Implications of Minimum and Maximum Length Scales in Cosmology

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We investigate the cosmological implications of the generalized and extended uncertainty principle (GEUP), and whether it could provide an explanation for the dark energy. The consequence of the GEUP is the existence of a minimum and a maximum length, which can in turn modify the entropy area law and also modify the Friedmann equation. The cosmological consequences are studied by paying particular attention to the role of these lengths. We find that the theory allows a cosmological evolution where the radiation- and matter-dominated epochs are followed by a long period of virtually constant dark energy, that closely mimics the ΛCDM model. The main cause of the current acceleration arises from the maximum length scale β, governed by the relation \( \Lambda \sim -\beta^{-1} W(\beta^{-1}) \). Using recent observational data (the Hubble parameters, type Ia supernovae, and baryon acoustic oscillations, together with the Planck or WMAP 9-year data of the cosmic microwave background radiation), we estimate constraints to the minimum length scale \( \alpha \lesssim 10^{-3} \) and the maximum length scale \( \beta \sim -10^{-2} \).

I. INTRODUCTION

The observation that the universe is accelerating [1,2] has generated extensive investigations aiming to establish its theoretical foundation. A promising possible explanation involves invoking the cosmological constant \( \Lambda \), which is related to the vacuum energy density. For consistency with existing observations, \( \Lambda \) must be very small, on the scale of \( \sim 10^{-120} \) orders of magnitude smaller than the Planck scale \( M_p \). However, the exact value gives rise to the “cosmological constant problem” [3]. Another possibility is a dynamic dark energy model [4,5], in which the cosmological constant varies dynamically. Arguably, observational data favor dynamical dark energy models over the standard ΛCDM model [7,8]. One possible way to describe dynamic dark energy models is as a generalization of Heisenberg's uncertainty principle.

Early applications of this principle concerned mainly black-hole thermodynamics [9,11], but more recently also cosmological topics, such as inflation [12,13], non-singular universe construction [14,15] and the dark energy model [16,17]. The underlying idea is that a generalization of the principle can modify the entropy-area relation in thermodynamics, thereby introducing corrections to the cosmological evolution equation. One well-known example is the “generalized uncertainty principle” (GUP) [15], \( \Delta x \Delta p \geq 1/2 + \alpha l_p^2/2 \Delta p^2 \), which allows the introduction of quantum-gravity into ordinary quantum mechanics via the deformation of the Heisenberg uncertainty principle. Such a deformation implies the existence of a minimum length, \( \Delta x_{\text{min}} \sim \alpha^{1/2} l_p \), and is expected to have been most apparent in the early universe or in the high-energy regime. Another possible generalization is the “extended uncertainty principle” (EUP) [19], \( \Delta x \Delta p \geq 1/2 + \beta / L_x \Delta x^2 \), where \( \beta \) is a dimensionless parameter and \( L_x \) is an unknown fundamental length scale. In contrast to the GUP, the EUP implies the existence of a minimum momentum with positive values of \( \beta \), \( \Delta p_{\text{min}} \sim \beta^{1/2} / L_x \), and is predicted to be most apparent at later times in the universe. However, as mentioned in [20], it is interesting that a positive cosmological constant can only result from negative values of \( \beta \), thus contradicting the EUP prediction of a minimum momentum. In such a case, a position measurement may not exceed an unknown length scale, i.e., the maximum length \( \Delta x_{\text{max}} \sim L_x / \beta^{1/2} \). By combining the EUP and GUP (GEUP) we obtain a more general form [18,21]

\[
\Delta x \Delta p \geq \frac{1}{2} \left( 1 + \alpha l_p^2 \Delta p^2 + \frac{\beta}{L_x^2} \Delta x^2 \right).
\]

This formulation of the GEUP [1] predicts the existence of both a minimum and a maximum length (with negative values of \( \beta \)). It is worth mentioning that these modified Heisenberg uncertainty principles (i.e., the GUP, EUP, or GEUP), can yield a correction to the Bekenstein–Hawking entropy of a black hole [22,23].

