N-dimensional Auto-Bäcklund Transformation

and Exact Solutions to n-dimensional Burgers System

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Abstract: N-dimensional Bäcklund transformation (BT), Cole-Hopf transformation and Auto-Bäcklund transformation (Auto-BT) of n-dimensional Burgers system are derived by using simplified homogeneous balance (SHB). By the Auto-BT, another solution of the n-dimensional Burgers system can be obtained provided that a particular solution of the Burgers system is given. Since the particular solution of n-dimensional Burgers system can be given easily by the Cole-Hopf transformation, then using the Auto-BT repeatedly, more solutions of n-dimensional Burgers system can be obtained successively.

Keywords: n-dimensional Burgers system; n-dimensional BT; n-dimensional Cole-Hopf transformation; n-dimensional Auto-BT; SHB; exact solution

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1. Introduction

The classical Burgers equation is the simplest nonlinear model in fluid dynamics[1] and has been widely used in the surface perturbations, acoustical waves, electromagnetic waves, density waves, population growth, magnetohydrodynamic waves[2-4], etc. The n-dimensional Burgers system in the form[5-7]

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} - \mu \Delta u_i = 0, \quad i = 1, 2, \cdots, n, \quad \Delta \equiv \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}, \quad (1)$$

with an irrotational condition:

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial u_j}{\partial x_i}, \quad i, j = 1, 2, \cdots, n, \quad i \neq j,$$

is an important generalization of the Burgers equation and was investigated recently by Yang Chen etc. In paper [5], where the authors have shown that the n-dimensional Burgers system (1) can be transformed into n-dimensional linear heat equation by a n-dimensional Cole-Hopf transformation.

In the present paper we will study further n-dimensional Burgers system (1) by using the idea of SHB[8], which means that an undetermined function $f = f(\varphi)$ and its derivatives

$$f_x = f'(\varphi)\varphi_x, \cdots,$$

that appearing in the original homogeneous balance(HB)[9-12], are replaced by a
logarithmic function $A \ln(\varphi)$ and its derivatives $A(\ln \varphi)' = A \frac{\varphi'}{\varphi}$, ..., respectively, where constant $A$ and the function $\varphi = \varphi(x,t)$ are to be determined.

The aim of this paper is to derive the n-dimensional BT for n-dimensional Burgers system (1), and from the BT to reason out the n-dimensional Cole-Hopf transformation and the n-dimensional Auto-BT for the n-dimensional Burgers system (1). By the Auto-BT, another exact solution of the Burgers system can be obtained provided a particular solution of it is given. Hence, once the particular solution is given, using the Auto-BT repeatedly, more exact solutions of the Burgers system can be obtained successively.

2. Derivation of n-dimensional Auto-BT

Considering the homogeneous balance between $u_i \frac{\partial u_j}{\partial x_i}$ and $\frac{\partial^2 u_j}{\partial x_i^2}$ in system (1) ($m + n + 1 = n + 2 \rightarrow m = 1$), according to SHB, we can suppose that the solution of the n-dimensional Burgers system (1) is of the form

$$u_i = A(\ln \varphi)' + u_i^{(0)} = A \frac{\varphi_x}{\varphi} + u_i^{(0)}, i = 1, 2, \cdots, n$$

(2)

where $u_i^{(0)} = u_i^{(0)}(x_1, x_2, \cdots, x_n, t), i = 1, 2, \cdots, n$, be a given particular solution of n-dimensional Burgers system (1), i.e.

$$\frac{\partial u_i^{(0)}}{\partial t} + \sum_{j=1}^{n} u_j^{(0)} \frac{\partial u_i^{(0)}}{\partial x_j} - \mu \Delta u_i^{(0)} = 0, i = 1, 2, \cdots, n$$

(3)

with the irrotational condition $\frac{\partial u_i^{(0)}}{\partial x_j} = \frac{\partial u_j^{(0)}}{\partial x_i}, i, j = 1, 2, \cdots, n, i \neq j$, constant $A$ and the function $\varphi = \varphi(x_1, x_2, \cdots, x_n, t)$ are to be determined later.

