Ising model on nonorientable surfaces: 
Exact solution for the Möbius strip and the Klein bottle

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Closed-form expressions are obtained for the partition function of the Ising model on an $M \times N$ simple-quartic lattice embedded on a Möbius strip and a Klein bottle for finite $M$ and $N$. The finite-size effects at criticality are analyzed and compared with those under cylindrical and toroidal boundary conditions. Our analysis confirms that the central charge is $c = 1/2$.

I. INTRODUCTION

There has been considerable recent interest [1–3] in studying lattice models on nonorientable surfaces, both as new challenging unsolved lattice-statistical problems and as a realization and testing of predictions of the conformal field theory [4]. In a recent paper [1] we have presented the solution of dimers on the Möbius strip and Klein bottle and studied its finite-size corrections. In this paper we consider the Ising model.

The Ising model in two dimensions was first solved by Onsager in 1944 [5] who obtained the close-form expression of the partition function for a simple-quartic $M \times N$ lattice wrapped on a cylinder. The exact solution for an $M \times N$ lattice on a torus, namely, with periodic boundary condition in both directions, was obtained by Kaufman 4 years later [6]. Onsager and Kaufman used spinor analysis to derive the solutions, and the solution under the cylindrical boundary condition was rederived later by McCoy and Wu [7] using the method of dimers. As far as we know, these are the only known solutions of the two-dimensional Ising model on finite lattices. Here, using the method of dimers, we derive exact expressions for the partition function of the Ising model on finite Möbius strips and Klein bottles. As we shall see, as a consequence of the Möbius topology, the solution assumes a form which depends on whether the width of the lattice is even or odd. However, all solutions yield the same bulk free energy. We also present results of finite-size analyses for corrections to the bulk solution, and compare with those deduced under other boundary conditions. Our explicit calculations confirm that the central charge is $c = 1/2$.

II. THE $2M \times N$ MÖBIUS STRIP

To begin with, we consider a $2M \times N$ simple-quartic Ising lattice $\mathcal{L}$ embedded on a Möbius strip, where $M, N$ are integers and $2M$ is the width of the strip. The example of a lattice $\mathcal{L}$ for $2M = 4, N = 5$ is shown in Fig. 1.

While we shall consider the case of a uniform reduced interaction $K$, to facilitate considerations it is convenient to let the $N$ vertical edges located in the middle of the strip to take on a different interaction $K_1$ as shown. By setting $K_1 = 0$ the Möbius strip reduces to an $M \times 2N$ strip with a “cylindrical” boundary condition, namely, periodic in one direction and free in the other, for which the partition function has been evaluated by McCoy and Wu [7]. By setting $K_1 = \infty$ the two center rows of spins coalesce into a single row with an (additive) interaction which in this case is $2K$. These are two key elements of our consideration.

![Diagram](https://example.com/diagram.png)

**FIG. 1.** A $4 \times 5$ Möbius strip $\mathcal{L}$. Vertices labeled $A, B, C, D$ are repeated sites.

Following standard procedures [7] we write the partition function of the Ising model on $\mathcal{L}$ as

$$Z_{2M,N}^{\text{Mob}}(K, K_1) = 2^{2MN} \cosh(K)^{2(2MN-N)} \cosh(K_1)^N \times G(z, z_1),$$

where $z = \tanh K$, $z_1 = \tanh K_1$, and

$$G(z, z_1) = \sum_{\text{closed polygons}} z^n z_1^{n_1}$$

is the generating function of all closed polygonal graphs on $\mathcal{L}$ with edge weights $z$ and $z_1$. Here, $n$ is the number...
of polygon edges with weight $z$ and $n_1$ the number of edges with weight $z_1$.

The generating function $G(z, z_1)$ is a multinomial in $z$ and $z_1$ and, due to the Möbius topology, the integer $n_1$ can take on any value in $\{0, N\}$. Thus, we have

$$G(z, z_1) = \sum_{n_1=0}^{N} T_{n_1}(z) z_1^{n_1},$$

where $T_{n_1}(z)$ are polynomials in $z$ with strictly positive coefficients.

