ITERATIONS OF CURVATURE IMAGES

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ABSTRACT. We study the iterations of a class of curvature image operators \( \Lambda^p \) introduced by the author in (J. Funct. Anal. 271 (2016) 2133–2165). The fixed points of these operators are the solutions of the \( L_p \) Minkowski problems with the positive continuous prescribed data \( \varphi \). One of our results states that if \( p \in (-n, 1) \) and \( \varphi \) is even, or if \( p \in (-n, -n+1] \), then the iterations of these operators applied to suitable convex bodies sequentially converge in the Hausdorff distance to fixed points.

1. Introduction

The setting here is \( n \)-dimensional Euclidean space. Let \( \varphi \in C(S^{n-1}) \) be a positive continuous function defined on the unit sphere. Suppose either \( \varphi \) is even (i.e., it takes the same value at antipodal points) and \( p \in (-n, 1) \), or \( p \in (-n, -n+1] \). Using an iteration method, we show that there exists a convex body \( K \) with support function \( h_K \) and curvature function \( f_K \) such that

\[
\varphi h_K^{1-p} f_K = \text{const}.
\]

While the existence of solutions to (1.1) has been known in this range of \( p \) since the work of Chou–Wang [11], we use a notion of generalized curvature image to add a novel existence method to the literature on the \( L_p \) Brunn-Minkowski theory.

Let us briefly recall the origin and the historical context of (1.1). For any \( x \) on the boundary of a convex body \( K \), \( \nu_K(x) \) is the set of all unit exterior normal vectors at \( u \). The surface area measure of \( K \), \( S_K \), is a Borel measure on the unit sphere defined by

\[
S_K(\omega) = \mathcal{H}^n(\nu_K^{-1}(\omega)) \quad \text{for all Borel sets } \omega \text{ of } S^{n-1}.
\]

Here, \( \mathcal{H}^n \) denotes the \( n \)-dimensional Hausdorff measure. If \( K \) has a positive continuous curvature function, then \( dS_K = f_K d\sigma \), where \( \sigma \) is the spherical Lebesgue measure.

The classical Minkowski problem is one of the corner stones of the Brunn-Minkowski theory. It asks what are the necessary and sufficient conditions on a Borel measure \( \mu \) on \( S^{n-1} \) in order to be the surface area measure of a convex body. The complete solution to this problem was
found by Minkowski, Aleksandrov and Fenchel and Jessen (see, e.g., Schneider [30]): A Borel measure \( \mu \) whose support is not contained in a closed hemisphere is the surface area measure of a convex body if and only if

\[
\int_{\mathbb{S}^{n-1}} u d\mu(u) = 0.
\]

Moreover, the solution is unique up to translations.

The \( L_p \) Minkowski asks what are the necessary and sufficient conditions on a Borel measure \( \mu \) on \( \mathbb{S}^{n-1} \), such that there exists a convex body \( K \) with support function \( h_K \), so that

\[
h_K^{1-p} dS_K = \gamma d\mu \quad \text{for some constant } \gamma > 0.
\]

This problem for \( p > 1 \) was put forward by Lutwak [25] almost a century after Minkowski’s original work and stems from the \( L_p \) linear combination of convex bodies. See [5, 6, 8–11, 19, 26, 30–32] regarding the \( L_p \) Minkowski problem and Lutwak et al. [27] for an application.

To motivate our iteration scheme, let us briefly recall a few observations from Lutwak [23]. Suppose \( K \) has its Santaló point at the origin. Then by Minkowski’s existence theorem (see, e.g., [7, pp. 60–67]), there exists a convex body \( \Lambda K \), uniquely determined up to translations, whose curvature function is given by

\[
f_{\Lambda K} = \frac{V(K)}{V(K^*)} \frac{1}{h_K^{n+1}}.
\]

Here, \( K^* \) is the polar body and \( V(\cdot) \) is the \( n \)-dimensional Lebesgue measure. We always choose \( \Lambda K \) such that its Santaló point is at the origin. The curvature image operator \( \Lambda \) was introduced by Petty [20]. See [30, Section 10.5] for the importance of the curvature image in affine differential geometry.

