1. Introduction

Among the different spatial discretization frameworks, the Discontinuous Galerkin Spectral Element Method (DGSEM) \cite{1,2} approximates a function as a set of discontinuous, high-order polynomials defined in a tessellation of the spatial domain. The DGSEM is a collocation method which represents the solution as a combination of Lagrange polynomials that shares the nodes with a certain quadrature rule, usually Gauss or Gauss-Lobatto.

Fisher and Carpenter showed in \cite{3} that diagonal-norm summation-by-parts (SBP) derivative operators can be rewritten in telescopic form. In this case, the action of the operator resembles a finite volume scheme, ensuring local conservation in the sense of Lax-Wendroff. In the context of the DGSEM, the derivative operator with Gauss-Lobatto nodes falls in this category, whereas the use of Gauss nodes leads to a generalized SBP operator \cite{4,5,6,7}.

In this work we show that the generalized formulation of Chan \cite{7} also admits a flux-differencing form and thus, from a more practical standpoint, entropy-stable schemes can be generated by introducing certain dissipative numerical fluxes at the interfaces of the sub-elements \cite{3,8,9}.

2. High-order DGSEM on Gauss nodes

Beginning with a one-dimensional grid where \( \xi_i \) and \( \omega_i \) are the nodes and weights of the Gauss quadrature rule of order \( N \) (\( i \in [0, N] \)) in the reference domain, \( E \), we define a complementary grid \( \tilde{\xi}_i \) (\( i \in [0, N + 1] \)) as in fig. 1.

\[
\tilde{\xi}_0 = -1, \quad \tilde{\xi}_i = \tilde{\xi}_{i-1} + \omega_i, \quad \tilde{\xi}_{N+1} = 1.
\]

Fisher and Carpenter showed in \cite{3} that SBP derivative operators, \( D \), can be rewritten in telescopic form. In this case, the action of the operator resembles a finite volume scheme,

\[
(\partial_{\xi} f_i) \approx \sum_{k=0}^{N} D_{ik} f_k = \frac{\tilde{f}_{i+1} - \tilde{f}_i}{\omega_i}.
\]
As stated in the introduction, the derivative operator of the DGSEM with Gauss-Lobatto nodes falls in this category, while we can use the developments of Chan [6][7] if Gauss nodes are used. Following Chan’s notation, the DGSEM discretization of a generic one-dimensional conservation law, \( u_t + f_x = 0 \), is,

\[
\mathbf{M} \mathbf{u} + \left[ \mathbb{I} + \mathbf{V}_f^T \mathbf{B} \right] \left( 2 \mathbf{Q} \circ \mathbf{F}^S \right) \cdot \mathbf{I} + \mathbf{V}_f^T \mathbf{B} \begin{bmatrix} f_L^* - f_L \\ f_R^* - f_R \end{bmatrix} = 0.
\]

(2)

The mass matrix, \( \mathbf{M} = \text{diag}(J \omega_i) \), contains the Jacobians and weights of the quadrature rule in each mesh element. \( \mathbf{V}_f \) represents the projection of the interior values to the faces, and \( \mathbf{B} = \text{diag}(-1, 1) \). \( \mathbf{Q} \) is a generalized SBP operator representing the integral of the derivative of the basis functions, \( f_L^* \) and \( f_R^* \) are the numerical fluxes at the element interfaces, and sub-indices with a tilde mean that the magnitude has been computed on so-called entropy projected variables. Finally, \( \mathbf{F}^S \) is the matrix with all the combinations of entropy-conservative fluxes (including the entropy-projected values at the interfaces), \( f_{ij}^S = f^S(u_i, u_j) \), and \( \circ(\cdot, \cdot) \) is the Hadamard product. The actual expression of all these variables can be found in [6].

