Hypercomplex representation of the Lorentz’s group

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Lorentz’s group represented by the hypercomplex system of numbers, which is based on dirac matrices, is investigated. This representation is similar to the space rotation representation by quaternions. This representation has several advantages. Firstly, this is reducible representation. That is why transformation of different geometrical objects (vectors, antisymmetric tensors of the second order and bispinors) are implemented by the same operators. Secondly, the rule of composition of two arbitrary Lorentz’s transformations has a simple form. These advantages strongly simplify finding a lot of the laws related to the Lorentz’s group. In particular they simplify investigation of the spin connection with Pauli-Lubanski pseudovector and Wigner little group.

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INTRODUCTION

Main properties of the Lorentz’s group have been investigated using infinitesimal transformations. However, finite transformations are necessary in a lot of practical application. Several consistent transformations of such kind are often necessary. In this case transformation progress depends on simplicity/complexity of the composition law of the parameters, which describe the transformation. This, in turn, depends on parameters choice. The simplest composition rule is Fedorov’s parametrization. He introduced matrix \( L \), which reflects Lorentz’s transformation of the 4-vector, in a form shown below. I.e. he parametrized this \( L \) matrix by complex 3D vector \( c = a + ib \).

\[
L(c) = 2 \begin{pmatrix}
  a_1a_1 + b_1b_1 + \frac{1-a^2-b^2}{2} & a_1a_2 + b_1b_2 - a_3 & a_1a_3 + b_1b_3 + a_2 & i(b_1 + \varepsilon_{1kl}a_kb_l) \\
a_2a_1 + b_2b_1 + a_3 & a_2a_2 + b_2b_2 + \frac{1-a^2-b^2}{2} & a_2a_3 + b_2b_3 - a_1 & i(b_2 + \varepsilon_{2kl}a_kb_l) \\
a_3a_1 + b_3b_1 - a_2 & a_3a_2 + b_3b_2 + a_1 & a_3a_3 + b_3b_3 + \frac{1-a^2-b^2}{2} & i(b_3 + \varepsilon_{3kl}a_kb_l) \\
-\imath(b_1 - \varepsilon_{1kl}a_kb_l) & -\imath(b_2 - \varepsilon_{2kl}a_kb_l) & -\imath(b_3 - \varepsilon_{3kl}a_kb_l) & \frac{1+a^2+b^2}{2}
\end{pmatrix}
\]

In this way he shows that two successive transformations \( L(c_1) \) and \( L(c_2) \) with parameters \( c_1 \) and \( c_2 \) can be changed by the one \( L(c) = L(c_2)L(c_1) \). Where parameter \( c \) is expressed by \( c_1 \) и \( c_2 \) as

\[
c = \frac{c_1 + c_2 + c_2 \times c_1}{1 - c_1 \cdot c_2}.
\]

However, use of matrix \( L(c) \) in relativistic quantum electrodynamics is inconvenient and is not widely used. Instead matrix exponent \( e^{L(c)} \) for Dirac’s bispinor transformation is widely used. In this case transformation parameter is antisymmetric tensor \( L_{\alpha\beta} \) of the second order presented in the matrix form \( L_{\alpha\beta}\sigma^{\alpha\beta} \). In fact, this representation is the hypercomplex number form. However, composition rule for these hypercomplex parameters for two arbitrary Lorentz’s transformations is not known. We will get this rule below and show that composition rule for hypercomplex parameters is similar to the Fedorov’s (1). But Lorentz’s transformations implement in traditional for relativistic quantum electrodynamics way.

It will be also shown that hypercomplex description allows to easily derive set of rules connected with Lorentz’s transformations.

DIRAC’S HYPERCOMPLEX NUMBERS

It is known, that unit matrix \( \hat{1} \), four Dirac’s matrices \( \gamma^\alpha \)

\[
\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = 2\eta^{\alpha\beta}\hat{1},
\]

\( \eta^{\alpha\beta} = \text{diag}(1,-1,-1,-1) \), and eleven products

\( \hat{i} = \gamma^0 \gamma^1 \gamma^2 \gamma^3 \), \( \pi^\alpha = \gamma^\alpha \hat{i} \), \( \sigma^{\alpha\beta} = \left( \gamma^\alpha \gamma^\beta - \gamma^\beta \gamma^\alpha \right)/2 \)

can be used as sixteen basic units of the hypercomplex numbers system \( \mathbb{C} \). It does not matter what the actual
presentation is. Frequently used are: standard Dirac-Pauli basis and chiral Weyl's basis.

Let us call numbers of this system

$$D = a \hat{1} + b \hat{i} + c_\alpha \gamma^\alpha + d_\alpha \pi^\alpha + f_{\alpha\beta} \sigma^{\alpha\beta} \quad (4)$$

as Dirac's numbers. Coefficients in front of basic units are complex numbers. Assume that coefficients $f_{\alpha\beta}$ in front of $\sigma^{\alpha\beta}$ are antisymmetric $f_{\alpha\beta} = -f_{\beta\alpha}$. It is possible because symmetrical parts have no contribution into the contraction with antisymmetric $\sigma^{\alpha\beta}$. As usually, the Greek indices take on values 0, 1, 2, 3, Latin indices - 1, 2, 3. The hypercomplex representation of the continuous proper Lorentz's transformations are discussed below. It will be shown that numbers $a\hat{1}$ and $ai$ under the continuous proper Lorentz's transformations behave as scalars, $a_\alpha \gamma^\alpha$ and $a_\alpha \pi^\alpha$ - as vectors, $a_{\alpha\beta} \sigma^{\alpha\beta}$ - as antisymmetric tensors of the second order. But under the discrete space reflection (inversion) like $\gamma^0 D \gamma^0$, numbers $a\hat{1}$ behave as scalars, $\gamma^0 a \hat{1} \gamma^0 = a\hat{1}$, numbers $ai$ as pseudoscalars, $\gamma^0 a i \gamma^0 = -ai$, numbers $a_\alpha \gamma^\alpha$ as vectors $\gamma^0 (a_\alpha \gamma^\alpha + a_\alpha k^\alpha) \gamma^0 = a_\alpha \gamma^\alpha - a_\alpha k^\alpha$, numbers $a_\alpha \pi^\alpha$ as pseudovectors $\gamma^0 (a_\alpha \pi^\alpha + a_\alpha k^\alpha) \gamma^0 = -a_\alpha \pi^\alpha + a_\alpha k^\alpha$. That is why we will use for these variables corresponding terms.

We can add, subtract and multiply Dirac numbers as any matrices. Table 1 can be useful in order to simplify multiplications. First multiplier must be taken from the left column of the table and the second - from the first raw. For example,

$$a_{\alpha\beta} \sigma^{\alpha\beta} b_{\mu} \pi^\mu = 2b^\alpha (a_\alpha^0 \gamma^\beta - a_{\alpha\beta} \pi^\beta).$$

Here $a_\alpha^0$ is a tensor dual to the $a_{\alpha\beta}$ one

$$a_\alpha^0 = \frac{1}{2} \varepsilon_{\alpha\beta\mu\nu} a^{\mu\nu}, \quad (6)$$

$\varepsilon_{\alpha\beta\mu\nu}$ - completely antisymmetric tensor, $\varepsilon_{0123} = 1$, $\varepsilon_{0123} = -1$.

| $a_\alpha \gamma^\alpha$ | $b_\mu \gamma^\mu$ | $b_\mu \sigma^{\mu\nu}$ | $b \hat{i}$ |
|------------------------|------------------|----------------------|-------------|
| $a_\alpha b^\alpha + \frac{1}{2} [a_\alpha b_\beta] \sigma^{\alpha\beta}$ | $ia_\alpha b^\alpha + \frac{1}{2} [a_\alpha b_\beta] \sigma^{\alpha\beta}$ | $2(a_\alpha b_\beta \gamma^\beta - a^\alpha b_\beta^\alpha \pi^\beta)$ | $ba_\alpha \pi^\alpha$ |
| $a_\alpha \pi^\alpha$ | $-ia_\alpha b^\alpha - \frac{1}{2} [a_\alpha b_\beta] \sigma^{\alpha\beta}$ | $a^\alpha b^\alpha + \frac{1}{2} [a_\alpha b_\beta] \sigma^{\alpha\beta}$ | $2(a_\alpha b_\beta \gamma^\beta + a^\alpha b_\beta^\alpha 
π^\beta)$ | $-ba_\alpha \gamma^\alpha$ |
| $a_{\alpha\beta} \sigma^{\alpha\beta}$ | $-2(b^\alpha a_{\alpha\beta} \gamma^\beta + b^\beta a_{\alpha\beta} \pi^\beta)$ | $2(b^\alpha a_{\alpha\beta}^0 \gamma^\beta - b^\beta a_{\alpha\beta} \pi^\beta)$ | $-2a_{\alpha\beta}^0 b_{\beta\alpha} + 2[a_\alpha \mu b_{\beta\mu}] \sigma^{\alpha\beta}$ | $ba_{\alpha\beta}^0 \sigma^{\alpha\beta}$ |
| $ai$ | $-ab_\mu \pi^\mu$ | $ab_\mu \gamma^\mu$ | $ab_\mu \sigma^{\mu\nu}$ | $-ab$ |

In the Table 1 unit vector $\hat{1}$ is omitted. Below we also omit $\hat{1}$ if it does not case misunderstanding.

Square brackets designate two operations. Firstly, the antisymmetrized tensor product that collates two 4-vectors to the antisymmetric tensor of the second order

$$[a_\alpha b_\beta] = a_\alpha b_\beta - b_\alpha a_\beta, \quad [a_\alpha b_\beta] = -[a_\beta b_\alpha]. \quad (7)$$

Secondly, antisymmetrized contraction that collates two antisymmetric tensor of the second order to the antisymmetric tensor of the second order

$$[a_{\alpha\beta} b_{\mu\nu}] = a_{\alpha\beta} b_{\mu\nu} - b_{\alpha\beta} a_{\mu\nu}, \quad [a_{\alpha\beta} b_{\mu\nu}] = -[a_{\beta\alpha} b_{\mu\nu}]. \quad (8)$$

Symbol “bullet” (●) designates operation that collates two antisymmetric tensor to the scalar and pseudoscalar:

$$a_{\alpha\beta}^\alpha \bullet b_{\alpha\beta} = a_{\alpha\beta}^\alpha b_{\alpha\beta} - ia_{\alpha\beta}^\alpha b_{\alpha\beta} = \nonumber$$

$$= a_{\alpha\beta}^\alpha b_{\alpha\beta} - \frac{1}{2} i \varepsilon_{\alpha\beta\mu\nu} a_{\alpha\beta} b_{\mu\nu}. \quad (9)$$

System of the numbers (4) contains several subsystems. Below we will use three of them.

