Non-Abelian Vortex in Four Dimensions as a Critical String on a Conifold

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1 Introduction

It is believed that confinement in QCD is due to formation of confining vortex strings.

*Nambu, Mandelstam and 't Hooft 1970’s:*  
**Confinement** is a dual Meissner effect upon condensation of monopoles.

Monopoles condense $\rightarrow$ electric Abrikosov-Nielsen-Olesen flux tubes are formed $\rightarrow$ electric charges are confined
Hadron spectrum is well described by linear Regge trajectories. However in all known examples the Regge trajectories show linear behavior only at asymptotically large spins.

Examples:

- Abrikosov-Nielsen-Olesen (ANO) vortex in weakly coupled Abelian-Higgs model
- Seiberg-Witten confinement in $\mathcal{N} = 2$ super-Yang-Mills theory

Length of the rotating string:

$$ L^2 \sim \frac{J}{T} $$

Transverse size of the string is given by the inverse mass of the bulk fields forming the string:

$$ m \sim g\sqrt{T} $$
String length $\gg$ its transverse size:

\[ mL \gg 1, \quad J \gg \frac{1}{g^2} \]

We expect
In the real world Regge trajectories are linear at $J \sim 1$

Can we find any example of 4D theory where confining string remains thin at $J \sim 1$?

Thin string condition:

$$T \ll m^2$$
2 Thin string regime

How the problem of ”thick” string is seen in the world sheet effective theory?

ANO string: Nambu-Goto action

\[ S_{NG} = T \int d^2 \sigma \left\{ \sqrt{h} + O \left( \frac{\partial^n}{m^n} \right) \right\} \]

where

\[ h = \text{det} (\partial_{\alpha} x^\mu \partial_{\beta} x_\mu) \]

*Polchinski-Strominger, 1991: Without higher derivative terms the world sheet theory is not UV complete*
Higher derivative terms at weak coupling, $g \ll 1$

$$O \left( \frac{\partial^n}{m^n} \right), \quad m \sim g\sqrt{T}$$

At $J \sim 1 \quad \partial \rightarrow \sqrt{T}$

Thus higher derivative terms

$$\rightarrow \left( \frac{T}{m^2} \right)^n$$

blow up at weak coupling!

*Polyakov:* string surface become ”crumpled”.
We want to find a regime in which the string remains thin. This means that the higher derivative corrections should be parametrically small. The low-energy world-sheet theory should be UV complete. This leads us to the following necessary conditions to have such a regime:

(i) The low-energy world-sheet theory on the string must be conformally invariant;
(ii) It must have the critical value of the Virasoro central charge.

- Bosonic string $D=26$
  - Superstring $D=10$
- ANO string in $D=4$ is not critical
Non-Abelian vortex strings

Non-Abelian strings were suggested in $\mathcal{N} = 2$ U(N) QCD:

- Hanany, Tong 2003
- Auzzi, Bolognesi, Evslin, Konishi, Yung 2003
- Shifman Yung 2004
- Hanany Tong 2004

$\mathbb{Z}_N$ Abelian string: Flux directed in the Cartan subalgebra, say for $SO(3) = SU(2)/\mathbb{Z}_2$

$$\text{flux} \sim \tau_3$$

Non-Abelian string: Orientational zero modes

Rotation of color flux inside SU(N).
Idea:

Non-Abelian string has more moduli than ANO string.

It has translational + orientational moduli

*Shifman and Yung, 2015:*

Non-Abelian vortex in $\mathcal{N} = 2$ supersymmetric QCD can behave as a critical superstring

We can fulfill the criticality condition: The solitonic non-Abelian vortex must have six orientational moduli, which, together with four translational moduli, will form a ten-dimensional space.
3 Non-Abelian vortex strings

For $U(N)$ gauge group in the bulk we have 2D $CP(N - 1)$ model on the string

$CP(N - 1) == U(1)$ gauge theory in the strong coupling limit

\[
S_{CP(N-1)} = \int d^2x \left\{ \left| \nabla_\alpha n^P \right|^2 + \frac{e^2}{2} \left( |n^P|^2 - 2\beta \right)^2 \right\},
\]

where $n^P$ are complex fields $P = 1, ..., N$,

Condition

\[
|n^P|^2 = 2\beta = \frac{4\pi}{g^2},
\]

imposed in the limit $e^2 \to \infty$
More flavors ⇒ semilocal non-Abelian string

The orientational moduli described by a complex vector \( n^P \) (here \( P = 1, \ldots, N \)),
\[
\tilde{N} = (N_f - N) \text{ size moduli are parametrized by a complex vector } \rho^K (K = N + 1, \ldots, N_f).
\]

The effective two-dimensional theory is sigma model with the target space \( O(-1)^{\oplus (N_f - N)}_{CP^1} \) (\( N = (2, 2) \) weighted CP model)

\[
S_{WCP} = \int d^2 x \left\{ |\nabla_\alpha n^P|^2 + |\tilde{\nabla}_\alpha \rho^K|^2 + \frac{e^2}{2} (|n^P|^2 - |\rho^K|^2 - 2\beta)^2 \right\},
\]

