Projections over Quantum Homogeneous

Odd-dimensional Spheres

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Abstract

We give a complete classification of isomorphism classes of finitely generated projective modules, or equivalently, unitary equivalence classes of projections, over the C*-algebra $C(S_q^{2n+1})$ of the quantum homogeneous sphere $S_q^{2n+1}$. Then we explicitly identify as concrete elementary projections the quantum line bundles $L_k$ over the quantum complex projective space $\mathbb{C}P^n_q$ associated with the quantum Hopf principal $U(1)$-bundle $S_q^{2n+1} \to \mathbb{C}P^n_q$.

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1 Introduction

In the theory of quantum/noncommutative geometry popularized by Connes [4], C*-algebras are often viewed as the algebra $C(X_q)$ of continuous functions on a virtual quantum space $X_q$, and finitely generated projective (left) $C(X_q)$-module $\Gamma(E_q)$ are viewed as virtual vector bundles over the quantum space $X_q$. The former viewpoint is motivated by Gelfand’s Theorem identifying all commutative C*-algebras as exactly function algebras $C_0(X)$ for locally compact Hausdorff spaces $X$, while the latter is motivated by Swan’s Theorem [24] characterizing all finitely generated projective $C(X)$-modules for a compact Hausdorff space $X$ as exactly the spaces $\Gamma(E)$ of continuous cross-sections of vector bundles $E$ over $X$.

As spheres and projective spaces provide fundamentally important examples for the classical study of topology and geometry, quantum versions of spheres and projective spaces have been developed and provide important examples for the study of quantum geometry. In particular, from the quantum group viewpoint [7, 27, 28], Soibelman, Vaksman, Meyer and others [26, 12, N, 21] introduced and studied the homogeneous odd-dimensional quantum sphere $S^{2n+1}_q$ and the associated quantum complex projective space $\mathbb{C}P^n_q$, and from the multipullback viewpoint, Hajac and his collaborators including Baum, Kaygun, Matthes, Nest, Pask, Sims, Szymański, Zieliński, and others [10, 9, 11] developed and studied the multipullback odd-dimensional quantum sphere $S^{2n+1}_H$ and the associated quantum complex projective space $\mathbb{P}^n(T)$.

As in the classical situation, the above mentioned quantum odd-dimensional spheres and their associated quantum complex projective spaces provide a quantum Hopf principal
$U(1)$-bundle, from which some associated quantum line bundles $L_k$, or rank-one projective modules over the quantum algebra of the complex projective space, for $k \in \mathbb{Z}$ are constructed [12, 11, 10, 11].

It is well known that classifying up to isomorphism all vector bundles over a space $X$ in the classical case or a finitely generated (left) projective module over a C*-algebra $C(X_q)$ in the quantum case is an interesting but difficulty task. A major challenge in such classification is the so-called cancellation problem [15, 16] which deals with determining whether the stable isomorphism between such objects determined by $K$-theoretic analysis can imply their isomorphism.

In this paper, we use the powerful groupoid approach to C*-algebras initiated by Renault [14] and popularized by Curto, Muhly, and Renault [5, 13] to study the C*-algebra structures of $C(S^{2n+1}_q)$ and $C(CP^n_q)$. In this framework, we work to get a complete classification of projections over $C(S^{2n+1}_q)$ up to equivalence, extending the result of Bach [2], and determine the canonical monoid structure on the collection of all equivalence classes of projections over $C(S^{2n+1}_q)$ with the diagonal sum ⊞ as its binary operation. Then we proceed to present a set of elementary projections that freely generate $K_0(C(CP^n_q))$ and represent the line bundles $L_k$ over $CP^n_q$ by concrete ⊞-sum of elementary projections. We mention that a similar study has been carried out for the multipullback quantum spheres $S^{2n+1}_H$ and the associated projective space $\mathbb{P}^n(T)$ in the paper [23, 22], and an interesting geometric study via Milnor construction is presented by Farsi, Hajac, Maszczyk, and Zieliński in [8] for $C(\mathbb{P}^2(T))$.

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2 Preliminary notations

In this paper, we use freely the basic techniques and manipulations for $K$-theory of C*-algebras, or more generally, Banach algebras, found in [3, 25]. Commonly widely used notations like $M_\infty (\mathcal{A})$, $GL_\infty (\mathcal{A})$, unitization $\mathcal{A}^+$, diagonal sum $P \oplus Q$ of elements $P, Q \in M_\infty (\mathcal{A})$, the identity component $G^0$ of a topological group $G$, the positive cone $K_0 (\mathcal{A})_+$ of $K_0 (\mathcal{A})$, $B (\mathcal{H})$, $K (\mathcal{H})$, and others will not be explained in details here, and we refer to the notations section in [23] for any need of further clarification.

By a projection (or an idempotent) over a C*-algebra $\mathcal{A}$, we mean a projection (or an idempotent) in the algebra $M_\infty (\mathcal{A})$ of all finite matrices with entries in $\mathcal{A}$. Two projections (or idempotents) $P, Q \in M_\infty (\mathcal{A})$ are called equivalent over $\mathcal{A}$, denoted as $P \sim_\mathcal{A} Q$, if there is an invertible $U \in GL_\infty (\mathcal{A})$ such that $UPU^{-1} = Q$.

We recall that the mapping $P \mapsto \mathcal{A}^n P$ induces a bijective correspondence between the equivalence (respectively, the stable equivalence) classes of idempotents over $\mathcal{A}$ and the isomorphism (respectively, the stable isomorphism) classes of finitely generated projective modules over $\mathcal{A}$ [3], where by a module over $\mathcal{A}$, we mean a left $\mathcal{A}$-module, unless otherwise specified.

We also recall that the $K_0$-group $K_0 (\mathcal{A})$ classifies idempotents over $\mathcal{A}$ up to stable equivalence. The classification of idempotents up to equivalence, appearing as the so-called can-
cellation problem, was popularized by Rieffel’s pioneering work \[15, 16\] and is in general an interesting but difficult question.

For a C*-algebra homomorphism \( h : A \rightarrow B \), we use the same symbol \( h \), instead of the more formal symbol \( M_\infty (h) \), to denote the algebra homomorphism \( M_\infty (A) \rightarrow M_\infty (B) \) that applies \( h \) to each entry of any matrix in \( M_\infty (A) \).

The set of all equivalence classes of idempotents, or equivalently, all unitary equivalence classes of projections, over a C*-algebra \( A \) is an abelian monoid \( \mathfrak{P}(A) \) with its binary operation provided by the diagonal sum \( \oplus \).

In the following, we use the notations \( \mathbb{Z}_{\geq k} := \{ n \in \mathbb{Z} | n \geq k \} \) and \( \mathbb{Z}_{\geq} := \mathbb{Z}_{\geq 0} \). In particular, \( \mathbb{N} = \mathbb{Z}_{\geq 1} \). We use \( I \) to denote the identity operator canonically contained in \( \mathcal{K}^+ \subset \mathcal{B}(\ell^2(\mathbb{Z}_{\geq})) \), and

\[
P_m := \sum_{i=1}^{m} e_{ii} \in M_m (\mathbb{C}) \subset \mathcal{K}
\]

to denote the standard \( m \times m \) identity matrix in \( M_m (\mathbb{C}) \subset \mathcal{K} \) for any integer \( m \geq 0 \) (with \( M_0 (\mathbb{C}) = 0 \) and \( P_0 = 0 \) understood). We also use the notation

\[
P_{-m} := I - P_m \in \mathcal{K}^+
\]

for integers \( m > 0 \), and take symbolically \( P_{-0} \equiv I - P_0 = I \neq P_0 \). This should not cause any trouble since we will not formally add up the subscripts of these \( P \)-projections without necessary clarification.
3 Quantum spaces as groupoid C*-algebras

In the following, we work with some concrete $r$-discrete (or étale) groupoids and use them to analyze and encode important structures of quantum $S_q^{2n+1}$ and quantum $CP^n_q$ in the context of groupoid C*-algebras. This groupoid approach to C*-algebras was popularized by the work of Curto, Muhly, and Renault [5, 13, 17] and shown to be useful in the study of quantum homogeneous spaces [19, 18, 20, 21]. We refer readers to Renault’s pioneering book [14] for the fundamental theory of groupoid C*-algebras.

We denote by $\mathbb{Z} := \mathbb{Z} \cup \{+\infty\}$ the discrete space $\mathbb{Z}$ with a point $+\infty$ canonically adjoined as a limit point at the positive end, and take $\mathbb{Z}_\geq := \{ n \in \mathbb{Z} | n \geq 0 \} \subset \mathbb{Z}$. (We could also take $\mathbb{Z}$ to be the one-point compactification of the discrete space \( \mathbb{Z} \) in this paper since essentially we work only with gorupoids restricted to a positive cone of their unit spaces.)

Let $\mathcal{F}^n := Z \times (\mathbb{Z}^n \ltimes \mathbb{Z}^n) |_{\mathbb{Z}_\geq}$ with $n \geq 1$ be the direct product of the group $\mathbb{Z}$ and the transformation group groupoid $\mathbb{Z}^n \ltimes \mathbb{Z}^n$ restricted to the positive “cone” $\mathbb{Z}_\geq^n$, where $\mathbb{Z}^n$ acts on $\mathbb{Z}^n$ componentwise in the canonical way.

As shown in [20], $C(S_q^{2n+1}) \simeq C^*(\mathcal{F}_n)$, where $\mathcal{F}_n$ is a subquotient groupoid of $\mathcal{F}^n$, namely, $\mathcal{F}_n := \tilde{\mathcal{F}}_n / \sim$ for the subgroupoid

$$\tilde{\mathcal{F}}_n := \{(z, x, w) \in \mathcal{F}^n | w_i = \infty \text{ with } 1 \leq i \leq n \text{ implies } x_i = -z - x_1 - x_2 - \ldots - x_{i-1} \text{ and } x_{i+1} = \ldots = x_n = 0\}$$
of $\mathcal{F}^n$, where $\sim$ is the equivalence relation generated by

$$
(z, x, w) \sim (z, x, w_1, ..., w_i = \infty, \infty, ..., \infty)
$$

for all $(z, x, w)$ with $w_i = \infty$ for an $1 \leq i \leq n$. The unit space of $\widetilde{\mathcal{F}}_n$ is $Z := \mathbb{Z}^n / \sim$ where $\mathbb{Z}^n_{\geq}$ is the unit space of $\widetilde{\mathcal{F}}_n \subset \mathcal{F}^n$ embedded in $\widetilde{\mathcal{F}}_n$ as the $\sim$-invariant subset $\{0\} \times \{0\} \times \mathbb{Z}^n_{\geq}$.