Imposing the equality of eq. (2) with a finite volume scheme,

\[
\mathbf{M} \mathbf{u} + \Delta \mathbf{f} = 0,
\]

(3)

where \( \Delta \mathbf{f} \) represents the flux-differencing formulation of eq. (1), we find an expression for the different subcell fluxes, \( \mathbf{f} \),

\[
\Delta \mathbf{f} = \left[ \mathbb{I} + \mathbf{V}_f^T \mathbf{B} \right] \left( 2 \mathbf{Q} \circ \mathbf{F}^S \right) \cdot \mathbf{I} + \mathbf{V}_f^T \mathbf{B} \begin{bmatrix} f_L^* - f_L \\ f_R^* - f_R \end{bmatrix}.
\]

(4)

With the additional constraints \( \bar{f}_0 = f_L^* \) and \( \bar{f}_{N+1} = f_R^* \), we can compute the value of \( \tilde{f}_{i+1} \) from \( \bar{f}_i \) and eq. (4), obtaining an expression equivalent to the one developed by Rueda-Ramírez et al. [10],

\[
\tilde{f}_{i+1} = \bar{f}_i + \sum_{k=0}^{N} \tilde{S}_{ik} f_k^S - k(-1) \left[ f_k^S - \sum_{k=0}^{N} l_k(-1) f_L^S + f_L^S \right] + l_i(1) \left[ f_k^S - \sum_{k=0}^{N} l_k(1) f_R^S + f_R^* \right], \quad i = 0, \ldots, N,
\]

(5)

where \( \tilde{S} = 2 \mathbf{Q} - \mathbf{B} = \mathbf{Q} - \mathbf{Q}^T \) is a new skew-symmetric derivative matrix that includes the inner boundary contributions. We remark that eq. (2) is recovered when \( \bar{f}_0 \) and \( \bar{f}_{i+1} \) are introduced in eq. (1) by construction. There is, however, a problem with eq. (5). For a set of \( N+1 \) Gauss nodes there are \( N+2 \) complementary staggered fluxes, but we have \( N+3 \) equations. The flux at the right face, \( \tilde{f}_{i+1} \), can be computed from \( \tilde{f}_{i+1}(i = N) \), but it also must be equal to \( f_R^* \). We can overcome this issue by proving that both equations are equivalent and our system is not over-constrained.

**Proof.** Since the complementary flux \( \tilde{f}_{i+1} \) is defined in terms of \( \bar{f}_i \), it is possible to define all the complementary fluxes in terms of the first one, \( \bar{f}_0 = f_L^* \),

\[
\tilde{f}_{i+1} = f_L^* + \sum_{a=0}^{i} \sum_{k=0}^{N} \hat{S}_{ak} f_k^S - \sum_{a=0}^{i} l_a(-1) \left( f_a^S - \sum_{k=0}^{N} l_k(-1) f_L^S + f_L^S \right) + \sum_{a=0}^{i} l_a(1) \left( f_a^S - \sum_{k=0}^{N} l_k(1) f_R^S + f_R^* \right),
\]

(6)

Now we set \( i = N \) and apply different simplifications to prove that eq. (6) is equivalent to \( \tilde{f}_{N+1} = f_R^* \) when \( i = N \). In this case, the third and fourth terms of the right-hand side can be further simplified by considering that, for the Lagrange interpolating polynomials, \( \sum_{a=0}^{N} l_a(x) = 1 \). Substituting this allows us to “exchange” \( f_L^* \) in the first term by \( f_R^* \),

\[
\tilde{f}_{N+1} = f_R^* + \sum_{a=0}^{N} \sum_{k=0}^{N} \hat{S}_{ak} f_k^S - \sum_{k=0}^{N} l_k(+1) f_R^S - l_k(-1) f_L^S + \sum_{a=0}^{N} l_a(1) f_a^S - l_a(-1) f_a^S.
\]
Written in this form we can now apply the symmetry property of the entropy-conservative numerical flux, $S_{ij} = S_{ji}$, and cancel the last two terms,

$$\tilde{f}_{N+1} = f_{R}^* + \sum_{\alpha=0}^{N} \sum_{k=0}^{N} \tilde{S}_{\alpha k} f_{ak}^S.$$  

Finally, we remark that $\tilde{S}_{ij} = -\tilde{S}_{ji}$ and thus, the product of the last term is the Hadamard product of the skew-symmetric matrix $\tilde{S}$ with the symmetric matrix $f_{ij}^S$. This is, in fact, another skew-symmetric matrix and the last term is also zero, $\tilde{f}_{N+1} = f_{R}^*$. \hfill \Box

Note that this proof is also applicable to the standard Gauss DGSEM scheme since the expression for the staggered fluxes is similar, simply lacking the terms containing entropy-projected quantities.

