Twisted partially pure spinors

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Abstract

Motivated by the relationship between orthogonal complex structures and pure spinors, we define twisted partially pure spinors in order to characterize spinorially subspaces of Euclidean space endowed with a complex structure.

1 Introduction

In this paper, we characterize subspaces of Euclidean space $\mathbb{R}^n$ endowed with an orthogonal complex structure by means of twisted spinors, which is a generalization of the relation between classical pure spinors and orthogonal complex structures on Euclidean space $\mathbb{R}^{2m}$. Recall that a classical pure spinor $\phi \in \Delta_{2m}$ is a spinor such that the (isotropic) subspace of complexified vectors $X - iY \in \mathbb{R}^{2m} \otimes \mathbb{C}$, $X, Y \in \mathbb{R}^{2m}$, which annihilate $\phi$ under Clifford multiplication

$$(X - iY) \cdot \phi = 0$$

is of maximal dimension, where $m \in \mathbb{N}$ and $\Delta_{2m}$ is the standard complex representation of the Spin group $Spin(2m)$ (cf. [5]). This means that for every $X \in \mathbb{R}^{2m}$ there exists a $Y \in \mathbb{R}^{2m}$ satisfying

$$X \cdot \phi = iY \cdot \phi.$$ 

By setting $Y = J(X)$, one can see that a pure spinor determines a complex structure on $\mathbb{R}^{2m}$. Geometrically, the two subspaces $TM \cdot \phi$ and $iTM \cdot \phi$ of $\Delta_{2m}$ coincide, which means $TM \cdot \phi$ is a complex subspace of $\Delta_{2m}$, and the effect of multiplication by the number $i = \sqrt{-1}$ is transferred to the tangent space $TM$ in the form of $J$.

The authors of [1, 6] investigated (the classification of) non-pure classical spinors by means of their isotropic subspaces. In [6], the authors noted that there may be many spinors (in different orbits under the action of the Spin group) admitting isotropic subspaces of the same dimension, and that there is a gap in the possible dimensions of such isotropic subspaces. In our Euclidean/Riemannian context, such isotropic subspaces correspond to subspaces of Euclidean space endowed with orthogonal complex structures. In this paper, we define twisted partially pure spinors...
spinors (cf. Definition 3.1) in order to establish a one-to-one correspondence between subspaces of Euclidean space (of a fixed codimension) endowed with orthogonal complex structures (and oriented orthogonal complements), and orbits of such spinors under a particular subgroup of the twisted spin group (cf. Theorem 3.1). By using spinorial twists we avoid having different orbits under the full twisted spin group and also the aforementioned gap in the dimensions.

The need to establish such a correspondence arises from our interest in developing a spinorial setup to study the geometry of manifolds admitting (almost) CR structures (of arbitrary codimension) and elliptic structures. Since such manifolds are not necessarily Spin nor Spin$, we are led to consider spinorially twisted spin groups, representations, structures, etc. Geometric and topological considerations regarding such manifolds will be presented in [4].

The paper is organized as follows. In Section 2 we recall basic material on Clifford algebras, spin groups and representations; we define the twisted spin groups and representations that will be used, and the space of anti-symmetric 2-forms and endomorphisms associated to twisted spinors; we also present some results on subgroups and branching of representations. In Section 3 we define partially pure spinors, deduce their basic properties and prove the main theorem, Theorem 3.1, which establishes the aforementioned one-to-one correspondence.

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2 Preliminaries

In this section, we briefly recall basic facts about Clifford algebras, the Spin group and the standard Spin representation [3]. We also define the twisted spin groups and representations, and the antisymmetric 2-forms and endomorphisms associated to a twisted spinor, and describe various inclusions of groups into (twisted) spin groups.

2.1 Clifford algebras

Let $Cl_n$ denote the Clifford algebra generated by the orthonormal vectors $e_1, e_2, \ldots, e_n \in \mathbb{R}^n$ subject to the relations

$$e_j e_k + e_k e_j = -2 \langle e_j, e_k \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^n$. Let

$$Cl_n = Cl_n \otimes \mathbb{C}$$

denote the complexification of $Cl_n$. The Clifford algebras are isomorphic to matrix algebras. In particular,

$$\mathbb{C}l_n \cong \begin{cases} 
\text{End}(\mathbb{C}^{2^k}), & \text{if } n = 2k, \\
\text{End}(\mathbb{C}^{2^k}) \oplus \text{End}(\mathbb{C}^2), & \text{if } n = 2k + 1,
\end{cases}$$

where

$$\Delta_n := \mathbb{C}^{2^k} = \underbrace{\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2}_{k \text{ times}}$$
is the tensor product of $k = \left\lceil \frac{n}{2} \right\rceil$ copies of $\mathbb{C}^2$. The map
$$\kappa : \mathcal{C}l_n \longrightarrow \text{End}(\mathbb{C}^{2^k})$$
is defined to be either the above mentioned isomorphism if $n$ is even, or the isomorphism followed by the projection onto the first summand if $n$ is odd. In order to make $\kappa$ explicit, consider the following matrices
$$Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$In terms of the generators $e_1, \ldots, e_n$, $\kappa$ can be described explicitly as follows,
$$e_1 \mapsto Id \otimes Id \otimes \ldots \otimes Id \otimes g_1,$$
$$e_2 \mapsto Id \otimes Id \otimes \ldots \otimes Id \otimes g_2,$$
$$e_3 \mapsto Id \otimes Id \otimes \ldots \otimes Id \otimes g_1 \otimes T,$$
$$e_4 \mapsto Id \otimes Id \otimes \ldots \otimes Id \otimes g_2 \otimes T,$$
$$\vdots$$
$$e_{2k-1} \mapsto g_1 \otimes T \otimes \ldots \otimes T \otimes T \otimes T,$$
$$e_{2k} \mapsto g_2 \otimes T \otimes \ldots \otimes T \otimes T \otimes T,$$
and, if $n = 2k + 1$,
$$e_{2k+1} \mapsto iT \otimes T \otimes \ldots \otimes T \otimes T \otimes T.$$
The vectors
$$u_{+1} = \frac{1}{\sqrt{2}}(1, -i) \quad \text{and} \quad u_{-1} = \frac{1}{\sqrt{2}}(1, i),$$form a unitary basis of $\mathbb{C}^2$ with respect to the standard Hermitian product. Thus,
$$\{u_{\varepsilon_1, \ldots, \varepsilon_k} = u_{\varepsilon_1} \otimes \ldots \otimes u_{\varepsilon_k} \mid \varepsilon_j = \pm 1, j = 1, \ldots, k\},$$is a unitary basis of $\Delta_n = \mathbb{C}^{2^k}$ with respect to the naturally induced Hermitian product. We will denote inner and Hermitian products by the same symbol $\langle \cdot, \cdot \rangle$ trusting that the context will make clear which product is being used.
Clifford multiplication is defined by
$$\mu_n : \mathbb{R}^n \otimes \Delta_n \longrightarrow \Delta_n,$$
$$x \otimes \psi \mapsto \mu_n(x \otimes \psi) = x \cdot \psi := \kappa(x)(\psi).$$A quaternionic structure $\alpha$ on $\mathbb{C}^2$ is given by
$$\alpha\left( \begin{array}{c} z_1 \\ z_2 \end{array} \right) = \left( \begin{array}{c} -\overline{z}_2 \\ \overline{z}_1 \end{array} \right),$$and a real structure $\beta$ on $\mathbb{C}^2$ is given by
$$\beta\left( \begin{array}{c} z_1 \\ z_2 \end{array} \right) = \left( \begin{array}{c} \overline{z}_1 \\ \overline{z}_2 \end{array} \right).
Real and quaternionic structures $\gamma_n$ on $\Delta_n = (\mathbb{C}^2)^{\otimes[n/2]}$ are built as follows

$$
\gamma_n = (\alpha \otimes \beta)^{\otimes 2k} \quad \text{if } n = 8k, 8k + 1 \quad \text{(real)},
$$

$$
\gamma_n = \alpha \otimes (\beta \otimes \alpha)^{\otimes 2k} \quad \text{if } n = 8k + 2, 8k + 3 \quad \text{(quaternionic)},
$$

$$
\gamma_n = (\alpha \otimes \beta)^{\otimes 2k+1} \quad \text{if } n = 8k + 4, 8k + 5 \quad \text{(quaternionic)},
$$

$$
\gamma_n = \alpha \otimes (\beta \otimes \alpha)^{\otimes 2k+1} \quad \text{if } n = 8k + 6, 8k + 7 \quad \text{(real)}.
$$

### 2.2 The Spin group and representation

The Spin group $\text{Spin}(n) \subset \text{Cl}_n$ is the subset

$$
\text{Spin}(n) = \{x_1x_2 \cdots x_{2l-1}x_{2l} \mid x_j \in \mathbb{R}^n, |x_j| = 1, l \in \mathbb{N}\},
$$

endowed with the product of the Clifford algebra. It is a Lie group and its Lie algebra is

$$
\text{spin}(n) = \text{span}\{e_ie_j \mid 1 \leq i < j \leq n\}.
$$

Recall that the Spin group $\text{Spin}(n)$ is the universal double cover of $\text{SO}(n), n \geq 3$. For $n = 2$ we consider $\text{Spin}(2)$ to be the connected double cover of $\text{SO}(2)$. The covering map will be denoted by

$$
\lambda_n : \text{Spin}(n) \rightarrow \text{SO}(n).
$$

Its differential is given by $\lambda_n(e_ie_j) = 2E_{ij}$, where $E_{ij} = e_i^* \otimes e_j - e_j^* \otimes e_i$ is the standard basis of the skew-symmetric matrices, and $e^*$ denotes the metric dual of the vector $e$. Furthermore, we will abuse the notation and also denote by $\lambda_n$ the induced representation on $\wedge^n \mathbb{R}^n$.

