LOW ENERGY CONSTANTS OF CHIRAL PERTURBATION THEORY
FROM THE INSTANTON VACUUM
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Abstract

In the framework of the instanton vacuum model we make expansion over the current mass \( m \) and number of colors \( N_c \) and evaluate \( O(1/N_c, m, m/N_c, m \ln m/N_c) \) corrections to the dynamical quark mass \( M \), the quark condensate \( \langle q\bar{q} \rangle \), the pion mass \( M_\pi \) and decay constant \( F_\pi \). We found the \( SU(2) \chi PT \) low-energy constants \( \tilde{l}_3, \tilde{l}_4 \) in a good correspondence with the phenomenology and lattice calculations.

Introduction. In QCD with massless \( u,d \) quarks the \( SU(2) \times SU(2) \) chiral symmetry is spontaneously broken and leads to appearance of the massless Goldstone particles – pions. In reality, \( u,d \) quark current masses are non-zero but small on the hadronic scale \( \sim 1 \text{ GeV} \).

The Chiral Perturbation Theory (\( \chi PT \)) was proposed in [1] for parameterization of the QCD hadronic correlators at low-energy region, where the expansion parameters are light quark current masses \( m \) and pion momenta \( p \). The basic tool is the phenomenological effective lagrangian, which has a form of the infinite series in these parameters. Naturally, the low-energy constants (LEC) of the series expansion are not fixed. Up to now they were extracted only from the experimental data. Recent progress in lattice calculations provide us with the estimates of LEC. The main problem of lattice evaluations is the still-large pion masses \( M_\pi \) available on the finite size lattices.

QCD instanton vacuum model, often referred as the instanton liquid model, provides a very natural nonperturbative explanation of the S\( \chi \)SB (see the reviews [2, 3]). It provides a consistent framework for description of the pions and thus may be used for evaluation of the LEC. Quasiclassical considerations show that it is energetically favourable to have humps of strong gluon fields (instantons) spread over 4-dimensional Euclidian space. Such fields do strongly modify the quark propagation due to the t’Hooft type quark-quark interactions in the background of the instanton vacuum field. This background is assumed as a superposition of \( N_+ \) instantons and \( N_- \) antiinstantons

\[ A_\mu(x) = \sum_{I=1}^{N_+} A^I_\mu(x_I, x) + \sum_{A=1}^{N_-} A^A_\mu(x_A, x), \]

where \( x = (\rho, z, U) \) are the (anti)instanton collective coordinates – size, position and color orientation. The most essential for the low-energy processes are the would-be quark zero modes, which result in a very strong attraction in the channels with quantum numbers of vacuum, appearance of the quark condensate and generation of the dynamical quark mass. The main parameters of the model are the average inter-instanton distance \( R \) and the average instanton size \( \rho \). The estimates of these quantities are \( \rho \approx 0.33 \text{ fm}, R \approx 1 \text{ fm} \) (phenomenological), \( \rho \approx 0.35 \text{ fm}, R \approx 0.95 \text{ fm} \) (variational) [2, 3], \( \rho \approx 0.36 \text{ fm}, R \approx 0.89 \text{ fm} \) (lattice) [4]-[8] and have \( \sim 10 – 15\% \) uncertainty. Recent computer investigations [9] of a current mass dependence of QCD observables within instanton liquid model show that the best correspondence with lattice QCD data is obtained for \( \rho \approx 0.32 \text{ fm}, R \approx 0.76 \text{ fm} \). While in the real world the number of colors is \( N_c = 3 \), since the pioneering work of t’Hooft it is assumed that one can consider \( N_c \)-counting as a useful tool, i.e. take the limit \( N_c \to \infty \) and neglect the \( 1/N_c \)-corrections. In the instanton vacuum model \( N_c \)-counting is naturally incorporated. The phenomenological set is popular since in the leading order (LO) it reproduces reasonable values for most of the physical quantities. This leads to rather consistent description of pions and nucleons in the chiral limit.

The main purpose of this work is evaluation of \( O(1/N_c, m, m/N_c, m/N_c \ln m) \) non-leading-order (NLO) corrections to different physical observables, which provide LECs. So,
we are dealing with double expansion over \( m \) and over \( 1/N_c \). There are several sources of such NLO corrections:

1. At pure gluonic sector of the instanton vacuum model the width of the instanton size distribution is \( \mathcal{O}(1/N_c) \). The account of the finite width leads to rather small corrections. In the following we will check the accuracy of \( \delta \)-function type of the instanton size distribution by direct evaluation of the finite width corrections.

