Geometric phase shift for detection of gravitational radiation

N V Mitskievich and A I Nesterov
Departamento de Física, Universidad de Guadalajara,
Guadalajara, Jalisco, México
E-mail: nmitskie@cencar.udg.mx, nesterov@cencar.udg.mx

Abstract

An effect of geometrical phase shift is predicted for a light beam propagating in the field of a gravitational wave. Gravitational radiation detection experiments are proposed using this new effect, the corresponding estimates being given.

PACS numbers: 04.80. +z
Keywords: Berry phase, Fermi coordinates, gravitational waves, right and left polarizations of light waves

1 Introduction

M. Berry (1984) has shown that a quantum system whose Hamiltonian \( H(\zeta) \) depends on some parameters \( \zeta \) and which evolves in time in such a way that during the evolution the state of the system traces out a closed curve \( C \) in the space of these parameters, the wave function can get an additional geometrical phase \( \theta(C) \). This geometric phase depends on motion of the system in the space of parameters.

Recently an effect of rotation of the polarization vector was predicted (Chiao and Wu 1986) for a linearly polarized laser beam travelling along a single helically wound optical fiber, the results having been immediately experimentally confirmed and their connection with Berry’s phase (Tomita and Chiao 1986) shown. Later it was found (Cai et al 1990) that this Berry’s phase has in fact a classical origin, and it arises from the intrinsic topological structure of Maxwell’s theory if the Minkowski space-time is considered as a background. If \( k \) is the wave vector of an electromagnetic wave and \( e(k) \) its complex polarization vector, then the condition \( k^2 = \omega^2(k) = \text{const} \) determines a sphere \( S^2 \). The angle \( \theta(k) \) of rotation of \( e(k) \), when it is parallelly transported along a null geodesic with the tangent four-vector \( k \), is defined by

\[
\delta \theta = i e_L(k) \cdot de_R(k),
\]

where

\[
e_R = 2^{-1/2}(e_1 + ie_2), \quad e_L = 2^{-1/2}(e_1 - ie_2), \quad e_L \times e_R = i k/\omega
\]

and \( e = e^{-i\theta} e_R \). The integral angle occurs to be \( \theta = \int \delta \theta \) where integration is performed over the path of \( k \) on \( S^2 \); thus \( \theta \) has a geometric origin. When the curve is a closed path \( C \) on \( S^2 \), the angle \( \theta(C) \) is given by \( \theta(C) = -\Omega(C) \), where \( \Omega(C) \) is the solid angle of the loop \( C \) with respect...
to the center of the sphere. The expression above is essentially a flat space-time expression (see (Cai et al 1990)), however applicable also to general relativity (cf. (Bildhauer 1990)).

In the second example of the Berry’s phase in optical experiments (Pancharatnam 1956), the state space is the Poincaré sphere which describes all possible polarization states of light. For this case the direction of the light propagation is fixed and a cycle of changes in polarization states corresponds to a closed curve on the Poincaré sphere (see e.g. (Bhandari and Manuel 1988, Simon et al 1988)). The phase $\theta(C)$ is known as Pancharatnam phase (Pancharatnam 1956) and given by

$$\theta(C) = -\frac{1}{2} \Omega(C).$$

Here we predict a similar effect of geometrical phase shift for light beams propagating in the field of a gravitational plane wave or pulse of gravitational radiation: 1) cyclically from and (after reflection) to an observer thus being closely related to the Pancharatnam phase; 2) along a circular fiber of radius $R_0$. We show that for a light beam orthogonal to the direction of propagation of a gravitational wave, this phase grows proportionally to $L/\lambda$ where $L$ is distance between the observer and reflecting system and $\lambda$, characteristic wavelength of the gravitational wave packet. In the second case if $\lambda = \pi R_0$, the resonance occurs and the phase shift grows proportionally to $m$, the number of revolutions of light. For a pulse of gravitational radiation, the relative phase shift is proportional to the characteristic amplitude of the pulse in either of these two cases (for preliminary results see (Mitskievich and Nesterov 1995)).

In this paper we use the space-time signature $(+1, -1, -1, -1)$; Greek indices run from 0 to 3 and Latin, from 1 to 3; it is essential to remember these notations when the integration by parts is performed (see (4) and, in the Appendix, (9)).