Substituting (2) into the left hand side of system (1), noticing that

$$\frac{\partial u_i}{\partial t} = A \frac{\partial}{\partial x_i} \left( \frac{\varphi_x}{\varphi} \right) + \frac{\partial u_i^{(0)}}{\partial t},$$

$$u_j \frac{\partial u_i}{\partial x_j} = \frac{1}{2} A^2 \frac{\partial}{\partial x_j} \left( \frac{\varphi_x}{\varphi} \right)^2 + A \frac{\partial}{\partial x_j} \left( u_j^{(0)} \frac{\varphi_x}{\varphi} \right) + u_j^{(0)} \frac{\partial u_i^{(0)}}{\partial x_j},$$

$$\Delta u_i = \frac{\partial}{\partial x_i} \left( \Delta \frac{\varphi}{\varphi} - \sum_{j=1}^{n} \left( \frac{\varphi_x}{\varphi} \right)^2 \right) + \Delta u_i^{(0)},$$

and using (3), we have
\[
\frac{\partial u_i}{\partial t} + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} - \mu \Delta u_i = A \frac{\partial}{\partial x_i} \left[ \varphi_i + \sum_{j=1}^{n} u_j^{(0)} \frac{\partial \varphi}{\partial x_j} - \mu \Delta \varphi \right] + \left( \frac{1}{2} A + \mu \right) \left( \sum_{j=1}^{n} \frac{\varphi_{x_j}}{\varphi} \right)^2 ,
\]

\[i = 1, 2, \cdots, n . \] (4)

In order to find \( A \), setting the coefficient of \( \sum_{j=1}^{n} \left( \frac{\varphi_{x_j}}{\varphi} \right)^2 \) to zero, yields

\[
\frac{1}{2} A + \mu = 0 , \text{ which implies that } A = -2\mu . \] (5)

Using (5) the expression (2) becomes

\[
u_i = -2\mu \frac{\varphi_{x_i}}{\varphi} + u_i^{(0)}, i = 1, 2, \cdots, n \] (6)

and the expression (4) becomes

\[
\frac{\partial u_i}{\partial t} + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} - \mu \Delta u_i = -2\mu \frac{\partial}{\partial x_i} \left[ \varphi_i + \sum_{j=1}^{n} u_j^{(0)} \frac{\partial \varphi}{\partial x_j} - \mu \Delta \varphi \right] = 0, \; i = 1, 2, \ldots, n \] (7)

provided the function \( \varphi = \varphi(x_1, x_2, \cdots, x_n, t) \) satisfies the equation

\[
\varphi_i + \sum_{j=1}^{n} u_j^{(0)} \frac{\partial \varphi}{\partial x_j} - \mu \Delta \varphi = 0 . \] (8)

Based on (6), (7) and (8), we come to the important conclusion that if \( u_i^{(0)} = u_i^{(0)}(x_1, x_2, \cdots, x_n, t), i = 1, 2, \cdots, n \), be a particular solution of the n-dimensional Burgers system (1) and the function \( \varphi = \varphi(x_1, x_2, \cdots, x_n, t) \) be a nonzero solution of Eq.(8), then the nonlinear transformation (6) satisfies exactly the n-dimensional Burgers system (1). The transformation (6) with Eq.(8) is called a BT of the n-dimensional Burgers system (1).

In order to find exact solution of the n-dimensional Burgers system (1), we consider two particular cases of the n-dimensional BT:

**The first case is that**

When \( u_i^{(0)} = 0, i = 1, 2, \cdots, n \), then (6), (7) and (8) become

\[
u_i = -2\mu \frac{\varphi_{x_i}}{\varphi}, i = 1, 2, \cdots, n , \] (6)'

\[
\frac{\partial u_i}{\partial t} + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} - \mu \Delta u_i = -2\mu \frac{\partial}{\partial x_i} \left[ \varphi_i - \mu \Delta \varphi \right] = 0, \; i = 1, 2, \cdots, n , \] (7)'}
\[ \varphi_i - \mu \Delta \varphi = 0, \quad (8)' \]
respectively.

