Pseudo-Harmonic Maps From Complete Noncompact Pseudo-Hermitian Manifolds To Regular Balls

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Abstract

In this paper, we give an estimate of sub-Laplacian of Riemannian distance functions in pseudo-Hermitian manifolds which plays a similar role as Laplacian comparison theorem, and deduce a prior horizontal gradient estimate of pseudo-harmonic maps from pseudo-Hermitian manifolds to regular balls of Riemannian manifolds. As an application, Liouville theorem is established under the conditions of nonnegative pseudo-Hermitian Ricci curvature and vanishing pseudo-Hermitian torsion. Moreover, we obtain the existence of pseudo-harmonic maps from complete noncompact pseudo-Hermitian manifolds to regular balls of Riemannian manifolds.

1 Introduction

Inspired by Eells-Sampson’s theorem, one natural problem is to consider the existence of harmonic maps from complete noncompact Riemannian manifold. It is always affirmative under some convexity conditions. For example, target manifolds have nonpositive sectional curvature (cf. [10, 18]) and the images of initial maps are regular balls (cf. [17]). Based on elliptic theory, these theorems have been studied for generalized harmonic maps (cf. [7, 19]). The pseudo-harmonic map is an analogue of the harmonic map in pseudo-Hermitian geometry. Let \((M, \theta)\) be a pseudo-Hermitian manifold of real dimension \(2m + 1\) and \((N, h)\) be a Riemannian manifold. The horizontal energy of a smooth map \(f : M \to N\) is defined by

\[
E_H(f) = \int_M |d_h f|^2 \theta \wedge (d\theta)^m
\]

where \(d_h f\) is the horizontal part of \(df\). The pseudo-harmonic map is a critical point of \(E_H\). Hence it locally satisfies the following Euler-Lagrange equation

\[
\tau^i_H(f) \triangleq \Delta_{h} f^i + \sum_{j,k} \Gamma^i_{jk}(f) \langle d_h f^j, d_h f^k \rangle = 0,
\]
where $\Gamma^i_{jk}$’s are Christoffel symbols of Levi-Civita connection in $(N, h)$. The sub-Laplacian is a subelliptic operator which enjoys similar local Sobolev theorems as elliptic theory. By heat flow method, the Eells-Sampson’s type theorem also holds for pseudo-harmonic maps (cf. [5, 20]). The Dirichlet problem of pseudo-harmonic maps to regular balls has also been solved by Jost-Xu (cf. [15]).

This paper will study the existence of pseudo-harmonic maps from complete noncompact pseudo-Hermitian manifolds to regular balls. Laplacian comparison theorem is an important tool to explore global problems in Riemannian geometry. Its extension to Sasakian geometry has been studied in [1, 3, 6, 16]. However, up to now, there is no satisfactory comparison theorem for a pseudo-Hermitian manifold, which is not Sasakian. For our purpose, we will give a new sub-Laplacian comparison theorem for a pseudo-Hermitian manifold. Note that the Riemannian distance associated with Webster metric has better regularity than the Carnot-Carathéodory-distance, and its variational theory is well studied in Riemannian geometry. By the index comparison theorem in Riemannian geometry, we can derive the following estimate of sub-Laplacian of Riemannian distance on pseudo-Hermitian manifolds.

**Theorem 3.6.** Suppose $(M^{2m+1}, \theta)$ is a complete pseudo-Hermitian manifold and $B_R(x_0)$ is the Riemannian geodesic ball of radius $R$ centered at $x_0 \in M$. If for some $k, k_1 \geq 0$,

$$R_* \geq -k, \quad |A|, |\text{div}A| \leq k_1, \quad \text{on } B_R(x_0),$$

where $R_*$ is the pseudo-Hermitian Ricci curvature and $A$ is the pseudo-Hermitian torsion, then there exists $C_3 = C_3(m)$ such that

$$\Delta_b r \leq C_3 \left( \frac{1}{r} + \sqrt{1 + k + k_1 + k_1^2} \right), \quad \text{on } B_R(x_0) \setminus \text{Cut}(x_0), \quad (1.3)$$

where $r$ is the Riemannian distance from $x_0$ and $\text{Cut}(x_0)$ is the cut locus of $x_0$.

Based on this sub-Laplacian comparison theorem, we will establish a local prior horizontal gradient estimate of pseudo-harmonic maps. Hence the solutions of Dirichlet problem on exhaustion regions can approach a global pseudo-harmonic map. Our main theorem is as follows

**Theorem 5.4.** Suppose that $(M, \theta)$ is a complete noncompact pseudo-Hermitian manifold and $(N, h)$ is a Riemannian manifold with sectional curvature bounded from above. Then there is a pseudo-harmonic map $f : M \to B_D(p_0)$ which is a regular ball of $N$.

Besides, we also obtain the following Liouville theorem.

**Theorem 4.5.** Let $(M, \theta)$ be a noncompact complete Sasakian manifold with nonnegative pseudo-Hermitian Ricci curvature and $(N, h)$ be a Riemannian manifold with sectional curvature bounded above. Then there is no nontrivial pseudo-Hermitian map from $M$ to any regular ball of $N$.

The similar Liouville theorem for harmonic case was deduced by Choi [9]. One may doubt whether the pseudo-harmonic map given by Theorem 5.4 is trivial. We will show an example whose domain is Sasakian with negative pseudo-Hermitian Ricci curvature.
2 Basic Notions

In this section, we present some basic notions of pseudo-Hermitian geometry and pseudo-harmonic maps. For details, readers may refer to [12, 24, 25]. Recall that a smooth manifold \( M \) of real dimension \( 2m + 1 \) is said to be a CR manifold if there exists a smooth rank \( n \) complex subbundle \( T_{1,0}M \subset TM \otimes \mathbb{C} \) such that

\[
T_{1,0}M \cap T_{0,1}M = \{0\} \tag{2.1}
\]

\[
[\Gamma(T_{1,0}M), \Gamma(T_{1,0}M)] \subset \Gamma(T_{1,0}M) \tag{2.2}
\]

where \( T_{0,1}M = \overline{T_{1,0}M} \) is the complex conjugate of \( T_{1,0}M \). Equivalently, the CR structure may also be described by the real subbundle \( HM = Re \{T_{1,0}M \oplus T_{0,1}M\} \) of \( TM \) which carries an almost complex structure \( J: HM \to HM \) defined by \( J(X + \overline{X}) = i(X - \overline{X}) \) for any \( X \in T_{1,0}M \). Since \( HM \) is naturally oriented by the almost complex structure \( J \), then \( M \) is orientable if and only if there exists a global nowhere vanishing 1-form \( \theta \) such that \( HM = Ker(\theta) \). Any such section \( \theta \) is referred to as a pseudo-Hermitian structure on \( M \).

The space of all pseudo-Hermitian structures is 1-dimensional. The Levi form \( L_\theta \) of a given pseudo-Hermitian structure \( \theta \) is defined by

\[
L_\theta(X,Y) = d\theta(X, JY) \quad \text{for any } X,Y \in HM.
\]

An orientable CR manifold \((M, HM, J)\) is called strictly pseudo-convex if \( L_\theta \) is positive definite for some \( \theta \). Such a quadruple \((M, HM, J, \theta)\) is called a pseudo-Hermitian manifold. For simplicity, we denote it by \((M, \theta)\).

For a pseudo-Hermitian manifold \((M, \theta)\), there exists a unique nowhere zero vector field \( \xi \), called the Reeb vector field, transverse to \( HM \) and satisfying \( \xi \wedge \theta = 1 \), \( \xi \wedge d\theta = 0 \). It gives a decomposition of the tangent bundle \( TM \):

\[
TM = HM \oplus \mathbb{R}_\xi \tag{2.3}
\]

which induces the projection \( \pi_H : TM \to HM \). Set \( G_\theta = \pi_H^* L_\theta \). Since \( L_\theta \) is a metric on \( HM \), it is natural to define a Riemannian metric

\[
g_\theta = G_\theta + \theta \otimes \theta \tag{2.4}
\]

which makes \( HM \) and \( \mathbb{R}_\xi \) orthogonal. Such metric \( g_\theta \) is called Webster metric, also denoted by \( \langle \cdot , \cdot \rangle \). By requiring \( J\xi = 0 \), the almost complex structure \( J \) can be extended to an endomorphism of \( TM \). The integrable condition \( \langle \cdot , \cdot \rangle \) guarantees that \( g_\theta \) is \( J \)-invariant. Clearly, \( \theta \wedge (d\theta)^m \) differs a constant with the volume form of \( g_\theta \). Henceforth it is always regarded as the canonical volume form in pseudo-Hermitian geometry.

