The Jacobi theta distribution

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Abstract

We form the Jacobi theta distribution through discrete integration of exponential random variables over an infinite inverse square law surface. It is continuous, supported on the positive reals, has a single positive parameter, is unimodal, positively skewed, and leptokurtic. Its cumulative distribution and density functions are expressed in terms of the Jacobi theta function. We describe asymptotic and log-normal approximations, inference, and a few applications of such distributions to modeling.

Keywords: Jacobi theta function, Jacobi theta distribution, Laplace transform, log-normal distribution, inverse-square law

1 Introduction

We describe a univariate continuous distribution called the Jacobi theta distribution supported on the positive reals that does not appear in the literature to the best of

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our knowledge (see e.g., Johnson et al. (1994)). The distribution is attained from the action of an infinite random measure $N$ with random weights $\{W_x\}$ and fixed atoms $N \geq 1$, where the weights are iid exponential random variables with common mean $m \in (0, \infty)$, on the test function $f(x) = 1/x^2$ for $x \in N \geq 1$, such that $Nf$ is a random variable having the Jacobi theta distribution. It is represented as the infinite sum

$$Nf = \sum_{x \geq 1} W_x/x^2$$

Its law is encoded in its Laplace transform

$$\alpha \mapsto \sqrt{\alpha m \pi} \text{csch}(\sqrt{\alpha m \pi})$$

The Jacobi theta distribution is continuous, has a single parameter $m$, is unimodal, positively skewed, and leptokurtic.

This note is organized as follows. In Section 2 we give the mathematical backdrop in terms of random measures. In Section 3 we give the main result of the existence of the distribution and state some of its properties. In Section 4 we show that the distribution may be approximated by asymptotic expansion and by the log-normal distribution. In Section 5 we give three applications to modeling data. In Section 6 we end with discussions and conclusions.

## 2 Background

We give background using notation and conventions from Cinlar (2011); Kallenberg (2017). Let $(\Omega, \mathcal{H}, \mathbb{P})$ be a probability space and let $(E, \mathcal{E})$ be a measurable space. A random measure is a transition kernel from $(\Omega, \mathcal{H})$ into $(E, \mathcal{E})$. Specifically the mapping $N : \Omega \times E \mapsto \mathbb{R}_\geq 0$ is a random measure if $\omega \mapsto N(\omega, A)$ is a random variable for each $A$ in $\mathcal{E}$ and if $A \mapsto N(\omega, A)$ is a measure on $(E, \mathcal{E})$ for each $\omega$ in $\Omega$. We denote $\mathcal{E}_\geq 0$ the set of non-negative $\mathcal{E}$-measurable functions.

The law of $N$ is uniquely determined by the Laplace functional $L$ from $\mathcal{E}_\geq 0$ into $[0, 1]$

$$L(f) = \mathbb{E}e^{-Nf} = \mathbb{E}\exp\left(\int_E N(dx)f(x)\right) \quad \text{for} \quad f \in \mathcal{E}_\geq 0 \quad (1)$$

The Laplace functional encodes all the information of $N$: its distribution, moments, etc. The distribution of $Nf$, denoted by $\eta$, i.e. $\eta(dx) = \mathbb{P}(Nf \in dx)$, is encoded by the Laplace transform, which may be expressed in terms of the Laplace functional

$$F(\alpha) = \mathbb{E}e^{-\alpha Nf} = \mathbb{E}e^{-N(\alpha f)} = L(\alpha f) \quad \text{for} \quad \alpha \in \mathbb{R}_\geq 0 \quad (2)$$

The moments of $Nf$ (if they exist) can be attained from the Laplace functional

$$\mathbb{E}(Nf)^n = (-1)^n \lim_{q \downarrow 0} \frac{\partial^n}{\partial q^n} L(qf) = \lim_{q \downarrow 0} \int_0^\infty \eta(dx)x^n e^{-qx} \quad \text{for} \quad n \in \mathbb{N}_1 \quad (3)$$
Let $D \subset E$ be a countable subset of $E$ and let $\{W_x : x \in D\}$ be an independency of non-negative random variables distributed $W_x \sim \nu_x$ with mean $m_x$ and variance $\sigma^2_x$. The random measure $N$ on $(E, \mathcal{E})$ formed as

$$N(A) = \int_E N(dx) \mathbb{1}_A(x) = \sum_{x \in D} W_x \mathbb{1}_A(x) \quad \text{for} \quad A \in \mathcal{E}$$

