On the $RO(Q)$-graded coefficients of Eilenberg–MacLane spectra

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Received: 7 June 2021 / Accepted: 20 August 2022
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Abstract
Let $Q$ denote the cyclic group of order two. Using the Tate diagram we compute the $RO(Q)$-graded coefficients of Eilenberg–MacLane $Q$-spectra and describe their structure as modules over the coefficients of the Eilenberg–MacLane spectrum of the Burnside Mackey functor. If the underlying Mackey functor is a Green functor, we also obtain the multiplicative structure on the $RO(Q)$-graded coefficients.

Keywords equivariant homotopy theory · equivariant homotopy groups · $RO(G)$ · graded homotopy groups

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Communicated by Daniel Davis.

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Published online: 05 October 2022
1 Introduction

Let $G$ be a finite group. In $G$-equivariant topology the role of ordinary cohomology is played by Bredon cohomology [2]. Whilst easy to define, making computations in Bredon theories is more complicated than in their non-equivariant analogues. Firstly, the coefficients of Bredon theories are of the form of a functor from the orbit category of $G$ to the category of abelian groups. Secondly, Bredon theories are more naturally $RO(G)$-graded (graded over the representation ring of $G$) rather than graded over the integers.

As shown in [15], a $\mathbb{Z}$-graded Bredon theory extends to an $RO(G)$-graded one if its coefficients are of the form of a Mackey functor. As in non-equivariant topology, the Bredon homology/cohomology with coefficients in a Mackey functor $M$ is represented by the Eilenberg–MacLane $G$-spectrum $HM$. Spectra of this form appear in various contexts in equivariant topology. For example, equivariant Eilenberg–MacLane spectra are 0-slices in the slice spectral sequence [12, Proposition 4.50].

The difficulties of computations in $RO(G)$-graded Bredon theories may be seen in calculations of $HM_G^{\ast} := \pi_\ast^G(HM)$, the $RO(G)$-graded $G$-homotopy groups of $HM$ (a five-pointed star indicates $RO(G)$-grading). This is equivalent to computing the $RO(G)$-graded Bredon homology and cohomology of a point with coefficients in $M$ and thus we will refer to $HM_G^n$ as the coefficients of $HM$. The groups $HM_G^n$ are zero for $n \in \mathbb{Z}$ unless $n = 0$, which resembles the non-equivariant case. However, if $V$ is not a trivial representation then $HM_G^V$ might be non-zero.

In this paper we use the Tate diagram to compute the $RO(Q)$-graded coefficients of Eilenberg–MacLane $Q$-spectra, where $Q$ is the cyclic group of order 2. We do this in three instances: as an $RO(Q)$-graded abelian group, as a module over the coefficients of the Eilenberg–MacLane spectrum associated to the Burnside Mackey functor, and finally as an $RO(Q)$-graded ring, when appropriate.

Tate diagram

The idea behind the Tate diagram is to decompose a $Q$-spectrum $X$ into computationally simpler pieces:

1. Borel completion $X^h$;
2. free $Q$-spectrum $Xh$;
3. singular spectrum $X^\Phi$;
4. Tate spectrum $X^t$,

which are connected by the following commutative diagram:

$$
\begin{array}{cccc}
X_h & \longrightarrow & X & \longrightarrow & X^\Phi \\
\downarrow & & \downarrow & & \downarrow \\
X_h & \longrightarrow & X^h & \longrightarrow & X^t.
\end{array}
$$

What makes computations by the Tate diagram feasible is that the rows are cofibre sequences and the right-hand square (known as the Tate square) is a homotopy pull-
back. Moreover, the coefficients of the spectra appearing in the bottom row may be computed by the homotopy orbit and homotopy fixed point and Tate spectral sequences, respectively. The foundational work on the Tate diagram is [9], where all of the details are discussed.

The computational strength of the Tate diagram has been proven in various contexts: for example Greenlees used it in [8] to compute the coefficients of the Eilenberg–MacLane $Q$-spectrum $H \mathbb{Z}$ as an $RO(Q)$-graded ring, Greenlees and Meier computed the coefficients of K-theory with reality $K \mathbb{R}$ in [10, Section 11] and Hu and Kriz used it to compute the coefficients of $H \mathbb{F}_p$ and the $Q$-equivariant Steenrod algebra in [14, Section 6]. It was also used to compute the coefficients of $H \mathbb{Z}$ over groups $C_p$ with $p$ prime by Zeng [18].

**$RO(Q)$-graded abelian group structure**

The first step is describing the $RO(Q)$-graded abelian group structure of $HM_\ast^Q$. We show that it is fully determined by the underlying Mackey functor $M$. This structure is given in Theorem 6.1, which can be informally stated as follows:

**Theorem** The $RO(Q)$-graded abelian group structure of $HM_\ast^Q$ may be presented by Fig. 1, where:

1. Every lattice point represents a $Q$-representation, the horizontal axis describes multiplicity of the trivial $Q$-representation and the vertical axis describes multiplicity of the sign representation.
2. The empty circle at the position $(0, 0)$ is the abelian group $M(\mathbb{Q}/\mathbb{Q})$.
3. The values of $HM_\ast^Q$ lying on the $x = 0$ axis are subgroups of $M(\mathbb{Q}/\mathbb{Q})$ given by the kernel of the restriction and the cokernel of the transfer.
4. The full dots in positions $(1, −1)$ and $(-1, 1)$ are $\mathbb{Z}[Q]$-submodules of $M(\mathbb{Q}/e)$ given by the kernel of the transfer and the cokernel of the restriction respectively, whereas the values lying on the red/blue lines above/below them are their subquotients.
5. The values lying in blue and red areas are respectively the group cohomology and homology of $Q$ with coefficients in $M(\mathbb{Q}/e)$.
6. All other values are zero.

**$H\underline{A}_\ast^Q$-module structure**

The category of $Q$-Mackey functors has a symmetric monoidal structure and commutative monoids with respect to this structure are called Green functors. If $M$ is a Green functor, the $Q$-spectrum $HM$ is a (naive) commutative ring $Q$-spectrum and its homotopy groups form an $RO(Q)$-graded commutative ring. The most fundamental example of a Green functor is the Burnside Mackey functor $\underline{A}$.

Every $Q$-Mackey functor $M$ is a module over $\underline{A}$. Since taking the Eilenberg–MacLane spectrum is a lax monoidal functor (see [9, Chapter 8]), the spectrum $HM$ is a module over $H\underline{A}$ and $HM_\ast^Q$ is a module over the $RO(Q)$-graded commutative ring $H\underline{A}_\ast^Q$.  

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We describe $H_{\mathbb{A}_*}^Q$ in Sect. 7. Its multiplicative structure is determined by two elements - $a$ and $u$. The first is an Euler class associated to the inclusion $S^0 \to S^\sigma$, whereas the second is the generator of $H_{\mathbb{A}_{2-2\sigma}}^Q$ and corresponds to the generator of $H_{\mathbb{A}_2^Q}(S^{2\sigma})$.

The action of $H_{\mathbb{A}_*}^Q$ on $HM_*^Q$ may be informally stated as follows:

**Observation** The $H_{\mathbb{A}_*}^Q$-module structure of $HM_*^Q$ is determined by the action of 3 elements - $a$, $u$ and $\omega$, where the latter is the class of $Q/e$ in the Burnside ring.

The action by $a$ on $HM_*^Q$ may be easily derived from the cofibre sequence

$$Q_+ \to S^0 \to S^\sigma.$$

However, a description of the action of $u$ requires more work. The detailed analysis of this action is possible due to the Tate diagram and the connection between $HM$ and its Borel completion. The details are given in Sect. 8. The action of the element $\omega$ is discussed in Sect. 9.

**$RO(Q)$-graded ring structure**

Finally, we describe the multiplicative structure of $HM$ when $M$ is a Green functor.

In Sects. 8 and 9 we show that most of this structure may be derived from the $H_{\mathbb{A}_*}^Q$-module structure. The first issue that we encounter is the graded commutativity. The sign rule for commutativity in $RO(Q)$-graded rings involves units in the Burnside
ring - thus not only \(-1\), but also \(1 - \omega\), where \(\omega\) is the class of \(Q/e\). For the precise statement, see Observation 9.1.

In Sect. 9 we show the following Theorem:

**Theorem (9.3)** If \(M\) is a Green functor then \(H M^Q\) is a strictly commutative ring, i.e., all signs coming from the graded commutativity rule are trivial.

Finally, at the end of Sect. 9 in Observation 9.8, we give a recipe for describing the multiplicative structure of \(H M^Q\) for any Green functor \(M\).

**Observation** If \(M\) is a \(Q\)-Green functor then the multiplicative structure of \(H M^Q\) is fully determined by its \(H A^Q\)-module structure and relations between elements of degrees \(1 - \sigma, \sigma - 1, 3 - 3\sigma, \) and \(3\sigma - 3\). These relations may be derived from the induced map of \(H M \to H M^h\).

The procedure of obtaining multiplicative structure is illustrated with a wide array of examples in Sects. 7 and 10.

**Contribution of the paper and related work**

Computations of coefficients of Eilenberg–MacLane spectra over \(Q\) already have a long history. Based on unpublished work of Stong, Lewis computed \(H A^Q\), where \(A\) is the Burnside Mackey functor for \(Q\) in [16, Section 2]. Calculations for the constant Mackey functor \(F_2\) by Caruso can be found in [4] and by Hu-Kriz in [14, Proposition 6.2]. The computations for the constant Mackey functor \(Z\) may be found in Dugger’s work [5, Appendix B].

The full \(RO(Q)\)-graded Mackey functor valued coefficients of \(H M\) for any Mackey functor \(M\) was given by Ferland in [7] and may also be found in Ferland–Lewis [6, Chapter 8]. Their computations are based on the cofibre sequence

\[
Q_+ \to S^0 \to S^\sigma.
\]

This sequence allows one to describe attaching maps of free cells, and thus to compute the homology of any representation sphere. Therefore this method may be described as the “cell method”. While being intuitive and allowing the computation of the \(RO(Q)\)-graded abelian group structure, this approach does not provide efficient methods of describing the multiplicative structure. We note that Ferland’s computations work for all cyclic groups of prime order, but in this paper we will restrict our attention to the group of order 2. From Ferland’s work one can derive the \(RO(Q)\)-graded abelian group structure and \(H A^Q\)-module structure of \(H M^Q\).

The aim of this paper is to show the computational strength of the Tate diagram on the example of computations of coefficients of equivariant Eilenberg–MacLane spectra. As demonstrated by Greenlees in his computations of \(HZ^Q\) in [8], the method based on the Tate diagram gives not only a good insight into the abelian group structure, but also into the multiplicative structure. In this paper we generalise Greenlees’s results to all \(Q\)-Mackey and Green functors. Therefore the main contribution of this paper
is the description of the multiplicative structure of $HM_Q^\ast$ and showing that the Tate diagram gives an algorithmic way to describe this structure.

The other strong computational technique in equivariant homotopy theory is the slice spectral sequence of Hill–Hopkins–Ravenel [12]. The idea behind it is a decomposition of a $G$-spectrum into a tower similar to the Postnikov tower, but retaining more equivariant information. However, one of the challenges that one may face while using the slice spectral sequence is to determine the filtration quotients. Therefore it needs additional techniques to support the calculations. An example of this is the fact that for any $Q$-spectrum $X$ the starting input for the slice spectral sequence is given by the spectrum $\Sigma^{-1}H\pi_-(X)$, the desuspension of the Eilenberg–MacLane spectrum of $\pi_-(X)$. In this paper we demonstrate that the computations based on the Tate diagram for simple equivariant objects, such as $HM$, may be carried out in a neat and algorithmic way and thus provide a useful method of doing auxiliary calculations for the slice spectral sequence.

Additionally, it is worth noting that equivariant homotopy theory has a rather small repository of computational examples and is in a constant need for developing methods of doing calculations. Therefore an auxiliary contribution of this paper is adding another piece to this development.

We note here that the “full information” is given by the Mackey functor valued $RO(Q)$-graded coefficients $HM_Q^\ast$, as given for example in [16]. However, the main computational effort is computing the $Q/Q$ level of the $RO(Q)$-graded Mackey functor $HM_Q^\ast$ and this is where the Tate method gives a hand. Therefore we focus our attention in this paper on computing $HM_Q^0$. The rest of the Mackey functor structure of $HM_Q^\ast$ may be easily deduced.

1.1 Notation and conventions

Throughout the whole paper $Q$ denotes the group of order 2 and $\gamma$ its non-trivial element. We denote the norm element of the group ring $\mathbb{Z}[Q]$ by $N$, i.e., $N = 1 + \gamma$. The Burnside ring of $Q$ is denoted by $A(Q)$ and we write $\omega$ for the the class of $Q/e$ in this ring.

If $M$ is a $\mathbb{Z}[Q]$-module, we denote its $Q$-fixed points by $M^Q$. We also put

$$M_Q = \frac{M}{(1 - \gamma)M}.$$

We denote $N$-torsion elements of $M$ by $M_N$, i.e., $M_N = \{ x \in M \mid Nx = 0 \}$. Two important $\mathbb{Z}[Q]$-modules are:

- $\mathbb{Z}$—the integers with trivial $Q$-action;
- $\tilde{\mathbb{Z}}$—the integers with sign action.

We denote the $n$-th Tate cohomology group of $Q$ with coefficients in $M$ by $\hat{H}^n(Q; M)$. These can be computed to be:

$$\hat{H}^n(Q; M) = \begin{cases} M^Q/NM & \text{if } n \in \mathbb{Z} \text{ even} \\ NM/(\gamma - 1)M & \text{if } n \in \mathbb{Z} \text{ odd}. \end{cases}$$
For details and a precise definition see [17, Definition 6.2.4] or [3, Chapter VI].

Underlined capital letters are used to denote Mackey functors. If we need to show the structure of some particular Mackey functor we use the Lewis diagram:

\[
\begin{array}{c}
M(Q/Q) \\
\text{res}_M \\
\end{array}
\]

\[
\begin{array}{c}
\text{tr}_M \\
M(Q/e) \\
\end{array}
\]

The map \(M(Q/Q) \rightarrow M(Q/e)\) is the \textit{restriction map} and will be denoted by \(\text{res}_M\). The map \(M(Q/e) \rightarrow M(Q/Q)\) is called the \textit{transfer map} and will be denoted by \(\text{tr}_M\).

