Quantum fluctuations, pseudogap, and the $T = 0$ superfluid density in strongly correlated d-wave superconductors

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I study the effect of Coulomb interaction on superconducting order in a d-wave lattice superconductor at $T = 0$ by considering the superconducting saddle point in the two dimensional t-J-U model with a repulsion $U$. The theory of low-energy superconducting phase fluctuations around this saddle point is derived in terms of the effective hard-core bosons (representing the density of spin-up electrons and the phase of the order parameter), interacting with the fluctuating density of spin-down electrons. Whereas the saddle-point value of the superconducting gap is found to continuously increase towards half filling, the phase stiffness at $T = 0$ has a maximum, and then decreases with further underdoping. Right at half filling the phase stiffness vanishes for large $U$. This argues that the pseudogap phenomenon of the type observed in cuprates is in principle possible without a development of any competing order, purely as a result of growing correlations in the superconducting state. Implications for the finite temperature superconducting transition and the effects of static disorder are discussed qualitatively.

I. INTRODUCTION

Pseudogap phenomenon has become one of the hallmarks of high temperature superconductivity \cite{1}; while the suppression of the single particle density of states is observed in many quantities at a high temperature $T^*$, which appears to increase towards half filling, the superconducting transition temperature $T_c$ together with the $T = 0$ superfluid density, at the same time continuously approaches zero \cite{2}. Being in dramatic contrast to the well understood behavior of the standard BCS superconductors \cite{2}, this puzzling phenomenon has prompted different explanations. These may be grouped into at least two conceptually separate camps. The first group postulates development of a second order parameter in the underdoped region, which competes with superconductivity and suppresses its transition temperature \cite{2, 3, 4, 5, 6, 7, 8, 9, 10}. The second interprets $T^*$ as a crossover temperature at which the Cooper pairs, loosely speaking, begin to form, and the lower $T_c$ as the point where the phase coherence finally sets in \cite{11, 12, 13}. The suppression of $T_c$ is then attributed to the gradual loss of carriers near the insulating state at half filling. There is a significant amount of possible empirical support for the latter point of view: d-wave symmetry of the pseudogap \cite{14}, specific heat measurements \cite{12}, heat transport \cite{16}, microwave conductivity \cite{17}, and the Nernst effect \cite{18}, may all be understood as directly or indirectly supporting the proposed superconducting origin of the pseudogap temperature. Some of the same measurements, however, may also be understood within a competing theory, and the physics of underdoped high temperature superconductors at present still remains controversial.

In this paper it is argued that the pseudogap phenomenon may in principle arise purely from strong Coulomb interactions in a d-wave superconductor and sufficiently near half filling, without any competing order. While a competing order is possible, and indeed in some materials may even be likely in the underdoped region, it seems not to be required simply by the existence of pseudogap. Furthermore, this is shown for a superconductor with only a weak attraction in the d-wave channel, so the mechanism behind the pseudogap considered here is different than in the real-space pairing approaches extensively discussed in the literature \cite{19, 20}. Deriving from Coulomb repulsion, it is similar in spirit, although still different in detail, to the one in the RVB theory \cite{21, 22}.

We consider what is probably the simplest model which contains the relevant physics, the t-J-U model in Eq. (1), with the standard exchange term rewritten as the pairing interaction and with a strong repulsion, and study the evolution of the $T = 0$ superconducting gap and the phase stiffness with doping. Independently of its magnitude, the repulsion does not affect the dependence of the mean-field d-wave superconducting (dSC) gap on the number of particles, as it only shifts the (unphysical) bare value of the chemical potential. As a result, the saddle point value of the superconducting gap (Figure 1) continuously increases towards half filling, as found in other similar calculations \cite{23}. It is intuitively clear however, that for a large repulsion $U(x)$ the superconducting order should eventually be weakened near half filling, since the Cooper pairs, although formed, will have little space left to move. It is less obvious in which exact fashion should this occur; for example, should the suppression be gradual or abrupt \cite{24}, or what its effect on quasiparticles should be \cite{25, 26}. It is possible to make the physics behind this ‘jamming’ effect quite explicit by introducing certain collective coordinates to describe the (quantum) phase fluctuations around the dSC saddle point: two densities of spin-up and spin-down electrons, and the phase of the superconducting order parameter. The crucial feature of our representation is that only one of the electron densities (say of spin-up) becomes the conjugate variable to the superconducting phase. This enables one to understand quantum ($T = 0$) phase fluctuations around the dSC saddle point within the theory of interacting bosons.
(representing jointly the density of spin-up electrons and the order parameter phase) moving in the fluctuating and interacting background provided by the density of the remaining spin-down electrons. A self-consistent calculation of the $T = 0$ phase stiffness in such an effective bosonic system yields then generically a non-monotonic behavior with doping (Figure 2). For $U$ large enough the phase stiffness at half filling vanishes, while the superconducting gap remains perfectly finite, and in fact large.

