A two-stage opinion formation model based on the extended XY-magnet interaction and socio-dynamic update mechanism

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Abstract. We introduce herein a two stage model of opinion formation in a community of individuals which share strong interests within a permanent pair, and attend also a persuasion social process with others to finalize their attitudes. In the first step, individuals behave as member of their couples and seek to optimize respective satisfaction. Then, they modify their attitudes inherited from the first stage throughout a persuasion process and social influence. The first stage consists in an extension of a classical XY magnet model, and the calculation in this step are performed using statistical mechanics tools. At the end of this phase, each opinion agent has adapted its support $O_1$ for the issue $F$ and the agreement within their pair $O = |O_1 + O_2|$. The evaluated quantities $O_x(i)$ depend on the magnetic-like system parameters $J$, $T$, $F$ and on the extra parameter $a(i)$ which embodies the dissimilarity of the system utility from the reference Hamiltonian. The set $\{O_x(i),O_x(i)\}$ represents the opinion state of the community straightaway after the first stage, and is presented hereto with respective histogram. We observed that histograms of $O_x(i)$ approximate to q-Gaussian distributions for moderate $F/T$ ratio, and approaches to power law distributions for low $F/T$ ratio. The histograms of the inner agreement $O(i)$ do not fit well to a given distribution, and therefore, the social comportment which we identify hereto with this set, is not a stationary quantity under this approach. Next, the opinion quantities $O_x(i)$ will pursue an update course as result of the persuasion process and social influence. We performed the calculation based on Deffuant and Heglesman models by using $O_x(a_{ij})_{x,y}$ as initial opinion values. We observed that the final opinion fragmentation resulted lower than when using standard assumptions of those models and also, the time to consensus was shorter. Next, for the special case where there are only two output final opinion values, the lower one needs more iteration time steps to converge. In our approach, usually the lowest level opinion converge slower than higher ones. Finally, we have implemented a modified preferential attachment model to realize a network of the linked nodes based on the opinions $O_x$ inherited from the early stage. We acknowledged the power law distribution of the grades of nodes, but in our case there are no disturbances in the queue of the histogram, which are common in the standard simulation for such networking process. As conclusion, having regard to every aspect and specifics, we believe that the proposed model would help on the understanding of the complexity of social conduct.

1. Introduction
Studying of social phenomena has become increasingly appealing to scientist from other fields [1]. In this context, using models and methods of physics has improved the quantitative analysis of complex processes as opinion dynamics, collective phenomena and social behaviour. So, in the case of the voter model introduced firstly in [2] and reviewed in [3] etc., the opinion takes binary values $s = \pm 1$ similarly
to the spins variables in the Ising model. Analogous considerations were encountered in the Pott’s model presented early in [4]. Various socio-dynamics models and methods have been discussed in references [5-10] etc., reinforcing the fact that physical approach to the complexity and social systems has been proven as fruitful. Stauffer has proposed in [12] the use of a particular utility function named unhappiness in the role of opinion state energy. Other specific analysis of social phenomena implemented directly physical kinetic quantities as in [14] for example. In this regard, we have considered the study of some opinion system like decision making about a question asked, taking attitude on an issue presented etc., if the first impact involves a very small groups of individuals say couples, small families, discussants who share strong interests. So, the interests, the satisfaction or the unhappiness could be set in the role of the utility or energy-like function. To typify this case, in [11] we have proposed an extension of the classical XY-magnet model that we analysed in [15] in another contexts. We called this model “early stage of opinion formation”. In this work, we pursue completing the full pattern of opinion formation in the community by proposing to use opinions agreed or adapted in the early stage as the starting configuration for the successive stage. This stage is analysed by considering the models introduced in [5], [6], and the analyses in the reviews discussed in the references [7], [10], [20].

