Research Article

Hermite-Hadamard and Schur-Type Inequalities for Strongly $h$-Convex Fuzzy Interval Valued Functions

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The concept of fuzzy theory was developed in 1965 and becomes an acknowledged research subject in both pure and applied mathematics and statistics, showing how this theory is highly applicable and productive in many applications. In the present study, we introduced the definition of fuzzy interval valued strongly $h$-convex function and investigated some of its properties.

1. Introduction

The theory of interval analysis is one of the top areas of research nowadays because of its enormous application in various fields especially in automatic error analysis [1], computer graphics [2], and neural network output optimization [3]. In [4], Moore et al. give the first monograph on interval analysis, and since then, a huge amount of work have been done in interval calculus; for example, in [5], Chalco Cano et al. studied the interval-valued functions using generalized Hukuhara derivative and presented applications of interval valued calculus. An efficient method for solving fuzzy optimization problems using interval valued calculus is presented in [6]. Costa et al. [7] calculated the possible conformations arising from uncertainty in the molecular distance geometry problem using constraint interval analysis. Interval vector spaces and interval optimizations are given in [8], while optimality conditions for generalized differentiable interval valued functions are presented in [9]. Several classical inequalities for interval valued functions have been established in [10].

To address the modern problems, the convexity has been generalized in number of ways and some interesting generalizations are strongly $(h,s)$-convex functions [11], strongly generalized convex functions [12], $p$-convex function [13], and many more. The authors introduce $h$-convex function in [14] as follows;

A function $\phi : I = [\mu_1, \mu_2] \rightarrow \mathbb{R}$ is known as $h$-convex if

$$\phi(\beta s + (1 - \beta)t) \leq h(\beta)\phi(s) + h(1 - \beta)\phi(t),$$

for all $s, t \in I = [\mu_1, \mu_2]$ and $\beta \in [0, 1].$

The authors introduce the strongly convex function in [15] as follows:

A function $\phi : I = [\mu_1, \mu_2] \rightarrow \mathbb{R}$ with modulus $\vartheta \geq 0$ is defined as

$$\phi(\beta s + (1 - \beta)t) \leq \beta\phi(s) + (1 - \beta)\phi(t) - \vartheta(1 - \beta)(s - t)^2,$$

for all $s, t \in I = [\mu_1, \mu_2]$ and $\beta \in [0, 1]$ which is called strongly convex.

The unification of above two definitions is defined in [16] as follows:

A function $\phi : I = [\mu_1, \mu_2] \rightarrow \mathbb{R}$ with modulus $\vartheta \geq 0$ defined as follows:

$$\phi(\beta s + (1 - \beta)t) \leq h(\beta)\phi(s) + h(1 - \beta)\phi(t) - \vartheta(1 - \beta)(s - t)^2,$$

for all $s, t \in I = [\mu_1, \mu_2]$ and $\beta \in [0, 1].$
for all \( s, t \in I = [\mu_1, \mu_2] \) and \( \beta \in [0, 1] \) which is known as strongly \( h \)-convex.

In recent years, mathematical inequalities for interval valued convex and nonconvex function got attention of many mathematician [17]. Bai et al. [10] established Hermite-Hadamard and Jensen-type inequalities for the class of interval valued nonconvex functions. In 2017, Costa [18] presented Jensen-type inequality for the class of fuzzy interval valued functions, and in the same year, Costa and Román-Flores [19] presented some integral inequalities for the same class of functions. Some Opial–type inequalities were studied in [20]. For other remarkable results, we refer [21] and references therein.

In this report, we proposed the definition of fuzzy interval valued strongly \( h \)-convex function. We investigated some of its properties and established Hermite-Hadamard and Schur-type inequalities for the proposed definition.

The paper is organized as follows: In Section 2, we will give some preliminaries and basic definitions, and we will established some basic properties. However, Section 3 is devoted for the establishment of main results like Hermite Hadamard and Schur-type inequalities.

