Spectral properties of Ruelle transfer operators for regular Gibbs measures and decay of correlations for contact Anosov flows

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Abstract. In this work we study strong spectral properties of Ruelle transfer operators related to a large family of Gibbs measures for contact Anosov flows. The ultimate aim is to establish exponential decay of correlations for Hölder observables with respect to a very general class of Gibbs measures. The approach invented by Dolgopyat [D1] and further developed in [St2] is substantially refined here, allowing to deal with much more general situations than before, although we still restrict ourselves to the uniformly hyperbolic case. A rather general procedure is established which produces the desired estimates whenever the Gibbs measure admits a Pesin set with exponentially small tails, that is a Pesin set whose preimages along the flow have measures decaying exponentially fast. We call such Gibbs measures regular. Recent results in [GSt] prove existence of such Pesin sets for hyperbolic diffeomorphisms and flows for a large variety of Gibbs measures determined by Hölder continuous potentials. The strong spectral estimates for Ruelle operators and well-established techniques lead to exponential decay of correlations for Hölder continuous observables, as well as to some other consequences such as: (a) existence of a non-zero analytic continuation of the Ruelle zeta function with a pole at the entropy in a vertical strip containing the entropy in its interior; (b) a Prime Orbit Theorem with an exponentially small error.

1 Introduction and Results

1.1 Introduction

The study of statistical properties of dynamical systems has a long history and has been the subject of a considerable interest due to their applications in statistical mechanics and thermodynamics. Many physical systems poses some kind of ‘strong hyperbolicity’ and are known to have or expected to have strong mixing properties. For example in the 70’s, due to works by Sinai, Bowen and Ruelle, it was already known that for Anosov diffeomorphisms exponential decay of correlations takes place for Hölder continuous observables (see e.g. the survey article [ChY]). However the continuous case proved to be much more difficult and it took more than twenty years until the breakthrough work of Dolgopyat [D1], where he established exponential decay of correlations for Hölder continuous potentials in two major cases: (i) geodesic flows on compact surfaces of negative curvature (with respect to any Gibbs measure); (ii) transitive Anosov flows on compact Riemann manifolds with $C^1$ jointly non-integrable local stable and unstable foliations (with respect to the Sinai-Bowen-Ruelle measure). The work of Dolgopyat was preceded (and possibly partially inspired) by that of Chernov [Ch1] who proved sub-exponential decay of correlations for Anosov flows on 3D Riemann manifolds (with respect to the Sinai-Bowen-Ruelle measure).

Dolgopyat’s work was followed by a considerable activity to establish exponential and other types of decay of correlations for various kinds of systems, most of the results dealing with the measure determined by the Riemann volume. Without trying to provide a comprehensive review of the literature, a sample of important works in this area is the following:

(a) The so called functional-analytic approach initiated by the work of Blank, Keller and Liverani [BKL] which involves the so called Ruelle-Perron-Frobenius operators $L_t g = \frac{g \circ \phi^{-t}}{|(\det d\phi_t)| \circ \phi^{-t}}$, $t \in \mathbb{R}$, was further developed by various authors, notably Liverani, Baladi, Tsujii, Gouëzel, and many others ([BaG], [BaT], [CL1], [CL2], [T]) - see e.g. the lectures of Liverani [L3] for a nice exposition of the main ideas. Using this method and also (generally speaking) “Dolgopyat’s cancellation mechanism” from [D1], Liverani [L2] proved exponential decay of correlations for $C^4$ contact Anosov flows with respect to the measure determined by the Riemann volume. Some finer results were obtained later by Tsujii [T] (for $C^3$ contact Anosov flows).

A similar approach, however studying Ruelle-Perron-Frobenius operators acting on currents, was used by Giulietti, Liverani and Pollicott in [GLP] where they proved some remarkable results.
For example, they established that for $C^\infty$ Anosov flows the Ruelle zeta function is meromorphic in the whole complex plane. In [GLP] the authors derived also (amongst other things) exponential decay of correlations for contact Anosov flows with respect to the measure of maximal entropy (generated by the potential $F = 0$) under a bunching condition (which implies that the stable/unstable foliations are $\frac{2}{3}$-H"older).

Various other results on decay of correlations for uniformly hyperbolic systems have been established using different methods as well, see e.g. [Ch1], [D2], [D3], [St1], [N], [ABV], [FMT], [OW1], [Wi], and the references there.

(b) The ideas of Dolgopyat were used for example in [BaV] and [AGY] to prove exponential decay of correlations for systems with infinite Markov partitions (with respect to SRB measures). For such systems a general approach was invented by L.-S. Young [Y1], [Y2] who introduced the so called "Young towers". This approach was later used by many authors in a variety of papers dealing with decay of correlations for diffeomorphisms and flows (uniformly and non-uniformly hyperbolic) - see e.g. [M1], [M2], [AM], and the historical remarks and references there. See also [Sa] for a different approach.

(c) Decay of correlations for hyperbolic systems with singularities (e.g. billiards) have been studied for a very long time. The first results in this area deal with the corresponding discrete dynamical system, generated by the billiard ball map from boundary to boundary. To my knowledge, these results were: the subexponential decay of correlations established for a very large class of dispersing billiards by Bunimovich, Sinai and Chernov [BSC] and the exponential decay of correlations for some classes of dispersing billiards in the plane and on the two-dimensional torus established by Young [Y1] and Chernov [Ch2] as consequences of their more general arguments. See also [CTY] and the comments and references there. Later on various other results were established both for the discrete and continuous dynamical systems, all of them dealing with SRB measures. Notably, Melbourne [M1] proved super-polynomial decay of correlations for Lorentz billiard flows with finite horizon on a two-dimensional torus, while Chernov [Ch4] established stretched exponential decay of correlations for such flows. More recently Baladi and Liverani [Bal] proved exponential decay of correlations for piecewise hyperbolic contact flows on three-dimensional manifolds. Finally, using techniques, methods and ideas from [D1], [L2], [Bal], [ChM], [DZ1] and [DZ2], in a remarkable recent paper Baladi, Demers and Liverani [BaDL] established exponential decay of correlations for Sinai billiards with finite horizon on a two-dimensional torus. See also [BNST] and the historical remarks and references in [BaDL].

(d) During the last several years there has been a lot of activity in applying methods and tools that are usually seen in the analysis of PDE’s and scattering theory to dynamical problems such as the study of decay of correlations, dynamical zeta functions and the distribution of Ruelle-Pollicott resonances. For example, Nonnenmacher and Zworski [NZ] established exponential decay of correlations for a class of $C^\infty$ flows which includes the $C^\infty$ contact Anosov flows, while Dyatlov and Zworski [DyZ2] gave a proof using microlocal analysis of the meromorphic continuation of the Ruelle zeta function for $C^\infty$ Anosov flows. Various other interesting and deep results have been obtained in [DDyZ], [DyFC], [DyG], [DyZ1], [DyZ3], [Fa5], [FaT1], [FaT2], [JZ]. See also the comments and references in those papers. To my knowledge, all works in this area deal with the measure determined by the Riemann volume.

(e) What concerns extension and further development of the ideas of Dolgopyat [D1], we should mention here our papers [St1] - [St3]. Strictly speaking the first result on exponential decay of correlations for billiard flows was the one in [St1] for open billiard flows in the plane, however the system in this case is uniformly hyperbolic and admits a finite Markov family. Higher-dimensional open billiards, under an additional condition, were considered in [St3]. The results

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1In [St1] the so called triple intersection property of cylinders on unstable manifolds was introduced, and it was proved in the case of open billiard flows in the plane. As a consequence, it was established that for such flows
in [St2] are much more general - they deal with Axiom A flows on basic sets having Lipschitz stable and unstable foliation and satisfying a certain non-integrability condition. E.g., as shown in [St5], contact Axiom A flows on basic sets satisfying a pinching condition (similar to the $1/4$-pinching condition for geodesic flows on manifolds of negative curvature) always satisfy the assumptions in [St2], and therefore they have exponential decay of correlations with respect to any Gibbs measure generated by a Hölder potential. In the present work we generalise the results and the developments in [St2] by far.

Ruelle transfer operators with two complex parameters were studied in [PeS5] and results in the spirit of these in [St2] were established for the same kind of Axiom A flows on basic sets as the one considered in [St2]. It seems this was the first time transfer operators depending on two parameters have been considered. It should be mentioned that the transition from one to two parameters is non-trivial.

In this work, as a consequence of the main result, we derive exponential decay of correlations for $C^5$ contact Anosov flows on Riemann manifolds $M$ of any dimension and with respect to any regular Gibbs measure on $M$, i.e. a Gibbs measure admitting a Pesin set with exponentially small tails (see Sect. 1.2 for the precise definition of the latter). We should stress again the fact that we deal with a large variety of Gibbs measures, not just SRB measures. It appears that so far the only results of this kind has been that of Dolgopyat [D1] for geodesic flows on $C^5$ compact surfaces and the one in [St2] for Axiom A flows on basic sets (under some additional assumptions).

In [D1] Dolgopyat developed a certain technique involving estimates of Laplace transforms of correlations functions (following previous works of Pollicott [Po1] and Ruelle [R3]) that leads more or less automatically to exponential decay of correlations for Hölder continuous potentials, once certain strong spectral properties of Ruelle transfer operators have been established. Given an Anosov flow $\phi_t : M \to M$ on a Riemann manifold $M$, consider a Markov family consisting of rectangles $R_i = [U_i, S_i]$, where $U_i$ and $S_i$ are pieces of unstable/stable manifolds at some $z_i \in M$, the first return time function

$$\tau : R = \bigcup_{i=1}^{k_0} R_i \to [0, \infty)$$

and the standard Poincaré map $\mathcal{P} : R \to R$ (see Sect. 2 for details). The shift map

$$\sigma : U = \bigcup_{i=1}^{k_0} U_i \to U,$$

given by $\sigma = \pi(U) \circ \mathcal{P}$, where $\pi(U) : R \to U$ is the projection along the leaves of local stable manifolds, defines a dynamical system which is essentially isomorphic to an one-sided Markov shift. Given a bounded function $f \in B(U)$, one defines the Ruelle transfer operator $L_f : B(U) \to B(U)$ by

$$(L_f h)(x) = \sum_{\sigma(y) = x} e^{f(y)} h(y).$$

Assuming that $f$ is real-valued and Hölder continuous, let $P_f \in \mathbb{R}$ be such that the topological pressure of $f - P_f \tau$ with respect to $\sigma$ is zero (cf. e.g. [PP]). Dolgopyat proved (for the type of flows he considered in [D1]) that for small $|a|$ and large $|b|$ the spectral radius of the Ruelle operator

$$L_{f-(P_f+a+ib)\tau} : C^\alpha(U) \to C^\alpha(U)$$

acting on $\alpha$-Hölder continuous functions ($0 < \alpha \leq 1$) is uniformly bounded by a constant $\rho < 1$. All Gibbs measures generated by Hölder potentials had the so called Federer property, and then an appropriate modification of the approach in [D1] could be used. Later, Naud [N] used a similar procedure in the case of geodesic flows on convex co-compact hyperbolic surfaces.
More general results of this kind were proved in [St2] for mixing Axiom A flows on basic sets under some additional regularity assumptions, amongst them – Lipschitzness of the so local stable holonomy maps \( \alpha \) (see Sect. 2).

Our main result in this paper is that for contact Anosov flows on a compact Riemann manifolds \( M \) correlations for Hölder continuous observables decay exponentially fast with respect to any regular Gibbs measure on \( M \).

It was proved recently in [GSt] that Pesin sets with exponentially small tails exist for Gibbs measures for Axiom A flows (and diffeomorphism) satisfying a certain condition, called exponential large deviations for all Lyapunov exponents (see Sect. 3 below). In fact, under such a condition, Pesin sets with exponentially small tails exist for any continuous linear cocycle over a transitive subshift of finite type (see Theorem 1.7 in [GSt]; see also Sect. 3 below). And it turns out that in this generality, exponential large deviations for all exponents is a generic condition (see Theorem 1.5 in [GSt]).

The main results mentioned above are in fact consequences of a more general result. Given \( \theta \in (0,1) \), the metric \( D_\theta \) on \( U \) is defined by \( D_\theta(x,y) = 0 \) if \( x = y \), \( D_\theta(x,y) = 1 \) if \( x, y \) belong to different \( U_j \)'s and \( D_\theta(x,y) = \theta^N \) if \( P^j(x) \) and \( P^j(y) \) belong to the same rectangle \( R_j \) for all \( j = 0, 1, \ldots, N - 1 \), and \( N \) is the largest integer with this property. Denote by \( \mathcal{F}_\theta(U) \) the space of all functions \( h : U \to \mathbb{C} \) with Lipschitz constants

\[
|h|_\theta = \sup \left\{ \frac{|h(x) - h(y)|}{D_\theta(x,y)} : x \neq y, x, y \in U \right\} < \infty.
\]

The central Theorem 1.3 below says that for sufficiently large \( \theta \in (0,1) \) and any real-valued function \( f \in \mathcal{F}_\theta(U) \) the Ruelle transfer operators related to \( f \) are eventually contracting on \( \mathcal{F}_\theta(U) \). A similar result holds for Hölder continuous functions on \( U \) – see Corollary 1.4 below.

In the proof of the central Theorem 1.3 we generalise significantly the approach of Dolgopyat [D1] and its development in [St2]. The general framework described in Sects. 6, 7 and 8 below is indeed rather general and is expected to work in a variety of other situations (and possibly for some non-uniformly hyperbolic systems, as well). The assumption of fundamental importance is the existence of a Pesin set with exponentially small tails. Certain technical troubles, such as the lack of regularity of the local stable/unstable manifolds and related local stable/unstable holonomy maps\(^3\) are overcome here by using the fact that the flow is contact and by using Lyapunov exponents on an appropriately chosen Pesin set. The most significant part in overcoming these difficulties is Sect. 9 below dealing with non-integrability matters - it is technical, lengthy and non-trivial. However, as far as ideas are concerned, the most significant ideas in this work are those in Sect. 6 and 7 below. See Sect 1.3 for some more details.

It has been well known since Dolgopyat’s paper [D1] that strong spectral estimates for Ruelle transfer operators as the ones described in Theorem 1.3 lead to deep results concerning zeta functions and related topics which are difficult to obtain by other means. For example, such estimates were fundamental in [PoS1], where the statements in Theorem 1.2 was proved for geodesic flows on compact surfaces of negative curvature. For the same kind of flows, fine and very interesting asymptotic estimates for pairs of closed geodesics were established in [PoS3], again by using the strong spectral estimates in [D1]. For Anosov flows with \( C^1 \) jointly non-integrable horocycle foliations full asymptotic expansions for counting functions similar to \( \pi(\lambda) \) however with some homological constraints were obtained in [An] and [PoS2]. In [PeS2] Theorem 1.3 above was used to obtain results similar to these in [PoS3] about correlations for pairs of

\(^2\)In general these are only Hölder continuous – see [Ha1], [Ha2].

\(^3\)E.g. the local stable holonomy maps are defined by sliding along local stable manifolds – as we mentioned earlier, in general these are only Hölder continuous. In [D1] and [St2] these were assumed to be \( C^1 \) and Lipschitz, respectively. Since the definition of Ruelle operators itself involves sliding along local stable manifolds, it appears to be a significant technical problem to overcome the lack of regularity in general.
closed billiard trajectories for billiard flows in $\mathbb{R}^n \setminus K$, where $K$ is a finite disjoint union of strictly convex compact bodies with smooth boundaries satisfying the so called ‘no eclipse condition’ (and some additional conditions as well). For the same kind of models and using Theorem 1.3 again, a rather non-trivial result was established in [PeS1] about analytic continuation of the cut-off resolvent of the Dirichlet Laplacian in $\mathbb{R}^n \setminus K$, which appears to be the first of its kind in the field of quantum chaotic scattering. In [PeS3], using the spectral estimate in [St2] and under the assumptions there, a fine asymptotic was obtained for the number of closed trajectories in $M$ with primitive periods lying in exponentially shrinking intervals $(x - e^{-\delta x}, x + e^{-\delta x}), \delta > 0, x \rightarrow +\infty$.

In [PeS4] a sharp large deviation principle was established concerning intervals shrinking with sub-exponential speed for the Poincaré map related to a Markov family for an Axiom A flow on a basic set $Λ$ satisfying the assumptions in [St2]. Finally, the spectral estimates in [PeS5], which we mentioned above, were used to derive some interesting applications: (i) combining them with some arguments from [PoS5], we proved the so called Hannay-Ozorio de Almeida sum formula for a basic set $Λ$ satisfying the assumptions in [St2]. Finally, the spectral estimates in [PeS5], which we mentioned above, were used to derive some interesting applications: (ii) for Axiom A flows on basic sets $Λ$ satisfying the assumptions in [St2] and for any Hölder continuous function $F : Λ \rightarrow \mathbb{R}$ there exists $ε > 0$ such that the counting function

$$\pi_F(T) = \sum_{\lambda(\gamma) \leq T} e^{\lambda_F(\gamma)} , \quad \lambda_F(\gamma) = \int_0^{\lambda(\gamma)} F(\phi_t(x_\gamma))dt,$$

where $γ$ is a primitive period orbit of the flow, $\lambda(\gamma)$ is the least period of $γ$, and $x_\gamma \in γ$, has the asymptotic

$$\pi_F(T) = li(e^{Pr(F)T})(1 + O(e^{-cT})) , \quad T \rightarrow \infty,$$

where $li(x) := \int_2^x \frac{1}{\log y} dy \sim \frac{x}{\log x}, x \rightarrow +\infty$. This generalised a result of Pollicott [Po2] for geodesic flows on compact manifolds of negative $\frac{1}{4}$-pinched curvature.

### 1.2 Statement of results

Let $ϕ : M \rightarrow M$ be a $C^2$ contact Anosov flow on a $C^2$ compact Riemann manifold $M$. Let $φ = ϕ_1$ be the time-one map of the flow, and let $m$ be an $φ$-invariant probability measure on $M$. A compact subset $P$ of $M$ will be called a Pesin set with exponentially small tails with respect to $m$ if $P$ is a Pesin set with respect to $m$ and for every $δ > 0$ there exist $C > 0$ and $c > 0$ such that

$$m(\{x \in L : \# \{j : 0 \leq j \leq n - 1 \text{ and } ϕ^j(x) \notin P\} \geq δn\}) \leq Ce^{-cn},$$

for all $n \geq 1$. The measure $m$ will be called regular if it admits a Pesin set with exponentially small tails. See Sect. 3 for a sufficient condition for the existence of Pesin sets with exponentially small tails. As explained below this sufficient condition is ‘generic’ in a certain sense.

The main result in this work is the following.

**Theorem 1.1.** Let $ϕ : M \rightarrow M$ be a $C^5$ contact Anosov flow, let $F_0$ be a Hölder continuous function on $M$ and let $m$ be the Gibbs measure determined by $F_0$ on $M$. Assume that $m$ is regular. Then for every $α > 0$ there exist constants $C = C(α) > 0$ and $c = c(α) > 0$ such that

$$\left| \int_M A(x)B(ϕ_t(x))\ dm(x) - \left( \int_M A(x)\ dm(x) \right) \left( \int_M B(x)\ dm(x) \right) \right| \leq Ce^{-ct}\|A\|_α\|B\|_α$$

for any two functions $A, B \in C^α(M)$.

We obtain this as a consequence of Theorem 1.3 below and the procedure described in [D1].

As we mentioned earlier, it appears that so far the only results concerning exponential decay of correlations for general Gibbs potentials have been that of Dolgopyat [D1] for geodesic flows.
on compact surfaces and the one in [St2] for Axiom A flows on basic sets (under some additional assumptions).

Next, consider the Ruelle zeta function

$$\zeta(s) = \prod_{\gamma} (1 - e^{-s\ell(\gamma)})^{-1}, \quad s \in \mathbb{C},$$

where $\gamma$ runs over the set of primitive closed orbits of $\phi_t : M \to M$ and $\ell(\gamma)$ is the least period of $\gamma$. Denote by $h_T$ the topological entropy of $\phi_t$ on $M$.

Using Theorem 1.3 below and an argument of Pollicott and Sharp [PoS1], one derives the following.\footnote{Instead of using the norm $\| \cdot \| : \|_{1,b}$ as in [PoS1], in the present case one has to work with $\| \cdot \|_{\theta,b}$ for some $\theta \in (0,1)$, and then one has to use the so called Ruelle’s Lemma in the form proved in [W]. This is enough to prove the estimate (2.3) for $\zeta(s)$ in [PoS1], and from there the arguments are the same.}

**Theorem 1.2.** Let $\phi_t : M \to M$ be a $C^2$ contact Anosov flow on a $C^2$ compact Riemann manifold $M$. Assume\footnote{It is still not proven that every Gibbs measure related to a contact Anosov flow is regular.} that there exists a Pesin set with exponentially small tails with respect to the Sinai-Bowen-Ruelle measure\footnote{This is known to be true under some standard pinching conditions – see e.g. the comments at the end of Sect. 1 in [GS1]. However we expect that this condition should be satisfied in much more general circumstances.}. Then:

(a) The Ruelle zeta function $\zeta(s)$ of the flow $\phi_t : M \to M$ has an analytic and non-vanishing continuation in a half-plane $\text{Re}(s) > c_0$ for some $c_0 < h_T$ except for a simple pole at $s = h_T$.

(b) There exists $c \in (0, h_T)$ such that

$$\pi(\lambda) = \#\{\gamma : \ell(\gamma) \leq \lambda\} = \text{li}(e^{h_T - \lambda}) + O(e^{\lambda})$$

as $\lambda \to \infty$, where $\text{li}(x) = \int_2^x \frac{du}{\log u} \sim \frac{x}{\log x}$ as $x \to \infty$.

Parts (a) and (b) were first established by Pollicott and Sharp [PoS1] for geodesic flows on compact surfaces of negative curvature (using [D1]), and then similar results were proved in [St2] for mixing Axiom A flows on basic sets satisfying certain additional assumptions (as mentioned above). Recently, using different methods, it was proved in [GLP] that: (i) for volume preserving three dimensional Anosov flows (a) holds, and moreover, in the case of $C^\infty$ flows, the Ruelle zeta function $\zeta(s)$ is meromorphic in $\mathbb{C}$ and $\zeta(s) \neq 0$ for $\text{Re}(s) > 0$; (ii) (b) holds for geodesic flows on $\frac{1}{3}$-pinched compact Riemann manifolds of negative curvature. These were obtained as consequences of more general results in [GLP].
Define the norm $\| \cdot \|_{\theta,b}$ on $\mathcal{F}_\theta(\hat{U})$ by

$$\| h \|_{\theta,b} = \| h \|_0 + \frac{|h|_\theta}{|b|},$$

where $\| h \|_0 = \sup_{x \in \hat{U}} |h(x)|$.

Given a real-valued function $f \in \mathcal{F}_\theta(\hat{U})$, set $g = g_f = f - P_f \tau$, where $P_f \in \mathbb{R}$ is the unique number such that the topological pressure $P_{\tau}(g)$ of $g$ with respect to $\tau$ is zero (cf. [PP]).

We say that Ruelle transfer operators related to $f$ are eventually contracting on $\mathcal{F}_\theta(\hat{U})$ if there exist constants $0 < \rho < 1$, $a_0 > 0$, $b_0 \geq 1$, $T_0 \geq 1$ and $C > 0$ such that if $a, b \in \mathbb{R}$ satisfy $|a| \leq a_0$ and $|b| \geq b_0$, then

$$\| L_{f-(P_f+a+ib)_\tau}^m h \|_{\theta,b} \leq C \rho^m \| h \|_{\theta,b}$$

for any integer $m \geq T_0 \log |b|$ and any $h \in \mathcal{F}_\theta(\hat{U})$.

This condition implies that the spectral radius of $L_{f-(P_f+a+ib)_\tau}$ on $\mathcal{F}_\theta(\hat{U})$ does not exceed $\rho$. It is also easy to see that it implies the following: for every $\epsilon > 0$ there exist constants $0 < \rho < 1$, $a_0 > 0$, $b_0 \geq 1$ and $C > 0$ such that if $a, b \in \mathbb{R}$ satisfy $|a| \leq a_0$ and $|b| \geq b_0$, then

$$\| L_{f-(P_f+a+ib)_\tau}^m h \|_{\theta,b} \leq C \rho^m |b|^\epsilon \| h \|_{\theta,b}$$

for any integer $m \geq 0$ and any $h \in \mathcal{F}_\theta(\hat{U})$.

The central result in the following.

**Theorem 1.3.** Let $\phi_t : M \to M$ be a $C^2$ contact Anosov flow on a $C^2$ compact Riemann manifold $M$, let $\mathcal{R} = \{R_i\}_{i=1}^{k_0}$ be a (pseudo-) Markov partition for $\phi_t$ as above and let $\tau : U \to U$ be the corresponding shift map. There exists a constant $0 < \hat{\theta} < 1$ such that for any $\theta \in (0, 1)$ and any real-valued function $f \in \mathcal{F}_\theta(\hat{U})$ which is induced on $\hat{U}$ by a Hölder continuous function $F_0$ on $M$ so that the Gibbs measure $\nu_{F_0}$ is regular, the Ruelle transfer operators related to $f$ are eventually contracting on $\mathcal{F}_\theta(\hat{U})$.

Here $\hat{\theta}$ is the minimal number in $(0, 1)$ such that the first-return time function $\tau \in \mathcal{F}_\theta(\hat{U})$.

A similar result for Hölder continuous functions (with respect to the Riemann metric) looks a bit more complicated, since in general Ruelle transfer operators do not preserve any of the spaces $C^\alpha(\hat{U})$. However, they preserve a certain ‘filtration’ $\cup_{0 < \alpha \leq a_0} C^\alpha(\hat{U})$. Here $\alpha > 0$ and $C^\alpha(\hat{U})$ is the space of all $\alpha$-Hölder complex-valued functions on $\hat{U}$.

Define the norm $\| \cdot \|_{\alpha,b}$ on $C^\alpha(\hat{U})$ by $\| h \|_{\alpha,b} = \| h \|_0 + \frac{|h|_\alpha}{|b|}$.

**Corollary 1.4.** Under the assumptions of Theorem 1.3, there exists a constant $a_0 > 0$ such that for any real-valued function $f \in C^{a_0}(\hat{U})$ the Ruelle transfer operators related to $f$ are eventually contracting on $\cup_{0 < \alpha \leq a_0} C^\alpha(\hat{U})$. More precisely, there exist constants $\hat{\beta} \in (0, 1]$, $0 < \rho < 1$, $a_0 > 0$, $b_0 \geq 1$, $C > 0$ and $T_0 > 0$ such that if $a, b \in \mathbb{R}$ satisfy $|a| \leq a_0$ and $|b| \geq b_0$, then for every integer $m \geq T_0 \log |b|$ and every $\alpha \in (0, a_0]$ the operator

$$L_{f-(P_f+a+ib)_\tau}^m : C^\alpha(\hat{U}) \to C^{\alpha \cdot \hat{\beta}}(\hat{U})$$

is well-defined and

$$\| L_{f-(P_f+a+ib)_\tau}^m h \|_{\alpha \cdot \hat{\beta},b} \leq C \rho^m \| h \|_{\alpha,b}$$

for every $h \in C^\alpha(\hat{U})$.

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Which is the way we define eventual contraction of Ruelle transfer operators in [ST2], and it agrees with the way the main result in [D1] is stated.
The maximal constant \( \alpha_0 \in (0,1] \) that one can choose above (which is determined by the minimal \( \hat{\theta} \) one can choose in Theorem 1.3) is related to the regularity of the local stable/unstable foliations. Estimates for this constant can be derived from certain bunching condition concerning the rates of expansion/contraction of the flow along local unstable/stable manifolds (see \[Ha1, Ha2, PSW]\). In the proof of Corollary 1.4 in Sect. 8 below we give some rough estimate for \( \alpha_0 \).

The above was first proved by Dolgopyat (\[D1\]) in the case of geodesic flows on compact Riemann manifolds with \( C^1 \) jointly non-integrable local stable and unstable foliations. For such flows Dolgopyat proved that the conclusion of Corollary 1.4 with \( \alpha_0 = 1 \) holds for the Sinai-Bowen-Ruelle potential \( F_0 = \log \det(d\phi_s)|_{E^u} \). More general results were proved in \[S2\] for mixing Axiom A flows on basic sets (again for \( \alpha_0 = 1 \)) under some additional regularity assumptions. As mentioned earlier, the results apply e.g. to \( C^2 \) mixing Axiom A flows on basic sets satisfying a certain pinching condition (similar to the 1/4-pinching condition for geodesic flows on manifolds of negative curvature).

### 1.3 Plan of the work and comments on the proof of the central Theorem 1.3

Sects. 2 and 3 contain some basic definitions and facts from hyperbolic dynamics and Pesin’s theory of Lyapunov exponents, respectively. Unlike previous works on Ruelle transfer operators, here we make a heavy use of Pesin’s theory of Lyapunov exponents.

Let \( F_0 : M \to \mathbb{R} \) be a Hölder continuous functions and let \( m \) be the Gibbs measure determined by \( F_0 \). Given a pseudo-Markov family \( R = \{R_i\}_{i=1}^{k_0} \) for \( \phi_t \) (see Sect. 2 for details), let

\[
\tau : R = \bigcup_{i=1}^{k_0} R_i \to [0, \infty) \quad \text{and} \quad \mathcal{P} : R \to R
\]

be the corresponding first return map and the Poincaré map. The measure \( m \) induces a Gibbs measure \( \mu \) on \( R \) (with respect to the Poincaré map \( \mathcal{P} \)) for the function

\[
F(x) = \int_0^{\tau(x)} F_0(\phi_s(x)) \, ds, \quad x \in R.
\]

The function \( F \) is Hölder and, by Sinai’s Lemma, is cohomologous to a Hölder function \( f : R \to \mathbb{R} \) which is constant on stable leaves in rectangles \( R_i \) in \( R \). Thus, \( \mu \) coincides with the Gibbs measure on \( R \) determined by \( f \).

In Sect. 4 we state several lemmas concerning the non-integrability of the flow due to the preservation of a contact form. Lemma 4.1 is Liverani’s Lemma B.7 from \[L2\] – see below. The rest of Sect. 4 has to do with Lyapunov exponents and all statements involve a certain fixed Pesin set \( P_0 \) with exponentially small tails and fixed constants \( \hat{\epsilon}_0, \hat{\delta}_0 > 0 \). Given an integer \( m \), let \( \Xi_m \) be the set of those Lyapunov regular points \( x \in R \) such that \( \mathcal{P}^j(x) \notin P_0 \) ‘relatively frequently’ for \( 0 \leq j < m \), more precisely

\[
\# \{j : 0 \leq j \leq m - 1, \mathcal{P}^j(x) \notin P_0 \} \geq \hat{\delta}_0 m.
\]

By the choice of \( P_0 \), the sets \( \Xi_m \) have exponentially small measure: \( \mu(\Xi_m) \leq C e^{-cm} \) for some constants \( C, c > 0 \). Lemmas 4.2 - 4.4 deal with unstable cylinders \( C \) in \( R \) that have common points with \( P_0 \setminus \Xi_m \). For such cylinders \( C \) we can estimate their diameters, or rather the diameter of their projections \( \hat{C} \) to a true unstable manifold, by using powers of the smallest ‘unstable’ Lyapunov exponent \( \lambda_1 > 1 \). We also get an important estimate for the temporal distance function (see Sect. 2):

\[
|\Delta(x, y') - \Delta(x, y'')| \leq C \text{diam}(\hat{C})(d(y', y''))^\beta
\]

for some constants \( C, \beta > 0 \) independent of \( C \) and \( m \), whenever \( x, z \in C, y', y'' \in W^s(z) \). This is the contents of Lemma 4.2. Its proof is given in Sect. 9.
The non-integrability Lemma 4.3 plays a very important role in the proof of the main result. It has to do with an important feature in the general construction of Dolgopyat’s contraction operators, although here the situation is significantly more complicated than the one in [DI1] (and also the one in [St2]). Given a large integer $N \gg 1$ for cylinders $C$ of length $m$ with $C \cap P_0 \setminus \Xi_m \neq \emptyset$ we construct families of points $y_1, y_2$ in $P^N(V) \cap W^s_t(Z)$, where $Z \in C \cap P_0 \setminus \Xi_m$ and $V$ is a small neighbourhood of $Z$ in $W^s_t(Z)$ such that

$$c \operatorname{diam}(\hat{C}) \leq |\Delta(x, \pi y_1(z)) - \Delta(x, \pi y_2(z))|$$

for some constant $c > 0$ independent of the cylinder $C$ and its length $m$, for all ‘appropriately positioned’ points $x, z \in C$. To prove Lemma 4.3 we use Liverani’s Lemma 4.1 which says that there exist constants $C_0 > 0, \vartheta > 0$ and $\epsilon_0 > 0$ such that for any $z \in M$, any $u \in E^u(z)$ and $v \in E^s(z)$ with $\|u\|, \|v\| \leq \epsilon_0$ we have

$$|\Delta(\exp_u^v(u), \exp_z^v(v)) - d\omega_z(u, v)| \leq C_0 \left(\|u\|^2 \|v\|^\vartheta + \|u\|^\vartheta \|v\|^2\right),$$

(1.1)

where $d\omega_z$ is the symplectic form defined by the contact form on $M$. We want to use this when $z \in P_0$, $v \neq 0$ is fixed and $\|u\|$ is small. Then however the right-hand-side of (1.1) is only $O(\|u\|^\vartheta)$ which is not good enough. As Liverani suggests in Remark B.8 in [L2], one might be able to improve the estimate pushing the points $x = \exp_z^u(u)$ and $y = \exp_z^v(v)$ forwards or backwards along the flow. We go forwards roughly until $\|d\omega_t(z) \cdot u\| \geq \|d\omega_t(z) \cdot v\|$ for some $t > 0$. Moreover we are only interested in directions $u \in E_1^u(z)$, where $E_1^u$ is the sub-bundle of $E^u$ corresponding to the smallest ‘unstable’ Lyapunov exponent $\lambda_1 > 1$. The proof of Lemma 4.3 is given in Sect. 9. We mentioned already, it is rather technical, lengthy and non-trivial.

As in [St2], here we work a lot with cylinders defined by the Markov family. In [St2] we worked under a certain regularity assumption – the so called regular distortion along unstable manifolds. A range of examples of flows having this property was described in [St5]. However it seems unlikely that it holds for any (contact) Anosov flow. Lemma 4.4 states that unstable cylinders $C$ of length $m$ with $C \cap P_0 \neq \emptyset$ and $P^m(C) \cap P_0 \neq \emptyset$ have properties similar to the properties considered in [St2]. Its proof is given in Sect. 10. Although it is technical and takes quite a bit of space and effort, it is using ideas similar to these in [St5], so, in some sense, it cannot be regarded as something that requires a significant intellectual effort.

Sect. 5 contains the main application of Lemma 4.3, namely Lemma 5.4 which provides estimates from below of differences of temporal distances in a form convenient for the estimates of contraction operators in Sect. 6.

As we have mentioned already, in the proof of Theorem 1.3 we use the general framework of Dolgopyat’s method from [DI1] and its development in [St2]. As in [DI1], we deal with the normalized operators

$$L_{ab} = L_{f^{(a)} - 1} \operatorname{br},$$

where

$$f^{(a)}(u) = f(u) - (P_f + a)\tau(u) + \ln h_a(u) - \ln h_a(\sigma(u)) - \ln \lambda_a,$$

$\lambda_a > 0$ being the largest eigenvalue of $L_{f^{(a+1)}} \tau$, and $h_a$ a particular corresponding positive eigenfunction (see Sect. 5.1 for details). Their real parts $\mathcal{M}_a = L_{f^{(a)}}$ satisfy $\mathcal{M}_a 1 = 1$. Now instead of dealing with these operators on some $C^\alpha(U)$, we consider them on the space $\mathcal{F}_\theta(\hat{U})$ of $\mathcal{D}_q$-Lipschitz functions on $\hat{U}$. We choose $\theta \in (0, 1)$ so that $\tau \in \mathcal{F}_\theta(\hat{U})$. The main benefit in working with $\mathcal{D}_q$ is that the local stable holonomy maps are isometries in this metric, and every $h \in \mathcal{F}_\theta(\hat{U})$ can be considered as a function on $R$ which is constant on stable leaves. Such $h$ has

---

8Liverani says that the best one can hope for is to get $o(\|u\|)$ in the right-hand-side of (1.1) if $\vartheta > \sqrt{3} - 1$, and this may be so, however we are interested in particular directions $u$ and for these without any restrictions on $\vartheta$ we succeed to get a bit more.
the same ‘trace’ on each $W^n_R(x)$, so we have the freedom to choose whichever unstable leaf is more convenient to work on.

Another simple thing that helps to avoid the lack of regularity is to approximate the partition $\mathcal{R} = \{ R_i \}_{i=1}^{k_0}$ (which we call a pseudo-Markov partition below) by a true (at least according to the standard definition, see [B]) Markov partition $\{ \tilde{R}_i \}_{i=1}^{k_0}$, where each $\tilde{R}_i$ is contained in a submanifold $D_i$ of $M$ of codimension one. We can take $D_i$ so that $U_i \cup S_i \subset D_i$. The shift along the flow determines a bi-Hölder continuous bijection $\tilde{\Psi} : R \rightarrow \tilde{R} = \bigcup_{i=1}^{k_0} \tilde{R}_i$, and whenever we need to measure the ‘size’ of a cylinder $C$ lying in some $W^n_R(x)$ we use $\text{diam}(\tilde{\Psi}(C))$, instead of $\text{diam}(C)$. Since the Poincaré map $\mathcal{P} : \tilde{R} = \bigcup_{i=1}^{k_0} \tilde{R}_i \rightarrow \tilde{R}$ is essentially Lipschitz, estimates involving $\text{diam}(\tilde{\Psi}(C))$ are much nicer. Roughly speaking, whenever we deal with Ruelle operators, measures and Gibbsian properties of measures, we work on $\tilde{R}$, and whenever we have to estimate distances and diameters in the Riemannian metric we use projections $\tilde{\Psi}$ in $\tilde{R}$.

As in [DT], the main result would follow if we show that, given $f \in \mathcal{F}_\theta(\tilde{U})$, there exist constants $C > 0$, $\rho \in (0, 1)$, $a_0 > 0$, $b_0 \geq 1$ and an integer $N \geq 1$ such that for $a, b \in \mathbb{R}$ with $|a| \leq a_0$ and $|b| \geq b_0$ and any $h \in \mathcal{F}_\theta(\tilde{U})$ we have

$$\int_U |L^{mN}_{ab} h|^2 \, \nu \leq C \rho^m \| h \|_{\theta, b}$$

for every positive integer $m$. Here $\nu$ is the Gibbs measure on $U$ determined by $g = f - P_f \tau$, which is naturally related to the Gibbs ($\mathcal{P}$-invariant) measure $\mu$ on $R$.

In order to prove (1.2) in [DT], Dolgopyat constructs for any choice of $a$ and $b$, a family of contraction operators $\mathcal{N}_J$, and the proof of (1.2) goes as follows. Given $h$ with (in his case) $\|h\|_{1, b} \leq 1$, define $h^{(m)} = L^{mN}_{ab} h$, $h^{(0)} = 1$ and $H^{(m)} = \mathcal{N}_J h^{(m-1)}$ for an appropriately chosen sequence of contraction operators $\mathcal{N}_J$, so that $|h^{(m)}| \leq H^{(m)}$ for all $m$. In [DT] (and [St2]) the contraction operators indeed contract in the $L^2$ norm so that

$$\int_U (\mathcal{N}_J H)^2 \, \nu \leq \rho \int_U H^2 \, \nu$$

for some constant $\rho \in (0, 1)$ independent of $a$, $b$, $J$ and $H$. Thus,

$$\int_U |L_{ab}^{mN} h|^2 \, \nu = \int_U |h^{(m)}|^2 \, \nu \leq \int_U (H^{(m)})^2 \, \nu \leq \rho^m.$$  

In the present work our contraction operators do not satisfy (1.3). Moreover we cannot deal immediately with functions $f \in \mathcal{F}_\theta(\tilde{U})$; instead we fix a sufficiently small $\theta_1 \in (0, \theta)$ and assume initially $f \in \mathcal{F}_{\theta_1}(\tilde{U})$. The general case is dealt with using approximations.

Even with $\theta$ replaced by a very small $\theta_1$, we cannot prove an analogue of (1.2); instead we establish that for every $s > 0$ there exist integers $N \geq 1$, $k = k(N) \geq 1$ and a constant $C = C(N) > 0$ such that

$$\int_U |L^{kN \log |b|} h| \, \nu \leq \frac{C}{|b|^s} \| h \|_{\theta_1, b}$$

for all $h \in \mathcal{F}_{\theta_1}(U)$. From this it follows by a relatively standard procedure (see Sect. 8) that for any $h \in \mathcal{F}_{\theta_1}(U)$ we have

$$\| L^{kN \log |b|} h \|_{\theta_1, b} \leq \frac{C}{|b|^s} \| h \|_{\theta_1, b}$$

for some (possibly different) constants $N$, $k$ and $C > 0$. The estimate in Theorem 1.3 is derived from (1.6) using another ‘standard procedure’.

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9 In fact, sometimes it is more convenient to use the projections $\Psi_i : R_i \rightarrow \tilde{R}_i = \bigcup_{z \in \mathcal{S}} \tilde{U}_i(z)$, where $\tilde{U}_i(z)$ is the part of the true unstable manifold $W^n_{\tilde{R}_i}(z)$ corresponding to $W^n_{R_i}(z)$ via the shift along the flow. This is particularly convenient when using Liverani’s Lemma 4.1.
Here we come to the most central part of this work – the construction of the contraction operators and the study of their main properties in Sects. 6 and 7. Having fixed a Pesin set $P_0$ with exponentially small tales, for any sufficiently large value of the parameter $b$ that appears in $L_{ab} = L_{f^p(J)}$, we construct several objects that depend on $b$. First, we construct in a special way a ‘sufficiently representative’ compact subset $P_0'$ of $P_0$ so that $\mu(P_0 \setminus P_0') \leq \frac{c}{|b|}$ for some constants $C, c > 0$ independent of $b$ and set $K_0 = \pi(U)(P_0')$, where $\pi(U) : R \to U$ is the projection along stable leaves in $R$. Then we choose a special essentially disjoint family of unstable maximal cylinders $C_1, \ldots, C_{m_0}$ with $\text{diam}(C_m) \leq \frac{C}{|b|}$ for some constant $C_1 > 0$ such that their projections $C_m' = \pi(U)(C_m)$ cover $K_0$. Here, $\hat{C}_m$ is the projection of $C_m$ along the flow to a true unstable manifold.

Next, given a large integer $N$, independent of $b$, we use Lemma 5.4 (consequence of Lemma 4.3) and construct a family of pairs of local inverses of the map $\sigma^N$:

$$v_i^{(\ell)} = v_i^{(\ell)}(Z_m, \cdot) : C_m' = \pi(U)(C_m) \to U, \quad \ell = 1, \ldots, \ell_0, \quad i = 1, 2,$$

where $Z_m$ is fixed point in $C_m \cap P_0'$ and $\sigma^N(v_i^{(\ell)}(Z_m, x)) = x$ for $x \in C_m'$. Then define ‘contraction’ operators $\mathcal{N}_j(u, b)$ for a family of symbols $J$, small $|a|$ and large $|b|$, very much as in [St2]. However proving some kind of contraction properties of these operators is non-trivial. To achieve this in Sect. 6.2 we define a special metric $D$ on $\hat{U}$ (depending on $b$ and the cylinders $C_m$) as follows. For any $u, u' \in \hat{U}$, let $\ell(u, u') \geq 0$ be the length of the smallest cylinder $Y(u, u')$ in $\hat{U}$ containing $u$ and $u'$. Set $D(u, u') = 0$ if $u = u'$. If $u \neq u'$ and there exists $p \geq 0$ with $\sigma^p(Y(u, u')) \subset C_m'$, $\ell(u, u') \geq p$, for some $m = 1, \ldots, m_0$, take the maximal $p$ with this property and the corresponding $m$ and set

$$D(u, u') = \frac{D_\theta(u, u')}{\text{diam}_\theta(C_m)},$$

where $\text{diam}_\theta(C_m)$ is the diameter of $C_m$ with respect to the metric $D_\theta$. Finally, if no $p$ exists as in the previous sentence, set $D(u, u') = 1$. Then, choosing appropriately a large constant $E > 0$, let $K_E$ be the set of all functions $H \in \mathcal{F}_\theta(\hat{U})$ such that $H > 0$ on $\hat{U}$ and

$$\frac{|H(u) - H(u')|}{H(u')} \leq E D(u, u')$$

for all $u, u' \in \hat{U}$ for which there exists an integer $p \geq 0$ with $\sigma^p(Y(u, u')) \subset C_m$ for some $m \leq m_0$ and $\ell(u, u') \geq p$. It turns out that $\mathcal{N}_j(K_E) \subset K_E$ for any of the contraction operators $\mathcal{N}_j$ defined earlier, and this turns out to be very important for the estimates that follow.

Next, consider the following assumption for points $u, u' \in \hat{U}$ contained in some cylinder $C'_m$ ($1 \leq m \leq m_0$), an integer $p \geq 0$ and points $v, v' \in \hat{U}$:

$$u, u' \in C'_m, \sigma^p(v) = v_i^{(\ell)}(u), \sigma^p(v') = v_i^{(\ell)}(u'), \ell(v, v') \geq N, \quad (1.7)$$

for some $i = 1, 2$. Assume $f \in \mathcal{F}_\theta(\hat{U})$, and denote by $K_h$ the family of all pairs $(h, H)$ such that $h \in \mathcal{F}_\theta(\hat{U})$, $H \in K_E$, $|h| \leq H$ on $\hat{U}$, and for any $u, u' \in \hat{U}$ contained in a cylinder $C'_m$ for some $m = 1, \ldots, m_0$, any integer $p \geq 0$ and any points $v, v' \in \hat{U}$ satisfying (1.7) for some $i = 1, 2$ and $\ell = 1, \ldots, \ell_0$ we have

$$|h(v) - h(v')| \leq E |b| \theta_2^{p+N} H(v') \text{diam}(\hat{C}_m).$$

\[10\] We believe that the scope of applicability of the arguments developed in Sects. 6-7 is significantly wider than what is actually stated as results in this paper.

\[11\] A different notation is used in Sect. 6; in fact the whole construction is more complicated than what we say here.

\[12\] They can only have common points at their boundaries.
Here \( \theta_2 \in (\theta, 1) \) is an appropriately chosen constant sufficiently close to 1.

We then succeed to derive that some kind of cancellation occurs in the actions of the operators \( \mathcal{N}_f \). More precisely, we prove that for small \( \|a\| \) and large \( \|b\| \), defining the cylinders \( C_m \) and the metric \( D \) as above, for any \( (h,H) \in \mathcal{K}_b \) there exists a contraction operator \( \mathcal{N}_f \) such that \( (L_{ab}^\theta h, \mathcal{N}_f H) \in \mathcal{K}_b \).

Perhaps the cancellation we have just mentioned looks a bit tiny and achieved under very special conditions, however it turns out it is enough to prove (1.5) and then (1.6). This is done in Sect. 7. Theorem 1.3 is derived from these in Sect. 8 using some ‘standard procedures’. It seems some of the latter are difficult to find in the literature, so in Sect. 8 we included detailed arguments on how exactly these procedures are done.

Acknowledgements. Thanks are due to Boris Hasselblatt, Dima Dolgopyat, Sebastian Gouëzel, Carlangelo Liverani, Yakov Pesin, Vesselin Petkov, Mark Pollicott and Amie Wilkinson who provided me with various useful information and/or made valuable comments on this paper at various stages during the work on it. Special thanks are due to Sebastian Gouëzel who pointed out to some errors in the initial version of the paper.

2 Preliminaries

Throughout this paper \( M \) denotes a \( C^2 \) compact Riemann manifold, and \( \phi_t : M \to M \) a \( C^2 \) Anosov flow on \( M \). That is, there exist constants \( C > 0 \) and \( 0 < \lambda < 1 \) such that there exists a \( d\phi_t \)-invariant decomposition \( T_x M = E^0(x) \oplus E^s(x) \oplus E^u(x) \) of \( T_x M \) \((x \in M)\) into a direct sum of non-zero linear subspaces, where \( E^0(x) \) is the one-dimensional subspace determined by the direction of the flow at \( x \), \( \|d\phi_t(u)\| \leq C \lambda^t \|u\| \) for all \( u \in E^s(x) \) and \( t \geq 0 \), and \( \|d\phi_t(u)\| \leq C \lambda^{-t} \|u\| \) for all \( u \in E^u(x) \) and \( t \leq 0 \).

For \( x \in M \) and a sufficiently small \( \epsilon > 0 \) let

\[
W^s_\epsilon(x) = \{y \in M : d(\phi_t(x), \phi_t(y)) \leq \epsilon \text{ for all } t \geq 0, d(\phi_t(x), \phi_t(y)) \to_{t \to \infty} 0 \},
\]

\[
W^u_\epsilon(x) = \{y \in M : d(\phi_t(x), \phi_t(y)) \leq \epsilon \text{ for all } t \leq 0, d(\phi_t(x), \phi_t(y)) \to_{t \to \infty} 0 \}
\]

be the (strong) stable and unstable manifolds of size \( \epsilon \). Then \( E^u(x) = T_{x_0} W^u_\epsilon(x) \) and \( E^s(x) = T_{x_0} W^s_\epsilon(x) \). Given \( \delta > 0 \), set \( E^u(x; \delta) = \{u \in E^u(x) : \|u\| \leq \delta\} \); \( E^s(x; \delta) \) is defined similarly.

It follows from the hyperbolicity of the flow on \( M \) that if \( \epsilon_0 > 0 \) is sufficiently small, there exists \( \epsilon_1 > 0 \) such that if \( x, y \in M \) and \( d(x,y) < \epsilon_1 \), then \( W^s_\epsilon(x) \) and \( \phi_{[-\epsilon_0,\epsilon_0]}(W^u_{\epsilon_0}(y)) \) intersect at exactly one point \( [x,y] \) (cf. [KII]). That is, there exists a unique \( t \in [-\epsilon_0,\epsilon_0] \) such that \( \phi_t([x,y]) \in W^u_{\epsilon_0}(y) \).

Setting \( \Delta(x,y) = t \), defines the so called temporal distance function \( \Delta \) ([KII], [D1], [CHI], [L1]).

For \( x, y \in M \) with \( d(x,y) < \epsilon_1 \), define \( \pi_y(x) = [x,y] = W^s_\epsilon(x) \cap \phi_{[-\epsilon_0,\epsilon_0]}(W^u_{\epsilon_0}(y)) \). Thus, for a fixed \( y \in M \), \( \pi_y : W \to \phi_{[-\epsilon_0,\epsilon_0]}(W^u_{\epsilon_0}(y)) \) is the projection along local stable manifolds defined on a small open neighbourhood \( W \) of \( y \) in \( M \). Choosing \( \epsilon_1 \in (0,\epsilon_0) \) sufficiently small, the restriction \( \pi_y : \phi_{[-\epsilon_1,\epsilon_1]}(W^u_{\epsilon_1}(x)) \to \phi_{[-\epsilon_0,\epsilon_0]}(W^u_{\epsilon_0}(y)) \) is called a local stable holonomy map \( \pi_y \).

Combining such a map with a shift along the flow we get another local stable holonomy map \( \mathcal{H}^u_{\epsilon_1} : W^u_{\epsilon_1}(x) \to W^u_{\epsilon_0}(y) \). In a similar way one defines local holonomy maps along unstable laminations.

We will say that \( A \) is an admissible subset of \( W^u_{\epsilon_1}(z) \) if \( A \) coincides with the closure of its interior in \( W^u_{\epsilon_1}(z) \). Admissible subsets of \( W^u_{\epsilon_1}(z) \) are defined similarly.