The restriction of $\kappa$ to $\text{Spin}(n)$ defines the Lie group representation

$$
\text{Spin}(n) \rightarrow GL(\Delta_n),
$$

which is, in fact, special unitary. We have the corresponding Lie algebra representation

$$
\text{spin}(n) \rightarrow \text{gl}(\Delta_n).
$$

**Remark.** For the sake of notation we will set

$$
\text{SO}(0) = \{1\}, \quad \text{SO}(1) = \{1\},
$$

$$
\text{Spin}(0) = \{\pm 1\}, \quad \text{Spin}(1) = \{\pm 1\},
$$

and

$$
\Delta_0 = \Delta_1 = \mathbb{C}
$$

a trivial 1-dimensional representation.

Clifford multiplication $\mu_n$ has the following properties:

- It is skew-symmetric with respect to the Hermitian product

  $$
  \langle x \cdot \psi_1, \psi_2 \rangle = -\langle \psi_1, x \cdot \psi_2 \rangle. \tag{1}\text{clifford-skew-symmetric}
  $$

- $\mu_n$ is an equivariant map of $\text{Spin}(n)$ representations.
• \( \mu_n \) can be extended to an equivariant map

\[
    \mu_n : \Lambda^*(\mathbb{R}^n) \otimes \Delta_n \rightarrow \Delta_n \\
    \omega \otimes \psi \mapsto \omega \cdot \psi,
\]

of \( \text{Spin}(n) \) representations.

At this point we will make the following convention. Consider the involution

\[
    F_{2m} : \Delta_{2m} \rightarrow \Delta_{2m} \\
    \phi \mapsto (-i)^m e_1 e_2 \cdots e_{2m} \cdot \phi,
\]

and let

\[
    \Delta_{2m}^\pm = \{ \phi \mid F_{2m}(\phi) = \pm \phi \}.
\]

This definition of positive and negative Weyl spinors differs from the one in \([3]\) by a factor \((-1)^m\). Nevertheless, we have chosen this convention so that the spinor \(u_1, \ldots, u_n\) is always positive and corresponds to the standard (positive) complex structure on \(\mathbb{R}^{2m}\).

### 2.3 Spinorially twisted Spin groups

Consider the following groups:

- By using the unit complex numbers \(\mathbb{U}(1)\), the Spin group can be twisted \([3]\)

\[
    \text{Spin}^c(n) = (\text{Spin}(n) \times \mathbb{U}(1))/\{\pm(1,1)\} = \text{Spin}(n) \times_{\mathbb{Z}_2} \mathbb{U}(1),
\]

with Lie algebra

\[
    \text{spin}^c(n) = \text{spin}(n) \oplus i\mathbb{R}.
\]

- In \([2]\) we have considered the twisted Spin group \(\text{Spin}^r(n)\), \(r \in \mathbb{N}\), defined as follows

\[
    \text{Spin}^r(n) = (\text{Spin}(n) \times \text{Spin}(r))/\{\pm(1,1)\} = \text{Spin}(n) \times_{\mathbb{Z}_2} \text{Spin}(r).
\]

The Lie algebra of \(\text{Spin}^r(n)\) is

\[
    \text{spin}^r(n) = \text{spin}(n) \oplus \text{spin}(r).
\]

- Here, we will also consider the following group

\[
    \text{Spin}^{c,r}(n) = (\text{Spin}(n) \times \text{Spin}^c(r))/\{\pm(1,1)\} \\
    = \text{Spin}(n) \times_{\mathbb{Z}_2} \text{Spin}^c(r),
\]

where \(r \in \mathbb{N}\), whose Lie algebra is

\[
    \text{spin}^{c,r}(n) = \text{spin}(n) \oplus \text{spin}(r) \oplus i\mathbb{R}.
\]

It fits into the exact sequence

\[
    1 \rightarrow \mathbb{Z}_2 \rightarrow \text{Spin}^{c,r}(n) \xrightarrow{\lambda_n \times \lambda_r \times \lambda_2} SO(n) \times SO(r) \times U(1) \rightarrow 1,
\]

where

\[
    (\lambda_n \times \lambda_r \times \lambda_2)([g,[h,z]]) = (\lambda_n(g),\lambda_r(h),z^2).
\]

**Remark.** For \(r = 0,1\), \(\text{Spin}^{c,r}(n) = \text{Spin}^c(n)\).
2.4 Twisted spin representations

Consider the following twisted representations:

- The Spin representation $\Delta_n$ extends to a representation of $\text{Spin}^c(n)$ by letting
  $$[g, z] \mapsto z\kappa_n(g) =: zg.$$

- The twisted $\text{Spin}^{c,r}(n)$ representation
  $$[g, [h, z]] \mapsto z\kappa_r(h) \otimes \kappa_n(g) =: zh \otimes g.$$

  which is also unitary with respect to the natural Hermitian metric.

- For $r = 0, 1$, the twisted spin representation is simply the $\text{Spin}^c(n)$ representation $\Delta_n$.

We will also need the map

$$\mu_r \otimes \mu_n : (\Lambda^* \mathbb{R}^r \otimes \Lambda^* \mathbb{R}^n) \otimes \mathbb{R} (\Delta_r \otimes \Delta_n) \rightarrow \Delta_r \otimes \Delta_n$$

$$\otimes (w_1 \otimes w_2) \otimes (\psi \otimes \varphi) \mapsto (w_1 \otimes w_2) \cdot (\psi \otimes \varphi) = (w_1 \cdot \psi) \otimes (w_2 \cdot \varphi).$$

As in the untwisted case, $\mu_r \otimes \mu_n$ is an equivariant homomorphism of $\text{Spin}^{c,r}(n)$ representations.

2.5 Skew-symmetric 2-forms and endomorphisms associated to twisted spinors

We will often write $f_{kl}$ for the Clifford product $f_k f_l$.

**Definition 2.1** [2] Let $r \geq 2$, $\phi \in \Delta_r \otimes \Delta_n$, $X, Y \in \mathbb{R}^n$, $(f_1 \ldots, f_r)$ an orthonormal basis of $\mathbb{R}^r$ and $1 \leq k, l \leq r$.

- Define the real 2-forms associated to the spinor $\phi$ by
  $$\eta_{kl}^\phi(X, Y) = \text{Re} \langle X \wedge Y \cdot f_k f_l \cdot \phi, \phi \rangle .$$

- Define the antisymmetric endomorphisms $\hat{\eta}_{kl}^\phi \in \text{End}^- (\mathbb{R}^n)$ by
  $$X \mapsto \hat{\eta}_{kl}^\phi(X) := (X \cdot \eta_{kl}^\phi)^\sharp,$$

  where $X \in \mathbb{R}^n$, $\cdot$ denotes contraction and $\sharp$ denotes metric dualization from 1-forms to vectors.

**Lemma 2.1** Let $r \geq 2$, $\phi \in \Delta_r \otimes \Delta_n$, $X, Y \in \mathbb{R}^n$, $(f_1 \ldots, f_r)$ an orthonormal basis of $\mathbb{R}^r$ and $1 \leq k, l \leq r$. Then

$$\text{Re} \langle f_k f_l \cdot \phi, \phi \rangle = 0,$$

$$\text{Re} \langle X \wedge Y \cdot \phi, \phi \rangle = 0,$$

$$\text{Im} \langle X \wedge Y \cdot f_k f_l \cdot \phi, \phi \rangle = 0,$$

$$\text{Re} \langle X \cdot \phi, Y \cdot \phi \rangle = \langle X, Y \rangle |\phi|^2 .$$
Proof. By using (1) twice
\[ \langle f_k f_l \cdot \phi, \phi \rangle = -\langle f_k f_l \phi, \phi \rangle. \]

For identity (2), recall that for \( X, Y \in \mathbb{R}^n \)
\[ X \wedge Y = X \cdot Y + \langle X, Y \rangle. \]
Thus
\[ \langle X \wedge Y \cdot \phi, \phi \rangle = -\langle X \wedge Y \cdot \phi, \phi \rangle. \]

Identities (3) and (4) follow similarly. \( \square \)

Remarks.
• For \( k \neq l \),
\[ \eta^\phi_{kl} = (\delta_{kl} - 1) \eta^\phi_{lk}. \]
• By (3), if \( k \neq l \),
\[ \eta^\phi_{kl}(X, Y) = \langle X \wedge Y \cdot f_k f_l \cdot \phi, \phi \rangle. \]

Lemma 2.2 Any spinor \( \phi \in \Delta_r \otimes \Delta_n \), \( r \geq 2 \), defines two maps (extended by linearity)
\[ \bigwedge^2 \mathbb{R}^r \longrightarrow \bigwedge^2 \mathbb{R}^n \]
\[ f_{kl} \mapsto \eta^\phi_{kl} \]
and
\[ \bigwedge^2 \mathbb{R}^r \longrightarrow \text{End}(\mathbb{R}^n) \]
\[ f_{kl} \mapsto \hat{\eta}^\phi_{kl}. \]

2.6 Subgroups, isomorphisms and decompositions
In this section we will describe various inclusions of groups into (twisted) spin groups.

