Generalized Elliptic Integrals and Applications

N.D. Bagis
email: nikosbagis@hotmail.gr

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Abstract

We use some general properties, presented in previous work, to evaluate special cases of integrals relating Rogers-Ramanujan continued fraction, eta function and elliptic integrals.

1 Introductory definitions and formulas

Let

$$\eta(\tau) := e^{\pi i \tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau})$$

(1)

denotes the Dedekind eta function which is defined in the upper half complex plane. It not defined for real $\tau$.

$$2F_1[a, b; c; x] := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{x^n}{n!}$$

(2)

is the Gauss Hypergeometric function, $(s)_k := \Gamma(s+k)/\Gamma(s)$ and $\Gamma(s) := \int_0^{\infty} e^{-t} t^{s-1} \, dt$ is Euler’s Gamma Function.

For $|q| < 1$, the Rogers Ramanujan continued fraction (RCF) is defined as

$$R(q) := \frac{q^{1/5} q^1 q^2 q^3 \ldots}{1+1+1+1+\ldots}$$

(3)

We also for $|q| < 1$ define

$$f(-q) := \prod_{n=1}^{\infty} (1 - q^n)$$

(4)

to be the eta function of Ramanujan. Also hold the following relations proved by Ramanujan

$$\frac{1}{R(q)} - 1 - R(q) = \frac{f(-q^{1/5})}{q^{1/5} f(-q^5)}$$

(5)
For the derivative of the (RRCF) (see [5]) holds

\[ R'(q) = 5^{-1} q^{-5/6} f(-q)^4 R(q)^{1/2} - 5 - R(q)^5 \]  

Some interesting results can be found if we consider the Appell $F_1$ hypergeometric function. This function is defined as

\[ F_1[a, b_1, b_2, c; x, y] := \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b_1)_{m} (b_2)_{n}}{m! n!} x^m y^n \]  

It is known that

\[ \int x^m (ax^2 + bx + c)^n \, dx = \frac{c^n x^{m+1}}{m+1} F_1 \left[ m+1, -n, -m+2; \frac{x}{\rho_1}, \frac{x}{\rho_2} \right] \]  

where \( \rho_{1,2} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a} \).

## 2 Evaluations of elliptic integrals and Rogers--Ramanujan continued fraction

From relation (7) we have multiplying both sides with \( G(R(q)) \) and integrating

\[ \int_a^b f(-q)^4 q^{-5/6} G(R(q)) \, dq = 5 \int_{R(a)}^{R(b)} \frac{G(x)}{x^{1/2} x^{-b} - 11 - x^5} \, dx \]  

Setting \( a = 0, b = R^{-1} \left( \frac{\sqrt[3]{-11 + 5\sqrt{5}}}{2} \right) \) in (10) and making the change of variables \( x \rightarrow w^{1/5}, w \rightarrow y/\rho_2 \) with \( \rho_2 = \frac{-11 + 5\sqrt{5}}{2} \), and \( w^{-1} - 11 - w \rightarrow t \) and then using the formula (see [13])

\[ 2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b) \Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} \, dt \]  

we get the following theorem

**Theorem 1.**

For every \( \nu > 0 \)

\[ \int_0^{R^{-1} \left( \frac{\sqrt[3]{-11 + 5\sqrt{5}}}{2} \right)} f(-q)^4 q^{-5/6} R(q)^{5\nu} \, dq = \]

\[ = \Gamma\left( \frac{5}{6} \right) \left( \frac{11 + 5\sqrt{5}}{2} \right)^{-\nu} \Gamma\left( \frac{1}{6} + \nu \right) \Gamma\left( 1 + \nu \right) 2F_1 \left[ \frac{1}{6}, \frac{1}{6} + \nu; 1 + \nu; \frac{11 - 5\sqrt{5}}{11 + 5\sqrt{5}} \right] \]  

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For certain values of $\nu$ we get closed form evaluations of (12). An example is $\nu = n + 1/2$, $n = 0, 1, 2, \ldots$, which a special case for $n = 0$ is

$$\int_0^{R(-1)} \left( \sqrt[5]{\frac{11+5\sqrt{5}}{2}} \right) f(-q)^4 q^{-5/6} R(q)^{5/2} dq =$$

$$= \frac{27 \Gamma \left( \frac{5}{6} \right) \Gamma \left( \frac{5}{3} \right) \sin \left( \frac{2}{3} \arctan \left( \sqrt[5]{\frac{11+5\sqrt{5}}{11+5\sqrt{5}}} \right) \right)}{2\sqrt{5\pi}}$$  \hspace{1cm} (13)

