REDUCING SUBSPACES ON THE ANNULUS

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Abstract. We study reducing subspaces for an analytic multiplication operator \( M_z \) on the Bergman space \( L^2_a(A_r) \) of the annulus \( A_r \), and we prove that \( M_z^n \) has exactly \( 2^n \) reducing subspaces. Furthermore, in contrast to what happens for the disk, the same is true for the Hardy space on the annulus. Finally, we extend the results to certain bilateral weighted shifts, and interpret the results in the context of complex geometry.

1. Introduction

Important themes in operator theory are determining invariant subspaces and reducing subspaces for concretely defined operators. Our goal in this note is to determine the reducing subspaces for a power of certain multiplication operators on natural Hilbert spaces of holomorphic functions on an annulus.

We begin with the Bergman space and Hardy space. Next, we consider a generalization to certain bilateral weighted shifts. Finally, we interpret our results in the context of complex geometry describing another approach to these questions.

The motivation for these questions arises from some earlier results of K. Zhu ([12]), M. Stessin and K. Zhu ([10]), and other researchers ([1], [8]). In these studies, the annulus is replaced by the open unit disk, and one considers \( M_z^n \) on the Hardy space \( H^2 \) or the Bergman space \( L^2_a \). In particular, the lattice of reducing subspaces of the \( n \)th power of the multiplication operator, \( M_z^n \) on \( L^2_a \), was shown to be discrete and have precisely \( 2^n \) elements. This contrasted with the case of the classic Toeplitz operator \( T_z^n \) on the Hardy space \( H^2 \) for which this lattice is infinite and isomorphic to the lattice of all subspaces of \( \mathbb{C}^n \). Thus, as is true for many other questions, the situations on the unit disk and annulus are different.

For \( 0 < r < 1 \), let \( A_r \) denote the annulus \( \{ z \in \mathbb{C} : r < |z| < 1 \} \) in the complex plane \( \mathbb{C} \). Let \( L^2(A_r) \) denote the usual \( L^2 \)-space for planar Lebesgue measure on \( A_r \) and \( L^2_a(A_r) \) be the closure of \( R(A_r) \) in \( L^2(A_r) \), where \( R(A_r) \) is the space of all rational functions with poles outside the closure of \( A_r \).

We let \( P_{L^2(A_r)} \) be the orthogonal projection of \( L^2(A_r) \) onto the Bergman space \( L^2_a(A_r) \).

For \( \varphi \) in \( H^\infty(A_r) \), the space of bounded holomorphic functions on \( A_r \), define the operator \( M_\varphi \) on \( L^2_a(A_r) \) so that

\[
M_\varphi(f) = \varphi f
\]

for \( f \) in \( L^2_a(A_r) \). We are concerned with determining the reducing subspaces of \( M_{z^n} = M_z^n \) for \( n \geq 2 \).

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2. Reducing Subspaces for $M_{z^n}(n \geq 2)$

We let $S_k$ denote the subspaces of $L^2_n(A_r)$ generated by $\{z^m \in L^2_n(A_r) : m = k(\mod n)\}$ for $0 \leq k < n$. To study reducing subspaces for the multiplication operator $M_{z^n}$ on $L^2_n(A_r)$, we will use these $n$ reducing subspaces $S_k(0 \leq k < n)$ for $M_{z^n}$. Note that

$$L^2_n(A_r) = S_0 \oplus S_1 \oplus \cdots \oplus S_{n-1},$$

and so for any $f \in L^2_n(A_r)$, we have a unique orthogonal decomposition

$$f = f_0 + f_1 + \cdots + f_{n-1},$$

where $f_k \in S_k(0 \leq k < n)$.

In this section, we will need the following well known fact [4]. For completeness, we provide a proof.

**Lemma 1.** If $M_F : S_k \to L^2_n(A_r)$ is a (bounded) multiplication operator by a function $F$ on $A_r$, then $F \in H^\infty(A_r)$ and $\|F\|_\infty \leq \|M_F\|$.

**Proof.** First, since $F$ is the quotient of two analytic functions $(F = (M_F z^k)/z^k)$, it is meromorphic on $A_r$. For a fixed $z \in A_r$, let $\lambda_z$ denote the point-evaluation functional on $L^2_n(A_r)$ defined by

$$\lambda_z(f) = f(z)$$

for $f \in L^2_n(A_r)$. Clearly, $\lambda_z$ is bounded, and for $f_k \in S_k$,

$$|F(z)\lambda_z(f_k)| = |F(z)f_k(z)| = |\lambda_z(M_F(f_k))| \leq \|\lambda_z\| \cdot \|M_F\| \cdot \|f_k\|.$$ 

It follows that $|F(z)| \cdot \|\lambda_z\| \leq \|\lambda_z\| \cdot \|M_F\|$ for any $z \in A_r$. Therefore,

$$|F(z)| \leq \|M_F\|$$

for any $z \in A_r$, and $F$ is analytic.

Another familiar result classifies bilateral shifts up to unitarily equivalence.

**Proposition 2.** [9] If $S$, $T$ are two bilateral weighted shifts with weight sequences $\{v_m\}$, $\{w_m\}$, and if there exists an integer $k$ such that

$$|v_m| = |w_{m+k}| \text{ for all } m,$$

then $S$ and $T$ are unitarily equivalent. Moreover, the converse is true.

**Lemma 3.** For $i$, $j$ such that $0 \leq i \neq j < n$, if $M_i = M_{z^n}|S_i$ and $M_j = M_{z^n}|S_j$, then $M_i$ and $M_j$ are not unitarily equivalent.

**Proof.** Let $e_k^i = \frac{z^{kn+i}}{\|z^{kn+i}\|}$ where $k$ is an integer. Then, $\{e_k^i : k \in \mathbb{Z}\}$ is an orthonormal basis for $S_i$.