Based on \((6)', (7)', \) and \((8)'\), we can say that if \( \varphi = \varphi(x_1, x_2, \ldots, x_n, t) \) is a solution of the n-dimensional linear heat equation \((8)'\), then the n-dimensional nonlinear transformation \((6)'\) satisfies exactly the n-dimensional Burgers system \((1)\). Thus \((6)'\) with \((8)'\) have made up the n-dimensional Cole-Hopf transformation for the n-dimensional Burgers system \((1)\), which coincides with that obtained in paper \([5]\).

**The second case is that**

When \( \varphi = u_i^{(0)}, i = 1, 2, \ldots, n \) then \((6), (7)\) and \((8)\) become

\[ u_i = -2\mu \left( \frac{u_i^{(0)}}{u_j^{(0)}} \right) + u_j^{(0)}, i = 1, 2, \ldots, n. \quad (6)^* \]

\[ \frac{\partial u_i}{\partial t} + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} - \mu \Delta u_i = -2\mu \frac{\partial}{\partial x_i} \left( \frac{\partial u_i^{(0)}}{\partial t} + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} - \mu \Delta u_i^{(0)} \right) = 0, i = 1, 2, \ldots, n. \quad (7)^* \]

\[ \frac{\partial u_i^{(0)}}{\partial t} + \sum_{j=1}^{n} u_j^{(0)} \frac{\partial u_i^{(0)}}{\partial x_j} - \mu \Delta u_i^{(0)} = 0, i = 1, 2, \ldots, n. \quad (8)^* \]
respectively.

Based on \((6)^*, (7)^*, \) and \((8)^*\), we can say that if \( u_i^{(0)} = u_i^{(0)}(x_1, x_2, \ldots, x_n, t), \)

\( i = 1, 2, \ldots, n, \) is a given particular solution of n-dimensional Burgers system \((1)\), then the n-dimensional nonlinear transformation \((6)^*\) is another solution of it. Thus \((6)^*\) with \((8)^*\) have made up the n-dimensional Auto-BT for the n-dimensional Burgers system \((1)\).

In the next section, we will use the Auto-BT to get more solutions of the n-dimensional Burgers system \((1)\), when a particular solution of the system is given.

### 3. Application of Auto-BT

It is well-known that the n-dimensional linear heat equation \((8)'\) has a solution

\[ \varphi(x_1, x_2, \ldots, x_n, t) = \frac{1}{(2\sqrt{\pi \mu t})^n} \exp \left[ -\frac{\sum_{j=1}^{n} x_j^2}{4\mu t} \right], \quad (9) \]

which is denoted by
\[ E(x_1, x_2, \cdots, x_n, t) = \frac{1}{(2\sqrt{\pi})^n} \exp \left[ -\frac{\sum_{j=1}^{n} x_j^2}{4\mu t} \right]. \]

Substituting (9) into the n-dimensional Cole-Hopf transformation (6)', we have an exact solution of n-dimensional Burgers system (1) as follows

\[ u_i^{(0)} = -2\mu \frac{E_{x_i}}{E} = \frac{x_i}{t}, \quad i = 1, 2, \cdots, n, \quad \text{(10)} \]

where

\[ E_{x_i} = \frac{1}{(2\sqrt{\pi})^n} \left( -\frac{x_i}{2\mu t} \right) \exp \left[ -\frac{\sum_{j=1}^{n} x_j^2}{4\mu t} \right]. \]

Substituting (10) which is called a seed solution into the n-dimensional auto-BT (6)**, we have another solution of n-dimensional Burgers system (1) as follows

\[ u_i^{(1)} = -2\mu \frac{u_i^{(0)}}{u_i^{(0)}} + u_i^{(0)} = \frac{x_i^2 - 2\mu t}{x_i t}, \quad i = 1, 2, \cdots, n. \quad \text{(11)} \]

Substituting (11) instead of \( u_i^{(0)} \) in (6)**, into (6)** we have another new solution of n-dimensional Burgers system (1), as follows

\[ u_i^{(2)} = -2\mu \frac{u_i^{(1)}}{u_i^{(1)}} + u_i^{(1)} = \frac{x_i^3 - 6\mu x_i t}{t(x_i^2 - 2\mu t)}, \quad i = 1, 2, \cdots, n. \quad \text{(12)} \]