2. Some Basic Properties of Interval Calculus

Let \( \mathbb{R}^+_2 \) and \( \mathbb{R}^+_1 \) be all positive intervals and all intervals. The algebraic operations \(+\) and \(\cdot\) are defined as \[ [s_1, s_2] + [t_1, t_2] = [s_1 + t_1, s_2 + t_2] \]

\[ \lambda[s_1, s_2] = \begin{cases} \lambda s_1, & \text{if } 0 \leq \lambda, \\ \lambda s_2, & \text{if } \lambda < 0. \end{cases} \]

A function \( F : V \subseteq \mathbb{R} \longrightarrow \mathbb{R}^+_1 \) with \( F(s) = [\phi_1(s), \phi_2(s)] \), where \( \phi_1, \phi_2 : V \longrightarrow \mathbb{R} \) are real functions and known as interval valued functions if \( \phi(s) \leq \phi(s), \) for all \( s \in V \).

For intervals \([s, s]\) and \([t, t]\), the distance \( d([s, s], [t, t]) = \max \{ |s - t|, |s - t| \} \) is the Hausdorff distance. Then \( (\mathbb{R}^+_1, d) \) is complete.

The tagged partition \( P \) of interval \([s, t]\) is the set of numbers \( \{s_i, t_i \}_{i=1}^m \) if

\[ s = s_0 < s_1 < \cdots < s_m = t \]

and if \( s_{i-1} \leq t_i \leq s_i \), for all \( i = 1, 2, \cdots, m \).

Moreover, letting \( \Delta s_i = s_i - s_{i-1} \) and \( \Delta t_i \leq \delta \) for every \( i \), the partition is said to be \( \delta \)-fine. The family of the all such \( \delta \)-fine partitions of \([s, t]\) is represented as \( P(\delta, [s, t]) \) [17].

Let \( X \) be the \( \delta \)-fine partitions of \([s, t] \); then the function \( \phi : [s, t] \longrightarrow \mathbb{R}^+_1 \) is said to be integral sum if

\[ S(\phi, X, \delta, [s, t]) = \sum_{i=1}^m \phi(t_i)(s_i - s_{i-1}). \]

**Proposition 1** (see [17]). The four arithmetic operators \((+, - , \cdot , \div)\) for \([t, t] \) and \([s, s]\) are defined as

\[ [t] + [s] = [t + s, t + s], \]

\[ [t] - [s] = [t - s, t - s], \]

\[ [t] \cdot [s] = \left[ \min \{ t \cdot s, t \cdot s, t \cdot s, t \cdot s \}, \max \{ t \cdot s, t \cdot s, t \cdot s, t \cdot s \} \right], \]

\[ \frac{[t]}{[s]} = \left[ \min \left\{ \frac{t}{s}, \frac{t}{s}, \frac{t}{s}, \frac{t}{s} \right\}, \max \left\{ \frac{t}{s}, \frac{t}{s}, \frac{t}{s}, \frac{t}{s} \right\} \right], \]

(7)

where \( 0 \not\in [s, s] \).

**Definition 2.** A function \( \phi : I = [\mu_1, \mu_2] \longrightarrow \mathbb{R} \) is called convex function if

\[ \phi(\beta s + (1 - \beta)t) \leq \beta \phi(s) + (1 - \beta)\phi(t), \]

(8)

for all \( s, t \in I = [\mu_1, \mu_2] \) and \( \beta \in [0, 1] \).

**Definition 3** (see [23]). Let a nonnegative function \( h : [a, b] \longrightarrow \mathbb{R} \), \( h \neq 0 \) and \( (0, 1) \subseteq [\mu_1, \mu_2] \). We say that a function \( \phi : [\mu_1, \mu_2] \longrightarrow \mathbb{R}^+_1 \) is interval \( h \)-convex, if for all \( s, t \in [\mu_1, \mu_2] \) and \( \beta \in (0, 1) \), we have

\[ h(\beta)\phi(s) + h(1 - \beta)\phi(t) \leq \phi(\beta s + (1 - \beta)t) \]

(9)

If the set inclusion (9) is reversed, then \( \phi \) is said to be interval \( h \)-concave, i.e., \( \phi \in SV(h, [\mu_1, \mu_2], \mathbb{R}) \).