is additive with fixed atoms of $D$ and random weights of $\{W_x\}$. The Laplace functional of $N$ is given by

$$L(f) = \mathbb{E} e^{-Nf} = \prod_{x \in D} \int_{\mathbb{R}_{\geq 0}} \nu_x(dz) e^{-zf(x)} = \prod_{x \in D} F_x(f(x)) \quad \text{for} \quad f \in \mathcal{E}_{\geq 0}$$

where $F_x$ is the Laplace transform of $\nu_x$ defined as

$$F_x(\alpha) = \int_{\mathbb{R}_{\geq 0}} \nu_x(dz) e^{-\alpha z} \quad \text{for} \quad \alpha \in \mathbb{R}_{\geq 0}$$

The Laplace transform $F$ of $Nf$ is expressed in terms of the Laplace functional $F(\bullet) = L(\bullet f)$.

$Nf$ is formed as

$$Nf = \int_E N(dx)f(x) = \sum_{x \in D} W_x f(x) \quad \text{for} \quad f \in \mathcal{E}_{\geq 0}$$

and has mean, variance, and second moment

$$\mathbb{E}Nf = \sum_{x \in D} m_x f(x)$$

$$\text{Var}Nf = \sum_{x \in D} \sigma^2_x f^2(x)$$

$$\mathbb{E}(Nf)^2 = \text{Var}Nf + (\mathbb{E}f)^2$$

and covariance of $f, g \in \mathcal{E}_{\geq 0}$

$$\text{Cov}(Nf, Ng) = \sum_{x \in D} \sigma^2_x f(x)g(x)$$

Hence, assuming at least one of the $\{W_x\}$ is non-degenerate, the covariance is zero if and only if the functions are disjoint.

A product random measure $M = N \times N$ on $(E \times E, \mathcal{E} \otimes \mathcal{E})$ can be defined as

$$Mf = \int_{E \times E} M(dx, dy)f(x, y)$$

$$= \int_{E \times E} N(dx)N(dy)f(x, y)$$

$$= \sum_{(x,y) \in D^2} W_x W_y f(x, y) \quad \text{for} \quad f \in (\mathcal{E} \otimes \mathcal{E})_{\geq 0}$$
with Laplace functional
\[ \mathbb{E}e^{-Mf} = \prod_{(x,y) \in D^2} F_{xy}(f(x, y)) \quad \text{for} \quad f \in (\mathcal{E} \otimes \mathcal{E})_{\geq 0} \]

where \( F_{xy} = F_x \otimes F_y \) is the Laplace transform of \( W_x W_y \) and \( \otimes \) is the convolution operator. For product functions \( f = g \times g \in (\mathcal{E} \otimes \mathcal{E})_{\geq 0} \), we have that \( Mf = (Ng)^2 \). This readily extends to \( n \)-products with \( M = \bigotimes_{x}^{n} N \) for \( f = \bigotimes_{x}^{n} g \), where we have \( Mf = (Ng)^n \). Therefore
\[ \mathbb{E}(Ng)^n = \sum_{(x,\ldots,y) \in D^n} \mathbb{E}(W_x \cdots W_y)g(x)\cdots g(y) \quad \text{for} \quad g \in \mathcal{E}_{\geq 0}, \quad n \geq 1 \quad (8) \]

## 3 Distribution

We define the Jacobi theta function.

**Definition 1** (Jacobi theta function). The Jacobi theta function is defined as
\[ \theta_2(z, q) = 2q^{1/4} \sum_{k=0}^{\infty} q^{k(k+1)} \cos((2k + 1)z) \quad (9) \]

Now we give the main result on the existence of the Jacobi theta distribution.

**Theorem 1** (Jacobi theta). Consider the random measure \( N \) \[ \square \] . Let \( D = \mathbb{N}_{\geq 1} \) with \( (E, \mathcal{E}) = (\mathbb{R}_{>0}, \mathcal{B}_{\mathbb{R}_{>0}}) \) and let \( \{W_x\} \) be an independency of exponential random variables with common mean \( m \in (0, \infty) \). Then for \( f \in \mathcal{E}_{\geq 0} \) as \( f(x) = 1/x^2 \), \( Nf \) has Laplace transform
\[ F(\alpha) = \sqrt{\alpha m \pi} \operatorname{csch}(\sqrt{\alpha m \pi}) \quad \text{for} \quad \alpha \in \mathbb{R}_{\geq 0} \]

cumulative distribution function
\[ \eta(Nf \leq x) = \sqrt{\frac{m \pi}{x}} \theta_2 \left( 0, e^{-\frac{mx^2}{x}} \right) \]

