Critical phenomena in $\mathcal{N} = 4$ SYM plasma

Alex Buchel

Department of Applied Mathematics
University of Western Ontario
London, Ontario N6A 5B7, Canada

Perimeter Institute for Theoretical Physics
Waterloo, Ontario N2J 2W9, Canada

Albert Einstein Minerva Center
Weizmann Institute of Science
Rehovot 76100, Israel

Abstract

Strongly coupled $\mathcal{N} = 4$ supersymmetric Yang-Mills plasma at finite temperature and chemical potential for an R-symmetry charge undergoes a second order phase transition. We demonstrate that this phase transition is of the mean field theory type. We explicitly show that the model is in the dynamical universality class of ‘model B’ according to the classification of Hohenberg and Halperine, with dynamical critical exponent $z = 4$. We study bulk viscosity in the mass deformed version of this theory in the vicinity of the phase transition. We point out that all available models of bulk viscosity at continuous phase transition are in conflict with our explicit holographic computations.

May 2010
1 Introduction

According to gauge theory/string theory correspondence of Maldacena [1] maximally supersymmetric $\mathcal{N} = 4$ $SU(N)$ Yang-Mills (SYM) theory is dual to string theory on $AdS_5 \times S^5$. In the planar limit ($g_{YM}^2 \to 0$, $N \to \infty$ with $\lambda \equiv g_{YM}^2 N$ kept fixed) and for large t’ Hooft coupling $\lambda \gg 1$ the strongly coupled SYM is described by classical type IIB supergravity on $AdS_5 \times S^5$, making it essentially soluble. The value of this holographic duality is that it can provide explicit tests of various phenomenological models invented to describe the dynamics of strongly coupled systems. The focus of this paper is the application of gauge/gravity duality to the transport properties of strongly coupled gauge theory plasma in the vicinity of the second order phase transitions.

In [2] it was argued that the only model for the critical behavior of the bulk viscosity in strongly coupled systems at continuous phase transitions not in conflict with explicit holographic computations was that proposed by Onuki [3]. Specifically, Onuki’s model predicts that close to the phase transition the bulk viscosity scales as

$$\zeta \propto |t|^{-2\nu + \alpha}, \quad (1.1)$$

where

$$t \equiv \frac{T}{T_c} - 1, \quad (1.2)$$

is the reduced temperature, $\nu$ and $\alpha$ are the usual static critical exponents of the continuous phase transition, and $z$ is a dynamical critical exponent. In this paper we would like to definitely answer the question as to whether or not (1.1) is realized in a strongly coupled gauge theory plasma with a holographic dual.

Our starting point is the best studied example of gauge theory/string theory duality, namely that of $\mathcal{N} = 4$ SYM plasma. This theory has an $SO(6) \sim SU(4)$ R-symmetry; thus one can turn on three independent chemical potentials (one for each of the $U(1)$’s in the Cartan subalgebra of the R-symmetry group). It is well known that $\mathcal{N} = 4$ SYM plasma at finite temperature $T$ and for a single $U(1)$ R-symmetry\(^1\) chemical potential $\mu$ undergoes a second order phase transition\(^2\) [4, 5, 6]. Moreover, recently [7], the

---

\(^1\)This is not the diagonal $U(1)$ of the $SU(4)$ R-symmetry.

\(^2\)As we show below, some of the static critical exponent first computed in [5] and since then widely used in the literature are incorrect. This issue could be traced back to the fact that the hyperscaling relation between static critical exponents is violated in this theory.
conductivity $\sigma_Q$ of this gauge theory plasma was shown to be finite on the critical line

$$\frac{\mu}{T}_{\text{critical}} = \frac{\pi}{\sqrt{2}} \quad (1.3)$$

As a result, the authors of [7] argued that the dynamical universality class of $\mathcal{N} = 4$ SYM plasma is that of 'model B' according to classification of Hohenberg and Halperine [8], with the dynamical critical exponent

$$z = 4 - \eta , \quad (1.4)$$

with $\eta$ being the anomalous static critical exponent. In this paper we confirm the identification made in [7], and compute $z$ for the $\mathcal{N} = 4$ SYM plasma.

Unfortunately, we can not use the $\mathcal{N} = 4$ SYM plasma directly to test Onuki’s prediction for the scaling of the bulk viscosity in the vicinity of the phase transition (1.1) — conformal invariance of the theory guarantees that the bulk viscosity must vanish for arbitrary chemical potential and the temperature. Thus, we need to deform the theory in such a way that we break the scale invariance. The simplest deformation one can consider is to give mass $M$ to fermions of $\mathcal{N} = 4$ SYM. If

$$M \ll T_{\text{critical}} , \quad (1.5)$$

it is sufficient to work to order $O\left(\frac{M^2}{T^2}\right)$. Although not necessary, one can think about above deformation (to the order specified) as that corresponding to deforming $\mathcal{N} = 4$ plasma to $\mathcal{N} = 2^*$ plasma [9, 10, 11].

The paper is organized as follows. We begin section 2 with presenting the effective action for the holographic dual of $\mathcal{N} = 4$ SYM plasma, deformed by a dimension-3 operator. We study this gravitational model at finite temperature and chemical potential to order $O\left(\frac{M^2}{T^2}\right)$: we (numerically) determine the background geometry, discuss the holographic renormalization of the theory, and compute the equilibrium thermodynamics. We present a highly nontrivial test on our analysis by demonstrating that the basic thermodynamic relations are satisfied. Appendix A contains necessary technical details. In section 3 we study hydrodynamic fluctuations in charged gauge theory plasma and their holographic dual — the lowest quasinormal modes of the (mass-deformed) Reissner-Nordström (RN) asymptotically $AdS_5$ black holes. We derive the speed of sound and the sound wave attenuation coefficient in a generic charged plasma. We find that the attenuation coefficient is sensitive to both the shear and the bulk viscosities of the plasma, as well as the plasma conductivity. Interestingly, the dependence
on a conductivity in the attenuation coefficient arises only at the fourth order in the parameter breaking the scale invariance, i.e., $\frac{M}{f}$. We derive (coupled) equations for the quasinormal modes in the deformed RN black hole background and explain how to decode from their spectrum the speed of sound waves and the bulk viscosity of the dual plasma. We point out the computational difficulty in the numerical analysis, intrinsic to finding the sound channel quasinormal modes in charged black hole backgrounds, and present a new method of computing the corresponding quasinormal modes. Appendix B is used to explain this new method in a simple setting of $\mathcal{N} = 4$ SYM plasma at finite temperature, but zero chemical potentials. As a highly nontrivial test of our analysis, we show that the speed of sound obtained from the thermodynamic analysis is in excellent agreement with the one extracted from the dispersion relation of the lowest sound channel quasinormal mode. In section 4 we discuss dynamical critical phenomena in $\mathcal{N} = 4$ SYM plasma. We review how the dynamical susceptibility can be used to compute static anomalous critical exponent $\eta$ and the critical exponent $\nu$ associated with the divergence of the correlation length in the vicinity of the transition, as well as the dynamical critical exponent $z$. We explain how the problem of finding the sound channel quasinormal mode can be adjusted to extract the dynamical susceptibility. We emphasize why the latter analysis can not be performed in the hydrodynamic limit. We present results for the critical exponents for $\mathcal{N} = 4$ SYM plasma and demonstrate that these exponents are robust against the mass deformation of $\mathcal{N} = 4$ plasma. We summarize and interpret all the results in concluding section 5.

2 Holographic dual of mass deformed $\mathcal{N} = 4$ plasma at equilibrium

Effective five-dimensional action describing the holographic dual to $\mathcal{N} = 4$ SYM deformed by an operator $\mathcal{O}_\Delta$ of dimension $\Delta$ takes form\(^3\)

$$S_5 = \frac{1}{16\pi G_5} \int_{\mathcal{M}_5} d^5 \xi \sqrt{-g} \mathcal{L}$$

$$= \frac{1}{16\pi G_5} \int_{\mathcal{M}_5} d^5 \xi \sqrt{-g} \left( R - \frac{1}{4} \phi^{4/3} F^2 - \frac{1}{3} \phi^{-2} (\partial \phi)^2 + 4\phi^{2/3} + 8\phi^{-1/3} + \delta \mathcal{L} \right),$$

(2.1)

\(^3\)The supergravity gauge coupling is chosen so that the asymptotic $AdS_5$ radius is one.
where $\delta L$ is a mass deformation
\[
\delta L = -\frac{1}{2} (\partial \chi)^2 - \frac{m^2}{2} \chi^2 + O(\chi^4). \tag{2.2}
\]
As usual, the mass of $\chi$ is related to the dimension $\Delta$ of the corresponding operator in the dual description
\[
\Delta(\Delta - 4) = m^2. \tag{2.3}
\]
In what follows we focus on $\Delta = 3$ ($m^2 = -3$) deformation.

The non-normalizable component $\lambda$ of $\chi$ near the (asymptotic) $AdS_5$ boundary is related to the coupling $M$ of operator $O_3$ deforming the Lagrangian $L_{CFT}$ of the $\mathcal{N} = 4$ conformal fixed point:
\[
L_{CFT} \rightarrow L_{CFT} - M O_3, \quad M \propto \lambda, \tag{2.4}
\]
where the precise definition of $\lambda$ and the relation between $M$ and $\lambda$ (up to an irrelevant $c$-number normalization of $O_3$) will be established later. Notice that the identification of (2.4) with the dual holographic action (2.1) can be established only to order $\lambda^2$, and thus $\chi^2$ in (2.2). The latter fact is emphasized by an $O(\chi^4)$ term in (2.2). The reason why this is so is best illustrated with the $\mathcal{N} = 2^*$ example of the holographic gauge theory/string theory correspondence [12]. The duality studied in [12] is one of the few examples where it is possible to match exactly the gravitational parameters with the corresponding gauge-theoretical ones. In this case, a simple deformation of the type (2.4) leads to a complicated potential for a supergravity scalar $\chi$ — nonetheless, up to order $M^2$, the dual supergravity deformation is unambiguously fixed by the scaling dimension of $O_\Delta$, as in (2.2).

We are interested in the critical phenomena in $\mathcal{N} = 4$ plasma (and its massive deformation) at finite temperature and chemical potential — thus as far as $M \ll T$, in particular for $T = T_c$, if is sufficient to work with the effective holographic description (2.1), (2.2).

### 2.1 Background

In this section we set up our notations for describing the background geometry of $\mathcal{N} = 4$ SYM plasma at finite temperature and $U(1)_R$ chemical potential, deformed by $O_3$ operator to quadratic order in its coupling.

Consider the following ansatz
\[
ds_5^2 = -c_1^2 \, dt^2 + c_2^2 \, d\vec{x}^2 + c_3^2 \, dr^2, \quad A_\mu = A_\mu \delta_\mu, \tag{2.5}
\]
where $\delta_\mu$ is a mass deformation

\[
\delta_\mu = -\frac{1}{2} (\partial \chi)^2 - \frac{m^2}{2} \chi^2 + O(\chi^4). \tag{2.6}
\]
where \( c_i = c_i(r), A = A(r), \phi = \phi(r), \chi = \chi(r) \). We find it convenient to introduce a new radial coordinate \( x \) as follows

\[
1 - x = \frac{c_1}{c_2},
\]

so that \( x \to 0_+ \) corresponds to an AdS boundary and \( x \to 1_- \) corresponds to a regular Schwarzschild horizon. Further, we introduce

\[
\phi = H^3, \quad c_2 = g \left( \frac{H}{H^3 - 1} \right)^{1/2},
\]

with \( g = g(x), H = H(x) \). We would like to construct background geometry perturbatively in \( \chi \), in other words we parametrize the background as

\[
g = g_0 + \lambda^2 g_2, \quad H = H_0 + \lambda^2 H_2, \quad A = A_0 + \lambda^2 A_2, \quad \chi = \lambda \chi_1,
\]

where the coefficient of the non-normalizable mode of \( \chi \), i.e., \( \lambda \), is introduced so that

\[
\chi_1 = x^{1/4} + \mathcal{O}(x^{1/2}), \quad \text{as} \quad x \to 0_+.
\]

- **To order \( \mathcal{O}(\lambda^0) \)** we find

\[
g_0 = \beta, \quad A_0 = \frac{\beta \sqrt{1 + \kappa}}{\kappa} \left( \frac{1}{H_0} - \frac{1}{1 + \kappa} \right),
\]

\[
H_0 = \frac{2(1 + \kappa) + (2x - x^2)\kappa^2 + \kappa \sqrt{x(2 - x)(2 + x\kappa)(2(1 + \kappa) - x\kappa)}}{2(1 + \kappa)},
\]

where the two constants \( \{\beta, \kappa\} \) are related to the temperature \( T \) and the chemical potential \( \mu \) of the R-charged black brane

\[
2\pi T \bigg|_{\lambda=0} = \beta \frac{\kappa + 2}{\sqrt{\kappa(1 + \kappa)}}, \quad \mu \bigg|_{\lambda=0} = \frac{\beta}{\sqrt{1 + \kappa}}.
\]

Note that the ratio

\[
\frac{2\pi T}{\mu} \bigg|_{\lambda=0} = \sqrt{\kappa} + \frac{2}{\sqrt{\kappa}}
\]

attains a minimum at \( \kappa(\lambda = 0) = 2 \), at which the black brane undergoes a second order phase transition [4, 5, 6].

- **To order \( \mathcal{O}(\lambda^1) \)** we have

\[
0 = \chi'' + \frac{H_0^3(1 + \kappa) + H_0^6(1 + \kappa) + H_0^8(2\kappa^2 - \kappa + 2\kappa^2x^2 - 1 - 4x\kappa^2) - 1 - \kappa}{(H_0^9 - 1)(H_0^3 - 1)(x - 1)(1 + \kappa)} \chi',
\]

\[
- \frac{H_0^7\kappa^2m^2(2\kappa^2x^2 - 2x\kappa^2 + H_0^3 - 2\kappa + \kappa^2 - 2) + \kappa + 1 + H_0^5\kappa}{(1 + \kappa)^2(H_0^9 - 1)^2} \chi_1.
\]
To order $O(\lambda^2)$ we have

\begin{align}
0 &= g''_2 + C_{11} g'_2 + C_{12} H'_2 + C_{13} A'_2 + C_{14} (\chi'_1)^2 + C_{15} g_2 + C_{16} H_2 + C_{17} \chi_1^2, \\
0 &= H''_2 + C_{21} g'_2 + C_{22} H'_2 + C_{23} A'_2 + C_{24} (\chi'_1)^2 + C_{25} g_2 + C_{26} H_2 + C_{27} \chi_1^2, \\
0 &= A''_2 + C_{31} g'_2 + C_{32} H'_2 + C_{33} A'_2 + C_{34} g_2 + C_{35} H_2,
\end{align}

(2.14)

where the coefficients $C_{ij}$ are collected in Appendix A.

2.2 Holographic renormalization and the boundary stress-energy tensor

In this section we carefully perform the holographic renormalization of (2.1) and extract the thermodynamic quantities, subject to the following constraints:

- the background geometry of the gauge theory dual to the gravitational action (2.1) is $R^{3,1}$;
- since (2.1) is expected to be valid only to order $O(\chi^2)$, we perform holographic renormalization only to this order in $\chi$, but exact in $\phi$.

Let $r$ be the position of the boundary, and $S_E^r$ be the Euclidean gravitational action on the cut-off space

$$\lim_{r \to \infty} S_E^r = S_E,$$

(2.15)

where $S_E$ is the Euclidean version of (2.1). Explicitly, using equations of motion,

$$S_E^r = \frac{1}{16\pi G_5} \int_{r_h}^r dr \int_{\partial M} d^4\xi \sqrt{h_E} \mathcal{L}_E = \frac{1}{16\pi G_5} \int_{r_h}^r dr \int_{\partial M} d^4\xi \sqrt{h_E} \mathcal{L}$$

$$= \frac{1}{8\pi G_5} \left( \int_{\partial M_5} d^4\xi \right) \int_{r_h}^r dr \left[ \frac{c_1 c_2' c_2}{c_3} \right] = \frac{1}{8\pi G_5} \left( \int_{\partial M_5} d^4\xi \right) \times \left[ \frac{c_1 c_2' c_2}{c_3} \right] \bigg|_{r_h},$$

(2.16)

where $r_h$ is a position of the regular Schwarzschild horizon of (2.5), $h_E$ is the induced metric on the boundary, and

$$\frac{1}{8\pi G_5} = \frac{N^2}{4\pi^2}.$$  

(2.17)

Notice that for a regular Schwarzschild horizon, the horizon contribution in (2.16) vanishes. Besides the standard Gibbons-Hawking term

$$S_{GH} = -\frac{1}{8\pi G_5} \int_{\partial M_5} d^4\xi \sqrt{h_E} \nabla_\mu n^\mu = -\frac{1}{8\pi G_5} \frac{[c_1 c_2']}{c_3} \int_{\partial M_5} d^4\xi,$$

(2.18)
we supplement the combined regularized action \((S'_E + S_{GH})\) by the appropriate boundary counterterms which are needed to get the finite action. These boundary counterterms must be constructed from the local metric and \(\{\phi, \chi\}\) scalar invariants\footnote{\label{footnote}}

\[
S^{\text{counter}} = \frac{1}{4\pi G_5} \int_{\partial M_5} d^4\xi \sqrt{h_E} \left( \alpha_1 + \alpha_2 \chi^2 + \alpha_3 \left(\ln \phi\right)^2 + \mathcal{O}(\chi^4) \right),
\]

(2.19)

\(\alpha_i\) are constant coefficients of the counterterms which are determined by the requirement of having a finite action. The counterterm \(\propto \alpha_3\) is finite, but is required to insure the first law of thermodynamics \cite{15} and the supersymmetry (at extremality) \cite{14} of the single-charge AdS\(_5\) RN black hole. Thus, the finiteness of the renormalized Euclidean action

\[
\mathcal{I}_E \equiv \lim_{r \to \infty} \left( S'_E + S_{GH} + S^{\text{counter}} \right), \quad |\mathcal{I}_E| < \infty,
\]

(2.20)

constraints

\[
\alpha_1 = \frac{3}{2}, \quad \alpha_2 = \frac{1}{8}, \quad \alpha_3 = \frac{1}{6}.
\]

(2.21)

These values of \(\alpha_i\) agree with those reported in \cite{17, 18, 14}.

