NON-TRAPPING ESTIMATES NEAR NORMALLY HYPERBOLIC TRAPPING

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Abstract. In this paper we prove semiclassical resolvent estimates for operators with normally hyperbolic trapping which are lossless relative to non-trapping estimates but take place in weaker function spaces. In particular, we obtain non-trapping estimates in standard $L^2$ spaces for the resolvent sandwiched between operators which localize away from the trapped set $\Gamma$ in a rather weak sense, namely whose principal symbols vanish on $\Gamma$.

1. Introduction

The purpose of this paper is to obtain lossless, relative to non-trapping, albeit weaker, in terms of function spaces, semiclassical estimates for pseudodifferential operators $P_h(z)$ with normally hyperbolic trapping for $z$ real. Thus, the main result is an estimate of the form

$$\|u\|_{\mathcal{H}_{h,\Gamma}} \leq C h^{-1} \|P_h(z)u\|_{\mathcal{H}_{h,\Gamma}^*},$$

with certain function spaces $\mathcal{H}_{h,\Gamma}$ and $\mathcal{H}_{h,\Gamma}^*,$ described below; away from the trapped set these are just standard $L^2$ spaces. As the main application of such estimates is in so-called $b$-spaces, e.g. Kerr-de Sitter spaces, for which the estimates follow from the semiclassical ones immediately in the presence of dilation invariance, we also prove their counterpart in the general, non-dilation-invariant, $b$-setting.

So at first we consider a family $P_h(z)$ of semiclassical pseudodifferential operators $P_h(z) \in \Psi_h(X)$ on a closed manifold $X$, depending smoothly on the parameter $z \in \mathbb{C}$, with normally hyperbolic trapping, with trapped set $\Gamma$, and assume that $P_h(z)$ is formally self-adjoint for $z \in \mathbb{R}$. Wunsch and Zworski [13] studied this setting, imposing other global assumptions, the most important one being adding complex absorption $W$ in such a way that all bicharacteristics outside $\Gamma$, in both the forward and backward directions, either enter the elliptic set of $W$ in finite time or tend to $\Gamma$, and in at least one of the two directions they tend to the elliptic set of $W$. The bicharacteristics tending to $\Gamma$ in the forward/backward directions are forward/backward trapped; denote by $\Gamma_-$, resp. $\Gamma_+$ the forward, resp. backward trapped set,\(^1\) and assume that these are smooth codimension one submanifolds of the semiclassical characteristic set $\Sigma_{h,z}$, intersecting transversally.

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\(^1\)In the notation of Wunsch and Zworski [13], $\Gamma_{\pm}$ are the backward/forward trapped sets for all (not necessarily null) bicharacteristics near the, say, zero level set of the semiclassical principal symbol $p_{h,z}$, $\Gamma_{\pm}^{\lambda}$ are the corresponding sets within the $\lambda$-level set of $p_{h,z}$. 

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In this normally hyperbolic setting Wunsch and Zworski [13] have shown polynomial semiclassical resolvent estimates

$$\|u\| \leq Ch^{-N}\|P_h(z)u\|, \ 0 < h < h_0,$$

(1.1)

in small strips $|\text{Im } z| \leq ch$, $c > 0$ sufficiently small, $N > 1$, and indeed for $z$ real, the loss is merely logarithmic, i.e. one has

$$\|u\| \leq Ch^{-1}(\log h^{-1})\|P_h(z)u\|, \ 0 < h < h_0,$$

(1.2)

where $\|\cdot\|$ is the $L^2$-norm. Dyatlov’s work [5] has since then improved the estimates in $\text{Im } z < 0$ (for instance, by making $c, N$ explicit).

We are concerned with improved estimates (for $z$ almost real) if one localizes $u$ and $P_h(z)u$ away from the trapping $\Gamma$ in a rather weak sense, such as by applying pseudodifferential operators with symbols vanishing at $\Gamma$. To place this into context, recall that Datchev and Vasy [3, 4] have shown that under our assumptions, with $\text{Im } z = O(h^\infty)$, if $A, B \in \Psi_h(X)$ with $\text{WF}_h(A) \cap \Gamma = \text{WF}_h(B) \cap \Gamma = \emptyset$, $B$ elliptic on $\text{WF}_h(A)$, then for all $M$ there is $N$ such that

$$\|Au\| \leq Ch^{-1}\|BP_h(z)u\| + C'h^M\|u\| + C''h^{-N}\|(\text{Id} - B)P_h(z)u\|.$$  

(1.3)

Thus, if $P_h(z)u$ is $O(h^{N-1})$ at $\Gamma$ (corresponding to the $\text{Id} - B$ term in the estimate), then on the elliptic set of $A$, hence off $\Gamma$ by appropriate choice of $A$, $u$ satisfies non-trapping semiclassical estimates:

$$\|AP_h(z)^{-1}Av\| \leq Ch^{-1}\|v\|,$$

with $A$ as above (take $B$ as above with $\text{WF}_h'(\text{Id} - B) \cap \text{WF}_h'(A) = \emptyset$). Here the $O(h^\infty)$ bound on $\text{Im } z$ arises from the a priori estimate, (1.1), and if $1 < N < 3/2$, e.g. as is on, and sufficiently near, the real axis, then one can take $\text{Im } z = O(h^{1+2N})$. The purpose of this paper is to improve this result by relaxing the conditions on $\text{WF}_h'(A)$ and $\text{WF}_h'(B)$ in (1.3).

The main point of the theorem below is thus that its estimate degenerates only at, as opposed to near, $\Gamma$. The proof given here is closely related to the proof of Wunsch and Zworski [13, Section 4], but being suboptimal in terms of the $L^2$-estimate, even though it is optimal (non-trapping) when a pseudodifferential operator with vanishing principal symbol at $\Gamma$ is applied from both sides, it can take place in a significantly simpler, standard, semiclassical pseudodifferential algebra. To set this up, let $Q_\pm \in \Psi_h^\infty(X)$ be self-adjoint and have symbols which are defining functions of $\Sigma_{h,z}$ near $\Gamma$, within the characteristic set $\Sigma_{h,z}$, say on a neighborhood of $\Gamma$. Let $Q_0 \in \Psi^0_h(X)$ be a semiclassical operator with $\text{WF}_h'(Q_0) \cap \Gamma = \emptyset$ which is elliptic on $O^c$ (and thus on a neighborhood of $O^c$), with real principal symbol for convenience. One considers normally isotropic spaces at $\Gamma$, denoted $H_{h, \Gamma}$, with squared norms given by

$$\|u\|^2_{H_{h, \Gamma}} = \|Q_0 u\|^2 + \|Q_+ u\|^2 + \|Q_- u\|^2 + h\|u\|^2;$$

this is just the standard $L^2$-space microlocally away from $\Gamma$ as one of $Q_+$, $Q_-$ or $Q_0$ is elliptic there, and it does not depend on the choice of $Q_0$ as on $O \setminus \Gamma$ one of

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2In the latter case by the Phragmén-Lindelöf theorem.
$Q_+$ and $Q_-$ is elliptic at every point. The dual space relative to $L^2$ is then

$$\mathcal{H}_{h,\Gamma}^* = h^{1/2}L^2 + Q_+L^2 + Q_-L^2 + Q_0L^2$$

(which is $L^2$ as a space, but with this norm); $P_h(z)u$ will then be measured in $\mathcal{H}_{h,\Gamma}^*$.

**Theorem 1.1.** Let $P = P_h(z), Q_{\pm}$ be as above, $\Im z = \mathcal{O}(h^2)$. Then

$$\|Q_+u\| + \|Q_-u\| \leq C^{-1}(Pu\|_{\mathcal{H}_{h,\Gamma}}^* + C' h^{1/2}\|u\|, \quad (1.4)$$

and thus by (1.2),

$$\|u\|_{\mathcal{H}_{h,\Gamma}} \leq C h^{-1}\|Pu\|_{\mathcal{H}_{h,\Gamma}^*}. \quad (1.5)$$

In fact, we also obtain a direct proof of (1.5) without using (1.2) at the end of Section 2, as well as the aforementioned $b$-estimates in Section 3, see Theorem 3.2.

2. Semiclassical resolvent estimates on the real line

2.1. Notation and definitions. We will review some definitions of semiclassical analysis, partially in order to fix our notation. For a general reference, see Zwartinski [14].

Let $X$ be a compact $n$-dimensional manifold without boundary, and fix a smooth density on $X$.

- For $u \in L^2(X)$, denote by $\|u\|$ its $L^2(X)$ norm; moreover, denote by $\langle \cdot , \cdot \rangle$ the (sesquilinear) inner product on $L^2(X)$.
- A family of functions $u = (u_h)_{h \in (0,1)}$ on $X$ is polynomially bounded if $\|u\| \leq Ch^{-N}$ for some $N$. If $k \in \mathbb{R}$, we say that $u \in \mathcal{O}(h^k)$ if $\|u\| \leq C_k h^k$, and $u \in \mathcal{O}(h^\infty)$ if $\|u\| \leq C_N h^N$ for every $N$.
- For $a = (a_h)_{h \in (0,1)} \in \mathcal{C}^\infty(T^*X)$, we say $a \in h^k \mathcal{S}^m(T^*X)$ if $a$ satisfies

$$|\partial_\alpha^\alpha \partial_\beta^\beta a_h(z, \zeta) \| \leq C_{\alpha \beta} h^k \langle \zeta \rangle^{-|\beta|}$$

for all multiindices $\alpha, \beta$ and all $N \in \mathbb{N}$ in any coordinate chart, where the $z$ are coordinates in the base and $\zeta$ coordinates in the fiber. We define the **seminormal quantization** $\mathcal{O}_h(a)$ of $a$ by

$$\mathcal{O}_h(a)u(z) = (2\pi h)^{-n} \int e^{iz\zeta/h} a(z, \zeta) \hat{u}(\zeta/h) d\zeta$$

for $u \in \mathcal{C}_c^\infty(X)$ supported in a chart and for general $u \in \mathcal{C}_c^\infty(X)$ by using a partition of unity. We write $\mathcal{O}_h(a) \in h^k \mathcal{S}_h^m(X)$. The quantization depends on the choice of partition of unity, but the resulting class of operators does not, modulo operators that have Schwartz kernel in $h^\infty \mathcal{C}(X^2)$. We say that $a$ is a **symbol** of $\mathcal{O}_h(a)$. The equivalence class of $a$ in $h^k \mathcal{S}^m(T^*X)/h^{k-1} \mathcal{S}^{m-1}(T^*X)$ is invariantly defined and is called the principal symbol of $\mathcal{O}_h(a)$. All operators below except $Q_0 \in \mathcal{S}_h^m(X)$ will in fact have compact microsupport in the sense that they are quantizations of symbols $a \in h^k \mathcal{S}^m(T^*X)$ satisfying in addition for all $N$

$$|\partial_\alpha^\alpha \partial_\beta^\beta a_h(z, \zeta) \| \leq C_N h^N \langle \zeta \rangle^{-N}$$

for $\zeta$ outside of a compact subset of $T^*X$. We denote the class of such symbols by $h^k \mathcal{S}(T^*X)$ and the corresponding class of operators by $h^k \mathcal{S}_h^m(X)$.

