Some Ideas to Test if a Polyhedron is Empty

Laurent Truffet  
IMT-A  
Dpt. Automatique-Produitique-Informatique  
Nantes, France  
email: laurent.truffet@imt-atlantique.fr

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Abstract

In this paper we develop a pure algebraic method which provides an algorithm for testing emptiness of a polyhedron.

Keywords. Moore-Penrose inverse, interval arithmetic.

1 Introduction

Testing if a polyhedron is empty is a fundamental issue for linear optimization theory. In the seminal work [6] a geometric approach was proposed to solve this problem. Since this work and to the best knowledge of the author only geometric approaches were proposed (see e.g. [5], [1] among many others). In this work we develop a method based only on algebraic considerations. The main concepts deal with Moore-Penrose inverse of a rectangular matrix (see Definition 1.1) and some results of arithmetic intervals [8]. This method provides an algorithm which seems to be new for testing emptiness of a polyhedron.

1.1 Main notations

For any integer $k \geq 1$ we defined the set $[k] := \{1, \ldots, k\}$.

Bold letters represent matrices or column vectors. $t(\cdot)$ denotes transpose operator. $\text{Mat}(\mathbb{R}, m, n)$ denotes the set of all real $m \times n$-matrices.

We denote by $0_{p,q} \in \text{Mat}(\mathbb{R}, p, q)$ the $p \times q$-matrix whose elements are all zero. We denote by $I_p \in \text{Mat}(\mathbb{R}, p, p)$ the $p \times p$-identity matrix. We denote $0$ (resp. $-\infty$) the column vector whose components are all 0 (resp. $-\infty$). The dimension is fixed by the context.

The natural order defined on $\mathbb{R}$ is denoted $\leq$. For any integer $p \geq 2$ this total order is extended to the partial order (called componentwise ordering), once again denoted $\leq$, defined on $p$-dimensional vectors as follows. If $x = t(x_1, \ldots, x_p)$ and $y = t(y_1, \ldots, y_p)$ then
\[ x \leq y \iff \forall i = 1, \ldots, p : \ x_i \leq y_i. \]  

(1)

The relation \( y \geq x \) means \( x \leq y \).

The vector \( e_i \) denotes the vector such that \( \forall j, e^i_j = 1 \) if \( j = i \) and 0 otherwise. Its dimension is determined by the context.

### 1.2 Problem statement

In this paper we consider the following set:

\[ \mathcal{P}(A, b) := \{ x \in \mathbb{R}^n : Ax \leq b \}. \]  

(2)

\( A \in \text{Mat}(\mathbb{R}, m, n) \) and \( b \in \mathbb{R}^m \).

And we wonder wether this set is empty or not ?

We make the following assumptions (see Section 3 for a discussion):

ASSUMPTION (A). Matrix \( A \) has no null row.

ASSUMPTION (B). We assume \( m > n \) and that \( A \) has full column-rank, that is \( \text{rk}(A) = n \).

### 1.3 Basic concepts and results

Our method is based on the following concepts, remarks and results.

**Definition 1.1 (Moore-Penrose inverse [7], [10])** The Moore-Penrose inverse of the matrix \( A \in \text{Mat}(\mathbb{R}, m, n) \) is the unique matrix \( A^+ \in \text{Mat}(\mathbb{R}, n, m) \) such that:

1. \( AA^+A = A \)
2. \( A^+AA^+ = A^+ \)
3. \( (AA^+) = AA^+ \)
4. \( (A^+A) = A^+A \)

A matrix which satisfies relations 1 and 2 will be called a \( \{1, 2\} \)-inverse.

We make the following remarks:

R1 \( \mathcal{P}(A, b) \neq \emptyset \iff \exists x, \exists c \leq b, Ax = c \)

R2 \( \exists x, Ax = c \iff AA^+c = c \), recalling that \( A^+ \in \text{Mat}(\mathbb{R}, n, m) \) denotes the Moore-Penrose inverse of \( A \).
Remark (R1) is obvious. Remark (R2) can be found in e.g. [4]. As a main consequence of (R1) and (R2) we have:

$$\mathcal{P}(A, b) \neq \emptyset \iff \exists c \leq b, A^t c = c.$$  \hfill (3)

Noticing that each component $c_i$ of $c$ belongs to the interval $C_i := [−\infty, b_i]$, $i = 1, \ldots, m$, we recall some results of interval arithmetic [8].

Let $−\infty \leq s \leq t \leq +\infty$ and $−\infty \leq s' \leq t' \leq +\infty$ we define the intervals $[s, t]$ and $[s', t']$ as the following sets: $[s, t] := \{a : s \leq a \leq t\}$ and $[s', t'] := \{a : s' \leq a \leq t'\}$.

- Interval addition. The addition of the intervals $[s, t]$ and $[s', t']$ is the new interval denoted by $[s, t] + [s', t']$ and defined by:

$$[s, t] + [s', t'] := \{a + a', a \in [s, t], a' \in [s', t']\} = [s + s', t + t].$$  \hfill (4)

Because the addition of reals is commutative and associative so is the interval addition. The interval $[0, 0]$ being its neutral element.