Write \( \Omega(K) \) for the affine surface area of \( K \) (Definition 3.4 with \( \varphi \equiv 1, p = -n \)). By a straightforward calculation,

\[
\Omega(\Lambda K)^{n+1} = n^{n+1} V(K)^n V(K^*).
\]

On the other hand, for any convex body \( L \) with the origin in its interior by the Hölder inequality, we have

\[
\Omega(L)^{n+1} \leq n^{n+1} V(L)^n V(L^*).
\]

Hence, using this inequality for \( L = K \) and \( L = \Lambda K \), we see

\[
\Omega(\Lambda K) \geq \Omega(K)
\]

\[
V(K) V(K^*) \leq \left( \frac{V(\Lambda K)}{V(K)} \right)^{n-1} V(\Lambda K) V((\Lambda K)^*).
\]
By Minkowski’s mixed volume inequality, we have
\[ V(\Lambda K) \leq V(K). \]
Moreover, using the affine isoperimetric inequality,
\[ V(\Lambda K)^{n-1} \geq \frac{\Omega(\Lambda K)^{n+1}}{n^{n+1}V(B)^2} \geq \frac{\Omega(K)^{n+1}}{n^{n+1}V(B)^2}, \]
where \( B \) denotes the unit ball. Therefore, we arrive at
\[ \left( \frac{\Omega(K)^{n+1}}{n^{n+1}V(B)^2} \right)^{\frac{1}{n-1}} \leq V(\Lambda K) \leq V(K). \]
Let us put \( \Lambda^i K := \Lambda \cdots \Lambda K \). By induction, we obtain
\[ V(\Lambda^{i-1} K) V((\Lambda^{i-1} K)^*) \leq \left( \frac{V(\Lambda^i K)}{V(\Lambda^{i-1} K)} \right)^{n-1} V(\Lambda^i K) V((\Lambda^i K)^*), \]
\[ \left( \frac{\Omega(K)^{n+1}}{n^{n+1}V(B)^2} \right)^{\frac{1}{n-1}} \leq V(\Lambda^i K) \leq V(K). \]
To sum up these observations, we have seen the curvature image under the operator \( \Lambda \) strictly increases (unless it is applied to an origin-centered ellipsoid; see Marini–De Philippis [22] regarding the fact that the only solutions of \( \Lambda K = K \) are origin-centered ellipsoids) the volume product functional, while \( \{ V(\Lambda^i K) \}_{i} \) is uniformly bounded above and below away from zero.

The previous observations motivate us to seek a curvature image operator \( \Lambda^\varphi_p \) (see Definition 3.2) that satisfies the following three rules.

1. The fixed points, \( \Lambda^\varphi_p L = L \), are solutions of (1.1).
2. The curvature image under \( \Lambda^\varphi_p \) strictly increases a "suitable" functional, unless \( \Lambda^\varphi_p \) is applied to a solution of (1.1).
3. There are uniform lower and upper bounds on the volume after applying any number of iteration.

Put \( (\Lambda^\varphi_p)^i K := \Lambda^\varphi_p \cdots \Lambda^\varphi_p K \). When \( \varphi \equiv 1 \), we use \( \Lambda_p \) in place of \( \Lambda^\varphi_p \).

**Theorem 1.1.** The following statements hold:

1. suppose either
   - \(-n < p < 1, \varphi \in C(S^{n-1}) \) is positive and even, and \( K \) is origin-symmetric, or
\( -n < p \leq -n + 1, \varphi \in C(S^{n-1}) \) is positive, \( K \) contains the origin in its interior and

\[
\int_{S^{n-1}} u \frac{1}{(\varphi h_{K}^{1-p})(u)} d\sigma = o.
\]

Then a subsequence of iterations \( \{(\Lambda^{p}_{i}K)_{i}\} \), converges in the Hausdorff distance, as \( i \to \infty \), to a convex body \( L \) such that

\[
\varphi h_{L}^{1-p} f_{L} = \text{const}.
\]

(2) If \( -n < p < 1 \), and \( K \) contains the origin in its interior and

\[
\int_{S^{n-1}} uh_{K}^{p-1}(u)d\sigma = o,
\]

then \( \{\Lambda^{p}_{i}K\}_{i} \) converges in the Hausdorff distance, as \( i \to \infty \), to an origin-centered ball.