We state and prove now the next

**Theorem 2.**

If $A = \{1/2, 1, 1+1/2, 2, 2+1/2, 3, \ldots\}$ and $u(q) = R(q)^{-5} - 11 - R(q)^5$, then

$$\int u^{(-1)}(x) u(q)^n f(-q)^5 dq =$$

$$= - \int_0^x \frac{t^{n-1}}{\sqrt{125 + 22t + t^2}} dt = \text{known function when } n \in A$$  \hspace{1cm} (14)

**Proof.**

From [5] we get easily the first equality of (14). For the second equality which is the evaluation part we have ($\rho_1$, $\rho_2$ are the roots of $125 + 22t + t^2 = 0$):

$$\int \frac{t^{n-1}}{\sqrt{125 + 22t + t^2}} dt = \int t^{n-1/2} (t - \rho_1)^{-1/2} (t - \rho_2)^{-1/2} dt$$

is of the form

$$I_{2m} = \int t^{2m} \{(A_1 t^2 + B_1)(A_2 t^2 + B_2)\}^{-1/2} dt$$  \hspace{1cm} (15)

and reduces to elliptic integrals of the first and second kinds, when $m$ is integer (see [4] and [14]).

Using the same methods of Theorems 1,2 one can prove that

$$\int_0^{\rho_1} x^m (ax^2 + bx + c)^n dx =$$

$$= c^n \rho_1^{m+1} \Gamma(m+1) \Gamma(n+1) \frac{2F_1 \left( m+1, -n; m+n+2; \frac{\rho_1}{\rho_2} \right)}{\Gamma(m+n+2)}$$  \hspace{1cm} (16)

where $\rho_1 < \rho_2$ are the roots of $ax^2 + bx + c = 0$. 

3
Another interesting formula is for $0 < \nu < 1$, (see [5]):

$$
\int_{0}^{1} \left( \frac{f(-q^{5})}{f(-q)} \right)^{6\nu+5} f(-q)^{4} q'\, dq = -\frac{\pi \csc(\nu \pi)}{11 \nu + 1} _2F_1 \left[ \frac{\nu + 1}{2}, \frac{\nu + 2}{2}; 1; -\frac{4}{11} \right]
$$

(17)

which also can be written as

$$
\int_{0}^{1} u(q)^{-5/6} f(-q)^{4} q^{-5/6} \, dq = \pi \csc(\nu \pi) 11^{-5/6} _2F_1 \left[ \frac{\nu + 1}{2}, \frac{\nu + 2}{2}; 1; -\frac{4}{11} \right]
$$

(18)

Continuing set $k_r = k$ and $k'_r = k' = \sqrt{1 - k^2}$. Then from (7) and the relations

$$
f(-q) = 2^{1/3} \pi^{-1/2} q^{-1/24} k^{1/12} k'^{1/3} K(k)^{1/2}
$$

(19)

and

$$
\frac{dq}{dk} = -\frac{q \pi^2}{2kk' K(k)^2}
$$

(20)

we get

$$
\frac{dR(q)}{dk} = -5^{-1} \cdot 2^{1/3} (kk')^{-2/3} R(q) \sqrt{R(q)^{-5} - 11 - R(q)^{5}}
$$

(21)

Which is clear that $R(q)$ can be expressed as a function of the singular modulus $k = k_r$.

By integration, we get if $q = e^{-\pi \sqrt{r}}$ with $r$ real positive

$$
\frac{3\sqrt{3}}{5} k^{1/3} _2F_1 \left[ \frac{1}{3}, \frac{1}{6}, \frac{7}{6}; k^2 \right] = \int_{0}^{R(q)} \frac{dx}{x \sqrt{x^{-5} - 11 - x^5}}
$$

(22)

Continuing, from (10) replacing were $f$ the Dedekind eta function $\eta$, we get

Theorem 3.

$$
\pi \int_{\sqrt{\pi}}^{+\infty} \eta(it/2)^4 \, dt = 3 \sqrt{2k_r} \cdot 2^{1/3} _2F_1 \left[ \frac{1}{3}, \frac{1}{6}, \frac{7}{6}; k_r^2 \right] = 5 \int_{0}^{R(q)} \frac{dx}{x \sqrt{x^{-5} - 11 - x^5}}
$$

(23)

Applications

1) If $r = 4$ then from the following evaluation of Ramanujan

$$
R(e^{-2\pi}) = -\frac{1 + \sqrt{5}}{2} + \sqrt{\frac{5 + \sqrt{5}}{2}}
$$

one gets

$$
\int_{0}^{+\infty} \eta(it/2)^4 \, dt = 5\pi^{-1} \int_{0}^{1/2 \sqrt{5} + \sqrt{1/2 \sqrt{5}} - 1} \frac{dx}{x \sqrt{x^{-5} - 11 - x^5}}
$$