Substituting (12) instead of \( u_i^{(0)} \) in (6)**, into (6)** we have another new solution of n-dimensional Burgers system (1), as follows

\[ u_i^{(3)} = -2\mu \frac{u_i^{(2)}}{u_i^{(2)}} + u_i^{(2)} = \frac{x_i^4 - 12\mu x_i^2 t + 12\mu^2 t^2}{x_i^2 t - 6\mu x_i^2 t^2}, \quad i = 1, 2, \cdots, n. \quad \text{(13)} \]

\[ \cdots \]

and so on. The solutions \( u_i^{(k)} (i = 1, 2, \cdots, n, k = 0, 1, 2, \cdots) \) are all rational functions.

In the above instance, it seems that the component \( u_i \) of solution for n-dimensional Burgers system (1) depends upon the corresponding variable \( x_i \) and \( t \) only. For simplicity, the solutions (11), (12) and (13) are considered as the functions of variables \( x, t \) and \( u \) and are shown in Fig.(1).

However, we have other examples that the component \( u_i \ (i = 1, 2, \cdots, n) \) of solution for n-dimensional Burgers system depends upon all variables \( x_1, x_2, \cdots, x_n, t \). For example, the
n-dimensional heat equation (8)' has a solution
\[ \varphi = 1 + e^{\omega} \cos \eta, \quad \omega = -\mu \left( \sum_{j=1}^{n} \alpha_j^2 \right), \quad \eta = \sum_{j=1}^{n} \alpha_j x_j, \quad \alpha_j \text{-arbitrary costant.} \]  

(14)

Substituting (14) into n-dimensional Cole-Hopf transformation (6)', we have a solution of n-dimensional Burgers system (1)
\[ u_i^{(0)} = 2 \mu \alpha_i \frac{e^{\omega} \sin \eta}{1 + e^{\omega} \cos \eta}, \quad \omega = -\mu \left( \sum_{j=1}^{n} \alpha_j^2 \right), \quad \eta = \sum_{j=1}^{n} \alpha_j x_j, \quad i = 1, 2, \ldots, n. \]  

(15)

Substituting (15) as a seed solution into n-dimensional Auto-BT (6)'', yields another solution of n-dimensional Burgers system (1)
\[ u_i^{(1)} = 2 \mu \alpha_i \frac{e^{\omega} \sin^2 \eta - \cos \eta - e^{\omega}}{(1 + e^{\omega} \cos \eta) \sin \eta}, \quad i = 1, 2, \ldots, n, \]  

(16)

\[ \omega = -\mu \left( \sum_{j=1}^{n} \alpha_j^2 \right), \quad \eta = \sum_{j=1}^{n} \alpha_j x_j, \quad \alpha_j \text{-arbitrary costants.} \]

The component \( u_i^{(1)} \) in (16) depends upon all variables \( x_1, x_2, \ldots, x_n \) and \( t \); and it is a periodic function in variable \( \eta \), which is a linear combination of space variables \( (x_1, x_2, \ldots, x_n) \).

For simplicity, the solutions (15) and (16) are considered as the functions of variables \( \eta, t \) and \( u \) and are shown in Fig. (2).
\[
\begin{align*}
    u_i^{(0)} &= -2\mu\alpha_i \frac{e^{\omega_i} \sinh \eta_i}{1 + e^{\omega_i} \cosh \eta_i}, \quad i = 1, 2, \ldots, n. \\
    \omega_i &= \mu \left( \sum_{j=1}^{n} \alpha_j^2 \right), \quad \eta_i = \sum_{j=1}^{n} \alpha_j x_j.
\end{align*}
\]

where \( \omega_i = \mu \left( \sum_{j=1}^{n} \alpha_j^2 \right), \quad \eta_i = \sum_{j=1}^{n} \alpha_j x_j \).

