Binary factors of shifts of finite type

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Abstract. We construct two new classes of topological dynamical systems; one is a factor of a one-sided shift of finite type while the second is a factor of the two-sided shift. The data are a finite graph which presents the shift of finite type, a second finite directed graph and a pair of embeddings of it into the first, satisfying certain conditions. The factor is then obtained from a simple idea based on binary expansion of real numbers. In both cases, we construct natural metrics on the factors and, in the second case, this makes the system a Smale space, in the sense of Ruelle. We compute various algebraic invariants for these systems, including the homology for Smale space developed by the author and the K-theory of various C*-algebras associated to them, in terms of the pair of original graphs.

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1. Introduction

In the subject of topological dynamical systems, a crucial part has been played by systems which display some type of hyperbolicity; that is, systems which have some type of local expanding or contracting/expanding behaviour. This began in the smooth category with the study of Anosov diffeomorphisms [16]. Smale’s seminal paper [32] showed that, even for smooth systems, the hyperbolic behaviour may be limited to the non-wandering set which may be far from being a submanifold. Motivated by this, David Ruelle gave a purely topological definition of hyperbolicity which he called a Smale space [30]. This broader framework includes shifts of finite type, which are highly combinatorial in nature (see §2 for the definition). Restricted to irreducible systems, these are precisely the Smale spaces whose underlying space is totally disconnected.

One of the fundamental notions for a topological dynamical system is that they may be coded by a Markov partition [16]. In a natural way, a Markov partition yields an almost-one-to-one factor map from a shift of finite type onto the given system. As noted by Adler in [1], the simplest example of this is binary expansion of real numbers: the space
of one-sided 0, 1-sequences maps onto the unit circle via the familiar formula sending a sequence $x_n, n \geq 1,$ to $\exp(2\pi i \sum_n x_n2^{-n}).$ Moreover, this map intertwines the dynamics of the left shift map on the sequences with the squaring map on the circle. (‘Binary’ may be changed to ‘decimal’ by replacing certain 2’s with 10’s.) One tends to regard decimal expansion as a bijection, but the two spaces here are very different topologically.

Returning to the general situation, we recall the following seminal result. In the form stated, it is due to Rufus Bowen [4], but it builds on the work of many others, including Sinai [31], Adler and Weiss [2] and others. Let $(X, d)$ be a compact metric space and $\varphi$ be a homeomorphism of $X$ such that $(X, \varphi, d)$ is an irreducible Smale space (see § 4 for the definition). Alternately, let $(X, \varphi)$ be the restriction of an Axiom A system to a basic set. Then there is an irreducible shift of finite type $(\Sigma, \sigma)$ and a continuous surjection $\pi : \Sigma \to X$ such that $\pi \circ \varphi = \sigma \circ \pi.$

The main goal of this paper is a kind of reverse-engineering of Bowen’s result, where the Smale space is the end product of the construction, rather than the initial object of interest. We begin with some combinatorial data: a pair of shifts of finite type associated with finite directed graphs $G$ and $H$ with a pair of embeddings of the latter into the former $\xi_0, \xi_1 : H \to G.$ We let $A_G$ and $A_H$ denote the adjacency matrices for the two graphs. We defer the details of the construction as well as certain hypotheses on our data to § 3 and focus on the properties of the resulting systems. In particular, the standing hypotheses are described in Definition 3.1.

As described in detail in § 2, we let $(X^+_G, \sigma)$ and $(X_G, \sigma)$ denote the one- and two-sided shifts, respectively, associated with the graph $G.$ Adding a basic idea modelled on binary expansion, we construct topological dynamical systems $(X^+_\xi, \sigma_\xi)$ and $(X_\xi, \sigma_\xi)$ along with factor maps $\pi_\xi : (X^+_G, \sigma) \to (X^+_\xi, \sigma_\xi)$ and $\pi_\xi : (X_G, \sigma) \to (X^+_\xi, \sigma_\xi).$ Both systems are continuous and surjective and the second is actually a homeomorphism. We also construct specific metrics $d_\xi$ on our two spaces having nice properties. Indeed, most of the effort lies in producing the metrics.

Let us summarize the main results on these systems. The first is a local expansiveness property for the first system.

**Theorem 3.17.** If $x, y$ in $X^+_\xi$ satisfy $d_\xi(x, y) \leq 2^{-2},$ then

$$2d_\xi(x, y) \leq d_\xi(\sigma_\xi(x), \sigma_\xi(y)) \leq 8d_\xi(x, y).$$

In fact, the map is actually a local homeomorphism.

**Theorem 3.19.** If $x, y$ are in $X^+_\xi$ with $d_\xi(x, \sigma_\xi(y)) \leq 2^{-1},$ then there is $z$ in $X^+_\xi$ with $d_\xi(z, y) \leq 2^{-1}d_\xi(x, \sigma_\xi(y))$ and $\sigma_\xi(z) = x.$ In particular, we have

$$X^+_\xi(\sigma_\xi(y), \epsilon) \subseteq \sigma_\xi(X^+_\xi(y, 2^{-1}\epsilon))$$

for any $\epsilon < 2^{-1}$ and the map $\sigma_\xi$ is open.

The actual geometric structure of $X^+_\xi$ is rather curious. We offer an intriguing picture of a single example in § 7, but we also note the following corollary.

**Corollary 7.7.** The connected subsets of $X^+_\xi$ are either points or circles and both occur.
The property analogous to locally expanding for the second system is that it possesses local coordinates of contracting and expanding directions. In short, it is a Smale space. (We review the definition in §4.)

**Theorem 4.2.** \((X_{\xi}, d_{\xi}, \sigma_{\xi})\) is a Smale space.

We also note the following (abridged version). Here, \(X^s(x)\) denotes the global stable set of the point \(x\) in a Smale space \(X\).

**Theorem 4.3.** The map \(\pi_{\xi} : (X_G, \sigma) \to (X_{\xi}, \sigma_{\xi})\) is s-bijective; that is, for every \(x\) in \(X_G\), \(\pi_{\xi}|_{X^s_G(x)}\) is a bijection from \(X^s_G(x)\) to \(X^s_{\xi}(\pi_{\xi}(x))\).

This means, in particular, that the local stable sets of \(X_{\xi}\) are Cantor sets.

There are several interesting invariants of a Smale space. First, one can construct a number of different \(C^*\)-algebras from a single Smale space. For any Smale space, \((X, d, \phi)\) and choice of a finite \(\phi\)-invariant set \(P \subseteq X\), there are \(C^*\)-algebras \(S(X, \phi, P)\) and \(U(X, \phi, P)\) based on stable and unstable equivalence, respectively. Each has a canonical isomorphism induced by \(\phi\) and we let \(R^s(X, \phi, P) = S(X, \phi, P) \rtimes_\phi \mathbb{Z}\) and \(R^u(X, \phi, P) = U(X, \phi, P) \rtimes_\phi \mathbb{Z}\) be the associated crossed product \(C^*\)-algebras (see §6).

This was done initially for shifts of finite type by Cuntz and Krieger [7] and later by Ruelle [29] for general Smale spaces. The K-theory groups of these \(C^*\)-algebras have been investigated quite thoroughly and provide interesting data. For shifts of finite type, these include Krieger's dimension group invariant as well as the Bowen–Franks groups [6]. There are, in fact, two dimension groups which are associated with right and left tail-equivalence. These are described in §5 but we note that for a shift of finite type associated with a finite directed graph \(G\), these have simple combinatorial descriptions in terms of \(G\) and we denote them by \(D^s(G)\) and \(D^u(G)\). We let \(A_G\) denote the adjacency matrix associated with \(G\). The matrices \(A_G\) and \(A_G^T\) induce automorphisms of \(D^s(G)\) and \(D^u(G)\), respectively.

In another direction, the author constructed a homology theory for Smale spaces [24] which generalizes Krieger's dimension group for shifts of finite type. Indeed, the dimension group is a key part of the construction. The existence of this theory, which provides a Lefschetz formula, was conjectured by Bowen [5]. For any Smale space \((X, d, \phi)\), there are groups \(H^s_N(X, \phi), H^u_N(X, \phi)\), \(N \in \mathbb{Z}\), and each has a canonical automorphism induced by \(\phi\).

It is worth noting that recent work of Projetti and Yamashita [21, 22] shows that there are very close relations between the K-theory of the \(C^*\)-algebras and the Smale space homology, as well the cohomology of certain groupoids.

One of the appeals of our construction of binary factors is that these invariants may be computed quite easily from the initial combinatorial data. The following summarizes our computation of the homology theory.

**Theorem 5.1.** Under the standing hypotheses, we have
\[
\begin{align*}
H^s_0(X_{\xi}, \sigma_{\xi}) & \cong D^s(G), & (\sigma_{\xi})^{-1} & \cong A_G, \\
H^s_1(X_{\xi}, \sigma_{\xi}) & \cong D^s(H), & (\sigma_{\xi})^{-1} & \cong A_H.
\end{align*}
\]
Binary factors of shifts of finite type

891

\[ H_0^u(X, \sigma) \cong D^u(G), \quad (\sigma)_* \cong A^T_G, \]
\[ H_0^u(X, \sigma) \cong D^u(H), \quad (\sigma)_* \cong A^T_H, \]
\[ H_k^s(X, \sigma) \cong 0, \quad k \neq 0, 1, \]
\[ H_k^u(X, \sigma) \cong 0, \quad k \neq 0, 1. \]

In the first two lines, we regard \( \sigma \) as an \( s \)-bijective factor map from \((X, \sigma)\) to itself and our description of the induced map on homology is interpreted via the isomorphism which precedes it. In the next two lines, we regard \( \sigma \) as a \( u \)-bijective factor map.

The next two results summarize our computations for the \( K \)-theory of the \( C^* \)-algebras.

**Theorem 6.1.** Under the standing hypotheses, we have the following statements.

1. \( K_0(S(X, \sigma), P_\xi)) \cong D^s(G) \) as ordered abelian groups and, under this isomorphism, the automorphism induced by \( \sigma \) is \( A^T_G \).
2. \( K_1(S(X, \sigma), P_\xi)) \cong D^s(H) \) and, under this isomorphism, the automorphism induced by \( \sigma \) is \( A^T_H \).
3. \( K_0(R^s(X, \sigma), P_\xi)) \cong \mathbb{Z}^G \oplus \ker(I - A^T_G : \mathbb{Z}^H \to \mathbb{Z}^H) \).
4. \( K_1(R^s(X, \sigma), P_\xi)) \cong \mathbb{Z}^G \oplus \ker(I - A^T_G : \mathbb{Z}^G \to \mathbb{Z}^G) \).

**Theorem 6.2.** Under the standing hypotheses, we have the following statements.

1. \( K_0(U(X, \sigma), P_\xi)) \cong D^s(G) \) as ordered abelian groups and, under this isomorphism, the automorphism induced by \( \sigma \) is \( A^{-1}_G \).
2. \( K_1(U(X, \sigma), P_\xi)) \cong D^s(H) \) and, under this isomorphism, the automorphism induced by \( \sigma \) is \( A^{-1}_H \).
3. \( K_0(R^s(X, \sigma), P_\xi)) \cong \mathbb{Z}^G \oplus \ker(I - A_H : \mathbb{Z}^H \to \mathbb{Z}^H) \).
4. \( K_1(R^s(X, \sigma), P_\xi)) \cong \mathbb{Z}^G \oplus \ker(I - A_H : \mathbb{Z}^G \to \mathbb{Z}^G) \).

A curious consequence of these computations and the Phillips–Kirchberg classification theorem for \( C^* \)-algebras is the following corollary.

**Theorem 6.5.** Let \((X, \varphi, d)\) be a mixing Smale space and \( Q \) be a finite \( \varphi \)-invariant subset of \( X \). There exist finite directed graphs, \( G, H \) and embeddings \( \xi^0, \xi^1 : H \to G \) satisfying the standing hypotheses such that \( R^s(X, \sigma, P_\xi) \cong R^s(X, \varphi, Q) \), for any \( P, a \)

finite \( \sigma \)-invariant subset of \( X_G \).

We remark that, just as in the case for the Cuntz–Krieger algebras, these isomorphisms do not usually respect the decomposition of these algebras as crossed products \( R^s(X, \varphi, Q) \cong S(X, \varphi, Q) \times \mathbb{Z} \) and are not equivariant for the associated circle actions. We also remark that these \( C^* \)-algebras are stable rather than unital and the appropriate classification result is used. To see this, we refer the reader to [11, Appendix A] for the fact that the stable and unstable \( C^* \)-algebras are stable, in the \( C^* \)-algebra sense. The conclusion for the Ruelle algebras then follows from the results of [13], which shows that stability is preserved under crossed products by actions of discrete groups.

Section 2 contains basic background information on shifts of finite type, or at least edge shifts for finite directed graphs. Section 3 is the construction of our factor system. At this
stage, we begin only with one-sided shifts of finite type and construct factors which are continuous topological dynamical systems, but not invertible. Much of the effort here is focused on the construction of a specific metric with nice properties. In §4, we continue the construction to produce invertible systems. This is a standard technique by inverse limits, but again, we construct a specific metric which makes the result a Smale space. We also review the definitions of a Smale space. Section 5 is devoted to the computation of the homology theory for our Smale spaces. Section 6 deals with the various \( C^* \)-algebras associated with our examples, including computations of their K-theory groups. In §7, we return to the factors of the one-sided shifts of finite type. Geometrically, shifts of finite type are not particularly interesting; at least, they are no more interesting than the Cantor ternary set. This not is true of our factors and we spend some time giving a description of their geometry. In particular, we show that the spaces can be embedded in \( \mathbb{R}^3 \). We also give a couple of simple examples, which can actually be embedded in the plane.

2. Preliminaries

In this section, we set out some well-known preliminaries on shift spaces. An excellent reference is the book of Lind and Marcus [18]. As discussed in the introduction, we are considering shifts of finite type. However, we will deal exclusively with edge shifts of finite directed graphs. These are dynamically equivalent, as a consequence of [18, Example 1.5.10 and Theorem 2.3.2].

By a finite directed graph, \( G \), we mean two finite sets \( G^0 \) (the vertex set) and \( G^1 \) (the edge set) along with maps \( i, t : G^1 \to G^0 \). A path of length \( n \) in \( G \) is a finite sequence \( p = (p_1, p_2, \ldots, p_n) \) in \( G^1 \) such that \( t(p_i) = i(p_{i+1}) \), for all \( 1 \leq i < n \). We define \( i(p) = i(p_1) \) and \( t(p) = t(p_n) \).

The adjacency matrix for the graph is denoted \( A_G \) and we regard it as the function on \( G^0 \times G^0 \) whose value at \( (v, w) \) is the number of edges \( e \) with \( i(e) = w \) and \( t(e) = v \). We also regard it as an endomorphism of \( \mathbb{Z}^{G^0} \) by matrix multiplication.

We say \( G \) is irreducible if, for every ordered pair of vertices \( v, w \), there is a path \( p \) with \( i(p) = v, t(p) = w \). Equivalently, \( G \) is irreducible if, for every \( (v, w) \), there is a positive integer \( k \) with \( A^k_G(v, w) \) positive. We say \( G \) is primitive if there is a positive integer \( k \) such that \( A^k_G(w, v) \) is positive, for all \( v, w \).

We let \( X_G^+ \) be the one-sided infinite path space: an element is a sequence \( (x_1, x_2, \ldots) \) in \( G^1 \) with \( t(x_n) = i(x_{n+1}) \) for all \( n \geq 1 \). The bi-infinite path space \( X_G \) is defined analogously with sequences indexed by the integers. We let \( \chi_G^+ \) denote the obvious map from \( X_G \) to \( X_G^+ \), which simply restricts the domain of the sequence.

The spaces \( X_G^+ \) and \( X_G \) both have canonical metrics. For \( x, y \) in \( X_G^+ \), we define

\[
d_G(x, y) = \inf \{2^{-n} | n \geq 0, x_i = y_i, 1 \leq i \leq n \}
\]

and for \( x, y \) in \( X_G \), we define

\[
d_G(x, y) = \inf \{2^{-n} | n \geq 0, x_i = y_i, 1 - n \leq i \leq n \}.
\]

Using the same notation should cause no confusion. Observe that \( \chi_G^+ \) is a contraction.
The spaces $X^+_G$ and $X_G$ both carry dynamical systems. We define $\sigma$ by $\sigma(x)_n = x_{n+1}$. Here, either $x$ is in $X^+_G$ and $n \geq 1$ or $x$ is in $X_G$ and $n$ is an integer, making $\sigma$ a self-map of either space. Both are continuous and surjective and the latter is a homeomorphism. We make the easy observation that $d_G(\sigma^n(x), \sigma^n(y)) = 2^n d_G(x, y)$, for $x, y \in X^+_G$.

Let $G, H$ be two finite directed graphs. By a graph homomorphism from $H$ to $G$, we mean a function, $\xi : H^0 \cup H^1 \rightarrow G^0 \cup G^1$ such that $\xi(H^0) \subseteq G^0$, $\xi(H^1) \subseteq G^1$ and satisfying $t_G \circ \xi|_{H^0} = \xi \circ t_H|_{H^0}$, $i_G \circ \xi|_{H^0} = \xi \circ i_H|_{H^0}$. It will usually not cause any confusion to drop the subscripts on $i, t$. We usually write such a function as $\xi : H \rightarrow G$.

A graph embedding is a graph homomorphism which is injective.

If $(X, d)$ is a metric space, $x$ is in $X$ and $\epsilon$ is positive, then we let $x(x, \epsilon)$ denote the ball centred at $x$ of radius $\epsilon$. If $A$ is any subset of $X$, we let $\text{Cl}(A)$ denote its closure. If $(X_1, d_1)$ and $(X_2, d_2)$ are metric spaces, we let

$$d_1 \times d_2((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2),$$

for all $x_1, y_1$ in $X_1$ and $x_2, y_2$ in $X_2$, which is a metric for the product space $X_1 \times X_2$.

3. Construction

In this section, we describe our basic construction.

**Definition 3.1.** For finite directed graphs, $G, H$, a pair of graph embeddings $\xi^0, \xi^1 : H \rightarrow G$ are said to satisfy:

1. (H0) if $\xi^0|_{H^0} = \xi^1|_{H^0}$;
2. (H1) if $\xi^0(H^1) \cap \xi^1(H^1)$ is empty;
3. (H2) if, for every $y$ in $H^1$, there is $x$ in $G^1$ with $t(x) = t(\xi^0(y))$, $i(x) = i(\xi^0(y))$ and $x \notin \xi^0(H^1) \cup \xi^1(H^1)$.

We say that $G, H, \xi = (\xi^0, \xi^1)$ satisfies the standing hypotheses if $G$ is primitive and $\xi$ satisfies hypotheses (H0), (H1) and (H2).

The first two conditions will be essential for our construction. The third is a convenience and is not always needed. It also seems likely that the hypothesis that $G$ is primitive can be weakened.

We observe a simple consequence of hypotheses (H0) and (H1): if $y$ is any edge in $H^1$, then $\xi^0(y)$ and $\xi^1(y)$ are distinct, but have the same initial and terminal vertices in $G^0$.

**Definition 3.2.** Let $\xi^0, \xi^1 : H \rightarrow G$ satisfy hypotheses (H0) and (H1) of Definition 3.1.

1. We let $\xi(H^1) = \xi^0(H^1) \cup \xi^1(H^1)$.
2. For $x$ in $\xi(H^1)$, we let $\varepsilon(x) = 0, 1$ be such that $x$ is in $\xi^{\varepsilon(x)}(H^1)$.
3. We let $G_\xi$ be the graph with $G_\xi^0 = G_0$ and $G_\xi^1$ obtained by identifying $\xi^0(y)$ with $\xi^1(y)$, for every $y$ in $H^1$. We also let $\tau_\xi$ denote the obvious quotient map from $G^1$ to $G_\xi^1$ as well as the associated map from $X_G^+$ to $X_{G_\xi}^+$ and the map from $X_G$ to $X_{G_\xi}$.

**Definition 3.3.** Let $\xi^0, \xi^1 : H \rightarrow G$ satisfy hypotheses (H0) and (H1) of Definition 3.1.

1. For $x$ in $X_G^+$, let $\kappa(x)$ be the number of positive integers $n$ such that $x_n$ is not in $\xi(H^1)$, allowing the values $0$ and $\infty$. 

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(2) For $0 \leq k \leq \infty$, let $X^+_k = \kappa^{-1}\{k\}$.
(3) For $x$ in $X^+_G$ with $\kappa(x) > 0$, we define $n(x)$ to be the least positive integer with $x_{n(x)}$ not in $\xi(H^1)$.

The following result is worth noting but its proof is trivial and we omit it.

