From conjugacy classes in the Weyl group to semisimple conjugacy classes

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To Bert Kostant with admiration

Abstract: Suppose $G$ is a connected complex semisimple group and $W$ is its Weyl group. The lifting of an element of $W$ to $G$ is semisimple. This induces a well-defined map from the set of elliptic conjugacy classes of $W$ to the set of semisimple conjugacy classes of $G$. In this paper, we give a uniform algorithm to compute this map. We also consider the twisted case.

Keywords: Algebraic groups, Weyl groups, elliptic conjugacy classes, semisimple conjugacy classes.

1. Introduction

Let $G$ be a connected complex semisimple group. Choose a Cartan subgroup $T \subset G$, let $N(T) = N_G(T)$ be the normalizer of $T$ in $G$, and let $W = N(T)/T$ be the Weyl group. We have the exact sequence

$$1 \rightarrow T \rightarrow N(T) \xrightarrow{p} W \rightarrow 1.$$  

(1.1)

In [2] we considered the question of whether this sequence splits, and more generally if $w \in W$, what can be said about the orders of elements of $p^{-1}(w)$. Here we consider a related problem.

Let $[W]$ be the set of conjugacy classes in $W$. Let $G^{ss}$ be the semisimple elements of $G$, and $[G^{ss}]$ the set of semisimple conjugacy classes. We have $N(T) \subset G^{ss}$. For $w \in W$ write $[w]$ for the $W$-conjugacy class of $w \in W$. Similarly for $g \in G^{ss}$ let $[g]$ be the $G$-conjugacy class of $g$. 

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Suppose $w \in W$ and $n_w \in p^{-1}(w) \subset N(T)$. For general $w$ there are many choices of $n_w$, even up to conjugacy. However in an important special case the choice of $n_w$ is unique up to conjugation. We say $w \in W$ is elliptic if it has no nontrivial fixed vectors in the reflection representation. It is well known that if $w$ is elliptic then any two elements of $p^{-1}(w) \subset N(T)$ are $T$-conjugate (see Lemma 1.1.3).

**Definition 1.2.** Suppose $w \in W$ is elliptic. Choose $n_w \in p^{-1}(w) \subset N(T)$, and define

$$
\Psi : [W^e] \to [G^{ss}] : [w] \to [n_w].
$$

This map is well defined on the level of conjugacy classes by Lemma 1.1.3.

This map has been studied by many people, including Reeder [20] and Reeder-Levy-Yu-Gross [12], for applications to representations of $p$-adic groups and number theory. For the exceptional groups the map $\Psi$ has been computed using case-by-case calculations: the papers [17], [4] and [12] cover all cases, with some overlaps, and include other information. The main result of this paper is a uniform algorithm to compute the map $\Psi$ which is free of any case-by-case considerations.

We give this algorithm in the next section, after first discussing the twisted case. We have implemented the algorithm in the *Atlas of Lie groups and Representations* software [1] (see the file *weyltosemisimple.at*). It can be used to compute $\Psi$ for any semisimple group, and Section 9 includes complete tables for the (twisted and untwisted) exceptional groups.

**1.1. The twisted case**

A pinning is a triple $(G, T, \{X_\alpha \mid \alpha \in \Pi\})$, where $\Pi$ is a set of simple roots of $T$ in $G$, and for each $\alpha \in \Pi$, $X_\alpha$ is an $\alpha$-weight vector. We say an automorphism of $G$ is distinguished if it preserves some pinning. An inner automorphism is distinguished if and only if it is trivial as any two pinnings are conjugate by a unique inner automorphism, and the outer automorphism group of $G$ is isomorphic to the group of automorphisms of a pinning.

Suppose $\Pi$ is a fixed set of simple roots, and $(G, T, \{X_\alpha \mid \alpha \in \Pi\})$ is a pinning. Suppose $\delta$ is an automorphism of $G$, preserving the pinning. Then $\delta$ induces an automorphism of $W$, which we also denote by $\delta$. We define

$$
\delta G = G \rtimes \langle \delta \rangle, \quad \delta W = W \rtimes \langle \delta \rangle.
$$

Let $N_{\delta G}(T)$ be the normalizer of $T$ in $\delta G$. Then $\delta W \simeq N_{\delta G}(T)/T$, and we write $p : N_{\delta G}(T) \to \delta W$. 

Let $\Delta = \Delta(T, G)$ be the root system, and $V = \mathbb{Q}(\Delta)$. This is a representation of $\delta W$ which we refer to as the reflection representation. We say $y \in \delta W$ is elliptic if it has no nontrivial fixed vectors in the reflection representation. Write $[y]$ for the $\delta W$-conjugacy class of $y$. We consider elements $y \in W\delta$, and write $[(W\delta)^e]$ for the $\delta W$-conjugacy classes of elliptic elements in $W\delta$.

From the identity $\delta^{-1}(\delta w)\delta = w(\delta w)w^{-1}$ it follows that two elements of $W\delta$ are $\delta W$-conjugate if and only if they are $W$-conjugate.

We say an element $y$ of $\delta G$ is semisimple if $\text{Ad}(y)$ is semisimple. Write $[y]$ for the $\delta G$-conjugacy class of $y \in \delta G$. We consider elements $y \in G\delta$, and write $[(G\delta)^{ss}]$ for the $G$-conjugacy classes of semisimple elements in $G\delta$. As in the case of $\delta W$, two elements of $G\delta$ are $\delta G$-conjugate if and only if they are $G$-conjugate.

**Lemma 1.1.3.** Suppose $w \in W\delta$ is elliptic. Then any two elements of $p^{-1}(w) \subset N_{\delta G}(T)$ are $T$-conjugate.

**Proof.** The result is known, e.g. see [7, Remark 4.1.1]. We include a proof for completeness.

Suppose $g \in p^{-1}(w) \subset N(T)\delta$. Then for $t \in T$,

$$tg(t)^{-1} = tw(t^{-1})g.$$  

Since $w$ is elliptic and $G$ is semisimple, the map $t \to tw(t^{-1})$ has finite kernel, so is surjective.

Now suppose $g_1, g_2 \in p^{-1}(w)$. Since $w \in W\delta$, $g_1, g_2 \in G\delta$. Write $g_1 = h_1\delta, g_2 = h_2\delta$ with $h_1, h_2 \in G$. By the previous discussion choose $t$ so that $tw(t^{-1}) = h_2h_1^{-1}$. Then $tg_1t^{-1} = g_2$. \hfill $\Box$

**Definition 1.1.4.** Suppose $w \in W\delta$ is elliptic. Choose $n_w \in p^{-1}(w) \subset N(T)\delta$, and define

$$\Psi : [(W\delta)^e] \to [(G\delta)^{ss}] : [w] \to [n_w].$$

This map is well defined on the level of conjugacy classes by Lemma 1.1.3.

If $w \in W\delta$ is not elliptic then the lifts $n_w$ are not all $G$-conjugate. Nevertheless by realizing $w$ as an elliptic element in a Levi subgroup one can define a canonical map from $[(W\delta)^e]$ to $[(G\delta)^{ss}]$. See Section 8.

We’ve stated the result over $\mathbb{C}$. The algorithm applies over any algebraically closed field of characteristic 0, and in general with a weak restriction on the characteristic. See Remark 2.7.
2. The algorithm

Assume $G$ is a semisimple reductive group defined over $\mathbb{C}$, and $\delta$ is an automorphism of $G$ of finite order. We fix a pinning $(G, T, \{X_\alpha \mid \alpha \in \pi\})$ that is preserved by $\delta$. We write $\Delta$ for the root system and $V = \mathbb{Q}\langle \Delta \rangle$. Write $\alpha^\vee$ for the coroot associated to $\alpha \in \Delta$, $\Delta^\vee$ the set of coroots, and $V^\vee = \mathbb{Q}\langle \Delta^\vee \rangle$. Write $\langle , \rangle$ for the canonical pairing between $V^\vee$ and $V$.

Suppose $w \in \delta W$. Let

\begin{enumerate}[(2.1)(a)]
  \item $\Gamma_w = \{0 \leq \theta \leq \pi \mid \text{such that } e^{i\theta} \text{ is an eigenvalue of } w \text{ on } V_\mathbb{C}\}$
\end{enumerate}

Write

\begin{enumerate}[(2.1)(b)]
  \item $\Gamma_w = \{\theta_1, \theta_2, \ldots, \theta_k\}$ with $0 \leq \theta_1 < \theta_2 < \cdots < \theta_k \leq \pi$.
\end{enumerate}

For $\theta \in \Gamma_w$ let

\begin{enumerate}[(2.1)(c)]
  \item $V(w, \theta) = \{v \in V_\mathbb{R} \mid w(v) + w^{-1}(v) = 2 \cos(\theta)v\}$.
\end{enumerate}

Then $V(w, \theta)_\mathbb{C}$ is the direct sum of the eigenspaces of $w$ on $V_\mathbb{C}$ with eigenvalues $e^{\pm i\theta}$.

For $1 \leq i \leq k$ set

$$F_i = \sum_{j=1}^{i} V(w, \theta_j)$$

and set $F_0 = 0, \theta_0 = 0$. This gives a filtration

\begin{enumerate}[(2.1)(d)]
  \item $0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k = V$
\end{enumerate}

with strict containments. For $0 \leq i \leq k$, set

\begin{enumerate}[(2.1)(e)]
  \item $\Delta_i = \{\alpha \in \Delta \mid \langle \alpha^\vee, F_i \rangle = 0\}$.
\end{enumerate}

Then each $\Delta_i$ is a root system, and set

\begin{enumerate}[(2.1)(f)]
  \item $W_i = W(\Delta_i)$.
\end{enumerate}

Thus we have

\begin{enumerate}[(2.1)(g)]
  \item $\Delta = \Delta_0 \supseteq \Delta_1 \supseteq \cdots \supseteq \Delta_k = \emptyset$, $W = W_0 \supseteq W_1 \supseteq \cdots \supseteq W_k = \{1\}$.
\end{enumerate}
By [14, Lemma 5.1] after conjugating by $W$ we may assume that all the $\Delta_i$ are standard, i.e., the corresponding Levi subgroups are standard Levi subgroups.