2 General results

A concise description of this phenomenon can be done using the Newman-Penrose formalism which is applicable to propagation of light in arbitrary media (we are interested here in the cases of (1) a vacuum and (2) an optical fibre, the both in a gravitational field). The real null Newman–Penrose vectors are $l = (k^0, k)$ (tangent to the light world line) and $n$ ($l \cdot n = 1$), the complex ones, $m$ and $\bar{m}$ ($m \cdot \bar{m} = -1$). We shall consider $m = e^{-i\theta} e_R$ to be the circular (right) polarization vector. When light propagates along optical fibers (not necessarily geodesically), the Newman–Penrose description is quite essential for differentiation along $l$, $D = \nabla_l$ being a Newman–Penrose operator. The corresponding equations reduce to

$$Dl = (\epsilon + \bar{\epsilon}) l - \kappa m - \kappa \bar{m},$$

$$Dn = -(\epsilon + \bar{\epsilon}) n + \pi m + \bar{\pi} \bar{m},$$

$$Dm = \pi l - \kappa \bar{m} + (\epsilon - \bar{\epsilon}) m.$$  

Coefficients in the right-hand side of these equations are in general complex functions (a bar denoting complex conjugation), $\epsilon - \bar{\epsilon}$ determining torsion and $\kappa$ curvature of the space-like trajectory of light (an optical fiber); cf. (Penrose and Rindler 1986, p. 169 ff). The change of polarization angle generally reads

$$\theta = i \int e_L \cdot D e_R \, d\eta + \int (\bar{\epsilon} - \epsilon) d\eta.$$
In a vacuum, a light beam propagates geodesically, thus $Dl = 0$, and equation of the polarization vector transport reads $Dm = 0$ (cf. [10]); these two equations yield also $Dn = 0$. For a planar optical fiber, $Dm = \pi l - \kappa n$. Hence in the both cases $\vec{m} \cdot Dm=0$, thus the integration being performed along the light world line $\Gamma$ canonically parametrized by $\eta$. For the left polarization one has to exchange subscripts $L$ and $R$ in (1), or, equivalently, to change the sign in the right-hand side of this equation.

Note that under the parallel transport along the space-like geodesics orthogonal to the observer’s world line $\gamma$, the vector of right (left) polarization $e_R (e_L)$ does not change, but it changes under the transport along $\Gamma$ (the null line of light whose characteristics are measured in course of the proposed experiment). This very fact makes the existence of nontrivial phase $\theta$ essential.

If we intend to consider an experiment of detection of gravitational radiation, it is natural to connect the Newman-Penrose frame with the Fermi coordinates $X^\mu = \delta^\mu_0 s + \delta^\mu_i \xi^i u$ (see e.g. (Manasse and Misner 1963, Misner et al 1973)). Here $s$ is proper time along the observer’s geodesic world line $\gamma$, $u$ being proper length parametrizing the (space-like) geodesic orthogonal to it, with a unit tangent vector $\xi$ on $\gamma$. In fact, $\xi$ describes the direction in which such a space-like geodesic goes, and it possesses only spatial components different from zero (the temporal coordinate $X^0 = T$ is directed along $\gamma$). In Fermi coordinates, components of the corresponding orthonormal tetrad $e^{(\nu)\mu}$ parallelly transported along the spacelike geodesic, are represented as expansions

$$e^{(0)}_{\mu} = \delta^\mu_0 - \frac{1}{2} R^\mu_{ij0} X^i X^j + \cdots,$$
$$e^{(p)}_{\mu} = \delta^\mu_p - \frac{1}{6} R^\mu_{ijp} X^i X^j + \cdots,$$

and the connection coefficients (Christoffel symbols) take the form

$$\Gamma^\mu_{\nu0} = R^\mu_{\nu0} X^i + \cdots,$$
$$\Gamma^\mu_{ij} = -\frac{2}{3} R^\mu_{(ij)k} X^k + \cdots.$$

Quantities of the type of $Q$ are taken on $\gamma$.

The radius of convergence of series is determined by conditions (Manasse and Misner 1963)

$$u_0 \ll \min \left\{ \frac{1}{\left| R^0_{\alpha\gamma\delta\beta} \right|^{1/2}}, \frac{|R^0_{\alpha\gamma\delta\beta}|}{\left| R^0_{\gamma\delta\beta,i} \right|} \right\}. \quad (2)$$

The first condition $u_0 \ll |R^0_{\alpha\gamma\delta\beta}|^{-1/2}$ determines the size of $V$ where the curvature has not yet caused spatial geodesics to cross each other. The second condition defines the domain where the curvature does not change essentially. For instance, for gravitational waves with wavelength $\eta$
the Riemann tensor is \( \sim A \exp(ik_\mu x^\mu)/\lambda^2 \), where \( A \) is the dimensionless amplitude. So equation (2) yields

\[
u_0 \ll \min\{\lambda/\sqrt{A}, \lambda\}.
\]

Generally it is assumed \( A \leq 10^{-18} \). This means that the size of \( V \) is restricted by \( u_0 \ll \lambda \). So the application of Fermi coordinates to the modern experiments may be very restrictive since \( \lambda \) is often supposed being in the order of 300 km. Thus for enlarging the range of validity by a factor \( 1/\sqrt{A} \) (which is about \( 10^9 \) in our example) it is necessary to take into account all derivatives of the Riemann tensor.