- Interval multiplication by a real. Let $z \in \mathbb{R}$ the multiplication of $z$ by the interval $[s, t]$ provides the new interval denoted $z \cdot [s, t]$ and defined by:

$$z \cdot [s, t] := \{za, a \in [s, t]\} = \begin{cases} [zs, zt] & \text{if } z > 0 \\ [zt, zs] & \text{if } z < 0 \\ [0, 0] & \text{if } z = 0 \end{cases}$$  \hfill (5)

- Linear combination of two intervals. Let $z, z' \in \mathbb{R}$. The linear combination $z \cdot [s, t] + z' \cdot [s', t']$ is the set defined as:

$$z \cdot [s, t] + z' \cdot [s', t'] := \{za + z'a', a \in [s, t], a' \in [s', t']\}.$$  \hfill (6)

Let $r \geq 1$, and let $I_i := [s_i, t_i], i = 1, \ldots, r$ be a series of $r$ intervals. Let $a := \{I_1, \ldots, I_r\}$ be the $r$-dimensional vector of the intervals $I_i, i = 1, \ldots, r$. Let us also define $s := \{s_1, \ldots, s_r\}$ and $t := \{t_1, \ldots, t_r\}$. We then have:

$$a = \{a : s \leq a \leq t\}.$$  \hfill (7)

An interval $[s, t]$ is thin if $s = t$. By extension we say that the interval vector $a$ is thin if $s = t$.

**Result 1.1 (Beeck)** For all $z \in \mathbb{R}^r$ we have:

$$\exists a \text{ s.t. } s \leq a \leq t \text{ and } ^t z a = 0 \iff \exists \theta \text{ s.t. } ^t z \cdot a.$$  \hfill (8)

Where

$$^t z \cdot a := z_1 \cdot I_1 + \ldots + z_r \cdot I_r,$$

with $z_1 \cdot I_1 + \ldots + z_r \cdot I_r$ that generalizes the linear combination of two intervals as follows: $z_1 \cdot I_1 + \ldots + z_r \cdot I_r := \{\sum_{i=1}^r z_i a_i, a_i \in I_i, i = 1, \ldots, r\}$. 

3
Proof. By definition of the set $z_1 \cdot I_1 + \ldots + z_r \cdot I_r$ we remark that:

$$^t z \cdot a = \{ ^t z a, \ a \in a \}.$$  \hfill (9)

And the proof is thus obvious. In fact, it is a very simplified version of Beeck’s Theorem (see e.g. [9] and references therein).

Let us also recall the following rules of interval calculus (see e.g. [9]):

AI1. Two arithmetical expressions which are equivalent in real arithmetic are equivalent in interval arithmetic when every variable occurs only once on each side.

AI2. If $f$ and $g$ are two arithmetical expressions of variables $x_1 \in I_1, \ldots, x_n \in I_n$ where $I_1, \ldots, I_n$ are given intervals, which are equivalent in real arithmetic then the inclusion $f(I_1, \ldots, I_n) \subseteq g(I_1, \ldots, I_n)$ holds if every variable $x_i$ occurs only once in $f$.

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2 Main results

Let us consider a matrix $A \in \text{Mat}(\mathbb{R}, m, n)$ satisfying ASSUMPTIONS (A) and (B) (see subsection 1.2), and a vector $b \in \mathbb{R}^m$.

As a direct consequence of ASSUMPTION (B) we can suppose that $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ with $A_1 := \begin{pmatrix} a_{1,..} \\ \vdots \\ a_{m-n,..} \end{pmatrix} \in \text{Mat}(\mathbb{R}, m-n, n)$ and $A_2 := \begin{pmatrix} a_{m-n+1,..} \\ \vdots \\ a_{m,..} \end{pmatrix} \in \text{Mat}(\mathbb{R}, n, n)$ assumed to be invertible. Its inverse is denoted $A_2^{-1}$. For all $i \in [m]$, $a_{i,*}$ denotes the $i$th row of matrix $A$.

Finally, let us recall that $A^+$ denotes the Moore-Penrose inverse of matrix $A$ (see Definition 1.1).

Lemma 2.1 We have the following two logical equivalences:

$$\forall c, \ AA^+ c = c \iff (AA^+ - I_m)c = 0 \iff Uc = 0$$

with:

$$U := \begin{pmatrix} I_{m-n} & -A_1A_2^{-1} \\ 0_{n,m-n} & 0_{n,n} \end{pmatrix}$$  \hfill (10)

Proof. The first equivalence is obvious. So, let us prove $(AA^+ - I_m)c = 0 \iff Uc = 0$. 


Using [3] we develop $A^+$ as follows:

$$A^+ = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}^+ = \begin{pmatrix} K^+ \ t A_1 \\ K^+ \ t A_2 \end{pmatrix},$$

with $K := A_1 A_1 \ t + A_2 A_2$.

Because $A_2$ is invertible, $A_2 A_2$ is symmetric invertible and thus $K$ is invertible. Hence, $K^+ = K^{-1}$.