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3One really has $Q_+^*$ and $Q_0^*$ in this formula, but the reality of the principal symbols assures that one may replace them by $Q_\pm$ and $Q_0$ modulo $hL^2$. See [7, Appendix A] for a general discussion of the underlying functional analysis; also see Footnote 12.
If \( A, B \in \Psi_h(X) \), then \([A, B] \in h\Psi_h(X)\), and its principal symbol is \( \frac{i}{\hbar}H_ab \), where we define the Hamilton vector field in a coordinate chart by
\[
H_a = (\partial_a \zeta)\partial_z - (\partial_z \zeta)\partial_a.
\]

By a bicharacteristic of \( A \) we mean an integral curve of the Hamilton vector field of the principal symbol of \( A \). We denote the integral curve passing through the point \( \rho \in T^*X \) by \( \gamma_{\rho} \), i.e. \( \gamma_{\rho}(0) = \rho \) and \( \gamma'_{\rho}(s) = H_a(\gamma_{\rho}(s)) \). We shall also write \( \phi^s(\rho) := \gamma_{\rho}(s) \) for the bicharacteristic flow.

For a polynomially bounded family \((u_h)_{h \in (0,1)}\) and \( k \in \mathbb{R} \cup \{\infty\} \), we say that \( u = \mathcal{O}(h^k) \) at a point \( \rho \in T^*X \) if there exists \( a \in S(T^*X) \) with \( a(\rho) \neq 0 \) such that \( ||\text{Op}_h(a)u|| = \mathcal{O}(h^k) \). We define the semiclassical wave front set \( \text{WF}_{\text{sc}} \) of \( \rho \) through the point \( A \) as the complement of the set of all \( \rho \in T^*X \) at which \( u = \mathcal{O}(h^\infty) \).

The microsupport of \( A = \text{Op}_h(a) \in h^k\Psi_h(X) \), denoted \( \text{WF}_{\text{sc}}(A) \), is the complement of the set of all \( \rho \in T^*X \) so that \( |\partial^a u| = \mathcal{O}(h^\infty) \) near \( \rho \) for every multiindex \( a \), in any (and therefore in every) coordinate chart.

For \( A \in h^k\Psi_h(X) \) with principal symbol \( a \in h^kS(T^*X) \), we say that \( A \) is elliptic at \( \rho \in T^*X \) if there is a constant \( C > 0 \) such that \( |a(\rho')| \geq Ch^k \) for \( \rho' \) near \( \rho \) and \( h \) sufficiently small. For a subset \( E \subseteq T^*X \), we say that \( A \) is elliptic on \( E \) if \( A \) is elliptic at each point of \( E \). If \( A \in h^k\Psi_h(X) \) is elliptic on \( E \subseteq T^*X \) and \( Au = f \) with \( u, f \) polynomially bounded and \( f = \mathcal{O}(1) \) on \( E \), then microlocal elliptic regularity states that \( u = \mathcal{O}(h^{-k}) \) on \( E \).

Let \( P \in \Psi_h(X) \) be a semiclassical operator. Let \( U \subseteq X \) denote an open subset so that the cotangent bundle over \( U \) contains what will be the trapped set, and place complex absorbing potentials in a neighborhood of \( U^c \). We recall the notion of normal hyperbolicity from [13]:

Define the backward, resp. forward, trapped set \( \bar{\Gamma}_- \), resp. \( \bar{\Gamma}_+ \), by
\[
\bar{\Gamma}_\pm = \{ \rho \in T^*X : \gamma_{\rho}(s) \notin T^*_\varphi X \text{ for all } \varphi \geq 0 \}.
\]

Let \( \Gamma_\pm = \bar{\Gamma}_\pm \cap p^{-1}(\lambda) \) be the backward/forward trapped set within the energy surface \( p^{-1}(\lambda) \), and define the trapped set \( \Gamma_\lambda := \Gamma_+ \cap \Gamma_- \). In the context of Theorem 1.1, we will only work within the characteristic set of \( p \), hence with \( \Gamma_\pm := \Gamma_{\pm,0} \) and \( \Gamma := \Gamma_0 \). We say that \( P \) is normally hyperbolically trapping if:

1. There exists \( \delta > 0 \) such that \( dp \neq 0 \) on \( p^{-1}(\lambda) \) for \( |\lambda| < \delta \);
2. \( \bar{\Gamma}_\pm \cap p^{-1}(\delta, \delta) \) are smooth codimension one submanifolds intersecting transversally at \( \bar{\Gamma} \cap p^{-1}(\pm, \delta) \);
3. the flow is hyperbolic in the normal directions to \( \Gamma_\lambda \) within the energy surface: There exist subbundles \( E^\pm_\lambda \) of \( T\bar{\Gamma}_\lambda(\Gamma_{\pm,0}) \) such that \( T\bar{\Gamma}_\lambda \Gamma_{\pm,0} = TT_\lambda \oplus E^\pm_\lambda \), where \( d\phi^s : E^\pm_\lambda \rightarrow E^\pm_\lambda \), and there exists \( \theta > 0 \) such that for

\(^4\text{See [13, 11] for details; the point here is that the relevant part of our analysis takes place microlocally near the trapped set, and the complex absorbing potentials allow us to 'cut off' the bicharacteristic flow in a neighborhood of the trapped set.}\)
with a similar result for $Q$ and gives
$$\text{H}_p \phi_{\pm} = \mp c_{\pm}^2 \phi_{\pm}$$
with $c_{\pm} > 0$ near $\Gamma$, and with $^5$
$$\{\phi_{+}, \phi_{-}\} > 0$$

near $\Gamma$. This is the only relevant feature of normal hyperbolicity for this paper; thus these identities and estimates could be taken as its definition for our purposes.

By shrinking $O$ if necessary we may assume that this Poisson bracket as well as $c_{\pm}$ have positive lower bounds on $O$. Then notice that
$$\text{H}_p \phi_{\pm}^2 = -2c_{\pm}^2 \phi_{\pm}^2, \quad \text{H}_p \phi_{\mp}^2 = 2c_{\pm}^2 \phi_{\pm}^2.$$

As indicated in the introduction, we consider normally isotropic spaces at $\Gamma$, denoted $H_{h, \Gamma}$, with squared norms given by
$$\|u\|^2_{H_{h, \Gamma}} = \|Q_0 u\|^2 + \|Q_+ u\|^2 + \|Q_- u\|^2 + h\|u\|^2;$$
we can take $Q_{\pm}$ with principal symbol $\phi_{\pm}$, while $Q_0$ is elliptic on $O^c$ with real principal symbol. This is just the standard $L^2$-space microlocally away from $\Gamma$ as one of $Q_+$, $Q_-$ or $Q_0$ is elliptic there, and it does not depend on the choice of $Q_0$ as on $O \setminus \Gamma$ one of $Q_+$ and $Q_-$ is elliptic at every point. Notice that in fact
$$(Q_+ - iQ_-)^*(Q_+ - iQ_-) = Q_+^* Q_+ + Q_-^* Q_- - i[Q_+, Q_-]$$

and if $B \in \Psi_h(X)$ with $WF_h(B) \subset O$ then
$$h\|Bv\|^2 \leq C \text{Re}(i[Q_+, Q_-]Bv, Bv) + Ch \|v\|^2,$$
$C > 0$, in view of $\{\phi_{+}, \phi_{-}\} > 0$ on $O$, so
$$Q_+^* Q_+ + Q_-^* Q_- = \frac{1}{2}(Q_+^* Q_+ + Q_-^* Q_- + (Q_+ - iQ_-)^*(Q_+ - iQ_-) + i[Q_+, Q_-])$$
shows that, for $h > 0$ small, the norm on $H_{h, \Gamma}$ is equivalent to just the norm
$$\|u\|^2_{H_{h, \Gamma}} = \|Q_0 u\|^2 + \|Q_+ u\|^2 + \|Q_- u\|^2.$$

As mentioned in the introduction, the dual space relative to $L^2$ is then
$$H_{h, \Gamma}^* = h^{1/2} L^2 + Q_+ L^2 + Q_- L^2 + Q_0 L^2.$$

Then $\Psi_h(X)$ acts on $H_{h, \Gamma}$, and thus on $H_{h, \Gamma}^*$, for $B \in \Psi_h(X)$ preserves $h^{-1/2} L^2$ and gives
$$\|Q_+ Bu\| \leq \|BQ_+ u\| + \|Q_+, B\| u\| \leq C\|Q_+ u\| + h\|u\|_{L^2},$$
with a similar result for $Q_-$ and $Q_0$. We remark that the notation $H_{h, \Gamma}$ is justified as the space depends only on $\Gamma$, not on the particular defining functions $\phi_{\pm}$ as any

$^5$These defining functions exist globally when $\Gamma_{\pm}$ is orientable; but even if $\Gamma_{\pm}$ is not such, the square is globally defined. There is only a minor change required below if $\phi_{\pm}$ are not well defined; see Footnote 6.
other defining functions would change $Q_{\pm}$ by an elliptic factor modulo an element of $h\Psi_{h}(X)$, whose contribution to the squared norm can be absorbed into $Ch^{2}\|u\|_{L^2}^2$, and thus dropped altogether (for $h$ small) in view of the equivalence of the two norms discussed above.