Covariant derivatives of the Fermi basis can be described as (see the Appendix)

\[
\nabla_\lambda e_\nu = -\int_0^u R^\mu_\nu_\lambda_\rho \xi^\rho d\tau + \frac{1}{u} \int_0^u d\tau \int_0^\tau \delta \xi^\rho d\tau' + O(R^2)
\]

where integration is performed along a space-like geodesic orthogonal to the Fermi observer world line \( \gamma \). This integral formula obviously includes all derivatives of the Riemann tensor. We shall apply equation (3) for calculation of the phase shift (1).

Using (3) we find from (1)

\[
\theta = -i \int_\Gamma e_L^\mu e_R^\nu \xi^\lambda d\lambda \int_0^u R^\mu_\nu_\lambda_\rho \xi^\rho d\tau + i \int_\Gamma \frac{1}{u} e_L^\mu e_R^\nu \xi^\lambda d\xi \int_0^u d\tau \int_0^\tau R^\mu_\nu_\lambda_\rho \xi^\rho d\tau' + O(R^2).
\]

Integrating by parts we obtain

\[
\theta = -i \int_\Gamma e_L^\mu e_R^\nu \xi^\lambda d\lambda \int_0^u R^\mu_\nu_0_\rho \xi^\rho d\tau + i \int_\Gamma \frac{1}{u} e_L^\mu e_R^\nu \xi^\lambda d\xi \int_0^u d\tau \int_0^\tau R^\mu_\nu_0_\rho \xi^\rho d\tau' + O(R^2).
\]

We consider now a plane weak gravitational wave whose metric tensor is usually written in synchronous coordinates,

\[
ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + h_{ab} dx^a dx^b,
\]

where \( a \) and \( b \) run from 1 to 2 while

\[
h_{ab} = h_{ab}(t-z), \quad h_{22} = -h_{11} = h_+, \quad h_{12} = h_\times.
\]

Using the definition of the Riemann tensor

\[
R^\mu_\nu_\lambda_\sigma = \frac{1}{2}(h^\mu_\nu_\lambda_\rho + h^\mu_\rho_\nu_\lambda - h^\mu_\nu_\rho_\lambda - h^\mu_\nu_\sigma_\rho),
\]

we find that in the linearized theory non-zero components of Riemann tensor are

\[
R_{3a0} = R_{0a0} = -R_{30a0} = \frac{1}{2} h_{ab}, \quad R_{\mu22\nu} = -R_{\mu11\nu}, \quad R_{\mu12\nu} = R_{\mu21\nu}
\]

where dot being derivative with respect to \( t \). Now the equation (4) is readily applicable, and two typical cases emerge: (A) parallelly (antiparallelly) propagating gravitational and light waves, (B) mutually orthogonal waves. Below we study the both cases.
2.1 Parallelly (antiparallelly) propagating gravitational and light waves

We assume that both waves propagate along $z$ axis. One can write $l = l^0(1, 0, 0, \pm 1)$, where $l^0 = dT/d\eta$; the upper sign corresponds to positive direction propagation of light and the lower sign, to negative one. Let us take

$$e_{R\mu} = \frac{1}{\sqrt{2}}(0, 1, i, 0), \quad e_{L\mu} = \frac{1}{\sqrt{2}}(0, 1, -i, 0).$$

Then equation (4) reduces to

$$\theta = -\int_0^T d\tau \int_0^{u(\tau)} R_{120b} \xi^b d\tau' \pm \int_0^T d\tau \int_0^{u(\tau)} R_{123b} \xi^b \tau' d\tau' + O(R^2). \quad (6)$$

Applying (5) we find $\theta = 0$ that means absence of the phase shift.

2.2 Orthogonally propagating gravitational and light waves

Let the gravitational wave propagate along $z$ axis and the light beam, in the $(x, y)$ plane at an angle $\phi$ to $x$ axis. We suppose

$$\xi = (0, \cos \phi, \sin \phi, 0), \quad l = l^0(1, \pm \cos \phi, \pm \sin \phi, 0), \quad l^0 = dT/d\eta, \quad (7)$$

$$e_{R\mu} = \frac{1}{\sqrt{2}}(0, \mp \sin \phi, \pm \cos \phi, i), \quad e_{L\mu} = \frac{1}{\sqrt{2}}(0, \mp \sin \phi, \pm \cos \phi, -i), \quad (8)$$

where the upper sign corresponds to propagation of light from the observer, and the lower one, to the observer. Then the phase shift (4) is given by

$$\theta = \pm \int_0^T d\tau \int_0^{u(\tau)} (R_{3120}(\cos^2 \phi - \sin^2 \phi) + 2R_{3220} \sin \phi \cos \phi) d\tau' + O(R^2) \quad (9)$$

where we used (5). It is convenient to rewrite this equation as

$$\theta = \mp \frac{1}{2} \int_0^T d\tau \int_0^{u(\tau)} (\tilde{h}_x \cos 2\phi + \tilde{h}_+ \sin 2\phi) d\tau + O(R^2). \quad (10)$$

