Motivated by the string theory corrections in low energy limit of both gauge and gravity sides, we consider three dimensional black holes in the presence of dilatonic gravity and Born-Infeld nonlinear electromagnetic field. We find that geometric behavior of the solutions is similar to the one of hyper-scaling violation metric, asymptotically. We also investigate thermodynamics of the solutions and show that the generalization to dilatonic gravity introduces novel properties into thermodynamics of the black holes which were absent in the Einstein gravity. Furthermore, we explore the possibility of tuning out part of the dilatonic effects using the Born-Infeld generalization.

I. INTRODUCTION

One of the main interests in considering a dilaton field is the fact that the low-energy limit of string theory precisely involves a massless scalar dilaton field. This in turn has motivated the scientific community to study dilaton gravity from different viewpoints. The scalar dilaton field has significant impacts on the casual structure as well as on the thermodynamical features of the charged black holes. In fact, the presence of dilaton field also affects the structure of spacetime geometry. The impacts are sometimes effective concerning the asymptotic behavior of dilaton solutions. In particular, it was proved that in presence of one or two Liouville-type dilaton potentials, black hole spacetimes are neither asymptotically flat nor (anti)-de Sitter \cite{1-7}. Nevertheless, in the case of three Liouville type dilaton potentials, it is possible to construct dilatonic black hole solutions in the background of (anti)-de Sitter (A)dS spacetime \cite{8, 9}. Also, the coupling of a dilaton field with other gauge fields may have profound effects on the resulting solutions \cite{10-12}. Dilaton fields can also be relevant to the construction of black holes with rather unconventional asymptotes. For example, charged Lifshitz black holes with arbitrary dynamical exponent can be sustained by the presence of at least two dilaton scalar fields \cite{13}. The extension of this dilatonic model with a nonlinear electrodynamics was considered in Refs. \cite{14-18}. Also, in this context, the holographic properties of black holes have been studied in Refs. \cite{19-21}. Recently, studies on neutron stars in the context of dilaton gravity \cite{22} as well as black holes in dilaton gravity’s rainbow \cite{23, 24} have been done.

In the present work, we will focus on three-dimensional dilaton gravity. Our motivations come from the fact that the discovery of the three-dimensional black hole (BTZ) \cite{25} and lower-dimensional gravity have gained a lot of interests in recent two decades \cite{26-34}. Indeed, the reasons for studying three dimensional gravity theories are multiple. For example, the near horizon geometry of three-dimensional solutions can serve as a worthwhile model to investigate some conceptual questions about the AdS/CFT correspondence \cite{35}. Moreover, the BTZ solution is a ground that offers many facets to explore. For example, the study of the BTZ black hole has improved our knowledge on gravitational systems and their interactions in three dimensions \cite{35}. It also opens the possible existence of gravitational Aharonov-Bohm effect due to the noncommutative BTZ black holes \cite{36}. The existence of specific relations between these black holes and effective action in string theory \cite{37, 38} are motivations. Concerning the black hole solutions that are BTZ-like, the current literatures contains a lot of them. For example, the existence of BTZ black holes/wormholes in the presence of the nonlinear electrodynamics has been investigated in Refs. \cite{39, 40} or in higher dimensions \cite{42, 43}. In addition, exact BTZ-like solutions were shown to arise in massive gravity \cite{44}, dilatonic gravity \cite{45, 47}, gravity’s rainbow \cite{48, 49}, new massive gravity \cite{50, 51}, Lifshitz gravity \cite{52}, massive gravity’s rainbow \cite{53} (see also Refs. \cite{54, 55} for more details). Moreover, thermal aspects have been explored where the existence of a phase transition between the BTZ black hole and thermal AdS space is possible \cite{61}. It is worth mentioning that three dimensional BTZ solution is also interesting form a quantum point of view \cite{62, 63}.

Here, we consider dilatonic BTZ black holes coupled with linear and nonlinear Born-Infeld electrodynamics. In addition to what we said for our motivations in the previous paragraphs, we are going to investigate the effects of
nonlinearity on the properties of the solutions. Although classical electrodynamics is well organized with the Maxwell equations (accompanying to Lorentz force), some of their shortcomings motivate one to consider nonlinear theory. One of the old successful theory of nonlinear electrodynamics is the so called Born-Infeld [68]. This abelian theory enjoys most of Maxwell properties and also its related electric field of a point like charge is regular everywhere. The foundations of this theory became firmly established when it is realized that one can obtain its Lagrangian from a class of low energy limit of string theory [69–74]. Hoffmann employed this type of nonlinear theory in context of Einstein gravity [75]. Then, different types of black holes in the presence of this electrodynamics have been studied in Refs. [76–90]. Based on the mentioned motivation, we discuss dilatonic black holes with Maxwell and Born-Infeld Einstein gravity [75]. Then, different types of black holes in the presence of this electrodynamics have been studied in Refs. [76–90].

II. BTZ BLACK HOLE SOLUTIONS IN DILATON-MAXWELL GRAVITY

Here, we consider the three-dimension action given by

\[ I = -\frac{1}{16\pi} \int d^3x \sqrt{-g} \left[ R - 4 (\nabla \Phi)^2 - V(\Phi) + L(h, \Phi) \right], \tag{1} \]

where \( \Phi \) is the dilaton field, \( V(\Phi) \) is a scalar potential, \( R \) denotes the scalar curvature and

\[ L(h, \Phi) = -e^{-4\alpha \Phi} h. \tag{2} \]

In the last expression, \( h \) stands for the Maxwell invariant \( h = h_{\mu\nu}h^{\mu\nu} \) where \( h_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) and \( \alpha \) represents the dilaton coupling parameter. The variation of the action \( (1) \) with respect to the metric tensor, the dilaton field \( (\Phi) \) and the gauge field \( (A_\mu) \), yields

\[ R_{\mu\nu} = 4 \left[ \partial_\mu \Phi \partial_\nu \Phi + \frac{1}{4} g_{\mu\nu} V(\Phi) \right] + 2 e^{-4\alpha \Phi} \left( h_{\mu\eta} h^{\eta\nu} - \frac{1}{2} g_{\mu\nu} h_{\lambda\eta} h^{\lambda\eta} \right), \tag{3} \]

\[ \nabla^2 \Phi = \frac{1}{8} \frac{\partial V(\Phi)}{\partial \Phi} + \frac{\alpha}{2} e^{-4\alpha \Phi} h_{\lambda\eta} h^{\lambda\eta}, \tag{4} \]

\[ 0 = \partial_\mu \left( \sqrt{-g} e^{-4\alpha \Phi} h^{\mu\nu} \right). \tag{5} \]