To evaluate $G(z, z_1)$, we again follow the usual procedure of mapping polygonal configurations on $L$ onto dimer configurations on a dimer lattice $L_D$ of $8MN$ sites, constructed by expanding each site of $L$ into a “city” of 4 sites [1]. The resulting $L_D$ for the $4 \times 5$ $L$ is shown in Fig. 2.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{The dimer lattice $L_D$ corresponding to the $4 \times 5$ Möbius strip.}
\end{figure}

As the deletion of all $z_1$ edges reduces the lattice to one with a cylindrical boundary condition solved in [3], we orient all edges of weights $z$ and 1 as in [3]. In addition, all $z_1$ edges are oriented in the direction as shown in Fig. 2. Then, we have the following:

**Theorem:**

*Let $A$ be the $8MN \times 8MN$ antisymmetric determinant defined by the lattice edge orientation shown in Fig. 2, and let*

$$\text{Pf}A(z, z_1) = \sqrt{\det A(z, z_1)}$$

(*4*)

*denote the Pfaffian of $A$. Then*

$$\text{Pf}A(z, z_1) = \sum_{n_1=0}^{N} \epsilon_{n_1} T_{n_1}(z) z_1^{n_1},$$

(*5*)

*where $\epsilon_{4m} = \epsilon_{4m+1} = 1$, $\epsilon_{4m+2} = \epsilon_{4m+3} = -1$ for any integer $m \geq 0$. Remark: Define

$$X_p = \sum_{m=0}^{[N/4]} T_{4m+p}(z) z^{4m+p}, \quad p = 0, 1, 2, 3,$$

(*6*)

*where $[N/4]$ is the integral part of $N/4$ so that $G(z, z_1) = X_0 + X_1 + X_2 + X_3$. It then follows from (*6*) that we have*

$$\text{Pf}A(z, \pm i z_1) = X_0 + X_2 \pm i(X_1 + X_3).$$

(*7*)

*As a consequence, we obtain*

$$G(z, z_1) = \frac{1}{2} \left[ (1 - i)\text{Pf}A(z, i z_1) + (1 + i)\text{Pf}A(z, -i z_1) \right],$$

(*8*)

*where, as evaluated in the next section, the Pfaffian is given by*

$$\text{Pf}A(z, z_1) = [z(1 - z^2)]^{MN} \times \prod_{n=1}^{N} \left[ \frac{\sinh(M+1)t(\phi_n) - c(z, z_1) \sinh M t(\phi_n)}{\sinh t(\phi_n)} \right],$$

(*9*)

*with

$$c(z, z_1) = \frac{z(1 + z^2 + 2z \cos \phi_n) + 2(-1)^n z_1 \sin \phi_n}{1 - z^2},$$

$$\cosh t(\phi) = \cosh 2K \coth 2K - \cos \phi,$$

$$\phi_n = (2n - 1)\pi/2N.$$  

(*10*)

*Here we have used the fact that $\prod_{n=1}^{N} \sinh t(\phi_n) = \prod_{n=1}^{2N} \cosh t(\phi_n)$ in the product in (*6*). Substituting these results into (*6*), we are led to the following explicit expression for the partition function,*

$$Z_{2M, N}^{\text{Mob}}(K, K) = \frac{1}{2} \left[ 2\sinh 2K \right]^{MN} \times \left[ (1 - i)F_+ + (1 + i)F_- \right],$$

(*11*)

*where

$$F_{\pm} = \prod_{n=1}^{N} \left[ e^{M t(\phi_n)} \left( \frac{e^{t(\phi_n)} - c(z, \pm i z)}{2 \sinh t(\phi_n)} \right) - e^{-M t(\phi_n)} \left( \frac{e^{-t(\phi_n)} - c(z, \pm i z)}{2 \sinh t(\phi_n)} \right) \right].$$

(*12*)

*This completes the evaluation of the Ising partition function for the $2M \times N$ Möbius strip. Note that we have $\cosh t(\phi_n) \geq 1$ so we can always take $t(\phi_n) \geq 0$. The leading contribution in (*12*) for large $M$ is therefore $\prod e^{M t(\phi_n)}$. For the $2 \times 5$ Möbius strip, for example, we find

$$\text{Pf}A(z, z_1) = 1 + z^{10} + 10z_1 z^5 - 5z_1 z^2 (1 + z^2 + z^4 + z^6) - 20z_1 z^5 + 5z_1 z^4 (1 + z^2) + 2z_1 z^5,$$

$$G(z, z_1) = 1 + z^{10} + 10z_1 z^5 + 5z_1 z^2 (1 + z^2 + z^4 + z^6) + 20z_1 z^5 + 5z_1 z^4 (1 + z^2) + 2z_1 z^5,$$
which can be verified by explicit enumerations.

We next prove the theorem.

Considered as a multinomial in \(z\) and \(z_1\), there exists a one-one correspondence between terms in the dimer generating function \(G(z, z_1)\) and (combinations of) terms in the Pfaffian. However, while all terms in \(G(z, z_1)\) are positive, terms in the Pfaffian do not necessarily possess the same sign. The crux of the matter is to find an appropriate linear combination of Pfaffians to yield the desired \(G(z, z_1)\). For this purpose it is convenient to compare an arbitrary term \(C_1\) in the Pfaffian with a standard one \(C_0\). We choose \(C_0\) to be one in which no \(z\) and \(z_1\) dimers are present.

