Gerbes and twisted orbifold quantum cohomology
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1. Introduction

An important part of stringy orbifold theory is the various twistings the theory possesses. Unfortunately, it is also the part of stringy orbifold theory that we understand the least. For example, for the untwisted theory, we have a rather complete conjectural answer to its structure and relation to crepant resolution. On the other hand, both the structure of twisted theory and its relation to desingularization are still mysterious at this moment. This article fills in one piece of the puzzle.

Recall that for any almost complex orbifold $X$, we can associate a Chen-Ruan orbifold cohomology ring $H^*_{CR}(X, \mathbb{C})$ as the summation of ordinary cohomology over all the sectors with appropriate degree shifting. There are two important factors of this ring. (1) There is a K-theoretic counterpart $K_{orb}(X)$ due to Adem-Ruan [AR]. (2) A precise relation between Chen-Ruan orbifold cohomology ring and the cohomology ring of its crepant resolution has been proposed [R1]. These two key aspects of the theory will always serve as benchmarks to our future constructions. Namely, any theory we constructed should have two properties: (1) it should be compatible with K-theory; (2) it should describe the ring structure of its crepant resolution or more generally desingularization. Here, a desingularization is obtained by first deforming the equation of a Gorenstein orbifold and then taking a crepant resolution. The miracle is that the right answer for one is often automatically the right answer for the other one. This gives us two powerful approaches to stringy orbifolds.

Historically, the earliest twisting is due to Vafa [V], [VW] in the case of a global quotient orbifold $X = Y/G$. Vafa’s twisting is a group cohomology class $\alpha \in H^2(G, S^1)$ called discrete torsion. The notion of discrete torsion was generalized to arbitrary orbifolds as a class in $H^2(\pi_{orb}^1(X), S^1)$ [R]. One can construct a twisted orbifold cohomology using discrete torsion [VW, R]. Its K-theoretic counterpart was constructed by Adem-Ruan [AR]. However, it fails badly of describing the cohomology of desingularization. To remedy the situation, a more general twisting was proposed by the author [R]. This new twisting is a

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flat line bundle $\mathcal{L}$ over the inertial orbifold satisfying a certain compatibility condition (see Definition 3.1). $\mathcal{L}$ is called an inner local system. The inner local system works well for the second task, i.e., describing the cohomology group of a desingularization. But Adem-Ruan’s construction of twisted orbifold K-theory fails to cover the case of an inner local system.

A more important problem is to twist orbifold quantum cohomology, which is unknown even for discrete torsion. Recall that the Cohomological Crepant Resolution Conjecture can be phrased as follows: the cup product of a crepant resolution $Y$ of Gorenstein orbifold $X$ is the Chen-Ruan product of $X$ and quantum corrections coming from Gromov-Witten invariants of exceptional rational curves. This conjecture was obtained by understanding the behavior of quantum cohomology when we deform a crepant resolution to orbifold, even though our initial goal is only to understand cohomology. Therefore, to understand even the ordinary ring structure of desingularization, one has to understand the quantum cohomology and its twisting. This is the main goal of this paper.

Back then, both problems seemed to be hopeless. The situation changed when Lupercio-Uribe introduced the notion of gerbes to orbifolds [LU1] (see also Tu-Xu-Laurent-Gengoux [TXL]). Lupercio-Uribe-Tu-Xu-Laurent-Gengoux constructed twisted K-theory using a gerbe on any groupoid, which is much more general than an orbifold. Their twisted K-theory generalizes Adem-Ruan’s twisted K-theory on orbifolds and twisted K-theory on smooth manifolds studied by Witten [W], Bouwknegt-Mathai [BM] Freed-Hopkins-Teleman [FHT] and others. The beauty of gerbes is that one can easily do differential geometry, which is precisely what we were doing for quantum cohomology. In this context, Lupercio-Uribe interpreted an inner local system as the holonomy line bundle on the inertial groupoid of a gerbe. In this article, we would like to go one more step further to use the gerbe to twist orbifold quantum cohomology. During the course of this work, some subtleties arise. In the theory of gerbes, there is a distinction between flat gerbes and non-flat gerbes. A flat gerbe has torsion characteristic class and is often referred as a torsion gerbe. It has a rather long history in classical geometry under the name of Brauer group. The flat gerbe or element of the Brauer group is precisely the data we are able to use to twist orbifold quantum cohomology. On a smooth manifold, our twisted orbifold quantum cohomology did not give any new information (see Corollary 6.2). However, an orbifold flat gerbe naturally contains discrete torsion. It gives an abundance of new invariants. Our construction does not work for non-flat gerbes. In many ways, non-flat
gerbes seem to fall into the realm of non-commutative geometry. A further understanding of twisted orbifold quantum cohomology may require a full-fledged theory of non-commutative quantum cohomology. The on-going development of geometry with B-field by Hitchin and others may provide another approach to this type of question.

Since a gerbe and its twisted K-theory can be constructed over a singular space much more general than an orbifold, a natural question is: Can we construct a (twisted) orbifold (quantum) cohomology for a general groupoid such that (1) it agrees with twisted K-theory rationally; (2) it describes the cohomology of its desingularization?

The main results of this paper were announced by the second author in 2001 at ICM Satellite conference on Stringy Orbifolds in Chengdu. For various reason, we were distracted by other projects. We apologize for such a long delay. During the preparation of this paper, we received an article of Lupercio and Uribe where there is some overlap between our section 4 and their paper [LU].

The paper is organized as follows. In section 2, we will review the basic definitions of orbifold and groupoid. In section 3, we will review the definition of gerbes and their holonomy. In section 4, we will show how to use the holonomy of a gerbe to twist orbifold GW-invariants. Some examples will be computed in the last section.

2. A review of orbifold Gromov-Witten invariants

We will review the construction of ordinary orbifold Gromov-Witten invariants due to Chen-Ruan [CR2]. We will only sketch the main construction and refer the detail to [CR2]. But we take this opportunity to steamline the definition.

From now on, we will use $X_o$ to denote a connected component of $X$. We will also assume that all intersections $U_{i_1 \cdots i_k} = U_{i_1} \cap \cdots \cap U_{i_k}$ are connected. Otherwise, we work component by component.

An orbifold atlas is defined in the same way.

**Definition 2.1 (Orbifold Atlas).** A $n$-dimensional smooth orbifold atlas on connected open cover $\{U_i\}_{i \in I}$ of $X$ is given by the following data:

1. Each $U_i$ is covered by an uniformizing system $(\tilde{U}_i, G_{\tilde{U}_i}, \pi_i)$ in the following sense. $\tilde{U}_i$ is smooth. $G_{\tilde{U}_i}$ is a finite group acting smoothly on $\tilde{U}_i$ and $\pi_i : \tilde{U}_i \to U_i$ is invariant under $G_{\tilde{U}_i}$ such that it induces a homeomorphism $\tilde{U}_i/G_{\tilde{U}_i} \cong U_i$. We call $U_i$ an orbifold chart.
Choose a component \( \pi_i^{-1}(U_{ij}) \_o \) and let \( G_{\pi_i^{-1}(U_{ij})} \_o \subset G_{\tilde{U}_i} \) be the subgroup fixing \( \pi_i^{-1}(U_{ij}) \_o \). Then \( (\pi_i^{-1}(U_{ij}) \_o, G_{\pi_i^{-1}(U_{ij})} \_o, \pi_i) \) is an uniformizing system of \( U_{ij} \) (called an induced uniformizing system). Other induced uniformizing system are of the form \( (g\pi_i^{-1}(U_{ij}) \_o, gG_{\pi_i^{-1}(U_{ij})} \_o, g^{-1}, \pi_i) \) for \( g \in G_{\tilde{U}_i} \). Namely, \( G_{\tilde{U}_i} \) acts transitively on the collection of induced uniformizing systems. In the same way, \( (\tilde{U}_j, G_{\tilde{U}_j}, \pi_j) \) induces a collection of uniformizing systems over \( U_{ij} \) acts transitively by \( G_{\tilde{U}_j} \).

(2): Tran\((U_i, U_j)\) is a collection of isomorphisms from \( (\pi_i^{-1}(U_{ij}) \_o, G_{\pi_i^{-1}(U_{ij})} \_o, \pi_i) \) to \( (\pi_j^{-1}(U_{ij}) \_o, G_{\pi_j^{-1}(U_{ij})} \_o, \pi_j) \). Here, the isomorphism is an isomorphism \( \lambda_{ij} : G_{\pi_i^{-1}(U_{ij})} \_o \to G_{\pi_j^{-1}(U_{ij})} \_o \) and an equivariant diffeomorphism \( \phi_{ij} : \pi_i^{-1}(U_{ij}) \_o \to \pi_j^{-1}(U_{ij}) \_o \). Moreover, all such isomorphisms are generated from a fixed one by the action of \( G_i \times G_j \) in an obvious way. Each isomorphism is called a transition map.

(3): Tran\((U_i, U_i)\) is generated by the identity. Tran\((U_j, U_i)\) = Tran\((U_i, U_j)^{-1}\) in the sense that each transition in Tran\((U_j, U_i)\) is the inverse of some transition of Tran\((U_i, U_j)\).

Over the triple intersection \( U_{ijk} \), each of \( U_i, U_j, U_k \) induces a uniformizing system and transitions restrict to them as well. Then, we require

(4): There is a multiplication such that \( (\phi_{jk}, \lambda_{jk}) \circ (\phi_{ij}, \lambda_{ij}) \) is the restriction of an element of Tran\((U_i, U_k)\).

Note that we do not require \( \tilde{U}_i \) to be connected.

If \( \mathcal{U}' \) is a refinement of \( \mathcal{U} \) satisfying (*), then there is an induced orbifold atlas over \( \mathcal{U}' \) in an obvious fashion. Two orbifold atlases are considered to be equivalent if their induced orbifold atlases are equivalent over a common refinement in an obvious fashion. Such an equivalence class is called an orbifold structure over \( X \). So we may choose \( \mathcal{U} \) to be arbitrarily fine.

Let \( x \in X \). By choosing a small neighborhood \( V_p \in \mathcal{U} \), we may assume that its uniformizing system \( \mathcal{V}(V_p) = (U_p, G_p) \) has the property that \( U_p \) is an \( n \)-ball centered at origin \( o \) and \( \pi_p^{-1}(p) = o \) where \( \pi_p : U_p \to V_p = U_p/G_p \) is the projection map. In particular, the origin \( o \) is fixed by \( G_p \). We called \( G_p \) the local group at \( p \). If \( G_p \) acts effectively for every \( p \), we call \( X \) an effective orbifold.

Recall Satake’s definition of orbifold map. A map \( f : X \to Y \) is an orbifold map iff locally \( f : U_i \to V_i \) can be lifted to an equivariant map \( \tilde{f}_i : \tilde{U}_i \to \tilde{V}_i \) with a homomorphism \( \lambda_i : G_{\tilde{V}_i} \to G_{\tilde{V}_i} \). Suppose that we
want to pull back an orbifold vector bundle from \( Y \). We can use a local lifting \( \tilde{f}_i \) to construct the local pull-back. But there is no reason that local pull-backs can be glued together. In order to glue them together, we have to impose the condition on transitions. Then, we obtain the notion of orbifold morphism. Now it is clear how we should impose our condition called compatible system.

**Definition 2.2.** Fix an underlying map \( f : X \to Y \). A compatible system consists of an orbifold atlas \((\tilde{U}_i, G_{\tilde{U}_i}), \text{Tran}(U_i, U_j)\) of \( X \) and an orbifold atlas \((\tilde{V}_j, G_{\tilde{V}_j}), \text{Tran}(V_i, V_j)\) of \( Y \) with the following additional properties:

(i): \( f \) maps a member of one atlas to a member of other atlases, i.e., \( f : U_i \to V_{\kappa(i)} \).

(ii): The local map in (i) can be lifted to \( \lambda_i : G_{\tilde{U}_i} \to G_{\tilde{V}_{\kappa(i)}} \) and an equivariant map \( \tilde{f}_i : \tilde{U}_i \to \tilde{V}_{\kappa(i)} \).

(iii): There is a map \( \lambda_{ij} : \text{Tran}(U_i, U_j) \to \text{Tran}(V_{\kappa(i)}, V_{\kappa(j)}) \) preserving the identity, inverse and multiplication.

(iv): \( \lambda_{ij}(g) \circ \tilde{f}_i = \tilde{f}_j \circ g \).

Suppose that \( \{V_\beta\} \) is a refinement of \( \{V_j\} \). Then, \( \{f^{-1}(V_\beta)\} \) is a refinement of \( \{U_i\} \). We can take a further refinement \( \{U_\alpha\} \) of \( \{f^{-1}(V_\beta)\} \). Then we still have property (i). Furthermore, the original compatible system induces compatible systems over \( \{U_\alpha\}, \{V_\beta\} \). We call this a refinement of compatible systems.