2. The back-reaction of the light quark determinants to the instanton vacuum properties is formally controlled by \( N_f/N_c \)-factor. It does not sizably change the distribution over \( N_+ + N_- \) but radically change the distribution over \( N_+ - N_- \). Any \( m_f = 0 \) leads to \( \delta \)-function type of the distribution. In the following we take \( N_+ = N_- \).

3. There are the quark-quark tensor interaction terms which are \( 1/N_c \)-suppressed and thus are absent in the \( N_c \)LO effective action. These terms correspond to nonplanar diagrams in old-fashioned diagrammatic technique.

4. The contribution of meson quantum fluctuations (meson loops) has to be taken into account. First we study the role of the meson loops which give the dominant contribution. At the end we also estimate the contributions of finite width of instanton size distribution, and above-mentioned tensor interaction term.

We consider parameters \( \rho, R \) free within their \( \sim 15\% \) uncertainty and fix them from the requirement \( F_{\pi,m=0} = 88 \text{MeV}, \langle \bar{q}q \rangle_{m=0} = (255 \text{MeV})^2 \) with account of \( N_c \)NLO corrections, as it is requested by \( \chi \)PT. We found the values \( \rho = 0.350 \text{fm}, \ R = 0.856 \text{fm} \) in agreement with the above-given estimates. Note that though the evaluation of the meson loop corrections in the instanton vacuum model is similar to the earlier meson loop evaluations \([10]-[13]\) in the NJL model, there are a few differences:

1. As it has been already mentioned, the meson loop corrections are not the only sources of \( 1/N_c \)-corrections in the instanton model.

2. Due to nonlocal form-factors there is no need to introduce independent fermion and boson cutoffs \( \Lambda_f, \Lambda_b \). The natural cutoff scale for all the loops (including meson loops) is the inverse instanton size \( \rho^{-1} \).

3. The quark coupling constant is defined through the saddle-point equation in the instanton model whereas it is a fixed external parameter in NJL.

The basic object we study are the correlators. The simplest correlators are: \( \langle \bar{q}q(m) \rangle = -F^2 B + \mathcal{O}(m), \int d^4 x e^{-ixq} \langle j^{a5}_\mu(x) j^{\mu\nu}_\nu(0) \rangle = F^2 \delta^{ab} \left( g_{\mu\nu} - \frac{q_{\mu}q_{\nu}}{q^2 + M^2_\pi} \right) + \mathcal{O}(q^2), F_\pi = F + \mathcal{O}(m), M^2_\pi = m^2_\pi + \mathcal{O}(m^2) \), where \( m^2_\pi = 2Bm \). The constants \( F, B \) define pion decay constant and quark condensate in the chiral limit, which are the \( \chi \)LO LECs. \( \chi \)PT provide proper way of the parameterization of the correlators (observables) in low energy region by means of \( \chi \)LO LECs \( F, B \) and \( \chi \)NLO LECs \( \tilde{l}_i \), defined in \([1]\). Important fact that bare constants \( \tilde{l}_i \) are renormalized by pion loops which lead to \( l_i \rightarrow \tilde{l}_i \). The correlators and observables of pion physics should be expressed in terms of \( \tilde{l}_i \).

There is a number of running experiments dedicated to the low-energy pion physics: \( \pi^+ \pi^- \), \( \pi K \) atoms @DIRAC@CERN, \( K \rightarrow \pi \pi e\nu \) @BNL E865, \( K^\pm \rightarrow \pi^\pm \pi^0 \pi^- \) @NA48/2, pion electromagnetic polarizabilities @Mainz Microtron MAMI, etc. and QCD lattice evaluations of LECs - the collaborations MILC, ETM, JLQCD, RBC/UKQCD, PACS-CS \([17]-[22]\).

We assume that the promising method of the calculation of LECs is the application of instanton vacuum model.