(24)
2) If we set \( v(\tau) = R(e^{-\pi \tau}) \), \( \tau = \sqrt{r} \), then

\[
-\pi \int_{+\infty}^{\tau} \eta(it/2)^4 dt = 5 \int_{0}^{\tau} \frac{dx}{x^5 - 11 - x^5}
\]

inverting \( v_\tau \) and derivating we get

\[
-\pi \eta \left( \frac{iv^{(-1)}(x)}{2} \right)^4 \frac{d}{dx} v^{(-1)}(x) = 5 \frac{1}{x^5 - 11 - x^5}
\]

Solving with respect to \( \frac{d}{dx} v^{(-1)}(x) \), we get

\[
\frac{d}{dx} v^{(-1)}(x) = \frac{-5\pi^{-1}}{\eta \left( \frac{iv^{(-1)}(x)}{2} \right)^4 x^5 - 11 - x^5}
\]

(25)

and as in Application 1 if we set \( x = -\frac{1 + \sqrt{5}}{2} + \sqrt{\frac{5 + \sqrt{5}}{2}} \), then the next evaluation is valid

\[
\frac{d}{dx} v^{(-1)} \left( \left( -\frac{1 + \sqrt{5}}{2} + \sqrt{\frac{5 + \sqrt{5}}{2}} \right) \right) =
\]

\[
= -5\pi^{-1} \eta (2i)^{-4} \sqrt{\frac{1}{10} \left( 1 + 3\sqrt{5} + 2\sqrt{10 + 2\sqrt{5}} \right)}
\]

Where the \( \eta (2i) \) can evaluated from (19) using the singular modulus \( k_4 = 3 - 2\sqrt{2} \) and the elliptic singular value \( K(k_4) \), in terms of algebraic numbers and values of the \( \Gamma \) function.

Let \( q = e^{-\pi \sqrt{r}} \) and \( F(x), m(x) \) be functions defined as

a)

\[
x = \int_{0}^{F(x)} \frac{dt}{t^5 - 11 - t^5} = -\frac{1}{5} \int_{+\infty}^{F(x)^{-5} - 11 - F(x)^5} \frac{dt}{t^{1/6} \sqrt{125 + 22t + t^2}} =
\]

\[
= \left( \frac{6}{5} h^{-1/6} F_1 \left[ \frac{1}{6}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{h}, \frac{11 - 2i}{h}, \frac{11 + 2i}{h} \right] \right)_{h=F(x)^{-5} - 11 - F(x)^5}
\]

(26)

and

b)

\[
x = \pi \int_{m(x)}^{+\infty} \frac{\eta (it/2)^4 dt}{m(x)}
\]

(27)

respectively.

Using the above functions
Theorem 4.
If \( q = e^{-\pi \sqrt{r}} \), \( r > 0 \) then
\[
R(q) = F \left( \frac{3}{3} \sqrt[2]{2k_r} \cdot 2F_1 \left[ \frac{1}{3}, \frac{1}{6}; \frac{7}{6}; k_r^2 \right] \right)
\]  \( (28) \)

Also the equation
\[
\int_0^y \frac{dt}{t\sqrt{t^5 - 11 - t^5}} = A
\]  \( (29) \)

have solution
\[
y = R \left( e^{-\pi \sqrt{m(a)}} \right)
\]  \( (30) \)

which also means that the function \( F \) have representation
\[
F(x) = R \left( e^{-\pi \sqrt{m(x)}} \right)
\]  \( (31) \)

and the solution of the equation \( m(x) = r \) is
\[
x = 3 \sqrt[3]{2k_r} \cdot 2F_1 \left[ \frac{1}{3}, \frac{1}{6}; \frac{7}{6}; k_r^2 \right]
\]  \( (32) \)

Also if the differential equation
\[
\frac{y'}{y^{\mu/6} (ay^2 + by + c)} = \frac{1}{x^{1/6} \sqrt{x^2 + 22x + 125}}
\]  \( (33) \)

have solution \( y = g(x) \) then \( g \) satisfies
\[
\int_0^{g(Y(q))} \frac{dx}{x^{\mu/6} \sqrt{ax^2 + bx + c}} = 3 \sqrt[3]{2k_r} \cdot 2F_1 \left[ \frac{1}{3}, \frac{1}{6}; \frac{7}{6}; k_r^2 \right]
\]

where \( Y(q) = R(q^2)^{-5} - 11 - R(q^2)^5 \). Hence we get the next

Proposition.
If \( g \) satisfies (33), then
\[
\int_0^{g \left( Y \left( e^{-\pi \sqrt{m(x)}} \right) \right)} \frac{dt}{t^{\mu/6} \sqrt{at^2 + bt + c}} = x
\]  \( (34) \)

