The hamiltonian reduction of the BRST complex and N=2 SUSY

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We study the nonunitary representations of $N = 2$ Super Virasoro algebra for the rational central charges $\hat{c} < 1$. The resolutions for the irreducible representations of $N = 2$ $SVir$ in terms of the ”2-d gravity modules” are obtained and their characters are computed. The correspondence between $N = 2$ nonunitary ”minimal” models and the Virasoro minimal models + 2-d gravity is shown at the level of states. We also define the hamiltonian reduction of the BRST complex of $\hat{sl}(N)/\hat{sl}(N)$ coset to the BRST complex of the W-gravity coupled to the W matter. The case $\hat{sl}(2)$ is considered explicitly. It leads to the presentation of $N = 2$ Super Virasoro algebra as the Lie algebra cohomology. Finally, we reveal the mechanism of the correspondence $\hat{sl}(2)/\hat{sl}(2)$ coset — 2-d gravity.
1. Introduction

(The reader who is only interested in \(N = 2\) nonuninitary representations and don’t care for cosets for the first reading may skip the first four paragraphs of the Introduction and the whole Section 2.)

In the first part of this paper we continue studying the relationships between the non-critical \(W_N\) strings and \(\widehat{\mathfrak{sl}}(N)\)-\(\widehat{\mathfrak{sl}}(N)_k\)-cosets started in [1]. Our main goal here is to explain the role of the hamiltonian reduction in the story. It is well known, that the hamiltonian reduction [2], [3] maps the representations of \(\widehat{\mathfrak{sl}}(N)_k\) to that of \(W_N\). (In a sense, we can consider this as a definition of \(W_N\)). Technically speaking it is achieved by taking the (semi-infinite) cohomology \(M_W\) of \(\widehat{N}_+(\mathfrak{sl}(N))\) (“twisted” by some character) of the given module \(M_{\mathfrak{sl}(N)}\). So defined, \(M_W\) has a natural structure of a \(W\)-algebra module [3]. When computing the spectrum of physical states in coset, we take the (semi-infinite) cohomology of the whole \(\widehat{\mathfrak{sl}}(N)\) acting on the tensor product of two \(\widehat{\mathfrak{sl}}(N)\) representations. (Usually one of them is irreducible (the matter sector) and another is a Wakimoto representation (Toda sector)). There is a suspicion, backed by the explicit calculation for \(\widehat{\mathfrak{sl}}(2)\) coset and \(Vir\)-gravity respectively, that the spectra of \(\widehat{\mathfrak{sl}}(N)\) cosets and \(W_N\) gravities must coincide [4] [5] [6][1]. To make the statement more precise, let us consider two modules, \(M_1\) and \(M_2\) of \(\widehat{\mathfrak{sl}}(N)\) (\(\widehat{\mathfrak{sl}}(2)\) in this example), and two modules \(M_1^{DS}\) and \(M_2^{DS}\) of \(W_N\) corresponding to the first pair by the (quantum Drinfeld-Sokolov) hamiltonian reduction. For definiteness, let \(M_1\) be an ”admissible” irreducible representation \((k_1 + N = \frac{p}{q} — \text{rational})\) and \(M_2\) be a Wakimoto (free fields) representation with the value of the central charge \(k_2 = -k_1 - 2N\).

Then, the \(\widehat{\mathfrak{sl}}(2)\)-BRST homology of \(M_1 \otimes M_2\) do coincide with \(W_2 = Vir\)-BRST homology of \(M_1^{DS} \otimes M_2^{DS}\). The similar result is true if we take two Wakimoto or one Wakimoto and one ”transposed” Wakimoto modules for \(\widehat{\mathfrak{sl}}(2)\) and two free boson Fock modules for \(Vir\) respectively.

Whereas the spectra of cosets can be found explicitly, it is quite difficult to do for \(W_N\)-gravity. Although it is really possible to construct a BRST complex in that case [7] [8], it is not at all clear why it is possible. Then, it appears that this complex is not very convenient for the direct computations.

Taken together, all these facts motivate a desire to define a procedure of hamiltonian reduction not only for one \(\widehat{\mathfrak{sl}}(N)\) module (quantum Drinfeld-Sokolov reduction), but also for the whole BRST complex. In Section 2 we address this issue and give a proper modification of the reduction procedure. Then we outline how it works for \(\widehat{\mathfrak{sl}}(2)\).
At this stage, quite naturally, the (topologically twisted) \( N = 2 \) superconformal algebra appears. \( (N = 2 \text{ SVir} \text{ for the basic example of } \hat{sl}(2).) \) We show that the ”reduced” \( \hat{sl}_k(2) \)-BRST complex, with an irreducible representation in the matter sector is just a direct sum of two copies of the irreducible representation of \( N = 2 \text{ SVir} \) with the central charge \( \hat{c} = \frac{k}{k + 2} \) as a vector space. The differential is given by the zero mode \( G^+_0 \) of the superconformal current. (This is proven in Section 4.1).

In Section 3 we present the necessary background material on the nonunitary representations of \( N = 2 \text{ SVir} \), following [9]. In fact, in [9] was considered only the case of irrational \( \hat{c} \), so here we have to generalize that results to the more complicated situation of rational \( \hat{c} \). We obtain a resolution for the irreducible representations of \( N = 2 \text{ SVir} \) in terms of the products \( L(Vir) \otimes F(Liouville) \otimes F_{gh} \) (a ”2-d gravity” resolution), where \( L(Vir), F(Liouville) \) and \( F_{gh} \) denote respectively a Virasoro irreducible representation (”the matter”), a free bosonic Fock space (”the Liouville field”) and a two fermions Fock space (”the diffeomorphisms ghosts”). Using this resolution we find in particular a character formula for the irreducible representations of \( N = 2 \text{ SVir} \).

This formula explicitly incorporates the Lian-Zuckerman states of the Virasoro \((p, q) \) (with \( \frac{p}{q} = k + 2 \)) minimal model coupled to gravity. Thus it explains for the representations the relations between \( N = 2 \text{ SVir} \) and 2d-gravity, found in [10],[11] for the chiral algebras. It also gives a piece of evidence in favour of the idea [12] that \( N = 2 \) minimal theory ”may already know about 2-d gravity”. Also it is nice to have an object which puts together an infinite number of LZ states for the given matter field.

Finally, in Section 4.2 we combine the BRST complex hamiltonian reduction of Sec. 2 and the resolutions of Sec. 3 to trace explicitly the mechanism identifying the physical states in \( \hat{sl}(2) \) coset and 2-d gravity. From the point of view of the \( N = 2 \text{ SVir} \) representations theory this last section may be viewed as the consistency check for the results we obtained in the Section 3, the ”2-d gravity resolution” in particular.

2. Hamiltonian reduction of \( \hat{sl}(N) \) BRST complex

2.1. Some motivations

In a sense, this section is the second Introduction. We wish to informally explain here what we mean by the hamiltonian reduction of the BRST complex. The technical details can be found in the next two sections. First, it seems natural to repeat some motivations from [1].
In [2],[3] the quantum Drinfeld-Sokolov (DS) hamiltonian reduction for Wakimoto and irreducible representations has been defined as the homology \( H^0 \) of the DS BRST complex, associated with the constraints
\[
e^\alpha(z) = 1 \text{ if } \alpha \text{ is a simple root} \\
e^\alpha(z) = 0 \text{ if } \alpha \text{ is not a simple root}
\]
(2.1)

Suppose now that we have a \( \hat{\mathfrak{sl}}(N) \)-BRST complex which computes the cohomology of the tensor product of two modules \( M_1, M_2 \). Here \( M_1 \) can be either an irreducible representation \( L_k(\hat{\mathfrak{sl}}(N)) \) or a Wakimoto representation \( \text{Wak}_k \) (or possibly a "transposed" Wakimoto representation \( \hat{\text{Wak}}_k \)) and \( M_2 \) is a Wakimoto representation. In principle, it is also interesting to consider other combinations. In the language of the \( \hat{\mathfrak{sl}}(N)/\hat{\mathfrak{sl}}(N) \) coset model the modules \( M_1, M_2 \) represent respectively the fields of the matter and of the Liouville-Toda sectors of the theory. The cohomology \( H^*_\text{BRST}(M_1 \otimes M_2) \), where \( M_1 \) runs over some specified set of representations with the fixed level \( k \) and \( M_2 \) runs over all Wakimoto representations with level \( -2N - k \) form the spectrum of physical states of the theory. Usually we consider rational levels \( k + N = \frac{p}{q} \) and restrict \( M_1 \) to the "admissible" irreducible representations of \( \hat{\mathfrak{sl}}(N) \).