**Definition 2.3** (Isomorphism of compatible systems). Two compatible systems given by \((\tilde{f}_i, \lambda_i, \lambda_{ij}), (\tilde{f}_i', \lambda_i', \lambda_{ij}')\) over the same orbifold atlas \((\tilde{U}_i, G_{\tilde{U}_i}), (\tilde{V}_j, G_{\tilde{V}_j})\) are said to be isomorphic if they differ by an automorphism of orbifold structure \((\tilde{V}_j, G_{\tilde{V}_j})\). Namely, there is an element \( \delta_i \in \text{Tran}(V_i, V_i) \) such that

\[
\tilde{f}' = \delta_i \circ \tilde{f}_i, \quad \lambda_i' = \delta_i \lambda_i \delta_i^{-1}, \quad \lambda_{ij}' = \delta_j \lambda_{ij} \delta_i^{-1}.
\]

For two arbitrary compatible systems over isomorphic orbifold atlases, by taking refinements and components if necessary, we can induce two compatible systems over the same orbifold atlas. Then the original ones are isomorphic iff the induced ones are isomorphic in the above sense. An orbifold morphism is a map with an isomorphism class of compatible system.

Chen-Ruan also developed a machinery to classify good maps. The key is an invariant they called the characteristic. The case we will use
is the global quotient orbifold denoted by the stacky notation \([X/G]\).

The characteristic can be interpreted as follows.

Suppose that \(f : Y \to [X/G]\) is a good map. We can pull back the \(G\)-bundle \(X \to X/G\) to obtain a \(G\)-bundle \(p : E \to Y\) and a \(G\)-map \(F : E \to X\). In fact, the equivalence class of a good map \(f\) is equivalent to the pair \((p, F)\) modulo bundle isomorphism. Namely, \((p, F) \sim (p', F')\) if \(p' = ph, F' = Fh\) for a bundle isomorphism \(h : E' \to E\).

Since \(G\) is a finite group, \(p : E \to Y\) is an orbifold cover. By covering space theory, \(E\) is determined by the conjugacy class of a homomorphism \(\rho : \pi_1^{orb}(Y, x_0) \to G\). We call \(\rho\) and its conjugacy class Chen-Ruan characteristic.

Consider the pairs:

\[
\wedge X = \{(p, (g)_{G_p})| p \in X, g \in G_p\},
\]

where \((g)_{G_p}\) is the conjugacy class of \(g\) in \(G_p\). If there is no confusion, we will omit the subscript \(G_p\) to simplify the notation. \(\wedge X\) has a natural orbifold structure (Proposition 2.4) and is called the inertia orbifold.

More generally, we can define the multisector

\[
\tilde{X}_k = \{(p, (g_1, \cdots, g_k)_{G_p})| p \in X, g_i \in G_p\},
\]

It is clear that \(\wedge X = \tilde{X}_1\). There are two classes of maps between multisectors.

\[
I : \tilde{X}_k \to \tilde{X}_k
\]

by

\[
I(p, (g_1, \cdots, g_k)_{G_p}) = (p, (g_1^{-1}, \cdots, g_k^{-1})_{G_p}),
\]

and

\[
e_{i_1, \cdots, i_l} : \tilde{X}_k \to \tilde{X}_l
\]

by

\[
e_{i_1, \cdots, i_l}(p, (g_1, \cdots, g_k)_{G_p}) = (p, (g_{i_1}, \cdots, g_{i_l})_{G_p}).
\]

Suppose that \(X\) has an orbifold structure \(U\) with orbifold atlas \((\tilde{U}_i, G_{\tilde{U}_i}, \pi_i), Tran(U_i, U_j)\).

**Proposition 2.4.** \(\tilde{X}_k\) is naturally an orbifold, with the orbifold atlas given by

\[
(\bigsqcup_{g \in G_k} \tilde{U}^g, G_U),
\]

where \(\tilde{U}^g = \tilde{U}^{g_1} \cap \tilde{U}^{g_2} \cap \cdots \cap \tilde{U}^{g_k}\). Here \(g = (g_1, \cdots, g_k)\), \(\tilde{U}^g\) stands for the fixed-point set of \(g\) in \(\tilde{U}\). When \(X\) is almost complex, \(\tilde{X}_k\) inherits an almost complex structure from \(X\), and when \(X\) is closed, \(\tilde{X}_k\) is a finite disjoint union of closed orbifolds.
Next, we would like to describe the connected components of $\tilde{X}_k$. Suppose that $p, q$ are in the same orbifold chart $U_i$ uniformized by $(\tilde{U}_i, G_{\tilde{U}_i}, \pi_i)$. Let $\tilde{p}, \tilde{q}$ be preimages of $p, q$ respectively. Then $G_p = G_{\tilde{p}}, G_q = G_{\tilde{q}}$ and both of them are subgroups of $G_{\tilde{U}_i}$. We say that $(g_1)_{G_p} \cong (g_2)_{G_q}$ if $h(q) = p, g_1 = h g_2 h^{-1}$ for some element $h \in G_{\tilde{U}_i}$. For two arbitrary points $p, q \in X$, we say $(g)_{G_p} \cong (g')_{G_q}$ if there is a sequence $(p_0, (g_0)_{G_{p_0}}), \cdots, (p_k, (g_k)_{G_{p_k}})$ such that $(p_0, (g_0)_{G_{p_0}}) = (p, (g)_{G_p}), (p_k, (g_k)_{G_{p_k}}) = (q, (g')_{G_q})$ and $p_i, p_{i+1}$ are in the same orbifold chart and $(g_i)_{G_{p_i}} \cong (g_{i+1})_{G_{p_{i+1}}}$. This defines an equivalence relation on $\{(g)_{G_p}\}$. In particular, it is possible that $(g)_{G_p} \cong (g')_{G_p}$ while $(g)_{G_p} \not\cong (g')_{G_p}$. Let $T_k$ be the set of equivalence classes. By abuse of the notation, we often use $(g)$ to denote the equivalence class to which $(g)_{G_q}$ belongs. It is clear that $\tilde{X}_k$ decomposes as a disjoint union of connected components

$$\tilde{X}_k = \bigsqcup_{(g) \in T_k} X_{(g)},$$

where

$$X_{(g)} = \{(p, (g')_{G_p}) | g' \in G^k_p, (g')_{G_p} \in (g)\}.$$

Let $T_k^o \subset T^k$ be such that $(g_1, \cdots, g_k) \in T_k^o$ has the property $g_1 \cdots g_k = 1$.

$$\overline{\mathcal{M}}_k(X) = \bigsqcup_{(g) \in T_k^o} X_{(g)}.$$

**Definition 2.5.** $X_{(g)}$ for $g \neq 1$ is called a twisted sector. $X_{(g)}$ is called a $k$-multi-sector or a $k$-sector. Furthermore, we call $X_{(1)} = X$ the nontwisted sector.

**Example 2.6.** Suppose that $X = Y/G$ is a global quotient. By the definition, $\bigwedge X = \bigsqcup_{g \in G} Y^g/G$ where $Y^g$ is the fixed-point set of elements $g \in G$. Equivalently, $\bigwedge X = \bigsqcup_{(g)} Y^g/C(g)$.

**Proposition 2.7.** Both the evaluation maps $e_{i_1, \ldots, i_l}$ and $I$ are orbifold morphisms.

Next, we extend the notion of orbifold morphism to the case in which the domain is a nodal orbifold Riemann surface.

Recall that a nodal curve with $k$ marked points is a pair $(\Sigma, z)$ consisting of a connected topological space $\Sigma = \bigsqcup \pi_{\Sigma_\nu} (\Sigma_\nu)$, where $\Sigma_\nu$ is a smooth complex curve and $\pi_{\Sigma_\nu} : \Sigma_\nu \rightarrow \Sigma$ is a continuous map, and
$z = (z_1, \cdots, z_k)$ consists of $k$ distinct points in $\Sigma$ with the following properties:

- For each $z \in \Sigma_\nu$, there is a neighborhood of it such that the restriction of $\pi_\nu : \Sigma_\nu \to \Sigma$ to this neighborhood is a homeomorphism to its image.
- For each $z \in \Sigma$, we have $\sum_\nu \#\pi_\nu^{-1}(z) \leq 2$.
- $\sum_\nu \#\pi_\nu^{-1}(z_i) = 1$ for each $z_i \in z$.
- The number of complex curves $\Sigma_\nu$ is finite.
- The set of nodal points $\{z | \sum_\nu \#\pi_\nu^{-1}(\pi_\nu(z)) = 2\}$ is finite.

A point $z \in \Sigma_\nu$ is called singular (or a node) if $\sum_\omega \#\pi_\omega^{-1}(\pi_\nu(z)) = 2$.

A point $z \in \Sigma_\nu$ is said to be a marked point if $\pi_\nu(z) = z_i \in z$. Each $\Sigma_\nu$ is called a component of $\Sigma$. Let $k_\nu$ be the number of points on $\Sigma_\nu$ which are either singular or marked, and $g_\nu$ be the genus of $\Sigma_\nu$; a nodal curve $(\Sigma, z)$ is called stable if $k_\nu + 2g_\nu \geq 3$ holds for each component $\Sigma_\nu$ of $\Sigma$.

**Definition 2.8.** A nodal orbicurve is a nodal marked curve $(\Sigma, z)$ with an orbifold structure as follows:

- The set $z_\nu$ of orbifold points of each component $\Sigma_\nu$ is contained in the set of marked points and nodal points $z$.
- A neighborhood of a marked point is uniformized by a branched covering map $z \to z_{m_i}$ with $m_i \geq 1$.
- A neighborhood of a nodal point (viewed as a neighborhood of the origin of $\{xy = 0\} \subset C^2$) is uniformized by a branched covering map $(x, y) \to (x^{n_j}, y^{n_j})$, with $n_j \geq 1$, and with group action $e^{2\pi i/n_j}(x, y) = (e^{2\pi i/n_j}x, e^{-2\pi i/n_j}y)$.

Here $m_i$ and $n_j$ are allowed to be equal to one, i.e., the corresponding orbifold structure is trivial there. We denote the corresponding nodal orbicurve by $(\Sigma, z, m, n)$ where $m = (m_1, \cdots, m_k)$ and $n = (n_j)$.

Once we have the definition of nodal orbicurve, we can extend the definition of compatible system and orbifold morphism word by word to the case where the domain is a nodal orbicurve.

First, recall that for every point $p \in \Sigma$, an orbifold morphism $f : \Sigma \to X$ induces a homomorphism $G_p \to G_{f(p)}$.

**Definition 2.9.** Let $(X, J)$ be an almost complex orbifold. An orbifold stable map into $(X, J)$ is a quadruple $(f, (\Sigma, z, m, n), \varphi_s), \xi)$ described as follows:
(1) \( f \) is a continuous map from the nodal orbicurve \((\Sigma, z, m, n)\) into \(X\) such that each \( f_\nu = f \circ \pi_\nu \) is a pseudo-holomorphic map from \(\Sigma_\nu\) into \(X\).

(2) \( \xi \) is an isomorphism class of compatible structures.

(3) Let \( k_\nu \) be the order of the set \( z_\nu \), namely the number of points on \(\Sigma_\nu\) which are special (i.e. nodal or marked); if \( f_\nu \) is a constant map, then \( 2g_\nu - 2 + k_\nu > 0 \).

(4) At any marked or nodal point \( p \) the induced homomorphism on the local group \( \lambda_p : G_p \to G_{f(p)} \) is injective.

Finally we observe that each \( C^\infty \) orbifold morphism from an orbifold nodal Riemann surface with \( k \) marked points into an orbifold \( X \) determines a point in the product of inertia orbifolds \((\wedge X)^k\) as follows: let the underlying continuous map be \( f \) and for each marked point \( z_i \), \( i = 1, \cdots, k \), let \( x_i \) be the positive generator of the cyclic local group at \( z_i \), and \( \lambda_{z_i} \) be the homomorphism determined by the given compatible system; then the determined point in \((\wedge X)^k\) is

\[
((f(z_1), (\lambda_{z_1}(x_1))G_{f(z_1)}), \cdots, (f(z_k), (\lambda_{z_k}(x_k))G_{f(z_k)})).
\]

Let \( x = (X_{g_1}, \cdots, X_{g_k}) \) be a connected component in \((\wedge X)^k\). We say that a good map with a compatible system is of type \( x \) if the above point it determines in \((\wedge X)^k\) lies in the component \( x \).