3 Some Remarks on a sextic equation

We define the function \( G(x) \) as
\[
x = \frac{-1}{5} \int_{+\infty}^{G(x)} \frac{dt}{t^{1/6} \sqrt{125 + 22t + t^2}}
\]  \( (35) \)
then $G(x) = F(x)^{-5} - 11 - F(x)^5$, and the solution of the sextic equation (see [8]):

$$\frac{b^2}{20a} + bX + aX^2 = C_1X^{5/3}$$

is

$$X = X_r = \frac{b}{250a} (R(q^2)^{-5} - 11 - R(q^2)^5) = \frac{b}{250a} \cdot G \left( 3 \cdot 5^{-1/3} \sqrt[3]{2k_{4r}} \cdot {}_2F_1 \left[ \frac{1}{3}, \frac{1}{6}, \frac{7}{6}, k_{4r} \right] \right)$$

(37)

where $j_r = 250C_1^3a^{-2}b^{-1}$ is Klein’s $j$–invariant. But from

$$j_r = \frac{16 \left( 1 + 14k_{4r}^2 + k_{4r}^4 \right)^3}{k_{4r}^4(1 - k_{4r}^2)^4}$$

(38)

(observe that (38) is always solvable in radicals with respect to $k_{4r}$), we lead to the following

**Theorem 5.**

A solution of (36) is

$$X = \frac{b}{250a} G \left( 3 \cdot 5^{-1/3} \sqrt[3]{2k_{4r}} \cdot {}_2F_1 \left[ \frac{1}{3}, \frac{1}{6}, \frac{7}{6}, t \right] \right)$$

(39)

where the $t$ is given from

$$250C_1^3a^{-2}b^{-1} = \frac{16(1 + 14t + t^2)^3}{t(1-t)^4}.$$  

(40)

Relation (39) is always solvable with respect to $t$. If we make the change of variable

$$t = \left( \frac{1 - \sqrt{1 - x^2}}{1 + \sqrt{1 - x^2}} \right)^2,$$

then (40) becomes solvable in radicals with respect to $x$ and hence to $t$ also.

From Theorem 5 and the definition of $G(x)$ it is clear that

$$-\frac{1}{5} \int_{+\infty}^{250X_rab^{-1}} \frac{dx}{x^{1/6} \sqrt{125 + 22x + x^2}} = 5^{-1}4^{-1/3}B \left( b_r, \frac{1}{6}, \frac{2}{3} \right)$$

where

$$250C_1^3a^{-2}b^{-1} = \frac{16(1 + 14b_r + b_r^2)^3}{b_r(1 - b_r)^4}.$$  

Setting $a = 1, b = 250$ we get the next
Proposition 1.
We define the function $\beta^* = \beta^*_r$ to be root of the equation
\[
\sqrt{\frac{B(1-x, \frac{1}{6}, \frac{2}{3})}{B(x, \frac{1}{6}, \frac{2}{3})}} = \sqrt{r}
\] (41)

Then $x = \beta^*_r$ is algebraic when $r \in Q^*_+$ and further for any $b_r$
\[
B(b_r, \frac{1}{6}, \frac{2}{3}) = \sqrt{4} \int_{x_r}^{+\infty} \frac{dx}{x^{1/6} \sqrt{125 + 22x + x^2}}
\] (42)

with
\[
C_1 \equiv \frac{16 (1 + 14b_r + b_r^2)^3}{b_r (1 - b_r)^4}
\] (43)

and $X_r$ is solution of
\[
3125 + 250X_r + X_r^2 = C_1X_r^{5/3}.
\] (44)

Also if $r_1 = 4^{-1}k(-1)\left(\beta_r^{1/2}\right)$ then $\beta_r = b_{r_1} = k_{4r_1}^2$ and $C_1 = j_{r_1}^{1/3}$ and
\[
X_{r_1} = R\left(e^{-2\pi\sqrt{r_1}}\right)^{-5} - 11 - R\left(e^{-2\pi\sqrt{r_1}}\right)^5
\] (45)

