CHOICE FUNCTION AND ITS RELATED CONJECTURE

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Abstract. We already have a subset sum problem which is a prototype of $NP$-complete problem in computer science. We give here a conjecture that a choice function could also give rise to an $NP$-complete problem. For this purpose we apply modular representation theory to computer science.

1. Introduction

Because choosing remarkable bases in our certain algebras seems to be important in arguing about $P$-problems, $NP$-problems, $NP$-hard problems, $NP$-complete problems, etc., we are very fond of modular representation theory of Lie algebras among other things.

Even though solving the Kim’s conjecture directly is itself an important problem, we think however that such a way could be more important as using Kim’s conjecture to solve a problem relating to $NP$-completeness.

We already gave an $NP$-problem coming from a sort of counting problem in [NWK-1] and [NWK-2] by making use

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of a choice function and a modular representation of Lie algebras.

In this paper we further conjecture that such a problem called a certain counting problem could become an \( NP \)-hard problem, and consequently an \( NP \)-complete problem, which shall be done in the final section 5 of this paper after recapitulating the proof in [NWK-1,2].

Before we construct such a conjecture, preliminaries and rigorous definitions under consideration shall be given in advance.

Section 2 explains some languages regarding computer science.

Afterwards section 3 and 4 remark on the modular Lie algebras and their representations because the bases of certain algebras associated with irreducible modules over classical modular Lie algebras are very crucial and important to construct our counting problem mentioned above.

We shall use conventions and notations in [NWK-1,2] unless otherwise stated.

2. PREREQUISITE LANGUAGE OF COMPUTER SCIENCE

We should be ready in this section for the argument relating to computer science in the later sections.
So we exhibit some definitions needed for our progression ahead.

Small $o$ and big $\mathcal{O}$ are often used in computer science.

**Definition 2.1.** In complexity theory we are usually concerned with functions from $\mathbb{Z}^{\geq 0}$ to itself.

So the letter $x$ shall denote usually the standard variable of these functions such as $e^x, ax^2 + bx + xc, ax + bx, \sin x, \ln x,$ etc.

In fact $\sqrt{x}$ or $\ln x$ or $(\ln x)^2 - 7\sqrt{x}$ may not produce nonnegative integers; nevertheless we shall usually think of them as such integers, and so $f(x)$ for a given function $f$ really means the greatest integer $\leq f(x)$. Hence we use the notation $[f(x)]$ to denote such an integer.

Suppose that $f_1$ and $f_2$ are functions from $\mathbb{Z}^{\geq 0}$ to itself. If we have an integer $R$ such that $f_1(x) \leq R \cdot f_2(x) \forall x \geq x_0$, then in this case we say that $f_1(x)$ is big oh of $f_2(x)$; whereas if the ratio $f_1(x)/f_2(x) \to 0$ as $x$ tends to $\infty$, we say that $f_1(x)$ is small oh of $f_2(x)$.

Informally speaking the notation $f_1(x) = \mathcal{O}(f_2(x))$ gives the meaning that $f_1$ grows slower than $f_2$. Hence $f_2(x) = \mathcal{O}(f_1(x))$ means the opposite situation, in which case the notation $f_1(x) = \Omega(f_2(x))$ is also used.

If both situations happen, then we simply write $f_1(x) = \Theta(f_2(x))$ instead of $f_1(x) = \Omega(f_2(x))$ and $f_1(x) = \mathcal{O}(f_2(x))$. Clearly the same rate of growth arises in this situation.

**[example]**
Suppose that $f(x)$ is a polynomial of degree $n$; then $f(x) = \ldots$
Θ(\(x^n\)) may be used in this case meaning that the polynomial’s first nonzro term captures the rate of growth of \(f(x)\).

Next if we suppose that \(R\) is an integer \(\geq 2\) and \(f(x)\) is any polynomial, then it is known that \(f(x) = \mathcal{O}(R^x)\), which probably becomes the important and most useful fact about growths of functions in complexity theory.

So we understand that the expression \(f(x) = \mathcal{O}(R^x)\) means informally that any exponential function with the base number \(\geq 2\) grows precisely faster after all than any exponential function, whose fact can be easily shown by dint of L’Hospital formula.

It follows in the same context as this that \(ln(x) = \mathcal{O}(x)\) and moreover \((ln.x)^p = \mathcal{O}(x)\) for any integer power \(p \geq 1\).

What is more important is what we are dealing with as the source of our consideration in connection with the required time. So we may say for example that it suffices to say that we are contented with the rate of time growth \(\mathcal{O}(x^2)\) in this paper.

Hence as a token of the problem having been solved satisfiably, such polynomial rates of growth shall be regarded as acceptable time requirements. Evidently a cause of concern in contrast with polynomial rates arises because of exponential rates or factorial rates such as \(x!, a^x\), etc.

If we encounter with a problem having such persistent rates of growth and if algorithm after algorithm we have invented
does not satisfy the problem within a polynomial time, then we are liable to say that there does not seem to be a practical useful solution which is amenable or tractable to the problem in hand.

So it is undoubtedly uncontroversial that we consider the dichotomy between two kinds of time bounds such as polynomial time bounds and nonpolynomial time bounds and that we identify the polynomial time algorithms with the intuitive concept of practical plausible computation matter.

[remark]
Some efficient computations in practice don’t happen as polynomial time ones and some computations don’t seem to be useful.

Hence linear programming is just an important subject that deals with the problems which give us examples for either kind of exception and polynomial time algorithms. For more references to linear programming and its simplex method, see 9.5.34 in [Pa].

We see that the simplex method is just considered as a widely used classical algorithm for this basic problem even though in the worst case the simplex method turns out to be exponential.

In practice the performance of the simplex method is to be done consistently superbly within a polynomial time, but the
ellipsoid algorithm has impractically slow performance in contrast with the simplex method.

Generally speaking it seems that the story of linear programming stands in favor of the methodology of complexity theory in spite of a subtle discussion arises as a little obstruction to the methodology;

in other words linear programming looks like an indication showing that polynomial time solvable problems with practical algorithms usually appear although the empirically good algorithm may not be coincident with the polynomial time algorithm.

For example $O(x^{100})$ or $O(x^{121})$ algorithm would produce limited practical values. But in contrast $O(7^{\sqrt{x}})$ or $O(x^{2lnx})$ algorithm could be efficient and useful empirically.

Well at any rate the polynomial time paradigm has some sustainable strong supports. Note that we usually use exponential functions rather than polynomial functions from the view point of the fact that the formers prevail the latters after all except for a finite set of instances of the problem and the former expressions may be simple.

However in practice such exceptional instances are those which are liable to arise within the universe confining us and moreover $O(x^{1000})$ algorithms or $O(x^{\sqrt{x}})$ algorithms seldom show up.
In fact it seems that exponential growth algorithms turn out to be impractical, although polynomial time growth algorithms usually possess reasonable constants $R$ and small powers as in the above example.

Against another criticism of our polynomial time paradigm that the paradigm checks only how the polynomial time algorithm in the least favorable instances behaves roughly, we insist that on the average the polynomial time algorithm may be satisfiable and the most inconvenient algorithms may occur because of a statistically insignificant fraction of the input sources.

If we analyze the expected performance of an algorithm as opposed to the worst case, then informative results usually happen. Nevertheless the input distribution of a problem is hard to perceive, i.e., we hardly know each possible instance’s probability with which the instance arises as an input to our feasible algorithm. Hence unfortunately we cannot implement a truly informative average case analysis.

So it becomes of little consolation or help to recognize that we have stumbled upon a statistically insignificant exception if we are eager to know just one special instance abysmally.

We may also study algorithms from a probabilistic point of view, say as in chapter 11[Pa]. So in such a situation we must clarify the source of randomness of inputs which exists within the algorithm itself rather than input’s lottery. The readers may also refer to the reference mentioned above for an interesting complexity theoretic treatment of average case
Criticism from everywhere may bother our choice of polynomial time algorithms as the mathematical concept which is expressed to capture the informal notion of practically efficient computation.

We note that mathematicians in any field of mathematics intend to capture an intuitive real life concept such as smooth function by a mathematical one such as $C^\infty$ excluding undesirable elements.

Notwithstanding considering the structure arising from this construction may still allow the notion to include certain undesirable specimens. After all a useful and elegant theory may come out from adopting polynomial worst case performance as our criterion of efficiency. Such an attempt usually could not happen without such a beautiful theory saying about practical meaningful computation.

As a matter of fact, polynomials usually have much mathematical handiness. For example they are much stable under various useful operations. Moreover a constant $\Theta(lnx)$ relates the logarithms of such polynomial functions. Besides such a constant could result in a convenient one in several opportunities such as space bounds argument.

For a while let’s think of the notion of space bounds or space requirements paying attention to the reachability problem and its search algorithm explained below.
Assume that $E$ is a set of edges and $V$ is a finite set of vertexes. By a graph we mean $G = (V, E)$ consisting of pairs of vertexes which are connected and directed. A famous so-called reachability problem is the following: If we are given two vertexes $n_1, n_k \in V$ of a graph $G$, then we ask ‘may we go from $n_1$ to $n_k$?’

We contend that the reachability problem may be solved by the so-called search algorithm described as follows:

At first we use a certain set of vertexes. We denote it by $V'$ which may vary depending on $n_1$ and $n_k$. In the first place we start with $V' = \{n_1\}$. We can mark or unmark each vertex in the sense that we may mark the vertex if and only if $n_i$ is in $V'$ at present or $n_i$ has belonged to $V'$ at some point in the past. We may mark $n_1$ in the first step.

According to every step iterated by this search algorithm, a vertex $n_i$ is selected and is got rid of from $V'$. Such a process should be repeated for every edge $n_i \rightarrow n_j$ out of $n_i$. If we continue marking, then $V'$ becomes empty at last.

Now we may answer the reachability problem saying no if $n_k$ is not marked and saying yes if $n_k$ is marked. Evidently the markings of the vertexes and the maintaining of the designated set $V'$ are basically required in this algorithm.

Here we may say that the search algorithm requires $O(x)$ space in the sense that we can have at most $x$ markings and $V'$ cannot also become bigger than $x$. 
Next we remind ourselves of a formal and systematic model, the so called Turing machine for representing arbitrary algorithms. In the next paragraph we define it precisely.

But roughly speaking, any algorithm can be efficiently simulated by the Turing machine although it has a weak appearance and looks clumsy. For this reason we look at the flow chart relating any algorithm to a relevant Turing machine and vice versa. Hereafter we shall use some programming languages including Turing machine.

The Turing machine has a string of symbols rather primitivesly as a single data structure. We permit the program to move a cursor right and left on the string of symbols, to write on the present position, and to branch depending on the value of the present symbol if we use the available operations.

We present here the exact definition of this rather primitive language together with others.

We say in principle that a mathematical problem belongs to the $P$-class if it is solvable in polynomial time by an ordinary (deterministic) Turing machine, while a mathematical problem belongs to the $NP$-class if it is solvable by a non-deterministic Turing machine whose time complexity is bounded by a polynomial function of input size.

In the following paragraph we shall give definitions of the Turing machine, $P$-class, and $NP$-class more or less comprehensibly and precisely.
**Definition 2.2.** The Turing machine should be in one of a finite number of internal states \( q_0, q_1, \ldots, q_m \) which are possible contents of a memory device and are fixed at first for this machine, being supplied with a tape divided into squares.

Such a tape must be infinite in both directions. Each square must be blank or any of the previously specified symbols

\( s_0, s_1, \ldots, s_n \) must be printed on each square with \( s_0 \) being blank. The number of nonblank squares is of course finite in any situation.

Suppose that we got a mapping \( \mu: Q \times M \rightarrow Q \times M \times \{R, L, 0\} \), where \( Q = \{q_0, q_1, \ldots, q_m\} \) and \( M = \{s_0, s_1, \ldots, s_n\} \). A Turing machine \( \tau \) is specified by the five tuple \( \{Q, M, \mu, q_0, F\} \), where \( F \) is a subset of \( Q \) called a set of final states and \( q_0 \) indicates the specified initial state. \( M \) is usually called the alphabet of \( \tau \).

If \( \mu(q, s) = (q', s', x) \), then we call the five tuple \( (q, s, q', s', x) \) an instruction for \( \mu \). The following actions must be satisfied by \( \tau \).

(i) In the beginning, \( \tau \) stands in the state \( q_0 \). As an input some sequence of symbols in \( M \) are written on the tape, and the machine stands on the square next to the leftmost nonblank symbol.

(ii) The machine chooses an instruction \( (q, s, q', s', x) \) with respect to the present state \( q \) and the scanned symbol \( s \). Afterwards the next state is rendered to \( q' \), \( s' \) replaces \( s \), and \( \tau \) moves one square left or right or stands still according as \( x \).
becomes $L$, $R$, or $0$.

(iii) Whenever the state becomes an element of $F$, $\tau$ stops moving. So the sequence of symbols on the tape is the result of the computation as the output.

Here in case there always exists at most one instruction associated with a given pair, we shall call $\tau$ deterministic; otherwise we shall call it nondeterministic.

Of course we may have other definitions of Turing machine.

We are already aware that an algorithm exists for a problem if and only if a deterministic Turing machine exists for a problem.

It is evident that the containment $P \subseteq NP$ holds. In 1971 Stephen A. Cook (1939-) asked the problem about the reverse containment. We call such a problem $P$ versus $NP$.

[example] We let the Turing machine $\tau$ defined as follows:

$\tau = \{Q, M, \mu, q_0, F\}, Q = \{q_0, k, k_0, k_1\}, M = \{\bar{0}, 1, s_0, s_1\}, F = \{q_0\}$, and $\mu$ is designated as in the table1 having 3 columns and 17 rows in the next page.

How is this program going to begin and operate? Of course the state is initially $q_0$. After a finite long string, we may initialize this string to the first leftmost nonblank symbol $s_1$.

Such a string is said to be the input of the Turing machine in general. In our situation of the above example, we can take
Table 1. example of $\tau$

| row   | column1 | column2 | column3          |
|-------|---------|---------|------------------|
| row1  | $q \in Q$ | $s \in M$ | $\mu(q, s)$     |
| row2  | $q_0$   | 0       | $(q_0, 0, R)$   |
| row3  | $q_0$   | 1       | $(q_0, 1, R)$   |
| row4  | $q_0$   | $s_0$   | $(k, s_0, L)$   |
| row5  | $q_0$   | $s_1$   | $(k, s_1, R)$   |
| row6  | $k$     | 0       | $(k_0, s_0, R)$ |
| row7  | $k$     | 1       | $(k_1, s_0, R)$ |
| row8  | $k$     | $s_0$   | $(k, s_0, 0)$   |
| row9  | $k$     | $s_1$   | $(k, s_1, 0)$   |
| row10 | $k_0$   | 0       | $(q_0, 0, L)$   |
| row11 | $k_0$   | 1       | $(q_0, 0, L)$   |
| row12 | $k_0$   | $s_0$   | $(q_0, 0, L)$   |
| row13 | $k_0$   | $s_1$   | $(k_0, s_1, R)$ |
| row14 | $k_1$   | 0       | $(q_0, 1, L)$   |
| row15 | $k_1$   | 1       | $(q_0, 1, L)$   |
| row16 | $k_1$   | $s_0$   | $(q_0, 1, L)$   |
| row17 | $k_1$   | $s_1$   | $(k_1, s_1, R)$ |

$\{0, 1, s_0\}$ as an input of this Turing machine.

It is notable that the output could be the same as the input in this example.

**Definition 2.3.** ($P$-problem, $NP$-problem)

By $NP$-problems we mean a class of problems solutions of which are hard to find but easy to verify and which can be solved by Non-Deterministic Turing machine in polynomial
By $P$-problems we mean a class of problems solutions of which are hard to find but which have an algorithm, i.e., which can be solved by Deterministic Turing machine in polynomial time.

**Definition 2.4.** (NP-hard, NP-complete) We say a problem $M$ is NP-complete in case $M$ is NP-hard and in case we may find an NP-problem $M'$ which is reducible to $M$ in polynomial time.

We say a problem $M$ is NP-hard in case all problems in NP-class are reducible to $M$ in polynomial time.

3. **MODULAR LIE ALGEBRAS**

Next in order to attain our purpose we have to connect the representation theory of modular Lie algebras with NP problems.

We begin with some definitions first prior to some preliminary results related to modular Lie algebra theory. Let $F$ be an algebraically closed field of nonzero characteristic $p$.

**Definition 3.1.** We shall call a mapping $[p] : L \rightarrow L$ via $a \rightarrow a^{[p]}$ a $p$-mapping in case that the following three conditions are satisfied

$$(i) \forall a \in L, ad(a^{[p]}) = (ada)^p,$$
(ii) $\forall c \in F, \forall a \in L, (ca)^{[p]} = c^p a^{[p]}$,

(iii) $\forall a, b \in L, (a + b) = a^{[p]} + b^{[p]} + \sum_{i=1}^{p-1} s_i(a, b),$

where $(ad(a \otimes X + b \otimes 1))^{p-1}(a \otimes 1) = \sum_{i=1}^{p-1} i s_i(a, b) \otimes X^{i-1} = \text{in } L \otimes_F F[X]$. We call such a pair $(L, [p])$ a restricted Lie algebra over $F$.

**Definition 3.2.** Let $(L, [p])$ be a restricted Lie algebra over $F$ and $\chi$ is a linear form of $L$, i.e., $\chi$ is an element of $L^*$. A representation of $L$, $\rho_{\chi} : L \rightarrow gl(V)$ is called a $\chi$-representation in case that $x \in L$,

$$\rho_{\chi}(x)^p - \rho_{\chi}(x^{[p]}) = \chi(x)^p id_V.$$

We call $\chi$ the $p$-character of the representation or of the corresponding module.

If $\chi \neq 0$ in particular, then $\rho_{\chi}$ is called a nonrestricted representation. Otherwise $\rho_0$ is called $p$-representation or restricted representation.

Let $L$ be a finite dimensional indecomposable restricted Lie algebra over $F$.

Suppose that $\rho$ is the corresponding representation to an $L$-module $V$.

**Proposition 3.3.** We let $(L, [p])$ be a restricted Lie algebra defined as above and $V$ a finite dimensional $L$-module over an algebraically closed field $F$ of nonzero characteristic $p$.

We then have some submodules $V_1, V_2, \cdots, V_s$ and some characters $\chi_1, \chi_2, \cdots, \chi_s$ satisfying that $V = \sum_{i=1}^{s} V_i$ and \{\rho(x)^p - \rho(x^{[p]}) - \chi_i(x)^p id\} |_{V_i}$ becomes nilpotent $\forall x \in L$. 
If $V$ is irreducible in particular, then there exists $\chi \in L^*$ such that $\rho(x)^p - \rho(x^{[p]}) = \chi(x)^p id_V$.

**Proof.** As we are well aware, we have that $x^p - x^{[p]} \in \mathfrak{Z}(u(L))$, $\forall x \in L$, which is just the center of the universal enveloping algebra of $L$.

