Iterative Decoding Performance Bounds for LDPC Codes on Noisy Channels

Chun-Hao Hsu and Achilleas Anastasopoulos
Electrical Engineering and Computer Science Department
University of Michigan, Ann Arbor, MI, 48109-2122
e-mail: {chhsu, anastas}@umich.edu

Abstract

The asymptotic iterative decoding performances of low-density parity-check (LDPC) codes using min-sum (MS) and sum-product (SP) decoding algorithms on memoryless binary-input output-symmetric (MBIOS) channels are analyzed in this paper. For MS decoding, the analysis is done by upper bounding the bit error probability of the root bit of a tree code by the sequence error probability of a subcode of the tree code assuming the transmission of the all-zero codeword. The result is a recursive upper bound on the bit error probability after each iteration. For SP decoding, we derive a recursively determined lower bound on the bit error probability after each iteration. This recursive lower bound recovers the density evolution equation of LDPC codes on the binary erasure channel (BEC) with inequalities satisfied with equalities. A significant implication of this result is that the performance of LDPC codes under SP decoding on the BEC is an upper bound of the performance on all MBIOS channels with the same uncoded bit error probability. All results hold for the more general multi-edge type LDPC codes.

1 Introduction

Low-density parity-check (LDPC) codes, first introduced by Gallager [1], have been shown to be powerful channel codes under iterative decoding by numerous simulations in the literature. However, the loopy graphical structure of the LDPC codes generally prohibits their iterative decoding performance from being exactly analyzed. This problem has been partially solved in [2] by considering the asymptotic average performance of an ensemble of LDPC codes when their codeword length goes to infinity. In particular, the authors in [2] proved that the performance of each code in the ensemble asymptotically approaches the average performance of the ensemble with loop-free local structure within finite iterations as the codeword length goes to infinity. This result gave birth to the density evolution (DE) method proposed in [2], and was used to provide the exact asymptotic performance of LDPC codes after an arbitrary number of iterations on memoryless binary-input output-symmetric (MBIOS) channels.

A major drawback of the DE method is that the evolved densities in general require an infinite dimensional description. Therefore, DE is not suitable for the derivation of analytical performance bounds. To solve this problem, several approaches have been proposed that track the evolution of the densities projected to some specific finite dimensional space. These include the Gaussian
approximation [3], the extrinsic information transfer (EXIT) chart [4], and the generalized EXIT (GEXIT) [5] chart methods. Unfortunately, since the EXIT and GEXIT chart methods still require numerical calculations, and the Gaussian approximation does not imply any upper or lower bounds on the exact performance, these results still can not be used to analytically bound the code performance.

In [6], the authors propose to map the evolved densities to Bhattacharyya parameters, which are further used to bound the bit error probability of the LDPC codes under sum-product (SP) decoding [7] on MBIOS channels. It turns out that an upper bound on the Bhattacharyya parameters can be obtained by a recursion involving only one-dimensional real numbers, so the whole result can be used to determine a guaranteed decoding capability for the LDPC codes. In this paper, we improve this result of [6] by showing that the same recursively determined upper bound on the bit error probability of the LDPC codes not only holds for the SP decoding, but also holds for the min-sum (MS) decoding [7]. This result is attained by upper bounding the probability of error of the root bit of a tree code by a sequence error probability of a subcode of the tree code, and then using the union bound. Therefore, the whole proof does not involve the Bhattacharyya parameters.

Then, we turn our attention to SP decoding and derive a recursive lower bound on the probability of bit error after each iteration. This recursive lower bound becomes exact and recovers the one-dimensional density evolution equation on the binary erasure channel (BEC). Exploiting the resemblance of our recursion to the DE equation on the BEC, we prove that LDPC codes under SP decoding on an MBIOS channel with uncoded bit error probability $P_0$ always have an equal or worse performance than that on the BEC with erasure probability $\epsilon = 2P_0$. This result should be compared with the result in [8], where a tighter recursive upper bound than that in [6] on the Bhattacharyya parameters associated with the outgoing message of bits (and hence an upper bound on the bit error probability) after each iteration is derived, which also exactly recovers the density evolution equation on the BEC. Therefore, the iterative decoding analysis of LDPC codes on the BEC can be used to bound the performance of LDPC codes from below and above on all MBIOS channels via our result and the result in [8]. Note also that, due the nature of the proofs of the main lemmas, this connection between BEC and MBIOS channels is also true for the general family of multi-edge type LDPC codes [9], including the irregular repeat-accumulate (IRA) codes [10, 11] and the low-density parity-check and generator matrix (LDPC-GM) codes [12].