Now, we have by block multiplication of matrices:

$$AA^+ - I_m = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \begin{pmatrix} K^+ \ t A_1 \\ K^+ \ t A_2 \end{pmatrix} - \begin{pmatrix} I_{m-n} & 0_{m-n,n} \\ 0_{n,m-n} & I_n \end{pmatrix}$$

$$= \begin{pmatrix} A_1 K^+ \ t A_1 - I_{m-n} & A_1 K^+ \ t A_2 \\ A_2 K^+ \ t A_1 & A_2 K^+ \ t A_2 - I_n \end{pmatrix}$$

Using block linear elimination we have (last row block multiplied by $A_1 A_2^{-1}$ and then substracting to first row block):

$$\begin{pmatrix} I_{m-n} & -A_1 A_2^{-1} \\ A_2 K^+ \ t A_1 & A_2 K^+ \ t A_2 - I_n \end{pmatrix}$$

Left multiplying the last row block by $K A_2^{-1}$ one has:

$$\begin{pmatrix} I_{m-n} & -A_1 A_2^{-1} \\ \ t A_1 & \ t A_2 - KA_2^{-1} \end{pmatrix}$$

Right multiplying the first row block by $-A_1$ and adding to the second row block we obtain:

$$\begin{pmatrix} I_{m-n} & -A_1 A_2^{-1} \\ 0_{n,m-n} & \ t A_1 A_2^{-1} + \ t A_2 - KA_2^{-1} \end{pmatrix}.$$ 

Now, we just have to note that:

$$\ t A_1 A_2^{-1} + \ t A_2 - KA_2^{-1} = (\ t A_1 A_1 + \ t A_2 A_2 - K) A_2^{-1}$$

$$= (K - K) A_2^{-1} = 0_{n,n}.$$ 

Conversely, assume that $Uc = 0$. This equality is equivalent to

$$c_1 = Rc_2,$$

with $c_1 := \ t (c_1, \ldots, c_{m-n}), c_2 := \ t (c_{m-n+1}, \ldots, c_m)$ and

$$R := A_1 A_2^{-1}. \quad (12)$$

Now, every $c$ such that $Uc = 0$ has the form: $Rc_2$, with $R := \begin{pmatrix} R \\ I_n \end{pmatrix}$. It remains to check that $AA^+ R = \hat{R}$. We develop the computation as follows.
\[ \mathbf{AA}^+ \mathbf{R} = \begin{pmatrix} A_1 \mathbf{K}^+ t_a & A_1 \mathbf{K}^+ t_a \mathbf{A}_2 \\ A_2 \mathbf{K}^+ t_a & A_2 \mathbf{K}^+ t_a \mathbf{A}_2 \end{pmatrix} \begin{pmatrix} \mathbf{R} \\ \mathbf{I}_n \end{pmatrix} \]

Now, we have:

\[ A_1 \mathbf{K}^+ t_a \mathbf{A}_1 \mathbf{R} + A_1 \mathbf{K}^+ t_a \mathbf{A}_2 = A_1 \mathbf{K}^+ t_a \mathbf{A}_1 \mathbf{A}_2^{-1} + A_1 \mathbf{K}^+ t_a \mathbf{A}_2 \]

\[ = A_1 \mathbf{K}^+ (t_a \mathbf{A}_1 + t_a \mathbf{A}_2 \mathbf{A}_2^{-1}) \]

\[ = A_1 \mathbf{K}^+ \mathbf{A}_2^{-1} \]

And

\[ A_2 \mathbf{K}^+ t_a \mathbf{A}_1 \mathbf{R} + A_2 \mathbf{K}^+ t_a \mathbf{A}_2 = A_2 \mathbf{K}^+ t_a \mathbf{A}_1 \mathbf{A}_2^{-1} + A_2 \mathbf{K}^+ t_a \mathbf{A}_2 \]

\[ = A_2 \mathbf{K}^+ (t_a \mathbf{A}_1 + t_a \mathbf{A}_2 \mathbf{A}_2^{-1}) \]

\[ = A_2 \mathbf{K}^+ \mathbf{A}_2^{-1} \]

Thus, the result is proved.

Let \( t_a \) be any row vector of matrix \( \mathbf{A}_1 \). Let \( \mathbf{A}_2 \) be any submatrix of \( \mathbf{A}_2 \) such that \( \mathbf{A}_2 \in \text{Mat} (\mathbb{R}, k, n) \) for some \( k \in [n] \). Let \( \mathbf{A} := \begin{pmatrix} t_a \\ \mathbf{A}_2 \end{pmatrix} \). Finally, let us denote \( \mathbf{c} := \begin{pmatrix} c \\ \mathbf{c}_2 \end{pmatrix} \) where \( c \in \mathbb{R} \) and \( \mathbf{c}_2 \in \mathbb{R}^k \). Then

**Lemma 2.2**

\[ \forall \mathbf{c}, (\mathbf{A} \mathbf{A}^+ - \mathbf{I}_{k+1}) \mathbf{c} = 0 \leftrightarrow \mathbf{U} \mathbf{c} = 0, \]

with

\[ \mathbf{U} := \begin{pmatrix} 1 & -t_a \mathbf{A}_2^+ \\ 0_{k,1} & 0_{k,k} \end{pmatrix}. \]

