On G-Drazin inverses of finite potent endomorphisms and arbitrary square matrices

Fernando Pablos Romo
Departamento de Matemáticas and Instituto Universitario de Física Fundamental y Matemáticas, Universidad de Salamanca, Salamanca, España

Abstract
The aim of this work is to extend to finite potent endomorphisms the notion of G-Drazin inverse of a finite square matrix. Accordingly, we determine the structure and the properties of a G-Drazin inverse of a finite potent endomorphism and, as an application, we offer an algorithm to compute the explicit expression of all G-Drazin inverses of a finite square matrix.

1. Introduction
For an arbitrary \((n \times n)\)-matrix \(A\) with entries in the complex numbers, the index of \(A\), \(i(A) \geq 0\), is the smallest integer such that \(\text{rk}(A^{i(A)}) = \text{rk}(A^{i(A) + 1})\). Given \(A \in \text{Mat}_{n \times n}(\mathbb{C})\) with \(i(A) = r\), H. Wang and X. Liu introduced in [1] the notion of ‘G-Drazin inverse’ of \(A\) as a solution \(X\) of the system

\[
AXA = A; \\
XA^{r+1} = A^r; \\
A^{r+1}X = A^r,
\]

where \(X\) is a \((n \times n)\)-matrix with entries in \(\mathbb{C}\).

Recently, C. Coll et al. have proved in [2] that a matrix \(X \in \text{Mat}_{n \times n}(\mathbb{C})\) is a solution of the system (1) if and only if \(X\) satisfies that

\[
AXA = A; \\
XA^r = A^rX.
\]
On the other hand, if $k$ is a field, $V$ is an arbitrary vector space over $k$ and $\varphi$ is an endomorphism of $V$, according to [3] we say that $\varphi$ is ‘finite-potent’ if $\varphi^n V$ is finite dimensional for some $n$.

During recent years, the author has extended the notions of Drazin inverse, Core-Moore-Penrose inverse and Drazin-Moore-Penrose inverses of finite square matrices to finite potent endomorphisms, and has offered several properties of these extensions [4–6]. In particular, all the results obtained for finite potent endomorphisms are also valid for finite square matrices.

The aim of this work is to extend to finite potent endomorphisms the notion of G-Drazin inverse of a finite square matrix. Indeed, we determine the structure of a G-Drazin inverse of a finite potent endomorphism and, in particular, we offer the explicit expression of all G-Drazin inverses of a finite square matrix.

The paper is organized as follows. In Section 2 we briefly recall the basic definitions of this work: the definition of finite potent endomorphisms with the decomposition of the vector space given by Argerami et al. [7]; the Jordan bases of nilpotent endomorphisms of infinite-dimensional vector spaces; the Drazin inverse of an $(n \times n)$-matrix and a finite potent endomorphism; the core-nilpotent decomposition of a finite potent endomorphism; and the basic properties of the G-Drazin inverses of an square matrix.

Section 3 is devoted to proving the existence of G-Drazin inverses of finite potent endomorphisms (Proposition 3.4) and to offering the explicit structure of these linear maps on arbitrary $k$-vector spaces (Lemma 3.10 and Corollary 3.11). Moreover, an explicit example of G-Drazin inverses of a finite potent endomorphism is given.

Finally, the goal of Section 4 is to apply the results of Section 3 to study the set $A\{GD\}$ of G-Drazin inverses of a square matrix $A \in \text{Mat}_{n \times n}(k)$, where $k$ is an arbitrary field. Accordingly, if the index of $A$ is $r$ and $\text{rk}(A^i)$ is the rank of $A^i$, we can write $v_i(A) = n - \text{rk}(A^i)$ for all $i \in \{1, \ldots, r\}$ and we check that there exists a bijection

$$k^{v_1(A) - v_r(A)} \times k^{[v_r(A) - v_1(A)]} \times k^{[v_1(A) - v_1(A)]} \sim A\{GD\},$$

from where we determine the explicit expression of all G-Drazin inverses of $A$.

## 2. Preliminaries

This section is added for the sake of completeness.

### 2.1. Finite potent endomorphisms

Let $k$ be an arbitrary field, and let $V$ be a $k$-vector space.

Let us now consider an endomorphism $\varphi$ of $V$. We say that $\varphi$ is ‘finite potent’ if $\varphi^n V$ is finite dimensional for some $n$. This definition was introduced by J. Tate in [3] as a basic tool for his elegant definition of Abstract Residues.