We will drop the subscripts if it is clear from the context which Mackey functor the notation refers to. For brevity we use \(V\) to denote the \(\mathbb{Z}[Q]\)-module \(M(Q/e)\).

Throughout this whole paper we work in the category of genuine \(Q\)-spectra. By “commutative ring \(Q\)-spectrum” we mean an \(E_\infty\)-ring in \(Q\)-spectra. In the literature these are also called naive commutative ring \(Q\)-spectra.

If \(X\) is a \(Q\)-spectrum we put \(X^Q_\star = \pi^Q_\star (X)\). The five-pointed star is used to indicate the grading over \(RO(Q)\)—the representation ring of \(Q\). This means that our grading is “two-dimensional”—all representations of \(Q\) may be written in the form \(xR + y\sigma\), where \(x, y\) are integers, \(R\) is a real trivial one-dimensional representation and \(\sigma\) is a real sign representation. We will abbreviate \(xR + y\sigma\) to \(x + y\sigma\). We sometimes refer to \(x\) in the gradation as a fixed degree and to \(y\) as a twisted degree. By the antidiagonal we mean the line \(y = -x\). Whenever we want to restrict our attention to \(\mathbb{Z}\)-grading, we will use an asterisk \(*\) instead.

We put \(X^e_\star := \pi^Q_\star (F(Q_+, X))\). Note that by the induction-restriction adjunction \(X^e_\star \) is the same as the \(\mathbb{Z}[Q]\)-module \(\pi_{x+y}(\text{res}_Q^e (X))\), which we abbreviate as \(\pi_{x+y}(X)\). A map \(\phi: X \rightarrow Y\) that induces an isomorphism \(X^e_\star \cong Y^e_\star\) will be called a nonequivariant equivalence.

Note that in some papers a different grading convention for \(X^Q_\star\) is used, e.g., by Dugger [5]. Our grading is related to this by \(\mathbb{R}^{x+y} = x + y\sigma\).

2 The Tate method

In this section we recall the Tate diagram method developed in [9].

Let \(EQ\) be a free contractible \(Q\)-space and let us define the \textit{isotropy separation sequence} to be the cofibre sequence:

\[EQ_+ \rightarrow S^0 \rightarrow \widetilde{EQ}.\]

Let \(X\) be a \(Q\)-spectrum. Let \(\epsilon\) be the map

\[\epsilon: X \cong F(S^0, X) \rightarrow F(EQ_+, X).\]
By smashing $\epsilon$ with the isotropy separation sequence we obtain the following diagram:

$$
\begin{array}{ccc}
EQ^+ \wedge X & \xrightarrow{\sim} & X \\
\downarrow & & \downarrow \\
EQ^+ \wedge F(EQ^+, X) & \xrightarrow{\sim} & F(EQ^+, X)
\end{array}
$$

The left-hand vertical map $EQ^+ \wedge \epsilon$ is an equivalence by the following \cite[Proposition 1.1]{9}:

**Lemma 2.1** Let $\phi : X \to Y$ be a map of $Q$-spectra which is a nonequivariant equivalence. Then the induced maps

$$
\phi \wedge EQ^+ : X \wedge EQ^+ \to Y \wedge EQ^+
$$

are $Q$-equivalences.

Since the map $\epsilon$ is a nonequivariant equivalence, the map $EQ^+ \wedge \epsilon$ is a $Q$-equivalence. Therefore the right-hand square is a homotopy pullback square.

We define

- $X^h := F(EQ^+, X)$,
- $X_h := EQ^+ \wedge X$,
- $X^t := F(EQ^+, X) \wedge \sim EQ = X^h \wedge \sim EQ$,
- $X^\Phi := \sim EQ \wedge X$.

We also put $X_{hQ} := (X_h)^Q$, respectively for $X^hQ$, $X^\Phi Q$ and $X^tQ$.

After renaming entries in the diagram (\*) we obtain the following commutative diagram, called the Tate diagram:

$$
\begin{array}{ccc}
X_h & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
X_h & \xrightarrow{v} & X^h \\
\downarrow & & \downarrow \\
X_h & \xrightarrow{\theta} & X^t
\end{array}
$$

The right-hand square in the Tate diagram is a homotopy pullback square of $Q$-spectra and is known as the Tate square. If additionally $X$ is a ring $Q$-spectrum, all of the corners of the Tate square are also ring $Q$-spectra and the square is a homotopy pullback of ring $Q$-spectra. The analogous statement holds for $X$-module $Q$-spectra (see \cite[Proposition 3.5]{9}).

We will need the following fact about $X_h$:

**Proposition 2.2** If $X$ is a ring $Q$-spectrum, $X_h$ is a module over $X^h$ in the homotopy category.
Proof By Proposition 2.1 the map $EQ_+ \wedge \epsilon$ is a $Q$-equivalence, so it has an inverse in the homotopy category. Denote this inverse by $\bar{\epsilon}$. Let $\mu: X \wedge X \to X$ be the multiplication in $X$. The $X^h$-module structure is given by the following composite:

$$F(EQ_+, X) \wedge (EQ_+ \wedge X) \xrightarrow{\bar{\epsilon} \wedge 1} X \wedge EQ_+ \wedge X \xrightarrow{twist} EQ_+ \wedge X \wedge X \xrightarrow{EQ_+ \wedge \mu} EQ_+ \wedge X.$$ 

Now we will describe an important multiplicative property of the spectra $X^\Phi$ and $X^t$. Let $a$ be the element of $\pi_{-\sigma}^Q(S^0)$ corresponding to the inclusion $S^0 \to S^\sigma$. The map $a \wedge 1: X \to S^\sigma \wedge X$ gives a multiplication by $a$ in $X$. If $a$ acts as an isomorphism on $X^Q_*$ we say that $X$ is $a$-periodic.

Lemma 2.3 If $X$ is a $Q$-spectrum, then

$$\pi_*^Q (X \wedge \widetilde{EQ}) = a^{-1}X^Q_*.$$ 

So the spectrum $X \wedge \widetilde{EQ}$ is $a$-periodic. In particular $X^\Phi$ and $X^t$ are $a$-periodic.

Proof The $Q$-space $\widetilde{EQ}$ has a model $S^{\infty \sigma}$. So $X \wedge \widetilde{EQ}$ may be seen as a homotopy colimit of the sequence:

$$X \xrightarrow{a} S^\sigma \wedge X \xrightarrow{a} S^{2\sigma} \wedge X \xrightarrow{a} \ldots.$$ 

After applying $\pi_*^Q$ we obtain the following sequence:

$$X^Q_* \xrightarrow{a} X^Q_*-\sigma \xrightarrow{a} X^Q_*-2\sigma \xrightarrow{a} \ldots$$

from which we get an identification

$$\pi_*^Q (X \wedge \widetilde{EQ}) \cong a^{-1}X^Q_*.$$ 

Thus $a$ acts as an isomorphism on $\pi_*^Q (X \wedge \widetilde{EQ})$. □

Let $M$ be a $Q$-Mackey functor. To compute $HM^Q_*$ we will use the following method, described in [8]:

1. Firstly we compute $HM^{hQ}_*$ and $(HM^{hQ})_*$ using the homotopy fixed point and homotopy orbit spectral sequences - details are in Sect. 4.
2. Using Lemma 2.3 we deduce $HM^{tQ}_*$ from $HM^{hQ}_*$ by inverting $a$, since $HM^t = HM^h \wedge \widetilde{EQ}$.
3. We infer $HM^{\Phi Q}_*$ from $HM^{tQ}_*$. Since both theories are $a$-periodic, we need only to compute $HM^{\Phi Q}_n$ for $n \in \mathbb{Z}$. We calculate them using the fact that fibres of $\epsilon$ and $\epsilon_t$ in the Tate diagram are equivalent.
4. Finally we deduce $HM^Q_*$ from the Tate diagram.
3 Structure of $HM^Q_{*\sigma}$

From now on let $M$ be a Mackey functor. We start with a description of the subgroup $HM^Q_{*\sigma}$ with fixed degree 0. In this case the computations follow from the cofibre sequence

$$Q^+ \longrightarrow S^0 \longrightarrow S^\sigma.$$  \hfill (†)

From this sequence we can deduce the following lemma, which gives a complete description of the multiplication by the element $a \in \pi^Q_{*\sigma}(S^0)$ on Eilenberg–MacLane spectra:

**Lemma 3.1** The map $a : HM^Q_{x+y\sigma} \to HM^Q_{x+(y-1)\sigma}$ is:

1. a monomorphism if $x - 1 = -y$;
2. an epimorphism if $x = -y$;
3. an isomorphism otherwise.

**Proof** After smashing (†) with $HM$ and applying $\pi^Q_{x+y\sigma}(-)$ we obtain an exact sequence:

$$HM^e_{x+y\sigma} \longrightarrow HM^Q_{x+y\sigma} \longrightarrow HM^Q_{x+(y-1)\sigma} \longrightarrow HM^e_{(x-1)+y\sigma}.$$  

Note that the outer groups are of the form $HM^e_{x+y\sigma} \cong \pi^Q_{x+y}(\text{res}^Q e HM)$. The spectrum $\text{res}^Q e HM$ is the Eilenberg–MacLane spectrum associated to the abelian group $V = M(Q/e)$. Thus $HM^e_{x+y\sigma} = 0$ unless $x = -y$ and the lemma follows. \( \square \)

**Remark 3.2** The same result can be found in [6, Lemma 8.7(b)].

Lemma 3.1 simplifies the calculations of the groups $HM^Q_{y\sigma}$ for $y \in \mathbb{Z}$. These are also derived from the cofibre sequence (†).

**Proposition 3.3** The groups $HM^Q_{y\sigma}$ are given by:

$$HM^Q_{y\sigma} = \begin{cases} 
\ker(\text{res}_M) & \text{if } y > 0 \\
\coker(\text{tr}_M) & \text{if } y < 0 \\
M(Q/Q) & \text{if } y = 0.
\end{cases}$$

Moreover, the multiplication $a : HM^Q_{0} \to HM^Q_{0}$ is the inclusion of $\ker(\text{res}_M)$ in $M(Q/Q)$ and $a : HM^Q_{0} \to HM^Q_{-\sigma}$ is the projection onto $\coker(\text{tr}_M)$.

**Proof** The case $y = 0$ is the definition of an Eilenberg–MacLane spectrum of a Mackey functor. So we need to prove the proposition for $y \neq 0$.

By Lemma 3.1 it is enough to prove the claim for $y = 1$ and $y = -1$. If we smash the cofibre sequence (†) with $HM$ and apply $\pi^Q_{0}(-)$ we obtain an exact sequence

$$HM^e_{0} \longrightarrow HM^Q_{0} \longrightarrow HM^Q_{-\sigma} \longrightarrow 0.$$
Thus \( H_{-\sigma}^Q = \text{coker}(\text{tr}_M) \) and the multiplication by \( a \) is the projection. Analogously, by applying \([- , H^Q M] \) to (†) we get that \( H_{-\sigma}^Q = \ker(\text{res}_M). \)

4 Homotopy fixed points, homotopy orbits and the Tate spectrum of \( HM \)

In this section we compute the coefficients of the \( Q \)-spectra \( HM^h \), \( HM_h \) and \( HM^t \). These are given in Propositions 4.2 and 4.8. However, firstly we need to give a brief description of the homotopy fixed point and homotopy orbit spectral sequences.

4.1 Homotopy fixed point and homotopy orbit spectral sequences

The space \( EQ \) used in the definitions of \( Q \)-spectra \( HM^h \), \( HM_h \) and \( HM^t \) (see Sect. 2) has a very convenient model of the form \( S(\infty \sigma) \). This space has a filtration by skeleta

\[
S(\sigma) \subset S(2 \sigma) \subset S(3 \sigma) \subset \ldots
\]

and consequently, if \( X \) is a \( Q \)-spectrum, we obtain a filtration of the spectra \( X_h \) and \( X^h \). The spectral sequences associated to these filtrations take the form

\[
E^{2}_{p,q} = H_{p}(Q; \pi_{q}(X)) \Rightarrow \pi_{p+q}(X_{hQ})
\]

and

\[
E^{2}_{p,q} = H_{p}(Q; \pi_{-q}(X)) \Rightarrow \pi_{-(p+q)}(X_{hQ}).
\]

The first spectral sequence is called the homotopy orbit spectral sequence and the second is known as the homotopy fixed point spectral sequence. Details of the construction may be found in [9, Chapter 10].

Both spectral sequences can be used to compute the \( RO(Q) \)-graded coefficients of \( X_h \) and \( X^h \). To this end we note that

\[
X^Q_{x+y\sigma} = [S^x+y\sigma, F(EQ_+, X)]^Q \cong [S^x, F(EQ_+, F(Sy\sigma, X))]^Q = \pi_x(F(Sy\sigma, X)^hQ).
\]

One may argue similarly for the homotopy orbits, and thus we can arrange the homotopy orbit and homotopy fixed point spectral sequences to be trigraded:

\[
E^{2}_{p,q}(y) = H_{p}(Q; \pi_{q}(F(Sy\sigma, X))) \Rightarrow \pi_{p+q}(F(Sy\sigma, X)_{hQ})
\]

and

\[
E^{2}_{p,q}(y) = H_{p}(Q; \pi_{-q}(F(Sy\sigma, X))) \Rightarrow \pi_{-(p+q)}(F(Sy\sigma, X)^hQ).
\]

Note that even though the spectral sequences are trigraded, the differentials live on the “layers” corresponding to the single value of \( y \). If \( X \) is a ring \( Q \)-spectrum then the
pairings

\[ S^{y_1} \wedge S^{y_2} \to S^{(y_1 + y_2)} \]

give the trigraded homotopy fixed point spectral sequence a multiplicative structure.

It will prove useful to have a description of the $E_1$-page. Since we will only use it for the homotopy fixed point spectral sequence, we omit here the description of the $E_1$-page of the homotopy orbit spectral sequence.