The superposition of two dimer configurations represented by \(C_0\) and \(C_1\) produces superposition polygons. Kasteleyn has shown that the two terms will have the same sign if all superposition polygons are oriented “clockwise-odd”, namely, there is an odd number of edges oriented in the clockwise direction.

Now since all \(z\) and 1 edges of \(E_D\) are oriented as in \(C_0\), terms in the Pfaffian with no \(z_1\) edges \((n_1 = 0)\) will have the same sign as \(C_0\). To determine the sign of a term when \(z_1\) edges are present, we associate a + sign to each clockwise-odd superposition polygon and a – sign to each clockwise-even superposition polygon. Then the sign of \(C_1\) relative to \(C_0\) is the product of the signs of all superposition polygons. The following elementary facts can be readily verified:

- i) Deformations of the borders of a superposition polygon always change \(m_1\), the number of its \(z_1\) edges, by multiples of 2.
- ii) The sign of a superposition polygon is reversed under border deformations which change \(m_1\) by 2.
- iii) Superposition polygons having 0 or 1 \(z_1\) edges have a sign +.
- iv) There can be at most one superposition polygon having an odd number of \(z_1\) edges (a property unique to nonorientable surfaces).

Let \(m_1 = 4m + p\), where \(m\) is an integer and \(p = 0,1,2,3\). Because of iv), we need only to consider the presence of at most one polygon having \(p = 1\) or \(3\). It now follows from i) and iii) that \(\epsilon_{4m} = \epsilon_{4m+1} = +\), and from i), ii) and iii) that \(\epsilon_{4m+2} = \epsilon_{4m+3} = –\). This establishes the theorem.

III. EVALUATION OF THE PFaffIAN

In this section we derive the expression (13).

From the edge orientation of \(E_D\) of Fig. 2, one finds that the \(8MN \times 8MN\) antisymmetric matrix \(A\) assumes the form

\[
A(z, z_1) = A_0(z) \otimes I_{2N} + A_+ (z) \otimes J_{2N} + A_-(z) \otimes J^T_{2N} + A_1(z_1) \otimes H_{2N},
\]

where \(A_0, A_+, A_-, A_1\) are \(4M \times 4M\) matrices, \(I_{2N}\) is the \(2N \times 2N\) identity matrix, and \(J_{2N}, H_{2N}\) are the \(2N \times 2N\) matrices

\[
J_{2N} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -1 & 0 & 0 & \cdots & 0 \\ \end{pmatrix}, \quad H_{2N} = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix},
\]

In addition, one has

\[
A_0(z) = a_{0,0} \otimes I_M + a_{0,1}(z) \otimes F_M + a_{0,-1}(z) \otimes F_M^T
\]

\[
A_\pm (z) = a_{\pm1,0}(z) \otimes I_M
\]

\[
A_1(z_1) = a(z_1) \otimes G_M,
\]

where \(F_M, G_M\) are \(M \times M\) matrices

\[
F_M = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \end{pmatrix}, \quad G_M = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \end{pmatrix}.
\]

\(F_M^T\) is the transpose of \(F_M\), and

\[
a_{0,0} = \begin{pmatrix} 0 & 1 & -1 & -1 \\ -1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix},
\]

\[
a(z_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & z_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
a_{1,0}(z) = \begin{pmatrix} 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
a_{0,1}(z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
a_{-1,0}(z) = a_{T_1,0}(z),
\]

\[
a_{0,-1}(z) = a_{T,0,1}(z).
\]

We use the fact that the determinant in (13) is equal to the product of the eigenvalues of the matrix \(A\). To evaluate the latter, we note that \(J_{2N}, J^T_{2N}\) and \(H_{2N}\) mutually commute so that they can be diagonalized simultaneously. This leads to the respective eigenvalues \(e^{i\phi}, e^{-i\phi}\) and \(i(-1)^{n+1}\) and the expression
where

\[
M(z, z_1; \phi_n) = A_0(z) + \frac{A_+ e^{i\phi_n} + A_- e^{-i\phi_n}}{1 + \lambda_z} + i(-1)^n A_1(z_1)
\]

is a $4M \times 4M$ matrix. Writing out explicitly, we have

\[
A(z, z_1; \phi_n) =
\begin{pmatrix}
    B(z) & a_{0,1}(z) & 0 \\
    a_{0,-1}(z) & B(z) & a_{0,1}(z) & 0 \\
    \vdots & \ddots & \ddots & \ddots \\
    0 & a_{0,-1}(z) & B(z) & a_{0,1}(z) & 0 & a_{0,-1}(z) & C(z, z_1)
\end{pmatrix}
\]

