EXACT GROWTH RATES OF SOLUTIONS OF DELAY–DOMINATED DIFFERENTIAL EQUATIONS WITH REGULARLY VARYING COEFFICIENTS

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Abstract. In this paper we determine the exact rate of growth of the solution of a deterministic delay differential equation in which the delayed term is regularly varying at infinity and dominates, and determine criteria to characterise this dominance. The preservation of growth rates using a uniform step size Euler scheme is also discussed.

1. Introduction

This paper examines the growth rate of $x(t) \to \infty$ as $t \to \infty$ of solutions of the delay differential equation

$$x'(t) = f(x(t)) + g(x(t-\tau)), \quad t > 0,$$  \hspace{1cm} (1.1)

We establish criteria on the size of $g$ relative to $f$ under which the solution of the delay equation does not grow like the solution of the ordinary differential equation $y'(t) = f(y(t))$. In broad terms, we focus on the cases when $g$ grows polynomially, and $f$ grows sublinearly, though the general theory extends to cover more rapidly growing $g$ as well, and even recovers the exact exponential growth and characteristic equation in the linear case.

In [4], we established general results for the exact rate of growth of solutions of (1.1) in which the delay term in some sense asymptotically dominates the instantaneous term. The general theorems are obtained by employing a constructive comparison principle (see Appleby [1] and Appleby and Buckwar [2], for example). The asymptotic results are restated here in Section 2.2, and their hypotheses explained. In these general theorems, the sufficient conditions which describe this dominance, as well as the rate of growth of solutions, depend on the existence of an auxiliary function $\phi$ obeying certain asymptotic properties. Apart from some examples, we do not attempt systematically in [4] to demonstrate that such an auxiliary function $\phi$ exists, nor did we indicate how it might be constructed.

In this paper, we show when $g$ is regularly varying at infinity with positive index, and $f$ is sufficiently small, that the rate of growth of solutions of (1.1) can be determined in the form

$$\lim_{t \to \infty} \frac{G(x(t))}{t} = \lambda > 0,$$

for some function $G$ that is known in terms of $g$. This is achieved because, for such classes of problems, the auxiliary function $\phi$ can be found, and the exact asymptotic
behaviour determined by applying general results. In addition, we show that for an explicit Euler scheme with uniform step size $h > 0$ that the asymptotic behaviour is preserved, in the sense that for every $h$ there exist $0 < \lambda(h) < +\infty$ such that

$$
\lim_{t \to \infty} \frac{G(x_h(t))}{t} = \lambda(h),
$$

and $\lim_{h \to 0} \lambda(h) = \lambda$, where $x_h$ is the extension to continuous time of the Euler scheme.

Statements and discussion of the main results in continuous–time, as well as examples, are given in Section 2. The preservation of the asymptotic rate of growth is considered in Section 3. Proofs of continuous time results are deferred to Section 4, with results for the discretisations being supplied in Section 5. We do not address here the asymptotic behaviour when the instantaneous term dominates.

## 2. Statement and Discussion of Main Continuous Time–Results

### 2.1. Notation and preliminary results: existence and non–explosion.

In this paper, $\mathbb{R}$ stands for the real numbers, $\mathbb{N}$ for the natural numbers and $\mathbb{Z}$ for the integers. A function $k : [0, \infty) \to (0, \infty)$ is said to be regularly varying at infinity with index $\alpha \in \mathbb{R}$ if $\lim_{x \to \infty} k(\lambda x)/k(x) = \lambda^\alpha$ for each $\lambda > 0$. We write $k \in \text{RV}_\infty(\alpha)$. The reader is referred to Bingham, Goldie and Teugels [7] for results on regularly varying functions. If $I$ and $J$ are subintervals of $\mathbb{R}$, the space $C(I; J)$ contains all continuous functions $\phi : I \to J$.

We make some hypotheses regarding our problem. Suppose

$$
f \in C((0, \infty); [0, \infty)) \text{ is locally Lipschitz continuous.} \quad (2.1)
$$

and obeys

$$
\int_1^\infty \frac{1}{f(u)} \, du = +\infty. \quad (2.2)
$$

We interpret this condition as being satisfied if $f$ is identically zero. Suppose also that

$$
g \in C((0, \infty); (0, \infty)). \quad (2.3)
$$

Let $\tau > 0$ and suppose that

$$
\psi \in C([-\tau, 0]; (0, \infty)), \quad (2.4)
$$

and consider the delay–differential equation given by

$$
x'(t) = f(x(t)) + g(x(t - \tau)), \quad t > 0; \quad x(t) = \psi(t), \quad t \in [-\tau, 0]. \quad (2.5)
$$

The following result then holds.

**Theorem 1.** Let $f$ obey (2.1), (2.2), $g$ obeys (2.3), and $\psi \in C([-\tau, 0]; (0, \infty))$ where $\tau > 0$. Then there is $x \in C([-\tau, \infty))$ which is the unique continuous solution of (2.5) and which moreover obeys

$$
\lim_{t \to \infty} x(t) = \infty. \quad (2.6)
$$

The condition (2.2) prevents a finite time explosion. Note that $g$ being positive forces $x$ to be increasing on $[0, \infty)$, and this ensures that (2.6) holds, because $\lim_{t \to \infty} x(t) =: L \in (0, \infty)$ forces $g(L) = 0$, a contradiction.
2.2. Statement and Discussion of General Comparison Results. Before we state our general comparison results, we first introduce some notation and auxiliary functions. Since \( \psi(t) > 0 \) for all \( t \in [-\tau, 0] \) we may define \( \psi^* := \max_{-\tau \leq s \leq 0} \psi(s) > 0 \). Suppose that \( \phi : (\psi^*, \infty) \to (0, \infty) \) is continuous, and define

\[
\Gamma(x) = \int_{\psi^*}^{x} \frac{1}{\phi(u)} \, du, \quad x > \psi^*.
\]  

(2.7)

Suppose that

\[
\lim_{x \to \infty} \Gamma(x) = +\infty.
\]  

(2.8)

Define also for \( c > 0 \) the function \( \Gamma_c \) given by

\[
\Gamma_c(x) = \frac{1}{c} \Gamma(x), \quad x > \psi^*.
\]  

(2.9)

In our first main result, which appears as Theorem 1 in \[4\], we claim that if the delayed term \( f \) is asymptotically dominated by the instantaneous term \( g \), then the solution of \( \text{(2.5)} \) behaves according to the ordinary differential equation \( z'(t) = \phi(z(t)) \).

**Theorem 2.** Suppose that \( f \) obeys \((2.1)\) and \((2.2)\). Let \( g \) be non-decreasing and obey \((2.3)\) and let \( \tau > 0 \) and \( \psi \in C([-\tau, 0]; (0, \infty)) \). Suppose that there exists a continuous function \( \phi \) such that \( \Gamma, \Gamma_c \) are defined by \((2.7)\) and \((2.9)\) respectively, and that \( \Gamma \) obeys \((2.8)\). Suppose also that

\[
\lim_{\varepsilon \to 0^+} \eta(\varepsilon) = \eta, \tag{2.10}
\]

and suppose that

\[
\lim_{x \to \infty} \frac{f(x)}{g(\Gamma^{-1}_c(\Gamma(\eta_c)(x) - \tau))} = 0, \quad \text{for every } \varepsilon \in (0, 1), \tag{2.11}
\]

\[
\limsup_{x \to \infty} \frac{g(x)}{\phi(\Gamma^{-1}_c(\Gamma(\eta_c)(x) + \tau))} = \bar{\eta}_c \in [0, \infty) \quad \text{for every } \varepsilon \in (0, 1). \tag{2.12}
\]

where

\[
\sup_{c \in (0, 1)} \bar{\eta}_c := \bar{\eta} < \eta. \tag{2.13}
\]

If \( x \) is the unique continuous solution of \((2.7)\), then

\[
\limsup_{t \to \infty} \frac{\Gamma(x(t))}{t} \leq \eta. \tag{2.14}
\]

We comment briefly on Theorem 2 and its hypotheses. First, we note that the existence of a function \( \phi \) obeying \((2.11)\) and \((2.12)\) is not assured by the theorem; the existence or construction of such a function must be achieved independently. However, it can be seen that \((2.12)\) describes an asymptotic relationship between \( \phi \) and \( g \) only, and this is what identifies candidates for \( \phi \). In the next section, we show that for a wide class of \( g \) that suitable \( \phi \) can be chosen. The condition \((2.11)\) characterises the fact that the instantaneous term \( f \) is dominated by the delayed term.

We now offer an improvement on Theorem 2. In it the condition \((2.11)\) is relaxed. In later examples we show that this enables asymptotic estimates to be extended to a wider class of problems.

**Theorem 3.** Suppose that \( f \) obeys \((2.1)\) and \((2.2)\). Let \( g \) be non-decreasing and obey \((2.3)\) and let \( \tau > 0 \) and \( \psi \in C([-\tau, 0]; (0, \infty)) \). Suppose that there exists a...
continuous function $\phi$ such that $\Gamma, \Gamma_c$ are defined by (2.7) and (2.9) respectively, and that $\Gamma$ obeys (2.8). Suppose also that (2.10) and suppose that $f$ obeys
\[
\lim_{x \to \infty} \frac{f(x)}{\phi(x)} = 0, \tag{2.15}
\]
and that $g$ and $\phi$ obey (2.12) where $\eta, \phi, \epsilon$ obey (2.18). If $x$ is the unique continuous solution of (2.5), then it obeys (2.14).

We now state a corresponding result which enables us to determine a lower bound on the rate of growth of solutions. It appeared as Theorem 2 in [4].

**Theorem 4.** Suppose that $f$ obeys (2.1) and (2.2). Let $g$ be non-decreasing and obey (2.3) and let $\tau > 0$ and $\psi \in C([−\tau, 0]; (0, \infty))$. Suppose that there exists a continuous function $\phi$ such that $\Gamma, \Gamma_c$ are defined by (2.7) and (2.9) respectively, and $\Gamma$ obeys (2.8). Suppose also that
\[
\lim_{\epsilon \to 0^+} \mu(\epsilon) = \mu, \tag{2.16}
\]
and that $g$ and $\phi$ obey
\[
\lim_{x \to \infty} \frac{g(x)}{\phi(\Gamma(\mu(x), \Gamma_c(x)) \mu(\epsilon) + \tau(1 - \epsilon)))} = \bar{\mu} \in (0, \infty] \quad \text{for every } \epsilon \in (0, 1), \tag{2.17}
\]
where
\[
\inf_{\epsilon \in (0, 1)} \tilde{\mu}_\epsilon := \bar{\mu} > \mu. \tag{2.18}
\]
If $x$ is the unique continuous solution of (2.5), then
\[
\lim_{t \to \infty} \inf_{x(t)} \frac{\Gamma(x(t))}{t} \geq \mu. \tag{2.19}
\]

As in Theorem 2, in which the condition (2.12) determines a relationship between $\phi$ and $g$, in Theorem 4 there is a corresponding and closely related condition (2.17) which describes the relationship between $g$ and $\phi$.

Contingent on other hypotheses being satisfied, we notice that the lower bound (2.10) and the upper bound (2.14) incorporate the same function $\Gamma$. Therefore, under certain conditions we may combine Theorems 2 and 4 to arrive at the exact asymptotic behaviour of $x$. This is the subject of the next result, which improves on a result in [4].

**Theorem 5.** Suppose $f$ obeys (2.1) and (2.2). Let $g$ be non-decreasing and obey (2.3) and let $\tau > 0$ and $\psi \in C([−\tau, 0]; (0, \infty))$. Suppose that there exists a continuous function $\phi$ such that $\Gamma, \Gamma_c$ are defined by (2.7) and (2.9), and that $\Gamma$ obeys (2.8). Suppose also that there is $\eta > 0$ such that $\mu(\epsilon) \to \eta$ and $\eta(\epsilon) \to \eta$ as $\epsilon \to 0$ and that $f, g, \phi$ obey (2.15), (2.12) and (2.17), where
\[
\sup_{\epsilon \in (0, 1)} \tilde{\eta}_\epsilon := \bar{\eta} < \eta, \quad \inf_{\epsilon \in (0, 1)} \tilde{\mu}_\epsilon := \bar{\mu} > \eta. \tag{2.20}
\]
If $x$ is the unique continuous solution of (2.5), then
\[
\lim_{t \to \infty} \frac{\Gamma(x(t))}{t} = \eta. \tag{2.21}
\]

Provided that a function $\phi$ can be found so that all the hypotheses of Theorem 5 are satisfied, the conclusion of Theorem 5 (viz., (2.21)) which describes an exact rate of growth, is sharp.
2.3. **Application to equations with regularly varying $g$.** We consider some cases in which the unknown auxiliary function $\phi$ (and therefore $\Gamma$) in Theorems 2, 4 can be constructed explicitly in terms of $g$. Essentially, our examples cover the cases where $g$ grows polynomially at either a sublinear or superlinear rate. First we consider the case where $g$ is in RV$_{\infty}(\beta)$ for $\beta \leq 1$ and $g(x)/x \to 0$ as $x \to \infty$.

**Theorem 6.** Let $f$ obey $2.7$, $2.8$. Let $g$ obey $2.3$ be non-decreasing and let $\tau > 0$ and $\psi \in C([-\tau, 0]; (0, \infty))$. Suppose $g \in$ RV$_{\infty}(\beta)$ for some $\beta \leq 1$, $\lim_{x \to \infty} g(x)/x = 0$, and $\lim_{x \to \infty} f(x)/g(x) = 0$. If $x$ is the unique continuous solution of (2.5), then

$$\lim_{t \to \infty} \frac{1}{t} \int_1^x \frac{1}{g(u)} \, du = 1. \tag{2.22}$$

This result is proven using Theorems 2 and 4; it recovers part (ii) of Theorem 2.2 in Appleby, McCarthy and Rodkina [3]. Next we consider the case where $g$ is in RV$_{\infty}(1)$ but in which $g(x)/x \to \infty$ as $x \to \infty$, and use Theorem 5 to determine the growth rate.

