ON ALGEBRAICALLY CLOSED FIELDS WITH A DISTINGUISHED SUBFIELD

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Abstract. This paper is concerned with the model-theoretic study of pairs $(K, F)$ where $K$ is an algebraically closed field and $F$ is a distinguished subfield of $K$ allowing extra structure. We study the basic model-theoretic properties of those pairs, such as quantifier elimination, model-completeness and saturated models. We also prove some preservation results of classification-theoretic notions such as stability, simplicity, NSOP$_1$, and NIP. As an application, we conclude that a PAC field is NSOP$_1$ iff its absolute Galois group is (as a profinite group).

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1. Introduction

In their study of pseudo-algebraically closed fields, or PAC fields (known at that time as regularly closed fields, for obvious reasons, see Definition 6.8) Cherlin, van den Dries and Macintyre [CvdDM80, CvdDM81] described elementary invariants for those fields. This was inspired by the work of Ax on pseudo-finite fields. Among those invariants is the elementary theory of the absolute Galois group of those fields in a suitable omega-sorted language, called the inverse system of the absolute Galois group. It was already clear to the authors of [CvdDM80, CvdDM81] that this invariant is an essential tool for the study of PAC fields. The intuition that the model theoretic complexity of the theory of PAC fields is mainly controlled by the theory of its absolute Galois group was confirmed by numerous results since then. For example, Chatzidakis [Cha19] proved that if the inverse system of the absolute Galois group of a PAC field is NSOP$_n$ ($n > 2$), then so is the theory of the field. Ramsey [Ram18] proved the corresponding results for NTP$_1$ and NSOP$_1$. It is a fact that the inverse system of the absolute Galois group of a field $F$ is interpretable in the theory of the pair $(K, F)$ for any algebraically closed field $K$ extending $F$. 

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This motivated our interest in the model-theoretic study of such pairs \((K, F)\).

The model-theoretic study of pairs of fields goes back to Tarski when he raised in \cite{Tarski51} the question of the decidability of the pair \((\mathbb{R}, \mathbb{R} \cap \mathbb{Q}_{\text{alg}})\) (the reals with a predicate for the reals algebraic over \(\mathbb{Q}\)). The (positive) answer was given by Robinson in \cite{Robinson59}, who gave a full set of axioms for the theories of \((\mathbb{R}, \mathbb{R} \cap \mathbb{Q}_{\text{alg}})\) and \((\mathbb{C}, \mathbb{Q}_{\text{alg}})\). The celebrated work of Morley and of Shelah in the 70s created a growing interest in classification of first-order theories, and in particular of theories of fields and their expansions. It was known since the 80’s that the theory of \((\mathbb{C}, \mathbb{Q}_{\text{alg}})\) is stable\(^1\) and Poizat \cite{Poizat83} generalized this result to a more general context: he gave a criterion for the stability of special pairs of elementary substructures \(N \preceq M\) (called “belle paires”), under a strong stability assumption on the theory of \(M\) (and \(N\)) called nfp, introduced by Keisler \cite{Keisler67}. This was later generalised to the context of simple theories \cite{Bryant83} with the notion of lovely pairs. Back to algebraically closed fields, Delon \cite{Delon12} introduced a language for quantifier elimination for proper pairs of algebraically closed fields (which are models of the theory of belles paires of algebraically closed fields) and proper pairs of algebraically closed valued fields. Recently, Martin-Pizarro and Ziegler \cite{MPZ20} proved that the theory of proper pairs of algebraically closed fields is equational, by a deep analysis of definable sets.

As was mentioned above, the main topic of this paper is another generalization of pairs of algebraically closed fields which are pairs \((K, F)\) where \(F\) is an arbitrary field, perhaps with some extra structure (in a language extending the language of rings), and \(K \supseteq F\) is an algebraically closed field, such that the degree of \(K\) over \(F\) is infinite. An early result about this theory was given by Keisler \cite{Keisler64}: if \(F\) and \(F'\) are two elementarily equivalent fields (not real-closed nor algebraically closed and without extra structure), then the pairs \((K, F)\) and \((K', F')\) are also elementarily equivalent, for any algebraically closed extensions \(K \supseteq F, K' \supseteq F'\). In \cite{Hils18}, Hils, Kamensky and Rideau gave a quantifier elimination result for the theory of the pairs \((K, F)\), which we also obtain in Theorem \ref{thm:qe} (we became aware of their work only after we finished writing our proof and we decided to keep it for completeness).

The purpose of this paper is twofold: (1) investigate the basic logical properties of the theory of such pairs and (2) prove preservation results of several classification-theoretic properties.

For (1), we discuss saturated models, completeness, quantifier elimination and model-completeness. For example, as we mentioned above we prove quantifier elimination for the theory of pairs \((K, F)\) (see Theorem \ref{thm:qe}) in a natural expansion of the language following Delon’s approach \cite{Delon12}. This allows us to isolate a condition implying the model-completeness of the theory of the pair \((K, F)\) which is weaker than the model completeness of the theory of \(F\) (see Theorem \ref{thm:modelcomp}). For (2), we prove preservation of several classification-theoretic properties: if the theory of \(F\) is \((\omega-/super)\) stable/NIP/simple/NSOP\(_1\), then so is the theory of the pair \((K, F)\) (see Corollaries \ref{cor:nip} and \ref{cor:nsop1} and Theorems \ref{thm:preservation_nip}, \ref{thm:preservation_nip}, \ref{thm:preservation_nip} and \ref{thm:preservation_nip}). In the case of NSOP\(_1\), we also identify Kim-independence for algebraically closed sets (see Proposition \ref{prop:kim}).

As immediate applications we conclude that:

1. The theory of a PAC field \(F\) in the language of rings is NSOP\(_1\) if and only if the theory of its Galois group is (see Proposition \ref{prop:pac}).

2. When \(F\) is pseudofinite in the language of rings, then the theory of the pair \((K, F)\) is simple.

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\(^1\)See the first sentence of \cite{Poizat83}.
In addition, we consider the theory $\mathbf{ACF}^I$ of a chain of algebraically closed fields ordered by some linear order $I$, and discuss its properties depending on the order type of $I$ (see Proposition 6.4).

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2. Preliminaries

In this section we present common definitions and results from fields and model theory. We will start by setting up some basic notation for the whole paper.

Notation 2.1. Whenever $A$ is a field, let $\overline{A}$ be its algebraic closure. Whenever $A$ and $B$ are subfields of a larger field, let $A.B$ be their field compositum. If $A$ is a field and $S$ is a set, then let $A(S)$ be the field extension of $A$ by the elements of $S$. Say that the set $S$ is algebraically independent over $A$ if each element $s \in S$ is algebraically independent over $A(S \setminus \{s\})$. If $R$ is a sub-ring of a larger field, then denote by $\text{Fr}(R)$ the field generated by $R$. Unless specified otherwise, all the fields will be subfields of a large algebraically closed field.

2.1. Linear disjointness.

Definition 2.2. Let $A$, $B$ and $C$ be fields with $C \subseteq A \cap B$.

1. Say that $A$ is linearly disjoint from $B$ over $C$ if whenever $a_0, \ldots, a_{n-1} \in A$ are linearly independent over $C$ they are also linearly independent over $B$. Denote this by $A \perp_C^I B$.

2. Say that $A$ is algebraically disjoint from $B$ over $C$ if whenever $a_0, \ldots, a_{n-1} \in A$ are algebraically independent over $C$, then they are also algebraically independent over $B$. This is the same as the non-forking independence in $\mathbf{ACF}$, which we will denote $A \perp^\mathbf{ACF}_C B$.

Fact 2.3 ([Mor96 Proposition 20.2]). Let $A$, $B$ and $C$ be fields with $C \subseteq A \cap B$. Construct a map $A \otimes_C B \rightarrow A[B]$ by mapping $a \otimes b \mapsto ab$. This map is an isomorphism iff $A \perp_C^I B$.

Fact 2.4. The following is a list of useful model theoretic properties that $\perp_C^I$ has inside $\mathbf{ACF}$. Let $A$, $B$, $C$, $D$, $A'$, $B'$ and $C'$ be fields with $C \subseteq A \cap B$, $C' \subseteq A' \cap B'$ and $B \subseteq D$.

- (Invariance) if $ABC \equiv A'B'C'$ and $A \perp_C^I B$, then $A' \perp_{C'}^I B'$.
- (Monotonicity) if $A \perp_C^I D$, then $A \perp_C^I B$.
- (Base monotonicity) if $A \perp_C^I D$, then $AB \perp_B^I D$.
- (Transitivity) if $A \perp_C^I B$ and $AB \perp_B^I D$, then $A \perp_C^I D$.
- (Symmetry) if $A \perp_C^I B$, then $B \perp_C^I A$.
- (Stationarity) if $A \equiv_C A'$ and $A \perp_C^I B$, then $A \equiv_B A'$.
- (Local character) for a finite tuple $a$, there exists a countable subfield $B_0 \subseteq B$, such that $B_0(a) \perp_{B_0}^I B_0$.

Proof. Invariance is trivial. Proofs for monotonicity, base monotonicity and transitivity can be found in [FJo8 Lemma 2.5.3], symmetry is proven in [FJo8 Lemma 2.5.1]. Stationarity follows directly from Fact 2.3 and quantifier elimination in $\mathbf{ACF}$.

Local character follows from [Lan72 Theorem III.7, Proposition III.6 and Theorem III.8], by setting $B_0$ to be the field of definition of the locus of $a$ over $B$. This
gives an even stronger result, as \( B_0 \) is finitely generated and not merely countable. For a more direct proof of local character, see Remark 5.2.

**Corollary 2.5.** Let \( A_0, B_0, C_0, A_1, B_1 \) and \( C_1 \) be fields with \( C_0 \subseteq A_0 \cap B_0, C_1 \subseteq A_1 \cap B_1 \), such that \( A_0 \downarrow_{C_0} B_0, A_1 \downarrow_{C_1} B_1 \). Suppose there are isomorphism \( f : A_0 \rightarrow A_1, g : B_0 \rightarrow B_1 \) such that \( f|_{C_0} = g|_{C_0} \). Then there is a unique isomorphism \( F : A_0.B_0 \rightarrow A_1.B_1 \) such that \( F|_{A_0} = f, F|_{B_0} = g \).

**Proof.** Consider \( A_0, A_1, B_0, B_1 \) as tuples, such that \( f \) and \( g \) match the tuples. Extend \( g \) to an automorphism \( \sigma \) arbitrarily. From invariance, by applying \( \sigma \) to \( A_0 \downarrow_{C_0} B_0 \), we get \( \sigma(A_0) \downarrow_{C_1} B_1 \). From stationarity \( \sigma(A_0) \equiv_B A_1 \), let \( \tau \) be an automorphism witnessing the equivalence. Let \( F = (\tau \circ \sigma)|_{A_0.B_0} \), we have \( F(A_0) = \tau(\sigma(A_0)) = A_1 \) and \( F(B_0) = \tau(\sigma(B_0)) = \tau(B_1) = B_1 \) as tuples. In particular, \( F : A_0.B_0 \rightarrow A_1.B_1 \) is an isomorphism, and from the way we chose the tuples \( F|_{A_0} = f \) and \( F|_{B_0} = g \). \( \square \)

**Definition 2.6.** A field extension \( A \subseteq B \) is called:
- regular if \( \overline{A} \downarrow_A^l B \),
- separable if \( A^{1/p} \downarrow_A^l B \), where \( p = \text{char}(A) > 0 \) and \( A^{1/p} \) is the field of \( p \)-th roots of all elements in \( A \) (if \( \text{char}(A) = 0 \), then all extensions are separable), and
- relatively algebraically closed if \( \overline{A} \cap B = A \).

**Fact 2.7.** Suppose \( A \subseteq B \) is a field extension.

1. \([\text{FJ08]}\) Lemma 2.6.4 The extension \( A \subseteq B \) is regular iff it is separable and relatively algebraically closed.

2. \([\text{FJ08]}\) Lemma 2.6.7 If the extension \( A \subseteq B \) is regular and \( C \) is a field extending \( A \) such that \( B \downarrow_A^ACF C \), then \( B \downarrow_A^l C \).

**Lemma 2.8.** If \( A \subseteq B \) is a regular field extension and \( \sigma : B \rightarrow B' \) is an isomorphism of fields, then \( \sigma(A) \subseteq B' \) is regular.

**Proof.** We can extend \( \sigma \) to the algebraic closure, \( \overline{\sigma} : B \rightarrow B' \). From \( \overline{A} \downarrow_A^l B \) we get by invariance \( \overline{\sigma(A)} \downarrow_{\sigma(A)}^l B' \). But \( \overline{\sigma(A)} = \sigma(A) \), so we have \( \overline{\sigma(A)} \downarrow_{\sigma(A)}^l B' \) as needed. \( \square \)

**Lemma 2.9.** If \( A \subseteq B \) is a regular field extension and \( S \) is a set algebraically independent over \( B \), then \( \overline{A(S)} \downarrow_A^l B \).

**Proof.** As \( S \) is algebraically independent over \( B \), we have \( \overline{A(S)} \downarrow_A^ACF B \). By Fact 2.7(2), \( \overline{A(S)} \downarrow_A^l B \). \( \square \)

### 2.2. Language of regular extensions.

In \([\text{Mac08]}\), Macintyre defines relations in the language of rings that are preserved in a field extension iff it is regular. We will present those relations, and use them to expand a theory of fields\(^2\) in such a way that the models are the same but for any two models \( M, N \), \( N \) extends \( M \) iff it is a regular field extension.

**Fact 2.10** \((\text{[Mac08] \S 4.7}]). Let \( A \subseteq B \) be a field extension.

1. The extension is relatively algebraically closed iff it preserves the relations \( \text{Sol}_n(x_0, \ldots, x_{n-1}) = \exists y(x_0 + x_1y + \cdots + x_{n-1}y^{n-1} + y^n = 0) \) for \( n \geq 1 \).

2. For \( p = \text{char}(A) \), the extension is separable iff it preserves the relations \( D_{n,p}(x_0, \ldots, x_{n-1}) = \exists y_0, \ldots, y_{n-1}(y_0^px_0 + \cdots + y_{n-1}^px_{n-1} = 0) \) for \( n \geq 1 \) (note that if \( p = 0 \), \( D_{n,p} \) is quantifier-free definable).

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\(^2\)By a theory of fields, we mean a theory in a language expanding the language of rings which contains all the fields axioms.
Corollary 2.11. Suppose $M$ and $N$ are fields. If $M \preceq N$, then $M \subseteq N$ is a regular extension.

Proof. The fact that $M \preceq N$ implies in particular that $M \subseteq N$ is a field extension that preserves $\text{Sol}_1$ and $D_{n,p}$ ($p = \text{char}(A)$). By Fact 2.10, the extension $M \subseteq N$ is relatively algebraically closed and separable, so by Fact 2.7(1) it is a regular extension.