We wish to have for the whole BRST complex something like what the DS reduction is for the single module. More precisely, ultimately we wish to obtain a \( \mathcal{W}_N \) BRST complex by this "something". From the first sight it seems natural to try reduction independently on each factor. So we would have to add two sets of the DS ghost-antighost pairs, labeled by the positive roots of \( \mathfrak{sl}(N) \) and consider the DS BRST operators \( Q_1, Q_2 \) acting respectively on \( M_1 \) and \( M_2 \), then take the product \( M_1^{DS} \otimes M_2^{DS} \otimes \{\hat{\mathfrak{sl}}(N) \text{ ghosts}\} \), where \( M_i^{DS} = H^0_{Q_i}(M_i \otimes \{\text{DS ghosts}\}_i) \) are the reduced modules. \( M_i^{DS} \) are the representations of \( \mathcal{W}_N \) algebra.

This procedure does not work because we have to require that the reduction BRST operator, which is \( Q_1 + Q_2 \) here, commute with \( \hat{\mathfrak{sl}}(N) \)-BRST operator, and it is not difficult to check, that there is no proper modification of \( Q_1 + Q_2 \), commuting with \( Q_{BRST} \). The other problem to deal with is what to do with the \( \hat{\mathfrak{sl}}(N) \) ghosts. They form a representation of \( \hat{\mathfrak{sl}}(N) \) (at level \( 2N \)) and the general ideology requires to reduce this representation also. (We must somehow obtain the \( \mathcal{W}_N \) ghosts!) After a short reasoning it seems very natural not to introduce special reduction ghosts at all and to try to make the reduction of the BRST complex using the \( \hat{\mathfrak{sl}}(2) \) ghosts themselves.

\[^1\text{mainly because in } Q_1 + Q_2 \text{ necessarily participate not only the symmetric combinations of currents like } J_1^a + J_2^a \text{ but also the antysymmetric ones like } J_1^a - J_2^a. \text{ The commutators of the latter with } Q_{BRST} \text{ cannot be compensated by adding extra terms with ghosts}\]
2.2. The basic definition

(If the reader is not interested in "general nonsense" and only wants to consider \(\hat{sl}(2)\) and 2-d gravity, he may skip this subsection and start from 2.3.)

Let us give a definition.

**Definition-Hypothesis.** Let \(M_1\) be either the irreducible or the Wakimoto representation at level \(k\), and \(M_2\) be the Wakimoto representation at level \(-k - 2N\) of the \(\hat{sl}(N)\) algebra. Then there exists a spectral sequence, converging (at the second term) to the \(\hat{sl}(N)\)-BRST cohomology \(H^*_{Q_{BRST}}(M_1 \otimes M_2)\) of the module \(M_1 \otimes M_2\). Denote by \(Q_R\) a differential in the first term of the spectral sequence. Then the second term of the spectral sequence, i.e. the complex \((H^*_R, Q_W)\) with the differential \(Q_W\) is quasiisomorphic (has the same cohomology) to the \(W_N\)-BRST complex of the module \(M_1^{DS} \otimes M_2^{DS}\) where the superscript "DS" denotes the standard (Drinfeld-Sokolov) hamiltonian reduction.

We say, that \(Q_R\) "makes the hamiltonian reduction of \(\hat{sl}(N)\)-BRST complex" and call \((H^*_R, Q_W)\) "the reduced BRST complex".

**Comment** Because of the required quasiisomorphism the cohomology computed by the spectral sequence are determined by the cohomology of \(W_N\)-BRST. One the other hand this is the \(\hat{sl}(n)\)-BRST cohomology just by the definition. Thus we see that the equivalence of spectra of cosets and \(W\)-gravity should follow from the hypothesis above.

2.3. An example of the construction.

We have formulated a general hypothesis. It may seem a little bit complicated and not very explicit. Now we wish to show how it can be proved for the algebra \(\hat{sl}(2)\). In this example the coincidence of spectra was known for some time from the explicit computation, which was fairly straightforward in this case. However we shall see that our construction is nontrivial already in this simplest case and gives rise to the interesting \(N=2\) supersymmetric structure.

Let us we consider a decomposition \(Q_{BRST} = Q_R + Q_W\) with

\[
Q_R = \oint c^+(E_1 + E_2) + c^0(H_1 + H_2 + 2(b_+ c^+ - b_- c^-))
\]

\[
Q_W = \oint c^- (F_1 + F_2 + c^+ b_-)
\]

(2.2)

It is convenient to rephrase our hypothesis in this particular case in the form of the

**Theorem.** Let \(M_1\) be either the irreducible or the Wakimoto representation at level \(k\), and \(M_2\) be the Wakimoto representation at level \(-k - 4\) of the \(\hat{sl}(2)\) algebra. Then
\( \widehat{\mathfrak{sl}}(2) \)-BRST complex for \( M_1 \otimes M_2 \) has a structure of the double complex with differentials \( (Q_R, Q_W) \) such that \( Q_R + Q_W = Q_{BRST} \). Consider a spectral sequence of this double complex whose first term has a differential \( Q_R \). Then the second term of the spectral sequence, i.e. the complex \((H^*_{Q_R}, Q_W)\) with the differential \( Q_W \) is quasiisomorphic (has the same cohomology) to the Virasoro-BRST complex of the module \( M_1^{DS} \otimes M_2^{DS} \) where the superscript "DS" denotes the standard (Drinfeld-Sokolov) Hamiltonian reduction.

In other words, there exists a Hamiltonian reduction of \( \widehat{\mathfrak{sl}}(2) \)-BRST complex”.

It is a straightforward calculation to check that the operators (2.2) do define the structure of the double complex on the \( \widehat{\mathfrak{sl}}_k(2) \) BRST complex. What is \( H^*_{Q_R}(\widehat{\mathfrak{sl}}_k(2) \otimes \text{Wakim}_{-4-k}) \) the \( Q_R \) cohomology of the chiral algebra \( \widehat{\mathfrak{sl}}_k(2) \otimes \text{Wakim}_{-4-k} \) here?

Doing the direct computation\(^2\) one convinces oneself that two currents

\[
G^+(z) = \frac{1}{2(k+2)}(c^-(F_1 + F_2 + c^+b_0) + 2\partial(c^-))
\]

\[
G^-(z) = b_-(E_1 - E_2)
\]

belong to \( H^*_{Q_R} \). Their OPE is

\[
G^+(z)G^-(0) = \frac{k/k+2}{z^3} + \frac{J(z)}{z^2} + \frac{T(z) + \partial J(z)}{z} + \text{regular terms}
\]

(2.4)

where

\[
J(z) := c_-(b^- - \frac{2}{\sqrt{2(k+2)}}(\partial\varphi(z) + \frac{1}{\sqrt{2(k+2)}}([E_1 + \beta]\gamma + :b_+ c^+ :)))
\]

\[
+ \frac{1}{2(k+2)}\{Q_R, b_0\}
\]

(2.5)

The operators \( J(z), T(z) \) also are the nontrivial elements of \( H^*_{Q_R}(\widehat{\mathfrak{sl}}_k(2) \otimes \text{Wakim}_{-4-k}) \). Moreover, it is true that \( T(z) \) is equivalent modulo \( Q_R \)-exact term to the twisted stress-energy of the coset and that four currents \( G^+(z), G^-(z), T(z), J(z) \) form a closed chiral algebra which is just a (topologically twisted) \( N=2 \) SuperVirasoro with the central charge

\[
\hat{c} = \frac{k}{k+2}
\]

(2.6)

This is not very surprising. For example, the similar phenomena was observed in [14] for the Kazama-Suzuki coset with \( k = -3 \) in our notations.