The boundary stress energy tensor \(T^{\mu \nu}\) is obtained from the variation of the full action

\[
S_{\text{tot}} = S'_E + S_{GH} + S^{\text{counter}},
\]

(2.22)

with respect to the (Minkowski) boundary metric \(\eta_{\mu \nu} = c_\mu^{-2} h_{\mu \nu}\)

\[
T^{\mu \nu} = 2\frac{\delta S_{\text{tot}}}{\delta \eta_{\mu \nu}} = \frac{c_1 c_2^2}{c_\mu^2} \frac{2}{\sqrt{-h}} \frac{\delta S_{\text{tot}}}{\delta h_{\mu \nu}} \bigg|_{r \to \infty}
\]

\[
= \frac{c_1 c_2^2}{8\pi G_5} \left[ -\Theta^{\mu \nu} + \Theta h^{\mu \nu} - 2\left\{ \alpha_1 + \alpha_2 \chi^2 + \alpha_3 \left(\ln \phi\right)^2 \right\} \right] h^{\mu \nu} + \mathcal{O}(\chi^4) \bigg|_{r \to \infty},
\]

(2.23)

where

\[
\Theta^{\mu \nu} = \frac{1}{2} \left( \nabla^{\mu} n^\nu + \nabla^{\nu} n^\mu \right), \quad \Theta = \text{Tr} \Theta^{\mu \nu}.
\]

(2.24)

### 2.3 Asymptotics of the background geometry

Within the background ansatz (2.5)-(2.7), it is straightforward to derive from (2.1) equations of motion for \(A, H, g, \chi\) and solve these equations perturbatively near the
boundary \( x \to 0_+ \), and near the horizon \( y = 1 - x \to 0_+ \). These equations enjoy exact scaling symmetry

\[
(A, g, H, \chi) \sim (\Lambda A, \Lambda g, H, \chi)
\]

for any constant \( \Lambda \).

### 2.3.1 The boundary \( x \to 0_+ \) asymptotics

We find

\[
A = a_0 + a_1 x^{1/2} + \left( \frac{a_1^2 h_1}{4 g_0^2} - 2 h_1 - \frac{1}{72} c_0^2 \right) a_1 x + \mathcal{O} \left( x^{3/2} \ln x \right), \tag{2.26}
\]

\[
g = g_0 \left( 1 + \frac{1}{24} c_0^2 x^{1/2} + \left( \frac{a_1^2 h_1 c_0^2}{96 g_0^2} - \frac{7 c_0^4}{1728} + \frac{1}{4} + \frac{h_1 c_0^2}{24} - \frac{a_1^2}{8 g_0^2} \right) x + \mathcal{O} \left( x^{3/2} \ln^2 x \right) \right), \tag{2.27}
\]

\[
H = 1 + h_1 x^{1/2} + \left( \frac{a_1^2 h_1 c_0^2}{4 g_0^2} + \frac{c_0^2}{9} \right) h_1 x + \mathcal{O} \left( x^{3/2} \ln x \right), \tag{2.28}
\]

\[
\chi = c_0 x^{1/4} \left( 1 + \left( c_10 + \frac{c_0^2}{24} \ln x \right) x^{1/2} + \mathcal{O} \left( x \ln x \right) \right). \tag{2.29}
\]

The general solution is determined by 6 parameters

\[
\left( a_0, a_1, h_1, g_0, c_0, c_{10} \right), \tag{2.30}
\]

of which one is the temperature (it can be thought roughly as a scaling parameter in (2.25)), and another other two \( \{a_0, c_0\} \) are the coefficients of the non-normalizable modes related to the \( U(1)_R \) chemical potential \( \mu = a_0 \) and the mass-deformation scale \( M \propto c_0 \) respectively.

The remaining parameters \( \{a_1, h_1, c_{10}\} \) correspond to the charge density \( \rho \) (conjugate to the chemical potential), and the expectation values of the dimension-2 \( \langle O_2 \rangle \propto h_1 \) and the dimension-3 \( \langle O_3 \rangle \propto c_0 c_{10} \) operators. To establish precise correspondence, we need to introduce a 'universal RG scale' — a universal (in a sense of being independent of any scales in the gauge theory: the temperature, the chemical potential, and the mass-deformation scale) radial coordinate:

\[
\hat{r} \equiv c_2^{-1} = \frac{3^{1/2} h_1^{1/2}}{g_0} x^{1/4} \left( 1 + \left( \frac{a_1^2 h_1}{8 g_0^2} + \frac{c_0^2}{72} \right) x^{1/2} + \mathcal{O} \left( x \ln x \right) \right). \tag{2.31}
\]
The precise values (up to a $c$-number normalization) of various mass scales and operator expectation values (VEV’s) can then be obtained as the coefficients of the non-normalizable/normalizable modes of the corresponding gravitational fields with a radial dependence given by (2.31):

$$M = c_0 \frac{g_0}{3^{1/2} h_1^{1/2}}, \quad \rho = -\frac{1}{8\pi G_5} a_1 \left( \frac{g_0}{3^{1/2} h_1^{1/2}} \right)^2, \quad \langle O_2 \rangle = h_1 \left( \frac{g_0}{3^{1/2} h_1^{1/2}} \right)^2,$$

$$\langle O_3 \rangle = c_0 \left( \frac{g_0}{3^{1/2} h_1^{1/2}} \right)^3 \left( c_{10} - \frac{a_1^2 h_1}{8g_0} \right) + \mathcal{O} \left( g_0^3 c_0^3 \right).$$

(2.32)

The factor of $(−8\pi G_5)^{-1}$ in the definition of $\rho$ is inserted to insure consistency of the thermodynamics of the dual gauge theory. Notice that the expectation value $\langle O_3 \rangle$ is determined only up to order $\mathcal{O}(c_0^2)$ — this is related to the fact that the effective action (2.1) is defined only to this order as well.

2.3.2 The horizon $y = 1 - x \to 0_+$ asymptotics

We find

$$A = a_2^h y^2 + \mathcal{O}(y^4),$$

(2.33)

$$g = g_0^h + g_2^h y^2 + \mathcal{O}(y^4),$$

(2.34)

$$H = h_0^h - \frac{(4(h_0^h)^6(a_2^h)^2 + (h_0^h)^4(a_2^h)^2(c_0^h)^2 + 4(h_0^h)^3(a_2^h)^2 + 8g_2^h g_0^h)h_0^h((h_0^h)^3 - 1)}{3(g_0^h)^2(h_0^h(c_0^h)^2 + 4)} y^2 + \mathcal{O}(y^4),$$

(2.35)

$$\chi = c_0^h - \frac{3(h_0^h)^6(c_0^h)^2 + 2g_2^h g_0^h}{(g_0^h)^2(h_0^h(c_0^h)^2 + 4)} y^2 + \mathcal{O}(y^4).$$

(2.36)

The most general solution is determined by 5 parameters

$$\left( a_2^h, g_0^h, g_2^h, h_0^h, c_0^h \right).$$

(2.37)

Note that once the temperature, the chemical potential and the deformation scale are fixed, we have $5 + 6 - 3 = 8$ parameters, which is the correct number to uniquely determine the solution.
2.4 Thermodynamics of the background geometry

Given (2.32), the thermodynamics of the mass-deformed theory must be studied under the constraint

\[ c_0 = \frac{3^{1/2}h_1^{1/2}}{g_0} \quad \text{M}, \]  

(2.38)

for a fixed M.

All the thermodynamic quantities are given to order \( O(c_0^2) \) or \( O((c_0 h_1)^2) \).

The temperature \( T \) and the chemical potential \( \mu \) of the background geometry is given by

\[ (2\pi T)^2 = \frac{(h_0^h(c_0^h)^2 + 4)(g_0^h)^4}{4((h_0^h)^3 - 1)\left((h_0^h)^3 + (a_2^2)^2 + 2g_0^h h_1^h\right)}, \quad \mu = a_0. \]  

(2.39)

The entropy density \( s \) is

\[ s = \frac{A_{hor}}{4G_5} = \frac{c_2^3}{4G_5}\bigg|_{y \to 0+} = \frac{1}{8\pi G_5} \frac{2\pi(g_0^h)^3(h_0^h)^{3/2}}{((h_0^h)^3 - 1)^{3/2}}. \]  

(2.40)

The regularized Euclidean action (2.20) (up to the space-time volume factor) has to be identified with the Gibbs free energy density \( \Omega \)

\[ \Omega = \left( \int_{\partial M_5} d^4\xi \right)^{-1} \mathcal{I}_E = \frac{1}{8\pi G_5} \left( -\frac{g_0^4}{9h_1^2} + \frac{g_0^2(a_1^2 h_1 - 8g_0^2c_{10})c_0^2}{288h_1^2} \right). \]  

(2.41)

From (2.23), the energy density \( \epsilon \) and the pressure \( P \) are given by

\[ \epsilon = \frac{1}{8\pi G_5} \left( \frac{g_0^4}{3h_1^2} + \frac{g_0^2(a_1^2 h_1 - 8g_0^2 c_{10})c_0^2}{288h_1^2} \right), \]
\[ P = \frac{1}{8\pi G_5} \left( \frac{g_0^4}{9h_1^2} - \frac{g_0^2(a_1^2 h_1 - 8g_0^2 c_{10})c_0^2}{288h_1^2} \right). \]  

(2.42)

Notice that \( \Omega = -P \), as required by the Minkowski space-time thermodynamics. Finally, from (2.32), the charge density is given by

\[ \rho = -\frac{1}{8\pi G_5} \frac{a_1g_0^2}{3h_1}. \]  

(2.43)

While not automatically satisfied, we expect the basic thermodynamic relations

\[ \Omega = \epsilon - Ts - \mu \rho, \quad d\epsilon = T \, ds + \mu \, d\rho, \quad dP = s \, dT + \rho \, d\mu, \]  

(2.44)

to hold.
2.5 Thermodynamics to order $\mathcal{O}(\lambda^0)$

The background geometry to this order is given by (2.10). Thus, we find (we do not need $c_{10}$)

$$\left(a_0, a_1, h_1, g_0, c_0\right) = \left(\frac{\beta}{\sqrt{1+\kappa}}, -2^{1/2}\beta, \frac{2^{1/2}\kappa}{3\sqrt{1+\kappa}}, \beta, 0\right),$$

(2.45)

and

$$\left(a_2^h, g_0^h, g_2^h, h_0^h, c_0^h\right) = \left(\frac{\beta}{\sqrt{1+\kappa(\kappa+2)}}, \beta, 0, (1+\kappa)^{1/3}, 0\right),$$

(2.46)

leading to (we use (2.17))

$$s = 4\pi^2(1+\kappa)^2T^3N^2, \quad \epsilon = 3P = \frac{6N^2T^4(1+\kappa)^3\pi^2}{(\kappa + 2)^4},$$

(2.47)

$$\rho = \frac{2\pi(1+\kappa)^2\kappa^{1/2}T^3N^2}{(\kappa + 2)^3}, \quad \frac{2\pi T}{\mu} = \sqrt{\kappa + \frac{2}{\sqrt{\kappa}}},$$

It is straightforward to verify the basic thermodynamic relations (2.44).

From (2.47) we see that $T\mu$ achieves a minimum at $\kappa = \kappa_c = 2$, corresponding to the critical temperature $T_c = \sqrt{2}\mu/\pi$ and the critical chemical potential $\mu_c = \pi T/\sqrt{2}$.

Introducing

$$t = \frac{T}{T_c} - 1, \quad \bar{\mu} = 1 - \frac{\mu}{\mu_c} \quad \implies \quad \bar{\mu} = \frac{t}{t+1},$$

(2.48)

we find

$$\Omega = \Omega_\pm(\mu, t) = -\frac{27N^2\mu^4}{32\pi^2} \left(1 + \frac{8}{3} t - \frac{16\sqrt{2}}{27} t^{3/2} + \frac{68}{27} t^2 + \mathcal{O}\left(t^{5/2}\right)\right),$$

$$\Omega = \Omega_\pm(T, \bar{\mu}) = -\frac{27N^2T^4\pi^2}{128} \left(1 - \frac{4}{3} \bar{\mu} + \frac{16\sqrt{2}}{27} \bar{\mu}^{3/2} + \frac{14}{27} \bar{\mu}^2 + \mathcal{O}\left(\bar{\mu}^{5/2}\right)\right),$$

(2.49)

where the signs in both expressions correlate. Thus for a given temperature $t$ there are two thermodynamic phases of the system, with $\Omega_-$ being the stable one. Specific heat of the $\Omega_+$ phase is negative, and thus this phase is thermodynamically unstable. In [16] it was argued that in the absence of the chemical potentials in holographic examples of gauge theory/string theory correspondence the thermodynamic instabilities show up
as instabilities in the propagation of sound waves. In RN plasma of interest here there are no instabilities in the sound channel; rather, we find that in the $\Omega_+$ phase the two-point correlation function of the order parameter (the charge density) in the vicinity of the critical point oscillates, instead of exponentially decaying. Additionally, the dynamical relaxation time in the $\Omega_+$ phase is negative, which explicitly demonstrates the instability of this phase (see section 4).

For $\Omega_-$ phase we find:

$$C = T \left( \frac{\partial s}{\partial T} \right) \bigg|_{\mu} \propto -\frac{\partial^2 \Omega_-(\mu, t)}{\partial t^2} \propto +t^{-1/2},$$

$$\rho = -\left( \frac{\partial \Omega_-}{\partial \mu} \right) \bigg|_{T} \propto -\frac{\Omega_-(T, \bar{\mu})}{\partial \bar{\mu}} \propto +\bar{\mu}^{1/2} \propto +t^{1/2},$$

$$\chi_T = \left( \frac{\partial \rho}{\partial \mu} \right) \bigg|_{T} \propto -\frac{\partial^2 \Omega_-(T, \bar{\mu})}{\partial \bar{\mu}^2} \propto +\bar{\mu}^{-1/2} \propto +t^{-1/2},$$

in the vicinity of the phase transition.

