Unification of Rare/Weak Detection Models using Moderate Deviations Analysis and Log-Chisquared P-values

Alon Kipnis
kipnisal@stanford.edu

Department of Statistics, Stanford University

Abstract:
Rare/Weak models for multiple hypothesis testing assume that only a small proportion of the tested hypotheses concern non-null effects and the individual effects are only moderately large, so that they generally do not stand out individually, for example in a Bonferroni analysis.

Such rare/weak models have been studied in quite a few settings, for example in some cases studies focused on underlying Gaussian means model for the hypotheses being tested; in some others, Poisson. It seems not to have been noticed before that such seemingly different models have asymptotically the following common structure: Summarizing the evidence each test provides by the negative logarithm of its P-value, previous rare/weak model settings are asymptotically equivalent to detection where most negative log P-values have a standard exponential distribution but a small fraction of the P-values might possibly have an alternative distribution which is moderately larger; we do not know which individual tests those might be, or even if there are any such. Moreover, the alternative distribution is noncentral chisquared on one degree of freedom.

We characterize the asymptotic performance of global tests combining these P-values in terms of the chisquared mixture parameters: the scaling parameters controlling heteroscedasticity, the non-centrality parameter describing the effect size whenever it exists, and the parameter controlling the rarity of the non-null effects. Specifically, in a phase space involving the last two parameters, we derive a region where all tests are asymptotically powerless. Outside of this region, the Berk-Jones and the Higher Criticism tests have maximal power. Inference techniques based on the minimal P-value, false-discovery rate controlling, and Fisher’s combination test have sub-optimal asymptotic phase diagrams. We provide various examples for multiple testing problems of the said common structure.

Our log-chisquared approximation for P-values is different from Bahadur’s log-normal approximation; the log-normal approximation is a large deviations phenomenon, while the effects we study appear instead on the moderate deviations scale. The log-normal approximation would be unsuitable for understanding Rare/Weak multiple testing models.

Keywords and phrases: hypothesis testing, multiple testing, P-value, heterogeneous mixture, heteroscedastic mixture, higher criticism, false discovery rate, Berk-Jones.
1. Introduction

Consider a multiple hypothesis testing situation, each test involves a different feature of the data where different features are independent. We are interested in testing a global null hypothesis against the following alternative: the non-null effects are concentrated in a small, but unknown, subset of the hypotheses. In the most challenging situation, effects are not only rare but also weak in the sense that the non-null test statistics are unlikely to provide evidence after Bonferroni correction. Rare/weak multiple hypothesis testing problems of this nature arise frequently in modern statistics and comprise a wide range of applications [21]. Specific examples include:

- **Sparse signal detection.** Suppose we are interested in intercepting a transmission that occupies few frequency bands out of potentially many, while the occupied bands are unknown to us [55]. The features are periodogram ordinates associated with individual frequency bands. Evidence for the presence of a signal can be gathered by testing each periodogram ordinate against the same exponential distribution.

- **Classification.** Classifying images or other high-dimensional signals frequently involves thousands or more features. We think about the typical response in each feature under a specific base class as the global null hypothesis. Testing against this null amounts to determining whether the tested signal is associated with the base class or not. A situation of wide interest is when inter-class discrimination is due to a small proportion of features out of potentially many, and we do not know which ones they are likely to be [20].

- **Detecting changes from a reference dataset.** Testing whether two high-dimensional datasets are simply two different realizations of the same data generating mechanism is a classical problem in statistics, computer science, and information theory [8, 2, 22]. This scenario is formulated as a two-sample testing problem; the null hypothesis states that both samples were obtained from the same high-dimensional parent distribution.

Applications as above and many others have motivated a significant body of work in rare/weak multiple testing settings through the past two decades, providing fruitful insights for signal detection, feature selection, and classification problems in high dimensions [37]. Specific examples of rare/weak multiple testing setting include normal mixtures [33, 36, 19, 1], binomial mixtures [47], linear regression model under Gaussian noise [3, 35], Poisson mixtures [5], heteroscedastic normal mixtures [13], general mixtures [14, 6], mixture of unknown distributions [16, 17, 6], and several two-sample settings [22].

In this paper, we study one rare/weak multiple testing setting that subsumes the vast majority of these previously studied ones. Our setting is not tied to a specific data-generating model. Instead, we model the behavior of a collection of P-values, each P-value summarizes the evidence of one test statistic against the global null. These P-values may be obtained either from one- or two-sample tests, and may represent responses over a variety of models. More generally,
advantages of modeling the distribution of the P-values rather than the data are discussed in [39, 40, 52, 11].

In our setting, the \(i\)-th test statistic yields the P-value \(p_i\), for \(i = 1, \ldots, n\). We further assume that \(p_i \sim \text{Unif}(0,1)\) under the global null, corresponding to the case where the model underlying the \(i\)-th test statistics has a continuous distribution (we relax this assumption later on). Consequently, \(-2\log(p_i) \sim \text{Exp}(2)\), where \(\text{Exp}(2)\) is the exponential distribution with mean 2, aka as the chisquared distribution with two degrees of freedom \(\chi^2_2\). Our model proposes the following alternative: Roughly \(n\epsilon\) of the P-values depart from their uniform distribution and instead obey

\[-2\log(p_i) \overset{D}{\approx} (\mu + \sigma Z)^2, \quad Z \sim N(0,1).\] (1)

Here \(\overset{D}{\approx}\) indicates a specific form of approximation in distribution that we formalize in Section 1.1 below. Leaving the details of this approximation aside for now, (1) says that \(-2\log(p_i)\) is approximately distributed as a scaled noncentral chisquared random variable (RV) over one degree of freedom with noncentrality parameter \(\mu\), and scaling parameter \(\sigma\). We focus on the case where the rarity parameter \(\epsilon\) vanishes while the intensity parameter \(\mu\) is only moderately large, making our global testing problem challenging; in some cases, impossible. As we shall see, in this regime the non-null effects are not only rare but are also weak in the sense that they generally do not stand out individually in a Bonferroni analysis.

A key insight of our analysis says that the log-chisquared approximation (1) is accurate for characterizing the moderate-deviations behavior of the involved statistics in the sense of [51] [18, Ch. 3.7]. Consequently, this approximation can be used to analyze the asymptotic power of tests in all previously studied rare/weak settings in which moderate deviations analysis applies [19, 13, 17, 3, 14, 5, 47, 22], allowing the study of these models under a unified setting we denote as the Rare Moderate Departures model.

The emergence of the log-chisquared approximation for P-values under the rare/weak multiple testing setting is somewhat surprising, as this approximation is different than the classical log-normal approximation of Bahadur [7, 28] and Lambert and Hall [40]. In Section 4, we show that our log-chisquared distribution fit the distribution of the P-values under moderate departures significantly better than the log-normal distribution, and that the log-normal approximation does not indicate the correct asymptotic performance of tests under our setting. To summarize this last point, we establish here that a rare/weak multiple hypothesis testing setting in which the departures are on the moderate scale corresponds to detecting a few noncentral chisquared signals against an exponential background rather than detecting a few normal signals, as one might have proposed in view of Bahadur’s approximation.

We also note that the logarithmic scoring scale for P-values is interesting in its own right. This scale goes back to Fisher, who initially suggested it as a method of ranking success in card guessing games [25]. For global testing, Fisher
proposed the statistic \[ F_n \equiv \sum_{i=1}^{n} -2 \log(p_i), \] (2)

which has a \( \chi^2_{2n} \) distribution under the global null. A test based on \( F_n \) is known to be effective in the presence of small effects distributed across the bulk of cases, but not effective under relatively rare and somewhat stronger but individually still weak as our model proposes; see a formal statement about the inadequacy of \( F_n \) in our setting in Theorem 1.6 below. The logarithmic scale for P-values is now standard in genome-wide association studies (GWAS) \[9, 48, 49\] and other areas \[42, 50\]. Our setting provides an explicit model for testing rare/weak effects in these applications: testing chisquared departures against an exponential background. A similar model arises in detecting the presence of rare/weak sinusoids in white noise based on the periodogram. For this setting, Fisher’s periodogram test is based on the largest periodogram ordinate \[26\] which is analogous to a Bonferroni analysis.

1.1. Rare Moderate Departures Setting

The description above depicts the following global hypothesis testing setting.

\[
H_0 : \quad -2 \log(p_i) \sim \text{Exp}(2), \quad i = 1, \ldots, n, \\
H_1(n) : \quad -2 \log(p_i) \sim (1 - \epsilon)\text{Exp}(2) + \epsilon Q_i^{(n)}, \quad i = 1, \ldots, n,
\]

(3)

where \( Q_i^{(n)} \) is a probability distribution specifying the non-null behavior of the \( i \)-th P-value.

We calibrate the rarity parameter \( \epsilon \) to \( n \) according to

\[ \epsilon = \epsilon_n \equiv n^{-\beta}, \]

(4)

where \( \beta \in (0, 1) \). This calibration proposes that, for an overwhelming majority of the individual tests, the response is indistinguishable under the null and alternative.

1.1.1. Log-Chisquared Approximation under Moderate Departures

We compare \( Q_i^{(n)} \) to the non-central and scaled chisquared distribution as in the right-hand side of (1), where the scaling parameter \( \sigma \) is fixed and the non-centrality parameter \( \mu \) is calibrated to \( n \) as in:

\[ \mu = \mu_n(r) \equiv \sqrt{2r \log(n)}. \]

(5)

Specifically, define the moderately perturbed scaled chisquared distribution

\[ \chi^2(r, \sigma) \overset{D}{=} (\mu_n(r) + \sigma Z)^2, \quad Z \sim \mathcal{N}(0, 1), \]

(6)
where $\overset{D}{=} \equiv$ indicates equality in distribution. For the sake of formalizing the approximation in (1), we introduce the function

$$
\alpha(q; r, \sigma) \equiv \left( \frac{\sqrt{q} - \sqrt{r}}{\sigma} \right)^2,
$$

and note that

$$
\lim_{n \to \infty} \frac{\log \Pr \left[ \chi^2(r, \sigma) \geq 2q \log(n) \right]}{\log(n)} = -\alpha(q; r, \sigma), \quad q > 0.
$$

Our log-chisquared approximation of $\{Q_i^{(n)}\}$ says that

$$
\lim_{n \to \infty} \max_{i=1,\ldots,n} \frac{\log \Pr \left[ Q_i^{(n)} \geq 2q \log(n) \right]}{\log(n)} = -\alpha(q; r, \sigma), \quad q \in (0, 1 + \gamma), \quad (8)
$$

for some $\gamma > 0$. Namely, we require that the moderate tail probability of $Q_i^{(n)}$ behaves as that of a non-central and scaled chisquared $\chi^2(r, \sigma)$. Henceforth, we refer to hypothesis testing problems of the form (3) under the condition (8) as the Rare Moderate Departures (RMD) model with log-chisquared parameters $(r, \sigma)$. A useful criterion for the validity of (8) is

$$
Q_i^{(n)} \overset{D}{=} (\mu_n(r) + \sigma Z)^2 (1 + o_p(1)), \quad n \to \infty, \quad (9)
$$

where $o_p(1)$ indicates a sequence of RVs tending to zero in probability uniformly in $i$ as $n \to \infty$.

