SINGULARITY THEORY IN CLASSICAL COSMOLOGY

GIAMPIERO ESPOSITO

*Istituto Nazionale di Fisica Nucleare, Sezione di Napoli, Gruppo IV,
Mostra d’Oltremare, Padiglione 20, 80125 Napoli

*Dipartimento di Scienze Fisiche,
Mostra d’Oltremare, Padiglione 19, 80125 Napoli

Summary. - This paper compares recent approaches appearing in the literature on the singularity problem for space-times with nonvanishing torsion.

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In the last few years, a new approach has been proposed by the author to singularity theory in classical cosmology with nonvanishing torsion, based on the definition of geodesics as curves whose tangent vector moves by parallel transport [1,2]. With the notation described in Refs. [1,2], our main result can be stated as follows.

**Theorem.** The $U_4$ space-time of the ECSK theory cannot be timelike geodesically complete if:

1. $\left[R(U,U) - 2\tilde{\omega}^2 \right] \geq 0$ for any nonspacelike vector $U$;
2. there exists a compact spacelike three-surface $S$ without edge;
3. the trace of the extrinsic curvature tensor of $S$ is either everywhere positive or everywhere negative [this tensor also plays a role in the theory of maximal timelike geodesics and partial Cauchy surfaces].

However, the generalization of singularity theorems to $U_4$ space-times has also been addressed in Ref. [3]. The differences between our work and these papers are:

1. The author of Ref. [3] does not define geodesics as autoparallel curves;
2. After a review of the approach in Ref. [4], he still makes a splitting so as to express $R(U,U) \equiv R_{ab}U^aU^b$ as the part formally identical to general relativity plus other contributions involving torsion;
3. He derives the Hawking-Penrose timelike convergence condition in $U_4$ space-time for a shear-free and convergence-free congruence, and he obtains $R(U,U) \geq 0$, since he separately requires (using the notation in Eqs. (5.2-3) of Ref. [1])

$$\frac{1}{4}\tilde{S}_{ab}\tilde{S}^{ab} \geq \left[ \frac{\theta^2}{3} + \frac{d\theta}{ds} \right].$$ (1)
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However, in the generalized Raychaudhuri equation, $R(U,U)$ and $2\tilde{\omega}^2$ occur with opposite signs, so that one has to require in general condition (1) of our theorem, if $\frac{d\theta}{ds}$ has to remain less than or equal to $-\frac{\theta^2}{3}$, as we explained in Refs. [1,2]. Thus, in our work (see also Ref. [5]), torsion explicitly appears in writing down $\tilde{\omega}^2$ appearing in condition (1) of the theorem stated above, whereas Eq. (5.8) of Ref. [1] is written as in Einstein’s theory, and the singularity theorem is proved under only three assumptions, as in general relativity. By contrast, in the work appearing in Ref. [3], one requires two separate conditions: $R(U,U) \geq 0$ and Eq. (1) of this note, instead of condition (1) of our theorem.

(4) The author of Ref. [3] does not consider the full extrinsic curvature tensor and conditions for maximal timelike geodesics, and he does not avoid the introduction of a modified energy-momentum tensor;

(5) The author of Ref. [3] studies the generalization of the Hawking-Penrose singularity theorem.

However, appendix A in the first paper of Ref. [3] deals with Hawking’s singularity theorem without causality assumptions in general relativity, and in the concluding sect. 4 of that paper it is emphasized that one could investigate singularities in space-times with torsion by looking at the completeness of autoparallels (which are there called nongeodesic curves). It is thus possible that the derivation of our theorem, first published in Ref. [1], will be studied independently by other authors in the near future.

Another interesting study of the inclusion of spin in the Raychaudhuri equation can be found in Ref. [6], where this equation is applied to the behaviour of an irrotational, unaccelerated fluid, and the development of singularities in the expansion is studied for
constant spin densities. The fundamental difference between our work and their work is the following. In Eq. (19) of Ref. [6], a Raychaudhuri equation is written for space-times with torsion where all covariant derivatives only contain Christoffel symbols. The spin connection has been separated from the covariant derivatives and explicitly included. However, as we already emphasized in Refs. [1,2], this split is not in agreement with the Hamiltonian treatment of theories with torsion. Hence we can repeat the remarks made in comparing our work with Ref. [3].

We should emphasize this paper is not aiming at criticizing Refs. [3,6], and we only hope to stimulate further research on singularity theory in classical cosmology with torsion. We are grateful to Dr. P. Scudellaro for bringing Ref. [3] to our attention.

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