Recall that in the classic theory of static critical phenomena one introduces critical exponents

$$\alpha, \beta, \gamma, \delta, \nu, \eta.$$ (2.52)

Once we identify the charge density $\rho$ of the RN plasma with the order parameter, and the chemical potential $\mu$ with the external (control) parameter we can read from (2.51) the following 4 critical exponents

$$\alpha, \beta, \gamma, \delta = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 2 \right).$$ (2.53)

Under a single scale hypothesis in a continuous critical phenomena only two static critical exponents are independent — there are 4 scaling relations:

$$\alpha + 2\beta + \gamma = 2, \quad \gamma = \beta(\delta - 1) = \nu(2 - \eta), \quad 2 - \alpha = \nu p,$$ (2.54)

where $p$ is the number of spatial dimensions of a critical system. Notice that given (2.53), the first two scaling relations in (2.54) are satisfied. One can use the rest of the scaling relations to determine the remaining static critical exponents:

$$\nu \bigg|_{\text{relation}} = \frac{1}{2}, \quad \eta \bigg|_{\text{relation}} = 1.$$ (2.55)

As we will see in section 4, the values of $(\nu, \eta)$ as in (2.55) are incorrect. We point only that the value of $\eta$ in (2.55) already raises suspicion: we consider here the critical
phenomena in RN plasma in the strict ‘t Hooft limit, i.e., as \( N \to \infty \) — in this limit we expect a mean-field criticality, leading to

\[
\eta \bigg|_{\text{mean-field}} = 0 .
\]

### 2.6 Thermodynamics to order \( \mathcal{O}(\lambda^2) \)

Since we are going to be interested in the deformation of the critical phenomena in (2.49) at order \( \mathcal{O}(\lambda^2) \) (or more precisely \( \mathcal{O}(M^2) \), see (2.38)) we have to solve (2.13) and (2.14) in the vicinity of \( \kappa = 2 \). Thus, we further expand

\[
\begin{align*}
g_2 &= \beta \sum_{n=0}^{\infty} (\kappa - 2)^n g_2^{(n)}, \quad H_2 = \sum_{n=0}^{\infty} (\kappa - 2)^n H_2^{(n)}, \quad A_2 = \beta \sum_{n=0}^{\infty} (\kappa - 2)^n A_2^{(n)}, \\
\chi_1 &= \sum_{n=0}^{\infty} (\kappa - 2)^n \chi_1^{(n)}.\end{align*}
\]

Clearly, \( \kappa = 2 \) is not a singular point of (2.13) and (2.14), thus (2.57) leads to a series of smooth ODE’s for \( \{g_2^{(n)}, H_2^{(n)}, A_2^{(n)}, \chi_1^{(n)}\} \). These ODE’s must be solved with the following boundary conditions:

\[
\begin{align*}
x \to 0_+ : & \quad g_2^{(n)} \to 0, \quad H_2^{(n)} \to 0, \quad A_2^{(n)} \to 0, \quad \chi_1^{(n)} \to \delta_0^{n} x^{1/4}, \\
x \to 1_+ : & \quad g_2^{(n)} \to \text{const}, \quad H_2^{(n)} \to \text{const}, \quad A_2^{(n)} \to 0, \quad \chi_1^{(n)} \to \text{const}.\end{align*}
\]

The normalization of the non-normalizable modes of \( A_2^{(n)} \) and \( \chi_1 \) near the boundary implies from (2.39) and (2.32) that

\[
\lambda = \frac{(k + 2)}{2^{3/4}(1 + \kappa)^{3/4} \pi} \frac{M}{T} \left( 1 + \mathcal{O}\left( \frac{M^2}{T^2} \right) \right).
\]

It is straightforward to construct asymptotic solutions for \( \{g_2^{(n)}, H_2^{(n)}, A_2^{(n)}, \chi_1^{(n)}\} \), subject to the boundary conditions (2.58) — with obvious modifications they take the form of (2.26)-(2.29) and (2.33)-(2.36). Thus we find:
for UV parameters (2.30):
\[
a_0 = \frac{\beta}{\sqrt{1 + \kappa}}, \quad a_1 = -\beta \sqrt{2} + \lambda^2 \beta \sum_{n=0}^{\infty} (\kappa - 2)^n a_1^{(n)}, \\
h_1 = \frac{\kappa \sqrt{2}}{3 \sqrt{1 + \kappa}} + \lambda^2 \sum_{n=0}^{\infty} (\kappa - 2)^n h_1^{(n)}, \quad g_0 = \beta, \quad c_0 = \lambda,
\]
(2.60)

for IR parameters (2.37):
\[
a_2^h = \frac{\beta}{(\kappa + 2) \sqrt{1 + \kappa}} + \lambda^2 \beta \sum_{n=0}^{\infty} (\kappa - 2)^n a_2^h^{(n)}, \quad g_0^h = \beta + \lambda^2 \sum_{n=0}^{\infty} (\kappa - 2)^n g_0^h^{(n)}, \\
g_2^h = \lambda^2 \sum_{n=0}^{\infty} (\kappa - 2)^n g_2^h^{(n)}, \quad h_0^h = (1 + \kappa)^{1/3} + \lambda^2 \sum_{n=0}^{\infty} (\kappa - 2)^n h_0^h^{(n)}, \\
c_0^h = \lambda \sum_{n=0}^{\infty} (\kappa - 2)^n c_0^h^{(n)},
\]
(2.61)

where \(\lambda\) is given by (2.59).

Notice that at each order \((n)\) there are eight parameters
\[
\left( a_1^{(n)}, h_1^{(n)}, c_{10}^{(n)}, a_2^{(n)}, g_0^{h(n)}, g_2^{h(n)}, h_0^{h(n)}, c_0^{h(n)} \right),
\]
(2.62)

which are uniquely determined by solving ODE’s for \(\{g_2^{(n)}, H_2^{(n)}, A_2^{(n)}, \chi_1^{(n)}\}\), subject to the boundary conditions (2.58).

For the first two orders — \(n = 0, 1\) — we collected these coefficients in Table 1.

Given (2.59), (2.60) and (2.61) we can compute following (2.42) and (2.43) thermodynamic potentials as a series in \((\kappa - 2)\) and to order \(O \left( \frac{M^2}{T^2} \right)\). Recall that the basic thermodynamic relations (2.44) are not automatically satisfied. Rather, they impose algebraic constraints on (2.62). We find:

- at order \(n = 0\):
\[
0 = 7 \sqrt{2} 3^{2/3} h_0^{h(0)} - \frac{15}{2} \sqrt{3} h_1^{(0)} - \frac{1}{4} \sqrt{2} 3^{1/3} (c_0^{h(0)})^2 + \frac{32}{3} \sqrt{2} g_2^{h(0)} - 10 \sqrt{2} g_0^{h(0)} + 8 \sqrt{6} a_2^{h(0)} + a_1^{(0)},
\]
(2.63)
Table 1: Coefficients of the normalizable modes of the background geometry.

| $n$  | 0             | 1             |
|------|---------------|---------------|
| $a_1^{(n)}$ | -0.165555569119159 | -0.01307814287910066 |
| $h_1^{(n)}$ | 0.11928166889409134 | 0.018530943268730934 |
| $c_{10}^{(n)}$ | -0.23071409088829617 | -0.01240068268619082 |
| $a_2^{h(n)}$ | -0.03250808756595229 | 0.005503802347302805 |
| $g_0^{h(n)}$ | 0.021412846345792092 | 0.0015030136731377015 |
| $g_2^{h(n)}$ | -0.007908684866787086 | -0.0009615292197374708 |
| $h_0^{h(n)}$ | 0.1485225060823788 | 0.042145056457606175 |
| $c_0^{h(n)}$ | 0.7464562054847809 | -0.0129139123278239 |

\[
0 = 12 \sqrt{3} h_1^{(0)} - \frac{3}{4} \sqrt{2} 3^{1/3} (h_0^{(0)})^2 + 14 \sqrt{2} 3^{2/3} h_0^{(0)} + \frac{1}{24} \sqrt{3} + 32 \sqrt{2} g_2^{h(0)} \\
- 18 \sqrt{2} g_0^{h(0)} - \frac{3}{8 \sqrt{2}} c_{10}^{(0)} + a_1^{(0)} + 24 \sqrt{6} a_2^{h(0)},
\]

(2.64)

\[
0 = -\frac{3}{16} \sqrt{2} c_{i0}^{(1)} + \frac{1}{16} \sqrt{2} c_{i0}^{(0)} + 4 \sqrt{3} h_1^{(1)} - \frac{21}{4} \sqrt{3} h_1^{(0)} - \frac{14}{3} \sqrt{6} a_2^{h(0)} - \frac{73}{18} \sqrt{2} 3^{2/3} h_0^{(0)} \\
+ \frac{7}{2} \sqrt{2} 3^{2/3} h_0^{h(1)} + \frac{23}{6} \sqrt{2} g_0^{h(0)} - 6 \sqrt{2} g_0^{h(1)} - \frac{56}{9} \sqrt{2} g_2^{h(0)} + \frac{7}{48} \sqrt{2} 3^{1/3} (c_0^{h(0)})^2 + a_1^{(1)},
\]

(2.65)

- at order $n = 1$:

\[
0 = -\frac{1}{6} a_1^{(0)} + \frac{21}{4} \sqrt{3} h_1^{(0)} - \frac{8}{3} \sqrt{2} g_2^{(0)} + \frac{11}{144} \sqrt{2} 3^{1/3} (c_0^{h(0)})^2 - \frac{169}{36} \sqrt{2} 3^{2/3} h_0^{h(0)} \\
+ \frac{25}{6} \sqrt{2} g_0^{h(0)} + \frac{32}{3} \sqrt{2} g_2^{h(1)} + 8 \sqrt{6} a_2^{h(1)} - 10 \sqrt{2} g_0^{h(1)} + 7 \sqrt{2} 3^{2/3} h_0^{h(1)} - \frac{15}{2} \sqrt{3} h_1^{(1)} \\
- \frac{1}{2} \sqrt{2} 3^{1/3} c_0^{h(0)} c_0^{h(1)} + a_1^{(1)},
\]

(2.66)

\[
0 = 14 \sqrt{2} 3^{2/3} h_0^{h(1)} - \frac{3}{2} \sqrt{2} 3^{1/3} c_0^{h(0)} c_0^{h(1)} + \frac{11}{48} \sqrt{2} 3^{1/3} (c_0^{h(0)})^2 - \frac{80}{9} \sqrt{2} 3^{2/3} h_0^{h(0)} \\
- 8 \sqrt{2} g_2^{h(0)} + \frac{15}{2} \sqrt{2} g_0^{h(0)} + \frac{1}{8} \sqrt{2} c_{10}^{(1)} + 32 \sqrt{2} g_2^{h(1)} - 18 \sqrt{2} g_0^{h(1)} - \frac{3}{8} \sqrt{2} c_{10}^{(1)} - \frac{1}{6} a_1^{(0)} \\
+ a_1^{(1)} + 24 \sqrt{6} a_2^{h(1)} + \frac{33}{4} \sqrt{3} h_0^{(0)} - 12 \sqrt{3} h_1^{(1)},
\]

(2.67)
\[ 0 = \frac{7}{48} \sqrt{2} 3^{1/3} c_0^{(0)} c_0^{(1)} + \frac{19}{6} \sqrt{2} g_0^{(1)} - \frac{1}{12} a_1^{(1)} + a_1^{(2)} + \frac{23}{18} \sqrt{6} a_2^{(0)} - \frac{7}{3} \sqrt{6} a_2^{(1)} \]
\[ - \frac{61}{24} \sqrt{3} h_0^{(0)} + \frac{31}{8} \sqrt{3} h_1^{(1)} - \frac{21}{4} \sqrt{3} h_1^{(2)} + \frac{67}{27} \sqrt{2} g_2^{(0)} - \frac{29}{16} \sqrt{2} g_0^{(0)} \]
\[ - \frac{215}{3456} \sqrt{2} 3^{1/3} (c_0^{(0)})^2 - \frac{28}{9} \sqrt{2} g_2^{(1)} + \frac{3235}{1296} \sqrt{2} 3^{2/3} h_0^{(0)} + \frac{1}{16} \sqrt{2} c_1^{(1)} - \frac{11}{384} \sqrt{2} c_1^{(0)} \]
\[ - \frac{3}{16} \sqrt{2} c_1^{(2)} - \frac{479}{144} \sqrt{2} 3^{2/3} h_0^{(1)} + \frac{7}{2} \sqrt{2} 3^{2/3} h_0^{(2)} - 6 \sqrt{2} g_0^{(2)}. \]

(2.68)

Given data in Table 1 we can verify (2.63)-(2.64) to agree with a relative accuracy of \(\sim 10^{-10}\), and (2.65)-(2.67) with a relative accuracy of \(\sim 10^{-7}\).

We are now in position to evaluate the free energy near the critical point. First of all, notice that \(\kappa_c, T_c\) or \(\mu_c\) will receive \(\mathcal{O}(\lambda^2)\) corrections:

\[ \kappa_c = 2 + \frac{M^2}{\pi^2 T^2} \left( \frac{512 \sqrt{6}}{81} g_2^{(0)} + \frac{320 \sqrt{2}}{9} a_2^{(0)} - \frac{184 \sqrt{2}}{27} 3^{1/6} h_0^{(0)} - \frac{8 \sqrt{2}}{81} 3^{5/6} (c_0^{(0)})^2 \right) \]
\[ + \frac{112 \sqrt{2}}{3} 3^{1/6} h_0^{(1)} + \frac{256 \sqrt{2}}{3} a_2^{(1)} + \frac{1024 \sqrt{6}}{27} g_2^{(1)} - \frac{128 \sqrt{6}}{9} g_0^{(1)} \]
\[ - \frac{16 \sqrt{2}}{9} 3^{5/6} (c_0^{(0)}) c_0^{(1)} \right) + \mathcal{O} \left( \frac{M^4}{T^4} \right), \]

(2.69)

\[ T_c = \frac{\sqrt{2}}{\pi} \mu \left( 1 + \frac{M^2}{\mu^2} \left( \frac{4}{9} \sqrt{6} g_0^{(0)} - \frac{32}{27} \sqrt{6} g_2^{(0)} - \frac{8}{3} \sqrt{2} a_2^{(0)} - \frac{7}{6} \sqrt{2} 3^{1/6} h_0^{(0)} \right) \right) \]
\[ + \frac{1}{36} \sqrt{2} 3^{5/6} (c_0^{(0)})^2 \right) + \mathcal{O} \left( \frac{M^4}{\mu^4} \right), \]

(2.70)

\[ \mu_c = \frac{\pi}{\sqrt{2} T} \left( 1 + \frac{M^2}{\pi^2 T^2} \left( \frac{8}{9} \sqrt{6} g_0^{(0)} + \frac{64}{27} \sqrt{6} g_2^{(0)} + \frac{16}{3} \sqrt{2} a_2^{(0)} + \frac{7}{3} \sqrt{2} 3^{1/6} h_0^{(0)} \right) \right) \]
\[ - \frac{1}{18} \sqrt{2} 3^{5/6} (c_0^{(0)})^2 \right) + \mathcal{O} \left( \frac{M^4}{T^4} \right), \]

(2.71)

with the mass deformation lowering\(^5\) the critical temperature for a fixed chemical potential. Once again, introducing

\[ t = \frac{T}{T_c} - 1, \quad \bar{\mu} = 1 - \frac{\mu}{\mu_c}, \]

(2.72)

\(^5\)We used results of Table 1.
we find
\[
\Omega_\pm (\mu, t) = -\frac{27N^2 \mu^4}{32 \pi^2} \left( 1 + s_\mu^0 \frac{M^2}{\mu^2} \right) \left( 1 \pm s_\mu^1 \frac{M^2}{\mu^2} \right) \frac{t^{1/2}}{2} \left( 1 + s_\mu^2 \frac{M^2}{\mu^2} \right)
\]
\[
\mp \frac{16 \sqrt{2}}{27} \left( 1 + s_\mu^3 \frac{M^2}{\pi^2 T^2} \right) t^{3/2} + \cdots + O\left( \frac{M^4}{\mu^4} \right),
\]
(2.73)
\[
\Omega_\pm (T, \tilde{\mu}) = -\frac{27N^2 T^4 \pi^2}{128} \left( 1 + s_\mu^0 \frac{M^2}{\pi^2 T^2} \right) \left( 1 \pm s_\mu^1 \frac{M^2}{\pi^2 T^2} \right) \frac{t^{1/2}}{2} \left( 1 + s_\mu^2 \frac{M^2}{\pi^2 T^2} \right) \tilde{\mu}
\]
\[
\mp \frac{16 \sqrt{2}}{27} \left( 1 + s_\mu^3 \frac{M^2}{\pi^2 T^2} \right) \tilde{\mu}^{3/2} + \cdots + O\left( \frac{M^4}{T^4} \right).
\]
(2.74)

For the first two corrections to (2.49) we have
\[
s_\mu^0 = -2 \ h_1^{(0)} - \frac{1}{5} + \frac{1}{18} \sqrt{6} \ c_{10}^{(0)},
\]
(2.75)
\[
\begin{align*}
s_\mu^1 &= -8 \sqrt{2} \ h_1^{(1)} + \frac{8}{3} \sqrt{2} \ h_1^{(0)} - \frac{16}{243} \ 3^{5/6} \ (c_0^{(h)})^2 - \frac{32}{27} \ 3^{5/6} \ c_0^{(h)} c_0^{(h)} + \frac{224}{9} \ 3^{1/6} \ h_0^{(h)} \\
&\quad - \frac{386}{81} \ h_0^{(h)} - \frac{256}{27} \sqrt{3} \ g_0^{(h)} + \frac{1024}{243} \sqrt{3} \ g_2^{(h)} + \frac{2048}{81} \sqrt{3} \ g_2^{(h)} + \frac{640}{27} \ a_2^{(h)} \\
&\quad + \frac{512}{9} \ a_2^{(h)} - \frac{2}{81} \sqrt{2} + \frac{4}{9} \sqrt{3} \ c_{10}^{(1)},
\end{align*}
\]
(2.76)
\[
\begin{align*}
s_\mu^0 &= -4 \ h_1^{(0)} + \frac{9}{9} \sqrt{6} c_{10}^{(0)} - \frac{1}{27} - \frac{32}{9} \sqrt{6} \ g_0^{(h)} + \frac{256}{27} \sqrt{6} \ g_2^{(h)} + \frac{28}{3} \ 3^{1/6} \ \sqrt{2} \ h_0^{(h)} \\
&\quad + \frac{64}{3} \sqrt{2} \ a_2^{(h)} - \frac{2}{9} \ 3^{5/6} \ \sqrt{2} \ (c_0^{(h)})^2,
\end{align*}
\]
(2.77)
\[
\begin{align*}
\begin{align*}
s_\mu^1 &= -16 \sqrt{2} \ h_1^{(1)} + \frac{16}{3} \sqrt{2} \ h_1^{(0)} - \frac{32}{243} \ 3^{5/6} \ (c_0^{(h)})^2 - \frac{64}{27} \ 3^{5/6} \ c_0^{(h)} c_0^{(h)} + \frac{448}{9} \ 3^{1/6} \ h_0^{(h)} \\
&\quad - \frac{736}{81} \ 3^{1/6} \ h_0^{(h)} - \frac{512}{27} \sqrt{3} \ g_0^{(h)} + \frac{2048}{243} \sqrt{3} \ g_2^{(h)} + \frac{4096}{81} \sqrt{3} \ g_2^{(h)} + \frac{1280}{27} \ a_2^{(h)} \\
&\quad + \frac{1024}{9} \ a_2^{(h)} - \frac{4}{81} \sqrt{2} + \frac{8}{9} \sqrt{3} \ c_{10}^{(1)}.
\end{align*}
\end{align*}
\]
(2.78)

It is straightforward to check, that given constraints from the basic thermodynamic relations (2.63)-(2.67)
\[
s_\mu^1 = s_\mu^1 = 0,
\]
(2.79)
which guarantees that the mass-deformed theory has the same static critical exponents \((\alpha, \beta, \gamma, \delta)\) as in (2.53).
3 Fluctuations in charged plasma and its holographic dual

3.1 Hydrodynamic modes in charged plasma

In this section we consider the propagation of hydrodynamic modes in four-dimensional charged relativistic plasma in the absence of background electro-magnetic fields.