1.1.2. Strong Log-Chisquared Approximation

The characterization of the information theoretic limit of detection in (3) require the following stronger form of log-chisquared approximation than (8).

$$
\lim_{n \to \infty} \max_{i=1,\ldots,n} \frac{\log \left( \frac{dQ_i^{(n)}}{d\chi^2(r, \sigma)} (2q \log(n)) \right)}{\log(n)} = 0, \quad q \in (0, 1 + \gamma), \quad (10)
$$

for some $\gamma > 0$. Namely, (10) says that the log-likelihood ratio between $Q_i^{(n)}$ and the moderately perturbed chisquared distribution $\chi^2(r, \sigma)$ grows at most sub-logarithmically uniformly over all coordinates for $q < 1 + \gamma$. The type of equivalence between $Q_i^{(n)}$ and $\chi^2(r, \sigma)$ described in (10) is similar to the setting of [14]. Henceforth, we refer to hypothesis testing problems of the form (3) under the condition (10) as the strong RMD. It follows from Lemma 5.3 below that (10) implies (8). Note that (10) holds whenever the distribution of each $Q_i^{(n)}$ has a density and satisfies (9).
1.2. Non-uniformly Distributed P-values

We extend our setting (3) to situations where \( p_1, \ldots, p_n \) are not uniformly distributed under the null by considering

\[
\begin{align*}
H_0^{(n)} & : -2 \log(p_i) \sim E_i^{(n)}, \quad i = 1, \ldots, n, \\
H_1^{(n)} & : -2 \log(p_i) \sim (1 - \epsilon)E_i^{(n)} + \epsilon Q_i^{(n)}, \quad i = 1, \ldots, n,
\end{align*}
\]  

instead of (3), where \( Q_i^{(n)} \) satisfies (8) and where the probability distribution \( E_i^{(n)} \) converges to \( \text{Exp}(2) \) in the sense that

\[
\lim_{n \to \infty} \max_{i=1,\ldots,n} \frac{\log \Pr[E_i^{(n)} \geq 2q \log(n)]}{\log(n)} = -q, \quad q > 0.
\]  

This extension of the RMD setting is particularly useful in situations when the distribution of the P-values under the null is only super uniform as in some discrete models, or when we consider asymptotic P-values rather than exact ones.

1.3. Asymptotic Power and Phase Transition

RMD models experience a phase transition phenomenon in the following sense: for some choice of the parameters \( r, \beta, \) and \( \sigma \), the two hypotheses are completely indistinguishable. In another region, some tests can asymptotically distinguish \( H_1^{(n)} \) from \( H_0 \) with probability tending to one.

Formally, for a given sequence of statistics \( \{T_n\}_{n=1}^{\infty} \), we say that \( \{T_n\}_{n=1}^{\infty} \) is asymptotically powerful if there exists a sequence of thresholds \( \{h_n\}_{n=1}^{\infty} \) such that

\[
\Pr_{H_0}(T_n < h_n) + \Pr_{H_1^{(n)}}(T_n \leq h_n) \to 0,
\]

as \( n \) goes to infinity. In contrast, we say that \( \{T_n\}_{n=1}^{\infty} \) is asymptotically powerless if

\[
\Pr_{H_0}(T_n < h_n) + \Pr_{H_1^{(n)}}(T_n \leq h_n) \to 1,
\]

for any sequence \( \{h_n\}_{n \in \mathbb{N}} \). The so-called phase transition curve is the boundary of the region in the parameter space \( (\beta, r) \) in which all tests are asymptotically powerless.

Define the would-be phase curve

\[
\rho(\beta; \sigma) \equiv \begin{cases} 
(2 - \sigma^2)(\beta - 1/2) & \frac{1}{2} < \beta < 1 - \frac{\sigma^2}{4}, \quad 0 < \sigma^2 < 2 \\
(1 - \sigma \sqrt{1 - \beta})^2 & 1 - \frac{\sigma^2}{4} \leq \beta < 1, \quad 0 < \sigma^2 < 2, \\
0 & \frac{1}{2} < \beta < 1 - \frac{1}{\sigma^2}, \quad 0 < \sigma^2 \geq 2 \\
(1 - \sigma \sqrt{1 - \beta})^2 & 1 - \frac{1}{\sigma^2} \leq \beta < 1, \quad \sigma^2 \geq 2.
\end{cases}
\]  

One side of the phase transition characterization is provided as follows.
The phase transition curve $\rho(\beta; \sigma)$ of (13) defines the detection boundary under in the Rare Moderate Departure models (3) and (11). For $r < \rho(\beta; \sigma)$, all tests are asymptotically powerless. For $r > \rho(\beta; \sigma)$, some tests, including Higher Criticism and the Berk-Johns, are asymptotically powerful.

Figure 1 depicts $\rho(\beta; \sigma)$ for three choices of $\sigma$. The function $\rho(\beta; \sigma)$ was first derived in [13] to describe the detection boundary of rare/weak heteroscedastic normal means; see the discussion in Section 4.1 below. Theorems 1.1 extends this result from [13] to rare/weak multiple testing model obeying the RMD formulation, several of which we discuss in Section 2 below.

1.4. Optimal Tests

To complete the phase transition characterization of RMD models initiated in Theorem 1.1, we consider two tests that are asymptotically powerful whenever $r > \rho(\beta; \sigma)$.

1.4.1. Higher Criticism Test

The Higher Criticism (HC) of the P-values $p_1, \ldots, p_n$ is defined as

$$
HC_n^* \equiv \max_{1 \leq i \leq n} \sqrt{n} \frac{i/n - p(i)}{\sqrt{p(i)(1 - p(i))}}.
$$
where \( p_{(i)} \) is the \( i \)-th order statistic of \( p_1, \ldots, p_n \), and \( 0 < \gamma_0 < 1 \) is a tunable parameter that has no effect on the asymptotic value of \( \text{HC}_n^* \) \([19]\). The HC test rejects \( H_0^{(n)} \) for large values of \( \text{HC}_n^* \).

**Theorem 1.2.** Consider the hypothesis testing problem (11) under the RMD conditions (8) and (12). If \( r > \rho(\beta; \sigma) \), then \( \text{HC}_n^* \) is asymptotically powerful.

### 1.4.2. Berk-Jones Test

Define the P-values

\[
\pi_i = \Pr(\text{Beta}(i, n - i + 1) < p_{(i)}), \quad i = 1, \ldots, n,
\]

where \( \text{Beta}(a, b) \) is the Beta distribution with shape parameters \( a, b > 0 \). The Berk-Jones (BJ) test statistic is defined as \([10]\)

\[
M_n \equiv \min \{ M_n^-, M_n^+ \}, \quad M_n^- \equiv \min_i \pi_i, \quad M_n^+ \equiv \min_i (1 - \pi_i).
\]

**Theorem 1.3.** Consider the hypothesis testing problems (3) under the RMD condition (8). If \( r > \rho(\beta; \sigma) \), then \( M_n \) is asymptotically powerful.

As opposed to other results in this paper, Theorem 1.3 is limited to the setting of (3). The challenge in extending this theorem to the setting of (11) is in characterizing the distribution of \( M_n \) under \( H_0^{(n)} \) of (11) in which the null behavior of each \( p_i \) is only approximately uniform. Indeed, when the \( p_i \)-s are uniform, such characterization relies on properties of the empirical uniform process \([30, 46]\).

### 1.5. Sub-optimal Tests

#### 1.5.1. Bonferroni and false-discovery rate controlling

Bonferroni and false-discovery rate (FDR) controlling methods are two popular approaches to inference based on multiple testing \([24]\). Starting with the P-values \( p_1, \ldots, p_n \), Bonferroni type inference uses the minimal P-value \( p_{(1)} \) as the test statistics. One rule for FDR controlling selection with control parameter \( q \) uses the minimal \( k^* \) P-values such that \( k^* \) is the largest integer \( k \) satisfying \( p_{(k)} \leq qk/n \), hence a test based on \( p_{(1)}, \ldots, p_{(k^*)} \) at the level \( \alpha \) takes the form

\[
\text{Reject } H_0^{(n)} \text{ if and only if } \min_{1 \leq i \leq n} \frac{p_{(i)}}{i/n} \leq h(\alpha, n), \quad (14)
\]

where \( h(\alpha, n) < 1 \) is a critical value designed to reject \( H_0^{(n)} \) with probability at most \( \alpha \) under \( H_0^{(n)} \).
For an RMD model, both procedures turn out to be asymptotically powerful (respectively, powerless) within the exact same region. The phase transition curve distinguishing powerfulness from powerlessness is given by

$$\rho_{\text{Bonf}}(\beta; \sigma) \equiv \begin{cases} 
(1 - \sigma \sqrt{1 - \beta})^2, & 1/2 < \beta < 1, \quad \sigma^2 < 2, \\
(1 - \sigma \sqrt{1 - \beta})^2, & 1 - \frac{1}{\sigma^2} \leq \beta < 1, \quad \sigma^2 > 2, \\
0, & \beta < 1 - \frac{1}{\sigma^2}, \quad \sigma^2 > 2.
\end{cases}$$

(15)

**Theorem 1.4.** Consider the hypothesis testing problem (11) under the RMD conditions (8) and (12). $T_n^{\text{Bonf}} = -\log(p_{(i)})$ is asymptotically powerless whenever $r < \rho_{\text{Bonf}}(\beta; \sigma)$ and asymptotically powerful whenever $r > \rho_{\text{Bonf}}(\beta; \sigma)$.

**Theorem 1.5.** Consider the hypothesis testing problem (11) under the RMD conditions (8) and (12). A test based on (14) is asymptotically powerless whenever $r < \rho_{\text{Bonf}}(\beta; \sigma)$ and asymptotically powerful whenever $r > \rho_{\text{Bonf}}(\beta; \sigma)$.

Theorems 1.4 and 1.5 imply that both Bonferroni and FDR type inference are asymptotically optimal for $\sigma < 2$ only when $\beta < 1/2$ or $(4 - \sigma^2)/4 < \beta$. This situation is similar to the case of the Gaussian means model studied in [19], implying that under weak heteroscedasticity and moderate rarity the evidence for discriminating $H_0(\mathbf{n})$ from $H_1(\mathbf{n})$ are not amongst sets of the form \{ $p_i : p_i < qk/n$, $q \in (0, 1)$, $k = 1, \ldots, n$ \}. Asymptotically, in this case, optimal discrimination is achieved by considering $P$-values in the much wider range \{ $p_i : p_i < n^{-(1-\delta)}$ \} for some $\delta > 0$. This range is considered by HC and BJ, but not by FDR or Bonferroni.

### 1.5.2. Fisher’s Combination Test

We conclude this section by noting that Fisher’s combination test (2) is asymptotically powerless for all $\beta > 1/2$, provided we strengthen condition (8) as follows:

$$\lim_{n \to \infty} \max_{i = 1, \ldots, n} \frac{Q_i^{(n)} \geq 2q \log(n)}{\log(n)} = q, \quad q > 0,$$

(16)

**Theorem 1.6.** Consider the hypothesis testing problem (11) under (16). A test based on $F_n$ of (2) is asymptotically powerless whenever $\beta > 1/2$.

Condition (16) requires the asymptotic approximation of $\Pr \left[ Q_i^{(n)} \geq 2q \log(n) \right]$ to hold over the entire moderate deviation scale, i.e., not only in the range $q \in (0, 1 + \gamma)$ as in (8). This extension is necessary to control the contribution to $F_n$ by very small $p_i$-s, as their number can be significant when $r > \rho_{\text{Bonf}}(\beta; \sigma)$.
1.6. Structure of this paper

In Section 2 we explore several rare/weak signal detection problems that conform to the RMD model formulation. Therefore, the asymptotic characterizations of global testing in these models provided by Theorems 1.1-1.6 apply. In Section 3 we compare between the underlying log-chisquared approximation of P-values and the classical log-normal approximation. A detailed discussion of our results and properties of RMD models is provided in Section 4. All proofs are in Section 5.

2. Examples of Rare/Weak Moderate Departures Models

We consider below various examples of rare/weak multiple testing settings that are carried under our RMD formulation. In most cases, these settings were previously studied, however, without the RMD formulation and without establishing all RMD model properties in Theorems 1.1-1.6. We note these earlier studies at the end of each example.

2.1. Normal Means

Consider the hypothesis testing problem

\[ H_0 : X_i \sim \mathcal{N}(0, 1), \quad \forall i = 1, \ldots, n \]
\[ H_1 : X_i \sim (1 - \epsilon)\mathcal{N}(0, 1) + \epsilon\mathcal{N}(\mu, \sigma^2), \quad \forall i = 1, \ldots, n. \]  

The relation between the model (17) to (3) is via the z-tests

\[ p_i = \Phi(X_i), \quad \Phi(x) \equiv \Pr(\mathcal{N}(0, 1) > x), \quad i = 1, \ldots, n. \]  

Standard facts about Mills’ ratio (see, e.g., [31]) imply

\[ -2 \log(\Phi(x)) \sim -2 \log \left( \frac{\phi(x)}{|x|} \right) = x^2(1 + o(1)), \]  

as \( x \to \infty \). Consequently, under \( H_1^{(n)} \), the distribution of \(-2 \log(p_i)\) is of the form

\[ (1 - \epsilon)\text{Exp}(2) + \epsilon Q_i^{(n)}, \]

where \( Q_i^{(n)} \) is a probability distribution obeying

\[ Q_i^{(n)} \overset{D}{=} (\sigma Z + \mu)^2(1 + o_p(1)), \quad Z \sim \mathcal{N}(0, 1), \]

as \( \mu \to \infty \). For \( \mu = \mu_n(r) \), the last evaluation implies that \( Q_i^{(n)} \) satisfies (8). Since each \( Q_i^{(n)} \) also has a density, the P-values of (18) correspond to the strong RMD model formulation with log-chisquared parameters \((r, \sigma)\).