**Theorem 7.** Let $f$ obey $2.7$, $2.8$. Let $g$ obey $2.3$ and be non-decreasing. Let $\tau > 0$ and $\psi \in C([-\tau, 0]; (0, \infty))$. Suppose $g \in$ RV$_{\infty}(1)$, $x \mapsto g(x)/x$ is asymptotic to a non-decreasing function, $\lim_{x \to \infty} g(x)/x = \infty$, and

$$\lim_{x \to \infty} \frac{f(x)}{x \log(g(x)/x)} = 0. \tag{2.23}$$

Define

$$G(x) = \int_1^x \frac{1}{u \log(1 + g(u)/u)} \, du, \quad x > 1. \tag{2.24}$$

Then the unique continuous solution $x$ of (2.5) obeys

$$\lim_{t \to \infty} \frac{G(x(t))}{t} = \frac{1}{\tau}. \tag{2.25}$$

With a slightly stronger hypothesis on $f$ we can obtain the same conclusion on the growth rate, but by an alternative proof.

**Theorem 8.** Let $f$ obey $2.7$, $2.8$. Let $g$ obey $2.3$ and be non-decreasing. Let $\tau > 0$ and $\psi \in C([-\tau, 0]; (0, \infty))$. Suppose $g \in$ RV$_{\infty}(1)$, $x \mapsto g(x)/x$ is asymptotic to a non-decreasing function, $\lim_{x \to \infty} g(x)/x = \infty$, and $\lim_{x \to \infty} f(x)/x = 0$. If $G$ is defined by (2.24), then the unique continuous solution $x$ of (2.5) obeys

$$\lim_{t \to \infty} \frac{G(x(t))}{t} = \frac{1}{\tau}. \tag{2.26}$$

The case where $g$ grows according to $g \in$ RV$_{\infty}(\beta)$ for some $\beta \leq 1$ with $g(x)/x$ tending to a zero limit is covered by Theorem 6.

The proof of Theorem 8 is facilitated by the following Lemma, which appears as Lemma 2.7 in Appleby, McCarthy and Rodkina [3]. It also motivates the choice of $\phi$ in Theorem 7.

**Lemma 1.** Let $h > 0$. Suppose $g \in C((0, \infty); (0, \infty))$, $g \in$ RV$_{\infty}(1)$, $g(y)/y \to \infty$ as $y \to \infty$, and there is a function $g_1$ with $g_1(y)/g(y) \to 1$ as $y \to \infty$ such that $y \mapsto g_1(y)/y$ is non-decreasing. If $y_{n+1} = y_n + hg(y_n)$, $n \geq 0$ and $y_0 = \xi > 0$, then $\lim_{n \to \infty} G(y_n)/n = 1$, where $G$ is defined by (2.24).

If $g(x)/x$ tends to a finite non-zero limit, we are in the standard linear case, but even this is recovered independently of the standard linear theory by applying Theorems 2 and 4.
Theorem 9. Let \( C > 0, \tau > 0 \) and suppose that \( \psi \in C([-\tau, 0]; (0, \infty)) \). Let \( x \) be the unique continuous solution of (2.5) with \( f(x)/x \to 0 \) and \( g(x)/x \to C \) as \( x \to \infty \). Then there is a unique \( \lambda > 0 \) such that \( \lambda = Ce^{-\lambda^\tau} \) and \( x \) obeys \( \lim_{x \to \infty} \log x(t)/t = \lambda \).

In the case when \( g \) has a power–like growth faster which is faster than linear, the rate of growth can be determined by means of Theorem 5.

Theorem 10. Suppose that \( f \) obeys (2.1) and (2.2). Let \( g \) obey (2.4) be non–decreasing and let \( \tau > 0 \) and \( \psi \in C([-\tau, 0]; (0, \infty)) \). Suppose also that there exists \( \beta > 1 \) such that \( \lim_{x \to \infty} \log g(x)/\log x = \beta and \)

\[
\lim_{x \to \infty} \frac{f(x)}{x \log x} = 0.
\]

Then the unique continuous solution \( x \) of (2.5) obeys

\[
\lim_{t \to \infty} \frac{\log x(t)}{t} = \frac{\log(\beta)}{\tau}.
\]

The proofs of all these results are postponed to Section 4.

2.4. Examples. We consider representatives example to which Theorem 5 can be applied. For simplicity, we set \( f \) to be identically zero.

Example 11. Suppose \( g \) obeys (2.3) and is non–decreasing, and there exists \( C_1 > 0 \) and \( \alpha \in (0, 1) \) such that \( \lim_{x \to \infty} g(x)/(x \exp((\log x)^{\alpha})) = C_1 \), and \( f(x) = 0 \) for all \( x \geq 0 \). Suppose \( \tau > 0 \) and \( \psi \) obeys (2.4). Then the unique continuous solution \( x \) of (2.5) obeys \( \lim_{t \to \infty} \log x(t)/t^{1/(1-\alpha)} = (\eta(1-\alpha)/\tau)^{1/(1-\alpha)} \).

To see this, we note that \( g \) obeys all the properties of Theorem 7. For \( x > e \) let \( \phi(x) = x \exp((\log x)^{\alpha}) \). Then \( \Gamma(x) = (\log(x)^{\alpha})/(\alpha) - 1 \). By Theorem 7 we have \( \lim_{t \to \infty} \Gamma(x(t))/t = 1/\tau \), which rearranges to give \( \lim_{t \to \infty} \log x(t)/t^{1/(1-\alpha)} = (\eta(1-\alpha)/\tau)^{1/(1-\alpha)} \).

We remark that the results can be applied to equations in which \( g \) grows more rapidly than a polynomial function; here again is a representative example, which was considered without supporting calculations in [4].

Example 12. Suppose \( g \) obeys (2.3) and is non–decreasing, and there exists \( C_1 > 0 \) and \( \alpha > 1 \) such that \( \lim_{x \to \infty} g(x)/(x \exp((\log x)^{\alpha}) = C_1 \), and \( f(x) = 0 \) for all \( x \geq 0 \). Suppose \( \tau > 0 \) and \( \psi \) obeys (2.4). Then the unique continuous solution \( x \) of (2.5) obeys \( \lim_{t \to \infty} \log x(t)/t = \log \alpha/\tau \).

To justify Example 12 set \( \phi(x) = (1 + x) \log(1 + x) \log_2(1 + x) \) for \( x > e^e \). With \( c := \log_2(1 + e^e) \), we have \( \Gamma(x) = (\log(1 + x) - c)/\eta \) and with \( \lambda = e^{\eta(\theta)} \), \( \Gamma^{-1}_\eta(\Gamma_\eta(x) + \theta) = \exp((\log(1 + x))^{\lambda}) - 1 \). Therefore we have \( \lim_{x \to \infty} \phi(\Gamma^{-1}_\eta(\Gamma_\eta(x) + \theta)) = \exp((\log(1 + x))^{\lambda}) \log x^{\lambda} \log_2 x = x \). Define \( \eta(\epsilon) = (1 + \epsilon) \log \alpha/\tau \) and \( \mu(\epsilon) = \log \alpha/(\tau(1 - \epsilon)^2) \). Then

\[
\lim_{x \to \infty} \frac{g(x)}{\phi(\Gamma^{-1}_{\eta(\epsilon)}(\Gamma_{\eta(\epsilon)}(x) + \mu(\epsilon)))} = 0
\]

and

\[
\lim_{x \to \infty} \frac{g(x)}{\phi(\Gamma^{-1}_{\mu(\epsilon)}(\Gamma_{\mu(\epsilon)}(x) + \mu(1 - \epsilon)))} = \infty.
\]

Since \( \eta(\epsilon), \mu(\epsilon) \to \log \alpha/\tau \) as \( \epsilon \to \infty \), from Theorem 5 we have \( \lim_{t \to \infty} \Gamma(x(t))/t = \log \alpha/\tau \), from which the result follows.
3. Preservation of Growth Rates under Discretisation

Let $N \in \mathbb{N}$, and suppose that $h = \tau/N$. Consider the discretisation of (2.5) according to

\begin{align*}
x_h(n + 1) &= x_h(n) + hf(x_h(n)) + hg(x_h(n - N)), \quad n \geq 0; \\
x_h(n) &= \psi(nh), \quad n = -N, \ldots, 0.
\end{align*}

(3.1a, 3.1b)

We also find it of interest to define a continuous time extension of $x_h$. If $(x_n)$ obeys (3.1), define $\bar{x}_h(t) = C([-\tau, \infty), (0, \infty))$ by $\bar{x}_h(t) = \psi(t)$, $t \in [-\tau, 0]$, so $\bar{x}_h$ takes the value $x_n(h)$ at time $nh$ for $n \geq 0$ and interpolates linearly between the values of $(x_n(h))$ at the times $\{0, h, 2h, \ldots\}$. As $h \to 0$, $\bar{x}_h$ approaches $x$ on any compact interval $[0, T]$ in the sense that $\lim_{h \to 0} \sup_{0 \leq t \leq T} |x(t) - \bar{x}_h(t)| = 0$ (see e.g., [6]).

3.1. General discrete comparison results. In this section we simply state our most general comparison results for the discretised equation. Later, we will apply these results to obtain concrete estimates of the growth of solutions of the discretised equation.

**Theorem 13.** Suppose that $f$ obeys (2.7) and (2.8). Let $g$ be non-decreasing and obey (2.3) and let $\psi \in C([-\tau, 0); (0, \infty))$. Suppose that there exists a continuous function $\phi$ such that $\Gamma$, $\Gamma_\epsilon$ are defined by (2.7) and (2.9) respectively, and that $\Gamma$ obeys (2.8). Suppose also that (2.10) and suppose that $f$ obeys (2.11), and that $g$ and $\phi$ obey (2.12) where $\bar{\eta}_\epsilon$ obeys (2.13). Suppose finally that $\phi$ and $f$ are non-decreasing. If $x_h$ is the unique solution of (3.1), then it obeys

$$\limsup_{n \to \infty} \frac{\Gamma(x_h(n))}{nh} \leq \eta.$$  

(3.3)

**Theorem 14.** Suppose that $f$ obeys (2.7) and (2.8). Let $g$ be non-decreasing and obey (2.3) and let $\tau > 0$ and $\psi \in C([-\tau, 0); (0, \infty))$. Suppose that there exists a continuous function $\phi$ such that $\Gamma$, $\Gamma_\epsilon$ are defined by (2.7) and (2.9) respectively, and $\Gamma$ obeys (2.8). Suppose also that (2.10) holds and that $g$ and $\phi$ obey

$$\liminf_{x \to \infty} \frac{g(x)}{\phi(\Gamma^{-1}_\epsilon(\Gamma_\epsilon(x) + (\tau + h)(1 - \epsilon)))} = \mu_\epsilon \in (0, \infty) \quad \text{for every } \epsilon \in (0, 1),$$

(3.4)

where (2.18) also holds. If $x_h$ is the unique solution of (3.1), then

$$\liminf_{n \to \infty} \frac{\Gamma(x_h(n))}{nh} \geq \mu.$$  

(3.5)

3.2. Preservation of growth rate for regularly varying $g$. In [3], it was shown that the uniform Euler scheme (3.1) and the continuous time extension $x_h$ preserves the rate of growth of the underlying continuous equation (2.5) in the case when $g$ is in $RV_\infty(\beta)$ for $\beta \leq 1$, and $g$ is sublinear. We extract here the relevant parts of Theorems 2.4 and 2.5 of [3].

**Theorem 15.** Let $f$ obey (2.7), (2.8). Let $g$ obey (2.3). Let $\tau > 0$ and $\psi \in C([-\tau, 0); (0, \infty))$. Let $\beta \leq 1$ and suppose $g$ is $RV_\infty(\beta)$, and $\lim_{x \to \infty} g(x)/x = 0$. If $\lim_{x \to \infty} f(x)/g(x) = 0$, then the unique solution $x_h$ of (2.7) obeys

$$\lim_{n \to \infty} \frac{1}{nh} \int_1^{x_h(n)} \frac{1}{g(s)} ds = 1.$$  

(3.6)
Moreover, if \( \bar{x}_h \) is the linear interpolant given by (3.2), then
\[
\lim_{t \to \infty} \frac{1}{t} \int_1^{\bar{x}_h(t)} \frac{1}{g(s)} \, ds = 1.
\]

In this paper, we demonstrate that the essential growth rate is preserved for all \( h > 0 \), and that the exact rate of growth is recovered in the limit as \( h \to 0^+ \), in a sense now made precise. We first consider the discrete analogue of Theorem 8.

**Theorem 16.** Let \( f \) obey (2.1), (2.2). Let \( g \) obey (2.3). Let \( \tau > 0 \) and \( \psi \in C([-\tau, 0]; (0, \infty)) \). Suppose \( g \) is asymptotic to a non-decreasing function, \( \lim_{x \to \infty} g(x)/x = \infty \), and \( \lim_{x \to \infty} f(x)/x = 0 \). If \( G \) is defined by (2.24), then the unique solution \( x_h \) of (3.1) obeys
\[
\lim_{n \to \infty} \frac{G(x_h(n))}{nh} = \frac{1}{\tau + h}.
\]

Moreover, if \( \bar{x}_h \) is the linear interpolant given by (3.2), then
\[
\lim_{t \to \infty} \frac{G(\bar{x}_h(t))}{t} = \frac{1}{\tau + h}.
\]

The proof is postponed to the final section. By comparing (2.25) and (3.8), it can be seen that the essential growth rate is recovered by the linear interpolant for all \( h > 0 \), and the exact rate is recovered in the limit as \( h \to 0^+ \).

The rate of growth is also recovered in the same manner in the case when \( g \) grows polynomially at a superlinear rate, as confirmed by the following discrete analogue of Theorem 10.