The remaining of this paper is structured as follows. In Section 2 we review the preliminary background on MBIOS channels and the asymptotic analysis of LDPC codes. Then, we present our asymptotic performance analysis of LDPC codes on MBIOS channels under MS and SP decoding in Section 3 and 4 respectively. Finally, we conclude this paper in Section 5.

## 2 Preliminaries

If $x \in \{0, 1\}$ and $y \in \mathbb{R}$ are the input and output symbols, respectively, of an MBIOS channel with conditional density function $f(y|x)$, then we have the following symmetry condition:

$$f(y|0) = f(-y|1), \quad \forall y \in \mathbb{R}$$

(1)

Note that all our results hold also when $y$ is a discrete random variable, in which case, we simply treat $f(y|x)$ as a conditional probability mass function and integration signs as summation signs wherever necessary. However, for convenience, we will assume $y$ to be a continuous random variable throughout this paper. To analyze the asymptotic average iterative decoding performance of a $(\lambda, \rho)$
irregular LDPC ensemble when the codeword length goes to infinity, where \( \lambda(x) \triangleq \sum_{i=1}^{\infty} \lambda_i x^{i-1} \), and \( \rho(x) \triangleq \sum_{i=1}^{\infty} \rho_i x^{i-1} \) are the standard variable and check node degree distributions, respectively, from the edge perspective as defined in [13], it is shown in [2] that we can as well consider the cycle-free case. In this case, the probability of bit error after \( l \) decoding iterations is the probability of decoding error of the root bit on the tree of \( l + 1 \) (variable node) levels whose construction is dictated by the degree distributions \( (\lambda, \rho) \) as in [13]. Due to the symmetry condition of MBIOS channels, we can also assume without loss of generality that the all-zero codeword is transmitted. Hence, in the following two sections, we will analyze the asymptotic MS and SP iterative decoding performance of the \( (\lambda, \rho) \) LDPC ensemble on MBIOS channels by considering the corresponding tree codes and assuming the transmission of the all-zero codeword.

3 Min-Sum Decoding Performance Analysis

Consider an arbitrary binary tree code whose codebook is \( C = \{c_0, c_1, \ldots, c_M\} \), where \( c_i = (c_{i1}, c_{i2}, \ldots, c_{in}) \) is a codeword of length \( n \) for all \( i \), and \( c_0 \) is the all-zero codeword. Let \( c_{i1} \) be the root bit of this tree for all \( i \), \( x = (x_1, x_2, \ldots, x_n) \) the transmitted codeword, and \( y = (y_1, y_2, \ldots, y_n) \) the received sequence from an MBIOS channel with conditional probability mass function \( p(y|x) \). When min-sum (MS) decoding is performed on this tree, it essentially performs maximum likelihood sequence detection (MLSQD) on the whole sequence to produce an estimate \( \hat{c} = (\hat{c}_1, \hat{c}_2, \ldots, \hat{c}_n) \). Therefore, if we define the decision region for the codeword \( c_i \) as

\[
\mathcal{U}_i \triangleq \bigcap_{k \neq i} \mathcal{U}_{ik},
\]

where

\[
\mathcal{U}_{ik} \triangleq \left\{ y \in \mathbb{R}^n \mid \prod_{j=1}^{n} p(y_j|c_{ij}) \geq \prod_{j=1}^{n} p(y_j|c_{kj}) \right\},
\]

then the probability of the root bit being in error under MS decoding assuming \( x = 0 \) is

\[
P_{b}^{MS} = Pr(\hat{c}_1 = 1|x = c_0) = Pr\left( y \in \bigcup_{i,c_{i1}=1} \mathcal{U}_i | x = c_0 \right).
\]

