A proof of the Baum–Connes conjecture for reductive adelic groups

Paul BAUM\textsuperscript{a}, Stephen MILLINGTON\textsuperscript{b}, and Roger PLYMEN\textsuperscript{c}

\textsuperscript{a} Department of Mathematics, Pennsylvania State University, University Park, PA16802, USA.
E-mail: baum@math.psu.edu
\textsuperscript{b} Department of Mathematics, University of Manchester, Manchester, M13 9PL, UK.
E-mail: stem@quick.freeserve.co.uk
\textsuperscript{c} Department of Mathematics, University of Manchester, Manchester, M13 9PL, UK.
E-mail: roger@ma.man.ac.uk

Abstract. Let $F$ be a global field, $A$ its ring of adeles, $G$ a reductive group over $F$. We prove the Baum–Connes conjecture for the adelic group $G(\mathbb{A})$.

Une démonstration de la conjecture de Baum–Connes pour les groupes réductifs adéliques

Résumé. Soit $F$ un corps global, $A$ son anneau d’adèles, $G$ un groupe réductif sur $F$. Nous démontrons la conjecture de Baum–Connes pour le groupe adélique $G(\mathbb{A})$.

Version Française Abrégée

Soit $X_1 \subset X_2 \subset X_3 \subset \ldots$ une suite croissante d’espaces topologiques. Nous pouvons donner sur la réunion $X = \bigcup X_n$ la topologie limite inductive: un ensemble est ouvert dans $X$ si et seulement si son intersection avec chaque $X_n$ est ouverte. Si
chaque $X_n$ est un espace $T_1$ et $E$ est une partie compacte de $X$, il existe $n$ tel que $E \subset X_n$.

Soit $G_1 \subset G_2 \subset G_3 \subset \ldots$ une suite croissante de groupes localement compacts, de base dénombrable, séparés tel que $G_n$ is un sous-groupe ouvert de $G_m$ pour $n \leq m$. Soit $G = \cup G_n$ dans la topologie limite inductive. Alors $G$ est un groupe localement compact, de base dénombrable, séparé. Une base pour la topologie sur $G$ est donnée par l’ensemble des parties ouvertes dans chaque $G_n$. Si un groupe $G$ est égale à la réunion d’une suite croissante de sous-groupes ouverts alors la topologie sur $G$ est la même que la topologie limite inductive donnée par ces sous-groupes.

Dans [9] Kasparov et Skandalis construisent un exemple universel localement compact. Plus précisément, soit $MG$ l’ensemble des mesures positives à support compact sur $G$ avec mesure totale dans l’intervalle $(1/2, 1]$ et muni de la topologie faible *. A l’aide de l’espace $MG$ nous construisons un espace pour lequel il est facile de calculer la $K$-théorie topologique d’une réunion d’une suite croissante de sous-groupes ouverts. $MG$ est un espace métrisable à base dénombrable avec espace quotient métrisable $MG/G$.

Si $G$ est la réunion d’une suite croissante de sous-groupes ouverts $G_n$ alors il existe une suite croissante fermée $MG_1 \subset MG_2 \subset \ldots$. L’ensemble $MG$ est égale à la réunion $\cup MG_n$. Maintenant nous donnons à $MG$ une nouvelle topologie, limite inductive des sous-espaces $MG_n$. L’action de $G$ reste continue avec cette nouvelle topologie. Remarquons que cet espace n’est pas toujours localement compact ou métrisable — il tombe en dehors de la classe des espaces propres au sens de [2]. Il est possible de donner une nouvelle définition de propre — plus faible que la définition dans [4] — pour laquelle cet espace est un exemple universel pour les actions propres (pour la nouvelle définition) de $G$. Dans cette Note, il suffit de démontrer le résultat suivant.

**Théorème 1.** L’espace $MG$, muni de la topologie inductive limite, est équivalent par $G$-homotopie à $MG$.

Nous démontrons aussi les résultats suivants.

