RIGIDITY FOR $p$-LAPLACIAN TYPE EQUATIONS ON COMPACT RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we obtain two rigidity results for $p$-Laplacian type equations on compact Riemannian manifolds by using of the carré du champ and nonlinear flow methods, respectively, where rigidity means that the PDE has only constant solution when a parameter is in a certain range. Moreover, an interpolation inequality is derived as an application.

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1. Introduction and main results

Throughout the paper we assume that $(M, g)$ is an $n$-dimensional compact Riemannian manifold without boundary and $dV$ the volume element induced by the Riemannian metric $g$. In [6] and [2] (or see [1]), Licois-Véron and Barky-Ledoux proved a rigidity result for equation (1.1) on compact Riemannian manifolds with $\text{Ric} \geq Kg(K > 0)$,

$$-\Delta v + \frac{\lambda}{q-2}(v - v^{q-1}) = 0,$$

if $\lambda$ satisfies

$$\lambda \leq (1 - \vartheta)\lambda_0 + \vartheta \frac{n}{n-1}K, \quad \vartheta = \frac{(n-1)^2(q-1)}{n(n+2) + q-1},$$

where $\lambda_0$ is the lowest positive eigenvalue of $-\Delta$. Both of them applied the Bochner formula or the carré du champ method.

Recently, Dolbeault-Esteban-Loss [4] established a rigidity result for equation (1.1) by nonlinear flow method. More precisely, if $\lambda \in (0, \Lambda)$ and for any $q \in (1, 2) \cup (2, 2^*)$, the equation (1.1) have a positive constant solution, where $2^* = \frac{2n}{n-2}$ and

$$\Lambda := \inf_{u \in H^2(M) \setminus \{0\}} \frac{(1 - \vartheta) \int_M (\Delta u)^2 dV + \frac{\vartheta n}{n-1} \int_M \|Q_g u\|^2 + \text{Ric}(\nabla u, \nabla u) dV}{\int_M |\nabla u|^2 dV}.$$

Here the quantity $Q_g u$ is defined by

$$Q_g u := \nabla \nabla u - \frac{g}{n} \Delta u - \frac{\vartheta}{\vartheta(n+3-q)} \left[ \nabla u \otimes \nabla u - \frac{g}{u} \frac{|\nabla u|^2}{n} \right],$$

where $\vartheta$ is defined in (1.2).

In recent preprints [9] and [8], the first author and coauthors obtain the rigidity results for $n$-Laplace equation with exponential nonlinearities on $n$-dimensional compact Riemannian manifolds and weighed equation (1.1) on weighted Riemannian manifolds via Bakry-Emery Ricci curvature.
Motivated by above works, the purpose of this paper is to study rigidity results for $p$-Laplacian type equation

$$\Delta_p v + \frac{\beta p-1}{2-p+\beta(q-p)}\left(v^{p-2+\beta p-1} - v^{p-1}\right) = 0,$$

on compact Riemannian manifolds by using of carré du champ and nonlinear flow methods, where $p$-Laplacian $\Delta_p$ is defined by $\Delta_p v := \text{div}(\nabla|v|^{p-2}\nabla v)$ and

$$\beta = \frac{2(p-1)(n(p-1)+p)}{(n(p-1)+p)(2p-1-q)+n(q-1)}.$$

Now, Let us state the main results of this paper. The first rigidity result is given by the method of carré du champ, which mainly depends on $p$-Bochner formula (see\cite{7})

$$\frac{1}{p} L(|\nabla u|^p) = |\nabla u|^{2p-4} \left( |\nabla \nabla u|_{\lambda}^2 + \text{Ric}(\nabla u, \nabla u) \right) + |\nabla u|^{p-2} \langle \nabla (\Delta_p u), \nabla u \rangle,$$

where $L$ is the linearized operator of $\Delta_p$ at the point $u$,

$$L(\psi) := \text{div} \left( |\nabla u|^{p-2} A(\nabla \psi) \right)$$

and $\nabla \nabla u$ denotes the Hessian of $u$, $|\nabla \nabla u|_{\lambda}^2 = A^i_k A^j_l \nabla_i u \nabla_j \nabla_k u$, $A$ is a 2-tensor defined by

$$A := \text{Id} + (p-2) \frac{\nabla u \otimes \nabla u}{|\nabla u|^2}.$$

Let $\lambda_1$ be the lowest positive eigenvalue of $-L$, that is $-L \varphi = \lambda_1 \varphi$.

**Theorem 1.1.** Let $(M, g)$ be an $n$-dimensional compact Riemannian manifold with $\text{Ric} \geq Kg (K > 0)$. If $\lambda$ is a positive parameter such that

$$\lambda \leq (1 - \theta) \lambda_1 + \theta \frac{nR}{n-1},$$

where

$$\theta = \frac{p^2(n-1)^2(q-1)}{4n(p-1)(n(p-1)+p)+(np-2n+p)^2(q-1)}$$

and

$$R = \frac{\int_M u^\gamma |\nabla u|^{2p-2} dV}{\int_M u^\gamma |\nabla u|^p dV} K, \quad \gamma = -p,$$

then for any $q \in (p, \frac{np}{n-p})$, the equation (1.3) has a unique constant solution, and equal to 1.