**Proposition 3.4.** Suppose that $G$, $H$, $\xi$ satisfy the standing hypotheses.
(1) The sets $X^+_k$, $0 \leq k \leq \infty$, are pairwise disjoint and, for any $0 \leq k < \infty$, $\bigcup_{j=0}^{k} X^+_j$ is closed in $X^+_G$.
(2) If $x, y$ in $X^+_G$ satisfy $\tau_{\xi}(x) = \tau_{\xi}(y)$, then $\kappa(x) = \kappa(y)$ and $n(x) = n(y)$ if $\kappa(x) > 0$.
(3) If $x$ is in $X^+_G$ with $0 < \kappa(x) < \infty$, then $\kappa(\sigma^n(x)(x)) = \kappa(x) - 1$.

The following is a summary of some convenient topological properties of the sets $X^+_k$, $k \geq 0$. Essentially, these are consequences of our property (H2) and $G$ being primitive.

**Proposition 3.5.** Suppose that $G$, $H$, $\xi$ satisfy the standing hypotheses.
(1) If $x$ is in $X^+_j$ and $j < k < \infty$, then there is a sequence $x^l, l \geq 1$, in $X^+_k$ which converges to $x$ in $X^+_G$.
(2) The closure of $X^+_k$ in $X^+_G$ is $\bigcup_{j=0}^{k} X^+_j$.
(3) If $x$ is in $X^+_G$, then there is a sequence $x^l, l \geq 1$, converging to $x$ in $X^+_G$ with $x^l$ in $X^+_l$, for all $l \geq 1$.
(4) The closure of $\bigcup_{k=0}^{\infty} X^+_k$ in $X^+_G$ is $X^+_G$.

**Proof.** We first observe that it follows from hypothesis (H2) and $G$ being primitive that $G^1 - \xi(H^1)$ is connected.

For $x$ in $X^+_j$, there are only finitely many $n$ with $x_n$ not in $\xi(H^1)$. Choose $n_0$ such that $x_n$ is in $\xi(H^1)$, for all $n \geq n_0$. For $l \geq 1$, we define $x^l$ in three segments, as follows. Define $x^l_n = x_n$, for $1 \leq n \leq n_0 + l$. Then choose $x^l_n, n_0 + l + k - j$ to be any path in $G^1 - \xi(H^1)$ going through the same vertices as $x_n$, by property (H2). Finally, set $x^l_n = x_n$ for $n > n_0 + l + k - j$. It is clear that this sequence has the desired properties.

The first part shows that the closure of $X^+_k$ contains $\bigcup_{j=0}^{k} X^+_j$. The reverse containment follows from the fact that the latter is closed, as we noted in Proposition 3.4.

As $G$ is primitive, we may find $l_0 \geq 1$ such that, for any ordered pair of vertices in $G^0$ and any $l \geq l_0$, there is a path of length $l$ between these two vertices. From property (H2), we may assume such a path lies in $G^1 - \xi(H^1)$. For $l > l_0$, we again define $x^l$ in three segments as follows. For the first segment, we set $x^l_n = x_n$, for $1 \leq n \leq l - l_0$. Let $j$ be the number of integers, $n$, between 1 and $l - l_0$ with $x_n$ not in $\xi(H^1)$. Obviously, $j \leq l - l_0$. For the second segment, $x^l_n, l - l_0 + n \leq 2l - l_0 - j$, we choose any path in $G^1 - \xi(H^1)$ from $t(x_{l-1})$ to any vertex of $\xi^0(H^0)$ having length $l - j \geq l_0$. Finally, we define the third segment $x^l_n, 2l - l_0 - j < n < \infty$, to be any path in $\xi(H^1)$. The points $x^l, 1 \leq l \leq l_0$, may be chosen arbitrarily from $X^+_l$. Again, it is clear that this sequence has the desired properties. The last statement follows at once.

We now give our two main definitions. The influence of binary expansion should be clear in the first.
Definition 3.6. Let $x$ be in $X_G^+$. 

1. If 
   
   $x = (x_1, x_2, \ldots, x_n, \xi^{1-i}(y_n), \xi^i(y_{n+1}), \xi^i(y_{n+2}), \ldots),$

   for some $n \geq 1$, $y_n, y_{n+1}, \ldots$ in $H^1$ and $i = 0, 1$, then we define 
   
   $x' = (x_1, x_2, \ldots, x_{n-1}, \xi^i(y_n), \xi^{1-i}(y_{n+1}), \xi^{1-i}(y_{n+2}), \ldots).$

2. If 
   
   $x = (x_1, x_2, \ldots, x_n, \xi^i(y_{n+1}), \xi^i(y_{n+2}), \ldots),$

   for some $n \geq 1$ and $i = 0, 1, y_{n+1}, y_{n+2}, \ldots$ in $H^1$, $x_n$ is not in $\xi(H^1)$, and $i = 0, 1$, then we define 
   
   $x' = (x_1, x_2, \ldots, x_n, \xi^{1-i}(y_{n+1}), \xi^{1-i}(y_{n+2}), \xi^{1-i}(y_{n+3}), \ldots).$

3. We define $x \sim_\xi x'$, whenever $x$, $x'$ are as above. Also, we define $x \sim_\xi x$, for every $x$ in $X_G^+$.

It is worth noting that the $n$ in parts (1) and (2) is unique (if it exists). The proofs of the following easy facts are left to the interested reader.

**Proposition 3.7**

1. $\sim_\xi$ is an equivalence relation and each equivalence class has either one or two elements.

2. If $x \sim_\xi x'$, then $\tau_\xi(x) = \tau_\xi(x')$.

3. For each $k \geq 0$, the set $X_k^+$ is $\sim_\xi$-invariant. That is, if $x$ is in $X_k^+$ and $x \sim_\xi x'$, then $x'$ is in $X_k^+$.

4. If $x \sim_\xi x'$, then $\sigma(x) \sim_\xi \sigma(x')$.

5. Applying $\sigma$ to the $\sim_\xi$-equivalence class of $x$ yields the $\sim_\xi$-equivalence class of $\sigma(x)$.

We can now define our main item of interest.

**Definition 3.8.** Let $G$, $H$ be finite directed graphs and $\xi^0, \xi^1 : H \rightarrow G$ be two graph embeddings satisfying hypotheses (H0) and (H1) of Definition 3.1. We define $X^+_G$ to be the quotient space of $X^+_G$ by the equivalence relation $\sim_\xi$, endowed with the quotient topology. We let $\pi_\xi : X^+_G \rightarrow X^+_G$ be the quotient map. It follows from part (2) of Proposition 3.7 that there is a function we denote by $\rho_\xi : X^+_G \rightarrow X^+_G$ satisfying $\rho_\xi \circ \pi_\xi = \tau_\xi$.

We also define $\sigma_\xi : X^+_G \rightarrow X^+_G$ by $\sigma_\xi(\pi_\xi(x)) = \pi_\xi(\sigma(x))$, for all $x$ in $X^+_G$, which is well defined by Proposition 3.7. We let $X^+_{G,k} = \pi_\xi(X^+_G)$, for each $k \geq 0$.

As indicated above, the space $X^+_G$ is given the quotient topology. It is not hard to use general topological results to show that $X^+_G$ is metrizable. That is not sufficient for our goals as we will need a metric with specific properties. The construction is rather subtle.

We begin the construction of a pseudo-metric, $d$, on $X^+_G$. We will show that $d(x, y) \leq 3d_G(x, y)$, for all $x, y$ in $X^+_G$. In addition, we will prove $d(x, y) = 0$ if and only if $x \sim_\xi y$, for any $x, y$ in $X^+_G$. The immediate consequence is that $d$ induces a metric on the quotient
space $X^+_\xi$, denoted by $d_\xi$, whose topology is the quotient topology. This means that an alternate definition of our space $X^+_G$ is as the metric space naturally induced by the pseudo-metric $d$ on $X^+_G$. The definition given in Definitions 3.6 and 3.8 seems more intuitive.

For $x$ in $X^+_G$, if $\kappa(x) = 0$, we define

$$\theta(x) = \exp \left( 2\pi i \sum_{j=1}^{\infty} \varepsilon(x_j) 2^{-j} \right).$$

If $\kappa(x) > 0$, we define

$$\theta(x) = \exp \left( 2\pi i \sum_{j=1}^{n(x)-1} \varepsilon(x_j) 2^{-j} \right).$$

Obviously, $\theta(x)$ lies on the unit circle, $\mathbb{T}$, in either case and is a $2^{n(x)-1}$th root of unity in the latter.

For $k \geq 0$, we define $\lambda_k : X^+_k \times X^+_k \to [0, \infty)$ inductively as follows. First, we set

$$\lambda_0(x, y) = d_T(\theta(x), \theta(y)) = \inf \{|t| \mid t \in \mathbb{R}, e^{2\pi it} \theta(x) = \theta(y)\},$$

for $x, y$ in $X^+_0$.

For $x, y$ in $X^+_k$ with $k > 0$, we define

$$\lambda_k(x, y) = |2^{-n(x)} - 2^{-n(y)}| + d_T(\theta(x), \theta(y)) + \delta((n(x), \theta(x)), (n(y), \theta(y))) 2^{-n(x)} \lambda_{k-1}(\sigma^{n(x)}(x), \sigma^{n(y)}(y)),$$

where we use $\delta(\cdot, \cdot)$ for the Kronecker delta function.

Let us make a couple of comments on this definition. First, the value does not really depend on the actual edges $x_j$ but on $\varepsilon(x_j)$, that is, which of the two embeddings is used. Second, the value of $\lambda_k(x, y)$ depends only on $\varepsilon(x_j)$, $1 \leq j \leq n(x)$ and $\varepsilon(y_j)$, $1 \leq j \leq n(y)$ unless those two finite sequences are equal. In that case, they are deleted and the computation is done using $\lambda_{k-1}$ on the rest of the sequence (times a factor of $2^{-n(x)}$). Roughly speaking, this is an attempt to compute the distance between the binary factor parts of $x$ and $y$.

**Lemma 3.9.** For $k \geq 0$, we have

$$d_T(\theta(x), \theta(y)) \leq d_G(x, y),$$

and

$$\lambda_k(x, y) \leq 2d_G(x, y),$$

for all $x, y$ in $X^+_k$.

**Proof.** We proceed by induction on $k$. For $k = 0$, suppose $d_G(x, y) = 2^{-m}$, for some $m \geq 0$, which implies $x_j = y_j$ for $1 \leq j \leq m$. We have
Binary factors of shifts of finite type

\[ d_T(\theta(x), \theta(y)) \leq \sum_{j=1}^{\infty} |\varepsilon(x_j) - \varepsilon(y_j)|2^{-j} \]

\[ \leq \sum_{j=m+1}^{\infty} |\varepsilon(x_j) - \varepsilon(y_j)|2^{-j} \]

\[ \leq 2^{-m} \]

\[ = d_G(x, y). \]

For \( k = 0 \), the second statement follows from the first as \( \lambda_0(x, y) = d_T(\theta(x), \theta(y)) \).

Now assume that \( k > 0 \) and again \( d_G(x, y) = 2^{-m} \), so \( x_j = y_j \) for \( 1 \leq j \leq m \). If either \( n(x) \leq m \) or \( n(y) \leq m \), then \( n(x) = n(y) \) and \(|2^{-n(x)} - 2^{-n(y)}| = 0\). In addition, we have \( \theta(x) = \theta(y) \) so \( d_T(\theta(x), \theta(y)) = 0 \).

Otherwise, we have \( n(x), n(y) > m \) and

\[ d_T(\theta(x), \theta(y)) \leq \left| \sum_{j=m+1}^{n(x)} \varepsilon(x_j)2^{-j} - \sum_{j=m+1}^{n(y)} \varepsilon(y_j)2^{-j} \right| \]

\[ \leq \sum_{j=m+1}^{\infty} 2^{-j} \]

\[ = 2^{-m} \]

\[ = d_G(x, y). \]

This establishes the first inequality. In addition, we note that

\[ |2^{-n(x)} - 2^{-n(y)}| \leq 2^{-m-1} < d_G(x, y). \]

Now, we consider \( \lambda_k(x, y) \). If either \( n(x) \neq n(y) \) or \( \theta(x) \neq \theta(y) \), then we have

\[ \lambda_k(x, y) = |2^{-n(x)} - 2^{-n(y)}| + d_T(\theta(x), \theta(y)) \leq d_G(x, y) + d_G(x, y). \]

In the case \( n(x) = n(y) = n \) and \( \theta(x) = \theta(y) \), from the definition and the induction hypothesis, we have

\[ \lambda_k(x, y) = 2^{-2n} \lambda_{k-1}(\sigma^n(x), \sigma^n(y)) \]

\[ \leq 2^{-2n} \cdot 2d_G(\sigma^n(x), \sigma^n(y)) \]

\[ \leq 2^{-2n} \cdot 2 \cdot 2^n d_G(x, y) \]

\[ < d_G(x, y). \]

We can now begin our definition of our pseudo-metric on \( X^+_G \). Initially, we treat the spaces \( X^+_k \) separately.

**Proposition 3.10**

(1) For each \( k \geq 0 \),

\[ d_k(x, y) = d_{G_k}(\tau_k(x), \tau_k(y)) + \lambda_k(x, y), \]

for \( x, y \) in \( X^+_k \), is a pseudo-metric on \( X^+_k \).
(2) For $x, y$ in $X^+_k$, we have
\[ d_{G}(\tau_\xi(x), \tau_\xi(y)) \leq d_k(x, y) \leq 3d_G(x, y). \]

(3) For $x, y$ in $X^+_k$, $d_k(x, y) = 0$ if and only if $x \sim_\xi y$.

Proof. For the first part, the $d_{G}$ term is clearly a pseudometric, so it suffices to show that $\lambda_k$ is also, which we do by induction on $k$. This is clear for $k = 0$. For $k > 0$, a simple induction argument shows that $\lambda_k$ is reflexive and symmetric. Now suppose that $x, y, z$ are in $X^+_k$. We first consider the case $(n(x), \theta(x)) \neq (n(y), \theta(y))$, where we have
\[ \lambda_k(x, y) = |2^{-n(x)} - 2^{-n(y)}| + d_T(\theta(x), \theta(y)) \leq |2^{-n(x)} - 2^{-n(z)}| + d_T(\theta(x), \theta(z)) \]
\[ + |2^{-n(z)} - 2^{-n(y)}| + d_T(\theta(z), \theta(y)). \]

If the first two terms are both zero, then their sum is certainly less than or equal to $\lambda_k(x, z)$. However, if either is non-zero, then their sum equals $\lambda_k(x, z)$, by definition. The third and fourth terms are done similarly.

Now suppose that $(n(x), \theta(x)) = (n(y), \theta(y))$. If $(n(z), \theta(z)) = (n(x), \theta(x))$, then
\[ \lambda_k(x, y) = 2^{-n(x)}\lambda_{k-1}(\sigma^{n(x)}(x), \sigma^{n(y)}(y)) \leq 2^{-n(x)}\lambda_{k-1}(\sigma^{n(x)}(x), \sigma^{n(z)}(z)) \]
\[ + 2^{-n(x)}\lambda_{k-1}(\sigma^{n(z)}(z), \sigma^{n(y)}(y)) = \lambda_k(x, z) + \lambda_k(z, y) \]
follows from the induction hypothesis.

We next consider the case $n(z) \neq n(x)$ and so $n(z) \neq n(y)$ also. For any positive integers $m \neq n$, we have
\[ |2^{-m} - 2^{-n}| = 2^{-\min\{m,n\}}|1 - 2^{-m-n}| \geq 2^{-\min\{m,n\}-1} \geq 2^{-n-1} \]
and so
\[ \lambda_k(x, z) \geq |2^{-n(x)} - 2^{-n(z)}| \geq 2^{-n(x)+1}. \]

A similar estimate holds for $\lambda_k(z, y)$.

However, we also have
\[ 2^{-2^{-n(x)}\lambda_{k-1}(\sigma^{n(x)}(x), \sigma^{n(y)}(y))} \leq 2^{-2^{-n(x)}}2d_G(\sigma^{n(x)}(x), \sigma^{n(y)}(y)) \]
\[ \leq 2^{1-n(x)}. \]
Together this yields
\[ \lambda_k(x, y) = 2^{-2^{-n(x)}\lambda_{k-1}(\sigma^{n(x)}(x), \sigma^{n(y)}(y))} \]
\[ \leq 2^{1-n(x)} \]
\[ = 2^{-2-n(x)} + 2^{-2-n(y)} \]
\[ \leq \lambda_k(x, z) + \lambda_k(z, y). \]
The final case to consider is \( n(x) = n(y) = n(z) \) and \( \theta(z) \neq \theta(x) \), so \( \theta(z) \neq \theta(y) \) also. Then, as \( \theta(x), \theta(y), \theta(z) \) are all \( 2^{n(x)-1} \)th roots of unity, we have
\[
d_{T}(\theta(x), \theta(z)), d_{T}(\theta(z), \theta(y)) \geq 2^{-n(x)}.
\]
It follows that
\[
\lambda_{k}(x, y) = 2^{-2-n(x)}\lambda_{k-1}(\sigma^{n(x)}(x), \sigma^{n(y)}(y)) \leq 2^{-1-n(x)} = 2^{-2-n(x)} + 2^{-2-n(y)} \leq d_{T}(\theta(x), \theta(z)) + d_{T}(\theta(z), \theta(y)) = \lambda_{k}(x, z) + \lambda_{k}(z, y).
\]

The second part is immediate from the second statement of Lemma 3.9.

For the third part, we consider the ‘if’ direction first. We proceed by induction on \( k \), beginning with \( k = 0 \). Suppose \( x \sim_{\xi} y \), where \( x, y \) are in \( X_{0}^{+} \). From part (2) of Proposition 3.4, we know \( \tau_{\xi}(x) = \tau_{\xi}(y) \). In addition, \( \varepsilon(x_{j}), j \geq 1 \) and \( \varepsilon(y_{j}), j \geq 1 \) are two 0, 1-sequences and the definition of \( \sim_{\xi} \) implies that these sequences are dyadic representations of the same real number, modulo the integers. This implies \( \theta(x) = \theta(y) \) and so
\[
d_{0}(x, y) = d_{T}(\theta(x), \theta(y)) + d_{G_{\xi}}(\tau_{\xi}(x), \tau_{\xi}(y)) = 0 + 0 = 0.
\]
Now assume the result is true for some \( k \) and let \( x \sim_{\xi} y \) be in \( X_{k+1}^{+} \). From Proposition 3.4, we have \( n(x) = n(y) \) and \( x_{i} = y_{i} \), for all \( 1 \leq i \leq n(x) \). It follows that \( \theta(x) = \theta(y) \) and
\[
d_{k+1}(x, y) = 2^{-2-n(x)}\lambda_{k}(\sigma^{k}(x), \sigma^{k}(y)) \leq 2^{-2-n(x)}d_{k}(\sigma^{k}(x), \sigma^{k}(y)).
\]
It follows from Proposition 3.4 that \( \sigma^{k}(x) \sim_{\xi} \sigma^{k}(y) \), so the conclusion follows from the induction hypothesis.

Now we consider the ‘only if’ direction. Suppose \( d_{0}(x, y) = 0 \), where \( x, y \) are in \( X_{0}^{+} \). This means that \( x_{i}, y_{i} \) are in \( \xi(H^{1}) \), for all \( i \geq 1 \). The hypothesis obviously implies that \( d_{G_{\xi}}(\tau_{\xi}(x), \tau_{\xi}(y)) = 0 \) so \( \tau_{\xi}(x_{i}) = \tau_{\xi}(y_{i}) \), for all \( i \geq 1 \). In other words, there is a path \( z \) in \( X_{H}^{+} \) such that \( \{x_{i}, y_{i}\} \subseteq \{\xi^{0}(z_{i}), \xi^{1}(z_{i})\} \) for all \( i \). The fact that \( \lambda_{0}(x, y) = 0 \) implies that \( \theta(x) = \theta(y) \) and this implies that the sequences \( \varepsilon(x_{i}), \varepsilon(y_{i}) \), \( i \geq 1 \) are the binary expansions of the same real number, modulo the integers. This implies \( x \sim_{\xi} y \).

We now consider \( k > 0 \) and \( d_{k}(x, y) = 0 \), where \( x, y \) are in \( X_{k}^{+} \). The first consequence is that \( \tau_{\xi}(x) = \tau_{\xi}(y) \). This implies that \( n(x) = n(y) = n \) and since \( x_{n(x)} \) and \( y_{n(y)} \) are not in \( \xi^{0}(H) \cup \xi^{1}(H) \), we have \( x_{n(x)} = y_{n(y)} \). In addition, there is a path \( h_{1}, \ldots, h_{n-1} \) such that \( x_{i}, y_{i} \) are in \( \{\xi^{0}(h_{i}), \xi^{1}(h_{i})\} \), for all \( 1 \leq i < n \). The fact that \( \lambda_{k}(x, y) = 0 \) implies that \( \theta(x) = \theta(y) \). The map which takes a 0, 1 sequence \( \varepsilon_{1}, \ldots, \varepsilon_{n-1} \) to \( \exp(2\pi \sum_{j=1}^{n-1} \varepsilon_{j}2^{-j}) \) is injective and so we conclude that \( \varepsilon(x_{i}) = \varepsilon(y_{i}) \), for all \( 1 \leq i < n \). It follows that \( x_{i} = y_{i} \), for \( 1 \leq i \leq n \). The fact \( \tau_{\xi}(x) = \tau_{\xi}(y) \) implies that \( \tau_{\xi}(\sigma^{n(x)}(x)) = \tau_{\xi}(\sigma^{n(y)}(y)) \) also and so we have
\[
d_{k-1}(\sigma^{n}(x), \sigma^{n}(y)) = \lambda_{k-1}(\sigma^{n}(x), \sigma^{n}(y)) = 2^{2+n}\lambda_{k}(x, y) = 0.
\]
By our induction hypothesis, $\sigma^n(x) \sim \xi \sigma^n(y)$. Together with the fact that $x_i = y_i$ for $1 \leq i \leq n$ implies $x \sim \xi y$.