We end this section of preliminaries by introducing the new concept of interval valued strongly \( h \)-convex function. Note that for interval \([s, s]\) and \([t, t]\), the inclusion \( t' \subseteq t' \) is defined by

\[ [s, s] \subseteq [t, t], \quad \text{if } t \leq s \leq t. \]

(10)

**Definition 4.** Let \( X \) be a normed space and \( D \subseteq X \) is convex.

\( h : (0, 1) \longrightarrow \mathbb{R} \) is a function. We call a function \( \phi : I = [\mu_1, \mu_2] \longrightarrow \mathbb{R} \) with modulus \( \theta \geq 0 \) is interval valued strongly \( h \)-convex function if

\[ \phi(\beta s + (1 - \beta)t) \geq h(\beta)\phi(s) + h(1 - \beta)\phi(t) - \theta (1 - \beta)(s - t)^2, \]

(11)

for all \( s, t \in I = [\mu_1, \mu_2] \) and \( \beta \in [0, 1] \).

The notion \( SX(h ; [\mu_1, \mu_2, \mathbb{R}^+_1]) \) is reserved for the class of interval valued strongly \( h \)-convex functions.

**Remark 5.**

(1) By taking \( \theta = 0 \), then the set inclusion (11) reduces to interval valued \( h \)-convex function [23].

(2) By taking \( \phi = \phi \), then the set inclusion (11) reduces to strongly \( h \)-convex function [16].

(3) By taking \( \phi = \phi \) and \( \theta = 0 \), then the set inclusion (11) reduces to \( h \)-convex function [14].
(4) By taking \( \phi = \tilde{\phi} \) and \( h(\beta) = \beta \), then the set inclusion (11) reduces to strongly-convex function [15]

(5) By taking \( \phi = \tilde{\phi}, h(\beta) = \beta \) and \( \theta = 0 \), then the set inclusion (11) reduces to classical convexity

(6) By taking \( \phi = \bar{\phi}, h(\beta) = 1/\beta \) and \( \theta = 0 \), then the set inclusion (11) reduces to Godunova- Levin function

3. Main Results

In this section, we establish the Hermite-Hadamard- and Schur-type inequalities for the proposed definition.

**Theorem 6.** Let \( X \) be a normed space and \( D \subseteq X \) is convex. \( h : (0, 1) \longrightarrow \mathbb{R} \) is a function with holding condition \( h(\beta) \geq \beta \) for each \( \beta \in [0, 1] \) if \( g : D \longrightarrow \mathbb{R} \) belongs to \( SX(h; (\mu_1, \mu_2, \mathbb{R}^*_1)) \); then \( \Phi : D \longrightarrow \mathbb{R} \) is defined as

\[
\Phi(s) = g(s) + \theta s^2,
\]

which also belongs to \( SX(h; (\mu_1, \mu_2, \mathbb{R}^*_1)) \).

**Proof.** Consider \( g \in SX(h; (\mu_1, \mu_2, \mathbb{R}^*_1)) \); then

\[
g([\beta t + (1 - \beta)s]) = \phi([\beta t + (1 - \beta)s]) + \theta(\beta t + (1 - \beta)s)^2 \geq h(\beta)g(t)
\]

\[
+ h(1 - \beta)g(t) + \theta(\beta t + (1 - \beta)s)^2 \geq h(\beta)\phi(t)
\]

\[
+ h(1 - \beta)\phi(s) + \theta(\beta t + (1 - \beta)s)^2 \geq h(\beta)\phi(t)
\]

\[
+ h(1 - \beta)\phi(s) + \theta(\beta t + (1 - \beta)s)^2 \geq h(\beta)\phi(t) + h(1 - \beta)\phi(s)
\]

\[
+ h(1 - \beta)\phi(s) - \beta(1 - \beta)(s - t)^2 + \theta(\beta t + (1 - \beta)s)^2 \geq h(\beta)\phi(t) + h(1 - \beta)\phi(s)
\]

\[
+ h(1 - \beta)\phi(s) - \beta(1 - \beta)(s - t)^2 + \theta(\beta t + (1 - \beta)s)^2 \geq h(\beta)\phi(t) + h(1 - \beta)\phi(s)
\]

(13)

which gives the desired result.

**Theorem 7.** Let \( X \) be a normed space and \( D \subseteq X \) is convex. \( h : (0, 1) \longrightarrow \mathbb{R} \) is a function with holding condition \( h(\beta) \leq \beta \) for each \( \beta \in [0, 1] \) if \( g : D \longrightarrow \mathbb{R} \) belongs to \( SX(h; (\mu_1, \mu_2, \mathbb{R}^*_1)) \); then \( g : D \longrightarrow \mathbb{R} \) is defined as

\[
g(s) = \phi(s) + \theta s^2,
\]

which also belongs to \( SX(h; (\mu_1, \mu_2, \mathbb{R}^*_1)) \).