Lemma 2.3 There exists a monomorphism \( h : \text{Spin}(2m) \times_{\mathbb{Z}_2} \text{Spin}(r) \longrightarrow \text{Spin}(2m + r) \) such that the following diagram commutes
\[
\begin{array}{ccc}
\text{Spin}(2m) \times_{\mathbb{Z}_2} \text{Spin}(r) & \xrightarrow{h} & \text{Spin}(2m + r) \\
\downarrow & & \downarrow \\
\text{SO}(2m) \times \text{SO}(r) & \hookrightarrow & \text{SO}(2m + r)
\end{array}
\]
Proof. Consider the decomposition
\[ \mathbb{R}^{2m+r} = \mathbb{R}^{2m} \oplus \mathbb{R}^r, \]
and let
\[ \text{Spin}(2m) = \left\{ \prod_{i=1}^{2s} x_i \in C_{2m+r} \mid x_i \in \mathbb{R}^{2m}, |x_i| = 1, s \in \mathbb{N} \right\} \subset \text{Spin}(2m+r), \]
\[ \text{Spin}(r) = \left\{ \prod_{j=1}^{2t} y_j \in C_{2m+r} \mid y_j \in \mathbb{R}^r, |y_j| = 1, t \in \mathbb{N} \right\} \subset \text{Spin}(2m+r). \]
It is easy to prove that
\[ \text{Spin}(2m) \cap \text{Spin}(r) = \{1, -1\}. \]
Define the homomorphism
\[ h : \text{Spin}(2m) \times_{\mathbb{Z}_2} \text{Spin}(r) \longrightarrow \text{Spin}(2m+r) \]
\[ [g, g'] \mapsto gg'. \]
If \([g, g'] \in \text{Spin}(2m) \times_{\mathbb{Z}_2} \text{Spin}(r)\) is such that
\[ gg' = 1 \in \text{Spin}(2m+r), \]
then
\[ g' = g^{-1} \in \text{Spin}(2m) \subset \text{Spin}(2m+r), \]
so that
\[ g, g' \in \text{Spin}(2m) \cap \text{Spin}(r) = \{1, -1\}. \]
Hence \([g, g'] = [1, 1]\) and \(h\) is injective. \(\square\)

Lemma 2.4 Let \(r \in \mathbb{N}\). There exists an monomorphism \(h : U(m) \times SO(r) \hookrightarrow \text{Spin}^{c,r}(2m+r)\) such that the following diagram commutes
\[
\begin{array}{ccc}
\text{Spin}^{c,r}(2m+r) & \cong & \text{Spin}^{c,r}(2m+r) \\
\downarrow & & \downarrow \\
U(m) \times SO(r) & \longrightarrow & SO(2m+r) \times SO(r) \times U(1)
\end{array}
\]
Proof. Suppose we have an orthogonal complex structure on \(\mathbb{R}^{2m} \subset \mathbb{R}^{2m+r}\)
\[ J : \mathbb{R}^{2m} \longrightarrow \mathbb{R}^{2m}, \quad J^2 = \text{Id}_{2m}, \quad \langle \cdot, \cdot \rangle = \langle J\cdot, J\cdot \rangle. \]
The subgroup of \(SO(2m+r)\) that respects both the orthogonal decomposition \(\mathbb{R}^{2m+r} = \mathbb{R}^{2m} \oplus \mathbb{R}^r\) and \(J\) is
\[ U(m) \times SO(r) \subset SO(2m) \times SO(r) \subset SO(2m+r). \]
There exists a lift

\[
\begin{array}{ccc}
Spin^c(2m) & \rightarrow & U(m) \\
\downarrow & & \downarrow \\
SO(2m) \times U(1) & \rightarrow & A \\
\end{array}
\]

and we can consider the diagram

\[
\begin{array}{ccc}
Spin(r) \times \mathbb{Z}_2 Spin(r) & \rightarrow & SO(r) \\
\downarrow & & \downarrow \\
SO(r) \times SO(r) & \rightarrow & SO(r) \times SO(r)
\end{array}
\]

We can put them together as follows

\[
Spin^c(2m) \times \mathbb{Z}_2 Spin^r(r) \cong Spin^r(2m) \times \mathbb{Z}_2 Spin^c(r) \hookrightarrow Spin(2m + r) \times \mathbb{Z}_2 Spin^c(r)
\]

where the last inclusion is due to Lemma\(^2\). It is easy to prove that the lift monomorphism \(U(m) \times SO(r) \rightarrow Spin^c(2m) \times \mathbb{Z}_2 Spin^r(r)\) exists and there is a natural isomorphism

\[
Spin^c(2m) \times \mathbb{Z}_2 Spin^r(r) \cong Spin^r(2m) \times \mathbb{Z}_2 Spin^c(r).
\]

\[\square\]

\[\langle\text{factorization}\rangle\]

**Lemma 2.5** Let \(r \in \mathbb{N}\). The standard representation \(\Delta_{2m+r}\) of \(Spin(2m + r)\) decomposes as follows

\[
\Delta_{2m+r} = \Delta_r \otimes \Delta_{2m}^+ \oplus \Delta_r \otimes \Delta_{2m}^-,
\]

with respect to the subgroup \(Spin(2m) \times \mathbb{Z}_2 Spin(r) \subset Spin(2m + r)\).

**Proof.** Consider the restriction of the standard representation of \(Spin(2m + r)\) to

\[
Spin(2m) \times \mathbb{Z}_2 Spin(r) \subset Spin(2m + r) \rightarrow Gl(\Delta_{2m+r}).
\]

By using the explicit description of a unitary basis of \(\Delta_{2m+r}\), we see that the elements of \(Spin(2m)\) act on the last \(m\) factors of

\[
\Delta_{2m+r} = \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{[r/2] \text{ times}} \otimes \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{m \text{ times}},
\]

as they do on \(\Delta_{2m} = \Delta_{2m}^+ \oplus \Delta_{2m}^-\). The elements of \(Spin(r)\) act as usual on the first \([r/2]\) factors of \(\Delta_r\), act trivially on \(\Delta_{2m}^+\), and act by multiplication by \((-1)\) on the factor \(\Delta_{2m}^-\).

\[\square\]
3 Twisted partially pure spinors

In order to simplify the statements, we will consider the twisted spin representation

$$\Sigma_r \otimes \Delta_n \subseteq \Delta_r \otimes \Delta_n.$$ 

where

$$\Sigma_r = \begin{cases} \Delta_r & \text{if } r \text{ is odd,} \\ \Delta_r^+ & \text{if } r \text{ is even,} \end{cases}$$

$n, r \in \mathbb{N}.$

Definition 3.1 Let $(f_1, \ldots, f_r)$ be an orthonormal frame of $\mathbb{R}^r$. A unit-length spinor $\phi \in \Sigma_r \otimes \Delta_n$, $r < n$, is called a twisted partially pure spinor if

- there exists a $(n - r)$-dimensional subspace $V^\phi \subset \mathbb{R}^n$ such that for every $X \in V^\phi$, there exists a $Y \in V^\phi$ such that $X \cdot \phi = i Y \cdot \phi$.
- it satisfies the equations

$$\langle f_k f_l \cdot \phi, \phi \rangle = 0,$$

for all $1 \leq k < l \leq r$.
- If $r = 4$, it also satisfies the condition

$$\langle f_1 f_2 f_3 f_4 \cdot \phi, \phi \rangle = 0.$$

Remarks.
1. The requirement $|\phi| = 1$ is made in order to avoid renormalizations later on.
2. The extra condition for the case $r = 4$ is fulfilled for all other ranks.
3. From now on we will drop the adjective twisted since it will be clear from the context.

3.1 Example of partially pure spinor

Lemma 3.1 Given $r, m \in \mathbb{N}$, there exists a partially pure spinor in $\Sigma_r \otimes \Delta_{2m+r}$.

Proof. Let $(e_1, \ldots, e_{2m}, e_{2m+1}, \ldots, e_{2m+r})$ and $(f_1, \ldots, f_r)$ be orthonormal frames of $\mathbb{R}^{2m+r}$ and $\mathbb{R}^r$ respectively. Consider the decomposition of Lemma 2.3

$$\Delta_{2m+r} = \Delta_r \otimes \Delta_{2m}^+ \oplus \Delta_r \otimes \Delta_{2m}^-,$$

corresponding to the decomposition

$$\mathbb{R}^{2m+r} = \text{span}\{e_1, \ldots, e_{2m}\} \oplus \text{span}\{e_{2m+1}, \ldots, e_{2m+r}\}.$$
Let
\[ \varphi_0 = u_1, \ldots, u_{r/2} \in \Delta_{2m}^+ \]
and
\[ \{ v_{\varepsilon_1, \ldots, \varepsilon_{[r/2]}} | (\varepsilon_1, \ldots, \varepsilon_{[r/2]}) \in \{ \pm 1 \}^{[r/2]} \} \]
be the unitary basis of the twisting factor \( \Delta_r = \Delta(\text{span}(f_1, \ldots, f_r)) \) which contains \( \Sigma_r \). Let us define the standard twisted partially pure spinor \( \phi_0 \in \Sigma_r \otimes \Delta_r \otimes \Delta_{2m}^+ \) by
\[
\phi_0 = \begin{cases} 
\frac{1}{\sqrt{2^{r/2}}}(\sum_{I \in \{ \pm 1 \}^{[r/2]}} v_I \otimes \gamma_r(u_I)) \otimes \varphi_0 & \text{if } r \text{ is odd,} \\
\frac{1}{\sqrt{2^{r/2}}}(\sum_{I \in \{ \pm 1 \}^{[r/2]}_+} v_I \otimes \gamma_r(u_I)) \otimes \varphi_0 & \text{if } r \text{ is even,}
\end{cases}
\]
where the elements of \( \{ \{ \pm 1 \}^{[r/2]}_+ \} \) contain an even number of \((-1)\).

Checking the conditions in the definition of partially pure spinor for \( \phi_0 \) is done by a (long) direct calculation as in [2]. For instance, taking \( n = 7, r = 3 \), we have
\[
\phi_0 = \frac{1}{\sqrt{2}}(v_1 \otimes \gamma_3(u_1) \otimes u_1 \otimes u_1 + v_{-1} \otimes \gamma_3(u_{-1}) \otimes u_1 \otimes u_1)
\]
where \( \gamma_3 \) is a quaternionic structure. We check that this \( \phi_0 \) is a partially pure spinor. Putting \( C = 1/\sqrt{2} \) and remembering that \( \gamma_3(u_\varepsilon) = -i\varepsilon u_{-\varepsilon} \), we get
\[
\phi_0 = iC(v_{-1} \otimes u_1 \otimes u_1 \otimes u_1 - v_1 \otimes u_{-1} \otimes u_1 \otimes u_1),
\]
which has unit length. Let \( \{ e_i \} \) be the standard basis of \( \mathbb{R}^7 \), so that
\[
e_1 \cdot \phi_0 = iC(v_{-1} \otimes u_1 \otimes u_1 \otimes g_1(u_1) - v_1 \otimes u_{-1} \otimes u_1 \otimes g_1(u_1)) = ie_2 \cdot \phi_0,
\]
and, similarly,
\[
e_3 \cdot \phi_0 = ie_4 \cdot \phi_0.
\]
So, \( \phi_0 \) induces the standard complex structure on \( V^{\phi_0} = \langle e_1, e_2, e_3, e_4 \rangle \). Let \( \{ f_i \} \) be the standard basis of \( \mathbb{R}^3 \). Similar calculations give
\[
\eta_{kl}^{\phi_0} = e_{4+k} \wedge e_{4+l},
\]
\[
(\eta_{kl}^{\phi_0} + f_{kl}) \cdot \phi_0 = 0,
\]
and
\[
\langle f_{kl} \cdot \phi_0, \phi_0 \rangle = 0.
\]
\[\square\]
3.2 Properties of partially pure spinors

**Lemma 3.2** The definition of partially pure spinor does not depend on the choice of orthonormal basis of \( \mathbb{R}^r \).