The superconducting $T_c$ is then non-monotonic because it is determined by the smaller of the two characteristic temperatures, one set by the superconducting gap, and the other by $T = 0$ superfluid density. It is not the bare superfluid density that vanishes near half-filling, however, but only the physical (renormalized) one that becomes reduced by quantum fluctuations. The bare one, indeed, only increases towards half-filling, since in a weakly coupled superconductor it is proportional to the remaining spin-down electrons. A self-consistent calculation of the order parameter phase) moving in the fluctuating and superfluid density. The discussion of the results in relation to strongly correlated electrons has been often invoked in the past to study non-monotonic time quantum mechanical action $S = \int_0^\beta dt L(\tau)$, \(\beta = 1/T\), and $L(\tau) = \sum_{x,\sigma} c_\sigma^\dagger(x,\tau)(\partial_\tau - \mu)c_\sigma(x,\tau) + H(\tau)$, with $c_\sigma(x,\tau)$ being the standard Grassman electronic variables, and the Hamiltonian

$$ H(\tau) = -t \sum_{\langle x,x' \rangle, \sigma = \pm} c_\sigma^\dagger(x,\tau)c_\sigma(x',\tau) + (1) \sum_{x,x'} (n_+(x,\tau) + n_-(x,\tau)) \frac{U(x-x')}{2} (n_+(x',\tau) + n_-(x',\tau)) + J \sum_{\langle x,x' \rangle} [\bar{S}(x,\tau) \cdot \bar{S}(x',\tau) - \frac{1}{4} \sum_{\sigma,\sigma'} n_\sigma(x,\tau)n_{\sigma'}(x',\tau)].$$

Here, $\bar{S}(x,\tau) = (1/2) \sum_{\alpha,\beta} c_\alpha^\dagger(x,\tau)\bar{\sigma}_{\alpha\beta}c_\beta(x,\tau)$ and $n_\sigma(x,\tau) = c_\sigma^\dagger(x,\tau)c_\sigma(x,\tau)$ are the standard electron spin and particle densities, $U(x), J > 0$, and $x$ labels sites of a two dimensional quadratic lattice. When $U(x) = U\delta_{x,x'}$, with $U \to \infty$ the model becomes equivalent to the standard t-J model as derivable from the underlying Hubbard model, since double occupancy then becomes completely suppressed. Here we will relax this condition, and consider a general, and finite repulsive interaction $U(x)$. We will however, be particularly interested in the regime of parameters $U(x) \gg t > J$, which corresponds to a weakly coupled, but a strongly correlated superconductor.

It is presently controversial whether the canonical t-J model indeed supports the d-wave superconducting ground state. Although interesting, this issue will be of little concern to us here. Our philosophy, shared by number of recent works, is to start from the experimental fact that cuprates are d-wave superconductors at least for a range of dopings, and take that as the point of departure for further exploration of the phase diagram. We will therefore regard the Hamiltonian (1) merely as a useful tool for introducing correlations into the postulated superconducting ground state.

The last ($\sim J$) term in the Hamiltonian can be rewrit-
ten to explicate its pairing nature as

\[ H_J = -\frac{J}{2} \sum_{(x,x')} B^\dagger(x,x',\tau')B(x,x',\tau), \]

with \( B(x,x') = c_-(x',\tau)c_+(x,\tau) - c_+(x',\tau)c_-(x,\tau) \) annihilating a singlet on a pair of neighboring sites. In the momentum space, \( H_J \) is a sum of the d-wave and the extended s-wave pairing terms \[22\]. In what follows we consider a purely d-wave saddle point, and neglect completely a possible s-wave component. Being small at the Fermi surface near half filling, the amplitude of the s-wave order parameter is expected to be completely suppressed by the dominate d-wave component \[22\].

We start by decoupling the \( H_J \) term using a complex Hubbard-Stratonovich field \( \Delta(x,x',\tau) \), and by introducing two density variables \( \rho_{\sigma}(x,\tau), \sigma = \pm \) as

\[ L(\tau) = \sum_{(x,x')} c_\sigma^\dagger(x,\tau)[(\partial_\tau - \mu + ij_\sigma(x))\delta_{x,x'} - t]\sigma(x',\tau) + \frac{1}{2} \sum_{x,x'} \rho(x,\tau)U(x-x')\rho(x',\tau) + \frac{1}{2} \sum_{x,x'} \Delta(x,x',\tau)[\Delta(x,x',\tau) + H.c.] \]

The variables \( j_\sigma(x,\tau) \) are the Lagrange multipliers that enforce the constraints \( \rho_{\sigma}(x,\tau) = n_{\sigma}(x,\tau) \[41\], and \( \rho(x,\tau) = \rho_+(x,\tau) + \rho_-(x,\tau) \). The d-wave superconducting saddle point is as usual given by \( \Delta(x,x',\tau) = +\Delta_0 \) if \( x' = x + 1 \) and \( \Delta(x,x',\tau) = -\Delta_0 \) if \( x' = x + 2 \), where \( \hat{1} \) and \( \hat{2} \) are the unit lattice vectors, and with the amplitude \( \Delta_0 \neq 0 \) at \( T = 0 \) being determined by the standard BCS gap equation

\[ 1 = J \int \frac{d^2k}{(2\pi)^2} \frac{\phi_\sigma^2(k)}{\sqrt{\epsilon(k) - \mu^2} + |\Delta_0^2\phi_\sigma^2(k)} \]

The evolution of the postulated superconducting ground state in presence of strong repulsion.

