Lagrangians with linear velocities within Riemann-Liouville fractional derivatives

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Abstract

Lagrangians linear in velocities were analyzed using the fractional calculus and the Euler-Lagrange equations were derived. Two examples were investigated in details, the explicit solutions of Euler-Lagrange equations were obtained and the recovery of the classical results was discussed.

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1 Introduction

The mathematical idea of fractional derivatives, which goes back to the seventeenth century, has represented the subject of interest for various mathematicians \cite{1, 2, 3}. The fractional calculus, which means the calculus of derivatives and integrals of any arbitrary real or complex order is gaining a considerable importance in various branches of science, engineering and finance \cite{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}.

As it was observed during the past three decades, the fractional differential calculus describes more accurately the physical systems \cite{4, 5, 6, 12}. Many applications of fractional calculus amount to replacing the time derivative in an evolution equation with a derivative of fractional order. This is not merely a phenomenological procedure providing an additional fit parameter. One of the problems encountered in this field is what kind of fractional derivatives will replace the integer derivative for a given problem \cite{1, 2, 3, 13}. Depending on the specified physical situation different authors have applied different derivatives \cite{6, 9}.

Nonconservative Lagrangian and Hamiltonian mechanics were investigated by Riewe within fractional calculus \cite{14, 15}. Besides, Lagrangian and Hamiltonian fractional sequential mechanics, the models with symmetric fractional derivative were studied in \cite{16, 17} and the properties of fractional differential forms were introduced \cite{18, 19}.

Recently, an extension of the simplest fractional variational problem and the fractional variational problem of Lagrange was obtained \cite{20}. A natural and interesting generalization of Agrawal’s approach \cite{20} is to apply the fractional

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calculus to the constrained systems [21, 22]. In the present work, we apply the
concept of fractional calculus to the Lagrangian with linear velocities.

The plan of this paper is as follows:
In Sec. 2 the Riemann-Liouville (RL) fractional derivatives were briefly presented.
In Sec. 3 the fractional Lagrangians with linear velocities were constructed and
their corresponded Euler-Lagrange were analyzed. Sec. 4 was devoted to our
conclusions.

2 Riemann - Liouville fractional derivatives

In this section the definitions of the right and left RL derivatives as well as their
basic properties are briefly presented.

RL fractional derivatives [3, 20] are defined as follows

\[
aD_t^n f(t) = \frac{1}{\Gamma(n - \alpha)} \left( \frac{d}{dt} \right)^n \int_a^t (t - \tau)^{n-\alpha-1} f(\tau) d\tau ,
\]

(1)

\[
tD_b^n f(t) = \frac{1}{\Gamma(n - \alpha)} \left( -\frac{d}{dt} \right)^n \int_t^b (\tau - t)^{n-\alpha-1} f(\tau) d\tau ,
\]

(2)

where the order \( \alpha \) fulfills \( n - 1 \leq \alpha < n \) and \( \Gamma \) represents the Euler’s gamma
function. The first derivative is called the RL left fractional derivative and in (2)
the expression of the RL right fractional is presented. If \( \alpha \) is any integer, the
relations to the usual derivative are obtained as follows

\[
aD_t^n f(t) = \left( \frac{d}{dt} \right)^\alpha , \quad tD_b^n f(t) = \left( -\frac{d}{dt} \right)^\alpha .
\]

(3)

Under the assumptions that \( f(t) \) is continuous and \( p \geq q \geq 0 \), the most
general property of RL fractional derivatives can be written as

\[
aD_t^p \left( aD_t^{-q} f(t) \right) = aD_t^{p-q} f(t).
\]

(4)

For \( p > 0 \) and \( t > a \) we obtain

\[
aD_t^p \left( aD_t^{-p} f(t) \right) = f(t),
\]

(5)

which means that the RL fractional differentiation operator is a left inverse to
the RL fractional integration operator of the same order. The relation (5) is
called the fundamental property of the RL fractional derivative. In addition, the
fractional derivative of a constant is not zero and the RL fractional derivative of
the power function \((t - a)^\nu\) is given by

\[
aD_t^p (t - a)^\nu = \frac{\Gamma(\nu + 1)}{\Gamma(-p + \nu + 1)} (t - a)^{\nu-p},
\]

(6)
where $\nu > -1$. The normal derivatives $\frac{d^\nu}{dt^\nu}$ and $\alpha D_t^\nu$ commute only if $f^{(j)}(a) = 0$, $j = 0, 1, \ldots, n - 1$ is fulfilled and two RL fractional derivative operators $\alpha D_t^\nu$ and $\alpha D_t^{\nu-1}$ commute only if $\left[\alpha D_t^{\nu-1} f(t)\right]_{t=a} = 0$, $j = 1, \ldots, m$ (7) and $\left[\alpha D_t^{\nu-1} f(t)\right]_{t=a} = 0$, $j = 1, \ldots, n$ (8).