The filtration $(\ast)$ gives a filtration of $X^h_{Q} \cong F_{Q}(E_{Q_{+}}, X)$, the spectrum of $Q$-equivariant maps from $E_{Q_{+}}$ to $X$. Therefore the $E_1$-page has the form

\[ E_1^{p,q} = \pi_{-q}(X). \]

However, there is a more useful description of $E_1^{p,q}$ given in [9, Chapter 9] which allows us also to determine differentials.

**Proposition 4.1** There is an isomorphism

\[ E_1^{p,q} = [Q_{+} \wedge S^p, \Sigma^{p+q} X]_{Q} \cong \text{Hom}_{\mathbb{Z}[Q]} \left( H_p(Q_{+} \wedge S^p), X_{-q} \right). \]

The differential

\[ d_1 : E_1^{p,q} \to E_1^{p+1,q} \]

is induced by the map $\partial_{+} : H_{p+1}(Q_{+} \wedge S^{p+1}) \to H_{p}(Q_{+} \wedge S^p)$, which is further induced in homology by the geometric boundary:

\[ \partial : Q_{+} \wedge S^{p+1} \to \Sigma S ((p + 1)\sigma)_{+} \to \Sigma (Q_{+} \wedge S^p). \]

Note that the complex

\[ \ldots \to H_{p+1}(Q_{+} \wedge S^{p+1}) \xrightarrow{\partial_{+}} H_{p}(Q_{+} \wedge S^p) \xrightarrow{\partial_{p}} H_{p-1}(Q_{+} \wedge S^{p-1}) \to \ldots \]

is the cellular complex of $S(\infty \sigma)$ and thus it is exact ($S(\infty \sigma)$ is contractible) with differentials given by the degrees of attaching maps. Therefore, since $H_{p}(Q_{+} \wedge S^p) \cong \mathbb{Z}[Q]$, the complex above gives us a 2-periodic $\mathbb{Z}[Q]$-resolution of $\mathbb{Z}$. Thus along the $q$-th row on the $E_1$-page we have the complex computing the group cohomology with coefficients in $X_{-q}$. This concludes the description of the $E_1$-page.

### 4.2 Calculations for Eilenberg–MacLane spectra

Now we specialise to the case $X = H_{M}$. Recall the notation $V := M(Q/e)$.

**Proposition 4.2** The $RO(Q)$-graded coefficients of $H_{M_{h}}$ and $H_{M^{\ell}}$ are given by:

1. \( (H_{M_{h}}^{Q})_{(y \sigma - y) + p} = H_{p}(Q; H^{y}(S^{y \sigma}, V)) \)
\( H_{(y \sigma - y) - p} = H^p(Q; H^y(S^{y \sigma}, V)). \)

**Proof** The proposition follows from the fact that the trigraded homotopy orbit and homotopy fixed point spectral sequences collapse on the second page. The coefficients of the group homology/cohomology on the second page are given by

\[
\pi_p(F(S^{y \sigma}, H^M)) \cong H^{-p}(S^{y \sigma}, V).
\]

The (reduced) singular cohomology \( H^{-p}(S^{y \sigma}, V) \) is 0 unless \(-p = y\). Thus for every \( y \) the \( E_2(y) \)-page consists of one row and all differentials are 0.

**Remark 4.3** Note that since \( Q \) acts on \( H^y(S^{y \sigma}, \mathbb{Z}) \) by the degree of \( \gamma \) as a map, it is isomorphic to \( \mathbb{Z} \) if \( y \) is even and \( \tilde{\mathbb{Z}} \) if \( y \) is odd. So by the Universal Coefficient Theorem \( H^y(S^{y \sigma}, V) \) is isomorphic to \( V \) if \( y \) is even and \( \tilde{V} := \tilde{\mathbb{Z}} \otimes V \) if \( y \) is odd.

The projective \( \mathbb{Z}[Q] \)-resolution of \( \tilde{\mathbb{Z}} \) is given by:

\[
\ldots \rightarrow \mathbb{Z}[Q] \xrightarrow{1-\gamma} \mathbb{Z}[Q] \xrightarrow{1+\gamma} \mathbb{Z}[Q] \xrightarrow{\tilde{\epsilon}} \tilde{\mathbb{Z}} \rightarrow 0.
\]

Here \( \tilde{\epsilon} \) is the ring map defined by \( \tilde{\epsilon}(\gamma) = -1 \).

Recall the 2-periodic \( \mathbb{Z}[Q] \)-resolution of \( \mathbb{Z} \):

\[
\ldots \rightarrow \mathbb{Z}[Q] \xrightarrow{1+\gamma} \mathbb{Z}[Q] \xrightarrow{1-\gamma} \mathbb{Z}[Q] \xrightarrow{\epsilon} \mathbb{Z} \rightarrow 0.
\]

Therefore from the resolution of \( \tilde{\mathbb{Z}} \) we can see that there are isomorphisms \( H^i(Q; \tilde{V}) \cong H^{i+1}(Q; V) \) and \( H_i(Q; \tilde{V}) \cong H_{i+1}(Q; V) \) for \( i \geq 1 \). This gives us four potentially non-zero values for \( (H_{M_Q^h})_* \) and four for \( H_{M_Q^h}^* \), as depicted in Figs. 2 and 3.

**Remark 4.4** Note that from Figs. 2 and 3 it is easy to see that both \( (H_{M_Q^h})_* \) and \( H_{M_Q^h}^* \) are \((2 - 2\sigma)\)-periodic. We will attribute this phenomenon to the multiplication by the generator of \( H_{A_{2-2\sigma}}^Q \) in Sect. 8.

Before proceeding to the description of \( H_{M_Q^h}^* \) we give a couple of observations on the structure of \( H_{M_Q^h}^* \) and \( H_{M_Q^h}^h \).

**Observation 4.5** We have that

\[
\pi_*^e(H_{M_Q^h}) \cong \pi_*^e(H_{M_Q^h}^h) \cong \pi_*^e(\mathbb{H}_M).
\]

In particular, the multiplication by \( a \) in \( (H_{M_Q^h})_* \) and \( H_{M_Q^h}^* \) can be described in the same way as in Lemma 3.1.

**Proof** We are going to prove the observation only for \( H_{M_Q^h}^* \), as the proof for \( H_{M_Q^h}^h \) follows analogously. Note that \( \text{res}_e^Q(E_{Q+}) \simeq S^0 \). By the definition of non-equivariant homotopy groups we have that
Fig. 2 Coefficients of $\text{HM}_h$

$$\left(\text{HM}_{hQ}\right)_{x+y\sigma}$$

Key: 
- $V/(1-\gamma)V$
- $V^Q/(1+\gamma)V$
- $V/(1+\gamma)V$
- $N V/(1-\gamma)V$

Fig. 3 Coefficients of $\text{HM}^h$

$$\text{HM}^{hQ}_{x+y\sigma}$$

Key: 
- $V^Q$
- $V^Q/(1+\gamma)V$
- $N V$
- $N V/(1-\gamma)V$
\[ \pi_n^e(H M_h^Q) = [Q_+ \land S^n, E Q_+ \land H M]^Q \]
\[ \cong [S^n, \text{res}_e^Q (E Q_+ \land H M)] \]
\[ \cong [S^n, \text{res}_e^Q (E Q_+ \land \text{res}_e^Q H M)] \]
\[ \cong [S^n, S^0 \land \text{res}_e^Q] = \pi_n^e(H M). \]

Here the second isomorphism comes from the fact that the restriction functor is symmetric monoidal. The statement about the multiplication by \(a\) follows now by the verbatim repetition of the proof of Lemma 3.1. \(\Box\)

**Proposition 4.6** The map \(f_0: V_Q = (HM_{hQ})_0 \to HM_0^Q = M(Q/Q)\) is the map induced by \(\text{tr}_M\) on \(V_Q\). The map \(e_0: M(Q/Q) = HM_0^Q \to HM_{hQ}^0 = V_Q\) is the restriction of \(\text{res}_M\) to the codomain \(V_Q\).

**Proof** Let \(i: Q_+ \to EQ_+\) be the inclusion of the 0-skeleton. Note that the map \(Q_+ \to S^0\) extends to the filtration \((\ast)\) of \(EQ_+\) and thus it factors as
\[ Q_+ \xrightarrow{i} EQ_+ \to S^0. \]

By smashing this sequence with \(HM\) and taking \(\pi_0^Q(-)\) we obtain the following commutative diagram:

```
\[
\begin{array}{ccc}
HM_0^e & \xrightarrow{\text{tr}_M} & HM_0^Q \\
\downarrow f_0 & & \downarrow \text{tr}_M \\
(HM_{hQ})_0 & & JM_0^Q
\end{array}
\]
```

By Proposition 4.2 this diagram is isomorphic to the following:

```
\[
\begin{array}{ccc}
V & \xrightarrow{\text{tr}_M} & M(Q/Q) \\
\downarrow f_0 & & \downarrow M(Q/Q) \\
V/(1-\gamma)V & & V/(1-\gamma)V
\end{array}
\]
```

The left diagonal arrow is the canonical projection. Thus the map \(f_0: (HM_{hQ})_0 \to HM_0^Q\) is the map induced by the transfer on \(V_Q\). Note that by the properties of Mackey functors \(\text{tr}_M\) always factors via \(V_Q\). We obtain the second assertion by using the dual argument. \(\Box\)

We now give a corollary, which will become useful in Sect. 8.

**Corollary 4.7** If \(y \geq 1\), then the map \(f_{y\sigma}: (HM_{hQ})_{y\sigma} \to HM_{hQ}^Q\) is induced by the transfer.
Proof By Lemma 3.1 and Observation 4.5 we need to prove only the case $y = 1$. Consider the following commutative diagram:

\[
\begin{array}{ccc}
(HM_{\mathbb{Q}})_{\sigma} & \xrightarrow{f_{\sigma}} & HM_{\mathbb{Q}} \\
\downarrow a & & \downarrow a \\
(HM_{\mathbb{Q}})_{0} & \xrightarrow{f_{0}} & HM_{0}^{\mathbb{Q}}.
\end{array}
\]

From Lemma 3.1 and Observation 4.5 we get that the vertical arrows in this diagram are inclusions. By Propositions 3.3 and 4.2 it is isomorphic to the following diagram:

\[
\begin{array}{ccc}
\frac{N V}{(1-\gamma) V} & \xrightarrow{f_{\sigma}} & \ker(\text{res}_{M}) \\
\downarrow a & & \downarrow a \\
V_{Q} & \xrightarrow{f_{0}} & M(Q/Q).
\end{array}
\]

Since the map $f_{0}$ is induced by the transfer in $M$ by Proposition 4.6, the map $f_{\sigma}$ is also induced by the transfer. The statement follows.

Proposition 4.8 The coefficients of the $\mathbb{Q}$-spectrum $HM^{t}$ are given by

\[
HM^{t}_{x+y\sigma} = \hat{H}^{-x}(Q; V),
\]

where $\hat{H}^{x}(Q; V)$ denotes the $x$-th Tate cohomology of $Q$ with coefficients in $V$ (see Sect. 1.1). Note that since $HM^{t}$ is $a$-periodic, $HM^{t}_{x+y\sigma}$ does not depend on $y$.

Proof By definition $X^{t} = X^{h} \wedge \widetilde{E^{1}}$. So by Lemma 2.3 we have that

\[
HM^{t}_{x} = a^{-1} HM^{h}_{x}.
\]

From this isomorphism we see in particular that $HM^{t}_{x+y\sigma} = HM^{h}_{x+y\sigma}$ below the antidiagonal, i.e., when $y < -x$. So if $x$ is even we obtain by Proposition 4.2 and $a$-periodicity of $HM^{t}$ that

\[
HM^{t}_{x+y\sigma} \cong HM^{t}_{x-(x+1)\sigma} = HM^{h}_{x-(x+1)\sigma} \cong V^{O}/N V.
\]

Analogously, if $x$ is odd we obtain that $HM^{t}_{x+y\sigma} \cong N V/(1-\gamma)V$.  

5 Geometric fixed points

In this section we describe the coefficients of the $\mathbb{Q}$-spectrum $HM^{b}$.
Theorem 5.1 The coefficients of $H M^\Phi$ are given by:

$$HM^\Phi_{x+y\sigma} = \begin{cases} 
\text{coker}(\text{tr}_M) & \text{if } x = 0 \\
(\ker(\text{tr}_M))^Q & \text{if } x = 1 \\
HM^i_{x+y\sigma} = \hat{H}^i(Q; V) & \text{if } x \geq 2 \\
0 & \text{otherwise}
\end{cases}$$

Note that the values of $HM^\Phi_{x+y\sigma}$ depend only on the fixed degree, which comes from Lemma 2.3. We prove Theorem 5.1 by a series of lemmas.

Lemma 5.2 $HM^Q_{y\sigma} = \text{coker}(\text{tr}_M)$ and $HM^\Phi_Q = (\ker(\text{tr}_M))^Q$.

Proof By Lemma 2.3 we have that $HM^\Phi_Q \cong HM^Q_0$, so we need to prove the claim only in this case. Analogously for $x = 1$. From the top row of the Tate diagram we obtain the following exact sequence:

$$HM_1^Q \xrightarrow{g_1} HM_1^\Phi \xrightarrow{f_0} (HM_0^h)^Q \xrightarrow{g_0} HM_0^\Phi \xrightarrow{f_0} (HM_0^h)^{-1}.$$

The outer groups in this sequence are zero: $HM_1^Q$ by the definition of an Eilenberg–MacLane spectrum and $(HM_0^h)^{-1}$ by Proposition 4.2. Thus $HM_0^\Phi$ and $HM_1^\Phi$ are respectively cokernel and kernel of the map $f_0$. Thus by Proposition 4.6 we obtain that $\text{coker}(f_0) = \text{coker}(\text{tr}_M)$ and $\ker(f_0) = \ker(\text{tr}_M)^Q$. $\square$

Lemma 5.3 If $x \geq 2$ then $HM^\Phi_{x+y\sigma} = HM^i_{x+y\sigma}$.

