta, \beta)}\right)^k \theta(v), \quad \tilde{\alpha}, \tilde{\beta} \in \tilde{\Lambda}
$$

will be relevant for us.

**Proposition 4.1** ([FMN2]). The space of non-abelian theta functions $H^0(\mathcal{M}_X(n), L^k)$ is naturally identified with

$$
H^+_k, n := \{ \theta \in H(\mathfrak{h}) : \theta \text{ satisfies (4.5) and } w\theta = \theta, \forall w \in W \}.
$$

We remark that since the quasi-periodicity condition in (4.5) does not depend on the first summand (i.e., $\tilde{\alpha}$) of the lattice $\tilde{\Lambda} \oplus \tau \tilde{\Lambda}$, the non-abelian theta functions in the proposition can be also considered to be Weyl invariant holomorphic functions on $T_C = \mathfrak{h} / \tilde{\Lambda}$, or equivalently, Ad–invariant holomorphic functions on $\text{SL}_n(\mathbb{C})$.

Motivated by the Segal–Bargmann–Hall or “coherent state” transform for Lie groups, we will now describe a way of obtaining such Ad–invariant holomorphic functions on $G = \text{SU}(n)$ starting from Ad–invariant distributions on the maximal compact subgroup $K = \text{SU}(n)$.

Let $\Lambda^+_W$ denote the set of dominant weights, in one to one correspondence with irreducible representations $R_\lambda$ of $K = \text{SU}(n)$. For $x \in K$, the expression

$$
f = \sum_{\lambda \in \Lambda^+_W} \text{tr}(A_\lambda R_\lambda),
$$

(4.6)

where $A_\lambda \in \text{End}(R_\lambda)$ are endomorphism–valued coefficients, defines a distribution under appropriate growth conditions on the operator norm of the $A_\lambda$ (see [FMN2]).

Let $c_\lambda \geq 0$ be the eigenvalue of $-\Delta_K$, where $\Delta_K$ is the Laplace-Beltrami operator on $K$ associated with the Ad-invariant inner product on $\mathfrak{su}_n$ for which the roots have squared length 2, on functions of the form $\text{tr}(A_\lambda R_\lambda(x))$, $A_\lambda \in \text{End}(R_\lambda)$.

Given a positive parameter $t > 0$, and $\tau \in \mathbb{H}$, the (generalized) coherent state transform (CST for short) is given by associating to a distribution $f$ as in (4.6) the holomorphic
function on $\text{SL}_n(\mathbb{C})$
\[
C_t f(g) := \sum_{\lambda \in \Lambda^+_m} e^{it\tau c_\lambda} \text{tr}(A_\lambda R_\lambda(g)).
\]

Recall from [Lo, FMN2] that non-abelian theta functions on $\mathcal{M}_X(n)$ are more conveniently described in terms of Weyl anti-invariant theta functions on $\mathfrak{h}$. Denote by $\theta^{-n}_n$ the unique (up to scale) $W$-anti-invariant theta function of level $n$ on $\mathfrak{h}$.

Let now $\rho$ be the Weyl vector given by half the sum of the positive roots and let $\sigma$ be the denominator of the Weyl character formula analytically continued to $\text{SL}_n(\mathbb{C})$. Let $\hat{\alpha}$ be the longest root in $\mathfrak{sl}_n(\mathbb{C})$ and let
\[
D_{k,n} := \{ \lambda \in \Lambda^+_W : \langle \lambda, \hat{\alpha} \rangle \leq k \}
\]
be the parameter space for integrable representations of the level $k$ affine Kac-Moody algebra $\tilde{\mathfrak{sl}}_n(\mathbb{C})_k$. Note that $\#D_{k,n} = \binom{n-1+k}{k}$, which equals the Verlinde number (4.4).

As seen in [FMN2], there is a ($\tau$-independent) finite-dimensional space of $\text{Ad}$-invariant distributions $V_{k,n}$ on $\text{SU}(n)$ which has an orthonormal basis labelled by the elements of $D_{k,n}$, such that the following holds:

**Theorem 4.2** ([FMN2]). Let $n > 2$ and $C^\infty(\text{SU}(n))^{\text{Ad}} \supset L^2(\text{SU}(n))$ denote the space of $\text{Ad}$-invariant distributions on $\text{SU}(n)$. Restricting the CST to $V_{k,n}$, we obtain
\[
V_{k,n} \hookrightarrow C(\text{SU}(n))^{\text{Ad}} \overset{C_t}{\longrightarrow} \mathcal{H}(\text{SL}_n(\mathbb{C}))^{\text{Ad}}.
\]
Moreover, the composition of maps
\[
\varphi_{k,\tau} \circ C_{k+n} : V_{k,n} \longrightarrow H^0(\mathcal{M}_X(n), L^k) \subset \mathcal{H}(\text{SL}_n(\mathbb{C}))^{\text{Ad}},
\]
where
\[
\varphi_{k,\tau}(f) = e^{\frac{||\rho||^2}{4\tau} t \sigma \rho} f,
\]
is an isomorphism; we identify a $W$-invariant theta function on $\mathfrak{h}$ with an $\text{Ad}$-invariant function on $\text{SL}_n(\mathbb{C})$ using (4.3) and (4.5).