*Proof.* If \( r = 0, 1 \), a partially pure spinor is a classical pure spinor for \( n \) even or the straightforward generalization of pure spinor for \( n \) odd [5, p. 336]. Suppose \( (f'_1, \ldots, f'_r) \) is another orthonormal frame of \( \mathbb{R}^r \), then

\[
f'_i = \alpha_{i1} f_1 + \cdots + \alpha_{ir} f_r,
\]

so that the matrix \( A = (\alpha_{ij}) \in SO(r) \). Let us denote

\[
\eta^{\phi}_{kl}(X, Y) := \text{Re} \left< X \wedge Y \cdot f_k f_l \cdot \phi, \phi \right>
\]

Thus,

\[
\eta^{\phi}_{kl} \cdot \phi = \sum_{1 \leq a < b \leq n} \eta^{\phi}_{kl}(e_a, e_b) e_a e_b \cdot \phi
\]

\[
= \sum_{1 \leq a < b \leq n} \text{Re} \left< e_a e_b \cdot \left( \sum_{s=1}^r \alpha_{ks} f_s \right) \left( \sum_{t=1}^r \alpha_{lt} f_t \right) \cdot \phi, \phi \right> e_a e_b \cdot \phi
\]

\[
= \sum_{1 \leq a < b \leq n} \sum_{s=1}^r \sum_{t=1}^r \alpha_{ks} \alpha_{lt} \text{Re} \left< e_a e_b \cdot f_s f_t \cdot \phi, \phi \right> e_a e_b \cdot \phi
\]

\[
= \sum_{s=1}^r \sum_{t=1}^r \alpha_{ks} \alpha_{lt} \eta^{\phi}_{st} (e_a, e_b) e_a e_b \cdot \phi
\]

\[
= \sum_{s=1}^r \sum_{t=1}^r \alpha_{ks} \alpha_{lt} \eta^{\phi}_{st} \cdot \phi
\]

\[
= - \sum_{s=1}^r \sum_{t=1}^r \alpha_{ks} \alpha_{lt} f_s f_t \cdot \phi
\]

\[
= - \left( \sum_{s=1}^r \alpha_{ks} f_s \right) \left( \sum_{t=1}^r \alpha_{lt} f_t \right) \cdot \phi
\]

\[
= - f'_k f'_l \cdot \phi.
\]

For the third part of the definition, note that

\[
\left< f'_k f'_l \cdot \phi, \phi \right> = \left< \left( \sum_{s=1}^r \alpha_{ks} f_s \right) \left( \sum_{t=1}^r \alpha_{lt} f_t \right) \cdot \phi, \phi \right>
\]

\[
= \sum_{s=1}^r \sum_{t=1}^r \alpha_{ks} \alpha_{lt} \left< f_s f_t \cdot \phi, \phi \right>
\]

\[
= 0.
\]

For \( r = 4 \), the volume form is invariant under \( SO(4) \), \( f'_1 f'_2 f'_3 f'_4 = f_1 f_2 f_3 f_4 \), and

\[
\left< f'_1 f'_2 f'_3 f'_4 \cdot \phi, \phi \right> = \left< f_1 f_2 f_3 f_4 \cdot \phi, \phi \right> = 0.
\]
Lemma 3.3 Given a partially pure spinor \( \phi \in \Sigma_r \otimes \Delta_n \), there exists an orthogonal complex structure on \( V^\phi \) and \( n - r \equiv 0 \) (mod 2).

**Proof.** By definition, for every \( X \in V^\phi \), there exists \( Y \in V^\phi \) such that
\[
X \cdot \phi = iY \cdot \phi,
\]
and
\[
Y \cdot \phi = i(-X) \cdot \phi.
\]
If we set
\[
J^\phi(X) := Y;
\]
we get a linear transformation \( J^\phi : V^\phi \to V^\phi \), such that \((J^\phi)^2 = -\text{Id}_{V^\phi}\), i.e. \( J^\phi \) is a complex structure on the vector space \( V^\phi \) and \( \dim_{\mathbb{R}}(V^\phi) \) is even. Furthermore, this complex structure is orthogonal. Indeed, for every \( X \in V^\phi \),
\[
X \cdot JX \cdot \phi = -i|X|^2 \phi,
\]
\[
JX \cdot X \cdot \phi = i|JX|^2 \phi,
\]
and
\[
(-2 \langle X, JX \rangle + i(|JX|^2 - |X|^2)) \phi = 0,
\]
i.e.
\[
\langle X, JX \rangle = \ 0 \\
|X| = |JX|.
\]
\[\square\]

Lemma 3.4 Let \( r \geq 2 \) and \( \phi \in \Sigma_r \otimes \Delta_n \) be a partially pure spinor. The forms \( \eta^\phi_{kl} \) are non-zero, \( 1 \leq k < l \leq r \).

**Proof.** Since \((f_k f_l)^2 = -1\), the equation
\[
\eta^\phi_{kl} \cdot \phi = -f_k f_l \cdot \phi \tag{5}
\]
implies
\[
\eta^\phi_{kl} \cdot f_k f_l \cdot \phi = \phi. \tag{6}
\]
By taking an orthonormal frame \((e_1, \ldots, e_n)\) of \( \mathbb{R}^n \) we can write
\[
\eta^\phi_{kl} = \sum_{1 \leq i < j \leq n} \eta^\phi_{kl}(e_i, e_j) e_i e_j.
\]
By (\(\square\)), and taking hermitian product with \( \phi \)
\[
1 = |\phi|^2
\]
\[
\begin{align*}
= & \left\langle \eta_{kl}^\phi \cdot f_k f_l \cdot \phi, \phi \right\rangle \\
= & \left\langle \sum_{1 \leq i < j \leq n} \eta_{kl}^\phi(e_i, e_j) e_i e_j \cdot f_k f_l \cdot \phi, \phi \right\rangle \\
= & \sum_{1 \leq i < j \leq n} \eta_{kl}^\phi(e_i, e_j) (e_i e_j \cdot f_k f_l \cdot \phi) \\
= & \sum_{1 \leq i < j \leq n} \eta_{kl}^\phi(e_i, e_j)^2.
\end{align*}
\]

\[\square\]

\textbf{Lemma 3.5} Let \( r \geq 2 \). The image of the map associated to a partially pure spinor \( \phi \in \Sigma_r \otimes \Delta_n \),

\[
\Lambda^2 \mathbb{R}^r \longrightarrow \text{End}(\mathbb{R}^n) \\
f_{kl} \mapsto \hat{\eta}_{kl}^\phi,
\]

forms a Lie algebra of endomorphisms isomorphic to \( \mathfrak{so}(r) \).

\textit{Proof.} Let \((e_1, \ldots, e_n)\) be an orthonormal frame of \( \mathbb{R}^n \). First, let us consider the following calculation for \( i \neq j, k \neq l, s \neq t \):

\[
\text{Re} \left( e_se_t \cdot \eta_{ij}^\phi \cdot f_k f_l \cdot \phi, \phi \right) = \text{Re} \left( e_se_t \cdot \left( \sum_{a<b} \eta_{ij}^\phi(e_a, e_b) e_a e_b \right) \cdot f_k f_l \cdot \phi, \phi \right)
\]

\[
= \text{Re} \sum_{a<b} \eta_{ij}^\phi(e_a, e_b) \langle e_s \cdot e_t \cdot e_a \cdot e_b \cdot f_k f_l \cdot \phi, \phi \rangle
\]

\[
= - \sum_{a<b} \eta_{ij}^\phi(e_s, e_b) \eta_{kl}^\phi(e_b, e_t) + \sum_{b} \eta_{kl}^\phi(e_s, e_b) \eta_{ij}^\phi(e_b, e_t)
\]

\[
= - \sum_{b} \left[ \hat{\eta}_{kl}^\phi \right]_{tb} \left[ \hat{\eta}_{ij}^\phi \right]_{bs} + \sum_{b} \left[ \hat{\eta}_{ij}^\phi \right]_{tb} \left[ \hat{\eta}_{kl}^\phi \right]_{bs}
\]

\[
= \left[ \hat{\eta}_{ij}^\phi, \hat{\eta}_{kl}^\phi \right]_{ts}
\]

is the entry in row \( t \) and column \( s \) of the matrix \([\hat{\eta}_{ij}^\phi, \hat{\eta}_{kl}^\phi] \).