**Theorem 17.** Let \( f \) obey (2.1), (2.2). Let \( g \) obey (2.3). Let \( \tau > 0 \) and \( \psi \in C([-\tau, 0]; (0, \infty)) \). Suppose that there exists \( \beta > 1 \) such that \( g \) obeys
\[
\lim_{x \to \infty} \frac{\log g(x)}{\log x} = \beta,
\]
and \( \lim_{x \to \infty} f(x)/x = 0 \). Then the unique solution \( x_h \) of (3.1) obeys
\[
\lim_{n \to \infty} \frac{\log_2 x_h(n)}{nh} = \frac{\log \beta}{\tau + h}.
\]

Moreover, if \( \bar{x}_h \) is the linear interpolant given by (3.2), then
\[
\lim_{t \to \infty} \frac{\log_2 \bar{x}_h(t)}{t} = \frac{\log \beta}{\tau + h}.
\]

Once again, by comparing (2.23) and (3.10), we see that the essential growth rate is recovered by the linear interpolant for all \( h > 0 \), and the exact rate is recovered in the limit as \( h \to 0^+ \). Again, we relegate the proof to the end.

### 4. Proof of Main Continuous–Time Results

In this section, we give the proofs of the main results from Section 2, with the exception of Theorem 8, whose proof is strongly based on that of Theorem 16. The proofs of these two results, along with Theorem 17 are given in Section 5.

#### 4.1. Proof of Theorem 11

Suppose that \( x \) has a finite interval of existence. Then there is a unique continuous solution of (2.25) on \([-\tau, T]\) where \( T \in (0, \infty) \) is such that
\[
\lim_{t \to T^-} x(t) = \infty.
\]
The limit is \( +\infty \) because the positivity of the initial condition, together with the non-negativity of \( f \) and \( g \) ensure that \( x'(t) \geq 0 \) for \( t \in [0, T) \).
We wish to rule out the possibility that $T < +\infty$. Suppose that $T \in (0, \tau]$. Clearly, if $g_1 = \max_{x \in [-\tau, 0]} g(x(s)) \geq 0$, we have
\[ x'(t) \leq f(x(t)) + g_1, \quad t \in [0, T). \]
Define $f_1(x) := f(x) + g_1$ for $x \geq 0$. Then, as $x(t) \to \infty$ as $t \to T^-$, we have
\[ \int_{x(0)}^{\infty} \frac{1}{f_1(x)} \, dx = \lim_{t \to T^-} \int_{0}^{t} \frac{x'(s)}{f_1(x(s))} \, ds \leq T. \]
However, (2.12) implies that $\int_{x(0)}^{\infty} 1/f_1(u) \, du = \infty$, which gives a contradiction. Hence $T > \tau$.
Suppose now that $x$ does not explode in $[0, n\tau]$, but does in $(n\tau, (n+1)\tau]$. This is true for $n = 1$. Clearly, if $g_n = \max_{x \in [(n-1)\tau, n\tau]} g(x(s)) \geq 0$, we have
\[ x'(t) \leq f(x(t)) + g_n, \quad t \in [n\tau, T). \]
Define $f_n(x) := f(x) + g_n$ for $x \geq 0$. Then, as $x(t) \to \infty$ as $t \to T^-$, we have
\[ \int_{x(n\tau)}^{\infty} \frac{1}{f_n(x)} \, dx = \lim_{t \to T^-} \int_{n\tau}^{t} \frac{x'(s)}{f_n(x(s))} \, ds \leq T - n\tau. \]
However, (2.12) implies that $\int_{x(n\tau)}^{\infty} 1/f_n(u) \, du = \infty$, which gives a contradiction. Hence $T > (n+1)\tau$. Since this is true for any $n \in \mathbb{N}$, it follows that $T = \infty$.
We have shown that (2.5) has interval of existence $[-\tau, \infty)$. Since $\psi(t) > 0$ for $t \in [-\tau, 0]$ and $f(x) \geq 0, g(x) \geq 0$ for all $x \geq 0$, we have that $x'(t) \geq 0$ for all $t \geq 0$. Therefore $x(t) = x(0) + \int_{0}^{t} f(x(s)) \, ds + \int_{0}^{t-\tau} g(x(s)) \, ds, \quad t \geq \tau,$
by the continuity of $f$ and $g$ we have
\[ \lim_{t \to \infty} \frac{1}{t} \int_{0}^{t} f(x(s)) \, ds = f(L), \quad \lim_{t \to \infty} \frac{1}{t} \int_{0}^{t-\tau} g(x(s)) \, ds = g(L). \]
Since $x(t)$ tends to the finite limit $L$, we get
\[ 0 = \lim_{t \to \infty} \frac{x(t)}{t} = \lim_{t \to \infty} \frac{\psi(0)}{t} + \frac{1}{t} \int_{0}^{t} f(x(s)) \, ds + \frac{1}{t} \int_{0}^{t-\tau} g(x(s)) \, ds = f(L) + g(L). \]
Since $g$ is positive and $f$ is nonnegative, we have $L = 0$, a contradiction. Hence $x$ obeys (2.6), as claimed.

4.2. Proof of Theorem 3 By (2.12) for every $\epsilon \in (0, 1)$ there exists $x_2(\epsilon) > 0$ such that for $x > x_2(\epsilon)$ we have
\[ g(x) < (\bar{\eta} + \epsilon)\phi(\Gamma_{\bar{\eta} + \epsilon}^{-1}(\Gamma_{\bar{\eta}}^1(x + \tau))), \quad x > x_2(\epsilon). \]
where the last inequality is a consequence of (2.13). Since $\bar{\eta} < \eta = \lim_{\epsilon \to 0^+} \eta(\epsilon)$, there exists $\epsilon' \in (0, 1)$ such that for $\epsilon < \epsilon'$, we have $\eta(\epsilon) > \bar{\eta} + \epsilon$. Thus for all $\epsilon < \epsilon' < 1$ we have
\[ g(x) < \eta(\epsilon)\phi(\Gamma_{\eta(\epsilon)}^{-1}(\Gamma_{\eta(\epsilon)}^1(x + \tau))), \quad x > x_2(\epsilon). \]

By (2.15) for every $\epsilon \in (0, 1)$ there exists an $x_1(\epsilon) > 0$ such that
\[ f(x) \leq c\eta(\epsilon)\phi(x), \quad x > x_1(\epsilon). \]
Define
\[ c(\epsilon) = \Gamma_{\eta(\epsilon)}(\psi^* + x_1(\epsilon) + x_2(\epsilon)) + (1 + \epsilon)\tau, \]
and define also
\[ x_\epsilon(t) = \Gamma_{\eta(\epsilon)}((1 + \epsilon)t + c(\epsilon)), \quad t \geq -\tau. \]
This function is well-defined since $c(\epsilon) > \Gamma_{\eta(\epsilon)}(\psi^*) + (1 + \epsilon)\tau$, so $c(\epsilon) - (1 + \epsilon)\tau > \Gamma_{\eta(\epsilon)}(\psi^*)$, or $x_\epsilon(t) > \psi^*$ for all $t \in [-\tau, 0]$. Since $c(\epsilon) > \Gamma_{\eta(\epsilon)}(x_1(\epsilon)) + (1 + \epsilon)\tau$ and $\Gamma_{\eta(\epsilon)}$ is increasing, $\Gamma_{\eta(\epsilon)}(c(\epsilon) - (1 + \epsilon)\tau) > x_1(\epsilon)$, so $x_\epsilon(t) > x_1(\epsilon)$ for all $t \geq -\tau$. Therefore by (4.2),

$$f(x_\epsilon(t)) \leq c(\epsilon)\phi(x_\epsilon(t)).$$

Also for $t \geq 0$, we have

$$g(x_\epsilon(t) - \tau) = g(\Gamma_{\eta(\epsilon)}^{-1}((1 + \epsilon)(t - \tau) + c(\epsilon))) = g(\Gamma_{\eta(\epsilon)}^{-1}((1 + \epsilon)t - \epsilon\tau + c(\epsilon)))$$

$$< g(\Gamma_{\eta(\epsilon)}^{-1}((1 + \epsilon)t - \tau + c(\epsilon))).$$

Now, because $c(\epsilon) > \Gamma_{\eta(\epsilon)}(x_2(\epsilon)) + \tau$, we have that the argument of $g$ on the righthand side exceeds $x_2(\epsilon)$ for all $t \geq 0$. Therefore by (4.1), we have

$$g(x_\epsilon(t) - \tau) < g(\Gamma_{\eta(\epsilon)}^{-1}((1 + \epsilon)t - \tau + c(\epsilon))).$$

Hence for $t \geq 0$

$$f(x_\epsilon(t)) + g(x_\epsilon(t) - \tau) < (1 + \epsilon)c(\epsilon)\phi(x_\epsilon(t)).$$

(4.5)

Now for $t > 0$, $\Gamma_{\eta(\epsilon)}(x_\epsilon(t)) = (1 + \epsilon)t + c(\epsilon)$, so $\Gamma_{\eta(\epsilon)}(x_\epsilon(t))x'_\epsilon(t) = (1 + \epsilon)$, or $x'_\epsilon(t) = (1 + \epsilon)\eta(\epsilon)\phi(x_\epsilon(t))$. Hence

$$x'_\epsilon(t) = (1 + \epsilon)\eta(\epsilon)\phi(\Gamma_{\eta(\epsilon)}^{-1}((1 + \epsilon)t + c(\epsilon))), \quad t > 0.$$  

(4.6)

Thus by (4.5) and (4.6) for $t > 0$ we have $x'_\epsilon(t) > f(x_\epsilon(t)) + g(x_\epsilon(t) - \tau)$. Now as $x_\epsilon(t) > \psi^* = \max_{s \in [-\tau, 0]} \psi(s)$, we have $x_\epsilon(t) > x(t)$ for $t \in [-\tau, 0]$ and $x'_\epsilon(t) > f(x_\epsilon(t)) + g(x_\epsilon(t) - \tau)$ for $t \geq 0$. Suppose that there is a $t_0 > 0$ such that $x_\epsilon(t) > x(t)$ for $t \in [-\tau, t_0]$. Then $x'_\epsilon(t_0) \leq x'_\epsilon(t_0)$. Then as $g$ is non-decreasing,

$$x'_\epsilon(t_0) \leq x'_\epsilon(t_0) = f(x(t_0)) + g(x(t_0) - \tau)$$

$$= f(x(t_0)) + g(x(t_0) - \tau) \leq f(x(t_0)) + g(x(t_0))$$

$$< x'_\epsilon(t_0),$$

a contradiction. Thus $x_\epsilon(t) > x(t)$ for all $t \geq -\tau$. Hence $\Gamma_{\eta(\epsilon)}(x(t)) < \Gamma_{\eta(\epsilon)}(x_\epsilon(t))$ for all $t \geq -\tau$. Hence

$$\Gamma_{\eta(\epsilon)}(x(t)) \leq \Gamma_{\eta(\epsilon)}(x_\epsilon(t)) = (1 + \epsilon)t + c(\epsilon), \quad t \geq -\tau.$$  

But $\Gamma(x(t)) = \eta(\epsilon)\Gamma_{\eta(\epsilon)}(x(t)) < (1 + \epsilon)\eta(\epsilon) + \eta(\epsilon)c(\epsilon)$. Therefore

$$\lim_{t \to \infty} \Gamma(x(t))/t \leq (1 + \epsilon)\eta(\epsilon).$$

Since $\epsilon > 0$ is arbitrary, and $\eta(\epsilon) \to \eta$ as $\epsilon \to 0$, we have (2.14).

4.3. Proof of Theorem 6 Suppose that $\phi(x) = g(x)$ for $x > 0$. Thus $\Gamma(x) = \eta^{-1} \int_0^x du/(u \phi(u))$. Let $z(t) = \Gamma_{\eta(t)}^{-1}(t)$ for $t \geq 0$. Then $z'(t) = \eta g(z(t))$ for $t > 0$ with $z(0) = \psi^*$. Thus $z'(t)/z(t) \to 0$ as $t \to \infty$. Therefore

$$\log \left( \frac{z(t)}{z(t) - \theta} \right) = \int_{t-\theta}^t \frac{z'(s)}{z(s)} ds \to 0 \quad \text{as} \quad t \to \infty,$$

so $\lim_{t \to \infty} z(t - \theta)/z(t) = 1$ for any $\theta \in \mathbb{R}$. Since $g \in \mathcal{RV}_\infty(\beta)$, we have

$$\lim_{t \to \infty} g(z(t - \theta))/g(z(t)) = 1.$$
Hence \( \lim_{t \to \infty} g(\Gamma^{-1}_\eta(t - \theta))/g(\Gamma^{-1}_\eta(t)) = 1 \). Since \( \Gamma^{-1}_\eta(t) \to \infty \) as \( t \to \infty \), we have

\[
\lim_{x \to \infty} \frac{g(x)}{\phi(\Gamma^{-1}_\eta(\Gamma(x) + \theta))} = \lim_{x \to \infty} \frac{g(x)}{g(\Gamma^{-1}_\eta(\Gamma(x) + \theta))} = 1. 
\]

(4.7)

Since this holds for every \( \eta > 0 \) and \( \theta \in \mathbb{R} \) it follows that (2.12) and (2.11) hold with \( \bar{\eta}_\theta = \bar{\mu}_\epsilon = 1 \). Let \( \rho \in (0, 1) \). Define \( \mu(\epsilon) = 1 - \rho \) and \( \eta(\epsilon) = 1 + \rho \). Then with \( \eta = 1 + \rho \) and \( \mu = 1 - \rho \), (2.10), (2.10), (2.15) and (2.18) hold. To prove (2.16), we note that

\[
\lim_{x \to \infty} \frac{f(x)}{\phi(x)} = \lim_{x \to \infty} \frac{f(x)}{g(x)} = 0.
\]

Since all the hypotheses of Theorems 2 and 4 hold, we have \( \limsup_{t \to \infty} \Gamma(x(t))/t \leq 1 + \rho \) and \( \liminf_{t \to \infty} \Gamma(x(t))/t \geq 1 - \rho \). Letting \( \rho \to 0 \), we have \( \lim_{t \to \infty} \Gamma(x(t))/t = 1 \), whence the result.