Previous studies of the setting (17) were conducted by Cai, Jeng, and Jin [13], which derived the optimal phase transition curve \( \rho(\beta; \sigma) \) and showed that it is attained by HC of the P-values (18). The homeostatic case \( \sigma^2 = 1 \) was initially studied by Ingster [33], Jin [36], and Donoho and Jin [19].
2.2. Two-Sample Normal Means

A two-sample version of (17) takes the form:

\[
H_0 : X_i, Y_i \sim N(\nu_i, 1), \quad i = 1, \ldots, n,
\]

\[
H_1^{(n)} : \begin{cases} X_i \sim N(\nu_i, 1), \\ Y_i \sim (1 - \epsilon)N(\nu_i, 1) + \epsilon N(\nu_i + \mu, \sigma^2) \end{cases}, \quad i = 1, \ldots, n, \tag{21}
\]

where \(\nu_1, \ldots, \nu_n\) is a sequence of unknown means. For this setting, consider the P-values

\[
p_i = 2\Phi\left(\frac{|Y_i - X_i|}{\sqrt{2}}\right). \tag{22}
\]

Notice that, with \(\tilde{Y}_i \sim N(\nu_i + \mu, \sigma^2)\) and \(X_i \sim N(\nu_i, 1)\), Mills’ ratio (19) implies

\[
-2 \log \left(2\Phi\left(\frac{\tilde{Y}_i - X_i}{\sqrt{2}}\right)\right) \overset{D}{=} \left(\sqrt{\frac{1 + \sigma^2}{2}}Z + \frac{\mu}{\sqrt{2}}\right)^2 (1 + o_p(1)), \quad Z \sim N(0, 1),
\]

as \(\mu \to \infty\). Therefore, under \(H_1^{(n)}\) we have that the distribution of \(-2 \log(p_i)\) is of the form

\[
(1 - \epsilon)\text{Exp}(2) + \epsilon Q_i^{(n)}, \tag{23}
\]

where \(Q_i^{(n)}\) is a probability distribution obeying

\[
Q_i^{(n)} \overset{D}{=} \left(\sqrt{\frac{1 + \sigma^2}{2}}Z + \frac{\mu}{\sqrt{2}}\right)^2 (1 + o_p(1)), \quad Z \sim N(0, 1),
\]

as \(\mu \to \infty\). It follows that with \(\mu\) calibrated to \(n\) as in (5), \(Q_i^{(n)}\) satisfies (8) with mean parameter \(\mu_n(r) = \mu_n(r)/\sqrt{2} = \sqrt{r \log(n)}\) and scaling parameter \(\sigma' = \sqrt{(1 + \sigma^2)/2}\), hence the P-values (22) corresponds to the strong RMD model formulation with log-chisquared parameters \((r/2, \sigma')\).

In order to derive a phase transition curve for this model, we start from (13), adjusting for the factor 2 scaling in the non-centrality parameter compared to (5), and substituting \(\sqrt{(1 + \sigma^2)/2}\) for the standard deviation. We obtain:

\[
\rho_{\text{two-sample}}(\beta; \sigma) \equiv \begin{cases} (3 - \sigma^2)(\beta - 1/2) & \frac{1}{2} < \beta < \frac{7 - \sigma^2}{8}, \quad 0 < \sigma^2 < 3, \\ 2 \left(1 - \frac{1 + \sigma^2}{2} \sqrt{1 - \beta}\right)^2 & \frac{7 - \sigma^2}{8} \leq \beta < 1, \quad 0 < \sigma^2 < 3, \\ 0 & \frac{1}{2} < \beta < \frac{\sigma^2 - 1}{\sigma^2 + 1}, \quad \sigma^2 \geq 3, \\ 2 \left(1 - \frac{1 + \sigma^2}{2} \sqrt{1 - \beta}\right)^2 & \frac{\sigma^2 - 1}{\sigma^2 + 1} \leq \beta < 1, \quad \sigma^2 \geq 3. \end{cases} \tag{24}
\]

Figure 2 depicts \(\rho_{\text{two-sample}}(\beta; \sigma)\) for several values of \(\sigma\).
Fig 2. Two-Sample Phase Diagram. The phase transition curve $\rho_{\text{two-sample}}(\beta; \sigma)$ of (24) defines the detection boundary for an asymptotically log chi-squared perturbation model (3). For $r < \rho_{\text{two-sample}}(\beta; \sigma)$, all tests are powerless. For $r > \rho_{\text{two-sample}}(\beta; \sigma)$, the Higher Criticism and the Berk-Jones tests are asymptotically powerful. The faint lines correspond to $2\rho(\beta; \sigma)$, where we have $\rho_{\text{two-sample}}(\beta; 1) = 2\rho(\beta; 1)$.

2.3. Poisson Means

Consider the hypothesis testing problem

\begin{align}
H_0 : & \quad X_i \overset{iid}{\sim} \text{Pois}(\lambda_i), \quad i = 1, \ldots, n, \\
H_1 : & \quad X_i \overset{iid}{\sim} (1 - \epsilon)\text{Pois}(\lambda_i) + \epsilon\text{Pois}(\lambda_i'), \quad i = 1, \ldots, n,
\end{align}

where $\lambda_1, \ldots, \lambda_n$ is a sequence of known means and where the $\{\lambda_i'\}$s are obtained by perturbing $\lambda_i$ upwards. For this model, we have the P-values

\begin{equation}
 p_i = \bar{P}(X_i; \lambda_i), \quad i = 1, \ldots, n,
\end{equation}

where $\bar{P}(x; \lambda_i) \equiv \Pr[\text{Pois}(\lambda_i) \geq x]$. We suppose that the Poisson means increase with $n$ such that

\begin{equation}
 (\min \lambda_i) / \log(n) \to \infty,
\end{equation}

and the perturbed means are given by

\begin{equation}
 \lambda_i' = \lambda_i + \mu_n(r) \sqrt{\lambda_i}, \quad i = 1, \ldots, n.
\end{equation}

Noting that $\log(n)/\lambda_i' \to 0$ and $\lambda_i' - \lambda_i \to \infty$, the behavior of $p_i$ under $H_1^{(n)}$ is obtained using a moderate deviation estimate of the RVs $\Upsilon_{\lambda_i'} \sim \text{Pois}(\lambda_i')$. This estimate leads to

\begin{equation}
 -2 \log \left( \bar{P}(\Upsilon_{\lambda_i'}; \lambda_i) \right) \overset{D}{=} (Z + \mu_n(r))^2 (1 + o_p(1)), \quad Z \sim \mathcal{N}(0, 1),
\end{equation}
which implies (8) with $\sigma = 1$ and $\mu = \mu_n$. The same moderate deviation estimate implies that the distribution of $-2 \log(p_i)$ under $H_0$ obeys (12), hence the Poisson means model of (25) corresponds to the RMD model formulation of (11) with log-chisquared parameters $(r, 1)$. Alternately, we may consider a randomized version of the P-values $p_1, \ldots, p_n$ of (26) as in

$$
\hat{p}_i = p_i^+ + (p_i - p_i^-) \cdot U_i, \quad U_i \overset{iid}{\sim} \text{Unif}(0, 1), \quad i = 1, \ldots, n,
$$

(30)

where $p_i^- \equiv \bar{P}(x_i + 1; \lambda_i)$. We have that $\hat{p}_1, \ldots, \hat{p}_n$ are uniformly distributed over $(0, 1)$ under $H_0$ while under $H_1$ their non-null mixture components satisfy (29). Consequently, the Poisson means model (25) with the randomized P-values (30) corresponds to the RMD model formulation of (3) with log-chisquared parameters $(r, 1)$.

Arias-Castro and Wang studied the Poisson Means model (25) in [5]. They derived the optimal phase transition $\rho(\beta; 1)$, the Bonferroni phase transition $\rho_{\text{Bonf}}(\beta; 1)$, and showed that a version of HC is asymptotically powerful whenever $r > \rho(\beta; 1)$.

### 2.4. Two-Sample Poisson Means

A two-sample version of (25) is given as:

$$
\begin{align*}
H_0^{(n)} : \quad X_i, Y_i &\overset{iid}{\sim} \text{Pois}(\lambda_i), \quad i = 1, \ldots, n. \\
h_1^{(n)} : \quad \begin{cases} &X_i \overset{iid}{\sim} \text{Pois}(\lambda_i) \\
&Y_i \overset{iid}{\sim} (1 - \epsilon)\text{Pois}(\lambda_i) + \epsilon\text{Pois}(\lambda'_i), \quad i = 1, \ldots, n.
\end{cases}
\end{align*}
$$

(31)

Here $\lambda_1, \ldots, \lambda_n$ is a sequence of unknown Poisson means that satisfy (27), while $\lambda'_1, \ldots, \lambda'_n$ are defined as in (28). We summarize the significance of the pair $(X_i, Y_i)$ associated with the $i$-th feature by the RV:

$$
p_i \equiv 2\Phi \left( \left| \sqrt{2Y_i} - \sqrt{2X_i} \right| \right),
$$

(32)

for $i = 1, \ldots, n$.

In order to analyze the behavior of $p_1, \ldots, p_n$ under $H_0^{(n)}$ and $h_1^{(n)}$, note that the transformed Poisson RV $\sqrt{X}, X \sim \text{Pois}(\lambda)$, is variance stable:

$$
2\sqrt{X} - 2\sqrt{X} \rightarrow \mathcal{N}(0, 1).
$$

Under $h_1^{(n)}$, (27) and (28) imply

$$
\sqrt{\lambda_i'}(1 + o(1)) = \sqrt{\lambda_i} \pm \mu_n(r)/2,
$$

(33)

where $o(1) \rightarrow 0$ as $n \rightarrow \infty$ uniformly in $i$. Consequently, with $\Upsilon_{\lambda_i'} \sim \text{Pois}(\lambda'_i)$,

$$
\sqrt{2\Upsilon_{\lambda_i'}} - \sqrt{2\Upsilon_i} = \sqrt{2\Upsilon_{\lambda_i'}} - \sqrt{2\Upsilon_i} - \left( \sqrt{2\Upsilon_i} - \sqrt{2\lambda_i} \right) + \left( \sqrt{2\lambda_i'} - \sqrt{2\lambda_i} \right) \overset{D}{=} \left( Z + \mu_n(r)/\sqrt{2} \right)(1 + o_p(1)),
$$

(33)

$$
Z \sim \mathcal{N}(0, 1),
$$

(33)
as \( n \to \infty \). By setting
\[
\pi_i \equiv 2\Phi \left( \sqrt{2} \lambda_i' - \sqrt{2} X_i \right),
\]
combining Mill’s ratio (19) and (33), we obtain
\[
-2 \log(\pi_i) \overset{D}{=} \left( Z + \mu_n(r)/\sqrt{2} \right)^2 (1 + o_p(1)).
\] (34)
The last evaluation shows that (9) holds with log-chisquared parameters \((r/2, 1)\). Because \( p_1, \ldots, p_n \) are only asymptotic P-values, in order to conclude that they conform to a RMD model with these log-chisquared parameters we must first verify that the distribution \( E_i^{(n)} \) of \( p_i \) obeys (12) under \( H_0^{(n)} \). This can be achieved using several approaches, one of which is by using approximations of the Poisson distribution involving high-order terms as in [38, Ch. 4.5]. Another approach is to show that (12) holds for the asymptotic P-values obtained from the standardized Poisson data
\[
p_i' \equiv 2\Phi \left( \frac{Y_i - X_i}{{\sqrt{2}} \lambda_i} \right),
\]
using the Berry-Esseen theorem as \( \lambda_i \to \infty \), then argue that the distribution of the variance-stabilized data is closer to the normal distribution than the standardized data [12]. Finally, one can also verify the convergence in (12) using Monte Carlo simulations as we demonstrate in Figure 3.

