Gauge theory of collective modes

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Abstract. The classical theory of Riemann ellipsoids is formulated naturally as a gauge theory based on a principal $G$-bundle $P$. The structure group $G = SO(3)$ is the vorticity group, and the bundle $P = GL_+(3, \mathbb{R})$ is the connected component of the general linear group. The base manifold is the space of positive-definite real $3 \times 3$ symmetric matrices, identified geometrically with the space of inertia ellipsoids. Non-holonomic constraints determine connections on the bundle. In particular, the trivial connection corresponds to rigid body motion, the natural Riemannian connection to irrotational flow, and the invariant connection to the falling cat. The curvature form determines the fluid’s field tensor which is an analogue of the familiar Faraday tensor. Associated $G$-bundles and the covariant derivative yield new quantum geometrical collective models that are a natural generalization of the Bohr model. These new geometric structures formulate the collective model as a Yang-Mills gauge theory.

1. Introduction
The general collective motion algebra $\mathfrak{gcm}(3)$ is a fifteen dimensional, noncompact subalgebra of the twenty-one dimensional symplectic Lie algebra. This subalgebra is a semi-direct sum of the nine-dimensional algebra of the general linear group $GL_+(3, \mathbb{R})$ and the six dimensional abelian ideal $\mathbb{R}^6$ generated by the mass quadrupole and monopole tensors. The ideal may be identified with the symmetric $3 \times 3$ real matrices, considered as a vector space.

The irreducible unitary representations of $\mathfrak{gcm}(3)$ are indexed by a non-negative integer $C$. The Bohr–Mottelson quantum model of nuclear rotational and vibrational states is the particular irreducible representation corresponding to $C = 0$. The primary advantage to the $\mathfrak{gcm}(3)$ characterization of the Bohr collective model is that it provides, without any approximation, the model’s microscopic foundation. This is achieved by viewing the algebra in three different ways.

First $\mathfrak{gcm}(3)$ is an abstract algebra of $6 \times 6$ matrices

$$\mathfrak{gcm}(3) = \left\{ S = \begin{pmatrix} X & -U \\ 0 & -X^t \end{pmatrix} \mid X, U \in M_3(\mathbb{R}), U^t = U \right\},$$

where $M_n(\mathbb{R})$ denotes the set of real $n \times n$ matrices and a right superscript $X^t$ indicates matrix transposition of $X$.

Secondly, a representation $\sigma$ of the matrix algebra $\mathfrak{gcm}(3)$ provides the algebra’s physical interpretation in quantum many-body physics. Let $(\hat{x}_{\alpha j}, \hat{p}_{\alpha j})$, $1 \leq j \leq 3$, denote the dimensionless Cartesian components of the position and momentum hermitian operators for particle $\alpha$ in a finite system of particles. They obey the canonical commutation relation,
\[ [\hat{x}_{\alpha j}, \hat{p}_{\beta k}] = i \delta_{\alpha \beta} \delta_{jk}. \] The set of hermitian one-body operators,

\[
\hat{Q}_{jk} = \sum_{\alpha} \hat{x}_{\alpha j} \hat{x}_{\alpha k} \\
\hat{N}_{jk} = \sum_{\alpha} \left( \hat{x}_{\alpha j} \hat{p}_{\alpha k} - \frac{i}{2} \delta_{jk} \right),
\]

close under commutation to form a Lie algebra. For \( A \) identical fermions these one-body operators act on the Hilbert space \( \mathcal{H} \) that is the antisymmetrized tensor product of \( A \)-copies of the single-particle space. For each matrix \( S \) in \( \mathfrak{gcm}(3) \), define the operator

\[
\sigma(S) = i \sum_{jk} X_{jk} \hat{N}_{jk} + \frac{i}{2} \sum_{jk} U_{jk} \hat{Q}_{jk}.
\]

When \( S \) is in \( \mathfrak{gcm}(3) \), the operator \( \sigma(S) \) is a skew-adjoint one-body operator on \( \mathcal{H} \). This set of operators is a reducible representation of \( \mathfrak{gcm}(3) \): \([\sigma(S_1), \sigma(S_2)] = \sigma([S_1, S_2])\).

