Photon helicity and quantum anomalies in curved spacetimes

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Abstract We compare several definitions of photon helicity (magnetic, electric, electromagnetic) present in literature. In curved spacetime quantum anomalies can spoil helicity conservation inducing - according to these definitions - different effects on photons: either rotation of linear polarization or production of circular polarization. We derive the Noether current associated with duality transformations starting from manifestly invariant Lagrangians in Minkowski spacetime.

Keywords Helicity · Duality · Quantum Anomalies.

1 Introduction

In absence of charges and currents Maxwell equations are invariant under rotations of electric and magnetic fields into each others (electromagnetic duality). This invariance is associated with the conservation of polarization properties of electromagnetic waves during propagation in free space. Already in the Sixties Lipkin [1] and Calkin [2] discussed the physical implications of this symmetry: for a free electromagnetic field the difference between the number of right and left circularly polarized photons is constant in time.
Quantum effects may induce violation of helicity conservation for photons propagating in curved spacetimes. After the first paper on this topic by Dolgov, Khriplovich and Zakharov [3] several publications appeared in the Eighties [4, 5, 6, 7, 8]. They discussed possible observational consequences of electromagnetism quantization in curved spacetimes focusing on the difference between the number of left-handed and right-handed photons [4] and on the rotation of the plane of linear polarization [8]. More recently several papers [9, 10, 11, 12, 13, 14] studied the breaking of the classical symmetry focusing the production of circular polarization in curved spacetimes.

Section II is dedicated to the comparison of the various definitions present in literature for the photon helicity: magnetic, electric and electromagnetic. We discuss the time dependence of the helicity densities for photons propagating in free space and the connection with the Stokes parameters (for a plane monochromatic wave and for the superposition of two waves of different frequencies). In Section III we examine the consequences of quantum anomalies in curved spacetimes: observable effects depend whether we focus on magnetic or electromagnetic helicity. If magnetic helicity is not conserved the main effect is a rotation of the plane on linear polarization. On the other side if electromagnetic helicity varies during propagation the degree of circular polarization is modified. We discuss in detail this two different effects in the case of the Kerr metric. In Section IV we confront and contrast various derivations of the Noether current associated with duality transformations starting from manifestly invariant Lagrangians. We conclude in Section V.

2 Maxwell equations and photon helicity

2.1 Minkowski spacetime

In absence of sources the Maxwell equations in a non conductive medium are [15]:

\[ \nabla \cdot B = 0, \quad \nabla \times E + \frac{\partial B}{\partial t} = 0, \quad (1) \]

\[ \nabla \cdot D = 0, \quad \nabla \times H - \frac{\partial D}{\partial t} = 0, \quad (2) \]

where electric field \(E\) and magnetic field \(H\) are related to electric induction \(D\) and magnetic induction \(B\) by the constitutive relations of the medium (e.g. in vacuum \(D = \epsilon_0 E\) and \(B = \mu_0 H\)).

We have decided, following [16, 17, 18, 19, 20], to introduce two potentials \(A\) and \(C\) relative, respectively, to magnetic induction \(B\) and electric induction \(D\).
In terms of the magnetic potential $A$:  
\[ B = \nabla \times A, \quad (3) \]

using Coulomb gauge condition $\nabla \cdot A = 0$, the scalar part of the potential is set equal to zero, therefore:
\[ E = \frac{\partial A}{\partial t}. \quad (5) \]

In terms of the electric potential $C$:  
\[ D = -\nabla \times C, \quad (4) \]

using Coulomb gauge condition $\nabla \cdot C = 0$, the scalar part of the potential is set equal to zero, therefore:
\[ H = -\frac{\partial C}{\partial t}. \quad (6) \]

The covariant formulation is easily obtained once we have introduced the magnetic and electric potential four vectors $A^\alpha = (0, A)$ and $C^\alpha = (0, C)$. We define the differential two-form $F$, the electromagnetic field strength, as $F \equiv dA$ ($A$ being the four-dimensional 1-form associate to $A$). Analogously, we introduce the differential two-form $G$ as $G \equiv dC$ ($C$ being a 1-form associated to $C$). This choice is done since the duality transformations has to be implemented on the true dynamical variables cfr. [16]. If one performs the duality transformations on the fields, derivatives of potentials, will incur in some technical problems (see Section IV). Therefore we obtain, following [20]:

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix}, \quad (7) \]

\[ G_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu = \begin{pmatrix} 0 & -H_x/c & -H_y/c & -H_z/c \\ H_x/c & 0 & -D_z & D_y \\ H_y/c & D_z & 0 & -D_x \\ H_z/c & -D_y & D_x & 0 \end{pmatrix}, \quad (8) \]

where $c \equiv 1/\sqrt{\mu_0\epsilon_0}$. The Hodge dual is defined as $^*F_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$, here $\epsilon_{\mu\nu\rho\sigma} \equiv \sqrt{-|g|} \epsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita symbol and $[\cdots]$ guarantees complete anti-symmetrization in the four indexes [22, pp. 88.97] [23, p. 174]:

\[ ^*F_{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z/c & -E_y/c \\ -B_y & -E_z/c & 0 & E_x/c \\ -B_z & E_y/c & -E_x/c & 0 \end{pmatrix}, \quad (9) \]

\[ ^*G_{\mu\nu} = \begin{pmatrix} 0 & D_x & D_y & D_z \\ -D_x & 0 & H_z/c & -H_y/c \\ -D_y & -H_z/c & 0 & H_x/c \\ -D_z & H_y/c & -H_x/c & 0 \end{pmatrix}. \quad (10) \]

If we assume propagation in in vacuum ($D = \epsilon_0 E$ and $B = \mu_0 H$) there is a simple relation between $^*F_{\mu\nu}$ and $G_{\mu\nu}$ ($^*G_{\mu\nu}$ and $F_{\mu\nu}$):
\[ *F_{\mu\nu} = -\sqrt{\frac{\mu_0}{\epsilon_0}} G_{\mu\nu}, \quad (11) \quad \]
\[ *G_{\mu\nu} = \sqrt{\frac{\epsilon_0}{\mu_0}} F_{\mu\nu}, \quad (12) \]

The following invariants can be explicitly evaluated:
\[ F_{\mu\nu} F^{\mu\nu} = -*F_{\mu\nu} *F^{\mu\nu} = 2 \left( B^2 - \frac{E^2}{c^2} \right), \quad (13) \]
\[ G_{\mu\nu} G^{\mu\nu} = -*G_{\mu\nu} *G^{\mu\nu} = 2 \left( D^2 - \frac{H^2}{c^2} \right), \quad (15) \]
\[ G_{\mu\nu} *G^{\mu\nu} = -4 D \cdot \frac{H}{c}. \quad (16) \]

The Maxwell equations in terms of the potential are:

once we have introduced the potential \( \mathbf{A} \) the first two Maxwell equations (Eqs. 1) are identically verified:
\[ dF = d(dA) = 0, \]