**Proof.** The proof is similar to that of Theorem 6.

**Example 8.** Let \( \beta = 1, \beta \in (0, 1) \). Then \( \phi : [-1, 1] \longrightarrow \mathbb{R} \) defined by \( \phi(s) = 1, s \in [-1, 1] \) is interval valued strongly \( h \)-convex with modulus \( \theta = 1 \). Because for every \( s, t \in [-1, 1] \) and \( \beta \in (0, 1) \), then from (11), we have

\[
\phi(\beta t + (1 - \beta)s) \geq \phi(t) + \theta(\beta t + (1 - \beta)s - t)^2,
\]

\[
\phi(\beta t + (1 - \beta)s) \leq \phi(t) + \theta(\beta t + (1 - \beta)s - t)^2,
\]

which implies that

\[
\left[ h\left( 1 \right) \phi(u) + \phi(v) \right] - \frac{9}{4}(u - v)^2 \leq \phi\left( \frac{1}{2} \right) \phi(u) + \phi(v)
\]

(20)

\[
\phi\left( \frac{1}{2} \right) \phi(u) + \phi(v) - \frac{9}{4}(u - v)^2,
\]

(21)

\[
\phi\left( \frac{1}{2} \right) \phi(u) + \phi(v) - \frac{9}{4}(u - v)^2.
\]

(22)
Substituting the values of \( u \) and \( v \) in (21), we get
\[
\frac{1}{2} \left( \frac{1}{2} \right) \left( \phi(\beta u_1 + (1 - \beta)u_2) + \phi((1 - \beta)u_1 + \beta u_2) \right)
- \frac{\theta}{4} \left( (2\beta - 1)u_1 + (1 - 2\beta)u_2 \right)^2 \leq h \left( \frac{1}{2} \right)
- \frac{\partial}{\partial h} \left( \phi(\beta u_1 + (1 - \beta)u_2) + \phi((1 - \beta)u_1 + \beta u_2) \right)
- \frac{\partial}{\partial h} \left( (2\beta - 1)u_1 + (1 - 2\beta)u_2 \right)^2.
\]
(23)

Integrating (23) with respect to \( h \) over \( (0, 1) \); we obtain
\[
\frac{1}{2} \left( \frac{1}{2} \right) \left( \int_0^{\mu_1} \phi(\beta u_1 + (1 - \beta)u_2) d\beta + \int_0^{\mu_2} \phi((1 - \beta)u_1 + \beta u_2) d\beta \right)
- \frac{\theta}{4} \left( (2\beta - 1)u_1 + (1 - 2\beta)u_2 \right)^2 \leq h \left( \frac{1}{2} \right)
- \frac{\partial}{\partial h} \left( \phi(\beta u_1 + (1 - \beta)u_2) + \phi((1 - \beta)u_1 + \beta u_2) \right)
- \frac{\partial}{\partial h} \left( (2\beta - 1)u_1 + (1 - 2\beta)u_2 \right)^2.
\]
(24)

Similarly substituting the values of \( u \) and \( v \) in (22), we get
\[
\frac{1}{2} \left( \frac{1}{2} \right) \left( \mu_1 + \mu_2 \right) \leq \mu_2 - \mu_1 \int_{\mu_1}^{\mu_2} \phi(x) dx - \frac{\theta}{12} \left( \mu_2 - \mu_1 \right).
\]
(25)

Combining (24) and (25), we obtain
\[
\frac{1}{2} \left( \frac{1}{2} \right) \left( \mu_1 + \mu_2 \right) \leq \mu_2 - \mu_1 \int_{\mu_1}^{\mu_2} \phi(x) dx - \frac{\theta}{12} \left( \mu_2 - \mu_1 \right) \leq \phi \left( \frac{\mu_1 + \mu_2}{2} \right),
\]
(26)

which gives the proof of left hand side.