It will be useful to have a characterization of Cauchy sequences in our pseudo-metric.

**Proposition 3.11**

1. A sequence $x^l, l \geq 1$, in $X_0^+$ is Cauchy in $d_0$ if and only if $\tau_\xi(x^l), l \geq 1$, is convergent in $X_{G_\xi}^+$ and $\theta(x^l), l \geq 1$, is convergent in $\mathbb{C}$.

2. For $k > 0$, a sequence $x^l, l \geq 1$, in $X_k^+$ is Cauchy in $d_k$ if and only if $\tau_\xi(x^l), l \geq 1$, is convergent in $X_{G_\xi}^+$ and either:
   
   a) $\lim l n(x^l) = \infty$ and $\theta(x^l), l \geq 1$, is convergent in $\mathbb{C}$; or
   
   b) the sequences $n(x^l)$ and $\theta(x^l)$ are eventually constant and $\sigma^n(x^l)(x^l), l \geq 1$, is Cauchy in $d_{k-1}$.

**Proof.** The first part is clear from the definition of $d_0$.

For the second part, we begin with the ‘only if’ direction. If $x^l, l \geq 1$, is Cauchy in $d_k$, then $\tau_\xi(x^l), l \geq 1$, is clearly Cauchy and hence convergent in $X_{G_\xi}$. In addition, both sequences $2^{-n(x^l)}$ and $\theta(x^l)$ are Cauchy. There are clearly two cases: either $n(x^l), l \geq 1$, tends to infinity or it is eventually constant. In the first case, we have case (a). In the second case, if $n(x^l) = n$ for all $l$ sufficiently large, then for such $l$, $\theta(x^l)$ lies in a finite set, so to be Cauchy, it must be eventually constant also, say with value $\theta$. Then for sufficiently large values of $l, m$, we have

$$d_k(x^l, x^m) = d_{G_\xi}(\tau_\xi(x^l), \tau_\xi(x^m)) + 2^{-2-n} \lambda_k - 1(\sigma^n(x^l), \sigma^n(x^m))$$

$$\geq 2^{-n} d_{G_\xi}(\tau_\xi(\sigma^n(x^l)), \tau_\xi(\sigma^n(x^m)))$$

$$+ 2^{-2-n} \lambda_k - 1(\sigma^n(x^l), \sigma^n(x^m))$$

$$\geq 2^{-2-n} d_{k-1}(\sigma^n(x^l), \sigma^n(x^m)).$$

It follows that $\sigma^n(x^l), l \geq 1$, is Cauchy in $d_{k-1}$.

Conversely, if $x^l, l \geq 1$, is any sequence in $X_k^+$, then, for any $l, m$, $d_k(x^l, x^m)$ is equal to either

$$d_{G_\xi}(\tau_\xi(x^l), \tau_\xi(x^m)) + |2^{-n(x^l)} - 2^{-n(x^m)}| + d_T(\theta(x^l), \theta(x^m))$$

or

$$d_{G_\xi}(\tau_\xi(x^l), \tau_\xi(x^m)) + 2^{-2-n} \lambda_k - 1(\sigma^n(x^l), \sigma^n(x^m)).$$

It is then immediate that $\tau_\xi(x^l), l \geq 1$, being convergent and either of the two conditions imply the sequence is Cauchy in $d_k$.

At this point, we have defined a pseudo-metric on $X_k^+$, for each value of $k$. Assuming condition (H2) and that $G$ is primitive, we also know that $X_k^+$ will have limit points in $X_j^+$, for every $0 \leq j < k$. We now need to show these different pseudo-metrics are compatible, in an obvious sense. The following concerns a special case ($j = 0$) but will be used in the proof of the general statement as well.
Lemma 3.12. Let $k \geq 1$ and suppose that $x^l, l \geq 1$, is a sequence in $X^+_k$ which converges to $x$ in $X^+_G$ and such that $n(x^l)$ tends to infinity as $l$ tends to infinity. Then $x$ is in $X^+_0$ and \(\lim_{l \to \infty} \theta(x^l) = \theta(x)\).

Proof. For any $n \geq 1$, we have $x_n = \lim l x^l$ and $x^l_n$ is in $\xi(H^1)$, provided $n(x^l) > n$. It follows that $x_n$ is in $\xi(H^1)$, for every $n$, so $x$ is in $X^+_0$.

If we fix $m \geq 1$, we may choose $l_0$ sufficiently large so that $n(x^l) > m$ and $x_l = x^l_l$, for all $1 \leq i \leq m$ and $l \geq l_0$. For such $l$, we have

\[
|\theta(x) - \theta(x^l)| = \left| \exp \left( 2\pi i \sum_{j=1}^{\infty} \epsilon(x_j)2^{-j} \right) - \exp \left( 2\pi i \sum_{j=1}^{n(x^l)-1} \epsilon(x^l_j)2^{-j} \right) \right|
\]

\[
= \left| \exp \left( 2\pi i \sum_{j=m+1}^{\infty} \epsilon(x_j)2^{-j} \right) - \exp \left( 2\pi i \sum_{j=m+1}^{n(x^l)-1} \epsilon(x^l_j)2^{-j} \right) \right|
\]

\[
\leq 2\pi \left| \sum_{j=m+1}^{\infty} \epsilon(x_j)2^{-j} - \sum_{j=m+1}^{n(x^l)-1} \epsilon(x^l_j)2^{-j} \right|
\]

\[
\leq 2\pi 2^{-m}.
\]

As $m$ was arbitrary, this completes the proof. 

The following summarizes the compatibility of the different pseudo-metrics.

Proposition 3.13. Suppose $0 \leq j < k$, the sequences $x^l, l \geq 1$, and $y^l, l \geq 1$, are in $X^+_k$, $x$ and $y$ are in $X^+_j$, $x^l, l \geq 1$, converges to $x$ and $y^l, l \geq 1$, converges to $y$ in $X^+_G$, then

\[
d_j(x, y) = \lim_{l \to \infty} d_k(x^l, y^l).
\]

Proof. We first consider the case $n(x^l), l \geq 1$, tends to infinity. The fact that $x^l, l \geq 1$, converges in $X^+_G$, along with part (2) of Proposition 3.10 shows that $x^l, l \geq 1$, is Cauchy in $d_k$. Proposition 3.11 shows that there are two cases to consider. We begin by assuming that part (a) holds so $n(x^l)$ tends to infinity while $\theta(x^l), l \geq 1$, converges.

Lemma 3.12 implies that $x$ is in $X^+_0$. Hence, we have $j = 0$. We now claim that the sequence $n(y^l)$ must also tend to infinity. By the same argument as above, $y^l, l \geq 1$, is Cauchy in $d_k$. If $n(y^l), l \geq 1$, is eventually constant, say $n$, then $y_n = \lim_l y^l_n$ is not in $\xi(H^1)$ which means that $y$ is not in $X^+_0$, which is a contradiction.

Now, we consider the case $\theta(x) \neq \theta(y)$. From Lemma 3.12, we see that $\theta(x^l) \neq \theta(y^l)$ for $l$ sufficiently large. In this case, we know that

\[
d_k(x^l, y^l) = d_{G^k}(\tau_{\xi}(x^l), \tau_{\xi}(y^l)) + |2^{-n(x^l)} - 2^{-n(y^l)}| + d_{T}(\theta(x^l), \theta(y^l)).
\]

The first term converges to $d_{G^k}(\tau_{\xi}(x), \tau_{\xi}(y))$, the second to zero and the third to $d_{T}(\theta(x), \theta(y))$. The desired equality follows from the definition of $d_0(x, y)$.

We next turn to the case $\theta(x) = \theta(y)$. Here, we have either

\[
d_k(x^l, y^l) = d_{G^k}(\tau_{\xi}(x^l), \tau_{\xi}(y^l)) + |2^{-n(x^l)} - 2^{-n(y^l)}| + d_{T}(\theta(x^l), \theta(y^l))
\]
or else

\[ d_k(x^l, y^l) = dG_\xi (\tau_\xi (x^l), \tau_\xi (y^l)) + 2^{-2-n(x^l)} \lambda_0 (\sigma^n(x^l), \sigma^n(y^l)). \]

In either case, the limit is

\[ dG_\xi (\tau_\xi (x), \tau_\xi (y)) = dG_\xi (\tau_\xi (x), \tau_\xi (y)) + d_\tau (\theta (x), \theta (y)) = d_0(x, y). \]

We are left to consider the case that \( n(x^l) \) is eventually constant. It follows that \( \theta(x^l) \) is also. Suppose these values are \( n \) and \( \theta \), respectively. It follows that \( n(x) = n, \theta(x) = \theta \) and \( j > 0 \). If the sequence \( n(y^l) \) is unbounded, then, arguing as before for \( n(x^l) \), we have \( j = 0 \), which is a contradiction. Hence, the sequences \( n(y^l) \) and \( \theta(y^l) \) are also eventually constant, say with values \( n' \) and \( \theta' \), respectively.

As \( x_i^l \) (\( y_i^l \)) converges to \( x_i \) (\( y_i \), respectively), for \( 1 \leq i \leq n \) \((1 \leq i \leq n', \) respectively), we know then that \( n(x) = n, \theta(x) = \theta, n(y) = n' \) and \( \theta(y) = \theta' \). Let us first assume that \( (n, \theta) \neq (n', \theta') \). In this case, we have, for \( l \) sufficiently large,

\[ d_k(x^l, y^l) = dG_\xi (\tau_\xi (x^l), \tau_\xi (y^l)) + |2^{-n} - 2^{-n'}| + d_\tau (\theta, \theta'), \]

which clearly converges to

\[ dG_\xi (\tau_\xi (x), \tau_\xi (y)) + |2^{-n} - 2^{-n'}| + d_\tau (\theta, \theta') = d_j(x, y). \]

The second case to consider is \( (n, \theta) = (n', \theta') \). In this case, we have, for \( l \) sufficiently large,

\[ d_k(x^l, y^l) = dG_\xi (\tau_\xi (x^l), \tau_\xi (y^l)) + 2^{-2-n} \lambda_{k-1} (\sigma^n(x^l), \sigma^n(y^l)) \]

\[ = dG_\xi (\tau_\xi (x^l), \tau_\xi (y^l)) - 2^{-2-n} dG_\xi (\tau_\xi (\sigma^n(x^l)), \tau_\xi (\sigma^n(y^l))) \]

\[ + 2^{-2-n} d_{k-1} (\sigma^n(x^l), \sigma^n(y^l)). \]

It is clear that the \( \sigma^n(x^l), \sigma^n(y^l), l \geq 1 \), and points \( \sigma^n(x), \sigma^n(y) \) satisfy the hypotheses for integer where \( 0 \leq j - 1 < k - 1 \), so by the induction hypothesis, the sequence above converges to

\[ dG_\xi (\tau_\xi (x), \tau_\xi (y)) - 2^{-2-n} dG_\xi (\tau_\xi (\sigma^n(x)), \tau_\xi (\sigma^n(y))) \]

\[ + 2^{-2-n} d_{j-1} (\sigma^n(x), \sigma^n(y)) \]

\[ = d_j(x, y). \]

We are now ready to unify our metrics \( d_k \) into a single pseudo-metric.

Let us take a moment to recall (one approach to) taking the completion of a metric space \((X, d)\) (see [17, p. 196]). A key step is showing that if \( x^l, y^l, l \geq 1 \), are two Cauchy sequences in \( X \) with respect to \( d \), then the sequence of real numbers, \( d(x^l, y^l), l \geq 1 \), is also Cauchy and hence convergent. This proof only needs symmetry and the triangle inequality, and works equally well if \( d \) is a pseudo-metric.

Let us describe how we will use this. For \( x, y \in X^+_k \), we have the inequality \( d_k(x, y) \leq 3d_G(x, y) \). If we choose any sequences \( x^l, y^l, l \geq 1 \), in \( X^+_k \), converging to \( x, y \) in \( X^+_k \), respectively, then both are Cauchy in \( d_G \) and hence also in \( d_k \). The argument outlined above then implies that \( \lim_{l \to \infty} d_k(x^l, y^l) \) exists and it can be shown to be independent of the choice of sequences. If \( x, y \) are actually in \( X^+_k \), then this limit agrees with \( d_k(x, y) \). We refer to this as an extension of \( d_k \). We use this repeatedly in the following theorem.
THEOREM 3.14. Assume that $G$, $H$, $\xi$ satisfies the standing hypotheses. There exists a pseudo-metric, $d$, on $X^+_k$ satisfying the following items:

1. $d_G(\tau_\xi(x), \tau_\xi(y)) \leq d(x, y) \leq 3d_G(x, y)$, for all $x$, $y$ in $X^+_G$;
2. $d(x, y) = 0$ if and only if $x \sim \xi y$, for all $x$, $y$ in $X^+_G$;
3. $d(x, y) = d_k(x, y)$, for all $x$, $y$ in $X^+_k$.

Proof. For $k \geq 0$, the space $X^+_k$ carries two pseudo-metrics, $d_G$ and $d_k$. The former dominates the latter (part (2) of Proposition 3.10), at least up to a factor of 3. In addition, it is clear that $d_G(\tau_\xi(x), \tau_\xi(y)) \leq d_k(x, y)$, for all $x$, $y$ in $X^+_k$. We also recall from Proposition 3.5 that the closure of $X^+_k$ in $X^+_G$ is $\text{Cl}(X^+_k) = \bigcup_{j=0}^k X^+_j$. It follows that $d_k$ extends to a pseudo-metric on $\text{Cl}(X^+_k)$, which we denote $\bar{d}_k$, as described above which is also dominated by $3d_G$ and bounded below by $d_G(\pi_\xi(\cdot), \pi_\xi(\cdot))$. In addition, it follows from Proposition 3.13 that the restriction of $\bar{d}_k$ to $X^+_j \times X^+_j$ equals $d_j$, for any $0 \leq j \leq k$.

It also follows that $\bar{d}_k$ equals $\bar{d}_j$ after extending the latter to $\bigcup_{i=0}^j X^+_i$.

We may then define a pseudo-metric, $d_\infty$, on $\bigcup_{j=0}^\infty X^+_j$ which is $\bar{d}_k$ on $\bigcup_{j=0}^k X^+_j$. This is also bounded by $3d_G$ and so extends continuously to a pseudo-metric $d$ on the closure of $\bigcup_{j=0}^\infty X^+_j$ which is $X^+_G$ (Proposition 3.5) and is bounded below by $d_\xi(\pi_\xi(\cdot), \pi_\xi(\cdot))$ and above by $3d_G$.

The first and third properties of $d$ in the conclusion are immediate. For the second, we first suppose that $x \sim \xi y$. It follows from the definition of $\sim \xi$ that $x$, $y$ lie in $X^+_k$, for some $0 \leq k < \infty$ and hence $d(x, y) = d_k(x, y) = 0$, by Proposition 3.10.

Now assume that $x$, $y$ are in $X^+_G$ and $d(x, y) = 0$. It follows that $\tau_\xi(x) = \tau_\xi(y)$ which, in turn, implies that $\kappa(x) = \kappa(y)$ by Proposition 3.4. If this is finite, say $k$, then $x$, $y$ are in $X^+_k$ and the conclusion follows from Proposition 3.10. We now assume $\kappa(x) = \infty$ and $\tau_\xi(x) = \tau_\xi(y)$ also implies that $n(x) = n(y)$. From Proposition 3.5, we may find sequences, $x^l$, $y^l$ in $X^+_l$ converging to $x$, $y$, respectively. This means that

$$0 = d(x, y) = \lim_{l} d(x^l, y^l) = \lim_{l} d_l(x^l, y^l).$$

There exists $l_0 \geq 1$ such that, for all $l \geq l_0$, $x_n = x^l_n$, $y_n = y^l_n$, for $1 \leq n \leq n(x)$. From this, it follows that $n(x^l) = n(y^l) = n(x)$ and $\theta(x) = \theta(x^l)$, $\theta(y) = \theta(y^l)$, for $l \geq l_0$. If $\theta(x^l)$ and $\theta(y^l)$ are distinct for all $l$, then

$$d_l(x^l, y^l) \geq d_T(\theta(x^l), \theta(y^l)) \geq 2^{-n(x)},$$

which is a contradiction. It follows that, for some $l$, we have $\theta(x) = \theta(x^l) = \theta(y^l) = \theta(y)$. Together with the fact that $\tau_\xi(x_n) = \tau_\xi(y_n)$, this implies that $x_n = y_n$, for $1 \leq n \leq n(x)$.

We note that for $l \geq l_0$, $\sigma^{n(x)}(x^l)$, $\sigma^{n(x)}(y^l)$ are in $X^+_{l-1}$ and we have

$$d_{l-1}(\sigma^{n(x)}(x^l), \sigma^{n(x)}(y^l)) = d_{G_\xi}(\tau_\xi \circ \sigma^{n(x)}(x^l), \tau_\xi \circ \sigma^{n(x)}(y^l)) + \lambda_{l-1}(\sigma^{n(x)}(x^l), \sigma^{n(x)}(y^l))$$

$$= 2^{n(x)}d_{G_\xi}(\tau_\xi(x^l), \tau_\xi(y^l)) + \lambda_{l-1}(\sigma^{n(x)}(x^l), \sigma^{n(x)}(y^l))$$

$$\leq 2^{n(x)}d_l(x^l, y^l).$$
After re-indexing the sequence, we can apply the same argument to \(\sigma^{n(x)}(x)\), \(\sigma^{n(x)}(y)\), \(\sigma^{n(x)}(x^{l-1})\), \(\sigma^{n(x)}(y^{l-1})\), which shows that \(x_n = y_n\), for \(1 \leq n \leq n(x) + n(\sigma^{n(x)})\). Continuing in this way, we see that \(x = y\), as desired. 

**COROLLARY 3.15.** The space topological \(X^+_\xi\) has a metric, \(d_\xi\), inducing the topology such that:

1. \(d_G(\tau_\xi(x), \tau_\xi(y)) \leq d_\xi(\pi_\xi(x), \pi_\xi(y)) \leq 3d_G(x, y)\), for all \(x, y \in X^+_G\);
2. \(d_\xi(\pi_\xi(x), \pi_\xi(y)) = d_k(x, y)\), for all \(x, y \in X^+_k\).

**THEOREM 3.16.**

1. The map \(\rho_\xi : X^+_\xi \to X^+_{G_\xi}\) satisfies
   \[
   d_G(\rho_\xi(x), \rho_\xi(y)) \leq d_\xi(x, y),
   \]
   for all \(x, y \in X^+_\xi\).
2. The map \(\theta : X^+_G \to T\) is constant on \(\sim_\xi\)-equivalence classes and so there is a map (also) denoted by \(\theta : X^+_\xi \to T\) which satisfies
   \[
   d_{T}(\theta(x), \theta(y)) \leq d_\xi(x, y),
   \]
   for all \(x, y \in X^+_\xi\).

**Proof.** It is a trivial consequence of the definition of \(d_k\) that the inequality holds with \(d_k\) on the right-hand side for \(x, y \in X^+_k\), for any \(k \geq 1\). The conclusion holds since \(d_\xi\) is the common extension of all \(d_k\).

As we mentioned earlier, it is an easy matter to see that \(X^+_\xi\) is metrizable. We have gone to considerable trouble to actually produce a metric. The reason is that it satisfies some nice properties and the following is the most important. We remark that the same property holds for the map \(\sigma\) on \((X^+_G, d_G)\).

**THEOREM 3.17.** If \(x, y \in X^+_\xi\) satisfy \(d_\xi(x, y) \leq 2^{-2}\), then
\[
2d_\xi(x, y) \leq d_\xi(\sigma_\xi(x), \sigma_\xi(y)) \leq 8d_\xi(x, y).
\]

**Proof.** We will prove the analogous statement for \(d_k, \sigma\) and \(x, y \in X^+_k\), for some fixed \(k \geq 0\).