We would like to find a compact representation of the set

\[
\mathcal{U} \triangleq \bigcup_{i,c_{i1}=1} \mathcal{U}_i,
\]

or at least a superset of \( \mathcal{U} \) so that we can bound \( P_{b}^{MS} \) from above. Note that such an upper bound is also a valid upper bound on the probability of root bit error \( P_{b}^{SP} \) when the SP decoding is performed on the tree since the SP decoding essentially performs optimal maximum likelihood decoding for each bit.

**Definition 1** Define the subcode \( C_r \) of \( C \) as the set of codewords in \( C \) such that

(i) the root bit is 1, and
(ii) each check node with parent variable node equal to 1 has exactly one child variable node equal to 1, and

(iii) each check node with parent variable node equal to 0 has all children variable nodes equal to 0.

We show how we can use this reduced codebook $C_r$ to characterize $U$ in the following lemma.

**Lemma 1** \( U \subset \bigcup_{i,c_i \in C_r} U_{i,c_i} \).

**Proof:** Given any \( y \in U = \bigcup_{i,c_i \in C_r} U_{i,c_i} \), there exists a \( k \) such that \( y \in U_k \) and \( c_{k1} = 1 \). To proceed with our proof, we first carry out the following labeling procedure on the codeword \( c_k \).

1. At the initial state, the root bit is labelled as “survivor”, and all the other variable nodes are unlabelled. Note that in this labeling procedure, the “survivor” variable nodes will always have value 1 in \( c_k \). We first consider the check nodes at the topmost level of the tree.

2. For every check node \( c \) at this level whose parent node is labelled as “survivor”, it must have at least one child variable node with value 1 in \( c_k \) since “survivor” nodes always have value 1 in \( c_k \). Choose an arbitrary child variable node of \( c \) with value 1 in \( c_k \), and label it as “survivor”. Then, label the subtrees emanating from the other unlabelled child variable nodes of \( c \) as “dropped”.

3. If there are no check nodes at the next lower level of the tree, stop. Otherwise, move to the check nodes at the next lower level and go back to 2.

As we can see after this labeling procedure, the check nodes with a “survivor” parent node all have exactly one “survivor” child node, and the ones with a “dropped” parent node have purely “dropped” child nodes. Therefore, if we let

\[
    c_m = \begin{cases} 
    0 & \text{for all “dropped” bits} \\
    1 = c_k & \text{for all “survivor” bits}
\end{cases} 
\tag{6}
\]

then we have \( c_m \in C_r \subset C \). In the following, we would like to prove that \( y \in U_{m0} \), which completes the proof. Let

\[
    c_l = \begin{cases} 
    c_k & \text{for all “dropped” bits} \\
    0 & \text{for all “survivor” bits}
\end{cases} 
\tag{7}
\]

i.e., let \( c_l \) be the bitwise XOR of \( c_k \) and \( c_m \). Then, since \( c_k \) and \( c_m \) are valid codewords in \( C \), so is \( c_l \). Moreover, since \( y \in U_k \) implies \( y \in U_{kl} \), we have from (6) that

\[
    \prod_{j=1}^{n} p(y_j | c_{k_j}) \geq \prod_{j=1}^{n} p(y_j | c_{l_j}) \Rightarrow \prod_{\text{all “survivor” bits } j} p(y_j | c_{k_j}) \geq \prod_{\text{all “survivor” bits } j} p(y_j | c_{l_j})
\]

\[
    \Rightarrow \prod_{\text{all “survivor” bits } j} p(y_j | c_{m_j}) \geq \prod_{\text{all “survivor” bits } j} p(y_j | 0)
\]

\[
    \Rightarrow \prod_{j=1}^{n} p(y_j | c_{m_j}) \geq \prod_{j=1}^{n} p(y_j | 0) \tag{8}
\]