\(^2\) It can be a good idea to use the Mathematica OPE package [13] to do this!
Thus $SVir \subset H^*_R(sl_k(2) \otimes \text{Wakim}_{-4-k})$. In fact, there is also an operator $c^0_0$ — the zero mode of the ghost $c^0(z)$ — which belongs to $H^*_R$. In the next sections we shall see using more complicated technique that actually

$$H^*_R(sl_k(2) \otimes \text{Wakim}_{-4-k}) = [N = 2 SVir] \oplus c^0_0(N = 2 SVir)$$

as a chiral algebra. For now let us assume this is true.

Similarly it can be shown (see Sec.4.1) that for the representations the cohomology

$$H^*_R(L_k(sl(2)) \otimes \text{Wakim}_{-4-k})$$

are given just by the direct sum of two copies (one is again shifted by $c^0_0$) of the irreducible representation $L(N = 2 SVir)$ of $N=2 SVir$:

$$H^*_R(L_k(sl(2)) \otimes \text{Wakim}_{-4-k}) = [L(N = 2 SVir) \oplus c^0_0L(N = 2 SVir)]$$

The crucial for the following observation is that the second differential $Q_W$ of our double complex is just a zero mode of the superconformal currents:

$$Q_W = G_0^+$$

Thus the the reduced BRST complex for $sl_k(2)$ can be expressed solely in terms of $N=2 SVir$:

$$\left(H^*_R(L_k(sl(2)) \otimes \text{Wakim}_{-4-k}), Q_W \right) = (L(N = 2 SVir) \oplus c^0_0L(N = 2 SVir), G_0^+)$$

Now we shall use the relation between $N=2 SVir$ and 2d gravity coupled to the minimal matter found in [10], [11]to establish the quasiisomorphism the main hypothesis claims. Let me remind here this relation.

Consider the BRST complex of 2d gravity coupled to the minimal matter with the central charge $c_M$. The chiral algebra of the matter sector is $Vir$, and that of the Liouville and ghost sectors are correspondingly $Heis$ and $Clif$ with the total central charge $-c_M$. Then

There is an embedding

$$N = 2 SVir \rightarrow Vir \otimes Heis \otimes Clif$$

of the chiral algebras and the corresponding map on the representations such that $G^{-}(z) \rightarrow b(z)$, $G^{+}(z) \rightarrow J_{BRST}(z)$, where $J_{BRST}(z)$ is a Vir-BRST current plus a total derivative term. In particular, $G_0^+ \rightarrow Q_{Vir}$ — a BRST operator of 2-d gravity.
We shall show in Sections 3,4 that the similar relation exists at the level of representations. In particular, for the irreducible representation \( L(N = 2 SVir) \) there exists a resolution in terms of the "2-d gravity" modules \( L(Vir) \otimes F(Liouv.) \otimes F_{gh} \), where \( L(Vir) \), as usual, denotes the irreducible representation (of Virasoro, this time), \( F(Liouv.) \) is a free boson Fock space and \( F_{gh} \) is the diffeomorphisms ghosts Fock space:

\[
0 \to L(N = 2 SVir) \to L(Vir) \otimes F(Liouv.) \otimes F_{gh} \to L'(Vir) \otimes F'(Liouv.) \otimes F_{gh} \to L''(Vir) \otimes F''(Liouv.) \otimes F_{gh} \to \cdots
\]

(2.12)

(This resolution is infinite in the most interesting cases, see Section 3.)

To complete the proof, we should substitute this resolution into the reduced BRST complex (2.10) to obtain the double complex of "2-d gravity" modules with two differentials. One of them comes from the resolution (2.12). The other one is just

\[
Q_W = G_0^+ = Q_{Vir}
\]

We are almost done now. The cohomology of the double complex are computed in the Sec. 4.2 where it is shown that they reduce to \( H^*_{Q_{Vir}}(L(Vir) \otimes F(Liouv.) \otimes F_{gh}) \) — the 2-d gravity BRST cohomology of the first term in the resolution (2.12). Before passing to the details of the proof let us summarize what we have learned and propose the possible generalization for the algebras \( \hat{sl}(N) \). First, we decomposed the Lie algebra BRST operator into two pieces, corresponding to the Borel subalgebra of \( sl(2) \) and its compliment (+some ghost terms corrections) to obtain a structure of a double complex. This decomposition for \( \hat{sl}(N) \) goes through if we take as a \( Q_R \) a proper piece of \( \hat{sl}(N) - Q_{BRST} \), corresponding to the maximal parabolic subalgebra of (the finite dimensional) \( sl(N) \).

Then we computed the cohomology of \( Q_R \) which turned to be the irreducible representation of \( N=2 SVir \). For \( \hat{sl}(N) \) it is likely the representation of \( N=2 SuperW \) algebra. The cohomology has the structure of a complex with differential \( Q_W = G_0^+ \).

Finally we used the map (2.11)[10], [11] to relate this complex to the BRST complex of 2d gravity coupled to the minimal matter. Such map into W-gravity+W minimal matter also exists for any \( N=2 SuperW \) algebra [11]. Then we show that the cohomology of the reduced BRST complex are the same as of the W-system.
3. Representations of N=2 SuperVirasoro

3.1. The bosonisation formulas.

In this subsection we recall the results from the representation theory of \( N = 2SVir \), obtained in [9]. It is important, that we actually need the nonunitary representations of \( N = 2SVir \), it follows from the formula (2.6) for the central charge (remember that \( k \) is just rational, not necessarily integer).

Let us introduce the basic notations. We have a system consisting of the Virasoro algebra (\( Vir \)) — a matter sector, a free bosonic field \( \phi \) with the background charge (\( Heis \)) — a Liouville sector and a pair of fermions \( b, c \) of spins 2,-1 (\( Clif \)) — the diffeomorphism ghosts ( in the brackets are the names of the corresponding chiral algebras). We require the total central charge be equal to zero. Then the currents

\[
J(z) = : cb : + \alpha_\pm \partial \phi \\
G^+(z) = : c [T_{Vir} + T_\phi + \frac{1}{2}T_{bc}] : -2\alpha_\pm \partial (c \partial \phi) + \frac{1}{2} (1 - 2\alpha^2) \partial^2 c \\
G^-(z) = b(z) \\
T = T_{Vir} + T_\phi + T_{bc} \\
T_\phi = -\frac{1}{4} : (\partial \phi)^2 : + \beta_0 \partial^2 \phi
\]

(3.1)

satisfy the OPE of (twisted) N=2 SVir chiral algebra. In fact these formulas give the embedding of N=2 SVir into the tensor product of three other chiral algebras \( Vir \otimes Heis \otimes Clif \) ( we used this fact in the previous section ). To describe the properties of this map it is technically convenient to bosonise the Virasoro algebra by the free field \( X \) with the background charge \( \alpha_0 \). In other words we embed \( Vir \) into the Heisenberg algebra generated by \( \partial X \) which we denote by \( Heis' \) to distinguish it from the Liouville \( Heis \).