Definition 2.12. Let $T$ be a theory of fields in a language $L$ expanding the language of rings. Define $L_{\text{reg}} = L \cup \{\text{Sol}_n\}_{n \geq 1} \cup \left\{\tilde{D}_{n,p}\right\}_{n \geq 1, p \in \text{Prime}, \{0\}}$ where $\text{Sol}_n$, $\tilde{D}_{n,p}$ are $n$-ary relations, and extend $T$ to $T_{\text{reg}}$ in $L_{\text{reg}}$ by defining $\text{Sol}_n$ as above and defining

$$\tilde{D}_{n,p} = D_{n,p} \land \left(\bigcup_{p} \left(1 + \cdots + 1 = 0\right)\right).$$

Lemma 2.13. Let $T$ be a theory of fields and let $Q, R \models T$ with $Q \subseteq R$ a substructure. By adding definable relations, $Q$ and $R$ can be expanded to models of $T_{\text{reg}}$. Then $Q$ is an $L_{\text{reg}}$-substructure of $R$ iff $Q \subseteq R$ is a regular field extension.

Proof. Let $p = \text{char}(Q)$. Note that by Facts 2.7 and 2.10, it is enough to prove that $Q$ is an $L_{\text{reg}}$-substructure of $R$ iff the extension $Q \subseteq R$ preserves $\text{Sol}_n$ and $\tilde{D}_{n,p}$ for all $n$. Indeed, this equivalence holds because $\tilde{D}_{n,p}$ is equivalent to $D_{n,p}$ and $\tilde{D}_{n,q}$ is trivially false for any prime $q \neq p$.

2.3. NSOP$_1$. In this subsection we will review the definition and basic properties of NSOP$_1$ theories.

We will work in a monster model $M$ (large, saturated) of a complete theory $T$.

Definition 2.14. A formula $\phi(x; y)$ has SOP$_1$ if there is a tree of tuples $(b_\eta)_{\eta \in 2^{<\omega}}$ such that

- for all $\eta \in 2^{<\omega}$, $\{\phi(x; b\eta_\alpha) \mid \alpha < \omega\}$ is consistent,
- for all $\eta \in 2^{<\omega}$, if $\nu \geq \eta \prec (0)$, then $\{\phi(x; b_\eta), \phi(b; a_{\eta\prec (1)})\}$ is inconsistent.

We say that a theory $T$ is SOP$_1$ if some formula has SOP$_1$ modulo $T$. Otherwise, $T$ is NSOP$_1$.

Definition 2.15. Let $A$ be a set and $a$ and $b$ tuples, say that $a$ is coheir independent of $b$ over $A$ if the type $\text{tp}(a/Ab)$ is finitely satisfiable in $A$, and denote $a \downarrow^u_A b$. A sequence $(a_i)_{i \in I}$ is an $A$-indiscernible coheir sequence if it is $A$-indiscernible and $a_i \downarrow^u_A a_{<i}$.

Using coheir-independence, we can use a different criterion for NSOP$_1$, due to [CR16] Theorem 5.7.

Fact 2.16 (Weak independent amalgamation). The theory $T$ is NSOP$_1$ iff given any model $M \models T$ and tuples $a_0b_0 \equiv_M a_1b_1$ such that $b_1 \downarrow^u_M b_0$ and $b_1 \downarrow^u_M a_i$ for $i = 0,1$, there exists a such that $ab_0 \equiv_M ab_1 \equiv_M a_0b_0$.

Kim-dividing, and its extension Kim-forking, were defined in [KR20], over arbitrary sets. For our purposes we will give a simplified definition, which we will call Kim$^u$-dividing, and define it only over models.

Definition 2.17. A formula $\phi(x, b)$ Kim$^u$-divides over a model $M$ if there exists an $M$-indiscernible coheir sequence $(b_i)_{i \in \omega}$ with $b \equiv_M b_i$, such that $\{\phi(x, b_i)\}_{i \in \omega}$ is inconsistent. A formula Kim$^u$-forks over $M$ if it implies a disjunction of Kim$^u$-dividing formulas over $M$.

A type Kim$^u$-divides (Kim$^u$-forks) over $M$ if it implies a Kim$^u$-dividing (Kim$^u$-forking) formula over $M$. Denote $a \downarrow^K_M b$ when the type $\text{tp}(a/Mb)$ does not Kim$^u$-fork over $M$. 
Remark 2.18. In this definition, $(b_i)_{i<\omega}$ is a Morley sequence in a restriction of a global coheir type. In the original definition of Kim-dividing, the global coheir type is replaced with a global invariant type. By Kim’s lemma for Kim-dividing [KR20, Theorem 3.16], those definitions are equivalent for NSOP$_1$ theories.

Remark 2.19. The type tp($a/Mb$) does not Kim$^n$-divide over $M$ iff for every $M$-indiscernible coheir sequence $(b_i)_{i<\omega}$ with $b \equiv_M b_i$, there exists $a'$ such that $ab \equiv_M a'b_i$ for every $i < \omega$.

Fact 2.20. Suppose $T$ is NSOP$_1$, then

1. [KR20, Theorem 3.16] If $\phi(x,b)$ Kim-divides over $M \models T$, then for every $M$-indiscernible coheir sequence $(b_i)_{i<\omega}$ with $b \equiv_M b_i$, $(\phi(x,b_i))_{i<\omega}$ is inconsistent.
2. [KR20, Proposition 3.19] Kim-dividing is equivalent to Kim-forking over models.
3. [KR20, Theorem 5.16] $\downarrow^K$ is symmetric over models.
4. [KR20, Corollary 5.17] Let $M \models T$, $a \downarrow^K_M b \iff acl(a) \downarrow^K_M b \iff a \downarrow^K acl(b)$.
5. [KR20, Proposition 8.8] $T$ is simple iff $\downarrow^K$ satisfies base monotonicity over models: if $M,N \models T$ and $M \subseteq N$, then $a \downarrow^K_M Nb$ implies $a \downarrow^K_N b$.
6. [KR20, Proposition 8.4] $T$ is simple iff $\downarrow^K = \downarrow^f$ over models.

3. Basic properties of ACF$_T$

In this section we will define and study the basic properties of ACF$_T$, the theory of algebraically closed fields with a distinguished subfield (in an arbitrary language). We will also consider expansions of the theory by definable relations and functions, that Delon defined to study pairs of ACF in [Del12].

3.1. Delon’s language.

Definition 3.1. Let $T$ be a theory of fields (not necessarily complete), in a language expanding the language of rings $L \supseteq L_{\text{rings}}$. Expand $L$ to the language $L^P = L \cup \{P\}$, with $P$ a unitary predicate, and expand ACF to ACF$_T$ in the language $L^P$ by adding the following axioms:

1. $P$ is a subfield of the universe, i.e. $P$ is closed under the ring operations (and contains $0,1$).
2. $P$ is a model of $T$. This can be achieved by taking all the axioms of $T$ and restricting the quantifiers to be over $P$ (see Remark 3.6).
3. For every $n$-ary function symbol $f \in L \setminus L_{\text{rings}}$, if $x_0, \ldots, x_{n-1} \in P$, then $f(x_0, \ldots, x_{n-1}) \in P$. Else, if some $x_i \notin P$, then we do not care about the value of $f(x_0, \ldots, x_{n-1})$, and we can set it arbitrarily to 0.
4. For every $n$-ary relation symbol $R \in L$ (equivalently $R \in L \setminus L_{\text{rings}}$ as $L_{\text{rings}}$ does not have any relation symbols), if some $x_i \notin P$, then $\neg R(x_0, \ldots, x_{n-1})$. That is, $R \subseteq P^n$.
5. The degree of the field extension of the universe over $P$ is infinite, i.e. the universe has infinite dimension as a vector space over $P$. By the Artin-Schreier theorem [AS27], it is enough to assert that the degree is at least 3.

Remark 3.2. The assumption that the degree of the universe over $P$ is infinite, that is, for $M \models ACF_T$, $[M : P_M] = \infty$, always holds when models of $T$ are not algebraically closed or real closed, because in that case $[P_M : P_M] = \infty$. When models of $T$ are algebraically closed, it simply means that $M \neq P_M$, i.e. $(M,P_M)$ is a proper pair. The only case excluded is when models of $T$ are real closed and $M = P_M$, but then $(P_M,P_M)$ is definable in $P_M$. 


Definition 3.3. Let $T$, $L$ be as above. Consider the following definable relations and functions over ACF:

- For $n \geq 1$, define the $n$-ary relation $l_n$ by $l_n(x_0, \ldots, x_{n-1})$ iff $x_0, \ldots, x_{n-1}$ are linearly independent over $P$.
- For $n \geq 1$, suppose we have $l_n(x_0, \ldots, x_{n-1})$ and $\neg l_{n+1}(x_0, \ldots, x_n)$. That is, $x_0, \ldots, x_{n-1}$ are linearly independent over $P$ and $x_n$ is in their span over $P$. Then there are unique $y_i \in P$ such that $x_n = y_0x_0 + \cdots + y_{n-1}x_{n-1}$. Define the $n+1$-ary function $f_{n,i}$ by $f_{n,i}(x_n; x_0, \ldots, x_{n-1}) = y_i$. If $x_0, \ldots, x_n$ do not satisfy this condition, then we do not care about the value of $f_{n,i}(x_n; x_0, \ldots, x_{n-1})$ and can set it arbitrarily to 0.

Expand ACF to $\text{ACF}^{ld}_{T}$ in the language $L^{ld} = L^P \cup \{l_n\}_{n \geq 1}$, by defining $l_n$ as above. Expand $\text{ACF}^{ld}_{T}$ to $\text{ACF}^f_T$ in the language $L^f = L^{ld} \cup \{f_{n,i}\}_{n>i \geq 0}$, by defining $f_{n,i}$ as above.

Notation 3.4. If $M \models \text{ACF}_T$, then let $P_M$ be the predicate $P$ in $M$ with the associated $L$-structure. If $A \subseteq M$ is a subset, then let $P_A = P_M \cap A$. This notation is used instead of the usual $P(M)$ and $P(A)$, because the notation $P(A)$ is reserved for the field extension of $P$ by $A$.

Definition 3.5. Call a formula $\phi(x) \in L^P$ bounded if every quantifier in $\phi$ is over $P$.

Remark 3.6. For a formula $\phi(x) \in L$ there is a corresponding bounded formula $\phi^f(x) \in L^P$ created by restricting every quantifier to be over $P$ and asserting $x \in P$. For $M \models \text{ACF}_T$, we have $\phi^f(M) = \phi(P_M)$.

3.2. Substructures and isomorphisms.

Lemma 3.7. Let $M \models \text{ACF}^f_T$ and $A \subseteq M$ a subset. Then $A$ is an $L^f$-substructure iff $P_A \subseteq P_M$ is an $L$-substructure, $A$ is a subring, $P_A$ is a subfield and $\text{Fra}(A) \downarrow_{P_A} P_M$.

Proof. Suppose $A \subseteq M$ is an $L^f$-substructure. We get that $P_A \subseteq P_M$ is an $L$-substructure, because for any function symbol $f \in L$ and $\overline{a} \in P_A$, $f(\overline{a}) \in A$ as $A \subseteq M$ is a substructure, and also $f(\overline{a}) \in P_A$ because of the axioms of ACF, so $f(\overline{a}) \in A \cap P_M = P_A$. It is clear that $A$ is a subring, and so is $P_A$, but for every $0 \neq a \in P_A$, $a^{-1} = f_{1,0}(1;a) \in P_A$, so $P_A$ is also a subfield. By [LaMa92] Chapter III, Criterion 1) to prove that $\text{Fra}(A) \downarrow_{P_A} P_M$, it is enough to show that if $a_0, \ldots, a_{n-1} \in A$ are linearly dependent over $P_M$, then they are linearly dependent over $P_A$. Suppose $a_0, \ldots, a_{n-1} \in A$ are linearly dependent over $P_M$. If $a_0 = 0$, then the tuple is trivially linearly dependent over $P_A$. Else, there is some maximal $1 \leq k < n$ such that $a_0, \ldots, a_{k-1}$ are linearly independent over $P_M$, so we have $l_k(a_0, \ldots, a_{k-1})$ and $\models \neg l_{k+1}(a_0, \ldots, a_k)$. Hence we can look at $p_i = f_{k,i}(a_k; a_0, \ldots, a_{k-1}) \in P_M$, which give us $a_k = p_{0}a_0 + \cdots + p_{k-1}a_{k-1}$. Because $A$ is a substructure, $p_i \in A$, so $p_i \in P_A$. Thus, $a_0, \ldots, a_{n-1}$ are linearly dependent over $P_A$.

In the other direction, suppose $A$ is a subring, $P_A$ is a subfield, $P_A \subseteq P_M$ is an $L$-substructure and $\text{Fra}(A) \downarrow_{P_A} P_M$. It follows that $\text{Fra}(A) \cap P_M = P_A$, and in particular $A \cap P_M = P_A$. For any function symbol $f \in L \setminus L_{\text{rings}}$ and $a_0, \ldots, a_{n-1} \in A$, if $a_0, \ldots, a_{n-1} \in P_A$, then $f(a_0, \ldots, a_{n-1}) \in P_A$ as $P_A \subseteq P_M$ is a substructure, and else we defined $f(a_0, \ldots, a_{n-1}) = 0 \in A$. It remains to check that $A$ is closed under $f_{n,i}$. Let $a_0, \ldots, a_n \in A$ and suppose $\models l_n(a_0, \ldots, a_{n-1})$, $\models \neg l_{n+1}(a_0, \ldots, a_n)$. Let $p_i = f_{n,i}(a_n; a_0, \ldots, a_{n-1})$, that is $p_i \in P_M$ and $a_n = p_0a_0 + \cdots + p_{n-1}a_{n-1}$. We know that $a_0, \ldots, a_{n-1}$ are linearly dependent over $P_M$, so by $\text{Fra}(A) \downarrow_{P_A} P_M$ they are linearly dependent over $P_A$. However, $a_0, \ldots, a_{n-1}$
must be linearly independent over $P_A$, as they are linearly independent over $P_M$, so $a_n$ can be written as a linear combination of $a_0,\ldots,a_{n-1}$ over $P_A$. This linear combination is in particular over $P_M$, but $a_n = p_0 a_0 + \cdots + p_{n-1} a_{n-1}$ is the unique linear combination over $P_M$, so we must have $p_0,\ldots,p_{n-1} \in P_A$, as needed.

**Corollary 3.8.** If $M \models \operatorname{ACF}_T$ and $A \subseteq M$ is an $L'$-substructure, then $\operatorname{Frac}(A) \subseteq M$ is an $L'$-substructure with $P_{\operatorname{Frac}(A)} = P_A$.

**Proof.** Lemma 3.7 implies that $\operatorname{Frac}(A) \mathrel{\downarrow}_{P_A} P_M$, and in particular $P_{\operatorname{Frac}(A)} = P_M \cap \operatorname{Frac}(A) = P_A$. Thus, $P_{\operatorname{Frac}(A)} \subseteq P_M$ is a subfield and an $L'$-substructure, $\operatorname{Frac}(A)$ is a subring (even subfield) and $\operatorname{Frac}(A) \mathrel{\downarrow}_{P_{\operatorname{Frac}(A)}} P_M$, so by Lemma 3.7 $\operatorname{Frac}(A) \subseteq M$ is an $L'$-substructure.