Proof Since both $HM^\Phi$ and $HM^i$ are a-periodic by Lemma 2.3, we need only to show that if $x \geq 2$, then $HM^\Phi_{x+y\sigma} = HM^i_{x+y\sigma}$ with twisted degree 0. Note that the fibres of $\epsilon : HM^\Phi_x \to HM^h$ and $\epsilon : HM^\Phi \to HM^i$ are equivalent, as they are both of the form $F(E_Q, HM)$. Let $F$ denotes this fibre. By applying the long exact sequence in homotopy to the fibration $F \to HM \to HM^h$ we get:

$$\ldots \to HM^Q_{m+1} \xrightarrow{\epsilon_*} HM^{hQ}_{m+1} \to F^Q_m \to HM^Q_m \xrightarrow{\epsilon_*} HM^{hQ}_m \to \ldots$$

But by the definition of Eilenberg–MacLane spectrum we have that $HM^Q_m = 0$ if $m \neq 0$ and $HM^{hQ}_m = 0$ for $m \geq 1$ by Proposition 4.2, so we get from this exact sequence that $F^Q_m = 0$ for $m \geq 1$. By applying the analogous long exact sequence to the fibration $F \to HM^\Phi \to HM^i$ we get that $HM^\Phi_m = HM^i_m$ for $m \geq 2$. $\square$

Lemma 5.4 $HM^\Phi_{x+y\sigma} = 0$ for $x < 0$.

Proof As above, by Lemma 2.3 we need only to show the statement for the twisted degree $y = 0$. Writing the long exact sequence in homotopy for the upper row of the Tate diagram yields:
... $\rightarrow (HM_{hQ})_m \rightarrow HM^{Q}_m \rightarrow HM^{\Phi Q}_m \rightarrow (HM_{hQ})_{m-1} \rightarrow HM^Q_{m-1} \rightarrow \ldots$.

But $HM^Q_m = 0$ for $m \neq 0$ and $(HM_{hQ})_m = 0$ for $m < 0$ by Proposition 4.2, so $HM^{\Phi Q}_m = 0$ for $m < 0$. \hfill \Box

**Proof of Theorem 5.1** Follows from Lemmas 5.2, 5.3 and 5.4. \hfill \Box

### 6 RO(Q)-graded abelian group structure of $HM^Q_x$

In this section we describe the structure of $HM^Q_*$ as a RO($Q$)-graded abelian group. It is given by the following theorem:

**Theorem 6.1** The RO($Q$)-graded abelian group structure of $HM^Q_*$ is given by:

1. $HM^Q_{y\sigma} = \begin{cases} 
\ker(\text{res}_M) & \text{if } y > 0 \\
\coker(\text{tr}_M) & \text{if } y < 0 \\
M(Q/Q) & \text{if } y = 0
\end{cases}$

2. $HM^Q_{1+y\sigma} = \begin{cases} 
\ker(\text{tr}_M) & \text{if } y = -1 \\
\ker(\text{tr}_M)Q & \text{if } y < -1 \\
0 & \text{if } y > -1
\end{cases}$

3. $HM^Q_{y\sigma-1} = \begin{cases} 
\coker(\text{res}_M) & \text{if } y = 1 \\
VQ/\text{im}(\text{res}_M) & \text{if } y > 1 \\
0 & \text{if } y < 1
\end{cases}$

4. If $x \geq 2$ then $HM^Q_{x+y\sigma} = HM^{hQ}_{x+y\sigma}$.
5. If $x \leq -2$ then $HM^Q_{x+y\sigma} = (HM_{hQ})_{x+y\sigma}$.

This data is presented in Fig. 4.

**Remark 6.2** Recall from the introduction that the same values can be found in [6, Theorem 8.1].

We are going to prove Theorem 6.1 in a series of lemmas. Firstly, let us note that the point (1) of the theorem is actually Proposition 3.3, so it is already proven.

We start with proving a lemma which describes the general shape of $HM^Q_*$—i.e., that it is zero below the antidiagonal on the half-plane $x < 0$ and above the antidiagonal on the half-plane $x > 0$: 

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Lemma 6.3 For $x < 0$, the groups $H^Q_{x+y\sigma}$ are zero if $y < -x$. If $x > 0$ then $H^Q_{x+y\sigma}$ is zero for $y > -x$.

Proof The groups $H^Q_{x}$ are zero if $x \neq 0$. Let $x > 0$. By Lemma 3.1 iterated multiplication by $a$ gives isomorphisms $H^Q_{x} = H^Q_{x+y\sigma}$ if $y > -x$. Thus we get that $H^Q_{x+y\sigma} = 0$ if $y > -x$. The case when $x < 0$ follows similarly. \[\square\]
Lemma 6.4 \( H M_{1+y\sigma}^Q \) is given by

\[
HM_{1+y\sigma}^Q = \begin{cases} 
\ker(\text{tr}_M) & \text{if } y = -1 \\
\ker(\text{tr}_M)Q & \text{if } y < -1 \\
0 & \text{if } y > -1.
\end{cases}
\]

**Proof** Firstly we note that the case \( y > -1 \) follows from Lemma 6.3, so it is proven.

Let \( y = -1 \). If we smash the cofibre sequence (†) with \( HM \) from Sect. 3 we obtain the following cofibre sequence:

\[
Q_+ \wedge HM \longrightarrow HM \longrightarrow S^\sigma \wedge HM.
\]

This cofibration gives us the following exact sequence in homotopy:

\[
HM_1^Q \longrightarrow HM_{1-\sigma}^Q \longrightarrow HM_0^\epsilon \xrightarrow{\text{tr}_M} HM_0^Q.
\]

By the definition of Eilenberg–MacLane spectrum we have that \( HM_1^Q = 0 \). Thus \( HM_{1-\sigma}^Q = \ker(\text{tr}_M) \).

Finally, let \( y < -1 \). By applying the long exact sequence in homotopy to the upper row of the Tate diagram we get

\[
(HM_{hQ})_{1+y\sigma} \longrightarrow HM_{1+y\sigma}^Q \longrightarrow HM_{1+y\sigma}^{\Phi Q} \longrightarrow (HM_{hQ})_{y\sigma}.
\]

By Proposition 4.2 the outer terms of the sequence above are 0, so \( HM_{1+y\sigma}^Q \cong HM_{1+y\sigma}^{\Phi Q} \). By Theorem 5.1 we get that \( HM_{1+y\sigma}^{Q} \cong \ker(\text{tr}_M)Q \).

Lemma 6.5 \( HM_{y\sigma-1}^Q \) is given by

\[
HM_{y\sigma-1}^Q \cong \begin{cases} 
\coker(\text{res}_M) & \text{if } y = 1 \\
V^Q / \text{im}(\text{res}_M) & \text{if } y > 1 \\
0 & \text{if } y < 1.
\end{cases}
\]

**Proof** The case \( y < 1 \) is proven by Lemma 6.3. For \( y = 1 \) we use a dual argument to the one given in the proof of Lemma 6.4. Thus we are left with one case.

Let \( y > 1 \). By applying the long exact sequences in homotopy to the Tate diagram we obtain the following commutative diagram:

\[
\begin{array}{ccc}
H M_{y\sigma}^{\Phi Q} & \longrightarrow & (H M_{hQ})_{y\sigma-1} \\
\epsilon_{**} & \downarrow & \downarrow \\
H M_{y\sigma}^{Q} & \longrightarrow & (H M_{hQ})_{y\sigma-1}
\end{array}
\]

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Here $\epsilon_{t*}$ is the map induced by $\epsilon_t$. The zero in the top right corner comes from the fact that $HM_{y\sigma-1}^{\Phi Q} = 0$ by Theorem 5.1. Note that the bottom arrow is a part of the following sequence:

$$HM_{y\sigma}^h Q \rightarrow HM_{y\sigma}^t Q \rightarrow (HM_{h Q})_{y\sigma-1} \rightarrow HM_{y\sigma-1}^h Q.$$ 

In this sequence the outer terms are zero by Proposition 4.2 if $y > 1$ and thus the map $HM_{y\sigma}^t Q \rightarrow (HM_{h Q})_{y\sigma-1}$ in the diagram (**) is an isomorphism. Thus we can deduce from this diagram that $HM_{y\sigma-1}^Q = \text{coker}(\epsilon_{t*})$. So it remains to describe the map $\epsilon_{t*} : HM_{y\sigma}^{\Phi Q} \rightarrow HM_{y\sigma}^t Q$.

Since both $HM^\Phi$ and $HM^t$ are $a$-periodic by Lemma 2.3, it is enough to identify this map for $y = 0$. It is described by the following diagram:

$$HM_0^Q = M(Q/Q) \rightarrow HM_0^{\Phi Q} = \text{coker}(\text{tr}_M)$$

$\xrightarrow{\text{res}_M}$

$$HM_0^h Q = V Q \rightarrow HM_0^t Q = V Q/(1+\gamma) V.$$ 

The map $\text{res}_M : M(Q/Q) \rightarrow V Q$ is the map $\text{res}_M$ with the codomain restricted to $V Q$. Note that by properties of Mackey functors we have that $\text{im}(\text{res}_M) \subset V Q$. Both horizontal maps are canonical projections. So $\epsilon_{t*}$ is the map induced by $\text{res}_M$. Thus we have:

$$HM_{y\sigma-1}^{\Phi Q} \cong \text{coker}(\epsilon_{t*}) = \frac{(V Q/\text{tr} M_{MM/(Q/Q)})}{\text{res}_M (M(Q/Q)/\text{tr} M(V))} \cong V Q / \text{im}(\text{res}_M).$$

$\square$

**Proof of Theorem 6.1** Follows from Lemmas 6.3, 6.4 and 6.5. $\square$

### 7 Examples

In this section we present two examples of the structure shown in previous sections. Since both examples are Green functors, we will also describe a multiplicative structure on the coefficients. The main example is the Burnside Mackey functor $A_*$, as we are going to build the multiplicative structure of other Green functors upon the knowledge of $H_{A*}^Q$. However, a big part of computations of the coefficients of $H_{A*}$ is the same as in the case of constant Mackey functor $Z$, coefficients of which are already computed by Tate method in [8]. Thus we begin the examples with computations of $H_{Z*}^Q$. 

[Springer]
7.1 Constant Mackey functor \( Z \)

The Mackey functor \( Z \) has the following form:

\[
\begin{array}{c}
\mathbb{Z} \\
\mathbb{Z}
\end{array}
\]

\( 1 \xrightarrow{2} \cdot \mathbb{Z} \).

Computations of \( H^Q_Z \) as a ring may be found in [5, Appendix B] and later on, using the Tate square technique, in [8]. From the latter we recall the following two lemmas [8, Lemma 2.1, Corollary 2.3, Lemma 2.5]:

**Lemma 7.1**

\[
\begin{align*}
H^hQ_Z &= BB[u, u^{-1}] \\
(H^hQ_Z)_* &= NB[u, u^{-1}] \\
H^{Q*}_Z &= F_2[a, a^{-1}][u, u^{-1}]
\end{align*}
\]

where \( BB = \mathbb{Z}[a]/2a, \) \( NB = \mathbb{Z} \oplus \Sigma^{-1+\sigma}F_2[a^{-1}] \) and \( |a| = -\sigma, \) \( |u| = 2 - 2\sigma \).

**Lemma 7.2** The coefficients of \( H^\Phi_Z \) are given by \( H^\Phi_Z = F_2[a, a^{-1}][u] \).

**Remark 7.3** Note that the element \( a \) in Lemmas 7.1 and 7.2 is the same as the element \( a \cdot 1 \) in the sense of Lemma 3.1, so there is no clash of the notation.

Now we are in a position to describe the structure of \( H^Q_Z \) as a ring.

**Theorem 7.4** (1) The \( RO(Q) \)-graded abelian group structure of \( H^Q_Z \) is given by:

\[
H^Q_Z = \begin{cases}
\mathbb{Z} & \text{if } y = -x \text{ and } x = 2n \text{ for } n \in \mathbb{Z} \\
\mathbb{Z}/2 & \text{if } y < -x \text{ and } x = 2n \text{ for } n \in \mathbb{Z}_{\geq 0} \\
\mathbb{Z}/2 & \text{if } y \geq -x \text{ and } x = -2n - 1 \text{ for } n \in \mathbb{Z}_{\geq 1} \\
0 & \text{else.}
\end{cases}
\]

(2) The multiplicative structure of \( H^Q_Z \) is characterized by the following properties:

- (a) it is strictly commutative;
- (b) red lines in Fig. 5 represent multiplication by \( a \in H^Q_Z_{-\sigma} \);
- (c) blue dashed lines represent multiplication by \( u \in H^Q_Z_{-2\sigma} \);
- (d) the map \( u : H^Q_Z_{2\sigma} \to H^Q_Z_{0} \) is multiplication by \( 2 \);
- (e) the groups \( H^Q_Z_{2n\sigma-2n} \) for \( n > 0 \) are generated by \( 2u^{-n} \).

**Proof** (1) This comes from Theorem 6.1:

- if \( x \leq -2 \) we have that \( H^Q_Z_{x+y\sigma} = (H^hQ_Z)_{x+y\sigma} \), so the structure is described by Lemma 7.1;
On the $RO(Q)$-graded coefficients...

Fig. 5 Coefficients of $H^*_\mathbb{Z}$

- if $x \geq 2$ then $H^*_\mathbb{Z}^Q = H^*_\mathbb{Z}^hQ$, so the structure is also described by Lemma 7.1;
- if $-1 \leq x \leq 1$ the structure comes from the following calculations:

$$
H^*_\mathbb{Z}^Q_{1-y} = \ker(tr_{\mathbb{Z}}) = 0
$$

$$
H^*_\mathbb{Z}^Q_{1-y} = \ker(tr_{\mathbb{Z}})_Q = 0 \text{ for } y > 1
$$

$$
H^*_\mathbb{Z}^Q_{y-1} = \coker(res_{\mathbb{Z}}) = 0
$$

$$
H^*_\mathbb{Z}^Q_{y+1} = V^Q/im(res_{\mathbb{Z}}) = 0 \text{ for } y > 1
$$

$$
H^*_\mathbb{Z}^Q_{y} = \ker(res_{\mathbb{Z}}) = 0 \text{ for } y > 0
$$
\[ H_{\mathbb{Z}/2 \sigma}^Q = \text{coker}(\text{tr}_Z) = \mathbb{Z}/2 \text{ for } y < 0. \]

(2) (a) Since \( \mathbb{Z} \) is a Green functor, \( H_{\mathbb{Z}}^Q \) is a commutative ring \( Q \)-spectrum and so \( H_{\mathbb{Z}/2 \sigma}^Q \) is a graded commutative ring. The graded commutativity rule is given as follows: if \( \alpha \in H_{\mathbb{Z}/2 \sigma}^Q x + y \sigma \) and \( \beta \in H_{\mathbb{Z}/2 \sigma}^Q x' + y' \sigma \) then

\[ \alpha \beta = (-1)^{xx'}(1 - \omega)^{yy'} \beta \alpha, \]

where \( \omega \) is the class of \( Q/e \) in \( A(Q) \) (see Observation 9.1 and Remark 9.2). However, since all entries except of the antidiagonal are either zero or \( \mathbb{Z}/2 \), the sign rule might give non-trivial sign only in \( \mathbb{Z}/2 \)’s on the antidiagonal. But they all lie in even fixed and twisted degrees, so the sign is always 1.