First, we calculate the weights of the operator $M_i$. Since

$$M_i(e_k^i) = \frac{z^{(k+1)n+i}}{\|z^{kn+i}\|} = \frac{\|z^{(k+1)n+i}\|}{\|z^{kn+i}\|} e_{k+1}^i,$$

the weights of $M_i$ are

$$\lambda_k = \frac{\|z^{(k+1)n+i}\|}{\|z^{kn+i}\|}$$

for $k \in \mathbb{Z}$.
Similarly, the weights of \( M_j \) are
\[
\mu_k = \frac{\|z^{(k+1)n+j}\|}{\|z^{kn+j}\|}
\]
for \( k \in \mathbb{Z} \).

Since \( \|z^n\|^2 = \frac{1}{n+1} - \frac{2^{(n+1)}n+1}{n+1} \), by Proposition 2 we conclude that \( M_i \) and \( M_j \) are not unitarily equivalent.

Recall that determining the reducing subspaces of \( M_{z^n} \) is equivalent to finding the projections in the commutant of \( M_{z^n} \) (5). Thus, in the following Proposition, we characterize every bounded linear operator \( T \) on \( L^2_\alpha(A_r) \) commuting with \( M_{z^n} \).

**Proposition 4.** A bounded linear operator \( T \) on \( L^2_\alpha(A_r) \) commutes with \( M_{z^n} \) if and only if there are functions \( F_i(0 \leq i < n) \) in \( H^\infty(A_r) \) such that
\[
T f = \sum_{i=0}^{n-1} F_i f_i,
\]
where \( f_i(0 \leq i < n) \) denotes the functions in equation (7).

**Proof.** \((\Leftarrow)\) Let \( M_{F_i} : L^2_\alpha(A_r) \to L^2_\alpha(A_r) \) be the multiplication operator defined by \( M_{F_i}(g) = F_i g \) for \( g \in L^2_\alpha(A_r) \). Then,
\[
\sup_{\|f\|=1} \|Tf\| \leq \sum_{i=0}^{n-1} \|F_i f_i\| \leq (\sum_{i=0}^{n-1} \|M_{F_i}\|)(\sup_{i=0,\ldots,n-1} \|f_i\|) \leq (\sum_{i=0}^{n-1} \|M_{F_i}\|) \|f\|.
\]
It follows that \( \|T\| < \infty \). Clearly, \( TM_{z^n} = M_{z^n} T \).

\((\Rightarrow)\) Assume that \( T \) is a (bounded) operator on \( L^2_\alpha(A_r) \) such that \( TM_{z^n} = M_{z^n} T \). Then, \( T^* \) commutes with \( M_{\lambda^{z^n} - z^n} \) for any \( \lambda \in A_r \). Clearly, for \( \lambda \in A_r \), \( \ker M_{\lambda^{z^n} - z^n} \) is generated by
\[
\{k_{\lambda \omega_k} : \omega_k = \exp(2\pi ik/n)(0 \leq k < n)\},
\]
where \( k_{\lambda \omega_k} \) is the Bergman kernel function at \( \lambda \omega_k \).

Since \( T^* k_{\lambda} \in \ker M_{\lambda^{z^n} - z^n} \), we have
\[
T^* k_{\lambda} = \sum_{k=0}^{n-1} \overline{a_k(\lambda)} k_{\lambda \omega_k},
\]
for uniquely determined complex numbers \( \{a_k(\lambda)\}_{k=0}^{n-1} \).

If \( f \in L^2_\alpha(A) \) and \( z \in A_r \), then, by equation (5),
\[
T f(z) = (T f, k_z) = (f, T^* k_z) = \sum_{k=0}^{n-1} a_k(z) f(z \omega_k).
\]
Since \( \omega_k^n = 1 \) for any \( 0 \leq k < n \),
\[
f_i(z \omega_k) = \omega_k^i f_i(z) \quad (0 \leq i, k < n),
\]
where \( f_i \) (\( 0 \leq i < n \)) is the function defined in equation (1).

Since \( \sum_{k=0}^{n-1} a_k(z) f(z \omega_k) = \sum_{k=0}^{n-1} a_k(z) f_0(z \omega_k) + \sum_{k=0}^{n-1} a_k(z) f_1(z \omega_k) + \cdots + \sum_{k=0}^{n-1} a_k(z) f_{n-1}(z \omega_k) \), (6) and (7) imply that
\[
T f(z) = \sum_{k=0}^{n-1} a_k(z) f_0(z) + \sum_{k=0}^{n-1} a_k(z) \omega_k f_1(z) + \cdots + \sum_{k=0}^{n-1} a_k(z) \omega_{k-1}^{n-1} f_{n-1}(z).
\]
For $0 \leq k < n$, a function $F_k$ on $A_r$ is defined by
\begin{equation}
F_k(z) = \sum_{i=0}^{n-1} a_i(z)\omega_i^k.
\end{equation}

Then, equation (8) implies that
\begin{equation}
Tf = \sum_{i=0}^{n-1} F_i f_i.
\end{equation}

To finish this proof, we have to show that $F_i(0 \leq i < n)$ is in $H^\infty(A_r)$. Since $F_k(z) = \frac{T(z^k)}{z^k}$ for $0 \leq k < n$, $F_k$ is analytic on $A_r$.

By Lemma 1, $\|F_k\|_\infty \leq \|M_{F_k}\|$; that is, $\|F_k\|_\infty < \infty$ for any $0 \leq k < n$.

An analogous result is known for Toeplitz operators on the open unit disk [9].

To prove this, we have to show that $F_i(0 \leq i < n)$ is in $H^\infty(A_r)$. Since $F_k(z) = \frac{T(z^k)}{z^k}$ for $0 \leq k < n$, $F_k$ is analytic on $A_r$.