It is remarkable that \((M, HM, G_\theta)\) could also be viewed as a sub-Riemannian manifold which satisfies the strong bracket generating hypothesis. The completeness is well settled under the Carnot-Carathéodory distance (cf. [23]). By definition, this distance is larger than Riemannian distance associated with the Webster metric \( g_\theta \) which implies that sub-Riemannian completeness is stronger than Riemannian one. In this paper, a pseudo-Hermitian manifold \((M, \theta)\) is called complete if it is complete associated with the Webster metric \( g_\theta \).
On a pseudo-Hermitian manifold, there exists a canonical connection $\nabla$, which is called Tanaka-Webster connection (cf. [12]), preserving the horizontal distribution, almost complete structure and Webster metric. Moreover, its torsion $T_{\nabla}$ satisfies

$$T_{\nabla}(X, Y) = 2d\theta(X, Y)\xi \quad \text{and} \quad T_{\nabla}(\xi, JX) + JT_{\nabla}(\xi, X) = 0. \tag{2.5}$$

The pseudo-Hermitian torsion, denoted by $\tau$, is a symmetric and traceless tensor defined by $\tau(X) = T_{\nabla}(\xi, X)$ for any $X \in TM$ (cf. [12]). Set

$$A(X, Y) = g_{\theta}(\tau(X), Y), \quad \text{for any} \ X, Y \in TM.$$ 

A pseudo-Hermitian manifold is Sasakian if $\tau \equiv 0$. Sasakian geometry plays important roles in Kähler geometry and Einstein metrics (cf. [1]).

Suppose that $(M, \theta)$ is a pseudo-Hermitian manifold of real dimension $2m + 1$. Let $R$ be the curvature tensor of the Tanaka-Webster connection. Set

$$R(X, Y, Z, W) = \langle R(Z, W)Y, X \rangle, \quad \text{for any} \ X, Y, Z, W \in TM.$$ 

Let $\{\eta_\alpha\}_{\alpha=1}^m$ be a local unitary frame of $T_{1,0}M$ and $R_{ABCD}$ be the components of $R$ under the frame $\{\eta_0 = \xi, \eta_\alpha, \eta_{\bar{\alpha}}\}$. Webster [25] derived the first Bianchi identity, i.e.

$$R_{\bar{\alpha}\beta\lambda\bar{\mu}} = R_{\bar{\alpha}\lambda\bar{\beta}\mu}.$$ 

The other components of $R$ can be solved by the pseudo-Hermitian torsion and its derivative. For example,

$$R_{\alpha\beta\lambda\mu} = 2i(A_{\beta\mu}\delta_{\alpha\lambda} - A_{\alpha\lambda}\delta_{\beta\mu}), \quad R_{\alpha\beta0\mu} = -A_{\beta\mu, \bar{\alpha}}, \quad R_{\alpha\beta\bar{\alpha}0\bar{\mu}} = A_{\alpha\bar{\mu}, \beta}$$

where $A_{\beta\mu, \bar{\alpha}}, A_{\alpha\bar{\mu}, \beta}$ are the components of $\nabla A$. Tanaka [24] defined the pseudo-Hermitian Ricci tensor $R_\ast$ by

$$R_\ast X = -i \sum_{\lambda=1}^m R(\eta_\lambda, \eta_{\bar{\lambda}})JX \quad \text{for any} \ X \in T_{1,0}M. \tag{2.6}$$

The pseudo-Hermitian scalar curvature is given by

$$s = \frac{1}{2} \text{trace}_{G_\theta} R_\ast. \tag{2.7}$$

In this paper, we will use Einstein summation contention when there is an repeated index. Denote $R_{\lambda\bar{\mu}} = R_{\alpha\lambda\bar{\mu}}$. Hence by the first Bianchi identity, $R_\ast \eta_0 = R_{\alpha\bar{\beta}}\eta_\beta$ and $s = R_{\alpha\bar{\alpha}}$.

Assume that $(N, h)$ is a Riemannian manifold. Let $\{\sigma^i\}$ be an local orthonormal frame of $T^*N$. Denote the Levi-Civita connection and the Riemannian curvature of $(N, h)$ by $\nabla^N$ and $\mathcal{R}^N$ respectively. Suppose that $f : M \to N$ is a smooth map. The pullback connection on the pullback bundle $f^*(TN)$ and the Tanaka-Webster connection induce a connection on $TM \otimes f^*(TN)$, also denoted by $\nabla$.

**Definition 2.1.** A smooth map $f : M \to N$ is called pseudo-harmonic if the tensor field

$$\tau_H(f) \triangleq \text{trace}_{G_\theta} \nabla_b df \equiv 0,$$

where $\nabla_b df$ is the restriction of $\nabla df$ onto $HM \times HM$. 

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Actually, pseudo-harmonic maps are the Dirichlet critical points of the horizontal energy (cf. [2, 12])

\[ E_H(f) = \frac{1}{2} \int_M |d_b f|^2 \theta \wedge (d\theta)^m \]  

(2.8)

where \( d_b f \) is the horizontal restriction of \( df \). The sub-Laplacian \( \Delta_b u \) of a smooth function \( u \) is defined by

\[ \Delta_b u = \text{trace}_{G_0} \nabla_b d_b u, \]  

(2.9)

which is viewed as the special case of \( \tau_H \) acting on functions.

**Lemma 2.2** (CR Bochner Formulas, cf. [5, 14, 21]). For any smooth map \( f : M \to N \), we have

\[ \frac{1}{2} \Delta_b |d_b f|^2 = |\nabla_b d_b f|^2 + \langle \nabla_b \tau_H(f), d_b f \rangle + 4i(f^i_\alpha f^i_\alpha - f^i_\beta f^i_\beta) \]

\[ + 2R_{\alpha\beta\delta} f^i_\delta f^i_\gamma - 2i(m - 2)(f^i_\alpha f^i_\beta A_{\alpha\beta} - f^i_\alpha f^i_\beta A_{\alpha\beta}) \]

\[ + 2(f^i_\alpha f^i_\beta f^i_\gamma f^i_\delta R^N_{ijkl} + f^i_\alpha f^i_\beta f^i_\gamma f^i_\delta R^N_{ijkl}) \]  

(2.10)

\[ \frac{1}{2} \Delta_b |f_0|^2 = |\nabla_0 f_0|^2 + \langle \nabla_{\xi \tau_H}(f), f_0 \rangle + 2f^i_\alpha f^j_\beta A_{\alpha\beta}^* f^k_\delta A_{\delta\alpha}^* + 2f^i_\alpha f^j_\beta A_{\alpha\beta}^* A_{\beta\alpha}^* + f^i_\alpha f^j_\beta A_{\alpha\beta} + f^i_\alpha f^j_\beta A_{\beta\alpha} \]  

(2.11)

where \( f^i_A \) and \( f^i_{AB} \) are the components of \( df \) and \( \nabla df \) respectively under the orthonormal coframe \( \{\theta, \theta^a, \theta^{\bar{a}}\} \) of \( T^*M \) and an orthonormal frame \( \{\sigma_i\} \) of \( T^*N \), and \( f_0 = df(\xi) \).