For each $i$ set

$$\Delta_i^+ = \Delta^+ \cap \Delta_i,$$

(2.1)(h)

$$\rho_i^\vee = \frac{1}{2} \sum_{\alpha \in \Delta_i^+} \alpha^\vee.$$ We define rational coweights $\{\lambda_0^\vee, \ldots, \lambda_k^\vee\}$ by downward induction. Set $\lambda_k^\vee = 0$, and for $0 \leq j \leq k - 1$ define

(2.1)(i) $$\lambda_j^\vee = \frac{d(\theta_{j+1} - \theta_j)}{2\pi} \rho_j^\vee + \lambda_{j+1}^\vee,$$

where $d$ is the order of $w$, and $\lambda_{j+1}^\vee$ is the element in the $W_j$-orbit of $\lambda_j^\vee$ which is dominant for $\Delta_j^+.$

**Theorem 2.2.** Suppose $w \in W\delta$ is of order $d$. Construct $\lambda_0^\vee$ by the algorithm. Then some lift $n_w$ of $w$ is $G$-conjugate to $\exp(2\pi \sqrt{-1} \lambda_0^\vee / d) \delta$.

In particular if $w \in (W\delta)^e$ then

(2.3) $$\Psi(|w|) = [\exp(2\pi \sqrt{-1} \lambda_0^\vee / d) \delta].$$

**Remark 2.4.** The algorithm uses the full sequence $(\theta_1, \ldots, \theta_k)$. One may replace the full sequence by any subsequence that is admissible in the sense of [14, subsection 5.2]. The proof of the Theorem applies with no change in this generality. The element $\exp(2\pi \sqrt{-1} \lambda_0^\vee / d) \delta \in \delta G$ obtained from an admissible subsequence of $(\theta_1, \ldots, \theta_k)$ is, in general, different from the element obtained here using the full sequence.

We will use the following well-known conjugacy result [21, Chapter 13, Prop. 2.5]. We include a proof since it plays a role in the subsequent Proposition.

**Proposition 2.5.** Suppose $W$ is a finite group all of whose characters take values in $\mathbb{Q}$. Suppose $w \in W$ has order $d$. If $(d, k) = 1$, then $w^k$ is conjugate to $w$.

It is well known that all representations of Weyl group take integral values [22, Theorem 8.5].
Proof. Let $F = \mathbb{Q}(\zeta_d)$ where $\zeta_d$ is a primitive $d^{th}$ root of unity. If $\pi$ is a representation of $W$, with character $\theta_{\pi}$, then

$$\theta_{\pi}(w) = \sum_{i=1}^{r} z_i,$$

where each $z_i$ is a $d^{th}$ root of unity in $F$. Then

$$\theta_{\pi}(w^k) = \sum_{i=1}^{r} z_i^k.$$

Since $(k, d) = 1$ the map $\zeta_d \to \zeta_d^k$ induces an automorphism $\tau$ of $F/\mathbb{Q}$. By assumption $\sum z_i \in \mathbb{Q}$, so

$$\theta_{\pi}(w^k) = \sum_{i=1}^{r} z_i^k = \sum_{i=1}^{r} \tau(z_i) = \tau \left( \sum_{i=1}^{r} z_i \right) = \sum_{i=1}^{r} z_i = \theta_{\pi}(w).$$

Since the characters separate conjugacy classes this implies $w$ is conjugate to $w^k$.

We need a slight generalization of this.

Proposition 2.6. Suppose $W$ is as in the previous Proposition, and $\delta$ is an automorphism of $W$ of finite order. Let $\delta W = W \rtimes \langle \delta \rangle$. Suppose $w \in W \delta \subset \delta W$ has order $d$. Suppose $(k, d) = 1$ and $w^k$ is also contained in $W \delta$. Then $w^k$ is conjugate to $w$.

Proof. Let $m$ be the order of $\delta$. By Clifford theory the characters of $\delta W$ are defined over $\mathbb{Q}(\zeta_m)$ where $\zeta_m$ is a primitive $m^{th}$ root of 1. Replace $\mathbb{Q}$ with $E = \mathbb{Q}(\zeta_m)$ in the proof of the previous Proposition, and consider the field $E(\zeta_d)$. The condition $w^k \in W \delta$ implies $k = 1 \pmod{m}$, so $\zeta_m^k = \zeta_m$ and the map $\zeta_d \to \zeta_d^k$ induces an automorphism of $E(\zeta_d)/E$. The rest of the proof goes through with minor changes.

Remark 2.7. Suppose $F$ is an algebraically closed field and $G$ is a connected semisimple algebraic group over $F$. Assume furthermore that the order of $\delta W$ is invertible in $F$. Then all of the elements of $N_G(T)$ are semisimple and a version of Theorem 2.2 holds in this setting. The only issue is to make sense of the right hand side of (2.3) over $F$. 

Suppose $\lambda_0^\vee$ is constructed as in the algorithm. Choose $m \in \mathbb{Z}$ so that $m\lambda_0^\vee \in X_*$, and choose a primitive $dm^{th}$ root of unity $\zeta_{dm} \in F$. Then

$$\Psi([w]) = [\delta(m\lambda_0^\vee)(\zeta_{dm})]$$

where we view $m\lambda_0^\vee$ as a one-parameter subgroup corresponding to $m\lambda_0^\vee : F^\times \to T(F)$. By Proposition 2.6 this is independent of the choices of $m$ and $\zeta_{dm}$.

3. Digression on good elements

The algorithm in Section 2 is motivated in part by the construction of the good elements in a given conjugacy class of $\delta W$. The good elements in $W$ were introduced by Geck and Michel in [10] and the notion was generalized to $\delta W$ by Geck, Kim and Pfeiffer in [9]. In this section we discuss this construction.

Let $B^+$ be the braid monoid associated with $(W,S)$. There is a canonical injection $j : W \to B^+$ identifying the generators of $W$ with the generators of $B^+$ and satisfying $j(w_1w_2) = j(w_1)j(w_2)$ for $w_1, w_2 \in W$ whenever $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$.

Now the automorphism $\delta$ induces an automorphism of $B^+$, which is still denoted $\delta$. Set $\delta B^+ = B^+ \rtimes \langle \delta \rangle$. Then $j$ extends in a canonical way to an injection $\delta W \to \delta B^+$, which we still denote by $j$. We will simply write $w$ for $j(w)$.

By definition, $w \in \delta W$ is a good element if there exists a strictly decreasing sequence $\Pi_0 \supseteq \Pi_1 \supseteq \cdots \supseteq \Pi_l$ of subsets of $\Pi$ and even positive integers $d_0, \cdots, d_l$ such that

$$w^d = w_0^{d_0} \cdots w_l^{d_l}.$$ 

Here $d$ is the order of $w$ and $w_i$ is the longest element of the parabolic subgroup of $W$ generated by $\Pi_i$.

It was proved in [10], [9] and [13] that for any conjugacy class of $W$, there exists a good minimal length element. In [14], the second and third-named authors gave a general proof, which also provides an explicit construction of good minimal length elements.

Now we recall the construction in [14]. Let $F_0, \ldots, F_k$ be as in (2.1)(d).

Let $A$ be a Weyl chamber and for $0 \leq i < k$, let $C_i(A)$ be the connected component of $V - \bigcup_{H_a:F_i \subset H_a} H$ containing $A$. We say $A$ is in good position with respect to $w$ if for any $i$, the closure $\overline{C_i(A)}$ contains some regular point of $F_{i+1}$.

By [14, Lemma 5.1], there exists some Weyl chamber $A$ that is in good position with respect to any given $w$. By definition, for any $x \in W$, the Weyl
chamber \( x(A) \) is in good position with respect to \( xw^{-1} \). In particular, for any conjugacy class of \( \delta W \), there exists an element \( w \) such that the dominant chamber is in good position with respect to \( w \). In this case, \( \Delta_i \) is the root system of the standard Levi subgroup of \( G \) associated to the subset \( \Pi_i := \Pi \cap \Delta_i \) of simple roots and \( W_i = W_{F_i} \) is a standard parabolic subgroup of \( W \) for any \( i \). We denote by \( W^{\Pi_i} \) (resp. \( \Pi_i W \)) the set of minimal length representatives in \( W/W_i \) (resp. in \( W_i \backslash W \)). We write \( ^{\Pi_i}W_i \) for \( \Pi_i W \cap W_i \). By [14, Proposition 2.2], we have the following good factorization of \( w \).

**Proposition 3.1.** Let \( w \in W \delta \subset \delta W \). Suppose that the dominant chamber is in good position with respect to \( w \). Then there are \( x_i \in W \) (\( 1 \leq i \leq k \)) so that

\[
w = \delta x_1 x_2 \cdots x_k,
\]

where for \( 1 \leq i \leq k \), we have \( x_i \in W_{i-1} \cap W^{\Pi_i} \).

Furthermore

\[
(\delta x_1 \cdots x_i)(\Pi_i) = \Pi_i \quad (1 \leq i \leq k).
\]

The following result is proved in [14, Theorem 5.3].

**Theorem 3.2.** Suppose \( w \in \delta W \), and the fundamental chamber is in good position with \( w \). Then we have the following equality in the Braid monoid associated with \((W,S)\):

\[
w^d = w_0^{d \theta_1/\pi} w_1^{d (\theta_2 - \theta_1)/\pi} \cdots w_{k-1}^{d (\theta_k - \theta_{k-1})/\pi},
\]

where \( d \) is the order of \( w \) in \( \delta W \), \( (\theta_1, \ldots, \theta_k) \) is the sequence consisting of the elements in \( \Gamma_w \) and \( w_i \) is the maximal element in the standard parabolic subgroup \( W_i \).

### 4. The regular case

We first study the regular elliptic elements. A similar discussion is in [20, Section 2.6].