4.4. Proof of Theorem 7. Since \( g \in R^\infty(1) \), it follows that there exists an increasing and continuously differentiable function \( \delta : [\psi^*, \infty) \to (0, \infty) \) with \( \delta(\psi^*) > ev^* \) such that \( \delta(x)/g(x) \to 1 \) as \( x \to \infty \) and \( x\delta'(x)/\delta(x) \to 1 \) as \( x \to \infty \).

Define \( \phi(x) = x \log(\delta(x)/x) \) for \( x \geq \psi^* \). Define \( \Gamma(x) = \int^x_{\psi^*} du/\phi(u) \) for \( x \geq \psi^* \). Since \( (g(x)/x)/\delta(x)/x \to 1 \) as \( x \to \infty \), we have \( \log(g(x)/x)/\log(\delta(x)/x) \to 1 \) as \( x \to \infty \). Therefore by L'Hôpital's rule, we have \( \Gamma(x)/G(x) \to 1 \) as \( x \to \infty \).

Define \( \Gamma_\eta(x) = \Gamma(x)/\eta \) and \( \delta_1(x) = \delta(x)/x \) for \( x \geq \psi^* \). Since \( x\delta'(x)/\delta(x) \to 1 \) as \( x \to \infty \), we have that \( \delta_1 \) is continuously differentiable and \( x\delta'(x)/\delta(x) \to 0 \) as \( x \to \infty \). Define \( y(t) = \Gamma^{-1}_\eta(t) \) for \( t \geq 0 \) and \( y(t) = \log(\delta_1(y(t))) \). Then \( y'(t) = \eta\delta(\eta(t)) = \eta y(t)\log(\delta_1(y(t))) = \eta y(t)\). Moreover since \( \Gamma_\eta(x) \to \infty \) as \( x \to \infty \), we have that \( y(t) \to \infty \) as \( t \to \infty \). Thus

\[
\lim_{x \to \infty} \frac{g(x)}{\phi(\Gamma^{-1}_\eta(\Gamma_\eta(x) + \theta))} = \lim_{x \to \infty} \frac{\delta(x)}{\phi(\Gamma^{-1}_\eta(\Gamma_\eta(x) + \theta))} = \lim_{t \to \infty} \frac{\delta(\Gamma^{-1}_\eta(t - \theta))}{\Gamma^{-1}_\eta(t) \log(\delta(\Gamma^{-1}_\eta(t)))}. 
\]

and therefore we have

\[
\lim_{x \to \infty} \frac{g(x)}{\phi(\Gamma^{-1}_\eta(\Gamma_\eta(x) + \theta))} = \lim_{t \to \infty} \frac{y(t - \theta)\delta_1(y(t - \theta))}{y(t)\log(\delta_1(y(t)))}.
\]

Since \( \log((y(t)/y(t - \theta)) = \int_{t-\theta}^t y'(s)/y(s) \, ds = \int_{t-\theta}^t \eta u(s) \, ds \). Hence

\[
\log \left( \lim_{x \to \infty} \frac{g(x)}{\phi(\Gamma^{-1}_\eta(\Gamma_\eta(x) + \theta))} \right) = \lim_{t \to \infty} \left\{ -\log(y(t)/y(t - \theta)) + u(t - \theta) - \log(u(t)) \right\}.
\]

Since \( \delta_1, y \) are continuously differentiable, so is \( u \), and we have

\[
u'(t) = \delta_1(y(t))y'(t)/\delta_1(y(t)) = \eta u(t) \cdot y(t) \delta_1'(y(t))/\delta_1(y(t)). \]

Since \( x\delta'(x)/\delta(x) \to 0 \) as \( x \to \infty \) and \( y(t) \to \infty \) as \( t \to \infty \), we have \( u'(t)/u(t) \to 0 \) as \( t \to \infty \). Also we have \( u(t) \to \infty \) as \( t \to \infty \). Therefore \( u(t - \theta)/u(t) \to 1 \) as \( t \to \infty \) and

\[
\lim_{t \to \infty} \int_{t-\theta}^t u(s) \, ds/u(t) = \theta,
\]

so

\[
\lim_{t \to \infty} \left\{ -\eta \frac{1}{u(t)} \int_{t-\theta}^t u(s) \, ds + \frac{u(t - \theta)}{u(t)} - \log(u(t)) \right\} = 1 - \eta \theta.
\]
Therefore we have
\[ \lim_{x \to \infty} \frac{g(x)}{\phi(\Gamma_{\eta}^{-1}(\Gamma_{\eta}(x) + \theta))} = \begin{cases} -\infty & \text{if } 1 - \eta \theta < 0 \\ +\infty & \text{if } 1 - \eta \theta > 0. \end{cases} \]

Therefore, with \( \eta(\epsilon) = (1 + \epsilon) / \tau \) and \( \mu(\epsilon) = (1 - \epsilon) / \tau \), we have
\[ \lim_{x \to \infty} \frac{g(z)}{\phi(\Gamma_{\mu(\epsilon)}^{-1}(\Gamma_{\mu(\epsilon)}(x) + \tau(1 - \epsilon))))} = 0, \quad \lim_{x \to \infty} \frac{g(x)}{\phi(\Gamma_{\mu(\epsilon)}^{-1}(\Gamma_{\mu(\epsilon)}(x) + \tau(1 - \epsilon))))} = \infty. \]

Since \( \mu(\epsilon), \eta(\epsilon) \to 1 / \tau \) as \( \epsilon \to 0 \), and we have \( \bar{\eta} = 0 =: \bar{\eta} < 1 / \tau \) and \( \bar{\mu} = +\infty =: \bar{\mu} > 1 / \tau \).

Next, note that (2.23) implies
\[ \lim_{x \to \infty} \frac{f(x)}{\phi(x)} = \lim_{x \to \infty} \frac{f(x)}{x \log(x)} = \lim_{x \to \infty} \frac{f(x)}{x \log(g(x)/x)} = 0. \]

Therefore by Theorem 5 we have \( \lim_{t \to \infty} \Gamma(x(t))/t = 1 / \tau \), and due to the fact that \( \lim_{t \to \infty} G(x(t))/\Gamma(x(t)) = 1 \), we get \( \lim_{t \to \infty} G(x(t))/t = 1 / \tau \), as required.

4.5. Proof of Theorem 9. Set \( \phi(x) = x \) for \( x \geq \psi^* \). Then \( \Gamma_{\eta}(x) = \eta^{-1} \log(x/\psi^*) \), \( \Gamma_{\eta}^{-1}(x) = \psi^* e^{\eta x} \), and \( \phi(\Gamma^{-1}_{\eta}(\Gamma_{\eta}(x) + \theta)) = x e^{\eta \theta} \). Thus \( \lim_{x \to \infty} \frac{g(x)}{\phi(\Gamma_{\eta}^{-1}(\Gamma_{\eta}(x) + \theta))} = \frac{1}{\eta} \). Set \( c(\nu) := \nu - Ce^{-\nu \tau} \). Then \( c(\nu) \) is increasing on \([0, \infty)\) and there is a unique \( \lambda > 0 \) such that \( c(\lambda) = 0 \), or \( \lambda = Ce^{-\lambda \tau} \). Let \( \sigma \in \mathbb{R} \) and \( \lambda_\sigma := \lambda(1 + \sigma) \).

For \( \sigma > 0 \), \( c(\lambda_\sigma) > 0 \) or \( \lambda_\sigma > Ce^{-\lambda \sigma \tau} \). Similarly, \( \lambda_\sigma < Ce^{-\lambda \sigma \tau} \). Define \( \eta(\epsilon) = \lambda_\sigma(1 + \epsilon) \). Then \( \eta(\epsilon) \to \lambda_\sigma =: \eta \) as \( \epsilon \to 0 \). Also \( \lim_{x \to \infty} \frac{g(x)}{\phi(\Gamma^{-1}_{\eta}(\Gamma_{\eta}(x) + \tau))} = Ce^{-\lambda_\sigma(1+\epsilon) \tau} =: \tilde{\eta}_\epsilon \). Then \( \sup_{x \in (0,1)} \tilde{\eta}_\epsilon = Ce^{-\lambda \sigma \tau} =: \tilde{\eta} \). But \( \eta = Ce^{-\lambda \sigma \tau} < \lambda_\sigma = \eta \). Finally, \( \tilde{f}(x)/\phi(x) = \tilde{f}(x)/x \to 0 \) as \( x \to \infty \), and so by Theorem 2 \( \limsup_{x \to \infty} \frac{\Gamma(x(t))/t \leq \lambda_\sigma \tau \leq \limsup_{x \to \infty} \log(x(t)/t \leq \lambda(1+\sigma)) \). Letting \( \sigma \to 0 \), \( \limsup_{x \to \infty} \log(x(t)/t \leq \lambda \). Define \( \mu(\epsilon) = \lambda_\sigma(1 - \epsilon) \). Then \( \lim_{x \to \infty} \mu(\epsilon) = \lambda_\sigma =: \mu \). Also \( \lim_{x \to \infty} \frac{g(x)}{\phi(\Gamma^{-1}_{\eta}(\Gamma_{\eta}(x) + \tau(1 - \epsilon)))} = Ce^{-\lambda_\sigma(1-\epsilon) \tau} =: \tilde{\mu}_\epsilon \). Then \( \inf_{x \in (0,1)} \tilde{\mu}_\epsilon = Ce^{-\lambda_\sigma \tau} =: \bar{\mu} \). Thus \( \bar{\mu} = Ce^{-\lambda_\sigma \tau} > \lambda_\sigma = \mu \). Thus by Theorem 3 \( \liminf_{t \to \infty} \Gamma(x(t))/t \geq \lambda_\sigma \), or \( \liminf_{t \to \infty} \log(x(t)/t \geq \lambda(1 - \sigma)) \). Letting \( \sigma \to 0 \), \( \liminf_{t \to \infty} \log(x(t)/t \geq \lambda \), whence the result.

4.6. Proof of Theorem 10 Define \( \phi(x) = (1 + x) \log(1 + x) \) for \( x \geq \psi^* \). Hence for \( \eta > 0 \) we have
\[ \Gamma_{\eta}(x) = \frac{1}{\eta} \log \left( \frac{\log(1 + x)}{\log(1 + \psi^*)} \right), \quad \Gamma_{\eta}^{-1}(x) = \exp(\log(1 + \psi^*) e^{\eta x}) - 1. \]

Thus \( \phi(\Gamma_{\eta}^{-1}(\Gamma_{\eta}(x) + \theta)) = e^{\eta \theta} (1 + x) e^{\eta x} \log(1 + x) \). Also \( \Gamma_{\eta}^{-1}(\Gamma_{\eta}(x) - \tau) = (1 + x) e^{-\eta \tau} - 1 \). Therefore
\[ \lim_{x \to \infty} \frac{g(x)}{\phi(\Gamma_{\eta}^{-1}(\Gamma_{\eta}(x) + \theta))} = e^{-\eta \theta} \lim_{x \to \infty} \frac{g(x)}{x (1 + x) e^{\eta x} \log(1 + x)} = \frac{g(x)}{x (1 + x) e^{\eta x} \log(1 + x)} = \lim_{x \to \infty} \frac{f(x)}{f(x)} \cdot \frac{\log(x)}{x \log(g(x)/x)} = 0. \]

Next, \( \eta(\epsilon) := \epsilon + \log(\beta)/\tau \). Then \( \lim_{\epsilon \to 0} \eta(\epsilon) = \log(\beta)/\tau =: \eta \), and so
\[ \lim_{x \to \infty} \frac{g(x)}{\phi(\Gamma_{\eta}(x(\psi(x) + \beta))))} = 0. \]

Therefore \( \bar{\eta} = 0, \bar{\mu} = 0 < \log(\beta)/\tau = \eta \). Next, as \( f(x)/x \log(x) \to 0 \) as \( x \to \infty \), we have
\[ \lim_{x \to \infty} \frac{f(x)}{\phi(x)} = \frac{f(x)}{(1 + x) \log(1 + x)} = 0. \]

By Theorem 3
\[ \limsup_{t \to \infty} \Gamma(x(t))/t \leq \eta, \]
or equivalently \( \limsup_{t \to \infty} \log \log x(t)/t \leq \log(\beta)/\tau \). We now obtain a lower bound. Define \( \mu(\epsilon) = \log(\beta)/\tau \) for \( \epsilon > 0 \). Then

\[
\lim_{x \to \infty} \frac{g(x)}{\phi(\Gamma^{-1}_{\mu(\epsilon)}(\Gamma_{\mu(\epsilon)}(x) + \tau(1 - \epsilon)))} = \beta^{-(1 - \epsilon)} \lim_{x \to \infty} \frac{g(x)}{(1 + x)^{\beta^{1 - \epsilon}} \log(1 + x)}.
\]

Therefore

\[
\lim_{x \to \infty} \frac{g(x)}{\phi(\Gamma^{-1}_{\mu(\epsilon)}(\Gamma_{\mu(\epsilon)}(x) + \tau(1 - \epsilon)))} = \infty
\]

, so \( \bar{\mu}_e = +\infty = \bar{\mu} = \log(\beta)/\tau \). By Theorem 4, \( \liminf_{t \to \infty} \Gamma(x(t))/t \geq \mu \), or \( \liminf_{t \to \infty} \log \log x(t)/t \geq \log \beta/\tau \), which proves (2.27).

5. Proof of Main Discrete-Time Results

In this section, we give the proofs of results from Section 3. We also give the proof of Theorem 8, which is greatly facilitated by the proof of Theorem 16.