Since we are interested in electrically charged static solution, we consider the following ansatz

\[ ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\varphi^2, \tag{6} \]

where \( f(r) \) and \( R(r) \) are two metric functions. Our study being dedicated to black holes with a radial electric field, the suitable choice of gauge potential is given by \( A_\mu = \delta^t_\mu A_0(r), \) where \( A_0 \) denotes the electric potential. As usual the direct integration of the Maxwell equation \( (5) \) permits to express the electric field \( E(r) \) as

\[ E(r) = \frac{q e^{4\alpha \Phi}}{r R(r)}, \tag{7} \]

where \( q \) is an integration constant which is related to the electric charge. Now, the Einstein field equations \( (3) \) can be re-arranged as

\[ eq_{tt} : \frac{1}{2} \left( f''(r) + \left( \frac{1}{r} + \frac{R'(r)}{R(r)} \right) f'(r) \right) + V(\Phi) = 0, \tag{8} \]

\[ eq_{rr} : eq_{tt} + \left[ \frac{R''(r)}{R(r)} + 2 \frac{R'(r)}{r R(r)} + 4 \Phi'^2(r) \right] f(r) = 0, \tag{9} \]

\[ eq_{\theta\theta} : 2 E^2(r) e^{-4\alpha \Phi(r)} \left[ \frac{R''(r)}{R(r)} + \frac{2 R'(r)}{r R(r)} \right] f(r) + \left[ \frac{1}{r} + \frac{R'(r)}{R(r)} \right] f'(r) + V(\Phi) = 0. \tag{10} \]

It is then easy to see that the substraction of Eq. \( (8) \) with Eq. \( (9) \) yields the following constraint

\[ \frac{R''(r)}{R(r)} + 2 \frac{R'(r)}{r R(r)} + 4 \Phi'^2(r) = 0. \tag{11} \]
In order to transform this equation into a differential equation for the dilaton field, we use the following judicious ansatz \[100\] for the metric function

\[ R(r) = e^{2\alpha \Phi(r)}. \]

This in turn implies that the dilaton scalar field can be determined to be

\[ \Phi(r) = \frac{\gamma}{2\alpha} \ln \left( \frac{b}{r} \right), \tag{12} \]

where \( b \) is an arbitrary constant and for convenience, we have defined

\[ \gamma = \frac{\alpha^2}{(\alpha^2 + 1)}. \]

In order to find analytic solutions, the Liouville-type dilation potential is chosen as

\[ V(\Phi) = 2\Lambda e^{4\alpha\Phi}, \tag{13} \]

where \( \Lambda \) is a free parameter that plays the role of a cosmological constant. Note that this kind of potential have been used in the context of Friedman-Robertson-Walker scalar field cosmology \[91\] as well as in the case of Maxwell-dilaton black holes \[92, 93\]. Finally, the remaining metric function is found to be

\[ f(r) = \frac{2q^2 (\alpha^2 + 1)^2}{\alpha^2} - m r^\gamma + \frac{2\Lambda r^2 (\alpha^2 + 1)^2}{\alpha^2 - 2} \left( \frac{b}{r} \right)^{2\gamma}, \tag{14} \]

where \( m \) is an integration constant related to the mass of black holes.

Before continuing our study, it is interesting to re-write the metric solution (6) in the "standard form" by defining the radial coordinate \( \rho = r R(r) \)

\[ ds^2 = -F(\rho) \, dt^2 + \frac{b^{2\gamma}}{(1 - \gamma)^2} \frac{d\rho^2}{\rho^{\gamma - 1} F(\rho)} + \rho^2 d\varphi^2, \]

with

\[ F(\rho) = \frac{2q^2 (\alpha^2 + 1)^2}{\alpha^2} - m \rho^{\gamma - \gamma} b^{\gamma - \gamma} + \frac{2\Lambda (\alpha^2 + 1)^2}{\alpha^2 - 2} \rho^2. \]

It is interesting to note that for \( \alpha \to 0 \) with \( q/\alpha \to 0 \), the solution reduces to the uncharged static BTZ black hole. Asymptotically for \( \rho \gg 1 \), the metric behaves as

\[ ds^2 \sim -\rho^2 dt^2 + \frac{d\rho^2}{\rho^{2(1-\alpha^2)}} + \rho^2 d\varphi^2, \quad \alpha^2 < 2, \]

or

\[ ds^2 \sim -\rho^2 dt^2 + \rho^2 d\rho^2 + \rho^2 d\varphi^2, \quad \alpha^2 > 2. \]

In both cases, the asymptotic behavior is similar to the one of the hyperscaling violation metric

\[ ds^2 = \frac{1}{r^{2\theta}} \left[ -r^2 dt^2 + \frac{d\rho^2}{r^2} + r^2 d\varphi^2 \right] \sim -\rho^{2(1-\alpha)} \, dt^2 + \frac{d\rho^2}{\rho^{1-\alpha}} + \rho^2 d\varphi^2, \]

where \( z \) is the Lifshitz dynamical exponent and \( \theta \) is the hyperscaling violating parameter. More precisely, for \( \alpha^2 < 2 \), the asymptotic metric corresponds to an hyperscaling violation metric with \( z = 1 \) and \( \theta = \alpha^2/(\alpha^2 - 1) \), and for \( \alpha^2 > 2 \), this corresponds to \( z = 2/\alpha^2 \) and \( \theta = (2 + \alpha^2)/\alpha^2 \).