**Théorème 2.** Soit $G$ la réunion des sous-groupes $G_n$ localement compacts, de base dénombrable, séparés, ouverts et tels que la conjecture de Baum-Connes est vraie pour chaque $G_n$. Alors la conjecture de Baum-Connes est vraie pour $G$.

Soit $F$ un corps global, c’est-à-dire ou bien une extension finie de $\mathbb{Q}$, ou bien une extension finie du corps de fonctions $\mathbb{F}_p(T)$.

**Théorème 3.** Soit $F$ un corps global, $\mathbb{A}$ son anneau d’adèles, $G$ un groupe
réductif sur $F$. Alors la conjecture de Baum-Connes est vraie pour le groupe adélique $G(\mathbb{A})$.

1. Introduction

Suppose we have an ascending sequence of topological spaces

$$X_1 \subset X_2 \subset X_3 \subset \ldots$$

Then we can give the union $X = \bigcup X_n$ the direct limit topology: a set is open in $X$ if and only if it has open intersection with each $X_n$. If each $X_n$ is a $T_1$ space then any compact subset of $X$ lies entirely within some $X_n$.

Suppose now that for each $n \in \mathbb{N}$ we have a locally compact, second countable and Hausdorff topological group $G_n$, such that $G_n$ is an open subgroup of $G_m$ for $n \leq m$. Let $G = \bigcup G_n$ and furnish this with the direct limit topology. Then $G$ is a locally compact, second countable, Hausdorff group in an obvious way. A basis for the topology on $G$ is given by the collection of open sets in each $G_n$. Furthermore if a group $G$ is equal to the union of an ascending sequence of open subgroups then the topology on $G$ is the same as the direct limit topology with respect to these subgroups; note this is not necessarily the case if the subgroups are not open.

We construct a topological space which is a direct limit and is homotopic to a universal example for $G$. Using this space we may express the topological $K$-theory of $G$ as a direct limit. The adelic groups fit into this framework and the problem reduces to proving the Baum-Connes conjecture for certain finite products.

Throughout we shall assume that all groups are locally compact, second countable and Hausdorff.

We would like to thank Paula Cohen, Georges Skandalis and Vincent Lafforgue for several valuable conversations. Paul Baum was partially supported by an NSF grant.

2. The space $MG$

In [9] Kasparov and Skandalis construct a locally compact universal example. Namely let $MG$ be the set of all compactly supported positive measures on $G$ with total measure in the interval $(1/2, 1]$, topologized with the weak* topology. We shall use $MG$ to construct a space for which it is easy to calculate the topological $K$-theory of an open ascending union of groups.

First we collect some useful properties of $MG$. $MG$ is second countable. This follows from the fact that $C_c(G)$ is separable, which may be seen using the Stone-Weierstrass approximation theorem together with the fact that $G$ is second countable, locally compact and Hausdorff. So using the Urysohn metrization theorem $MG$ is metrizable.
Suppose $K \subset MG$ is compact. We shall show $GK$ is closed. Let $g_i k_i$ be a convergent sequence in $GK$ with limit $y$. We show that $y \in GK$. Let $U$ be a neighbourhood of $y$ with compact closure.

Since the action is proper (by [6] there is no ambiguity in our use of proper as $MG$ is locally compact) we have \( \{g \in G : gK \cap \bar{U} \neq \emptyset\} \) is compact. But because $g_i k_i$ is convergent this compact set must contain an infinite number of $g_i$ and so $g_i$ has a limit point $g$. Similarly $k_i$ all lie in the compact $K$ so must have a limit point $k$. It is now clear that $y = gk$ as required.

Using the fact that $MG$ is normal it is now easy to show that the quotient space is Hausdorff and locally compact. Also from the above it is clearly second countable. So using the Urysohn metrization theorem we can conclude that the quotient space is metrizable.