Let $\pi_n$ denote the faithful $\ast$-representation of the groupoid C*-algebra $C^* (\mathcal{F}_n)$ canonically constructed on the Hilbert space $\ell^2 (Z \times \mathbb{Z}^n_{\geq}) = \ell^2 (Z) \otimes \ell^2 (\mathbb{Z}^n_\geq)$ built from the open dense orbit $\mathbb{Z}^n_{\geq}$ in the unit space $Z$ of $\mathcal{F}_n$. For practical purposes, we often identify $C^* (\mathcal{F}_n)$ with the concrete operator algebra $\pi_n (C^* (\mathcal{F}_n))$ without making explicit distinction. Note that by restricting $\mathcal{F}_n$ to the open subset $\mathbb{Z}^n_{\geq}$, we get the groupoid $\mathcal{F}_n|_{\mathbb{Z}^n_{\geq}} \cong Z \times ((\mathbb{Z}^n \ltimes \mathbb{Z}^n)|_{\mathbb{Z}^n_{\geq}})$

with

$$
C^* (\mathcal{F}_n|_{\mathbb{Z}^n_{\geq}}) \cong C^* (Z) \otimes C^* ((\mathbb{Z}^n \ltimes \mathbb{Z}^n)|_{\mathbb{Z}^n_{\geq}}) \cong C^* (\mathbb{T}) \otimes K (\ell^2 (\mathbb{Z}^n_{\geq}))
$$

under the representation $\pi_n$, where $C^* (Z) \cong C (\mathbb{T})$ acts on $\ell^2 (Z) \cong L^2 (\mathbb{T})$ by multiplication operators, and hence $C^* (\mathbb{T}) \otimes K (\ell^2 (\mathbb{Z}^n_{\geq}))$ can be viewed as a closed ideal of $\pi_n (C^* (\mathcal{F}_n)) \cong C^* (\mathcal{F}_n)$.

Note that the $\mathbb{Z}$-component of $\mathcal{F}^n \equiv Z \times ((\mathbb{Z}^n \ltimes \mathbb{Z}^n)|_{\mathbb{Z}^n_{\geq}}$ gives a grading on $C^* (\mathcal{F}_n)$, decomposing it into (a completion of) a direct sum of some subspaces index by $\mathbb{Z}$. More precisely, $\widetilde{\mathcal{F}}_n$ is the union of the pairwise disjoint close and open sets

$$
\left( \widetilde{\mathcal{F}}_n \right)_k := \left\{ (k, x, w) \in \widetilde{\mathcal{F}}_n \mid (j, k) \in \mathbb{Z}^n \times \mathbb{Z}^n_{\geq} \right\}
$$

with $k \in \mathbb{Z}$ which are invariant under the equivalent relation $\sim$, so $\mathcal{F} \equiv \widetilde{\mathcal{F}}_n / \sim$ is the union
of the pairwise disjoint close and open sets

$$(\mathfrak{F}_n)_k := (\tilde{\mathfrak{F}}_n)_k / \sim$$

and hence $C^* (\mathfrak{F}_n)$ is the closure of the (algebraic) direct sum $\oplus_{k \in \mathbb{Z}} C_c (\mathfrak{F}_n)_k$ where $C_c (\mathfrak{F}_n)_k := C_c ((\tilde{\mathfrak{F}}_n)_k)$. In fact, the groupoid character $[(k, x, w)] \in \mathfrak{F}_n \mapsto t^k \in \mathbb{T}$ for any fixed $t \in \mathbb{T} \equiv U(1)$ defines an isometric *-automorphism of $L^1 (\mathfrak{F}_n)$ and hence a C*-algebra automorphism $\rho (t)$ of $C^* (\mathfrak{F}_n)$, sending $\delta_{[(k, x, w)]}$ to $t^k \delta_{[(k, x, w)]}$. Clearly $\rho : t \mapsto \rho (t)$ defines a $U(1)$-action on $C^* (\mathfrak{F}_n)$. The degree-$k$ spectral subspace $C^* (\mathfrak{F}_n)_k$ of $C^* (\mathfrak{F}_n)$ under the action $\rho$, i.e. the set consisting of all elements $a \in C^* (\mathfrak{F}_n)$ with $(\rho (t)) (a) = t^k a$ for all $t \in \mathbb{T}$, is a closed linear subspace of $C^* (\mathfrak{F}_n)$ containing $C_c (\mathfrak{F}_n)_k$. Clearly $C^* (\mathfrak{F}_n)_k \cap C^* (\mathfrak{F}_n)_{k'} = 0$ for any $k \neq k'$. The integration operator

$$\Lambda_k : a \in C^* (\mathfrak{F}_n) \mapsto a_k := \int_{\mathbb{T}} t^{-k} (\rho (t)) (a) \, dt \in C^* (\mathfrak{F}_n)_k \subset C^* (\tilde{\mathfrak{F}}_n)$$

is a well-defined continuous projection onto $C^* (\mathfrak{F}_n)_k$ and eliminates $C^* (\mathfrak{F}_n)_{l}$ for all $l \neq k$, where $\mathbb{T}$ is endowed with the standard Haar measure. Indeed for any $s \in \mathbb{T}$,

$$(\rho (s)) (a_k) = \int_{\mathbb{T}} t^{-k} (\rho (t)) (\rho (s) a) \, dt = s^k \int_{\mathbb{T}} (st)^{-k} (\rho (st)) (a) \, dt$$

$$= s^k \int_{\mathbb{T}} t^{-k} (\rho (t)) (a) \, dt = s^k a_k,$$

and for any $b \in C^* (\mathfrak{F}_n)_{l}$,

$$\Lambda_k (b) = \int_{\mathbb{T}} t^{-k} (\rho (t)) (b) \, dt = \int_{\mathbb{T}} t^{-k} t^l b dt = \left( \int_{\mathbb{T}} t^{l-k} dt \right) b = \delta_{k,l} b.$$

So $\Lambda_k$’s are mutually orthogonal projections in the sense that $\Lambda_k \circ \Lambda_l = \delta_{kl} \text{id}$. With the (algebraic) sum $\sum_{k \in \mathbb{Z}} C_c (\mathfrak{F}_n)_k$ clearly dense in $C^* (\mathfrak{F}_n)$ and $C_c (\mathfrak{F}_n)_k \subset C^* (\mathfrak{F}_n)_k$, we see that
$C_c(\mathcal{F}_n)_k = C^* (\mathcal{F}_n)_k$ by applying the projection operator $\Lambda_k$ to any sequence in $\sum_{l \in \mathbb{Z}} C_c (\mathcal{F}_n)_l$ converging to an element of $C^* (\mathcal{F}_n)_k$. Furthermore we note that clearly $C_c (\mathcal{F}_n)_k C_c (\mathcal{F}_n)_l \subset C_c (\mathcal{F}_n)_{k+l}$ and $C^* (\mathcal{F}_n)_k C^* (\mathcal{F}_n)_l \subset C^* (\mathcal{F}_n)_{k+l}$ for all $k, l \in \mathbb{Z}$, i.e. $C_c (\mathcal{F}_n)$ and $C^* (\mathcal{F}_n)$ are graded algebras (up to completion).

Recall that the group $U(1) \equiv \mathbb{T}$ acts on $C(S_q^{2n+1})$ by sending the standard generators $u_{n+1,m} \in C(SU_q(n+1))$, $1 \leq m \leq n+1$, of $C(S_q^{2n+1})$ to $tu_{n+1,m}$ for each group element $t \in \mathbb{T} \subset \mathbb{C}$. This $U(1)$-action, denoted as $\tau_t$ for $t \in \mathbb{T}$, decomposes $C(S_q^{2n+1})$ into spectral subspaces $C(S_q^{2n+1})_k$ of degree $k \in \mathbb{Z}$ consisting of element $a \in C(S_q^{2n+1})$ satisfying $\tau_t(a) = t^k a$ for all $t \in \mathbb{T}$. Each $u_{n+1,m}$ is in the degree-1 spectral subspace $C(S_q^{2n+1})_1$. On the other hand, under the identification of $C(S_q^{2n+1})$ with $C^* (\mathcal{F}_n)$ established in the work of [19, 20], each $u_{n+1,m}$ faithfully represented as $t_{n+1} \otimes \gamma^{\otimes n+1-m} \otimes \alpha^* \otimes 1^\otimes m-2$ is identified with an element in $C_c(\mathcal{F}_n)_1 = C^*(\mathcal{F}_n)_1$. So the grading on $C^* (\mathcal{F}_n)$ by $C^* (\mathcal{F}_n)_k$ coincides with the grading on $C(S_q^{2n+1})$ by $C(S_q^{2n+1})_k$, i.e. $C(S_q^{2n+1})_k = C^*(\mathcal{F}_n)_k$.

The degree-0 spectral subspace $C(S_q^{2n+1})_0$, or equivalently, the $U(1)$-invariant subalgebra $(C(S_q^{2n+1}))^{U(1)}$ of $C(S_q^{2n+1})$ can be naturally called the algebra of quantum $\mathbb{C}P^n$, denoted as $C(\mathbb{C}P^n_q)$. The embedding $C(\mathbb{C}P^n_q) \subset C(S_q^{2n+1}) \equiv C^*(\mathcal{F}_n)$, or virtually the quantum quotient map $S_q^{2n+1} \to \mathbb{C}P^n_q$, is a quantum analogue of the Hopf principal $U(1)$-bundle $S^{2n+1} \to \mathbb{C}P^n$. Furthermore the degree-$k$ spectral subspaces $C(S_q^{2n+1})_k \equiv C^*(\mathcal{F}_n)_k$ become the quantum line bundles, denoted $L_k$, over $\mathbb{C}P^n_q$ associated with the quantum principal $U(1)$-bundle $S_q^{2n+1} \to \mathbb{C}P^n_q$. Note that in the context of groupoid $C^*$-algebras, $C(\mathbb{C}P^n_q) \equiv C(S_q^{2n+1})_0$ is canonically identified with the groupoid $C^*$-algebra $C^*((\mathcal{F}_n)_0)$, where $(\mathcal{F}_n)_0$
is clearly an open and closed subgroupoid of $\mathfrak{F}_n$. It is easy to see that the unit space of $(\mathfrak{F}_n)_0 \subset \mathfrak{F}_n$ is the same unit space $Z \equiv \mathbb{Z}_2^i/\sim$ that $\mathfrak{F}_n$ has.