which proves that \( y \in U_{m0} \) as desired.
This lemma shows that we can upper bound $P_b^{MS}$ by the probability of MLSqD error $P_s^{MLSqD}$ on the reduced codebook $\mathcal{C}_r$, assuming that the all-zero codeword is transmitted. One way to proceed from here is to use union bound and the fact (see [14, Theorem 7.5] for a proof) that

$$Pr(y \in \mathcal{U}_b | x = c_0) \leq D^{w(c_i)}, \forall i$$

where $w(c_i)$ denotes the Hamming weight of $c_i$, and $D$ is the Bhattacharryya parameter associated with the MBIOS channel $p(y|x)$, to further upper bound $P_s^{MLSqD}$. For this purpose, we would like to introduce the weight enumerator $N_i(x)$ of the reduced codebook $\mathcal{C}_i$ of the tree code $G_l$ of level $l+1$ associated with a randomly drawn code from the $(\lambda, \rho)$ LDPC ensemble. Let $A_i$ be the number of codewords of weight $i$ in $\mathcal{C}_i$. We define $N_i(x)$ by $N_i(x) \triangleq \sum_{i=1}^{\infty} A_i x^i$. Moreover, let $\bar{N}_i(x)$ denote the expected value of $N_i(x)$ averaged over the whole $(\lambda, \rho)$ LDPC ensemble. We have the following lemma.

**Lemma 2** $\bar{N}_0(x) = x$ and

$$\bar{N}_1(x) = \lambda(\rho' (1) \bar{N}_{l-1}(x)), \forall l \geq 1$$

where $\rho'(x)$ denotes the derivative of $\rho(x)$.

**Proof:** It is obvious that $\bar{N}_0(x) = x$. To prove the recursion for $\bar{N}_l(x)$, first consider the subtree emanating from the $i$th check node $c_i$, immediately below the root bit. Let $Z_i^{(0)}(x)$ denote the weight enumerator of the reduced codebook of this subtree. Since, the root bit is 1 for all codewords in $\mathcal{C}_l$, there is exactly one child subtree with root 1 emanating from $c_i$ for all codewords in $\mathcal{C}_l$. Therefore, if $c_i$ has degree $d_c$, then we have

$$Z_i^{(0)}(x) = (d_c - 1)N_{l-1}(x) \Rightarrow \bar{Z}_l^{(0)}(x) = \sum_{i=1}^{\infty} (i - 1)N_{l-1}(x) \rho_i = \rho'(1)\bar{N}_{l-1}(x).$$

Similarly, if the root bit has degree $d_v$, then we have

$$N_l(x) = \prod_{i=1}^{d_c-1} Z_i^{(0)}(x) \Rightarrow \bar{N}_l(x) = \sum_{j=1}^{d_v} \prod_{i=1}^{d_c-1} Z_i^{(0)}(x) \lambda_j = \sum_{i=j}^{d_v} \prod_{i=1}^{d_c-1} Z_i^{(0)}(x) \lambda_j = \lambda (\bar{Z}_l^{(1)}(x)).$$

where the second equality follows from the fact that the subtrees emanating from different $c_i$’s are generated independently, and the third equality follows from the fact that $\bar{Z}_i^{(0)}(x)$ does not depend on $i$ as shown in [11]. Combining [11] and [12], the lemma is proved.

Using this lemma and the union bound on $P_s^{MLSqD}$, we have the following theorem.

**Theorem 2** Given any $(\lambda, \rho)$ LDPC ensemble, let $P_i^{MS}$ and $P_i^{SP}$ be its asymptotic (as the codeword length approaches infinity) average bit error probability after $l$ iterations under MS and SP decoding, respectively, on an MBIOS channel with Bhattacharryya parameter $D$. If we define the sequence $\{z_l\}_{l=0}^{\infty}$ by $z_0 = D$, and $z_l = \lambda(\rho'(1)z_{l-1})$, for all $l \geq 1$, then we have $P_i^{SP} \leq P_i^{MS} \leq z_l$, for all $l \geq 0$.