Secondly, we prove that the endomorphisms \( \hat{\eta}_{kl}^\phi \) satisfy the commutation relations of \( \mathfrak{so}(r) \):

1. If \( 1 \leq i, j, k, l \leq r \) are all different,

\[
[\hat{\eta}_{kl}^\phi, \hat{\eta}_{ij}^\phi] = 0. \tag{7} \text{eq:}[kl,ij]=0
\]

2. If \( 1 \leq i, j, k \leq r \) are all different,

\[
[\hat{\eta}_{ij}^\phi, \hat{\eta}_{jk}^\phi] = -\hat{\eta}_{ik}^\phi. \tag{8} \text{eq:}[ij,jk]=-ik
\]

To prove (7), note that by (5),

\[
\eta_{ij}^\phi \cdot f_k f_l \cdot \phi = \eta_{kl}^\phi \cdot f_i f_j \cdot \phi, \tag{9} \text{eq:identity1}
\]
by (7)
\[
\text{Re} \left( e_s e_t \cdot \eta_{ij}^\phi \cdot f_k f_l \cdot \phi, \phi \right) = \left[ \hat{\eta}_{ij}, \hat{\eta}_{kl} \right]_{ts},
\]
\[
\text{Re} \left( e_s e_t \cdot \eta_{kl}^\phi \cdot f_i f_j \cdot \phi, \phi \right) = \left[ \hat{\eta}_{kl}, \hat{\eta}_{ij} \right]_{ts},
\]
and by (9) and the anticommutativity of the bracket,
\[
\left[ \hat{\eta}_{ij}, \hat{\eta}_{kl} \right] = 0.
\]

To prove (8), note that by (5)
\[
f_i f_j \cdot \eta_{jk}^\phi \cdot \phi = f_i f_k \cdot \phi
\]
and
\[
f_j f_k \cdot \eta_{ij}^\phi \cdot \phi = -f_i f_k \cdot \phi
\]
so that
\[
f_j f_k \cdot \eta_{ij}^\phi \cdot \phi = f_i f_j \cdot \eta_{jk}^\phi \cdot \phi - 2f_i f_k \cdot \phi.
\]
Thus,
\[
\text{Re} \left( e_s e_t \cdot \eta_{ij}^\phi \cdot f_j f_k \cdot \phi, \phi \right) = \text{Re} \left( e_s e_t \cdot \eta_{jk}^\phi \cdot f_i f_j \cdot \phi, \phi \right) - 2\eta_{ik}^\phi (e_s, e_t)
\]
and by (7)
\[
\left[ \hat{\eta}_{ij}, \hat{\eta}_{jk} \right] = \left[ \hat{\eta}_{jk}, \hat{\eta}_{ij} \right] - 2\hat{\eta}_{ik}^\phi,
\]
i.e.
\[
\left[ \hat{\eta}_{ij}, \hat{\eta}_{jk} \right] = -\hat{\eta}_{ik}^\phi.
\]

Thirdly, we will prove, in five separate cases, that the set of endomorphisms \{\hat{\eta}_{kl}^\phi\} is linearly independent. For \(r = 0, 1\) there are no endomorphisms. For \(r = 2\) it is obvious since there is only one non-zero endomorphism. For \(r = 3\), suppose
\[
0 = \alpha_{12} \hat{\eta}_{12}^\phi + \alpha_{13} \hat{\eta}_{13}^\phi + \alpha_{23} \hat{\eta}_{23}^\phi,
\]
where \(\alpha_{12} \neq 0\). Take the Lie bracket with \(\hat{\eta}_{13}^\phi\) to get
\[
0 = \alpha_{12} \hat{\eta}_{23}^\phi - \alpha_{23} \hat{\eta}_{12}^\phi,
\]
i.e.
\[
\hat{\eta}_{23}^\phi = \frac{\alpha_{23}}{\alpha_{12}} \hat{\eta}_{12}^\phi.
\]

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We can also consider the bracket with $\hat{\eta}_{23}^\phi$,

$$0 = -\alpha_{12}\hat{\eta}_{13}^\phi + \alpha_{13}\hat{\eta}_{12}^\phi,$$

so that

$$\hat{\eta}_{13}^\phi = \frac{\alpha_{13}}{\alpha_{12}}\hat{\eta}_{12}^\phi.$$

By substituting in the original equation we get

$$0 = (\alpha_{12}^2 + \alpha_{13}^2 + \alpha_{23}^2)\hat{\eta}_{12}^\phi,$$

which gives a contradiction.

Now suppose $r \geq 5$ and that there is a linear combination

$$0 = \sum_{k<l} \alpha_{kl}\hat{\eta}_{kl}^\phi.$$

Taking successive brackets with $\hat{\eta}_{13}^\phi$, $\hat{\eta}_{12}^\phi$, $\hat{\eta}_{34}^\phi$ and $\hat{\eta}_{45}^\phi$ we get the identity

$$\alpha_{12}\hat{\eta}_{15}^\phi = 0,$$

i.e. $\alpha_{12} = 0$. Similar arguments give the vanishing of every $\alpha_{kl}$.

For $r = 4$, suppose there is a linear combination

$$0 = \alpha_{12}\eta_{12}^\phi + \alpha_{13}\eta_{13}^\phi + \alpha_{14}\eta_{14}^\phi + \alpha_{23}\eta_{23}^\phi + \alpha_{24}\eta_{24}^\phi + \alpha_{34}\eta_{34}^\phi.$$

Multiply by $-\phi$

$$0 = (\alpha_{12}f_{12} + \alpha_{13}f_{13} + \alpha_{14}f_{14} + \alpha_{23}f_{23} + \alpha_{24}f_{24} + \alpha_{34}f_{34}) \cdot \phi.$$

Multiply by $-f_{12}$

$$0 = (\alpha_{12} - \alpha_{13}f_{23} - \alpha_{14}f_{24} + \alpha_{23}f_{13} + \alpha_{24}f_{14} - \alpha_{34}f_{1234}) \cdot \phi.$$

Now, take hermitian product with $\phi$

$$0 = \langle (\alpha_{12} - \alpha_{13}f_{23} - \alpha_{14}f_{24} + \alpha_{23}f_{13} + \alpha_{24}f_{14} - \alpha_{34}f_{1234}) \cdot \phi, \phi \rangle$$

$$= \alpha_{12}|\phi|^2 - \alpha_{34} \langle f_{1234} \cdot \phi, \phi \rangle$$

$$= \alpha_{12}.$$

Similar arguments give the vanishing of the other coefficients.

\(\square\)

\textbf{Lemma 3.6} Let $r \geq 2$ and $\phi \in \Sigma_r \otimes \Delta_n$ be a partially pure spinor. Then

$$V^\phi \subseteq \bigcap_{1 \leq k < l \leq r} \ker \hat{\eta}_{kl}^\phi.$$
Proof. Let \( 1 \leq k < l \leq r \) be fixed and \( X \in V^\phi \). Since \( \mathbb{R}^n = V^\phi \oplus (V^\phi)^\perp \) and \( J^\phi \) is a complex structure on \( V^\phi \), there exists a basis \( \{ e_1, e_2, \ldots, e_{2m-1}, e_{2m} \} \cup \{ e_{2m+1}, \ldots, e_{2m+r} \} \) such that

\[
V^\phi = \text{span}(e_1, e_2, \ldots, e_{2m-1}, e_{2m}),
\]

\[
(V^\phi)^\perp = \text{span}(e_{2m+1}, \ldots, e_{2m+r}),
\]

\[
J^\phi(e_{2j-1}) = e_{2j},
\]

\[
J^\phi(e_{2j}) = -e_{2j-1},
\]

where \( m = (n - r)/2 \) and \( 1 \leq j \leq m \). Note that

\[
\hat{\eta}^\phi_{kl}(e_{2j-1}) = \sum_{a=1}^{n} \text{Re} \left< e_{2j-1} \wedge e_a \cdot f_{kl} \cdot \phi, \phi \right> e_a
\]

\[
= -\sum_{a \neq 2j-1}^{n} \text{Re} \left< f_{kl} \cdot e_a e_{2j-1} \cdot \phi, \phi \right> e_a
\]

\[
= -\sum_{a \neq 2j-1}^{n} \text{Re} \left< f_{kl} \cdot e_a (iJ^\phi(e_{2j-1})) \cdot \phi, \phi \right> e_a
\]

\[
= \sum_{a \neq 2j-1}^{n} \text{Im} \left< e_a e_{2j-1} \cdot f_{kl} \cdot \phi, \phi \right> e_a
\]

\[
= -\text{Im} \left< f_{kl} \cdot \phi, \phi \right> e_{2j}
\]

\[
= 0.
\]

\[ \square \]

Lemma 3.7 Let \( r \geq 2 \) and \( \phi \in \Sigma_r \otimes \Delta_n \) be a partially pure spinor. Then \( (V^\phi)^\perp \) carries a standard representation of \( \mathfrak{so}(r) \), and an orientation.

Proof. By Lemma 3.5 \( \mathfrak{so}(r) \) is represented non-trivially on \( \mathbb{R}^n = V^\phi \oplus (V^\phi)^\perp \) and, by Lemma 3.6 it acts trivially on \( V^\phi \). Thus \( (V^\phi)^\perp \) is a nontrivial representation of \( \mathfrak{so}(r) \) of dimension \( r \). \( \square \)

Remark. The existence of a partially pure spinor implies \( r \equiv n \pmod{2} \). In this case, let \( (e_1, \ldots, e_n) \) and \( (f_1, \ldots, f_r) \) be orthonormal frames for \( \mathbb{R}^n \) and \( \mathbb{R}^r \) respectively,

\[
v_n = e_1 \cdots e_n, \quad v_r = f_1 \cdots f_r,
\]

and

\[
F : \Sigma_r \otimes \Delta_n \to \Sigma_r \otimes \Delta_n \quad \phi \mapsto (-i)^{n/2}i^{r/2}v_n \cdot v_r \cdot \phi.
\]

Note that \( i^{r/2}v_r \) acts as \( (-1)^{r/2}\text{Id}_{\Sigma_r} \) on \( \Sigma_r \) and that \( (-i)^{n/2}v_n \) determines the decomposition \( \Delta_n = \Delta_n^+ \oplus \Delta_n^- \). Thus we have that

\[
\Sigma_r \otimes \Delta_n = (\Sigma_r \otimes \Delta_n)^+ \oplus (\Sigma_r \otimes \Delta_n)^-,
\]

and we will call elements in \( (\Sigma_r \otimes \Delta_n)^+ \) and \( (\Sigma_r \otimes \Delta_n)^- \) positive and negative twisted spinors respectively.

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Definition 3.2 Let \( n \) be even, \( \mathbb{R}^n \) be endowed with the standard inner product and orientation, and \( \text{vol}_n \) denote the volume form. Let \( V, W \) be two orthogonal oriented subspaces such that \( \mathbb{R}^n = V \oplus W \). Furthermore, assume \( V \) admits a complex structure inducing the given orientation on \( V \). The oriented triple \( (V, J, W) \) will be called positive if given (oriented) orthonormal frames \( (v_1, J(v_1), \ldots, v_m, J(v_m)) \) and \( (w_1, \ldots, w_r) \) of \( V \) and \( W \) respectively,

\[
v_1 \wedge J(v_1) \wedge \ldots \wedge v_m \wedge J(v_m) \wedge w_1 \wedge \ldots \wedge w_r = \text{vol}_n,
\]

and negative if

\[
v_1 \wedge J(v_1) \wedge \ldots \wedge v_m \wedge J(v_m) \wedge w_1 \wedge \ldots \wedge w_r = -\text{vol}_n.
\]

Lemma 3.8 If \( r \) is even, a partially pure spinor \( \phi \) is either positive or negative. Furthermore, a partially pure spinor \( \phi \) is positive (resp. negative) if and only if the corresponding oriented triple \( (V^\phi, J^\phi, (V^\phi)'^\perp) \) is positive (resp. negative).