The second rigidity result takes advantage of nonlinear flow of porous medium type. Let us introduce $p$-porous medium flow,

$$\partial_t u = u^{p-\beta} \left( \Delta_p u + \kappa \frac{|\nabla u|^p}{u} \right),$$

(1.9)
\[ \kappa = p - 1 + \beta(q - p), \]  
where

\[ Q_u := |\nabla u|^{p-2} \nabla u - \frac{a}{n} \Delta u \]

which can be viewed as a \( p \)-Laplacian version in [4]. Next, recalling \( \theta \) is given in (1.7), define

\[ Q_u := \frac{p(n-1)(q-1)}{\theta[(n(p-1)+(p-1-q)+n(q-1))] \left( |\nabla u|^{p-2} \nabla u \otimes \nabla u - \frac{p-1}{n} |\nabla u|^p u \right)^{\frac{1}{\theta}}, \]

where \( a \) is the inverse of \( A \), define

\[ \Lambda_* := \inf_{u \in W^{2,p}(M) \setminus \{0\}} \frac{(1 - \theta) \int_M (\Delta u)^2 dV + \frac{\theta n}{n-1} \int_M |Q_u|^2_A + |\nabla u|^{2p-4} \text{Ric}(\nabla u, \nabla u)| dV}{\int_M u^{2p-4} |\nabla u|^p dV}, \]

where \( m = \max\{p, 2p-2\} \). If \( 0 < \theta < 1 \), then

\[ 1 < q < p^*, \quad p^* = \frac{np}{n-p}. \]

**Theorem 1.2.** Assume that \( \Lambda_* > 0 \). For any \( q \in (1, p) \cup (p, p^*) \), if \( 0 < \lambda < \Lambda_* \), then the equation (1.3) on compact Riemannian manifolds has a unique constant solution, which equal to 1.

**Remark 1.3.** Taking \( p = 2 \) in Theorem 1.1 and 1.2, such rigidity results have been proved by Licois-Véron [6] (Barky-Ledoux [4]) and Dolbeault-Esteban-Loss [4].

**Remark 1.4.** It is clear that we get the same \( \theta \) through different methods in Theorem 1.1 and Theorem 1.2; however, \( \theta \) in Theorem 1.2 is more naturally acquired. Moreover, conditions in Theorem 1.2 are weaker than Theorem 1.1. The Ricci curvature is not required to be positive or negative in Theorem 1.2 instead of Ricci curvature must be positive in Theorem 1.1.

Finally, we can establish an interpolation inequality as an application of rigidity results.

**Theorem 1.5.** Assume that \( \Lambda_* > 0 \). For any \( q \in (1, p) \cup (p, p^*) \), if \( \lambda < \Lambda_* \) and \( \nu = u^\beta \), the following inequality holds:

\[ \|\nabla \nu\|_{L^p(M)}^p \geq \frac{p\beta^{p-2}}{s} \frac{\lambda}{q-s} \left( \|\nu\|_{L^p(M)}^p - \|\nu\|_{L^p(M)}^{p^*} \right), \forall \nu \in W^{1,p}(M), \]

where \( \beta \) is defined in (1.4) and \( s = \frac{p^*-2+2p}{\beta} \).

**Remark 1.6.** When the manuscript was finished, we find that some interpolation inequalities in \( W^{1,p}(\mathbb{S}^1) \) are derived based on the carré du champ methods in preprint [5].

2. **Proof of Theorem 1.1**

In order to prove Theorem 1.1, we need to the following Lemmas.

**Lemma 2.1.** If \( \beta \) is defined in (1.4) and \( \gamma \neq p \), let \( u^\beta \) be a solution to equation (1.3), we get

\[ \alpha \int_M u^{2\beta} |\nabla u|^p dV + \sigma \int_M (L(u^{1+\gamma}))^2 dV \]

\[ = \frac{\beta(p-1)}{2\gamma} \int_M \left[ u^{2\beta} J + u^{2\beta} |\nabla u|^{2p-4} \text{Ric}(\nabla u, \nabla u) \right] dV + \left( 1 - \frac{1}{p^*} \right) \lambda \int_M u^{2\beta} |\nabla u|^p dV, \]

where

\[ \alpha = \frac{p}{n-1}, \quad \sigma = \frac{p-1}{n-1} \]

and

\[ \Lambda := \frac{1}{2\gamma} \int_M (\Delta u)^2 dV \]
where \( p^* = \frac{np}{n-p} \), \( \mathcal{L} \) is the linearized operator defined in (1.6),

\[
J := |\nabla u|^{2p-4}\|\nabla \nabla u\|_\lambda^2 - \frac{1}{n}(\Delta_p u)^2, \tag{2.2}
\]

and \( \alpha, \sigma \) are two constants given by

\[
\alpha = \left(1 - \frac{1}{p^*}\right)\left(\beta + 1\right)(p-1)\left(\beta(q-p) - (p-1) - \frac{2\gamma}{p}\right) - \frac{\gamma(p\beta(1-q) + 2\gamma)}{2p^2} \right) \frac{\beta(q-1)[(2p-3)\gamma - 2p(p-1)]}{2p^2}, \tag{2.3}
\]

\[
\sigma = \frac{p}{n(p-1)^2(\gamma + p)^2} \left[n(p-1) + p + \frac{p^2\beta(q-1)}{2\gamma}(n-1)\right]. \tag{2.4}
\]

**Proof.** Set \( v = u^{-\beta} \) be a solution to equation (1.3), then \( u \) satisfies

\[
-\Delta_p u + (p-1)(\beta + 1)\frac{\|\nabla u\|^p}{u} + \frac{\lambda}{2 - p + \beta(q-p)}(u^{p-1+\beta(p-q)} - u) = 0. \tag{2.5}
\]

Multiplying (2.5) by \( u^{2p-4}\|\nabla u\|^p \) and \( u^{2p}\Delta_p u \) respectively and integrating by parts, which give the following identities

\[
\int_M u^{2p-4}\Delta_p u|\nabla u|^p dV = (p-1)(\beta + 1) \int_M u^{2p-2}\|\nabla u\|^p dV + \frac{\lambda}{2 - p + \beta(q-p)} \int_M u^{p-2+\beta(p-q)+\frac{2p}{p}} |\nabla u|^p dV
\]

\[
- \frac{\lambda}{2 - p + \beta(q-p)} \int_M u^{2p}\|\nabla u\|^p dV \tag{2.6}
\]

and

\[
\int_M u^{2p}\Delta_p u^2 dV = (p-1)(\beta + 1) \int_M u^{2p-4}\Delta_p u|\nabla u|^p dV
\]

\[
+ \frac{\lambda}{2 - p + \beta(q-p)} \int_M \left(u^{p-1+\beta(p-q)} u^{\frac{2p}{p}} - u^{\frac{2p}{p}}\right) \Delta_p u dV
\]

\[
= (p-1)(\beta + 1) \int_M u^{2p-2}\Delta_p u|\nabla u|^p dV + \frac{(1 + \frac{2p}{p})\lambda}{2 - p + \beta(q-p)} \int_M u^{2p}\|\nabla u\|^p dV
\]

\[
- \frac{(p-1 + \beta(p-q) + \frac{2p}{p})\lambda}{2 - p + \beta(q-p)} \int_M u^{p-2+\beta(p-q)+\frac{2p}{p}} |\nabla u|^p dV. \tag{2.7}
\]