**Lemma 3.9.** Let $M,N \models \operatorname{ACF}_T$ and let $A \subseteq M$, $B \subseteq N$ be $L'$-substructures. A map $\sigma : A \to B$ is an $L'$-isomorphism iff $\sigma$ is an isomorphism of rings such that $\sigma(P_A) = P_B$ and $\sigma|_{P_A} : P_A \to P_B$ is an $L'$-isomorphism.

**Proof.** If $\sigma$ is an $L'$ isomorphism, then it is clearly an isomorphism of rings, $\sigma(P_A) = P_B$ because $\sigma$ preserves $P$ and $\sigma|_{P_A} : P_A \to P_B$ is a $L'$-isomorphism because $L'$ expands $L$ on $P$. For the other direction, we need to show that $\sigma$ preserves $l_n, f_{n,i}$. Let $a_0,\ldots,a_{n-1} \in A$ with $\models l_n(a_0,\ldots,a_{n-1})$. Suppose we have $\models \neg l_n(\sigma(a_0),\ldots,\sigma(a_{n-1}))$, i.e. $\sigma(a_0),\ldots,\sigma(a_{n-1})$ are linearly dependent over $P_N$. Lemma 3.7 implies that $\operatorname{Frac}(B) \mathrel{\downarrow}_{P_B} P_N$, so $\sigma(a_0),\ldots,\sigma(a_{n-1})$ are also linearly dependent over $P_B$. Thus, there are $q_0,\ldots,q_{n-1} \in P_B$ not all zero, such that $q_0 \sigma(a_0) + \cdots + q_{n-1} \sigma(a_{n-1}) = 0$. By applying $\sigma^{-1}$ we get $\sigma^{-1}(q_0 a_0 + \cdots + \sigma^{-1}(q_{n-1}) a_{n-1} = 0$, however $\sigma^{-1}(q_0),\ldots,\sigma^{-1}(q_{n-1}) \in P_A$, in contradiction to $\models l_n(a_0,\ldots,a_{n-1})$. The other direction follows from symmetry. Now suppose we have $a_0,\ldots,a_n \in A$ with $\models l_n(a_0,\ldots,a_{n-1})$ and $\models \neg l_{n+1}(a_0,\ldots,a_L)$. By the first part, we also have $\models l_{n+1}(\sigma(a_0),\ldots,\sigma(a_{n-1}))$ and $\models l_{n+1}(\sigma(a_0),\ldots,\sigma(a_{n-1}))$. Let $p_i = f_{n,i}(a_0,\ldots,a_{n-1}) \in P_A$, $a_n = p_0 a_0 + \cdots + p_{n-1} a_{n-1}$. Apply $\sigma$ to get $\sigma(a_n) = \sigma(p_0) \sigma(a_0) + \cdots + \sigma(p_{n-1}) \sigma(a_{n-1})$, but $\sigma(p_0),\ldots,\sigma(p_{n-1}) \in P_B$, so by uniqueness $\sigma(p_n) = f_{n,i}(\sigma(a_0),\sigma(a_0),\ldots,\sigma(a_{n-1}))$.

**Lemma 3.10.** Let $M,N \models \operatorname{ACF}_T$. By adding definable relations and functions, $M$ and $N$ can be expanded to models of $\operatorname{ACF}^id$, $\operatorname{ACF}'$. With those expansions, the following are equivalent:

1. $M \subseteq N$ is an $L'$-substructure.
2. $M \subseteq N$ is an $L^id$-substructure.
3. $M \subseteq N$ is a subfield, $P_M \subseteq P_N$ is an $L$-substructure and $M \mathrel{\downarrow}_{P_M} P_N$.

**Proof.** 1 $\implies$ 2: $L^id$ is a restriction of $L'$.

2 $\implies$ 3: It is clear that $M \subseteq N$ is a subfield and $P_M \subseteq P_N$ as sets. For every quantifier free formula $\phi(\overline{x}) \in L$ and $\overline{a} \in P_M$, $\models \phi(\overline{a}) \iff M \models \phi(\overline{a}) \iff N \models \phi(\overline{a}) \iff P_N \models \phi(\overline{a})$, so $P_M$ is an $L'$-substructure of $P_N$.

Let $a_0,\ldots,a_{n-1} \in M$ be linearly independent over $P_M$, $\models l_n(a_0,\ldots,a_{n-1}) \implies N \models l_n(a_0,\ldots,a_{n-1})$, so $a_0,\ldots,a_{n-1}$ are linearly independent over $P_N$. Thus, $M \mathrel{\downarrow}_{P_M} P_N$.

3 $\implies$ 1: Let $M'$ be the $L'$-structure with the same underlying set as $M$, but with structure induced as a subset of $N$. Note that $M' \subseteq N$ is really an $L'$-substructure, from Lemma 3.7. To prove that $M$ is an $L'$-substructure of $N$, we need to show that $M$ and $M'$ have the same structure, that is that the identity map $id : M \to M'$ is an $L'$-isomorphism. We know that $M$ is a subfield of $N$, so $id : M \to M'$ is a field isomorphism. From $M \mathrel{\downarrow}_{P_M} P_N$ we get that
$P_{M'} = M \cap P_N = P_M$ and $P_M$ is an $L$-substructure of $P_N$, so $id|_{P_M} : P_M \to P_{M'}$ is an $L$-isomorphism. Lemma 3.9 implies that $id$ is an $L'$-isomorphism. □

3.3. Saturated models. We will study saturated models of $ACF_T$. Note that $\kappa$-saturated models of $ACF_T$ are the same as $\kappa$-saturated models of $ACF_{id}^T$ or $ACF_T^f$, because $\{l_n\}_{n>1}$ and $\{f_{n,i}\}_{n,i>0}$ are definable in $ACF_T$. A full characterization of $\kappa$-saturated models will be given in Proposition 4.11.

Lemma 3.11. If $M \models ACF_T$ is $\kappa$-saturated, then $P_M$ is a $\kappa$-saturated model of $T$.

Proof. Follows from Remark 3.6 by relativizing each formula in the type we wish to realize to $P$. □

For the next result, we will need the following algebraic technical lemma, whose proof is left as an exercise to the reader.

Fact 3.12. Suppose $F$ is a field and $t$ is transcendental over $F$. For every $n$, $[F(t) : F(t^n)] = n$.

Lemma 3.13. If $M \models ACF_T$ is $\kappa$-saturated, then $\text{trdeg}(M/P_M) \geq \kappa$.

Proof. Let $S \subseteq M$ be an algebraically independent set over $P_M$. Suppose $|S| < \kappa$, we want to prove that there is some $a \in M$ such that $a \notin P_M(S)$. Consider the partial type over $S$

$$\Sigma(x) = \{ \forall \bar{y} \in P \ (q(x, \bar{y}) = 0 \rightarrow \forall x' q(x', \bar{y}) = 0) \mid q(x, \bar{y}) \in Q[x, \bar{y}, S] \}$$

where $Q$ is the prime field ($\mathbb{F}_p$ or $\mathbb{Q}$), $x$ is a single variable and $\bar{y}$ is a tuple of variables. Let $\Sigma_n(x)$ contain all formulas in $\Sigma(x)$ where the degree of $q(x, \bar{y})$ in $x$ is $\leq n$. We will show that $a \not\models \Sigma_n(x)$ iff $[P_M(S, a) : P_M(S)] > n$ and that $\Sigma_n(x)$ is satisfiable in $M$. From compactness and saturation ($|S| < \kappa$), we will get that $\Sigma(x)$ is satisfied by some $a \in M$. But then $[P_M(S, a) : P_M(S)] > n$ for all $n$, so $a \not\in P_M(S)$.

Suppose $a \not\models \Sigma_n(x)$. If $[P_M(S, a) : P_M(S)] \leq n$, then there is some non-zero polynomial $r(x) \in P_M(S)[x]$ of degree $\leq n$ such that $r(a) = 0$. The coefficients of $r(x)$ are rational functions in $S$ over $P_M$. By multiplying by the denominators, we can assume the coefficients are polynomials in $S$ and $P_M$, so $r(x) = q(x, \bar{p})$ for $q(x, \bar{y}) \in Q[x, \bar{y}, S]$ and $\bar{p} \in P_M$. However, because $q(a, \bar{p}) = r(a) = 0$, we get from $a \not\models \Sigma_n(x)$ that $r(x)$ is constant zero.

Now suppose $[P_M(S, a) : P_M(S)] > n$. Let $q(x, \bar{y}) \in Q[x, \bar{y}, S]$ of degree $\leq n$ in $x$ and $\bar{p} \in P_M$, such that $q(a, \bar{p}) = 0$. The polynomial $q(x, \bar{p})$ is over $P_M(S)$, has degree $\leq n$ and has $a$ as root, but $[P_M(S, a) : P_M(S)] > n$, so $q(x, \bar{p})$ must be constant zero. Hence $a \not\models \Sigma_n(x)$.

To prove that $\Sigma_n(x)$ is satisfiable for every $n$, we need to prove that there is some $a \in M$ such that $[P_M(S, a) : P_M(S)] > n$. Split into three cases.

1. $S = \emptyset$, $M \not\models P_M$: Take some $a \in M \setminus P_M$ and we are done.
2. $S = \emptyset$, $M = P_M$: The axioms of $ACF_T$ (Definition 3.1) imply that $P_M : P_M = \infty$. By Kei64 Lemma 3.1, there exists some $a \in P_M$ such that $[P_M(a) : P_M] > n$.
3. $S \neq \emptyset$: Take some $s_0 \in S$ and define $F = P_M(S \setminus \{s_0\})$. Because $M$ is algebraically closed, there exists an $n + 1$-th root $a = s_0^{1/n} \in M$. We know that $s_0$ is transcendental over $F$, so $a$ is also transcendental over $F$. Fact 3.12 implies that $[F(a) : F(s_0)] = n + 1$, where $F(s_0) = P_M(S)$ and $F(a) = P_M(S, a)$, as needed. □
Lemma 3.14. Suppose \( \text{trdeg}(M/P_M) \geq \kappa \) (in particular, if \( M \) is \( \kappa \)-saturated) and let \( A, A' \subseteq M \) be subsets with \( |A|, |A'| < \kappa \). If \( f : P_M(A) \to P_M(A') \) is an isomorphism of fields that restricts to an \( L \)-automorphism \( f|_{P_M} \), then \( f \) can be extended to an automorphism of \( M \).

Proof. From transitivity of transcendental degree

\[
\text{trdeg}(M/P_M) = \text{trdeg}(M/P_M(A)) + \text{trdeg}(P_M(A)/P_M),
\]

and \( \text{trdeg}(P_M(A)/P_M) \leq |A| < \kappa \), so \( \text{trdeg}(M/P_M(A)) = \text{trdeg}(M/P_M) \). Similarly, \( \text{trdeg}(M/P_M(A')) = \text{trdeg}(M/P_M) \). Let \( S, S' \subseteq M \) be transcendence basis of \( M \) over \( P_M(A), P_M(A') \) respectively, \( |S| = \text{trdeg}(M/P_M) = |S'| \). Extend \( f \) to an automorphism of fields \( \sigma : M \to M \), by mapping \( S \mapsto S' \) and extending to the algebraic closure arbitrarily. The restriction \( \sigma|_{P_M} = f|_{P_M} \) is an \( L \)-automorphism of \( P \), so Lemma 3.9 implies that \( \sigma \) is an \( L^P \)-automorphism. \( \square \)

4. Quantifier elimination and more

4.1. Completions. Keisler [Kei64] proved that \( \text{ACF}_T \) is complete when \( T \) is a complete theory in the language of rings. We generalize this by allowing the language of \( T \) to be arbitrary.

In his proof, Keisler used special models. We will instead use saturated models, which simplifies the proof, but requires an additional set-theoretic assumption (namely, the generalized continuum hypothesis). There are standard techniques from set theory that ensures the generalized continuum hypothesis from some point on while fixing a fragment of the universe (so this does not affect questions of e.g., completeness of a given theory), see [HK21], and we will use this freely.

Proposition 4.1. If \( T \) is a complete theory of fields, then \( \text{ACF}_T \) is complete.

Proof. It is enough to show that if \( M, N \models \text{ACF}_T \) are saturated models of the same cardinality \( \kappa \), then they are isomorphic (see the discussion above the proposition). By Lemma 4.14, \( P_M, P_N \models T \) are \( \kappa \)-saturated, and in particular \( |P_M| = |P_N| = \kappa \). Because \( T \) is complete, [CK90, Theorem 5.1.13] implies that there is a \( L \)-isomorphism \( \sigma_0 : P_M \to P_N \). By Lemma 3.13 \( \text{trdeg}(M/P_M) = \text{trdeg}(N/P_N) = \kappa \). Let \( S \subseteq M, S' \subseteq N \) be transcendence basis over \( P_M, P_N \) respectively, \( |S| = |S'| = \kappa \). We can extend \( \sigma_0 \) to an isomorphism of fields \( \sigma_1 : M \to N \), by mapping \( S \mapsto S' \) and extending to the algebraic closure arbitrarily. The restriction \( \sigma_1|_{P_M} \) is an \( L \)-isomorphism, so by Lemma 3.9 \( \sigma_1 \) is an \( L^P \)-isomorphism. \( \square \)

4.2. Quantifier elimination. Our proof of quantifier elimination will be essentially the same as Delon’s [Del12, Proposition 14]. One difference is that the criterion used by Delon to prove quantifier elimination assumes a countable language, so we will need a slightly generalized criterion.

In [HKR18], Hils, Kamensky and Rideau proved the same result in a similar fashion. Our proof was derived independently, as we were not aware of their work during the research.

We will need the following fact, which follows from [Hod93, Theorem 8.4.1].

Fact 4.2. A theory \( T \) has quantifier elimination iff for any two models \( M, N \models T \) such that \( N \) is \( |M|^{+} \)-saturated and any substructures \( A \subseteq M \) and \( A' \subseteq N \) with an isomorphism \( \sigma : A \to A' \), \( \sigma \) can be extended to an embedding \( M \to N \).

Theorem 4.3. If \( T \) has quantifier elimination, then \( \text{ACF}_T \) has quantifier elimination.

Proof. Let \( M, N \models \text{ACF}_T \) such that \( N \) is \( |M|^{+} \)-saturated. Let \( A \subseteq M, A' \subseteq N \) be \( L^1 \)-substructures with isomorphism \( \sigma : A \to A' \). By Corollary 3.8 \( \text{Frac}(A) \subseteq M \),
Frac($A'$) \subseteq N are $L'$-substructures with $P_{\text{Frac}(A)} = P_A$, $P_{\text{Frac}(A')} = P_{A'}$. We can extend $\sigma$ to an isomorphism of fields Frac($A$) $\to$ Frac($A'$) that will have the same restriction $P_A \to P_{A'}$, and so by Lemma 3.9 would still be an $L'$-isomorphism. Thus, we can assume without loss of generality that $A$ and $A'$ are subfields. By (3.11) $P_N$ is $|M|^+$-saturated, and in particular $|P_M|^+$-saturated. The restriction $\sigma|_{P_A} : P_A \to P_{A'}$ is an isomorphism of $L$-structures from Lemma 3.9, so quantifier elimination and Fact 4.2 imply that we can extend $\sigma|_{P_A}$ to an embedding $\sigma_0 : P_M \to P_N$.