(20)

with

\[
B(z) = \begin{pmatrix}
    0 & 1 + ze^{i\phi_n} & -1 & -1 \\
    -(1 + ze^{-i\phi_n}) & 0 & 1 & -1 \\
    1 & -1 & 0 & 1 \\
    1 & 1 & -1 & 0
\end{pmatrix}
\]

(22)

The evaluation of $\det A_M(z, z_1; \phi_n)$ can be carried out by using a recursion procedures introduced in [10] for a self-dual Ising model. Specifically, let $B_M = \det A_M(z, z_1; \phi_n)$ and $D_M$ the determinant of the matrix $A_M(z, z_1; \phi_n)$ with the fourth row and fourth column removed. Then by expanding the determinants one finds the recursion relation (which is the same as that in [10])

\[
\begin{pmatrix}
    B_M \\
    D_M
\end{pmatrix} = \begin{pmatrix}
    a_{11} & a_{12} \\
    a_{21} & a_{22}
\end{pmatrix} \begin{pmatrix}
    B_{M-1} \\
    D_{M-1}
\end{pmatrix}, \quad M \geq 2,
\]

(23)

where

\[
\begin{align*}
    a_{11} &= 1 + z^2 - 2z \cos \phi_n, \\
    a_{12} &= 2izz_1 \sin \phi_n, \\
    a_{21} &= 2iz \sin \phi_n, \\
    a_{22} &= z^2(1 + z^2 + 2z \cos \phi_n),
\end{align*}
\]

and the initial condition (which is different from [10])

\[
\begin{align*}
    B_1 &= B_1(z, z_1) \\
    &= 1 - 2z \cos \phi_n + z^2 - 2(-1)^n z z_1 \sin \phi_n, \\
    D_1 &= D_1(z, z_1) \\
    &= 2iz \sin \phi_n - i(-1)^n z_1 (1 + 2z \cos \phi_n + z^2).
\end{align*}
\]

(24)

(25)

This leads to the solution

\[
\begin{align*}
    B_M &= B_1 \lambda_+^M - \lambda_-^M \left( a_{22} B_1 - a_{12} D_1 \right) \frac{\lambda_+^{M-1} - \lambda_-^{M-1}}{\lambda_+ - \lambda_-}, \\
    D_M &= D_1 \lambda_+^M - \lambda_-^M \left( a_{11} D_1 - a_{21} B_1 \right) \frac{\lambda_+^{M-1} - \lambda_-^{M-1}}{\lambda_+ - \lambda_-},
\end{align*}
\]

(26)

where $\lambda_{\pm} = (1 - z^2)e^{\pm i\phi_n}$ are the eigenvalues of the $2 \times 2$ matrix in (23). After some algebraic manipulation, this yields the expression [1] quoted in the preceding section.

\section{IV. THE $(2M - 1) \times N$ MÖBIUS STRIP}

We consider a $(2M - 1) \times N$ Möbius strip in this section. In order to make use of results of the preceding sections, we start from the $2M \times N$ strip of Sec. 2, and let spins in the two center rows of the strip (the $M$th and $(M + 1)$th rows) having interactions $K_0 = K/2$. The example of a $4 \times 5$ lattice with these interactions is shown in Fig. 3. Then, by taking $K_1 = \infty$ ($z_1 = 1$) as described in Sec. 1, this lattice reduces to the desired $(2M - 1) \times N$ Möbius strip of a uniform interaction $K$.

Following this procedure, we have

\[
Z^{\text{Mob}}_{2M-1,N}(K) = 2^{(2M-1)N} \left( \cosh K \right)^{4(M-1)N} \times \cosh^{2N}(K/2) G(z, z_0, z_1) \bigg|_{z_1 = 1},
\]

(27)

where $z_0 = \tanh(K/2)$, and $G(z, z_0, z_1)$ is the generating function of closed polygons on the $2M \times N$ Möbius net with edge weights as described in the above.

The generating function $G(z, z_0, z_1)$ can be evaluated as in the previous sections. In place of [6], [19] and [22], we now have

\[
\text{Pf}(z, z_0, z_1) = \sqrt{\det A(z, z_0, z_1)} \prod_{n=1}^{2N} \sqrt{\det A_M(z, z_0, z_1; \phi_n)}
\]

(28)

with

\[
A_M(z, z_0, z_1; \phi_n) =
\begin{pmatrix}
    B(z) & a_{0,1}(z) & 0 \\
    a_{0,-1}(z) & B(z) & a_{0,1}(z) & 0 \\
    \vdots & \ddots & \ddots & \ddots \\
    0 & a_{0,-1}(z) & B(z) & a_{0,1}(z) & 0 & a_{0,-1}(z) & C(z_0, z_1)
\end{pmatrix}
\]