The first order hydrodynamic equations of motion in Minkowski space-time are simply the conservation laws for the stress-energy tensor and the $U(1)$ current:

$$\partial_{\nu} T^{\mu\nu} = 0, \quad \partial_{\mu} J^{\mu} = 0. \tag{3.1}$$

One can do the standard decomposition of the stress tensor,

$$T^{\mu\nu} = \epsilon u^{\mu} u^{\nu} + P \Delta^{\mu\nu} + \Pi^{\mu\nu}, \tag{3.2}$$

where

$$\Delta^{\mu\nu} = \eta^{\mu\nu} + u^{\mu} u^{\nu}, \quad \Pi_{\mu\nu} u^{\nu} = 0, \tag{3.3}$$

and $\epsilon$ and $P$ are the energy density and the pressure respectively. The dissipative term $\Pi^{\mu\nu}$ is given by

$$\Pi^{\mu\nu} = -\eta \sigma^{\mu\nu} - \zeta \Delta^{\mu\nu} (\partial_{\alpha} u^{\alpha}), \tag{3.4}$$

where

$$\sigma^{\mu\nu} = \Delta^{\mu\alpha} \Delta^{\nu\beta} (\partial_{\alpha} u_{\beta} + \partial_{\beta} u_{\alpha}) - \frac{2}{3} \Delta^{\mu\nu} \Delta^{\alpha\beta} (\partial_{\gamma} u^{\gamma}), \tag{3.5}$$

and $\eta$ and $\zeta$ are the shear and the bulk viscosities. Note that $\Pi^{\mu\nu}$ is, by definition, zero at local equilibrium. The current $J^{\mu}$ is given by

$$J^{\mu} = \rho u^{\mu} + \nu^{\mu}, \tag{3.6}$$

where $\nu^{\mu}$ is the dissipative part satisfying $u^{\mu} \nu_{\mu} = 0$:

$$\nu^{\mu} = \sigma_{Q} \Delta^{\mu\nu} \left( -\partial_{\nu} \mu + \frac{\mu}{T} \partial_{\nu} T \right). \tag{3.7}$$

In this expression, $T$ is the temperature, $\mu$ is the chemical potential, and $\sigma_{Q}$ is the conductivity coefficient. We would like to study fluctuations around the equilibrium state in which

$$u^{\mu} = (1, 0, 0, 0), \quad T = \text{const.}, \quad \mu = \text{const.}. \tag{3.8}$$

As an independent set of variables we will choose the three spatial components of the velocity $\delta u^{1} = \delta u^{x}$, $\delta u^{2} = \delta u^{y}$, $\delta u^{3} = \delta u^{z}$, as well as $\delta T$ and $\delta \mu$. As usual, all
perturbations are of the plane-wave form $\exp(-i\omega t + iqz)$. We find that the relevant fluctuations of $T^{\mu\nu}$ are

$$
\delta T^{tt} = \delta \epsilon = \left(\frac{\partial \epsilon}{\partial \mu}\right)_T \delta \mu + \left(\frac{\partial \epsilon}{\partial T}\right)_\mu \delta T,
$$

$$
\delta T^{ti} = (\epsilon + P)\delta u^i,
$$

$$
\delta T^{xx} = -\eta \partial_z \delta u^x, \quad \delta T^{yz} = -\eta \partial_z \delta u^y,
$$

$$
\delta T^{zz} = \delta P - \left(\frac{4}{3}\eta + \zeta\right) \partial_z \delta u^z = \left(\frac{\partial P}{\partial \mu}\right)_T \delta \mu + \left(\frac{\partial P}{\partial T}\right)_\mu \delta T - \left(\frac{4}{3}\eta + \zeta\right) \partial_y \delta u_y.
$$

Similarly, we obtain the following fluctuations of the current

$$
\delta J^t = \delta \rho = \left(\frac{\partial \rho}{\partial \mu}\right)_T \delta \mu + \left(\frac{\partial \rho}{\partial T}\right)_\mu \delta T,
$$

$$
\delta J^x = \rho \delta u^x, \quad \delta J^y = \rho \delta u^y,
$$

$$
\delta J^z = \rho \delta u^z + \sigma Q \left(-\partial_z \delta \mu + \frac{\mu}{T} \partial_z \delta T\right).
$$

Substituting these expressions into equations of motion (3.1) and performing a Fourier transformation we get the following system of equations

$$
0 = \omega \left(\frac{\partial \epsilon}{\partial \mu}\right)_T \delta \mu + \left(\frac{\partial \epsilon}{\partial T}\right)_\mu \delta T - q(\epsilon + P)\delta u^z,
$$

$$
0 = \omega (\epsilon + P)\delta u^z - q \left(\frac{\partial P}{\partial \mu}\right)_T \delta \mu + \left(\frac{\partial P}{\partial T}\right)_\mu \delta T + iq^2 \left(\frac{4}{3}\eta + \zeta\right) \delta u^z,
$$

$$
0 = \omega \left(\frac{\partial \rho}{\partial \mu}\right)_T \delta \mu + \left(\frac{\partial \rho}{\partial T}\right)_\mu \delta T - q \rho \delta u^z + iq^2 \sigma Q \left(\delta \mu - \frac{\mu}{T} \delta T\right),
$$

where we assembled equations in the decoupled sets. The three equations (3.11) describe the propagation of sound waves in charged plasma, while the pair (3.12) describes the propagation of the shear modes of different polarizations.

In the shear channel, (3.12), the dispersion relation takes form

$$
\omega \equiv iq^2D = -iq^2 \frac{\eta}{\epsilon + P} = -iq^2 \frac{\eta}{T} \frac{T s}{Ts - \mu \rho}.
$$
Notice that even though in a holographic gauge theory plasma we study here the ratio of the shear viscosity to the entropy density is universal \[19\]

\[
\frac{\eta}{s} = \frac{1}{4\pi},
\]

(3.14)
a combination involving the diffusive constant \(T D\) has a nontrivial dependence on \(\frac{T}{\mu}\) and \(\frac{M}{\mu}\).

In the sound channel, (3.11), the dispersion relation takes form

\[
\omega = \pm c_s q - i \Gamma q^2 + \mathcal{O}(q^3),
\]

(3.15)
where the speed of sound \(c_s\) and the attenuation \(\Gamma\) are given by

\[
c_s^2 = \left( (\epsilon + P) \frac{\partial (P, \rho)}{\partial (T, \mu)} + \rho \frac{\partial (\epsilon, P)}{\partial (T, \mu)} \right) \left( (\epsilon + P) \frac{\partial (\epsilon, \rho)}{\partial (T, \mu)} \right)^{-1},
\]

(3.16)

\[
\Gamma = \frac{2\eta}{3(\epsilon + P)} \left( 1 + \frac{3\zeta}{4\eta} \right) - \frac{\sigma Q}{2T} \left( \frac{\partial P}{\partial \rho} \right)_\epsilon \left( (\epsilon + P) \frac{\partial (P, \rho)}{\partial (T, \mu)} + \rho \frac{\partial (\epsilon, P)}{\partial (T, \mu)} \right)^{-1} \times
\]

\[
\left( (\epsilon + P) \left( \frac{\partial \rho}{\partial \ln \mu} \right)_T + \left( \frac{\partial \rho}{\partial \ln T} \right)_\mu \right) - \rho \left( \frac{\partial \epsilon}{\partial \ln \mu} \right)_T + \left( \frac{\partial \epsilon}{\partial \ln T} \right)_\mu \right) \right).
\]

(3.17)

It is instructive to analyze (3.15) for conformal theories and for theories with softly broken scale invariance, i.e., when

\[
\frac{M}{T} \ll 1, \quad \frac{M}{\mu} \ll 1,
\]

(3.18)
where \(M\) is a conformal symmetry breaking scale.

- In a conformal hydrodynamics

\[
\zeta = 0, \quad \epsilon = 3 P, \quad \epsilon \equiv T^4 \mathcal{E} \left( \frac{\mu}{T} \right), \quad \rho \equiv T^3 \mathcal{R} \left( \frac{\mu}{T} \right),
\]

(3.19)
where \(\mathcal{E}\) and \(\mathcal{R}\) are functions of the dimensionless ratio \(\frac{\mu}{T}\), thus

\[
\frac{\partial (P, \rho)}{\partial (T, \mu)} = \frac{1}{3} \frac{\partial (\epsilon, \rho)}{\partial (T, \mu)}, \quad \frac{\partial (\epsilon, P)}{\partial (T, \mu)} = \frac{1}{3} \frac{\partial (\epsilon, \epsilon)}{\partial (T, \mu)} = 0,
\]

\[
\left( \frac{\partial \rho}{\partial \ln \mu} \right)_T + \left( \frac{\partial \rho}{\partial \ln T} \right)_\mu = T^3 \frac{\mu}{T} \mathcal{R}' + 3T^3 \mathcal{R} + T^4 \left( -\frac{\mu}{T^2} \right) \mathcal{R}' = 3\rho,
\]

\[
\left( \frac{\partial \epsilon}{\partial \ln \mu} \right)_T + \left( \frac{\partial \epsilon}{\partial \ln T} \right)_\mu = T^4 \frac{\mu}{T} \mathcal{E}' + 4T^4 \mathcal{E} + T^5 \left( -\frac{\mu}{T^2} \right) \mathcal{E}' = 4\epsilon,
\]

(3.20)
\[
\left( \frac{\partial P}{\partial \rho} \right)_\epsilon = \frac{\partial (P, \epsilon)}{\partial (\rho, \epsilon)} = \frac{1}{3} \frac{\partial (\epsilon, \epsilon)}{\partial (\rho, \epsilon)} = 0.
\]
As a result, we find for the speed of sound and the sound wave attenuation
\[ c^2_s \bigg|_{\text{CFT}} = \frac{1}{3}, \quad \Gamma \bigg|_{\text{CFT}} = \frac{2\eta}{3(\epsilon + P)} = \frac{2}{3T}s \frac{T_s}{T_s - \mu \rho}. \] (3.21)

Once again, notice that \( \rho T \) has a nontrivial dependence on \( \frac{\mu}{T} \), even though the shear viscosity ratio is universal (3.14)\(^6\).

- When the scale invariance is softly broken (3.18), we find
\[
c^2_s = \frac{1}{3} + \mathcal{O}\left(\frac{M^2}{T^2}\right), \quad \Gamma = \frac{2\eta}{3(\epsilon + P)} \left(1 + \frac{3\zeta}{4\eta}\right) + \frac{1}{T} \times \frac{\sigma_Q}{T} \times \mathcal{O}\left(\frac{M^2}{T^2}\right),
\] (3.22)

where, given (3.20), we used
\[
\left(\frac{\partial P}{\partial \rho}\right)_{\epsilon} = T \times \mathcal{O}\left(\frac{M^2}{T^2}\right),
\]
\[
(\epsilon + P) \left(\left(\frac{\partial \rho}{\partial \ln \mu}\right)_T + \left(\frac{\partial \rho}{\partial \ln T}\right)_{\mu}\right) - \rho \left(\left(\frac{\partial \epsilon}{\partial \ln \mu}\right)_T + \left(\frac{\partial \epsilon}{\partial \ln T}\right)_{\mu}\right)
\] = \(T^7 \times \mathcal{O}\left(\frac{M^2}{T^2}\right).
\] (3.23)

In (3.22) and (3.23) we suppressed dependence on \( \frac{\mu}{T} \) in \( \mathcal{O}(M^2) \) terms. What is important for the later discussion is that the contribution of the term \( \propto \sigma_Q \) to the attenuation \( \Gamma \) is of order \( \mathcal{O}(M^4) \).

### 3.2 Sound waves of mass-deformed RN plasma

We can use (3.16) and (3.17) to compute the speed of sound and the bulk viscosity of the mass-deformed RN plasma (2.1) to order \( \mathcal{O}(\lambda^2) \).

We find it convenient to introduce
\[
\omega \equiv \frac{\omega}{2\pi T}, \quad q \equiv \frac{q}{2\pi T},
\] (3.24)

where \( \{\omega, q = |\bar{q}|\} \) are the frequency and the momentum of the sound mode. Furthermore, we parametrize the sound wave dispersion relation (3.15) as
\[
w = \frac{q}{\sqrt{3}} \beta_1 - \frac{q^2}{3} \beta_2 + \mathcal{O}(q^3),
\]
\[
\beta_1 \equiv \sum_{n=0}^{1} \left\{ \lambda^{2n} \beta_{1,n} \right\}, \quad \beta_2 = \sum_{n=0}^{1} \left\{ \lambda^{2n} \beta_{2,n} \right\}.
\] (3.25)

\(^6\)In conformal holographic hydrodynamics this was first established in [20].
In this parametrization \( \beta_1 = \beta_2 = 1 \) for a conformal plasma with vanishing chemical potential \([21]\). Using (2.42), (2.43), (2.59), (3.14) and introducing
\[
a_0 = \frac{\beta}{\sqrt{1 + \kappa}}, \quad a_1 = -\beta \sqrt{2} + \lambda^2 \beta a_{12}(\kappa),
\]
\[
h_1 = \frac{\kappa \sqrt{2}}{3 \sqrt{1 + \kappa}} + \lambda^2 h_{12}(\kappa), \quad g_0 = \beta, \quad c_0 = \lambda,
\]
\[
c_{10} = c_{10}(\kappa),
\]
for the UV parameters (2.30), and
\[
a_2^h = \frac{\beta}{(\kappa + 2) \sqrt{1 + \kappa}} + \lambda^2 \beta a_{22}^h(\kappa), \quad g_0^h = \beta + \lambda^2 \beta g_{02}^h(\kappa),
\]
\[
g_2^h = \lambda^2 \beta g_{22}^h(\kappa), \quad h_0^h = (1 + \kappa)^{1/3} + \lambda^2 h_{02}^h(\kappa),
\]
\[
c_0^h = \lambda c_{01}^h(\kappa),
\]
for the IR parameters (2.37), we find from (3.15)-(3.17)
\[
\beta_{1,0} = \pm 1, \quad \beta_{2,0} = \frac{\kappa + 2}{2 \kappa + 2},
\]
\[
\beta_{1,1} = -\frac{\kappa \sqrt{2}}{144 \sqrt{1 + \kappa}} + \frac{1}{12} c_{10}(\kappa),
\]
while \( \beta_{2,1} \) directly determines the bulk viscosity to the shear viscosity ratio, since contribution to the attenuation proportional to the conductivity \( \sigma_Q \) vanishes to order \( \mathcal{O}(\lambda^2) \), see (3.22), and the conductivity itself is finite at criticality \([7]\),
\[
\frac{\zeta}{\eta} = \Delta(\kappa) \lambda^2 + \mathcal{O}(\lambda^4) = \Delta(\kappa) \frac{(\kappa + 2)^2}{2^{3/2}(1 + \kappa)^{3/2} \pi^2} \frac{M^2}{T^2} + \mathcal{O} \left( \frac{M^4}{T^4} \right),
\]
where
\[
\Delta(\kappa) = \frac{8(1 + \kappa)}{3(\kappa + 2)} \beta_{2,1} - \frac{4(1 + \kappa)^{1/2} 2^{1/2}}{\kappa} h_{12}(\kappa) + \frac{4}{3}(\kappa + 2)(1 + \kappa)^{1/2} a_{22}^h(\kappa)
\]
\[
- \frac{20}{3} g_{22}^h(\kappa) + \frac{4(\kappa + 2)^2}{3(1 + \kappa)} g_{22}^h(\kappa) + \frac{2(5 \kappa + 4)}{(1 + \kappa)^{1/3} \kappa} h_{02}^h(\kappa) - \frac{1}{6}(1 + \kappa)^{1/3} \left( c_{01}^h(\kappa) \right)^2.
\]