**Proof:** From Lemma 2 we see that $z_l = \bar{N}_l(D)$ for all $l$. Now, the lemma follows from the union bound on $P_s^{MLSqD}$ as discussed above.

A similar result to Theorem 2 is established in [6, Lemma 1]. However, Theorem 2 differs from [6, Lemma 1] in two aspects. First, Theorem 2 holds for both MS and SP decoding while [6, Lemma 1] holds only for the SP decoding. Second, we did not keep track of the evolution of the Bhattacharryya parameters, which are used in [6] to bound $P_i^{SP}$ for all $l$. 

\[\]
4 Sum-Product Decoding Performance Analysis

Let

\[ m = \log \frac{f(y|0)}{f(y|1)} \]  

be the log-likelihood ratio (LLR) of the input variable \( x \) given the output variable \( y \) of the MBIOS channel \( f(y|x) \). Moreover, let \( M \) be the random variable whose realization is \( m \) assuming \( x = 0 \), and \( g(m) \) be the probability density function (pdf) of \( M \). Then the symmetry condition (11) becomes

\[ g(-m) = e^{-m}g(m), \quad \forall m \in \mathbb{R} \]  

Define the probability of error \( P_e(M) \) under ML decoding of the LLR \( M \) as follows

\[ P_e(M) = \int_{-\infty}^{0} g(m) dm. \]  

We have the following lemma.

Lemma 3

\[ E \left[ \tanh \frac{|M|}{2} \right] = 1 - 2P_e(M) \]  

Proof:

\[ E \left[ \tanh \frac{|M|}{2} \right] = \int_{0}^{\infty} \left( \tanh \frac{m}{2} \right) [g(m) + g(-m)] \, dm \]

\[ = \int_{0}^{\infty} \left( \frac{1 - e^{-m}}{1 + e^{-m}} \right) [g(m) + e^{-m}g(m)] \, dm \]

\[ = \int_{0}^{\infty} (1 - e^{-m}) g(m) \, dm \]

\[ = \int_{0}^{\infty} g(m) \, dm - \int_{-\infty}^{0} g(m) \, dm \]

\[ = 1 - 2P_e(M) \]  

Now, consider the SP decoding on a tree code used on an MBIOS channel. As shown in [2], all the SP decoding messages can be represented by the LLR’s and satisfy the symmetry condition (14). Assuming that the all-zero codeword is transmitted, we have the following lemma describing the relationship between the probability of ML decoding errors associated with the incoming and outgoing messages of a check node. Note that in the following, we will use capital letters to denote random variables, whose realizations are denoted by the corresponding lower-case letters.

Lemma 4 Let \( c \) be a check node of degree \( d_c \). Furthermore, let \( M_c \) be the outgoing message on an edge, and \( M_1, M_2, \ldots, M_{d_c-1} \) the incoming messages from the other edges. Assuming all the incoming messages are independent with each other, we have

\[ 1 - 2P_e(M_c) = \prod_{i=0}^{d_c-1} [1 - 2P_e(M_i)] \]
Proof: It follows from Lemma \ref{lemma1} the fact that under SP decoding \cite{2},
\begin{equation}
\tanh \frac{|m_c|}{2} = \prod_{i=1}^{d_c-1} \tanh \frac{|m_i|}{2}
\end{equation}
and the independence of the incoming messages. 
A similar relationship for a variable node is also derived. However, since the derivation is more complicated than the one for a check node, we first consider the simple case where the number of incoming messages is 2.