Numbers $x = a\hat{1} + ib$ form subsystem, isomorphic to the complex numbers. Let us call these numbers as $i$-complex numbers. Consequently, let us call $\bar{x} = a\bar{1} - ib$ numbers $i$-conjugate to the $x = a\hat{1} + ib$ numbers. We can use $i$-complex numbers in the same way as complex numbers. In particular

$$ii = -1, \quad \sqrt{-1} = \pm i. \quad (10)$$

Below we will use $i$-complex numbers and $i$-complex vectors.

Four basic units $\hat{1}, \sigma^{kl}$, where $k, l = 1, 2, 3$ form another subsystem from the numbers:

$$q = a\hat{1} + f_{kl} \sigma^{kl} \quad (11)$$

This subsystem is isomorphic to the quaternions. Multiplication table for the basic units $\hat{1}, \sigma^{23}, \sigma^{31}, \sigma^{12}$
Also numbers coincidence with the quaternion multiplication table:

| × | σ⁻²³ | σ⁻¹² | σ⁻¹¹ | σ⁻¹⁰ |
|---|---|---|---|---|
| σ⁻²³ | -1 | σ¹２ | -σ⁻¹¹ | -σ⁻¹² |
| σ⁻¹² | -σ⁻¹¹ | -1 | σ⁻¹⁰ | σ⁻¹¹ |
| σ⁻¹¹ | σ⁻¹² | σ⁻¹⁰ | -1 | -σ⁻¹¹ |
| σ⁻¹⁰ | -σ⁻¹² | σ⁻¹¹ | σ⁻¹⁰ | -1 |

Table 2 can be useful for multiplication numbers (16)

| b₁ | b₂γμ | hmnσmn |
|----|----|----|
| a₁ | ab₁ | ab₂γμ |
| aαγα | bαγα | aαbαＩ + ½([aαbβ] - i[aαbβ])σkl |
| aklσkl | bklσkl | -2bklαγl + iεklαβδklαγδkl |

Table 2 based on the 8 units with 3-vectors

D = uÎ + zαγα + wklσkl, k,l = 1, 2, 3. (16)

Here u, zα,wkl are complex coefficients, u = uÎ + u∅k etc. Coefficients u, zα, wkl are connected with a, b, cα, dα, fαβ in (4) as

u = a + ib,

zα = cα - i dα, (17)

wkl = fkl - iεklαβfαβ = fkl - ifαβ.

Table 2 can be useful for multiplication numbers (16) based on the 8 units with complex coefficients. In the Table and below

[σαβ] = 1/2εσαββσαββ. (20)

It is more convenient to use electric E and magnetic field B instead of the electromagnetic field tensor Fαβ in the electrodynamics. Similar to this we can get some advantages of the Dirac’s numbers (16) based on 8 units, if we use those parts of 4-vectors 3αγα or 4-tensors multiplication by i converts σ^lm into σ^0k because

\[ iσ^{αβ} = \frac{1}{2}σ^{αβ},σ^{μν}. \] (13)

In particular, iσ^lm = 1/2 σ^lm σ^αβ. For example,

\[ iσ^{23} = -σ^{01}, \quad iσ^{31} = -σ^{02}, \quad iσ^{12} = -σ^{03}. \] (14)

That is why instead of the subsystem (12) with 8 units we can use system (11) with 4 units 1, σ^kl, but with i-complex coefficients u and wkl:

\[ d = uÎ + wklσkl, \quad k,l = 1, 2, 3. \] (15)

This subsystem is isomorphic to the biquaternions. In the same way multiplication by i converts γ^α into π^α, iγ^α = -π^α, that is why there is no necessity to use 16 basic units in the Dirac’s system of numbers (4). It is enough to use 8 units 1, γ^α, γ^23, γ^31, γ^12, but with i-complex coefficients:

wklσkl, which behave as scalars and 3-vectors as a part of 8 units under the action of the space rotation, instead of 4-vectors or 4-tensors. Let us write 3αγα and wklσkl in the form

\[ zαγα = z₀γ₀ + z_kγ^k = z₀γ₀ - z_kγ^k \equiv z₀γ₀ - zγ, \quad (21) \]

\[ wklσkl = 2(w_{23}σ_{23} + w_{31}σ_{31} + w_{12}σ_{12}) = \]

\[ = 2(w_{23}σ_{23} + w_{31}σ_{31} + w_{12}σ_{12}) ≡ 2wφ \] (22)

in order to separate these parts. In such 3D i-complex notation number (16) has a form

\[ D = uÎ + z₀γ₀ - zγ + 2wφ. \] (23)

As we will see later, under the space rotation quantities uÎ and z₀γ₀ are transformed as i-complex scalars, and quantities zγ and wφ as 3D vectors with i-complex components. Multiplication of z₀γ₀, zγ and wφ is reduced to the 3D dot product and cross product. But we need to remember that i commute with φ and anticommute with γ. Multiplication rules are cited in
Parallel to the $r$ component of the vector $a$ ($\langle a \frac{r}{r} \rangle$) is not changed, but the perpendicular one rotates around $r$ in positive direction on the angle $2r$. Positive direction of the vector $r$ can be determine by the right-hand rule. Thus, expression $e^{rq}a e^{-rq}$ describes the rotation of the $a$ vector. Vector $r$ - parameter of this transformation. This rotation around the $r$ vector in positive direction. Angle magnitude is equal to the $2r$.

Convenience of the quaternion rotation description is that it gives us simple rule of two rotation composition. Let us write exponent $e^{rq}$ in the following way

$$e^{rq} = \cos r (1 + \frac{rq}{r} \tan r) =$$

$$= \cos r (1 + \rho q) = \frac{1 + \rho q}{\sqrt{1 + \rho \cdot \rho}}.$$  \hspace{1cm} (29)

Here

$$\rho = \frac{r}{r} \tan r.$$ \hspace{1cm} (30)

By replacing $r$ by $\rho$ we change the rotation parametrization. Vector $\rho$ becomes the rotation parameter. Vector $a$ rotates around $\rho$ in positive direction and the $\rho$ length equal to the tangent of the half-angle rotation:

$$\tan r = \sqrt{\rho \cdot \rho}.$$ \hspace{1cm} (31)

As we will see composition rule becomes very simple if parameter $\rho$ is used. Let us describe the first rotation by exponent $e^{r_1q}$ and the following rotation by exponent $e^{r_2q}$. Then their composition can be written as multiplication of two exponents $e^{r_1q}$ and $e^{r_2q}$:

$$e^{r_2q}e^{r_1q} = \frac{(1 + \rho_2q)(1 + \rho_1q)}{\sqrt{1 + \rho_2 \cdot \rho_2}(1 + \rho_1 \cdot \rho_1)} \left(1 + \frac{\rho_1 + \rho_2 + \rho_2 \times \rho_1}{1 - \rho_1 \cdot \rho_2}\right).$$ \hspace{1cm} (32)

Let us set

$$\rho = \frac{\rho_1 + \rho_2 + \rho_2 \times \rho_1}{1 - \rho_1 \cdot \rho_2}.$$ \hspace{1cm} (33)

It is easy to see that

$$\rho \cdot \rho = \frac{(1 + \rho_2 \cdot \rho_2)(1 + \rho_1 \cdot \rho_1)}{(1 - \rho_1 \cdot \rho_2)^2} - 1.$$ \hspace{1cm} (34)

That is why

$$e^{r_2q}e^{r_1q} = \frac{1 + \rho q}{\sqrt{1 + \rho \cdot \rho}} = e^{rq}.$$ \hspace{1cm} (35)

Here

$$r = \rho \frac{r}{\tan r}, \quad \tan r = \sqrt{\rho \cdot \rho}.$$ \hspace{1cm} (36)
Thus, parameter $\rho$ of the general transformation $e^{\rho q}$, which is the result of rotation $e^{\rho_1 q}$ and further rotation $e^{\rho_2 q}$, is determined by the composition rule (33). We can see that this rule differs from the rule (1) by transformation parameter only. Parameter $\rho$ is real number while parameter $c$ is complex number.

Let us proceed to the Lorentz’s transformation. Hypercomplex units $\sigma_{\alpha\beta}$ commutators coincide with the infinitesimal Lorentz’s group operator commutators. That is why in hypercomplex description proper Lorentz’s transformations can be presented as

$$D' = e^{i\frac{1}{2}L_{\alpha\beta}\sigma_{\alpha\beta}^a}De^{-i\frac{1}{2}L_{\alpha\beta}\sigma_{\alpha\beta}^a}. \tag{37}$$

Exponent $e^{iL_{\alpha\beta}\sigma_{\alpha\beta}^a}$ is determined by the expansion

$$e^{iL_{\alpha\beta}\sigma_{\alpha\beta}^a} = 1 + \frac{1}{2}L_{\alpha\beta}\sigma_{\alpha\beta}^a + \frac{1}{2!2}L_{\alpha\beta}\sigma_{\alpha\beta}^aL_{\alpha\beta}\sigma_{\alpha\beta}^a + \ldots \tag{38}$$

Product $(L_{\alpha\beta}\sigma_{\alpha\beta}^aL_{\alpha\beta}\sigma_{\alpha\beta}^a)/4$ in (38) is $i$-complex number:

$$\frac{1}{2}L_{\alpha\beta}\sigma_{\alpha\beta}^a\frac{1}{2}L_{\alpha\beta}\sigma_{\alpha\beta}^a =$$

$$= -\frac{1}{2}(L_{\alpha\beta}L_{\alpha\beta} - iL_{\alpha\beta}L_{\alpha\beta}^\circ) = -\frac{1}{2}L_{\alpha\beta}^2 \equiv L^2. \tag{39}$$

Correspondingly for the exponent (38) we get

$$e^{i\frac{1}{2}L_{\alpha\beta}\sigma_{\alpha\beta}^a} = \cosh L + \sinh L \frac{1}{2}L_{\alpha\beta}\sigma_{\alpha\beta}^a =$$

$$= \cosh L(1 + \frac{\tanh L \frac{1}{2}L_{\alpha\beta}\sigma_{\alpha\beta}^a}{\sqrt{1 + \frac{1}{2}L_{\alpha\beta}^2}}). \tag{40}$$