Taking into account that $X^\mu = x^\mu + O(h)$ and $h = h(t - z)$ ($h$ being $h_x$ or $h_\times$), one can write (10) as

$$\theta = \mp \frac{1}{2} \int_0^T (\tilde{h}_x(\tau) \cos 2\phi + \tilde{h}_+(\tau) \sin 2\phi) u(\tau) d\tau + O(R^2). \quad (11)$$

Let us define $\theta = \alpha_+ \sin 2\phi + \alpha_\times \cos 2\phi$, where $\alpha_+$ and $\alpha_\times$ correspond to the two independent polarization modes of gravitational wave (see e.g. Misner et al 1973). Then (for $\alpha_+$ or $\alpha_\times$)

$$\alpha = \mp \frac{1}{2} \int_0^T \tilde{h}(\tau) u(\tau) d\tau + O(R^2), \quad (12)$$

$h$ being $h_+$ or $h_\times$. Integration by parts with $du = \pm d\tau$ yields

$$\alpha = \frac{1}{2} \left( h(\tau) \mp \tilde{h}(\tau) u(\tau) \right)_0^T$$
where, as above, the upper sign corresponds to propagation of (right polarized) light from the observer, and the lower one, to the observer. For the left polarized light the overall sign of the right-hand side of the both last equations should be changed.

Let us consider an experiment when a circularly polarized electromagnetic wave is travelling between the observer and a reflecting system in the \((x, y)\) plane, the gravitational wave propagating in the positive direction of \(z\) axis. If (say, right) polarization does not change in course of reflection, the phase shift is

\[
\Delta \alpha_1 = \frac{1}{2} [h(2T) - h(0) - 2T \dot{h}(T)],
\]

and if polarization changes to the left one,

\[
\Delta \alpha_2 = \frac{1}{2} [2h(T) - h(2T) - h(0)]
\]

(the effects for an initially left polarized beam have inverse signs).

For multiple reflections we have

\[
\Delta \alpha_1(N) = \frac{1}{2} [h((N + 1)T) + h(NT) - h(T) - h(0) - 2T \sum_{m=1}^{N} \dot{h}(mT)]
\]

\[
\Delta \alpha_2(N) = \frac{1}{2} [h(NT) - h((N + 1)T) + h(T) - h(0)],
\]

where \(N\) is the number of reflections.

For a plane monochromatic gravitational wave, \(h(t - z) = A \cos(\omega(t - z) + \delta)\) where \(A\) is a dimensionless amplitude. In this case \((15), (16)\) are rewritten as

\[
\Delta \alpha_1(N) = A(\Lambda - \sin \Lambda) \frac{\sin(N \Lambda/2) \sin(N + 1 \Lambda + \delta)}{\sin(\Lambda/2)},
\]

\[
\Delta \alpha_2(N) = 2A \sin(\Lambda/2) \sin(N \Lambda/2) \cos(N + 1/2 \Lambda + \delta).
\]

Here \(\Lambda = 2\pi L/\lambda\), \(L\) being distance between the observer and reflecting system, and \(\lambda\) the gravitational wavelength.

The average relative phase shift between the right and left polarized light, is

\[
\sqrt{\langle \Delta \theta_1^2 \rangle} = A \left| (\Lambda - \sin \Lambda) \frac{\sin(N \Lambda/2)}{\sin(\Lambda/2)} \right|,
\]

\[
\sqrt{\langle \Delta \theta_2^2 \rangle} = 2A \left| \sin(\Lambda/2) \sin(N \Lambda/2) \right|,
\]

\(A = \sqrt{A_+^2 + A_-^2}\) being the dimensionless amplitude of an unpolarized gravitational wave, and averaging is performed with respect to all polarizations and phases \(\delta_+\) and \(\delta_-\).

We shall consider here no phase change in course of reflection since this is the only way to obtain a sensible integral effect. If \(\Lambda \gg 1\), then, for instance for \(N = 1\), we obtain

\[
\sqrt{\langle \Delta \theta^2 \rangle} = A\Lambda.
\]
We see that effectively the dimensionless amplitude $A$ grows by a factor $\Lambda$. When $N \gg 1$, we find that maximum of phase shift

$$\sqrt{\langle \Delta \theta^2 \rangle} = 2\pi m N A$$

occurs for $\Lambda = 2\pi m$ where $m$ is integer. If $\Lambda \ll 1$, we obtain

$$\sqrt{\langle \Delta \theta^2 \rangle} = 2A \left| \sin \left( \frac{N\Lambda}{2} \right) \right|,$$  (23)

and relative shift takes its maximum value,

$$\sqrt{\langle \Delta \theta^2 \rangle} = 2A,$$  (24)

when $N\Lambda = \pi$.