The connection between thermodynamics and gravity was first investigated by Bardeen, Carter, and Hawking [24]. There has since then been an abundant literature on, e.g., the Rindler space-time [25] and the Friedmann–Robertson–Walker (FRW) universe [26]. With regard to the Rindler space-time, Jacobson found that the Einstein equation can be derived from the thermodynamic relation between heat, entropy, and temperature: \( dQ = TdS \), where \( dQ \) is the energy flux and \( T \) is the Unruh temperature, which are detected by an accelerating observer located just within the local Rindler causal horizons. The FRW universe, on the other hand, assumes that the apparent horizon \( r_A \) has an associated entropy \( S = A/4G \) and a temperature \( T = \kappa/2\pi \) in Einstein gravity, where \( A \) and \( \kappa \) are, respectively, the area and surface gravity of the apparent horizon. Akbar and Cai derived the differential form of the Friedmann equation

\[d^2a/dt^2 = -4\pi G \rho_a - 4\pi G \Lambda a^2 - \frac{2H}{a} dH/dt.
\]
for a FRW universe from the first law of thermodynamics at the apparent horizon, i.e., \( dE = TdS + WdV \), where \( E \) is the total energy density of matter existing within the apparent horizon, \( V \) is the volume contained within the apparent horizon, and the work density \( W = (\rho - p)/2 \) is a function of the energy density \( \rho \) and the pressure \( p \) of matter in the universe. A modified Friedmann equation was recently suggested \([27]\) based on a corrected entropy formula that is potentially useful within the context of cosmology.

The purpose of the present study is to consider cosmology within the framework of the GEUP, i.e., by considering the minimum and maximum lengths, and to compare the results with observations. The GEUP involves two parameters that can be constrained by measurements.

This paper is organized as follows: In Section II, we investigate the influence of the GEUP on thermodynamics, and obtain a corrected Friedmann equation for the FRW universe. In Section III, we investigate the effects of the GEUP on thermodynamics within the framework of the GEUP, i.e., by considering the minimum and maximum lengths, and to compare the results with observations. The GEUP involves two parameters that can be constrained by measurements. Section IV presents observational constraints on our model parameters. Section V closes with discussions and concluding remarks.

II. MODIFIED FRIEDMANN EQUATIONS

This section presents calculations of the modified Friedmann equations within the framework of the GEUP to describe cosmological effects. The outcome of the GEUP \([1]\) is the modified momentum uncertainty

\[
\Delta p \geq \frac{\Delta x}{\alpha l_p^2} \left( 1 - \sqrt{1 - \frac{\alpha^2}{\beta}} \Delta x^2 \right),
\]

where the Taylor expansion calculated at \( \alpha = \beta = 0 \). As noted in \([22]\), the Heisenberg uncertainty principle \( \Delta p > 1/\Delta x \) can be rewritten in terms of a lower bound to the energy \( (E > 1/\Delta x) \), which in the case of the GEUP becomes

\[
E \geq \frac{1}{2\Delta x} \left( 1 + \frac{\alpha l_p^2}{4\Delta x^2} + \frac{\beta}{L^2} \Delta x^2 \right).
\]

When a black hole absorbs or emits a classical particle of energy \( E \) and size \( R \), the minimal change in the surface area of the black hole is \( \Delta A_{\text{min}} \geq 8\pi l_p E R \). Arguably, the size of a quantum particle cannot be smaller than \( \Delta x \) \([28]\), which would imply the existence of a finite bound \( \Delta A_{\text{min}} \geq 8\pi l_p E \Delta x \). Thus, considering the GEUP, we obtain

\[
\Delta A_{\text{min}} \geq 4\pi l_p \left( 1 + \frac{\alpha l_p^2}{4\Delta x^2} + \frac{\beta}{L^2} \Delta x^2 \right).
\]