In the next section we study sound channel quasinormal modes of mass-deformed RN black holes (2.5) and compute \( \beta_1 \) and \( \beta_2 \) in (3.25) to order \( \mathcal{O}(\lambda^2) \) inclusive, at \( \kappa = 2 \). Since \( c_{10} \big|_{\kappa=2} = c_{10}^{(0)} \) in Table 1, (3.28) would provide a highly nontrivial test on
a consistency of our thermodynamic and hydrodynamic analysis. Moreover, given that

\[ h_{1,2}^{\kappa=2} = h_1^{(0)}, \quad a_{2,2}^{\kappa=2} = a_2^{h(0)}, \quad g_0^{h(0)}, \quad g_0^{(0)}, \quad \beta_2^{(0)} = \beta_2^{h(0)}, \quad \beta_2^{h(0)}, \quad \beta_2^{(0)}, \quad \beta_2^{h(0)} \]

as presented in Table 1 are finite, any possible divergence in the bulk viscosity at criticality, i.e., at \( \kappa = \kappa_c \) (2.69), would arise from the divergence of \( \beta_{2,1} = \beta_{2,1}(\kappa) \), as \( \kappa \to 2 \). The bulk viscosity to the shear viscosity ratio to leading order in \( M^2 T^2 \) at criticality is then computed from (3.29):

\[
\lim_{T \to T_c} \frac{\zeta}{\eta} = \lambda^2 \lim_{\kappa \to 2} \Delta(\kappa) + \mathcal{O}(\lambda^4) = \frac{2^{5/2}}{3^{3/2} \pi^2} \frac{M^2}{T_c^2} \lim_{\kappa \to 2} \Delta(\kappa) + \mathcal{O}\left(\frac{M^4}{T_c^4}\right). \tag{3.32}
\]

### 3.3 Fluctuations of the deformed RN black hole

To determine dispersion relation of the sound channel quasinormal mode we have to analyze fluctuation in the background geometry \( \{g_{\mu\nu}, A_\mu, \phi, \chi\} \) (2.5):

\[
\begin{align*}
    g_{\mu\nu} &\to g_{\mu\nu} + h_{\mu\nu}, \\
    A_\mu &\to A_\mu + \delta A_\mu, \\
    \phi &\to \phi + \delta \phi, \\
    \chi &\to \chi + \delta \chi.
\end{align*} \tag{3.33}
\]

We choose the gauge

\[
h_{tr} = h_{x_3 r} = h_{rr} = 0, \quad \delta A_r = 0. \tag{3.34}
\]

Additionally, we assume that all the fluctuations depend only on \( (t, x_3, r) \), i.e., we have an \( O(2) \) rotational symmetry in the \( x_1 x_2 \) plane. At a linearized level, the fluctuations \( \{h_{\mu\nu}, \delta A_\mu, \delta \phi, \delta \chi\} \) of different helicities with respect to this symmetry will decouple from each other. The sound channel quasinormal mode corresponds to helicity-zero fluctuations [22]:

\[
\{h_{tt}, h_{aa} = h_{xx} + h_{yy}, h_{x_3 x_3}, h_{x_3 x_3}, \delta A_t, \delta A_{x_3}, \delta \phi, \delta \chi\}. \tag{3.35}
\]
We introduce

\[ h_{tt} = e^{-i\omega t + iq x_3} c_1^2 H_{tt}, \]
\[ h_{tx_3} = e^{-i\omega t + iq x_3} c_2^2 H_{tx_3}, \]
\[ h_{aa} = e^{-i\omega t + iq x_3} c_2^2 H_{aa}, \]
\[ h_{x_3x_3} = e^{-i\omega t + iq x_3} c_2^2 H_{x_3x_3}, \]
\[ \delta A_t = e^{-i\omega t + iq x_3} A_t, \]
\[ \delta A_{x_3} = e^{-i\omega t + iq x_3} A_{x_3}, \]
\[ \delta \phi = e^{-i\omega t + iq x_3} p, \]
\[ \delta \chi = e^{-i\omega t + iq x_3} c, \]

where \( \{ H_{tt}, H_{tx_3}, H_{aa}, H_{x_3x_3}, A_t, A_{x_3}, p, c \} \) are functions of a radial coordinate only.

From the effective action (2.1) it is straightforward to derive 8 second order differential equations of motion for the fluctuations, and 4 first order differential constraints associated with fixing the gauge invariance as in (3.34). Altogether we expect \( 8 - 4 = 4 \) independent gauge-invariant combinations of fluctuations. Indeed, analyzing the transformations of (3.36) under the residual gauge and diffeomorphism transformations it is straightforward to construct these combinations:

\[ Z_H = 4q \omega H_{tz} + 2 H_{zz} - H_{aa} \left( 1 - \frac{q^2 c_1^2}{\omega^2 c_2^2} \right) + 2q^2 \omega^2 c_2^2 H_{tt}, \]
\[ Z_A = A_t + \frac{A'}{[\ln c_2]^2} H_{aa}, \]
\[ Z_p = p - \frac{\phi'}{[\ln c_2]^2} H_{aa}, \]
\[ Z_c = c - \frac{\chi'}{[\ln c_2]^2} H_{aa}. \]

(3.37)

With somewhat tedious analysis we can verify that equations of motion for the gauge-invariant fluctuations \( \{ Z_H, Z_A, Z_p, Z_c \} \) decouple.

The spectrum of quasinormal modes is determined [22] by imposing on \( \{ Z_H, Z_A, Z_p, Z_c \} \) an incoming wave boundary condition at the horizon, and requiring vanishing of the non-normalizable modes for \( \{ Z_H, Z_A, Z_p, Z_c \} \) near the boundary. In the hydrodynamic limit \( \omega \to 0, q \to 0 \) with \( \frac{\omega}{q} \) kept fixed, this leads to the following perturbative
expansions

\[ Z_H = (1 - x)^{-in} \left( z_{H,0} + i q \ z_{H,1} + \mathcal{O}(q^2) \right), \]
\[ Z_A = (1 - x)^{-in} \left( z_{A,0} + i q \ z_{A,1} + \mathcal{O}(q^2) \right), \]
\[ Z_c = (1 - x)^{-in} \left( z_{c,0} + i q \ z_{c,1} + \mathcal{O}(q^2) \right), \]

with the following boundary conditions on \( \{ z_{H,i}, z_{A,i}, z_{c,i} \} \):

\[ \lim_{x \to 1^-} z_{H,0} = 1, \quad \lim_{x \to 1^-} z_{H,1} = 0, \quad \lim_{x \to 1^-} z_{A,i} = \lim_{x \to 1^-} z_{p,i} = \lim_{x \to 1^-} z_{c,i} = \text{finite}, \]
\[ z_{H,i} = \mathcal{O}(x), \quad z_{A,i} = \mathcal{O}(x^{1/2}), \quad z_{p,i} = \mathcal{O}(x^{1/2}), \quad z_{c,i} = \mathcal{O}(x^{3/4}), \]

as \( x \to 0_+ \).

To leading order in the hydrodynamic approximation, wave functions of the gauge-invariant fluctuations \( \{ z_{H,0}, z_{A,0}, z_{p,0}, z_{c,0} \} \) satisfy the following equations

\[ 0 = z_{H,0}'' + C_{101} \ z_{H,0}' + C_{102} \ z_{A,0}' + C_{103} \ z_{p,0}' + C_{104} \ z_{c,0}' + C_{105} \ z_{H,0} + C_{106} \ z_{A,0} \]
\[ + C_{107} \ z_{p,0} + C_{108} \ z_{c,0}, \]
\[ 0 = z_{A,0}'' + C_{201} \ z_{H,0}' + C_{202} \ z_{A,0}' + C_{203} \ z_{p,0}' + C_{204} \ z_{c,0}' + C_{205} \ z_{H,0} + C_{206} \ z_{A,0} \]
\[ + C_{207} \ z_{p,0} + C_{208} \ z_{c,0}, \]
\[ 0 = z_{p,0}'' + C_{301} \ z_{H,0}' + C_{302} \ z_{A,0}' + C_{303} \ z_{p,0}' + C_{304} \ z_{c,0}' + C_{305} \ z_{H,0} + C_{306} \ z_{A,0} \]
\[ + C_{307} \ z_{p,0} + C_{308} \ z_{c,0}, \]
\[ 0 = z_{c,0}'' + C_{401} \ z_{H,0}' + C_{402} \ z_{A,0}' + C_{403} \ z_{p,0}' + C_{404} \ z_{c,0}' + C_{405} \ z_{H,0} + C_{406} \ z_{A,0} \]
\[ + C_{407} \ z_{p,0} + C_{408} \ z_{c,0}, \]

where connection coefficients \( C_{ioj} \) are complicated nonlinear functionals of the background fields \( \{ g, A, \phi, \chi \} \) with explicit dependence on \( x \) and \( \beta_{12} \equiv \beta_1^2 \), and implicit dependence on \( \kappa \) and \( \lambda \):

\[ C_{ioj} = C_{ioj} \left[ \{ g, A, \phi, \chi \}; \ x; \ \beta_{12} \right]. \]

To leading order in the hydrodynamic approximation, wave functions of the gauge-invariant fluctuations \( \{ z_{H,1}, z_{A,1}, z_{p,1}, z_{c,1} \} \) satisfy equations identical to (3.40), apart
from the source terms \( \{J_H, J_A, J_p, J_c\} \):

\[
0 = z_{H,1}'' + C_{101} z_{A,1}' + C_{102} z_{A,1}' + C_{103} z_{p,1}' + C_{104} z_{c,1}' + C_{105} z_{H,1} + C_{106} z_{A,1} + C_{107} z_{p,1} + C_{108} z_{c,1} + J_H,
\]

\[
0 = z_{A,1}'' + C_{201} z_{H,1}' + C_{202} z_{A,1}' + C_{203} z_{p,1}' + C_{204} z_{c,1}' + C_{205} z_{H,1} + C_{206} z_{A,1} + C_{207} z_{p,1} + C_{208} z_{c,1} + J_A,
\]

\[
0 = z_{p,1}'' + C_{301} z_{H,1}' + C_{302} z_{A,1}' + C_{303} z_{p,1}' + C_{304} z_{c,1}' + C_{305} z_{H,1} + C_{306} z_{A,1} + C_{307} z_{p,1} + C_{308} z_{c,1} + J_p,
\]

\[
0 = z_{c,1}'' + C_{401} z_{H,1}' + C_{402} z_{A,1}' + C_{403} z_{p,1}' + C_{404} z_{c,1}' + C_{405} z_{H,1} + C_{406} z_{A,1} + C_{407} z_{p,1} + C_{408} z_{c,1} + J_c,
\]

with

\[
J_H = C_{111} z_{H,0}' + C_{112} z_{A,0}' + C_{113} z_{p,0}' + C_{114} z_{c,0}' + C_{115} z_{H,0} + C_{116} z_{A,0} + C_{117} z_{p,0} + C_{118} z_{c,0},
\]

\[
J_A = C_{211} z_{H,0}' + C_{212} z_{A,0}' + C_{213} z_{p,0}' + C_{214} z_{c,0}' + C_{215} z_{H,0} + C_{216} z_{A,0} + C_{217} z_{p,0} + C_{218} z_{c,0},
\]

\[
J_p = C_{311} z_{H,0}' + C_{312} z_{A,0}' + C_{313} z_{p,0}' + C_{314} z_{c,0}' + C_{315} z_{H,0} + C_{316} z_{A,0} + C_{317} z_{p,0} + C_{318} z_{c,0},
\]

\[
J_c = C_{411} z_{H,0}' + C_{412} z_{A,0}' + C_{413} z_{p,0}' + C_{414} z_{c,0}' + C_{415} z_{H,0} + C_{416} z_{A,0} + C_{417} z_{p,0} + C_{418} z_{c,0}.
\]

The new connection coefficients \( C_{ij} \) are complicated nonlinear functionals of the background fields \( \{g, A, \phi, \chi\} \) with explicit dependence on \( x \) and \( \beta_{12}, \beta_2 \), and implicit dependence on \( \kappa \) and \( \lambda \):

\[
C_{ij} = C_{ij} \left[ \{g, A, \phi, \chi\}; \ x; \ \{\beta_{12}, \beta_2\} \right]. \tag{3.44}
\]

Explicit expressions for \( C_{ikj} \) are available from the author upon the request.

It is straightforward to determine the asymptotic expansions (satisfying (3.39)) and thus set up the boundary value problem what would determine \( \{\beta_{12}, \beta_2\} \), along with the coefficients of the normalizable modes for \( \{z_{H,i}; z_{A,i}; z_{p,i}; z_{c,i}\} \). We found that, unfortunately, directly solving the resulting boundary value problem (as it was done for example in [23]) is not possible with reasonable computational resources. In the
rest of this section we explain the origin of the problem and outline the solution\(^7\).

Recall that (3.40) and (3.42) were obtained in the hydrodynamic limit from the full quasinormal mode equations for \(\{Z_H, Z_A, Z_p, Z_c\}\). The latter equations have a structure identical to that of (3.40). For example, we have

\[
0 = Z''_H + C_{H,1} Z'_H + C_{H,2} Z'_A + C_{H,3} Z'_p + C_{H,4} Z'_c + C_{H,5} Z_H + C_{H,6} Z_A + C_{H,7} Z_p + C_{H,8} Z_c, \tag{3.45}
\]

with

\[
C_{H,i} = C_{H,i} \left[ \{g, A, \phi, \chi\}; \ x; \ \{w, q\} \right]. \tag{3.46}
\]

Explicit expressions for \(C_{H,i}\) show that (some of) these coefficients have a simple pole at

\[
0 = w^2 - (1 - x)^2 q^2. \tag{3.47}
\]

The residues of these poles are always proportional to the derivative of the background gauge potential\(^8\) \(A'\), or \((A')^2\), (see (2.5)), and does not vanish when either (or both) \(\phi = 1\) and \(\chi = 0\). In other words, these poles always occur in studies of the sound channel quasinormal modes of charged black holes in asymptotic \(AdS_5\) geometry. Since the speed of sound squared \(c_s^2\) in holographic plasma with a UV conformal fixed point is bounded by \([26, 27]\)\(^9\)

\[
c_s^2 \leq \frac{1}{3}, \tag{3.48}
\]

from (3.47) we always expect to encounter a singularity in the connection coefficients of the differential equations describing the propagation of the sound channel quasinormal modes of the charged black holes inside the range of the integration, \(i.e.,\) for \(x \in (0, 1)\). Of course, the singularity in the coefficients of the differential equation does not imply that the solutions are singular — in our case we find that they are not — however, the presence of such singularities poses technical difficulties for the boundary value problem one has to solve. Actually, the issue is even more complicated: the successive hydrodynamic approximations produce equations ((3.40) and (3.42) in our case) which connection coefficients involve successive derivatives with respect to \(c_s\). Thus, while some of the coefficients \(C_{i0j}\) in (3.40) have a simple pole inside the range

---

\(^7\)A simple explicit example explaining the use of a new computational technique is presented in Appendix B.

\(^8\)This explains why such poles are absent in the analysis of [24] or [25].

\(^9\)See [28] for an exception.
of integration, some of the coefficients $C_{i1j}$ in (3.43) have a double pole! Finally, a perturbative expansion in the deformation parameter $\lambda$, which is needed to extract $\beta_{2,1}$ and ultimately the bulk viscosity of the charged plasma, see (3.32), produces yet higher order poles — here we need to deal with the third order poles in the connection coefficients inside the range of the integration in the boundary value problem. We find that numerical techniques for solving the boundary value problem developed in [29] become unreliable once the connection coefficients of the differential equations have a second or higher order poles. A new approach is needed.