Combining (2.6) with (2.7), we obtain

\[
\lambda \int_M u^{\frac{2p}{p}} |\nabla u|^p dV = \left[\beta(1-q) + \frac{2\gamma}{p}\right] \int_M u^{\frac{2p}{p}} \Delta_p u|\nabla u|^p dV + \int_M u^{\frac{2p}{p}} (\Delta_p u)^2 dV
\]

\[
- (p-1)(\beta + 1) \left(p - 1 + \beta(p-q) + \frac{2\gamma}{p}\right) \int_M u^{\frac{2p}{p}} |\nabla u|^{2p} dV. \tag{2.8}
\]

Multiplying the \( p \)-Bochner formula (1.5) by \( u^{\frac{2p}{p}} \) and integrating by parts, then we get

\[
\int_M \left[u^{\frac{2p}{p}} J + u^{\frac{2p}{p}} \|\nabla u\|^{2p-4}\text{Ric} (\nabla u, \nabla u) \right] dV \tag{2.9}
\]
Direct calculation shows that
\[
\left( u^{1+\frac{2}{p}} \right)^2 = \frac{2γ(2γ - p)(p - 1)}{p^2} \int_M u^{2γ - 2} |\nabla u|^2 dV + \frac{2γ(2γ - p)(p - 1)}{p^3} \int_M u^{2γ - 2} |\nabla u|^2 dV + \frac{n - 1}{n} \int_M u^{2γ}(Δ_p u)^2 dV,
\]
where \( J \) is defined in (2.2). In fact,
\[
\int_M u^{2γ} L(|\nabla u|) dV = \frac{2γ(2γ - p)(p - 1)}{p} \int_M u^{2γ - 1} Δ_p u|\nabla u|^p dV + \frac{2γ(2γ - p)(p - 1)}{p^2} \int_M u^{2γ - 2} |\nabla u|^2 dV
\]
and
\[
\int_M u^{2γ} |\nabla u|^{p - 2} \nabla u \cdot \nabla (Δ_p u) dV = -\int_M u^{2γ}(Δ_p u)^2 dV - \frac{2γ}{p} \int_M u^{2γ - 1} Δ_p u|\nabla u|^p dV.
\]
Thus, by (2.10), if \( γ \neq -p \), there holds
\[
\int_M u^{2γ - 1} Δ_p u|\nabla u|^p dV = \frac{P_1}{2γ(2γ + p)(p - 1)^2} \left( L\left(u^{1+\frac{2}{p}}\right)\right)^2 - \frac{γ}{2p} u^{2γ - 2} |\nabla u|^2 - \frac{P}{2γ} u^{2γ} (Δ_p u)^2.
\]
Putting (2.11) into (2.8) and (2.9), we obtain
\[
\lambda \int_M u^{2γ} |\nabla u|^p dV = \frac{βp(p - 1)}{2γ} \int_M u^{2γ}(Δ_p u)^2 dV - \frac{p^2(βq - 1) - 2β}{2γ(γ + p)^2(p - 1)^2} \int_M \left( L\left(u^{1+\frac{2}{p}}\right)\right)^2 dV
\]
\[
+ \left( β + 1)(p - 1) \left( βq - 1 - (p - 1) - \frac{2γ}{P} - \frac{γ(βq - 1 - γ)}{2p^2} \right) \int_M u^{2γ - 2} |\nabla u|^2 dV
\]
and
\[
\int_M \left[ u^{2γ} J + u^{2γ} |\nabla u|^{2p - 4} \text{Ric}(\nabla u, \nabla u) \right] dV = \frac{p(2p - 1)}{(γ + p)^2(p - 1)^2} \int_M \left( L\left(u^{1+\frac{γ}{p}}\right)\right)^2 dV + \frac{2p - 3}{p^3} \frac{γ^2 - 2p(p - 1)γ}{p^3} \int_M u^{2γ - 2} |\nabla u|^2 dV
\]
\[
- \left( 1 - \frac{1}{p^2} \right) \int_M u^{2γ}(Δ_p u)^2 dV.
\]
Eliminating the term \( \int_M u^{2γ}(Δ_p u)^2 dV \) between (2.12) and (2.13), we get (2.1).

\[\square\]

**Lemma 2.2.** According to definition of \( L \), the following inequality holds
\[
\int_M (L(ψ))^2 dV \geq λ_1 \int_M |∇ψ|^2 |\nabla u|^{p - 2} dV,
\]
where \( λ_1 \) is the lowest positive eigenvalue of \( -L \) and \( |∇ψ|^2 = A^{i,j} ∇_iψ ∇_jψ \).
Proof. Let \( \lambda_1 \) be the lowest positive eigenvalue of \(-L\) such that
\[
- L(\psi - \bar{\psi}) = \lambda_1 (\psi - \bar{\psi}), \quad \text{where} \quad \bar{\psi} = \int_M \psi \, dV. \tag{2.15}
\]
Multiplying (2.15) by \((\psi - \bar{\psi})\) and integrating by parts,
\[
\lambda_1 \int_M (\psi - \bar{\psi})^2 \, dV = - \int_M L(\psi - \bar{\psi})(\psi - \bar{\psi}) \, dV = \int_M |\nabla \psi|^2 |\nabla u|^{p-2} \, dV. \tag{2.16}
\]
By Cauchy-Schwarz inequality
\[
\left( \int_M |\nabla \psi|^2 |\nabla u|^{p-2} \, dV \right)^2 \leq \int_M (L(\psi))^2 \, dV \int_M (\psi - \bar{\psi})^2 \, dV
\]
yields
\[
\int_M (L(\psi))^2 \, dV \geq \frac{\left( \int_M |\nabla \psi|^2 |\nabla u|^{p-2} \, dV \right)^2}{\int_M (\psi - \bar{\psi})^2 \, dV}. \tag{2.17}
\]
Using (2.16) and (2.17), we obtain (2.14).