The general collective motion group \( \text{GCM}(3) \) is a connected Lie matrix group given by exponentiation of \( \mathfrak{gcm}(3) \),

\[
\text{GCM}(3) = \left\{ g = \begin{pmatrix} x & -x \cdot U \\ 0 & (x^t)^{-1} \end{pmatrix} \mid x \in \text{GL}_+(3, \mathbb{R}), U \in M_3(\mathbb{R}), U^t = U \right\}.
\]

The connected general linear group \( \text{GL}_+(3, \mathbb{R}) \) consists of the \( 3 \times 3 \) real matrices with positive determinant; it is a subgroup of \( \text{GCM}(3) \) given by all \( g = \text{diag}(x, x^t x^{-1}) \in \text{GCM}(3) \) for \( x \in \text{GL}_+(3, \mathbb{R}) \). The representation \( \sigma \) determines a unitary reducible representation of the Lie group \( \text{GCM}(3) \) on the quantum many-fermion Hilbert space \( \mathcal{H} \).

Thirdly, the Mackey inducing construction determines all the irreducible unitary representations of \( \text{GCM}(3) \). For each nonnegative integer \( C \), the Hilbert space of a \( \text{GCM}(3) \) irrep \( \pi \) is

\[
H^C = \left\{ \Psi : \text{GL}_+(3, \mathbb{R}) \to \mathbb{C}^{2^C+1} \mid \begin{array}{c} (i) \Psi(g R) = \Psi(g) D^C(R), \text{for } g \in \text{GL}_+(3, \mathbb{R}), R \in \text{SO}(3) \\ (ii) \int_{\text{GL}_+(3, \mathbb{R})} \| \Psi(g) \|^2 \, d\nu(g) < \infty \end{array} \right\},
\]

where \( D^C \) denotes the \( 2^C+1 \) dimensional unitary irreducible representation of \( \text{SO}(3) \), and \( d\nu(g) \) is the invariant measure on \( \text{GL}_+(3, \mathbb{R}) \). The action of \( \text{GL}_+(3, \mathbb{R}) \) on \( H^C \) is given by

\[
(\pi(x) \Psi)(g) = \Psi(x^{-1} g), \text{for } x, g \in \text{GL}_+(3, \mathbb{R}).
\]

The \( Q \) tensor acts as a multiplication operator,

\[
(\pi(Q_{ij}) \Psi)(g) = (g^t g)_{ij} \Psi(g), \text{for } g \in \text{GL}_+(3, \mathbb{R}).
\]

Every unitary irreducible representation of \( \text{GCM}(3) \) is equivalent to one of these representations on \( H^C \) for some non-negative integer \( C \). Two irreducible representations defined by two different integers \( C \) are inequivalent.

Each \( \text{GCM}(3) \) irrep \( C \) determines a collective model. The \( C = 0 \) irrep is indistinguishable from the Bohr model, including both quadrupole and monopole collective motion.
2. What GCM(3) irreps are possible physical models?

The answer to this question is the solution to a well-defined group theoretical problem: What GCM(3) irreducible unitary representations $C$ are part of the reduction of the GCM(3) reducible representation $\sigma$ on many-fermion Hilbert space $H$?

In a study of the unitary $\text{sp}(3,\mathbb{R})$ discrete series representations and their $\text{gcm}(3)$ decomposition, Rowe and Repka [4] proved a reciprocity theorem that precisely answers the question. Consider a symplectic unitary irrep in the reduction of $H$. This irrep is labeled by the quantum numbers, $N_0(\lambda_0,\mu_0)$, of a “starting” $\text{U}(3)$ irrep. The symplectic irrep is a reducible representation of the subgroup $\text{GCM}(3) \subset \text{Sp}(3,\mathbb{R})$. The reciprocity theorem asserts that the number of times an infinite-dimensional GCM(3) unitary irrep, labeled by $C$, occurs in the reduction of one infinite-dimensional symplectic irrep equals the multiplicity of the angular momentum in the reduction of the finite-dimensional SU(3) irrep $(\lambda_0,\mu_0)$. For example, the leading symplectic irrep for Neon 20 has $(\lambda_0,\mu_0) = (8,0)$; consequently, its GCM(3) decomposition is a direct sum of the $C=0,2,4,6,8$ irreps.

Thus, except for systems of only a few fermions, every GCM(3) irrep is physically allowed. Whether or not a particular GCM(3) irrep $C$ is discovered in its pure form for some isotope depends on the Hamiltonian. In general, a nuclear wave function will be a linear combination of many GCM(3) irreps with various values of $C$.

3. Riemann ellipsoids

In Fall 1987, I visited Jerry Draayer at LSU during a sabbatical year. We mainly collaborated on upgrading a symplectic model code to include symplectic tensor operators, especially the hexadecapole E4 operator, and I thoroughly enjoyed the intellectually stimulating environment of Jerry’s group and department during my Baton Rouge visit. Norman Lebovitz, an astrophysicist at Chicago, gave a LSU departmental seminar about the Riemann ellipsoid model. Lebovitz had collaborated with Chandrasekhar in the 1960s about the foundations of this classical model and its applications to rotating galaxies and stars [5, 6, 7, 8]. During the seminar it became apparent to me that the Riemann ellipsoid model is the classical version of the GCM(3) quantum model [9]. The GCM(3) matrix group is the same, but it acts in the Riemann model as a group of canonical transformations of many-particle phase space.