Let \( D \) be a submanifold of \( M \) of codimension one such that \( \text{diam}(D) \leq \epsilon \) and \( D \) is transversal to the flow \( \phi_t \). Assuming that \( \epsilon > 0 \) is sufficiently small, the projection \( \text{pr}_D : \phi_{[-\epsilon,\epsilon]}(D) \to D \) along the flow is well-defined and smooth. Given \( x, y \in D \), set \( (x,y)_D = \text{pr}_D([x,y]) \). A subset

\footnote{In fact in [D1] and [L1] a different definition for \( \Delta \) is given, however in the important case (the only one considered below) when \( x \in W^u_\epsilon(z) \) and \( y \in W^s_\epsilon(z) \) for some \( z \in M \), these definitions coincide with the present one.}

\footnote{In a similar way one can define holonomy maps between any two sufficiently close local transversals to stable laminations; see e.g. [PSW].}
$\tilde{R}$ of $D$ is called a rectangle if $\langle x, y \rangle_D \in \tilde{R}$ for all $x, y \in \tilde{R}$. The rectangle $\tilde{R}$ is called proper if $\tilde{R}$ coincides with the closure of its interior in $D$. For any $x \in \tilde{R}$ define the stable and unstable leaves through $x$ in $\tilde{R}$ by $W^s_\tilde{R}(x) = \text{pr}_D(W^s_\epsilon(x) \cap \phi^{-\epsilon}(D)) \cap \tilde{R}$ and $W^u_\tilde{R}(x) = \text{pr}_D(W^u_\epsilon(x) \cap \phi^\epsilon(D)) \cap \tilde{R}$. For a subset $A$ of $D$ we will denote by $\text{Int}_D(A)$ the interior of $A$ in $D$.

Let $\tilde{R} = \{\tilde{R}_i\}_{i=1}^{k_0}$ be a family of proper rectangles, where each $\tilde{R}_i$ is contained in a submanifold $D_i$ of $M$ of codimension one. We may assume that each $\tilde{R}_i$ has the form

$$\tilde{R}_i = \langle U_i, S_i \rangle_{D_i} = \{(x, y)_{D_i} : x \in U_i, y \in S_i\},$$

where $U_i \subset W^u_\epsilon(z_i)$ and $S_i \subset W^s_\epsilon(z_i)$, respectively, for some $z_i \in M$. Moreover, we can take $D_i$ so that $U_i \cup S_i \subset D_i$. Set $\tilde{R} = \bigcup_{i=1}^{k_0} \tilde{R}_i$. We will denote by $\text{Int}(\tilde{R}_i)$ the interior of the set $\tilde{R}_i$ in the topology of the disk $D_i$. The family $\tilde{R}$ is called complete if there exists $\chi > 0$ such that for every $x \in M$, $\phi_\chi(x) \in \tilde{R}$ for some $t \in (0, \chi]$. The Poincaré map $\tilde{P} : \tilde{R} \to \tilde{R}$ related to a complete family $\tilde{R}$ is defined by $\tilde{P}(x) = \phi_{\tau(x)}(x) \in \tilde{R}$, where $\tau(x) > 0$ is the smallest positive time with $\phi_{\tau(x)}(x) \in \tilde{R}$. The function $\tau$ is called the first return time associated with $\tilde{R}$. A complete family $\tilde{R} = \{\tilde{R}_i\}_{i=1}^{k_0}$ of rectangles in $M$ is called a Markov family of size $\chi > 0$ for the flow $\phi_t$ if: (a) diam$(\tilde{R}_i) < \chi$ for all $i$; (b) for any $i \neq j$ and any $x \in \text{Int}(\tilde{R}_i) \cap \tilde{P}^{-1}(\text{Int}(\tilde{R}_j))$ we have $W^s_{\tilde{R}_i}(x) \subset \tilde{P}^{-1}(W^s_{\tilde{R}_j}(\tilde{P}(x)))$ and $\tilde{P}(W^u_{\tilde{R}_i}(x)) \supset W^u_{\tilde{R}_j}(\tilde{P}(x))$; (c) for any $i \neq j$ at least one of the sets $\tilde{R}_i \cap \phi_{[0, \chi]}(\tilde{R}_j)$ and $\tilde{R}_j \cap \phi_{[0, \chi]}(\tilde{R}_i)$ is empty.

The existence of a Markov family $\tilde{R}$ of an arbitrarily small size $\chi > 0$ for $\phi_t$ follows from the construction of Bowen [3].

Following [2] and [11], we will now slightly change the Markov family $\tilde{R}$ to a pseudo-Markov partition $\mathcal{R} = \{R_i\}_{i=1}^{k_0}$ of pseudo-rectangles $R_i = [U_i, S_i] = \{[x, y] : x \in U_i, y \in S_i\}$, where $U_i$ and $S_i$ are as above. Set $R = \bigcup_{i=1}^{k_0} R_i$. Notice that $\text{pr}_D(R_i) = \tilde{R}_i$ for all $i$. Given $\xi = [x, y] \in R$, set $W^s_R(\xi) = W^s_\epsilon(x) \cap U_i$ and $W^s_R(\xi) = W^s_\epsilon(y) \cap S_i$. The corresponding Poincaré map $\mathcal{P} : \mathcal{R} \to \mathcal{R}$ is defined by $\mathcal{P}(x) = \phi_{\tau(x)}(x) \in \mathcal{R}$, where $\tau(x) > 0$ is the smallest positive time with $\phi_{\tau(x)}(x) \in \mathcal{R}$. The function $\tau$ is the first return time associated with $\mathcal{R}$. The interior $\text{Int}(R_i)$ of a rectangle $R_i$ is defined by $\text{pr}_D(\text{Int}(R_i)) = \text{Int}(\tilde{R}_i)$. In a similar way one can define $\text{Int}^n(A)$ for a subset $A$ of some $W^s_{\tilde{R}_i}(x)$ and $\text{Int}^n(A)$ for a subset $A$ of some $W^s_{\tilde{R}_i}(x)$.

We may and will assume that the family $\mathcal{R} = \{R_i\}_{i=1}^{k_0}$ has the same properties as $\tilde{R}$, namely: (a’) diam$(R_i) < \chi$ for all $i$; (b’) for any $i \neq j$ and any $x \in \text{Int}(R_i) \cap \mathcal{P}^{-1}(\text{Int}(R_j))$ we have $\mathcal{P}(\text{Int}(W^s_{R_i}(x))) \subset \text{Int}^n(W^s_{\epsilon_i}(\mathcal{P}(x)))$ and $\mathcal{P}(\text{Int}(W^u_{R_i}(x))) \supset \text{Int}^n(W^u_{\epsilon_i}(\mathcal{P}(x)))$; (c’) for any $i \neq j$ at least one of the sets $R_i \cap \phi_{[0, \chi]}(R_j)$ and $R_j \cap \phi_{[0, \chi]}(R_i)$ is empty. Define the matrix $A = (A_{ij})_{i,j=1}^{k_0}$ by $A_{ij} = 1$ if $\mathcal{P}(\text{Int}(R_i)) \cap \text{Int}(R_j) \neq \emptyset$ and $A_{ij} = 0$ otherwise. According to [11] (see section 2 there), we may assume that $\mathcal{R}$ is chosen in such a way that $A_{M_0} > 0$ (all entries of the $M_0$-fold product of $A$ by itself are positive) for some integer $M_0 > 0$. In what follows we assume that the matrix $A$ has this property.

Notice that in general $\mathcal{P}$ and $\tau$ are only (essentially) Hölder continuous. However there is an obvious relationship between $\mathcal{P}$ and the (essentially) Lipschitz map $\tilde{P}$, and this will be used below.

From now on we will assume that $\tilde{R} = \{\tilde{R}_i\}_{i=1}^{k_0}$ is a fixed Markov family for $\phi_t$ of size $\chi < \epsilon_0/2 < 1$ and that $\mathcal{R} = \{R_i\}_{i=1}^{k_0}$ is the related pseudo-Markov family. Set

$$U = \bigcup_{i=1}^{k_0} U_i$$

and $\text{Int}^n(U) = \bigcup_{j=1}^{k_0} \text{Int}^n(U_j)$.

The shift map $\sigma : U \to U$ is given by $\sigma = \pi(U) \circ \mathcal{P}$, where $\pi(U) : R \to U$ is the projection along stable leaves. Notice that $\tau$ is constant on each stable leaf $W^s_{\tilde{R}_i}(x) = W^s_{\epsilon_i}(x) \cap R_i$. For any
integer \( m \geq 1 \) and any function \( h : U \to \mathbb{C} \) define \( h_m : U \to \mathbb{C} \) by

\[
h_m(u) = h(u) + h(\sigma(u)) + \ldots + h(\sigma^{m-1}(u)).
\]

Denote by \( \tilde{U} \) (or \( \tilde{R} \)) the core of \( U \) (resp. \( R \)), i.e. the set of those \( x \in U \) (resp. \( x \in R \)) such that \( \mathcal{P}_m(x) \in \text{Int}(R) = \bigcup_{i=1}^k \text{Int}(R_i) \) for all \( m \in \mathbb{Z} \). It is well-known (see [12]) that \( \tilde{U} \) is a residual subset of \( U \) (resp. \( R \)) and has full measure with respect to any Gibbs measure on \( U \) (resp. \( R \)). Clearly in general \( \tau \) is not continuous on \( U \), however \( \tau \) is essentially Hölder on \( U \), i.e. there exist constants \( L > 0 \) and \( \alpha > 0 \) such that \( |\tau(x) - \tau(y)| \leq L (d(x, y))^\alpha \) whenever \( x, y \in U_i \) and \( \sigma(x), \sigma(y) \in U_j \) for some \( i, j \). The same applies to \( \sigma : U \to U \). Throughout we will mainly work with the restrictions of \( \tau \) and \( \sigma \) to \( \tilde{U} \). Set \( U_i = U_i \cap \tilde{U} \). For any \( A \subset M \), let \( \tilde{A} \) be the set of all \( x \in A \) whose trajectories do not pass through boundary points of \( R \).

Let \( B(\tilde{U}) \) be the space of bounded functions \( g : \tilde{U} \to \mathbb{C} \) with its standard norm \( ||g||_0 = \sup_{x \in \tilde{U}} |g(x)| \). Given a function \( g \in B(\tilde{U}) \), the Ruelle transfer operator \( L_g : B(\tilde{U}) \to B(\tilde{U}) \) is defined by

\[
(L_g h)(u) = \sum_{\sigma(v) = u} e^{g(v)} h(v).
\]

Given \( \alpha > 0 \), let \( C^\alpha(\tilde{U}) \) denote the space of the essentially \( \alpha \)-Hölder continuous functions \( h : \tilde{U} \to \mathbb{C} \), i.e. such that there exists \( L \geq 0 \) with \( |h(x) - h(y)| \leq L (d(x, y))^\alpha \) for all \( i = 1, \ldots, k_0 \) and all \( x, y \in U_i \). The smallest \( L > 0 \) with this property is called the \( \alpha \)-Hölder exponent of \( h \) and is denoted \( ||h||_\alpha \). Set \( ||g||_\alpha = ||g||_0 + ||g||_\alpha \).

The hyperbolicity of the flow implies the existence of constants \( c_0 \in (0, 1] \) and \( \gamma_1 > \gamma > 1 \) such that

\[
c_0 \gamma^m d(x, y) \leq d(\tilde{P}_m(x)), \tilde{P}_m(y)) \leq \frac{\gamma^m}{c_0} d(x, y)
\]

for all \( x, y \in \tilde{R} \) such that \( \tilde{P}_j(x), \tilde{P}_j(y) \) belong to the same \( \tilde{R}_i \) for all \( j = 1, \ldots, m \).

Throughout this paper \( \alpha_1 \in (0, 1] \) will denote the largest constant such that \( \tau \in C^{\alpha_1}(\tilde{U}) \) and the local stable/unstable holonomy maps are uniformly \( \alpha_1 \)-Hölder. We will also need to fix a constant \( \tilde{\alpha}_1 \in (0, 1) \) (take e.g. the largest again) such that the projection \( \tilde{\Psi} : R \to \tilde{R} \) along stable leaves is \( \tilde{\alpha}_1 \)-Hölder.

3 Lyapunov exponents and Lyapunov regularity functions

Let \( M \) be a \( C^2 \) Riemann manifold, and let \( \phi_t \) be a \( C^2 \) Anosov flow on \( M \). Let \( F_0 \) be a Hölder continuous real-valued function on \( M \) and let \( m \) be the Gibbs measure generated by \( F_0 \) on \( M \). Then \( m(\mathcal{L}) = 1 \), where \( \mathcal{L} \) is the set of all Lyapunov regular points of \( \varphi = \phi_1 \) (see [PT] or section 2.1 in [BP]). There exists a subset \( \mathcal{L} \) of \( \mathcal{L} \) with \( m(\mathcal{L}) = 1 \) such that the positive Lyapunov exponents

\[
\chi_1 < \chi_2 < \ldots < \chi_{\tilde{k}}
\]

of \( \varphi \) are constant on \( \mathcal{L} \). For \( x \in \mathcal{L} \), let

\[
E^u(x) = E^u_1(x) \oplus E^u_2(x) \oplus \ldots \oplus E^u_{\tilde{k}}(x)
\]

be the \( d\phi_t \)-invariant decomposition of \( E^u(x) \) into subspaces of constant dimensions \( n_1, \ldots, n_{\tilde{k}} \) with \( n_1 + n_2 + \ldots + n_{\tilde{k}} = n^u = \dim(E^u(x)) \). We have a similar decomposition for \( E^s(x), x \in \mathcal{L} \). If the flow is contact, we have \( n^s = \dim(E^s(x)) = n^u \).

Set \( \lambda_i = e^{\chi_i} \) for all \( i = 1, \ldots, \tilde{k} \). Fix an arbitrary constant \( \beta \in (0, 1] \) such that

\[
\lambda_j^\beta < \lambda_{j+1} \quad , \quad 1 \leq j < \tilde{k}.
\]
Take \( \dot{\epsilon} > 0 \) so small that
\[
e^{8\dot{\epsilon}} < \lambda_1, \quad e^{8\dot{\epsilon}} < \lambda_j/\lambda_{j-1} \quad (j = 2, \ldots, \tilde{k}).
\] (3.1)

Some further assumptions about \( \dot{\epsilon} \) will be made later. Set
\[
1 < \nu_0 = \lambda_1 e^{-8\dot{\epsilon}} < \mu_j = \lambda_j e^{-\dot{\epsilon}} < \lambda_j < \nu_j = \lambda_j e^{\dot{\epsilon}}
\] (3.2)
for all \( j = 1, \ldots, \tilde{k} \).

**Fix \( \dot{\epsilon} > 0 \) with the above properties and set** \( \epsilon = \dot{\epsilon}/4 \). There exists a Lyapunov \( \epsilon \)-regularity function \( R = R_\epsilon : \mathcal{L} \rightarrow (1, \infty) \), i.e. a function with
\[
e^{-\epsilon} \leq \frac{R(\varphi(x))}{R(x)} \leq e^\epsilon, \quad x \in \mathcal{L},
\] (3.3)
such that
\[
\frac{1}{R(x)} e^{\epsilon n} \leq \frac{\|d\varphi^n(x) \cdot v\|}{\lambda_1^n \|v\|} \leq R(x) e^{\epsilon n}, \quad x \in \mathcal{L}, \quad v \in E^n_u(x) \setminus \{0\}, \quad n \geq 0.
\] (3.4)

We will discuss these functions in more details later.

For \( x \in \mathcal{L} \) and \( 1 \leq j \leq d \) set
\[
\tilde{E}^u_j(x) = E^u_1(x) \oplus \cdots \oplus E^u_{j-1}(x), \quad \tilde{E}^u_j = E^u_j(x) \oplus \cdots \oplus E^u_d(x).
\]

Also set \( \tilde{E}^u_1(x) = \{0\} \) and \( \tilde{E}^u_{k+1}(x) = E^u(x) \). For any \( x \in \mathcal{L} \) and any \( u \in E^n_u(x) \) we will write \( u = (u^{(1)}, u^{(2)}, \ldots, u^{(k)}) \), where \( u^{(i)} \in E^u_{i}(x) \) for all \( i \). We will denote by \( \| \cdot \| \) the norm on \( E^n_u(x) \) generated by the Riemann metric.

It follows from the general theory of non-uniform hyperbolicity (see [15], [15]) that for any \( j = 1, \ldots, \tilde{k} \) the invariant bundle \( \{ \tilde{E}^u_j(x) \}_{x \in \mathcal{L}} \) is uniquely integrable over \( \mathcal{L} \), i.e. there exists a continuous \( \varphi \)-invariant family \( \{ \tilde{W}^u_{r_{\tilde{t}(x)}}(x) \}_{x \in \mathcal{L}} \) of \( C^2 \) submanifolds \( \tilde{W}^u_{r_{\tilde{t}(x)}}(x) = \tilde{W}^u_{r_{\tilde{t}(x)}}(x) \) of \( M \) tangent to the bundle \( \tilde{E}^u_{j} \) for some Lyapunov \( \dot{\epsilon}/2 \)-regularity function \( \tilde{r} = \tilde{r}_{\epsilon/2} : \mathcal{L} \rightarrow (0,1) \).

Moreover, with \( \beta \in (0,1] \) as in the beginning of this section, for \( j > 1 \) it follows from Theorem 6.6 in [15] and (3.1) that there exists an \( \varphi \)-invariant family \( \{ \tilde{W}^u_{r_{\tilde{t}(x)}}(x) \}_{x \in \mathcal{L}} \) of \( C^{1+\beta} \) submanifolds \( \tilde{W}^u_{r_{\tilde{t}(x)}}(x) = \tilde{W}^u_{r_{\tilde{t}(x)}}(x) \) of \( M \) tangent to the bundle \( \tilde{E}^u_{j} \). (However this family is not unique in general.) For each \( x \in \mathcal{L} \) and each \( j = 2, \ldots, \tilde{k} \) fix an \( \varphi \)-invariant family \( \{ \tilde{W}^u_{r_{\tilde{t}(x)}}(x) \}_{x \in \mathcal{L}} \) with the latter properties. Then we can find a Lyapunov \( \dot{\epsilon} \)-regularity function \( r = r_{\epsilon} : \mathcal{L} \rightarrow (0,1) \) and for any \( x \in \mathcal{L} \) a \( C^{1+\beta} \) diffeomorphism
\[
\Phi^u_x : E^n_u(x; r(x)) \rightarrow \Phi^u_x(E^n_u(x; r(x))) \subset W^u_{r_{\tilde{t}(x)}}(x)
\]
such that
\[
\Phi^u_x(\tilde{E}^u_j(x; r(x))) \subset \tilde{W}^u_{r_{\tilde{t}(x)}}(x), \quad \Phi^u_x(\tilde{E}^u_j(x; r(x))) \subset \tilde{W}^u_{r_{\tilde{t}(x)}}(x)
\] (3.5)
for all \( x \in \mathcal{L} \) and \( j = 2, \ldots, \tilde{k} \). Moreover, since for each \( j > 1 \) the submanifolds \( \tilde{W}^u_{r_{\tilde{t}(x)}}(x) \) and \( \exp^u_x(\tilde{E}^u_j(x; r(x))) \) of \( W^u_{r_{\tilde{t}(x)}}(x) \) are tangent at \( x \) of order \( 1 + \beta \), we can choose \( \Phi^u_x \) so that the diffeomorphism
\[
\Psi^u_x = (\exp^u_x)^{-1} \circ \Phi^u_x : E^n_u(x; r(x)) \rightarrow \Psi^u_x(E^n_u(x; r(x))) \subset E^n_u(x; r(x))
\]
is \( C^{1+\beta} \)-close to identity. Thus, replacing \( R(x) \) with a larger regularity function if necessary, we may assume that
\[
\|\Psi^u_x(u) - u\| \leq R(x)\|u\|^{1+\beta}, \quad \|(\Psi^u_x)^{-1}(u) - u\| \leq R(x)\|u\|^{1+\beta}
\] (3.6)
for all $x \in \mathcal{L}$ and $u \in E^u(x;\hat{r}(x))$, and also that
\begin{equation}
\|d\Phi_x^u(u)\| \leq R(x) \quad \|d\Phi_x^u(u)^{-1}\| \leq R(x), \quad x \in \mathcal{L}, \ u \in E^u(x;\hat{r}(x)).
\end{equation}
Finally, again replacing $R(x)$ with a larger regularity function if necessary, we may assume that
\begin{equation}
\|\Phi_x^u(v) - \Phi_x^u(u) - d\Phi_x^u(u) \cdot (v - u)\| \leq R(x) \|v - u\|^{1+\beta} \quad x \in \mathcal{L}, \ u, v \in E^u(x;\hat{r}(x)),
\end{equation}
and
\begin{equation}
\|d\Phi_x^u(u) - \text{id}\| \leq R(x) \|u\|^{\beta}, \quad x \in \mathcal{L}, \ u \in E^u(x;\hat{r}(x)).
\end{equation}
In a similar way one defines the maps $\Phi_x^*$ and we will assume that $\hat{r}(x)$ is chosen so that these maps satisfy the analogues of the above properties.

For any $x \in \mathcal{L}$ consider the $C^{1+\beta}$ map (defined locally near 0)
\begin{equation}
\hat{\varphi}_x = (\Phi_x^u)^{-1} \circ \varphi \circ \Phi_x^u : E^u(x) \to E^u(\varphi(x)).
\end{equation}
It is important to notice that
\begin{equation}
\hat{\varphi}_x^{-1}(\hat{E}_x^u(\varphi(x);r(\varphi(x))) \subset \hat{E}_x^u(x;\hat{r}(x)), \quad \hat{\varphi}_x^{-1}(\hat{E}_x^u(\varphi(x);r(\varphi(x))) \subset \hat{E}_x^u(x;\hat{r}(x))
\end{equation}
for all $x \in \mathcal{L}$ and $j > 1$.

Given $y \in \mathcal{L}$ and any integer $j \geq 1$ we will use the notation
\begin{equation}
\hat{\varphi}_j = \hat{\varphi}_{\varphi^{-1}(y)} \circ \cdots \circ \hat{\varphi}(y) \circ \hat{\varphi}, \quad \hat{\varphi}_y = (\hat{\varphi}_{\varphi^{-1}(y)})^{-1} \circ \cdots \circ (\hat{\varphi}_{\varphi^{-2}(y)})^{-1} \circ (\hat{\varphi}_{\varphi^{-1}(y)})^{-1},
\end{equation}
at any point where these sequences of maps are well-defined.

It is well known (see e.g. the Appendix in [LY1] or Sect. 3 in [PS]) that there exists a Lyapunov $\hat{\epsilon}$-regularity functions $\Gamma = \Gamma_{\hat{\epsilon}} : \mathcal{L} \to [1, \infty)$ and $r = r_{\hat{\epsilon}} : \mathcal{L} \to (0, 1)$ and for each $x \in \mathcal{L}$ a norm $\| \cdot \|_{x}^{r}$ on $T_x M$ such that
\begin{equation}
\|v\| \leq \|v\|_{x}^{r} \leq \Gamma(x)\|v\|, \quad x \in \mathcal{L}, \ v \in T_x M,
\end{equation}
and for any $x \in \mathcal{L}$ and any integer $m \geq 0$, assuming $\hat{\varphi}_x^j(u), \hat{\varphi}_x^j(v) \in E^u(\varphi^j(x), r(\varphi^j(x)))$ are well-defined for all $j = 1, \ldots, m$, the following hold:
\begin{align}
\mu_1^m \|u - v\|_{x}^{r} & \leq \|\hat{\varphi}_x^m(u) - \hat{\varphi}_x^m(v)\|_{x}^{r}, \quad u, v \in \hat{E}_x^u(x;\hat{r}(x)), \quad \mu_2^m \|u - v\|_{x}^{r} \leq \|\hat{\varphi}_x^m(u) - \hat{\varphi}_x^m(v)\|_{x}^{r}, \quad u, v \in \hat{E}_x^u(x;\hat{r}(x)), \\
\mu_3^m \|v\|_{x}^{r} & \leq \|d\hat{\varphi}_x^m(u) \cdot v\|_{x}^{r} \leq \nu_d^m \|v\|_{x}^{r}, \quad x \in \mathcal{L}, \ u \in E^u(x;\hat{r}(x)), \ v \in E^u(x), \\
\mu_4^m \|v\|_{x}^{r} & \leq \|d\hat{\varphi}_x^m(0) \cdot v\|_{x}^{r} \leq \nu_j^m \|v\|_{x}^{r}, \quad x \in \mathcal{L}, \ v \in E^u(x).
\end{align}
We will also use the norm
\begin{equation}
|u| = \max\{\|u^{(i)}\| : 1 \leq i \leq \tilde{k}\}.
\end{equation}
Clearly,
\begin{equation}
\|u\| = \|u^{(1)} + \ldots + u^{(\tilde{k})}\| \leq \sum_{i=1}^{\tilde{k}} \|u^{(i)}\| \leq \tilde{k} \|u\|.
\end{equation}
Taking the regularity function $\Gamma(x)$ appropriately, we have $|u| \leq \Gamma(x)\|u\|$, so

$$\frac{1}{k}\|u\| \leq |u| \leq \Gamma(x)\|u\| , \quad x \in \mathcal{L} , \ u \in E^u(x). \quad (3.15)$$

Next, Taylor's formula (see also section 3 in [PS]) implies that there exists a Lyapunov $\hat{\varepsilon}$-regularity function $D = D_{\hat{\varepsilon}} : \mathcal{L} \to [1, \infty)$ such that for any $i = \pm 1$ we have

$$\|\phi^i_z(v) - \phi^i_z(u) - d\phi^i_z(u) \cdot (v - u)\| \leq D(x)\|v - u\|^{1+\beta} , \quad x \in \mathcal{L} , \ u, v \in E^u(x; r(x)), \quad (3.16)$$

and

$$\|d\phi^i_z(u) - d\phi^i_z(0)\| \leq D(x)\|u\|^\beta , \quad x \in \mathcal{L} , \ u \in E^u(x; r(x)). \quad (3.17)$$

Finally, we state here a Lemma from [St4] which will be used several times later.

**Lemma 3.1.** (Lemma 3.3 in [St4]) There exist a Lyapunov $6\hat{\varepsilon}$-regularity function $L = L_{6\hat{\varepsilon}} : \mathcal{L} \to [1, \infty)$ and a Lyapunov $7\varepsilon/\beta$-regularity function $r = r_{7\varepsilon/\beta} : \mathcal{L} \to (0,1)$ such that for any $x \in \mathcal{L}$, any integer $p \geq 1$ and any $v \in E^u(z, r(z))$ with $\|\phi^i_z(v)\| \leq r(x)$, where $z = \varphi^{-p}(x)$, we have

$$\|w^p_{(1)} - v^p_{(1)}\| \leq L(x)w_p|^{1+\beta},$$

where $v_p = \phi^p_z(v) \in E^u(x)$ and $w_p = d\phi^p_z(0) \cdot v \in E^u(x)$. Moreover, if $|v_p| = \|v^p_{(1)}\| \neq 0$, then $1/2 \leq \|w^p_{(1)}\|/\|v^p_{(1)}\| \leq 2$.

**Remark.** Notice that if $v \in E^u_1(z, r(z))$ in the above lemma, then $v_p, w_p \in E^u_1(x)$, so $\|w_p - v_p\| \leq L(x)\|w_p\|^{1+\beta}$.

Let $F_0 : M \to \mathbb{R}$ be a Hölder continuous functions as in Sect. 3.1 and let $m$ be the Gibbs measure determined by $F_0$. Let $\mathcal{R} = \{ R_i \}_{i=1}^{k_0}$ be a pseudo-Markov family for $\phi_t$ as in Sect. 2, and let $\tau : R = \cup_{i=1}^{k_0} R_i \to [0,1/2]$ and $\mathcal{P} : R \to R$ be the corresponding first return map and the Poincaré map. As before fix constants $0 < \tau_0 \leq \hat{\tau}_0 \leq 1/2$ so that

$$\tau_0 \leq \tau(x) \leq \hat{\tau}_0 , \quad x \in R.$$ 

The Gibbs measure $m$ induces a Gibbs measure $\mu$ on $R$ (with respect to the Poincaré map $\mathcal{P}$) for the function

$$F(x) = \int_0^{\tau(x)} F_0(\phi_s(x)) \, ds , \quad x \in R.$$ 

The function $F$ is Hölder and, using Sinai’s Lemma, it is cohomologous to a Hölder function $f : R \to \mathbb{R}$ which is constant on stable leaves in rectangles $R_i$ in $R$. Thus, $\mu$ coincides with the Gibbs measure determined by $f$. For every continuous function $H$ on $M$ we then have (see e.g. [PP])

$$\int_M H \, dm = \frac{\int_R \left( \int_0^{\tau(x)} H(\phi_s(x)) \, ds \right) \, d\mu(x)}{\int_R \tau \, d\mu}. \quad (3.18)$$

Given a Lyapunov regularity function $R_\varepsilon$ with (3.3) and (3.4), any set of the form

$$Q_p(\varepsilon) = \{ x \in \mathcal{L} : R_\varepsilon(x) \leq \varepsilon^p \}$$

is called a Pesin set. Given $p > 0, \varepsilon > 0, \delta > 0$ and an integer $n \geq 1$ set

$$\Xi_n = \Xi_n(p, \varepsilon, \delta) = \{ x \in \mathcal{L} \cap R : \# \{ j : 0 \leq j \leq n - 1 \text{ and } \mathcal{P}^j(x) \notin Q_p(\varepsilon) \} \geq \delta n \} \quad . \quad (3.19)$$
Definition. (GS) Consider a log-integrable linear cocycle $M$ above a transformation $(T, \mu)$, with Lyapunov exponents $\lambda_1 \geq \ldots \geq \lambda_d$. We say that $M$ has exponential large deviations for all exponents if, for any $i \leq d$ and any $\epsilon > 0$, there exists $C > 0$ such that, for all $n \geq 0$,

$$
\mu\{x : \log \|A^i M^{n}(x)\| - n(\lambda_1 + \ldots + \lambda_i) \geq n\epsilon\} \leq C e^{-n/C}.
$$

(3.20)

The following theorem, which is a special case of Theorem 1.7 in [GS], shows that if $d\varphi$ has exponential large deviations for all exponents, then most points in $\mathcal{L}$ return exponentially often to some Pesin set.

**Theorem 3.2.** (GS) Assume that $d\varphi$ has exponential large deviations for all exponents with respect to $\mu$. Let $\epsilon_0 > 0$ and $\delta_0 > 0$. Then there exist $p_0 > 0$, $C > 0$ and $c > 0$ such that

$$
\mu\left(\left\{x \in \mathcal{L} \cap R : \exists \{j : 0 \leq j \leq n-1 \text{ and } P^{j}(x) \notin Q_{p_0}(\epsilon_0)\} \geq \delta_0 n\right\}\right) \leq C e^{-cn},
$$

for all $n \geq 1$. Thus, there exist constants $p > 0$, $C' > 0$ and $c' > 0$ such that

$$
\mu(\Xi_n(p_0, \epsilon_0, \delta_0)) \leq C' e^{-c'n}
$$

(3.21)

for all $n \geq 1$.

Clearly, if (3.21) holds for $p_0$, then it will hold with $p$ replaced by any $p \geq p_0$.

As established in [GS] (see Theorem 1.5 there), for a transitive subshift of finite type $T$ on a space $\Sigma$, if $\mu$ is a Gibbs measure for a Hölder-continuous potential and $M$ is a continuous linear cocycle on a vector bundle $E$ above $T$, each of the following conditions is sufficient for $M$ to have exponential large deviations for all exponents: (i) if all its Lyapunov exponents coincide; (ii) if there is a continuous decomposition of $E$ as a direct sum of subbundles $E = E_1 \oplus \ldots \oplus E_k$ which is invariant under $M$, such that the restriction of $M$ to each $E_i$ has exponential large deviations for all exponents; (iii) more generally, if there is an invariant continuous flag decomposition $\{0\} = F_0 \subseteq F_1 \subseteq \ldots \subseteq F_k = E$, such that the cocycle induced by $M$ on each $F_i/F_{i+1}$ has exponential large deviations for all exponents; (iv) if the cocycle $M$ is locally constant in some trivialization of the bundle $E$ (this is equivalent to the existence of invariant continuous holonomies which are commuting); (v) if the cocycle $M$ admits invariant continuous holonomies, and if it is pinching and twisting in the sense of Avila-Viana [AV]; (vi) if the cocycle $M$ admits invariant continuous holonomies, and the bundle is 2-dimensional.

It follows from the above and Theorem 9.18 in [V] that generic linear cocycles have exponential large deviations for all exponents. Moreover, amongst fiber bunched cocycles, generic cocycles in the Hölder topology also have exponential large deviations for all exponents.

In what follows sometimes it will be more convenient to use sets of the form

$$
\hat{\Xi}_k(p, \epsilon, \delta) = \{x \in \mathcal{L} : \exists \{s \in \mathbb{Z} : 0 \leq s < k \text{ and } \varphi^s(x) \notin Q_p(\epsilon)\} \geq \delta k\}.
$$

These sets are naturally related to sets of the form $\Xi_n(p, \epsilon, \delta)$. More precisely, using the fact that $\tau_0 \leq \tau(x) \leq \tau_0$ for all $x \in R$, it is easy to check that

$$
\Xi_m(p_0, \epsilon_0, \delta_0) \subset \Xi_m(p_0 - \epsilon_0, \delta_0/(\tau_0))(\tau_0)
$$

(3.22)

and

$$
\Xi_m(p_0 + \epsilon_0, \delta_0/\tau_0) \subset \Xi_m(p_0, \epsilon_0, \delta_0)
$$

(3.23)

\footnote{Which are the most frequently met Hölder cocycles.}
for all integers \( m \geq 1 \). Indeed, to check (3.22), let \( z_0 \in \Xi_m(p_0, \hat{\epsilon}_0, \hat{\delta}_0) \). Then
\[
\sharp \{ j : 0 < j < m - 1, \, R_\epsilon(P^j(z_0)) > e^{p_0} \} \geq \hat{\delta}_0 m.
\]
For every \( j < m \) with \( R_\epsilon(P^j(x)) > e^{p_0} \), let \( s = [T_j(z_0)] \). Then \( s < j \hat{\tau}_0 < m \hat{\tau}_0 \) and
\[
R_\epsilon(\phi^s(z_0)) \geq e^{-\epsilon} R_\epsilon(P^s(z_0)) > e^{(\epsilon - \epsilon) p_0} \geq e^{p_0 - \epsilon p_0}.
\]

The number of \( j \)'s with \( j < m \) and \( s = [T_j(z_0)] \) is not more than \( 1/\tau_0 \), so
\[
\sharp \{ s : 0 < s < m \hat{\tau}_0, \, R_\epsilon(\phi^s(z_0)) > e^{p_0 - \epsilon p_0} \} \geq \frac{1}{\tau_0} \frac{\hat{\delta}_0}{\hat{\tau}_0} \geq \frac{\hat{\delta}_0}{\hat{\tau}_0} m \hat{\tau}_0,
\]
which shows that \( z_0 \) belongs to the right-hand-side of (3.22).

In a similar way one proves (3.23).

Finally we will make an important remark concerning Lyapunov regularity functions on \( \mathcal{L} \) (see Sect. 3.1). Consider a fixed Lyapunov regularity function \( R_\epsilon(x) \) satisfying (3.3) and (3.4). A Lyapunov \( \epsilon \)-regularity function \( H(x), \, x \in \mathcal{L} \), will be called a large (resp. small) canonical \( \epsilon \)-regularity function if there exist constants \( p > 0 \) and \( H_0 > 0 \) such that
\[
1 \leq H(x) \leq H_0 (R_\epsilon(x))^p \quad \text{(resp. } 1 \geq H(x) \geq \frac{1}{H_0 (R_\epsilon(x))^p} \text{)}
\]
for all \( x \in \mathcal{L} \).

**Proposition 3.3.** The function \( r_\epsilon \) in Sect. 3.1 can be chosen to be a small canonical \( \epsilon \)-regularity function, while \( \Gamma(x) \) in (3.10), (3.15) and \( D(x) \) in (3.16), (3.17) can be chosen to be large canonical \( \epsilon \)-regularity functions. Moreover, we can always choose \( r_\epsilon(x) \) so that \( r_\epsilon(x) \leq \frac{1}{(R_\epsilon(x))^p} \) for all \( x \in \mathcal{L} \cap \mathcal{R} \).

This follows e.g. from the arguments in Sect. 3 in [PS]. Moreover, following the arguments in [St4], the function \( L(x) \) in Lemma 3.1 is a large canonical regularity function, and similarly the arguments in Sect. 9 show that all regularity functions constructed there are canonical regularity functions.

### 4 Non-integrability of Anosov flows

#### 4.1 Choice of constants, sets of Lyapunov regular points

In what follows we assume that \( \mathcal{R} = \{ R_i \}_{i=1}^{k_0} \) is a fixed Markov partition for \( \phi_t \) on \( M \) of size \( < 1/2 \) and \( \mathcal{R} = \{ R_i \}_{i=1}^{k_0} \) is the related pseudo-Markov family as in Sect. 2. We will use the notation associated with these from Sect. 2, and we will assume that for any \( i = 1, \ldots, k_0 \), \( z_i \) is chosen so that \( z_i \in \text{Int}^u(W^u_{R_i}(z_i)) \). For any \( x \in R \), any \( y \in \tilde{R} \) and \( \delta > 0 \) set
\[
B^u(x, \delta) = \{ y \in W^u_{R_i}(x) : d(x, y) < \delta \} \quad \text{and} \quad \tilde{B}^u(y, \delta) = \{ z \in W^u_{\tilde{R}_i}(z) : d(z, y) < \delta \}.
\]
In a similar way define \( B^s(x, \delta) \). For brevity sometimes we will use the notation
\[
U_i(z) = W^u_{\tilde{R}_i}(z)
\]
for \( z \in R_i \).

Fix constants \( 0 < \tau_0 < \hat{\tau}_0 < 1 \) so that \( \tau_0 \leq \tau(x) \leq \hat{\tau}_0 \) for all \( x \in R \) and \( \tau_0 \leq \hat{\tau}(x) \leq \hat{\tau}_0 \) for all \( x \in \tilde{R} \).
Let $\alpha_1 > 0$ be as in Sect. 2, and let $f$ be an essentially $\alpha_1$-Hölder continuous potential on $R$. Set $g = f - P_f \tau$, where $P_f \in \mathbb{R}$ is chosen so that the topological pressure of $g$ with respect to the Poincaré map $P: R = \cup_{i=1}^{k_0} R_i \rightarrow R$ is 0. Let $\mu = \mu_g$ be the Gibbs measure on $R$ determined by $g$, then $\mu(R) = 1$. We will assume that $f$ (and therefore $g$) depends on forward coordinates only, i.e. it is constant on stable leaves of $R_i$ for each $i$.

Given an unstable leaf $W = W^u_{R_i}(z)$ in some rectangle $\tilde{R}_i$ and an admissible sequence $i = i_0, \ldots, i_m$ of integers $i_j \in \{1, \ldots, k_0\}$, the set

$$C_W[i] = \{ x \in W : \tilde{P}_{i_j}(x) \in \tilde{R}_{i_j}, \; j = 0, 1, \ldots, m \}$$

will be called a cylinder of length $m$ in $W$ (or an unstable cylinder in $\tilde{R}$ in general). When $W = U_i$ we will simply write $C[i]$. In a similar way one defines cylinders $C_V[i]$, where $V = W^u_{\tilde{R}_i}(z)$ is an unstable leaf in some rectangle $\tilde{R}_i$.

Let

$$\text{pr}_D: \cup_{i=1}^{k_0} \phi_{[-\epsilon, \epsilon]}(D_i) \rightarrow \cup_{i=1}^{k_0} D_i$$

be the projection along the flow, i.e. for all $i = 1, \ldots, k_0$ and all $x \in \phi_{[-\epsilon, \epsilon]}(D_i)$ we have $\text{pr}_D(x) = \text{pr}_D(x)$ (see Sect. 2). For any $z \in R$ denote by $\tilde{U}(z)$ the part of the unstable manifold $W^u_{R_i}(z)$ such that $\text{pr}_D(\tilde{U}(z)) = W^u_{\tilde{R}_i}(z)$. The shift along the flow determines a bi-Hölder continuous bijections

$$T_z: W^u_{\tilde{R}_i}(z) \rightarrow \tilde{U}(z) \quad \text{and} \quad \Psi: W^u_{\tilde{R}_i}(z) \rightarrow W^u_{\tilde{R}_i}(z)$$

for all $i$. These define bi-Hölder continuous bijections

$$\Psi: R \rightarrow \tilde{R} = \cup_{i=1}^{k_0} \tilde{R}_i,$$

where $\tilde{R}_i = \cup_{z \in S_i} \tilde{U}(z)$ and $\Psi|_{W^u_{\tilde{R}_i}(z)} = (T_z)|_{W^u_{\tilde{R}_i}(z)}$ for $z \in S_i$, and $\tilde{\Psi}: R \rightarrow \tilde{R}$. Notice that there exists a global constant $C > 1$ such that $\frac{1}{C}d(x, y) \leq d(T_z(x), T_z(y)) \leq C d(x, y)$ for any $z \in \tilde{R}$ and any $x, y \in W^u_{\tilde{R}_i}(z)$.

4.2 Non-integrability

Throughout we assume that $\phi_t$ is a $C^2$ contact Anosov flow on $M$ with a $C^2$ invariant contact form $\omega$. Then the two-form $d\omega$ is $C^1$, so there exists a constant $C_0 > 0$ such that

$$|d\omega_x(u, v)| \leq C_0 ||u|| ||v||, \; u, v \in T_x M, \; x \in M.$$  \hspace{1cm} (4.1)

Moreover, there exists a constant $\theta_0 > 0$ such that for any $x \in M$ and any $u \in E^u(x)$ with $||u|| = 1$ there exists $v \in E^s(x)$ with $||v|| = 1$ such that $|d\omega_x(u, v)| \geq \theta_0$.

The main ingredient in this section is the following lemma of Liverani (Lemma B.7 in [L2]) which significantly strengthens a lemma of Katok and Burns ([KB]).

**Lemma 4.1.** ([L2]) Let $\phi_t$ be a $C^2$ contact flow on $M$ with a $C^2$ contact form $\omega$. Then there exist constants $C_0 > 0$, $\vartheta > 0$ and $\bar{\epsilon}_0 > 0$ such that for any $z \in M$, any $x \in W^u_{\bar{\epsilon}_0}(z)$ and any $y \in W^s_{\bar{\epsilon}_0}(z)$ we have

$$|\Delta(x, y) - d\omega_x(u, v)| \leq C_0 \left( ||u||^2 ||v||^\vartheta + ||u||^\vartheta ||v||^2 \right),$$  \hspace{1cm} (4.2)

where $u \in E^u(z)$ and $v \in E^s(z)$ are such that $\exp^u(z) = x$ and $\exp^s(z) = y$.

\[\text{If the initial potential } F \text{ on } R \text{ is } \alpha^2\text{-Hölder, applying Sinai’s Lemma (see e.g. [PP]) produces an } \alpha\text{-Hölder potential } f \text{ depending on forward coordinates only.}\]
Note. Actually Lemma B.7 in [L2] is more precise with a particular choice of the constant \( \vartheta \) determined by the (uniform) Hölder exponents of the stable/unstable foliations and the corresponding local holonomy maps. However in this paper we do not need this extra information.

From now on we will assume that \( C_0 > 0 \), \( \vartheta > 0 \) and \( \hat{\varepsilon}_0 \) from Theorem 3.2 satisfy 
\[ \hat{\varepsilon}_0 \in (0, \varepsilon_0/4) \], (4.1) and (4.2). As in Sect. 3, set
\[ Q_p(\varepsilon) = \{ x \in \mathcal{L} : R_\varepsilon(x) \leq e^p \} \]
for all \( \varepsilon \in (0, \varepsilon_0) \) and \( p > 0 \). Then
\[ Q_0(\varepsilon_0) \subset Q_1(\varepsilon_0) \subset \ldots \subset Q_n(\varepsilon_0) \subset \ldots \]
and \( \cup_{p=0}^\infty Q_p(\varepsilon_0) = \mathcal{L} \). Fix an integer \( p_0 \geq 1 \) so large that \( \mu(Q_{p_0}(\varepsilon_0)) > 1 - \delta \) for some small appropriately chosen \( \delta > 0 \) (to be determined later). Set
\[ \mathcal{L}_0 = \cup_{p=0}^\infty Q_p(\varepsilon_0). \]
Then \( \mu(\mathcal{L}_0 \cap R) = 1 \). Set
\[ P_0 = Q_{p_0}(\varepsilon_0) \subset \hat{P}_0 = Q_{p_0+\hat{\varepsilon}_0}(\varepsilon_0). \tag{4.3} \]
Then the Lyapunov regularity function \( R_\varepsilon(x) \) is bounded by some constant on \( \hat{P}_0 \) (and therefore on \( P_0 \) as well), and according to Proposition 3.3, we may assume
\[ R_\varepsilon(x) \leq R_0, \quad r(x) \geq r_0, \quad \Gamma(x) \leq \Gamma_0, \quad L(x) \leq L_0, \quad D(x) \leq D_0 \tag{4.4} \]
for all \( x \in \hat{P}_0 \) for some positive constants \( R_0, \Gamma_0, L_0, D_0 \geq 1 \) and \( r_0 > 0 \). We fix \( r_0 > 0 \) so that \( r_0 \leq \frac{1}{R_0} \).

It follows easily from the properties of Markov families\(^{17}\) that there exists a constant \( r_1 > 0 \) such that for every \( i \) and every \( x \in \partial R_i \) there exists \( y \in R_i \) such that \( \text{dist}(y, \partial R_i) \geq r_1 \) and \( d(x, y) < r_0/2 \). Fix a constant \( r_1 < \frac{r_0}{2R_0} \) with this property.

We will show below that for Lyapunov regular points \( x \in \mathcal{L}_0 \) the estimate (4.2) can be improved what concerns the involvement of \( u \) for certain choices of \( u \) and \( v \). More precisely, we will show that choosing \( v \) in a special way, \( \Delta(x, y) \) becomes a \( C^1 \) function of \( x = \exp^u_x(u) \) with a non-zero uniformly bounded derivative in a certain direction.

We will now state two Main Lemmas. Their proofs, both using Liverani’s Lemma 4.1, are given in Sect. 8.

Let \( \hat{\varepsilon}_0 > 0 \) and \( \hat{\delta}_0 > 0 \) be given constants and let \( \Xi_m = \Xi_m(p_0, \hat{\varepsilon}_0, \hat{\delta}_0) \) be as in Sect. 3.

Lemma 4.2. There exist constants \( C_1 > 0 \) and \( \beta_1 \in (0, 1) \) with the following properties:
(a) For any unstable cylinder \( \mathcal{C} \) in \( R \) of length \( m \) with \( \mathcal{C} \cap P_0 \setminus \Xi_m = \emptyset \) and any \( z \in \mathcal{C} \) we have
\[ \frac{1}{C_1 \lambda_1^p} \leq \text{diam}(\overline{\Psi}(\mathcal{C})) \leq \frac{C_1 e^{2p}}{\lambda_1^p}, \tag{4.5} \]
where \( p = [\tau_m(z)] \).

(b) For any unstable cylinder \( \mathcal{C} \) of length \( m \) in \( R \) with \( \mathcal{C} \cap P_0 \setminus \Xi_m = \emptyset \), any \( \hat{x}_0, \hat{z}_0 \in \mathcal{C} \) and any \( \hat{y}_0, \hat{b}_0 \in W_R(\hat{z}_0) \) we have
\[ |\Delta(\hat{x}_0, \hat{y}_0) - \Delta(\hat{x}_0, \hat{b}_0)| \leq C_1 \text{diam}(\overline{\Psi}(\mathcal{C})) (d(\hat{y}_0, \hat{b}_0))^{\beta_1}. \]

In particular,
\[ |\Delta(\hat{x}_0, \hat{y}_0)| \leq C_1 \text{diam}(\overline{\Psi}(\mathcal{C})) (d(\hat{y}_0, \hat{z}_0))^{\beta_1} \leq C_1 \text{diam}(\overline{\Psi}(\mathcal{C})). \]

\(^{17}\) Easy proof by contradiction.
Fix a constant $C_1 > 0$ with properties in Lemma 4.3. We take $C_1 \geq C_0$. Set

$$\beta_0 = \frac{1}{\sqrt{1 + \theta_0^2/(64C_1^2)}}. \quad (4.6)$$

Next, fix an integer $\ell_0 = \ell_0(\delta) \geq 1$ so large that we can find unit vectors $\eta_1, \eta_2, \ldots, \eta_{\ell_0}$ in $\mathbb{R}^{\mathbb{N}}$ such that for any unit vector $\xi \in \mathbb{R}^{\mathbb{N}}$ there exists $j$ with $\langle \xi, \eta_j \rangle \geq \beta_0$. Then fix measurable families $\eta_1(x), \eta_2(x), \ldots, \eta_{\ell_0}(x)$ $(x \in \mathcal{L}_0)$ of unit vectors in $\mathbb{E}_1(x)$ such that for any $x \in \mathcal{L}_0$ and any $\xi \in \mathbb{E}_1(x)$ with $\|\xi\| = 1$ there exists $j$ with $\langle \xi, \eta_j(x) \rangle \geq \beta_0$.

Recall the projections $T_z : W^{u}_R(z) \rightarrow \hat{U}(z) \subset W^{u}_0(z)$ for $z \in R$.

The following lemma is derived from the non-integrability of the flow which stems from the fact that the flow is contact. It will play a very essential role in proving that our contraction operators actually do have some contraction properties due to certain cancelations provided by property (4.9) below.

**Lemma 4.3.** Let $\phi_t$ be a $C^2$ contact Anosov flow on $M$. Let $\eta_1(x), \eta_2(x), \ldots, \eta_{\ell_0}(x)$ $(x \in \mathcal{L}_0)$ be families of unit vectors in $\mathbb{E}_1(x)$ as above, and let $\kappa \in (0,1)$ be a constant. Then there exist constants $\epsilon'' > 0$, $0 < \delta'' < \delta'$ (depending on $\kappa$ in general) and $\delta_0 \in (0,1)$ with the following properties:

(a) For any integer $m \geq 1$ and any $Z \in P_0 \setminus \Xi_m$ there exist families of points $y_{\ell}(Z) \in B^s(Z, \delta')$ $(\ell = 1, \ldots, \ell_0)$ such that if $C$ is a cylinder of length $m$ in $W^s_R(Z)$ with $Z \in C$, then for any $x_0 \in T_Z(C)$, $z_0 \in T_Z(C \cap P_0)$ of the form $x_0 = \Phi^s_Z(u_0)$, $z_0 = \Phi^s_Z(w_0)$ such that

$$d(x_0, z_0) \geq \kappa \text{diam}(C''), \quad (4.7)$$

where $C'' = T_z(C)$, and

$$\left\langle \frac{w_0 - u_0}{\|w_0 - u_0\|}, \eta_{\ell}(Z) \right\rangle \geq \frac{\beta_0}{2R_0} \quad (4.8)$$

for some $\ell = 1, \ldots, \ell_0$, then we have

$$\beta_0 \delta_0 \kappa \text{diam}(C'') \leq |\Delta(x_0, \pi_{d_1}(z_0)) - \Delta(x_0, \pi_{d_2}(z_0))| \quad (4.9)$$

for any $d_1 \in B^s(y_{\ell}(Z), \delta'')$ and $d_2 \in B^s(Z, \delta'')$.