The other two equations (Eqs. 2) are:
\[ \partial_\mu F^{\mu\nu} = 0. \]

or \( d^*F = 0 \) using differential forms. Corresponding to the Lagrangian density:
\[ \mathcal{L}_F = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu}. \]

In analogy with with fluid mechanics and plasma physics different gauge invariants pseudoscalar quantities can be introduced. They describe the linking numbers of the magnetic and electric lines \[ 24 \], the knottedness of the vortex lines \[ 17,25,26 \]. For example, in fluid dynamics, if \( \mathbf{v} \) is a vector field describing the fluid flow velocity, \( \nabla \times \mathbf{v} \) is the vorticity, and \( \mathbf{v} \cdot (\nabla \times \mathbf{v}) \) describes the knottedness of the vortex lines. For the electromagnetic field the quantities more frequently used are:

- **Magnetic helicity** is defined using the vector potential \( \mathbf{A} \):
\[ \mathcal{H}_{\text{mag}} = \int_{\mathbb{R}^3} \mathbf{h}^0_{\text{mag}} d^3\mathbf{x} \]
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\[ \frac{1}{2} \sqrt{\frac{\epsilon_0}{\mu_0}} \int_{\mathbb{R}^3} A \cdot (\nabla \times A) \, d^3x = \frac{1}{2} \sqrt{\frac{\epsilon_0}{\mu_0}} \int_{\mathbb{R}^3} A^i * F_0^i \, d^3x. \quad (17) \]

- Electric helicity is defined in terms of the vector potential \( \mathbf{C} \):

\[ \mathcal{H}_{el} \equiv \int_{\mathbb{R}^3} h_0^0 d^3x = \frac{1}{2} \sqrt{\frac{\mu_0}{\epsilon_0}} \int_{\mathbb{R}^3} \mathbf{C} \cdot (\nabla \times \mathbf{C}) \, d^3x = \frac{1}{2} \sqrt{\frac{\mu_0}{\epsilon_0}} \int_{\mathbb{R}^3} C^i * G_0^i \, d^3x. \quad (18) \]

- Summing these two terms we obtain the definition of the electromagnetic helicity:

\[ \mathcal{H}_{em} \equiv \int_{\mathbb{R}^3} (h_{mag}^0 + h_{el}^0) \, d^3x \]

\[ = \frac{1}{2} \int_{\mathbb{R}^3} \left[ \sqrt{\frac{\epsilon_0}{\mu_0}} \mathbf{A} \cdot (\nabla \times \mathbf{A}) + \sqrt{\frac{\mu_0}{\epsilon_0}} \mathbf{C} \cdot (\nabla \times \mathbf{C}) \right] \, d^3x \]

\[ = \frac{1}{2} \int_{\mathbb{R}^3} \left( \frac{\epsilon_0}{\mu_0} A^i * F_0^i + C^i * F_0^i \right) \, d^3x \]

\[ = \frac{1}{2} \int_{\mathbb{R}^3} \left( -A^i G_{0i} + \sqrt{\frac{\mu_0}{\epsilon_0}} C^i * G_0^i \right) \, d^3x \]

\[ = \frac{1}{2} \int_{\mathbb{R}^3} \left( A^i G_{0i} - C^i F_0^i \right) \, d^3x. \quad (19) \]

\[ = \frac{1}{2} \int_{\mathbb{R}^3} \left( -A^i G_{0i} + \sqrt{\frac{\mu_0}{\epsilon_0}} C^i * G_0^i \right) \, d^3x \]

\[ = \frac{1}{2} \int_{\mathbb{R}^3} \left( A^i G_{0i} - C^i F_0^i \right) \, d^3x \]

\[ = \frac{1}{2} \int_{\mathbb{R}^3} \left( A^i G_{0i} - C^i F_0^i \right) \, d^3x. \quad (20) \]

In order to clarify the physical meaning of these quantities we compare them to the Stokes parameters for some simple electromagnetic waves. For a plane monochromatic wave propagating in \( z \) direction with electric field:

\[ E = \left( E_+ e^{i \delta_+} \frac{x + i y}{\sqrt{2}} + E_- e^{i \delta_-} \frac{x - i y}{\sqrt{2}} \right) \exp(ikz - \omega t), \quad (24) \]

here \( k = \sqrt{\mu_0 \epsilon_0 \omega} \). We can easily evaluate the Stokes parameters, describing the polarization properties: \[15\] [Sect 7.2] [27][p.348]:

\[ I \equiv \langle E_+^* (t) E_+ (t) \rangle + \langle E_-^* (t) E_- (t) \rangle = E_+^2 + E_-^2, \quad (25) \]

\[ Q \equiv \langle E_+^* (t) E_+ (t) \rangle - \langle E_-^* (t) E_- (t) \rangle = 2E_+ E_- \cos (\delta_- - \delta_+), \quad (26) \]

\[ U \equiv \langle E_+^* (t) E_- (t) \rangle + \langle E_-^* (t) E_+ (t) \rangle = 2E_+ E_- \sin (\delta_- - \delta_+), \quad (27) \]

\[ V \equiv -i \left( \langle E_+^* (t) E_+ (t) \rangle - \langle E_-^* (t) E_- (t) \rangle \right) = E_+^2 - E_-^2, \quad (28) \]

the plane of linear polarization has a constant orientation angle:

\[ \alpha = \frac{1}{2} \arctan \frac{U}{Q} = \frac{\delta_- - \delta_+}{2}. \quad (29) \]
Starting from the definition of the electric field we derive:

\[
A = \frac{i}{\omega} \left( E_+ e^{i\delta_x} \frac{x + iy}{\sqrt{2}} + E_- e^{i\delta_x} \frac{x - iy}{\sqrt{2}} \right) \exp\left(i k z - i \omega t \right),
\]

(30)

\[
B = \frac{i k}{\omega} \left( E_+ e^{i\delta_x} \frac{x + iy}{\sqrt{2}} - E_- e^{i\delta_x} \frac{x - iy}{\sqrt{2}} \right) \exp\left(i k z - i \omega t \right),
\]

(31)

\[
C = -\frac{k}{\mu_0 \omega^2} \left( E_+ e^{i\delta_x} \frac{x + iy}{\sqrt{2}} - E_- e^{i\delta_x} \frac{x - iy}{\sqrt{2}} \right) \exp\left(i k z - i \omega t \right),
\]

(32)

and evaluate the helicity densities:

\[
h_{\text{mag}} = \frac{1}{2} \frac{\epsilon_0}{\sqrt{\mu_0 \epsilon_0}} \text{Re}[A] \cdot \text{Re}[B] = \frac{\epsilon_0 (E_+^2 - E_-^2)}{4 \omega} = \frac{\epsilon_0}{4 \omega} V,
\]

(33)

\[
h_{\text{el}} = \frac{1}{2} \frac{\mu_0}{\epsilon_0} \text{Re}[C] \cdot \text{Re}[D] = \frac{\mu_0}{\epsilon_0} (E_+^2 - E_-^2) = \frac{\mu_0}{4 \omega} V,
\]

(34)

\[
h_{\text{an}} = \frac{1}{2} \left( \frac{\epsilon_0}{\mu_0} \text{Re}[A] \cdot \text{Re}[B] - \frac{\mu_0}{\epsilon_0} \text{Re}[C] \cdot \text{Re}[D] \right)
\]

\[
= \frac{\epsilon_0}{\mu_0} (E_+^2 - E_-^2) = \frac{\epsilon_0}{2 \omega} V.
\]

(35)

Note that, in this particular case, there is a simple relation connecting the Stokes parameter \(V\) describing circular polarization and helicity densities.

