Monopoles and Harmonic Maps

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September 1999

Abstract
Recently Jarvis has proved a correspondence between $SU(N)$ monopoles and rational maps of the Riemann sphere into flag manifolds. Furthermore, he has outlined a construction to obtain the monopole fields from the rational map. In this paper we examine this construction in some detail and provide explicit examples for spherically symmetric $SU(N)$ monopoles with various symmetry breakings. In particular we show how to obtain these monopoles from harmonic maps into complex projective spaces. The approach extends in a natural way to monopoles in hyperbolic space and we use it to construct new spherically symmetric $SU(N)$ hyperbolic monopoles.
I. Introduction

This paper is concerned with static $SU(N)$ BPS monopoles, which are topological solitons in a Yang-Mills-Higgs gauge theory. The Bogomolny equation, which describes all static monopoles, is integrable and so a variety of techniques are available for studying monopoles, including twistor methods. Despite this fact there are still only a limited number of known explicit monopole solutions, though the integrability of the Bogomolny equation allows many features of monopoles, such as the dimensions of their moduli spaces, to be determined.

An example where the integrability of the Bogomolny equation can be used to prove results on monopoles is the correspondence proved by Jarvis [1], between monopoles and rational maps from the Riemann sphere into flag manifolds. The rational map arises as the scattering data, along half-lines from the origin, of a linear operator constructed from the monopole fields. Furthermore, in proving the correspondence Jarvis outlines an ‘inverse scattering’ procedure whereby the monopole fields can be reconstructed from the rational map. It is this construction which is the focus of this paper. The construction involves solving a nonlinear partial differential equation which is equivalent to the Bogomolny equation, but for which the boundary conditions are given in terms of the rational map. This is the main point of the construction, since for the original Bogomolny equation it is not at all clear how to specify boundary conditions on the fields so as to obtain a unique monopole solution. We perform the construction explicitly for several examples of $SU(N)$ monopoles with spherical symmetry and a variety of symmetry breakings. The solutions are obtained from harmonic maps of the plane into $\mathbb{CP}^{N-1}$, with the degrees of the harmonic maps related to the topological charges of the monopoles.

Perhaps we should make it clear at this point that there are several approaches to studying spherically symmetric monopoles [2, 3, 4, 5] and the main aim of this paper is not the construction of new monopole solutions, but rather to gain a better understanding of the correspondence between monopoles and rational maps. In particular we study the construction of monopole solutions from the rational map data, and $SU(N)$ monopoles with spherical symmetry are a good vehicle for this.

The construction of monopoles from rational maps has a natural generalization to monopoles in hyperbolic space. Using this approach we construct explicit solutions for spherically symmetric $SU(N)$ hyperbolic monopoles. As far as we are aware these multimonomopole solutions are new. As we shall see, the construction of hyperbolic monopoles has a simplifying feature in comparison to the Euclidean case, and therefore a useful way to obtain the Euclidean solutions is as the zero curvature limit of the hyperbolic ones.
II. SU(N) Monopoles

BPS monopoles are finite energy solutions to the Bogomolny equation

$$D_i \Phi = -\frac{1}{2} \varepsilon_{ijk} F^{jk}$$

where $D_i = \partial_i + [A_i,]$ is the covariant derivative with $A_i$, for $i = 1, 2, 3$, an $su(N)$-valued gauge potential with gauge field $F_{jk} = \partial_j A_k - \partial_k A_j + [A_j, A_k]$. The Higgs field, $\Phi$, is an $su(N)$-valued scalar field for which nontrivial asymptotic boundary conditions are imposed so that topological solitons exist. More precisely, there is a choice of gauge such that in a given direction the Higgs field for large radius $r$ is given by

$$\Phi = i \Phi_0 - \frac{i}{r} \Phi_1 + O\left(\frac{1}{r^2}\right)$$

where $\Phi_0 = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$, with the trace-free condition requiring that $\lambda_1 + \lambda_2 + \ldots + \lambda_N = 0$, and we choose the ordering such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$. $\Phi_1$ is another diagonal matrix, $\Phi_1 = \frac{1}{2} \text{diag}(n_1, n_2 - n_1, \ldots, n_{N-1} - n_{N-2}, -n_{N-1})$, and it can be shown that the numbers $n_1, n_2, \ldots, n_{N-1}$ are always integers.

$\Phi_0$ is the vacuum expectation value of $\Phi$ and it breaks the symmetry group from $SU(N)$ to a residual symmetry group $J$, given by the isotropy group of $\Phi_0$. The Higgs field on the two-sphere at infinity defines a map from $S^2$ to the coset space of vacua $SU(N)/J$, so that when $\pi_2(SU(N)/J)$ is non-trivial then all solutions have a topological characterization.

If the $\lambda_i$ are all distinct then the residual symmetry group is the maximal torus, that is, $J = U(1)^{N-1}$, and this is known as maximal symmetry breaking. In this case

$$\pi_2 \left(\frac{SU(N)}{U(1)^{N-1}}\right) = \pi_1(U(1)^{N-1}) = \mathbb{Z}^{N-1}$$

so the monopoles are associated with $N - 1$ integers, which are called the topological charges, and are precisely the integers $n_1, n_2, \ldots, n_{N-1}$ appearing in $\Phi_1$.

In contrast the case of minimal symmetry breaking is when all but one of the $\lambda_i$ coincide, so the residual symmetry group is $U(N - 1)$. Since

$$\pi_2 \left(\frac{SU(N)}{U(N - 1)}\right) = \mathbb{Z}$$

there is only one topological charge in this case and the remaining integers are called magnetic weights. The simplest way to distinguish the topological charges from the magnetic weights is to examine the expression for the energy of the monopole.

The condition (2) guarantees that the configuration has finite energy

$$E = \frac{1}{4\pi} \int -\text{tr} \left(\frac{1}{2} F_{ij}^2 + (D_i \Phi)^2\right) d^3x.$$
The energy depends only on the topological charges and the asymptotic eigenvalues of $\Phi$, in fact
$$E = (\lambda_1 - \lambda_2)n_1 + (\lambda_2 - \lambda_3)n_2 + \ldots + (\lambda_{N-1} - \lambda_N)n_{N-1}.$$  
(6)
From this expression it can be seen that the difference $\lambda_j - \lambda_{j+1}$ determines the mass of the monopole of type $j$, of which there are $n_j$ in the given solution. In the minimal symmetry breaking case, where $\lambda_2 = \lambda_3 = \ldots = \lambda_N$, then all but the first type of monopole becomes massless, so that $n_1$ remains a topological charge but the remaining integers become magnetic weights and do not contribute to the value of the energy. Note that we cannot distinguish in a gauge invariant way between $n_2 - n_1$, ... $n_{N-1} - n_{N-2}$ and $-n_{N-1}$, and so when we refer to values of magnetic weights it is understood that this equivalence should be applied.

For intermediate cases of symmetry breaking the residual symmetry group is $J = U(1)^r \times K$, where $K$ is a rank $N - r - 1$ semisimple Lie group, the exact form of which depends on how the $\lambda_i$ coincide with each other. Such monopoles have $r$ topological charges.