Another possible experiment involves a scheme similar to that of (Braginski and Menskii 1971) but with measurement of the geometric phase shifts for right and left polarizations (and not the frequency shift as in [15]) of light travelling along a circular fiber of radius $R_0$. Let us consider the case when a plane gravitational wave is propagating in the positive direction of $z$ axis, the fiber lies in $(x, y)$ plane and the light propagates in counter-clockwise direction. We suppose

$$\xi = (0, \cos \phi, \sin \phi, 0), \quad l = l^0 (1, -\sin \phi, \cos \phi, 0), \quad \dot{t}^0 = dT/d\eta,$$  (25)

$$e_R^\mu = \frac{1}{\sqrt{2}} (0, \cos \phi, \sin \phi, i), \quad e_L^\mu = \frac{1}{\sqrt{2}} (0, \cos \phi, \sin \phi, -i).$$  (26)

Then the phase shift is given by

$$\theta = R^0 \int_0^T R_{3ab} \xi^a (\tau) \xi^b (\tau) d\tau + O(R^2) = -\frac{R_0}{2} \int_0^T \dot{h}_{ab} (\tau) \xi^a (\tau) \xi^b (\tau) d\tau + O(R^2),$$  (27)

where we used equation (5). Defining $\theta = \alpha_+ + \alpha_\times$ we obtain

$$\alpha_+ = \frac{R_0}{2} \int_0^T \dot{h}_+ (\tau) \cos (2\omega_0 \tau) d\tau + O(R^2);$$

$$\alpha_\times = -\frac{R_0}{2} \int_0^T \dot{h}_\times (\tau) \sin (2\omega_0 \tau) d\tau + O(R^2),$$

where $\omega_0 = 2\pi/T_0$, and $T_0$ is the period of revolution of light. Integration by parts with $h(\tau) = A \cos(\omega \tau + \delta)$ where, as above, $A$ is dimensionless amplitude, yields

$$\alpha_+ = A_+ \left( \frac{\omega}{2\omega_0} \right) \left[ \frac{\sin((\omega_0 - \omega/2)T) \cos((\omega_0 - \omega/2)T - \delta_+) - \sin((\omega_0 + \omega/2)T) \cos((\omega_0 + \omega/2)T - \delta_+)}{1 - 2(\omega_0/\omega)} \right];$$

$$\alpha_\times = A_\times \left( \frac{\omega}{2\omega_0} \right) \left[ \frac{\sin((\omega_0 + \omega/2)T) \sin((\omega_0 + \omega/2)T + \delta_\times) - \sin((\omega_0 - \omega/2)T) \sin((\omega_0 - \omega/2)T - \delta_\times)}{1 + 2(\omega_0/\omega)} \right].$$
For left polarized light, the overall sign of the right-hand side of the both last equations should be changed. The average relative phase shift between right and left polarized light, is

\[
\sqrt{< \Delta \theta^2 >} = \frac{A \omega}{2\sqrt{2} \omega_0} \left[ \sin^2\left(\frac{\omega_0 + \omega/2}{2}\right)T \right]
\]

\[
+ 2 \frac{\sin(\omega_0 - \omega/2)T \sin(\omega_0 + \omega/2)T \cos(2\omega T)}{1 - (2\omega_0/\omega)^2} + \frac{\sin^2(\omega_0 - \omega/2)T}{(1 - 2(\omega_0/\omega))^2} \right]^{1/2}.
\]

(30)

When the gravitational wavelength \( \eta \) is equal to \( \pi R_0 \), a resonance occurs leading to \( \sqrt{< \Delta \theta^2 >} = \sqrt{2} \pi mA \) where \( m \) is the number of revolutions of light. It is clear that such a detector is effective for high frequency gravitational radiation, the corresponding \( R_0 \) being around 100 km for \( 10^3 \) Hz. A considerably smaller size of detector could be achieved for a toroidal winding of the fiber, this case being currently under consideration. However measurements of the phase shift in a fiber (and in any media other than a good vacuum) could be virtually impossible due to the random fluctuations, so that we consider here these cases to the end of completeness only.