Now we can start our proof.

Proof of Theorem 1.1. Set \( \psi = u^{1+\frac{1}{p}} \) in Lemma 2.2, it follows that
\[
\int (L(u^{1+\frac{1}{p}}))^2 \, dV \geq \frac{(\gamma + p)^2 (p - 1)^2}{p^2} \int u^{2\frac{2}{p}} |\nabla u|^p \, dV. \tag{2.18}
\]
With the notations in Lemma 2.1, we can find a couple \((\beta, \gamma)\) such that
\[
\alpha \geq 0, \quad \sigma \geq 0, \quad \frac{\beta}{\gamma} \leq 0. \tag{2.19}
\]
By (2.18) and (2.1), we get
\[
\alpha \int_M u^{2\frac{2}{p} - 2|\nabla u|^{2p} \, dV
\leq \frac{\beta p (q - 1)}{2\gamma} \int_M u^{2\frac{2}{p}} J \, dV + \frac{\beta p (q - 1)}{2\gamma} R \int_M u^{2\frac{2}{p}} |\nabla u|^p \, dV - \sigma \frac{(\gamma + p)^2 (p - 1)^2}{p^2} \lambda_1 \int_M u^{2\frac{2}{p}} |\nabla u|^p \, dV
+ \left( 1 - \frac{1}{p} \right) \lambda \int M u^{2\frac{2}{p}} |\nabla u|^p \, dV
\leq \frac{\beta p (q - 1)}{2\gamma} \int_M u^{2\frac{2}{p}} J \, dV + \left[ 1 - \frac{1}{p} \right] (\lambda - \lambda_1) + \frac{\beta p (q - 1)}{2\gamma} \left( R - \frac{n - 1}{n} \lambda_1 \right) \int_M u^{2\frac{2}{p}} |\nabla u|^p \, dV, \tag{2.20}
\]
where we use the definitions of \(\sigma\) in (2.4) and \(R\) in (1.8).

Set
\[
X = \frac{\beta}{\gamma}, \quad \eta = \frac{p - 1}{\gamma} + \frac{1}{p}, \quad \bar{\alpha} = \frac{np}{[n(p - 1) + p]^{\gamma}}. 
\]
On the one hand, \( \sigma \geq 0 \) in (2.4) implies
\[
- \frac{2(n(p-1)+p)}{p^2(n-1)(q-1)} \leq X \leq 0. \tag{2.21}
\]

On the other hand, \( \alpha \geq 0 \) is equivalent to the following inequality
\[
\overline{\alpha} = -\eta^2 + \frac{(q-1)(np-2n+p) + 2(1-p)(n(p-1)+p)}{n(p-1)+p} \eta X + \frac{(n-1)(q-1)}{2(n(p-1)+p)} X + (p-1)(q-p)X^2 \geq 0.
\]

The derivative of \( \overline{\alpha} \) with respect to \( \eta \) is
\[
\frac{d\overline{\alpha}}{d\eta} = -2 \left[ \eta - \frac{(q-1)(np-2n+p) + 2(1-p)(n(p-1)+p)}{2(n(p-1)+p)} X \right].
\]

Thus, \( \overline{\alpha} \) takes the maximum value, when
\[
\eta = \eta_0 = \frac{(q-1)(np-2n+p) + 2(1-p)(n(p-1)+p)}{2(n(p-1)+p)} X.
\]

That is
\[
\overline{\alpha}(\eta_0, X) = \left[ (p-1)(q-p) + \left( \frac{(q-1)(np-2n+p) + 2(1-p)(n(p-1)+p)}{2(n(p-1)+p)} \right)^2 \right] X^2 + \frac{(n-1)(q-1)}{2(n(p-1)+p)} X.
\]

We set \( \overline{\alpha}(\eta_0, X) = 0 \), its negative root is
\[
X_0 = -\frac{2(n-1)(n(p-1)+p)}{4n(p-1)(n(p-1)+p) + (q-1)(np-2n+p)^2}.
\]

By (2.21), one get
\[
X_0 \geq -\frac{2(n(p-1)+p)}{p^2(n-1)(q-1)},
\]
which is equivalent to \( q \leq p^* \).

Putting the specific value of \( X = \frac{\rho}{\gamma} = X_0 \) into the last term in (2.20), we obtain
\[
\left[ \left( 1 - \frac{1}{p^*} \right) (\lambda - \lambda_1) - \frac{p(q-1)(n-1)(n(p-1)+p)}{4n(p-1)(n(p-1)+p) + (q-1)(np-2n+p)^2} \left( R - \frac{n-1}{n} \lambda_1 \right) \right] = \left( 1 - \frac{1}{p^*} \right) \left[ \lambda - \left( (1-\theta)\lambda_1 + \theta \frac{nR}{n-1} \right) \right], \tag{2.22}
\]
where \( \theta \) is defined in (1.7) and \( R \) is defined in (1.8).
The Cauchy-Schwarz inequality implies (see [7])
\[
J = ||\nabla u||^{2p-4}_{L^4} ||\nabla u||_{L^4}^2 - \frac{1}{n} (\Delta u)^2 \geq 0. \tag{2.23}
\]

According to (2.22) and (2.23), when \( q \in (p, p^*) \) and
\[
\lambda \leq (1-\theta)\lambda_1 + \theta \frac{nR}{n-1},
\]
in (2.20), we get
\[
\alpha \int_M u^{2p-2} |\nabla u|^{2p} dV \leq 0.
\]
According to the assumption of \( \alpha \geq 0 \) in (2.19), then the equation (1.3) has a unique constant solution, and equal to 1.