Let $B = \sigma_0(P_M) \subseteq P_N$. By Lemma 3.4 $A \downarrow_{P_M} P_M$ and $A' \downarrow_{P_{A'}} P_N$, in particular by monotonicity $A' \downarrow_{P_{A'}} P_M$. The field isomorphisms $\sigma : A \to A'$ and $\sigma_0 : P_M \to B$ both restrict to the same isomorphism $P_A \to P_{A'}$, so there is a unique field isomorphism $\sigma_1 : A.P_M \to A'.B$ such that $\sigma_1|_A = \sigma$, $\sigma_1|_{P_M} = \sigma_0$, by Corollary 2.5.

Let $S \subseteq M$ be a transcendental basis of $M$ over $A.P_M$, $|S| \leq |M|$. From Lemma 3.13 trdeg($N/P_N$) $\geq |M|^+$ and $|A'| = |A| \leq |M|$, so there exists $S' \subseteq N$ algebraically independent over $A'.P_N$ with $|S| = |S'|$. Let $M' = \overline{A'.B(S')} \subseteq N$. Quantifier elimination implies that the substructure $B \subseteq P_N$ is elementary, so by Corollary 2.11 $B \subseteq P_N$ is regular. We also know that $A' \downarrow_{P_{A'}} P_N$, so by base monotonicity $A'.B \downarrow_{B} P_N$ and by Lemma 2.4 $\overline{A'.B(S')} \downarrow_{B} P_N$, where $\overline{A'.B(S')} = M'$. Thus, $M' \subseteq N$ is a substructure, with $P_M = B$, from Lemma 3.4.

We also have $M = \overline{A.P_M(S)}$, so we can extend $\sigma_1 : A.P_M \to A'.B$ to $\sigma_2 : M \to M'$ by mapping $S \mapsto S'$ arbitrarily and extending to the algebraic closure. In particular, $\sigma_2|_{P_M} = B = P_{M'}$ and $\sigma_2|_{P_M} = \sigma_0$ is an isomorphism of $L'$-structures, so $\sigma_2$ is an isomorphism of $L'$-$S'$-structures by Lemma 3.8. Thus, $\sigma_2$ is an embedding of $M$ into $N$ that extends $\sigma$.

\textbf{Corollary 4.4 (Dalpayrat Theorem 1)].} $\text{ACF}^f_{\text{ACVF}}$ eliminates quantifiers.

\textbf{Corollary 4.5.} $\text{ACF}^f_{\text{RCF}}$ eliminates quantifiers, where $\text{RCF}$ is the theory of real closed fields in the language $L_{\text{rings}} \cup \{ \leq \}$.

\textbf{Corollary 4.6.} Let $ACVF$ be the theory of algebraically closed valued fields in the divisibility language, that is the language of rings with a binary relation $x|y$ signifying $x < y$. $ACVF$ eliminates quantifiers, so $\text{ACF}^f_{\text{ACVF}}$ eliminates quantifiers (by Corollary 5.35 it is also NIP).

From quantifier elimination, we can deduce a couple of important corollaries. Both corollaries will rely on expanding a theory $T$ to the Morleyization, which has quantifier elimination, as defined below.

\textbf{Definition 4.7.} For a theory $T$, the \textit{Morleyization} $T_{\text{Mor}}$ of $T$ is an expansion of $T$ by relations $R_\psi(x)$ for any $\psi(x) \in L$, such that $T_{\text{Mor}} \vdash \forall x(R_\psi(x) \leftrightarrow \psi(x))$.

\textbf{Corollary 4.8.} Every formula $\phi(x) \in L^P$ is equivalent modulo $\text{ACF}_T$ to a bounded formula, that is a formula where every quantifier is over $P$ (see Definition 5.2).

\textit{Proof.} Consider the Morleyization $T_{\text{Mor}}$ and the theory $\text{ACF}^f_{\text{ACF}_{\text{Mor}}}$, which has quantifier elimination by Theorem 4.3. In particular, $\phi(x)$ is equivalent to a quantifier free formula $\phi_0(x) \in L^f_{\text{Mor}}$ modulo $\text{ACF}^f_{\text{ACF}_{\text{Mor}}}$. Replace all occurrences of $l_n$, $f_{n,i}$ in $\phi_0(x)$ with the formulas defining them, to get an equivalent formula $\phi_1(x) \in L^P_{\text{Mor}}$. The formulas defining $l_n$, $f_{n,i}$ are bounded, so $\phi_1(x)$ is bounded.

For any formula $\psi(y) \in L$ consider the bounded formula $\psi^P(y) \in L^P$ created from Remark 5.6. The axioms of $\text{ACF}_{\text{Mor}}$ (Definition 5.1) imply that $\text{ACF}_{\text{Mor}} \vdash \forall y(R_\psi(y) \leftrightarrow \psi^P(y))$. Replace each predicate $R_\psi(y)$ in $\phi_1(x)$ by the corresponding
ψ^P(y), to get a bounded formula φ_2(x) ∈ L^P which is equivalent to φ(x) modulo ACF_T.

Remark 4.9. In that case that L is the language of rings, Corollary [4.8 follows from CZ01] Proposition 2.1], because ACF has nfcp and P_M is small in any model M ⊨ ACF_T (as witnessed in a saturated extension, by Lemma 3.4.3).

Corollary 4.10. Let M, N ⊨ ACF_T^L and let A ⊆ M, B ⊆ N be substructures. Then σ : A → B is a partial elementary map from M to N iff σ : A → B is an isomorphism of rings such that σ(P_A) = P_B and σ|_{P_A} : P_A → P_B is a partial elementary map from P_M to P_N.

Proof. Suppose σ : A → B is a partial elementary map from M to N in ACF_T^L. Then σ is in particular an isomorphism, so σ(P_A) = P_B. The restriction σ|_{P_A} is a partial elementary map from P_M to P_N in T, because for every formula φ(x) ∈ T, we can apply Remark 3.3 to get φ^P(\bar{x}) ⊨ ACF_T, such that φ(P_B) = φ^P(B) = σ(φ^P(A)) = σ(φ(P_A)).

For the other direction, suppose σ : A → B is an isomorphism of rings such that σ(P_A) = P_B and σ|_{P_A} : P_A → P_B is a partial elementary map in P_M to P_N in T. In particular, P_M and P_N have the same theory, so we can assume that T is the complete theory T = Th(P_M) = Th(P_N). Let T_Mor be the Morleyzation of T, T_Mor has quantifier elimination. We can expand the language of P_M and P_N by definable relations to get P_M, P_N ⊨ T_Mor. With this expanded language M, N ⊨ ACF_T^{T_Mor}. The expansion is only relational, so we can still consider A and B as substructure. The restriction σ|_{P_A} is a partial elementary map in T, so it is an isomorphism in T_Mor, and thus by Lemma 3.9 σ is an isomorphism in ACF_T^{T_Mor}. By Proposition 4.1 and Theorem 4.3 ACF_T^{T_Mor} is complete and eliminates quantifiers, so σ is a partial elementary map in ACF_T^{T_Mor}. In particular, it is a partial elementary map in ACF_T.

Using this result on elementary maps, we can now show that Lemmas 3.11 and 3.13 fully characterize the saturated models of ACF_T.

Proposition 4.11. Suppose κ > |L|, then N ⊨ ACF_T is κ-saturated iff P_N ⊨ T is κ-saturated and trdeg(N/P_N) ≥ κ.

Proof. The first direction, if N ⊨ ACF_T is κ-saturated, then P_N ⊨ T is κ-saturated and trdeg(N/P_N) ≥ κ, is proved in Lemmas 3.11 and 4.13. For the other direction, we will prove κ-homogeneity and κ*-universality. By expanding the language with definable relations and functions, we can assume N ⊨ ACF_T^L. Let A, B ⊆ N and let σ : A → B be a partial elementary map in N with σ(A) = B, such that |A| = |B| < κ. Without loss of generality, we can assume that A, B ⊆ N are L'-substructures, and by Corollary 3.8 we can also assume they are subfields. Corollary 4.10 implies that σ|_{P_A} : P_A → P_B is a partial elementary map in P_N. We know that P_N is κ-homogeneous and |P_A| = |P_B| < κ, so we can extend σ|_{P_A} to an automorphism σ_0 : P_N → P_N in T.

We have A ⊨_{P_A} P_N and B ⊨_{P_B} P_N from Lemma 3.4 and the field isomorphisms σ and σ_0 restrict to the same isomorphism P_A → P_B, so by Corollary 2.6 they can be jointly extended to an isomorphism of fields σ_1 : A.P_N → B.P_N. From Lemma 3.14 σ_1 can be extended to an automorphism of fields σ_2 : N → N. Lemma 3.9 implies that σ_2 is an L'-automorphism because σ_2|_{P_N} = σ_0 is an automorphism in T, and σ_2 extends σ as needed.

Now Let M ⊨ ACF_T with |M| ≤ κ, by expanding the language we can assume M ⊨ ACF_T^L. We have P_M ⊨ T with |P_M| < κ, so by κ*-universality of P_N there
exists an elementary embedding \( \tau_0 : P_M \to P_N \). Let \( B = \tau_0(P_M) \). We have \( B \leq P_N \), and in particular from Corollary 2.11 \( B \subseteq P_N \) is a regular extension. Let \( S \) be a transcendental basis of \( M \) over \( P_M \), \( |S| \leq \kappa \) and \( \text{trdeg}(N/P_N) \geq \kappa \), so there exists \( S_0 \subseteq N \) algebraically independent over \( P_N \) with \( |S_0| = |S| \). We can extend \( \tau_0 \) to an embedding \( \tau : M \to N \) by mapping \( S \to S_0 \) arbitrarily and extending to the algebraic closure. Let \( M_0 = \tau_1(M) = B(S_0) \). From Lemma 2.9 \( \overline{B(S_0)} \downarrow_P P_N \), so by Lemma 3.10 \( M_0 \subseteq N \) is an \( L^f \)-substructure with \( P_{M_0} = B \). We have that \( \tau_1 : M \to M_0 \) is an isomorphism of fields with \( \tau_1|_{P_M} = \tau_0 : P_M \to P_{M_0} \), an elementary embedding, so by Corollary 4.10 \( \tau_1 \) is an elementary embedding. 

4.3. Model completeness. In [Del12, Corollary 15], Delon proved that \( \text{ACF}^\text{ld}_{\text{ACF}} \) is model complete. We can show that if \( T \) is model complete, then \( \text{ACF}^\text{ld}_{\text{PSF}} \) is model complete, but in fact we only need a weaker condition — that regular extensions in \( T \) are elementary.

Theorem 4.12. The following are equivalent:

1. \( \text{ACF}^\text{ld}_T \) is model complete.
2. \( \text{ACF}^\text{ld}_T \) is model complete.
3. For any \( Q, R \models T \) such that \( Q \subseteq R \) is a substructure, if \( Q \subseteq R \) is a regular extension, then \( Q \leq R \).
4. \( \text{T}_{\text{reg}} \) (Definition 4.14) is model complete.

Proof. 1 \( \iff \) 2: Let \( M, N \models \text{ACF}^\text{ld}_T \) with \( M \subseteq N \) an \( L^f \)-substructure. We can expand \( M \) and \( N \) uniquely to models of \( \text{ACF}^\text{ld}_T \), by Lemma 3.10 \( M \subseteq N \) is an \( L^f \)-substructure. \( \text{ACF}^\text{ld}_T \) is model complete, so \( M \leq N \) in \( L^f \), in particular \( M \subseteq N \) in \( L^f \).

2 \( \implies \) 3: Let \( Q, R \models T \) with \( Q \subseteq R \) a regular extension. We will construct \( M, N \models \text{ACF}^\text{ld}_T \) such that \( P_M = Q, P_N = R \) and \( M \subseteq N \). We would have liked to take \( M = \overline{Q} \), but then we may have \( |M : Q| < \infty \), so we should make \( M \) a bit larger. Let \( s \) be a new element, transcendental over \( R \). The subfield \( Q \subseteq R \) is regular, so by Lemma 2.9 \( \overline{Q(s)} \downarrow_P R \). Define \( M = \overline{Q(s)} \), \( Q \subseteq M \) is not an algebraic extension so in particular \( |M : Q| = \infty \). We have \( M \models \text{ACF}^\text{ld}_T \), where we define \( P_M = Q \). Similarly, define \( N = \overline{R(s)} \), \( N \models \text{ACF}^\text{ld}_T \) with \( P_N = R \). We know that \( P_M \subseteq P_N \) is an \( L \)-substructure and \( M \downarrow_{P_M} P_N \), so by Lemma 3.10 \( M \subseteq N \) is an \( L^f \)-substructure. Model completeness implies \( M \leq N \), and in particular \( P_M \leq P_N \), because for every formula \( \phi(\bar{x}) \in L \) we have \( P_M \models \phi(\bar{a}) \iff M \models \phi^P(\bar{a}) \iff N \models \phi^P(\bar{a}) \iff P_N \models \phi(\bar{a}) \) for every \( \bar{a} \in P_M \), where \( \phi^P \) is given by Remark 3.6.

3 \( \implies \) 4: Let \( Q, R \models T_{\text{reg}} \) be such that \( Q \subseteq R \) is an \( L_{\text{reg}} \)-extension. By Lemma 2.13 \( Q \subseteq R \) is a regular field extension, so \( Q \leq R \) in \( L \) by assumption. Because \( L_{\text{reg}} \) is an expansion by definable relations, \( Q \leq R \) also in \( L_{\text{reg}} \).

4 \( \implies \) 1: Let \( M, N \models \text{ACF}^\text{ld}_T \) and suppose \( M \subseteq N \) is a substructure. Lemma 3.10 implies that \( P_M \subseteq P_N \) is an \( L \)-substructure and \( M \downarrow_{P_M} P_N \). However, \( M \) is algebraically closed, so by monotonicity \( P_M \downarrow_{P_N} P_N \), that is \( P_M \subseteq P_N \) is a regular extension. Extending \( P_M \) and \( P_N \) to models \( T_{\text{reg}} \), we see by Lemma 2.13 that \( P_M \subseteq P_N \) is an \( L_{\text{reg}} \)-extension, so \( P_M \leq P_N \) by assumption. The inclusion map \( M \to N \) restricts to the elementary inclusion \( P_M \to P_N \), so by Corollary 4.10 \( M \leq N \).

Corollary 4.13 ([Del12, Corollary 15]). \( \text{ACF}^\text{ld}_{\text{ACF}} \) is model complete.