Note. From the above Proposition we get also that for any $X > 0$ we can find $b = \theta(X)$, where $\theta(X)$ is an algebraic function of $X$ and can be calculated explicit from
\[
\int_{X}^{+\infty} \frac{dx}{x^{1/6} \sqrt{125 + 22x + x^2}} = \frac{1}{\sqrt{4}}B(b, \frac{1}{6}, \frac{2}{3})
\] (46)

Theorem 6.
If $F$ is arbitrary function and the algebraic singular equation
\[
\frac{F(1-x)}{F(x)} = \sqrt{r}
\] (47)

have root $x = \alpha_r$ then we can define $r_0$ as
\[
r_0 = K \left(\frac{\sqrt{1-\alpha_r}}{\sqrt{\alpha_r}}\right)^2
\] (48)

If the computated $r_0$ gives the value of $j_{r_0}$ in radicals then we can evaluate the value of $\alpha_r$ from the solvable equation
\[
\frac{256(x + (1-x)^2)^3}{x^2(1-x)^2} = j_{r_0}.
\] (49)
More precisely one algebraic solution will be $x = \alpha_r$.

The above elementary theorem is for numerical purposes and is a complete change of base from $F$ to $K$ and the evaluation of $\alpha_r$ by the class invariant $j_{r_0}$. Note also that the theorem holds for every $F$ not necessary modular base.

**Theorem 7.**

If

$$X(r) = R \left( e^{-2\pi \sqrt{r}} \right)^{-5} - 11 - R \left( e^{-2\pi \sqrt{r}} \right)^5$$

then

$$X'(r) = -\pi \frac{\eta(i \sqrt{r})^4}{\sqrt{r}} X(r)^{1/6} \sqrt{125 + 22X(r)} + X(r)^2$$

**Proof.**

From Theorem 3 and definition (b) relation (27), we have that

$$m(r) = -\frac{2^{5/3}}{3} B \left( k_r^2, 1/6, 2/3 \right)$$

and

$$\frac{d}{dr} B \left( k_r^2, 1/6, 2/3 \right) = \pi \sqrt{\eta(i \sqrt{r}/2)^4}$$

But

$$-\frac{1}{5} \int_{-\infty}^{X(r/4)} \frac{dt}{t^{1/6} \sqrt{125 + 22t + t^2}} = \frac{1}{5} \sqrt{4} B \left( k_r^2, 1/6, 2/3 \right)$$

Differentiating we get the result.

3) Continuing the investigation of equation (36) we have if $X = X_r = Y(q)$, then

$$C_1 Y(q)^{5/3} = a Y(q)^2 + b Y(q) + b^2 / (20a)$$

hence, from (9) with

$$j_r = 250 C_1^3 a^{-2} b^{-1}$$

and

$$Y(q) = R(q^2)^{-5} - 11 - R(q^2)^5$$

then

$$\frac{d}{dx} \left( \frac{3x^{8/3}}{25000} F_1 \left[ \frac{8}{3}, 1, \frac{11}{3}, -x, 25(5 - 2\sqrt{5}) ; \frac{-x}{25(5 + 2\sqrt{5})} \right] \right)_{x = Y(q)} = j_r^{-1/3}$$

Hence if $j_r = j(q)$, then

$$\int_0^x j \left( Y^{(-1)}(w) \right) dw = \frac{3x^{8/3}}{25000} F_1 \left[ \frac{8}{3}, 1, \frac{11}{3}, -x, 25(5 - 2\sqrt{5}) ; \frac{-x}{25(5 + 2\sqrt{5})} \right]_{Y(q)} (50)$$
But as we will see in Section 3 below, we have \(\nu = 1\) and 
\[ P_n(\nu, x) = \sum_{k=0}^{n} x^k, \]
for all numbers \(n\). Hence if 
\[ \phi(x) = _2F_1 \left[ 1, \frac{8}{3}; \frac{11}{3}; x \right] \]
then 
\[ \int_{0}^{x} j \left( Y^{(-1)}(w) \right)^{1/3} dw = \]
\[ = \frac{3x^{8/3}}{25000(x - 1)} \left[ -\phi \left( \frac{5 + 2\sqrt{5}}{125} x \right) + x\phi \left( \frac{5 + 2\sqrt{5}}{125} x^2 \right) \right]. \]  
(51)

The function \(\phi\) can be evaluated using elementary functions. For \(x = 1\) we get the next special value 
\[ \int_{0}^{1} j \left( Y^{(-1)}(w) \right)^{1/3} dw = \frac{100}{719} \left( 60 + \sqrt{5} \right) - 5 \cdot _2F_1 \left[ 1, \frac{8}{3}; \frac{11}{125}; \frac{1}{2} \left( 5 + 2\sqrt{5} \right) \right] \]