2. An overview of the early stage of opinion formation
In [11] we have considered the special case of the opinion system where a conjoint interest portrays the duos of individuals. In this case, the typical quantity representing the state of the system is the sum of opinion vectors \( \mathbf{O} = \mathbf{O}_1 + \mathbf{O}_2 \), figure 1. The unhappiness, which plays the role of the energy or the Hamiltonian of the system, is proposed the scalar quantity:

\[
U_u(\mathbf{O}; \mathbf{F}) = -\frac{1}{2} (\mathbf{O}^2 - 2) - \mathbf{F} \mathbf{O} + \alpha \frac{1}{2} (\mathbf{O}^2 - 2) \mathbf{F} \mathbf{O}
\]

where \( \mathbf{F} \) is the exterior field, \( j = \pm 1 \) denotes ferromagnetic (FM) and anti-ferromagnetic (AFM) interaction, all in similarity with XY magnet vector model [15]. The novel parameter \( \alpha \) embodies the pairing between inner and outer satisfactions. The magnitude of the opinion vectors is assumed one unit, and the field \( \mathbf{F} \) is supposed to be aligned to the x-axis whereas the y-direction is named “no interest direction”. We performed similar calculation developed in [15] by implementing some additional steps in [11] due to the presence of the extra term in (1). The average opinion induced is obtained

\[
< \mathbf{O}_x > = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{\beta \alpha^2}{\sqrt{2} \alpha} I_0(\beta FO(1 - \frac{\alpha}{2}) \mathbf{O}^2 - 2) \frac{\beta \alpha^2}{\sqrt{2} \alpha} \int_0^{\infty} \frac{\beta \alpha^2}{\sqrt{2} \alpha} I_0(\beta FO(1 - \frac{\alpha}{2}) \mathbf{O}^2 - 2) d\mathbf{O}_2
\]

where \( I_{0,1} (\ldots) \) are the incomplete Bessel functions of the first kind of orders 0 or 1 respectively.

Figure 1. Opinion vectors.

The calculations were performed numerically for different sets of parameters \( J, \alpha \) and quantities \( \mathbf{F}, \mathbf{T} \). Now, by denoting the opinion states by quantities \( \{ \mathbf{O}_x, \mathbf{O} \} \), the averaged \( \mathbf{O}_x \) in the community would give the rate of the support on the issue or exterior field \( \mathbf{F} \) whereas the averaged magnitude \( \mathbf{O} \) would report the a kind of social comportment. Respective histograms give the pattern of the opinion in the
community after the first stage according to our approach. The configuration resulting from the updating process applied signifies the final or representative community attitude.

2.1. Averaged support of the community for the exterior field after the early stage

Assuming that the society is made up of a given number of the opinion entities introduced above, we have analysed the average \( x \)-component of the opinion immediately after the early stage. In upper frame of the figure 2 we have shown the opinion as function of \( \alpha \) for FM and AFM interaction respectively, keeping \( F, T \) unchanged. In this calculation, the parameter \( \alpha \) is drawn from uniform distribution in a given segment, where the physical condition \( O_x(\alpha) \leq 1 \) has been satisfied. For specific ratio \( F/T \) in the case of FM interaction, the quantity \( O_x \) has a specific dependency on parameter \( \alpha \) as shown by blue and red curves in the upper frame of figure 2. So, for FM interaction, the opinion of the community straightaway after the early stage, result in few categories or levels of the agreement. From the curves blue and red in figure 1 we distinguish 4 levels. A better analysis would be provided by histograms. We observe that For low T/F ratio the histogram appears to fit to a q-Gaussian.

The formulae used herein is

\[
(1 - (1 - q) \cdot \frac{(x-\mu)^2}{b^2})^{\frac{1}{1-q}}
\]

where we used \( x-\mu \) instead of \( x \). q-Gaussian distribution have been proposed in [16] and detailed in [21] account for the effects of correlation and other nonlinearity effects that leads to the nonstationary state. Note that for \( q \to 1 \) the q-Gaussian reduces to the classical Gaussian, whereas for \( q>>1 \) it approaches to a power law function. The finding herein suggests that in our model, strengthening of the exterior field or weakening of disturbing effects leads to a more stationary distribution of the opinions after the first stage. For a given interval \( F/T \) the opinion
would be categorised in few groups in the FM type communities. If real inner conditions embodied in parameter $\alpha$ impose boundaries, the resulting pattern shows more specific. Up here we do not go deeper in concrete sociologic significance of those conditions but qualitatively they rely with a willingness to correlate local and global attitudes. By a careful it resulted that the above histograms should include an extra ‘zero support’ category. So, formula (2) predicts that the states $[J, \alpha]$ where $1 - \frac{\alpha J}{2} [O^2 - 2] = 0$ have zero components along $F$ as seen directly by putting Bessel function values $I_1(0) = 0$ and $I_0(0) = 1$ in equation (2). However we obtained that the fraction of individuals with their opinion in such special conditions is small. We will reconsider it below. The histograms of the $x$-components after early stage can be discussed in the framework of the observations and findings in real social systems. We identify some similarities of histograms obtained herein for mixed FM+AFM society and some findings for electoral results reported in [17].