In 2007, M. Argerami et al. showed in [7] that an endomorphism $\varphi$ is finite potent if and only if $V$ admits a $\varphi$-invariant decomposition $V = U_\varphi \oplus W_\varphi$ such that $\varphi|_{U_\varphi}$ is nilpotent, $W_\varphi$ is finite dimensional and $\varphi|_{W_\varphi} : W_\varphi \sim W_\varphi$ is an isomorphism.
Indeed, if \( k[x] \) is the algebra of polynomials in the variable \( x \) with coefficients in \( k \), we may view \( V \) as an \( k[x] \)-module via \( \varphi \), and the explicit definition of the above \( \varphi \)-invariant subspaces of \( V \) is:

- \( U_\varphi = \{ v \in V \text{ such that } x^m v = 0 \text{ for some } m \} \);
- \( W_\varphi = \{ v \in V \text{ such that } p(x)v = 0 \text{ for some } p(x) \in k[x] \text{ relative prime to } x \} \).

Note that if the annihilator polynomial of \( \varphi \) is \( x^m \cdot p(x) \) with \( (x, p(x)) = 1 \), then \( U_\varphi = \text{Ker } \varphi^m \) and \( W_\varphi = \text{Ker } p(\varphi) \).

Hence, this decomposition is unique. In this paper we shall call this decomposition the \( \varphi \)-invariant AST-decomposition of \( V \).

For a finite potent endomorphism \( \varphi \), a trace \( \text{tr}_V(\varphi) \in k \) may be defined as \( \text{tr}_V(\varphi) = \text{tr}_{W_\varphi}(\varphi|_{W_\varphi}) \)

This trace has the following properties:

1. if \( V \) is finite dimensional, then \( \text{tr}_V(\varphi) \) is the ordinary trace;
2. if \( W \) is a subspace of \( V \) such that \( \varphi W \subset W \) then
   \[
   \text{tr}_V(\varphi) = \text{tr}_W(\varphi) + \text{tr}_{V/W}(\varphi);
   \]
3. if \( \varphi \) is nilpotent, then \( \text{tr}_V(\varphi) = 0 \).

Usually, \( \text{tr}_V \) is named ‘Tate’s trace’.

It is known that in general \( \text{tr}_V \) is not linear; that is, it is possible to find finite potent endomorphisms \( \theta_1, \theta_2 \in \text{End}_k(V) \) such that

\[
\text{tr}_V(\theta_1 + \theta_2) \neq \text{tr}_V(\theta_1) + \text{tr}_V(\theta_2).
\]

Moreover, with the previous notation, and using the AST-decomposition of \( V \), D. Hernández Serrano and the author of this paper have offered in [8] a definition of a determinant for finite potent endomorphisms as follows:

\[
\det^k_V(1 + \varphi) := \det^k_{W_\varphi}(1 + \varphi|_{W_\varphi}).
\]

This determinant satisfies the following properties:

- if \( V \) is finite dimensional, then \( \det^k_V(1 + \varphi) \) is the ordinary determinant;
- if \( W \) is a subspace of \( V \) such that \( \varphi W \subset W \), then
  \[
  \det^k_V(1 + \varphi) = \det^k_W(1 + \varphi) \cdot \det^k_{V/W}(1 + \varphi);
  \]
- if \( \varphi \) is nilpotent, then \( \det^k_V(1 + \varphi) = 1 \).

For details readers are referred to [3, 8–10].

### 2.2. Jordan bases of nilpotent endomorphisms of infinite-dimensional vector spaces

Let \( V \) be a vector space over an arbitrary field \( k \) and let \( f \in \text{End}_k(V) \) be a nilpotent endomorphism.
If $n$ is the nilpotency index of $f$, according to the statements [11], setting $W_i^f = \Ker f^i / [\Ker f^{i-1} + f(\Ker f^{i+1})]$ with $i \in \{1, 2, \ldots, n\}$, $\alpha_i(V, f) = \dim_k W_i^f$ and $S_{\alpha_i}(V, f)$ a set such that $\# S_{\alpha_i}(V, f) = \alpha_i(V, f)$ with $S_{\alpha_i}(V, f) \cap S_{\alpha_j}(V, f) = \emptyset$ for all $i \neq j$, one has that there exists a family of vectors $\{v_{s_i}\}$ that determines a Jordan basis of $f$:

$$B = \bigcup_{s_i \in S_{\alpha_i}(V, f), 1 \leq i \leq n} \{v_{s_i}, f(v_{s_i}), \ldots, f^{i-1}(v_{s_i})\}. \quad (3)$$

Moreover, if we write $H_{s_i}^f = \langle v_{s_i}, f(v_{s_i}), \ldots, f^{i-1}(v_{s_i}) \rangle$, the basis $B$ induces a decomposition

$$V = \bigoplus_{s_i \in S_{\alpha_i}(V, f), 1 \leq i \leq n} H_{s_i}^f. \quad (4)$$

For a different method to construct Jordan bases of nilpotent endomorphisms of infinite-dimensional vector spaces readers can see [12].