(29)
Then (8) becomes
\[ G(z, z_0, z_1) = \frac{1}{2} \left[ (1 - i) \text{Pf} A(z, z_0, iz_1) + (1 + i) \text{Pf} A(z, z_0, -iz_1) \right]. \] (30)

The evaluation of \( \det A_M(z, z_0, z_1; \phi_n) \) can again be done recursively. Define as before \( B_M = \det A_M(z, z_0, z_1; \phi_n) \) and \( D_M \) the determinant of \( A_M \) with the fourth row and column removed, one obtains again the recursion relations (23) and arrives at precisely the same solution (26), but now with a different initial condition
\[ B_1 = B_1(z_0, z_1), \quad D_1 = D_1(z_0, z_1), \] (31)
where the functions \( B_1 \) and \( D_1 \) are defined in (23). After some algebra, this leads to
\[ \text{Pf} A(z, z_0, z_1) = [z(1 - z^2)]^{(M-1)N} \times \prod_{n=1}^{N} \left[ c_1 \sinh M t(\phi_n) - c_2 \sinh (M - 1)t(\phi_n) \right], \] (32)
where
\[ c_1 = \frac{2z_0}{z} \left( 1 - z \left[ \cos \phi_n + (-1)^n z_1 \sin \phi_n \right] \right), \]
\[ c_2 = 2z_0 \left( 1 + z \left[ \cos \phi_n + (-1)^n z_1 \sin \phi_n \right] \right). \] (33)

The substitution of (32) into (30) and (27) now completes the evaluation of the partition function for a \((2M-1) \times N\) Möbius strip.

V. THE KLEIN BOTTLE

The Ising model on a Klein bottle can be considered similarly. We consider first a \(2M \times N\) lattice \( \mathcal{L} \), constructed by connecting the upper and lower edges of the Möbius strip of Fig. 1 in a periodic fashion with \( N \) extra vertical edges. As in the case of the Möbius strip, it is convenient to let the extra edges have interactions \( K_2 \). The solution for a uniform interaction \( K \) is obtained at the end by setting \( K_1 = K_2 = K \).

The Ising partition function for the Klein bottle now assumes the form
\[ Z_{2M,N}^{\text{Klin}}(K, K_1, K_2) = 2^{2MN} (\cosh K)^{4MN - 2N} \times (\cosh K_1 \cosh K_2)^N G^{\text{Klin}}(z, z_1, z_2), \] (34)
where
\[ G^{\text{Klin}}(z, z_1, z_2) = \sum_{\text{closed polygons}} z^n z_1^{n_1} z_2^{n_2}. \] (35)

This generates all closed polygons on the \(2M \times N\) lattice \( \mathcal{L} \) with edge weights \( z = \tanh K, z_1 = \tanh K_1, \) and \( z_2 = \tanh K_2 \). The desired partition function is given by
\[ Z_{2M,N}^{\text{Klin}}(K, K, K) = 2^{2MN} (\cosh K)^{4MN} G^{\text{Klin}}(z, z, z). \] (36)

Again, it is necessary to first write \( G^{\text{Klin}}(z, z_1, z_2) \) as a multinomial in \( z, z_1, z_2 \) in the form of
\[ G^{\text{Klin}}(z, z_1, z_2) = \sum_{m,n=0}^{N} T_{m,n}(z) z_1^m z_2^n, \] (37)
where \( T_{m,n}(z) \) are polynomials in \( z \) with strictly positive coefficients.

The evaluation of \( G^{\text{Klin}}(z, z_1, z_2) \) parallels that of \( G(z, z_1) \) for the Möbius strip. One first maps the lattice \( \mathcal{L} \) into a dimer lattice \( \mathcal{L}_D \) by expanding each site into a city of 4 sites as shown in Fig. 2. Orient all \( K \) and \( K_1 \) edges of \( \mathcal{L}_D \) as shown, and orient all \( K_2 \) edges in the same (downward) direction as the \( K_1 \) edges. Then this defines an \( 8MN \times 8MN \) antisymmetric matrix obtained by adding an extra term to \( A(z, z_1) \) given by (14), namely,
\[ A^{\text{Klin}}(z, z_1, z_2) = A(z, z_1) + b(z_2) \otimes G'_M \otimes H_{2N}. \] (38)

Here,
\[ b(z_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -z_2 \end{pmatrix}, \]
\[ G'_M = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \] (39)