(b) There exists an integer $N_0 \geq 1$ such that for any integer $N \geq N_0$, any integer $m \geq 1$ and any $Z \in P_0 \setminus \Xi_m$ there exist families of points

$$y_{\ell,1}(Z), y_{\ell,2}(Z) \in \mathcal{P}^N(B^u(Z; \epsilon'')) \cap B^s(Z, \delta') \quad , \quad \ell = 1, \ldots, \ell_0,$$

such that if $C$ is a cylinder of length $m$ in $W^u_R(Z)$ with $Z \in C$, $x_0 \in T_Z(C)$ and $z_0 \in T_Z(C \cap P_0)$ have the form $x_0 = \Phi^u_Z(u_0)$, $z_0 = \Phi^u_Z(w_0)$ and (4.7) and (4.8) hold for some $\ell = 1, \ldots, \ell_0$, then (4.9) holds for any $d_1 \in B^s(y_{\ell,1}(Z), \delta'')$ and $d_2 \in B^s(y_{\ell,2}(Z), \delta'').$

### 4.3 Regular distortion of cylinders

In [St5] we established some nice properties concerning diameters of cylinders for Axiom A flows on basic sets satisfying a pinching condition which we called regular distortion along unstable manifolds. In [St4] something similar was established for Anosov flows with Lipschitz local stable holonomy maps. It seems unlikely that any Anosov flow will have such properties, however it turns out that for general Anosov flows something similar holds for ‘sufficiently regular’ cylinders in $R$. More precisely we have the following.
Lemma 4.4. (a) There exists a constant $0 < \rho_1 < 1$ such that for any unstable leaf $W$ in $R$, any cylinder $C_W[i] = C_W[i_0, \ldots, i_m]$ in $W$ and any sub-cylinder $C_W[i'] = C_W[i_0, i_1, \ldots, i_{m+1}]$ of $C_W[i]$ of co-length $1$ such that $C_W[i'] \cap P_0 \neq \emptyset$ and $\mathcal{P}^{m+1}(C_W[i']) \cap P_0 \neq \emptyset$ we have

$$\rho_1 \text{diam}(\tilde{\Psi}(C_W[i])) \leq \text{diam}(\tilde{\Psi}(C_W[i'])).$$

(b) For any constant $\rho' \in (0, 1)$ there exists an integer $q' \geq 1$ such that for any unstable leaf $W$ in $R$, any cylinder $C_W[i] = C_W[i_0, \ldots, i_m]$ of length $m$ in $W$ and any sub-cylinder $C_W[i'] = C_W[i_0, i_1, \ldots, i_{m+1}, \ldots, i_{m+q'}]$ of $C_W[i]$ of co-length $q'$ such that $C_W[i'] \cap P_0 \neq \emptyset$ and $\mathcal{P}^{m+q'}(C_W[i']) \cap P_0 \neq \emptyset$ we have

$$\text{diam}(\tilde{\Psi}(C_W[i'])) \leq \rho' \text{diam}(\tilde{\Psi}(C_W[i])).$$

(c) There exist an integer $q_0 \geq 1$ and a constant $\rho_1 \in (0, 1)$ such that for any unstable leaf $W$ in $R$ and any cylinder $C_W[i] = C_W[i_0, \ldots, i_m]$ in $W$ such that $C_W[i] \cap P_0 \neq \emptyset$ and $\mathcal{P}^m(C_W[i]) \cap P_0 \neq \emptyset$ there exist points $z, x \in C_W[i]$ such that if $C_W[i'] = C_W[i_0, i_1, \ldots, i_{m+1}, \ldots, i_{m+q_0}]$ is the sub-cylinder of $C_W[i]$ of co-length $q_0$ containing $x$ then $d(z, y) \geq \rho_1 \text{diam}(\tilde{\Psi}(C_W[i]))$ for all $y \in C_W[i']$.

(d) We can choose the constant $\rho_1 \in (0, 1)$ from part (a) such that for any unstable leaf $W$ in $R$, any cylinder $C_W[i] = C_W[i_0, \ldots, i_m]$ in $W$ and any sub-cylinder $C_W[i'] = C_W[i_0, i_1, \ldots, i_{m+1}]$ of $C_W[i]$ of co-length $1$ such that there exists $x \in C_W[i']$ with $\mathcal{P}^m(x) \cap P_0$ we have

$$\rho_1 \text{diam}((\Phi_x)^{-1}(C_W[i])) \leq \text{diam}((\Phi_x)^{-1}(C_W[i'])).$$

Clearly, all statements in Lemma 4.4 remain true replacing $P_0$ by $\hat{P}_0$ and slightly changing the constants $\rho', q', q_0$ and $\rho_1$.

Notice that in the last part (d) we do not require $C_W[i'] \cap P_0 \neq \emptyset$, however this time we measure diameters ‘upstairs’ in the tangent bundle $E^u(x)$.

Lemma 4.4 will be used essentially in the proof of the main result in Sects. 5-8 below. Its proof is given in Sect. 9.

5 Construction of a ‘contraction set’ $K_0$

5.1 Normalized Ruelle operators and the metric $D_\theta$

Let the constants $C_0 > 0$, $c_0 > 0$, $1 < \gamma < \gamma_1$ be as in Sects. 2 and 4, so that (2.1) and (4.1) hold.

Fix a constant $\theta$ such that

$$\frac{1}{\gamma \alpha_1} = \hat{\theta} \leq \theta < 1,$$

(5.1)

where $\alpha_1 > 0$ is the constant chosen at the end of Sect. 2.

Recall the metric $D_\theta$ on $\hat{U}$ and the space $\mathcal{F}_\theta(\hat{U})$ from Sect. 1.1. In the same way we define the distance $D_\theta(x, y)$ for $x, y \in W \cap \hat{R}$. Lemma 5.2 below shows that $\tau \in \mathcal{F}_\theta(\hat{U})$. For a non-empty subset $A$ of $U$ (or some $W^u_R(x)$) let $\text{diam}_\theta(A)$ be the diameter of $A$ with respect to $D_\theta$.

Let $f \in \mathcal{F}_\theta(\hat{U})$ be a fixed real-valued function and let $g = f - P_f \tau$, where $P_f \in \mathbb{R}$ is such that $\text{Pr}_x(g) = 0$. Since $f$ is a Hölder continuous function on $\hat{U}$, it can be extended to a Hölder continuous function on $R$ which is constant on stable leaves.

Set $F^{(a)} = f - (P_f + a)\tau$. By Ruelle-Perron-Frobenius’ Theorem (see e.g. chapter 2 in [PP]) for any real number $a$ with $|a|$ sufficiently small, as an operator on $\mathcal{F}_\theta(\hat{U})$, $L_{F^{(a)}}$ has a largest eigenvalue $\lambda_a$ and there exists a (unique) regular probability measure $\hat{\nu}_a$ on $\hat{U}$ with $L_{F^{(a)}}^* \hat{\nu}_a = \lambda_a \hat{\nu}_a$, i.e. $\int L_{F^{(a)}} H d\hat{\nu}_a = \lambda_a \int H d\hat{\nu}_a$ for any $H \in \mathcal{F}_\theta(\hat{U})$. Fix a corresponding (positive) eigenfunction $h_a \in \mathcal{F}_\theta(\hat{U})$ such that $\int h_a d\hat{\nu}_a = 1$. Then $d\nu = h_0 d\hat{\nu}_0$ defines a $\sigma$-invariant probability measure $\nu$
on $U$. Since $\Pr_\sigma(f - P_f \tau) = 0$, it follows from the main properties of pressure (cf. e.g. chapter 3 in [PP]) that $|\Pr_\sigma(F(a))| \leq ||\tau||_0 |a|$. Moreover, for small $|a|$ the maximal eigenvalue $\lambda_a$ and the eigenfunction $h_a$ are Lipschitz in $a$, so there exist constants $a_0 > 0$ and $C > 0$ such that $|h_a - h_0| \leq C|a|$ on $\hat{U}$ and $|\lambda_a - 1| \leq C|a|$ for $|a| \leq a_0$.

For $|a| \leq a_0$, as in [D1], consider the function

$$f(a)(u) = f(u) - (P_f + a)\tau(u) + \ln h_a(u) - \ln h_a(\sigma(u)) - \ln \lambda_a$$

and the operators

$$L_{ab} = L_{f(a) - 1b\tau} : \mathcal{F}_\theta(\hat{U}) \rightarrow \mathcal{F}_\theta(\hat{U}) , \quad \mathcal{M}_a = L_{f(a)} : \mathcal{F}_\theta(\hat{U}) \rightarrow \mathcal{F}_\theta(\hat{U}).$$

One checks that $\mathcal{M}_a 1 = 1$ and $||L_{ab}^m h(u)|| \leq (\mathcal{M}_a^m |h|)(u)$ for all $u \in \hat{U}$, $h \in \mathcal{F}_\theta(\hat{U})$ and $m \geq 0$. It is also easy to check that $L_{f(0)}^\nu = \nu$, i.e. $\int L_{f(0)} H \, d\nu = \int H \, d\nu$ for any $H \in \mathcal{F}_\theta(\hat{U})$.

Since $g$ has zero topological pressure with respect to the shift map $\sigma : U \rightarrow U$, there exist constants $0 < c_1 \leq c_2$ such that for any cylinder $C = C^u[i_0, \ldots, i_m]$ of length $m$ in $U$ we have

$$c_1 \leq \frac{\nu(C)}{\theta^{C(y)}} \leq c_2 , \quad y \in C ,$$

(see e.g. [PP] or [P2]).

We now state some basic properties of the metric $D_\theta$ that will be needed later.

**Lemma 5.1.** (a) For any cylinder $C$ in $U$ the characteristic function $\chi_C$ of $C$ on $U$ is Lipschitz with respect to $D_\theta$ and $\text{Lip}_\theta(\chi_C) \leq 1/\text{diam}_\theta(C)$.

(b) There exists a constant $C_2 > 0$ such that if $x, y \in \hat{U}_i$ for some $i$, then

$$|\tau(x) - \tau(y)| \leq C_2 D_\theta(x, y).$$

That is, $\tau \in \mathcal{F}_\theta(\hat{U})$. Moreover, we can choose $C_2 > 0$ so that

$$|\tau_m(x) - \tau_m(y)| \leq C_2 D_\theta(\sigma^m(x), \sigma^m(y))$$

whenever $x, y \in \hat{U}_i$ belong to the same cylinder of length $m$.

(c) There exist constants $C_2 > 0$ and $\alpha_2 > 0$ such that for any $z \in R$, any cylinder $C$ in $W^u_R(z)$ and any $x, y \in C$ we have $d(\Psi(x), \Psi(y)) \leq C_2 D_\theta(x, y)$ and $\text{diam}_\theta(C) \leq C_2 (\text{diam}(\Psi(C)))^{\alpha_2}$. Moreover, we can take $\alpha_2 > 0$ so that $1/(\gamma_1)^{\alpha_2} = \theta$.

**Proof.** (a) Let $C$ be a cylinder in $U$ and let $x, y \in \hat{U}$. If $x, y \in C$ or $x \notin C$ and $y \notin C$, then $\chi_C(x) - \chi_C(y) = 0$. Assume that $x \in C$ and $y \notin C$. Let $D_\theta(x, y) = \theta^{N+1}$ and let $C'$ be a cylinder of length $N$ containing both $x$ and $y$. Since $x \in C$, as well, and $x$ is an interior point of $C$, we must have $C \subset C'$. Thus, $\text{diam}_\theta(C) \leq D_\theta(x, y)$. This gives

$$|\chi_C(x) - \chi_C(y)| = 1 = \frac{\text{diam}_\theta(C)}{\text{diam}_\theta(C)} \leq \frac{1}{\text{diam}_\theta(C)} D_\theta(x, y),$$

which proves the assertion.

(b), (c) Assume $x \neq y$ and let $C$ be the cylinder of largest length $m$ containing both $x$ and $y$. Set $\tilde{x} = \Psi(x), \tilde{y} = \Psi(y) \in \bar{R}$. Then $D_\theta(x, y) = \theta^{m+1}$. On the other hand, (2.1) and (5.1) imply

$$|\tau(x) - \tau(y)| \leq |\tau|_{C_1} (d(\tilde{x}, \tilde{y}))^{\alpha_1} \leq \frac{\text{Const}_{\gamma_1^{\alpha_1}}}{m} \leq \text{Const} \theta^m \leq C_2 D_\theta(x, y)$$

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for some global constant $C_2 > 0$. The above also shows that $d(\tilde{x}, \tilde{y}) \leq \text{Const} \, \theta^m \leq C_2 D_\theta(x, y)$, which proves half of part (c). The second part of (c) follows by using a similar estimate and the other half of (2.1).

Next, assume that $x, y$ belong to the same cylinder $C$ of length $m$. Let $\mathcal{P}_j(x), \mathcal{P}_j(y) \in R_{ij}$ for all $j = 0, 1, \ldots, m$. Assume that $D_\theta(x, y) = \theta^p$, where $x' = \sigma^m(x)$ and $y' = \sigma^m(y)$. Then $D_\theta(x, y) = \theta^{m+p-1}$ and moreover $D_\theta(\sigma(x), \sigma(y)) = \theta^{m-j+p-1}$ for all $j = 0, 1, \ldots, m - 1$. Now (2.1) and (5.1) imply

$$|\tau(\sigma(x)) - \tau(\sigma(y))| \leq |\tau|_{a_1} (d(\sigma(x), \sigma(y)))^{\alpha_1} \leq \text{Const} (d(\Psi(\sigma(x)), \Psi(\sigma(y))))^{\alpha_1} \leq \frac{1}{c_0 \gamma^{m-j+p}} (\tilde{P}^{m+p-j}(\Psi(\sigma(x)), \tilde{P}^{m+p-j}(\Psi(\sigma(y))))^{\alpha_1} \leq \text{Const} \, \theta^{m-j+p} \leq \text{Const} \, \theta^{m-j-1} D_\theta(x', y').$$

So

$$|\tau_m(x) - \tau_m(y)| \leq \sum_{j=0}^{m-1} |\tau(\sigma^j(x)) - \tau(\sigma^j(y))| \leq \text{Const} \, D_\theta(x', y') \sum_{j=0}^{m-1} \theta^{m-j+1} \leq \text{Const} \, D_\theta(x', y'),$$

which proves the statement. ■

It follows from Lemma 5.1 that $\tau \in \mathcal{F}_\theta(\tilde{U})$, so assuming $f \in \mathcal{F}_\theta(\tilde{U})$, we have $h_a \in \mathcal{F}_\theta(\tilde{U})$ for all $|a| \leq a_0$. Then $f^{(a)} \in \mathcal{F}_\theta(\tilde{U})$ for all such $a$. Moreover, using the analytical dependence of $h_a$ and $\lambda_a$ on $a$ and assuming that the constant $a_0 > 0$ is sufficiently small, there exists $T = T(a_0)$ such that

$$T \geq \max \{ \|f^{(a)}\|_0, |f^{(a)}|_\theta, |\tau|_\theta \} \quad (5.3)$$

for all $|a| \leq a_0$. Fix $a_0 > 0$ and $T > 0$ and with these properties. Taking the constant $T > 0$ sufficiently large, we have $\|f^{(a)} - f^{(0)}\|_0 \leq T |a|$ on $\tilde{U}$ for $|a| \leq a_0$.

The following Lasota-Yorke type inequality is similar to that in [DI], and the corresponding one in [ST] (although we now use a different metric) and its proof is also very similar. We include a proof in the Appendix for completeness.

**Lemma 5.2.** There exists a constant $A_0 > 0$ such that for all $a \in \mathbb{R}$ with $|a| \leq a_0$ the following holds: If the functions $h$ and $H$ on $\tilde{U}$, the constant $B > 0$ and the integer $m \geq 1$ are such that $H > 0$ on $\tilde{U}$ and $|h(v) - h(v')| \leq B H(v') \, D_\theta(v, v')$ for any $i$ and any $v, v' \in \tilde{U}_i$, then for any $b \in \mathbb{R}$ with $|b| \geq 1$ we have

$$|L^m_{ab} h(u) - L^m_{ab} h(u')| \leq A_0 \left[ B \, \theta^m \left( M^m_{a} H(u') + |b| \, (M_{a} H)(u') \right) \right] D_\theta(u, u')$$

whenever $u, u' \in \tilde{U}_i$ for some $i = 1, \ldots, k_0$. ■

**Remark.** It follows from the proof of this lemma that the constant $A_0$ depends only on $\|f\|_\theta$ and some global constants, e.g. $c_0$ and $\gamma$ in (2.1).

### 5.2 First step – using Lemmas 4.2 and 4.3

Let the constants $c_1$ and $c_2$ be as in (5.2). **Fix constants** $\rho \in (0, 1)$ and $q_0 \geq 1$ such that Lemma 4.4(a), (b), (c) and (d) hold with $\rho' = \rho_1/8$ and $q' = q_0$, both for the set $P_0$ and when $P_0$ is replaced by $P_0$. In what follows we will use the entire set-up and notation from Sect. 4, e.g. the subsets $P_0$ and $\tilde{P}_0$ of $\mathcal{L}_0$, the numbers $r_0 \geq r_1 > 0$, $R_0 > 1$, etc., satisfying (4.3), (4.4), etc. Let $\eta_1(x), \eta_2(x), \ldots, \eta_{k_0}(x) (x \in M)$ be families of unit vectors in $E^1_i(x)$ as in the text.
just before Lemma 4.3, and let \( \epsilon'' \in (0, \epsilon') \), \( 0 < \delta'' < \delta' \), \( \delta_0 > 0 \) (depending on the choice of \( \kappa \)), \( \beta_1 \in (0, 1) \), \( C_1 > 0 \) be constants with the properties described in Lemmas 4.3 and 4.2.

Let \( E > 1 \) be a constant – we will see later how large it should be, and let \( \epsilon_1 > 0 \) be a constant with

\[
0 < \epsilon_1 \leq \min \left\{ \frac{1}{32C_0R_0}, \frac{1}{4EC_1R_0^2} \right\}. \tag{5.4}
\]

Having fixes \( \theta \in [\hat{\theta}, 1) \) in (5.1), now **fix a constant** \( \theta_1 \in (0, \theta) \) with

\[
0 < \theta_1 = \frac{1}{\gamma_1} = \theta^{\alpha_2} < \theta,
\]

recalling the choice of \( \alpha_2 > 0 \) in Lemma 5.1(c), and set

\[
\theta_2 = \max \{ \theta, 1/\gamma^2 \alpha \beta_1 \}.
\]

Fix a constant \( A > 0 \) such that the maps \( \tilde{\Psi}^{-1} \circ T_z \) and \( T_z^{-1} \circ \Psi \) are \( A \)-Lipschitz for any \( z \in R \). We will also assume that \( A > 0 \) is so large that for any \( i = 1, \ldots, k_0 \) and any \( y, x, \bar{x} \in R \) we have

\[
\text{diam}(\tilde{\Psi}(W_R^{\tilde{\mu}}(y))) \leq A \text{diam}(\tilde{\Psi}(W_R^{\tilde{\mu}}(\bar{x}))).
\]

Then fix an integer \( N_0 \geq 1 \) with the property described in Lemma 4.3(b), and then take \( N \geq N_0 \) such that

\[
\gamma^N > \frac{1}{\delta''}, \quad \theta^N < \frac{\beta_0^2 \delta_0 \epsilon_1}{256E}, \quad \theta_2^N < \frac{\delta_0 \epsilon_1}{64EAR_0^2}, \tag{5.5}
\]

where \( \beta_1 > 0 \) is the constant from Lemma 4.2. A few additional conditions on \( N \) will be imposed later. Set

\[
\hat{\delta} = \frac{\beta_0 \delta_0 \epsilon_1}{16R_0 C_0}, \tag{5.6}
\]

where \( \beta_0 > 0 \) is defined by (4.6).

**Lemma 5.3.** Let \( C \) be an unstable cylinder in \( R \) of length \( m \geq 1 \) such that \( C \cap P_0 \neq \emptyset \) and \( P^m(C) \cap P_0 \neq \emptyset \).

(a) There exist sub-cylinders \( D \) and \( D_1 \) of \( C \) of co-length \( q_0 \) such that

\[
d(\tilde{\Psi}(y), \tilde{\Psi}(x)) \geq \rho_1 \frac{1}{2} \text{diam}(\tilde{\Psi}(C))
\]

for all \( y \in D_1 \) and \( x \in D \). Moreover, we can take one of the sub-cylinders, e.g. \( D \), so that it contains \( z \).

(b) There exists an integer \( q_1 \geq q_0 \) such that for any sub-cylinder \( C_1 \) of \( C \) of co-length \( q_1 \) with \( P^m(C_1) \cap P_0 \neq \emptyset \) we have

\[
\text{diam}(\tilde{\Psi}(C_1)) \leq \min \left\{ \frac{\rho_1}{8}, \frac{\hat{\delta}}{8C_1} \right\} \text{diam}(\tilde{\Psi}(C)).
\]

**Proofs.** (a) Take \( z, x \in C \) as in Lemma 4.4(c), and let \( D \) and \( D_1 \) be the sub-cylinders of \( C \) of co-length \( q_0 \) containing \( z \) and \( x \), respectively. By Lemma 4.4(b) and the choice of \( q_0 \) and \( \rho' = \rho_1/8 \) it follows that \( \text{diam}((\tilde{\Psi}(D)) \leq \frac{\rho_1}{8} \text{diam}(\tilde{\Psi}(C)) \). Next, by the choice of \( z, x \) in Lemma 4.4(c), for any \( y \in D_1 \) we have

\[
d(\tilde{\Psi}(y), \tilde{\Psi}(z)) \geq \rho_1 \text{diam}(\tilde{\Psi}(C)).
\]

Then

\[
d(\tilde{\Psi}(x), \tilde{\Psi}(y)) \geq d(\tilde{\Psi}(y), \tilde{\Psi}(z)) - d(\tilde{\Psi}(x), \tilde{\Psi}(z)) \geq \rho_1 \text{diam}(\tilde{\Psi}(C)) - \frac{\rho_1}{8} \text{diam}(\tilde{\Psi}(C)) > \frac{\rho_1}{2} \text{diam}(\tilde{\Psi}(C))
\]

for any \( y \in D_1 \) and any \( x \in D \).

(b) This follows from Lemma 4.4(b): take \( q_1 = q_0^r \) for some sufficiently large integer \( r \geq 1 \).
5.3 Main estimates for temporal distances

We will use Lemma 4.3 with \( \kappa = \hat{\rho}/2 \), where \( \hat{\rho} = \rho_{1}^{\frac{\rho_{1}}{8C_{2}}} \), \( \rho_{1} \) being the constant from Lemma 4.4.

Define \( \hat{\delta} \) by (5.6). We will also use the integers \( N_{0} \geq 1 \) and the constants \( \epsilon'' > 0 \) and \( \delta' > \delta'' > 0 \) from Lemma 4.3.

Assume the integer \( n_{0} \geq 1 \) is chosen so large that for any \( z \in R \) and any unstable cylinder \( C \) of length \( \geq n_{0} \) in \( R \) we have \( \text{diam}(\tilde{\Psi}(C)) \leq \epsilon'' \) and \( \text{diam}(T_{z}(C)) \leq \epsilon'' \) for any \( z \in C \).

Given \( m \geq n_{0} \) and \( Z \in P_{0} \setminus \Xi_{m} \), let

\[
y_{\ell,1}(Z) \in B^{n}(Z, \delta') \cap \mathcal{P}^{N}(B^{n}(Z, \epsilon'')), \quad y_{\ell,2}(Z) \in B^{n}(Z, \delta') \cap \mathcal{P}^{N}(B^{n}(Z, \epsilon'')),
\]

\((\ell = 1, \ldots, \ell_{0})\) be families of points satisfying the requirements of Lemma 4.3(b).

**Lemma 5.4.** For any \( m \geq n_{0} \), any point \( Z \in P_{0} \setminus \Xi_{m} \), any integer \( N \geq N_{0} \), any \( \ell = 1, \ldots, \ell_{0} \) and any \( i = 1, 2 \) there exists a (Hölder) continuous map

\[
B^{n}(Z', \epsilon'') \ni x \mapsto v_{i}^{(\ell)}(Z, x) \in U,
\]

where \( Z' = \pi^{(U)}(Z) \in U \), such that \( s^{N}(v_{i}^{(\ell)}(Z, x)) = x \) for all \( x \in B^{n}(Z, \epsilon'') \) and the following property holds:

For any cylinder \( C \) in \( W_{s}^{u}(Z) \) of length \( m \) with \( Z \in C \) and \( \mathcal{P}^{m}(Z) \in P_{0} \) there exist sub-cylinders \( D \) and \( D_{1} \) of \( C \) of co-length \( q_{1} \) and \( \ell = 1, \ldots, \ell_{0} \) such that \( Z \in D \) and for any points \( x \in D \) and \( z \in D_{1} \), setting \( x' = \pi^{(U)}(x), z' = \pi^{(U)}(z) \), we have \( d(T_{Z}(x), T_{Z}(z)) \geq \frac{\hat{\delta}}{2} \text{diam}(T_{Z}(C)) \) and

\[
I_{N,\ell}(x', z') = |\varphi_{\ell}(Z, x') - \varphi_{\ell}(Z, z')| \geq \hat{\delta} \text{diam}(\tilde{\Psi}(C)),
\]

where

\[
\varphi_{\ell}(Z, x) = \tau_{N}(v_{1}^{(\ell)}(Z, x)) - \tau_{N}(v_{2}^{(\ell)}(Z, x)).
\]

Moreover, \( I_{N,\ell}(x', z') \leq C_{1} \text{diam}(\tilde{\Psi}(C)) \) for any \( x, z \in C \), where \( C_{1} > 0 \) is the constant from Lemma 4.2.

**Proof.** Fix for a moment \( Z \in P_{0} \setminus \Xi^{(m)} \), \( N \geq N_{0} \) and \( \ell = 1, \ldots, \ell_{0} \). Assume \( Z \in R_{i_{0}} \). Using Lemma 4.3, there exist points \( y_{\ell,1} = y_{\ell,1}(Z), y_{\ell,2} = y_{\ell,2}(Z) \in W_{s}^{u}(Z) \) such that the property (b) in Lemma 4.3 holds.

Given \( i = 1, 2 \), there exists a cylinder \( L_{i}^{(\ell)} = L_{i}^{(\ell)}(Z) \) of length \( N \) in \( W_{s}^{u}(Z) \) so that

\[
\mathcal{P}^{N} : L_{i}^{(\ell)} \longrightarrow W_{R_{i_{0}}}^{u}(y_{\ell, i})
\]

is a bijection; then it is a bi-Hölder homeomorphism. Consider its inverse and its Hölder continuous extension \( \mathcal{P}^{-N} : W_{R_{i_{0}}}^{u}(y_{\ell, i}) \longrightarrow L_{i}^{(\ell)} \).

Set \( M_{i}^{(\ell)} = M_{i}^{(\ell)}(Z) = \pi^{(U)}(L_{i}^{(\ell)}(Z)) \subset U \); this is then a cylinder of length \( N \) in \( U_{i_{0}} \). Define the maps

\[
\tilde{v}_{i}^{(\ell)}(Z, \cdot) : U_{i_{0}} \longrightarrow L_{i}^{(\ell)} \subset B^{n}(Z, \epsilon'') \quad , \quad v_{i}^{(\ell)}(Z, \cdot) : U_{i_{0}} \longrightarrow M_{i}^{(\ell)} \subset U
\]

by

\[
\tilde{v}_{i}^{(\ell)}(Z, y) = \mathcal{P}^{-N}(\pi_{y_{\ell, i}}(y)) \quad , \quad v_{i}^{(\ell)}(Z, y) = \pi^{(U)}(\tilde{v}_{i}^{(\ell)}(Z, y)).
\]

Then

\[
\mathcal{P}^{N}(\tilde{v}_{i}^{(\ell)}(Z, y)) = \pi_{y_{\ell, i}}(y) = W_{s}^{u}(y) \cap W_{R_{i_{0}}}^{u}(y_{\ell, i}), \quad \tag{5.7}
\]

and

\[
\mathcal{P}^{N}(v_{i}^{(\ell)}(Z, y)) = W_{s}^{u}(y) \cap \mathcal{P}^{N}(M_{i}^{(\ell)}) = \pi_{d_{\ell, i}}(y), \quad \tag{5.8}
\]

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where $d_{\ell,i} = d_{\ell,i}(Z) \in W^s_R(Z)$ is such that $\mathcal{P}^N(M_1^{(\ell)}) = W^s_R(d_{\ell,i})$. Next, there exist $x' \in M_1^{(\ell)}$ and $y' \in L_1^{(\ell)}$ with $\mathcal{P}^N(x') = d_{\ell,i}$ and $\mathcal{P}^N(y') = y_{\ell,i}$. Since stable leaves shrink exponentially fast, using (2.1) and (5.5) we get
\[ d(d_{\ell,i}, y_{\ell,i}) \leq \frac{1}{c_0 \gamma^N} d(x', y') \leq \frac{1}{\gamma^N} < \delta''. \]

Thus, $d_{\ell,1}, d_{\ell,2}$ satisfy the assumptions and therefore the conclusions of Lemma 4.3(b).

Let $C$ be a cylinder in $W^u_{R_0}(Z)$ of length $m$ with $Z \in C \cap P_0 \setminus \Xi_m$ and $\mathcal{P}^m(Z) \in P_0$. Set $C'' = T_Z(C)$ and $\tilde{C} = \Psi(C)$. By the choice of the constant $C_0$, we have
\[ \frac{1}{C_0} \text{diam}(\tilde{C}) \leq \text{diam}(C'') \leq C_0 \text{diam}(\tilde{C}). \]

Let $D$ be the sub-cylinder of $C$ of co-length $q_1$ containing $Z$.

Next, by Lemma 5.3(a), there exists a sub-cylinder $D_1$ of $C$ of co-length $q_1$ such that $d(\tilde{\Psi}(y), \tilde{\Psi}(x)) \geq \frac{\rho_1}{2} \text{diam}(\tilde{C})$ for all $y \in D_1$ and $x \in D$. Thus,
\[ d(T_Z(y), T_Z(x)) \geq \frac{\rho_1}{2C_0^2} \text{diam}(C'') \geq \hat{\rho} \text{diam}(C'') \]
for all $y \in D_1$ and $x \in D$.

Let $x \in D$, $x \neq Z$. Set $x_0 = T_Z(x)$ and let $x_0 = \Phi_Z^u(u_0)$, where $u_0 \in E^u(Z)$. Since $x \neq Z$ we have $u_0 \neq 0$. By the choice of the constant $\beta_0$ and the family of unit vectors $\{\eta_i(Z)\}_{i=1}^{m_0}$, there exists some $\ell = 1, \ldots, \ell_0$ such that
\[ \left\langle \frac{u_0}{\|u_0\|}, \eta_i(Z) \right\rangle \geq \beta_0. \]

Moreover, $d(x_0, Z) \geq \hat{\rho} \text{diam}(T_Z(C))$. It then follows from Lemma 4.3(b) with $\kappa = \hat{\rho}$ and (5.9) that
\[ \frac{\beta_0 \delta \hat{\rho}}{R_0} \text{diam}(T_z(C)) \leq |\Delta(x_0, d_{\ell,1}) - \Delta(x_0, d_{\ell,2})|. \quad (5.10) \]
(In the present situation, since $d_{\ell,1}, d_{\ell,2} \in W^s_R(Z)$, we have $\pi_{d_{\ell,i}}(Z) = d_{\ell,i}$ for $i = 1, 2$.)

Consider the projections of $x, Z$ to $U$ along stable leaves: $x' = \pi(U)(x) \in U_i$, $Z' = \pi(U)(Z) \in U_i$, where as before $R_{i_0}$ is the rectangle containing $Z$ (and therefore $C$). We have
\[
I_{N,\ell}(x', Z') = \left| \tau_N(v_1^{(\ell)}(Z, x')) - \tau_N(v_2^{(\ell)}(Z, x')) - [\tau_N(v_1^{(\ell)}(Z, Z')) - \tau_N(v_2^{(\ell)}(Z, Z'))] \right|
= \left| \tau_N(v_1^{(\ell)}(Z, x')) - \tau_N(v_1^{(\ell)}(Z, Z')) - [\tau_N(v_2^{(\ell)}(Z, x')) - \tau_N(v_2^{(\ell)}(Z, Z'))] \right|
= \Delta(\mathcal{P}^N(v_1^{(\ell)}(Z, x')) - \mathcal{P}^N(v_1^{(\ell)}(Z, Z'))) - \Delta(\mathcal{P}^N(v_2^{(\ell)}(Z, x')) - \mathcal{P}^N(v_2^{(\ell)}(Z, Z')))
= \Delta(x', d_{\ell,1}(Z')) - \Delta(x', d_{\ell,2}(Z'))
= |\Delta(x, d_{\ell,1}) - \Delta(x, d_{\ell,2})|. \]

We claim that the latter is the same as the right-hand-side of (5.10). Indeed, let $\Delta(x, d_{\ell,1}) = s_1$ and $\Delta(x, d_{\ell,2}) = s_2$. Then $\phi_{s_1}([x, d_{\ell,1}]) \in W^u_{R_0}(d_{\ell,1})$ and $\phi_{s_2}([x, d_{\ell,2}]) \in W^u_{R_0}(d_{\ell,2})$. Let $\phi_s(x_0) = x$. It is then straightforward to see that $\Delta(x_0, d_{\ell,1}) = s + s_1$ and $\Delta(x_0, d_{\ell,2}) = s + s_2$. Thus,
\[ |\Delta(x_0, d_{\ell,1}) - \Delta(x_0, d_{\ell,2})| = |(s + s_1) - (s + s_2)| = |s_1 - s_2| = |\Delta(x, d_{\ell,1}) - \Delta(x, d_{\ell,2})|. \]

Combining this with (5.10) gives
\[ I_{N,\ell}(x', Z') \geq \frac{\beta_0 \delta \hat{\rho}}{R_0} \text{diam}(T_z(C)) \geq 2\hat{\delta} \text{diam}(\Psi(C)). \]
For arbitrary \(x, z \in \mathcal{C}\), setting \(x' = \pi^{(U)}(x)\), \(z' = \pi^{(U)}(z)\), the above calculation and Lemma 4.2 give
\[
I_{N,\ell}(x', z') = |\Delta(x, \pi_{d,1}(z)) - \Delta(x, \pi_{d,2}(z))| \leq C_1 \text{diam}(\bar{\Psi}(\mathcal{C})).
\]
The same argument shows that for any \(z \in \mathcal{D}\), using Lemma 5.3(b) and the fact that \(Z \in \mathcal{D}\), we have
\[
I_{N,\ell}(z', Z') = |\Delta(z, d_{\ell,1}) - \Delta(z, d_{\ell,2})| \leq C_1 \text{diam}(\bar{\Psi}(\mathcal{D})) \leq \frac{\hat{\delta}}{8} \text{diam}(\bar{\Psi}(\mathcal{C})).
\]
Similarly, for any \(z \in \mathcal{D}_1\) and \(z_0 = T_Z(z)\) we have
\[
I_{N,\ell}(z', Z') = |\Delta(z_0, d_{\ell,1}) - \Delta(z_0, d_{\ell,2})| \leq C_1 \text{diam}(\bar{\Psi}(\mathcal{D}')) \leq \frac{\hat{\delta}}{8} \text{diam}(\bar{\Psi}(\mathcal{C})).
\]
Since \(\Delta(x, \pi_y(z)) = \Delta(x, y) - \Delta(z, y)\) for any \(y \in W'_\ell(Z)\), it follows that
\[
I_{N,\ell}(x', z') = I_{N,\ell}(x', Z') - I_{N,\ell}(z', Z') \geq 2\hat{\delta} \text{diam}(\bar{\Psi}(\mathcal{C})) - \hat{\delta} \text{diam}(\bar{\Psi}(\mathcal{C})) = \hat{\delta} \text{diam}(\bar{\Psi}(\mathcal{C})).
\]
This completes the proof of the lemma. \(\blacksquare\)

6 Contraction operators

We use the notation and the set-up from Sect. 5.

6.1 Choice of cylinders, definition of the contraction operators

Recall the constants \(R_0\) from (4.4) and \(\epsilon_1\) with (5.4). Below we will consider certain unstable cylinders \(\mathcal{C}\) in \(R\) such that
\[
\hat{\rho} \frac{\epsilon_1}{R_0 |b|} \leq \text{diam}(\bar{\Psi}(\mathcal{C})) \leq \frac{R_0 \epsilon_1}{|b|}, \quad (6.1)
\]
where, as before,
\[
\hat{\rho} = \frac{\rho_1}{8C_0} \in (0, 1),
\]
where \(\rho_1 \in (0, 1)\) is the constant from Lemma 4.4. Then by Lemma 5.1, if \(\ell\) is the length of such \(\mathcal{C}\), then
\[
\frac{-\log C_2 - \alpha_2 \log(R_0 \epsilon_1)}{|\log \hat{\theta}|} + \frac{\alpha_2}{|\log \hat{\theta}|} \log |b| \leq \ell \leq \frac{-\log C_2 - \log(\hat{\rho} \epsilon_1 / R_0)}{|\log \hat{\theta}|} + \frac{1}{|\log \hat{\theta}|} \log |b|,
\]
where \(\hat{\theta} \in (0, 1)\) is given by (5.1). Thus, there exists a global constant \(B > 1\) (independent of \(b\)) such that if \(|b| \geq 10\), then
\[
\frac{1}{B} \log |b| \leq \ell \leq B \log |b|, \quad (6.2)
\]

**Fix a constant** \(B > 1\) with this property. Later we may have to impose some further requirements on \(B\). Then take \(A_0 > 0\) as in Lemma 5.2 such that \(A_0 \geq \frac{2B}{|\log \hat{\theta}|} \).

Next, **fix constants** \(p_0 > 0, \hat{\epsilon}_0, \hat{\delta}_0 \in (0, 1)\) as in Sect. 3 so that
\[
\frac{\hat{\epsilon}_0 \hat{\delta}_0}{\lambda_{\ell, d}^2} \leq e^{\hat{\epsilon}}.
\]

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(This particular choice will be used later in Sect. 9.3.2.) Recall the sets $\Xi_n = \Xi_n(p_0, \hat{\epsilon}_0, \hat{\delta}_0)$ defined by (3.19), and set

$$\Omega_B^{(n)} = \mathcal{L} \setminus \bigcup_{n/B \leq \ell \leq Bn} \Xi_\ell.$$  \hspace{1cm} (6.3)

It follows from Theorem 3.2 that, choosing the constants $C_3, c_3 > 0$ appropriately, we have

$$\mu(R \setminus \Omega_B^{(n)}) \leq C_3 e^{-c_3 n/B}$$  \hspace{1cm} (6.4)

for all $n \geq 1$.  

**Fix integers** $d \geq 1$ and $t_0$ such that

$$c_0 \gamma^d > \frac{1}{\rho}, \quad t_0 \geq \frac{1}{\beta_1 \log \gamma} \left| \log \frac{4C_1 A^2 R_0^2}{\beta_0 \delta_0 \hat{\rho}^{q_1}} \right| + \left| \log c_0 \right| / \log \gamma, \quad s_0 = \frac{1}{|\log \rho_1|} \left| \log \frac{\beta_0 \delta_0 \hat{\rho}^{q_1}}{2C_1 A^5 R_0^3} \right|,$$  \hspace{1cm} (6.5)

where $q_1 \geq 1$ is the constant from Lemma 5.3(b), while $c_0 > 0$ and $\gamma > 1$ are the constants from the end of Sect. 2.

Let $N \geq N_0$ be as in Sect. 5. **Assume also that** $N > t_0 + s_0$. Choose two other constants $0 \leq \delta_1 < \delta_2$ so small that

$$\delta_1 < \frac{\hat{\delta}_0}{100 dB}, \quad \delta_2 = (2d + 3)B \delta_1 < \frac{1}{2}.$$  \hspace{1cm} (6.6)

Set

$$\mu_0 = \mu_0(N, \theta) = \min \left\{ \frac{\theta^{2N+2d}}{6e^{T/(1-\theta)}}, \frac{1}{10 e^{2T/(1-\theta)}} \sin^2 \left( \frac{\hat{\delta} \hat{\rho}_1}{8R_0} \right), \frac{\theta^{s_0 + t_0}}{100 e^{T/(1-\theta)}} \right\},$$  \hspace{1cm} (6.7)

and

$$b_0 = b_0(N, \theta) = \max \left\{ \theta^{-N}, \left( \frac{2C_0^{d/3}}{c_0 \delta_0} \right)^{1/\alpha_1}, R_0 \left( \frac{3C_5 T e^{T/(1-\theta)}}{(1-\theta)} \right)^{1/\alpha_2} \right\},$$  \hspace{1cm} (6.8)

where $\alpha_2 > 0$ is as in Lemma 5.1(c).

Throughout the rest of Sect. 6, $b$ will be a **fixed real number** with $|b| \geq b_0$. Set

$$\hat{b} = |\log |b||.$$  \hspace{1cm} (6.9)

For every $z \in \mathcal{L} \cap R$ denote by $C(z)$ the **maximal cylinder** in $W^{n}_R(z)$ with

$$\text{diam}((\Phi_z)^{-1}(C(z))) \leq \epsilon_1 / |b|.$$  \hspace{1cm} (6.10)

If $\mathcal{P}^{m_z}(C(z)) \cap P_0 \neq \emptyset$, where $m_z$ is the length of $C(z)$, then the maximality and Lemma 4.4(d) imply $\text{diam}((\Phi_z)^{-1}(C(z))) \geq \hat{\rho} \epsilon_1 / |b|$. Define the subset $P_1 = P_1(b)$ of $\mathcal{L}$ by

$$P_1 = \{ z \in \mathcal{L} \cap R : \mathcal{P}^{m_z}(z) \in P_0 \},$$

and set

$$K_0 = \pi^{(U)}(P_1 \cap P_0 \cap \Omega_B^{(\hat{b})}).$$

Next, we define an important family of cylinders in $R$ and $U$ and some sub-cylinders of theirs that will play an important role throughout Sects. 6 and 7.

**Definition 6.1 (Choice of cylinders):** For any $u \in K_0$ amongst the cylinders $C(z)$ with $z \in P_1 \cap P_0 \cap \Omega_B^{(\hat{b})}$ and $\pi^{(U)}(z) = u$, there is **one of maximal length** (and so of smallest $\text{diam}_\theta$). Choose one of these – it has the form $C(Z(u))$ for some $Z(u) \in P_1 \cap P_0 \cap \Omega_B^{(\hat{b})}$ with $\pi^{(U)}(Z(u)) = u$.  

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Set $C'(u) = \pi^{(U)}(C(Z(u)))$. It follows from this choice that for any $z \in R$ with $\pi^{(U)}(z) = u$ we have $C'(u) \subset \pi^{(U)}(C(z))$.

Since the lengths of the cylinders $C'(u)$ are bounded above and $K_0 \subset \bigcup_{u \in K_0} C'(u)$, there exist finitely many different cylinders $C'_m = C'(u_m)$ for some $m = 1, \ldots, m_0$ such that

$$K_0 \subset \bigcup_{m=1}^{m_0} C'_m.$$ Different cylinders have no common interior points, so $C'_m \cap C'_{m'} \cap \widehat{U} = \emptyset$ for $m \neq m'$. For each $m$, set $C_m = C(Z(u_m))$; then

$$Z_m = Z(u_m) \subset C_m \cap P_1 \cap P_0 \cap \Omega_B^{(b)}$$ is so that $\pi^{(U)}(Z_m) = u_m$. According to the definitions of the cylinders $C(z)$, $C_m = C(Z_m)$ is a maximal closed cylinder in $W^u_R(Z_m)$ with diam$(\Phi_z)^{-1}(C_m)) \leq \epsilon_1/|b|$. Let $D_1, \ldots, D_j$ be the list of all closed unstable cylinders in $R$ which are sub-cylinders of co-length $q_1$ of some $C_m$. Here $q_1 \geq 1$ is the constant from Lemma 5.3(b). Set $D_j' = \pi^{(U)}(D_j) \subset U$. Re-numbering the cylinders $D_j$ if necessary, we may assume there exists $j_0 \leq j$ such that $D_1, \ldots, D_{j_0}$ is the list of all sub-cylinders $D_j$ with $D_j \cap P_1 \cap P_0 \cap \Omega_B^{(b)} \neq \emptyset$.

From the choice of the cylinders $C_m$ and Lemmas 4.4 and 5.3, and using (3.7) as well and the fact that $R(x) \leq R_0$ on $P_0$, we get:

$$\hat{\rho} \frac{\epsilon_1}{R_0|b|} \leq \text{diam}(\tilde{\Psi}(C_m)) \leq \frac{R_0\epsilon_1}{|b|} , \quad 1 \leq m \leq m_0. \quad (6.11)$$

If $\ell_m$ is the length of the cylinder $C_m$, it follows from (6.2) that

$$\frac{1}{B} \log |b| \leq \ell_m \leq B \log |b| , \quad m = 1, \ldots, m_0. \quad (6.12)$$

Set

$$V_b = \bigcup_{j=1}^{j_0} D_j' \subset U. \quad (6.13)$$

It follows from the construction that $K_0 \subset V_b$.

We are now ready to define an important family of contraction operators. For any $\ell = 1, \ldots, \ell_0$, $i = 1, 2$ and $j = 1, \ldots, j_0$, consider the unique $m = 1, \ldots, m_0$ with $D_j \subset C_m$, and set

$$v_i^{(\ell)} = v_i^{(\ell)}(Z_m, \cdot) , \quad X_{i,j}^{(\ell)} = v_i^{(\ell)}(D_j') \subset U.$$ where $v_i^{(\ell)}(Z_m, \cdot)$ is the map from Lemma 5.4 for the integer $N$. We will consider this map only on $C'_m$. By Lemma 5.1(a), the characteristic function

$$\omega_{i,j}^{(\ell)} = \chi_{X_{i,j}^{(\ell)}} : \widehat{U} \rightarrow [0, 1]$$

of $X_{i,j}^{(\ell)}$ belongs to $\mathcal{F}_\theta(\widehat{U})$ and $\text{Lip}_\theta(\omega_{i,j}^{(\ell)}) \leq 1/\text{diam}_\theta(X_{i,j}^{(\ell)})$.

A subset $J$ of the set

$$\Pi(b) = \{ (i,j,\ell) : 1 \leq i \leq 2 , 1 \leq j \leq j_0 , 1 \leq \ell \leq \ell_0 \}$$

will be called representative if for every $j = 1, \ldots, j_0$ there exists at most one pair $(i, \ell)$ such that $(i,j,\ell) \in J$, and for any $m = 1, \ldots, m_0$ there exists $(i,j,\ell) \in J$ such that $D_j \subset C_m$. Let $J(b)$ be the family of all representative subsets $J$ of $\Pi(b)$.
Given \( J \in \mathcal{J}(b) \), define the function \( \omega = \omega_J(b) : \hat{U} \rightarrow [0, 1] \) by

\[
\omega = 1 - \mu_0 \sum_{(i,j,t) \in J} \omega_{i,j}^{(t)}.
\]

Clearly \( \omega \in \mathcal{F}_0(\hat{U}) \) and \( \frac{1}{2} \leq 1 - \mu_0 \leq \omega(u) \leq 1 \) for any \( u \in \hat{U} \). Define the \textit{contraction operator}

\[
\mathcal{N} = \mathcal{N}_J(a, b) : \mathcal{F}_0(\hat{U}) \rightarrow \mathcal{F}_0(\hat{U}) \quad \text{by} \quad \mathcal{N}h = \mathcal{M}_a^\mathcal{N}(\omega_J \cdot h).
\]

### 6.2 Main properties of the contraction operators

First, we derive an important consequence of the construction of the cylinders \( \mathcal{C}_m \) and \( \mathcal{D}_j \).

**Lemma 6.2.** If \( \sigma^p(\mathcal{D}_j') \subset \mathcal{C}_k^e \) for some \( p \geq 0 \), \( j \leq j_0 \) and \( k \leq m_0 \), then \( p \leq t_0 \), where \( t_0 \) is as in \((6.5)\). Moreover, the co-length of \( \sigma^p(\mathcal{D}_j') \) in \( \mathcal{C}_k^e \) does not exceed the constant \( s_0 \) from \((6.5)\).

**Proof.** Assume \( \mathcal{D} = \sigma^p(\mathcal{D}_j') \subset \mathcal{C}_k^e \) for some \( p > 0 \), \( j \leq j_0 \) and \( k \leq m_0 \). From the assumptions we get \( \pi_{Z_k}(\mathcal{D}) \subset \mathcal{C}_k \subset W_R^u(Z_k) \).

According to the choice of the sub-cylinders \( \mathcal{D}_j \), there exists \( Z \in \mathcal{D}_j \cap P_t \cap P_0 \cap \Omega_0^b \). Then using Lemma 4.3(a) with \( \kappa = 1/2 \) and Lemma 4.4(c) and an appropriately chosen \( X \in \mathcal{T}_Z(\mathcal{D}_j) \subset W^u_\epsilon(Z) \) with

\[
d(X, Z) \geq \frac{1}{2} \text{diam}(\mathcal{T}_Z(\mathcal{D}_j)) \geq \frac{\beta_0 \delta_0}{2AR_0} \text{diam}(\mathcal{T}_Z(\mathcal{D}_j)).
\]

we can find points \( d_1, d_2 \in W^u_\epsilon(Z) \) such that

\[
|\Delta(X, \pi_{d_1}(Z)) - \Delta(X, \pi_{d_2}(Z))| \geq \frac{\beta_0 \delta_0}{2R_0} \text{diam}(\mathcal{T}_Z(\mathcal{D}_j))
\]

that is

\[
|\Delta(X, d_1) - \Delta(X, d_2)| \geq \frac{\beta_0 \delta_0}{4AR_0^2} \frac{\beta^q \epsilon_1}{|b|}.
\]

Let \( T = \pi_T(Z) \); then \( z = \mathcal{P}^p(Z) = \phi_T(Z) \). Next, consider the points \( d'_1 = \phi_T(d_i) \in W^u_R(z) \) \((i = 1, 2)\) and \( x = \phi_T(X) \in W^u_\epsilon(z) \subset \phi_T(W^u_\epsilon(X)) \). It follows from \((6.15)\) and the properties of temporal distance that

\[
|\Delta(x, d'_1) - \Delta(x, d'_2)| \geq \frac{\beta_0 \delta_0}{4AR_0^2} \frac{\beta^q \epsilon_1}{|b|}.
\]

while \((2.1)\) yields

\[
d(d'_1, d'_2) = d(\mathcal{P}^p(d_1), \mathcal{P}^p(d_2)) \leq \frac{1}{c_0 \gamma^p} d(d_1, d_2) \leq \frac{1}{c_0 \gamma^p}.
\]

Let \( y \in W^u_R(z) \) be such that \( \mathcal{C}_k \subset W^u_R(y) \). Since \( \mathcal{P}^p(\mathcal{D}_j') \subset \mathcal{C}_k^e \), we have \( y \in \mathcal{C}_k \). Using this again, for the point \( x' = \pi_y(x) \in W^u_{\epsilon(c)}(y) \) we have \( \phi_T(x') \in \mathcal{C}_k \) for some \( t \in \mathbb{R} \), so \( x'' = T_y(\phi_t(x')) \in \mathcal{T}_y(\mathcal{C}_k) \). Moreover it is easy to see, using just the definition of the temporal distance function and the fact that \( x'' = \phi_s(x') \) for some \( s \in \mathbb{R} \), that

\[
|\Delta(x, d'_1) - \Delta(x, d'_2)| = |\Delta(x', d'_1) - \Delta(x', d'_2)| = |\Delta(x'', d'_1) - \Delta(x'', d'_2)|.
\]

This and \((6.16)\) give

\[
|\Delta(x'', d'_1) - \Delta(x'', d'_2)| \geq \frac{\beta_0 \delta_0 \beta^q \epsilon_1}{4AR_0^2} \frac{1}{|b|}.
\]


Combining the latter with $\text{diam}(T_y(C_k)) \leq AR_0\epsilon_1/|b|$, $y, x'' \in T_y(C_k)$, $d'_1, d'_2 \in W_R^y(y)$, since $C_k \cap P_1 \cap \Omega_B^{(b)} \neq \emptyset$, Lemma 4.3(b) implies
\[
\frac{\beta_0\delta_0\rho^{n_1}}{4AR_0^2} \frac{\epsilon_1}{|b|} \leq |\Delta(x'', d'_1) - \Delta(x'', d'_2)| \leq C_1\text{diam}(T_y(C_k)) (d(d'_1, d'_2))^{\beta_1} \leq C_1AR_0 \frac{\epsilon_1}{|b|} (d(d'_1, d'_2))^{\beta_1}.
\]
This and (6.17) give
\[
\left( \frac{\beta_0\delta_0\rho^{n_1}}{4C_1A^2R_0^2} \right)^{1/\beta_1} \leq \frac{1}{c\gamma'},
\]
so $p \leq t_0$, where $t_0 > 0$ is the integer from (6.5).