For the model (31), Donoho and Kipnis [22] proposed to use P-values of exact binomial testing as in
\[
p_i' \equiv \Pr \left( \left| \text{Bin}(X_i + Y_i, 1/2) - \frac{X_i + Y_i}{2} \right| \leq \frac{|X_i - Y_i|}{2} \right),
\] (35)
which have several advantages over (32) in practice. Our RMD formulation shows that the optimal phase transition curves of both collections of P-values under (21) are identical and given by \( \rho_{\text{two-sample}}(\beta; 1) \).

2.5. Two-sample t-Testing

Consider two populations and a set of \( n \) independent features. Denote by \( \nu_{*,i} \) the unknown mean of the \( i \)-th feature in population \( * \), where \( * \in \{x, y\} \) and \( i \in \{1, \ldots, n\} \). Suppose that we have \( n_{*,i} \) independent samples of feature \( i \) from population \( * \), and that we are interesting in testing the global null
\[
H_0 : \nu_{x,i} = \nu_{y,i} \quad \forall i = 1, \ldots, n.
\] (36)
This scenario arises in several high-dimensional learning problems where the goal is to test whether two classes are distinguishable under certain transformations [32]; see also the recent works [44, 41].
Fig 3. Average of \(\max_{q \in [0,0.05]} |\log(n^{-1}\# \{i, -2 \log(p_i) > 2q \log(n)\})/\log(n) - q|\) over 1,000 Monte Carlo simulations for different values of \(n\), where \(p_1, \ldots, p_n\) are the asymptotic P-values obtained from the variance-stabilized Poisson RVs in (32) with \(\lambda = n^{1/3}\). The convergence of this average to 0 implies that the distribution of \(p_1, \ldots, p_n\) obeys (12). The dashed lines indicate 2 standard error intervals.

Given the data \(\{X_{i,j}, Y_{i,k} : 1 \leq j \leq n_{x,i}, 1 \leq k \leq n_{y,i}\}_{1 \leq i \leq n}\), we summarize the evidence provided by the \(i\)-th feature using a \(t\)-test against \(H_{0,i} : \nu_{x,i} = \nu_{y,i}\). Specifically, with

\[
\bar{X}_i = \frac{1}{n_{x,i}} \sum_{j=1}^{n_{x,i}} X_{i,j}, \quad \bar{Y}_i = \frac{1}{n_{y,i}} \sum_{k=1}^{n_{y,i}} Y_{i,k},
\]

and

\[
\begin{align*}
    s^2_{x,i} &\equiv \frac{1}{n_{x,i} - 1} \sum_{j=1}^{n_{x,i}} (X_{i,j} - \bar{X}_i)^2, \\
    s^2_{y,i} &\equiv \frac{1}{n_{y,i} - 1} \sum_{k=1}^{n_{y,i}} (Y_{i,k} - \bar{Y}_i)^2,
\end{align*}
\]

set

\[
T_i = \frac{\bar{X}_i - \bar{Y}_i}{\sqrt{s^2_{x,i} + s^2_{y,i}/n_{y,i}}}.
\]

Denote by \(T_\kappa\) the \(t\)-distribution with \(\kappa\) degrees of freedom. Under standard conditions ensuring asymptotic normality of \(X_i, Y_i, s^2_{x,i}, \) and \(s^2_{y,i}\),

\[
p_i \equiv \Pr\left(|T_i| \geq |T_{n_{x,i} + n_{y,i} - 2}|\right), \quad i = 1, \ldots, n,
\]

are asymptotic P-values for (36). Similarly to the two-sample normal means model (21), we consider an alternative in which roughly \(n\epsilon_n\) of the features’ mean obey \(|\nu_{x,i} - \nu_{y,i}| = \mu_n\) while \(\nu_{x,i} = \nu_{y,i}\) for the remaining ones. This situation can be written as

\[
H_1^{(n)} : |\nu_{x,i} - \nu_{y,i}| = \begin{cases} 
    \mu_n & i \in I \\
    0 & i \notin I
\end{cases}, \quad 1_{i \in I} \sim \text{Bernoulli}(\epsilon_n), \quad i = 1, \ldots, n.
\]
The following proposition provides conditions under which the P-values of (39) conform to a RMD model with log-chisquared parameters (r, 1).

**Proposition 2.1.** For $i = 1, \ldots, n$, suppose that:

(i) $E [|X_{ij}|^3] < \infty$ and $E [|Y_{ij}|^3] < \infty$.
(ii) As $n \to \infty$, $n_{x,i} + n_{y,i} \to \infty$ while

$$\min_{i=1,\ldots,n} \frac{n_{x,i} + n_{y,i}}{(\log(n))^3} \to \infty.$$  

(iii) There exists $c_{x,i}, c_{y,i} > 0$ such that $c_{x,i} \leq n_{x,i} / n_{y,i} \leq c_{y,i}$.

Suppose that $\mu = \mu_n(r)$. Then $p_1, \ldots, p_n$ are of the form (11) with $E_i^{(n)}$ and $Q_i^{(n)}$ probability distributions obeying (12) and (8), respectively.

We note that Delaigle, Hall, and Jin considered a high-dimensional version of the hypothesis testing problem defined by (36) and (40) in [17], for which they also derived a region where HC is asymptotically powerful.

3. Log-Chisquared versus Log-Normal

In this section, we compare our log-chisquared approximation for P-values under the alternative hypothesis to the classical log-normal approximation of Bahadur [7, 28] and Lambert and Hall [40].

3.1. The Log-normal Approximation

Informally, suppose that the alternative hypothesis is characterized by a parameter $\theta$, and that $a_n$ is a sequence tending to infinity with $n$ describing the ‘cost’ of attaining new data. Bahadur’s log-normal approximation says that, under some conditions, a P-value $\pi$ under the alternative $H_1(\theta, a_n)$ obeys

$$\frac{\log(\pi) + a_n c(\theta)}{\sqrt{a_n}} \overset{D}{\to} N(0, \tau^2(\theta)),$$

as $n \to \infty$. It is convenient to write (41) as

$$\log(\pi) \overset{D}{\to} N(a_n c(\theta), a_n \tau^2(\theta)).$$

In the terminology of [40], $c(\theta)$ is Bahadur’s half-slope describing the asymptotic behavior of the test’s size, i.e., the rate at which $\pi$ goes to zero. Note that the asymptotic behavior of the test’s power is determined by both $\tau(\theta)$ and $c(\theta)$.

3.2. Formal Comparison

It is well-recognized that (42) is a *large deviation* estimate of the test statistic in the following sense: if this statistic satisfies a large deviation principle, then
$c(\theta)$ is a transformation of its rate function [53, 29, 54]. In contrast, in all RMD models of Section 2 the alternative hypotheses correspond to situations where the deviation of each test statistic from its null is moderate in the sense of [51]. Consequently, the log-normal approximation of (42) is too rough to correctly indicate the asymptotic power of tests under the RMD formulation.

To formally illustrate this last point, consider the homoscedastic sparse normal means model (the model (17) with $\sigma = 1$):

$$H_0^{(n)} : X_i \overset{iid}{\sim} \mathcal{N}(0, 1),$$
$$H_1^{(n)} : X_i \overset{iid}{\sim} (1 - \epsilon)\mathcal{N}(0, 1) + \epsilon\mathcal{N}(\mu, 1), (43)$$

with the $P$-values $p_i = \Phi(X_i)$. Under $H_1^{(n)}$, the distribution of $p_i$ is of the form

$$-2\log p_i \sim (1 - \epsilon)\text{Exp}(2) + \epsilon Q_i^{(n)}, (44)$$

where the probability distribution $Q_i^{(n)}$ is the subject of our approximation. Suppose that $\mu$ and $\epsilon$ are calibrated to $n$ as in (5) and (4), respectively. In this paper we propose the log-chisquared approximation

$$Q_i^{(n)} \overset{D}{\approx} (\mu_n(r) + Z)^2. (45)$$

On the other hand, we have

$$(\mu + Z)^2 = (\mu^2 + 2\mu Z)(1 + o_p(1)) = (\mu^2 + 2\mu Z)(1 + o_p(1)), \quad \mu \to \infty,$$

implying the log-normal approximation:

$$Q_i^{(n)} \overset{D}{\approx} \mathcal{N}(\mu_n^2(r), 4\mu_n^2(r)) = \mathcal{N}(2r \log(n), 8r \log(n)). (46)$$

In particular, in the notation of (42) we have $\theta = r, a_n = \log(n), c(\theta) = 2r$, and $\tau^2(\theta) = 8r$. Next, observe that the theoretical engine underlying the success of HC and the BJ tests is the behavior of

$$\Pr(\pi_i < n^{-q}, -2\log(p_i) \sim Q_i^{(n)},$$

as $n \to \infty$. With $Q_i^{(n)}$ as in (45),

$$\Pr(\pi_i < n^{-q}) = \Pr(-2\log(p_i) > 2q \log(n))$$
$$\sim \Pr \left( (\mu_n(r) + Z)^2 \geq 2q \log(n) \right)$$
$$\sim \Pr \left( Z \geq \sqrt{\log(n)} \left( \sqrt{2q} - \sqrt{2r} \right) \right). (47)$$

A standard evaluation of the behavior of HC under $H_1^{(n)}$ uses (47) to show that it is asymptotically powerful for $r > \rho(\beta; 1)$ [14, 22]. On the other hand, with $Q_i^{(n)}$ as in (46),

$$\Pr(\pi_i < n^{-q}) = \Pr(-2\log(p_i) > 2q \log(n))$$
$$\sim \Pr \left( \mu_n(r)^2 + 2\mu_n(r)Z \geq 2q \log(n) \right)$$
$$= \Pr \left( Z \geq \sqrt{\log(n)} \frac{q - r}{\sqrt{2r}} \right). (48)$$
Since \[
\frac{q - r}{\sqrt{2r}} \geq \sqrt{2q} - \sqrt{2r}, \quad q \geq r > 0,
\]
using the log-normal approximation in a formal exercise of would-be power analysis of HC by replacing (47) with (48), incorrectly predicts that HC is powerful for some \( r < \rho(\beta; 1) \).

### 3.3. Empirical Comparison

The roughness of Bahadur’s log-normal approximation under moderate departures compare to the log-chisquared approximation can also be seen in finite samples using Monte Carlo simulations. In each simulation, we sample data \( x_1, \ldots, x_n \) independently from \( N(\mu_n(r), 1) \) and consider \( \pi_i = \Phi(x_i) \) as a P-values for \( H_0 : X_i \overset{iid}{\sim} N(0, 1), i = 1, \ldots, n \). Figure 4 illustrates the results of one simulation with \( n = 1,000 \) (top panels) and one simulation with \( n = 100,000 \) (bottom panels), while the departure intensity parameter \( r = 1 \) is fixed in both cases. The panels on the left show the histogram of \( \{-2 \log(p_i)\}_{i=1}^{n} \) with the density of the normal distribution \( N(\hat{\mu}, \hat{\sigma}^2) \) and the density of the noncentral chisquared distribution \( \chi^2(\hat{\lambda}) \), where \( \hat{\mu} \) and \( \hat{\sigma}^2 \) are the standard mean and variance estimates and \( \hat{\lambda} = \hat{\mu} - 2 \) is the non-centrality estimate. The middle and right panels illustrate QQ-plots of the empirical distribution of \( \{-2 \log(p_i)\}_{i=1}^{n} \) against \( \chi^2(\hat{\lambda}) \) and \( N(\hat{\mu}, \hat{\sigma}^2) \), respectively, showing the better fit of the empirical distribution of \( \{-2 \log(p_i)\}_{i=1}^{n} \) to \( \chi^2(\hat{\lambda}) \). Figure 5 illustrates the results of 1,000 Monte Carlo simulations with many configurations of \( n \) and \( r \). For each configuration, we conducted a goodness-of-fit test using the Kolmogorov-Smirnov (KS) statistic \(^1\) of \( \{-2 \log(p_i)\}_{i=1}^{n} \) against \( \chi^2(\hat{\lambda}) \) and \( N(\hat{\mu}, \hat{\sigma}^2) \). The test against the normal (respectively, chisquared) rejects when the KS statistic exceeds its simulated .95-th quantile under the null obtained by sampling 10,000 from the normal (chisquared) distribution.