(b) The multiplication by \( a \) is described by Lemma 3.1.

(c) Firstly recall from Sect. 2 that the map \( \epsilon_* : H_{\mathbb{Z}_h}^Q \to H_{\mathbb{Z}}^Q \) is a ring map. Thus for \( x \geq 0 \) multiplication by the element \( u \) is described by Lemma 7.1. By Proposition 2.2 we have that \( (H_{\mathbb{Z}_h}^Q)_* \) is a module over \( H_{\mathbb{Z}}^Q \), so the claim follows from Lemma 7.1.

(d) We have the following commutative diagram:

\[
\begin{array}{ccc}
(H_{\mathbb{Z}_h}^Q)_{2\sigma - 2} & \xrightarrow{u} & (H_{\mathbb{Z}_h}^Q)_0 \\
\downarrow & & \downarrow f_0 \\
H_{\mathbb{Z}/2 \sigma - 2}^Q & \xrightarrow{u} & H_{\mathbb{Z}/2 \sigma - 2}^Q.
\end{array}
\]

By Lemma 7.1 the upper horizontal arrow is an isomorphism. By Theorem 6.1 the left vertical arrow is an isomorphism, so multiplication \( u : H_{\mathbb{Z}/2 \sigma - 2}^Q \to H_{\mathbb{Z}/2 \sigma - 2}^Q \) is up to isomorphism the same as the right vertical map \( f_0 \). By Proposition 4.6 this map is induced by the transfer. Thus if \( \alpha \in H_{\mathbb{Z}/2 \sigma - 2}^Q \) we obtain that \( u \cdot \alpha = 2 \alpha \in H_{\mathbb{Z}/2 \sigma - 2}^Q \).

(e) Let \( \theta \) be the generator corresponding to 1 in \( H_{\mathbb{Z}/2 \sigma - 2}^Q \cong \mathbb{Z} \). By the previous point \( u \cdot \theta = 2 \), so \( \theta = 2u^{-1} \). Since \( H_{\mathbb{Z}/2 \sigma - 2}^Q \cong (H_{\mathbb{Z}_h}^Q)_{x + y \sigma} \) if \( x \leq -2 \) and multiplication by \( u \) is an isomorphism on \( (H_{\mathbb{Z}_h}^Q)_* \), we get that \( H_{\mathbb{Z}/2 \sigma - 2n}^Q \) is generated by \( 2u^{-n} \) for \( n > 0 \).

\[ \square \]

Remark 7.5 The ring \( H_{\mathbb{Z}}^Q \) has a more concise description as:

\[ H_{\mathbb{Z}}^Q \cong BB[u] \oplus u^{-1} \cdot NB[u^{-1}] \]

with \( BB \) as before and \( NB = 2 \mathbb{Z} \oplus \Sigma^{-1 + \sigma} \mathbb{F}_2[a^{-1}] \). The structure of \( BB \)-module of \( NB \) is as suggested by the notation. See [8, Corollary 2.6].
7.2 Burnside Mackey functor

The second example is the Burnside Mackey functor $A$. It is the most important example—since we will build our understanding of multiplicative structure of Eilenberg–MacLane spectra upon the knowledge of the graded ring structure of $H \mathbb{A}^Q_\ast$.

The Mackey functor $A$ has the following form:

$$\mathbb{Z}[\omega]/\omega^2 - 2\omega$$

with the transfer given by multiplication by $\omega$ and the restriction given by evaluating at $\omega = 2$. Note that the $Q/Q$-level of $A$ is in fact the Burnside ring $\mathbb{A}(Q)$ with $\omega$ being the class of $Q/e$. The Mackey functor valued coefficients of $A$ may be found in [16, Section 2] and [6, Proposition 1.7(b)].

The coefficients of $H \mathbb{A}^h$, $H \mathbb{A}^h$ and $H \mathbb{A}^l$ depend only on $\mathbb{A}(Q/e) = \mathbb{Z}$ (see Propositions 4.2 and 4.8), which is the same as $\mathbb{Z}(Q/e)$. So $H \mathbb{A}^h \mathbb{A}^Q \simeq H \mathbb{Z}^h \mathbb{A}^Q$, analogously for $(H \mathbb{Z}^h \mathbb{A})^Q$ and $H \mathbb{A}^l \mathbb{A}^Q$. Thus Lemma 7.1 gives a description for this entries of the Tate diagram.

Now we need to compute the coefficients of $H \mathbb{A}^\Phi$:

**Lemma 7.6**

$H \mathbb{A}^\Phi \mathbb{A}^Q \simeq \mathbb{Z}[a, a^{-1}][u]/2u$.

**Proof** By Theorem 5.1 we have that $H \mathbb{A}^\Phi \mathbb{A}^Q \simeq \mathbb{Z}$ and $H \mathbb{A}^\Phi \mathbb{A}^Q \simeq 0$. So by $a$-periodicity of $H \mathbb{A}^\Phi$ (see Lemma 2.3) we obtain that $H \mathbb{A}^\Phi \mathbb{A}^Q \simeq \mathbb{Z}[a, a^{-1}]$. Since the map $\epsilon_{t^\ast} : H \mathbb{A}^\Phi \mathbb{A}^Q_x \rightarrow H \mathbb{A}^\Phi \mathbb{A}^Q_{x+y}$ induced by $\epsilon_{t}$ is a ring map and it is an isomorphism if $x \geq 2$, the result follows by Lemma 7.1.

**Theorem 7.7** (1) The $RO(Q)$-graded abelian group structure of $H \mathbb{A}^Q_\ast$ is given by:

$$H \mathbb{A}^Q_\ast \mathbb{A}^Q \mathbb{A}^Q \simeq \begin{cases} \mathbb{A}(Q) & \text{if } x = y = 0 \\ \mathbb{Z} & \text{if } x = 0 \text{ and } y \neq 0 \\ \mathbb{Z} & \text{if } x \text{ even and } y = -x \\ \mathbb{Z}/2 & \text{if } x \text{ odd, } x \leq -3 \text{ and } y > x \\ \mathbb{Z}/2 & \text{if } x \text{ even, } x \geq 2 \text{ and } y < x \\ 0 & \text{else} \end{cases}$$

(2) The multiplicative structure of $H \mathbb{A}^Q_\ast$ is given by the following properties:

(a) it is strictly commutative;
(b) red lines on Fig. 6 represent multiplication by $a$;

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(c) blue dashed lines represent multiplication by $u$, the generator of $H_{-2-s\sigma}^Q$ corresponding to $1$;
(d) if $\tau$ is a generator of $H_{-2\sigma}^Q$, then $a\tau = \omega - 2$;
(e) $u : H_{-2\sigma}^Q \to H_{-2-2\sigma}^Q$ is the transfer map in $\mathbb{A}$ and $u : H_{-2\sigma}^Q \to H_{-2-2\sigma}^Q$ is the restriction;
(f) for $n > 0$ the group $H_{-2\sigma-2n}^Q$ is generated by $\omega u^{-n}$.

In particular, the subring consisting of entries for $x \geq 0$ and $y \leq 0$ is a truncated polynomial algebra

$$\mathbb{A}(Q)[a, u]/a\omega, 2au.$$  

The data above is presented in Fig. 6.

Proof We provide here only the proof of points 1 and 2a, since the rest is completely analogous to the case of $H\mathbb{Z}$ given in Theorem 7.4.

(1) If $-1 \leq x \leq 1$ then the statement comes from the following calculations:

$$\ker(\text{res}_\mathbb{A}) = (\omega - 2) \cong \mathbb{Z} \text{ as an abelian group}$$
$$\text{coker}(\text{res}_\mathbb{A}) = 0$$
$$\ker(\text{tr}_\mathbb{A}) = 0$$
$$\text{coker}(\text{tr}_\mathbb{A}) = \mathbb{Z}.$$  

The rest follows from Lemma 7.1 in the same way as in the proof of Theorem 7.4.

(2) (a) Firstly note that $\mathbb{A}$ is a Green functor, so $H\mathbb{A}$ is a commutative ring $\mathbb{Q}$-spectrum. Recall the graded commutativity rule from the proof of Theorem 7.4, part 2a: if $\alpha \in H_{x+y\sigma}^Q$ and $\beta \in H_{x'+y'\sigma}^Q$ then

$$\alpha\beta = (-1)^{xx'}(1 - \omega)^{yy'}\beta\alpha.$$  

Since all non-zero entries have even fixed degree, the first unit is always 1. So we need to show that $1 - \omega$ also acts as 1 in all cases. To this end we need to show that this claim holds only on the antidiagonal and on the $y$-axis, as all other non-zero entries are $\mathbb{Z}/2$.

On the antidiagonal all non-zero entries are in even twisted degrees and so $1 - \omega$ acts as 1. For $y > 0$ we have that $H_{y\sigma}^Q = \ker(\text{res}_\mathbb{A}) = (\omega - 2)$. Multiplication by $\omega$ on this ideal gives zero, since $\omega(\omega - 2) = \omega^2 - 2\omega = 0$ in $\mathbb{A}(Q)$. So $1 - \omega$ acts as 1 on $H_{y\sigma}^Q$ if $y > 0$. Finally, let $y < 0$. In this case we have that $H_{y\sigma}^Q = \text{coker}(\text{tr}_\mathbb{A}) = \mathbb{A}(Q)/\omega$, so $\omega$ acts as 0 and $1 - \omega$ as 1.  

$\square$
Sections 2 and 7 suggest that there exist some patterns in the coefficients of Eilenberg–MacLane \( Q \)-spectra. One of them is a repetition along the vertical lines, which by Lemma 3.1 we can attribute to multiplication by \( a \). The other one may be seen in the antidiagonal direction; we are going to describe it in this section.

Let \( u \) be the generator of \( H\mathbb{A}^Q_{2-2\sigma} \cong \mathbb{Z} \).

**Theorem 8.1** For any Mackey functor \( M \) the map \( u: HM^Q_{(x+y)\sigma} \rightarrow HM^Q_{(x+2)+(y-2)\sigma} \) is:

1. the map induced by the transfer \( V_Q \rightarrow M(Q/Q) \) if \( x = -2 \) and \( y = 2 \);
(2) the map induced by the transfer \( \frac{N}{(1 - \gamma)V} \to \ker(\text{res}_M) \) if \( x = -2 \) and \( y > 2 \);
(3) the restriction map \( M(Q/Q) \to V^Q \) if \( x = y = 0 \);
(4) the map induced by the restriction \( \text{coker}(\text{tr}_M) \to V^Q/NV \) if \( x = 0 \) and \( y < 0 \);
(5) the inclusion \( \ker(\text{tr}_M) \to \frac{N}{V} \) if \( x = 1 \) and \( y = -1 \);
(6) the map \( \ker(\text{tr}_M) \to \frac{N}{V}/(1 - \gamma)V \) induced by the inclusion from Point 4 if \( x = 1 \) and \( y < -1 \);
(7) multiplication by \( 1 - \gamma \) if \( x = -1 \) and \( y = 1 \);
(8) the zero map if \( x = -1 \) and \( y > 1 \);
(9) the projection \( V/NV \to \text{coker}(\text{res}_M) \) if \( x = -3 \) and \( y = 3 \);
(10) the projection \( V^Q/NV \to V^Q/\text{im}(\text{res}_M) \) induced by projection from Point 9 if \( x = -3 \) and \( y > 3 \);
(11) the zero map if \( x < 0 \) and \( y < -x \) or \( x \geq 0 \) and \( y > -x \);
(12) an isomorphism otherwise.

Before proving this theorem we give a couple of preparatory lemmas. Note that since \( HM \) is a module over \( HA \) we have that \( HM^{h} \) is a module over \( HA^{h} \) (see [9, Proposition 3.5]). The equivalence

\[ \epsilon : EQ^+ \wedge X \to EQ^+ \wedge F(EQ^+, X) \]

gives \( HM^{h} \) an \( HA^{h} \)-module structure in the homotopy category by the following composite:

\[ \xymatrix{ EQ^+ \wedge F(EQ^+, HA) \wedge HM \ar[r]^-{\bar{\epsilon} \wedge HM} & EQ^+ \wedge HA \wedge HM \ar[r] & EQ^+ \wedge HM. } \]

Here \( \bar{\epsilon} \) is the inverse of \( \epsilon \) in the homotopy category. The domain of the composite is \( HA^{h} \wedge HM^{h} \) after applying the appropriate twist map.

**Lemma 8.2** For every Mackey functor \( M \) the modules \( HM^{hQ} \) and \( (HM^{hQ})^{\ast} \) are \( u \)-periodic, i.e., \( u \) acts as a unit.