By Lemma 1, $\|F_k\|_\infty \leq \|M_{F_k}\|$; that is, $\|F_k\|_\infty < \infty$ for any $0 \leq k < n$.

Proposition 5. For $0 \leq k < n$, let $M_k = M^\infty|S_k$.

If $B = (B_{ij})_{(n \times n)}$ is a projection such that
\begin{equation}
\begin{pmatrix}
M_0 & 0 & \cdots & 0 & 0 \\
0 & M_1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & M_{n-1}
\end{pmatrix}
B
= B
\begin{pmatrix}
M_0 & 0 & \cdots & 0 & 0 \\
0 & M_1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & M_{n-1}
\end{pmatrix},
\end{equation}

then there are holomorphic functions $\varphi_{ij}(0 \leq i, j < n)$ in $H^\infty(A_r)$ such that

\begin{equation}
B_{ij} = M_{\varphi_{ij}}.
\end{equation}

Moreover, $\varphi_{ii}$ is a real-valued constant function on $A_r$ for $0 \leq i < n$, and $\varphi_{ij} \equiv 0$ for $i \neq j$.

Proof. Since the operator $B$ commutes with $M_z$, by Proposition 4 and equation (9), we have

\begin{equation}
Bf = \sum_{i=0}^{n-1} \varphi_i f_i,
\end{equation}

where $\varphi_i(z) = \sum_{k=0}^{n-1} a_k(z)\omega_k^i$ and hence $\varphi_i \in H^\infty(A_r)$.

Let $a_k(z) = \sum_{i=0}^{n-1} a_{ki}(z)$ where $a_{ki} \in S_i(i = 0, 1, \ldots, n-1)$. Then,

\begin{equation}
B = \begin{pmatrix}
\sum_{k=0}^{n-1} a_{k0} & \sum_{k=0}^{n-1} a_{k1} & \cdots & \sum_{k=0}^{n-1} a_{kn-1} \\
\sum_{k=0}^{n-1} a_{k1} & \sum_{k=0}^{n-1} a_{k2} & \cdots & \sum_{k=0}^{n-1} a_{kn-1} \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{k=0}^{n-1} a_{k(n-1)} & \sum_{k=0}^{n-1} a_{k(n-2)} & \cdots & \sum_{k=0}^{n-1} a_{kn-1}
\end{pmatrix}.
\end{equation}

It follows that

\begin{equation}
B = (M_{\varphi_{ij}})_{i,j=0}^{n-1},
\end{equation}

where $M_{\varphi_{ij}} : S_j \rightarrow S_i$ is the multiplication operator defined by

\begin{equation}
M_{\varphi_{ij}}(f) = \varphi_{ij} f_j
\end{equation}

for $f_j \in S_j$. By Lemma 1, $\varphi_{ij}(0 \leq i, j < n)$ is in $H^\infty(A_r)$. 

Since $M_{\varphi_{ij}} : S_l \to S_l$ and $B$ is a projection, $M_{\varphi_{ij}}^* = M_{\varphi_{ij}}$. Thus, $\varphi_{ii}$ is a real-valued holomorphic function and hence $\varphi_{ii}$ is a constant function.

We now prove that $\varphi_{ij} = 0$ if $i \neq j$. Suppose that there are $l$ and $k$ in $\{0, 1, \ldots, n - 1\}$ such that $l \neq k$ and $\varphi_{lk} \neq 0$. By equation (10),

$$M_l M_{\varphi_{lk}} M_k = M_{\varphi_{lk}} M_k M_{\varphi_{kl}} M_l,$$

Thus, equation (11) implies that

$$M_l M_{\varphi_{lk}} = M_{\varphi_{lk}} M_k = M_{\varphi_{lk}} M_l,$$

Since $M_{\varphi_{kl}} = M_{\varphi_{kl}}^*$, $M_{\varphi_{lk}} M_{\varphi_{kl}} : S_l \to S_l$ is a self-adjoint operator commuting with $M_l$. Then, in the same way as for $\varphi_{ii}$, we conclude that $M_{\varphi_{lk}} M_{\varphi_{kl}} = M_{\varphi_{lk}} M_{\varphi_{kl}}$ is a constant multiple of the identity operator; that is,

$$M_{\varphi_{lk}} = c_l I_{S_l} \quad \text{for } 0 \leq l < n,$$

where $I_{S_l}$ is the identity operator on $S_l$. Note that $c_l > 0$, since $M_{\varphi_{lk}} M_{\varphi_{kl}}$ is positive and $\varphi_{lk} \neq 0$.

Equations (11) and (13) imply that $M_k$ and $M_l$ are unitarily equivalent which is a contradiction by Lemma 3.

Finally, it is time to determine the reducing subspaces of the multiplication operators $M_{z^n}(n \geq 2)$.

**Theorem 6.** For a given $n \geq 2$, the multiplication operator $M_{z^n} : L^2(A_r) \to L^2(A_r)$ has $2^n$ reducing subspaces with minimal reducing subspaces $S_0, \ldots, S_{n-1}$.

**Proof.** Since $M_{z^n}$ and $M_{z^n} S_0 \oplus M_{z^n} S_1 \oplus \cdots \oplus M_{z^n} S_{n-1}$ are unitarily equivalent, it is enough to consider the reducing subspaces of $M_{z^n} S_0 \oplus M_{z^n} S_1 \oplus \cdots \oplus M_{z^n} S_{n-1}$.

By Proposition 5 if $B = (B_{ij})_{(n \times n)}$ is a projection satisfying equation (10), then

$$B = \begin{pmatrix} c_0 & 0 & \cdots & 0 \\ 0 & c_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_{n-1} \end{pmatrix},$$

where $c_i (0 \leq i < n)$ are real numbers. Since $B^2 = B$, it follows that $c_i = 0, 1$ for $0 \leq i < n$.