Following [22] we say an element \( w \in \delta W \) is regular if it has a regular eigenvector. We say \( w \) is \( d \)-regular if the corresponding eigenvalue has order \( d \) (\( d \) turns out to be independent of the choice of regular eigenvector). Following [12] we say element \( w \) is \( \mathbb{Z} \)-regular if \( \langle w \rangle \) acts freely on \( \Delta \). It is proved in [12, Prop. 1] that \( \mathbb{Z} \)-regularity implies regularity. The converse does not hold in general.

**Proposition 4.1.** Suppose \( w \in W \delta \) is elliptic and regular. Let \( \theta \in \Gamma_w \) such that \( V(w, \theta) \) contains a regular vector of \( V \). Then \([n_w] = [\exp(\sqrt{-1} \theta \rho^\vee) \delta] \).
Remark 4.2. The case where $w$ is $\mathbb{Z}$-regular is proved in [12, Prop. 12].

Proof. Let $\zeta = \exp(\sqrt{-1}\theta)$ and let $\tau = \delta \exp(\sqrt{-1}\rho') = \delta \rho'(\zeta)$. Let $d, m$ be the orders of $\zeta \in \mathbb{C}^\times$ and $\tau$ respectively. Let $\xi$ be a primitive $m$-th root of unity such that $\zeta = \xi^{m/d}$. We set $\nu = \tau^d = \delta^d$. Let $g, t$ be the Lie algebras of $G$ and $T$ respectively. We denote by $g^i$ and $t^i$ the subalgebras of $\iota$-fixed points of $g$ and $t$ respectively. Then $g^i$ is also a semisimple Lie algebra and $t^i$ is a Cartan subalgebra of $g^i$.

The automorphism $\tau$ gives a periodic grading: $g = \bigoplus_{i \in \mathbb{Z}/m\mathbb{Z}} g_i$, where $g_i$ is the $\xi^i$-eigenspace of $g$ for $\tau$. Then we have $g^i = \bigoplus_{i \in \mathbb{Z}/d\mathbb{Z}} g_i^i$, where $g_i^i = g_{im/d}$ is the $\zeta^i$-eigenvalue for $\tau$. This is an $N$-regular periodic grading of $g_i^i$, see [19, Section 3].

Let $\mathfrak{c}' \subseteq g_1^i$ be a Cartan subspace. By the construction in [12, Subsection 3.1], there is a $\tau$-stable Cartan subalgebra $\mathfrak{s}'$ of $g^i$ containing $\mathfrak{c'}$. Let $s$ be the centralizer of $\mathfrak{s}'$ in $g$. As $\mathfrak{s}'$ is conjugate to $t'$, $s$ is also a Cartan subalgebra of $g$ fixed by $\tau$. Let $g \in G$ such that $t = \text{Ad}(g) s$, and set $\varepsilon = g\tau g^{-1}$. Then $\varepsilon \in G\delta$ fixes $t$ and lies in $N\delta$, where $N$ is the normalizer of $T$ in $G$. Let $t(n_w, \zeta)$ and $t(\varepsilon, \zeta)$ be the $\zeta$-eigenvalues of $t$ for $n_w$ and $\varepsilon$ respectively. Thanks to [22, Theorem 6.4 (iv)] and the ellipticity of $w$, to show $n_w$ and $\varepsilon$ are conjugate, it suffices to show

$$\dim t(n_w, \zeta) = \dim t(\varepsilon, \zeta).$$

Notice that $t(\varepsilon, \zeta) = \text{Ad}(g) \mathfrak{c}'$.

Let $v \in V(w, \theta)$ be a regular point. We may assume that $v$ is (strictly) dominant. Since $w^d(v) = v$ and $v$ is strictly dominant, one has $w^d = \delta^d = \iota$ and hence $w \in W^\varepsilon$, where $W^\varepsilon$ is the subgroup of $\varepsilon$-fixed points of $W$. Notice that $t(n_w, \zeta) = t'(n_w, \zeta)$ and that $W^\varepsilon$ is the Weyl group of $t'$ in $g_i^i$. Let $t'(n_w, \zeta)$ be the $\zeta$-eigenspace of $t'$ for $n_w$. Then [22, Theorem 6.4 (ii)] says

$$\dim t(n_w, \zeta) = \dim t'(n_w, \zeta) = a(d, \delta),$$

where $a(d, \delta)$ is defined in [22, Section 6] with respect to $W^\varepsilon$ and $\delta$.

On the other hand, applying [19, Theorem 3.3 (v)] to the $N$-regular periodic grading $g_i^i = \bigoplus_{i \in \mathbb{Z}/d\mathbb{Z}} g_i^i$, one deduces that

$$\dim t(\varepsilon, \zeta) = \dim c' = a(d, \delta).$$

The proof is finished.

Corollary 4.3. Suppose $w \in W\delta$ is elliptic and $d$-regular. Then

$$[n_w] = [\exp(2\pi\sqrt{-1}\rho'/d)\delta].$$

This follows from the fact that $V(w, 2\pi/d)$ contains a regular vector.
5. The general case

We prove Theorem 2.2. We first collect some facts needed for the proof.

Suppose \( w \in \delta W \), and the dominant chamber \( \mathcal{C} \) is in good position with \( w \) (see Section 3). Let \( w = \delta x_1 x_2 \cdots x_k \) be the good factorization of \( w \) as in Proposition 3.1, and define \( \Delta_i, W_i, \lambda_i^\vee \) \( (0 \leq i \leq k) \) as in (2.1)(e-i). Also define \( \delta_0 = \delta \) and

\[
\delta_i = \delta x_1 \ldots x_i \quad (1 \leq i \leq k).
\]

Recall (Proposition 3.1) \( \delta_i(\Pi_i) = \Pi_i \) \( (0 \leq i \leq k) \).

**Lemma 5.1.** For all \( 0 \leq i \leq k \) we have \( \delta_i(\lambda_i^\vee) = \lambda_i^\vee \). In particular \( \delta(\lambda_0^\vee) = \lambda_0^\vee \).

**Proof.** We proceed by downward induction on \( i \). If \( i = k \) then \( \lambda_k^\vee = 0 \) so there is nothing to prove. Assume \( \delta_{i+1}(\lambda_{i+1}^\vee) = \lambda_{i+1}^\vee \) for some \( 0 \leq i \leq k - 1 \). We prove \( \delta_i(\lambda_i^\vee) = \lambda_i^\vee \). Since \( \delta_i(\Delta_i^\pm) = \Delta_i^\pm \), we have

\[
\delta_i(\rho_i^\vee) = \rho_i^\vee.
\]

Let \( y_i \in W_i \) such that \( \lambda_{i+1}^\vee = y_i(\lambda_{i+1}^\vee) \). Then

\[
\lambda_{i+1}^\vee = y_i \delta_{i+1} y_i^{-1}(\lambda_{i+1}^\vee) = y_i \delta_i x_{i+1} y_i^{-1}(\lambda_{i+1}^\vee).
\]

Note that \( y_i \delta_i x_{i+1} y_i^{-1} \in W_i \delta_i \) and that \( \lambda_{i+1}^\vee \) is dominant for \( \Delta_i^\pm \). Therefore, \( \delta_i \) fixes \( \lambda_{i+1}^\vee \) and by (2.1)(i) also fixes \( \lambda_i^\vee \) as desired. \( \square \)

**Lemma 5.2.** Suppose \( x \in W_\delta \) has order \( d \). Then for \( v \in V \), \( n(x) \exp(2\pi \sqrt{-1}v) \) is conjugate to \( n(x) \exp(\frac{2\pi \sqrt{-1}}{d} \sum_{i=0}^{d-1} x^i(v)) \) by an element in \( T \).

**Proof.** Let

\[
t = \exp\left(\frac{2\pi \sqrt{-1}}{d} \sum_{k=0}^{d-1} k x^{-k}(v)\right) \in T.
\]

Then \( t^{-1} n(x) \exp(2\pi \sqrt{-1}v)t = n(x) \exp(2\pi \sqrt{-1}v) \sum_{i=0}^{d-1} x^i(v) \) as desired. \( \square \)

Consider the element \( \delta_1 = \delta x_1 \). We may view this as an automorphism of \( W_1 \) and the corresponding Levi subgroup \( L_1 \). We will need to conjugate \( n(\delta_1) \) to an element of \( T_\delta \). Note that \( \delta_1 \) is not necessarily an elliptic element of \( \delta W \), so its liftings are not all conjugate in \( G \). For the purpose of our argument, we fix a set of lifts \( n(x) \in N(T) \) for \( x \in \delta W \) such that

1. \( n(s_\alpha)^2 = \exp(\pi \sqrt{-1} \alpha^\vee) \) for \( \alpha \in \Pi; \)
2. \( n(x x') = n(x) n(x') \) if \( \ell(xx') = \ell(x) + \ell(x') \).