The charged dilatonic BTZ black holes are different from the charged BTZ black holes in Einstein gravity. The existence of the dilaton may exchange the role of the mass with the charge and vice et versa. Indeed, in the Einstein-Maxwell gravity, the mass is associated to the constant term of the metric function while the charge term appears in the structural metric function with a function depending on the radial coordinate. As one can note from the expression (14), in the dilatonic case, this is exactly the opposite that occurs. In addition, it is worth mentioning that
the obtained charged dilatonic BTZ solution does not reduce to the charged BTZ black hole solution in the absence of dilaton field. It is expected and comes from the difference between polynomial functions and logarithmic one. This behavior is the same as comparison between higher dimensional charged black holes and three dimensional case.

We are looking for the curvature singularity, in order to confirm the black hole interpretation of the solutions. For this purpose, we calculate the Ricci and Kretschmann scalars. These latter are obtained as

\[ R = \frac{4q^2}{r^2} - \frac{m \gamma}{r^{(2+\gamma)/(2+\gamma)}} + \frac{4 \Lambda \left( 2 \alpha^2 - 3 \right) \left( \frac{b}{r} \right)^{2 \gamma}}{\alpha^2 - 2} , \]

\[ R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} = \frac{16q^4}{r^4} + \frac{3 \gamma^2 m^2}{r^{2(\alpha^2+2)/(\alpha^2+1)}} + \frac{32 \left( \alpha^4 - 2 \alpha^2 + \frac{3}{4} \right) \Lambda^2 \left( \frac{b}{r} \right)^{4 \gamma}}{(\alpha^2 - 2)^2} - \frac{32 q^2 \left( \alpha^2 - 1 \right) \Lambda^{2 \gamma}}{(\alpha^2 - 2) r^{2(\alpha^2+1)/(\alpha^2+1)}} - \frac{8 m q^2}{r^{(2+2 \gamma)/(\alpha^2+1)}} - \frac{8 \gamma m \left( 2 \alpha^2 - 1 \right) \Lambda^{2 \gamma}}{(\alpha^2 - 2) r^{(3 \alpha^2+2)/(\alpha^2+1)}} . \]

Calculations show that for finite values of radial coordinate, the Ricci and Kretschmann scalars are finite. Also, for very small and very large values of \( r \), we have

\[ \lim_{r \to 0^+} R = \infty, \]

\[ \lim_{r \to 0^+} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} = \infty, \]

\[ \lim_{r \to \infty} R \propto \frac{4 \Lambda \left( 2 \alpha^2 - 3 \right) \left( \frac{b}{r} \right)^{2 \gamma}}{\alpha^2 - 2} , \]

\[ \lim_{r \to \infty} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} \propto \frac{32 \left( \alpha^4 - 2 \alpha^2 + \frac{3}{4} \right) \Lambda^2 \left( \frac{b}{r} \right)^{4 \gamma}}{(\alpha^2 - 2)^2} . \]

The above equation (17) confirms that there is an essential singularity located at \( r = 0 \), and Eq. (18) for \( \alpha = 0 \), the asymptotic behavior of solutions is \((A)dS \) (\( \lim_{r \to \infty} R \propto 6 \Lambda \) and \( \lim_{r \to \infty} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} \propto 12 \Lambda^2 \)), while for nonzero \( \alpha \), the asymptotic behavior of solutions is not that of \((A)dS \).

For more details regarding the behavior of the metric function, we plot \( f(r) \) versus \( r \) in Fig. 1. As one can see, this solution may contain real positive roots, and therefore, the singularity can be covered with an event horizon and interpreted as a black hole.

### A. Thermodynamical properties

In this section, we are going to calculate the thermodynamic and the conserved quantities of the solutions and then examine the first law of thermodynamics.
In order to calculate the Hawking temperature, we employ the definition of surface gravity. In doing so, we have that
\[
T = \frac{1}{2\pi} \sqrt{-\frac{1}{2} (\nabla_{\mu} \chi_{\nu}) (\nabla^\mu \chi^\nu)},
\]
where \( \chi = \partial/\partial t \) is the Killing vector. The Hawking temperature for this black hole can be written as
\[
T_+ = -\frac{(\alpha^2 + 1)}{2\pi r_+} \left[ q^2 + \Lambda r_+^2 \left( \frac{b}{r_+} \right)^2 \right]^{\gamma},
\]
where \( r_+ \) is the event horizon of black hole which is the largest real root of metric function, that is \( f(r = r_+) = 0 \). In addition, the electric potential \( U \) is defined by the gauge potential in the following form \[94\]
\[
U = A_{\mu} \chi^\mu \mid_{r \to \text{reference}} - A_{\mu} \chi^\mu \mid_{r = r_+},
\]
and for our solutions, we obtain
\[
U = \frac{q}{\gamma} \left( \frac{b}{r_+} \right)^\gamma.
\]
Also, one can use the area law of entropy in Einstein gravity to obtain the entropy of this black hole. Based on this law, the black hole’s entropy equals to one-quarter of horizon area \[95–97\]. Therefore, the entropy is
\[
S = \frac{\pi r_+}{2} \left( \frac{b}{r_+} \right)^\gamma.
\]
In order to obtain the electric charge of the black holes, one can calculate the flux of the electromagnetic field at infinity. The electric charge is obtained as
\[
Q = \frac{q}{2}.
\]
Regarding the timelike Killing vector (\( \xi = \partial/\partial t \)), one can show that the finite mass can be obtained as
\[
M = \frac{m}{8} (1 - \gamma) b^\gamma.
\]
By evaluating the metric function on the largest root of the solution, one is able to extract the geometrical mass \( (m) \) and insert into the total mass \[25\]. This leads to following relation
\[
M = \frac{(\alpha^2 + 1) \Lambda r_+^2}{4(\alpha^2 - 2)} \left( \frac{b}{r_+} \right)^{3\gamma} + \frac{q^2 (\alpha^2 + 1)}{4\alpha^2} \left( \frac{b}{r_+} \right)^\gamma.
\]
Using the relations obtained for the entropy \[23\] and the total electric charge \[24\], one is able to find a Smarr-like formula as
\[
M(S, Q) = \frac{(\alpha^2 + 1) \Lambda (\frac{2S}{\alpha})^{2(\alpha^2+1)} b^{-2\alpha^2}}{4(\alpha^2 - 2)} \left( \frac{\pi b}{2S} \right)^{3\alpha^2} + \frac{Q^2 (\alpha^2 + 1)}{\alpha^2} \left( \frac{\pi b}{2S} \right)^{\alpha^2}.
\]
Now, we can check the validity of first law of thermodynamics. In order to achieve this task, we note that
\[
dM(S, Q) = \left( \frac{\partial M(S, Q)}{\partial S} \right)_Q dS + \left( \frac{\partial M(S, Q)}{\partial Q} \right)_S dQ,
\]
and it is a matter of check to show that following equalities hold
\[
T = \left( \frac{\partial M}{\partial S} \right)_Q \quad \& \quad U = \left( \frac{\partial M}{\partial Q} \right)_S.
\]
The above relations confirm that the first law of thermodynamics is valid, namely
\[
dM = TdS + UdQ.
\]
FIG. 2: \(M\) versus \(r_+\), for \(b = 1, q = 1, \Lambda = -2\) (continuous line), \(\Lambda = -1\) (dotted line), \(\Lambda = 0\) (dashed line) and \(\Lambda = 1\) (dashed-dotted line).
Left panels: \(\alpha = 1\); Right panels: \(\alpha = 2\).