**Proof** By Lemma 7.1 this is true for \( HA^{h} \).

By [9, Proposition 8.4] the pairing

\[ HA^{h} \wedge HM^{h} \to HM^{h} \]

gives the pairing in the group cohomology:

\[ H^{\ast}(Q; \mathbb{Z}) \otimes H^{\ast}(Q; V) \to H^{\ast}(Q; V) \]

Since \( HA^{hQ}_{2-2u} = H^{0}(Q; \mathbb{Z}) \cong \mathbb{Z} \) and \( u \) is a generator, the statement follows for \( HM^{hQ} \). By the discussion preceding the lemma \( (HM^{hQ})^{\ast} \) is a module over \( HA^{hQ} \), so the second part follows analogously. \( \square \)
Note that if \( X \) is a \( Q \)-spectrum, its homotopy groups \( X^Q_* \) and \( X^e_* \) form an \( RO(Q) \)-graded Mackey functor denoted by \( X^* \). We investigate here the Mackey functor structure of two entries of \( HM^\bullet_\sigma \), namely \( HM^\bullet_{1-\sigma} \) and \( HM^\bullet_{\sigma-1} \). Recall the notation \( \tilde{V} = \tilde{Z} \otimes V \).

**Lemma 8.3** The Mackey functor structure of \( HM^\bullet_{\sigma-1} \) is given by:

\[
\text{coker}(\text{res}_M) \quad N \left\{ \begin{array}{c}
\pi \\
\tilde{V},
\end{array} \right. \]

where \( \pi \) is the map induced by projection of \( V \) onto \( \text{coker}(\text{res}_M) \).

**Proof** By Proposition 4.1 we have that:

\[
HM^\partial_{\sigma-1} = [Q_+ \wedge S^{\sigma-1}, HM]^Q
\]
\[
\cong [Q_+ \wedge S^{-1}, F(S^\sigma, HM)]^Q
\]
\[
\cong \text{Hom}_{\mathbb{Z}[Q]} \left( H_{-1}(Q_+ \wedge S^{-1}), \pi_{-1}(F(S^\sigma, HM)) \right)
\]
\[
\cong H^1(S^\sigma, V) \cong \tilde{V}.
\]

The last isomorphism comes from Remark 4.3.

From Theorem 6.1 we get that \( HM^0_{\sigma-1} = \text{coker}(\text{res}_M) \).

Recall the cofibre sequence (†) from Sect. 3:

\[
Q_+ \longrightarrow S^0 \longrightarrow S^\sigma.
\]

After smashing this sequence with \( \text{HM} \) and applying \([S^{\sigma-1}, -]^Q\) we obtain the following exact sequence of abelian groups, where \( \text{tr} \) denotes the transfer in \( HM^\bullet_{\sigma-1} \):

\[
HM^\epsilon_{\sigma} \longrightarrow HM^0_{\sigma} \longrightarrow HM^0_{\sigma-1} \longrightarrow HM^\epsilon_{\sigma-1} \overset{\text{tr}}{\longrightarrow} HM^0_{\sigma-1} \longrightarrow HM^0_{\sigma-1}.
\]

The outer terms are zero, so from this exact sequence and Theorem 6.1 we can read that the underlying map of abelian groups of \( \text{tr} \) is the projection of \( V \) onto \( \text{coker}(\text{res}_M) \).

Let \( \tilde{x} \in \tilde{V} \) and \( x \in V \) be its underlying element in \( V \). Then the transfer of \( HM^\epsilon_{\sigma-1} \) is given by

\[
\tilde{x} \mapsto x + \text{im}(\text{res}_M).
\]

Note that this map satisfies the condition \( \text{tr}(\tilde{x}) = \text{tr}(\gamma \tilde{x}) \) for transfer in a Mackey functor.
Finally, by the definition of a Mackey functor we have that \( \text{res}(\text{tr}(\tilde{x})) = N\tilde{x} \). Thus the restriction in \( HM_{\sigma - 1}^Q \) has to be the map

\[
x + \text{im}(\text{res}_M) \mapsto N\tilde{x}.
\]

An easy calculation shows that this map is well-defined and satisfies the required properties. \( \square \)

**Lemma 8.4** *The Mackey functor structure of \( HM_{1-\sigma}^\bullet \) is given by:*

\[
\ker(\text{tr}_M) \quad i \left\{ \begin{array}{c}
N \\
\tilde{V}
\end{array} \right\}
\]

where \( i \) is the inclusion of \( \ker(\text{tr}_M) \) in \( (\tilde{V})^Q = NV \).

**Proof** This is analogous to the proof of Lemma 8.3. Note that the identification \( (\tilde{V})^Q = NV \) follows from the following calculation: \( \tilde{x} \in (V)^Q \) if and only if \( \tilde{x} - \gamma \tilde{x} = 0 \). But the last equality on the underlying element in \( V \) gives that \( x + \gamma x = 0 \), i.e., \( x \in NV \). \( \square \)

The following generalisation of Proposition 4.6 will be also of use here:

**Proposition 8.5** *The map \( f_V : (HM_{hQ})_V \to HM_{hQ}^V \) is the map induced by the transfer in the Mackey functor \( HM_{V}^\bullet \). The map \( \epsilon_V : HM_{hQ}^V \to HM_{V}^Q \) is the map induced by the restriction in the Mackey functor \( HM_{V}^\bullet \).*

**Proof** Follows analogously to the proof of Proposition 4.6. \( \square \)

**Proof of Theorem 8.1** (1) Multiplication by \( u \) on \( (HM_{hQ})_\bullet \) and \( HM_{\sigma}^Q \) gives us the following commutative diagram:

\[
\begin{array}{ccc}
(HM_{hQ})_{y\sigma - 2} & \xrightarrow{u} & (HM_{hQ})_{(y-2)\sigma} \\
\downarrow f_{y\sigma - 2} & & \downarrow f_{(y-2)\sigma} \\
HM_{\sigma}^Q_{y\sigma - 2} & \xrightarrow{u} & HM_{\sigma}^Q_{(y-2)\sigma}.
\end{array}
\]

The left vertical arrow is an isomorphism by Theorem 6.1 and the top arrow is an isomorphism by Lemma 8.2. Thus the right vertical arrow is up to isomorphism the same as the bottom arrow.

If \( y = 2 \), the right vertical arrow is the map \( f_0 \). Thus by Proposition 4.6 the bottom arrow is the map induced by transfer on \( V_Q \).

(2) We will use the same commutative diagram as in Point 1. If \( y > 2 \), the right vertical arrow is the map \( f_{(y-2)\sigma} \). By Corollary 4.7, this is the map \( NV/(1-\gamma)V \to \ker(\text{res}_M) \) induced by \( \text{tr}_M \). Similarly as in Point 1, the right vertical arrow is the same as the bottom arrow, which is multiplication by \( u \).
(3) Follows analogously to Point 1 by considering the map $\epsilon : HM \to HM^h$.

(4) By Lemma 3.1 we need to prove the statement only for $y = -1$. By Theorem 7.7 the ring $H^Q_{\Lambda^*}$ is strictly commutative, so the following diagram is commutative:

$$
\begin{array}{ccc}
HM_0^Q & \xrightarrow{u} & HM_{2-2\sigma}^Q \cong V^Q \\
\downarrow^a & & \downarrow^a \\
HM_{-\sigma}^Q \cong \text{coker}(\text{tr}_M) & \xrightarrow{u} & HM_{2-3\sigma}^Q \cong V^Q/NV.
\end{array}
$$

By Lemma 3.1 the vertical arrows in this diagram are projections. From the previous point we get that the top arrow is the restriction map $\text{res}_M$, thus the bottom arrow is the map induced by the restriction $\text{coker}(\text{tr}_M) \to V^Q/NV$.

(5) Note that three corners of the diagram displaying multiplication by $u$ are isomorphic:

$$
\begin{array}{ccc}
HM_{1-\sigma}^Q & \xrightarrow{u} & HM_{3-3\sigma}^Q \\
\downarrow^{\epsilon_{1-\sigma}} & & \downarrow^{\epsilon_{3-3\sigma}} \\
HM_{1-\sigma}^h & \xrightarrow{u} & HM_{3-3\sigma}^h.
\end{array}
$$

The right vertical arrow is an isomorphism by Theorem 6.1 and the bottom arrow is an isomorphism by Lemma 8.2. Thus the top arrow is the same as the left vertical arrow, which by Proposition 8.5 is induced by the restriction of the Mackey functor $HM_{1-\sigma}^\bullet$. So it is the inclusion $\ker(\text{tr}_M) \to (\tilde{V})^Q = N V$.

(6) By the same argument as in Point 5 we have that $u : HM_{1+y\sigma}^Q \to HM_{3+(y-2)\sigma}^Q$ can be identified with $\epsilon_{1+y\sigma} : HM_{1+y\sigma}^Q \to HM_{1+y\sigma}^h$. To identify this map, we use the following diagram:

$$
\begin{array}{ccc}
HM_{1+y\sigma}^Q & \xrightarrow{u} & HM_{1+y\sigma}^\Phi \\
\downarrow^{\epsilon_{1+y\sigma}} & & \downarrow^\phi \\
HM_{1+y\sigma}^h & \xrightarrow{u} & HM_{1+y\sigma}^\Phi.
\end{array}
$$

(*)

If we apply the long exact sequence in homotopy to the top and bottom rows of the Tate diagram we obtain:

$$
\cdots \to (HM_{hQ})_{1+y\sigma} \to HM_{1+y\sigma}^Q \to HM_{1+y\sigma}^\Phi \to (HM_{hQ})_{\gamma\sigma} \to \cdots
$$

and

$$
\cdots \to (HM_{hQ})_{1+y\sigma} \to HM_{1+y\sigma}^h \to HM_{1+y\sigma}^\Phi \to (HM_{hQ})_{\gamma\sigma} \to \cdots.
$$
Since \((HM_{hQ})_{1+y\sigma} = (HM_{hQ})_{y\sigma} = 0\) by Proposition 4.2, the top and bottom horizontal arrows in the diagram \((\ast)\) are isomorphisms. Thus \(u\) acts on \(HM_{Q}^{1+y\sigma}\) in the same way as \(\zeta\), so we need to identify this map.

Note that by \(a\)-periodicity of \(HM^{\Phi}\) and \(HM^{I}\) (see Lemma 2.3) it is enough to identify this map for \(y = 0\). Recall from the proof of Lemma 5.3 that the fibre \(F\) of the map \(\epsilon_{t}: HM^{\Phi} \to HM^{I}\) is 0-coconnective, in particular \(F_{Q}^{1} = 0\). Thus by Proposition 4.8 and Theorem 5.1 we have that \(\zeta\) is an inclusion

\[\ker(tr_{M})_{Q} \hookrightarrow N V/(1 - \gamma)V.\]

(7) The proof of this point needs the following diagram:

\[
\begin{array}{c}
HM_{\sigma-1}^{Q} \xrightarrow{u} HM_{1-\sigma}^{Q} \\
\downarrow \epsilon_{\sigma-1} \quad \downarrow \epsilon_{1-\sigma} \\
HM_{\sigma-1}^{hQ} \xrightarrow{u} HM_{1-\sigma}^{hQ}
\end{array}
\]

By Proposition 8.5 the left vertical map \(\epsilon_{\sigma-1}\) is the map induced by the restriction of the Mackey functor \(HM_{(-\sigma)}^{\bullet-1}\). From Theorem 6.1 and Lemma 8.3 we get that this is the map \(\ker(res_{M}) \to (\tilde{V})^{Q} = N V\). This map can be identified with the multiplication by \(1 + \sigma\) in \(\tilde{V}\), thus the multiplication by \(1 - \gamma\) in \(V\).

Similarly, by Proposition 8.5 the right vertical map is induced by the restriction of the Mackey functor \(HM_{(-\sigma)}^{\bullet}\), so by Lemma 8.4 it is an inclusion \(\ker(res_{M}) \to N V\). Since by Lemma 8.2 the bottom arrow is an isomorphism, the top arrow is a multiplication by \(1 - \gamma\).

(8) By Lemma 6.3 the codomain of \(u: HM_{y\sigma-1}^{Q} \to HM_{(y-2)\sigma+1}^{Q}\) is zero when \(y > 1\).

(9) We consider the following commutative diagram:

\[
\begin{array}{c}
(HM_{hQ})_{y\sigma-3} \xrightarrow{u} (HM_{hQ})_{(y-2)\sigma-1} \\
\downarrow \\
HM_{y\sigma-3}^{Q} \xrightarrow{u} HM_{(y-2)\sigma-1}^{Q}
\end{array}
\]

We use an analogous reasoning as in Point 1 to show that the right arrow is the same as the bottom arrow.

If \(y = 3\), then by Proposition 8.5 and Lemma 8.3 the vertical left map is induced by the transfer in \(HM_{(-\sigma)}^{\bullet-1}\). Therefore it is a projection \(V/NV \to \ker(res_{M})\).

(10) For this point we consider the same diagram as in the previous point together with the observation that the bottom arrow is the same as the right vertical arrow. Thus our goal is to describe the right vertical arrow \(f_{(y-2)\sigma-1}\) if \(y > 3\). Note that by Lemma 3.1 we need only to prove the statement for \(y = 4\).
By Lemma 3.1 and Observation 4.5 the following diagram is commutative with vertical maps being inclusions (compare with the proof of Corollary 4.7):

\[
\begin{array}{ccc}
(HM_hQ)_{2\sigma-1} & \xrightarrow{f_{2\sigma-1}} & HM_{2\sigma-1}^Q \\
\downarrow a & & \downarrow a \\
(HM_hQ)_{\sigma-1} & \xrightarrow{f_{\sigma-1}} & HM_{\sigma-1}^Q.
\end{array}
\]

Therefore the top map is the restriction of the projection \( V/NV \to \text{coker}(\text{res}_M) \) to the domain \( V^Q/NV \) and codomain \( V^Q/\text{im}(\text{res}_M) \).

(11) Follows from Lemma 6.3. In the case \( x < 0 \) and \( y < -x \) the multiplication by \( u \) starts in the left half-plane below the antidiagonal where all entries are zero, therefore it is the zero map. If \( x \geq 0 \) and \( y > -x \), then the multiplication by \( u \) lands in the right half-plane above the antidiagonal, where all entries are zero.

(12) If \( x \leq -2 \) or \( x \geq 2 \) then by Theorem 6.1 we have that \( HM^Q_{x+y\sigma} \) is isomorphic to respectively \( (HM_hQ)_{x+y\sigma} \) and \( HM^hQ_{x+y\sigma} \). Since by Lemma 8.2 both \( (HM_hQ)_\bullet \) and \( HM^hQ_\bullet \) are \( u \)-periodic, the statement follows.

\[\square\]

### 9 Commutativity

In Sect. 7 we have seen that both examples share one feature - the coefficients of both spectra are strictly commutative rings, i.e., the sign coming from the swap of factors is always trivial. In this section we show that this happens for all Green functors. Throughout this section, let \( M \) be a Green functor. Recall that the class of \( Q/e \) in \( A(Q) \) is denoted by \( \omega \).