Next, let $s$ be the co-length of $\sigma^p(D'_j)$ in $C_k$. Denote by $Q$ the cylinder in $W_R^y(z)$ such that $Q \parallel C_k$, i.e. $\pi(U)(Q) = \pi(U)(C_k)$. Then $\sigma^p(D_j)$ is a sub-cylinder of $Q$ of co-length $s$, so $D_j$ is a sub-cylinder of co-length $s$ of $Q' = \sigma^{-p}(Q)$. Since $Z \in D_j \cap P_1 \cap P_0 \cap \Omega_B^{(b)}$, it follows from Lemma 4.3(a) with $\kappa = 1/2$ that there exist $x_0 \in T_Z(Q')$ and $y_1, y_2 \in W_R^x(Z)$ such that
\[
\frac{\beta_0\delta_0}{2AR_0^2} \text{diam}(T_Z(Q')) \leq |\Delta(x_0, y_1) - \Delta(x_0, y_2)|.
\]
Setting $x'_0 = \phi_T(x_0) \in T_Z(Q)$ and $y'_i = \phi_T(y_i) \in W_R^y(z)$, $i = 1, 2$, we have
\[
|\Delta(x_0, y_1) - \Delta(x_0, y_2)| = |\Delta(x'_0, y'_1) - \Delta(x'_0, y'_2)|,
\]
so
\[
\frac{\beta_0\delta_0}{2AR_0^2} \text{diam}(T_Z(Q')) \leq |\Delta(x'_0, y'_1) - \Delta(x'_0, y'_2)|.
\]
As above, denoting by $x''_0 \in T_{Z_k}(C_k)$ the unique point such that $x''_0 \in W^{sc}(x'_0)$, and using Lemma 4.3(b), we get
\[
\frac{\beta_0\delta_0}{2AR_0^2} \text{diam}(T_Z(Q')) \leq |\Delta(x'_0, y'_1) - \Delta(x'_0, y'_2)| = |\Delta(x''_0, y'_1) - \Delta(x''_0, y'_2)| \leq C_1\text{diam}(T_{Z_k}(C_k)),
\]
so
\[
\frac{\beta_0\delta_0}{2AR_0^2} \text{diam}(T_Z(Q')) \leq \frac{C_1AR_0 \epsilon_1}{|b|} \leq \frac{C_1A^2R_0 \rho^{n_1} \epsilon_1}{\rho^{n_1}} \frac{A}{|b|} \leq \frac{C_1A^2R_0^2}{\rho^{n_1}} \text{diam}(T_Z(D_j)).
\]
On the other hand, it follows from Lemma 4.4(b) with $\rho' = \rho_1$ that
\[
\text{diam}(T_Z(D_j)) \leq A^2\rho_1^s \text{diam}(T_Z(Q')).
\]
Thus,
\[
\frac{\beta_0\delta_0}{2AR_0^2} \text{diam}(T_Z(Q')) \leq \frac{C_1A^4R_0^2}{\rho^{n_1}} \rho_1^s \text{diam}(T_Z(Q')),
\]
so
\[
\frac{\beta_0\delta_0\rho^{n_1}}{2C_1A^6R_0^2} \leq \rho_1^s,
\]
which implies $s \geq s_0$, the constant from (6.5). \\\n
Given $u, u' \in \widehat{U}$, we will denote by $\ell(u, u') \geq 0$ \textit{the length of the smallest cylinder $Y(u, u')$ in $\widehat{U}$ containing $u$ and $u'$}.

\textbf{Definition 6.3.} Define the \textbf{distance} $D(u, u')$ for $u, u' \in \widehat{U}$ by\footnote{Clearly $D$ depends on the cylinders $C_m$ and therefore on the parameter $b$ as well.} 18

(i) $D(u, u') = 0$ if $u = u'$;
(ii) Let \( u \neq u' \), and assume there exists \( p \geq 0 \) with \( \sigma_p(Y(u, u')) \subset C'_m, \ell(u, u') \geq p \), for some \( m = 1, \ldots, m_0 \). Take the maximal \( p \) with this property and the corresponding \( m \) and set
\[
\mathcal{D}(u, u') = \frac{D_\theta(u, u')}{\text{diam}_\theta(C_m)};
\]

(iii) Assume \( u \neq u' \), however there is no \( p \geq 0 \) with the property described in (ii). Then set \( \mathcal{D}(u, u') = 1 \).

Notice that \( \mathcal{D}(u, u') \leq 1 \) always. Some other properties of \( \mathcal{D} \) are contained in the following, part (b) of which needs Lemma 6.2.

**Lemma 6.4.** Assume that \( u, u' \in \hat{U}, u \neq u', \) and \( \sigma^N(v) = u, \sigma^N(v') = u' \) for some \( v, v' \in \hat{U} \) with \( \ell(v, v') \geq N \). Assume also that there exists \( p \geq 0 \) with \( \sigma_p(Y(u, u')) \subset C'_m, \ell(u, u') \geq p \), for some \( m = 1, \ldots, m_0 \).

(a) We have \( \mathcal{D}(v, v') = \theta^N \mathcal{D}(u, u') \).

(b) Assume in addition that \( \omega_J(v) < 1 \) and \( \omega_J(v') = 1 \) for some \( J \in J(b) \). Then \( p \leq t_0 \) and
\[
|\omega_J(v) - \omega_J(v')| \leq \frac{\mu_0}{\theta^{t_0 + s_0}} \mathcal{D}(u, u').
\]

**Proof.** (a) Let \( p \) be the maximal integer with the given property and let \( m \leq m_0 \) correspond to \( p \). Then \( \sigma^{p+N}(Y(v, v')) \subset C'_m, \ell(v, v') \geq p + N, \) and \( p + N \) is the maximal integer with this property. Thus,
\[
\mathcal{D}(v, v') = \frac{D_\theta(v, v')}{\text{diam}_\theta(C_m)} = \theta^N \frac{D_\theta(u, u')}{\text{diam}_\theta(C_m)} = \theta^N \mathcal{D}(u, u').
\]

(b) \( \omega_J(v) < 1 \) means that \( v \in X^\ell_{i,j} \) for some \( (i, j, \ell) \in J \), and so \( v = v^\ell_{i,j}(u) \) for some \( u \in \mathcal{D}'_j \). Then \( u = \sigma^N(v) \). If \( u' \in \mathcal{D}'_j \), then \( v'' = v^\ell_{i,j}(u') \in X^\ell_{i,j} \) and \( \sigma^N(v'') = u' \), so we must have \( v'' = v' \), which implies \( \omega_J(v') = \omega_J(v^\ell_{i,j}(u')) = 1 \), a contradiction. This shows that \( u' \notin \mathcal{D}'_j \), and so
\[
D_\theta(u, u') \geq \text{diam}_\theta(\mathcal{D}'_j).
\]

Since \( u \in \mathcal{D}'_j, u' \notin \mathcal{D}'_j \) and \( \ell(u, u') \geq p \), it follows that \( \sigma^p(u) \in \sigma^p(\mathcal{D}'_j) \) and \( \sigma^p(u') \notin \sigma^p(\mathcal{D}'_j) \). On the hand, by assumption, \( \sigma^p(u), \sigma^p(u') \in C'_m \). Thus, the cylinder \( \sigma^p(\mathcal{D}'_j) \) must be contained in \( C'_m \). Now Lemma 6.2 gives \( p \leq t_0 \) and the co-length \( s \) of \( \sigma^p(\mathcal{D}'_j) \) in \( C'_m \) is \( s \leq s_0 \). If \( \ell_m = \text{length}(C_m) \), and \( \ell = \text{length}(\mathcal{D}'_j) \) we have \( \ell - p - s = \text{length}(\sigma^p(\mathcal{D}'_j)) - s = \text{length}(C_m) = \ell_m \). Hence
\[
\ell = \ell_m + p + s \leq \ell_m + t_0 + s_0, \text{ and using } \mathcal{D}(u, u') = \frac{D_\theta(u, u')}{\text{diam}_\theta(C_m)}, \text{ we get }
\]
\[
|\omega(v) - \omega(v')| = \mu_0 = \mu_0 \frac{D_\theta(u, u')}{\text{diam}_\theta(\mathcal{D}'_j)} \leq \mu_0 \frac{D_\theta(u, u')}{\text{diam}_\theta(\mathcal{D}'_j)} = \mu_0 \frac{D_\theta(u, u')}{\theta^t} \leq \mu_0 \frac{D_\theta(u, u')}{\theta^{t_0 + s_0}} \mathcal{D}(u, u').
\]

This proves the lemma. \( \blacksquare \)

Given \( E > 0 \) as in Sect. 5.2, let \( \mathcal{K}_E \) be the set of all functions \( H \in \mathcal{F}_\theta(\hat{U}) \) such that \( H > 0 \) on \( \hat{U} \) and
\[
\frac{|H(u) - H(u')|}{H(u')} \leq E \mathcal{D}(u, u')
\]
for all \( u, u' \in \hat{U} \) for which there exists an integer \( p \geq 0 \) with \( \sigma^p(Y(u, u')) \subset C_m \) for some \( m \leq m_0 \) and \( \ell(u, u') \geq p \).
Using Lemma 6.4 we will now prove the main lemma in this section, which makes it possible to use an inductive procedure involving the contraction operators \( \mathcal{N}_j \).

**Lemma 6.5.** For any \( J \in J(b) \) we have \( \mathcal{N}_j(\mathcal{K}_E) \subset \mathcal{K}_E \).

**Proof.** Let \( u, u' \in \hat{U} \) be such that there exists an integer \( p \geq 0 \) with \( \omega^p(Y(u, u')) \subset \mathcal{C}_m \) for some \( m = 1, \ldots, m_0 \) and \( \ell(u, u') \geq p \).

Given \( v \in \hat{U} \) with \( \sigma^N(v) = u \), let \( C[i] = C[i_0, \ldots, i_N] \) be the cylinder of length \( N \) containing \( v \). Set \( \hat{C}[i] = C[i] \cap \hat{U} \). Then \( \sigma^N(\hat{C}[i]) = \hat{U}_i \). Moreover, \( \sigma^N : \hat{C}[i] \rightarrow \hat{U}_i \) is a homeomorphism, so there exists a unique \( v' = v'(v) \in \hat{C}[i] \) such that \( \sigma^N(v') = u' \). Then

\[
D_\theta(\sigma^i(v), \sigma^i(v'(v))) = \theta^{N-j} D_\theta(u, u')
\]

for all \( j = 0, 1, \ldots, N - 1 \). Also \( D_\theta(v, v'(v)) = \theta^N D_\theta(u, u') \) and \( D(v, v'(v)) = \theta^N D(u, u') \). Using (5.3), we get

\[
|f^a_N(v) - f^a_N(v')| \leq \sum_{j=0}^{N-1} |f^a_i(\sigma^j(v)) - f^a_i(\sigma^j(v'))| \leq \sum_{j=0}^{N-1} |f^a_i| \theta^{N-j} D_\theta(u, u') \leq \frac{T}{1-\theta} D_\theta(u, u'). \tag{6.18}
\]

Let \( J \in J(b) \) and let \( H \in \mathcal{K}_E \). Set \( \mathcal{N} = \mathcal{N}_J \). We will show that \( \mathcal{N}H \in \mathcal{K}_E \).

Using the above and the definition of \( \mathcal{N} = \mathcal{N}_J \), and setting \( v' = v'(v) \) for brevity, we get

\[
\begin{align*}
\frac{|(\mathcal{N}H)(u) - (\mathcal{N}H)(u')|}{\mathcal{N}H(u')} &\leq \frac{\left| \sum_{\sigma^N v = u} e^{f^a_N(v)} \omega(v)H(v) - \sum_{\sigma^N v = u} e^{f^a_N(v'(v))} \omega(v'(v))H(v'(v)) \right|}{\mathcal{N}H(u')} \\
&\leq \frac{\left| \sum_{\sigma^N v = u} e^{f^a_N(v)} [\omega(v)H(v) - \omega(v')H(v')] \right|}{\mathcal{N}H(u')} + \frac{\left| \sum_{\sigma^N v = u} \left| e^{f^a_N(v)} - e^{f^a_N(v')} \right| \omega(v')H(v') \right|}{\mathcal{N}H(u')} \\
&\leq \frac{\left| \sum_{\sigma^N v = u} \left| e^{f^a_N(v)} - f^a_N(v') \right| e^{f^a_N(v')} \omega(v)H(v) \right|}{\mathcal{N}H(u')} + \frac{\left| \sum_{\sigma^N v = u} \left| e^{f^a_N(v)} - f^a_N(v') \right| \omega(v')H(v') \right|}{\mathcal{N}H(u')} \\
&\leq \frac{\left| e^{f^a_N(v)} - f^a_N(v') \right| e^{f^a_N(v')} \omega(v)H(v)}{\mathcal{N}H(u')} + \frac{\left| e^{f^a_N(v)} - f^a_N(v') \right| \omega(v')H(v')}{\mathcal{N}H(u')} \leq \frac{T}{1-\theta} D_\theta(u, u').
\end{align*}
\]

By the definition of \( \omega \), either \( \omega(v) = \omega(v') \) or at least one of these numbers is \( < 1 \). Using Lemma 6.4 we then get \( |\omega(v) - \omega(v')| \leq \frac{\mu_0}{\theta^{t_0+\theta}} D(u, u') \). Apart from that \( H \in \mathcal{K}_E \) implies

\[
|H(v) - H(v')| \leq E H(v') D(v, v') = E H(v') \theta^N D(u, u'),
\]

while

\[
\left| e^{f^a_N(v)} - f^a_N(v') \right| \leq e^{T/(1-\theta)} \frac{T}{1-\theta} D_\theta(u, u').
\]

Thus,

\[
\frac{|(\mathcal{N}H)(u) - (\mathcal{N}H)(u')|}{\mathcal{N}H(u')} \leq e^{T/(1-\theta)} \frac{\mu_0}{\theta^{t_0+\theta}} \sum_{\sigma^N v = u} e^{f^a_N(v')} D(u, u') H(v')
\]

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Assume that the points \( \ell \) for some \( \theta \).

Clearly, \( \ell \) and \( \theta \) will denote \( \tilde{u} \), \( \tilde{v} \), \( \hat{u} \), \( \hat{v} \).

Recall the numbers \( \theta \), \( \theta \) and taking \( \theta \) and assuming \( (6.7) \) and Lemma 5.1, and assuming

\[
2e^{T/(1-\theta)} \theta^N \leq 1/3 \quad \text{and} \quad e^{T/(1-\theta)} C_2 (R_0 \varepsilon_1 / |b|)^{\alpha_2} \frac{T}{(1-\theta)} \leq \frac{1}{3} \leq \frac{E}{3}.
\]

The latter follows from \( |b| \geq b_0 \) and \( (6.8) \). Hence \( NH \in \mathcal{K}_E \).

6.3 Return times’ estimates

Recall the numbers \( \theta, \theta \in (0, 1) \) defined in the beginning of Sect. 5.2. Then using the proof of Lemma 5.1(c) and taking \( C_2 > 0 \) sufficiently large we have\(^{19}\)

\[
diam \theta(C) \leq C_2 \text{diam}(\tilde{\Psi}(C)) \quad (6.19)
\]

for any cylinder \( C \) in \( U \).

Throughout the rest of this section we assume that \( f \in \mathcal{F}_{\theta} (\hat{U}) \).

Given a cylinder \( C_m \), recall the point \( Z_m \in \mathcal{C}_m \) chosen in Sect. 6.1. For points \( u, u' \in U \) we will denote \( \tilde{u} = T_{Z_m}(\pi_{Z_m}(u)) \) and \( \tilde{u}' = T_{Z_m}(\pi_{Z_m}(u')) \); these are then points on the true unstable manifold \( \hat{W}_u(Z_m) \). In this section we will frequently work under the following assumption for points \( u, u' \in \hat{U} \) contained in some cylinder \( C_m = \pi(U)(C_m) \) \( (1 \leq m \leq m_0) \), an integer \( p \geq 0 \) and points \( v, v' \in \hat{U} \):

\[
u, u' \in C_m, \quad \sigma^p(v) = v^{(\ell)}_i(u), \quad \sigma^p(v') = v^{(\ell)}_i(u'), \quad \ell(v, v') \geq N, \quad (6.20)
\]

for some \( i = 1, 2 \). From \( (6.20) \) we get \( \ell(v, v') \geq N + p \) and \( \sigma^{N+p} (v) = u, \sigma^{N+p} (v') = u' \). We will use the notation

\[
\hat{C}_m = \hat{\Psi}(C_m) \subset \hat{R}.
\]

The following estimate plays a central role in this section.

**Lemma 6.6.** There exists a global constant \( C_3 > 0 \) independent of \( b \) and \( N \) such that if the points \( u, u' \in \hat{U} \), the cylinder \( C_m \), the integer \( p \geq 0 \) and the points \( v, v' \in \hat{U} \) satisfy \( (6.20) \) for some \( i = 1, 2 \) and \( \ell = 1, \ldots, \ell_0 \), and \( w, w' \in \hat{U} \) are such that \( \sigma^N w = v, \sigma^N w' = v' \) and \( \ell(w, w') \geq N \), then

\[
|\tau_N(w) - \tau_N(w')| \leq C_3 \theta_2^{p+N} \text{diam}(\hat{C}_m).
\]

**Proof.** Assume that the points \( u, u', v, v', w, w' \) and the cylinder \( C \) satisfy the assumptions in the lemma. Clearly, \( \ell(w, w') \geq p + 2N \) and

\[
\tau_N(w) - \tau_N(w') = [\tau_{p+2N}(w) - \tau_{p+2N}(w')] - [\tau_{p+N}(v) - \tau_{p+N}(v')]. \quad (6.21)
\]

Recall the construction of the map \( v^{(\ell)}_i \) from the proof of Lemma 5.4. In particular by \( (5.8) \), \( \mathcal{P}^N (v^{(\ell)}_i (u)) = \pi_{d_{\ell_i}} (u) \), where we set \( d_{\ell_i} = d_{\ell_i} (Z_m) \in W^s_R (Z_m) \) for brevity. Since \( \sigma^p(v) =

\(^{19}\)Notice that for \( (6.19) \) choosing \( \theta_1 \) with \( \theta_1^2 \leq \theta \) would be enough. However in the beginning of Sect. 6.1 we imposed a stronger condition on \( \theta_1 \) which will be used later on (see the end of the proof of Theorem 1.3 in Sect. 8).
and (2.1) for points on local stable manifolds, i.e. going back along the flow, we get

\[ C_\alpha \]

the local stable/unstable holonomy maps are uniformly

\[ P \]

Lemma 6.7. This, (6.21), Lemma 4.2 and the above estimate yield

\[ W \]

\[ \tau \]

\[ \phi \]

\[ \pi \]

\[ \Delta \]

\[ d \]

\[ v \]

\[ u \]

\[ \tilde{u} \]

\[ \tilde{u}' \]

\[ \phi_{t(u)}(\tilde{u}) \]

\[ \phi_{t(u')}(\tilde{u}') \]

\[ (d', d'') \]

\[ \tau_{p+N}(v) - \tau_{p+N}(v') = \Delta(P^{p+N}(v), P^{p+N}(v')) = \Delta(u, \pi_{d'}(u')) = \Delta(\tilde{u}, \pi_{d'}(\tilde{u}')) \]

\[ = \Delta(\tilde{u}, \pi_{d'}(\tilde{u}')) + t(u) - t(u') , \]

and similarly

\[ \tau_{p+2N}(w) - \tau_{p+2N}(w') = \Delta(P^{p+2N}(w), P^{p+2N}(w')) = \Delta(\tilde{u}, \pi_{d''}(\tilde{u}')) = \Delta(\tilde{u}, \pi_{d''}(\tilde{u}')) + t(u) - t(u') . \]

This, (6.21), Lemma 4.2 and the above estimate yield

\[ |\tau_N(w) - \tau_N(w')| = |[\Delta(\tilde{u}, \pi_{d'}(\tilde{u}')) + t(u) - t(u')] - [\Delta(\tilde{u}, \pi_{d''}(\tilde{u}')) + t(u) - t(u')]| \]

\[ = |\Delta(\tilde{u}, \pi_{d'}(\tilde{u}')) - \Delta(\tilde{u}, \pi_{d''}(\tilde{u}'))| \leq C_1 \text{diam}(\tilde{C}_m) (d(d', d''))^\beta \]

\[ \leq C_1 C_3 \theta_2^{p+N} \text{diam}(\tilde{C}_m) . \]

This proves the lemma. ■

Set \( M_1 = M_0 + a_0 \) (see Sect. 2 for the choice of \( M_0 \)) and let

\[ E_1 = 2C_4 e^{C_4} \quad \text{where} \quad C_4 = \frac{T_0 C_2}{1 - \theta} + M_1 C_3 , \]

and \( C_3 > 0 \) is the constant from Lemma 6.6. Assume \( N \) is so large that

\[ \theta_2^N e^{C_4} \leq \frac{1}{2} . \]

Denote by \( \mathcal{K}_0 \) the set of all \( h \in \mathcal{F}_0(\tilde{U}) \) such that \( h \geq 0 \) on \( \tilde{U} \) and for any \( u, u' \in \tilde{U} \) contained in some cylinder \( \mathcal{C}_m' = \pi(\tilde{U})(\tilde{C}_m) \) \( (1 \leq m \leq m_0) \), any integer \( p \geq 0 \) and any points \( v, v' \in \tilde{U} \) satisfying \( (6.20) \) for some \( i = 1, 2 \) and \( \ell = 1, \ldots, \ell_0 \) we have

\[ |h(v) - h(v')| \leq E_1 \theta_2^{p+N} h(v') \text{diam}(\tilde{C}_m) . \]  \hspace{1cm} (6.22)

We are going to show that the eigenfunctions \( h_a \in \mathcal{K}_0 \) for \( |a| \leq a_0 \) (see Sect. 5.1). This will be derived from the following.

**Lemma 6.7.** For any real constant \( s \) with \( |s| \leq M_1 \) we have \( L^N_{\ell-s \tau}(\mathcal{K}_0) \subset \mathcal{K}_0 \) for all integers \( q \geq N \).
Proof. We will use Lemma 6.6 and a standard argument.

Assume that \(u, u' \in \tilde{U}\), the cylinder \(\tilde{C}_m\) in \(U\), the integer \(p \geq 0\) and the points \(v, v' \in \tilde{U}\) satisfy (6.20) for some \(i = 1, 2\) and \(\ell = 1, \ldots, \ell_0\), and \(w, w' \in \tilde{U}\) are such that \(\sigma^N w = v\), \(\sigma^N w' = v'\) and \(\ell(w, w') \geq N\); then \(w' = w'(w)\) is uniquely determined by \(w\).

Using \(f \in F_{\theta_1}(\tilde{U})\), the choice of \(\theta_1\) and (6.19), we get
\[
|f_N(w) - f_N(w')| \leq \frac{T_0}{1 - \theta_1} D_{\theta_1}(v, v') = \frac{T_0}{1 - \theta_1} \theta^{p+N}_1 D_{\theta_1}(u, u') \\
\leq \frac{T_0}{1 - \theta_1} \theta^{p+N}_1 \text{diam}_{\theta_1}(\mathcal{C}_m) \leq C_2 \theta^{p+N}_2 \text{diam}(\tilde{C}_m),
\]

where \(T_0 = |f|_\theta\) and \(C_2 = C_2 T_0/(1 - \theta_1)\). This and Lemma 6.6 imply
\[
|(f - s\tau)_N(w) - (f - s\tau)_N(w')| \leq C_4 \theta^{p+N}_2 \text{diam}(\tilde{C})
\]
for all \(s \in \mathbb{R}\) with \(|s| \leq M_1\), where \(C_4 > 0\) is as above.

Thus, given \(s\) with \(|s| \leq M_1\) and \(h \in K_0\) we have:
\[
\begin{align*}
&\left|\left((L_{p, s, h}^N)\right)(v) - \left((L_{p, s, h}^N)\right)(v')\right| = \left|\sum_{\sigma^N w = v} e^{(f - s\tau)_N(w)} h(w) - \sum_{\sigma^N w = v} e^{(f - s\tau)_N(w'(w))} h(w'(w))\right| \\
&\leq \left|\sum_{\sigma^N w = v} e^{(f - s\tau)_N(w)} [h(w) - h(w')]}\right| + \sum_{\sigma^N w = v} \left|e^{(f - s\tau)_N(w)} - e^{(f - s\tau)_N(w')}\right| h(w') \\
&\leq \sum_{\sigma^N w = v} e^{(f - s\tau)_N(w) - (f - s\tau)_N(w')} e^{(f - s\tau)_N(w')} E_1 \theta^{p+2N}_2 \text{diam}(\tilde{C}_m) h(w') \\
&+ \sum_{\sigma^N w = v} \left|e^{(f - s\tau)_N(w) - (f - s\tau)_N(w') - 1}\right| e^{(f - s\tau)_N(w')} h(w') \\
&\leq E_1 \theta^{p+2N}_2 \text{diam}(\tilde{C}_m) e^{C_4} (L_{p, s, h}^N)(v') + e^{C_4} C_4 \theta^{p+N}_2 \text{diam}(\tilde{C}_m) (L_{p, s, h}^N)(v') \\
&\leq E_1 \theta^{p+N}_2 \text{diam}(\tilde{C}_m) (L_{p, s, h}^N)(v'),
\end{align*}
\]
since \(e^{C_4} C_4 \leq E_1/2\) and \(\theta^{p+N}_2 \leq 1/2\) by the choice of \(N\). Hence \(L_{p, s, h}^N \in K_0\). ■

Lemma 6.7 is required to prove the following.

Corollary 6.8. For any real constant \(a\) with \(|a| \leq a_0\) we have \(h_a \in K_0\).

Proof. Let \(|a| \leq a_0\). Since the constant function \(h = 1 \in K_0\), it follows from Lemma 6.7 that \(L_{p, a}^N \in K_0\) for all \(m \geq 0\). Now the Ruelle-Perron-Frobenius Theorem (see e.g. [PP]) and the fact that \(K_0\) is closed in \(F_{\theta}(\tilde{U})\) imply \(h_a \in K_0\). ■

6.4 Estimates of \(L_{ab}^N\) by contraction operators

We will now define a class of pairs of functions similar to \(K_0\) however involving the parameter \(b\). We continue to assume that \(f \in F_{\theta_1}(\tilde{U})\).

Denote by \(K_b\) the set of all pairs \((h, H)\) such that \(h \in F_{\theta_1}(\tilde{U})\), \(H \in K_E\), and the following two conditions are satisfied:

(i) \(|h| \leq H\) on \(\tilde{U}\),

(ii) for any \(u, u' \in \tilde{U}\) contained in a cylinder \(\tilde{C}_m = \pi(U)(\tilde{C}_m)\) for some \(m = 1, \ldots, m_0\), any integer \(p \geq 0\) and any points \(v, v' \in \tilde{U}\), satisfying (6.20) for some \(i = 1, 2\) and \(\ell = 1, \ldots, \ell_0\) we have
\[
|h(v) - h(v')| \leq E |b| \text{diam}(\tilde{C}_m).
\]
Recall that here \( \tilde{C}_m = \overline{\Psi}(C_m) \).

Our aim in this section is to prove the following.

**Lemma 6.9.** Choosing \( E > 1 \) and \( \mu_0 \) as in Sect. 5.2 and assuming \( N \) is sufficiently large, for any \( |a| \leq h_0, \) any \( |b| \geq b_0 \) and any \( (h, H) \in K_b \) there exists \( J \in J(b) \) such that \( (L^N_{ab}h, N_{J}H) \in K_b \).

To prove this we need the following lemma, whose proof is essentially the same as that of Lemma 14 in [D1]. For completeness we prove it in the Appendix.

**Lemma 6.10.** Let \( (h, H) \in K_b \). Then for any \( m \leq m_0 \), any \( j = 1, \ldots, j_0 \) with \( D_j \subset C_m \), any \( i = 1, 2 \) and \( \ell = 1, \ldots, \ell_0 \) we have:

\[
\frac{1}{2} \leq \frac{H(v^{(\ell)}_i)(u')}{H(v^{(\ell)}_i)(u'')} \leq 2 \text{ for all } u', u'' \in D'_j;
\]

(b) Either for all \( u \in D'_j \) we have \( |h(v^{(\ell)}_i)(u)| \leq \frac{2}{3}H(v^{(\ell)}_i)(u) \), or \( |h(v^{(\ell)}_i)(u)| \geq \frac{1}{3}H(v^{(\ell)}_i)(u) \) for all \( u \in D'_j \).

**Proof of Lemma 6.9.** The constant \( E_1 > 1 \) from Sect. 6.3 depends only on \( C_4 \), and we take \( N \) so large that \( E_1 \theta_2^N \leq 1/4 \); then \( C_4 \theta_2^N \leq 1/2 \) holds too.

Let \( |a| \leq a_0, \) \( |b| \geq b_0 \) and \( (h, H) \in K_b \). We will construct a representative set \( J \in J(b) \) such that \( (L^N_{ab}h, N_{J}H) \in K_b \).

Consider for a moment an arbitrary (at this stage) representative set \( J \). We will first show that \( (L^N_{ab}h, N_{J}H) \) has property (ii).

Assume that the points \( u, u' \), the cylinder \( C_m \) in \( U \), the integer \( p \geq 0 \) and the points \( v, v' \) satisfy (6.20) for some \( i = 1, 2 \) and \( \ell = 1, \ldots, \ell_0 \).

From the definition of \( f^{(a)} \), for any \( w, w' \) with \( \sigma N = v, \sigma N(w') = v' \) and \( \ell(w, w') \geq N \) we have

\[
f^{(a)}_N(w) = f_N(w) - (P + a)\tau_N(w) + (\ln h_a - \ln h_a \circ \sigma)_N(w) - N\lambda_a.
\]

Since \( h_a \in K_0 \) by Corollary 6.8,

\[
|\ln h_a(w) - \ln h_a(w')| \leq \frac{|h_a(w) - h_a(w')|}{\min\{|h_a(w)|, |h_a(w')|\}} \leq E_1 \theta_2^{p+2N} \text{ diam}(\tilde{C}_m),
\]

and similarly, \( |\ln h_a(v) - \ln h_a(v')| \leq E_1 \theta_2^{p+2N} \text{ diam}(\tilde{C}_m) \). Using this and Lemma 6.6, as in the proof of Lemma 6.7 we get

\[
|f^{(a)}_N(w) - f^{(a)}_N(w')| \leq C_4 \theta_2^{p+2N} \text{ diam}(\tilde{C}_m) + 2E_1 \theta_2^{p+2N} \text{ diam}(\tilde{C}_m) \leq (C_4 + 2E_1) \theta_2^{p+2N} \text{ diam}(\tilde{C}_m) \leq 1,
\]

(6.24)

by the choice of \( N \).

Hence for any \( a \) and \( b \) with \( |a| \leq a_0 \) and \( |b| \geq b_0 \), using (6.24) and Lemma 6.6, we get

\[
|L^N_{ab}h(w) - (L^N_{ab}h)(w')| = \left| \sum_{\sigma N = v} e^{f^{(a)}_N - \cdot \ln h_a}(w) h(w) - \sum_{\sigma N = v} e^{f^{(a)}_N - \cdot \ln h_a}(w') h(w') \right|
\]

\[
\leq \left| \sum_{\sigma N = v} e^{f^{(a)}_N - \cdot \ln h_a}(w) [h(w) - h(w')] \right| + \left| \sum_{\sigma N = v} e^{f^{(a)}_N - \cdot \ln h_a}(w) - e^{f^{(a)}_N - \cdot \ln h_a}(w') \right| |h(w')|
\]

\[
\leq \sum_{\sigma N = v} e^{f^{(a)}_N - f^{(a)}_N}(w) E|b|\theta_2^{p+2N} \text{ diam}(\tilde{C}_m) H(w').
\]

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\[\begin{align*}
&+ \sum_{\sigma^N_{u'=v}} \left| e^{i\sigma_N^a(\theta)}(u) - e^{i\sigma_N^a(\theta)}(u') - 1 \right| e^{i\sigma_N^a}(u')H(u') \\
\leq& \ eE|b|\theta_2^{p+2N} \text{diam}(\bar{C}_m)(M_3^N H)(v') + e(C_4 + 2E_1 + C_3|b|)\theta_2^{p+2N} \text{diam}(\bar{C}_m)(M_3^N H)(v') \\
\leq& \ [2eE\theta_2^N + 2e(C_4 + 2E_1 + C_3)]|b|\theta_2^{p+2N} \text{diam}(\bar{C}_m)(N_J H)(v') \leq E|b|\theta_2^{p+2N} \text{diam}(\bar{C}_m)(N_J H)(v'),
\end{align*}\]

assuming \(2e\theta_2^N \leq 1/2\) and \(2e(C_4 + 2E_1 + C_3) \leq E/2\). Thus, \((L_{ab}^N h, N_J H)\) has property (ii).

So far the choice of \(J\) was not important. We will now construct a representative set \(J\) so that \((L_{ab}^N h, N_J H)\) has property (i), namely

\[|L_{ab}^N h|(u) \leq (N_J H)(u), \quad u \in \hat{U}. \quad (6.25)\]

Define the functions \(\psi_{\ell}, \gamma_{\ell}^{(1)}, \gamma_{\ell}^{(2)} : \hat{U} \rightarrow \mathbb{C}\) by

\[\psi_{\ell}(u) = e^{i\sigma_N^a(\theta)}(v_1^\ell(u))h(v_1^\ell(u)) + e^{i\sigma_N^a(\theta)}(v_2^\ell(u))h(v_2^\ell(u)),\]

\[\gamma_{\ell}^{(1)}(u) = (1 - \mu_0)e^{i\sigma_N^a(\theta)}(v_1^\ell(u))H(v_1^\ell(u)) + e^{i\sigma_N^a(\theta)}(v_2^\ell(u))H(v_2^\ell(u)),\]

while \(\gamma_{\ell}^{(2)}(u)\) is defined similarly with a coefficient \((1 - \mu_0)\) in front of the second term.

Notice that (6.25) is trivially satisfied for \(u \notin V_0\) for any choice of \(J\).

Consider an arbitrary \(m = 1, \ldots, m_0\). We will construct \(j \leq j_0\) with \(D_j \subset C_m\), and a pair \((i, \ell)\) for which \((i, j, \ell)\) will be included in \(J\).

Recall the functions \(\varphi_{\ell,m}(u) = \varphi_{\ell}(Z_m, u), \ u \in U, \) from Lemma 5.4.

**Case 1.** There exist \(j \leq j_0\) with \(D_j \subset C_m, \ i = 1, 2\) and \(\ell \leq \ell_0\) such that the first alternative in Lemma 6.10(b) holds for \(D_j, i\) and \(\ell\). For such \(j\), choose \(i = i_j\) and \(\ell = \ell_j\) with this property and include \((i, j, \ell)\) in \(J\). Then \(\mu_0 \leq 1/4\) implies \(|\psi_{\ell}(u)| \leq \gamma_{\ell}^{(i)}(u)\) for all \(u \in D_j\), and regardless how the rest of \(J\) is defined, (6.25) holds for all \(u \in D_j\), since

\[\begin{align*}
|L_{ab}^N h|(u) &\leq \sum_{\sigma^N_{u'=v}} e^{i\sigma_N^a(\theta)}(v)h(v) + |\psi_{\ell}(u)| \\
&\leq \sum_{\sigma^N_{u'=v}} e^{i\sigma_N^a(\theta)}|h(v)| + \gamma_{\ell}^{(i)}(u) \\
&\leq \sum_{\sigma^N_{u'=v}} e^{i\sigma_N^a(\theta)}|\omega_J(v)H(v) + \left[e^{i\sigma_N^a(\theta)}(v_1^\ell(u))\omega_J(v_1^\ell(u))H(v_1^\ell(u)) + e^{i\sigma_N^a(\theta)}(v_2^\ell(u))\omega_J(v_2^\ell(u))H(v_2^\ell(u))\right] \\
&\leq (N_J H)(u). \quad (6.26)
\end{align*}\]

**Case 2.** For all \(j \leq j_0\) with \(D_j \subset C_m, \ i = 1, 2\) and \(\ell \leq \ell_0\) the second alternative in Lemma 6.10(b) holds for \(D_j, i\) and \(\ell,\) i.e.

\[|h(v_1^\ell(u))| \geq \frac{1}{4} H(v_1^\ell(u)) > 0 \quad (6.27)\]

for any \(u \in C_m\).

Let \(u, u' \in C_m\), and let \(i = 1, 2\). Using (6.23) and the assumption that \((h, H) \in K_h\), and in particular property (ii) with \(p = 0, \ v = v_1^\ell(u)\) and \(v' = v_1^\ell(u')\), and assuming e.g.

\[\min\{|h(v_1^\ell(u))|, |h(v_1^\ell(u'))|\} = |h(v_1^\ell(u'))|,\]
such that for all $u \in C'_m$, with values in $[0, \pi/6]$ and a constant $\lambda^{(m)}_i$ such that
\[
   h(v^{(l)}_i(u)) = e^{i(\lambda^{(m)}_i + \theta^{(m)}_i(u))} |h(v^{(l)}_i(u))|, \quad u \in C'_m.
\]
Fix an arbitrary $u_0 \in C'_m$ and set $\lambda^{(m)} = |b| \varphi_{\ell,m}(u_0)$. Replacing e.g. $\lambda^{(m)}_2$ by $\lambda^{(m)}_2 + 2r \pi$ for some integer $r$, we may assume that $|\lambda^{(m)}_2 - \lambda^{(m)}_1 + \lambda^{(m)}| \leq \pi$.

Using the above, $\theta \leq 2 \sin \theta$ for $\theta \in [0, \pi/3]$, and some elementary geometry yields
\[
   |\theta_i^{(m)}(u) - \theta_i^{(m)}(u')| \leq 2 \sin |\theta_i^{(m)}(u) - \theta_i^{(m)}(u')| < 16 E A R_0 \theta_2^N \epsilon_1
\]
for all $u, u' \in C'_m$.

The difference between the arguments of the complex numbers $e^{ib\tau_N(v^{(l)}_i(u))} h(v^{(l)}_i(u))$ and $e^{ib\tau_N(v^{(l)}_i(u'))} h(v^{(l)}_i(u'))$ is given by the function
\[
   \Gamma_\ell(u) = [b \tau_N(v^{(l)}_i(u)) + \theta^{(m)}_i(u) + \lambda^{(m)}_2] - [b \tau_N(v^{(l)}_i(u)) + \theta^{(m)}_i(u) + \lambda^{(m)}_1]
   = (\lambda^{(m)}_2 - \lambda^{(m)}_1) + |b| \varphi_{\ell,m}(u) + (\theta^{(m)}_2(u) - \theta^{(m)}_1(u)).
\]

It follows from Lemma 5.4 that there exist $j, j' \leq j_0$ with $j \neq j'$, $D_j, D_{j'} \subset C_m$, and $\ell = 1, \ldots, \ell_0$ such that for all $u \in D_j$ and $u' \in D_{j'}$ we have
\[
   \frac{\delta \rho \epsilon_1}{R_0 |b|} \leq \delta \text{ diam}(\Psi(C_m)) \leq |\varphi_{\ell,m}(u) - \varphi_{\ell,m}(u')| \leq C_1 \text{ diam}(\Psi(C_m)) \leq C_1 \frac{R_0 \epsilon_1}{|b|}. \quad (6.28)
\]

Fix $\ell_m = \ell$ with this property. Then for $u \in D_j'$ and $u' \in D_{j'}$, we have
\[
   |\Gamma_\ell(u) - \Gamma_\ell(u')| \geq |b| |\varphi_{\ell,m}(u) - \varphi_{\ell,m}(u')| - |\theta_1^{(m)}(u) - \theta_1^{(m)}(u')| - |\theta_2^{(m)}(u) - \theta_2^{(m)}(u')|
   \geq \frac{\delta \rho \epsilon_1}{R_0} - 32 E A \theta_2^N A R_0 \epsilon_1 > 2 \epsilon_3,
\]
since $32 E A \theta_2^N < \frac{\delta \rho}{2 R_0}$ by (5.5), (5.6), where
\[
   \epsilon_3 = \frac{\delta \rho \epsilon_1}{4 R_0}.
\]

Thus, $|\Gamma_\ell(u) - \Gamma_\ell(u')| \geq 2 \epsilon_3$ for all $u \in D_j'$ and $u' \in D_{j'}$. Hence either $|\Gamma_\ell(u)| \geq \epsilon_3$ for all $u \in D_j'$ or $|\Gamma_\ell(u')| \geq \epsilon_3$ for all $u' \in D_{j'}$.

Assume for example that $|\Gamma_\ell(u)| \geq \epsilon_3$ for all $u \in D_j'$. On the other hand, (6.28) and the choice of $\epsilon_1$ imply that for any $u \in C'_m$ we have
\[
   |\Gamma_\ell(u)| \leq |\lambda^{(m)}_2 - \lambda^{(m)}_1 + \lambda^{(m)}| + |b| |\varphi_{\ell}(u) - \varphi_{\ell}(u_0)| + |\theta_2^{(m)}(u) - \theta_1^{(m)}(u)|
   \leq \pi + C_1 \epsilon_1 R_0 + 16 E A \theta_2^N \epsilon_1 R_0 < \frac{3 \pi}{2}.
\]
Thus, $\epsilon_3 \leq |\Gamma_\ell(u)| < \frac{3 \pi}{2}$ for all $u \in D'_j$. 

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Hence, we see that for \( u \in \mathcal{D}_j' \) the difference \( \Gamma_{\ell}(u) \) between the arguments of the complex numbers \( e^{ib\tau_N(v_1^{(t)}(u))}h(v_1^{(t)}(u)) \) and \( e^{ib\tau_N(v_2^{(t)}(u))}h(v_2^{(t)}(u)) \), defined as a number in the interval \([0, 2\pi)\), satisfies
\[
\Gamma_{\ell}(u) \geq \epsilon_3 \quad , \quad u \in \mathcal{D}_j'.
\]

As in [D1] it follows from Lemma 6.10 that either \( H(v_1^{(t)}(u)) \geq H(v_2^{(t)}(u))/4 \) for all \( u \in \mathcal{D}_j' \) or \( H(v_2^{(t)}(u)) \geq H(v_1^{(t)}(u))/4 \) for all \( u \in \mathcal{D}_j' \). Indeed, fix an arbitrary \( u' \in \mathcal{D}_j' \) and assume e.g. \( H(v_1^{(t)}(u')) \geq H(v_2^{(t)}(u')) \). Then for any \( u \in \mathcal{D}_j' \) using Lemma 6.10(a) twice we get
\[
H(v_1^{(t)}(u)) \geq (H(v_1^{(t)}(u'))/2) \geq (H(v_2^{(t)}(u')))/2 \geq H(v_2^{(t)}(u))/4.
\]

Similarly, if \( H(v_2^{(t)}(u')) \geq H(v_1^{(t)}(u')) \), then \( H(v_1^{(t)}(u)) \geq H(v_2^{(t)}(u))/4 \) for all \( u \in \mathcal{D}_j' \).

Now assume e.g. that \( H(v_1^{(t)}(u)) \leq H(v_2^{(t)}(u))/4 \) for all \( u \in \mathcal{D}_j' \). As in [D1] (see also [St2]) we will show that \( |\psi_{\ell}(u)| \leq \gamma_{\ell}^{(1)}(u) \) for all \( u \in \mathcal{D}_j' \). Given such \( u \), consider the points
\[
z_1 = e^{f_N^{(a)}+ib\tau_N}h(v_1^{(t)}(u)) \quad , \quad z_2 = e^{f_N^{(a)}+ib\tau_N}h(v_2^{(t)}(u))
\]
in the complex plane \( \mathbb{C} \), and let \( \varphi \) be the smallest angle between \( z_1 \) and \( z_2 \). It then follows from the above estimates for \( \Gamma_{\ell}(u) \) that \( \epsilon_3 \leq \varphi \leq 3\pi/2 \). Moreover, (6.24) and (6.27) imply
\[
\frac{|z_1|}{|z_2|} = e^{f_N^{(a)}(v_1^{(t)}(u))-f_N^{(a)}(v_2^{(t)}(u))} \frac{|h(v_1^{(t)}(u))|}{|h(v_2^{(t)}(u))|} \leq \frac{H(v_1^{(t)}(u))}{H(v_2^{(t)}(u))}/4 \leq 16.
\]
This yields
\[
|z_1 + z_2| \leq (1 - t)|z_1| + |z_2|,
\]
where we can take e.g.
\[
t = \frac{1 - \cos(\epsilon_3)}{20}.
\]

Indeed, we have
\[
|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2(z_1, z_2) \leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2|(1 - \alpha),
\]
where \( \alpha = 1 - \cos \epsilon_3 \). Thus, (6.29) will hold if
\[
|z_1|^2 + |z_2|^2 + 2|z_1||z_2|(1 - \alpha) \leq (1 - t)^2|z_1|^2 + |z_2|^2 + 2(1 - t)|z_1||z_2|,
\]
that is if
\[
(1 - (1 - t)^2)|z_1| + 2|z_2|(1 - \alpha) \leq 2(1 - t)|z_2|,
\]
which equivalent to
\[
|z_1| \leq 2\frac{\alpha - t}{t(2 - t)}|z_2|.
\]
If \( t = \alpha/20 \), then \( 16 < 2\frac{\alpha - t}{t(2 - t)} = \frac{38}{2-\alpha/20} \), so the above inequality holds. This proves (6.29) with the given choice of \( t \).

Since \( \mu_0 \leq t \) by (6.7), we have \( |\psi_{\ell}(u)| \leq \gamma_{\ell}^{(1)}(u) \) for all \( u \in \mathcal{D}_j' \). Now set \( j_m = j, \ell_m = \ell \) and \( i_m = 1 \), and include \((i_m, j_m, \ell_m)\) in the set \( J \). Then \( \mathcal{D}_{j_m} \subset \mathcal{C}_m \) and as in the proof of (6.26) we deduce that (6.25) always holds for \( u \in \mathcal{D}_{j_m}' \).

This completes the construction of the set \( J = \{(i_m, j_m, \ell_m) : m = 1, \ldots, m_0 \} \subset J(b) \) and also the proof of (6.25) for all \( u \in V_b \). As we mentioned in the beginning of the proof, (6.25) always holds for \( u \in U \setminus V_b \).
7 \( L^1 \) contraction estimates

Here we obtain \( L^1 \)-contraction estimates for large powers of the contraction operators using the properties of these operators on \( K_0 \), the strong mixing properties of the shift map \( \mathcal{P} : R \rightarrow R \) and the Pesin set \( P_0 \) with exponentially small tails. We continue to use the notation from Sects. 5 and 6.

For any \( J \in J(b) \) set

\[
W_J = \bigcup \{ \mathcal{D}'_{ij} : (i, j, \ell) \in J \text{ for some } i, \ell \} \subset V_b.
\]

Using Lemma 6.4 and the class of functions \( \mathcal{K}_E \) we will now prove the following important estimates.\(^{20}\)

**Lemma 7.1.** Let \( f \in \mathcal{F}_{\theta_1}(\hat{U}) \).

(a) There exists a global constant \( C_5 > 0 \), independent of \( b \) and \( N \), such that for any \( H \in \mathcal{K}_E \) and any \( J \in J(b) \) we have

\[
\int_{V_b} H^2 \, d\nu \leq C_5 \int_{W_J} H^2 \, d\nu. \tag{7.1}
\]

(b) Assuming that \( a_0 > 0 \) is sufficiently small, for any \( H \in \mathcal{K}_E \) and any \( J \in J(b) \) we have

\[
\int_{V_b} (\mathcal{N}_J H)^2 \, d\nu \leq \rho_3 \int_{V_b} \mathcal{L}^N_{J(0)}(H^2) \, d\nu, \tag{7.2}
\]

where

\[
\rho_3 = \rho_3(N) = \frac{e^{a_0 NT}}{1 + \frac{\mu a e^{-NT}}{C_5}} < 1.
\]

**Proofs.** (a) Let \( H \in \mathcal{K}_E \) and \( J \in J(b) \). Consider an arbitrary \( m = 1, \ldots, m_0 \). There exists \((i_m, j_m, \ell_m) \in J \) such that \( \mathcal{D}_{j_m} \subset \mathcal{C}_m \). It follows from (5.2) that there exists a global constant \( \omega_0 \in (0, 1) \) such that

\[
\frac{\nu(\mathcal{D}_{j_m}')}{\nu(\mathcal{C}_m')} \geq 1 - \omega_0.
\]

Since \( H \in \mathcal{K}_E \), for any \( u, u' \in \mathcal{C}_m' \) we have

\[
\frac{|H(u) - H(u')|}{H(u')} \leq E \mathcal{D}(u, u') \leq E,
\]

so \( H(u)/H(u') \leq 1 + E \leq 2E \). Thus, if \( L_1 = \max_{\mathcal{C}_m} H \) and \( L_2 = \min_{\mathcal{C}_m} H \) we have \( L_1/L_2 \leq 2E \). This gives

\[
\int_{\mathcal{C}_m'} H^2 \, d\nu \leq L_1^2 \nu(\mathcal{C}_m') \leq \frac{4E^2}{1 - \omega_0} \int_{\mathcal{D}_{j_m}'} H^2 \, d\nu.
\]

Hence

\[
\int_{V_b} H^2 \, d\nu \leq \sum_{m=1}^{m_0} \int_{\mathcal{C}_m'} H^2 \, d\nu \leq \frac{4E^2}{1 - \omega_0} \sum_{m=1}^{m_0} \int_{\mathcal{D}_{j_m}'} H^2 \, d\nu \leq C_5 \int_{W_J} H^2 \, d\nu,
\]

with \( C_5 = \frac{4E^2}{1 - \omega_0} \), since \( \bigcup_{m=1}^{m_0} \mathcal{D}_{j_m}' = W_J \). This proves (7.1).

(b) Let again \( H \in \mathcal{K}_E \) and \( J \in J(b) \). By Lemma 6.4, \( \mathcal{N}_J H \in \mathcal{K}_E \), while the Cauchy-Schwartz inequality implies

\[
(\mathcal{N}_J H)^2 = (\mathcal{M}^N_0 \omega H)^2 \leq (\mathcal{M}^N_0 \omega_0^2) (\mathcal{M}^N_0 H^2) \leq (\mathcal{M}^N_0 \omega_J) (\mathcal{M}^N_0 H^2) \leq \mathcal{M}^N_0 H^2.
\]

\(^{20}\)This should be regarded as the analogue of Lemma 12 in \[D1\] (and Lemma 5.8 in \[St2\]).
Notice that if \( u \notin W_J \), then \( \omega_J(u) = 1 \). Let \( u \in W_J \); then \( u \in D'_J \) for some (unique) \( j \leq j_0 \), and there exists a unique \((i(j), j, \ell(j)) \in J\). Set \( i = i(j), \ell = \ell(j) \) for brevity. Then \( v_i^{(e)}(u) \in X_i^{(e)} \), so \( \omega_i^{(e)}(v_i^{(e)}(u)) = 1 \), and therefore \( \omega(v_i^{(e)}(u)) \leq 1 - \mu_0 \omega_i^{(e)}(v_i^{(e)}(u)) = 1 - \mu_0 \). In fact, if \( \sigma^N(v) = u \) and \( \omega(v) < 1 \), then \( \omega_i^{(e)}(v) = 1 \) for some \((i', j', \ell') \in J\), so \( v \in X_i^{(e)} \). Then \( u = \sigma^N(v) \in \sigma^N(X_i^{(e)}) = D'_J \). Thus, we must have \( j' = j \), and since for a given \( j \), there is only one element \((i(j), j, \ell(j)) \) in \( J \), we must have also \( i' = i(j) \) and \( \ell' = \ell(j) \). Assuming e.g. that \( i = 1 \), this implies \( v = v_1^{(e)}(u) \).

Thus,

\[
(M_a^N \omega_J)(u) = \sum_{\sigma^N v = u, v \neq v_1^{(e)}(u)} e_i^{(a)}(v) + e_j^{(a)}(v_1^{(e)}(u)) \omega_J(v_1^{(e)}(u))
\]

\[
= \sum_{\sigma^N v = u, v \neq v_1^{(e)}(u)} e_i^{(a)}(v) + (1 - \mu_0) e_i^{(a)}(v_1^{(e)}(u))
\]

\[
= \sum_{\sigma^N v = u} e_i^{(a)}(v) - \mu_0 e_i^{(a)}(v_1^{(e)}(u)) \leq (M_a^N 1)(u) - \mu_0 e^{-NT} = 1 - \mu_0 e^{-NT}.
\]

This holds for all \( u \in W_J \), so

\[
(N_J H)^2 \leq (1 - \mu_0 e^{-NT}) (M_a^N H^2)
\]

on \( W_J \). Using this and part (a) we get:

\[
\int_{V_b} (N_J H)^2 \, d\nu = \int_{V_b \setminus W_J} (N_J H)^2 \, d\nu + \int_{W_J} (N_J H)^2 \, d\nu
\]

\[
\leq \int_{V_b \setminus W_J} (M_a^N H)^2 \, d\nu + (1 - \mu_0 e^{-NT}) \int_{W_J} (M_a^N H)^2 \, d\nu
\]

\[
= \int_{V_b} (M_a^N H)^2 \, d\nu - \mu_0 e^{-NT} \int_{W_J} (M_a^N H)^2 \, d\nu
\]

\[
\leq \int_{V_b} (M_a^N H)^2 \, d\nu - \mu_0 e^{-NT} \int_{V_b} (N_J H)^2 \, d\nu
\]

\[
\leq \int_{V_b} (M_a^N H)^2 \, d\nu - \frac{\mu_0 e^{-NT}}{C_5} \int_{V_b} (N_J H)^2 \, d\nu.
\]

From this and

\[
(M_a^N H)^2 \leq (M_a^N 1)^2(M_a^N H^2) \leq M_a^N H^2 = L_f^{N} (e_i^{(e)} - J_i^{(0)} H^2) \leq e_{a_0} NT (L_f^{N} H^2),
\]

we get

\[
(1 + \mu_0 e^{-NT} / C_5) \int_{V_b} (N_J H)^2 \, d\nu \leq \int_{V_b} (M_a^N H)^2 \, d\nu \leq e_{a_0} NT \int_{V_b} L_f^{N} H^2 \, d\nu.
\]

Thus (7.2) holds with \( \rho_3 = \frac{e_{a_0} NT}{1 + \mu_0 e^{-NT} C_5} > 0 \). Taking \( a_0 = a_0(N) > 0 \) sufficiently small, we have \( \rho_3 < 1 \). □

We can now prove that iterating sufficiently many contraction operators provides an \( L^1 \)-contraction on \( U \).