Lemma 5 Let \( v \) be a variable node. Furthermore, let \( M_v \) be the outgoing message on an edge, and \( M_1 \) and \( M_2 \) the incoming messages from either the other edges or the channel. Assuming \( M_1 \) and \( M_2 \) are the only incoming messages and independent with each other, we have
\begin{equation}
2P_e(M_v) \geq \prod_{i=1}^{2} [2P_e(M_i)]
\end{equation}
Proof: From Lemma \ref{lemma1} we have
\begin{equation}
2P_e(M_v) \geq \prod_{i=1}^{2} [2P_e(M_i)]
\end{equation}
\begin{align}
\Leftrightarrow & 1 - E \left[ \tanh \frac{|M_v|}{2} \right] \geq \left( 1 - E \left[ \tanh \frac{|M_1|}{2} \right] \right) \left( 1 - E \left[ \tanh \frac{|M_2|}{2} \right] \right) \\
\iff & E \left[ \tanh \frac{|M_v|}{2} \right] \leq E \left[ \tanh \frac{|M_1|}{2} \right] + E \left[ \tanh \frac{|M_2|}{2} \right] - E \left[ \tanh \frac{|M_1|}{2} \right] E \left[ \tanh \frac{|M_2|}{2} \right]
\end{align}
Since under SP decoding, we have from \cite{2} that \( m_v = m_1 + m_2 \), the left hand side of (21) becomes
\begin{align}
& \int_{m_1 + m_2 \geq 0} [1 - e^{-(m_1 + m_2)}] g_1(m_1)g_2(m_2)dm_1dm_2 \\
& = \int_{m_1 \geq 0, m_2 \geq 0} [1 - e^{-(m_1 + m_2)}] g_1(m_1)g_2(m_2)dm_1dm_2 \\
& \quad + \int_{m_1 \geq -m_2, m_2 \leq 0} [1 - e^{-(m_1 + m_2)}] g_1(m_1)g_2(m_2)dm_1dm_2 \\
& \quad + \int_{m_2 \geq -m_1, m_1 \leq 0} [1 - e^{-(m_1 + m_2)}] g_1(m_1)g_2(m_2)dm_1dm_2
\end{align}
where \( g_1 \) and \( g_2 \) are the pdf’s of \( M_1 \) and \( M_2 \), respectively. On the other hand, the right hand side becomes
\begin{align}
& \int_{m_1 \geq 0} (1 - e^{-m_1}) g_1(m_1)dm_1 + \int_{m_2 \geq 0} (1 - e^{-m_2}) g_2(m_2)dm_2 \\
& \quad - \int_{m_1 \geq 0, m_2 \geq 0} (1 - e^{-m_1}) (1 - e^{-m_2}) g_1(m_1)g_2(m_2)dm_1dm_2 \\
& = \int_{m_1 \geq 0, m_2 \geq 0} [1 - e^{-(m_1 + m_2)}] g_1(m_1)g_2(m_2)dm_1dm_2 \\
& \quad + \int_{m_1 \geq 0, m_2 < 0} (1 - e^{-m_1}) g_1(m_1)g_2(m_2)dm_1dm_2 + \int_{m_2 \geq 0, m_1 < 0} (1 - e^{-m_2}) g_1(m_1)g_2(m_2)dm_1dm_2
\end{align}
Now, since
\[ \int_{m_i \geq 0, m_j < 0} (1 - e^{-m_i}) g_i(m_i) g_j(m_j) dm_i dm_j \geq \int_{m_i \geq -m_j, m_j < 0} (1 - e^{-m_i}) g_i(m_i) g_j(m_j) dm_i dm_j \]
\[ \geq \int_{m_i \geq -m_j, m_j < 0} (1 - e^{-m_i} e^{-m_j}) g_i(m_i) g_j(m_j) dm_i dm_j \]

for \((i, j)\) equals \((1, 2)\) and \((2, 1)\), (21) is true, and the lemma is proved. ■

Armed with the previous lemma, we can proceed to the general case where the variable node has degree \(d_v\).

**Corollary 1** Let \(v\) be a variable node of degree \(d_v\). Furthermore, let \(M_0\) be the incoming message from the channel, \(M_v\) the outgoing message on an edge, and \(M_1, M_2, \ldots, M_{d_v-1}\) the incoming messages from the other edges. Assuming all the incoming messages are independent with each other, we have

\[ 2P_e(M_v) \geq \prod_{i=0}^{d_v-1} [2P_e(M_i)] \] (25)

**Proof:** It follows directly from Lemma 5 and induction. ■

For a tree code used on an MBIOS channel under SP decoding, all the incoming messages to the variable and check nodes are independent with each other. Hence, Lemma 4 and Corollary 1 can be used to characterize the evolution of the probability of error associated with the messages after processes of the variable and check nodes on a tree, and imply the following theorem.