An example of a numerical solution of the saddle-point Eqs. \((4)-(5)\) is presented in Figure 1, for parameters \( t = 150\,meV \) and \( J = 100\,meV \). Even with such a relatively large \( J \) the superconductor is effectively still in the weak-coupling regime, as evidenced by the chemical potential, for example, changing very little by the opening of the superconducting gap. In the rest of the paper we will therefore be concerned exclusively with such a weakly-coupled superconducting state. Figure 1 should of course be understood only as an illustration, as a more realistic calculation should include at least a sizable next-nearest-neighbor hopping \( t' \). It does demonstrate, however, two expected and generic features: a) the gap at a given electron density is unaffected by the repulsion \( U(x) \), which only shifts the bare value of the chemical potential \( \mu \), b) the saddle-point value of the gap is a monotonically increasing function of the electron density. The mean-field analysis by itself would therefore suggest an increasing superconducting \( T_c \) towards half filling, in blatant contradiction to the generic behavior of underdoped cuprates. This conclusion will be overturned however by the inclusion of quantum phase fluctuations introduced by the large repulsion, as discussed next.

### III. QUANTUM FLUCTUATIONS

To this end, consider the fluctuations of the phase of the order parameter, \( \Delta(x,x + \hat{a},\tau) = (-)^a\Delta_0 e^{-i\theta(x,\tau)} \), where \( \hat{a} = \hat{1}, \hat{2} \). To reduce the algebraic complexity we will neglect the fluctuations of the amplitude of the order parameter, which should be justified at \( T = 0 \). We are also assuming a single phase \( \theta(x,\tau) \) for both links \((x, x + \hat{i})\) and \((x, x + \hat{j})\) emanating from the site \( x \). In doing so we are neglecting the fluctuations towards the possible s-wave component of the superconducting order, in accord with the expectation that it is suppressed by the dominate d-wave order \[39\].

As usual, at this stage one would like to integrate out fermions, to be left with an effective action for the collective variables. To perform this step, we first introduce new Grassman variables as \( c_\sigma^\dagger(x,\tau) = e^{i\theta(x,\tau)}a_\sigma^\dagger(x,\tau) \), \( c_\sigma(x,\tau) = e^{-i\theta(x,\tau)}a_\sigma(x,\tau) \), \( c_\sigma^\dagger(x,\tau) = a_\sigma^\dagger(x,\tau) \), and \( c_\sigma(x,\tau) = a_\sigma(x,\tau) \). Note that in contrast to previous works \[42\], \[8\], the entire superconducting phase here is 'absorbed' into particles of a single spin-projection, arbitrarily chosen to be up \[14\]. This guarantees that for all possible configurations of the phase, including vortices, new Grassman variables \( a \) and \( a^\dagger \) are single valued and satisfy the standard fermionic boundary conditions at \( \tau = \beta \). Although the above change of variables is not unique in accomplishing this it probably is the simplest.

Next, as usual, introduce the deviations from the saddle-point values of the fields as \( \delta j_\sigma(x,\tau) = \delta\rho_\sigma(x,\tau) + nU(q = 0), \delta\rho_\sigma(x,\tau) = \delta\rho_\sigma(x,\tau) + n/2, \sigma = \pm \), and then
shift $\delta j_{\pm}(x, \tau) - i\partial_\tau \theta(x, \tau) \to \delta j_{\pm}(x, \tau).$ The Lagrangian then becomes $L(\tau) = L_\rho(\tau) + L_\rho(\tau), \text{ with}$

$$L_\rho(\tau) = \sum_{(x,x'),\sigma} a^\dagger_\sigma(x, \tau) [\partial_\tau - \mu + \delta j_\sigma(x, \tau)] a_\sigma(x', \tau),$$

$$-\mu \delta j_\sigma(x, \tau) + \sum_{(x,x')} \pm \Delta_0 [a^\dagger_\sigma(x, \tau) a_\sigma(x', \tau) - e^{-i(\theta(x, \tau) - \theta(x', \tau))}]$$

$$a^\dagger_\sigma(x, \tau) a^\dagger_\sigma(x', \tau) + H.c.,$$

$$L_\rho(\tau) = -\sum_{x,\sigma} \delta j_\sigma(x, \tau) \rho_\sigma(x, \tau) + \frac{1}{2}$$

$$\int \frac{d^2q}{(2\pi)^3} \int \frac{d^2q}{(2\pi)^3} \delta \rho((x-y), \nu) \rho_\sigma(x, \tau) - \int \delta \rho(x, \tau) \partial_\tau \theta(x, \tau).$$

The shift of one of the Lagrange multipliers serves to isolate the imaginary-time derivative of the phase into the last term in $L_\rho$, and thus promote the density of spin-up electrons $\rho_+$ to the status of \textit{conjugate variable} to the superconducting phase. It is important to note that it is not $\rho_+(x, \tau) + \rho_-(x, \tau)/2$ that plays the role of the conjugate variable, as it would have been obtained if the two projections of spin were treated equally, and half of the superconducting phase absorbed into each species. That, often invoked transformation is allowed only for the trivial boundary condition $\theta(x, \beta) = \theta(x, 0) + 2\pi n$, with $n = 0$, and with the vortices in the phase being forbidden. For this trivial boundary condition the last term in Eq. 7 becomes independent of the average particle density, and any effect of the proximity to commensurate, density, which is our main subject, is lost. To have any commensuration effects on the superfluid response it is thus paramount to allow for the non-trivial (in the above sense) phase configurations.