Then, in place of theorem (3), we now have
\[ \text{Pf} A^{\text{Klin}}(z, z_1, z_2) = \sum_{m,n=0}^{N} \epsilon_m \epsilon_n T_{m,n}(z) z_1^m z_2^n, \] (40)
from which one obtains in a similar manner the result
\[ G^{\text{Klin}}(z, z_1, z_2) = \frac{1}{2} \left[ \text{Pf} A^{\text{Klin}}(z, iz_1, -iz_2) + \text{Pf} A^{\text{Klin}}(z, -iz_1, iz_2) - i \text{Pf} A^{\text{Klin}}(z, iz_1, iz_2) + i \text{Pf} A^{\text{Klin}}(z, -iz_1, -iz_2) \right]. \] (41)
\[ \text{Pf} A^{\text{Kin}}(z, z_1, z_2) = \sqrt{\det A^{\text{Kin}}(z, z_1, z_2)} = \prod_{n=1}^{2N} \sqrt{\text{det} A^{\text{Kin}}_M(z, z_1, z_2; \phi_n)}, \]  
\[ \text{where} \quad A^{\text{Kin}}_M(z, z_1, z_2; \phi_n) = A_M(z, z_1; \phi_n) + i(-1)^n b(z_2) \otimes G'_M. \]

Now we expand \text{det} A^{\text{Kin}}_M in \( z_2 \). Since setting \( z_2 = 0 \) the determinant is precisely \( B_M \) and the term linear in \( z_2 \), the \( (4,4) \) element of the determinant, is by definition \( D_M \), one obtains
\[ \text{det} A^{\text{Kin}}_M(z, z_1, z_2; \phi_n) = B_M + i(-1)^n z_2 D_M, \quad M \geq 2, \]

where \( B_M \) and \( D_M \) have been computed in [27]. This leads to
\[ \text{Pf} A^{\text{Kin}}(z, z_1, z_2) = \left(1 + \frac{z_1 z_2}{z^2}\right)^N \left[z(1 - z^2)\right]^{MN} \prod_{n=1}^{N} \left[\frac{\sinh(Mt + 1) - c(z, z_1, z_2) \sinh M t}{\sinh t}\right], \]

where
\[ c(z, z_1, z_2) = \frac{1}{z(1 - z^2)(z^2 + z_1 z_2)} \left[(1 + z^2)(z^4 + z_1 z_2) + 2 z_2 (z^4 - z_1 z_2) \cos \phi_n + 2(-1)^n (z_1 + z_2) z^3 \sin \phi_n\right]. \]

Setting \( z_1 = z_2 = z \) in (44) and using (45), we obtain after some algebra
\[ G^{\text{Kin}}(z, z, z) = \left[z(1 - z^2)\right]^{MN} \prod_{n=1}^{N} \left(2 \cosh M t(\phi_n) + \text{Im} \sum_{n=1}^{N} \left(\frac{\sinh M t(\phi_n)}{\sinh t(\phi_n)} - D(\phi_n)\right)\right], \]

where
\[ D(\phi_n) = \frac{1}{z(1 - z^2)} \left[(1 - z^4) - 2 z(1 + z^2) \cos \phi_n - 4 t (-1)^n z^2 \sin \phi_n\right], \]

and \( \text{Im} \) denotes the imaginary part. The substitution of (47) into (44) now completes the evaluation of the partition function for a \( 2M \times N \) Klein bottle.

For the \( 2 \times 2 \) Klein bottle, for example, we find
\[ \text{Pf} A^{\text{Kin}}(z, z_1, z_2) = 1 + z^4 + 4(z_1 + z_2) z^2 - 2(z_1^2 + z_2^2) z^2 + 2 z_1 z_2 (1 + z^2)^2 - 4 z_1 z_2 (z_1 + z_2) z^2 - z_1^2 z_2^2 (1 + z^4), \]
\[ G^{\text{Kin}}(z, z_1, z_2) = 1 + z^4 + 4(z_1 + z_2) z^2 + 2(z_1^2 + z_2^2) z^2 + 2 z_1 z_2 (1 + z^2)^2 + 4 z_1 z_2 (z_1 + z_2) z^2 + z_1^2 z_2^2 (1 + z^4), \]

which can be verified by explicit enumerations.

For a \( (2M - 1) \times N \) Klein bottle we can proceed as before by first considering a \( 2M \times N \) Klein bottle with interactions \( K, K_1, K_2 \) and, within the center two rows, interactions \( K_0 = K/2 \) as shown in Fig. 3. This is followed by taking \( K_1 \to \infty \) and \( K_2 = K \). Thus, in place of (44), we have
\[ Z_{2M-1,N}^{\text{Kin}}(K) = 2(2M-1)^N (\cosh K)^{(4M-3)N} \times \cosh^2 (K/2) G^{\text{Kin}}(z, z_0, 1, z), \]