**Remark 2.10.** If \( f : \Sigma \to X \) is a pseudo-holomorphic map whose image intersects the singular locus of \( X \) at only finitely many points, then there is a unique choice of orbifold structure on \( \Sigma \) together with a unique \((\tilde{f}, \xi)\), where \( \tilde{f} \) is a good map with an isomorphism class of compatible systems \( \xi \) whose underlying continuous map is \( f \). If the image of \( f \) lies completely inside the singular locus, there could be different choices, and they are regarded as different points in the moduli space.

**Definition 2.11.**

(1) An orbifold \( X \) is symplectic if there is a closed 2-form \( \omega \) on \( X \) whose local liftings are non-degenerate.

(2) A projective orbifold is a complex orbifold which is a projective variety as an analytic space.

**Proposition 2.12.** Suppose that \( X \) is a symplectic or projective orbifold. The moduli space of orbifold stable maps \( \overline{M}_{p,k}(X, J, A, x) \) is a compact metrizable space under a natural topology, whose “virtual dimension” is \( 2d \), where

\[
d = c_1(TX) \cdot A + (\dim_{\mathbb{C}} X - 3)(1 - g) + k - \iota(x).
\]
Here $\iota(x) := \sum_{i=1}^{k} \iota(g_i)$ for $x = (X_{(g_1)}, \cdots, X_{(g_k)})$.

For any component $x = (X_{(g_1)}, \cdots, X_{(g_k)})$, there are $k$ evaluation maps

$$e_i : \overline{\mathcal{M}}_{g,k}(X, J, A, x) \to X_{(g_i)}, \quad i = 1, \cdots, k.$$ 

$e_i$ has a natural compatible system to make it a good map. For any set of cohomology classes $\alpha_i \in H^*(X_{(g_i)}; \mathbb{C}) \subset H^*_\text{orb}(X; \mathbb{C})$, $i = 1, \cdots, k$, the orbifold Gromov-Witten invariant is defined as

$$\Psi^{X, J}_{(g,k,A,x)}(\alpha_1^{l_1}, \cdots, \alpha_k^{l_k}) = \prod_{i=1}^{k} c_1(L_i)^{l_i} e_i^* \alpha_i[\overline{\mathcal{M}}_{g,k}(X, J, A, x)]^\text{vir},$$

where $L_i$ is the line bundle generated by the cotangent space of the $i$-th marked point. The virtual fundamental cycle $[\overline{\mathcal{M}}_{g,k}(X, J, A, x)]^\text{vir}$ is defined as the fundamental cycle of a certain orbifold $S\text{ CR2}$.

The inertial orbifold admits another interpretation as the space of constant loops. Then it is naturally a subset of the free loop space. We shall sketch this construction due to Chen [C] (see [LU2] for a groupoid description).

Let $\Omega X$ be the space of orbifold morphisms from $S^1$ with trivial orbifold structure to $X$. $\Omega X$ is the analog of the free loop space of a smooth manifold. We need the following important lemma.

**Lemma 2.13** (Lemma 3.15 in [C]). Let $X = Y/G$ be a global quotient. Then, $\Omega X = P(Y, G)/G$, where

$$P(Y, G) = \{ (\gamma, g) : [0, 1] \to Y, g \in G, \gamma(1) = g\gamma(0) \}.$$ 

Here, $G$ acts on $P(Y, G)$ by $h(\gamma, g) = (h \circ \gamma, h^{-1}gh)$.

We call $\tau$ a constant loop if the underlying map is constant. Suppose that the image is $p \in X$. Let $U_p/G_p$ be the orbifold chart at $p$. By the Lemma, $\Omega U_p/G_p = P(U_p, G_p)/G$. In particular, $\tau$ is an equivalence class of a pair $(\gamma, g)$ where $\text{im} (\gamma) = p$. Under the action of $G$, we naturally identify it as $(p, (g)_{G_p})$. Therefore, the space of constant loop is precisely the inertia orbifold $\tilde{X}$.

Suppose that $f : \Sigma \to X$ is an orbifold stable map. We take a real blow-up of $\Sigma$ at all the marked points to obtain a Riemann surface with boundary $\Sigma^\dagger$. $\Sigma^\dagger$ can be understood as follows. We remove the marked point $x_i$. A neighborhood of a puncture point $x_i$ is biholomorphic to $S^1 \times [0, \infty)$. Hence, we can view $\Sigma^\dagger$ as a manifold with cylindrical end and $x_i$ is replaced by a circle $S_\infty$ attached at $\infty$. Another way to interpret the evaluation map is that

$$e_i(f) = f(S_\infty).$$
This description is important later in our construction.

3. Gerbes and their holonomy

After reviewing the construction of orbifold quantum cohomology in the last section, we are ready to touch upon the main topic of this article-twisting. The earliest twisting from physics is discrete torison by Vafa [V]. However, discrete torison is too restrictive to describe interesting examples. Therefore, a more general twisting is needed. For this purpose, the second author introduced the notion of inner local system. Roughly speaking, an inner local system is a flat orbifold line bundle over the inertial orbifold \( \tilde{X}_1 \) satisfying certain compatibility conditions. Later, Lupercio-Uribe introduced the concept of gerbe to orbifolds. The holonomy line bundle of a gerbe with connection is naturally an inner local system. However, not all inner local systems are induced in this way [AP]. In this section, we will study the relation between a gerbe and its holonomy in detail.

3.1. Inner local system. Recall that for \( (g_1, \ldots, g_k) \in T_k \), there are \( k + 1 \) evaluation maps

\[
e_i : X_{(g_1, \ldots, g_k)} \rightarrow X_{(g_i)}, \quad i \leq k,
\]

and

\[
e_{k+1} : X_{(g_1, \ldots, g_k)} \rightarrow X_{(g_1 \cdots g_k)}.
\]

Now we introduce the notion of inner local system for an orbifold.

**Definition 3.1.** Suppose that \( X \) is an orbifold (almost complex or not). An inner local system \( \mathcal{L} = \{L(g)\}_{g \in T_1} \) is an assignment of a flat complex orbifold line bundle

\[
L(g) \rightarrow X(g)
\]

to each sector \( X(g) \) satisfying the following compatibility conditions (1-4).

(1): \( L(1) \) is a trivial orbifold line bundle with a fixed trivialization.

(2): There is a nondegenerate pairing \( L(g) \otimes \overline{L(g^{-1})} \rightarrow \mathbb{C} = L(1) \).

(3): There is a multiplication

\[
e_1^*L(g_1) \otimes e_2^*L(g_2) \rightarrow e_3^*L(g_1g_2)
\]

over \( X_{(g_1g_2)} \) for \( (g_1, g_2) \in T_2 \).

(4): \( \theta \) is associative in the following sense. For \( (g_1, g_2, g_3) \in T_3 \), the evaluation maps \( e_i : X_{(g_1g_2g_3)} \rightarrow X_{(g_i)} \) factor through

\[
P = (P_1, P_2) : X_{(g_1g_2g_3)} \rightarrow X_{(g_1g_2)} \times X_{(g_1g_2g_3)}.
\]
Let \( e_{12} : X_{(g_1,g_2,g_3)} \to X_{(g_1,g_2)} \). We first use \( P_1 \) to define
\[
\theta : e_1^*L_{(g_1)} \otimes e_2^*L_{(g_2)} \to e_{12}^*L_{(g_1,g_2)}.
\]

Then, we can use \( P_2 \) to define a product
\[
\theta : e_{12}^*L_{(g_1,g_2)} \otimes e_3^*L_{(g_3)} \to e_4^*L_{(g_1,g_2,g_3)}.
\]

Taking the composition, we define
\[
\theta(\theta(e_1^*L_{(g_1)}, e_2^*L_{(g_2)}), e_3^*L_{(g_3)}): e_1^*L_{(g_1)} \otimes e_2^*L_{(g_2)} \otimes e_3^*L_{(g_3)} \to e_4^*L_{(g_4)}.
\]

On the other hand, the evaluation maps \( e_i \) also factor through
\[
P' : X_{(g_1,g_2,g_3)} \to X_{(g_1,g_2,g_3)} \times X_{(g_2,g_3)}.
\]

In the same way, we can define another triple product
\[
\theta(e_1^*L_{(g_1)}, \theta(e_2^*L_{(g_2)}, e_3^*L_{(g_3)})): e_1^*L_{(g_1)} \otimes e_2^*L_{(g_2)} \otimes e_3^*L_{(g_3)} \to e_4^*L_{(g_4)}.
\]

Then, we require the associativity
\[
\theta(\theta(e_1^*L_{(g_1)}, e_2^*L_{(g_2)}), e_3^*L_{(g_3)}) = \theta(e_1^*L_{(g_1)}, \theta(e_2^*L_{(g_2)}, e_3^*L_{(g_3)})).
\]

If \( X \) is a complex orbifold, we assume that \( L_{(g)} \) is holomorphic.

**Definition 3.2.** Given an inner local system \( \mathcal{L} \), we define the twisted orbifold cohomology
\[
H^*_CR(X, \mathcal{L}) = \oplus_{(g)} H^{*-2\kappa(g)}(X_{(g)}, L_{(g)}).
\]

**Definition 3.3.** Suppose that \( X \) is a closed complex orbifold and \( \mathcal{L} \) is an inner local system. We define Dolbeault cohomology groups
\[
H^{p,q}_{CR}(X, \mathcal{L}) = \oplus_{(g)} H^{p-\kappa(g), q-\kappa(g)}(X_{(g)}; L_{(g)}).
\]

**Proposition 3.4.** If \( X \) is a Kähler orbifold, we have the Hodge decomposition
\[
H^k_{CR}(X, \mathcal{L}) = \oplus_{k=p+q} H^{p,q}_{CR}(X, \mathcal{L}).
\]

**Proof:** Note that each sector \( X_{(g)} \) is a Kähler orbifold. The proposition follows by applying the ordinary Hodge theorem with twisted coefficients to each sector \( X_{(g)} \). \( \square \)
3.2. **Basics on gerbes and connections.** The original motivation for the introduction of gerbes to orbifolds by Lupercio-Uribe is to understand inner local systems conceptually. Let’s start from the definition of a gerbe on a smooth manifold. We follow closely the exposition of [H].

Let’s suppose $X$ is a smooth manifold and $\{U_\alpha\}_\alpha$ an open cover. Recall the definition of line bundle. It can be described by transition functions $g_{\alpha\beta}: U_{\alpha\beta} = U_\alpha \cap U_\beta \to S^1$ satisfying the conditions

$$g_{\alpha\alpha} = 1, g_{\beta\alpha} = g_{\alpha\beta}^{-1}, (\delta g)_{\alpha\beta\gamma} = g_{\alpha} g_{\beta\gamma} = 1.$$ 

In terms of cohomological language, $g_{\alpha\beta}$ is a Čech 1-cocycle of the sheaf of $S^1$-valued functions $C^\infty(S^1)$. Two sets of transition functions induce isomorphic line bundles iff they induce the same class in $H^1(X, C^\infty(S^1))$.

A gerbe is a generalization of a line bundle. It is defined as a Čech 2-cocycle of sheaf of $S^1$-valued function $C^\infty(S^1)$ over some open cover $U$. Two gerbes are equivalent if they induced the same cocycle over a common refinement. They are isomorphic if they induced the same cohomology class in $H^2(X, C^\infty(S^1))$.

In terms of local data, they are functions $g_{\alpha\beta\gamma}: U_\alpha \cap U_\beta \cap U_\gamma \to S^1$ defined on the threefold intersections satisfying

$$g_{\alpha\beta\gamma} = g_{\alpha\gamma\beta}, g_{\beta\alpha\gamma}, g_{\gamma\beta\alpha}^{-1} = 1$$

and the cocycle condition

$$(\delta g)_{\alpha\beta\gamma\eta} = g_{\beta\gamma\eta} g_{\alpha\gamma\eta} g_{\alpha\beta\gamma} g_{\alpha\beta\gamma}^{-1} = 1$$

on the four-fold intersections $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\eta$. It also defines a class in $H^3(X; \mathbb{Z})$; Consider the long exact sequence of cohomology

$$\cdots \to H^i(X, C^\infty(\mathbb{R})) \to H^i(X, C^\infty(S^1)) \xrightarrow{\partial} H^{i+1}(X, \mathbb{Z}) \to \cdots$$

derived from the exact sequence of sheaves

$$0 \to \mathbb{Z} \to C^\infty(\mathbb{R}) \to C^\infty(S^1) \to 1.$$ 

Recall that $\tau_1([g_{\alpha\beta}]) \in H^2(X, \mathbb{Z})$ is the first Chern class of the corresponding line bundle. In the same way, the characteristic class of a gerbe is $\tau_2([g_{\alpha\beta\gamma}])$. It is well-known that $C^\infty(\mathbb{R})$ is a fine sheaf; we get $H^2(X, C^\infty(S^1)) \cong H^3(X, \mathbb{Z})$. We might say that a gerbe is determined topologically by its characteristic class. Furthermore, we can tensor them using the product of cocycles.
We call a gerbe \( g = \{ g_{\alpha\beta\gamma} \} \) a trivial gerbe if \( g = \delta f \) is a coboundary for some 1-cochain \( f \). \( f \) is called a trivialization of \( g \). In terms of local data, \( f \) is defined by functions
\[
f_{\alpha\beta} = f_{\beta\alpha} : U_{\alpha} \cap U_{\beta} \rightarrow S^1
\]
on the twofold intersections such that
\[
g_{\alpha\beta\gamma} = f_{\alpha\beta} f_{\beta\gamma} f_{\gamma\alpha}
\]
Hence, \( g \) is represented as a coboundary \( \delta f = g \).