With respect to the finite dimensional abelian Lie algebra $C := \langle \{ \rho(x)^p - \rho(x^{[p]}) | x \in L \} \rangle_F$ which is the $F$- algebra generated by $\rho(x)^p - \rho(x^{[p]}) \forall x \in L$, we can decompose $V$ into weight spaces.

We thus have $V = \bigotimes_i V_{\phi_i}$ with $\phi_i \in Map(C, F[X])$, where $V_{\phi_i} = \{ v \in V | \forall c \in C, \exists n(c,v) \in \mathbb{N} s.t. (\phi_i(c)\rho(c))^{n(c,v)}(v) = 0 \}$.

Because $F$ is algebraically closed, it follows that $\phi_i \in Map(C, F)$ actually. Obviously $V_{\phi_i}$ are submodules since $[C, \rho(L)] = 0$ is trivial. Putting $\chi_i(x) := \phi_i(\rho(x)^p - \rho(x^{[p]}))^{\frac{1}{p}}$, we perceive that $\{ \rho(x)^p - \rho(x^{[p]}) - \chi_i(x)^p id \} | V_{\phi_i}$ is nilpotent for each $x \in L$. Because of abelian $C$, we get $\phi_i \in C^*$ and because the map $f : x \mapsto \rho(x)^p - \rho(x^{[p]})$ is semilinear, i.e.,

$$f(\alpha x + y) = \alpha^p f(x) + f(y) \forall x, y \in L, \forall \alpha \in F,$$

we see that $\chi_i$ becomes linear. Due do Schur’s lemma, the latter claim is clear.

□

**Definition 3.4.** Let $F$ be an algebraically closed field of nonzero characteristic $p$. By the classical type Lie algebras over $F$, we mean analogues over $F$ of the so called 9 types of simple
Lie algebras over the complex number field $\mathbb{C}$:

$$A_l(l \geq 1), B_l(l \geq 2), C_l(l \geq 3), D_l(l \geq 4), G_2, F_4, E_6, E_7, \text{ and } E_8.$$ 

In other words any Lie algebra of classical type over $F$ is isomorphic to the Lie algebra $\sum_{j=1}^{n} \mathbb{Z} e_j \otimes \mathbb{Z} F$ for some Chevalley basis $\{e_1, \cdots, e_n\}$ of the type above.

Next we would like to supplement Zassenhaus' theory related to the background work on modular representation theory. We assume for a while that $F$ is an algebraically closed field of nonzero characteristic $p$ until it is stated otherwise.

**Definition 3.5.** We let $\mathcal{O}(L)$ denote the $p$-center for a fixed restricted Lie algebra $L$ of dimension $n$ over $F$, i.e., $\mathcal{O}(L) := \langle \{x^p - x^{[p]} | x \in L\} \cup 3(L) >_F \rangle$ is the subalgebra of $u(L)$ generated by $3(L)$ and $1$ and $\{x^p - x^{[p]} | x \in L\}$, where $3(L)$ indicates the center of $L$ and $u(L)$ denotes the universal enveloping algebra of $L$.

We denote the center of $u(L)$ by $3(u(L))$ and simply by $3$ if there is no confusion. We are well aware that $3$ becomes an affine algebra as an integral domain.

We thus may apply the commutative algebra theory or the elementary algebraic geometry theory to this structure $3$. There is an affine variety associated with this affine algebra. We shall call this variety the Zassenhaus variety.

**Proposition 3.6.** We have $3 := 3(u(L)) = \mathcal{O}(L)[s_1, \cdots, s_k]$, where $s_i$'s are integral over $\mathcal{O}(L)$ as elements of $u(L)$. 
Proof. Easy to know but the readers may refer to theorem 1 in [ZH] or chapter 6 in [SF].

\[\square\]

**Proposition 3.7.** If the mapping \( h : \mathcal{O}(L)[X_1, \cdots, X_k] \rightarrow \mathcal{O}(L)[s_1, \cdots, s_k] \) is an evaluation algebra homomorphism, then we have the following:

(i) Ker \( h \) just becomes a prime ideal of \( \mathcal{O}(L)[X_1, \cdots, X_k] \).

(ii) Any point of \( \nu(Kerh) \) in \( F^{n+k} \) becomes a normal point of \( \nu(Kerh) = \nu(\sqrt{Kerh}) \neq \phi \), where \( \sqrt{Kerh} \) indicates the nilradical of \( Kerh \) and \( \nu \) indicates the affine variety cofunctor.

**Proof.** (i) Since \( Z \) becomes an integral domain, the identical relations \( Z := Z(u(L)) = \mathcal{O}(L)[s_1, \cdots, s_k] \)

\[\simeq_{alg} \mathcal{O}(L)[X_1, \cdots, X_k]/Kerh \] proves our assertion.

(ii) We get \( \mathcal{I}(\nu(Kerh)) = \sqrt{Kerh} = Kerh \) by virtue of (i).

We thus have \( Z(u(L)) = \mathcal{O}(L)[X_1, \cdots, X_k]/\mathcal{I}(\nu(Kerh)). \) Because \( Z \) is integrally closed, each local ring \( R_x \) also becomes integrally closed.

Hence every point of \( \nu(Kerh) \) becomes a normal point, i.e., \( R_x \) becomes a normal ring. \[\square\]

**Proposition 3.8.** Suppose that \( V \) is an irreducible algebraic variety and \( x \) is a simple point of \( V \); then we have:
(i) $R_x$ becomes a regular (Noetherian) local ring and so $x$ is a normal point, i.e., $R_x$ is normal.

(ii) But the converse is not true.

Proof. (i) is clear from chapter 1, Linear algebraic groups written by James E. Humphreys. (ii) holds by the above proposition considering the figure of node of the polynomial $(x - c)(x - 2)^2 - y^2 = 0$ for $c \leq 2$ in $\mathbb{R}^2 \subset \mathbb{C}^2$.

□

Proposition 3.9. We obtain the following under the same notation as above:

(i) $\nu(\ker h)$ becomes a normal irreducible algebraic variety.

(ii) Suppose that $\ker h$ is principal with $\tan X_x \subsetneq F^{n+1}, \forall x \in X := \nu(\ker h)$; then we claim that $X$ becomes a smooth irreducible algebraic variety, in other words all points of $X$ are simple and so normal, where the geometric tangent space of $X$ at $x$ is denoted by $\tan X_x$.

Proof. (i) is also evident from propositions (3.7), (3.8) above.

(ii) Since we know that $X$ is irreducible, we have only to prove the latter assertion. We have that each point $x \in X$ becomes a simple point because $\dim_F \tan X_x = n$ from the fact $\tan X_x = \nu(\sum_{i=1}^{n+1} \frac{\partial f}{\partial x_i}(x)(X - x_i))$ with $\ker h = \langle f \rangle$ for some irreducible polynomial $f \in \ker h$ and from the fact $\dim X = n$ according to proposition 3.6. Normal point is already proven in the preceding propositions.

□

Definition 3.10. Let $\chi \in L^*$ and let $I_\chi$ denote the ideal of $u(L)$ generated by $(l^p - l^{[p]} - \chi(l)^p) \in \mathfrak{J}(u(L)), \forall l \in L$. We
shall call $u_\chi(L) := u(L)/I_\chi$ the \textit{reduced enveloping algebra} of $L$ associated with the $p$-character $\chi$. If $\chi = 0$ in particular, then $u_\chi(L)$ is called the \textit{restricted enveloping algebra} of $L$.

**Definition 3.11.** C. Chevalley (1909-1984) found that the classification of simple algebraic groups over $F$ reduces essentially to that of simple Lie algebras over $\mathbb{C}$ and that the nine types of simple Lie algebras over $\mathbb{C}$ are induced from these nine types of simple Lie groups over $\mathbb{C}$, which are denoted by the same symbols as in definition 3.4.

We usually classify the simple algebraic groups over $F$ into infinite family of four classical groups of types $A_l, B_l, C_l, D_l$ along with five exceptional types $E_6, E_7, E_8, F_4$, and $G_2$.

We shall call an algebraic group over $F$ \textit{simply connected} when this algebraic group has its Lie algebra with the basis consisting essentially of a Chevalley basis. Suppose that $G$ denotes a simply connected algebraic group over $F$ with its Lie algebra $\mathfrak{g} := L(G)$.

It is known that $\mathfrak{g}$ has a triangular decomposition like $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ as in characteristic zero. Here $\mathfrak{h}$ indicates a \textit{Cartan subalgebra} of $\mathfrak{g}$ which is defined to be the Lie algebra of a maximal torus of $G$, whereas $\mathfrak{n}^-$ and $\mathfrak{n}^+$ indicates the respective sums of negative and positive root spaces.

We put $\mathfrak{B} = \mathfrak{h} \oplus \mathfrak{n}^+$ and call it a \textit{Borel subalgebra} of $\mathfrak{g}$. It is wellknown that $\mathfrak{B} = L(B)$ for some \textit{Borel subgroup} $B$ of $G$, which is defined to be a connected maximal solvable subgroup of $G$. 
Definition 3.12. Suppose that $G$ is an algebraic group over $F$ with its Lie algebra $\mathfrak{g} = L(G)$. $G$ has a unique largest connected normal solvable subgroup automatically closed, which is just the identity component denoted by $R(G)$ and called the \textit{radical} of $G$.

We shall denote by $R_u(G)$ the normal subgroup of $R(G)$ consisting of all its unipotent elements and call it the \textit{unipotent radical} of $G$.

If a connected algebraic group $G$ of positive dimension has no closed connected normal subgroup except the identity, i.e., $R(G) =$ identity, then we call the group $G$ \textit{semisimple}.

A semisimple algebraic group $G$ has a Borel subgroup of the form $B = T \cdot R_u(B)$ with $T$ a fixed maximal torus. It is known that a Borel subgroup $B$ containing $T$ has another Borel subgroup $B^-$ called opposite $B$ relative to $T$ satisfying $B \cup B^- = T$. So we get $B^- = T \cdot R_uB^-$, which is known to be unique.

Suppose that we are given a simple and simply connected algebraic group $G$. Putting $\mathfrak{B} := L((T \cdot R_u)B), \mathfrak{U} := L(R_u(B)), \mathfrak{h} := L(T)$, and $\mathfrak{U}^- := L(R_u(B^-))$, we let $\mathfrak{B}^*, \mathfrak{h}^*$ and $\mathfrak{U}^*$ denote $\{ \bar{\alpha} \in \mathfrak{g}^* | \bar{\alpha}(\mathfrak{U}) = 0 \}, \{ \bar{\alpha} \in \mathfrak{g}^* | \bar{\alpha}(\mathfrak{U} \oplus \mathfrak{U}^-) = 0 \}$ and $\{ \bar{\alpha} \in \mathfrak{g}^* | \bar{\alpha}(\mathfrak{h} \oplus \mathfrak{U}) = 0 \}$ respectively.

It is known that any $\bar{\alpha} \in \mathfrak{g}^*$ gets a unique Jordan decomposition $\bar{\sigma} = \bar{\sigma}_s + \bar{\sigma}_n$, where the linear functionals $\bar{\sigma}_s$ and $\bar{\sigma}_n$ must satisfy the following conditions:

$\exists g \in G$ such that
(i) $g \cdot \alpha_s \in \overline{h}^*$, (ii) $g \cdot \alpha_n \in \overline{U}^*$, and (iii) $(g \cdot \alpha_s)(h_\alpha) \neq 0 \implies (g \cdot \alpha_n)(x_\alpha) = (g \cdot \alpha_n)(x_{-\alpha}) = 0$, where $\{x_\alpha, x_{-\alpha}, h_\alpha\}$ denotes the canonical basis of $sl_2(F)$ for any root $\alpha$ of the root system $\Phi$ of $T$ in $G$ and the action of $g$ is coadjoint.

So due to the above properties, any $p$-character $\chi$ of an irreducible representation $\rho_\chi$ has a unique Jordan decomposition like $\chi = \chi_s + \chi_n$ such that $\chi$ is contained in the $G$-orbit of $\overline{B}^*$. We thus may assume that the associated $p$-character $\chi$ of $\rho_\chi$ always vanishes on the nilradical $n^+ = \mathfrak{U} \subset B$. For any given character $\chi \in \mathfrak{g}^*$, we shall consider linear functionals $\lambda : \mathfrak{h} \to F$ called weights such that $\forall h \in \mathfrak{h}, \lambda(h)^p - \lambda(h^{[p]}) = \chi(h)^p$ holds.

Since $\chi(\mathfrak{U}) = 0$, we can put $\chi(\mathfrak{U}) = 0$ and naturally extend such a $\lambda$ to a Lie algebra homomorphism $\lambda : \mathfrak{B} \to F$. An algebra homomorphism $\lambda : u(\mathfrak{B}) \to F$ may still arise by extending this $\lambda$.

Letting $W(\chi)$ be the set of all such weights, we obviously perceive that the cardinality $\#(W(\chi))$ of $W(\chi)$ is equal to $p^l$ with $\dim \mathfrak{h} = l$.

**Proposition 3.13.** As in definition 3.11 and 3.12, we let $L(G) = \mathfrak{g}$ be of classical type. Suppose that $\chi$ is a character in $\overline{\mathfrak{B}}^*$ as above.

We then have a $1 - 1$ correspondence between weights $\lambda \in W(\chi)$ and $1$-dimensional $\mathfrak{B}_\chi$-modules, where $\mathfrak{B}_\chi$ denotes the
subalgebra of \( \mathfrak{U}_\chi(\mathfrak{g}) \) generated by \( \mathfrak{B} \) and 1.

For a given weight \( \lambda \), let \( F_\chi \) be the corresponding 1-dimensional \( \mathfrak{B}_\chi \)-module and put \( Z_\chi(\lambda) := \mathfrak{U}_\chi(\mathfrak{g}) \otimes_{\mathfrak{B}_\chi} F_\lambda \). We then have \( \dim Z_\chi(\lambda) = p^m \) and any irreducible \( \mathfrak{U}_\chi(\mathfrak{g}) \)-module becomes the quotient of some \( Z_\chi(\lambda) \) in addition.

**Proof.** Refer to proposition 1.5 in modular representation theory of Lie algebras by E.M. Friedlander and B.J. Parshall, American Journal of Math, 110 (1988) 1055-1094, or proposition 3.1.20 in [KY-4].

**Definition 3.14.** Let \( F \) be any field of nonzero characteristic and let \( R \) be a commutative algebra over \( F \) with a unit element. Let \( R' \) be another ring.

We say that \( R' \) is an \( R \)-ring if the following conditions are satisfied:

\( Au \) must be defined in \( R' \); namely \( Au \in R', \forall A \in R, \forall u \in R' \) with the following properties from (i) to (6).

(i) If \( A \equiv A' \) and \( u \equiv u' \) in \( R \) and \( R' \) respectively, then \( Au = A'u' \),

(ii) \( A(u + v) = Au + Av \),

(iii) \( (A + A')u = Au + A'u \),

(iv) \( (AA')u = A(A'u) \),
(v) $A(uv) = (Au)v = u(Av)$,

(vi) $1_R \cdot u = u, \forall A, A' \in R, \forall u, u', v \in R'$ and for the unity $1_R$ in $R$.

Suppose that $M$ is an $R$-module and let $\theta$ be an $F$- algebra homomorphism of $R$ onto another $F$- algebra $\theta R$.

Obviously we can define a homomorphism $\theta^M$ of $R$- module $M$ onto a $\theta R$- module $\theta^M M$ in as general a sense as possible.

We consider the module $\theta^M M$ which is generated by $\{\theta^M u|u \in M\}$ and subject to such defining relations as

$$\theta^M(u + v) = \theta^M u + \theta^M v, \forall u, v \in M \text{ and } \theta^M(Au) = \theta(A) \cdot \theta^M(u), \forall u \in M, \forall A \in R.$$ 

Next if $M$ is an $R$- ring, then $M$ is automatically an $R$- module. If we define a natural multiplication in $\theta^M M$ by the recipe $\theta^M(u) \cdot \theta^M(v) = \theta^M(uv), \forall u, v \in M$, then we easily see that $\theta^M M$ becomes a $\theta R$- ring.

If the homomorphism $\theta^M : M \to \theta^M M$ is equipped with a ring homomorphism at the same time as well as a module homomorphism in a general sense above, then we call this homomorphism a specialization of the $R$-ring $M$ over $\theta$.

**Proposition 3.15.** Let $L$ be a finite dimensional restricted Lie algebra of dimension $n$ over an algebraically closed field $F$ of nonzero characteristic $p$.

We then have that $u(L)$ is a free $\mathfrak{Z}(u(L))$-module and its rank is $p^{2m}$, where $Q(u(L)) = (\mathfrak{Z}(u(L)) - \{0\})^{-1}$ and $[Q(u(L)) : \mathfrak{Z}(u(L)) - \{0\}]$. 

Q(3(u(L))) = p^{2m}. Here we call Q(u(L)) the quotient algebra of u(L), which turns out to be a division algebra.

Proof. For the Q(3)-vector space Q(u(L)), we may choose a basis of elements in u(L). So our proposition is evident considering the Poincare-Birkoff-Witt theorem and elementary algebraic geometry.

Next putting \( \mathcal{O}_1(L) = \text{alg} < \{x^p - x^{[p]} \mid \forall x \in L\} > \), which is the subalgebra of u(L) generated by all \( x^p - x^{[p]}, \forall x \in L \), we obtain \( p^{\dim L} = p^n = [Q(u(L)) : Q(3)][Q(3) : Q(\mathcal{O}_1(L))] \) with \( [Q(u(L)) : Q(3)] = p^{2m} \).

\[ \square \]

**Proposition 3.16.** We have the following:

(i) We may extend any specialization \( \theta \) over \( F \) of the center \( 3 \) of \( u(L) \) onto an extension field \( \theta 3 \) of \( F \) to a specialization \( \theta u(L) \) of \( u(L) \) over \( F \) onto an algebra \( \theta u(L)u(L) \) of rank not greater than \( p^{2m} \) over \( \theta 3 \).

(ii) There is a general element of \( u(L) \) over \( 3 \) whose minimal polynomial is mapped by \( \theta \) onto a multiple of the minimal polynomial of the corresponding general element of \( \theta u(L) \) over \( \theta 3 \).

(iii) The algebra \( \theta u(L)u(L) \) of maximal dimension of an indecomposable representation \( \theta u(L) \) is separable over \( \theta 3 \) if and only if it becomes centrally simple of dimension \( p^{2m} \) over \( \theta 3 \).

If this is the case, then \( \theta \) just maps any minimal polynomial of \( u(L) \) over \( 3 \) onto a minimal polynomial of \( \theta u(L)u(L) \) over
\( \theta \mathfrak{Z} \).

(iv) The discriminant ideal over \( \mathfrak{Z} \) defined by H. Zassenhaus never vanish.

**Proof.** (i) Because \( \theta \mathfrak{Z} \) becomes an extension field of \( F \), we see that \( \theta u(L)u(L) \) has a dimension over \( \theta \mathfrak{Z} \) obviously.