and density
\[ \eta(dx) = \frac{\sqrt{m \pi}}{2x^{5/2}} \left( 2m^2 \frac{\theta_2 \left( 0, e^{-\frac{mx^2}{x}} \right)}{4} + 2e^{-\frac{mx^2}{x}} \sum_{k \geq 1} k(k+1) \left( e^{-\frac{mx^2}{x}} \right)^{k(k+1)-3/4} \right) - x \theta_2 \left( 0, e^{-\frac{mx^2}{x}} \right) dx \]
Proof. The Laplace transform is computed as
\[
\mathbb{E}e^{-\alpha N_f} = \mathbb{E}e^{-\sum_{x \in D} \alpha W_x f(x)} = \prod_{x \in D} \mathbb{E}e^{-\alpha W_x f(x)} = \prod_{x \in D} \int_0^\infty \int_0^\infty dze^{-z/m_x}e^{-\alpha z f(x)} = \prod_{x \in D} \frac{1}{1 + \alpha m_x f(x)}
\]
which specialized for \(m_x = m\) gives the result. The inverse Laplace transform follows from noting that
\[
L^{-1}(\frac{\text{csch} \sqrt{\alpha}}{\sqrt{\alpha}})(x) = \frac{2}{\sqrt{\pi}x} \sum_{k \geq 0} e^{-\frac{(2k+1)^2}{4x}} = \theta_2(0, e^{-1/x})
\]
so that
\[
L^{-1}(\sqrt{\alpha \text{csch}(\sqrt{\alpha})})(x) = \frac{\partial}{\partial x} L^{-1}(\frac{\text{csch} \sqrt{\alpha}}{\sqrt{\alpha}})(x)
\]
giving the cumulative distribution function and density upon substitution of \(x \leftarrow x/(m\pi^2)\). The derivative follows from the derivative of \(\theta_2\) in the second coordinate. □

The frequency spectrum of \(N_f\) is given by \(C(\omega) = F(-i\omega)\) for \(\omega \in \mathbb{R}\) with magnitude squared
\[
|C(\omega)|^2 = |\omega| m\pi^2 \text{csch}(\sqrt{-i\omega m\pi}) \text{csch}(\sqrt{i\omega m\pi}) \quad \text{for} \quad \omega \in \mathbb{R}
\]
and the phase spectrum is given by \(P(\omega) = \arctan(\mathfrak{I}(C(\omega))/\mathfrak{R}(C(\omega)))\). Both are plotted below in Figure 1 for \(m = 7\).
The statistics follow and are expressed in terms of the Riemann zeta function. Both skewness and kurtosis are constant.

**Corollary 1** (Statistics). The mean, variance, and second moment are

\[ \mathbb{E} N_f = m\zeta(2) = m\pi^2/6 \]
\[ \text{Var} N_f = m^2\zeta(4) = m^2\pi^4/90 \]
\[ \mathbb{E}(N_f)^2 = 7m^2\pi^4/180 \]

with constant signal to noise ratio \( \mathbb{E} N_f / \sqrt{\text{Var} N_f} = \sqrt{5/2} \approx 1.58 \), where \( \zeta \) is the Riemann zeta function. The skewness is

\[ \text{Skewness}(N_f) = 4\sqrt{10}/7 \approx 1.81 \]

and the kurtosis is

\[ \text{Kurtosis}(N_f) = 57/7 \approx 8.14 \]

Next we show that the moments are finite at all orders and that the Jacobi theta
distribution is uniquely defined by its moments.

**Proposition 1** (Finite characterizing moments). \( N_f \) has finite moments of all orders

\[ \mathbb{E}(N_f)^n < \infty \quad \text{for} \quad m > 0, \quad n \geq 1 \]

with moment generating function

\[ M(t) = F(t) < \infty \quad \text{for} \quad t \in \mathbb{R} \]

where \( F \) is the Laplace transform of \( N_f \).

**Proof.** The finiteness of all moments follows from (8), where

\[
\mathbb{E}(N_f)^n = \sum_{(x,\ldots,y) \in D^n} \mathbb{E}(W_x \cdots W_y) f(x) \cdots f(y)
\]

\[
\leq \sum_{(x,\ldots,y) \in D^n} \mathbb{E}W_x^n f(x) \cdots f(y)
\]

\[
= n!(m\zeta(2))^n < \infty \quad \text{for} \quad m > 0, \quad n \geq 1
\]

For the mgf, we have that \( M(t) = F(-t) = F(t) \) for \( t \in \mathbb{R} \). \( \square \)

Below in Figure 2 we show the densities for \( m = 1, \cdots, 7 \).