Suppose that \( f_1, f_2 \) are two different trivializations of \( g \). Then \( \delta(f_1^{-1} f_2) = 1 \). Hence \( h = f_1 f_2^{-1} \) is a 1-cocycle and hence defines a line bundle.

A connection will consist of a pair \( (A_{\alpha\beta}, F_\alpha) \) where \( A_{\alpha\beta} \) are 1-forms over the double intersections \( A_{\alpha\beta} \), such that
\[
i A_{\alpha\beta} + i A_{\beta\gamma} + i A_{\gamma\alpha} = g_{\alpha\beta\gamma}^{-1} dg_{\alpha\beta\gamma}
\]
and the 2-forms \( F_\alpha \) are defined over \( U_\alpha \) such that \( F_\alpha - F_\beta = dA_{\alpha\beta} \). Note that we define a global 3-form \( G \) such that \( G|_{U_\alpha} = F_\alpha \). This 3-form \( G \) is called the curvature of the gerbe connection.

When the curvature \( G \) vanishes we say that the connection on the gerbe is flat. Therefore, \( dF_\alpha = 0 \). Since \( U_\alpha \) is contractible, we can find \( B_\alpha \) such that \( F_\alpha = dB_\alpha \). Then, on \( U_\alpha \cap U_\beta \),
\[
F_\beta - F_\alpha = dA_{\alpha\beta} = d(B_\beta - B_\alpha).
\]
This implies that
\[
A_{\alpha\beta} - B_\beta + B_\alpha = df_{\alpha\beta}.
\]
From the definition of connection
\[
i A_{\alpha\beta} + i A_{\beta\gamma} + i A_{\gamma\alpha} = g_{\alpha\beta\gamma}^{-1} dg_{\alpha\beta\gamma}.
\]
Hence,
\[
d(i f_{\alpha\beta} + i f_{\beta\gamma} + i f_{\gamma\alpha} - \log g_{\alpha\beta\gamma}) = 0.
\]
Let
\[
c_{\alpha\beta\gamma} = e^{i f_{\alpha\beta}} e^{i f_{\beta\gamma}} e^{i f_{\gamma\alpha}} g_{\alpha\beta\gamma}.
\]
\( c_{\alpha\beta\gamma} \) is constant. It is clear that \( c_{\alpha\beta\gamma} \) is a 2-cocycle differing from \( g_{\alpha\beta\gamma} \) by a coboundary \( e^{i f_{\alpha\beta}} e^{i f_{\beta\gamma}} e^{i f_{\gamma\alpha}} \). Since it is constant, \( c_{\alpha\beta\gamma} \) represents a Čech class in \( H^2(X, S^1) \) which we call the holonomy of the connection.

Next, we check that \( \{ c_{\alpha\beta\gamma} \} \) is independent of the choice of \( B_\alpha, f_{\alpha\beta} \) as a Čech cohomology class. This is the analogue of the fact that one can use a flat connection on a line bundle to change the transition function to be constant. If we have different \( B'_\alpha \), then \( d(B_\alpha - B'_\alpha) = 0 \) and hence we can write \( B_\alpha - B'_\alpha = df_\alpha \). Let
\[
f'_{\alpha\beta} = f_{\alpha\beta} - f_\alpha + f_\beta.
\]
\[ df'_{\alpha\beta} = A_{\alpha\beta} - B_{\beta} + B_{\alpha} - B_{\alpha} + B'_{\alpha} + B'_{\beta} - B'_{\beta} = A_{\alpha\beta} - B'_{\beta} + B'_{\alpha}. \]

Then
\[ c'_{\alpha\beta\gamma} = e^{i f'_{\alpha\beta}} e^{i f'_{\beta\gamma}} e^{i f'_{\gamma\alpha}} g_{\alpha\beta\gamma} = c_{\alpha\beta\gamma}. \]

If we have a different choice
\[ A_{\alpha\beta} - B_{\beta} + B_{\alpha} = df_{\alpha\beta}, \]
\[ \tilde{f}_{\alpha\beta} = f_{\alpha\beta} + \lambda_{\alpha\beta} \]
where \( \lambda_{\alpha\beta} \) is a constant function. Then,
\[ \tilde{c}_{\alpha\beta\gamma} = e^{i \tilde{f}_{\alpha\beta}} e^{i \tilde{f}_{\beta\gamma}} e^{i \tilde{f}_{\gamma\alpha}} g_{\alpha\beta\gamma} = e^{i \lambda_{\alpha\beta}} e^{i \lambda_{\beta\gamma}} e^{i \lambda_{\gamma\alpha}}. \]

Namely, it differs by a coboundary in the constant sheaf \( S^1 \).

For a line bundle, when the holonomy is trivial we get a covariant constant trivialization of the bundle. If the holonomy of a gerbe is trivial, then \( c_{\alpha\beta\gamma} \) is a coboundary, so that there are constants \( k_{\alpha\beta} \in S^1 \) such that
\[ c_{\alpha\beta\gamma} = k_{\alpha\beta} k_{\beta\gamma} k_{\gamma\alpha}. \]

Let
\[ h_{\alpha\beta} = k_{\alpha\beta} e^{-i f_{\alpha\beta}}. \]

Then,
\[ h_{\alpha\beta} h_{\beta\gamma} h_{\gamma\alpha} = g_{\alpha\beta\gamma} \]
and so we have a trivialization of the gerbe, which we call a flat trivialization.

Suppose that the line bundle is given by a Čech cocycle \( g_{\alpha\beta} \). Recall that a connection is a 1-form \( A_\alpha \) on \( U_\alpha \) such that
\[ iA_\beta - iA_\alpha = g_{\alpha\beta}^{-1} dg_{\alpha\beta}. \]

A section is \( f_\alpha : U_\alpha \to S^1 \) such that \( f_\alpha = g_{\alpha\beta} f_\beta \). It is covariant constant iff it satisfies the equation \( df_\alpha = iA_\alpha f_\alpha \). If we write \( f_\alpha = e^{i p_\alpha} \), then \( dp_\alpha = A_\alpha \). Therefore, a necessary condition is \( F_\alpha = dA_\alpha = 0 \), i.e., the connection is flat. In the case of a gerbe, \( dA_{\alpha\beta} \neq 0 \) in general. We have to allow the freedom to choose \( B_\alpha \) such that \( d(A_{\alpha\beta} - B_\beta + B_\alpha) = 0 \). Hence, the trivialization \( h_{\alpha\beta} \) satisfies a modified equation
\[ dh_{\alpha\beta} = i(A_{\alpha\beta} - B_\beta + B_\alpha) h_{\alpha\beta}. \]

Suppose that we have a second flat trivialization \( h'_{\alpha\beta} \); then \( g_{\alpha\beta} = h'_{\alpha\beta}/h_{\alpha\beta} \) defines a line bundle \( L \). Moreover,
\[ iB_\beta - iB_\alpha - iA_{\alpha\beta} = d \log h_{\alpha\beta}, \]
\[ iB'_{\beta} - iB'_{\alpha} - iA_{\alpha\beta} = d \log h'_{\alpha\beta}. \]

Hence,
\[ i(B' - B)_\beta - i(B' - B)_\alpha = d \log g_{\alpha\beta}. \]
and \( A_\alpha = (B' - B)_\alpha \) defines a connection on \( L \). By the definition of \( B_\alpha \) and \( B'_\alpha \),
\[
F_\alpha = dB_\alpha = dB'_\alpha.
\]
Hence the curvature \( dA_\alpha = 0 \). Thus, the difference of two flat trivializations of a gerbe is a flat line bundle. One can show that the converse is also true.

3.3. String connection. Recall that a connection on a line bundle induces a holonomy map \( \text{Hol} : \Omega X \to S^1 \). The holonomy of a connection on a gerbe has similar property. One way to understand it is via its analogy to topological quantum field theory. Recall that topological quantum field theory can be described as follows. For any oriented \( d \)-dimensional manifold \( D \), we associate a Hilbert space \( \mathcal{H}_D \). For any cobordism \( W \) such that \( \partial W = D_1 \cup -D_2 \), we associate a homomorphism \( \theta_W : \mathcal{H}_{D_1} \to \mathcal{H}_{D_2} \). \( \theta_W \) satisfies the gluing axiom. Suppose that \( \partial W_{12} = D_1 \cup -D_2, \partial W_{23} = D_2 \cup -D_3 \). We can glue \( W_{12}, W_{23} \) along \( D_2 \) to obtain \( W_{13} \). Then the gluing axiom is \( \theta_{13} = \theta_{23} \circ \theta_{12} \). The analogy for a gerbe is called a string connection. It contains the following ingredients:

(i) Let \( l : S^1 \to X \) be a smooth map. Since \( S^1 \) is one-dimensional, the pull-back of a gerbe with connection to the circle is flat and has trivial holonomy. Thus we have flat trivializations. For each \( l \), we associate the moduli space of flat trivializations \( \mathcal{L}_l \). \( \mathcal{L}_l \) is analogous to \( \mathcal{H}_D \). Recall that we identify flat trivializations if they differ by a flat line bundle with trivial holonomy. Then, for each loop we have a space which is acted on freely and transitively by the moduli space of flat line bundles \( H^1(S^1, S^1) \cong S^1 \). Hence, \( \mathcal{L}_l \) is isomorphic to \( S^1 \). In other words we have a principal \( S^1 \)-bundle \( \mathcal{L} \) over the free loop space \( \Omega X \).

We will pay special attention to the space of constant loops. Since \( X \) is embedded in \( \Omega X \) as the space of constant loops, it is interesting to compute the restriction of \( \mathcal{L} \) over \( X \). Suppose that \( f : S^1 \to X \) is a constant map. Then \( f \) is the composition of \( p : S^1 \to pt \) and \( i_f : pt \to X \). The pull-back gerbe \( i_f^*g_{\alpha\beta\gamma} \) is obviously trivial. Furthermore, \( i_f^*F_\alpha = 0, i_f^*A_{\alpha\beta} = 0 \). Any trivialization of \( i_f^*g_{\alpha\beta\gamma} \) is a flat trivialization. A key observation is that the flat line bundle over a point is trivial as well. Therefore, the pull-back gerbe with connection by \( i_f \) fixes a unique flat trivialization. Its pull-back by the projection map \( p : S^1 \to pt \) defines a canonical element \( s_f \in \mathcal{L}_f \) and hence a canonical section of \( \mathcal{L}|_X \). Hence, \( \mathcal{L}|_X \) is trivial with a canonical trivialization. Therefore, we obtain
Lemma 3.5. \( \mathcal{L}|_X \) is independent of the connection of the gerbe. Furthermore, it is trivial with a canonical trivialization.

(ii) Suppose that \( f : \Sigma \to X \), where \( \Sigma \) is a closed Riemann surface. Then the pull-back connection of the gerbe \( (f^*F, f^*A_{\alpha,\beta}) \) is flat. Its holonomy \( \text{Hol}_f = \{c_{\alpha\beta\gamma}\} \) is a cohomology class in \( H^2(\Sigma, S^1) \). Since \( H^2(\Sigma, S^1) = \text{Hom}(H_2(\Sigma, \mathbb{Z}), S^1) \), its evaluation on the fundamental class of \( \Sigma \) naturally identifies it as a complex number.