### 3 Discussions and conclusions

We would like to discuss the possibility of detecting gravitational radiation based on the proposed new effect. Let us consider the case \( \Lambda \gg 1 \). From (21) we obtain \( \sqrt{< \Delta \theta^2 >} \approx 2 \cdot 10^{-5} AL\nu \), where \( \nu \) is characteristic frequency of the gravitational radiation in Hz and \( L \) is the distance in km. It is clear that such a detector is effective for high frequency gravitational radiation \( (\nu \sim 10^4 \text{ Hz}) \). If, for instance, the reflecting system is placed on the surface of moon, we have \( \sqrt{< \Delta \theta^2 >} \sim 8A\nu \). For the 5-million-kilometer-long Laser Interferometer Space Antenna (LISA), which would fly in heliocentric orbit (see Thorn 1995a,b), our estimation of phase shift is \( \sqrt{< \Delta \theta^2 >} \sim 10^2 A\nu \).

Let us compare the experiment proposed here, with the standard experiments involving resonant antennae directed to the Virgo cluster and tuned to some 3000 Hz. In this case \( A \sim 10^{-20} \) (see the corresponding data in (Douglas and Braginsky 1979, Thorne 1995a,b)). So the geometrical phase detector with the base Earth – Moon treats this radiation as if it had an effective magnitude of some \( 10^{-16} \) and for LISA as if it had an effective magnitude of some \( 10^{-14} \).

For experiments using a laboratory size apparatus, \( \Lambda \ll 1 \), we see that \( \sqrt{< \Delta \theta^2 >} \sim 2A |\sin(N\Lambda/2)| \), and the relative shift takes its maximum value when \( N\Lambda = \pi \). For LIGO/VIRGO \( (L \sim 3 \cdot 10^5 \text{ cm}) \) interferometers we find that the correspondent frequency of the gravitational wave is \( \nu \sim 3 \cdot 10^4/N \). It is known that for \( L \approx 10^5 \text{ cm} \) one could expect about 300 reflections (Douglas and Braginsky 1979) which is sufficient for detection of continuous waves with \( 100 \text{ Hz} < \nu < 10^4 \text{ Hz} \). The advanced LIGO interferometers are expected to have their optimal sensitivity at \( \nu \sim 100 \text{ Hz} \), and rather good sensitivity all the way from \( \nu \sim 10 \text{ Hz} \) to \( \nu \sim 500 \text{ Hz} \) (Thorne 1995a,b).

If we are interested in detection of a burst of gravitational radiation with \( L \ll \lambda \) (let its maximum be at the moment of time \( N\tau/2 \) and characteristic duration \( \tau = N\tau \)), then (15) yields \( \sqrt{< \Delta \theta^2 >} \sim A \) for \( N \gg 1 \) or \( N = 1 \), where \( A \) is characteristic dimensionless amplitude of the pulse. Realistic estimates for \( L \) and \( N \) (LIGO/VIRGO) are \( L \sim 3 \cdot 10^5 \text{ cm} \), \( 10 < N < 300 \), while the characteristic frequency is \( 100 \text{ Hz} < \nu < 10^4 \text{ Hz} \).
Measuring of this effect consists of a comparison of interference patterns for the both circulary polarizations of a light beam. Similar measurements of a phase shift between opposite senses of circularly polarized light were performed using a nonplanar Mach–Zehnder interferometer (Chiao et al 1988); a very clear exposition of theoretical and experimental details concerning observation of phase shifts due to geometrical and topological effects, see in (Chiao 1990), naturally, without dealing with any gravitational effects.

It is worth making a comment on the order of smallness of the dimensionless amplitude of the gravitational wave $A$ in connection with the limits of short gravitational waves. This order is most invariantly attributed to the space-time curvature which is calculated using the metric tensor, see (5): $\text{Riem} \sim A/\lambda^2$. Thus when $\lambda \to 0$, $A$ should also tend to zero if the curvature keeps its order of magnitude unchanged, and in the limit of short gravitational wavelengths (geometric optics in the gravitational sense) the apparent divergence of the predicted effect at small $\lambda$’s (see equations (17), (21)), is merely spurious: $\sqrt{\langle \Delta \theta^2 \rangle} \sim \text{Riem} \lambda L$.

The effect we predict in this paper, makes it in principle possible to detect gravitational waves using not an interferometer as a whole, but only one of its arms, since there is a fundamental difference in propagation of the left and right polarizations along one and the same null line of the light. One has merely to separate the light of these different polarizations after it has returned from its travel, then to transform the (circular) polarization of one of the resulting beams to the opposite one, and finally to observe the interference fringes after mixing the beams.

We think that this effects reflects an interaction between photon’s spin and the space-time curvature, which is closely related to the well-known Papapetrou-Mathisson effect.

Acknowledgments

This essay was selected for an Honorable Mention by the Gravity Research Foundation, 1994. The work was supported by CONACYT Grant No. 1626P-E9507.