Several key insights emerge from the group theoretic perspective on the Riemann model. In the GCM(3) set-up, the Riemann phase space is a co-adjoint orbit of GCM(3). The Riemann dynamical equations are a finite-dimensional Lax system, a considerable mathematical simplification of the Riemann equations [10]. By the way, the classical Lax system can only be derived and proven equivalent to the Riemann dynamical equations by using a simple expression from nuclear theory for the collective kinetic energy that Riemann and Chandrasekhar had overlooked.

An invariant of the Lax system is the GCM(3) Casimir which is a constant function on a co-adjoint orbit. This invariant is the Riemann fluid’s Kelvin circulation and its value is $C$, a nonnegative real number. Only the integral values of $C$ determine a phase space that satisfies the Bohr-Sommerfeld conditions for quantization. The quantized Hilbert space associated with the co-adjoint orbit is the same space $H^C$ carrying a GCM(3) irrep. Thus the physical interpretation of the constant $C$ as the Kelvin circulation comes from the classical Riemann model.

The Riemann model has another interesting aspect in which the circulation and vorticity are related to the angular momentum. The classical Riemann model introduces a real parameter $f$ that determines this relationship in a so-called S-type ellipsoid for which the axis of rotation coincides with a principal axis. This raises a question: What is the fundamental significance of this relationship?
4. Differential geometry of the Riemann model

The relationship between the vortex velocity and the angular velocity determines a differential geometric connection on the group GCM(3). To achieve this understanding, the Lie group GCM(3) is viewed as a principal G bundle [11].

An ellipsoid is defined by a positive-definite real symmetric $3 \times 3$ matrix $q$; the points $X = (X_1, X_2, X_3)$ of its surface in Euclidean space are solutions to the quadratic equation

$$\sum_{ij} a_{ij}^{-1} X_i X_j = 1.$$  \hspace{1cm} (8)

In particular, when $q = \text{diag}(a_1^2, a_2^2, a_3^2)$, the surface is an ellipsoid with major axis half-lengths equal to $a_1, a_2, a_3$ and the ellipsoid's principal axes are aligned with the Cartesian axes. The space of all ellipsoidal surfaces is identified with the six dimensional manifold of such matrices $q$:

$$Q = \{ q \in M_3(\mathbb{R}) \mid q^t = q, q > 0 \}.$$  \hspace{1cm} (9)

The configuration space for Riemann ellipsoids is the general linear group $P = \{ \xi \in M_3(\mathbb{R}) \mid \det \xi > 0 \} = GL_+(3, \mathbb{R})$, because only linear motions are allowed.

The general linear group transforms an ellipsoid into another ellipsoid that is deformed and rotated. If $q \in Q$ and $\xi \in P$, then this spatial transformation corresponds to $q \mapsto \xi q \xi^t$. In particular, the unit sphere with $q = I$ is changed into the ellipsoid with $q = \xi \xi^t$. Thus, the natural mapping $\pi$ from the group $P$ onto the ellipsoidal space $Q$

$$\pi : P \rightarrow Q \hspace{1cm} \xi \mapsto q = \xi \xi^t$$  \hspace{1cm} (11)

is surjective (onto).

The group $P$ also acts on itself. The group elements can be multiplied in two ways, on the left or on the right. Left multiplication of elements $\xi \in P$ by $r \in SO(3)$

$$\xi \mapsto L_r \xi = r \xi$$  \hspace{1cm} (12)

corresponds to physical rotations $r$ in three-dimensional Euclidean space. In contrast, right multiplication of $\xi \in P$ by $g \in G = SO(3)$,

$$\xi \mapsto R_g \xi = \xi g^{-1},$$  \hspace{1cm} (13)

corresponds to vortex motion. Although the groups $G$ and $SO(3)$ are mathematically isomorphic, their distinct actions on $P$ imply different physical interpretations.

With respect to left multiplication by the rotation group element $r \in SO(3)$, the ellipsoid with inertia tensor $q = \pi(\xi) = \xi \xi^t$ is transformed into the rotated ellipsoid with inertia tensor $\pi(L_r \xi) = r \xi r^t = rqr^t$. But right multiplication by the vorticity group element $g \in G$ leaves the inertia ellipsoid invariant, $\pi(R_g \xi) = \xi g^{-1} g \xi^t = q$, since $g^{-1} = g^t$. This invariance is expressed more elegantly by the composition of mappings

$$\pi \circ R_g = \pi, \text{ for all } g \in G.$$  \hspace{1cm} (14)

Hence $G$ is a group of motions internal to the ellipsoidal surface.