Here $\Lambda_{\alpha\beta} = \frac{\tanh L \frac{1}{2}L_{\alpha\beta}}{\sqrt{1 + \frac{1}{2}L_{\alpha\beta}^2}}$, $L = \sqrt{L^2}$, $L^2$ and $L$ are $i$-complex numbers. In the definition $L$ either root sign can be used because $L/L$ is an even function. Obviously,

$$e^{i\frac{1}{2}L_{\alpha\beta}\sigma_{\alpha\beta}^a}e^{-i\frac{1}{2}L_{\alpha\beta}\sigma_{\alpha\beta}^a} = \left(\cosh L + \frac{1}{2}L_{\alpha\beta}\sigma_{\alpha\beta}^a \sinh L\right) \times$$

$$\times \left(\cosh L - \frac{1}{2}L_{\alpha\beta}\sigma_{\alpha\beta}^a \sinh L\right) = \cosh^2 L - \sinh^2 L = 1. \tag{41}$$

With a help of table (1) and exponent (40) it is easy to show that expression (37) is the proper Lorentz’s transformations. For this we must present scalar and pseudo-scalar as numbers $a$ and $ai$, vectors and pseudo-vectors as numbers $a\gamma^a, a\sigma^a$, and antisymmetric tensors of the second order as numbers $a_{\alpha\beta}\sigma_{\alpha\beta}^a$. In fact, this operation does not change numbers $a$ and $ai$:

$$a' = e^{i\frac{1}{2}L_{\alpha\beta}\sigma_{\alpha\beta}^a}a \equiv e^{-i\frac{1}{2}L_{\alpha\beta}\sigma_{\alpha\beta}^a}a = e^{i\frac{1}{2}L_{\alpha\beta}\sigma_{\alpha\beta}^a}e^{-i\frac{1}{2}L_{\alpha\beta}\sigma_{\alpha\beta}^a}a = a, \tag{42}$$

$$ai' = e^{i\frac{1}{2}L_{\alpha\beta}\sigma_{\alpha\beta}^a}ai \equiv e^{-i\frac{1}{2}L_{\alpha\beta}\sigma_{\alpha\beta}^a}ai = ai. \tag{43}$$

and does not change quadratic forms $a^a a_a = a_0^2 - a_1^2 - a_2^2 - a_3^2$, which are connected with vectors and pseudo-vectors:

$$a' a' = a'_a a'_a = e^{4\frac{1}{2}L_{\alpha\beta}\sigma_{\alpha\beta}^a}a_a a_a e^{-4\frac{1}{2}L_{\alpha\beta}\sigma_{\alpha\beta}^a} =$$

$$= e^{4\frac{1}{2}L_{\alpha\beta}\sigma_{\alpha\beta}^a}a_a a_a e^{-4\frac{1}{2}L_{\alpha\beta}\sigma_{\alpha\beta}^a} = a^a a_a. \tag{44}$$

$$a' ai' = a'_a ai = a'_a ai = a_0 a_1. \tag{45}$$

Also does not change quadratic forms $a^a a_{\alpha\beta}$ and $a^{\alpha\beta} a_{\alpha\beta}^\circ$, which are connected with antisymmetric tensors:

$$2a_{\alpha\beta} a_{\alpha\beta} = 2(a^a a_{\alpha\beta} - i\alpha^a a^a a_{\alpha\beta}) = -a'_{\alpha\beta} a^a a_{\alpha\beta} =$$

$$= -e^{-4\frac{1}{2}L_{\alpha\beta}\sigma_{\alpha\beta}^a}a_{\alpha\beta} a_{\alpha\beta} e^{-4\frac{1}{2}L_{\alpha\beta}\sigma_{\alpha\beta}^a} =$$

$$= e^{-4\frac{1}{2}L_{\alpha\beta}\sigma_{\alpha\beta}^a}a_{\alpha\beta} a_{\alpha\beta} e^{-4\frac{1}{2}L_{\alpha\beta}\sigma_{\alpha\beta}^a} =$$

$$= e^{-4\frac{1}{2}L_{\alpha\beta}\sigma_{\alpha\beta}^a}a_{\alpha\beta} a_{\alpha\beta} e^{-4\frac{1}{2}L_{\alpha\beta}\sigma_{\alpha\beta}^a} =$$

$$= e^{-4\frac{1}{2}L_{\alpha\beta}\sigma_{\alpha\beta}^a}a_{\alpha\beta} a_{\alpha\beta} e^{-4\frac{1}{2}L_{\alpha\beta}\sigma_{\alpha\beta}^a} =$$

$$= 2a^a a_{\alpha\beta} = 2a_{\alpha\beta} a_{\alpha\beta} = 2a^a a_{\alpha\beta} = 2a_{\alpha\beta} a_{\alpha\beta} = 2a_{\alpha\beta} a_{\alpha\beta}. \tag{46}$$

Quadratic forms $a^a a_{\alpha\beta}$ and $a^{\alpha\beta} a_{\alpha\beta}^\circ$ are counterparts of the well-known from electrodynamics scalar and pseudo-scalar invariants $E^2 - c^2 B^2$ and $E \cdot B$.

Conservation law of the $a_{\alpha\beta} a^a$ proves that transformation (37) is Lorentz’s transformation of the vectors and pseudo-vectors. In turn, conservation law of the $a_{\alpha\beta} a^a$ and $a_{\alpha\beta} a_{\alpha\beta}^\circ$ proves that transformation (37) is Lorentz’s transformation of the antisymmetric tensors. Six components of the antisymmetric tensor $L_{\alpha\beta}\sigma_{\alpha\beta}^a$ are parameters of this transformation.

Instead of the $L_{\alpha\beta}$ we can use dimensionless variable $L_{\alpha\beta} = \frac{1}{mc}F_{\alpha\beta}\delta t$, which is proportional to the electromagnetic field tensor $F_{\alpha\beta}$. $\delta t$ - infinitesimal proper time interval of the particle of mass $m$ and charge $q$. In this case with linear accuracy on $\delta t$ terms for the 4-impulse transformations of the particle we get [? ]

$$p_{\alpha}' \gamma^\alpha = e^{\frac{1}{2}L_{\alpha\beta}\sigma_{\alpha\beta}^a}p_{\alpha} \gamma^\alpha e^{-\frac{1}{2}L_{\alpha\beta}\sigma_{\alpha\beta}^a} =$$

$$= (1 + \frac{1}{2}L_{\alpha\beta}\sigma_{\alpha\beta}^a) p_{\alpha} \gamma^\alpha (1 - \frac{1}{2}L_{\alpha\beta}\sigma_{\alpha\beta}^a) =$$

$$= (p_{\alpha} + L_{\alpha\beta} p^\beta) \gamma^\alpha = (p_{\alpha} + \frac{q}{mc} F_{\alpha\beta} \gamma^\alpha) =$$

$$= p_{\alpha} + \delta p_{\alpha} \gamma^\alpha. \tag{47}$$

Change of the momentum in this case is the same as its change under the action of the Lorentz force:

$$\frac{dp_{\alpha}}{dt} = \frac{q}{mc} F_{\alpha\beta} p^\beta. \tag{48}$$

In other words, the moment of the charged particle in the electromagnetic field experiences the sequence of infinitesimal Lorentz transformations. Parameters of
these transformations $L_{\alpha \beta}$ are proportional to the electromagnetic field tensor $F_{\alpha \beta}$. We can also treat motion of any particle with constant mass as sequence of infinitesimal Lorentz transformations, because in this case quantity $p_{\alpha}p^{\alpha}$ conserves.

Let us return to the main topic of the investigation. It is also clear that $e^{\frac{1}{2}L_{\alpha \beta}\sigma^{\alpha \beta}} \psi$ is Lorentz's transformation of the Dirac bispinors $\psi$, because it does not change bilinear combination $\psi \psi$. Really because of

$$ (L_{\alpha \beta}\sigma^{\alpha \beta})^{1/2} \gamma^0 = -\gamma^0 L_{\alpha \beta} \sigma^{\alpha \beta}, \quad (49) $$

we get

$$ \bar{\psi}' = \psi' \gamma^0 = (e^{\frac{1}{2}L_{\alpha \beta}\sigma^{\alpha \beta}} \psi) \gamma^0 = \psi \gamma^0 e^{\frac{1}{2}L_{\alpha \beta}\sigma^{\alpha \beta}} = \bar{\psi} e^{-\frac{1}{2}L_{\alpha \beta}\sigma^{\alpha \beta}}. \quad (50) $$

That is why

$$ \bar{\psi}' \psi' = \bar{\psi} e^{-\frac{1}{2}L_{\alpha \beta}\sigma^{\alpha \beta}} e^{\frac{1}{2}L_{\alpha \beta}\sigma^{\alpha \beta}} \psi = \bar{\psi} \psi. \quad (51) $$

We have seen, therefore, that (37) implements an arbitrary proper Lorentz's transformation of the vectors and antisymmetric tensors of the second order. And $e^{\frac{1}{2}L_{\alpha \beta}\sigma^{\alpha \beta}} \psi$ implements an arbitrary proper Lorentz's transformation of the bispinors. Transformation of all these quantities implements by the same exponents $e^{\pm \frac{1}{2}L_{\alpha \beta} \sigma^{\alpha \beta}}$. This considerably simplify treatment in a lot of cases.