B. Thermodynamic behavior

Here, the main goal in this section is to specificize the effects on the coupling constants of the problem on the thermodynamic behavior of the solution, particularly for the mass, the temperature and the heat capacity.

1. Mass/Internal energy

The mass of the black holes is usually interpreted as the internal energy of the system. Nevertheless, in presence of a cosmological constant, the mass can also be viewed as an enthalpy with the cosmological constant playing the role of the thermodynamical pressure. Here, we do not consider such possibility and instead we will regard the mass as the internal energy.

First of all, the mass as defined in (26) requires the coupling constant \(\alpha \neq \pm \sqrt{2}\). Now, since the dilatonic parameter \(b > 0\), the \(q^2 -\) part of the mass expression is always positive. On the other hand, it is known that the constant \(\Lambda\) plays the role of a cosmological constant in the dilaton gravity. Therefore, it could be negative (for adS case) or positive (for dS case). Considering these two options, one finds that the \(\Lambda-\)contribution of the mass expression is positive provided that

\[
\Lambda > 0 \quad \& \quad \alpha > \sqrt{2},
\]

\[
\Lambda < 0 \quad \& \quad \alpha < \sqrt{2}.
\]

Under these conditions, to absence of roots and the positivity of the internal energy are ensured. In contrast, a negative value of the \(\Lambda-\)contribution of the mass expression can yield the existence of a root and a region of negativity for the internal energy. In such case, the root of the internal energy is obtained as

\[
r_+\big|_{M=0} = \left( -\frac{q^2 (\alpha^2 - 2)}{\Lambda \alpha^2} \right)^{\frac{1}{2\alpha}} b^{-\alpha^2}. \tag{31}\]

Evidently, the root of the internal energy is a decreasing function of \(\Lambda\), while it is an increasing function of the electric charge. In order to elaborate our results, we have plotted a series of diagrams (see Fig. 2). Finally, considering positive internal energy as a condition for having black holes, one can conclude that physical black hole solutions are present in range of \(0 < r_+ < r_+|_{M=0}\). Later, we will add some other restrictions which are imposed by temperature and heat capacity to complete our picture for our solutions to describe physical black holes.
2. Temperature

In classical thermodynamics of black holes, one of the conditions for having physical solutions is the positivity of the temperature. This highlights the importance of the roots of temperature. It is a matter of calculation to show that for these black holes, we have the following root for temperature

\[ r_+|_{T=0} = \left( -\frac{q^2}{\Lambda} \right)^{\frac{\alpha^2+1}{2}} b^{-\alpha^2}. \]  

(32)

Here, the root of the temperature is a decreasing function of \( \Lambda \), while it is an increasing function of the electric charge. Considering the possibility of having both positive and negative values for \( \Lambda \), the positive valued root only exists for \( \Lambda < 0 \). On the other hand, for positive values of the cosmological constant (\( \Lambda > 0 \)), the temperature will be negative. This indicates that in the classical thermodynamics of black holes, physical solutions exist only for \( \Lambda < 0 \) with the following condition

\[ \Lambda < -\frac{q^2}{r_+^2} \left( \frac{b}{r_+} \right)^{-2\gamma}. \]

Let us summarize what we have found by studying the temperature (in classical thermodynamics of black holes). First of all, physical solutions only exist for negative values of \( \Lambda \). Also, we found an upper limit on the values of \( \Lambda \) which is obtained by the condition of having positive temperature. Condition that bridges the values of the electric charge, \( q \) with \( \Lambda \). For completeness, we also present the following diagrams for the temperature in term of \( r_+ \) (see Fig. 3). In case of the absence of root, the temperature is negative valued everywhere. On the other hand, in presence of root, the positive valued temperature only exists for \( r_+|_{T=0} < r_+ \). By suitable choices of different parameters, the temperature could also acquires an extremum (maximum). It is a matter of calculation to show that this extremum is obtained as

\[ r_+|_{T=T_{Maximum}} = \left( \frac{q^2 (\alpha^2 + 1)}{\Lambda (\alpha^2 - 1)} \right)^{\frac{\alpha^2+1}{2}} b^{-\alpha^2}. \]  

(33)

Later, we will show that this extremum coincides with the divergencies of the heat capacity.
3. Heat Capacity

The study of the heat capacity is of important from two perspectives. Firstly, their discontinuities represent thermodynamical phase transition points. Secondly, the sign of the heat capacity determines whether system is in thermally stable/instable state.