(3) If \( p = -n \) and \( K \) has its Santaló point at the origin, then there exists a sequence of volume-preserving transformations \( \ell_{i} \), such that \( \{\ell_{i}A^{i}K\}_{i} \) converges in the Hausdorff distance, as \( i \to \infty \), to an origin-centered ball.

Remark 1.2. Each convex body after a suitable translation satisfies the required integral condition in the theorem (see, e.g., [15, Lemma 3.1]).

Iterations methods in convex geometry were previously applied in [12, 28] and as smoothing tools in [16, 17] to prove local uniqueness of fixed points of a certain class of operators. Also to deduce the asymptotic behavior of a class of curvature flows in [13, 15, 18] and to prove a stability version of the Blaschke–Santaló inequality in the plane [14] we used some properties of the curvature image operators. We mention that the unique convex body of maximal affine perimeter contained in a given two-dimensional convex body is (up to translations) a curvature image body (see Bárány [2, 3]). Moreover, Schneider [29] proved that in any dimension a curvature image body uniquely possess the maximal affine surface area among all convex bodies contained in it.

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2. Background and notation

A compact convex set with non-empty interior is called a convex body. The set of convex bodies is denoted by $K$. Write $K_o, K_e$, respectively, for the set of convex bodies containing the origin in their interiors and the origin-symmetric convex bodies. We write $C^+(\mathbb{S}^{n-1})$ for the set of positive continuous functions and $C^+_e(\mathbb{S}^{n-1})$ for the set of positive continuous even functions on the unit sphere.

The support function of a convex body $K$ is defined as

$$h_K(u) = \max_{x \in K} \langle x, u \rangle.$$  

For a convex body $K$ with the origin $o$ in its interior, the polar body $K^*$ is defined by

$$K^* = \{y : \langle x, y \rangle \leq 1 \ \forall x \in K\}.$$  

For $x \in \text{int } K$, we set $K^x = (K - x)^*$. The Santaló point of $K$, denoted by $s$, is the unique point in $\text{int } K$ such that

$$V(K^s) \leq V(K^x) \ \forall x \in \text{int } K.$$  

Moreover, the Blaschke–Santaló inequality states that

$$V(K)V(K^s) \leq V(B)^2.$$  

with equality if and only if $K$ is an ellipsoid.

Let $K, L$ be two convex bodies and $0 < a < \infty$. The Minkowski sum $K + aL$ is defined by $h_{K+aL} = h_K + ah_L$ and the mixed volume of $K, L$ is defined by

$$V_1(K, L) = \frac{1}{n} \lim_{a \to 0} \frac{V(K + aL) - V(K)}{a}.$$  

Corresponding to each $K$, there is a unique Borel measure $S_K$ on the unit sphere such that

$$V_1(K, L) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_L dS_K \text{ for any convex body } L.$$  

If the boundary of $K$, $\partial K$, is $C^2$-smooth and strictly convex, then $S_K$ is absolutely continuous with respect to spherical Lebesgue measure $\sigma$ and $dS_K/d\sigma$ is the reciprocal Gauss curvature.

The Minkowski mixed volume inequality states that

$$V_1(K, L)^n \geq V(K)^n V(L),$$  

and equality holds if and only if $K$ and $L$ are homothetic.