Let \( \pi_{(1,1)} \nabla_b d_b f \) be the \((1,1)\)-part of \( \nabla_b d_b f \) and

\[ \pi_{(1,1)} \nabla_b d_b f = \nabla_b d_b f - \pi_{(1,1)} \nabla_b d_b f \]

which is orthogonal to \( \pi_{(1,1)} \nabla_b d_b f \). The commutation relation (cf. [5, 21])

\[ f^i_{\alpha\beta} - f^i_{\beta\alpha} = 2i f^i_0 \delta^{\alpha\beta}_{\alpha\beta} \]  

(2.12)

shows that

\[ |\pi_{(1,1)} \nabla_b d_b f|^2 \geq 2 \sum_{\alpha=1}^{m} f^i_{\alpha\bar{\alpha}} f^i_{\bar{\alpha}\alpha} \]

\[ = \frac{1}{2} \sum_{\alpha=1}^{m} \left[ |f^i_{\alpha\bar{\alpha}} + f^i_{\bar{\alpha}\alpha}|^2 + |f^i_{\alpha\bar{\alpha}} - f^i_{\bar{\alpha}\alpha}|^2 \right] \]

\[ \geq \frac{1}{2} \sum_{\alpha=1}^{m} |f^i_{\alpha\bar{\alpha}} - f^i_{\bar{\alpha}\alpha}|^2 \]

\[ = 2m |f_0|^2. \]  

(2.13)

Combining with Lemma 2.2, we have the following lemma.
Lemma 2.3. Suppose that \((M^{2m+1}, \theta)\) is a pseudo-Hermitian manifold with
\[
R_* \geq -k, \text{ and } |A|, |\text{div} A| \leq k_1 \tag{2.14}
\]
and \((N, h)\) is a Riemannian manifold with sectional curvature
\[
K^N \leq \kappa \tag{2.15}
\]
for \(k, k_1, \kappa \geq 0.\) Then there exists \(C_1 = C_1(k, k_1)\) such that for any pseudo-harmonic map \(f : M \rightarrow N,\) we have
\[
\Delta_b|d_bf|^2 \geq (2 - \epsilon)|\nabla_b d_bf|^2 + 2m\epsilon|f_0|^2 + \epsilon|\pi^1_{(1,1)} \nabla_b d_bf|^2 - \epsilon_1|\nabla_b f_0|^2 - (C_1 + 16\epsilon_1^{-1})|d_bf|^2 - 2\kappa|d_bf|^4 \tag{2.16}
\]
and
\[
\Delta_b|f_0|^2 \geq 2|\nabla_b f_0|^2 - 2\kappa|f_0|^2|d_bf|^2 - C_1|\pi^1_{(1,1)} \nabla_b d_bf|^2 - C_1|f_0|^2 - C_1|d_bf|^2 \tag{2.17}
\]
where \(\epsilon\) and \(\epsilon_1\) are any positive number. In particular, if \(k = 0\) and \(k_1 = 0,\) then \(C_1 = 0.\)

**Proof.** For (2.16), due to (2.13), Cauchy inequality and the identity
\[
i(f^i_\alpha f^i_0 - f^i_\alpha f^i_0) = -\langle \nabla_b f_0, d_bf \circ J \rangle,
\]
it suffice to prove that
\[
f^i_\alpha f^j_\beta f^k_\alpha k R^N_{ijkl} + f^i_\alpha f^j_\beta f^k_\alpha l R^N_{ijkl} \geq -\frac{1}{2} \kappa |d_bf|^4. \tag{2.18}
\]

Set
\[
df(\eta_\alpha) = t_\alpha + it'_\alpha.
\]
Hence due to sectional curvature \(K^N \leq \kappa,\) a direct calculation shows that
\[
f^i_\alpha f^j_\beta f^k_\alpha k R^N_{ijkl} + f^i_\alpha f^j_\beta f^k_\alpha l R^N_{ijkl}
= 2(\langle R^N(t_\beta, t_\alpha) t_\beta, t_\alpha \rangle + \langle R^N(t_\beta', t'_\alpha) t_\beta', t'_\alpha \rangle + \langle R^N(t'_\beta, t_\alpha) t'_\beta, t_\alpha \rangle + \langle R^N(t'_\beta', t'_\alpha) t'_\beta', t'_\alpha \rangle)
\geq -2\kappa \sum_{\alpha, \beta = 1}^m (|t_\alpha|^2 |t_\beta|^2 + |t'_\alpha|^2 |t'_\beta|^2 + |t_\alpha|^2 |t'_\beta|^2 + |t'_\alpha|^2 |t_\beta|^2)
= -2\kappa \left( \sum_{\alpha = 1}^m (|t_\alpha|^2 + |t'_\alpha|^2) \right) \left( \sum_{\beta = 1}^m (|t_\beta|^2 + |t'_\beta|^2) \right)
\]
which, combining with
\[
|d_bf|^2 = 2 \sum_{\alpha = 1}^m \langle df(\eta_\alpha), df(\eta_\alpha) \rangle = 2 \sum_{\alpha = 1}^m \langle t_\alpha + it'_\alpha, t_\alpha - it'_\alpha \rangle = 2 \sum_{\alpha = 1}^m (|t_\alpha|^2 + |t'_\alpha|^2)
\]
yields (2.18).

Similarly, (2.17) follows from the following process
\[
f^i f_{\alpha} f^j f_{\bar{\alpha}} f^l P^{N}_{ijkl} = \langle R^N(t_{\alpha} - it'_{\alpha}, f_0)(t_{\alpha} + it'_{\alpha}), f_0 \rangle = \langle R^N(t_{\alpha}, f_0) t_{\alpha}, f_0 \rangle + \langle R^N(t'_{\alpha}, f_0) t'_{\alpha}, f_0 \rangle \\
\geq -\kappa |f_0|^2 \left( \sum_{\alpha=1}^{m} (|t_{\alpha}|^2 + |t'_{\alpha}|^2) \right) \\
= -\frac{1}{2} \kappa |f_0|^2 |d_b f|^2.
\]

At the end of this Section, we briefly recall Folland-Stein space. Let \((M, \theta)\) be a pseudo-Hermitian manifold and \(\Omega \subset M\). For any \(k \in \mathbb{N}\) and \(p > 1\), the Folland-Stein space \(S^p_k(\Omega)\) is given by
\[
S^p_k(\Omega) = \{ u \in L^p(\Omega) \mid \nabla^l_b u \in L^p(\Omega), l = 0, 1, \ldots, k \}
\]
where \(\nabla^l_b u\) is the horizontal restriction of \(\nabla^l u\) and its \(S^p_k\)-norm is defined by
\[
||u||_{S^p_k(\Omega)} = \sum_{l=0}^{k} ||\nabla^l_b u||_{L^p(\Omega)}.
\]

Under this generalized Sobolev space, the interior regularity theorem of subelliptic equations will behave as elliptic ones.

**Theorem 2.4** (Theorem 3.17 in [12], Theorem 16 in [22]). Suppose that \((M, \theta)\) is a pseudo-Hermitian manifold and \(\Omega \subset M\). Assume that \(u, v \in L^1_{\text{loc}}(\Omega)\) and \(\Delta_b u = v\) in the distribution sense. For any \(\chi \in C^\infty_0(\Omega),\) if \(v \in S^p_k(\Omega)\) with \(p > 1\) and \(k \in \mathbb{N}\), then \(\chi u \in S^p_{k+2}(\Omega)\) and
\[
||\chi u||_{S^p_{k+2}(\Omega)} \leq C_\chi \left( ||u||_{L^p(\Omega)} + ||v||_{S^p_k(\Omega)} \right)
\]
where \(C_\chi\) only depends on \(\chi\).

Due to the commutation relation (2.12)
\[
u_{\alpha\beta} - u_{\beta\alpha} = 2iu_0 \delta_{\alpha\beta},
\]
Reeb derivatives can be controlled by horizontal derivatives with double times. Hence Folland-Stein space may be embedded into some classical Sobolev space.

**Theorem 2.5** (Theorem 19.1 in [13]). Suppose that \((M, \theta)\) is a pseudo-Hermitian manifold and \(\Omega \subset M\). Then for any \(k \in \mathbb{N}\) and \(p > 1\),
\[
S^p_k(\Omega) \subset L^p_{k/2}(\Omega)
\]
where \(L^p_{k/2}(\Omega)\) is the classical Sobolev space. Moreover, for any \(r \in \mathbb{N}\) and \(p > \dim M\), there exists \(k \in \mathbb{N}\) such that
\[
S^p_k(\Omega) \subset C^{r, \alpha}(\Omega).
\]
3 Sub-Laplacian Comparison Theorem

This section will estimate the lower bound of sub-Laplacian of Riemannian distance function under some conditions of pseudo-Hermitian Ricci curvature and pseudo-Hermitian torsion, which plays a similar role as Laplacian comparison theorem in Riemannian geometry.

Suppose that \((M^{2m+1}, \theta)\) is a complete noncompact pseudo-Hermitian manifold. Let \(r\) be the Riemannian distance with respect to Webster metric \(g_\theta\) from a reference point \(x_0 \in M\). We formulate all Riemannian symbols with "\(\hat{\cdot}\)" to distinguish with ones in pseudo-Hermitian geometry, such as Levi-Civita connection \(\hat{\nabla}\) and Riemannian curvature tensor \(\hat{R}\). Lemma 1.3 in [12] shows the relation of Tanaka-Webster connection and Levi-Civita connection associated with Webster metric:

\[
\hat{\nabla} = \nabla - (d\theta + A) \otimes \xi + \tau \otimes \theta + 2\theta \odot J \tag{3.1}
\]

where \(2\theta \odot J = \theta \otimes J + J \otimes \theta\). Hence the sub-Laplacian of \(r\) can also be calculated by Levi-Civita connection as follows:

\[
\Delta_b r = \text{trace}_{g_\theta} \hat{\text{Hess}}(r) \big|_{HM \times HM} \tag{3.2}
\]

where \(\hat{\text{Hess}}\) is the Riemannian Hessian.