Substituting the bosonized matter stress energy

\[
T_{Vir}(z) = \frac{1}{4} : (\partial X)^2 : + \alpha_0 X \\
\beta_0^2 - \alpha_0^2 = 1, \quad \alpha_\pm = \alpha_0 \pm \beta_0 \\
c_{Vir} = 1 - 24\alpha_0^2
\]

(3.2)

into (3.1) we finally obtain the bosonisation prescriptions for N=2 SVir we need. Unlike the standard bosonisation [15],[16], [17] the formulas for \( G^+(x) \) and \( G^-(z) \) are very asymmetric. Comparing (3.2) with (2.6) we see that

\[
\alpha_- = \frac{-1}{\sqrt{k + 2}}
\]

(3.3)
(it is a nonstandard notation for $\hat{sl}(2)_k$!). For the representations, we take a Fock space
\[ F_{\alpha\beta} = F_{\alpha}(X) \otimes F_{\beta}(\phi) \otimes F_{gh} \]  
(3.4)

$F_{\alpha}(X)$ and $F_{\beta}(\phi)$ here are the standard Fock modules of $Heis'$ and $Heis$ with vacuums $|\alpha>$ and $|\beta>$ respectively and $F_{gh}$ is a ghosts Fock space (a $Clif$ Verma module) with the vacuum vector $|0>$ annihilated by
\[ c_n |0> = 0 \quad n > 1, \quad b_n |0> = 0 \quad n > -2 \]  
(3.5)

In $F_{gh}$ we take a vector $|0>_{phys} = c_1 |0>$ and define the N=2 vacuum as $\Omega = |\alpha> \otimes |\beta> \otimes |0>_{phys}$. This procedure is well known in string theory. Here we use it to endow the free field Fock space $F_{\alpha\beta}$ with a structure of a highest weight N=2 $SVir$ module:
\[ L_n \Omega = J_n \Omega = G_n^- \Omega = 0, \quad n \geq 0 \]
\[ G_n^+ \Omega = 0, \quad n > 0 \]
\[ L_0 \Omega = \Delta \Omega = (-1 + \alpha(\alpha - 2\alpha_0) - \beta(\beta - 2\beta_0)) \]
\[ J_0 \Omega = Q \Omega = (1 + 2\alpha_- \beta)\Omega \]  
(3.6)

It is convenient to rewrite the formula for the conformal weight as
\[ \Delta(\alpha, \beta) = (\alpha + \beta - \alpha_+)(\alpha - \beta - \alpha_-) \]  
(3.7)

There are two screening operators in our bosonisation. One of them is just $E = \oint : e^{\alpha_+ X(z)} :$. It comes from the bosonisation of the $Vir$ matter. The other one is $F_1 = \oint : b(z) e^{-\frac{\alpha_+}{2}(X(z) + \phi(z))} :$ [11],[18],[19]. It is fermionic and local to itself:
\[ F_1^2 = 0 \]  
(3.8)

Together, $E$ and $F_1$ form a quantum superalgebra $u_q(n_+(sl(2|1)))$ with $q = e^{\pi i \alpha_+^2}$. Namely they satisfy the Serre relation:
\[ E^2 F_1 - (q + q^{-1}) EF_1 E + F_1 E^2 = 0 \]  
(3.9)

(As usual, the left hand side of (3.9)is to be understood as a part of some formal polynomial in screenings acting on the appropriate state.)

There is a simple but important remark to me made here. The basic object for the correspondence N=2 $SVir \rightarrow \{2 - d gravity\}$ above is the Virasoro algebra itself, not the
free field $X(z)$ which is just a useful tool for describing this correspondence. Bosonising
the Virasoro algebra, we could choose a screening operator $E(-) = \oint e^{\alpha - X(z)}$ instead
of $E(+) = \oint e^{\alpha + X(z)}$. The vertex operators corresponding to $E(-)$, $F$ are local with
respect to each other, so we may simply set

$$E(-)F - FE(-) = 0$$  \hfill (3.10)

Hence the algebraic structure produced by the pair $E(-)$, $F$ is much simpler and therefore
much less powerful for the purposes of the representation theory then the structure pro-
duced by $E(+)$, $F$. Therefore in this subsection we’d better stick to the latte r. In fact, we
will be able to take advantage of the simplicity of (3.10) later, \textit{when we already know the
representation theory of N=2 SVir}.

Using very rigid conditions (3.8), (3.9) we can classify the irreducible representations of
$\text{N}=2 SVir$ according to the types of the free fields resolution they have. By such resolution
we mean the complex of the free field Fock spaces with the cohomology being nontrivial
only in the zero degree where it is represented by the irreducible representation. Having
a complex we compute its Euler characteristics (character valued, as usual) which turns
to be a character of the irreducible $L_{\alpha,\beta}$. It is convenient to deal with the normalized
characters

$$\tilde{\chi}_{\alpha,\beta} = \frac{\chi_{\alpha,\beta}}{\chi(F_{\alpha,\beta})}$$

$$\chi(F_{\alpha,\beta}) = Tr_{F_{\alpha,\beta}}(q^{L_0} x^{2J_0})$$  \hfill (3.11)

\textbf{3.2. The case of irrational $\hat{c}$}

First let me describe the classification for the generic (irrational) values of $\hat{c}$. Depending
on $(\alpha, \beta)$, the irreducible representation $L_{\alpha,\beta}$ may belong to either of the four following
types.

\textbf{Case I}. $(\alpha, \beta)$ is generic, the module $F_{\alpha,\beta}$ is irreducible. The corresponding complex
is therefore trivial, $\tilde{\chi}_{\alpha,\beta} = 1$.

\textbf{Case II}.

$$\alpha_n m = \alpha_+ \frac{1 - n}{2} + \alpha_- \frac{1 - m}{2}$$  \hfill (3.12)

$\beta$ is generic. This case essentially reduces to the well known theory for the Virasoro algebra.
The map $E^n$ is surjective, its kernel is a submodule in $F_{\alpha,\beta}$ generated by the highest weight
vector. It gives

$$L(N = 2 SVir) = L_m n(Vir) \otimes F_\beta \otimes F_{gh}$$  \hfill (3.13)
This is the only map from $F_{\alpha\beta}$ and the only map to $F_{\alpha+n\alpha+ \beta}$; there is no maps to $F_{\alpha\beta}$ or from $F_{\alpha+n\alpha+ \beta}$. Therefore the submodule in $F_{\alpha\beta}$ generated by the highest weight vector is irreducible. The quotient of $F_{\alpha\beta}$ by this submodule is also irreducible and coincides with $F_{\alpha+n\alpha+ \beta}$. The character is

$$\tilde{\chi}_{\alpha\beta} = 1 - q^{nm} \quad (3.14)$$

This was the typical example of the argument to be used in this sort of constructions.

\textbf{Case III$^-$}.

$$\alpha - \beta = -\alpha - l \quad (3.15)$$

When $l \geq -1$ the map $F_1$ sends the highest weight vector of $F_{\alpha,\beta}$ to a nonzero element, which generate in $\mathcal{F}_{\alpha-\alpha,\beta-\alpha+}$ a proper submodule $\mathcal{S}\mathcal{F}_{\alpha-\alpha,\beta-\alpha+}$. There is no other maps into $\mathcal{F}_{\alpha-\alpha,\beta-\alpha+}$, so $\mathcal{F}_{\alpha-\alpha,\beta-\alpha+}$ is the only proper submodule. Therefore it must coincide with the kernel of the map $F_1$ from $\mathcal{F}_{\alpha-\alpha,\beta-\alpha+}$ to $\mathcal{F}_{\alpha-\alpha,\beta-\alpha+}$. This means that the diagram $\text{III}_-$ is exact — the image of the incoming arrow coincide with the kernel of the outgoing arrow.