We need a formula for the cycle defined by these lifts.
Lemma 5.3 ([16, Lemma 2.1A]). Given $x, y \in \delta W$ let
\[
S = \Delta^+ \cap y^{-1}(\Delta^-) \cap y^{-1}x^{-1}(\Delta^+)
\]
and
\[
\gamma = \sum_{\beta \in S} \beta^\vee.
\]
Then
\[
n(x)n(y) = n(xy) \exp(\pi \sqrt{-1} \gamma^\vee).
\]

Lemma 5.4. Suppose $x, y \in \delta W$ satisfy $\ell(xy) = \ell(x) + \ell(y)$. Let $z = xyx^{-1}$.
Then
\[
n(z) = n(x)n(y)n(x)^{-1} \exp\left(\pi \sqrt{-1} \sum_{\beta \in \Delta^+ \cap x(\Delta^-) \cap z^{-1}(\Delta^-)} \beta^\vee\right).
\]

Proof. By Lemma 5.3
\[
n(z) = n(xyx^{-1})
= n(xy)n(x^{-1}) \exp\left(-\pi \sqrt{-1} \sum_{\beta \in \Delta^+ \cap x(\Delta^-) \cap z^{-1}(\Delta^-)} \beta^\vee\right)
= n(x)n(y)n(x)^{-1} \exp\left(-\pi \sqrt{-1} \sum_{\beta \in \Delta^+ \cap x(\Delta^-) \cap z^{-1}(\Delta^+)} \beta^\vee\right)
= n(x)n(y)n(x)^{-1} \exp\left(\pi \sqrt{-1} \sum_{\beta \in \Delta^+ \cap x(\Delta^-) \cap z^{-1}(\Delta^-)} \beta^\vee\right).
\]
The last equality follows from $n(x^{-1}) = n(x)^{-1} \exp(\pi \sqrt{-1} \sum_{\beta \in \Delta^+ \cap x(\Delta^-)} \beta^\vee)$, which follows from another application of Lemma 5.3. \qed

5.1. A technical lemma

In the setting of the beginning of this section, we focus our attention on $\Delta_1$, with simple roots $\Pi_1$, and the element $\delta_1 \in \delta W$ which preserves $\Pi_1$. Recall that $L_1$ is the standard Levi subgroup corresponding to $\Pi_1$.
Recall (2.1(i)):
\[
\lambda^\vee_1 = \frac{d(\theta_2 - \theta_1)}{2\pi} \rho_1^\vee + \lambda_2^\vee.
\]
We say $v \in V^{\delta_1}$ is a general point of $V^{\delta_1}$ if $\langle \alpha, v \rangle = 0$ for some root $\alpha \in \Delta$ implies $\langle \alpha, V^{\delta_1} \rangle = 0$. We can find a general point of $V^{\delta_1}$ in a sufficiently small
neighborhood of $\lambda_1^C$. Furthermore, since the (open) $\Pi_1$-dominant chamber intersects $V^{\delta_1}$ we may assume $v$ is strictly $\Pi_1$-dominant. Given $v$ let $z \in W$ be the (unique) minimal element such that $z^{-1}(v) = \pi$, the unique dominant $W$-conjugate of $v$. Then $z^{-1}(\lambda_1^C) = \lambda_1^C$.

Set $y = z^{-1}\delta_1 z$. Then $y(\pi) = \pi$, and (since $\pi$ is dominant and $\delta$ fixes the dominant chamber) $\delta(\pi) = \pi$.

Set
\[
\Delta' = \{ \alpha \in \Delta \mid \langle \alpha, V^y \rangle = 0 \}.
\]
Then $\Delta'$ is the root system of a Levi factor $L'$; we write $W_{\Delta'}$ for its Weyl group. Note that, since $v$ is a general point of $V^{\delta_1}$, $\Delta' = \{ \alpha \in \Delta \mid \langle \alpha, \pi \rangle = 0 \}$. Then (since $\delta(\pi) = \pi$), $\delta(\Delta') = \Delta'$. Also $y \in W_{\Delta', \delta}$, and by the definition of $z$ we have
\[
\ell(zy) = \ell(z) + \ell(y).
\]

**Lemma 5.6.** Let $d$ be the order of $y$. Then
\[
\Delta^- - z^{-1}(\Delta_1) - \Delta' = \cup_{i=-d}^{d-1} y^{-i}(\Delta^- \cap z^{-1}(\Delta^+) \cap z^{-1}\delta_1^{-1}(\Delta^-)).
\]
Moreover, each root of $\Delta^- - z^{-1}(\Delta_1) - \Delta'$ lies in exactly $\frac{d\theta_1}{2\pi}$ of the sets $y^{-i}(\Delta^- \cap z^{-1}(\Delta^+) \cap z^{-1}\delta_1^{-1}(\Delta^-))$ for $0 \leq i \leq d - 1$.

**Remark 5.7.** The proof uses a similar method of counting root hyperplanes as in [14, Lemma 2.1].

**Proof.** Let $v_0$ be a general point of $V(\delta_1, \theta_1)$. Then $z^{-1}(v_0)$ is a general point of $V(y, \theta_1)$. For $i \in \mathbb{Z}$, set $v_i = y^{-i}z^{-1}(v_0)$. Since $V(y, \theta_1)_C$ is the sum of eigenspaces of $y$ of eigenvalues $e^{\pm i \theta_1}$, the points $v_i$ for $i \in \mathbb{Z}$ are contained in the subspace $D$ spanned by $v_0$ and $v'_1$. In particular $\dim D \leq 2$.

We first consider the case where $\dim D = 1$. In this case, we have $\theta_1 = \pi$ and hence $w = -1$. Then $\Delta_1 = \emptyset$, $\delta_1 = w$, $\Delta' = \Delta$ and $z = 1$. One checks that $\Delta^- - z^{-1}(\Delta_1) - \Delta' = \emptyset = \Delta^- \cap z^{-1}(\Delta^+) \cap z^{-1}\delta_1^{-1}(\Delta^-)$. So the statement follows.

Now we consider the case where $\dim D = 2$. Let $S \subseteq D$ the circle containing $v_i$ for $i \in \mathbb{Z}$. Note that $v_i = v'_j$ if $(i - j)\theta_1/(2\pi) \in \mathbb{Z}$.

Set $\mathcal{B} = \Delta^- \cap z^{-1}(\Delta^+) \cap z^{-1}\delta_1^{-1}(\Delta^-)$. Then
\[
(a) \quad \mathcal{B} = \{ \beta \in \Delta^-; H_\beta \text{ separates } z^{-1}(\mathcal{C}) \text{ from } \mathcal{C} \text{ and } y^{-1}z^{-1}(\mathcal{C}) \},
\]
where $\mathcal{C}$ is the dominant chamber and $H_\beta \subseteq V$ denotes the root hyperplane of $\beta$. 
Figure 1: This is an illustration for the proof of Lemma 5.6. Here $d = 5$ and $\theta_1 = \frac{4\pi}{5}$. The straight line is the intersection $H_\beta \cap D$.

As $z(\Delta') > 0$, we see $\Delta' \cap B = \emptyset$. Since $y \in W'\delta = \delta W'$, we have $y^{-i}(B) \subseteq \Delta^- - \Delta'$. Let $\beta \in y^{-i}(B)$. By (a), $H_\beta$ separates $y^{-i}z^{-1}C$ from $y^{-i-1}z^{-1}C$. This means $\beta \notin z^{-1}(\Delta_1)$ as $\delta_1(\Pi_1) = \Pi_1$. So

$$y^{-i}(B) \subseteq \Delta^- - z^{-1}(\Delta_1) - \Delta'.$$

Let $\beta \in \Delta^- - z^{-1}(\Delta_1) - \Delta'$. If $v'_k \in H_\beta$ for some $k$, then $z^{-1}(V_{\delta_1}) \subseteq H_\beta$ since $v'_k = y^{-k}z^{-1}(v_0)$ is a general point of $z^{-1}(V_{\delta_1})$. This means $\beta \in z^{-1}(\Delta_1)$, a contradiction. So $v'_i \notin H_\beta$ for any $i \in \mathbb{Z}$. Thus, $H_\beta \cap S = \{\pm p\}$ for some $p \in S$. Let $S^+, S^-$ be the two connected components of $S - \{\pm p\} = S - H_\beta$ such that $\bar{v}, S^+$ are on the same side of $H_\beta$.

Let $C$ be the closure of $C$. Recall that $v$ is a general point of $V^{\delta_1}$. As $\beta \notin \Delta'$ and $\bar{v} \in C$, $\bar{v} \notin H_\beta$ and

(b) $y^{-i}(\bar{v}) = \bar{v}$, $y^{-i}(C)$ are on the same side of $H_\beta$ for $i \in \mathbb{Z}$.

Similarly, one has

(c) $v'_i, y^{-i}z^{-1}(C)$ are on the same side of $H_\beta$ for $i \in \mathbb{Z}$.

Combining (a), (b) and (c), we have

(d) $\beta \in y^{-i}(B)$ if and only if $v'_i \in S^-$ and $v'_{i+1} \in S^+$.

Notice that the acute arc $(v'_i, v'_{i+1})$ is of angle $0 < \theta_1 < \pi$. It follows that the number of integers $0 \leq i \leq d - 1$ satisfying the condition (d) is exactly $d\theta_1/2\pi$. The proof is finished. \qed
Example 5.8. Let $W$ be of type $B_3$ and $\Pi = \{1, 2, 3\}$ such that $\alpha_3$ is the unique short simple root and $s_{13} = s_{31}$. Let $w = s_{1232}$. Then $\theta_1 = \pi/2$, $d = 4$ and $w = \delta_1 w_1$, where $\delta_1 = s_{1232}$, $w_1 = s_3$, $\Pi_1 = \{3\}$ and $V^{\delta_1} = \mathbb{R} \alpha_3$. We take $v = \alpha_3$. Hence $z = s_{21}$, $y = s_{23}$ and $\Delta' = \{2, 3\}$. One checks that each root of $\Delta - z^{-1}(\Delta_1) - \Delta' = \{-\alpha_1, -\alpha_1 + \alpha_2, -\alpha_1 + \alpha_2 + 2\alpha_3, -(\alpha_1 + 2\alpha_2 + 2\alpha_3)\}$ appears exactly one of the sets $\{-y^i (\alpha_1 + \alpha_2)\}$ for $0 \leq i \leq 3$.

5.2. Proof of Theorem 2.2

We argue by induction on the cardinality of $\Delta$. If $\Delta = \emptyset$, then the statement is trivial. Suppose that $\Delta \neq \emptyset$ and that the statement is true for any proper subsystem of $\Delta$.

Recall from the beginning of Section 5 we have $\delta_1 = \delta x_1$, and this satisfies $\delta_1(\Pi_1) = \Pi_1$. Let $L_1$ be the Levi subgroup of $G$ with root system $\Delta_1$, with derived group $L_{1, \text{der}}$. The conjugation action of $n(\delta_1)$ preserves $L_{1, \text{der}}$. Then $n(\delta_1) \in L_{1, \text{der}} \rtimes \langle n(\delta_1) \rangle \subset \delta G$.

