Abstract: We consider the edge statistics of Dyson Brownian motion with deterministic initial data. Our main result states that if the initial data has a spectral edge with rough square root behavior down to a scale $\eta_0 \geq N^{-2/3}$ and no outliers, then after times $t \gg \sqrt{\eta_0}$, the statistics at the spectral edge agree with the GOE/GUE. In particular we obtain the optimal time to equilibrium at the edge $t = N^{\epsilon}/N^{1/3}$ for sufficiently regular initial data. Our methods rely on eigenvalue rigidity results similar to those appearing in [36, 40], the coupling idea of [15] and the energy estimate of [14].

1 Introduction

One of the guiding principles of random matrix theory is that of universality. This states that the limiting behavior of the eigenvalues of large random matrices is universal. Said differently, universality is the observation that often the eigenvalue fluctuations of two seemingly unrelated random matrix ensembles converge, as the size of the matrices $N \to \infty$, to the same limiting object.

Perhaps the most fundamental class of random matrix ensembles are Wigner matrices. These are $N \times N$ symmetric matrices $W$ with independent (up to the constraint $W = W^*$) centered entries of identical variances. We consider two symmetry classes of Wigner matrices; real symmetric, in the case that the entries are real; or complex Hermitian in the case that the entries are complex. The prototypical examples of real symmetric and complex Hermitian Wigner matrices are the Gaussian Orthogonal and Unitary ensembles (GOE/GUE). These are constructed by taking the entries to be standard real or complex Gaussians, respectively. In the case of the GOE/GUE, the limiting eigenvalue behavior can be computed explicitly. The universality conjecture for Wigner matrices can be rephrased as the fact that these formulas are true, in the limit $N \to \infty$, for all Wigner matrices regardless of the details of the distribution of the matrix elements, and depend only on the symmetry class (real symmetric or complex Hermitian) of the ensemble.

There has been significant progress on the understanding of bulk universality for random matrix ensembles. Bulk universality refers to the behavior of eigenvalues contained in the interior of the support of the eigenvalue density. Bulk universality for Wigner matrices of all symmetry classes was proven in the works [20,22–24,27,29]. Parallel results were established in certain cases in [44,45], with the key result being a “four moment comparison theorem.”

One of the major contributions of the works [20,22–24,27,29] was to establish a general, robust framework within which to establish bulk universality for random matrix ensembles. This three-step strategy is as follows.

1. Prove a local law for the random matrix ensemble under consideration. This local law is used to establish high probability rigidity estimate for the eigenvalue locations.

2. Given a random matrix ensemble $H$, establish bulk universality for a Gaussian divisible ensemble of the form $H + \sqrt{t}G$, where $G$ is a GOE/GUE matrix, and $t = o(1)$ is interpreted as time.

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3. A density argument comparing the eigenvalue statistics of a general ensemble to a Gaussian divisible ensemble for which universality was established in the previous step.

The first step is model-dependent, and gives strong a-priori estimates crucial to the next two steps. The third step is a perturbation argument, typically based on comparing the expectation of the Green’s functions, or a use of the Ito lemma.

For the present article, the second step is most relevant. It is based on an analysis of the local ergodicity of *Dyson Brownian motion*. In the seminal work [19], Dyson introduced a matrix-valued stochastic process and calculated the resulting eigenvalue dynamics. At each fixed time \( t \), the eigenvalues of this matrix-valued Brownian motion are distributed as the eigenvalues of a Gaussian divisible ensemble. Crucially, he found that the eigenvalues satisfy a closed system of stochastic differential equations, for which the GOE/GUE is the equilibrium measure.

While global eigenvalue statistics reach equilibrium under DBM only for long times \( t \gg 1 \), Dyson conjectured that local statistics reach equilibrium after a much shorter time. The works [24, 25, 28] established that if Dyson Brownian motion is started from a Wigner matrix, then the bulk GOE statistics are reached in a very short time \( t \sim N^{-1} \). This is one of the crucial elements in proving bulk universality for Wigner matrices. Moreover, it provides a crucial conceptual understanding of the origin of universality - that of the local ergodicity of Dyson Brownian motion.

More recently, there has been much success in applying the above three-step strategy to the bulk statistics of random matrix ensembles which go beyond the Wigner class. As in the case of Wigner matrices, one of the important ingredients has been a study of the behavior of Dyson Brownian motion for more general initial data [26, 34, 35]. The main contribution of these works is to establish bulk universality for the gap statistics [26, 35] and correlation functions [34] of Dyson Brownian motion with very general deterministic or random initial data, going far beyond the class of Wigner matrices. In terms of the three-step strategy outlined above and in the context of obtaining bulk universality, this takes care of the second step for many random matrix ensembles.

While the third step is quite robust and applies in most cases, obtaining a local law is typically model dependent and challenging. We mention here some of the recent works on local laws and bulk universality for random matrix ensembles going beyond the Wigner class. The adjacency matrices of \( d \)-regular graphs were considered in [11, 12], and other sparse random graph ensembles in [1, 20, 22, 30, 32]. A very general class of Wigner-type matrices were studied in a series of papers [2, 5, 6], and matrices with correlated entries were studied in [3, 4, 17]. An additive model of random matrices was studied in [8–10, 18]. The proof of the local ergodicity of Dyson Brownian motion [26, 34, 35] for general initial data has been an important element in obtaining the eigenvalue universality in many of these works.

So far, the above discussion has been limited to the behaviour of the bulk eigenvalues, or those confined to the interior of the spectrum. A natural question is also to investigate the behavior of the extremal eigenvalues of random matrices. In the case of the Gaussian ensembles, Tracy and Widom [46, 47] calculated

\[
\lim_{N \to \infty} \mathbb{P} \left[ N^{2/3} (\lambda_N - 2) \leq s \right] = F_\beta(s)
\]

where \( \lambda_N \) is the largest eigenvalue of a GOE/GUE matrix and \( F_\beta \) can be characterized in terms of Painlevé equations and \( \beta = 1, 2 \) respectively for the real symmetric and complex Hermitian ensembles.

Edge universality was first obtained by Soshnikov for a wide class of Wigner ensembles via the moment method [43]. This method required symmetry of the distribution of the individual matrix elements. This requirement was partially removed in [42]. A different approach to edge universality based on direct comparison to Gaussian ensembles was developed in [29, 44]. The work [29] proves edge universality for general Wigner matrices under only a high finite moment condition. This latter condition was partially removed in [20]. In the work [41] it was shown that finiteness of the fourth moment is an essentially optimal condition for edge universality of Wigner matrices. An almost-optimal necessary condition was obtained earlier in [7].

In contrast with the existing work on bulk universality of Wigner matrices and the three step strategy outlined above, these works make no use of Dyson Brownian motion. As shown in [29], edge universality can be proven by directly comparing the eigenvalue statistics of an arbitrary Wigner matrix to that of the corresponding Gaussian ensemble. To put it another way, the Green function
comparison theorem can be used even if the Gaussian component is large, i.e., \( t \) order 1. This is in contrast to the bulk where it is required that \( t \ll 1 \) in order to match Green’s functions.

The work [14] implemented the above three-step strategy to prove edge universality of generalized Wigner matrices, which are ensembles in which the matrix element variances are allowed to vary. Such ensembles cannot be directly matched to the GOE/GUE and so it was required to establish first edge universality for a Gaussian divisible ensemble. The approach there uses a certain random walk representation of the correlation functions as well as the uniqueness of Gibbs measures of local log gasses. Moreover, the work [14] establishes the edge universality of \( \beta \)-ensembles for general potentials and \( \beta \geq 1 \).

The three-step strategy offers an attractive route to analyze the edge universality of general random matrix models. For general ensembles the moment method may fail, especially if the spectral edge is not an extremal eigenvalue - for example, models whose limiting spectral density has multiple intervals of support, or adjacency matrices of random graphs which may have outlying eigenvalues separated from the density of states.

In the present work we analyze the local ergodicity of the edge statistics of Dyson Brownian motion for general initial data \( V \). Our main result is that if the initial data has an edge with square root behavior down to a scale \( \eta_s \gtrsim N^{-2/3} \), then the distribution of the first eigenvalue at that spectral edge is given by the Tracy-Widom law in the limit \( N \to \infty \) for \( t \gg \sqrt{\eta_s} \). The assumption on “square root behavior” is quantified in terms of the imaginary part of the Stieltjes transform of \( V \).

In particular, this result solves the DBM component of applications of the three-step strategy to edge universality of general random matrix ensembles. In a joint work with J. Huang [33], we apply the main result of the present paper to analyze the edge statistics of sparse random matrix ensembles.

Another approach to edge universality has been developed by Lee-Schnelli in the works [37–39]. In this approach, the edge statistics of ensembles are calculated by direct comparison of the Green’s function to the GOE via a continuous interpolation. To rephrase this in our language, the change in expectation of Green’s function elements under the DBM flow are carefully calculated for long times \( t \gg 1 \). In the work [29] it was noted that naive power counting arguments can be improved using higher order cancellations in the trace of the Green’s function. The works [37–39] can be interpreted as systematically extending this cancellation to arbitrarily high orders.

The methodology and scope of [37–39] are unrelated to the present work. Our work concerns models of the form \( A + \gamma B \) for GOE \( B \) and arbitrary \( A \), whereas in [37–39], \( A \) is a random matrix with independent entries (up to \( \gamma_B = A^\gamma \)), such as a Wigner or sparse random matrix. We allow \( \gamma = N^{-c} \) and find the optimal size which can be viewed as Dyson’s conjecture of local ergodicity for edge statistics.

A short-time edge universality result for general initial data is useful for random matrix theory. Our result is applied in a joint work with J. Huang on the edge statistics of sparse Erdős-Rényi matrices [33]. In [33], Green’s function methods are used to uncover a Gaussian shift to the position of the extremal eigenvalues in the regime that the edge probability satisfies \( N^{-7/9} \ll p \ll N^{-2/3} \). After subtracting this shift, the edge statistics are compared to a Gaussian divisible ensemble; the present work implies that the latter has Tracy-Widom fluctuations. This implies that the eigenvalue gaps near the edge have the same distribution as the GOE, and that at \( p = N^{-2/3} \), the extremal eigenvalues converge to a sum of Gaussian and Tracy-Widom distribution.

Our analysis of DBM is based around the coupling idea of [15] and matching idea of [35]. At a time \( t_0 \gg \sqrt{\eta_s} \) we find that the free convolution, which gives the macroscopic eigenvalue density of DBM, has a square root behavior at the edge. We re-scale and shift the DBM so that this edge matches the semicircle density of the GOE. For times \( t \geq t_0 \), the Dyson Brownian motion is then coupled (as in [15]) to DBM with initial data the matching GOE ensemble. Under this coupling, the difference between these two stochastic eigenvalue locations obeys a simple discrete parabolic equation.

In order to analyze the discrete parabolic equation, it is convenient to localize our analysis by introducing a “short-range approximation” to Dyson Brownian motion. Coupling the short-range approximations instead of the full DBMs leads to a parabolic equation whose heat kernel has rapid off-diagonal decay. Such finite speed estimates first appeared in [16]. These rapid decay estimates offer several advantages. Due to the fact that the free convolution law varies on the scale \( t^2 \) near the
edge, the DBM may only be matched to a GOE matrix locally, and not globally. The finite speed estimates then allow for a cut-off of these non-matching elements. Moreover, as our assumptions on \( V \) are only local, it is possible that the DBM evolution away from the edge is irregular - the finite speed estimates ensure that this does not affect the behavior at the edge.

Our use of the short-range approximation and the finite speed estimates of [16] ensure that we do not rely on level repulsion estimates. Level repulsion estimates were proven for DBM in the bulk in [35]. Near the edge, these kinds of results for Dyson Brownian motion are unknown and do not appear to be a direct generalization of the method in the bulk.

Our analysis of the resulting parabolic equation is based around the energy estimate of [14]. The input of this into our work is that the \( F^2 \) norm of the solution to the parabolic equation is much smaller than \( N^{-2/3} \) for times \( t \gg N^{-1/3} \). Hence, we find that the edge statistics of DBM match those of the coupled GOE ensemble down to scale \( N^{-2/3} \), yielding the universality.

An additional input of our work is rigidity for Dyson Brownian motion. Rigidity for long times \( t \gg 1 \) were established in [36] via matrix methods, and in the bulk this method was extended to short times in [35]. In [31] rigidity in the bulk was established using purely dynamical methods. In this work we need to establish rigidity at the edge; we chose to do so using matrix methods which is a straightforward extension of [36, 40]. The purely dynamical approach does not appear to be easily extended to the edge.

The remainder of the work is organized as follows. In 2 we define precisely our model and state our main results. Our main contribution is in Section 3, which is our analysis of DBM via coupling as outlined above. Section 4 contains an auxiliary calculation needed in Section 3. The local deformed law is proven in Sections 5 and 6. Finally we analyze the regularity of the deformed semicircle law in Section 7.

1.1 Asymptotic notation

The fundamental large parameter in our paper is \( N \), the size of our matrices, and all asymptotic notation is wrt \( N \).

We use \( C \) to denote generic \( N \)-independent constants, the value of which may change from line to line in proofs.

For two possibly \( N \)-dependent nonnegative parameters \( X, Y \) we use the notation

\[
X \asymp Y
\]

to denote the fact that there is a constant \( C > 0 \) so that

\[
\frac{1}{C} X \leq Y \leq C X.
\]

For two possibly \( N \)-dependent parameters \( X \) and \( Y \), with \( X \) possibly complex and \( Y \) nonnegative, the notation

\[
X \leq O(Y), \quad X = O(Y)
\]

means that \( |X| \leq CY \) for some constant \( C > 0 \). The notation \( X = Y + O(Z) \) means \( X - Y = O(Z) \).

2 Definition of model and main results

In this paper we will consider models of the form

\[
H_t := V + \sqrt{t} G
\]

where \( V \) is a deterministic diagonal matrix, and \( G \) is a GOE matrix. We define the Stieltjes transform of \( V \) by

\[
m_V(z) = \frac{1}{N} \sum_i \frac{1}{V_i - z}.
\]

WLOG we assume that \( V_i \) are in increasing order. We consider the following class of \( V \).
Definition 2.1. Let \( \eta_\ast \) be a parameter satisfying
\[
\eta_\ast := N^{-\phi_\ast}
\] (2.3)
for some \( 0 < \phi_\ast \leq 2/3 \). We say that \( V \) is \( \eta_\ast \)-regular if
1. There is a constant \( C_V > 0 \) such that
\[
\frac{1}{C_V} \frac{\eta}{\sqrt{|E| + \eta}} \leq \text{Im}[m_V(E + i\eta)] \leq C_V \frac{\eta}{\sqrt{|E| + \eta}},
\] (2.4)
and
\[
\frac{1}{C_V} \frac{\eta}{\sqrt{|E| + \eta}} \leq \text{Im}[m_V(E + i\eta)] \leq C_V \sqrt{|E| + \eta},
\] (2.5)
2. There are no \( V_i \) in the interval \([-1, -\eta_\ast] \).
3. We have \(||V|| \leq N^{C_V} \) for some \( C_V > 0 \).

There are many possible reformulations of the above assumptions. We summarize a few of these observations in the remarks below.

Remark.
1. The motivation for the upper and lower bounds of assumption 1 is as follows. If \( m(z) \) is the Stieltjes transform of a measure with density \( \rho(z) \) such that \( \rho(z) \approx 1_{\{z \geq 0\}} \sqrt{\pi} \) then the estimates of assumption 1 holds for \( \text{Im}[m(z)] \) all \( z = E + i\eta \) near 0.
2. The first two assumptions 1, 2 are equivalent (up to constants) to the estimates of assumption 1 holding, as well as the estimate (2.4) holding in the larger regime \( 10 \geq \eta \geq 0 \) and \(-1 \leq E \leq \eta_\ast \).
3. The second assumption together with only the estimate (2.5) imply that (2.4) holds on a slightly smaller domain \(-1 + c \leq E \leq 0 \), any \( c > 0 \). For clarity, we have chosen to list both estimates of assumption 1 regardless of this redundancy.
4. The choice of the interval \([-1, 1] \) and the 10 in 10 \( \geq \eta \) in the first two assumptions plays no role, it is only important that the estimates hold in an order 1 interval near 0. We just use the above for notational simplicity.
5. Our main result below, Theorem 2.2, in fact holds under weaker assumptions regarding the width of the interval on which we assume estimates for \( V \) - that is, the width of the interval may go to 0 with \( N \). For simplicity of proofs we have not explored the optimal assumption.
6. The above set-up is for an extremal eigenvalue at a left edge. Of course, one can consider also a right edge, etc.

We now state our main result. We denote the eigenvalues of \( H_t \) by \( \lambda_i \).

Theorem 2.2. Let \( V \) be \( \eta_\ast \)-regular. Let \( t \) satisfy \( N^{-\varepsilon} \geq t \geq N^{\varepsilon} \eta_\ast \). Recall that \( V_i \) are indexed in increasing order. Let \( i_0 \) be the index of the first \( V_i \geq -1/2 \). Fix \( k \) a nonnegative integer, and let \( F : \mathbb{R}^{k+1} \to \mathbb{R} \) be a test function such that
\[
||F||_\infty \leq C, \quad ||\nabla F||_\infty \leq C.
\] (2.6)
There are deterministic scaling factors \( \gamma_0 \) and \( E_- \) depending only on \( V \) such that,
\[
\left| \mathbb{E}[F(\gamma_0 N^{2/3}(\lambda_{i_0} - E_-), \cdots , N^{2/3} \gamma_0 (\lambda_{i_0+k} - E_-))] 
- \mathbb{E}(\text{GOE})[F(N^{2/3}(\mu_1 + 2), \cdots , N^{2/3}(\mu_{1+k} + 2))] \right| \leq N^{-c}
\] (2.7)
for some \( c > 0 \). The latter expectation is with respect to the eigenvalues \( \mu_i \) of a GOE. The scaling factor \( \gamma_0 \approx 1 \) and the magnitude of \( E_- \) is bounded above.

In the next subsection we define the scaling factors \( \gamma_0 \) and \( E_- \). They are defined in terms of the free convolution of \( V \) and the semicircle distribution.
2.1 Free convolution

The eigenvalue density of $H_t$ is described by the free convolution of $V$ with the semicircle distribution at time $t$, which we denote by $\rho_{\text{fc},t}$. The free convolution is well studied and we refer the reader to, e.g., [13] for its properties. It is defined by its Stieltjes transform which is the unique solution to the following fixed point equation, that has the property $|m(z)| \sim |1/z|$, 

$$m_{\text{fc},t} = m_V(z + tm_{\text{fc},t}(z)) = \frac{1}{N} \sum_i \frac{1}{V_i - z - tm_{\text{fc},t}(z)}.$$  \hfill (2.8)

For $t > 0$, the density $\rho_{\text{fc},t}$ is analytic and may be recovered by the Stieltjes inversion formula,

$$\rho_{\text{fc},t}(E) = \lim_{\eta \to 0} \frac{\text{Im}[m_{\text{fc},t}(E + i\eta)]}{\pi}.$$  \hfill (2.9)

The following lemma will be proven in Section 7. It follows from Lemma 7.8.

**Lemma 2.3.** Let $V$ be $\eta_\ast$-regular and $N^{-\epsilon} \geq t \geq N^\epsilon \eta_\ast$. The support of the restriction of $\rho_{\text{fc},t}$ to $[-3/4, 3/4]$ consists of a single interval of the form $[E_-(t), 3/4]$ where 

$$|E_-(t)| \leq C t \log(N)$$  \hfill (2.10)

for a constant $C$. For $|E - E_-| \leq ct^2$ we have the expansion

$$\rho_{\text{fc},t}(E) = 1_{\{E \geq E_\ast\}} \gamma_0^{-1/2} \sqrt{E - E_-} \left(1 + t^{-2}(E - E_-)\right)$$  \hfill (2.11)

where the scaling factor $\gamma_0$ is defined by

$$\gamma_0 := \left(\frac{3}{N} \sum_i \frac{1}{(V_i - E_- - tm_{\text{fc},t}(E_-))^3}\right)^{-1/3} \approx 1.$$  \hfill (2.12)

The scaling factors $\gamma_0$ and $E_-(t)$ are those given in the statement of Theorem 2.2.

2.1.1 Free convolution properties and associated notation

In this short subsection we summarize some of the key properties of $m_{\text{fc},t}$. We also introduce some notation which will be used when dealing with the free convolution. Define

$$\kappa := |E - E_-|, \quad \xi(z) := z + tm_{\text{fc},t}(z),$$  \hfill (2.13)

and

$$g_i := \frac{1}{V_i - z - tm_{\text{fc},t}}, \quad R_k := \frac{1}{N} \sum_i g_i^k$$  \hfill (2.14)

The following lemma is proved in Section 7. It follows from Lemma 7.9.

**Lemma 2.4.** Let $V$ and $t$ be as above. We have

$$\text{Im}[m_{\text{fc},t}] \approx \frac{\eta}{\sqrt{\kappa + \eta}}, \quad -3/4 \leq E \leq E_-,$$  \hfill (2.15)

and

$$\text{Im}[m_{\text{fc},t}] \approx \sqrt{\kappa + \eta}, \quad E_- \leq E \leq 3/4.$$  \hfill (2.16)

We will also use the notion of overwhelming probability.

**Definition 2.5.** We say that an event $F$ or possibly a family of events $F_u$ with $u$ in some index set $\mathcal{I}$, hold with overwhelming probability if

$$\inf_{u \in \mathcal{I}} \mathbb{P}[F_u] \geq 1 - N^{-D}$$  \hfill (2.17)

for any $D > 0$, for large enough $N$. We say that an event $F_1$ holds with overwhelming probability on an event $F_2$ if

$$\mathbb{P}[F_1 \cap F_2] \leq N^{-D}$$  \hfill (2.18)

for any $D > 0$ for large enough $N$. We also have a similar notion as above for families of events $F_{i,u}, u \in \mathcal{I}$.
3 DBM calculations

In this section we fix two time scales

\[ t_0 = \frac{N^{\omega_0}}{N^{1/3}}, \quad t_1 = \frac{N^{\omega_1}}{N^{1/3}}, \quad (3.1) \]

with

\[ 0 < \omega_1 < \omega_0/100. \quad (3.2) \]

The purpose of the introduction of these scales is to first “regularize” the global eigenvalue density that \( H_t \) follows. For times \( t_0 \leq t \leq t_0 + t_1 \) we use a coupling idea of [15], and couple the DBM process started from \( H_{t_0} \) to a GOE ensemble. For times \( t_0 \leq t \leq t_0 + t_1 \) we will have to track the evolution of the edge of the ensemble, but the scaling factor (i.e., the size of the eigenvalue gaps) will remain approximately constant due to the fact that \( t_1 \ll t_0 \).

Recall the definition of \( t_0 \) as in the statement of Theorem 2.2. For \( t \geq 0 \) we define the process \( \lambda_i \) by

\[ d\lambda_i = \frac{dB_i - t_0 \lambda_i + 1}{\sqrt{N}} + \frac{1}{N} \sum_j 1 \frac{1}{\lambda_i - \lambda_j} dt, \quad (3.3) \]

with initial data

\[ \lambda_i(0) = \lambda_i(\gamma_0 H_{t_0}), \quad (3.4) \]

where \( \gamma_0 \) is as defined in Section 2.1. Note that for each time \( t \), the process \( \{\lambda_i(t)\}_i \) is distributed like the eigenvalues of the matrix \( \gamma_0 V + \sqrt{\gamma_0 t_0} tG \) where \( G \) is a GOE matrix. Above \( \{B_i\}_{1 \leq i \leq N} \) are standard independent Brownian motions. In terms of the free convolution law, the edge of the \( \lambda_i(t) \) is given by

\[ E_\lambda(t) := \gamma_0 E_\mu(t_0 + t^2/\gamma_0^2). \quad (3.5) \]

Note that \( \gamma_0 \) is fixed - it was chosen from time \( t_0 \), but the edge continues to evolve in time.

The purpose of the re-scaling by \( \gamma_0 \) is so that constant scaling the square root in (2.11) is changed to 1 at time \( t_0 \), matching the semicircle.

We now define \( \mu_i \) as the unique strong solution to the SDE,

\[ d\mu_i = \frac{dB_i}{\sqrt{N}} + \frac{1}{N} \sum_j 1 \frac{1}{\mu_i - \mu_j} dt, \quad (3.6) \]

with initial data \( \mu_i(0) \) being distributed like the eigenvalues of a GOE matrix independent of \( H_{t_0} \). Then for each time \( t \), the \( \{\mu_i(t)\}_i \) are distributed like the eigenvalues of \( \sqrt{T+G} \) for \( G \) a GOE matrix, and their edge is given by

\[ E_\mu(t) = -2\sqrt{T+t}. \quad (3.7) \]

The main result of this section is the following.

**Theorem 3.1.** Let \( t_1 \) be as above. With overwhelming probability, we have

\[ |(\lambda_{i_0+i-1}(t_1) - E_\lambda(t_1)) - (\mu_i - E_\mu(t_1))| \leq \frac{1}{N^{2/3+c}} \quad (3.8) \]

for a \( c > 0 \) and for any finite \( 1 \leq i \leq K \).

Note that the Brownian motions for \( \mu_1 \) and \( \lambda_{i_0} \) are identical. At this point, we want to take the difference of the \( \mu_i \) and \( \lambda_j \), but we need to pad each system with dummy particles so that the difference is defined for all indices \( i \).

More precisely, we let the system \( \{x_i\}_{1 \leq i \leq N} \) of \( 2N + 1 \) particles be defined by

\[ dx_i = \frac{dB_i}{\sqrt{N}} + \frac{1}{N} \sum_j 1 \frac{1}{x_i - x_j} dt, \quad (3.9) \]
with initial data
\[
x_i(0) = \begin{cases} 
-3N^{CV} + iN, & -N \leq i \leq 1 - i_0 \\
\lambda_{i+i_0-1}(0), & 2 - i_0 \leq i \leq N + 1 - i_0 \\
3N^{CV} + iN, & N + 2 - i_0 \leq i \leq N 
\end{cases}
\] (3.10)
and \( y_i \) be defined by
\[
dy_i = \frac{dB_i}{\sqrt{N}} + \frac{1}{N} \sum_j \frac{1}{y_i - y_j} dt,
\] (3.11)
with initial data
\[
y_i(0) = \begin{cases} 
-3N^{CV} + iN, & -N \leq i \leq 0 \\
\mu_i(0), & 1 \leq i \leq N 
\end{cases}
\] (3.12)
The following follows from Appendix C of [34], and we omit the proof.