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**Figure 2:** Density \( \eta \) of Jacobi theta distribution for \( m = 1, \cdots, 7 \).
The CDF may be used to furnish an estimator for \( m \) given data.

**Proposition 2 (Estimator).** Let \( \{X_i = 1, \cdots, n\} \) be an independency of data with empirical distribution function \( F_n \). Let \( u \in (0,1) \) be a cdf value at the point \( x = F_n^{-1}(u) \), e.g., \( u = 1/2 \). Then \( m \) may be estimated by numerically finding the root of the following equation

\[
\sqrt{\frac{m\pi}{x}} \theta_2 \left( 0, e^{-\frac{mx^2}{4}} \right) = u
\]

**4 Approximations**

We discuss how the Jacobi theta distribution may be approximated through asymptotics and the log-normal distribution.

**4.1 Asymptotic**

**Proposition 3 (First-order).** The Jacobi theta distribution cumulative distribution function may be approximated to first-order as

\[
\mathbb{P}(Nf \leq x) \simeq 2\sqrt{\frac{m\pi}{x}} e^{-\frac{mx^2}{4}} \quad \text{for} \quad x \leq \frac{m\pi^2}{2}
\]

with total mass \( \mathbb{P}(Nf \leq \frac{m\pi^2}{2}) = 2\sqrt{\frac{2}{e\pi}} \simeq 0.968 \) and density function

\[
\eta(dx) \simeq \sqrt{\frac{\pi}{2x^2}} e^{-\frac{mx^2}{4x}} \left( \frac{\pi^2 m^2 - 2x}{2x^2} \right) \sqrt{\frac{m}{x}} dx \quad \text{for} \quad x \in (0,\frac{m\pi^2}{2}]
\]

**Proof.** The proof follows from the first-order truncation of representation [9]. \( \square \)

Higher-order approximations are admitted in view of [9].

We show the cumulative distribution function of the Jacobi theta distribution for \( m = 7 \) against the asymptotic approximation on the given support.
Figure 3: Cumulative distribution function of Jacobi theta distribution for \( m = 7 \) and asymptotic approximation

The asymptotic approximation may be used to furnish an estimator for \( m \) given data.

**Proposition 4** (Estimator). Let \( \{X_i = 1, \cdots, n\} \) be an independence of Jacobi theta random variables with empirical distribution function \( F_n \) converging almost surely to \( F \). Let \( u \in (0, 2\sqrt{\frac{2}{\pi}}] \subset (0, 1) \) be an argument to \( F^{-1} \) such that \( F^{-1}(u) \) is closely approximated by \( F_n^{-1}(u) \), e.g., \( u = 1/2 \). Then \( m \) is estimated as

\[
m_n = F_n^{-1}(u) \left( \frac{-2W_{-1}(-\pi u^2/8)}{\pi^2} \right)
\]

where \( W \) is the product-logarithm function.

**Proof.** We solve the equation \( 2\sqrt{\frac{\max x_n}{\pi}} e^{-\frac{m_n^2}{4x_n}} = u \) for \( m_n \), where \( x_n = F_n^{-1}(u) \).

\[\square\]

### 4.2 Log-normal

The Jacobi theta distribution can be approximated by the log-normal distribution \( \text{LogNormal}(\mu, \sigma) \). Matching first and second moments, we have

\[
\mu = \log \left( \frac{1}{6} \sqrt{\frac{5}{\pi}} \pi^2 m \right)
\]

(10)

\[
\sigma = \sqrt{\log \left( \frac{7}{5} \right)}
\]

(11)
which gives approximate density

\[ \tilde{\eta}(dx) = \exp\left( -\frac{\left( \frac{1}{2} \sqrt{\frac{\pi}{2}} \sigma x - \log(x) \right)^2}{2 \log(\frac{5}{2})} \right) \frac{dx}{x \sqrt{2 \pi \log(\frac{5}{2})}} \]

and cumulative distribution function

\[ \tilde{\eta}(t \leq x) = \frac{1}{2} (1 + \text{erf}\left( \frac{\log x - \mu}{\sigma \sqrt{2}} \right)) \]

The skewness is \(17\sqrt{2/5}/5 \simeq 2.15\) and the kurtosis is \(7631/625 \simeq 12.21\), both independent of \(m\).