Basically, we need to reformulate the boundary value problem in such a way that connection coefficients of the corresponding differential equations have simple poles inside the range of integration, at worst. First, even though we are after the hydrodynamics to order $O(\lambda^2)$ only, we treat (3.40) and (3.42) exactly in $\lambda$. The boundary value problem would then determine

$$\beta_1 = \beta_1(\lambda), \quad \beta_2 = \beta_2(\lambda).$$

(3.49)

From the data sets for small $\lambda$ we can extract $\beta_{1,0}, \beta_{1,1}, \beta_{2,0}, \beta_{2,1}$ (see (3.25)). Such a step insures that $C_{i0j}$ have simple poles at most, while $C_{i1j}$ have double poles at most. In particular, the boundary value problem for (3.40) is amenable to the treatment of [29]. Second, the hydrodynamic origin of the second order poles in $C_{i1j}$ implies that the residues of the second order poles in $C_{i1j}$ and $\frac{\partial C_{i0j}}{\partial \beta_{12}}$ must be proportional to each other. Indeed, for each value of indexes $\{i, j\}$ we find

$$C_{i1j} = -2\beta_2 \frac{\sqrt{\beta_{12}}}{\sqrt{3}} \frac{\partial}{\partial \beta_{12}} C_{i0j} + \tilde{C}_{i1j}, \quad \tilde{C}_{i1j} = \tilde{C}_{i1j} \left[\{g, A, \phi, \chi\}; \ x; \ \beta_{12}\right],$$

(3.50)

with $\tilde{C}_{i1j}$ having at most simple poles for $x \in (0, 1)$. Given (3.50), the boundary value problem (3.42) can be reformulated as follows.

- We represent

$$z_{H,1} = -2\beta_2 \frac{\sqrt{\beta_{12}}}{\sqrt{3}} \frac{\partial}{\partial \beta_{12}} \tilde{z}_{H,0} + \tilde{z}_{H,1},$$
$$z_{A,1} = -2\beta_2 \frac{\sqrt{\beta_{12}}}{\sqrt{3}} \frac{\partial}{\partial \beta_{12}} \tilde{z}_{A,0} + \tilde{z}_{A,1},$$
$$z_{p,1} = -2\beta_2 \frac{\sqrt{\beta_{12}}}{\sqrt{3}} \frac{\partial}{\partial \beta_{12}} \tilde{z}_{p,0} + \tilde{z}_{p,1},$$
$$z_{c,1} = -2\beta_2 \frac{\sqrt{\beta_{12}}}{\sqrt{3}} \frac{\partial}{\partial \beta_{12}} \tilde{z}_{c,0} + \tilde{z}_{c,1}.$$
The wave functions \( \{ \hat{z}_{H,0}, \hat{z}_{A,0}, \hat{z}_{p,0}, \hat{z}_{c,0} \} \) satisfy exactly the same equations as \( \{ z_{H,0}, z_{A,0}, z_{p,0}, z_{c,0} \} \), i.e., (3.40), with \( \beta_{12} \) being treated as an extra free parameter and the only change in the boundary conditions being

\[
\lim_{x \to 0^+} \hat{z}_{H,0} = \text{finite} \equiv Z_0(\beta_{12}). \tag{3.52}
\]

Clearly, imposing the Dirichlet boundary condition \( Z_0(\beta_{12}) = 0 \) would determine the value of \( \beta_{12} \) which would identify all \( \hat{z}_{\cdots,0} \) with \( z_{\cdots,0} \):

\[
\left. \hat{z}_{\cdots,0} \right|_{Z_0(\beta_{12})=0} = z_{\cdots,0}, \tag{3.53}
\]

where \( \cdots \) stand for any of \( \{ H, A, p, c \} \). For each value of \( \lambda \), we can evaluate \( Z_0 \) for some set of \( \beta_{12} \) and compute

\[
Z'_0 \equiv \left. \frac{\partial}{\partial \beta_{12}} Z_0(\beta_{12}) \right|_{Z_0(\beta_{12})=0}. \tag{3.54}
\]

The boundary value problem(s) implementing this procedure involve solving ODE’s with simple poles at most in the connection coefficients inside the range of integration.

- Given (3.51) and the definition of \( \tilde{z}_{\cdots,1} \), it is straightforward to see that \( \tilde{z}_{\cdots,1} \) would satisfy equations identical to (3.42), except that now the source terms (3.43) would be constructed from the connection coefficients \( \tilde{C}_{i1j} \). For example,

\[
\mathcal{J}_H \to \tilde{\mathcal{J}}_H \equiv \tilde{C}_{111} z'_{H,0} + \tilde{C}_{112} z'_{A,0} + \tilde{C}_{113} z'_{p,0} + \tilde{C}_{114} z'_{c,0} + \tilde{C}_{115} z_{H,0} + \tilde{C}_{116} z_{A,0} + \tilde{C}_{117} z_{p,0} + \tilde{C}_{118} z_{c,0}. \tag{3.55}
\]

By construction, this final boundary value problem would involve ODE’s with simple poles at most in the connection coefficients inside the range of integration. However, since it does not depend on \( \beta_2 \) (see (3.50)), to find a solution we must modify\(^{10} \) the Dirichlet condition at the boundary for \( \tilde{z}_{H,1} \):

\[
\lim_{x \to 0^+} \tilde{z}_{H,1} = \text{finite} \equiv Z_1. \tag{3.56}
\]

- Ultimately, \( z_{H,1} \) must satisfy the Dirichlet condition at the boundary (3.39). Using (3.51), (3.52), (3.54) and (3.56) this leads to

\[
-2 \beta_2 \frac{\sqrt{\beta_{12}}}{\sqrt{3}} Z'_0 + Z_1 = 0, \tag{3.57}
\]

\(^{10}\text{This is the only modification in the boundary conditions.}\)
which determines $\beta_2$ as
\[
\beta_2 = \frac{\sqrt{3}}{2\sqrt{\beta_2}} \frac{Z_1}{Z_0}.
\] (3.58)

Thus, we succeeded in reformulating the boundary value problem for computing $\beta_2$ in such a way that it does not involve numerical integration of ODE’s with connection coefficients having singularities stronger than simple poles for $x \in (0, 1)$.

We emphasize once again that the procedure described above is necessary only in computing sound channel quasinormal modes for charged black holes in asymptotic $AdS_5$ geometry. On the other hand, it is generic, and can be applied to AdS-Schwarzschild black holes as well. In Appendix B we demonstrate the new method for the computation of the dispersion relation of the sound waves in strongly coupled $\mathcal{N} = 4$ SYM plasma.

### 3.4 Mass-deformed RN plasma transport at criticality

In this section we present results for the speed of sound and the bulk viscosity of the mass-deformed RN plasma in the vicinity of the second order phase transition. We perform numerical analysis for a fixed $\kappa = 2$ as a function of (a small) mass-deformation parameter $\lambda$, see (2.59). In general (see (2.69))

\[
\kappa_c = \kappa_c(\lambda) = 2 + O(\lambda^2).
\] (3.59)

However, setting $\kappa = 2$ is still sufficient to extract the leading correction to the speed of sound
\[
\lim_{T \to T_c} \left( c_s^2 \right) + O(\lambda^4) = \frac{1}{3} \left( \beta_1(\kappa = 2, \lambda) \right)^2,
\] (3.60)

and the leading contribution to the bulk viscosity near the vicinity of the phase transition, see (3.32):
\[
\lim_{T \to T_c} \left( \lambda^{-2} \times \frac{\zeta}{\eta} \right) + O(\lambda^2) = \lim_{\lambda \to 0} \Delta(\kappa = 2, \lambda).
\] (3.61)

#### 3.4.1 The speed of sound

First, we verify that in the conformal limit the speed of sound is independent of the chemical potential, see (3.21). We find
\[
\left| (\beta_1(\kappa, \lambda = 0))^2 - 1 \right| \sim 10^{-14} \cdots 10^{-11}, \quad \kappa \in [0.4, 2].
\] (3.62)
Figure 1: (Colour online) Deviation of the speed of sound \(1 - \beta_1^2\), see (3.25), in mass-deformed RN plasma from its conformal value as a function of the mass-deformation parameter \(\lambda\) at \(\kappa = 2\), see (2.59). The blue dots represent results obtained from the holographic hydrodynamic equations (3.40), and the solid red line represents thermodynamic prediction, see (3.28).

Figure 1 presents the results for the deviation of the speed of sound in RN plasma from the conformal value as a function of the mass-deformation parameter \(\lambda\): the blue dots are obtained from directly solving the holographic hydrodynamic equations to leading order (3.40), while the solid red line\(^\text{11}\)

\[
(1 - \beta_1^2) \bigg|_{\text{red}} \equiv -2\beta_{1,1} \lambda^2 = 0.061132(8),
\]

is the thermodynamic prediction for this deviation, valid in the limit \(\lambda \to 0\). The results are in excellent agreement: for instance, for \(\lambda = 0.01\) we find

\[
\left| \frac{(1 - \beta_1^2)_{\text{red}}}{(1 - \beta_1^2)_{\text{blue}}} - 1 \right| \approx 5 \times 10^{-6}.
\]

3.4.2 The bulk viscosity

Scale invariance of the model at \(\lambda = 0\) predicts (3.28)

\[
\left( \beta_2(\lambda = 0) \right)_{\text{prediction}} = \beta_{2,0}(\kappa = 2) = \frac{2}{3}.
\]

\(^{11}\)We used (3.28) and the result of Table 1: \(c_{10}(\kappa = 2) = c_{10}^{(0)}\).
Figure 2: (Colour online) Blue dots represent $Z_0(\beta_{12})$ (see (3.52)) for the mass-deformation parameter $\lambda = 0.01$. $\beta_{12}^* \equiv (\beta_1(\lambda = 0.01))^2$. The solid red line represents the best linear fit to $Z_0$.

Explicit computation of the attenuation coefficient $\beta_2$ following the method explained in section 3.3 yields

$$\left| \frac{\beta_2(\lambda = 0)}{\beta_2(\lambda = 0)_{\text{prediction}}} - 1 \right| \approx 2 \times 10^{-6}. \quad (3.66)$$

Before we present a general plot for the bulk viscosity in RN plasma, we discuss in some details results for $\lambda = 0.01$. Blue dots in Figure 2 represent the results for $Z_0(\beta_{12})$ (see (3.52)) for values of $\beta_{12}$ in the vicinity of $\beta_{12}^* \equiv (\beta_1(\lambda = 0.01))^2$ as evaluated from solving the hydrodynamic equations at leading order (3.40). We present the data as a function of $(\beta_{12} - \beta_{12}^*)$ since, following the discussion around (3.52), we expect

$$Z_0(\beta_{12} = \beta_{12}^*) = 0. \quad (3.67)$$

The solid red line represents the best linear fit to the hydrodynamic $Z_0$ data. Explicitly, we find

$$Z_0 \bigg|_{\text{red}} = -8. \times 10^{-14} - 1.400011(9) \ (\beta_{12} - \beta_{12}^*). \quad (3.68)$$

From (3.68) we determine

$$Z_0' \bigg|_{\lambda=0.01} = -1.400011903986774. \quad (3.69)$$

Next, we evaluate $Z_1$ as explained in section 3.3:

$$Z_1 \bigg|_{\lambda=0.01} = -1.0777132662397266, \quad (3.70)$$
Figure 3: (Colour online) Blue dots represent $\beta_2(\lambda)$ (see (3.58)) at $\kappa = 2$. The solid red line represents the best quadratic fit to the first 10 blue dots.

which following (3.58) computes $\beta_2$:

$$\beta_2 \bigg|_{\lambda=0.01} = 0.6666585596689777. \tag{3.71}$$

Figure 3 presents results (blue points) for $\beta_2(\lambda)$, were we simply iterated the procedure described above for $\beta_2(\lambda = 0.01)$. The solid red line represents the best quadratic fit (with the linear term absent) to the first 10 blue points:

$$\beta_2 \bigg|_{red} = 0.666666(3) - 0.081002(5) \lambda^2. \tag{3.72}$$

Note that $\beta_2(\lambda = 0)$ in this fit is remarkably close to exact CFT value (3.65). We can now compute $\beta_{2,1}$ in (3.25) as

$$\beta_{2,1} = \frac{1}{2} \frac{d^2}{d\lambda^2} \beta_2 \bigg|_{red} \quad \implies \quad \beta_{2,1} = -0.081002(5). \tag{3.73}$$

Finally, using results in Table 1, we find from (3.30)

$$\Delta = 0.224133(3) + 2\beta_{2,1}, \tag{3.74}$$

thus, from (3.29)

$$\frac{\zeta}{\eta} \approx 0.062128(2) \lambda^2 + O(\lambda^4). \tag{3.75}$$
Curiously, since
\[
\frac{1}{3} - c_s^2 \equiv \frac{1}{3}(-2\beta_{1,1}) = \frac{1}{3}(1 - \beta_{12}^* ) = 0.020377(6) \lambda^2 + \mathcal{O}(\lambda^4) ,
\] (3.76)
the bulk viscosity bound for holographic gauge theory plasma proposed in [13]
\[
\frac{\zeta}{\eta} \geq 2 \left( \frac{1}{3} - c_s^2 \right) ,
\] (3.77)
is satisfied in the vicinity of the phase transition of mass-deformed RN plasma.

To summarize, we find:
\[
\frac{\zeta}{\eta} = 3.0488(5) \left( \frac{1}{3} - c_s^2 \right) + \mathcal{O} \left( \left( \frac{1}{3} - c_s^2 \right)^2 \right) ,
\] (3.78)
for the bulk viscosity of the mass-deformed RN plasma in the vicinity of the second order phase transition.

### 4 Dynamical critical phenomena in RN plasma

In this section we study the response of conformal and mass-deformed RN plasma to inhomogeneous and time-dependent variation of the chemical potential
\[
\mu \to \mu + \delta \mu(t, \vec{x}) , \quad \delta \mu(t, \vec{x}) = \int \frac{d^3 k}{(2\pi)^3} \int \frac{d\omega}{2\pi} e^{i\vec{k} \cdot \vec{x} - i\omega t} \mu_{\omega, \vec{k}} .
\] (4.1)

At a linearized level the variation of the chemical potential would produce a corresponding variation in the charge density, \( \delta \rho(t, \vec{x}) \) (\( \rho_{\omega, \vec{k}} \) for the Fourier components). As in conventional theory of dynamical critical phenomena [8], we introduce the dynamical susceptibility \( \chi_{\omega, \vec{k}} \),
\[
\chi_{\omega, \vec{k}} = \left. \frac{\rho_{\omega, \vec{k}}}{\mu_{\omega, \vec{k}}} \right|_T , \quad \lim_{(\omega, \vec{k}) \to 0} \chi_{\omega, \vec{k}} = \chi_T = \left. \left( \frac{\partial \rho}{\partial \mu} \right) \right|_T .
\] (4.2)

The fluctuation-dissipation theorem states that
\[
G(\omega, \vec{k}) = \frac{2T}{\omega} \text{Im} \chi_{\omega, \vec{k}} ,
\] (4.3)
where \( G(\omega, \vec{k}) \) is a Fourier transform of the charge density variation two-point correlation function
\[
G(t, \vec{x}) = \langle \delta \rho(t, \vec{x}) \delta \rho(0, \vec{0}) \rangle_{\mu=0} .
\] (4.4)
Furthermore, the equal-time correlation function
\[ G(\vec{k}) \equiv G(\omega = 0, \vec{k}), \] (4.5)
is related to the static susceptibility
\[ \chi_{\vec{k}} \equiv \chi_{\omega = 0, \vec{k}}, \] (4.6)
by the equipartition theorem
\[ G(\vec{k}) = T \chi_{\vec{k}}. \] (4.7)

In the vicinity of (but not at) the critical point \( t = \frac{T}{T_c} - 1 \to 0 \) \((t \neq 0)\) the equal-time correlation function \( G(\vec{x}) \) decays exponentially
\[ G(\vec{x}) \propto e^{-|\vec{x}|/\xi}, \] (4.8)
where \( \xi \) is the correlation length, implying that \( G(\vec{k}) \), and through the equipartition relation (4.7) the static susceptibility \( \chi_{\vec{k}} \), have a pole at
\[ k^2 \propto -\xi^{-2}. \] (4.9)

Right at the critical point, \( t = 0 \), the equal-time correlation function has a power-law decay
\[ G(\vec{x}) \propto |\vec{x}|^{-p+2-\eta} \Rightarrow G(\vec{k}) \propto |\vec{k}|^{-2+\eta}, \] (4.10)
where \( p = 3 \) is the number of spatial dimensions and \( \eta \) is the anomalous scaling exponent.

The theory of dynamical critical phenomena [8] predicts that in the vicinity of the continuous phase transition and for \( |\vec{k}| \sim \xi^{-1} \) the full dynamical susceptibility \( \chi_{\omega, \vec{k}} \) will develop a pole at
\[ \omega \propto -i\xi^{-z}, \] (4.11)
with \( z \) being the dynamical critical exponent of the system. The frequency in (4.11) defines a relaxation time \( \tau^{-1} \)
\[ \tau^{-1} \equiv i\omega \propto \xi^{-z}, \] (4.12)
characterizing the equilibration time scale of the dynamical system.