For more general optical fields there is not direct proportionality between helicity density and the Stokes parameter \(V\) [28]. If we consider the superposition of two monochromatic waves with opposite circular polarization and different frequencies \((\omega_1 \neq \omega_2)\) propagating in \(z\) direction with electric field:

\[
E = E_0 \exp\left(i k_1 z - i \omega_1 t \right) \frac{x + iy}{\sqrt{2}} + E_0 \exp\left(i k_2 z - i \omega_2 t \right) \frac{x - iy}{\sqrt{2}},
\]

(36)

here \(k_{1,2} = \sqrt{\mu_0 \epsilon_0 \omega_{1,2}}\). The Stokes parameters are:

\[
I = 2 E_0^2,
\]

(37)

\[
Q = 2 E_0^2 \cos\left(\Delta \omega t - \Delta \omega z/v\right),
\]

(38)

\[
U = -2 E_0^2 \sin\left(\Delta \omega t - \Delta \omega z/v\right),
\]

(39)

\[
V = 0,
\]

(40)

with \(\Delta \omega \equiv (\omega_2 - \omega_1) / 2\); the plane of linear polarization slowly rotates of an angle \(\alpha\) with frequency \(\Delta \omega\):

\[
\alpha = \frac{1}{2} \arctan \left( \frac{U}{Q} \right) = -\Delta \omega \left( t - \frac{z}{c} \right).
\]

(41)

Once we have evaluated:

\[
A = -i E_0 \left( \frac{\exp\left(i k_1 z - i \omega_1 t \right) x + iy}{\omega_1 \sqrt{2}} + \frac{\exp\left(i k_2 z - i \omega_2 t \right) x - iy}{\omega_2 \sqrt{2}} \right),
\]

(42)
we obtain these expressions for helicities [25, 29]:

\[ h_{\text{mag}} = \frac{\epsilon_0 E_0^2}{\mu_0} \left( \frac{1}{\omega_1} - \frac{1}{\omega_2} \right) \left[ 1 + \cos \left( (k_1 + k_2)z - (\omega_1 + \omega_2)t \right) \right] , \] (45)

\[ h_{\text{el}} = \frac{\epsilon_0 E_0^2}{\mu_0} \left( \frac{1}{\omega_1} - \frac{1}{\omega_2} \right) \left[ 1 + \cos \left( (k_1 + k_2)z - (\omega_1 + \omega_2)t \right) \right] , \] (46)

\[ h_{\text{em}} = \frac{\epsilon_0 E_0^2}{\mu_0} \left( \frac{1}{\omega_1} - \frac{1}{\omega_2} \right) \cos^2 \left( \frac{k_1 + k_2}{2} z - \frac{\omega_1 + \omega_2}{2} t \right) , \] (47)

and considering the average over time of electromagnetic helicity:

\[ \langle h_{\text{em}} \rangle = \frac{\epsilon_0 E_0^2}{2} \left( \frac{1}{\omega_1} - \frac{1}{\omega_2} \right) . \] (49)

In this case, superposition of two monochromatic waves, electromagnetic helicity density is not related to the Stokes parameter \( V \).

Time evolution of the helicity densities is easily obtained from the Maxwell equations [30, 31]:

\[ \frac{\partial h_{\text{mag}}^0}{\partial t} + \nabla \cdot \left( \sqrt{\epsilon_0 \mu_0} E \times A \right) = -2 \sqrt{\epsilon_0 \mu_0} E \cdot B , \] (50)

\[ \frac{\partial h_{\text{el}}^0}{\partial t} + \nabla \cdot \left( \sqrt{\mu_0 / \epsilon_0} H \times C \right) = 2 \sqrt{\mu_0 / \epsilon_0} H \cdot D = 2 \sqrt{\epsilon_0 \mu_0} E \cdot B , \] (51)

\[ \frac{\partial h_{\text{em}}^0}{\partial t} + \nabla \cdot \left( \sqrt{\epsilon_0 \mu_0} E \times A + \sqrt{\mu_0 / \epsilon_0} H \times C \right) = 0 . \] (52)

Integrating over tridimensional space:

\[ \frac{d H_{\text{mag}}}{dt} = -2 \sqrt{\epsilon_0 / \mu_0} \int_{\mathbb{R}^3} E \cdot B d^3 x , \] (53)

\[ \frac{d H_{\text{el}}}{dt} = 2 \sqrt{\mu_0 / \epsilon_0} \int_{\mathbb{R}^3} H \cdot D d^3 x = 2 \sqrt{\epsilon_0 / \mu_0} \int_{\mathbb{R}^3} E \cdot B d^3 x , \] (54)

\[ \frac{d H_{\text{em}}}{dt} = 0 . \] (55)

we note that \( H_{\text{em}} \) is constant in time, while \( H_{\text{mag}} \) and \( H_{\text{el}} \) are not conserved, in general.

Helicities are explicitly defined in terms of the vector potentials, however it was shown in [17] that only gauge-invariant transverse pieces (\( A^\perp \) and \( C^\perp \))
contribute when integration over all space is performed, therefore the integrated quantities are gauge invariant. Conservation of electromagnetic helicity has a clear physical interpretation even in quantum theory: the number of right circularly polarized photons minus the number of left circularly polarized photons is conserved \[2,24\].

2.2 Curved spacetime

Electrodynamics in curved spacetime described by the metric $g_{\mu\nu}$ can be obtained replacing ordinary derivatives $\partial_{\mu}$ by covariant derivatives $\nabla_{\mu}$. The two tensors are now defined as:

\[
F_{\mu\nu} = \nabla_{\mu} A_{\nu} - \nabla_{\nu} A_{\mu}.
\]

The Lagrangian density:

\[
\mathcal{L}_F = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu},
\]

induces the following equations:

\[
\nabla_{\mu} F^{\mu\nu} = 0.
\]

\[
G_{\mu\nu} = \nabla_{\mu} C_{\nu} - \nabla_{\nu} C_{\mu}.
\]

The Lagrangian density:

\[
\mathcal{L}_G = -\frac{1}{4\epsilon_0} G_{\mu\nu} G^{\mu\nu},
\]

induces the following equations:

\[
\nabla_{\mu} C^{\mu\nu} = 0.
\]

Helicity densities definitions are unchanged, but the definition of the Hodge dual contains now the determinant of the metric $g$ and in order to obtain total helicity we have to integrate on a curved volume element:

\[
H \equiv \int_{\Sigma^3} h^0 d^3r.
\]

where $d^3r$ is the volume element in curved tridimensional space.