Then $\Delta_1 \not\subset \Delta$. Let $\lambda_{0, L_1}^{\vee}$ be the element defined for $L_1$ with the angle sequence $\theta_2, \theta_3, \ldots$, i.e.

$$\lambda_{0, L_1}^{\vee} = \frac{d \theta_2}{2\pi} \rho_1^\vee + \lambda_2^\vee$$

Then by the inductive hypothesis applied to $L_{1, \text{der}}$ we have

$$(5.10)(a) \quad n(w) \sim_G n(\delta_1) \exp\left(\frac{2\pi \sqrt{-1} \lambda_{0, L_1}^{\vee}}{d}\right),$$

where $\lambda_{0, L_1}^{\vee}$ is given by (5.9). Define $y$ and $z$ as at the beginning of Section 5.1. Then (5.5) holds so we can apply Lemma 5.4 to conclude

$$(5.10)(b) \quad n(w) \sim_G n(z)^{-1} n(\delta_1) \exp\left(\frac{2\pi \sqrt{-1} \lambda_{0, L_1}^{\vee}}{d}\right)n(z)$$

$$= n(y) t \exp\left(\frac{2\pi \sqrt{-1} z^{-1}(\lambda_{0, L_1}^{\vee})}{d}\right),$$

where

$$(5.10)(c) \quad t = \exp\left(\pi \sqrt{-1} \sum_{\beta \in \Delta^+ \cap \pi(\Delta) \cap \delta_1^{-1}(\Delta^-)} z^{-1}(\beta^\vee)\right)$$
From conjugacy classes in the Weyl group

$$\exp\left(\pi \sqrt{-1} \sum_{\beta \in \Delta^+ \cap z^{-1}(\Delta^+) \cap z^{-1}\Delta^-} \beta^\vee\right).$$

(5.10)(d)

As in Section 5.1 let $\Delta' = \{\alpha \in \Delta \mid \langle \alpha, V^y \rangle = 0\}$. This is the root system of a Levi factor $L'$, with Weyl group $W_{\Delta'}$.

Lemma 5.11. We have $z^{-1}(\Delta_1^+) \subseteq \Delta^+$ and $z^{-1}(\Delta_1) \cap \Delta' = \emptyset$.

Proof. Notice that $\Delta_1' = \{\alpha \in \Delta \mid \langle \alpha, \bar{v} \rangle = 0\}$. For $\gamma \in \Delta_1^+$ we have

$$\langle z^{-1}(\gamma), \bar{v} \rangle = \langle z^{-1}(\gamma), z^{-1}(v) \rangle = \langle \gamma, v \rangle > 0,$$

where the last inequality follows from the fact that $v$ is strictly $\Pi_1$-dominant. So $z^{-1}(\gamma) > 0$ and $z^{-1}(\gamma) \notin \Delta'$ as desired.

Let $\gamma^\vee$ be the summand in (d). By Lemma 5.6 and Lemma 5.11, with $c = \frac{w_1}{2\pi} \in \mathbb{Z}$ we have

$$\sum_{j=0}^{d-1} y^j \gamma^\vee = 2c(-\rho^\vee + z^{-1}\rho_1^\vee + \rho_{L_1}^\vee).$$

(5.12a)

Since (d) is unchanged if we replace $\gamma^\vee$ with $-\gamma^\vee$, by Lemma 5.2 we see $n(y)t$ is $T$-conjugate to

$$n(y) \exp\left(\frac{2\pi \sqrt{-1}c(\rho^\vee - z^{-1}\rho_1^\vee - \rho_{L_1}^\vee)}{d}\right)$$

and therefore

$$n(w) \sim n(y) \exp\left(\frac{2\pi \sqrt{-1}c(\rho^\vee - z^{-1}\rho_1^\vee - \rho_{L_1}^\vee)}{d}\right) \exp\left(\frac{2\pi \sqrt{-1}z^{-1}(\lambda_0^\vee)}{d}\right).$$

(5.12b)

Lemma 5.13. No roots in $\Delta'$ vanish identically on $V(y, \theta_1)$.

Proof. Let $\alpha \in \Delta'$. We have to show that $\langle \alpha, V(y, \theta_1) \rangle \neq \{0\}$. Assume otherwise. Noticing that $\delta_1 = w(x_2 \cdots x_k)^{-1}$ and $x_2 \cdots x_k \in W_1$ fixes each point of $V(w, \theta_1)$, we have $V(w, \theta_1) \subseteq V(\delta_1, \theta_1) = z^{-1}(V(y, \theta_1))$. Since $\langle \alpha, V(y, \theta_1) \rangle = \{0\}$, we conclude that $z(\alpha) \in \Delta_1$, which contradicts Lemma 5.11.

Therefore we can apply Proposition 4.1 to conclude $n(y) \sim_{L'} \delta \times \exp\left(\frac{2\pi \sqrt{-1}c}{d}\rho_{L_1}^\vee\right)$. Using the fact that $\rho^\vee - z^{-1}\rho_1^\vee - \rho_{L_1}^\vee$ and $z^{-1}(\lambda_0^\vee)$ are
central in $L'$, we can insert this in (5.12)(c) to give
\[
n(w) \sim \delta \exp \left( \frac{2\pi \sqrt{-1} c}{d} \rho_{L'}^\vee \right) \exp \left( \frac{2\pi \sqrt{-1} c (\rho_1^\vee - z^{-1}(\rho_1^\vee) - \rho_{L'}^\vee)}{d} \right) \\
\times \exp \left( \frac{2\pi \sqrt{-1} z^{-1}(\lambda_{0,L_1}^\vee)}{d} \right) \\
= \delta \exp \left( \frac{2\pi \sqrt{-1} (cp_1^\vee + z^{-1}(\lambda_{0,L_1}^\vee - cp_1^\vee))}{d} \right)
\]

By (5.9) and the definition of $c$ we compute:
\[
z^{-1}(\lambda_{0,L_1}^\vee - cp_1^\vee) = z^{-1} \left( \frac{d \theta_2}{2\pi \rho_1^\vee} - \frac{d \theta_1}{2\pi \rho_1^\vee} \right) = z^{-1}(\lambda_1^\vee) = \lambda_1^\vee.
\]

Inserting this we conclude
\[
n(w) \sim \delta \exp \left( \frac{2\pi \sqrt{-1} (cp_1^\vee + \lambda_1^\vee)}{d} \right) = \delta \exp \left( \frac{2\pi \sqrt{-1} \lambda_0^\vee}{d} \right).
\]

This completes the proof of Theorem 2.2. \qed

6. Kac diagrams

We consider a connected complex semisimple group $G$, with Cartan subgroup $T$, root system $\Delta$, simple roots $\Pi$ and Weyl group $W$. We are also given an automorphism $\delta$ of $G$, possibly trivial, of finite order $n$, preserving a pinning. The semisimple conjugacy classes of $G^\delta$ of finite order are parametrized by their Kac diagrams. We summarize the statements here, and refer to [8], [15], [18] and [20] for details.

Let $X^*$, respectively $X_*$, be the lattice of characters (resp. co-characters) of $T$. Let $R = \mathbb{Z}(\Delta) \subset X^*$ be the root lattice, and $\Delta^\vee \subset R^\vee \subset X_*$ the co-roots and co-root lattice.

Recall that $V = X_* \otimes \mathbb{Q}$. Then $\delta$ acts on $V$, and we also write $\delta$ for the transpose action on $V^* = \text{Hom}_{\mathbb{Q}}(V, \mathbb{Q})$. We write $\delta$ as a superscript to denote fixed points. We identify $(V^\delta)^*$ with $(V^*)^\delta$, and we view this as a subset of $\text{Lie}(T)$.

We can write $T = (T^\delta)^0(1 - \delta)T$, where $^0$ denotes identity component, and $(1 - \delta)T = \{ t \delta(t^{-1}) \mid t \in T \}$. Both groups are (connected) tori. It follows easily from this that every element of $G^\delta$ is $G$-conjugate to an element of $(T^\delta)^0$. 
Define
\[(6.1)\quad e(\gamma^\vee) = \exp(2\pi \sqrt{-1} \gamma^\vee) \quad (\gamma^\vee \in V^\delta).\]

This map is surjective onto the elements of finite order in \((T^\delta)^0\).

Although we will make no use of this fact, it is interesting to note that if \(G\) is simple then \(G^\delta\) and \(T^\delta\) are connected.

Let \(\Pi_\delta\) the set of orbits of \(\delta\) on \(\Pi\). The map \(\Pi \ni \alpha \mapsto \alpha|_{T^\delta}\) identifies \(\Pi_\delta\) with a set of simple roots of \(G^\delta\). Write \(\Pi_\delta = \{\alpha_1, \ldots, \alpha_r\}\).

Let \(\Delta_\delta\) be the root system of \((T^\delta)^0 \subset (G^\delta)^0\), and let \(\Delta_\delta^\vee = \{\alpha^\vee \mid \alpha \in \Delta_\delta\} \subset V^\delta\) be the canonical coroot defined by ([3, Chapter 6, Section 1.1]).

Set \(R_\delta^\vee = \mathbb{Z}\langle \Delta_\delta^\vee \rangle\), the coroot lattice of \(\Delta_\delta\). Then \(X_*(X^\delta)^0 = (X^\delta)_\delta^\vee\) and \(R_\delta^\vee = (R^\vee)^\delta\).

Let \(W_\delta\) be the Weyl group of \(\Delta_\delta\).

Define projection \(P : V \rightarrow V^\delta\) by \(P(v) = \frac{1}{n} \sum_{i=0}^{n-1} \delta^i v\). Then \(P(X_*)\) is a lattice containing \(X_\delta^\vee\) of finite index, and \(W_\delta\) acts on \(P(X_*)\) and \(P(R^\vee)\).

Define:
\[(6.2)\quad \hat{W}_\delta = W_\delta \ltimes P(X_*),\]
\[\hat{W}_\delta^\vee = W_\delta \ltimes P(R^\vee).\]

Then \(\hat{W}_\delta\) is an affine Weyl group. Let \(\overline{C}\) be a fundamental domain for the action of \(\hat{W}_\delta\) on \(V^\delta\). Furthermore \(\hat{W}_\delta\) is an extended affine Weyl group, and
\[\Omega = \hat{W}_\delta / \hat{W}_\delta^\vee \simeq P(X_*) / P(R^\vee)\]
is a finite group which acts naturally on \(\overline{C}\).