where \( z_0 = \tanh(K/2) \) and \( G(z, z_0, z_1, z_2) \) generates polygonal configurations on the \( 2M \times N \) lattice. Then, as in the above, we find
\[ G^{\text{Kin}}(z, z_0, z_1, z_2) = \frac{1}{2} \left[ \text{Pf} A^{\text{Kin}}(z, z_0, i z_1, -i z_2) + \text{Pf} A^{\text{Kin}}(z, z_0, -i z_1, i z_2) \right. \]
\[ - i \text{Pf} A^{\text{Kin}}(z, z_0, i z_1, i z_2) + i \text{Pf} A^{\text{Kin}}(z, z_0, -i z_1, -i z_2)\],

where \( \text{Pf} A^{\text{Kin}}(z, z_0, z_1, z_2) \) is found to be given by the right-hand side of (52), but with
\[ c_1 = (1 + z_0^2)(1 - z_1 z_2) - 2 z_0 (1 + z_1 z_2) \cos \phi_n - 2(-1)^n (z_1 + z_2) z_0 \sin \phi_n, \]
\[ c_2 = \frac{1}{z(1 - z^2)} \left[z^2 + z_1 z_2\right] \left[(1 - z_0) z^2 + (z - z_0)^2\right] \]
\[ + 2(z - z_0)(1 - z_0) z z_2 \cos \phi_n + (-1)^n (z^2 - z_1 z_2) \sin \phi_n\]

expressions which are valid for arbitrary \( z, z_0, z_1, \) and \( z_2 \). For \( z_0 = \tanh(K/2) \), the case we are considering, (52) reduces to
\[ c_1 = \frac{2z_0}{z} \left[1 - z_1 z_2 - z(1 + z_1 z_2) \cos \phi_n + (-1)^n (z_1 + z_2) \sin \phi_n\right], \]
\[ c_2 = \frac{2z_0}{z^2} \left[z^2 + z_1 z_2 + z(z^2 - z_1 z_2) \cos \phi_n + (-1)^n (z_1^2 z_2 + z^2) \sin \phi_n\right], \]

which reduces further to (53) after setting \( z_2 = 0 \). The explicit expression for the partition function is now obtained by substituting (51) into (50).

VI. THE BULK LIMIT AND FINITE-SIZE CORRECTIONS

In the thermodynamic limit, our solutions of the Ising partition function give rise to a bulk free energy
\[ f_{\text{bulk}}(K) = \lim_{M,N \to \infty} \frac{1}{2MN} \ln Z(K) \]  

identical to that of the Onsager solution \[^3\]. Here, \( Z(K) \) 
is any one of the 4 partition functions. Indeed, using the 
solution \[^11\) for the \( 2M \times N \) Möbius strip, for example, 
one obtains

\[
\begin{align*}
    f_{\text{bulk}}(K) &= C(K) + \lim_{N \to \infty} (2N)^{-1} \sum_{n=1}^{N} t(\phi_n) \\
    &= C(K) + \frac{1}{2\pi} \int_{0}^{\pi} d\phi \ t(\phi) \\
    &= C(K) + \frac{1}{2\pi^2} \int_{0}^{\pi} d\phi \int_{0}^{\pi} d\theta \\
    &\quad \ln \left[ 2 \cosh 2K \cot 2K - 2(\cos \theta + \cos \phi) \right],
\end{align*}
\]

where \( C(K) = \ln(2 \sinh 2K) / 2 \). This expression is the 
Onsager solution (steps leading to the last line of (55) 
can be found in \[^11\) \). It is well-known that \( f_{\text{bulk}}(K) \) 
is singular at the critical point \( \sinh 2K_c = 1 \) or \( 2K_c = 
\ln(\sqrt{2} + 1) \).

For large \( M \) and \( N \), one can use the Euler-MacLaurin 
summation formula to evaluate corrections to the bulk 
free energy, an analysis first carried out by Ferdinand and 
Fisher \[^12\) for the Kaufman solution of the Ising model 
on a torus. Generally, for large \( M \) and \( N \), we expect to have

\[
\ln Z_{2M,N}(K) = 2MN f_{\text{bulk}}(K) + N c_1(\xi, K) \\
+ 2M c_2(\xi, K) + c_3(\xi, K) + \cdots
\]

where \( \xi = N/2M \) is the aspect ratio of the lattice. For 
the purpose of comparing with the conformal field theory 
\[^4\), it is of particular interest to analyze corrections at the 
critical point. Following \[^13\) as well as similar analyses 
for dimer systems \[^1,13\), we have carried out such analyses 
for our solutions as well as for the solution of the 
Ising model under cylindrical boundary conditions \[^14\).