Corollary 4.14. \( \text{ACF}^\text{ld}_{\text{PSF}} \) is model complete, where \( \text{PSF} \) is the theory of pseudo-finite fields in the language of rings (see Proposition 6.9 for a proof).
Remark 4.15. $\text{ACF}_{\text{ACF}}$ is not model complete. By \cite[page 207]{TZ12}, the pregeometry of an algebraically closed field $K$ of transcendence degree at least 4 over its prime field with algebraic independence is not modular: there are algebraically closed subfields $A, B \subseteq K$ such that $A \not\cong_{\text{ACF}} B$. Define
\[
M = A, \quad N = K, \\
P_M = A \cap B, \quad P_N = B.
\]
It is clear that $M \subseteq N$ is an $L^P$-substructure, however if $M \preceq N$, then Lemma 3.10 would imply that $A \cong_{\text{ACF}} B$, and in particular $A \cong_{\text{ACF}} B$, a contradiction.

5. Classification and Independence

In this section we will assume that $T$ is complete (Proposition 4.1 implies that $\text{ACF}_T$ is also complete) and we will work inside a monster model $M \models \text{ACF}_T$. Denote $P := P_M$.

Assuming $T$ is NSOP$_1$, we will define an independence relation $\downarrow^*$ on $M$ and prove that it implies Kim-dividing (in fact, Kim“ dividing, see Definition 2.17). With this result, we will prove that $\text{ACF}_T$ is NSOP$_1$ and that under certain conditions $\downarrow^*$ is the Kim-independence. We will then expand this result to simplicity and $\lambda$-stability.

We will also prove that stability lifts from $T$ to $\text{ACF}_T$ using a different approach, by counting types. This approach will let us extend the result to $\lambda$-stability.

Finally, we will prove that NIP lifts from $T$ to $\text{ACF}_T$.

5.1. Kim-dividing.

Definition 5.1. Call a subfield $A \subseteq M$ $D$-closed (D for Delon’s language) if it is closed under the functions $f_n$, or equivalently if $A \downarrow^i_{P_A} P$. For a set $B \subseteq M$, denote by $\langle B \rangle_D$ the $D$-closure of $B$, that is the smallest field containing $B$ and closed under $f_n$.

Remark 5.2. We have the following remarks on $D$-closure:

- In \cite[Definition 3.1]{NPZ20}, the condition $D$-closed was called $P$-special.
- If $A \subseteq M$ is definably closed in $L^P$, then it is $D$-closed. In particular, for every $A \subseteq M$, $\text{dcl}(A)$ and $\text{acl}(A)$ are $D$-closed.
- $D$-closure gives a shorter proof of local character of $\downarrow^i$ (see Fact 2.3).

Suppose $a$ is finite and $P$ is an infinite field. Let $A = \langle a \rangle_P$ be the $D$-closure of $a$ inside the pair of fields $(P(a), P)$. Consider $P_{A} = P \cap A$, which is countable. We have $P_A(a) \subseteq A$, so by monotonicity $P_A(a) \downarrow^i_{P_A} P$.

Lemma 5.3. Suppose $A, B, C \subseteq M$ are subfields with $C \subseteq A \cap B$. If $A$ is $D$-closed, then $A.P \downarrow^i_{C.P} B.P$ if and only if $A \downarrow^i_{C.P_A} B.P$. By symmetry, if $B$ is $D$-closed, then $A.P \downarrow^i_{C.P} B.P$ if and only if $A.P \downarrow^i_{C.P_B} B$. Furthermore, if both $A$ and $B$ are $D$-closed, then $A.P \downarrow^i_{C.P} B.P$ implies $A.B \downarrow^i_{P_A.P_B} P$, i.e. $P_{A.B} = P_A.P_B$ and $A.B$ is $D$-closed.

Proof. If $A \downarrow^i_{C.P_A} B.P$, then $A.P \downarrow^i_{C.P} B.P$ from base monotonicity. On the other hand, if $A.P \downarrow^i_{C.P} B.P$, then because $A \downarrow^i_{P_A} P$ implies $A \downarrow^i_{C.P_A} C.P$ from base monotonicity, we get from transitivity that $A \downarrow^i_{C.P_A} B.P$. For the furthermore part, we know from $A \downarrow^i_{P_A} P$ and $A.P \downarrow^i_{C.P} B.P$ that $A \downarrow^i_{C.P_A} B.P$. By base monotonicity, $A.B \downarrow^i_{B.P_A} B.P$. Also, from $B \downarrow^i_{P_B} P$ and base monotonicity, $B.P_A \downarrow^i_{P_A.P_B} P$, thus by transitivity $A.B \downarrow^i_{P_A.P_B} P$. \qed

Definition 5.4. Let $M \preceq M$ and $A, B \subseteq M$ be small $D$-closed subfields, such that $M \subseteq A \cap B$. Define $A \downarrow^i_{M} B$ if
(1) $P_A \downarrow_{P_B}^{K} P_B$ in $P$.

(2) $A.P \downarrow_{M.P}^{I} B.P$.

**Lemma 5.5.** Let $A, B, C \subseteq \mathbb{M}$ be small subsets with $C \subseteq A \cap B$. If $A \downarrow_{C}^{u} B$, then:

1. $P_A \downarrow_{P_B}^{u} P_B$ in $P$.
2. If $A$, $B$ and $C$ are subfields and $B$ is $D$-closed, then $A.P \downarrow_{C.P}^{I} B.P$.

In particular, if $M \subseteq \mathbb{M}$ and $A$ and $B$ are $D$-closed with $M \subseteq A \cap B$, then $A \downarrow_{M}^{u} B$ implies $A \downarrow_{M}^{u} B$.

**Proof.** For point (1), suppose $P \models \phi(a, b)$ for some formula $\phi(x, y) \in L$, $a \in P_A$ and $b \in P_B$. Let $\phi^c(x, y) \in L^P$ be as in Remark 5.6. We have $M \models \phi^c(a, b)$. By $A \downarrow_{C}^{u} B$ there is some $c \in C$ such that $M \models \phi^c(c, b)$. Thus, $c \in P \cap C = P_C$, and we have $P \models \phi(c, b)$.

For point (2), by Lemma 5.3, it is enough to prove $A.P \downarrow_{C.P}^{I} B$. Let $\sum_i u_i b_i = 0$ for $u_i \in A.P$ and $b_i \in B$ such that the $u_i$ are not all equal to 0. We can write $u_i = f_i(\bar{a}_i, \bar{p}_i)$ for $f_i \in C(\bar{x}, \bar{y})$ rational functions, $\bar{a}_i \in A$ and $\bar{p}_i \in P$. Assume that $f_i$ are polynomials by multiplying by all denominators. We have

$$\models \sum_i f_i(\bar{a}_i, \bar{p}_i) b_i = 0 \land \bigvee_i f_i(\bar{a}_i, \bar{p}_i) \neq 0,$$

and in particular

$$\models \exists \bar{y}_i \in P, \sum_i f_i(\bar{a}_i, \bar{y}_i) b_i = 0 \land \bigvee_i f_i(\bar{a}_i, \bar{y}_i) \neq 0.$$

From $A \downarrow_{C}^{u} B$, there are $\bar{e}_i \in C$ such that

$$\models \exists \bar{y}_i \in P, \sum_i f_i(\bar{e}_i, \bar{y}_i) b_i = 0 \land \bigvee_i f_i(\bar{e}_i, \bar{y}_i) \neq 0.$$

Let $\bar{q}_i \in P$ witness the existence, and let $v_i = f_i(\bar{e}_i, \bar{q}_i) \in C.P$. We have $\sum_i v_i b_i = 0$ and $v_i$ are not all equal to 0. Moreover, $B \downarrow_{P_B}^{I} P$, so by base monoticity $B \downarrow_{C.P_B}^{I} C.P$, thus there are $w_i \in C.P_B$, not all equal to 0, such that $\sum_i w_i b_i = 0$, as needed.

The “in particular” part follows from the definition of $\downarrow_{C}^{u}$, because $P_A \downarrow_{P_B}^{u} P_B$ implies $P_A \downarrow_{P_B}^{K} P_B$ (see [4E21] Fact 3.10).

**Lemma 5.6.** Let $A, B, C \subseteq \mathbb{M}$ be small subsets with $C \subseteq A \cap B$ and let $(B_i)_{i<\omega}$ be a $C$-indiscernible coheir sequence such that $B \equiv_A B_i$ in $ACF_T$, then $(P(B_i))_{i<\omega}$ is a $P_C$-indiscernible coheir sequence such that $P_B \equiv_{P_B} P_B$, $P_B$.

**Proof.** For every formula in $P$, we can restrict all quantifiers and free variables to be over $P$ to get a formula in $M$ with the same definable set. This proves that $(P(B_i))_{i<\omega}$ is $P_C$-indiscernible and $P_B \equiv_{P_B} P_B$, $P_B$ in $P$. From Lemma 5.3, $(P_B)_{i<\omega}$ is a $P_C$-indiscernible coheir sequence in $P$, where $P_{B_i}$ is enumerated as $(\beta_i(b))_{b \in P_B}$. Because $T$ is $\text{NSOP}_1$ and $P_A \downarrow_{P_B}^{K} P_B$ in $P$, Fact 2.20(2) implies that there exists $Q \subseteq P$ such that $P_A P_B \equiv_{P_B} Q P_B$, in $P$ for all $i < \omega$, where we consider all the above fields as tuples. More explicitly, let $p((x_\alpha)_{\alpha \in P_A}, (x_b)_{b \in P_B}) = \text{tp}((a)_{a \in P_A}, (b)_{b \in P_B}/P_M)$,
then let \((a')_{a \in PA}\) be a realization of \(\bigcup_{i < \omega} p((x_{a})_{a \in PA}, (\beta_{i}(b))_{b \in PB})\), and let \(Q\) be \(\{a' \mid a \in PA\}\). As \((a)_{a \in PA}(b)_{b \in PB} \equiv_{PM} (a')_{a \in PA}(\beta_{i}(b))_{b \in PB}\) in \(P\), by saturation there are automorphisms \(\gamma_{i}\) of \(P\) mapping \(PA_{P}B_{P}\) to \(QP_{B}\), extending \(\beta_{i} |_{P}\) (so fixing \(P_{M}\) pointwise) such that \(\gamma_{i}(a) = a'\) for all \(a \in PA\). In particular, the restrictions \(\gamma_{i} |_{PA} : PA \to Q\) are the same for every \(i < \omega\). Name this restriction \(\alpha_{0} : PA \to Q\).

Let \(S \subseteq A\) be a transcendence basis of \(A\) over \(M_{PA}\). Lemma 5.13 implies that \(\text{trdeg}(M_{P}/P) = |M_{P}|\), so there exists some \(S'\) algebraically independent over \(B_{P}\) with \(|S'| = |S|\). Define \(A' = MQ(S')\). From Lemma 3.7 \(M \downarrow_{P_{M}} P_{M}\), so from monotonicity \(M \downarrow_{P_{M}} PA_{M}\) and \(M \downarrow_{P_{M}} Q\). Thus, from stationarity of \(\downarrow\), we can extend \(\alpha_{0} : PA \to Q\) to an isomorphism of fields \(M_{PA} \to MQ\) preserving \(M\) pointwise. Map \(S \mapsto S'\) arbitrarily and extend arbitrarily to the algebraic closure, to get an isomorphism of fields \(\alpha : A \to A'\). This give us a way to consider \(A'\) as a tuple.

Let \(i < \omega\). We know that \(B \downarrow_{P_{M}} B_{M}\) and \(B_{M} \downarrow_{P_{M}} P_{M}\), the field isomorphisms \(\beta_{i} : B \to B_{i}\) and \(\gamma_{i} : P \to P_{i}\) both restrict to the same isomorphism \(P_{B} \to P_{B_{i}}\), so from Corollary 2.5 they can be jointly extended to an isomorphism of fields \(\sigma_{i,0} : B_{P} \to B_{i}P\). From \(A_{P} \downarrow_{M_{P}} B_{M}\) and Lemma 5.3 we get that \(A \downarrow_{M} B_{P}\). We would like to prove that also \(A' \downarrow_{M} B_{i}P\). We know that \(A\) is algebraically closed, so \(M_{PA} \subseteq B_{P}\) is regular. Applying Lemma 2.8 with \(\sigma_{i,0}\), we get that \(MQ \subseteq B_{P}\) is regular. The set \(S'\) is algebraically independent over \(B_{P}\), so from Lemma 2.9 \(MQ(S') \downarrow_{M} MQ_{M}(S') \downarrow_{M} B_{i}P\), where \(MQ(S') = A'\).

The isomorphisms of fields \(\alpha : A \to A'\) and \(\sigma_{i,0} : B_{P} \to B_{i}P\) restrict to the same isomorphism \(M_{PA} \to MQ_{P}\), which acts as \(\alpha_{0}\) on \(PA\) and preserves \(M\) pointwise. Thus, from Corollary 2.5 they can be jointly extended to an isomorphism of fields \(\sigma_{i,1} : A_{B}B_{P} \to A'B_{i}P\). By Lemma 5.14 \(\sigma_{i,1}\) can be extended to \(\sigma_{i,2}\) an \(L^{P}\)-automorphism of \(M\). The automorphism \(\sigma_{i,2}\) maps \(AB \to A'B_{i}\) and extends \(\alpha\) and \(\beta_{i}\) (in particular fixes \(M\) pointwise). Let \(q((x_{a})_{a \in A}, (x_{b})_{b \in B}) = \text{tp}((a)_{a \in A}, (b)_{b \in B}/M)\). We get that \(((\alpha(a))_{a \in A}\) realizes \(\bigcup_{i < \omega} q((x_{a})_{a \in A}, (\beta_{i}(b))_{b \in B})\) as required. \(\square\)

5.2. NSOP\(_{1}\), simplicity.

**Remark 5.8.** In a general theory \(T\), if \(A \downarrow_{\omega} B\), then acl\((AC) \downarrow_{\text{acl}(C)} \text{acl}(BC)\). Indeed, by extension, for some \(A' \equiv_{BC} A\) we have \(A' \downarrow_{C} \text{acl}(BC)\), and by applying an automorphism taking \(A'\) to \(A\) and fixing \(BC\) we get that \(A \downarrow_{\omega} \text{acl}(BC)\). By base monotonicity, \(A \downarrow_{\text{acl}(C)} \text{acl}(BC)\).

Suppose that \(\models \phi(d, b)\) where \(\phi(x, y)\) is a formula over acl\((C)\), \(d \in \text{acl}(AC)\) and \(b \in \text{acl}(BC)\). Let \(\psi(x, z)\) be a formula over \(C\) and \(a \in A\) be such that \(\psi(x, a)\) is algebraic, say of size \(n\), and \(\models \psi(d, a)\), that is
\[\models \exists^{\leq n}x \psi(x, a) \land \exists x (\phi(x, b) \land \psi(x, a)).\]
As \(A \downarrow_{\text{acl}(C)} \text{acl}(BC)\), there exists \(c \in \text{acl}(C)\) such that \(\psi(x, c)\) is of size at most \(n\) and \(\models \exists x (\phi(x, b) \land \psi(x, c))\), let \(e\) witness the existence. The fact that \(\models \psi(e, c)\) implies that \(c \in \text{acl}(C)\), and we have \(\models \phi(e, b)\), so \(\text{acl}(AC) \downarrow_{\text{acl}(C)} \text{acl}(BC)\).