Proof. We must prove that either \( \phi \in (\Sigma_r \otimes \Delta_n)^+ \) or \( \phi \in (\Sigma_r \otimes \Delta_n)^- \). Since \( \phi \) is a partially pure spinor, there exist frames \( (e'_1, \ldots, e'_m) \) and \( (e''_{2m+1}, \ldots, e''_{2m+r}) \) of \( V^\phi \) and \( (V^\phi)'^\perp \) respectively such that

\[
e'_{2j} = J(e'_{2j-1}) \quad \text{and} \quad \eta^\phi_{kl} = e''_{2m+k} \wedge e''_{2m+k},
\]

where \( 1 \leq j \leq m \) and \( 1 \leq k < l \leq r \). Now,

\[
e'_1 \wedge e'_2 \wedge \ldots \wedge e'_m \wedge e''_{2m+1} \wedge \ldots \wedge e''_{2m+r} = \pm \text{vol}_n.
\]

Then,

\[
(-i)^{n/2}r/2 \text{vol}_n \cdot \text{vol}_r \cdot \phi = \pm (-i)^{n/2}r/2 e'_1 e'_2 \cdots e'_m e''_{2m+1} \cdots e''_{2m+r} \cdot f_1 \cdots f_r \cdot \phi
\]

\[
= \pm (-i)^{n/2}r/2 e'_1 J(e'_1) \cdots e''_{2m-1} J(e''_{2m-1}) \eta^\phi_{12} \cdots \eta^\phi_{r-3,r-2} \cdot f_1 \cdots f_{r-1, r} \cdot \eta^\phi_{r-3,r} \cdot \phi
\]

\[
= \pm (-i)^{n/2}r/2 e'_1 J(e'_1) \cdots e''_{2m-1} J(e''_{2m-1}) \eta^\phi_{12} \cdots \eta^\phi_{r-3,r} \cdot f_1 \cdots f_{r-3, r} \cdot \phi
\]

\[
= \pm (-i)^{n/2}r/2 e'_1 J(e'_1) \cdots e''_{2m-3} J(e''_{2m-3}) e''_{2m-1} (-i e''_{2m-1}) \cdot \phi
\]

\[
= \pm (-1)^m (-i)^{n/2+m} r/2 \phi
\]

i.e. \( \phi \in (\Sigma_r \otimes \Delta_n)^\pm \).

3.3 Orbit of a partially pure spinor

Lemma 3.9 Let \( \phi \in \Sigma_r \otimes \Delta_n \) be a partially pure spinor. If \( g \in \text{Spin}^{c,r}(n) \), then \( g(\phi) \) is also a partially pure spinor.

Proof. Let \( g \in \text{Spin}^{c,r}(n) \) and \( \lambda_n^{c,r}(g) = (g_1, g_2, g_3) \in SO(n) \times SO(r) \times U(1) \). First, suppose \( X, Y \in V^\phi \),

\[
X \cdot \phi = iY \cdot \phi.
\]
Apply $g$ on both sides

$$g_1(X) \cdot g(\phi) = ig_1(Y) \cdot g(\phi).$$

which means that $g_1$ maps $V^{\phi}$ into $V^{g(\phi)}$ injectively. On the other hand, any pair of vectors $X, Y \in V^{g(\phi)}$ such that

$$X \cdot g(\phi) = iY \cdot g(\phi),$$

are the image under $g_1$ of some vectors $X, Y \in \mathbb{R}^n$, i.e.

$$g_1(X) \cdot g(\phi) = ig_1(Y) \cdot g(\phi).$$

Apply $g^{-1}$ on both sides to get

$$X \cdot \phi = iY \cdot \phi,$$

so that $X, Y \in V^{\phi}$, i.e. $V^{g(\phi)} = g_1(V^{\phi})$. Moreover,

$$J^{g(\phi)} = g_1 \big|_{V^{\phi}} \circ J^{\phi} \circ (g_1 \big|_{V^{\phi}})^{-1}.$$

Now, let $e'_a = g_1^{-1}(e_a)$ and $f'_k = g_2^{-1}(f_k)$, so that

$$\eta^{g(\phi)}_{kl} \cdot g(\phi) = \sum_{1 \leq a < b \leq n} \eta^{g(\phi)}_{kl}(e'_a, e'_b)e'_a e'_b \cdot g(\phi)$$

$$= \sum_{1 \leq a < b \leq n} \langle g_1(e'_a)g_1(e'_b) \cdot g_1(f'_k)g_1(f'_l), g(\phi) \rangle g_1(e'_a)g_1(e'_b) \cdot g(\phi)$$

$$= \sum_{1 \leq a < b \leq n} \langle e'_a e'_b \cdot f'_k f'_l, \phi \rangle g(e'_a e'_b \cdot \phi)$$

$$= g \left( \sum_{1 \leq a < b \leq n} \eta^{g(\phi)}_{kl}(e'_a, e'_b)e'_a e'_b \cdot \phi \right)$$

$$= g (\eta^{g(\phi)}_{kl} \cdot \phi)$$

$$= g (-f'_k f'_l \cdot \phi)$$

$$= -f'_k f'_l \cdot g(\phi),$$

and

$$\langle f'_k f'_l \cdot g(\phi), g(\phi) \rangle = \langle g(f'_k f'_l \cdot \phi), g(\phi) \rangle$$

$$= \langle f'_k f'_l \cdot \phi, \phi \rangle$$

$$= 0.$$

For $r = 4$, note that the volume form is invariant under $SO(4)$

$$\langle f_1 f_2 f_3 f_4 \cdot g(\phi), g(\phi) \rangle = \langle f_1 f_2 f_3 f_4 \cdot \phi, \phi \rangle = 0.$$ 

\[\square\]
Lemma 3.10 Let $\phi \in \Sigma_r \otimes \Delta_n$ be a partially pure spinor. The stabilizer of $\phi$ is isomorphic to $U(m) \times SO(r)$.

Proof. Let $g \in Spin^{c,r}(n)$ be such that $g(\phi) = \phi$ and $\lambda^{r}_{\phi}(g) = (g_{1}, g_{2}, g_{3}) \in SO(n) \times SO(r) \times U(1)$. It can be checked easily that

$$
\begin{align*}
[g_{1}, J^{\phi}] &= 0 \\
g_{1}(V^{\phi}) &= V^{\phi} \\
g_{1}|_{V^{\phi}} &\in U(V^{\phi}, J^{\phi}) \cong U(m).
\end{align*}
$$

Clearly, $g_{1}((V^{\phi})^{\perp}) = (V^{\phi})^{\perp}$.

As in Lemma 3.6, one can prove

$$
\eta_{kl}^{\phi} = \sum_{2m+1 \leq a < b \leq 2m+r} \eta_{kl}^{\phi}(e_{a}, e_{a})e_{a}e_{b} \in \Lambda^{2}(V^{\phi})^{\perp},
$$

where $(e_{1}, \ldots, e_{2m+r})$ is an oriented frame of $V^{\phi} \oplus (V^{\phi})^{\perp}$. Furthermore,

$$
g_{1}(\eta_{kl}^{\phi}) = \eta_{kl}^{\phi},
$$

where $f'_{k} = g_{2}(f_{k})$. Now, we have that

$$
\begin{align*}
f_{k}f_{l} &\xrightarrow{g_{2}} f'_{k}f'_{l} \\
\downarrow &\downarrow \\
\eta_{kl}^{\phi} &\xrightarrow{h_{2}} \eta_{kl}^{\phi}
\end{align*}
$$

for the diagram

$$
\begin{array}{ccc}
\mathfrak{so}(r) & \xrightarrow{\Lambda^{2}\mathbb{R}^{r}} & \Lambda^{2}\mathbb{R}^{r} \\
\downarrow & \downarrow \\
\mathfrak{so}(r) & \xrightarrow{\Lambda^{2}(V^{\phi})^{\perp}} & \Lambda^{2}(V^{\phi})^{\perp}
\end{array}
$$

where the vertical arrows are Lie algebra isomorphisms and the horizontal arrows correspond to $g_{2}$ and $h_{2}$ acting via the adjoint representation of $SO(r)$. Thus, $h_{2}$ and $g_{2}$ correspond to each other under the isomorphism $\Lambda^{2}(V^{\phi})^{\perp} \cong \Lambda^{2}\mathbb{R}^{r}$ given by $f_{kl} \mapsto \eta_{kl}^{\phi}$.