**Lemma 1.** If $H$ is an open subgroup of $G$ then $MH$ is a closed subspace of $MG$.

**Proof.** A basis for topology is given by the sets

\[
(\mu; f_1, \ldots, f_n, \varepsilon) = \left\{ \lambda \in MG : \left| \int f_k \, d\mu - \int f_k \, d\lambda \right| < \varepsilon, \text{ for } k = 1, \ldots, n \right\}
\]

where $\mu \in MG$, $f_i \in C_c(G)$, and $\varepsilon > 0$.

Clearly $MH \subset MG$. It is also clear that the open basis sets $(\lambda; f_1, \ldots, f_k, \varepsilon)$ of $MH$ are simply

\[
\left\{ \lambda \in MG : \left| \int f^G_k \, d\mu - \int f^G_k \, d\lambda \right| < \varepsilon, \text{ for } k = 1, \ldots, n \right\} \cap MH
\]

where each $f^G_k$ is defined to be $f_k$ extended to zero on $G - H$; giving a continuous function as $H$ is clopen in $G$. So $MH$ is a topological subspace of $MG$.

Now take any $\lambda \notin MH$. Then there is some compact $K \subset G - H$ with $\lambda(K) = \varepsilon > 0$. Then the open set $(\lambda, f_K, \varepsilon)$ is a neighbourhood of $\lambda$ which does not contain any element of $MH$, where $f_K$ is a compactly supported function which is 1 on $K$ and zero on $H$. So we see that $MG - MH$ is open and hence $MH$ is closed.

If $G$ is the union of an ascending sequence of open subgroups $G_n$ then there is a closed ascending sequence $MG_1 \subset MG_2 \subset \ldots$. If we take the union $\cup MG_n$ it is easy to see that as a set this is the same as $MG$. Using the direct limit topology we can then think of $MG$ as being retopologized with respect to this ascending sequence. The action of $G$ remains continuous under this new topology. Note that this space may not be locally compact or metrizable and so falls outside the class of spaces we admit as proper in the sense of [2]. However it is possible to give a new definition of proper—which is weaker than that in [2]—in which this space may be shown to
be a universal example for proper (in this new sense) actions of $G$. However for our purposes it suffices to prove the following.

**Theorem 1.** $M\!G$ retopologized in this fashion is $G$-homotopy equivalent to $\overline{M\!G}$ in the weak* topology.

**Proof.** We write $(M\!G)_\text{lim}$ to denote $M\!G$ with the direct limit topology. The direct limit topology is finer than the original topology: this is because each $M\!G_n$ is closed. So the identity map $(M\!G)_\text{lim} \to M\!G$ is obviously continuous and $G$-equivariant. We must construct a map in the other direction.

Take any point $\mu \in M\!G$ then as in [2, 1.3] there is a triple $(U_\mu, H_\mu, \rho_\mu)$, where $U_\mu$ is a $G$-invariant open neighbourhood of $\mu$ (in the weak* topology), $H_\mu$ is a compact subgroup of $G$, and $\rho_\mu$ is a $G$-equivariant map from $U_\mu$ to $G/H_\mu$. As $M\!G$ is a universal example there is a point $\lambda \in M\!G$ which is fixed by $H_\mu$. Because the action remains continuous when we change topology the map $\sigma_\mu : G/H_\mu \to (M\!G)_\text{lim}$ defined by $\sigma_\mu(gH_\mu) = g\lambda$ is well defined and $G$-equivariant. Define $\Phi_\mu = \sigma_\mu \circ \rho_\mu : U_\mu \to (M\!G)_\text{lim}$. Note that this is a continuous (and $G$-equivariant) map from $U$ with the subspace topology obtained from the weak* topology.

We may cover $M\!G$ with such open neighbourhoods and thus get a cover of $\overline{M\!G}/G$ by open sets $\pi(U_\mu)$. As the quotient space is paracompact let $\Psi_\mu : \overline{M\!G}/G \to [0,1]$ be a locally finite partition of unity subordinate to this cover.

The map we require is then given by

$$\Xi : M\!G \to (M\!G)_\text{lim}, \quad \Xi(\nu) = \sum_\mu \Psi_\mu(\nu)\Phi_\mu(\nu)$$

The argument is then completed by noting that any two $G$-maps from $M\!G$ to itself in either topology are $G$-homotopy equivalent by taking their weighted sums on the interval $[0,1]$.