Instead of hypercomplex numbers (4) with 16 basic units we can use numbers (16) with 8 units and $i$-complex coefficients. In this case expression for the exponent (38) and subsequent expressions become simpler, because instead of tensor $L_{\alpha \beta} \sigma^{\alpha \beta}$ with 6 basic units $\sigma^{\alpha \beta}$ and real $L_{\alpha \beta}$ we must use tensor $l_{kl} \sigma^{kl}$ with 3 basic units $\sigma^{kl}$ and $i$-complex $l_{kl}$. Proper Lorentz's transformations are:

$$ D' = e^{\frac{1}{2}l_{kl} \sigma^{kl}} D e^{-\frac{1}{2}l_{kl} \sigma^{kl}}, \quad (52) $$

where $\frac{1}{2}l_{kl} \sigma^{kl} = l_{23} \sigma^{23} + l_{31} \sigma^{31} + l_{12} \sigma^{12}$. Exponent $e^{\frac{1}{2}l_{kl} \sigma^{kl}}$ equal

$$ e^{\frac{1}{2}l_{kl} \sigma^{kl}} = \cosh l + \frac{\sinh \frac{1}{2} l_{kl} \sigma^{kl}}{l} = \cosh (1 + \frac{1}{2} \frac{\lambda_{kl} \sigma^{kl}}{l}), \quad (53) $$

Here $l$ — $i$-complex number

$$ l = \sqrt{\frac{1}{2} l_{kl} \sigma^{kl} l_{kl} \sigma^{kl}} = \sqrt{\frac{1}{2} l_{kl} l_{kl}}, \quad (54) $$

and $\lambda_{kl} = l_{kl} \frac{\tan h l}{l}$. It is never mind which sign to take in front of $l$ because $\sinh l/l$ is even function. Expression (53) is simpler than (40) because regular tensor contraction $l_{kl} l_{kl}$ of $i$-complex parameters $l_{kl}$ is used instead of (39), which determines $i$-complex number $L_{\alpha \beta}$.

Vectors and pseudo-vectors are represented by $z_{\alpha} \gamma^{\alpha}$ numbers (with $i$-complex $z_{\alpha}$) and tensors are represented by $w_{kl} \sigma^{kl}$ (with $i$-complex $w_{kl}$) when expression (52) is used for proper Lorentz's transformations.

Further simplifications for the exponent (38) and subsequent expressions we can achieve if we use 3D $i$-complex notation (23). In this case exponent power in (38) is

$$ \frac{L_{\alpha \beta} \sigma^{\alpha \beta}}{2} = \frac{l_{kl} \sigma^{kl}}{2} = l_{23} \sigma^{23} + l_{31} \sigma^{31} + l_{12} \sigma^{12} \equiv l, \quad (55) $$

Proper Lorentz's transformation (37) is presented as product:

$$ D' = e^{l \sigma} D e^{-l \sigma} \quad (56) $$

with $e^{\pm l \sigma}$ exponent

$$ e^{l \sigma} = \bar{\psi} e^{-\frac{1}{2}l_{kl} \sigma^{kl}} e^{\frac{1}{2}l_{kl} \sigma^{kl}} \psi = \bar{\psi} \psi. \quad (57) $$

Here $l$ and $\lambda$ = $i$-complex 3D-vectors, and $l$ — is $i$-complex scalar:

$$ l = r + i b, \quad (58) $$

$$ l = \sqrt{l_{kl} l_{kl}} = \sqrt{r \cdot r - b \cdot b}, \quad (59) $$

$$ \lambda = \frac{\tan h l}{l}. \quad (60) $$

Use of expression (23) assumes that 4-vectors and 4-pseudo-vectors are represented by numbers $a_0 \gamma^0 - a^\alpha$, $\gamma^\alpha$ (with $i$-complex coefficients) and antisymmetric tensors of the second order is presented by $a_{\alpha \beta}$ numbers (with $i$-complex coefficients).

Use of these expressions gives us definite expressions for arbitrary Lorentz's transformations, in particular — rotations and boosts.

**Vectors and pseudo-vectors transformation**

Proper Lorentz's transformations change vectors and pseudo-vectors in the same way. That is why we can treat them jointly. Let us discuss change of the hypercomplex number

$$ z_{\alpha} \gamma^{\alpha} = z_0 \gamma^0 + z_1 \gamma^1 + z_2 \gamma^2 + z_3 \gamma^3 = z_0 \gamma^0 - z_1 \gamma^1 - z_2 \gamma^2 - z_3 \gamma^3 = z_0 \gamma^0 - z \gamma \quad (61) $$

by (56). Here $z_{\alpha}$ is $i$-complex components of the $i$-complex 4-vector $z_{\alpha}$. Numbers $z_{\alpha} \gamma^{\alpha}$ are vectors if $z_{\alpha}$ are
real and \( z_\alpha \gamma^\alpha \) are pseudo-vectors if \( z_\alpha \) are \( i \)-complex.

\[
\begin{align*}
\frac{\partial}{\partial t'} &= \frac{\partial}{\partial t} + \frac{v}{c} \left( \frac{\partial}{\partial z'} - \frac{v}{c} \frac{\partial}{\partial r} \right)
\end{align*}
\]

Coefficients in front of \( z_\alpha \) are real. That is why transformation of the real and complex part of the \( i \)-complex vector \( z_\alpha \) is independent and similar. Just reminder, that vector \( z_\alpha \) describes vectors and pseudo-vectors.

In general case expression (62) (when \( l = i \cdot \) - \( i \)-complex value) describes arbitrary proper Lorentz’s transformations of vectors and pseudo-vectors. In particular it can be rotations and boosts.

Rotations

When \( b = 0 \) and \( l = r \), then \( l = \sqrt{r_2 r_2} = \sqrt{-r \cdot r} = i \sqrt{r \cdot r} = i r, \sinh l = i \sin r, \cosh l = \cos r \), matrix \( e^{ibz} \) is unitary; \( (e^{ibz})^\dagger = e^{-ibz} = (e^{ibz})^{-1} \). Transformation (62) becomes

\[
\begin{align*}
\frac{\partial}{\partial t'} &= \frac{\partial}{\partial t} + \frac{v}{c} \left( \frac{\partial}{\partial z'} - \frac{v}{c} \frac{\partial}{\partial r} \right)
\end{align*}
\]

I.e.

\[
\begin{align*}
z_0' &= z_0, \quad (64) \\
z' &= \frac{r}{r} \left( \frac{r \cdot z}{r} \right) + \left[ z - \frac{r}{r} \left( \frac{r \cdot z}{r} \right) \right] \cos 2r + \left( \frac{r \cdot z}{r} \right) \sin 2r. \quad (65)
\end{align*}
\]

As we can see this transformation does not change time component of the \( z_0 \) 4-vector \( z^\alpha \) and rotate its spatial part \( z \) around the the \( r \) vector in positive direction on angle \( 2r \).

Boosts

When \( r = 0 \) and \( l = i b \), then \( l_2 = i b_2, l = \sqrt{l_2 l_2} = \sqrt{b \cdot b} = b, \sinh l = \sinh b, \cosh l = \cosh b, \) matrix \( e^{ibz} \) is hermitian: \( (e^{ibz})^\dagger = e^{-ibz} \). Transformation (62) becomes

\[
\begin{align*}
\frac{\partial}{\partial t'} &= \frac{\partial}{\partial t} + \frac{v}{c} \left( \frac{\partial}{\partial z'} - \frac{v}{c} \frac{\partial}{\partial r} \right)
\end{align*}
\]

Let us set up \( \frac{b}{v} = \frac{\xi}{v} \), tanh \( 2b = \frac{\xi}{v} \). In this case expression (66) describes transformation of the contravariant components of the vector \( z^\alpha \) under the move from the reference frame \( S \) to the reference frame \( S' \), which is moving with relative velocity \( v \). Also \( \cosh 2b = 1/\sqrt{1 - \xi^2/v^2}, \sinh 2b = \xi/v/\sqrt{1 - \xi^2/v^2} \) and instead of (66) we get

\[
\begin{align*}
\frac{\partial}{\partial t'} &= \frac{\partial}{\partial t} + \frac{v}{c} \left( \frac{\partial}{\partial z'} - \frac{v}{c} \frac{\partial}{\partial r} \right)
\end{align*}
\]

Thus

\[
\begin{align*}
z_0' &= z_0 \left( \frac{\cos \frac{v}{c} \left( \frac{r \cdot z}{r} \right) \cos 2r + \left( \frac{r \cdot z}{r} \right) \sin 2r}{1 - \xi^2/v^2} \right) \quad (68)
\end{align*}
\]

This is veil-known \( z^\alpha \) vector transformation caused by boost, i.e. transformation from reference frame \( S \) to the reference frame \( S' \).

Tensor transformation

Using \( F_{\alpha \beta} \) numbers we can represent antisymmetric tensor of the second order as

\[
\begin{align*}
\frac{1}{2} F_{\alpha \beta} \sigma^{\alpha \beta} =
\end{align*}
\]

Here usual designations are used for the electromagnetic field tensor

\[
\begin{align*}
F^{01}, F^{02}, F^{01} = - E_x, - E_y, - E_z, \\
F^{23}, F^{31}, F^{12} = - c B_x, - c B_y, - c B_z,
\end{align*}
\]
also $i$-complex number is used

$$F = -(iE + cB). \quad (71)$$

For the Lorentz's transformation of the antisymmetric tensor of the second order the same expression (56) is used as well as for the vector transformation. Really

$$\frac{1}{2} F_{\alpha'\beta}' \sigma^{\alpha\beta} = F' \zeta = -(iE' + cB') \zeta = e^{i\xi} \frac{1}{2} F_{\alpha'\beta} \sigma^{\alpha\beta} e^{-i\xi} =$$

$$(cosh \frac{l}{l} + l \zeta) F \zeta(cosh \frac{l}{l} - l \zeta sinh \frac{l}{l}) =$$

$$= \left\{ -\frac{l}{l} \left( \frac{F \cdot l}{l} \right) + \left[ F + \frac{l}{l} \left( \frac{F \cdot l}{l} \right) \right] \right\} \cosh 2l + \frac{l}{l} \times F \sinh 2l \zeta. \quad (72)$$

Here the convenient equality is used

$$c \zeta a c \zeta = [a c^2 - 2c(a \cdot c)] \zeta. \quad (73)$$

So, expression $e^{i\xi} \frac{1}{2} F_{\alpha'\beta} \sigma^{\alpha\beta} e^{-i\xi}$ changes $i$-complex number $F$ in the following way:

$$F' = \frac{l}{l} \left( F \cdot \frac{l}{l} \right) + \left[ F + \frac{l}{l} \left( \frac{F \cdot l}{l} \right) \right] \cosh 2l + \frac{l}{l} \times F \sinh 2l. \quad (74)$$

For the rotations, when $l = r, \ l = ir$ we get

$$F' = \frac{r}{r} \left( F \cdot \frac{r}{r} \right) + \left[ F + \frac{r}{r} \left( F \cdot \frac{r}{r} \right) \right] \cos 2r + \frac{r}{r} \times F \sin 2r. \quad (75)$$

This is rotation around the vector $r$ on positive angle $2r$ of vectors $E$ and $cB$.