Since $h_{1}$ is unitary with respect to $J$, there is a frame $(e_{1}, \ldots, e_{2m})$ of $V^{\phi}$ such that

$$
e_{2j} = J(e_{2j-1})
$$

and $h_{1}$ is diagonal with respect to the unitary basis $\{e_{2j-1} - ie_{2j} \mid j = 1, \ldots, m\}$, i.e.

$$
h_{1}(e_{2j-1} - ie_{2j}) = e^{i\theta_{j}}(e_{2j-1} - ie_{2j})
$$

where $0 \leq \theta_{j} < 2\pi$. On the other hand, there is a frame $(f_{1}, \ldots, f_{r})$ of $\mathbb{R}^{r}$ such that

$$
g_{2} = R_{\varphi_{1}} \circ \cdots \circ R_{\varphi_{[r/2]}}
$$
where $R_{\varphi_k}$ is a rotation by an angle $\varphi_k$ on the plane generated by $f_{2k-1}$ and $f_{2k}$, $1 \leq k \leq \lfloor r/2 \rfloor$. Now, since the endomorphisms $\hat{\eta}^\phi_{kl}$ span an isomorphic copy of $\mathfrak{so}(r)$, there is a frame $(e_{2m+1}, \ldots, e_{2m+r})$ of $(V^\phi)^\perp$ such that

$$\hat{\eta}^\phi_{kl} = e_{2m+k} \wedge e_{2m+l},$$

$1 \leq k < l \leq r$. Since the adjoint representation of $SO(r)$ is faithful

$$h_2 = R'_{\varphi_1} \cdots R'_{\varphi_{\lfloor r/2 \rfloor}}$$

where $R'_{\varphi_k}$ is a rotation by an angle $\varphi_k$ on the plane generated by $e_{2m+2k-1}$ and $e_{2m+2k}$, $1 \leq k \leq \lfloor r/2 \rfloor$. Thus,

$$g = \pm \left[ \prod_{j=1}^{m} (\cos(\theta_j/2) - \sin(\theta_j/2)e_{2j-1}e_{2j}) \cdot \prod_{k=1}^{\lfloor r/2 \rfloor} (\cos(\varphi_k/2) - \sin(\varphi_k/2)\eta^\phi_{2k-1,2k}) \right] \cdot \prod_{k=1}^{\lfloor r/2 \rfloor} (\cos(\varphi_k/2) - \sin(\varphi_k/2)f_{2k-1}f_{2k}), e^{i\theta/2}.$$

Now,

$$\phi = g(\phi)$$

$$= \pm e^{i\theta/2} \prod_{j=1}^{m} (\cos(\theta_j/2) - \sin(\theta_j/2)e_{2j-1}e_{2j})$$

$$\cdot \prod_{k=1}^{\lfloor r/2 \rfloor} (\cos(\varphi_k/2) - \sin(\varphi_k/2)\eta^\phi_{2k-1,2k}) \cdot \prod_{k=1}^{\lfloor r/2 \rfloor - 1} (\cos(\varphi_k/2) - \sin(\varphi_k/2)\eta^\phi_{2k-1,2k})$$

$$\cdot \prod_{k=1}^{\lfloor r/2 \rfloor} (\cos(\varphi_k/2) - \sin(\varphi_k/2)f_{2k-1}f_{2k})(\cos(\varphi_{\lfloor r/2 \rfloor}/2) - \sin(\varphi_{\lfloor r/2 \rfloor}/2)\eta^\phi_{2[\lfloor r/2 \rfloor]-1,2[\lfloor r/2 \rfloor]}) \cdot (\phi)$$

$$= \pm e^{i\theta/2} \prod_{j=1}^{m} (\cos(\theta_j/2) - \sin(\theta_j/2)e_{2j-1}e_{2j})$$

$$\cdot \prod_{k=1}^{\lfloor r/2 \rfloor - 1} (\cos(\varphi_k/2) - \sin(\varphi_k/2)\eta^\phi_{2k-1,2k}) \cdot \prod_{k=1}^{\lfloor r/2 \rfloor - 1} (\cos(\varphi_k/2) - \sin(\varphi_k/2)f_{2k-1}f_{2k})$$

$$(\cos(\varphi_{\lfloor r/2 \rfloor}/2) - \sin(\varphi_{\lfloor r/2 \rfloor}/2)f_{2[\lfloor r/2 \rfloor]-1}f_{2[\lfloor r/2 \rfloor]} \cdot (\cos(\varphi_{\lfloor r/2 \rfloor}/2) + \sin(\varphi_{\lfloor r/2 \rfloor}/2)f_{2[\lfloor r/2 \rfloor]-1}f_{2[\lfloor r/2 \rfloor]} \cdot (\phi)$$

$$= \pm e^{i\theta/2} \prod_{j=1}^{m} (\cos(\theta_j/2) - \sin(\theta_j/2)e_{2j-1}e_{2j})$$

$$\cdot \prod_{k=1}^{\lfloor r/2 \rfloor - 1} (\cos(\varphi_k/2) - \sin(\varphi_k/2)\eta^\phi_{2k-1,2k}) \cdot \prod_{k=1}^{\lfloor r/2 \rfloor - 1} (\cos(\varphi_k/2) - \sin(\varphi_k/2)f_{2k-1}f_{2k})(\phi).$$
\[ \pm e^{i\theta/2} \prod_{j=1}^{m} (\cos(\theta_j/2) - \sin(\theta_j/2)e_{2j-1}e_{2j})(\phi) \]
\[ = \pm e^{i\theta/2} \prod_{j=1}^{m} (\cos(\theta_j/2) + i\sin(\theta_j/2)e_{2j-1}(iJ(e_{2j-1}^{-1}))(\phi) \]
\[ = \pm e^{i\theta/2} \prod_{j=1}^{m} (\cos(\theta_j/2) + i\sin(\theta_j/2)e_{2j-1}e_{2j})(\phi) \]
\[ = \pm e^{i\theta/2} \prod_{j=1}^{m} (\cos(\theta_j/2) - i\sin(\theta_j/2))(\phi) \]
\[ = \pm e^{i\theta/2} \prod_{j=1}^{m} e^{-i\theta_j/2}(\phi) \]
\[ = \pm e^{i\theta - \sum_{j=1}^{m} \theta_j}(\phi). \]

This means
\[ e^{i\theta - \sum_{j=1}^{m} \theta_j} = \pm 1 \]
i.e.
\[ \det_{\mathbb{C}}(h_1) = e^{i\sum_{j=1}^{m} \theta_j} \]
\[ = e^{i\theta}. \]

Thus we have found that
\[ \lambda^{c,r}_n(g) = ((h_1, h_2), h_2, \det_{\mathbb{C}}(h_1)), \]
which is in the image of the horizontal row in the diagram of Lemma 2.4
\[ Spin^{c,r}(n) \]
\[ \xrightarrow{\mathcal{F}} \]
\[ U(m) \times SO(r) \rightarrow SO(n) \times SO(r) \times U(1) \]

\[ \square \]

**Remark.** Note that for any spinor \( \phi \in \Sigma \otimes \Delta_n \), \( g \in Spin^{c,r}(n) \), \( \lambda^{c,r}_n(g) \in SO(n) \times SO(r) \times U(1) \),
\[ \eta^{g(\phi)}_{kl}(X,Y) = \langle X \wedge Y \cdot f_k, f_l \cdot g(\phi), g(\phi) \rangle \]
\[ = \langle g_1(X') \wedge g_1(Y') \cdot g_2(f'_k)g_2(f'_l) \cdot g(\phi), g(\phi) \rangle \]
\[ = \langle g(X' \wedge Y' \cdot f'_k, f'_l \cdot g(\phi), g(\phi) \rangle \]
\[ = \langle X' \wedge Y' \cdot f'_k, f'_l \cdot g(\phi) \rangle \]
\[ = \eta^{\bar{g}(\phi)}_{kl}(X', Y'), \]

for \( X' = g_1^{-1}(X), Y' = g_1^{-1}(Y) \in \mathbb{R}^n, f'_k = g_2^{-1}(f_k) \). Thus, the matrices representing \( \eta^{g(\phi)}_{kl} \) (with respect to some basis) are conjugate to the matrices representing \( \eta^{\bar{g}(\phi)}_{kl} \).
Lemma 3.11 Let $\phi, \psi \in \Sigma_r \otimes \Delta_n$ be partially pure spinors and $\text{Spin}^c(r)$ the standard copy of this group in $\text{Spin}^{c,r}(n)$. Then, $\psi \in \text{Spin}^c(r) \smallsetminus \phi$ if and only if they generate the same oriented triple $(V^\phi, J^\phi, (V^\phi)\perp) = (V^\psi, J^\psi, (V^\psi)\perp)$.

Proof. Suppose $\psi = g(\phi)$ for some $g \in \text{Spin}^c(r) \subset \text{Spin}^{c,r}(n)$, and let $\lambda_n^r(g) = (g_2, e^{i\theta})$. Such an element induces

$$\langle X \wedge Y \cdot f_k f_l \cdot g(\phi), g(\phi) \rangle = \langle X \wedge Y \cdot f'_k f'_l \cdot \phi, \phi \rangle$$

for $f'_k = g_2^{-1}(f_k)$, i.e.

$$\eta^g_{kl}(X, Y) = \eta^\phi_{kl}(X, Y),$$

so that they span the same copy of $\mathfrak{so}(r)$ in $\text{End}^-(\mathbb{R}^n)$,

$$\text{span}^1(\eta^g_{kl}(\phi)) = \text{span}(\eta^\phi_{kl}) \cong \mathfrak{so}(r) \subset \text{End}^-(\mathbb{R}^n).$$

Thus, by Lemma 3.9, the partially pure spinors $\phi$ and $g(\phi)$ determine the same oriented triple $(V^g(\phi), J^g(\phi), (V^g(\phi))\perp) = (V^\phi, V^\psi, (V^\psi)\perp)$.

Conversely, assume $(V^\phi, J^\phi, (V^\phi)\perp) = (V^\psi, J^\psi, (V^\psi)\perp)$, and consider the subalgebras of

$$\mathfrak{so}(r)_{\phi} = \text{span}(\eta^\phi_{kl}) + f_{kl}$$

$$\mathfrak{so}(r)_{\psi} = \text{span}(\eta^\psi_{kl}) + f_{kl}.$$ 

There exist frames $(e_{2m+1}, \ldots, e_{2m+r})$ and $(e_{2m+1}', \ldots, e_{2m+r}')$ of $(V^\phi)^\perp$ and $(V^\psi)^\perp$ respectively, such that

$$\eta^\phi_{kl} = e_{2m+k} \wedge e_{2m+l},$$

$$\eta^\psi_{kl} = e'_{2m+k} \wedge e'_{2m+l}.$$

Let $A = (a_{kl}) \in \text{SO}(r)$ the matrix such that

$$A : e_{2m+k}' \mapsto a_{k1}e_{2m+1}' + \cdots + a_{kr}e_{2m+r}' = e_{2m+k}$$

$1 \leq k < l \leq r$. The induced transformation maps

$$A : e_{2m+k}' \wedge e_{2m+l}' \mapsto e_{2m+k} \wedge e_{2m+l},$$

and set

$$A^T : f_k \mapsto a_{k1}f_1 + \cdots + a_{kr}f_r = f'_k,$$

and

$$A^T : f_k \wedge f_l \mapsto f'_k \wedge f'_l.$$ 

Consider

$$\langle e_{2m+p} \wedge e_{2m+q} \cdot f_k f_l \cdot \psi, \psi \rangle = \left\langle \left( \sum_{s=1}^r a_{ps} e_{2m+s}' \right) \wedge \left( \sum_{t=1}^r a_{qt} e_{2m+t}' \right) \cdot f_k f_l \cdot \psi, \psi \right\rangle$$
More precisely, under this subgroup

$$\begin{align*}
\text{But there is only a 1-dimensional summand in the decomposition of } \Sigma^r_n & \text{ since the induced transformation by } A = \text{exp}(\eta^e_{kl}) \sim e_{2m+k} \wedge e_{2m+t}.
\end{align*}$$