**Theorem 3** For the \((\lambda, \rho)\) LDPC ensemble used on an MBIOS channel, the probability of bit error \(P_l\) associated with the outgoing message of any variable node after the \(l\)th decoding iteration asymptotically satisfies

\[ 2P_l \geq 2P_0 \lambda (1 - \rho (1 - 2P_{l-1})) \] (26)

where \(P_0\) is the uncoded bit error probability under ML decoding of the channel.

**Proof:** As discussed in Section 2, the probability of error associated with the outgoing message of any variable node after the \(l\)th decoding iteration is, asymptotically as the codeword length goes to infinity, the one of the root variable node of a tree of level \(l + 1\). Since all the variable and check nodes in the tree have exactly the same degree distributions \(\lambda\) and \(\rho\), respectively, and the channel is memoryless, all the incoming messages from child nodes to a parent node are independent and identically distributed. Hence, if we let \(c\) be a check node in the tree, \(M_c\) its outgoing message, and \(M_v\) one of its incoming message, then we have from Lemma 4 that

\[ 2P_e(M_c) = 1 - \sum_{i=1}^{\infty} [1 - 2P_e(M_v)]^{(i-1)} \rho_i = 1 - \rho (1 - 2P_e(M_v)) \] (27)

Similarly, if we let \(v\) be a variable node in the tree, \(M_c\) its outgoing message, \(M_0\) the incoming message from the channel, and \(M_v\) one of its incoming messages from its child nodes, then we have from Corollary 1 that

\[ 2P_e(M_v) \geq 2P_e(M_0) \sum_{i=1}^{\infty} [2P_e(M_c)]^{(i-1)} \lambda_i = 2P_e(M_0) \lambda (1 - 2P_e(M_c)) \] (28)
Combining (27), (28), and the fact that $\rho$ is a monotonically increasing function (since all $\rho_i$’s are nonnegative), the theorem is proved. Notice that on the binary erasure channel (BEC) of erasure probability $\epsilon$, the probability of uncoded bit error of this channel is $\epsilon/2$. Hence, (26) is satisfied with equality, and recovers the well-known DE equation

$$x_l = \epsilon \lambda (1 - \rho (1 - x_{l-1})), \quad x_0 = \epsilon$$

(29)

where $x_l$ is the bit erasure probability after the $l$th iteration on the BEC. Using this fact, we have the following corollary.

**Corollary 2** For any $(\lambda, \rho)$ LDPC ensemble, its asymptotic SP decoding performance on the MBIOS channel with uncoded bit error probability $P_0$ is always worse than its asymptotic SP decoding performance on the BEC with erasure probability $\epsilon \leq 2P_0$.

**Proof:** From (26) and (29), we have $x_l \leq 2P_l$ for all $l \geq 0$ when $\epsilon \leq 2P_0$. Since the erasure probability is two times the bit error probability, the corollary is proved.

Notice that, Lemma 4 and Corollary 1 can also be used in the more general family of multi-edge type LDPC codes [9], including the irregular repeat-accumulate (IRA) codes [10, 11] and the low-density parity-check and generator matrix (LDPC-GM) codes [12], to produce similar results. Hence, Corollary 2 is not restricted to the irregular LDPC codes, but also holds for the general multi-edge type LDPC codes.

5 Conclusion

In this paper, we analyze the asymptotic performance of LDPC codes under MS and SP decoding on MBIOS channels. This is done by upper bounding the bit error probability of the root bit of the tree code associated with the $(\lambda, \rho)$ LDPC ensemble assuming the all-zero codeword is transmitted. When MS decoding is performed on this tree code, we upper bound the probability of the root bit being in error by the probability of sequence error under ML decoding of a subcode of the tree code. A recursive equation describing the evolution of the weight enumerator of this subcode after each iteration is then derived and used in a union bound to bound the ML decoded sequence error of this subcode. As a result, we obtain a recursive upper bound on the bit error probability as a function of the number of iterations for the LDPC codes under MS decoding on MBIOS channels. Note that these upper bounds are also upper bounds for the SP decoding since SP decoding is optimal on the bit error probability for tree codes. This result is very similar to [6, Lemma 1] with the difference being that we establish it not only for the SP decoding, but also for the MS decoding, and that we obtain it via a totally different approach.