3. Proof of Theorem 1.2

Let us define two \( A \)-tree free quantities on \((M, g)\)
\[
Bu := |\nabla u|^{p-2} \nabla \nabla u - \frac{a}{n} \Delta_p u \tag{3.1}
\]
and
\[
Gu := |\nabla u|^{p-2} \frac{\nabla u \otimes \nabla u}{u} - \frac{p-1}{n} \frac{|\nabla u|^{p}}{u} a, \tag{3.2}
\]
where \( \nabla \nabla u \) is the Hessian of \( u \) and \( a = (a_{ij}) = g_{ij} - \frac{p-2}{p-1} \frac{\nabla u \otimes \nabla u}{|\nabla u|^2} \) is the inverse matrix of \( A^{ij} \). If \( T_{ij} \) and \( S_{ij} \) are two tensors, we use the Einstein summation convention and
\[
[T, S]_A := A^{ik} A^{jl} T_{ij} S_{kl}, \quad ||T||_A^2 = [T, T]_A, \tag{3.3}
\]
in particular, when \( p = 2 \), \( a = g \), \( A = g^{-1} \) and
\[
[T, S]_g := g^{ik} g^{jl} T_{ij} S_{kl}, \quad ||T||_g^2 = [T, T]_g.
\]

Using the notations of (3.1)-(3.3), we need the following Lemmas to prove Theorem 1.2.

Lemma 3.1. Assume \( n \geq 2 \), the following identity holds
\[
\int_M (\Delta_p u)^2 dV = \frac{n}{n-1} \int_M ||Bu||_A^2 dV + \frac{n}{n-1} \int_M |\nabla u|^{2p-4} \text{Ric}(\nabla u, \nabla u) dV. \tag{3.4}
\]

Proof. Directly computation shows that
\[
||Bu||_A^2 = \left| |\nabla u|^{p-2} \nabla \nabla u - \frac{a}{n} \Delta_p u \right|_A^2 = |\nabla u|^{2p-4} ||\nabla u||_A^2 - \frac{1}{n} (\Delta_p u)^2, \tag{3.5}
\]
in fact,
\[
|\nabla u|^{p-2} \text{tr}_A (\nabla \nabla u) = |\nabla u|^{p-2} A^{ij} \nabla_i \nabla_j u = \Delta_p u.
\]
Integrating \( p \)-Bochner formula (1.5) on \( M \) yields
\[
\int_M (\Delta_p u)^2 dV = \int_M |\nabla u|^{2p-4} ||\nabla u||_A^2 dV + \int_M |\nabla u|^{2p-4} \text{Ric}(\nabla u, \nabla u) dV, \tag{3.6}
\]
where we use the formula of integration by parts,
\[
\int_M |\nabla u|^{p-2} \langle \nabla (\Delta_p u), \nabla u \rangle dV = - \int_M \nabla \cdot (|\nabla u|^{p-2} \nabla u) \Delta_p u dV = - \int_M (\Delta_p u)^2 dV.
\]
Combining (3.5) with (3.6), we get (3.4). \(\Box\)
Lemma 3.2. If \( u \) is a positive function, then we have
\[
\int_M \Delta_p u \left| \nabla u \right|^p \frac{dV}{u} = \frac{n(p-1)}{n(p-1)+p} \int_M \left| \nabla u \right|^{2p} \frac{dV}{u^2} - \frac{np}{(p-1)(n(p-1)+p)} \int_M [Bu, Gu]_A dV. 
\] (3.7)

Proof. Integrating by parts implies that
\[
\int_M \Delta_p u \left| \nabla u \right|^p \frac{dV}{u} = \int_M \left| \nabla u \right|^{2p} \frac{dV}{u^2} - \int_M \left| \nabla u \right|^{2p-4} \left[ \nabla \nabla u, \frac{\nabla \otimes \nabla u}{u} \right]_g dV
\]
where
\[
[Bu, Gu]_A = A^{ik} A^{jl} [Bu]_{ij} [Gu]_{kl} = A^{ik} A^{jl} \left| \nabla u \right|^{p-2} \nabla \nabla u - \frac{a}{n} \Delta_p u \left| \nabla u \right|^{p-2} \nabla \otimes \nabla u - \frac{p-1}{n} \left| \nabla u \right|^p A
\]
\[
= (p-1)^2 \left| \nabla u \right|^{2p-4} \left[ \nabla \nabla u, \frac{\nabla \otimes \nabla u}{u} \right]_g - \frac{p-1}{n} \Delta_p u \frac{\nabla u}{u^p}
\]
By collecting the same items in (3.8), we obtain (3.7).

Lemma 3.3. Assume \( u \) be a smooth positive solution to equation (1.9), the functional \( u \mapsto \int_M u^\beta dV \) remains constant with respect to \( t \).

Proof. By computing the derivative of \( t \) and the definition of \( \kappa \) in (1.10) yield
\[
\frac{d}{dt} \int_M u^\beta dV = \beta q \int_M u^{p-1+\beta(q-\beta)} \left( \Delta_p u + \kappa \frac{\nabla u}{u} \right) dV
\]
\[
= \beta q \int_M \left( u^{p-1+\beta(q-\beta)} \right) \Delta_p u dV + \beta q \kappa \int_M u^{p-2+\beta(q-\beta)} |\nabla u|^p dV
\]
\[
= -\beta q[p-1+\beta(q-\beta)] \int_M u^{p-2+\beta(q-\beta)} |\nabla u|^p dV + \beta q \kappa \int_M u^{p-2+\beta(q-\beta)} |\nabla u|^p dV = 0.
\]