We first consider \(k = 0\) and note that \(\sigma(X^+_0) = X^+_0\). For \(x, y \in X^+_0\), we have
\[
d_0(x, y) = d_G(\tau_\xi(x), \tau_\xi(y)) + d_T(\theta(x), \theta(y)).
\]
Our hypothesis implies both terms on the right are at most \(2^{-2}\). This implies that \(\tau_\xi(x_1) = \tau_\xi(y_1)\) and hence
\[
d_G(\tau_\xi(\sigma(x)), \tau_\xi(\sigma(y))) = d_G(\sigma \circ \tau_\xi(x), \sigma \circ \tau_\xi(y)) = 2d_G(\tau_\xi(x), \tau_\xi(y))
\]
\[
\]
and \( d_T(\theta(x), \theta(y)) \leq 2^{-2} \) and hence
\[
d_T(\theta(\sigma(x)), \theta(\sigma(y))) = d_T(\theta(x)^2, \theta(y)^2) = 2d_T(\theta(x), \theta(y)).
\]
It follows that \( d_0(\sigma(x), \sigma(y)) = 2d_0(x, y) \) and we are done.

Now assume \( k \geq 1 \). First, we note that \( d_k(x, y) \leq 2^{-2} \) implies that \( d_{G_\xi}(\tau_\xi(x), \tau_\xi(y)) \leq 2^{-2} \) so \( \tau_\xi(x_i) = \tau_\xi(y_i) \), for \( i = 1, 2 \). It follows that \( n(x) = n(y) \) or both are at least 3.

Our first case is \( n(x) = n(y) = 1 \). It follows that \( \sigma(x), \sigma(y) \) are both in \( X_{k-1}^+ \). We have \( n(x) = n(y) = 1 \) so
\[
d_\xi(x, y) = d_k(x, y)
= d_{G_\xi}(\tau_\xi(x), \tau_\xi(y)) + \lambda_k(x, y)
= d_{G_\xi}(\tau_\xi(x), \tau_\xi(y)) + 2^{-3}\lambda_{k-1}(\sigma(x), \sigma(y))
= 2^{-1}d_{G_\xi}(\tau_\xi(\sigma(x)), \tau_\xi(\sigma(y))) + 2^{-3}\lambda_{k-1}(\sigma(x), \sigma(y)).
\]
The right-hand side is clearly bounded above by
\[
2^{-1}d_{G_\xi}(\tau_\xi(\sigma(x)), \tau_\xi(\sigma(y))) + 2^{-1}\lambda_{k-1}(\sigma(x), \sigma(y)) = 2^{-1}d_{k-1}(\sigma(x), \sigma(y))
\]
and below by
\[
8^{-1}d_{G_\xi}(\tau_\xi(\sigma(x)), \tau_\xi(\sigma(y))) + 8^{-1}\lambda_{k-1}(\sigma(x), \sigma(y)) = 8^{-1}d_{k-1}(\sigma(x), \sigma(y)).
\]
Next, we consider \( n(x) = n(y) = 2 \). Also, \( \theta(x), \theta(y) \) are both \( \pm 1 \). If they are distinct, we have
\[
d_\xi(x, y) \geq \lambda_k(x, y) \geq d_T(\theta(x), \theta(y)) \geq 2^{-1},
\]
which is a contradiction. Hence, we have \( \theta(x) = \theta(y) \). So we have \( x, y \) are still in \( X_k^+ \), \( n(\sigma(x)) = n(x) - 1 = n(\sigma(y)) \) and \( \theta(\sigma(x)) = \theta(x)^2 = \theta(y)^2 = \theta(\sigma(y)) \) and
\[
d_\xi(x, y) = d_k(x, y)
= d_{G_\xi}(\tau_\xi(x), \tau_\xi(y)) + \lambda_k(x, y)
= 2^{-1}d_{G_\xi}(\tau_\xi(\sigma(x)), \tau_\xi(\sigma(y))) + 2^{-n(x)-2}\lambda_{k-1}(\sigma^{n(x)}(x), \sigma^{n(y)}(y))
= 2^{-1}(d_{G_\xi}(\tau_\xi(\sigma^{n(x)}(x)), \tau_\xi(\sigma^{n(x)}(y)))
+ 2^{-1}2^{-n(x)-1}\lambda_{k-1}(\sigma(\sigma(x)), \sigma(\sigma(y))))
= 2^{-1}(d_{G_\xi}(\tau_\xi(\sigma^{n(x)}(x)), \tau_\xi(\sigma^{n(x)}(y)))
+ 2^{-1}\lambda_k(\sigma(x), \sigma(y)))
= 2^{-1}d_k(\sigma(x), \sigma(y)).
\]
Finally, we consider the case \( n(x), n(y) > 2 \). Here again, we have \( n(\sigma(x)) = n(x) - 1, n(\sigma(y)) = n(y) - 1, \theta(\sigma(x)) = \theta(x)^2 \) and \( \theta(\sigma(y)) = \theta(y)^2 \). Let us first assume that \( (n(x), \theta(x)) \neq (n(y), \theta(y)) \). We claim that \( (n(\sigma(x)), \theta(\sigma(x))) \neq (n(\sigma(y)), \theta(\sigma(y))) \). If they are equal, then \( n(x) - 1 = n(y) - 1 \) so \( n(x) = n(y) \). It follows that \( \theta(x) \neq \theta(y) \), while \( \theta(x)^2 = \theta(y)^2 \), which implies \( \theta(x) = -\theta(y) \). This in turn implies that \( d_k(x, y) \geq 2^{-1}d_k(\sigma(x), \sigma(y)). \)
\(d_T(\theta(x), \theta(y)) = 2^{-1}\), which is a contradiction. From the fact that \(2^{-2} \geq d_k(x, y) \geq d_T(\theta(x), \theta(y))\), we see that \(d_T(\theta(x)^2, \theta(y)^2) = 2d_T(\theta(x), \theta(y))\).

Now we have
\[
d_\xi(x, y) = d_k(x, y)
= d_{G_\xi}(\tau_\xi(x), \tau_\xi(y)) + \lambda_k(x, y)
= d_{G_\xi}(\tau_\xi(x), \tau_\xi(y)) + |2^{-n(x)} - 2^{-n(y)}| + d_T(\theta(x), \theta(y))
= 2^{-1}d_{G_\xi}(\tau_\xi(\sigma(x)), \tau_\xi(\sigma(y))) + 2^{-1}|2^{1-n(x)} - 2^{1-n(y)}|
+ 2^{-1}d_T(\theta(x)^2, \theta(y)^2)
= 2^{-1}d_k(\sigma(x), \sigma(y))
= 2^{-1}d_\xi(\sigma(x), \sigma(y)).
\]

The final case is \((n(x), \theta(x)) = (n(y), \theta(y))\). Here, we have \((n(\sigma(x)), \sigma(\theta(x))) = (n(x) - 1, \theta(x)^2) = (n(\sigma(y)), \sigma(\theta(y)))\) and
\[
d_\xi(x, y) = d_k(x, y)
= d_{G_\xi}(\tau_\xi(x), \tau_\xi(y)) + \lambda_k(x, y)
= d_{G_\xi}(\tau_\xi(x), \tau_\xi(y)) + 2^{-n(x)} - 2^{-n(y)}\lambda_{k-1}(\sigma_n(x), \sigma_n(y))
= 2^{-1}d_{G_\xi}(\tau_\xi(\sigma(x)), \tau_\xi(\sigma(y)))
+ 2^{1-n(x)} - 2^{1-n(y)}\lambda_{k-1}(\sigma_n(x), \sigma_n(y))
= 2^{-1}d_{G_\xi}(\tau_\xi(\sigma(x)), \tau_\xi(\sigma(y))) + \lambda_k(\sigma(x), \sigma(y))
= 2^{-1}d_k(\sigma(x), \sigma(y))
= 2^{-1}d_\xi(\sigma(x), \sigma(y)).
\]

We finish this section with an alternate version of the local expanding property which will be useful in the later sections.

**Lemma 3.18.** If \(x, y\) are in \(X_G^+\) satisfy \(x_1 = y_1\) and \(d_\xi(\sigma(x), \sigma(y)) \leq 2^{-1}\), then \(d_\xi(x, y) \leq 2^{-1}d_\xi(\sigma(x), \sigma(y))\).

**Proof.** By Proposition 3.5, we may find sequences \(x^k, k \geq 0\), and \(y^k, k \geq 0\), converging to \(x\) and \(y\), respectively, and \(x^k, y^k\) are in \(X_k^+\). So it suffices to prove the statement for \(x, y\) in \(X_k^+\), for \(k \geq 2\).

We consider three cases separately. The first is when \(x_1 = y_1\) is not in \(\xi(H^1)\). This implies that \(n(x) = n(y) = 1, \theta(x) = \theta(y) = 1\) and \(n(\sigma(x)) = n(\sigma(y)) = n(x) - 1\). It also means that \(\sigma(x), \sigma(y)\) are in \(X_{k-1}^+\). It follows that
\[
d_\xi(\sigma(x), \sigma(y)) = d_{G_\xi}(\tau_\xi(\sigma(x)), \tau_\xi(\sigma(y))) + \lambda_{k-1}(\sigma(x), \sigma(y))
= 2d_{G_\xi}(\tau_\xi(x), \tau_\xi(y)) + 2^3\lambda_k(x, y)
\geq 2d_\xi(x, y).
\]

The second case is \(x_1 = y_1\) is in \(\xi(H^1)\), which implies that \(n(x), n(y) > 1\), and \((n(x), \theta(x)) \neq (n(y), \theta(y))\). We have \(n(\sigma(x)) = n(x) - 1\) and \(n(\sigma(y)) = n(y) - 1\),...
\(\theta(\sigma(x)) = \theta(x)^2, \theta(\sigma(y)) = \theta(y)^2\). It also means that \(\sigma(x), \sigma(y)\) are in \(X^+_G\). We have
\[
d_\xi(\sigma(x), \sigma(y)) = d_{G, \xi}(\tau_\xi(\sigma(x)), \tau_\xi(\sigma(y))) + \lambda_k(\sigma(x), \sigma(y))
\]
\[
= 2d_{G, \xi}(\tau_\xi(x), \tau_\xi(y)) + |2^{-n(x)+1} - 2^{-n(y)+1}|
\]
\[
+ d_T(\theta(x)^2, \theta(y)^2)
\]
\[
= 2d_{G, \xi}(\tau_\xi(x), \tau_\xi(y)) + 2|2^{-n(x)} - 2^{-n(y)}|
\]
\[
+ d_T(\theta(x)^2, \theta(y)^2).
\]

The hypothesis that \(d_\xi(\sigma(x), \sigma(y)) \leq 2^{-1}\) implies that \(d_T(\theta(x)^2, \theta(y)^2) \leq 2^{-1}\). The fact that \(x_1 = y_1\) implies \(\epsilon(x_1) = \epsilon(y_1)\). Together, these imply that \(d_T(\theta(x)^2, \theta(y)^2) = 2d_T(\theta(x), \theta(y))\). Putting this in the last line above yields \(d_\xi(\sigma(x), \sigma(y)) = 2d_\xi(x, y)\).

The final case to consider is that \(x_1 = y_1\) is in \(\xi(H^1)\), which implies that \(n(x), n(y) > 1\), and \((n(x), \theta(x)) = (n(y), \theta(y))\). Again, we know \(\sigma(x), \sigma(y)\) are in \(X^+_G\). Here, we have \(n(\sigma(x)) = n(x) - 1 \equiv n(\sigma(y)) = n(y) - 1\), and \(\theta(\sigma(x)) = \theta(x)^2 = \theta(\sigma(y)) = \theta(y)^2\). We have
\[
d_\xi(\sigma(x), \sigma(y)) = d_{G, \xi}(\tau_\xi(\sigma(x)), \tau_\xi(\sigma(y))) + \lambda_k(\sigma(x), \sigma(y))
\]
\[
= 2d_{G, \xi}(\tau_\xi(x), \tau_\xi(y)) + 2^{-2-(n(x)-1)}\lambda_{k-1}(\sigma^{n(x)-1}(\sigma(x)), \sigma^{n(y)-1}(\sigma(y)))
\]
\[
= 2(\xi(X^+_G) + 2^{-2-n(x)}\lambda_k(\sigma^{n(x)}(x), \sigma^{n(y)}(y)))
\]
\[
= 2d_\xi(x, y).
\]

**Theorem 3.19.** If \(x, y\) are in \(X^+_G\) with \(d_\xi(x, \sigma_\xi(y)) \leq 2^{-1}\), then there is \(z\) in \(X^+_G\) with \(d_\xi(z, y) \leq 2^{-1}d_\xi(x, \sigma_\xi(y))\) and \(\sigma_\xi(z) = x\). In particular, we have
\[
X^+_G(\sigma_\xi(x), \epsilon) \subseteq \sigma_\xi(X^+_G(\sigma(x), 2^{-1}\epsilon))
\]
and the map \(\sigma_\xi\) is open.

**Proof.** Write \(x = \pi_\xi(x')\), for some \(x'\) in \(X^+_G\). Let \(y'\) be in \(X^+_G\) with \(\pi_\xi(y') = y\). As \(\epsilon \leq 2^{-3}\), we know that \(d_{G, \xi}(\tau_\xi(x'), \tau_\xi(\sigma(y'))) \leq 2^{-3}\) also. This implies that \(\tau_\xi(x'_1) = \tau_\xi(y'_2)\), which in turn implies that \(i(x'_1) = i(y'_2) = i(y'_1)\). It follows that \(z' = (y'_1, x'_1, x'_2, \ldots)\) is in \(X^+_G\). Applying Lemma 3.18 to \(z', y'\) gives
\[
d_\xi(y, z) = d_\xi(y', z') \leq 2^{-1}d_\xi(\sigma(y'), \sigma(z')) = 2^{-1}d_\xi(\sigma(y), x).
\]

4. **Smale spaces**

The results of §3 showed a construction of a compact metric space \((X^+_G, d_\xi)\) along with a map \(\sigma_\xi : X^+_G \to X^+_G\) which is continuous, surjective and open. It also satisfies a local expansiveness condition in Theorem 3.17. Our goal in this section is to replace this system with another where the dynamics is actually a homeomorphism. This is a standard construction via inverse limits. Without giving the precise definition for the moment, this introduces a new component to the space where the dynamics is contracting. In short, we
are going to build a Smale space. Two references for Smale spaces are the works of Ruelle [30] and Putnam [24].

Let us begin with recalling the definition of a Smale space. First of all, \((X, d)\) is a compact metric space and \(\varphi\) is a homeomorphism of \(X\). We assume that there are constants \(\epsilon_X > 0, 0 < \lambda < 1\) and a continuous map 
\[
\cdot, \cdot : \{(x, y) \mid x, y \in X, d(x, y) \leq \epsilon_X\} \to X,
\]
which satisfies the following statements:

- **B1** \([x, x] = x;\)
- **B2** \([x, [y, z]] = [x, z];\)
- **B3** \([[x, y], z] = [x, z];\)
- **B4** \([\varphi(x), \varphi(y)] = \varphi[x, y].\)

for all \(x, y, z\) in \(X\), whenever both sides of an equation are defined, and finally:

- **C1** if \([x, y] = y\), then \(d(\varphi(x), \varphi(y)) \leq \lambda d(x, y);\)
- **C2** if \([x, y] = x\), then \(d(\varphi^{-1}(x), \varphi^{-1}(y)) \leq \lambda d(x, y),\)

for \(x, y\) with \(d(x, y) \leq \epsilon_X\).

We say that \((X, d, \varphi)\) is a Smale space.

We now consider our specific case of interest. We will usually work under the standing hypotheses. It seems likely that it is sufficient for \(G\) to be irreducible. That is only used in an essential way in Proposition 3.5, but it seems possible that some other hypothesis on \(\xi\) may be needed for the construction of the last section to work.

**Definition 4.1.** Let \(G, H, \xi\) satisfy the standing hypotheses. We define \(X_\xi\) to be the inverse limit of the system
\[
X_\xi^+ \xleftarrow{\sigma_\xi} X_\xi^+ \xleftarrow{\sigma_\xi} \cdots.
\]

That is,
\[
X_\xi = \{(x^0, x^1, \ldots) \mid x^n \in X_\xi^+, x^n = \sigma_\xi(x^{n+1}), n \geq 0\}.
\]

We also define \(\sigma_\xi : X_\xi \to X_\xi\) by \(\sigma_\xi(x)^n = \sigma_\xi(x^n)\), for \(n \geq 0\) and \(x^n, n \geq 1\) in \(X_\xi\). It is a simple matter to check that the inverse is given by
\[
\sigma_\xi^{-1}(x)^n = x^{n+1}, \quad n \geq 0,
\]
for any \(x\) in \(X_\xi\).

Finally, we define a metric, also denoted \(d_\xi\), on \(X_\xi\) by
\[
d_\xi(x, y) = \sup\{2^{-n}d_\xi(x^n, y^n) \mid n \geq 0\},
\]
for \(x = (x^n)_{n \geq 0}, y = (y^n)_{n \geq 0}\) in \(X_\xi\).

We remark that if we replace \(X_\xi^+, \sigma_\xi\) by \(X^+_G, \sigma_G\), the result is \(X_G, \sigma_G\) in a canonical way by identifying \(x\) in \(X_G\) with the sequence \(x^n = \chi^+_G(\sigma^{-n}(x)) = (x_{1-n}, x_{2-n}, \ldots)\), for \(n \geq 0\). Pursuing this remark a little further, it is a fairly simple matter to check that the map which sends \(x\) in \(X_G\) to the sequence \(\pi_\xi(\chi^+_G(\sigma^{-n}(x))), n \geq 0\), in \(X_\xi\) is well defined. We denote the map by \(\pi_\xi\). It is immediate that \(\pi_\xi \circ \sigma = \sigma_\xi \circ \pi_\xi\).
The system \((X_G, d_G, \sigma)\) is a Smale space with constants \(\epsilon_{X_G} = \lambda = 2^{-1}\) and bracket defined as \([x, y] = (\ldots, y_{-1}, y_0, x_1, \ldots)\), for \(d_G(x, y) \leq 2^{-1}\), which ensures \(x_0 = y_0, x_1 = y_1\).

To see that \((X_\xi, d_\xi, \sigma_\xi)\) is a Smale space, we need to define the bracket map. Let \(x = (x_n)_{n \geq 0}, y = (y_n)_{n \geq 0}\) be in \(X_\xi\) and assume that \(d_\xi(x, y) \leq 2^{-1}\).

We inductively define \(z^n, n = 0, 1, 2, 3, \ldots\) in \(X_\xi^+\) satisfying:

(1) \(d_\xi(z^n, y^n) \leq 2^{-n-1};\)

(2) \(\sigma_\xi(z^{n+1}) = z^n.\)

Begin with \(z^0 = x^0\). Observe the first condition holds for \(n = 0\) and

\[
d_\xi(x^0, y^0) = d_\xi(x, y) \leq 2^{-1}.\]

To define \(z^{n+1}\), we have \(d_\xi(z^n, \sigma_\xi(y^{n+1})) = d_\xi(z^n, y^n) \leq 2^{-n-1}\). By Theorem 3.19, we may find \(z^{n+1}\) in \(X_\xi^+\) with \(\sigma_\xi(z^{n+1}) = z^n\) and \(d_\xi(z^{n+1}, y^{n+1}) \leq 2^{-1}d_\xi(z^n, y^n) \leq 2^{-n-2}\). So \(z = (z^0, z^1, \ldots)\) lies in \(X_\xi\).

We use the constants \(\epsilon_{X_\xi} = \lambda = 2^{-1}\). Notice that \(d_\xi(x, y) \leq 2^{-1}\) implies that \(d_\xi(x^0, y^0) \leq 2^{-1}\), which ensures that \([x, y]\) is defined.

We will not verify the axioms B1–B4. As for axiom C1, suppose that \(z = [x, y] = y\). It follows that \(y^0 = x^0\) and so

\[
d(\sigma_\xi(x), \sigma_\xi(y)) = \sup\{2^{-n}d_\xi(\sigma_\xi(x^n), \sigma_\xi(y^n)) \mid n \geq 0\}
= \sup\{2^{-n}d_\xi(\sigma_\xi(x^n), \sigma_\xi(y^n)) \mid n \geq 1\}
= \sup\{2^{-n}d_\xi(x^{n-1}, y^{n-1}) \mid n \geq 1\}
= 2^{-1}\sup\{2^{-m}d_\xi(x^m, y^m) \mid m \geq 0\}
= 2^{-1}d_\xi(x, y).
\]

We also verify axiom C2: if \(z = [x, y] = x\), then we have

\[
d(\sigma_\xi^{-1}(x), \sigma_\xi^{-1}(y)) = \sup\{2^{-n}d_\xi(x^{n+1}, y^{n+1}) \mid n \geq 0\}
= \sup\{2^{-n}2^{-1}d_\xi(x^n, y^n) \mid n \geq 0\}
= 2^{-1}d_\xi(x, y).
\]

**Theorem 4.2.** With constants \(\epsilon_{X_\xi} = \lambda = 2^{-1}\) and bracket map as defined above, \((X_\xi, d_\xi, \sigma_\xi)\) is a Smale space.

**Theorem 4.3**

(1) The map \(\pi_\xi : (X_G, d_G, \sigma) \to (X_\xi, d_\xi, \sigma_\xi)\) is a factor map.