4 Modular equations and singular values of Beta functions

From relations (8) and (9) we have
\[ \int \frac{x^\mu}{(ax^2 + bx + c)^\nu} dx = c^{-\nu} x^{\mu+1} \sum_{n=0}^{\infty} \frac{(\nu)_n}{(\mu + 2)_n} \frac{1}{\mu + 1} \sum_{l=0}^{\infty} \frac{(\nu)_l (\nu)_n - l}{l! (n - l)!} x^n \]
If we denote \(P_n(x)\) as
\[ P_n(\nu, x) = _2F_1 \left[ -n, \nu; 1 - n - \nu; x \right] \]  
(52)
then
\[ \sum_{l=0}^{\infty} \frac{(\nu)_l (\nu)_n - l}{l! (n - l)!} \frac{1}{\rho_2^{n-l}} = \rho_1^{-n} (\nu)_n \rho_2^n \sum_{\lambda=0}^{\nu} \frac{(\nu)_{\lambda}}{(1 - n - \nu)_{\lambda}} = \rho_1^{-n} (\nu)_n \rho_2^n \]
hence if \(C_{n,k} = \text{Binomial}[n, k] = \binom{n}{k}\), and
\[ \int_{0}^{x} t^\mu \frac{1}{(at^2 + bt + c)^\nu} dt = c^{-\nu} x^{\mu+1} \sum_{n=0}^{\infty} \frac{(\nu)_n}{(n + \mu + 1)\rho_1^n} \frac{1}{n!} x^n = \]
\[ = c^{-\nu} x^{\mu+1} \sum_{n=0}^{\infty} \frac{(\nu)_n}{(n + \mu + 1)\rho_1^n} \left[ \sum_{\lambda=0}^{\nu} \frac{C_{n,\lambda}(\nu)_\lambda}{(1 - n - \nu)_{\lambda}} \left( \frac{-\rho_1}{\rho_2} \right)^\lambda \right] x^n \]  
(53)
From the above we obtain the next Ramanujan-type (like π) formula

**Theorem 8.**
Let \( \rho_1, \rho_2 \) be the roots of \( ax^2 + bx + c = 0 \), then

\[
\int_0^x \frac{t^n}{(at^2 + bt + c)^\nu} \, dt = c^{-\nu} x^{\mu+1} \sum_{n=0}^{\infty} \left[ \sum_{\lambda=0}^{n} \frac{C_{n,\lambda} \cdot (\nu)_{\lambda} \left( -\frac{\rho_1}{\rho_2} \right)^\lambda}{(1-n-\nu)_{\lambda} n!} \right] \frac{\nu}{n!} \frac{(x/\rho_1)^n}{n + \mu + 1}
\]

\( (54) \)

**Examples.**
1) For \( m = 0, \nu = 1/2, \) and \( a = b = c = 1 \), we get

\[
\log \left( 1 + \frac{2}{\sqrt{3}} \right) = \sum_{n=0}^{\infty} \left[ \sum_{\lambda=0}^{n} \frac{n! \cdot C_{n,\lambda} \cdot \left( \frac{1}{2} \right)_{\lambda} \left( 1 - i\sqrt{3} \right)^\lambda}{\left( \frac{1}{2} - n \right)_{\lambda} n!} \right] \frac{\left( \frac{1}{2} \right)_n (\sqrt{3} - i)^n}{(n+1)!}
\]

\( (55) \)

2) Let now \( \mu = -1/2, \nu = 1/2, a = 1 \) and \( \rho_1 = -\rho_2 = p, x = p^2 \). Then

\[
\frac{1}{p} \int_0^{p^2} \frac{dt}{\sqrt{t(1-t/p)(1+t/p)}} = 2 \cdot \frac{\Gamma \left( \frac{1}{4} \right)}{\Gamma \left( \frac{1}{2} \right)} \frac{\Gamma \left( \frac{1}{4} \right)}{\Gamma \left( \frac{1}{2} \right)} = \frac{1}{p} \int_0^{p^2} \frac{dt}{\sqrt{t(1-t/p)(1+t/p)}} = \frac{\sqrt{p}}{2} \frac{\left( \frac{1}{4} \right)_n \left( \frac{1}{2} \right)_n \left( \frac{5}{4} \right)_n \left( \frac{1}{2} \right)_n}{\left( \frac{1}{2} \right)_n \left( \frac{1}{2} \right)_n}
\]

\( (56) \)

**Conjecture.**
The functions \( \sqrt{B(z, \alpha, \beta)} \), where \( \{\alpha, \beta\} \in \{\{\frac{1}{2}, \frac{1}{2}\}, \{\frac{1}{2}, \frac{3}{2}\}, \{\frac{1}{2}, \frac{1}{4}\}, \{\frac{1}{2}, \frac{1}{6}\}, \{\frac{1}{2}, \frac{1}{6}\}, \{\frac{1}{2}, \frac{1}{6}\}\} \) are modular bases.