3 Quantum anomalies: helicity non-conservation in curved spacetimes

In the Eighties - after the first study of Dolgov, Khriplovich, and Zakharov [3] - several papers [4,5,6,7,8] tried to estimate the effects of electromagnetic field quantization in curved spacetimes. They focused in particular on magnetic helicity $h_{\text{mag}}$, see Eq. (17), deriving a relation between vacuum expectation value $\langle \nabla_{\mu} h_{\text{mag}}^{\mu} \rangle$ and $R_{\alpha\beta\mu\nu} \ast R^{\alpha\beta\mu\nu}$ the Chern-Pontryagin invariant (or Hirzebruch signature); where $R_{\alpha\beta\mu\nu}$ is the Riemann tensor and $\ast R^{\alpha\beta\mu\nu}$ its dual.

More recently Agullo et al. [9,10,11,12,13,14] focused on electromagnetic helicity $h_{\text{em}}$, see Eq. (23). As we remembered in the previous Section, $h_{\text{em}}^{\mu}$ is a classical conserved current ($\nabla_{\mu} h_{\text{em}}^{\mu} = 0$), unlike $h_{\text{mag}}^{\mu}$ and $h_{\text{em}}^{\mu}$ which are not
conserved. They showed that the vacuum expectation value for $\nabla_\mu h^\mu_{\text{em}}$ could be different from zero if the theory is quantized in curved metric [12]:

$$\langle \nabla_\mu h^\mu_{\text{em}} \rangle = -\frac{\hbar}{96\pi^2} R^\alpha_\alpha R^\beta_\beta.$$  \hspace{1cm} (59)

Therefore total electromagnetic helicity $H_{\text{em}}$ is not constant anymore for photons propagating in a spacetime with a nonzero Chern-Pontryagin invariant:

$$\frac{d}{dt} \langle H_{\text{em}} \rangle = -\frac{\hbar}{96\pi^2} \int_{\Sigma^3} R^\alpha_\alpha R^\beta_\beta d^3r.$$ \hspace{1cm} (60)

Helicity variation over a finite interval of time is proportional to:

$$\Delta \langle H_{\text{em}} \rangle \propto \int_{t_1}^{t_2} \int_{\Sigma^3} R^\alpha_\alpha R^\beta_\beta \sqrt{-g} d^4x.$$ \hspace{1cm} (61)

The term “photon chiral anomaly” is used when the classical symmetry is not conserved due to the quantization of the electromagnetic field in curved spacetime.

In the following part of this Section we show that observable effects are different whether we focus on magnetic or electromagnetic helicity non-conservation.

### 3.1 Non-conservation of magnetic helicity

Several papers [6,8,32,33] estimated the observable effects of non-conservation of magnetic helicity. We have already shown that is not conserved even at classical level, see Eq. (53):

$$\frac{dH_{\text{mag}}}{dt} \propto -\int_{\mathbb{R}^3} E \cdot B \, d^3r \propto \int_{\mathbb{R}^3} F_{\mu\nu} \ast F_{\mu\nu} d^4r.$$ \hspace{1cm} (62)

Therefore the quantization in curved spacetime induces a term $\langle F_{\mu\nu} \ast F_{\mu\nu} \rangle \propto -R^\alpha_\alpha R^\beta_\beta$. When light propagates in a region with $R^\alpha_\alpha R^\beta_\beta \neq 0$ polarization is modified since $F_{\mu\nu} \ast F_{\mu\nu}$ acquires a nonzero vacuum expectation value. The effects can be estimated, following [6,7,8], introducing a “conventional pseudoscalar field” $\phi(x)$ coupled to photons via the term:

$$\mathcal{L}_\phi = \frac{1}{2} g_{\phi} F_{\mu\nu} \ast F^{\mu\nu}.$$ \hspace{1cm} (63)

In this case the main effect on the propagation of photons is the rotation of the plane of linear polarization (birefringence), where the angle is proportional to the variation of the pseudoscalar field $\phi(x)$ along the line of sight [34,35,36]. Also the degree of circular polarization may vary, but the effect is subdominant [37,38,39].

The evolution of $\phi(x)$ is obtained from the equation of motion:

$$\nabla_\mu \nabla^\mu \phi(x) = \frac{g_{\phi}}{4} F_{\mu\nu} \ast F^{\mu\nu} \propto -R^\alpha_\alpha R^\beta_\beta.$$ \hspace{1cm} (64)
once we have specified the metric and evaluated $R_{\alpha\beta\mu\nu} \ast R^{\alpha\beta\mu\nu}$.

If we consider, for example, the Kerr Metric [40]:

$$ds^2 = -\left(1 - \frac{2Gmr}{r^2 + a^2 \cos^2 \theta}\right) dt^2 + \frac{4Gmra}{r^2 + a^2 \cos^2 \theta} dt d\phi$$

$$+ \frac{r^2 + a^2 \cos^2 \theta}{r^2 - 2Gmr + a^2} dr^2 + (r^2 + a^2 \cos^2 \theta) d\theta^2$$

$$+ \left(\frac{r^2 + a^2}{r^2 + a^2 \cos^2 \theta}\right) \sin^2 \theta d\phi^2,$$

(65)

where $a \equiv \frac{J}{m}$ is angular momentum per unit mass, we can evaluate:

$$^*R \cdot R = -1536aG^2m^2 \cos \theta \left[\frac{r^5}{(r^2 + a^2 \cos^2 \theta)^5} - \frac{r^3}{(r^2 + a^2 \cos^2 \theta)^3}\right]$$

$$+ \frac{3r}{16 (r^2 + a^2 \cos^2 \theta)^4},$$

(66)

in agreement with [40] [p. 356], and we find the following solution of Eq. (64):

$$\phi(r, \theta) = \frac{5}{384} g_\phi \frac{a \cos \theta}{4\pi^2 (Gm)^2} \left[\frac{4(Gm)^2}{r^2} + \frac{8(Gm)^3}{r^3} + \frac{72(Gm)^4}{5r^4}\right]$$

$$= \frac{5}{384\pi^2} \frac{g_\phi ha \cos \theta}{Gm} + O\left(\frac{1}{r^4}\right),$$

(67)

see also Eq. (12) of [8].

The amount of rotation of linear polarization depends from the physical properties of the Kerr black hole and is in general extremely small. Moreover this effect must be compared with the classical General Relativity effect due to photon propagation in a curved spacetime: gravitational Faraday rotation (or Skrotskii effect), see [41,42]. An estimate of this classical effect is given in Eq. (31) of [43]:

$$\Omega_{\text{Kerr}}^{\text{SK}} = -\frac{\pi G^2 m^2 a \cos \theta}{4 c^8 r_{\text{min}}^4}.$$

(68)

3.2 Non-conservation of electromagnetic helicity

Quantization of electromagnetic field in curved spacetime can spoil electromagnetic helicity conservation (photon chiral anomaly). Here the observable effects of the anomaly - see [10,11,12,13] and in particular [14] - are directly connected to the variation of electromagnetic helicity, see Eq. (61):

$$\langle \mathcal{H}_{\text{em}}(t_1) \rangle - \langle \mathcal{H}_{\text{em}}(t_2) \rangle \propto \int_{t_1}^{t_2} \int_{\Sigma^3} R_{\alpha\beta\mu\nu} \ast R^{\alpha\beta\mu\nu} \sqrt{-g} d^4x.$$

(69)
If the integral in the right term is different from zero, then $H_{em}$ is not conserved. The difference between the numbers of right circularly polarized photons and left circularly polarized photons changes: the degree of circular polarization is not conserved. The observable effect associated with this formulation of the quantum anomaly is not related with a change in linear polarization angle, but with a variation of the degree of circular polarization.