Lemma 3.4. If \( u \) is a smooth positive solution to equation (1.9), then
\[
\frac{1}{p^\beta q} \frac{d}{dt} F[u] = - \int_M \left( (\Delta_p u)^2 + (\kappa + (\beta - 1)(p-1)) \Delta_p u \frac{|\nabla u|^p}{u} + \kappa(\beta - 1)(p-1) \frac{|\nabla u|^{2p}}{u^2} \right) dV
\]
\[
+ \lambda \int_M u^{2p-4} |\nabla u|^p dV,
\] (3.9)
where the functional \( F[u] \) is defined by
\[
F[u] := \int_M |\nabla (u^\beta)|^p dV + \frac{p^\beta q \lambda}{(p^\beta + p-2)(2-p+\beta(q-\beta))} \left[ \int_M u^{p\beta+2p-2} dV - \left( \int_M u^{\beta q} dV \right)^{p/q} \right].
\]
Proof. By equation \((1.9)\) and integrating by parts,
\[
\frac{d}{dt} \int_M |\nabla (u^\theta)|^p \, dV = -p\beta \int_M \Delta_p (u^\theta) u^{p-1+\beta(1-p)} \left( \Delta_p u + \kappa \frac{|\nabla u|^p}{u} \right) \, dV
\]
\[
= -p\beta^p \int_M \left[ \Delta_p u + (\beta - 1)(p-1) \frac{|\nabla u|^p}{u} \right] \left( \Delta_p u + \kappa \frac{|\nabla u|^p}{u} \right) \, dV
\]
\[
= -p\beta^p \int_M \left[ (\Delta_p u)^2 + (\kappa + (\beta - 1)(p-1)) \Delta_p u \frac{|\nabla u|^p}{u} + \kappa(\beta - 1)(p-1) \frac{|\nabla u|^{2p}}{u^2} \right] \, dV
\]
and
\[
\frac{d}{dt} \int_M u^{\beta p - 2} \, dV = (p\beta + p - 2) \int_M u^{\beta p - 3} u^{p-4} \left( \Delta_p u + \kappa \frac{|\nabla u|^p}{u} \right) \, dV
\]
\[
= (p\beta + p - 2)(\kappa - 2p + 3) \int_M u^{2p-4}|\nabla u|^p \, dV
\]
\[
= (p\beta + p - 2)(2 - p + \beta(q - p)) \int_M u^{2p-4}|\nabla u|^p \, dV,
\]
where \(\kappa = p - 1 + \beta(q - p)\). Combining this with Lemma \(3.3\), we finish the proof of Lemma \(3.4\). \(\square\)

For any \(\theta \in (0, 1]\), we can rewrite \((3.9)\) as
\[
\frac{1}{p\beta^p} \frac{d}{dt} \mathcal{F}[u] = -(1 - \theta) \int_M (\Delta_p u)^2 \, dV - \mathcal{G}[u] + \lambda \int_M u^{2p-4}|\nabla u|^p \, dV,
\]
where
\[
\mathcal{G}[u] := \int_M \left[ \theta(\Delta_p u)^2 + (\kappa + (\beta - 1)(p-1)) \Delta_p u \frac{|\nabla u|^p}{u} + \kappa(\beta - 1)(p-1) \frac{|\nabla u|^{2p}}{u^2} \right] \, dV.
\]
Let us set
\[
Q^\theta_u := Bu - \frac{p(n-1)}{2\theta p - 1}[\kappa + (\beta - 1)(p-1)] Gu,
\]
where \(Bu\) and \(Gu\) are defined in \((3.1)\) and \((3.2)\), respectively.

**Lemma 3.5.** Assume that \(n \geq 2\), any positive function \(u\) satisfies
\[
\mathcal{G}[u] = \frac{\theta n}{n-1} \int_M \left[ \|Q^\theta_u\|^2_A + |\nabla u|^{2p-4} \text{Ric}(\nabla u, \nabla u) \right] \, dV - \mu \int_M \frac{|\nabla u|^{2p}}{u^2} \, dV,
\]
where
\[
\mu = \frac{p^2(n-1)^2[\kappa + (\beta - 1)(p-1)]^2}{4\theta(n(p - 1) + p^2)} - \kappa(\beta - 1)(p-1) - [\kappa + (\beta - 1)(p-1)] \frac{n(p-1)}{n(p-1) + p}.
\]

**Proof.** Combining Lemma \(3.1\) with Lemma \(3.2\), we have
\[
\mathcal{G}[u] = \frac{\theta n}{n-1} \int_M \|Bu\|^2_A \, dV - \frac{p(n-1)}{\theta p - 1}[\kappa + (\beta - 1)(p-1)] \int_M \frac{[Bu, Gu]_A}{dV}
\]
\[
+ \left[ \kappa(\beta - 1)(p-1) + [\kappa + (\beta - 1)(p-1)] \frac{n(p-1)}{n(p-1) + p} \right] \int_M \frac{|\nabla u|^{2p}}{u^2} \, dV
\]
With above notations, one has the identities

\[
\|Q_{a}u\|_{A}^{2} = \left\| Bu - \frac{p(n - 1)}{2\theta(p - 1)[n(p - 1) + p]}[\kappa + (\beta - 1)(p - 1)]Gu \right\|_{A}^{2} \\
= \|Bu\|_{A}^{2} - \frac{p(n - 1)}{\theta(p - 1)[n(p - 1) + p]}[\kappa + (\beta - 1)(p - 1)] [Bu, Gu]_{A} \\
+ \frac{p^{2}(n - 1)^{3}}{4\theta^{2}n(p - 1) + p} [\kappa + (\beta - 1)(p - 1)]^{2} \frac{\|u\|^{2p}}{u^{2}}
\]

and

\[
\|Gu\|_{A}^{2} = \left\| \nabla u \right\|^{2p - 2} - \frac{p - 1}{n} \frac{\|\nabla u\|^{p}}{u} = \frac{(n - 1)(p - 1)^{2}}{n} \frac{\|u\|^{2p}}{u^{2}}.
\]