**3. K-theory**

**Lemma 2.** If $G$ is the union of an ascending sequence of open subgroups $G_n$ then

$$K_j(C^*_r(G)) = \lim_{\longrightarrow} K_j(C^*_r(G_n))$$

**Proof.** We show that $C^*_r(G) = \lim_{\longrightarrow} C^*_r(G_n)$. We may include $C_c(G_n) \hookrightarrow C_c(G)$ for any $n$ by extending functions to zero. This gives a continuous function as $G_n$ is clopen. Taking $\varepsilon > 0$ and $x \in C^*_r(G)$ we can get $f \in C_c(G)$ which is $\varepsilon$ close to $x$. But any compact set in $G$ must lie totally within some $G_n$. Hence $f \in C_c(G_n)$ for some $n$ and so $\bigcup_n C^*_r(G_n)$ is dense in $C^*_r(G)$. We have

$$C^*_r(G) = \lim_{\longrightarrow} C^*_r(G_n) \Rightarrow K_j(C^*_r(G)) = \lim_{\longrightarrow} K_j(C^*_r(G_n))$$
Lemma 3. If $G$ is the union of an ascending sequence of open subgroups $G_n$ then

$$K_j^\text{top}(G) = \lim_{\rightarrow} K_j^\text{top}(G_n)$$

Proof. We have shown that $(\underline{MG})_\lim$ is $G$-homotopic to the universal example $\underline{MG}$:

$$K_j^\text{top}(G) = K_j^G(\underline{MG}) = K_j^G((\underline{MG})_\lim)$$

By definition 3.13 in [2] we have

$$K_j^G((\underline{MG})_\lim) = \lim_{\rightarrow G\text{-compact}} K_j^G(Z).$$

Any $Z \subset (\underline{MG})_\lim$ is $G$-compact if and only if it is the $G$-saturation of a compact set. But any compact subset of $(\underline{MG})_\lim$ lies within some $\underline{MG}_n$. So $Z \subset G \cdot \underline{MG}_n$, for some $n$. We get

$$K_j^G(\underline{MG}) = \lim_n \lim_{Z \subset G \cdot \underline{MG}_n} K_j^G(Z) = \lim_n K_j^G(G \cdot \underline{MG}_n) = \lim_n K_j^G(G \times_{G_n} \underline{MG}_n)$$

We now appeal to Proposition 5.14 in [3] which tells us

$$K_j^G(G \times_{G_n} \underline{MG}_n) = K_j^{G_n}(\underline{MG}_n) = K_j^\text{top}(G_n)$$

as the action of $G_n$ on $\underline{MG}_n$ is proper.

Theorem 2. Let $G$ be the union of open subgroups $G_n$ such that the Baum-Connes conjecture is true for each $G_n$. Then the Baum-Connes conjecture is true for $G$.

Proof. Taking $m \leq n$ we check that the following diagram commutes

$$
\begin{array}{ccc}
K_j^\text{top}(G_m) & \overset{\mu_{G_m}}{\longrightarrow} & K_j(C^*_r(G_m)) \\
\downarrow & & \downarrow \\
K_j^\text{top}(G_n) & \overset{\mu_{G_n}}{\longrightarrow} & K_j(C^*_r(G_n))
\end{array}
$$

and use the fact that, for any $G_m$-compact $X$, the map $\mu$ factorizes as follows:

$$KK^j_{G_m}(C_0(X), \mathbb{C}) \overset{j_{G_m}}{\longrightarrow} KK^j(C_0(X) \rtimes G_m, C^*_r(G_m)) \overset{\#1}{\longrightarrow} KK^j(\mathbb{C}, C^*_r(G_m))$$

As the $\mu$ map is defined by direct limits we get the result by using Lemmas 2 and 3.