Figure 3. Illustrative histograms for the averaged support on issue F.

2.2. Properties of the agreement in the society after early stage

The behaviour of magnitude $O$ appears more complicated. First, let consider the special state of the type ‘no interest’. By putting Bessel function values $I_1(0) = 0$ and $I_0(0) = 1$ in equation (2) it results $O_x = 0$. The magnitude $O$ in those special states has the value

$$O = \arg \left(1 - \frac{\alpha J}{2} [O^2 - 2] = 0\right) \equiv \frac{\pm \sqrt{\alpha (\alpha + 1)}}{\alpha}$$

(3)

The “no interest states” correspond to specific inner condition represented by parameter $\alpha$. respectively, equation (3) produces physical magnitude $0 \leq O \leq 2$ if $1 < \alpha < 2.5$ for FM case and $-2.5 < \alpha < 1$ for AFM interaction, figure 4. Herein we do not analyse the condition that brings parameter $\alpha$ in specific
zones, so the immediate results is the fact that those specific states are characterised by high inner agreement in the pair.

Finally we analysed the histogram of \( O(i) \) which we related to the social comportment for the community. From figure 1, a state having \( O_x \) component along \( F \) can have the magnitude of the opinion in the range \([O_{\min}, O_{\max}]\) following geometric relationship between vectors. If we assume only non-negative \( y \)-values, the physical angles shown in figure 1 would fall in the segment \([0, \pi]\) and consequently, \( O_{\min} = 2^\frac{1}{2} \cdot O_x \cdot \left(\frac{1}{O_x + 1}\right)^\frac{1}{2}\). However in our first proposal and calculation in [12] we performed the integration in full range of angles in \([0, 2\pi]\), therefore, \( O_{\min} = O_x \). By selecting a random value in the interval \([O_{\min}, O_{\max}]\) for both choices, we realised that respective histogram shows significant departure from a regular distribution. Up here we summarize that the quantity embodied in \( \{O(i)\} \) is quite un-stationary under the assumption provided in this work, and in those circumstances social comportment in the end of early stage is not known.

3. The second stage of the opinion in society and clustering process

Now, consider the case in which the members of the community discuss with others and readjust their attitude as result of the influential conversation and persuasion. They have adapted already a certain support on the issue \( F \) as result of the early stage process, but not definitely. In principle, their attitudes will affect others or would be modified as result of socio-dynamic processes. This dynamics has been addressed mathematically by implementing the so called update mechanisms in specific models as described in the review [7], for example. We underline that in respective rendering techniques therein, the initializing opinion values were taken from the uniform distribution, usually in the segment \([0, 1]\) as, for example, in [5] or [6] etc. Here we propose to use the results of the “early stage” instead.

3.1. Clusters and consensus according to continuous update mechanism

We propose to use the idea introduced above in the well-known models introduced by Deffuant in [5] and Heglesman-Kreuse [6]. In Deffuant update, in each happenstance of contacts between two agents \((i,j)\) with sufficiently close opinions, opinions are alternated as to be more consensual. By simply using inherited \( O_x \) values from the early stage as initial configuration, the updating rule introduced in [5] takes the form:

\[
O_{x,i,j}(t + 1) = \begin{cases} 
O_{x,i,j}(t) + \mu \left(O_{x,i}(t) - O_{x,j}(t)\right), & |O_{x,i} - O_{x,j}| < \varepsilon \\
O_{x,i,j}(t), & \text{otherwise}
\end{cases}
\]
Parameters $\varepsilon$ and $\mu$ represent the society tolerance and the willingness to compromise. After sufficient time of iteration the process (4) would produce several final value for low tolerance $\varepsilon$. Note that for fixed parameters $\{\varepsilon, \mu\}$ the outputs of Deffuant update remain unchanged. Conversely, social behaviour is highly dynamical and complex. In the model presented hereto, the initial opinion configuration depends on the set of the parameters $\{J, F, T, \alpha\}$ involved in the calculation (2). Consequently the outcome of the process (5) depend on local parameters $\{J, \alpha\}$, exterior quantities $\{F, T\}$ adding to the original model parameters $\{\varepsilon, \mu\}$. Basically the outputs on this case would be the final opinion configuration, the time to consensus (or cluster formation) and the size of the group having a certain final opinion.

Hence, underlining some evidences would clarify the presences of novelties. By performing calculation (5), we observed that the number of opinion values in the output is always smaller than when using standard initializing opinion configuration given in [5]. In the figure 5 are shown the results for $\varepsilon=0.3$, $\mu=0.5$ for each initializing configuration respectively.

![Deffuant update](image)

**Figure 5**: Final opinion clusters resulting from Deffuant update. $T=1$, $F=1$; $\varepsilon=0.15$, $\mu=0.5$.

Next, the iteration time steps needed to reach the consensus are usually smaller in our case. It is interesting to underline that in the case of two final averaged opinion output the low level agreement takes longer finalise as seen in the figure 6.

Similar conclusions have been obtained by applying Heglesman-Kreuse model introduced in [6] and discussed in [19], [1], [7] etc. We applied the following version of the update rule for this model

$$O_{x,i}(t+1) = O_{x,i}(t) \ast \sigma + (1 - \sigma) \ast \frac{\sum_{j} N_{\varepsilon} a_{ij} O_{x,j}}{\sum_{j} N_{\varepsilon} a_{ij}}$$

In (6) $j$ indexes all $N_{\varepsilon}$ opinion that meet the condition $O_{x,i} - O_{x,j}\leq \varepsilon$, and $\sigma$ marks the level of self-confidence. Note that in original paper [6], and [19] the network topology is part of the model. However, leaving this part of the model out of the focus does not affect our conclusions.
Figure 6. Low level agreements converge slower.

It resulted that the fragmentation of the final opinion of the society predicted by our calculation is remarkably lower than the one resulting from using standard initial opinion configuration. In figure 7 are shown those results for an illustrative selection of the set \([J, F, T, \alpha]\) and keeping the original model parameters \(\{\varepsilon, \mu\}\) the same in both cases. Also, we see that the number of opinion clusters obtained by using our approach was smaller than the one resulting from the standard model for a large range of parameters \([J, F, T, \alpha]\). Next, the size of the opinion group (number of participant) revealing high support on \(F\) resulted larger than the low support group. We believe that the varieties of the outcomes for specific conditions \([J, \alpha]\) and exterior effects \([F, T]\), pronounce the capability of the proposed approach to explain complicated decision making process and social behaviour. However, evidences from socio-metric measurement or other data are needed for advocating this benefit.
4. The preferential attachment update modelling

Let consider a community of individuals seeking to establish links or relationship after they have adapted a certain level of support on the issue F form the early stage. Suppose that in this case, the dominant process is the preferential attachment mechanisms discussed thoroughly in [20] and discussed in the reviews [7] etc. Therein has been stated that the probability of establishing a link between a new coming node (i) with a network node (j) is proportional to the grade of the node (the number of node’s links $L_j$). Again, we assume that the values of opinion nodes are inherited from the early stage, which differs from the binary one used in the original model. Therefore, to implement the preferential attachment update in our case, we introduce the affinity of the nodes $a_{ij}$ which bounces the probability that two nodes (i, j) could be associated (e.g., start the conversation) for a possible linkage together. We propose it by following relation

$$a_{ij} = \begin{cases} 1 & O_x(i) - O_x(j) \leq \varepsilon \\ 0 & O_x(i) \cdot O_x(j), O_x(i) - O_x(j) > \varepsilon \end{cases}$$