For the particular case of the Kerr metric, discussed in the previous subsection, we have:

$$\int R_{\alpha\beta\mu\nu} \star R^{\alpha\beta\mu\nu} \sqrt{-g} d^4x \propto \int_0^\pi \cos \theta \sin \theta \left[ r^2 + a^2 \frac{1 + \cos(2\theta)}{2} \right] d\theta = 0. \quad (70)$$

Therefore, since the integral over all space is zero - due to symmetry reasons - in this case there are no observable effects related to the quantum anomaly.

In order to have a non zero effect a metric with a nonzero Chern-Pontryagin integrated term should be considered. In del Rio et al. [14] some estimates are numerically derived for non-stationary spacetimes.

4 Dual symmetry and photon helicity density

In order to clarify the helicity definition that should be used to study quantum anomalies we compare various derivations of the Noether current associated to invariance under duality transformations. Several authors pointed out that transformations has to be implemented on the true dynamical variables cfr. [16,21]. Moreover there is a further argument against duality transformations implemented at the level of the fields. In fact vector fields and pseudovectors will me mixed, in this way, generating inconsistencies as well [44]. Therefore we define the transformations at the level of the potentials [20]:

$$A \rightarrow A \cos \theta + \sqrt{\frac{\mu}{\epsilon}} C \sin \theta, \quad C \rightarrow C \cos \theta - \sqrt{\frac{\epsilon}{\mu}} A \sin \theta, \quad (71)$$

and for the tensors $F_{\mu\nu}$ and $G_{\mu\nu}$:

$$F_{\mu\nu} \rightarrow F_{\mu\nu} \cos \theta + \sqrt{\frac{\mu}{\epsilon}} G_{\mu\nu} \sin \theta, \quad G_{\mu\nu} \rightarrow G_{\mu\nu} \cos \theta - \sqrt{\frac{\epsilon}{\mu}} F_{\mu\nu} \sin \theta, \quad (72)$$

Taking the time derivative and the curl of Eqs. (71) we obtain the well known duality transformations [15, p. 274]:

$$E \rightarrow E \cos \theta + \sqrt{\frac{\mu}{\epsilon}} H \sin \theta, \quad H \rightarrow H \cos \theta - \sqrt{\frac{\epsilon}{\mu}} E \sin \theta, \quad (73)$$

$$D \rightarrow D \cos \theta + \sqrt{\frac{\mu}{\epsilon}} B \sin \theta, \quad B \rightarrow B \cos \theta - \sqrt{\frac{\epsilon}{\mu}} D \sin \theta. \quad (74)$$
Maxwell equations, Eqs. (1-2), are manifestly invariant under such transformations for a real angle $\theta$. In vacuum ($D = \epsilon_0 E$ and $B = \mu_0 H$):

$$
E \to E \cos \theta + \frac{B}{\sqrt{\mu_0 \epsilon_0}} \sin \theta, \quad B \to B \cos \theta - \frac{E}{\sqrt{\mu_0 \epsilon_0}} \sin \theta
$$

(75)

$$
F_{\mu\nu} \to F_{\mu\nu} \cos \theta - *F_{\mu\nu} \sin \theta, \quad G_{\mu\nu} \to G_{\mu\nu} \cos \theta + *G_{\mu\nu} \sin \theta.
$$

(76)

Note that for $\theta = \pi/2$ they simply exchange electric and magnetic fields:

$$
E \to B \sqrt{\mu_0 \epsilon_0}, \quad B \to -\sqrt{\mu_0 \epsilon_0} E,
$$

(77)

or a field with its dual:

$$
F_{\mu\nu} \to -*F_{\mu\nu}, \quad G_{\mu\nu} \to *G_{\mu\nu}.
$$

(78)

It was well known, already at the end of Nineteenth century (Heaviside-Larmor symmetry), that in free space Maxwell equations are invariant under an exchange of $E$ and $B$. Later the properties under duality transformations were studied also at the level of the Lagrangian densities, Eqs. (56-57) [2,16,45]:

$$
L_F \to L_F \cos^2 \theta + L_G \sin^2 \theta - \frac{1}{2\sqrt{\mu_0 \epsilon_0}} F_{\mu\nu} G^{\mu\nu} \sin \theta \cos \theta
$$

$$
= L_F \cos (2\theta) + \frac{1}{4\mu_0} F_{\mu\nu} *F^{\mu\nu} \sin (2\theta)
$$

(79)

$$
L_G \to L_G \cos^2 \theta + L_F \sin^2 \theta + \frac{1}{2\sqrt{\mu_0 \epsilon_0}} F_{\mu\nu} G^{\mu\nu} \sin \theta \cos \theta
$$

$$
= L_G \cos (2\theta) + \frac{1}{4\epsilon_0} G_{\mu\nu} *G^{\mu\nu} \sin (2\theta)
$$

(80)

where we have used the relations $*F_{\mu\nu} = -\sqrt{\mu_0 / \epsilon_0} G_{\mu\nu}, *G_{\mu\nu} = \sqrt{\epsilon_0 / \mu_0} F_{\mu\nu}$, Eqs. (11-12), and the Bianchi identities. Even if the Maxwell equations are invariant under duality transformations, $L_F$ and $L_G$ are not manifestly invariant under transformations, see Eq. (76) [16,46]. If we consider an infinitesimal transformation:

$$
L_F \to L_F + \frac{1}{\mu_0} \nabla_\mu (A_\nu *F^{\mu\nu}) \theta,
$$

(81)

$$
L_G \to L_G + \frac{1}{\epsilon_0} \nabla_\mu (C_\nu *G^{\mu\nu}) \theta,
$$

(82)

the Lagrangians change by a total derivative term. Sometimes it is speculated that dual invariance of the Maxwell theory should implies also the current conservation [17 p. 24][49 p. 24]:

$$
\nabla_\mu (A_\nu *F^{\mu\nu}) \equiv 0 \implies \nabla_0 h_{mag}^0 + \nabla_i h_{mag}^i = 0,
$$

(83)

$$
\nabla_\mu (C_\nu *G^{\mu\nu}) \equiv 0 \implies \nabla_0 h_{el}^0 + \nabla_i h_{el}^i = 0,
$$