(2) For each \(x, y\) in \(X_G\) such that \(d_G(x, y) \leq 2^{-3}\), both \([x, y]\) and \([\pi_\xi(x), \pi_\xi(y)]\) are defined and \(\pi_\xi[x, y] = [\pi_\xi(x), \pi_\xi(y)]\).

(3) Points \(x, y\) in \(X_G\) satisfy \(\pi_\xi(x) = \pi_\xi(y)\) if and only if exactly one of the following hold:

(a) \(x = y;\)

(b) there exists \(i = 0, 1\) and \(z\) in \(X_H\) such that \(\xi^i(z_n) = x_n, \xi^{1-i}(z_n) = y_n\), for all integers \(n;\)
5. Homology

(c) there is an integer m, i = 0, 1 and z in X_H such that:
   (i) \( x_n = y_n \), for all \( n < m \);
   (ii) \( x_m = y_m \) is not in \( \xi(H^1) \) or \( \xi^{1-i}(z_m) = x_m, \xi^i(z_m) = y_m \); and
   (iii) \( \xi^i(z_n) = x_n, \xi^{1-i}(z_n) = y_n \), for all integers \( n > m \).

(4) The map \( \pi_\xi \) is s-bijective; that is, for every \( x \) in \( X_G \), \( \pi_\xi|X^+_G(x) \) is a bijection from \( X^+_G(x) \) to \( X^+_\xi(\pi_\xi(x)) \).

Proof. The first part is easy, as we have already noted.

For the second part, if \( d_G(x, y) \leq 2^{-3} \), then \( [x, y] \) is defined. In addition, by Corollary 3.15, \( d_\xi(\pi_\xi(x), \pi_\xi(y)) \leq 3 \cdot 2^{-3} < 2^{-1} \), so \( [\pi_\xi(x), \pi_\xi(y)] \) is also defined.

We define, for \( n \geq 0 \), \( z^n = \pi_\xi(y_{-n}, \ldots, y_{-1}, x_0, x_1, \ldots) \) in \( X^+_\xi \). It is clear that \( z^0 = \pi_\xi(\chi^+_G(x)) \) and \( \sigma_\xi(z^n) = z^{n-1} \), for all \( n \geq 1 \). We claim that \( d_\xi(z^n, y^n) \leq 2^{-n-1} \), which we show by induction. The case \( n = 0 \) is simply the fact that \( d_\xi(\pi_\xi(x), \pi_\xi(y)) \leq 2^{-1} \), which we have already established. For any \( n \geq 1 \), Lemma 3.18 and the induction hypothesis show that

\[
d_\xi(z^n, y^n) \leq 2^{-1}d_\xi(z^{n-1}, y^{n-1}) \leq 2^{-n-1}.
\]

It follows from the definition of the bracket on \( X_\xi \) that \( z = [\pi_\xi(x), \pi_\xi(y)] \). However, we have

\[
(\pi_\xi[x, y])^n = \pi_\xi([x, y]^n) = \pi_\xi(\chi^+_G(\sigma^{-n}[x, y])) = z^n,
\]

by definition.

For the third part, it is clear that if \( x, y \) satisfy conditions (b) or (c), then \( \chi^+_G(\sigma^{-n}(x)) \sim_\xi \chi^+_G(\sigma(y)) \), for all integers \( n \), and it follows that \( \pi_\xi(x)^n = \pi_\xi(y)^n \), for all integers \( n \).

Conversely, if \( \pi_\xi(x) = \pi_\xi(y) \), then \( \chi^+_G(\sigma^{-n}(x)) \sim_\xi \chi^+_G(\sigma^{-n}(y)) \), for all integers \( n \). According to the definition of \( \sim_\xi \), one possibility is that there is a unique path \( z^n \) in \( X^+_H \) and \( i_n = 0, 1 \) such that

\[
\chi^+_G(\sigma^{-n}(x)) = \xi^{i_n}(z^n), \quad \chi^+_G(\sigma^{-n}(y)) = \xi^{1-i_n}(z^n).
\]

Let us first suppose that this holds for all integers \( n \). It follows \( i_{n+1} = i_n \) and \( \sigma(z^{n+1}) = z^n \). We then define \( z \) in \( X_H \) by \( z_n = z_n^0 \), for all \( n \geq 1 \), and \( z_n = z_{n-1}^{-1} \), for all \( n \leq 0 \), and part (b) holds with \( i = i_1 \). Otherwise, the condition fails for some integer \( n \). It is a simple matter to check that part (c) holds in this case.

For the last part, if \( \pi_\xi(x) = \pi_\xi(y) \), then by part (3), \( x \) and \( y \) are not stably equivalent (that is, right tail-equivalent) unless \( x = y \). This shows that \( \pi_\xi|X^+_G(x) \) is injective. The surjectivity follows from the fact that \( G \) primitive implies \( (X_G, \sigma) \) is non-wandering (see [18, Proposition 2.2.14]), and [24, Theorem 2.5.8].

In [24], a homology theory for (non-wandering) Smale spaces is described. Specifically, given a non-wandering Smale space \( (X, d, \varphi) \) and integer \( n \), there are two countable abelian groups, \( H^n_d(X, \varphi) \) and \( H^n_d(X, \varphi) \). The former invariant is covariant for \( s \)-bijective factor maps and contravariant for \( u \)-bijective factor maps, while the latter is covariant for
u-bijective factor maps and contravariant for s-bijective factor maps. As such, the map \( \varphi \) induces a pair of automorphisms of each. The aim of this section is to compute these invariants for the systems \((X_\xi, d_\xi, \sigma_\xi)\) we have constructed.

Let us begin with some preliminary notions. If \( G \) is any finite directed graph, we may consider the free abelian group on the vertex set \( G^0 \), denoted \( \mathbb{Z}G^0 \). This comes with two canonical endomorphisms:

\[
\gamma^{s}_{G}(v) = \sum_{i(e) = v} t(e), \quad \gamma^{u}_{G}(v) = \sum_{t(e) = v} i(e),
\]

for any \( v \) in \( G^0 \). This is needed at various places in the theory and in our proofs below. However, for the statements of the results, we can use the more familiar group \( \mathbb{Z}G^0 \), which is isomorphic to \( \mathbb{Z}G^0 \) in an obvious way. Under this isomorphism, the map \( \gamma^{s}_{G} \) becomes multiplication by the matrix \( AG \), while \( \gamma^{u}_{G} \) is multiplication by its transpose. Then we define \( D^{s}(G) \) as the inductive limit of the sequence:

\[
\mathbb{Z}G^0 \xrightarrow{A} \mathbb{Z}G^0 \xrightarrow{A} \mathbb{Z}G^0 \xrightarrow{A} \ldots
\]

Similarly, the invariant \( D^{u}(G) \) is computed in a similar way, but with the transpose, \( A^T \), replacing \( AG \).

With this notation, we can summarize our results with the following theorem.

**Theorem 5.1.** Let \( G, H, \xi \) satisfy the standing hypotheses. We have

\[
\begin{align*}
H^s_0(X_\xi, \sigma_\xi) & \cong D^s(G), \quad (\sigma_\xi)^{-1}_* \cong A_G, \\
H^s_1(X_\xi, \sigma_\xi) & \cong D^s(H), \quad (\sigma_\xi)^{-1}_* \cong A_H, \\
H^u_0(X_\xi, \sigma_\xi) & \cong D^u(G), \quad (\sigma_\xi)^*_* \cong A^T_G, \\
H^u_1(X_\xi, \sigma_\xi) & \cong D^u(H), \quad (\sigma_\xi)^*_* \cong A^T_H, \\
H^s_k(X_\xi, \sigma_\xi) & \cong 0, \quad k \neq 0, 1, \\
H^u_k(X_\xi, \sigma_\xi) & \cong 0, \quad k \neq 0, 1.
\end{align*}
\]

In the first two lines, we regard \( \sigma_\xi \) as an s-bijective factor map from \((X_\xi, \sigma_\xi)\) to itself and our description of the induced map on homology is interpreted via the description which precedes it. In the next two lines, we regard \( \sigma_\xi \) as a u-bijective factor map.

In fact, we will prove only the first, second and fifth parts as the other three are quite similar.

The starting point for computation of the homology theory for a general Smale space, \((X, \varphi)\), is to find an s/u-bijective pair: Smale spaces \((Y, \psi)\) and \((Z, \xi)\) such that \( Y^u(y) \) and \( Z^s(z) \) are totally disconnected, for all \( y \) in \( Y \) and \( z \) in \( Z \), along with an s-bijective factor map \( \pi_s : (Y, \psi) \to (X, \varphi) \) and a u-bijective factor map \( \pi_u : (Z, \xi) \to (X, \varphi) \). Usually, the collection \((Y, \psi, \pi_s, Z, \xi, \pi_u)\) is written as \( \pi \).

Given such an s/u-bijective pair, one proceeds to define dynamical systems as fibred products: for \( L, M \geq 0 \),

\[
\Sigma_{L,M}(\pi) = \{(y^0, \ldots, y^L, z^0, \ldots, z^M) \in Y^{L+1} \times Z^{M+1} | \pi_s(y^l) = \pi_u(z^m), \text{ all } l, m\}.
\]
Each of these systems is a shift of finite type and $\Sigma_{L,M}(\pi)$ carries an obvious action of the group $S_{L+1} \times S_{M+1}$.

The homology $H^s_\pi$ is computed from a double complex whose groups are denoted $C^s_{Q,A}(\pi)_{L,M}$. These are obtained by considering the dimension groups $D^s(\Sigma_{L,M}(\pi))$ and taking first subgroups and then quotient groups obtained from the actions of $S_{M+1}$ and $S_{L+1}$, respectively. This is an analogy with full and alternating complexes in algebraic topology.

**Theorem 5.2.** Let $G, H, \xi$ satisfy the standing hypotheses. Then $(X_G, \sigma, \pi_\xi, X_\xi, \sigma_\xi, \text{id}_{X_\xi})$ is an s/u-bijective pair for $(X_\xi, \sigma_\xi)$, which we denote by $\pi_\xi$. Moreover, we have $\Sigma_{0,0}(\pi_\xi) = X_G$ and $C^s_{Q,A}(\pi_\xi)_{0,0} \cong D^4(G)$.

The next step is to find a graph which gives a symbolic presentation for $\Sigma_{0,0}(\pi_\xi)$. This requires that, for any $x, y$ in $X_G$ satisfying $t(x_0) = t(y_0)$, then $\pi_\xi(x, y) = [\pi_\xi(x), \pi_\xi(y)]$, meaning that both sides are defined. Unfortunately, $G$ will not suffice but this is not a major obstacle.

We review the standard notion of the higher block coding of $G$ (see [18, §2.3]). We let $G^K$ be the set of all paths of length $K$ in $G$, for any $K \geq 2$, and define $i, t : G^K \to G^{K-1}$ by

$$i(x_1, \ldots, x_K) = (x_1, \ldots, x_{K-1}), \quad t(x_1, \ldots, x_K) = (x_2, \ldots, x_K),$$

for $(x_1, \ldots, x_K)$ in $G^K$. In this way, we interpret $G^K, G^{K-1}, i, t$ as a graph. For any $1 \leq k \leq K$, the map sending $(x_1, \ldots, x_K)$ in $G^K$ to $x_k$ induces a homeomorphism between $X_{G^K}$ and $X_G$ which intertwines the shift maps. It will not be necessary for us to write this map explicitly: we simply regard $X_{G^K}$ and $X_G$ as equal, in an obvious way.

We use $K = 7$ and $k = 4$. The point is that, if $x, y$ are in $X_G$ and $t(x_0) = t(y_0)$, then regarding them as elements in $X_G^7$ means $x_{[-2,3]} = y_{[-2,3]}$, which implies $d(x, y) \leq 2^{-3}$ and so part (2) of Theorem 4.3 applies and $G^7$ satisfies the conditions of [24, Definition 2.6.8].

We remark that the map $t : G^6 \to G^0$ induces an isomorphism between $D^4(G^7)$ and $D^4(G)$.

Our two maps are $\pi_s = \pi_\xi$ and $\pi_u$ being the identity on $X_\xi$. Due to part (3) of Theorem 4.3, we see that $\#\pi_s(x) \leq 2 = L_0$ and obviously $\#\text{id}^{-1}(x) \leq 1 = M_0$, for all $x$ in $X_\xi$, so by [24, Theorem 5.1.10], our complex consists of only two non-zero groups, $C^s_{Q,A}(\pi_\xi)_{0,0}$ and $C^s_{Q,A}(\pi_\xi)_{1,0}$, and a single homomorphism from the latter to the former. So what are these groups, how do we compute them and what is the homomorphism between them? The first is given already in Theorem 5.2. In the case $L = M = 0$, the permutation groups are both trivial and $C^s_{Q,A}(\pi_\xi)_{0,0} = D^4(G)$.

We now come to the second group, $C^s_{Q,A}(\pi_\xi)_{1,0}$, where it will be necessary to use our higher block presentation. The system $\Sigma_{(1,0)}(\pi_\xi)$ of [24] consists of pairs

$$\Sigma_{(1,0)}(\pi_\xi) = \{(x^0, x^1) \mid x^0, x^1 \in X_G, \pi_\xi(x^0) = \pi_\xi(x^1)\}.$$

Fortunately, Theorem 4.3 gives a nice description of these. The fact that our map is regular means that this is the shift of finite type associated with an obvious subgraph of $G^7_1 \subseteq G^7 \times G^7$, whose vertex set is $G^6_1 \subseteq G^6 \times G^6$, with obvious maps $i, t$. (This suppresses
LEMMA 5.4. from a technical point in its computation is the following lemma. Our notation is slightly different — v and it follows that the inductive limit only uses that part. However, when we quotient by equivalent, then all in Kξ

LEMMA 5.3. We have C_{Q,A}(\pi_\xi)_1 \cong D^7(H).

We are now left to consider the boundary map between the two groups and a key technical point in its computation is the following lemma. Our notation is slightly different from [24], so we give a self-contained statement here.

LEMMA 5.4. For the regular s-bijective factor map \pi_\xi : (X_G, \sigma) \to (X_\xi, \sigma_\xi), the constant K_\xi = 0 satisfies the condition of [24, Lemma 2.7.2]. That is, if x^1, x^2, y^1, y^2 are all in X_G with \pi_\xi(x^1) = \pi_\xi(x^2), \pi_\xi(y^1) = \pi_\xi(y^2), x^1_k = y^1_k, for k \geq k_0, and x^2, y^2 stably equivalent, then x^2_k = y^2_k, for k \geq k_0, also.
**Proof.** The statement is trivial if either \( x^1 = x^2 \) or \( y^1 = y^2 \). It remains to consider when both pairs are distinct. The situation of \( x \neq y, \pi_\xi(x) = \pi_\xi(y) \) with \( x = x^1, y = x^2 \) is described explicitly in parts 3(b) and 3(c) of Theorem 4.3. The rest of the proof is done by checking the different cases, which we leave to the reader.

**Lemma 5.5.** The boundary map

\[
d^i_Q : C^i_{Q,A}(\pi_\xi)_{1,0} \to C^i_{Q,A}(\pi_\xi)_{0,0}
\]

is the zero map.

**Proof.** Via the isomorphism of Lemma 5.3, this group is generated by classes of the form \((\xi^0(y), \xi^1(y))\), \(y \in H^6\). Using the formula given in [24, Definitions 4.2.1 and 4.2.10], we have

\[
d^i_Q(\xi^0(y), \xi^1(y)) = \xi^0(y) - \xi^1(y),
\]

for each \( y \in H^6 \), where the term on the right is interpreted as an element of \( \mathbb{Z}G^6 \). We know that the map \( t \) induces an isomorphism between \( D^*(G^7) \) and \( D^*(G) \) and we have

\[
t(\xi^0(y) - \xi^1(y)) = t(\xi^0(y)) - t(\xi^1(y)) = 0,
\]

by hypothesis (H0).

The computations of the homology groups, \( H_*^Q(X_\xi, \sigma_\xi) \), as summarized in Theorem 5.1, follow from Theorem 5.2 and Lemmas 5.3 and 5.5.

Let us mention an alternate proof of these results. This is based on two sets of results. The first are those of Proietti and Yamashita [21, 22], who show that the Smale space homology here agrees with the groupoid homology as studied by Matui. Further, Matui has recently adapted the results which are used in the next section for the \( K \)-theory of the \( C^* \)-algebras to the case of groupoid homology: see [19]. These require the unit space to be totally disconnected and so only apply to parts (1), (2) and (5) of Theorem 5.1.

6. **Groupoids, \( C^* \)-algebras and \( K \)-theory**

The goal of this section is to describe the stable and unstable equivalence relations (or groupoids) of our Smale space, \((X_\xi, d_\xi, \sigma_\xi)\), their associated \( C^* \)-algebras and their \( K \)-theory.

We begin with a short review of what is involved for general Smale space. For any Smale space, \((X, d, \varphi)\), two points \( x, y \) in \( X \) are stably equivalent if

\[
\lim_{n \to +\infty} d(\varphi^n(x), \varphi^n(y)) = 0.
\]

Unstable equivalence is defined by simply replacing both occurrences of \( \varphi \) by \( \varphi^{-1} \). We define

\[
G^s(X, \varphi) = \{(x, y) \in X \times X \mid \lim_{n \to +\infty} d(\varphi^n(x), \varphi^n(y)) = 0\},
\]

\[
G^u(X, \varphi) = \{(x, y) \in X \times X \mid \lim_{n \to +\infty} d(\varphi^{-n}(x), \varphi^{-n}(y)) = 0\}.
\]
Each is an equivalence relation and has a natural topology. It will not be necessary to describe this here in detail, but we refer the reader to [23]. In addition, if \((X, \varphi)\) is non-wandering, then, as groupoids, each has a Haar system. Again, we do not need a detailed description. For any \(x\) in \(X\), we let \(X^s(x)\) and \(X^u(x)\) be the equivalence classes of \(x\) in the two equivalence relations. Each carries a natural topology which is finer than the relative topology of \(X\) but makes each locally compact Hausdorff.

Generally speaking, étale groupoids [27] are much easier to deal with and, thanks to the work of Muhly, Renault and Williams [20], our groupoids are equivalent to étale groupoids by reducing to an abstract transversal. With the slightly stronger hypothesis that \((X, \varphi)\) is irreducible, we select a finite set \(P\) with the property that \(\varphi(P) = P\) (that is, consisting of periodic points), we let 

\[
X^u(P) = \bigcup_{p \in P} X^u(p), \quad X^s(P) = \bigcup_{p \in P} X^s(p)
\]

and these function as natural transversals to \(G^s(X, \varphi), G^u(X, \varphi)\), respectively. Note that these spaces are given locally compact topologies which are finer than the relative topology of \(X\). Let us give a brief description of the former. First, the collection of local unstable sets provides a topology for the unit space of \(G^s(X, \varphi)\), which is \(X^u(P)\). Furthermore, if \(x\) and \(y\) are in the space and are stably equivalent, there is a positive integer \(n\) such that \(\varphi^n(x)\) and \(\varphi^n(y)\) lie in the same local stable set. It follows that the function \(z \mapsto \varphi^{-n}[\varphi^n(z), \varphi^n(y)]\) is defined on some local unstable set of \(x\) and maps each \(z\) to a point stably equivalent to \(z\). This is a typical open set in the base for the topology. We refer the reader to [26].

We denote the reduction to these transversals by

\[
G^s(X, \varphi, P) = G^s(X, \varphi) \cap (X^u(P) \times X^u(P)), \\
G^u(X, \varphi, P) = G^u(X, \varphi) \cap (X^s(P) \times X^s(P)).
\]

Each is an étale equivalence relation. Different choices of \(P\) yield equivalent groupoids. In [26], it is shown that these groupoids are amenable, so we do not need to make a distinction between the full and reduced \(C^*\)-algebras. We define

\[
S(X, \varphi, P) = C^*(G^s(X, \varphi, P)), \\
U(X, \varphi, P) = C^*(G^u(X, \varphi, P)).
\]

The map which sends a continuous function of compact support, \(f\), on either of the two groupoids, to \(f \circ (\varphi \times \varphi)^{-1}\) extends to an automorphism of the respective \(C^*\)-algebra, which is denoted by \(\varphi\). The Ruelle \(C^*\)-algebras are defined as the associated crossed-products:

\[
R^s(X, \varphi, P) = S(X, \varphi, P) \rtimes_{\varphi} \mathbb{Z}, \\
R^u(X, \varphi, P) = U(X, \varphi, P) \rtimes_{\varphi} \mathbb{Z}.
\]

Let us review some basic properties of these \(C^*\)-algebras. If \((X, \varphi)\) is mixing, then \(S(X, \varphi, P)\) and \(U(X, \varphi, P)\) are separable, simple, finite, stable, satisfy the universal coefficient theorem (UCT) [26] and have finite nuclear dimension [9]. If \((X, \varphi)\) is irreducible,
then $R^s(X, \varphi, P)$ and $R^a(X, \varphi, P)$ are separable, simple, purely infinite, stable, satisfy the UCT [26] and have finite nuclear dimension [9]. All of these $C^*$-algebras come under the Elliott classification scheme. (We include a small remark: the results of [9] need the hypothesis that the $C^*$-algebras contain a projection. This was later shown to hold in general [10], but, in our case, we will explicitly provide clopen subsets of the unit spaces of both $G^s(X_\xi, \sigma_\xi, P_\xi)$ and $G^a(X_\xi, \sigma_\xi, P_\xi)$ which show this holds.)