**Light quarks in instanton vacuum \([23]-[29]\).** Zero-mode approximation for the quark propagator in a single instanton field - \( S_i(x,y) \approx \frac{|\Phi_{0i}|^2}{m} \) + \( \frac{1}{10} \), \((i\hat{\partial} + gA_i)\Phi_{0i} = 0)\)
is working well at \( m \Rightarrow 0 \) but wrong beyond the chiral limit. Our extension [23]-[29] of zero-mode approximation beyond the chiral limit:

\[
S_i = S_0 - S_0 \frac{|\Phi_{0i}|}{(\Phi_{0i}|\hat{p}S_0\hat{p}|\Phi_{0i})} \hat{p}S_0, \quad S_0 = \frac{1}{\hat{p} + im}, \quad S_i|\Phi_{0i}) = \frac{1}{im}|\Phi_{0i}), \quad \langle \Phi_{0i}|S_i = \langle \Phi_{0i}|\frac{1}{im} \tag{1}
\]

Then, quark propagator in instanton media and in the presence of the external fields \( \hat{V} = s + p\gamma_5 + \hat{v} + \hat{a}\gamma_5 \) become:

\[
\tilde{S} - \tilde{S}_0 = -\tilde{S}_0 \sum_{i,j} \hat{p} |\phi_{0i}\rangle \langle \phi_{0j}| \hat{p} \tilde{S}_0 \quad \langle \phi_{0i}| = \frac{1}{\hat{p} + V + im}, \quad L_i(x, z_i) = P \exp \left(i \int_{z_i}^x dy \mu(y) + a\mu(y)\gamma_5 \right)
\]

From this one the low-frequencies part of quark determinant is:

\[
\ln \tilde{\text{Det}}_{\text{low}} = \text{Tr} \int dm \tilde{S}(m) = \ln \text{det} \langle \phi_{0i}|\hat{p} \tilde{S}_0 \tilde{g}_\sigma \hat{p}|\phi_{0j}\rangle \tag{3}
\]

Averaging of \( \tilde{\text{Det}}_{\text{low}} \) over instantons by means of fermionization leads to the partition function \( Z_N \) in terms of constituent quarks \( \psi \) (in the following \( N_f = 2 \) case). Then,

\[
Z_N = \int d\lambda_+d\lambda_-D\psi D\bar{\psi} e^{-S}, \quad S = \sum_{\pm} \left( N_\pm \ln \frac{K}{\lambda_\pm} - N_\pm + \psi^\dagger (i\hat{\partial} + \hat{V} + im)\psi + \lambda_\pm Y^{\pm}_2 \right) \tag{4}
\]

\[
Y^{\pm}_2 = \int d\rho D(p) \left( \alpha^2 \text{det} J^{\pm} + \beta^2 \text{det} J^{\pm}_{\mu\nu} \right), \quad \frac{\beta^2}{\alpha^2} := \frac{1}{8N_c^2} \frac{2N_c}{2N_c - 1} = \frac{1}{8N_c - 4} = O \left( \frac{1}{N_c} \right)
\]

\[
J^{\pm}_{\mu\nu} = \psi^\dagger \frac{1}{2} \left( \gamma^\pm L + \gamma^\mp L \right) \sigma_\mu \sigma_\nu \frac{1}{2} \gamma^5 L \psi.
\]

Here the dynamical quark-quark interaction coupling \( \lambda_\pm \) is due to the exponentiation in \( Z_N \) by using of Stirling-like formula. Further step - the bosonization in terms of mesons is obvious procedure for \( N_f = 2 \) case. The integration over fermions provide the action:

\[
S = -N \ln \lambda + 2 \sum_i \left( \Phi_i^2 + \frac{1}{2} \Phi_{i,\mu\nu}^2 \right) - Tr \log \left[ \hat{p} + \hat{V} + im + i\lambda^0 \bar{L} F(p) \left( \alpha \Phi_i \Gamma_i + \frac{1}{2} \beta \Phi_{i,\mu\nu} \sigma_{\mu\nu} \Gamma_i \right) F(p) L^{-1} \right] \tag{5}
\]

The mesons \( \Phi_i, \Phi_{i,\mu\nu} \) are chiral doublets: \( (\sigma, \bar{\sigma}), (\bar{\sigma}, \eta) \) and \( (\sigma_{\mu\nu}, \bar{\sigma}_{\mu\nu}) \).