The entropy can be approximated as

\[ H \simeq \log_2(\sigma \sqrt{2\pi} \exp(\mu + 1/2)) = \log_2 \left( \frac{1}{3} \sqrt{\frac{5e}{7}} \pi^{5/2} m \sqrt{\log\left(\sqrt{\frac{7}{5}}\right)} \right) \sim O(\log_2(m)) \]

In Figure 4 we show the cumulative distribution function of the Jacobi theta distribution for \(m = 7\) against the matching log-normal distribution. The approximation is very good.

Figure 4: Cumulative distribution function of Jacobi theta distribution for \(m = 7\) and log-normal approximation
4.3 Maximum likelihood

**Proposition 5 (MLE).** Let $X = \{X_i : i = 1, \cdots, n\}$ be a collection of data. Then the log-normal maximum likelihood estimator with parameters $\mu$ (10) and $\sigma$ (11) for $m \in (0, \infty)$ of the Jacobi theta distribution is given by

$$
\hat{m} = \frac{6}{\pi^2} \sqrt{\frac{7}{5}} \left( \prod_{i=1}^{n} X_i \right)^{1/n} \sim \text{LogNormal}(\mu - \sigma^2/(2n) + \log \left( \frac{6}{\pi^2} \sqrt{\frac{7}{5}} \right), \sigma/\sqrt{n})
$$

with mean and variance

$$
\mathbb{E}(\hat{m}) = \frac{6}{\pi^2} \sqrt{\frac{7}{5}} \exp \mu = m
$$

$$
\text{Var}(\hat{m}) = m^2 \left( \exp\left( \frac{\sigma^2}{n} \right) - 1 \right) \xrightarrow{n \text{ large}} m^2 \left( \frac{\sigma^2}{n} \right)
$$

**Proof.** The parameter $m$ can be estimated using the log-normal maximum likelihood estimator of $\mu$

$$
\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} \log(X_i) \xrightarrow{n \text{ large}} \text{Gaussian}(\mu, \sigma^2/n)
$$

The log-normal distribution follows from the MLE of the log-normal distribution. \hfill \Box

5 Applications

The Jacobi theta distribution can be applied to any problem where the random variable of interest is an infinite superposition of exponential random variables with inverse quadratically scaled means relative to the positive integers. Superpositions at integer points correspond to discrete integration over an inverse square law surface. Inverse square laws are found in many applications, including gravitation, electric fields and forces, intensity of light, radiation from a source, intensity of sound, and so on. Exponential distributions arise as the maximum entropy distribution on the positive reals with fixed mean.

5.1 Maximum likelihood estimation

Consider a random sample of data $X = \{X_i : i = 1, \cdots, n\}$ distributed according to the Jacobi theta distribution with parameter $m \in (0, \infty)$. We estimate the parameter $m$ using the exact and asymptotic CDF methods and the log-normal approximation. We take $m = 7$, $n = 10^2$ and generate $10^4$ samples of $X$ as $\{X^j : j = 1, \cdots, 10^4\}$. For each $X^j$ we estimate $m$ using the three estimators. We show the histogram distributions of the estimators below in Figure 5. The log-normal estimator has the smallest variance.
Consider an equispaced grid of locations of interferers transmitting radio-frequency waves with constant radial spacing $d > 0$. The transmitted powers are denoted $\{P_x : x = 1, 2, \cdots\}$ and are distributed $\text{Exponential}(4\pi d^2 \lambda)$ where $\lambda > 0$. The sphere’s surface area is $4\pi r^2$ and the power at the origin from interferer $x$ is $I_x = P_x/r^2$. Therefore, the total power received at the origin $I$ from the field of interferers is Jacobi theta distributed with $m = 1/(4\pi d^2 \lambda)$, having mean

$$\mathbb{E} I = \mathbb{E} \sum_x I_x = \frac{\zeta(2)}{4\pi \lambda d^2} = \frac{\pi}{24\lambda d^2}$$

and variance

$$\text{Var} I = \frac{\zeta(4)}{16\pi^2 \lambda^2 d^4} = \frac{\pi^2}{1440\lambda^2 d^4}$$

In Figure 6, we show $z = 10^3$ random locations on a disk of radius $z$ with unit radial spacing. The distribution of points on the plane concentrates towards the origin, with point density

$$\frac{1}{\pi r} \sim O(r^{-1})$$
Figure 6: $10^3$ random locations with constant radial spacing