However the rank of \( u(L) \) over \( \mathfrak{Z} \) is nothing but \( p^{2m} \) by virtue of the preceding proposition. Hence we have that the rank of \( \theta u(L)u(L) \) over \( \theta \mathfrak{Z} \) must be at most \( p^{2m} \).

(ii) By virtue of Brauer- Albert theorem and P-B-W theorem, we have general elements \( \alpha \) and \( \alpha' \) which are conjugate to each other such that
\[
\{ \alpha^i(\alpha')^j | 0 \leq i, j < p^m \}
\]
constitutes a \( Q(\mathfrak{Z}) \) basis for \( Q(u(L)) \) and such that the minimal polynomial of \( \alpha \) over \( \mathfrak{Z} \) becomes of the form \( P(X) = X^{p^m} + \sum_{i=1}^{p^m} P_i(x_1, \cdots, x_n)X^{p^m-i} \), where \( P_i \) denotes a homogeneous polynomial of degree \( i \) contained in \( \mathfrak{Z}[x_1, \cdots, x_n] \) for an \( F \)-basis \( \{ x_1, \cdots, x_n \} \) of \( L \).

It follows that \( P(\theta \alpha) = 0 \) is obtained from \( P(\alpha) = 0 \), whence the minimal polynomial polynomial of \( \theta \alpha \) over \( \theta \mathfrak{Z} \) divides \( \theta P \).

(iii) Recall that the \( \theta \mathfrak{Z} \)-algebra \( \theta u(L)u(L) \) is defined to be separable if for every field extension \( K \) of \( \theta \mathfrak{Z} \), the algebra \( \theta u(L)u(L) \otimes_{\theta \mathfrak{Z}} K \) is semisimple. Because an irreducible \( \theta u(L) \)-module gives rise to an irreducible \( u(L) \)-module by pull-back, we see that its maximal dimension is \( p^m \).
So by the Jacobson’s density theorem the algebra $\theta^u(L)u(L)$ is separable over $\theta\mathfrak{Z}$ if and only if it becomes centrally simple of dimension $p^{2m}$ over $\theta\mathfrak{Z}$.

No doubt the latter assertion holds by (i).

(iv) In the proof of (i), we note that $\alpha$ is a primitive element of a maximal separable subfield of $Q(u(L))$.

So the minimal polynomial of $\alpha$ over $Q(\mathfrak{Z})$ becomes of degree $p^m$. We see that the second highest coefficient of this minimal polynomial is nothing but $tr(\alpha)$.

We now generalize such a linear functional $tr$ to any $R$-ring $M$ in case $M$ is of rank $r$ over $R$ as a free $R$-module. We know that the discriminant ideal is defined to be the ideal $\mathfrak{D}_{M,R,tr}$ of $R$ generated by the set of all the determinants $|tr(u_i, v_j)|$, where $u_i, v_j \in M$ and $1 \leq i, j \leq r$.

If we use the Laplace development of the determinant concerned, then we have that for the given $R$-basis $\{b_1, b_2, \cdots, b_r\}$ of $M$, $\mathfrak{D}_{M,R,tr} = R|tr(b_i, b_j)|$.

So it is clear that $\mathfrak{D}_{M,R,tr} \neq 0$ if and only if $tr(\cdot, \cdot) := tr(\cdot, \cdot, \cdots, \cdot)$ is nondegenerate.\[It is known that if the quotient algebra $Q(M)$ over $Q(R)$ is defined and is separable over $Q(R)$, then the trace form of $M$ over $R$ is nondegenerate.\]

So we have $\mathfrak{D}_{u(L),M,tr} \neq 0$ because $Q(u(L))$ in particular becomes centrally simple and hence separable over $Q(\mathfrak{Z})$.\]
**Proposition 3.17.** We assume that $\theta$ is given as a specialization over $F$ of $\mathfrak{F}$ onto an extension field $\theta \mathfrak{F}$ of $F$;

we then have $\theta(\mathfrak{D}_{u(L)}3,\theta_{tr}) \neq 0$ if and only if $\theta^{u(L)} u(L)$ becomes centrally simple of dimension $p^2m$ over $\theta \mathfrak{F}$ for an indecomposable representation $\theta^{u(L)}$.

*Proof.* ($\Leftarrow$) By virtue of the preceding proposition, any minimal polynomial of $u(L)$ over $\mathfrak{F}$ is mapped onto a minimal polynomial of $\theta^{u(L)} u(L)$ over $\theta \mathfrak{F}$ under the hypothesis that $\theta^{u(L)} u(L)$ becomes centrally simple of dimension $p^2m$ over $\theta \mathfrak{F}$, in which case we have $\theta(\mathfrak{D}_{u(L)},\theta_{tr}) = \mathfrak{D}_{\theta^{u(L)} u(L)},\theta_{tr}$ holds inevitably.

Because $\theta^{u(L)} u(L)$ is separable over $\theta \mathfrak{F}$, we have $\mathfrak{D}_{\theta^{u(L)} \theta \mathfrak{F},\theta_{tr}} \neq 0$. We should remember that the second highest coefficient of the minimal polynomial of an element over a field is just the trace of the element.

($\Rightarrow$) Suppose that $\theta(\mathfrak{D}_{u(L)}3,\theta_{tr}) \neq 0$. We have immediately that $tr(ab) = tr(ba), tr((ab)c) = tr(a(bc)), \forall a, b, c \in u(L)$ and $tr(a^p b^p) = (tr(ab))^p$. Furthermore we have that $\forall u, v, w \in \theta^{u(L)} u(L)$, and $\theta tr(uv) = \theta tr(vu), \theta tr((uv)w) = \theta tr(u(vw))$ and $\theta tr(u^p v^p) = \{\theta tr(uv)\}^p$ since $\theta$ is a specialization.

By the given condition we have $\theta(\mathfrak{D}_{u(L)},\theta_{tr}) = \mathfrak{D}_{\theta^{u(L)} \theta \mathfrak{F},\theta_{tr}} \neq 0$, so that $\theta_{tr}$ turns out to be a nondegenerate symmetric bilinear form $\overline{\theta_{tr}}$ on $\theta^{u(L)} u(L) / \mathfrak{A}$ over $\theta \mathfrak{F}$.
satisfying $\theta(x, y) = \{\theta(x, y)\}^p$, and hence $\theta(x, y) = \{\theta(x, y)\}^p$, $\forall u, v \in \theta(u(L))u(L)/\mathfrak{A}$ and the dimension of $\theta(u(L))$ over $\mathfrak{A}$ equals $p^2m$, where $\mathfrak{A}$ indicates a two-sided ideal of $\theta(u(L))u(L)$ consisting of all element $u \in \theta(u(L))u(L)$ which satisfies the condition $\theta(x, y) = 0, \forall v \in \theta(u(L))u(L)$.

Moreover it is easy to see that $\dim \theta(u(L))u(L)/\mathfrak{A} = p^2m$ over $\mathfrak{A}$ considering the proofs of the above propositions.

Next for any element $\bar{x}$ of the Jacobson (or Wedderburn) radical of $\theta(u(L))$, $\bar{x}^j = 0$ for some $j$. Hence we have $\{\theta(x, y)\}^p = \theta(x, y) = 0$, and so $\forall v \in \theta(u(L))u(L)/\mathfrak{A}$, $\theta(x, y) = 0$.

We thus get $\bar{x} = 0$, and hence it follows that $\theta(u(L))u(L)$ is semisimple over $\mathfrak{A}$. Such an argument still holds for any extension of the ground field $\theta(u(L))u(L)/\mathfrak{A}$, and so $\theta(u(L))u(L)/\mathfrak{A}$ also becomes semisimple over $\mathfrak{A}$, i.e., $\theta(u(L))u(L)/\mathfrak{A}$ is separable over $\mathfrak{A}$.

By the way we know from the preceding proposition that the degree of a minimal polynomial of $\theta(u(L)) \leq p^m$, which holds a fortiori for $\theta(u(L))u(L)/\mathfrak{A}$.

From proposition 3.16(iii), we have that $\theta(u(L))/\mathfrak{A}$ becomes separable if and only if it is centrally simple of the same dimension $p^2m$ over $\mathfrak{A}$.

Hence we get $\mathfrak{A} = 0$, i.e., $\theta(u(L))u(L)$ over $\mathfrak{A}$ must be centrally simple of dimension $p^2m$. 
We had better also remember that if the determinant $|\text{tr}(b_i, b_j)|$ as in the proof of proposition 3.16, is nonzero, then $\{b_1, b_2, \cdots, b_r\}$ becomes a basis of the free $R$-module $M$. \hfill \square

**Corollary 3.18.** An irreducible representation $\theta^{u(L)}$ of the Lie algebra $L$ has its representation space of dimension $p^{2m}$ if and only if $\theta^{u(L)}(\mathfrak{D}_{u(L),3,\text{tr}}) \neq 0$.

**Proof.** By virtue of the preceding propositions 3.16 and 3.17, our claim is straightforward. \hfill \square

Now suppose that $V$ is any finite dimensional irreducible $L$-module and that $V$ has its associated irreducible representation $\rho_\chi: u(L) \to \text{End}_F(V)$ with a character $\chi \in L^*$ such that for any $x \in L$, $\rho_\chi(x)^p - \rho_\chi(x[x^p]) = \chi(x)^p \text{id}_V$.

We should note that $\rho_\chi(u(L))$ becomes dense in $\text{End}_\Delta(V)$ by dint of Jacobson’s density theorem, where $\Delta := \text{End}_{u(L)}(V)$. By Schur’s lemma we have $\Delta = F$.

By the general theory of modular Lie algebra $[Q(u(L)) : Q(\mathfrak{Z}(u(L)))] = p^{2m}$ for some integer $m$ with $2m \leq n$.

Here $\mathfrak{Z} := \mathfrak{Z}(U(L))$ denotes the center of $U(L)$ as before and $Q$ denotes the relevant quotient algebra of noncommutative algebras $U(L)$ and the commutative algebra $\mathfrak{Z}(U(L))$ respectively. So it follows that $\rho_\chi(u(L)) \cong u(L)/\ker \rho_\chi \cong M_m(F)$, the full matrix algebra $[SF]$.

Hence it is sometimes very convenient for us to express the basis of this factor algebra related to classical type Lie algebra by the representatives of the form: $\{\otimes_{i=1}^{2m}(B_i + A_i)^j : 0 \leq j \leq p - 1\}$, where $B_i$ is an element in the CSA of $L$ and $A_i$'s are
elements of $u(L)\forall i = 1, \cdots, 2m$. We shall call such a form
Lee’s basis of the Lie algebra concerned.

4. FOUR KINDS OF POINTS

Now let $L$ be a finite dimensional restricted Lie algebra with
a $p$- mapping $[p]$ over an algebraically closed field of nonzero
characteristic $p$ and with a basis $\{x_1, x_2, \cdots, x_n\}$ which is cen-
terless and indecomposable.

Suppose further that the center $Z(u(L))$ of $u(L)$ has the
Noether normalization form $F[x_1 - x_1^{[p]}, x_2 - x_2^{[p]}, \cdots, x_n - x_n^{[p]}, s_1, s_2, \cdots, s_k],$
where $s_i$’s are algebraic over the field generated by alge-
braically independent elements $\{x_i - x_i^{[p]} : 1 \leq i \leq n\}$.

Any maximal ideal of $u(L)$ must contain a certain ideal of
the form $\{\sum_{i=1}^{n} u(L)(x_i - x_i^{[p]} - \xi_i)\} + \sum_{j=1}^{k} u(L)(s_j - \mu_j)$,
where $\xi$’s and $\mu_j$’s are some constants in $F$.

We defined four kinds of points in [KY-4] and shall exhibit
some explanation therein as follows:

Definition 4.1. We consider four possible cases and give
names associated with points $(\xi_1, \cdots, \xi_n, \mu_1, \cdots, \mu_k)$ on Zassen-
haus variety obtained from the center $Z := Z(u(L))$ of $u(L)$.

[I] If all $\xi_i = 0$, then there may exist finitely many left
maximal ideals $\rho_l$ containing $\{\sum_{i=1}^{n} u(L)(x_i - x_i^{[p]} - 0)\} + \sum_{j=1}^{k} u(L)(s_j - \mu_j)$, so that $u(L)/\rho_l$ becomes $p$- represen-
tation modules for $L$ with dimension $\leq p^m$. We shall call such a
point \((0, \cdots, 0, \mu_1, \cdots, \mu_k)\) a \(p\)-point.

In particular if it is a \(p\)-point and its associated irreducible module has dimension \(p^m\), then we call the point a regular \(p\)-point.

[II] If not all \(\xi_i\)'s are zero, i.e., \(\xi_i \neq 0\) for some \(i\), then there are finitely many left maximal ideals \(\rho_l\) containing \(\{\sum_{i=1}^n u(L)(x_i - x_i^{[p]} - \xi_i)\} + \sum_{j=1}^k u(L)(s_j - \mu_j)\), so that \(u(L)/\rho_l\) become irreducible modules for \(L\) with dimension \(\leq p^m\).

If all these irreducible modules become of dimension \(p^m\), then they are isomorphic \(L\)-irreducible modules.

In this case we call such a point \((\xi_1, \cdots, \xi_n, \mu_1, \cdots, \mu_k)\) a regular point. We shall denote the set of all regular points by \(R(L,p,\chi)\).

For all left maximal ideals \(\rho_l\) containing \(\{\sum_{i=1}^n u(L)(x_i - x_i^{[p]} - \xi_i)\} + \sum_{j=1}^k u(L)(s_j - \mu_j)\), \(u(L)/\rho_l\) may have dimension less than \(p^m\) and are possibly nonisomorphic.

We shall call such a point subregular-point. We shall denote the set of all subregular points by \(S(L,p,\chi)\). We should keep track of the relation of a point and its associated character \(\chi\).

For our examples of such points mentioned above, we recapitulate important facts relating to \(A_l, B_l\), and \(D_l\)-type modular Lie algebras.
The $A_l$-type Lie algebra over $\C$ has its root system $\Phi=\{\epsilon_i - \epsilon_j | 1 \leq i \neq j \leq l+1\}$, where $\epsilon_i$'s are orthonormal unit vectors in the Euclidean space $\R^{l+1}$. The base of $\Phi$ is equal to $\{\epsilon_i - \epsilon_{i+1} | 1 \leq i \leq l\}$.

We let $L$ be an $A_l$-type simple Lie algebra over an algebraically closed field of characteristic $p \geq 7$.

For a root $\alpha \in \Phi$, we put $g_\alpha := x_\alpha^{p-1} - x_{-\alpha}$ and $w_\alpha := (h_\alpha + 1)^2 + 4x_{-\alpha}x_\alpha$.

We have seen from [KC] and [KY-4] that any $A_l$-type modular Lie algebra over $F$ becomes a Park’s Lie algebra. However we would like to specify the proof or prove it in a different way when $\chi(H) \neq 0$ for a CSA $H$ of $L$.

**Proposition 4.2.** Suppose that $\chi$ is a character of any simple $L$-module with $\chi(h_\alpha) \neq 0$ for some $\alpha \in$ the base of $\Phi$, where $h_\alpha$ is an element in the Chevalley basis of $L$ such that $Fx_\alpha + Fx_{-\alpha} + Fh_\alpha = \mathfrak{sl}_2(F)$ with $[x_\alpha, x_{-\alpha}] = h_\alpha \in H$ (a CSA of $L$).

We then have that the dimension of any simple $L$-module with character $\chi = p^m = p^{\frac{(n-l)}{2}}$, where $n = \dim L = 2m + l$ for $H$ with $\dim H = l$.

**Proof.** If $\chi(x_\alpha) \neq 0$ or $\chi(x_{-\alpha}) \neq 0$, then our assertion is evident from [KC],[KY-4]. So we may assume that $\chi(x_\alpha) = \chi(x_{-\alpha}) = 0$ but $\chi(h_\alpha) \neq 0$.

Furthermore we may put $\alpha = \epsilon_1 - \epsilon_2$ without loss of generality since all roots are conjugate under the Weyl group of $\Phi$. 
Since the case for \( l = 1 \) is trivial, we may assume \( l \geq 2 \).
For \( i = 1,2,\ldots \), we put \( B_i := b_{i1}h_{\epsilon_1-\epsilon_2} + \cdots + b_{il}h_{\epsilon_l-\epsilon_{l+1}} \) as in [KC],[KY-4] and we put \( \mathfrak{B} := \{(B_1 + A_{\epsilon_1-\epsilon_2})^{i_1} \otimes (B_2 + A_{\epsilon_2-\epsilon_1})^{i_2} \otimes (\otimes_{j=3}^{l+1}(B_j + A_{\epsilon_j-\epsilon_{j-1}})^{i_j}) \otimes (\otimes_{j=3}^{l+1}(B_{2l-2+j} + A_{\epsilon_{2l-3+j}-\epsilon_j})^{i_{2l-3+j}} \otimes \cdots \otimes (B_{2m-1} + A_{\epsilon_{2l-2m+1}-\epsilon_{l+1}})^{i_{2m-1}} \otimes (B_{2m} + A_{\epsilon_{2l-2m+2}-\epsilon_j})^{i_{2m}} \} \) for \( 0 \leq i_j \leq p - 1 \),

where we set

\[
A_{\epsilon_1-\epsilon_2} = g_\alpha = g_{\epsilon_1-\epsilon_2} = x_{\epsilon_1-\epsilon_2}^{p-1} - x_{\epsilon_2-\epsilon_1},
\]

\[
A_{\epsilon_2-\epsilon_1} = c_{\epsilon_2-\epsilon_1} + (h_\alpha + 1)^2 + 4^{-1}x_{\alpha}x_{\alpha},
\]

\[
A_{\epsilon_1-\epsilon_3} = g_\alpha^2(c_{\epsilon_1-\epsilon_3} + x_{\epsilon_2-\epsilon_3}x_{\epsilon_3-\epsilon_2} \pm x_{\epsilon_1-\epsilon_3}x_{\epsilon_3-\epsilon_1}),
\]

\[
A_{\epsilon_3-\epsilon_1} = g_\alpha^3(c_{\epsilon_3-\epsilon_1} + x_{\epsilon_2-\epsilon_3}x_{\epsilon_3-\epsilon_2} \pm x_{\epsilon_3-\epsilon_1}x_{\epsilon_1-\epsilon_3}) \text{ or } x_{\epsilon_3-\epsilon_4}(c_{\epsilon_3-\epsilon_1} + x_{\epsilon_3-\epsilon_2}x_{\epsilon_2-\epsilon_3} \pm x_{\epsilon_3-\epsilon_1}x_{\epsilon_3-\epsilon_2}),
\]

\[
A_{\epsilon_3-\epsilon_2} = g_\alpha^4(c_{\epsilon_3-\epsilon_2} + x_{\epsilon_2-\epsilon_3}x_{\epsilon_3-\epsilon_2} \pm x_{\epsilon_3-\epsilon_2}x_{\epsilon_3-\epsilon_1}) \text{ (if } j = 3 \text{) or } x_{\epsilon_4-\epsilon_3}(c_{\epsilon_3-\epsilon_2} + x_{\epsilon_3-\epsilon_2}x_{\epsilon_2-\epsilon_3} \pm x_{\epsilon_3-\epsilon_2}x_{\epsilon_3-\epsilon_1}),
\]