(84)
and therefore $\mathcal{H}_{mag}$ and $\mathcal{H}_{el}$ should be constant in time. But we have already seen in Section 2 that $\mathcal{H}_{mag}$ and $\mathcal{H}_{el}$ are not conserved in general, Eqs. (53-54). The dual-asymmetric definitions for helicity are not satisfactory in the general case [46, p. 6]. Even if we remember that the Noether’s theorem can be applied also if the action is changed by a surface term [48, p. 18] we obtain these expression for the conserved currents:

$$h_\mu^F = \partial \frac{\mathcal{L}_F}{\nabla_\mu A_\nu} \delta A_\nu - \frac{1}{\mu_0} A_\nu \ast F^\mu\nu$$
$$= - \frac{F^{\mu\nu}}{\mu_0} \frac{\mu_0}{\epsilon_0} C_\nu - \frac{1}{\mu_0} A_\nu \ast F^\mu\nu = \frac{-1}{\sqrt{\mu_0 \epsilon_0}} \left( F^\mu\nu C_\nu + \frac{\epsilon_0}{\mu_0} A_\nu \ast F^\mu\nu \right)$$

$$h_\mu^G = \partial \frac{\mathcal{L}_G}{\nabla_\mu C_\nu} \delta C_\nu - \frac{1}{\epsilon_0} C_\nu \ast G^\mu\nu$$
$$= \frac{G^{\mu\nu}}{\epsilon_0} \frac{\epsilon_0}{\mu_0} A_\nu - \frac{1}{\epsilon_0} C_\nu \ast G^\mu\nu = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \left( G^\mu\nu A_\nu - \frac{\mu_0}{\epsilon_0} C_\nu \ast G^\mu\nu \right)$$

If we restrict ourselves to Minkowski spacetime the time components are:

$$h_0^F = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \left( F_{0i} C^i + \frac{\epsilon_0}{\mu_0} A_0 \ast F_{0i} \right) = 2 h_0^{em},$$

$$h_0^G = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \left( -G_{0i} A^i + \frac{\mu_0}{\epsilon_0} C^i \ast G_{0i} \right) = 2 h_0^{em}.$$
Eq. (11), but to use the method of method of Lagrange multipliers and add to the Lagrangian a new term invariant under duality transformations:

\[ \mathcal{L}_{\text{mult}} \equiv \mathcal{L}_{\text{dual}} + \lambda \left( \frac{\mu_0}{\epsilon_0} G_{\mu\nu}^* G^{\mu\nu} + F_{\mu\nu}^* F^{\mu\nu} \right) \rightarrow \mathcal{L}_{\text{mult}} . \]  

(90)

The condition \( \partial \mathcal{L}_{\text{mult}} / \partial \lambda = 0 \) corresponds to the equation:

\[ \frac{\mu_0}{\epsilon_0} G_{\mu\nu}^* G^{\mu\nu} = -F_{\mu\nu}^* F^{\mu\nu} , \]  

(91)

which is verified if:

\[ \sqrt{\frac{\mu_0}{\epsilon_0}} G_{\mu\nu} = \mp F_{\mu\nu}^* , \text{ and its dual: } \sqrt{\frac{\mu_0}{\epsilon_0}} G_{\mu\nu}^* = \pm F_{\mu\nu} . \]  

(92)

If we consider the covariant derivatives:

\[ \sqrt{\frac{\mu_0}{\epsilon_0}} \nabla_\mu G_{\mu\nu} = \mp \nabla_\mu F_{\mu\nu} \implies \nabla_\mu G^{\mu\nu} = 0 , \]  

(93)

\[ \sqrt{\frac{\mu_0}{\epsilon_0}} \nabla_\mu G_{\mu\nu}^* = \pm \nabla_\mu F_{\mu\nu} \implies \nabla_\mu F^{\mu\nu} = 0 , \]  

(94)

where we have used the Bianchi identities. Imposing the condition \( \partial \mathcal{L}_{\text{mult}} / \partial \lambda = 0 \) corresponds to verify equations of motion (on-shell solutions). If we now apply the Noether’s theorem to this Lagrangian (90) we obtain the correct expression for electromagnetic helicity density \( h_{\text{em}}^0 \), Eq. (23). The problem here is that Eq. (91) has (92) as one solution, which is not unique. Therefore the equations of motion derived from (90) contain Maxwell electromagnetism as solution, but, in general, their solution space can be bigger than electromagnetism.

In Lagrangian formalism, the more formal approach to derive electromagnetic helicity is to consider the Lagrangian introduced by Pasti, Sorokin, and Tonin [51,52,53]:

\[ \mathcal{L}_{\text{PST}} \equiv \mathcal{L}_{\text{dual}} - \frac{1}{4} \partial_{\alpha} \partial^{\alpha} a F^{\mu\rho} \rightarrow \mathcal{L}_{\text{PST}} , \]  

(95)

where \( a(x) \) is an auxiliary scalar field and \( F^{\mu\rho} \) is:

\[ F^{\mu\rho} \equiv G^{\mu\nu} G_{\nu\rho} + G^{\mu\nu}^* F_{\nu\rho} + G^{\mu\nu}^* G_{\nu\rho} + G^{\mu\nu} F_{\nu\rho} + G^{\mu\nu} G_{\nu\rho} + G^{\mu\nu} F_{\nu\rho} . \]  

(96)

This Lagrangian is manifestly Lorentz covariant, and \( \mathcal{L}_{\text{PST}} \rightarrow \mathcal{L}_{\text{PST}} \) under transformations of Eq. (72) - invariance under duality transformations. The Euler-Lagrangian equations associated to \( \mathcal{L}_{\text{PST}} \) are [54]:

\[ \partial_\mu F^{\mu\nu} = 0 , \text{ and } : \partial_\mu G^{\mu\nu} = 0 , \]  

(97)

\footnote{The authors are deeply grateful to Dmitri Sorokin for a very helpful discussion on this topic.}
where $F^{\mu\nu}$ and $G^{\mu\nu}$ are not independent, but related by the conditions contained in the Lagrangian:

$$G_{\mu\nu} + \ast F_{\mu\nu} = 0, \quad \text{and} \quad F_{\mu\nu} - \ast G_{\mu\nu} = 0,$$

(98)
corresponding to the relations derived in Eq. (11). Note that in discussing $\mathcal{L}_{\text{PST}}$ we assume $\mu_0 = \epsilon_0 = c = 1$ and restrict ourselves to Minkowski spacetime.