**Lemma 6.3.** If \(G\) is adjoint then \(\Omega \simeq \pi_1(G^\delta)\).

**Proof.** If \(\delta = 1\) this is standard. In general since \(G\) is adjoint it reduces easily to the simple case and then a case-by-case check. In fact in the twisted cases \(\Omega\) is trivial except for \(2A_{2n+1}\) and \(2D_n\), in which case it has order 2. \(\square\)

**Lemma 6.4.** Every semisimple conjugacy class of finite order in \(G^\delta\) is of the form \([e(\gamma^\vee)\delta]\) for some \(\gamma^\vee \in V^\delta\).

In particular suppose \(\gamma^\vee, \tau^\vee \in V^\delta\). Then \([e(\gamma^\vee)\delta] = [e(\tau^\vee)\delta]\) if and only if there exists \(w \in \hat{W}_\delta\) such that \(w^\gamma^\vee = \tau^\vee\).

Suppose \(\gamma^\vee, \tau^\vee \in \overline{C}\). Then \([e(\gamma^\vee)\delta] = [e(\tau^\vee)\delta]\) if and only if there exists \(\omega \in \Omega\) such that \(\omega(\gamma^\vee) = \tau^\vee\).
Sketch of proof. We’ve already discussed the first assertion. The second follows from the calculation that
\[ e(\mu^\vee)e(\gamma^\vee)\delta e(-\mu^\vee) = e((1-\delta)\mu^\vee + \gamma^\vee)\delta \quad (\mu^\vee, \gamma^\vee \in V^\delta) \]
and the fact that
\[ [(1-\delta)V + X_\ast]^{\delta} = P(X_\ast). \]
The final assertion is standard.

We now apply the theory of affine Weyl groups to describe \( C \) when \( G \) is simple. If \( \zeta \) is a root of unity let
\[ g[\zeta] = \{ X \in g \mid \text{Ad}(g)(X) = \zeta X \}. \]
Then \( g[\zeta] \) is \( T^\delta \) invariant. Let \( \alpha_0 \) be the lowest weight of \( T^\delta \) acting on \( g[\zeta] \). Then
\[ -\alpha_0 = \begin{cases} \text{highest root of } \Delta_\delta, & \text{if } \delta = 1; \\ \text{highest short root of } \Delta_\delta, & \text{if } \delta \neq 1, \Pi_\delta \neq \emptyset; \\ 2\text{*highest short root of } \Delta_\delta, & \text{if } \delta \neq 1, \Pi_\delta = \emptyset. \end{cases} \]
The last case occurs if \( G \) is of type \( A_{2n} \) and \( \delta \) has order 2 (type \( ^2A_{2n} \)). Set \( \Pi_\delta = \{ \alpha_0, \alpha_1, \ldots, \alpha_r \} \), and let \( \gamma_1^\vee, \ldots, \gamma_r^\vee \in V^\delta \) be the corresponding fundamental co-weights. Define integers \( c_0, c_1, \ldots, c_r \) by \( c_0 = 1 \) and
\[ \sum_{i=0}^r c_i \alpha_i = 0. \]

We define the affine Dynkin diagram \( \tilde{D}(G, \delta) \) of \( (G, \delta) \) to be the Dynkin diagram of \( \Pi_\delta \). We equip each node with its label \( c_i \). See [18, Reference Chapter, Table 6] for a list of these diagrams.

The automorphism group of \( \tilde{D}(G, \delta) \) is isomorphic to the automorphism group of \( \overline{C} \). The group \( P(X_\ast) \) acts by translation on \( V^\delta \), and this induces an action of \( \Omega \) on \( \overline{C} \) and \( \tilde{D}(G, \delta) \).

Let \( \tilde{\alpha}_0 \) be the affine function
\[ \tilde{\alpha}_0 = \alpha_0 + \frac{1}{n}. \]
We define the affine coordinates of \( \gamma^\vee \in V^\delta \) to be
\[ (\tilde{\alpha}_0(\gamma^\vee), \alpha_1(\gamma^\vee), \ldots, \alpha_r(\gamma^\vee)). \]
The affine coordinates \( (a_0, \ldots, a_r) \) of a point in \( V^\delta \) satisfy \( \sum_{i=0}^r c_i a_i = \frac{1}{n} \).
For the fundamental domain $\mathcal{C}$ we take points whose affine coordinates $(a_0, \ldots, a_r)$ satisfy $a_i \geq 0$ ($0 \leq i \leq r$).

**Definition 6.5.** A Kac diagram for $(G, \delta)$ is a vector $D = [a_0, \ldots, a_r]$ where each $a_i$ is a non-negative integer and $\gcd\{a_0, \ldots, a_r\} = 1$. Set $d(D) = \sum_{i=0}^r a_i c_i$, $n = \text{order}(\delta)$, and define

$$e(D) = e\left( \frac{n}{d(D)} \sum_{i=1}^r a_i \gamma_i^\vee \right) \delta.$$

Note that $\frac{n}{d(D)} \sum_{i=1}^r a_i \gamma_i^\vee \in \mathcal{C}$. Here is the conclusion.

**Proposition 6.6.** Suppose $g \delta \in G \delta$ satisfies $g \delta \in Z(G)$. Then there is a Kac diagram $D$, with $d(D) = d$, such that $[g \delta] = [e(D)]$.

If $D, E$ are Kac diagrams then $[e(D)] = [e(E)]$ if and only if there exists $\omega \in \Omega$ satisfying $\omega(D) = E$.

We examine the role of the group $\Omega$ more closely. If $z \in Z(G \delta)$ the map $[g \delta] \to [z g \delta]$ is a well defined map of conjugacy classes in $G \delta$. Via the Proposition this induces an action of $Z(G \delta)$ on Kac diagrams.

Note that the Kac diagrams for $G$ are independent of isogeny; the only role that isogeny plays is in the action of $\Omega$. So suppose $D$ is a Kac diagram.

View it as giving a conjugacy class of finite order in $G_{\text{sc}} \delta$ where $G_{\text{sc}}$ is the simply connected cover of $G$. Thus $Z((G_{\text{sc}}) \delta)$ acts on Kac diagrams. Recall $\Omega \simeq \pi_1((G_{\text{sc}}) \delta)$, which is a quotient of $Z((G_{\text{sc}}) \delta)$. This is compatible with Proposition 6.6: if $z \in Z((G_{\text{sc}}) \delta)$ and $D$ is a Kac diagram then

$$[e(zD)] = [p(z)e(D)],$$

where $p : Z((G_{\text{sc}}) \delta) \to Z(G \delta)$.

**Lemma 6.7.** The orbits of $Z((G_{\text{sc}}) \delta)$ and $\text{Aut}(\tilde{D}(G, \delta))$ on the nodes of $\tilde{D}(G, \delta)$ are the same.

**Proof.** The nodes with label 1 are in bijection with $Z((G_{\text{sc}}) \delta)$, so the action of $Z((G_{\text{sc}}) \delta)$ on these nodes is simply transitive and the result is immediate. The remaining nodes follow from a case–by–case check.

**Proposition 6.8.** Suppose $w \delta \in W \delta$ is elliptic, and $n(w \delta) \in G \delta$ is a representative of $w$. Then the Kac diagram of $n(w \delta)$ is fixed by $\text{Aut}(\tilde{D}(G, \delta))$.

**Proof.** Suppose $w \delta \in W \delta$, with representative $g \delta \in G \delta$. If $z \in ZG \delta$ then $z g \delta$ is also a representative of $w \delta$, so if $w \delta$ is elliptic then $[z g \delta] = [g \delta]$. Therefore the Kac diagram of $[g \delta]$ is fixed by $Z(G \delta)$, hence by $Z((G_{\text{sc}}) \delta)$ (which acts by projection), and hence by $\text{Aut}(\tilde{D}(G, \delta))$ by Lemma 6.7.
For example in (untwisted) type $A_n$ every node has label 1, so the only Kac diagram which is fixed by all automorphisms has all labels 1, which corresponds to the Coxeter element. This proves the well-known fact that this is the only elliptic conjugacy class in this case.

7. Some applications to elliptic conjugacy classes

In this section, we make a digression and discuss some applications to elliptic conjugacy classes of finite Weyl groups.

**Proposition 7.1.** Assume that $W$ is irreducible. Then every elliptic $W$-conjugacy class of $W\delta$ is stable under the diagram automorphisms of $W$ which commute with $\delta$.

**Remark 7.2.** In fact, the result is true for any finite Coxeter group (see [11, Theorem 3.2.7] when $\delta = \text{id}$ and [13, Theorem 7.5] in general). The original proof (even for finite Weyl groups) is based on a characterization of elliptic conjugacy classes via the characteristic polynomials and length functions. Such characterization is established via a laborious case-by-case analysis on the elliptic conjugacy classes with the aid of computer for exceptional groups. Now we give a general proof for finite Weyl groups via the Kac diagram.

**Proof.** This follows easily from Proposition 6.8. If $\tau$ is an automorphism of $W$, commuting with $\delta$, it induces an automorphism of $^W\delta$, preserving $W\delta$. Let $G$ be the corresponding simply connected group. Then $\tau$ lifts to an automorphism, also denoted $\tau$, of $G$. If $w\delta \in W\delta$ is elliptic, so is $\tau(w\delta)$, and $[\tau(w\delta)] = [w\delta]$ if and only if $[\tau(n(w\delta))] = [n(w\delta)]$, where $n(w\delta)$ is a representative in $G\delta$ of $w\delta$, i.e. $\tau(n(w\delta))$ and $n(w\delta)$ have the same Kac diagram.

Now the result follows from Proposition 6.8.

Proposition 7.1 is used in an essential way to prove that in finite Weyl groups, elliptic conjugacy classes never fuse.