III. Rational Maps

In this section we briefly review the recent correspondence proved by Jarvis [1], between $SU(N)$ monopoles and rational maps from the Riemann sphere into flag manifolds. Actually the correspondence proved in [1] is more general than this and is valid for all compact semisimple gauge groups $G$, but in this paper we shall only be concerned with the simplest case of $G = SU(N)$.

The first step is to introduce polar coordinates, so that a point of $\mathbb{R}^3$ is given by a distance $r$ from the origin and a direction determined by a point $z$ on the Riemann sphere around the origin. In terms of the usual angular coordinates $\theta, \varphi$ this is simply $z = e^{i\varphi} \tan(\theta/2)$.

The Jarvis map is obtained by considering Hitchin’s equation
$$\left(D_r - i\Phi, D_{\bar{z}} \right) = 0$$  
(7)
for the complex $N$-vector $s$, along each radial half-line from the origin out to infinity, with the direction of the half-line determined by the value of $z$.

For the moment we shall assume that we are dealing with maximal symmetry breaking. From the boundary conditions we see that since at spatial infinity $\Phi$ is in the gauge orbit of $\Phi_0$ then in the $N$-dimensional solution space there is a one-dimensional subspace generated by the solution which decays at the fastest rate as $r \to \infty$. Now evaluate this solution at $r = 0$. This procedure has thus determined a line in $\mathbb{C}^N$ for each value of $z$. The next step is to consider how this line varies with the direction $z$, and the analysis shows that it varies holomorphically. The crucial ingredient here is that the Bogomolny equation implies that $\left[D_r - i\Phi, D_{\bar{z}} \right] = 0$, so that the operator in equation (7) commutes with the covariant derivative in the angular direction $D_{\bar{z}}$. It can be shown that the degree of this holomorphic map into $\mathbb{C}P^{N-1}$ is precisely the topological charge $n_1$, and hence the map is rational. Note that if we apply a gauge transformation then the map will be transformed by
multiplication by a constant element of $SU(N)$, corresponding to the gauge transformation evaluated at the origin, so that we consider only the equivalence classes of such maps.

Now we repeat the above process but this time we consider the two-dimensional solution space generated by the solution which decays fastest and the solution which decays the next fastest. In the same way as above this will now define a holomorphic plane in $\mathbb{C}^N$ (ie. a space spanned by two holomorphic lines), which of course will contain the holomorphic line we have already described. The degree of this plane is equal to the topological charge $n_2$, and so again the map is rational. Proceeding in this way we finally arrive at the rational map $R : \mathbb{CP}^1 \mapsto F(\mathbb{C}^N)$, where $F(\mathbb{C}^N)$ denotes the space of total flags in $\mathbb{C}^N$. This is a series of vector subspaces $0 \subset V_1 \subset V_2 \subset ... \subset V_{N-1} \subset \mathbb{C}^N$, where $V_i$ has dimension $i$, which is clearly the structure we have just described.

In the above discussion the degrees refer to the elements of the homotopy group $\pi_2(F(\mathbb{C}^N)) = \mathbb{Z}^{N-1}$, and are given by the highest powers which occur in some holomorphic polynomials, as described later. For a detailed discussion of rational maps into flag manifolds and their relationship to monopoles the interested reader may find it useful to consult refs. [6, 7].

For symmetry breaking which is not maximal the picture is similar, except that now the rational map will not be into the space of total flags, since the exponential decay of some of the solutions will be the same and hence some of the subspaces $V_i$ will be missing from the flag. This of course corresponds to the fact that there will now be fewer topological charges, and these correspond to the degrees of the maps into the vector spaces which remain in the flag.

Because the construction of the rational map from the monopole does not break the rotational symmetry of $\mathbb{R}^3$ it is a very useful approach for studying monopoles with symmetries. For the case of $SU(2)$ there is only one vector subspace; the space of lines in $\mathbb{C}^2$. Thus the rational map is $R : \mathbb{CP}^1 \mapsto \mathbb{CP}^1$, that is, a rational map between Riemann spheres, and its degree is the sole topological charge. By explicit construction of some symmetric maps the existence of various $SU(2)$ monopoles with special symmetries has been proved [8].

**IV. Constructing the Monopole**

The proof of the correspondence between monopoles and rational maps [1] involves constructing the monopole from the rational map. The starting point is to write the Bogomolny equation (1) in terms of the coordinates $r, z, \bar{z}$ and observe that a (complex) gauge can always be chosen so that

$$\Phi = -iA_r = -\frac{i}{2}H^{-1}\partial_r H, \quad A_z = H^{-1}\partial_z H, \quad A_{\bar{z}} = 0$$

(8)

where $H \in SL(N, \mathbb{C})$ is a Hermitian matrix.

The Bogomolny equation is then equivalent to the single equation for $H$

$$\partial_r \left( H^{-1} \partial_r H \right) + \frac{(1 + |z|^2)^2}{r^2} \partial_{\bar{z}} \left( H^{-1} \partial_z H \right) = 0$$

(9)
which we shall refer to as the Jarvis equation. Jarvis [1] then proves that solutions of this equation are determined by the rational map, which specifies the asymptotic boundary conditions on $H$ for large $r$. The analysis presented in [1] is complicated and is not very suitable for attempting to implement the construction explicitly, so in this section we shall present a more explicit prescription for determining the boundary conditions on $H$ in terms of the rational map.

For simplicity in this section we shall restrict to the case of $SU(2)$ monopoles. With a choice of normalization for the Higgs field we have the boundary conditions on the monopole as

$$
\Phi = \Phi_\infty (1 - \frac{n}{2r} + O\left(\frac{1}{r^2}\right)) \tag{10}
$$

where $\Phi_\infty$ is in the gauge orbit of $i\sigma_3 = \text{diag}(i, -i)$, and $n$ is the topological charge.

Any $2 \times 2$ Hermitian matrix $H$, which has unit determinant, can always be written in the form

$$
H = \exp\{g(P - \frac{1}{2})\} \tag{11}
$$

where $g$ is a real function and $P$ is a $2 \times 2$ Hermitian projector, that is, $P^\dagger = P = P^2$. A motivation for introducing projectors is that it is a useful formulation for dealing with similar equations that arise in the context of Skyrmions [3]. Examining the asymptotic boundary condition (10) for large $r$ we see that the magnitude of the Higgs field at infinity is a constant and moreover the direction of the Higgs field in the $su(2)$ algebra is independent of the radius to leading order in $1/r$. Comparing this behaviour with equation (8) for the Higgs field in terms of $H$, we find that the leading order behaviour for large $r$ is that the profile function $g$ is independent of the angular coordinates $z, \bar{z}$ and the projector $P$ is a function only of the angular coordinates. We are now going to examine the large $r$ behaviour of the solution, so we use the above leading order result and set $g(r)$ and $P(z, \bar{z})$.