On the other hand, the quantum complex projective space $U(n)_q \setminus SU(n+1)_q$ has been formulated and studied by researchers from the viewpoint of quantum homogeneous space $\mathbb{N}$. The author showed in [21] that such a quantum space can be concretely realized by the C*-subalgebra generated by $u_{n+1,i}^* u_{n+1,j}$ with $1 \leq i, j \leq n + 1$ in $C(\mathbb{S}_q^{2n+1})$, and then identified this C*-algebra with the groupoid C*-algebra $C^*(\mathfrak{T}_n)$ of the subquotient groupoid $\mathfrak{T}_n := \tilde{\mathfrak{T}}_n/\sim$ of $\mathbb{Z}^n \ltimes \mathbb{Z}^n|_{\mathbb{Z}_2^n}$, where

$$\tilde{\mathfrak{T}}_n := \{(x, w) \in \mathbb{Z}^n \ltimes \mathbb{Z}^i|_{\mathbb{Z}_2^n} : w_i = \infty \text{ with } 1 \leq i \leq n \text{ implies } x_i = -x_1 - x_2 - ... - x_{i-1} \text{ and } x_{i+1} = ... = x_n = 0\}$$

is a subgroupoid of $\mathbb{Z}^n \ltimes \mathbb{Z}^i|_{\mathbb{Z}_2^n}$ and $\sim$ is the equivalence relation generated by

$$(x, w) \sim (x, w_1, ..., w_i = \infty, \infty, ..., \infty)$$

for all $(x, w)$ with $w_i = \infty$ for an $1 \leq i \leq n$. It is easy to see that $[(0, x, w)] \in (\mathfrak{F}_n)_0 \mapsto [(x, w)] \in \mathfrak{T}_n$ is a well-defined homeomorphic groupoid isomorphism, and hence $C^*(\mathfrak{T}_n) \cong C^* ((\mathfrak{F}_n)_0)$. So the quantum homogeneous space $U(n)_q \setminus SU(n+1)_q$ coincides with the quantum complex projective space $\mathbb{C}P^n_q$ defined above, and the results obtained in [21] are valid for our study of the quantum complex projective space $\mathbb{C}P^n_q$. 

10
4 Projections over $C\left(\mathbb{S}^{2n+1}_q\right)$

In [21], taking the groupoid C*-algebra approach, we established an inductive family of short exact sequences of C*-algebras

$$0 \to C\left(\mathbb{T}\right) \otimes \mathcal{K}\left(\ell^2\left(\mathbb{Z}^n\right)\right) \to C\left(\mathbb{S}^{2n+1}_q\right) \to C\left(\mathbb{S}^{2n-1}_q\right) \to 0.$$ 

However for the purpose of classification of projections over $C\left(\mathbb{S}^{2n+1}_q\right)$, it turns out that another inductive family of short exact sequences constructed below is more convenient.

Under the homeomorphic groupoid isomorphism

$$\gamma : [(z, x, w)] \in \tilde{\mathcal{F}}_n/\sim \mapsto [(z + x_1, x, w)] \in \tilde{\mathcal{F}}'_n/\sim' =: \mathcal{F}'_n.$$ 

Note that the groupoid C*-algebra $C^*\left(\mathcal{F}'_n\right)$ also has a faithful *-representation $\pi'_n$ canonically constructed on the Hilbert space $\ell^2\left(\mathbb{Z} \times \mathbb{Z}^n_\geq\right) = \ell^2\left(\mathbb{Z}\right) \otimes \ell^2\left(\mathbb{Z}^n_\geq\right)$ built from the open dense orbit $\mathbb{Z}^n_\geq$ in the unit space of $\mathcal{F}'_n$. 

11
Let \( m(k) \) denote \((m,\ldots,m) \in \mathbb{Z}^k\). Note that \( \{\infty\} \times \mathbb{Z}_\geq^{n-1}/\sim \) is a closed invariant subset of the unit space \( Z \equiv \mathbb{Z}_\geq^n/\sim \) of \( \mathfrak{F}_n \) such that with a singleton unit space, 

\[
\mathfrak{F}_n|\{\{\infty(n)\}\} \equiv \{(z,-z,0^{(n-1)},\infty(n)) : z \in \mathbb{Z}\} \cong \mathbb{Z}
\]
as a group and \( \gamma(\mathfrak{F}_n|\{\{\infty(n)\}\}) = \{(0,0^{(n)},\infty(n))\} \). On the other hand, the complement of \( \{\{\infty(n)\}\} \) in \( Z \) is the open invariant subset \( O := (\mathbb{Z}_\geq \times \mathbb{Z}_\geq^{n-1})/\sim \) such that \( w_1 \neq \infty \) for all \([z,x,w] \in \mathfrak{F}_n|_O \) and hence in \( \gamma([z,x,w]) = [(z+x_1,x,w)] \), there is no non-trivial condition from the definition of \( \mathfrak{F}_n'/\sim' \) imposed on \((x_1,w_1)\), while the non-trivial conditions from the definition of \( \mathfrak{F}_n'/\sim' \) imposed on the other components of \( \gamma([z,x,w]) \) match those in defining \( \mathfrak{F}_{n-1} \). That is to say, by rewriting \( x_1,w_1 \) as the first two components of \( \gamma([z,x,w]) \), we have a homeomorphic groupoid isomorphism from \( \mathfrak{F}_n|_O \) onto the groupoid

\[
(Z \ltimes Z)|_{Z_\geq} \times \mathfrak{F}_{n-1},
\]
which then induces a C*-algebra isomorphism

\[
\gamma_* : C^*(\mathfrak{F}_n|_O) \to C^*((Z \ltimes Z)|_{Z_\geq}) \otimes C^*(\mathfrak{F}_{n-1}) \subset C^*(\mathfrak{F}_n).
\]

Note that \( \pi'_n = \pi_0 \otimes \pi_{n-1} \) on \( \gamma_*(C^*(\mathfrak{F}_n|_O)) \) where \( \pi_0 : C^*((Z \ltimes Z)|_{Z_\geq}) \to \mathcal{K}(\ell^2(Z_\geq)) \) is the well-known canonical faithful representation, and the faithful representation

\[
\pi'_n \circ \gamma_* : C^*(\mathfrak{F}_n) \to \mathcal{B}(\ell^2(Z \ltimes Z_\geq))
\]
restricts to an isomorphism \( C^*(\mathfrak{F}_n|_O) \cong \mathcal{K}(\ell^2(Z_\geq)) \otimes C(S^2n-1) \). So these invariant subsets \( \{\{\infty(n)\}\} \) and \( O \) give rise to a short exact sequence

\[
0 \to \mathcal{K}(\ell^2(Z_\geq)) \otimes C(S^2n-1) \cong C^*(\mathfrak{F}_n|_O) \to C(S^2n+1) \xrightarrow{\eta} C^*(\mathfrak{F}_n|_{\{\{\infty(n)\}\}}) \cong C(\mathbb{T}) \to 0.
\]
The set \( T := \{ [(z, x, w)] \in \mathfrak{F}_n : x_1 = 1 = -z, x_2 = \ldots = x_n = 0 \} \) is a compact open sub-set of \( \mathfrak{F}_n \), corresponding to the set \( \{ [(0, x, w)] \in \mathfrak{F}_n : x = (1, 0^{(n-1)}) \} \subset \mathfrak{F}_n' \) under the isomorphism \( \gamma \), and its characteristic function \( \chi_T \in C_c (\mathfrak{F}_n) \subset C^* (\mathfrak{F}_n) \) determines the operator

\[
\pi_n' (\gamma^* (\chi_T)) = \mathcal{S} \otimes \text{id} \in \mathcal{T} \otimes \pi_{n-1} (C (S_q^{2n-1})) \subset \mathcal{B} (\ell^2 (\mathbb{Z}_2)) \otimes \mathcal{B} (\ell^2 (\mathbb{Z} \times \mathbb{Z}_2^{n-1}))
\]

where \( \mathcal{S} \) is the unilateral shift operator generating the Toeplitz algebra \( \mathcal{T} \) with \( \sigma (\mathcal{S}) = \text{id}_\mathcal{T} \) for the symbol map \( \sigma \) in the short exact sequence \( 0 \to \mathcal{K} \to \mathcal{T} \to C (\mathbb{T}) \to 0 \). Since the quotient map \( \eta : C^* (\mathfrak{F}_n) \to C^* (\mathfrak{F}_n |\{[\infty]n\}\}) \equiv C^* (\mathbb{Z}) \) restricts \( \chi_T \) to \( \delta_1 \in C_c (\mathbb{Z}) \equiv C_c (\mathfrak{F}_n |\{[\infty]n\}\}) \) yielding the function \( \text{id}_\mathcal{T} \in C (\mathbb{T}) \equiv C^* (\mathbb{Z}) \), we get

\[
C (S_q^{2n+1}) \subset \mathcal{T} \otimes \pi_{n-1} (C (S_q^{2n-1})) \equiv \mathcal{T} \otimes C (S_q^{2n-1})
\]

being the sum of \( \mathcal{K} \otimes C (S_q^{2n-1}) \) and \( \mathcal{S} \otimes 1_{C (S_q^{2n-1})} \), which coincides with a description of \( C (S_q^{2n+1}) \) in [26].

The above surjective \( C^* \)-algebra homomorphism \( C (S_q^{2n+1}) \to C (\mathbb{T}) \) facilitates the notion of rank for an equivalence class of idempotent \( P \in M_\infty (C (S_q^{2n+1})) \) over \( C (S_q^{2n+1}) \), namely, the well-defined classical rank of the vector bundle over \( \mathbb{T} \) determined by the idempotent \( \eta (P) \) over \( C (\mathbb{T}) \).