**Theorem 7.3.** Let $W$ be a finite Weyl group and $\mathcal{O}$ be a $W$-conjugacy class of $W\delta$. Let $J \subset I$ with $\delta(J) = J$ and $\mathcal{O} \cap W_J\delta$ contains an elliptic element of $W_J\delta$. Then $\mathcal{O} \cap W_J\delta$ is a single conjugacy class of $W_J$.

This result was first proved in [11, Theorem 3.2.11] when $\delta = \text{id}$ and in [6, Theorem 2.3.4] in general. The strategy is to first reduce to the case where $W$ is irreducible, then to reduce to the case where $W_J$ is irreducible. Note that the different $W_J$-conjugacy classes in $\mathcal{O} \cap W_J\delta$ are obtained from one another by diagram automorphisms of $W_J$. The final (and crucial) step in [11] and [6] was to use the characterization of elliptic conjugacy classes
to deduce that the intersection is a single $W_J$-conjugacy class. Now the final step may be replaced by Proposition 7.1, the proof of which is simpler than the characterization of elliptic conjugacy classes.

8. Non-elliptic elements

For $w \in W$, we denote by $\text{supp}(w)$ the support of $w$, i.e., the set of simple reflections that occur in some (or equivalently, any) reduced expression of $w$. We define

$$\text{supp}(w\delta) := \bigcup_{i \in \mathbb{Z}} \delta^i(\text{supp}(w)).$$

By [11, §3.1] and [13, §7], a conjugacy class of $\delta W$ is elliptic if and only if it does not intersect with $\delta W_J = W_J \rtimes \langle \delta \rangle$ for any proper $\delta$-stable subset $J$ of $\Pi$, in other words, $\text{supp}(w) = \Pi$ for any $w$ in the conjugacy class.

We have the following result (see [11, Corollary 3.1.11] for untwisted conjugacy classes and [6, Proposition 2.4.1] in the general case).

**Proposition 8.1.** Let $w_1, w_2 \in W\delta$ be minimal length elements in the same conjugacy class. Let $J_i = \text{supp}(w_i)$ for $i = 1, 2$. Then there exists $x \in J_2 W J_1 \cap W\delta$ with $x J_1 x^{-1} = J_2$.

Consider the set $\mathcal{P}_\delta$ of pairs $(J, D)$, where $J \subset \Pi$ is a $\delta$-stable subset and $D \subset W_J \delta$ is an elliptic conjugacy class of $\delta W_J$. The equivalence relation on $\mathcal{P}_\delta$ is defined by $(J, D) \sim (J', D')$ if there exists $x \in J' W J \cap W\delta$ such that $x J_1 x^{-1} = J'$ and $xDx^{-1} = D'$.

Combining Theorem 7.3 with Proposition 8.1, we have

**Theorem 8.2.** The map

$$C \mapsto \{(J, C \cap W_J \delta) \mid J = \text{supp}(w) \text{ for some } w \text{ of minimal length in } C\}$$

induces a bijection from $[W\delta] \to \mathcal{P}_\delta/\sim$.

For untwisted conjugacy classes, the statement is obtained by Geck and Pfeiffer in [11, Theorem 3.2.12]. The general case is proved in a similar way.

Let $C \in [\delta W]$ and $w \in C$. In general the lifts of $w$ to $\delta G$ are not $G$-conjugate. However there is a reasonable canonical choice of this lifting, defined as follows.

Without loss of generality, we assume that $C \subset W\delta$. Let $(J, D) \in \mathcal{P}_\delta$ be an element corresponds to $C$. Let $L_J$ be the standard Levi subgroup corresponds to $J$. Apply the algorithm of Section 2 to $(\delta L_J, D)$ to construct a conjugacy class $\Psi_J(D)$ in $\delta L_J$, and thus (by acting by $G$) a conjugacy class $\widetilde{\Psi}_J(D)$ in $\delta G$. 


Proposition 8.3. The map

\[ [W_\delta] \to [\delta G^{ss}], \quad C \mapsto \tilde{\Psi}_J(D) \]

is well-defined.

Proof. Let \((J,D), (J',D')\) be elements in \(P_\delta\) that correspond to \(C\). By Theorem 8.2, there exists \(x \in J_2WJ_1 \cap W_\delta\) with \(xJ_1x^{-1} = J_2\). As discussed in [2, Section 2] the Tits group provides a section \(\sigma : W \to N(T)\) satisfying \(\delta(\sigma(w)) = \sigma(\delta(w))\). In particular, \(\sigma(x)\) is \(\delta\)-stable. Since \(xJx^{-1} = J'\), we have \(\sigma(x)L_J\delta(\sigma(x))^{-1} = \sigma(x)L_J\sigma(x)^{-1} = L_{J'}\). Since \(xDx^{-1} = D'\), we have \(\sigma(x)\Psi_J(D)\sigma(x)^{-1} = \Psi_{J'}(D')\). Hence \(\tilde{\Psi}_J(D) = \tilde{\Psi}_{J'}(D')\). \(\square\)

9. Tables

For each exceptional group we list representatives of the elliptic conjugacy classes in \(W\), their Kac diagrams, and some other information.

We use the Bourbaki numbering of the simple roots [3]. Each table is preceded by the affine Dynkin diagram with the labels of the nodes.

(1) Name: name of the elliptic conjugacy class, as in [5] and [11].
(2) \(d\): order of the elements in the conjugacy classes.
(3) Kac diagram: with respect to the given affine Dynkin diagram.
(4) Centralizer: type of the derived group of the centralizer of the nilpotent element.
(5) good: \(\nu^d\) in the braid monoid (see Theorem 3.2). Here \(\Delta_S\) is the long element of the Weyl group \(W_S\) of the Levi factor defined by \(S\), and \(\Delta\) is the long element of the Weyl group of \(W\).

These tables were computed using the algorithm of Section 2.

Alternatively one can compute the Kac diagram in many cases using standard techniques, starting with the result for regular elements. The remaining cases require a number of case-by-case arguments, for example see [12, Section 8]. This is how the tables in [4], [12], and [17] were computed.

Example 9.1. Consider the conjugacy class \(E_8(a_7)\) of \(W(E_8)\) [11, Table B.6]. We take the following representative

\[ w = 2343654231435426543178 \]

of order \(d = 12\) and length 22. Let \(\zeta\) be a primitive 12\(^{th}\) root of unity. The eigenvalues of \(w\) are \(\{\zeta^k \mid k = 1, 2, 5, 7, 10, 11\}\). The dimension of the
eigenspaces are 1, 2, 1, 1, 2, 1, respectively. In the notation of the algorithm we have
\[ \Gamma_w = \{ \theta_1, \theta_2, \theta_3 \} = \{ 2\pi/12, 4\pi/12, 10\pi/12 \} = \frac{2\pi}{12} \ast \{ 1, 2, 5 \}. \]
Note that
\[ \frac{d(\theta_2 - \theta_1)}{2\pi} = \frac{d(\theta_1 - \theta_0)}{2\pi} = 1. \]
We have
\[ 0 = F_0 \subset F_1 \subset F_2 \subset F_3 = V, \]
where the \( F_\ell \) have dimensions 0, 2, 6 and 8, respectively.
In particular \( F_1 = V(w, 2\pi/12) \) is two-dimensional. The set \( \Delta_1 \) of roots vanishing on this space is a standard Levi subgroup of type \( D_4 \), with simple roots \( \{ 2, 3, 4, 5 \} \). It turns out that \( \Delta_2 = \emptyset \), so we have
\[ \Delta = \Delta_0 = E_8 \supset \Delta_1 = D_4 \supset \Delta_2 = \Delta_3 = \emptyset. \]
The algorithm gives the following elements in turn:
\[ \lambda_3^\vee = \lambda_2^\vee = 0, \]
\[ \lambda_1^\vee = \frac{d(\theta_2 - \theta_1)}{2\pi} \rho_1^\vee = \rho_1^\vee. \]
Next find \( w \in W(\Delta_0) = W \) so that \( w\rho_1^\vee \) is dominant, and then set
\[ \lambda_0^\vee = \frac{d(\theta_1 - \theta_0)}{2\pi} \rho^\vee + w\rho_1^\vee = \rho^\vee + w\rho_1^\vee. \]
In fundamental weight coordinates we have
\[ \rho_1^\vee = (-3, 1, 1, 1, 1, -3, 0, 0), \]
\[ w\rho_1^\vee = (0, 0, 0, 0, 0, 0, 1, 1), \]
\[ \lambda_0^\vee = \rho^\vee + w\rho_1^\vee = (1, 1, 1, 1, 1, 1, 2, 2), \]
\[ \lambda_0^\vee/12 = (1, 1, 1, 1, 1, 1, 2, 2)/12. \]
This element is dominant but not in the fundamental alcove; its affine coordinates are
\[ (1, 1, 1, 1, 1, 1, 2, 2, -22)/12. \]
Applying the affine Weyl group takes this to the element \( (0, 0, 1, 0, 1, 0, 0, 1)/12 \), or affine coordinates \( (0, 0, 1, 0, 1, 0, 0, 1, 1)/12 \). The corresponding Kac coordinates are therefore \( (0, 0, 1, 0, 1, 0, 0, 1, 1) \). Note that the sum of the coefficients
times the corresponding labels is $1 \times 4 + 1 \times 5 + 1 \times 2 + 1 \times 1 = 12$. See the corresponding line in the $E_8$ table. Compare [12, Section 8].

