Radiative corrections to the cross section of $e^- + p \rightarrow \nu + n$ and the crossed processes.

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Abstract

Born cross section and the radiative corrections to its lowest order are considered in the framework of QED with structureless nucleons including the emission of virtual and real photons. Result is generalized to take into account radiative corrections in higher orders of perturbation theory in the leading and next-to leading logarithmic approximation. Crossing processes are considered in the leading approximation.

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I. INTRODUCTION

The process of neutron $\beta$ decay is rather completely investigated experimentally. Its theoretical description has not been sufficiently studied up to now, due to uncertainties connected with effects of strong interactions which arise of calculations of the radiative corrections (RC). Research into neutrino-deuterium inelastic scattering was carried out in experimental set-ups like the Sudbury Neutrino Observatory (SNO)\cite{4}. The increasing accuracy of experiments and the importance of adequate theoretical description of the neutrino sector of Standard Model (SM) is the motivation of our paper. It is devoted to calculation of RC in the framework of QED with the point-like nucleons and omission the electro-weak effects. The last ones were considered in a series of papers (see\cite{1} and the references therein).

Below we consider the process

$$e(p_e) + P(p) \rightarrow \nu(p_\nu) + N(p_n),$$

and the crossed processes

$$\nu(p_\nu) + N(p_n) \rightarrow e(p_e) + P(p), (\nu n),$$

$$\bar{\nu}(p_\nu) + P(p) \rightarrow e^+(p_e) + N(p_n), (\nu p).$$

Our work is performed in terms of the old formulation of weak processes which is valid for the intermediate region of energies of electron $E$ (we suppose proton to be at rest -Laboratory Frame)

$$m = m_e \ll E \ll M_Z.$$

II. BORN CROSS SECTION. KINEMATICS

The matrix element of the process $ep \rightarrow \nu n$ in the Born approximation has the form:

$$M = \frac{G}{\sqrt{2}} V_{ud} \bar{u}_\nu(p_\nu) \gamma_\alpha(1 + \gamma_5)u(p_e) \bar{u}_n \gamma_\alpha(1 + \rho \gamma_5) u(p),$$

with $G, V_{ud}$ being the Fermi constant, the Cabbibo-Kobayashi-Maskawa quark-mixing matrix element and $\rho = g_A/g_V = 1.25$ being the ratio of axial and vector coupling constants.

The cross-section has the form

$$d\sigma = \frac{1}{8s} \sum |M_{ij}|^2 d\Gamma,$$
with
\[\sum |M_{if}|^2 = 8G_F^2|V_{ud}|^2 S_0(p_e, p),\]
\[S_0(p_e, p) = s^2(1 + \rho)^2 + u^2(1 - \rho)^2 - 2M^2(s + u)(1 - \rho^2);\]  \(6\)
where
\[s = 2ME; \quad u = -2p_ep_n = -2E(E_n - P_ne), \quad p_e^2 = m^2, \quad p^2 = p_n^2 = M^2,\]
and the phase volume
\[d\Gamma = \frac{d^3p_e d^3p_n}{2E_n 2E_n}(2\pi)^{-2}\delta^4(p_e + p - p_e - p_n).\]  \(7\)
Performing the integration on neutrino momentum we write the phase volume as
\[d\Gamma = \frac{c d\alpha}{2\pi} \frac{M E(M + E)^2}{[(M + E)^2 - E^2c^2]^2};\]  \(8\)
where \(E-\) is the energy of initial electron, \(c = \cos \theta\) and \(\theta\) is the angle in Laboratory Frame between the 3-momenta of electron and neutron. The energy \(E_n\) and the value of 3-momentum of neutron \(P_n\) are:
\[E_n = M \frac{(E + M)^2 + E^2c^2}{(E + M)^2 - E^2c^2}; \quad P_n = M \frac{2E(E + M)c}{(E + M)^2 - E^2c^2}.\]  \(9\)
Another form of phase volume can be used for investigating the neutron energy distribution
\[d\Gamma = \frac{dE_n}{8\pi E}.\]  \(10\)