**Theorem 5.9.** If \(T\) is NSOP\(_{1}\), then \(AC_{F}T\) is NSOP\(_{1}\).

**Proof.** We will use Fact 2.9. Let \(M \preceq M\) and suppose \(A_{0}\), \(A_{1}\), \(B_{0}\) and \(B_{1}\) are such that \(A_{0}B_{0} \equiv_{M} A_{1}B_{1}\) in \(AC_{F}T\), \(B_{1} \downarrow_{M} B_{0}\) and \(B_{1} \downarrow_{M} A_{i}\) for \(i = 0, 1\). By Remark 5.8, we can assume that \(A_{1} = \text{acl}(A_{M})\), \(B_{1} = \text{acl}(B_{M})\), and in particular they are all D-closed and algebraically closed.

From \(B_{0} \downarrow_{M} A_{0}\), we get using Lemma 5.5 that \(B_{0} \downarrow_{M} A_{0}\). However, \(T\) is NSOP\(_{1}\), so Fact 2.9 implies that \(\downarrow^{*}P\) is symmetric, thus \(\downarrow^{*}\) is also
symmetric and we have $A_0 \downarrow^*_{M} B_0$. By Proposition 5.7, $tp(A_0/B_0)$ does not Kim"-divide over $M$. Extend the pair $(B_0, B_1)$ to a coheir sequence $(B_i)_{i<\omega}$ (to do that, first extend $tp(B_1/MB_0)$ to a global type which is finitely satisfiable in $M$, and then generate a Morley sequence in that type; see [KR20, §3.1]). By the definition of Kim"-dividing (Definition 2.17), we get that there exists $A \subseteq M$ such that $A_0B_0 \equiv_M AB_0 \equiv_M AB_1$ in $ACF_T$. □

Corollary 5.10. The theory of $\omega$-free PAC fields was shown to be non-simple by Chatzidakis [Cha99], as it is PAC and unbounded, and NSOP$_1$ by Chernikov and Ramsey [CR10]. Thus, $ACF_{\omega$-free PAC} is NSOP$_1$ and non-simple as the theory of $\omega$-free PAC fields is interpretable in $ACF_{\omega$-free PAC}.

Now we will show that in NSOP$_1$ theories, Kim-independence is $\downarrow^*$ for certain sets.

Proposition 5.11. Assume $T$ is NSOP$_1$. Let $M \subseteq \mathbb{N}$ and let $A,B \subseteq \mathbb{N}$ be small $D$-closed subfields with $M \subseteq A \cap B$. Then $A \downarrow^*_{M} B$ implies $A \downarrow^*_M B$. If either $A$ or $B$ are algebraically closed as fields, then also $A \downarrow^*_{M} B$ implies $A \downarrow^*_M B$.

Proof. We will first prove that $A \downarrow^*_{M} B$ implies $A \downarrow^*_M B$. Suppose $A \downarrow^*_{M} B$, we need to prove that $P_A \downarrow^*_{P_M} P_B$ in $P$ and $A.P \downarrow^*_{M,P} B.P$. Take an arbitrary $M$-indiscernible coheir sequence $(B_i)_{i<\omega}$, with $B \equiv_M B_i$ in $ACF_T$. The theory $T$ is NSOP$_1$, so $ACF_T$ is also NSOP$_1$ from Theorem [3.9] By Remark 2.19 and Fact 2.20(2) there exists $A' \subseteq M$ such that $AB \equiv_M A'B_i$ in $ACF_T$. In particular, by $A \equiv_M A'$ in $ACF_T$ there exists an automorphism $\sigma$ of $M$ mapping $A'$ to $A$ and preserving $M$ pointwise. Letting $B'_i = \sigma(B_i)$, $(B'_i)_{i<\omega}$ is an $M$-indiscernible coheir sequence with $B \equiv_A B'_i$ in $ACF_T$. By Lemma 5.9, $(P_B')_{i<\omega}$ is a $P_M$-indiscernible coheir sequence with $P_B \equiv_{\mathcal{P}_A} P_{B'}$ in $P$. Because $T$ is NSOP$_1$, Fact 2.20(1) implies that $P_A \downarrow^*_{P_M} P_B$ in $P$.

To prove that $A.P \downarrow^*_{M,P} B.P$, it is enough to prove that $A \downarrow^*_{M.P} A.P$ by Lemma 5.8. Let $\mathfrak{a} \in A$ be a finite tuple and suppose it is linearly dependent over $B.P$. Because $A \downarrow^*_{M} B$, we can construct an uncountable $M$-indiscernible coheir sequence $(B_i)_{i<\omega_1}$, with $B \equiv_A B_i$ in $ACF_T$. Let $\sigma_i \in Aut(\mathbb{M}/A)$ be an automorphism mapping $B$ to $B_i$. We know that $\sigma_i$ preserves $P$ setwise, so by applying $\sigma_i$ we get that $\mathfrak{a}$ is linearly dependent over $B_i.P$. By local character, there is some countable subfield $C \subseteq acl(B_{<\omega_1}).P$ such that $C(\mathfrak{a}) \downarrow^*_{C.P} acl(B_{<\omega_1}).P$. Because $C$ is countable, there is some $i < \omega_1$ such that $C \subseteq acl(B_{<i}).P$. By Remark 5.8, we have $B_i \downarrow^*_{M} acl(B_{<i})$, so Lemma 5.8 implies that $B_i.P \downarrow^*_{M,P} acl(B_{<i}).P$, and in particular from monotonicity $B_i.P \downarrow^*_{M,P} M.P.C$. However, the fact that $C(\mathfrak{a}) \downarrow^*_{C.P} acl(B_{<\omega_1}).P$ also implies, using monotonicity, base monotonicity and symmetry, that $B_i.P.C \downarrow^*_{M.P.C} M.P.C(C(\mathfrak{a}))$, so by transitivity $B_i.P \downarrow^*_{M,P} M.P.C(\mathfrak{a})$. The tuple $\mathfrak{a}$ is linearly dependent over $B_i.P$, so it is linearly dependent over $M.P$. However, $A$ is $D$-closed so $A \downarrow^*_{P_A} P$ and by base monotonicity $A \downarrow^*_{M,P.A} M.P$. Thus, $\mathfrak{a}$ is linearly dependent over $M.P.A$, as needed.

If $A$ is algebraically closed and $A \downarrow^*_M B$, then from Proposition 5.7, $tp(A/B)$ does not Kim"-divide over $M$. $ACF_T$ is NSOP$_1$, so by Remark 2.18 Kim"-dividing is the same as Kim-dividing, and by Fact 2.20(2) Kim-dividing is the same as Kim-forking, thus $A \downarrow^*_M B$. The case where $B$ is algebraically closed follows from symmetry of $\downarrow^*$ and $\downarrow^*$ (Fact 2.20(3)). □

Remark 5.12. The proof of Proposition 5.11 was inspired by the proof of [BYPV03, Proposition 7.3]

Theorem 5.13. If $T$ is simple, then $ACF_T$ is simple.
Suppose $A, B$ to Kim-independence having base monotonicity. Let $A \⫋ M$ and $N \subseteq M$ submodels, such that $M \subseteq A, M \subseteq N \subseteq B$. Suppose $A \downarrow^K_M B$, we want to prove $A \downarrow^K N B$. Without loss of generality we can assume that $A$ and $B$ are acl-closed.

By Proposition 5.11, $A \downarrow^K_M B$ implies $A \downarrow^K N B$. We have $A.P \downarrow^K_M B.P$ and by monotonicity $A.P \downarrow^K_M N.P$, so from Lemma 5.3 $N.A$ is D-closed. Since $B$ is D-closed and algebraically closed as a field, by Proposition 5.11 it is enough to prove $N.A \downarrow^K B$. By base monotonicity of linear disjointness, $A.P \downarrow^K_M B.P$ implies $N.A.P \downarrow^K_M B.P$. We know that $T$ is simple, so by base monotonicity of Kim-independence in $P$, $PA \downarrow^K_{P_A} PB$ implies $PN.PA \downarrow^K_{P_N} PB$. 

Corollary 5.14. ACF$_{PSF}$ is simple, where PSF is the theory of pseudo-finite fields (see Proposition 6.12 for an alternative proof).

5.3. Stability. There are a few ways to prove that if $T$ is stable, then ACF$_T$ is stable. The first option, continuing in the path of the previous results, is using a Kim-Pillay style characterization on non-forking independence, which in simple theories is the same as Kim-independence over models.

The second option is a more direct approach, by counting types. The second option will give us a stronger result, that if $T$ is $\lambda$-stable, then so is ACF$_T$, which will let us extend to super-stability and $\omega$-stability. Even though the second option is strictly stronger than the first, we will also show the first, to complete the picture on Kim-independence.

A third way to prove stability, is by proving the existence of saturated models of certain cardinalities. This could be done using the characterization of saturated models of ACF$_T$ found in Proposition 4.11 but we will not expand on it here.

Remark 5.15. When the predicate has no extra structure, stability can also be deduced from [CZ01 Corollary 5.4] (which cites [Pi05], probably meaning Proposition 3.1 there), which is a much more general statement: if $M$ is strongly minimal and $A$ is some subset of $M$ such that the induced structure on $A$ is stable, then $(M, A)$ is stable.

Theorem 5.16. If $T$ is stable, then ACF$_T$ is stable.

Proof. Suppose $T$ is stable, in particular $T$ is simple so Theorem 5.13 implies that ACF$_T$ is simple. [KR20] Proposition 8.4] says that in simple theories, non-forking independence over models is the same as Kim-independence. To show that ACF$_T$ is stable, it is enough to show that non-forking independence has stationarity over models ([Cas11 Theorem 12.22]). Let $A, A'$ and $B$ be small subsets such that $M \subseteq A \cap A' \cap B$. Suppose $A \downarrow^K_M B, A' \downarrow^K_M B$ and $A \equiv_M A'$. Without loss of generality we can assume $A, A'$ and $B$ are acl-closed. Let $\alpha : A \rightarrow A'$ be an $L^T$-elementary map fixing $M$ pointwise. We want to extend $\alpha$ to an automorphism fixing $B$ pointwise.

By Corollary 4.10 $\alpha|_{P_A}$ is an $L$-elementary map in $P$, and by Proposition 5.11 $P_A \downarrow^K_{P_A} P_B$ and $P_A \downarrow^K_{P_A} P_B$ in $P$. We know that $T$ is stable, so by stationarity $P_A \equiv_{P_B} P_{A'}$, i.e., $(a)_{a \in P_A} \equiv_{P_B} (a(a))_{a \in P_{A'}}$. Let $\sigma_0$ be an automorphism of $P$ mapping $P_A$ to $P_{A'}$ extending $\alpha|_{P_A}$ and preserving $P_B$ pointwise. We have $B \downarrow^{P_B} P_B$, so by stationarity of linear disjointness we can extend $\sigma_0$ to $\sigma_1 : B.P \rightarrow B.P$ preserving $B$ pointwise. By Proposition 5.11 and Lemma 5.3 $A \downarrow^K_{M.P_A} B.P$ and $A' \downarrow^K_{M.P_{A'}} B.P$, so by Corollary 2.5 we can extend $\sigma_1$ and $\alpha$ to $\sigma_2 : A.B.P \rightarrow$
Extend $\sigma_2$ to $\sigma_3$, an automorphism of $M$, using Lemma 3.14. Since $\sigma_3$ extends $\sigma$ and fixes $B$ pointwise we are done. \hfill $\Box$

**Corollary 5.17.** ACF$_{SCF}$ is stable, where SCF is the theory of separably closed fields.

To prove stability by counting types, we will need to show that $P$ is stably embedded in $M$.

**Definition 5.18.** A set $Q \subseteq M^n$ which is definable over the empty set is called stably embedded if for every $n$, if $D \subseteq M^m$ is definable, then $D \cap Q^n$ is definable with parameters from $Q$.

**Fact 5.19** ([Cha99, Appendix, Lemma 1]). For $Q \subseteq M$ as above, if every automorphism of the induced structure on $Q$ lifts to an automorphism of $M$, then $Q$ is stably embedded.

**Remark 5.20.** The precise formulation of the above fact is more general but requires extra assumptions on $T$, namely that $T = T^{eq}$ and that the language is countable. However, those assumptions are not used in the proof of the direction we cited.

**Lemma 5.21.** The induced structure on $P$ as a subset of $M$ is the same (up to interdefinability) as the intrinsic L-structure of $P$.

**Proof.** If $A \subseteq P^n$ is definable in $P$ by a formula $\phi \in L$, then we can construct by Remark 3.6 a formula $\phi^P \in L^P$ that defines $A$ in $M$.

In the other direction, if $A \subseteq P^n$ is definable in $M$ by a formula $\psi \in L^P$, then we can assume by Corollary 4.8 that $\psi$ is bounded. Remove any occurrence of $P$ in $\psi$, by replacing $x \in P$ with a tautology $(x = x)$, to get a formula in $L$ that defines $A$ in $P$.

This can also be deduced from Lemma 3.14 using compactness (since Lemma 3.14 implies that if $a, b \in P$ and $a \equiv b$ in $L$, then $a \equiv b$ in $L^P$ which implies the lemma using e.g. [TZ12, Lemma 3.1.1]). \hfill $\Box$

From Fact 5.19 and Lemmas 3.14 and 5.21 we conclude the following:

**Corollary 5.22.** $P$ is stably embedded in $M$.

**Remark 5.23.** It follows from a simple compactness argument that if $P$ is even uniformly stably embedded, that is, for any formula $\phi(x, y)$ there exists a formula $\psi(x, z)$ such that for every $b \in M$ there is $c \in P$ with $\phi(P, b) = \psi(P, c)$.

**Theorem 5.24.** If $T$ is $\lambda$-stable, then ACF$_T$ is $\lambda$-stable.

**Proof.** Suppose $T$ is $\lambda$-stable, we can assume that $|T| \leq \lambda$ by replacing $T$ with an interdefinable theory (see e.g. [TZ12, Exercise 5.2.6]). Let $C \subseteq M$ be a subset with $|C| \leq \lambda$, we need to prove that $|S^1_{\text{ACF}_T}(C)| \leq \lambda$, where $S^1_{\text{ACF}_T}(C)$ is the space of types in one variable over $C$. First we will prove that all elements in $M \setminus P(C)$ have the same type over $C$ in ACF$_T$. Suppose $a_0, a_1 \in M \setminus P(C)$, that is both $a_0$ and $a_1$ are transcendental over $P(C)$. There is an isomorphism of fields $P(C, a_0) \rightarrow P(C, a_1)$ given by fixing $P(C)$ pointwise and mapping $a_0 \mapsto a_1$. By Lemma 3.14 we can extend this map to an automorphism of $M$, so $a_0 \equiv_C a_1$ in ACF$_T$.