The heat capacity is given by

\[ C_Q = T \left( \frac{\partial^2 M}{\partial S^2} \right)_{Q} = T \left( \frac{\partial S}{\partial T} \right)_{Q} = T \left( \frac{\partial T}{\partial r} \right)_{Q} , \tag{34} \]

where by using Eqs. (20) and (23), this expression becomes

\[ C_Q = -\pi r_+ \left( \frac{b}{r_+} \right)^\gamma \left[ q^2 + \Lambda r_+^2 \left( \frac{b}{r_+} \right)^{2\gamma} \right] \left( \alpha^2 + 1 \right) \]

\[ 2q^2 (3\alpha^2 - 1) - 2 (\alpha^2 - 1) \Lambda r_+^2 \left( \frac{b}{r_+} \right)^{2\gamma} . \tag{35} \]

It is a matter of calculation to show that the root and divergence points of the heat capacity are given respectively by

\[ r_+ (C_Q = 0) = \left( -\frac{q^2}{\Lambda} \right)^{\frac{\alpha^2+1}{2}} b^{-\alpha^2} , \tag{36} \]

\[ r_+ (C_Q \to \infty) = \left( -\frac{q^2 (\alpha^2 + 1)}{\Lambda (\alpha^2 - 1)} \right)^{\frac{\alpha^2+1}{2}} b^{-\alpha^2} . \tag{37} \]

Evidently, both the heat capacity and the temperature share the same roots. Therefore, the arguments given for the root of the temperature stand for the heat capacity as well. In addition, the extremum for the temperature \[ \frac{\partial T}{\partial r} = 0 \]

and divergence point of heat capacity are same. In order to have positive valued divergence point for the heat capacity, the following conditions should be satisfied

\[ \Lambda > 0 \quad & \quad \alpha < 1 , \]

\[ \Lambda < 0 \quad & \quad \alpha > 1 . \]

In the previous section, we have shown that only for negative values of the \( \Lambda \), our solutions enjoy a positive temperature. Consequently, for negative values of \( \Lambda \) and \( \alpha > 1 \), our solutions will develop a phase transition in their thermodynamical structure. In order to have a positive heat capacity which implies stable solutions, the denominator and numerator of the heat capacity must be of the same sign, that is

\[ q + \Lambda r_+^2 \left( \frac{b}{r_+} \right)^{2\gamma} < 0 \quad & \quad 2q^2 (\alpha^2 + 1) + 2 (\alpha^2 - 1) \Lambda r_+^2 \left( \frac{b}{r_+} \right)^{2\gamma} > 0 , \]

or

\[ q + \Lambda r_+^2 \left( \frac{b}{r_+} \right)^{2\gamma} > 0 \quad & \quad 2q^2 (\alpha^2 + 1) + 2 (\alpha^2 - 1) \Lambda r_+^2 \left( \frac{b}{r_+} \right)^{2\gamma} < 0 . \]

In order to have a better picture of the behavior of the heat capacity, we have plotted various diagrams (see Fig. 4). It is explicitly shown that there are three possible cases for the heat capacity: I) The heat capacity has no root and is negative everywhere. II) The heat capacity has only one root, in which after that, the heat capacity is positive and the solutions are stable. III) Finally, the heat capacity enjoys one root and one divergence point in its structure. In this case, before the root and after the divergency, the heat capacity is negative and solutions are thermally unstable. Whereas, only between the root and divergence point, the system is thermally stable. To end this section, we would like to add a comment. Previously, it was shown that the heat capacity in the context of BTZ-Λ-Maxwell theory enjoys the existence of root only for negative values of \( \Lambda \), whereas, the divergence point was only observed for positive values of the \( \Lambda \). Here, we see that the generalization to dilaton gravity has a significant effect on such behavior. Here, the negative values of \( \Lambda \) could enjoy both root and divergence in their structure while the branch of the positive \( \Lambda \) was ruled out due to the absence of positive valued temperature. Thermodynamically speaking, the generalization to dilaton gravity provided black holes with more complexity in their thermodynamical structure on level of the introduction of new phase transition point which was absent in the previous case. This highlights the differences between these two theories.
FIG. 4: $C_Q$ versus $r_+$, for $b = 1$, $q = 1$, $\Lambda = -2$ (continuous line), $\Lambda = -1$ (dotted line), $\Lambda = 0$ (dashed line) and $\Lambda = 1$ (dashed-dotted line).
Left panels: $\alpha = 1$; Right panels: $\alpha = 2$.

III. BTZ BLACK HOLE SOLUTIONS IN DILATON-BORN-INFELD GRAVITY

We now turn in the derivation of the dilatonic-BI-BTZ black holes where the Lagrangian of the BI-dilaton part is given by

$$L(h, \Phi) = 4\beta^2 e^{4\alpha\Phi} \left( 1 - \sqrt{1 + \frac{e^{-8\alpha\Phi} h}{2\beta^2}} \right),$$  \hspace{1cm} (38)

where $\beta$ is the BI parameter. It is notable that, in the limit $\beta \to \infty$, the Lagrangian BI, reduces to the standard Maxwell field coupled to a dilaton field as $L(h, \Phi) = -e^{4\alpha\Phi} h$. Varying the action (1) with respect to the metric tensor $g_{\mu\nu}$, the dilaton field $\Phi$ and the gauge field $A_\mu$, we obtain the following field equations

$$R_{\mu\nu} = 4 \left[ \partial_\mu \Phi \partial_\nu \Phi + \frac{1}{4} g_{\mu\nu} V(\Phi) \right] - 4e^{-4\alpha\Phi} \partial_Y L(Y) h_{\mu\eta} h_{\nu}^\eta + 4\beta^2 e^{4\alpha\Phi} \left[ 2\partial_Y L(Y) - L(Y) \right] g_{\mu\nu},$$  \hspace{1cm} (39)

$$\nabla^2 \Phi = \frac{1}{8} \frac{\partial V(\Phi)}{\partial \Phi} + 2\alpha\beta^2 e^{4\alpha\Phi} \left[ 2Y \partial_Y L(Y) - L(Y) \right],$$  \hspace{1cm} (40)

$$0 = \partial_\mu \left( \sqrt{-g} e^{-4\alpha\Phi} \partial_Y L(Y) h^{\mu\nu} \right),$$  \hspace{1cm} (41)

where we can obtain the electric field as

$$E(r) = \frac{dA(r)}{dr} = \frac{qe^{4\alpha\Phi}}{rR(r)\sqrt{1 + \Gamma}}.$$  \hspace{1cm} (45)
where we have defined $\Gamma = \frac{e^{2\Phi}}{\sqrt{r^2 \mu^2(r)}}$. For latter convenience, we chose a Liouville-type dilaton potential defined by $V(\Phi) = 2\Lambda e^{4\alpha\Phi}$ with the ansatz $R(r) = e^{2\alpha\Phi(r)}$. After some algebraic calculations, we obtain the following differential equations

$$\frac{\alpha^2 f(r) - r (1 + \alpha^2) f'(r)}{r^2 (1 + \alpha^2)^2} + 4 \left( \beta^2 - \frac{\Lambda}{2} \right) \left( \frac{b}{r} \right)^{2\gamma} - 4\beta^2 \left( \frac{b}{r} \right)^{2\gamma} \sqrt{1 + \frac{q^2}{r^2 \beta^2 (\frac{b}{r})^{2\gamma}}} = 0, \quad (46)$$

$$2 \left( \frac{e^{2\alpha\Phi(r)}}{r} \right)' + r \left( \frac{e^{2\alpha\Phi(r)}}{r} \right)'' + 4r \left( \frac{\Phi'(r)}{r} \right) e^{2\alpha\Phi(r)} = 0. \quad (47)$$