The routine calculation implies that

\[
G[u] = \frac{\partial n}{n - 1} \int_{M} \left[ \|Q_{a}u\|_{A}^{2} + \frac{\|\nabla u\|^{p}}{u} \right] dV - \frac{p^{2}(n - 1)^{2}[\kappa + (\beta - 1)(p - 1)]^{2}}{4\theta^{2}[n(p - 1) + p]^{2}} \int_{M} \frac{\|u\|^{2p}}{u^{2}} dV \\
+ \left[ \kappa(\beta - 1)(p - 1) + [\kappa + (\beta - 1)(p - 1)] \frac{n(p - 1)}{n(p - 1) + p} \right] \int_{M} \frac{\|\nabla u\|^{2p}}{u^{2}} dV \\
+ \frac{\partial n}{n - 1} \int_{M} \|\nabla u\|^{2p - 4} \text{Ric}(\nabla u, \nabla u) dV
\]

where \(\mu\) is defined in (3.12). Then we finish the proof of Lemma 3.5

**Proposition 3.6.** Assume \(n \geq 2, q \in (1, p) \cup (p, p^{*})\), \(\beta\) and \(\theta\) are given by (3.18) and (3.19). If \(u\) is a smooth positive solution to equation (1.9), we get

\[
\frac{1}{p\beta^{p}} \frac{d}{dt} F[u] \leq (\lambda - \Lambda) \int_{M} u^{2p - 4} |\nabla u|^{p} dV,
\]

where \(\Lambda^{*}\) is defined in (1.12).

**Proof.** According to (3.10) and (3.11), we obtain

\[
\frac{1}{p\beta^{p}} \frac{d}{dt} F[u] = - (1 - \theta) \int_{M} (\Delta_{M} u) dV - \frac{\partial n}{n - 1} \left[ \int_{M} \|Q_{a}u\|_{A}^{2} dV + \int_{M} \|\nabla u\|^{2p - 4}\text{Ric}(\nabla u, \nabla u) dV \right] \\
+ \mu \int_{M} \frac{\|\nabla u\|^{2p}}{u^{2}} dV + \lambda \int_{M} u^{2p - 4} |\nabla u|^{p} dV.
\]

Now, using \(\kappa = p - 1 + \beta(q - p)\), \(\mu\) can be rewritten as

\[
\mu = \left[ \frac{p^{2}(n - 1)^{2}(q - 1)^{2}}{4\theta[n(p - 1) + p]^{2}} - (p - 1)(q - p) \right] \beta^{2} - \frac{(p - 1)[n(p - 1) + p](2p - 1 - q) + n(q - 1)]}{n(p - 1) + p} \beta \\
+ (p - 1)^{2}.
\]

(3.16)
Note that, unless
\[
\frac{p^2(n-1)^2(q-1)^2}{4\theta(n(p-1)+p)^2} = (p-1)(q-p),
\]
the coefficient \( \mu \) is quadratic in terms of \( \beta \). Thus, \( \mu \) takes the extremum value, when
\[
\beta = \frac{2\theta(n(p-1)+p)(p-1)(n(p-1)+p)(2p-1-q) + n(q-1)]}{p^2(n-1)^2(q-1)^2 - 4\theta(p-1)(n(p-1)+p)^2}.
\]

Our goal is to choose proper constants \( \theta \) and \( \beta \) such that \( \mu = 0 \). Set \( a \) and \( b \) be the coefficients of \( \mu \) in (3.16) and \( b = 2\sqrt{a}(p-1) \), then
\[
\mu(\beta) = a\beta^2 - b\beta + (p-1)^2 = \left( \sqrt{a}\beta - (p-1) \right)^2 = 0.
\]
Thus,
\[
\beta = \frac{p-1}{\sqrt{a}} = \frac{2(p-1)}{b},
\]
that is
\[
\beta = \frac{2(p-1)(n(p-1)+p)}{(n(p-1)+p)(2p-1-q) + n(q-1)},
\]
where
\[
q \neq \frac{(n(p-1)+p)(2p-1-q) - n}{(n(p-1)+p) - n}.
\]

According to the discriminant \( \Delta = b^2 - 4a(p-1)^2 = 0 \), we obtain
\[
\theta = \frac{p^2(n-1)^2(q-1)}{4n(p-1)(n(p-1)+p) + (np-2n+p)^2(q-1)}.
\]
Therefore, if we choose \( \beta \) and \( \theta \) in (3.18) and (3.19), then \( \mu = 0 \) and \( \Omega_\alpha u = \Omega_\beta^\mu u \). According to the definition of \( \Lambda_\alpha \) in (1.12) and the fact \( \mu = 0 \), we obtain the desired inequality (3.14). \( \square \)

**Proof of Theorem 1.2.** If \( v = u^\rho \) is a solution to (1.3), then \( u \) satisfies the equation
\[
\Delta u + (\beta - 1)(p-1)\frac{|\nabla u|^p}{u} + \frac{\lambda}{2-p+\beta(q-p)}(u^\kappa - u^{2p-3}) = 0.
\]
Multiplying (3.20) by \( \left( \Delta u + \kappa \frac{|\nabla u|^p}{u} \right) \) and applying the fact that
\[
\int_M u^\kappa \left( \Delta u + \kappa \frac{|\nabla u|^p}{u} \right) d\nu = 0,
\]
we have
\[
0 = - \int_M \left( \Delta u + (\beta - 1)(p-1)\frac{|\nabla u|^p}{u} \right) - \frac{\lambda}{2-p+\beta(q-p)}(u^{2p-3} - u^\kappa) \left( \Delta u + \kappa \frac{|\nabla u|^p}{u} \right) d\nu
\]
\[
= - \int_M \left( \Delta u \right)^2 + (\kappa + (\beta - 1)(p-1))\Delta u \frac{|\nabla u|^p}{u} + \kappa(\beta - 1)(p-1)\frac{|\nabla u|^{2p}}{u^2} \right) d\nu + \lambda \int_M u^{2p-4}|\nabla u|^p d\nu
\]
\[
= \frac{1}{p\beta^p} \frac{d}{dt} \int_M u^p \leq (\lambda - \Lambda_\alpha) \int_M u^{2p-4}|\nabla u|^p d\nu,
\]
where we use the results of Lemma 3.4 and Proposition 3.6.

Thus, \( u \) is a constant and equal to 1 if \( \lambda < \Lambda_\alpha \). \( \square \)
So far, we have only obtained rigidity result with the power law nonlinearities, in fact, we can extend our results to the case of general nonlinearities.