**Remark 4.3.** Note that the map $\varphi_{k,\tau}$ is well defined only on $C_{k+n}^{-1} (V_{k,n})$, since these holomorphic functions are divisible by $\theta^{-n}_n$ [Lo, FMN2].

Let $\tau_2 = \text{Im}(\tau) > 0$, and define the hermitian inner product on $H^0(\mathcal{M}_X(n), L^k)$ by
\[
\langle \langle F_1, F_2 \rangle \rangle := \int_{q^{-1}(\mathfrak{h}_0)} T_1 F_2 |q^* \theta^{-n}_n|^2 d\nu_{\frac{\tau_2}{k+n}}
\]
(see [AdPW]), where $q$ was defined in equation (4.2), $d\nu_{\frac{\tau_2}{k+n}}$ denotes what is known as the heat kernel measure of $\text{SL}_n(\mathbb{C})$, at time $\frac{\tau_2}{k+n}$, and $\mathfrak{h}_0 \subset \mathfrak{h}$ is a fundamental domain for the action of the semi-direct product $W \ltimes (\Lambda \oplus \tau \tilde{\Lambda})$.

**Theorem 4.4** ([FMN2]). The map $\varphi_{k,\tau} \circ C_{k+n} : V_{k,n} \longrightarrow H^0(\mathcal{M}_X(n), L^k)$ is a unitary isomorphism.
Let us now consider the (slightly different) case $n = 2$, which will be especially relevant in the next section, and for which the distributions in the previous theorem can be written in a simple way. For simplicity, we will state the result only for even level, which is the case we will need.

The space $V_{2k,2} \subset C(SU(2))'^{Ad}$ is the $k$-dimensional $\mathbb{C}$-span of the distributions

$$\psi_{j,2k}(x) = \frac{1}{\sigma} \sum_{n \in \mathbb{Z}} \left( e^{2\pi i(j+2kn)x} - e^{-2\pi i(j+2kn)x} \right) \in C^\infty(SU(2))', j = 1, \ldots, k$$

(see [FMN2]).

Consider the basis of level $2k + 4$ theta functions for the elliptic curve $X$, namely

$$\{\theta_{j,2k+4}\}_{0 \leq j < 2k+4},$$

with

$$\theta_{j,2k+4}(z) = \sum_{m \in \mathbb{Z}} \exp \left( \frac{\pi i}{2k+4} \left( j + (2k+4)m \right)^2 + 2\pi i (j + (2k+4)m)z \right), \quad 0 \leq j < 2k+4.$$ 

The Weyl anti-invariant theta function of level 4 on $X$ is given by $\theta_{-4} = \theta_{1,4} - \theta_{3,4}$.

Recall from [Lo, FMN2] that the space of Weyl invariant theta functions of level $2k$ on $X$ can be conveniently described in terms of Weyl anti-invariant theta functions of level $2k + 4$,

$$H^0(X, L_{2k}^0)^- \cong H^0(X, L_{2k}^0)^+ / \theta_4^-,$$

where $\theta_4^-$ is the (unique up to nonzero multiplicative constant) Weyl anti-invariant theta function of level 4 on $X$. The bundle of conformal blocks (of level $k$) over $M_1$, which is associated to the moduli space of semistable rank two vector bundles with trivial determinant on $X$, has a natural hermitian structure which is easily expressed in terms of theta functions in $H^0(X, L_{2k}^0)^-$ as described above [AdPW, FMN2].

**Theorem 4.5.** The composition of maps

$$V_{2k,2} \xrightarrow{\varphi_{k,\tau}} C_{1/2}^{1} (V_{2k,2}) \xrightarrow{\varphi_{k,\tau}} H^0(M_X(2), L^{2k}) \subset \mathcal{H}(SL_2(\mathbb{C}))^{Ad}$$

where $\varphi_{k,\tau}(f) = e^{2\pi i \tau \sigma/\theta_4} f$, is an isomorphism. The image of the natural basis $\{\psi_{j,2k}\}_{j=1,\ldots,k}$ is given by

$$\{\vartheta_{j,2k}\}_{j=1,\ldots,k},$$

where $\vartheta_{j,2k} = (\theta_{j,2k+4} - \theta_{2k+4-j,2k+4})/\theta_4^-$. [FMN2].