The set of equivalence classes of idempotents \( P \in M_\infty (C (S_q^{2n+1})) \) equipped with the binary operation \( \Box \) becomes an abelian graded monoid

\[
\mathfrak{P} (C (S_q^{2n+1})) = \sqcup_{r=0}^{\infty} \mathfrak{P}_r (C (S_q^{2n+1}))
\]

where \( \mathfrak{P}_r (C (S_q^{2n+1})) \) is the set of all (equivalence classes of) idempotents over \( C (S_q^{2n+1}) \) of
rank $r$, and
\[
\mathfrak{P}_r \left( C \left( \mathbb{S}_q^{2n+1} \right) \right) \oplus \mathfrak{P}_l \left( C \left( \mathbb{S}_q^{2n+1} \right) \right) \subset \mathfrak{P}_{r+l} \left( C \left( \mathbb{S}_q^{2n+1} \right) \right)
\]
for $r, l \geq 0$. Clearly $\mathfrak{P}_0 \left( C \left( \mathbb{S}_q^{2n+1} \right) \right)$ is a submonoid of $\mathfrak{P} \left( C \left( \mathbb{S}_q^{2n+1} \right) \right)$.

Now we can proceed to classify up to equivalence all projections over $C \left( \mathbb{S}_q^{2n+1} \right)$ by induction on $n$, extending the result obtained in [2] for the case of $n = 1$.

First we define some standard basic projections

\[
P_{j,k} := \begin{cases} 
1_T \otimes ((\otimes^{j-1}P_1) \otimes P_k \otimes (\otimes^{n-j}I)) \in C \left( \mathbb{T} \right) \otimes (\mathbb{K}^+) \otimes^n, & \text{if } k > 0 \text{ and } 1 \leq j \leq n \\
1_T \otimes (\boxplus k I \otimes^n) \equiv 1_T \otimes (\boxplus k (\otimes^n I)) \in M_k \left( C \left( \mathbb{T} \right) \otimes (\mathbb{K}^+) \otimes^n \right), & \text{if } k \geq 0 \text{ and } j = 0
\end{cases}
\]

where $I$ stands for the unit of $\mathbb{K}^+$. Note that $P_{0,0} = 0$. (For the convenience of argument, we also use the symbol $P_{j,k}$ for the case of $n = 0$, by taking $(\mathbb{K}^+) \otimes^0 := C$ and noting that $P_{0,k} = 1_T \otimes (\boxplus k I) \in M_k \left( C \left( \mathbb{T} \right) \otimes (\mathbb{K}^+) \otimes^n \right)$ for $k \geq 0$ makes sense, while $P_{j,k}$ with $1 \leq j \leq n$ does not exist when $n = 0$.)

We note that the basic projection $P_{j,k}$ with $j \geq 1$ is implemented by the characteristic function $\chi_{A_{j,k}}$ of the compact open subset

\[
A_{j,k} := \left( \left\{ 0 \right\} \times \left\{ 0 \right\} \times^n \times \left\{ 0 \right\} \times^{j-1} \times \left\{ 0, 1, \ldots, k-1 \right\} \times \mathbb{Z}^{n-j} \right) / \sim
\]

of $\mathbb{F}_n$ under both representations $\pi_n$ and $\pi'_n \circ \gamma_*$. So each $P_{j,k}$ with $j \geq 1$ is a projection in $C \left( \mathbb{S}_q^{2n+1} \right)$. On the other hand, $P_{0,k} = \boxplus k \tilde{I}$ is the identity projection in $M_k \left( C \left( \mathbb{S}_q^{2n+1} \right) \right)$, where $\tilde{I}$ is the identity element of $C \left( \mathbb{S}_q^{2n+1} \right)$.

Recall that in the inductive family of short exact sequences

\[
0 \rightarrow C \left( \mathbb{T} \right) \otimes \mathbb{K} \left( \ell^2 \left( \mathbb{Z}^n \right) \right) \rightarrow C \left( \mathbb{S}_q^{2n+1} \right) \rightarrow C \left( \mathbb{S}_q^{2n-1} \right) \rightarrow 0.
\]
for $C \left( S_{q}^{2n+1} \right)$ found in [21], the quotient map $\mu_n : C \left( S_{q}^{2n+1} \right) \to C \left( S_{q}^{2n-1} \right)$ is implemented by the restriction map

$$C^* (\mathcal{F}_n) \to C^* \left( \mathcal{F}_n \big|_{\mathbb{Z}_{\geq}^{n-1} \times \{\infty}\} / \sim \right) \cong C^* (\mathcal{F}_{n-1}).$$

For any $n \in \mathbb{N}$, a projection $P$ over $C \left( S_{q}^{2n+1} \right)$ annihilated by $M_\infty (\mu_n)$ is a projection in $M_\infty \left( C (\mathbb{T}) \otimes K (\ell^2 (\mathbb{Z}_q^n)) \right)$ and hence has a well-defined finite operator-rank $d_n (P) \in \mathbb{Z}_{\geq}$, namely, the rank of the projection operator $P (t) \in M_\infty \left( K (\ell^2 (\mathbb{Z}_q^n)) \right)$, independent of $t \in \mathbb{T}$.

If $P$ is not annihilated by $\mu_n$, then we assign $d_n (P) := \infty$. Note that $d_n (P)$ depends only on the equivalence class of $P$ over $C \left( S_{q}^{2n+1} \right)$. In the degenerate case of $n = 0$, for a projection $P$ over $C \left( S_{q}^{1} \right) \equiv C (\mathbb{T})$, we define $d_0 (P)$ to be the finite rank of projection $P (t) \in M_\infty (\mathbb{C})$, independent of $t \in \mathbb{T}$.

Now for a projection $P$ over $C \left( S_{q}^{2n+1} \right)$, we define for $0 \leq l \leq n$,

$$\rho_l (P) := d_1 \left( (\mu_{l+1} \circ \cdots \circ \mu_{n-1} \circ \mu_n) (P) \right) \text{ if } l = n \equiv d_n (P)$$

which depends only on the equivalence class of $P$ over $C \left( S_{q}^{2n+1} \right)$ and gives us a well-defined monoid homomorphism

$$\rho_l : \left( \mathfrak{P} \left( C \left( S_{q}^{2n+1} \right) \right) , \boxplus \right) \to (\mathbb{Z}_{\geq} \cup \{\infty\} , +).$$

It is easy to verify that

$$\rho_l (P_{j,k}) := \begin{cases} \infty, & \text{if } n - j > n - l, \text{ i.e. } j < l \\ k, & \text{if } j = l \\ 0, & \text{if } n - j < n - l, \text{ i.e. } j > l \end{cases}$$
which shows that these projections $P_{j,k}$ are mutually inequivalent over $C\left(\mathbb{S}^{2n+1}_q\right)$ because $P_{j,k}$’s with different indices $(j, k)$ are distinguished by the collection of homomorphisms $\rho_0, \rho_1, \ldots, \rho_n$.

**Theorem 1.** $\mathfrak{P}\left(C\left(\mathbb{S}^{2n+1}_q\right)\right)$ for $n \geq 0$ is the disjoint union of

$$
\mathfrak{P}_0\left(C\left(\mathbb{S}^{2n+1}_q\right)\right) = \{[P_{0,0}]\} \cup \{[P_{j,k}] : k > 0 \text{ and } 1 \leq j \leq n\},
$$

containing pairwise distinct $[P_{j,k}]$’s indexed by $(j, k)$, and

$$
\mathfrak{P}_k\left(C\left(\mathbb{S}^{2n+1}_q\right)\right) = \{[P_{0,k}]\} \quad \text{a singleton for all } k > 0,
$$

and its monoid structure is explicitly determined by that

$$
[P_{j,k}] \boxtimes [P_{j',k'}] = \begin{cases} 
[P_{j,k}], & \text{if } 0 \leq j < j' \text{ and } k, k' > 0, \\
[P_{j,k+k'}], & \text{if } j = j' \geq 0.
\end{cases}
$$

So $[P_{j,k}] = 0$ in $K_0\left(C\left(\mathbb{S}^{2n+1}_q\right)\right)$ if $1 \leq j \leq n$.

**Proof.** Knowing that $P_{j,k}$ are mutually inequivalent, we only need to show that any projection over $C\left(\mathbb{S}^{2n+1}_q\right)$ is equivalent to one of these $P_{j,k}$’s and verify the stated monoid structure.

We prove by induction on $n \geq 0$.

When $n = 0$, $C\left(\mathbb{S}^{2n+1}_q\right) = C\left(\mathbb{T}\right)$ and it is well known from algebraic topology about vector bundles over $\mathbb{T}$ that isomorphism classes of (complex) vector bundles over $\mathbb{T}$ are faithfully represented by trivial vector bundle, i.e. $\mathfrak{P}_0\left(C\left(\mathbb{T}\right)\right) = \{0\} \equiv \{[P_{0,0}]\}$ while $\mathfrak{P}_k\left(C\left(\mathbb{T}\right)\right) = \{[P_{0,k}]\}$ for $k > 0$. Then the statements of this theorem for $n = 0$ are clearly verified.