We are now in a position to obtain exact solutions. Considering the equations (46) and (47), we can obtain the general solutions as

$$f(r) = \frac{2 (\alpha^2 + 1)^2 (\Lambda - 2\beta^2) r^2}{\alpha^2 - 2} \left( \frac{b}{r} \right)^{2\gamma} - 4\beta^2 \left( \frac{\alpha^2 + 1}{\alpha^2 - 2} \right) \left( \frac{b}{r} \right)^{2\gamma} r^2 H_1 + \frac{4q^2 (\alpha^2 + 1)^2}{\alpha^2} H_2, \quad (48)$$

$$\Phi(r) = \frac{\gamma}{2\alpha} \ln \left( \frac{b}{r} \right), \quad (49)$$

in which $H_1$ and $H_2$ are the following hypergeometric functions

$$H_1 = 2F1 \left[ \left[ \frac{1}{2}, \frac{\alpha^2 - 2}{2} \right], \left[ \frac{\alpha^2}{2} \right], -\Gamma \right]$$

$$H_2 = 2F1 \left[ \left[ \frac{1}{2}, \frac{\alpha^2}{2} \right], \left[ \frac{\alpha^2 + 2}{2} \right], -\Gamma \right].$$

It is notable that, in the absence of a BI field ($\beta \to \infty$), the solutions (48) reduce to the charged dilatonic BTZ black hole solutions (see Eq. (13)).

Calculation of the Kretschmann scalar shows that it is finite for nonzero $r$ and its behavior for very small and very large values of $r$ can be reported as

$$\lim_{r \to 0^+} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} = \infty, \quad (50)$$

$$\lim_{r \to \infty} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} \propto \frac{16\Lambda^2 (2\alpha^4 - 4\alpha^2 + 3)}{(\alpha^2 - 2)^2} \left( \frac{b}{r} \right)^{4\gamma}. \quad (51)$$

The equation (50) confirms that there is an essential singularity located at $r = 0$, while the equation. (51) shows that in the presence dilaton field ($\alpha \neq 0$), the asymptotic behavior of the solutions is not that of (A)dS. It is notable that, in the absence of dilaton field ($\alpha = 0$), the asymptotic behavior of the solutions is (A)dS. We plot the obtained metric function Eq. (18) in figure 5. This figure shows that the metric function may contain real positive roots, and so, the curvature singularity can be covered with an event horizon and interpreted as a black hole.

### A. Thermodynamical properties

We can obtain the Hawking temperature by using Eq. (19) for this black hole as

$$T = \frac{(\alpha^2 + 1) r_+}{2\pi} \left( \frac{b}{r_+} \right)^{2\gamma} \left[ \frac{2\beta^2 (1 - H_{1_+}) - \Lambda}{\alpha^2 q r_+} \right] H_3 - \frac{\alpha^2 H_{2_+}}{(\alpha^2 + 2)} + \frac{\alpha^2 \Gamma H_4}{(\alpha^2 + 2)}, \quad (52)$$

Notice that $H_{1_+} = H_1|_{r = r_+}$ and $H_{2_+} = H_2|_{r = r_+}$, and also that $H_3$ and $H_4$ are given by

$$H_3 = 2F1 \left[ \left[ \frac{3}{2}, \frac{\alpha^2}{2} \right], \left[ \frac{\alpha^2 + 2}{2} \right], -\Gamma_+ \right], \quad (53)$$

$$H_4 = 2F1 \left[ \left[ \frac{3}{2}, \frac{\alpha^2 + 2}{2} \right], \left[ \frac{\alpha^2 + 4}{2} \right], -\Gamma_+ \right]. \quad (54)$$
FIG. 5: $f(r)$ versus $r$, for $b = 0.3, \Lambda = -1$.
Left up panel: for $m = 4, \alpha = 1, \beta = 0.5, q = 0.490$ (dashed line), $q = 0.518$ (continuous line) and $q = 0.550$ (dotted line).
Right up panel: for $m = 4, \alpha = 1, \beta = 0.5, \gamma = 0.615$ (continuous line) and $\beta = 0.850$ (dotted line).
Right down panel: for $m = 4, \beta = 0.5, \alpha = 1.000$ (dashed line), $\alpha = 1.021$ (continuous line) and $\alpha = 1.040$ (dotted line).
Left down panel: for $q = 0.5, \beta = 0.5, \alpha = 1, m = 4.10$ (dashed line), $m = 3.86$ (continuous line) and $m = 3.60$ (dotted line).