\( \Delta x \) is the position uncertainty of a photon which can be associated with the black hole radius, \( \Delta x = 2r_s \), where \( r_s \) is the Schwarzschild radius. Given the surface area of the black hole, \( A = 4\pi r_s^2 \), the relation between \( A \) and \( \Delta x \) can be expressed as \( \Delta x^2 = A/\pi \). Substituting this equation into \([5]\), the minimal area change becomes

\[
\Delta A_{\text{min}} \geq 4\pi l_p \lambda \left( 1 + \frac{\alpha l_p^2}{4A} + \frac{\beta}{\pi l_p^2} \right),
\]

where \( \lambda \) is the calibration factor that is determined from the Bekenstein-Hawking entropy formula. The entropy of the black hole is assumed to depend on its surface area. Also, given that the entropy increases by a factor of \( \ln 2 \) at least, regardless of the value of the area, we have

\[
\frac{dS}{dA} = \frac{\Delta S_{\text{min}}}{\Delta A_{\text{min}}} = \frac{1}{4\pi l_p^2} \left( 1 + \frac{\alpha l_p^2}{4A} + \frac{\beta}{\pi l_p^2} \right)^{-1},
\]

where \( \ln 2/\lambda = \pi \), as mentioned above. Integrating \([7]\), the GEUP-corrected entropy is

\[
S = \frac{A}{4\pi l_p^2} \left( 1 - \frac{\alpha l_p^2}{4A} \ln \left( \frac{A}{4\pi l_p^2} \right) - \frac{\beta}{2\pi l_p^2} \right).
\]

We note that the modified Bekenstein-Hawking entropy \([8]\) arises from the existence of the minimum and maximum lengths.

Based on the “apparent horizon” approach \([26]\), we derive the modified Friedmann equations with the modified entropy \([5]\) applied to the first law of thermodynamics, \( dE = TdS + WdV \). Thus, we considered that space-time geometry is characterized by the FRW metric

\[
ds^2 = -dt^2 + a^2 \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right),
\]

where \( a \) is a scaling factor of our universe, and the values of the spatial curvature constant \( k = +1, 0, \) or \(-1\) correspond, respectively, to a closed, flat, or open universe. Using spherical symmetry, the metric \([9]\) can be rewritten as

\[
ds^2 = h_{ab} dx^a dx^b + \tilde{r}^2 d\Omega^2,
\]

where \( x^0 = t \), \( x^1 = r \) and \( \tilde{r} = ar \), and the two-dimensional metric \( h_{ab} = \text{diag}(-1, a^2/(1 - kr^2)) \). In the FRW universe, a dynamic horizon always exists because it is a local quantity of space-time, which is a marginally trapped surface with vanishing expansion. It is determined by the relation \( h_{ab} \partial_a \tilde{\rho} \partial_b \tilde{\rho} \), which yields the radius of the apparent horizon

\[
\tilde{r}_{\Lambda}^2 = \frac{1}{H^2 + k/a^2},
\]

where \( H = \dot{a}/a \) is the Hubble parameter. By assuming that matter in the FRW universe forms a perfect fluid
with four-velocity $u^\mu$, the energy-momentum tensor can be written
\[ T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}, \]
where $\rho$ is the energy density of the perfect fluid and $p$ is its pressure. The energy conservation law, $\nabla_\mu T^{\mu\nu} = 0$, yields the continuity equation
\[ \dot{\rho} + 3H(\rho + p) = 0. \]  

According to the main results of [26, 27], applying the first law of thermodynamics to the apparent horizon of the FRW universe yields the corresponding Friedmann equations
\[ \frac{8\pi G}{3} \rho = -16\pi G \int \frac{S'(A)}{A^2} dA, \]
\[ -\pi(\rho + p) = S'(A) \left( \dot{H} - \frac{k}{a^2} \right), \]
where $A$ is the area of the apparent horizon, given by
\[ A = 4\pi \tilde{r}_A^2 = \frac{4\pi}{H^2 + \frac{k}{a^2}}. \]