3. Subcell limiting

The existence of an underlying flux-differencing formula for the Gauss-DGSEM enables the use of the subcell limiting strategies presented by Rueda-Ramírez et al. [9] to improve the robustness of the method, e.g., in the presence of shocks. In particular, we propose a hybrid scheme obtained as a convex combination of the high-order Gauss-DGSEM with a first-order FV method,

$$m_i u_i = \tilde{f}_i - \tilde{f}_{i+1}, \quad (7)$$

with $m_i$ the entries of the diagonal mass matrix. The individual fluxes are given as $\tilde{f}_i = \alpha_i f_i^{FN} + (1 - \alpha_i) \bar{f}_i$, where $f_i^{FN}$ is a robust first-order approximation of the flux and $\alpha_i$ is a so-called blending coefficient, which is selected such that the resulting scheme exhibits some desired properties, e.g., positivity, non-oscillatory behavior, etc.

The main difference of (7) with the hybrid LGL-DGSEM/FV methods in [8, 9] is that the Gauss-DGSEM does not share the same boundary fluxes with the first-order FV scheme. As a result, the inter-element fluxes have to be obtained as a convex combination of low- and high-order fluxes to obtain a robust scheme.

4. Results

This section includes some numerical results that confirm the theoretical developments of sections 2 and 3. In both cases we solve the Euler equations using a split-form with Chandrashekar’s two-point flux [11] and LLF numerical fluxes. In section 4.1 the non-dissipative term uses the same two-point flux, whereas we employ a simple average in section 4.2.

4.1. Convergence

We test the numerical accuracy of the subcell approach by comparing it against the split-form formulation obtained by Chan for Gauss nodes [7]. Considering the solution to the one-dimensional Euler equations,

$$\rho = 2 + \sin \pi (x - t), \quad u = 1, \quad p = 1,$$

we integrate them until $t = 0.7$ with a CFL = 0.125 and using the 5-stage, 4th-order Runge-Kutta algorithm described by Carpenter and Kennedy [12]. The convergence results are shown in fig. 2, comparing the new approach against the baseline from J. Chan.

4.2. Sedov blast

To illustrate the shock-capturing capacity of the hybrid DGSEM/FV method, we simulate a Sedov blast problem describing the evolution of a blast wave expanding from an initial concentration of density and pressure. For the initial condition, we assume a gas in rest, $v_1(t = 0) = v_2(t = 0) = 0$, with a Gaussian distribution of density and pressure,

$$\rho(t = 0) = \rho_0 + G(r; \sigma_p), \quad p(t = 0) = p_0 + (\gamma - 1)G(r; \sigma_p), \quad G(r; \sigma) = \frac{1}{4\pi \sigma^2} \exp \left( -\frac{1}{2} \frac{r^2}{\sigma^2} \right), \quad r^2 = x^2 + y^2, \quad (8)$$

where we choose $\sigma_p = 0.25$ and $\sigma_p = 0.15$. Furthermore, the ambient density is set to $\rho_0 = 1$ and the ambient pressure to $p_0 = 10^{-1}$. We complement the simulation domain, $\Omega = [-1, 1]^2$, with periodic boundary conditions, and
tessellate it using $K = 64^2$ quadrilateral elements. We represent the solution with polynomials of degree $N = 3$ and run the simulation until $t = 1$.

To avoid non-physical oscillations in the vicinity of shocks, we use an element-wise blending coefficient $\alpha$ with the modal shock sensor of Hennemann et al. [8]. At the element boundaries, we use the high-order DG flux for simplicity, such that the combination of the two schemes [7] is completely element-local. Figure 3 illustrates the density and blending coefficient in the domain obtained with the hybrid DGSEM/FV method using Gauss and Gauss-Lobatto nodes at time $t = 0.9$.

Fig. 3: Density and blending coefficient, $\alpha$, for the Sedov blast problem obtained with the hybrid DGSEM/FV method at $t = 0.90$ using Gauss and Gauss-Lobatto nodes.

5. Conclusions

We have presented in this work a novel flux-differencing expression of the entropy-conservative formulation of J. Chan [6, 7] in terms of staggered fluxes. This approach can be applied explicitly to the DGSEM framework with Gauss nodes. We have also proved that this telescopic formulation of the derivative operators exists for split-forms with and without entropy-projected variables.

When applied to a simple test case, we have shown that the convergence properties of the flux-differencing method match those of the entropy-conserving split-from used as the baseline. This allowed us to apply some subcell limiting strategies already developed for the DGSEM with Gauss-Lobatto nodes, although more work is needed in this aspect.

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