Computing the Higgs field we obtain

$$
\Phi = -\frac{i}{2} H^{-1} \partial_r H = -\frac{i}{2} g'(P - \frac{1}{2}) \tag{12}
$$

with magnitude

$$
\|\Phi\|^2 = -\frac{1}{2} \text{tr}(\Phi^2) = \frac{g'^2}{16} = 1 - \frac{n}{r} + O\left(\frac{1}{r^2}\right). \tag{13}
$$

Integrating this equation for $g$ we obtain (there is a choice of sign here that we shall discuss below)

$$
g = -4r + 2n \log r + O(1). \tag{14}
$$

On substituting the form (11) into equation (8) and using the asymptotic expression (14) we obtain the result that

$$
e^{4r - 2(n+1)(1 + |z|^2)^2}[PP_{\bar{z}\bar{z}} + P_{z}P_{\bar{z}}] + O\left(\frac{1}{r^2}\right) = 0 \tag{15}
$$
where subscripts denote partial differentiation. The coefficient of the growing term in (13) must therefore vanish and we find the equation satisfied by $P$ is

$$(PP_z)_z = 0. \quad (16)$$

The equation $PP_z = 0$ gives the instanton solutions of the $\mathbb{C}P^1$ $\sigma$-model in the plane (see eg. ref [10]) and clearly these will satisfy equation (16). Furthermore, as we prove in the appendix, this gives all solutions of equation (13).

All instanton solutions of the $\mathbb{C}P^1$ $\sigma$-model are given by

$$P = \frac{ff^\dagger}{|f|^2} \quad (17)$$

where $f$ is a 2-component column vector whose entries are holomorphic functions of $z$. Note that the multiplication of $f$ by an overall factor does not change the projector $P$, so that $f$ is an element of $\mathbb{C}P^1$.

Substituting the asymptotic behaviour (14) into equation (12) we obtain the expression for the Higgs field on the two-sphere at infinity

$$\Phi_\infty = i(2P - 1). \quad (18)$$

The topological charge, $n$, is the winding number of this map, which is equal to the degree of the holomorphic vector $f(z)$ which is used to construct the projector via (17). Thus we conclude that the boundary condition on $H$ is determined in this simple and explicit way in terms of the degree $n$ rational map $f(z) : \mathbb{C}P^1 \mapsto \mathbb{C}P^1$.

Note that (17) and (18) give us an explicit expression for the Higgs field at infinity in terms of the rational map. Naively one may think that this does not contain very much information, since for example it is always possible to choose a (singular) gauge in which the Higgs field at infinity is diagonal and constant. However, the important point is that our expression is given in an explicit known gauge, and therefore we have removed the gauge freedom and are left with the physical information in the Higgs field: the fact that it is rational.

To be precise, we have not yet proved an equivalence between the rational map $f$ and the one introduced by Jarvis [1]. To prove this equivalence we shall now show that $f$ is indeed the map obtained as the scattering data.

In a unitary gauge there is a basis of solutions to Hitchin’s equation (7) which have the leading order large $r$ behaviour

$$s \sim e^{-\lambda_j r} v_j \quad (19)$$

where $\lambda_j$ is an eigenvalue of $-i\Phi_\infty$ and $v_j$ is the corresponding eigenvector. In the $SU(2)$ case, when $\lambda_1 = -\lambda_2 = 1$, the scattering map is determined by the decaying solution or more fundamentally by the solution associated with the $\lambda_1 = 1$ eigenspace. Recall that the scattering map is obtained by evaluating this solution at the origin $r = 0$. Now, in the

\[1\] We thank Nick Manton for suggesting this analysis.
Hitchin’s equation is trivialised to $\partial_r s = 0$, so the solutions are $r$ independent and hence the scattering map is the eigenvector of $-i\Phi_\infty$ with eigenvalue one. Thus all that remains to be shown is that $f$ is the eigenvector of $-i\Phi_\infty$ with eigenvalue one. Using the explicit expression (18) and the definition of the projector (17), this is elementary as

$$-i\Phi_\infty f = (2P - 1)f = \left(\frac{2ff^\dagger}{|f|^2} - 1\right)f = f.$$  \hfill (20)

On a minor point, it is worth while making a comment about the choice of sign made in equation (14) when taking the square root and integrating equation (13). If the opposite choice of sign is made then following through the analysis we find that the boundary condition is determined by an anti-holomorphic map. Thus this choice of sign is merely an orientation and determines whether we wish monopoles to correspond to holomorphic or anti-holomorphic rational maps.

The construction of a monopole from its rational map is now clear. Choose a rational map $f(z)$ and calculate the associated projector (17). Then compute the solution of the Jarvis equation (9) satisfying the boundary condition that for large $r$

$$H \sim \exp(r(2 - 4P)).$$  \hfill (21)

Obviously this construction is not easy to implement explicitly in practice, since it still requires the solution of a nonlinear partial differential equation. In this sense it is not as powerful as say the ADHMN construction [11], which reduces the problem to solving a set of nonlinear matrix ordinary differential equations plus a further linear system of ordinary differential equations. The advantage is that for the construction discussed here the data is free, in that any rational map is allowed, whereas in the ADHMN construction the Nahm data must satisfy complicated constraints (including the aforementioned set of nonlinear ordinary differential equations). Thus even using the ADHMN construction very few explicit examples of monopole solutions are known. There is always an inherent difficulty associated with solving the monopole equations and the difference between these two alternative constructions is whether the main difficulty resides in performing the construction or specifying the data upon which the construction is performed.

There are simplifying special cases for which we are able to perform the construction explicitly, the easiest example being the rational map $f = (1, z)^t$, which corresponds to the spherically symmetric $SU(2)$ 1-monopole. In this case the asymptotic behaviour, $g(r)$ and $P(z, \bar{z})$, is valid for all $r$ and substituting (11) into the Jarvis equation gives the following ordinary differential equation for the profile function

$$g'' + \frac{2}{r^2} \left(1 - e^g\right) = 0.$$  \hfill (22)

The large $r$ boundary condition $g \sim -4r$, together with the condition $g(0) = 0$, which is required for $H$ to be well-defined at the origin, determines the unique solution of (22) as

$$g = 2\log(2r/\sinh2r).$$  \hfill (23)
This gives the well-known 1-monopole solution and comparing the asymptotic expansion of (23) with equation (14) we verify that \( n = 1 \), so we see explicitly that the topological charge is determined as the degree of the rational map and there is no freedom in the profile function once the map has been specified.

In the following section we provide some explicit examples of solutions to the Jarvis equation, corresponding to spherically symmetric SU(\( N \)) monopoles with various symmetry breakings. We present the rational maps and describe how the solutions of the Jarvis equation are obtained from these in terms of harmonic maps into \( \mathbb{CP}^{N-1} \).

V. Harmonic Maps and Spherical Monopoles

In the first part of this section we briefly review some facts that we shall need about harmonic maps of the \( \mathbb{CP}^{N-1} \) \( \sigma \)-model in the plane. These results can be found in, for example, ref. [10].