---

3We only need the “easy” direction of Lemma 5.21 i.e. that the L-structure is a reduct of the induced structure.
It remains to show that there are at most \( \lambda \) types in \( P(C) \). Any element of \( P(C) \) solves some non-zero polynomial of the form \( q(x; b, c) \) with \( b \in P^n \) and \( c \in C^n \), and in particular satisfies
\[
\phi(x; c) = \exists y \in P \ (q(x; y, c) = 0 \land \exists x' q(x'; y, c) \neq 0).
\]
Thus, any type in \( P(C) \) contains some formula \( \phi(x; c) \) as above. There are at most \( \lambda \) formulas in \( L^P \) with parameters from \( C \), so it is enough to prove that there are at most \( \lambda \) types that contain any given formula \( \phi(x; c) \) as above.

First of all, \( P \) is stably embedded in \( M \) (Corollary 5.22), so every \( C \)-definable subset of \( P^n \) in \( \text{ACF}_T \) is also definable in \( \text{ACF}_T \) with parameters from \( P \). Let \( D \subseteq P \) be the set of all the parameters needed to define every \( C \)-definable subset of \( P^n \). There are at most \( \lambda \) definable subsets of \( P^n \) over \( C \), so \( |D| \leq \lambda \).

Let \( \varnothing \subseteq S^1_\lambda(\text{ACF}_T(C)) \) be the set of types implying \( \phi(x; c) \). We will construct a map \( \rho : [\varnothing] \to S^1_\lambda(D) \) such that \( \rho \) has finite fibers. Because \( T \) is \( \lambda \)-stable, \( |S^1_\lambda(D)| \leq \lambda \), so this will imply \( |[\varnothing]| \leq \lambda \) as needed.

For any type \( p(x) \in [\varnothing] \), choose some realization \( a \models p \). In particular, \( \models \phi(a; c) \), so we can choose some \( b \in P^n \) such that \( q(x; b, c) \) is non-zero and \( q(a; b, c) = 0 \).

Define \( \rho(p) = tp^T(b/D) \). Suppose \( p_0, p_1 \in [\varnothing] \) and \( \rho(p_0) = \rho(p_1) \), that is, if \( a_i, b_i \) are the specific elements we chose for \( p_i \) \((i = 0, 1)\), then \( b_0 \equiv_D b_1 \) in \( T \).

There is an automorphism of \( P \) over \( D \) mapping \( b_0 \mapsto b_1 \), which can be extended by Lemma 5.14 to an automorphism of \( M \) over \( D \), so \( b_0 \equiv_D b_1 \) in \( \text{ACF}_T \).

We want to prove that \( b_0 \equiv_C b_1 \) in \( \text{ACF}_T \). Suppose \( b_0 \) belongs to some \( C \)-definable set, we can assume that it is a subset of \( P^n \) because \( b_0 \in P^n \). By the construction of \( D \), this \( C \)-definable subset of \( P^n \) is also \( D \)-definable in \( \text{ACF}_T \), so \( b_1 \) belongs to it as \( b_0 \equiv_D b_1 \).

Let \( \sigma \in \text{Aut}(M/C) \) be an automorphism mapping \( b_0 \) to \( b_1 \). In particular \( q(\sigma(a_0); b_1, c) = 0 \), thus \( a_0 \) has the same type over \( C \) as a root of \( q(x; b_1, c) \), specifically \( \sigma(a_0) \). It follows that every type in the fiber of \( \rho(p) \) is a type over \( C \) of a root of \( q(x; b_1, c) \), however \( q(x; b_1, c) \) is non-zero, so it has only finitely many roots. Thus, \( \rho \) has finite fibers. \( \square \)

We can apply Theorem 5.24 to specific \( \lambda \)'s to give another proof of Theorem 5.18.

We also get the following corollaries:

**Corollary 5.25.** If \( T \) is superstable, then \( \text{ACF}_T \) is superstable.

**Corollary 5.26.** If \( T \) is \( \omega \)-stable, then \( \text{ACF}_T \) is \( \omega \)-stable.

**Corollary 5.27.** \( \text{ACF}_{\text{ACF}} \) is \( \omega \)-stable, see Proposition 6.2 for an extended application of this result.

**Remark 5.28.** By [Poi83], \( \text{ACF}_{\text{ACF}} \) is a belle pair (see there for the definition), and it is stable. In [BYPV03], the notion of belle pairs was expanded to lovely pairs and a description of non-forking independence was given. When considering pairs of \( \text{ACF} \), the description of non-forking independence in Proposition 5.14 is slightly different from the description given in [BYPV03, Proposition 7.3] — instead of the condition \( A.P \downarrow_{M, P} B.P \) they have \( A.P \downarrow_{M, P} \text{ACF} B.P. \) However, in this case the conditions are equivalent, as can be seen in [MPZ20, Corollary 6.2].

### 5.4 NIP

We will prove that if \( T \) is NIP, then \( \text{ACF}_T \) is NIP. First we will define the notions of a NIP formula, type and theory, and present some basic facts based on [Sim15] and [KST14].

**Definition 5.29.** Suppose that \( T \) is some theory. A formula \( \phi(x, y) \) has the independence property (IP) if there is a sequence \( (a_i)_{i \in \omega} \) (in a model of \( T \)) such that for every \( s \subseteq \omega \) the set \( \{ \phi(a_i, y) \mid i \in s \} \cup \{ \neg \phi(a_i, y) \mid i \not\in s \} \) is consistent.
A partial type $\pi(x)$ has IP if there is a formula $\phi(x, y)$ and a sequence $(a_i)_{i<\omega}$ of realizations $a_i \models \pi(x)$ such that for every $s \subseteq \omega$ the set $\{\phi(a_i, y) \mid i \in s\} \cup \{\neg \phi(a_i, y) \mid i \notin s\}$ is consistent. Otherwise, $\pi(x)$ is NIP.

The theory $T$ has IP if some formula has IP, or equivalently the type $x = x$ has IP. Otherwise, $T$ is NIP.

Fact 5.30 ([Sim15] Lemma 2.7). A formula $\phi(x, y)$ has IP iff there is an indiscernible sequence $(a_i)_{i<\omega}$ and a tuple $b$ such that $\models \phi(a_i, b)$ iff $i$ is even.

Fact 5.31 ([Sim15] Proposition 2.11). A theory $T$ is NIP iff no formula $\phi(x, y)$ with $|y| = 1$ has IP.

Fact 5.32 ([KS14] Proposition 2.6). Suppose $\pi(x)$ is a partial NIP type over $A$ and $B$ is a set of realizations of $\pi(x)$. If $I = (a_i)_{i<|I|^+}$ is an $A$-indiscernible sequence, then some end segment of $I$ is indiscernible over $AB$.

First we need to show that $P$ is NIP as in Definition 5.29.

Lemma 5.33. If $T$ is NIP, then $P$ is NIP, i.e. the partial type $x \in P$ is NIP.

Proof. Suppose $x \in P$ has IP. Then there are a sequence $(a_i)_{i<\omega}$ with $a_i \in P$ and a formula $\phi(x, y)$, such that for every $s \subseteq \omega$, there exists $b_s \in M$ such that $M \models \phi(a_i, b_s)$ iff $i \in s$. By Remark 5.28 $P$ is uniformly stably embedded in $M$, so there exists a formula $\psi(x, z) \in L^P$ and parameters $c_s \in P$ for every $s \subseteq \omega$, such that $\phi(P, b_s) = \psi(P, c_s)$, and in particular $M \models \psi(a_i, c_s)$ iff $i \in s$.

The induced structure on $P$ is interdefinable with the internal $L$-structure of $P$ (Lemma 5.24), so there is some formula $\psi'(x, z) \in L$ that defines the same set in $P$ as $\psi(x, z)$, in particular $P \models \psi'(a_i, c_s)$ iff $i \in s$. The formula $\psi'(x, y)$ has IP in $P = T$, in contradiction to $T$ being NIP.

Theorem 5.34. If $T$ is NIP, then $ACF_T$ is NIP.

Proof. Suppose $ACF_T$ has IP, by Fact 5.31 there is some $\phi(x, y)$ with $|y| = 1$ that has IP. Using Fact 5.30 and compactness, there is an indiscernible sequence $I = (a_i)_{i<|I|^+} \subseteq M$ and some $c \in M$ such that $M \models \phi(a_i, c)$ iff $i$ is even.

First consider the case where $c$ is transcendental over $P(I)$. In particular, $c$ is transcendental over $P(a_0)$ and $P(a_1)$. There is an automorphism mapping $a_0$ to $a_1$, as they have the same type over the empty set. Apply this automorphism on $c$ to get $c'$ which is transcendental over $a_1$. Both $c$ and $c'$ are transcendental over $P(a_1)$, so by Lemma 5.14 $c$ and $c'$ have the same type over $P(a_1)$ in $ACF_T$. This is a contradiction, as we have $\models \phi(a_1, c')$ and $\models \neg \phi(a_1, c)$.

Now consider the case where $c$ is algebraic over $P(I)$. There is some finite subsequence $I_0 \subseteq I$ and some finite tuple $b \in P$, such that $c$ is algebraic over $I_0b$. Let $I' \subseteq I$ be some end segment starting after $I_0$; note that $I'$ is indiscernible over $I_0$. As $P$ is NIP (Lemma 5.33), by Fact 5.32 there is an end segment $I'' \subseteq I'$ that is indiscernible over $I_0b$. It follows that $I''$ is also indiscernible over acl($I_0b$), and in particular over $c$, a contradiction.

Corollary 5.35. Let $ACVF$ be the theory of algebraically closed valued fields in the divisibility language, that is the language of rings with a binary relation $x|y$ signifying $v(x) < v(y)$. $ACVF$ is NIP, so $ACVF_{ACVF}$ is NIP.

Remark 5.36. One could also use a counting type approach to prove preservation of NIP, similar to the proof of Theorem 5.24. This would require working in a generic extension of ZFC such that $\text{ded}(\kappa)^{T_{\omega}} < 2^\kappa$ for some infinite cardinal $\kappa$ where $\text{ded}(\kappa)$ is the supremum of cardinalities of linear orders with a dense subset of size $\leq \kappa$). For an expanded explanation of this approach, see [Shi90] Theorem II.4.10 and [Adl07] Corollary 24.
Alternatively, one could also apply more general results, i.e., [CSI15] Corollary 2.5 and [JS20] Proposition 2.5, but we chose to give a direct argument.

6. Applications

In this section we will apply the above results to specific theories.

6.1. Tuples of algebraically closed fields. In this section we will consider (perhaps infinite) chains of algebraically closed fields, which, for the finite case, is a particular case of beaux uples in the sense of [BP88]. The main result of this section is Proposition 6.4, which classifies the theories of such chains based on the order type of the chain.

Definition 6.1. For any ordered set \( I \), define \( L^I = L_{\text{rings}} \cup \{ P_i \}_{i \in I} \) with \( P_i \) unitary predicates and define the theory ACF\( ^I \) expanding ACF in \( L^I \), such that:

1. Each \( P_i \) is an algebraically closed field, that is strictly contained in the model.
2. For \( i < j \), \( P_i \subsetneq P_j \).

In particular, ACF\( ^n \) is the theory of algebraically closed fields \( M \), with \( n \) algebraically closed subfields \( P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_{n-1} \subsetneq M \).

Proposition 6.2. Let \( I \) be any ordered set.

1. The completions of ACF\( ^I \) are given by fixing the characteristic, ACF\( ^I_p \).
2. Every completion of ACF\( ^I \) is stable.

Proof. We will first prove for \( I = n \), by induction on \( n \). For \( n = 0 \), ACF\( ^0 = ACF \), and indeed the completions of ACF are given by fixing the characteristic and every completion ACF\( _p \) is stable. Suppose it is true for \( n \). We have ACF\( ^{n+1} = ACF_{ACF_n} \), where we denote the added predicate by \( P_n \). By Proposition 4.1, the completions of ACF\( ^{n+1} \) are given by completions of ACF\( ^n \), which are given by fixing the characteristic. Furthermore, ACF\( ^{n+1} = ACF_{ACF_n} \), so by Theorem 5.16 every completion ACF\( ^{n+1}_p \) is stable.

Now consider a general ordered set \( I \) and fix a characteristic ACF\( ^I_p \). Let \( \phi \) be a sentence in \( L^I \) and let \( I_\phi \subset I \) be the subset of indexes \( i \in I \) such that \( P_i \) appears in \( \phi \). Let \( i_0 \) be finite, suppose \( I_\phi = \{ i_0 < \cdots < i_{n-1} \} \). ACF\( ^n_p \) is complete, so by renaming the predicates \( P_0, \ldots, P_{n-1} \) to \( P_i^0, \ldots, P_i^{n-1} \), we get that ACF\( ^I_p \) is complete. Thus, ACF\( ^I_p \models \phi \) or ACF\( ^I_p \models \neg \phi \), but ACF\( ^I_p \) is a restriction of ACF\( ^I_p \), so ACF\( ^I_p \) is stable. The completions ACF\( ^I_p \) are all the completions of ACF\( ^I \), because any completion has to fix a characteristic so it must extend some ACF\( ^I_p \).

We need to show that every completion ACF\( ^I_p \) is stable. If \( \phi \in L^I \) was a formula witnessing instability in ACF\( ^I_p \), then it would witness instability in ACF\( ^I_p \), which would imply that ACF\( ^n_p \) is unstable for \( n = |I_\phi| \).

We will further classify the stability of ACF\( ^I_p \) (when is it \( \omega \)-stable, superstable or totally transcendental) based on the order type of \( I \). In the case that \( I \) is an ordinal, we will need the following lemma.

Lemma 6.3. Let \( \alpha \) be an ordinal and \( M \models ACF^\alpha \). Any \( L^\beta \)-automorphism of \( P_\beta \) for \( \beta < \alpha \) can be extended to an \( L^\alpha \)-automorphism of \( M \).

Proof. Let \( \sigma_\beta \) be an automorphism of \( P_\beta \), we will construct by transfinite induction on \( \beta \leq \gamma < \alpha \) automorphisms \( \sigma_\gamma \) of \( P_\gamma \), such that if \( \beta \leq \gamma' < \gamma < \alpha \), then \( \sigma_\gamma \) extends \( \sigma_\gamma' \).
Let \( \beta \leq \gamma < \alpha \) and suppose we constructed \( \sigma_{\gamma'} \) for \( \beta \leq \gamma' < \gamma \). Let \( \sigma_{\gamma'} \) be the union of \( \{ \sigma_{\gamma'} \}_{\beta \leq \gamma' < \gamma} \), \( \sigma_{\gamma'} \) is a field automorphism of \( P_{\gamma'} = \bigcup_{\beta \leq \gamma'} P_{\beta} \) (if \( \gamma = \gamma' + 1 \) is a successor ordinal, then \( \sigma_{\gamma'} = \sigma_{\gamma} \)). Let \( S \) be a transcendence basis of \( P_{\gamma} \) over \( P_{\gamma'}, \) extend \( \sigma_{\gamma} \) to a field automorphism \( \sigma_{\gamma} \) by fixing \( S \) pointwise and extending to the algebraic closure. For every \( \gamma' < \gamma \), \( \sigma_{\gamma} \) preserves \( P_{\gamma} \) setwise, so \( \sigma_{\gamma} \) is an \( L^1 \)-automorphism.