Define \( \tilde{b} \) by (6.9), and let \( 0 < \delta_1 < \delta_2 \) be as in (6.6). Set

\[
\Lambda_N(b) = \left\{ x \in L \cap R : \# \left\{ j : 0 \leq j < N \tilde{b}, \mathcal{P}^j(x) \notin P_1 \cap P_0 \cap \Omega^{(\delta)}_B \right\} \geq \frac{\delta_2}{N} N \tilde{b} \right\}.
\]

(7.3)
In what follows we will use the estimate (6.4). Apart from that it follows from Theorem 3.2
that there exist constants \( C_0' = C_0'(N) > 0 \) and \( c_0' = c_0'(N) > 0 \) such that
\[
\mu(\Xi_n(p_0, \epsilon_0, \delta_1/N)) \leq C_0' e^{-c_0'n}, \quad n \geq 1.
\] (7.4)

The following lemma will play a significant role in showing that some cancellation occurs in
the actions of the contraction operators \( N_f \).

**Lemma 7.2.** Let \( s > 0 \) be a constant. There exist constants \( C_6 > 0 \) and \( N_0 \geq 1 \), independent of \( b \), such that
\[
\mu(\Lambda_N(b)) \leq \frac{C_6}{|b|^s}
\]
for all \( b \) with \( |b| \geq b_0 \) and all integers \( N \geq N_0 \).

**Proof.** Set \( m = \hat{b} \). We claim that
\[
\Lambda_N(b) \subset \Xi_{Nm}(p_0, \epsilon_0, \delta_1/N) \cup Y,
\] (7.5)
where
\[
Y = \cup_{j=0}^{Nm} \mathcal{P}^{-j}(R \setminus \Omega_B^{(\hat{b})}).
\] (7.6)

Set \( \Xi = \Xi_{Nm}(p_0, \epsilon_0, \delta_1/N) \) for brevity. Assume that there exists \( x \in \Lambda_N(b) \setminus (\Xi \cup Y) \). Then
\( x \notin Y \), so \( \mathcal{P}^j(x) \notin R \setminus \Omega_B^{(\hat{b})} \) for all \( j \in [0, Nm] \), i.e.
\[
\mathcal{P}^j(x) \in \Omega_B^{(\hat{b})}, \quad 0 \leq j \leq Nm.
\] (7.7)

Next, \( x \notin \Xi \) gives
\[
\# \{j : 0 \leq j < Nm, \mathcal{P}^j(x) \notin P_0\} < \frac{\delta_1}{N} Nm.
\] (7.8)

Now \( x \in \Lambda_N(b) \) and (7.7) imply \( \mathcal{P}^j(x) \notin P_1 \cap P_0 \) for at least \( \frac{\delta_1}{N} Nm \) values of \( j = 0, 1, \ldots, Nm - 1 \).
Notice that by (6.6), \( \frac{\delta_1}{N} Nm = \frac{(2d+3)B\delta_1}{N} Nm \). This and (7.8) now yield
\[
\# \{j : 0 \leq j < Nm, \mathcal{P}^j(x) \notin P_1\} \geq \frac{(2d+2)B\delta_1}{N} Nm.
\] (7.9)

We need to extend (7.8) a bit. Using (7.7) for \( j = Nm \) and (6.3), we get \( \mathcal{P}^{Nm}(x) \notin \Xi_{Bm}(p_0, \epsilon_0, \delta_1) \), which means that
\[
\# \{i : 0 \leq i < Bm, \mathcal{P}^{Nm+i}(x) \notin P_0\} < \delta_1 Bm = \frac{\delta_1 B}{N} Nm.
\]

Combining the latter with (7.8) gives
\[
\# \{j : 0 \leq j < (N + B)m, \mathcal{P}^j(x) \notin P_0\} < \frac{2B\delta_1}{N} Nm.
\] (7.10)

Given \( y = \mathcal{P}^j(x) \) for some \( j = 0, 1, \ldots, Nm - 1 \), let \( m_j = m_y \) be the length of the maximal
cylinder \( C(y) \) in \( W^u_R(y) \) with \( \text{diam}((\Phi_y)^{-1}(C(y))) \leq \epsilon_1/|b| \). Then by (6.2), \( m_j \leq B\delta_1 = Bm \).
Moreover, if \( y \notin P_1 \), then \( \mathcal{P}^{m_j}(y) \notin P_0 \), i.e. \( \mathcal{P}^{m_j+i}(x) \notin P_0 \). By (7.7), \( y = \mathcal{P}^j(x) \in \Omega_B^{(\hat{b})} \), and
then by (6.3), \( y \notin \Xi_{\ell} \) for all \( \ell \in [m/B, Bm] \). Using the latter with \( \ell = m/B, \) the definition of \( \Xi_{\ell} \) yields \( \mathcal{P}^i(y) \in P_0 \) for some \( i \in [(1 - \delta_0)m/B, m/B] \). Since \( m_j \geq m/B \) by (6.2), we have
\( (1 - \delta_0)m/B \leq i \leq m_j \). Now we can apply Lemma 4.4(d) using the maximality of the cylinder
\( C(y) \), and the points \( y \) and \( \mathcal{P}^i(y) \in P_0 \) and derive that \( \text{diam}((\Phi_y)^{-1}(C(y))) \geq \rho_1 \epsilon_1/|b| \). More precisely,
let \( C_1 \) be the unstable cylinder in \( W^u(y) \) of length \( i - 1 \); this then < \( m_j \), the length of \( C(y) \). Thus \( C(y) \subset C_1 \) and the maximality of \( C(y) \) implies \( \text{diam}(\Phi_y^{-1}(C_1)) > \epsilon_1/|b| \). Since \( P^i(y) \in \mathcal{P}_0 \), it follows from Lemma 4.4(d) and \( \hat{\rho} < \rho_1 \) that

\[
\text{diam}(\Phi_y^{-1}(C(y))) \geq \rho_1 \text{diam}(\Phi_y^{-1}(C_1)) > \rho_1 \frac{\epsilon_1}{|b|}.
\]

On the other hand, by the choice of the constant \( d \) we have \( c_0 \gamma^d \rho_1 > 1 \), therefore

\[
\text{diam}(\Phi_y^{-1}(C(y))) (P^d(C(y))) > \epsilon_1/|b|.
\]

Thus, \( C(P^d(y)) = C(P^{j+d}(x)) \) is a proper sub-cylinder of \( P^d(C(y)) \) and therefore its length \( m_{j+d} \) is strictly larger that the length \( m_j - d \) of \( P^d(C(y)) \), i.e. \( m_{j+d} > m_j - d \), so \( m_j < m_{j+d} + (j + d) \).

Moreover, for \( j \in [0, Nm) \) we have \( m_j + j \in [0, (N + B)m) \).

By (7.9) there are at least \( \frac{(2d+2)Bk}{dN} \) values of \( j = 0, 1, \ldots, Nm - 1 \) with \( P^j(x) \notin \mathcal{P}_1 \), and so \( P^{m_j+j}(x) \notin \mathcal{P}_0 \). From those \( j \)'s we get a strictly increasing sub-sequence with at least \( \frac{(2d+2)Bk}{dN} \) members which is a contradiction with (7.10).

This proves (7.5). Now using (6.4), (7.4) and \( m = \hat{b} = \lfloor \log |b| \rfloor \) we get

\[
\mu(A_N(b)) \leq C_6 \rho_1 e^{-\gamma \hat{b}/b} + N \hat{b} \mu(R \setminus \Omega_\hat{B}) \leq \frac{C_6}{|b|^{N \rho_1}} + N \hat{b} \rho_3 e^{-c_3 \hat{b}/b}.
\]

Since \( \hat{b} > 1 \), assuming \( N_0 \geq 1 \) is sufficiently large and \( N \geq N_0 \), we have \( N \hat{b} \rho_3 e^{-c_3 \hat{b}/b} \leq e^{-c_3 \hat{b}/b} \).

Thus, for such \( N \),

\[
\mu(A_N(b)) \leq \frac{C_6}{|b|^{N \rho_1}} + C_3 e^{-c_3 \hat{b}/b} = \frac{C_6}{|b|^{N \rho_1}} + \frac{C_3}{|b|^{N \rho_1}} e^{-c_3 \hat{b}/b} \leq \frac{C_6 + C_3}{|b|^{N \rho_1}}.
\]

assuming \( N_0 c_6 \geq s \) and \( \frac{N_0 c_3}{2B} \geq s \). This proves the lemma.  

Set

\[
\rho_3 = \frac{e^{\alpha_0 N T}}{1 + \frac{\mu_0 e^{-N T}}{C_5}} < 1, \quad R = e^{\alpha_0 N T} > 1, \quad \hat{h} = \rho_3 \chi_{V_b} + R \chi_{U \setminus V_b},
\]

and notice that \( \rho_3 \) is as in Lemma 7.1. We will assume \( \alpha_0 = \alpha_0(N) > 0 \) is chosen so small that

\[
8\alpha_0 N T < \log \left(1 + \frac{\mu_0 e^{-N T}}{2C_5}\right), \quad 4\alpha_0 N T < c_6.
\]

Recall the constants \( \theta_1 \) and \( \theta_2 \) from the beginning of Sect. 5.2.

After the comprehensive study of contraction operators in Sects. 6 and 7 so far, we can now prove that the contraction operators \( \mathcal{N}_J \) do have some contraction properties.

**Lemma 7.3.** Let \( f \in \mathcal{F}_{\theta_1}(\hat{U}) \) and let \( s \geq 1 \) be a constant. There exist global constants \( N_0 \geq 0 \), \( C_7 > 0 \) and \( C_8 > 0 \) such that for any \( N \geq N_0 \) there exist constants \( k = k(N) \geq 1 \), \( a_0 = a_0(N) > 0 \) and \( b_0 = b_0(N) \geq 1 \) such that for any \( |a| \leq a_0 \) and \( |b| \geq b_0 \) we have the following:

**(a)** For any sequence \( J_1, J_2, \ldots, J_r \ldots \) of representative subsets of \( J(b) \), setting \( H^{(0)} = 1 \) and \( H^{(r+1)} = \mathcal{N}_{J_r}(H^{(r)}) \) (\( r \geq 0 \)) we have

\[
\int_U (H^{(k)}b)^2 d\nu \leq \frac{C_7}{|b|^{10s}}.
\]

(7.12)
(b) For all \( h \in \mathcal{F}_{\theta_1}(U) \) we have

\[
\|L_{ab}^{2kN_0}h\|_0 \leq \frac{C_8}{|b|^s} \|h\|_{\theta_1,b}.
\]  

(7.13)

**Remark.** Notice that in general the operator \( L_{ab} \) does not have to preserve the space \( \mathcal{F}_{\theta_1}(\hat{U}) \). Indeed, the function \( f^{\alpha} \) involves \( \tau \) which is in \( \mathcal{F}_{\theta}(\hat{U}) \), however not necessarily in \( \mathcal{F}_{\theta_1}(\hat{U}) \). So, in the left-hand-side of (7.13) we just have the sup-norm of a function in \( \mathcal{F}_{\theta}(\hat{U}) \).

**Proof of Lemma 7.3.** (a) Set \( \omega_r = \omega_{J_r} \), \( W_r = W_{J_r} \) and \( N_r = N_{J_r} \). Since \( H^{(0)} = 1 \in \mathcal{K}_E \), it follows from Lemma 6.5 that \( H^{(r)} \in \mathcal{K}_E \) for all \( r \geq 1 \).

Using \( L_{j(\alpha)}^N((\hat{h} \circ \sigma^N)H) = \hat{h} (L_{j(\alpha)}^N H) \) and Lemma 7.1(b) we get

\[
\int_U (H^{(m)})^2 d\nu = \int_{V_b} (H^{(m)})^2 d\nu + \int_{U \setminus V_b} (H^{(m)})^2 d\nu \\
\leq \rho_3 \int_{V_b} L_{j(\alpha)}^N (H^{(m-1)})^2 d\nu + e^{a_0NT_0} \int_{U \setminus V_b} L_{j(\alpha)}^N (H^{(m-1)})^2 d\nu \\
= \int_U \hat{h} (L_{j(\alpha)}^N H)^2 d\nu = \int_U L_{j(\alpha)}^N ((\hat{h} \circ \sigma^N) (H^{(m-1)})^2 d\nu \\
= \int_U (\hat{h} \circ \sigma^N) (H^{(m-1)})^2 d\nu.
\]

Similarly,

\[
\int_U (\hat{h} \circ \sigma^N) (H_{m-1})^2 d\nu \leq \int_U (\hat{h} \circ \sigma^{2N}) (\hat{h} \circ \sigma^N) (H^{(m-2)})^2 d\nu.
\]

Continuing by induction and using \( H^{(0)} = 1 \), we get

\[
\int_U (H^{(m)})^2 d\nu \leq \int_U (\hat{h} \circ \sigma^{mN}) (\hat{h} \circ \sigma^{(m-1)N}) \cdots (\hat{h} \circ \sigma^{2N}) (\hat{h} \circ \sigma^N) d\nu. \tag{7.14}
\]

Let \( s > 0 \) be a constant. Using Lemma 7.2, choose the constants \( C_6 > 0 \) and \( N_0 \geq 1 \) so that for \( N \geq N_0 \) we have

\[
\mu(\Lambda_N(b)) \leq C_6/|b|^{12s}.
\]

Given an integer \( k \geq 1 \), set \( m = k\hat{b} \) and let \( 0 < \delta_1 < \delta_2 \) be as in (6.6). Let \( N \geq N_0 \). Set

\[
W = \{ x \in U : x \in \sigma^{-jN}(U \setminus V_b) \}
\]
for at least \( \delta_2m \) values of \( j = 0, 1, \ldots, m - 1 \).

Since \( K_0 \subset V_b, x \in \sigma^{-jN}(U \setminus V_b) \) implies \( x \in \sigma^{-jN}(U \setminus K_0) \).

Notice that

\[
(\pi^{(U)})^{-1}(W) \subset \Lambda_N(b), \tag{7.15}
\]

the set defined by (7.3). Indeed, if \( x \in W \) and \( y \in \hat{W}_R(x) \), then for any \( j = 0, 1, \ldots, m - 1 \) with \( \sigma^{jN}(x) \notin K_0 = \pi^{(U)}(P_1 \cap P_0 \cap \Omega_B) \), since \( \pi^{(U)}(P_j \cap \Omega_B) = \sigma^{jN}(x) \), we have \( P_j \cap \Omega_B \notin \hat{W}_R(x) \).

Thus, the latter holds for at least \( \delta_2m \) values of \( j = 0, 1, \ldots, m - 1 \), so \( \pi^{(U)} \notin P_1 \cap P_0 \cap \Omega_B \) for at least \( \delta_2m = kN\hat{b} \) values of \( i = 0, 1, \ldots, N\hat{b} - 1 \). It follows from (7.3) that \( y \in \Lambda_N(b) \). This proves (7.15), and now Lemma 7.2 implies

\[
\nu(W) \leq \frac{C_6}{|b|^{12s}}. \tag{7.16}
\]
Notice that if \( x \in U \setminus W \), then \( x \in \sigma^{-jN}(V_b) \) for at least \((1 - \delta_2)m\) values of \( j = 0, 1, \ldots, m - 1\), so \((\hat{h} \circ \sigma^j)(x) = \rho_3\) for that many \( j \)'s. Thus, assuming \( a_0 = a_0(N)\) is taken small enough so that \( a_0NT < c_0/2\), and using \( \log(1 + x) > 1 + x/2 \) for \( 0 < x < 1 \), (7.11) and (7.16) yield

\[
\int_U (H^{(m)})^2 d\nu \leq \int_{U \setminus W} \prod_{j=1}^m (h \circ \sigma^j) d\nu + \int_W \prod_{j=1}^m (h \circ \sigma^j) d\nu
\]

\[
\leq \rho_3^{(1 - \delta_2)m} R^{\delta_2m} + R^m \nu(W) \leq (\rho_3^{1 - \delta_2} R^{\delta_2})^m + \frac{C_6 R^m}{|b|^{12s}}
\]

\[
\leq \left( e^{(1-\delta_2)\log \rho_3 + \delta_2 a_0 NT} \right)^{k \log |b|} + \frac{C_6 e^{a_0 NT k \log |b|}}{|b|^{12s}}
\]

\[
\leq \left( e^{(1-\delta_2)a_0 NT - \frac{1}{2} (1-\delta_2) \log (1 + \mu_0 e^{-NT/C_5}) + \delta_2 a_0 NT} \right)^{k \log |b|} + \frac{C_6}{|b|^{12s-a_0 NT k}}
\]

\[
\leq \left( e^{a_0 NT - \frac{1}{4} \log (1 + \mu_0 e^{-NT/C_5})} \right)^{k \log |b|} + \frac{C_6}{|b|^{12s-a_0 NT k}}.
\]

Now choose

\[
k = \left\lfloor \frac{100 s C_5 e^{NT}}{\mu_0} \right\rfloor,
\]

and assume that

\[
a_0 \leq \frac{\mu_0}{100 C_5^2 N T e^{NT}}.
\]

Then the above yields

\[
\int_U (H^{(m)})^2 d\nu \leq \left( e^{-\frac{\mu_0}{8 C_5 e^{NT}}} \right)^{k \log |b|} + \frac{C_6}{|b|^{12s-a_0 NT k}}
\]

\[
\leq \frac{1}{|b|^{\frac{k \log |b|}{8 C_5 e^{NT}}} + \frac{C_6}{|b|^{12s-a_0 NT k}}} \left( \frac{100 s C_5 e^{NT}}{\mu_0} \right) + 1
\]

\[
\leq \frac{1}{|b|^{10s}} + \frac{C_6}{|b|^{11s-a_0 NT k}} \leq 1 + \frac{C_6}{|b|^{10s}}.
\]

Thus, we can take \( C_7 = 1 + C_6 \).

(b) Let \( h \in \mathcal{F}_{\theta_1}(\hat{U}) \) be such that \( ||h||_{\theta_1, b} \leq 1 \). Then \( |h(u)| \leq 1 \) for all \( u \in \hat{U} \) and \( |h|_{\theta_1} \leq |b| \).

Assume that the points \( u, u' \), the cylinder \( C = C_m \) in \( U \), the integer \( p \geq 0 \) and the points \( v, v' \in \hat{U}_1 \) satisfy (6.20) for some \( i = 1, 2 \). Then, using (6.19) and \( |h|_{\theta_1} \leq |b| \) we get

\[
|h(v) - h(v')| \leq |b| D_{\theta_1} (v, v') = |b| \theta_1^{p+N} D_{\theta_1} (u, u') \leq |b| \theta_1^{p+N} \text{diam}_4 (C)
\]

\[
|b| \theta_1^{p+N} C_2 \text{diam}(\hat{\Psi}(C)) \leq E |b| \theta_1^{p+N} \text{diam}(\hat{\Psi}(C)),
\]

since \( C_2 \leq E \). Thus, \((h, 1) \in \mathcal{K}_b \). Set \( h^{(m)} = L_{ab}^m h \) for \( m \geq 0 \). Define the sequence of functions \( \{H^{(m)}\} \) recursively by \( H^{(0)} = 1 \) and \( H^{(m+1)} = N_{J_m} H^{(m)} \), where \( J_m \in J(b) \) is chosen by induction as follows. Since \((h^{(0)}, H^{(0)}) \in \mathcal{K}_b \), using Lemma 6.9 we find \( J_0 \in J(b) \) such that for \( h^{(1)} = L_{ab}^N h^{(0)} \) and \( H^{(1)} = N_{J_m} H^{(0)} \) we have \((h^{(1)}, H^{(1)}) \in \mathcal{K}_b \). Continuing in this way we construct by induction an infinite sequence of functions \( \{H^{(m)}\} \) with \( H^{(0)} = 1 \), \( H^{(m+1)} = N_{J_m} H^{(m)} \) for all \( m \geq 0 \), such that \((h^{(m)}, H^{(m)}) \in \mathcal{K}_b \).

Next, a choose \( a_0 \) and \( k \geq 1 \) as in part (a) and set \( m = kb \). Then part (a) implies

\[
\int_U (H^{(m)})^2 d\nu \leq \frac{C_7}{|b|^{10s}}.
\]
Hence
\[ \int_U |L_{ab}^{mN} h|^2 \, d\nu = \int_U |h^{(m)}|^2 \, d\nu \leq \int_U (H^{(m)})^2 \, d\nu \leq \frac{C_7}{|h|^{10s}}. \]

From this it follows that for any \( h \in \mathcal{F}_{\hat{\theta}_1}(\hat{U}) \) we have
\[ \int_U |L_{ab}^{mN} h|^2 \, d\nu \leq \frac{C_7}{|h|^{10s}} \|h\|^2_{\hat{\theta}_1, b}, \]
and so
\[ \int_U |L_{ab}^{mN} h| \, d\nu \leq \frac{\sqrt{C_7}}{|h|^{5s}} \|h\|_{\hat{\theta}_1, b}. \]
\tag{7.17} \]

We will now use a standard procedure (see \cite{Di}) to derive an estimates of the form (7.13) from (7.17).

**7.4 Standard procedure:** First, recall from the Perron-Ruelle-Frobenius Theorem (see e.g. \cite{PP}) that there exist global constants \( C_9 \geq 1 \) and \( \rho_4 \in (0, 1) \), independent of \( b \) and \( N \), such that
\[ \|L_{f(0)}^n w - h_0 \int_U w \, d\nu\| \leq C_9 \rho_4^n \|w\|_{\theta} \]
\tag{7.18} \]
for all \( w \in \mathcal{F}_{\theta}(\hat{U}) \) and all integers \( n \geq 0 \), where \( h_0 > 0 \) is the normalised eigenfunction of \( L_{f - \tau} \) in \( \mathcal{F}_{\theta}(\hat{U}) \) (see the beginning of Sect. 5.1).

Given \( h \in \mathcal{F}_{\hat{\theta}_1}(\hat{U}) \) with \( \|h\|_{\theta, 1, b} \leq 1 \), we have \( |h|_{\theta} \leq |h|_{\hat{\theta}_1} \leq |b| \), so using Lemma 5.2 with \( H = 1 \) yields
\[ |L_{ab}^r h|_{\theta} \leq A_0 \|b|^{\theta r} + \|b| \leq 2A_0|b| \]
\tag{7.19} \]
for any integer \( r \geq 0 \).

Choose again \( a_0 \) and \( k \geq 1 \) as in the proof of part (a) and set \( m = k\hat{b} \). Then (7.17) holds. Write \( \rho_4 = e^{-\beta_3} \) for some global constant \( \beta_3 > 0 \). Given \( h \in \mathcal{F}_{\hat{\theta}_1}(\hat{U}) \) with \( \|h\|_{\theta, 1, b} \leq 1 \), we have
\[ |L_{ab}^{2mN} h| = |L_{ab}^{mN} (L_{ab}^{mN} h)| \leq M_a^{mN} |L_{ab}^{mN} h| = L_{f(0)}^{mN} \left( e^{f^{(a)} - f^{(0)}_{mN}} L_{ab}^{mN} h \right) \]
\[ \leq \left( L_{f(0)}^{mN} \left( e^{f^{(a)} - f^{(0)}_{mN}} \right)^2 \right)^{1/2} \left( L_{f(0)}^{mN} |L_{ab}^{mN} h|^2 \right)^{1/2}. \]

For the first term in this product (5.3) implies
\[ \left( L_{f(0)}^{mN} \left( e^{f^{(a)} - f^{(0)}_{mN}} \right)^2 \right)^{1/2} \leq e^{a_0 NTm} \leq e^{a_0 NTk|\log |b| = |b|^{a_0 NTk}. \]

By the choice of \( k \) and \( a_0 \),
\[ a_0 NTk \leq a_0 NT \left( \frac{100sC_5e^{NT}}{\mu_0} + 1 \right) \leq \frac{\mu_0 NT}{100sC_5e^{NT}} \frac{100sC_5e^{NT}}{\mu_0} + a_0 NT < s + \frac{1}{2}, \]
since \( a_0 NT < 1/2 \). Thus,
\[ \left( L_{f(0)}^{mN} \left( e^{f^{(a)} - f^{(0)}_{mN}} \right)^2 \right)^{1/2} \leq |b|^{s + 1/2}. \]

For the second term, using (7.18) with \( w = |L_{ab}^{mN} h| \), we get
\[ L_{f(0)}^{mN} L_{ab}^{mN} h \leq \int_U |L_{ab}^{mN} h| \, d\nu + C_9 \rho_4^{mN} \|L_{ab}^{mN} h\|_{\theta}. \]
By (7.19), \( \| L_{ab}^{mN} h \|_\theta \leq 2A_0 |b| \), so by (7.17),
\[
L_{f(0)}^{mN} |L_{ab}^{mN} h|^2 \leq \frac{C_8'}{|b|^{5s}} + 2A_0 C_9 |b| \rho_4^{mN}.
\]
Now
\[
\rho_4^{mN} \leq e^{-\beta_3 N \log |b|} = \frac{1}{|b|^{N\beta_3}} < \frac{1}{|b|^{5s+1}},
\]
assuming \( N\beta_3 > 5s + 1 \), so we get
\[
L_{f(0)}^{mN} |L_{ab}^{mN} h|^2 \leq \frac{C_8''}{|b|^{5s}}.
\]
Combining the estimates of the two terms and using \( s \geq 1 \), we get
\[
|L_{ab}^{2mN} h| \leq |b|^{s+1/(2(C_8''/|b|^{5s})^{1/2}) \leq \frac{C_8}{|b|^s},
\]
assuming \( a_0 = a_0(N) > 0 \) is chosen sufficiently small, e.g. \( a_0 NT < s/2 \) is enough. ■

8 Proofs of the main results

Here we prove Theorems 1.3 and 1.1 and Corollary 1.4.

Proof of Theorem 1.3. Let \( \theta_1 = \theta_1(\theta) \in (0, \theta] \) and \( \theta_2 = \theta_2(\theta) \in [\theta, 1) \) be as in the beginning of Sect. 5.2.

We will again assume that \( f \in F_{\theta_1}(\hat{U}) \); the general case \( f \in F_{\theta}(\hat{U}) \) will be done later using an approximation procedure.

Let \( \hat{\theta} \leq \theta < 1 \), where \( \hat{\theta} \) is as in (5.1). Set
\[
s = \frac{2}{\alpha_2},
\]
where \( \alpha_2 > 0 \) is the constant from Lemma 5.1(c), and recall that \( \theta_1^{\alpha_2} = \theta_2 \). Next, choose \( N_0 \geq 1 \) as in Lemma 7.2, replacing \( s \) by \( 12s \), so that \( \mu(\Lambda_N(b)) \leq C_6 / |b|^{12s} \), as in the proof of Lemma 7.3.

Let \( N \geq N_0 \). Choose \( k = k(N) \), \( a_0 = a_0(N) \), \( b_0 = b_0(N) \), \( \rho_1 = \rho_4(N) \in (0, 1) \), \( C_7, C_8 > 0 \) as in Lemma 7.3. Then (7.12) and (7.13) hold.

Let \( |a| \leq a_0 \) and \( |b| \geq b_0 \), and let \( h \in F_{\hat{\theta}_1}(\hat{U}) \) be such that \( \| h \|_{\theta, b} \leq 1 \). Then \( |h(u)| \leq 1 \) for all \( u \in \hat{U} \) and \( |h|_{\theta} \leq |b| \).

Take the smallest integer \( p \) so that \( \theta^p \leq 1 / |b|^2 \). It is known (see e.g. the end of Ch. 1 in [PP]) that there exists \( h' \in F_{\theta_1}(\hat{U}) \) which is constant on cylinders of length \( p \) so that \( \| h - h' \|_0 \leq |h|_{\theta} \theta^p \).

Then \( \| h - h' \|_0 \leq 1 / |b| \) and so \( \| h' \|_0 \leq 2 \), and it follows easily from this that
\[
\| h' \|_{\theta_1} \leq \frac{4}{\theta_1^{p-1}} \leq \frac{4}{\theta_1^{(p-1)/\alpha_2}} \leq C_{10} |b|^{2/\alpha_2}.
\]
Thus, \( \| h' \|_{\theta_1, b} \leq 2C_{10} |b|^{2/\alpha_2 - 1} \) and (7.13) gives
\[
\| L_{ab}^{2kN_0} h' \|_{\theta_1, b} \leq \frac{C_8}{|b|^s} 2C_{10} |b|^{2/\alpha_2 - 1} \leq \frac{2C_{10}C_8}{|b|^{2/\alpha_2 - 2/\alpha_2 + 1}} = \frac{2C_{10}C_8}{|b|},
\]
so
\[
\| L_{ab}^{2kN_0} h' \|_0 \leq \frac{C_{10}}{|b|}.
\]
for some global constant $C_{10} > 0$. Therefore

$$\|L_{ab}^{2kN\hat{b}}h\|_0 \leq \|L_{ab}^{2kN\hat{b}}h'\|_0 + \|L_{ab}^{2kN\hat{b}}(h - h')\|_0 \leq \frac{C_{10}}{|b|} + \frac{1}{|b|} \leq \frac{2C_{10}}{|b|}.$$  

Next, using Lemma 5.2, and writing $\theta = e^{-\beta_5}$ for some constant $\beta_5 > 0$, we get

$$|L_{ab}^{2kN\hat{h}} h|_\theta = |L_{ab}^{2kN}(L_{ab}^{-2kN\hat{h}}) h|_\theta \leq A_0 \left[ |b| \theta^{2kN\hat{h}} + |b| \|L_{ab}^{2kN\hat{h}}h\|_0 \right] \leq A_0 \left[ |b| \frac{1}{|b|^{2kN\beta_5}} + |b| \frac{2C_{10}}{|b|} \right] \leq C_{11}',$$

assuming $2N\beta_5 \geq 1$. This yields

$$\|L_{ab}^{2kN\hat{h}}h\|_{\theta,b} \leq \frac{C_{11}}{|b|} \|h\|_{\theta,b}$$

for all $h \in \mathcal{F}_\theta(\hat{U})$, where $C_{11} > 0$ is a global constant.

Let $n \geq 4kN\hat{b}$ be an arbitrary integer. Writing $n = r(4kN\hat{b}) + \ell$ for some $\ell = 0,1,\ldots, 4kN\hat{b} - 1$, and using the above $r$ times we get

$$\|L_{ab}^{r4kN\hat{h}}h\|_{\theta,b} \leq \frac{1}{|b|^r} \|h\|_{\theta,b}.$$  

As before, using Lemma 5.2 with $H = 1$ and $B = |L_{ab}^{r4kN\hat{h}}h|_\theta$, implies

$$|L_{ab}^n h|_\theta = |L_{ab}^\ell(L_{ab}^{r4kN\hat{h}}) h|_\theta \leq A_0 \left[ |L_{ab}^{r4kN\hat{h}}h|_\theta |h| + \|L_{ab}^{r4kN\hat{h}}h\|_0 \right],$$

so

$$\frac{1}{|b|} |L_{ab}^n h|_\theta \leq 2A_0 \|L_{ab}^{r4kN\hat{h}}h\|_{\theta,b} \leq \frac{2A_0}{|b|^r} \|h\|_{\theta,b}.$$  

This and $\|L_{ab}^n h\|_0 \leq \|L_{ab}^{r4kN\hat{h}}h\|_0 \leq \frac{1}{|b|^r} \|h\|_{\theta,b}$ give

$$\|L_{ab}^n h\|_{\theta,b} \leq \frac{3A_0}{|b|^r} \|h\|_{\theta,b} = 3A_0 e^{-r \log |b|} \|h\|_{\theta,b}.$$  

We have $r \geq (r + 1)/2$ for all $r \geq 1$, so the above implies

$$\|L_{ab}^n h\|_{\theta,b} \leq 3A_0 e^{-\frac{(r+1)\log |b|}{2}} \|h\|_{\theta,b} \leq 3A_0 e^{-\frac{(r+1)4kN\hat{b}}{8kN}} \|h\|_{\theta,b} \leq 3A_0 \rho_6^n \|h\|_{\theta,b}, \quad (8.1)$$

where $\rho_6 = e^{-1/(8kN)} \in (0,1)$.

Thus, (8.1) holds for all $h \in \mathcal{F}_\theta(\hat{U})$ and all integers $n \geq 4kN\hat{b} = 4kN \log |b|$. Finally, recall the eigenfunction $h_a \in \mathcal{F}_\theta(\hat{U})$ for the operator $L_{f-(P_f+a)^r}$ from Sect. 5.1. It is known that $\|h_a\|_\theta \leq \text{Const}$ for bounded $a$, e.g. for $|a| \leq a_0$. It now follows from

$$L_{ab}^n(h/h_a) = \frac{1}{\lambda^n h_a} L_{f-(P_f+a+ib)^r}^n h$$

and the above estimate that there exist constants $0 < \rho < 1$, $a_0 > 0$, $b_0 \geq 1$ and $C > 0$ such that if $a, b \in \mathbb{R}$ satisfy $|a| \leq a_0$ and $|b| \geq b_0$, then

$$\|L_{f-(P_f+a+ib)^r}^n h\|_{\theta,b} \leq C \rho^n \|h\|_{\theta,b}$$

for any integer $n \geq 4kN \log |b|$ and any $h \in \mathcal{F}_\theta(\hat{U})$. So, we can just set $T_0 = 4kN$. 

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This completes the proof of Theorem 1.3 under the assumption that \( f \in \mathcal{F}_{\theta_1}(\bar{U}) \). The case \( f \in \mathcal{F}_{\bar{U}}(\bar{U}) \) follows by using an approximation procedure. To our knowledge this has not been done anywhere in details, and the argument involved is not trivial, so we will sketch it for completeness.

**Sketch of the proof of Theorem 1.3 for arbitrary** \( f \in \mathcal{F}_{\bar{U}}(\bar{U})$: We will use again the constants from the beginning of Sect. 6.1, including \( \theta_1, \theta_2, \) etc. Fix \( B, N \) as before and define \( \tilde{b} \) by (6.3). Let \( |a| \leq a_0 \) and \( |b| \geq b_0 \), where \( b_0 \) is given by (6.8).

Let \( f \in \mathcal{F}_{\bar{U}}(\bar{U}) \) be an arbitrary real-valued function. Take the minimal integer \( t = t(b) > 0 \) so that

\[
\theta^{t+1} \leq \frac{2A_0}{\log |b|},
\]  

(8.3)

where \( A_0 \) is the constant from Lemma 5.2. There exists a real-valued \( f^{(t)} \) depending only on \( t \) coordinates such that

\[
\| f - f^{(t)} \|_0 \leq |f| \| \theta^t \| \leq T \theta^t
\]

(see the end of Ch. 1 in [PP]), where \( T \) is as in (5.3). Then \( f^{(t)} \in \mathcal{F}_{\theta_1}(\bar{U}), \| f^{(t)} \|_0 \leq 2T \) and

\[
|f^{(t)}|_\theta \leq \frac{4}{\theta^t} \leq \frac{2\log |b|}{A_0}, \quad |f^{(t)}|_{\overline{\theta}_1} \leq \frac{4}{\theta_1^t}.
\]

Let \( \lambda_{at} \) be the largest eigenvalue of

\[
F^{(at)} = f^{(t)} - (P_t + a)\tau,
\]

where \( P_t = P_{f^{(t)}} \), and let \( h_{at} \in \mathcal{F}_{\bar{U}}(\bar{U}) \) be a corresponding (positive) eigenfunction such that

\[
\int h_{at} d\nu_{at} = 1,
\]

where \( \nu_{at} \) is the unique regular probability measure on \( \bar{U} \) with \( (F^{(at)})^*\nu_{at} = \nu_{at} \).

For \( |a| \leq a_0 \), as in [DI], consider the function

\[
f^{(at)}(u) = f^{(t)}(u) - (P_t + a)\tau(u) + \ln h_{at}(u) - \ln h_{at}(\sigma(u)) - \ln \lambda_{at}
\]

and the operators

\[
L_{at} = L_{f^{(at)} - 1} \mathcal{B} : \mathcal{F}_{\theta}(\bar{U}) \longrightarrow \mathcal{F}_{\theta}(\bar{U}), \quad \mathcal{M}_{at} = L_{f^{(at)}} : \mathcal{F}_{\bar{U}}(\bar{U}) \longrightarrow \mathcal{F}_{\bar{U}}(\bar{U}).
\]

Then \( \mathcal{M}_{at} = 1 = 1 \) and \( |(L_{at}^m h)(u)| \leq (\mathcal{M}_{at}^m |h|)(u) \) for all \( u \in \bar{U} \).

Using part of the proof of Lemma 4.1 in [PeS], one shows that \( |h_{at}|_\theta \leq \text{Const} \ |f^{(t)}|_\theta \) for some global constant \( \text{Const} > 0 \). Thus, \( |f^{(at)}|_\theta \leq \text{Const} \ |f^{(t)}|_\theta \), and it is also clear that \( \| f^{(at)} \|_0 \leq \text{Const} \).

Next, define the set \( K_0 \), cylinders \( \mathcal{C}_m \) and their sub-cylinders \( \mathcal{D}_j \) and the function \( \omega = \omega_j \) as in Sect. 6.1 and consider the operator \( \mathcal{N}^{(t)} = \mathcal{N}_j^{(t)} \) on \( \mathcal{F}_{\theta}(\bar{U}) \) defined by

\[
\mathcal{N}^{(t)}(h) = \mathcal{M}_{at}^N(h) = L_{f^{(at)}}^N h.
\]

It is important to notice that

\[
e^{\| f^{(t)} \|_\theta} \operatorname{diam}(\mathcal{C}_m) \leq \frac{1}{0^{|1/A_0|}}.
\]  

(8.4)

provided we took the constants \( A_0 \) in Lemma 5.2 and \( B \) in Sect. 6.1 so that \( A_0 \geq \frac{4B}{\log \theta} \). Indeed, for the length \( \ell_m \) of \( \mathcal{C}_m \) we have (6.2), so

\[
e^{\| f^{(t)} \|_\theta} \operatorname{diam}(\mathcal{C}_m) \leq e^{\frac{2\log |b|}{\theta} \ell_m} = |b|^{\frac{2}{A_0}} e^{-\frac{\ell_m |\log \theta|}{2}} \leq |b|^{\frac{2}{A_0}} e^{-\left(|\log \theta|/B \right) \log |b|} = |b|^{\frac{2}{A_0} - \frac{1}{A_0} \log |b|} \leq |b|^{-1/A_0},
\]

and
which proves (8.4).

Then we define the metric $D(u,u')$ on $\hat{U}$ and the class of positive functions $K_E$ as in Sect. 6.2. Now with the above one easily shows that Lemma 6.5 is valid in the form $N^{(t)}(K_E) \subset K_E$. Indeed, the main observation to make to prove this is that, given $u,u' \in \hat{U}$ such that there exists an integer $p \geq 0$ with $\mathcal{S}^p(Y(u,u')) \subset C_0^m$ for some $m \leq m_0$ and $\ell(u,u') \geq p$, then for any integer $k \geq 1$, if $v,v'(v) \in U$ satisfy $\sigma^k(v) = u$, $\sigma^k(v') = u'$ and belong to the same cylinder of length $k$, then

$$|f^{(at)}_k(v) - f^{(at)}_k(v')| \leq \sum_{j=0}^{m-1} |f^{(at)}| \theta^{m-j} D_{\theta}(u,u') \leq \text{Const} \ |f^{(t)}| \theta D_{\theta}(u,u')$$

$$\leq \text{Const} \ |f^{(t)}| \theta \text{diam}_{\theta}(C_m) \leq \text{Const}.$$ (8.5)

With this observation, a simple modification of the proof of Lemma 6.5 gives $N^{(t)}H \in K_E$ for every $H \in K_E$.

Next, we define the class of functions $K_0$ as in Sect. 6.3 and prove the analogue of Lemma 6.7: $L_{f^{(t)}-a\tau}(K_0) \subset K_0$ for all $s$ with $|s| \leq M_1$ and all integers $q \geq N$. To prove this, the choice of $\theta_1$ is important; it implies (using Lemma 5.1 and $\theta_1^2 = \theta$)

$$\text{diam}_{\theta_1}(C_m) = \theta_1^{\ell_m} \leq \theta_1^{\ell_m/2} (\theta_1^{\ell_m})^{1/\alpha_2} \leq \theta_1^{\ell_m/2} \text{diam}(C_m).$$

Then, assuming $u,u',v,v',w,w'$ are as in the proof of Lemma 6.7, we derive

$$|f^{(at)}_N(w) - f^{(at)}_N(w')| \leq \text{Const} \ |f^{(t)}| \theta_1^{p+N} \text{diam}_{\theta_1}(C_m) \leq \text{Const} \ \frac{4}{\theta_1^{p+N}} \theta_1^{\ell_m/2} \text{diam}(C_m)$$

$$\leq \text{Const} \ \theta_2^{p+N} \theta_1^{\ell_m/2-t} \text{diam}(C_m) \leq \text{Const} \ \theta_2^{p+N} \text{diam}(C_m) \leq 1,$$ (8.6)

since $t << \ell_m/2$. Now the rest of the proof of Lemma 6.7 is the same, and as a consequence one gets (as in Corollary 6.8) that the eigenfunctions $h_{at}$ belong to $K_0$.

Finally, the arguments in Sect. 6.4 can be repeated with very little change – the main one is that in the first estimate of $|L^{(at)}_{ab}(h)(v) - L^{(at)}_{ab}(h')(v')|$ one has to use (8.5) again. This proves the analogue of Lemma 6.9, where the operator $L^{(at)}_{ab}$ is replaced by $L_{ab}$.

We will now prove Lemma 6.9 in its original form under the present assumption that $f \in \mathcal{F}_\theta(\hat{U})$.

**Lemma 8.1.** Assume $f \in \mathcal{F}_\theta(\hat{U})$. Choosing $E > 1$ and $\mu_0$ as in Sect. 5.2 and assuming $N$ is sufficiently large, for any $|a| \leq a_0$, any $|b| \geq b_0$ and any $(h,H) \in K_b$ there exists $J \in \mathcal{J}(b)$ such that $(L^{(at)}_{ab}(h),\mathcal{N}_J H) \in K_b$.

**Proof.** Consider the function

$$\zeta = f^{(a)}_N - f^{(at)}_N \in \mathcal{F}_\theta(\hat{U}).$$

Notice that for any $u \in \hat{U}$ and any function $h$ on $\hat{U}$ we have

$$(\mathcal{M}^n_{a}(h))(u) = \sum_{\sigma^n v = u} e^{f^{(a)}_N(v)}(v)(h(v)) = \sum_{\sigma^n v = u} e^{f^{(at)}_N(v)}(f^{(at)}_N - f^{(a)}_N)(v)(h(v)) = (\mathcal{M}^n_{a}(e^\zeta h))(u).$$

Thus, $\mathcal{M}^n_{a}h = \mathcal{M}^n_{a}(e^\zeta h)$, and similarly one observes that $L^{(at)}_{ab}h = L^{(at)}_{ab}(e^\zeta h)$.

We will now repeat the argument from the proof of Lemma 6.9.

Let $|a| \leq a_0$, $|b| \geq b_0$ and $(h,H) \in K_b$. We will construct a representative set $J \in \mathcal{J}(b)$ such that $(L^{(at)}_{ab}(h),\mathcal{N}_J H) \in K_b$. Given an arbitrary representative set $J$, we will first show that $(L^{(at)}_{ab}(h),\mathcal{N}_J^{(t)}(e^\zeta H))$ has property (ii).

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Assume that the points $u, u'$, the cylinder $C_m$ in $U$, the integer $p \geq 0$ and the points $v, v' \in \hat{U}$ satisfy (6.20) for some $i = 1, 2$ and $\ell = 1, \ldots, \ell_0$. Since $h_a, h_{at} \in K_0$, we have

$$|\ln h_a(w) - \ln h_a(w')| \leq \frac{|h_a(w) - h_a(w')|}{\min\{|h_a(w)|, |h_a(w')|\}} \leq E_1 \theta_2^{p+2N} \text{diam}(\bar{C}_m),$$

and similarly, $|\ln h_{at}(v) - \ln h_{at}(v')| \leq E_1 \theta_2^{p+N} \text{diam}(\bar{C}_m)$. By (8.6),

$$|f_N^{(at)}(w) - f_N^{(at)}(w')| \leq \text{Const} \theta_2^{p+N} \text{diam}(\bar{C}_m) \leq 1,$$

(8.7) assuming $N$ is chosen appropriately. Thus, using (8.6) and a similar but simpler estimate for $|f_N^{(a)}(w) - f_N^{(a)}(w')|$, we get

$$|\zeta(w) - \zeta(w')| = |(f^{(a)} - f^{(at)})_N(w) - (f^{(a)} - f^{(at)})_N(w')|$$

$$\leq C_{12}' \theta_2^{p+N} \text{diam}(\bar{C}_m) < C_{12}'$$

(8.8)

for some global constant $C_{12}' > 0$. This implies

$$|e^{\zeta(w)} - e^{\zeta(w') - 1}| \leq \text{Const} \ |\zeta(w) - \zeta(w')| \leq C_{12} \theta_2^{p+N} \text{diam}(\bar{C}_m) < C_{12}$$

(8.9)

for some global constant $C_{12} > 0$.

Hence for any $a$ and $b$ with $|a| \leq a_0$ and $|b| \geq b_0$ we have:

$$|(L_{ab}^N h)(v) - (L_{ab}^N h)(v')| = |(L_{ab}^N (e^{\zeta} h))(v) - (L_{ab}^N (e^{\zeta} h))(v')|$$

$$= \left| \sum_{\sigma N w = v} e^{(f_N^{(at)} - m\tau_N)(w)} e^{\zeta(w)} h(w) - \sum_{\sigma N w = v} e^{(f_N^{(at)} - m\tau_N)(w')} e^{\zeta(w')} h(w')(w) \right|$$

$$\leq \left| \sum_{\sigma N w = v} e^{(f_N^{(at)} - m\tau_N)(w)} e^{\zeta(w)} [h(w) - h(w')] + \sum_{\sigma N w = v} e^{(f_N^{(at)} - m\tau_N)(w')} \left[ e^{\zeta(w)} - e^{\zeta(w')} \right] h(w') \right|$$

$$+ \sum_{\sigma N w = v} \left| e^{(f_N^{(at)} - m\tau_N)(w)} - e^{(f_N^{(at)} - m\tau_N)(w')} \right| e^{\zeta(w')} |h(w')|$$

$$\leq \sum_{\sigma N w = v} e^{(f_N^{(at)}(w) - f_N^{(at)}(w'))} e^{f_N^{(at)}(w')} E|b| \theta_2^{p+2N} \text{diam}(\bar{C}_m) e^{C_{12}'} e^{\zeta(w')} H(w')$$

$$+ \sum_{\sigma N w = v} e^{(f_N^{(at)}(w) - f_N^{(at)}(w'))} e^{f_N^{(at)}(w')} C_{12} \theta_2^{p+N} \text{diam}(\bar{C}_m) e^{\zeta(w')} H(w')$$

$$+ \sum_{\sigma N w = v} e^{(f_N^{(at)} - m\tau_N)(w) - (f_N^{(at)} - m\tau_N)(w') - 1} e^{f_N^{(at)}(w')} e^{\zeta(w')} H(w')$$

$$\leq e^{1+C_{12}'} E|b| \theta_2^{p+2N} \text{diam}(\bar{C}_m) M_{at}^N(e^{\zeta} H)(v') + C_{13} \theta_2^{p+N} \text{diam}(\bar{C}_m) M_{at}^N(e^{\zeta} H)(v')$$

$$+ C_{13} |b| \theta_2^{p+N} \text{diam}(\bar{C}_m) M_{at}^N(e^{\zeta} H)(v')$$

$$\leq [2e^{1+C_{12}'} E\theta_2^{N} + 2C_{13}] |b| \theta_2^{p+N} \text{diam}(\bar{C}_m) N_{J(t)}(e^{\zeta} H)(v')$$

for some global constant $C_{13} > 0$. Assuming $2e^{1+C_{12}'} \theta^{N} \leq 1/2$ and $2C_{13} \leq E/2$, we get

$$|(L_{ab}^N h)(v) - (L_{ab}^N h)(v')| \leq E \ |b| \theta_2^{p+N} \text{diam}(\bar{C}_m) N_{J(t)}(e^{\zeta} H)(v') = E \ |b| \theta_2^{p+N} \text{diam}(\bar{C}_m) (N_{J} H)(v'),$$

so, $(L_{ab}^N h, N_{J} H)$ has property (ii).

Now we will construct $J$ so that $|L_{ab}^N h|(u) \leq (N_{J} H)(u)$ for all $u \in \hat{U}$, which is equivalent to

$$|L_{ab}^N (e^{\zeta} h)|(u) \leq (N_{J} (e^{\zeta} H))(u)$$

(8.10)
for all \( u \in \tilde{U} \).

Define the functions \( \psi_{\ell}, \gamma_{\ell}^{(1)}, \gamma_{\ell}^{(2)} : \tilde{U} \to \mathbb{C} \) as in the proof of Lemma 6.9. Notice that

\[
\psi_{\ell}(u) = e^{(f^{(at)} + b\tau_{\ell})(v^{(f)}(u))} (e^\epsilon h)(v^{(f)}(u)) + e^{(f^{(at)} + b\tau_{\ell})(v^{(i)}(u))} (e^\epsilon h)(v^{(f)}(u)),
\]

\[
\gamma_{\ell}^{(1)}(u) = (1 - \mu_{\ell}) e^{(f^{(at)}(v^{(i)}(u)))} (e^\epsilon H)(v^{(f)}(u)) + e^{(f^{(at)}(v^{(i)}(u)))} (e^\epsilon H)(v^{(f)}(u)),
\]

and similarly for \( \gamma_{\ell}^{(2)}(u) \). We will use again the functions \( \varphi_{\ell,m}(u) = \varphi_{\ell}(Z_{m}, u), u \in U \), from Sect. 5.3.

As before (8.10) is trivially satisfied for \( u \notin V_{b} \) for any choice of \( J \).

Consider an arbitrary \( m = 1, \ldots, m_{0} \). We will construct \( j \leq j_{0} \) with \( D_{j} \subset C_{m, i} \), and a pair \((i, \ell)\) for which \( (i, j, \ell) \) will be included in \( J \).

**Case 1.** There exist \( j \leq j_{0} \) with \( D_{j} \subset C_{m, i} \), \( i = 1, 2 \) and \( \ell \leq \ell_{0} \) such that the first alternative in Lemma 6.10(b) holds for \( \bar{D}_{j}, i \) and \( \ell \). This case is dealt with exactly as in the proof of Lemma 6.9.

**Case 2.** For all \( j \leq j_{0} \) with \( D_{j} \subset C_{m, i} \), \( i = 1, 2 \) and \( \ell \leq \ell_{0} \) the second alternative in Lemma 6.10(b) holds for \( \bar{D}_{j}, i \) and \( \ell \), i.e.

\[
|h(v^{(f)}(u))| \geq \frac{1}{4} H(v^{(f)}(u)) > 0
\]  

(8.11)

for any \( u \in \tilde{C}_{m} \).