In the rest of this section we analyze dynamical susceptibility of the strongly coupled conformal RN plasma. Following up the poles in static susceptibility (see (4.9)) in the vicinity of the phase transition we determine the (static) scaling exponent \( \nu \) of the correlation length: \( \xi \propto t^{-\nu} \). The scaling of the pole in the dynamical susceptibility (see (4.11)) determines the dynamical critical exponent of the RN plasma. Finally, we comment on dynamical critical phenomena in mass-deformed RN plasma.
4.1 Computation of $\chi_{\omega, \vec{k}}$ in holographic dual

Without the loss of generality we can assume that

$$k^i = q \delta^i_3. \quad (4.13)$$

In dual gravitational description the variation of the chemical potential $\mu_{\omega, k}$ translates into the variation of the non-normalizable mode of the fluctuation of the bulk vector field $A_t$ (3.36), correspondingly the variation of the non-normalizable mode in the gauge-invariant fluctuation $Z_A$ (3.37). Up to an overall factor$^{12}$ (see (2.32)), the variation in the charge density $\rho_{\omega, k}$ is the normalizable component of $A_t$ (or $Z_A$ for the gauge-invariant fluctuation). The fluctuations of the dual gravitational background were analyzed extensively in section 3. As we study here the dynamical critical phenomena of the conformal RN plasma, we can consistently set $Z_c = 0$. We are left with the linear coupled system of fluctuations $\{Z_H, Z_A, Z_p\}$ (3.37). There are two important differences in analysis of this system of fluctuations compare to the one in section 3, relevant for the computation of the sound wave dispersion relation:

- In both cases $\{Z_H, Z_A, Z_p\}$ must satisfy an incoming boundary conditions at the horizon. In case of the sound waves, we had to impose the vanishing of all the non-normalizable modes for $\{Z_H, Z_A, Z_p\}$ at the boundary (see (3.39)). As a result, solving the boundary value problem determined the dispersion relation for the sound waves (3.25)

$$w = w(q). \quad (4.14)$$

In computing the dynamical susceptibility, both $w$ and $q$ are independent; nonetheless, the boundary value problem on the gauge-invariant fluctuations has a solution because the non-normalizable component of $Z_A$ at the boundary is nonzero now.

- Transport coefficients are encoded in the perturbative in $q$ expansion of the sound wave dispersion relation (3.15). Thus, it was sufficient to implement the perturbative (hydrodynamic) expansion for the fluctuations (3.38) and the dispersion relation (3.25). In study of the dynamical critical phenomena of strongly coupled RN plasma we are interested in the poles (for complex $w, q$) of the dynamical susceptibility $\chi_{w,q}$. Thus, we can not do the computations perturbatively in $w, q$. Indeed, since the system of equations of motion for the fluctuations $\{Z_H, Z_A, Z_p\}$ is linear, without the loss of generality we can set the non-normalizable component of $Z_A$ near the boundary,

$^{12}$This factor is finite in the vicinity of the transition.
namely $\mu_{w,q}$ to one:

$$\mu_{w,q} = 1.$$  \hfill (4.15)

From (4.2) we have

$$\chi_{w,q} = \rho_{w,q},$$  \hfill (4.16)

thus any poles in the dynamical susceptibility must come from the poles in the normalizable component of $Z_A$ near the boundary, and as such they must be non-perturbative in $w, q$.

We can summarize now the boundary value problem whose solution would determine the dynamical susceptibility. Introducing\footnote{The $w$– and $q$–dependent rescaling are for convenience in further analysis.}

$$Z_H = (1 - x)^{-i w} w^{-2} z_h(x, w, q),$$

$$Z_A = (1 - x)^{-i q} q^{-2} z_A(x, w, q),$$

$$Z_p = (1 - x)^{-i q} q^{-2} z_p(x, w, q),$$  \hfill (4.17)

the equations of motion for $\{z_h, z_A, z_p\}$ are solved with the following boundary conditions:

$$\lim_{x \to 1^-} z_H = \lim_{x \to 1^-} z_A = \lim_{x \to 1^-} z_p = \text{finite},$$

$$z_H = O(x), \quad z_A = 1 + Z(w, q) x^{1/2} + O(x), \quad z_p = O(x^{1/2}), \quad \text{as } x \to 0^+. $$  \hfill (4.18)

The normalizable component of $z_A$ near the boundary $Z$ is proportional to the dynamical susceptibility:

$$\chi_{w,q} \propto Z(w, q).$$  \hfill (4.19)

### 4.2 Static and dynamical susceptibilities of RN plasma

As discussed in section 2.5, a second order phase transition in strongly coupled RN plasma happens for

$$\kappa = 2 \iff \frac{T}{\mu} = \frac{\sqrt{2}}{\pi}. $$  \hfill (4.20)

We do not provide the technical details of the analysis of the boundary value problem (4.17)-(4.19) and present only the results. We point out that the coefficients of the differential equations for $\{z_h, z_A, z_p\}$ are non-singular as $\kappa \to 2$; thus the appearance of poles in $Z$ in this limit is not obvious.
Figure 4: (Colour online) The scaling (blue dots) of the inverse of the static susceptibility $\chi_{w=0,q=0}$ in the vicinity of the critical point. The solid red line is a quadratic fit to the data.

Figure 4 shows the inverse of the static susceptibility at $q = 0$ (blue dots) in the vicinity of the critical point. The solid red line represents the best quadratic fit to the data:

$$Z^{-1}_{\text{fit}} = 4.74744 \cdot 10^{-8} + 0.125116 (\kappa - 2) - 0.0156467 (\kappa - 2)^2 + \mathcal{O}((\kappa - 2)^3).$$  (4.21)

The red line (4.21) intersects the $\kappa$ axis at

$$\kappa_c = 1.999999(6),$$  (4.22)

in excellent agreement with the expected value $\kappa_c = 2$. Thus we reproduce the thermodynamic result for the static susceptibility

$$\chi_{w=0,q=0} = \chi_T \propto Z \propto \frac{1}{\kappa - \kappa_c} \propto \kappa - \kappa_c \ll \kappa_c,$$  (4.23)

where we used relation (2.50) between $\kappa$ and the reduced temperature $t$.

Figure 5 presents the poles (blue dots) of the static susceptibility at $q = q_*$ in the vicinity of the critical point:

$$\chi_{w=0,q_*}^{-1} = 0.$$  (4.24)

The solid red line represents the best quadratic fit to the data:

$$q_{*,\text{fit}}^2 = -2.32509 \cdot 10^{-6} + 0.11873 (\kappa - 2) - 0.0347648 (\kappa - 2)^2 + \mathcal{O}((\kappa - 2)^3).$$  (4.25)
Figure 5: (Colour online) Poles of the static susceptibility in the vicinity of the critical point: $\chi_{m=0,a=q_*}^{-1} = 0$. The solid red line is a quadratic fit to the data.

Notice that in the stable phase, i.e., for $\kappa \leq 2$, in the vicinity of the phase transition the poles in the static susceptibility are for purely imaginary momenta, which implies the exponential decay of the charge density two-point correlation function (4.8). Furthermore, from (4.9) we identify the correlation length as

$$\left(2\pi T_c \xi \right)^2 \propto q_*^{-2} \propto \frac{1}{\kappa - \kappa_c} \propto +t^{-1/2}, \quad 0 < \kappa_c - \kappa \ll \kappa_c, \quad (4.26)$$

where we used the results of the fit (4.25) and the relation between $\kappa$ and the reduced temperature $t$ (2.50). From (4.26) we extract the (static) critical exponent $\nu$:

$$\xi \propto t^{-\nu} \propto t^{-1/4} \quad \Rightarrow \quad \nu = \frac{1}{4}. \quad (4.27)$$

Given that the static critical exponent $\alpha = \frac{1}{2}$, (4.27) implies that the hyperscaling relation is violated

$$2 - \alpha \neq p \nu, \quad (4.28)$$

where $p = 3$ stands for the number of spatial dimensions of the system.

Figure 6 shows the inverse of the static susceptibility as a function of $q$ (blue dots) right at the critical point $\kappa = 2$. The solid red line represents the best quadratic (in $q^2$) fit to the data

$$(Z_{fit}^{crit})^{-1} = -1.57468 \cdot 10^{-8} - 1.06109 \cdot q^2 + 3.84182 \cdot q^4 + O(q^6). \quad (4.29)$$
\[
(Z^{crit})^{-1} \propto (\chi^{crit}_{\omega=0,q})^{-1}
\]

Figure 6: (Colour online) The scaling (blue dots) of the inverse of the static susceptibility \(\chi^{crit}_{\omega=0,q}\) at the critical point, \(\kappa = 2\). The solid red line is a quadratic (in \(q^2\)) fit to the data.

The red line (4.29) intersects the \(q^2\) axis at
\[
q_c^2 = -1.57468 \cdot 10^{-8},
\]
in excellent agreement with the expected value \(q_c^2 = 0\) (4.10). The data implies
\[
\chi^{crit}_{\omega=0,q} \propto Z^{crit} \propto q^{-2} \iff \chi^{crit}_{\omega=0,q} \propto q^{-2+\eta},
\]
which determines the anomalous critical exponent \(\eta\) as
\[
\eta = 0.
\]

Figure 7 presents the poles in the dynamical susceptibility at \(\omega = \omega_*\) in the vicinity of the critical point
\[
\chi^{-1}_{\omega=\omega_*,q} = 0,
\]
for select values of the momenta \(q\):
\[
q^2 = \{10^{-6}, 10^{-5}, 10^{-4}, 10^{-3}\} \sim \{\text{blue, green, orange, black}\}.
\]
The results of the analysis clearly show that as \(q \to 0\) the values of \(i\frac{\omega_*}{q^2}\) tend to a universal profile
\[
\lim_{q \to 0} i \frac{\omega_*}{q^2} = 2.79163 \cdot 10^{-6} - 0.333392(\kappa - 2) + 0.0278087(\kappa - 2)^2 + O((\kappa - 2)^3),
\]
Figure 7: (Colour online) Poles of the dynamical susceptibility in the vicinity of the critical point, $\chi^{-1}_{m=w,q} = 0$ for a set of momenta values $q^2 = 10^{-6}$ (blue dots), $10^{-5}$ (green dots), $10^{-4}$ (orange dots) and $10^{-3}$ (black dots). The solid red line is a quadratic fit to $\frac{i w^*}{q^2}$ at $q^2 = 10^{-6}$.

which is presented by the solid red line on Figure 7\textsuperscript{14}. Given (4.35) we can determine the critical scaling of the relaxation time (see (4.12)) of strongly coupled RN plasma

\[(2\pi T_c \tau)^{-1} \equiv i w_* \propto q^2 \cdot (\kappa - \kappa_c) \propto (2\pi T_c q^2)^2 \cdot (2\pi T_c \xi)^{-4} \propto (2\pi T_c \xi)^{-4}, \quad (4.36)\]

where we wrote the $q$ dependence as $\propto q^2 \xi$ and used (4.26). Thus,

\[\tau \propto \xi^2 \propto \xi^4 \quad \Rightarrow \quad z = 4. \quad (4.37)\]

In [7] Maeda, Natsuume and Okamura argued that the strongly coupled RN plasma at criticality should be identified with the 'model B' according to classification of [8]. As such, the dynamical critical exponent $z$ in this model is predicted to be [8]

\[z = 4 - \eta. \quad (4.38)\]

Since $\eta = 0$ (4.32) for the strongly coupled RN plasma, we explicitly confirm the conclusion of [7].

4.3 Universality class of the mass-deformed RN plasma

In section 2.5 we computed the static critical exponents ($\alpha, \beta, \gamma, \delta$) of the conformal RN plasma, see (2.53). Further, in previous section we determined the remaining static

\textsuperscript{14}In practice we used the best quadratic fit to $i \frac{w^*}{q^2}$ at $q^2 = 10^{-6}$.\footnote{In practice we used the best quadratic fit to $i \frac{w^*}{q^2}$ at $q^2 = 10^{-6}$.}
Figure 8: (Colour online) Poles of the static susceptibility of mass-deformed RN plasma in the vicinity of the critical point: \( \chi_{w=0,q=q_c}^{-1} = 0 \). The solid red line is a quadratic fit to the data.

critical exponents \((\nu, \eta)\) of this theory, and determined its dynamical critical exponent \(z\).

Notice that the mass deformation of the \( \mathcal{N} = 4 \) SYM plasma modifies its infrared (hydrodynamic) properties — it generates a non-vanishing bulk viscosity. Thus, one might worry that it is a relevant deformation at criticality and potentially might change the universality class of the theory. We show here that this is not the case: the universality class of the mass deformed \( \mathcal{N} = 4 \) SYM is the same as that of the conformal theory.

We already argued that some of the static critical exponents of the mass deformed theory are unchanged (see section 2.6):

\[
(\alpha, \beta, \gamma, \delta) = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)
\]  

(4.39)

To determine the remaining critical exponents \((\nu, \eta, z)\) we have to generalize the computation of the dynamical susceptibility as explained in section 4.1 to include the fluctuation \(Z_c\) in addition to \(\{Z_H, Z_A, Z_p\}\). In analogy to (4.17) we introduce

\[
Z_c = (1 - x)^{-iw} q^{-2} z_c(x, w, q),
\]

(4.40)

and solve equations of motion for \(\{z_h, z_A, z_p, z_c\}\) with boundary conditions (4.18) sup-
(\mathcal{Z}^{crit})^{-1} \propto (\chi^{crit}_{w=0,q})^{-1}

Figure 9: (Colour online) The scaling (blue dots) of the inverse of the static susceptibility of mass deformed RN plasma $\chi^{crit}_{w=0,q}$ at the critical point, $\tilde{\kappa} = \tilde{\kappa}_c$, (4.43). The solid red line is a quadratic (in $q^2$) fit to the data.

\[ \lim_{x \to 1^-} z_c = \text{finite}, \quad \text{and} \quad z_c = \mathcal{O}(x^{3/4}), \quad \text{as} \quad x \to 0^+. \quad (4.41) \]

As before, up to an overall constant, the dynamical susceptibility is identified with the normalizable mode of $z_A$ near the asymptotic $AdS_5$ boundary, see (4.19). We present the results of such analysis for one specific choice of $M/T_c$, namely,

\[ \frac{M}{T_c} \approx 10^{-1} \times \frac{\pi \sqrt{3}}{2} = 0.272(1). \quad (4.42) \]

We use approximate sign since relation (2.59), which was used to obtain (4.42), receives corrections of order $M^2/T_c^2$.

Figure 8 is equivalent to Figure 5 for the conformal RN plasma. Clearly, the slope of the solid red line (representing the quadratic fit to the data) is finite at $q^2_\star = 0$, which, much like in (4.25)-(4.27), implies that the critical exponent $\nu = \frac{1}{4}$. Note a technical detail: $\tilde{\kappa} = \kappa + \mathcal{O}(M^2/T_c^2)$; the precise relation is not important for the purpose of extracting the critical exponent. From the intersection of the solid red line the $\tilde{\kappa}$ axis we find

\[ \tilde{\kappa}_c = 2.0099887(0). \quad (4.43) \]
Figure 10: (Colour online) Poles of the dynamical susceptibility of mass deformed RN plasma in the vicinity of the critical point, $\chi_{\text{mass, } q}^{-1} = 0$ for a set of momenta values $q^2 = 10^{-6}$ (the solid red line quadratic fit), $10^{-5}$ (blue dots), and $10^{-3}$ (green dots). The vertical dashed blue line denotes $\tilde{\kappa} = \tilde{\kappa}_c$, see (4.43).

Figure 9 is equivalent to Figure 6 for the conformal RN plasma. It represents the scaling of the inverse static susceptibility of the mass-deformed RN plasma at criticality. As in (4.31), we conclude that $\eta = 0$ in this case.

Finally, Figure 10 is equivalent to Figure 7 for the conformal RN plasma. As in (4.35)-(4.37) we conclude that $z = 4$ for the dynamical critical exponent of mass-deformed RN plasma.

5 Conclusions

In this paper we presented extensive analysis of the critical phenomena in superconformal $\mathcal{N} = 4$ SYM gauge theory plasma at finite temperature and a single $U(1) \subset SU(4)$ R-symmetry chemical potential. From the explicit analysis of the dynamical susceptibility near criticality we extracted the static critical exponents $(\nu, \eta)$ and identified the static universality class of the model\(^{15}\):

$$
(\alpha, \beta, \gamma, \delta, \nu, \eta) = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 2, \frac{1}{4}, 0 \right). \quad (5.1)
$$

\(^{15}\)This corrects conclusions of [5].
We explicitly computed the dynamical critical exponent of the theory
\[ z = 4, \] (5.2)
and confirmed the identification of the \( \mathcal{N} = 4 \) SYM plasma dynamical universality class with that of 'model B' according to classification of [8], originally made in [7].

We demonstrated that although a deformation of the \( \mathcal{N} = 4 \) SYM theory by a dimension-3 operator is relevant in the infrared — in particular, it generates the non-zero bulk viscosity in the effective hydrodynamic description of the mass-deformed plasma — the static and the dynamical universality classes of the theory remains unchanged: (5.1) and (5.2).