Once we constructed \( \sigma_{\gamma} \) for every \( \beta \leq \gamma < \alpha \), we can construct \( \sigma_{\alpha} \), an \( L^\alpha \)-automorphism of \( M \), in a similar fashion: take \( \sigma_{<\alpha} \) the union of \( \{ \sigma_{\gamma} \}_{\beta \leq \gamma < \alpha} \), fix a transcendence basis of \( M \) over \( P_{<\alpha} \) pointwise and extend to the algebraic closure.

\( \square \)

### Proposition 6.4

**For an ordered set** \( I \):

1. If \( I \) is finite, or countable and well-ordered, then every completion of \( \text{ACF}^I \) is \( \omega \)-stable.
2. If \( I \) is uncountable and well-ordered, then every completion of \( \text{ACF}^I \) is totally transcendental, and in particular superstable, but not \( \omega \)-stable.
3. If \( I \) is not well-ordered, then no completion of \( \text{ACF}^I \) is superstable.

**Proof.** Fix a completion \( \text{ACF}^I \) (by Proposition 6.2).

1. The theory \( \text{ACF}^I \) depends only on the order type of \( I \), up to renaming predicates, so it is enough to prove for \( I = \alpha \) a finite or countable ordinal. We will prove that \( \text{ACF}^\alpha \) is \( \omega \)-stable by transfinite induction on \( \alpha < \omega_1 \). For \( \alpha = 0 \), \( \text{ACF}^0 = \text{ACF} \) is \( \omega \)-stable. If \( \text{ACF}^\alpha \) is \( \omega \)-stable, then note that \( \text{ACF}^{\alpha+1} = \text{ACF}_{\text{ACF}^\alpha} \) where we name the added predicate \( P_{\alpha} \), so by Corollary 5.26 \( \text{ACF}^{\alpha+1} \) is \( \omega \)-stable.

2. Suppose that \( \alpha \) is a countable limit ordinal and for every \( \beta < \alpha \), \( \text{ACF}^\beta \) is \( \omega \)-stable, the proof that \( \text{ACF}^\alpha \) is \( \omega \)-stable will be similar to the proof of Theorem 5.24. Let \( M \models \text{ACF}^\alpha \) be a monster model and let \( C \subseteq M \) be a countable subset. Denote \( P_{<\alpha} = \bigcup_{\beta < \alpha} P_\beta \). First we will show that every two elements in \( M \setminus P_{<\alpha} \) have the same type over \( C \). Let \( a_0, a_1 \in M \setminus P_{<\alpha} \), for every \( \beta < \alpha \), \( a_0 \) and \( a_1 \) are transcendental over \( P_\beta \) so by Lemma 5.13 there is an automorphism of \( M \setminus L^{\beta+1} \) preserving \( P_\beta \) and mapping \( a_0 \mapsto a_1 \). Thus, \( a_0 \equiv_C a_1 \) in \( L^{\beta+1} \) for every \( \beta < \alpha \), so \( a_0 \equiv_C a_1 \) in \( L^\alpha \), as every formula in \( L^\alpha \) belongs to some \( L^{\beta+1} \) where \( \beta \) is the largest ordinal such that \( P_\beta \) appears in the formula.

Now we will show that there at most countably many types over \( C \) realized in \( P_{<\alpha}(C) \). Any element \( a \in P_{<\alpha}(C) \) solves some non-zero polynomial of the form \( q(x; b, c) = 0 \) with \( b \in P_{<\alpha} \) and \( c \in C^m \). There is some \( \beta < \alpha \) such that \( b \in P_\beta \), in particular \( a \) satisfies

\[ \phi(x; c) = \exists y \in P_\beta \ (q(x; y, c) = 0 \land \exists x' q(x'; y, c) \neq 0). \]

Thus, any type in \( P_{<\alpha}(C) \) contains some formula \( \phi(x; c) \) as above. There are countably many formulas in \( L^\alpha \) with parameters from \( C \), so it is enough to prove that there are at most countably many types that contain any given formula \( \phi(x; c) \) as above.

First of all, \( P_\beta \) is stably embedded in \( M \) (every automorphism of \( P_\beta \) can be extended to an automorphism of \( M \) so we can use Fact 5.19 alternatively, \( \text{ACF}^\alpha \) is stable so every definable subset is stably embedded), so every \( C \)-definable subset of \( P_\beta \) is also definable in \( \text{ACF}^\alpha \) with parameters from \( P_\beta \). Let \( D \subseteq P_\beta \) be the set of all the parameters needed to define every \( C \)-definable subset of \( P_\beta \). There are at most countably many definable subsets of \( P_\beta \) over \( C \), so \( D \) is countable.
Let \([\phi] \subseteq S_1^{\text{ACF}_0}(C)\) be the set of types implying \(\phi(x;c)\) as above, we will construct a map \(\rho : [\phi] \rightarrow S_n^{\text{ACF}_0}(D)\) such that \(\rho\) has finite fibers. Because \(\text{ACF}_0^\omega\) is \(\omega\)-stable, \(|S_n^{\text{ACF}_0}(D)|\) is countable, so this will imply that \([\phi]\) is countable as needed.

For any type \(p(x) \in [\phi]\), choose some realization \(a \models p\). In particular, \(\models \phi(a;c)\), so we can choose some \(b \in P_n\) such that \(q(x;b,c)\) is non-zero and \(q(a;b,c) = 0\). Define \(\rho_p = \{\}^{\text{ACF}_0^\omega}[b/D]\). Suppose \(p_0, p_1 \in [\phi]\) and \(\rho(p_0) = \rho(p_1)\), that is, if \(a_i\) and \(b_i\) are the specific elements we chose for \(p_i\) \((i = 0, 1)\), then \(b_0 \equiv_D b_1\) in \(\text{ACF}_0^\omega\).

There is an automorphism of \(\text{ACF}_0^\omega\) over \(D\) mapping \(b_0 \mapsto b_1\), which can be extended by Lemma 6.3 to an automorphism of \(M\) over \(D\), so \(b_0 \equiv_D b_1\) in \(\text{ACF}_0^\omega\). We want to prove that \(b_0 \equiv_C b_1\) in \(\text{ACF}_0^\omega\). Suppose \(b_0\) belongs to some \(C\)-definable set, we can assume that it is a subset of \(P_n\) because \(b_0 \in P_n\). By the construction of \(D\), this \(C\)-definable subset of \(P_n\) is also \(D\)-definable in \(\text{ACF}_0^\omega\), so \(b_1\) belongs to it as \(b_0 \equiv_D b_1\) in \(\text{ACF}_0^\omega\).

Let \(\sigma \in \text{Aut}(\mathbb{M}/C)\) be an automorphism mapping \(b_0 \mapsto b_1\). In particular, \(q(\sigma(a_0); b_1, c) = 0\), thus \(a_0\) has the same type over \(C\) as a root of \(q(x; b_1, c)\), specifically \(\sigma(a_0)\). It follows that every type in the fiber of \(\rho(p_1)\) is a type over \(C\) of a root of \(q(x; b_1, c)\), however \(q(x; b_1, c)\) is non-zero, so it has only finitely many roots. Thus, \(\rho\) has finite fibers.

(2) Suppose \(I\) is uncountable and well-ordered. If \(\text{ACF}_0^\omega\) was not totally transcendental, there would be a binary tree of consistent formulas \(\{\phi_s(x; c_s)\}_{s \in 2^{<\omega}}\) (see [TZ12 Definition 5.2.5]). Let \(I_0 \subseteq I\) be the finite or countable subset of indexes \(i \in I\) such that \(P_i\) appears in some formula \(\phi_s\). The tree \(\{\phi_s(x; c_s)\}_{s \in 2^{<\omega}}\) is also a binary tree of consistent formulas in \(\text{ACF}_0^\omega\), so \(\text{ACF}_0^\omega\) is not totally transcendental. However, a subset of a well-ordered set is also well-ordered, so by the previous part \(\text{ACF}_0^\omega\) is \(\omega\)-stable and in particular totally transcendental.

However, \(\text{ACF}_0^\omega\) can not be \(\omega\)-stable, as it is not interdefinable with a theory in a countable language — each \(P_i\) for \(i \in I\) is a distinct definable set.

(3) Note that an ordered set \(I\) is well-ordered iff \(I\) does not contain an infinite descending chain. If \(I\) is not well-ordered, let \((i_k)_{k<\omega} \subseteq I\) be a descending chain, then \((P_{i_k})_{k<\omega}\) is a descending chain of definable subfields in \(\text{ACF}_0^\omega\). Considering only the additive group structure, \((P_{i_k})_{k<\omega}\) is a descending chain of definable subgroups each of infinite index in the previous one, so \(\text{ACF}_0^\omega\) is not superstable (see e.g. [TZ12 Exercise 8.6.10]).

6.2. Complete system of a Galois group. For a profinite group \(G\) one can associate a structure \(S(G)\), called the complete system of \(G\), in a multi-sorted language. This definition is due to [CvdDM81], we will present the definition as given in [Ram18 Definition 7.1.6].

**Definition 6.5.** Suppose \(G\) is a profinite group. Let \(\mathcal{N}(G)\) be the collection of open normal subgroups of \(G\). Define

\[
S(G) = \coprod_{N \in \mathcal{N}(G)} G/N.
\]

Let \(L_G\) be the language with a sort \(X_n\) for each \(n < \omega\), two binary relation symbols \(\leq, C\) and a ternary relation \(P\). We regard \(S(G)\) as an \(L_G\)-structure in the following way:

- The coset \(gN\) is in the sort \(X_n\) iff \([G : N] \leq n\).
- \(gN \leq hM\) iff \(N \subseteq M\).
- \(C(gN, hM)\) iff \(N \subseteq M\) and \(gM = hM\).
- \(P(g_1N_1, g_2N_2, g_3N_3)\) iff \(N_1 = N_2 = N_3\) and \(g_1g_2N_1 = g_3N_1\).
Note that we do not require the sorts to be disjoint (see [Cha08 §1] for a discussion on the syntax of this structure).

For a field $F$, let $G(F) = \text{Gal}(\overline{F}/F)$ be the absolute Galois group of $F$, which is profinite. In [Ram18 Corollary 7.2.7], Ramsey proved that if $F$ is a PAC field such that $\text{Th}(S(G(F)))$ is NSOP$_1$, then $\text{Th}(F)$ is NSOP$_1$. We will prove the other direction, using the following fact, proved in [Cha02 Proposition 5.5].

**Fact 6.6.** $S(G(F))$ is interpretable in $(K,F)$ where $K$ is any algebraically closed field extending $F$.

**Proposition 6.7.** Let $F$ be a PAC field. Then $\text{Th}(F)$ is NSOP$_1$ iff $\text{Th}(S(G(F)))$ is NSOP$_1$.

*Proof.* The left to right direction is [Ram18 Corollary 7.2.7]

For the right to left direction, let $K \supseteq F$ be a large enough algebraically closed extension, $(K,F) \models \text{ACF}_{\text{Th}(F)}$. From Theorem 5.3 ACF$_{\text{Th}(F)}$ is NSOP$_1$, but from Fact 6.6 $S(G(F))$ is interpretable in $(K,F)$, so $\text{Th}(S(G(F)))$ is NSOP$_1$. $\square$

**6.3. Pseudo finite fields.** Pseudo finite fields were first studied in [Ax68], we will give the definition from [TZ12].

**Definition 6.8.** Suppose $F$ is a field. We say that $F$ is pseudo-algebraically closed if every absolutely irreducible variety over $F$ has an $F$-rational point, or equivalently if it is existentially closed in every regular extension. We say that $F$ is pseudo-finite if it is perfect, pseudo-algebraically closed and 1-free (has exactly one extension of degree $n$ for every $n$). Being pseudo-algebraically closed or pseudo-finite is an elementary property [TZ12 Corollary B.4.3, Remark B.4.12], so there are first-order theories PAC, PSF of pseudo-algebraically closed, pseudo-finite fields respectively.

**Proposition 6.9.** ACF$_{\text{PSF}}$ is model complete.

*Proof.* If $Q$ and $R$ are pseudo-finite fields such that $Q \subseteq R$ is a relatively algebraically closed extension, then $Q \cap R = Q$, then $Q \leq R$ [JA08 Proposition 20.10.2]. In particular, if $Q \subseteq R$ is a regular extension, then it is relatively algebraically closed, so $Q \preceq R$. Thus, by Theorem 6.12 ACF$_{\text{PSF}}$ is model complete. $\square$

**Proposition 6.10.** Every completion of ACF$_{\text{PSF}}$ is simple.

*Proof.* By Proposition 6.11 completions of ACF$_{\text{PSF}}$ are given by completions of PSF, which are simple by TZ12 Corollary 7.5.6, so the result follows from Theorem 6.13.

We will give another more direct proof using ACFA, the model companion of difference fields, which is simple [Kim14 Example 2.6.9].

Let $(M,P) \models \text{ACF}_{\text{PSF}}$. We will show that there is an automorphism $\sigma \in \text{Gal}(\overline{T}/P)$ such that $\text{Fix}(\sigma) := \{a \in \overline{T} \mid \sigma(a) = a\} = P$. Consider $P_n$ the unique cyclic extension of degree $n$ of $P$ and $\sigma_n$ a generator of $\text{Gal}(P_n/P)$. The fixed field of $\sigma_n$ is $P$, so the inverse limit of $\sigma_n$ is an automorphism of $\overline{P}$ whose fixed field is $P$.

By [As14 Corollary 1.2], we can embed $(\overline{T},\sigma)$ into $(N,\sigma')$ a model of ACFA, with $\text{Fix}(\sigma') = P$. The structure $(N,P)$ is a reduct of $(N,\sigma')$, so it is simple. The structures $(M,P)$, $(N,P)$ and $(\overline{T},P)$ are models of ACF$_{\text{PSF}}$, and they can be uniquely expanded to models of ACF$_{\text{PSF}}$. Lemma 6.10 implies that $(\overline{T},P) \subseteq (M,P)$, $(\overline{T},P) \subseteq (N,P)$ are substructures in ACF$_{\text{PSF}}$, because they all share the same predicate. However, Proposition 6.9 says that ACF$_{\text{PSF}}$ is model complete, so those are elementary substructures. In particular, they are elementary substructures in ACF$_{\text{PSF}}$. Because $(N,P)$ is simple and $(P,P) \preceq (N,P)$, we get that $(P,P)$ is simple. But also $(\overline{P},P) \preceq (M,P)$, so $(M,P)$ is simple. $\square$
7. Questions

There are several questions that arose in our work, which we did not address in this paper.

**Question 7.1.** What other classification properties can we lift from $T$ to $\text{ACF}_T$? $\text{NTP}_2$, $\text{NSOP}_n$ (for $n \geq 2$)?

**Question 7.2.** What results still hold when we replace $\text{ACF}$ in $\text{ACF}_T$ with a different theory of fields? $\text{SCF}$, $\text{ACVF}$? The theory of dense pairs of $\text{ACVF}$ was studied in [Del12].

**Question 7.3.** What results still hold when we replace $\text{ACF}$ in $\text{ACF}_T$ with any strongly minimal theory? See Remark 5.15.

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