### 2.3. Drazin inverse of finite potent endomorphisms

#### 2.3.1. Drazin inverse of $(n \times n)$-matrices

Let $A \in \text{Mat}_{n \times n}(\mathbb{C})$.

**Definition 2.1:** The ‘index of $A$’, $i(A) \geq 0$, is the smallest integer such that $\text{rk}(A^{i(A)}) = \text{rk}(A^{i(A)+1})$.

In 1958, given a matrix $A \in \text{Mat}_{n \times n}(\mathbb{C})$ with $i(A) = r$, M. P. Drazin [13] showed the existence of a unique $(n \times n)$-matrix $A^D$ satisfying the equations:

- $A^{r+1}A^D = A^r$ for $r = i(A)$;
- $A^DAA^D = A^D$;
- $A^DA = AA^D$.

The Drazin inverse $A^D$ also verifies that

- $(A^D)^D = A$ if and only if $i(A) \leq 1$;
- if $A^2 = A$, then $A^D = A$.

#### 2.3.2. Drazin inverse of finite potent endomorphisms

Let $V$ be an arbitrary $k$-vector space and let $\varphi \in \text{End}_k(V)$ be a finite potent endomorphism of $V$. Let us consider the AST-decomposition $V = U_\varphi \oplus W_\varphi$ induced by $\varphi$ (Subsection 2.1).

**Definition 2.2:** We shall call ‘index of $\varphi$’, $i(\varphi)$, to the nilpotent order of $\varphi_{|U_\varphi}$.

In [6] (Lemma 3.2) is proved that for finite-dimensional vector spaces this definition of index coincides with Definition 2.1. Note that $i(\varphi) = 0$ if and only if $V$ is a finite-dimensional vector space and $\varphi$ is an automorphism.
For each finite potent endomorphism $\phi$ there exists a unique finite potent endomorphism $\phi^D$ that satisfies that:

1. $\phi^{r+1} \circ \phi^D = \phi^r$;
2. $\phi^D \circ \phi \circ \phi^D = \phi^D$;
3. $\phi^D \circ \phi = \phi \circ \phi^D$,

where $r$ is the index of $\phi$.

The map $\phi^D$ is the Drazin inverse of $\phi$ and is the unique linear map such that:

$$\phi^D(v) = \begin{cases} (\phi_{|W_\phi})^{-1} & \text{if } v \in W_\phi \\ 0 & \text{if } v \in U_\phi \end{cases}.$$ 

Moreover, $\phi^D$ satisfies the following properties:

- $(\phi^D)^D = \phi$ if and only if the $i(\phi) \leq 1$;
- $\phi = \phi^D$ if and only if $\phi_{|U_\phi} = 0$ and $(\phi_{|U_\phi})^2 = \text{Id}_{|W_\phi}$;
- $\text{tr}_V(\phi + \phi^D) = \text{tr}_V(\phi) + \text{tr}_V(\phi^D)$;
- if $\psi$ is a projection finite potent endomorphism, then $\psi^D = \psi$.

### 2.4. CN decomposition of a finite potent endomorphism

Given a finite potent endomorphism $\phi \in \text{End}_k(V)$, there exists a unique decomposition $\phi = \phi_1 + \phi_2$, where $\phi_1, \phi_2 \in \text{End}_k(V)$ are finite potent endomorphisms satisfying that:

- $i(\phi_1) \leq 1$;
- $\phi_2$ is nilpotent;
- $\phi_1 \circ \phi_2 = \phi_2 \circ \phi_1 = 0$.

According to [4, Theorem 3.2], one has that $\phi_1 = \phi \circ \phi^D \circ \phi$, which is the core part of $\phi$. Also, $\phi_2$ is named the nilpotent part of $\phi$.