The above properties of RL fractional derivatives lead us to the conclusion that there are many substantial differences from the usual derivatives and therefore the solutions of the differential fractional equations contain more information than the classical ones.

### 3 Fractional Euler-Lagrange equations for Lagrangians with linear velocities

Let $J[q^1, \cdots, q^n]$ be a functional of the form

$$
\int_a^b L\left(t, q^1, \cdots, q^n, \alpha D_t^\alpha q^1, \cdots, \alpha D_t^\alpha q^n, \beta D_b^\beta q^1, \cdots, \beta D_b^\beta q^n\right) dt
$$

defined on the set of functions $q^i(t)$, $i = 1, \cdots, n$ which have continuous left RL fractional derivative of order $\alpha$ and right RL fractional derivative of order $\beta$ in $[a, b]$ and satisfy the boundary conditions $q^i(a) = q_a^i$ and $q^i(b) = q_b^i$. A necessary condition for $J[q^1, \cdots, q^n]$ to admit an extremum for given functions $q^i(t)$, $i = 1, \cdots, n$ is that $q^i(t)$ satisfy Euler-Lagrange equations [20]

$$
\frac{\partial L}{\partial q^j} + \alpha D_t^\alpha \frac{\partial L}{\partial \alpha D_t^\alpha q^j} + \beta D_b^\beta \frac{\partial L}{\partial \beta D_b^\beta q^j} = 0, \quad j = 1, \cdots, n.
$$

In the following we consider the Lagrangian with linear velocities

$$
L = a_j \left(q^i\right) \dot{q}^i - V\left(q^i\right),
$$

where $a_j(q^i)$ and $V(q^i)$ are functions of their arguments.

The first step is to construct the corresponding fractional generalization of the Lagrangian given by (11). The fractional Lagrangian is not unique, in other words there are several possibilities to replace the time derivative with fractional ones. The requirement is to obtain the same Lagrangian expression if the order $\alpha$ is 1. Having in mind the above considerations, for $0 < \alpha \leq 1$, we propose two fractional Lagrangians. The first one is as follows
\[ L' = a_j \left( q^i \right) a D_\alpha^q q^j - V \left( q^i \right). \] 

From (10) and (12), the corresponding Euler-Lagrange equations emerge as

\[ \frac{\partial a_j (q^i)}{\partial q^k} a D_\alpha^q q^j + a D_\alpha^q a_k (q^i) - \frac{\partial V (q^i)}{\partial q^k} = 0. \] 

The second Lagrangian is given by

\[ L' = -a_j \left( q^i \right) t D_\alpha^q q^j - V \left( q^i \right). \]

Using (10) and (14) the corresponding Euler-Lagrange equations become

\[ \frac{\partial a_j (q^i)}{\partial q^k} t D_\alpha^q q^j + a D_\alpha^q a_k (q^i) + \frac{\partial V (q^i)}{\partial q^k} = 0. \]

### 3.1 Examples

**A.** To illustrate our approach, let us consider the following Lagrangian

\[ L = \dot{q}^1 q^2 - \dot{q}^2 q^1 - (q^1 - q^2) q^3 \]

which is a gauge invariant [23]. In this case we proposed the corresponding fractional Lagrangian to be as

\[ L' = a D_\alpha^q q^2 - \left( a D_\alpha^q q^2 \right) q^1 - (q^1 - q^2) q^3. \]

Using (13), the Euler-Lagrange equations corresponding to (17) become

\[ q^1 = q^2, \quad -a D_\alpha^q q^2 - q^3 + t D_\alpha^q q^2 = 0, \quad a D_\alpha^q q^1 + q^3 - t D_\alpha^q q^1 = 0. \]

The solution of (18) is given as follows

\[ q^1 = q^2, \]

\[ q^3 = (-a D_\alpha^q + t D_\alpha^q) q^1. \]

From (19) and (20) we conclude that the classical solution is obtained if \( \alpha \to 1 \).