**Observation 9.1** If \( M \) is a Green functor, then \( HM^Q_\bullet \) is an RO\((Q)\)-graded ring satisfying the following graded commutativity rule: if \( \alpha \in HM^Q_{x+y\sigma} \) and \( \beta \in HM^Q_{x'+y'\sigma} \) then

\[
\alpha\beta = (-1)^{xx'}(1 - \omega)^{yy'} \beta\alpha.
\]

Therefore \( HM^Q_\bullet \) is an RO\((Q)\)-graded commutative ring.

**Remark 9.2** Note that we already used the graded commutativity rule in the proofs of Theorems 7.4 and 7.7. For details on the graded commutativity rule in equivariant homotopy theory see [1, Section 6] or [14, Lemma 2.12] specifically for the case of \( Q \).

The Burnside ring \( \mathbb{A}(Q) \) acts on \( HM^Q_\bullet \) as a 0th \( Q \)-homotopy group of the sphere spectrum. By the associativity of a smash product this is given by the action on \( HM^Q_0 \):

\[
\pi^Q_0(S^0) \otimes (HM^Q_0 \otimes HM^Q_V) = (\pi^Q_0(S^0) \otimes HM^Q_0) \otimes HM^Q_V \to HM^Q_V.
\]
By the definition of a box product of Mackey functors we have that $\omega$ acts on $\mathcal{M}(Q/Q)$ as $\tr_{\mathcal{M}}(1)$: 

$$\omega \cdot 1 = \tr_{\mathcal{M}}(1) \cdot 1 = \tr_{\mathcal{M}}(1 \otimes (\res_{\mathcal{M}}(1))) = \tr_{\mathcal{M}}(1).$$

We used here relations in the box product of Mackey functors. Details may be found in [16, Section 1]. The last equality is obtained by the fact that the restriction in a Green functor is a ring homomorphism, so in particular it preserves the identity.

Recall that $V$ denotes $\mathcal{M}(Q/e)$. Note that if $\mathcal{M}$ is a Green functor then $V$ is a $\mathcal{M}(Q/Q)$-algebra with action given by $\res_{\mathcal{M}}$. From this we can deduce that if $v \in V$ then 

$$\omega v = \res_{\mathcal{M}}(\tr_{\mathcal{M}}(1))v = 2v,$$

i.e., $\omega$ acts on $V$ as multiplication by 2. The last equality here comes from the fact that if $\mathcal{M}$ is a Green functor, then $\gamma$ needs to act on $V$ as a unitary ring homomorphism - thus $\res_{\mathcal{M}}(\tr_{\mathcal{M}}(1)) = (1 + \gamma)1 = 2$.

**Theorem 9.3** If $\mathcal{M}$ is a Green functor, then $H_{\mathcal{M}}^{Q}$ is a strictly commutative ring.

Before giving a proof of Theorem 9.3 we prove a couple of preparatory lemmas.

**Lemma 9.4** For any Mackey functor $\mathcal{M}$ the groups $H_{\mathcal{M}}^{Q}(x+y,\sigma)$ are 2-torsion if $x \neq 0$ and $x \neq -y$.

**Proof** Note that if the conditions of Lemma 6.3 hold (i.e., $x + y\sigma$ lies below the antidiagonal on the half-plane $x < 0$ or above the antidiagonal on the half-plane $x > 0$) then $H_{\mathcal{M}}^{Q}(x+y,\sigma)$ is zero, so the statement is trivially satisfied.

We will consider four cases depending on the value of $x$.

1. $x \geq 2$ and $y < -x$. Assume firstly that $y$ is even. By Theorem 6.1 and Proposition 4.2 we have that 

   $$H_{\mathcal{M}}^{Q}(x+y,\sigma) \cong H^{-x-y}(Q; V).$$

   Note that the group cohomology $H^{p}(Q; V)$ is 2-torsion for $p > 0$ (this can be easily deduced from [17, Theorem 6.2.2]). By the assumption $-x - y \geq 1$, so the statement holds. If $y$ is odd, we have that 

   $$H_{\mathcal{M}}^{Q}(x+y,\sigma) \cong H^{-x-y}(Q; \tilde{V}).$$

   Here we use the fact that $H^{p}(Q; \tilde{V}) \cong H^{p+1}(Q; V)$ (see Sect. 4). Thus the claim is proven in this case.

2. $x \leq -2$ and $y > -x$. In this case we proceed analogically to the previous point, using the fact that $H_{\mathcal{M}}^{Q}(x+y,\sigma)$ is isomorphic to the group homology.

3. $x = 1$ and $y < -x$. Then $H_{\mathcal{M}}^{Q}(1+y,\sigma) \cong \ker(tr_{\mathcal{M}})$. Let $\alpha \in \ker(tr_{\mathcal{M}})$. Then $(1+\gamma)\alpha = \res_{\mathcal{M}}(\tr_{\mathcal{M}}(\alpha)) = 0$, so $\alpha = -\gamma \alpha$. Thus $2\alpha = \alpha - \gamma \alpha = (1-\gamma)\alpha = 0$. 

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(4) \(x = -1\) and \(y > -x\). Then \(HM^Q_{y\sigma} \cong V^Q / \text{im}(\text{res}_M)\). Let \(\alpha \in V^Q / \text{im}(\text{res}_M)\). Then \(2\alpha = (1 + \gamma)\alpha = \text{res}_M(\text{tr}_M(\alpha)) = 0\).

\[\square\]

**Lemma 9.5** The subring \(HM^Q_{x-y\sigma}\) is strictly commutative.

**Proof** Note that unless \(x = 0\) the groups \(HM^Q_{x-x\sigma}\) are submodules or subquotients of \(V\), thus \(\omega\) acts on them as \(2\). Let \(\alpha \in HM^Q_{x-x\sigma}\) and \(\beta \in HM^Q_{x'-x'\sigma}\). By the sign rule the only possibility when a non-trivial sign might occur is when both \(x\) and \(x'\) are odd. In this case we have

\[\alpha\beta = (-1)^{xx'}(1 - 2)^{xx'}\beta\alpha = \beta\alpha.\]

\[\square\]

**Lemma 9.6** The subring \(HM^Q_{x\sigma}\) is strictly commutative.

**Proof** Since the fixed degree is zero we need only to check the possible sign coming from the multiplication by \(1 - \omega\). We are going to prove that \(\omega\) acts as \(0\) unless \(y \neq 0\). Consider two cases:

- \(y > 0\). Then \(HM^Q_{y\sigma} \cong \ker(\text{res}_M)\) by Theorem 6.1. Let \(m \in \ker(\text{res}_M)\). We have that

\[\omega \cdot m = \text{tr}(1) \cdot m = \text{tr}(1 \cdot \text{res}(m)) = 0.\]

So \(\omega\) acts as \(0\).

- \(y < 0\). Then \(HM^Q_{y\sigma} \cong \text{coker}(\text{tr}_M)\). Thus \(\text{tr}_M(1) = 0\) and \(\omega\) acts as \(0\).

Now let \(m \in HM^Q_{y\sigma}\) and \(n \in HM^Q_{y'\sigma}\). If any of \(y\), \(y'\) is zero or even then the statement trivially holds. Thus let both \(y\) and \(y'\) be odd. Then

\[\alpha\beta = (1 - \omega)^{yy'}\beta\alpha = ((1 - \omega)\beta)\alpha = \beta\alpha.\]

**Proof of Theorem 9.3** Let \(\alpha \in HM^Q_{x+y\sigma}\) and \(\beta \in HM^Q_{x'+y'\sigma}\). Then \(\alpha\beta \in HM^Q_{(x+x')+y+y'\sigma}\). Consider the following cases depending on the degree of the product \(\alpha\beta\):

1. \(x + x' \neq 0\) and \(x + x' \neq -(y + y')\). Then the product \(\alpha\beta\) does not lie on either the antidiagonal or the axis \(x = 0\). So by Lemma 9.4 the product \(\alpha\beta\) belongs to a 2-torsion group, thus \(\alpha\beta = \beta\alpha\).
2. \(x = x' = 0\). Then we are in the situation of Lemma 9.6, so the statement holds.
3. \(x = -y\) and \(x' = -y'\). In this case we use Lemma 9.5.
4. \(x, x' \neq 0\) and \(x + x' = 0\). If both \(\alpha\) and \(\beta\) lie on the antidiagonal then the claim is proven by the previous point. Without loss of generality assume that \(x \neq -y\), i.e., \(\alpha\) does not lie on the antidiagonal. By the proof of Lemma 9.6 the possible sign coming from \((1 - \omega)^{yy'}\) is trivial since \(\omega\) acts trivially on groups lying on \(x = 0\) axis. Note that \(\alpha\) belongs to a 2-torsion group, so \(\alpha\beta\) is also 2-torsion and the statement follows.
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(5) \( x + x' = -(y + y') \), i.e., \( \alpha \beta \) lies on the antidiagonal. Then we use the same argument as in the previous point.

Observation 9.7 By Theorem 7.7 we know that in order to describe the \( H_{\mathbb{A}^Q} \)-module structure of \( HM^Q \) for any Mackey functor \( M \) we need to describe an action of three elements—\( a, u \) and \( \omega \). Thus we have already described this structure:

(1) the action of \( a \) is described by Lemma 3.1;
(2) the action of \( u \) is described by Theorem 8.1;
(3) the action of \( \omega \) is described in this Section.

Observation 9.8 Let \( M \) be a Green functor. A big part of the multiplicative structure may be derived from the \( H_{\mathbb{A}^Q} \)-module structure and commutativity described in this Section—we know that there are elements \( a_{HM} = a \cdot 1 \in HM^Q_{-\sigma} \) and \( u_{HM} \in HM^Q_{-2\sigma} \) and their multiplicative relations are described by \( H_{\mathbb{A}^Q} \)-module structure.

This gives a full description of the multiplication in the even fixed degrees. For the odd fixed degrees we proceed as follows:

(1) We need to consider only the elements lying on the antidiagonal, as the relations for other elements follows from Lemma 3.1.
(2) Since the multiplication map \( u : HM^Q_{x+y\sigma} \to HM^Q_{(x+2)+(y-2)\sigma} \) is an isomorphism if \( x \geq 3 \) or \( x \leq -4 \), we can restrict our attention to the elements of degrees \( 1 - \sigma, 3 - 3\sigma, \sigma - 1 \) and \( 3\sigma - 3 \).
(3) Relations between these elements can be inferred from the ring maps \( \epsilon_* : HM^Q \to HM^h \) and \( g_* : HM^Q \to HM^\Phi \).

Examples of such computations are in Sect. 10.

10 Further examples

10.1 Constant Mackey functor \( \mathbb{F}_2 \)

We start this section with the constant Mackey functor \( \mathbb{F}_2 \). It has the following structure:

\[
\begin{array}{c}
\mathbb{F}_2 \\
\mathcal{L} \\
\mathbb{F}_2
\end{array}
\]

Computations of the coefficients of \( HF_{\mathbb{F}_2} \) which are built on the unpublished work of Stong appear in [4, 7], also in work of Hu-Kriz in [14, Proposition 6.2].

Remark 10.1 To express the RO\((Q)\)-graded ring structure of \( HF_{\mathbb{F}_2}^Q \) we will use here the notation from [11, Remark 2.1]. Therefore we define the following \( \mathbb{F}_2[\lambda] \)-module:

\[
\mathbb{F}_2[\lambda]_{\lambda^\infty} := \operatorname{colim}_k \mathbb{F}_2[\lambda]_{\lambda^k}.
\]
This is a $\mathbb{F}_2[\lambda]$-module consisting entirely of elements that are infinitely divisible by $\lambda$. By $\frac{\mathbb{F}_2[\lambda]}{\lambda^\infty}\{\theta\}$ we denote the $\mathbb{F}_2[\lambda]$-module consisting of elements of the form $\frac{\theta}{\lambda^k}$ for $k \geq 1$. Note that in particular $\theta \notin \frac{\mathbb{F}_2[\lambda]}{\lambda^\infty}\{\theta\}$.

**Theorem 10.2** The $RO(Q)$-graded abelian group structure and the multiplicative structure of $HF_2^*$ are given by:

$$(HF_2^*)_* \cong \mathbb{F}_2[a, \lambda] \oplus \bigoplus_{s \geq 0} \frac{\mathbb{F}_2[\lambda]}{\lambda^\infty}\{\frac{\theta}{a^s}\}.$$  

with $|\theta| = \sigma - 1$, $|\lambda| = 1 - \sigma$ and $|a| = -\sigma$.

This data is presented in Fig. 7.

**Remark 10.3** Note that in the theorem above $\theta$ is a “virtual” element which does not describe any existing element of $(HF_2^*)_*$. Also note that the product of any two elements from the second summand is zero. The theorem in this form appeared in [11, Section 2.1].

In order to prove this theorem we will need to describe the multiplicative structure of $(HF_2^*)_h^Q$ and coefficients of $(HF_2^*)_h$.

**Lemma 10.4**

$$(HF_2^*)_h^Q \cong \mathbb{F}_2[a][\lambda, \lambda^{-1}].$$

**Proof** The $RO(Q)$-graded abelian group structure follows from Proposition 4.2. To see the multiplicative structure we are going to describe the $E_1$-page of the trigraded homotopy fixed point spectral sequence, following the discussion given in Sect. 4.1.

Fix $y$. Recall from Sect. 4.1 that for $X = HF_2^*$:

$$E_1^{p,q}(y) = \text{Hom}_Q(H_p(Q_+ \wedge S^p), \pi_{-q}(F(S^{\sigma}, HF_2^2)))$$

where $\pi_{-q}(F(S^{\sigma}, HF_2^2))$ denotes the $-q$-th homotopy group of the underlying naive spectrum of $F(S^{\sigma}, HF_2^2)$.