III. VIRTUAL PHOTON EMISSION AND CONTRIBUTION COUNTER TERM

Consider now the one-loop virtual radiative correction: the doubled interference of Born amplitude with one with the emission of virtual photon taken into account. As in Born amplitude we suggest that the loop momentum as well as the external ones are much small compared to the \(W\)-boson mass. Only one triangle Feynman diagram is relevant. The corresponding contribution to the summed up spin states of the square of matrix element is
\[\Delta \sum |M_{virt}|^2 = (G_F V_{ud}/\sqrt{2})^2 \frac{8\alpha}{\pi} \int \frac{d^4k}{i\pi^2} \frac{T}{(0)(1)(2)},\]
\[(0) = k^2 - \lambda^2; \quad (1) = (p - k)^2 - m^2 + i0; \quad (2) = (p + k)^2 - M^2 + i0,\]  \(11\)
where $k$ is the loop 4-momentum, $\lambda$ is the fictitious photon mass, and

\[ T = \frac{1}{4} Tr \hat{p}_\nu \gamma_\alpha (1 + \gamma_5) (\hat{p}_e - \hat{k}) \gamma_\mu \hat{p}_\nu \gamma_\eta (1 + \gamma_5) \]

\[ \frac{1}{4} Tr (\hat{p}_n + M) \gamma_\alpha (1 + \rho \gamma_5) (\hat{p} + \hat{k} + M) \gamma_\mu (\hat{p} + M) \gamma_\eta (1 + \rho \gamma_5). \]  

(12)

The next step is to perform the loop momentum integration. To avoid ultraviolet divergences we must introduce the cut-off parameter $|k^2| < \Lambda^2$. It is natural to choose the cut-off parameter to be equal to the $W$-boson mass:

\[ \Lambda = M_W. \]

Using the Feynman trick to join the denominators we obtain the expression

\[ \int \frac{d^4k}{i\pi^2} \int_0^1 dx \int_0^1 dy \frac{T}{[(k - yp_x)^2 - D]^3}, \]

with $D = y^2 p_x^2 + (1 - x) \lambda^2 + i0, p_x^2 = x^2 m^2 + (1 - x)^2 M^2 - sx (1 - x)$.

The computation of the traces leads to:

\[ T = A_0 + A_e(2kp_e) + A_n(2kp_n) + A_p(2kp) + A_{en}(2kp_e)(2kp_n) + A_{ep}(2kp_e)(2kp) + A_{pn}(2kp)(2kp_n) + A_{ee}(2kp_e)^2 + A_{nn}(2kp_n)^2 + A_{kk}k^2, \]  

(13)

with

\[ A_0 = 2s S_0; \quad A_e = 2(s + M^2)[s(1 + \rho)^2 - u(1 - \rho)^2]; \]

\[ A_n = 2s[u(1 - \rho)^2 - M^2(1 - \rho^2)]; \]

\[ A_p = -2s^2(1 + \rho)^2 + 2(2s + u)M^2(1 - \rho^2); \]

\[ A_{np} = u(1 - \rho)^2; \quad A_{ee} = (s + u)(1 - \rho)^2 - 4M^2 \rho(1 - \rho); \]

\[ A_{kk} = -2u(u + s)(1 - \rho)^2 - 4s^2(1 + \rho)^2 + 4M^2(s + u)(1 + \rho - 2\rho^2); \]

\[ A_{ep} = u(1 - \rho)^2 - 4M^2(1 - \rho^2); \]

\[ A_{en} = -(2s + u)(1 - \rho)^2 + 4M^2 \rho(1 - \rho). \]  

(14)
Further integration (we are interested only in the real part) gives

\[ [i; i_x; i_x b; i_x x; i_{x x b}; i_{x x b b}] = \int_0^1 \frac{dx}{P_x^2} [1; x, 1 - x; x^2; x(1 - x); (1 - x)^2] = \]

\[ -\frac{1}{s} L; -\frac{1}{s} L + \frac{1}{a} L_M; -\frac{1}{a} L_M; \]

\[ -\frac{1}{s} L + \frac{1}{a} + \frac{s + 2M^2}{a^2} L_M; -\frac{1}{a} - \frac{M^2}{a^2} L_M; \]

\[ \frac{1}{a} - \frac{s}{a^2} L_M, \] (15)

\[ i_p = \int_0^1 dx \ln \frac{p_x^2}{M^2} = -2 + \frac{s}{a} L_M, \] (16)

with \( L = \ln \frac{4E^2}{m^2} \) being the so-called "large logarithm", \( L_M = \ln \frac{2E}{M} \) and \( a = s + M^2 \). Finally,

\[ I = \int \frac{d^4k}{i\pi^2 (0)(1)(2)} = -\int_0^1 \int_0^1 \frac{dxdy}{D} = -\int_0^1 \frac{dx \ln \frac{p_x^2}{M^2}}{2P_x^2}. \] (17)