### 4. Discussion

#### 4.1. Heteroscedasticity in RMD models

The phase transition described by \( \rho(\beta; \sigma) \) can be seen as the result of two distinct phenomena: (i) location shift controlled by \( r \), and (ii) heteroscedasticity controlled by \( \sigma^2 \). Roughly speaking, increasing the effect of either (i) or (ii) eases detection and reduces the phase transition curve, as seen in Figure 1. We refer to [13] for a comprehensive discussion on the effect of (ii) on the phase transition curve. In view of our results, the aforementioned discussion from [13] is relevant to any RMD model – not only to the rare/weak normal means model of [13] (see Section 2.1 above).

\(^1\)For the normal model this is know as the Lilliefors test [43].
Fig 4. Comparing log-normal and log-chisquared approximations to moderately perturbed P-values $\pi_1, \ldots, \pi_n$. Here $\pi_i \sim \Phi(X_i)$, $X_i \sim \mathcal{N}(\sqrt{2r \log(n)}, 1)$, with $n = 10^3$ (top) and $n = 10^5$ (bottom). Left: histogram of $\{-2 \log(\pi_i)\}_{i=1}^n$. Middle: QQ-plots of the empirical distribution of $\{-2 \log(\pi_i)\}_{i=1}^n$ against the noncentral chi-squared distribution. Right: QQ-plots of the empirical distribution of $\{-2 \log(\pi_i)\}_{i=1}^n$ against the normal distribution.

Fig 5. Rejection rate in Kolmogorov-Smirnov (KS) Goodness of fit testing at significance level .05, assessing the fit of the empirical distribution of moderately perturbed P-values to the normal and noncentral chi-squared distributions, respectively. The rejection rate is the fraction of cases out of 1,000 in which the KS statistics exceeds its .95-th quantile under the null. Consequently, the smaller the rejection rate, the better the fit. Left: rejection rate versus perturbation intensity parameter $r$; $n = 1,000$ is fixed. Right: rejection rate versus sample size $n$; $r = 2$ is fixed.
Comparing the effect of heteroscedasticity in one- versus two- sample settings, we see that
\[ \rho_{\text{two-sample}}(\beta; 1) = 2\rho(\beta; 1), \]
an observation first made in [22]. Interestingly, as shown in Figure 2, this relation between \( \rho_{\text{two-sample}}(\beta; \sigma) \) and \( \rho(\beta; \sigma) \) does not hold when \( \sigma \neq 1 \). Specifically, detection in the two-sample homeostatic setting (\( \sigma = 1 \)) asymptotically requires twice the effect size. On the other hand, compared to the one-sample case, more than twice the effect size is needed for overdispersed mixtures (\( \sigma > 1 \)) and less for underdispersed ones (\( \sigma < 1 \)).

4.2. More General Conditions for Indetectability

The proof of Theorem 1.1 in Section 5.2 reveals that powerlessness of all tests for (3) is attained provided:

\[ \max_{i=1,\ldots,n} \mathbb{E}_{X \sim \text{Exp}(2)} \left[ L_i^{(n)}(X) \mathbb{1}_{\{X \leq 2q \log(n)\}} \right] \geq 1 - n^{-\alpha(q; r, \sigma) + o(1)}, \tag{49a} \]

and

\[ \max_{i=1,\ldots,n} \mathbb{E}_{X \sim Q_i^{(n)}} \left[ L_i^{(n)}(X) \mathbb{1}_{\{X \leq 2q \log(n)\}} \right] \leq n^{\max_{y \in [0, q]} \left\{ -2\alpha(q; r, \sigma) + y \right\} + o(1)}, \tag{49b} \]

for all \( q \in (0, 1 + \gamma] \) for some \( \gamma > 0 \), where

\[ L_i^{(n)}(x) \equiv \frac{dQ_i^{(n)}}{d\text{Exp}(2)}(x) \tag{50} \]
is the likelihood ratio between the mixture components. Obvious extensions of (49) exist for (11). Conditions (49) provide some intuition for the difficulty in testing \( H_0 \) versus \( H_1^{(n)} \) of (3) resulting in indetectability for \( r < \rho(\beta; \sigma) \). Indeed, (49a) guarantees that deviations on the moderate scale are not larger in expectation than deviations of non-central chisquared from exponential, while (49b) ensures that for such deviations the likelihood ratio remains relatively small under the alternative. The net effect of these two constraints is a limited success for the likelihood ratio test, especially whenever

\[ 1 - \alpha(1; r, \sigma) < \beta \quad \text{and} \quad \max_{q \in [0, 1]} \left\{ 1 + q \frac{1}{2} - \alpha(q; r, \sigma) \right\} < \beta, \]

which occur when \( r < \rho(\beta; \sigma) \).

For the HC test, Donoho and Kipnis [23] considered the condition

\[ \max_{i} \Pr_{X_i \sim Q_i^{(n)}}[X_i > 2q \log(n)] = n^{-\alpha'(q; r) + o(1)}, \]
for some continuous, non-negative bivariate function $\alpha'(q;r)$ that is increasing in $q$ and decreasing in $r$. They showed that HC of $p_1, \ldots, p_n$ is powerless in the region

$$\Xi_{\text{HC}} = \left\{ (r, \beta) : \max_{q \in [0,1]} \left( \frac{1+q}{2} - \alpha'(q;r) \right) < \beta \right\}.$$  

This region coincide with \{$(r, \beta) : r < \rho(\beta;\sigma)$\} in the RMD setting for which we have $\alpha'(q;r) = \alpha(q;r,\sigma)$.

### 4.3. Other Generalizations of Rare/Weak Models

Cai and Wu [14] considered general rare/weak detection models characterized by the asymptotic behavior of the likelihood ratio between the mixture components on the moderate deviation scale, similarly to (10). For rare/weak departures from the exponential distribution, our setting generalizes that of [14] by relaxing the likelihood ratio condition (10) to the range $q \in [0, 1 + \gamma]$, and allowing for non-identically distributed coordinates and approximate null distribution.

Arias-Castro and Wang [6] provided another important generalization of rare/weak detection models when the null distribution is symmetric. For this case, they considered non-parametric HC- and Bonferroni- type tests and showed that these tests have interesting optimality properties. We anticipate that our RMD formulation applies to the setting of [6] when the non-symmetric behavior of an individual test statistic under the alternative hypothesis is at the scale of the moderate deviation. We leave the application of our formulation to the setting of [6] as future work.

### 5. Proofs

#### 5.1. Technical Lemmas

**Lemma 5.1.** Suppose that the distribution $E_i^{(n)}$ has a density for every $i = 1, \ldots, n$ and that $\{E_i^{(n)}\}$ satisfy (12). Then

$$\lim_{n \to \infty} \max_{i=1,\ldots,n} \frac{\log \left( \frac{dE_i^{(n)}}{dx} (2q \log(n)) \right)}{\log(n)} = -q. \quad (51)$$

**Proof.** We have

$$n^{-q+o(1)} = \max_{i=1,\ldots,n} \Pr \left[ E_i^{(n)} \geq 2q \log(n) \right]$$

$$= \max_{i=1,\ldots,n} \int_{2q \log(n)}^{\infty} \frac{dE_i^{(n)}}{dx}(x)dx$$

$$= \max_{i=1,\ldots,n} 2 \log(n) \int_{q}^{\infty} \frac{dE_i^{(n)}}{dx}(2 \log(n)y)dy.$$
Differentiating both sides with respect to $q$ leads to
\[- \log(n) n^{-q+o(1)} = -2 \log(n) \max_{i=1,\ldots,n} \frac{dE_i^{(n)}}{dx}(2 \log(n)q),\]
which implies (51).

We require the following lemma from [14], providing a particular version of Laplace’s principle.

**Lemma 5.2.** [14, Lemma 3] Let $(X,\mathcal{F},\nu)$ be a measure space. Let $F : X \times \mathbb{R}_+ \to \mathbb{R}_+$ be measurable. Assume that
\[
\lim_{M \to \infty} \frac{\log F(x,M)}{M} = f(x)
\]
holds uniformly in $x \in X$ for some measurable $f : X \to \mathbb{R}$. If
\[
\int_X \exp(M_0 f(x)) d\nu(x) < \infty
\]
for some $M_0 > 0$, then
\[
\lim_{M \to \infty} \frac{1}{M} \log \int_X F(x,M)d\nu(x) = \esssup_{x \in X} f(x).
\]

**Lemma 5.3.** Suppose that $\{Q_i^{(n)}\}_{i=1}^n$ satisfy (8), $\{E_i^{(n)}\}_{i=1}^n$ satisfy (12) and each $E_i^{(n)}$ has a density. Set
\[
L_i^{(n)}(x) \equiv \frac{dQ_i^{(n)}}{dE_i^{(n)}}(x).
\]
Assume that
\[
\lim_{n \to \infty} \max_{i=1,\ldots,n} \frac{\log \left( \frac{dQ_i^{(n)}}{dE_i^{(n)}}(2q \log(n)) \right)}{\log(n)} = 0, \quad q \in (0,a),
\]
for some $a > 0$. Then
\[
\lim_{n \to \infty} \max_{i=1,\ldots,n} \frac{\mathbb{E}_{X \sim Q_i^{(n)}} \left[ L_i^{(n)}(X) 1_{X \leq 2q \log(n)} \right]}{\log(n)} = \max_{y \in [0,a]} \left\{ -2\alpha(y;r,\sigma) + y \right\},
\]
for $q \in (0,a)$. 

Proof. We have

\[ E_{X \sim Q(n)} \left[ L_i^{(n)}(X) 1_{\{X \leq 2q \log(n)\}} \right] = E_{X \sim E_i^{(n)}} \left[ (L_i^{(n)})^2(X) 1_{\{X \leq 2q \log(n)\}} \right] \]

\[ = \int_{0}^{2q \log(n)} \left( \frac{dQ(n)}{dE_i^{(n)}}(x) \right)^2 E_i^{(n)}(dx) \]

\[ = 2 \log(n) \int_{0}^{q} \left( \frac{dQ(n)}{dE_i^{(n)}}(2 \log(n) y) \right)^2 E_i^{(n)}(2 \log(n) dy) \]

\[ = \log(n) \int_{0}^{q} \left( \frac{dQ(n)}{dE_i^{(n)}}(2 \log(n) y) \right)^2 e^{-y \log(n)(1+o(1))} dy \]

\[ = \log(n) \int_{0}^{q} n^{-2\alpha(y;r,\sigma)+y+o(1)} \cdot n^{o(1)} \cdot n^{-y} dy = \int_{0}^{q} n^{-2\alpha(y;r,\sigma)+y+o(1)} dy, \]

(57)

where (57) follows from the change of variables \( x = 2y \log(n) \), (58) follows from Lemma 5.1, and (59) follows from (55). We now apply Lemma 5.2 to (58) with \( X = (0, q] \), \( M = \log(n) \), \( F(x; M) = n^{-2\alpha(x; r, \sigma)+x+o(1)} \), \( f(x) = -2\alpha(x; r, \sigma) + x \), and \( \nu \) the Lebesgue measure. We obtain:

\[ \lim_{n \to \infty} \frac{\log \left( E_{X \sim Q(n)} \left[ L_i^{(n)}(X) 1_{\{X > 2q \log(n)\}} \right] \right)}{\log(n)} = \max_{y \in [0, q]} \left\{ -2\alpha(y; r, \sigma) + y \right\}. \]

\[ \square \]

5.2. Proof of Theorem 1.1

We consider the truncated likelihood ratio method of [35, 34] summarized by the following lemma, proof of which is provided in Section 5.3 below.