5.1. Proof of Theorem 13. By (2.12) for every \( \epsilon \in (0, 1) \) there exists \( x_2(\epsilon) > 0 \) such that for \( x > x_2(\epsilon) \) we have

\[
g(x) < (\bar{\eta} + \epsilon)\phi(\Gamma^{-1}_{\eta(\epsilon)}(\Gamma_{\eta(\epsilon)}(x) + \tau)) \leq (\bar{\eta} + \epsilon)\phi(\Gamma^{-1}_{\eta(\epsilon)}(\Gamma_{\eta(\epsilon)}(x) + \tau)),
\]

where the last inequality is a consequence of (2.13). Since \( \bar{\eta} < \eta = \lim_{x \to 0^+} \eta(\epsilon) \), there exists \( \epsilon' \in (0, 1) \) such that for \( \epsilon < \epsilon' \), we have \( \eta(\epsilon) > \bar{\eta} + \epsilon \). Thus for all \( \epsilon < \epsilon' < 1 \) we have

\[
g(x) < \eta(\epsilon)\phi(\Gamma^{-1}_{\eta(\epsilon)}(\Gamma_{\eta(\epsilon)}(x) + \tau)), \quad x > x_2(\epsilon).
\] (5.1)

By (2.15) for every \( \epsilon \in (0, 1) \) there exists an \( x_1(\epsilon) > 0 \) such that

\[
f(x) \leq \epsilon\eta(\epsilon)\phi(x), \quad x > x_1(\epsilon).
\] (5.2)

Define

\[
c(\epsilon) = \Gamma_{\eta(\epsilon)}(\psi^* + x_1(\epsilon) + x_2(\epsilon)) + (1 + \epsilon)\tau,
\] (5.3)

and define also

\[
x_\epsilon(n) = \Gamma^{-1}_{\eta(\epsilon)}((1 + \epsilon)n\tau + c(\epsilon)), \quad n \geq -N.
\] (5.4)

This function is well-defined since \( c(\epsilon) > \Gamma_{\eta(\epsilon)}(\psi^*) + (1 + \epsilon)\tau \), so \( c(\epsilon) - (1 + \epsilon)\tau > \Gamma_{\eta(\epsilon)}(\psi^*), \) or \( x_{\epsilon}(n) > \psi^* \) for all \( n \in \{-N, \ldots, 0\} \). Since \( c(\epsilon) > \Gamma_{\eta(\epsilon)}(x_1(\epsilon)) + (1 + \epsilon)\tau \) and \( \Gamma_{\eta(\epsilon)} \) is increasing, \( \Gamma^{-1}_{\eta(\epsilon)}(c(\epsilon) - (1 + \epsilon)\tau) > x_1(\epsilon) \), so \( x_{\epsilon}(n) > x_1(\epsilon) \) for all \( n \geq -N \). Therefore by (5.2), \( f(x_{\epsilon}(n)) \leq \epsilon\eta(\epsilon)\phi(x_{\epsilon}(n)) \) for \( n \geq 0 \). Also for \( n \geq 0 \), we have

\[
g(x_{\epsilon}(n - N)) = g(\Gamma^{-1}_{\eta(\epsilon)}((1 + \epsilon)(n - N)\tau + c(\epsilon)))
\]

\[
\leq g(\Gamma^{-1}_{\eta(\epsilon)}((1 + \epsilon)n\tau - \tau + c(\epsilon)))
\]

\[
< g(\Gamma^{-1}_{\eta(\epsilon)}((1 + \epsilon)nh - \tau + c(\epsilon))).
\]

Now, because \( c(\epsilon) > \Gamma_{\eta(\epsilon)}(x_2(\epsilon)) + \tau \), we have that the argument of \( g \) on the righthand side exceeds \( x_2(\epsilon) \) for all \( t \geq 0 \). Therefore by (5.1), we have

\[
g(x_{\epsilon}(n - N)) < g(\Gamma^{-1}_{\eta(\epsilon)}((1 + \epsilon)n\tau - \tau + c(\epsilon)))
\]

\[
< \eta(\epsilon)\phi(\Gamma^{-1}_{\eta(\epsilon)}(\Gamma_{\eta(\epsilon)}(\Gamma^{-1}_{\eta(\epsilon)}((1 + \epsilon)nh - \tau + c(\epsilon)) + \tau))
\]

\[
= \eta(\epsilon)\phi(\Gamma^{-1}_{\eta(\epsilon)}((1 + \epsilon)nh - \tau + c(\epsilon)) + \tau)
\]

\[
= \eta(\epsilon)\phi(\Gamma^{-1}_{\eta(\epsilon)}((1 + \epsilon)nh + c(\epsilon))
\]

\[
= \eta(\epsilon)\phi(x_{\epsilon}(n)).
\]

Hence

\[
f(x_{\epsilon}(n)) + g(x_{\epsilon}(n - N)) < (1 + \epsilon)\eta(\epsilon)\phi(x_{\epsilon}(n)), \quad n \geq 0.
\] (5.5)
Now for $n \geq 0$, $\Gamma_{\eta}(x_{n+1}(n)) = (1 + \epsilon)n h + c(\epsilon)$, so
\[
\Gamma_{\eta}(x_{n+1}) - \Gamma_{\eta}(x_{n}) = (1 + \epsilon)h.
\]
Since $\Gamma_{\eta}$ is in $C^{1}$ and $(x_{n}(n))_{n \geq 0}$ is an increasing sequence, there exists $\xi(n) \in [x_{n}(n), x_{n}(n+1)]$ such that
\[
\Gamma_{\eta}(x_{n+1}(n)) = \Gamma_{\eta}(x_{n}(n)) + \Gamma'_{\eta}(\xi(n))(x_{n+1}(n) - x_{n}(n)).
\]
Therefore we have
\[
(1 + \epsilon)h = \Gamma'_{\eta}(\xi(n))(x_{n+1}(n) - x_{n}(n)) = \frac{1}{\eta(\epsilon)}\frac{1}{\phi(\xi(n))}(x_{n+1}(n) - x_{n}(n)).
\]
Thus as $\phi$ is non-decreasing, as $\xi(n) \geq x_{n}(n)$, we have
\[
x_{n+1}(n) = x_{n}(n) + (1 + \epsilon)\eta(\epsilon)\phi(\xi(n)) = x_{n}(n) + (1 + \epsilon)\eta(\epsilon)\phi(x_{n}(n)), \quad n \geq 0.
\]
Thus by (5.5) and (5.6) for $n \geq 0$ we have
\[
x_{n+1}(n) \geq x_{n}(n) + (1 + \epsilon)\eta(\epsilon)\phi(x_{n}(n)) > x_{n}(n) + hf(x_{n}(n)) + hg(x_{n}(n) - N).
\]
Now as $x_{n}(n) > \psi^* = \max_{n \in \{-N, \ldots, 0\}} \psi(nh)$, we have $x_{n}(n) > x_{h}(n)$ for $n \in \{N, \ldots, 0\}$.

Suppose that there is a $n_{0} \geq 1$ such that $x_{n}(n) > x_{h}(n)$ for $t \in \{-N, \ldots, n_{0} - 1\}$
\[
x_{n}(n_{0}) \leq x_{h}(n_{0}).
\]
Therefore $x_{n}(n_{0}) - x_{n}(n_{0} - 1) \leq x_{h}(n_{0}) - x_{n}(n_{0} - 1)$. Since $f$ and $g$ are non-decreasing,
\[
x_{n}(n_{0}) - x_{n}(n_{0} - 1) \leq x_{h}(n_{0}) - x_{n}(n_{0} - 1)
\]
\[
= hf(x_{n}(n_{0} - 1)) + hg(x_{n}(n_{0} - 1) - N))
\]
\[
\leq hf(x_{n}(n_{0} - 1)) + hg(x_{n}(n_{0} - N))
\]
\[
\leq hf(x_{n}(n_{0} - 1)) + hg(x_{n}(n_{0} - 1) - N))
\]
\[
< x_{n}(n_{0}) - x_{n}(n_{0} - 1),
\]
a contradiction.

Thus $x_{n}(n) > x_{h}(n)$ for all $n \geq -N$. Hence $\Gamma_{\eta}(x_{h}(n)) < \Gamma_{\eta}(x_{n}(n))$ for all $n \geq -N$. Hence
\[
\Gamma_{\eta}(x_{h}(n)) < \Gamma_{\eta}(x_{n}(n)) = (1 + \epsilon)n h + c(\epsilon), \quad n \geq -N.
\]
But $\Gamma(x_{h}(n)) = \eta(\epsilon)\Gamma_{\eta}(x_{h}(n)) < (1 + \epsilon)\eta(\epsilon)n h + \eta(\epsilon)c(\epsilon)$. Therefore
\[
\lim_{n \to \infty} \frac{\Gamma(x_{h}(n))}{nh} \leq (1 + \epsilon)\eta(\epsilon).
\]
Since $\epsilon > 0$ is arbitrary, and $\eta(\epsilon) \to \eta$ as $\epsilon \to 0$, we have (3.3).

5.2. Proof of Theorem 14. Suppose first that $\bar{\mu}$ is finite. Then by (2.17) for every $\epsilon \in (0, 1)$ there exists $x_{3}(\epsilon) > 0$ such that for $x \geq x_{3}(\epsilon)$
\[
g(x) > \bar{\mu}(1 - \epsilon)\phi(\Gamma_{\bar{\mu}}(\Gamma_{\mu}(x) + (\tau + h)(1 - \epsilon)))
\]
\[
\geq \bar{\mu}(1 - \epsilon)\phi(\Gamma_{\bar{\mu}}^{-1}(\Gamma_{\mu}(x) + (\tau + h)(1 - \epsilon)))
\]
\[
> \mu(\epsilon)\phi(\Gamma_{\mu}(x) + (\tau + h)(1 - \epsilon))),
\]
where the penultimate inequality is a consequence of (2.18), and the last inequality holds for all $\epsilon < \epsilon'$, because for such $\epsilon$ we have $\mu(\epsilon) < (1 - \epsilon)\bar{\mu}$. This holds for the following reason.

By (2.16), there exists $\epsilon_{1} \in (0, 1)$ such that $\epsilon \in (0, \epsilon_{1})$ implies $-\epsilon < \mu(\epsilon) - \mu < \mu$. Since $\mu < \bar{\mu}$, it follows that there exists $\epsilon_{2} \in (0, 1)$ such that $\epsilon < \epsilon_{2}$ implies $\bar{\mu} > (1 + \epsilon)\mu/(1 - \epsilon)$. Hence for all $\epsilon < \epsilon' := \epsilon_{1} \land \epsilon_{2}$, we have $\mu(\epsilon) < \mu(1 + \epsilon) < (1 - \epsilon)\bar{\mu}$.
Thus for all $0 < \epsilon < \epsilon' < 1$, and $x > x_3(\epsilon)$ we have
\[
g(x) > \mu(\epsilon) \phi(\Gamma_{\mu(\epsilon)}^{-1}(\Gamma_{\mu(\epsilon)}(x) + (\tau + h)(1 - \epsilon))) = x > x_3(\epsilon).
\] (5.7)

When $\mu(x) = +\infty$, because $\mu(x)$ is finite, (5.7) is trivial.

Define $g_3(\epsilon) = \Gamma_{\mu(\epsilon)}(x_3(\epsilon)) + (\tau + h)(1 - \epsilon)$. Then for $y > g_3(\epsilon)$, if we define $x = \Gamma_{\mu(\epsilon)}^{-1}(y - (\tau + h)(1 - \epsilon))$, for $x > x_3(\epsilon)$ we have that $y > g_3(\epsilon)$. Thus by (5.7)
\[
g(\Gamma_{\mu(\epsilon)}^{-1}(y - (\tau + h)(1 - \epsilon))) > \mu(\epsilon) \phi(\Gamma_{\mu(\epsilon)}^{-1}(y)), \quad y > g_3(\epsilon).
\] (5.8)

Next let $N_0(\epsilon) = \inf\{n > 0 : x_h(n) \geq x_3(\epsilon)\}$ and define $N_1 > N_0$ such that
\[
(1 - \epsilon)(\tau + h)\Gamma_{\mu(\epsilon)}(x_h(N_0)) \leq \Gamma_{\mu(\epsilon)}(x_h(N_1)).
\]

Define
\[
x_e(n) = \Gamma_{\mu(\epsilon)}^{-1}((1 - \epsilon)(n - N_1)h + \Gamma_{\mu(\epsilon)}(x_h(N_0))), \quad n \geq N_1.
\] (5.9)

Therefore for $n \geq N_1 + N$ we have
\[
(1 - \epsilon)(n + 1 - N_1)h + \Gamma_{\mu(\epsilon)}(x_h(N_0)) \geq (1 - \epsilon)(\tau + h) + \Gamma_{\mu(\epsilon)}(x_h(N_0)) \geq (1 - \epsilon)(\tau + h) + \Gamma_{\mu(\epsilon)}(x_3(\epsilon)) = y_3(\epsilon).
\]

Setting $y = (1 - \epsilon)(n + 1 - N_1)h + \Gamma_{\mu(\epsilon)}(x_h(N_0))$ in (5.8) yields
\[
g(\Gamma_{\mu(\epsilon)}^{-1}((1 - \epsilon)(n - N_1 - N)h + \Gamma_{\mu(\epsilon)}(x_h(N_0)))) > \mu(\epsilon) \phi(\Gamma_{\mu(\epsilon)}^{-1}(1 - \epsilon)(n - N_1 - N)h + \Gamma_{\mu(\epsilon)}(x_h(N_0))), \quad n \geq N_1 + N.
\]

By (5.9) we have
\[
g(x_e(n + 1) - x_e(n)) \geq \mu(\epsilon) \phi(x_e(n + 1),), \quad n \geq N_1 + N.
\] (5.10)

Therefore by (5.10) for $n \geq N_1 + N$, and the fact that
\[
\Gamma_{\mu(\epsilon)}(x_e(n)) = (1 - \epsilon)(n - N_1)h + \Gamma_{\mu(\epsilon)}(x_h(N_0)),
\]

we have
\[
\Gamma_{\mu(\epsilon)}(x_e(n + 1)) - \Gamma_{\mu(\epsilon)}(x_e(n)) = (1 - \epsilon)h.
\]