The diagram $\text{III}_-$ has already a natural structure of the complex. The graded components are just the Fock spaces at the vertices and the differentials are given by the arrows. This complex is infinite in both directions. It is exact, so its cohomology is trivial. To obtain the resolution of $L_{\alpha\beta}$ one cuts the diagram by the arrow going from $F_{\alpha\beta}$ to obtain two complexes with the equal cohomology (so there are two resolutions in fact), one can use either of them. The character is

$$\tilde{\chi}_{\alpha\beta} = \frac{1}{1 + x^{-1}q^{l+1}} \quad (3.16)$$

We see that for $l \geq -1$ the formula (3.16)can naturally be interpreted as a character of the representation with the highest weight $(\Delta_{\alpha\beta}, q_{\alpha\beta})$. But for $l < -1$ the identical transformation $(3.16) \rightarrow \frac{xq^{-l-1}}{1+xq^{-l-1}}$ shows that the character we compute now correspond to the weight $(\Delta_{\alpha+\alpha,\beta+\alpha}, q_{\alpha+\alpha,\beta+\alpha})$. The reason for this phenomena is simple. For $l < -1$ the map $F_1$ kills the highest weight vector of $F_{\alpha\beta}$ and sends some vector $w \in F_{\alpha\beta}$ to the highest vector of $\mathcal{F}_{\alpha-\alpha,\beta-\alpha+}$. Hence each Fock space has one cosingular vector $w_{\alpha\beta}$ and the irreducible representation is a submodule of $F_{\alpha\beta}$ generated by the highest weight vector of the Fock space$^3$. Now, to obtain a resolution of $L_{\alpha\beta}$ we should cut the diagram $\text{III}_-$ by the arrow coming into $F_{\alpha\beta}$. The character is

$$\tilde{\chi}_{\alpha\beta} = \frac{1}{1 + xq^{-l-1}} \quad (3.17)$$

$^3$ Compare to $l \geq -1$ when the irreducible representation was a quotient of the Fock space.
Case $III_+$

$$\alpha + \beta = \alpha_-(l + 1) + \alpha_+ \quad (3.18)$$

One can repeat everything that has been said about $III_-$. The only subtlety here is to check that the composition of two consequent maps in the diagram is zero. The reader should convince oneself it is true using (3.9) and simple $q$-polynomial identities. The formulas for the characters are given by the same formulas (3.16), (3.17).

Case $IV_-$ — the conditions II and $III_-$ ($IV_-$) are met simultaneously.

$$\alpha_{nm} = \alpha_+ \frac{1 - n}{2} + \alpha_- \frac{1 - m}{2}$$
$$\beta_{nml} = \alpha_+ \frac{1 - n}{2} + \alpha_- \frac{1 - m + 2l}{2} \quad (3.19)$$

We denote $F_{\alpha_{nm} \beta_{nml}}$ by $F_{nml}$. We know everything already about the maps in the diagram. First consider the resolution of the representation with $n \geq 0$, i.e., belonging to the left column in Fig.1. To obtain a resolution we should again cut the diagram by the horizontal line crossing the arrow above $(\alpha \beta)$ for $l \geq -1$ or below $(\alpha \beta)$ for $k < -1$. Keeping the upper half we end up with a "ladder" shown in the Fig.2. The structure of the complex $\{C^r, d_r\}_{r \geq 0}$ is given by

$$C^0 = F_{n+1 \ m \ l}, \ C^r = F_{n+1 + r \ m \ l} \oplus F_{-(n+r) \ m \ l}$$
$$d_0 = E^{n+1} \oplus F_1, \ d_r = \begin{pmatrix} F_1 & 0 \\ E^{n+1 + r} & \ x_{n+r}EF_1 - F_1E \end{pmatrix}$$
$$x_{l+1} = (q + q^{-1}) - \frac{1}{xl}, \ x_0 = q + q^{-1} \quad (3.20)$$

The character is

$$l \geq -1 \ \tilde{\chi}_{\alpha \beta} = \frac{1 - q^{m(n+1)} + q^{m-l-1}(1 - q^{mn})}{(1 + x^{-1}q^{l+1})(1 + qx^{m-l-1})}$$
$$l < -1 \ \tilde{\chi}_{\alpha \beta} = \frac{1 - q^{mn} + q^{m-l-1}(1 - q^{m(n-1)})}{(1 + qx^{-l-1})(1 + qx^{m-l-1})} \quad (3.21)$$

Case $IV_+$ — the conditions II and $III_+$ are met simultaneously.

$$\alpha_{nm} = \alpha_+ \frac{1 - n}{2} + \alpha_- \frac{1 - m}{2}$$
$$\beta_{nml} = \alpha_+ \frac{1 + n}{2} + \alpha_- \frac{-1 + m + 2(l + 1)}{2} \quad (3.22)$$

One just repeats what was said about $IV_+$ (probably it is better to take a bottom half of the cut diagram to construct a resolution of the representations with $n > 0$ in this case). The formulas for the characters (3.22) are applicable.
Note that the equation

\[ \Delta(\alpha, \beta) = (\alpha + \beta - \alpha_+)(\alpha - \beta - \alpha_-) = 0 \]  

(3.24)

has two branches of solutions corresponding to either \(III_\pm\) or \(IV_\pm\) with \(l = -1\).

It is important to note that the resolutions in the cases \(I - IV\) above can be rewritten in terms of modules \(L_\alpha \otimes F_\beta \otimes F_{gh}\) instead of \(F_\alpha \otimes F_\beta \otimes F_{gh}\), where \(L_\alpha\) is the irreducible representation of the Virasoro algebra.

Indeed, for the case II we have shown this explicitly in (3.13). For the cases \(I, III\) this is trivial because the irreducible and free field representations are the same object: \(L_\alpha = F_\alpha\). In the case IV we can compute the cohomology of the free field resolution using the spectral sequence, associated with the ”vertical” filtration, shown in Fig.2. The first term of this spectral sequence computes the ”horizontal” (in Fig.2) cohomology, which gives exactly \(L_\alpha \otimes F_\beta \otimes F_{gh}\). Then the second term represents the resolution we are after:

\[ 0 \rightarrow L_{nml} \rightarrow L_{nm} \otimes F(\beta_{nml}) \otimes F_{gh} \rightarrow L_{n+1 \; m} \otimes F(\beta_{n+1 \; ml}) \otimes F_{gh} \rightarrow \cdots \]  

(3.25)

Existence of such resolution in terms of irreducible representations of Virasoro is not surprising at all, because it is the Virasoro algebra, not a free field \(X_z\) which is basic in the correspondence between Supervirasoro and 2-d gravity, so everything has to be expressible in terms of it. This fact should be viewed as a counterpart for the representations of the map (2.11)between the chiral algebras.

In such form, the resolutions above can be generalized to the rational values of \(\hat{c}\).

3.3. The case of rational \(\hat{c}\)

Up to now we dealt with irrational \(k\). But as (2.6)shows, to consider the most interesting ”minimal” cosets we must take \(k\) rational! The free field resolution becomes more complicated in this case, comparing to what we had in \(I - IV\). The reason for it is that for \(k + 2 = \frac{p}{q}\), \(p, q\) — integer numbers, the bosonic screening \(E\) becomes nilpotent:

\[ E^q = 0 \]  

(3.26)

It results essentially in that there appear more maps among the free Fock spaces than there were for irrational central charges. Easy to see, that it changes the diagrams only for the \((\alpha, \beta)\) pairs where \(\alpha\) satisfies the integrality condition given by (3.12), i.e., for the cases \(II, IV_\pm\). Thus the diagrams of maps in Fig.1 remain the same for the cases \(I, III\).
The proper modification of the diagram $IV$ in Fig.1 is shown in the Fig.3. Comparing Fig.1, Fig.3, and the Felder resolution for the irreducible "discrete" representation of the Virasoro algebra we see that passing to the rational values of $\hat{c}$ in $N=2 SVir$ effectively amounts to using the correct input for the Virasoro piece for the rational central charge $c_M$.