Performing the integration on Feynman parameters we obtain

\[ ReI = \frac{1}{2s} \left[ \frac{1}{2} L^2 + 2L \ln \frac{m}{\lambda} - L^2_M - \pi^2 - 2Li_2(1 + \frac{M^2}{s}) \right], \] (18)

with the Euler dilogarithm defined as

\[ Li_2(z) = -\int_0^z \frac{dx}{x} \ln(1 - x). \] (19)

The result is:

\[ 2 \sum M_B^* M_{\text{virt}} = \]

\[ -\frac{\alpha}{\pi} S_0[-L + L \ln \frac{m}{\lambda} + \frac{1}{4} L^2 - \frac{1}{2} L^2_m - \pi^2 - Li_2(1 + \frac{M^2}{s}) - ALA + K_V], \] (20)

where \( L_A = \ln(M_V^2/M^2) \) and

\[ A = \frac{1}{2S_0} \left[ 4s^2(1 + \rho)^2 + u^2(1 - \rho)^2 - 5M^2(s + u)(1 - \rho^2) \right]. \] (21)

A rather complicated expression for \( K_V \) is given in Appendix.
Since we work within the unrenormalized theory, we must take into account the fermion (of mass $m$) wave function renormalization:

$$Z = 1 - \frac{\alpha}{2\pi} \frac{1}{2} \ln \frac{\Lambda^2}{m^2} + \ln \frac{\lambda^2}{m^2} + \frac{9}{4}.$$  \hspace{1cm} (22)

Keeping in mind to apply this procedure for both electron and proton we obtain the contribution to the summed on spin states of the matrix element squared

$$\Delta |M|^2_c = -\frac{\alpha}{2\pi} S_0 [L_\Lambda + \frac{9}{2} - \frac{1}{2} L + L_M].$$ \hspace{1cm} (23)

### IV. REAL SOFT AND HARD PHOTON EMISSION

A standard way to take into account the emission of additional soft (in the Laboratory frame) real photon consists in calculation of the 3-dimensional integral

$$\frac{d\sigma_{soft}}{d\sigma_B} = -\frac{4\pi\alpha}{16\pi^3} \int \frac{d^3k_1}{\omega_1} \left( \frac{p_e}{p_k} - \frac{p}{p_k} \right)^2 |_{\omega_1 < \Delta E}, \quad \Delta E \ll E. \hspace{1cm} (24)$$

The standard calculation leads to

$$d\sigma_{soft} = d\sigma_B \frac{\alpha}{\pi} [(L - 1) \ln \frac{\Delta E}{E} + (L - 2) \ln \frac{m}{\lambda} + \frac{1}{4} L^2 + 1 - \xi_2], \quad \xi = \frac{\pi^2}{3}. \hspace{1cm} (25)$$

The total sum including the Born cross section and corrections arising from the wave function renormalization, and ones arising from taking into account emission of virtual and soft real photons is free from infrared singularities

$$\frac{d\sigma_B + d\sigma_{soft} + d\sigma_{virt+c}}{d\sigma_B} = 1 + \frac{\alpha}{2\pi} [P_\Delta (L - 1) + \frac{\alpha}{\pi} K_{svc} - \frac{\alpha}{\pi} A L_\Lambda] - \frac{\alpha}{\pi} \ln \frac{\Delta E}{E}, \hspace{1cm} (26)$$

with

$$P_\Delta = 2 \ln \frac{\Delta E}{E} + \frac{3}{2}. \hspace{1cm} (27)$$

We note that the term $(L - 1)P_\Delta$ containing the "large logarithm" can be associated with the kernel of the Altarely-Lipatov-Parisi evolution equation of twist-2 operators.

Results obtained here for virtual and soft real photons emission are in agreement with ones obtained in [9, 10].