A more interesting case is the case of a Riemann surface with boundaries. Suppose that \( f : \Sigma \to X \), where \( \Sigma \) is a Riemann surface with boundary \( t_i \) with a fixed orientation-preserving parameterization \( \delta_i : S^1 \to t_i \). Let \( l_i = f \circ \delta_i \). Since \( H^2(\Sigma, S^1) = H^3(\Sigma, \mathbb{Z}) = 0 \), the pull-back gerbe is trivial and its holonomy is trivial as well. A flat trivialization restricts to a flat trivialization on each \( t_i \). Namely, it induces an element \( \sigma \) in \( \prod_i L_{l_i} \). A different flat trivialization of \( \Sigma \) differs by a flat line bundle of \( \tau \) of \( \Sigma \). It restricts to a flat line bundle \( \tau_i \) over each boundary circle viewed as a standard \( S^1 \) via \( \delta_i \). Recall that \( \pi_1(\Sigma) = \{\lambda_1, \cdots, \lambda_2g, l_1, \cdots, l_k | \prod_i [\lambda_2i-1, \lambda_2i]l_1 \cdots l_k = 1\} \).

Hence, \( \tau_1 \cdots \tau_k = 1 \). Hence, different flat trivializations induce elements differing by a multiplication of \( (\tau_1, \cdots, \tau_k) \) of \( \tau \in S^1 \) with \( \tau_1 \cdots \tau_k = 1 \). Suppose that \( L_1, \cdots, L_k \) are \( k \)-circle bundles. \( L_1 \otimes \cdots \otimes L_k \) can be constructed as follows. Let \( H \) be the \( (S^1)^k \)-bundle with fiber \( \prod_i (L_i)_x \). Then \( \otimes_i L_i = H \times S^1/(S^1)^k \) via the product homomorphism \( (S^1)^k \to S^1 \). From the previous construction,

Lemma 3.6 (Theorem 6.2.4 in [B]). The pull-back gerbe on \( \Sigma \) induces a canonical element \( \tilde{\theta}_\Sigma = \otimes L_{l_i} \), or a trivialization \( \theta_\Sigma : \otimes L_{l_i} \to S^1 \).

Note that if we reverse the orientation of a boundary circle \( l_i \) to obtain \( \bar{l}_i \), then \( L_{\bar{l}_i} = L_{l_i}^* \).

(iii) \( \theta \) has a decomposition property as follows. We decompose \( \Sigma \) along a circle \( \Sigma = \Sigma_1 \cup_{S^1} \Sigma_2 \) where \( \Sigma_1, \Sigma_2 \) are glued along boundary circles \( l, \bar{l} \). Let \( l_i, \bar{l}_i \) be the corresponding loop. Then \( l_i = \bar{l}_i \) and there is a canonical isomorphism \( L_{l_i} \otimes L_{\bar{l}_i} \cong S^1 \). Hence \( \otimes_i L_{l_i} \cong \otimes_i L_{l_i} \otimes L_{l_i} \otimes L_{\bar{l}_i} \). Under this identification, it is clear that

**Gluing axiom:**

\[ \theta_\Sigma = \theta_{\Sigma_1} \otimes \theta_{\Sigma_2}. \]
θ admits another interpretation closely analogous to topological quantum field theory. We can view Σ as a cobordism between incoming circles $l_i$ (with opposite orientation from the boundary orientation) and outgoing circles $l_j$. Then, $\theta_\Sigma$ can also be interpreted as an element of $\text{Hom}(\otimes_i L_{l_i}, \otimes_j L_{l_j})$. Then the gluing axiom corresponds to the usual gluing.

One application of $\theta_\Sigma$ is to define a connection on $\mathcal{L} \to \Omega X$. Take a path in the loop space

$$F : [0, 1] \times S^1 \to X.$$  

Applying the above Lemma, we obtain a canonical isomorphism between $\mathcal{L}_{\{0\} \times S^1}$ and $\mathcal{L}_{\{1\} \times S^1}$. This can be viewed as the parallel transport of a connection over $\mathcal{L}$. Recall that a section generated by the parallel transport from a point is precisely a covariant constant section. From our construction, the restriction of a flat trivialization on $[0, 1] \times S^1$ to each $\{t\} \times S^1$ gives a covariant constant section.

**Lemma 3.7.** The canonical section $s$ of $\mathcal{L}|_X$ is a covariant constant section.

**Proof:** Suppose that $F : [0, 1] \times S^1 \to X$ is a path of constant loops. Then $F$ is the composition of the projection to the first factor $p_1 : [0, 1] \times S^1 \to [0, 1]$ and a path of $X$, $i_F : [0, 1] \to X$. We first use $i_F$ to pull back a gerbe with its connection to $[0, 1]$. Such a pull-back is flat with trivial holonomy. Then, we construct a flat trivialization on $[0, 1]$. Now, we pull it back to $[0, 1] \times S^1$ to obtain a flat trivialization $s_F$ over $[0, 1] \times S^1$. It is clear that the restriction of $s_F$ to $\{t\} \times S^1$ is $s_{lt}$ for $l_t = F(t, \cdot)$. By the definition, $s_t = s_{lt}$ is a covariant constant section along the path.

**4. Gerbe on orbifold**

The previous construction has been generalized to orbifolds by Lupercio-Uribe [LU1], [LU2]. It is amazing that $\mathcal{L} \to X$ starts to become non-trivial on an orbifold! Therefore, it is more interesting to study gerbes on orbifold than on smooth manifolds!

4.1. **Basics.** Lupercio-Uribe’s construction is carried out for an arbitrary groupoid. The precise definition of gerbe over an orbifold is not important for us. Therefore, instead of giving a long technically correct definition, let me motivate the definition of groupoid from orbifold. We first start from a smooth manifold where one can view a groupoid as a
language to formalize the construction of an open cover. Let
\[ G_0 = \bigsqcup U_\alpha, G_1 = \bigsqcup U_{\alpha\beta}. \]
In the language of groupoids, \( G_0 \) is called the space of objects and \( G_1 \) is called the space of arrows. There are two maps
\[ s : U_{\alpha\beta} \to U_\alpha, t : U_{\alpha\beta} \to U_\beta. \]
s, t are called the source map and target map. Consider the fiber product
\[ G_2 = G_1 \times_s G_1 = \{(x, y); t(x) = s(y)\}. \]
Using an open cover, it is not hard to see that \( G_2 = \bigsqcup U_{\alpha\beta\gamma} \). There are also source and target maps \( s, t : G_2 \to G_0 \) by \( s(x, y) = s(x), t(x, y) = y \). In the language of open covers, it corresponds to inclusion maps
\[ s : U_{\alpha\beta\gamma} \to U_\alpha, t : U_{\alpha\beta\gamma} \to U_\gamma. \]
There is an additional multiplication map \( m : G_2 \to G_1 \) corresponding to the inclusion \( U_{\alpha\beta\gamma} \to U_{\alpha\gamma} \). To complete the definition of groupoid, we also need an identity \( e : G_0 \to G_1 \) and an inverse \( i : G_1 \to G_1 \). In our set-up, \( e : U_\alpha \to U_{\alpha\alpha}, I : U_{\alpha\beta} \to U_{\beta\alpha} \) are identity maps. These structure maps satisfy several obvious compatibility conditions for which we refer to [LU1]. We often use \( \mathcal{G} = \{ G_1 \xrightarrow{s,t} G_0 \} \) to denote the groupoid. The process of taking a refinement of an open cover is called Morita equivalence in groupoid language.

One can go on to construct
\[ G_n = G_{n-1} \times_s G_1. \]
It corresponds to the disjoint union of \((n + 1)\)-fold intersections.

With the above correspondence, we can state Lupercio-Uribe’s definition of gerbes over a groupoid as a function \( g : G_2 \to S^1 \) satisfying the obvious cocycle condition generalizing the condition on smooth manifolds. If two gerbes \( g_1, g_2 \) differ by a coboundary, we call them equivalent. An equivalence class of gerbes is a Čech cohomology class of the sheaf \( C^\infty(S^1) \) over the so-called classifying space \( B\mathcal{G} \) of the groupoid. Furthermore, we have a long exact sequence
\[ H^2(B\mathcal{G}, C^\infty(\mathbb{R})) \to H^2(B\mathcal{G}, C^\infty(S^1)) \to H^3(B\mathcal{G}, \mathbb{Z}). \]
It is different from the smooth case in that the characteristic class \( \tau([g]) \) is an integral cohomology class of the classifying space \( B\mathcal{G} \) instead of its space of orbits \( |\mathcal{G}| \). For an orbifold, we can always choose its groupoid representative \( \mathcal{G} \) with the property that the components of \( G_0 \) are contractible. Such a kind of groupoid is called a fine groupoid. Over
a fine groupoid, $C^\infty(\mathbb{R})$ is a fine sheaf. In this case, the equivalence class of gerbes is still classified by its characteristic class.

Over a groupoid a connection on a gerbe is a pair $(A, F)$ where $A$ is a one-form on $G_1$ and $F$ is a two-form on $G_0$ satisfying the condition.

$$t^*F - s^*F = dA, \quad i\pi^*_1A + i\pi^*_2A + i\pi^*I^*A = g^{-1}dg.$$ 

Now, to extend the definition of gerbe to orbifolds, we just have to associate a groupoid to orbifolds which was done by Moerdijk-Pronk [MP]. To this purpose, we just have to construct $G_0, G_1$ and $s, t$.

Let $X$ be an orbifold and $(\tilde{U}_i, G_{\tilde{U}_i}), Tran(U_i, U_j)$ be an orbifold atlas. We simply define $G_0 = \sqcup_i \tilde{U}_i, G_1 = \sqcup_{ij} Tran(U_i, U_j)$. $s, t$ are natural projections

$$s : Tran(U_i, U_j) \rightarrow \tilde{U}_i, \quad t : Tran(U_i, U_j) \rightarrow \tilde{U}_j.$$ 

We call the above groupoid an orbifold groupoid.

An orbifold morphism corresponds to a Morita equivalence of morphisms between orbifold groupoids. An obvious and important fact is that

**Remark 4.1.** An orbifold morphism between orbifolds pulls back a gerbe with connection to a gerbe with connection.

In particular, if $f : Y \rightarrow X$ is a smooth orbifold morphism from a smooth manifold $Y$ (viewed with trivial orbifold structure) and $[g]$ is a gerbe with connection $(A, F)$ on $X$, then $f^*[g], (f^*A, f^*F)$ is a gerbe with connection on a smooth manifold $Y$ even though we started from an orbifold $X$. Therefore, the previous construction on the holonomy line bundle $L$ goes through trivially. However, its restriction to the inertia orbifold is no longer trivial.

We first look at the case of discrete torsion for a global quotient orbifold. The inner local system has been constructed in [R1]. We would like to show that it agrees with the holonomy line bundle from the gerbe induced by discrete torsion. Recall that discrete torsion is a two-cocycle $\alpha : G \times G \rightarrow S^1$. Being a cocyle means

$$\alpha_{g,1} = \alpha_{1,g} = 1, (\delta\alpha)_{g,h,k} = \alpha_{h,k}\alpha^{-1}_{gh,k}\alpha_{g,hk}\alpha^{-1}_{g,h} = 1.$$ 

The groupoid presentation of the global quotient orbifold $X/G$ is a translation groupoid with $G_0 = X, G_1 = X \times G$ and $s(x, g) = x, t(x, g) = gx$. We will use stacky notation $[X/G]$ to denote this groupoid structure. One can check that $G_2 = X \times G \times G$. $\alpha$ induces a gerbe on $G$ in the obvious way. Furthermore, we can choose a flat connection with $F = 0, A = 0$. Recall that the inertia orbifold is $[(\sqcup gX^g)/G]$ and let
Recall [R1] that we can define an inner local system on $[X/G]$ as follows. Consider the trivial bundle $\sqcup_g X^g \times C_g$ where we use $C_g$ to denote the fiber $C$ associated to $X^g$ and $1_g$ to denote the element $1$ in $C_g$. Then, we define an action of $g : C_h \to C_{ghg^{-1}}$ by
\[ g(1_h) = \gamma_{g,h} 1_{ghg^{-1}}. \]

Let $L_\alpha$ be the quotient of the trivial bundle under the above action.

**Theorem 4.2.**
\[ L|\wedge[X/G] = L_\alpha. \]

**Proof:** We start with some algebraic preliminaries. If a 2-cocycle $\alpha$ can be expressed as a coboundary $\alpha_{g,h} = \rho_g \rho_h \rho_{gh}^{-1}$, we call $\rho$ a flat trivialization of $\alpha$.

A 2-cocycle $\alpha$ corresponds to an equivalence class of group extensions
\[ 1 \to S^1 \to \tilde{G}_\alpha \to G \to 1 \]

The group $\tilde{G}_\alpha$ can be given the structure of a compact Lie group, where $S^1 \to \tilde{G}_\alpha$ is the inclusion of a closed subgroup. The elements in the extension group can be represented by pairs $\{(g,a) \mid g \in G, a \in S^1\}$ with the product $(g_1,a_1)(g_2,a_2) = (g_1g_2, \alpha_{g_1g_2}a_1a_2)$.