\[
A_{\epsilon_j-\epsilon_3} = g_\alpha^5(c_{\epsilon_j-\epsilon_3} + x_{\epsilon_2-\epsilon_3}x_{\epsilon_3-\epsilon_2} \pm x_{\epsilon_j-\epsilon_3}x_{\epsilon_3-\epsilon_1}) \text{ (if } j = 3 \text{) or } x_{\epsilon_j-\epsilon_4}(c_{\epsilon_j-\epsilon_3} + x_{\epsilon_j-\epsilon_2}x_{\epsilon_2-\epsilon_3} \pm x_{\epsilon_j-\epsilon_3}x_{\epsilon_3-\epsilon_2}),
\]

\[
A_{\epsilon_2-\epsilon_4} = x_{\epsilon_3-\epsilon_4}(c_{\epsilon_2-\epsilon_4} + x_{\epsilon_2-\epsilon_4}x_{\epsilon_4-\epsilon_2} \pm x_{\epsilon_1-\epsilon_4}x_{\epsilon_4-\epsilon_1}),
\]

\[
A_{\epsilon_4-\epsilon_2} = x_{\epsilon_4-\epsilon_3}(c_{\epsilon_4-\epsilon_2} + x_{\epsilon_4-\epsilon_2}x_{\epsilon_2-\epsilon_4} \pm x_{\epsilon_4-\epsilon_1}x_{\epsilon_1-\epsilon_4}),
\]

\[
A_{\epsilon_1-\epsilon_j} = x_{\epsilon_3-\epsilon_j}(c_{\epsilon_1-\epsilon_j} + x_{\epsilon_1-\epsilon_j}x_{\epsilon_j-\epsilon_1} \pm x_{\epsilon_3-\epsilon_j}x_{\epsilon_j-\epsilon_2}).
\]
\[ A_{\epsilon_j - \epsilon_1} = x_{\epsilon_j - \epsilon_3}^2 (c_{\epsilon_j - \epsilon_1} + x_{\epsilon_1 - \epsilon_j} x_{\epsilon_3 - \epsilon_1} \pm x_{\epsilon_2 - \epsilon_j} x_{\epsilon_3 - \epsilon_2}), \]

\[ A_{\epsilon_i - \epsilon_j} = x_{\epsilon_i - \epsilon_j}^2 \text{ or } x_{\epsilon_i - \epsilon_j}^3 \text{ for other roots } \epsilon_i - \epsilon_j, \]

where signs are chosen so that they may commute with \( x_\alpha \) and \( c_\beta \) are chosen so that \( A_{\epsilon_2 - \epsilon_1} \) and parentheses are invertible in \( U(L)/\mathfrak{m}_\chi \) for the kernel \( \mathfrak{m}_\chi \) in \( U(L) \) of any given simple representation of \( L \) with the character \( \chi \).

We may see without difficulty that \( \mathfrak{B} \) is a linearly independent set in \( U(L) \) by virtue of P-B-W theorem.

We shall prove that a nontrivial linearly dependent equation leads to absurdity. We assume first that we have a dependence equation which is of least degree with respect to \( h_{\alpha_j} \in H \) and the number of whose highest degree terms is also least.

In case it is conjugated by \( x_\alpha \), then there arises a nontrivial dependence equation of lower degree than the given one, which contradicts to our assumption.

Otherwise we have to prove that

(i) \( x_{\epsilon_l - \epsilon_k} K + K' \in \mathfrak{m}_\chi \) with \( l, k \neq 1, 2 \)

(ii) \( g_\alpha K + K' \in \mathfrak{m}_\chi \)

lead to a contradiction, where both \( K \) and \( K' \) commute with \( x_{\pm \alpha} \) modulo \( \mathfrak{m}_\chi \). In particular \( K \) commute with \( g_\alpha \).
For the case (i), we may change it to the form $x_\alpha K + K'' \in M_\chi$ for some $K''$ commuting with $x_\alpha = x_{e_1 e_2}$ modulo $M_\chi$.

So we have $x_\alpha^p K + x_\alpha^{p-1} K'' \equiv 0$, thus $x_\alpha^{p-1} K'' \equiv 0$.

Subtracting from this $x_{-\alpha} x_\alpha K + x_{-\alpha} K'' \equiv 0$, we get

$$-x_{-\alpha} x_\alpha K + g_\alpha K'' \equiv 0.$$  

Recall here that $g_\alpha$ is invertible and $w_\alpha$ belongs to the center of $U(sl_2(F))$ according to [RS].

So we get $4^{-1} \{(h_\alpha + 1)^2 - w_\alpha\} K + g_\alpha K'' \equiv 0$, and hence

$$(*) g_\alpha^{p-1} 4^{-1} \{(h_\alpha + 1)^2 - w_\alpha\} K + c K'' \equiv 0$$

is obtained and from the start equation we have

$$(**) c x_\alpha K + c K'' \equiv 0,$$  

where $g_\alpha^p - c \equiv 0$.

Subtracting $(**)$ from $(*)$, we have $4^{-1} g_\alpha^{p-1} \{(h_\alpha + 1)^2 - w_\alpha\} K - c x_\alpha K \equiv 0$.

Multiplying this equation by $g_\alpha^{1-p}$ to the right, we obtain

$4^{-1} g_\alpha^{p-1} \{(h_\alpha + 1)^2 - w_\alpha\} g_\alpha^{1-p} K - c x_\alpha g_\alpha^{1-p} K \equiv 0$

We thus have $4^{-1} \{(h_\alpha + 1 - 2)^2 - w_\alpha\} K - x_\alpha g_\alpha K \equiv 0$.

So it follows that $4^{-1} \{(h_\alpha - 1)^2 - w_\alpha\} K + x_\alpha x_{-\alpha} K \equiv 0$.

Next multiplying $x_{-\alpha}^{p-1}$ to the right of this last equation, we obtain $\{(h_\alpha - 1)^2 - w_\alpha\} K x_{-\alpha}^{p-1} \equiv 0$. Now multiply $x_\alpha$ in turn consecutively to the left of this equation until it becomes of
the form

\((a\text{ nonzero polynomial of degree } \geq 1 \text{ with respect to } h_\alpha)K \in \mathcal{M}_\chi,\)

which comes from the fact that the intersection of \(\mathcal{M}_\chi\) and the commutative algebra
\(\mathfrak{Z}(u(L))[h_\alpha, w_\alpha]\) becomes a prime ideal of \(\mathfrak{Z}(u(L))[h_\alpha, w_\alpha]\), which is generated by \(h_\alpha, w_\alpha\) over \(\mathfrak{Z}(u(L))\).

By making use of conjugation by \(x_\alpha\) and subtraction consecutively, we are led to a contradiction \(K \in \mathcal{M}_\chi\).

Finally for the case (ii), we consider \(K + g_\alpha^{-1}K' \in \mathcal{M}_\chi\). So we have \(x_\alpha K + x_\alpha g_\alpha^{-1}K' \equiv 0 \mod \mathcal{M}_\chi\).

By analogy with the argument as in the case (i), we obtain a contradiction \(K \in \mathcal{M}_\chi\). \(\square\)

Next we note first that the orthogonal Lie algebra of \(B_l\)-type with rank \(l\), i.e., the \(B_l\)-type Lie algebra over \(\mathbb{C}\) has its root system \(\Phi=\{\pm \epsilon_i \text{ (of squared length 1); } \pm (\epsilon_i \pm \epsilon_j) \text{ (of squared length 2)} | 1 \leq i \neq j \leq l \}\), where \(\epsilon_i, \epsilon_j\) are linearly independent orthonormal unit vectors in \(\mathbb{R}^l\) with \(l \geq 2\).

The base of \(\Phi\) equals \(\{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \cdots, \epsilon_{l-1} - \epsilon_l, \epsilon_l\}\).

For an algebraically closed field \(F\) of prime characteristic \(p \geq 7\), the \(B_l\)-type Lie algebra \(L\) over \(F\) is just the analogue over \(F\) of the \(B_l\)-type simple Lie algebra over \(\mathbb{C}\).
In other words the $B_l$-type Lie algebra over $F$ is isomorphic to the Chevalley Lie algebra of the form $\sum_{i=1}^n \mathbb{Z}c_i \otimes \mathbb{Z} F$,

where $n= \dim_F L$ and $x_\alpha = \text{some } c_i$ for each $\alpha \in \Phi$, $h_\alpha = \text{some } c_j$ with $\alpha$ some base element of $\Phi$ for a Chevalley basis $\{c_i\}$ of the $B_l$-type Lie algebra over $\mathbb{C}$.

Of course we are well aware that $\mathfrak{sl}_2(F) = Fx_\alpha + Fx_{-\alpha} + Fh_\alpha$, where $h_\alpha = [x_\alpha, x_{-\alpha}]$.

**Proposition 4.3.** Let $\alpha$ be any root in the root system $\Phi$ of $L$. If $\chi(x_\alpha) \neq 0$, then $\dim_F \rho_\chi(U(L)) = p^{2m}$, where $[Q(U(L)): Q(3)] = p^{2m} = p^{n-l}$ with $3$ the center of $U(L)$ and $Q$ denotes the quotient algebra.

So the simple module corresponding to this representation has $p^m$ as its dimension and the Lee’s basis for this simple module is obtained as follows.

We meet with 2 cases of root length.

(I) Suppose that $\alpha$ is a short root. Since all roots of a given length are conjugate under the Weyl group of $\Phi$, we may put $\alpha = \epsilon_1$ without loss of generality.

Let us put $B_i = b_{i1}h_{\epsilon_1} + b_{i2}h_{\epsilon_2} + \cdots + b_{i,l-1}h_{\epsilon_{l-1}} - \epsilon_i + b_i h_{\epsilon_i}$ for $i = 1, 2, \ldots, 2m$, where $(b_{i1}, \cdots, b_{il}) \in F^l$ are chosen so that arbitrary $(l+1) - B_i$'s are linearly independent in $\mathbb{P}^l(F)$, the $B_i$ below becomes an $F$-linearly independent set in $U(L)$ if necessary and $x_\alpha B_i \neq B_i x_\alpha$ with $\alpha = \epsilon_1$. 
Let $\mathfrak{M}_\chi$ be the kernel of the irreducible representation $\rho_\chi : L \to \mathfrak{gl}(V)$ of the restricted Lie algebra $(L, [\rho])$ associated with any given irreducible $L$-module $V$ with a character $\chi$.

In $U(L)/\mathfrak{M}_\chi$ we give a basis as $\mathfrak{B}:=\{(B_1 + A_{\epsilon_1})^{i_1} \otimes (B_2 + A_{-\epsilon_1})^{i_2} \otimes (B_3 + A_{\epsilon_1 - \epsilon_2})^{i_3} \otimes (B_4 + A_{-(\epsilon_1 - \epsilon_2)})^{i_4} \otimes \cdots \otimes (B_{2l} + A_{-(\epsilon_l - \epsilon_{l-1})})^{i_{2l}} \otimes (B_{2l+1} + A_{\epsilon_l})^{i_{2l+1}} \otimes (B_{2l+2} + A_{-\epsilon_l})^{i_{2l+2}} \otimes \epsilon_j^{2m} (B_j + A_{\alpha_j})^{i_j} \}$ for $0 \leq i_j \leq p - 1$,

where we put

$A_{\epsilon_1} = x_{\epsilon_1},$

$A_{-\epsilon_1} = c_{-\epsilon_1} + (h_{\epsilon_1} + 1)^2 + 4x_{-\epsilon_1}x_{\epsilon_1},$

$A_{-\epsilon_1 \pm \epsilon_2} = x_{\epsilon_1 \pm \epsilon_2}(c_{-\epsilon_1 \pm \epsilon_2} + x_{-\epsilon_1 \pm \epsilon_2}x_{-(\epsilon_1 \pm \epsilon_2)} \pm x_{\epsilon_1 \pm \epsilon_2}x_{-(\epsilon_1 \pm \epsilon_2)} \pm x_{\epsilon_1 \pm \epsilon_2}x_{-(\epsilon_1 \pm \epsilon_2)}),$

$A_{-\epsilon_1 \pm \epsilon_j} = x_{-\epsilon_1 \pm \epsilon_j}(c_{-\epsilon_1 \pm \epsilon_j} + x_{\epsilon_1 \pm \epsilon_j}x_{-(\epsilon_1 \pm \epsilon_j)} \pm x_{\epsilon_1 \pm \epsilon_j}x_{-(\epsilon_1 \pm \epsilon_j)} \pm x_{\epsilon_1 \pm \epsilon_j}x_{-(\epsilon_1 \pm \epsilon_j)}),$

$A_{\pm \epsilon_2} = x_{\epsilon_2 \pm \epsilon_2}^2(c_{\pm \epsilon_2} + x_{\epsilon_2}x_{-\epsilon_2} \pm x_{\epsilon_1 + \epsilon_2}x_{-(\epsilon_1 + \epsilon_2)} \pm x_{\epsilon_2 - \epsilon_1}x_{\epsilon_1 - \epsilon_2}),$

$A_{\epsilon_j} = x_{\epsilon_2 + \epsilon_j}(c_{\epsilon_j} + x_{\epsilon_j}x_{-\epsilon_j} \pm x_{\epsilon_1 + \epsilon_j}x_{-(\epsilon_1 + \epsilon_j)} \pm x_{\epsilon_j - \epsilon_1}x_{\epsilon_1 - \epsilon_j}),$

$A_{-\epsilon_j} = x_{\epsilon_2 - \epsilon_j}(c_{-\epsilon_j} + x_{-\epsilon_j}x_{\epsilon_j} \pm x_{\epsilon_1 - \epsilon_j}x_{-(\epsilon_1 - \epsilon_j)} \pm x_{-\epsilon_j - \epsilon_1}x_{\epsilon_1 + \epsilon_j}),$

with the sign chosen so that they commute with $x_\alpha$ and with $c_\beta \in F$ chosen so that $A_{-\epsilon_1}$ and parentheses( ) are invertible. For any other root $\beta$, we put $A_\beta = x_\beta^2$ or $x_\beta^3$ if possible. Otherwise we make use of the parentheses( ) again used for
designating \( A_{-\beta} \). So in this case we put \( A_\beta = x_\gamma^2 \) or \( x_\gamma^3 \) attached to these ( ) so that \( x_\alpha \) may commute with \( A_\beta \).

(II) Suppose that \( \alpha \) is a long root; then we may put \( \alpha = \epsilon_1 - \epsilon_2 \) since all roots of the same length are conjugate under the Weyl group of \( \Phi \).

Similarly as in (I), we put \( B_i := \) the same as in (I) except that \( \alpha = \epsilon_1 - \epsilon_2 \) this time instead of \( \epsilon_1 \).

In \( U(L)/M_{\chi} \) we have a basis \( \mathcal{B} := \{(B_1 + A_{\epsilon_1 - \epsilon_2})^{i_1} \otimes (B_2 + A_{-(\epsilon_1 - \epsilon_2)})^{i_2} \otimes \cdots \otimes (B_{2l-2} + A_{-(\epsilon_l-1 - \epsilon_l)})^{i_{2l-2}} \otimes (B_{2l-1} + A_{\epsilon_l})^{i_{2l-1}} \otimes (B_{2l} + A_{-\epsilon_l})^{i_{2l}} \otimes (\otimes_{j=2l+1}^{2m} (B_j + A_{\alpha_j})^{i_j})\mid 0 \leq i_j \leq p - 1\}\),

where we put

\[
A_{\epsilon_1 - \epsilon_2} = x_\alpha = x_{\epsilon_1 - \epsilon_2},
\]

\[
A_{\epsilon_2 - \epsilon_1} = c_{\epsilon_2 - \epsilon_1} + (h_{\epsilon_1 - \epsilon_2} + 1)^2 + 4x_{\epsilon_2 - \epsilon_1}x_{\epsilon_1 - \epsilon_2}
\]

\[
A_{\epsilon_2 \pm \epsilon_3} = x_{\epsilon_2 \pm \epsilon_3}(c_{\epsilon_2 \pm \epsilon_3} + x_{\epsilon_2 \pm \epsilon_3}x_{-(\epsilon_2 \pm \epsilon_3)} \pm x_{\epsilon_1 \pm \epsilon_3}x_{-(\epsilon_1 \pm \epsilon_3)}),
\]

\[
A_{\epsilon_2 \pm \epsilon_k} = x_{\epsilon_2 \pm \epsilon_k}(c_{\epsilon_2 \pm \epsilon_k} + x_{\epsilon_2 \pm \epsilon_k}x_{-(\epsilon_2 \pm \epsilon_k)} \pm x_{\epsilon_1 \pm \epsilon_k}x_{-(\epsilon_1 \pm \epsilon_k)})
\]

if \( k \neq 1 \),

\[
A_{\epsilon_2} = x_{\epsilon_2}(c_{\epsilon_2} + x_{\epsilon_2}x_{-(\epsilon_2)} \pm x_{\epsilon_1}x_{-(\epsilon_1)}),
\]

\[
A_{-\epsilon_1} = x_{-\epsilon_2}(c_{-\epsilon_1} + x_{-\epsilon_1}x_{\epsilon_1} \pm x_{-\epsilon_2}x_{\epsilon_2}),
\]
\[ A_{-(\epsilon_1 \pm \epsilon_3)} = x_{-(\epsilon_1 \pm \epsilon_3)}(c_{-(\epsilon_1 \pm \epsilon_3)} + x_{\epsilon_2 \pm \epsilon_3} x_{-(\epsilon_2 \pm \epsilon_3)} \pm x_{\epsilon_1 \pm \epsilon_3} x_{-(\epsilon_1 \pm \epsilon_3)}), \]
\[ A_{-(\epsilon_1 \pm \epsilon_k)} = x_{-(\epsilon_1 \pm \epsilon_k)}(c_{-(\epsilon_1 \pm \epsilon_k)} + x_{\epsilon_2 \pm \epsilon_k} x_{-(\epsilon_2 \pm \epsilon_k)} \pm x_{\epsilon_1 \pm \epsilon_k} x_{-(\epsilon_1 \pm \epsilon_k)}), \]
\[ A_{\epsilon_l} = x_{\epsilon_l}^2, \]
\[ A_{-\epsilon_l} = x_{-\epsilon_l}^2, \]

with the signs chosen so that they may commute with \( x_\alpha \) and with \( c_\beta \in F \) chosen so that \( A_{\epsilon_2 - \epsilon_1} \) and parentheses are invertible.
For any other root \( \beta \), we put \( A_\beta = x_\beta^3 \) or \( x_\beta^4 \) if possible.

Otherwise we make use of the parentheses( ) again used for designating \( A_{-\beta} \). So in this case we put \( A_\beta = x_\gamma^2 \) or \( x_\gamma^3 \) attached to these ( ) so that \( x_\alpha \) may commute with \( A_\beta \).