The emission of the real hard photon matrix element has the form

$$M^\gamma = \frac{G}{\sqrt{2}} V_{ud} \bar{u}_\nu(p_\nu) \gamma_\alpha (1 + \gamma_5) [\tilde{p}_e - \hat{k} + m] \hat{e} u(p_e) \cdot \bar{u}(p_\nu) \gamma_\alpha (1 + \rho \gamma_5) u(p)$$

$$- \bar{u}_\nu(p_\nu) \gamma_\alpha (1 + \gamma_5) u(p_e) \cdot \bar{u}(p_\nu) \gamma_\alpha (1 + \rho \gamma_5) \tilde{p} - \hat{k} + m \hat{e} u(p).$$ \hspace{1cm} (28)
We extract the terms which correspond to the collinear kinematics of photon emission (photon momentum is collinear to the electron one) and the noncollinear ones. The contribution first one contains the large logarithm $L$, and thus include electron mass singularities. The contribution second of one is finite in the mass electron zero limit

$$\sum |M^\gamma|^2 = \sum |M^\gamma|^2_{\text{coll}} + \sum |M^\gamma|^2_{\text{necoll}}. \quad (29)$$

The first term (in agreement with the prescription of quasi-real electron method \cite{2} is:

$$\frac{1}{16} \sum |M^\gamma|^2_{\text{coll}} = \left( G \sqrt{2} V_{ud} \right)^2 S_0(p_e(1-x), p) \left[ \frac{1 + (1-x)^2}{x(1-x)} \frac{1}{p_e k} - \frac{m^2}{(p_e k)^2} \right], \quad (30)$$

with $x = \frac{\omega}{E}$ energy fraction of hard photon.

The relevant contribution to the cross section is

$$d\sigma_h = \frac{\alpha}{\pi} \int_0^1 dx \left[ \frac{1 + (1-x)^2}{2x} (L-1) + K_h d\sigma_B(p_e(1-x), p) + \frac{\alpha}{\pi} \ln \Delta d\sigma_B(p_e, p) \right],$$

$$\Delta = \frac{\Delta E}{E} \ll 1. \quad (31)$$

V. CROSS SECTION IN THE LEADING AND NEXT TO LEADING APPROXIMATION

Let us write down the results obtained above in the Born approximation and their corrections connected with emission of virtual and soft photons as:

$$d\sigma_B + d\sigma_{vs} = d\sigma_B [1 + \frac{\alpha}{2\pi} (L-1)(2 \ln \Delta + \frac{3}{2}) + \frac{\alpha}{\pi} K_{vs} - \frac{\alpha}{\pi} \ln \Delta]. \quad (32)$$

Keeping in mind the result for the contribution of hard real photon emission (see previous section) the total sum can be written in the form:

$$d\sigma = \int_0^1 D(x, L) d\sigma_B(p_e(1-x), p)(1 + \frac{\alpha}{\pi} K)d x,$$

$$\quad (33)$$

with the non-singlet electron Structure function defined as \cite{5}:

$$D(x, L) = \delta(x) + \frac{\alpha}{2\pi} (L-1) P^{(1)}(x) + ..., \quad (34)$$

and the kernel of the evolution equation of twist-2 operator

$$P^{(1)}(x) = \lim_{\Delta \to 0} [\delta(x)(2 \ln \Delta + \frac{3}{2}) + \frac{1 + (1-x)^2}{x} \theta(x - \Delta)]. \quad (35)$$
The smoothed form of the electron structure function is

\[ D(x, L) = bx^{b-1} \left[ 1 + \frac{3}{4}b \right] - b \left( 1 - \frac{1}{2}x \right) + O(b^2), \]  

(36)

\[ b = \frac{2\alpha}{\pi}(L - 1); \quad \int_0^1 D(x, L) dx = 1. \]

The so-called K-factor

\[ K = K_{EW} + K_{vsc} + K_h \]

is the smooth function, finite in the limit of zero electron mass \( m \to 0 \). We put this contribution in the form \( K_{EW} = (K_{EW})_\Lambda + (K_{EW})_f \), where

\[ (K_{EW})_\Lambda = \left[ -\frac{1}{2} + A \right] \ln \left( \frac{M_W^2}{M^2} \right), \]  

(37)

and the part which does not depend on \( M_W^2/M^2 \) is included in \( (K_{EW})_f \). Numerically it is much smaller than \( (K_{EW})_\Lambda \) and is not touched upon here. For details see [1].

The value \( K_h \) depends on experimental conditions of photon detection and will not conditional here.

The function \( K_{vsc} \) is presented in the Table.