**Remark 9.2.** In [11, Table B.6] there is a different representative for this conjugacy class. Although this representative is good, it turns out the positive chamber is not in good position for this element (both “good” and “good position” are defined in Section 3), and in particular the Levi subgroups defined by (2.1)(g) are not standard.

\[ G_2 : \begin{array}{c}
\circ \\
1 \\
2 \\
3
\end{array} \]

| $w$ | $d$ | Kac diagram | good | Centralizer |
|-----|-----|-------------|------|-------------|
| 12  | 6   | 111         | $\Delta^2$ | *           |
| 1212| 3   | 110         | $\Delta^2$ | $A_1$       |
| $w_0$| 2   | 010         | $\Delta^2$ | $2A_1$      |

\[ ^3D_4 : \begin{array}{c}
\circ \\
1 \\
2 \\
3
\end{array} \]

| $w$ | $d$ | Kac diagram | good | Centralizer |
|-----|-----|-------------|------|-------------|
| 12  | 12  | 111         | $\Delta^2$ | *           |
| 132132| 6 | 010         | $\Delta^2 \Delta^1_{23}$ | $2A_1$ |
| 1323 | 6   | 101         | $\Delta^2$ | $A_1$       |
| 13213423 | 3 | 001         | $\Delta^2$ | $A_2$ |

\[ F_4 : \begin{array}{c}
\circ \\
1 \\
2 \\
3
\end{array} \overset{>}{\rightarrow} \begin{array}{c}
\circ \\
4 \\
2
\end{array} \]

| Name | $d$ | Kac diagram | good | Centralizer |
|------|-----|-------------|------|-------------|
| $F_4$ | 12  | 11111      | $\Delta^2$ | *           |
| $B_4$ | 8   | 11101      | $\Delta^2$ | $A_1$       |
| $F_4(a1)$ | 6 | 10101     | $\Delta^2$ | $2A_1$      |
| $D_4$ | 6   | 11100      | $\Delta^2 \Delta^1_{34}$ | $A_2$ |
| $C_3 + A_1$ | 6 | 01010     | $\Delta^2 \Delta^1_{12}$ | $3A_1$ |
| $D_4(a1)$ | 4 | 10100     | $\Delta^2$ | $A_1 + A_2$ |
| $A_3 + A_1$ | 4(8) | 02010     | $\Delta^2 \Delta^2_{23}$ | $3A_1$ |
| $A_2 + A_2$ | 3 | 00100     | $\Delta^2$ | $2A_2$      |
| $4A_1$ | 2 | 01000     | $\Delta^2$ | $A_1 + C_3$ |

The conjugacy class $A_3 + \tilde{A}_1$ of $W$ has order 4, but its lift to a semisimple conjugacy class has order 8 [2, Theorem B].
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\[ ^2E_6 : \]

\[ \begin{array}{cccc}
w & d & \text{Kac diagram} & \text{good} & \text{Centralizer} \\
1254 & 18 & 11111 & \Delta^2 & \ast \\
123143 & 12 & 11011 & \Delta^2 & A_1 \\
45423145 & 10 & 01011 & \Delta^2 \Delta_3^5 & 2A_1 \\
1231431543165431 & 6 & 11000 & \Delta^2 \Delta_3^2 \Delta_2^356 & B_3 \\
425423456542345 & 6 & 00100 & \Delta^2 \Delta_2^4 \Delta_2^356 & 2A_2 \\
23423465423456 & 6 & 00011 & \Delta^2 \Delta_3^2 \Delta_2^4 & A_3 \\
12314354231365431 & 4 & 00010 & \Delta^2 & A_1 + B_2 \\
w_0 & 2 & 00001 & \Delta^2 & C_4 \\
\end{array} \]

\[ E_6 : \]

\[ \begin{array}{cccc}
\text{Name} & d & \text{Kac diagram} & \text{good} & \text{Centralizer} \\
E_6 & 12 & 1 & \Delta^2 & \ast \\
& & 1 & \Delta^2 & A_1 \\
E_6(a1) & 9 & 1 & \Delta^2 & 3A_1 \\
& & 0 & \Delta^2 & 3A_1 \\
E_6(a2) & 6 & 1 & \Delta^2 & 4A_1 \\
& & 0 & \Delta^2 & 3A_1 \\
A_5 + A_1 & 6 & 1 & \Delta^2 \Delta_2^4 & 4A_1 \\
& & 0 & \Delta^2 & 3A_1 \\
3A_2 & 3 & 0 & \Delta^2 & 3A_1 \\
\end{array} \]
### Kac Diagrams for $E_7$

#### Centralizer Information

| Name       | d | Kac diagram | good | Centralizer |
|------------|---|-------------|------|-------------|
| $E_7$      | 18| $\begin{array}{c}1 \\
                      1111111 \end{array}$ | $\Delta^2$ | *           |
| $E_7(a1)$  | 14| $\begin{array}{c}1 \\
                      1110111 \end{array}$ | $\Delta^2$ | $A_1$       |
| $E_7(a2)$  | 12| $\begin{array}{c}1 \\
                      1101011 \end{array}$ | $\Delta^2\Delta^2_{257}$ | $2A_1$      |
| $E_7(a3)$  | 30| $\begin{array}{c}1 \\
                      3212123 \end{array}$ | $\Delta^2\Delta^4_{24}$ | *           |
| $D_6 + A_1$| 10| $\begin{array}{c}1 \\
                      0101010 \end{array}$ | $\Delta^2\Delta^8_{24}$ | $4A_1$      |
| $A_7$      | 8 | $\begin{array}{c}0 \\
                      0101010 \end{array}$ | $\Delta^2\Delta^2_{257}\Delta^2_{13}$ | $2A_1 + A_2$ |
| $E_7(a4)$  | 6 | $\begin{array}{c}0 \\
                      1001001 \end{array}$ | $\Delta^2$ | $2A_2 + A_1$ |
| $D_6(a2) + A_1$ | 6 | $\begin{array}{c}1 \\
                      0100100 \end{array}$ | $\Delta^2\Delta^4_{13}$ | $2A_1 + A_3$ |
| $A_5 + A_2$| 6 | $\begin{array}{c}0 \\
                      0010100 \end{array}$ | $\Delta^2\Delta^2_{2345}$ | $3A_2$      |
| $D_4 + 3A_1$| 6 | $\begin{array}{c}1 \\
                      0001000 \end{array}$ | $\Delta^2\Delta^4_{24567}$ | $2A_3$      |
| $2A_3 + A_1$| 4 | $\begin{array}{c}0 \\
                      0001000 \end{array}$ | $\Delta^2\Delta^2_{257}$ | $2A_3 + A_1$ |
| $7A_1$    | 2 | $\begin{array}{c}1 \\
                      0000000 \end{array}$ | $\Delta^2$ | $A_7$       |
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\[ E_8 : \]

\[ \begin{array}{cccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array} \]

| Name   | d  | Kac diagram | good  |
|--------|----|-------------|-------|
| \( E_8 \) | 30 | 11111111    | \( \Delta^2 \)  |
| \( D_8(a2) \) | 30 | 12102030    | \( \Delta^6 \Delta^4 \Delta_{2456} \Delta_{30}^2 \)  |
| \( E_8(a1) \) | 24 | 11011111    | \( \Delta^2 \)  |
| \( E_8(a2) \) | 20 | 11010111    | \( \Delta^2 \)  |
| \( E_7.A_1 \) | 18 | 11101010    | \( \Delta^2 \Delta_{24}^8 \)  |
| \( E_8(a4) \) | 18 | 01010111    | \( \Delta^2 \Delta_{24}^4 \)  |
| \( E_8(a5) \) | 15 | 10101011    | \( \Delta^2 \)  |
| \( D_8 \) | 14 | 10101010    | \( \Delta^2 \Delta_{2}^{18} \)  |
| \( E_8(a3) \) | 12 | 10100101    | \( \Delta^2 \)  |
| \( E_8(a7) \) | 12 | 01010011    | \( \Delta^2 \Delta_{2345}^2 \)  |
| \( E_6 + A_2 \) | 12 | 00100100    | \( \Delta^2 \Delta_{2345}^6 \)  |
| \( D_5(a1) + A_3 \) | 12 | 01002000    | \( \Delta^2 \Delta_{123456}^2 \)  |
| \( D_8(a1) \) | 12 | 00101010    | \( \Delta^2 \Delta_{34578}^4 \)  |
| Name       | d   | Kac diagram | good          | Centralizer          |
|------------|-----|-------------|---------------|----------------------|
| $E_7(a2) + A_1$ | 12  | 0           | $\Delta^2\Delta_{2345}\Delta_{24}$ | $2A_1 + A_3$         |
| $E_8(a6)$  | 10  | 0           | $\Delta^2$   | $2A_1 + 2A_2$        |
| $D_6 + 2A_1$ | 10  | 0           | $\Delta^2\Delta_{2345}\Delta_{34}$ | $2A_3$               |
| $A_8$      | 9   | 0           | $\Delta^2\Delta_{34}$ | $A_1 + 3A_2$         |
| $A_1 + A_7$ | 8   | 0           | $\Delta^2\Delta_{2345}\Delta_{1367}$ | $A_1 + 2A_3$         |
| $E_8(a3)$  | 8   | 0           | $\Delta^2$   | $2A_1 + 2A_3$        |
| $D_8(a8)$  | 6   | 0           | $\Delta^2$   | $A_3 + A_4$          |
| $E_7(a4) + A_1$ | 6   | 0     | $\Delta^2\Delta_{2345}\Delta_{1367}$ | $A_1 + A_5$         |
| $E_6(a2) + A_2$ | 6   | 0     | $\Delta^2\Delta_{1367}$ | $A_3 + A_4$         |
| $2D_4$     | 6   | 0           | $\Delta^2\Delta_{1367}$ | $A_3 + D_4$         |
| $D_4 + 4A_1$ | 6   | 0     | $\Delta^2\Delta_{1367}$ | $A_7$               |
| $A_1 + A_2 + A_5$ | 6   | 0     | $\Delta^2\Delta_{2345}\Delta_{1367}$ | $A_1 + A_2 + A_5$  |
| $2A_4$     | 5   | 0           | $\Delta^2$   | $2A_4$               |
| $2A_3 + 2A_1$ | 4   | 0     | $\Delta^2\Delta_{2345}$ | $A_1 + A_7$         |
| $2D_4(a1)$ | 4   | 0           | $\Delta^2$   | $A_3 + D_5$          |
| $4A_2$     | 3   | 0           | $\Delta^2$   | $A_8$                |
| $8A_1$     | 2   | 0           | $\Delta^2$   | $D_8$                |
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