Unlike the diagrams shown in Fig.1, the diagram in Fig.3 does not admit a natural structure of the complex. Essentially this is because of presence of the (shaded in Fig.3) rows, corresponding to the boundary of the Kac table for Virasoro. To obtain the complex (actually, the double complex), one throws away these lines to come up with a picture, shown in Fig.4 — it is a commutative diagram and the composition of any two consequent vertical or horizontal arrows is zero. Now we can use a spectral sequence, similar to what we used before for the complex in the Fig.2. The "horizontal" cohomology give the resolution (here $L_{nml}$ denotes the irreducible representation of $N = 2 SVir$, corresponding to $(\alpha_{nm}, \beta_{nml})$, whereas $L_{nm}$ denotes the irreducible representation of $Vir$, corresponding to $\alpha_{nm}$)

\[0 \to L_{nml} \to L_{nm} \otimes F(\beta_{nml}) \otimes F_{gh} \to L_{(n+1)m} \otimes F(\beta_{(n+1)m}) \otimes F_{gh} \to \cdots\]
\[\to L_{(q-1)m} \otimes F(\beta_{(q-1)m}) \otimes F_{gh} \to L_{(q-2)m} \otimes F(\beta_{(q-2)m}) \otimes F_{gh} \to \cdots\]
\[\to L_{1m} \otimes F(\beta_{1m}) \otimes F_{gh} \to L_{1m} \otimes F(\beta_{1m}) \otimes F_{gh} \to \cdots\]  

(3.27)

(This is $IV_-$ case, the similar resolution exists for $IV_+$.) Starting from $L_{(q-1)m} \otimes F(\beta_{(q-1)m}) \otimes F_{gh}$ the resolution becomes periodic (with the period $2(q - 1)$) in its Virasoro sector. Passing to the next period shifts the Liouville charge $\beta$ by the amount $\alpha_p$ (cf. the charges of the first and the last a Fock spaces in (3.27)). Note also that in (3.27) participate only the representations $L_{nm}$ from the principal Kac table (i.e. with $0 < n < q$, $0 < m < p$).

The "throwing away" procedure that we use seems a little bit ad hoc. In fact, it can be understood using the same logic as we used in the previous section for irrational $\hat{c}$. Another way to obtain the resolution (3.27) is explained in the next section.

Let us compute the characters of the representations of the type $IV$ here, using the resolution (3.27). The formulas are (unlike the previous subsection, these are just the usual
\[ \chi_{nml} = x^{-2\alpha - \beta_{nml}} \chi_{bc}(t, x) \sum_{r=1}^{q-1} (-1)^{r-n} x^{r-n} \chi_{Vir}^{Vir}(t) \left( \sum_{s=0}^{\infty} \sum_{r \geq n, s=1} \sum_{r < n, s=1} x^{2sq} t^{\Delta^+(s)} \right) \]

\[ \Delta^+(s) = -\beta_{rm(l+sp)}(\beta_{rm(l+sp)} - \alpha_+ + \alpha_-) \]

\[ \Delta^-(s) = -\beta_{rm(l-m+sp)}(\beta_{rm(l-m+sp)} - \alpha_+ + \alpha_-) \]

— for the \( IV_- \) case. Because the symbol \( q \) is reserved already for the index of the minimal model, we denoted the modular parameter by \( t \). \( \chi^{Vir}_{ir}(t) \) and \( \chi_{bc}(t, x) \) stand for the charaters of the Virasoro irreducible representation \( L_{rm} \) the \((b, c)\) ghosts Fock space (the latter is independent of \( n, m, l \)) respectively.

It should be noted that the formulas for the characters were also obtained by Kac and Wakimoto and, independently, in [17]. Both groups used the different methods, which did not allow to see the role of the Virasoro characters.

It is particularly interesting to consider the representations \( IV_+ \) with \( l = jp \) and \( IV_- \) with \( l = -1+jp \), where \( j \) is any integer number. For definiteness let us consider the latter.

Note, that for such \( l \) the charges \( \beta \) of the Liouville Fock spaces coupled to the Virasoro irreducible representation \( L_{mn}(Vir) \) in the resolution (3.27) are exactly the same as in the Lian-Zuckerman’s papers [20][21][22] on the spectrum of 2-d gravity; it means that these liouvlles are the ”dressing fields” of the discrete states. More precisely, the weights \( \Delta^{\pm}(s) \) in (3.28) are related to the weights of the singular vectors in the Verma module \( M_{mn}(Vir) \) (whose quotient is \( L_{mn}(Vir) \)) by the formula:

\[ \Delta^\pm(s) + \Delta_{mn}(-j - s) = 1 \]
In this formula, $\Delta_{nm}(i)$ denotes the weight of the singular vector in $M_{mn}(Vir)$, and $i$ is the ghost number of the LZ state, corresponding to this singular vector.

It is interesting, that the the infinite number of LZ states (with the ghost numbers $\leq -j$) are combined in the single object — the character of the irreducible representation of $N = 2$ SuperVirasoro algebra. It demonstrates the relationship between 2-d gravity and $N = 2$ $SVir$ very explicitly at the level of representations. In Section 4.2 we shall learn more about this.

4. Details of computations and proof of the equivalence theorem

4.1. Reduction of the BRST complex

Now we can compute the cohomology $H^*_{Q_R}(L_k \otimes W_{ak-k-4})$. It is convenient to take the Felder resolution of the irreducible representation $L_k$ of $\hat{sl}(2)$ in terms of free Fock modules generated by the currents $\partial x(z)$, $\beta_M(z)$, $\gamma_M(z)^4$, using a usual screening operator

$$E^{(-)}_{sl(2)} = \oint : \beta_M(z)e^{\sqrt{2}\alpha_- x(z)} :$$ (4.1)

In principle, there is another screening

$$E^{(+)}_{sl(2)} = \oint : (\beta_M(z))^{-(k+2)}e^{\sqrt{2}\alpha_+ x(z)} :$$ (4.2)

Although the notion of the rational powers of $\beta_M(z)$ can be justified if we make bosonisation of the $\beta_M, \gamma_M$ pair, we prefer to deal with the "conventional" choice (4.1).

We know that under the standard (Drinfeld-Sokolov) hamiltonian reduction these two screenings go to the two screenings

$$E^{(\pm)}_{Vir} = \oint : e^{\alpha_{\pm} x(z)} :$$ (4.3)

of the Virasoro algebra. Thus we can anticipate that it is also so in our reduction scheme, which means that the "conventional" screening $E^{(-)}_{sl(2)}$ must go to $E^{(-)}_{Vir}$ and the "unconventional" one $E^{(+)}_{sl(2)}$ must go to $E^{(+)}_{Vir}$.

As the Toda-Liouville sector is already bosonized, we use the subscript "T" for its $\beta, \gamma$ system, so this sector is generated by the currents $\partial \varphi(z), \beta_T(z), \gamma_T(z)$. 

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Let us decompose the reduction BRST operator $Q_R$ as

$$Q_R = \hat{Q}_R + c^0(0)\mathcal{H}_0$$

$$\mathcal{H}_0 = (H_1 + H_2 + 2 : b_+c^+ : -2 : b_-c^- :)_0$$

and note that there is a relation

$$\mathcal{H}_0 = \{Q_R, b_0(0)\}$$

Then the usual argument shows that on the cohomology

$$\mathcal{H}_0|_{H^*_{Q_R}} = 0$$

must hold. Moreover, we see that the operator $c^0(0)$ is $Q_R$-nontrivial so it maps the $Q_R$ cohomology to itself:

$$|\lambda \in H^*_{Q_R} \rightarrow c^0(0)|\lambda \in H^*_{Q_R}$$

(This “doubling” of $Q_R$ cohomology will be important in relations to 2-d gravity.)