For the Lorentz boost when $l = ib, l = b, \ b = \frac{v}{c}$, tanh $2b = \frac{v}{c}$, the $i$-complex vector $F$ is transformed as

$$F' = \frac{v}{v} \left( F \cdot \frac{v}{v} \right) + \frac{F - \frac{v}{v} \left( F \cdot \frac{v}{v} \right)}{\sqrt{1 - \frac{v^2}{c^2}}} + i \frac{\frac{v}{v} \times F}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (76)$$

Consequently, transformation of the fields $E$ and $cB$

$$E' = \frac{E + \frac{v}{v} \times cB + v \left( E \cdot \frac{v}{v} \right) \left( 1 - \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \right)}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (77)$$

$$cB' = \frac{cB - \frac{v}{v} \times E + v \left( cB \cdot \frac{v}{v} \right) \left( 1 - \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \right)}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (78)$$

coincide with the well-known electric and magnetic fields boost.

So, we see that antisymmetric tensor of the second order can be presented as hypercomplex numbers (70). Their Lorentz transformation is provided by the same (56) expression as for vectors.

**LORENTZ TRANSFORMATIONS COMPOSITION**

The hypercomplex representation allows us to find the simple rule of the two Lorentz’s transformations composition. I.e. to find the Lorentz’s transformation what is equivalent to two successive Lorentz’s transformations. Those two transformations (the first with parameter $L'$ and the second with parameter $L''$) can be written as multiplication of two exponent $e^{\frac{i}{2} L'_{\alpha'\beta} \sigma^{\alpha\beta}}$ and $e^{\frac{i}{2} L''_{\alpha'\beta} \sigma^{\alpha\beta}}$:

$$e^{\frac{i}{2} L'_{\alpha'\beta} \sigma^{\alpha\beta}} e^{\frac{i}{2} L''_{\alpha'\beta} \sigma^{\alpha\beta}} =$$

$$= \frac{1 + \frac{1}{2} \Lambda''_{\alpha'\beta} \sigma^{\alpha\beta} (1 + \frac{1}{2} \Lambda'_{\alpha'\beta} \sigma^{\alpha\beta})}{\sqrt{1 + \frac{1}{2} \Lambda''_{\alpha'\beta} \sigma^{\alpha\beta} (1 + \frac{1}{2} \Lambda'_{\alpha'\beta} \sigma^{\alpha\beta})}} \times$$

$$\times \left( 1 + \frac{1}{2} \Lambda_{\alpha'\beta} + \frac{\Lambda''_{\alpha'\beta} \Lambda''_{\alpha'\beta}}{1 - \frac{1}{2} \Lambda''_{\alpha'\beta} \sigma^{\alpha\beta}} \right). \quad (79)$$

Let us set up

$$\Lambda_{\alpha'\beta} = \frac{\Lambda'_{\alpha'\beta} + \Lambda''_{\alpha'\beta} + \Lambda''_{\alpha'\beta} \Lambda''_{\alpha'\beta}}{1 - \frac{1}{2} \Lambda''_{\alpha'\beta} \sigma^{\alpha\beta}}. \quad (80)$$

It is easy to see that

$$\Lambda_{\alpha'\beta} \cdot \Lambda_{\alpha'\beta} = \frac{1 + \frac{1}{2} \Lambda''_{\alpha'\beta} \sigma^{\alpha\beta} (1 + \frac{1}{2} \Lambda'_{\alpha'\beta} \sigma^{\alpha\beta})}{(1 - \frac{1}{2} \Lambda''_{\alpha'\beta} \sigma^{\alpha\beta})^2} - 1. \quad (81)$$

Expression (79) becomes

$$e^{\frac{i}{2} L_{\alpha'\beta} \sigma^{\alpha\beta}} e^{\frac{i}{2} L_{\alpha'\beta} \sigma^{\alpha\beta}} = \frac{1 + \frac{1}{2} \Lambda_{\alpha'\beta} \sigma^{\alpha\beta}}{\sqrt{1 + \frac{1}{2} \Lambda_{\alpha'\beta} \sigma^{\alpha\beta}}} = e^{\frac{i}{2} L_{\alpha'\beta} \sigma^{\alpha\beta}}. \quad (82)$$

Here

$$L_{\alpha'\beta} = \Lambda_{\alpha'\beta} \frac{L}{\tanh L}, \quad \tanh L = \sqrt{\frac{1}{2} \Lambda_{\alpha'\beta} \sigma^{\alpha\beta}}. \quad (83)$$

Thus the result of the two Lorentz transformations (with parameters $\Lambda'_{\alpha'\beta}$ and $\Lambda''_{\alpha'\beta}$) is Lorentz transformation with parameter $\Lambda_{\alpha'\beta}$. And expression (80) is the rule of Lorentz transformations parameters $\Lambda_{\alpha'\beta}$ composition.

Two successive Lorentz transformations (52), the first
In this case parameter composition rule (85) becomes
\[
e^{\frac{1}{2}l_{kl} \sigma_{kl}} e^{\frac{1}{2}l_{kl} \sigma_{kl}} = \frac{(1 + \frac{1}{2} \lambda^{'}_{kl} \sigma_{kl})(1 + \frac{1}{2} \lambda^{''}_{kl} \sigma_{kl})}{\sqrt{(1 + \frac{1}{2} \lambda^{'}_{kl} \lambda^{''}_{kl})(1 + \frac{1}{2} \lambda^{'}_{kl} \lambda^{''}_{kl})}} \times \\
\left(1 + \frac{1}{2} \lambda^{''}_{kl} + \lambda^{''}_{kl} + [\lambda^{''}_{km} \lambda^{m}_{l}] \sigma_{kl} \right).
\]
(84)

if we use hypercomplex numbers (16) with 8 basic units with \(i\)-complex coefficients.

Let us set up
\[
\lambda_{kl} = \lambda^{'}_{kl} + \lambda^{''}_{kl} + \lambda^{''''}_{km} \lambda^{m}_{l} \sigma_{kl} - \frac{1}{2} \lambda^{''}_{kl} \lambda_{kl}.
\]
(85)

It is easy to see that
\[
\lambda^{'}_{kl} \lambda_{kl} = \frac{(1 + \frac{1}{2} \lambda^{''}_{kl} \lambda^{'''}_{kl})(1 + \frac{1}{2} \lambda^{''}_{kl} \lambda^{''}_{kl})}{(1 - \frac{1}{2} \lambda^{''}_{kl} \lambda^{''}_{kl})^2} - 1.
\]
(86)

Expression (84) becomes
\[
e^{\frac{1}{2}l_{kl} \sigma_{kl}} e^{\frac{1}{2}l_{kl} \sigma_{kl}} = \frac{1 + \frac{1}{2} \lambda_{kl} \sigma_{kl}}{\sqrt{1 + \frac{1}{2} \lambda_{kl} \lambda_{kl}}} = e^{\frac{1}{2}l_{kl} \sigma_{kl}}.
\]
(87)

Here
\[
l_{kl} = \lambda_{kl} \frac{l}{\tanh l}, \quad \tanh l = \sqrt{- \frac{1}{2} \lambda^{'}_{kl} \lambda_{kl}}.
\]
(88)

In this case parameter composition rule (85) becomes easier, because instead of (9), which determines the \(i\)-complex number \(\Lambda^{'''}_{\alpha \beta} \cdot \lambda_{kl}^{'''}\), the regular tensor contraction \(\lambda^{'''}_{kl} \lambda^{'''}_{kl}\) of the \(i\)-complex tensor coordinates of the tensor parameters have to be used.

Use of the \(i\)-complex 3D vector notifications (23), (56) lead to the further composition rule simplifications:
\[
e^{\frac{1}{2}l_{kl} \sigma_{kl}} = \frac{(1 + \Lambda^{'''}_{kl} \Lambda^{'''}_{kl})(1 + \Lambda^{'''}_{kl} \Lambda^{'''}_{kl})}{\sqrt{(1 + \Lambda^{'''}_{kl} \Lambda^{'''}_{kl})(1 + \Lambda^{'''}_{kl} \Lambda^{'''}_{kl})}} \times \\
\left(1 + \frac{1}{2} \lambda^{'''}_{kl} + \lambda^{'''}_{kl} + [\lambda^{'''}_{km} \lambda^{m}_{l}] \sigma_{kl} \right).
\]
(90)

Thus, expression (89) becomes
\[
e^{\frac{1}{2}l_{kl} \sigma_{kl}} e^{-\frac{1}{2}l_{kl} \sigma_{kl}} = \frac{1 + \Lambda^{'''}_{kl} \Lambda^{'''}_{kl}}{\sqrt{1 + \Lambda^{'''}_{kl} \Lambda^{'''}_{kl}}} = e^{\Lambda^{'''}_{kl}}.
\]

Here
\[
l = \frac{l}{\tanh l}, \quad \tanh l = \sqrt{- \frac{1}{2} \lambda^{'}_{kl} \lambda_{kl}}.
\]
(93)

As we can see, use of the composition rule (90) in this case require regular dot-product and cross-product of \(i\)-complex 3D vectors only.

Composition rule (90) coincide with Fedorov’s rule (1), but \(\lambda\) in (90) is a \(i\)-complex 3D coefficients of the hypercomplex numbers \(\lambda \Lambda\) (i.e. \(4 \times 4\) matrices). Lorentz transformations are performed by \(e^{\pm+i\lambda}\) exponents - usual for the relativistic quantum mechanics way.