A. Harmonic Maps

The harmonic map (or \( \sigma \)-model) equations for the \( \mathbb{CP}^{N-1} \) model are given by

\[
[P_{zz}, P] = 0
\]  

(24)

where \( P \) is an \( N \times N \) Hermitian projector.

As stated earlier, one set of solutions to these equations are the instantons given by

\[
P(f) = \frac{ff^\dagger}{|f|^2}
\]  

(25)

where \( f(z) \) is an \( N \)-component column vector which is a holomorphic function of \( z \) and whose degree is equal to the topological charge of the \( \sigma \)-model. Another set of solutions are the anti-instantons, which have the same form but this time \( f \) is an anti-holomorphic function, and then the \( \sigma \)-model topological charge is minus the degree of \( f \).

For \( N = 2 \) these are all the finite action solutions, but for \( N > 2 \) there are other non-instanton solutions. These can be described by introducing the operator \( \Delta \) defined by its action on any vector \( f \in \mathbb{CP}^N \) as

\[
\Delta f = \partial_z f - \frac{f (f^\dagger \partial_z f)}{|f|^2}
\]  

(26)

and then define further vectors \( \Delta^k f \) by induction: \( \Delta^k f = \Delta(\Delta^{k-1} f) \).

To proceed further we note the following useful properties of \( \Delta^k f \) when \( f \) is holomorphic:

\[
(\Delta^k f)^\dagger \Delta^l f = 0, \quad k \neq l \quad \tag{27}
\]

\[
\partial_z (\Delta^k f) = -\Delta^{k-1} f \frac{|\Delta^k f|^2}{|\Delta^{k-1} f|^2}, \quad \partial_z \left( \frac{\Delta^{k-1} f}{|\Delta^{k-1} f|^2} \right) = \frac{\Delta^k f}{|\Delta^{k-1} f|^2}. \quad \tag{28}
\]
These properties either follow directly from the definition of $\Delta$ or are easy to prove [10]. It is also convenient to define projectors $P_k$ corresponding to the family of vectors $\Delta^k f$ as

$$P_k = P(\Delta^k f), \quad k = 0, \ldots, N - 1.$$ (29)

Applying $\Delta$ a total of $N - 1$ times to a holomorphic vector gives an anti-holomorphic vector, so that a further application of $\Delta$ gives the zero vector and hence no corresponding projector.

The projectors $P_k$ are solutions of the harmonic map equations (24) and all solutions can be found in this way by starting with an appropriate holomorphic vector $f$. In the $\mathbb{C}P^1$ case the operator $\Delta$ converts a holomorphic vector to an anti-holomorphic vector, that is, instantons to anti-instantons and these are all the solutions in this case.

Note that the projectors obtained from this sequence always satisfy the relation $\sum_{k=0}^{N-1} P_k = 1$.

For connecting harmonic maps with monopoles it is useful to recall the following interpretation of the non-instanton solutions [10]. From a holomorphic vector $f$ form the exterior product of $f$ and its derivatives as

$$h^k = f \wedge \partial_z f \wedge \ldots \wedge \partial_z^k f, \quad k = 0, \ldots, N - 1.$$ (30)

Thus $h^k$ is holomorphic, though it is an element of a larger dimensional space; it may be represented as a totally anti-symmetric tensor with $k + 1$ indices. With this notation it may then be shown that

$$\bar{h}^{k-1} \cdot h^k \simeq \Delta^k f$$ (31)

where $\cdot$ denotes the summation over all the indices of $h^{k-1}$ and all but the first index of $h^k$. Here $\simeq$ denotes that two vectors are equal up to an overall factor, which is the important equivalence since we are dealing with elements of projective spaces. Equation (31) leads to the relation

$$\deg(\Delta^k f) = \deg(h^k) - \deg(h^{k-1})$$ (32)

where the left-hand side is defined as the $\sigma$-model topological charge of the projector $P_k = P(\Delta^k f)$, and $\deg(h^k)$ is the highest power of $z$ which occurs in the holomorphic tensor $h^k$. Thus the non-instanton solutions may be interpreted as special mixtures of instantons and anti-instantons.

**B. Spherical Monopoles**

In section IV we saw that the rational map for the spherically symmetric $SU(2)$ 1-monopole is given by $f(z) = (1, z)^t$. This map is spherically symmetric in the sense that a rotation in $\mathbb{R}^3$, which is realised as an $SU(2)$ Möbius transformation

$$z \mapsto \bar{z} = \frac{\alpha z + \beta}{-\beta z + \bar{\alpha}}, \quad \text{with} \quad |\alpha|^2 + |\beta|^2 = 1$$ (33)
can be compensated by a constant $SU(2)$ gauge transformation. Explicitly
\[ f(\bar{z}) \simeq \begin{pmatrix} \bar{\alpha} & -\bar{\beta} \\ \beta & \alpha \end{pmatrix} f(z). \] (34)

This is the only spherically symmetric map into $\mathbb{C}P^1$ which has positive degree and hence there are no more spherically symmetric $SU(2)$ monopoles. For $SU(N)$ we first require spherically symmetric maps into $\mathbb{C}P^{N-1}$ and these are given by
\[ f = (f_0, ..., f_j, ..., f_{N-1})^t, \text{ where } f_j = z^j \binom{N-1}{j} \] (35)
and $\binom{N-1}{j}$ denote the binomial coefficients. It can be shown that these maps are spherically symmetric by an explicit presentation of the compensating transformation as in (34). Of course there are other spherically symmetric maps which are obtained by embedding the above maps and setting all other entries to be zero.

For a spherically symmetric $SU(N)$ monopole we require a rational map into the space of total flags $F(\mathbb{C}^N)$, which has spherical symmetry. Thus we need an explicit representation of the holomorphic line, the holomorphic plane (which contains the line), etc. As we shall see in more detail below, we take each $k$-dimensional subspace to be the space spanned by the vectors $f, \partial_z f, ..., \partial_z^{k-1} f$, where $f$ is the spherical map (35). Note that these are precisely the spaces $h^{k-1}$ defined in (30). Thus the topological charges of the monopole, $n_k$, are given by
\[ n_k = \deg(h^{k-1}), \quad k = 1, ..., N - 1. \] (36)
Hence from (32) it is clear that the monopole topological charges are therefore not equal to the $\sigma$-model topological charges of the harmonic maps from which we shall create them. The exception to this statement is the case $N = 2$, where all the harmonic maps are instantons and then the only degree is $\deg(h^0)$ which in this case is equal to the $\sigma$-model topological charge.