For the proof of right hand side, take
\[
\frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} \phi(u) du = \int_0^1 \phi((1 - \beta)\mu_1 + \beta \mu_2) d\beta.
\]
(27)

Using the definition of interval valued strongly \( h \)-convex function, we obtain
\[
\frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} \phi(u) du \geq \int_0^1 h(1 - \beta)\phi(\mu_1) + h(\beta)\phi(\mu_2)
- \frac{\theta}{6} (1 - \beta)^2 (\mu_1 - \mu_2)^2 d\beta \geq [\phi(\mu_1) + \phi(\mu_2)] \int_0^1 h(\beta) d\beta
- \frac{\theta}{6} (\mu_2 - \mu_1)^2 \frac{1}{6},
\]
(28)

which gives right hand side.

Combining (26) and (28), we obtain desire result.

**Remark 10.**

1. By taking \( \theta = 0 \), then (17) becomes Hermite Hadamard-type inequality for interval valued \( h \)-convex function [23].
2. By substituting \( \phi = \phi \), then (17) becomes Hermite Hadamard-type inequality for strongly \( h \)-convex function [16].
3. By substituting \( \phi = \phi \) and \( h(t) = t \), then (17) becomes Hermite Hadamard-type inequality for strongly convex function [15].
4. For \( \phi = \phi \), \( \theta = 0 \) and \( h(t) = t \), then (17) becomes classical Hermite Hadamard-type inequality.

**Example 11.** Consider \( h(t) = t \) for \( t \in [0, 1] \), \( [\mu_1, \mu_2] = [-1, 1] \), and \( \phi : [\mu_1, \mu_2] \rightarrow \mathbb{R}^+ \) be defined by \( \phi(t) = [t^2, 4 - e^t] \) and \( \theta = 1 \); then, we have
\[
\frac{1}{2h(1/2)} \phi \left( \frac{\mu_1 + \mu_2}{2} \right) = \phi(0) = \left[ \frac{1}{3}, \frac{10}{3} \right],
\]
(29)
\[
\frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} \phi(t) dt = \left[ \frac{1}{3}, 4 - e - e^{-1} \right].
\]
(30)

Also,
\[
[\phi(\mu_1) + \phi(\mu_2)] \int_0^1 h(x) dx - \frac{\theta}{6} (\mu_2 - \mu_1)^2 = \left[ \frac{1}{3}, 22 - 3(e + e^{-1}) \right].
\]
(31)

Returning to (29), (30), and (31), we deduce
\[
\left[ \frac{1}{3}, \frac{10}{3} \right] \supseteq \left[ \frac{1}{3}, 4 - e - e^{-1} \right] \supseteq \left[ \frac{1}{3}, 22 - 3(e + e^{-1}) \right].
\]
(32)

Consequently, Theorem 9 is verified.

### 3.2. Interval Schur-Type Inequality

**Theorem 12.** Consider \( g : I = [u_1, u_2] \rightarrow \mathbb{R} \) be the interval valued strongly \( h \)-convex function and a nonnegative super multiplicative function \( h : J \rightarrow \mathbb{R} \) for all \( s_1, s_2, s_3 \in I = [u_1, u_2] \) and \( s_1 < s_2 < s_3 \) such that \( s_3 - s_1, s_3 - s_2, s_2 - s_1 \in J \); then the following inequality holds:
\[
h(s_3 - s_1)g(s_2) + h(s_2 - s_1)g(s_3) - h(s_3 - s_2)g(s_1) - 3g(s_1 - s_2)(s_2 - s_1),
\]
(33)

where \( (0, 1) \subseteq J \) iff \( g \) is interval valued strongly \( h \)-convex function.

**Proof.** Let \( s_1, s_2, s_3 \in I \) be numbers which satisfy assumptions of the proposition. Then \( (s_3 - s_2, s_1) \in (0, 1) \subseteq J \), \( (s_3 - s_2) / s_3 - s_1 \in (0, 1) \subseteq J \), and
\[ \frac{s_3 - s_2}{s_3 - s_1} + \frac{s_2 - s_1}{s_3 - s_1} = 1. \]  

(34)

Also,

\[ h(s_3 - s_2) = h\left(\frac{s_3 - s_2}{s_3 - s_1}(s_3 - s_1)\right) \geq h\left(\frac{s_3 - s_2}{s_3 - s_1}\right) h(s_3 - s_1), \]

(35)

\[ h(s_2 - s_1) \geq h\left(\frac{s_2 - s_1}{s_3 - s_1}\right) h(s_3 - s_1). \]  