Substituting (17) as a seed solution into n-dimensional Auto-BT (6)* yields a new solution of n-dimension Burger system (1)

\[
\begin{align*}
    u_i^{(1)} &= -2\mu\alpha_i \frac{e^{\omega_i} \sinh^2 \eta_i + \cosh \eta_i \sinh \eta_i}{\left(1 + e^{\omega_i} \cosh \eta_i \right) \sinh \eta_i}, \quad i = 1, 2, \cdots, n, \\
    \omega_i &= \mu \left( \sum_{j=1}^{n} \alpha_j^2 \right), \quad \eta_i = \sum_{j=1}^{n} \alpha_j x_j, \quad \alpha_j -\text{arbitrary costants.}
\end{align*}
\]

The component \( u_i^{(1)} \) in (18) depends upon all variables \( x_1, x_2, \ldots, x_n \) and \( t \) as well, for simplicity, the solutions (15) and (16) are considered as the functions of variables \( \eta, t \) and \( u \) and are shown in Fig.(3).

![Fig. 3. Plots of the solutions (17) and (18) with \( \mu = 1, \omega = 1, \alpha = 1 \).](image)

4. Conclusion

Using the SHB we have derived out the n-dimensional BT for the n-dimensional Burger system. The BT involves a given particular solution \( u_i^{(0)} (i = 1, 2, \ldots, n) \) of the n-dimensional Burger system as well as the solution \( \phi = \phi(x_1, x_2, \ldots, x_n, t) \) of Eq.(8). If \( u_i^{(0)} (i = 1, 2, \ldots, n) \) then Eq.(8) becomes classical n-dimensional heat equation, the BT becomes the well known Cole-Hopf transformation; if \( u_i^{(0)} \neq 0 \), and taking \( \phi = u_i^{(0)} \), then Eq.(8) becomes right the n-dimension Burgers system for \( u_i^{(0)} \). Thus the BT becomes the Auto-BT of the n-dimension Burgers system. Using Cole-Hopf transformation, it is easy to get a particular solution of the n-dimension Burgers system. Taking the particular solution as a seed solution of n-dimension Burgers system and using Auto-BT repeatedly, more solutions of the n-dimension Burgers system can be obtained successively.
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References
[1] J. M. Burgers, Mathematical examples illustrating relations occurring in the theory of turbulent fluid motion, Verhand. Kon. Neder. Akad. Wetenschappen, Afd. Natuurkunde, Eerste Sect. 17 (1939) 1-53.
[2] R. A. Kraenkel, J. G. Pereira, M.A. Manna, Nonlinear surface-wave excitations in the Bernard Marangoni system, Phys. Rev. A 46 (1992)4786-4790.
[3] M. J. Ablowitz, S. De Lillo, The Burgers equation under deterministic and stochastic forcing, Physica D 92 (1996) 245-259.
[4] V. Gurarie, A. Migdal, Instantons in the Burgers equation, Phys. Rev. E 54(1996) 4908-4914.
[5] Y. Chen, E. G. Fan, M. W. Yuen, The Hopf-Cole transformation, topological solitons and multiple fusion solutions for the n-dimensional Burgers system, Physics Letters A, 380(2016)9-14.
[6] S. F. Shen, J. Zhang and Z. L. Pan, New Exact Solution of (N+1)-Dimensional Burgers System, Commun. Theor. Phys. 43 (2005) 389–390.
[7] C. H. Chan, M. Czubak, Regularity of solutions for the critical N-dimensional Burgers’ equation, Ann. I. H. Poincaré – AN 27 (2010) 471–501.
[8] M. L. Wang and X. Z. Li, Simplified Homogeneous balance method and its application to the Whitham-Broer-Kaup model equations, Journal of Applied Mathematics and Physics, 2(2014)823-827.
[9] M. L. Wang, Solitary wave solutions for variant Boussinesq equations, Physics Letters A 199(1995)169-172.
[10] M. L. Wang, Exact solutions for a compound KdV-Burgers equation, Physics Letters A, 213 (1996) 279-287.
[11] M. L. Wang,Y. B. Zhou, Z. B. Li, Applications of a homogeneous balance method to exact solutions of nonlinear equations in mathematicl physics, Physics Letters A, 216( 1996) 67-75.
[12] J. L. Zhang, Y. M. Wang, M. L. Wang, Z. D. Fang, New Application of the Homogeneous Balance Principle. Chinese Physics, 12(2003) 245-250.