The degree of the map (35) is $N - 1$ and hence it is easy to calculate the degree of $h^k$ from (30) which, after taking into account the anti-symmetry, gives
\[ n_k = \deg(h^{k-1}) = k(N - k), \quad k = 1, ..., N - 1. \] (37)
Thus we have computed the monopole topological charges and now it remains to construct the corresponding solution of the Jarvis equation. The $SU(N)$ generalization of the $SU(2)$ form given in (11) is to take a sum of the $N - 1$ projectors
\[ H = \exp\{g_0(P_0 - \frac{1}{N}) + g_1(P_1 - \frac{1}{N}) + ... + g_{N-2}(P_{N-2} - \frac{1}{N})\} \] (38)
where $g_k(r)$ for $k = 0, ..., N - 2$, are profile functions. Recall that the projector $P_{N-1}$ is a linear combination of the other projectors plus the identity matrix, which is why it is not included in the above formula. The profile functions satisfy the regularity condition
$g_k(0) = 0$, and have a linear growth in $r$ for large $r$, the coefficients of which determine the
symmetry breaking pattern. Once the symmetry breaking is specified the profile functions
are, of course, uniquely determined; since there is a one-to-one correspondence between
monopoles and rational maps. We shall illustrate this explicitly in the following with some
examples.

C. SU(3) Examples

For $N = 3$, with symmetry breaking to $U(1) \times U(1)$, the charges (37) are $(n_1, n_2) = (2, 2)$. From (35) the holomorphic line is given by $f = (1, \sqrt{2}z, z^2)^t$ and the plane is spanned
by $f$ and $f_z$. The SU(3) case has a simplifying feature, in that the holomorphic plane in $\mathbb{C}^3$
can be specified by giving a line orthogonal to the plane; which will then be anti-
holomorphic. This line is given by

$$f_\perp = f \times f_z = \sqrt{2}(\bar{z}^2, -\sqrt{2}\bar{z}, 1)^t (39)$$

which is clearly anti-holomorphic and by construction is orthogonal to the holomorphic
plane, that is, $f_\perp^t f = f_\perp^t f_z = 0$. By inspection of (39) the plane has degree two and clearly
has spherical symmetry (compare the structure of $f_\perp$ and $f$). Hence in this case it is simple
to see that the charge is $(2, 2)$. However, as an illustration of the general formalism we
shall also present this example in terms of the notation described above. Thus we find

$$h^0 = \begin{pmatrix} 1 \\ \sqrt{2}z \\ z^2 \end{pmatrix}, \quad h^1 = \begin{pmatrix} 0 & \sqrt{2} & 2z \\ -\sqrt{2} & 0 & \sqrt{2}z^2 \\ -2z & -\sqrt{2}z^2 & 0 \end{pmatrix} (40)$$

giving $(n_1, n_2) = (\deg(h^0), \deg(h^1)) = (2, 2)$.

Taking the $h_k$ from (40) we construct the associated projectors, using (31), and insert
these into the form for $H$ given in (38). This gives a solution of the Jarvis equation provided
the profile functions satisfy the ordinary differential equations

$$-g_0'' + \frac{2}{r^2} (e^{g_0} - 1) + \frac{2}{r^2} (e^{g_1} - 1) = 0$$

$$-g_1'' - \frac{2}{r^2} (e^{g_0} - 1) + \frac{4}{r^2} (e^{g_1} - 1) = 0. (41)$$

The Higgs field is given in terms of the solution of the Jarvis equation by (8) and the
eigenvalues and topological charges can simply be read off by restricting to a given radial
line, say $z = 0$, which gives $\Phi = \frac{1}{6} \text{diag}(g'_1 - 2g'_0, g'_0 - 2g'_1, g'_0 + g'_1)$.

Each profile function has an asymptotic expansion of the form

$$g_k = -\alpha_k r + \beta_k \log r + \log \gamma_k + O\left(\frac{1}{r}\right) (42)$$

with the $\alpha_k$ determined by the vacuum expectation value of the Higgs field. Comparing
with (2) for this case we have that

$$\lambda_1 = \frac{2\alpha_0 - \alpha_1}{6}, \quad \lambda_2 = \frac{2\alpha_1 - \alpha_0}{6}, \quad \lambda_3 = -\frac{\alpha_0 + \alpha_1}{6}, \quad n_1 = \frac{2\beta_0 - \beta_1}{3}, \quad n_2 = \frac{\beta_0 + \beta_1}{3}. (43)$$
It is simple to verify that the topological charge is \((2, 2)\) without resorting to an explicit solution of the profile function equations. Maximal symmetry breaking implies that \(\alpha_0 > \alpha_1 > 0\), so that the terms in \(\Pi\) which contain exponentials of profile functions do not contribute to the leading order behaviour which is \(O(1/t)\). The coefficients of this leading order term then simply give that \(\beta_0 = 4, \beta_1 = 2\), which when substituted into \(\Pi\) confirms that \((n_1, n_2) = (2, 2)\).

The explicit solutions for the profile functions can be obtained, for example, if we choose \(\Phi_0 = \text{diag}(2, 0, -2)\), then the solution is \(g_0 = 2g_1 = 2g\), where \(g\) is the 1-monopole profile function defined in \((23)\).

If we now consider the case of minimal symmetry breaking then the topological charges which survive will be unchanged, but the magnetic weights will not be given by the topological charges which do not survive. As an example consider the symmetry breaking to \(U(1) \times SU(2)\) given by \(\Phi_0 = \text{diag}(1, -\frac{1}{2}, -\frac{1}{2})\). From \(\Pi\) this corresponds to setting \(\alpha_0 = 3, \alpha_1 = 0\). The previous analysis of the profile function equations must now be modified to take into account the fact that exponentials of profile functions may now contribute to leading order (this happens whenever any of the \(\alpha_i\) coincide or are zero, and corresponds to changing the symmetry breaking pattern). In this case it is easy to see that \(\Pi\) requires that \(\beta_0 = 3, \beta_1 = 0, \gamma_1 = \frac{1}{2}\) which gives the values \((n_1, [n_2]) = (2, [1])\), where we have used the notation that square brackets denote magnetic weights rather than topological charges.

The profile function equations that we obtain are related to those derived from the ansatz based approach of Bais et al \([3, 2]\) and the methods employed there can be adapted to solve for the profile functions explicitly. This method requires a careful limiting procedure to be taken to deal with non-maximal symmetry breaking. In section VI we shall see that the solutions for monopoles in hyperbolic space are obtained without the need for this limiting procedure and the Euclidean case can then be obtained from the natural limit in which the curvature of hyperbolic space tends to zero.

For this example the solution is (see section VI)

\[
\begin{align*}
g_0 & = \log \frac{81 r^4}{4 \left[(-3r - 1)e^{-r} + e^{2r}\right] \left[(3r - 1)e^{r} + e^{-2r}\right]} \\
g_1 & = \log \frac{9 r^2 \left[(-3r - 1)e^{-r} + e^{2r}\right]}{2 \left[(3r - 1)e^{r} + e^{-2r}\right]^2}
\end{align*}
\]

and it can be checked that the asymptotic properties are as stated above.

Spherically symmetric monopoles of lower charge, such as the \((1, 1)\) monopole, can be obtained in a similar way by embedding the spherically symmetric maps \([33]\) of lower degree.