Therefore, the reducing subspaces of $M_{z^n} S_0 \oplus M_{z^n} S_1 \oplus \cdots \oplus M_{z^n} S_{n-1}$ are

$$c_0 S_0 \oplus c_1 S_1 \oplus \cdots \oplus c_{n-1} S_{n-1}, \quad \text{with } c_i = 0, 1.$$ 

Thus, this theorem is proven.

□

3. REDUCING SUBSPACES FOR $T_{z^n}$

J.A. Ball ([1]) and E. Nordgren ([8]) studied the problem of determining reducing subspaces for an analytic Toeplitz operator on the Hardy space $H^2(\mathbb{D})$ of the open unit disk.

In this section, for $n \geq 2$, we determine the reducing subspaces for the analytic Toeplitz operator $T_{z^n}$ on the Hardy space $H^2(A_r)$ of the annulus $A_r$. Note that, for $T_{z^n}$ on $H^2(\mathbb{D})$, the problem has an easy but sufficient answer, since $T_{z^n}$ and $z \otimes I_{\mathbb{C}^n}$ are unitarily equivalent.
Recall that the Hardy space $H^2(A_r)$ is the closure of $R(A_r)$ in $L^2(m)$, where $m$ is linear Lebesgue measure on $\partial A_r$.

Let $S_k$ denote the subspaces of $H^2(A_r)$ generated by $\{z^m \in H^2(A_r) : m = k(\text{mod } n)\}$ for $0 \leq k < n$. In the same way as in Section 2, we will use these $n$ reducing subspaces $S_k$ for the Toeplitz operator $T_{z^n} : H^2(A_r) \to H^2(A_r)$ defined by
\[
T_{z^n}(f) = z^n f,
\]
for $n \geq 2$. Note that
\[
H^2(A_r) = S_0 \oplus S_1 \oplus \cdots \oplus S_{n-1},
\]
and so for any $f \in H^2(A_r)$, we have a unique orthogonal decomposition
\[
f = f_0 + f_1 + \cdots + f_{n-1},
\]
where $f_k \in S_k (0 \leq k < n)$.

**Proposition 7.** For $i, j$ such that $0 \leq i \neq j < n$, if $T_i = T_{z^n} | S_i$ and $T_j = T_{z^n} | S_j$, then $T_i$ and $T_j$ are not unitarily equivalent.

**Proof.** Note that
\[
\|z^n\|^2_{H^2(A_r)} = \frac{1}{2\pi} \int_0^{2\pi} |e^{i\theta}|^2 d\theta + \frac{1}{2\pi} \int_0^{2\pi} r^{2n}|e^{i\theta}|^2 d\theta = 1 + r^{2n}.
\]

Then, in the same way as in Lemma 3, the result is proven.

Determining the reducing subspaces of $T_{z^n}$ is equivalent to finding projections in the commutant of $T_{z^n}$. Since $T_{z^n}$ and $T_{z^n} | S_0 \oplus T_{z^n} | S_1 \oplus \cdots \oplus T_{z^n} | S_{n-1}$ are unitarily equivalent, we consider the commutant of $T_{z^n} | S_0 \oplus T_{z^n} | S_1 \oplus \cdots \oplus T_{z^n} | S_{n-1}$ in the following Proposition.

Since we also have a kernel function in this case, a description similar to that of the commutant of $M_{z^n}$ on the Bergman space $L^2_\alpha(A_r)$ is obtained;

**Proposition 8.** A bounded linear operator $T$ on $H^2(A_r)$ commutes with $T_{z^n}$ if and only if there are functions $G_i (0 \leq i < n)$ in $H^\infty(A_r)$ such that
\[
Tf = \sum_{i=0}^{n-1} G_if_i,
\]
where $f_i$ denotes the functions in equation (15).

By Proposition 7, $T_{z^n} | S_i$ and $T_{z^n} | S_j$ are not unitarily equivalent where $0 \leq i \neq j < n$. Thus, in the same way as in Proposition 5, we characterize a projection which is in the commutant of $T_{z^n} | S_0 \oplus T_{z^n} | S_1 \oplus \cdots \oplus T_{z^n} | S_{n-1}$;

**Proposition 9.** For $0 \leq k < n$, let $T_k = T_{z^n} | S_k$.

If $F = (F_{ij})_{(n \times n)}$ is a projection such that
\[
\begin{pmatrix}
T_0 & 0 & \cdots & 0 & 0 \\
0 & T_1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & T_{n-1}
\end{pmatrix}
F = F
\begin{pmatrix}
T_0 & 0 & \cdots & 0 & 0 \\
0 & T_1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & T_{n-1}
\end{pmatrix},
\]
then there are holomorphic functions $\varphi_{ij} (0 \leq i, j < n)$ in $H^\infty(A_r)$ such that
\[
F_{ij} = T_{\varphi_{ij}},
\]
Moreover, $\varphi_{ii}$ is a real-valued constant function on $A_r$ for $0 \leq i < n$, and $\varphi_{ij} \equiv 0$ if $i \neq j$.

Since $T_\infty$ and $T_\infty|S_0 \oplus T_\infty|S_1 \oplus \cdots \oplus T_\infty|S_{n-1}$ are unitarily equivalent, we have the following result:

**Theorem 10.** For a given $n \geq 2$, the Toeplitz operator $T_\infty : H^2(A_r) \to H^2(A_r)$ has $2^n$ reducing subspaces with minimal reducing subspaces $S_0, \ldots, S_{n-1}$.

4. Reducing Subspaces for Bilateral Weighted Shifts

Note that the multiplication operator $M_z$ on the Bergman space $L^2_\alpha(A_r)$ and the Toeplitz operator $T_z$ on the Hardy space $H^2(A_r)$ are both bilateral weighted shifts. Moreover, in Section 2 and Section 3 we showed that the lattice of reducing subspaces for the operators $(M_z)^n (= M_{z^n})$ on the Bergman space $L^2_\alpha(A_r)$ and $(T_z)^n (= T_{z^n})$ on the Hardy space $H^2(A_r)$, both have $2^n$ elements for $n \geq 2$. Thus, it is natural to ask the following question.