Let \( u, u' \in \tilde{C}_{m} \), and let \( i = 1, 2 \). Using (6.20) and the assumption that \((h, H) \in K_{b}\), and in particular property (ii) with \( p = 0 \), \( v = v^{(f)}(u) \) and \( v' = v^{(f)}(u') \), and also (8.8) and (8.9) with \( p = 0 \), and assuming e.g.

\[
\min\{|e^{\zeta(v^{(f)}(u))} h(v^{(f)}(u))|, |e^{\zeta(v^{(f)}(u'))} h(v^{(f)}(u'))|\} = |e^{\zeta(v^{(f)}(u))} h(v^{(f)}(u'))|,
\]

we get

\[
\frac{|e^{\zeta(v^{(f)}(u))} h(v^{(f)}(u)) - e^{\zeta(v^{(f)}(u'))} h(v^{(f)}(u'))|}{\min\{|e^{\zeta(v^{(f)}(u))} h(v^{(f)}(u))|, |e^{\zeta(v^{(f)}(u'))} h(v^{(f)}(u'))|\}} \\
\leq \frac{|e^{\zeta(v^{(f)}(u))} - e^{\zeta(v^{(f)}(u'))}| |h(v^{(f)}(u'))|}{|e^{\zeta(v^{(f)}(u))} h(v^{(f)}(u'))|} + \frac{|e^{\zeta(v^{(f)}(u))} h(v^{(f)}(u)) - h(v^{(f)}(u'))|}{|e^{\zeta(v^{(f)}(u))} h(v^{(f)}(u'))|} \\
\leq |e^{\zeta(v^{(f)}(u))} - e^{\zeta(v^{(f)}(u'))}| - 1 + C_{14} \frac{|h(v^{(f)}(u)) - h(v^{(f)}(u'))|}{|h(v^{(f)}(u'))|} \\
\leq C_{14} \theta_{2}^{N} \text{diam}(\tilde{C}_{m}) + \frac{E|b| \theta_{2}^{N} H(v^{(f)}(u'))}{|h(v^{(f)}(u'))|} \text{diam}(\tilde{C}_{m}) \\
\leq (C_{14} + 4E|b|) \theta_{2}^{N} A R_{0} e \frac{1}{|b|} < 5E \theta_{2}^{N} e, 
\]

assuming \( E \geq C_{14} A R_{0} \). So, the angle between the vectors \( e^{\zeta(v^{(f)}(u))} h(v^{(f)}(u)) \) and \( e^{\zeta(v^{(f)}(u'))} h(v^{(f)}(u')) \) in \( \mathbb{R}^{2} \) is \( < 10 E A R_{0} \theta_{2}^{N} e \) by [5,5]. Since \( e^{\zeta(v^{(f)}(u))} \) and \( e^{\zeta(v^{(f)}(u'))} \) are real numbers, the arguments of the complex numbers \( e^{\zeta(v^{(f)}(u))} h(v^{(f)}(u)) \) and \( e^{\zeta(v^{(f)}(u'))} h(v^{(f)}(u')) \) are the same as those of \( h(v^{(f)}(u)) \) and \( h(v^{(f)}(u')) \).

As before, for each \( i = 1, 2 \) we can choose a real continuous function \( \theta_{i}^{(m)}(u), u \in C_{m}' \), with values in \([0, \pi/6]\) and a constant \( \lambda_{i}^{(m)} \) such that

\[
h(v^{(f)}(u)) = e^{i(\lambda_{i}^{(m)} + \theta_{i}^{(m)}(u))} |h(v^{(f)}(u))|, \\
u \in C_{m}'.
\]
Fix an arbitrary \( u_0 \in \mathcal{C}_m \) and set \( \lambda^{(m)} = |b|\varphi_{\ell,m}(u_0) \), and assume again that \( |\lambda_2^{(m)} - \lambda_1^{(m)} + \lambda^{(m)}| \leq \pi \). Then
\[
|\theta_i^{(m)}(u) - \theta_i^{(m)}(u')| \leq 2\sin |\theta_i^{(m)}(u) - \theta_i^{(m)}(u')| < 16ER_0\theta_2^N\epsilon_1
\]
for all \( u, u' \in \mathcal{C}_m \).

As in the proof of Lemma 6.9, the difference between the arguments of the complex numbers \( e^{ib\tau_N(v_1^{(\ell)}(u))}(e^{\xi}h)(v_1^{(\ell)}(u)) \) and \( e^{ib\tau_N(v_2^{(\ell)}(u))}(e^{\xi}h)(v_2^{(\ell)}(u)) \) is given by the function
\[
\Gamma_\ell(u) = [b\tau_N(v_2^{(\ell)}(u)) + \theta_2^{(m)}(u) + \lambda_2^{(m)}] - [b\tau_N(v_1^{(\ell)}(u)) + \theta_1^{(m)}(u) + \lambda_1^{(m)}] = (\lambda_2^{(m)} - \lambda_1^{(m)}) + |b|\varphi_{\ell,m}(u) + (\theta_2^{(m)}(u) - \theta_1^{(m)}(u)),
\]
and as before we prove that there exist \( j \leq j_0 \) and \( \ell \leq \ell_0 \) such that \( \epsilon_3 \leq |\Gamma_\ell(u)| < \frac{3\pi}{2} \) for all \( u \in \mathcal{D}_j' \).

As in the proof of Lemma 6.9 either \( H(v_1^{(\ell)}(u)) \geq H(v_2^{(\ell)}(u))/4 \) for all \( u \in \mathcal{D}_j' \) or \( H(v_2^{(\ell)}(u)) \geq H(v_1^{(\ell)}(u))/4 \) for all \( u \in \mathcal{D}_j' \). Assume e.g. that \( H(v_1^{(\ell)}(u)) \leq H(v_2^{(\ell)}(u))/4 \) for all \( u \in \mathcal{D}_j' \). We will show that \( |\psi_\ell(u)| \leq \gamma_\ell^{(1)}(u) \) for all \( u \in \mathcal{D}_j' \). Given such \( u \), let \( \varphi \) be the smaller angle between the vectors
\[
z_1 = e^{(f_1^{(\alpha)} + 4b\tau_N)(v_1^{(\ell)}(u))}(e^{\xi}h)(v_1^{(\ell)}(u)), \quad z_2 = e^{(f_2^{(\alpha)} + 4b\tau_N)(v_2^{(\ell)}(u))}(e^{\xi}h)(v_2^{(\ell)}(u))
\]
in the complex plane \( \mathbb{C} \); then \( \epsilon_3 \leq \varphi \leq 3\pi/2 \). Moreover, (8.8) and (8.9) imply
\[
\left| \frac{z_1}{z_2} \right| = e^{f_1^{(\alpha)}(v_1^{(\ell)}(u)) - f_2^{(\alpha)}(v_2^{(\ell)}(u))} e^{\xi(v_1^{(\ell)}(u)) - \xi(v_2^{(\ell)}(u))} \frac{|h(v_1^{(\ell)}(u))|}{|h(v_2^{(\ell)}(u))|} \leq (2C_{12})16 = 32C_{12}.
\]
As in the proof of Lemma 6.9 this yields
\[
|z_1 + z_2| \leq (1 - t)|z_1| + |z_2|,
\]
for some constant \( t \in (0, 1) \) depending on \( \epsilon_3 \) and \( C_{11} \). Assuming that the constant \( \mu_0 \leq t \), we have \( |\psi_\ell(u)| \leq \gamma_\ell^{(1)}(u) \) for all \( u \in \mathcal{D}_j' \). Now set \( j_m = j \), \( \ell_m = \ell \) and \( i_m = 1 \), and include \( (i_m, j_m, \ell_m) \) in the set \( J \). Then \( \mathcal{D}_{j_m} \subset \mathcal{C}_m \) and we deduce that (8.10) holds on \( \mathcal{D}_{j_m} \).

Next, we proceed with what is done in Sect. 7. First, we prove parts (a) and (b) of Lemma 7.1 assuming \( f \in \mathcal{F}_\theta(\bar{U}) \). Part (a) goes without a change. In part (b) one proves that
\[
\int_{V_b} (N_JH)^2 \, d\nu \leq \rho_3 \int_{V_b} L^N_{J(0)}(H^2) \, d\nu,
\]
for any \( H \in \mathcal{K}_E \) and any \( J \in J(b) \), where \( \rho_3 = \rho_3(N) < 1 \) is possibly a slightly larger constant, and again \( a_0 = a_0(N) > 0 \) is chosen sufficiently small. The proof of this uses the same lines as the ones in the proof of Lemma 7.1(b) combined with the fact that \( ||\zeta_N||_0 = ||f_N^{(a)} - f_N^{(at)}|| \leq N \text{Const} \) for some global constant Const > 0.

Then, using the analogue of Lemma 7.1 (with \( f \in \mathcal{F}_\theta(\bar{U}) \)) and Lemma 7.2 one proves Lemma 7.3 in the same form – the difference is that now \( f \in \mathcal{F}_\theta(\bar{U}) \) compared to the previous stronger assumption \( f \in \mathcal{F}_{\theta_1}(\bar{U}) \). This gives the estimate (7.13) in exactly the same form under this more general assumption. And then one just needs to repeat the argument from the proof of Theorem 1.3 (the same as under the assumption \( f \in \mathcal{F}_{\theta_1}(\bar{U}) \)).

\textbf{Proof of Theorem 1.1.} This follows from the procedure described in [DI] (see Sect. 4 and Appendix 1 there). ■
Proof of Corollary 1.4. Let again $\hat{\theta}$ be as in (5.1). Choose the constants $C > 0$, $\varrho \in (0, 1)$, $a_0 > 0$ and $b_0 \geq 1$, $T_0 > 0$ as in the proof of Theorem 1.3. Let $\theta \leq \theta < 1$. As in the proof of Lemma 5.1, $(d(x,y))^{\alpha} \leq \text{Const } D_{\theta}(x,y)$ will always hold assuming $1/\gamma^{\alpha} \leq \theta$, i.e. $\alpha \geq \frac{\log \theta}{\log \gamma}$. Here $1 < \gamma < \gamma_1$ are the constants from (2.1). Then for such $\alpha$ we have $|h|_{\alpha} \leq \text{Const } |h|_{\alpha}$.

Recall that $\alpha_1 = \frac{\log \theta}{\log \gamma} > 0$ by (5.1), and that $\alpha_1 \in (0, 1]$ is chosen such that the local stable holonomy maps on $\tilde{R}$ are uniformly $\alpha_1$-Hölder, i.e. there exists a constant $C_{15} > 0$ such that for any $z, z' \in \tilde{R}_i$ for some $i = 1, \ldots, k_0$ and any $x, y \in W^u_R(z)$ for the projections $x', y' \in W^u_R(z')$ of $x, y$ along stable leaves we have $d(x', y') \leq C_{15} (d(x,y))^{\alpha_1}$.

Let $\alpha \in (0, \alpha_1]$; then $\alpha = \frac{\log \theta}{\log \gamma}$ for some $\theta \in [\hat{\theta}, 1)$. As above this gives $|h|_{\alpha} \leq C'_{16} |h|_{\alpha}$ for any $h \in C^{\alpha}(\tilde{U})$.

Assume that for a given $h \in C^{\alpha}(\tilde{U})$ we have $\|h\|_{\alpha,b} \leq 1$; then $\|h\|_0 \leq 1$ and $|h|_{\alpha} \leq |b|$, so $|h|_{\alpha} \leq C'_{16} |b|$ and therefore $\|h\|_{\alpha,b} \leq C'_{16} + 1$. As in (8.1),

$$\|L_{ab}^n h\|_{\alpha,b} \leq 3A_0 \rho^n_0 , \quad n \geq 4kN\hat{b} = 4kN \lfloor \log |b| \rfloor,$$

so in particular

$$\|L_{ab}^n h\|_0 \leq 3A_0 \rho^n_0 \quad (8.14)$$

for all $n \geq 4kN\hat{b}$.

Next, one needs to repeat part of the arguments from the proof of Theorem 1.3 above.

First, one needs a version of Lemma 5.2(b) for functions $w \in C^{\alpha}(\tilde{U})$. Given an integer $m \geq 0$ and $u, u' \in U_i$ for some $i = 1, \ldots, k_0$, notice that of $\sigma^m(v) = u, \sigma^m(v') = u'$ and $v' = v'(v)$ belongs to the cylinder of length $m$ containing $v$, then

$$|w(\sigma^j v) - w(\sigma^j (v'))| \leq |w|_{\alpha} (d(\sigma^j v, \sigma^j (v')))^{\alpha} \leq \frac{|w|_{\alpha}}{\epsilon^{\sigma^j (m-j)}} (d(\tilde{\sigma}^{m-j}(\sigma^j v), \tilde{\sigma}^{m-j}(\sigma^j (v')))^{\alpha} \leq C''_{16} \frac{|w|_{\alpha}}{\epsilon^{\sigma^j (m-j)}} (d(u, u'))^{\alpha \alpha_1}.$$

This implies

$$|w_m(v) - w_m(v')| \leq C_{17} |w|_{\alpha} (d(u, u'))^{\alpha \alpha_1} \quad (8.16)$$

This is true for $w = f, w = \tau$. Now repeating the argument in the proof of Lemma 5.2(b), for $|a| \leq a_0$ and $w \in C^{\alpha}(\tilde{U})$ we get

$$\|(L_{f-(P+a)\tau}^m w)(u) - (L_{f-(P+a)\tau}^m w)(u')\| \leq \sum_{\sigma^m v = u} e^{A(P+a)(m+1)} w(v) - \sum_{\sigma^m v = u} e^{A(P+a)(m+1)} w(v'(v)) |w(v')| \leq C_{17} \left( \frac{|w|_{\alpha}}{\epsilon^{\sigma^j m}} + |w|_0 \right) (L_{f-(P+a)\tau}^m 1)(u) (d(u, u'))^{\alpha \alpha_1}.$$

In particular this shows that $L_{f-(P+a)\tau}^m w \in C^{\alpha \alpha_1}(\tilde{U})$ for all $w \in C^{\alpha}(\tilde{U})$ and all integers $m \geq 0$.

Since $w = 1 \in C^{\alpha}(\tilde{U})$, it now follows from Perron-Ruelle-Frobenius Theorem that the eigenfunction $h_\alpha \in \tilde{C}^{\alpha \alpha_1}(\tilde{U})$ and so $f^{(a)} \in C^{\alpha \alpha_1}(\tilde{U})$ for all $|a| \leq a_0$. Moreover, taking $a_0$ sufficiently small, we may assume that $\|h_\alpha\|_{\alpha \alpha_1} \leq C_{18} = \text{Const}$ for all $|a| \leq a_0$. Using (8.16) with $w = f_a$ and $\alpha$ replaced by $\alpha \alpha_1$, we get

$$|f_a^m(v) - f_a^m(v')| \leq C''_{18} (d(u, u'))^{\alpha \alpha_1^2},$$
and also

$$|f^{(a)}(v) - f^{(a)}(v')| \leq C''_{18} \rho_{14}^m (d(u, u'))^{\alpha_1^2}.$$  

Now, using standard arguments, for $h \in C^\alpha(\widehat{U})$ we get

$$|L_{ab}^m h(u) - L_{ab}^m h(u')| = \left| \sum_{\sigma^m v = u} \left( e^{f^{(a)}_m(v)} - \nu \tau h(v) - e^{f^{(a)}_m(v') - \nu \tau h(v')} \right) \right| \leq \left| \sum_{\sigma^m v = u} \left( e^{f^{(a)}_m(v)} - e^{f^{(a)}_m(v') - \nu \tau h(v')} \right) \right| \leq \left| \sum_{\sigma^m v = u} e^{f^{(a)}_m(v)} \right| \left| h(v) - h(v') \right| + \left| \sum_{\sigma^m v = u} e^{f^{(a)}_m(v) - \nu \tau h(v')} \right| \left| h(v') \right|.$$

Using (8.15) with $w = h$ and $j = 0$ gives $|h(v) - h(v')| \leq C_{17} |h|_{\alpha} (d(u, u'))^{\alpha_1}$. Moreover,

$$\left| e^{f^{(a)}_m(v)} - e^{f^{(a)}_m(v') - \nu \tau h(v')} - 1 \right| \leq C_{18}^m |h| (d(u, u'))^{\alpha_1^2}.$$

Thus,

$$|L_{ab}^m h(u) - L_{ab}^m h(u')| \leq C_{18} |\rho|^m |h|_{\alpha} + |b| \left| h\right|_{0} (d(u, u'))^{\alpha_1^2}.$$

(8.17)

Since $|h|_{\alpha} \leq |b|$ and $|h|_{0} \leq 1$, this gives $|L_{ab}^m h|_{\alpha_1^4} \leq \text{Const} |b|$ for all $m \geq 0$. Using (8.14) and (8.17) with $h$ replaced by $L_{ab}^m h$ and $\alpha$ replaced by $\alpha_1^4 \leq \alpha_0$, we get

$$\left| (L_{ab}^m h)(u) - (L_{ab}^m h)(u') \right| \leq \text{Const} \left| \rho \right|^m \left| L_{ab}^m h \right|_{\alpha_1^4} + |b| \left| L_{ab}^m h \right|_{0} (d(u, u'))^{\alpha_1^4} \leq C_{19} \rho_{T}^m |b| + |b| \left| h\right|_{6} (d(u, u'))^{\alpha_1^4}$$

for $m \geq 4kN\hat{b}$. Thus, $\left\| L_{ab}^m h \right\|_{\alpha_1^4} \leq C_{19} \rho_{T}^m$ for all $m \geq 4kN\hat{b}$ and all $h \in C^\alpha(\widehat{U})$ with $\left\| h\right\|_{\alpha} \leq 1$. Since

$$L_{f-(P_1 + \alpha + \hat{b})r} h = \frac{1}{h_{a}} L_{ab}^m (h_{a} h),$$

it is now easy to get

$$\left\| L_{f-(P_1 + \alpha + \hat{b})r} h \right\|_{\alpha_1^4} \leq C_{19} \rho_{T}^m \left\| h\right\|_{\alpha, b}$$

for all $m \geq 4kN\hat{b}$ and all $h \in C^\alpha(\widehat{U})$. Setting $\alpha_0 = \alpha_1$ and $\hat{\beta} = \alpha_1^4$ proves the assertion.

9 Temporal distance estimates on cylinders

Here we prove Lemmas 4.2 and 4.3.

9.1 A technical lemma

Notice that in Lemma 4.1 the exponential maps are used to parametrize $W^u_e(z)$ and $W^s_e(z)$. The particular choice of the exponential maps is not important, however it is important that these maps are $C^2$. So, we cannot use the maps $\Phi^u_e$ and $\Phi^s_e$ defined in Sect. 3. In order to use Lemma 4.1 we will need in certain places to replace the local lifts $\varphi_p$ of the iterations $\varphi^p$ of the map $\varphi$ by slightly different maps.

For any $x \in \mathcal{L}$ consider the $C^2$ map

$$\tilde{\varphi}_x = (\exp^u_{\varphi(x)})^{-1} \circ \varphi \circ \exp^u_x : E^u(x; r(x)) \rightarrow E^u(\varphi(x), \tilde{r}(\varphi(x))).$$
Proof. It is enough to show that \( d\omega \) vanishes on every stable/unstable manifold of a point on \( M \). For every \( R \) and the fact that \( d\omega \) corresponding to the Lyapunov exponent \( n \) into subspaces of dimensions \( n \), \( 1 \), \( 2 \) maps \( \tilde{\varphi} \) at any point where these sequences of maps are well-defined. In a similar way one defines the maps \( \hat{\varphi} \) and their iterations on \( E^\alpha(x; r(x)) \).

Following the notation in Sect. 3 and using the fact that the flow \( \hat{\varphi} \) is contact, the negative Lyapunov exponents over \( L \) are

\[
- \log \lambda_1 > - \log \lambda_2 > \ldots > - \log \lambda_k.
\]

Fix \( \hat{\epsilon} > 0 \) as in Sect. 3, assuming in addition that

\[
\hat{\epsilon} \leq \frac{\log \lambda_1}{100} \min \{ \beta, \theta \}, \quad \hat{\epsilon} \leq \frac{\log \lambda_1 (\log \lambda_2 - \log \lambda_1)}{4 \log \lambda_1 + 2 \log \lambda_2}.
\]

For \( x \in L \) we have an \( \varphi \)-invariant decomposition

\[
E^\alpha(x) = E^\alpha_1(x) \oplus E^\alpha_2(x) \oplus \ldots \oplus E^\alpha_k(x)
\]

into subspaces of dimensions \( n_1, \ldots, n_k \), where \( E^\alpha_i(x) \) is the \( d\varphi \)-invariant subbundle corresponding to the Lyapunov exponent \(-\log \lambda_i \). For the Lyapunov \( \hat{\epsilon} \)-regularity function \( R = R_\hat{\epsilon} : L \rightarrow (1, \infty) \), chosen as in in Sect. 3 (see also Sect. 4), we have

\[
\frac{1}{R(x) e^{m \hat{\epsilon}}} \leq \frac{\|d\varphi^m(x) \cdot v\|}{\lambda_i^{-m} \| v \|} \leq R(x) e^{m \hat{\epsilon}}, \quad x \in L, \ v \in E^\alpha_i(x) \setminus \{0\}, \ m \geq 0.
\]  

(9.1)

We will also assume that the set \( P_0 \) is as in (4.3), and the regularity functions \( R_\epsilon(x), \ r(x), \ \Gamma(x), \ L(x), \ D(x) \) satisfy (4.4).

For the contact form \( \omega \) it is known (see e.g. Sect. in [KH] or Appendix B in [Li]) that \( \omega \) vanishes on every stable/unstable manifold of a point on \( M \), while \( d\omega \) vanishes on every weak stable/unstable manifold. For Lyapunov regular points we get a bit of extra information.

Lemma 9.1. For every \( x \in L \) and every \( u = (u^{(1)}, \ldots, u^{(k)}) \in E^u(x; r(x)) \) and \( v = (v^{(1)}, \ldots, v^{(k)}) \in E^s(x; r(x)) \) we have

\[
d\omega_x(u, v) = \sum_{i=1}^k d\omega_x(u^{(i)}, v^{(i)}).
\]  

(9.2)

Proof. It is enough to show that \( d\omega_x(u^{(i)}, v^{(j)}) = 0 \) if \( i \neq j \). Let e.g. \( i < j \). Using (3.4), (4.1), (9.1) and the fact that \( d\omega \) is \( d\varphi \)-invariant, for \( m \geq 0 \) and \( x_m = \varphi^m(x) \) we get

\[
|d\omega_x(u^{(i)}, v^{(j)})| = |d\omega_{x_m}(d\varphi^m(x) \cdot u^{(i)}, d\varphi^m(x) \cdot v^{(j)})| \\
\leq C \|d\varphi^m(x) \cdot u^{(i)}\| \|d\varphi^m(x) \cdot v^{(j)}\| \\
\leq C R^2(x) \|u^{(i)}\| \|v^{(j)}\| \frac{(\lambda_i e^{2\hat{\epsilon} m})^m}{\lambda_j^m}.
\]

Since \( \lambda_i e^{2\hat{\epsilon} m} \leq \lambda_j \), the latter converges to 0 as \( m \to \infty \), so \( d\omega_x(u^{(i)}, v^{(j)}) = 0 \).

The case \( i > j \) is considered similarly by taking \( m \to -\infty \).
9.2  Proof of Lemma 4.2(a)

We will consider cylinders $\mathcal{C}$ of length $m \geq 1$ in $\bar{R}$ with $\mathcal{C} \cap P_0 \setminus \Xi_m \neq \emptyset$ (instead of considering cylinders $\mathcal{C}$ in $R$) with corresponding obvious changes in the estimates we need to prove.

Let $\mathcal{C}$ be a cylinder of length $m$ in $\bar{R}$. Fix an arbitrary $z_0 \in \mathcal{C} \cap P_0 \setminus \Xi_m$. Given $x_0 \in \mathcal{C}$, write $x_0 = \Phi_{z_0}^u(\xi_0) = \exp_{z_0}^u(\xi_0)$ for some $\xi_0, \tilde{\xi}_0 \in E^u(z_0)$ with $\tilde{\xi}_0 = \Psi_{z_0}^u(\xi_0)$. Then $\|\xi_0\| \leq R_0 \text{diam}(\mathcal{C})$.

Set $\mathcal{C}' = \tilde{\Psi} \circ \Psi^{-1}(\mathcal{C}) \subset \bar{R}$, $T = \tilde{\tau}_m(z_0)$ and $p = [T]$, so that $p \leq T < p + 1$. Moreover, $m\tau_0 \leq \tilde{\tau}_m(z_0) \leq m\tau_0$ gives $\frac{p}{\tau_0} \leq m \leq \frac{p+1}{\tau_0} \leq \frac{2p}{\tau_0}$.

Since $m$ is the length of $\mathcal{C}'$, $\tilde{P}^m(\mathcal{C}')$ contains a whole unstable leaf of a proper rectangle $\bar{R}_j$. Moreover, $z_0 \in \mathcal{C} \setminus \Xi_m$ shows that there exists an integer $m'$ with $m(1 - \delta_0) \leq m' \leq m$ such that $z = \tilde{P}^m(z_0) \in P_0$. Let $z \in \bar{R}_i$. By the choice of the constant $r_1 > 0$ (see Sect. 4.2), there exists $y \in R_i$ such that $B^{u}(y, r_1) \subset W^u_{\tilde{R}_i}(z)$ and $d(z, y) < r_0/2$. In particular, for every point $b' \in B^{u}(y, r_1)$ there exists $b \in \mathcal{C}$ with $\tilde{P}^m(b) = b'$. Set $p' = [\tilde{\tau}_m(z_0)]$; then $\varphi_{p'}^u(z_0) \in \tilde{P}^\prime$.

To estimate $p - p'$, notice that, as above, $\frac{p'}{\tau_0} \leq m' \leq \frac{2p'}{\tau_0}$, so the relationship between $m'$ and $m$ implies

$$p - p' \leq \tilde{\tau}_m(z_0) - \tilde{\tau}_{m'}(z_0) + 1 \leq (m - m')\tilde{\tau}_0 + 1 \leq \delta_0 m\tilde{\tau}_0 + 1 \leq \frac{3\delta_0\tilde{\tau}_0}{\tau_0} p,$$

assuming that $m$ (and so $p$) is sufficiently large. Thus, $p(1 - 3\delta_0\tilde{\tau}_0/\tau_0) \leq p' \leq p$, where

$$3\delta_0\tilde{\tau}_0/\tau_0 < 1,$$

and $z_{p'} = \varphi_{p'}^u(z_0) \in \tilde{P}_0$, so $r(z_{p'}) \geq r_0$ (by (4.3)). Clearly, $p' \geq \tilde{\tau}_{m'}(z_0)$. Then for every $b \in W^{u}_{\tilde{R}_i}(z_{p'})$ there exists $b \in \mathcal{C}$ with $\varphi_{p'}^u(b) = b'$. Consider an arbitrary $\zeta_{p'} \in E^u(z_{p'}; r_1/R_0)$ such that $\|\zeta_{p'}^{(1)}\| \geq r_1/R_0$, and set $\zeta = \tilde{\varphi}_{z_{p'}}^u(\zeta_{p'})$. Then $x = \Phi_{z_0}^u(\zeta) \in \mathcal{C}$, so

$$\text{diam}(\mathcal{C}) \geq d(z_0, x) \geq \frac{||\zeta||}{R_0} \geq \frac{||\zeta^{(1)}||}{R_0 R_0}.$$

On the other hand, Lemma 3.5 in [14] (see Lemma 10.1 below) and $\mu_1^{p'} \leq \mu_1^p < \lambda_1^p$ give

$$\frac{1}{\Gamma_0} \geq \frac{||\zeta^{(1)}||}{R_0} \geq \frac{||\zeta_{p'}^{(1)}||}{\Gamma_0 \mu_1^{p'}} \geq \frac{r_1/R_0}{\Gamma_0 \mu_1^p} > \frac{r_1}{R_0 \Gamma_0 \lambda_1^p},$$

hence $\text{diam}(\mathcal{C}) \geq \frac{c_3}{\lambda_1^p}$, where $c_3 = \frac{r_1}{R_0 \Gamma_0 \lambda_1^p} \geq 1$.

This proves the left-hand-side inequality in (4.5) with $C_1 = 1/c_3$. The other inequality in (4.5) follows by a similar (in fact, easier) argument, using the above estimate of $p - p'$. We omit the details.

9.3  Proof of Lemma 4.2(b)

Let $\mathcal{C}$ be a cylinder of length $m$ in $R$ such that there exists $\hat{z} \in \mathcal{C} \cap P_0 \setminus \Xi_m$. Set $\tilde{\mathcal{C}} = \psi(\mathcal{C})$.

Let $\hat{x}_0, \tilde{z}_0 \in \mathcal{C}$, $\tilde{y}_0, \tilde{b}_0 \in W^s_{\tilde{R}_1}(\tilde{z}_0)$. We can assume that $\mathcal{C}$ is the smallest cylinder containing $\hat{x}_0$ and $\tilde{z}_0$; otherwise we will replace $\mathcal{C}$ by a smaller cylinder.

It is enough to consider the case when $\hat{z}_0 = \hat{z}$. Indeed, assuming the statement is true with $\hat{z}_0$ replaced by $\tilde{z}$, consider arbitrary points $\hat{x}_0, \tilde{z}_0 \in \mathcal{C}$. Set \{y\} = $W^u_{\hat{R}}(\tilde{y}_0) \cap W^u_{\hat{R}}(\hat{z})$ and \{b\} = $W^s_{\hat{R}}(\hat{b}_0) \cap W^s_{\hat{R}}(\hat{z})$. Since the local unstable holonomy maps are uniformly Hölder, there exist (global) constants $C' > 0$ and $\beta' > 0$ such that $d(y, b) \leq C'(d(\tilde{y}_0, \tilde{b}_0))^{\beta'}$. Thus, using the assumption,

$$|\Delta(\hat{x}_0, y) - \Delta(\hat{x}_0, b)| \leq C_1 \text{diam}(\mathcal{C})(d(y, b))^{\beta_1} \leq C_1(C')^{\beta_1} \text{diam}(\mathcal{C})(d(\tilde{y}_0, \tilde{b}_0))^{\beta' \beta_1}.$$
A similar estimate holds for $|\Delta(\hat{z}_0, y) - \Delta(\hat{z}_0, b)|$, so

$$|\Delta(\hat{x}_0, \hat{y}_0) - \Delta(\hat{x}_0, \hat{b}_0)| = |(\Delta(\hat{x}_0, y) - \Delta(\hat{z}_0, y) - (\Delta(\hat{x}_0, b) - \Delta(\hat{z}_0, b))|$$

$$\leq |\Delta(\hat{x}_0, y) - \Delta(\hat{x}_0, b)| + |\Delta(\hat{z}_0, y) - \Delta(\hat{z}_0, b)|$$

$$\leq 2C_1(C')^{\beta_3}\text{diam}(C)(d(\hat{y}_0, \hat{b}_0))^{\beta_3}. \tag{9.3}$$

So, from now on we will assume that $\hat{z}_0 = \hat{z} \in C \cap P_0 \setminus \Xi_m$. Then $R(\hat{z}_0) \leq R_0$, $r(\hat{z}_0) \geq r_0$, etc. Set $x_0 = \Psi(\hat{x}_0)$, $z_0 = \Psi(\hat{z}_0)$, $y_0 = \Psi(\hat{y}_0) \in R$, $b_0 = \Psi(\hat{b}_0)$, and then write $x_0 = \Phi^u_{z_0}(\xi_0) = \exp_{z_0}^u(\xi_0)$ for some $\xi_0, \tilde{\xi}_0 \in E^u(z_0)$ with $\tilde{\xi}_0 = \Psi_{z_0}(\xi_0)$. Then $\|\xi_0\|, \|\tilde{\xi}_0\| \leq R_0\text{diam}(C)$. Similarly, write $y_0 = \exp_{x_0}^s(\tilde{y}_0) = \Phi_{x_0}^s(\eta_0)$ for some $\eta_0, \tilde{\eta}_0 \in E^s(z_0)$ with $\tilde{\eta}_0 = \Psi_{x_0}(\eta_0)$. By (3.6),

$$\|\tilde{v}_0 - v_0\| \leq R_0\|v_0\|^{1 + \beta}, \quad \|\tilde{\xi}_0 - \xi_0\| \leq R_0\|\xi_0\|^{1 + \beta}, \quad \|\tilde{\eta}_0 - \eta_0\| \leq R_0\|\eta_0\|^{1 + \beta}. \tag{9.3}$$

### 9.3.1 Pushing forward

Set $p = [\hat{r}_m(z_0)]$; then (15.5) holds. Set $q = [p/2]$. We will in fact assume that $q = p/2$; the difference with the case when $p$ is odd is insignificant. For any integer $j \geq 0$ set $z_j = \varphi^j(z_0)$, $x_j = \varphi^j(x_0)$, $y_j = \varphi^j(y_0)$ and also

$$\hat{\xi}_j = d\hat{\varphi}_{z_0}^j(0) \cdot \xi_j, \quad \hat{\xi}_j = \hat{\varphi}_{z_0}^j(\hat{\xi}_0), \quad \tilde{\xi}_j = d\hat{\varphi}_{z_0}^j(0) \cdot v_0, \quad v_j = \tilde{\varphi}_{z_0}^j(v_0), \quad \tilde{\xi}_j = d\hat{\varphi}_{z_0}^j(\tilde{\xi}_0).$$

Notice that $\hat{\xi}_0 = \Psi_{z_0}^u(\xi_0)$, $\tilde{\xi}_0 = \Psi_{z_0}^u(v_0)$, and also

$$\tilde{\xi}_0 = \Psi_{z_0}^s(\hat{\xi}_j), \quad \Phi_{z_0}^s(\hat{\xi}_j) = x_j, \quad \tilde{\xi}_0 = \Psi_{z_0}^s(\hat{\xi}_j), \quad \eta_j = \Psi_{x_0}^s(\eta_0), \quad \tilde{\eta}_j = \Psi_{x_0}^s(\eta_0).$$

so by (3.6),

$$\|\xi_j - \tilde{\xi}_j\| \leq R(z_i)\|\xi_j\|^{1 + \beta}, \quad \|v_j - \tilde{v}_j\| \leq R(z_i)\|v_j\|^{1 + \beta}, \quad \|\eta_j - \tilde{\eta}_j\| \leq R(z_i)\|\eta_j\|^{1 + \beta}. \tag{9.4}$$

Moreover, $\exp_{z_j}^s(\hat{\xi}_j) = \varphi^j(\exp_{z_0}^s(\xi_0)) = \varphi^j(x_0) = x_j$, $\exp_{z_j}^s(\tilde{v}_j) = y_j$ and $\exp_{z_j}^s(\tilde{\eta}_j) = b_j$, so Lemma 4.1 implies

$$|\Delta(x_j, y_j) - d\omega_{z_j}(\xi_j, \tilde{v}_j)| \leq C_0 \left[\|\hat{\xi}_j\|^{2\theta} + \|\hat{\xi}_j\|^{\theta} \|\tilde{\xi}_j\|^{2} \right] \tag{9.5}$$

and similarly

$$|\Delta(x_j, b_j) - d\omega_{z_j}(\xi_j, \tilde{\eta}_j)| \leq C_0 \left[\|\hat{\xi}_j\|^{2\theta} + \|\hat{\xi}_j\|^{\theta} \|\tilde{\eta}_j\|^{2} \right] \tag{9.5}$$

for every integer $j \geq 0$. From (9.4) one gets

$$|d\omega_{z_j}(\xi_j, \tilde{v}_j) - d\omega_{z_j}(\xi_j, v_j)| \leq 2C_0 R(z_i)\|\xi_j\| \|v_j\| \|\xi_j\|^{\beta} \|v_j\|^{\beta},$$

$$|d\omega_{z_j}(\xi_j, \tilde{\eta}_j) - d\omega_{z_j}(\xi_j, \eta_j)| \leq 2C_0 R(z_i)\|\xi_j\| \|\eta_j\| \|\xi_j\|^{\beta} \|\eta_j\|^{\beta},$$

and also $\|\hat{\xi}_j\| \leq 2\|\xi_j\|$, $\|\tilde{v}_j\| \leq 2\|v_j\|$ and $\|\tilde{\eta}_j\| \leq 2\|\eta_j\|$. Indeed, from (9.4),

$$\|\hat{\xi}_j\| \leq \|\xi_j\|((1 + R(z_i))\|\xi_j\|^{\beta}) \leq \|\xi_j\|((1 + R_0 r_0 / r_0^2) \leq \|\xi_j\|((1 + R_0 r_0) \leq 2\|\xi_j\|),$$

since $r_0 \leq 1/R_0$. Similarly, $\|\tilde{v}_j\| \leq 2\|v_j\|$ and $\|\tilde{\eta}_j\| \leq 2\|\eta_j\|$.

Using these, it follows from (9.5) that

$$|\Delta(x_j, y_j) - d\omega_{z_j}(\xi_j, v_j)| \leq 2C_0 R(z_i)\|\xi_j\| \|v_j\| \|\xi_j\|^{\beta} \|v_j\|^{\beta} + 8C_0 \left[\|\hat{\xi}_j\|^{2\theta} + \|\xi_j\|^{\theta} \|v_j\|^{2} \right]. \tag{9.6}$$
and similarly

\[ |\Delta(x_j, b_j) - d\omega_{z_j}(\xi_j, \eta_j)| \leq 2C_0 R(z_j) \|\xi_j\| \|\eta_j\| (\|\xi_j\|^\beta + \|\eta_j\|^\beta) + 8C_0 \left( \|\xi_j\|^2 \|\eta_j\|^\beta + \|\xi_j\|^\beta \|\eta_j\|^2 \right). \]  

(9.7)

for every integer \( j \geq 0 \).

We will be estimating \( |\Delta(x_0, y_0) - d\omega_{z_0}(\xi_0, v_0)| \). Since \( \Delta \) is \( \varphi \)-invariant and \( d\omega \) is \( d\varphi \)-invariant we have

\[ \Delta(x_0, y_0) = \Delta(x_j, y_j), \quad d\omega_{z_0}(\xi_0, v_0) = d\omega_{z_j}(\xi_j, \hat{v}_j), \]

and also \( \Delta(x_0, b_0) = \Delta(x_j, b_j) \) and \( d\omega_{z_0}(\xi_0, \eta_0) = d\omega_{z_j}(\xi_j, \hat{\eta}_j) \) for all \( j \). (Notice that \( d\varphi(x) = d\varphi(x) \) for all \( x \in M \).)

It follows from \( z_0 \notin \Xi_m \) and (3.23) that \( z_0 \notin \widehat{\Xi}_{m\tau_0}(p_0 + \hat{e}_0, \hat{\xi}_0, \hat{\eta}_0/\tau_0) \), so there exists an integer \( \ell \) with

\[ q - (m\tau_0)\frac{\hat{\tau}_0}{\tau_0} \leq \ell \leq q \]

such that \( \varphi^\ell(z_0) \in \widehat{P}_0 \). \textbf{Fix an \( \ell \) with this property.} As in Sect. 9.2, we have \( \frac{p}{\tau_0} \leq m \leq \frac{2p}{\tau_0} \).

Combined with \( q = p/2 \), this gives \( (m\tau_0)\frac{\hat{\tau}_0}{\tau_0} \leq (4\pi\tau_0)\frac{\hat{\tau}_0}{\tau_0} \). Thus, we have

\[ (1 - \hat{\delta}_0)q \leq \ell \leq q, \quad z_\ell = \varphi^\ell(z_0) \in P_0, \]

(9.8)

where

\[ \hat{\delta}_0 = \frac{4\pi\tau_0}{\tau_0^2} \hat{\delta}_0 < 1. \]

It then follows from Lemma 3.1, the choice of \( L_0 \) and \( \|\xi_\ell\| \leq r(z_\ell) \) (since \( \ell \leq q = p/2 \); see also Sect. 9.3.2 below) that

\[ \|\xi_\ell^{(1)} - \xi_\ell^{(1)}\| \leq L_0 \|\xi_\ell\|^{1+\beta}. \]

(9.9)

Apart from that, using Lemma 10.7(b) below, backwards for stable manifolds, with \( a = d\hat{\varphi}_{z_\ell}\xi_\ell(0) \cdot v_\ell \in E^s(z_0) \), \( b = d\hat{\varphi}_{z_\ell}(0) \cdot \eta_\ell \in E^u(z_0) \), since \( v_\ell = \hat{\varphi}_{z_\ell}(v_\ell) \) and \( \eta_\ell = \hat{\varphi}_{z_\ell}(\eta_\ell) \), it follows that

\[ \|(a_\ell^{(1)} - b_\ell^{(1)}) - (v_\ell^{(1)} - \eta_\ell^{(1)}))\| \leq L_0 \left( \|v_\ell - \eta_\ell\|^{1+\beta} + \|\eta_\ell\|^\beta \|v_\ell - \eta_\ell\| \right) \leq 2L_0 \|v_\ell - \eta_\ell\|. \]

Thus,

\[ \|d\hat{\varphi}_{z_\ell}^{-\ell}(0) \cdot (v_\ell^{(1)} - \eta_\ell^{(1)}) - (v_\ell^{(1)} - \eta_\ell^{(1)})\| \leq 2L_0 \|v_\ell - \eta_\ell\|. \]

(9.10)

In what follows we denote by \( \text{Const} \) a global constant (depending on constant like \( C_0 \), \( L_0 \), \( R_0 \) however independent of the choice of the cylinder \( C \), the points \( x_0, z_0, y_0, b_0 \), etc.) which may change from line to line.

Using (9.9), (9.10) and the above remarks, we obtain

\[ |d\omega_{z_\ell}(\xi_\ell, v_\ell - \eta_\ell)| \]

\[ \leq |d\omega_{z_\ell}(\xi_\ell^{(1)}, v_\ell^{(1)} - \eta_\ell^{(1)})| + C_0 \sum_{i=2}^{\hat{k}} \|\xi_{\ell i}^{(i)}\| (\|v_{\ell i}^{(i)}\| + \|\eta_{\ell i}^{(i)}\|) \]

\[ \leq |d\omega_{z_\ell}(\xi_\ell^{(1)}, v_\ell^{(1)} - \eta_{\ell i}^{(1)})| + \text{Const} \|\xi_{\ell i}\|^{1+\beta}\|v_{\ell i}^{(1)} - \eta_{\ell i}^{(1)}\| + C_0 \sum_{i=2}^{\hat{k}} \|\xi_{\ell i}^{(i)}\| (\|v_{\ell i}^{(i)}\| + \|\eta_{\ell i}^{(i)}\|) \]

\[ \leq |d\omega_{z_\ell}(d\hat{\varphi}_{z_\ell}(0) \cdot \xi_0^{(1)} - \eta_{\ell i}^{(1)})| + \text{Const} \|\xi_{\ell i}\|^{1+\beta}\|v_{\ell i}^{(1)} - \eta_{\ell i}^{(1)}\| + C_0 \sum_{i=2}^{\hat{k}} \|\xi_{\ell i}^{(i)}\| (\|v_{\ell i}^{(i)}\| + \|\eta_{\ell i}^{(i)}\|) \]
\[ = |d\omega_{20}(\xi_0^{(1)}, d\hat{\phi}_{\varepsilon}(0) \cdot (v_{\ell}^{(1)} - \eta_{\ell}^{(1)}))| + \text{Const} \|\xi_{\ell}\|^{1+\beta} \|v_{\ell}^{(1)} - \eta_{\ell}^{(1)}\| + C_0 \sum_{i=2}^{k} \|\xi_{\ell}^{(i)}\| \left(\|v_{\ell}^{(i)}\| + \|\eta_{\ell}^{(i)}\|\right) \]
\[ \leq |d\omega_{20}(\xi_0^{(1)}, v_0^{(1)} - \eta_0^{(1)})| + 2C_0 L_0 \|\xi_0\| \|v_0 - \eta_0\| + \text{Const} \|\xi_{\ell}\|^{1+\beta} \|v_{\ell}^{(1)} - \eta_{\ell}^{(1)}\| \]
\[ + C_0 \sum_{i=2}^{k} \|\xi_{\ell}^{(i)}\| \left(\|v_{\ell}^{(i)}\| + \|\eta_{\ell}^{(i)}\|\right) \]
\[ \leq \text{Const} \text{diam}(C) \|v_0 - \eta_0\| + \text{Const} \|\xi_{\ell}\|^{1+\beta} \|v_{\ell}^{(1)} - \eta_{\ell}^{(1)}\| + C_0 \sum_{i=2}^{k} \|\xi_{\ell}^{(i)}\| \left(\|v_{\ell}^{(i)}\| + \|\eta_{\ell}^{(i)}\|\right). \quad (9.11) \]

**9.3.2 Estimates for \(\|\xi_{\ell}\|, \|v_{\ell}\| \) and \(\|\eta_{\ell}\|\)**

We will now use the choice of \(\ell\) to estimate \(\|\xi_{\ell}\|, \|v_{\ell}\|\) and \(\|\eta_{\ell}\|\) by means of \(\|\xi_0\|, \|v_0\|\) and \(\|\eta_0\|\).

We will first estimate \(\|\xi_q\|, \|v_q\|\) and \(\|\eta_q\|\).

Using the definition of \(\xi_j, p = 2q, z_0 \in P_0\) and (3.12) we get
\[
\|\xi_q\| \leq \|\xi_q\|_{L^p} \leq \frac{\|\xi_p\|_{L^p}}{\mu_1^{-q}} \leq \frac{\Gamma(z_p)e^{q\xi_q} \|\xi_p\|}{\lambda_1^q} \leq \frac{\Gamma_0 e^{q\xi_q} \|\xi_p\|}{\lambda_1^q}. 
\]

Since \(\Phi^{u}_{z_p}(\xi_p) = x_p\) and \(d(x_p, z_p) \leq \text{diam}(\tilde{R}_i)\), we get \(\|\xi_p\| \leq R(z_p) d(x_p, z_p) \leq R_0 e^{p\varepsilon} r_1 \leq R_0 e^{p\varepsilon}\). Thus,
\[
\|\xi_q\| \leq R_0 \Gamma_0 e^{q\xi_q} \frac{\|\xi_p\|}{\lambda_1^q}. \quad (9.12)
\]

Using (3.12) again (on stable manifolds) and \(\|v_0\| \leq 2\delta^\prime / R_0 < 1\), we get
\[
\|v_q\| = \|v_q\|_{L^p} \leq \frac{\|v_0\|_{L^p}}{\mu_1^q} \leq \frac{\Gamma_0 e^{q\xi_q} \|v_0\|}{\lambda_1^q} \leq \frac{\Gamma_0 e^{q\xi_q}}{\lambda_1^q}. \quad (9.13)
\]

Similarly, \(\|\eta_q\| \leq \frac{\Gamma_0 e^{q\xi_q}}{\lambda_1^q}\).

Next, it follows from (4.15) that \((\lambda_1)^{2q} \geq c_3 / \text{diam}(C)\), so
\[
q \geq \frac{1}{2 \log \lambda_1} \log \frac{c_3}{\text{diam}(C)}. \quad (9.14)
\]

This and (9.12) give
\[
\|\xi_q\| \leq R_0 \Gamma_0 (\lambda_1 e^{-5\varepsilon})^{-q} = R_0 \Gamma_0 e^{-q \log(\lambda_1 e^{-5\varepsilon})} < 1. \quad (9.15)
\]

since \(\log 1 < 1\). Similarly, (9.13) yields
\[
\|v_q\| \leq \Gamma_0 (\lambda_1 e^{-\varepsilon})^{-q} \leq \Gamma_0 e^{-q \log(\lambda_1 e^{-\varepsilon})} \leq \Gamma_0 c_3 (\text{diam}(C)) \frac{\log(1/\lambda_1)}{2 \log \lambda_1}.
\]

The same estimate holds for \(\|\eta_q\|\).

We need similar estimates, however with \(q\) replaced by \(\ell\). Since \(q - \ell \leq \delta_0 q\) by (9.8), as in (9.15) one obtains
\[
\|\xi_{\ell}\| \leq \|\xi_{\ell}\|_{L^p} \leq \|\xi_q\|_{L^p} \leq \Gamma_0 e^{q\xi_q} \|\xi_q\| \leq \frac{R_0 \Gamma_0^2}{c_3} (\text{diam}(C)) \frac{\log(1/\lambda_1)}{2 \log \lambda_1}.
\]
Since $\lambda_k^{q_0} < \epsilon^q$ by the choice of $\delta_0$ in Sect. 6.1, we have $\lambda_k^{q-\ell} \leq \lambda_k^{q_0} + \epsilon^q$, and therefore

$$\|v_{\ell}\| \leq \Gamma(z_\ell) e^{(q-\ell)\lambda_k^{q-\ell}} \|v_{q}\| \leq \Gamma_0 e^{3q\epsilon^q} \|v_{q}\| \leq \Gamma_0^2 (\lambda_1 e^{-4\epsilon})^{-q} \leq \frac{\Gamma_0^2}{c_3} (\text{diam}(C))^{\frac{\log \lambda_1 - 6\epsilon}{2\log \lambda_1}},$$

and again the same estimate holds for $\|\eta_{\ell}\|$. Thus, taking the constant $C'' > 0$ so large that $C'' \geq R_0 \Gamma_0^2 / c_3$, we get

$$\|v_{\ell}\|, \|\eta_{\ell}\|, \|\xi_{\ell}\| \leq C'' (\text{diam}(C))^{\frac{\log \lambda_1 - 6\epsilon}{2\log \lambda_1}}.$$

Using these we get the following estimates for the terms in (9.11):

$$\|\xi_{\ell}\| \|v_{\ell}\| (\|\xi_{\ell}\|^{\beta} + \|v_{\ell}\|^{\beta}) \leq 2 (C'')^3 (\text{diam}(C))^{(2+\beta)\frac{\log \lambda_1 - 6\epsilon}{2\log \lambda_1}} \leq 2 (C'')^3 (\text{diam}(C))^{1+\hat{\beta}},$$

where we choose

$$0 < \hat{\beta} = \min \left\{ \frac{1}{4} \min \{\beta, \vartheta\}, \frac{\log \lambda_2 - \log \lambda_1}{2 \log \lambda_1} \right\}, \quad (9.16)$$

and we use the assumption $\epsilon \leq \frac{\log \lambda_1}{100} \min \{\beta, \vartheta\}$. Then

$$(2 + \beta) \frac{\log \lambda_1 - 6\epsilon}{2\log \lambda_1} \geq 1 + \hat{\beta}$$

and also

$$(2 + \vartheta) \frac{\log \lambda_1 - 6\epsilon}{2\log \lambda_1} \geq 1 + \hat{\beta},$$

which is used in the next estimate. Similarly,

$$\|\xi_{\ell}\|^{1+\beta} \|v_{\ell}\| \leq (C'')^3 (\text{diam}(C))^{1+\hat{\beta}},$$

and

$$\|\xi_{\ell}\|^{2} \|v_{\ell}\|^{\vartheta} + \|\xi_{\ell}\|^{\beta} \|v_{\ell}\|^{\beta} \leq 2 (C'')^3 (\text{diam}(C))^{1+\hat{\beta}}. \quad (9.17)$$

Next, for any $\xi = \xi^{(1)} + \xi^{(2)} + \ldots + \xi^{(k)} \in E^u(z)$ or $E^s(z)$ for some $z \in M$ set $\xi^{(2)} = \xi^{(2)} + \ldots + \xi^{(k)}$, so that $\xi = \xi^{(1)} + \xi^{(2)}$. Using Lemma 3.5 in [SL4] (see Lemma 10.1 below), $p - \ell = 2q - \ell \geq q$ and the fact that $\|\xi_{\ell}\| \leq \|\xi_{q}\| \leq R_0 r_1 \leq R_0$, we get

$$\|\xi^{(2)}_{\ell}\|^{2} \|v_{\ell}\|^{q} \leq \frac{\Gamma_0 \|\xi^{(2)}_{\ell}\|^{2}}{\mu_2^q} \leq \frac{\Gamma_0 \|\xi_{\ell}\|^{2}}{\mu_2^q} \leq \frac{\Gamma_0 R_0}{\mu_2^q}.$$ 

Similarly, using Lemma 3.5 in [SL4] (backwards for the map $f^{-1}$ on stable manifolds), $z_0 \in P_0$, $v_0 = v_{j,1}(z_0) \in E^s(z_0, r_0^1)$ and the fact that $\|v_0\| \leq \delta', 1 < q$, we get

$$\|\xi^{(2)}_{\ell}\|^{2} \|v_{\ell}\|^{q} \leq \frac{\Gamma_0 \|v_0\|^{q}}{\mu_2^{q(1-\delta'_0)}} \leq \frac{\Gamma_0}{\mu_2^{q(1-\delta'_0)}}.$$ 

Hence for $i \geq 2$ we have

$$\|\xi^{(i)}_{\ell}\| \leq \|\xi^{(2)}_{\ell}\| \leq \frac{\Gamma_0 R_0}{\mu_2^q},$$

and similarly $\|v^{(i)}_{\ell}\| \leq \frac{\Gamma_0}{\mu_2^{q(1-\delta'_0)}}$. Using these estimates, (9.14), $\mu_2 = \lambda_2 e^{-\hat{\epsilon}}$, and the assumptions about $\hat{\epsilon}$, we get

$$\|\xi^{(i)}_{\ell}\| \|v^{(i)}_{\ell}\| \leq \Gamma_0^2 R_0 (\lambda_2 e^{-2\hat{\epsilon}})^{-2q} = \frac{\Gamma_0^2 R_0}{\mu_2^q} e^{-2q \log(\lambda_2 e^{-2\hat{\epsilon}})} \leq \frac{\Gamma_0^2 R_0}{\mu_2^q} e^{-\frac{\log(\lambda_2 e^{-2\hat{\epsilon}})}{\log \lambda_1} \log \frac{c_3}{\text{diam}(C)}} \leq \frac{\Gamma_0^2 R_0}{\mu_2^q} \left( \frac{\text{diam}(C)}{c_3} \right) \leq C'' (\text{diam}(C))^{1+\hat{\beta}},$$

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using $\hat{\beta} \leq \frac{\log \lambda_2 - \log \lambda_1}{2 \log \lambda_1}$ by (9.16) and assuming $C'' \geq \Gamma^2_0 R_0/(c_3)^{\log \lambda_2/\log \lambda_1}$. Then

$$\frac{\log \lambda_2 - 2\ell}{\log \lambda_1} - 1 = \frac{\log \lambda_2 - 2\ell - \log \lambda_1}{\log \lambda_1} \geq \hat{\beta}.$$  

### 9.3.3 Final estimate

Using (9.11) and the above estimates for $\|\xi_\ell\|$, $\|v_\ell\|$, $\|\xi_\ell^{(i)}\| \|v_\ell^{(i)}\|$, we obtain

$$|d\omega_{z_\ell}(\xi_\ell, v_\ell - \eta_\ell)| \leq \text{Const} \ \text{diam}(C) \|v_0 - \eta_0\| + \text{Const} \ (\text{diam}(C))^{1+\hat{\beta}}.$$ 

Next, using (9.6) and (9.7) with $j = \ell$ and the previous estimate we get

$$|\Delta(x_0, y_0) - \Delta(x_0, b_0)| = |\Delta(x_\ell, y_\ell) - \Delta(x_\ell, b_\ell)| 
\leq |d\omega_{z_\ell}(\xi_\ell, v_\ell - \eta_\ell)| + \text{Const} \ (\text{diam}(C))^{1+\hat{\beta}} 
\leq \text{Const} \ \text{diam}(C) \|v_0 - \eta_0\| + \text{Const} \ (\text{diam}(C))^{1+\hat{\beta}} \quad (9.18)$$

Next, we consider two cases.