**D. SU(4) Examples**

For maximally broken SU(4) the charge, from \([37]\), is \((3, 4, 3)\) and the associated profile
function equations are
\begin{align*}
-g''_0 + \frac{3}{r^2} \left( e^{g_0 - g_1} - 1 \right) + \frac{3}{r^2} \left( e^{g_2} - 1 \right) &= 0 \\
-g''_1 + \frac{3}{r^2} \left( e^{g_0 - g_1} - 1 \right) + \frac{4}{r^2} \left( e^{g_1 - g_2} - 1 \right) + \frac{3}{r^2} \left( e^{g_2} - 1 \right) &= 0 \\
-g''_2 + \frac{4}{r^2} \left( e^{g_1 - g_2} - 1 \right) + \frac{6}{r^2} \left( e^{g_2} - 1 \right) &= 0.
\end{align*}

As for the $SU(3)$ case it is a simple task to confirm the topological charge by a leading order analysis of this set of equations. For the choice $\Phi_0 = \text{diag}(3, 1, -1, -3)$, corresponding to equal monopole masses, the explicit solution is $g_0/3 = g_1/2 = g_2 = g$, where $g$ is given by (23).

There are several possible symmetry breakings and in each case it is a simple matter to determine both the topological charges and magnetic weights by an analysis of equations (15).

For $\Phi_0 = \text{diag}(1, \frac{1}{2}, \frac{3}{2}, -2)$ the symmetry breaking is $U(1) \times SU(2) \times U(1)$ and the charge is $(3, [3], 3)$. The corresponding explicit solution is
\begin{align*}
g_0 &= \log \frac{625 r^6}{[25 e^{-2r} - (30r - 24) e^{-r} + 4 r e^r][25 e^{2r} - (30r + 24) e^r - e^{4r}]} \\
g_1 &= \log \frac{25 r^4 [25 e^{2r} - (30r + 24) e^r - e^{4r}]}{2 [-25 e^{-2r} - (30r - 24) e^{-r} + 4 r e^r][6 e^{-2r} + (5r - 6) e^{3r} - (5r + 6) e^{-3r} + 6 e^{2r}]} \\
g_2 &= \log \frac{50 r^2 [6 e^{-2r} + (5r - 6) e^{3r} - (5r + 6) e^{-3r} + 6 e^{2r}]}{[-25 e^{-2r} - (30r - 24) e^{-r} + 4 r e^r][6 e^{-2r} + (5r + 6) e^{3r} - (5r + 6) e^{-3r} + 6 e^{2r}]}.
\end{align*}

Choosing $\Phi_0 = \text{diag}(\frac{2}{3}, \frac{3}{4}, -\frac{1}{4}, -\frac{5}{4})$ gives the symmetry breaking $SU(2) \times U(1) \times U(1)$ with charge ([2], 4, 3) and solution
\begin{align*}
g_0 &= \log \frac{256 r^6}{9 [(4r + 3) e^{-3r/2} + e^{5r/2} - 4 e^{r/2}][4r - 3] e^{3r/2} - e^{-5r/2} + 4 e^{-r/2}]} \\
g_1 &= \log \frac{16 r^4 [(4r - 3) e^{3r/2} - e^{-5r/2} + 4 e^{-r/2}]}{3 [(4r + 3) e^{-3r/2} + e^{5r/2} - 4 e^{r/2}][4r - 1] e^r + (4r - 1) e^{-r} + e^{-3r} + e^{3r}]} \\
g_2 &= \log \frac{16 r^2 [(-4r - 1) e^r + (4r - 1) e^{-r} + e^{-3r} + e^{3r}]}{3 [(4r + 3) e^{-3r/2} + e^{5r/2} - 4 e^{r/2}][4r - 1] e^r + (4r - 1) e^{-r} + e^{-3r} + e^{3r}]}.
\end{align*}

By taking two pairs of eigenvalues to be equal, for example $\Phi_0 = \text{diag}(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$, the symmetry is broken to $SU(2) \times U(1) \times SU(2)$. In this case the charge is ([2], 4, [2]) and the profile functions are given by
\begin{align*}
g_0 &= \log \frac{r^6}{9 (r \cosh r - \sinh r)^2}, \quad g_1 = \log \frac{r^4}{3 (\sinh^2 r - r^2)}, \quad g_2 = \log \frac{r^2 (\sinh^2 r - r^2)}{3 (r \cosh r - \sinh r)^2}.
\end{align*}
Finally, minimal symmetry breaking to \( U(1) \times SU(3) \) occurs when three eigenvalues coincide, say \( \Phi_0 = \text{diag}(3, -1, -1, -1) \) and this gives a charge \((3, [2], [1])\) with solution
\[
\begin{align*}
g_0 &= \log \frac{1024 r^6}{9 \left[ e^{r} (8r^2 - 4r + 1) - e^{-3r} \right] \left[ e^{3r} - e^{-r} (8r^2 + 4r + 1) \right]} \\
g_1 &= \log \frac{16 r^4 \left[ e^{3r} - e^{-r} (8r^2 + 4r + 1) \right]}{3 \left[ e^{r} (8r^2 - 4r + 1) - e^{-3r} \right] \left[ e^{2r} (2r^2 - r) + e^{-2r} (2r^2 + r) \right]} \\
g_2 &= \log \frac{64 r^2 \left[ e^{2r} (2r^2 - r) + e^{-2r} (2r^2 + r) \right]}{3 \left[ e^{r} (8r^2 - 4r + 1) - e^{-3r}\right]^2}.
\end{align*}
\] (49)

VI. Hyperbolic Monopoles

Hyperbolic monopoles are solutions of the Bogomolny equation (1) in which Euclidean space \( \mathbb{R}^3 \) is replaced by hyperbolic 3-space, which we denote by \( \mathbb{H}^3_{\kappa} \), where \(-\kappa^2\) is the curvature of hyperbolic space. They were first studied by Atiyah [12], who observed that \( S^1 \) invariant instantons can be interpreted as hyperbolic monopoles. Often hyperbolic monopoles turn out to be easier to study than the Euclidean case and we shall see this explicitly in the following. It has long been expected that in the limit as the curvature of hyperbolic space tends to zero then Euclidean monopoles are recovered, but only recently has this been rigorously established [13]. In this section we shall adapt the methods of section V to the hyperbolic case to obtain spherically symmetric \( SU(N) \) monopoles. The \( SU(2) \) 1-monopole solution has been obtained before [14], as a circle invariant instanton, but we believe that all our multi-monopole solutions are new. By explicitly taking the zero curvature limit we recover the Euclidean monopole solutions and explain why this is a simpler way to obtain these solutions than to consider the Euclidean case from the beginning.

Perhaps the most familiar description of \( \mathbb{H}^3_{\kappa} \) is as the interior of the unit ball. In terms of angular coordinates \( z, \bar{z} \) and radial coordinate \( \rho \in [0, 1) \) the metric is
\[
ds^2 = \frac{4}{\kappa^2(1 - \rho^2)^2} (d\rho^2 + \rho^2 \frac{4dzd\bar{z}}{(1 + |z|^2)^2}) = dr^2 + \frac{\sinh^2(\kappa r)}{\kappa^2} \frac{4dzd\bar{z}}{(1 + |z|^2)^2}, \tag{50}
\]
where we have introduced \( r \), the hyperbolic distance from the origin, through the relation \( \rho = \tanh(\kappa r/2) \).