In principle the fermions can now be integrated out. The resulting action will be a functional of the phase $\theta(x, \tau)$ and the two densities $\rho_\pm(x, \tau)$ of spin-up and spin-down electrons. The fact that $\rho_+$ and $\theta$ form a pair of conjugate variables suggests a particular change of variables, so that the action becomes a functional of the \textit{bosonic} variable $\Psi = \sqrt{\rho_+}e^{i\theta}$, $\Psi^*$, and $\rho_-$. This is the central idea of this paper. In the rest we will try to approximately determine the functional $S[\Psi, \Psi^*, \rho_-]$, and then use it to compute the stiffness for the superconducting phase fluctuations at $T = 0$. Alternatively, the lattice action in Eqs. 6 and 7 can be studied numerically.

In practice however, the integration over fermions is performable analytically only for small phase gradients and a small $\delta j_\sigma$. After a straightforward but a rather involved algebra one finds $S_F = S_F + \int_0^\beta d\tau L_\rho(\tau), \text{ with}$

$$S_F = \frac{1}{2} \int \frac{d^2q}{(2\pi)^3} R(\vec{q}, \nu) M(\vec{q}, \nu) R(\vec{q}, \nu) + O(R^4),$$

$$\frac{n}{2} \int_0^\beta d\tau \sum_{x,\sigma} \delta j_\sigma(x, \tau) + O(R^4), \text{ where}$$

$$R^I(\vec{q}, \nu) = (\delta j_+, \nu, -i\theta(\vec{q}, \nu, \delta j_-, \nu), \text{ and the}$$

$3 \times 3$ matrix $M$ is symmetric with the elements

$$M_{11}(\vec{q}, \nu) = M_{33}(\vec{q}, \nu) = \int \frac{d^2q}{(2\pi)^3} G(\vec{k}, \omega) G(\vec{k} + \vec{q}, \omega + \nu),$$

$$M_{13}(\vec{q}, \nu) = -\int \frac{d^2q}{(2\pi)^3} F(\vec{k}, \omega) F(\vec{k} + \vec{q}, \omega + \nu),$$

$$M_{12}(\vec{q}, \nu) = \int \frac{d^2q}{(2\pi)^3} G(\vec{k} + \vec{q}, \omega + \nu) [\nu - \vec{k}, \omega] \nu - \vec{k}, \omega] + (\vec{k} + \vec{q}) - \Delta(\vec{k}) G(\vec{k}, \omega),$$

$$M_{23}(\vec{q}, \nu) = \int \frac{d^2q}{(2\pi)^3} F(\vec{k} + \vec{q}, \omega + \nu) - \nu - \vec{k}, \omega] \nu - \vec{k}, \omega] + (\vec{k} + \vec{q}) - \Delta(\vec{k}) G(\vec{k}, \omega),$$

and finally

$$M_{22}(\vec{q}, \nu) = \int \frac{d^2q}{(2\pi)^3} [\nu - \vec{k}, \omega] \nu - \vec{k}, \omega] + (\vec{k} + \vec{q}) - \Delta(\vec{k}) G(\vec{k}, \omega),$$

with $G(\vec{k}, \omega) = \frac{(-i\omega - \vec{k} \vec{p})}{\omega^2 + \Delta^2(\vec{k})} + (\vec{e}(\vec{k} - \vec{p})^2), \text{ and}$$

$$F(\vec{k}, \omega) = \frac{\Delta(\vec{k}, \omega)}{\omega^2 + \Delta^2(\vec{k})} + (\vec{e}(\vec{k} - \vec{q})^2), \text{ and}$$

$$\Delta(\vec{k}) = \Delta_0 \rho_0(\vec{k}). \text{ Note that Eqs. (9)-(13) are similar, but not identical to those that can be found in [12], for example. This is due to a different change of fermionic variables employed there which treats spin-up and spin-down symmetrically, as discussed earlier. In particular, we have $M_{12} \neq M_{23}, \text{ and as a result the two densities are in principle coupled differently to the gradients of the phase. This serves to compensate for the asymmetry between $\rho_+$ and $\rho_-$ in $L_\rho$, and to insure that if solved exactly, the theory would lead to equal average densities of up and down particles.}$

The elements of $M(\vec{q}, \nu)$ can be easily evaluated for a weakly coupled superconductor with $\Delta_0 \ll t$. Assuming a spherical Fermi surface for convenience, we find $M_{11}(\vec{q}, \nu) = -\hat{N}/4 + O(q^2, \nu^2), M_{12}(\vec{q}, \nu) = -\hat{N}/4 + O(q^2, \nu^2), M_{13}(\vec{q}, \nu) = M_{33}(\vec{q}, \nu) = O(q^2), \text{ and}$$

$M_{22}(\vec{q}, \nu) = (e_F / 4\pi)(1 + O((\Delta_0/e_F)^2)) q^2 + O(q^4, \nu^2), \text{ where} \hat{N} \text{ is the density of states at the Fermi level } e_F$
with \( m^* = 1/(2a^2) \) being an effective electron mass, and \( a \) the lattice spacing.