We say $K$ has a positive continuous curvature function $f_K$ if

$$V_1(K, L) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_L f_K d\sigma \text{ for any convex body } L.$$
Write $\mathcal{F}$ for set of convex bodies with positive continuous curvature functions, and put
\[ \mathcal{F}_o = \mathcal{K}_o \cap \mathcal{F}, \quad \mathcal{F}_e = \mathcal{K}_e \cap \mathcal{F}. \]

### 3. Curvature image operators

**Assumption 3.1.** Suppose one of the following cases occurs.

1. $-n < p \neq 1 < \infty$, $\varphi \in C^+_c(S^{n-1})$ and $K \in \mathcal{K}_o$.
2. $-n < p \leq -n + 1$, $\varphi \in C^+_c(S^{n-1})$, $K \in \mathcal{K}_o$ and
   \[ \int_{S^{n-1}} \frac{u}{\left(\varphi h^{-p}_K\right)(u)} d\sigma = o. \]
3. $-n \leq p \neq 1 < \infty$, $K \in \mathcal{K}_o$ and
   \[ \int_{S^{n-1}} uh^{p-1}_K(u)d\sigma = o. \]

**Definition 3.2.** Under the Assumption 3.1, the curvature image $\Lambda_{\varphi}^p K$ of $K$ is defined as the unique convex body whose curvature function is
\[ f_{\Lambda_{\varphi}^p K} = \frac{V(K)}{1/n} \frac{h^{p-1}_K}{\varphi} \]
and its support function satisfies
\[ \int_{S^{n-1}} \frac{u}{\left(\varphi h^{-p}_{\Lambda_{\varphi}^p K}\right)(u)} d\sigma = o. \]

**Remark 3.3.** By Minkowski’s existence theorem and [15, Lemma 3.1], there exists a unique convex body that satisfies (3.1) and (3.2). The integral condition (3.2) for $p = -n$ and $\varphi \equiv 1$ says the curvature image has its Santaló point at the origin.

In view of $V_1(\Lambda_{\varphi}^p K, K) = V(K)$ and Minkowski’s mixed volume inequality, we have
\[ V(K) \geq V(\Lambda_{\varphi}^p K). \]
Moreover, equality holds if and only if $\Lambda_{\varphi}^p K = K$.

**Definition 3.4.** Suppose $\varphi \in C^+_c(S^{n-1})$. For $K \in \mathcal{K}_o$, we define
\[ \mathcal{A}_{\varphi}^p(K) = \begin{cases} V(K) \left(\int_{S^{n-1}} \frac{h_K}{\varphi} d\sigma\right)^{-\frac{n}{p}}, & 0 \neq p \in [-n, \infty), \\ V(K) \exp\left(\frac{\int_{S^{n-1}} \log h_K d\sigma}{\frac{1}{n} \int_{S^{n-1}} \frac{1}{\varphi} d\sigma}\right), & p = 0. \end{cases} \]
Let $K \in \mathcal{F}$. Define
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• for \( p \in [-n, \infty) \setminus \{0, 1\} \):

\[
\mathcal{B}_p^{\phi}(K) = V(K)^{1-n} \left( \int_{S^{n-1}} \varphi^{-p} f_K^{p-1} d\sigma \right)^{\frac{n(p-1)}{p}},
\]

• for \( p = 0 \):

\[
\mathcal{B}_0^{\phi}(K) = V(K)^{1-n} \exp \left( \int_{S^{n-1}} \log(\varphi f_K) d\theta \right) \left( \int_{S^{n-1}} \frac{1}{\varphi} d\sigma \right)^n,
\]

where \( d\theta = \frac{1}{\int_{S^{n-1}} \frac{1}{\varphi} d\sigma} d\sigma \).

For \( K \in \mathcal{F} \), define

\[
\Omega_p^{\phi}(K) = \begin{cases} 
\int_{S^{n-1}} \varphi^{-p} f_K^{p-1} d\sigma, & p \in [-n, \infty) \setminus \{0, 1\}, \\
\exp \left( \int_{S^{n-1}} \frac{1}{\varphi} \log(\varphi f_K) d\sigma \right), & p = 0.
\end{cases}
\]

A straightforward calculation shows that

\[
B_p^{\phi}(\Lambda_p^{\phi} K) = n^n \left( \frac{V(K)}{V(\Lambda_p^{\phi} K)} \right)^{n-1} A_p^{\phi}(\Lambda_p^{\phi} K).
\]

For \( L \in \mathcal{F} \) and \( x \in \text{int} \, L \), by the Hölder and Jensen inequalities

\[
\begin{cases}
B_p^{\phi}(L) \leq n^n A_p^{\phi}(L - x), & p \in [-n, 1) \\
B_p^{\phi}(L) \geq n^n A_p^{\phi}(L - x), & p > 1.
\end{cases}
\]

From now onward, we only focus on the case \( p \in (-n, 1) \) and establish the desired properties mentioned in the introduction. For \( p > 1 \), the second inequality in (3.5) is in the wrong direction and hence \( \Lambda_p^{\phi} \) does not exhibit the same behavior as \( \Lambda \) does.