**Question:** Let $H$ be a separable Hilbert space, and $S : H \to H$ be a bilateral weighted shift. Then, for a given $n \geq 2$, does the operator $S^n$ have a discrete lattice of $2^n$ reducing subspaces?

In [10], Stessin and Zhu answered this question for powers of unilateral weighted shifts generalizing the earlier results for $T_z$ on $H^2$ and $M_z$ on $L^2_\alpha$. That some condition is necessary is shown by considering the weighted shift with weights $(\cdot \cdot \cdot, \frac{1}{2}, \frac{1}{2}, 2, \cdot \cdot \cdot)$. In this case $T^2 = I$ and hence the lattice of reducing subspaces consists of all subspaces.

In this section, we generalize their results finding hypotheses to answer this question in the affirmative for certain bilateral weighted shifts with spectrum $A_r$.

Let $\{\beta(m)\}$ be a two-sided sequence of positive numbers such that

$$\sup_m \lambda_m = \sup_m \beta(m+1)/\beta(m) < \infty.$$  

We consider the space of two-sided sequences $f = \{\hat{f}(m)\}$ such that

$$\|f\|^2 = \|f\|_{\beta}^2 = \sum |\hat{f}(m)|^2(\beta(m))^2 < \infty.$$  

We shall use the notation

$$f(z) = \sum \hat{f}(m)z^m,$$

whether or not the series converges for any (complex) value of $z$. We shall denote this space as $L^2(\beta)$ for the Laurent series case.

Recall that these spaces are Hilbert spaces with the inner product

$$\langle f, g \rangle = \sum \hat{f}(m)\overline{g(m)}(\beta(m))^2.$$  

Let $M_z : L^2(\beta) \to L^2(\beta)$ be the linear transformation defined by

$$(M_z f)(z) = \sum \hat{f}(m)z^{m+1}.$$  

By [17], $M_z$ is bounded [21]. (Note that $\{\lambda_m\}$ are the weights.) If $g_k(z) = z^k$, then $\{g_k\}$ is an orthogonal basis for $L^2(\beta)$.

We let $S_k$ denote the subspace of $L^2(\beta)$ generated by

$$\{g_m \in L^2(\beta) : m = k(\text{mod } n)\}.$$
for $0 \leq k < n$. To study the reducing subspaces for the operator $M_z^n : L^2(\beta) \to L^2(\beta)$ defined by
\begin{equation}
(M_z^n f)(z) = \sum \hat{f}(m) z^{m+n} (f \in L^2(\beta)),
\end{equation}
we will use the $n$ reducing subspaces $S_k (0 \leq k < n)$ for $M_z^n$. Note that
\[ L^2(\beta) = S_0 \oplus S_1 \oplus \cdots \oplus S_{n-1}, \]
and so for any $f \in L^2(\beta)$, we have a unique orthogonal decomposition
\begin{equation}
f = f_0 + f_1 + \cdots + f_{n-1},
\end{equation}
where $f_k \in S_k (0 \leq k < n)$.

Consider the multiplication of formal Laurent series, $fg = h$:
\begin{equation}
(\sum \hat{f}(m) z^m)(\sum \hat{g}(m) z^m) = \sum \hat{h}(m) z^m,
\end{equation}
where, for all $m$,
\begin{equation}
\hat{h}(m) = \sum_k \hat{f}(k) \hat{g}(m-k).
\end{equation}

In general, we will assume that the product [22] is defined only if all the series [23] are absolutely convergent. $L^\infty(\beta)$ denotes the set of formal Laurent series $\phi(z) = \sum \hat{\phi}(m) z^m (-\infty < m < \infty)$ such that $\phi L^2(\beta) \subset L^2(\beta)$.

If $\phi \in L^\infty(\beta)$, then the linear transformation of multiplication by $\phi$ on $L^2(\beta)$ will be denoted by $M_\phi$.

**Proposition 11.** If $A$ is a bounded operator on $L^2(\beta)$ that commutes with $M_z$, then $A = M_\phi$ for some $\phi \in L^\infty(\beta)$.

**Proposition 12.** For $\phi \in L^\infty(\beta)$, $M_\phi$ is a bounded linear transformation, and the matrix $(a_{mk})$ of $M_\phi$, with respect to the orthogonal basis $\{g_k\}$, is given by
\begin{equation}
a_{mk} = \hat{\phi}(m-k).
\end{equation}

**Proposition 13.** The operator $M_z$ on $L^2(\beta)$ is unitarily equivalent to the operator $\tilde{M}_z$ on $L^2(\tilde{\beta})$ if and only if there is an integer $k$ such that
\[ \beta(n+k+1) = \lambda_{n+k} = \tilde{\beta}(n+1) = \frac{\beta(n+1)}{\beta(n)} \]
for all $n$. Equivalently, $L^2(\tilde{\beta}) = z^k L^2(\beta)$, and
\begin{equation}
||f||_1 = \left\| z^k f \right\| (f \in L^2(\tilde{\beta})),
\end{equation}
where $||f||_1$ denotes the norm of $f$ in $L^2(\tilde{\beta})$.

**Lemma 14.** If $\beta_0(k) = \beta(nk)$, then $M_0 = M_{z^n}|_{S_0}$ is unitarily equivalent to $M_z$ on $L^2(\beta_0)$.

**Proof.** Let $T : S_0 \to L^2(\beta_0)$ be the linear transformation defined by
\begin{equation}
T(z^{nm}) = z^m,
\end{equation}
where $m \in \mathbb{Z}$.