Hence there is $\xi(n) \in [x_e(n), x_e(n + 1)]$ such that
\[
x_e(n + 1) - x_e(n) = (1 - \epsilon)h \mu(\epsilon) \phi(\xi(n)).
\]

Since $\phi$ is non-decreasing and $\xi(n) \leq x_e(n + 1)$, we have
\[
x_e(n + 1) = x_e(n) + (1 - \epsilon)h \mu(\epsilon) \phi(\xi(n)) \leq x_e(n) + (1 - \epsilon)h \mu(\epsilon) \phi(x_e(n + 1)).
\]

Therefore by (5.10), we get for $n \geq N_1 + N$
\[
x_e(n + 1) \leq x_e(n) + (1 - \epsilon)h \mu(\epsilon) \phi(x_e(n + 1))< x_e(n) + h(1 - \epsilon)g(x_e(n + N)) \leq x_e(n) + h f(x_e(n)) + h(1 - \epsilon)g(x_e(n - N)) < x_e(n) + h f(x_e(n)) + h g(x_e(n - N)).
\]

Now for $n \in \{N_1, \ldots, N_1 + N\}$ we have
\[
x_e(n) \leq x_e(N_1 + N) = \Gamma_{\mu(\epsilon)}^{-1}((1 - \epsilon)\tau + \Gamma_{\mu(\epsilon)}(x_h(N_0))) < \Gamma_{\mu(\epsilon)}^{-1}((1 - \epsilon)(\tau + h) + \Gamma_{\mu(\epsilon)}(x_h(N_0))) \leq \Gamma_{\mu(\epsilon)}^{-1}(\Gamma_{\mu(\epsilon)}(x_h(N_1))) = x_h(N_1) \leq x_h(n),
\]

where we used at the last step the fact that $x_h(\epsilon)$ is increasing on $\{N_1, \ldots, N_1 + N\} \subset \{N, N + 1, \ldots\}$. Therefore we have $x_e(n) < x_h(n)$ for $n \in \{N_1(\epsilon), \ldots, N_1(\epsilon) + N\}$, and also $x_e(n + 1) < x_h(n) + h f(x_e(n)) + h g(x_e(n - N))$ for $n \geq N_1 + N$. 

Suppose that there is a \( n_1 \geq N_1(\epsilon) + N + 1 \) such that \( x_\epsilon(n) < x_\mu(n) \) for \( n \in \{N_1(\epsilon), \ldots, n_1\} \) and \( x_\epsilon(n_1) \geq x_\mu(n_1) \). Therefore \( x_\epsilon(n_1) - x_\epsilon(n_1 - 1) \geq x_\mu(n_1) - x_\mu(n_1 - 1) \). Then as \( f \) and \( g \) are non-decreasing,

\[
x_\epsilon(n_1) - x_\epsilon(n_1 - 1) \geq x_\mu(n_1) - x_\mu(n_1 - 1)
\]

\[
= hf(x_\epsilon(n_1 - 1)) + hg(x_\epsilon(n_1 - 1 - N))
\]

\[
\geq hf(x_\epsilon(n_1 - 1)) + hg(x_\epsilon(n_1 - 1 - N))
\]

\[
> x_\epsilon(n_1) - x_\epsilon(n_1 - 1),
\]

a contradiction. Thus \( x_\epsilon(n) < x_\mu(n) \) for all \( n \geq N_1 \). Hence \( \Gamma_\mu(\epsilon)(x_\mu(n)) > \Gamma_\mu(\epsilon)(x_\epsilon(n)) \) for all \( n \geq N_1(\epsilon) \). Hence

\[
\Gamma_\mu(\epsilon)(x_\mu(n)) > \Gamma_\mu(\epsilon)(x_\epsilon(n)) = (1 - \epsilon)(n - N_1) + \Gamma_\mu(\epsilon)(x_\epsilon(N_0)), \quad n \geq N_1(\epsilon).
\]

But \( \Gamma(x_\mu(n)) = \mu(\epsilon)\Gamma_\mu(\epsilon)(x_\mu(n)) \) \( > (1 - \epsilon)\mu(\epsilon) + \mu(\epsilon)\Gamma_\mu(\epsilon)(x_\epsilon(N_0)) \). Therefore

\[
\lim_{n \to \infty} \frac{\Gamma(x_\mu(n))}{nh} \geq (1 - \epsilon)\mu(\epsilon).
\]

Since \( \epsilon > 0 \) is arbitrary, and \( \mu(\epsilon) \to \mu \) as \( \epsilon \to 0 \), we have (3.5).

5.3. Proof of Theorem 16 Let \( j \geq N \). Summing across both sides of (3.1) yields

\[
x_\mu(j + 1) = x_\mu(j - N) + h \sum_{n=j-N}^{j} f(x_\mu(n)) + h \sum_{n=j-N}^{j} g(x_\mu(n - N)).
\]

Let \( \epsilon(\tau + h) < 1/2 \). Since \( x_\mu(n) \to \infty \) as \( n \to \infty \) and \( f(x)/x \to 0 \) as \( x \to \infty \), there exists \( N_1(\epsilon) \) such that \( f(x_\mu(n)) \leq \epsilon x_\mu(n) \) for all \( n \geq N_1(\epsilon) \). Hence for \( j \geq N_1(\epsilon) \) we have

\[
x_\mu(j + 1) \leq x_\mu(j - N) + h \sum_{n=j-N}^{j} \epsilon x_\mu(n) + h \sum_{n=j-N}^{j} g(x_\mu(n - N))
\]

\[
\leq x_\mu(j - N) + h(N + 1)\epsilon x_\mu(j) + h \sum_{n=j-N}^{j} g(x_\mu(n - N))
\]

\[
\leq x_\mu(j - N) + h(N + 1)\epsilon x_\mu(j + 1) + h \sum_{n=j-N}^{j} g(x_\mu(n - N)).
\]

Hence for \( j \geq N_1(\epsilon) \) we have

\[
x_\mu(j + 1) \leq \frac{1}{1 - (\tau + h)\epsilon} x_\mu(j - N) + \frac{h}{1 - (\tau + h)\epsilon} \sum_{n=j-N}^{j} g(x_\mu(n - N)).
\]

Since \( g \) is in \( RV_\infty(1) \), \( x \mapsto g(x)/x \) is asymptotic to a non-decreasing function, there exists \( g_0 \) such that \( g_0 \) is non-decreasing, \( g_0(x) \to \infty \) as \( x \to \infty \) and \( g_0(x)/g(x)/x \to 1 \) as \( x \to \infty \). Therefore \( g_1 \) defined by \( g_1(x) := xg_0(x) \) is increasing and is in \( RV_\infty(1) \). Since \( x_\mu(n) \to \infty \) as \( n \to \infty \), for every \( \epsilon > 0 \) there exists \( N_2(\epsilon) \geq N_1(\epsilon) \) such that \( g(x_\mu(n - N)) < (1 + \epsilon)g_1(x_\mu(n - N)) \). Thus for \( j \geq N_2(\epsilon) \) we have

\[
h \sum_{n=j-N}^{j} g(x_\mu(n - N)) \leq h(1 + \epsilon) \sum_{n=j-N}^{j} g_1(x_\mu(n - N))
\]

\[
\leq h(N + 1)(1 + \epsilon)g_1(x_\mu(j - N)).
\]

Hence

\[
h \sum_{n=j-N}^{j} g(x_\mu(n - N)) \leq (\tau + h)(1 + \epsilon)g_1(x_\mu(j - N)), \quad j \geq N_2(\epsilon).
\]
Let $N_3 = \max(N_1, N_2)$. Then for $j \geq N_3$, we have

$$x_h(j+1) \leq x_h(j-N) + \left( \frac{1}{1 - (\tau + h)\epsilon} - 1 \right) x_h(j-N) + \frac{(\tau + h)(1 + \epsilon)}{1 - (\tau + h)\epsilon} g_1(x_h(j-N)).$$

Define $x_h^*(n) = x_h(n(N+1))$ for $n \geq 0$. Therefore for $n \geq N_3$ we have

$$x_h^*(j+1) \leq x_h^*(j) + \left( \frac{1}{1 - (\tau + h)\epsilon} - 1 \right) x_h^*(j) + \frac{(\tau + h)(1 + \epsilon)}{1 - (\tau + h)\epsilon} g_1(x_h^*(j)).$$

Define

$$g_\epsilon(x) = \left( \frac{1}{1 - (\tau + h)\epsilon} - 1 \right) x + \frac{(\tau + h)(1 + \epsilon)}{1 - (\tau + h)\epsilon} g_1(x), \quad x > 0. \quad (5.11)$$

Then $g_\epsilon$ is in $RV_\infty(1)$, $x \mapsto g_\epsilon(x)/x$ is positive and non-decreasing, and $g_\epsilon(x)/x \rightarrow \infty$ as $x \rightarrow \infty$. Moreover

$$x_h^*(n+1) \leq x_h^*(n) + g_\epsilon(x_h^*(n)), \quad n \geq N_3(\epsilon).$$

Next, define

$$y_\epsilon(n+1) = y_\epsilon(n) + g_\epsilon(y_\epsilon(n)), \quad n \geq N_3(\epsilon) \quad \text{and} \quad y_\epsilon(N_3) = 2x_h^*(N_3(\epsilon)).$$

Since $g_\epsilon$ is increasing, it follows that $x_h^*(n) \leq y_\epsilon(n)$ for all $n \geq N_3(\epsilon)$. Define

$$H_\epsilon(x) = \int_1^x \frac{1}{u \log(1 + g_\epsilon(u)/u)} \, du, \quad x \geq 0.$$

Then by applying Lemma 14 to $(y_\epsilon)$, we have that

$$\lim_{n \rightarrow \infty} \frac{H_\epsilon(y_\epsilon(n))}{n} = 1.$$

Since $H_\epsilon$ is increasing, and $x_h^*(n) \leq y_\epsilon(n)$ for all $n \geq N_3(\epsilon)$, we have by the definition of $x_h^*$, that

$$\lim_{n \rightarrow \infty} \frac{H_\epsilon(x_h(n(N+1)))}{n} = \lim_{n \rightarrow \infty} \frac{H_\epsilon(x_h^*(n))}{n} \leq \lim_{n \rightarrow \infty} \frac{H_\epsilon(y_\epsilon(n))}{n} = 1.$$

Now by L’Hôpital’s rule and (5.11)

$$\lim_{x \rightarrow \infty} \frac{H_\epsilon(x)}{G(x)} = \lim_{x \rightarrow \infty} \frac{\log(1 + g(x)/x)}{\log(1 + g_\epsilon(x)/x)} = \lim_{x \rightarrow \infty} \frac{1}{\frac{\log(1 + g(x)/x)}{1 - (\tau + h)\epsilon} + \frac{(\tau + h)(1 + \epsilon)}{1 - (\tau + h)\epsilon} g_1(x)}.$$

Since $g(x)/g_1(x) \rightarrow 1$ as $x \rightarrow \infty$, we have that

$$\lim_{x \rightarrow \infty} \frac{\log(1 + g(x)/x)}{\log(1 + g_1(x)/x)} = 1.$$

Therefore

$$\lim_{x \rightarrow \infty} \frac{H_\epsilon(x)}{G(x)} = 1.$$ Hence

$$\lim_{n \rightarrow \infty} \frac{\log(1 + g(x_h(n(N+1))))}{n} \leq 1. \quad (5.12)$$

For every $n \in \mathbb{N}$ there exists $j = j(n) \geq 1$ such that $n(N+1) \leq j < (n+1)(N+1)$. Since $G$ is increasing, and $(x_h(n))_{n \geq 0}$ is increasing, we have

$$\frac{G(x_h(j))}{jh} \leq \frac{G(x_h((n+1)(N+1)))}{jh} \leq \frac{G(x_h((n+1)(N+1)))}{n(N+1)h} = \frac{1}{\tau + h} \frac{G(x_h((n+1)(N+1)))}{n+1} \cdot \frac{n+1}{n}.$$

By (5.12), we have

$$\lim_{j \rightarrow \infty} \frac{G(x_h(j))}{jh} \leq \frac{1}{\tau + h}.$$
which gives the desired upper limit in (5.7).

To get a lower bound, since \( f(x) \geq 0 \), we have \( x_h(n+1) \geq x_h(n) + h g(x_h(n-N)) \) for \( n \geq 0 \). Since \( x_h(n) \to \infty \) as \( n \to \infty \), for every \( \epsilon \in (0, 1) \) there exists \( N_2(\epsilon) \geq N \) such that \( g(x_h(n-N)) > (1-\epsilon)g_1(x_h(n-N)) \). Let \( N_5(\epsilon) = \max(N_2(\epsilon), N_1(\epsilon)) \). Let \( y_h^{(1)} \) be defined by

\[
y_h^{(1)}(n+1) = y_h^{(1)}(n) + h(1-\epsilon)g_1(y_h^{(1)}(n-N)), \quad n \geq N_5(\epsilon);
\]

\[
y_h^{(1)}(n) = x_h(n)/2, \quad n = N_5(\epsilon) - N, \ldots, N_5(\epsilon).
\]

Then we have for \( n \geq N_5(\epsilon) \) the inequality \( x_h(n+1) \geq x_h(n) + h(1-\epsilon)g_1(x_h(n-N)) \).

Hence \( y_h^{(1)}(n) \leq x_h(n) \) for \( n \geq N_5(\epsilon) - N \). Clearly \((y_h^{(1)}(n))_{n \geq N_5(\epsilon)}\) is increasing and \( y_h^{(1)}(n) \to \infty \) as \( n \to \infty \).