**Lemma 3.2**. **With overwhelming probability, the following estimates hold.** We have,
\[
\sup_{0 \leq t \leq 1} \sup_{2 - i_0 \leq i \leq N + 1 - i_0} |x_i(t) - \lambda_{i+i_0-1}(t)| \leq \frac{1}{N^{100}},
\] (3.13)
and
\[
\sup_{0 \leq t \leq 1} \sup_{1 \leq i \leq N} |y_i(t) - \mu_i(t)| \leq \frac{1}{N^{100}},
\] (3.14)
and
\[
\sup_{0 \leq t \leq 1} x_{1-i_0}(t) \leq -2N^{CV}
\]
\[
\inf_{0 \leq t \leq 1} x_{N+2-i_0}(t) \geq 2N^{CV}
\]
\[
\sup_{0 \leq t \leq 1} y_0(t) \leq -2N^{CV}.
\] (3.15)

### 3.1 Interpolation

While we would like to take the difference \( x_i - y_i \) directly, the equation that this function satisfies is quite singular. Instead, we introduce an interpolation that results in a better equation. A similar interpolation appeared in [34].

We define the following interpolating processes for \( 0 \leq \alpha \leq 1 \).
\[
dz_i(t, \alpha) = \frac{dB_i}{\sqrt{N}} + \frac{1}{N} \sum_i \frac{1}{z_i(t, \alpha) - z_j(t, \alpha)} dt
\] (3.16)
with initial data
\[
z_i(0, \alpha) := ax_i(0) + (1 - \alpha)y_i(0).
\] (3.17)
We define
\[
m_N(t, \alpha) = \frac{1}{N} \sum_i \frac{1}{z_i(t, \alpha) - z}.
\] (3.18)
Define \( \rho_{x,t} \) to be the free convolution of \( \gamma_0 \) with the semicircle at time \( t_0 + t/\gamma_0^2 \). The edge is given by \( E_x(t) = E_\lambda(t) \) as above. Denote \( \rho_{y,t} \) to be the semicircle at time \( \sqrt{1 + t} \) with edge \( E_y(t) = E_\mu(t) \) as above.

By our choice of \( \gamma_0 \) (see Lemma 7.8) we have
\[
\rho_{x,0}(E + E_x(0)) = \rho_{x,0}(E + E_y(t)) \left( 1 + \mathcal{O}\left( \left| \frac{E}{t_0^2} \right| \right) \right), 0 \leq E \leq ct_0^2
\] (3.19)
for some \( c > 0 \). Let \( \gamma_{x,t}(t) \) and \( \gamma_{y,t}(t) \) be the quantiles - more precisely, we define them by
\[
\frac{i}{N} = \int_{-1/2}^{\gamma_{x,t}(t)} \rho_{x,t}(E) dE = \int_{-10}^{\gamma_{y,t}(t)} \rho_{y,t}(E) dE.
\] (3.20)
By the local law estimates of Section 5 (as well as a stochastic continuity argument as in Appendix B of [34] to pass from fixed times $t$ to all times) we know that there is a $k_\ast \asymp N$ so that

$$\sup_{0 \leq t \leq 10t_\ast} |z_i(t, 1) - \gamma_{x,i}(t)| \leq \frac{N^\varepsilon}{i^{1/3}N^{2/3}}, \quad 1 \leq i \leq k_\ast \quad (3.21)$$

with overwhelming probability, for any $\varepsilon > 0$, and a similar estimate for $z_i(t, 0) = y_i(t)$. An easy calculation using (3.19) gives that

$$|\gamma_{x,i}(0) - E_x(0)| - (\gamma_{y,i}(0) - E_y(0))| \leq C \frac{\varepsilon^{3/2}}{N^{2+\varepsilon}N^{2/3}}, \quad (3.22)$$

for $1 \leq i \leq N^{6w_0/5}/C$.

### 3.1.1 Construction of a density for interpolating ensembles

We will need measures with well-behaved square root densities that give the eigenvalue density of the interpolating ensembles. For this, we construct the following measures. First, let $\rho_{x,0}$ and $\rho_{y,0}$ as above.

Define the eigenvalue counting functions near 0 by

$$n_x(E) = \int_{-1/2}^{E} d\rho_{x,0}(E'), \quad n_y(E) = \int_{-10}^{E} d\rho_{y,0}(E'), \quad (3.23)$$

and then the eigenvalue counting functions by

$$n_x(\varphi_x(s)) = s, \quad n_y(\varphi_y(s)) = s. \quad (3.24)$$

Define now

$$\varphi(s, \alpha) := \alpha \varphi_x(s) + (1 - \alpha) \varphi_y(s) \quad (3.25)$$

on the domain

$$\varphi(s, \alpha) : [0, k_\ast/N] \rightarrow [\alpha E_x(0) + (1 - \alpha) E_y(0), \alpha \varphi_x(k_\ast/N) + (1 - \alpha) \varphi_y(k_\ast/N)] := F_\alpha. \quad (3.26)$$

Define now the inverse function $n(E, \alpha) : F_\alpha \rightarrow [0, k_\ast/N]$ by

$$n(\varphi(s, \alpha), \alpha) = s, \quad (3.27)$$

and finally the density $\rho(E, \alpha)$ on the interval $F_\alpha$ by

$$\rho(E, \alpha) := \frac{d}{dE} n(E, \alpha). \quad (3.28)$$

From the inverse function theorem we see that

$$\rho(E, \alpha) = (\alpha(\rho_x(n(E, \alpha)))^{-1} + (1 - \alpha)(\rho_y(n(E, \alpha)))^{-1})^{-1} \quad (3.29)$$

from which we can deduce that

$$\rho(E + E_-(\alpha), \alpha) = \rho_y(E + E_y(0)) \left(1 + \mathcal{O}\left(|E|/t_0^2\right)\right), \quad 0 \leq E \leq d_0^2. \quad (3.30)$$

Using these, we need to construct measures $\mu(E, \alpha)$ to which $m_N(0, \alpha)$ are close. We construct these as follows.

$$d\mu(E, \alpha) = \rho(E, \alpha) 1_{\{E \in F_\alpha\}} dE + \frac{1}{N} \sum_{i \leq 0} \delta_{\varphi_{x,0}(0)}(E) + \frac{1}{N} \sum_{i > k_\ast} \delta_{\varphi_{x,0}(E)}(E). \quad (3.31)$$

The motivation for this definition is as follows. We would like to take a deterministic density that matches $z_i(0, \alpha)$ with which to take a free convolution with. However, we have no real control on the density away from 0, so we can only find a deterministic density near 0 - this is the role of $\rho(E, \alpha)$ -
this density matches $z_i(0, \alpha)$ near 0. For the remaining particles which are an order 1 distance away from our point of interest we just take $\delta$ functions. While this part of the measure is random, we can control the effect that it has on deterministic quantities that we need.

We let now $\rho_t(E, \alpha)$ be the free convolution of $\mu(\alpha)$ with the semicircle at time $t$, and denote the Stieltjes transform by $m_t(z, \alpha)$. The properties of these measures are studied in Section 7.3. In particular, they have a square root density which we denote by $\rho_t(E, \alpha)$ with an edge which we denote by $E_-(t, \alpha)$.

With $\mu$ as constructed above, it is not hard to see that the difference $m_N(z, 0, \alpha) - m_0(z, \alpha)$ obeys the estimates outlined at the start of Section 6. Let $\gamma_i(t, \alpha)$ be the classical eigenvalue locations wrt $\rho_t(E, \alpha)$. To be more precise, they are defined by

$$\frac{i}{N} = \int_{E_-(t, \alpha)}^{\gamma_i(t, \alpha)} \rho_t(E, \alpha) dE.$$  (3.32)

As a consequence of Section 6 we have the following.

**Lemma 3.3.** There is an $i_* \sim N$ so that the following estimates hold. We have,

$$\sup_{0 \leq \alpha \leq 1} \sup_{0 \leq t \leq \alpha t_1} |z_i(t, \alpha) - \gamma_i(t, \alpha)| \leq \frac{N^2}{N^{2/3} \alpha^{1/3}},$$  (3.33)

for $1 \leq i \leq i_*$, for any $\varepsilon > 0$ and with overwhelming probability.

While the measures $\rho_t(E, \alpha)$ are random, they obey estimates wrt deterministic quantities with overwhelming probability. In particular, we see from Section 7.3 (more precisely, Lemmas 7.11 and 7.12) that

**Lemma 3.4.** For all $0 \leq E \leq cN^{-2} t_0^2$ we have

$$\rho_t(E + E_-(t, \alpha)) = \rho_{y,t}(E + E_{y,t}) \left(1 + O(N^\varepsilon t/t_0)\right),$$  (3.34)

and

$$|\text{Re}[m_t(E + E_-(t, \alpha)) - m_t(E_-(t, \alpha))] - \text{Re}[m_{y,t}(E + E_{y,t}) - m_{y,t}(E_{y,t})]| \leq C \frac{|E|N^\varepsilon}{t_0},$$  (3.35)

and for $cN^{-2} t_1 t_0 \leq E \leq 0$,

$$|\text{Re}[m_t(E + E_-(t, \alpha)) - m_t(E_-(t, \alpha))] - \text{Re}[m_{y,t}(E + E_{y,t}) - m_{y,t}(E_{y,t})]| \leq C \frac{|E|^{1/2} (t_1)^{1/2} N^\varepsilon}{t_0^{1/2}}.$$  (3.36)

as well as

$$|\gamma_i(\alpha, t) - \gamma_i(0, t)| \leq N^2 \frac{2/3 t}{N^{2/3} t_0}.$$  (3.37)

Additionally, we have the following estimates, as proven in Appendix E.

**Lemma 3.5.** We have,

$$|E_-(t, 1) - E_{x,t}| \leq N^\varepsilon \left(t^3 + \frac{t}{N^{1/2}}\right)$$  (3.38)

and

$$|E_-(t, 0) - E_{y,t}| \leq N^\varepsilon \left(t^3 + \frac{t}{N^{1/2}}\right).$$  (3.39)
3.2 Short-range approximation

It will be convenient to introduce the shifted \( z_i(t, \alpha) \),

\[
  \tilde{z}_i(t, \alpha) := z_i(t, \alpha) - E_-(t, \alpha),
\]

which obey the SDE

\[
  d\tilde{z}_i(t, \alpha) = \frac{dB_i}{\sqrt{N}} + \frac{1}{N} \sum_j \frac{1}{\tilde{z}_i(t, \alpha) - \tilde{z}_j(t, \alpha)} dt + \text{Re}[m_\ell(E_-(t, \alpha), \alpha)] dt.
\]  

We also introduce the corresponding

\[
  \tilde{\gamma}_i(t, \alpha) := \gamma_i(t, \alpha) - E_-(t, \alpha).
\]

We now construct a “short-range” set of indices \( \mathcal{A} \subseteq [[-N, N]] \times [[-N, N]] \). We choose \( \mathcal{A} \) to be symmetric, i.e., \( (i, j) \in \mathcal{A} \) iff \( (j, i) \in \mathcal{A} \). The definition of \( \mathcal{A} \) requires the choice of

\[
  \ell := N^{\omega_p}.
\]

We let

\[
  \mathcal{A} := \left\{ (i, j) : i, j > 0, |i - j| \leq \ell(10\ell^2 + \ell^{2/3} + j^{2/3}) \right\} \bigcup \left\{ (i, j) : i, j > i_*/2 \right\} \bigcup \left\{ (i, j) : i, j \leq 0 \right\}.
\]

The following is not essential, but it will simplify notation. It is an exercise.

Lemma 3.6. For each \( i \), the set \( \{ j : (i, j) \in \mathcal{A} \} \) is an interval of natural numbers.

It will be convenient to introduce the following short-range summation notation. We define

\[
  \sum_{j}^{\mathcal{A}, (i)} := \sum_{j : (i, j) \in \mathcal{A}}, \quad \sum_{j}^{\mathcal{A}^c, (i)} := \sum_{j : (i, j) \not\in \mathcal{A}}.
\]

We also need the integral analogs of the above. For each \( i \), we define the interval

\[
  \mathcal{I}_i(\alpha, t) := [\tilde{\gamma}_-(-\alpha, t), \tilde{\gamma}_+(-\alpha, t)]
\]

where for each \( i \), \( \{ j : (i, j) \in \mathcal{A} \} := \{ [j_-, j_+] \} \). We remark that we are only going to use the classical eigenvalue locations \( \gamma_i(\alpha, t) \) for indices \( 1 \leq i \leq cN \) for some small \( c > 0 \cdot \) for such locations, the measure \( \rho_\ell(E, \alpha) \) is well-behaved (i.e., here it has a square root density). In particular, the behavior of the above intervals is relatively tame.

The short-range approximation to \( \tilde{z} \) is the process \( \tilde{z} \) defined as the solution to the following SDE. It requires the choice of an additional parameter \( N^{\omega_p} \). We introduce the notation

\[
  \mathcal{J}(\alpha, t) := [-1/2, \tilde{\gamma}^\omega_{\lfloor \alpha \rfloor/4}(\alpha, t)],
\]

where \( i_* \) is as in Lemma 3.3. Recall,

\[
  i_* = N.
\]

For \( i \leq 0 \) we let

\[
  d\tilde{z}_i(t, \alpha) = \frac{dB_i}{\sqrt{N}} + \frac{1}{N} \sum_j^{\mathcal{A}, (i)} \frac{1}{\tilde{z}_i(t, \alpha) - \tilde{z}_j(t, \alpha)} dt + \frac{1}{N} \sum_j^{\mathcal{A}^c, (i)} \frac{1}{\tilde{z}_i(t, \alpha) - \tilde{z}_j(t, \alpha)} dt + \text{Re}[m_\ell(E_-(t, \alpha), \alpha)],
\]

for \( 1 \leq i \leq N^{\omega_p} \),

\[
  d\tilde{z}_i(t, \alpha) = \frac{dB_i}{\sqrt{N}} + \frac{1}{N} \sum_j^{\mathcal{A}, (i)} \frac{1}{\tilde{z}_i(t, \alpha) - \tilde{z}_j(t, \alpha)} dt + \int_{\mathcal{I}_i(\alpha, t)} \frac{1}{\tilde{z}_i(t, \alpha) - E} \rho_\ell(E + E_-(t, 0), 0) dE dt + \text{Re}[m_\ell(E_-(t, 0), 0)] dt.
\]
for $N^{\omega_A} \leq i \leq i_*/2$,

$$
\frac{d\hat{z}_i(t, \alpha)}{\sqrt{N}} + \frac{1}{N} \sum_j A^{(i)}(\hat{z}_i(t, \alpha) - \hat{z}_j(t, \alpha)) + \int \frac{1}{\cal Z} \frac{E}{E_{\ell}(E + E_-(t, \alpha), \alpha)} dE
+ \sum_{j > 2^m/4, j \leq 0} \frac{1}{\hat{z}_j(t, \alpha) - \hat{z}_j(t, \alpha)} + \text{Re}[m_\ell(E_-(t, \alpha), \alpha)] dt
$$

(3.51)

and for $i_*/2 \leq i \leq N$,

$$
d\hat{z}_i(t, \alpha) = \frac{dE_1}{\sqrt{N}} + \frac{1}{N} \sum_j A^{(i)}(\hat{z}_i(t, \alpha) - \hat{z}_j(t, \alpha)) + \frac{1}{N} \sum_j A^{(i)}(\hat{z}_i(t, \alpha) - \hat{z}_j(t, \alpha)) + \text{Re}[m_\ell(E_-(t, \alpha), \alpha)],
$$

(3.52)

with initial data

$$
\hat{z}_i(0, \alpha) = \hat{z}_i(0, \alpha).
$$

(3.53)

We pause to review the hierarchy of scales that we have introduced. We have

$$
t_0 = \frac{N^{\omega_0}}{N^{1/3}}, \quad t_1 = \frac{N^{\omega_1}}{N^{1/3}}, \quad \ell = N^{\ell}, \quad N^{\omega_A}
$$

(3.54)

and

$$
0 \ll \omega_1 \ll \omega_\ell \ll \omega_A \ll \omega_0.
$$

(3.55)

The purpose of the scale $\ell$ is to cut-off the effect of the initial data far from the edge - that is, for large $i$, we do not have $\hat{z}_i(t = 0, \alpha = 1) \sim \hat{z}_i(t = 0, \alpha = 0)$, whereas for small $i$ they do in fact match. Secondly, we will want to differentiate in $\alpha$. The quantities that depend on $\alpha$, i.e., $\rho_\ell(E, \alpha)$ are approximately $\alpha$ independent near $E = 0$ (after accounting for the drift $\partial_t E_-(t, \alpha)$). This regularity scale comes $\omega_0$, and so we fix a scale $\omega_A \ll \omega_0$ on which we replace the $\alpha$-dependent quantities by $\alpha$-independent quantities.

We now show that $\hat{z}_i(t, \alpha)$ is a good approximation to $\hat{z}_i(t, \alpha)$. Make the following choices of parameters: choose $\omega_1 \ll \omega_\ell/10, \omega_\ell < \omega_A/20$ and $\omega_A < \omega_0/20$. With these choices, the error term on the RHS of (3.56) is $\ll N^{-2/3}$.

**Lemma 3.7.** With overwhelming probability,

$$
\sup_{0 \leq \alpha \leq 1} \sup_i \sup_{0 \leq t \leq 10t_1} |\hat{z}_i(t, \alpha) - \hat{z}_i(t, \alpha)| \leq \frac{N^{\varepsilon}}{N^{2/3}} \left( \frac{N^{\omega_1}}{N^{2\varepsilon}} + \frac{N^{\omega_1}}{N^{1/6}} + \frac{N^{2\omega_A/3 + \omega_1}}{N^{\omega_0}} + \frac{N^{\omega_A/3 + 2\omega_1}}{N^{\omega_0}} \right)
$$

(3.56)

for any $\varepsilon > 0$.

**Remark.** The important error on the RHS is the term $N^{\omega_1}/N^{2\omega_1}$. More precisely, it is important that $\omega_\ell$ appear with a power strictly greater than 1, for later estimates.

**Proof.** Let $v_i = \hat{z}_i - \hat{z}_i$. We have the equation

$$
\partial_t v = B_1 v + \mathcal{V}_1 v + \zeta,
$$

(3.57)

where $B_1$ is the operator defined by

$$
(B_1 v)_i = \frac{1}{N} \sum_j A^{(i)}(\hat{z}_i(\alpha) - \hat{z}_j(\alpha))(\hat{z}_i(\alpha) - \hat{z}_j(\alpha)),
$$

(3.58)

the operator $\mathcal{V}_1$ is diagonal: $(\mathcal{V}_1 v)_i = \mathcal{V}_1(i)v_i$, where for $\mathcal{V}_1(i) = 0$ for $i \leq 0$ or $i \geq i_*/2$ and for $1 \leq i \leq N^{\omega_A}$,

$$
\mathcal{V}_1(i) = -\int_{\cal Z(0,t)} \frac{\rho_\ell(E + E_-(t,0),0)(\hat{z}(\alpha,t) - E)(\hat{z}(\alpha,t) - E)}
$$

(3.59)
and for $N^{\omega_A} \leq i \leq i_*/2$, 

$$V_1(i) = - \int_{I_i^* \cap \mathcal{J}(a,t)} \frac{\rho_t(E + E_-(t,\alpha), \alpha)}{(\tilde{z}_i(t,\alpha) - E)(\tilde{\zeta}_1(t,\alpha) - E)}.$$

(3.60)

In particular

$$V_1 \leq 0,$$

(3.61)

and so the semigroup of $B_1 + V_1$ is a contraction on every $\ell^p$ space. Hence, for the difference we have,

$$v(t) = \int_0^t \mathcal{U}^{B_1 + V_1}(s,t)\zeta(s)ds,$$

(3.62)

and so

$$\|v(t)\|_\infty \leq t \sup_{0 \leq s \leq t} \|\zeta(s)\|_\infty,$$

(3.63)

and so we must estimate $\|\zeta\|_\infty$.

The error term $\zeta$ is given by $\zeta_i = 0$ for $i \leq 0$ and $i \geq i_*/2$, and for $1 \leq i \leq N^{\omega_A}$, it is given by

$$\zeta_i := \int_{I_i^* \cap \mathcal{J}(a,t)} \frac{\rho_t(E + E_-(t,\alpha), \alpha)}{(\tilde{z}_i(t,\alpha) - E)} - \frac{1}{N} \sum_{j \leq 3i_*/4} \frac{1}{\tilde{z}_i(t,\alpha) - \tilde{z}_j(t,\alpha)} + \text{Re}[m_i(E_-(t,\alpha), \alpha)] - \text{Re}[m_i(E_-(t,0),0)],$$

(3.64)

and for $N^{\omega_A} \leq i \leq i_*/2$ by

$$\zeta_i := \int_{I_i^* \cap \mathcal{J}(a,t)} \frac{\rho_t(E + E_-(t,\alpha), \alpha)}{(\tilde{z}_i(t,\alpha) - E)} - \frac{1}{N} \sum_{j \leq 3i_*/4} \frac{1}{\tilde{z}_i(t,\alpha) - \tilde{z}_j(t,\alpha)}.$$

(3.65)

We need to estimate $\|\zeta\|_\infty$. This term is controlled by

$$\left| \int_{\mathcal{J}(a,t) \cap I_i^*} \frac{\rho_t(E + E_-(t,\alpha), \alpha)}{(\tilde{z}_i(t,\alpha) - E)} - \frac{1}{N} \sum_{0 < j < 3i_*/4} \frac{1}{\tilde{z}_i(t,\alpha) - \tilde{z}_j(t,\alpha)} \right| \leq \frac{N^e}{N^{5/3}} \sum_{0 < j < 3i_*/4} \frac{1}{(\zeta_i - \zeta_j)^2 j^{1/3}} \leq \frac{C N^e}{N^{1/3}} \sum_{0 < j < 3i_*/4} \frac{1}{(\zeta_i - \zeta_j)^2 j^{1/3}}.$$

(3.66)

We first estimate,

$$\sum_{j > i} \frac{A^{\epsilon,\alpha}(i)}{(i-j)^2 j^{1/3}} \leq C \sum_{j > i} \frac{j^{1/3}}{(i-j)^2}$$

$$\leq C \sum_{j > i} \frac{j^{1/3}}{(i-j)^2} + \frac{1}{(i-j)^5/3}$$

$$\leq C \frac{i^{1/3}}{\ell(\ell^2 + i^{2/3})} + \frac{C}{(\ell(\ell^2 + i^{2/3}))^{2/3}}.$$

(3.67)

We also have

$$\sum_{j < i} \frac{A^{\epsilon,\alpha}(i)}{(i-j)^2 j^{1/3}} \leq C \sum_{j < i/2} \frac{j^{2/3}}{((i+\ell(\ell^2 + i^{2/3}))^2 j^{1/3} + C \sum_{i > j > i/2} \frac{j^{1/3}}{(i-j)^2}}$$

$$\leq C \frac{i^{4/3}}{(i+\ell(\ell^2 + i^{2/3}))^2} + C \frac{i^{1/3}}{\ell(\ell^2 + i^{2/3})} \leq C \frac{i^{1/3}}{\ell(\ell^2 + i^{2/3})}.$$
Hence,
\[ \frac{CN^\varepsilon}{N^{1/3}} \sum_{0<j<3\varepsilon/4} \frac{j^{2/3} + j^{2/3}}{(i-j)^2/j^{1/3}} \leq \frac{CN^\varepsilon}{N^{1/3}N^{2\omega^\varepsilon}}. \] (3.69)

In conclusion, for \( i \geq N^{\omega^\varepsilon} \),
\[ |\zeta_i| \leq N^\varepsilon \left( \frac{1}{N^{1/3}N^{2\omega^\varepsilon}} \right) \] (3.70)
for any \( \varepsilon > 0 \) with overwhelming probability. We now need to bound \( \zeta_i \) for \( 1 \leq i \leq N^{\omega^\varepsilon} \). We rewrite it as follows.

\[ \zeta_i = \left( \int_{I_i(a,t)} \rho_i(E + E_-(\alpha, t), \alpha) - \frac{1}{N} \sum_{j} \frac{A^{\varepsilon}(i)}{\hat{z}_i(t, \alpha) - \hat{z}_j(t, \alpha)} \right) + \text{Re}[\text{Re}e^{\eta_1}] \right) \right) + \left( \int_{I_i(a,t)} \rho_i(E + E_-(\alpha, t), \alpha) - \frac{1}{N} \sum_{j} \frac{A^{\varepsilon}(i)}{\hat{z}_i(t, \alpha) - \hat{z}_j(t, \alpha)} \right) \right) \) (3.71)

For the term \( B_1 \) we rewrite it as
\[ B_1 = \left( \int_{J_i(a,t)} \rho_i(E + E_-(\alpha, t), \alpha) - \frac{1}{N} \sum_{j} \frac{A^{\varepsilon}(i)}{\hat{z}_i(t, \alpha) - \hat{z}_j(t, \alpha)} \right) + \left( \int_{J_i(a,t)} \rho_i(E + E_-(\alpha, t), \alpha) - \frac{1}{N} \sum_{j} \frac{A^{\varepsilon}(i)}{\hat{z}_i(t, \alpha) - \hat{z}_j(t, \alpha)} \right) \) (3.72)
\[ B_2 = \left( \int_{J_i(a,t)} \rho_i(E + E_-(\alpha, t), \alpha) - \frac{1}{N} \sum_{j} \frac{A^{\varepsilon}(i)}{\hat{z}_i(t, \alpha) - \hat{z}_j(t, \alpha)} \right) + \left( \int_{J_i(a,t)} \rho_i(E + E_-(\alpha, t), \alpha) - \frac{1}{N} \sum_{j} \frac{A^{\varepsilon}(i)}{\hat{z}_i(t, \alpha) - \hat{z}_j(t, \alpha)} \right) \) (3.73)

The term (3.72) was handled above and is bounded by \( N^\varepsilon/(N^{1/3+2\omega^\varepsilon}) \). Now we estimate the term (3.73) we write. Fix for the moment an auxiliary scale \( \eta_1 \). We write (3.73) as
\[ \left( \int_{J_i(a,t)} \rho_i(E + E_-(\alpha, t), \alpha) - \frac{1}{N} \sum_{j} \frac{A^{\varepsilon}(i)}{\hat{z}_i(t, \alpha) - \hat{z}_j(t, \alpha)} \right) + \left( \int_{J_i(a,t)} \rho_i(E + E_-(\alpha, t), \alpha) - \frac{1}{N} \sum_{j} \frac{A^{\varepsilon}(i)}{\hat{z}_i(t, \alpha) - \hat{z}_j(t, \alpha)} \right) \) (3.74)