**Case 1.** $\text{diam}(C) \leq \|v_0 - \eta_0\|^{\vartheta/2}$. Then (9.18) immediately implies

$$|\Delta(x_0, y_0) - \Delta(x_0, b_0)| \leq \text{Const} \ \text{diam}(C) \|v_0 - \eta_0\|^{\hat{\beta}\vartheta/2}.$$ 

**Case 2.** $\text{diam}(C) \geq \|v_0 - \eta_0\|^{\vartheta/2}$. Set $\{X'\} = W^u_R(y_0) \cap W^s_R(x_0)$ and $X = \phi_{\Delta(x_0, y_0)}(X')$. Then $X \in W^{\alpha}_{\varepsilon_0}(y_0)$ and it is easy to see that

$$|\Delta(x_0, y_0) - \Delta(x_0, b_0)| = |\Delta(X, b_0)|.$$ 

We have $X = \exp^u_{y_0}(\tilde{t})$ and $b_0 = \exp^s_{y_0}(\tilde{s})$ for some $\tilde{t} \in E^u(y_0)$ and $\tilde{s} \in E^s(y_0)$. Clearly $\|\tilde{t}\| \leq \text{Const}$ . Using Liverani’s Lemma (Lemma 4.1) we get

$$|\Delta(X, b_0)| \leq C_0[|d\omega_{y_0}(\tilde{t}, \tilde{s})| + \|\tilde{t}\|^2\|\tilde{s}\|^\vartheta + \|\tilde{t}\|^\vartheta\|\tilde{s}\|^2] \leq \text{Const} \ \|\tilde{s}\|^\vartheta.$$ 

However, $\|\tilde{s}\| \leq \text{Const} \ d(y_0, b_0) \leq \text{Const} \ \|v_0 - \eta_0\|$, so

$$|\Delta(X, b_0)| \leq \text{Const} \ \|v_0 - \eta_0\|^\vartheta \leq \text{Const} \ \text{diam}(C) \|v_0 - \eta_0\|^\vartheta/2.$$ 

This proves the lemma. ■

### 9.4 Proof of Lemma 4.3

#### 9.4.1 Set-up – choice of some constants and initial points

Choosing a constant $\ell' \in (0, r_0/2)$ sufficiently small, for any $z \in M$ and any $z' \in B^u(z, \ell')$ the local unstable holonomy map $\mathcal{H}_{z'}^u: W^u_{\ell'}(z) \to W^s_{\ell'}(z')$ is well defined and uniformly Hölder continuous. Replacing $\ell'$ by a smaller constant if necessary, by (3.7) for $z \in P_0$ and $z' \in P_0 \cap B^u(z, \ell')$ the pseudo-holonomy map

$$\hat{\mathcal{H}}_{z'}^u = (\Phi_{z'}^s)^{-1} \circ \mathcal{H}_{z'}^u \circ \Phi_{z}^s: E^s(z; \ell') \to E^s(z'; r_0)$$

is uniformly Hölder continuous, as well. Thus, there exist constants $C' > 0$ and $\beta'' > 0$ (depending on the set $P_0$) so that for $z, z'$ as above we have

$$\|\hat{\mathcal{H}}_{z'}^u(u) - \hat{\mathcal{H}}_{z'}^u(v)\| \leq C'\|u - v\|^\beta'', \quad u, v \in E^s(z; \ell').$$  

(9.19)
We will assume $\beta'' \leq \beta$, where $\beta \in (0, 1]$ is the constant from Sect. 3.

Fix arbitrary constants $\delta' > 0$ with
\[
(\delta')^{\beta''} < \frac{\beta_0 \kappa \theta_0}{16L_0 C_1 R_0 \Gamma_0^2},
\] (9.20)
s\_0 with $0 < s_0 < \delta'/(2R_0^2)$ and $\delta''$ with
\[
0 < \delta'' < \min \left\{ \frac{\delta'}{3R_0}, \frac{\beta_0 \delta_0 \kappa}{100R_0^3 L_0 C_1^2}, \frac{s_0 \theta_0 c_0}{4C_1 \gamma_1 R_0^2} \right\},
\] (9.21)
Then set
\[
\delta_0 = \frac{s_0 \theta_0}{16R_0} > 0.
\]

Next, assuming $\beta'' > 0$ is taken sufficiently small and $C'' > 0$ sufficiently large, for any $j = 1, \ldots, \ell_0$ there exists a Lipschitz family of unit vectors $\eta_j(Z, z) \in E^i_1(z)$, $Z \in \tilde{P}_0$, $z \in B^u(Z, r_0/2) \cap \tilde{P}_0$, such that $\eta_j(Z, Z) = \eta_j(Z)$ and for any $v \in E^s(Z)$ we have
\[
|\omega_z(\eta_j(Z, z), \tilde{H}_Z^s(v)) - \omega_Z(\eta_j(Z), v)| \leq C'd(Z, z)\|v\|^{\beta''}.
\]

Fix a constant $\epsilon'' \in (0, \epsilon'/2)$ so small that $C'(\epsilon'')^{\beta''} < \delta'$. Then
\[
|\omega_z(\eta_j(Z, z), \tilde{H}_Z^s(v)) - \omega_Z(\eta_j(Z), v)| \leq \delta''\|v\|^{\beta''}, \quad Z \in \tilde{P}_0, \quad z \in B^u(Z, r_0) \cap \tilde{P}_0.
\] (9.22)

Using the symbolic coding provided by the Markov family $\{R_i\}$ it is easy to see that there exists an integer $N_0 \geq 1$ such that for any integer $N \geq N_0$ we have $P^N(B^u_{\epsilon'}(z)) \cap B^s(z', \delta'') \neq \emptyset$ for any $z, z' \in R$ (see the notation in the beginning of Sect. 4).

Fix for a moment $Z \in P_0$. Given $j = 1, \ldots, \ell_0$, since $\eta_j(Z) \in E^i_1(Z)$, by Lemma 9.1 and the choice of $\theta_0 > 0$ (see Sect. 4.2), there exists $\tilde{v}_j(Z) \in E^s(Z)$ with
\[
d\omega_Z(\eta_j(Z), \tilde{v}_j(Z)) \geq \theta_0, \quad \|\tilde{v}_j(Z)\| = 1.
\]

Fix a vector $\tilde{v}_j(Z)$ with the above property for every $j$.

Set
\[
v_j(Z) = \frac{s_0}{R_0} \tilde{v}_j(Z) \in E^i_1(Z), \quad y_j(Z) = \Phi_Z^s(v_j(Z)) \in W^s_{s_0}(Z).
\] (9.23)
Then $s_0/R_0^2 \leq d(Z, y_j(Z)) \leq s_0$. Since $d\omega_Z(\eta_j(Z), v_j(Z)) \geq s_0\theta_0/R_0$, by (4.1),
\[
|d\omega_Z(\eta_j(Z), v)| \geq \frac{s_0\theta_0}{2R_0}, \quad v \in E^s(Z), \quad \|v - v_j(Z)\| \leq \frac{s_0\theta_0}{2C_0R_0}.
\] (9.24)

Fix an arbitrary $N \geq N_0$. It follows from the above that for each $Z \in P_0$, each $i = 1, 2$ and each $j = 1, \ldots, \ell_0$ there exists
\[
y_{j,1}(Z) \in P^N(B^u(Z, \epsilon')) \cap B^s(y_j(Z), \delta'') \text{ and } y_{j,2}(Z) \in P^N(B^u(Z, \epsilon')) \cap B^s(Z, \delta'').
\] (9.25)
Fix points $y_{j,i}(Z)$ with these properties; then $y_{j,i}(Z) \in W^s_{s_0}(Z)$. We have
\[
y_{j,i}(Z) = \Phi_Z^s(w_{j,i}(Z)) \quad \text{for some } w_{j,i}(Z) \in E^s(Z)
\]
\[21\]Uniform continuity is enough.
\[22\]E.g. define $\eta_j(Z, z) = \frac{(\Phi_Z^s)^{-1}\Phi_Z^s(r_{y_j(Z)}(z)/2)}{\|((\Phi_Z^s)^{-1}\Phi_Z^s(r_{y_j(Z)}(z)/2))\|}$. 

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such that $w_{j,i}(Z) \in (\Phi^u_Z)^{-1}(B^s(y_{j,i}(Z), \delta''))$. For $z \in B^u(Z, \epsilon')$ set

$$w_{j,i}(Z, z) = \tilde{\Phi}^u_z(w_{j,i}(Z)) \in E^s(z).$$  \hspace{1cm} \text{(9.26)}

Notice that

$$\Phi^u_z(w_{j,i}(Z, z)) = \pi_{y_{j,i}(Z)}(z).$$ \hspace{1cm} \text{(9.27)}

Given $Z \in P_0$ and $z \in B^u(Z, \epsilon') \cap \overline{P}_0$, $d(y_j(Z), y_{j,1}(Z)) \leq \delta''$ implies

$$\|w_{j,1}(Z) - v_j(Z)\| \leq \delta'' R_0,$$

In particular,

$$\frac{s_0}{2R_0} \leq \|w_{j,1}(Z)\| \leq \frac{2s_0}{R_0}.$$  \hspace{1cm} \text{(9.28)}

Apart from that, $\|w_{j,2}(Z)\| \leq \frac{\delta''}{R_0}$. Now (9.19) gives

$$\|w_{j,2}(Z, z)\| = \|\tilde{\Phi}^u_z(w_{j,2}(Z)) - \tilde{\Phi}^u_z(0)\| \leq C' \|w_{j,2}(Z)\| \beta'' \leq C' \left( \frac{\delta''}{R_0} \right)^{\beta''} \leq \frac{s_0}{4R_0^2}. \hspace{1cm} \text{(9.29)}$$

A similar estimate holds for $w_{j,1}(Z, z)$, so we get

$$\|w_{j,2}(Z, z)\| \leq \frac{s_0}{2R_0^2} \leq \|w_{j,1}(Z, z)\| \leq 2s_0 R_0 \hspace{1cm} \text{for some } Z \in P_0, \ z \in B^u(Z, \epsilon') \cap \overline{P}_0.$$  \hspace{1cm} \text{(9.29)}

Next, (9.24) implies

$$|d\omega_Z(\eta_j(Z), w_{j,1}(Z))| \geq \frac{s_0 \theta_0}{2R_0},$$

while (9.22) yields

$$|d\omega_Z(\eta_j(Z, z), w_{j,1}(Z, z))| \geq \frac{s_0 \theta_0}{2R_0} - \delta' s_0 \left( \frac{4s_0}{R_0} \right)^{\beta''} \geq \frac{s_0 \theta_0}{4R_0},$$

and therefore

$$|d\omega_Z(\eta_j(Z, z), w_{j,1}(Z, z))| \geq \frac{4\delta_0}{2} \hspace{1cm} \text{for some } Z \in P_0, \ z \in B^u(Z, \epsilon'') \cap \overline{P}_0. \hspace{1cm} \text{(9.30)}$$

To finish with this preparatory section, let $C$ be a cylinder of length $m$ in $R$ such that $C \cap P_0 \setminus \Xi_m \neq \emptyset$, let $Z \in C \cap P_0 \setminus \Xi_m$, $Z_0 = \Psi(Z)$, and let $x_0 \in \Psi(C)$, $z_0 \in \Psi(C)$ have the form

$x_0 = \Phi^u_{z_0}(u_0), \ z_0 = \Phi^u_{z_0}(w_0), \ \text{where}$

$$d(x_0, z_0) \geq \kappa \text{diam}(\Psi(C))$$ \hspace{1cm} \text{(9.31)}$$

for some $\kappa \in (0, 1]$, and

$$\left( \frac{w_0 - u_0}{\|w_0 - u_0\|}, \eta_j(Z_0) \right) \geq \beta_0$$ \hspace{1cm} \text{(9.32)}$$

for some $j = 1, \ldots, \ell_0$. Fix $\kappa$ and $j$ with these properties. Set $\tilde{C} = \tilde{\Psi}(C)$. Then $Z_0, z_0 \in \tilde{C} \setminus \overline{P}_0$. By the assumption on $m$, diam($\tilde{C}$) < $\epsilon''$, so $z_0 \in B^u(Z_0, \epsilon'')$. Let $z_0 = \phi_{t_0}(z)$ for some $z \in C$ and $t_0 \in (-\chi, \chi)$. Set

$$x_0 = \Phi^u_{z_0}(\xi_0), \ \ \ v_0 = d\phi_{t_0}(z) \cdot w_{j,1}(Z, z_0) \in E^u(z_0; r_0/R_0),$$

for some $\xi_0 \in E^u(z_0; r_0/R_0)$; then $\|\xi_0\| \leq R_0 \text{diam}(\tilde{C})$.  \hspace{1cm} \text{(9.33)}
9.4.2 Estimates for $|d\omega_0(\xi_0^{(1)}, v_0^{(1)})|

Recall that $z_0 = \Phi^u_{Z_0}(w_0)$. Since $Z_0, z_0 \in \hat{P}_0$ and $\|w_0\| \leq R_0\varepsilon'' < r_0/R_0$, the map

$$Q = (\Phi^u_{Z_0})^{-1} \circ \Phi^u_{Z_0} : E^u(z_0) \rightarrow E^u(z_0)$$

is well-defined and $C^{1+\beta}$. Using $d(\Phi^u_{z_0})^{-1}(z_0) = \text{id}$, $Q(w_0) = 0$ and $Q(u_0) = \xi_0$, we get

$$dQ(w_0) = d(\Phi^u_{z_0})^{-1}(z_0) \circ d\Phi^u_{Z_0}(w_0) = d\Phi^u_{Z_0}(w_0).$$

Now (3.8) implies

$$\|\xi_0 - d\Phi^u_{Z_0}(w_0) \cdot (u_0 - w_0)\| \leq 10R_0^3\|u_0 - w_0\|^{1+\beta}. \quad (9.33)$$

[Proof of (9.33): Using $C^2$ coordinates in $W^u_{r_0}(Z_0)$, we can identify $W^u_{r_0}(Z_0)$ with an open subset $V$ of $\mathbb{R}^n$ and regard $\Phi^u_{Z_0}$ and $\Phi^u_{z_0}$ as $C^{1+\beta}$ maps on $V$ whose derivatives and their inverses are bounded by $R_0$. By Taylor’s formula (3.8),

$$\Phi^u_{Z_0}(u_0) - \Phi^u_{Z_0}(w_0) = d\Phi^u_{Z_0}(w_0) \cdot (u_0 - w_0) + \eta,$$

for some $\eta \in \mathbb{R}^n$ with $\|\eta\| \leq R_0\|u_0 - w_0\|^{1+\beta}$. Hence

$$d(\Phi^u_{z_0})^{-1}(z_0) \cdot (\Phi^u_{Z_0}(u_0) - \Phi^u_{Z_0}(w_0)) = d\Phi^u_{Z_0}(w_0) \cdot (u_0 - w_0) + \eta.$$

Since $Z_0 \in P_0$, by (3.9),

$$\|d\Phi^u_{Z_0}(w_0) - \text{id}\| = \|d\Phi^u_{Z_0}(w_0) - d\Phi^u_{Z_0}(0)\| \leq R_0\|w_0\|^{1+\beta},$$

so $\|d\Phi^u_{Z_0}(w_0)\| \leq 2R_0$. Using Taylor’s formula again,

$$Q(u_0) - Q(w_0) = (\Phi^u_{z_0})^{-1}(\Phi^u_{Z_0}(u_0)) - (\Phi^u_{Z_0})^{-1}(\Phi^u_{Z_0}(w_0)) = d(\Phi^u_{z_0})^{-1}(z_0) \cdot (\Phi^u_{Z_0}(u_0) - \Phi^u_{Z_0}(w_0)) + \zeta,$$

for some $\zeta$ with

$$\|\zeta\| \leq R_0\|\Phi^u_{Z_0}(u_0) - \Phi^u_{Z_0}(w_0)\|^{1+\beta} \leq R_0 \left(2R_0\|w_0 - u_0\| + R_0\|w_0 - u_0\|^{1+\beta}\right)^{1+\beta} \leq 9R_0^3\|u_0 - w_0\|^{1+\beta}.$$

Thus,

$$\xi_0 = Q(u_0) - Q(w_0) = d\Phi^u_{Z_0}(w_0) \cdot (u_0 - w_0) + \eta + \zeta,$$

where $\|\eta + \zeta\| \leq (R_0 + 9R_0^3)\|u_0 - w_0\|^{1+\beta} \leq 10R_0^3\|u_0 - w_0\|^{1+\beta}$.]

Next, by (4.8) the direction of $w_0 - u_0$ is close to $\eta_j(Z_0)$. More precisely, let

$$w_0 - u_0 = t\eta_j(Z_0) + u$$

for some $t \in \mathbb{R}$ and $u \perp \eta_j(Z_0)$. Then for $s = t/\|w_0 - u_0\|$ we have

$$\frac{w_0 - u_0}{\|w_0 - u_0\|} = s\eta_j(Z_0) + \frac{u}{\|w_0 - u_0\|},$$

so

$$s = \left\langle \frac{w_0 - u_0}{\|w_0 - u_0\|}, \eta_j(Z_0) \right\rangle \geq \beta_0,$$

and therefore $t = s\|w_0 - u_0\| \geq \beta_0\|w_0 - u_0\|$. Moreover,

$$\|u\|^2 = \|w_0 - u_0 - t\eta_j(Z_0)\|^2 = \|w_0 - u_0\|^2 - 2t\langle w_0 - u_0, \eta_j(Z_0) \rangle + t^2$$

$$= \|w_0 - u_0\|^2 \left(1 - 2s \left\langle \frac{u}{\|w_0 - u_0\|}, \eta_j(Z_0) \right\rangle + s^2 \right) = \|w_0 - u_0\|^2(1 - 2s^2 + s^2)$$

$$= \|w_0 - u_0\|^2(1 - s^2) \leq (1 - \beta_0^2)\|w_0 - u_0\|^2,$$

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and therefore $\|u\| \leq \sqrt{1 - \beta^2} \|w_0 - u_0\|$

Since $v_0 = d\phi_{t_0} \cdot w_{j,1}(Z, z_0) = \tilde{H}^u_Z(w_{j,1}(Z))$, it follows from (9.24) with $z = z_0$ and $w = v_0$ that

$$|d\omega_{z_0}(\eta_j(Z, z_0), v_0)| \geq 8\delta_0,$$

while (9.29) gives

$$s_0/(2R_0^3) \leq \|v_0\| \leq 2s_0 R_0 \leq 2\delta'/R_0.$$

Using $d\Phi^u_{z_0}(0) = \text{id}$ and (3.9), we have

$$\|d\Phi^u_{z_0}(w_0) - \text{id}\| \leq R_0 \|w_0\|^\beta \leq R_0 (R_0 e'')^\beta \leq R_0^2 (e'')^\beta.$$ 

Moreover, $\beta_0^2 (1 + \theta_0^2/(64C_0)^2) = 1$, so $\beta_0^2 \theta_0^2 = (64C_0)^2 (1 - \beta_0^2)$, and therefore

$$4C_0 \sqrt{1 - \beta_0^2} = \beta_0 \theta_0/16.$$ 

The above, (9.29), (9.23), (9.21), (9.22), $\|v_0^{(1)}\| \leq |v_0| \leq \|v_0\|$, Lemma 9.1 and the fact that $\eta_j(z_0) \in E^u_\gamma(z_0)$ imply

$$|d\omega_{z_0}(\xi_0^{(1)}, v_0^{(1)})|$$

$$= |d\omega_{z_0}(\xi_0, v_0^{(1)})| \geq |d\omega_{z_0}(d\Phi^u_{z_0}(w_0) \cdot (u_0 - w_0), v_0^{(1)})| - |d\omega_{z_0}(\xi_0 - d\Phi^u_{z_0}(w_0) \cdot (u_0 - w_0), v_0^{(1)})|$$

$$\geq \|u_0 - w_0\| \left[ |d\omega_{z_0}(\eta_j(z_0), v_0)| - |d\omega_{z_0}(\eta_j(z_0) - \eta_j(z_0), v_0^{(1)})| \right]$$

$$\geq \|u_0 - w_0\| \left[ 8\beta_0 \delta_0 - \sqrt[3]{1 - \beta_0^2} \delta'/R_0 - 40C_0 R_0^3 \delta' \right]$$

Combining this with (4.7) and (3.7) gives

$$|d\omega_{z_0}(\xi_0^{(1)}, v_0^{(1)})| \geq \frac{4\beta_0 \delta_0 \kappa}{R_0} \text{diam}(\tilde{C}).$$

Next, set $\tilde{\xi}_0 = \Psi^u_{z_0}(\xi_0) \in E^u(z_0)$. Then

$$\exp_{z_0}^u(\tilde{\xi}_0) = \Phi^u_{z_0}(\xi_0) = x_0,$$

and

$$\frac{\kappa}{R_0} \text{diam}(\tilde{C}) \leq \|\xi_0\| \leq R_0 \text{diam}(\tilde{C}).$$

Setting $\tilde{v}_0 = \Psi^s_{z_0}(v_0) \in E^s(z_0)$ and $y_0 = \exp_{z_0}^s(\tilde{v}_0)$, using $v_0 = w_{j,1}(Z, z_0)$, (9.25) and (9.27), we get

$$y_0 = \exp_{z_0}^s(\tilde{v}_0) = \Phi^s_Z(w_{j,1}(Z, z_0)) = \pi_{y_{j,1}}(z_0) \in B^s(z_0, \delta'').$$

We will now prove that

$$|\Delta(x_0, y_0)| \geq \frac{2\beta_0 \delta_0 \kappa}{R_0} \text{diam}(\tilde{C}).$$

From this and Lemma 4.2(b), (4.9) follows easily for $d_1 \in B^s(y_1^{(j)}(Z), \delta'')$ and $d_2 \in B^s(Z, \delta'')$, using the choice of $\delta''$.  

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It follows from (3.6), \( \|v_0\| \leq r_0/R_0 \) and \( \|\xi_0\| \leq r_0/R_0 \) that
\[
\|\tilde{v}_0 - v_0\| \leq R_0 \|v_0\|^{1+\beta}, \quad \|\tilde{\xi}_0 - \xi_0\| \leq R_0 \|\xi_0\|^{1+\beta},
\]
and in particular \( \|\tilde{v}_0\| \leq 2\|v_0\| \) and \( \|\tilde{\xi}_0\| \leq 2\|\xi_0\| \leq 2R_0 \text{diam}(\tilde{C}) \).

As in Sect. 9.3.1, set \( p = \tau_m(z_0), q = p/2 \), and for \( j \geq 0 \) define \( z_j = \varphi^j(z_0), x_j = \varphi^j(x_0), y_j = \varphi^j(y_0), \xi_j = d\varphi_{z_0}^j(0) \cdot \xi_0 \), etc. in the same way. By the choice of \( \epsilon'' > 0 \) all estimates in Sect. 9.3.1 hold without change. Choosing an arbitrary \( z \in \mathcal{C} \cap P_0 \setminus \Xi_m \), as before we find \( j \geq 0 \) with \( \mathcal{P}^j(z) \in P_0 \) such that (9.8) holds for \( \ell = [\tilde{\tau}_j(\Psi(z))] \) and \( r(z_\ell) \geq r_0 \). Fix \( \ell \) with these properties; then (9.9) and (9.10) hold again.

We need an estimate from below for \( |d\omega_{z_\ell}(\xi_\ell, v_\ell)| \) similar to (9.11). Instead of using Lemma 10.7 this time it is enough to use Lemma 3.1. Since \( v_\ell = \varphi_\ell^\ell(v_0) \in E^\ell(z_\ell) \) and \( z_0 \in P \) implies \( L(z_0) \leq L_0 \), for \( w = d\varphi_\ell^{-\ell}(0) \cdot v_\ell \), using Lemma 3.1, we get
\[
\|v_\ell(0) - w(0)\| \leq L_0(z_0)\|v_\ell\|^{1+\beta} \leq L_0\|v_0\|^{1+\beta}.
\]
(9.39)

As in the proof of (9.11) we will now use the estimates in Sect. 9.3.2. It follows from Lemma 9.1, (9.9) and (9.35) that
\[
|d\omega_{z_\ell}(\xi_\ell, v_\ell)| \geq \sum_{i=2}^{\tilde{k}} |d\omega_{z_\ell}(\xi_\ell^{(i)}, v_\ell^{(i)})| - C_0 L_0 \|\xi_\ell\|^{1+\beta} \|v_\ell\| - C_0 \sum_{i=2}^{\tilde{k}} \|\xi_\ell^{(i)}\| \|v_\ell^{(i)}\|
\]
\[
= \sum_{i=2}^{\tilde{k}} |d\omega_{z_\ell}(\varphi_\ell^{-\ell}(0) \cdot \xi_\ell^{(i)}, \varphi_\ell^{-\ell}(0) \cdot v_\ell^{(i)})| - C_0 L_0 \|\xi_\ell\|^{1+\beta} \|v_\ell\| - C_0 \sum_{i=2}^{\tilde{k}} \|\xi_\ell^{(i)}\| \|v_\ell^{(i)}\|
\]
\[
= \sum_{i=2}^{\tilde{k}} |d\omega_{z_\ell}(\xi_0^{(i)}, v_0^{(i)})| - C_0 L_0 \|\xi_0\|^{1+\beta} \|v_0\| - C_0 \sum_{i=2}^{\tilde{k}} \|\xi_\ell^{(i)}\| \|v_\ell^{(i)}\|
\]
\[
\geq \sum_{i=2}^{\tilde{k}} |d\omega_{z_\ell}(\xi_0^{(i)}, v_0^{(i)})| - C_0 L_0 R_0 \text{diam}(\tilde{C}) \|v_0\|^{1+\beta} - \text{Const} \ (\text{diam}(\tilde{C}))^{1+\beta}.
\]
Combining this with (9.6) and (9.30) gives
\[
|\Delta(x_0, y_0)| = |\Delta(x_\ell, y_\ell)| \geq |d\omega_{z_\ell}(\xi_\ell, v_\ell)| - 8C_0 R_0 \|\xi_\ell\| \|v_\ell\|((\|\xi_\ell\|^\beta + \|v_\ell\|^\beta)
\]
\[
-8C_0 \left[ \|\xi_\ell\|^2 \|v_\ell\|^\theta + \|\xi_\ell\| \|v_\ell\|^2 \right]
\]
\[
\geq |d\omega_{z_\ell}(\xi_0^{(1)}, v_0^{(1)})| - C_0 L_0 R_0 \text{diam}(\tilde{C}) \|v_0\|^{1+\beta} - \text{Const} \ (\text{diam}(\tilde{C}))^{1+\beta}
\]
\[
\geq \frac{4\beta_0 \delta_0 \kappa}{R_0} \text{diam}(\tilde{C}) - C_0 L_0 R_0 \text{diam}(\tilde{C}) \|v_0\|^{1+\beta} - C'' \text{diam}(\tilde{C})^{1+\beta}
\]
for some constant \( C'' > 0 \). Now assume
\[
(2\epsilon'')^\beta \leq \frac{\beta_0 \delta_0 \kappa}{R_0 C''},
\]
and recall that \( \|v_0\| \leq \delta' \) and \( \text{diam}(\tilde{C}) \leq 2\epsilon'' \). By (9.29), \( \|v_0\| \leq 2s_0 \), while (9.20) implies \( \|v_0\|^\beta \leq (\epsilon'')^\beta < (\delta'')^\beta \leq \frac{\beta_0 \theta_0}{16L_0 C_0 R_0} \). Thus, using (8.21),
\[
C_0 L_0 R_0 \kappa \text{diam}(\tilde{C}) \|v_0\|^{1+\beta} \leq C_0 L_0 R_0 \text{diam}(\tilde{C}) \|v_0\|^{1+\beta} \leq C_0 L_0 R_0 \kappa \text{diam}(\tilde{C}) \|v_0\|^{1+\beta} \leq \text{diam}(\tilde{C}) \frac{\beta_0 \delta_0 \kappa}{R_0},
\]
and therefore \( \Delta(x_0, y_0) \geq \frac{2\beta_0 \delta_0 \kappa}{R_0} \text{diam}(\tilde{C}) \). This proves (9.38). \( \blacksquare \)
10 Regular distortion for Anosov flows

In this section we prove Lemma 4.4. Here we do not need to assume that the flow $\phi_t$ is contact.

10.1 Expansion along $E_1^u$

Let again $M$ be a $C^2$ complete Riemann manifold and $\phi_t$ be a $C^2$ Anosov flow on $M$. Set

$$\hat{\mu}_2 = \lambda_1 + \frac{2}{3}(\lambda_2 - \lambda_1), \quad \hat{\nu}_1 = \lambda_1 + \frac{1}{3}(\lambda_2 - \lambda_1).$$

Then $\hat{\mu}_2 < \mu_2 e^{-\epsilon}$ and $\lambda_1 < \nu_1 < \hat{\nu}_1 < \hat{\mu}_2 < \mu_2 < \lambda_2$. For $\epsilon > 0$, apart from (3.1), we assume in addition that

$$e^{\epsilon} \leq \frac{2\lambda_2}{\lambda_2 + \hat{\mu}_2}.$$

For a non-empty set $X \subset E^u(x)$ set

$$\ell(X) = \sup\{|u| : u \in X\}.$$

Given $z \in L$ and $p \geq 1$, setting $x = \varphi^p(z)$, define

$$B^u_p(z, \delta) = \{u \in E^u(z) : \|\hat{\varphi}^p_u(u)\| \leq \delta\}.$$

Fix for a moment $x \in L$ and an integer $p \geq 1$, set $z = \varphi^{-p}(x)$ and given $v \in E^u(z; r(z))$, set

$$z_j = \varphi^j(z), \quad v_j = \hat{\varphi}^j_u(v) \in E^u(z_j), \quad w_j = d\hat{\varphi}^j_u(0) \cdot v \in E^u(z_j)$$

for any $j = 0, 1, \ldots, p$ (assuming that these points are well-defined).

For any $v = v^{(1)} + v^{(2)} + \ldots + v^{(k)} \in E^u(x)$ with $v^{(j)} \in E^u_j$, set $\hat{v}^{(2)} = v^{(2)} + \ldots + v^{(k)} \in \tilde{E}^u_2(x)$.

Lemma 10.1. Assume that the regularity function $\hat{r} \leq r$ satisfies

$$\hat{r}(x) \leq \min\left\{\left(\frac{1/\hat{\mu}_2 - 1/\lambda_2}{6T^2(x)D(x)}\right)^{1/\beta}, \left(\frac{1/\lambda_1 - 1/\hat{\nu}_1}{6e^{3t}T^2(x)D(x)}\right)^{1/\beta}\right\}$$

for all $x \in L$. Then for any $x \in L$ and any $V = V^{(1)} + \tilde{V}^{(2)} \in E^u(x; \hat{r}(x))$, setting $y = \varphi^{-1}(x)$ and $U = \hat{\varphi}^{-1}_y(V)$, we have

$$\|\tilde{U}^{(2)}\|_y' \leq \frac{\|\tilde{V}^{(2)}\|_x'}{\hat{\mu}_2},$$

and

$$\|U^{(1)}\|_y' \geq \frac{\|V^{(1)}\|_x'}{\hat{\nu}_1}.$$}

Moreover, if $V, W \in E^u(x; \hat{r}(x))$ and $W^{(1)} = V^{(1)}$, then for $S = \hat{\varphi}^{-1}_y(W)$ we have

$$\|\tilde{U}^{(2)} - \tilde{S}^{(2)}\|_y' \leq \frac{\|\tilde{V}^{(2)} - \tilde{W}^{(2)}\|_x'}{\hat{\mu}_2},$$

and, if $\tilde{W}^{(2)} = \tilde{V}^{(2)} \in E^u(x; \hat{r}(x))$ and $S = \hat{\varphi}^{-1}_y(W)$ again, then

$$\|U^{(1)} - S^{(1)}\|_y' \geq \frac{\|V^{(1)} - W^{(1)}\|_x'}{\hat{\nu}_1}.$$}

Proof. The estimates (10.2) and (10.3) follow from Lemma 3.5 in [SL4], while the proofs of (10.4) and (10.5) are similar. We will prove (10.4) for completeness.

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Assuming $V, W \in E^u(x; r(x))$ and $V^{(1)} = W^{(1)}$, by (3.16),

$$\|\hat{\varphi}^{-1}_x(V) - \hat{\varphi}^{-1}_x(W) - d\hat{\varphi}^{-1}_x(W) \cdot (V - W)\| \leq D(x)\|V - W\|^{1+\beta},$$

so

$$\hat{\varphi}^{-1}_x(V) - \hat{\varphi}^{-1}_x(W) = d\hat{\varphi}^{-1}_x(W) \cdot \left(\hat{V}^{(2)} - \hat{W}^{(2)}\right) + \xi$$

for some $\xi \in E^u(y)$ with $\|\xi\| \leq D(x)\|\hat{V}^{(2)} - \hat{W}^{(2)}\|^{1+\beta}$. By (3.17),

$$\|d\hat{\varphi}^{-1}_x(W) \cdot \left(\hat{V}^{(2)} - \hat{W}^{(2)}\right) - d\hat{\varphi}^{-1}_x(0) \cdot \left(\hat{V}^{(2)} - \hat{W}^{(2)}\right)\| \leq D(x)\|W\|^\beta \|\hat{V}^{(2)} - \hat{W}^{(2)}\|,$$

so

$$d\hat{\varphi}^{-1}_x(W) \cdot \left(\hat{V}^{(2)} - \hat{W}^{(2)}\right) = d\hat{\varphi}^{-1}_x(0) \cdot \left(\hat{V}^{(2)} - \hat{W}^{(2)}\right) + \eta,$$

with $\|\eta\| \leq D(x)\|W\|^\beta \|\hat{V}^{(2)} - \hat{W}^{(2)}\|$. Now we get

$$U - S = \hat{\varphi}^{-1}_x(V) - \hat{\varphi}^{-1}_x(W) = d\hat{\varphi}^{-1}_x(0) \cdot \left(\hat{V}^{(2)} - \hat{W}^{(2)}\right) + \xi + \eta,$$

which yields

$$\bar{U}^{(2)} - \bar{S}^{(2)} = d\hat{\varphi}^{-1}_x(0) \cdot \left(\hat{V}^{(2)} - \hat{W}^{(2)}\right) + \bar{\xi}^{(2)} + \bar{\eta}^{(2)}.$$

This, (3.10) and (3.14) imply

$$\|\bar{U}^{(2)} - \bar{S}^{(2)}\|_y \leq \|d\hat{\varphi}^{-1}_x(0) \cdot \left(\hat{V}^{(2)} - \hat{W}^{(2)}\right)\|_y + \Gamma(y)(\|\xi\| + \|\eta\|)$$

$$\leq \|\hat{V}^{(2)} - \hat{W}^{(2)}\|_x \left(\frac{1}{\mu_2} + \Gamma(y)D(x)\|\hat{V}^{(2)} - \hat{W}^{(2)}\|^\beta + \|W\|^\beta\right)$$

$$\leq \|\hat{V}^{(2)} - \hat{W}^{(2)}\|_x \left(\frac{1}{\mu_2} + \Gamma(x)e^\delta D(x)3\mu^\beta(x)\right).$$

By (10.1) and (3.2),

$$\frac{1}{\mu_2} + \Gamma(x)e^\delta D(x)3\mu^\beta(x) \leq \frac{e^\delta}{\lambda_2} + \frac{1}{\mu_2} - \frac{1}{\lambda_2} = \frac{e^\delta\lambda_2 + \hat{\mu}_2}{2\lambda_2\hat{\mu}_2} < \frac{1}{\mu_2},$$

since $e^\delta < \frac{2\lambda_2}{\lambda_2 + \hat{\mu}_2}$. This proves (10.4).\[\square\]

Next, for any $y \in \mathcal{L}$, $\epsilon \in (0, r(y)]$ and $p \geq 1$ set

$$\hat{B}^{u,1}_p(y, \epsilon) = \hat{B}^{u}_p(u; \epsilon) \cap E^u_1(y).$$

Replacing the regularity function with a smaller one, we may assume that

$$(\Gamma(x))^{1+\beta} L(x)(\hat{r}(x))^{\beta} \leq \frac{1}{100n_1} \quad , \quad x \in \mathcal{L}, \quad (10.6)$$

where $n_1 = \dim(E^u_1(x))$.

The proof of the following lemma is similar to the proof of Proposition 3.2 in [ST4].

**Lemma 10.2** Let $z \in \mathcal{L}$ and $x = \varphi^p(z)$ for some integer $p \geq 1$, and let $\epsilon \in (0, \hat{r}(x)]$. Then

$$\ell(\hat{B}^{u,1}_p(z, \epsilon)) \leq 2\kappa \Gamma^4(x) \ell(\hat{B}^{u,1}_p(z, \epsilon)). \quad (10.7)$$

Moreover for any $\epsilon' \in (0, \epsilon]$ there exists $u \in \hat{B}^{u,1}_p(z, \epsilon')$ with

$$\|u\| \geq \frac{\epsilon'}{2\kappa e \Gamma^3(x)} \ell(\hat{B}^{u}_p(z, \epsilon)) \quad \text{and} \quad \|\varphi^p_*(u)\| \geq \epsilon'/2. \quad (10.8)$$
Proof. Let z ∈ ℒ and x = ϕ^p(z) for some integer p ≥ 1. Let v = (v^1, v^2) ∈ ̃B^u_p(z, ε) be such that ∥v∥ is the maximal possible, i.e. ∥v∥ = ℓ( ̃B^u_p(z, ε)). Set

\[ V = (V^1, V^2) = ϕ^p_z(v), \quad W = (W^1, W^2) = dϕ^p_z(0) \cdot v. \]

Then ∥V∥ ≤ ∥V∥ ≤ Γ(x)ε, and by Lemma 3.1 and (10.6),

\[ ∥W^1(1) - V^1(1)∥ \leq L(x)|V|^{1+β} \leq \frac{1}{100} ∥V∥ < \frac{ε}{100}, \]

so

\[ ∥W^1∥ \leq ∥V^1∥ + L(x)|V|^{1+β} \leq ∥V^1∥ + \frac{ε}{100}. \]

**Case 1.** ∥v^2(2)∥ ≥ ∥v^1(1)∥. Take U = (U^1, 0) ∈ E^u_1(x, ε) such that ∥U^1∥ ≥ ε/2. Then u = ̃ϕ^p_x(U) ∈ ̃B^u_p(z, ε) ∩ E^u_1(z). Applying (10.3) p times gives ∥u^1∥ ≥ 2^p ∥v∥ ≥ 2^p ∥v^1∥. Similarly, applying (10.2) p times for V = ̃ϕ^p_z(v) we get

\[ ∥v∥ \leq Γ(z)|v| = Γ(z) ∥v^2∥ ≤ 2Γ^2(x) ∥v^1∥ \]

Thus,

\[ ∥v∥ \leq Γ(z)|v| = Γ(z) ∥v^2∥ \leq 2Γ^2(x) ∥v^1∥ \]

which proves (10.7), since ̃ϕ^p_x : E^u_1(z) → E^u_1(z) is a homeomorphism.

To prove (10.8) in this case, use the above argument however this time choose U = (U^1, 0) ∈ E^u_1(x, ε') with ∥U^1∥ ≥ ε'/2. Then as above we derive ∥u^1∥ ≥ 2ε'/2 and

\[ ∥v∥ \leq 2Γ^2(x) ∥v^1∥ \leq 2Γ^2(x) ∥v∥ \]

which proves (10.8).

**Case 2.** ∥v^2(2)∥ < ∥v^1(1)∥. Set

\[ u = \frac{ε'}{2εΓ(x)} v^1 \in E^u_1(z). \]

We will now check that u ∈ ̃B^u_p(z, ε'). Indeed, by Lemma 3.1 and (10.6),

\[ ∥ϕ^p_z(u) - dϕ^p_z(0) \cdot u∥ \leq L(x)|ϕ^p_z(u)|^{1+β} \leq \frac{∥ϕ^p_z(u)∥}{100}, \]

so

\[ ∥ϕ^p_z(u)∥ \leq ∥dϕ^p_z(0) \cdot u∥ + L(x)|ϕ^p_z(u)|^{1+β} \]

\[ \leq ∥dϕ^p_z(0) \cdot u∥ + L(x)(ϕ^p_z(u))^{1+β} \]

\[ \leq ∥dϕ^p_z(0) \cdot u∥ + \frac{1}{100} ∥ϕ^p_z(u)∥. \]
Lemma 10.3. There exist a regularity function $\hat{r}(x) < 1$ ($x \in \mathcal{L}$) such that:

(a) For any $x \in \mathcal{L}$ and any $0 < \delta \leq \hat{r}(x)$ we have

$$
\ell \left( \hat{B}_p^{u,1}(\varphi^{-p}(x), \epsilon) \right) \leq 16n_1 \frac{\epsilon^p}{\delta} \ell \left( \hat{B}_p^{u,1}(\varphi^{-p}(x), \delta) \right)
$$

(10.9)

for any integer $p \geq 1$.

(b) For any $x \in \mathcal{L}$ and any $0 < \epsilon \leq \hat{r}(x)$ and any $\rho \in (0,1)$, for any $\delta$ with

$$
0 < \delta \leq \frac{\rho \epsilon}{16n_1}
$$

we have

$$
\ell \left( \hat{B}_p^{u,1}(\varphi^{-p}(x), \delta) \right) \leq \rho \ell \left( \hat{B}_p^{u,1}(\varphi^{-p}(x), \epsilon) \right)
$$

(10.10)

for any integer $p \geq 1$.

Theorem 10.4. There exist a regularity function $\hat{r}(x) < 1$ ($x \in \mathcal{L}$) such that:

This gives

$$
\|\mathcal{Q}^p_z(u)\| \leq \|d\mathcal{Q}^p_z(0) \cdot u\| = \frac{\epsilon'}{2\epsilon\Gamma(x)} \|d\mathcal{Q}^p_z(0) \cdot v^{(1)}\|
$$

$$
= \frac{\epsilon'}{2\epsilon\Gamma(x)} \|W^{(1)}\| \leq \frac{\epsilon'}{2\epsilon\Gamma(x)} \left( \|v^{(1)}\| + \frac{\epsilon}{100} \right)
$$

$$
\leq \frac{\epsilon'}{2\epsilon\Gamma(x)} \left( \Gamma(x)\epsilon + \frac{\epsilon}{100} \right) < \epsilon'.
$$

Thus, $u \in \hat{B}_p^{u,1}(z, \epsilon')$. Since

$$
\|u\| = \frac{\epsilon'}{2\epsilon\Gamma(x)} \|v^{(1)}\| = \frac{\epsilon'}{2\epsilon\Gamma(x)} |v| \geq \frac{\epsilon'|v|}{2\epsilon k \Gamma(x)},
$$

taking $\epsilon' = \epsilon$, proves (10.7).

To prove (10.8), let now $u \in \hat{B}_p^{u,1}(z, \epsilon')$ be such that $\|u\|$ is the maximal possible, and set $U = \mathcal{Q}^p_z(u) \in E_1^u(x, \epsilon')$. It follows from the previous argument that $\|u\| \geq \frac{\epsilon'}{2\epsilon k \Gamma(x)} \ell (\hat{B}_p^{u,1}(z, \epsilon))$. It remains to show that $\|U\| \geq \epsilon'/2$. If $\|U\| < \epsilon'/2$, then by Lemma 3.1 and (10.6),

$$
\|U - d\mathcal{Q}^p_z(0) \cdot u\| \leq \frac{\|U\|}{100} < \frac{\epsilon'}{200},
$$

so $\|d\mathcal{Q}^p_z(0) \cdot u\| < 2\epsilon'/3$. Setting $\hat{u} = tu$ for some $t > 1$, $t$ close to 1, we get

$$
\|\mathcal{Q}^p_z(\hat{u}) - d\mathcal{Q}^p_z(0) \cdot (\hat{u})\| \leq \|\mathcal{Q}^p_z(\hat{u})\| / 100,
$$

so

$$
\|\mathcal{Q}^p_z(\hat{u})\| \leq \frac{100}{99} \|d\mathcal{Q}^p_z(0) \cdot \hat{u}\| \leq \frac{100}{99} \cdot \frac{2t\epsilon'}{3} < \epsilon',
$$

if $t$ is sufficiently close to 1. Thus, $\hat{u} \in \hat{B}_p^{u,1}(z, \epsilon')$ for $t > 1$, $t$ close to 1. However $\|\hat{u}\| = t\|u\| > \|u\|$, contradiction with the choice of $u$. Hence we must have $\|U\| \geq \epsilon'/2$, which proves (10.8). ■
(a) For any \( x \in \mathcal{L} \) and any \( 0 < \delta \leq \epsilon \leq \hat{r}(x) \) we have
\[
\ell \left( \hat{B}_p^u(z, \epsilon) \right) \leq \frac{32\kappa_1 \Gamma^4(x) \epsilon}{\delta} \ell \left( \hat{B}_p^u(z, \delta) \right)
\]
for any integer \( p \geq 1 \), where \( z = \varphi^{-p}(x) \).

(b) For any \( x \in \mathcal{L} \), any \( 0 < \epsilon \leq \hat{r}(x) \), any \( \rho \in (0,1) \) and any \( \delta \) with
\[
0 < \delta \leq \frac{\rho \epsilon}{32\kappa_1 \Gamma^3(x)}
\]
we have
\[
\ell \left( \hat{B}_p^u(z, \delta) \right) \leq \rho \ell \left( \hat{B}_p^u(z, \epsilon) \right)
\]
for all integers \( p \geq 1 \), where \( z = \varphi^{-p}(x) \).

(c) For any \( x \in \mathcal{L} \), any \( 0 < \epsilon' < \epsilon \leq \hat{r}(x)/2 \), any \( 0 < \delta < \frac{\epsilon'}{100n_1} \) and any integer \( p \geq 1 \), setting \( z = \varphi^{-p}(x) \), there exists \( u \in \hat{B}_p^{-1}(z, \epsilon') \) such that for every \( v \in E^u(z) \) with
\[
\| \hat{\varphi}_u^p(u) - \hat{\varphi}_u^p(v) \| \leq \delta
\]
we have
\[
\| v \| \geq \frac{\epsilon'}{4\kappa \Gamma^4(x)} \ell(\hat{B}_p^u(z, \epsilon)).
\]

Using Lemma 10.3, we will now prove Theorem 10.4. The proof of Lemma 10.3 is given in the next sub-section. In fact, part (c) above is a consequence of Lemmas 3.1 and 10.2 and does not require Lemma 10.3.

**Proof of Theorem 10.4.** Choose the function \( \hat{r}(x) \) as in Lemma 10.3.

(a) Let \( 0 < \delta < \epsilon \leq \hat{r}(x) \). Given an integer \( p \geq 1 \), set \( z = \varphi^{-p}(x) \). Then Lemmas 10.2 and 10.3 imply
\[
\ell(\hat{B}_p^u(z, \epsilon)) \leq 2\kappa \Gamma^4(x) \ell(\hat{B}_p^{-1}(z, \epsilon)) \leq 2\kappa \Gamma^4(x) 16n_1 \frac{\epsilon}{\delta} \ell(\hat{B}_p^{-1}(z, \delta))
\]
\[
\leq 32\kappa_1 \Gamma^4(x) \frac{\epsilon}{\delta} \ell(\hat{B}_p^u(z, \delta)).
\]

(b) Let \( x \in \mathcal{L} \) and \( 0 < \epsilon \leq \hat{r}(x) \). Given \( \rho \in (0,1) \), set \( \rho' = \frac{\rho}{2\kappa \Gamma^3(x)} < \rho \). By Lemma 10.3(b), if \( 0 < \delta \leq \frac{\rho'}{100n_1} \), then (10.10) holds with \( \rho \) replaced by \( \rho' \) for any integer \( p \geq 1 \) with \( z = \varphi^{-p}(x) \). Using this and Lemma 10.2 we get
\[
\ell(\hat{B}_p^u(z, \delta)) \leq 2\kappa \Gamma^3(x) \ell(\hat{B}_p^{-1}(z, \delta)) \leq 2\kappa \Gamma^3(x) \rho' \ell(\hat{B}_p^u(z, \epsilon)) = \rho \ell(\hat{B}_p^u(z, \epsilon)),
\]
which completes the proof.

(c) We will prove the following:

(d) For any \( x \in \mathcal{L} \), any \( 0 < \epsilon' < \epsilon \leq \hat{r}(x)/2 \), any \( 0 < \delta < \frac{\epsilon'}{100n_1} \) and any integer \( p \geq 1 \), setting \( z = \varphi^{-p}(x) \), there exists \( u \in \hat{B}_p^{-1}(z, \epsilon') \) such that for every \( v \in E^u(z) \) with
\[
\| \hat{\varphi}_u^p(u) - \hat{\varphi}_u^p(v) \| \leq \delta / \Gamma(x)
\]
we have
\[
\| v \| \geq \frac{\epsilon'}{4\kappa \Gamma^3(x)} \ell(\hat{B}_p^u(z, \epsilon)).
\]
As one can see, the only difference between (d) and (c) is in the condition involving \( \delta \); in (d) \( \delta \) is replaced by \( \delta / \Gamma(x) \). Once (d) is proved, replacing the regularity function \( \hat{r}(x) \) by \( \hat{r}(x) / \Gamma(x) \) and multiplying all constants appearing in (d) by \( \Gamma(x) \) will prove (c).

So, we will prove (d) now.

Given \( x \in \mathcal{L} \), \( z = \varphi^{-p}(x) \), let \( \epsilon' \), \( \epsilon \) and \( \delta \) be as in the assumptions in (d). Let \( u \in \hat{B}^{u,1}_p(z, \epsilon') \) be such that \( \|u\| \) is the maximal possible. By Lemma 10.2, for \( U = \varphi^p_z(u) \in E^n_1(x) \) we have \( \epsilon' / 2 \leq \|U\| \leq \epsilon' \). Setting \( W = d \varphi^p_z(0) \cdot u \in E^n_1(x) \), Lemma 3.1 and (10.6) give

\[
\|W - U\| \leq L(x)\|U\|^{1+\beta} \leq \frac{\|U\|}{100n_1} \leq \frac{\epsilon'}{100n_1},
\]

so \( \|W\| \leq \frac{40\epsilon'}{100} \).