**Proof.** Refer to proposition 2.1 in [KY-6]. \(\square\)

**Proposition 4.4.** Let \( \chi \) be a character of any simple \( L \)-module with \( \chi(h_\alpha) \neq 0 \) for some \( \alpha \in \Phi \), where \( h_\alpha \) is an element in the Chevalley basis of \( L \) such that \( Fx_\alpha + Fh_\alpha + Fx_{-\alpha} = \mathfrak{sl}_2(F) \) with \( [x_\alpha, x_{-\alpha}] = h_\alpha \in H \).

We then have that any simple \( L \)-module with character \( \chi \) is of dimension \( p^m = p^{\frac{n-l}{2}} \), where \( n = \dim L = 2m + l \) for a CSA \( H \) with \( \dim H = l \) and the Lee’s basis for this simple module is given as follows.
Let $\mathcal{M}_\chi$ be the kernel of this irreducible representation; so $\mathcal{M}_\chi$ is a certain (2-sided) maximal ideal of $U(L)$.

(I) Assume first that $\alpha$ is a short root; then we may put $\alpha = \epsilon_1$ without loss of generality since all roots of a given length are conjugate under the Weyl group of the root system $\Phi$.

If $\chi(x_{\epsilon_1}) \neq 0$ or $\chi(x_{-\epsilon_1}) \neq 0$, then our assertion is evident from proposition 2.1 in [KY-4].

So we may let $x_{\epsilon_1}^p \equiv x_{-\epsilon_1}^p \equiv 0$ modulo $\mathcal{M}_\chi$.

We let $B_i := b_{i1} h_{\epsilon_1} - h_{\epsilon_2} + b_{i2} h_{\epsilon_2} - \cdots + b_{i,l-1} h_{\epsilon_{l-1}} - \epsilon_l + b_{il} h_{\epsilon_l}$ for $i = 1, 2, \cdots, 2m$, where $(b_{i1}, b_{i2}, \cdots, b_{il}) \in F^l$ are chosen so that any $(l + 1) - B_i$'s are linearly independent in $F^l$ and $\mathcal{B}$ below becomes an $F$-linearly independent set in $U(L)$ if necessary and $x_{\alpha} B_i \neq B_i x_{\alpha}$ for $\alpha = \epsilon_1$.

In $U(L)/\mathcal{M}_\chi$ we claim that we have a basis

\[
\mathcal{B} := \{(B_1 + A_{\epsilon_1})^{i_1} \otimes (B_2 + A_{-\epsilon_1})^{i_2} (B_3 + A_{\epsilon_1 - \epsilon_2})^{i_3} \otimes (B_4 + A_{-(\epsilon_1 - \epsilon_2)})^{i_4} \otimes \cdots \otimes (B_{2l} + A_{-(\epsilon_{l-1} - \epsilon_l)})^{i_{2l}} \otimes (B_{2l+1} + A_{\epsilon_l})^{i_{2l+1}} \otimes (B_{2l+2} + A_{-\epsilon_l})^{i_{2l+2}} \otimes (\otimes_{j=2l+3}^{2m} (B_{j} + A_{\alpha_j})^{i_j}) | 0 \leq i_j \leq p - 1\},
\]

where we put

$A_{\epsilon_1} = g_{\alpha} = g_{\epsilon_1},$

$A_{-\epsilon_1} = c_{-\epsilon_1} + (h_{\epsilon_1} + 1)^2 + 4x_{-\alpha} x_{\alpha},$

$A_{-\epsilon_1 \pm \epsilon_2} = x_{\epsilon_1 \pm \epsilon_2} (c_{-\epsilon_1 \pm \epsilon_2} + x_{-\epsilon_1 \pm \epsilon_2} x_{\epsilon_1 \pm \epsilon_2} \pm x_{\epsilon_1 \pm \epsilon_2} x_{-\epsilon_2} \pm x_{\epsilon_1 \pm \epsilon_2} x_{-\epsilon_1 \pm \epsilon_2}),$
A_{\epsilon_1+\epsilon_2} = x^{2}_{\epsilon_3-\epsilon_2} \left( c_{\epsilon_1+\epsilon_2} + x_{-\epsilon_1-\epsilon_2} x_{\epsilon_1+\epsilon_2} \pm x_{-\epsilon_1+\epsilon_2} x_{\epsilon_1-\epsilon_2} x_{-(\epsilon_1-\epsilon_2)} \right),

A_{-\epsilon_1+\epsilon_2} = x_{-\epsilon_2+\epsilon_1} \left( c_{-\epsilon_1+\epsilon_2} + x_{-\epsilon_1+\epsilon_2} x_{-\epsilon_1+\epsilon_2} \pm x_{-\epsilon_1-\epsilon_2} x_{\epsilon_1-\epsilon_2} x_{-(\epsilon_1-\epsilon_2)} \right),

A_{\epsilon_1+\epsilon_2} = x^{2}_{-\epsilon_2-\epsilon_1} \left( c_{\epsilon_1+\epsilon_2} + x_{-\epsilon_1-\epsilon_2} x_{\epsilon_1+\epsilon_2} \pm x_{-\epsilon_1+\epsilon_2} x_{\epsilon_1-\epsilon_2} x_{-(\epsilon_1-\epsilon_2)} \right),

A_{\pm\epsilon_2} = x^{2}_{\epsilon_3-\epsilon_2} \left( c_{\pm\epsilon_2} + x_{\epsilon_2} x_{-\epsilon_2} \pm x_{\epsilon_1+\epsilon_2} x_{-\epsilon_1-\epsilon_2} + x_{\epsilon_2-\epsilon_1} x_{\epsilon_1-\epsilon_2} \right),

A_{\pm\epsilon_2} = x_{\epsilon_2+\epsilon_2} \left( c_{\pm\epsilon_2} + x_{\epsilon_2} x_{-\epsilon_2} \pm x_{\epsilon_1+\epsilon_2} x_{-\epsilon_1-\epsilon_2} + x_{\epsilon_2-\epsilon_1} x_{\epsilon_1-\epsilon_2} \right),

with the sign chosen so that they commute with \( x_\alpha \) and with \( c_\alpha \in F \) chosen so that \( A_{-\epsilon_1} \) and parentheses are invertible. For any other root \( \beta \) we put \( A_\beta = x^2_\beta \) or \( x^3_\beta \) or \( x^4_\beta \) if possible.

Otherwise attach to these sorts the parentheses( ) used for designating \( A_{-\beta} \) so that \( A_{\gamma} \forall \gamma \in \Phi \) may commute with \( x_\alpha \).

(II) Next we assume that \( \alpha \) is a long root; we may thus let \( \alpha = \epsilon_1 - \epsilon_2 \) without loss of generality.

As we have done before, we let \( B_i \) be defined as in (I).

We have a Lee’s basis in \( U(L)/\mathcal{M}_\chi \) as \( \mathcal{B} := \{(B_1 + A_{\epsilon_1-\epsilon_2})^{i_1} \otimes (B_2 + A_{-(\epsilon_1-\epsilon_2)})^{i_2} \otimes \cdots \otimes (B_{2l-2} + A_{-(\epsilon_{l-1}-\epsilon_l)})^{i_{2l-2}} \otimes (B_{2l-1} + A_{\epsilon_l})^{i_{2l-1}} \otimes (B_{2l} + A_{-\epsilon_l})^{i_{2l}} \otimes (\otimes_{j=2l+1}^{2m} (B_j + A_{\alpha_j})^{i_j}) | 0 \leq i_j \leq p-1 \}, \)

where we put

\( A_{\epsilon_1-\epsilon_2} = g_\alpha = g_{\epsilon_1-\epsilon_2} \),
\[A_{e_2-e_1} = g_\alpha^2 \left( c_{e_2-e_1} + (h_{e_1-e_2} + 1)^2 + 4x_{e_2-e_1}x_{e_1-e_2} \right),\]

\[A_{e_1\pm e_3} = \pm \left( c_{e_1\pm e_3} + x_{e_2\pm e_3}x_{-(e_2\pm e_3)} \pm x_{e_1\pm e_3}x_{-(e_1\pm e_3)} \right),\]

\[A_{-e_1-e_3} = \pm \left( c_{-e_1-e_3} + x_{e_1+e_3}x_{-e_1-e_3} \pm x_{e_2+e_3}x_{-e_2-e_3} \right),\]

\[A_{e_3-e_1} = \left( c_{e_3-e_1} + x_{e_2-e_3}x_{e_3-e_2} \pm x_{e_1-e_3}x_{e_3-e_1} \right),\]

\[A_{e_1} = x_{e_3} \left( c_{e_2} + x_{e_2-e_3}x_{e_3-e_2} \pm x_{e_1-e_3}x_{e_3-e_1} \right),\]

\[A_{-e_2} = x_{-e_3} \left( c_{-e_2} + x_{e_2-e_3}x_{e_3-e_2} \pm x_{e_1-e_3}x_{e_3-e_1} \right),\]

\[A_{e_1} = x_{e_3} \left( c_{e_1} + x_{e_2-e_3}x_{e_3-e_2} \pm x_{e_1-e_3}x_{e_3-e_1} \right),\]

\[A_{-e_1} = x_{-e_3} \left( c_{-e_1} + x_{e_2-e_3}x_{e_3-e_2} \pm x_{e_1-e_3}x_{e_3-e_1} \right),\]

\[A_{e_2\pm e_j} = x_{-e_j}^{1+2} \left( c_{e_2\pm e_j} + x_{e_2\pm e_j}x_{-(e_2\pm e_j)} \pm x_{e_1\pm e_j}x_{-(e_1\pm e_j)} \right),\]

\[A_{-e_2-e_j} = x_{-e_j}^{3} \left( c_{-e_2-e_j} + x_{e_2+e_j}x_{-e_2-e_j} \pm x_{e_1+e_j}x_{-e_1-e_j} \right),\]

\[A_{e_j-e_2} = x_{e_j} \left( c_{e_j-e_2} + x_{e_2-e_j}x_{e_3-e_2} \pm x_{e_1-e_j}x_{e_3-e_1} \right),\]

\[A_{e_1\pm e_j} = x_{e_3-e_j}^{1+2} \left( c_{e_1\pm e_j} + x_{e_2\pm e_j}x_{-(e_2\pm e_j)} \pm x_{e_1\pm e_j}x_{-(e_1\pm e_j)} \right),\]

\[A_{-e_1-e_j} = x_{e_3-e_j}^{3} \left( c_{-e_1-e_j} + x_{e_2+e_j}x_{-e_2-e_j} \pm x_{e_1+e_j}x_{-e_1-e_j} \right),\]

\[A_{e_j-e_1} = x_{e_j-e_3} \left( c_{e_j-e_1} + x_{e_2-e_j}x_{e_3-e_2} \pm x_{e_1-e_j}x_{e_3-e_1} \right),\]

with the signs chosen so that they may commute with \(x_\alpha\) and with \(c_\beta \in F\) chosen so that \(A_{e_2-e_1}\) and parentheses are
invertible.

For any other root \( \beta \), we put \( A_\beta = x_\beta^3 \) or \( x_\beta^4 \) if possible.

Otherwise we make use of the parentheses( ) again used for designating \( A_{-\beta} \). So in this case we put \( A_\beta = x_\gamma^2 \) or \( x_\gamma^3 \) attached to these ( ) so that \( x_\alpha \) may commute with \( A_\beta \).

Proof. Obvious from chapter5 in [KY-4]. \( \square \)

Now we turn to the classical Lie algebra of \( D_l \)-type. We let \( L \) be any modular Lie algebra of \( D_l \)-type over any algebraically closed field \( F \) of characteristic \( p \geq 7 \).

We note first that the Lie algebra of \( D_l \)-type with rank \( l \), i.e., the \( D_l \)-type Lie algebra over \( \mathbb{C} \) has its root system \( \Phi = \{ \pm (\epsilon_i \pm \epsilon_j) | i \neq j \} \), where \( \epsilon_i, \epsilon_j \) are linearly independent orthonormal unit vectors in \( \mathbb{R}^l \) with \( l \geq 4 \). The base of \( \Phi \) is equal to \( \{ \epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \cdots, \epsilon_{l-1} - \epsilon_l, \epsilon_{l-1} + \epsilon_l \} \).

**Proposition 4.5.** We let \( \alpha \) be any root in the root system \( \Phi \) of \( L \). If \( \chi(x_\alpha) \neq 0 \), then \( \dim_F \rho_\chi(U(L)) = p^{2m} \), where \( [Q(U(L)) : Q(\mathfrak{z}(U(L))) = p^{2m} = p^{n-l} \) with \( \mathfrak{z} \) the center of \( U(L) \) and \( Q \) denotes the quotient algebra.

So the simple module corresponding to this representation has \( p^m \) as its dimension and the Lee’s basis for this simple module is given as follows.

Let \( \mathfrak{M}_\chi \) be the kernel of this simple representation with character \( \chi \). We may put \( \alpha = \epsilon_1 - \epsilon_2 \) since only long roots exist
and all roots of the same length are conjugate under the Weyl group of $\Phi$.

We put $B_i := b_{i1} h_{\epsilon_1 - \epsilon_2} + b_{i2} h_{\epsilon_2 - \epsilon_3} + \cdots + b_{i,l-1} h_{\epsilon_{l-1} - \epsilon_l} + b_{il} h_{\epsilon_{l-1} + \epsilon_l}$, where $(b_{i1}, b_{i2}, \cdots, b_{il}) \in F^l$ are chosen so that any $(l+1)-B_i$'s are linearly independent in $\mathbb{P}^l(F)$, the $\mathcal{B}$ below becomes an $F$-linearly independent set in $U(L)$ if necessary and $x_\alpha B_i \neq B_i x_\alpha$ for $\alpha = \epsilon_1 - \epsilon_2$.

In $U(L)/\mathcal{M}_\chi$ we have a basis $\mathcal{B} := \{(B_1 + A_{\epsilon_1 - \epsilon_2})^{i_1} \otimes (B_2 + A_{-(\epsilon_1 - \epsilon_2)})^{i_2} \otimes \cdots \otimes (B_{2l-2} + A_{-(\epsilon_{l-1} - \epsilon_l)})^{i_{2l-2}} \otimes (B_{2l-1} + A_{\epsilon_{l-1} + \epsilon_l})^{i_{2l-1}} \otimes (B_{2l} + A_{-(\epsilon_{l-1} + \epsilon_l)})^{i_{2l}} \otimes (\otimes_{j=2l+1}^{p}(B_j + A_{\alpha_j})^{i_j})|0 \leq i_j \leq p-1\}$, where we put

$$A_{\epsilon_1 - \epsilon_2} = x_\alpha = x_{\epsilon_1 - \epsilon_2},$$

$$A_{\epsilon_2 - \epsilon_1} = c_{\epsilon_2 - \epsilon_1} + (h_{\epsilon_1 - \epsilon_2} + 1)^2 + 4x_{\epsilon_2 - \epsilon_1}x_{\epsilon_1 - \epsilon_2}$$

$$A_{\epsilon_2 \pm \epsilon_3} = x_{\pm \epsilon_1 - \epsilon_2}(c_{\epsilon_2 \pm \epsilon_3} + x_{\epsilon_2 \pm \epsilon_3}x_{-(\epsilon_2 \pm \epsilon_3)} \pm x_{\epsilon_1 \pm \epsilon_3}x_{-(\epsilon_1 \pm \epsilon_3)},$$

$$A_{-(\epsilon_1 \pm \epsilon_3)} = (x_{-\epsilon_1 - \epsilon_2}^2 \text{ or } x_{-\epsilon_1 - \epsilon_2}^3)(c_{-(\epsilon_1 \pm \epsilon_3)} + x_{\epsilon_2 \pm \epsilon_3}x_{-(\epsilon_2 \pm \epsilon_3)} \pm x_{\epsilon_1 \pm \epsilon_3}x_{-(\epsilon_1 \pm \epsilon_3)},$$

$$A_{\epsilon_2 \pm \epsilon_k} = x_{\epsilon_3 \pm \epsilon_k}(c_{\epsilon_2 \pm \epsilon_k} + x_{\epsilon_2 \pm \epsilon_k}x_{-(\epsilon_2 \pm \epsilon_k)} \pm x_{\epsilon_1 \pm \epsilon_k}x_{-(\epsilon_1 \pm \epsilon_k)}$$

if $k \neq 1$,

$$A_{-(\epsilon_1 \pm \epsilon_k)} = x_{-(\epsilon_2 \pm \epsilon_k)}(c_{-(\epsilon_1 \pm \epsilon_k)} + x_{\epsilon_2 \pm \epsilon_k}x_{-(\epsilon_2 \pm \epsilon_k)} \pm x_{\epsilon_1 \pm \epsilon_k}x_{-(\epsilon_1 \pm \epsilon_k)}),$$

with the signs chosen so that they may commute with $x_\alpha$ and with $c_\beta \in F$ chosen so that $A_{\epsilon_2 - \epsilon_1}$ and parentheses are invertible.
For any other root $\beta$, we put $A_\beta = x_\beta^2$ or $x_\beta^3$ or $x_\beta^4$ if possible.

Otherwise we make use of the parentheses( ) again used for designating $A_{-\beta}$. So in this case we put $A_\beta = x_\gamma^2$ or $x_\gamma^3$ attached to these ( ) so that $x_\alpha$ may commute with $A_\beta$.

Proof. Also evident from [KY-4].

Proposition 4.6. We let $\chi$ be a character of any simple $L$-module with $\chi(h_\alpha) \neq 0$ for some $\alpha \in \Phi$, where $h_\alpha$ is an element in the Chevalley basis of $L$ such that $Fx_\alpha + Fh_\alpha + Fx_{-\alpha} = \mathfrak{sl}_2(F)$ with $[x_\alpha, x_{-\alpha}] = h_\alpha \in H$ (a CSA of $L$).

We may let $\alpha = \epsilon_1 - \epsilon_2$ without loss of generality since all roots are long and are conjugate under the Weyl group of $\Phi$.

As we have done before, we let $B_i$ be defined as in proposition 4.3.