Let’s recall the Index Lemma in Riemannian geometry (cf. [11] in page 212).

Lemma 3.1 (Index Lemma). Let \(\gamma : [0, a] \to M\) be a Riemannian geodesic without conjugate points to \(\gamma(0)\) in \((0, a]\) and \(X\) be a Jacobi field along \(\gamma\) with \(X \perp \dot{\gamma}\) and \(X(0) = 0\). If \(V \in \Gamma(TM)|_{\gamma}\) with \(V(0) = 0, V(a) = X(a)\) and \(V \perp \dot{\gamma}\). Then

\[
I_a(X, X) \leq I_a(V, V) \tag{3.3}
\]

where

\[
I_a(V, V) = \int_0^a \left( |\hat{\nabla}_\gamma V|^2 - \langle \hat{R}(V, \dot{\gamma})\dot{\gamma}, V \rangle \right) dt
\]

Now let \(\gamma : [0, a] \to M\) be such a geodesic and \(\{e_B(a)\}_{B=1}^{2m}\) be an orthonormal basis of \(HM|_{\gamma(a)}\). Set

\[
e_B^\perp(a) = e_B(a) - \langle e_B(a), \nabla r \rangle \nabla r \in TM|_{\gamma(a)}
\]

which is perpendicular to \(\dot{\gamma}(a) = \nabla r|_{\gamma(a)}\). Since \(\hat{\text{Hess}}(r)(\nabla r, \cdot) = 0\), then

\[
\Delta_b r|_{\gamma(a)} = \sum_{B=1}^{2m} \hat{\text{Hess}}(r)(e_B(a), e_B(a)) = \sum_{B=1}^{2m} \hat{\text{Hess}}(r)(e_B^\perp(a), e_B^\perp(a)) \tag{3.4}
\]

Using the Riemannian exponential map, we could extend \(e_B^\perp(a)\) as a Jacobi field \(U_B\) along \(\gamma\) with

\[
U_B(0) = 0, U_B(a) = e_B^\perp(a), [U_B, \dot{\gamma}] = 0.
\]
Hence we find
\[
\tilde{\text{Hess}}(r)(e^B_B(a), e^B_B(a)) = \text{Hess}(r)(U_B(a), U_B(a))
\]
\[
= \langle U_B, \tilde{\nabla}_B \nabla r \rangle|_{\gamma(a)} = \langle U_B, \tilde{\nabla}_3 U_B \rangle|_{\gamma(a)} = \int_0^a \frac{d}{dt} \langle U_B, \tilde{\nabla}_3 U_B \rangle dt
\]
\[
= \int_0^a \left( |\tilde{\nabla}_3 U_B|^2 + \langle U_B, \tilde{\nabla}_3 \tilde{\nabla}_3 U_B \rangle \right) dt = I_a(U_B, U_B),
\]
where the last equation is due to the Jacobi equation. Hence
\[
\Delta_b r|_{\gamma(a)} = \sum_{B=1}^{2m} I_a(U_B, U_B). \tag{3.5}
\]

Lemma 3.2. Let \(e^r_B(t)\) be the parallel extension of \(e^B_B(a)\) along \(\gamma\) with respect to Tanaka-Webster connection. Suppose the curvature along \(\gamma\) satisfies
\[
2m \sum_{B=1}^{2m} \langle \tilde{R}(e^r_B, \nabla r) \nabla r, e^B_B \rangle \geq -\hat{k} \tag{3.6}
\]
and the pseudo-Hermitian torsion is bounded, i.e.
\[
|A| \leq k_1, \tag{3.7}
\]
for some for \(\hat{k}, k_1 \geq 0\). Then there is a constant \(C_2 = C_2(m)\) such that
\[
\Delta_b r|_{\gamma(a)} \leq C_2 \left( \frac{1}{a} + \sqrt{1 + k_1 + k_1^2 + \hat{k}} \right). \tag{3.8}
\]

Proof. Due to \(3.1\), we have
\[
\tilde{\nabla}_3 e^r_B = -[d\theta(\dot{\gamma}, e^r_T) + A(\dot{\gamma}, e^r_B)]\xi + \theta(\dot{\gamma}) J e^r_B = -[g_B(J \dot{\gamma}, e^B_T) + A(\dot{\gamma}, e^B_B)]\xi + \theta(\dot{\gamma}) J e^B_B
\]
which implies that
\[
\sum_{B=1}^{2m} \left| \tilde{\nabla}_3 e^B_B \right|^2 = 2m |\theta(\dot{\gamma})|^2 + \sum_{B=1}^{2m} \left[ |g_B(J \dot{\gamma}, e^B_B)|^2 + 2g_B(J \dot{\gamma}, e^B_B) A(\dot{\gamma}, e^B_B) + |A(\dot{\gamma}, e^B_B)|^2 \right]
\]
\[
\leq 2m + 2A(\dot{\gamma}, J \dot{\gamma}) + \sum_{B=1}^{m} |A(\dot{\gamma}, e^B_B)|^2 \leq 2m + 2k_1 + k_1^2.
\]
Set
\[
e^r_B(t) = e^B_B(t) - \langle e^B_B(t), \nabla r \rangle \nabla r \perp \dot{\gamma}, \quad V_B(t) = \frac{s_\kappa(t)}{s_\kappa(a)} e^r_B(t),
\]
where
\[
s_\kappa(t) = \frac{1}{\sqrt{\kappa}} \sinh(\sqrt{\kappa}t) \quad \text{and} \quad \kappa = \frac{1}{4m} (4m + 4k_1 + 2k_1^2 + \hat{k}).
\]
Lemma 3.3. For $X, Y \in TM$, we have

$$\sum_{B=1}^{2m} \langle \hat{R}(e_B, X)Y, e_B \rangle = \sum_{B=1}^{2m} \langle R(e_B, X)Y, e_B \rangle - 3\langle \pi_H X, \pi_H Y \rangle + \langle \tau X, \tau Y \rangle + (2m - |\tau|^2)\theta(X)\theta(Y) + \text{div} \tau \theta(Y) \tag{3.10}$$
Proof. By (3.9) and $e_B \in HM$, we have

$$ \sum_{B=1}^{2m} \langle \hat{R}(e_B, X) Y, e_B \rangle = \sum_{B=1}^{2m} \langle R(e_B, X) Y, e_B \rangle + \sum_{B=1}^{2m} \langle (L e_B \wedge LX) Y, e_B \rangle + \sum_{B=1}^{2m} 2d\theta(e_B, X) \langle J Y, e_B \rangle $$

$$+ \sum_{B=1}^{2m} \theta(Y) \langle S(e_B, X), e_B \rangle + \sum_{B=1}^{2m} 2\theta(Y) \langle (\theta \wedge \omega)(e_B, X), e_B \rangle \quad (3.11) $$

Now we see each term in the right side except the first one. Note that

$$ \sum_{B=1}^{2m} \langle (L e_B \wedge LX) Y, e_B \rangle = \sum_{B=1}^{2m} \langle L e_B, Y \rangle \langle LX, e_B \rangle - \langle LX, Y \rangle \langle L e_B, e_B \rangle \quad (3.12) $$

On one hand, since $LX$ is horizontal and

$$ \langle L e_B, Y \rangle = \langle e_B, \tau Y \rangle - \langle e_B, J Y \rangle, $$

then we find

$$ \sum_{B=1}^{2m} \langle L e_B, Y \rangle \langle LX, e_B \rangle = \langle LX, \tau Y \rangle - \langle LX, J Y \rangle $$

$$= \langle \tau X, \tau Y \rangle + \langle J X, \tau Y \rangle - \langle \tau X, J Y \rangle - \langle J X, J Y \rangle $$

$$= \langle \tau X, \tau Y \rangle - \langle \pi H X, \pi H Y \rangle. \quad (3.13) $$

Here the last equation is due to $\tau J + J \tau = 0$ by (2.5). On the other hand,

$$ \langle L e_B, e_B \rangle = \text{trace}_{G_0} \tau + \text{trace}_{G_0} J = 0. \quad (3.14) $$