By the local law we have
\[ |A_4| \leq \frac{N^\varepsilon}{N\eta_1} \] (3.76)
with overwhelming probability. The term \( A_2 \) is bounded by
\[ |A_2| \leq \eta_1 C \text{Im}[m_N(\hat{z}_i(t, \alpha) + i, \alpha)] \leq \eta_1 C \] (3.77)
with overwhelming probability. Similarly, \( |A_1| \leq \eta_1 C \). A similar calculation to above yields
\[ |A_3| \leq \frac{N^\varepsilon}{N\eta_1}. \] (3.78)
We optimize and choose $\eta_i = N^{-1/2}$. Hence, we see that

$$|B_1| \leq N^\varepsilon \left( \frac{1}{N^{1/3}N^{2\omega_\ell}} + \frac{1}{N^{1/2}} \right).$$  \hfill (3.79)

From Lemma 3.4 we see

$$|B_2| \leq \frac{N^\varepsilon}{N^{1/3}} \frac{N^{2\omega_\ell/3}}{N^{\omega_\ell}} + \frac{N^\varepsilon N^{\omega_1/2}}{N^{1/3}N^{\omega_\ell/2}}.$$  \hfill (3.80)

Here we used the fact that the largest index $j_+(i)$ such that $(j, i) \in A$ can be bounded by

$$j_+(i) \leq C(\ell^3 + i) \leq C N^{\omega_\ell}.$$  \hfill (3.81)

We rewrite $B_3$ as

$$B_3 = \left( \int_{I_\alpha(t)} \rho_t(E + E_-(\alpha, t), \alpha) \frac{\bar{z}_i(t, \alpha) - E}{\bar{z}_i(t, \alpha) - E} \right) - \left( \int_{I_\alpha(t)} \rho_t(E + E_-(0, t), 0) \frac{\bar{z}_i(t, \alpha) - E}{\bar{z}_i(t, \alpha) - E} \right) + \left( \int_{I_\alpha(t)} \rho_t(E + E_-(0, t), 0) \frac{\bar{z}_i(t, \alpha) - E}{\bar{z}_i(t, \alpha) - E} \right) =: D_1 + D_2.$$  \hfill (3.82)

Starting with $D_2$ we first see that

$$|I_\alpha(t) \Delta I_\alpha(0, t)| \leq \frac{N^\varepsilon N^{\omega_\ell}(N^{2\omega_\ell} + i^{2/3})}{N^{2/3}N^{\omega_\ell}}$$  \hfill (3.83)

where $\Delta$ is symmetric difference, and we used (3.37). On the above symmetric difference, the integrand is bounded by

$$\left| \rho_t(E + E_-(0, t), 0) \frac{\bar{z}_i(t, \alpha) - E}{\bar{z}_i(t, \alpha) - E} \leq \frac{N^{1/3}(\ell + i^{1/3})}{\ell(\ell^2 + i^{2/3})}, \right.$$  \hfill (3.84)

and so

$$|D_2| \leq \frac{C N^\varepsilon N^{\omega_\ell/3}N^{\omega_1}}{N^{1/3}N^{\omega_\ell}},$$  \hfill (3.85)

with overwhelming probability. For $D_1$, the integral is a principal value so we have to do some minor case analysis to deal with the logarithmic singularity. First assume $i \geq N^5$ for a $\delta < \omega_\ell/10$. Then in particular we know that $\bar{z}_i(t, \alpha)$ is at least distance $N^{-2}$ away from the boundary of $I_\alpha(t, \alpha)$, and also that $|\bar{z}_i(t, \alpha)| > N^{-2}$. Then, using (3.34) we have

$$|D_1| \leq \frac{C N^\varepsilon N^{\omega_\ell}(\ell^2 + i^{2/3})}{N^{1/3}N^{\omega_\ell}} \int_{I_\alpha(t) \cap |E - \bar{z}_i(t, \alpha)| > N^{-50}} \frac{1}{|\bar{z}_i(t, \alpha) - E|} \left( \int_{|\bar{z}_i(t, \alpha) - E| < N^{-50}} \left| \frac{\rho_t(E + E_-(\alpha, t), \alpha) - \rho_t(E + E_-(0, t), 0)}{\bar{z}_i(t, \alpha) - E} \right| \right).$$  \hfill (3.86)

For the second term, we have on the domain of integration that $|\rho_t(E + E_-(\alpha, t), \alpha)| \leq C|\bar{z}_i(t, \alpha)|^{-1/2} \leq C N$, and so

$$\left| \int_{|\bar{z}_i(t, \alpha) - E| < N^{-50}} \frac{\rho_t(E + E_-(\alpha, t), \alpha) - \rho_t(E + E_-(0, t), 0)}{\bar{z}_i(t, \alpha) - E} \right| \leq N^{-10}. \hfill (3.87)$$

Therefore, in the case that $i > N^5$ we have

$$|D_1| \leq \frac{C N^\varepsilon N^{\omega_\ell}(N^{2\omega_\ell}/3)}{N^{1/3}N^{\omega_\ell}} \hfill (3.88)$$

for any $\varepsilon > 0$ with overwhelming probability. We now consider $i \leq N^5$. First of all, if $|\bar{z}_i(t, \alpha)| \geq N^{-100}$, then the above argument goes through and we obtain the same bound for $D_1$. So we assume that
\[ |\hat{z}_i(t, \alpha)| \leq N^{-100}. \] We break the integral up into a few pieces (note that in this case \( 0 \in \mathcal{I}_i(\alpha, t) \)). Denote the integrand by \( G(E) \). We write \( D_1 \) as

\[
D_1 = \int_{\mathcal{I}_i(\alpha, t) \cap E > N^{-50}} G(E) + \int_{3\hat{z}_i/2 < E < N^{-50}} G(E) + \int_{\hat{z}_i/2 < E < 3\hat{z}_i/2} G(E) + \int_{0 \leq E \leq \hat{z}_i/2} G(E)
\]

\[ =: I_1 + I_2 + I_3 + I_4. \] (3.89)

The term \( I_1 \) can be handled as in the case \( i > N^\delta \). For \( I_2 \), we just use that the integrand is bounded by \( C|E|^{-1/2} \) and \( |I_2| \leq CN^{-25} \). If \( \hat{z}_i(\alpha, t) \leq 0 \) then \( I_3 = I_4 = 0 \). So we consider the case that \( \hat{z}_i(\alpha, t) > 0 \). For \( I_3 \) that \( |\rho_t(E - E_{-}(t, \alpha), \alpha)| \leq C|\hat{z}_i(\alpha, t)|^{-1/2} \) on the domain of integration of \( I_3 \) to obtain \( |I_3| \leq C|\hat{z}_i(t, \alpha)|^{1/2} \leq CN^{-25} \). For \( I_4 \) we bound the integrand \( |G(E)| \leq C|E|^{-1/2} \) and obtain \( |I_4| \leq CN^{-25} \).

This proves that

\[
|D_1| \leq \frac{CN^{\varepsilon} N^{\omega_1} (N^{2\omega_A/3})}{N^{1/3} N^{\omega_0}}
\] (3.90)

for any \( i \). We get the claim.

The above implies the following, due to the fact that the choices of our parameters ensure that the RHS of (3.56) is bounded by

\[
\frac{1 - N^{\varepsilon} N^{\omega_1}}{N^{2/3} N^{2\omega_A/3}}
\] (3.91)
i.e., the first error term is the largest.

**Lemma 3.8.** Let \( i \leq N^{3\omega_A + \delta} \) for \( \delta < \omega_f - \omega_1 \). Then,

\[
\sup_{0 \leq t \leq 10\varepsilon_1} |\hat{z}_i(\alpha, t) - \hat{\gamma}_i(\alpha, t)| \leq \frac{N^{\varepsilon}}{N^{2/3} N^{1/3}}
\] (3.92)

with overwhelming probability.

### 3.3 Differentiation

Let \( u_i(t, \alpha) := \partial_\alpha \hat{z}_i(t, \alpha) \). We see that \( u \) satisfies the equation,

\[
\partial_t u = \mathcal{L} u + \zeta^{(0)},
\] (3.93)

where the operator \( \mathcal{L} \) is as follows, and \( \zeta^{(0)} \) is a forcing term as follows. First,

\[
\mathcal{L} = \mathcal{B} + \mathcal{V}
\] (3.94)

where

\[
(Bu)_i = \frac{1}{N} \sum_{j=1}^{A_i} \frac{u_j - u_i}{(\hat{z}_i(\alpha, t) - \hat{z}_j(\alpha, t))^2}, \quad (Vu)_i = V_i u_i
\] (3.95)

for \( 1 \leq i \leq N^{\omega_A} \),

\[
V_i = -\int_{\mathcal{I}_i(0,t)} \frac{\rho_t(E + E_-(0, t), 0)}{\hat{z}_i(\alpha, t) - E}^2
d\] (3.96)

for \( N^{\omega_A} \leq i \leq i_\alpha/2 \),

\[
V_i = -\int_{\mathcal{I}_i(\alpha,t)} \frac{\rho_t(E + E_i(\alpha, t), \alpha)}{\hat{z}_i(\alpha, t) - E}^2
d\] (3.97)

and \( V_i \) is 0 otherwise. The error term \( \zeta^{(0)} \) is 0 unless \( i \leq 0 \) or \( i \geq N^{\omega_A} \), and it comes from the \( \partial_\alpha \) derivative hitting all the other terms. It is not too hard to check that

\[
|\zeta^{(0)}_i| \leq N^{\delta}
\] (3.98)

\[ 
16
\]
for some $C > 0$, with overwhelming probability. We also make the definition

$$
B_{ij} = -\frac{1}{N} \frac{1}{(\tilde{z}_i(t, \alpha) - \tilde{z}_j(t, \alpha))^2}.
$$

(3.99)

Additionally, we introduce the $\ell^p$ norms,

$$
||u||_p = \left( \sum_i |u_i|^p \right)^{1/p}, \quad ||u||_\infty = \max_i |u_i| = \lim_{p \to \infty} ||u||_p.
$$

(3.100)

### 3.4 Long range cut-off

First, we have the following finite speed of propogation estimate. It follows from Lemma 4.3.

**Lemma 3.9.** For all small $\delta > 0$, we have the following. Let $a \leq N^{3\omega_1 + \delta}$ and $b \geq N^{3\omega_1 + 2\delta}$. Then

$$
\sup_{0 \leq s \leq t \leq 10t_1} \mathcal{U}^c_{ab}(s, t) + \mathcal{U}^c_{bc}(s, t) \leq N^{-D}
$$

(3.101)

for any $D > 0$ with overwhelming probability.

Fix now a small $\delta_1 > 0$. Define $v_i$ to be the solution to

$$
\partial_t v = \mathcal{L}v, \quad v_i(0) = u_i(0)1 \{1 \leq i \leq N^{3\omega_1 + \delta_1}\}.
$$

(3.102)

By Lemma 3.9 and the fact that $\mathcal{U}^c_{ij} = 0$ for $i \geq 1$ and $j \leq 0$ or $i \leq 0$ and $j \geq 1$, we immediately see the following.

**Lemma 3.10.** We have

$$
\sup_{0 \leq t \leq 10t_1} \sup_{1 \leq i \leq \ell} |u_i(t) - v_i(t)| \leq N^{-100}.
$$

(3.103)

### 3.5 Energy estimate

We require the following energy estimate.

**Lemma 3.11.** Let $\delta_1 > 0$ be small. Let $w \in \mathbb{R}^N$, $w_i = 0$ for $i \geq \ell^3 N^{\delta}$ or $i \leq 0$. Then for all $\varepsilon > 0$ and all $\eta > 0$ there is a constant $C$ (independent of $\varepsilon$ and $\eta$) s.t. for $0 \leq t \leq 2t_1$,

$$
||\mathcal{U}^c(0, t)w||_\infty \leq C(p, \eta) \left( \frac{N^{C\eta + \varepsilon} \eta^{3(1-6\eta)/p}}{N^{1/3} t} \right)^{3(1-6\eta)/p} ||w||_p.
$$

(3.104)

For its proof we need the following Sobolev-type inequality from [14].

**Lemma 3.12.** For all $\eta > 0$ there exists a $c_\eta > 0$ s.t.

$$
\sum_{i \neq j \in \mathbb{Z}_+} \frac{(u_i - u_j)^2}{|i^{2/3} - j^{2/3}|^{2-\eta}} \geq c_\eta \left( \sum_{i \in \mathbb{Z}_+} |u_i|^p \right)^{2/p}.
$$

(3.105)

The above lemma is used in the following which is proved via the Nash method, from which Lemma 3.11 follows. It is very similar to that appearing in [14].

**Lemma 3.13.** Let $0 \leq s \leq t \leq t_1$. Let $\delta_1 > 0$ satisfy

$$
0 < \delta_1 < \omega_t - \omega_1.
$$

(3.106)

Let $w$ be a vector s.t. $w_i = 0$ for $i \geq \ell^3 N^{\delta_1}$ and $i \leq 0$. Let $\eta > 0$ and $\varepsilon > 0$. There is a $C > 0$ independent of $\varepsilon$ and $\eta$ and a constant $c_\eta$ s.t. the following hold with overwhelming probability for all $0 \leq s \leq t \leq 5t_1$.

$$
||\mathcal{U}^c(s, t)w||_2 \leq \left( \frac{1}{(c_\eta N - C\eta - \epsilon N^{1/3}(t-s))^{3/2}} \right)^{1-6\eta} ||w||_1
$$

(3.107)

and

$$
|| (\mathcal{U}^c(s, t))^T w ||_2 \leq \left( \frac{1}{(c_\eta N - C\eta - \epsilon N^{1/3}(t-s))^{3/2}} \right)^{1-6\eta} ||w||_1
$$

(3.108)
Proof. We start with (3.107). This is a modification of the argument that uses the usual Nash approach. We provide all the details for completeness. Fix $\delta_2$ and $\delta_3$ s.t.

$$0 < \delta_1 < \delta_2 < \delta_3 < \omega - \omega_1.$$  \hfill (3.109)

For notational simplicity we just do $s = 0$. Fix $\eta > 0$. Let

$$p = \frac{3}{1 + \eta}. \hfill (3.110)$$

Assume that

$$||w(0)||_1 = 1. \hfill (3.111)$$

We can also assume that

$$||w(u)||_p \geq \frac{1}{N^{100}}, \quad 0 \leq u \leq t$$

or else

$$||w(t)||_p \leq \frac{1}{N^{95}} \hfill (3.113)$$

by the contraction property. We have

$$||w(u)||^2_p \leq \sum_{i,j \in \mathbb{Z}^+} \frac{\left( w_i - w_j \right)^2}{|i^{2/3} - j^{2/3}|^{2-\eta}} \leq \sum_{(i,j) \in A} \frac{\left( w_i - w_j \right)^2}{|i^{2/3} - j^{2/3}|^{2-\eta}} + C \sum_{i,j \in \mathbb{Z}^+} \frac{\left( w_i - w_j \right)^2}{|i^{2/3} - j^{2/3}|^{2-\eta}}. \hfill (3.114)$$

We have

$$\sum_{(i,j) \in A} \frac{(w_i - w_j)^2}{|i^{2/3} - j^{2/3}|^{2-\eta}} \leq \sum_{i \text{ or } j \leq \ell^\eta} \frac{(w_i - w_j)^2}{|i^{2/3} - j^{2/3}|^{2-\eta}} + \sum_{i,j \geq \ell^\eta} \frac{(w_i - w_j)^2}{|i^{2/3} - j^{2/3}|^{2-\eta}}. \hfill (3.115)$$

By Lemma 3.9,

$$\sum_{i,j \geq \ell^\eta} \frac{(w_i - w_j)^2}{|i^{2/3} - j^{2/3}|^{2-\eta}} \leq \frac{1}{N^{200}}. \hfill (3.116)$$

If $(i,j) \in A$ and $i$ or $j$ is less than $\ell^\eta N^{\delta_2}$ then both $i$ and $j$ are less than $\ell^\eta N^{\delta_3}$. By Lemma 3.8 we have

$$|\mathbf{\bar{z}}_i - \mathbf{\bar{z}}_j| \leq \frac{N^\eta |i^{2/3} - j^{2/3}|}{N^{2/3}} \hfill (3.117)$$

for all $\varepsilon > 0$ for such $i$ and $j$. Therefore,

$$\sum_{i \text{ or } j \leq \ell^\eta N^{\delta_2}} \frac{(w_i - w_j)^2}{|i^{2/3} - j^{2/3}|^{2-\eta}} \leq -N^{-2/3} N^{\varepsilon + C} \sum_{i,j} B_{ij} (w_i - w_j)^2 = -N^{-2/3} N^{\varepsilon + C} \langle w, Bw \rangle. \hfill (3.118)$$

(recall $B_{ij}$ are negative). Similarly,

$$\sum_{i,j} A_{ij} \frac{w_i^2}{|i^{2/3} - j^{2/3}|^{2-\eta}} \leq \sum_{i \leq \ell^\eta N^{\delta_3}} \sum_{j} A_{ij} \frac{w_i^2}{|i^{2/3} - j^{2/3}|^{2-\eta}} + \frac{1}{N^{200}} \leq -N^{C\eta} N^{-1/3} \sum_{i \leq \ell^\eta N^{\delta_3}} w_i^2 \mathbf{\bar{v}}_i + \frac{1}{N^{200}} \leq -N^{C\eta} N^{-1/3} \langle w, \mathbf{\bar{v}}w \rangle + \frac{1}{N^{200}}. \hfill (3.119)$$
From (3.116), (3.118) and (3.119) we obtain

\[ ||w(u)||_p^2 \leq -N^{\varepsilon+C\eta}N^{-1/3} < w, \mathcal{L}w > + \frac{1}{10} ||w(u)||_p^2 \]  

(3.120)

Therefore

\[ \partial_u||w(u)||_2^2 = \langle w, \mathcal{L}w \rangle \]

\[ \leq -c_\eta N^{-1/3}N^{-\varepsilon-C\eta}||w(u)||_p^2 \]

\[ \leq -c_\eta N^{-1/3}N^{-\varepsilon-C\eta}||w(s)||_{2^3}^\frac{2-4\eta}{3} \]

\[ \leq -c_\eta N^{-1/3}N^{-\varepsilon-C\eta}||w(s)||_{2^3}^\frac{2-4\eta}{3} \]  

(3.121)

where in the second inequality we used Holder and in the last we used the \( \ell^1 \) contractivity of \( U^\mathcal{L} \).

From this we obtain

\[ ||w(t)||_2 \leq \left( \frac{1}{(c_\eta N^{-C\eta-\varepsilon} N^{1/3t})^{3/2}} \right)^{1-6\eta} ||w(0)||_1 \]  

(3.122)

as desired.

The proof of (3.108) is identical and follows by duality. One considers the function

\[ w(u) = U^\mathcal{L}(u,t)w \]  

(3.123)

which satisfies

\[ \partial_u w(u) = \mathcal{L}(u)w(u). \]  

(3.124)

Note that the only inputs in the proof of (3.107) are time-independent lower bounds on \( \mathcal{L} \) and Lemma 3.9 which holds for both \( U^\mathcal{L} \) and \( (U^\mathcal{L})^T \).

**Proof of Lemma 3.11.** Let

\[ 0 < \delta_1 < \delta_2 < \omega_\ell - \omega_1. \]  

(3.125)

Let \( \chi_2(i) = 1 \}_{1 \leq i \leq \ell_1 N^{\delta_2}} \). Let \( v \) have \( ||v||_1 = 1 \). We have,

\[ \langle \mathcal{U}^\mathcal{L}(0,t)w, v \rangle = \langle w, (\mathcal{U}^\mathcal{L})^Tv \rangle = \langle w, (\mathcal{U}^\mathcal{L})^T \chi_2 v \rangle + \langle w, (\mathcal{U}^\mathcal{L})^T (1 - \chi_2) v \rangle. \]  

(3.126)

We have by Lemma 3.9,

\[ |\langle w, (\mathcal{U}^\mathcal{L})^T (1 - \chi_2) v \rangle| \leq \frac{1}{N^{100}} ||w||_2 ||v||_1. \]  

(3.127)

By Cauchy-Schwartz and Lemma 3.13

\[ |\langle w, (\mathcal{U}^\mathcal{L})^T \chi_2 v \rangle| \leq ||w||_2 ||(\mathcal{U}^\mathcal{L})^T \chi_2 v||_2 \]

\[ \leq ||w||_2 \left( \frac{1}{(c_\eta N^{-C\eta-\varepsilon} N^{1/3t})^{3/2}} \right)^{1-6\eta} ||v||_1. \]  

(3.128)

Hence,

\[ ||\mathcal{U}^\mathcal{L}(0,t)w||_\infty \leq \left( \frac{1}{(c_\eta N^{-C\eta-\varepsilon} N^{1/3t})^{3/2}} \right)^{1-6\eta} ||w||_2 \]  

(3.129)

and so by the semigroup property,

\[ ||\mathcal{U}^\mathcal{L}(0,t)w||_\infty \leq \left( \frac{1}{(c_\eta N^{-C\eta-\varepsilon} N^{1/3t})^{3/2}} \right)^{1-6\eta} ||w||_1. \]  

(3.130)

The rest follows from interpolation. \( \square \)
3.6 Proof of Theorem 3.1
For notational simplicity we just do $i = 1$. By Lemmas 3.5 and 3.7 we have,
\begin{equation}
|\lambda_0(t_1) - E_\lambda(t_1) - (\mu_1(t_1) - E_\mu(t_1))| \leq |\hat{z}_1(1, t_1) - \hat{z}_1(0, t_1)| + \frac{1}{N^{2/3 + \varepsilon}},
\end{equation}
with overwhelming probability. By the definition of $u_1(t)$ we have
\begin{equation}
\hat{z}_1(1, t_1) - \hat{z}_1(0, t_1) = \int_0^\alpha u_1(t_1, \alpha) d\alpha.
\end{equation}
By Markov’s inequality and Lemma 3.10 we have with overwhelming probability,
\begin{equation}
\int_0^1 u_1(t_1, \alpha) d\alpha \leq N^{-10} + \int_0^1 v_1(t_1, \alpha) d\alpha.
\end{equation}
Note that by (3.22) we see that
\begin{equation}
||v_1(0)||_4 \leq \frac{N^{2\varepsilon_1}}{N^{2/3}}
\end{equation}
for any $\varepsilon_1 > 0$, with overwhelming probability. Hence by Lemma 3.11 and Markov again we get that
\begin{equation}
\int_0^1 v_1(t_1, \alpha) d\alpha \leq \frac{N^{2\varepsilon_1}}{N^{2/3}} \frac{1}{N^{2/3}}
\end{equation}
with overwhelming probability. This yields the claim.

4 Finite speed calculations
The following method for getting bounds on $U^L$ originates from [16].

Lemma 4.1. Let $L$ and $U^L$ be as above. Fix a small $\delta > 0$ satisfying $\delta < \omega_\ell - \omega_1$. Let $\omega_1 < \omega_\ell$. For any $a \geq N^{3\omega_\ell + \delta}$ and $b \leq N^{3\omega_\ell + \delta}/2$ and fixed $s$ we have
\begin{equation}
\sup_{s \leq t \leq 10t_1} U^L_{ab}(s, t) + U^L_{ba}(s, t) \leq N^{-D}
\end{equation}
for any $D > 0$.

Proof. For notational simplicity we take $s = 0$. Let $\psi(x)$ be a function as follows. We let
\begin{equation}
\psi = -x, \quad |x| \leq \frac{N^{2\omega_\ell} N^{6/3}}{N^{2/3}},
\end{equation}
and
\begin{equation}
\psi'(x) = 0, \quad |x| > 2 \frac{N^{2\omega_\ell} N^{6/3}}{N^{2/3}},
\end{equation}
and demand that $|\psi(x) - \psi(y)| \leq |x - y|$ and $|\psi'(x)| \leq 1$, that $\psi$ be decreasing, and
\begin{equation}
|\psi''(x)| \leq C \frac{N^{2/3}}{N^{2\omega_\ell + 2\delta/3}}.
\end{equation}
We consider a solution of
\begin{equation}
\partial_t f = L f
\end{equation}
with initial data $f_i(0) = \delta_{p^*}$ for any $p^* \geq N^\delta N^{3\omega_\ell} =: p$. We consider the function
\begin{equation}
F(t) := \sum_k f_k^2 e^{\psi(\hat{z}_k(t, \alpha)) - \gamma_k(t, \alpha))} =: \sum_k f_k^2 \phi_k^2 =: \sum_k v_k^2
\end{equation}
This is obeying the equation

\[
dF = - \sum_{(i,j) \in \mathcal{A}} B_{ij} (v_i - v_j)^2 dt - \sum_i \mathcal{V}_i v_i^2 dt \tag{4.7}
\]
\[
+ \sum_{(i,j) \in \mathcal{A}} B_{ij} v_i v_j \left( \frac{\phi_i}{\phi_j} + \frac{\phi_j}{\phi_i} - 2 \right)
\]
\[
+ \nu \sum_i v_i^2 \psi' \left( \hat{z}_i - \hat{\gamma}_p \right) d(\hat{z}_i - \hat{\gamma}_i) \tag{4.8}
\]
\[
+ \sum_i v_i^2 \left( \frac{\nu^2}{N} \psi' \left( \hat{z}_i - \hat{\gamma}_p \right)^2 + \frac{\nu}{N} \psi'' \left( \hat{z}_i - \hat{\gamma}_p \right) \right) dt. \tag{4.9}
\]

We define a stopping time \( \tau \) as follows. We can take \( \tau_1 \) to be a stopping time so that for \( t < \tau_1 \) the rigidity estimates of Lemma 3.7 and (3.33) hold, for a small \( \varepsilon > 0 \) with \( \varepsilon < \delta/1000 \). With overwhelming probability, \( \tau \geq 10 t_1 \). Take \( \tau_2 \) to be the first time that \( F \geq 5 \), and then \( \tau = \tau_1 \wedge \tau_2 \wedge 10 t_1 \). We want to prove \( \tau = 10 t_1 \) with overwhelming probability. In the remainder of the proof we work with times \( t < \tau \).