If $f \in S_0$, then $f(z) = \sum_m \hat{f}(nm) z^{nm}$ and, by equation [18],
\begin{equation}
||f||^2_{S_0} = \sum_m |\hat{f}(nm)|^2 (\beta(nm))^2 = \sum_m |\hat{f}(nm)|^2 (\beta_0(m))^2 = ||Tf||^2_{L^2(\beta_0)}.
\end{equation}
Therefore, $T$ is an isometry. Clearly, for a given $g = \sum_m \hat{g}(m)z^m \in L^2(\beta_0)$, there is an element $f = \sum_m \hat{f}(nm)z^{nm} \in S_0$ such that $T(f) = g$, where $\hat{g}(m) = \hat{f}(nm)$; that is, $T$ is onto. It follows that $T$ is unitary.

Clearly,

$$M_z T = TM_0.$$  

Therefore, $M_0 = M_z^n|S_0$ is unitarily equivalent to $M_z$ on $L^2(\beta_0)$.

We focus on the bilateral shift operator $M_z$ on $L^2(\beta)$ with monotonically increasing weights $\{\lambda_n\}$. If the weights $\{\lambda_n\}$ of $M_z$ on $L^2(\beta)$ satisfy

$$|\lambda_n| \leq c r^n \text{ for some } c > 0 \text{ and } \lim_{n \to \infty} \lambda_n = 1,$$

then $\sigma(M_z)$ is the annulus $A_r$ of $\beta$.

We will call such operators $M_z$ a \textit{monotonic-$A_r$ weighted shift}.

First, we obtain the analogue of Proposition 4 and Proposition 5 in the case of monotonic-$A_r$ weighted shifts.

\textbf{Proposition 15.} For $0 \leq i < n$, let $M_i = M_z^n|S_i$, and assume that $M_z$ is a monotonic-$A_r$ weighted shift. If $P = (P_{ij})_{(n \times n)}$ is a projection such that

$$\begin{pmatrix} M_0 & 0 & \cdots & 0 & 0 \\ 0 & M_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & M_{n-1} \end{pmatrix} P = P \begin{pmatrix} M_0 & 0 & \cdots & 0 & 0 \\ 0 & M_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & M_{n-1} \end{pmatrix},$$

then there are elements $\varphi_{ij}(0 \leq i, j < n)$ in $L^\infty(\beta)$ such that $P_{ij} = M_{\varphi_{ij}}$.

Moreover, $\varphi_{ii}$ is a positive constant function for $0 \leq i < n$, and $\varphi_{ij} \equiv 0$ for $i \neq j$.

\textbf{Proof.} For a given $0 \leq i < n$, define a sequence of positive numbers $\{\beta_i(k)\}$ by $\beta_i(k) = \beta(nk + i)$ for $k \in \mathbb{Z}$, and let $T_i : S_i \to L^2(\beta_i)$ be the linear transformation defined by

$$T_i(z^{nm+i}) = z^m,$$

where $m \in \mathbb{Z}$.

If $f \in S_i$, then $f(z) = \sum_m \hat{f}(nm+i)z^{nm+i}$ and, by equation (18),

$$\|T_i f\|^2_{L^2(\beta_i)} = \sum_m |\hat{f}(nm+i)|^2 (\beta_i(m))^2 = \sum_m |\hat{f}(nm+i)|^2 (\beta(nm+i))^2 = \|f\|_{S_i}.$$  

Thus, $T_i$ is isometric, and for a given $g = \sum_m \hat{g}(m)z^m \in L^2(\beta_i)$, there is an element $f = \sum_m \hat{f}(nm+i)z^{nm+i} \in S_i$ such that $T_i(f) = g$, where $\hat{g}(m) = \hat{f}(nm+i)$; that is, $T_i$ is unitary.

Since $M_z T_i = T_i M_z$, we have $M_i = T_i^{-1} M_z T_i$. Hence, $M_i P_{ii} = P_{ii} M_i$ implies that $T_i^{-1} M_z P_{ii} T_i = P_{ii} T_i^{-1} M_z T_i$ and so $M_z(T_i P_{ii} T_i^{-1}) = (T_i P_{ii} T_i^{-1}) M_z$. By Proposition 11

$$T_i P_{ii} T_i^{-1} = M_{\varphi_{ii}},$$

for some $\varphi_{ii} \in L^\infty(\beta_i)$. Thus, $P_{ii}$ is unitarily equivalent to the linear transformation $M_{\varphi_{ii}}$ for some $\varphi_{ii} \in L^\infty(\beta_i)$.  

Therefore, $M_0 = M_z^n|S_0$ is unitarily equivalent to $M_z$ on $L^2(\beta_0)$.
By Proposition 12 since \( M_{\varphi_{ii}}(0 \leq i < n) \) is self-adjoint, for any integers \( m \) and \( p \),

\[
\hat{\varphi}_{ii}(m - p) = \overline{\varphi}_{ii}(p - m),
\]

and

\[
(M_{\varphi_{ii}}(z^p), z^m) = (z^p, M_{\varphi_{ii}}(z^m)).
\]

Equations (32) and (33) imply that

\[
\hat{\varphi}_{ii}(m - p)\beta(m)^2 = \overline{\varphi}_{ii}(p - m)\beta(p)^2 = \hat{\varphi}_{ii}(m - p)\beta(p)^2.
\]

In equation (35), if \( m \neq p \), without loss of generality, we assume that \( m < p \). Then, by equation (34),

\[
\hat{\varphi}_{ii}(m - p) = \hat{\varphi}_{ii}(m - p)\frac{\beta(p)^2}{\beta(m)^2} = \hat{\varphi}_{ii}(m - p)\frac{\beta(m+1)^2\beta(m+2)^2\cdots\beta(p)^2}{\beta(m)^2}\frac{1}{\beta(p-1)^2}.
\]