The radial scattering analysis proceeds as in the Euclidean case with the upshot that the Jarvis equation (9) in hyperbolic space becomes [13, 1]
\[
\partial_r \left( H^{-1} \partial_r H \right) + \frac{\kappa^2(1 + |z|^2)^2}{\sinh^2(\kappa r)} \partial_z \left( H^{-1} \partial_z H \right) = 0. \tag{51}
\]
Note that in the zero curvature limit, \( \kappa \to 0 \), the Euclidean equation (9) is recovered.
Solutions of (51) can be obtained using the form (38), with the same harmonic maps, but leading to modified equations for the profile functions. The equations for the monopole fields in terms of $H$ are still given by (8), but with $r$ now being hyperbolic distance. Hence the asymptotic boundary conditions remain the same as in the Euclidean case and together with the requirement that the profile functions vanish at the origin this determines a unique solution for any given choice of vacuum expectation value $\Phi_0$.

For the $SU(2)$ 1-monopole there is just one profile function, which must satisfy the equation
\[ g'' + \frac{2\kappa^2}{\sinh^2(\kappa r)}(1 - e^g) = 0. \] (52)

If we again normalize the Higgs field to have unit magnitude then the boundary conditions on the profile function are $g(0) = 0$ and $g(r) \sim -4r$ for large $r$. The solution is
\[ g = 2 \log \frac{(2 + \kappa)\sinh(\kappa r)}{\kappa \sinh((2 + \kappa)r)} \] (53)

which gives the known $SU(2)$ hyperbolic 1-monopole [14].

A. SU(3) Examples

The profile function equations for the $SU(3)$ charge (2, 2) hyperbolic monopole are

\[-g_0'' \frac{\sinh^2(\kappa r)}{\kappa^2} + 2 (e^{g_0 - g_1} - 1) + 2 (e^{g_1} - 1) = 0 \]
\[-g_1'' \frac{\sinh^2(\kappa r)}{\kappa^2} - 2 (e^{g_0 - g_1} - 1) + 4 (e^{g_1} - 1) = 0. \] (54)

For equal monopole masses, with $\Phi_0 = \text{diag}(2, 0, -2)$, the solution is $g_0 = 2g_1 = 2g$, with $g$ given by (53). For general $\Phi_0$, including minimal symmetry breaking, we now describe how the solution to (54) can be obtained using Hirota’s method.

Introducing the tau-functions $\tau_0, \tau_1$ via the transformation
\[ g_0 = \log \frac{\sinh^4(\kappa r)}{\tau_0 \tau_1 \kappa^4}, \quad g_1 = \log \frac{\tau_0 \sinh^4(\kappa r)}{\tau_1^2 \kappa^2} \] (55)

converts equation (54) into Hirota bilinear form
\[ \mathcal{D}^2 \tau_i, \tau_i + 4 \tau_{i+1} \tau_{i-1} = 0, \quad i = 0, 1 \] (56)

where we have defined $\tau_{-1} = \tau_2 = 1$, and $\mathcal{D}$ is the Hirota derivative defined by [15]
\[ \mathcal{D}^m \alpha, \beta = (\partial_r - \partial_{\tilde{r}})^m \alpha(r) \beta(\tilde{r})|_{r=\tilde{r}}. \] (57)

The Hirota derivative has many special properties which make the construction of solutions to bilinear equations such as (56) an elegant procedure. In particular from (57) it is clear that its action on exponential functions takes the simple form
\[ \mathcal{D}^m e^{\alpha r} e^{\alpha_2 r} = (\alpha_1 - \alpha_2)^m e^{(\alpha_1 + \alpha_2)r} \] (58)
Using this property, together with the bilinear form of the equation, means that it is a simple task to find solutions which are finite sums of exponential functions; in the context of integrable soliton equations, such as the KdV equation, solutions of bilinear equations which are a finite sum of exponentials correspond to multi-solitons \[15\].

Note that the bilinear equations (56) are independent of the curvature, \(-\kappa^2\), so in particular these equations are the ones which also arise for Euclidean monopoles. However, the transformation (55) involves \(\kappa\) which means that the boundary conditions on the tau-functions are \(\kappa\) dependent, and this is of crucial importance. For hyperbolic monopoles, ie. \(\kappa \neq 0\), the solutions satisfying the required boundary conditions are always given as a simple sum of exponentials, whereas for Euclidean monopoles the boundary conditions (except for the case of maximal symmetry breaking) mean that the solutions are not so simple and involve a sum of products of exponentials and polynomials. By taking the limit \(\kappa \to 0\) of a hyperbolic solution the more complicated Euclidean solutions are obtained and this is perhaps the most natural method to construct Euclidean monopoles.

As an example, consider the minimal symmetry breaking of \(SU(3)\) obtained from \(\Phi_0 = \text{diag}(1,-\frac{1}{2},-\frac{1}{2})\). As discussed in section V this choice of \(\Phi_0\) corresponds to the large \(r\) boundary conditions \(g_i \sim -\alpha_i r\), with \(\alpha_0 = 3, \alpha_1 = 0\). Comparing this with the transformation (55) gives the large \(r\) boundary conditions

\[
\tau_0 \sim A_0 e^{(2+2\kappa)r}, \quad \tau_1 \sim A_1 e^{(1+2\kappa)r}
\]

for some constants \(A_i\). The requirement that \(g_i(0) = 0\) gives the conditions at the origin that \(\tau_0 = r^2 + ..., \tau_1 = r^2 + ...\) as \(r \to 0\). Using the leading order behaviour (59) together with the properties of the Hirota derivative it is a simple task to find the following explicit solution

\[
\tau_0 = \frac{2\kappa e^{(2+2\kappa)r} - (3 + 4\kappa)e^{-r} + (3 + 2\kappa)e^{(1+2\kappa)r}}{(3 + 2\kappa)(3 + 4\kappa)\kappa}, \quad \tau_1 = \frac{(3 + 2\kappa)e^{(1+2\kappa)r} - (3 + 4\kappa)e^{r} + 2\kappa e^{(2+2\kappa)r}}{(3 + 2\kappa)(3 + 4\kappa)\kappa}.
\]

As claimed above, we see that there is only an exponential dependence on \(r\); this corresponds to the fact that hyperbolic monopoles approach the vacuum value exponentially, rather than algebraically like Euclidean monopoles.

Taking the limit \(\kappa \to 0\) this solution becomes

\[
\tau_0 = \frac{2}{9}(e^{2r} - (3r + 1)e^{-r}), \quad \tau_1 = \frac{2}{9}((3r - 1)e^{r} + e^{-2r})
\]

so we see the emergence of the algebraic factors. Substituting (61) into (57) we obtain the Euclidean monopole solution given by (44).