**BOOST COMPOSITION**

Let us treat boost composition, which is particular but important case of Lorentz transformations composition. Assume that we have two boosts. The first with parameter
\[
\lambda_1(i b_1) = i \frac{b_1}{b_1} \tanh b_1 = i \frac{v_1}{c} \frac{1}{\sqrt{1 - \frac{v_1^2}{c^2}}}
\]
and the second with parameter
\[
\lambda_2(i b_2) = i \frac{b_2}{b_1} \tanh b_2 = i \frac{v_2}{c} \frac{1}{\sqrt{1 - \frac{v_2^2}{c^2}}}.
\]
(95)

According to (90), parameter transformation, which is the result of the two successive above transformations is
\[
\lambda(i l) = \frac{l}{l} \tanh l = \\
i \left(\frac{b_1}{b_1} \tanh b_1 + \frac{b_2}{b_2} \tanh b_2\right) + (\frac{b_1}{b_1} \times \frac{b_2}{b_2}) \tanh b_1 \tanh b_2
\]
(96)

Because \(\lambda(l)\) is a \(i\)-complex number the resulting transformation is neither boost nor rotation. As any Lorentz transformation it can be represented as result of rotation and boost:
\[
e^{\frac{1}{2}l_{kl} \sigma_{kl}} e^{-\frac{1}{2}l_{kl} \sigma_{kl}} = \frac{1 + \Lambda^{'''}_{kl} \Lambda^{'''}_{kl}}{\sqrt{1 + \Lambda^{'''}_{kl} \Lambda^{'''}_{kl}}} = e^{\Lambda^{'''}_{kl}}
\]
(97)

or as result of boost and rotation:
\[
e^{\frac{1}{2}l_{kl} \sigma_{kl}} e^{-\frac{1}{2}l_{kl} \sigma_{kl}} = e^{i b_{kl} \Lambda^{'''}_{kl}} e^{-i b_{kl} \Lambda^{'''}_{kl}}
\]
(98)

Two successive boosts \(e^{i b_{kl} \Lambda^{'''}_{kl}}\) and \(e^{i b_{kl} \Lambda^{'''}_{kl}}\) can be found by multiplication both parts of equations (97) and (98) by their Hermitian conjugate expressions
\[
(e^{\Lambda^{'''}_{kl}})^{\dagger} = e^{-i \Lambda^{'''}_{kl}} = e^{i \Lambda^{'''}_{kl}} e^{i b_{kl} \Lambda^{'''}_{kl}},
\]
(99)
\[
(e^{\Lambda^{'''}_{kl}})^{\dagger} = e^{-i \Lambda^{'''}_{kl}} = e^{i b_{kl} \Lambda^{'''}_{kl}} e^{-i \Lambda^{'''}_{kl}}.
\]
(100)
Equation (97) multiply from the right and equation (98) from the left:
\[ e^{i \mathbf{k} \mathbf{e} - i \mathbf{k} e} = e^{ibc} e^{r \mathbf{e} - r \mathbf{e} e} e^{ibc} = e^{2ibc}, \]
\[ e^{-i \mathbf{k} e} = e^{ibc} e^{-r \mathbf{e} e} e^{ibc} = e^{2ibc}. \]
Transformation parameters \( e^{i \mathbf{k} e} = e^{2ibc} \) and \( e^{-i \mathbf{k} e} = e^{2ibc} \), according to the (90), is
\[ \lambda(i2b) = \frac{b}{b'} \tanh 2b = \frac{i}{c} \mathbf{v} = c \left( \frac{\lambda(I) - \hat{\lambda}(I) - \hat{\lambda}(I) \times \lambda(I)}{1 + \lambda(I) \cdot \hat{\lambda}(I)} \right) = 2 \frac{\lambda_1(1 + \lambda_2 \cdot \lambda_2) + \lambda_2(1 - \lambda_1 \cdot \lambda_1) - 2 \lambda_2(\lambda_1 \cdot \lambda_2)}{(1 - \lambda_1 \cdot \lambda_1)(1 - \lambda_2 \cdot \lambda_2) - 4(\lambda_1 \cdot \hat{\lambda}_2)} = \frac{\frac{w_1}{c} \sqrt{1 - \frac{v_{\parallel}^2}{c^2} + \frac{w_1}{c} \left[ 1 + \frac{w_1}{c} \left( 1 + \frac{w_1}{c} \right)^{-1} \right]}}{1 + \frac{w_1}{c} \cdot \frac{w_2}{c}}, \]
\[ \lambda(i2b') = \frac{i}{b'} \tanh 2b' = \frac{i}{c} \mathbf{v}' = c \left( \frac{\lambda(I) - \hat{\lambda}(I) - \hat{\lambda}(I) \times \lambda(I)}{1 + \lambda(I) \cdot \hat{\lambda}(I)} \right) = 2 \frac{\lambda_2(1 + \lambda_1 \cdot \lambda_1) + \lambda_1(1 - \lambda_2 \cdot \lambda_2) - 2 \lambda_1(\lambda_1 \cdot \hat{\lambda}_2)}{(1 - \lambda_1 \cdot \lambda_1)(1 - \lambda_2 \cdot \lambda_2) - 4(\lambda_1 \cdot \hat{\lambda}_2)} = \frac{\frac{w_2}{c} \sqrt{1 - \frac{v_{\parallel}^2}{c^2} + \frac{w_1}{c} \left[ 1 + \frac{w_1}{c} \left( 1 + \frac{w_1}{c} \right)^{-1} \right]}}{1 + \frac{w_1}{c} \cdot \frac{w_2}{c}}. \]

Here \( \mathbf{v} \) и \( \mathbf{v}' \) are boost velocities represented by the exponents \( e^{ibc} \) и \( e^{ibc} \). These two velocities are equal by magnitude
\[ \frac{v}{c} = \frac{v'}{c} = \sqrt{1 - \frac{v_{\parallel}^2}{c^2} \left( 1 - \frac{v_{\parallel}^2}{c^2} \right)^{-1}} \]
but have different directions.

Expressions (103)-(104) determine, in particular, relativistic rule of velocity addition. Really, particle, which is at rest in reference frame \( S \) has velocity \( \mathbf{v}_1 \) in reference frame \( S - \mathbf{v}_1 \), which is moving with velocity \( -\mathbf{v}_1 \) relative to the \( S \). Assume that we also have reference frame \( S - \mathbf{v}_2 \), which is moving with velocity \( -\mathbf{v}_2 \) relative to the reference frame \( S - \mathbf{v}_1 \). The same particle in this reference frame has velocity which is opposite to the velocity of the reference frame \( S - \mathbf{v}_2 \) relative to the reference frame \( S \). We represent transaction from \( S \) to \( S - \mathbf{v}_2 \) (using (97)) as rotation and successive boost. Rotation does not change velocity of the particle, which is at rest in reference frame \( S \), but it gets velocity (103) due to boost. Thus, particle which has velocity \( \mathbf{v}_1 \) in the reference frame \( S - \mathbf{v}_1 \), has velocity
\[ \frac{v}{c} = \frac{v}{c} = \sqrt{1 - \frac{v_{\parallel}^2}{c^2} + \frac{v}{c} \left[ 1 + \frac{w_1}{c} \left( 1 + \frac{w_1}{c} \right)^{-1} \right]}, \]
\[ = \frac{w_1 + w_2}{c} \times \left( \frac{w_1}{c} \right) \left( 1 + \frac{w_1}{c} \right)^{-1} \]
in the reference frame \( S - \mathbf{v}_2 \), which is moving with velocity \( -\mathbf{v}_2 \) relative to the reference frame \( S - \mathbf{v}_1 \). This is well-known relativistic velocity addition rule [4].

We can represent transformation from reference frame \( S \) to the reference frame \( S - \mathbf{v}_2 \) as boost and successive rotation. After boost particle has velocity (104), but this velocity differs from the final velocity by rotation \( e^{ibc} \).

Note also, that particle, which has velocity \( \mathbf{v}_2 \) in the reference frame \( S - \mathbf{v}_2 \), velocity (104) is velocity in the frame \( S - \mathbf{v}_1 \). Reference frame \( S - \mathbf{v}_1 \) is moving with velocity \( -\mathbf{v}_1 \) relative to the \( S - \mathbf{v}_2 \). Thus, switch of the particle velocity and the reference frame velocity is rotation of the calculated velocity represented by the exponent \( e^{ibc} \).

Let us calculate boost parameters represented in (97)-(98) by exponents \( e^{ibc} \) и \( e^{ibc} \) and rotation parameters represented by exponents \( e^{ic} \) и \( e^{ic} \). Parameters \( \lambda(i\mathbf{b}) \) and \( \lambda(i\mathbf{b}') \) of the boosts \( e^{ibc} \) и \( e^{ibc} \) differ from the parameters \( \lambda(i\mathbf{b}) \) и \( \lambda(i\mathbf{b}') \) of the boosts \( e^{ibc} \) и \( e^{ibc} \) by multiplier
\[ \frac{\lambda(i\mathbf{b})}{\lambda(i\mathbf{b}')} = \frac{1 + \sqrt{1 + \lambda(i\mathbf{b}) \cdot \lambda(i\mathbf{b})}}{1 + \sqrt{1 + \lambda(i\mathbf{b}') \cdot \lambda(i\mathbf{b}')}}. \]
\[\frac{\lambda(i\mathbf{b}')}{\lambda(i\mathbf{b})} = \frac{1 + \sqrt{1 + \lambda(i\mathbf{b}) \cdot \lambda(i\mathbf{b})}}{1 + \sqrt{1 + \lambda(i\mathbf{b}') \cdot \lambda(i\mathbf{b}')}}. \]

Thus, boost parameters are represented by exponents \( e^{ibc} \) и \( e^{ibc} \) are equal
\[ \lambda(i\mathbf{b}) = \frac{b}{b'} \tanh b = \frac{\lambda_1(1 + \lambda_2 \cdot \lambda_2) + \lambda_2[1 - \lambda_1 \cdot \lambda_1 - 2 \lambda_1 \cdot \lambda_2]}{1 + (\lambda_1 \cdot \lambda_1)(\lambda_2 \cdot \lambda_2) - 2 \lambda_1 \cdot \lambda_2}, \]
\[ \lambda(i\mathbf{b}') = \frac{b'}{b} \tanh b' = \frac{\lambda_2(1 + \lambda_1 \cdot \lambda_1) + \lambda_1[1 - \lambda_2 \cdot \lambda_2 - 2 \lambda_1 \cdot \lambda_2]}{1 + (\lambda_1 \cdot \lambda_1)(\lambda_2 \cdot \lambda_2) - 2 \lambda_1 \cdot \lambda_2}. \]

With a help of parameters \( \lambda(i\mathbf{b}) \) and \( \lambda(i\mathbf{b}') \) we can find rotation parameters represented in (97)-(98) by exponents \( e^{ic} \) и \( e^{ic} \). Let us multiply (97) by \( e^{-ibc} \) from the left, multiply (98) by \( e^{-ibc} \) from the right and use (90):
\[ \lambda(r) = \frac{r}{c} \tanh r = \frac{\lambda(\mathbf{r}) - \lambda(\mathbf{r}) \cdot \lambda(\mathbf{r})}{1 + \lambda(\mathbf{r}) \cdot \lambda(\mathbf{r})}, \]
\[ \lambda(r') = \frac{r'}{c} \tanh r' = \frac{\lambda(\mathbf{r'}) - \lambda(\mathbf{r'}) \cdot \lambda(\mathbf{r'})}{1 + \lambda(\mathbf{r'}) \cdot \lambda(\mathbf{r'})}. \]
As we can see rotation is the same in both cases.