At long distances \((q, \nu \to 0)\) one can therefore neglect the matrix elements \( M_{12} \) and \( M_{23} \) that couple the fluctuating phase to the Lagrange multipliers. The integration over \( \delta j_{\pm} \) then gives the action in terms of the densities and the phase to be

\[
S = \int \frac{dq}{(2\pi)} \frac{d\nu}{2\pi} \left\{ \frac{e_F}{8\pi} q^2 \theta(q, \nu) \theta(-q, -\nu) + \frac{1}{2} \int d\tau \delta \rho(q, \nu) \delta \rho(-q, -\nu) + \nu \theta(q, \nu) \rho_{-}(q, \nu) \right\},
\]

Eq. (17) therefore implies a vanishing uniform spin susceptibility and a finite compressibility (for short-range interaction) in a singlet superconductor at \( T = 0 \). Note also that for non-interacting electrons \((U = J = 0)\) one recovers \( \chi_s = \chi_c = -2M_{11}(q, \nu) \), which is then the familiar Lindhard response.

We may gain further insight into the nature of quasi-interactions \( V_{++} \) and \( V_{-+} \) by computing them at small \( q \). In the first approximation one may neglect the anisotropy of the d-wave gap and perform the calculation for the simpler s-wave superconductor. The result is

\[
M_{13}(q, \nu) - M_{11}(q, \nu) \approx \frac{e_F}{6\pi\Delta_0^2} q^2(1 + O(\nu^2)).
\]

We checked numerically that the result for d-wave superconductor is essentially the same, except for the anisotropy in the coefficient of the \( q^2 \)-term. Taking the d-wave nature of the gap into account yields a maximum of \( M_{13} - M_{11} \) at a finite \( q \) that connects the gap nodes, so that the Stoner instability of a metal in a dSC becomes replaced by an analogous RPA instability towards the spin density wave [15]. For the present purposes it suffices to note that the quasi-interactions induced by the fermion integration are long-ranged, but \( weak \) for a weakly coupled superconductor, i. e. \( V_{++} \sim \Delta_0^2/e_F \). In particular, this means that the electrons with the opposite spin, although paired in the momentum space by construction, may to a good approximation be considered as independent in real space, and hence to be interacting primarily via \( U(x) \).

IV. EFFECTIVE BOSONIC HAMILTONIAN AND THE PHASE STIFFNESS

If one would integrate out the Gaussian density fluctuations in Eq. 14 the remaining action for the phase, apart from the linear time derivative term, describes the standard collective mode in a weakly coupled superconductor. To capture the effects of nearly commensurate density on the superfluid response one needs to go beyond this hydrodynamic regime [16]. What is needed is the bosonic lattice action \( S[\Psi, \Psi^*, \rho_{\pm}] \), which would correspond to Eqs. 6 and 7, and which at long length scales would reduce to Eq. 14. The simplest candidate satisfying these requirements is:

\[
L_B(\tau) = \sum_x \sum_{\langle x, x' \rangle} \Psi^*(x, \tau) \partial_\tau \Psi(x, \tau) - t_B \sum_{\langle x, x' \rangle} \Psi^*(x, \tau) \Psi(x', \tau) + \frac{1}{2} \int d\tau' \sum_{x, x'} |\Psi(x, \tau)|^2 (V_{++}(x - x', \tau - \tau') + U(x - x') \delta(\tau - \tau')) |\Psi(x', \tau')|^2.
\]
FIG. 2: The phase stiffness $K/4t_B$ in the effective bosonic theory for phase fluctuations for $U=0$, $U/4t_B=1.8$, $U/4t_B=5$, and $t_B/U=0$ (top to bottom) vs. doping $x=1-n$.

$$+ \int d\tau \sum_{x,x'} |\Psi(x,\tau)|^2 (U(x-x')\delta(\tau-\tau')$$

$$+ V_{+-}(x-x',\tau-\tau')\rho_-(x',\tau')$$

$$- \mu_B \sum_x (|\Psi(x,\tau)|^2 + \rho_-(x,\tau))$$

$$+ \frac{1}{2} \int d\tau \sum_{x,x'} \rho_-(x')V_{--}(x-x',\tau-\tau')$$

$$+ U(x-x')\delta(\tau-\tau')\rho_-(x',\tau'),$$

with $\Psi(x,\tau) = \sqrt{\rho(x,\tau)e^{i\delta(x,\tau)}}$, $t_B = e_F/(4\pi)$ and $\mu_B = \chi_e^{-1}(0,0) n/2$. The reader is invited to check that when expanded in powers of the density and phase and their derivatives, the leading order terms in $L_B$ indeed reproduce the Eq. (14).