We are now ready to prove Theorem 1.1. Note that this theorem in particular implies the main result of [3] in this setting, in that the estimates are of the same kind, except that in [3] $Pu$ is assumed to be microlocalized away from, and $u$ is estimated microlocally away from, $\Gamma$.

We also remark that the microlocal version of the two estimates of the theorem is that given any neighborhood $O'$ of $\Gamma$ with closure in $O$, there exist $B_{0} \in \Psi_{h}(X)$ elliptic at $\Gamma$, $B_{1}, B_{2} \in \Psi_{h}(X)$ with $WF_{h}(B_{2}) \cap \Gamma_{+} = \emptyset$, $WF_{h}(B_{j}) \subset O'$ for $j = 0, 1, 2$ such that

\[
\|B_{0}Q_{+}u\| + \|B_{0}Q_{-}u\| \leq h^{-1}\|B_{1}Pu\|_{H_{u}^{\epsilon,\Gamma}} + \|B_{2}u\|_{L^2} + C'h^{1/2}\|u\|_{L^2},
\]

respectively

\[
\|B_{0}u\|_{H_{u},\Gamma} \leq h^{-1}\|B_{1}Pu\|_{H_{u}^{\epsilon,\Gamma}} + \|B_{2}u\|_{L^2} + C'h\|u\|_{L^2};
\]

see (2.8). The theorem is proved by controlling the $B_{2}u$ term using the backward non-trapped nature of $\Gamma_{-} \setminus \Gamma$.

\textbf{Proof of Theorem 1.1.} We first prove (1.4), which proves (1.5) by (1.2). Then we modify the proof slightly to show (1.5) directly, and in particular prove a weaker version of the Wunsch-Zworski estimate (1.2), namely

\[
\|u\|_{L^2} \leq C'h^{-2}\|Pu\|_{L^2}.
\]

Let $\chi_{0}(t) = e^{-1/t}$ for $t > 0$, $\chi_{0}(t) = 0$ for $t \leq 0$, $\chi \in C_{c}^\infty([0, \infty))$ be identically 1 near 0 with $\chi' \leq 0$, and indeed with $\chi' = -\chi_{1}^{2}$, $\chi_{1} \geq 0$, $\chi_{1} \in C_{c}^\infty([0, \infty))$, and let $\psi \in C_{c}^\infty(\mathbb{R})$ be identically 1 near 0. Let

\[
a = \chi_{0}(\phi_{+}^{2} - \phi_{-}^{2} + \kappa)\chi(\phi_{+}^{2})\psi(p),
\]

$\kappa > 0$ small. Notice that on supp $a$, if $\chi$ is supported in $[0, R]$, $\phi_{+}^{2} \leq R$, $\phi_{-}^{2} \leq \phi_{+}^{2} + \kappa = R + \kappa$, so $a$ is localized near $\Gamma$ if $R$ and $\kappa$ are taken sufficiently small. Then

\[
\frac{1}{4}H_{p}(a^{2}) = \frac{1}{4} - \frac{c_{1}^{2}\phi_{+}^{2} + c_{2}^{2}\phi_{-}^{2}}{0}\chi_{0}(\phi_{+}^{2} - \phi_{-}^{2} + \kappa)\chi(\phi_{+}^{2})\psi(p)^{2} \chi_{0}(\phi_{+}^{2} - \phi_{-}^{2} + \kappa)\chi(\phi_{+}^{2})\psi(p)^{2}.
\]

Now $\chi'_{0} \geq 0$, so the two terms have opposite signs. Let\(^6\)

\[
a_{\pm} = \phi_{\pm}\sqrt{\chi_{0}\chi_{0}'}(\phi_{+}^{2} - \phi_{-}^{2} + \kappa)\chi(\phi_{+}^{2})\psi(p),
\]

and

\[
e_{-} = c_{+}\phi_{+}\chi_{1}(\phi_{+}^{2})\chi_{0}(\phi_{+}^{2} - \phi_{-}^{2} + \kappa)\psi(p);
\]

then

\[
\frac{1}{4}H_{p}(a^{2}) = -c_{1}^{2}a_{+}^{2} - c_{2}^{2}a_{-}^{2} + c_{-}^{2}.
\]

\(^{6}\)If $\phi_{\pm}$ is not defined globally, $a_{\pm}$ are not defined as stated. (The term $e_{-}^{2}$ need not have a sign, so this issue does not arise for it; see the Weyl quantization argument below.) However, $a_{\pm}$ need not be real below, so as long as one can choose $\psi_{\pm}$ complex valued with $|\psi_{\pm}|^{2} = \phi_{\pm}^{2}$, replacing the first factor of $\phi_{\pm}$ with $\psi_{\pm}$ in the definition of $a_{\pm}$ allows one to complete the argument in general.
Here

\[ \text{supp } e_- \subset \text{supp } a, \]
\[ \text{supp } e_- \cap \Gamma_+ = \emptyset, \]

with the last statement following from \( \phi_+^2 \) taking values away from 0 on \( \text{supp } \chi_1 \); see Figure 1.

\[ \text{Figure 1. Supports of the commutant } a \text{ and the error term } e_- \text{ in the positive commutator argument of the non-trapping estimate near the trapped set } \Gamma, \text{ Theorem 1.1. The support of } a \text{ is indicated in light gray; on } \text{supp } a \setminus \text{supp } e_- \text{, darker colors correspond to larger values of } a. \text{ Also shown are the forward, resp. backward, trapped set } \Gamma_-, \text{ resp. } \Gamma_+, \text{ and the bicharacteristic flow nearby. The figure already suggests that } H(p(a^2)) \text{ is non-positive away from } \text{supp } e_-, \text{ and actually negative away from } \text{supp } e_- \cup \Gamma; \text{ see equation (2.3)}. \]

One then takes \( A \in \Psi_h(X) \) with principal symbol \( a \), and with \( \text{WF}_h'(A) \subset \text{supp } a \), \( A_\pm \in \Psi_h(X) \) with principal symbols of \( a_\pm \), and with \( \text{WF}_h'(A_\pm) \subset \text{supp } a_\pm \), \( C_\pm \) have symbol \( c_\pm \) and with \( \text{WF}_h'(C_\pm) \subset \text{supp } c_\pm \); one similarly lets \( E_- \in \Psi_h(X) \) have principal symbol \( e_- \), and wave front set in the support of \( e_- \). This gives that

\[ \frac{i}{4h}[P, A^*A] = -(C_+ A_+)^*(C_+ A_+) - (C_- A_-)^*(C_- A_-) + E_-^* E_- + hF, \quad (2.4) \]

for some \( F \in \Psi_h(X) \) with \( \text{WF}_h'(F) \subset \text{supp } a \).

Thus

\[ \frac{i}{4h}\langle [P, A^*A]u, u \rangle = -\|C_+ A_+ u\|^2 - \|C_- A_- u\|^2 + \|E_- u\|^2 + h\langle Fu, u \rangle. \]

Expanding the left hand side gives

\[ \langle PA^*Au, u \rangle - \langle A^*APu, u \rangle \]
\[ = \langle Au, APu \rangle - \langle APu, Au \rangle + \langle (P - P^*)A^*Au, u \rangle. \]
As we are assuming that $P - P^*$ is $O(h^2)$ near $\Gamma$, we may also assume that this holds on $\text{supp} \alpha$, thus the last term is $O(h^2)\|u\|^2$. Thus,

$$\|C_+A_+u\|^2 + \|C_-A_-u\|^2 \leq \|E_-u\|^2 + h^{-1} \langle APu, Au \rangle + C_1 \hbar \|u\|^2. \quad (2.5)$$

Now, by the duality of $\mathcal{H}_{h,\Gamma}$ and $\mathcal{H}_{h,\Gamma}^*$ relative to the $L^2$ inner product,

$$\langle APu, Au \rangle \leq \|APu\|_{\mathcal{H}_{h,\Gamma}^*} \|Au\|_{\mathcal{H}_{h,\Gamma}} \leq \frac{\hbar}{2} \|Au\|_{\mathcal{H}_{h,\Gamma}}^2 + \frac{1}{2\hbar} \|APu\|_{\mathcal{H}_{h,\Gamma}^*}^2.$$

Further, for $\epsilon > 0$ small, $\epsilon\|Q_+Au\|^2$ can be estimated in terms of $\|C_+A_+u\|^2 + O(h)\|u\|^2$, as can be seen by comparing the principal symbols, in particular using the ellipticity of $C_+$ on $\text{supp} \alpha$. One can thus absorb $\frac{\hbar}{2} \|Au\|_{\mathcal{H}_{h,\Gamma}}^2$ into the left hand side of (2.5). This shows

$$\|C_+A_+u\|^2 + \|C_-A_-u\|^2 \leq C \|E_-u\|^2 + C h^{-2} \|APu\|_{\mathcal{H}_{h,\Gamma}^*}^2 + C h \|u\|^2,$$

which together with the non-trapping control of $E_-u$ (the region $\text{supp} e_-$ is disjoint from $\Gamma_+$, so it is backward non-trapped and thus $E_-u$ is controlled by $Pu$ microlocalized off $\Gamma_+$, thus by $Q_+Pu$, modulo higher order in $h$ terms in $Pu$) proves the first part of Theorem 1.1. Thus, if we have a bound $\|u\| \leq C'h^{-1-s}\|Pu\|_{L^2}$, $0 < s < 1/2$, and thus $\hbar \|u\|^2 \leq C'h^{-1-2s}\|Pu\|_{L^2}^2 \leq C'h^{-1-2s}\|Pu\|_{\mathcal{H}_{h,\Gamma}^*}^2$, this implies a non-trapping estimate:

$$\|u\|_{\mathcal{H}_{h,\Gamma}} \leq C'h^{-1}\|Pu\|_{\mathcal{H}_{h,\Gamma}^*}.$$

This completes the proof of Theorem 1.1.