Substituting the modified entropy [8] into the modified Friedmann equations (14) and (15), we obtain
\[ H^2 + \frac{k}{a^2} = \frac{1}{3M_p^2} (\rho + \rho_x), \]
\[ \dot{H} - \frac{k}{a^2} = -\frac{1}{2M_p^2} (\rho + p + \rho_x + p_x), \]
where $\rho_x$ and $p_x$ are, respectively, the energy density and pressure originating from the GEUP. We interpret $\rho_x$ and $p_x$ as the “dark energy density” and the “dark pressure”, respectively:
\[ \rho_x = \tilde{\alpha} \left( H^2 + \frac{k}{a^2} \right)^2 + M_p^4 \tilde{\beta} \ln \left( \frac{H^2 + \frac{k}{a^2}}{8\pi G M_p^2} \right), \]
\[ \rho_x + p_x = -\frac{2}{3} \left( 2\tilde{\alpha} \left( H^2 + \frac{k}{a^2} \right)^2 + M_p^4 \tilde{\beta} \right) \left( H - \frac{k}{a^2} \right), \]
where $\tilde{\alpha} \equiv \frac{3\alpha}{2\pi G}$ and $\tilde{\beta} \equiv \frac{12\beta}{M_p^4}$. For conciseness, and $\rho_x$ and $p_x$ satisfy the energy conservation law
\[ \dot{\rho}_x + 3H(\rho_x + p_x) = 0. \]

Note that setting $\beta = 0$ in [1] amounts to the GUP model, with the corresponding dark energy density $\sim \tilde{\alpha} H^4$. In this case, the energy density, which scales like $H^4$, cannot explain the acceleration of the present universe. Even if the energy density were proportional to $H^2$, it would not account for the present acceleration either [29, 30]. This is due to the energy density decreasing very quickly. GUP alone can therefore not explain the current accelerating universe. However, by further imposing a maximum length, the energy density acquires a logarithmic term $\sim \ln H$. Conceptually, the exponent of $\ln H$ is nearly zero, so that the change in $\ln H$ is not large (i.e., $\ln H \sim$ constant). It is therefore possible to explain the acceleration of the universe via the $\ln H$ term, derived from the maximum length. Indeed, observations suggest that the dark energy density is almost constant at present.

### III. Cosmology

This section analyzes the evolution equations by assuming that the universe, at each stage of its existence, is dominated by a barotropic perfect fluid with a constant equation-of-state parameter $w_i = \rho_i/p_i (i = m, r)$ and, later, by $\rho_x$. The evolution equation is then
\[ H^2 + \frac{k}{a^2} = \frac{1}{3M_p^2} (\rho_r + \rho_m + \rho_x). \]

#### A. Early-time approximation

It is assumed that, in the early stages of the universe, the energy densities of matter and of the dark energy were negligibly small compared to that of radiation:
\[ \rho_x \ll \rho_r, \quad \rho_m \ll \rho_r. \]

The Hubble parameter $H$ is hence given by
\[ 3M_p^2 H^2 \simeq \frac{\rho_{r,0} a^4}{a^4}, \]
where $\rho_{r,0}$ is the present value of the radiation energy density. At these early stages, the first term dominated in [19], i.e., $\rho_x \sim \tilde{\alpha} H^4$, so that the conditions [23] yield the constraint
\[ \tilde{\alpha} \rho_r \ll 1. \]