**Theorem 3.7.** Let \( f \) be a Lipschitz increasing function such that

\[
\frac{\beta}{2 - p + \beta(q - p)} \left( f'(v) - (q - 1) \frac{f(v)}{v} \right) \leq 0, \quad \forall v > 0. \tag{3.21}
\]

Assume that \( \Lambda_* > 0 \). For any \( q \in (1, p) \cup (p, p^*) \), if \( \lambda \in (0, \Lambda_*) \), then the equation

\[
-\Delta_p v + \frac{\beta^{p-1} \lambda}{2 - p + \beta(q - p)} \left( v^{\frac{p-2+\beta(q-1)}{p}} - f(v) \right) = 0 \tag{3.22}
\]

has a unique positive constant solution \( c \), which satisfies \( f(c) = c^{\frac{p-2+\beta(q-1)}{p}} \).

**Proof.** If \( v = u^\beta \) is a solution to (3.22), which satisfies equation

\[
-\Delta_p u - (\beta - 1)(p - 1) \frac{\nabla u|^p}{u} + \frac{\lambda}{2 - p + \beta(q - p)} \left( u^{2p-3} - \frac{f(u)}{u} u^{p-1+\beta(2-p)} \right) = 0.
\]

Multiplying this equation by \( (\Delta_p u + \kappa \frac{\nabla u|^p}{u}) \) and integrating by parts, we get

\[
0 = - \int_M \left( \Delta_p u + (\beta - 1)(p - 1) \frac{\nabla u|^p}{u} \right) u - \frac{\lambda}{2 - p + \beta(q - p)} u^{2p-3} \left( \Delta_p u + \kappa \frac{\nabla u|^p}{u} \right) dV
\]

Multiplying this equation by \( \left( \Delta_p u + \kappa \frac{\nabla u|^p}{u} \right) \) and integrating by parts, we get

\[
0 = - \int_M \left( \Delta_p u + (\beta - 1)(p - 1) \frac{\nabla u|^p}{u} \right) u - \frac{\lambda}{2 - p + \beta(q - p)} u^{2p-3} \left( \Delta_p u + \kappa \frac{\nabla u|^p}{u} \right) dV
\]

Multiplying this equation by \( \left( \Delta_p u + \kappa \frac{\nabla u|^p}{u} \right) \) and integrating by parts, we get

\[
0 = - \int_M \left( \Delta_p u + (\beta - 1)(p - 1) \frac{\nabla u|^p}{u} \right) u - \frac{\lambda}{2 - p + \beta(q - p)} u^{2p-3} \left( \Delta_p u + \kappa \frac{\nabla u|^p}{u} \right) dV
\]

Multiplying this equation by \( \left( \Delta_p u + \kappa \frac{\nabla u|^p}{u} \right) \) and integrating by parts, we get

\[
0 = - \int_M \left( \Delta_p u + (\beta - 1)(p - 1) \frac{\nabla u|^p}{u} \right) u - \frac{\lambda}{2 - p + \beta(q - p)} u^{2p-3} \left( \Delta_p u + \kappa \frac{\nabla u|^p}{u} \right) dV
\]

where we use (3.21) and Proposition 3.6. In fact,

\[
- \frac{\lambda}{2 - p + \beta(q - p)} \int_M \frac{f(u)}{u} u^{p-1+\beta(2-p)} \left( \Delta_p u + \kappa \frac{\nabla u|^p}{u} \right) dV
\]

\[
= - \frac{\lambda}{2 - p + \beta(q - p)} \int_M \frac{f(u)}{u} u^{p-1+\beta(2-p)} \Delta_p u dV - \frac{\lambda \kappa}{2 - p + \beta(q - p)} \int_M \frac{f(u)}{u} u^{p-2+\beta(2-p)} |\nabla u|^p dV
\]

\[
= \frac{\lambda \beta}{2 - p + \beta(q - p)} \int_M f'(u) u^{p-2+\beta(2-p)} |\nabla u|^p dV + \frac{\lambda(p - 1)(1 - \beta)}{2 - p + \beta(q - p)} \int_M \frac{f(u)}{u} u^{p-2+\beta(2-p)} |\nabla u|^p dV
\]

where \( \kappa = p - 1 + \beta(q - p) \). Therefore, if \( \lambda < \Lambda_* \), \( u \) is a constant. \( \square \)
4. Proof of Theorem 1.5

Proof of Theorem 1.5 According to (3.14) in Proposition 3.6, we know the functional $\mathcal{F}[u]$ is non-increasing, if $\lambda \leq \Lambda_*$. Integrating on both sides of (3.14) implies that

$$\mathcal{F}[u(t, \cdot)]_0^\infty \leq p\beta^p(\lambda - \Lambda_*) \int_0^\infty \left[ \int_M u_2^{p-4} |\nabla u|^p dV \right] dt,$$

if $\lambda < \Lambda_*$, we obtain $\int_0^\infty \int_M u_2^{p-4} |\nabla u|^p dV dt$ is finite. Therefore, when $t \to \infty$, $\nabla u$ converges to 0 and $u(t, \cdot)$ converges to a constant. Thus, for any $t \geq 0$, there holds

$$\mathcal{F}[u(t = 0, \cdot)] \geq \lim_{t \to \infty} \mathcal{F}[u(t, \cdot)] = 0.$$

Hence, we have

$$\int_M |\nabla (u^\beta)|^p dV + \frac{p\beta^p \lambda}{(p\beta + p - 2)(2 - p + \beta(q - p))} \left[ \int_M u^{p\beta + p - 2} dV - \left( \int_M u^\beta dV \right)^{p/q} \right] \geq 0.$$

Taking $v = u^\beta$ and $s = \frac{p-2+\beta p}{\beta}$, we obtain the interpolation inequality (1.13)

$$\|\nabla v\|_{L^p(M)}^p \geq \frac{p\beta^{p-2}}{s} \frac{\lambda}{q - s} \left( \|v\|_{L^q(M)}^q - \|v\|_{L^p(M)}^q \right).$$

□

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