In $U(L)/\mathfrak{m}_\chi$, we have a Lee’s basis as $\mathfrak{B} := \{(B_1 + A_{\epsilon_1 - \epsilon_2})^{i_1} \otimes (B_2 + A_{-(\epsilon_1 - \epsilon_2)})^{i_2} \otimes \cdots \otimes (B_{2l-2} + A_{-(\epsilon_1 - \epsilon_l)})^{i_{2l-2}} \otimes (B_{2l-1} + A_{\epsilon_1 - \epsilon_l})^{i_{2l-1}} \otimes (B_{2l} + A_{-(\epsilon_1 + \epsilon_l)})^{i_{2l}} \otimes (\otimes_{j=2l+1}^{2m} (B_j + A_{\alpha_j})^{i_j}) | 0 \leq i_j \leq p - 1 \}$,

where we put

$A_{\epsilon_1 - \epsilon_2} = g_\alpha = g_{\epsilon_1 - \epsilon_2},$

$A_{\epsilon_2 - \epsilon_1} = c_{\epsilon_2 - \epsilon_1} + (h_{\epsilon_1 - \epsilon_2} + 1)^2 + 4x_{\epsilon_2 - \epsilon_1}x_{\epsilon_1 - \epsilon_2},$
\( A_{\epsilon_2 \pm \epsilon_3} = g^{2or3}_\alpha c_{\epsilon_2 \pm \epsilon_3} + x_{\epsilon_2 \pm \epsilon_3} x_{-(\epsilon_2 \pm \epsilon_3)} \pm x_{\epsilon_1 \pm \epsilon_3} x_{-(\epsilon_1 \pm \epsilon_3)} \),

\( A_{-(\epsilon_1 \pm \epsilon_3)} = g^{4or5}_\alpha (c_{-(\epsilon_1 \pm \epsilon_3)} + x_{\epsilon_1 \pm \epsilon_3} x_{-(\epsilon_1 \pm \epsilon_3)} \pm x_{\epsilon_2 \pm \epsilon_3} x_{-(\epsilon_2 \pm \epsilon_3)} )\),

\( A_{\epsilon_2 \pm \epsilon_k} = x_{\epsilon_3 \pm \epsilon_k} (c_{\epsilon_2 \pm \epsilon_k} + x_{\epsilon_2 \pm \epsilon_k} x_{-(\epsilon_2 \pm \epsilon_k)} \pm x_{\epsilon_1 \pm \epsilon_k} x_{-(\epsilon_1 \pm \epsilon_k)} )\),

\( A_{-(\epsilon_1 \pm \epsilon_k)} = x^2_{-(\epsilon_3 \pm \epsilon_k)} (c_{-(\epsilon_1 \pm \epsilon_k)} + x_{\epsilon_2 \pm \epsilon_k} x_{-(\epsilon_2 \pm \epsilon_k)} \pm x_{\epsilon_1 \pm \epsilon_k} x_{-(\epsilon_1 \pm \epsilon_k)} )\),

with the signs chosen so that they may commute with \( x_{\alpha} \) and with \( c_{\beta} \in F \) chosen so that \( A_{\epsilon_2 - \epsilon_1} \) and parentheses are invertible.

For any other root \( \beta \), we put \( A_{\beta} = x^3_\beta \) or \( x^4_\beta \) if possible.

Otherwise we make use of the parentheses ( ) again used for designating \( A_{-\beta} \). So in this case we put \( A_{\beta} = x^3_\gamma \) or \( x^4_\gamma \) attached to these ( ) so that \( x_{\alpha} \) may commute with \( A_{\beta} \).

**Proof.** Also evident from chapter 6 in [KY-4].

\[ \square \]

**[Remark]** We have seen up to now that classical modular Lie algebras of \( D_l, B_l, A_l \)-types have no subregular point under mild conditions.

The modular Lie algebra of \( C_l \)-type also has no subregular point under mild conditions referring to [KY-6].

So we cannot help believing that all classical Lie algebras including exceptional types behave like this, so we conjectured such a similar result in [KWa].
In particular we are fond of the $A_l$-type out of these types because of the simplicity of its root system.

**Proposition 4.7.** Let $L$ be any Lie algebra of $A_l$-type over an algebraically closed field $F$ of characteristic $p \geq 7$. Then $L$ becomes a Hypo- Lie algebra.

**Proof.** Refer to proposition (4.2) of this paper and the chapter 4 in [KY-4].

Because we proved that all modular $A_l$-type Lie algebras for characteristic $p \geq 7$ have no subregular point, i.e., $S(L, p, \chi) = \phi$, we need to find out some example of nontrivial indecomposable modular Lie algebra which has no subregular point.

By making use of the fact that $sl_2(F)$ has no subregular point for $p > 2$ (refer to [RS],[KY-4]), we can extend this one to $L_5$ which is explained below.

We shall show that a certain indecomposable Lie algebra over $F$ has a subregular point even for $p \geq 7$, whose fact is notable related to Kim’s conjecture [KY-6].

**Proposition 4.8.** Let $L_5$ be a Lie algebra generated by

$x_{\epsilon_j}, x_{\epsilon_2-\epsilon_j}, h_{\epsilon_2-\epsilon_j}, x_{\epsilon_1-\epsilon_2}, x_{\epsilon_1-\epsilon_j}, (j \neq 1, 2)$ over an algebraically closed field $F$ of characteristic $p > 2$, where all these generators are included in classical $A_l$-type Lie algebra. Suppose that $\chi$ is a character of $L_5$ s.t. $\chi(x_{\epsilon_j}) \neq 0$ or $\chi(x_{\epsilon_2-\epsilon_j}) \neq 0$ or $\chi(h_{\epsilon_2-\epsilon_j}) \neq 0$.

Then we claim that $L_5$ is centerless and indecomposable with $S(L_5, p, \chi) \neq \phi$. 
Proof. Because $L_5$ is a proper sub-Lie algebra of an $A_l$-type Lie algebra, it is easy to see that $L_5$ is a centerless and indecomposable modular Lie algebra. Let $\langle x_{\epsilon_1-\epsilon_2}, x_{\epsilon_1-\epsilon_j} \rangle_F$ denote the sub-Lie algebra of $L$ generated by $x_{\epsilon_1-\epsilon_2}$ and $x_{\epsilon_1-\epsilon_j}$ over $F$.

It is just a proper ideal of $L_5$, so that we have a projective homomorphism $\pi$ of Lie algebras

$$\pi : L_5 \to L_5/\langle x_{\epsilon_1-\epsilon_2}, x_{\epsilon_1-\epsilon_j} \rangle_F.$$

Since we have $[Q(U(L_5)) : Q(3(U(L_5)))] \geq p^3$, it follows that $[Q(U(L_5)) : Q(3(U(L_5)))] = p^4$, and so the maximal dimension of irreducible $L_5$-modules is $p^2$.

Hence evidently there exists a subregular point for this $L_5$, i.e., $S(L_5, p, \chi) \neq \phi$ in view of definitions in section 3.

□

We remark here that the codomain of the projective homomorphism $\pi$ is isomorphic to $sl_2(F)$ and the Lie algebra $L_5$ over $F$ in the domain has dimension 5.

In [KWa] we made a remark conjecturing that some extended Lie algebras of classical modular Lie algebras must be Hypo-Lie algebras relating to subregular points.

If we have regular points almost everywhere except for a finite number of subregular points and $p$-points related to a Lie algebra $L$ included in a classical modular Lie algebra and if we can find Lee’s bases associated to all these regular points, then we called $L$ a Hypo-Lie algebra.
So in connection with such concepts, the Lie algebra $L_5$ above in this paper is noway Hypo- Lie algebra.

We thus conclude that $L_5$ closely related to $A_l$- type is a nontrivial centerless indecomposable Lie algebra having infinitely many subregular points over an algebraically closed field of characteristic $p > 2$.

5. CONJECTURE

Now in this section we come back to our main story related to a conjecture.

Because the $B_l$- type Lie algebra looks like a general prototype out of all classical Lie algebras and we used it already in [NWK], so we would like to still use it preferably.

For a while we fix an algebraically closed field $F$ of characteristic $p \geq 7$ unless otherwise specified until we remark on our conjecture related to $NP$-completeness.

As we are well aware, there exists the universal Casimir element $s \in u(L)$ for any $B_l$- type classical Lie algebra $L$. Let the rank $l$ of $L$ is $l \geq 3$, i.e., the dimension of CSA $\geq 3$.

We may express the universal Casimir element $s$ as $s = \sum_{\alpha \in \Phi^+} a_\alpha x_\alpha x_{-\alpha} + \sum_{i=1}^l b_i h_{\alpha_i} + \sum_{i,j} a_{ij} h_{\alpha_i} h_{\alpha_j}$, where $0 \neq b_i \in F$, $a_\alpha, a_{ij} \in F$, $H$ is a Cartan subalgebra (abbreviated as CSA) with a basis $\{h_{\alpha_i} | 1 \leq i \leq l\}$, and $\{x_\alpha, \alpha \in \Phi; h_{\alpha_i}, 1 \leq i \leq l\}$ is the standard Chevalley basis of $L$ with a root system $\Phi$ including $\Phi^+$ as a set of positive roots.
Here we rearrange the Chevalley basis as
\[ \{ h_{\alpha_1}, h_{\alpha_2}, \ldots, h_{\alpha_l}, x_{\alpha_1}, x_{-\alpha_1}, \ldots, x_{\alpha_l}, x_{-\alpha_l}, x_{\alpha_{l+1}}, x_{-\alpha_{l+1}}, \ldots, x_{\alpha_m}, x_{-\alpha_m} \} \]
where \( \dim L = n = l + 2m \), and \( l = \text{rank} L = \dim H \) over \( F \).

Clearly we have \( x^{[p]}_{\alpha_i} = x^{[p]}_{-\alpha_i} = 0 \) for \( 1 \leq i \leq m \) and \( h^{[p]}_{\alpha_j} = h_{\alpha_j} \) for \( 1 \leq j \leq l \). We shall denote the center of \( u(L) \) simply by 3 as before.

**Proposition 5.1.** We have the following in the quotient algebra \( Q(u(L)) \):

(i) \( \dim Q(3)(h_{\alpha_1}, \ldots, h_{\alpha_l}) = p^l \), (ii) \( Q(3)(h_{\alpha_1}, \ldots, h_{\alpha_l}) \) becomes a Galois field over \( Q(3) \).

**Proof.** Since \( Q(u(L)) \) becomes a central simple Artinian algebra over \( Q(3) \) which is actually a division algebra, we obtain (i) immediately from the Noether-Skolem theorem and from the identities such as \( h_{\alpha_i} x_{\alpha_i} = x_{\alpha_i} (h_{\alpha_i} + 2) \) for \( i = 1, 2, \cdots, l \) and

\[ [Q(3)(h_{\alpha_1}, h_{\alpha_2}, \ldots, h_{\alpha_l}) : Q(3)] = [Q(3)(h_{\alpha_1}, h_{\alpha_2}, \ldots, h_{\alpha_l}) : Q(3)(h_{\alpha_1}, h_{\alpha_2}, \ldots, h_{\alpha_{l-1}})] \]

\[ \cdots \]

\[ [Q(3)(h_{\alpha_i}) : Q(3)] \] for \( i = 1, 2, \cdots, l \).

(ii) is an immediate consequence of (i) considering that each bracket [ ] in the above factorization gives rise to \( p \)-distinct conjugates of \( h_{\alpha_i} \) for \( j = 1, 2, \cdots, l \).
Proposition 5.2. We may obtain the irreducible polynomial \( \text{Irr}(s, \mathcal{O}(L)) \) of \( s \) over the \( p \)-center \( \mathcal{O}(L) \) by expanding out the following norm

\[
(*) \quad N_{Q(\mathbb{Z})}^{Q(3)}(s_i - (b_1 h_{\alpha_1} + \cdots + b_l h_{\alpha_l} + \sum_{i,j} a_{ij} h_{\alpha_i} h_{\alpha_j})) = N_{Q(3)}^{Q(3)}(\sum_{\alpha \in \Phi^+} a_{\alpha} x_{\alpha} x_{-\alpha}).
\]

Moreover we have \( \deg(\text{Irr}(s, \mathcal{O}(L))) = p^l \) and \( s \) becomes separable over \( \mathcal{O}(L) \) and so over \( Q(\mathcal{O}(L)) \).

Proof. First thing we should perceive that

the left hand side of (\( \ast \)) = \( s^{p^l} + \hat{a}_1 s^{p^l-1} + \cdots + \hat{a}_{p^l-1} s + \hat{a}_{p^l} \)

for some \( \hat{a}_i \in Q(\mathbb{Z})(h_{\alpha_1}, h_{\alpha_2}, \cdots, h_{\alpha_l}) \) for \( i = 1, 2, \ldots, p^l \).

Here we contend that \( \hat{a}_i \in Q(\mathcal{O}(L)) \) in fact. We choose any distinct \( p^l \)-elements \( s_i \in \mathcal{O}(L) \) and take norms such as

\[
N_{Q(3)}^{Q(3)}(s_i - (b_1 h_{\alpha_1} + \cdots + b_l h_{\alpha_l} + \sum_{i,j} a_{ij} h_{\alpha_i} h_{\alpha_j})) = N_{Q(3)}^{Q(3)}(s_i - s + (\sum_{\alpha \in \Phi^+} a_{\alpha} x_{\alpha} x_{-\alpha})).
\]

We know that the Noether-Skolem theorem allows us to extend every automorphism of \( Q(\mathbb{Z})(h_{\alpha_1}, \cdots, h_{\alpha_l}) \) to an inner automorphism of \( Q(u(L)) \).

So if we conjugate this by some elements in \( Q(u(L)) \), then we obtain \( p^l \)-distinct automorphisms of \( Q(\mathbb{Z})(h_{\alpha_1}, \cdots, h_{\alpha_l}) \) over \( Q(\mathbb{Z}) \).

Next we need to note that \( [Q(\mathcal{O}(L))(h_{\alpha_1}, \cdots, h_{\alpha_l}) : Q(\mathcal{O}(L))] = [Q(3)(h_{\alpha_1}, \cdots, h_{\alpha_l}) : Q(3)] = p^l \) and that isomorphisms of \( Q(\mathbb{Z})(h_{\alpha_1}, \cdots, h_{\alpha_l}) \) over \( Q(3) \) are the same as those of \( Q(\mathcal{O}(L))(h_{\alpha_1}, \cdots, h_{\alpha_l}) \)
over $Q(\mathcal{O}(L))$.

So we obtain

$$N_{Q(3)}^{Q(\mathcal{O}(L))}(h_{\alpha_1}, \ldots, h_{\alpha_l}) \{s_i - (b_1 h_{\alpha_1} + \cdots + b_l h_{\alpha_l} + \sum_{i,j} a_{i,j} h_{\alpha_i} h_{\alpha_j})\} =$$

$$N_{Q(\mathcal{O}(L))}^{Q(3)}(h_{\alpha_1}, \ldots, h_{\alpha_l}) \{s_i - (b_1 h_{\alpha_1} + \cdots + b_l h_{\alpha_l} + \sum_{i,j} a_{i,j} h_{\alpha_i} h_{\alpha_j})\} \in Q(\mathcal{O}(L)),$$

which should boil down to the form $N_{Q(3)}^{Q(\mathcal{O}(L))}(h_{\alpha_1}, \ldots, h_{\alpha_l}) \{s_i - s + (\sum_{\alpha \in \Phi^+} a_{\alpha} x_\alpha x_{-\alpha})\}$.

On the other hand, if we put

$$s_i^{(p)} + \hat{b}_1 s_i^{(p-1)} + \cdots + \hat{b}_{p-1} s_i + \hat{b}_p = k_i$$

for some $k_i \in Q(\mathcal{O}(L))$ and $\hat{b}_j \in Q(\mathcal{O}(L))$ for $j = 1, 2, \ldots, p'$,

then we have a system in indeterminates $\hat{b}_j$ of $p'$-nonhomogeneous linear equations with the determinant of coefficients:

$$\det A = \begin{vmatrix}
1 & s_1 & \cdots & s_1^{p'-1} \\
\vdots & \ddots & \ddots & \vdots \\
1 & s_p & \cdots & s_p^{p'-1}
\end{vmatrix} = \prod_{i < j} (s_j - s_i) \neq 0,$$

which turns out to be the so called Vandermonde determinant.

If we make use of the Cramer’s formula, then we may have the solutions $\hat{b}_j \in Q(\mathcal{O}(L))$ of the system above. So we get
\( \hat{a}_j = \hat{b}_j \) for \( j = 1, 2, \cdots, p^l \), and hence \( \hat{a}_j \in Q(\mathcal{O}(L)) \) follows.

By the way, since \( s \) is integral over \( \mathcal{O}(L) \), we have that

the right hand side of \((\star) = N^{Q(3)}_{Q(3)}(h_{\alpha_1}, \cdots, h_{\alpha_l}) (\sum_{\alpha \in \Phi^+} a_{\alpha} x_{\alpha} x_{-\alpha}) \) also belongs to \( Q(\mathcal{O}(L)) \) which is secured by the following lemma.

\[ \square \]

**Lemma 5.3.** If we suppose that \( s \) satisfies an algebraic equation of the form

\[ f(X) := X^{p_l} + \hat{a}_1 X^{p_l-1} + \cdots + \hat{a}_{p_l-1} X + \hat{a}_{p_l} = 0 \]

with \( \hat{a}_j \in Q(\mathcal{O}(L)) \) for \( 1 \leq j \leq p_l - 1 \), which is the same form as in the above argument for the left hand side of norm equation \((\star)\),

then we have that \( \hat{a}_{p_l} \) also belongs to \( Q(\mathcal{O}(L)) \).

**Proof.** Thanks to proposition 5.1, we may have \( p_l \) distinct automorphisms \( \sigma_i \) of \( Q(3)(h_{\alpha_1}, \cdots, h_{\alpha_l}) \) over \( Q(3) \), which are represented by inner automorphisms of \( Q(L) \).

Next the integral equation of \( s \) over \( \mathcal{O}(L) \) is just \( Irr(s, Q(\mathcal{O}(L))) \) itself because \( \mathcal{O}(L) \) becomes the Noether normalization of \( 3 \) and hence a unique factorization domain.

Now the map \( s \mapsto \sigma_i(b_1 h_{\alpha_1} + \cdots + b_l h_{\alpha_l} + \sum_{i,j} a_{ij} h_{\alpha_i} h_{\alpha_j}) + \)

\[ \sigma_j(\sum_{\alpha \in \Phi^+} a_{\alpha} x_{\alpha} x_{-\alpha}) \text{ for } 1 \leq i, j \leq p^l \]

gives rise to an isomorphism of the algebra \( Q(\mathcal{O}(L))(h_{\alpha_1}, \cdots, h_{\alpha_l})(s) \) over \( Q(\mathcal{O}(L)) \).
Hence the irreducible polynomial $\text{Irr}(s, Q(\mathcal{O}(L)))$ must divide the polynomial $f(X)$, and thus we have that $f(X) = \text{Irr}(s, Q(\mathcal{O}(L))) \cdot g(X)$ for some unique $g(X) := X^{p^j - p'^j} + c_1 \cdot X^{p^j - p'^j - 1} + \cdots + c_{p^j - p'^j}$ with $c_j \in Q(3)$ and $p^j - p'^j \geq 0$.

Seeing that $c_j$’s are uniquely determined only by the coefficients of $\text{Irr}(s, Q(\mathcal{O}(L)))$ and those of terms $X^{p^j}, X^{p^j - 1}, \ldots, X^{p^j - p'^j}$ in $f(X)$ and that those coefficients belong to $Q(\mathcal{O}(L))$, we know that our claim is proven.

Now coming back to our main proof, we must still show that the algebraic equation of $s$ obtained just above is the $\text{Irr}(s, Q(\mathcal{O}(L)))$, which is also the integral equation of $s$ over $\mathcal{O}(L)$. Such a fact is attributed to the following another lemma.