The action in Eq. (21) represents bosonic particles hopping on a lattice and interacting with a fluctuating background provided by the density of spin-down electrons. Although the amplitude of our bosonic field is proportional to the density of spin-up electrons only, its phase is still the full superconducting phase and therefore the bosons have the electromagnetic charge $2e$. Since $e_F \sim n$, hopping amplitude for the bosons is $t_B = e_F/(4\pi n) \sim t$, and is approximately doping independent. Since the change of fermionic variables leading to Eq. (7) broke the spin up-down symmetry, although the chemical potential $\mu_\pm$ is the same for both the bosons and the spin-down particles in $L_B$, two densities in our approximation will not automatically be equal as they should. One may easily correct this by allowing for two different chemical potentials $\mu_+ \neq \mu_-$, and adjust them so that the average densities of bosons and spin-down electrons become equal, as of course they are in the full theory in Eqs. 6 and 7. Furthermore, assuming the Coulomb interaction $U(x) \sim 1/x$, for a weakly coupled superconductor we may neglect the quasi-interactions $V_{+-}$ and $V_{++}$, which are weak, and also less singular than $U(x)$ at $x = 0$ [18]. To simplify the calculation that follows, after neglecting the quasi-interactions we may further re-place the realistic Coulomb interaction by the hard-core repulsion between the particles. To make the calculation a little more interesting however, we will assume the hard-core repulsion only between the same spin particles [19], and allow for some finite on-site repulsion $U$ between the particles with opposite spin. The single parameter $U$ then effectively measures the degree of correlations at the superconducting saddle point. For $U = \infty$ then all the particles interact via hard-core repulsion.

The phase stiffness from Eq. (21) is therefore

$$K = t_B |\langle \Psi(x,\tau) \rangle|^2. \quad (22)$$

Within the local approximation discussed above we may compute $K$ in the effective bosonic action in Eq. (21) by decoupling the hopping term with a Hubbard-Stratonovich field $\Phi(x,\tau)$, where $\langle \Phi(x,\tau) \rangle = 2t_B \langle \Psi(x,\tau) \rangle$, and by making yet another saddle-point approximation. Assuming that at such a saddle point both $\Phi(x,\tau)$ and $\rho_-(x,\tau)$ are independent of the imaginary time, the 'mean-field' bosonic free energy becomes (see Appendix)

$$F_{B,MF} = \frac{1}{4} \sum_{x,x'} \Phi^*(x)(t_B^{-1})_{x,x'} \Phi(x') \quad (23)$$

$$-T \sum_x \ln Tr Exp \left\{ -\frac{1}{T} \frac{1}{2} \Phi(x) \Psi^\dagger(x) + \frac{1}{2} \Phi^*(x) \Psi(x) - \mu_+ \Psi^\dagger(x) \Psi(x) - U \Psi^\dagger(x) \Psi(x) \rho_-(x) \right\},$$

where the trace is to be evaluated over four possible states: empty site $|0\rangle$, boson $\Psi^\dagger(x)|0\rangle$, spin-down electron $|-\rangle$, and boson and spin-down electron $\Psi^\dagger(x)|-\rangle$. Since we assumed a static saddle point, the temperature $T$ appears explicitly in $F_B$, and at the end we need to impose the limit $T \rightarrow 0$. $\Phi(x)$ takes the value that minimizes $F_{B,MF}$, and the chemical potentials $\mu_\pm$ are to be tuned so that the densities of bosons and of spin down electrons are both equal to half the electron density:

$$\frac{\partial F_{B,MF}}{\partial \Phi(x)} = 0, \quad (24)$$

$$\frac{n}{2} = -\frac{1}{M} \frac{\partial F_{B,MF}}{\partial \mu_+}, \quad (25)$$

$$\frac{n}{2} = -\frac{1}{M} \frac{\partial F_{B,MF}}{\partial \mu_-}. \quad (26)$$

Evaluating the trace in $F_{B,MF}$ yields:

$$F_{B,MF} = \frac{1}{4} \sum_{x,x'} \Phi^*(x)(t_B^{-1})_{x,x'} \Phi(x') \quad (27)$$

$$-\frac{M\mu_+}{2} - T \sum_x \ln \sum_{i=1}^{4} e^{\lambda_i},$$
interaction. When $\lambda \ll U$, one finds

$$\frac{1}{4t_B} = \frac{((\mu_+^2 + |\Phi|^2)^{-1/2}e^{\lambda_1} + ((U - \mu_+)^2 + |\Phi|^2)^{-1/2}e^{\lambda_3}}{e^{\lambda_1} + e^{\lambda_3}},$$

and

$$\frac{n}{2} = \frac{1}{2} \left[ 1 + \sqrt{\frac{\mu_+^2 + |\Phi|^2}{e^{\lambda_1} + e^{\lambda_3}}} \right].$$

where we have also assumed a uniform $\Phi(x) = \Phi \neq 0$.