When SP decoding performance is considered, we derive a recursive lower bound on the probability of bit error as a function of the number of iterations. Note that this is also a lower bound for the MS decoding due to the optimality of SP decoding on the bit error probability. More significantly, this recursion recovers the DE equation on the BEC for LDPC codes with the lower bound being an exact equality. This further implies that the SP decoding performance of LDPC codes on the BEC can serve as a upper bound of the performance on all MBIOS channels with the same probability of uncoded bit error. This result is also true for the more general multi-edge type LDPC codes, including IRA and LDPC-GM codes, since the main ingredient in the proof, i.e., Lemma 4 and Corollary 1, can also be utilized for these codes.
References

[1] R. G. Gallager, “Low density parity check codes,” IEEE Trans. Information Theory, vol. 8, pp. 21–28, Jan. 1962.

[2] T. J. Richardson and R. L. Urbanke, “The capacity of low-density parity-check codes under message-passing decoding,” IEEE Trans. Information Theory, vol. 47, no. 2, pp. 599–618, Feb. 2001.

[3] T. J. Richardson S.-Y. Chung and R. L. Urbanke, “Analysis of sum-product decoding of low-density parity-check codes using a Gaussian approximation,” IEEE Trans. Information Theory, vol. 47, no. 2, pp. 657–670, Feb. 2001.

[4] S. ten Brink, “Convergence behavior of iteratively decoded parallel concatenated codes,” IEEE Trans. Communications, vol. 49, no. 10, pp. 1727–1737, Oct. 2001.

[5] C. Measson, A. Montanari, and R. Urbanke, “Why we can not surpass capacity: The matching condition,” in Proc. Allerton Conf. Commun., Control, Comp., Monticello, IL, Sept. 2005.

[6] M. Lentmaier, D. Truhachev, K. Zigangirov, and D. Costello, “An analysis of the block error probability performance of iterative decoding,” IEEE Trans. Information Theory, vol. 51, no. 11, pp. 3834–3855, Nov. 2005.

[7] N. Wiberg, Codes and Decoding on General Graphs, Ph.D. thesis, Linköping University, Linköping, Sweden, 1996.

[8] H. Jin and T. Richardson, “Block error iterative decoding capacity for LDPC codes,” in Proc. International Symposium on Information Theory, Adelaide, Australia, Sept. 2005, pp. 52–56.

[9] T. Richardson and R. Urbanke, “Multi-edge type LDPC codes,” 2004, [Online]. Available: http://lthcwww.epfl.ch/papers/multiedge.ps.

[10] H. Jin, A. Khandekar, and R. J. McEliece, “Irregular repeat-accumulate codes,” in Proc. International Symposium on Turbo Codes and Related Topics, Brest, France, Sept. 2000, pp. 1–8.

[11] H. D. Pfister, I. Sason, and R. Urbanke, “Capacity-achieving ensembles for the binary erasure channel with bounded complexity,” IEEE Trans. Information Theory, vol. 51, no. 7, pp. 2352–2379, July 2005.

[12] C.-H. Hsu and A. Anastasopoulos, “Capacity-achieving codes with bounded graphical complexity on noisy channels,” in Proc. Allerton Conf. Commun., Control, Comp., Monticello, IL, Sept. 2005.

[13] T. J. Richardson, M. A. Shokrollahi, and R. L. Urbanke, “Design of capacity-approaching irregular low-density parity-check codes,” IEEE Trans. Information Theory, vol. 47, no. 2, pp. 619–637, Feb. 2001.

[14] R. J. McEliece, The Theory of Information and Coding, 2nd ed., Cambridge University Press, Cambridge, UK, 2002.