In fact, as mentioned earlier, a slight change of point of view proves Theorem 1.1 directly. To see this, we use the Weyl quantization\(^7\) when choosing $a, a_\pm, c_\pm, e_-; \text{since we are on a manifold, this requires identifying functions with half-densities via trivialization of the half-density bundle by the Riemannian metric; this identification preserves self-adjointness.} \text{We also write } P_{h,\zeta} \text{ as the Weyl quantization of } p_0 + h p_1 \text{ with } p_0, p_1 \text{ real modulo } O(h^2). \text{Then the principal symbol calculation above holds with } p_0 \text{ in place of } p, \text{ and with } p_1 \text{ included it yields additional terms}

$$\frac{1}{4} H_p(a^2) = - (c_+^2 \phi_+^2 + c_-^2 \phi_-^2 - h \phi_+ H_p \phi_+ + h \phi_- H_p \phi_-) \times (\chi_0 \chi_0'(\phi_+^2 - \phi_-^2 + \kappa)\chi(\phi_+^2)^2 \psi(p)^2

- (c_+^2 \phi_+^2 - h \phi_+ H_p \phi_+) (\chi' \phi_0(\phi_+^2 - \phi_-^2 + \kappa)^2 \psi(p)^2).

Now, (2.4) becomes

$$\frac{i}{4h} [P, A^* A] = - (C_+ A_+)^*(C_+ A_+) - (C_- A_-)^*(C_- A_-)

+ h(A_+^* G_+ + G_+^* A_+ + A_-^* G_- + G_-^* A_-) + E + h^2 F, \quad (2.6)$$

with $G_\pm$ being the Weyl quantization of

$$g_\pm = \pm \frac{1}{2}(H_{p_\pm}(\phi_{\pm}) \sqrt{(\chi_0 \chi_0')(\phi_+^2 - \phi_-^2 + \kappa)}\chi(\phi_+^2) \psi(p),

\text{and with } F \in \Psi_h(X) \text{ with}

\text{WF}_h(F) \subset \text{supp} \alpha.$$

---

\(^7\)In fact, the Weyl quantization is irrelevant. It is straightforward to see that if $A \in \Psi_h(X)$ and if the principal symbol of $A$ is real then the real part of the subprincipal symbol is defined independently of choices. This is all that is needed for the argument below.
Correspondingly, (2.5) becomes
\[ \|C_+ A_+ u\|_2^2 + \|C_- A_- u\|_2^2 \leq |\langle Eu, u \rangle| + h^{-1}|\langle APu, Au \rangle| + 2h\|A_+ u\|_1 \|G_+ u\|_2 + 2h\|A_- u\|_1 \|G_- u\|_2 + C_1 h^2 \|u\|_2^2, \] 
\tag{2.7}

The terms with \( G_{\pm} \) on the right hand side can be estimated by
\[ \epsilon \|A_{\pm} u\|_1^2 + \epsilon^{-1} h^2 \|G_{+} u\|_1^2 + \epsilon \|A_{-} u\|_1^2 + \epsilon^{-1} h^2 \|G_{-} u\|_1^2, \]
and for \( \epsilon > 0 \) sufficiently small, the \( \|A_{\pm} u\|_1^2 \) terms can now be absorbed into the left hand side of (2.7). Proceeding as above yields
\[ \|C_+ A_+ u\|_2^2 + \|C_- A_- u\|_2^2 \leq C|\langle Eu, u \rangle| + Ch^{-2} \|APu\|_{H^1, \tau}^2 + Ch^2 \|u\|_2^2. \] 
\tag{2.8}

Together with the non-trapping for the \( E \) term this gives the global estimate
\[ \|u\|_{H^1, \tau}^2 \leq Ch^{-2} \|Pu\|_{H^1, \tau}^2 + Ch^2 \|u\|_2^2, \]
and now the last term on the right hand side can be absorbed in the left hand side for sufficiently small \( h \), giving the estimate (1.5).

3. Non-trapping estimates in non-dilation invariant settings

We now transfer Theorem 1.1 into the \( b \)-setting; the discussion in the previous section is essentially the dilation invariant special case of this,\(^8\) though in the \( b \)-setting there is additional localization near the boundary.

3.1. Notation and definitions. For a general reference for \( b \)-analysis, see Melrose [6].

Let \( M \) be an \( n \)-dimensional compact manifold with boundary \( X \).

- Let \( \mathcal{V}_b(M) \) be the Lie algebra of \( b \)-vector fields on \( M \), i.e. of vector fields on \( M \) which are tangent to \( X \). Elements of \( \mathcal{V}_b(M) \) are sections of a natural vector bundle on \( M \), namely the \( b \)-tangent bundle \( \mathcal{V}_b(M) \); in local coordinates \((\tau, x)\) near \( X \), the fibers of \( \mathcal{V}_b(M) \) are spanned by \( \tau \partial_\tau \) and \( \partial_x \). The fibers of the dual bundle \( \mathcal{V}_b^*(M) \), called \( b \)-cotangent bundle, are spanned by \( \frac{dx}{\tau} \) and \( dx \).

It is often convenient to consider the fiber compactification \( \mathcal{V}_b^*(M) \) of \( \mathcal{V}_b^*(M) \), where the fibers are replaced by their radial compactification. The new boundary of \( \mathcal{V}_b^*(M) \) at fiber infinity is the \( b \)-cosphere bundle \( \mathcal{V}_b^*(M) \); it still possesses the compactification of the ‘old’ boundary \( \mathcal{V}_b^*(X) \), see Figure 2. \( \mathcal{V}_b^*(M) \) is naturally the quotient of \( \mathcal{V}_b^*(M) \) by the \( \mathbb{R}^+ \)-action of dilation in the fibers of the cotangent bundle. Many sets that we will consider below are conic subsets of \( \mathcal{V}_b^*(M) \), and we will often view them as subsets of \( \mathcal{V}_b^*(M) \).

- For \( a \in \mathcal{V}_b^*(M) \), we say \( a \in \mathcal{V}_b^*(M) \) if \( a \) satisfies
  \[ |\partial_\tau^\alpha \partial_\zeta^\beta a(\tau, \zeta)| \leq C_{\alpha, \beta} (\zeta)^{m-|\beta|} \]
for all multiindices \( \alpha, \beta \) in any coordinate chart, where \( \tau \) are coordinates in the base and \( \zeta \) coordinates in the fiber; more precisely, in local coordinates \((\tau, x)\) near \( X \), we take \( \zeta = (\sigma, \xi) \), where we write \( b \)-covectors as
  \[ \sigma \frac{d\tau}{\tau} + \sum_j \xi_j dx_j. \]

\(^8\)See [11, Section 3.1] for a discussion of the relationship between \( b \)- and semiclassical analysis.
Define the bicharacteristics completely analogously to the semiclassical setting. We define bicharacteristics in a coordinate chart, by

\[ T M \sa \mathcal{D}'(\mathbb{R}^n) \rightarrow \mathbb{C} \]

where the \( \tau' \)-integral is over \([0, \infty)\), and \( \phi \in C_c^\infty((-1/2, 1/2)) \) is identically 1 near 0.\(^9\) For general \( u \), define \( \text{Op}(a)u \) using a partition of unity. We write \( \text{Op}(a) \in \Psi^m(M) \). We say that \( a \) is a symbol of \( \text{Op}(a) \). The equivalence class of \( a \) in \( \mathcal{S}(b T^*M)/\mathcal{S}^{m-1}(b T^*M) \) is invariantly defined on \( b T^*M \) and is called the principal symbol of \( \text{Op}(a) \). We will tacitly assume that all our operators have homogeneous principal symbols.

- If \( A \in \Psi^m(M) \) and \( B \in \Psi^m(M) \), then \( [A, B] \in \Psi^{m_1+m_2-1}(M) \), and its principal symbol is \( \frac{1}{2} \text{H}_a b \equiv \frac{1}{2} \{ a, b \} \), where the Hamilton vector field \( \text{H}_a \) of the principal symbol \( a \) of \( A \) is the extension of the Hamilton vector field from \( T^*M \) to \( b T^*M \), which is a homogeneous degree \( m - 1 \) vector field on \( b T^*M \) tangent to the boundary \( b T^*_X M \). In local coordinates \((\tau, x, \sigma, \xi)\) on \( b T^*M \) as above, this has the form

\[
\text{H}_a = (\partial_\sigma a)(\tau \partial_\tau) - (\tau \partial_\tau a) \partial_\sigma + \sum_j \left( (\partial_\xi_j a) \partial_{x_j} - (\partial_{x_j} a) \partial_\xi_j \right). \tag{3.1}
\]