4.2. **Non-abelian theta functions on $\mathcal{M}_P$.** Let $\Lambda$ be the coroot lattice of $sl_2(\mathbb{C})$, and let $\mathfrak{h}$, as before, be the Cartan subalgebra. The abelian variety from the previous subsection is now $M = X \otimes \Lambda \cong X$.

From Proposition 2.2, we have

$$\mathcal{M}_P \cong \text{Pic}^0(X)^d / \Gamma_d \cong M^d / \Gamma_d,$$

where $\Gamma_d = \mathbb{Z}^d_2 \rtimes \Sigma_d$. We have the isomorphism of abelian varieties

$$M^d = (\mathfrak{h} / \Lambda \oplus \tau \Lambda)^d.$$
Let $p : M^d \rightarrow M^d/\Gamma_d$ be the natural projection. From above, the pull-back of the determinant line bundle by $p$ gives the line bundle $\bigotimes_{i=1}^d (q_i^* L_0)^2$ over $M^d$. Therefore, non-abelian theta functions of level $k$ on $\mathcal{M}_P$ will be described by $\Gamma_d$ invariant products of level $2k$ theta functions on each of the factors $\mathfrak{h}/\Lambda \oplus \tau \Lambda$.

The analog of diagram (4.3) is now

$$
\begin{array}{c}
\text{SL}_2(\mathbb{C})^d \rightarrow T^d_{c}/W^d \leftarrow T^d_{c} = (\mathfrak{h}/\Lambda)^d \\
\mathcal{M}_P \xrightarrow{\varrho} \mathcal{M}_X(2)^d \leftarrow M^d \leftarrow \mathfrak{h}^d.
\end{array}
$$

Let $K$ be a compact Lie group. The CST on $K^d$ equipped with the product metric can be applied to $\Sigma_d$-invariant functions. Since the averaged heat kernel measures are the product of the $d$ measures on each of the factors of $K$, we have the following commutative diagram

$$
\begin{array}{c}
(C^\infty(K^d)^{\Sigma_d} \hookrightarrow C^\infty(K^d)^{t} \\
\downarrow C^\otimes_d_t \downarrow C^\otimes_d_t \\
\mathcal{H}_d(G)^{\Sigma_d} \hookrightarrow \mathcal{H}_d(G).
\end{array}
$$

where the CST for $K^d$ is given by

$$
C^\otimes_d_t = C_t \otimes \cdots \otimes C_t,
$$

in terms of the CST $C_t$ for $K$.

**Definition 4.6.** Let $V_k \subset C^\infty((SU(2)^d)^{\Sigma_d}$ be the vector space with basis $\{\Psi_{J,k}\}_{J=\{j_1,\ldots,j_d\},1 \leq j_i \leq k}$, where

$$
\Psi_{J,k} = \sum_{\sigma \in \Sigma_d} \psi_{j_{\sigma_1},2k} \otimes \cdots \otimes \psi_{j_{\sigma_d},2k},
$$

and the distributions $\psi_{j,2k} \in C^\infty(SU(2))^t$ are given in (4.10).

Let $\varphi^\otimes_{k,\tau} = \otimes \varphi_{k,\tau}$ be defined on $C^\otimes_{k,\tau}(V_k) \subset \mathcal{H}_d(\text{SL}_2(\mathbb{C})^d)$, where $\varphi_{k,\tau}$ was defined in (4.8).

**Theorem 4.7.** The CST $C^\otimes_{k,\tau}$ establishes an isomorphism between the space $V_k$ of distributions on $SU(2)^d$ and the space of non-abelian theta functions $H^0(\mathcal{M}_P,\xi^k)$ of level $k$, meaning the map

$$
\varphi^\otimes_{k,\tau} \circ C^\otimes_{k,\tau} : V_k \rightarrow H^0(\mathcal{M}_P,\xi^k)
$$

is an isomorphism. The image of the natural basis (Definition 4.6) is given by $\{\phi_{J,k}\}_{J=\{j_1,\ldots,j_d\},1 \leq j_i \leq k}$, where