4. Reductive adelic groups
Let \( \mathbb{F}_p(T) \) denote the field of rational functions in the indeterminate \( T \) with coefficients in \( \mathbb{F}_p \). Now let \( F \) be a global field (an \( \mathbb{A} \)-field in the sense of Weil \([14, \text{p.}41]\)). The field \( F \) is a finite algebraic extension of \( \mathbb{Q} \) or a finite algebraic extension of \( \mathbb{F}_p(T) \). We denote by \( \mathbb{A} \) its ring of adeles, the restricted product of all the completions \( F_v \), as in \([14, \text{p.}59]\). The field \( F \) is a discrete cocompact subfield of the non-discrete locally compact semisimple commutative ring \( \mathbb{A} \). Let \( G \) be a reductive group over \( F \) and let \( G(\mathbb{A}) \) denote the group of adelic points in the algebraic group \( G \). The field \( F \) has at most a finite number of infinite places; it has at least one if it is of characteristic 0, and none otherwise. A place of a global field of characteristic 0 is infinite if and only if it lies above the place \( \infty \) of \( \mathbb{Q} \), see \([14, \text{p.}45]\). Let \( S_\infty \) be the finite set of infinite places of \( F \) and let \( F_\infty = \prod_{v \in S_\infty} F_v \). If \( v \) is an imaginary place, then the complex reductive group \( G(F_v) \) has also the structure of real reductive group. We will assume that each local group \( G(F_v) \) with \( v \in S_\infty \) is a connected Lie group. Then \( G(F_\infty) \) is a connected real reductive group. Choose an ordering \( v_1, v_2, \ldots \) of the finite places, and let

\[
H_n = \prod_{v \leq v_n} G(F_v) \times \prod_{v > v_n} G(\mathfrak{a}_v), \quad G_n = G(F_\infty) \times H_n
\]

The groups \( G_n \) form an inductive system and \( G(\mathbb{A}) = \bigcup G_n \). The standard locally compact topology on \( G(\mathbb{A}) \) is described in \([4, \text{p.}293]\). This topology coincides with the direct limit topology on \( G(\mathbb{A}) \) following the open subgroups \( G_n \).

**Theorem 3.** Let \( F \) be a global field, \( \mathbb{A} \) its ring of adeles, \( G \) a reductive group over \( F \). Then the Baum–Connes conjecture is true for the adelic group \( G(\mathbb{A}) \).

**Proof.** Consider the commutative diagram

\[
\begin{array}{ccc}
K_i^{\text{top}}(G(F_\infty)) \otimes_{\mathbb{Z}} K_j^{\text{top}}(H_n) & \xrightarrow{\mu_{G(F_\infty)} \otimes_{\mathbb{Z}} \mu_{H_n}} & K_i C^*_r(G(F_\infty)) \otimes_{\mathbb{Z}} K_j C^*_r(H_n) \\
\downarrow & & \alpha \downarrow \\
K_i^{\text{top}}(G_n) & \xrightarrow{\mu_{G_n}} & K_{i+j} C^*_r(G_n)
\end{array}
\]

in which each vertical map is an external product.

We know that \( \mu_{G(F_\infty)} \) is an isomorphism by \([12, 13]\). The “finite” product \( H_n \) admits a 4-tuple satisfying the condition (HC) of Lafforgue \([10, \text{Defin.}1.1]\). Therefore, by Proposition 1.3 in \([10]\), for \( t \in \mathbb{R}_+ \) sufficiently large, the Banach space \( S_t(H_n) \) is a good completion of \( C_t(H_n) \) and is a subalgebra of \( C^*_r(H_n) \) dense and stable under holomorphic functional calculus. If \( v \) is a finite place then the Euclidean building \( \beta G(F_v) \) is a model of the universal example for the local group \( G(F_v) \). Now a Euclidean building is a weakly geodesic and strongly bolic metric space \([8, 9]\). Let \( X_n \) denote the set of vertices in the product building \( \beta G(F_{v_1}) \times \cdots \times \beta G(F_{v_n}) \). Then
admits a metric \(d_n\) such that \((X_n, d_n)\) is a weakly geodesic, strongly bolic and uniformly locally finite metric space on which \(H_n\) acts isometrically, continuously and properly. By the fundamental result of Lafforgue [1], the Baum-Connes conjecture holds for \(H_n\) and so \(\mu_{H_n}\) is an isomorphism. So \(\mu_{G(F_{\infty})} \otimes \mu_{H_n}\) is an isomorphism.