\( P = 1, \ldots, N, \quad K = N + 1, \ldots, N_f. \)

The fields \( n^P \) and \( \rho^K \) have charges +1 and −1 with respect to the auxiliary U(1) gauge field
\[
e^2 \to \infty
\]
4 From non-Abelian vortices to critical strings

String theory

\[
S = \frac{T}{2} \int d^2 \sigma \sqrt{h} h^{\alpha \beta} \partial_\alpha x^\mu \partial_\beta x_\mu \\
+ \int d^2 \sigma \sqrt{h} \left\{ h^{\alpha \beta} \left( \tilde{\nabla}_\alpha \tilde{n}_P \nabla_\beta n^P + \nabla_\alpha \tilde{\rho}_K \tilde{\nabla}_\beta \rho^K \right) \\
+ \frac{e^2}{2} \left( |n^P|^2 - |ho^K|^2 - 2\beta \right)^2 \right\} + \text{fermions},
\]

where \( h^{\alpha \beta} \) is the world sheet metric. It is independent variable in the Polyakov formulation.
What about necessary conditions for thin string?

- **Conformal invariance**

\[ b_{WCP} = N - \tilde{N} = 0 \Rightarrow N = \tilde{N}, \quad N_f = 2N \]

- **Critical dimension = 10**

Number of orientational + size degrees of freedom

\[ = 2(N + \tilde{N} - 1) = 2(2N - 1) \]

\[ 4 + 2(2N - 1) = 4 + 6 = 10, \quad \text{for } N = 2 \]

Our string is BPS so we have \( \mathcal{N} = (2, 2) \) supersymmetry on the world sheet.

For these values of \( N \) and \( \tilde{N} \) the target space of the weighted \( CP(2, 2) \) model is a non-compact Calabi-Yau manifold studied by Witten and Vafa, namely conifold.
Given that for non-Abelian vortex low energy world sheet theory is critical

Conjecture:

Thin string regime

\[ T \ll m^2 \]

is actually satisfied at strong coupling \( g_c^2 \sim 1 \).

\[ m(g) \to \infty, \quad g^2 \to g_c^2 \]

Higher derivative corrections can be ignored
Strings in the $U(N)$ theories are stable; they cannot be broken. Thus, we deal with the closed string.

For closed string moving on Calabi-Yau manifold $\mathcal{N} = (2, 2)$ world sheet supersymmetry ensures $\mathcal{N} = 2$ supersymmetry in 4D.

This is expected since we started with 4D QCD with $\mathcal{N} = 2$ supersymmetry.

Type IIB string is a chiral theory and breaks parity while Type IIA string theory is left-right symmetric and conserves parity.

Our bulk theory conserves parity $\Rightarrow$ we have Type IIA superstring
There is self-duality in 4D bulk theory

\[ \tau \to \tau_D = -\frac{1}{\tau}, \quad \tau = \frac{4\pi i}{g^2} + \frac{\theta_{4D}}{2\pi}, \]

We conjectured that the string becomes thin at \( g^2 \to g_c^2 \sim 1 \).

It is natural to expect that \( g_c^2 = 4\pi = \text{self-dual point} \).

\[ m^2 \to T \times \begin{cases} 
    g^2, & g^2 \ll 1 \\
    \infty, & g^2 \to 4\pi \\
    16\pi^2/g^2, & g^2 \gg 1
\end{cases} \]

In 2D theory on the string self-dual point is \( \beta = 0 \)

Conifold develops conical singularity.
5 4D Graviton

Our goal:

Study massless states of closed string propagating on

\[ R_4 \times Y_6, \quad Y_6 = \text{conifold} \]

and interpret them as hadrons in 4D \( \mathcal{N} = 2 \) QCD.

Massless 10D graviton

\[
\delta G_{\mu\nu} = \delta g_{\mu\nu}(x) g_6(y), \quad \delta G_{\mu i} = B_\mu(x) g_i(y), \quad \delta G_{ij} = g_4(x) \delta g_{ij}(y)
\]

Lichnerowicz equation

\[
(\partial_\mu \partial^\mu + \Delta_6) g_4(x) g_6(y) = 0
\]
Expand $g_6(y)$ in eigenfunctions

$$-\Delta_6 g_6(y) = \lambda_6 g_6(y), \quad \lambda_6 = \text{mass}$$

Consider massless states $\lambda_6 = 0$

$$-\Delta_6 g_6(y) = 0.$$

Solutions of this equation for Calabi-Yau manifolds are given by elements of Dolbeault cohomology $H^{(p,q)}(Y_6)$, where $(p, q)$ denotes numbers of holomorphic and anti-holomorphic indices in the form. The dimensions of these spaces $h^{(p,q)}$ are called Hodge numbers for a given $Y_6$. 
For 4D graviton $g_6(y)$ is scalar

$$-D_i \partial^i g_6 = 0$$

The only solution is

$$g_6(y) = \text{const}$$

Non-normalizable on non-compact conifold $Y_6$.