VI. CROSSED PROCESSES. DISCUSSION.

The differential cross section of process (2) \( e^+(p_e) + N(p_n) \to P(p) + \bar{\nu}(p_\nu) \) can be obtained from the results given above by replacement \( s \leftrightarrow -u \) with the same definitions of kinematic invariants.

The cross section of the process \((\nu n), (\nu p) \) (see (2)) in the leading logarithmical approximation (LLA) has the form

\[ \frac{d\sigma}{dy} d\sigma^i(y, s) = \int_y^1 \frac{dx}{x} d\sigma_B^i \left( \frac{p_e}{x} \right) D \left( \frac{y}{x}, L \right), i = (\nu p), (\nu n), \]  

(38)

where \( E_\nu, yE_\nu \) are energies of the initial neutrino and the final lepton.

The relevant cross sections in the Born approximation have a form (5) the squared matrix element given in (6) and phase volume defined (7,8).

One can be convinced in the fulfillment of the Kinoshita-Lee-Nauenberg theorem [7, 8] about cancellation of mass singularities in the fraction of final electron differential cross sections integrated by energy:

\[ \int_0^1 dy d\sigma(y) = \int_0^1 dx d\sigma_B \left( \frac{p_e}{x} \right) \int_0^x \frac{dy}{x} D \left( \frac{y}{x}, L \right) = \int_0^1 dx d\sigma_B \left( \frac{p_e}{x} \right). \]  

(39)
So the dependence on \( L \) disappears.

The order of value of the total cross sections considered above

\[
\sigma \sim 10 \text{pb} \, E(\text{GeV})
\]

is too small to be measured in modern experiments.

Above, we supposed to simplify the structure of hadron weak current. This really it must be considered as a model approximation. In reality, the induced axial and vector nucleon formfactors must be taken into account \[11\]. Nevertheless, the form cross section (33) remains valid.

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**APPENDIX A: K-FACTOR**

The relevant loop momentum integrals are

\[
(I, I_\mu, I_{\mu\nu}, I_{kk}) = \int \frac{d^4 k}{i\pi^2} \frac{1}{(0)(1)(2)}. \tag{A1}
\]

Feynman joining denominators procedure leads to (scalar 3-denominator integral \( I \) is given above):

\[
I_{\mu} = -p_{e\mu}^i x + p_{\mu}^i \bar{x};
\]

\[
I_{\mu\nu} = \frac{1}{4} g_{\mu\nu}[L_\Lambda - i_p - \frac{1}{2}] - \frac{1}{2} [p_{e\mu} p_{e\nu} I_{xx} + p_{\mu} p_{\nu} i_{xx} - (p_{e\mu} p_{e\nu} + p_{e\nu} p_{e\mu}) I_{x\bar{x}}];
\]

\[
I_{kk} = L_\Lambda - i_p - 1. \tag{A2}
\]

The values of the contributions to the \( K_{svc} = -K_V + K_S + K_C \)-factor are:

\[
K_S = 1 + \xi_2; \quad K_C = -\frac{9}{4} - \frac{1}{2} L_M, \tag{A3}
\]

and

\[
K_V = -\frac{1}{2} L_M^2 - 3 \xi_2 - L i_2 \left( 1 + \frac{M^2}{s} \right) + \frac{1}{2 S_0 a} \left[ B_1 + \frac{1}{a} L_M B_M \right]. \tag{A4}
\]
Fig. 1: Dependence of $K_{svc}$ on $(s/M^2)$ for $u = -s$ (dashed line) and $u = 0$ (solid line) (see Appendix).

where

$$B_1 = -M^4(s + u)(1 - \rho)(7 + 11\rho) + (s + M^2)u^2(1 - \rho)^2 + 8s^3(1 + \rho)^2 +$$

$$M^2s^2(1 + \rho)(-3 + 19\rho) - 9suM^2(1 - \rho^2);$$

$$B_M = -4M^6(s + u)(1 - \rho^2) + s^2u^2(1 - \rho)^2 + 4s^4(1 + \rho)^2 +$$

$$u^2(s + M^2)M^2(1 - \rho)^2 - M^2s^2u(1 - \rho)(7 + 5\rho) +$$

$$+ M^2s^3(1 + \rho)(-1 + 13\rho) - 10M^4su(1 - \rho^2) + M^4s^2(-7 + 2\rho + 13\rho^2). \quad (A5)$$

In Fig. 1 the dependence of $K_{svc}(s, u)$ is presented as a function of $s$ for fixed values of $u$.

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