$$
\phi_{J,k}(z_1,\ldots,z_d) = \sum_{\sigma \in \Sigma_d} \vartheta_{j_{\sigma_1},2k}(z_1) \cdots \vartheta_{j_{\sigma_d},2k}(z_d).
$$
Proof. Recall that the determinant line bundle $\xi$ on the moduli space of parabolic bundle $\mathcal{M}_P$, satisfies $\xi \cong \otimes_{i=1}^d (q_i^* L_0)^2$, where $q_i : M^d \to M$ is the projection on the $i$th factor. Therefore, elements in $H^0(\mathcal{M}_P, \xi^k)$ are given by $\Gamma_d$ invariant theta functions of level $2k$ on $M^d$. From Theorem 4.5 it follows that applying $\varphi_{k,\tau}^{\otimes d} \circ C_{\chi+2}^{\otimes d}$ to $\Psi_{J,k}$ we get $\phi_{J,k}$. □

Remark 4.8. We see that the Verlinde number is equal to the dimension of the space of degree $k$ polynomials in $d$ variables, which is consistent with $\xi \cong O(1)$ on $\mathcal{M}_P \cong \mathbb{P}^d$.

Remark 4.9. Non-abelian theta functions $H^0(\mathcal{M}_P, \xi^k)$ of level $k$ can therefore be described as the image, by a coherent state transform, of the finite dimensional space of distributions on the compact group $SU(2)^d$. In particular, following [FMN1, FMN2], in this case the CST can also be interpreted as the parallel transport of a unitary connection on the bundle of conformal blocks over $\mathcal{M}_{0,4}$.

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References

[AdPW] S. Axelrod, S. Della Pietra and E. Witten, Geometric quantization of Chern-Simons theory, Jour. Diff. Geom. 33 (1991), 787–902.
[At1] M. F. Atiyah, On the Krull–Schmidt theorem with application to sheaves, Bull. Soc. Math. Fr. 84 (1956), 307–317.
[At2] M. F. Atiyah, Vector bundles over an elliptic curve, Proc. London Math. Soc. 7 (1957), 412–452.
[BMN] T. Baier, J. Mourão and J. P. Nunes, Quantization of abelian varieties: distributional sections and the transition from Kähler to real polarizations, Jour. Funct. Anal. 258 (2010), 3388–3412.
[BBN] V. Balaji, I. Biswas and D. S. Nagaraj, Principal bundles over projective manifolds with parabolic structure over a divisor, Tohoku Math. Jour. 53 (2001), 337–367.
[Bi1] I. Biswas, Parabolic bundles as orbifold bundles, Duke Math. Jour. 88 (1997), 305–325.
[Bi2] I. Biswas, Determinant line bundle on moduli space of parabolic bundles, Ann. Global Anal. Geom. 40 (2011), 85–94.
[BR] I. Biswas and N. Raghavendra, Determinants of parabolic bundles on Riemann surfaces, Proc. Indian Acad. Sci. (Math. Sci.) 103 (1993), 41–71.
[FMN1] C. Florentino, J. Mourão and J. P. Nunes, Coherent state transforms and abelian varieties, Jour. Funct. Anal. 192 (2002), 410–424.
[FMN2] C. Florentino, J. Mourão and J. P. Nunes, Coherent state transforms and vector bundles on elliptic curves, Jour. Funct. Anal. 204 (2003), 355–398.
[Go] W. M. Goldman, The symplectic nature of fundamental groups of surfaces, Adv. Math. 54 (1984), 200–225.
[Ha] B. Hall, The Segal-Bargmann coherent state transform for compact Lie groups, Jour. Funct. Anal. 122 (1994), 103–151.
[Hi] N. J. Hitchin, Flat connections and geometric quantization, Comm. Math. Phys. 131 (1990), 347–380.
[La] Y. Laszlo, About $G$-bundles over elliptic curves, Ann. Inst. Fourier 48 (1998), 413–424.
[Lo] E. Looijenga, Root systems and elliptic curves, Invent. Math. 38 (1976), 17–32.
M. Maruyama and K. Yokogawa, Moduli of parabolic stable sheaves, *Math. Ann.* **293** (1992), 77–99.

V. B. Mehta and C. S. Seshadri, Moduli of vector bundles on curves with parabolic structures, *Math. Ann.* **248** (1980), 205–239.

M. S. Narasimhan and C. S. Seshadri, Stable and unitary vector bundles on a compact Riemann surface, *Ann. of Math.* **82** (1965), 540–567.

D. G. Quillen, Determinants of Cauchy-Riemann operators on Riemann surfaces, *Funct. Anal. Appl.* **19** (1985), 37–41.

L. A. Takhtajan and P. Zograf, The first Chern form on moduli of parabolic bundles, *Math. Ann.* **341** (2008), 113–135.