These are modular bases which I manage to find with a quick view. I suppose that exist more of them. Also these bases are represented from Gauss hypergeometric functions. I think that these bases are not like the cubic the 4th the fifth modular bases which related with the complete elliptic integral of the first kind \( K \) (see [6]).

**Example 3.**
Consider the equation

\[
\frac{\sqrt{B \left( \frac{1}{2} - \beta, \frac{1}{6}, \frac{1}{6} \right)}}{\sqrt{B \left( \frac{1}{6}, \frac{1}{6}, \frac{1}{6} \right)}} = \sqrt{r}
\]
then some solutions extracted with numerical methods of Mathematica are

\[
\begin{align*}
\beta_2 &= \frac{1}{4} \left( 2 - \sqrt{3} \right) \\
\beta_3 &= \frac{1}{4} \left( 2 - \sqrt{3 \left( 3 - \sqrt{3} \right)} \right) \\
\beta_{3/2} &= \frac{1}{8} \left( 4 - \sqrt{-9 + 9\sqrt{5} - 3\sqrt{150 - 66\sqrt{5}}} \right) \\
\beta_4 &= \frac{1}{8} \left( 4 - \sqrt{-9 + 9\sqrt{5} + 3\sqrt{150 - 66\sqrt{5}}} \right) \\
\beta_5 &= \frac{1 + 3 \cdot 2^{1/3} - 3 \cdot 2^{2/3}}{8 + 4 \cdot \sqrt{3 \cdot (1 - 2^{1/3} + 2^{2/3})}}
\end{align*}
\]

and have the property

\[
\frac{B(\beta_n^r, \frac{1}{\sigma}, \frac{1}{\sigma})}{B(\beta_r, \frac{1}{\sigma}, \frac{1}{\sigma})} = \text{rational}, \quad (57)
\]

when \( n, r \in \mathbb{N} \).

Consider the function

**Definition.**

\[
B_\alpha(x) := \sqrt{B(x, \alpha, \alpha)} = \sqrt{\int_0^x (t - t^2)^{\alpha - 1} dt}, \quad \text{where } 0 < \alpha < 1 \text{ and } x > 0
\]

Then by changing of variable \( t \to 1 - w \), we have

\[
B_\alpha^2(x) + B_\alpha^2(1 - x) = \int_0^1 (w(1 - w))^{\alpha - 1} dw = \frac{\Gamma(\alpha)^2}{\Gamma(2\alpha)}
\]

Setting now \( x = \beta_r \), where

\[
\frac{B_\alpha(1 - \beta_r)}{B_\alpha(\beta_r)} = \sqrt{r} \quad (58)
\]

we conclude that

\[
B_\alpha(\beta_r) = \sqrt{\frac{\Gamma(\alpha)^2}{\Gamma(2\alpha)(r + 1)}} \quad (59)
\]
From Theorem 8 with \( a = 1, b = -2, c = 1, \mu = -5/6, \nu = 5/12 \) and \( r = 5, x = \beta_5 \), we have the next formula for the constant \( B(\beta_5, \frac{1}{6}, \frac{1}{6}) = \frac{\Gamma(\frac{1}{6})}{\sqrt[6]{6} \Gamma(\frac{1}{3})} \):

\[
\frac{\Gamma(\frac{1}{6})^2}{6\Gamma(\frac{1}{4})} = \sum_{n=0}^{\infty} \left( \sum_{\lambda=0}^{n} (-1)^\lambda C_{n,\lambda} \left( \frac{5}{12} \right)_\lambda \right) \frac{5}{12} n (\beta_5)^{n+1/6} \tag{60}
\]

In general holds the following very interesting theorem

**Theorem 9.**
If \( \beta_r \) is solution of (58) then

\[
\frac{\Gamma(\alpha)^2}{(r+1)\Gamma(2\alpha)} = \sum_{n=0}^{\infty} \left( \sum_{\lambda=0}^{n} (-1)^\lambda C_{n,\lambda} \left( \frac{1-\alpha}{2} \right)_\lambda \right) \frac{1-\alpha}{2} n \beta_r^{n+\alpha} \tag{61}
\]

**Example 4.**
Let

\[
\psi(x) := \sqrt{2 \arcsin(x)}
\]

Then the equation

\[
\frac{\psi(1-s_r)}{\psi(s_r)} = \sqrt{r}, \ r \in \mathbb{R}^+
\]

is equivalent to

\[
2i + y^{-1} - y - y^{-r} - y^r = 0 \tag{63}
\]

with \( y = is + \sqrt{1-s^2}, s = s_r \).