Lemma 5.4. Consider testing

\[ H_0^{(n)} : (X_1, \ldots, X_n) \sim P_0^{(n)}, \quad i = 1, \ldots, n, \]

versus

\[ H_1^{(n)} : (X_1, \ldots, X_n) \sim P_1^{(n)}, \quad i = 1, \ldots, n, \]

for \( P_1^{(n)} \) that is absolutely continuous with respect to \( P_0^{(n)} \). Denote by \( L_n = \frac{dP_1^{(n)}}{dP_0^{(n)}} \) the likelihood ratio between \( P_1^{(n)} \) and \( P_0^{(n)} \). Suppose that there exists a sequence of sets \( A^{(n)} \subset \mathbb{R}^n \) such that

\[ 1 - E_{H_0^{(n)}} \left[ L_n(X_1, \ldots, X_n) 1_{(X_1, \ldots, X_n) \in A^{(n)}} \right] \leq o(1) \]

(62)
while
\[ \mathbb{E}_{H_0^{(n)}} \left[ L_n^2(X_1, \ldots, X_n) \mathbf{1}_{(X_1, \ldots, X_n) \in A^{(n)}} \right] \leq 1 + o(1). \] (63)

For any sequence of tests \( \psi^{(n)} : \mathbb{R}^n \to \{0, 1\} \),
\[ \liminf_{n \to \infty} \left\{ \mathbb{E}_{H_0^{(n)}} \left[ \psi^{(n)}(X_1, \ldots, X_n) \right] + \mathbb{E}_{H_1^{(n)}} \left[ 1 - \psi^{(n)}(X_1, \ldots, X_n) \right] \right\} \geq 1. \]

To prove Theorem 1.1, first note that for \( r < \rho(\beta; \sigma) \), there exists \( \delta > 0 \) such that
\[ \max_{q \in [0, 1]} \left( \frac{1 + q}{2} - \alpha(q; r, \sigma) \right) + \delta - \beta < 0. \] (64)

In particular,
\[ 1 - \beta - \alpha(1; r, \sigma) \leq -\delta, \] (65)
and, by continuity of \( q \to \alpha(q; r, \sigma) \), there exists \( \eta \in (0, \gamma) \) such that
\[ 1 - 2\beta + \max_{q \in [0, 1+\eta]} \{ q - 2\alpha(q; r, \sigma) \} < -\delta. \] (66)

We now use Lemma 5.4 with
\[ P_0^{(n)} = \prod_{i=1}^{n} E_i^{(n)}, \quad P_1^{(n)} = \prod_{i=1}^{n} \left[ (1 - \epsilon)E_i^{(n)} + \epsilon Q_i^{(n)} \right], \]
where \( \{ E_i^{(n)} \} \) satisfy (12), and
\[ A^{(n)} = \prod_{i=1}^{n} \{ X_i \leq 2(1 + \eta) \log(n) \}, \]
for some fixed \( \delta > 0 \) satisfying (64). We have
\[ \tilde{L}_n = \prod_{i=1}^{n} \tilde{L}_i^{(n)}(X_i) \mathbf{1}_{\{ X_i \leq 2(1 + \eta) \log(n) \}}, \] (67)
where
\[ \tilde{L}_i^{(n)}(x) \equiv (1 - \epsilon_n) + \epsilon_n L_i^{(n)} = 1 + \epsilon_n (L_i^{(n)}(x) - 1), \] (68)
and
\[ L_i^{(n)}(x) \equiv \frac{dQ_i^{(n)}}{dE_i^{(n)}}(x). \]
Henceforth, all expectations are with respect to \( X_i \sim E^{(n)}_i \) unless otherwise specified. For the first moment, since \( \mathbb{E} \left[ L_i^{(n)}(X_i) \right] = 1 \), we have

\[
\mathbb{E} \left[ \bar{L}_n \right] = \prod_{i=1}^{n} (1 - a_{n,i}),
\]

where,

\[
a_{n,i} \equiv \mathbb{E} \left[ \bar{L}_i^{(n)}(X_i) \mathbf{1}_{\{X_i > 2(1+\eta) \log(n)\}} \right].
\]

Consider

\[
a_{n,i} = \Pr_{X_i \sim E^{(n)}_i} \left[ X_i \geq 2(1+\eta) \log(n) \right] + \epsilon_n \mathbb{E} \left[ \left( L_i^{(n)}(X_i) - 1 \right) \mathbf{1}_{\{X_i > 2 \log(n)\}} \right]
\]

\[
\leq \Pr_{X_i \sim E^{(n)}_i} \left[ X_i \geq 2(1+\eta) \log(n) \right] + \epsilon_n \mathbb{E} \left[ L_i^{(n)}(X_i) \mathbf{1}_{\{X_i > 2 \log(n)\}} \right]
\]

\[
= n^{-(1+\eta)+o(1)} + n^{-\beta} - 1 + o(1),
\]

where the last transition follows from (12) and (8). It follows from (65) that \( a_{n,i} = o(1/n) \), hence (69) converges to 1 and the first moment condition of Lemma 5.4 holds.

As for the second moment, we have

\[
\mathbb{E} \left[ \bar{L}_n^2 \right] = \prod_{i=1}^{n} \mathbb{E} \left[ \left( (1 - \epsilon_n)^2 + 2\epsilon_n(1 - \epsilon_n)L_i^{(n)}(X_i) + \epsilon_n^2 (L_i^{(n)}(X_i))^2 \right) \mathbf{1}_{\{X_i \leq 2(1+\eta) \log(n)\}} \right]
\]

\[
= \prod_{i=1}^{n} \left( (1 - \epsilon_n)^2 + 2\epsilon_n(1 - \epsilon_n)\mathbb{E} \left[ L_i^{(n)}(X_i) \right] + \epsilon_n^2 \mathbb{E} \left[ (L_i^{(n)}(X_i))^2 \mathbf{1}_{\{X_i \leq 2(1+\eta) \log(n)\}} \right] \right)
\]

\[
\leq \prod_{i=1}^{n} \left( 1 - \epsilon_n^2 + \epsilon_n^2 \mathbb{E} \left[ (L_i^{(n)}(X_i))^2 \mathbf{1}_{\{X_i \leq 2(1+\eta) \log(n)\}} \right] \right) \leq \prod_{i=1}^{n} (1 + b_{n,i}),
\]

where \( b_{n,i} \equiv \epsilon_n^2 + \epsilon_n^2 \mathbb{E} \left[ (L_i^{(n)}(X_i))^2 \right] \mathbf{1}_{\{X_i \leq 2(1+\eta) \log(n)\}} \).

By (56),

\[
\mathbb{E}_{X_i \sim E^{(n)}_i} \left[ (L_i^{(n)}(X_i))^2 \mathbf{1}_{\{X_i \leq 2(1+\eta) \log(n)\}} \right] = \mathbb{E}_{X_i \sim Q_i^{(n)}} \left[ L_i^{(n)}(X_i) \mathbf{1}_{\{X_i \leq 2(1+\eta) \log(n)\}} \right]
\]

\[
= n^{\max_{\delta} \{q-2\alpha(q,r,\sigma)\} + \delta + o(1)}.
\]

It follows from (65) that

\[
n \cdot b_{n,i} = n^{1-2\beta} + n^{1-2\beta+\max_{\delta} \{q-2\alpha(q,r,\sigma)\} + o(1)} < n^{1-2\beta + n^{-\delta}},
\]

which implies \( b_{n,i} = o(1/n) \) as \( \beta > 1/2 \). We conclude that (71) converges to 1 hence the second moment condition of Lemma 5.4 holds. \( \square \)
5.3. Proof of Lemma 5.4

Set
\[ \tilde{L}_n \equiv \tilde{L}_n(X_1, \ldots, X_n) \equiv L_n(X_1, \ldots, X_n)1_A(n)(X_1, \ldots, X_n). \]

Conditions (62) and (63) imply
\[ \mathbb{E}_{H_0^{(n)}}[\tilde{L}_n] = \left( \mathbb{E}_{H_0^{(n)}}[\tilde{L}_n^2] - 1 \right) - 2 \left( \mathbb{E}_{H_0^{(n)}}[\tilde{L}_n] - 1 \right) \leq o(1), \]
hence \( \tilde{L}_n(X) \to 1 \) in probability under \( H_0^{(n)} \). Next, for some \( \psi^{(n)}: \mathbb{R}^n \to \{0, 1\} \) and \( \epsilon > 0 \),
\[ \mathbb{E}_{H_0^{(n)}}[\psi^{(n)}] + \mathbb{E}_{H_1^{(n)}}[1 - \psi^{(n)}] = \mathbb{E}_{H_0^{(n)}}[\psi^{(n)} + L_n(1 - \psi^{(n)})] \geq \mathbb{E}_{H_0^{(n)}}[\psi^{(n)} + \tilde{L}_n(1 - \psi^{(n)})] \geq \mathbb{E}_{H_0^{(n)}}[\psi^{(n)} + \tilde{L}_n(1 - \psi^{(n)})] \geq (1 - \epsilon) \mathbb{P}[|\tilde{L}_n - 1| < \epsilon] = (1 - \epsilon)(1 + o(1)). \]

As \( \epsilon > 0 \) is arbitrary, we have that
\[ \liminf_{n \to \infty} \{ \mathbb{E}_{H_0^{(n)}}[\psi^{(n)}] + \mathbb{E}_{H_1^{(n)}}[1 - \psi^{(n)}] \} \geq 1. \]

5.4. Proof of Theorem 1.2

The proof requires the following Lemma from [22].

Lemma 5.5. [22, Lem. 5.7] Let \( F_n(t) \) be the normalized sum of \( n \) independent random variables. Fix \( q \in (0, 1] \) and \( \beta > 0 \). Suppose that
\[ \mathbb{E}[F_n(n^{-q} - q + o(1))] = n^{-q+o(1)}(1 - n^{-\beta}) + n^{-\beta}n^{-\alpha(q)+o(1)}, \]
for two non-negative functions \( \delta(\cdot) \) and \( \gamma(\cdot) \) with \( \delta(q) < q \) and
\[ \delta(q) + \beta < \gamma(q). \]

For a positive sequence \( \{a_n\}_{n=1}^{\infty} \) obeying \( a_n n^{-q} \to 0 \) for any \( \eta > 0 \), we have
\[ \mathbb{P}[n^{\gamma(q)}(F_n(n^{-q}) - n^{-q}) \leq a_n] \to 0, \quad n \to \infty. \]
It follows from [19, Thm. 1.1] that

\[ \Pr_{H_0} \left[ HC_n^* \leq \sqrt{4 \log \log(n)} \right] \to 0. \]  
(72)

Therefore, it is enough to show that

\[ \Pr_{H_1^{(n)}} \left[ HC_n \geq \log(n) \right] \to 0. \]  
(73)

Set

\[ F_n(t) \equiv \frac{1}{n} \sum_{i=1}^{n} 1_{p_i \leq t}, \]

and note that (12) and (8) imply

\[ E_{H_1^{(n)}} \left[ F_n(n^{-q}) \right] = n^{-q+o(1)}(1 - n^{-\beta}) + n^{-\beta} n^{-\alpha(q;r,\sigma)+o(1)}. \]

Since

\[ HC_n^* = \sup_{1/n \leq u \leq 1} \sqrt{n} \frac{F_n(u) - u}{\sqrt{u(1-u)}} \]  
(74)

almost surely,

\[ HC_n^* \geq \sqrt{n} \frac{F_n(t) - t}{\sqrt{t(1-t)}}, \quad t \in [1/n, 1). \]  
(75)

Therefore, setting \( t_n = n^{-q} \) for \( q \leq 1 \), we obtain:

\[
\Pr_{H_1^{(n)}} [HC_n^* \leq \log(n)] \leq \Pr_{H_1^{(n)}} \left( \sqrt{n} \frac{F_n(t_n) - t_n}{\sqrt{t_n(1-t_n)}} \leq \log(n) \right) \leq \Pr_{H_1^{(n)}} \left[ n^{\frac{q+1}{2}} (F_n(t_n) - t_n) \leq \log(n) \right].
\]  
(76)

Apply Lemma 5.5 to (76) with \( \delta(q) = \alpha(q;r,\sigma) \), \( \gamma(q) = (q + 1)/2 \), and \( a_n = \log(n) \) to conclude that (76) goes to zero as \( n \to \infty \). Theorem 1.2 follows.