The mean-field Eqs. (30)-(32) can be solved for any $U$, but we focus here only on the limits of large and small interaction. When $U = 0$, one finds

$$K = \frac{\Phi^2}{4t_B} = n(2-n)4t_B + O(U).$$

The phase stiffness of the non-interacting ($U = 0$) weakly coupled dSC is a monotonically decreasing function of doping $x = 1 - n$, behaving in essentially the same way as the saddle-point amplitude $\Delta_0$, except for being much larger. This is the familiar non-correlated BCS limit. In the strongly interacting case $U/t_B \gg 1$, on the other hand,

$$\frac{K}{4t_B} = 2n(1-n) + 2n^2(1-n)\frac{4t_B}{U} + O(\frac{t_B}{U})^2.$$

As $n \to 0$ the superfluid density of course, vanishes for any $U$. For strong interaction, however, as $n \to 1$, $K \to 0$ for $U/4t_B > 2$. The order parameter is a decreasing function of $U$, and its evolution with interaction is presented on Fig. 2.

Physical mechanism behind the suppression of the phase stiffness near half filling is simple: although the number of bosons per site near $n = 1$ is only $\sim 1/2$, the remaining sites are largely occupied with spin-down particles. For $U/4t_B > 2$ and right at half filling this completely blocks the motion of bosons and brings the stiffness to zero. For a small $U$, however, it is advantageous for bosons to be in the superfluid state even at half-filling, since there is only a small energy cost to hop through a site occupied with a spin down electron. At $U = 0$, of course, spin-down electrons are (almost) invisible to bosons (with $V_{\perp}$ neglected), and the stiffness at half filling is actually the largest.

V. DISCUSSION

Our main result is that for a strong repulsion $U$, the $T = 0$ superfluid density becomes strongly suppressed near half filling, and in fact vanishes right at for large $U$. This is demonstrated for a weakly-coupled superconductor, in which the electrons constituting a Cooper pair may be considered essentially uncorrelated in real space. The gist of the method is the introduction of two separate density fields for spin up and down electrons, only one of which is grouped together with the superconducting phase into a bosonic variable, while the other provides the interacting fluctuating background. Vanishing of the phase stiffness towards half filling is then obtained without invoking any competing order parameters or the real-space pairing.

That the found behavior of the superfluid density implies a pseudogap behavior and the nonmonotonic superconducting $T_c$ with doping is qualitatively seen as follows. In principle, the superconducting $T_c$ is determined by the lower of two characteristic temperatures, the mean-field $T_{BCS} \propto \Delta_0$, and $T_g \propto K$ at which the phase coherence would dissipate even with the amplitude $\Delta_0$ held fixed. Since the characteristic energy scale for $T_g$ is the hopping parameter $t$, and for a weakly coupled superconductor $\Delta_0 \ll t$, for a weak interaction $U T_g \gg T_{BCS}$, and it is the $T_{BCS}$ that determines $T_c$. For strong $U$ however, this will remain true only above some doping $x_{opt}$. For the parameters used in Fig. 1 $x_{opt} < 0.5$, since at $x \approx 0.5$ the gap becomes rather small while the superfluid density there peaks. For $x \ll x_{opt}$, $T_c$ is the hopping parameter $t$, and for a weakly coupled superconductor $\Delta_0 \ll t$, for a weak interaction $U T_g \gg T_{BCS}$, and it is the $T_{BCS}$ that determines $T_c$. One can still see the remnant of $T_{BCS}$ as the crossover temperature where the density of states becomes suppressed, just as it would be in a true superconductor. This divorce of $T_{BCS}$ and the true superconducting $T_c$ is an unavoidable consequence of strong repulsion, and may be taken as an operating definition of a strongly correlated d-wave superconductor.

Our $T = 0$ calculation has another interesting consequence worth mentioning. While a strong $U$ implies the existence of the optimal doping $x_{opt}$ at which $T_c$ peaks, at $T = 0$ there is nothing particular happening at that doping at all. In fact, as doping is increased above $x_{opt}$, $T_c$ that will determine $T_{opt}$. One can still see the remnant of $T_{BCS}$ as the crossover temperature where the density of states becomes suppressed, just as it would be in a true superconductor. This divorce of $T_{BCS}$ and the true superconducting $T_c$ is an unavoidable consequence of strong repulsion, and may be taken as an operating definition of a strongly correlated d-wave superconductor.

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order to our bosonic description of quantum phase fluctuations in underdoped regime would be expected to lead to precisely such a behavior, as suggested by the studies of superfluid-insulator transition \cite{24}. Recalling that electrons, and consequently our effective bosons, in reality interact via long-range Coulomb interaction suggests the universality class of two-dimensional dirty bosons with \( z = 1 \), and \( v \approx 1 \) \cite{25}, in accord with a number of different experiments in underdoped cuprates \cite{26}. In the overdoped regime, on the other hand, \( \Delta_0 \) becomes small and the pair breaking effect of disorder will bring it, together with \( T_c \) and the superfluid density, to zero at another critical doping \( x_o \) \cite{27}. The precise mechanism of this suppression seems also in agreement with the present, albeit somewhat limited, experimental results on overdoped cuprates \cite{28}. This would also explain why \( T = 0 \) superfluid density starts to decrease for \( x > x_{\text{opt}} \), where it would still be increasing according to our picture, if there was no disorder.