where $\Gamma_+ = \Gamma|_{r=r_+}$. The electric potential $U$ is obtained

$$U = \frac{q}{\gamma} \left( \frac{b}{r_+} \right) \gamma H_{2+}. \quad (55)$$

Using the area law of entropy in Einstein gravity, and also calculating the flux of electromagnetic field at infinity we can obtain the entropy and the electric charge of this black hole as

$$S = \frac{\pi r_+}{2} \left( \frac{b}{r_+} \right) \gamma, \quad (56)$$

$$Q = \frac{q}{2}. \quad (57)$$

According to the mentioned method for calculation of the total finite mass of metric presented in Eq. 50, we can obtain the total mass as

$$M = \frac{m}{8} (1 - \gamma) b^7, \quad (58)$$

which does not depend on the nonlinearity and on the electromagnetic field directly since both the nonlinearity and the electromagnetic field vanish for $r \to \infty$. Following the steps of the previous sections, the total mass of the black
hole solution is obtained by evaluating the metric function on its largest root

$$M(r_+) = \frac{(\alpha^2 + 1)}{2(\alpha^2 - 2)} \left( \frac{b}{r_+} \right)^7 \left\{ \left( \frac{\Lambda}{2} - \beta^2 \right) r_+^2 + \beta^2 \left[ r_+^2 \left( \frac{b}{r_+} \right)^2 H_1 + \frac{q^2 (\alpha^2 - 2)}{\alpha^2 \beta^2} H_{2+} \right] \right\},$$

yielding as well to a Smarr-like formula given by

$$M(S, Q) = \frac{(\alpha^2 + 1)}{2(\alpha^2 - 2)} \left( \frac{\pi b}{2S} \right) a^2 \left\{ \left( \frac{\Lambda}{2} - \beta^2 \right) \left( \frac{2S}{b^2 a^2} \right)^2 (\alpha^2 + 1) + \beta^2 \left[ \left( \frac{2S}{\pi} \right)^2 H_{1sQ} + \frac{4Q^2 (\alpha^2 - 2)}{\alpha^2 \beta^2} H_{2sQ} \right] \right\},$$

where $H_{1sQ} = H_{1s} \bigg|_{r_+ = b(\frac{2S}{\pi})^{2/3}, \ q = 2Q}$ and $H_{2sQ} = H_{2s} \bigg|_{r_+ = b(\frac{2S}{\pi})^{2/3}, \ q = 2Q}$. As a direct consequence, the first law of thermodynamics holds

$$dM = TdS + UdQ,$$

with

$$dM(S, Q) = \left( \frac{\partial M(S, Q)}{\partial S} \right)_Q dS + \left( \frac{\partial M(S, Q)}{\partial Q} \right)_S dQ,$$

and

$$T = \left( \frac{\partial M}{\partial S} \right)_Q \quad \& \quad U = \left( \frac{\partial M}{\partial Q} \right)_S.$$

**B. Thermodynamic behavior**

In this section, we would like to stress how the Born-Infeld theory can modify the thermodynamic behavior of the black holes. Our main motivation is to distinguish the differences between the Maxwell and Born-Infeld theories in the thermodynamic context of charged BTZ-dilatonic black holes.

1. **Mass/Internal energy**

As in the Maxwell case, the condition $\alpha \neq \sqrt{2}$ must also be taken into consideration. This is not surprising since this condition is only originated from the dilatonic part of the action. Interestingly enough, one can see from the expression of the mass that for

$$\Lambda = 2\beta^2,$$

the effects of the $\Lambda$–term are canceled by the nonlinearity term and since the other two terms ($q$ and other $\beta$ terms) are positive valued, the mass is positive everywhere without any root for this case. This is one of the most important contributions of the generalization to BI field which is not seen in the context of Maxwell case. Due to the complexity of the mass relation, it is not possible to obtain the root of the mass analytically. Therefore, we employ some numerical method and plot the following diagrams (see left and middle panels of Fig. [4]). It could be seen that similar behaviors to those in Maxwell case are observed here too. This means that the mass of these black holes could enjoy the existences of root and two regions of positivity and negativity or it could be positive valued everywhere. In case of the existence of root, the mass is only positive valued before the root. The place of this root is a function of the $\alpha$ and $\beta$ term. Therefore, we can separate the effects of different terms (except the $q$–term) in the mass into two categories: $\alpha > \sqrt{2}$ and $\alpha < \sqrt{2}$. We give the details for $\alpha > \sqrt{2}$ since the opposite holds for the case $\alpha < \sqrt{2}$ (except for the $q$–term). For $\alpha > \sqrt{2}$, the $q$–term and one of the $\beta$–terms (the one which is coupled with the electric charge) have positive contributions on the total value of the mass. Whereas, the other $\beta$–term has always negative contribution. The effects of the $\Lambda$–term depends on the choices of $\Lambda$ itself. For negative $\Lambda$, the effect of this term is toward decreasing mass while the opposite is observed for positive $\Lambda$. The existence of a root for positive $\Lambda$ depends on following condition

$$\Lambda < 2\beta^2.$$
FIG. 6: $M$ versus $r_+$, for $b = 1$, $q = 1$, $\beta = 0.5$, $\Lambda = -2$ (continuous line), $\Lambda = -1$ (dotted line), $\Lambda = 0$ (dashed line) and $\Lambda = 1$ (dashed-dotted line).

Left panels: $\alpha = 1$; Middle panels: $\alpha = 2$.
Right panels: for $\Lambda = -2$, $\alpha = 2$, $\beta = 0.5$ (continuous line), $\beta = 1$ (dotted line), $\beta = 1.5$ (dashed line) and Maxwell case : $\beta \to \infty$ (dashed-dotted line).

If the mentioned condition is satisfied, it is possible to find a root for the obtained mass, otherwise, the mass is always positive valued without any root. The situation for a negative $\Lambda$—term is different and depends on the following condition

$$
\left(\frac{\Lambda}{2} - \beta^2\right) r_+^2 > \beta^2 \left[r_+^2 \left(\frac{b}{r_+}\right)^{2\gamma} H_1 + q^2 \left(\frac{b^2 - 2 \alpha^2}{\alpha^2 \beta^2}\right) H_2\right],
$$

which highly depends on choices of the nonlinearity parameter, $\beta$.

Here, we see that the effects of the nonlinearity parameter, hence the BI generalization on the properties of the mass are significant. The presence of $\beta$ provided an extra degree of freedom, and because of that the mass of the solutions may have different behaviors. In the next sections, we will give further details regarding this matter.

2. Temperature

Obtaining the root and extremum point of temperature is not possible analytically. As before, we will use some numerical method and we plot the following diagrams (see left and middle panels of Fig. 7). Evidently, depending on the choices of different parameters: I) The temperature could be completely negative which in turn implies that the solutions are not physical. II) The temperature could have one root and before it, the temperature is negative valued. III) Having one root and one maximum in which the maximum is located after the root and only after root the temperature is positive. The places of the root and extremum depend on the choices of the nonlinearity parameter (see right panel of Fig. 7). The generalization to nonlinear electromagnetic field provided us with extra terms in the temperature which eventually modified the root, the regions of negativity/positivity and the extremum of temperature. In addition, this generalization results into one more degree of freedom which could be used to tune out the effects of some part of the dilaton gravity. This option was not possible in the context of Maxwell theory.