Now consider $g \in Spin^c(r) \subset Spin^{r,n}$ such that $\lambda^{r,n}_n(g) = (1, A, 1) \in SO(n) \times SO(r) \times U(1)$. Then

$$\begin{align*}
\delta_{pk}\delta_{ql} &= \langle e_{2m+p} \wedge e_{2m+q} :: f_{k} f_{l} \cdot \psi, \psi \rangle \\
&= \langle g(e_{2m+p} \wedge e_{2m+q} :: f_{k} f_{l} \cdot \psi), g(\psi) \rangle \\
&= \langle e_{2m+p} \wedge e_{2m+q} :: A(f_{k}) A(f_{l}) \cdot g(\psi), g(\psi) \rangle \\
&= \langle e_{2m+p} \wedge e_{2m+q} :: f_{k} f_{l} \cdot g(\psi), g(\psi) \rangle,
\end{align*}$$

so that

$$\begin{align*}
\mathfrak{so}(r)^g(\psi) &= \text{span}(\eta^g_{kl} \cdot f_{kl}) = \text{span}((\eta^g_{kl} + f_{kl}) = \mathfrak{so}(r)^{\phi}.
\end{align*}$$

This implies that $g(\psi)$ and $\phi$ share the same stabilizer

$$\begin{align*}
U(V^\phi, J^\phi) \times \text{exp} (\mathfrak{so}(r)^\phi) &= U(V^g(\psi), J^g(\psi)) \times \text{exp} (\mathfrak{so}(r)^g(\psi)) \cong U(m) \times SO(r).
\end{align*}$$

But there is only a 1-dimensional summand in the decomposition of $\Sigma_r \otimes \Delta_n$ under this subgroup. More precisely, under this subgroup

$$\begin{align*}
\Sigma_r \otimes \Delta_n & = \Sigma_r \otimes \Delta_r \otimes \Delta_{2m},
\end{align*}$$

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Lemma 3.12  

• If $r$ is odd, the group $\text{Spin}^{c,r}(n)$ acts transitively on the set of partially pure spinors in $\Sigma_r \otimes \Delta_n$.

• If $r$ is even, the group $\text{Spin}^{c,r}(n)$ acts transitively on the set of positive partially pure spinors in $(\Sigma_r \otimes \Delta_n)^+$. 

Proof. Suppose that $r$ is odd. Note that the standard partially pure spinor $\phi_0$ satisfies the conditions

\[
\begin{align*}
  e_{2j-1}e_{2j} \cdot \phi_0 &= i\phi_0, \\
  e_{2m+k}e_{2m+l} \cdot \phi &= -f_{kl} \cdot \phi, \\
  \langle f_{kl} \cdot \phi, \phi \rangle &= 0,
\end{align*}
\]

where $(e_1, \ldots, e_n)$ and $(f_1, \ldots, f_r)$ are the standard oriented frames of $\mathbb{R}^n$ and $\mathbb{R}^r$ respectively. There exist frames $(e'_1, \ldots, e'_{2m})$ and $(e'_{2m+1}, \ldots, e'_{2m+r})$ of $V^\phi$ and $(V^\phi)^\perp$ respectively such that

\[
e'_{2j} = J(e'_{2j-1}) \quad \text{and} \quad \eta_{kl}^\phi = e'_{2m+k} \wedge e'_{2m+l},
\]

$1 \leq k < l \leq r, 1 \leq j \leq m$. Call $g'_1 \in O(n)$ the transformation of $\mathbb{R}^n$ taking the new frame to the standard one. Define $g_1 \in SO(n)$ as follows

\[
\begin{align*}
g_1 &= g'_1, & \text{if } e'_1 \wedge \ldots \wedge e'_{2m+r} = \text{vol}_n, \\
g_1 &= -g'_1, & \text{if } e'_1 \wedge \ldots \wedge e'_{2m+r} = -\text{vol}_n.
\end{align*}
\]

Then $(g_1, 1, 1) \in SO(n) \times SO(r) \times U(1)$ has two preimages $\pm \tilde{g} \in \text{Spin}^{c,r}(n)$. By Lemma 3.9, $\tilde{g}(\phi)$ is a partially pure spinor. We will check that $\tilde{g}(\phi)$ satisfies (10) as $\phi_0$ does. Indeed,

\[
e_{2j-1}e_{2j} \cdot \tilde{g}(\phi) = g'_1(e'_{2j-1})g'_1(e'_{2j}) \cdot \tilde{g}(\phi) = (\pm g_1(e'_1))(\pm g_1(e'_{2j})) \cdot \tilde{g}(\phi) = g_1(e'_1)g_1(e'_{2j}) \cdot \tilde{g}(\phi) = \tilde{g}(e'_{2j-1}e'_{2j} \cdot \phi) = \tilde{g}(i\phi) = i\tilde{g}(\phi),
\]

and

\[
e_{2m+k}e_{2m+l} \cdot \tilde{g}(\phi) = g'_1(e'_{2m+k})g'_1(e'_{2m+l}) \cdot \tilde{g}(\phi) = (\pm g_1(e'_{2m+k}))(\pm g_1(e'_{2m+l})) \cdot \tilde{g}(\phi) = g_1(e'_{2m+k})g_1(e'_{2m+l}) \cdot \tilde{g}(\phi) = \tilde{g}(e'_{2m+k}e'_{2m+l} \cdot \phi) = \tilde{g}(-f_{kl} \cdot \phi)
\]
\[
\begin{align*}
&= -\lambda_2(\tilde{g})(f_k)\lambda_2(\tilde{g})(f_l) \cdot \tilde{g}(\phi) \\
&= -f_k f_l \cdot \tilde{g}(\phi),
\end{align*}
\]
since \(\lambda_2(\tilde{g}) = 1\). Similarly,
\[
\langle f_k f_l \cdot \tilde{g}(\phi), \tilde{g}(\phi) \rangle = \langle \lambda_2(\tilde{g})(f_k)\lambda_2(\tilde{g})(f_l) \cdot \tilde{g}(\phi), \tilde{g}(\phi) \rangle \\
= \langle \tilde{g}(f_k f_l \cdot \phi), \tilde{g}(\phi) \rangle \\
= \langle f_k f_l \cdot \phi, \phi \rangle \\
= 0.
\]
Thus, \(\tilde{g}(\phi)\) generates the same oriented triple \((V\tilde{g}(\phi), J\tilde{g}(\phi), (V\tilde{g}(\phi))^\perp) = (V\phi_0, J\phi_0, (V\phi_0)^\perp)\) as \(\phi_0\) which, by Lemma 3.11 concludes the proof for \(r\) odd.

The case for \(r\) even is similar. \(\square\)

**Theorem 3.1** Let \(\mathbb{R}^n\) be endowed with the standard inner product and orientation. Given \(r \in \mathbb{N}\) such that \(r < n\), the following objects are equivalent:

1. A (positive) triple consisting of a codimension \(r\) vector subspace endowed with an orthogonal complex structure and an oriented orthogonal complement.
2. An orbit \(\text{Spin}^c(r) \cdot \phi\) for some (positive) twisted partially pure spinor \(\phi \in \Delta_n \otimes \Sigma_r\).

**Proof.** Given a codimension \(r\) vector subspace \(D\) endowed with an orthogonal complex structure, \(\dim_{\mathbb{R}}(D) = 2m, n = 2m + r\). By Lemma 2.5

\[
\Delta_n \cong \Delta(D^\perp) \otimes \Delta(D).
\]

Let us define

\[
\Sigma_r \cong \begin{cases} 
\Delta(D^\perp) & \text{if } r \text{ is odd,} \\
\Delta(D^\perp)^+ & \text{if } r \text{ is even,}
\end{cases}
\]

so that

\[
\Sigma_r \otimes \Delta_n
\]

contains the standard twisted partially pure spinor \(\phi_0\) of Lemma 3.1.

The proof of the converse is the content of Subsection 3.2. \(\square\)

Let \(\tilde{S}\) denote the set of all partially pure spinors of rank \(r\)

\[
\tilde{S} = \frac{\text{Spin}^c(n)}{U(m) \times \text{SO}(r)}
\]

Consider

\[
S = \frac{\tilde{S}}{\text{Spin}^c(r)}
\]

where \(\text{Spin}^c(r)\) is the canonical copy of such a group in \(\text{Spin}^c(n)\). Thus we have the following expected result.
Corollary 3.1 The space parametrizing subspaces with orthogonal complex structures of codimension $r$ in $\mathbb{R}^n$ with oriented orthogonal complements is

$$S \cong \frac{SO(n)}{U(m) \times SO(r)}.$$

References

1. Charlton, P., *The Geometry of Pure Spinors, with Applications*. PhD thesis, Department of Mathematics, The University of Newcastle, Australia, 1998.
2. Espinosa, M.; Herrera, R.: Spinorially twisted Spin structures, I: curvature identities and eigenvalue estimates. Preprint (2014), arXiv:1409.6246
3. Friedrich, Th.: Dirac operators in Riemannian geometry, Graduate studies in mathematics, Volume 25, American Mathematical Society.
4. Herrera, R.; Nakad, R.: Spinorially twisted spin structures, III: CR structures. (in progress)
5. Lawson, H. B., Jr.; Michelsohn, M.-L.: Spin geometry. Princeton Mathematical Series, 38. Princeton University Press, Princeton, NJ, 1989. xii+427 pp. ISBN: 0-691-08542-0
6. Trautman A.; Trautman, K.: Generalized pure spinors. Journal of Geometry and Physics 15 (1994), 121.