**Proof.** Let us denote \( \mathbf{\delta} := \mathbf{A}_2^+ \mathbf{a} \), and \( \mathbf{\gamma} := \mathbf{a} - \mathbf{A}_2 \mathbf{\delta} \). Because \( \mathbf{A}_2 \) is a \( k \times n \)-submatrix of the invertible matrix \( \mathbf{A}_2 \), the matrix \( \mathbf{A}_2^+ \) is the right inverse of \( \mathbf{A}_2 \), thus \( \mathbf{A}_2 \mathbf{A}_2^+ = \mathbf{I}_k \). So, we are in the case where \( \mathbf{\gamma} = 0 \). And we apply [2, section 4] to the block matrix \( \mathbf{A} \) to obtain:

\[ \mathbf{A}^+ = \left( h^{-1} \mathbf{A}_2^+ t \mathbf{A}_2^+ \mathbf{a} - h^{-1} \mathbf{A}_2^+ t \mathbf{A}_2^+ t \mathbf{a} \mathbf{A}_2^+ \right), \]

where \( h := 1 + t \mathbf{v} \mathbf{v}^t \) with \( t \mathbf{v} := t \mathbf{A}_2^+ \mathbf{a} \) and the matrix \( \mathbf{A}_2^+ \) is defined as the matrix \( t (\mathbf{A}_2^+) = (t \mathbf{A}_2^+)^t \). Now, we have:

\[ \mathbf{\hat{A}} \mathbf{A}^+ = \begin{pmatrix} h^{-1} t \mathbf{v} & t \mathbf{v}^t - h^{-1} t \mathbf{v}^t \mathbf{A}_2^+ t \mathbf{a}^t \\ h^{-1} \mathbf{A}_2 \mathbf{A}_2^+ t \mathbf{v} & \mathbf{A}_2 \mathbf{A}_2^+ t \mathbf{v}^t - h^{-1} \mathbf{A}_2 \mathbf{A}_2^+ t \mathbf{v}^t \end{pmatrix}. \]
Then, left multiplying the first row by 

\(-\mathbf{R}v\)

Noticing that \(\tilde{a}\) recalling that \(\tilde{\mathbf{c}}\)

And the result is proved.

Conversely, the system of equations \(\tilde{\mathbf{U}}\mathbf{c} = \mathbf{0}\) is equivalent to \(c = ^{\mathsf{T}}v\tilde{\mathbf{c}}_2 = ^{\mathsf{T}}a\tilde{\mathbf{A}}_2\tilde{\mathbf{c}}_2\). Thus, we have to prove that matrix \(\tilde{\mathbf{R}} := \left(\begin{array}{c} ^{\mathsf{T}}v \\ I_k \end{array}\right)\) satisfies: \(\tilde{\mathbf{A}}\tilde{\mathbf{R}} = \tilde{\mathbf{R}}\).

We then, have:

\[
\tilde{\mathbf{A}}\tilde{\mathbf{R}} = \left(\begin{array}{c c c}
^{\mathsf{T}}v_h \mathbf{v} & ^{\mathsf{T}}v_h - ^{\mathsf{T}}v_h^{\mathsf{T}}\tilde{\mathbf{A}}_2^{\mathsf{T}}a^{\mathsf{T}}v \\
^{\mathsf{T}}v_h^{\mathsf{T}}\tilde{\mathbf{A}}_2^{\mathsf{T}}v & \tilde{\mathbf{A}}_2^{\mathsf{T}}\tilde{\mathbf{A}}_2 - h^{-1}\tilde{\mathbf{A}}_2^{\mathsf{T}}a^{\mathsf{T}}v
\end{array}\right) \left(\begin{array}{c} I_k \\
0_{k,k}
\end{array}\right).
\]

Noticing that \(v = ^{\mathsf{T}}\tilde{\mathbf{A}}_2a\) we have:

\[
h^{-1}^{\mathsf{T}}vv^{\mathsf{T}}v - h^{-1}^{\mathsf{T}}v\tilde{\mathbf{A}}_2^{\mathsf{T}}a^{\mathsf{T}}v = h^{-1}^{\mathsf{T}}vv^{\mathsf{T}}v + ^{\mathsf{T}}v - h^{-1}^{\mathsf{T}}v\mathbf{v}^{\mathsf{T}}v = ^{\mathsf{T}}v.
\]

Noticing that \(\tilde{\mathbf{A}}_2\tilde{\mathbf{A}}_2 = I_k\) we have:

\[
h^{-1}\tilde{\mathbf{A}}_2\tilde{\mathbf{A}}_2^{\mathsf{T}}v^{\mathsf{T}}v + \tilde{\mathbf{A}}_2\tilde{\mathbf{A}}_2 - h^{-1}\tilde{\mathbf{A}}_2\tilde{\mathbf{A}}_2^{\mathsf{T}}v^{\mathsf{T}}v = h^{-1}v^{\mathsf{T}}v + I_k - h^{-1}v^{\mathsf{T}}v = I_k.
\]

And the result is proved. \(\square\)

For \(i \in [m]\) we define:

\[
\mathcal{L}_i := \{\mathbf{x} \in \mathbb{R}^n : a_i \cdot \mathbf{x} \leq b_i\}, \quad (14)
\]

recalling that \(a_i\) denotes the \(i\)th row vector of matrix \(\mathbf{A}\).