**Dynamical quark mass** [24], [27], [28]. For evaluation of the partition function \( Z_N \), it is used the effective action [14] \( \Gamma_{\text{eff}}[m, \lambda, \sigma] \), defined as:

\[
Z_N[m] = \int d\lambda Z_N[m, \lambda] = \int d\lambda \exp(-\Gamma_{\text{eff}}[m, \lambda, \sigma]) \tag{6}
\]

where the vacuum field \( \sigma \) and the coupling \( \lambda \) are the solutions of the Eqs.

\[
\frac{\partial \Gamma_{\text{eff}}[m, \lambda, \sigma]}{\partial \sigma} = 0, \quad \frac{\partial \Gamma_{\text{eff}}[m, \lambda, \sigma]}{\partial \lambda} = 0. \tag{7}
\]

In the \( N_c \)LO \( \Gamma_{\text{eff}}[m, \lambda, \sigma] = S[m, \lambda, \sigma] \). Meson fluctuations provide NLO term given by

\[
\Gamma_{\text{eff}}^{\text{mes}}[m, \lambda, \sigma] = \frac{1}{2} \text{Tr} \ln \left( 4\delta_{ij} - \frac{1}{\sigma^2} \text{Tr} \frac{M(p)}{\hat{p} + im(p)} \Gamma_i \frac{M(p)}{\hat{p} + im(p)} \Gamma_j \right), \tag{8}
\]
where \( \mu(p) = m + M(p) \) and we introduced the dynamical quark mass \( M(p) = MF^2(p) \); \( M = \frac{(2\pi\rho)^2}{2\rho^0} \). The dynamical quark mass \( M(p) \) and the coupling \( \lambda \) are the solutions of the vacuum and saddle-point Eqs. The numerical solution of the Eqs. is

\[
M(m) = 0.36 - 2.36 m - \frac{m}{N_c}(0.808 + 4.197 \ln m)
\]

(9)

Here and in the following \( M \) and \( m \) are given in GeV.

**Quark condensate** [24], [27], [28]. The quark condensate \( \langle \bar{q}q \rangle \) characterizes the \( \Sigma\chi SB \) and is given by Eq. \( \langle \bar{q}q \rangle = \frac{\partial \ln Z_N}{\partial m} \). Numerical evaluation gives

\[
- \langle \bar{q}q \rangle (m) = ((0.00497 - 0.0343m)N_c + +(0.00168 - 0.0494m - 0.0580m \ln m)) [\text{GeV}^3]
\]

(10)

**Quarks in external axial-vector field and pion properties** [28]. External axial-vector isovector field \( a_\mu = a_\mu^i \tau_i/2 \) generate nonzero vacuum pion field \( \vec{u} \) and we have an additional vacuum equation:

\[
\frac{\partial \Gamma_{\text{eff}}[m, \lambda, \vec{u}, \vec{a}_\mu]}{\partial \vec{u}} = 0.
\]

(11)

We may easily get that the shifts of \( \sigma, \lambda \) contribute only to \( \mathcal{O}(a^4) \)-terms and thus may be safely omitted. The total vacuum meson fields are represented as \( \Phi_{\text{vac}} = \sigma U, \ U = u_0 + i\vec{\tau} \vec{u}, \ U^\dagger U = UU^\dagger = 1. \) In the NLO one has to take into account the fluctuations of the meson fields \( \Phi = \Phi_{\text{vac}} + \Phi' \). Now \( \Gamma_{\text{eff}}[m, \lambda, \vec{u}, \vec{a}_\mu] = S[m, \lambda, \vec{u}, \vec{a}_\mu] + \Gamma_{\text{mes}}^{\text{eff}}[m, \lambda, \vec{u}, \vec{a}_\mu] \), where

\[
\Gamma_{\text{mes}}^{\text{eff}}[m, \lambda, \vec{u}, \vec{a}_\mu] = \frac{1}{2} \text{Tr} \ln \frac{\delta^2 S[m, \lambda, \sigma, \vec{u}, \vec{a}_\mu, \Phi]}{\delta \phi^i \delta \phi^j} |_{\phi = 0} = \Gamma_{\text{eff}}^{\text{mes}}[m, \lambda] + \Delta \Gamma_{\text{eff}}^{\text{mes}}[m, \lambda, \vec{u}, \vec{a}_\mu]
\]

(12)