Now assume that the statements hold for $C\left(\mathbb{S}^{2n-1}_q\right)$. We need to show that they also hold for $C\left(\mathbb{S}^{2n+1}_q\right)$.
Since any complex vector bundle over $\mathbb{T}$ is trivial, any idempotent over $C(\mathbb{T})$ is equivalent to the standard projection $1 \otimes P_m \in C(\mathbb{T}) \otimes M_\infty(\mathbb{C})$ for some $m \in \mathbb{Z}_\geq$. So for any nonzero idempotent $P \in M_\infty(C(S_q^{2n+1}))$ over $C(S_q^{2n+1})$, there is some $U \in GL_\infty(C(\mathbb{T}))$ such that
\[ U\eta(P)U^{-1} = 1 \otimes P_m = \eta\left(\boxplus^m \tilde{I}\right) \]
for some $m \in \mathbb{Z}_\geq$ where $\tilde{I}$ is the identity element of $C(S_q^{2n+1})$ viewed as the identity element in $(\mathcal{K} \otimes C(S_q^{2n+1}))^+ \subset C(S_q^{2n+1})$, and hence $VPV^{-1} - \boxplus^m \tilde{I} \in M_\infty(\mathcal{K} \otimes C(S_q^{2n-1}))$ for any lift $V \in GL_\infty(C(S_q^{2n+1}))$ (which exists) of $U \boxplus U^{-1} \in GL_0(\mathcal{T})$ along $\eta$. Replacing $P$ by the equivalent $VPV^{-1}$, we may assume that $P \in \left(\boxplus^m \tilde{I}\right) + M_{r-1}(\mathcal{K} \otimes C(S_q^{2n-1}))$ for some large $r \geq m + 1$. Now since $M_\infty(\mathbb{C})$ is dense in $\mathcal{K}$, there is an idempotent $Q \in \left(\boxplus^m \tilde{I}\right) + M_{r-1}(\mathbb{C} \otimes C(S_q^{2n-1}))$ sufficiently close to and hence equivalent to $P$ over $C(S_q^{2n+1})$ for some large $N$. So replacing $P$ by $Q$, we may assume that
\[ K := P - \boxplus^m \tilde{I} \in M_{r-1}(\mathbb{C} \otimes C(S_q^{2n-1})) \, . \]
Rearranging the entries of $P \equiv K + \boxplus^m \tilde{I} \in M_{r-1}(C(S_q^{2n+1})) \subset M_r(C(S_q^{2n+1}))$ via conjugation by the unitary
\[ U_{r,N} := \sum_{j=1}^{r-1} e_{jj} \otimes ((S \otimes \text{id})^*)^N + e_{rj} \otimes \left((S \otimes \text{id})^{(j-1)N} P_N\right) + e_{rr} \otimes (S \otimes \text{id})^{(r-1)N} \]
\[ \in M_r(\mathbb{C}) \otimes C(S_q^{2n+1}) \equiv M_r(C(S_q^{2n+1})) \]
we get the idempotent
\[ U_{r,N}PU_{r,N}^{-1} \equiv U_{r,N}(P \boxplus 0)U_{r,N}^{-1} = \left(\boxplus^m \tilde{I}\right) \boxplus \left(\boxplus^{r-1-m} 0\right) \boxplus R \]
for some idempotent

\[ R \in M_{(r-1)N} \left( C \left( \mathbb{S}^{2n-1}_q \right) \right) \subset \mathcal{K} \otimes C \left( \mathbb{S}^{2n-1}_q \right) \subset C \left( \mathbb{S}^{2n+1}_q \right) \]

which has rank at least \( m \) as an idempotent over \( C \left( \mathbb{S}^{2n-1}_q \right) \) since it contains \( m \) copies of the identity element \( \tilde{I}' \) of \( C \left( \mathbb{S}^{2n-1}_q \right) \) as diagonal \( \oplus \)-summands, relocated from the \( N \)-th diagonal entry of each of the \( m \) copies of \( \tilde{I} \) in \( P \).

Now by the induction hypothesis, the idempotent \( R \in M_{(r-1)N} \left( C \left( \mathbb{S}^{2n-1}_q \right) \right) \) is equivalent over \( C \left( \mathbb{S}^{2n-1}_q \right) \) to \( P'_{j,k} \) (denoting a standard projection \( P_{j,k} \) over \( C \left( \mathbb{S}^{2n-1}_q \right) \)) with \( 0 \leq j \leq n-1 \) and \( k > 0 \), which is identified with

\[
P_{j+1,k} \equiv \begin{cases} P_1 \otimes P'_{j,k} \in P_1 \otimes C \left( \mathbb{S}^{2n-1}_q \right) \subset \mathcal{K} \otimes C \left( \mathbb{S}^{2n-1}_q \right) \subset C \left( \mathbb{S}^{2n+1}_q \right), & \text{if } j > 0 \vspace{0.5em} \\ P_k \otimes \tilde{I}' \in P_k \otimes C \left( \mathbb{S}^{2n-1}_q \right) \subset \mathcal{K} \otimes C \left( \mathbb{S}^{2n-1}_q \right) \subset C \left( \mathbb{S}^{2n+1}_q \right), & \text{if } j = 0 \end{cases}
\]

i.e. \( WRW^{-1} = P'_{j,k} \equiv P_{j+1,k} \) for some invertible \( W \in M_{N'} \left( C \left( \mathbb{S}^{2n-1}_q \right) \right) \) with \( N' \geq (r-1)N \).

Note that if \( m > 0 \), then \( R \) has a positive rank as an idempotent over \( C \left( \mathbb{S}^{2n-1}_q \right) \) and hence \( j = 0 \). Since

\[
W + \left( \tilde{I} - P_{N'} \otimes \tilde{I}' \right) \in (\mathcal{K} \otimes C \left( \mathbb{S}^{2n-1}_q \right))^+ \subset C \left( \mathbb{S}^{2n+1}_q \right),
\]

we get \( \left( \boxplus^m \tilde{I} \oplus (\boxplus^{r-1-m} 0) \right) \boxplus R \) equivalent over \( C \left( \mathbb{S}^{2n+1}_q \right) \) to the projection

\[
\left( \left( \boxplus^m \tilde{I} \oplus (\boxplus^{r-1-m} 0) \right) \boxplus P_{j+1,k} \right) \quad \text{where } j = 0 \text{ if } m > 0.
\]

If \( m = 0 \), then clearly \( P \) is equivalent over \( C \left( \mathbb{S}^{2n+1}_q \right) \) to \( P_{j+1,k} \in \mathcal{K} \otimes C \left( \mathbb{S}^{2n-1}_q \right) \) with \( j + 1 > 0 \) and hence is of rank 0. (We assumed \( P \) nonzero, so \( k > 0 \).)
If $m \in \mathbb{N}$ and hence $j = 0$, then $P_{j+1,k} = P_{1,k} \equiv P_k \otimes \tilde{I}$ and we can rearrange entries of 

$$
\left( \left( \bigoplus m \tilde{I} \right) \oplus (\bigoplus_{r=1}^{r-1-m} 0) \right) \oplus P_{1,k}
$$

via conjugation by the unitary

$$
U_l := e_{11} \otimes (S^k \otimes \text{id}) + e_{1r} \otimes P_k + \sum_{j=2}^{r-1} e_{jj} \otimes \tilde{I} + e_{rr} \otimes (S^k \otimes \text{id})^\ast
$$

$$
\in M_r (\mathbb{C}) \otimes C(\mathbb{S}^{2n+1}) \equiv M_r (C(\mathbb{S}^{2n+1}))
$$

to get $P$ equivalent over $C(\mathbb{S}^{2n+1})$ to

$$
U_l \left( \left( \left( \bigoplus m \tilde{I} \right) \oplus (\bigoplus_{r=1}^{r-1-m} 0) \right) \oplus P_{1,k} \right) U_l^{-1} = \left( \bigoplus m \tilde{I} \right) \oplus (\bigoplus_{r=1}^{r-m} 0) \equiv \bigoplus m \tilde{I} \equiv P_{0,m}
$$

which is of rank $m \in \mathbb{N}$.

So we have proved the description of the sets $\mathfrak{P}_k (C(\mathbb{S}^{2n+1}))$ in the theorem. It remains to verify the monoid structure of $\mathfrak{P} (C(\mathbb{S}^{2n+1}))$.

By specializing the above analysis for $P \equiv K + \bigoplus m \tilde{I}$ to the case of $K = (\bigoplus m 0) \oplus P_{j+1,k}$, we have already established that

$$
P_{0,m} \oplus P_{j+1,k} \equiv \left( \bigoplus m \tilde{I} \right) \oplus P_{j+1,k} \sim_{C(\mathbb{S}^{2n+1})} P_{0,m}
$$

for all $m \in \mathbb{N}$ and $j + 1 > 0$, while $[P_{0,k}] \oplus [P_{0,k'}] = [P_{0,k+k'}]$ is obvious.

On the other hand, by induction hypothesis,

$$
P'_{j,k} \oplus P'_{j',k'} \sim_{C(\mathbb{S}^{2n+1})} \begin{cases} 
P'_{j,k}, & \text{if } 0 \leq j < j' \\
\sum_{j=2}^{r-1} e_{jj} \otimes \tilde{I}, & \text{if } j = j' \geq 0
\end{cases}.
$$

Now by applying $P_1 \otimes \cdot$ to both sides of this equivalence, we get

$$
P_{j+1,k} \oplus P_{j+1,k'} \sim_{C(\mathbb{S}^{2n+1})} \begin{cases} 
P_{j+1,k}, & \text{if } 1 \leq j + 1 < j' + 1 \\
P_{j+1,k+k'}, & \text{if } j + 1 = j' + 1 \geq 1
\end{cases}.
$$
since if an invertible $U \in M_N \left( C \left( S_q^{2n-1} \right) \right)$ with $N$ sufficiently large conjugates an idempotent $P$ over $C \left( S_q^{2n-1} \right)$ to an idempotent $Q$, then

$$
(P_N \otimes U_{ij})_{i,j=1}^N + \boxplus^N \left( \tilde{I} - P_N \otimes \tilde{I}' \right) \in M_N \left( C \left( S_q^{2n+1} \right) \right)
$$

is an invertible conjugating $P_1 \otimes P$ to $P_1 \otimes Q$.

Now we have established all the monoid structure rules for $\mathfrak{P} \left( C \left( S_q^{2n+1} \right) \right)$.

\[\square\]

**Remark.** The last part of the above proof about the monoid structure of $\mathfrak{P} \left( C \left( S_q^{2n+1} \right) \right)$ can be avoided by applying the injective monoid homomorphism $\rho$ of the following Corollary 2 to both sides of the equivalence relations describing the monoid structure.

**Corollary 1.** All projections over $C \left( S_q^{2n+1} \right)$ of strictly positive rank are trivial. The cancellation law holds for projections of rank $\geq 1$, but fails for projections of rank 0.