We note that \( \psi' \left( \hat{z}_i - \hat{\gamma}_i \right) = 0 \) unless \( i \leq CN^{3\omega_\mathcal{A} + \delta} \) for some \( C > 0 \). Moreover, if \( i \leq CN^{3\omega_\mathcal{A} + \delta} \) and \( (i, j) \in \mathcal{A} \), then \( j \leq C'N^{3\omega_\mathcal{A} + \delta} \). From this we see that the nonzero terms in the sum in (4.8) have both \( i, j \leq CN^{3\omega_\mathcal{A} + \delta} \). For such terms we have

\[
|\hat{z}_i - \hat{z}_j| \leq \frac{\ell^2 N^{5/3}}{N^{2/3}}. \tag{4.11}
\]

Hence, if

\[
\frac{\nu \ell^2 N^{5/3}}{N^{2/3}} \leq C, \tag{4.12}
\]

then

\[
\sum_{(i,j) \in \mathcal{A}} B_{ij} v_i v_j \left( \frac{\phi_i}{\phi_j} + \frac{\phi_j}{\phi_i} - 2 \right) \leq C \nu^2 N \sum_i v_i^2 \sum_{j: (j,i) \in \mathcal{A}} 1_{\{\phi_j \neq \phi_i\}} \leq \frac{\nu^2 \ell^2 N^{25/3}}{N} F(t). \tag{4.13}
\]

The term (4.10) is easily bounded by

\[
\left| \sum_i v_i^2 \left( \frac{\nu^2}{N} \psi' \left( \hat{z}_i - \hat{\gamma}_p \right)^2 + \frac{\nu}{N} \psi'' \left( \hat{z}_i - \hat{\gamma}_p \right) \right) \right| \leq C \left( \frac{\nu^2}{N} + \frac{\nu}{N^{1/3} N^{2\omega_\mathcal{A} + 28/3}} \right) F(t). \tag{4.14}
\]

Now we deal with (4.9). Recall that \( \psi' \left( \hat{z}_i - \hat{\gamma}_p \right) \neq 0 \) only if \( i \leq CN^{3\omega_\mathcal{A} + \delta} \ll N^{\omega_\mathcal{A}} \). Hence, for such \( i \) we have

\[
d(\hat{z}_i(t, \alpha) - \hat{\gamma}_p(t, \alpha)) = \frac{dB_i}{\sqrt{N}} + \frac{1}{N} \sum_{j}^{A^{(i)}} \hat{z}_j(t, \alpha) - \hat{z}_j(t, \alpha) + \int_{\mathcal{I} \cap (0,t)} \frac{1}{\hat{z}_i(t, \alpha) - E} \rho_i(E - E_-(0,t), 0) dE
\]
\[
+ \Re[m_t(E_-(t, 0), 0)] + \Re[m_t(\hat{\gamma}_p(\alpha, t) + E_-(-\alpha, \gamma_p(t, \alpha)), 0)] - \Re[m_t(E_-(\alpha, \gamma_p(t, \alpha))]. \tag{4.15}
\]

By the definition of \( \tau \) and the BDG inequality, we have with overwhelming probability,

\[
\sup_{0 \leq t \leq \tau} \int_0^t \sum_i v_i^2 \psi' \left( \hat{z}_i - \hat{\gamma}_p \right) \nu \frac{dB_i}{\sqrt{N}} \leq C N^\varepsilon \left( \frac{\nu^2 N^{\omega_\mathcal{A}}}{N^{4/3}} \right)^{1/2}. \tag{4.16}
\]

Next,

\[
\frac{\nu}{N} \sum_{(i,j) \in \mathcal{A}} \frac{\psi' \left( \hat{z}_i - \hat{\gamma}_p \right) v_i^2}{\hat{z}_i - \hat{z}_j} = \frac{\nu}{2N} \sum_{(i,j) \in \mathcal{A}} \frac{\psi' \left( \hat{z}_i - \hat{\gamma}_p \right) (v_i^2 - v_j^2)}{\hat{z}_i - \hat{z}_j} + \frac{\nu}{2N} \sum_{(i,j) \in \mathcal{A}} v_i^2 \psi' \left( \hat{z}_i - \hat{\gamma}_p \right) \frac{\hat{z}_i - \hat{z}_j}{\hat{z}_i - \hat{z}_j}. \tag{4.17}
\]
The latter sum is bounded by
\[
\frac{\nu}{2N} \sum_{(i,j) \in A} v_i^2 \psi'(\tilde{z}_i - \hat{\gamma}_p) \frac{v_j^2}{\tilde{z}_j - \tilde{z}_j} \leq C \frac{\nu}{N^{1/3}(N^{2\omega}\nu^{2\delta/3})} \sum_{i} v_i^2 \sum_{j} A_i A_j 1_{\{\psi'(\tilde{z}_i - \hat{\gamma}_p) \neq \psi'(\tilde{z}_j - \hat{\gamma}_p)\}} \\
\leq C \frac{\nu N^\omega \nu^{2\delta/3}}{N^{1/3}} F(t).
\] (4.18)

For the first term we use the Schwarz inequality, obtaining
\[
\frac{\nu}{2N} \sum_{(i,j) \in A} \psi'(\tilde{z}_i - \hat{\gamma}_p)(v_i^2 - v_j^2) \frac{v_i^2}{\tilde{z}_i - \tilde{z}_j} = \frac{\nu}{2N} \sum_{(i,j) \in A} \psi'(\tilde{z}_i - \hat{\gamma}_p)(v_i - v_j)(v_i + v_j) \\
\leq \frac{1}{100N} \sum_{(i,j) \in A} (v_i - v_j)^2 \frac{v_i^2}{(\tilde{z}_i - \tilde{z}_j)^2} \\
+ C \frac{\nu^2}{2N} \sum_{(i,j) \in A} \psi'(\tilde{z}_i - \hat{\gamma}_p)^2(v_i^2 + v_j^2).
\] (4.19)

The second sum is bounded by
\[
C \frac{\nu^2}{2N} \sum_{(i,j) \in A} \psi'(\tilde{z}_i - \hat{\gamma}_p)^2(v_i^2 + v_j^2) \leq C \frac{\nu^2 N^{3\omega}\nu^{25/3}}{N} F(t).
\] (4.20)

In summary, we have estimated
\[
\frac{\nu}{N} \sum_{(i,j) \in A} \frac{\psi'(\tilde{z}_i - \hat{\gamma}_p)v_i^2}{\tilde{z}_i - \tilde{z}_j} \leq \frac{1}{100} \sum_{(i,j) \in A} B_{ij}(v_i - v_j)^2 + F(t) C \left( \frac{\nu N^\omega \nu^{2\delta/3}}{N^{1/3}} + \frac{\nu^2 N^{3\omega}\nu^{25/3}}{N} \right).
\] (4.21)

Returning to (4.15) we write
\[
\int_{I_{\epsilon}(0,t)} \frac{1}{\bar{z}_i(t,\alpha) - E} \rho_t(E + E_-(0,t),0) dE \\
+ \text{Re}[m_t(E_-(0,t),0)] + \text{Re}[m_t(\hat{\gamma}_p(\alpha,t) + E_-(\alpha,t),\alpha)] - \text{Re}[m_t(E_-(\alpha,t),\alpha)] \\
= \left( \int_{I_{\epsilon}(0,t)} \frac{1}{\bar{z}_i(t,\alpha) - E} \rho_t(E + E_-(0,t),0) dE + \text{Re}[m_t(\hat{\gamma}_p(\alpha,t) + E_-(0,t),0)] \right) \\
+ \left[ \text{Re}[m_t(E_-(0,t),0)] + \text{Re}[m_t(\hat{\gamma}_p(\alpha,t) + E_-(\alpha,t),\alpha)] \\
- \text{Re}[m_t(E_-(\alpha,t),\alpha)] - \text{Re}[m_t(\hat{\gamma}_p(\alpha,t) + E_-(0,t),0)] \right] =: A_1 + A_2.
\] (4.22)

By (3.35) we have
\[
|A_2| \leq N^\varepsilon N^{2\omega + 25/3} \frac{N^{1/3} N^{\varepsilon}}{N_{1/3}}.
\] (4.23)

Using \(\rho_t(\hat{\gamma}_2(\alpha,t) + E_-(0,t),0) \leq CN^{\omega}\nu N^{5/3}/N^{1/3}\) we see that
\[
|A_1| \leq \left( \int_{E \geq \hat{\gamma}_2(\alpha,t)} \left( \frac{1}{\bar{z}_i(t,\alpha) - E} - \frac{1}{\bar{\gamma}_p(\alpha,t) - E} \right) \rho_t(E + E_-(0,t),0) \right) + C \frac{N^{\omega + \delta/3}}{N^{1/3}}.
\] (4.24)

It is easy to see that the first term is bounded by
\[
\left( \int_{E \geq \hat{\gamma}_2(\alpha,t)} \left( \frac{1}{\bar{z}_i(t,\alpha) - E} - \frac{1}{\bar{\gamma}_p(\alpha,t) - E} \right) \rho_t(E + E_-(0,t),0) \right) \\
\leq C|\bar{z}_i(\alpha,t) - \bar{\gamma}_p(\alpha,t)| \frac{\text{Im}[m_t(E_-(0,t) + i\gamma_p(\alpha,t),0)]}{\gamma_p(\alpha,t)} \\
\leq C \frac{\sqrt{\gamma_p(\alpha,t)}}{\gamma_p(\alpha,t)} \leq C N^{\omega + \delta/3} \frac{N^{1/3}}{N^{1/3}}.
\] (4.25)
Collecting the above, we see that, using \( \omega_1 \leq \omega_t/2 \), and \( \omega_t < \omega_0/10 \),

\[
\sup_{0 \leq s \leq \tau} F(s) \leq \left( \frac{\nu^2 N^{4\omega_t+2\delta/3}}{N^{4/3}} + \frac{\nu N^{\omega_t+\omega_t+\delta/3}}{N^{2/3}} \right) + F(0).
\]  

(4.26)

under the condition (4.12). Hence if we choose

\[
\nu = \frac{N^{2/3}}{N^{2\omega_t+2\delta/3}} N^{\varepsilon_1}
\]

(4.27)

for \( \varepsilon_1 < \delta/10 \), then we see by continuity that \( \tau = 10t_1 \) with overwhelming probability, and so

\[
\sup_{0 \leq s \leq 10t_1} F(s) \leq 5
\]

(4.28)

with overwhelming probability. Now if \( i \leq N^{3\omega_t+\delta}/2 \), we see that

\[
\nu|\tilde{z}_i(t, \alpha) - \tilde{\gamma}_p(t, \alpha)| \geq cN^{\varepsilon_1}
\]

(4.29)

with overwhelming probability.

This yields,

\[
\mathcal{U}_{p, i}^L(0, t) \leq N^{-D}
\]

(4.30)

for such \( i, p \) with overwhelming probability. The proof for \( \mathcal{U}_{p, i}^L \) is the same; instead, use \( \psi \to -\psi \) and set the initial condition to be \( f_j = \delta_j \).

\[ \square \]

It will be convenient to establish the above estimate holding simultaneously for all \( 0 \leq s \leq t \leq 10t_1 \). We will require the following.

**Lemma 4.2.** Let \( u_i \) be a solution of

\[
\delta_t u = \mathcal{L} u,
\]

(4.31)

with \( u_i(0) \geq 0 \). Then for \( 0 \leq t \leq 10t_1 \) we have

\[
\frac{1}{2} \sum_i u_i(0) \leq \sum_i u_i(t) \leq \sum_i u_i(0)
\]

(4.32)

with overwhelming probability.

**Proof.** We see that

\[
\delta_t \sum_i u_i = \sum_i \mathcal{V}_i u_i.
\]

(4.33)

With overwhelming probability,

\[
-\mathcal{V}_i \leq C \frac{N^{1/3}}{N^{\omega_t}}.
\]

(4.34)

The claim follows from applying Gronwall to

\[
\delta_t \sum_i u_i \geq - \left( C \frac{N^{1/3}}{N^{\omega_t}} \right) \sum_i u_i.
\]

(4.35)

\[ \square \]

**Lemma 4.3.** Let \( \delta, \varepsilon > 0 \). Let \( a \leq N^{3\omega_t+\delta}/2 \) and \( b \geq N^{3\omega_t+\delta+\varepsilon} \). Then,

\[
\sup_{0 \leq s \leq t \leq 10t_1} \mathcal{U}_{ab}^L(s, t) + \mathcal{U}_{ma}^L(s, t) \leq N^{-D}
\]

(4.36)

with overwhelming probability.

**Proof.** We have,

\[
\mathcal{U}_{mi}^L(0, t) \geq \mathcal{U}_{ma}^L(s, t) \mathcal{U}_{ab}^L(0, s).
\]

(4.37)

By the previous lemma, we have that \( \sum_i \mathcal{U}_{ab}^L(0, s) \geq 1/2 \). By the first finite speed estimate we know that \( \mathcal{U}_{ab}^L \leq N^{-100} \) for any \( i \leq N^{3\omega_t+\delta+\varepsilon}/2 \). Hence there is an \( i_* \geq N^{3\omega_t+\delta} \) so that \( \mathcal{U}_{ab}^L(0, s) \geq 1/(4N) \). But then also by the first finite speed estimate,

\[
\mathcal{U}_{mi}^L(0, t) \leq N^{-D}.
\]

(4.38)

Hence we get \( \mathcal{U}_{ab}^L(s, t) \leq N^{-D+2} \). The estimate for \( \mathcal{U}_{ba}^L(s, t) \) is similar.

\[ \square \]
5 Local law for $t \geq N^{1/3}$

In this section we are going to prove a local law for $H_t$. We will use notation introduced in Section 2.1.1, i.e., $\kappa, \xi$, etc.

Let us denote the matrix elements of $H_t$ by

$$(H_t)_{ij} = V_i \delta_{ij} + \sqrt{t} h_{ij}. \quad (5.1)$$

We will also use the notation

$$t = \frac{N^\omega}{N^{1/3}}, \quad \omega > 0. \quad (5.2)$$

Fix a $\sigma > 0$. Define the domain

$$D_\sigma := \left\{ E + i \eta : 3/4 \geq E \geq E_-, \sqrt{\kappa + \eta} \geq \frac{N^\sigma}{N \eta} \right\} \cup \left\{ E + i \eta : -3/4 \leq E \leq E_-, \eta \geq N^\sigma / N^{2/3} \right\}. \quad (5.3)$$

The main theorem of this section is the following. The derivation of rigidity estimates such as (3.21) from such an estimate is standard - we refer, to, e.g., [21, 31].

**Theorem 5.1.** Let $\sigma > 0$. For any $\varepsilon > 0$, we have with overwhelming probability that the following estimates hold for all $z \in D_\sigma$. First, for $E \leq E_-$ we have

$$|m_N(z) - m_{\xi, t}(z)| \leq N^\varepsilon \left( \frac{1}{N(\kappa + \eta)} + \frac{1}{(N \eta)^2 (\kappa + \eta)} \right). \quad (5.4)$$

For $E \geq E_-$ we have,

$$|m_N(z) - m_{\xi, t}(z)| \leq \frac{N^\varepsilon}{N \eta}. \quad (5.5)$$

We introduce now some notation and estimates used in the proof. The Schur complement formula gives,

$$G_{ii}(z) := \frac{1}{V_i - z - t \lambda_{\xi, t} + Z_i} \quad (5.6)$$

where

$$Z_i = t(m_N - m_{\xi, t}) + \sqrt{t} h_{ii} + t A_i + t B_i + t (m_N^{(i)} - m_N)$$

$$= t(m_N - m_{\xi, t}) + t (m_N^{(i)} - m_N) + Q_i G_{ii}^{-1}. \quad (5.7)$$

Here we defined,

$$A_i = \sum_{j \neq k} h_{ij} G_{jk}^{(i)} h_{ki}, \quad B_i = \sum_j (h_{ij}^2 - 1/N) G_{jj}^{(i)}. \quad (5.8)$$

Let

$$\Lambda := |m_N - m_{\xi, t}|, \quad \Phi := \sqrt{\frac{t}{N}} + \frac{1}{N \eta} \sqrt{\frac{\text{Im}[m_{\xi, t}] + \Lambda}{N \eta}}. \quad (5.9)$$

We have the following estimates for the above parameters. They are standard, see, e.g., [35].

**Lemma 5.2.** We have,

$$t|m_N^{(i)} - m_N| \leq \frac{t}{N \eta} \text{Im}[G_{ii}] \leq \frac{t}{N \eta} \quad (5.10)$$

For any $\varepsilon > 0$ we have with overwhelming probability that

$$t|A_i| + t|B_i| \leq N^\varepsilon t \left( \frac{\text{Im}[m_N^{(i)}]}{N \eta} \right)^{1/2} \leq C N^\varepsilon \left( \frac{t}{N \eta} + \frac{1}{N \eta} \sqrt{\frac{\Lambda + \text{Im}[m_{\xi, t}]}{N \eta}} \right). \quad (5.11)$$

We have,

$$|Q_i G_{ii}^{-1}| \leq N^\varepsilon \left( \frac{t}{N} + \frac{1}{N \eta} \sqrt{\frac{\Lambda + \text{Im}[m_{\xi, t}]}{N \eta}} + \frac{t}{N \eta} \right). \quad (5.12)$$

Due to how some quantities appearing in the self-consistent equation for $m_N$ behave, it will be notationally convenient to split the proof of the above theorem into two parts. We will first consider $z \in D_\sigma$ s.t. $E \geq E_- - C_1 t^2$ for any (fixed) $C_1 > 0$. After this we sketch how the proof is modified to deal with the remaining part of $D_\sigma$. 

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5.1 Proof for $E \geq E_\pm - t^2$

In this section we prove the following.

**Proposition 5.3.** Fix $C_1 > 0$. *Theorem 5.1 holds in the domain*

$$\mathcal{D}_1 := \mathcal{D}_\sigma \cap \{ z = E + i\eta : E \geq E_\pm - C_1 t^2 \}. \quad (5.13)$$

In preparation we note the behaviour of a few parameters appearing in the proof. The proofs are provided in Lemma 7.5. First, we note that for $z \in \mathcal{D}_1$, that there is a $c > 0$ so that

$$|V_i - \xi| \geq c(t^2 + \eta + t t[m_{fc,t}]) = t^2 + \eta + t|E - E_-|^{1/2} 1_{(E \geq E_-)}. \quad (5.14)$$

For $z \in \mathcal{D}_1$ we have,

$$\frac{1}{N} \sum_{i} |g_i|^p \leq C \frac{t + \eta^{1/2} + t t[m_{fc,t}]}{(t^2 + \mathrm{Im}[\xi])^p} \leq C \frac{t + \mathrm{Im}[m_{fc,t}]}{(t^2 + \mathrm{Im}[\xi])^{p-1}} \quad (5.15)$$

The following is an immediate consequence of Lemma 5.2 and the definition of the spectral domain $\mathcal{D}_1$. Its role is to provide a sufficient condition under which we can Taylor expand the Schur complement formula for $G_{ii}$.

**Lemma 5.4.** On the event

$$\Lambda \leq \frac{\mathrm{Im}[m_{fc,t}]}{\log(N)^2}, \quad (5.16)$$

we have for any $\varepsilon > 0$,

$$|Z_i| \leq N^\varepsilon \left( t^2 (N^{-3\omega/2} + N^{-\sigma/2}) + t \mathrm{Im}[m_{fc,t}] N^{-\sigma/2} \right) + \frac{\mathrm{Im}[m_{fc,t}] t + t^2}{\log(N)^2}. \quad (5.17)$$

and

$$\frac{1}{2} |g_i| \leq |G_{ii}| \leq 2 |g_i|, \quad (5.18)$$

with overwhelming probability.

5.1.1 Self-consistent equation

In this subsection we derive the following self-consistent equation. Many arguments are similar to those appearing in [35].

**Proposition 5.5.** On the event

$$\Lambda \leq \frac{\mathrm{Im}[m_{fc,t}]}{\log(N)^2} \quad (5.19)$$

we have, for any $\varepsilon > 0$ with overwhelming probability,

$$\left| (1 - t R_2) (m_N - m_{fc,t}) + t^2 R_3 (m_N - m_{fc,t}) \right|^2 \leq \frac{1}{N} \sum_{i} |g_i|^2 Q_i(G_{ii}^{-1}) + N^\varepsilon \left( \frac{\Lambda + \mathrm{Im}[m_{fc,t}]}{N \eta} + \frac{1}{(N \eta)^2} \right) \frac{t}{t^2 + \mathrm{Im}[\xi]} + \Lambda^2 \frac{t}{\log(N)^2} + \frac{t}{\log(N)^2} \quad (5.20)$$

$$\leq N^\varepsilon \left( \frac{1}{N \eta} + \sqrt{\frac{\Lambda + \mathrm{Im}[m_{fc,t}]}{N \eta}} \right) \frac{t^2}{t^2 + \mathrm{Im}[\xi]} + \Lambda^2 \frac{t}{\log(N)^2} + \frac{t}{\log(N)^2} \quad (5.21)$$

**Proof.** Due to Lemma 5.4, we may, with overwhelming probability, Taylor expand the Schur complement formula (5.6) in powers of $Z_i$. We arrive at

$$m_N = m_{fc,t} + \frac{1}{N} \sum_{i} g_i^2 Z_i + \frac{1}{N} \sum_{i} g_i^3 (Z_i)^2 + O \left( \frac{1}{N} \sum_{i} |g_i|^4 |Z_i|^3 \right). \quad (5.22)$$
We split the first order term as
\[
\frac{1}{N} \sum_i g_i^2 Z_i = \frac{1}{N} \sum_i t g_i^2 (m_N - m_{\text{fc},t}) + \frac{1}{N} \sum_i g_i^2 Q_i (G_{ii}^{-1}) + \frac{1}{N} \sum_i g_i^2 t (m_N^{(i)} - m_N).
\] (5.23)
Using (5.10), (5.18) and (5.14) we obtain,
\[
\left| \frac{1}{N} \sum_i g_i^2 t (m_N^{(i)} - m_N) \right| \leq C \frac{1}{N^{1/2}} \sum_i |t g_i| \text{Im}[G_{ii}]
\leq C \frac{\Lambda + \text{Im}[m_{\text{fc},t}]}{N \eta} \frac{t}{t^2 + \text{Im}[\xi]}.
\] (5.24)
For the other term we write
\[
\frac{1}{N} \sum_i g_i^2 Q_i [G_{ii}^{-1}] = \frac{1}{N} \sum_i g_i^2 \sqrt{\gamma_i} + \frac{1}{N} \sum_i g_i^2 t (A_i + B_i).
\] (5.25)
For the first term we just calculate the variance and find that with overwhelming probability,
\[
\left| \frac{1}{N} \sum_i g_i^2 \sqrt{\gamma_i} \right| \leq N^\varepsilon \left( \frac{1}{N^{1/3}} \sum_i |t g_i|^4 \right)^{1/2} \leq C N^\varepsilon \frac{(t^2 + t \text{Im}[m_{\text{fc},t}])^{1/2}}{N(t^2 + \text{Im}[\xi])^{3/2}}
\leq C N^\varepsilon \left( \frac{t}{t^2 + \text{Im}[\xi]} \frac{\text{Im}[m_{\text{fc},t}]}{N \eta} + \frac{(t \text{Im}[m_{\text{fc},t}])^{1/2}}{N(t^2 + \text{Im}[\xi])^{3/2}} \right),
\] (5.26)
where we used (5.15). For the second sum of (5.25), we estimate, again using (5.15),
\[
\left| \frac{1}{N} \sum_i g_i^2 t (A_i + B_i) \right| \leq N^\varepsilon \left( \frac{1}{N \eta} + \sqrt{\frac{\Lambda + \text{Im}[m_{\text{fc},t}]}{N \eta}} \right) t^2 + t \text{Im}[m_{\text{fc},t}].
\] (5.27)
For the second order term we have by Cauchy-Schwarz
\[
\frac{1}{N} \sum_i g_i^2 Z_i^2 = \frac{1}{N} \sum_i g_i^2 (m_N - m_{\text{fc},t})
+ O \left( \frac{1}{N} \sum_i t^2 |g_i|^3 \frac{\Lambda^2}{\log(N)^2} + \log(N)^2 \frac{1}{N} \sum_i |g_i|^3 (|Q_i (G_{ii}^{-1})|^2 + t^2 |m_N - m_N^{(i)}|^2) \right)
\] (5.28)
We estimate,
\[
\frac{1}{N} \sum_i t^2 |g_i|^3 \frac{\Lambda^2}{\log(N)^2} \leq C \frac{\Lambda^2}{\log(N)^2} \frac{t}{t^2 + \text{Im}[\xi]}.
\] (5.29)
On $D_1$ we have,
\[
t |g_i| |m_N - m_N^{(i)}| \leq \frac{t}{(N \eta)(t^2 + t \text{Im}[\xi])} \leq 1
\] (5.30)
and so,
\[
\frac{1}{N} \sum_i |g_i|^2 t |m_N - m_N^{(i)}|^2 \leq \frac{1}{N} \sum_i |g_i|^2 t |m_N - m_N^{(i)}| \leq C \frac{\Lambda + \text{Im}[m_{\text{fc},t}]}{N \eta} \frac{t}{t^2 + \text{Im}[\xi]}.
\] (5.31)
Next, using (5.12) we find with overwhelming probability,
\[
\frac{1}{N} \sum_i |g_i|^3 |Q_i (G_{ii})|^2 \leq N^\varepsilon \left( \frac{t}{N} + \frac{t^2}{(N \eta)^2} + \frac{t^2 (\Lambda + \text{Im}[m_{\text{fc},t}])}{N \eta} \right) \frac{t + \text{Im}[m_{\text{fc},t}]}{(t^2 + \text{Im}[\xi])^2}
\leq N^\varepsilon \frac{t^2}{(N(t^2 + \text{Im}[\xi])^2) + \left[ \frac{\Lambda + \text{Im}[m_{\text{fc},t}]}{N \eta} + \frac{1}{(N \eta)^2} \right] \frac{t}{(t^2 + \text{Im}[\xi])}}
\leq CN^\varepsilon \left[ \frac{\Lambda + \text{Im}[m_{\text{fc},t}]}{N \eta} + \frac{1}{(N \eta)^2} \right] \frac{t}{t^2 + \text{Im}[\xi]}.
\] (5.32)
In the last line we used the fact that \( \text{Im}[m_{\ell, t}] / \eta \geq t/(t^2 + \text{Im}[\xi]) \) on \( D_1 \). Since we have \( |Z_i||g_i| \leq C/(\log(N))^2 \) with overwhelming probability, we easily obtain,

\[
\frac{1}{N} \sum_i |g_i|^2 |Z_i|^3 \leq \frac{C}{N} \sum_i t^2 |g_i|^3 \frac{\Lambda^2}{(\log(N))^2} + \frac{1}{N} \sum_i |g_i|^3 (|Q_i| [G_i^{-1}])^2 + t^2 |m_N - m_N^{(i)}|^2
\]

\[
\leq C \frac{\Lambda^2 t}{\log(N)^2 t^2 + \text{Im}[\xi]} + N^\varepsilon \left[ \frac{\Lambda + \text{Im}[m_{\ell, t}]}{N\eta} + \frac{1}{(N\eta)^2} \right] t^2 + \text{Im}[\xi].
\]  

(5.33)

The claim follows.

\[\square\]

### 5.1.2 Weak local law

Before establishing the optimal estimate of Proposition 5.3, we establish the following weaker estimate.