Substituting (3.13) and (3.14) into (3.12), the result is

$$ \sum_{B=1}^{2m} \langle (L e_B \wedge LX) Y, e_B \rangle = \langle \tau X, \tau Y \rangle - \langle \pi H X, \pi H Y \rangle. \quad (3.15) $$

For the third term in (3.11), we have

$$ \sum_{B=1}^{2m} 2d\theta(e_B, X) \langle J Y, e_B \rangle = \sum_{B=1}^{2m} 2\langle J e_B, X \rangle \langle J Y, e_B \rangle = -2\langle \pi H X, \pi H Y \rangle. \quad (3.16) $$

For the fourth term in (3.11), by the formula of $S$, we have

$$ \sum_{B=1}^{2m} \langle S(e_B, X), e_B \rangle = \sum_{B=1}^{2m} \langle (\nabla e_B \tau) X, e_B \rangle - \sum_{B=1}^{2m} \langle (\nabla X \tau) e_B, e_B \rangle = \text{div} \tau(X) \quad (3.17) $$
since \( \tau \) is traceless. For the fifth term, by the definition of \( \mathcal{O} \), we have
\[
\sum_{B=1}^{2m} 2\langle (\theta \wedge \mathcal{O})(e_B, X), e_B \rangle = \sum_{B=1}^{2m} -\langle \theta(X)\mathcal{O}(e_B), e_B \rangle \\
= \sum_{B=1}^{2m} -\theta(X)\langle (\tau^2 + 2J\tau - I)(e_B), e_B \rangle \\
= \theta(X)(2m - |\tau|^2)
\] (3.18)
due to
\[
- \sum_{B=1}^{2m} \langle J\tau(e_B), e_B \rangle = \sum_{B=1}^{2m} \langle \tau Je_B, e_B \rangle = \sum_{\alpha=1}^{m} \langle \tau Je_{\alpha}, e_{\alpha} \rangle + \langle \tau J^2 e_{\alpha}, Je_{\alpha} \rangle = 0.
\]
By substituting \( (3.15), (3.16), (3.17) \) and \( (3.18) \) to \( (3.11) \), we get \( (3.10) \).

Tanaka \[24\] obtained the following version of first Bianchi identity of \( R \):
\[
S(R(X,Y)Z) = 2S(d\theta(X,Y)\tau(Z)).
\] (3.19)
where \( S \) stands for the cyclic sum with respect to \( X, Y, Z \) \( \in \text{HM} \). One can prove it by applying Riemannian first Bianchi identity to \( (3.9) \).

**Lemma 3.4.** For any \( X, Y \in TM \), we have
\[
\langle R_4 X, Y \rangle = \sum_{B=1}^{2m} \langle R(e_B, \pi_H X)\pi_H Y, e_B \rangle - 2(m - 1)A(X, JY),
\] (3.20)

**Proof.** Since \( JX \) is horizontal, we can use the first Bianchi identity \( (3.19) \) and obtain
\[
- i \sum_{\alpha=1}^{m} R(\eta_\alpha, \eta_\bar{\alpha})JX - i \sum_{\alpha=1}^{m} R(\eta_\bar{\alpha}, JX)\eta_\alpha - i \sum_{\alpha=1}^{m} R(JX, \eta_\alpha)\eta_\bar{\alpha} \\
= - i \sum_{\alpha=1}^{m} 2d\theta(\eta_\alpha, \eta_\bar{\alpha})\tau JX - i \sum_{\alpha=1}^{m} 2d\theta(\eta_\bar{\alpha}, JX)\tau \eta_\alpha - i \sum_{\alpha=1}^{m} 2d\theta(JX, \eta_\alpha)\tau \eta_\bar{\alpha} \\
= 2m\tau JX - 2 \sum_{\alpha=1}^{m} \tau J(\langle \eta_\bar{\alpha}, X \rangle \eta_\alpha + \langle \eta_\alpha, X \rangle \eta_\bar{\alpha}) \\
= 2(m - 1)\tau JX.
\] (3.21)
On the other hand, note that
\[
i \sum_{\alpha=1}^{m} R(\eta_\bar{\alpha}, JX)\eta_\alpha + i \sum_{\alpha=1}^{m} R(JX, \eta_\alpha)\eta_\bar{\alpha} = - i \sum_{\alpha=1}^{m} R(JX, \eta_\alpha)\eta_\alpha + i \sum_{\alpha=1}^{m} R(JX, \eta_\bar{\alpha})\eta_\alpha \\
= - J \left( \sum_{\alpha=1}^{m} R(JX, \eta_\alpha)\eta_\alpha + R(JX, \eta_\bar{\alpha})\eta_\alpha \right) \\
= - J \left( \sum_{B=1}^{2m} R(JX, e_B)e_B \right)
\] (3.22)
Substituting (3.22) into (3.21), we obtain

$$\langle R_eX, Y \rangle = \sum_{B=1}^{2m} \langle R(e_B, JX)JY, e_B \rangle + 2(m-1)A(JX, Y).$$

By replacing $X, Y$ by $JX, JY$, the proof is finished.

For any $Y \in HM$, using (3.9), we have

$$\sum_{B=1}^{2m} \langle \hat{R}(e_B, \xi)Y, e_B \rangle = \sum_{B=1}^{2m} \langle R(e_B, \xi)Y, e_B \rangle$$

and

$$\sum_{B=1}^{2m} \langle \hat{R}(e_B, Y)\xi, e_B \rangle = \sum_{B=1}^{2m} \langle S(e_B, Y), e_B \rangle = \text{div} \tau(Y).$$

Applying the symmetric property of Riemannian curvature, we get

$$\sum_{B=1}^{2m} \langle R(e_B, \xi)Y, e_B \rangle = \text{div} \tau(Y).$$ (3.23)

Combing Lemma 3.3, Lemma 3.4 and (3.23), we obtain the following lemma.

**Lemma 3.5.** For any $X, Y \in TM$, we have

$$\sum_{B=1}^{2m} \langle \hat{R}(e_B, X)Y, e_B \rangle = \langle R_eX, Y \rangle + 2(m-1)A(X, JY) + \langle \tau X, \tau Y \rangle - 3\langle \pi_H X, \pi_H Y \rangle$$

$$+ (2m - |\tau|^2) \theta(X)\theta(Y) + \text{div} \tau(X)\theta(Y) + \text{div} \tau(Y)\theta(X)$$ (3.24)

Using Lemma 3.2, we obtain the following sub-Laplacian estimate of Riemannian distance function.

**Theorem 3.6.** Suppose $(M^{2m+1}, \theta)$ is a complete pseudo-Hermitian manifold and $B_R(x_0)$ is the geodesic ball of radius $R$ centered at $x_0$. If

$$R_* \geq -k, \text{ and } |A|, |\text{div}A| \leq k_1, \text{ on } B_R(x_0),$$

then there exists $C_3 = C_3(m)$ such that

$$\Delta_b r \leq C_3 \left( \frac{1}{r} + \sqrt{1 + k + k_1 + k_1^2} \right), \text{ on } B_R(x_0) \setminus \text{Cut}(x_0)$$ (3.25)

where $\text{Cut}(x_0)$ is the cut locus of $x_0$. 

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4 Horizontal Gradient Estimates and Liouville Theorem

Suppose that \((M^{2n+1}, \theta)\) is a complete noncompact pseudo-Hermitian manifold. Let \(r\) be the Riemannian distance function from \(x_0 \in M\) associated with the Webster metric \(g_\theta\) and \(B_R\) be the geodesic ball of radius \(R\) centered at \(x_0\). Assume that

\[ R_* \geq -k, \text{ and } |A|, |\text{div} A| \leq k_1, \quad \text{on } B_{2R} \]

for some \(R \geq 1\). Choose a cut-off function \(\varphi \in C^\infty([0, \infty))\) such that

\[ \varphi|_{[0,1]} = 1, \quad \varphi|_{[2,\infty)} = 0, \quad -C'_4 |\varphi|^2 \leq \varphi' \leq 0, \]

where \(C'_4\) is a universal constant. By defining \(\chi(r) = \varphi(\frac{r}{R})\) and using Theorem 3.6 we find that

\[ \frac{|\nabla_b \chi|^2}{\chi^2} \leq \frac{C_4}{R^2}, \quad \Delta_b \chi \geq -\frac{C_4}{R}, \quad \text{on } B_{2R} \setminus \text{Cut}(x_0), \] (4.1)

where \(C_4 = C_4(m, k, k_1)\).