In particular expression (109) represents Thomas rotation [3]. In this case in a reference frame \( S \) during small time interval \( \delta t \) particle changes its velocity from \( \mathbf{v} \) to \( \mathbf{v} + \delta \mathbf{v} \). This velocity change is equivalent to the
velocity change of the reference frame in the opposite direction. We can represent the velocity change to be done in two steps. At first, reference frame get the velocity \( v \). In this reference frame particle becomes unmovable in \( S \). Then reference frame get the velocity \(- (v + \delta v)\). Particle in this reference frame get velocity \( v + \delta v \). These two reference frames \( S \) boosts with velocities \( v_1 = v \) and \( v_2 = -(v + \delta v) \) can be substituted by one boost with velocity \( v' \) (104) and successive rotation \( e^{r \varsigma} \) with parameter \( \lambda(r') \) (109). As a first approximation by \( \delta v \) the parameter is

\[
\lambda(r') \approx \frac{r' \delta \varphi}{r'} = -\frac{v \times \delta v}{c^2(1 + \frac{v^2}{c^2}) - v^2} = \frac{\delta v \times v}{2v^2} \left( \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right).
\]

Consequently, reference frame connected to the particle has precession relative to the \( S \) in opposite direction with angular velocity

\[
\omega = \frac{r' \delta \varphi}{r' \delta t} = \frac{v}{v^2} \frac{\delta v \times \delta t}{\delta t} \left( \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right) = \frac{v \times a}{v^2} \left( \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right),
\]

where \( a = \frac{\delta v}{\delta t} \) — particle acceleration. This is Thomas precession.

**SPIN, PAULI–LUBANSKI PSEUDOVECTOR AND WIGNER LITTLE GROUP**

Representation of the Lorentz transformation by hypercomplex numbers clarify the spin connection with Pauli–Lubanski pseudovector and Wigner little group.

It is well known [3] that fermion, which is at rest and has spin \( \pm \hbar/2 \) projections on the \( n \), is described by bispinors \( \psi_{\pm}(n) \). These bispinors are proper \( \frac{\hbar}{2} i n \varsigma \) operator vectors:

\[
\frac{\hbar}{2} n \varsigma \psi_{\pm}(n) = \frac{\hbar}{2} \psi_{\pm}(n).
\]

Here \( n \varsigma = n_{23} \sigma^{23} + n_{31} \sigma^{31} + n_{12} \sigma^{12} \), \( n \varsigma n \varsigma = -1 \). Normalized to 1 bispinors \( \psi_{\pm}(n) \) for the fermion with positive energy can be represented as

\[
\psi_{\pm}(n) = e^{r \varsigma} \psi(e_z) = \frac{1 - (\pm n \varsigma)(e_z \varsigma)}{\sqrt{2(1 \pm n_z)}} \psi(e_z) = e^{\frac{i n \varsigma}{\hbar} \cdot n} = \frac{1 \pm i n \varsigma}{\sqrt{2(1 \pm n_z)}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.
\]

I.e. we can represent them as a result of bispinor rotation

\[
\psi(e_z) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},
\]

which describes fermion with spin projection along \( z \) axis equal \( \hbar/2 \). Bispinor \( \psi(e_z) \) is the proper vector of the operator \( \frac{\hbar}{2} i e_z \varsigma \): \( \frac{\hbar}{2} i e_z \varsigma \psi(e_z) = \frac{\hbar}{2} i \sigma^{12} \psi(e_z) = \frac{\hbar}{2} \psi(e_z) \).

Exponent

\[
e^{r \varsigma} = \cos r + \frac{r \varsigma}{r} \sin r = \sqrt{\frac{1 + e_z \cdot n}{2}} + \frac{(e_z \cdot n) \varsigma}{\sqrt{1/(e_z \cdot n)^2}} \sqrt{\frac{1 - e_z \cdot n}{2}} = \frac{1 - n \varsigma e_z \varsigma}{\sqrt{2(1 + e_z \cdot n)}}
\]

represents rotation, which align unit vector \( e_z \) along unit vector \( n \):

\[
e^{r \varsigma} e_z \gamma e^{-r \varsigma} = n \gamma.
\]

Bispinors \( \psi_{\pm}(n) \) describe fermions at rest. There is no change of their characteristics after reference frame rotation about \( n \) axis. After this rotation bispinors \( \psi_{\pm}(n) \) get unitary multipliers only, but no change of the structure:

\[
\psi_{\pm}(n) = e^{n \varsigma} \psi_{\pm}(n) = e^{\frac{i n \varsigma}{\hbar} \cdot n} \psi_{\pm}(n).
\]

There are more complex changes of the bispinors \( \psi_{\pm}(n) \) after rotation about another axis. That is, fermions at rest, which described by bispinors \( \psi_{\pm}(n) \) obey the symmetry with respect to the group transformation rotation about the \( n \) axis. The infinitesimal operators \( \frac{\hbar}{2} i n \varsigma \) of these transformations coinside (within the multiplier) with operator \( \frac{\hbar}{2} i n \varsigma \). For this operator bispinors \( \psi_{\pm}(n) \) are proper vectors. This approach shed light on the connection between proper operator \( \frac{\hbar}{2} i n \varsigma \) bispinors and those symmetry transformations \( e^{n \varsigma} \), for which operator \( \frac{\hbar}{2} i n \varsigma \) is infinitesimally small (within the multiplier).

Note also, that rotation \( e^{n \varsigma} \), as any spatial rotation does not change linear momentum of the fermion at rest.

In order to turn to the moving fermions it is enough to move from the proper reference frame into the frame \( S_{-v} \), which is moving with velocity \( -v \) relative to it. In the reference frame \( S_{-v} \) fermion has 4-impuls \( p_0 = mc/\sqrt{1 - v^2/c^2}, \ p = mv/\sqrt{1 - v^2/c^2} \). In order to come up with equation (112) in the reference frame \( S_{-v} \), apply
boost operator $e^{ibc}$ to the both parts of it:

$$
\frac{\hbar}{2} e^{ibc} n\xi e^{-ibc} e^{ibc} \psi_{n}(n) = \frac{\hbar}{2} i S\xi e^{ibc} \psi_{n}(n) = + \frac{\hbar}{2} e^{ibc} \psi_{n}(n).
$$

Here $b = -v/|v|$, tanh $2|b| = v/e$. As we can see bispinors

$$
e^{ibc} \psi_{n}(n) = \left[ \frac{p_{0} + mc}{2mc} \right]^{\frac{1}{2}} \frac{1 \mp in}{\sqrt{2}(1 \mp in)} \psi(e_{z}) (121)
$$

are the proper bispinors of the operator

$$
\frac{\hbar}{2} i S\xi = \frac{\hbar}{2} e^{ibc} n\xi e^{-ibc} = \left[ \frac{p_{0} + mc}{2mc} \right]^{\frac{1}{2}} \frac{1 \mp in}{\sqrt{2}(1 \mp in)} \psi(e_{z}) (122)
$$

Obviously, that bispinor $e^{ibc} \psi_{n}(n)$ represents bispinor $\psi_{n}(n)$ in the reference frame $S_{-\nu}$. And operator $\frac{\hbar}{2} i S\xi$ represents operator $\frac{\hbar}{2} i n\xi$ in the reference frame $S_{\nu}$. Operator $\frac{\hbar}{2} i S\xi$ can be written in routine for the moment operator form. We can represent tensor $\frac{\hbar}{2} i n\xi$ as product of spacelike pseudo-vector $n_{a} \pi^{a} = 0\pi^{a} - n\pi$ and timelike linear momentum $p_{a} \gamma^{a} = mc\gamma^{0} - 0\gamma$ in the proper fermion reference frame:

$$
\frac{\hbar}{2} n\xi = \frac{\hbar}{2} i mc p_{a} \gamma^{a} = \frac{\hbar}{2} \left( -1 \right) mc n_{a} \pi^{a} mc\gamma^{0} (123)
$$

$n_{a} \pi^{a} = -n\pi$ is Pauli–Lubanski pseudovector in a proper fermion reference frame $\Sigma$. It’s direction coincide with the rotation symmetry axis of the fermion at rest. Using (121)-(122), for the operator $\frac{\hbar}{2} i S\xi$ in the frame $S_{-\nu}$ we obtain

$$
\frac{\hbar}{2} i S\xi = \frac{\hbar}{2} i mc p_{a} \gamma^{a} = \frac{\hbar}{2} \left( -1 \right) mc n_{a} \pi^{a} mc\gamma^{0} = \frac{\hbar}{2} \left( -1 \right) mc n_{a} \pi^{a} mc\gamma^{0} (124)
$$

Here

$$
\frac{\hbar}{2} i S\xi \sigma_{\alpha\beta} = \frac{\hbar}{2} i mc (p_{0} s - s_{0} p) s r_{r} (s \times p) (125)
$$

Pauli–Lubanski pseudovector in a reference frame $S_{-\nu}$,

$$
s_{0} = \frac{n \cdot p}{mc}, \quad s = n + \frac{n \cdot p}{mc} \frac{p}{p_{0} + mc}, (126)
$$

$$
S = \frac{1}{mc} (p_{0} s - s_{0} p) + i s \times p, (127)
$$

$$
S_{\alpha\beta} = \frac{1}{mc} [s_{\alpha} p_{\beta}]^{\gamma} = \frac{1}{mc} \varepsilon_{\alpha\beta\mu\nu} (s^{\mu} p^{\nu} - s^{\nu} p^{\mu}) (128)
$$

It is easy to show, that $s_{\alpha} p^{\alpha} = s_{0} p^{0} - s \cdot p = 0$, and

$$
\frac{\hbar}{2} i S\xi \sigma_{\alpha\beta} = \frac{\hbar}{2} mc s_{\alpha} \pi^{\alpha} s_{\beta} p_{\gamma} ^{\beta} \text{ or } s^{\alpha} = \frac{\hbar}{2mc} s_{\alpha} \pi^{\alpha} S_{\mu\nu} p_{\beta} (129)
$$

Usually this expression is used for the definition of the Pauli–Lubanski pseudovector.