In our particular case for $(X_\xi, d_\xi, \sigma_\xi)$, we can actually go a little further and the first step is to make a particular choice for $P$, as we describe.

Under the standing hypotheses, we select a periodic point as follows. We have already observed that $G^1 - \xi(H^1)$ is also primitive. We select a cycle $C^1 \subseteq G^1 - \xi(H^1)$ of minimal length. This implies that $r|_{C^1}$ is injective. We let $C^0 = i(C^1)$. We let $P \subseteq X_G$ be the finite set of infinite paths which simply repeat the cycle $C^1$. We note that an element $p$ of $P$ is uniquely determined in $P$ by $i(p_1)$ in $C^0$. Also notice that $\pi_\xi$ is a bijection from $P$ to $P_\xi = \pi_\xi(P)$ from part (3) of Theorem 4.3.

The $s$-bijective map $\pi_\xi : X_G \to X_\xi$ induces maps at the level of groupoids and also $C^*$-algebras:

$$(\pi_\xi)^* : S(X_\xi, \sigma_\xi, P_\xi) \to S(X_G, \sigma, P),$$

$$(\pi_\xi)_* : U(X_G, \sigma, P) \to U(X_\xi, \sigma_\xi, P_\xi).$$

The $C^*$-algebras $S(X_G, \sigma, P)$ and $U(X_G, \sigma, P)$ are both AF-algebras, their $K$-zero groups are $D^a(G)$ and $D^s(G)$, respectively, while their $K_1$-groups are trivial.

We now state the two main results of this section. Of course, they could easily be assembled into a single result, but we separate them as the proofs are rather long and also rather different, so they will be divided into two subsections.

**Theorem 6.1.** If $G, H, \xi$ satisfy the standing hypotheses, we have the following items.

1. $K_0(S(X_\xi, \sigma_\xi, P_\xi)) \cong D^u(G)$ as ordered abelian groups and, under this isomorphism, the automorphism induced by $\sigma_\xi$ is $A^T_G$.
2. $K_1(S(X_\xi, \sigma_\xi, P_\xi)) \cong D^u(H)$ and, under this isomorphism, the automorphism induced by $\sigma_\xi$ is $A^T_H$.
3. $K_0(R^s(X_\xi, \sigma_\xi, P_\xi)) \cong \mathbb{Z}G^0/(I - A^T_H)\mathbb{Z}G^0 \oplus \ker(I - A^T_H : \mathbb{Z}H^0 \to \mathbb{Z}H^0)$.
4. $K_1(R^s(X_\xi, \sigma_\xi, P_\xi)) \cong \mathbb{Z}H^0/(I - A^T_H)\mathbb{Z}H^0 \oplus \ker(I - A^T_H : \mathbb{Z}G^0 \to \mathbb{Z}G^0)$.

**Theorem 6.2.** If $G, H, \xi$ satisfy the standing hypotheses, we have the following items.

1. $K_0(U(X_\xi, \sigma_\xi, P_\xi)) \cong D^s(G)$ as ordered abelian groups and, under this isomorphism, the automorphism induced by $\sigma_\xi$ is $A^{-1}_G$.
2. $K_1(U(X_\xi, \sigma_\xi, P_\xi)) \cong D^s(H)$ and, under this isomorphism, the automorphism induced by $\sigma_\xi$ is $A^{-1}_H$.
3. $K_0(R^u(X_\xi, \sigma_\xi, P_\xi)) \cong \mathbb{Z}G^0/(I - A_G)\mathbb{Z}G^0 \oplus \ker(I - A_G : \mathbb{Z}H^0 \to \mathbb{Z}H^0)$.
4. $K_1(R^u(X_\xi, \sigma_\xi, P_\xi)) \cong \mathbb{Z}H^0/(I - A_G)\mathbb{Z}H^0 \oplus \ker(I - A_G : \mathbb{Z}G^0 \to \mathbb{Z}G^0)$.
Remark 6.3. Both results are undoubtedly true if the hypothesis that $G$ is primitive is weakened to $(X_G, \sigma_G)$ being non-wandering. However, our proofs rely heavily on results of two papers [12, 25] and which require the respective equivalence relations $G^s(X_G, \sigma, P)$ and $G^u(X_G, \sigma, P)$ to have all equivalence classes dense and this is equivalent to $G$ being primitive. It would not be difficult to adjust these results to the more general situation, but it would complicate matters.

Remark 6.4. The alert reader may be somewhat perplexed by the switching of ‘s’ and ‘u’, between the two sides of the first two parts of both results. This is due to an historical anomaly: the superscript on $G^s$ was chosen for ‘stable’ equivalence and it seems logical to use $S$ for its $C^*$-algebra. However, the elements of the $K_0$-group of this $C^*$-algebra are realized by characteristic functions of clopen subsets of its unit space, which is $X^u(P)$. So the notation for $D^u(G)$ was chosen to coincide with this interpretation.

Let us mention one other groupoid, without going into great detail. It follows from Theorems 3.17 and 3.19 that the map $\sigma_\xi$ of $X_\xi^+$ is locally expanding and also a local homeomorphism. We can associate to it its Deaconu–Renault groupoid, as follows. For each $x, y$ in $X_\xi^+$ and positive integers $m, n$ satisfying $\sigma^m_\xi(x) = \sigma^n_\xi(y)$, we consider the triple $(x, m - n, y)$. The set of all such triples is a groupoid with the product $(x, k, y) \cdot (x', k', y')$ defined when $y = x'$ and result $(x, k + k', y')$. It can be given a topology which is étale [8]. The associated groupoid $C^*$-algebra has the nice feature of being unital, and it is also equivalent to $R^u(X_\xi, \sigma_\xi, P_\xi)$.

The result of our K-theory computations has an interesting consequence that our construction exhausts all possible Ruelle algebras from mixing Smale spaces, as we describe below.

We begin with some simple observations. Let $(X, \varphi, d)$ be a mixing Smale space and $Q$ be a finite $\varphi$-invariant subset of $X$. The $C^*$-algebras $R^s(X, \varphi, Q)$ and $R^u(X, \varphi, Q)$ are Spanier–Whitehead duals of each other (see [15, Definition 4.1 and Theorem 1.1]). As noted in [15, §4], this implies that their $K$-groups are finitely generated. Proietti and Yamashita have recently shown that $K_*(S(X, \varphi))$ and $K_*(U(X, \varphi))$ are finite rank ([22, Theorem 5.1]) and, as explained in [15, §4], this implies that $K_0(R^s(X, \varphi, Q))$ and $K_1(R^s(X, \varphi, Q))$ have the same rank. Putting all of this together, we conclude that there is an integer $k \geq 0$ and finite abelian groups $G_0$ and $G_1$ such that

$$K_0(R^s(X, \varphi, Q)) \cong \mathbb{Z}^k \oplus G_0,$$
$$K_1(R^s(X, \varphi, Q)) \cong \mathbb{Z}^k \oplus G_1.$$  

We add that another consequence of [22, Theorem 5.1], as described in [15, §4], is that

$$R^s(X, \varphi, Q) \cong R^u(X, \varphi, Q).$$

**Corollary 6.5.** Let $(X, \varphi, d)$ be a mixing Smale space and $Q$ be a finite $\varphi$-invariant subset of $X$. There exist finite directed primitive graphs, $G, H$ and embeddings $\xi^0, \xi^1 : H \to G$ satisfying hypotheses (H0), (H1) and (H2) such that $R^s(X_\xi, \sigma_\xi, P_\xi) \cong R^u(X, \varphi, Q)$, for any $P$, a finite $\sigma$-invariant subset of $X_G$.

We begin the proof with a simple algebraic result.
LEMMA 6.6. Let \( G \) be a finitely generated abelian group and \( d_0, M_0 \) be positive integers. There exist \( d \geq d_0 \) and a \( d \times d \) integer matrix \( A \) such that

\[
\mathbb{Z}^d/(I - A)\mathbb{Z}^d \cong G
\]

and each entry of \( A \) is greater than or equal to \( M_0 \).

Proof. From the classification theorem for finitely generated abelian groups ([14, Theorem 2.2, p. 76]), we know that there are integers \( k \geq 0, j_1, \ldots, j_l \geq 2, l \geq 0 \) such that

\[
G \cong \mathbb{Z}^k \oplus (\mathbb{Z}/j_1 \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/j_l \mathbb{Z}).
\]

Let \( D \) be the diagonal matrix constructed as follows. First, list \( k \) zeros on the diagonal. Follow this by \( j_1, \ldots, j_l \). Then continue with 1’s so the the size of \( D \), which we call \( d \), is at least \( d_0 \).

It is immediate that \( \mathbb{Z}^d/\mathbb{Z}^d \cong G \). Observe that this isomorphism still holds if we multiply \( D \) on either side by an integer matrix of determinant one. In particular, it remains true if we add a multiple of a row or column to another. Using column operations, the 1 in the \( d, d \)-entry of \( D \) can then be used to ensure row \( d \) has all positive integer entries. Then using row operations, we can ensure every row, other than row \( d \), has entries at least \( M_0 \). Finally, we add row 1 to row \( d \) and the resulting matrix, matrix \( B \), has all entries at least \( M_0 \). Then \( A = B + I \) satisfies the desired conclusion.

Corollary 6.5. We apply Lemma 6.6 to the group \( G = K_1(R^s(X, \varphi, Q)) \) with \( d_0 = M_0 = 1 \) to obtain a \( d \times d \) matrix \( A \). We apply Lemma 6.6 a second time to the group \( \text{Tor}(K_0(R^s(X, \varphi, Q))) \), with \( d_0 = d \) and \( M_0 = \max\{2A(i, j) + 1 \mid 1 \leq i, j \leq d\} \) to obtain a \( d' \times d' \) matrix \( B \).

We let \( H \) be the directed graph with \( d \) vertices and adjacency matrix \( A^T \) and \( G \) be the directed graph with \( d' \) vertices and adjacency matrix \( B^T \). As \( d' \geq d \), we may find an embedding of \( H^0 \) in \( G^0 \) and from the choice of \( B \), we can find embeddings \( \xi^0, \xi^1 \) of \( H^1 \) in \( G^1 \) which satisfy hypotheses (H0), (H1) and (H2).

We note that as

\[
\mathbb{Z}^{d'}/(I - B)\mathbb{Z}^{d'} \cong \text{Tor}(K_0(R^s(X, \varphi, Q))),
\]

which is finite, \( \ker(I - B) \) is free abelian and rank zero, so is trivial. If we let \( k \) be the rank of \( K_1(R^s(X, \varphi, Q)) \), then \( \ker(I - A) \) also has rank \( k \) and is free abelian. We conclude from Theorem 6.1 that

\[
K_0(R^s(X, \varphi, Q)) \cong \mathbb{Z}^k \oplus \text{Tor}(K_0(R^s(X, \varphi, Q))) \cong K_0(R^s(X_\xi, \sigma_\xi, P_\xi)),
\]

\[
K_1(R^s(X, \varphi, Q)) \cong K_1(R^s(X_\xi, \sigma_\xi, P_\xi)).
\]

As noted in [15], these \( C^* \)-algebras fall under the Kirchberg–Phillips classification theorem, so \( R^s(X, \varphi, Q) \cong R^s(X_\xi, \sigma_\xi, P_\xi) \).

6.1. Stable equivalence. In this subsection, we focus on the \( C^* \)-algebras \( S(X_\xi, \sigma_\xi, P_\xi) \) and \( R^s(X_\xi, \sigma_\xi, P_\xi) \).
**Lemma 6.7**

(1) The set

\[ Y^u_G(P) = \{ y \in X^u_G(P) \mid y_n = p_n, n \leq 0, \text{ some } p \in P \} \]

is a compact open subset of \( X^u_G(P) \). Its relative topology from \( X^u_G(P) \) agrees with the relative topology from \( X_G \). If \( x \) is any point of \( X_G \), then \( x \) is stably equivalent to some \( y \) in \( Y^u_G(P) \). Finally, we have \( \sigma^{-1}(Y^u_G(P)) \subseteq Y^u_G(P) \).

(2) The set \( Y^u_\xi(P) = \pi_\xi(Y^u_G(P)) \) is a compact open subset of \( X^u_\xi(P) \). Its relative topology from \( X^u_\xi(P) \) agrees with the relative topology from \( X_\xi \). If \( x \) is any point of \( X^u_\xi(P) \), then \( x \) is stably equivalent to some \( y \) in \( Y^u_\xi(P) \). Finally, we have \( \sigma^{-1}(Y^u_\xi(P)) \subseteq Y^u_\xi(P) \).

**Proof.** For the first part, the statements about the topology are standard. As \( G \) is primitive, we may have \( l > 0 \) such that there is a path of length \( l \) between any two vertices of \( G \). If \( x \) in \( X_G \), let \( p \) be any point of \( P \). Define \( y \) by \( y_n = p_n, n \leq 0, y_1, \ldots, y_l \) is any path from \( t(p_0) \) to \( t(x_l) \), and \( y_n = x_n, n > l \). So \( y \) is stably equivalent to \( x \) and lies in \( Y^u_G(P) \).

For the second part, \( Y^u_\xi(P) \) is compact since \( Y^u_G(P) \) is. Moreover, \( Y^u_G(P) \) is clearly invariant under \( \sim_\xi \) and \( \pi_\xi \) is continuous, so \( Y^u_G(P) = \pi^{-1}_\xi(Y^u_\xi(P)) \), which implies that \( Y^u_\xi(P) \) is open also. The remaining statements are clear. \( \square \)

The following proposition is an immediate consequence.

**Proposition 6.8.** Let us denote by \( G^3(\xi, P) \) the reduction of the groupoid \( G^3(X_\xi, \sigma_\xi) \) to \( Y^u_\xi(P) \). It is étale, equivalent to \( G^3(X_\xi, \sigma_\xi) \) and also \( G^3(X_\xi, \sigma_\xi, P_\xi) \), in the sense of Muhly, Renault and Williams [20]. In particular, we have containments and a commutative diagram

\[
\begin{align*}
C^*(G^3(\xi, P)) & \subseteq C^*(G^3(X_\xi, \sigma_\xi, P_\xi)) \\
\sigma^{-1}_\xi & \quad \sigma^{-1}_\xi \\
C^*(G^3(\xi, P)) & \subseteq C^*(G^3(X_\xi, \sigma_\xi, P_{\xi^1})).
\end{align*}
\]

The proof of Theorem 6.1 involves an application of the results of [12]. Let us provide a brief discussion of the set-up there. It will be fairly similar to the one we consider here, but there are a couple of differences which we need to address.

We begin with two Bratteli diagrams, \((V, E)\), \((W, F)\), and two embeddings of the latter into the former satisfying conditions analogous to hypotheses (H0) and (H1). Section 3 of [12] describes the construction of a quotient of the path space of the Bratteli diagram \( X_E \), denoted \( X_\xi \). The equivalence relation of tail equivalence on \( X_E \), denoted \( R_E \), then descends to an étale equivalence relation, denoted \( R_\xi \). It is then shown (Theorem 1.1) that the \( K_0(C^*(R)) \) is isomorphic to the dimension group of the Bratteli diagram \((V, E)\), while \( K_1(C^*(R)) \) is isomorphic to that of \((W, F)\).

The first point to note is that in our current situation, we will be using \((V, E)\) and \((F, W)\) stationary diagrams given by the matrices \( G \) and \( H \), respectively. That is, at least approximately, \( V_n = G^0, E_n = G^1, W_n = H^0, F_n = H^1 \), for all \( n \).
There is a second minor problem with the convention of [25] that the Bratteli diagram begins with $V_0$ as a single vertex. This is also solved easily: we let $V_0$ be a single vertex, $V_1 = t(C)$, our selected minimal cycle, and $E_1$ to have one edge from $V_0$ to each vertex $v$ of $V_1$ which is simply the unique edge $e$ of $C$ with $t(e) = v$. Then, for $n \geq 2$, we inductively define $E_n = i^{-1}(V_{n-1})$ and $V_n = t(E_n)$. As we assume that $G$ is primitive, there will be some $n_0 \geq 1$ such that $V_n = G^0$ and $E_n = G^1$, for all $n \geq n_0$. With these definitions, it is immediate that the path space $X_E$ of [12] coincides with $Y^u_G(P)$, the space $X_\xi$ of [12] coincides with $Y^u_\xi(P)$, the groupoid $R_E$ coincides with the reduction of $G^s(X_G, \sigma, P)$ to $Y^u_G(P)$ and the groupoid $R_\xi$ coincides with $G^s(\xi, P)$.

There is a somewhat different solution to this problem for $(W, F)$. We simply drop the assumption on the start and say $W_n$ is only defined for $n \geq n_0$, the two embeddings $\xi^0, \xi^1$ are the given ones. The results of [12] still hold verbatim.

For the third and fourth parts of Theorem 6.1, we use the Pimsner–Voiculescu exact sequence [28]. For notational purposes, we use $S$ instead of $S(X_\xi, \sigma_\xi, P_\xi)$ and $R^s = S \rtimes_{\sigma_\xi} \mathbb{Z}$:

$$
\begin{array}{ccc}
K_0(S) & \xrightarrow{id-(\sigma_\xi)_*} & K_0(S) \\
\downarrow & & \downarrow \\
K_1(R^s) & \xleftarrow{\text{id-(\sigma_\xi)_*}} & K_1(S)
\end{array}
$$

This immediately yields two short exact sequences:

$$
0 \to K_*(S)/(\text{id} - (\sigma_\xi)_*)K_*(S) \to K_*(R^s) \to \ker(\text{id} - (\sigma_\xi)_*) \to 0.
$$

We know that $K_0(S) \cong D^u(G)$, which is an inductive limit of groups $\mathbb{Z}G^0$. This means that there is a natural map we denote $i_1 : \mathbb{Z}G^0 \to D^u(G)$ as the first group in the inductive limit. This intertwines multiplication by $A^T_G$ with the automorphism $A^T_G$ and it is an easy exercise in algebra to show that this induces isomorphisms:

$$
\ker(I - A^T_G : \mathbb{Z}G^0 \to \mathbb{Z}G^0) \cong \ker(\text{id} - (\sigma_\xi)_*),
$$

$$
\mathbb{Z}G^0/(I - A^T_G)\mathbb{Z}G^0 \cong D^u(G)/(\text{id} - A^T_G)D^u(G).
$$

The quotients in the two exact sequences above are now seen to be subgroups of $\mathbb{Z}G^0$ and hence are finitely generated and free. It follows that the two sequences both split. This completes the proof of Theorem 6.1.

6.2. Unstable equivalence. In this subsection, we focus on the $C^*$-algebras $U(X_\xi, \sigma_\xi, P_\xi)$ and $R^u(X_\xi, \sigma_\xi, P_\xi)$.

**Lemma 6.9**

1. The set

$$
Y^u_G(P) = \{y \in X^u_G(P) \mid y_n = p_n, n \geq -1, \text{some } p \in P\}
$$

is a compact open subset of $X^s_G(P)$. Its relative topology from $X^s_G(P)$ agrees with the relative topology from $X_G$. If $x$ is any point of $X_G$, then $x$ is unstably equivalent to some $y$ in $Y^u_G(P)$. Finally, we have $\sigma(Y^u_G(P)) \subseteq Y^u_G(P)$.
(2) The set $Y^\xi_G(P) = \pi^\xi(Y^\xi_G(P))$ is a compact open subset of $X^\xi_G(P)$. Its relative topology from $X^\xi_G(P)$ agrees with the relative topology from $X^\xi_G$. If $x$ is any point of $X^\xi_G(P)$, then $x$ is unstably equivalent to some $y$ in $Y^\xi_G(P)$. Finally, we have $\sigma^\xi(Y^\xi_G(P)) \subseteq Y^\xi_G(P)$.

Proof. The first part is all easy topological facts while the second follows as $\pi^\xi$ is a homeomorphism when restricted to $X^\xi_G(P)$. □

The next result follows immediately from the last.

**Proposition 6.10.** Let us denote by $G^u(\xi, P)$ the reduction of the groupoid $G^u(X^\xi, \sigma^\xi)$ to $Y^\xi_G(P)$. It is étale equivalent to $G^u(X^\xi, \sigma^\xi)$ and also to $G^u(X^\xi, \sigma^\xi, P^\xi)$, in the sense of Muhly, Renault and Williams [20]. In particular, we have containments and a commutative diagram:

$$
\begin{array}{ccc}
C^*(G^u(\xi, P)) & \subseteq & C^*(G^u(X^\xi, \sigma^\xi, P^\xi)) \\
\sigma^\xi & & \sigma^\xi \\
C^*(G^u(\xi, P)) & \subseteq & C^*(G^u(X^\xi, \sigma^\xi, P^\xi)).
\end{array}
$$

We begin our analysis with a technical result on the structure of $G^u(\xi, P)$.