Note that big-bang nucleosynthesis (BBN) imposes an upper bound on any source of additional energy density present in the universe at the time of BBN [29, 31], giving the upper limit for that source $\rho_x$ in the GEUP model
\[ \left( \frac{\rho_x}{\rho_r} \right)_{BBN} \lesssim \mathcal{O}(0.1), \]
where the BBN epoch is $a \sim 10^{-10}$. Thus, the conditions [23] and [25] are naturally satisfied by the BBN constraint. According to [19] and [20], the dark energy and pressure are given by
\[ \rho_x \sim \frac{\tilde{\alpha}}{9} H^2, \quad p_x \sim \frac{5\tilde{\alpha}}{27} H^2. \]

We calculate the equation-of-state parameter during the radiation-dominated epoch as
\[ w_x = \frac{p_x}{\rho_x} \sim \frac{5}{3}. \]
B. Late-time approximation

This subsection considers the later stages of the universe, which are dominated by the dark energy:

$$\rho_r \ll \rho_x, \quad \rho_m \ll \rho_x, \quad (29)$$

The Hubble parameter $H$ is given by

$$3M_p^2 H^2 \simeq \rho_x. \quad (30)$$

In this case, the second term is dominant in $\rho_x \sim \beta \ln H$, because the first term $\sim H^4$ decreases rapidly. The dark energy density $\rho_x$ can then be rewritten

$$\rho_x \sim \beta \ln \rho_x. \quad (31)$$

In order to satisfy (31), the dark energy density should be almost constant for a given constant value of $\beta$ during this dark energy dominated epoch. Its value can be obtained as

$$\rho_x \sim -\beta W \left(-\frac{1}{\beta}\right), \quad (32)$$

where $W$ is the Lambert function, defined as the solution to the equation $W(x)e^{W(x)} = x$. We note that, according to (32), the dark energy density becomes essentially static during this dark energy dominated epoch. This makes the corresponding pressure approximately $p_x \simeq -\rho_x$ because $\rho_x \simeq 0$. In addition, we consider only negative values of $\beta$, as estimated from observations, in order that (32) yield a positive dark energy density.

Some comments are in order at this point. Firstly, one can notice that the dark energy density in each epoch, resulting from the GEUP, mostly depends on the minimum length scale $\alpha$ or the maximum length scale $\beta$, according to (27) and (32), respectively. These results are consistent with a previous study [32]. Secondly, Fig. 1 shows that the dark energy density decreases as $a^{-8}$ during the early epoch, and remains almost constant subsequently. Thus, our numerical results are consistent with approximate analytical solutions. Finally, a notable feature is that the equation-of-state parameter $w_x$ dynamically crosses over the value $-1$ (phantom crossing), with its values being slightly negative at present (Fig. 2), consistent with observations [7, 8].

IV. OBSERVATIONAL CONSTRAINTS

This section discusses the parameter estimation for our model, by Markov Chain Monte Carlo (MCMC) simulations and using the most recently available cosmological data, to investigate whether or not it can be distinguished from the ΛCDM model. For this purpose, we used the recent observational data, e.g., from type
Ia supernovae (SN), baryon acoustic oscillations (BAO) imprinted in the large-scale structure of galaxies, cosmic microwave background radiation (CMB), and Hubble parameters \([H(z)]\). The likelihood distributions for the model parameters were derived using the maximum-likelihood method. This method involves exploring the parameter space covered by the vector \(\theta\) in random directions. The choice of parameters that is favored by the observational data is determined by deciding whether to accept or reject a given randomly chosen parameter vector in successive iterations. This decision is made using the probability function \(P(\theta|D) \propto \exp(-\chi^2/2)\), where \(D\) denotes the data and \(\chi^2 = \chi^2_H(z) + \chi^2_{SN} + \chi^2_{BAO} + \chi^2_{CMB}\) is the sum of individual chi-squares for the \(H(z)\), SN, BAO, and CMB data. Full details are provided in Section 4 of [33]. For numerical analysis, it is convenient to rewrite the evolution equation (22) in terms of \(N \equiv \ln \alpha\) as follows:

\[
\dot{H}^2 = \Omega_r h^2 e^{-4N} + \Omega_m h^2 e^{-3N} + \Omega_x h^2 (N) + \Omega_k h^2 e^{-2N},
\]