Under duality transformations:

$$\delta \mathcal{L}_{\text{PST}} = \frac{\partial \mathcal{L}_{\text{PST}}}{\partial (\partial_\mu A_\nu)} \delta (\partial_\mu A_\nu) + \frac{\partial \mathcal{L}_{\text{PST}}}{\partial (\partial_\mu C_\nu)} \delta (\partial_\mu C_\nu)$$

$$= \partial_\mu \left[ -\frac{1}{2} F^{\mu\nu} \delta A_\nu - \frac{1}{2} G^{\mu\nu} \delta C_\nu \right],$$

(99)
where we have used the two Euler-Lagrange equations, and the relations Eq. (98). Since the Lagrangian $\mathcal{L}_{\text{PST}}$ is invariant under duality transformations, $\delta \mathcal{L}_{\text{PST}} = 0$, therefore we have:

$$\partial_\mu \left[ -\frac{1}{2} F^{\mu\nu} \delta A_\nu - \frac{1}{2} G^{\mu\nu} \delta C_\nu \right] = 0.$$

(100)
The electromagnetic helicity density is obtained considering the time component and the definition of duality transformations, see Eq. (71):

$$h^{0\text{em}} = -\frac{1}{2} F^{0\nu} \delta A_\nu - \frac{1}{2} G^{0\nu} \delta C_\nu$$

(101)
$$= -\frac{1}{2} \eta^{0\sigma} F_{\sigma\nu} \delta A^\nu - \frac{1}{2} \eta^{0\sigma} G_{\sigma\nu} \delta C^\nu$$

(102)
$$= \frac{1}{2} \left( -\eta^{0\sigma} F_{\sigma\nu} C^\nu + \eta^{0\sigma} G_{\sigma\nu} A^\nu \right)$$

(103)
$$= \frac{1}{2} \left( -F^{0\nu} C_\nu + G^{0\nu} A_\nu \right).$$

(104)
It coincides with the expression for the electromagnetic helicity density defined in Section II, see the integrand of Eq. (23). Since $\partial_0 h^0 + \partial_i h^i = 0$ we easily verify, integrating over the volume, that total electromagnetic helicity is conserved:

$$\frac{\partial}{\partial t} \int_{\Sigma^3} h^{0\text{em}} d^3 r = 0,$$

(105)
$h^{0\text{em}}$ is the Noether’s charge density associated to $\mathcal{L}_{\text{PST}}$ under duality transformations. It can be easily generalized to curved spacetimes replacing ordinary derivatives $\partial_\mu$ with covariant derivatives $\nabla_\mu$ and considering a general metric $g_{\mu\nu}$. An example of Noether’s charge density calculation associated to manifestly duality invariant symmetry for a curved spacetime can be found in [12].
5 Conclusions

We started this work discussing different definitions of helicity present in literature. Following the advice in [16], and the more recent refs. [17,18,19,20], we introduced two independent potential vectors $A^\mu$ and $C^\mu$, respectively for the magnetic induction $B$ and the electric induction $D$, in order to have a Lagrangian function manifestly duality invariant.

The relation with the Stokes parameters and their time evolution were derived starting from the Maxwell equations. We considered two kinds of violation of helicity conservation: magnetic helicity and electromagnetic helicity. The former is associated with rotation of linear polarization, the latter with circular polarization. Electromagnetic helicity has to be considered when we focus on photon chiral anomaly. In Section IV, we described several invariant Lagrangians of the electromagnetic field. These Lagrangians provide the same conserved current associated with the duality transformations. We highlighted only $L_{\text{PST}}$ reproduces Maxwell equations. Its relative density charge is the electromagnetic helicity. Therefore, quantum anomalies are associated with non-conservation of circular polarization [9,10,11,12,13,14], rather than with rotation of linear polarization [6,7,8]. Moreover we notice that even if $R_{\alpha\beta\gamma\delta}^* R_{\alpha\beta\gamma\delta} \neq 0$ locally, but the total space integral $\int_{\Sigma} R_{\alpha\beta\gamma\delta}^* R_{\alpha\beta\gamma\delta} d^3r$ is zero, no measurable effect can be detected as for the Kerr spacetime. In order to produce a non-null effect, we have to consider spacetimes with a non-trivial Pontryagin invariant: $\int_{\Sigma} R_{\alpha\beta\gamma\delta}^* R_{\alpha\beta\gamma\delta} d^3r \neq 0$; this request excludes mirror symmetric spacetimes [14]. The Chern-Pontryagin class is usually non-zero when it is evaluated over certain gravitational instantons, which happens to be solutions of the Euclidean Einstein’s equations only. One way to consider more physical cases, than gravitational instantons, is to study transient phenomena as highlighted in [14].

A future project could be to investigate the electromagnetic field in a chiral medium and the helicity conservation in it. The main point would be to go beyond, both in flat and curved space, the vacuum relations between the electric field induction and the electric field as well as the magnetic induction field and the magnetic field.