It is convenient to choose the ”light-cone” coordinates in the $\beta, \gamma$ sector:

$$\beta_+ = \frac{1}{\sqrt{2}}(\beta_M + \beta_T)$$
$$\beta_- = \frac{1}{\sqrt{2}}(\beta_M - \beta_T)$$
$$\gamma_+ = \frac{1}{\sqrt{2}}(\gamma_M + \gamma_T)$$
$$\gamma_- = \frac{1}{\sqrt{2}}(\gamma_M - \gamma_T)$$

To compute $H^*_{Q_R}$ let us use a double complex with differentials

$$d_1 = \sqrt{2} \oint \left(c^+\beta_+ - \sqrt{2}c^0(\beta_+\gamma_+ - b_+c^+)\right)$$
$$d_2 = \oint c^0(-\sqrt{2}\alpha_+(\partial\varphi + \partial x) - 2 : \beta_-\gamma_- : -2 : b_-c^- :$$

Thus we have managed to completely separate in $d_1$ and $d_2$ two subsets of fields: $\{c^+(z), b_-(z), \beta_+(z), \gamma_+(z)\}$ and $\{c^0(z), b_0(z), c^-(z), b_-(z), x(z), \varphi(z), \beta_-(z), \gamma_-(z)\}$. It means that our double complex is in fact a direct product of two complexes with differentials respectively $d_1$ and $d_2$. Their cohomology can be computed independently of each other. Let us compute the cohomology of $d_1$ first. It is easy to see that in $H^*_{d_1}$ two
systems $\beta_+, \gamma_+$ and $b_+, c^+$ "cancel" each other. It means that there are no excitations along these four directions in the "physical" space (i.e. $H^*_{d_1}$). This is an example of the famous Kugo-Ojima quartet decoupling mechanism. To compute $H^*_{d_2}$ we need to bosonise the $\beta_-, \gamma_-$ system in terms of two free bosons $\psi(z), \chi(z)$:

$$
\begin{align*}
: \beta_-\gamma_- : (z) &= \partial \psi(z) \\
\beta_-(z) &= e^{(\psi+i\chi)}
\end{align*}
$$

(4.10)

To be precise, following [23], the Hilbert space $F_{\beta,\gamma}$ of the $\beta_-, \gamma_-$ system is represented by the (0-) cohomology of the screening operator

$$
F = \oint : \exp(-i\chi(z)) : 
$$

(4.11)

acting on the free Fock modules

$$
\bigoplus F(\psi) \otimes F(\chi)
$$

(4.12)

(so it is a free field resolution). Now we have a double complex again: one differential is $d_2$ which acts now by the "free field" formula

$$
d_2 = \oint \left( -\sqrt{2}\alpha_+(\partial \varphi(z) + \partial x(z)) - 2\partial \psi(z) : -2 : b_- c^- : \right)
$$

(4.13)

and the other one is $F$. To compute the cohomology of this double complex let us find $H^*_{d_2}$ first. It is easy to do because it is actually the $U(1)$ cohomology. The representatives of $H^*_{d_2}$ can be written as:

$$
\begin{align*}
X(z) &= \sqrt{2}x(z) - \alpha_+(\psi + i\chi) \\
\phi(z) &= \sqrt{2}\varphi(z) + \alpha_+(\psi + i\chi) \\
B(z) &= b_- e^{(\psi+i\chi)}(z) \\
C(z) &= c^- e^{-(\psi+i\chi)}(z)
\end{align*}
$$

(4.14)

Together they form a space

$$
\bigoplus F(X) \otimes F(\phi) \otimes F(B, C)
$$

(4.15)

Now we should compute $H^*_{F}(H^*_{d_2})$ (the second term of the spectral sequence of the double complex). If we only need the $Q_R$ cohomology of two Wakimoto modules, we are done, and the answer is represented by the free field resolution

$$
F_{\alpha\beta} \rightarrow F_{\alpha - \alpha_+\beta - \alpha_+} \rightarrow F_{\alpha - \alpha_+\beta - \alpha_+} \rightarrow \cdots
$$

(4.16)
The charges $\alpha, \beta$ in terms of the weights $J_M, J_T$ of the Wakimoto modules are given by $\alpha = \alpha_- J_M, \beta = -\alpha_- J_T$. The differential in (4.16) is $F$.

However, if we are interested in the $Q_R$ cohomology of the product of the irreducible representation times Wakimoto module, we should remember about the screening $E_{\hat{sl}(2)}$ and the Felder resolution we made. Note, that the $\hat{sl}(2)$ screenings $E_{\hat{sl}(2)}(\pm)$ can be represented by $E_{Vir} = \oint : e^{\alpha \pm X(z)} :$ just as we thought (cf. (4.1), (4.2)):

$$
: e^{\alpha_+ X(z)} := e^{\sqrt{2} \alpha_+ x(z) - (k+2)(\psi + i\chi)} := (\beta_M)^{-k+2} : e^{\sqrt{2} (k+2) x(z)} : \\
: e^{\alpha_- X(z)} := e^{\sqrt{2} \alpha_- x(z) + (\psi + i\chi)} := \beta_M : e^{- \sqrt{2} (k+2) x(z)} :
$$

(By the triviality of $H^*_d$ we may simply set $\beta_-(z) = \beta_M(z)$ at the level of the chiral algebra. In (4.17) we use this relation.)

Also, adding the trivial piece

$$
\{d_2, b_0(z)\} = -\sqrt{2} \alpha_+ (\partial \varphi(z) + \partial x(z)) - 2 \partial \psi(z) : -2 : b_- c^- : \quad (4.18)
$$

to $i\chi(z)$ and using (4.10), (4.14) we see that $F$ (4.11) can be represented as

$$
F = \oint : e^{\sqrt{2} \alpha_+ (X(z) + \phi(z))} := \oint B(z) : e^{- \sqrt{2} (X(z) + \phi(z))} :
$$

so actually there is no abuse of notations in (4.11) and we may think of $F$ just as of the fermionic screening from the representation theory of N=2 $SVir$ which we introduced in Sec 3.

Thus, finally, we are left with the cohomology of the double complex (the last one in this story) with two differentials. One of them is just $F$ and another one comes from the differential of the Felder resolution and is given by the certain powers of the screening $E^-$. This ”ultimate” double complex is shown in the Fig.5.

It is interesting to compare this complex with the double complex in the Fig.4 because we anticipate that their cohomology are the same (and give the irreducible representation of N=2 Supervirasoro algebra). Although these two look very similar, there are some differences. The basic distinction between them is that the ”horizontal” differentials are ”made” of two different $Vir$-screenings: $E^{(\pm)}$ for the complex in the Fig.4 and $E^{(-)}$ for that one in the Fig.5.

They represent two different choices of the Felder resolutions for the same irreducible representation of Virasoro. As $E^{(-)}$ and $F$ commute (cf. (3.10)), the vertical differential
in Fig.5 is always \( F \). The relations between \( E^{(+)} \) and \( F \) are more involved. As a result in the Fig.4 the vertical differential is either \( F \) (denoted by \( F_1 \)) or a combination like \( xE^{(+)}F + FE^{(+)} \) (denoted by \( F_2 \)), or \( FE^{(+)}F \), depending on the place in the complex. Then, the weights \( \alpha \) corresponding to the principal Virasoro Kac table representations are concentrated along one column in Fig.4. There are no weights from the boundary of the Kac table (we ”threw them away”). On the other hand, the boundary weights are present in the Fig.5 and the weights from the interior of the principal Kac table are situated along the ”shifted” vertical segments, shaded in the picture. (A shift occurs each time we pass through the horizontal line corresponding to the boundary weight.)

These differences result in the difference in computation of the cohomology. In both cases one can use the ”vertical” filtration of the double complex to compute the ”horizontal” cohomology first. In the case of the complex in the Fig.5 we immediately had the Kac table irreducible representations of Virasoro times some Fock spaces of ghosts and Liouville field all along one column. The ”vertical” differential was induced either from \( F \) or \( xE^{(+)}F + FE^{(+)} \) or \( FE^{(+)}F \). The latter combination appeared between the rows where we have thrown away the ”boundary row”.