Recall \(\mathcal{C}_i = (-\infty, b_i], \ i \in [m]\). And let us define the vector of the intervals \(\mathcal{C}_i, \ i \in [m]\) by \(\mathbf{c} := ^{\mathsf{T}}(\mathcal{C}_1, \ldots, \mathcal{C}_m)\). That is

\[
\mathbf{c} = \{c : -\infty \leq c \leq b\}. \quad (15)
\]

For \(i \in [m-n]\) we define \(B_i := \{j \in [m] : u_{i,j} \neq 0\}\) recalling that \(\mathbf{U} = [u_{i,j}]\) is defined by (10).
Theorem 2.1 For all \(i \in [m - n]\) we have:

\[ 0 \in u_{i, \cdot} \iff \cap_{j \in B_i} L_j \neq \emptyset. \]

**Proof.** Without loss of generality we can assume that \(i = m - n\). So, that:

\[ u_{m-n,} = (0_{1,m-n-1}, 1, -a_{m-n}, A_2^{-1}). \]

Which could be rewritten as:

\[ u_{m-n,} = (0_{1,m-n-1}, 1, -a_{m-n}, A_2^{-1}\tilde{\Pi}). \quad (16) \]

Where \(\tilde{\Pi} = [\tilde{\pi}_{i,j}] \in \text{Mat}(\mathbb{R}, n, n)\) such that: \(\tilde{\pi}_{i,j} = 1\) if \(i = j\) and \(u_{m-n,i} \neq 0\), and 0 otherwise.

Now, let us define \(\Pi = [\pi_{i,j}] \in \text{Mat}(\mathbb{R}, m, m)\) by:

\[ \pi_{i,j} := \begin{cases} 0 & \text{if } i \neq j \text{ or } i = j \leq m - n - 1 \\ 1 & \text{if } i = j = m - n \\ \tilde{\pi}_{i,j} & \text{otherwise.} \end{cases} \]

By definition of \(\Pi\), we remark that:

\[ \cap_{j \in B_i} L_j \neq \emptyset \iff \exists x, \Pi A x \leq \Pi b \\
\iff \exists x, \exists c \leq b, \Pi A x = \Pi c \\
\iff \exists c \leq b, (\Pi A) (\Pi A)^+ \Pi c = \Pi c \\
\iff \exists c \leq b, (\Pi A)(\Pi A)^+ (\Pi - \Pi)c = 0. \]

Let \(k := |B_{m-n}|\) be the number of elements of the set \(B_{m-n}\). By renumbering the lines of matrix \(A\) we can assume that

\[ \Pi A = \begin{pmatrix} 0_{m-n-1,n} \\ a_{m-n,} \\ \Pi A_2 \end{pmatrix}, \]

with

\[ \Pi = \begin{pmatrix} I_k & 0_{k,n-k} \\ 0_{n-k,k} & 0_{k,n-k} \end{pmatrix}. \]

Because of the expression of vector \(u_{m-n,}\) (see \((16)\)) we define \(X \in \text{Mat}(\mathbb{R}, n, n)\) as \(X := A_2^{-1}\tilde{\Pi}\). It is easy to check that \(X\) is a \(\{1, 2\}\)-inverse (see Definition 1.1) of the matrix \(\Pi A_2\) such that: \((\Pi A_2)X = \Pi\). The latter equation means that because \(\Pi A_2 = \begin{pmatrix} A_2 & 0_{n-k,n} \\ 0_{n-k,n} & 0_{n-k,n-k} \end{pmatrix}\) where \(A_2\) is a submatrix of \(A_2\), we have \(X = \begin{pmatrix} A_2^+ & 0_{k,n-k} \end{pmatrix}\) where \(A_2^+\) is the right inverse of \(A_2\).

Noticing matrix \(\Pi A\) has the form \(\begin{pmatrix} 0_{m-n-1,n} & Y \\ 0_{n-k,n} \end{pmatrix}\) with \(Y := \begin{pmatrix} a_{m-n,} \\ A_2 \end{pmatrix}\), its Moore-Penrose inverse is then the matrix \(\begin{pmatrix} 0_{m-n-1,m-n-1} & 0_{m-n-1,k+1} & 0_{m-n-1,n-k} \\ 0_{k+1,m-n-1} & YY^+ & 0_{k+1,n-k} \\ 0_{n-k,m-n-1} & 0_{n-k,k+1} & 0_{n-k,n-k} \end{pmatrix}\). Hence,

\[ \Pi A (\Pi A)^+ = \begin{pmatrix} 0_{m-n-1,m-n-1} & 0_{m-n-1,k+1} & 0_{m-n-1,n-k} \\ 0_{k+1,m-n-1} & YY^+ & 0_{k+1,n-k} \\ 0_{n-k,m-n-1} & 0_{n-k,k+1} & 0_{n-k,n-k} \end{pmatrix}. \]
By definition of matrix $\Pi$ we then have:

$$\Pi A(\Pi A)^+\Pi - \Pi = \begin{pmatrix} 0_{m-n-1, m-n-1} & 0_{m-n-1, k+1} & 0_{m-n-1, k} \\ 0_{k+1, m-n-1} & YY^+ - I_{k+1} & 0_{k+1, n-k} \\ 0_{n-k, m-n-1} & 0_{n-k, k+1} & 0_{n-k, n-k} \end{pmatrix}.$$

We can focus our attention on matrix $Y = \begin{pmatrix} a_{m-n-1} \\ A_2 \end{pmatrix}$ and apply Lemma 2.2 to obtain that the system of equations $(\Pi A(\Pi A)^+\Pi - \Pi)c = 0$ is equivalent to $u_{m-n}, c = 0$. Hence the result is now proved.