Let \(i\) be the mod 2 dimension of the symmetric space of \(G(F_{\infty})\). Then \(K_*C^*_r(G(F_{\infty}))\) is concentrated in degree \(i\), and is a free abelian group [1], [12], [13]. So, by the Kunneth theorem [3, 23.1.3], \(\alpha\) is an isomorphism. Therefore \(\mu_{G_n}\) is surjective.

But \(G_n\) has a \(\gamma\) element namely the external product \(\gamma_{G(F_{\infty})} \# \gamma_{H_n}\), where \(\gamma_{H_n} = \gamma_{v_1} \# \ldots \# \gamma_{v_n}\). The individual \(\gamma\)-elements \(\gamma_{v_1}, \gamma_{v_2}, \ldots, \gamma_{v_n}\) were constructed in [9]. If \(v\) is a finite place of \(F\) then the affine building \(\beta G(F_v)\) is a model of the universal example \(EG(F_v)\). The \(\gamma\)-element \(\gamma_v\) for \(G(F_v)\) was constructed in [7, p.310]. The \(\gamma\)-element \(\gamma_v\) is an element in the Kasparov ring \(R(G_v) = KK_{G_v}(\mathbb{C}, \mathbb{C})\). This implies the injectivity of \(\mu_{G_n}\).

Therefore \(\mu_{G_n}\) is an isomorphism. So the Baum-Connes conjecture is true for each open subgroup \(G_n\). Now apply Theorem 2 to the adelic group \(G(A) = \lim_{\rightarrow} G_n\).

References

[1] Baum P., Connes A., Geometric K-theory for Lie groups and foliations, L’Enseignement Math. 46 (2000) 3–32.
[2] Baum P., Connes A., Higson N., Classifying space for proper actions and K-theory of group C*-algebras, Contemporary Math. 167 (1994) 241–291.
[3] Blackadar, B., K-theory for operator algebras, Springer, Berlin, 1986.
[4] Bump D., Automorphic forms and representations, Cambridge Studies in Advanced Math. 55 (1997).
[5] Chabert J., Echterhoff S., Permanence properties of the Baum-Connes conjecture, Preprint, 2000.
[6] Chabert J., Echterhoff S., Mayer R., Deux remarques sur la conjecture de Baum-Connes, Preprint, 2000.
[7] Kasparov G.G., Skandalis G., Groups acting on buildings, operator K-theory and Novikov’s conjecture, K–Theory 4 (1991) 303 – 337.
[8] Kasparov G.G., Skandalis G., Groupes boliques et conjecture de Novikov, C. R. Acad. Sci. Paris, Série I 319 (1994) 815 – 820.
[9] Kasparov G.G., Skandalis G., Groups acting properly on bolic spaces and the Novikov conjecture, Preprint, 1999.
[10] Lafforgue V., Espaces de Schwartz, Preprint, 1998.
[11] Lafforgue V., Une démonstration de la conjecture de Baum-Connes pour les groupes réductifs sur un corps p-adique et pour certains groupes discrets possédant la propriété T, C. R. Acad. Sci. Paris, Série I 327 (1998) 439–444.
[12] Lafforgue V., Compléments à la démonstration de la conjecture de Baum-Connes pour certains groupes possédant la propriété $T$, C. R. Acad. Sci. Paris, Série I 328 (1999) 203–208.

[13] Wassermann A., Une démonstration de la conjecture de Connes-Kasparov pour les groupes de Lie linéaires connexes réductifs, C. R. Acad. Sci. Paris, Série I 304 (1987) 559–562.

[14] Weil A., Basic Number Theory, Springer, Berlin 1973.