5.5. Proof of Theorem 1.4

Let

\[ \eta \equiv 1 - \alpha(1; r, \sigma) - \beta. \]  
(77)

The condition \( r > \rho_{\text{Bonf}}(\beta; \sigma) \) implies \( \eta > 0 \). By continuity of \( q \to \alpha(q;r,\sigma) \), there exits \( \delta > 0 \) such that

\[ 1 - \alpha(1+\delta; r, \sigma) - \beta \geq \eta/2. \]  
(78)
For the statistic $p(1) \equiv \min_{i=1,...,n} p_i$, we show that, along the sequence of thresholds $a_n = n^{-(1+\eta/2)}$, we have $\Pr_{H_0} (p(1) > a_n) \to 1$ while $\Pr_{H_1} (p(1) > a_n) \to 0$. Indeed,

$$\Pr_{H_0} \left[ p(1) \leq a_n \right] = 1 - \prod_{i=1}^{n} \Pr_{H_0} [p_i > a_n]$$

$$= 1 - \left( 1 - a_n \cdot n^{o(1)} \right)^n$$

$$= 1 - \left( 1 - n^{-(1+\eta/2)+o(1)} \right)^n \to 0,$$

where (79) follows from (12). On the other hand,

$$\Pr_{H_1} \left[ p(1) \leq a_n \right] = 1 - \prod_{i=1}^{n} \Pr_{H_1} [p_i > a_n]$$

$$= 1 - \prod_{i=1}^{n} \left( 1 - \Pr_{H_1} [p_i \leq a_n] \right),$$

(80)

hence it is enough to show that $\Pr_{H_1} [p_i \leq a_n] > n^{-1+\eta/2+o(1)}$ uniformly in $i$.

For $i = 1,\ldots,n$ let $X_i$ be a RV with law $-2 \log(X_i) \overset{D}{=} Q_i^{(n)}$. We have:

$$\Pr_{H_1} [p_i \leq a_n] = (1 - \epsilon n) a_n \cdot n^{o(1)} + \epsilon_n \Pr [X_i \leq a_n]$$

$$\geq \epsilon_n \Pr [X_i \leq a_n]$$

$$= n^{-\beta - o(1+\delta;r,\sigma) + o(1)}$$

$$\geq n^{-1+\eta/2+o(1)},$$

(81)

(82)

where (81) follows from (8), and (82) follows from (78).

\section*{5.6. Proof of Theorem 1.3}

The proof is similar to the proof of [46, Thm. 4.4]. In particular we use:

**Lemma 5.6.** [46, Cor. A1] Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence converging to infinity. Let $\mu_n$, $\sigma_n^2$, and $f_n$ denote the mean, variance and density of Beta($\alpha_n, n - \alpha_n + 1$), respectively. Let $g(n)$ be any positive function satisfying $g(n) = o(\min\{\alpha_n, n - \alpha_n\})$ as $n \to \infty$. Then,

$$f_n(\mu_n + \sigma_n \cdot t) \geq \frac{e^{-t^2/2}}{\sqrt{2\pi \cdot \sigma_n}} \left( 1 - \frac{t^3}{\sqrt{g(n)}} - \frac{1}{g(n)} \right)$$

(83)

Recall that $M_n^- = \min_{i=1,\ldots,n} \pi_i$, where

$$\pi_i = \Pr \left[ \text{Beta}(i, n-i+1) \leq p(i) \right], \quad i = 1,\ldots,n.$$
We use the sequence $t_n = 1/n$ to separate $H_0$ from $H_1^{(n)}$. The limiting distribution of $M_n$ under $H_0$ satisfies (see [30] and [46, Thm 4.1]),

$$\Pr_{H_0} \left[ M_n \leq \frac{x}{2 \log(n) \log \log(n)} \right] \to 1 - e^{-x},$$

from which it follows that

$$\Pr_{H_0} \left[ M_n \leq t_n \right] \to 0. \quad (84)$$

For $X \sim \text{Beta}(i, n - i + 1)$, set

$$\mu_i \equiv \mathbb{E}[X] = \frac{i}{n + 1}, \quad \sigma^2_i \equiv \text{Var}[X] = \frac{i(n - 1 + 1)}{(n + 1)^2(n + 2)},$$

hence, for $x \in \mathbb{R}$,

$$\frac{\mu_i - x}{\sigma_i} = \sqrt{n} \frac{i/n - x}{\sqrt{\frac{1}{n} \left( 1 - \frac{i}{n} \right)}} \left( 1 + o(1) \right). \quad (85)$$

The proof of Theorem 1.2 in Section 5.4 implies in particular

$$\Pr_{H_1^{(i)}} \left[ \max_{i = 1, \ldots, n} \sqrt{n} \frac{i/n - p(i)}{\sqrt{\frac{1}{n} \left( 1 - \frac{i}{n} \right)}} \geq \log(n) \right] \to 1. \quad (86)$$

Together with (85), the last display implies that for any $\delta > 0$ there exists $n_0(\delta)$ and $i^* \in \{1, \ldots, n\}$ such that

$$\tau^* \equiv \frac{\mu_{i^*} - p(i^*)}{\sigma_{i^*}} \geq \sqrt{2 \log(n)}, \quad (87)$$

with probability at least $1 - \delta$. Denote by $f_i$ the density $f_i : [0, 1] \to \mathbb{R}^+$ of $\text{Beta}(i, n - i + 1)$. We have

$$\pi_{i^*} = \int_0^{p(i^*)} f_{i^*}(x)dx$$

$$= \sigma_{i^*} \int_{-\mu_{i^*}/\sigma_{i^*}}^{\tau^*} f_{i^*}(\mu_{i^*} + \sigma_{i^*}t)dt$$

$$\leq \sigma_{i^*} \int_{-\infty}^{\tau^*} f_{i^*}(\mu_{i^*} + \sigma_{i^*}t)dt$$

$$\leq \int_{\tau^*}^{\infty} \frac{1 + o(1)}{\sqrt{2\pi}} e^{-x^2/2}dx$$

$$\leq \frac{1 + o(1)}{\sqrt{2\pi}} e^{-\tau^{*2}/2}, \quad (88)$$

$$= (1 - \Phi(\tau^*))(1 + o(1)) \sim \frac{1}{\tau^*} e^{-\tau^*/2}, \quad (89)$$
where (88) follows from Lemma 5.6 and (89) is due to Mills’ ratio. Consequently,
\[ \pi_{i^*} \leq n^{-1}, \quad \text{for} \quad n \geq n_0(\delta), \] (90)
with probability at least \(1 - \delta\). Hence, for \(n \geq n_0(\delta)\),
\[ \Pr_{H_1^{(n)}}[M_n \leq t_n] \geq \Pr_{H_1^{(n)}}[M_n^* \leq n^{-1}] \geq \Pr_{H_1^{(n)}}[\pi_{i^*} \leq n^{-1}] \geq 1 - \delta. \] (91)
Together with (84), the last display implies that the sequence of thresholds \(t_n = 1/n\) perfectly separates \(H_0\) from \(H_1^{(n)}\). \(\square\)

5.7. Proof of Theorem 1.5

The proof is similar to the proof of Theorem 1.4 in [19]. The main idea is to establish the following claims:

(i) Inference based on FDR thresholding ignores P-values in the range \((n^{-q}, 1]\), for \(q < 1\).

(ii) When \(r < \rho_{\text{Bonf}}(\beta; \sigma)\), P-values smaller than \(n^{-q}\) under \(H_1^{(n)}\) are as frequent as under \(H_0^{(n)}\).

In order to establish (i) and (ii), define, for an interval \(I \subset (0, 1)\),
\[ T_I \equiv \min_{i : \ w(i) \in I} \frac{w(i)}{n}. \] (92)

For some \(q > 0\) and a sequence \(\{a_n\}_{n=1}^\infty\) of threshold values with \(\lim \inf_{n \to \infty} a_n = 0\),
\[ \left| \Pr_{H_0^{(n)}}[\text{FDR rejects}] - \Pr_{H_1^{(n)}}[\text{FDR rejects}] \right| = \left| \Pr_{H_0^{(n)}}[T_{(0,1]} < a_n] - \Pr_{H_1^{(n)}}[T_{(0,1]} < a_n] \right| \leq \Pr_{H_1^{(n)}}[T_{(n^{-q},1]} < a_n] + \Pr_{H_0^{(n)}}[T_{(n^{-q},1]} < a_n] \] (93)
\[ + \left| \Pr_{H_1^{(n)}}[T_{(0,n^{-q}]} < a_n] - \Pr_{H_0^{(n)}}[T_{(0,n^{-q}]} < a_n] \right|. \] (94)

Note that the terms in (93) are associated with (i) while (94) is associated with (ii).

The following lemma implies that the terms in (93) vanish as \(n \to \infty\).

Lemma 5.7. Assume that \(r < \rho_{\text{Bonf}}(\beta; \sigma)\). For any \(0 < a < 1\) and \(q < 1\),
\[ \Pr_{H_1^{(n)}}[T_{(n^{-q},1]} < a] \to 0. \] (95)
The proof of Lemma 5.7 is in Section 5.8 below. We now focus on the term (94). Let \( I \subset \{1, \ldots, n\} \) be a random set such that \( i \in I \) with probability \( \epsilon_n = n^{-\beta} \). An equivalent way of specifying \( H^{(n)}_1 \) is
\[
-2 \log(p_i) \sim \begin{cases} Q^{(n)}_i & i \in I \\ \text{Exp}(2) & i \notin I, \end{cases} \quad i = 1, \ldots, n. \tag{96}
\]
For \( i = 1, \ldots, n \), let \( X_i \) be a RV satisfying \(-2 \log(X_i) \sim Q^{(n)}_i \). Choose \( r < q < 1 \) such that
\[
1 - \alpha(q; r, \sigma) - \beta + \delta < 0 \tag{97}
\]
for some \( \delta > 0 \), which is possible since \( r < \rho_{\text{ord}}(\beta; \sigma) < 1 \). Consider the event:
\[
E^q_n \equiv \{ p_i \leq n^{-q} \text{ for some } i \in I \}.
\]
Conditioned on the event \( |I| = M \), we have
\[
\Pr[E^q_n \mid |I| = M] = \Pr \left[ \min_{i=1,\ldots,n} X_i \leq n^{-q} \mid |I| = M \right] 
\leq 1 - \left( 1 - n^{-\alpha(q; r, \sigma) + o(1)} \right)^M \tag{98}
\leq M \cdot n^{-\alpha(q; r, \sigma) + o(1)} \tag{99}
\]
where (98) follows from (8) and (99) follows from the inequality \( M \cdot \log(1 + x) > \log(1 + M x), \ x \geq -1 \). As \( M \sim \text{Bin}(n, \epsilon_n) \), we have \( \Pr[M < n^{1+\delta/2-\beta}] = \Pr[M < n^{1+\delta/2-\beta}] \rightarrow 1 \). Consequently, for any \( \epsilon \),
\[
\Pr \left[ M \cdot n^{-\alpha(q; r, \sigma) + o(1)} > \epsilon \right] \leq o(1) + 1_{\{n^{1+\delta/2-\beta-\alpha(q; r, \sigma) + o(1)} > \epsilon\}} \rightarrow 0
\]
where the last transition is due to (97). It follows that \( \Pr[E^q_n] \to 0 \). From here, since
\[
\Pr_{H^{(n)}_1} \left[ T_{[0,n^{-q}]} < a_n \mid (E^q_n)^C \right] = \Pr_{H^{(n)}_1} \left[ T_{[0,n^{-q}]} < a_n \mid (E^q_n)^C \right],
\]
we get
\[
\Pr_{H^{(n)}_1} \left[ T_{[0,n^{-q}]} < a_n \right] = \Pr \left[ ((E^q_n)^C) \Pr_{H^{(n)}_1} \left[ T_{[0,n^{-q}]} < a_n \mid (E^q_n)^C \right] 
+ \Pr[E^q_n] \Pr_{H^{(n)}_1} \left[ T_{[0,n^{-q}]} < a_n \mid E^q_n \right]
\right.
= \Pr_{H^{(n)}_1} \left[ T_{[0,n^{-q}]} < a_n \mid (E^q_n)^C \right] (1 + o(1)) + o(1)
\]
so that (94) vanished as well.
5.8. Proof of Lemma 5.7

Let $F_n(t) \equiv \frac{1}{n} \sum_{i=1}^{n} 1_{p_i \leq t}$ be the empirical CDF of $p_1, \ldots, p_n$. Note that $i/n = F_n(p(i))$, hence
\[ \frac{p(i)}{i/n} \leq a \iff F_n(p(i)) \geq p(i)/a. \]  
(100)