**Proof.** The only equivalence class of projection of a fixed rank $k > 0$ is the trivial projection $[P_{0,k}] = \left[ \boxplus^k \tilde{I} \right]$ classified above. By counting the rank, it is clear that if $\boxplus^k \tilde{I}$ and $\boxplus^{k'} \tilde{I}$ are stably equivalent, then $k = k'$. So the cancellation law holds for projections of rank $\geq 1$.

On the other hand, since for any distinct pairs $(j, k)$ and $(j', k')$ with $1 \leq j, j' \leq n$ and $k, k' > 0$, $[P_{j,k}] \neq [P_{j',k'}]$ but

$$
[P_{j,k}] \boxplus [P_{0,1}] = [P_{0,1}] = [P_{j',k'}] \boxplus [P_{0,1}],
$$

the cancellation law fails for such rank-0 projections $P_{j,k}$ and $P_{j',k'}$.

\[\square\]
Corollary 2. The monoid \( \mathfrak{P}(C(S_q^{2n+1})) \) is a submonoid of \( \prod_{0 \leq l \leq n} \mathbb{Z}_\geq \) via the injective monoid homomorphism

\[
\rho: P \in \mathfrak{P}(C(S_q^{2n+1})) \mapsto \prod_{0 \leq l \leq n} \rho_l(P) \in \prod_{0 \leq l \leq n} \mathbb{Z}_\geq.
\]

**Proof.** \( \rho \) is injective since we already know that \( \rho_l \)'s can distinguish the standard projections \( P_{j,k} \) which have been shown to constitute the whole monoid \( \mathfrak{P}(C(S_q^{2n+1})) \).

\( \square \)

5 Generating Projections of \( K_0(C(CP_q^n)) \)

In this section, we present a set of elementary projections over \( C(CP_q^n) \), whose \( K_0 \)-classes form a set of free generators of the abelian group \( K_0(C(CP_q^n)) \). We remark that a fascinating geometric construction of free generators of \( K_0(C(CP_q^n)) \) has been found by D’Andrea and Landi in [6].

As discussed before, by restricting to the degree-0 part of the groupoid \( \mathfrak{F}_n \) consisting of exactly those \( [(z, x, w)] \) with \( z = 0 \), we get a subgroupoid \( (\mathfrak{F}_n)_0 \) which realizes \( C(CP_q^n) \) as a groupoid C*-algebra \( C^*((\mathfrak{F}_n)_0) \). Roughly speaking, \( (\mathfrak{F}_n)_0 \) can be extracted from \( \mathfrak{F}_n \) by simply ignoring or removing the \( z \)-component of the elements \( [(z, x, w)] \). Note that if \( [(0, x, w)] \in (\mathfrak{F}_n)_0 \) and \( w_1 = \infty \), then \( x_1 = 0 \) by the defining condition on \( \mathfrak{F}_n \). Furthermore since clearly \( \pi_n(C^*((\mathfrak{F}_n)_0)) \subset \text{id}_{\ell^2(\mathbb{Z})} \otimes \mathcal{B}(\ell^2(\mathbb{Z}^n_\geq)) \), we will ignore the factor \( \text{id}_{\ell^2(\mathbb{Z})} \) and view \( \pi_n|_{C^*((\mathfrak{F}_n)_0)} \) as a faithful representation of \( C^*((\mathfrak{F}_n)_0) \) on \( \ell^2(\mathbb{Z}_\geq^n) \) instead of on \( \ell^2(\mathbb{Z} \times \mathbb{Z}^n_\geq) \).

In [21], by considering the closed invariant subset \( \left( \mathbb{Z}_\geq^{n-1} \times \{\infty\} \right) / \sim \) (i.e. \( \{w : w \in \mathbb{Z}_\geq^{n-1} \times \{\infty\}\} \))
even though $\mathbb{Z}_n^{n-1} \times \{\infty\}$ is not really $\sim$-invariant in the unit space $\mathbb{F}_n$ of $\mathbb{S}_n \subset \mathbb{F}_n$ and its complement $O_0$ in the unit space $Z$ of $(\mathbb{S}_n)_0$ (and of $\mathbb{S}_n$ as well), we get the following short exact sequence

$$0 \to K_{\ell^2(\mathbb{Z}_n^{n-1})} \cong C^* ((\mathbb{S}_n)_0|_{O_0}) \to C (\mathbb{C}P^n_q) \xrightarrow{\nu} C^* (\mathbb{S}_n|_{(\mathbb{Z}_n^{n-1} \times \{\infty\})/\sim}) \cong C (\mathbb{C}P^n_{q-1}) \to 0$$

with $(\mathbb{S}_n)_0|_{O_0} \cong (\mathbb{Z}^n \times \mathbb{Z}^n)|_{\mathbb{Z}_n^2}$. Thus we get the following 6-term exact sequence

$$\begin{array}{cccc}
\mathbb{Z} = K_0 (K (\ell^2(\mathbb{Z}_n^{n-1}))) & \to & K_0 (C (\mathbb{C}P^n_q)) & \xrightarrow{\nu} K_0 (C (\mathbb{C}P^n_{q-1})) \\
\uparrow & & \downarrow & \\
K_1 (C (\mathbb{C}P^n_{q-1})) & \leftarrow & K_1 (C (\mathbb{C}P^n_q)) & \leftarrow K_1 (K (\ell^2(\mathbb{Z}_n^{n-1}))) = 0.
\end{array}$$

By an induction on $n \geq 1$, we can establish $K_0 (C (\mathbb{C}P^n_q)) \cong \mathbb{Z}^{n+1}$ and $K_1 (C (\mathbb{C}P^n_q)) = 0$. In fact, in the case of $n = 1$, we have $K_i (C (\mathbb{C}P^0_q)) = K_i (\mathbb{C}) \cong \delta_0; \mathbb{Z}$ and hence $K_0 (C (\mathbb{C}P^n_q)) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $K_1 (C (\mathbb{C}P^n_q)) = 0$. For $n > 1$, the induction hypothesis $K_1 (C (\mathbb{C}P^n_{q-1})) = 0$ and $K_0 (C (\mathbb{C}P^n_{q-1})) \cong \mathbb{Z}^n$ forces

$$K_0 (C (\mathbb{C}P^n_q)) \cong K_0 (K) \oplus K_0 (C (\mathbb{C}P^n_{q-1})) \cong \mathbb{Z} \oplus \mathbb{Z}^n = \mathbb{Z}^{n+1}$$

and also $K_1 (C (\mathbb{C}P^n_q)) = 0$ in the above 6-term exact sequence.

The above induction can be refined to get the following stronger result. First we note that

$$P_{j,k} \equiv 1_\mathbb{T} \otimes ((\otimes^{j-1}P_1) \otimes P_k \otimes (\otimes^{n-j}I)) \in C (\mathbb{S}_q^{2n+1}) \subset C (\mathbb{T}) \otimes \mathcal{B} (\ell^2(\mathbb{Z}_n^{n-1}))$$

with $0 < j \leq n$ is a projection in $C (\mathbb{C}P^n_q) \subset C (\mathbb{S}_q^{2n+1})$, and can be identified with

$$((\otimes^{j-1}P_1) \otimes P_k \otimes (\otimes^{n-j}I)) \in C (\mathbb{C}P^n_q) \subset \mathcal{B} (\ell^2(\mathbb{Z}_n^{n-1})).$$
From now on, we view $P_{j,k}$ with $0 < j \leq n$ as the latter elementary tensor product lying in $C(\mathbb{C}P^n_q)$. On the other hand, clearly the trivial projection $P_{0,k}$ of rank $k$ over $C(\mathbb{S}^{2n+1}_q)$ is also a trivial projection of rank $k$ over $C(\mathbb{C}P^n_q)$.

**Theorem 2.** The standard projections $P_{j,1} \equiv (\otimes^j P_1) \otimes (\otimes^{n-j} I)$ over $C(\mathbb{C}P^n_q)$ with $0 \leq j \leq n$ are inequivalent over $C(\mathbb{C}P^n_q)$ and their equivalence classes (over $C(\mathbb{C}P^n_q)$, not over $C(\mathbb{S}^{2n+1}_q)$) form a set of free generators of $K_0(C(\mathbb{C}P^n_q)) \cong \mathbb{Z}^{n+1}$.

**Proof.** Since $P_{j,1}$ are inequivalent over $C(\mathbb{S}^{2n+1}_q)$, they are clearly inequivalent over the subalgebra $C(\mathbb{C}P^n_q)$. Now we prove by induction on $n \geq 1$ that $[P_{j,1}]$ with $0 \leq j \leq n$ form a set of free generators of $K_0(C(\mathbb{C}P^n_q))$.

For $n = 1$, it is well-known that $K_0(\mathbb{K}(\ell^2(\mathbb{Z}_2)))^+ \cong C(\mathbb{C}P^1_q)$ has $[P_1] \equiv [P_{1,1}]$ and $[I] \equiv [P_{0,1}]$ as free generators of its $K_0$-group $K_0(\mathbb{K}(\ell^2(\mathbb{Z}_2)))^+ \cong \mathbb{Z}^2$.

For $n > 1$, $K_0(C(\mathbb{C}P^{n-1}_q)) \cong \mathbb{Z}$ has $[\otimes^n P_1] \equiv [P_{n,1}]$ as a free generator, while by induction hypothesis, $K_0(C(\mathbb{C}P^{n-1}_q)) \cong \mathbb{Z}^n$ has $[P'_{j,1}] \equiv [(\otimes^j P_1) \otimes (\otimes^{n-1-j} I)]$ with $0 \leq j \leq n-1$ as free generators. Now with $\nu_* ([P_{j,1}]) \equiv \nu_* ([P'_{j,1} \otimes I]) = [P'_{j,1}]$ for all $0 \leq j \leq n-1$, it is easy to see from the above 6-term exact sequence that $[P_{j,1}]$ for $0 \leq j \leq n-1$ together with $[P_{n,1}]$ form a set of free generators of $K_0(C(\mathbb{C}P^n_q)) \cong \mathbb{Z}^{n+1}$.