**B.** Let us consider the second Lagrangian given by

\[ L = \dot{q}^1 q^2 + \dot{q}^3 q^4 - V(q^2, q^3, q^4), \]

where \( V(q^2, q^3, q^4) = \frac{1}{2} [(q^4)^2 - 2q^2 q^3]\). We observe that (21) is a second class constrained system in Dirac’s classification [21].
We propose the fractional generalization of (21) to be as follows

\[ L' = -\left[ (t D_b^a q^1)^2 + (t D_b^a q^3) q^4 + V(q^2, q^3, q^4) \right]. \]  

(22)

From (15) and (22) the Euler-Lagrange equations are given by

\[ a D_t^\alpha q^2 = 0 , \]  

(23)

\[ t D_b^\alpha q^1 + q^3 = 0 , \]  

(24)

\[ a D_t^\alpha q^4 + q^2 = 0 , \]  

(25)

\[ t D_b^\alpha q^3 - q^4 = 0 . \]  

(26)

From (23), we conclude that the solution for \( q^2(t) \) has the form

\[ q^2(t) = C_1 (t - a)^{a-1} . \]  

(27)

From (25) and (27) an equation for \( q^4(t) \) is obtain as follows

\[ a D_t^\alpha q^4 = -C_1 (t - a)^{a-1} . \]  

(28)

The solution of (28) has the form

\[ q^4(t) = C_2 (t - a)^{a-1} - \frac{C_1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{a-1} (t - a)^{a-1} d\tau . \]  

(29)

Using (26) and (29), the solution of \( q^3(t) \) becomes

\[ q^3(t) = C_3 (t - a)^{a-1} + \frac{C_2}{\Gamma(\alpha)} \int_t^b (t - \tau)^{a-1} (a - \tau)^{a-1} d\tau \]

\[ -\frac{C_1}{\Gamma(\alpha)^2} \int_t^b (t - \tau)^{a-1} \int_a^\tau (\tau - \eta)^{a-1} (\eta - a)^{a-1} d\eta d\tau . \]  

(30)

Finally, the equation (24) together with (30) give the solution for \( q^1(t) \) as follows

\[ q^1(t) = C_4 (b - t)^{a-1} - \frac{C_3}{\Gamma(\alpha)} \int_t^b (t - \tau)^{a-1} (b - \tau)^{a-1} d\tau \]

\[ -\frac{C_2}{\Gamma(\alpha)^2} \int_t^b (t - \tau)^{a-1} \int_t^b (\tau + \eta)^{a-1} (a + \eta)^{a-1} d\eta d\tau \]

\[ + \frac{C_1}{\Gamma(\alpha)^3} \int_t^b (\sigma - t)^{a-1} \int_t^\sigma (\sigma - \eta)^{a-1} \int_a^\eta (\tau - \eta)^{a-1} (\eta - a)^{a-1} d\tau d\eta d\sigma . \]  

(31)
Here $C_1, C_2, C_3$ and $C_4$ are constants. If $\alpha \to 1$, $a \to 0$, $b \to 1$, then the standard solutions

$$q^1(t) = \frac{C'_4 t^3}{6} - \frac{C'_3 t^2}{2} + C'_2 t + C'_1, \quad q^2(t) = C'_4, \quad q^3(t) = \frac{C'_4 t^2}{2} - C'_3 t + C'_2, \quad q^4(t) = -C'_4 t + C'_3$$

are recovered if we redefine the constants from (27), (29), (30) and (31) as

- $C'_1 = C_4 - C_3 - \frac{C_2}{2} + \frac{C_1}{3}$
- $C'_2 = -\frac{C_1}{2} + C_2 + C_3$
- $C'_3 = C_2$
- $C'_4 = C_1$

(32)

4 Conclusion

The Lagrangians with linear velocities were investigated using left and right RL fractional derivatives. The corresponding fractional Lagrangians were proposed and the fractional Euler-Lagrange equations were obtained. Although the fractional Lagrangians contain only left or right derivatives, both derivatives are involved in Euler-Lagrange equations and both played an important role in finding the solutions. The exact solutions of the Euler-Lagrange equations were obtained for two examples corresponding to first and second-class constrained systems. A “gauge invariance” was reported for the first example. The solutions of the investigated examples depend on the limits $a$ and $b$ and the limiting procedure recovered the standard results.

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