- We define bicharacteristics completely analogously to the semiclassical setting.
- The microsupport \( \text{WF}_b(A) \subset b T^*M \setminus \partial \) of \( A = \text{Op}(a) \in \Psi^m(M) \) is the complement of the set of all \( \rho \in b T^*M \setminus \partial \) such that \( a \) is rapidly decaying in a conic neighborhood around \( \rho \). Note that \( \text{WF}_b(A) \) is conic, hence we will also view it as a subset of \( b S^*M \).
- Fix a b-density on \( M \), which is locally of the form \( a \left| \frac{d\tau}{\tau} \right| \), \( a > 0 \).
- Define the b-Sobolev space \( H_b^k(M) \) for \( k \in \mathbb{Z}_{\geq 0} \) by

\[
H_b^k(M) = \{ u \in L^2(M) : X_1 \cdots X_k u \in L^2(M), X_1, \ldots, X_j \in \mathcal{V}_b(M) \},
\]

and for general \( k \in \mathbb{R} \) by duality and interpolation. Moreover, define the weighted b-Sobolev spaces \( H_b^{s, \alpha}(M) := \tau^\alpha H_b^s(M) \) for \( s, \alpha \in \mathbb{R} \), where \( \tau \) is

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\(^9\)The cutoff \( \phi \) ensures that these operators lie in the ‘small b-calculus’ of Melrose, in particular that such quantizations act on weighted b-Sobolev spaces, defined below.
a boundary defining function, i.e. \( \tau = 0 \) at \( X \) and \( d\tau \neq 0 \) there. Every \( b \)-pseudodifferential operator \( A \in \Psi^m_b(M) \) defines a bounded map \( A: H^{s,\alpha}_b(M) \to H^{s-m,\alpha}_b(M), s, \alpha \in \mathbb{R}. \)

- For \( A \in \Psi^m_b(M) \) with principal symbol \( a \in S^m(bT^*M) \), we say that \( A \) is elliptic at \( \rho \in bT^*M \setminus o \) if there is a constant \( C > 0 \) such that \( |a(z,\zeta)| \geq C|\zeta|^m \) for \((z, \zeta)\) in a conic neighborhood of \( \rho \). The characteristic set \( \Sigma \) is the complement (in \( bT^*M \setminus o \)) of the set of all \( \rho \) at which \( A \) is elliptic.

- For \( u \in H^{\infty,\alpha}_b(M) \), define its \( H^{s,\alpha}_b \) wave front set \( \WF^s_b(u) \subset bT^*M \setminus o \) as the complement of the set of all \( \rho \in bT^*M \setminus o \) for which there exists \( a \in S^0(bT^*M) \) elliptic at \( \rho \) such that \( \Op(a)u \in H^{s,\alpha}_b(M) \). In particular, \( \WF^s_b(u) = \emptyset \) if and only if \( u \in H^{s,\alpha}_b(M) \).

- Microlocal elliptic regularity states that if \( Au = f \) with \( A \in \Psi^m_b(M), u, f \in H^{\infty,\alpha}_b(M), \rho \notin \WF^{s-m,\alpha}_b(f) \) and \( A \) is elliptic at \( \rho \), then \( \rho \notin \WF^{s,\alpha}_b(u) \).

- If \( A \in \Psi^m_b(M) \) has a principal symbol with non-positive imaginary part, \( u, f \in H^{\infty,\alpha}_b(M), Au = f, \) moreover \( \rho \notin \WF^{s,\alpha}_b(u) \) and \( \gamma_\rho([0,T]) \cap \WF^{s-m+1,\alpha}_b(f) = \emptyset \) for some \( T > 0 \), then the propagation of singularities states that \( \gamma_\rho(T) \notin \WF^{s,\alpha}_b(u) \).

3.2. Setup, statement and proof of the result. Suppose \( \mathcal{P} \in \Psi^m_b(M), \mathcal{P}-\mathcal{P}^* \in \Psi^{m-2}_b(M) \). Let \( p \) be the principal symbol of \( \mathcal{P} \), which is thus a homogeneous degree \( m \) function on \( bT^*M \setminus o \), which we assume to be real-valued. Let \( \hat{p} \) denote a homogeneous degree \(-1\) defining function of \( bS^*M \). Then the rescaled Hamilton vector field

\[
V = \hat{p}^{m-1}H_p
\]

is a \( C^\infty \) vector field on \( bT^*M \) away from the 0-section, and it is tangent to all boundary faces. The characteristic set \( \Sigma \) is the zero-set of the smooth function \( \hat{p}^mp \) in \( bS^*M \). We will, somewhat imprecisely, refer to the flow of \( V \) in \( \Sigma \subset bS^*M \) as the Hamilton, or (null)bicharacteristic flow; its integral curves, the (null)bicharacteristics, are reparameterizations of those of the Hamilton vector field \( H_p \), projected by the quotient map \( bT^*M \setminus o \to bS^*M \).

We first work microlocally near the trapped set, namely assume that

1. \( \Gamma \subset \Sigma \cap bS^*_X M \) is a smooth submanifold disjoint from the image of \( T^*X \setminus o \)
   (so \( \tau D_\tau \) is elliptic near \( \Gamma \)),
2. \( \Gamma_+ \) is a smooth submanifold of \( \Sigma \cap bS^*_X M \) in a neighborhood \( U_1 \) of \( \Gamma \),
3. \( \Gamma_- \) is a smooth submanifold of \( \Sigma \) transversal to \( \Sigma \cap bS^*_X M \) in \( U_1 \),
4. \( \Gamma_+ \) has codimension 2 in \( \Sigma \), \( \Gamma_- \) has codimension 1,
5. \( \Gamma_+ \) and \( \Gamma_- \) intersect transversally in \( \Sigma \) with \( \Gamma_+ \cap \Gamma_- = \Gamma \),
6. the rescaled Hamilton vector field \( V = \hat{p}^{m-1}H_p \) is tangent to both \( \Gamma_+ \) and \( \Gamma_- \), and thus to \( \Gamma \).

We assume that \( \Gamma_+ \) is backward trapped for the Hamilton flow (i.e. bicharacteristics in \( \Gamma_+ \) near \( \Gamma \) tend to \( \Gamma \) as the parameter goes to \( -\infty \)), i.e. is the unstable manifold of \( \Gamma \), while \( \Gamma_- \) is forward trapped, i.e. is the stable manifold of \( \Gamma \), see Figure 3; indeed, we assume a quantitative version of this. (There is a completely analogous statement if \( \Gamma_+ \) is forward trapped and \( \Gamma_- \) is backward trapped: replacing \( \mathcal{P} \) by \( -\mathcal{P} \) preserves all assumptions, but reverses the Hamilton flow.) To state this, let \( \phi_- \) be a defining function of \( \Gamma_- \), and let \( \phi_+ \in C^\infty(bS^*_M) \) be a defining function of \( \Gamma_+ \) in \( bS^*_X M \); thus \( \Gamma_+ \) is defined within \( bS^*M \) by \( \tau = 0, \phi_+ = 0 \). Notice
that $V$ being tangent to $\mathbb{S}_X^* M$ (due to (3.1)) implies that $V\tau$ is a multiple of $\tau$; we assume that, near $\Gamma$,

\[ V\tau = -c_0^2 \tau, \quad c_0 > 0; \]

this is consistent with the stability of $\Gamma_\pm$. By the tangency requirement, with

\[ \bar{p}_0 = \rho^m p_0, \]

\[ V\phi_- = c_-^2 \phi_- + \nu_- \bar{p}_0, \quad V\phi_+ = -c_+^2 \phi_+ + \beta_+ \tau + \nu_+ \bar{p}_0, \]

with $c_\pm > 0$ smooth near $\Gamma$, $\beta_+, \nu_\pm$ smooth near $\Gamma$ and

\[ \{\phi_+, \phi_-\} > 0 \] (3.4)

near $\Gamma$. However, if we merely assume the normal hyperbolicity within $\mathbb{S}_X^* M$ as in [13, Section 1.2], [13, Lemma 4.1], as corrected in [12], actually gives such defining functions $\phi_0^\pm$ within $\mathbb{S}_X^* M$ (i.e. letting $\tau = 0$); taking an arbitrary extension in case of $\phi_+^\pm$, and an extension which is a defining function in case of $\Gamma_-$, all the requirements above are satisfied. Let $U_0 \subset \overline{U_0} \subset U_1$ be a neighborhood of $\Gamma$ such that the Poisson bracket in (3.4) as well as $c_\pm$ have positive lower bounds.

There is an asymmetry between the roles of $\phi_\pm$ and $\tau$, and thus we consider the parabolic defining function

\[ \rho_+ = \phi_+^2 + M\tau \]
for $\Gamma_+, M > 0$, to be chosen. Then, near $\Gamma$,
\begin{align}
\hat{\rho}_+ &= V \rho_+ = -2c_+^2 \phi_+^2 + 2\beta_+ \phi_+ \tau + 2\nu_+ \phi_+ \hat{p}_0 - Mc_+^2 \tau \\
&= -2c_+^2 \phi_+^2 - (Mc_+^2 - 2\beta_+ \phi_+) \tau + 2\nu_+ \phi_+ \hat{p}_0 \\
&\leq -\tilde{c}_+^2 \rho_+ + 2\nu_+ \phi_+ \hat{p}_0, \quad \hat{c}_+ > 0,
\end{align}
if $M > 0$ is chosen sufficiently large, consistently with the forward trapped nature of $\Gamma_-$. (Here the term with $\hat{p}_0$ is considered harmless as one essentially restricts to the characteristic set, $\hat{p}_0 = 0$.) Also, note that one can use\textsuperscript{10} the reciprocal $\tilde{\rho} = |\sigma|^{-1}$ of the principal symbol $\sigma$ of $\tau D_\tau$ as the local defining function of $b^*S^s M$ as fiber-infinity in $b^*T^* M$ near $\Gamma$; then
\[ V \tilde{\rho} = \check{\alpha} \tilde{\rho} \sigma \]
for some $\check{\alpha}$ smooth in view of (3.1).