□

It is of interest to point out that these projections $P_{j,1}$ freely generating $K_0(C(\mathbb{C}P^n_q))$ are actually lying inside $C(\mathbb{C}P^n_q) \equiv M_1(C(\mathbb{C}P^n_q)) \subset M_\infty(C(\mathbb{C}P^n_q))$ and they form an increasing finite sequence of projections.
6 Quantum line bundles over $C\left(\mathbb{C}P^n_q\right)$

In this section, we identify the quantum line bundles $L_k \equiv C\left(\mathbb{S}^{2n+1}_q\right)_k$ of degree $k$ over $C\left(\mathbb{C}P^n_q\right)$ with a concrete (equivalence class of) projection described in terms of the basic projections. We remark that an intriguing noncommutative geometric study of these line bundles in comparison with Adam’s classical results on $\mathbb{C}P^n$ has been successfully accomplished by Arici, Brain, and Landi in [1]. (The degree convention is different in the ±-sign.)

To distinguish between ordinary function product and convolution product, we denote the groupoid $\mathrm{C}^*$-algebraic (convolution) multiplication of elements in $C_c(\mathcal{G}) \subset C^*(\mathcal{G})$ by $\ast$, while omitting $\ast$ when the elements are represented as operators or when they are multiplied together pointwise as functions on $\mathcal{G}$. We also view $C_c(\mathfrak{F}_n)$ or $C_c((\mathfrak{F}_n)_k)$ (also abbreviated as $C_c(\mathfrak{F}_n)_k$) as left $C_c(\mathfrak{F}_n)_0$-modules with $C_c(\mathfrak{F}_n)$ carrying the convolution algebra structure as a subalgebra of the groupoid $\mathrm{C}^*$-algebra $C^*(\mathfrak{F}_n)$. Similarly, for a closed subset $X$ of the unit space of $\mathfrak{F}_n$, the inverse image $\mathfrak{F}_n \upharpoonright_X$ of $X$ under the source map of $\mathfrak{F}_n$ or its grade-$k$ component $\left(\mathfrak{F}_n \upharpoonright_X\right)_k \equiv (\mathfrak{F}_n)_k \upharpoonright_X$ gives rise to a left $C_c((\mathfrak{F}_n)_0)$-module $C_c((\mathfrak{F}_n \upharpoonright_X)_k)$ or $C_c(\mathfrak{F}_n \upharpoonright_X)_k$.

For $k \in \mathbb{Z}_{\geq}$, the characteristic function $\chi_{B_k} \in C_c((\mathfrak{F}_n)_0)$ of the compact open set

$$B_k := \{(0,0^{(n)},w) \in (\mathfrak{F}_n)_0 : w_1 \geq k\}$$

is a projection over $C^*((\mathfrak{F}_n)_0) \equiv C\left(\mathbb{C}P^n_q\right)$ which is represented under $\pi_n$ as $P_{-k} \otimes (\otimes^{n-1}I)$, and

$$C_c((\mathfrak{F}_n)_0) \ast \chi_{B_k} = C_c((\mathfrak{F}_n)_0 \upharpoonright_{B_k})$$
where $B_k \subset (\mathcal{F}_n)_0$ in the notation $|B_k|$ is canonically viewed as a subset of the unit space of $(\mathcal{F}_n)_0$.

For $k \leq 0$, it is straightforward to check that

$$[(k, x, w)] \in (\mathcal{F}_n)_k \mapsto [(0, x_1 + k, x_2, \ldots, x_n, w_1 - k, w_2, \ldots, w_n)] \in (\mathcal{F}_n)_0 |_{B_k}$$

well defines a bijective homeomorphism. For example, for $w_1 = \infty$, we have $x_1 = -k$ on the domain side and $x_1 + k = 0$ on the range side of this map, matching the implicit constraints imposed on $(\mathcal{F}_n)_k$ and $(\mathcal{F}_n)_0$. Furthermore since any $[(k, x, w)] \in (\mathcal{F}_n)_k$ and its image $[(0, x_1 + k, x_2, \ldots, x_n, x_1 - k, x_2, \ldots, x_n)]$ share the same target element $[x + w] \in \mathbb{Z}_n^\geq / \sim$, it induces a left $C_c((\mathcal{F}_n)_0)$-module isomorphism

$$C_c((\mathcal{F}_n)_k) \rightarrow C_c((\mathcal{F}_n)_0 |_{B_k}) \equiv C_c((\mathcal{F}_n)_0) * \chi_{B_k}$$

which extends to a left $C(\mathbb{C} P_q^n)$-module isomorphism

$$L_k \equiv \overline{C_c((\mathcal{F}_n)_k)} \cong C(\mathbb{C} P_q^n) \left( P_{-|k|} \otimes (\otimes^{n-1}I) \right),$$

i.e. the quantum line bundle $L_k$ for $k \leq 0$ is the finitely generated left projective module determined by the projection $P_{-|k|} \otimes (\otimes^{n-1}I)$ over $C(\mathbb{C} P_q^n)$.

For $k > 0$, the situation is much more complicated. We first define the closed open set

$$(\mathcal{F}_n)_{k,j} := \left( \mathcal{F}_n |_{\{0\}^j \times \mathbb{Z}_n^{n-j}} / \sim \right)_k \equiv \left\{ [(k, x, w)] \in (\mathcal{F}_n)_k : w \in \{0\}^j \times \mathbb{Z}_n^{n-j} \right\}$$

with each $C_c((\mathcal{F}_n)_{k,j})$ a left $C_c((\mathcal{F}_n)_0)$-module. Note that

$$C_c((\mathcal{F}_n)_{0,j}) = C_c((\mathcal{F}_n)_0) * \chi_{\{0\}^j \times \mathbb{Z}_n^{n-j}} / \sim$$

25
with \( \chi_{((0)^j \times \mathbb{Z}^{n-j})/\sim} \) represented under \( \pi_n \) as the projection \((\otimes^j P_1) \otimes (\otimes^{n-j} I)\) over \( C(\mathbb{C}P^n_q) \).

Now the left \( C_c((\mathfrak{F}_n)_0) \)-module \( C_c((\mathfrak{F}_n)_{k,j}) \) can be decomposed as

\[
C_c((\mathfrak{F}_n)_{k,j}) = C_c \left( \left( \mathfrak{F}_n \uparrow ((0)^j \times \mathbb{Z}^{n-j})/\sim\right)_k \right) \oplus \bigoplus_{l=0}^{k-1} C_c \left( \left( \mathfrak{F}_n \uparrow ((0)^j \times (l) \times \mathbb{Z}^{n-j-l})/\sim\right)_k \right).
\]

It is straightforward to check that

\[
[(k, x, w)] \in \left( \mathfrak{F}_n \uparrow ((0)^j \times \mathbb{Z}^{n-j})/\sim\right)_k \mapsto [(0, x_1, \ldots, x_j, x_{j+1} + k, x_{j+2}, \ldots, x_n, 0^{(j)}), w_{j+1} - k, w_{j+2}, \ldots, w_n] \in \left( \mathfrak{F}_n \uparrow ((0)^j \times \mathbb{Z}^{n-j})/\sim\right)_0 \equiv (\mathfrak{F}_n)_{0,j}\]

well defines a bijective homeomorphism. For example, we are considering only \( w \) with \( w_1 = \ldots = w_j = 0 < \infty \), while for \( w_{j+1} = \infty \), we have \( x_{j+1} = -k - x_1 - \cdots - x_j \) on the domain side and \(-x_1 - \cdots - x_j - (x_{j+1} + k) = 0\) on the range side of this map, matching the implicit constraints imposed on \((\mathfrak{F}_n)_k\) and \((\mathfrak{F}_n)_0\). Furthermore since any \([(k, x, w)]\) and its image under this bijection share the same target element \([x + w] \in \mathbb{Z}^n_{\leq 0} / \sim\), it induces a left \( C_c((\mathfrak{F}_n)_0) \)-module isomorphism

\[
C_c \left( \left( \mathfrak{F}_n \uparrow ((0)^j \times \mathbb{Z}^{n-j})/\sim\right)_k \right) \rightarrow C_c \left( (\mathfrak{F}_n)_0 \right) = C_c ((\mathfrak{F}_n)_0) \ast \chi_{((0)^j \times \mathbb{Z}^{n-j})/\sim}
\]

which extends to a left \( C(\mathbb{C}P^n_q) \)-module isomorphism

\[
C_c \left( \left( \mathfrak{F}_n \uparrow ((0)^j \times \mathbb{Z}^{n-j})/\sim\right)_k \right) \cong C(\mathbb{C}P^n_q) \left( (\otimes^j P_1) \otimes (\otimes^{n-j} I) \right).
\]

On the other hand, for any \( 0 \leq l \leq k - 1 \),

\[
[(k, x, w)] \in \left( \mathfrak{F}_n \uparrow ((0)^j \times (l) \times \mathbb{Z}^{n-j-l})/\sim\right)_k \mapsto [(k - l, x_1, \ldots, x_j, x_{j+1} + l, x_{j+2}, \ldots, x_n, 0^{(j+1)}), w_{j+2}, \ldots, w_n] \in \left( \mathfrak{F}_n \uparrow ((0)^{j+1} \times \mathbb{Z}^{n-j-l})/\sim\right)_{k-l} \equiv (\mathfrak{F}_n)_{k-l,j+1}
\]

26
well defines a bijective homeomorphism which preserves the target element $[x + w]$ and hence induces a left $C_c((\mathcal{F}_n)_0)$-module isomorphism

$$C_c\left(\left(\mathcal{F}_n \upharpoonright \{0\}^* \times \{0\} \times \mathbb{Z}_{\geq -j} / \sim\right)_{\sim k}\right) \rightarrow C_c\left(\mathcal{F}_n_{k-l,j+1}\right).$$