For the \( 2M \times N \) Möbius strip, for example, one starts 
with the explicit expression \[^11\) of the partition function, 
and uses the Euler-MacLaurin formula to evaluate 
corrections to the bulk free energy. The analysis is lengthy, 
even at the critical point \( K_c \). We shall give details elsewhere 
and quote here only the results:

\[
\begin{align*}
    c_1(\xi, K_c) &\equiv c_1^{\text{Mob}} = I - K_c = -0.087 618 \ldots, \\
    c_2(\xi, K_c) &= 0, \\
    c_3(\xi, K_c) &= -\frac{1}{2} \ln 2 + \frac{1}{12} \ln \left[ \frac{2\vartheta_2^3(0|\xi)}{\vartheta_2(0|\xi)\vartheta_4(0|\xi)} \right] \\
    &\quad + \frac{1}{2} \ln \left[ 1 + \frac{\vartheta_2(0|\xi/2) - \vartheta_4(0|\xi/2)}{2\vartheta_3(0|\xi)} \right],
\end{align*}
\]

where

\[
I = \frac{1}{2\pi} \int_{0}^{\pi} \ln \left( \sqrt{2}\sin \phi + \sqrt{1+\sin^2 \phi} \right) d\phi = 0.353 068 \ldots
\]

and \( \vartheta_i(u|\tau) \), \( i = 2, 3, 4 \), are the Jacobi theta functions \[^4\)
\[
\begin{align*}
    \vartheta_2(u|\tau) &= \sum_{n=1}^{\infty} q^{(n-\frac{1}{2})^2} \cos(2n-1)u, \\
    \vartheta_3(u|\tau) &= 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos 2nu, \\
    \vartheta_4(u|\tau) &= 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2nu,
\end{align*}
\]

with \( q = e^{i\pi \tau} \). For the \( 2M \times N \) Klein bottle, we find

\[
\begin{align*}
    c_1(\xi, K_c) &= 0, \\
    c_2(\xi, K_c) &= 0, \\
    c_3(\xi, K_c) &= \frac{1}{6} \ln \left[ \frac{2\vartheta_2^3(0|2\xi)}{\vartheta_2(0|2\xi)\vartheta_4(0|2\xi)} \right] \\
    &\quad + \ln \left[ 1 + \frac{\vartheta_2(0|2\xi)}{2\vartheta_3(0|2\xi)} \right].
\end{align*}
\]

If one takes the limit of \( N \to \infty \) \( (M \to \infty) \) while 
keeping \( M \) (\( N \) finite, one obtains

\[
\begin{align*}
    \lim_{N \to \infty} \frac{1}{N} \ln Z_{2M,N}(K_c) &= 2M f_{\text{bulk}}(K_c) + c_1 \\
    &\quad + \Delta_1/2M + O(1/M^2), \\
    \lim_{M \to \infty} \frac{1}{2M} \ln Z_{2M,N}(K_c) &= N f_{\text{bulk}}(K_c) + c_2 \\
    &\quad + \Delta_2/N + O(1/N^2),
\end{align*}
\]

where \( c_1, c_2, \Delta_1, \Delta_2 \) are constants. We list our results in 
the table below. Also listed are values for the Ising model 
with toroidal boundary conditions taken from \[^12\), and 
values for the cylindrical boundary conditions computed 
using the solution of \[^4\). Our values of \( \Delta_1 \) and \( \Delta_2 \) imply 
a central charge

\[
c = 1/2
\]

for the Ising model. This is consistent with the conformal field theory prediction \[^4\).

| Möbius Klein | Cylindrical Toroidal |
|--------------|-------------------|
| \( c_1 \)   | \( c_1^{\text{Mob}} \) | \( c_1^{\text{Mob}} \) |
| \( c_2 \)   | 0                 | 0                 |
| \( \Delta_1 \) | \( \pi/48 \) | \( \pi/12 \) |
| \( \Delta_2 \) | \( \pi/48 \) | \( \pi/12 \) |
VII. SUMMARY

We have solved and obtained close-form expressions for the partition function of an Ising model on finite Möbius strips and Klein bottles with a uniform interaction $K$. The solution assumes different forms depending on whether the width of the lattice is even or odd. For a $2M \times N$ Möbius strip, where $2M$ is its width, the partition function $Z_{2M,N}^{\text{Mob}}(K)$ is given by (11) with $F_\pm$ given by (12). For a $(2M-1) \times N$ Möbius strip we employ a trick by first considering a $2M \times N$ lattice and then “fusing” it into the desired lattice by coalescing two rows of spins. The resulting partition function $Z_{2M-1,N}^{\text{Mob}}(K)$ is given by (27) in which the generating function $G(z, z_0, z_1)$ is computed using (51). All solutions yield the same Onsager bulk free energy (55).

We have also presented results of finite-size analyses of our solutions as well as those of the Ising model under cylindrical and toroidal boundary conditions at criticality. The analyses yield a central charge $c = 1/2$ in agreement with the conformal field prediction [4].

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