Although in the present work we were concerned with \( T = 0 \), our picture of the quantum superconductor-insulator transition then that the universality class of the transition at \( T = T_c \) in underdoped regime will be of Berezinskii-Kosterlitz-Thouless (BKT) type. Indeed, under the assumption of a finite superconducting gap, vortex unbinding becomes the only known mechanism for the loss of phase coherence in two dimensions. Possible BKT nature of the transition is supported by the observed large Nernst effect above \( T_c \) in underdoped cuprates \cite{13}, and the measurements of the microwave conductivity \cite{17}. It was argued recently \cite{29} however, that having the BKT transition well below \( T^* \) set by a large amplitude \( \Delta_0 \) requires a competing order parameter. The argument is as follows: the condensation energy per unit area of the BCS superconductor is \( \sim N \Delta_0^2 \), where \( N \) is the density of states at the Fermi level. Since the coherence length is \( \nu \xi / \Delta_0 \) one finds that the core energy of a vortex is \( \sim e_F \), the Fermi energy, which is much larger than the BCS superconducting \( T_c \). This is why the superconducting transition of a weakly coupled superconductor is essentially mean-field in character, with an unobservably narrow critical region. It seems then that a competing order developing in the vortex core is necessary to bring the core energy down, so that vortices may proliferate at a low \( T_c \). That this, however, does not necessarily follow, one may see by realizing that the above core energy \( \sim e_F \) is nothing but the bare stiffness for the phase fluctuations. The role of the competing order parameter would then be to reduce this stiffness with underdoping. But as we argued, this seems perfectly possible without any ordering, simply from the repulsion which suppresses phase coherence even at \( T = 0 \). As long as the \( T = 0 \) stiffness is small, the BKT transition temperature will also be small \cite{30}. The problem with the argument in favor of the competing order is that it ignores possible quantum fluctuations arising from interactions, which are on the other hand, the main point of the present work.

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VII. APPENDIX: DERIVATION OF THE BOSONIC MEAN-FIELD ENERGY

Here we present the details of the derivation of the Eq. (23) for the bosonic mean-field energy. Within the local approximation discussed right above Eq. (22), the partition function for the effective bosonic system becomes

\[
Z_B = \int D[\Psi^*, \Psi] D\rho_- e^{-\int_0^\beta d\tau L_B},
\]

with the bosonic Lagrangian

\[
L_B(\tau) = \sum_x \Psi^*(x, \tau) \partial_\tau \Psi(x, \tau) - t_B \sum_{\langle x, x' \rangle} \Psi^*(x, \tau) \Psi(x', \tau) + U' \sum_x |\Psi(x, \tau)|^2 (|\Psi(x, \tau)|^2 - 1) + U \sum_x |\Psi(x, \tau)|^2 \rho_-(x, \tau) - \sum_x (\mu_+ |\Psi(x, \tau)|^2 + \mu_- \rho_-(x, \tau)) + U' \sum_x \rho_-(x, \tau) (\rho_- (x, \tau) - 1),
\]

where we have written the hard core interaction \( U' \) \((U' \to \infty)\) explicitly. This way one can still write the standard coherent state representation of the partition function. One can then derive Eq. (23) as follows. First, decouple the hopping term by introducing the Hubbard-Stratonovich complex field \( \Phi(x, \tau) \):

\[
e^{\int d\tau t_B \sum_{\langle x, x' \rangle} \Phi^*(x, \tau) \Phi(x', \tau)} \propto \int D[\Phi^*, \Phi] \exp \left\{ e^{-\int d\tau \sum_{x, x'} \Phi^*(x, \tau)(t_B^{-1})_{x, x'} \Phi(x, \tau) + \sum_x \Phi^*(x, \tau) \Phi(x, \tau) + c.c.} \right\}.
\]

Next, calculate the partition function in the saddle point approximation. Assuming that at the saddle point \( \Phi(x, \tau) = \Phi(x) \) and \( \rho_-(x, \tau) = \rho_-(x) \), i.e. the static configuration, in the hard-core limit \( U' \to \infty \) the partition function becomes

\[
Z_{B,MF} = e^{-\sum_x (\mu_+ \Phi(x) \hat{b}^\dagger(x) + \mu_- \Phi(x) \hat{b}(x)) T \rho_-} e^{\int \beta d\tau \sum_x \Phi^*(x)(t_B^{-1})_{x, x'} \Phi(x')} - T e^{\beta \mu_+ \sum_x \Phi(x) \hat{b}^\dagger(x) + \beta \mu_- \sum_x \Phi(x) \hat{b}(x) + \beta \mu_- \sum_x \rho_-(x)},
\]

where the mean-field Hamiltonian is

\[
\hat{H}_{MF} = \sum_x (\Phi(x) \hat{b}^\dagger(x) + \Phi^*(x) \hat{b}(x)) - T \rho_- \sum_x (\mu_+ \hat{b}^\dagger(x) \hat{b}(x) + \mu_- \rho_-(x)) + U \sum_x \hat{b}^\dagger(x) \hat{b}(x) \rho_-(x),
\]
and the trace is to be evaluated over the four states as described below Eq. (23). Since $H_{MF}$ is local it factorizes into a product over the lattice sites, and thus the trace can be easily computed. Finally, rescaling $\Phi \to \Phi/2$ leads to the mean-field free energy as displayed in Eq. (23).

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