Appendix

Here we shall obtain the formula (equation (3) in the text)

$$\nabla_\sigma e(\nu) = - \int_0^u R^\mu_{\nu\lambda\rho} \xi^\rho d\tau + \frac{1}{u} \int_0^u d\tau \int_0^\tau d\tau' \delta_i^j R^\mu_{\nu i\lambda\rho} \xi^\rho + O(R^2).$$

Let us start with the Taylor expansion for tetrad in a world tube surrounding the world line $\gamma$ of inertial Fermi observer:

$$e(\nu) = \delta^\mu_\nu + \frac{d e(\nu)}{du} u + \frac{1}{2!} \frac{d^2 e(\nu)}{du^2} u^2 + \frac{1}{3!} \frac{d^3 e(\nu)}{du^3} u^3 + \cdots,$$

$u$ being the canonical parameter along spacelike geodesics orthogonal to $\gamma$. Using in Fermi coordinates the equation of parallel propagation

$$\frac{de(\nu)}{du} + \Gamma^\mu_{\nu\lambda} e(\nu)^\lambda \xi^i = 0,$$

we rewrite equation (1) as

$$e(\nu) = \delta^\mu_\nu - \frac{1}{2!} \Gamma^\mu_{\nu i,l} X^i X^l - \frac{1}{3!} \Gamma^\mu_{\nu i,l,p} X^i X^l X^p + \cdots + O(R^2).$$
The expansion of the connection coefficients is given by

\[ \Gamma^\mu_{\nu\lambda} = \Gamma^\mu_{\nu\lambda;i} X^i + \frac{1}{2!} \Gamma^\mu_{\nu\lambda,i,l} X^i X^l + \cdots. \]  

(34)

Applying equations (3), (4) we obtain the following series for the spatial covariant derivatives of tetrad

\[ \nabla \partial_i e^\mu_{(\nu)} = 0 \Gamma^\mu_{\nu i,k} X^k + 0 \Gamma^\mu_{\nu(i,k)} X^k + \frac{1}{2!} (0 \Gamma^\mu_{\nu i,k,l} - 0 \Gamma^\mu_{\nu(i,k,l)}) X^k X^l + \cdots, \]  

(35)

Now using the relations

\[ \frac{1}{2} 0 R^\mu_{\nu ki} X^k = (0 \Gamma^\mu_{\nu i,k} - 0 \Gamma^\mu_{\nu(i,k)}) X^k, \]

\[ \frac{2}{3} 0 R^\mu_{\nu ki,l} X^k X^l = (0 \Gamma^\mu_{\nu i,k,l} - 0 \Gamma^\mu_{\nu(i,k,l)}) X^k X^l, \]

\[ \frac{3}{4} 0 R^\mu_{\nu ki,l,m} X^k X^l X^m = (0 \Gamma^\mu_{\nu i,k,l,m} - 0 \Gamma^\mu_{\nu(i,k,l,m)}) X^k X^l X^m, \] etc

one can write the expansion (5) as

\[ \nabla \partial_i e^\mu_{(\nu)} = \frac{1}{2} 0 R^\mu_{\nu ki} X^k + \frac{2}{3} 0 R^\mu_{\nu ki,l} X^k X^l + \frac{3}{4} 0 R^\mu_{\nu ki,l,m} X^k X^l X^m + \cdots \]

\[ + \frac{n}{(n+1)!} 0 R^\mu_{\nu l_1 l_2 \cdots l_n} X^{l_1} X^{l_2} \cdots X^{l_n} + \cdots + \mathcal{O}(R^2). \]  

(36)

It is convenient to present this series in the form

\[ \nabla \partial_i e^\mu_{(\nu)} = \sum_{n=0}^{\infty} \frac{d^n}{du^n} \left( R^\mu_{\nu ki} \xi^k \right) \frac{u^{n+1}}{(n+2)(n+1)!} + \mathcal{O}(R^2) \]  

(37)

Straightforward calculation yields the following integral representation of equation (7):

\[ \nabla \partial_i e^\mu_{(\nu)} = -\frac{1}{u} \int_0^u R^\mu_{\nu ik} \xi^k d\tau + \mathcal{O}(R^2). \]  

(38)

Integrating by part we find

\[ \nabla \partial_i e^\mu_{(\nu)} = -\int_0^u R^\mu_{\nu ik} \xi^k d\tau + \frac{1}{u} \int_0^u d\tau \int_0^\tau d\tau' R^\mu_{\nu ik} \xi^k + \mathcal{O}(R^2). \]  

(39)