We carefully studied the propagation of the sound waves in non-conformal charged plasma. We confirmed the computation of the speed of sound from the thermodynamic analysis with the direct result extracted from the quasinormal mode analysis of the holographic dual. We showed that the bulk viscosity of the mass-deformed \( \mathcal{N} = 4 \) SYM plasma remains finite at criticality, and satisfies the bulk viscosity bound [13] (at least to order \( \mathcal{O}(M^2/T^2) \)). Our computations challenge the Onuki’s model [3] for the behavior of bulk viscosity near criticality. Indeed, the latter model predicts
\[ \zeta \bigg|_{Onuki} \propto |t|^{-2z+\alpha} \propto |t|^{-1/2}, \] (5.3)
for the mass-deformed \( \mathcal{N} = 4 \) SYM plasma, in contradiction with the finite result we obtained. Thus, when combined with analysis in [2], it appears that as of now, there is no model of transport at continuous phase transitions that is not in conflict with direct (first principle) holographic computations. It is important to use gauge theory/string theory correspondence to develop a consistent model.

In order to explain holographic computations of transport at criticality it might be necessary to generalize the framework of near-equilibrium relaxation. Specifically, it appears necessary to formulate the theory of dynamical critical phenomena in which different non-equilibrium correlators relax to equilibrium with different dynamical critical exponents.

\section*{Acknowledgments}

I would like to thank Ofer Aharony, Micha Berkooz and Rob Myers for valuable discussions. I further thank Rob Myers for comments on the manuscript. I would like
to thank Mitchel Institute for Fundamental Physics and Astronomy and Weizmann Institute for hospitality during the various stage of this project. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research & Innovation. I gratefully acknowledge further support by an NSERC Discovery grant and support through the Early Researcher Award program by the Province of Ontario.

A Coefficients $C_{ij}$

\[
C_{11} = \frac{1}{(x-1)(H_0^6 - 1)^2(2 + H_0^3)(1 + \kappa)} (H_0^{15}(1 + \kappa) - H_0^{12}(1 + \kappa) - 2H_0^9
\times (3\kappa^2(x - 1)^2 + \kappa + 1) - 2H_0^6(5\kappa^2(x - 1)^2 - \kappa - 1) - H_0^3(8\kappa^2(x - 1)^2 - \kappa - 1)
- 1 - \kappa).
\]

\[
C_{12} = \frac{3(2H_0^3 + 1)\beta}{2(x-1)(H_0^3 - 1)(2 + H_0^3)H_0}.
\]

\[
C_{13} = \frac{(2H_0^3 + 1)H_0^3\kappa}{\sqrt{1 + \kappa(x-1)(H_0^3 - 1)(2 + H_0^3)}}.
\]

\[
C_{14} = \frac{\beta}{4(2 + H_0^3)}.
\]

\[
C_{15} = -\frac{2(2H_0^3 + 1)\kappa^2H_0^3}{(2 + H_0^3)(H_0^3 - 1)(2 + H_0^3)(1 + \kappa)}.
\]

\[
C_{16} = \frac{2H_0^3(5H_0^6 - 2)\beta(2H_0^3 + 1)\kappa^2}{(2 + H_0^3)(H_0^3 - 1)^2(2 + H_0^3)(1 + \kappa)}.
\]

\[
C_{17} = \frac{(2H_0^3 + 1)(1 + \kappa + H_0^3((x-1)^2\kappa^2 - 2 - 2\kappa) + H_0^6(1 + \kappa)\kappa^2H_0^3\beta m^2)}{12(1 + \kappa)^2(H_0^3 - 1)^2(2 + H_0^3)}.
\]

\[
C_{21} = \frac{2H_0^9}{3\beta(H_0^3 - 1)(2 + H_0^3)(x-1)(1 + \kappa)}(3H_0^9(1 + \kappa) + H_0^6(8\kappa^2(x - 1)^2 - 3 - 3\kappa)
+ H_0^3(4\kappa^2(x - 1)^2 - 3 - 3\kappa) + 3 + 3\kappa).
\]

\[
C_{22} = \frac{1}{3(2 + H_0^3)(H_0^6 - 1)^2(x-1)(1 + \kappa)}(3H_0^{15}(1 + \kappa) + H_0^{12}(8\kappa^2(x - 1)^2 - 3\kappa - 3)
+ 2H_0^9(11\kappa^2(x - 1)^2 - 3 - 3\kappa) + 2H_0^9(11\kappa^2(x - 1)^2 + 3\kappa + 3) + H_0^3(3\kappa + 3)
+ 20\kappa^2(x - 1)^2 - 3 - 3\kappa).
\]
\[ C_{23} = \frac{2H_0^4 \kappa}{3\beta(1 + \kappa)^{3/2}(2 + H_0^6)(H_0^6 - 1)(H_0^6 + 1)(x - 1)}(3H_0^9(1 + \kappa) - H_0^6(2\kappa^2(x - 1)^2 - 3 - 3\kappa) - H_0^3(3 + 3\kappa + 4\kappa^2(x - 1)^2) - 3 - 3\kappa). \] (A.10)

\[ C_{24} = -\frac{H_0(H_0^3 - 1)}{6(2 + H_0^3)}. \] (A.11)

\[ C_{25} = -\frac{4H_0^4 \kappa^2}{3(2 + H_0^6)(H_0^6 - 1)^2(H_0^6 + 1)\beta(1 + \kappa)^2}(3H_0^9(1 + \kappa) - H_0^6(2\kappa^2(x - 1)^2 - 3 - 3\kappa) - H_0^3(3 + 3\kappa + 4\kappa^2(x - 1)^2) - 3 - 3\kappa). \] (A.12)

\[ C_{26} = -\frac{4H_0^4 \kappa^2}{9(1 + \kappa)^2(H_0^6 - 1)^3(H_0^6 - 1)(2 + H_0^3)(A.11)}(18 + 18\kappa + 2H_0^3(8\kappa^2(x - 1)^2 - 9\kappa - 9)
- H_0^3(32\kappa^2(x - 1)^2 + 63\kappa + 63) + 9H_0^9(7 + 7\kappa + 4\kappa^2(x - 1)^2) + H_0^{12}(45 + 45\kappa
+ 32\kappa^2(x - 1)^2) - H_0^{15}(45\kappa + 45 - 2\kappa^2(x - 1)^2)). \] (A.13)

\[ C_{27} = \frac{\kappa^2 H_0^3 m^2(1 + \kappa + H_0^3(\kappa^2(x - 1)^2 - 2 - 2\kappa) + H_0^6(1 + \kappa))}{6(1 + \kappa)^2(H_0^6 - 1)^2(H_0^6 - 1)(2 + H_0^3)}. \] (A.14)

\[ C_{31} = \frac{4\kappa(1 - x)}{\sqrt{1 + \kappa(H_0^6 - 1)}}. \] (A.15)

\[ C_{32} = \frac{6\beta \kappa(1 - 2H_0^3)(1 - x)}{\sqrt{1 + \kappa(H_0^6 - 1)(H_0^6 - 1)H_0}}. \] (A.16)

\[ C_{33} = -\frac{1}{(x - 1)(H_0^6 - 1)^2(1 + \kappa)}(1 + \kappa + 4H_0^3\kappa^2(x - 1)^2 - 2H_0^6(\kappa + 1 + 2(x - 1)^2\kappa^2)
+ 4H_0^9\kappa^2(x - 1)^2 + H_0^{12}(1 + \kappa)). \] (A.17)

\[ C_{34} = -\frac{8\kappa^3 H_0^3(x - 1)^2}{(1 + \kappa)^{3/2}(H_0^6 - 1)^2(H_0^6 + 1)}. \] (A.18)

\[ C_{35} = \frac{8\kappa^3 H_0^3 \beta(x - 1)^2(-4H_0^3 + 4H_0^6 + 2 + H_0^6)}{(1 + \kappa)^{3/2}(H_0^6 - 1)(H_0^6 - 1)^3}. \] (A.19)

**B  Sound of \( \mathcal{N} = 4 \) plasma via the new technique**

The technique for computing the holographic sound wave dispersion relation developed in section 3.3 is rather complicated. Thus, we believe that it warrants a simple explicit example. This example is being provided by an \( AdS_5 \) Schwarzschild black hole,
holographically dual to strongly coupled $\mathcal{N} = 4$ plasma at finite temperature and zero chemical potentials. The latter is realized as a special case of the background (2.5) with

$$A \equiv 0, \quad \phi \equiv 1, \quad \chi \equiv 0. \quad \text{(B.1)}$$

In this case the only gauge invariant fluctuations are those of $z_{H,0}$ and $z_{H,1}$. The analog of (3.40) is:

$$0 = z''_{H,0} + \frac{\beta_{12} + 3x^2 - 6x + 1}{(1-x)(x^2 - 2x + 3 - \beta_{12})} z'_{H,0} + \frac{4}{x^2 - 2x + 3 - \beta_{12}} z_{H,0}, \quad \text{(B.2)}$$

and the analog of (3.42) is

$$0 = z''_{H,1} + \frac{\beta_{12} + 3x^2 - 6x + 1}{(1-x)(x^2 - 2x + 3 - \beta_{12})} z'_{H,1} + \frac{4}{x^2 - 2x + 3 - \beta_{12}} z_{H,1} + \mathcal{J}_H$$

$$\mathcal{J}_H = -\frac{2\beta_{12}^{1/2}}{\sqrt{3(x-1)(x^2 - 2x + 3 - \beta_{12})^2}} \left( \beta_{12}^2 + 4\beta_{12}x - 2\beta_{12}x^2 - 6\beta_{12} - 4x^3 + 8\beta_2 x 
+ x^4 + 9 + 10x^2 - 12x - 4\beta_2 - 4\beta_2 x^2 \right) z'_{H,0} + \frac{4\beta_{12}^{1/2}(x^2 - 2x - \beta_{12} - 2\beta_2 + 3)}{\sqrt{3(x^2 - 2x + 3 - \beta_{12})^2}} z_{H,0}. \quad \text{(B.3)}$$

Note that since we expect $\beta_{12} = 1$, there are no poles in the connection coefficients in (B.2) inside the range of integration $x \in (0, 1)$ — the would-be poles are at

$$0 = x^2 - 2x + 3 - \beta_{12}. \quad \text{(B.4)}$$

Nonetheless, $\mathcal{J}_H$ in (B.3) contains factors of $(x^2 - 2x + 3 - \beta_{12})^2$ in the denominators of the connection coefficients. As explained in section 3.3 their origin can be traced back to the hydrodynamic expansion (3.25).

According to discussion around (3.52), solving (B.3) while treating $\beta_{12}$ as a free parameter and imposing the horizon boundary condition as in (3.39) computes $\hat{z}_{H,0}$:

$$\hat{z}_{H,0} = \left. \frac{1 - \beta_{12} + 2x - x^2}{2 - \beta_{12}} \right|_{\mathcal{Z}_0(\beta_{12})=0} = \frac{1 - \beta_{12}}{2 - \beta_{12}}. \quad \text{(B.5)}$$

Solving

$$\mathcal{Z}_0(\beta_{12} = \beta^*_{12}) = 0 \quad \Rightarrow \quad \beta^*_{12} = 1, \quad \text{(B.6)}$$

determines the speed of sound $c_s^2 = \frac{\beta^*_{12}}{3} = \frac{1}{3}$, and following (3.53) computes $z_{H,0}$:

$$z_{H,0} = \left. \hat{z}_{H,0} \right|_{\mathcal{Z}_0(\beta_{12})=0} = 2x - x^2. \quad \text{(B.7)}$$
Using (B.5) we compute

$$Z'_0 = \frac{d}{d\beta_{12}} Z_0(\beta_{12}) \bigg|_{Z_0(\beta_{12})=0} = -1.$$  \hspace{1cm} (B.8)

To proceed further, we introduce $\tilde{z}_{H,1}$ following (3.51). As explained around equation (3.55), $\tilde{z}_{H,1}$ satisfies the following equation

$$0 = \tilde{z}_{H,1}'' + \frac{3x^2 - 6x + 2}{(1 - x)(x^2 - 2x + 2)} \tilde{z}_{H,1}' + \frac{4}{x^2 - 2x + 2} \tilde{z}_{H,1} + \tilde{J}_H$$

$$\tilde{J}_H = \frac{2}{\sqrt{3}(1 - x)} \tilde{z}_{H,0}' + \frac{4}{\sqrt{3}(x^2 - 2x + 2)} z_{H,0},$$  \hspace{1cm} (B.9)

where we substituted $\beta_{12} = \beta_{12}^* = 1$. Notice that $\tilde{J}_H$ does not contain factors of $(x^2 - 2x + 2)^2$ in the denominators of the connection coefficients anymore. Using (B.7) we can solve for $\tilde{z}_{H,1}$, subject to the horizon boundary condition (3.39):

$$\tilde{z}_{H,1} = \frac{2}{\sqrt{3}} (2x - x^2) - \frac{2}{\sqrt{3}} \Rightarrow Z_1 = -\frac{2}{\sqrt{3}}.$$  \hspace{1cm} (B.10)

Finally, from (3.58) we compute the attenuation coefficient

$$\beta_2 = 1,$$  \hspace{1cm} (B.11)

which is the expected answer leading to a zero bulk viscosity for the $\mathcal{N} = 4$ SYM.

References

[1] J. M. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1999)] [arXiv:hep-th/9711200].

[2] A. Buchel and C. Pagnutti, Nucl. Phys. B 834, 222 (2010) [arXiv:0912.3212 [hep-th]].

[3] A. Onuki, Phys. Rev. E 55 403 (1997).

[4] K. Behrndt, M. Cvetic and W. A. Sabra, Nucl. Phys. B 553, 317 (1999) [arXiv:hep-th/9810227].

[5] R. G. Cai and K. S. Soh, Mod. Phys. Lett. A 14, 1895 (1999) [arXiv:hep-th/9812121].
[6] M. Cvetic and S. S. Gubser, JHEP 9907, 010 (1999) [arXiv:hep-th/9903132].

[7] K. Maeda, M. Natsuume and T. Okamura, Phys. Rev. D 78, 106007 (2008) [arXiv:0809.4074 [hep-th]].

[8] P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. 49, 435 (1977).

[9] K. Pilch and N. P. Warner, Nucl. Phys. B 594, 209 (2001) [arXiv:hep-th/0004063].

[10] A. Buchel, A. W. Peet and J. Polchinski, Phys. Rev. D 63, 044009 (2001) [arXiv:hep-th/0008076].

[11] N. J. Evans, C. V. Johnson and M. Petrini, JHEP 0010, 022 (2000) [arXiv:hep-th/0008081].

[12] A. Buchel, S. Deakin, P. Kerner and J. T. Liu, Nucl. Phys. B 784, 72 (2007) [arXiv:hep-th/0701142].

[13] A. Buchel, Phys. Lett. B 663, 286 (2008) [arXiv:0708.3459 [hep-th]].

[14] J. T. Liu and W. A. Sabra, Phys. Rev. D 72, 064021 (2005) [arXiv:hep-th/0405171].

[15] A. Buchel and L. A. Pando Zayas, Phys. Rev. D 68, 066012 (2003) [arXiv:hep-th/0305179].

[16] A. Buchel, Nucl. Phys. B 731, 109 (2005) [arXiv:hep-th/0507275].

[17] V. Balasubramanian and P. Kraus, Commun. Math. Phys. 208, 413 (1999) [arXiv:hep-th/9902121].

[18] A. Buchel, Nucl. Phys. B 708, 451 (2005) [arXiv:hep-th/0406200].

[19] P. Benincasa, A. Buchel and R. Naryshkin, Phys. Lett. B 645, 309 (2007) [arXiv:hep-th/0610145].

[20] D. T. Son and A. O. Starinets, JHEP 0603, 052 (2006) [arXiv:hep-th/0601157].

[21] G. Policastro, D. T. Son and A. O. Starinets, JHEP 0212, 054 (2002) [arXiv:hep-th/0210220].

51
[22] P. K. Kovtun and A. O. Starinets, Phys. Rev. D 72, 086009 (2005) [arXiv:hep-th/0506184].

[23] A. Buchel and C. Pagnutti, Nucl. Phys. B 816, 62 (2009) [arXiv:0812.3623 [hep-th]].

[24] P. Benincasa, A. Buchel and A. O. Starinets, Nucl. Phys. B 733, 160 (2006) [arXiv:hep-th/0507026].

[25] A. Buchel, Phys. Rev. D 72, 106002 (2005) [arXiv:hep-th/0509083].

[26] P. M. Hohler and M. A. Stephanov, Phys. Rev. D 80, 066002 (2009) [arXiv:0905.0900 [hep-th]].

[27] A. Cherman, T. D. Cohen and A. Nellore, Phys. Rev. D 80, 066003 (2009) [arXiv:0905.0903 [hep-th]].

[28] A. Buchel and C. Pagnutti, Nucl. Phys. B 824, 85 (2010) [arXiv:0904.1716 [hep-th]].

[29] O. Aharony, A. Buchel and P. Kerner, Phys. Rev. D 76, 086005 (2007) [arXiv:0706.1768 [hep-th]].