Thus, since \( \lambda_k = \frac{\beta(k+1)}{\beta(k)} \) for any \( k \),

\[
\hat{\varphi}_{ii}(m - p) = \hat{\varphi}_{ii}(m - p)\lambda_m^2\lambda_{m+1}^2\cdots\lambda_{p-1}^2.
\]

Since \( M_z \) is a monotonic-\( A_r \) weighted shift, by equation (35), we conclude that \( \hat{\varphi}_{ii}(m - p) = 0 \) if \( p \neq m \). Clearly, \( \hat{\varphi}_{ii}(0) \) is a real number by equation (32). Thus, \( \varphi_{ii} \) is a real-valued constant function: that is,

\[
M_{\varphi_{ii}} = c_iI_H
\]

for some \( c_i \in \mathbb{R} \). By equations (31) and (35), \( P_{ii} = M_{\varphi_{ii}} \).

Finally, if \( P_{lk} \neq 0 \) for some \( 0 \leq l < k < n \), in the same way as the proof of Proposition 5, we have that \( M_k \) and \( M_l \) are unitarily equivalent which is a contradiction, since the weights are distinct, the weights for \( M_k \) and \( M_l \) are completely different. Hence, \( M_k \) and \( M_l \) can’t be unitarily equivalent for any \( 0 \leq k < n \) by Proposition 13. Therefore, \( P_{ij} = 0 \) if \( i \neq j \).

\[\Box\]

In the next Theorem, we discuss the reducing subspaces of the bilateral weighted shift operator \( M_{z^n} \) on \( L^2(\beta) \) for a monotonic-\( A_r \) weighted shift \( M_z \). Since \( M_{z^n} \) and \( M_{z^n}|S_0 \oplus M_{z^n}|S_1 \oplus \cdots \oplus M_{z^n}|S_{n-1} \) are unitarily equivalent, we have the following result.

**Theorem 16.** If \( M_z \) is a monotonic-\( A_r \) weighted shift, then the operator \( M_{z^n} : L^2(\beta) \rightarrow L^2(\beta) \) has \( 2^n \) reducing subspaces for \( n \geq 2 \).

Although we could state hypothesis for a version of Theorem 16 in terms of the weights as M. Stessin and K. Zhu [10] do for the case of unilateral weighted shifts, we state one concrete result which generalizes Theorem 6 and Theorem 10.

An operator \( T \) is said to be **hyponormal** if \( [T^*, T] = T^*T - TT^* \geq 0 \) and a **strict hyponormal** if \( \ker[T^*, T] = \{0\} \). One concrete application of Theorem 16 is the Corollary 17.

**Corollary 17.** If \( M_z \) on \( L^2(\beta) \) is a strict hyponormal operator such that \( \sigma(M_z) = A_r \), then \( M_{z^n} \) has \( 2^n \) reducing subspaces for \( n \geq 2 \).

**Proof.** The operator \( M_z \) is a strict hyponormal if and only if \( \lambda_n < \lambda_{n+1} \) for all \( n \).

Since all subnormal weighted shift operators which are not isometric are strictly hyponormal, our earlier Theorems (Theorem 6 and Theorem 10) follow from Theorem 16.
5. Kernel Function Point of View

In this section, we also assume that the shift operator $M_z$ on $L^2(\beta)$ is invertible. Then, $\sigma(M_z)$ is the annulus $A = \{z \in \mathbb{C} : r(M_z^{-1})^{-1} \leq |z| \leq r(M_z)\}$, where $r(M_z)(r(M_z^{-1}))$ denotes the spectral radius of $M_z(M_z^{-1})$, respectively [3]. In this section, we focus on the shift operator $M_z$ on $L^2(\beta)$ with monotonic weights $\{\lambda_n\}$. In this section, we assume that the weights $\{\lambda_n\}$ of $M_z$ on $L^2(\beta)$ are monotonic satisfying

$$\lim_{n \to -\infty} \frac{\lambda_n}{r^n} = 1 \text{ and } \lim_{n \to \infty} \lambda_n = 1.$$ 

By a Laurent polynomial we mean a finite linear combination of the vectors $\{g_n\}(-\infty < n < \infty)$. Recall that for a complex number $\omega$, $\lambda_\omega$ denotes the functional of evaluation at $\omega$, defined on Laurent polynomials by $\lambda_\omega(p) = p(\omega)$.

**Definition 18.** $\omega$ is said to be a bounded point evaluation on $L^2(\beta)$ if the functional $\lambda_\omega$ extends to a bounded linear functional on $L^2(\beta)$.

In this section, the hypotheses on the weights imply that every point $\omega$ in $A_r$ is a bounded point evaluation. Thus, we have the reproducing kernel $k_\omega$ for $L^2(\beta)$ associated with the point $\omega \in A_r$.

**Lemma 19.** If $M_F : S_k \to L^2(\beta)$ is a (bounded) multiplication operator by a function $F$ on $A_r$, then $F \in H^\infty(A_r)$ and $\|F\|_\infty \leq \|M_F\|$.

**Proof.** Since every point $\omega$ in $A_r$ is a bounded point evaluation, it is proven in the same way as in Lemma 1.

**Proposition 20.** A bounded linear operator $T$ on $L^2(\beta)$ commutes with $M_{z^n}$ if and only if there are functions $\phi_i(0 \leq i < n)$ in $H^\infty(A_r)$ such that

$$Tf = \sum_{i=0}^{n-1} \phi_i f_i,$$

where $f_i(0 \leq i < n)$ denotes the functions in equation (37).

**Proof.** In the same way as in Proposition 1 we have analytic functions $\phi_i(0 \leq i < n)$ on $A_r$ satisfying equation (37).