**Proposition 5.6.** Let \( \varepsilon > 0 \). With overwhelming probability we have the following estimates for every \( z \in D_1 \). For \( \kappa + \eta \geq t^2 \),

\[
|m_N - m_{\ell, t}| \leq N^\varepsilon \sqrt{\frac{t + \text{Im}[m_{\ell, t}]}{N\eta}}.
\]  

(5.34)

For \( \kappa + \eta \leq t^2 \),

\[
|m_N - m_{\ell, t}| \leq N^\varepsilon \frac{t^{2/3}}{(N\eta)^{1/3}}.
\]  

(5.35)

We start with the following.

**Proposition 5.7.** Fix \( \varepsilon > 0 \). The following holds with overwhelming probability on the event

\[
\Lambda \leq \frac{t + \text{Im}[m_{\ell, t}]}{(\log(N))^2}.
\]  

(5.36)

If \( \kappa + \eta \geq t^2 \), then

\[
\Lambda \leq 2N^\varepsilon \sqrt{\frac{t + \text{Im}[m_{\ell, t}]}{N\eta}} \leq CN^\varepsilon N^{-\sigma/2}(t + \text{Im}[m_{\ell, t}]).
\]  

(5.37)

If \( \kappa + \eta \leq t^2 \) then we have the following dichotomy. Either,

\[
\Lambda \geq c\sqrt{\kappa + \eta},
\]  

(5.38)

or

\[
\Lambda \leq N^\varepsilon \left( \frac{t}{\sqrt{\kappa + \eta}} \sqrt{\frac{\text{Im}[m_{\ell, t}]}{N\eta}} + \frac{t^2}{\kappa + \eta} \frac{1}{N\eta} \right).
\]  

(5.39)

If \( \kappa + \eta \leq t^2 \) then,

\[
\Lambda \leq N^\varepsilon \frac{t^{2/3}}{(N\eta)^{1/3}} + (\kappa + \eta)^{1/2} + N^\varepsilon t^{1/2} \left( \frac{\text{Im}[m_{\ell, t}]}{(N\eta)^{1/4}} + \frac{1}{(N\eta)^{1/2}} \right).
\]  

(5.40)

**Proof.** First suppose that \( \kappa + \eta \geq t^2 \). From (5.21) we get the estimate

\[
\Lambda \leq \frac{\Lambda}{2} + N^\varepsilon \sqrt{\frac{t + \text{Im}[m_{\ell, t}]}{N\eta}},
\]  

(5.41)

and so

\[
\Lambda \leq 2N^\varepsilon \sqrt{\frac{t + \text{Im}[m_{\ell, t}]}{N\eta}} \leq CN^\varepsilon N^{-\sigma/2}(t + \text{Im}[m_{\ell, t}]).
\]  

(5.42)
Now we assume that $|E - E_-| + \eta \leq t^2$. In this regime $|1-tR_2| = \sqrt{\kappa + \eta}/t$ and so we obtain from (5.21),

$$\Lambda \leq C \frac{\Lambda^2}{\sqrt{\kappa + \eta}} + \frac{tN^\varepsilon}{\sqrt{\kappa + \eta}} \left( \frac{1}{N\eta} + \sqrt{\frac{\Lambda + \text{Im}[m_{t,t}]}{N\eta}} \right).$$  \hspace{1cm} (5.43)

Hence, either

$$\Lambda \geq c \sqrt{\kappa + \eta}$$  \hspace{1cm} (5.44)

or

$$\Lambda \leq N^\varepsilon \left( \frac{t}{\sqrt{\kappa + \eta}} \sqrt{\frac{\text{Im}[m_{t,t}]}{N\eta}} + \frac{t^2}{\kappa + \eta} \frac{1}{N\eta} \right).$$  \hspace{1cm} (5.45)

The quadratic estimate from (5.21) gives,

$$\Lambda^2 \leq tN^\varepsilon \frac{\Lambda^{1/2}}{(N\eta)^{1/2}} + \Lambda(\kappa + \eta)^{1/2} + C \frac{\Lambda^2}{\log(N)^2}$$

$$+ tN^\varepsilon \left( \sqrt{\frac{\text{Im}[m_{t,t}]}{N\eta}} + \frac{1}{N\eta} \right),$$

which implies

$$\Lambda \leq N^\varepsilon t^{2/3} (N\eta)^{1/3} + \sqrt{\kappa + \eta} + N^\varepsilon t^{1/2} \left( \frac{\text{Im}[m_{t,t}]}{(N\eta)^{1/4}} + \frac{1}{(N\eta)^{1/2}} \right).$$  \hspace{1cm} (5.47)

The claim is proven. \hfill \square

**Proof of Proposition 5.6.** We follow the usual proof of such weak local laws, which is a continuity argument in $\eta$. What we have to check is that the estimate we obtain at each scale on $\Lambda$ is much smaller than $t + \text{Im}[m_{t,t}]$.

Fix an energy $E$. The estimate for $\eta \geq 1$ is standard. Fix then a sequence $\eta_k = 1 - k/N^2$, of cardinality less than $N^2$. First we estimate for $\eta_k$ with $\kappa + \eta_k \geq t^2$. By the continuity of $\Lambda$ in $z$ we see that the estimate (5.37) at $\eta_k$ implies that (5.36) holds at $\eta_{k+1}$. Then by Proposition 5.7 that (5.37) also holds at $\eta_{k+1}$ with overwhelming probability. Hence we obtain (5.37) for $\kappa + \eta \geq t^2$.

Now let us consider $\kappa + \eta \leq t^2$. Let $\eta^* = \eta^*(E)$ be the first time that

$$\kappa + \eta = N^\alpha \frac{t^{4/3}}{(N\eta)^{2/3}}.$$  \hspace{1cm} (5.48)

for a small $\alpha > 0$ that we choose later.

We may suppose that the estimate (5.35) holds at $\eta_{k-1}$ (if $\eta_{k-1} = t^2$ then instead (5.34) holds, but this is estimate is the same order as (5.34) at this scale). On the domain $D_1$ we have

$$\frac{1}{N\eta} \leq N^{-\sigma/2}(t + \sqrt{\kappa + \eta})$$  \hspace{1cm} (5.49)

and so we see that (5.36) holds at $\eta_k$, as well as

$$\Lambda \leq 2 \frac{N^\varepsilon t^{2/3}}{(N\eta_k)^{1/3}} \leq 2N^\varepsilon N^{-\sigma/2} \sqrt{\kappa + \eta},$$  \hspace{1cm} (5.50)

by the definition of $\eta_k$. Proposition 5.7 implies that with overwhelming probability, either

$$\Lambda \geq c\alpha$$  \hspace{1cm} (5.51)

or

$$\Lambda \leq N^\delta \left( \frac{t\text{Im}[m_{t,t}]}{\alpha^{1/2}(N\eta)^{1/2}} + \frac{t^2}{\alpha N\eta} \right) \leq N^\delta \frac{t^{2/3}}{(N\eta)^{1/3}}.$$  \hspace{1cm} (5.52)
for any $\delta > 0$. However, due to (5.50) the latter estimate holds.

By iteration, we obtain that (5.35) holds for $\eta \geq \eta^s$ with overwhelming probability. Now consider $\eta \leq \eta^s$. Suppose that the estimate
\[
\Lambda \leq N^{\alpha + \epsilon} t^{2/3} (N\eta)^{1/3}
\] (5.53)
holds at $\eta_{k-1}$, for any $\epsilon > 0$ with overwhelming probability. This implies that
\[
\Lambda \leq N^{-\sigma/4}(t + \text{Im}[m_{\ell_\ell}]).
\] (5.54)
provided we choose $a < \sigma/10$, which we can do. Then, this estimate implies that (5.36) holds at $\eta_k$. Then, we see that (5.40) holds, which implies that,
\[
\Lambda \leq N^{\alpha + \delta} t^{2/3} (N\eta)^{1/3},
\] (5.55)
with overwhelming probability, for any $\delta > 0$, at $\eta_k$. Hence, we obtain the claim on our spectral domain $D_1$.

5.1.3 Fluctuation averaging lemma

We have the following improved bound for one of the error terms appearing in the local law for $m_N$. For a deterministic control parameter $\gamma$ we define
\[
\Phi := \sqrt{t/N} + t\sqrt{\frac{\text{Im}[m_{\ell_\ell}]}{N\eta}} + \gamma.
\] (5.56)

**Lemma 5.8.** Fix $z \in D_1$. Suppose that $\Lambda \leq \gamma$ with overwhelming probability, where $\gamma$ is a deterministic control parameter satisfying
\[
\frac{1}{N\eta} \leq \gamma \leq t + \frac{\text{Im}[m_{\ell_\ell}]}{\log(N)^2}.
\] (5.57)

For any even $p \geq 2$ we have,
\[
E\left[\frac{1}{N} \sum_i g_i Q_i [G^{-1}_\ell G^{-1}_\ell]\right]^p \leq N^\varepsilon \left(\frac{1}{N\eta} \frac{\text{Im}[m_{\ell_\ell}]}{t + \text{Im}[m_{\ell_\ell}]} + \gamma\right)^p
\] (5.58)
for any $\varepsilon > 0$.

**Proof.** By (B.6) and the estimates
\[
|g_i| \leq \frac{C}{t^2 + \text{Im}[\xi]}, \quad \frac{1}{N} \sum_i |g_i|^2 \leq \frac{t + \text{Im}[m_{\ell_\ell}]}{t^2 + \text{Im}[\xi]},
\] (5.59)
we obtain
\[
E\left[\frac{1}{N} \sum_i g_i^2 Q_i [G^{-1}_\ell G^{-1}_\ell]\right]^p
\]
\[
\leq \max_{0 \leq s \leq p} \max_{0 \leq t \leq (p+s)/2} \frac{1}{N^p} \left(\sqrt{\frac{t}{N}} + t\sqrt{\frac{\text{Im}[m_{\ell_\ell}]}{N\eta}} + \gamma\right)^{p+s} \frac{1}{(\text{Im}[\xi] + t^2)^{2p+s}} (N(t^2 + \text{Im}[\xi])(t + \text{Im}[m_{\ell_\ell}]))^l
\]
\[
\leq \max_{0 \leq s \leq p} \frac{1}{N^{p/2}} \left(\sqrt{\frac{t}{N}} + t\sqrt{\frac{\text{Im}[m_{\ell_\ell}]}{N\eta}} + \gamma\right)^{p+s} \frac{N^{s/2}(t + \text{Im}[m_{\ell_\ell}])^{(p+s)/2}}{(\text{Im}[\xi] + t^2)^{3p/2+s/2}}.
\]
\[
\leq C \frac{1}{(N(\text{Im}[\xi] + t^2))^p}
\]
\[
+ \max_{0 \leq s \leq p} \left[\frac{t(\text{Im}[m_{\ell_\ell}])^1}{N\eta^{1/2}(\text{Im}[\xi] + t^2)^{3/2}}\right]^s \left[\frac{t(\text{Im}[m_{\ell_\ell}])^1}{\eta^{1/2}(\text{Im}[\xi] + t^2)^{1/2}}\right]^s.
\] (5.60)
The first term is bounded by
\[
\frac{1}{(N \text{Im}[\xi] + t^2)^p} \leq C \frac{1}{(N\eta)^p} \left( \frac{\text{Im}[m_{fc,t}]}{t + \text{Im}[m_{fc,t}]} \right)^p \leq C \frac{1}{(N\eta)^p} \left( \frac{\text{Im}[m_{fc,t}] + \gamma}{t + \text{Im}[m_{fc,t}] + \gamma} \right)^p. \tag{5.61}
\]
In the first inequality we used $\text{Im}[m_{fc,t}]/\eta \geq c/t$ which holds on $D_1$. For the second term,
\[
\max_{0 \leq s \leq p} \left[ \frac{t(\text{Im}[m_{fc,t}] + \gamma)^{1/2}(t + \text{Im}[m_{fc,t}])^{1/2}}{N\eta^{1/2}(\text{Im}[\xi] + t^2)^{3/2}} \right]^p \leq \left[ \frac{t(\text{Im}[m_{fc,t}] + \gamma)^{1/2}(t + \text{Im}[m_{fc,t}])^{1/2}}{N\eta^{1/2}(\text{Im}[\xi] + t^2)^{3/2}} \right]^p \leq C \left( \frac{1}{N\eta} \frac{\text{Im}[m_{fc,t}] + \gamma}{t + \text{Im}[m_{fc,t}] + \gamma} \right)^p. \tag{5.62}
\]
This yields the claim.

5.1.4 Proof of Proposition 5.3

The proof is by an iteration procedure on $\gamma$. We may assume that,
\[
\Lambda \leq \gamma = \frac{N\varepsilon t^{2/3}}{(N\eta)^{1/3}}, \tag{5.63}
\]
with overwhelming probability. Let us first consider the case $\kappa + \eta \geq t^2$. By (5.20) and (5.58) we have,
\[
\Lambda \leq \frac{1}{N\eta} + N\varepsilon \left( \frac{\Lambda + \text{Im}[m_{fc,t}]}{N\eta} + \frac{1}{(N\eta)^2} \right) \frac{1}{t + \text{Im}[m_{fc,t}]} + CA^2 \frac{1}{t + \text{Im}[m_{fc,t}]} \tag{5.64}
\]
Using that $\Lambda \ll t + \text{Im}[m_{fc,t}]/\log(N)^2$ and $(N\eta)^{-1} \leq C(t + \text{Im}[m_{fc,t}])$ on $D_1$ we see that
\[
\Lambda \leq \frac{N\varepsilon}{N\eta} \tag{5.65}
\]
for any $\varepsilon > 0$ with overwhelming probability. Now we consider $\kappa + \eta \leq t^2$. We see that In this case we see that
\[
|(1 - tR_2)(m_N - m_{fc,t}) + t^2R_3(m_N - m_{fc,t})|^2 \leq \frac{N\varepsilon \text{Im}[m_{fc,t}] + \gamma}{N\eta} \frac{1}{t} + \frac{N\varepsilon}{(N\eta)^2} \frac{1}{t} + \frac{\Lambda^2}{\log(N)^2 t}. \tag{5.66}
\]
for any $\varepsilon > 0$ with overwhelming probability. Hence we get the estimates
\[
\Lambda \leq C \frac{\Lambda^2}{\alpha} + N\varepsilon \frac{\gamma + \text{Im}[m_{fc,t}]}{N\eta \alpha} + \frac{N\varepsilon}{(N\eta)^2 \alpha} \tag{5.67}
\]
where we denoted
\[
\alpha := (\kappa + \eta)^{1/2}. \tag{5.68}
\]
Hence, we see that either
\[
\Lambda \geq c \alpha \tag{5.69}
\]
or
\[
\Lambda \leq N\varepsilon \frac{\gamma + \text{Im}[m_{fc,t}]}{N\eta \alpha} + \frac{N\varepsilon}{(N\eta)^2 \alpha}. \tag{5.70}
\]
On the other hand if we estimate the quadratic term in (5.66), we see that
\[
\Lambda^2 \leq C\alpha \Lambda + N\varepsilon \frac{\text{Im}[m_{fc,t}] + \gamma}{N\eta} + N\varepsilon \frac{1}{(N\eta)^2}. \tag{5.71}
\]
Let \( \eta_\alpha \) be the first time that
\[
\frac{\gamma + \text{Im}[m_{c,t}]}{N\eta\alpha} + \frac{1}{(N\eta)^2\alpha} = N^{-4\varepsilon}\frac{\alpha}{2}.
\] (5.72)
By a similar continuity argument as in the proof of Proposition 5.6 we get
\[
\Lambda \leq CN^{\varepsilon}\frac{\gamma + \text{Im}[m_{c,t}]}{N\eta\alpha} + C N^{\varepsilon} N^{N\varepsilon}\frac{\gamma}{N\eta} + CN^{\varepsilon} \frac{1}{(N\eta)}.
\] (5.73)
for \( \eta \geq \eta_\alpha \), for any \( \varepsilon > 0 \) with overwhelming probability. For \( \eta < \eta_\alpha \) we get
\[
\Lambda \leq \frac{N^{4\varepsilon}}{N\eta} N^{4\varepsilon} \sqrt{\frac{\gamma}{N\eta}} + N^{4\varepsilon} \sqrt{\frac{\text{Im}[m_{c,t}]}{N\eta}} \leq \frac{N^{10\varepsilon}}{N\eta} + N^{10\varepsilon} \sqrt{\frac{\gamma}{N\eta}}
\] (5.74)
where we used \( \text{Im}[m_{c,t}] \leq C\alpha \) and then \( \alpha^2 \leq N^{5\varepsilon}/(N\eta)^2 + N^{5\varepsilon}\gamma/(N\eta) \) which follows from (5.72) and using \( \text{Im}[m_{c,t}] \leq C\alpha \) and the Schwarz inequality. By iterating this argument finitely many times we see that we derive
\[
|m_N - m_{c,t}| \leq \frac{N^{\varepsilon}}{N\eta}
\] (5.75)
with overwhelming probability for any \( \varepsilon > 0 \) and \( z \in D_1 \).
For \( E \leq E_- \), note that \( \alpha \ll (N\eta)^{-1} \) in \( D_1 \). Hence, we immediately derive from (5.66) and the fact that \( \gamma \leq N^{\varepsilon}/(N\eta) \) that
\[
\Lambda \leq \frac{N^{2\varepsilon}}{(N\eta)^2\sqrt{\kappa + \eta}} + \frac{N^{2\varepsilon}}{N(\kappa + \eta)}.
\] (5.76)
This completes the proof. \( \square \)

5.2 Proof for \( E \leq E_- - t^2 \)

We now complete the proof of Theorem 5.1 by proving the following. Let
\[
D_2 := \{ E + i\eta : E \leq E_- - C_1 t^2 \} \cap D_\sigma.
\] (5.77)

**Proposition 5.9.** The estimates of Theorem 5.1 hold in \( D_2 \).

As the proof is very similar to Proposition 5.3 we only give the important changes to the proof in the following subsections.

We will use the following a-priori estimates, which are used instead of (5.14) and (5.15). We have for \( z \in D_1 \),
\[
|V_i - \xi| \geq c(t^2 + \kappa + \eta),
\] (5.78)
and
\[
\frac{1}{N} \sum_i |g_i|^p \leq \frac{C}{(\kappa + \eta + t^2)^{p-3/2}}.
\] (5.79)
These follow from Lemma 7.6.

5.2.1 Self-consistent equation

Instead of Lemma 5.4 we have the following which is an easy consequence of Lemma 5.2

**Lemma 5.10.** On the event
\[
\Lambda \leq \frac{t}{\log(N)^2}
\] (5.80)
we have with overwhelming probability,
\[
|Z_i| \leq \frac{C\frac{t}{\log(N)^2}}{|G_{ii}| \leq 2|g_i|}.
\] (5.81)
and
\[
\frac{1}{2} |g_i| \leq |G_{ii}| \leq 2|g_i|.
\] (5.82)
In the place of Proposition 5.5 we have,

**Proposition 5.11.** On the event

\[ \Lambda \leq \frac{t}{\log(N)^2}, \]  

we have with overwhelming probability, for any \( \varepsilon > 0 \),

\[ |(1 - tR_2)(m_N - m_{\text{fc},t})| \leq C \frac{\Lambda^2}{t} + N^\varepsilon \left( \frac{1}{(N\eta)^2(t + \sqrt{\kappa + \eta})} + \frac{1}{N(\kappa + \eta + t^2)} \right) + \frac{1}{N} \sum_i g_i^2 Q_i[G_{ii}^{-1}] \]

\[ \leq C \frac{\Lambda^2}{t} + N^\varepsilon \left( \frac{1}{N^{1/2}(t + \sqrt{\eta + \kappa})^{1/2}} + \sqrt{\frac{\text{Im}[m_{\text{fc},t}] + \Lambda}{N\eta}} + \frac{1}{N\eta} \right). \]  

(5.84)

Its proof is identical to that of Proposition 5.5, except that we do not need to keep the term which is second order in \( \Lambda^2 \) as we always have \(|1 - tR_2| \approx 1 \) on \( \mathcal{D}_2 \).

**5.2.2 Weak local law**

From Proposition 5.11 we immediately see that we have with overwhelming probability on the event \( \Lambda \leq t/\log(N)^2 \) that

\[ \Lambda \leq \frac{N^\varepsilon}{N^{1/2}(\kappa + \eta + t^2)^{1/2}} + \frac{N^\varepsilon}{N\eta}, \]  

(5.85)

for any \( \varepsilon > 0 \). The estimate on the RHS is \( \ll t \), and so by a similar proof to Proposition 5.6 we get

**Proposition 5.12.** Let \( \varepsilon > 0 \). With overwhelming probability,

\[ \Lambda \leq \frac{N^\varepsilon}{N^{1/2}(\kappa + \eta + t^2)^{1/2}} + \frac{N^\varepsilon}{N\eta}, \]  

(5.86)

on \( \mathcal{D}_2 \).

**5.2.3 Fluctuation averaging lemma**

Analogously to Lemma 5.8 we have,

**Lemma 5.13.** Fix \( z \in \mathcal{D}_2 \). Suppose that \( \Lambda \leq \gamma \) with overwhelming probability, where \( \gamma \) is a deterministic control parameter satisfying

\[ \frac{1}{N\eta} \leq \gamma \leq \frac{t + \text{Im}[m_{\text{fc},t}]}{\log(N)^2}. \]  

(5.87)

For any even \( p \geq 2 \) we have,

\[ \mathbb{E} \left[ \frac{1}{N} \sum_i g_i^2 Q_i[G_{ii}^{-1}] \right]^p \leq N^\varepsilon \left( \frac{1}{N(t^2 + \kappa + \eta)} + \frac{\gamma}{N\eta(t + \sqrt{\kappa + \eta})} \right)^p, \]  

(5.88)

for any \( \varepsilon > 0 \).

**Proof.** We proceed as in the proof of Lemma 5.8. Applying (B.6) and (5.78), (5.79) we obtain,

\[ \mathbb{E} \left[ \frac{1}{N} \sum_i g_i^2 Q_i[G_{ii}^{-1}] \right]^p \leq \max_{0 \leq s \leq p} \max_{0 \leq \ell \leq (p+s)/2} N^\varepsilon \left( \sqrt{\frac{t}{N}} + \sqrt{\frac{\text{Im}[m_{\text{fc},t}] + \gamma}{N\eta}} \right)^{p+s} \frac{1}{N^{\ell}} \frac{N^\ell}{(t^2 + \eta + \kappa)^{s+2p-3\ell/2} N^{\ell/p}} \]

\[ \leq \max_{0 \leq s \leq p} C N^\varepsilon \frac{N^{s/2}}{N^{s/2}} \left( \sqrt{\frac{t}{N}} + \sqrt{\frac{\text{Im}[m_{\text{fc},t}] + \gamma}{N\eta}} \right)^{p+s} \frac{1}{(t + \sqrt{\kappa + \eta})^{s+5p/2}} \]

\[ \leq C N^\varepsilon \left( \frac{t}{N^{1/2}(t + \sqrt{\kappa + \eta})^{1/2}} \right)^p \left( \frac{\left( \text{Im}[m_{\text{fc},t}] + \gamma \right)^{1/2}}{N^{1/2}(t + \sqrt{\kappa + \eta})^{1/2}} \right)^s. \]  

(5.89)
For the second term, we estimate
\[
\max_{0 \leq s \leq \rho} \left[ \frac{t(\text{Im}[m_{t,s,x}]) + \gamma)^{1/2}}{N\eta^{1/2}(t + \sqrt{\kappa + \eta})^{3/2}} \right]^p \leq \left[ \frac{\text{Im}[m_{t,s,x}]) + \gamma)^{1/2}}{N\eta^{1/2}(t + \sqrt{\kappa + \eta})^{3/2}} \right]^p + \left[ \frac{(\text{Im}[m_{t,s,x}]) + \gamma^{1/2}}{N\eta\sqrt{\kappa + \eta}} \right]^p \leq C \left[ \frac{1}{N(\kappa + \eta)} + \frac{\gamma}{N\eta\sqrt{\kappa + \eta}} \right]^p
\]
which yields the claim.

5.2.4 Proof of Proposition 5.9

Let \( \Lambda \geq \gamma \) with \( \gamma \) as in Lemma 5.13. From Lemma 5.13 and Proposition 5.11, we see that
\[
\Lambda \leq N^\varepsilon \left( \frac{1}{N(\kappa + \eta)} + \frac{1}{(N\eta)^2\sqrt{\kappa + \eta}} + \frac{\gamma}{(N\eta)\sqrt{\kappa + \eta}} \right),
\]
with overwhelming probability, for any \( \varepsilon > 0 \). Iterating this, we see that \( \Lambda \leq N^{\varepsilon/\eta} \) with overwhelming probability on \( \mathcal{D}_2 \). Then, applying the above estimate again with \( \gamma = N^{\varepsilon/\eta} \) we conclude the Proposition.