**Lemma 4.3.** There is a 1-1 correspondence between the set of characters of $\tilde{G}_\alpha$ which restrict to scalar multiplication on the central $S^1$ and the set of flat trivializations of $\alpha$.

**Proof:** If $\psi : \tilde{G}_\alpha \to S^1$ is such a character then we define an associated trivialization of $\alpha$ via $\rho(g) = \psi(g,1)$. Note that $\rho(g h) = \psi(gh,1) = \alpha_{g,h}^{-1}\psi(gh,\alpha_{g,h}) = \alpha_{g,h}^{-1}\psi((g,1)(h,1)) = \alpha_{g,h}^{-1}\rho(g)\rho(h)$. Conversely, given a flat trivialization $\rho : G \to S^1$, we simply define $\psi(g,a) = a\rho(g)$. Note that
\[ \psi((g,a)(h,b)) = \psi(gh,\alpha_{g,h}ab) = a\rho(g)\rho(h) = a\rho(g)b\rho(h) = \psi(g,a)\psi(h,b). \]

We have proved the lemma \(\square\)

Now, we come back to the proof of theorem. Any element $g \in G$ generates an abelian subgroup $\langle g \rangle$. It pulls back the 2-cocycle $\alpha$ and we can define the corresponding group extension $\langle \tilde{g} \rangle_\alpha$. It is well known that any 2-cocycle of a finite cyclic abelian group is a coboundary. Hence, we have a nontrivial set of flat trivializations and hence a set of characters of $\langle \tilde{g} \rangle_\alpha$. For any $h \in G$, $h$ sends $\langle g \rangle$ to $\langle hgh^{-1} \rangle$. We would like to calculate the action of $h$ on the set of
characters (hence flat trivializations). Given a character \( \phi \) for \( \prec g \succ \alpha \) and a lifting \((h, a)\) of \( h \), the action is defined by the formula

\[
(h, a) \phi(x, b) = \phi((h, a)(x, b)(h, a)^{-1}).
\]

\[
(h, a)(x, b)(h, a)^{-1} = (hx, \alpha_{h,x}ab)(h^{-1}, \alpha_{h,h^{-1}}^{-1}a^{-1}) = (hxh^{-1}, \alpha_{h,x}h^{-1}\alpha_{h,h^{-1}}b) = (hxh^{-1}, \alpha_{h,h^{-1}}^{-1}b).
\]

Recall that \( \wedge[X/G] = (\sqcup_g X^g \times \{g\})/G \). It can be interpreted as follows. Let \( f : S^1 \to [X/G] \) be a constant good map with Chen-Ruan characteristic \((g)\). Then, it factors through the abelian orbifold \( [X/\prec g \succ] \) which is equivalent to the previous one. This is the action of \( G \) on \( \sqcup_g X^g \times \{g\} \). Now we consider the space of constant morphisms to \([X/\prec g \succ]\), which is parameterized by \( X^g \times \{g\} \). It admits a flat gerbe from \( G \) through the embedding \( \prec g \succ \to G \) (denoted by \( \alpha_{<g>} \)), since \( \prec g \succ \) is an abelian group and such a gerbe is trivial. Pick a flat trivialization \( \rho \) of \( \alpha_{<g>} \). \( \rho \) defines a section \( s_\rho \) of \( L_{X^g \times \{g\}} \). We can use the same argument as in the smooth case to show that \( s_\rho \) is covariant constant. Therefore, \( L_{X^g \times \{g\}} \) is trivial as a flat line bundle. However, it does not have a canonical flat trivialization since a different flat trivialization of \( \alpha_{<g>} \) will define a different flat trivialization of \( L_{X^g \times \{g\}} \). Using the calculation above, we conclude that the action of \( G \) on \( L_{X^g \times \{g\}} \) is via the character \( \gamma_{g,h} \).

We have proved the theorem.

**Theorem 4.4** (Lupercio-Uribe). \( L|X| \) is flat.

**Proof:** Lupercio-Uribe proved their theorem using a generalization of Brylinski’s relevant formula in the smooth case. Here, we use our previous analysis to give a direct, geometric proof. The question is local. Therefore, we can assume that our orbifold is a global quotient \([\mathbb{R}^n/G]\) with gerbe and connection given by \((\alpha^0, F, A)\). It follows that \( \alpha^0 = \alpha\beta \) where \( \alpha : G \times G \to S^1 \) is a 2-cocyle and \( \beta \) is a coboundary over \([\mathbb{R}^n/G]\) and \((\alpha^0, F, A) = (\alpha, 0, 0) + (\beta, F, A)\). Let \( L^0, L \) and \( L' \) represent the holonomy line bundles of \((\alpha^0, F, A)\), \((\alpha, 0, 0)\) and \((\beta, F, A)\) restricted to the inertia orbifold respectively. It is clear that \( L^0 = L \otimes L' \) and , fixing any path in the inertia orbifold, the parallel transport on \( L^0 \) is the tensor product of those on \( L \) and \( L' \). Since \( \beta \) is a coboundary , there is a flat trivialization of \( L' \). The fact that \( L \) is flat implies that \( L^0 \) is flat by Theorem 4.2.
4.2. Holonomy on an orbifold Riemann surface. If we only consider maps from a smooth Riemann surface, the construction in the smooth case can be readily generalized to the case of orbifolds. A more interesting case is the case when $f: \Sigma \to X$ is a good map from an orbifold Riemann surface $\Sigma$. In orbifold quantum cohomology, we have to consider its generalization where $\Sigma$ is a nodal orbifold Riemann surface.

Unfortunately, many things go wrong and we do not have a straightforward generalization of string connection. One of the critical facts for an oriented smooth Riemann surface is $H^2(\Sigma, \mathbb{Z}) = \mathbb{Z}$ generated by its fundamental class $\sigma$. We use this fact to interpret the holonomy of a gerbe with a connection as a number in $\mathbb{S}^1$. We start our discussion from the following computation of $H^2(B\Sigma, \mathbb{Z})$ for an orbifold Riemann surface. Indeed, a certain subtlety arises.

Let $\Sigma$ be an orientable orbifold Riemann surface, with singular points $\{x_1, \ldots, x_m\}$ and corresponding multiplicities $\{k_1, \ldots, k_m\}$. Note that the underlying topological space $|\Sigma|$ of the orbifold $\Sigma$ is a topological surface. Let $B\Sigma$ be the corresponding classifying space of the orbifold.

Given an action of the group $G$ on space $X$, let $EG$ be a free $G$ space which is contractible. One can utilize the Borel construction $X_G = EG \times_G X$. Define $H^*_G(X, \mathbb{Z}) = H_*(EG \times_G X, \mathbb{Z})$. It is well known that there is an action of $S^1$ on a 3-manifold $M$ such that $[M/S^1]$ is the given orbifold $\Sigma$. It is well-known that up to a weak homotopy equivalence $B\Sigma \cong M_G$. Hence, $H_2(B\Sigma, \mathbb{Z}) = H^1_S(M, \mathbb{Z})$.

There is a canonical map $\pi_X: X_G \to X/G$ and thus a homomorphism $H^*_G(X, \mathbb{Z}) \to H_*(X/G, \mathbb{Z})$ which is known to be an isomorphism if the action of $G$ on $X$ is free.

**Theorem 4.5.** Let $\Sigma$ be an oriented orbifold Riemann surface. $H_2(B\Sigma, \mathbb{Z}) = \mathbb{Z}$. However, $\pi_M*$ does not map the generator to the fundamental class $\sigma$. Instead, $\pi_M*(e) = \pm r\sigma$ where $e \in H_2(B\Sigma, \mathbb{Z})$ is the generator and $r$ is the least common multiple of the $k_i$.

**Proof:** All the coefficients are integers without further specification.

Choose small open discs $D_i$ centered around $x_i$ such that their closures $B_i$ are disjoint. Let $V_i = f^{-1}(B_i)$, $V = \bigcup_i V_i$, $V_i^* = f^{-1}(D_i)$, $V^* = \bigcup_i V_i^*$ and $U = M - V^*$. Note that the actions of $S^1$ on $U$ and $U \cap V$ are free.

Now $H^2_{S^1}(U) \cong H_2(U/S^1, \mathbb{Z}) = 0$ since the action is free and $U/S^1$ is a smooth surface with boundary. $H_j(V/S^1, \mathbb{Z}) = 0$ for $j > 0$ since $V/S^1$ is the disjoint union of closed discs. It is well known that the local model of a singular point of an orbifold Riemann surface is the quotient of a disc by the action of a cyclic group. It follows that
$H^1_{S^1,Z}(V) = \bigoplus_i H_j(B\mathbb{Z}_k, \mathbb{Z})$ and $H^2_{S^1,Z}(V) = 0$, $H^1_{S^1,Z}(V) = \bigoplus_i \mathbb{Z}_k$, since $H_2(B\mathbb{Z}_k, \mathbb{Z}) = 0$ for all positive integers.

One has the following commutative diagram by naturality of Mayer-Vietoris sequence

$$
\begin{array}{c}
H^2_{S^1}(M, \mathbb{Z}) \\ \pi_{M*} \downarrow \\
H^2(M/S^1, \mathbb{Z}) \\
\end{array}
\rightarrow
\begin{array}{c}
H^1(U \cap V, \mathbb{Z}) \\
h_1 \downarrow \\
H_1((U \cap V)/S^1, \mathbb{Z}) \\
\end{array}
\rightarrow
\begin{array}{c}
H^1(U, \mathbb{Z}) \oplus H^1_{S^1}(V, \mathbb{Z}) \\
h_2 \oplus 0 \\
H_1(U/S^1, \mathbb{Z})
\end{array}
$$

Now the homomorphisms $h_1, h_2$ are isomorphisms by the remark before the statement of the Theorem. A diagram chase argument shows that the map $p_{M*}$ is given by $a \mapsto \pm rb$ where $a$ is the generator of $H^2_{S^1}(M, \mathbb{Z}) \cong \mathbb{Z}$, $b$ is the generator of $H_2(M/S^1, \mathbb{Z}) \cong \mathbb{Z}$ and $r$ is the least common multiple of the $k_i$.

It is clear that any gerbe connection $(F, A)$ on an orbifold Riemann surface is flat for dimension reasons. Therefore, we can define its holonomy $Hol(F, A)$ as a class in $H^2(B\Sigma, S^1)$ where $S^1$ means the constant sheaf. One can evaluate $Hol(F, A)$ on the generator $e \in H_2(B\Sigma, \mathbb{Z})$ to obtain a number $Hol(F, A)(e) \in S^1$. However, the geometric evaluation is over the fundamental class $\sigma$. By the previous theorem, $p_{M*}(e) \neq \sigma$. Hence, $Hol(F, A)(\sigma)$ is undefined.

Now, we take a different point of view of the same phenomenon. Consider the gerbe connection over the local 2-dimensional orbifold disc $[D/\mathbb{Z}_k]$. The connection as well as its holonomy is trivial. A choice of flat trivialization restricts to a flat trivialization on the boundary circle $\partial(D/\mathbb{Z}_k)$. Namely, we obtain an element of $\mathcal{L}_{\partial(D/\mathbb{Z}_k)}$. However, the space of flat line bundles on $[D/\mathbb{Z}_k]$ is non-trivial. In fact, it is parameterized by $\mathbb{Z}_k$. They induce a set of $\mathbb{Z}_k$-points on $\mathcal{L}_{\partial(D/\mathbb{Z}_k)}$. Now, we go back to the closed orbifold Riemann surface $\Sigma$ with orbifold points at $(z_1, \cdots, z_l)$ of multiplicity $k_1, \cdots, k_l$ and $f : \Sigma \rightarrow X$. We decompose $\Sigma$ as the disjoint union of orbifold discs $[D_i/\mathbb{Z}_{k_i}]$ and $V = \Sigma - \sqcup_i [D_i/\mathbb{Z}_{k_i}]$. Then the flat trivialization at $V$ specifies an element on $\mathcal{L}_{l_i}$ of each boundary circle $l_i$. The gluing law indicates that we should associate a set of $k_1 \cdots k_l$-numbers in $S^1$.