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References

1. D. M. Lipkin, “Existence of a New Conservation Law in Electromagnetic Theory,” Journal of Mathematical Physics, 5, 696 (1964).
2. M. G. Calkin, “An Invariance Property of the Free Electromagnetic Field,” American Journal of Physics, 33, 958 (1965).
3. A. D. Dolgov, I. B. Khriplovich and V. I. Zakharov, “Chiral Boson Anomaly in a Gravitational Field,” JETP Lett. 45, 651 (1987) [Pisma Zh. Eksp. Teor. Fiz. 45, 511 (1987)].
4. A. D. Dolgov, I. B. Khriplovich, A. I. Vainshtein and V. I. Zakharov, “Photonic Chiral Current and Its Anomaly in a Gravitational Field,” Nucl. Phys. B 315, 138 (1989).
5. A. I. Vainshtein, A. D. Dolgov, V. I. Zakharov and I. B. Khriplovich, “Chiral Photon Current And Its Anomaly In A Gravitational Field,” Sov. Phys. JETP 67, 1326 (1988) [Zh. Eksp. Teor. Fiz. 94, 54 (1988)].
6. A. D. Dolgov, I. B. Khriplovich and V. I. Zakharov, “Macroscopic Manifestations Of The Chiral Anomaly In Gravitational Field,” Sov. Phys. JETP 67, 237 (1988) [Nucl. Phys. B 309, 591 (1988)] [Zh. Eksp. Teor. Fiz. 94, 45 (1988)].
7. M. Reuter, “The Chiral Anomaly Of Antisymmetric Tensor Fields,” Phys. Rev. D 37, 1456 (1988).
8. M. Reuter, “A Mechanism generating axion hair for Kerr black holes,” Class. Quant. Grav. 9, 751 (1992).
9. I. Agullo, J. Navarro-Salas and A. Landete, “Electric-magnetic duality and renormalization in curved spacetimes,” Phys. Rev. D 90, no.12, 124067 (2014) arXiv:1409.6406 [gr-qc].
10. I. Agullo, A. del Rio and J. Navarro-Salas, “Electromagnetic duality anomaly in curved spacetimes,” Phys. Rev. Lett. 118, no. 11, 111301 (2017) arXiv:1607.08879 [gr-qc].
11. I. Agullo, A. del Rio and J. Navarro-Salas, “Gravity and handedness of photons,” Int. J. Mod. Phys. D 26, 1742001 (2017) arXiv:1705.07082 [gr-qc].
12. I. Agullo, A. del Rio and J. Navarro-Salas, “Classical and quantum aspects of electromagnetic duality rotations in curved spacetimes,” Phys. Rev. D 98, no. 12, 125001 (2018) arXiv:1810.08085 [gr-qc].
13. I. Agullo, A. del Rio and J. Navarro-Salas, “On the Electric-Magnetic Duality Symmetry: Quantum Anomaly, Optical Helicity, and Particle Creation,” Symmetry 10, no. 12, 763 (2018) arXiv:1812.08211 [gr-qc].
14. A. del Rio, N. Sanchis-Gual, V. Mewes, I. Agullo, J. A. Font and J. Navarro-Salas, Phys. Rev. Lett. 124, no.21, 211301 (2020) arXiv:2002.01593 [gr-qc].
15. J. D. Jackson, “Classical Electrodynamics,” Wiley 1998.
16. S. Deser and C. Teitelboim, “Duality Transformations of Abelian and Nonabelian Gauge Fields,” Phys. Rev. D 13, 1592-1597 (1976).
17. S. M. Barnett, R. P. Cameron, and A. M. Yao, “Duplex symmetry and its relation to the conservation of optical helicity,” Phys. Rev. A 86 (2012), 013845 doi:10.1103/PhysRevA.86.013845.
18. R. P. Cameron, S. M. Barnett and A. M. Yao, “Optical helicity, optical spin and related quantities in electromagnetic theory,” New J. Phys. 14, 053050 (2012).
19. R. P. Cameron, “On the ‘second potential’ in electrodynamics,” Journal of Optics, 16 (2014), 015708.
20. R. P. Cameron and S. M. Barnett, “Electric-magnetic symmetry and Noether’s theorem,” Journal of Optics, 14 (2012), 123019.
21. S. Deser “Black hole electromagnetic duality,” AIP Conf. Proc. 400, 1, 437-445 (1997), arXiv:9701157[gr-qc].
22. C. W. Misner, K. S. Thorne and J. A. Wheeler, “Gravitation,” San Francisco 1973.
23. B. A. Dubrovin, A. T. Fomenko and S. P. Novikov, “Modern Geometry — Methods and Applications,” Springer 1973.
24. J. L. Trueba and A. F. Rañada, “The electromagnetic helicity,” European Journal of Physics, 17 (1996), 141.
25. F. Crimin, N. Mackinson, J. B. Goette and S. M. Barnett, “Optical helicity and chirality: conservation and sources,” Applied Sciences, 9 (2019), 828.
26. L. V. Poulikakos, J. A. Dionne and A. García-Etxarri, “Optical Helicity and Optical Chirality in Free Space and in the Presence of Matter,” Symmetry, 11 (2019), 1113.
27. L.A. Mandel and E. Wolf, “Optical Coherence and Quantum Optics,” Cambridge University Press 1995.
28. R. P. Cameron, S. M. Barnett and A. M. Yao, “Optical helicity of interfering waves,” Journal of Modern Optics, 61 (2014), 25.
29. N. Mackinson, “On the differences between helicity and chirality,” Journal of Optics, 21 (2019), 125402.
30. M. Elbistan, C. Duval, P. Horvathy and P. M. Zhang, “Duality and helicity: a symplectic viewpoint,” Phys. Lett. B 761, 265-268 (2016) arXiv:1608.01131 [math-ph].
31. M. Elbistan, “Optical helicity and Hertz vectors,” Phys. Lett. A 382, 1897-1902 (2018) arXiv:1802.10485 [physics.optics].
32. B. A. Campbell, M. J. Duncan, N. Kaloper and K. A. Olive, “Axion hair for Kerr black holes,” Phys. Lett. B 251, 34 (1990).
33. M. J. Duncan, N. Kaloper and K. A. Olive, “Axion hair and dynamical torsion from anomalies,” Nucl. Phys. B 387, 215 (1992).
34. S. M. Carroll, G. B. Field and R. Jackiw, “Limits on a Lorentz and Parity Violating Modification of Electrodynamics,” Phys. Rev. D 41, 1231 (1990).
35. S. M. Carroll and G. B. Field, “The Einstein equivalence principle and the polarization of radio galaxies,” Phys. Rev. D 43, 3789 (1991).
36. D. Harari and P. Sikivie, “Effects of a Nambu-Goldstone boson on the polarization of radio galaxies and the cosmic microwave background,” Phys. Lett. B 289, 67 (1992).
37. D. S. Lee and K. W. Ng, “Photon production of axionic cold dark matter,” Phys. Rev. D 61, 085003 (2000) [arXiv:hep-ph/9909282 [hep-ph]].
38. F. Finelli and M. Galaverni, “Rotation of Linear Polarization Plane and Circular Polarization from Cosmological Pseudo-Scalar Fields,” Phys. Rev. D 79, 063002 (2009) [arXiv:0802.3210 [astro-ph]].
39. S. Alexander, E. McDonough, A. Pullen and B. Shapiro, “Physics Beyond The Standard Model with Circular Polarization in the CMB and CMB-21cm Cross-Correlation,” JCAP 01, 032 (2020) [arXiv:1911.01418 [astro-ph.CO]].
40. I. Ciufolini, J. A. Wheeler, “Gravitation and Inertia,” Princeton University Press, Princeton (1995).
41. T. Piran and P. Safier, “A gravitational analogue of Faraday rotation,” Nature 318, 271 (1985).
42. M. Sereno, “Gravitational Faraday rotation in a weak gravitational field,” Phys. Rev. D 69, 087501 (2004) [astro-ph/0401295].
43. M. Sereno, “Detecting gravito-magnetism with rotation of polarization by a gravitational lens,” Mon. Not. Roy. Astron. Soc. 356, 381 (2005) [astro-ph/0410015].
44. P. Aschieri, S. Ferrara and S. Zumino, “Duality Rotations in Nonlinear Electrodynamics and in Extended Supergravity” Riv. Nuovo Cim. 31, 625-709 (2006) [arXiv:0807.4509 [hep-th]].
45. S. Deser, “Off-shell electromagnetic duality invariance,” J. Phys. A: Math. Gen. 15, 1053 (1982).
46. K. Y. Bliokh, A. Y. Beksheev and F. Nori, “Dual electromagnetism: Helicity, spin, momentum, and angular momentum,” New J. Phys. 15, 033026 (2013).
47. Y. M. Shnir, “Magnetic Monopoles,” Springer 2005.
48. M. E. Peskin and D. V. Schroeder, “An Introduction to quantum field theory”.
49. A. B. S, S. Mandal and S. Banerjee, “Characteristics of interaction between Gravitons and Photons,” [arXiv:2001.10196 [gr-qc]].
50. J. Bernabéu and J. Navarro-Salas, “A Non-Local Action for Electrodynamics: Duality Symmetry and the Aharonov-Bohm Effect, Revisited,” Symmetry 11, no.10, 1191 (2019).
51. P. Pasti, D. P. Sorokin and M. Tonin, “Duality symmetric actions with manifest space-time symmetries,” Phys. Rev. D 52, 4277-4281 (1995) [arXiv:hep-th/9506109 [hep-th]].
52. P. Pasti, D. P. Sorokin and M. Tonin, “On Lorentz invariant actions for chiral p forms,” Phys. Rev. D 55, 6292-6298 (1997) [arXiv:hep-th/9611100 [hep-th]].
53. A. Mazzitella, C. R. Preitschopf and D. P. Sorokin, “Duality of selfdual actions,” Nucl. Phys. B 589, 438-452 (1999) [arXiv:hep-th/9805110 [hep-th]].
54. A. Manta, “Electromagnetic Duality and its Physical Implications,” 2020.