and

\[
\Omega_x h^2 (N) = \frac{\alpha}{\dot{H}(N)} \left( \frac{\dot{H}^2 - \Omega_k h^2 e^{-2N}}{4\pi M_p^2 h^2} \right),
\]

where we have introduced the following dimensionless quantities:

\[
\dot{H}^2 = \frac{H^2 h^2}{H_0^2}, \quad \Omega_r = \frac{\rho_r,0}{3M_p^2 H_0^2}, \quad \Omega_m = \frac{\rho_m,0}{3M_p^2 H_0^2}, \quad \Omega_k = -\frac{k}{3M_p^2 H_0^2}, \quad \alpha = \frac{\dot{H}_0^2}{3M_p^2 h^2}, \quad \beta = \frac{\beta H_0^2}{3M_p^2 h^2}.
\]

\(H_0\) is the present value of the Hubble parameter, usually expressed as \(H_0 = 100 \ h \ \text{km s}^{-1} \ Mpc^{-1}\), and \(\Omega_r\) and \(\Omega_m\) are, respectively, the radiation- and matter-density parameters in the present universe. For the radiation density, we used \(\Omega_r h^2 = 4.17587 \times 10^{-5}\) for WMAP9 and \(\Omega_r h^2 = 4.17893 \times 10^{-5}\) for PLANCK [33]. Notice that the background dynamics is completely determined by the set of parameters \((\log_{10} \alpha, \beta, \Omega_m, \Omega_k)\). We also need the baryon density parameter \(\Omega_b\) to compare our model with the BAO and CMB data, giving five free parameters in total: \(\theta = (\log_{10} \alpha, \beta, \Omega_b h^2, \Omega_m h^2, \Omega_k h^2)\). It should be noted that the Hubble constant \((H_0)\) is no longer a free parameter because it can be derived from the equations starting from a given set of chosen parameters. We take the priors for the free parameters as follows: \(\log_{10} \alpha = [-50, -37]\), \(\beta = [-0.0020, -0.0010]\), \(\Omega_b h^2 = [0.015, 0.030]\), \(\Omega_m h^2 = [0.11, 0.15]\) and \(\Omega_k h^2 = [-0.1, 0.1]\).

We explored the allowed parameter ranges of our dark energy model by using the recent observational data within the MCMC method. In the calculation, we used \(\log \alpha, \beta, \Omega_m h^2, \Omega_b h^2\) and \(\Omega_k h^2\) as free parameters. Table 2 summarizes the parameter mean values and 68% confidence limits, and Fig. 3 shows the marginalized likelihood distributions of the parameters. We can see that the result obtained with the Planck data show slightly tighter constraints on the model parameters. Parameter \(\alpha\), within the BBN constraint (26) \((\log \alpha < -37)\), is uniformly distributed because it is uncorrelated with any of the other model parameters. In other words, the parameter estimation results are relatively unaffected by the presence of the minimum length scale \(\alpha\). The constraints on \(\log \alpha\) in Table 2 are therefore only cited as BBN. The best-fit locations in the parameter space are

\[
\begin{align*}
(\Omega_m h^2, \Omega_b h^2, \Omega_k, \log_{10} \alpha, \beta) & = (0.1359, 0.02447, -0.0037, -38.3, -0.00121), \\
(\Omega_m h^2, \Omega_b h^2, \Omega_k, \log_{10} \alpha, \beta) & = (0.1438, 0.02395, -0.0006, -41.4, -0.00119),
\end{align*}
\]

with a minimum chi-square of \(\chi^2_{\text{min}} = 584.278\) for the \(H(z) + \text{SN+BAO+WMAP9}\), and