Let \( n \geq N_5(\epsilon) + N \). Then as \( y_h^{(1)} \) is increasing, we have

\[
y_h^{(1)}(n+1) = y_h^{(1)}(n) + h(1-\epsilon)g_1(y_h^{(1)}(n-N)) \geq y_h^{(1)}(n-N) + h(1-\epsilon)g_1(y_h^{(1)}(n-N)).
\]

Therefore for \( n \geq N_5(\epsilon) + N \) we have

\[
\log y_h^{(1)}(n+1) \geq \log \left( \frac{g_1(y_h^{(1)}(n-N))}{y_h^{(1)}(n-N)} \right) + \log y_h^{(1)}(n-N) + \log h(1-\epsilon) + \frac{y_h^{(1)}(n-N)}{g_1(y_h^{(1)}(n-N))},
\]

and so

\[
\log y_h^{(1)}(n+1) \geq \log y_h^{(1)}(n-N) + \log(h(1-\epsilon)) + \log g_0(y_h^{(1)}(n-N)).
\]

Define \( u(n) := \log y_h^{(1)}(n) \) for \( n \geq N_5(\epsilon) \). Then \((u(n))_{n \geq N_5(\epsilon)}\) is increasing and tends to infinity as \( n \to \infty \), and with \( \gamma_0(x) := \log(h(1-\epsilon)) + \log g_0(e^x) \), we have

\[
u(n+1) \geq u(n-N) + \gamma_0(u(n-N)), \quad n \geq N_5(\epsilon) + N.
\]

Since \( g_0 \) is non-decreasing, so is \( \gamma_0 \), and moreover \( \gamma_0(x) \to \infty \) as \( x \to \infty \). Since \( g_0 \) is in \( RV_{\infty}(0) \), there is \( g_3 \) in \( RV_{\infty}(0) \) which is also in \( C^1 \) such that \( g(x)/g_3(x) \to 1 \) as \( x \to \infty \), \( x g_3'(x)/g_3(x) \to 0 \) as \( x \to \infty \). Clearly for \( x^* \) sufficiently large we have \( g_3(e^x) > e \) for all \( x > x^* \), and so we may define

\[
G_3(x) = \int_{x^*}^x \frac{1}{g_3(e^u)} \ du.
\]

Then \( G_3'(x) = 1/g_3(e^x) > 0 \) for \( x > x^* \) and since \( g_3 \) is in \( C^1 \) we have

\[
G_3''(x) = -\frac{d}{dx} \log g_3(e^x) \cdot \frac{1}{(\log g_3(e^x))^2} = -\frac{1}{g_3(e^x)} g_3'(e^x) e^x \cdot \frac{1}{(\log g_3(e^x))^2}.
\]

Since there \( u(n) \to \infty \), there is \( N_6 \) such that \( u(n) > x^* \) for \( n \geq N_6 \). Let \( N_7(\epsilon) = \max(N_5(\epsilon), N_6) + N \). Then for \( n \geq N_7(\epsilon) \) we have \( G_3(u(n+1)) \geq G_3(u(n-N)) + \gamma_0(u(n-N-N)) \) and so by Taylor’s theorem, there exists \( \xi_n \in [u(n-N), u(n-N-N)] \) such that

\[
G_3(u(n+1)) \\
\geq G_3(u(n-N)) + \gamma_0(u(n-N-N)) \\
= G_3(u(n-N)) + G_3'(u(n-N)) \gamma_0(u(n-N)) + \frac{1}{2} G_3''(\xi_n) \gamma_0^2(u(n-N)),
\]

for \( n \geq N_7(\epsilon) \). Next, with \( \eta_n := g_3'(e^\xi_n) e^{\xi_n}/g_3(e^{\xi_n}) \) and using the fact that \( x g_3'(x)/g_3(x) \to 0 \) as \( x \to \infty \), we have that \( \eta_n \to 0 \) as \( n \to \infty \). Define for \( n \geq N_7(\epsilon) \)
the sequence
\[ \delta(n) := \frac{\log(h(1 - \epsilon)) + \log g_0(e^{u(n-N)})}{\log g_3(e^{u(n-N)})} - 1 = \frac{1}{2^m} \left( \frac{\log(h(1 - \epsilon)) + \log g_0(e^{u(n-N)})}{\log g_3(e^{\xi_n})} \right)^2. \]

so that
\[ G_3(u(n + 1)) \geq G_3(u(n - N)) + 1 + \delta(n), \quad n \geq N_7(\epsilon). \]

Since \( \xi_n \to \infty \) as \( n \to \infty \) and \( g_3(x)/g_0(x) \to 1 \) as \( x \to \infty \) we have that for every \( \epsilon \in (0, 1) \) there exists \( N_8(\epsilon) \) such that \( \log g_3(e^{\xi_n}) > \log(1 - \epsilon) + \log g_0(e^{\xi_n}) \) for all \( n \geq N_8(\epsilon) \) and so for \( n \geq N_9(\epsilon) = \max(N_8(\epsilon), N_7(\epsilon)) + N \) and so
\[
\frac{(\log(h(1 - \epsilon)) + \log g_0(e^{u(n-N)}))^2}{(\log g_3(e^{\xi_n}))^2} \leq \frac{(\log(h(1 - \epsilon)) + \log g_0(e^{u(n-N)}))^2}{(\log(1 - \epsilon) + \log g_0(e^{\xi_n}))^2}.
\]

Since \( g_0 \) is increasing and \( \xi_n \geq u(n - N) \) we have \( \log g_0(e^{\xi_n}) \geq \log g_0(e^{u(n-N)}) \).

Hence
\[
\frac{(\log(h(1 - \epsilon)) + \log g_0(e^{u(n-N)}))^2}{(\log g_3(e^{\xi_n}))^2} \leq \frac{(\log(h(1 - \epsilon)) + \log g_0(e^{\xi_n}))^2}{(\log(1 - \epsilon) + \log g_0(e^{\xi_n}))^2}.
\]

Therefore
\[
\limsup_{n \to \infty} \frac{(\log(h(1 - \epsilon)) + \log g_0(e^{u(n-N)}))^2}{(\log g_3(e^{\xi_n}))^2} \leq 1,
\]
and so \( \delta(n) \to 0 \) as \( n \to \infty \). Let \( z(n) = G_3(u(n)) \). Note that \( z \) is increasing and \( z(n) \to \infty \) as \( n \to \infty \). Then we have \( z(n+1) \geq z(n - N) + 1 + \delta(n) \). Let \( j \in \{0, \ldots, N\} \). Define \( z^*_j(n) = z((N + 1)n + j) \). Then
\[
z^*_j(n) = z(Nn + n + j - 1 + 1)
\geq z(Nn + n + j - 1 - N) + 1 + \delta(Nn + n + j - 1)
= z^*_j(n - 1) + 1 + \delta(Nn + n + j - 1).
\]

Now for \( n \geq n' \) we have
\[
\sum_{m=n'}^n z^*_j(m) \geq \sum_{m=n'}^n z^*_j(m - 1) + n - n' + 1 + \sum_{m=n'}^n \delta(Nm + m + j - 1),
\]
so
\[
z^*_j(n) \geq \frac{z^*_j(n' - 1)}{n} + 1 + \frac{n' - 1}{n} + \frac{1}{n} \sum_{m=n'}^n \delta(Nm + m + j - 1).
\]

Since \( \delta(n) \to 0 \) as \( n \to \infty \), we have \( \liminf_{n \to \infty} z^*_j(n)/n \geq 1 \). Therefore
\[
\liminf_{n \to \infty} \frac{z((N + 1)n + j)}{n(N + 1)} \geq \frac{1}{N + 1} \quad \text{for each } j = 0, \ldots, N.
\]

Hence
\[
\liminf_{n \to \infty} \frac{G_3(\log g_0^{(1)}(n))}{n} = \liminf_{n \to \infty} \frac{G_3(u(n))}{n} = \liminf_{n \to \infty} \frac{z(n)}{n} \geq \frac{1}{N + 1}.
\]

Since \( x_h(n) \geq y_h^{(1)}(n) \) for \( n \geq N_5(\epsilon) - N \), and \( G_3 \) is increasing, we have
\[
\liminf_{n \to \infty} \frac{G_3(\log x_h(n))}{nh} \geq \liminf_{n \to \infty} \frac{G_3(\log g_0^{(1)}(n))}{nh} \geq \frac{1}{N + 1} \cdot \frac{1}{N + 1} \quad \text{for each } n \geq N_5(\epsilon) - N.
\]

Now
\[
G_3(\log x) = \int_{x^*}^{\log x} \frac{1}{\log g_3(e^v)} dv = \int_{x^*}^x \frac{1}{u \log g_3(u)} du =: G_4(x). \quad (5.14)
\]
Since $g_3(x)/g_0(x) \to 1$ as $x \to \infty$ and each belongs to $\text{RV}_\infty(0)$, we have that
\[
\lim_{x \to \infty} \frac{\log g_0(x)}{\log g_3(x)} = 1.
\]
Similarly, as $(1 + g(x)/x)/g_0(x) \to 1$ as $x \to \infty$ and $g_0$ is in $\text{RV}_\infty(0)$,
\[
\lim_{x \to \infty} \frac{\log(1 + g(x)/x)}{\log g_0(x)} = 1.
\]
Using these limits and L'Hôpital's rule, we arrive at
\[
\lim_{x \to \infty} \frac{G_4(x)}{G(x)} = \lim_{x \to \infty} \frac{\log(1 + g(x)/x)}{\log g(x)} = \lim_{x \to \infty} \frac{\log(1 + g(x)/x)}{\log g_0(x)} \cdot \lim_{x \to \infty} \frac{\log g_0(x)}{\log g_3(x)} = 1.
\]
Since $x_h(n) \to \infty$ as $n \to \infty$ and (5.13) and $G_4$ is defined by (5.14), by using the last limit, we get
\[
\lim_{n \to \infty} \inf \frac{G(x_h(n))}{nh} = \lim_{n \to \infty} \inf \frac{G(x_h(n))}{G_4(x_h(n))} \cdot \frac{G_4(x_h(n))}{nh} = \lim_{n \to \infty} \inf \frac{G_4(x_h(n))}{nh} = \frac{1}{\tau + h}.
\]
which is the lower limit in (5.7).

In order to prove (3.8), notice for any $t > 0$ that there exists $n \geq 0$ such that $nh \leq t < (n + 1)h$. Also as the linear interpolant $\bar{x}_h$ defined by (3.2), we have $x_h(n) \leq \bar{x}_h(t) \leq x_h(n + 1)$. Therefore
\[
\frac{G(\bar{x}_h(t))}{t} \leq \frac{G(x_h(n) + 1)}{nh} = \frac{G(x_h(n))}{(n + 1)h} \cdot \frac{n + 1}{n}.
\]
Therefore by (5.7), we have
\[
\limsup_{t \to \infty} \frac{G(\bar{x}_h(t))}{t} \leq \frac{1}{\tau + h}.
\]
(5.15)
To get the lower bound, we observe that for $nh \leq t < (n + 1)h$, we have
\[
\frac{G(\bar{x}_h(t))}{t} \geq \frac{G(x_h(n))}{nh} \cdot \frac{nh}{(n + 1)h} = \frac{G(x_h(n))}{n} \cdot \frac{n}{n + 1}.
\]
Therefore by (5.7), we have
\[
\liminf_{t \to \infty} \frac{G(\bar{x}_h(t))}{t} \geq \frac{1}{\tau + h}.
\]
Combining this limit with (5.15) yields (3.8).