Let \( v = (v^{(1)}, v^{(2)}) \in E^n(z) \) be such that for \( V = \varphi^p_z(v) \) we have \( \|V - U\| \leq \delta / \Gamma(x) \). Then \( |V - U| \leq \delta \), so \( \|V^{(1)} - U^{(1)}\| \leq \delta \) and \( \|V^{(2)}\| \leq \delta \). Moreover, \( \|V\| \leq \|U\| + \delta / \Gamma(x) \leq \epsilon' + \delta / \Gamma(x) \).

Set \( S = d \varphi^p_z(0) \cdot v \); then \( S^{(1)} = d \varphi^p_z(0) \cdot v^{(1)} \). By Lemma 3.1 and (10.7),

\[
\|S^{(1)} - V^{(1)}\| \leq L(x)|V|^{1+\beta} \leq L(x)(\Gamma(x))^{1+\beta}\|V\|^{1+\beta} \leq \frac{\|V\|}{100n_1} \leq \frac{\epsilon' + \delta}{100n_1},
\]

so

\[
\|S^{(1)} - W^{(1)}\| \leq \|S^{(1)} - V^{(1)}\| + \|V^{(1)} - U^{(1)}\| + \|U^{(1)} - W^{(1)}\| \leq \frac{\epsilon' + \delta}{100n_1} + \delta + \frac{\epsilon'}{100n_1} < \frac{\epsilon'}{30n_1}.
\]

Choose an orthonormal basis \( e_1, \ldots, e_{n_1} \) in \( E^n_1(x) \) such that \( W = W^{(1)} = c_1 e_1 \) for some \( c_1 \in [\epsilon'/3, \epsilon'] \). Let \( S^{(1)} = \sum_{i=1}^{n_1} d_i e_i \). Then the above implies \( |d_1 - c_1| \leq \frac{\epsilon'}{30n_1} \) and \( |d_i| \leq \frac{\epsilon'}{30n_1} \) for all \( i = 2, \ldots, n_1 \).

Notice that for any \( i = 1, \ldots, n_1 \),

\[
u' = d \varphi^{-p}_z(0) \cdot (\epsilon' e_i / 2) \in \hat{B}^{u,1}_p(z, \epsilon').
\]

Indeed, by Lemma 3.1 and (10.6),

\[
\|d \varphi^p_z(u') - d \varphi^p_z(0) \cdot u'\| \leq \frac{\|\epsilon' e_i/2\|}{100n_1} = \frac{\epsilon'}{200n_1},
\]

so

\[
\|d \varphi^p_z(u')\| \leq \|d \varphi^p_z(0) \cdot u'\| + \frac{\epsilon'}{200n_1} = \frac{\epsilon'}{2} + \frac{\epsilon'}{200n_1} < \epsilon'.
\]

By the choice of \( u \), this implies \( \|u'\| \leq \|u\| \), so \( d \varphi^{-p}_z(0) \cdot e_i \| \leq \frac{2\|u\|}{\epsilon'} \) for all \( i = 1, \ldots, n_1 \).

The above yields

\[
|d_1 d \varphi^{-p}_z(0) \cdot e_1| \geq |c_1 d \varphi^{-p}_z(0) \cdot e_1| - |(d_1 - c_1) d \varphi^{-p}_z(0) \cdot e_1| \geq \|u\| - \frac{\epsilon'}{30n_1} \frac{2\|u\|}{\epsilon'} = \|u\| \left(1 - \frac{1}{15n_1}\right).
\]

Moreover, for \( i \geq 2 \) we have

\[
|d_i d \varphi^{-p}_z(0) \cdot e_i| \leq \frac{\epsilon'}{30n_1} \cdot \frac{2\|u\|}{\epsilon'} = \frac{\|u\|}{15n_1}.
\]

Hence

\[
\|\nu^{(1)}\| = \|d \varphi^{-p}_z(0) \cdot S^{(1)}\| = \left\| \sum_{i=1}^{n_1} d_i d \varphi^{-p}_z(0) \cdot e_i \right\| \geq \|d_1 d \varphi^{-p}_z(0) \cdot e_1\| - \sum_{i=2}^{n_1} |d_i d \varphi^{-p}_z(0) \cdot e_i| \geq \|u\| \left(1 - \frac{1}{15n_1}\right) - n_1 \frac{\|u\|}{15n_1} > \frac{\|u\|}{2}.
\]
Combining this with Lemma 10.2 gives, \( \|v\| \geq |v| \geq \|v^{(l)}\| > \frac{\|u\|}{2} \geq \frac{e'}{4k1G(x)} \ell(\hat{B}^u_p(z, \epsilon)). \)

What we actually need later is the following immediate consequence of Theorem 10.4 which concerns sets of the form

\[ B^u_T(z, \epsilon) = \{ y \in W^u_T(z) : d(\phi_T(y), \phi_T(z)) \leq \epsilon \}, \]

where \( z \in \mathcal{L}, \epsilon > 0 \) and \( T > 0 \).

**Corollary 10.5.** There exist an \( \hat{\epsilon} \)-regularity function \( \hat{\tau}(x) < 1 \) (\( x \in \mathcal{L} \)) and a global constant \( L_1 \geq 1 \) such that:

(a) For any \( x \in \mathcal{L} \) and any \( 0 < \delta \leq \epsilon \leq \hat{\tau}(x) \) we have

\[ \text{diam}(B^u_T(z, \delta)) \leq L_1 \Gamma^3(x)R(z) \frac{\epsilon}{\delta} \text{diam}(B^u_T(z, \epsilon)) \]

for any \( T > 0 \), where \( z = \phi_{-T}(x) \).

(b) For any \( x \in \mathcal{L} \), any \( 0 < \epsilon \leq \hat{\epsilon}(x) \), any \( \rho \in (0, 1) \) and any \( \delta \) with \( 0 < \delta \leq \frac{\rho\epsilon}{L_1 \Gamma^3(x)R(z)} \) we have

\[ \text{diam}(B^u_T(z, \delta)) \leq \rho \text{diam}(B^u_T(z, \epsilon)) \]

for all \( T > 0 \), where \( z = \phi_{-T}(x) \).

(c) For any \( x \in \mathcal{L} \), any \( 0 < \epsilon' \leq \epsilon \leq \hat{\epsilon}(x) \), any \( 0 < \delta \leq \frac{\epsilon'}{100n_1} \) and any \( T > 0 \), for \( z = \phi_{-T}(x) \) there exists \( z' \in B^u_T(z, \epsilon') \) such that for every \( y \in B^u_T(z', \delta) \) we have

\[ d(z, y) \geq \frac{\epsilon'}{L_1 \epsilon \Gamma^3(x)R(z)} \text{diam}(B^u_T(z, \epsilon)). \]

### 10.2 Linearization along \( E^u_1 \)

Here we prove Lemma 10.3 using arguments similar to these in the proofs of Theorem 3.1 and Lemma 3.2 in [St5].

We use the notation from Sect. 10.1. Let \( \hat{\tau}(x), x \in \mathcal{L} \), be as in Lemma 10.1.

**Proposition 10.6.** There exist regularity functions \( \hat{\tau}_1(x) \leq \hat{\tau}(x) \) and \( L(x), x \in \mathcal{L} \), such that:

(a) For every \( x \in \mathcal{L} \) and every \( u \in E_1^u(x; \hat{\tau}_1(x)) \) there exists

\[ F_x(u) = \lim_{p \to \infty} d\hat{\varphi}^p(f_{-p}(x))(0) \cdot \hat{\varphi}^{-p}(u) \in E_1^u(x; \hat{\tau}(x)). \]

Moreover, \( \|F_x(u) - u\| \leq L(x) \|u\|^{1+\beta} \) for any \( u \in E_1^u(x, \hat{\tau}_1(x)) \) and any integer \( p \geq 0 \).

(b) The maps \( F_x : E_1^u(x; \hat{\tau}_1(x)) \to F_x(E_1^u(x; \hat{\tau}_1(x))) \subset E_1^u(x; \hat{\tau}(x)) \) (\( x \in \mathcal{L} \)) are uniformly Lipschitz. More precisely,

\[ \|F_x(u) - F_x(v) - (u - v)\| \leq C_1 \left( \|u - v\|^{1+\beta} + \|v\|^{\beta} \cdot \|u - v\| \right), \quad x \in \mathcal{L}, u, v \in E_1^u(x; \hat{\tau}_1(x)). \]

Assuming that \( \hat{\tau}_1(x) \) is chosen sufficiently small, this yields

\[ \frac{1}{2} \|u - v\| \leq \|F_x(u) - F_x(v)\| \leq 2 \|u - v\|, \quad x \in \mathcal{L}, u, v \in E_1^u(x; \hat{\tau}_1(x)). \]

(c) For any \( x \in M \) and any integer \( q \geq 1 \), setting \( x_q = \varphi^{-q}(x) \), we have

\[ d\hat{\varphi}^q_{x_q}(0) \circ F_{x_q}(v) = F_x \circ \hat{\varphi}^q_{x_q}(v). \]
for any $v \in E^u_1(x_q; \hat{r}_1(x_q))$ with $\|\hat{\varphi}^g_{z_q}(v)\| \leq \hat{r}_1(x)$.

As in [S15] this is derived from the following lemma. Part (b) below is a bit stronger than what is required here, however we need it in this form for the proof of Lemma 4.2 in Sect. 8.

**Lemma 10.7.** There exist regularity functions $\hat{r}_1(x)$ and $L(x)$, $x \in \mathcal{L}$ with the following properties:

(a) If $x \in \mathcal{L}$, $z = \varphi^p(x)$ and $\|\varphi_z^p(v)\| \leq r(x)$ for some $v \in E^u_1(z; \hat{r}_1(z))$ and some integer $p \geq 1$, then $\|d\varphi_z^p(0) \cdot v\| \leq 2\|\varphi_z^p(v)\|$ and

\[\|d\varphi_z^p(0) \cdot v - \varphi_z^p(v)\| \leq L(x) \|\varphi_z^p(v)\|^{1+\beta}.\]

Similarly, if $\|d\varphi_z^p(0) \cdot v\| \leq \hat{r}_1(x)$ for some $v \in E^u_1(z)$ and some integer $p \geq 1$, then $\|\varphi_z^p(v)\| \leq 2\|d\varphi_z^p(0) \cdot v\|$ and

\[\|\varphi_z^p(v) - d\varphi_z^p(0) \cdot v\| \leq L(x) \|d\varphi_z^p(0) \cdot v\|^{1+\beta}.\]

(b) For any $x \in \mathcal{L}$ and any integer $p \geq 1$, setting $z = \varphi^{-p}(x)$, the map

\[F^p_z = d\varphi_z^p(0) \circ (\varphi_z^p)^{-1} : E^u(x; \hat{r}_1(x)) \rightarrow E^u(x; \hat{r}(x)),\]

is such that

\[\left\| \left[ (F^p_z(a))^{(1)} - (F^p_z(b))^{(1)} \right] - [a^{(1)} - b^{(1)}] \right\| \leq L(x) \left( \|a - b\|^{1+\beta} + \|b\|^\beta \cdot \|a - b\| \right) \quad (10.11)\]

for all $a, b \in E^u(x; \hat{r}_1(x))$. Moreover,

\[\frac{1}{2} \|a - b\| \leq \|d\varphi_z^p(0) \cdot [(\varphi_z^p)^{-1}(a) - (\varphi_z^p)^{-1}(b)]\| \leq 2\|a - b\|, \quad a, b \in E^u_1(x; \hat{r}_1(x)). \quad (10.12)\]

**Proof of Lemma 10.7.** Set $\hat{r}_1(x) = \hat{r}(x)/2$, $x \in \mathcal{L}$.

Part (a) follows from Lemma 3.1 (see also the Remark after the lemma).

(b) Let $x \in \mathcal{L}$ and $z = \varphi^{-p}(x)$ for some integer $p \geq 1$. Given any $a, b \in E^u(x; \hat{r}_1(x))$, set $v = \varphi_z^{-p}(a)$ and $\eta = \varphi_z^{-p}(b)$. Then $\|\varphi_z^p(v)\| = \|a\| \leq \hat{r}(x)$ and $\|\varphi_z^p(\eta)\| = \|b\| \leq \hat{r}(x)$. Set $z_j = \varphi^j(z)$, $v_j = \varphi_z^j(v) \in E^u(z_j)$, $w_j = d\varphi_z^j(0) \cdot v \in E^u(z_j)$, $\eta_j = \varphi_z^j(\eta) \in E^u(z_j)$ and $\zeta_j = d\varphi_z^j(0) \cdot v \in E^u(z_j)$. Clearly $v_p = a$ and $\eta_p = b$. We need to prove that

\[\| (w_p^{(1)} - \zeta_p^{(1)}) - (v_p^{(1)} - \eta_p^{(1)}) \| \leq L(x) \left( \|v_p - \eta_p\|^{1+\beta} + \|\eta_p\|^\beta \cdot \|v_p - \eta_p\| \right).\]

Next, we use the argument from the proof of Lemma 3.1 (which is Lemma 3.3 in [S13]) with a small modification.

By (3.12),

\[\|v_k\| \leq \|\varphi_z^k(v)\| \leq \frac{1}{\mu_{1-k}} \|\varphi_z^p(v)\|^{1-k} \quad \text{and} \quad \|\eta_k\| \leq \|\varphi_z^k(\eta)\| \leq \frac{1}{\mu_{1-k}} \|\varphi_z^p(\eta)\|^{1-k},\]

and also

\[\|v_k - \eta_k\| \leq \|v_k - \eta_k\| \leq \frac{1}{\mu_{1-k}} \|v_p - \eta_p\|^{1-k} \quad (10.13)\]

for all $k = 0, 1, \ldots, p$.

It follows from (3.16) that

\[\|\hat{\varphi}_z(v) - \hat{\varphi}_z(\eta) - d\hat{\varphi}_z(0) \cdot (v - \eta)\| \leq D(z) \left( \|v - \eta\|^{1+\beta} + \|\eta\|^\beta \cdot \|\eta - \eta\| \right),\]

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so \( v_1 - \eta_1 = d\hat{\phi}_z(0) \cdot (v - \eta) + u_1 \) for some \( u_1 \in E^u(z_1) \) with \( \| u_1 \| \leq D(z) \| \| v - \eta \|^{1+\beta} + \| \eta \|^{\beta} \| v - \eta \| \).

Hence
\[
d\hat{\phi}_z(0) \cdot (v_1 - \eta_1) = d\hat{\phi}_z^2(0) \cdot (v - \eta) + d\hat{\phi}_z(0) \cdot u_1.
\]

Using (3.16) again,
\[
\| \hat{\phi}_z(v_1) - \hat{\phi}_z(v_1) - d\hat{\phi}_z(0) \cdot (v_1 - \eta_1) \| \leq D(z_1) \left( \| v_1 - \eta_1 \|^{1+\beta} + \| \eta_1 \|^{\beta} \| v_1 - \eta_1 \| \right),
\]
so \( v_2 - \eta_2 = d\hat{\phi}_z(0) \cdot (v_1 - \eta_1) + u_2 \) for some \( u_2 \in E^u(z_2) \) with
\[
\| u_2 \| \leq D(z_1) \| v_1 - \eta_1 \|^{1+\beta} + \| \eta_1 \|^{\beta} \| v_1 - \eta_1 \|,
\]
which gives \( v_2 - \eta_2 = (w_2 - \zeta_2) + u_2 + d\hat{\phi}_z(0) \cdot u_1 \).

Assume that for some \( k = 1, \ldots, p - 1 \) we have
\[
v_k - \eta_k = (w_k - \zeta_k) + u_k + d\hat{\phi}_{z_{k-1}}(0) \cdot u_{k-1} + \ldots + d\hat{\phi}_z^{k-1}(0) \cdot u_1,
\]
where \( u_j \in E^u(z_j) \) and
\[
\| u_j \| \leq D(z_{j-1}) \left( \| v_{j-1} - \eta_{j-1} \|^{1+\beta} + \| \eta_{j-1} \|^{\beta} \| v_{j-1} - \eta_{j-1} \| \right)
\]
for all \( j = 1, \ldots, k - 1 \). Then
\[
d\hat{\phi}_z(v_k - \eta_k) = (w_{k+1} - \zeta_{k+1}) + d\hat{\phi}_{z_k}(0) \cdot u_k + d\hat{\phi}_{z_{k-1}}^2(0) \cdot u_{k-1} + \ldots + d\hat{\phi}_z^k(0) \cdot u_1.
\]

By (3.16),
\[
\| \hat{\phi}_z(v_k) - \hat{\phi}_z(v_k) - d\hat{\phi}_z(0) \cdot (v_k - \eta_k) \| \leq D(z_k) \left[ \| v_k - \eta_k \|^{1+\beta} + \| \eta_k \|^{\beta} \| v_k - \eta_k \| \right],
\]
so \( v_{k+1} - \eta_{k+1} = d\hat{\phi}_z(0) \cdot (v_k - \eta_k) + u_{k+1} \) for some \( u_{k+1} \in E^u(z_{k+1}) \) with
\[
\| u_{k+1} \| \leq D(z_k) \left( \| v_k - \eta_k \|^{1+\beta} + \| \eta_k \|^{\beta} \| v_k - \eta_k \| \right).
\]

Thus, (10.14) holds for all \( k = 1, \ldots, p \).

It follows from (10.14) with \( k = p \) that
\[
v_p^{(1)} - \eta_p^{(1)} = (w_p^{(1)} - \zeta_p^{(1)}) + u_p^{(1)} + d\hat{\phi}_{z_{p-1}}(0) \cdot u_{p-1}^{(1)} + \ldots + d\hat{\phi}_z^{p-1}(0) \cdot u_1^{(1)}.
\]

Next, (10.13) implies
\[
\| u_j \|_{z_j} \leq \Gamma(z_j) \| u_j \| \leq \Gamma(z_j) D(z_{j-1}) \left[ \| v_{j-1} - \eta_{j-1} \|^{1+\beta} + \| \eta_{j-1} \|^{\beta} \| v_{j-1} - \eta_{j-1} \| \right]
\]
\[
\leq \Gamma(z_j) \frac{D(z_{j-1})}{\mu_1^{(1+\beta)(p-j+1)}} \left[ \left( \| v_p - \eta_p \|_{z_p} \right)^{1+\beta} + \left( \| \eta_p \|_{z_p} \right)^{\beta} \| v_p - \eta_p \|_{z_p} \right],
\]
so by (3.14),
\[
\| d\hat{\phi}_z^{p-j}(0) \cdot u_j^{(1)} \| \leq \| d\hat{\phi}_z^{p-j}(0) \cdot u_j^{(1)} \|_{z_j} \leq \left( \nu_1 \right)^{p-j} \| u_j^{(1)} \|_{z_j}^{p-j}
\]
\[
\leq \left( \nu_1 / \mu_1^{1+\beta} \right)^{p-j} \left[ \left( \| v_p - \eta_p \|_{z_p} \right)^{1+\beta} + \left( \| \eta_p \|_{z_p} \right)^{\beta} \| v_p - \eta_p \|_{z_p} \right]
\]
\[
= \Gamma(z_j) D(z_{j-1}) \left( \nu_1 / \mu_1^{1+\beta} \right)^{p-j} \left[ \| v_p - \eta_p \|^{1+\beta} + \| \eta_p \|^{\beta} \| v_p - \eta_p \| \right]
\]
\[
\leq \Gamma(x)^{2+\beta} D(x) \left( e^{2} \nu_1 / \mu_1^{1+\beta} \right)^{p-j} \left[ \| v_p - \eta_p \|^{1+\beta} + \| \eta_p \|^{\beta} \| v_p - \eta_p \| \right]
\]
for all \( j = 1, \ldots, p \). Using the above, choosing \( L(x) \) appropriately and setting \( \gamma_2 = e^{2\nu_1/\mu_1^{1+\beta}} < 1 \), we get
\[
\|(v_p^{(1)} - \eta_p^{(1)}) - (w_p^{(1)} - \zeta_p^{(1)})\| \leq L(x) \left[ \|v_p - \eta_p\|^{1+\beta} + \|\eta_p\|^{1+\beta} \|v_p - \eta_p\| \right],
\]
which proves (10.11).

Notice that when \( a, b \in E^n_1(x) \) we have \( v_p, \eta_p, w_p, \zeta_p \in E^n_1(x) \) as well, so (10.11) and the choice of \( \hat{r}(x) \) and \( L(x) \) imply
\[
\|(v_p - \eta_p) - (w_p - \zeta_p)\| \leq \|v_p - \eta_p\| 3L(x)(\hat{r}(x))^\beta \leq \frac{1}{2} \|v_p - \eta_p\|,
\]
which proves (10.13).

**Proof of Proposition 10.6.** Let \( L(x) \) and \( r(x) (x \in \mathcal{L}) \) be as in Lemma 10.7. Fix an arbitrary \( x \in \mathcal{L} \) and set \( x_j = \varphi^{-j}(x) \) for any integer \( p \geq 0 \). In what follows we use the maps \( F_y^p \) (\( y \in \mathcal{L}, \ p \geq 1 \)) from Lemma 10.7, and also the notation from the proof of Lemma 10.7.

(a) Given \( u \in E^n_1(x; \hat{r}(x)) \) and \( p \geq 0 \), set \( u_p = F_x^p(u) \in E^n_1(x; \hat{r}(x)) \). To show that the sequence \( \{u_p\} \) is Cauchy, consider any \( q > p \) and set \( v = \hat{\varphi}_x^{-p}(u) \in E^n_1(x_p, \hat{r}_1(x_p)) \). By (3.12) we have
\[
\|v\|_{F_x} \leq \frac{\|u\|_x'}{\mu_1^q},
\]
Set \( v_{q-p} = d\hat{\varphi}_x^{(q-p)}(0) \cdot (\hat{\varphi}_x^{(q-p)}(v)) \). From Lemma 10.7 we know that \( \|v_{q-p} - v\| \leq L(x_p) \|v\|^{1+\beta} \), i.e.
\[
\|d\hat{\varphi}_x^{(q-p)}(0) \cdot (\hat{\varphi}_x^{(q-p)}(v)) - v\| \leq L(x_p) \|v\|^{1+\beta}.
\]
Applying \( d\hat{\varphi}_x^{p}(0) \) to the above and using (3.14), we get
\[
\|u_q - u_p\| = \|d\hat{\varphi}_x^{q}(0) \cdot (\hat{\varphi}_x^{q}(u)) - d\hat{\varphi}_x^{p}(0) \cdot v\| \leq \Gamma(x) \|d\hat{\varphi}_x^{p}(0) \cdot (v_{q-p} - v)\|_{x'}
\leq L(x_p)\Gamma(x)(\nu_1)^p \|v\|_{x_p}^{1+\beta}
\leq \Gamma(x) L(x)(\nu_1 \epsilon^f/\mu_1^{1+\beta})^p \|u\|_{x'}^{1+\beta} \leq \Gamma(x)^{2+\beta} L(x) \gamma_2^p \|u\|^{1+\beta},
\]
where as above, \( \gamma_2 = e^{2\nu_1/\mu_1^{1+\beta}} < 1 \). Thus, \( \{u_p\} \) is Cauchy, so there exists \( F_x(u) = \lim_{p \to \infty} u_p \). Moreover, letting \( q \to \infty \) in the above gives
\[
\|F_x(u) - u_p\| \leq \Gamma(x)^{2+\beta} L(x) \gamma_2^p \|u\|^{1+\beta}, \quad u \in E^n_1(x; \hat{r}_1(x)) , \ p \geq 0.
\]
(b) Given \( u, v \in E^n_1(x; \hat{r}_1(x)) \) and \( p \geq 0 \), it follows from Lemma 10.7(c) that
\[
\|(u_p - v_p) - (u - v)\| \leq L(x) \|u - v\|^{1+\beta} + \|v\|^{1+\beta} \|u - v\|.
\]
Letting \( p \to \infty \), proves the desired estimate.

(c) Let \( v \in E^n_1(x, \hat{r}_1(x)) \) be such that \( \|\hat{\varphi}_x(v)\| \leq \hat{r}_1(x) \). It is enough to show that
\[
d\hat{\varphi}_x(0) \circ F_x(v) = F_x \circ \hat{\varphi}_x(v).
\]
Set \( u = \hat{\varphi}_x(v) \). For any integer \( p \geq 0 \) we have \( v_p = d\hat{\varphi}_x^{p}(0) \cdot \hat{\varphi}_x^{-p}(v) \), so
\[
d\hat{\varphi}_x(0)(v) = d\hat{\varphi}_x^{p+1}(0) \cdot \hat{\varphi}_x^{-p}(v) = d\hat{\varphi}_x^{p+1}(0) \cdot \hat{\varphi}_x^{-(p+1)}(u) = u_{p+1}.
\]
Letting \( p \to \infty \) gives \( d\hat{\varphi}_x(0)(F_x(v)) = F_x(u) \), which is exactly (10.16). ■
For $z \in \mathcal{L}$, $\epsilon \in (0, \hat{r}_1(z)]$ and an integer $p \geq 0$ set

$$\tilde{B}^{u,1}_p(z, \epsilon) = F_z(\tilde{B}^{u,1}_p(z, \epsilon)) \subset E^u_1(z; \hat{r}(z)).$$

Then, using Proposition 10.6(c) we get

$$d\tilde{\varphi}^{-1}_x(0)(\tilde{B}^{u,1}_{p+1}(x, \delta)) \subset \tilde{B}^{u,1}_p(\varphi^{-1}(x), \delta), \quad x \in \mathcal{L}, \ p \geq 1. \quad (10.17)$$

Indeed, if $\eta \in \tilde{B}^{u,1}_{p+1}(x, \delta)$, then $\eta = F_x(v)$ for some $v \in \tilde{B}^{u,1}_p(x, \delta)$, and then clearly $w = \tilde{\varphi}^{-1}_x(v) \in \tilde{B}^{u,1}_p(x, \delta)$. Setting $y = \varphi^{-1}(x)$, by Proposition 10.6(c), $\eta = F_x(v) = F_x(\tilde{\varphi}(y)) = d\tilde{\varphi}(0) \cdot (F_y(w))$, so $d\tilde{\varphi}^{-1}_x(0) \cdot \eta = F_y(w) \in \tilde{B}^{u,1}_p(y, \delta)$. Moreover, locally near 0 we have an equality in (10.17), i.e. if $\delta' \in (0, \delta)$ is sufficiently small, then $d\tilde{\varphi}^{-1}_x(0)(\tilde{B}^{u,1}_{p+1}(x, \delta)) \supset \tilde{B}^{u,1}_p(\varphi^{-1}(x), \delta').$

To prove part (a) of Lemma 10.3 we have to establish the following lemma which is similar to Lemma 4.4 in [St5] (see also the Appendix in [St5]).

**Lemma 10.8.** Let $x \in \mathcal{L}$ and let $0 < \delta \leq \epsilon \leq \hat{r}_1(x)$. Then

$$\ell \left( \tilde{B}^{u,1}_p(\varphi^{-p}(x), \epsilon) \right) \leq 8n_1 \frac{\epsilon}{\delta} \ell \left( \tilde{B}^{u,1}_p(\varphi^{-p}(x), \delta) \right)$$

for any integer $p \geq 0$, where $n_1 = \dim(E^u_1(x)).$

**Proof of Lemma 10.8.** Choose an orthonormal basis $e_1, e_2, \ldots, e_{n_1}$ in $E^u_1(x)$ and set $u_i = \frac{\epsilon}{4} e_i$.

Consider an arbitrary integer $p \geq 1$ and set $z = \varphi^{-p}(x)$. Given $v \in \tilde{B}^{u,1}_p(x, \epsilon)$, we have $v = F_z(w)$ for some $w \in \tilde{B}^{u,1}_p(z, \epsilon)$. Then $\|\hat{\varphi}^p_z(w)\| \leq \epsilon$. Now it follows from Proposition 10.6 that

$$\|d\hat{\varphi}^p_z(0) \cdot v\| = \|d\hat{\varphi}^p_z(0) \cdot F_z(w)\| = \|F_z(\hat{\varphi}^p_z(w))\| \leq 2\|\hat{\varphi}^p_z(w)\| \leq 2\epsilon.$$

So, $u = d\hat{\varphi}^p_z(0) \cdot v \in E^u_1(x, 2\epsilon)$. We have $u = \sum_{s=1}^{n_1} c_s u_s$ for some real numbers $c_s$ and $u = \sum_{s=1}^{n_1} \frac{\epsilon}{4} e_s$, so $\sqrt{\sum_{s=1}^{n_1} c_s^2} = \frac{\epsilon}{4}\|u\|$ and therefore $|c_s| \leq 8\epsilon \frac{1}{\delta}$ for all $s = 1, \ldots, n_1$.

By (10.17), $v_j = d\hat{\varphi}^p_z(0) \cdot u_j \in \tilde{B}^{u,1}_p(z, \delta)$. Indeed, since $\|u_j\| \leq \frac{\epsilon}{4}$, we have $u_j = F_z(u'_j)$ for some $u'_j \in E^u_1(x, \delta/2)$. Set $v'_j = \hat{\varphi}^p_z(u'_j)$; then $\|\hat{\varphi}^p_z(v'_j)\| \leq \frac{\epsilon}{4}$, so $v'_j \in \tilde{B}^{u,1}_p(x, \delta/2)$ and therefore $v_j = F_z(v'_j) \in \tilde{B}^{u,1}_p(z, \delta)$. Using Proposition 10.6(b), we get

$$d\hat{\varphi}^p_z(0) \cdot u_j = d\hat{\varphi}^p_z(0) \cdot F_z(u'_j) = F_z(\hat{\varphi}^p_z(u'_j)) = F_z(v'_j) = v_j.$$

It now follows that

$$\|v\| = \|d\hat{\varphi}^p_z(0) \cdot u\| = \left\| \sum_{s=1}^{n_1} c_s d\hat{\varphi}^p_z(0) \cdot u_s \right\| \leq n_1 \frac{\epsilon}{4} \max_{1 \leq s \leq n_1} \|v_s\| \leq 8n_1 \frac{\epsilon}{\delta} \ell(\tilde{B}^{u,1}_p(z, \delta)).$$

Therefore $\ell \left( \tilde{B}^{u,1}_p(z, \epsilon) \right) \leq 8n_1 \frac{\epsilon}{\delta} \ell \left( \tilde{B}^{u,1}_p(z, \delta) \right).$ \hfill \blacksquare

**Lemma 10.9.** Let $x \in \mathcal{L}$ and let $0 < \epsilon \leq \hat{r}_1(x)$ and $p \in (0, 1)$. Then for any $\delta$ with $0 < \delta \leq \frac{1}{8n_1} \epsilon$ we have

$$\ell \left( \tilde{B}^{u,1}_p(\varphi^{-p}(x), \delta) \right) \leq \rho \ell \left( \tilde{B}^{u,1}_p(\varphi^{-p}(x), \epsilon) \right)$$

for any integer $p \geq 0$.

**Proof of Lemma 10.9.** We will essentially repeat the argument in the proof of Lemma 10.8.

Fix $x \in \mathcal{L}$, $0 < \epsilon \leq \hat{r}_1(x)$ and $p \in (0, 1)$. Set $u_i = \frac{\epsilon}{4} e_i$ and let $0 < \delta \leq \frac{\epsilon}{8n_1}$. Then given an integer $p \geq 1$, set $z = \varphi^{-p}(x)$. Given $v \in \tilde{B}^{u,1}_p(z, \epsilon)$, as before we have $u = d\hat{\varphi}^p_z(0) \cdot v \in E^u_1(z, 2\delta).$
Then $u = \sum_{s=1}^{n_1} c_s u_s$ for some real numbers $c_s$ and $u = \sum_{s=1}^{n_1} \frac{r}{s} c_s$, so again we get $|c_s| \leq 8\delta \frac{1}{\epsilon}$ for all $s = 1, \ldots, n_1$. As in the proof of the Lemma 10.8 we get $v_j = d\hat{x}^{-p}(0) \cdot u_j \in \hat{B}_p^{n_1}(z, \delta)$, so

$$\|v\| = \|d\hat{x}^{-p}(0) \cdot u\| = \left\| \sum_{s=1}^{n_1} c_s d\hat{x}^{-p}(0) \cdot u_s \right\| \leq n_1 8\delta \frac{1}{\epsilon} \max_{1 \leq s \leq n_1} \|v_s\| \leq \rho \ell(\hat{B}_p^{n_1}(z, \delta)).$$

Thus, $\ell(\hat{B}_p^n(z, \delta)) \leq \rho \ell(\hat{B}_p^n(z, \epsilon))$. □

### 10.3 Consequences for cylinders in Markov partitions

Here we prove Lemma 4.4 using arguments similar to these in Sect. 4 in [St5]. We use the notation from Sect. 4.

Let $\hat{r}(x)$ be the canonical $\epsilon$-regularity function from Theorem 10.4 and Corollary 10.5. Here $\epsilon \in (0, \hat{\epsilon}]$ is some constant depending on $\hat{\epsilon}$. Then (see the end of Sect. 3.2) there exists a constant $\hat{r}_0 > 0$ such that $\hat{r}(x) \geq \hat{r}_0$ for all $x \in P_0$ and $x \in \hat{P}_0$. (See also the end of Sect. 3.) **Fix $\epsilon$ and $\hat{r}_0$ with these properties.**

Let $B > 0$ be a Lipschitz constant for the projection along the flow

$$\psi : \bigcup_{i=1}^{k_0} \phi_{[-\epsilon, \epsilon]}(D_i) \longrightarrow \bigcup_{i=1}^{k_0} D_i,$$

i.e. for all $i = 1, \ldots, k_0$ and all $x \in \phi_{[-\epsilon, \epsilon]}(D_i)$ we have $\psi(x) = \text{pr}_{D_i}(x)$.

Let $c_0, \gamma$ and $\gamma_1$ be the constants from (2.1). Next, assuming that the constant $\epsilon > 0$ is chosen so that $\epsilon^\gamma / \gamma < 1$, fix an integer $d_0 \geq 1$ such that

$$\frac{2k\Gamma_0^2 \epsilon^{2\gamma} \Gamma_1}{\gamma_0} < (\mu_1 \epsilon)^d_0,$$

$$\frac{1}{c_0 (\gamma \epsilon^{-\gamma})^d_0} < \frac{\hat{r}_0}{2}. \tag{10.18}$$

Set

$$\hat{r}_0' = \frac{\hat{r}_0 e^{-(d_0+1)\epsilon}}{B}. \tag{10.19}$$

**Proof of Lemma 4.4.** First note the following. Let $z \in \hat{R}_j$ be such that $\hat{P}_d^{d_0+1}(z) \in \hat{P}_0$. Then $z \in C_V[\hat{r}']$ for some $\hat{r}' = [i_0, \ldots, i_{d_0+1}]$ with $i_0 = j$, where $V = W_R'(z)$. Set $i = [i_0, \ldots, i_{d_0}]$. We claim that

$$C_V[i] \subset B_V(z, r'_0) \quad \text{and} \quad r(z) \geq r'_0. \tag{10.20}$$

Indeed, by (2.1) and (10.18), $\text{diam}(C_V[i]) \leq \frac{1}{c_0 \Gamma^\gamma_0 + \epsilon^\gamma} < r'_0/2$. On the other hand, $\hat{r}(x)$ is a Lyapunov $\hat{\epsilon}$-regularity function and $y = \hat{P}_d^{d_0+1}(z) \in \hat{P}_0$ and the definition of $\hat{P}_0$ show that $\hat{r}(y) \geq \hat{r}_0$. Also recall that $0 < \tau(x) \leq 1$ for all $x \in \hat{R}$ by the choice of the Markov family. Now using (10.19), we get

$$\hat{r}(z) \geq \hat{r}(y) e^{-\tau_0 d_0+1(z)\epsilon} \geq \hat{r}_0 e^{-(d_0+1)\epsilon} = r'_0 > 2 \text{diam}(C_V[i]).$$

This proves (10.20).

(a) Assume that $m > d_0$, and let $i = [i_0, i_1, \ldots, i_m]$ and $i' = [i_0, i_1, \ldots, i_{m+1}]$ be admissible sequences. Let $C = C[i]$ and $C' = C[i']$ be the corresponding cylinders in $\hat{R}$. Assume that there exists $z \in C' \cap P_0$ with $\hat{P}_d^{m+1}(z) \in P_0$. Fix such a point $z \in C'$; then $y = \hat{P}_d^{m+1}(z) \in P_0$ and $\hat{P}_d^j(z) \in \hat{R}_j$ for all $j = 0, 1, \ldots, m+1$.

Set $\hat{P}_d^{m-d_0}(z) = x$, $V = W_R'(x)$. Since $\hat{P}_d^{d_0+1}(x) = y \in \hat{P}_0$, we have $\hat{r}(y) \geq \hat{r}_0$, so $\hat{r}(x) \geq r_0$.

Consider the cylinders

$$\hat{C}' = C_V[i_{m-d_0}, i_{m-d_0+1}, \ldots, i_m, i_{m+1}] \subset \hat{C} = C_V[i_{m-d_0}, i_{m-d_0+1}, \ldots, i_m] \subset V.$$
Since $\tilde{P}_{d_0+1}(x) = y$, using (10.20) we get $\tilde{C} \subset B_V(x, r_0)$. On the other hand it is easy to see using (2.1) that $\tilde{C}' \supset B_V(x, c_0r_1/\gamma_1^{d_0+1})$.

We will now use Corollary 10.5(a) with $x$ and $z$ as above, $T = \tau_{m-d_0}(z) > 0$ and

$$0 < \delta = \delta_3 = \frac{c_0r_1}{B_1\gamma_1^{d_0+1}} < \epsilon = r_0'.$$

This, combined with (10.20), gives

$$\text{diam}(B_T^u(z\delta_3)) \geq \frac{\delta_3}{B_1\Gamma_1^3 R_0 r_0} \text{diam}(B_T^u(z, r_0)).$$

However, using the above information about $\tilde{C}$ and $\tilde{C}'$, as in the proof of Proposition 3.3 in [S2], one easily observes that $C' \supset B_T^u(z, \delta_3)$ and $C \subset B_T^u(z, Br_0')$. Thus,

$$\text{diam}(C') \geq \frac{\delta_3}{B_1\Gamma_1^3 R_0 r_0} \text{diam}(C).$$

This proves part (a) for $m > p_0$. Since there are only finitely many cylinders of length $\leq p_0$, it follows immediately that there exists $p_1 \in (0, \frac{\delta_3}{BL_1\Gamma_1^3 R_0 r_0})$ which satisfies the requirements of part (a).

(b) Let $\rho' \in (0, 1)$. It follows from Corollary 10.5(b) that for $z \in \tilde{R} \cap \mathcal{L}$ with $\Phi_T(z) \in \tilde{P}_0$ for some $T > 0$ we have

$$\text{diam}(B_T^u(z, B\delta)) \leq \rho' \text{diam}(B_T^u(z, r_1/B)), \quad (10.21)$$

provided $0 < \delta \leq \delta_4 = \frac{\rho' r_1}{B_2^2 L_1 \Gamma_1^3 R_0}$. Fix an integer $q' \geq 1$ so large that

$$\frac{1}{c_0 \gamma^{q'}} \leq \delta_4. \quad (10.22)$$

Consider the cylinders

$$C = C[i] = C[i_0, \ldots, i_m] \supset C' = C[i'] = C[i_0, i_1, \ldots, i_{m+1}, \ldots, i_{m+q'}]$$

in some unstable leaf in $\tilde{R}$. Assume that there exists $z \in C' \cap \tilde{P}_0$ with $\mathcal{P}^{m+q'}(z) \in \tilde{P}_0$. Set $T = \tau_{m+q'}(z)$. Then (10.22) holds. Set $x = \tilde{P}^m(z)$, $V = W^u_R(x)$; then $\tilde{P}'(x) \in \tilde{P}_0$, so $r(x) = r_0$. Consider the cylinder $\tilde{C} = C_V[i_m, i_{m+1}, \ldots, i_{m+q'}] \subset V$. Since $x \in \tilde{C}$ and diam$(\tilde{C}) \leq \frac{1}{c_0 \gamma^{q'}}$, (10.22) implies $\tilde{C} \subset B_V(x, \delta_4)$.

Next, we have $C[i'] \subset B_T^u(z, B\delta_4)$. Indeed, if $u \in C[i']$, then $\tilde{P}^m(u) \in \tilde{C}$, so $d(\tilde{P}^m(z), \tilde{P}^m(u)) < \frac{1}{c_0 \gamma^{q'}} < \delta_4$. Thus, $C[i'] \subset B_T^u(z, B\delta_4)$ and therefore by (10.22),

$$\text{diam}(C[i']) \leq \text{diam}(B_T^u(z, B\delta_4)) \leq \rho' \text{diam}(B_T^u(z, r_1/B)).$$

On the other hand, $W^u_{r_1}(x) \subset \tilde{R}_{i_m}$ (see Sect. 4.2 for the choice of $r_1 > 0$), so $B_T^u(z, Br_1) \subset C[i]$, and therefore $\text{diam}(C[i']) \leq \rho' \text{diam}(C[i])$.

(c) Let again $m > d_0$, let $\iota = [i_0, i_1, \ldots, i_m]$ be an admissible sequence, let $C = C_W[i]$ be the corresponding cylinder in an unstable leaf $W$ in $\tilde{R}$. Let $z \in C \cap \tilde{P}_0$ and let $\tilde{P}^m(z) = z' \in \tilde{P}_0$. Set $z'' = \tilde{P}^{m-d_0}(z)$, $V = W^u_{\tilde{R}}(z'')$. If $z' = \phi_T(z)$ and $z'' = \phi_{\tau}(z)$; then $\phi_{T-t}(z'') = z'$, so $T - t = \tau_{d_0}(z'') < d_0$. Thus, $\tilde{r}(z'') \geq \tilde{r}(z') e^{-d_0} \geq \tilde{r}_0 e^{-d_0} > r_0$. As in part (a), for the cylinder $\tilde{C} = C_V[i_{m-d_0}, i_{m-d_0+1}, \ldots, i_m] \subset V$, we have

$$z'' \in B_V(z'', c_0 \tilde{r}_0 / \gamma_1^{d_0}) \subset \tilde{C} \subset \tilde{P}^{m-d_0}(C) \subset B_V(z'', r_0).$$
Setting $\epsilon = c_0 \rho_0 / \gamma_1^d < \epsilon = B r'$, it follows from Corollary 10.5(c) that for $0 < \delta_5 = \frac{\epsilon}{100 \mu_1}$ there exists $x \in B^0_i(z, \epsilon')$ such that for every $y \in W_1(z)$ with $d(\phi_i(y), \phi_i(x)) \leq \delta_5$ we have

$$d(z, y) \geq \frac{\epsilon'}{L_1 \epsilon \Gamma_2 \hat{\gamma}_0 \gamma_1^d} \text{diam}(B^0_1(z, Br')) \geq \frac{c_0 \rho_0}{L_1 Br' \Gamma_2 \hat{\gamma}_0 \gamma_1^d} \text{diam}(C),$$

(10.23)

since $C \subset B^0_i(z, Br')$.

Take the integer $q_0 \geq 1$ so large that

$$\frac{1}{c_0 \gamma_1^{d_0} + q_0} < \frac{\delta_5}{B} = \frac{\epsilon}{100 B r_1},$$

where $n_1 = \text{dim}(E^u_1)$.

Let $x \in C$ and let

$$C' = C[t'] = C[i_0, i_1, \ldots, i_{m+1}, \ldots, i_{m+q_0}]$$

be the sub-cylinder of $C$ of co-length $q_0$ containing $x$. Then for the cylinder

$$\tilde{C}' = C_V[i_{m-d_0}, i_{m-d_0+1}, \ldots, i_m, i_{m+q_0}] \subset V$$

we have $\tilde{P}^{m-d_0}(x) \in \tilde{C}'$ and

$$\text{diam}(\tilde{C}') < \frac{1}{c_0 \gamma_1^{d_0} + q_0} < \frac{\delta_5}{B}.$$ 

Since for any $y \in C'$ we have $\tilde{P}^{m-d_0}(y) \in \tilde{C}'$, it follows that $d(\tilde{P}^{m-d_0}(x), \tilde{P}^{m-d_0}(y)) < \delta_5 / B$ and therefore $d(\phi_i(x), \phi_i(y)) < \delta_5$. Thus, $y$ satisfies (10.23). This proves the assertion with

$$\rho_1 = \frac{c_0 \rho_0}{L_1 Br' \Gamma_2 \hat{\gamma}_0 \gamma_1^d}.$$

(d) This follows from Theorem 10.4(a).

11 Appendix: Proofs of some technical lemmas

Proof of Lemma 5.2. (a) Let $u, u' \in \hat{U}_i$ for some $i = 1, \ldots, k_0$ and let $m \geq 1$ be an integer. Given $v \in \hat{U}$ with $\sigma^m(v) = u$, let $C[i] = C[i_0, \ldots, i_m]$ be the cylinder of length $m$ containing $v$. Set $\tilde{C}[i] = C[i] \cap \hat{U}_i$. Since the sequence $i = [i_0, \ldots, i_m]$ is admissible, the Markov property implies $i_{m-i} = i$ and $\sigma^m(\tilde{C}[i]) = \hat{U}_i$. Moreover, $\sigma^m : \tilde{C}[i] \rightarrow \hat{U}_i$ is a homeomorphism, so there exists a unique $v' = v'(v) \in \tilde{C}[i]$ such that $\sigma^m(v') = u'$. Clearly,

$$D_\theta(\sigma^j(v), \sigma^j(v'(v))) = \theta^{m-j} D_\theta(u, u'), \quad j = 0, 1, \ldots, m - 1.$$ 

Consequently, using (5.3),

$$|f^a_m(v) - f^a_m(v')| \leq \sum_{j=0}^{m-1} |f^a(\sigma^j(v)) - f^a(\sigma^j(v'))| \leq \sum_{j=0}^{m-1} \text{Lip}_\theta(f^a) \theta^{m-j} D_\theta(u, u')$$

$$\leq \frac{\theta T}{1 - \theta} D_\theta(u, u').$$

Also notice that $D_\theta(v, v'(v)) \leq \theta^m D_\theta(u, u')$.

Using the above, the definition of $\mathcal{M}_a$, and the fact that $\mathcal{M}_a 1 = 1$ (hence $\mathcal{M}_a^m 1 = 1$), we get

$$\frac{|(\mathcal{M}_a^n H)(u) - (\mathcal{M}_a^n H)(u')|}{\mathcal{M}_a^n H(u')} = \left| \sum_{\sigma^m v = u} e^{f^a_m(v)} H(v) - \sum_{\sigma^m v' = u} e^{f^a_m(v'(v))} H(v'(v)) \right|\mathcal{M}_a^n H(u').$$

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\[
\begin{align*}
&\leq \frac{\sum_{\sigma^m v = u} e^{f_m(a)}(v) \left( H(v) - H(v') \right)}{\mathcal{M}_a^m H(u')} + \frac{\sum_{\sigma^m v = u} e^{f_m(a)}(v) - e^{f_m(a)}(v')} {H(v')} H(v') \\
&\leq \frac{\sum_{\sigma^m v = u} e^{f_m(a)}(v) B H(v') D(v, v')}{\mathcal{M}_a^m H(u')} + \frac{\sum_{\sigma^m v = u} e^{f_m(a)}(v) - e^{f_m(a)}(v')}{1 - \theta} H(v') \\
&\leq B \theta^m D \left( u, u' \right) + e^{\frac{\theta T}{1 - \theta}} \| \frac{\theta T}{1 - \theta} D \left( u, u' \right) \| \leq A_0 \left[ B \theta^m + \frac{\theta T}{1 - \theta} \right] D(u, u'),
\end{align*}
\]

provided \( A_0 \geq e^{\frac{\theta T}{1 - \theta}}. \)

(b) Let \( m \geq 1 \) be an integer and \( u, u' \in \widehat{U}_j \) for some \( i = 1, \ldots, k. \) Using the notation \( v' = v'(v) \) from part (a) above, where \( \sigma^m v = u \) and \( \sigma^m v' = u' \), we get

\[
\begin{align*}
&\left| L_{ab}^m h(u) - L_{ab}^m h(u') \right| = \left| \sum_{\sigma^m v = u} \left( e^{f_m(a)}(v) - e^{f_m(a)}(v') \right) h(v) \right| \\
&\leq \left| \sum_{\sigma^m v = u} e^{f_m(a)}(v) h(v) - h(v') \right| + \left| \sum_{\sigma^m v = u} e^{f_m(a)}(v) - e^{f_m(a)}(v') \right| h(v') \\
&\leq \left| \sum_{\sigma^m v = u} e^{f_m(a)}(v) h(v) - h(v') \right| + \left| \sum_{\sigma^m v = u} e^{f_m(a)}(v) - e^{f_m(a)}(v') \right| + \left| e^{f_m(a)}(v') \right| h(v').
\end{align*}
\]

We have \( |h(v) - h(v')| \leq B H(v') D \left( v, v' \right). \) Also, using elementary inequalities one checks that \( e^{x+y} - 1 \leq e^x(|x| + |y|) \) for real \( x \) and \( y. \) By Lemma 5.1 and (5.3), \( |\tau_m(v) - \tau_m(v')| \leq T D(u, v'). \) Assuming \( A_0 \geq e^{\frac{\theta T}{1 - \theta}} \max\{1, \frac{2 \theta T}{1 - \theta}\} \) and \( |b| \geq 1, \) this and (5.3) give

\[
\begin{align*}
&\left| e^{f_m(a)}(v) - e^{f_m(a)}(v') + \right. \left. \frac{\theta T}{1 - \theta} B \theta^m D \left( u, u' \right) \right| \leq A_0 |b| D(u, u').
\end{align*}
\]

Next,

\[
\begin{align*}
&\sum_{\sigma^m v = u} e^{f_m(a)}(v) h(v) - h(v') \leq \sum_{\sigma^m v = u} e^{f_m(a)}(v) - e^{f_m(a)}(v') B H(v') D(v, v') \\
&\leq B \theta^m e^{\frac{\theta T}{1 - \theta}} D \left( u, u' \right) (\mathcal{M}_a^m H)(u'),
\end{align*}
\]

and therefore

\[
\begin{align*}
&\left| L_{ab}^m h(u) - L_{ab}^m h(u') \right| \leq A_0 \left[ B \theta^m (\mathcal{M}_a^m H)(u') + \frac{\theta T}{1 - \theta} (\mathcal{M}_a^m H)(u') \right] D(u, u').
\end{align*}
\]

This proves the assertion. \( \qed \)

**Proof of Lemma 6.10**. (a) Let \( u, u' \in \mathcal{D}'_j \) for some \( j \leq j_0. \) Let \( \mathcal{D}_j \subset C_m, \) \( m \leq m_0. \) Then for \( v = v_i^{(t)}(u) \in X_{i,j} \) and \( v' = v_i^{(t)}(v') \in X_{i,j}^{(t)}, \) we have \( \ell(v, v') \geq N \) and \( \sigma^N v, \sigma^N v' \in C_m. \) This and \( H \in K_E \) imply

\[
|\ln H(v) - \ln H(v')| \leq \frac{|H(v) - H(v')|}{\min\{H(v), H(v')\}} \leq E D(v, v') = E \theta^m D(u, u') \leq E \theta^m < \ln 2,
\]
Hence $|\ln H(x') - \ln H(x'')| \leq \ln 2$, so $\frac{1}{2} \leq \frac{H(v'(i))(u')}{{H(v'(i))(u'')}} \leq 2$.

(b) Consider the case when for some $v \in X_{i,j}^{(e)}$ we have $|h(v)| \geq \frac{2}{3}H(v)$. Fix $v$ with this property and consider an arbitrary $v' \in X_{i,j}^{(e)}$. It follows from (ii) in Sect. 6.4 that

$$|h(v') - h(v)| \leq E|b| \theta_2^N H(v) \diam(\Psi(C_m)) \leq E|b| \theta_2^N H(v) \frac{E}{|b|} = E_1 \theta_2^N H(v).$$

Using $2H(v) \geq H(v')$ which follows from (a), one obtains

$$|h(v')| \geq |h(v)| - E_1 \theta_2^N H(v) \geq \left(\frac{3}{4} - E_1 \theta_2^N\right) H(v) \geq \frac{1}{4} H(v'),$$

since $E_1 \theta_2^N \leq 1/4$ by (5.5). Thus, in this case the second alternative in (b) holds for all $v \in X_{i,j}^{(e)}$.

In the same way one shows that if $|h(v)| \leq \frac{1}{4}H(v)$ for some $v \in X_{i,j}^{(e)}$, then the first alternative in (b) holds. ■

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