Suppose that \((N, h)\) is a Riemannian manifold with sectional curvature

\[ K^N \leq \kappa \]

for some \(\kappa \geq 0\). Denote the Riemannian distance function from \(p_0 \in N\) by \(\rho\). Let \(B_D = B_D(p_0)\) be a regular ball of radius \(D\) around \(p_0\), that is \(D < \frac{\pi}{2\sqrt{\kappa}}\) and \(B_D\) lies inside the cut locus of \(p_0\) where \(\frac{\pi}{2\sqrt{\kappa}} = +\infty\) if \(\kappa = 0\). Set

\[ \phi(t) = \begin{cases} \frac{1-\cos(\sqrt{\kappa} t)}{\sqrt{\kappa}}, & \kappa > 0 \\ \frac{t^2}{2}, & \kappa = 0 \end{cases} \]

and

\[ \psi(q) = \phi \circ \rho(q). \]

Obviously, \(\phi\) is an increasing function and \(\psi\) is at least \(C^2\) in the cut locus of \(p_0\). Moreover, Hessian comparison theorem shows that

\[ \text{Hess } \psi \geq \cos(\sqrt{\kappa} \rho) \cdot h. \] (4.2)

**Lemma 4.1.** For any \(0 < D < \frac{\pi}{2\sqrt{\kappa}}\), there exist \(\nu \in [1, 2), b > \phi(D)\) and \(\delta > 0\) only depending on \(D\) such that

\[ \nu \frac{\cos(\sqrt{\kappa} t)}{b - \phi(t)} - 2\kappa > \delta, \quad \forall t \in [0, D] \] (4.3)
Proof. For the case $\kappa > 0$, it suffices to find $\nu \in [1, 2)$ and $b > \phi(D)$ such that

$$\phi(D) < b < \inf_{s \in [\phi(D)]} \left( \frac{\nu}{2\kappa} + \left(1 - \frac{\nu}{2}\right)s \right),$$

(4.4)

which is obvious due to $\phi(D) < \frac{1}{\kappa}$.

The case $\kappa = 0$ is obvious by choosing $\nu = 1$. \hfill \Box

Assume that $f : B_{2R}(x_0) \subset M \to B_D(p_0)$ is a pseudo-harmonic map. By (4.2), we have the following estimate:

**Lemma 4.2.** Let $\nu, b, \delta$ be given in Lemma 4.1. Then

$$\nu \frac{\Delta_b \psi \circ f}{b - \psi \circ f} - 2\kappa|d_b f|^2 \geq \delta|d_b f|^2$$

(4.5)

To estimate $|d_b f|^2$, we consider the following auxiliary function

$$\Phi_{\mu \chi} = |d_b f|^2 + \mu \chi|f_0|^2$$

where $\mu$ will be determined later.

**Lemma 4.3.** Suppose $\mu$ and $\epsilon$ satisfy

$$C_1 \mu \leq \epsilon \leq 1.$$ 

If $\chi(x) \neq 0$ and $\Phi_{\mu \chi}(x) \neq 0$, then at $x$, we have

$$\Delta_b \Phi_{\mu \chi} \geq \frac{1 - \epsilon}{2} \frac{\|\nabla_b \Phi_{\mu \chi}\|^2}{\Phi_{\mu \chi}} - 2\kappa|d_b f|^2 \Phi_{\mu \chi}$$

$$+ (2m\epsilon - C_1 \mu \chi - 4\nu^{-1} \mu \chi^{-1} |\nabla_b \chi|^2 + \mu |\Delta_b \chi|) |f_0|^2$$

$$- [C_1 + C_1 \mu \chi + 16(\epsilon \mu \chi)^{-1}] |d_b f|^2$$

(4.6)

**Proof.** Using (2.16) and (2.17) with $\epsilon_1 = \epsilon \mu \chi$, we have

$$\Delta_b \Phi_{\mu \chi} = \Delta_b(|d_b f|^2 + \mu \chi |f_0|^2)$$

$$\geq (2 - \epsilon)(|\nabla_b d_b f|^2 + \mu \chi |\nabla_b f_0|^2) + 4\mu\langle \nabla_b \chi \otimes f_0, \nabla_b f_0 \rangle - 2\kappa \Phi_{\mu \chi} |d_b f|^2$$

$$+ [2m\epsilon - C_1 \mu \chi + \mu \Delta_b \chi] |f_0|^2 - [C_1 + C_1 \mu \chi + 16(\epsilon \mu \chi)^{-1}] |d_b f|^2$$

(4.7)

By Cauchy inequality, we have the following estimate

$$|\nabla_b \Phi_{\mu \chi}|^2 = |\nabla_b (|d_b f|^2 + \mu \chi |f_0|^2)|^2$$

$$= |\nabla_b (d_b f + \sqrt{\mu \chi} f_0 \otimes \theta, d_b f + \sqrt{\mu \chi} f_0 \otimes \theta)|^2$$

$$= 4 \left| \begin{array}{c} d_b f + \sqrt{\mu \chi} f_0 \otimes \theta, \nabla_b d_b f + \sqrt{\mu \chi} \nabla_b f_0 \otimes \theta + \sqrt{\mu \chi} \frac{\nabla_b \chi}{2\sqrt{\chi}} \otimes f_0 \otimes \theta \end{array} \right|^2$$

$$\leq 4 |d_b f + \sqrt{\mu \chi} f_0 \otimes \theta|^2 \cdot \left| d_b d_b f + \frac{\mu \chi \nabla_b \chi}{2\sqrt{\chi}} \otimes f_0 \otimes \theta \right|^2$$

$$= 4 \Phi_{\mu \chi} \left( |\nabla_b d_b f|^2 + \mu \chi |\nabla_b f_0|^2 + \frac{\mu |\nabla_b \chi|^2}{4 \chi} |f_0|^2 + \mu \langle \nabla_b f_0, \nabla_b \chi \otimes f_0 \rangle \right)$$
which, using Cauchy inequality again, implies that
\[
(2 - \epsilon)(|\nabla_b \Omega_l| + \mu \chi|\nabla \Omega_l|) + 4 \mu(|\nabla_b \Omega_l| + \nabla \Omega_l)
\]
\[
= (2 - \epsilon)(|\nabla_b \Omega_l| + \mu \chi|\nabla \Omega_l|) + \epsilon \mu \chi|\nabla \Omega_l|
\]
\[
\geq 1 - \epsilon \frac{|\nabla_b \Phi_{\mu \chi}|^2}{\Phi_{\mu \chi}} - \frac{1}{2} + \frac{1 + \epsilon}{\epsilon} \mu \frac{|\nabla_b \chi|^2}{\chi} |\nabla \Omega_l|^2
\]
\[
\geq 1 - \epsilon \mu \frac{|\nabla_b \Phi_{\mu \chi}|^2}{\Phi_{\mu \chi}} - \left(1 - \epsilon + \frac{1 + \epsilon}{\epsilon} \mu \frac{|\nabla_b \chi|^2}{\chi} |\nabla \Omega_l|^2
\]
\[
\geq 1 - \epsilon \mu \frac{|\nabla_b \Phi_{\mu \chi}|^2}{\Phi_{\mu \chi}} - 4 \epsilon^{-1} \mu \frac{|\nabla_b \chi|^2}{\chi} |\nabla \Omega_l|^2
\] (4.8)
due to $\epsilon \leq 1$ and
\[
\frac{1 - \epsilon}{2} + \frac{(1 + \epsilon)^2}{\epsilon} \leq \frac{1 - \epsilon}{\epsilon} + \frac{(1 + \epsilon)^2}{\epsilon} = 2 \epsilon^{-1} + \epsilon + 1 \leq 4 \epsilon^{-1}.
\]
Submitting (4.8) to (4.7), we finished the proof. \qed