Since $\phi_i(z) = \frac{T(z^i)}{z}$ for $0 \leq i < n$, by Lemma 19, $\phi_i \in H^\infty(A_r)$.

**Proposition 21.** For $0 \leq k < n$, let $M_k = M_{z^n}|S_k$.

If $B = (B_{ij})_{(n \times n)}$ is a projection such that

$$\begin{pmatrix} M_0 & 0 & \cdots & 0 & 0 \\ 0 & M_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & M_{n-1} \end{pmatrix} B = B \begin{pmatrix} M_0 & 0 & \cdots & 0 & 0 \\ 0 & M_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & M_{n-1} \end{pmatrix},$$

then there are holomorphic functions $\varphi_{ij}(0 \leq i, j < n)$ in $H^\infty(A_r)$ such that

$$B_{ij} = M_{\varphi_{ij}}.$$

Moreover, $\varphi_{ii}$ is a real-valued constant function on $A_r$ for $0 \leq i < n$, and $\varphi_{ij} \equiv 0$ for $i \neq j$. 

Proof. Since the the weight \( \{ \lambda_n \} \) of \( M_z \) on \( L^2(\beta) \) is monotonic, by Proposition\(^\text{13}\) \( M_k \) and \( M_l \) are not unitarily equivalent for any \( 0 \leq k \neq l < n \). Thus, by the same way in Proposition\(^\text{5}\) it is proven. \( \square \)

**Theorem 22.** For \( 0 \leq i < n \), let \( M_i = M_{z^n}\big|_{S_i} \). Then the bilateral weighted shift operator \( M_{z^n} : L^2(\beta) \rightarrow L^2(\beta) \) has \( 2^n \) reducing subspaces for \( n \geq 2 \).

**Proof.** In the same way in Theorem\(^\text{6}\) it is proven. \( \square \)

### 6. A Complex Geometric Point of Views

The adjoint of a hyponormal weighted shift with spectrum equal to the closure of \( A_r \), for \( 0 < r < 1 \) and essential spectrum equal to \( \partial A_r \) belongs to a very special class of operators, \( B_1(A_r) \). Recall that, for a bounded domain \( \Omega \) in \( \mathbb{C} \) and a positive integer \( n \), the \( B_n(\Omega) \)-class was introduced by M. Cowen and the first author in \( \text{[3]} \) and consists of those bounded operators on a Hilbert space \( H \) that satisfy:

1. \( \text{ran} (T - \omega) \) is closed for \( \omega \in \Omega \),
2. \( \text{dim ker} (T - \omega) = n \) for \( \omega \in \Omega \), and
3. \( \bigvee_{\omega \in \Omega} \ker(T - \omega) = H \).

The operators \( M_z^* \) and \( T_z^* \) as well as the adjoints of the bilateral weighted shifts \( M_{z^k} \) defined in the previous sections with \( \sigma(M_z) = \overline{A_r} \) and \( \sigma_z = \partial A_r \) belong to \( B_1(A_r) \), while their \( n \)th powers, \( M_{z^n}^*, T_{z^n}^* \) and \( M_{z^n}^* \), belong to \( B_n(A_r) \).

All operators \( T \) in \( B_1(A_r) \) have a kernel function, \( k_z \), and \( \ker(T^n - \omega) \) is the span of \( \Gamma_\omega = \{ k_{\lambda\omega_k} : \omega_k = \exp(2\pi ik/n)(0 \leq k < n) \} \), where \( \overline{\lambda} = \omega \). Thus, a holomorphic frame for the hermitian holomorphic bundle \( E_T \) canonically defined by \( T^n \) is given by the sums of the appropriate functions in \( \lambda \) analogous to the subspace decomposition into powers of \( z, z^k \), where \( k \equiv i \text{ (modulo } n \text{)}, and \( 0 \leq i < n \). In the general case, these sections don’t correspond to reducing subspaces since these sections being pairwise orthogonal can be shown to be equivalent to \( T \) being a weighted shift.

Operators in the commutant of \( T^n \) correspond to anti-holomorphic bundle maps, which have a matrix representation once a anti-holomorphic frame is chosen for \( E_T \). That is what was accomplished as a first step in the earlier sections. Reducing subspaces correspond to projection-valued anti-holomorphic bundle maps and are determined by the value at a single point. Again, that is the result proved in each of the three cases in which the bundle \( E_T \) is presented as the orthogonal direct sum of \( n \) anti-holomorphic line bundles.

The question of whether there are other reducing subspaces is equivalent to the issue of representing this bundle as a different orthogonal direct sum. These bundles all have canonical Chern connections and hence a corresponding curvature. The fact that the operators obtained by restricting \( T^n \) to one of these reducing subspaces corresponds to the fact that the curvature has distinct eigenvalues at some point in \( A_r \). This is a straight calculation in the case of the disk but much less so for the annulus.

If we take a general \( T \) in \( B_1(A_r) \), it seems that the lattice of reducing subspaces has \( 2^k \) elements for \( 0 < k \leq n \). That is the case for Toeplitz operators on the Hardy space \( H_\omega^2(A_r) \), where \( \omega \in A_r \) and the measure used to define \( H_\omega^2(A_r) \) is harmonic measure on \( A_r \) for the point \( \omega \). It is not clear just how to settle the
general case, however, since calculating the curvature is probably not feasible. (Note in this case $T_z$ is not a bilateral weighted shift.) Thus, we need to develop other techniques to settle this question.

A more general question concerns operators $T$ in $B_n(\Omega)$ for more general $\Omega$. For $T_{z} \otimes I_{\mathbb{C}^n}$ on $H^2(D) \otimes \mathbb{C}^n$, the lattice of reducing subspaces is continuous and infinite with no discrete part. Does this happen for any other examples besides $T_{\varphi} \otimes I_{\mathbb{C}^n}$ where $\varphi$ is in $H^\infty(\Omega)$?

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