**Theorem 2.2** The polyhedron $\mathcal{P}(A, b)$ is not empty iff

$$\mathfrak{B} : \forall k, 0 \in ^t kU \cdot c.$$

**Proof.** Using the characterization of non emptiness of polyhedron $\mathcal{P}(A, b)$ (see (3)) and Lemma 2.1 we have:

$$\mathcal{P}(A, b) \neq \emptyset \iff \exists \mathfrak{A} : \exists c \leq b, UC = 0,$$

recalling that $U = \begin{pmatrix} I_{m-n} & -R \\ 0_{n,m-n} & 0_{n,n} \end{pmatrix}$ with $R = A_1 A_2^{-1}$. Thus, we have to prove: $\mathfrak{A} \iff \mathfrak{B}$.

By application of Result 1.1 $\mathfrak{A} \Rightarrow \mathfrak{B}$.

Let us prove $\mathfrak{B} \Rightarrow \mathfrak{A}$ by absurd. Thus, assume $\mathfrak{B}$ and $\overline{\mathfrak{A}} : \forall c \leq b, \exists i, u_i, c \neq 0$.

Let $c \leq b$ then by $\overline{\mathfrak{A}}$ there exists $i \in [m]$ such that $u_i, c \neq 0$. But, take $k = e^i$, then by $\mathfrak{B}$ we have: $0 \in ^t e^i U \cdot c = u_i \cdot c$ which means that $\exists c^1 \leq b$ such that $u_i, c^1 = 0$ (see Result 1.1). Let us define $C := [-\infty, b_1] \times \cdots \times [-\infty, b_m]$.

The vector $c^1$ is $\leq b$ and such that $\exists i \in [m]$ with $u_i, c^1 \neq 0$ (by $\overline{\mathfrak{A}}$). But by $\mathfrak{B}$ there exists $c^2$ such that $u_i, c^2 = 0$. So, we construct a $[m] \times C$-valued series

$$\sigma := \{(i_0 = i, e^0 = ^t c), (i_1, e^1), (i_2, e^2), (i_3, e^3), \ldots \}$$

which has the following property $\forall n \geq 0$:

(p). $u_{i_n}, c^n \neq 0$ and $u_{i_n}, c^{n+1} = 0$.

The set $[m] \times C$ is the cartesian product of compact spaces thus it is a compact space. Hence the sequence $\sigma$ admits a subsequence $\varphi. \sigma := \{(i_{\varphi(k)}, e^k(k)), k \in \mathbb{N}\}$ with $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ strictly increasing and such that $\lim_{k \rightarrow \infty} \varphi(k) = \infty$. And the subsequence $\varphi. \sigma$ admits a limit $(i, \ell) := \lim_{k \rightarrow \infty} (i_{\varphi(k)}, e^k(k))$. The point $(i, \ell)$ is an element of $[m] \times C$ which must satisfy property (p) by construction. Thus, we obtain a contradiction. And the result is now proved. □
Remark 2.1 Due to the structure of the matrix \( U \) we can restrict our attention to all \( m - n \) dimensional vectors \( k' \) such that

\[
0 \in \langle k' \rangle G \cdot \mathbf{e}, \tag{17}
\]

where matrix \( G \) is:

\[
G := \left( \begin{array}{cc} I_{m-n} & -R \end{array} \right). \tag{18}
\]

The main problem is then to enumerate only the relevant vectors \( k' \) such that \( 0 \in \langle k' \rangle G \). The following result adresses this problem.

Let \( \mathcal{B} := \{ e^i \} \) be the canonical basis of \( \mathbb{R}^{m-n} \). Let us define

\[
k'(j, i, i') := -r'_{i',j} e^i + r_{ij} e^{i'}, \tag{19}
\]

for all \( j \in [n], i \in [m - n - 1] \) and \( i' = i + 1 \) to \( m - n \).

Let us define

\[
\ker'(R) := \{ k' : \langle k' \rangle R = \langle 0 \rangle \}. \tag{20}
\]

The set \( \mathcal{K} \) denotes a basis of \( \ker'(R) \) if \( \ker'(R) \neq \{ 0 \} \) and \( \{ 0 \} \) otherwise.

Let us define \( b_1 = \left( \begin{array}{c} b_1 \\ \vdots \\ b_{m-n} \end{array} \right) \) and \( b_2 = \left( \begin{array}{c} \vdots \\ b_{m-n+1} \end{array} \right) \). Let us denote \( (b_1)\perp \) a basis of the set \( \{ k' : \langle k' \rangle b_1 = 0 \} \). And let us denote \( (Rb_2)\perp \) a basis of the set \( \{ k' : \langle k' \rangle (Rb_2) = 0 \} \).