The first term was calculated before and we have to calculate now second one. Collecting the terms \( a_\mu a_\nu, \ a_\mu \partial_\nu u_i \) and \( \partial_\sigma u_i \partial_\mu u_j \), we show that in agreement with chiral symmetry \( \Gamma_{\text{eff}} = F_{aa}^2 a_\mu^2 + F_{uu}^2 (\partial_\mu \vec{u})^2 + 2 F_{aa}^2 \vec{a}_\mu \partial_\mu \vec{u} + F_{uu}^2 M_\pi^2 \vec{a}^2 + \mathcal{O}(a^3, \vec{u}^3, m^2) \), where the constants \( F_{ij} \) differ only beyond chiral limit: \( F_{aa}^2 - F_{uu}^2 = 2 (F_{aa}^2 - F_{uu}^2) \sim m \). Then, one can get that the two-point axial-isovector currents correlator has a form:

\[
\int d^4x e^{-iq \cdot x} \langle j_{\mu A i}(x) j_{\nu A j}(0) \rangle = \delta_{ij} F_\pi^2 \left( \delta_{\mu \nu} - \frac{q_\mu q_\nu}{q^2 + M_\pi^2} \right) + \mathcal{O}(q^2)
\]

(13)

We see that \( M_\pi \) has a meaning of pion mass and \( F_\pi \) - pion decay constant.

**Pion decay constant** \( F_\pi \) **and mass** \( M_\pi \) **from** \( \Gamma_{\text{eff}} \) [28]. Finally numerical calculations lead to

\[
F_\pi^2 = N_c \left( \left( 2.85 - \frac{0.869}{N_c} \right) \right) - \left( 3.51 + \frac{0.815}{N_c} \right) m - \frac{44.25}{N_c} m \ln m + \mathcal{O}(m^2) \cdot 10^{-3} [\text{GeV}^2]
\]

(14)

and

\[
M_\pi^2 = m \left( \left( 3.49 + \frac{1.63}{N_c} \right) + m \left( 15.5 + \frac{18.25}{N_c} + \frac{13.5577}{N_c} \ln m \right) + \mathcal{O}(m^2) \right)
\]

(15)

Chiral log theorems [11] provided the test of all of the numerical calculations above. We found that all nonanalytical \( m \ln m \) terms in \( \langle \bar{q}q \rangle \), \( F_\pi \) and \( M_\pi \) very well correspond them.

**Instanton finite width and tensor terms contributions to** \( F_\pi, M_\pi \) [28]. The main effect of the finite width is the change of the \( p \)-dependence of dynamical quark mass
For the estimations we take
\[ \delta \rho^2 = \langle \rho^2 \rangle - \langle \rho \rangle^2 \approx \frac{0.5599 \text{GeV}}{N_c} \] which leads to the \( \approx 5\% \) for \( F_\pi^2 \) and \( \approx 2.6\% \) for \( \langle \bar{q}q \rangle \) corrections. They are negligible one. Tensor terms do not affect \( F_\pi \) and \( M_\pi \), since they lead to the \( O(q^2) \)-correction to the axial currents correlator.

LEC \( \bar{l}_3 \) and \( \bar{l}_4 \) provide \( \chi \)NLO terms in \( F_\pi(m) \) and \( M_\pi(m) \) as

\[
M_\pi^2 = m_\pi^2 \left( 1 - \frac{m_\pi^2}{32\pi^2 F_\pi^2} \bar{l}_3 + \mathcal{O}(m_\pi^4) \right), \quad F_\pi^2 = F^2 \left( 1 + \frac{m_\pi^2}{8\pi^2 F^2} \bar{l}_4 + \mathcal{O}(m_\pi^4) \right),
\]

and can be extracted from our previous estimations. They are compared with lattice calculations \[17]-\[22\] and phenomenological estimations \[1\] at the following Figs.\[1\], where left panel represent \( \bar{l}_3 \) and right one represent \( \bar{l}_4 \).

**Conclusion and outlook.** We established a reliable theoretical framework for the evaluation the \( \chi \)PT low-energy constants, which provide the understanding of pion physics in QCD. The calculated constants \( \bar{l}_3, \bar{l}_4 \) are in reasonable agreement with lattice results and phenomenological estimates. The calculations of all other constants and the extension to the \( N_f = 3 \) case are on the way.

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