6 Local law for \( 0 \leq t \leq N^{1/3} \) and regular initial data

In this section we want to prove a local law for
\[
H_t := V + \sqrt{t}G
\]
in the regime \( 0 \leq t \leq N^{-\varepsilon} \), provided that \( V \) already obeys a local law. That is, we consider \( V \) such that for any \( \varepsilon > 0 \) and \( \sigma > 0 \), \( V \) obeys the estimates
\[
|m_V - \hat{m}| \leq \frac{N^\varepsilon}{N\eta}, \quad 0 \leq E \leq 1, \quad 10 \geq |E| + \eta \geq \frac{N^\sigma}{N\eta},
\]
and
\[
|m_V - \hat{m}| \leq N^\varepsilon \left( \frac{1}{N(|E| + \eta)} + \frac{1}{(N\eta)^2\sqrt{|E| + \eta}} \right), \quad -1 \leq E \leq 0, \quad 10 \geq \eta \geq N^{\sigma-2/3},
\]
where \( \hat{m} \) is the Stieltjes transform of a law \( \hat{\rho}(x) \) so that for \( |x| \leq 1 \), we have
\[
\hat{\rho}(x) = 1_{\{|x| \geq 0\}} \sqrt{x}.
\]
We will denote the free convolution of the semicircle at time \( t \) with \( \hat{\rho} \) by \( \hat{\rho}_{t,c} \) and its Stieltjes transform by \( \hat{m}_{t,c} \). By Section 7, \( \hat{\rho}_{t,c} \) behaves like a square root, and we denote the edge by \( \hat{E}_- \). We will abuse notation slightly and denote
\[
\kappa := |E - \hat{E}_-|.
\]

We want to prove the following theorem. Let \( \mathcal{D}_\sigma \) be as in Section 5.

**Theorem 6.1.** For any \( \sigma > 0 \), \( V \) as above, \( \varepsilon > 0 \) and \( \varepsilon_1 > 0 \), the following estimates hold with overwhelming probability, for any \( 0 \leq t \leq N^{-\varepsilon_1} \). First,
\[
|m_N - \hat{m}_{t,c}| \leq \frac{N^\varepsilon}{N\eta}
\]
for \( z \in \mathcal{D}_\sigma \) and \( E \geq \hat{E}_- \). For \( E \leq \hat{E}_- \) we have
\[
|m_N - \hat{m}_{t,c}| \leq N^\varepsilon \left( \frac{1}{N(\kappa + \eta)} + \frac{1}{(N\eta)^2\sqrt{\kappa + \eta}} \right).
\]
Let $\sigma > 0$. Suppose that we want to prove the above result on $D_\sigma$. Then in the regime

$$N^{-\epsilon} \geq t \geq \frac{N^{\sigma/100}}{N^{1/3}},$$

we know that Theorem 5.1 holds, except that $m_N$ is close to $m_{fc,t}$ which is the free convolution of $V$ and the semicircle distribution, and not $\hat{m}_{fc,t}$, which is the free convolution of $\hat{\rho}$ and the semicircle. From Appendix D we see that the difference $\hat{m}_{fc,t} - m_{fc,t}$ obeys the stated estimates of Theorem 6.1. Hence, we only need to prove Theorem 6.1 in the case that

$$0 \leq t \leq \frac{N^{\sigma/100}}{N^{1/3}}$$

on the domain $D_\sigma$. This is the content of the remainder of Section 6.

### 6.1 Proof for $E \geq \hat{E} - N^{-2/3+\sigma}$

Similarly to Section 5 it is useful to split the proof of Theorem 6.1 into two cases. The first is the following.

**Proposition 6.2.** Let $D_\sigma$ as above. Let $0 \leq t \leq N^{\sigma/100}/N$. The estimates of Theorem 6.1 hold on $
\hat{D}_1 := D_\sigma \cap \{E + i\eta : E \geq \hat{E} - N^{-2/3+\sigma}\}.

On $\hat{D}_1$ we use the following estimates.

$$|V_i - \xi| \geq c(\eta + t\sqrt{\kappa + \eta}),$$

and

$$\frac{1}{N} \sum_i |g_i|^p \leq C \frac{\sqrt{\kappa + \eta}}{(\eta + t\sqrt{\kappa + \eta})^{p-1}}$$

which follows from Section 7.3.1.

The following plays the role of Lemma 5.4.

**Lemma 6.3.** On the event

$$\Lambda \leq \frac{\text{Im}[\hat{m}_{fc,t}]}{\log(N)^2}$$

we have

$$|Z_i| \leq C \frac{\eta + t\sqrt{\kappa + \eta}}{\log(N)^2}$$

and

$$\frac{1}{2} |g_i| \leq |G_{ii}| \leq 2|g_i|$$

with overwhelming probability.

**Proof.** We just need to check the statement on $Z_i$ (for the $G_{ii}$ statement recall (6.11)). Recall $Z_i = t(m_N - \hat{m}_{fc,t}) + t(m_N^{(i)} - m_N) + t(A_i + B_i) + \sqrt{\tilde{h}}_{ii}$. We only need to estimate $\sqrt{\tilde{h}}_{ii}$, as the bounds for the other terms are immediate from Lemma 5.2 and $(N\eta) \leq N^{-\sigma}\sqrt{\kappa + \eta}$. With overwhelming probability,

$$\sqrt{\tilde{h}}_{ii} \leq N^{\sigma/4} t^{1/2} \leq N^{\sigma/4} \frac{t}{N\eta} + \frac{\eta}{\log(N)^2}$$

and the claim follows.
6.1.1 Self consistent equation

We have the following.

**Proposition 6.4.** On the event

\[ \Lambda \leq \frac{\text{Im}[\hat{m}_{fc,t}]}{\log(N)^2} \]  

(6.17)

we have with overwhelming probability, for any \( \varepsilon > 0 \),

\[
| (1 - tR_2)(m_N - \hat{m}_{fc,t}) | \leq C \frac{\Lambda^2}{\sqrt{\kappa + \eta}} + \left| \frac{1}{N} \sum_i g_i^2 Q_i [G_{ii}^{-1}] \right| + \frac{N^\varepsilon}{N\eta} \\
\leq C \frac{\Lambda^2}{\sqrt{\kappa + \eta}} + C N^\varepsilon + N^\varepsilon \sqrt{\frac{\text{Im}[\hat{m}_{fc,t}]}{N\eta}}. 
\]  

(6.18)

**Proof.** We can write,

\[ m_N - \hat{m}_{fc,t} = \frac{1}{V_i - \xi} - \hat{m}(\xi) + \frac{1}{N} \sum_i g_i^2 Z_i + O \left( \frac{N^\varepsilon}{N\eta} \right). \]  

(6.19)

Note that by assumption,

\[ \left| \frac{1}{V_i - \xi} - \hat{m}(\xi) \right| \leq \frac{N^\varepsilon}{N\eta}. \]  

(6.20)

For the remaining term we have,

\[ \frac{1}{N} \sum_i g_i^2 Z_i = \frac{1}{N} \sum_i g_i^2 (m_N - \hat{m}_{fc,t}) + \frac{1}{N} \sum_i g_i^2 Q_i [G_{ii}^{-1}] + O \left( \frac{1}{N\eta} \right). \]  

(6.21)

For the last term we write,

\[ \frac{1}{N} \sum_i g_i^2 Q_i [G_{ii}^{-1}] = \frac{1}{N} \sum_i g_i^2 \sqrt{h_{ii}} + \frac{1}{N} \sum_i g_i^2 (A_i + B_i). \]  

(6.22)

By a variance calculation the first term is \( O(N^\varepsilon/(N\eta)) \) for any \( \varepsilon > 0 \). The second term is bounded by

\[
\left| \frac{1}{N} \sum_i g_i^2 t(A_i + B_i) \right| \leq N^\varepsilon \sqrt{\frac{\text{Im}[\hat{m}_{fc,t}]}{N\eta}} + \frac{1}{N} \sum_i |g_i|^2 t \leq C N^\varepsilon \sqrt{\frac{\text{Im}[\hat{m}_{fc,t}]}{N\eta}}. 
\]  

(6.23)

using Lemma 5.2 and (6.12). \( \square \)

6.1.2 Weak local law

From Proposition 6.4 and the fact that \( |1 - tR_2| \approx 1 \) on \( \mathcal{D}_\sigma \) (due to \( t|R_2| \leq Ct/\sqrt{\kappa + \eta} \approx 1 \)), we see that with overwhelming probability on the event \( \Lambda \leq \sqrt{\kappa + \eta}/\log(N)^2 \) we have

\[
\Lambda \leq N^\varepsilon \sqrt{\frac{\text{Im}[\hat{m}]}{N\eta}}. 
\]  

(6.24)

On \( \hat{D}_2 \) the RHS is \( \ll \sqrt{\kappa + \eta} \) and so we conclude the following.

**Proposition 6.5.** For any \( \varepsilon > 0 \) we have with overwhelming probability on \( \hat{D}_1 \) that

\[
\Lambda \leq N^\varepsilon \sqrt{\frac{\text{Im}[\hat{m}_{fc,t}]}{N\eta}}. 
\]  

(6.25)
6.1.3 Fluctuation averaging lemma

We have,

**Lemma 6.6.** Suppose that \( \Lambda \leq \gamma \) with overwhelming probability where

\[
\frac{1}{N\eta} \leq \gamma \leq \frac{\sqrt{\kappa + \eta}}{\log(N)^2}.
\]

Then,

\[
\mathbb{E} \left| \frac{1}{N} \sum_i g_i^2 Q_i \left[ C_{ii} \right] \right|^p \leq N^\varepsilon \frac{1}{(N\eta)^p}.
\]

for any \( \varepsilon > 0 \).

**Proof.** Similarly to before, the moment in question is bounded by

\[
\max_{0 \leq s \leq p} \max_{0 \leq l \leq (p+s)/2} \left( \sqrt{\frac{t}{N}} + t \frac{\sqrt{\text{Im}[m_{i,l}]} + \gamma}{N\eta} \right)^{p+s} \frac{1}{(\eta + t\sqrt{\kappa + \eta})^{s+2p-2}} \frac{1}{N^{p-1}} \left( \frac{\sqrt{\kappa + \eta}}{\eta + t\sqrt{\kappa + \eta}} \right)^l
\]

\[
\leq \max_{0 \leq s \leq p} \frac{N^{(p+s)/2}(\eta + t\sqrt{\kappa + \eta})^s 2pN^p}{(\eta + t\sqrt{\kappa + \eta})^{s+2p-2}} + \max_{0 \leq s \leq p} \frac{N^{(p+s)/2}(\eta + t\sqrt{\kappa + \eta})^s 2pN^p}{(\eta + t\sqrt{\kappa + \eta})^{s+2p-2}}
\]

\[
+ \max_{0 \leq s \leq p} \frac{\left( \text{Im}[m_{i,l}] + \gamma \right)^{1/2} t(\sqrt{\kappa + \eta})^{1/2}}{(N\eta)^{1/2} N^{1/2} (\eta + t\sqrt{\kappa + \eta})^{3/2}} \left( \frac{t(\text{Im}[m_{i,l}] + \gamma)^{1/2}(\sqrt{\kappa + \eta})^{1/2}}{(N\eta)^{1/2} (\eta + t\sqrt{\kappa + \eta})^{1/2}} \right)^s
\]

(6.28)

where we just bounded the max over \( l \) by the sum of the term with \( l = 0 \) and the term with \( l = (p+s)/2 \) and then subsequently split the \( (t/N)^{1/2} \) and \( \sqrt{\text{Im}[m_{i,l}]} + \gamma/N\eta \) in two. It is immediate that the first term is bounded by \( (N\eta)^{-p} (N\eta\sqrt{\kappa + \eta})^{-(p+s)/2} \leq (N\eta)^{-p} \). The contribution from the second term is

\[
\frac{\left( \text{Im}[m_{i,l}] + \gamma \right)^{1/2} t(\sqrt{\kappa + \eta})^{1/2}}{(N\eta)^{1/2} N^{1/2} (\eta + t\sqrt{\kappa + \eta})^{3/2}} \leq \frac{1}{(N\eta)^p}
\]

\[
(6.29)
\]

using \( \gamma \leq \sqrt{\kappa + \eta} \). It is immediate that the third term on the second last line of (6.28) is bounded by \( (N\eta)^{-p} \). It is easy to see that the term on the last line of (6.28) is less than \( C(N\eta)^{-p} \) by considering the cases \( s = 0 \) and \( s = p \) separately and using \( \gamma \leq \sqrt{\kappa + \eta} \).

\[
(6.29)
\]

---

6.1.4 Proof of Proposition 6.2

From Proposition 6.4 and Lemma 6.6 we immediately see that

\[
\Lambda \leq \frac{N^\varepsilon}{N\eta}
\]

(6.30)

with overwhelming probability.

\[
(6.30)
\]

6.2 Proof for \( E \leq \hat{E}_- - N^{-2/3+\sigma} \)

We now define

\[
\mathcal{D}_2 := \mathcal{D}_\sigma \cap \left\{ E + i\eta : E \leq \hat{E}_- - N^{-2/3+\sigma} \right\}
\]

(6.31)

The goal of this section is to prove the following.

**Proposition 6.7.** The estimates of Theorem 6.1 holds on \( \mathcal{D}_2 \).
The estimates of Lemma 7.4 hold in the set-up of the present section. Hence, we see that there is a $C > 0$ so that for $z \in \mathcal{D}_2$ that if $\kappa \geq C\eta$, then $|\text{Re}[\xi]| \geq c\kappa$ for some $c > 0$. Therefore, we see that for any $\varepsilon > 0$ we have
\[
\left| \frac{1}{N} \sum_i \frac{1}{V_i - \xi} - \tilde{m}(\xi) \right| \leq N^\varepsilon \left( \frac{1}{N(\kappa + \eta)} + \frac{1}{(N\eta)^2 \sqrt{\kappa + \eta}} \right). \tag{6.32}
\]
From Section 7.3.1 we conclude
\[
\frac{1}{N} \sum_i |g_i|^p \leq \frac{C}{(\kappa + \eta)^{p-3/2}}. \tag{6.33}
\]
Note that
\[
|V_i - \xi| \geq c(\kappa + \eta) \geq c\eta. \tag{6.34}
\]
The following is an easy consequence of the fact that $t^2 \ll \eta$ on $\mathcal{D}_2$.

**Lemma 6.8.** We have with overwhelming probability on the event
\[
\Lambda \leq N^{-1/3} \tag{6.35}
\]
that
\[
|Z_i| \leq \frac{\eta}{\log(N)^{1/2}}. \tag{6.36}
\]

### 6.2.1 Self-consistent equation

Similar to above we can derive the following.

**Proposition 6.9.** With overwhelming probability on the event
\[
\Lambda \leq N^{-1/3}, \tag{6.37}
\]
we have
\[
|(1 - tR_2)(m_N - \tilde{m}_{i,t})| \leq \left| \frac{1}{N} \sum_i g_i^2 Q_i[G_{ii}^{-1}] \right| + C \Lambda^2 \left( \frac{1}{\sqrt{\eta}} + N^\varepsilon \left( \frac{1}{N(\kappa + \eta)} + \frac{1}{(N\eta)^2 \sqrt{\kappa + \eta}} \right) \right) \leq N^\varepsilon \left( \frac{1}{N^{1/2}(\kappa + \eta)^{1/2}} + \frac{1}{N\eta} \right) \tag{6.38}
\]
for any $\varepsilon > 0$ with overwhelming probability.

For the second inequality we are just using that with overwhelming probability,
\[
\left| \frac{1}{N} \sum_i g_i^2 Q_i[G_{ii}] \right| \leq N^\varepsilon \left( \frac{t^{1/2}}{(N(\kappa + \eta))^{1/2}} + \frac{t}{\sqrt{\kappa + \eta}} \sqrt{\text{Im}[\tilde{m}_{i,t}] + \Lambda \frac{N\eta}{N}} \right) \leq CN^\varepsilon \frac{1}{N^{1/2}(\kappa + \eta)^{1/2}}. \tag{6.39}
\]

### 6.2.2 Weak local law

By Proposition 6.9, we see that with overwhelming probability on the event
\[
\Lambda \leq N^{-1/3}, \tag{6.40}
\]
we have
\[
\Lambda \leq N^\varepsilon \left( \frac{1}{N^{1/2}(\kappa + \eta)^{1/2}} + \frac{1}{N\eta} \right) \leq N^{-\sigma/10}N^{-1/3}. \tag{6.41}
\]
Hence, we conclude the following.

**Proposition 6.10.** With overwhelming probability we have on $\mathcal{D}_2$ that
\[
\Lambda \leq N^\varepsilon \left( \frac{1}{N^{1/2}(\kappa + \eta)^{1/2}} + \frac{1}{N\eta} \right). \tag{6.42}
\]
6.2.3 Fluctuation averaging lemma

We now have the following fluctuation averaging lemma.

**Lemma 6.11.** Suppose that \( \Lambda \leq \gamma \) with overwhelming probability, where \( \gamma \) obeys

\[
\frac{1}{N\eta} \leq \gamma \leq N^{-1/3}.
\]  

(6.43)

Then we have

\[
E \left| \frac{1}{N} \sum_i q_i^2 Q_i [G^{-1}_{ii}] \right|^p \leq N^\varepsilon \left( \frac{1}{N(k + \eta)} + \frac{\gamma}{N\eta \sqrt{k + \eta}} \right)^p.
\]  

(6.44)

for any \( \varepsilon > 0 \).

**Proof.** By (B.6) we have,

\[
E \left| \frac{1}{N} \sum_i q_i^2 Q_i [G^{-1}_{ii}] \right|^p \leq N^\varepsilon \max_{0 \leq s \leq p} \frac{\max \left( \sqrt{\frac{t}{N}} + t \sqrt{\frac{\text{Im}[\hat{m}_{fc,t}]}{N\eta}} + \gamma \right)^{p+s}}{(k + \eta)^{s+2p-3/2}} \frac{1}{N^{p-l}}
\]  

(6.45)

In the second inequality we used that the maximum occurs at \( l = (p+s)/2 \). Clearly the first term is bounded by \( CN^\varepsilon (N(k + \eta))^{-p} \). For the second, we see that since \( t \leq \sqrt{\eta} \) and \( \gamma \leq N^{-1/3} \) that the maximum occurs at \( s = 0 \). Then,

\[
\max_{0 \leq s \leq p} \frac{t^{p+s} (\text{Im}[\hat{m}_{fc,t}])^{(p+s)/2}}{N\eta^{p/2}(k + \eta)^{s+5p/4}} \leq C \left( \frac{(\text{Im}[\hat{m}_{fc,t}])^{1/2}}{N\eta^{1/2}(k + \eta)^{3/4}} \right)^p \leq C \left( \frac{1}{N(k + \eta)} + \frac{\gamma}{N\eta \sqrt{k + \eta}} \right)^p
\]  

(6.46)

6.2.4 Proof of Proposition 6.7

Starting with \( \gamma \) as in Proposition 6.10 we have by Proposition 6.9 and Lemma 6.11 that

\[
\Lambda \leq N^\varepsilon \left( \frac{1}{N(k + \eta)} + \frac{1}{(N\eta)^2 \sqrt{k + \eta}} + \frac{\gamma}{N\eta \sqrt{k + \eta}} \right).
\]  

(6.47)

for any \( \varepsilon > 0 \) with overwhelming probability. Hence by iteration we obtain that \( \Lambda \leq N^\varepsilon/(N\eta) \) with overwhelming probability. Taking this choice of \( \gamma \) we then get the claim.

7 Analysis of free convolution law

Let \( V \) be \( \eta_* - \text{regular} \) as in Definition 2.1. We will consider

\[
t := \frac{N\omega}{N^{1/3}}
\]  

(7.1)

with \( \phi_{\omega}/2 > 1/3 - \omega > 0 \).

Recall our definition of \( m_{fc,t} \) which satisfies

\[
m_{fc,t}(z) = m_V(\xi)
\]  

(7.2)

where

\[
\xi(z) = z + tm_{fc,t}(z).
\]  

(7.3)
Define the map
\[ F(\xi) = \xi - t \int \frac{d\mu_V(x)}{x - \xi}, \tag{7.4} \]
so that
\[ z = F(\xi). \tag{7.5} \]

An important role is played by the contour \( \gamma \) which we define as the image of \( \mathbb{R} \) under the map \( E \rightarrow \xi(E) \).

A useful observation is that since
\[ \text{Im}[m_{fc,t}] = \left(1 - t \int \frac{d\mu_V(x)}{|x - \xi|^2}\right)^{-1} \text{Im}[m_V(\xi)] \tag{7.6} \]
we have,
\[ \int \frac{d\mu_V(x)}{|x - \xi|^2} \leq \frac{1}{t} \tag{7.7} \]
from which the inequality
\[ |m_{fc,t}| \leq t^{-1/2} \tag{7.8} \]
follows by Cauchy-Schwartz. Hence, we see that \( \text{Im}[\xi(E)] = 0 \) for \( E \in [-3/4, -1/2] \).

By Lemma C.1 it is easy to check that there is a unique solution \( \xi_- \in [-3/4, 3/4] \) to the equation
\[ 1 = t \int \frac{d\mu_V}{(x - \xi_-)^2}. \tag{7.9} \]

Moreover, we see that
\[ -\xi_- \approx t^2. \tag{7.10} \]

Let \( E_- \) be such that \( \xi(E_-) = \xi_- \). For \( E \geq E_- \), \( \xi(E) \) has non-trivial imaginary part, equalling \( t\rho_{fc,t}(E) \).

We will write
\[ \xi(E) := a + bi \tag{7.11} \]
for \( a, b \in \mathbb{R} \). In general, \( a \) is a strictly increasing function of \( E \), and \( a \) and \( b \) solve
\[ 1 = t \int \frac{d\mu_V}{(x - a)^2 + b^2}. \tag{7.12} \]

We denote \( a_- = \text{Re}[\xi_-] = \xi_- \). Our first goal is to get qualitative behaviour of the contour \( \gamma \). We have,

**Lemma 7.1.** For \( 3/4 \geq a \geq a_- \), we have
\[ b \approx t|a - a_-|^{1/2}. \tag{7.13} \]

**Proof.** We first consider \( a \) near \( a_- \). First, by (7.10) and Lemma C.1 we see that for a small \( c > 0 \) we have
\[ |F^{(k)}(\xi)| \leq \frac{C}{t^{2k-2}}, \quad |\xi - \xi_-| \leq ct^2 \tag{7.14} \]
for any \( k \geq 2 \). Moreover, we see that
\[ |F^{(k)}(\xi_-)| = \frac{C}{t^{2k-2}} \tag{7.15} \]
for \( k \geq 2 \). Note that they are real numbers. Hence, for \( |\xi - \xi_-| \leq ct^2 \), we can expand
\[ F(\xi) - F(\xi_-) = \frac{F''(\xi_-)}{2}(\xi - \xi_-)^2 + \frac{F'''(\xi_-)}{6}(\xi - \xi_-)^3 + \mathcal{O}(t^{-6}|\xi - \xi_-|^4) \tag{7.16} \]

39
The LHS equals \( z - E_- \). We can set \( z = E \) and invert this, obtaining first that

\[
    \xi - \xi_- = \frac{2(E - E_-)}{F''(\xi_-)} \left( 1 - \frac{F'''(\xi_-)}{3F''(\xi_-)}(\xi - \xi_-) + \mathcal{O}(t^{-4}|\xi - \xi_-|^2) \right),
\]

and then back-substituting once more we obtain

\[
    \xi - \xi_- = \frac{2(E - E_-)}{F''(\xi_-)} \left( 1 - \frac{2(E - E_-)}{3F''(\xi_-)} \right) + \mathcal{O}(t^{-4}|\xi - \xi_-|^2). \tag{7.18}
\]

Taking real and imaginary parts (note that \( F''(\xi_-) \) is negative) we see that

\[
|a - a_-| \approx |E - E_-|, \quad b = t|E - E_-|^{1/2} \approx t|a - a_-|^{1/2}, \tag{7.19}
\]

for \( |\xi - \xi_-| \leq ct^2 \).

Now, a straightforward calculation using Lemma C.1 and (7.12) shows that there is a small \( c > 0 \) so that \( b \approx t|a - a_-| \) for \( -ct^2 \leq a \leq 3/4 \).

It remains to consider the region \( a_- + c_1 t^2 \leq a \leq -c_1 t^2 \) for a small \( c_1 > 0 \). That is, we need to prove that \( b \approx t^2 \) here. Note that the upper bound \( b \leq Ct^2 \leq C|a - a_-|^{1/2} \) follows immediately from (C.3). For the lower bound we compute

\[
0 = \frac{1}{t} - \int \frac{d \mu_V(x)}{2(x-a)^2 + b^2} + \int \frac{d \mu_V(x)}{(x-a_-)^2} - \int \frac{d \mu_V(x)}{(x-a)^2 + b^2}
\]

\[
= \int \frac{(x-a)^2 + b^2 - (x-a_-)^2}{(x-a)^2 + b^2} d \mu_V(x)
\]

\[
= \int \frac{1}{(x-a_-)^2} \frac{(x-a_-)^2}{(x-a)^2 + b^2} d \mu_V(x). \tag{7.20}
\]

We rearrange this to get

\[
\frac{b^2}{a - a_-} = \int \frac{2x-a}{(x-a)^2 + b^2} d \mu_V(x). \tag{7.21}
\]

Note that for \( x \in \text{supp}(\mu_V) \) we have \( |x - a| \geq ct^2 \) and since \( b \leq ct^2 \) we get

\[
(x-a)^2 + b^2 \geq (x-a)^2 \geq (x-a_-)^2. \tag{7.22}
\]

We need to get a lower bound on the numerator of (7.21) and an upper bound on the denominator. For the numerator, we have

\[
\int \frac{2x-a-a_-}{(x-a_-)^2((x-a)^2 + b^2)} d \mu_V(x) = \int_{x \geq 1/2} \frac{2x-a-a_-}{(x-a_-)^2((x-a)^2 + b^2)} d \mu_V(x)
\]

\[
+ \int_{x \leq -1/2} \frac{2x-a-a_-}{(x-a_-)^2((x-a)^2 + b^2)} d \mu_V(x)
\]

\[
\geq ct^2 \int_{x \geq 1/2} \frac{1}{(x-a_-)^2((x-a)^2 + b^2)} d \mu_V(x) - C
\]

\[
\geq ct^2 \int_{x \geq 1/2} \frac{1}{(x-a_-)^2} d \mu_V(x) - 2C
\]

\[
\geq ct^2 \int_{\mathbb{R}} \frac{1}{(x-a_-)^4} d \mu_V(x) - 2C \tag{7.23}
\]
In the first inequality we used that the integrand is bounded in the region $x \leq -1/2$ and that for $x \geq -1/2$ that $x - a \geq x - a_- \geq ct^2$ for $x \in \text{supp}(\mu_V)$. In the third inequality we again used the fact that the integrand is bounded for $x \leq -1/2$. In the fourth inequality we use (7.22) and in the final inequality we use that the integral is order $t^{-3}$.