Theorem 2.3 If for all \( i \in [m - n] \):

\[
0 \in \langle e^i \rangle G \cdot \mathbf{e},
\]

and

\[
\forall k' \in \mathcal{K}, \: 0 \in \langle k' \rangle G \cdot \mathbf{e},
\]

and

\[
\forall k' \in (b_1)\perp, \: 0 \in \langle k' \rangle G \cdot \mathbf{e},
\]

and

\[
\forall k' \in (Rb_2)\perp, \: 0 \in \langle k' \rangle G \cdot \mathbf{e},
\]

and for all \( j \in [n], i \in [m - n - 1] \) and \( i' = i + 1 \) to \( m - n \):

\[
0 \in \langle k'(j, i, i') \rangle G \cdot \mathbf{e}.
\]

Then

\[
\forall k', 0 \in \langle k' \rangle G \cdot \mathbf{e}.
\]

Proof. First, let us remark that by definition of \( \mathbf{e} \) (see (15)) we have:

\[
\langle k' \rangle G \cdot \mathbf{e} = \begin{cases} 
[-\infty, \langle k' \rangle Gb] & \text{if } \langle k' \rangle G \geq \langle 0 \rangle \\
\langle k' \rangle Gb, +\infty & \text{if } \langle k' \rangle G \leq \langle 0 \rangle \\
[-\infty, +\infty] & \text{otherwise}.
\end{cases}
\]
Noticing that \( t'k'G \leq t'0 \iff -t'k'G \geq t'0 \) we can focus our attention on the cone:

\[ \mathcal{G} := \{ k' : t'k'G \geq t'0 \} \]

If \( \mathcal{G} = \{ 0 \} \) then the result is obviously true. Thus, let us assume that the cone \( \mathcal{G} \neq \{ 0 \} \). In this case we have:

\[ \forall k' \in \mathcal{G}, 0 \in t'k'G \cdot \iff 0 \in \cap k' \in \mathcal{G} ( \left[ \begin{array}{c} -\infty \\ t'k'Gb \end{array} \right] ) \iff \forall k' \in \mathcal{G}, \ t'k'Gb \geq 0 \]

The last equivalence is then equivalent to \( \min_{k' \in \mathcal{G}} (t'k'Gb) \geq 0 \).

Now, let us remark that:

\[ f(k') := t'k'Gb = t'k'b_1 - t'k'Rb_2 \]

Due to the structure of matrix \( G \) the inequality \( t'k'G \geq t'0 \) implies \( k' \geq 0 \).

The different possibilities for the choice of vector \( k' \in \mathcal{G} \) in the function \( f \) are as follows:

1. \( k' = e^i, i \in [m - n] \). In such case \( f = g_i, b \).
2. \( k' \in K \) or \( k' \in (Rb_2)\perp \). In this case \( f = t'k'b_1 \).
3. \( k' \in (b_1)\perp \). And then, \( f = -t'k'Rb_2 \).
4. Finally, we can eliminate variables \( b_{m-n+j}, j \in [n] \), between the rows \( i \) and \( i' \) of matrix \( R \) for \( i \in [m - n - 1], i' = i + 1, \ldots, m - n \). In this case \( f = t'k'(j, i, i')Gb \).

Hence, the result. \( \square \)

Based on the previous results definitions and notations we provide the following algorithm for testing if a polyhedron is empty or not.

**Algorithm**

- **Inputs:** \( A \in \text{Mat}(\mathbb{R}, m, n) \) satisfying ASSUMPTIONS (A) and (B), and vector \( b \in \mathbb{R}^n \)
- **Output:** answer to the question “is \( \mathcal{P}(A, b) := \{ x \in \mathbb{R}^n : Ax \leq b \} \) empty?”

0. Put matrix \( A \) in the form \( \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \) with \( A_2 \) invertible.

1. Compute \( A_2^{-1} \).
2. Compute \( G := (I_{m-n} - R) \), with \( R := A_1A_2^{-1} \).
3. Compute \((b_1)^\perp\) a basis of the set \(\{k': \, ^t k' b_1 = 0\}\).
\[
(b_1)^\perp := ((b_1)^\perp \cup (-b_1)^\perp) \cap \mathcal{G}.
\]

4. Compute \((Rb_2)^\perp\) a basis of the set \(\{k': \, ^t k'(Rb_2) = 0\}\).
\[
(Rb_2)^\perp := ((Rb_2)^\perp \cup (-Rb_2)^\perp) \cap \mathcal{G}.
\]

5. Compute \(\mathcal{K}\) a basis of \(\ker(R)\).
\[
\mathcal{K}_+ := (\mathcal{K} \cup (-\mathcal{K})) \cap \mathcal{G}.
\]

6. For \(k' \in (b_1)^\perp \cup (Rb_2)^\perp \cup \mathcal{K}_+\)
   - if \(0 \notin \, ^t k' \mathcal{G} \cdot e\) exit: polyhedron \(\mathcal{P}(A, b) = \emptyset\)
   EndFor

7. For \(i = 1\) to \(m - n\)
   - if \(0 \notin \, ^te' \mathcal{G} \cdot e\) exit: polyhedron \(\mathcal{P}(A, b) = \emptyset\)
   EndFor