Consequently,
\[ \Pr_{H_1(n)} \left[ T_{(n-q,1)} \leq a \right] \leq \sup_{t > n^{-q} H_1(n)} \Pr_{H_1(n)} \left[ F_n(t) \geq t/a \right] \]
\[ = \sup_{t > n^{-q} H_1(n)} \Pr_{H_1(n)} \left[ nF_n(t) \geq nt/a \right] \]
\[ = \sup_{t > n^{-q} H_1(n)} \Pr \left[ nF_n(t) \geq \mathbb{E}_{H_1(n)} \left[ nF_n(t) \right] (1 + \kappa) \right], \]  
(101)

where
\[ \kappa = \kappa(n, a, t) = \frac{t}{a \mathbb{E}[F_n(t)]} - 1. \]  
(102)

Let $U_i \sim \text{Unif}(0, 1)$ and $-2 \log(X_i) \sim Q_1^{(n)}$, for $i = 1, \ldots, n$. Using the parameterization $t_n = n^{-q'}$, $q' \leq q < 1$,
\[ \mathbb{E}_{H_1(n)} \left[ F_n(t_n) \right] = \frac{1}{n} \sum_{i=1}^{n} \Pr_{H_1(n)} \left[ p_i \leq n^{-q'} \right] \]
\[ = (1 - \epsilon_n) \left[ U_i \leq n^{-q'} \right] + \epsilon_n \Pr \left[ X_i \leq n^{-q'} \right] \]
\[ = 1 - \epsilon_n + n^{-\alpha(q', r, \sigma)(1 + o(1)) - \beta}, \]  
(103)

where the last transition follows from (8). Since $\beta + \alpha(q', r, \sigma) \leq \beta + \alpha(1; r, \sigma) < 1$, the last display implies $\mathbb{E}_{H_1(n)} \left[ F_n(t_n) \right] / t_n \to 1$. It follows that
\[ \sup_{t > n^{-q}} \frac{E_{H_1(n)} \left[ F_n(t_n) \right]}{t} = 1 + o(1). \]  
(104)

Since $a < 1$, there exists $\eta > 0$ such that $\kappa \geq 1/a - 1 + \eta > 0$ for all $n \geq n_0(q)$ large enough. Using Chernoff’s inequality [45, Ch. 4] in (101), we obtain
\[ \Pr_{H_1(n)} \left[ T_{(n-q,1)} \leq a \right] \leq \sup_{t > n^{-q}} \exp \left\{ -\frac{n}{2a} \inf_{t > n^{-q}} E_{H_1(n)} \left[ F_n(t) \right] \right\} \]
\[ \leq \exp \left\{ -\frac{n}{2a} \inf_{t > n^{-q}} E_{H_1(n)} \left[ F_n(t) \right] \right\} \]
\[ = \exp \left\{ -\frac{1}{2a} n^{1-\beta + \alpha(q; r, \sigma) + o(1) - \beta} \right\} \to 0, \]

where the last transition follows because $r < \rho_{\text{conf}}(\beta; \sigma)$ implies $\beta + \alpha(q; r, \sigma) \leq \beta + \alpha(1; r, \sigma) < 1$. \qed
5.9. Proof of Theorem 1.6

We consider first the case where \( \{-2 \log(p_i)\} \) follow (3); the contrast between this case and the general case helps illustrate the necessity of (16).

Since \( F_n \sim \chi^2_{2n} \), under the null in (3) we have
\[
E \left[ F_n | H_0^{(n)} \right] = 2n, \quad \text{Var} \left[ F_n | H_0^{(n)} \right] = 4n. \quad (107)
\]

As \( F_n \) is asymptotically normal, it is enough to show that
\[
E \left[ F_n | H_1^{(n)} \right] \sim 2n \left(1 + o(\frac{1}{\sqrt{n}})\right), \quad \text{Var} \left[ F_n | H_1^{(n)} \right] \sim 4n \left(1 + o(1)\right). \quad (108)
\]
For \( X \sim \chi^2(r, \sigma) \), we have
\[
E[X] = \mu_n(r)^2 + \sigma^2, \quad E[X^2] = \mu_n(r)^4 + 4\mu_n^2\sigma^2 + 3\sigma^4, \quad (109)
\]
hence it follows that with \( Q_i^{(n)} = \chi^2(r, \sigma) \),
\[
2n \leq E \left[ F_n | H_1^{(n)} \right] = 2n(1 - \epsilon_n) + n \cdot \epsilon_n \left(2r \log(n) + \sigma^2\right) = 2n(1 + o(1/\sqrt{n})),
\]
where in the last transition we used that \( \beta > 1/2 \). Similarly, we have
\[
E \left[ F_n^2 | H_1^{(n)} \right] = 4n \left(1 + o(1/\sqrt{n})\right), \quad \text{hence (108) holds in this case.}
\]
For the general case of (11) under (16), first note that
\[
E_{X \sim E_i^{(n)}}[X] = \int_0^\infty \Pr \left[ X \geq x | X \sim E_i^{(n)} \right] dx
= 2 \log(n) \int_0^\infty \Pr \left[ X \geq 2y \log(n) | X \sim E_i^{(n)} \right] dy
= 2 \log(n) \int_0^\infty e^{-y \log(n)(1+o(1))} dy = 2 + o(1), \quad (110)
\]
and
\[
E_{X \sim E_i^{(n)}}[X^2] = \int_0^\infty x \Pr \left[ X \geq x | X \sim E_i^{(n)} \right] dx
= (2 \log(n))^2 \int_0^\infty y \Pr \left[ X \geq 2y \log(n) | X \sim E_i^{(n)} \right] dy
= (2 \log(n))^2 \int_0^\infty ye^{-y \log(n)(1+o(1))} dy = 4 + o(1).
\]
It follows that
\[
E \left[ F_n | H_0^{(n)} \right] = 2n(1 + a_n), \quad \text{and} \quad \text{Var} \left[ F_n | H_0^{(n)} \right] = 4n(1 + o(1)), \quad (111)
\]
where \( a_n \to 0 \). Next, notice that

\[
E_{X \sim Q_i^{(n)}} [X] = \int_0^\infty \Pr \left[ X \geq x | X \sim Q_i^{(n)} \right] dx
\]

\[
= 2 \log(n) \int_0^\infty \Pr \left[ X \geq 2y \log(n) | X \sim Q_i^{(n)} \right] dy
\]

\[
= \int_0^\infty n^{-\alpha(y;r,\sigma)+o(1)} dy,
\]  

(112)

\[
= n^{o(1)},
\]  

(113)

where (112) follows from (16) and (113) follows since

\[
\int_0^\infty n^{-\alpha(y;r,\sigma)} dy = \frac{\sigma^2 n^{-\frac{\pi r}{\log(n)}} + 2\sigma \sqrt{\frac{\pi r}{\log(n)}} \Phi \left( \frac{\sqrt{2r \log(n)}}{\sigma} \right)}{o(1)}.
\]

From (110) and because \( \beta > 1/2 \),

\[
2n \leq E \left[ F_n | H_1^{(n)} \right] = 2n(1 - \epsilon_n)(1 + a_n) + \epsilon_n n^{1+o(1)}
\]

\[
\leq 2n(1 + a_n) + o(1/\sqrt{n})
\]

Similarly,

\[
E_{X \sim Q_i^{(n)}} [X^2] = \int_0^\infty x \Pr \left[ X \geq x | X \sim Q_i^{(n)} \right] dx
\]

\[
= (2 \log(n))^2 \int_0^\infty y \Pr \left[ X \geq 2y \log(n) | X \sim Q_i^{(n)} \right] dy
\]

\[
= (2 \log(n))^2 \int_0^\infty y \cdot n^{-\alpha(y;r,\sigma)+o(1)} dy = n^{o(1)},
\]

where in the last transition we used that

\[
\int_0^\infty y \cdot n^{-\alpha(y;r,\sigma)} dy = o(1),
\]

as can be deduced from the analytic expression of this integral. We obtain

\[
4n \leq E \left[ F_n^2 | H_1^{(n)} \right] = 4n(1 - \epsilon_n)(1 + o(1)) + n \cdot \epsilon_n \cdot n^{o(1)}
\]

(114)

\[
= 4n(1 + o(1)).
\]  

(115)

Evaluations similar to those in (111) and (115) imply that \( F_n \) satisfies the conditions of the Lyapunov central limit theorem for sums of independent but perhaps non-identically distributed RVs. Consequently, \( F_n \) is asymptotically normal both under \( H_0^{(n)} \) and \( H_1^{(n)} \). Since we have

\[
\frac{E \left[ F_n | H_1^{(n)} \right] - E \left[ F_n | H_0^{(n)} \right]}{\sqrt{\text{Var} \left[ F_n | H_0^{(n)} \right]}} \to 0, \quad \text{and} \quad \frac{\text{Var} \left[ F_n | H_0^{(n)} \right]}{\text{Var} \left[ F_n | H_1^{(n)} \right]} \to 1,
\]

a standard argument (e.g., [4, Lem. B.2]) implies that \( F_n \) is asymptotically powerless. \( \square \)
5.10. Proof of Proposition 2.1

The proof requires the characterization of the $t$-statistic under moderate deviations in the form provided in [15]:

**Lemma 5.8.** [15, Theorem 1.2] In the notation of Section 2.5, set $X(j) = X_{1,j}$, $Y(j) = Y_{1,j}$, $\bar{X} = \bar{X}_1$, $\bar{Y} = \bar{Y}_1$, $\nu_x = \nu_{x,1}$, $\nu_y = \nu_{y,1}$, $s_1 = s_{x,1}$, $s_2 = s_{y,1}$, $n_x = n_{x,1}$, $n_y = n_{y,1}$, $n = n_x + n_y$, and

$$T_n = \frac{\bar{X} - \bar{Y} - (\nu_x - \nu_y)}{\sqrt{\frac{s_x^2}{n_x} + \frac{s_y^2}{n_y}}}.$$  \hfill (116)

Assume $E[|X(1)|^3] < \infty$ $E[|Y(1)|^3] < \infty$ and that there are $0 < c_x < c_y < \infty$ such that $c_x \leq n_x/n_y \leq c_y$. Then

$$\frac{\Pr (T_n \geq u)}{\Phi(u)} \to 1$$  \hfill (117)

uniformly in $u \in (0, o(n^{1/6}))$.

Lemma 5.8 says that for deviations smaller than $n^{1/6}$, the $t$-statistic \hfill (116) behaves asymptotically as a standard normal RV.

Denote $t(x;n) \equiv \Pr (|x| \geq |T_n|)$ and $z(x) \equiv \Pr (|x| \geq N(0,1))$. Note that the standard convergence of the $t$-distribution to the normal distribution implies

$$\sup_{q > 0} \left| t(\sqrt{q \log(n)}; n) - z(\sqrt{q \log(n)}) \right| = o(n^{-1}).$$  \hfill (118)

We now consider the distribution of $p_i$ under $H_1^{(n)}$. Abusing previous notation, suppose that $p_i$ is associated with the case $\nu_x \neq \nu_y$. For $q > r$ we get

$$\Pr [-2 \log(p_i) \geq 2q \log(n)] = \Pr [\Pr (|T_i| > |T_{n-2}|) \leq n^{-q}]$$

$$= \Pr [t(|T_i|; n-2) \leq n^{-q}]$$

$$= \Pr [z(|T_i|) + o(n^{-1}) \leq n^{-q}]$$

$$= \Pr [|T_i| > \sqrt{2q \log(n)} + o(1)]$$

$$= \bar{\Phi} \left( \sqrt{2q \log(n)} \right) (1 + o(1))$$

$$= \bar{\Phi} \left( 2q \log(n) - \sqrt{2r \log(n)} \right) (1 + o(1))$$

$$\leq n^{-\alpha(\sqrt{q-r} + o(1))} (1 + o(1)) = n^{-\alpha(\epsilon^r, r)+o(1)},$$

where (119) follows from (118), (120) is due to properties of the quantile function of the normal distribution as implied by Mills’ ratio, (121) follows from Lemma 5.8 as Condition (iii) implies $\sqrt{2q \log(n)} = o(n^{1/6})$ for $q > 0$, and (122)
follows from Mills’ ratio. The last display implies (8). Likewise, under $H_0^{(n)}$, we have

$$\Pr \left[ -2 \log(p_i) \geq 2q \log(n) \right] = n^{-q+o(1)},$$

so that (12) holds.

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