Similar to the normally isotropic spaces in the semiclassical setting, we introduce spaces which are \textit{normally isotropic} at $\Gamma$\textsuperscript{11}. Concretely, let $Q_\pm \in \Psi^0_0(M)$ have principal symbol $\phi_\pm$ as before, $\hat{\rho}_0 \in \Psi^0_0(M)$ have principal symbol $\hat{\rho}_0$ and let $Q_0 \in \Psi^0_0(M)$ be elliptic, with real principal symbol for convenience, on $U_0^\Gamma$ (and thus nearby). Define the (global) $b$-normally isotropic spaces at $\Gamma$ of order $s$, $\mathcal{H}_{b, \Gamma}^s$, by the norm
\[ \|u\|^2_{\mathcal{H}_{b, \Gamma}^s} = \|Q_0 u\|^2_{H_b^s} + \|Q_+ u\|^2_{H_b^s} + \|Q_- u\|^2_{H_b^s} + \|\tau^{1/2} u\|^2_{H_b^{1/2}} + \|\hat{P}_0 u\|^2_{H_b^{-s}} + \|u\|^2_{H_b^{-s+1/2}}, \]
and let $\mathcal{H}_{b, \Gamma}^{s, -s}$ be the dual space relative to $L^2$, which is thus\textsuperscript{12}
\[ Q_0 H_b^{-s} + Q_+ H_b^{-s} + Q_- H_b^{-s} + \tau^{1/2} H_b^{1/2} + \hat{P}_0 H_b^{-s} + H_b^{s+1/2}. \]
Note that microlocally away from $\Gamma$, $\mathcal{H}_{b, \Gamma}^s$ is just the standard $H_b^s$ space while $\mathcal{H}_{b, \Gamma}^{s, -s}$ is $H_b^{-s}$ since at least one of $Q_0, Q_\pm, \tau$ is elliptic. Moreover, $\Psi_0^0(M) \ni A : \mathcal{H}_{b, \Gamma}^s \to \mathcal{H}_{b, \Gamma}^{-s}$ is continuous since $[Q_+, A] \in \Psi^{-1}_0(M)$ etc.; the analogous statement also holds for the dual spaces. Further, the last term in (3.7) can be replaced by $\|u\|^2_{H_b^{1/2}}$ as $i[Q_+, Q_-] = B^* B + R$, $B \in \Psi^{-1/2}_0(M)$, $R \in \Psi^{-2}_0(M)$, using the same argument as in the semiclassical setting (however, it cannot be dropped altogether unlike in the semiclassical setting!).

\textbf{Remark 3.1.} The notation $\mathcal{H}_{b, \Gamma}^s(M)$ is justified for the space is independent of the particular defining functions $\phi_\pm$ chosen; near $\Gamma$ any other choice would replace $\phi_\pm$.

\textsuperscript{10}Indeed, in the semiclassical setting, after Mellin transforming this problem, $|\sigma|^{-1}$ plays the role of the semiclassical parameter $h$, which in that case \textit{commutes} with the operator.

\textsuperscript{11}Note that $b^*T^* M$ is not a symplectic manifold (in the natural way) since the symplectic form on $b^*T^* M$ does not extend smoothly to $b^*T^* M$. Thus, the word ‘normally isotropic’ is not completely justified; we use it since it reflects that in the analogous semiclassical setting, see [13], the set $\Gamma$ is symplectic, and the origin in the symplectic orthocomplement $(T_o \Gamma)^{\perp}$ of $T_o \Gamma$, which is also symplectic, is isotropic within $(T_o \Gamma)^{\perp}$.

\textsuperscript{12}We refer to [7, Appendix A] for a general discussion of the underlying functional analysis. In particular, Lemma A.3 there essentially gives the density of $C^\infty(M)$ in $\mathcal{H}_{b, \Gamma}^s(M)$: one can simply drop the subscript ‘$e$’ in the statement of that lemma to conclude that $H_{b, \Gamma}^\infty(M)$ (so in particular $H_{b, \Gamma}^{s, -s}(M)$) is dense in $\mathcal{H}_{b, \Gamma}^s(M)$, and then the density of $C^\infty(M)$ in $H_{b, \Gamma}^s(M)$ for any $s^*$ completes the argument. The completeness of $\mathcal{H}_{b, \Gamma}^s(M)$ follows from the continuity of $\Psi^0_0(M)$ on $H_{b}^{s, -s+1/2}(M)$.\]
by smooth non-degenerate linear combinations plus a multiple of $\tau$ and of $\hat{p}$, denote these by $\hat{Q}_\pm$, and thus the corresponding $\hat{Q}_\pm$ can be expressed as

$$B_+Q_++B_-Q_-+B_0\tau+\hat{B}\hat{p}+B_0Q_0+R, \quad B_\pm, B_0, \hat{B} \in \Psi^0_b(M), \; R \in \Psi^{-1}_b(M),$$

so the new norm can be controlled by the old norm, and conversely in view of the non-degeneracy.

Our result is then:

**Theorem 3.2.** With $\mathcal{P}, \mathcal{H}^s_{b, \Gamma}, \mathcal{H}^{s,s}_{b, \Gamma}$ as above, for any neighborhood $U$ of $\Gamma$ and for any $N$ there exist $B_0 \in \Psi^0_b(M)$ elliptic at $\Gamma$ and $B_1, B_2 \in \Psi^0_b(M)$ with $WF'_b(B_j) \subset U$, $j = 0, 1, 2$, $WF'_b(B_2) \cap \Gamma_+ = \emptyset$ and $C > 0$ such that

$$\|B_0u\|_{H^s_{b, \Gamma}} \leq \|B_1\mathcal{P}u\|_{H^{s,s-m+1}_{b, \Gamma}} + \|B_2u\|_{H^s_{b, \Gamma}} + C\|u\|_{H^{s,N}_{b, \Gamma}}, \quad (3.8)$$

i.e. if all the functions on the right hand side are in the indicated spaces: $B_1\mathcal{P}u \in \mathcal{H}^{s,s-m+1}_{b, \Gamma}$, etc., then $B_0u \in \mathcal{H}^s_{b, \Gamma}$, and the inequality holds.

The same conclusion also holds if we assume $WF'_b(B_2) \cap \Gamma_+ = \emptyset$ instead of $WF'_b(B_2) \cap \Gamma_+ = \emptyset$.

Finally, if $r < 0$, then, with $WF'_b(B_2) \cap \Gamma_+ = \emptyset$, (3.8) becomes

$$\|B_0u\|_{H^s_{b, \Gamma}} \leq \|B_1\mathcal{P}u\|_{H^{s-r-m+1}_{b, \Gamma}} + \|B_2u\|_{H^s_{b, \Gamma}} + C\|u\|_{H^{s,N-r}_{b, \Gamma}}, \quad (3.9)$$

while if $r > 0$, then, with $WF'_b(B_2) \cap \Gamma_+ = \emptyset$,

$$\|B_0u\|_{H^s_{b, \Gamma}} \leq \|B_1\mathcal{P}u\|_{H^{s-r-m+1}_{b, \Gamma}} + \|B_2u\|_{H^s_{b, \Gamma}} + C\|u\|_{H^{s,N-r}_{b, \Gamma}}, \quad (3.10)$$

**Remark 3.3.** Note that the weighted versions (3.9)-(3.10) use standard weighted $b$-Sobolev spaces; this corresponds to non-trapping semiclassical estimates if the subprincipal symbol has the correct, definite, sign at $\Gamma$.

**Proof.** We may assume that $U \subset U_0$ is disjoint from a neighborhood of $WF'_b(Q_0)$, and thus ignore $Q_0$ in the definition of $\mathcal{H}^s_{b, \Gamma}$ below.

We first prove that there exist $B_0, B_1, B_2$ as above and $B_3 \in \Psi^0_b(M)$ with $WF'_b(B_3) \subset U$ such that

$$\|B_0u\|_{H^s_{b, \Gamma}} \leq \|B_1\mathcal{P}u\|_{H^{s,s-m+1}_{b, \Gamma}} + \|B_2u\|_{H^s_{b, \Gamma}} + \|B_3u\|_{H^{s-1}_{b, \Gamma}} + C\|u\|_{H^{s-N}_{b, \Gamma}}. \quad (3.11)$$

An iterative argument will then prove the theorem.

The proof is a straightforward modification of the construction in the semiclassical setting above, replacing $\phi_+^2$ by $\phi_+^2 + M\tau$, $M > 0$ large, in accordance with (3.5).

We start by pointting out that for any $\hat{B}_0 \in \Psi^0_b(M)$ and any $\hat{B}_3 \in \Psi^0_b(M)$ elliptic on $WF'_b(\hat{B}_0)$, $\|\hat{P}_0\hat{B}_0u\|_{H^s_{b, \Gamma}} \leq C\|\hat{B}_0\mathcal{P}u\|_{H^{s-m}_{b, \Gamma}} + C'\|\hat{B}_3u\|_{H^{s-1}_{b, \Gamma}}$, by simply using that $\hat{P}_0$ is an elliptic multiple of $P$ modulo $\Psi^1_b(M)$. Since $\|\hat{B}_0\mathcal{P}u\|_{H^{s-m}_{b, \Gamma}} \leq C\|\hat{B}_0\mathcal{P}u\|_{H^{s-r}_{b, \Gamma}}$, the $\hat{P}_0$ contribution to $\|\hat{B}_0u\|_{H^s_{b, \Gamma}}$ in (3.11) is thus automatically controlled.