Next we slightly alter the problem by changing the spacing of the points and restricting to $[0, t]$. Altering the points changes the distribution, yet the mean and variance may be readily attained. Consider the grid on $[0, t]$ formed by $D_t = \{\sqrt{tx} : x = 1, \cdots, t\}$. Then $I$ on $[0, t]$ is denoted $I_t$

$$I_t = \sum_{z=\sqrt{tx} \in D_t} I_z$$

with mean and variance

$$\mathbb{E}I_t = \sum_{z=\sqrt{tx} \in D_t} \frac{1}{4\pi \lambda d^2 z^2} = \sum_{z=\sqrt{tx} \in D_t} \frac{1}{4\pi \lambda d^2 tx} = \frac{H_t/t}{4\pi \lambda d^2}$$

$$\text{Var}I_t = \sum_{z=\sqrt{tx} \in D_t} \frac{1}{(4\pi \lambda d^2 z^2)^2} = \sum_{z=\sqrt{tx} \in D_t} \frac{1}{(4\pi \lambda td^2 z^2)^2} = \frac{H_t^{(2)}/t^2}{(4\pi \lambda d^2)^2}$$

where $H_t$ is the $t$-th Harmonic number. Note that $H_t/t < \zeta(2)$ for $t \geq 1$ and thus the interference at the origin is reduced with the altered spacing. The distribution of points on the plane is constant, with point density $1/\pi$. We show $10^3$ random locations with linear radial spacing below in Figure 7.
Suppose we have a transmitter at some location $z$ with power $P_z \sim \text{Exponential}(4\pi d^2 \lambda z)$ superimposed with interfering field $I$ comprised of interferers $\{P_x\}$ as before, with Jacobi theta distribution with parameter $m = 1/(4\pi d^2 \lambda)$. We are interested in the mean and variance of the random variable of $Q_z = P_z / I$—the signal interference noise ratio (SINR)—which may be calculated through the ratio distribution using the log-normal approximation as

\[
\mathbb{E} Q_z \simeq \frac{21}{10\pi^4 d^4 \lambda m z} \sim O(z^{-1})
\]
\[
\mathbb{V} \text{ar} Q_z \simeq \frac{3969}{500\pi^8 d^4 \lambda^2 m^2 z^2} \sim O(z^{-2})
\]

and $\mathbb{E} Q_z / \sqrt{\mathbb{V} \text{ar} Q_z} \simeq \sqrt{5}/3 \simeq 0.75$. We plot $\mathbb{P}(Q_z > t)$—the coverage probability—for $z = 1, m \in \{3, 5, 7, 9\}, \lambda = 1/100$, and $d = 1$ below in Figure 8.

Figure 7: $10^3$ random locations with linear radial spacing

![Figure 7: 10^3 random locations with linear radial spacing](image)
Figure 8: Signal interference noise ratio for $m \in \{3, 5, 7, 9\}$, $\lambda = 1/100$, $d = 1$ at location $z = 1$

### 5.3 Economic gravity model

Consider an equispaced grid of locations of countries relative to a country at the origin with spacing $d$. For each location $x$, there exist two independent random variables: a fraction coefficient $U_x$ uniformly taking values in $(0, 1)$ and gross domestic product $G_x$, assumed to be Gamma$(2, \lambda)$ distributed. The gravity model of bilateral trade flows with the origin is described by $T_x = U_x G_x / d_x^2$, where $d_x > 0$ is distance. Then the trade flow at $x$ is given by $T_x = U_x G_x \sim \text{Exponential}(\lambda)$ and the total trade flows with the origin is given by

$$T = \sum_x \frac{U_x G_x}{(dx)^2}$$

which has the Jacobi theta distribution with parameter $m = 1/(\lambda d^2)$.

### 5.4 Electric field

Consider an equispaced grid of locations of point charges with spacing $d > 0$. The point charges are $\{Q_x : x = 1, 2, \cdots\}$ and are distributed Exponential$(4\pi\varepsilon_0 d^2 \lambda)$. Then the total charge of the electric field at the origin has the Jacobi theta distribution with parameter $m = 1/(4\pi\varepsilon_0 d^2 \lambda)$.

### 6 Discussion and conclusions

We describe a continuous univariate distribution supported on the positive reals based on integration of random measures composed of exponential random variables across
an inverse-square surface. The Jacobi theta distribution is single-parameter, positively skewed, and leptokurtic. The cumulative distribution and density functions are expressed in terms of the Jacobi theta function, and asymptotic and log-normal approximations enabling tractable calculations and exact maximum likelihood inference.

7 Acknowledgements

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