5.4. Proof of Theorem 8 Let $N \in \mathbb{N}$ and set $h = \tau/N$. Let $j \geq N$. Integrating over $[(j - N)h, (j + 1)h]$ yields
\[
x((j + 1)h) = x((j - N)h) + \int_{(j - N)h}^{(j + 1)h} f(x(s)) \, ds + \int_{(j - N)h}^{(j + 1)h} g(x(s - Nh)) \, ds.
\]
Let $\epsilon(\tau + h) < 1/2$. Since $x(t) \to \infty$ as $t \to \infty$ and $f(x)/x \to 0$ as $x \to \infty$, there exists $T_1(\epsilon) > \tau$ such that $f(x(s)) \leq \epsilon x(s)$ for all $s \geq T_1(\epsilon)$. Let $N_1(\epsilon)$ be an integer such that $N_1(\epsilon)h > T_1(\epsilon)$. Then for $j \geq N_1(\epsilon)$, and using the fact that $x$ is increasing, we have
\[
x((j + 1)h) \leq x((j - N)h) + \int_{(j - N)h}^{(j + 1)h} \epsilon x(s) \, ds + \int_{(j - N)h}^{(j + 1)h} g(x(s - Nh)) \, ds
\leq x((j - N)h) + h(N + 1)\epsilon x((j + 1)h) + \int_{(j - N)h}^{(j + 1)h} g(x(s - Nh)) \, ds
\leq x((j - N)h) + (\tau + h)\epsilon x((j + 1)h) + \int_{(j - N)h}^{(j + 1)h} g(x(s - Nh)) \, ds.$
Hence for $j \geq N_1(\epsilon)$ we have
\[
x((j+1)h) \leq x((j-N)h) + \left( \frac{1}{1 - (\tau + h)\epsilon} - 1 \right) x((j-N)h) \\
+ \frac{1}{1 - (h + \tau)\epsilon} \int_{(j-N)h}^{(j+1)h} g(x(s - Nh)) \, ds.
\]
Since $g$ is in $RV_\infty(1)$, $x \mapsto g(x)/x$ is asymptotic to a non-decreasing function, there exists $g_0$ such that $g_0$ is non-decreasing, $g_0(x) \to \infty$ as $x \to \infty$ and $g_0(x)/g(x)/x \to 1$ as $x \to \infty$. Therefore $g_1$ defined by $g_1(x) := xg_0(x)$ is increasing and is in $RV_\infty(1)$. Since $x(t) \to \infty$ as $t \to \infty$, for every $\epsilon > 0$ there exists $T_2(\epsilon) \geq \tau$ such that $g(x(t-\tau)) < (1 + \epsilon)g_1(x(t-\tau))$ for all $t \geq T_2(\epsilon)$. Let $N_2(\epsilon)$ be an integer such that $N_2(\epsilon)h > T_2(\epsilon)$. Thus for $j \geq N_2(\epsilon) + N$ we have $j \geq N_2(\epsilon)h + Nh > T_2 + \tau \geq 2\tau = 2Nh$, so as $x$ is increasing on $[0, \infty)$ we have
\[
\int_{(j-N)h}^{(j+1)h} g(x(s-Nh)) \, ds \leq (1 + \epsilon) \int_{(j-N)h}^{(j+1)h} g_1(x(s-Nh)) \, ds \\
\leq h(N+1)(1+\epsilon)g_1(x((j+1-N)h)).
\]
Let $N_3(\epsilon) = \max(N_1(\epsilon), N_2(\epsilon) + N)$. Then for $j \geq N_3(\epsilon)$ we have
\[
x((j+1)h) \leq x((j-N)h) + \left( \frac{1}{1 - (\tau + h)\epsilon} - 1 \right) x((j-N)h) \\
+ \frac{(h + \tau)(1 + \epsilon)}{1 - (h + \tau)\epsilon} g_1(x((j+1-N)h)),
\]
which, as $x$ is increasing, implies
\[
x((j+1)h) \leq x((j+1-N)h) + \left( \frac{1}{1 - (\tau + h)\epsilon} - 1 \right) x((j+1-N)h) \\
+ \frac{(h + \tau)(1 + \epsilon)}{1 - (h + \tau)\epsilon} g_1(x((j+1-N)h)), \quad j \geq N_3(\epsilon).
\]
Define $x^n_h(n) = x(nNh)$ for $n \geq -1$. Therefore for $n \geq N_3$, and since $N \geq 1$ we have
\[
x^n_h(j+1) \leq x^n_h(j) + \left( \frac{1}{1 - (\tau + h)\epsilon} - 1 \right) x^n_h(j) + \frac{(h + \tau)(1 + \epsilon)}{1 - (h + \tau)\epsilon} g_1(x^n_h(j)).
\]
The proof now continues as in the proof of Theorem 16, where $\tau$ is replaced by $\tau + h$. Proceeding in this manner we arrive at
\[
\limsup_{n \to \infty} G(x(nNh)) \frac{G(x(nNh))}{n} \leq 1. \tag{5.16}
\]
For every $t > 0$ there exists $n \in \mathbb{N}$ such that $nNh \leq t < (n+1)Nh$. Since $G$ is increasing, and $x$ is increasing, we have
\[
\frac{G(x(t))}{t} \leq \frac{G(x((n+1)Nh))}{t} \leq \frac{G(x((n+1)Nh))}{nNh} = \frac{1}{\tau} \frac{G(x((n+1)Nh))}{n+1} \cdot \frac{n+1}{n}.
\]
By (5.16), we have
\[
\limsup_{t \to \infty} \frac{G(x(t))}{t} \leq \frac{1}{\tau},
\]
and therefore the desired upper limit in (2.26).
To get a lower bound, since $f(x) \geq 0$, we have
\[
x((n+1)h) \geq x(nh) + \int_{nh}^{(n+1)h} g(x(s-Nh)) \, ds, \quad n \geq 0.
\]
Since \( x(t) \to \infty \) as \( t \to \infty \), for every \( \epsilon \in (0, 1) \) there exists \( T_4(\epsilon) \geq \tau \) such that
\[ g(x(t-\tau)) > (1-\epsilon)g(t) \] Let \( N_8(\epsilon) \) be an integer such that \( N_8(\epsilon)_h > T_4(\epsilon) \). Let \( N_5(\epsilon) = \max(N_4(\epsilon), N_1(\epsilon)) \). Thus for \( n \geq N_5(\epsilon) \) we have \( nh \geq N_5(\epsilon) \geq \max(T_4(\epsilon), \tau) \), so as \( x \) is increasing on \([0, \infty)\) we have
\[
x((n+1)h) \geq x(nh) + (1-\epsilon) \int_{nh}^{(n+1)h} g_4(x(s-Nh)) \, ds
\]
\[
\geq x(nh) + (1-\epsilon)h g_4(x(nh-Nh)).
\]

Then with \( x_h(n) := x(nh) \), we have the inequality
\[
x_h(n+1) \geq x_h(n) + (1-\epsilon)h g_4(x_h(n-N)), \quad n \geq N_5(\epsilon).
\]

Let \( y_h^{(1)} \) be defined by
\[
y_h^{(1)}(n+1) = y_h^{(1)}(n) + h(1-\epsilon)g_4(y_h^{(1)}(n-N)), \quad n \geq N_5(\epsilon);
\]
\[
y_h^{(1)}(n) = x(nh)/2, \quad n = N_5(\epsilon) - N, \ldots, N_5(\epsilon).
\]

Hence \( y_h^{(1)}(n) \leq x(nh) \) for \( n \geq N_5(\epsilon) - N \). The proof now proceeds exactly as in Theorem 16 and we arrive at the analogue of (5.13), namely
\[
\liminf_{n \to \infty} \frac{G_4(\log x(nh))}{nh} \geq \liminf_{n \to \infty} \frac{G_4(\log y_h^{(1)}(n))}{nh} \geq \frac{1}{Nh + h} = \frac{1}{\tau + h}, \quad (5.17)
\]
where we have used the fact that \( x(nh) = x_h(n) \). By (5.14), we have \( G_4(\log x) = G_4(x) \), so once again we have that \( \lim_{x \to \infty} G_4(x)/D(x) = 1 \). Since \( x(nh) \to \infty \) as \( n \to \infty \), (5.17) holds, and \( G_4 \) is defined by (5.14), by using the last limit, we get
\[
\liminf_{n \to \infty} \frac{G(x(nh))}{nh} = \liminf_{n \to \infty} \frac{G(x(nh))}{nh} \frac{G_4(x(nh))}{G_4(x(nh))} = \liminf_{n \to \infty} \frac{G_4(\log x(nh))}{nh} \geq \frac{1}{\tau + h}.
\]

Now, for every \( t > 0 \) there exists \( h \) such that \( nh \leq t < (n+1)h \). Since \( x \) is increasing and \( G \) is increasing, we have
\[
\frac{G(x(t))}{t} \geq \frac{G(x(nh))}{t} \geq \frac{G(x(nh))}{(n+1)h} = \frac{G(x(nh))}{nh} \cdot \frac{n}{n+1}.
\]

Therefore
\[
\liminf_{t \to \infty} \frac{G(x(t))}{t} \geq \liminf_{n \to \infty} \frac{G(x(nh))}{nh} \geq \frac{1}{\tau + h}.
\]

Letting \( h \to 0 \) yields
\[
\liminf_{t \to \infty} \frac{G(x(t))}{t} \geq \frac{1}{\tau},
\]
which is the lower limit in (2.20).

5.5. Proof of Theorem 17 Let \( j \geq N \). Summing across both sides of (5.1) yields
\[
x_h(j+1) = x_h(j-N) + h \sum_{n=j-N}^{j} f(x_h(n)) + h \sum_{n=j-N}^{j} g(x_h(n-N)).
\]

Let \( \epsilon(\tau + h) < 1/2 \). Since \( x_h(n) \to \infty \) as \( n \to \infty \) and \( f(x)/x \to 0 \) as \( x \to \infty \), there exists \( N_1(\epsilon) \) such that \( f(x_h(n)) \leq \epsilon x_h(n) \) for all \( n \geq N_1(\epsilon) \). Hence for \( j \geq N_1(\epsilon) \)
we have
\[ x_h(j + 1) \leq x_h(j - N) + h \sum_{n=j-N}^{j-1} c x_h(n) + h \sum_{n=j-N}^{j} g(x_h(n - N)) \]
\[ \leq x_h(j - N) + h(N + 1) c x_h(j) + h \sum_{n=j-N}^{j} g(x_h(n - N)) \]
\[ \leq x_h(j - N) + h(N + 1) c x_h(j + 1) + h \sum_{n=j-N}^{j} g(x_h(n - N)). \]

Hence for \( j \geq N_1(\epsilon) \) we have
\[ x_h(j + 1) \leq \frac{1}{1 - (\tau + h) \epsilon} x_h(j - N) + \frac{h}{1 - (\tau + h) \epsilon} h \sum_{n=j-N}^{j} g(x_h(n - N)). \]

Since \( \log g(x)/\log x \to \beta \) as \( x \to \infty \), and \( x_h(n) \to \infty \) as \( n \to \infty \), for every \( \epsilon > 0 \) there exists \( N_2(\epsilon) \geq N \) such that \( g(x_h(n - N)) < x_h(n - N)^{\beta + \epsilon} \). Thus for \( j \geq N_2(\epsilon) + N \) we have
\[ h \sum_{n=j-N}^{j} g(x_h(n - N)) \leq h \sum_{n=j-N}^{j} x_h(n - N)^{\beta + \epsilon} \]
\[ \leq h(N + 1) x_h(j - N)^{\beta + \epsilon}. \]

Hence
\[ h \sum_{n=j-N}^{j} g(x_h(n - N)) \leq (\tau + h) x_h(j - N)^{\beta + \epsilon}, \quad j \geq N_2(\epsilon). \]

Let \( N_3(\epsilon) = \max(N_1(\epsilon), N_2(\epsilon) + N) \). Then as \( 1 - (\tau + h) \epsilon > 1/2 \), for \( j \geq N_3(\epsilon) \) we have
\[ x_h(j + 1) \leq 2 x_h(j - N) + 2(\tau + h) x_h(j - N)^{\beta + \epsilon}. \]

Define \( x_h^*(n) = x_h(n(N + 1)) \) for \( n \geq -1 \). Therefore for \( n \geq N_3(\epsilon) \) we have
\[ x_h^*(j + 1) \leq x_h((j + 1)(N + 1)) \leq 2 x_h(j(N + 1)) + 2(\tau + h) x_h(j(N + 1))^{\beta + \epsilon} \]
\[ = 2 x_h^*(j) + 2(\tau + h) x_h^*(j)^{\beta + \epsilon}. \]

Thus
\[ \log x_h^*(j + 1) \leq \log 2(\tau + h) + (\beta + \epsilon) \log x_h^*(j) + \log \left( 1 + \frac{x_h^*(j)}{(\tau + h) x_h^*(j)^{\beta + \epsilon}} \right). \]

Thus we have, with \( u(n) = \log x_h^*(n) \), and all \( n > N_3(\epsilon) \), the inequality
\[ u(n + 1) \leq (\beta + 2\epsilon) u(n). \]

Thus there exists \( K(\epsilon) > 0 \) such that \( u(n) \leq K(\epsilon)(\beta + 2\epsilon)^n \) for \( n \geq N_3(\epsilon) \). Thus
\[ \frac{1}{n} \log u(n) \leq \frac{1}{n} \log K(\epsilon) + \log(\beta + 2\epsilon). \]

Therefore
\[ \limsup_{n \to \infty} \frac{\log_2 x_h(n(N + 1))}{n(N + 1) h} = \limsup_{n \to \infty} \frac{\log_2 x_h^*(n)}{n(N + 1) h} \leq \frac{\log(\beta + 2\epsilon)}{(N + 1) h} = \frac{\log(\beta + 2\epsilon)}{\tau + h}. \]

Letting \( \epsilon \downarrow 0 \), we arrive at
\[ \limsup_{n \to \infty} \frac{\log_2 x_h(n(N + 1))}{n(N + 1) h} \leq \frac{\log(\beta)}{\tau + h}. \] (5.18)
For every $n \in \mathbb{N}$ there exists $j = j(n) \geq 1$ such that $n(N+1) \leq j < (n+1)(N+1)$. Since $(x_h(n))_{n \geq 0}$ is increasing, we have
\[
\frac{\log_2 x_h(j)}{jh} \leq \frac{\log_2 x_h((n+1)(N+1))}{jh} \leq \frac{\log_2 x_h((n+1)(N+1)))}{n(N+1)h} + \frac{1}{n(N+1)} 
\]
which gives the desired upper limit.

By (5.18), we have
\[
\limsup_{j \to \infty} \frac{\log_2 x_h(j)}{jh} \leq \frac{\log \beta}{\tau + h},
\]
which gives the desired upper limit.

Since $f(x) \geq 0$ we have
\[x_h(n+1) \geq x_h(n) + hg(x_h(n - N)) \geq hg(x_h(n - N))\]
and since $x_h(n) \to \infty$ as $n \to \infty$ and $\log g(x)/\log x \to \beta$ as $x \to \infty$, it follows that for every $\epsilon < \beta$ there exists $N_0(\epsilon)$ such that $hg(x_h(n - N)) \geq x_h(n - N)^{\beta - \epsilon} > \epsilon$ for $n \geq N_0(\epsilon)$. Hence for $n \geq N_0(\epsilon)$ we have
\[x_h(n+1) \geq x_h(n - N))^{\beta - \epsilon}.
\]
Therefore with $u(n) = \log x_h(n)$, we have that
\[u(n+1) = \log x_h(n+1) \geq (\beta - \epsilon)u(n - N).
\]
Therefore, there exists $k(\epsilon) > 0$ such that $u(n) \geq k(\epsilon)(\beta - \epsilon)^n/(N+1)$ for $n \geq N_0(\epsilon)$. Therefore
\[
\frac{1}{n} \log u(n) \geq \frac{1}{n} \log k(\epsilon) + \frac{1}{N+1} \log(\beta - \epsilon).
\]
Hence
\[
\liminf_{n \to \infty} \frac{\log_2 x_h(n)}{nh} \geq \frac{\log(\beta - \epsilon)}{(N+1)h} = \frac{\log(\beta - \epsilon)}{\tau + h}.
\]
Letting $\epsilon \downarrow 0$, we get
\[
\liminf_{n \to \infty} \frac{\log_2 x_h(n)}{nh} \geq \frac{\log \beta}{\tau + h},
\]
and so combining this with the other limit we get
\[
\liminf_{n \to \infty} \frac{\log_2 x_h(n)}{nh} = \frac{\log \beta}{\tau + h},
\]
as required.

The proof that (3.10) follows from (3.9) is identical in all regards to the proof of Theorem 16 that (3.8) follows from (6.7), and is therefore omitted.

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