For the integral in the denominator of (7.21) we use (7.22) to prove that

$$\int \frac{1}{(x-a_-)^2((x-a)^2+b^2)} \, d\mu_V(x) \leq C \int \frac{1}{(x-a_-)^4} \, d\mu_V(x),$$

(7.24)

which yields $b^2 \geq ct^2(a - a_-)$ and completes the proof.

In order to complete our calculation of the contour $\gamma$ we need the following.

**Lemma 7.2.** We have $|a - a_-| \approx |E - E_-|$.

**Proof.** We already know from (7.19) that this holds for $|a - a_-| \leq ct^2$ for a small $c > 0$. The claim will therefore be proved by showing that

$$\frac{da}{dE} \approx 1$$

(7.25)

for $|a - a_-| \geq ct^2$.

We calculate

$$\frac{da}{dE} = \text{Re} \left[ 1 - t \int \frac{1}{(x - \xi)^2} \, d\mu_V(x) \right].$$

(7.26)

Since $1 = t \int |x - \xi|^{-2} \, d\mu_V(x)$ the numerator of (7.26) is positive and the denominator is less than 2, so

$$\frac{da}{dE} \geq \frac{1}{2} \text{Re} \left[ 1 - t \int \frac{1}{(x - \xi)^2} \, d\mu_V(x) \right].$$

(7.27)

Clearly,

$$\frac{da}{dE} \leq \left( \text{Re} \left[ 1 - t \int \frac{1}{(x - \xi)^2} \, d\mu_V(x) \right] \right)^{-1}.$$  

(7.28)

Hence, the claim will follow by proving

$$\text{Re} \left[ 1 - t \int \frac{1}{(x - \xi)^2} \, d\mu_V(x) \right] \geq c > 0$$

(7.29)

We can write

$$\text{Re} \left[ 1 - t \int \frac{1}{(x - \xi)^2} \, d\mu_V(x) \right] = \left( 1 - t \int \frac{1}{|x - \xi|^2} \, d\mu_V(x) \right) + 2b^2 t \int \frac{1}{|x - \xi|^4} \, d\mu_V(x)$$

$$= 2b^2 t \int \frac{1}{|x - \xi|^4} \, d\mu_V(x).$$

(7.30)

By Lemma C.1 we have for $a \geq 0$,

$$b^2 t \int \frac{1}{|x - \xi|^4} \, d\mu_V(x) = \frac{tb^2(a + b)^{1/2}}{b^3} \approx \frac{(a + b)^{1/2}}{|a - a_-|^{1/2}} \approx 1$$

(7.31)

where we used Lemma 7.1 in the second last step. For $a_- + ct^2 \leq a \leq 0$ we have

$$b^2 t \int \frac{1}{|x - \xi|^4} \, d\mu_V(x) = \frac{tb^2}{(|a| + b)^{5/2}} \approx \frac{t^3|a - a_-|}{(|a - a_-|^{1/2}t + |a|)^{5/2}} \approx 1.$$  

(7.32)

The claim follows.
7.1 Estimates on the map $\xi$ and control of self-consistent equation coefficients

The following result will be useful.

Lemma 7.3. The following holds for $|E| \leq 1/2$ and $0 \leq \eta \leq 10$. We have,

$$
|1 - t \int \frac{1}{(x - \xi)^2} d\mu_V(x)| = \min \left\{ \sqrt{\frac{|E - E_-| + \eta}{t}}, 1 \right\}
$$

(7.33)

Proof. We denote $\xi = a + bi$. Unlike in the previous subsection we do not restrict $\xi$ to lie on the contour $\gamma$. First assume that $|a - a_-| + b \geq ct^2$. We can write

$$
\text{Re} \left[ 1 - t \int \frac{1}{(x - \xi)^2} d\mu_V \right] = \left(1 - t \int \frac{1}{|x - |x - \xi|^2} d\mu_V(x) \right) + 2b^2t \int \frac{1}{|x - |x - \xi|^4} d\mu_V(x).
$$

(7.34)

The term in the brackets on the RHS is always positive. For $a \geq 0$ we have by Lemma C.1,

$$
t \int \frac{1}{|x - |x - \xi|^2} d\mu_V(x) = \frac{t(a^{1/2} + b^{1/2})}{b} =: Q
$$

(7.35)

and

$$
tb^2 \int \frac{1}{|x - |x - \xi|^2} d\mu_V(x) = Q.
$$

(7.36)

Hence,

$$
\text{Re} \left[ 1 - t \int \frac{1}{(x - \xi)^2} d\mu_V \right] \geq \max \{1 - CQ, cQ\} \geq c.
$$

(7.37)

Now we assume $a \leq 0$. We have,

$$
t \int \frac{1}{|x - |x - \xi|^2} d\mu_V(x) \approx \frac{t}{(|a| + b)^{1/2}},
$$

(7.38)

and so there is a constant $C$ s.t. if $|a|^{1/2} + b^{1/2} \geq Ct$ then

$$
\left(1 - t \int \frac{1}{|x - |x - \xi|^2} d\mu_V(x) \right) \geq c.
$$

(7.39)

Now we assume that $c_1t^2 \leq |a - a_-| + |b| \leq C_1t^2$, for given $c_1$ and $C_1$, as well as $a \leq 0$. Assume first that $a \geq a_-$. In this regime we must have $b \geq ct|a - a_-|^{1/2}$, as $b$ must lie above the contour $\gamma$ in $C$. Hence the assumption $c_1t^2 \leq |a - a_-| + |b|$ implies $b \geq ct^2$ for another $c > 0$ depending on $c_1$. Hence,

$$
b^2t \int \frac{1}{|x - |x - \xi|^2} d\mu_V(x) \approx \frac{b^2t}{(|a| + b)^{5/2}} \geq \frac{b^2t}{(|a_-| + |a - a_-| + b)^{5/2}} \geq \frac{ct^5}{(t^2 + t^2)^{5/2}} \geq c.
$$

(7.40)

We postpone the case $c_1t^2 \leq |a - a_-| + |b| \leq C_1t^2$ and $a \leq a_-$. First, we consider the case $|a - a_-| + b \leq ct^2$ for a small $c$. Then we can expand

$$
1 - t \int \frac{1}{(x - \xi)^2} d\mu_V(x) = E^n(\xi_-)(\xi - \xi_-) \left(1 + O(t^{-2}|\xi - \xi_-|)\right).
$$

(7.41)

Hence for a small enough $c$,

$$
\left|1 - \int \frac{1}{(x - \xi)^2} d\mu_V(x)\right| \approx t^{-2}(|a - a_-| + b) \approx t^{-1}(|E - E_-| + \eta)^{1/2}, \quad |\xi - \xi_-| \leq ct^2.
$$

(7.42)

We now return to the postponed case above. Since $|a - a_-| + b \geq c_1t^2$ we have that $b \geq c_1/2t^2$ if $|a - a_-| \leq c_1/2t^2$. Then calculation (7.40) applies, and so we can instead work in the regime $c_1t^2 \leq |a - a_-| + b \leq C_1t^2$ and $a \leq a_- - c_1t^2$. Then,

$$
1 - t \int \frac{d\mu_V}{(x - a)^2} + b^2 \geq 1 - t \int \frac{d\mu_V}{(x - a)^2} \geq 1 - t \int \frac{d\mu_V}{(x - (a_- - c_2t^2))^2} - Ct
$$

(7.43)
for any \( c_2 > 0 \) such that \( c_2 < c_1 \). The \( Ct \) appearing above is a bound for the contribution of the integral from \( x \in \text{supp}(\mu_V) \) such that \( x \leq -1 \). If we take \( c_2 \) small enough so that the estimate proved in the expansion above holds, then

\[
1 - t \int \frac{d\mu_V}{(x - (a_- - c_2t^2))^2} \asymp 1.
\]  

(7.44)

This completes the proof.

\[ \square \]

Again we write \( \xi = a + bi \) for general \( E + i\eta \) with \( |E| \leq 1/2 \) and \( 0 \leq \eta \leq 10 \). First we remark that the proof of Lemma 7.3 and (7.26) immediately yield that

\[
\frac{da}{dE} \asymp 1, \quad \kappa + \eta \geq ct^2
\]

(7.45)

for any \( c > 0 \). By the Cauchy-Riemann equations we also reach the same conclusion for \( \frac{db}{d\eta} \). Since

\[
\xi'(z) = \frac{1}{1 - t\int \frac{d\mu_V(x)}{(x - \xi)^2}}
\]

(7.46)

we then see that

\[
\left| \frac{da}{d\eta} \right| + \left| \frac{db}{dE} \right| \leq C
\]

(7.47)

for \( \kappa + \eta \geq ct^2 \).

Hence, we have proved the following.

**Lemma 7.4.** We have in the region \( \kappa + \eta \geq ct^2 \), for any \( c > 0 \), that

\[
\frac{da}{dE} \asymp 1, \quad \frac{db}{d\eta} \asymp 1.
\]

(7.48)

In the same region we have

\[
\left| \frac{da}{d\eta} \right| + \left| \frac{db}{dE} \right| \leq C.
\]

(7.49)

The above implies

\[
|a| + |a - a_-| + b \leq C(t^2 + \eta + \kappa).
\]

(7.50)

We also have,

\[
t\sqrt{\kappa + \eta} \asymp |\xi - \xi_-|,
\]

(7.51)

for \( \kappa + \eta \leq ct^2 \) for some small \( c > 0 \).

The estimate (7.51) follows from (7.17). The following lemma is needed for the proof of the local law.

**Lemma 7.5.** Consider \( z \in \mathcal{D}_\sigma \). In the region \( E \geq E_- - t^2 \), we have

\[
|V_i - \xi| \geq c(t^2 + \eta + t\text{Im}[m_{i,\xi}]),
\]

(7.52)

and

\[
\int \frac{d\mu_V(x)}{|x - \xi|^p} \leq C \frac{t + \sqrt{\kappa + \eta}}{(t^2 + \text{Im}[\xi])^{p-1}}.
\]

(7.53)

On \( \mathcal{D}_1 \) we have,

\[
t + \sqrt{\kappa + \eta} \leq C(t + \text{Im}[m_{i,\xi}]).
\]

(7.54)

**Proof.** First, note that \( |V_i - \xi| \geq \text{Im}[\xi] = \eta + t\text{Im}[m_{i,\xi}] \). If \( \eta + t\text{Im}[m_{i,\xi}] \leq c_1t^2 \) then \( \kappa + \eta \leq Cc_1t^2 \) for another \( C > 0 \). Choosing \( c_1 \) sufficiently small we see that \( |\xi - \xi_-(E_-)| \leq ct^2 \) for any small \( c \), from Lemma 7.4. Since \( \xi_-(E_-) \leq -ct^2 \) we see that \( |V_i - \xi| \geq ct^2 \).
For the next estimate, first suppose that $\text{Im}[\xi] \le c t^2$ for a small $c > 0$. Choosing $c$ small enough we see that, as above, $a \le -ct^2$ and so

$$
\int \frac{d\mu_V(x)}{|x - \xi|^p} \le C \frac{1}{|a + b|^{p-3/2}} \le C \frac{1}{(t^2 + \text{Im}[\xi])^{p-3/2}} \le C \frac{t}{(t^2 + \text{Im}[\xi])^{p-1}}.
$$

We may assume that $\text{Im}[\xi] \ge c t^2$ for a small $c > 0$. Then, noting that we always have (C.2) as an upper bound, no matter the sign of $a$, we get

$$
\int \frac{d\mu_V(x)}{|x - \xi|^p} \le C \frac{\sqrt{|a| + b}}{(\text{Im}[\xi])^{p-1}} \le C \frac{\sqrt{|a| + b}}{(t^2 + \text{Im}[\xi])^{p-1}}.
$$

Using Lemma 7.4 to bound the numerator, we complete the proof of (7.53). The estimate (7.54) is easy.

We also have the following.

**Lemma 7.6.** For $z \in D_\sigma$ and $E \le E_- - t^2$ we have,

$$
|V_\xi - \xi| \ge c(t^2 + \kappa + \eta)
$$

and

$$
\int \frac{d\mu_V(x)}{|x - \xi|^p} \le \frac{C}{(\kappa + \eta + t^2)^{p-3/2}}.
$$

**Proof.** For the first estimate, as argued in the proof of Lemma 7.5, we see that if $\kappa + \eta \le c t^2$ for a small $c > 0$, then we already arrive at $|V_\xi - \xi| \ge c t^2$. Since $\text{Im}[\xi] \ge \eta$ we then get that $|V_\xi - \xi| \ge c(t^2 + \eta)$. Then by Lemma 7.4, we note that there is a $C > 0$ so that if $E \le E_- - C\eta - t^2$, then $a \le -c_1 \kappa$ for another $c_1 > 0$. Hence, $|V_\xi - \xi| \ge C\kappa$ for such $E$, which completes (7.57).

Building on this observation, we see that there is a $C > 0$ so that if $E \le E_- - C\eta - t^2$, then

$$
\int \frac{d\mu_V(x)}{|x - \xi|^p} \le \frac{1}{C \kappa^{p-3/2}} \le \frac{1}{(\kappa + \eta + t^2)^{p-3/2}}
$$

where we applied (C.3). So we can assume that $\kappa \le C\eta + t^2$. If $\eta \ge t^2$, then the desired estimate immediately follows from (C.2) and $|a + b| \le C(t^2 + \eta + \kappa)$. In the case $\eta \le t^2$ then the result follows from (C.2) and just counting powers of $t$. 

We also have the following.

**Lemma 7.7.** There is a $c_1 > 0$ so that if $\kappa + \eta \le c_1 t^2$, then

$$
\left| \int \frac{d\mu_V(x)}{|x - \xi|^3} \right| \le \frac{1}{t^3}
$$

**Proof.** First, note that the claim follows for

$$
\int_{x \ge -1/2} \frac{d\mu_V(x)}{|x - \xi|^3}
$$

by Lemma C.1. Then note that for $x \ge -1/2$ and $x \in \text{supp}(\mu_V)$, that

$$
\left|\text{Re}\left[\frac{1}{|x - \xi|^3}\right]\right| \le \frac{1}{|x - \xi|^3},
$$

by direct calculation, if we take $c_1 > 0$ sufficiently small. This yields the claim.
7.2 Qualitative properties of $\rho_{fc,t}$ and $m_{fc,t}$

We first prove the following.

**Lemma 7.8.** We have the following for $|E| \leq 3/4$. The density $\rho_{fc,t}$ satisfies

$$\rho_{fc,t}(E) = |E - E_-|^{1/2} 1_{(E \geq E_-)}.$$  \hfill (7.63)

Moreover, for $|E - E_-| \leq ct^2$ we have,

$$\rho_{fc,t}(E) = \sqrt{\frac{|E - E_-|}{t^2 F''(\xi_-)}} \left(1 + O\left(\frac{|E - E_-|}{t^2}\right)\right)$$ \hfill (7.64)

We have also

$$|t^2 F''(\xi_-)| \approx 1.$$ \hfill (7.65)

**Proof.** We have already proved (7.63), because $\text{Im}[\xi] = t\rho_{fc,t}(E)$. Equation (7.64) follows from continued back-substitution in (7.17) (note that the correction to the $|E - E_-|^{1/2}$ term above is $|E - E_-|^{3/2}$ instead of $|E - E_-|$ - this is due to all the coefficients in (7.17) being real). The final estimate is a consequence of the fact that $-\xi_- \approx t^2$ and Lemma C.1. \hfill \Box

Now since $m_{fc,t}$ has a square root behaviour, we get the following.

**Lemma 7.9.** We have for $E \geq E_-$,

$$\text{Im}[m_{fc,t}] \approx \sqrt{\kappa + \eta}$$ \hfill (7.66)

and for $E \leq E_-$,

$$\text{Im}[m_{fc,t}] = \frac{\eta}{\sqrt{\kappa + \eta}}.$$ \hfill (7.67)

We have the equality

$$\partial_z m_{fc,t}(z) = \left(1 - t \int \frac{d\mu(x)}{(x - x_0)^2}\right)^{-1} \int \frac{d\mu(x)}{(x - x)^2}$$ \hfill (7.68)

from which, using Lemma 7.3 and (7.7) we conclude that

$$|\partial_z m_{fc,t}| \leq C \max \left\{ \frac{1}{\sqrt{\kappa + \eta}}, \frac{1}{t} \right\}.$$ \hfill (7.69)

Combining this with the trivial estimate

$$|\partial_z m_{fc,t}| \leq \frac{\text{Im}[m_{fc,t}]}{\eta}$$ \hfill (7.70)

we obtain

**Lemma 7.10.** For $\kappa + \eta \leq t^2$ we have

$$|\partial_z m_{fc,t}| \leq \frac{C}{\sqrt{\kappa + \eta}}.$$ \hfill (7.71)

For $\kappa + \eta \geq t^2$ we have for $E \geq E_-$,

$$|\partial_z m_{fc,t}| \leq C \frac{\sqrt{\kappa + \eta}}{t^{\sqrt{\kappa + \eta + \eta}},}$$ \hfill (7.72)

and for $E \leq E_-$,

$$|\partial_z m_{fc,t}| \leq \frac{C}{\sqrt{\kappa + \eta}}$$ \hfill (7.73)
7.3 Comparison of free convolutions of matching measures

In this section the set-up is the following. We have two measures $\mu_1$ and $\mu_2$ that have densities on the interval $[-1, 1]$ such that,

$$\rho_1(x) = \rho_2(x) \left(1 + O\left(\frac{|x|}{t_0^2}\right)\right), \quad 0 \leq x \leq ct_0^2,$$

(7.74)

and $\rho_1(x) = \rho_2(x) = 0$ for $-1 \leq x \leq 0$ and $\rho_2(x) = \sqrt{x}$.

We assume that for $|x| + \eta \leq ct_0^2$ that we have

$$|\partial_x m_i(z)| \leq \frac{C}{\sqrt{|x| + \eta}}.$$

(7.75)

We consider the free convolutions $m_{1,t}$ and $m_{2,t}$ for $0 \leq t \leq t_0 N^{-\varepsilon_0}$, for some $\varepsilon_0 > 0$. Denote the maps $\xi_i = z + tm_{i,t}(z)$, as well as the points $\xi_i, -$ as above. Our goal is to compare the densities $\rho_{i,t}$ to each other.

Since the $\rho_i(x)$ are continuous densities behaving like a square root, the qualitative behaviour of Definition 2.1 holds down to $\eta_\ast = 0$. Hence, the analysis of the previous subsection goes through, and the contours $\gamma_i = \xi_i(\mathbb{R})$ have the same qualitative behaviour, i.e., there is a $E_{-,i}$ at which they leave the real line, and for $E \geq E_{-,i}$ we have

$$|a_i - a_{-,i}| = |E - E_{-,i}|, \quad b_i \approx t|E - E_{-,i}|^{1/2},$$

(7.76)

etc. Moreover, it is not hard to check that the estimate

$$|\partial_x m_{i,t}(z)| \leq \frac{C}{\sqrt{|E - E_{-,i}| + \eta}}$$

(7.77)

using the methods of the previous section, as well as (7.75).

We first check that the $a_{-,i}$ are close. Using the equation that defines them we get

$$0 = \int \frac{d\mu_1(x)}{(x - a_{-,1})^2} - \int \frac{d\mu_2(x)}{(x - a_{-,2})^2} = (a_{-,1} - a_{-,2}) \int \frac{(2x - a_{-,1} - a_{-,2})}{(x - a_{-,1})^2(x - a_{-,2})^2} d\mu_1(x) + \left(\int \frac{d\mu_1(x)}{(x - a_{-,1})^2} - \int \frac{d\mu_2(x)}{(x - a_{-,2})^2}\right)$$

(7.78)

It is easy to check using the assumption (7.74) that

$$\left|\int \frac{d\mu_1(x)}{(x - a_{-,2})^2} - \int \frac{d\mu_2(x)}{(x - a_{-,2})^2}\right| \leq \frac{C}{t_0}$$

(7.79)

We bound below the factor multiplying $(a_{-,1} - a_{-,2})$ by

$$\left|\int \frac{(2x - a_{-,1} - a_{-,2})}{(x - a_{-,1})^2(x - a_{-,2})^2} d\mu_1(x)\right| = \left|\int \frac{d\mu_1(x)}{(x - a_{-,1})^2(x - a_{-,2})} + \int \frac{d\mu_1(x)}{(x - a_{-,1})(x - a_{-,2})}\right|$$

(7.80)

Each term on the RHS is positive and is order $t^{-3}$. Hence,

$$|a_{-,1} - a_{-,2}| \leq Ct_0^2 \frac{t}{t_0}.$$

(7.81)

It this then easy to see that,

$$\left|F_1^{(k)}(\xi_{-,1}) - F_2^{(k)}(\xi_{-,2})\right| \leq C \frac{t}{t_0} \frac{1}{t^{2(k-1)}}$$

(7.82)

Consider now the expansion

$$E - E_{-,i} = \sum_{k=2}^m \frac{F_1^{(l)}(\xi_{-,i})(\xi_i - \xi_{-,i})^j}{j!} + \mathcal{O}\left(|\xi_i - \xi_{-,i}|^{m+1}t^{-2m}\right).$$

(7.83)
By repeated back-substitution, as in the proof of Lemma 7.1 we see that for any $\delta_0 > 0$ we have, for $0 \leq x \leq N^{-\delta_0} t^2$, that
\[
a_1(x + E_{-1}) - a_{-1} = (a_2(x + E_{-2}) - a_{-2})(1 + O(t/t_0 + N^{-D})),
b_1(x + E_{-1}) = b_2(x + E_{-2})(1 + O(t/t_0 + N^{-D})).
\] (7.84)
for any $D > 0$, if we take $m$ large enough (but finite), depending on $\delta_0$.

We make the choice $\delta_0 < \varepsilon_0/100$. (7.85)

We now need to deal with
\[
t^2 N^{-\delta_0} \leq x \leq t_0^2.
\] (7.86)
Since we know that $da/dE$ is increasing we can parameterize $b = b(a)$. We then want to determine the difference $b_1(a) - b_2(a)$ beyond their natural scale. We have the equation
\[
0 = \int \frac{d\mu_1(x)}{(x-a)^2 + b_1^2} - \int \frac{d\mu_2(x)}{(x-a)^2 + b_2^2} = (b_2 - b_1)(b_1 + b_2) \int \frac{d\mu_1(x)}{((x-a)^2 + b_1^2)((x-a)^2 + b_2^2)} + \int \frac{d\mu_1(x) - d\mu_2(x)}{(x-a)^2 + b_2^2}.
\] (7.87)
By (7.74) we have the estimate,
\[
\left| \int \frac{d\mu_1(x) - d\mu_2(x)}{(x-a)^2 + b_2^2} \right| \leq C \frac{1}{t_0}.
\] (7.88)
To lower bound the integral multiplying the $b$'s, we note that since $|a - a_{-1}| \geq ct^2 N^{-\delta_0}$ and $|a - a_{-2}| \geq ct^2 N^{-\delta_0}$, and that by our choice of $\delta_0$ that $ct^2 N^{-\delta_0} \gg |a_{-1} - a_{-2}|$, we have $|a - a_{-1}| \approx |a - a_{-2}|$. This also implies $b_1 \approx b_2$. We have the lower bound
\[
\left| \int \frac{d\mu_1(x)}{((x-a)^2 + b_1^2)((x-a)^2 + b_2^2)} \right| \geq \frac{c_1 |a_{\geq 0}|}{(|a| + |a - a_{-1}|^{1/2} t)^{5/2}} + \frac{c_1 |a_{\geq 0}| (|a| + |a - a_{-1}|^{1/2} t)^{1/2}}{(t_0 |a - a_{-1}|^{1/2} t)^3}
\] (7.89)
Hence, using $|a| 1_{\{a \leq 0\}} \leq Ct^2$, and that $|a - a_{-1}| \approx |a| + t |a - a_{-1}|^{1/2}$ when $a \geq 0$,
\[
|b_1 - b_2| \leq \frac{C 1_{\{a \leq 0\}} (|a| + |a - a_{-1}|^{1/2} t)^{5/2}}{t_0 |a - a_{-1}|^{1/2} t} + \frac{C 1_{\{a \geq 0\}} (t |a - a_{-1}|^{1/2} t)^2}{t_0 (|a - a_{-1}|^{1/2} t + |a|)^2}
\leq \frac{C 1_{\{a \leq 0\}} t^5}{t_0 N^{-\delta_1} t^2} + \frac{C 1_{\{a \leq 0\}} (|a - a_{-1}|^{1/2} t)^2}{t_0 t_0 |a - a_{-1}|^{1/2} t} + \frac{C 1_{\{a \geq 0\}} t^2 |a - a_{-1}|^{1/2}}{t_0 t_0}.
\] (7.90)

We now want to use this to show that $da_1/dE$ and $da_2/dE$ are close for $a$ at least $ct^2 N^{-\delta_1}$ away from $a_{-1}$ and $a_{-2}$. This means that we need to study the function
\[
1 - t \int \frac{1}{(x-\xi_1)^2} d\mu_1(x).
\] (7.91)
Fix $a$. We write
\[
\left(1 - t \int \frac{d\mu_1(x)}{(x-\xi_1)^2}\right) - \left(1 - t \int \frac{d\mu_2(x)}{(x-\xi_2)^2}\right) = \left(t \int \frac{1}{(x-\xi_1)^2} - \frac{1}{(x-\xi_2)^2} d\mu_2(x)\right) + \left(t \int \frac{1}{(x-\xi_2)^2} d\mu_2(x) - d\mu_1(x)\right).
\] (7.92)
As above, we have
\[
\left| t \int \frac{1}{(x-\xi_1)^2} d\mu_2(x) - d\mu_1(x) \right| \leq C \frac{t}{t_0}.
\] (7.93)
For the other term we write it as
\[
\left( t \int \frac{1}{(x - \xi_2)^2} - \frac{1}{(x - \xi_1)^2} \right) \, \mu_2(x) = t(b_1 - b_2) \int \frac{(x - \xi_1) + (x - \xi_2)}{(x - \xi_1)^2(x - \xi_2)^2} \, d\mu_2(x). \tag{7.94}
\]