Set
\[
F_{\mu \chi} = \frac{\Phi_{\mu \chi}}{(b - \psi \circ f)^\nu}
\]
where $\nu \in [1, 2)$ and $b$ are determined in Lemma 4.1. The $\epsilon$ in Lemma 4.3 will be chosen as
\[
\epsilon = \frac{1}{\nu} - \frac{1}{2} \leq 1
\] (4.9)
and $\mu$ satisfy
\[
C_1 \mu \leq \epsilon.
\] (4.10)
Let $x$ be a maximum point of $\chi F_{\mu \chi}$ on $B_{2R}$ which is nonzero. Assume that $r$ is smooth at $x$. Otherwise we can modify the distance function $r$ as [8]. Hence at $x$, we have
\[
0 = \nabla_b \ln(\chi F_{\mu \chi}) = \frac{\nabla_b \chi}{\chi} + \frac{\nabla_b \Phi_{\mu \chi}}{\Phi_{\mu \chi}} + \frac{\nabla_b (\psi \circ f)}{b - \psi \circ f},
\] (4.11)
\[
0 \geq \Delta_b \ln(\chi F_{\mu \chi}) = \frac{\Delta_b \chi}{\chi} - \frac{|\nabla_b \chi|^2}{\chi^2} + \frac{\Delta_b \Phi_{\mu \chi}}{\Phi_{\mu \chi}} - \frac{|\nabla_b \Phi_{\mu \chi}|^2}{\Phi_{\mu \chi}^2}
\]
\[
+ \frac{\Delta_b (\psi \circ f)}{b - \psi \circ f} + \frac{|\nabla_b (\psi \circ f)|^2}{(b - \psi \circ f)^2}.
\] (4.12)
By (4.6), (4.12) becomes
\[
0 \geq \frac{\Delta_b \chi}{\chi} - \frac{|\nabla_b \chi|^2}{\chi^2} - \frac{1}{2} + \frac{\epsilon}{2} \frac{|\nabla_b \Phi_{\mu \chi}|^2}{\Phi_{\mu \chi}^2} - 2 \kappa |\nabla_b f|^2 + \frac{\Delta_b (\psi \circ f)}{b - \psi \circ f} + \frac{|\nabla_b (\psi \circ f)|^2}{(b - \psi \circ f)^2}
\]
\[
+ \left(2m \epsilon - C_1 \mu \chi \right) \frac{|\nabla_b \chi|^2}{\chi} \frac{|\nabla \chi|^2}{\chi} - \left[C_1 + C_1 \mu \chi + 16(\mu \chi)^{-1}\right] \frac{|\nabla f|^2}{\Phi_{\mu \chi}^2}.
\] (4.13)
Using (4.11) and Cauchy inequality, we have at \( x \)
\[
-\frac{1 + \epsilon}{2} \frac{\nabla \Phi_{\mu \chi}^2}{\Phi_{\mu \chi}^2} \geq -\frac{1 + \epsilon}{2} (1 + \epsilon^{-1}) \frac{\nabla \chi^2}{\chi^2} - \frac{1 + \epsilon}{2} (1 + \epsilon_2) \nu^2 \frac{\nabla \Phi_{\mu \chi}^2}{(b - \psi \circ f)^2}. 
\tag{4.14}
\]

Due to the choice \((4.9)\) of \( \epsilon \), we can take
\[
\epsilon_2 = \frac{2}{\nu(1 + \epsilon)} - 1 = \frac{2 - \nu}{2 + \nu} > 0
\]
and then
\[
\frac{1 + \epsilon}{2} (1 + \epsilon_2) \nu^2 = \nu, \quad \frac{1 + \epsilon}{2} (1 + \epsilon_2^{-1}) = \frac{2 + \nu}{\nu(2 - \nu)}. 
\tag{4.15}
\]

Substituting (4.14), (4.2) to (4.13), we have at \( x \)
\[
0 \geq \frac{\Delta_b \chi}{\chi} - \left(1 + \frac{2 + \nu}{\nu(2 - \nu)} \right) \frac{\nabla \chi^2}{\chi^2} + \nu \frac{\Delta_b \psi \circ f}{b - \psi \circ f} - 2\kappa |d_b f|^2 
+ \left(2m\epsilon - C_1 \mu \chi + \mu \Delta_b \chi - 4\epsilon^{-1} \mu \frac{\nabla \chi^2}{\chi} \right) \frac{|f_0|^2}{\Phi_{\mu \chi}} - [C_1 + 1 + 16(\epsilon \mu)^{-1}] \frac{|d_b f|^2}{\Phi_{\mu \chi}}. 
\tag{4.16}
\]

The estimates (4.1) and Lemma 4.2 yield that
\[
0 \geq -\frac{C_\nu}{\chi^R} + \delta |d_b f|^2 + \left(2m\epsilon - C_1 \mu \chi - \frac{\mu C_\nu}{R} \right) \frac{|f_0|^2}{\Phi_{\mu \chi}} - [C_1 + 1 + 16(\epsilon \mu)^{-1}] \frac{|d_b f|^2}{\Phi_{\mu \chi}},
\]
where \( C_\nu = C_\nu(\nu, C_4) \) and \( \delta \) is given by Lemma 4.1. By definition of \( \Phi_{\mu \chi} \),
\[
|f_0|^2 = \mu^{-1} \chi^{-1} (\Phi_{\mu \chi} - |d_b f|^2),
\]
which, together with (4.16), shows at \( x \),
\[
0 \geq \frac{1}{\chi} \left(2m\epsilon \mu^{-1} - C_1 - \frac{2C_\nu}{R} \right) + \left[\delta \chi \Phi_{\mu \chi} - 2m\epsilon \mu^{-1} - [C_1 + 1 + 16(\epsilon \mu)^{-1}] \right] \frac{|d_b f|^2}{\chi \Phi_{\mu \chi}}. 
\tag{4.17}
\]

To make the first bracket of the last line in (4.17) nonnegative, we can choose sufficiently small \( \mu \) such that
\[
\epsilon \mu^{-1} = C_1 + \frac{2C_\nu}{R},
\]
which makes (4.10) right. Hence
\[
(\chi \Phi_{\mu \chi})(x) \leq C_5 \delta^{-1}, 
\tag{4.18}
\]
where
\[
C_5 = (2m + 1)C_1 + \frac{4mC_\nu}{R} + \frac{C_1}{2C_1 + 4C_\nu R^{-1}} + \frac{64\nu^2}{(2 - \nu)^2} \left( C_1 + \frac{2C_\nu}{R} \right), 
\tag{4.19}
\]

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which implies
\[
\max_{B_{2R}(x_0)} \chi F_{\mu \chi} \leq \frac{\chi \Phi_{\mu \chi}}{(b - \psi \circ f)\nu}(x) \leq \frac{C_5}{\delta(b - \phi(D))\nu}.
\]
(4.20)

This shows that
\[
\max_{B_{2R}(x_0)} |d_b f|^2 \leq b^\nu \cdot \max_{B_{2R}(x_0)} F_{\mu \chi} \leq \frac{C_5b^\nu}{\delta(b - \phi(D))\nu}.
\]
(4.21)

Let’s summarize the results as follows.

**Theorem 4.4.** Let \((M^{2m+1}, \theta)\) be a noncompact complete pseudo-Hermitian manifold and \((N, h)\) be a Riemannian manifold with sectional curvature \(K^N \leq \kappa\) for some \(\kappa \geq 0\). On \(B_{2R}(x_0) \subset M\),
\[
R_* \geq -k \quad \text{and} \quad |A|, |\text{div} A| \leq k_1.
\]
(4.22)

Assume that \(f : B_{2R}(x_0) \subset M \to B_D(p_0) \subset N\) is pseudo-harmonic where \(B_D(p_0)\) is a regular ball in \(N\). Then the horizontal energy density \(|d_b f|\) on \(B_{2R}(x_0)\) is uniformly bounded by \(m, k, k_1, \kappa, D\) and \(R\).

Lemma 2.3 says that if \(k = 0\) and \(k_1 = 0\), then \(C_1 = 0\). According to (4.19), we find that
\[
C_5 = \left(4m + \frac{128\nu^2}{m(2 - \nu)^2}\right) \frac{C_\nu}{R},
\]
which implies that
\[
\max_{B_{2R}(x_0)} |d_b f|^2 \leq \left(4m + \frac{128\nu^2}{m(2 - \nu)^2}\right) \frac{C_\nu b^\nu}{R\delta(b - \phi(D))\nu} \to 0, \text{ as } R \to \infty.
\]

Hence we have the following Liouville theorem of pseudo-harmonic maps.

**Theorem 4.5.** Let \((M, \theta)\) be a noncompact complete Sasakian manifold with nonnegative pseudo-Hermitian Ricci curvature and \((N, h)\) be a Riemannian manifold with sectional curvature bounded above. Then there is no nontrivial pseudo-Hermitian map from \(M\) to any regular ball of \(N\).