From the expression (127) for $S_{\alpha\beta}$ we can see that structure of the tensor of the 4-momentum in classical mechanics

$$
M_{\alpha\beta} = x_{\alpha} p_{\beta} - x_{\beta} p_{\alpha} (130)
$$

is similar to the structure of the tensor

$$
\frac{\hbar}{mc} [s_{\alpha} p_{\beta}] = \frac{\hbar}{mc} s_{\alpha} p_{\beta} - \frac{\hbar}{mc} s_{\beta} p_{\alpha}, (131)
$$

dual to the $hS_{\alpha\beta}$. In this sense pseudo-vector $s_{\alpha} h/mc$ is the inner (proper) space-like pseudo-vector, which has Compton length $h/mc$ and direction along rotational symmetry axis of the fermion at rest.

Tensor of 4D momentum $M_{\alpha\beta}$ is similar to the $hS_{\alpha\beta}$ tensor, but to $h \Sigma_{\alpha\beta}$, which is dual to it. The reason is that, $x_{\alpha}$ is 4D-vector, and $s_{\alpha}$ is 4D pseudovector.

The tensors $hS_{\alpha\beta}$ and $M_{\alpha\beta}$ duality is reflected in similarity of the structure of the 3D pseudo-vector part of the tensor $hS_{\alpha\beta}$

$$
\frac{\hbar}{mc} (p_{0} s - s_{0} p) s r_{r} (s \times p) (132)
$$

to the structure of the 3D vector part of the tensor $M_{\alpha\beta}$

$$
p x_{\alpha} - x_{\alpha} p_{\beta} (133)
$$

Whereas they have similar structures the (131) part contributes to the angular momentum conservation law while the part (132) contributes to the law of the center mass motion. And vise versa, whereas the structure of the 3D vector part of the tensor $hS_{\alpha\beta}$

$$
\frac{\hbar}{mc} p s r_{r} (s \times p) (134)
$$

is similar to the structure of the 3D pseudo-vector part of the tensor $M_{\alpha\beta}$

$$
x \times p, (135)
$$
the (133) part contributes to the law of the center mass motion while the part (134) contributes to the angular momentum conservation law. Again, it is because $\mathbf{x}$ is a vector and $\mathbf{s}$ - pseudo-vector.

Note also, that pseudo-vector $s^\alpha$ is equal to the bilinear combination $i\bar{\psi}\pi^\alpha\psi$ of the bispinors (113)

$$s^\alpha = i\bar{\psi}\pi^\alpha\psi,$$  \hspace{1cm} (136)

Also tensor $S^{\alpha\beta}$ is bilinear combination $i\bar{\psi}\sigma^{\alpha\beta}\psi$:

$$S^{\alpha\beta} = i\bar{\psi}\sigma^{\alpha\beta}\psi.$$  \hspace{1cm} (137)

Operator $\hat{s}S\mathbf{\epsilon}$, similar to the $\hat{n}\mathbf{\epsilon}$ (in proper fermion reference frame), coincide within the multiplier with infinitesimal operator of Lorentz transformation which results in the multiplication of bispinor $e^{ibc}\psi_\pm$ by $e^{\hat{s}S\mathbf{\epsilon}}$. In order to prove it let us write (118) in the reference frame $S_{-\nu}$:

$$e^{ibc}\hat{\bar{\psi}}\mathbf{n}\mathbf{\epsilon}e^{-ibc}\psi_\pm = e^{\hat{s}S\mathbf{\epsilon}}e^{ibc}\psi_\pm.$$  \hspace{1cm} (138)

The product $e^{ibc}\hat{\bar{\psi}}\mathbf{n}\mathbf{\epsilon}e^{-ibc}$ can be written as

$$e^{ibc}\hat{\bar{\psi}}\mathbf{n}\mathbf{\epsilon}e^{-ibc} = e^{ibc}(\cos\frac{x}{2} + n\mathbf{\epsilon}\sin\frac{\varphi}{2})e^{-ibc} =$$

$$= \cos\frac{\varphi}{2} + \frac{ibc n\mathbf{\epsilon}}{S\mathbf{\epsilon}}\sin\frac{\varphi}{2} = \cos\frac{\varphi}{2} + S\mathbf{\epsilon}\sin\frac{\varphi}{2} = \hat{s}S\mathbf{\epsilon}.$$  \hspace{1cm} (139)

As we can see, exponent $\hat{s}S\mathbf{\epsilon}$ in (137) results bispinor transformation $e^{ibc}\psi_\pm$, that caused multiplication by unitary multiplier $e^{\hat{s}S\mathbf{\epsilon}}$ only. Infinitesimal operator $\hat{s}S\mathbf{\epsilon}$ of this transformation within the multiplier coincide with operator $\hat{\bar{\psi}}\mathbf{n}\mathbf{\epsilon}$. For this operator bispinor $e^{ibc}\psi_\pm$ is a proper bispinor. $S$ is a $i$-complex number. That is why transformation (137) with exponent $\hat{s}S\mathbf{\epsilon}$ is neither rotation nor boost.

There is a property of the transformation with exponent $\hat{s}S\mathbf{\epsilon}$ is multiplication of bispinor $e^{ibc}\psi_\pm$ by unitary multiplier. Also it does not change linear momentum of the fermion represented by this bispinor:

$$e^{\hat{s}S\mathbf{\epsilon}}p_\alpha\gamma^\alpha e^{-\hat{s}S\mathbf{\epsilon}} = p_\alpha\gamma^\alpha.$$  \hspace{1cm} (140)

It is obvious, because (139) represents in the reference frame $s_{-\nu}$ spatial rotation $e^{\hat{s}n\mathbf{\epsilon}}p_0\gamma^0 e^{-\hat{s}n\mathbf{\epsilon}}$ 4D linear momentum of the fermion in its own proper reference frame. This rotation does not change 4D linear momentum of fermion at rest:

$$e^{\hat{s}n\mathbf{\epsilon}}p_0\gamma^0 e^{-\hat{s}n\mathbf{\epsilon}} = p_0\gamma^0.$$  \hspace{1cm} (141)

Transformation to the reference frame $S_{-\nu}$

$$e^{ibc}\hat{\bar{\psi}}\mathbf{n}\mathbf{\epsilon}e^{-ibc}e^{ibc}\bar{\psi}\gamma^0 e^{-ibc}e^{ibc}\hat{\bar{\psi}}\mathbf{n}\mathbf{\epsilon}e^{-ibc} =$$

$$= e^{ibc}\bar{\psi}\gamma^0 e^{-ibc}.$$  \hspace{1cm} (142)

leads to (139). Here $p_\alpha\gamma^\alpha = e^{ibc}p_0\gamma^0 e^{-ibc} = 4D$ linear momentum $p_\alpha\gamma^\alpha = (mc^2 - mv\gamma)(\sqrt{1 - v^2})$ in the reference frame $S_{-\nu}$.

Obviously that not only spatial rotation $e^{\hat{s}n\mathbf{\epsilon}}$ about the $\mathbf{n}$ axis on angle $\varphi$ does not change 4D linear momentum of fermion at rest but also rotation $e^{\hat{s}s}$ about any axis $\mathbf{\epsilon}$ on any angle $\varphi$ does not change it. Correspondingly, not only transformation (139) does not change linear momentum of the moving fermion but any transformation $e^{S(x)\gamma^\alpha}e^{-S(x)\gamma^\alpha}$, which contains arbitrary 3D pseudo-vector $\mathbf{x}$ instead of Pauli-Lubanski pseudo-vector $\mathbf{n}$ (see expression (121) for $S(x)\gamma^\alpha$):

$$S(x)\gamma^\alpha = e^{ibc}2x\mathbf{n}\mathbf{\epsilon}e^{-ibc} =$$

$$= \frac{1}{mc}(p_0x - \frac{x \cdot p}{p_0 + mc}p + i\mathbf{x} \times \mathbf{p})\gamma^\alpha =$$

$$= \frac{1}{mc}(p_0y - y_0p) + i\mathbf{y} \times \mathbf{p})\gamma^\alpha.$$  \hspace{1cm} (143)

where

$$y_0 = \frac{x \cdot p}{mc}, \quad y = x + \frac{x \cdot p}{mc}p.$$  \hspace{1cm} (144)

We can use simple calculations in order to prove equation (139) is truly (genuinely) when use operator $e^{\pm S(x)\gamma^\alpha}$:

$$e^{S(x)\gamma^\alpha}p_\alpha\gamma^\alpha e^{-S(x)\gamma^\alpha} =$$

$$\frac{(\cos x + S\mathbf{\epsilon}\sin x)}{x}p_\alpha\gamma^\alpha (\cos x - S\mathbf{\epsilon}\sin x) = p_\alpha\gamma^\alpha.$$  \hspace{1cm} (145)

Because spatial rotations form three-parameter subgroup of the Lorentz group also transformations represented by exponents $e^{S(x)\gamma^\alpha}$ form three-parameter subgroup of the Lorentz group with parameters $\mathbf{x}$. It is little Wigner’s group $S_3$, which belongs to $p_\alpha$ linear momentum. Any transformation of this group does not change linear momentum $p_\alpha$. Those transformations, for which $\mathbf{x} = \mathbf{n} \frac{\hat{s}s}{\epsilon}$, not only does not change linear momentum but change bispinor in a special way. They multiply them by unitary $e^{\hat{s}S\mathbf{\epsilon}}$ only.

**SUMMARY**

Lorentz transformations are important for all branches of physics because they reflect the structure of spacetime. Thus, the way of easy implementation of these transformations is also important. Lorentz’s transformations are 4D rotations in spacetime. That is why this easy implementation must be similar to the rotations in in 3D and 2D spaces.

The way to get this easy implementation is to represent rotation operator as exponent with imagine power: a) for 3D case as exponent with vector-quaternion power b) for 4D spacetime as exponent with hypercomplex power,
which is the antisymmetric tensor of the second order in a hypercomplex representation.
Hypercomplex numbers are not commutative. So, even finding the exponent power, which is the product of the two exponents becomes a complex problem. In this paper the rule of finding such power for the Dirac numbers in an explicit form is presented (expressions (80), (85), (90)). This rule simplifies a lot consideration of the problems, which needs combination of several Lorentz’s transformations. Effectiveness of this rule demonstrated for several problems, connected with the Lorentz’s transformations.

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