For \( a \leq 0 \) we have, using our bounds on \( b_1 - b_2 \) and Lemma C.1,
\[
|t(b_1 - b_2) \int \frac{(x - \xi_1) + (x - \xi_2)}{(x - \xi_1)^2(x - \xi_2)^2} \, d\mu_2(x)| \leq C \frac{N^{25} t}{t_0} \frac{t^2 |a - a_{-1}|^{1/2}}{(|a| + t |a - a_{-1}|^{1/2})^{3/2}} \leq C \frac{N^{25} t}{t_0} \tag{7.95}
\]
where we used \(|a| + t |a - a_{-1}|^{1/2} \geq ct^2\). For \( a \geq 0 \) we have
\[
|t(b_1 - b_2) \int \frac{(x - \xi_1) + (x - \xi_2)}{(x - \xi_1)^2(x - \xi_2)^2} \, d\mu_2(x)| \leq C \frac{N^{25} t |a - a_{-1}|^{1/2} t^{1/2} |a - a_{-1}|^{1/4}}{t^2 |a|} \leq C \frac{N^{25} t}{t_0}. \tag{7.96}
\]

From this we get that for \(|a - a_{-1}| \geq ct^2 N^{-\delta},
\[
\left| \frac{dE_1}{da} - \frac{dE_2}{da} \right| \leq C \frac{N^{55} t}{t_0}. \tag{7.97}
\]

From our earlier expansion we see that for \( x \leq ct^2 N^{-\delta},
\[
(E_1(x + a_{-1}) - E_{-1}) = (E_2(x + a_{-2}) - E_{-2}) \left(1 + \mathcal{O}(t/t_0)\right). \tag{7.98}
\]
We then use (7.97) and get
\[
(E_1(x + a_{-1}) - E_{-1}) = (E_2(x + a_{-2}) - E_{-2}) \left(1 + \mathcal{O}\left(N^{55} t/t_0\right)\right), \tag{7.99}
\]
for \( 0 \leq x \leq t_0^2 N^{-10\delta}. \) This easily implies an estimate on \( a_1 \) and \( a_2. \) Indeed, define the functions
\[
f(x) = a_1(x + E_{-1}) - a_{-1}, \quad g(x) = a_2(x + E_{-2}) - a_{-2}. \tag{7.100}
\]
We know that for \( x \leq ct^2 \) that
\[
f(x) = g(x) \left(1 + \mathcal{O}(t/t_0)\right). \tag{7.101}
\]
Assume that \( x \geq ct^2. \) Both \( f \) and \( g \) are bijections of some intervals \( I_{f/g} \to J_{f/g} \) and for their inverses we have
\[
f^{-1}(y) = g^{-1}(y) \left(1 + \mathcal{O}(N^{55} t/t_0)\right). \tag{7.102}
\]
We write
\[
f(x) - g(x) = g(g^{-1}(f(x))) - g(f^{-1}(f(x))) = \frac{g(g^{-1}(f(x)) - g(f^{-1}(f(x)))}{g^{-1}(f(x)) - f^{-1}(f(x))} (g^{-1}(f(x)) - f^{-1}(f(x))). \tag{7.103}
\]
For the quotient we have the bound
\[
\left| g(g^{-1}(f(x)) - g(f^{-1}(f(x)))) \right| \leq C N^{25}, \tag{7.104}
\]
which is a result of the fact that
\[
\left| \frac{da_1}{dE} \right| \leq C N^{26_0} \tag{7.105}
\]
for \( E \geq E_{-} + N^{-\delta_0} t^2. \) Therefore, we obtain
\[
f(x) = g(x) \left(1 + \mathcal{O}(N^{75} t/t_0)\right). \tag{7.106}
\]
Now, we write
\[
b_1(x + E_{-1}) - b_2(x + E_{-2}) = (b_1(x + E_{-1}) - b_2(E)) + (b_2(E) - b_2(x + E_{-2})) \tag{7.107}
\]
where $E$ is chosen so that $a_2(E) = a_1(x + E_{-1})$. Then, by the above bounds on $b_1(a) - b_2(a)$ we get

$$|b_1(x + E_{-1}) - b_2(E)| \leq C t \frac{\sqrt{N^{5\delta_0} t}}{t_0}. \quad (7.108)$$

By (7.99), we see that

$$|E - (x + E_{-2})| \leq \frac{C x N^{5\delta_0} t}{t_0}. \quad (7.109)$$

Hence, since $|\partial_E \rho_l(E + E_{-i})| \leq C E^{-1/2}$ (by (7.77)) we see that

$$b_2(E) - b_2(x + E_{-2}) \leq C t \sqrt{x} \frac{N^{5\delta_0} t}{t_0}. \quad (7.110)$$

We have therefore proved the following.

**Lemma 7.11.** Let $\varepsilon > 0$ and $t, t_0$ as above. For $0 \leq x \leq c N^{-2\varepsilon} t_0^2$ we have

$$\rho_{t,1}(x + E_{-1}) = \rho_{t,2}(x + E_{-2}) (1 + \mathcal{O}(N^\varepsilon t/t_0)) \quad (7.111)$$

and

$$|\text{Re}[m_{t,1}(x + E_{-1}) - m_{t,1}(E_{-1})] - \text{Re}[m_{t,2}(x + E_{-2}) - m_{t,2}(E_{-2})]| \leq C \frac{x N^\varepsilon}{t_0}. \quad (7.112)$$

We also need an estimate similar to (7.112) for $x \leq 0$. That is, we have

**Lemma 7.12.** Let $\varepsilon > 0$ and $t, t_0$ as above. For $-c N^{-2\varepsilon} t_0^2(t_1/t_0) \leq x \leq 0$ we have,

$$|\text{Re}[m_{t,1}(x + E_{-1}) - m_{t,1}(E_{-1})] - \text{Re}[m_{t,2}(x + E_{-2}) - m_{t,2}(E_{-2})]| \leq C \frac{|x|^{1/2}(t_1)^{1/2} N^\varepsilon}{t_0^{1/2}}. \quad (7.113)$$

**Proof.** Fixing a scale $\eta \leq N^{-2\varepsilon} t_0^2$ we can estimate the quantity by

$$|\text{Re}[m_{t,1}(x + E_{-1}) - m_{t,1}(E_{-1})] - \text{Re}[m_{t,2}(x + E_{-2}) - m_{t,2}(E_{-2})]|$$

$$\leq \left| \int_{E \geq \eta, E < -1/2} \left( \frac{1}{E - x} - \frac{1}{E} \right) \rho_{t,1}(E + E_{-1}) \right|$$

$$+ \left| \int_{E \geq \eta, E < -1/2} \left( \frac{1}{E - x} - \frac{1}{E} \right) \rho_{t,2}(E + E_{-2}) \right|$$

$$+ \left| \int_{0 \leq E \leq \eta} (\rho_{t,2}(E + E_{-2}) - \rho_{t,1}(E + E_{-2})) \left( \frac{1}{E} + \frac{1}{E + x} \right) \right| =: A_1 + A_2 + A_3. \quad (7.114)$$

By the square root behaviour of the densities we have,

$$|A_1| + |A_2| \leq C \frac{|x|}{\eta^{1/2}}. \quad (7.115)$$

For $A_3$ we use (7.111) and find

$$|A_3| \leq C N^\varepsilon \frac{t}{t_0} \eta^{1/2}. \quad (7.116)$$

We obtain the estimate by choosing $\eta = |x|^{1/2}(t_0/t_1)^{1/2}$. \hfill \Box
7.3.1 Self-consistent equation coefficients

Recall the definition of $\hat{D}_1$. It is clear that on this domain, that $\text{Im}[m_{fc, t}] \geq c\sqrt{\kappa + \eta}$, and so we get

$$|V_i - \xi| \geq \text{Im}[\xi] \geq \eta + ct\sqrt{\kappa + \eta}. \quad (7.117)$$

In the set-up of Section 6 the analogous estimates to Lemma 7.4 hold, and so in general we have

$$|\xi| \leq C(t^2 + \kappa + \eta) \leq C(\kappa + \eta) \quad (7.118)$$

where the second inequality holds due to the definition of $\hat{D}_1$. Hence,

$$\frac{1}{N} \sum_i |g_i|^p \leq C \frac{|\xi|^{1/2}}{\text{Im}[\xi]^{p-1}} \leq C \frac{\sqrt{\kappa + \eta}}{(\eta + t\sqrt{\kappa + \eta})^{p-1}}. \quad (7.119)$$

We now consider $\hat{D}_2$. Since the estimates of Lemma 7.4 hold, we have that there is a $C > 0$ so that if $\kappa \geq C\eta$, then $\text{Re}[\xi] \leq -c\kappa$. Hence, we have (6.32). This also proves that

$$|V_i - \xi| \geq c(\kappa + \eta), \quad (7.120)$$

as well as

$$\frac{1}{N} \sum_i |g_i|^p \leq \frac{C}{(\kappa + \eta)^{p-3/2}}, \quad (7.121)$$

for $\kappa \geq C\eta$. On the other hand if $\kappa \leq C\eta$, then,

$$\frac{1}{N} \sum_i |g_i|^p \leq C \frac{|\xi|^{1/2}}{\eta^{p-1}} \leq C \frac{1}{(\kappa + \eta)^{p-3/2}}. \quad (7.122)$$

A Large deviations estimates

Let $X_i$ be a family of independent random variables obeying

$$\mathbb{E}[X_i] = 0, \quad \mathbb{E}[|X_i|^2] = 1, \quad \mathbb{E}[|X_i|^p] \leq C_p, \quad p \geq 2. \quad (A.1)$$

We have the following estimates, see, e.g., [21].

**Lemma A.1.** Let $X_i$ and $Y_i$ be random variables obeying (A.1). Let $b_i$ and $a_{ij}$ be deterministic. We have for any $\varepsilon > 0$ and $D > 0$ and $N$ large enough,

$$\mathbb{P}\left[\sum_i b_i X_i \geq N^\varepsilon \left(\sum_i |b_i|^2\right)^{1/2}\right] \leq N^{-D}, \quad (A.2)$$

$$\mathbb{P}\left[\sum_{ij} a_{ij} X_i Y_j \geq N^\varepsilon \left(\sum_{ij} |a_{ij}|^2\right)^{1/2}\right] \leq N^{-D}, \quad (A.3)$$

$$\mathbb{P}\left[\sum_{i\neq j} X_i a_{ij} X_j \geq N^\varepsilon \left(\sum_{i\neq j} |a_{ij}|^2\right)^{1/2}\right] \leq N^{-D}. \quad (A.4)$$

B Fluctuation averaging lemma

We record here the following fluctuation averaging lemma. As it is very similar to estimates appearing in [35], we do not give a proof. The proof of the main estimate, (B.6), is very similar to the proof given there.
Proposition B.1. Suppose that \( \gamma \) is a deterministic parameter such that
\[
|m_N - m_{\mathcal{C}, t}| \leq \gamma \tag{B.1}
\]
with overwhelming probability, and
\[
\frac{1}{N\eta} \leq \gamma \leq \frac{t + \sqrt{\kappa + \eta} + N^{-1/3}}{\log(N)^2}. \tag{B.2}
\]
Then, with overwhelming probability, for any \( \varepsilon > 0 \),
\[
|Q_{\text{t}}[(G_{ii}^{(\text{T})})^{-1}]| \leq N^\varepsilon \left( \sqrt{\frac{t}{N} + t \sqrt{\frac{\gamma + \text{Im}[m_{\mathcal{C}, t}]]}{N\eta}} \right), \tag{B.3}
\]
and
\[
\frac{1}{2} |g_i| \leq |G_{ii}^{(\text{T})}| \leq 2 |g_i|, \tag{B.4}
\]
and,
\[
|G_{ij}| \leq N^\varepsilon |g_i| |g_j| \left( \sqrt{\frac{t}{N} + t \sqrt{\frac{\text{Im}[m_{\mathcal{C}, t}] + \gamma}{N\eta}} \right) \tag{B.5}
\]
uniformly over \( |T| \leq C \) for any fixed \( C > 0 \).
Moreover, for any even \( p > 0 \) we have, for any \( \varepsilon > 0 \),
\[
\mathbb{E} \left| \frac{1}{N} \sum_i g_i^2 Q_i [G_{ii}^{-1}] \right|^p \leq N^\varepsilon \max_{0 \leq s \leq p} \max_{0 \leq t \leq (p+s)/2} \left( \sqrt{\frac{t}{N} + t \sqrt{\frac{\text{Im}[m_{\mathcal{C}, t}] + \gamma}{N\eta}} \right)^{s+2p-2t} \frac{1}{N^{p-t}} \left( \frac{1}{N} \sum_i |g_i|^2 \right)^t. \tag{B.6}
\]

C \ Im[m] \ analysis

Let \( \mu_Y \) be a measure whose Stieltjes transform obeys the assumptions of Section 2. Define the domain \( \mathcal{D}_\ast \) by
\[
\mathcal{D}_\ast := \{ E + \imath \eta : -3/4 \leq E \leq 0, 2\eta_\ast \leq \eta \leq 10 \} \cup \{ E + \imath \eta : 0 \leq E \leq 3/4, \eta^{1/2} \sqrt{|E| + \eta} \leq \eta \leq 10 \} \cup \{ E + \imath \eta : -3/4 \leq E \leq -2\eta_\ast, 0 \leq \eta \leq 10 \}. \tag{C.1}
\]
First it is clear that the estimates of Definition 2.1 hold in the domain \( \mathcal{D}_\ast \). We want to prove the following.

Lemma C.1. Let \( \mu_Y \) be as above. For any \( p \geq 2 \) we have the following for \( a + \imath b \in \mathcal{D}_\ast \). If \( a \geq 0 \),
\[
\int \frac{d\mu_Y(x)}{|x - a - \imath b|^p} \asymp \frac{\sqrt{a + b}}{b^{p-1}}. \tag{C.2}
\]
If \( a \leq 0 \) then,
\[
\int \frac{d\mu_Y(x)}{|x - a - \imath b|^p} \leq \frac{1}{(|a| + b)^{p-3/2}}. \tag{C.3}
\]
Proof. The upper bounds are immediate. We first prove the lower bound of (C.2). Fix a \( C_\ast > 0 \). We have,
\[
\int \frac{d\mu_Y(x)}{|x - a - \imath b|^p} \geq \int_{|x - a| \leq C_\ast b} \frac{d\mu_Y(x)}{|x - a - \imath b|^p} \geq \frac{c}{b^{p-2}} \int_{|x - a| \leq C_\ast b} \frac{d\mu_Y(x)}{|x - a - \imath b|^2}. \tag{C.4}
\]
for a \( c > 0 \) depending on \( C_\ast \). We then have,
\[
\int_{|x-a| \leq C_\ast b} \frac{d\mu_V(x)}{|x-a-b|^2} = \int \frac{d\mu_V(x)}{|x-a-b|^2} - \int_{|x-a| > C_\ast b} \frac{d\mu_V(x)}{|x-a-b|^2} \\
\geq \frac{1}{b} \left( \text{Im}[m_V(a+b)] - \frac{4}{C_\ast} \text{Im}[m_V(a+C_\ast b/2i)] \right) \\
\geq \frac{1}{b} \left( c_1 \sqrt{a+b} - \frac{C_1 \sqrt{a+C_\ast b}}{C_\ast} \right)
\]
(C.5)
where \( c_1 \) and \( C_1 \) only depend on the assumptions on \( \mu_V \). Hence by taking \( C_\ast \) large enough depending only on \( c_1 \) and \( C_1 \), we see that
\[
\int_{|x-a| \leq C_\ast b} \frac{d\mu_V(x)}{|x-a-b|^2} \geq c \frac{\sqrt{a+b}}{b}
\]
which in view of the above yields the lower bound of (C.2).

The above argument also gives the lower bound of (C.3) in the regime \(-b \leq a \leq 0\). Note that in this case, the RHS of (C.3) is the same order as (C.2). In particular, we obtain (C.3) for all \( a+b \in \mathcal{D}_\ast \) such that \(-2 \eta_\ast \leq a \leq 0\) (for such \( a \) we have \( b \geq 2 \eta_\ast \) by the definition of \( \mathcal{D}_\ast \)).

It remains to prove the lower bound of (C.3) in the case that \( a \leq -2 \eta_\ast \) and \( |a| \geq b \). Fix again a \( C_\ast > 0 \). We have
\[
\int \frac{d\mu_V}{|x-a-b|^p} \geq \int_{x \leq C_\ast |a|} \frac{d\mu_V}{|x-a-b|^p} \geq \frac{c}{|a|^{p-2}} \int_{x \leq C_\ast |a|} \frac{d\mu_V}{(x-a)^2}
\]
for a \( c > 0 \) depending on \( C_\ast \). We also used the fact that \( |a| \geq b \) to observe that \( |x-a-b| \geq |x-a| \) on the support of \( \mu_V \) since \( |a| \geq 2 \eta_\ast \). We now have,
\[
\int_{x \leq C_\ast |a|} \frac{d\mu_V(x)}{(x-a)^2} \geq \int_{x \geq C_\ast |a|} \frac{d\mu_V(x)}{(x-a)^2} - 2 \int \frac{d\mu_V(x)}{(x-C_\ast a)^2} \\
\geq \frac{c}{|a|^{1/2}} \frac{C_\ast}{|a|^{1/2}}.
\]
(C.8)
Choosing \( C_\ast \) large enough yields the claim.

\[\square\]

**D Free convolution continuity**

In this section we consider two measures \( \mu_1 \) and \( \mu_2 \) and denote the free convolution of each with the semicircle by \( m_{1,t} \) and \( m_{2,t} \). We estimate the difference \( m_{1,t} - m_{2,t} \) under the assumption that \( m_1 - m_2 \) is small.

We assume that the restriction of \( \mu_2 \) to \([-1, 1]\) has a density \( \rho_2(x) \) such that
\[
\rho_2(x) \asymp 1_{[x \geq 0]} \sqrt{x}.
\]
(D.1)

We assume the following estimates hold for any \( \delta, \varepsilon > 0 \). For any \( 1 \geq E \geq 0 \) and \( \sqrt{|E| + \eta} \geq N^\delta/(N \eta) + N^\delta/N^{1/3} \) we have,
\[
|m_1(z) - m_2(z)| \leq \frac{N^\varepsilon}{N \eta}.
\]
(D.2)

For \(-1 \leq E \leq 0 \) and \( \eta \geq N^\delta/N^{2/3} \) we have
\[
|m_1(z) - m_2(z)| \leq N^\varepsilon \left( \frac{1}{N(|E| + \eta)} + \frac{1}{(N \eta)^2 \sqrt{|E| + \eta}} \right).
\]
(D.3)

We denote \( \xi_t = z + tm_{i,t} \) and we let \( E_- \) be the edge of \( \rho_{2,t} \), and define \( \kappa = |E - E_-| \).
Lemma D.1. Let $\mu_1$ and $\mu_2$ as above. Let $\mathcal{D}_\sigma$ be as in Section 5 and $t$ satisfy
\[
\frac{N^\omega}{N^{1/3}} \leq t \leq N^{-\omega}
\]
for $\omega > 0$. For any $\varepsilon > 0$ the following estimates hold on $\mathcal{D}_\sigma$. First, for $E \geq E_-$ we have
\[
|m_{1,t} - m_{2,t}| \leq \frac{N^\varepsilon}{N \eta}.
\]
For $E \leq E_-$ we have,
\[
|m_{1,t} - m_{2,t}| \leq N^\varepsilon \left( \frac{1}{N(k + \eta)} + \frac{1}{(N\eta)^2 \sqrt{\kappa + \eta}} \right).
\]

Proof. Define $\Lambda = |m_{1,t} - m_{2,t}|$. For $\eta$ order 1, the estimates on $\Lambda$ follow easily. First let us consider $E \geq E_-$. Suppose that the estimate
\[
\Lambda \leq \frac{t + \text{Im}[m_{2,t}]}{\log(N)^2}
\]
holds. This assumption assures that $|x - \xi_2| > |\xi_1 - \xi_2|$ for $x$ in the support of $\mu_2$. We then write
\[
m_{1,t} - m_{2,t} = \left( \int \frac{d\mu_1}{x - \xi_1} - \int \frac{d\mu_2}{x - \xi_1} \right) + \left( \int \frac{d\mu_2}{x - \xi_1} - \int \frac{d\mu_2}{x - \xi_2} \right).
\]
Expanding the second term and estimating the first by $N^\varepsilon/N(\text{Im}[\xi_1])$ leads to
\[
|(1 - tR_2)(m_{1,t} - m_{2,t}) + t^2R_3(m_{1,t} - m_{2,t})^2| \leq \frac{N^\varepsilon}{N\text{Im}[\xi_1]} + Ct^3 \Lambda^3 \frac{t + \text{Im}[m_{2,t}]}{(t^2 + \text{Im}[\xi_2])^2}.
\]
Since $|1 - tR_2| = 1$ for $\kappa + \eta \geq t^2$ we can conclude that
\[
|m_{1,t} - m_{2,t}| \leq \frac{N^\varepsilon}{N \eta}, \quad E \geq E_-, \quad \kappa + \eta \geq t^2.
\]
We can now suppose that $E_- \leq E \leq E_- + t^2$ and $\eta \leq t^2$. Suppose that
\[
\Lambda \leq \frac{\sqrt{\kappa + \eta}}{\log(N)^2}.
\]
Note that when $\eta = t^2$ we know that this is the case. Then in this case, (D.9) leads to
\[
\Lambda \leq C \frac{\Lambda^2}{\sqrt{\kappa + \eta}} + \frac{N^\varepsilon}{N \eta}.
\]
Since $(N\eta) \ll \sqrt{\kappa + \eta}$ for $E \geq E_-$, we conclude that $\Lambda \leq N^\varepsilon/(N\eta)$ in the regime $E \geq E_-$. Now we consider the regime $E \leq E_-$. First, we observe that the estimate $\Lambda \leq N^\varepsilon/(N\eta)$ in the regime $E_- - t^2 \leq E \leq E_-$ and $\eta \geq ct^2$ follows from the above argument. We then check the regime $E_- - t^2 \leq E \leq E_- \text{ and } \eta \leq ct^2$. If we take $c > 0$ small enough, we can ensure that
\[
\text{Re}[\xi_2] \leq -c_1 t^2
\]
for another $c_1 > 0$. Hence, if $\Lambda \leq \sqrt{\kappa + \eta}/\log(N)^2$ we see that
\[
|m_1(\xi_1) - m_2(\xi_1)| \leq N^\varepsilon \left( \frac{1}{Nt^2} + \frac{1}{(N\eta)^2 t} \right).
\]
and so we obtain by a similar argument to above that the estimate $\Lambda \leq \sqrt{\kappa + \eta}/(\log(N))^2$ implies that
\[
|(1 - tR_2)(m_{1,t} - m_{2,t}) + t^2R_3(m_{1,t} - m_{2,t})^2| \leq N^\varepsilon \left( \frac{1}{Nt^2} + \frac{1}{(N\eta)^2 t} \right) + C\Lambda^3 t^2.
\]
We have that $|1 - tR_2| = \sqrt{\kappa + \eta/t}$, and so we see that

$$\Lambda \leq C \frac{\Lambda^2}{\sqrt{\kappa + \eta}} + CN^\varepsilon \left( \frac{1}{N(\kappa + \eta)} + \frac{1}{(N\eta)^2 \sqrt{\kappa + \eta}} \right).$$  \hspace{1cm} (D.16)

The second term is $\ll \sqrt{\kappa + \eta}$ and so we see that so far we have proven the desired estimates as long as $E \geq E_\varepsilon - t^2$.

Finally, to do the regime $E \leq E_\varepsilon - t^2$ we first observe that there is a $C > 0$ so that if $\kappa \geq C\eta$ then $-\Re[\xi_2] \geq C\kappa$. So, we see that the estimate $\Lambda \leq t/\log(N)^2$ implies

$$|m_{1,i}(\xi_1) - m_{2,i}(\xi_1)| \leq N^\varepsilon \left( \frac{1}{N(\kappa + \eta)} + \frac{1}{(N\eta)^2 \sqrt{\kappa + \eta}} \right).$$  \hspace{1cm} (D.17)

Hence, arguing as above we see that the estimate $\Lambda \leq t/\log(N)^2$ implies that

$$\Lambda \leq C \frac{\Lambda^2}{t} + N^\varepsilon \left( \frac{1}{N(\kappa + \eta)} + \frac{1}{(N\eta)^2 \sqrt{\kappa + \eta}} \right).$$  \hspace{1cm} (D.18)

This is enough to complete the proof. \hfill \square

### E Interpolating convolution measure properties

We prove Lemma 3.5. This follows from the following general estimate. If we have two measures $\mu_1, \mu_2$ that have, when restricted to $[-1, 1]$ a density $\rho(x)$ that behaves like

$$\rho(x) \approx \sqrt{x}1_{\{x \geq 1\}},$$  \hspace{1cm} (E.1)

and moreover

$$|m_{1}(z) - m_{2}(z)| \leq \frac{N^\varepsilon}{N\eta}$$  \hspace{1cm} (E.2)

for any $\varepsilon > 0$ and $\eta \geq N^{-2/3+\sigma}$. Denote the $\xi$ maps at time $t$ by $\xi_1$ and $\xi_2$, the edges $E_1, E_2$, and $\xi_{i,-} = \xi_i(E_i)$.

Subtracting the defining equations for the $\xi_{i,-}$ we easily see

$$|\xi_{1,-} - \xi_{2,-}| \leq Ct^3.$$  \hspace{1cm} (E.3)

Next, we estimate

$$|E_1 - E_2| \leq |\xi_{1,-} - \xi_{2,-}| + t|m_{1}(\xi_{1,-}) - m_{2}(\xi_{1,-})| + t|m_{2}(\xi_{2,-} - m_{2}(\xi_{1,-})|. $$  \hspace{1cm} (E.4)

The first term is bounded by $Ct^3$ and since $|m_{1}^i(E)| \leq t^{-1}$ for all $E$ such that $-E = t^2$, we see that

$$t|m_{2}(\xi_{2,-} - m_{2}(\xi_{1,-})| \leq Ct^3.$$  \hspace{1cm} (E.5)

For the last term, since the measures $\mu_1 = \mu_2$ on $[-1, 1]$ we see that,

$$|[m_{1}(\xi_{1,-}) - m_{2}(\xi_{1,-})] - [m_{1}(\xi_{1,-} + iN^{-1/2}) - m_{2}(\xi_{1,-} + iN^{-1/2})] | \leq N^{-1/2}$$  \hspace{1cm} (E.6)

and by assumption,

$$|m_{1}(\xi_{1,-} + iN^{-1/2}) - m_{2}(\xi_{1,-} + iN^{-1/2})| \leq \frac{N^\varepsilon}{N^{1/2}}.$$  \hspace{1cm} (E.7)

Hence,

$$|E_1 - E_2| \leq N^\varepsilon (t^3 + \frac{t}{N^{1/2}}).$$  \hspace{1cm} (E.8)
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