On the other hand, in the case shown in the Fig.6, the horizontal cohomology give the irreducible representations for the rows corresponding to the weights from the interior of the principal Kac table and zeros for the rows corresponding to the weights from the boundary of the Kac table. We obtain zero cohomology for the ”boundary” rows because each such row represents two (left- and right-sided) glued together resolutions of the same irreducible representation, so the resulting horizontal complex is exact. The nontrivial cohomology are concentrated along the ”shifted segments”.

The ”vertical” cohomology gives again the resolution (3.27). To see this we should first recall that the three pairs of indeces \( (n, m) \), \( (n + q, m + p) \) and \( (q - n, p - m) \) describe the same Virasoro weight. Note then that the ”knight move” differential \( d_2 \) is nontrivial for this spectral sequence. It is a ”connecting differential” — it acts from the top of one ”segment” to the bottom of another ”shifted segment”, mending all these segments together in one complex which is just (3.27).

Now to get \( H^*_{QR}(L_k \otimes Wak_{-k-4}) \) we recall about the \( c^0 \) zero mode ”doubling”. (Here also, as in the case of \( H^*_{QR}(Wak_k \otimes Wak_{-k-4} \) there could be a problem with the ”knight move” differential of the main spectral sequence, but it is zero here for the same reason as there.) Thus we have

\[
H^*_{QR}(L_k \otimes Wak_{-k-4}) = \left[ L(N = 2 SVir) \oplus c^0_0 L(N = 2 SVir) \right] \tag{4.20}
\]
Let the $sl(2)$ weights (spins) of the modules $L_k$ (the matter) and $Wak_{-k-4}$ (the "Toda-Liouville") in (4.20) be respectively $J_M$ and $J_T$. Then the parameters $(\alpha, \beta)$ of the irreducible representation $L_{\alpha,\beta}(N = 2 SVir)$ there are given by

$$\begin{align*}
\alpha &= -\alpha_- J_M \\
\beta &= \alpha_- J_T
\end{align*}$$

(4.21)

(remember that $\alpha_- = -\frac{1}{\sqrt{k+2}}$)

In particular, let $L_k(J_M)$ be the admissible irreducible representation of $\widehat{sl}(2)$, i.e.

$$J_M = \frac{m - 1}{2} + \frac{(k + 2)(1 - n)}{2}$$

(4.22)

and let the weight $J_T$ of $Wak_{-4-k}(J_T)$ be such that it dresses properly a singular vector (having weight $J_M'$) in the Verma module $M_k(J_M)$, which means that $J_T + J_M' = -1$. Then the representation $L_{\alpha,\beta}(N = 2 SVir)$ in (4.20) in the notations of Section 3 is $L_{nmnl}$ of $IV_-$ type with $l = -jp$, where $j$ is the ghost number of the corresponding $\widehat{sl}(2)$-BRST state.

Now, to prove (2.7) (it was promised in the Sec.2), we take the vacuum representation of $\widehat{sl}(2)$ which is a VOA for $\widehat{sl}(2)$ and, looking at (4.20), recall that the vacuum representation of $N=2 SVir$ algebra we see in the right hand side is a VOA for the latter. (In the more physical language it just means that by the $\text{Operators} \rightarrow \text{States}$ correspondence the chiral algebras $N = 2 SVir$ and $\widehat{sl}(2)$ are related to the "descendants of the unity operator" which form the corresponding vacuum representations.)

4.2. Equivalence of the spectra of coset and 2-d gravity.

Finally, we compute the cohomology $H^*_QW$ of the reduced $\widehat{sl}(2) -$ BRST complex $H^*_QR(L_k \otimes Wak_{-k-4})$ and show they coincide with the 2-d gravity BRST cohomology of $L^{DS}_k \otimes Wak^{DS}_{-k-4}$\textsuperscript{5}. For definiteness, we do it for this choice of $\widehat{sl}(2)$ representations, but it can easily be done also for other choices, mentioned in Section 2.

To do the actual computation, let us use the equivalence of two complexes:

$$(H^*_Q(R(L_k(\widehat{sl}(2)) \otimes Wak_{-4-k}), Q_W) = (L(N = 2 SVir) \oplus c_0L(N = 2 SVir), G_0^+)$$

(4.23)

\textsuperscript{5} As above, the superscript "DS" denotes the standard (Drinfeld-Sokolov) reduction.
Suppose that the spin of $L_k(\widehat{sl}_k(2))$ is given by (4.22) and the spin of the Wakimoto module $W_{-4-k}$ is ”dressing” (see the discussion after (4.22)). Then the $N = 2$ irreducible representation $L(N = 2 SVir)$ in (4.23) is $L_{nml}$ with $l = -jp$.

Now, take the ”2-d gravity” resolution (3.27) of $L_{nml}$. On the ”2-d gravity” modules $L(Vir) \otimes F(Liouv.) \otimes F_{gh}$ the action of the differential $G_0^+$ is given by the formula (2.13): $Q_W = G_0^+ = Q_{Vir}$. So we have a double complex. One differential there is $Q_{Vir}$ and the other one comes from the resolution (3.27).

The cohomology of this double complex, by our construction, compute the $\widehat{sl}(2)$ BRST cohomology of $L_k(\widehat{sl}_k(2) \otimes W_{-4-k} - 4 - k)$ — the physical states of the coset model. At this stage we have to assume that we know all the physical states either for the coset model or for 2-d gravity. Then we shall be able to find the spectrum of states for the other theory. Suppose for definiteness that we know the spectrum of 2-d gravity coupled to $(q, p)$ — minimal matter

It means that in our double complex we can compute the ”gravitational” cohomology $H_{Q_{Vir}}^*$ first — it gives the first term of the spectral sequence. Recalling that in (3.27) for the representations $IV_-$ with $l = -jp$ the charges of the liouvilles are just right to dress the ”discrete states”, we end up with the situation shown in the Fig.6. It is important to remember that we compute the usual, i.e. ”absolute” cohomology. It is well known that the BRST cohomology of the properly dressed irreducible representation are given by two elements at the adjacent ghost numbers. This ”Virasoro doubling” is due to the zero mode of the diffeomorphisms ghost $C_0$. Thus all nontrivial cohomology states in the first term of the spectral sequence (the Lian-Zuckerman states) are concentrated along the ”shifted segments” according to their ghost numbers. Each time we pass half-period $p - 1$ (recall the periodic structure of (3.27)), we get a shift by -1 of the ghost number. At the ghost number zero there are two states, and at each positive ghost number, there are four states. Note the ”$\widehat{sl}(2)$ doubling” due to the zero mode $c_0^0$.

To compute the second term of the spectral sequence, we need the properties of the differential $d$ of the resolution (3.27). This operator was studied in [19]. Remember that it is induced either from the fermionic screening $F$ (when it acts ”within one half-period”) or from $FEF$ (when it acts ”between two half-periods”). Thus it changes the ghost number by -1 or by -2 units respectively. Its action on the LZ states is shown by the arrows.

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6 Using the standard homological algebra, it is easy to go the other way — i.e. to obtain the 2-d gravity spectrum from the spectrum of the coset.
We see that the cohomology are concentrated in the same two degrees as the cohomology $H^*_Q \left( L_{nm} \otimes F(\beta_{nml}) \right)$ and also have two elements, now due to the "\( \hat{\mathfrak{sl}}(2) \) doubling". Of course, this is the correct answer for $H^*_{Q_{BRST}}(L_k \otimes \text{Wak}_{-k-4})$. The nontrivial thing that happens is that the "Virasoro doubling" gets transformed into the "\( \hat{\mathfrak{sl}}(2) \) doubling".

Now we should only note that $L_{nm} \otimes F(\beta_{nml})$ — the first term of the resolution (3.27), — is nothing else but the Drinfeld-Sokolov reduced $(L_k)^{DS} \otimes (\text{Wak}_{-k-4})^{DS}$. Thus we have shown, that the coset- and 2-d gravity cohomology are the same thing, basically because they both describe the $G_0^+$ cohomology of the $N = 2 \text{SVir}$ irreducible representation $L_{nml}$.

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