**Lemma 6.11.** Let $x, y$ be in $Y^\xi_G(P)$ and assume that $\pi^\xi(x)$ and $\pi^\xi(y)$ are unstably equivalent in $(X^\xi, \sigma^\xi)$. Then either:

1. There exists $N < 0$ such that $x_n = y_n$, for all $n \leq N$; or
2. There exists $N < 0, i = 0, 1$ and $z$ in $X_H$ such that $x_n = \xi^i(z_n), y_n = \xi^{1-i}(z_n)$, for all $n \leq N$.

Proof. We recall that in a general Smale space $(X, \varphi, d)$, if points $x, y$ are unstably equivalent, then there exists $n_0 < 0$ such that $\varphi^{n_0}(x)$ and $\varphi^{n_0}(y)$ are in the same local unstable set, meaning that $d(\varphi^{n_0}(x), \varphi^{n_0}(y)) < \epsilon_X$ and $[\varphi^{n_0}(x), \varphi^{n_0}(y)] = \varphi^{n_0}(x)$ (see [24, Proposition 2.1.11]). It follows by induction that for all $n \leq n_0$, we have $d(\varphi^n(x), \varphi^n(y)) < \epsilon_X \lambda^n - n_0$. In our case, this means that

$$
d^\xi(\sigma^\xi_n(\pi^\xi(x)), \sigma^\xi_n(\pi^\xi(y))) < 2^{-3+n-n_0},
$$

for all $n \leq n_0$. From the definition of the metric $d^\xi$ on the inverse limit space $X^\xi$, we have $d^\xi(\sigma^\xi_n(\pi^\xi(x)), \sigma^\xi_n(\pi^\xi(y))) < 2^{-3+n-n_0}$. Recalling the definition of $\pi^\xi$ on $X_G$, we have $d^\xi(\pi^\xi(\chi^+_G(\sigma^n(x))), \pi^\xi(\chi^+_G(\sigma^n(y)))) < 2^{-3+n-n_0}$.

It follows from Theorem 3.9 that we have

$$
d_G^\xi(\tau^\xi(\chi^+_G(\sigma^n(x))), \tau^\xi(\chi^+_G(\sigma^n(y)))) < 2^{-3+n-n_0+n},
$$

$$
d^\xi(\theta(\chi^+_G(\sigma^n(x)))\theta(\chi^+_G(\sigma^n(y)))) < 2^{-3+n-n_0+n},
$$

for $n \leq n_0$. The first of these inequalities immediately implies that $\tau^\xi(x_n) = \tau^\xi(y_n)$, for all $n < n_0$. If $x_n$ is not in $\xi(H^1)$, it follows that $x_n = y_n$. 
Any $x$ in $Y_G^+(P)$ is right-tail equivalent to a point in $P$ so, for any integer $n$, there is a least $m \geq n$ such that $x_m$ is not in $\xi(H^1)$. We note then that

$$\theta(\chi_G^+(\sigma^n(x))) = \exp \left( 2\pi i \sum_{j=1}^{m-n-1} \varepsilon(x_{n+j})2^{-j} \right).$$

As a consequence of the sum being finite, if the quantity above equals one, then $\varepsilon(x_k) = 0$, for $n + 1 \leq k < m$.

For $0 \leq t < 1$, we let $\exp(2\pi it)^{1/2} = \exp(\pi it)$. Observe that

$$\theta(\chi_G^+(\sigma^{-1}(x))) = \exp(\varepsilon(x_n)\pi i)\theta(\chi_G^+(\sigma^n(x)))^{1/2}$$

$$= (-1)^{\varepsilon(x_n)}\theta(\chi_G^+(\sigma^n(x)))^{1/2}.$$

Suppose for some $n < n_0 - 1$, we have $x_{n+1} = y_{n+1}$. We will show $x_n = y_n$ also. There are several cases to consider. First, suppose that $x_{n+1} = y_{n+1}$ is not in $\xi(H^1)$. If $x_n$ is also not in $\xi(H^1)$, then $x_n = y_n$. If $x_n$ is in $\xi(H^1)$, then $\theta(\chi_G^+(\sigma^n(x))) = \theta(\chi_G^+(\sigma^{-1}(y))) = \exp(2\pi i\varepsilon(x_n))$. The second inequality above then implies that $\varepsilon(x_n) = \varepsilon(y_n)$ and hence $x_n = y_n$ also.

Now, suppose $x_{n+1} = y_{n+1}$ is in $\xi(H^1)$. If $\varepsilon(x_{n+1}) = 0$, then both $\chi_G^+(\sigma^n(x))$ and $\chi_G^+(\sigma^n(y))$ lie in $\{\exp(2\pi it) | 0 \leq t < 1\}$. If $\varepsilon(x_{n+1}) = 1$, then both $\chi_G^+(\sigma^n(x))$ and $\chi_G^+(\sigma^n(y))$ lie in $\{\exp(2\pi it) | 1 \leq t < \infty\}$. In either case, we have

$$d_T(\theta(\chi_G^+(\sigma^n(x)))^{1/2}, \theta(\chi_G^+(\sigma^n(y)))^{1/2}) = 2^{-1}d_T(\theta(\chi_G^+(\sigma^n(x))), \theta(\chi_G^+(\sigma^n(y))))$$

and hence

$$2^{-2-n_0+n} > d_T(\theta(\chi_G^+(\sigma^n(x))), \theta(\chi_G^+(\sigma^n(y))))$$

$$= d_T(\theta(\chi_G^+(\sigma^n(x)))^{1/2}, \theta(\chi_G^+(\sigma^n(y)))^{1/2}(-1)^{\varepsilon(x_n), \varepsilon(y_n)}$$

$$\geq d_T(\theta(\chi_G^+(\sigma^n(x)))^{1/2}, \theta(\chi_G^+(\sigma^n(x)))^{1/2}(-1)^{\varepsilon(x_n), \varepsilon(y_n)}$$

$$- d_T(\theta(\chi_G^+(\sigma^n(x)))^{1/2}, \theta(\chi_G^+(\sigma^n(y)))^{1/2}(-1)^{\varepsilon(y_n)}$$

$$= d_T((-1)^{\varepsilon(x_n), \varepsilon(y_n)}$$

$$- 2^{-1}d_T(\theta(\chi_G^+(\sigma^n(x))), \theta(\chi_G^+(\sigma^n(y))))$$

$$\geq d_T((-1)^{\varepsilon(x_n), \varepsilon(y_n)} - 2^{-4-n_0+n}).$$

From which it follows that $\varepsilon(x_n) = \varepsilon(y_n)$ and so $x_n = y_n$.

We have shown that if $x_{n+1} = y_{n+1}$, for some $n < n_0 - 1$, then $x_n = y_n$ also. It follows by induction that we are in case (1).

It remains to consider the case $x_n \neq y_n$, which implies $x_n$ is in $\xi(H^1)$ and $\varepsilon(x_n) \neq \varepsilon(y_n)$, for all $n < n_0$. Without loss of generality, assume $\varepsilon(x_n) = 0$, $\varepsilon(y_n) = 1$, for some $n < n_0$. We will show that if $\varepsilon(x_{n-1}) = 1$, $\varepsilon(y_{n-1}) = 0$, then $x_{n-2} = y_{n-2}$, which contradicts our hypothesis. The only remaining possibility is case (2), so this will complete the proof.
Under the conditions $\varepsilon(x_n) = 0, \varepsilon(y_n) = 1, \varepsilon(x_{n-1}) = 1, \varepsilon(y_{n-1}) = 0$, we have $\theta(\chi_G^+(\sigma^{n-2}(x)))$ lies in $\{\exp(2\pi it) | 2^{-1} \leq t < 1 - 2^{-2}\}$, while $\theta(\chi_G^+(\sigma^{n-2}(y)))$ lies in $\{\exp(2\pi it) | 2^{-2} \leq t < 2^{-1}\}$. It follows that

$$d_T(\theta(\chi_G^+(\sigma^n(x))))^{1/2}, \theta(\chi_G^+(\sigma^n(y)))^{1/2} = 2^{-1}d_T(\theta(\chi_G^+(\sigma^n(x))), \theta(\chi_G^+(\sigma^n(y))))$$

and the same calculation as before shows that $\varepsilon(x_{n-2}) = \varepsilon(y_{n-2})$ and hence $x_{n-2} = y_{n-2}$.

The proof of Theorem 6.2 involves an application of the results of [25]. Let us provide a brief discussion of the set-up there. It will be fairly similar to the one we consider here, but there are a couple of differences which we need to address.

We begin with two Bratteli diagrams, $(V, E)$, $(W, F)$ and two embeddings of the latter into the former satisfying conditions analogous to hypotheses (H0) and (H1). Section 2 of [25] describes the construction of an étale equivalence relation, $R$, on the path space of the diagram $(V, E)$, $X_E$, which contains tail equivalence, $R_E$, as an open subequivalence relation. It is then shown that the $K_0(C^*(R))$ is isomorphic to the dimension group of the Bratteli diagram $(V, E)$, while $K_1(C^*(R))$ is isomorphic to that of $(W, F)$.

The first point to note is that in our current situation, we will be using $(V, E)$ and $(W, F)$ stationary diagrams given by the matrices $G$ and $H$, respectively. That is, at least approximately, $V_n = G^0, E_n = G^1, W_n = H^0, F_n = H^1$, for all $n$. The first minor annoyance is that, because we are studying unstable or left-tail equivalence, our graph $G$ is going in the wrong direction. (As we were also looking at right-tail equivalence in the last subsection, this was kind of inevitable.) This can be repaired easily by simply looking at the opposite graphs of $G$ and $H$; that is, simply reverse the maps $i, t$. This means that the dimension groups of $(V, E)$ and $(W, F)$ will be isomorphic to $D^s(G)$ and $D^s(H)$, respectively.

There is a second minor problem with the convention of [25] that the Bratteli diagram begins with $V_0$ and $W_0$ as a single vertex. This is also solved easily exactly as we did for the stable case.

The directional reversal does pose some notational problems. The simplest solution is the following. We observe the following: if $y$ is any path in $Y_G^+(P)$, then letting $\tilde{y}_n = y_{-n}$, for $n \geq 1$, defines an infinite path in $X_E$. In fact, this association is a homeomorphism between $Y_G^+(P)$ and $X_E$ which induces an isomorphism between $G^u(X_G, \sigma, Y_G^+(P))$ and $R_E$. Furthermore, bearing in mind that $\pi_\xi$ is an homeomorphism between $Y_G^+(P)$ and its image in $X_\xi$, Lemma 6.11 shows that $\pi_\xi$ induces a bijection between $R$, as described in [25], and $G^u(\xi, P)$. We will suppress this map and simply write our sequences as indexed by $n \leq 1$.

One technical issue remains: as a consequence of Lemma 6.11, we know that, under this identification, $G^u(\xi, P)$ and $R$ agree as sets, but the former is given a topology based on the Smale space structure, while the latter was given a rather ad hoc topology in [25]. We must check these coincide. This is the content of the following. We recall from [26] and the discussion at the beginning of this section that the unstable groupoid $G^u(X_\xi, \sigma_\xi, P_\xi)$ has a basis for its topology consisting of sets of the form $\{(z, \sigma^{-n}(y), \sigma^{-n}(z))\}$, where $z$ runs over some small local stable set in $X_\xi^+(P_\xi)$.
LEMMA 6.12. Let \((x, y)\) be in \(R\), as above. There exists a positive integer \(n\) and a compact, open neighbourhood \(U\) of \((x, y)\) in \(R\) such that

\[
\pi_{\xi} \times \pi_{\xi}(U) = \{(z, \sigma^n(y), \sigma^{-n}(z)) \mid z \in \pi_{\xi}(r(U))\}
\]

is a compact, open subset of \(G^n(\xi, p)\) and \(\pi_{\xi}(r(U))\) is a compact, open subset of \(Y^\xi_{\xi}(P)\).

Proof. We first consider the case that \((x, y)\) is in \(R_E\), meaning that \(x, y\) are themselves unstably equivalent in \(X_G\). We can then choose \(n \geq 3\) such that \(x_i = y_i\), for all \(i \leq -n + 3\). We define \(U\) to be the set of all pairs \((z, z')\) such that \(z_i = z'_i\), for \(i \leq -n, z_i = x_i, z'_i = y_i\), for all \(i \geq -n\). Notice that \(d_G(\sigma^{-n}(y), \sigma^{-n}(z)) \leq 2^{-3}\) and \([\sigma^{-n}(y), \sigma^{-n}(z)] = \sigma^{-n}(z')\), for all such \(z, z'\). The first desired property of \(U\) follow at once. The last statements follow from the fact that \(\pi_{\xi}\) is a homeomorphism on \(Y^\xi_{\xi}(P)\).

The second case is that \((x, y)\) is not in \(R_E\). By Lemma 6.11, we may find \(n \geq 1, j = 0, 1\) and a path \(w\) in \(X_H\) such that \(x_i = \xi^j(w_i), y_i = \xi^{1-j}(w_i)\), for all \(i \leq -n + 3\). As before, let \(U\) be the collection of all pairs \(z, z'\) such that \(z_i = x_i, z'_i = y_i\), for all \(i \geq -n\). If we let \(p = x_{-n+1}, \ldots, x_0, q = y_{-n+1}, \ldots, y_0\), then the set \(\delta_{p,q}^{1,1-j}\), as defined following [25, Lemma 2.4], is precisely \(U\) and hence is a compact, open subset of \(R\). If \(z, z'\) is in \(U\), let \(z''\) be the unique element of \(X_G\) with \(z'' \neq z, z'' \equiv \xi z\). It follows that \(d_G(\sigma^{-n}(y), \sigma^{-n}(z)) \leq 2^{-3}\) and we have

\[
\pi_{\xi}[\sigma^{-n}(y), \sigma^{-n}(z'')] = [\pi_{\xi}\sigma^{-n}(y), \pi_{\xi}\sigma^{-n}(z'')] = [\sigma^{-n}\pi_{\xi}(y), \sigma^{-n}\pi_{\xi}(z'')] = [\sigma^{-n}\pi_{\xi}(y), \sigma^{-n}\pi_{\xi}(z)].
\]

The first part of the conclusion follows. The last properties are as before. \(\square\)

The computation of the groups \(K_*(U(\xi_\xi, \sigma_\xi, P_\xi))\), as stated in Theorem 6.2, is an immediate consequence of [25, Theorem 1.1] and the construction given there of \(R\).

We next turn to the claim of the maps on these groups induced by \(\sigma_\xi\). First, because of the commutative diagram of Proposition 6.10, it suffices to compute the map induced by \(\sigma_\xi\), which is an endomorphism of \(C^*(G^n(\xi, P))\). For the \(K_0\) group, the results of [25] actually show that the inclusion of \(C^*(R_E)\) in \(C^*(R)\) induces an isomorphism on \(K_0\). The former is an AF-algebra with stationary Bratteli diagram, so the induced map is given by the matrix \((A^*_E)^{-1}\), as claimed.

For \(K_1\), let \(x\) be an element of \(Y^\xi_{\xi}(P)\) such that, for some \(n \leq -1, \ i(x_n)\) is in \(\xi^0(H^0)\). Say \(i(x_n) = \xi^0(w), w \in W_n = H^0\). Letting \(p = (x_n, \ldots, x_0)\), [25, Remark 3.6] gives an explicit description of a partial isometry, \(v_p\), in \(C^*(R)\), so that \(v_p + (1 - v^*_p v_p)\) is a unitary. We may regard \(w\) as determining an element of \(Z_{W_n}\) and hence an element of \(D^\xi(H)\), the map sending \(w\) to the class of \(v_p\), is an explicit, well-defined group isomorphism between \(D^\xi(H)\) and \(K_1(C^*(R))\).

Moreover, it is an easy exercise to check that if we let \(q = (x_{n-1}, \ldots, x_{-1})\), then \(v_q = \sigma_\xi(v_p)\) and it follows that the isomorphism intertwines the automorphism \((A^*_H)^{-1}\) and \((\sigma_\xi)_w\).

The proofs of the last two parts of Theorem 6.2 are completely analogous to those of Theorem 6.1 and we omit the details.
7. **Realizations**

Our goal in this section is to provide a more concrete description of the space $X_{\xi}^+$, more specifically, we give an explicit embedding of it into $\mathbb{R}^3$. We also examine a couple of simple examples more closely.

**Definition 7.1.** For $k \geq 0$, define $\zeta_k : X_{\xi}^+ \to \mathbb{C}$ inductively by setting $\zeta_0 = \theta$ on $X_{\xi}^+$ and $$\zeta_k(x) = (1 - 2^{-n(x)})\theta(x) + 2^{-3-n(x)}\zeta_{k-1}(\sigma^n(x)(x)),$$

for $x$ in $X_{\xi}^+$ and $k \geq 1$.

A remark is probably in order on the factor $1 - 2^{-n(x)}$: this is normalized so that, if $n(x) = 1$, then the first term is zero. Ultimately, when we extend the definition of $\zeta_k$ to a map $\zeta$ on all of $X_{\xi}^+$, this will have the effect that $\zeta(x) = 0$ exactly when $x_n$ is not in $\xi(H^1)$, for any $n \geq 1$.

**Lemma 7.2**

1. For $x$ in $X_{\xi}^+$, $k \geq 0$, we have $|\zeta_k(x)| \leq 1$.
2. For any $x, y$ in $X_{\xi}^+$, $k \geq 0$, we have $|\zeta_k(x) - \zeta_k(y)| \leq 8d_k(x, y)$.
3. There is a unique continuous map $\zeta_X : X_{\xi}^+ \to \mathbb{C}$ such that:
   a. for $x$ in $X_{\xi}^+$, $k \geq 0$, $\zeta_X(x) = \zeta_k(x)$;
   b. for $x, y$ in $X_{\xi}^+$,
      $$|\zeta_X(x) - \zeta_X(y)| \leq 8d_{\xi}(x, y);$$
   c. for $x$ in $X_{\xi}^+$, $|\zeta_X(x)| \leq 1$.

**Proof.** We prove the first statement by induction. It obviously holds for $k = 0$. Let $k \geq 1$, assume the statement is true for $k-1$ and $x$ be in $X_{\xi}^+$. We have

$$|\zeta_k(x)| \leq 1 - 2^{-n(x)} + 2^{-3-n(x)}|\zeta_{k-1}(\sigma^n(x)(x))|$$

$$\leq 1 - 2^{-n(x)} + 2^{-3-n(x)} < 1.$$

We prove the second part by induction on $k$. For $k = 0$, we have

$$2\pi d_0(x, y) = 2\pi d_{\xi}(\theta(x), \theta(y)) \geq |\theta(x) - \theta(y)| = |\zeta_0(x) - \zeta_0(y)|,$$

for all $x, y$ in $X_{\xi}^+$.

We now assume the result holds for $k - 1$, with $k \geq 1$. For $x, y$ in $X_{\xi}^+$, we consider two cases separately. The first is that $(n(x), \theta(x)) \neq (n(y), \theta(y))$. In this case, we have have

$$d_k(x, y) = d_{\xi}(\tau_\xi(x), \tau_\xi(y)) + |2^{-n(x)} - 2^{-n(y)}| + d_{\xi}(\theta(x), \theta(y)).$$

We claim that this is bounded below by $2^{-n}$, where $n = \min\{n(x), n(y)\}$. Without loss of generality, we assume $n(x) \leq n(y)$. First consider the case $n(x) < n(y)$, where we have

$$d_k(x, y) \geq |2^{-n(x)} - 2^{-n(y)}| \geq 2^{-n} - 2^{-n-1} = 2^{-1-n}.$$

We are left to consider $n(x) = n(y)$ and $\theta(x) \neq \theta(y)$. Here, we have

$$d_k(x, y) \geq d_{\xi}(\theta(x), \theta(y)) \geq 2^{-n},$$

since $\theta(x)$ and $\theta(y)$ are distinct $2^{n-1}$th roots of unity.
However, we have

\[ |\zeta_k(x) - \zeta_k(y)| \leq |(1 - 2^{1-n(x)})\theta(x) - (1 - 2^{1-n(y)})\theta(y)| + |2^{-2-n(x)}\zeta_{k-1}(\sigma^{n(x)}) - 2^{-2-n(x)}\zeta_{k-1}(\sigma^{n(y)})|. \]

For the first term, we use the triangle inequality as

\[ |(1 - 2^{1-n(x)})\theta(x) - (1 - 2^{1-n(y)})\theta(y)| \leq |(1 - 2^{1-n(x)})\theta(x) - (1 - 2^{1-n(y)})\theta(y)| + |2^{1-n(x)} - 2^{1-n(y)}| + |\theta(x) - \theta(y)| \leq 2^{1-n(x)} - 2^{1-n(y)} + 2\pi d_{T}(\theta(x), \theta(y)) \leq 2 \pi d_{k}(x, y). \]

For the second term, as \( \zeta_{k-1} \) is in the unit disc, this is bounded by

\[ 2^{-2-n(x)} + 2^{-2-n(y)} \leq 2^{-1-n} \leq d_{k}(x, y) \]

from our claim above. The conclusion follows as \( 2\pi + 1 \leq 8. \)

It remains for us to consider the case \((n(x), \theta(x)) = (n(y), \theta(y))\). Here, we make use of the induction hypothesis in estimating

\[ |\zeta_k(x) - \zeta_k(y)| = |2^{-3-n(x)}\zeta_{k-1}(\sigma^{n(x)}(x)) - 2^{-3-n(y)}\zeta_{k-1}(\sigma^{n(y)}(x))| \]

\[ = 2^{-3-n(x)}|\zeta_{k-1}(\sigma^{n(x)}(x)) - \zeta_{k-1}(\sigma^{n(y)}(x))| \]

\[ \leq 2^{-3-n(x)}8d_{k-1}(\sigma^{n(x)}(x), \sigma^{n(y)}(y)) \]

\[ = 2^{-n(x)}[d_{G_{\xi}}(\sigma^{n(x)}(x), \sigma^{n(y)}(y)) + \lambda_{k-1}(\sigma^{n(x)}(x), \sigma^{n(y)}(y))]] \]

\[ \leq d_{G_{\xi}}(x, y) + 2\lambda_{k}(x, y) \]

\[ \leq 8d_{k}(x, y). \]

The third part is an immediate consequence of the first two and the definition of \( d_{\xi} \).