Now let us calculate the temporal covariant derivative \( \nabla \partial_0 e^\mu_{(\nu)} \). From equations (2)–(4) we easily obtain

\[ \nabla \partial_0 e^\mu_{(\nu)} = 0 \Gamma^\mu_{\nu 0,k} X^k + \frac{1}{2!} (0 \Gamma^\mu_{\nu 0,k,l} - 0 \Gamma^\mu_{\nu k,0,l}) X^k X^l + \frac{1}{3!} (0 \Gamma^\mu_{\nu 0,k,l,m} - 0 \Gamma^\mu_{\nu k,0,l,m}) X^k X^l X^m + \cdots + \mathcal{O}(R^2), \]  

(40)
where we have taken into account that $\Gamma^\mu_{\nu0} = 0$. Obviously,

$$
\nabla_\partial e_{(\nu)}^\mu = \frac{0}{2!} \frac{0}{3!} \frac{0}{n!} \not{R}^{\mu \nu} \not{X}^k \not{X}^l \not{X}^m + \ldots
$$

or

$$
\nabla_\partial e_{(\nu)}^\mu = \sum_{n=0}^{\infty} \frac{d^n}{du^n} \left( \frac{0}{n+1} \not{R}^{\mu \nu} \not{\xi}^k \right) du^n + O(R^2),
$$

(41)

The correspondent integral representation reads

$$
\nabla_\partial e_{(\nu)}^\mu = - \int_0^u \not{R}^{\mu \nu} \not{\xi}^k d\tau + O(R^2).
$$

(42)

Finally one can combine equations (9), (13) and write

$$
\nabla_\partial e_{(\nu)}^\mu = - \int_0^u \not{R}^{\mu \nu} \not{\xi}^p d\tau + \frac{1}{u} \int_0^u d\tau \int_0^{\tau'} d\tau' \delta^\lambda_{\gamma} \not{R}^{\mu \nu} \not{\xi}^p + O(R^2).
$$

(44)

Noting that the transformation from arbitrary coordinate system to the Fermi one takes the form $x^\mu = \Lambda^\mu_\nu X^\nu + O(\Gamma)$ (or $x^\mu = \xi^\mu u + O(\Gamma)$ ) one can write Eq(14) in an arbitrary coordinate system as

$$
\nabla_\partial e_{(\nu)}^\mu = - \int_0^u \not{R}^{\sigma \gamma \delta \rho} (\Lambda^{-1})^{\mu \sigma} \Lambda^\gamma_\nu \Lambda^\delta_\lambda \not{\xi}^\rho d\tau
$$

$$
+ \frac{1}{u} \int_0^u d\tau \int_0^{\tau'} d\tau' \not{R}^{\sigma \gamma \delta \rho} (\Lambda^{-1})^{\mu \sigma} \Lambda^\gamma_\nu \Lambda^\delta_\lambda \not{\xi}^\rho + O(R^2).
$$

(45)
References

Berry M V 1984 *Proc. Roy. Soc.* A **392** 45

Bhandari R and Manuel J 1988 *Phys. Rev. Lett.* **60** 1210

Bildhauer S 1990 *Class. Quantum Gravity* **7** 2367

Braginsky V B and Menskii M B 1971 *Sov. Phys. — JETP Lett.* **13**, 417

Cai Y Q, Papini G and Wood W R 1990 *On Berry’s Phase for Photons and Topology in Maxwell’s Theory* (University of Regina Preprint)

Chiao R Y and Wu J S 1986 *Phys. Rev. Lett.* **51** 933

Chiao R Y, Antaramian A, Ganga K M, Jiao H, Wilkinson S R and Nathel H 1988 *Phys. Rev. Lett.* **60**, 1214

Douglass D H and Braginsky V B 1979 *Gravitational-radiation experiments*, in *General Relativity. An Einstein Centenary Survey*, edited by S W Hawking and W Israel (Cambridge University Press, Cambridge).

Manasse F K and Misner C W 1963 *J. Math. Phys.* **4** 735

Misner C W, Thorne K S and Wheeler J A 1973 *Gravitation* (W.H. Freeman, San Francisco).

Mitskievich N V and Nesterov A I 1995 *Gen. Relat. Grav.* **27** 361

Pancharatnam S 1956 *Proc. Ind. Acad. Sci. Ser. A* **44** 247

Simon R D, Kimble H J and Sudarshan E C 1988 *Phys. Rev. Lett.* **61** 19

Penrose R and Rindler W 1986 *Spinors and Space-time, Vol. 2* (Cambridge University Press, Cambridge)

Thorne K S 1995a *Gravitational waves*, Preprint [gr-qc/9506086](http://arxiv.org/abs/gr-qc/9506086)

Thorne K S 1995b *Gravitational waves from compact bodies*, Preprint [gr-qc/9506084](http://arxiv.org/abs/gr-qc/9506084)

Tomita A and Chiao R Y 1986 *Phys. Rev. Lett.* **51** 937