So let $\chi(t) = e^{-t/t}$ for $t > 0$, $\chi_0(t) = 0$ for $t \leq 0$, with $\gamma > 0$ (large) to be specified, $\chi \in C^\infty_c((0, \infty))$ be identically 1 near 0 with $\chi' \leq 0$, and indeed with $\chi' = -\lambda t^2$, $\chi_1 \geq 0$, $\chi_1 \in C^\infty_c((0, \infty))$, and let $\psi \in C^\infty_c(\mathbb{R})$ be identically 1 near
0. As we use the Weyl quantization, we write \( P \) as the Weyl quantization of \( p = p_0 + \hat{\rho}p_1 \), with \( \hat{\rho}p_1 \) of order \( m - 1 \). Let

\[
a = \hat{\rho}^{-s+(m-1)/2}\chi_0(\rho_+ - \phi^2_+ + \kappa)\chi(\rho_+)\psi(\hat{\rho}^m p),
\]

\( \kappa > 0 \) small. Notice that on \( \text{supp} a \), if \( \chi \) is supported in \([0, R]\),

\[
\rho_+ \leq R; \quad \phi^2_- \leq \rho_+ + \kappa = R + \kappa,
\]

so \( a \) is localized near \( \Gamma \) if \( R \) and \( \kappa \) are taken sufficiently small. In particular, the argument of \( \chi_0 \) is bounded above by \( R + \kappa \), so given any \( M_0 > 0 \) one can take \( F > 0 \) large so that

\[
\chi_0'\chi_0 - M_0\chi_0^2 = b'\chi_0\chi_0,
\]

with \( b \geq 1/2, C^\infty \), on the range of the argument of \( \chi_0 \).

In fact, we also need to regularize, namely introduce

\[
a_\epsilon = (1 + \epsilon\hat{\rho}^{-1})^{-2}a, \quad \epsilon \in [0, 1],
\]

which is a symbol of order \( s - (m - 1)/2 - 2 \) for \( \epsilon > 0 \), and is uniformly bounded in symbols of order \( s - (m - 1)/2 \) as \( \epsilon \) varies in \([0, 1]\). In order to avoid more cumbersome notation below, we ignore the regularizer and work directly with \( a \); since the regularizer gives the same kind of contributions to the commutator as the weight \( \hat{\rho}^{-s+(m-1)/2} \), these contributions can be dominated in exactly the same way.

Then, with \( p = p_0 + \hat{\rho}p_1 \) as above, \( W = \hat{\rho}^{-m-2}H_{\hat{\rho}p_1} \), which is a smooth vector field near \( \mathbb{H}S^*M \) as \( \hat{\rho}p_1 \) is order \( m - 1 \), noting \( W\hat{\rho} = \alpha_1\tau \hat{\rho} \) similarly to (3.6), and \( W\tau = \alpha_{0,1} \) the tangency of \( W \) to \( \tau = 0 \),

\[
\frac{1}{4}H_p(a^2) = -(-\hat{\rho}_+/2 + c_2^2\phi^- + \nu - \phi^-\tilde{\rho}_0 - \hat{\rho}\phi_+(W\phi_+) - \hat{\rho}\tilde{\rho}M_{\alpha_{0,1}}\tau + \hat{\rho}\phi_-(W\phi_-)) \\
\times \hat{\rho}^{-2s}(\chi_0\chi_0')(\rho_+ - \phi^2_+ + \kappa)\chi(\rho_+)\psi(\hat{\rho}^m p)^2 \\
+ \frac{1}{4}(-2s + m - 1)\hat{\rho}^{-2s}(\tilde{\alpha} + \hat{\rho}\tilde{\alpha}_1)\tau \chi_0(\rho_+ - \phi^2_+ + \kappa)\chi(\rho_+)\psi(\hat{\rho}^m p)^2 \\
+ \frac{1}{2}\hat{\rho}^{-2s}(\hat{\rho}_+ + \hat{\rho}W\rho_+)(\chi')(\rho_+)(\rho_+ - \phi^2_+ + \kappa)^2\chi(\rho_+)^2(\psi')^{2}(\hat{\rho}^m p)^2 \\
+ \frac{m}{2}(\hat{\rho} + \hat{\rho}\tilde{\alpha}_1)\hat{\rho}^{-2s}(\hat{\rho}^m p)\tau \chi_0(\rho_+ - \phi^2_+ + \kappa)\chi(\rho_+)^2(\psi')^{2}(\hat{\rho}^m p).
\]

(3.14)

A key point is that the second term on the right hand side, given by the weight \( \hat{\rho}^{-2s+m-1} \) being differentiated, can be absorbed into the first by making \( F > 0 \) large so that \( \hat{\rho}_+ \chi_0(\rho_+ - \phi^2_+ + \kappa) \) dominates

\[
| -2s + m - 1||\tilde{\alpha}|\tau \chi_0(\rho_+ - \phi^2_+ + \kappa)
\]

on \( \text{supp} a \), which can be arranged as \( | -2s + m - 1||\tilde{\alpha}|\tau \) is bounded by a sufficiently large multiple of \( \hat{\rho}_+ \) there. Thus,

\[
\frac{1}{4}H_p(a^2) = -c_2^2a_+^2 - c_2^2a_-^2 - a_0^2 + 2g_0a_0 + 2g_0a_0 + a_0 + \hat{\epsilon} + 2a_0jwp + 2a_0jwp
\]

(3.15)

with

\[
a_\pm = \hat{\rho}^{-s}a_\pm \sqrt{(\chi_0\chi_0')(\rho_+ - \phi^2_+ + \kappa)\chi(\rho_+)\psi(\hat{\rho}^m p)},
\]

---

\(^{13}\)Again, the Weyl quantization is irrelevant: If \( A \in \Psi^m_0(X) \) and the principal symbol of \( A \) is real then the real part of the subprincipal symbol is defined independently of choices, which suffices below.
\[
\begin{align*}
a_\theta &= \hat{\rho}^{-s} \tau^{1/2} \left( M(c_\rho^2/2) - \beta_+ \phi_+ - \hat{\rho} \delta_\theta \right)(\rho_0' \chi_0' (\rho_+ - \phi_-^2 + \kappa)
\end{align*}
\]

\[
- \frac{1}{4}(-2s + m - 1)(\alpha + \hat{\rho} \alpha_1)\chi_0 (\rho_+ - \phi_-^2 + \kappa)^2 \right)^{1/2} \chi(\rho_+) \psi(\hat{\rho} m p),
\]

\[
g_\pm = \pm \frac{1}{2} \hat{\rho}^{-s+1} (\langle W \phi_\pm \rangle - \nu_\pm \hat{\rho} m p) \left( \chi_0 (\rho_+ - \phi_-^2 + \kappa)^2 \right) \chi(\rho_+) \psi(\hat{\rho} m p),
\]

\[
e = -\frac{1}{2} \hat{\rho}^{-2s}(\rho_+ + \hat{\rho} W \rho_+) \chi_0 (\rho_+ - \phi_-^2 + \kappa)^2 \psi(\hat{\rho} m p)^2,
\]

\[
\hat{e} = \frac{m}{2} \hat{\rho}^{-2s}(\rho_+ p) (\hat{\rho} + \hat{\rho} \alpha_1) \tau \chi_0 (\rho_+ - \phi_-^2 + \kappa)^2 \chi(\rho_+) \psi(\hat{\rho} m p),
\]

\[
j_\pm = \pm \frac{1}{2} \rho_\pm \hat{\rho}^{-s+m} \left( \chi_0 (\rho_+ - \phi_-^2 + \kappa)^2 \right) \chi(\rho_+) \psi(\hat{\rho} m p),
\]

the square root in \(a_\theta\) is that of a non-negative quantity and is \(C^\infty\) for \(M\) large (so that \(\beta_+ \phi_+\) can be absorbed into \(M(c_\rho^2/2)\)) and \(F\) large (so that a small multiple of \(\chi_0'\) can be used to dominate \(\chi_0\), as discussed earlier, and

\[
\text{supp } e \subset \text{supp } a, \quad \text{supp } e \cap \Gamma_+ = \emptyset,
\]

\[
\text{supp } \hat{e} \subset \text{supp } a, \quad \text{supp } \hat{e} \cap \Sigma = \emptyset.
\]

This gives, with the various operators being Weyl quantizations of the corresponding lower case symbols,

\[
\frac{1}{4} [\mathcal{P}, A^* A] = - (C_+ A_+)^* (C_+ A_+) - (C_- A_-)^* (C_- A_-) - A_\theta \delta_\theta
\]

\[
+ \delta_0 G_+ + A^*_+ G_- + G^*_+ A_- + A^*_- G_-
\]

\[
+ E + \hat{E} + A^*_+ J_+ \mathcal{P} + \mathcal{P}^* J_+^* A_+ + A^*_- J_- \mathcal{P} + \mathcal{P}^* J_-^* A_- + F
\]

\[(3.16)\]

where now \(A \in \Psi^{-(m-1)/2}_b(M), A_\pm, A_\theta \in \Psi^{0}_b(M), G_\pm \in \Psi^{0}_b(M), E \in \Psi^{2}_b(M), \)

\(\hat{E} \in \Psi^{2}_b(M), J_\pm \in \Psi^{-(m-2)}_b(M), F \in \Psi^{2}_b(M)\) with \(WF'_b(F) \subset \text{supp } a\).