**No 4D graviton == good news!**

We do not have gravity in our 4D $\mathcal{N} = 2$ QCD
6  Kahler form deformations

Consider 4D scalar fields

Lichnerowicz equation on $Y_6$

$$D_k D^i \delta g_{ij} + 2 R_{ikjl} \delta g^{kl} = 0.$$ 

Solutions = Kahler form deformations or complex structure deformations.

Kahler form deformations = variations of 2D coupling $\beta$

$D$-term condition in weighted CP(2,2) model

$$|n_P|^2 - |\rho^K|^2 = \beta, \quad P = 1, 2, \quad K = 1, 2$$

Resolved conifold
The effective action for $\beta(x)$ is

$$S(\beta) = T \int d^4x \, h_\beta (\partial_\mu \beta)^2,$$

where

$$h_\beta = \int d^6y \sqrt{g} g^{li} \left( \frac{\partial}{\partial \beta} g_{ij} \right) g^{jk} \left( \frac{\partial}{\partial \beta} g_{kl} \right)$$

Using explicit Calabi-Yau metric on resolved conifold we get

$$h_\beta = (4\pi)^3 \frac{5}{6} \int dr \, r = \infty$$

$\beta$ - non-normalizable mode
Physical nature of non-normalizable modes

_Gukov, Vafa, Witten 1999_: Non-normalizable moduli = coupling constants in 4D

- 4D metric do not fluctuate. It is fixed to be flat. ”Coupling constants.”

- 2D coupling $\beta$ is related to 4D coupling $g^2$. Fixed. Non-dynamical.

Another option:

Large $y_i \Rightarrow$ large $n^P$ and $\rho^K$

Non-normalizable modes are not localized on the string.

_Unstable states_. Decay into massless perturbative states.

Higgs branch: $\dim \mathcal{H} = 4N\tilde{N} = 16$. 
7 Deformation of the complex structure

$D$-term condition

$$|n^P|^2 - |ho^K|^2 = \beta, \quad P = 1, 2, \quad K = 1, 2$$

Construct U(1) gauge invariant "mesonic" variables"

$$w^{PK} = n^P \rho^K.$$

$$\det w^{PK} = 0$$

Take $\beta = 0$

Complex structure deformation $\Rightarrow$ Deformed conifold

$$\det w^{PK} = b$$
\( b \) – complex modulos

The effective action for \( b(x) \) is

\[
S(\beta) = T \int d^4 x \ h_b (\partial_\mu b)^2 ,
\]

where

\[
h_b = \int d^6 y \sqrt{g} g^{li} \left( \frac{\partial}{\partial b} g_{ij} \right) g^{jk} \left( \frac{\partial}{\partial \bar{b}} g_{kl} \right)
\]

Using explicit Calabi-Yau metric on deformed conifold we get

\[
h_b = (4\pi)^3 \frac{4}{3} \log \frac{T^2 L^4}{|b|}
\]

For Type IIA string \( b \) is a part of hypermultiplet. Another complex scalar \( \tilde{b} \) comes from 10D 3-form.

\[
S(b) = T \int d^4 x \left\{ |\partial_\mu b|^2 + |\partial_\mu \tilde{b}|^2 \right\} \log \frac{T^4 L^8}{|b|^2 + |\tilde{b}|^2}
\]
8 Monopole-monopole baryon

Weak coupling

Strings in the $U(N)$ theories are stable; they cannot be broken. Thus, we deal with the closed string.

Quarks are condensed in the bulk theory. Therefore, monopoles are confined.

In $U(N)$ gauge theories the confined monopoles are junctions of two non-Abelian vortex strings.

Monopole-antimonopole meson  Monopole-monopole baryon

Stringy states are massive, with mass $\sim \sqrt{T}$.
Strong coupling

Global group of the 4D QCD:

\[ SU(2) \times SU(2) \times U(1) \]

U(1) - ”baryonic” symmetry.

\(b\)-hypermultiplet: (1, 1, 2)

Logarithmically divergent norm == Marginal stability at \(\beta = 0\)

\(b\)-state can decay into massless bi-fundamental (screened) quarks living on the Higgs branch.
9 Conclusions

• In $\mathcal{N} = 2$ supersymmetric QCD with gauge group U(2) and $N_f = 4$ quark flavors non-Abelian BPS vortex behaves as a critical fundamental superstring.

• Massless closed string state $b$ associated with deformations of the complex structure of the conifold $\Rightarrow$ monopole-monopole baryon.

• Successful tests of our gauge-string duality:
  • $\mathcal{N} = 2$ supersymmetry in 4D QCD
  • Absence of graviton and unwanted vector fields.
  • Massless monopole-monopole baryon is present only at $\beta = 0$ and cannot be continued to weak coupling.