So summarizing, we get the isomorphism relation

$$(*) \quad C_c\left(\mathcal{F}_n_{k,j}\right) \cong C_c\left(\mathcal{F}_n_{0,j}\right) \oplus \bigoplus_{l=0}^{k-1} C_c\left(\mathcal{F}_n_{k-l,j+1}\right) \equiv C_c\left(\mathcal{F}_n_{0,j}\right) \oplus \bigoplus_{l=1}^{k} C_c\left(\mathcal{F}_n_{l,j+1}\right)$$

which is recursive in the sense that the right hand side contains terms with either $k$ decreased or $j$ increased. So repeated application of this recursive expansion can lead to a direct sum of terms of the form $C_c\left(\mathcal{F}_n_{0,m}\right)$ or the form $C_c\left(\mathcal{F}_n_{l,n}\right)$, where

$$C_c\left(\mathcal{F}_n_{0,m}\right) \cong C\left(\mathbb{C}P^n\right)\left((\otimes^m P_1) \otimes (\otimes^{n-m} I)\right)$$

while

$$[(l, x, w)] \equiv [(l, x, 0^{(n)})] \in (\mathcal{F}_n \upharpoonright \{0\}^\times / \sim)_l \equiv (\mathcal{F}_n)_{l,n} \mapsto [(0, x, 0^{(n)})] \in (\mathcal{F}_n)_{0,n}$$

well defines a bijective homeomorphism which induces a left $C_c((\mathcal{F}_n)_0)$-module isomorphism

$$C_c\left(\mathcal{F}_n_{l,n}\right) \rightarrow C_c\left(\mathcal{F}_n_{0,n}\right)$$

extending to a left $C\left(\mathbb{C}P^n\right)$-module isomorphism

$$C_c\left(\mathcal{F}_n_{l,n}\right) \cong C\left(\mathbb{C}P^n\right)\left(\otimes^n P_1\right).$$

**Theorem 3.** For $n \geq 1$, the quantum line bundle $L_k \equiv C\left(\mathbb{S}_q^{2n+1}\right)_{k}$ of degree $k \in \mathbb{Z}$ over $C\left(\mathbb{C}P^n_q\right)$ is isomorphic to the finitely generated projective left module over $C\left(\mathbb{C}P^n_q\right)$.
determined by the projection $P_{-|k|} \otimes (\otimes^{n-1}I)$ if $k \leq 0$ (with $P_0 := I$ understood), and the projection

$$\bigoplus_{j=0}^{n} \left( \bigoplus_{j=0}^{C_{j}^{k+j-1}} (\otimes^{j} P_1) \otimes (\otimes^{n-j}I) \right)$$

if $k > 0$, where $C_j^k$ denotes the combinatorial number $(k! / (j!(k-j)!))$.

Proof. Having already taken care of the case of $k \leq 0$ in the above discussion, we only need to consider the case of $k > 0$.

First we establish by induction on $l$ that

$$\text{(**) } C_c \left( (\mathcal{F}_n)_{k,0} \right) \cong \left( \bigoplus_{j=0}^{l-1} \left( \bigoplus_{j=0}^{C_{j}^{k+j-1}} C_c \left( (\mathcal{F}_n)_{0,j} \right) \right) \bigoplus \left( \bigoplus_{m=1}^{k} \left( \bigoplus_{l-1}^{C_{l-1}^{k-m+l-2}} C_c \left( (\mathcal{F}_n)_{m,l} \right) \right) \right) \right)$$

Indeed for $l = 1$, (** ) becomes

$$C_c \left( (\mathcal{F}_n)_{k,0} \right) \cong C_c \left( (\mathcal{F}_n)_{0,0} \right) \bigoplus \left( \bigoplus_{m=1}^{k} C_c \left( (\mathcal{F}_n)_{m,1} \right) \right),$$

which is the same as the established recursive relation (*) with $j = 0$. For $n \geq l > 1$, by the induction hypothesis for $l - 1$ and the recursive relation (*), we get

$$C_c \left( (\mathcal{F}_n)_{k,0} \right) \cong \left( \bigoplus_{j=0}^{l-2} \left( \bigoplus_{j=0}^{C_{j}^{k+j-1}} C_c \left( (\mathcal{F}_n)_{0,j} \right) \right) \bigoplus \left( \bigoplus_{m=1}^{k} \left( \bigoplus_{l-2}^{C_{l-2}^{k-m+l-2}} C_c \left( (\mathcal{F}_n)_{m,l-1} \right) \right) \right) \right)$$

and

$$\left( \bigoplus_{j=0}^{l-2} \left( \bigoplus_{j=0}^{C_{j}^{k+j-1}} C_c \left( (\mathcal{F}_n)_{0,j} \right) \right) \bigoplus \left( \bigoplus_{m=1}^{k} \left( \bigoplus_{l-2}^{C_{l-2}^{k-m+l-2}} C_c \left( (\mathcal{F}_n)_{m,1} \right) \right) \right) \right)$$

and

$$\left( \bigoplus_{j=0}^{l-2} \left( \bigoplus_{j=0}^{C_{j}^{k+j-1}} C_c \left( (\mathcal{F}_n)_{0,j} \right) \right) \bigoplus \left( \bigoplus_{m=1}^{k} \left( \bigoplus_{l-2}^{C_{l-2}^{k-m+l-2}} C_c \left( (\mathcal{F}_n)_{m,1} \right) \right) \right) \right)$$

and

$$\left( \bigoplus_{j=0}^{l-2} \left( \bigoplus_{j=0}^{C_{j}^{k+j-1}} C_c \left( (\mathcal{F}_n)_{0,j} \right) \right) \bigoplus \left( \bigoplus_{m=1}^{k} \left( \bigoplus_{l-2}^{C_{l-2}^{k-m+l-2}} C_c \left( (\mathcal{F}_n)_{m,1} \right) \right) \right) \right)$$

and

$$\left( \bigoplus_{j=0}^{l-2} \left( \bigoplus_{j=0}^{C_{j}^{k+j-1}} C_c \left( (\mathcal{F}_n)_{0,j} \right) \right) \bigoplus \left( \bigoplus_{m=1}^{k} \left( \bigoplus_{l-2}^{C_{l-2}^{k-m+l-2}} C_c \left( (\mathcal{F}_n)_{m,1} \right) \right) \right) \right)$$

28
\( L_l \equiv \left( \bigoplus_{j=0}^{l-1} \left( \bigoplus_{c_j} C_{c} \left( (\mathfrak{F}_n)_{0,j} \right) \right) \bigoplus \left( \bigoplus_{i=1}^{k} \left( \sum_{m=1}^{k} C_{c} \left( (\mathfrak{F}_n)_{i,l} \right) \right) \right) \right) \)

\( \equiv \left( \bigoplus_{j=0}^{l-1} \left( \bigoplus_{c_j} C_{c} \left( (\mathfrak{F}_n)_{0,j} \right) \right) \bigoplus \left( \bigoplus_{m=1}^{k} \left( C_{c} \left( (\mathfrak{F}_n)_{m,l} \right) \right) \right) \right) \)

where

\[
\sum_{m=1}^{k} C_{l-2}^{k-m+l-2} = C_{l-2}^{l-2} + C_{l-2}^{l-1} + C_{l-2}^{l} + C_{l-2}^{l+1} + \cdots + C_{l-2}^{k+l-3} \\
= C_{l-2}^{l-1} + C_{l-2}^{l-1} + C_{l-2}^{l} + C_{l-2}^{l+1} + \cdots + C_{l-2}^{k+l-3} \\
= C_{l-2}^{l-1} + C_{l-2}^{l-1} + \cdots + C_{l-2}^{k+l-3} = C_{l-1}^{l} + C_{l-2}^{l} + \cdots + C_{l-2}^{k+l-3} \\
= \cdots = C_{l-1}^{k+l-3} + C_{l-2}^{k+l-3} = C_{l-1}^{k+l-2}
\]

and similarly

\[
\sum_{m=1}^{k} C_{l-2}^{k-m+l-2} = C_{l-2}^{l-2} + C_{l-2}^{l-1} + \cdots + C_{l-2}^{k-i+l-2} = C_{l-1}^{k-i+l-1}.
\]

Thus (**) holds for \( n \geq l > 1 \), concluding the inductive proof of (**).

Now by (**) for \( l = n \), we get

\[
L_n \equiv C_{c} \left( (\mathfrak{F}_n)_{k,0} \right) \cong \left( \bigoplus_{j=0}^{n-1} \left( \bigoplus_{c_j} C_{c} \left( (\mathfrak{F}_n)_{0,j} \right) \right) \bigoplus \left( \bigoplus_{m=1}^{k} \left( \sum_{m=1}^{k} C_{c} \left( (\mathfrak{F}_n)_{m,n} \right) \right) \right) \right)
\]

\[
= \left( \bigoplus_{j=0}^{n-1} \left( \bigoplus_{c_j} C \left( (\mathbb{C}P^n_q) \left( (\otimes^j P_1) \otimes (\otimes^{n-j} I) \right) \right) \right) \bigoplus \left( \bigoplus_{m=1}^{k} \left( \sum_{m=1}^{k} C \left( (\mathbb{C}P^n_q) \left( \otimes^n P_1 \right) \right) \right) \right)
\]

\[
= \left( \bigoplus_{j=0}^{n-1} \left( \bigoplus_{c_j} C \left( (\mathbb{C}P^n_q) \left( (\otimes^j P_1) \otimes (\otimes^{n-j} I) \right) \right) \right) \bigoplus \left( \sum_{m=1}^{k} C \left( (\mathbb{C}P^n_q) \left( \otimes^n P_1 \right) \right) \right)
\]

\[
= \left( \bigoplus_{j=0}^{n-1} \left( \bigoplus_{c_j} C \left( (\mathbb{C}P^n_q) \left( (\otimes^j P_1) \otimes (\otimes^{n-j} I) \right) \right) \right) \bigoplus \left( \sum_{m=1}^{k} C \left( (\mathbb{C}P^n_q) \left( \otimes^n P_1 \right) \right) \right)
\]

29
\[= \bigoplus_{j=0}^{n} \left( \bigoplus_{j=0}^{C_{k+j-1}} C \left( C P_{q}^{n} \right) \left( \bigotimes^{j} P_{1} \right) \otimes \left( \bigotimes^{n-j} I \right) \right) \]

where again

\[\sum_{m=1}^{k} C_{k-m+n}^{n-1} = C_{n-1}^{n-1} + C_{n-1}^{n-1} + C_{n-1}^{n-1} + \cdots + C_{n-1}^{n-1} = C_{n}^{n-1}.\]

Thus \(L_{k}\) for \(k > 0\) is implemented by the projection

\[\bigoplus_{j=0}^{n} \left( \bigoplus_{j=0}^{C_{k+j-1}} \left( \bigotimes^{j} P_{1} \right) \otimes \left( \bigotimes^{n-j} I \right) \right).\]

\[\square\]

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