8. For \(j = 1\) to \(n\)
   - For \(i = 1\) to \(m - n\)
     * For \(i' = i + 1\) to \(m - n\)
       \(\cdot k' := -r_{i',j}e' + r_{i,j}e''\)
       \(\cdot\) if \(0 \notin \, ^t k' \mathcal{G} \cdot e\) exit: polyhedron \(\mathcal{P}(A, b) = \emptyset\)
     * EndFor
   - EndFor

EndFor

3 Discussion about ASSUMPTIONS (A) and (B)

3.1 ASSUMPTION (A)

Independently of the definitions of a polyhedron (see subsection 3.2), if there exists a row \(a_{i,.} = \, ^t 0\) then the set \(\mathcal{L}_i\) (see (14)) is thus defined as
\[
\mathcal{L}_i = \{x \in \mathbb{R}^n : \, ^t 0 x \leq b_i\}.
\]

Then if \(b_i < 0\) \(\mathcal{L}_i = \emptyset\) and thus the polyhedron \(\mathcal{P}(A, b) = \emptyset\). Otherwise the inequality \(\, ^t 0 x \leq b_i\) can be removed.
3.2 ASSUMPTION (B)

In this subsection we discuss several definitions of polyhedron which appear in linear programming. We consider a matrix \( \tilde{A} \in \text{Mat}(\mathbb{R}, \tilde{m}, \tilde{n}) \) whose rank is \( \text{rk}(\tilde{A}) = r \). We assume that \( r \neq \tilde{n} \). And we consider a vector \( \tilde{b} \in \mathbb{R}^{\tilde{m}} \).

- If the polyhedron is defined as the following set:
  \[
P(\tilde{A}, \tilde{b}) := \{ x \in \mathbb{R}^\tilde{n} : \tilde{A}x \leq \tilde{b}, x \geq 0 \},
  \]
  then, we have:
  \[
  \tilde{A}x \leq \tilde{b}, x \geq 0 \iff Ax \leq b,
  \]
  with: \( A := \begin{pmatrix} \tilde{A} & -I_{\tilde{n}} \end{pmatrix} \in \text{Mat}(\mathbb{R}, \tilde{m} + \tilde{n}, \tilde{n}) \) and \( b := \begin{pmatrix} \tilde{b} \\ 0 \end{pmatrix} \in \mathbb{R}^{\tilde{m} + \tilde{n}} \). It is clear that \( m := \tilde{m} + \tilde{n} > n := \tilde{n} \) and that \( A \) has full column-rank (equal to \( n \)).

- If the polyhedron is defined as the following set:
  \[
P(\tilde{A}, \tilde{b}) := \{ x \in \mathbb{R}^\tilde{n} : \tilde{A}x = \tilde{b}, x \geq 0 \},
  \]
  then, we have:
  \[
  \tilde{A}x = \tilde{b}, x \geq 0 \iff Ax \leq b,
  \]
  with: \( A := \begin{pmatrix} \tilde{A} & -\tilde{A} \\ -I_{\tilde{n}} \end{pmatrix} \in \text{Mat}(\mathbb{R}, 2 \tilde{m} + \tilde{n}, \tilde{n}) \) and \( b := \begin{pmatrix} \tilde{b} \\ -\tilde{b} \\ 0 \end{pmatrix} \in \mathbb{R}^{2 \tilde{m} + \tilde{n}} \). Once again, it is clear that \( m := 2 \tilde{m} + \tilde{n} > n := \tilde{n} \) and that \( A \) has full column-rank (equal to \( n \)).

- Finally, if a polyhedron is defined as the set
  \[
P(\tilde{A}, \tilde{b}) := \{ x \in \mathbb{R}^\tilde{n} : \tilde{A}x \leq \tilde{b} \}.
  \]
Writing every element \( x \) of \( \mathbb{R}^\tilde{n} \) as: \( x = x_+ - x_- \) with \( x_+, x_- \geq 0 \) one has:

\[
\tilde{A}x \leq \tilde{b} \iff \begin{cases} \tilde{A}(x_+ - x_-) \leq \tilde{b} \\ x_+, x_- \geq 0 \end{cases} \equiv \begin{pmatrix} \tilde{A} & -\tilde{A} \\ -I_{\tilde{n}} & 0_{\tilde{n}, \tilde{n}} \end{pmatrix} \begin{pmatrix} x_+ \\ 0 \end{pmatrix} \leq \begin{pmatrix} \tilde{b} \\ 0 \end{pmatrix}.
\]

Let \( m := \tilde{m} + 2 \tilde{n} \) and \( n := 2 \tilde{n} \). Then, the matrix \( A := \begin{pmatrix} \tilde{A} & -\tilde{A} \\ -I_{\tilde{n}} & 0_{\tilde{n}, \tilde{n}} \end{pmatrix} \in \text{Mat}(\mathbb{R}, m, n) \) such that its submatrix \( A_2 := \begin{pmatrix} -I_{\tilde{n}} & 0_{\tilde{n}, \tilde{n}} \\ 0_{\tilde{n}, \tilde{n}} & -I_{\tilde{n}} \end{pmatrix} \) is clearly invertible. And thus, \( A \) has full column rank \( n \) with \( m > n \).
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