**Lemma 5.4.** In this lemma only,

let $E$ be a finite extension field of arbitrary field $F$ with nonzero characteristic $p$ and let $\alpha, \beta, \gamma$ be elements of $E - F$ with $[F(\alpha)(\gamma) : F] = p^n$.

Suppose that $f(X) := \text{Irr}(\alpha, F) = (X - \sigma_1(\alpha)) \cdots (X - \sigma_{p^m}(\alpha))$ for some distinct $\sigma_i(\alpha) \in F(\alpha)$ for $i = 1, 2, \ldots, p^m \leq p^n$ with $\sigma_i \in \text{isomorphisms of E over F}$ and that $\beta = \alpha + \gamma$ is an element such that $\forall \sigma_i, \beta$ satisfies $\sigma_i(\beta) = \beta$.

Suppose further that $\Pi_{i=1}^{p^m} \sigma_i(\gamma) \in F$ and $g(X) := \Pi_{i=1}^{p^m} (X - \sigma_i(\alpha)) - \Pi_{i=1}^{p^m} \sigma_i(\gamma) \in F[X]$. 
Then we have \( g(X) = \text{Irr}(\beta, F) \) which is separable over \( F \).

**Proof.** It is not difficult to see that \( \gamma \not\in F(\alpha) \), and hence there exists at least \( p^{m+1} \)-distinct isomorphisms of \( F(\alpha) \ni \beta \) over \( F \).

Next we consider a field lattice diagram as follows:

\[
F(\alpha)(\gamma) \ni \beta
\]

at least \( p \)-dimensional \( | \\
F(\alpha)
\)

\[
p^m \text{-dimensional} \mid F
\]

Given any nontrivial isomorphism \( \tau \) of \( F(\alpha)(\gamma) \) over \( F(\alpha) \), we must have \( \tau(\beta) = \tau(\alpha + \gamma) = \tau(\sigma_i(\alpha) + \sigma_i(\gamma)) = \sigma_i(\alpha) + \tau(\gamma) \) for some isomorphism \( \bar{\tau} \) of \( F(\alpha)(\gamma) \) over \( F(\gamma) \) and \( \forall i \) with \( 1 \leq i \leq p^m \).

Because \( F(\alpha) \) is a Galois extension of \( F \), we see that there are at least \( p^m \)-distinct conjugates of \( \beta \) over \( F \). However the degree of \( g(X) \) becomes \( \deg g(X) = p^m \) and clearly \( g(\beta) = 0 \), so that \( \text{Irr}(\beta, F) \) also has the same degree \( p^m \). In addition no doubt \( g(\chi) \) becomes separable over \( F \). \( \Box \)

Now so as to finish our proof of the main proposition 5.2, we put

\[
F := Q(\mathcal{O}(L)), \beta := s, \alpha := b_1h_{\alpha_1} + \cdots + b_lh_{\alpha_l} + \sum_{i,j} a_{ij}h_{\alpha_i}h_{\alpha_j}, \gamma := \sum_{\alpha \in \Phi^+} a_{\alpha}x_{\alpha}x_{-\alpha}
\]

and make use of automorphisms \( \{ \sigma_i | i = 1, 2, \ldots, p^l \} \) of \( Q(3)(h_{\alpha_1}, \ldots, h_{\alpha_l}) \) over \( Q(3) \) which are the same automorphisms of \( Q(\mathcal{O}(L))(h_{\alpha_1}, \ldots, h_{\alpha_l}) \)
over $Q(\mathcal{O}(L))$. Finally applying the preceding lemma, we are done after all.

**Proposition 5.5.** We obtain $\mathfrak{z} = \mathcal{O}(L)[s]$.

*Proof.* According to proposition 5.2, we must have $Q(\mathfrak{z}) = Q(\mathcal{O}(L)[s])$. Since $\mathfrak{z}$ becomes a finitely generated $\mathcal{O}(L)$-module and $\mathcal{O}(L)$ and $\mathcal{O}(L)[s]$ is completely closed in $\mathfrak{z}$, i.e., any nontrivial quotients of $\mathcal{O}(L)[s]$ is not contained in $\mathfrak{z} - \mathcal{O}(L)[s]$. We can explain this explicitly as follows.

Assume first that some $\mu \in \mathfrak{z} - \mathcal{O}(L)[s]$ satisfies an equation of the form $\mu \cdot f = \hat{f}$ for some distinct polynomials $f, \hat{f}$ in $F[x_{\alpha_1}, x_{-\alpha_1}, h_{\alpha_1} - h_{\alpha_1}, \cdots, x_{\alpha_i}, x_{-\alpha_i}, h_{\alpha_i} - h_{\alpha_i}, x_{\alpha_{i+1}}, x_{-\alpha_{i+1}}, \cdots, x_{\alpha_m}, x_{-\alpha_m}, s]$

satisfying that a nontrivial fraction $\mu = \hat{f}/f$ is reduced and that $\mu, s$ are integral over $\mathcal{O}(L)$.

Noticing that $x_{\alpha_1}, x_{-\alpha_1}, h_{\alpha_1} - h_{\alpha_1}, \cdots, x_{\alpha_i}, x_{-\alpha_i}, h_{\alpha_i} - h_{\alpha_i}, x_{\alpha_{i+1}}, x_{-\alpha_{i+1}}, \cdots, x_{\alpha_m}, x_{-\alpha_m}$ are all algebraically independent variables and that the fraction $\hat{f}/f$ should always be defined for all these variables, we perceive that the denominator $f$ must be 1, a contradiction to our assumption. So we obtain our required result.

□

**Proposition 5.6.** Suppose in particular that $L$ is $B_2$-type Lie algebra over an algebraically closed field of characteristic $p \geq 7$;
then we have that the quotient algebra $Q(u(L))$ becomes a crossed product in the Brauer group $B(Q(\mathfrak{g}))$ of $Q(\mathfrak{g})(\alpha)(x_{\alpha_1}x_{-\alpha_1} + \cdots, x_{\alpha_m}x_{-\alpha_m})$ with $\alpha$ as in the proof of proposition 5.4.

Proof. We must notice first that $\dim L = 5 \times \text{rank } L$.

So we see immediately that $Q(u(L))$ becomes a crossed product in the Brauer group $B(Q(\mathfrak{g}))$ of $Q(\mathfrak{g})(\alpha)(x_{\alpha_1}x_{-\alpha_1} + \cdots, x_{\alpha_m}x_{-\alpha_m})$ keeping in mind the above propositions. □

Now from here on, we fix our field $F$ to be algebraically closed field of characteristic $p \geq 7$.

Furthermore we assume that this field $F$ is a $p$-adic field with its valuation ring $A$ whose maximal ideal is denoted by $p$.

For the purpose of constructing our serious conjecture, relating to $NP$-hardness we consider only such a field henceforth.

We let $L$ be any $B_l$-type Lie algebra with rank $L= l \geq 3$ over an algebraically closed field $F$ of a $p$-adic field of characteristic $p \geq 7$. Let $A$ be the valuation ring of $F$ with $p$ the maximal ideal of $A$ definitely once and for all.

The Steinberg module $V$ associated with a regular $p$- point arises from an irreducible representation $\rho_{\chi} : u(L) \to \text{End}_F(V)$ with $\chi = 0$ and $\dim V = p^n$, where $\dim L = n = 2m + l$.

We shall use such a particular module for our conjecture as in [NWK-1,2].
Proposition 5.7. Let notations be as in the reference [NWK-1,2].

Then the counting problem in the reference [NWK-1,2] is an \(\text{NP}\)-problem.

**Proof.** We made use of a choice function which gives rise to a system of nonhomogeneous linear equations including arbitrary random variables of cardinality \(p^{2m}\), where \([Q(U(L)) : Q(\mathcal{Z}(U(L)))] = p^{2m}\).

Here \(L\) is an \(n\)-dimensional \(B_l\)-type classical Lie algebra of rank \(l\) with \(n = l + 2m\) over an algebraically closed field of a \(p\)-adic field of characteristic \(p \geq 7\) and \(Q\) indicates a quotient algebra of a noncommutative algebra.

Let \(s\) be the universal Casimir element of \(u(L)\).

Any maximal ideal \(m_\chi\) of \(U(L)\) associated with a character \(\chi \in L^*\) must contain an ideal of the form

\[
\{ \sum_{i=1}^{n} u(L)(\tilde{x}_i - \tilde{x}_i^{[p]} - \xi_i) \} + u(L)(s - \xi_{n+1}),
\]

where \(\xi_j\)'s are numbers in \(F\), and \(\tilde{x}_i\) represents the basis elements of \(L\) written as ordered elements like

\[x_{\alpha_1}, x_{\alpha_{-1}}, h_{\alpha_1}, \cdots, x_{\alpha_i}, x_{-\alpha_i}, h_{\alpha_i}, x_{\alpha_{i+1}}, x_{-\alpha_{i+1}}, \cdots, x_{\alpha_m}, x_{-\alpha_m}.
\]

Moreover \(\text{dim}_F u(L)/m_\chi \leq p^{n-l}\) considering \([Q(u(L)) : Q(\mathcal{Z}(u(L)))] = p^{n-l} = p^{2m}\).

For the purpose of proving our claim, we may make use our earlier propositions in section 3. However we would like to take,
a regular $p$-point associated with a $B_l$-type classical Lie algebra, which was used earlier in [NWK-1,2].

We see the Steinberg module $V$ over $L$ has the maximal dimension $p^m$ with its associated point $(0, \cdots, 0, \xi_{n+1})$ and with its associated character $\chi = 0$.

So the Steinberg module is closely connected with the restricted representation. We may assume $\xi_{i+1} = 0$ without loss of generality.

We put $S_{\alpha_j} := Fx_{\alpha_j} + Fx_{-\alpha_j} + h_{\alpha_j}$ for $1 \leq i \leq m$, $g_{\alpha_i} := x_{\alpha_i}^{p-1} - x_{-\alpha_i}$ for $1 \leq i \leq l$ and $w_{\alpha_i} := (h_{\alpha_i} + 1)^2 + 4x_{-\alpha_i}x_{\alpha_i}$ for $1 \leq i \leq m$.

There exists $w_{\alpha_j}$ for $1 \leq j \leq m$ with $2m = n - l$ such that $w_{\alpha_j}$ acts as zero for each irreducible quotient $S_{\alpha_j}$-module for the composition series of $S_{\alpha_j}$-module $V$.

In this case $g_{\alpha_j}$ is invertible in $u(L)/m_\chi$. We exhibit the basis of $u(L)/m_\chi$ as follows.

As a preliminary step we put $B_i := b_{i_1}h_{\epsilon_{i_1}} + b_{i_2}h_{\epsilon_{i_2}} + \cdots + b_{i_l}h_{\epsilon_{i_l}}$ for $l \geq 2$ and for $1 \leq i \leq 2m$, where $(b_{i_1}, \cdots, b_{i_l}) \in F^l$ is chosen so that any $(l-1)B_j$’s are linearly independent in $P^l(F)$ and the $B$ below becomes an $F$-linearly independent set in $u(L)$ if necessary for this particular $\alpha_j$ and $B_i \cdot x_{\alpha_j} \not\equiv x_{\alpha_j} \cdot B_i$ modulo $m_\chi$.

(I) We assume first $\alpha_j$ is a long root.
We may assume $\alpha_j = \epsilon_1 - \epsilon_2$ without loss of generality.

Next we put $\mathcal{B} := \{(B_1 + A_{\epsilon_1-\epsilon_2})^{i_1} \otimes (B_2 + A_{-(\epsilon_1-\epsilon_2)})^{i_2} \otimes \cdots \otimes (B_{2i-2} + A_{-(\epsilon_1-\epsilon_2)})^{i_{2l-2}} \otimes (B_{2l-1} + A_{\epsilon_1})^{2l-1} \otimes (B_{2l} + A_{-\epsilon_1})^{i_2} \otimes (\otimes_{j=2l+1}^{2m} (B_j + A_{\alpha_j})^{i_j})\}$ for $0 \leq i_j \leq p - 1$,

where we designate

$A_{\epsilon_1-\epsilon_2} = g_{\epsilon_1-\epsilon_2}$,

$A_{\epsilon_2-\epsilon_1} = g_{\epsilon_1-\epsilon_2}^2 (C_{\epsilon_2-\epsilon_1} + w_{\epsilon_1-\epsilon_2})$, 

$A_{\epsilon_1 \pm \epsilon_3} = g_{\epsilon_1-\epsilon_2}^{3or4} (C_{\epsilon_1 \pm \epsilon_3} + x_{\epsilon_1 \pm \epsilon_3} x_{-(\epsilon_1 \pm \epsilon_3)} \pm x_{\epsilon_2 \pm \epsilon_3} \cdot x_{-(\epsilon_2 \pm \epsilon_3)}$, 

$A_{-\epsilon_1-\epsilon_3} = g_{\epsilon_1-\epsilon_2}^5 (C_{-\epsilon_1-\epsilon_3} + x_{\epsilon_1 + \epsilon_3} \cdot x_{-\epsilon_1-\epsilon_3 \pm x_{\epsilon_2 + \epsilon_3} \cdot x_{-\epsilon_2-\epsilon_3}}$, 

$A_{\epsilon_3-\epsilon_1} = (C_{\epsilon_3-\epsilon_1} + x_{\epsilon_1 - \epsilon_3} \cdot x_{-(\epsilon_1 - \epsilon_3)} \pm x_{\epsilon_2 - \epsilon_3} \cdot x_{-(\epsilon_2 - \epsilon_3)}$, 

$A_{\epsilon_2} = x_{\epsilon_3} (C_{\epsilon_2} + x_{\epsilon_1 - \epsilon_3 \cdot x_{-(\epsilon_1 - \epsilon_3)} \pm x_{\epsilon_2 - \epsilon_3} \cdot x_{-(\epsilon_2 - \epsilon_3)}$, 

$A_{-\epsilon_2} = x_{\epsilon_3}^2 (C_{-\epsilon_2} + x_{\epsilon_1 - \epsilon_3 \cdot x_{-(\epsilon_1 - \epsilon_3)} \pm x_{\epsilon_2 - \epsilon_3} \cdot x_{-(\epsilon_2 - \epsilon_3)}$, 

$A_{\epsilon_1} = x_{\epsilon_3}^2 (C_{\epsilon_2-\epsilon_3} + x_{\epsilon_1 - \epsilon_3 \cdot x_{-(\epsilon_1 - \epsilon_3)} \pm x_{\epsilon_2 - \epsilon_3} \cdot x_{-(\epsilon_2 - \epsilon_3)}$, 

$A_{-\epsilon_1} = x_{-\epsilon_3} (C_{-\epsilon_1} + x_{\epsilon_1 - \epsilon_3 \cdot x_{-(\epsilon_1 - \epsilon_3)} \pm x_{\epsilon_2 - \epsilon_3} \cdot x_{-(\epsilon_2 - \epsilon_3)}$, 

$A_{\epsilon_2 \pm \epsilon_j} = x_{-\epsilon_j}^{1or2} (C_{\epsilon_2 \pm \epsilon_j} + x_{\epsilon_2 \pm \epsilon_j \cdot x_{-(\epsilon_2 \pm \epsilon_j) \pm x_{\epsilon_1 \pm \epsilon_j} \cdot x_{-(\epsilon_1 \pm \epsilon_j)}}$, 

$A_{-\epsilon_2-\epsilon_j} = x_{-\epsilon_j}^3 (C_{-\epsilon_2-\epsilon_j} + x_{\epsilon_2 + \epsilon_j} \cdot x_{-(\epsilon_2 + \epsilon_j) \pm x_{\epsilon_1 + \epsilon_j} \cdot x_{-(\epsilon_1 + \epsilon_j)}$, 

$A_{\epsilon_j-\epsilon_2} = x_{\epsilon_j} ((C_{\epsilon_j-\epsilon_2} + x_{\epsilon_2-\epsilon_j} \cdot x_{-(\epsilon_2-\epsilon_j) \pm x_{\epsilon_1-\epsilon_j} \cdot x_{-(\epsilon_1-\epsilon_j)}}$,
\[ A_{\epsilon_1 \pm \epsilon_j} = x_{\epsilon_3 - \epsilon_j}^{1\text{or}2} \left( C_{\epsilon_1 \pm \epsilon_j} + x_{\epsilon_2 \pm \epsilon_j} \cdot x_{-(\epsilon_2 \pm \epsilon_j)} \pm x_{-(\epsilon_1 \pm \epsilon_j)} \right), \]

\[ A_{-\epsilon_1 - \epsilon_j} = x_{\epsilon_1 - \epsilon_j}^{2\text{or}3} \left( C_{-\epsilon_1 - \epsilon_j} + x_{\epsilon_1 + \epsilon_j} \cdot x_{-\epsilon_1 - \epsilon_j} \pm x_{\epsilon_2 + \epsilon_j} \cdot x_{-\epsilon_2 - \epsilon_j} \right), \]

\[ A_{\epsilon_j - \epsilon_1} = x_{\epsilon_j - \epsilon_3} \left( C_{\epsilon_j - \epsilon_1} + x_{\epsilon_1 - \epsilon_j} \cdot x_{-(\epsilon_1 - \epsilon_j)} \pm x_{\epsilon_2 - \epsilon_j} \cdot x_{-(\epsilon_2 - \epsilon_j)} \right), \]

and \( A_\alpha = x_\alpha^2 \) or \( x_\alpha^3 \) or \( x_\alpha^4 \)

suitably for other roots \( \alpha \) so that they all have distinct and different weights with respect to \((adH)\)-module \( u(L)\),

where \( H \) is the cartan subalgebra consisting of all linear combination of \( h_\alpha \) for any root \( \alpha \). Here \( C_\alpha \in F \) are chosen so that parentheses are invertible and signs are chosen so that they commute with \( x_{\alpha_j} = x_{\epsilon_1 - \epsilon_2} \).