These properties show that $P$ is a principal G bundle where $G$ is the structure or gauge group, $Q$ is the base manifold, $\pi$ is the projection of the bundle onto the base manifold, and $\pi$
is right G-invariant. In Figure a time-dependent curve $\gamma(t)$ is drawn in the base manifold $Q$. As the time evolves, the curve describes the evolution of the orientation and deformation of a rotating ellipsoid. Above the base manifold are the fibres $G$ of the bundle, indicated as straight lines in the figure. A point $\xi$ of the bundle depends both on the deformation $q = \pi(\xi)$ and the internal vorticity measured by a structure group element.

As the ellipsoid is “cranked,” in the usual nuclear physics sense, the system responds with internal vortex dynamics. This response determines a unique lift of the curve $\gamma$ in the base manifold $Q$ to a curve $\tilde{\gamma}$ in the bundle $P$. In differential geometry, the curve $\tilde{\gamma}$ is called the horizontal lift of $\gamma$ and it is mathematically equivalent to specifying a differential connection for the bundle [12].

5. Connections on the $G$ principal bundle

A tangent vector to $\gamma$ in the base manifold determines the angular velocity $\vec{\omega}$, and the horizontal lift determines the vortex velocity $\vec{\lambda}$. In concrete terms, the ratio of the components of these velocities define the Christoffel symbols for the connection:

$$\Gamma_i^a(q) = \delta_i^a \left( \frac{\lambda_i}{\omega_i} \right).$$

(15)
For an ellipsoid with principal axes half-lengths \(a_1, a_2, a_3\), the special rotational modes for rigid, irrotational and the “falling cat” define the three connections

\[
\Gamma_k^i = \begin{cases} 
0 & \text{rigid} \\
\frac{2a_i a_j}{(a_i^2 + a_j^2)} & \text{irrotational} \\
\frac{(a_i^2 + a_j^2)/(2a_i a_j)} & \text{falling cat},
\end{cases}
\]

where \(i, j, k\) are cyclic. A “falling cat” has internal vortex motion, yet no change in the angular momentum. The irrotational flow connection is associated with the Riemannian geometry on the bundle; the falling cat is the G-invariant connection. For an S-type ellipsoid the connection is a linear interpolation between the special rigid and irrotational flow connections.

What equations determine the connection? The answer must be independent of the coordinate system, and the natural geometrical object for the equations is the bundle curvature. One equation that the curvature satisfies is the Bianchi identity. A second equation relates the exterior derivative of the curvature tensor to the non-holonomic internal forces of the system.

The analogy with classical theory of electricity and magnetism is helpful. This is an Abelian gauge theory with space-time as the base manifold and \(U(1)\) as the structure group. The curvature form is the Faraday tensor and the Bianchi identity is equivalent to two of Maxwell’s equations, \(\nabla \cdot B = 0\) and \(\nabla \times E + \partial B/\partial t = 0\). The other two of Maxwell’s equations involve the current and are equivalent to a second independent equation for the Faraday tensor.

6. Covariant derivatives and the quantum model

The quantization of a gauge theory for a principal G bundle is achieved by constructing its associated bundles [11]. The domain of the wave functions is the base manifold and the wave functions are vector-valued. The vector values are in an irreducible representation of the structure group. For each irreducible representation of the structure group \(G \simeq SO(3)\), an inequivalent associated bundle and a different quantization can be defined. An associated bundle for the Riemann model defines the space \(H^C\) of vector-valued wave functions.

A new ingredient of the bundle theory is that the classical connection determines the quantum covariant derivative. The Laplacian defined by the covariant derivative yields the collective kinetic energy. For irrotational flow, \(C = 0\), and the Riemannian connection, the Laplacian is the familiar Bohr collective kinetic energy operator. But, when \(C \neq 0\) and the connection is not Riemannian, a significant generalization of the Bohr collective model is attained.

Associated bundles and the covariant derivative are the essential structure of Yang-Mills theory. Thus the Bohr model is an example of a Yang-Mills theory. Indeed, the relationship is rather close because the Riemann bundle theory has the structure group SO(3) while the Yang-Mills electroweak theory uses the structure group U(2).

In a forthcoming article, the associated bundle theory for the Riemann model will be presented in some detail.

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