Now the rule of link establishing takes the form

$$p(i; j) = a(i; j) \cdot \frac{L(ij)}{\sum_{k=1}^{N} L(k)}$$

Next, we count the link going to the node (i) or (j) according to an ad–hoc rule which gives the priority to the highest grade node to attract the other node, but at the same time it let some opportunity for less linked node to attract the other also. So instead of $L_j$ we have $L_i$ or $L_j$ in the notation $L_{i;j}$ in (8). This is just a refining step, and doesn’t affect the outcome. Neither does this additional procedure aim directed
nor undirected graph alternative. In summary, we obtained that the distribution of the size of clusters is a power law curve with exponent usually smaller than -1.8, and has fitted better a q-Gaussian with q<2 for a large set of the values of F, T. There are not disturbances from this behaviour in the queue. For comparison, we used the uniformly disturbed opinions in [0, 1] in the assessment by implementing the formula (8). In this case we observe that histogram of the data produced shows a departure from the power law and in the high links node formed a queue not fitted in the distribution, figure 8.

![Figure 8. Distribution of cluster size follows a q-Gaussian pdf](image)

Note that in [20] the respective exponent is fixed, -3. According to [16] and [21], the q-parameter in q-Gaussian distribution measures the distance from classical Gaussian and identifies the level of the non-stationary of the state. Accordingly we acknowledged various level of non-stationarity for the distribution evaluated after implementation of the formula (8) depending on the early stage parameter values. Again, this finding is in accordance with the diversity of the stationarity level of the distributions obtained in the real networks or systems.

**Conclusions**

We propose to use a two-stage model for opinion formation in the case when the community can be viewed as an ensemble of couples of individuals which seek to optimize the utility in the pair and participate a persuasion processes to finalize their social conduct. The first stage introduced recently has been based on amended XY-magnet model. In the end of the early stage, agents have adapted their support $O_x(J, F, T, \alpha) \leq 1$ to the exterior issue F. Next, the society is assumed to be the community a number of such opinion pair which are under the same {F, T} exterior effects but in different local parameter {J, \alpha}. The attitude of the community is assessed using the early stage results and analysed based on model parameters. The histogram of the opinion component $Q_x$ straightway after the early stage is found highly sensitive to the rapport F/T for FM interaction case. It has the power law shape with
negative exponent for small rapport $F/T$ and for moderate $F/T$ it approaches to a non-stationary q-Gaussian. It resulted that for some specific magnitude $O$, the vector of opinion has zero component along the issue $F$ belonging to “no interest” state. The histogram of the inner agreement $O$ does not fit sufficiently with common distribution, therefore, under our approach, respective social quantity is unstationary. More interesting findings were observed when applying some update mechanism by using the output of the early stage as initializing configuration instead of uniform values in $[0,1]$ used in standard models. The final opinion configuration calculated by Deffuant or Heglesman methods by setting values $\{O_x(t)\}$ as initial opinion configuration. We obtained that the final opinion in the society resulted to be less fragmented than when using strictly the standard models assumption. Also, the consensus is reached sooner in our approach, and we observed that the unification of the lower consensual value of the opinion takes more time steps than the higher one. Usually the higher support group size is greater than the lower one, etc. But basically, all the above properties and behaviours depend in the local parameters $[J, \alpha, \varepsilon]$, in the exterior effects $[T, F]$ beside the original model parameters, the society tolerance $\varepsilon$ and willingness to compromise $\mu$. In this sense, our approach seems to address better the multiplicity and complexity of the social attitudes and conducts. Next, we proposed to analyse the link establishment process using the opinions realised in the end of the first stage and applying a slightly modified preferential attachment mechanism. The adaption is realised by multiplying the linkage probabilities of the original model with the linking affinity that is 1 if differences of opinions $O_x$ is smaller than the society tolerance and the production of the opinions $O_{x,i} \cdot O_{x,j}$ otherwise. We obtained that the distribution of the grades of the nodes fitted to a decreasing power law function with no apparent distortion in the queue. In general, the outcomes of proposed model are typically system-dependent that may be more suitable to explain various dynamics and complex nature of social behaviour.

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