**Lemma 3.5.** Suppose Assumption 3.1 holds and \( p < 1 \). We have the following.

1. \( A_p^{\phi}(K) \leq \left( \frac{V(\Lambda_p^{\phi} K)}{V(K)} \right)^{n-1} A_p^{\phi}(\Lambda_p^{\phi} K) \leq A_p^{\phi}(\Lambda_p^{\phi} K) \).
2. If \( p \neq 0 \), then

\[
\Omega_p^{\phi}(K)^{\frac{n(p-1)}{p(n-1)}} \leq \Omega_p^{\phi}(\Lambda_p^{\phi} K)^{\frac{n(p-1)}{p(n-1)}}.
\]

If \( p = 0 \), then

\[
\Omega_0^{\phi}(K) \leq \Omega_0^{\phi}(\Lambda_0^{\phi} K).
\]

3. If \( p \neq 0 \), then

\[
c_p^{\phi} \Omega_p^{\phi}(K)^{\frac{n(p-1)}{p(n-1)}} \leq V((\Lambda_p^{\phi})^i K) \leq V(K).
\]
If \( p = 0 \), then
\[
c_0^p \Omega^p_0(K) \leq V((\Lambda^p_0)^i K) \leq V(K).
\]

Proof. Inequalities of (1) and (2) follow from (3.3), (3.4), and (3.5) applied to \( L = K \) and \( L = \Lambda^p_0 K \).

In view of [1, Theorem 9.2], for each \( L \in \mathcal{K} \), there exists \( e_p \in \text{int} L \) such that
\[
\mathcal{A}^1_p(L - e_p) \leq \mathcal{A}^1_p(B).
\]

Therefore, for \( p \neq 0 \), due to (3.5) we see
\[
\mathcal{B}^p_0(L) \leq c_{p,\varphi} \quad \text{for any convex body } L \in \mathcal{F}.
\]

In particular, for \( p \neq 0 \), owing to (2), this last inequality yields
\[
c_{p,\varphi} V(\Lambda^p_0 K) \geq \Omega^p_0(\Lambda^p_0 K)^{\frac{n(p-1)}{n}} \geq \Omega^p_0(K)^{\frac{n(p-1)}{n}}.
\]

Now, (3) follows by induction.

The proof for the case \( p = 0 \) is similar and it follows from the following inequality. By (3.5), the Jensen and Blaschke–Santaló inequality, for any convex body \( L \in \mathcal{F} \) we have
\[
\mathcal{B}^p_0(L) \leq n^n \mathcal{A}^p_0(L - s) \leq n^n V(L) \frac{\int_{S^{n-1}} \frac{1}{\varphi k^p_1} d\sigma}{\int_{S^{n-1}} \frac{1}{\varphi} d\sigma} \leq c_0^p,
\]
where \( s \) is the Santaló point of \( L \). \( \square \)

4. Passing to a Limit

In this section, we give the proof of Theorem 1.1. First we consider the case \( p \neq -n \). By Lemma 3.5, the operator \( \Lambda^p_0 \) satisfies the three principals mentioned in the introduction. Therefore, by [15, Theorem 7.4], there are constants \( a, b \), such that
\[
a \leq h((\Lambda^p_0)^i)_K \leq b \quad \forall i.
\]