We have that $H_p(Q_+ \wedge S^p) \cong \mathbb{Z}[Q]$ as a $\mathbb{Z}[Q]$-module and

$$\pi_{-q}(F(S^{\sigma}, HF_2^2)) \cong \begin{cases} \mathbb{Z}/2 & \text{if } y = q \\ 0 & \text{else}. \end{cases}$$

The differentials on the $E_1$-page are the differentials computing the group cohomology, thus they are all 0. So the spectral sequence collapses on the $E_1$-page.

The trigraded homotopy fixed point spectral sequence is multiplicative, so we can describe the $E_1$-page as an algebra as follows:

$$E_1^{*,*}(*) \cong \mathbb{F}_2[a][\lambda, \lambda^{-1}].$$
The green lines represent multiplication by $\lambda$. Red lines, as before, represent multiplication by $a$ with $|a| = (1, -1, -1)$ and $|\lambda| = (0, -1, -1)$. Since all differentials are 0, we get that

\[ (H\mathbb{F}_2)^Q_{x+y\sigma} \cong \mathbb{F}_2[a][\lambda, \lambda^{-1}] \]

with $|a| = -\sigma$ and $|\lambda| = 1 - \sigma$.

**Corollary 10.5**

\[ (H\mathbb{F}_2)^Q_{tQ} \cong \mathbb{F}_2[a][\lambda, \lambda^{-1}] \]
\[ (H\mathbb{F}_2)^hQ_{hQ} \cong \mathbb{F}_2[a^{-1}][\lambda, \lambda^{-1}] \]
Proof The first statement follows from the fact that $H_{\mathbb{F}_2}^P \simeq H_{\mathbb{F}_2}^h \wedge \widetilde{E Q}$ and from Lemma 8.2. The second part follows from the long exact sequence in homotopy for the cofibre sequence

$$\left( H_{\mathbb{F}_2} \right)_h \rightarrow H_{\mathbb{F}_2}^h \rightarrow H_{\mathbb{F}_2}^t$$

where the map $\left( \left( H_{\mathbb{F}_2} \right)_h^Q \right)_\ast \rightarrow \left( H_{\mathbb{F}_2} \right)_h^Q$ is multiplication by $N$, so zero in this case. $\square$

Proof of Theorem 10.2 The $RO(Q)$-graded abelian group structure follows from the Theorem 6.1. Following Observation 9.8 we know that the multiplicative structure of $(H_{\mathbb{F}_2})_\ast^Q$ is described by the following elements:

- $a = a_{H_{\mathbb{F}_2}}$ — for the properties of multiplication by $a$ see Lemma 3.1;
- $u = u_{H_{\mathbb{F}_2}}$ — see Sect. 8;
- $\lambda \in (H_{\mathbb{F}_2})_{1-\sigma}^Q$.

We need to check three relations that do not follow directly from previous sections — i.e., we need to prove that:

1. $\lambda^2 = u$.
2. If $\eta$ is the generator of $H_{\mathbb{A}_2}^Q_{2\sigma-2}$ then $H_{\mathbb{M}_3}^Q_{3\sigma-3}$ is generated by $\lambda^{-1}\eta$.
3. $\eta^2 = 0$.

The rest of the structure will follow.

We firstly prove point (1), thus we are going to prove that $\lambda^2 = u$. Consider the diagram expressing multiplication by $\lambda$:

$$\begin{array}{ccc}
(H_{\mathbb{F}_2})_{1-\sigma}^Q & \xrightarrow{\lambda} & (H_{\mathbb{F}_2})_{2-2\sigma}^Q \\
\downarrow_{\epsilon_{1-\sigma}} & & \downarrow_{\epsilon_{2-2\sigma}} \\
(H_{\mathbb{F}_2})_{1-\sigma}^h & \xrightarrow{\lambda} & (H_{\mathbb{F}_2})_{2-2\sigma}^h 
\end{array}$$

By Theorem 6.1, the right vertical arrow is an isomorphism, so does the bottom horizontal arrow by Lemma 10.4. From Lemma 8.4 we deduce that the left vertical arrow is also an isomorphism. Thus the top vertical arrow is an isomorphism and $\lambda^2 = u$.

Now proceed to the point (2). Let $\eta$ be a generator of $(H_{\mathbb{F}_2})_{2\sigma-2}^Q$ and $\eta'$ a generator of $(H_{\mathbb{F}_2})_{3\sigma-3}^Q$. Similar argument as above applied to the diagram

$$\begin{array}{ccc}
\left( (H_{\mathbb{F}_2})_h^Q \right)_{3\sigma-3} & \xrightarrow{\lambda} & \left( (H_{\mathbb{F}_2})_h^Q \right)_{2\sigma-2} \\
\downarrow_{f_{\sigma-1}} & & \downarrow_{f_{2\sigma-2}} \\
(H_{\mathbb{F}_2})_{3\sigma-3}^Q & \xrightarrow{\lambda} & (H_{\mathbb{F}_2})_{2\sigma-2}^Q 
\end{array}$$
together with Corollary 10.5 shows that \( \lambda^{-1} \eta = \eta' \).

From the relations above we derive the point (3), i.e., that \( \eta^2 = 0 \). Note that \( \lambda \eta = 0 \). By multiplying the relation from the previous point \( \eta = \lambda \eta' \) by \( \eta \) and using commutativity, we get that \( \eta^2 = 0 \).

The statement in the Theorem is obtained by putting \( \eta = \frac{\theta}{\lambda} \). \( \square \)

10.2 The norm of \( \mathbb{F}_2, N_e^\mathbb{F}_2 \)

We continue the examples section with the Mackey functor \( N_e^\mathbb{F}_2 \). It has the form

\[
\begin{array}{c}
\mathbb{Z}/4 \\
\downarrow^2 \\
\mathbb{Z}/2.
\end{array}
\]

This Mackey functor appears in the work of Hill in [13], where he computes the Bredon homology with coefficients in it of spaces of the form \( \Omega^\sigma \Sigma^\sigma X \). We will follow the notation from this paper and denote this Mackey functor by \( B \). For us \( B \) is an example of an interesting feature - it has only two zero columns, which are in fixed degrees 1 and \(-1\).

Since \( B(Q/e) = \mathbb{F}_2(Q/e) \), Lemma 10.4 and Corollary 10.5 describe also \( H B_{hQ}^k \), \( (H B_{hQ})_\ast \), and \( H B_{tQ}^k \). Most of the proofs follow in the analogous way as in the previous examples, so we will only comment on how to get the multiplicative structure of \( H B_{Q}^\ast \).

Lemma 10.6

\[
H B_{h\Phi}^Q \cong \mathbb{F}_2[a, a^{-1}] \oplus \lambda^2 \mathbb{F}_2[a, a^{-1}, \lambda]
\]

with \( |a| = -\sigma \) and \( |\lambda| = 1 - \sigma \).

Theorem 10.7 The RO\((Q)\)-graded abelian group structure and the multiplicative structure of \( H B_{Q}^\ast \) is given by:

\[
H B_{Q}^\ast \cong \frac{\mathbb{Z}/4[a]}{2a} [u, u\lambda] \oplus \frac{\mathbb{Z}_4[a, u, u\lambda]}{(a^\infty, u^\infty, (u\lambda)^\infty)} \{2\}.
\]

Here \( |a| = -\sigma \), \( |\lambda| = 1 - \sigma \), \( |u| = 2 - 2\sigma \), \( u = \lambda^2 \) and the second direct summand is a \( \mathbb{Z}_4[a, u, u\lambda] \)-module consisting of elements of the form

\[
\frac{2}{a^k u^l (u\lambda)^m}
\]

for \( k, l, m \geq 0 \) not simultaneously equal to 0 (see Remark 10.1). This data is presented in Fig. 8.
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**Proof** We are going to comment only on the multiplicative structure. By Observation 9.8 we know that it can be described by the following elements:

- $a_{HB} \in HB_{\cdot \sigma}$;
- $u_{HB} \in HB_{2-2\sigma}$;
- a generator of $HB_{3-3\sigma}$.

and their inverses. From the map $\epsilon_*: HB_*^Q \rightarrow HB_*^{hQ}$ we can deduce that the last element may be described as $\lambda^3 = u\lambda$. Thus the result follows. □
We close the examples section with an example of a Mackey functor with a non-trivial action on the $Q/e$-level. This is the fixed points Mackey functor of $\tilde{\mathbb{Z}}$ and its structure is given by:

$$H\tilde{\mathbb{Z}}^Q_{x+y\sigma}$$

Key: ● $\mathbb{Z}$, ○ $\mathbb{Z}/2$

Fig. 9 Coefficients of $H\tilde{\mathbb{Z}}$

10.3 The Mackey functor $\tilde{\mathbb{Z}}$

We close the examples section with an example of a Mackey functor with a non-trivial action on the $Q/e$-level. This is the fixed points Mackey functor of $\tilde{\mathbb{Z}}$ and its structure is given by:

$$\begin{array}{c}
0 \\
\downarrow \kappa \\
\tilde{\mathbb{Z}}. \\
\end{array}$$
We denote this Mackey functor by \( \tilde{Z} \). Since \( \gamma \) does not act on the \( Q/e \)-level as a unitary ring homomorphism, it is not a Green functor. Therefore we describe only the \( H_{A^Q_*} \)-module structure of \( H_{\tilde{Z}^Q_*} \).

**Lemma 10.8**

\[
\left( H_{\tilde{Z}^Q_*} \right)^* \cong \left( H_{\tilde{Z}^Q_*} \right)^*_{+1-\sigma}
\]

\[
H_{\tilde{Z}^Q_*} \cong H_{\tilde{Z}^Q_*}^*_{+1-\sigma}.
\]

**Proof** Follows from the fact that \( \tilde{Z} \cong Z \) as \( Z[Q] \)-modules and Proposition 4.2. □

The \( H_{A^Q_*} \)-module structure of \( H_{\tilde{Z}^Q_*} \) is depicted in Fig. 9. Multiplication by \( u \) is represented by blue dashed lines, whereas multiplication by \( a \) is given by red dashed lines. Note that since the \( Q \)-action on the \( Q/e \)-level of \( \tilde{Z} \) is non-trivial, entries on \( 1-\sigma \) and \( \sigma-1 \) spots are different from the rest of the \( x = 1 \) and \( x = -1 \) columns, respectively.

**Acknowledgements** I would like to thank John Greenlees for the idea and continuous supervision of this work. I am also very grateful to Luca Pol and Jordan Williamson for numerous discussions, comments and corrections. The final version of this paper owes its readability to their countless suggestions and patient reading of previous versions. This paper also significantly improved as a result of the comments and suggestions made by the anonymous referee, to whom my thanks are also due.

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**References**

1. Adams, J.F.: Prerequisites (on equivariant stable homotopy) for Carlsson’s lecture, Algebraic topology, Aarhus, (Aarhus, 1982), Lecture Notes in Math., **1984**, vol. 1051, pp. 483–532. Springer, Berlin (1982)
2. Bredon, G.E.: Equivariant Cohomology Theories, Lecture Notes in Mathematics, vol. 34. Springer, Berlin (1967)
3. Brown, K.S.: Cohomology of Groups, Graduate Texts in Mathematics, vol. 87. Springer, New York (1982)
4. Caruso, J.L.: Operations in equivariant \( \mathbb{Z}/p \)-cohomology. Math. Proc. Camb. Philos. Soc. **126**(3), 521–541 (1999)
5. Dugger, D.: An Atiyah-Hirzebruch spectral sequence for \( KR \)-theory. \( K \)-Theory **35**(3-4), 213–256 (2006)
6. Ferland, K.K., Gaunce, L.L., Jr.: The \( RO(G) \)-graded equivariant ordinary homology of \( G \)-cell complexes with even-dimensional cells for \( G = \mathbb{Z}/p \). Mem. Am. Math. Soc. **167**(794), vii+129 (2004)
7. Ferland, K.K.: On the \( RO(G) \)-graded equivariant ordinary cohomology of generalized \( G \)-cell complexes for \( G = \mathbb{Z}/p \). ProQuest LLC, Ann Arbor, MI, Thesis (Ph.D.)–Syracuse University (1999)
8. Greenlees, J.P.C.: Four Approaches to Cohomology Theories with Reality, An Alpine Bouquet of Algebraic Topology, Contemporary Mathematics, vol. 708, pp. 139–156. American Mathematical Society, Providence (2018)
9. Greenlees, J.P.C., May, J.P.: Generalized Tate cohomology. Mem. Am. Math. Soc. **113**(543), viii–178 (1995)
10. Greenlees, J.P.C., Meier, L.: Gorenstein duality for real spectra. Algebraic Geom. Topol. **17**(6), 3547–3619 (2017)
11. Guillou, B.J., Hill, M.A., Isaksen, D.C., Ravenel, D.C.: The cohomology of $C_2$-equivariant $A(1)$ and the homotopy of $koC_2$. Tunis. J. Math **2**(3), 567–632 (2020)
12. Hill, M.A., Hopkins, M.J., Ravenel, D.C.: On the nonexistence of elements of Kervaire invariant one. Ann. Math. (2) **184**(1), 1–262 (2016)
13. Hill, M.A.: On the algebras over equivariant little disks, arXiv e-prints (2017). arXiv:1709.02005
14. Hu, P., Kriz, I.: Real-oriented homotopy theory and an analogue of the Adams–Novikov spectral sequence. Topology **40**(2), 317–399 (2001)
15. Lewis, G., May, J.P., McClure, J.: Ordinary $RO(G)$-graded cohomology. Bull. Am. Math. Soc. (N.S.) **4**(2), 208–212 (1981)
16. Lewis, Jr., L.G.: The $RO(G)$-graded equivariant ordinary cohomology of complex projective spaces with linear $\mathbb{Z}, p$ actions, Algebraic topology and transformation groups (Göttingen,: Lecture Notes in Math., 1988, vol. 1361, pp. 53–122. Springer, Berlin (1987)
17. Weibel, C.A.: An Introduction to Homological Algebra, Cambridge Studies in Advanced Mathematics, vol. 38. Cambridge University Press, Cambridge (1994)
18. Zeng, M.: Equivariant Eilenberg-MacLane spectra in cyclic $p$-groups, arXiv Mathematics e-prints (2018). arXiv:1710.01769

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