3. Heat Capacity

Our final study in this section is devoted to the heat capacity of the nonlinearly charged solutions. By taking a closer look at Eq. (34), one can see that the heat capacity contains temperature and the derivations of entropy and temperature with respect to the horizon radius. The conditions regarding the roots and positivity/negativity of the temperature were given in the last section. The derivation of the entropy with respect to the horizon radius does not produce any singular point and it does not contain terms which could contribute to negativity/positivity of the heat
FIG. 7: $T$ versus $r_+$, for $b = 1$, $q = 1$, $\beta = 0.5$, $\Lambda = -2$ (continuous line), $\Lambda = -1$ (dotted line), $\Lambda = 0$ (dashed line) and $\Lambda = 1$ (dashed-dotted line).

Left panels: $\alpha = 1$; Middle panels: $\alpha = 2$.

Right panels: for $\Lambda = -2$, $\alpha = 2$, $\beta = 0.5$ (continuous line), $\beta = 1$ (dotted line), $\beta = 1.5$ (dashed line) and Maxwell case: $\beta \to \infty$ (dashed-dotted line).

capacity. Therefore, we focus our attention on

$$
\left( \frac{\partial T}{\partial r_+} \right)_Q = \left( \frac{\alpha^2 - 1}{\pi} \right) \left( \frac{h}{r_+} \right)^{2\gamma} + \left( \frac{\alpha^2 - 1}{\pi} \beta^2 \left( \frac{h}{r_+} \right)^{2\gamma} H_{1+} - \frac{(2\alpha^2 - 1)}{\pi \alpha^2 r_+^2} H_{3+} - \frac{\alpha^2}{\pi \alpha^2 r_+^2} H_{2+} + \frac{(2\alpha^2 + 3)}{\pi \alpha^2 r_+^2} H_{4+} - 3H_5 \right)
$$

(65)

where $H_5$ and $H_6$ are defined as follows

$$
H_5 = \binom{\alpha^2 + 2}{\alpha^2 + 4} \binom{\alpha^2 + 2}{\alpha^2 + 4}, -\Gamma_+
$$

(67)

$$
H_6 = \binom{\alpha^2 + 2}{\alpha^2 + 4} \binom{\alpha^2 + 2}{\alpha^2 + 4}, -\Gamma_+
$$

(68)

Similar to other quantities, here, we are able to tune out the effects of $\Lambda$—term by suitable choices of the nonlinearity parameter. In other words, it is possible to cancel out the effects of $\Lambda$ in the heat capacity of BI solutions by choosing $\Lambda = 2\beta^2$. Once more we point it out that such possibility is present in BI generalization while it is not seen in the linear Maxwell theory. The presence of nonlinear electromagnetic field provided a complicated system of terms in the heat capacity. Unfortunately, such complication does not allow us to extract divergence points of the heat capacity analytically. We have plotted the following diagrams (see left and middle panels of Fig. 8). Depending on the choices of different parameters, one of the following cases would take place for the heat capacity; I) Two states of stable and unstable which are separated by a root. In the previous section, it was shown that the root of the temperature, hence the heat capacity, is a function of nonlinearity parameter. II) The heat capacity could be negative everywhere without any root or divergency. In this case the solutions are unstable but according to the results of previous section, the temperature is also negative which indicates that the solutions are not physical. III) The heat capacity could enjoy one root and one divergency. The divergency points out the existence of a phase transition. Before the root, the heat capacity is also negative. Therefore, the only physically stable solutions exist between the root and divergency of the heat capacity. The plotted diagram for the variation of the nonlinearity parameter (see right panel of Fig. 8), $\beta$, shows that the location of divergence is a function of this parameter. This indicates that the region in which physical stable solutions exist, depends on the choices of $\beta$. This highlights another significant effect of the nonlinearity parameter.
FIG. 8: $C_Q$ versus $r_+$, for $b = 1, q = 1, \beta = 0.5, \Lambda = -2$ (continuous line), $\Lambda = -1$ (dotted line), $\Lambda = 0$ (dashed line) and $\Lambda = 1$ (dashed-dotted line).
Left panels: $\alpha = 1$; Right panels: $\alpha = 2$.
Right panels: for $\Lambda = -2, \alpha = 2, \beta = 0.5$ (continuous line), $\beta = 1$ (dotted line), $\beta = 1.5$ (dashed line) and Maxwell case: $\beta \to \infty$ (dashed-dotted line).

IV. CONCLUSION

The paper at hand regarded BTZ black holes in the presence of two generalizations which are motivated by string theory: dilaton gravity and Born-Infeld nonlinear electromagnetic field.

First, the solutions in the presence of dilaton gravity were extracted and their thermodynamical properties were studied. It was shown that in comparison to the Einstein gravity, here, the mass of these black holes could enjoy the existence of root and a region of negative mass/internal energy. In addition, specific limits for the dilaton parameter and $\Lambda$ were obtained and it was shown that thermodynamically speaking, only for specific region of dilaton parameter, physical solutions exist. Therefore, although generalization to dilaton gravity provided us with new properties for the solutions, at the same time, it imposed specific limits on them as well. In other words, introducing new properties into solutions by dilatonic generalization was obtained at the cost of harder restrictions on the solutions.

Next, Born-Infeld generalization was implied to the action. It was shown that this generalization provides the possibility of canceling a part of dilatonic contribution by suitable choices of parameters. In addition, it was shown that although some of the Maxwell conditions for having thermodynamically physical solutions stand for this case as well, the general behavior of the solutions including phase transition point, region of stable solutions, conditions of having physical solutions were modified due to the contributions of BI theory.

The study conducted here could be employed to investigate aspects such as superconductor properties, holographical principles and entropy spectrum. Specially, it is interesting to study the central charges of 1+1 theories and understand the effects of the dilatonic gravity and BI generalization in this context. In addition, as it is known, we can discuss dyon solutions of our dilatonic setup with two horizons. We address these subjects in the forthcoming works.

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