After this point the calculations repeat the semiclassical argument: First using \(\mathcal{P} - \mathcal{P}^* \in \Psi^{m-2}_b(M),\)

\[
\|C_+ A_+ u\|^2 + \|C_- A_- u\|^2 + \|A_\theta u\|^2
\]

\[
\leq \|\langle E u, u \rangle\| + \|\langle \hat{E} u, u \rangle\| + \|\langle A \mathcal{P} u, A u \rangle\| + 2 \|A_+ u\| \|G_+ u\| + 2 \|A_- u\| \|G_- u\|
\]

\[
+ 2 \|J_+ \mathcal{P} u, A_+ u\| + 2 \|J_- \mathcal{P} u, A_- u\| + C_1 \|\hat{F}_1 u\|^2_{H^{-1}_b} + C_1 \|u\|^2_{H^{-N}_b},
\]

\[(3.17)\]

where we took \(\hat{F}_1 \in \Psi^{0}_b(M)\) elliptic on \(WF'_b(F)\) and with \(WF'_b(\hat{F}_1)\) near \(\Gamma\). Noting that \(WF'_b(\hat{E}) \cap \Sigma = \emptyset,\) the elliptic estimates give

\[
\|\langle E u, u \rangle\| \leq C \|B_1 \mathcal{P} u\|^2_{H^{-m}_b} + C \|u\|^2_{H^{-N}_b}
\]

if \(B_1 \in \Psi^{0}_b(M)\) is elliptic on \(\text{supp } \hat{e}\). Let \(\Lambda \in \Psi^{(m-1)/2}_b(M)\) be elliptic with real principal symbol \(\lambda\), and let \(\Lambda^- \in \Psi^{-(m-1)/2}_b(M)\) be a parametrix for it so that \(\Lambda \Lambda^- - \text{Id} = R_0 \in \Psi^{-\infty}_b(M)\). Then

\[
\|\langle A \mathcal{P} u, A u \rangle\| \leq \|\langle \Lambda^- A \mathcal{P} u, \Lambda^* A u \rangle\| + \|\langle R_0 A \mathcal{P} u, A u \rangle\|
\]

\[
\leq \frac{1}{2} \|\langle \Lambda^- A \mathcal{P} u\|^2_{H^{-1}_b} + \frac{c}{2} \|\Lambda^* A u\|^2_{H^0_{b, r}} + C' \|u\|^2_{H^{-N}_b}
\]
As $\Lambda^* A \in \Psi^s_b(M)$, for sufficiently small $\epsilon > 0$, $\frac{\xi}{\epsilon} \Lambda^* Au\xi_{R,\epsilon}^2$ can be absorbed into $\|C_+ A u\|^2 + \|C_- A u\|^2 + \|A_0 u\|^2$ plus $\|\bar{B}_0 \bar{P}_0 u\|^2_{H^s_b}$, and as discussed above, the latter already has the control required for (3.11). On the other hand, taking $B_1 \in \Psi^0_b(M)$ elliptic on $WF^1_b(A)$, as $\Lambda^* A \in \Psi^s_{b-m+1}(M)$,

$$\|\Lambda^* A P u\|^2_{H^s_{b,\epsilon}} \leq C'' \|B_1 P u\|^2_{H^s_{b,\epsilon}} + C'' \|u\|^2_{H^{-N}_{b,\epsilon}}.$$  

Similarly, to deal with the $J_{\pm}$ terms on the right hand side of (3.17), one writes

$$|\langle J_{\pm} P u, A_{\pm} u \rangle| \leq \frac{1}{2 \epsilon} \left( \|B_1 P u\|^2_{H^s_{b,\epsilon}} + C'' \|u\|^2_{H^{-N}_{b,\epsilon}} \right) + \frac{\epsilon}{2} \|A_{\pm} u\|^2_{L^2}$$

$$\leq \frac{1}{2 \epsilon} \left( \|B_1 P u\|^2_{H^s_{b,\epsilon}} + C'' \|u\|^2_{H^{-N}_{b,\epsilon}} \right) + \frac{\epsilon}{2} \|A_{\pm} u\|^2_{L^2},$$

while the $G_{\pm}$ terms can be estimated by

$$\epsilon \|A_{\pm} u\|^2 + \epsilon^{-1} \|G_{\pm} u\|^2 + \epsilon \|A_{\pm} u\|^2 + \epsilon^{-1} \|G_{\pm} u\|^2,$$

and for $\epsilon > 0$ sufficiently small, the $\|A_{\pm} u\|^2$ terms in both cases can be absorbed into the left hand side of (3.17) while the $G_{\pm}$ into the error term. This gives, with $\tilde{F}_2$ having properties as $\tilde{F}_1$,

$$\|C_+ A u\|^2 + \|C_- A u\|^2 + \|A_0 u\|^2$$

$$\leq \|E u, \psi\| + C\|B_1 P u\|^2_{H^s_{b,\epsilon}} + C_2 \|\tilde{F}_2 u\|^2_{H^s_{b,\epsilon}} + C_2 \|u\|^2_{H^{-N}_{b,\epsilon}}.$$  

By the remark before the statement of the theorem, if $B_0 \in \Psi^0_b(M)$ is such that $\chi_0 (\rho_+ - \phi^2 + \kappa) \chi (\rho_+ \psi(p) > 0$ on $WF^1_b(B_0)$, $\|B_0 u\|^2_{H^s_{b,\epsilon}}$ can be added to the left hand side at the cost of changing the constant in front of $\|\tilde{F}_2 u\|^2_{H^s_{b,\epsilon}} + \|u\|^2_{H^{-N}_{b,\epsilon}}$ on the right hand side. Taking such $B_0 \in \Psi^0_b(M)$, and $B_1$ elliptic on $WF^1_b(A)$ as before, $B_2 \in \Psi^0_b(M)$ elliptic on $WF^1_b(E)$ but with $WF^1_b(B_2)$ disjoint from $\Gamma$, we conclude that

$$\|B_0 u\|^2_{H^s_{b,\epsilon}} \leq C\|B_1 P u\|^2_{H^s_{b,\epsilon}} + C\|B_2 u\|^2_{H^s_{b,\epsilon}} + C\|\tilde{F}_2 u\|^2_{H^s_{b,\epsilon}} + C\|u\|^2_{H^{-N}_{b,\epsilon}},$$

proving (3.11), up to redefining $B_1$ by multiplication by a positive constant. Recall that unless one makes sufficient a priori assumptions on the regularity of $u$, one actually needs to regularize, but as mentioned after (3.13), the regularizer is handled in exactly the same manner as the weight.

Now in general, with $\chi$ as before, but supported in $[0, 1]$ instead of $[0, \bar{R} + \epsilon]$, writing $\chi_R = \chi(./R)$, letting $\alpha = \alpha_{R,\kappa}$ to emphasize its dependence on these quantities, when $\bar{R}$ and $\kappa$ are decreased, $\text{supp} \alpha_{R,\kappa}$ also decreases in $\Sigma$ in the strong sense that $0 < \bar{R} < \bar{R}'$, $0 < \kappa < \kappa'$ implies that $\alpha_{R',\kappa'}$ is elliptic on $\text{supp} \alpha_{R,\kappa}$ within $\Sigma$, and indeed globally if the cutoff $\psi$ is suitably adjusted as well. Thus, if $u \in H^{-N}_{b,\epsilon}$, say, one uses first (3.11) with $s = -N + 1$, and with $B_3$ given by the proof above, so the $B_3 u$ term is a priori bounded, to conclude that $B_0 u \in H^s_{b,\epsilon}$, and the estimate holds, so in particular, $u$ is in $H^{-N+1/2}_{b,\epsilon}$ microlocally near $\Gamma$ (concretely, on the elliptic set of $B_0$). Now one decreases $\kappa$ and $\bar{R}$ by an arbitrarily small amount and

\footnote{The point being that $\Lambda^* C_+ C_+ A - \epsilon \Lambda^* AQ_+ Q_+ A^* A$ has principal symbol $c^2 a_{+}^2 - \epsilon a_+^2 \phi_+^2 \lambda^2$ which can be written as the square of a real symbol for $\epsilon > 0$ small in view of the main difference in vanishing factors in the two terms being that $\lambda_0^0$ is replaced by $\chi_0$ in $a$, and thus the corresponding operator can be expressed as $\tilde{C}^* \tilde{C}$ for suitable $\tilde{C}$, modulo an element of $\Psi^{2s-1}_{b,\epsilon}$, with the latter contributing to the $H^{s-1}_{b,\epsilon}$ error term on the right hand side of (3.11).}
applies (3.11) with \( s = -N + 3/2 \); the \( B_3 u \) term is now a priori bounded by the microlocal membership of \( u \) in \( H_b^{-N+1/2} \), and one concludes that \( B_0 u \in H_b^{-N+3/2} \), so in particular \( u \) is microlocally in \( H_b^{-N+1} \). Proceeding inductively, one deduces the first statement of the theorem, (3.8).

If one reverses the role of \( \Gamma_+ \) and \( \Gamma_- \) in the statement of the theorem, one simply reverses the roles of \( \rho_+ = \phi_+^2 + M \tau \) and \( \phi_- \) in the definition of \( a \) in (3.12). This reverses the signs of all terms on the right hand side of (3.14) whose sign mattered below, and thus the signs of the first three terms on the right hand side of (3.16), which then does not affect the rest of the argument.

In order to prove (3.9), one simply adds a factor \( \tau^{-2r} \) to the definition of \( a \) in (3.12). This adds a factor \( \tau^{-2r} \) to every term on the right hand side of (3.16), as well as an additional term

\[
\frac{r}{2} \tau^{-2r} \tilde{\rho}^{-2s} c_0^2 \chi_0 (\rho_+ - \phi_-^2 + \kappa)^2 \chi (\rho_+)^2 \psi (p)^2,
\]

which for \( r < 0 \) has the same sign as the terms whose sign was used above, and indeed can be written as the negative of a square. Thus (3.15) becomes

\[
\frac{1}{4} H_p(a^2) = -c_+^2 a_+^2 - c_-^2 a_-^2 - a_r^2 - a_r^2
+ 2g_+ a_+ + 2g_- a_- + e + \tilde{e} + 2j_+ a_+ + 2j_- a_-
\tag{3.18}
\]

with

\[
a_r = \sqrt{\frac{-r}{2}} \tau^{-r} \tilde{\rho}^{-s} c_0 \chi_0 (\rho_+ - \phi_-^2 + \kappa) \chi (\rho_+) \psi (p),
\]

and all other terms as above apart from the additional factor of \( \tau^{-r} \) in the definition of \( a_\pm \), etc. Since \( a_r \) is actually elliptic at \( \Gamma \) when \( r \neq 0 \), this proves the desired estimate (and one does not need to use the improved properties given by the Weyl calculus!).

When the role of \( \Gamma_+ \) and \( \Gamma_- \) is reversed, there is an overall sign change, and thus \( r > 0 \) gives the advantageous sign; the rest of the argument is unchanged. \( \square \)

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