**Proof.**
Equation (62) is written as

\[
\arcsin(1-x) = r \arcsin(x) \text{ or } \sin(r^{-1} \arcsin(1-x)) = x
\]

which is similar with Tchebychev polynomials \( T_r(x) = \cos(r \arccos(x)), r > 0. \)

Using the identity

\[
(cos(x) + i \sin(x))^r = \sum_{n=0}^{\infty} C_{r,n} i^n \sin(x)^n \left( \sqrt{1-\sin(x)^2} \right)^{r-n} =
\]

\[
= \cos(nx) + i \sin(nx)
\]

and setting \( \sin(x) = w \) one can arrive to

\[
\left( \sqrt{1-w^2} - iw \right)^r - \left( \sqrt{1-w^2} + iw \right)^r + 2i(1-w) = 0 \tag{64}
\]

which is (63).
Set now $\xi = \frac{1}{2}(-i - \sqrt{3})$ and for every $r > 0$

$$x_r = 2^{-1/r} \left( \frac{1 - 2i\xi - \xi^2 + \sqrt{1 - 4i\xi - 2\xi^2 + 4i\xi^3 + \xi^4}}{\xi} \right)^{\frac{1}{r}}$$

If

$$x = \frac{-i(1 - x^2)}{2x_r},$$

and solve the equation $x = \sin(t)$ with respect to $t$ (i.e. $t = \text{arcsin}(x)$). Finally we get that $t = \sin\left(\frac{\pi}{4 + 2r}\right)$ is solution of

$$\frac{\psi(1 - 2t^2)}{\psi(t)} = \sqrt{r}$$

Another interesting note is for the function

$$m(R) = -\frac{1}{4}e^{-\frac{i\pi}{4R}} \left( -1 + e^{\frac{i\pi}{4R}} \right)^2 = \sin\left( \frac{\pi}{2(R + 1)} \right)^2$$

hold the following

**Example 5.**

Let

$$\psi^{*}(x) = \sqrt{B_{1/2}\left( x, \frac{1}{2}, \frac{1}{2} \right)} = \sqrt{\text{arcsin} \left( x^{1/2} \right)}$$

then $m(R)$ is the solution of

$$\frac{\psi^{*}(1 - x)}{\psi^{*}(x)} = \sqrt{R}, \ R \in \mathbb{R}_{+}^{*}$$

and holds the following modular equation

$$m(R + 1) = \frac{1 - \sqrt{1 - m\left( \frac{R}{2} \right)}}{2}$$

**Example 6.**

If $m(R)$ is given form (67), then

$$\frac{\pi}{R + 1} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{1}{2})_n}{n! \ (n + 1/2)} m(R)^{n+1/2}$$

**Proof.**

$$B \left( m(R), \frac{1}{2}, \frac{1}{2} \right) = \frac{\pi}{R + 1} = \sum_{n=0}^{\infty} \frac{(-1)^{n} C_{n,n} \left( \frac{1}{2} \right)_{n}}{(\frac{1}{4} - n)_{n}} \frac{(\frac{1}{4})_n m(R)^{n+1/2}}{n! (n + 1/2)}$$
The above is a Ramanujan-type $\pi$ formula of arbitrary large precision since any higher order values of $m(R)$ can always evaluated from (65) in radicals. Moreover is

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_{-n} m(R)^{n+1/2}}{\left(\frac{1}{2}\right)_{-n} n!} \cdot n + 1/2 = 2 \arcsin \left(\sqrt{m(R)}\right)$$

**Note.** Formulas similar to (67) can given if we consider the expansion of

$$\sin(nt) = P_n \left(\sqrt{1 - \sin(t)^2}, \sin(t)\right)$$

For example with $n = 6$ we get

$$p_6(y) = P_6(x, y) = 6x^5y - 20x^3y^3 + 6xy^5, \quad x = \sqrt{1 - y^2}$$

and for this $n$ the equation $p_6(t) = P_6(\sqrt{1 - t^2}, t) = a$ is solvable. Hence one can get a formula

$$t = \sin \left(\frac{\theta}{6}\right) = p_6^{-1} (\sin(\theta))$$

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