**B. SU(4) Examples**
The \(SU(4)\) equations are

\[
\begin{align*}
-g_0' \frac{\sinh^2(\kappa r)}{\kappa^2} &+ 3 \left( e^{g_0-g_1} - 1 \right) + 3 \left( e^{g_2} - 1 \right) = 0 \\
-g_1' \frac{\sinh^2(\kappa r)}{\kappa^2} &- 3 \left( e^{g_0-g_1} - 1 \right) + 4 \left( e^{g_1-g_2} - 1 \right) + 3 \left( e^{g_2} - 1 \right) = 0 \\
-g_2' \frac{\sinh^2(\kappa r)}{\kappa^2} &- 4 \left( e^{g_1-g_2} - 1 \right) + 6 \left( e^{g_2} - 1 \right) = 0.
\end{align*}
\]

(62)

The solution \(g_0/3 = g_1/2 = g_2 = g\), with \(g\) given by (53), corresponds to maximal symmetry breaking with \(\Phi_0 = \text{diag}(3, 1, -1, -3)\).

To obtain the solution for arbitrary \(\Phi_0\) we introduce the tau-functions as

\[
\begin{align*}
g_0 &= \log \frac{\sinh^6(\kappa r)}{\tau_0 \tau_2 K^6}, & g_1 &= \log \frac{\tau_0 \sinh^4(\kappa r)}{\tau_1 \tau_2 K^4}, & g_2 &= \log \frac{\tau_1 \sinh^2(\kappa r)}{\tau_2 K^2}
\end{align*}
\]

(63)

which transforms the equation into the Hirota form

\[
D^2 \tau_i, \tau_i + 2(1 + i)(3 - i)\tau_{i+1} \tau_{i-1} = 0, \quad i = 0, 1, 2
\]

(64)

where \(\tau_{-1} = \tau_3 = 1\).

As an example we give the solution for minimal symmetry breaking with \(\Phi_0 = \text{diag}(3, -1, -1, -1)\), which is

\[
\begin{align*}
\tau_0 &= 3 \frac{\kappa^2 e^{(3+3\kappa)r} - (3\kappa^2 + 5\kappa + 2) e^{(-1+\kappa)r} + (3\kappa^2 + 8\kappa + 4) e^{(-1-\kappa)r} - (\kappa^2 + 3\kappa + 2) e^{(-1+3\kappa)r}}{8(1+\kappa)(2+\kappa)(2+3\kappa)\kappa^2} \\
\tau_1 &= 3 \frac{(2 + \kappa)\cosh((2 + 4\kappa)r) - (4 + 4\kappa)\cosh((2 + 2\kappa)r) + (2 + 3\kappa)\cosh(2r)}{8(1 + \kappa)(2 + \kappa)(2 + 3\kappa)\kappa^2} \\
\tau_2 &= 3 \frac{(\kappa^2 + 3\kappa + 2) e^{(1+3\kappa)r} - (3\kappa^2 + 8\kappa + 4) e^{(1+\kappa)r} + (3\kappa^2 + 5\kappa + 2) e^{(1-\kappa)r} - \kappa^2 e^{(3+3\kappa)r}}{8(1 + \kappa)(2 + \kappa)(2 + 3\kappa)\kappa^2}.
\end{align*}
\]

Taking the zero curvature limit results in

\[
\begin{align*}
\tau_0 &= \frac{3}{32} \left( e^{3r} - (8r^2 + 4r + 1) e^{-r} \right) \\
\tau_1 &= \frac{3r}{16} \left( (2r - 1) e^{2r} + (2r + 1) e^{-2r} \right) \\
\tau_2 &= \frac{3}{32} \left( (8r^2 - 4r + 1) e^r - e^{-3r} \right)
\end{align*}
\]

(65)

which is the Euclidean monopole solution given in (49).

**VII. Conclusion**
We have studied in some detail the construction of $SU(N)$ monopoles from scattering data which consists of a rational map of the Riemann sphere into a flag manifold. Explicit solutions have been obtained in the case of spherical symmetry and we have shown how these solutions involve harmonic maps of the plane into $\mathbb{C}P^{N-1}$. This approach was generalized to the case of hyperbolic monopoles and new spherically symmetric solutions found, whose zero curvature limit was investigated explicitly.

The Jarvis equation is integrable, but in this paper we have made no use of the Lax pair. The precise description of the boundary conditions in terms of the rational map makes this a very convenient formulation of the Bogomolny equation and it may prove useful to undertake a classical inverse scattering study. An alternative, which is currently under investigation, is the numerical solution of the Jarvis equation, which is more promising than a numerical solution of the Bogomolny equation since the boundary conditions can be specified in a simple manner to ensure the existence of a unique solution.

Acknowledgements

Many thanks to Conor Houghton, Stuart Jarvis, Nick Manton, Michael Singer and Wojtek Zakrzewski for useful discussions. PMS acknowledges the EPSRC for an Advanced Fellowship and the grant GR/L88320.

Appendix

Let $P(z, \bar{z})$ be a $2 \times 2$ Hermitian projector. In this appendix we prove that the only solutions of the equation

\[(PP_z)_{\bar{z}} = 0 \quad \text{(A.1)}\]

are the $\sigma$-model instantons given by

\[PP_z = 0. \quad \text{(A.2)}\]

Let $F = PP_z$, then using the fact that $P$ is a projector, which is a solution of (A.1), it is clear that $F$ satisfies the following properties

\[F_{\bar{z}} = 0 \quad \text{(A.3)}\]
\[PF = F \quad \text{(A.4)}\]
\[FP = 0. \quad \text{(A.5)}\]

Taking (A.4) with (A.3) gives that $F^2 = 0$, so that it has at most rank one and can be written as

\[F = uw^\dagger \quad \text{(A.6)}\]
where $u$ and $w$ are two orthogonal column vectors, that is $w^\dagger u = 0$. Substituting this expression for $F$ into (A.3) and multiplying both sides by $w$ leads to the result that $w^\dagger u_z = 0$, that is, $u_z$ is orthogonal to $w$. But since we already know that $u$ is orthogonal to $w$ and they are elements of a two-dimensional vector space then this implies that $u$ and $u_z$ are parallel. Thus $u$ must have the form $u(z, \bar{z}) = g(z, \bar{z})\tilde{u}(z)$, where $\tilde{u}$ is a holomorphic vector and $g$ is some function.

$P$ is a Hermitian projector so it may be written as

$$P = \frac{vv^\dagger}{v^\dagger v} \quad \text{(A.7)}$$

for some 2-component column vector $v$. Substituting the expressions (A.6) and (A.7) into property (A.4) shows that $u$ and $v$ are parallel, that is, $v = \lambda u$ for some function $\lambda$. Now using the earlier factorization of $u$ we obtain $v = \lambda \tilde{u}$, so that finally we arrive at the result that

$$P = \frac{\tilde{u}\tilde{u}^\dagger}{\tilde{u}^\dagger\tilde{u}} \quad \text{(A.8)}$$

As we have already shown that $\tilde{u}$ is a holomorphic vector then this is an instanton solution of equation (A.2) (see for example ref. [10]) and the required result is proved.
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