5. ORBIFOLD QUANTUM COHOMOLOGY TWISTED BY A FLAT GERBE

5.1. String connection on an orbifold and orbifold stable maps.

Recall the compatibility condition of an inner local system over each $X_{(g_1, g_2)} : \theta_{(g_1, g_2)} : e^* \xi_{\mathcal{L}}(g_1) \otimes e^* \xi_{\mathcal{L}}(g_2) \otimes e^* \xi_{\mathcal{L}}((g_1g_2)^{-1}) \cong 1$. The purpose of this condition is as follows. $X_{(g_1, g_2)} = X_{(g_1g_2, g_3)}$ with $g_3 = (g_1g_2)^{-1}$ can be identified as the moduli space of degree zero genus zero maps with three marked points $\overline{\mathcal{M}}_{0,3}(X, J, 0, x)$. The evaluation maps at marked
points are $e_i$. Let $\alpha_i \in H^*(X(g_i), L(g_i))$. The trivialization $\theta_{(g_1, g_2)}$ maps $e_i^*\alpha_1 \wedge e_j^*\alpha_2 \wedge e_k^*\alpha_3$ to an ordinary cohomology class of $X_{(g_1, g_2, g_3)}$ and hence can be integrated. The latter property allows us to define the twisted orbifold product. To carry out the same construction for orbifold quantum cohomology, we have to construct the trivialization $\theta$ over $\overline{M}_{g,k}(X, J, A, x)$ for general $g, k, A$. This can be accomplished by Theorem 5.1.

Suppose that $f : \Sigma \to X$ is an orbifold morphism. We take a real blow-up at the marked points to obtain a Riemann surface with boundary $\Sigma^\dagger$. Let $l_\infty$ be the corresponding boundary cycle. It is clear that each $f : \Sigma \to X$ induces a morphism $f^\dagger : \Sigma^\dagger \to X$. It is clear that $f^\dagger = f^\dagger(l_\infty)$ is a constant loop. Moreover, we have an identification $f_\infty = e_i(f)$.

Next, the holonomy $\theta_\Sigma = \theta_{\Sigma^\dagger}$ is interpreted as

$$\theta_{\Sigma^\dagger} : \otimes i \leq i \leq k e_i^*L \to S^1.$$ 

Next, we extend the above discussion to orbifold stable maps. Suppose that $f : \Sigma \to X$ is an orbifold stable map where $\Sigma$ is a marked orbifold nodal Riemann surface. For simplicity, we assume that $\Sigma = \Sigma_1 \wedge \Sigma_2$ joining at point $p \in \Sigma_1, q \in \Sigma_2$. We observe that $f_{q,\infty}$ is the same as $f_{p,\infty}$ with the reversed orientation. Hence, $L_{f_{p,\infty}} = L_{f_{q,\infty}}^*$. Suppose that the marked points on $\Sigma_1$ are $x_1, \ldots, x_l$ and the marked points on $\Sigma_2$ are $x_{l+1}, \ldots, x_k$. We have

$$\theta_{\Sigma_1} : \otimes 1 \leq i \leq l e_i^*L \otimes e_p^*L, \quad \theta_{\Sigma_2} : \otimes l+1 \leq j \leq k e_j^*L \otimes e_q^*L.$$ 

Using the canonical isomorphism $e_p^*L \otimes e_q^*L \to S^1$, we obtain

$$\theta_\Sigma : \otimes 1 \leq i \leq k e_i^*L \to S^1.$$ 

The analogue of the gluing law is the following theorem.

**Theorem 5.1 (gluing law).** $\theta_{\Sigma^\dagger}$ is continuous with respect to the degeneration of orbifold stable maps, i.e., it induces a continuous trivialization of $\otimes i e_i^*L$ over $\overline{M}_{g,k}(A, X, J, x)$.

**Proof:** $\theta$ is clearly continuous over each stratum of $\overline{M}_{g,k}(A, X, J, x)$. We only have to check that it is continuous with respect to the degeneration of orbifold stable maps. It is enough to discuss the case of creation of a new nodal point. Suppose that $(f_n, (\Sigma_n, z_n), \xi_n)$ converges to $(f_0, (\Sigma_0, z_0), \xi_0)$ and $p \in \Sigma_0$ is the nodal point. It is instructive to see the degeneration of $\xi_n$ to $\xi_0$. Recall the construction in [CR2].
Locally, 
\[ f_n : W_{t_n} \to (V_p - \{p\})/G_p, \]
where \( W_t = \{xy = t; |x|, |y| < \epsilon\} \) and \((V_p, G_p, \pi_p)\) is a uniformizing system of \( p \in X \). The key is to construct the lifting \( \tilde{f}_n \) mapping into \( V_p \).

By Lemma 2.2.4 of [CR2], \( \xi_n \) determines a characteristic \( \theta_n : \pi_1(W_{t_n}) \to G_p \).

Suppose that \( g \) is the image of a generator and \( m \) is the order of \( g \). Then \( \theta_n \) determines a covering \( \tilde{W}_{t_n}^m \to W_{t_n} \). The argument of [CR2, Lemma 2.2.4] constructs a lifting \( \tilde{f}_n : \tilde{W}_{t_n}^m \to V_p \). Then the convergence of \( f_n \) as a good map is interpreted as convergence of ordinary maps \( \tilde{f}_n \) to \( f_0 : \tilde{W}_0^m \to V_p \), which gives a natural compatible system \( \xi_0 \) at \( p \). Note that \( \Sigma_0 \) acquires a natural orbifold structure at the nodal point \( p \), whose uniformizing system is given by \((\tilde{W}_0^m, Z_m)\), where the action of \( Z_m \) on \( \tilde{W}_0^m \) is the limit of the action on \( W_{t_n}^m \).

Let \( S_n^1 \subset W_{t_n} \) be the circle given by \(|x| = \frac{|t_n|}{2\epsilon}\) with complex orientation of \( x \). \( S_n^1 \) converges to a constant loop supported at \( p \). On the other hand, the same \( S^1 \) with opposite orientation can be expressed as \(|y| = \frac{|t_n|}{2\epsilon}\) (denoted by \( S_n^{1*} \)). \( S_n^{1*} \) converges to the constant loop support at \( q \). We decompose \( \Sigma_n \) along \( S_n^1 \) as \( \Sigma_n = \Sigma_n^1 \cup \Sigma_n^2 \). Then \( \Sigma_n^1 \) converges to \( \Sigma_1 \) and \( \Sigma_n^2 \) converges to \( \Sigma_2 \). The above construction implies that \( f_n|_{S_n^1} \) converges to \((f_0)_{\infty} \). Moreover, \( f_n|_{S_n^{1*}} \) as a good map converges to \((f_0)_{\infty} \). By the gluing axiom, \( \theta_{\Sigma_n} : \otimes_l e_l^* L \to S^1 \) can be decomposed as the product of \( \theta_{\Sigma_n^1} : \otimes_1 e_l^* L \otimes L_{f_n|_{S_n^1}} \) and \( \theta_{\Sigma_n^2} : \otimes_l e_l^* L \otimes L_{f_n|_{S_n^{1*}}} \) using canonical trivialization \( L_{f_n|_{S_n^1}} \otimes L_{f_n|_{S_n^{1*}}} \). It is clear that \( \theta_{\Sigma_n} \) converges to \( \theta_{\Sigma_\infty} \). □

5.2. **Twisted orbifold Gromov-Witten invariants.** So far, we have not yet brought in the flatness condition. Recall that \( \otimes_l e_l^* L \) in our construction is used as the coefficient system or flat line bundle. Therefore, we also need to construct a flat trivialization. This requires the assumption of flatness of the gerbe.

**Theorem 5.2.** For a flat gerbe, \( \theta_{\Sigma_\infty} \) is a flat trivialization.

**Proof:** A flat bundle is completely determined by its holonomy around a loop. Even though \( \overline{M}_{g,k}(A, X, J, \mathbf{x}) \) is not a smooth manifold in general, we can still discuss the flat trivialization of a flat bundle. Namely, it is enough to determine if the trivialization is flat around
each loop. Since flatness is a local condition, it is enough to prove that it is flat along a curve \( f : [0, 1] \to \overline{\mathcal{M}}_{g,k}(A, X, J, x) \).

Recall that there is a universal family

\[
\overline{U}_{g,k}(A, X, J, x) \to \overline{\mathcal{M}}_{g,k}(A, X, J, x)
\]
as an orbifold fiber bundle whose fiber is the domain of orbifold stable maps modulo automorphisms. The pullback \( \pi : f^*U_{g,k}(A, X, J, x) \to [0, 1] \) is an orbifold Riemann surface bundle. The total space \( f^*U_{g,k}(A, X, J, x) \) is an orbifold 3-manifold with boundary. Each marked point defines a section \( s_i \). For simplicity, we first assume that there is no nodal point. Now, we take a real blow-up along the image of \( s_i \) to obtain \( \pi^\dagger : f^*U_{g,k}(A, X, J, x)^\dagger \) \( \to [0, 1] \). Then we replace each \( s_i \) by \( S^1 \times [0, 1] \) (denoted by \( S^1_i \times [0, 1] \)). It is clear that \( \pi^\dagger(\cdot) \) is the real blow-up of \( \pi^{-1}(\cdot) \) along the marked points, where \( S^1_i \times \{ t \} \) is precisely the infinity circle associated to the \( i \)-th marked point of \( \Sigma_t = \pi^{-1}(\cdot) \).

A moment of thought tells that \( f^*U_{g,k}(A, X, J, x)^\dagger \) is an oriented 3-manifold with the boundary given by

\[
\partial f^*U_{g,k}(A, X, J, x)^\dagger = \pi^\dagger(0) \cup \cup_i S^1_i \times [0, 1] \cup \pi^\dagger(1).
\]

Furthermore, the identification happens precisely at the infinity circles corresponding to marked points on \( \pi^\dagger(0), \pi^\dagger(1) \). Furthermore, there is an evaluation \( e : U_{g,k}(A, X, J, x) \to X \) whose restriction to the \( i \)-th marked point defines \( e_i \). It induces an evaluation map

\[
e_i^\dagger : f^*U_{g,k}(A, X, J, x)^\dagger \to X
\]
whose restriction to each infinity circle defines the corresponding evaluation map \( e_i^\dagger = f_i^\infty \). Since the gerbe is flat, the holonomy around the boundary \( \partial f^*U_{g,k}(A, X, J, x)^\dagger \) is zero. Recall that the restriction of a flat trivialization of \( S^1_i \times [0, 1] \) to its boundary defines parallel transport from \( L_{f_i^\infty} \) to \( L_{f_i^\infty} \). By our definition, the restriction of the flat trivialization to the boundary of \( \pi^\dagger(0), \pi^\dagger(1) \) defines elements \( \theta_0 \in \otimes L_{f_i^\infty}, \theta_1 \in \otimes L_{f_i^\infty} \) respectively. The property that the total holonomy around \( \partial f^*U_{g,k}(A, X, J, x)^\dagger \) is zero can be interpreted as the statement that parallel transport maps \( \theta_0 \) to \( \theta_1 \). Then we prove that \( \theta \) is flat.

If we have nodal points, we do a real blow-up along the nodal point. It creates an additional boundary component. However, it is clear that the total holonomy of the additional components cancel each other. The above argument still applies.

\( \square \)
Remark 5.3. The proof of the above theorem depends critically on the flatness of the gerbe.

Now, we are ready to construct twisted orbifold GW-invariants. Using $e_i$, we can pull back $\mathcal{L}$ (now as an orbifold vector bundle) to define the tensor product $\otimes_i e_i^* \mathcal{L}$. Then $\theta$ provides a flat trivialization

$$
\theta : \otimes_i e_i^* \mathcal{L} \to \mathbb{C},
$$

continuous with respect to the topology of $\overline{\mathcal{M}}_{g,k}(A, X, J, \mathbf{x})$. Suppose that $\mathbf{x} = \prod_i X_{(g_i)}$. It induces a homomorphism

$$
\otimes_i H^*(X_{(g_i)}, \mathcal{L}_{(g_i)}) \to H^*(\overline{\mathcal{M}}_{g,k}(A, X, J, \mathbf{x}), \mathbb{C}),
$$

by

$$(\beta_1, \cdots, \beta_k) \to \theta^*(\wedge_i e_i^* \alpha_i),$$

where

$$
\theta^* : H^*(\overline{\mathcal{M}}_{g,k}(A, X, J, \mathbf{x}), \otimes e_i^* \mathcal{L}_{(g_i)}) \to H^*(\overline{\mathcal{M}}_{g,k}(A, X, J, \mathbf{x}), \mathbb{C})
$$

is the isomorphism induced by $\theta$.

Definition 5.4. The orbifold GW-invariants twisted by a flat gerbe are defined as

$$
\Psi^{X,J}_{(g,k, A, x, (A,F))}(\alpha_1^{l_1}, \cdots, \alpha_k^{l_k}) = \theta^*(\prod_{i=1}^k c_1(L_i)^t_i e_i^* \alpha_i) \overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})^{vir},
$$

where $L_i$ is the line bundle generated by the cotangent space of the $i$-th marked point and $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})^{vir}$ is the virtual fundamental cycle constructed in [CR2].