Due to the monotonicity of \( \mathcal{A}^p_0 \) under \( \Lambda^p_0 \),
\[
\lim_{i \to \infty} \mathcal{A}^p_0((\Lambda^p_0)^i K) \text{ exists and is positive}
\]
Thus, in view of (3.3) and
\[
\mathcal{A}^p_0((\Lambda^p_0)^i K) \leq \left( \frac{V((\Lambda^p_0)^{i+1} K)}{V((\Lambda^p_0)^i K)} \right)^{n-1} \mathcal{A}^p_0((\Lambda^p_0)^{i+1} K),
\]
we arrive at
\[
\lim_{i \to \infty} \frac{V((\Lambda^p_0)^{i+1} K)}{V((\Lambda^p_0)^i K)} = 1.
\]
By (4.1) and the Blaschke selection theorem, for a subsequence \( \{i_j\} \):

\[
\lim_{j \to \infty} (\Lambda_p^\varphi)^{i_j} K = L \in \mathcal{K}_o,
\]

From the continuity of \( \Lambda_p^\varphi \) (cf. [15, Theorem 7.6]), it follows that

\[
\lim_{j \to \infty} (\Lambda_p^\varphi)^{i_j+1} K = \Lambda_p^\varphi L.
\]

Consequently, we must have

\[
V(L) = \lim_{j \to \infty} V((\Lambda_p^\varphi)^{i_j} K) = \lim_{j \to \infty} V((\Lambda_p^\varphi)^{i_j+1} K) = V(\Lambda_p^\varphi L).
\]

Now, the equality case of (3.3) implies that \( \Lambda_p^\varphi L = L \) and hence,

\[
\varphi h_L^{1-p} f_L = \frac{V(L)}{\frac{1}{n} \int_{S_{n-1}} \frac{V_L}{\varphi} d\sigma}.
\]

Regarding the case \(-n < p < 1\) and \( \varphi \equiv 1 \), first note that due to the result of [4] the limiting shapes are origin-centered balls. To show that in fact they are the same ball, note that due to the monotonicity of the volume under \( \Lambda_p \), the limits have the same volume.

Finally, regarding the third claim of Theorem 1.1, \( p = -n \), recall that \( \{V(\Lambda^i K)\} \) is uniformly bounded above and below. For each \( i \), by Petty [21] (see also [33, Theorem 5.5.14]), we can find \( \ell_i \in SL(n) \) such that \( \ell_i \Lambda^i K \) is in a minimal position, that is, its surface area is minimal among its volume-preserving affine transformations. Therefore, for a subsequence \( \{i_j\} \), we have

\[
\lim_{j \to \infty} \ell_{i_j} \Lambda^{i_j} K = L \in \mathcal{K}_o,
\]

and \( L \) has its Santaló point at the origin (in fact, this follows from \( s(\ell_i \Lambda^i K) = \ell_i s(\Lambda^i K) = o \) and that \( s \) is a continuous map with respect to the Hausdorff distance). In particular, by [24, (7.12)] and the continuity of the curvature image operator we obtain

\[
\lim_{j \to \infty} \ell_{i_j} \Lambda^{i_j+1} K = \lim_{j \to \infty} \Lambda(\ell_{i_j} \Lambda^{i_j} K) = \Lambda L.
\]

Meanwhile by the monotonicity of the volume product and its upper bound due to the Blaschke-Santaló inequality, we have

\[
V(L) = \lim_{j \to \infty} V(\ell_{i_j} \Lambda^{i_j} K) = \lim_{j \to \infty} V(\ell_{i_j} \Lambda^{i_j+1} K) = V(\Lambda L).
\]

Therefore, \( \Lambda L = L \). By [22], \( L \) is an origin-centered ellipsoid. Since this ellipsoid is in a minimal position, it has to be a ball. The limit is independent of the subsequences as in the case (2) of the theorem. The proof of the theorem is finished.
5. Questions

(1) Let $p > 1$ and $\varphi \in C^+_e(S^{n−1})$. It would be of interest to find a curvature operator whose iterations applied to any $K \in \mathcal{K}_e$ converge to the solution of the $L_p$ Minkowski problem with the prescribed even data $\varphi$.

(2) Is the limit in Theorem 1.1 independent of the subsequence?

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