**Lemma 7.3**

1. For any \( x \) in \( X^+_G \) with \( \kappa(x) > 0 \), we have \( \zeta_\xi(x) = (1 - 2^{1-n(x)})\theta(x)2^{-3-n(x)}\zeta_\xi(\sigma^{n(x)}(x)) \).

2. For any \( x, y \) in \( X^+_G \) with \( \kappa(x), \kappa(y) > 0 \), if \( \zeta_\xi(x) = \zeta_\xi(y) \) and \( \tau_\xi(x_m) = \tau_\xi(y_m) \), for all \( 1 \leq m \leq n(x) \), then \( x_m = y_m \), for all \( 1 \leq m \leq n(x) \).

3. Let \( n \) be a positive integer. For any \( x, y \) in \( X^+_G \) with \( \kappa(x), \kappa(y) > 0 \) and \( x_n = y_n \) which is not in \( \xi(H^1) \), if \( \zeta_\xi(x) = \zeta_\xi(y) \) and \( \tau_\xi(x_m) = \tau_\xi(y_m) \), for all \( 1 \leq m \leq n \), then \( x_m = y_m \), for all \( 1 \leq m \leq n \).
Proof. The equation clearly holds on each $X_k^+$ by the definition of $\xi_\epsilon$. The conclusion follows from Proposition 3.5 and the facts that $n(\cdot), \theta$ and $\xi$ are all continuous.

For the second part, the second hypothesis implies that $n(y) = n(x) = n$. Moreover, if $n = 1$, then $x_1, y_1$ are not in $\xi(H^1)$ and we are done. It remains to consider the case $n > 1$.

We will first show that $\sin(\pi 2^{-n}) - 2^{-2^{-n}} > 0$ for $n \geq 1$, by induction. For $n = 1$, we have

$$\sin(\pi 2^{-n}) - 2^{-2^{-n}} = 1 - 2^{-3} > 0.$$  

Now assume this is positive for some $n \geq 1$. Making use of the formula $\sin(t) = 2 \sin(t/2) \cos(t/2)$, we have

$$\sin(\pi 2^{-(n+1)}) - 2^{-2^{-(n+1)}} = \sin(\pi 2^{-n-1}) - 2^{-3-n}$$

$$= 2^{-1}(\cos(\pi 2^{-n-1}))-1 \sin(\pi 2^{-n}) - 2^{-3-n}$$

$$\geq 2^{-1}(\sin(\pi 2^{-n}) - 2^{-2-n})$$

$$> 0$$

by the induction hypothesis.

We may choose $l_0$ sufficiently large so that, for $l \geq l_0$, we have

$$|\xi_\epsilon(x) - \xi_l(x^l)| < 2^{-1}(\sin(\pi 2^{-n}) - 2^{-2^{-n}}),$$

$$|\xi_\epsilon(y) - \xi_l(y^l)| < 2^{-1}(\sin(\pi 2^{-n}) - 2^{-2^{-n}}),$$

$$x^l_m = x_m,$$

$$y^l_m = y_m,$$

for all $1 \leq m \leq n$. It follows that, for such $l$,

$$(1 - 2^{1-n})|\theta(x) - \theta(y)| = (1 - 2^{1-n})|\theta(x^l) - \theta(y^l)|$$

$$\leq |\xi_l(x^l) - \xi_l(y^l)| + 2^{-2^{n}}$$

$$\leq |\xi_l(x^l) - \xi_\epsilon(x)| + |\xi_\epsilon(y) - \xi_l(y^l)| + 2^{-2^{n}}$$

$$< \sin(\pi 2^{-n}).$$

So we have

$$|\theta(x) - \theta(y)| < (1 - 2^{1-n})^{-1} \sin(\pi 2^{-n}) \leq 2 \sin(\pi 2^{-n}).$$

Both $\theta(x)$ and $\theta(y)$ are among the $2^{n-1}$th roots of unity and it is an elementary exercise that the minimum distance between any two is $2 \sin(\pi 2^{-n})$. Hence, we see $\theta(x) = \theta(y)$, which implies that $\epsilon(x_m) = \epsilon(y_m)$, for all $1 \leq m < n$. Together with the fact that $\tau_\epsilon(x_m) = \tau_\epsilon(y_m)$, for $1 \leq m \leq n$, implies $x_m = y_m$, for $1 \leq m \leq n$, as claimed.

We prove the third part using the strong form of induction on $n$. The statement is trivially true for $n = 1$. Now suppose $n > 1$. As $x_n = y_n$ is not in $\xi(H^1)$, we know $n(x), n(y) \leq n$. The rest of the hypothesis implies $n(x) = n(y)$. If $n(x) = n(y) = n$, then the conclusion is simply the result of the second part. We are left to consider $n > n(x) = n(y)$. In this case, the second part immediately implies that $x_m = y_m$ for $1 \leq m \leq n(x)$. In addition, $\sigma^n(x)(x), \sigma^n(y)(y)$ satisfy the hypotheses for integer $n - n(x) < n$ and so, by
the induction hypothesis, $\sigma^n(x)(x)_m = \sigma^n(y)(y)_m$, for all $1 \leq m \leq n - n(x)$. This implies the conclusion.

\begin{theorem}
Under the standing hypotheses, the map from $X^*_\xi$ to $X^*_G \times \mathbb{C}$ sending $\pi_\xi(x)$, $x \in X^*_G$, to the pair $(\tau_\xi(x), \zeta_\xi(x))$ is well defined, continuous and injective.
\end{theorem}

\begin{proof}
The facts that the map is well defined and continuous follows immediately from Theorem 3.7 and part (2) of Lemma 7.2. It remains for us to prove that it is injective. It suffices to show that, for any $x, y$ in $X^+_G$, if $\tau_\xi(x) = \tau_\xi(y)$ and $\zeta_\xi(x) = \zeta_\xi(y)$, then $x \sim_\xi y$.

The first hypothesis implies that $\kappa(x) = \kappa(y)$. If $\kappa(x) = \kappa(y) = 0$, then $x, y$ are in $X^+_0$, so by definition, $\zeta_\xi(x) = \theta(x), \zeta_\xi(y) = \theta(y)$ and hence $d_0(x, y) = 0$ so $x \sim_\xi y$.

We now assume $\kappa(x) = \kappa(y) > 0$ is finite. Let $n$ be the largest positive integer such that $x_n$ is not in $\xi(H^1)$. Since $\tau_\xi(x) = \tau_\xi(y)$, this is also the largest integer such that $y_n$ is not in $\xi(H^1)$ and $n = y_n$. It follows from part (2) of Lemma 7.3 that $x_m = y_m, 1 \leq m \leq n$. It is then obvious that $\tau_\xi(\sigma^n(x)) = \sigma^n(\tau_\xi(x)) = \tau_\xi(\sigma^n(y)) = \sigma^n(\tau_\xi(y))$. We also note that $\kappa(\sigma^n(x)) = 0 = \kappa(\sigma^n(y))$. We now conclude if $\kappa(x)$ is infinite, we may repeat this argument to show that $x \sim_\xi y$ from the first case.

Finally, if $\kappa(x) = \kappa(y)$ is infinite, there are infinitely many $n$ for which the hypotheses of part (3) of Lemma 7.3 hold and we conclude $x = y$.

Under our hypotheses, $X^+_G$ is homeomorphic to the Cantor ternary set, so we conclude the following corollary.

\begin{corollary}
Under the standing hypotheses, the space $X^*_\xi$ can be embedded in $\mathbb{R} \times \mathbb{C}$.
\end{corollary}

We also observe the following corollary, which already appeared in [12]. We provide a different proof here which employs our metric.

\begin{corollary}
Recall that we have factor maps
\[
(X^*_G, \sigma_G) \xrightarrow{\pi_\xi} (X^*_\xi, \sigma_\xi) \xrightarrow{\rho_\xi} (X^*_G, \sigma_G)
\]
and $\rho_\xi \circ \pi_\xi = \tau_\xi$.

Under the standing hypotheses, for each $x$ in $X^+_G$, $\rho_\xi^{-1}(\tau_\xi(x))$ is:

1. A finite collection of pairwise disjoint circles if $\kappa(x)$ is finite;
2. $2^m$ points if $m = \#(n | x_n \in \xi(H^1))$ is finite;
3. Totally disconnected if $\kappa(x)$ is infinite.
\end{corollary}

\begin{proof}
If we apply the map of Theorem 7.4 from $X^*_\xi$ to $X^*_G \times \mathbb{C}$ and restrict it to $\rho_\xi^{-1}(\tau_\xi(x))$, it is a homeomorphism to its image, which is $\{\tau_\xi(x)\} \times \zeta_\xi(\tau_\xi^{-1}(\tau_\xi(x)))$. Hence, it suffices for us to prove $\zeta_\xi(\tau_\xi^{-1}(\tau_\xi(x)))$ is as described, in each case.

For $x$ in $X^*_G$, define $I(x)$ to be the set of positive integers such that $x_n$ is not in $\xi(H^1)$, while $t(x_n)$ is in $\xi^0(H^0)$. For any $n$ in $I(x)$, we define $P(x, n)$ to be the set of all paths $p = (p_1, \ldots, p_n)$ in $G$ such that $\tau_\xi(p_m) = \tau_\xi(x_m)$, for all $1 \leq m \leq n$, which is obviously
a finite set. For such a path $p$, let $C(p, x)$ denote the set of $y$ in $X_G^+$ such that $y_m = p_m$, for $1 \leq m \leq n$ and $\tau_\xi(y_m) = \tau_\xi(x_m)$, for $m > n$.

If $\kappa(x) = 0$, then it is a simple matter to check that $\xi_0(\tau_\xi^{-1}(|\tau_\xi(x)|)) = T$.

If $0 < \kappa(x) < \infty$, then $I(x)$ is finite. Let $n$ be its maximum element. It is clear that $\tau_\xi^{-1}\{\tau_\xi(x)\} = \bigcup_{p \in P(x,n)} C(p, x)$. Again, an easy computation shows that, for each $p$, $\xi_\xi(C(p, x))$ is a circle. Part (3) of Lemma 7.3 proves that they are pairwise disjoint, for different values of $p$. This proves the first statement.

In the second case, we have $\tau_\xi^{-1}\{\tau_\xi(x)\}$ is finite, so its image under $\xi_\xi$ is also.

For the third case, it suffices to consider the case when $x_n$ is in $\xi(H^1)$, for infinitely many $n$ and not in $\xi(H^1)$ for infinitely many $n$. In this case, $I(x)$ is infinite.

Let $n$ be in $I(x)$. It is an immediate consequence of part (3) of Lemma 7.3 that the collections of sets $\xi_\xi(C(p, x))$, $p \in P(x, n)$ are pairwise disjoint. Their union is $\xi_\xi(\tau_\xi^{-1}\{\tau_\xi(x)\})$ Each is evidently closed and hence open as well. Moreover, if $y, z$ are in $\tau_\xi^{-1}\{\tau_\xi(x)\}$ and $y_l \neq z_l$, for some $l$, then they are in distinct elements of the partition for all $n > l$. This completes the proof.

**COROLLARY 7.7.** Under the standing hypotheses, the connected subsets of $X_\xi^+$ are either points or circles and both occur.

**Proof.** If $C$ is a connected subset of $X_\xi$, then $\rho_\xi(C)$ is a connected subset of $X_G\xi$ and hence is a single point, say $x$. So $C$ is a subset of $\rho_\xi^{-1}\{x\}$ and the conclusion follows from Corollary 7.6. The fact that both circles and single points occur follows from the easy fact that both cases (1) and (2) occur in Corollary 7.6. \qed

We finish this section by looking at a couple of specific examples. The first is instructive even if it does not satisfy hypothesis (H2).

**Example 7.8.** Suppose $G^0 = H^0$ contains a single vertex, $H^1$ contains a single edge and $G^1$ contains exactly two edges. There is essentially only one choice for $\xi$. The space $X_G^+$ may be identified with $\{0, 1\}^\mathbb{N}$ in an obvious way. Then $X_\xi$ is the unit circle and $\pi_\xi$ is binary expansion.

Rather more generally (as noted in [12]), if $G^0 = H^0$ and $G^1 = H^1 \times \{0, 1\}$ with $\xi^i$ being the identity map on $H^0$ and $\xi^i(x) = (x, i)$, for $x$ in $H^1$, then $X_\xi^+$ is homeomorphic to $X^+_H \times T$. This can be seen from the first part of Proposition 3.11 and the fact that $X_k^+$ is empty, for $k \geq 1$.

**Example 7.9.** Suppose $G^0 = H^0$ contains a single vertex, $H^1$ contains a single edge and $G^1$ contains exactly three edges. There is essentially only one choice for $\xi$. The space $X_G^+$ may be identified with $\{0, 1, 2\}^\mathbb{N}$ in an obvious way. Letting $A = \{x \in X_1^+ | n(x) \leq 5\}$, Figure 1 is a picture of $\xi_0(X_1^+) \cup \xi_1(A)$.

Hopefully, the reader can see how to sketch all of $\xi_1(X_1^+)$. To get an idea of $\xi_2(X_2^+)$, we suggest the reader verifies the following easy result:

$$\bigcup_{j=0}^{k} \xi_\xi(X_j^+) = \bigcup_{j=0}^{k} c + r \xi_\xi(X_0^+ \cup X_1^+),$$
where the union is over all \( c \) in \( \mathbb{C} \) and positive real numbers \( r \) such that \( c + rT \subseteq \bigcup_{k=0}^{k-1} \xi(X^+_j) \). That is, for every \( k \), the space \( \bigcup_{j=0}^{k} \xi(X^+_k) \) is a union of circles. To get the next one, one replaces each circle, \( c + rT \) in the current one, by \( c + r\xi(X^+_0 \cup X^+_1) \).

Finally, \( \xi(X^+_k) \) is the closure of the union of \( \xi(X^+_k) \), over all \( k \geq 0 \).

We leave it as an exercise to check that, in this case, \( \xi \) alone is injective so \( X^+_\xi \) can be embedded in the plane.

It appears that this set is an example of a fractal with condensation set (see [3, pp. 91–94]). Probably a caution is in order: the embedding to the plane we have given depends on several parameters which were chosen rather arbitrarily. The underlying iterated function system would be affected by these choices and perhaps in a rather bad way.

**Question 7.10.** In which cases can \( X^+_\xi \) be embedded in the plane?

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REFERENCES

[1] R. Adler. Symbolic dynamics and Markov partitions. *Bull. Amer. Math. Soc. (N.S.)* 35(1) (1998), 1–56.
[2] R. Adler and B. Weiss. *Similarity of Automorphisms of the Torus (Memoirs of the American Mathematical Society, 98).* American Mathematical Society, Providence, RI, 1970.
[3] M. F. Barnsly. *Fractals Everywhere,* 2nd edn. Academic Press, Cambridge, MA, 1993.
[4] R. Bowen. Markov partitions for Axiom A diffeomorphisms. *Amer. J. Math.* 3 (1970), 725–747.
[5] R. Bowen. *On Axiom A Diffeomorphisms (CBMS Regional Conference Series in Mathematics, 71).* American Mathematical Society, Providence, 1978.
[6] R. Bowen and J. Franks. Homology for zero-dimensional nonwandering sets. *Ann. of Math. (2)* 106(1) (1977), 73–92.
[7] J. Cuntz and W. Krieger. A class of $C^*$-algebras and topological Markov chains. *Invent. Math.* 56 (1980), 251–268.
[8] V. Deaconu. Groupoids associated with endomorphisms. *Trans. Amer. Math. Soc.* 347 (1995), 251–268.
[9] R. Deeley and K. R. Strung. Nuclear dimension and classification of $C^*$-algebras associated to Smale spaces. *Trans. Amer. Math. Soc.* 370(5) (2018), 3467–3485.
[10] R. J. Deeley, M. Goffeng and A. Yashinski. Smale space $C^*$-algebras have nonzero projections. *Proc. Amer. Math. Soc.* 148(4) (2020), 1625–1639.
[11] R. J. Deeley and A. Yashinski. The stable $C^*$-algebra of a Wieler solenoid: inductive limits and $K$-theory. *Ergod. Th. & Dynam. Sys.* 40 (2020), 2368–2398.
[12] M. Haslehurst. Some examples of factor groupoids. *Preprint,* 2022, arXiv:2204.13888v2.
[13] J. Hjelmborg and M. Rørdam. On stability of $C^*$-algebras. *J. Funct. Anal.* 155(1) (1998), 153–171.
[14] T. W. Hungerford. *Algebra (Graduate Texts in Mathematics, 73).* Springer, New York, 1974.
[15] J. Kaminker, I. F. Putnam and M. F. Whittaker. K-theoretic duality for hyperbolic dynamical systems. *J. Reine Angew. Math.* 730 (2017), 257–288.
[16] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems (Encyclopedia of Mathematics and its Applications, 54).* Cambridge University Press, Cambridge, 1995.
[17] J. L. Kelley. *General Topology (Graduate Texts in Mathematics, 27).* Springer, Berlin, 1955.
[18] D. Lind and B. H. Marcus. *An Introduction to Symbolic Dynamics and Coding.* Cambridge University Press, Cambridge, 1995.
[19] H. Matui. Long exact sequences of homology groups of étale groupoids. *Discrete Contin. Dyn. Syst.* 42 (2022), 5239–5271.
[20] P. S. Muhly, J. N. Renault and D. P. Williams. Equivalence and isomorphism for groupoid $C^*$-algebras. *J. Operator Theory* 17 (1987), 3–22.
[21] V. Proietti and M. Yamashita. Homology and K-theory of dynamical systems. II. Smale spaces with totally disconnected transversal. *Preprint,* 2021, arXiv:2104.10938.
[22] V. Proietti and M. Yamashita. Homology and K-theory of dynamical systems. III. Beyond totally disconnected case. *Preprint,* 2022, arXiv:2207.03118.
[23] I. F. Putnam. $C^*$-algebras from Smale spaces. *Canad. J. Math.* 48 (1996), 175–195.
[24] I. F. Putnam. *A Homology Theory for Smale Spaces (Memoirs of the American Mathematical Society, 232).* American Mathematical Society, Providence, RI, 2014.
[25] I. F. Putnam. Some classifiable groupoid $C^*$-algebras with prescribed $K$-theory. *Math. Ann.* 370(3–4) (2018), 1361–1387.
[26] I. F. Putnam and J. S. Spielberg. The structure of $C^*$-algebras associated with hyperbolic dynamical systems. *J. Funct. Anal.* 163 (1999), 279–299.
[27] J. Renault. *A Groupoid Approach to $C^*$-Algebras (Lecture Notes in Mathematics, 793).* Springer, Berlin, 1980.
[28] M. Rørdam, F. Larsen, N. J. Laustsen. *An Introduction to $K$-theory for $C^*$-Algebras (London Mathematical Society Student Texts, 49).* Cambridge University Press, Cambridge, 2000.
[29] D. Ruelle. Noncommutative algebras for hyperbolic diffeomorphisms. *Invent. Math.* 93 (1988), 1–13.
[30] D. Ruelle. *Thermodynamic Formalism* (Cambridge University Press, Cambridge, 2004).
[31] Y. Sinai. Markov partitions and C-diffeomorphisms. *Funct. Anal. Appl.* 2 (1968), 64–89.
[32] S. Smale. Differentiable dynamical systems. *Bull. Amer. Math. Soc. (N.S.)* 73 (1967), 747–817.