(II) Next we assume \( \alpha_j \) is a short root. Without loss of generality we may put \( \alpha_j = \epsilon_1 \). Likewise as in (I), we suggest a basis of \( u(L)/m_\chi \) like the following.

\[ \mathfrak{B} := \left\{ (B_1 + A_{\epsilon_1})^{i_1} \otimes (B_2 + A_{-\epsilon_1})^{i_2} \otimes (B_3 + A_{\epsilon_1 - \epsilon_2})^{i_3} \otimes (B_4 + A_{-(\epsilon_1 - \epsilon_2)})^{i_4} \otimes \cdots \otimes (B_{2l} + A_{-(\epsilon_1 - \epsilon_l)})^{i_{2l}} \otimes (B_{2l+1} + A_{\epsilon_l})^{i_{2l+1}} \otimes (B_{2l+2} + A_{-\epsilon_l})^{i_{2l+2}} \otimes (\otimes_{j=2l+3}^{2m} (B_l + A_{\alpha_j})^{i_j}) \right\} \]

for \( 0 \leq i_j \leq p - 1 \),

where \( A_{\epsilon_1} = g_{\epsilon_1} \),

\[ A_{-\epsilon_1} = (C_{-\epsilon_1} + w_{\epsilon_1}) \],
\[ A_{-\epsilon_1 \pm \epsilon_2} = x_{\epsilon_3 \pm \epsilon_2} (C_{-\epsilon_1 \pm \epsilon_2} + x_{-\epsilon_1 \pm \epsilon_2} \cdot x_{-(-\epsilon_1 \pm \epsilon_2)} \pm x_{\pm \epsilon_2} \cdot x_{-(\epsilon_1 \pm \epsilon_2)}), \]

\[ A_{\epsilon_1 + \epsilon_2} = x_{\epsilon_3 - \epsilon_2}^2 (C_{\epsilon_1 + \epsilon_2} + x_{-\epsilon_1 - \epsilon_2} \cdot x_{\epsilon_1 + \epsilon_2} \pm x_{-\epsilon_2} \cdot x_{\epsilon_1 - \epsilon_2} \cdot x_{\epsilon_2 - \epsilon_1}), \]

\[ A_{-\epsilon_1 \pm \epsilon_j} = x_{-\epsilon_2 \pm \epsilon_j} (C_{-\epsilon_1 \pm \epsilon_j} + x_{-\epsilon_1 \pm \epsilon_j} \cdot x_{-(\epsilon_1 \pm \epsilon_j)} \pm x_{\pm \epsilon_j} \cdot x_{-(\pm \epsilon_j)}), \]

\[ A_{\epsilon_1 + \epsilon_j} = x_{\epsilon_2 - \epsilon_j}^2 (C_{\epsilon_1 + \epsilon_j} + x_{-\epsilon_j - \epsilon_1} \cdot x_{\epsilon_1 + \epsilon_j} \pm x_{-\epsilon_j} \cdot x_{\epsilon_1 - \epsilon_j} \cdot x_{-(\epsilon_1 - \epsilon_j)}), \]

\[ A_{\pm \epsilon_2} = x_{\epsilon_3 \pm \epsilon_2}^2 (C_{\pm \epsilon_2} + x_{\epsilon_2} \cdot x_{-\epsilon_2} \pm x_{\epsilon_1 + \epsilon_2} \cdot x_{-(\epsilon_1 + \epsilon_2)} \pm x_{\epsilon_3 - \epsilon_1} \cdot x_{-(\epsilon_3 - \epsilon_1)}), \]

\[ A_{\epsilon_j} = x_{\epsilon_2 + \epsilon_j} (C_{\epsilon_j} + x_{\epsilon_1} \cdot x_{-\epsilon_1} \pm x_{\epsilon_1 + \epsilon_j} \pm x_{-(\epsilon_1 + \epsilon_j)} \pm x_{\epsilon_j - \epsilon_1} \cdot x_{-(\epsilon_j - \epsilon_1)}), \]

\[ A_{-\epsilon_j} = x_{\epsilon_2 - \epsilon_j} (C_{-\epsilon_j} + x_{-\epsilon_j} \cdot x_{\epsilon_j} \pm x_{\epsilon_1 - \epsilon_j} \cdot x_{-(\epsilon_1 - \epsilon_j)} \pm x_{-\epsilon_j} - \epsilon_1 \cdot x_{\epsilon_1 + \epsilon_j}, \]

and \( A_\alpha = x_\alpha^2 \) or \( x_\alpha^3 \) or \( x_\alpha^4 \) for the remaining roots \( \alpha \).

We proved in the references [KY-4],[NWK-1,2] that \( \mathfrak{B} \)'s really form a basis of the factor algebra \( u(L)/m_\chi \) under consideration.

However we can exhibit its proof differently from those in detail as below.
We built up the bases $\mathcal{B}$ in both cases of the factor algebra $u(L)/m_\chi$ above

with the sign chosen so that they commute with $x_\alpha$ and with $c_\alpha \in F$ chosen so that $A_{\epsilon_2-\epsilon_1}$ and parentheses are invertible. For any other root $\beta$ we put $A_\beta = x_\beta^2$ or $x_\beta^3$ if possible.

Otherwise we may attach to these sorts the parentheses( ) used for designating $A_{-\beta}$ so that $A_\gamma \forall \gamma \in \Phi$ may commute with $x_\alpha$.

We shall prove that $\mathcal{B}$ is a basis in $U(L)/M_\chi$.

By virtue of P-B-W theorem, it is not difficult to see that $\mathcal{B}$ is evidently a linearly independent set over $F$ in $U(L)$. Furthermore $\forall \beta \in \Phi, A_\beta \notin M_\chi$(see detailed proof below).

We shall prove that a nontrivial linearly dependent equation leads to absurdity.

We assume first that there is a dependence equation which is of least degree with respect to $h_\alpha \in H$ and the number of whose highest degree terms is also least.

In case it is conjugated by $g_{\alpha_j}$, then there arises a nontrivial dependence equation of lower degree than the given one, which contradicts our assumption.

Otherwise it reduces to one of the following forms:
(i) \( x_{\pm \epsilon_j} K + K' \in \mathcal{M}_\chi \),

(ii) \( x_{\pm \epsilon_j \pm \epsilon_k} K + K' \in \mathcal{M}_\chi \),

(iii) \( g_{\epsilon_1 - \epsilon_2} K + K' \in \mathcal{M}_\chi \),

where \( K, K' \) commute with \( x_\alpha \) and \( x_{-\alpha} \) modulo \( \mathcal{M}_\chi \). Here we assumed first that \( \alpha_j \) is a long root, and we may put \( \alpha_j = \alpha \) for brevity.

By making use of proofs of propositions in [KY-7] and proposition 2.1 in [KY-6], we may reduce (i) and (ii) to the equation of the form

\[ x_{\epsilon_1 - \epsilon_2} K + K' \in \mathcal{M}_\chi, \]

where \( K \) commute with \( x_{\pm (\epsilon_1 - \epsilon_2)} \) and \( K' \) commute with \( x_{\epsilon_1 - \epsilon_2} \) modulo \( \mathcal{M}_\chi \).

We have \( x_{\epsilon_1 - \epsilon_2}^p K + x_{\epsilon_1 - \epsilon_2}^{p-1} K' \equiv 0 \), so we get \( x_{\epsilon_1 - \epsilon_2}^{p-1} K' \equiv 0 \).

Subtracting \( x_{\epsilon_2 - \epsilon_1} x_{\epsilon_1 - \epsilon_2} K + x_{\epsilon_2 - \epsilon_1} K' \equiv 0 \) from this equation, we obtain \( -x_{\epsilon_2 - \epsilon_1} x_{\epsilon_1 - \epsilon_2} K + g_\alpha K' \equiv 0 \). We should remember that \( g_\alpha \) is invertible in \( U(L)/\mathcal{M}_\chi \) by virtue of [RS].

By the way we use \( w_\alpha := (h_\alpha + 1)^2 + 4 x_{-\alpha} x_\alpha \in \) the center of \( U(\mathfrak{sl}_2(F)) \). Hence we have \( -4^{-1} \{ w_\alpha - (h_\alpha + 1)^2 \} K + g_\alpha K' \equiv 0 \). So we obtain

\[ 4^{-1} g_\alpha^{p-1} \{ (h_\alpha + 1)^2 - w_\alpha \} + c K' \equiv 0 \cdots (∗) \]
and from the start equation we get
\[ cx_\alpha K + cK' \equiv 0 \cdots (**). \]

Subtracting (** from (*), we get \(4^{-1}g_\alpha^{-1}\{(h_\alpha + 1)^2 - w_\alpha\}K - cx_\alpha K \equiv 0\). Multiplying this equation by \(g_\alpha^{1-p}\) to the right, we have
\[4^{-1}g_\alpha^{p-1}\{(h_\alpha + 1)^2 - w_\alpha\}g_\alpha^{1-p}K - cx_\alpha g_\alpha^{1-p}K \equiv 4^{-1}g_\alpha^{p-1}\{(h_\alpha + 1)^2 - w_\alpha\}g_\alpha^{1-p}K + x_\alpha x_\alpha K \equiv 0.\]

Conjugation of the brace of this equation \((p - 1)\)-times by \(g_\alpha\) gives rise to \(4^{-1}\{(h_\alpha - 1)^2 - w_\alpha\}Kx_\alpha^{p-1} \equiv 0\) modulo \(\mathcal{M}_\chi\).

Now we multiply \(x_\alpha\) to the left of this equation consecutively until it becomes of the form
\[(a \text{ nonzero polynomial of degree } \geq 1 \text{ with respect to } h_\alpha)K \equiv 0 \text{ modulo } \mathcal{M}_\chi.\]

If we make use of conjugation and subtraction consecutively, then we arrive at a contradiction \(K \equiv 0\).

Next for the case (iii), we change it to the form \((iii)'K + g_\alpha^{-1}K' \in \mathcal{M}_\chi.\)

We thus have an equation
\[ x_{\epsilon_1 - \epsilon_2}K + x_{\epsilon_1 - \epsilon_2}g_{\epsilon_1 - \epsilon_2}K' \equiv 0 \text{ modulo } \mathfrak{M}_\chi. \] According to the above argument, we are also led to a contradiction \( K \in \mathfrak{M}_\chi \).

For a short root \( \alpha \), the proof is very similar to the one just above.

Hence any coset of \( u(L)/m_\chi \) must be of the form : (a linear combination of all elements of \( \mathfrak{B} + m_\chi \)).

Now we let \( v \) be the nonarchimedean valuation of \( F \) and \( w \) the additive valuation of rank 1 satisfying that \( v(x) = p^{-w(x)}, \forall x \in F \).

For any given \( x = (x_1, x_2, \cdots, x^{p^{2m}}) \in F^{p^{2m}} \), where \( x_i \) 's are coefficients of the above linear combination, we put \( v(x) := \sup(v(x_i)) \) and we put \( U_\lambda := \{ x \in A : \lambda \geq 0, v(x) \leq p^{-\lambda} \} \), which is an ideal of \( A \).

We let \( G_{U_\lambda} := \{ x \in F^{p^{2m}} : x_i \in U_\lambda, 1 \leq p^{2m} \} = \{ x \in F^{p^{2m}} : v(x) \leq p^{-\lambda} \} \).

Next we consider an infinite tower of \( G_{U_\lambda} \) 's such as \( G_{U_1} \supset G_{U_2} \supset G_{U_3} \supset \cdots \)

and pore over the infinite family of sets as follows: \( S_1 := G_{U_1} - G_{U_2}, S_2 := G_{U_2} - G_{U_3}, \cdots, S_n := G_{U_n} - G_{U_{n+1}}, \cdots \).

It is obvious that \( G_{U_1} = \bigcup_{i=1}^{\infty} S_i \). We put here \( S'_i := \{ x \in S_i : v(x) = p^{-i} \} \) for \( i \in \mathbb{N} \) (the natural number system).

Here at this juncture by the axiom of choice we may get a choice function \( \psi \) of the family \( \{ S'_i : i \geq 1 \} \).
satisfying that $\psi(S'_i) \in S'_i$.

We consider a counting problem asking whether or not a given particular element $x$ of $u(L)$ is contained in a coset of the form $\psi(S'_i) + m_\chi$ and how many coefficients are there in $e \in m_\chi$ modulo $\sum_i u(L)(x^p_i - x^{[p]}_i)$ having their absolute values equal to $v(x)$.

First thing in order to solve this problem we must check whether $x + e$ with some $e \in m_\chi$ becomes of the form $\psi(S'_i)$.

In the mean time we may delete any term having a factor $x^p_i - x^{[p]}_i$ for some $1 \leq i \leq n$ no matter where they may exist in $x$, $e$, or $\psi(S'_i)$ in terms of P-B-W basis.

Hence we let bars in the expression, say $\bar{x} + \bar{e} \in \overline{\psi(S'_i)}$ indicate the expression after such deletion in preference to the original counting problem.

We must show that it requires at most a polynomial time of the input size to check whether or not $\bar{x} + \bar{e} \in \overline{\psi(S'_i)}$.

Next we may define $\forall x \in u(L), v(\bar{x}) := \max \{v(c_j) : \bar{x} = \sum c_j b_j\}$ with $c_j \in F$ for P-B-W basis $b_j$ of $u(L)$. We consider first that if $\bar{x} \in \overline{\psi(S'_i)} + \bar{e}$, then we should get necessarily $v(\overline{\psi(S'_i)}) \leq v(\bar{x})$.

Let $c_i$ be coefficients of $\beta'_i$s of $\mathfrak{B}$ for $1 \leq i \leq p^{2m}$. Further let $C_{j_1j_2\cdots j_n}$ be coefficients of the term $x_{j_1j_2\cdots j_n}$ of a linear combination $x$ of elements of the form $x_{j_1j_2\cdots j_n} = b_1^{j_1}b_2^{j_2}\cdots b_n^{j_n}$,
where \( b_1, \ldots, b_n \) are basis elements of \( L \) and \( 0 \leq j_k \leq p - 1 \).

We can rearrange the element \( s, \sum c_i \beta_i \), and \( \bar{e} = \sum e_{j_1j_2 \cdots j_n} x_{j_1j_2 \cdots j_n} \) with respect to P-B-W basis in order to get a system of linear equations of the form

\[
C_{j_1j_2 \cdots j_n} = \sum_{i=1}^{p^2m} C'_{ij_1 \cdots j_n} \cdot C_{ij_1 \cdots j_n} + \sum e'_{j_1 \cdots j_n} \cdot C_{ij_1 \cdots j_n}
\]

with \( C'_{ij_1 \cdots j_n}, e'_{j_1 \cdots j_n} \) integers in \( \{1, 2, \ldots, p-1\} \).

We should note here that such integers arise due to rearranging \( \psi(S_i) \) and \( \bar{e} \) according to P-B-W basis.

Next the substitution of \( C_{ij_1 \cdots j_n} \) by \( \psi(S'_i) \) gives rise to a system of nonhomogeneous linear equations in indeterminates \( e_{j_1 \cdots j_n} \).

Obviously this system has an algorithm by the Cramer’s formula. We perceive that the input time is at most \( p^n \) which is the number of elements of P-B-W basis.

So as to put forth flow charts for this algorithm, we need only polynomial time in view of the fact that the following system

\[
\begin{cases}
\sum_{i=1}^{p^2m} C'_{ij_1 \cdots j_n} \psi_{ij_1 \cdots j_n} \\
\sum_{i=1}^{p^2m} C'_{i(p-1,\ldots,p-1)} \psi_{i(p-1,\ldots,p-1)}
\end{cases}
\]

\[
\begin{align*}
 a_{11}x_1 + \cdots + a_{1p^n}x_{p^n} &= C_{j_1 \cdots j_n} - \sum_{i=1}^{p^2m} C'_{ij_1 \cdots j_n} \\
 a_{p^n1}x_1 + \cdots + a_{p^n p^n}x_{p^n} &= C_{p-1,\ldots,p-1} - \sum_{i=1}^{p^2m} C'_{i(p-1,\ldots,p-1)}
\end{align*}
\]

is solvable if and only if \( v(x_i) \leq v(x), \forall i \leq p^n \).
where \( a_{ij} \) represents \( e'_{j_1 \cdots j_n} \), and \( x_1, \ldots, x_{p^n} \) represents \( e_{j_1 \cdots j_n} \) respectively with order \( j_1 \cdots j_n \leq j'_1 \cdots j'_n \) defined if and only if \( j_1 \leq j'_1, \ldots, j_{k-1} \leq j'_{k-1} \) and \( j_k < j'_k \) for some \( k \leq p^n \).

So it requires at most a polynomial time of the input size to check whether or not \( \bar{x} \in \psi(S'_i) + e \).

Moreover it is evident that within a big fixed constant time of the input size, the number of coefficients in \( e \in m_\chi \) modulo \( \sum u(L)(x_i^p - x_i^{[p]}) \) suitable for our bid can be easily detected by virtue of Cramer’s formula.

Hence the given counting problem is evidently contained in \( NP \)-class.

Here we should recollect that an algorithm of a problem in \( NP \)-class means at most polynomial time logical processes regarding input sizes for the individual check of solutions of the problem,

whereas an algorithm of a problem in \( P \)-class means at most polynomial time logical processes regarding the input sizes for the general solution of the problem.

We must also note that algorithm should be determined only by input data with other data finite and definitely determined and at most countable.
We may take an example for this matter. Consider the problem asking if a positive integer is a prime number or not.

It is already known that any specific integer can be checked by a calculator in order to know whether it is prime or not within at most polynomial time regarding the number of digits representing the integer.

So we know that the problem is contained in the \( NP \)-class.

However we must still compute to know whether or not the algorithm for the solution is generally determined within at most polynomial time regarding the input sizes consisting of information only about any given integer.

It means that we have to determine whether or not the problem is in the \( P \)-class. But it is well known that the problem is contained in the \( P \)-class, which was proved by the Indian Institute of Technology.

Finally, we insist that our built-up problem cannot belong to the \( P \)-class.

Suppose now that the counting problem under consideration is contained in the \( P \)-class. If we consider the cardinality of \( \hat{n}(x) := \# \{ j_1 \cdots j_n : v(e'_{j_1 \cdots j_n} \cdot e_{j_1 \cdots j_n}) = v(x) \} \), then the cardinality must be determined only by the input data of \( x \).

However by the former system (*) of nonhomogeneous linear equations, we see that \( \hat{n}(x) \) is determined not only by the data \( C_{j_1 \cdots j_n} \) of \( \bar{x} \) but also by the coefficients \( \psi_{ij_1 \cdots j_n} \) which are
random variables in the field $F$ regardless of $x$ except for $v(x)$.

Note also that the field $F$ is an uncountable set, so we cannot find out clearly any element in $F$ in polynomial time of the input size.

So if we identify the coefficients of $\bar{x}$ with $\bar{x}$ with respect to P-B-W basis for brevity, then the coefficients time becomes the input time $\bar{x}$ and the output time $f(\bar{x})$ is obviously not less than any exponential function time $R^{\bar{x}}$ in terms of definition 2.1, i.e., $R^{\bar{x}} = \mathcal{O}(f(\bar{x}))$.

In fact it may take an infinite time even to find out a particular choice function out of all choice functions because the ground field $F$ is uncountable.

Hence we meet with a contradiction. So our given counting problem is not contained in the $P$-class. Since it is known that $P$-class $\subseteq NP$-class, it turns out that $P$-class is properly contained in $NP$-class.

Hence we conclude that $P \neq NP$ after all. 

\[ \square \]

It may or may not be true that the counting problem is equivalent to the subset sum problem.

In other words the subset sum problem may or may not reduce to the counting problem in polynomial time.
However it looks like we may conjecture the following, the conjecture being compared to the subset sum problem.

**[conjecture]**
We conjecture that the counting problem under consideration could be an $NP$- complete problem.

We would like to find out the heuristic reason for this conjecture somehow.

We know that the subset sum problem requires at most the time
\[
\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n
\]
for the input time $n$.

We don’t know for now whether or not we can reduce the time required. If it is possible to say that the counting problem requires generally at least $2^n$ for the input time $n$, then we might say that the latter algorithm is absolutely harder than the former one.

So this could be roughly the heuristic reason for the conjecture above.

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