(36)

Assume that \( h(s_3 - s_1) \geq 0 \) and \( g \) is interval valued strongly \( h \)-convex function with modulus \( \theta \geq 0 \). Let \( s_1, s_2, s_3 \in I = [u_1, u_2] \); then substituting \( \beta = (s_3 - s_2)/s_3 - s_1 \), \( s = s_1 \), and \( t = s_3 \) in (11) yields that

\[
\begin{align*}
g\left(\frac{s_3 - s_2}{s_3 - s_1}s_1 + \left(1 - \frac{s_3 - s_2}{s_3 - s_1}\right)s_3\right) & \geq g\left(\frac{s_3 - s_2}{s_3 - s_1}\right) g(s_1) \\
& + h\left(1 - \frac{s_3 - s_2}{s_3 - s_1}\right) g(s_3) - \theta\left(\frac{s_3 - s_2}{s_3 - s_1}\right)^2 (s_1 - s_3)^2,
\end{align*}
\]

(37)

implying

\[ g(s_2) \geq h\left(\frac{s_3 - s_2}{s_3 - s_1}\right) g(s_1) + h\left(\frac{s_2 - s_1}{s_3 - s_1}\right) g(s_3) - \theta(s_3 - s_2)(s_2 - s_1). \]  

(38)

Now, we write the above set inclusion as

\[
\begin{align*}
g(s_2) & \geq h\left(\frac{s_3 - s_2}{s_3 - s_1}\right) g(s_1) + h\left(\frac{s_2 - s_1}{s_3 - s_1}\right) g(s_3) - \theta(s_3 - s_2)(s_2 - s_1), \\
& \geq h\left(\frac{s_3 - s_2}{s_3 - s_1}\right) g(s_1) + h\left(\frac{s_2 - s_1}{s_3 - s_1}\right) g(s_3) - \theta(s_3 - s_2)(s_2 - s_1).
\end{align*}
\]

(39)

Using the condition (35), we obtain

\[
\begin{align*}
g(s_2) & \leq \frac{h(s_3 - s_2)}{h(s_3 - s_1)} g(s_1) + \frac{h(s_2 - s_1)}{h(s_3 - s_1)} g(s_3) - \theta(s_3 - s_2)(s_2 - s_1), \\
g(s_2) & \geq \frac{h(s_3 - s_2)}{h(s_3 - s_1)} g(s_1) + \frac{h(s_2 - s_1)}{h(s_3 - s_1)} g(s_3) - \theta(s_3 - s_2)(s_2 - s_1).
\end{align*}
\]

(40)

Now, we write the above inequalities in set inclusion form

\[ h(s_3 - s_2) g(s_1) \supseteq h(s_3 - s_2) g(s_1) + h(s_2 - s_1) g(s_3) - \theta(s_3 - s_2)(s_2 - s_1) h(s_3 - s_1), \]  

(41)

as required. Conversely suppose that (33) holds, and we insert \( s_1 = u, s_2 = \beta u + (1 - \beta)v, \) and \( s_3 = v \) in (33); we get

\[ h(v - u)g(\beta u + (1 - \beta)v) \geq h(\beta(v - u)) g(u) \]

\[ + h((v - u)(1 - \beta)) g(v) - \theta(1 - \beta) h(v - u)(v - u)^2. \]  

(42)

Use condition (35), and simplify the above inclusion set yields that \( g \) is interval valued strongly \( h \)-convex, which completes the proof.

Remark 13. If we take \( \theta = 0 \) and \( g = g \) in Theorem 12, then, we obtain Schur-type inequality for \( h \)-convex function [24].

4. Concluding Remarks

In this report, we introduced the fuzzy interval valued strongly \( h \)-convex function. We also investigated some of its properties. Moreover, we established Hermite-Hadamard and Schur-type inequalities for the proposed definition. Our results are more general than many existing results. It is interesting for the researcher to find other types of inequalities for the proposed class of functions in the setting of various fractional integrals.

Data Availability

All data required for this research are included within the paper.

Conflicts of Interest

The authors do not have any competing interests.

Authors’ Contributions

Putian Yang proved the results and arranged the funding for this paper, and Shiqing Zhang wrote the paper and analyzed the results.

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