### 5 Global Existence Theorem

Jost and Xu [15] studied the minimizing sequence of Dirichlet problem of sub-elliptic harmonic maps and obtained the existence theorem under some convexity condition. Their results [15] seem to depend on the global fields which satisfy the Hörmander condition and the noncharacteristic assumption of the boundary. But the weak existence of Dirichlet problem and the interior continuity of weak solutions can be generalized to any sub-Riemannian manifolds with smooth boundaries, such as pseudo-Hermitian manifolds. We formulate the results as follows:
Theorem 5.1 (Theorem 1 and Theorem 2 in [15]). Suppose that \((M, \theta)\) is a pseudo-Hermitian manifold with smooth boundary and \((N, h)\) is a Riemannian manifold with sectional curvature \(K^N \leq \kappa\) for some \(\kappa \geq 0\). Let \(B_D = B_D(p_0) \subset N\) be a regular ball. If \(\varphi \in S^2_1(M, N)\) satisfies \(\varphi(M) \subset B_D(p_0)\), then there exists a weak pseudo-harmonic map \(f \in C(M, N) \cap S^2_M(M, N)\) with

\[
f - \varphi \in S^2_{1,0}(M, N)
\]

and

\[
f(M) \subset B_D(p_0).
\]

Remark 5.2. Note that \(B_D(p_0)\) can be covered by a normal coordinate \(\{z^i\}\) and thus it can be viewed as an open set of \(\mathbb{R}^n\) where \(n = \text{dim } N\). Hence the notion

\[
f^i = z^i \circ f \quad \text{and} \quad \Gamma_{jk}^i \text{'s are Christoffel symbols of Levi-Civita connection in } (N, h).
\]

Since the Euler-Lagrange equations of pseudo-harmonic maps are quasilinear sub-elliptic systems, these weak solutions will be interior smooth due to [26] by Xu-Zuily.

Theorem 5.3 (Theorem 1.1 in [26]). Suppose that \((M, \theta)\) is a pseudo-Hermitian manifold (with or without boundary) and \((N, h)\) is a Riemannian manifold. Let \(f : M \to N\) be a weak pseudo-harmonic map and \(f \in S^2(M, N)\). If \(f\) is continuous inside \(M\), then \(f \in C^\infty(M, N)\).

Now we consider the global existence of pseudo-harmonic maps to regular balls. Suppose that \((M, \theta)\) is a complete noncompact pseudo-Hermitian manifold and \((N, h)\) is a Riemannian manifold with sectional curvature \(K^N \leq \kappa\) for some \(\kappa \geq 0\). Let \(B_D(p_0) \subset N\) be a geodesic ball lying in the cut locus of \(p_0\) and \(D < \frac{\pi}{2\kappa}\). Assume that \(\varphi : M \to B_D(p_0)\) with \(\varphi(x_0) = p_0\).

We can choose a smooth exhaustion \(\{\Omega_i\}\) of \(M\) such that \(B_{2i}(x_0) \subset \Omega_i\). Theorem 5.1 and Theorem 5.3 guarantee that there is a smooth pseudo-harmonic map \(f_i : \Omega_i \to B_D(p_0)\). One can find the constants \(k(i)\) and \(k_1(i)\) such that

\[
R_s \big|_{B_{2i}(x_0)} \geq -k(i), \quad \text{and} \quad |A|_{B_{2i}(x_0)}; \ |\text{div } A|_{B_{2i}(x_0)} \leq k_1(i).
\]

Hence fixed \(i\), for \(j \geq i\), Theorem 4.4 controls the interior horizontal gradient of \(f_j\) on \(B_i(x_0)\):

\[
\max_{B_i(x_0)} |d_b f_j|^2 \leq C_0(i), \quad (5.3)
\]

where \(C_0(i)\) only depends on \(k(i), k_1(i), D, \kappa, i\). Arzelà-Ascoli theorem yields that by taking subsequence, \(f_j\) will uniformly converge to some continuous map in \(B_i(x_0)\) as \(j \to \infty\). By diagonalization, some subsequence of \(\{f_i\}\) will internally closed uniformly converge to a continuous map \(f : M \to B_D(p_0)\) as \(i \to \infty\). Moreover, \(f\) is a weak solution of (5.1) and thus is smooth pseudo-harmonic by Theorem 5.3.
Theorem 5.4. Suppose that \((M, \theta)\) is a complete noncompact pseudo-Hermitian manifold and \((N, h)\) is a Riemannian manifold with sectional curvature bounded from above. Let \(B_D(p_0) \subset N\) be a regular ball. Then there is a pseudo-harmonic map \(f : M \to B_D(p_0)\).

It is notable that the pseudo-harmonic map given by Theorem 5.4 will depend on the initial map. By Theorem 4.5, it is always trivial if the domain has nonnegative pseudo-Hermitian Ricci curvature. At the end of this paper, we will give a nontrivial example when the domain has negative pseudo-Hermitian Ricci curvature. One model of Sasakian space form with constant negative pseudo-Hermitian sectional curvature is the Riemannian submersion

\[ \pi : B^n_C \times \mathbb{R} \to B^n_C \]

where \(B^n_C \subset \mathbb{C}^n\) is the complex ball with Bergman metric \(\omega\) (cf. Example 7.3.22 in [4]). Let \(\omega_0\) be the canonical Kähler form on \(\mathbb{C}^n\). Since the identity \(I\) of \(B^n_C\) is a holomorphic map from \(B^n_C\) to \(\mathbb{C}^n\), then it is also a harmonic map from \((B^n_C, \omega)\) to \((\mathbb{C}^n, \omega_0)\). The lift of \(I\) is denoted by \(\tilde{I}\) such that

\[ \tilde{I} = I \circ \pi : B^n_C \times \mathbb{R} \to \mathbb{C}^n. \]

Then by the composition rule,

\[ \nabla d\tilde{I} = \hat{\nabla} dI(d\pi, d\pi) + dI(\hat{\nabla} d\pi) \quad (5.4) \]

where the Levi-Civita connections of \((B^n_C, \omega)\) and \((\mathbb{C}^n, \omega_0)\) are both denoted by \(\hat{\nabla}\). Suppose that \(\nabla\) is the Tanaka-Webster connection of \(B^n_C \times \mathbb{R}\). Their relation is given by (cf. Lemma 1.3 in [12])

\[ \hat{\nabla} = \nabla - d\theta \otimes \xi + 2\theta \otimes J \quad (5.5) \]

where \(2\theta \otimes J = \theta \otimes J + J \otimes \theta\). Assume that \(\{e_B\}_{B=1}^{2n}\) is a orthonormal frame in \((B^n_C, \omega)\) with \(e_{\alpha+n} = Je_\alpha\) for \(1 \leq \alpha \leq n\) and \(\tilde{e}_B\) is the horizontal lift of \(e_B\). On one hand, the relation (5.5) guarantees that

\[ \tau_H(\tilde{I}) = \sum_{B=1}^{2n} (\nabla_{\tilde{e}_B} d\tilde{I})(\tilde{e}_B) \]

\[ = \sum_{B=1}^{2n} \nabla_{\tilde{e}_B} (d\tilde{I}(\tilde{e}_B)) - \sum_{B=1}^{2n} d\tilde{I}(\nabla_{\tilde{e}_B} \tilde{e}_B) \]

\[ = \sum_{B=1}^{2n} \nabla_{\tilde{e}_B} (d\tilde{I}(\tilde{e}_B)) - \sum_{B=1}^{2n} d\tilde{I}(\hat{\nabla}_{\tilde{e}_B} \tilde{e}_B) \]

\[ = \sum_{B=1}^{2n} (\hat{\nabla}_{\tilde{e}_B} d\tilde{I})(\tilde{e}_B). \quad (5.6) \]

On the other hand, by the relation of Levi-Civita connection and metric, we have

\[ \sum_{B=1}^{2n} d\pi (\nabla_{\tilde{e}_B} \tilde{e}_B) = \sum_{B=1}^{2n} \hat{\nabla}_{e_B} e_B \]
which implies that
\[ \sum_{i=1}^{2n} \left( \hat{\nabla}_{\bar{e}_B} d\pi \right) (\bar{e}_B) = 0. \] (5.7)
Taking the horizontal trace of (5.4) and using (5.6), (5.7), we obtain that
\[ \tau_H(\bar{I}) = \sum_{i=1}^{2n} \left( \hat{\nabla}_{e_B} dI \right) (e_B) = 0, \]
since $I$ is harmonic. Hence $\bar{I}$ is nontrivial pseudo-harmonic. But the image of $\bar{I}$ is exactly the unit ball in $\mathbb{C}^n$ which is a regular ball. So this is a nontrivial pseudo-harmonic example when the domain has negative pseudo-Hermitian Ricci curvature.

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