A standard argument in Gromov-Witten theory will show that our twisted orbifold GW-invariants satisfy standard axioms [CR2]. In particular, it implies that there is an associative quantum multiplication on $H^*_{orb}(X, \mathcal{L})$.

6. Computation

The current treatment of gerbes in the literature is usually abstract. One of the author’s goals for this article is to be as concrete as possible. In this section, we will try to figure out how far we can go to compute holonomy inner local system explicitly. We will divide this section into smooth versus orbifold cases.
6.1. **Smooth case.** When $X$ is smooth, the holonomy line bundle $\mathcal{L}|_X$ is canonically trivial. Hence $\otimes_i e_i^*\mathcal{L}|_X$ is canonically trivial. Recall that for a stable map $f : \Sigma \to X$, $\theta_f$ is another trivialization $\otimes_i e_i^*\mathcal{L}|_X \cong S^1$. Therefore, the difference of two trivializations is a number in $S^1$. By abuse of notation, we still denote it by $\theta_f$. On the other hand, we can also associate the holonomy $\text{Hol}_f \in S^1$. The key theorem in the smooth case is

**Theorem 6.1.** Suppose that $X$ is smooth and $\mathcal{L}|_X$ is trivialized by its canonical trivialization. Then $\theta_f = \text{Hol}_f$.

**Proof:** Let $g$ be the cocycle representing the gerbe. First, we assume that the stable map $f : \Sigma \to X$ has domain $\Sigma$ as an irreducible Riemann surface. We take a real blow-up to obtain $\Sigma^!$. There is an obvious map $\pi : \Sigma^! \to \Sigma$ by contracting $l_i$ to $x_i$. Let $f^! = f \circ \pi$. $f^*g$ with its connection is flat on $\Sigma$. We can define its holonomy $\text{Hol}_f \in H^2(\Sigma, S^1)$. Since $H^2(\Sigma, S^1) = S^1$. We use $\text{Hol}_f$ to denote its Čech cocycle or the number through the above isomorphism without any confusion. From the construction of $\text{Hol}_f$, $f^*g = \text{Hol}_f \delta r$. Namely, they differ by a coboundary. Furthermore, we are allowed to change $r$ by a constant cochain. Therefore, we can choose $r$ in such fashion that $r|_z$ is a flat trivialization of $f^*g|_z$. Therefore, $s_{f_i} = \pi^*r|_{x_i}$, where $f_i$ is the restriction of $f^!$ to the boundary circle $l_i$. We use $f^!$ to pull back the gerbe represented by the cocycle $g$. Since $H^2(\Sigma^!, S^1) = 0$, $\pi^*\text{Hol}_f$ is trivial. Choose a trivialization $\pi^*\text{Hol}_f = \delta h$ where $h$ is constant. A flat trivialization of $(f^!)*g$ is of the form $\delta (h^*r)$. Then $\theta_f$ is the image of $((h^*r)|_{l_1}, \ldots, (h^*r)|_{l_k})$ in $\otimes_i \mathcal{L}_{f_i}$ and $\theta_f = \prod_i h_i s_{f_i}$. We claim that $\prod_i h_i$ is $\text{Hol}_f$ through the canonical isomorphism $P : H^2(\Sigma, S^1) \cong S^1$. This canonical isomorphism is defined by the evaluation on the fundamental class of $\Sigma$. Let’s consider the isomorphism induced by $\pi$

$$\pi^* : H^2(\Sigma, \mathbb{Z}, S^1) \to H^2(\Sigma^!, \partial \Sigma^!, S^1)$$

and the isomorphism induced by the inclusion $(\Sigma, \emptyset) \subseteq (\Sigma, \mathbb{Z})$

$$H^2(\Sigma, \mathbb{Z}, S^1) \to H^2(\Sigma, S^1).$$

Note that $\text{Hol}_f|_z = 1$ and hence can be viewed as an element of $H^2(\Sigma, \mathbb{Z}, S^1)$ as well as $H^2(\Sigma^!, \partial \Sigma^!, S^1)$ as its pull-back by $\pi$. Therefore, the image of $\text{Hol}_f$ under $P$ can also be obtained by the evaluation of $\pi^*\text{Hol}_f$ on the relative fundamental class of $(\Sigma^!, \partial \Sigma^!)$ which is precisely $\prod_i h_i$ since $\langle \pi^*\text{Hol}_f, [\Sigma^!, \partial \Sigma^!] \rangle = \langle \delta h, [\Sigma^!, \partial \Sigma^!] \rangle = \langle h, [\partial \Sigma^!] \rangle = \prod_i h_i$. 


Now, we consider the general case in which $\Sigma$ may have more than one component. Then, we apply previous argument for each component. By the gluing axiom, $\theta_f$ is multiplicative. $\text{Hol}_f$ is obviously multiplicative. Hence, the theorem is true for multi-components $\Sigma$ as well. \(\square\)

Suppose that we have a flat connection $(A, F)$. Then we have the global holonomy $\text{Hol} \in H^2(X, S^1)$. It is clear that $\text{Hol}_f = \text{Hol}(f_*[\Sigma])$. Then we prove that

**Corollary 6.2.** Suppose that $(g, F, A)$ is a flat gerbe. Under the canonical trivialization of $\mathcal{L}|_{X}$, the twisted GW-invariant

$$
\Psi^{X,j}_{(g,k,A,x,(A,F))}(\alpha_1^{l_1}, \cdots, \alpha_k^{l_k}) = \text{Hol}(A)\Psi^{X,j}_{(g,k,A,x)}(\alpha_1^{l_1}, \cdots, \alpha_k^{l_k}).
$$

**6.2. Discrete torsion.** Suppose that we have a global quotient orbifold $\left[ X/G \right]$ and a discrete torsion $\alpha : G \times G \rightarrow S^1$. Recall that we can express the inertia orbifold as a global quotient $\left[ X/G \right] = \left( \cup_g X^g \times \{g\} \right)/G$. Furthermore, the holonomy line bundle

$$
\mathcal{L}_{\wedge [X/G]} = \left( \cup_g X^g \times \{g\} \right) \times \gamma \mathbb{C}.
$$

We would like to express the moduli space of orbifold stable maps as a $G$-global quotient as well. This has already been done in [JJK]. Let’s briefly review their construction. As we mentioned previously, by pulling back via the $G$-bundle $X \rightarrow X/G$, an orbifold stable map is equivalent to a $G$-orbifold cover $E \rightarrow \Sigma$ and a $G$-map $\phi : E \rightarrow X$. A $G$-stable map has additional data $\tilde{z}_i$-a lifting of the marked point $z_i \in \Sigma$. Consider the isotropy subgroup $G_{\tilde{z}_i} \subset G$. For dimensional reasons $G_{\tilde{z}_i} \cong \mathbb{Z}_k$ for some $k$. $\tilde{z}_i$ uniquely determines an element $g_i$ as the generator of $G_{\tilde{z}_i}$. $g_i$ has a geometric interpretation as the monodromy of a small loop around $z_i$. It is independent of the lifting of small loop because a different lifting will conjugate $g_i$ by an element of $G_{\tilde{z}_i}$, which is abelian. Now, the evaluation map $e_i$ is lifted to

$$
\tilde{e}_i : \Sigma \rightarrow \Sigma, \phi, \tilde{z}_1, \cdots, \tilde{z}_k = (\phi(\tilde{z}_i), g_i).
$$

$G$ acts on $G$-stable maps by its action on $\tilde{z}_i$ and $\tilde{e}_i$ is $G$-equivariant. Let $\mathcal{M}^G_{g,k}(A, X, J)$ be the moduli space of $G$-stable maps. We can apply virtual fundamental cycle techniques for $\mathcal{M}^G_{g,k}$ to obtain a $G$-orbifold GW-invariant. The orbifold GW-invariant is the invariant part of the $G$-orbifold GW-invariant.

We can use $\tilde{e}_i$ to pull back $\mathcal{L}_{\wedge [X/G]}$ to form a flat line bundle

$$
\mathcal{M}^G_{g,k}(A, X, J) \times_{\gamma_{g_1} \cdots \gamma_{g_k}} (\otimes_i \mathbb{C}_{g_i}).$$
By our construction the holonomy of $\Sigma$ should give a flat trivialization of this flat line bundle and we would like to write it down explicitly.

Without loss of generality, we assume that $\Sigma$ is irreducible. We take a real blow-up $\Sigma^\dagger$ at all the marked points. There results a real blow-up $E^\dagger$ of $E$ at all the preimage points of $z_i$. $G$ acts on $E^\dagger$ freely and $\Sigma^\dagger = E^\dagger/G$. Using the translation groupoid representation $[E^\dagger/G]$ of $\Sigma^\dagger$, we can pull back $\alpha$ to a flat gerbe on $\Sigma^\dagger$. The general theory tells us that this flat gerbe is trivial. Furthermore, a flat trivialization restricts to a flat trivialization on each boundary circle. Let’s study the induced gerbe on $\Sigma^\dagger$ from the point of view of the Chen-Ruan characteristic. Fix a base point $x_0$ in the interior of $\Sigma^\dagger$ and choose a lifting $\tilde{x}_0 \in E^\dagger$. It defines the CR-characteristic $\rho : \pi_1(\Sigma^\dagger, x_0) \to G$. In fact, $E^\dagger = E_{\text{univ}} \times_\rho G$ where $E_{\text{univ}}$ is the universal cover. We can use $\rho$ to pull back $\alpha$ to a 2-cocycle $\tilde{\alpha}$ of $\pi_1(\Sigma^\dagger, x_0)$. The induced gerbe on $\Sigma^\dagger$ is given by $\tilde{\alpha}$. Since $E_{\text{univ}}$ is contractible,

$$H^2(\pi_1(\Sigma^\dagger, x_0), S^1) \cong H^2(\Sigma^\dagger, S^1) \cong 0.$$ 

Therefore, $\tilde{\alpha}$ is a coboundary. Choose a coboundary $\delta h$.

Let’s go back to the local monodromy $g_i$ at $\tilde{z}_i$. Indeed, it is the monodromy along the boundary circle associated with $z_i$. We would like to embed $<g_i>$ in $\pi_1(\Sigma^\dagger, x_0)$ as a subgroup. This can be done as follows. Choose a path $d_i$ from $\tilde{x}_0$ to the boundary circle associated with $\tilde{z}_i$ with the end point $x_i$, then go around the boundary circle to $g_i x_i$ and go back to $g_i \tilde{x}_0$ along $g_i d_i^{-1}$. Its projection $l_i$ on $\Sigma^\dagger$ is a loop at $x_0$ whose lifting defines $\rho(l_i) = g_i$. Then, we map $g_i$ to $\gamma_i$. It can be shown that a different path $d_i'$ conjugates $l_i$ by an element of the image of $\pi_1(E^\dagger, \tilde{x}_0)$. The image of $\pi_1(E^\dagger, \tilde{x}_0)$ is precisely the kernel of $\rho$. It is clear that $\tilde{\alpha}_{<g_i>} = \alpha_{<g_i>} = \delta h_{<g_i>}$. Therefore, we can use $h_{<g_i>}$ to trivialize $\mathcal{L}_{X^n \times \{g_i\}}$. Hence, the trivialization given by holonomy on $\Sigma^\dagger$ corresponds to the pull-back of the trivial bundle $\mathcal{L}_{X^n \times \{g_i\}}$. We still have to show that $\gamma_{g_1} \cdots \gamma_{g_k} = 1$ in order to descend to a trivial bundle over $\mathcal{M}_{g,k}(A, [X/G], J)$. This follows directly from the CR-characteristic $\rho$. Recall that $\pi_1(\Sigma^\dagger, x_0)$ has generators $l_1, \cdots, l_k, \mu_1, \cdots, \mu_{2g}$ with relation $\prod_j [\mu_{2j-1}, \mu_{2j}] l_1 \cdots l_k = 1$. Therefore,

$$1 = \gamma_{\prod_j [\mu_{2j-1}, \mu_{2j}]} l_1 \cdots l_k = \prod_j [\gamma_{\mu_{2j-1}}, \gamma_{\mu_{2j}}] \gamma_{l_1} \cdots \gamma_{l_k} = \gamma_{l_1} \cdots \gamma_{l_k}.$$ 

Then we can apply the previous construction to the $G$-orbifold Gromov-Witten invariants constructed in [JKK]. Thus, we proved
Theorem 6.3. The twisted orbifold Gromov-Witten invariant of a discrete torsion is the $G$-invariant part of the $G$-orbifold Gromov-Witten invariant, under the action twisted by the discrete torsion.

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