\[
\begin{align*}
(\Omega_m h^2, \Omega_b h^2, \Omega_k, \log_{10} \alpha, \beta) & = (0.1359, 0.02447, -0.0037, -38.3, -0.00121), \\
(\Omega_m h^2, \Omega_b h^2, \Omega_k, \log_{10} \alpha, \beta) & = (0.1438, 0.02395, -0.0006, -41.4, -0.00119),
\end{align*}
\]

with \(\chi^2_{\text{min}} = 590.312\) for \(H(z) + \text{SN+BAO+PLANCK}\).
To assess the goodness of fit of our model, Table I displays the parameter constraints for the $\Lambda$CDM model and lists, for each case, the value of the minimum reduced chi-square ($\chi^2_\nu$). This is defined as $\chi^2_\nu = \chi^2_{\nu,\text{min}}/\nu$, where $\nu = N - n - 1$ is the number of degrees of freedom and $N$ and $n$ are the numbers of data points and free model parameters, respectively. In our analysis, $N = 621$, and $n = 5$ for our model and $n = 4$ for the $\Lambda$CDM model. Our model fits the observational data slightly better, with smaller values of $\chi^2_{\nu,\text{min}}$ and $\chi^2_\nu$. We note that, for our model to be compatible with observations, log$_{10}$ $\hat{\alpha}$ must be smaller than $\sim -37$ (BBN constraint) and $\beta$ should be close to $\sim -10^{-3}$. Identifying the unknown maximum length scale $L_x$ with the Hubble radius $1/H_0$ of the current universe, $\chi^2 (L_x)$ gives

$$\alpha \lesssim 10^{81}, \quad \beta = \hat{\beta}/2 \sim -10^{-2}.$$  

(38)

The BBN does not impose a strong constraint on the minimum length scale $\alpha$ compared with other results (i.e., $\alpha \lesssim 10^{34}$ from the electroweak length scale $\chi^2_\nu$), $\alpha \lesssim 10^{36}$ from measurement of the Lamb shift $\chi^2_\nu$, and $\alpha \lesssim 10^{50}$ from measurement of Landau levels $\chi^2_\nu$.

V. CONCLUSION

To conclude, we investigated the cosmological implications of minimum and maximum lengths in the FRW universe, giving special attention to the potential significance of these lengths in relation with the dark energy. We found that the theory is consistent with the long period of acceleration which the universe is presently undergoing, a phase that closely mimics the $\Lambda$CDM model, in which the acceleration of the universe is due to the existence of the maximum length scale $\beta$ governed by the relation $\Lambda \sim -\beta^{-1}W(-\beta^{-1})$. A detailed numerical analysis, comparing various available data, predicts that the maximum length, governed by $\beta$, is of the order of $\sim 10^{-2}$, and the minimal length, governed by $\alpha$, is of the order of $\lesssim 10^{81}$.

Some interesting properties of cosmological evolution within the GEUP framework arise from the existence of the minimum and maximum lengths. The minimum length introduces a correction term in the Friedmann equation, with a corresponding energy density that scales as $\sim \hat{\alpha}H^4$. This is likely to have played an important role predominantly in the early universe. On the other hand, the energy density arising from the maximum length is given, in practice, by an intriguing relation $\sim \hat{\beta}\ln H$, which began significant mostly in the later universe. With regard to dynamics, the $\hat{\beta}\ln H$ term increases very slowly (as compared to a power-law behavior for $H$) and eventually becomes dominant in the present epoch. The minimal length scale $\alpha$ does not affect the current acceleration of the universe as long as the BBN constraint is satisfied. Even setting $\alpha = 0$ can explain the present acceleration of the universe via the existence of the maximum length only, but its equation-of-state parameter will be always in phantom phase as the universe expands. However, by imposing a minimal length, the equation-of-state parameter starts from $w_\gamma \simeq 5/3$ and cross the phantom divide in the intermediate state between the radiation- and matter-dominated epochs (Fig. 2), and the equation-of-state parameter at present is in the phantom phase, as allowed by observations.

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