Moreover, one has that

$$\phi = \phi_1 \iff U_\phi = \text{Ker} \phi \iff W_\phi = \text{Im} \phi \iff (\phi^D)^D = \phi \iff i(\phi) \leq 1. \quad (5)$$

### 2.5. G-Drazin inverses of a square matrix

Given $A \in \text{Mat}_{n \times n}(\mathbb{C})$ with $i(A) = r$, H. Wang and X. Liu introduced in [1] the notion of ‘G-Drazin inverse’ of $A$ as a solution $X$ of the system

$$AXA = A;$$
$$XA^{r+1} = A';$$
$$A^{r+1}X = A',$$

where $X$ is a $(n \times n)$-matrix with entries in $\mathbb{C}$. 

Recently, C. Coll et al. have proved in [2] that a matrix \( X \in \text{Mat}_{n \times n}(\mathbb{C}) \) is a solution of the system (7) if and only if \( X \) satisfies that

\[
AXA = A; \\
XA^r = A^rX.
\]

Usually, the set of G-Drazin inverses of a matrix \( A \) is denoted by \( A\{GD\} \) and a G-matrix inverse of \( A \) is denoted by \( A^{GD} \). We should note that G-Drazin inverses are considered for bounded linear operators.

If \( J \) is the Jordan matrix associated with \( A \in \text{Mat}_{n \times n}(\mathbb{C}) \), such that \( A = B \cdot J \cdot B^{-1} \), with \( B \) being a non-singular matrix and

\[
J = \begin{pmatrix} J_1 & 0 \\ 0 & J_0 \end{pmatrix},
\]

\( J_0 \) and \( J_1 \) being the parts of \( J \) corresponding to zero and non-zero eigenvalues respectively, it is known that a G-Drazin inverse is

\[
A^{GD} = B \cdot \begin{pmatrix} J_1^{-1} & 0 \\ 0 & J_0^{-1} \end{pmatrix} : B^{-1},
\]

where is \( J_0^{-1} \) is a generalized inverse of \( J_0 \) (1-inverse).

### 3. G-Drazin inverses of finite potent endomorphisms

The aim of this section is to generalize the definition and the main properties of the G-Drazin inverses of a matrix \( A \) to finite potent endomorphisms.

Let \( k \) be a field and let \( V \) be an arbitrary \( k \)-vector space.

**Definition 3.1:** Given a finite potent endomorphism \( \varphi \in \text{End}_k(V) \), we say that an endomorphism \( \varphi^{GD} \in \text{End}_k(V) \) is a G-Drazin inverse of \( \varphi \) when it satisfies that

\[
\varphi \circ \varphi^{GD} \circ \varphi = \varphi; \\
\varphi^{GD} \circ \varphi^r = \varphi^r \circ \varphi^{GD},
\]

where \( i(\varphi) = r \).

**Lemma 3.2:** Let \( \varphi \in \text{End}_k(V) \) be a finite potent endomorphism with \( i(\varphi) = r \) and let \( V = W_\varphi \oplus U_\varphi \) be the AST-decomposition determined by \( \varphi \). If \( f \in \text{End}_k(V) \) is an endomorphism such that \( f \circ \varphi^r = \varphi^r \circ f \), then \( W_\varphi \) and \( U_\varphi \) are invariant under the action of \( f \).

**Proof:** Let \( \varphi = \varphi_1 + \varphi_2 \) be the CN-decomposition of \( \varphi \).

Since \( i(\varphi) = r \), bearing in mind that \( \varphi^r = \varphi_1^r \), \( (\varphi_1^r)_{|W_\varphi} \in \text{Aut}_k(W_\varphi) \) and \( \text{Im} \varphi_1^r = W_\varphi \), if \( w \in W_\varphi \) and \( \varphi_1^r(w') = w \), then

\[
f(w) = (f \circ \varphi_1^r)(w') = (\varphi_1^r \circ f)(w') \in W_\varphi.
\]

Accordingly, \( W_\varphi \) is \( f \)-invariant.
Moreover, if \( u \in U_\varphi \), then
\[
0 = (f \circ \varphi^\dagger_1)(u) = (\varphi^\dagger_1 \circ f)(u)
\]
and we deduce that \( f(u) \in \text{Ker} \varphi^\dagger_1 = U_\varphi \). Hence, \( U_\varphi \) is also \( f \)-invariant.

**Corollary 3.3:** If \( \varphi^{GD} \in \text{End}_k(V) \) is a \( G \)-Drazin inverse of a finite potent endomorphism \( \varphi \in \text{End}_k(V) \), with \( i(\varphi) = r \) and \( \text{AST-decomposition} \ V = W_\varphi \oplus U_\varphi \), then \( W_\varphi \) and \( U_\varphi \) are invariant under the action of \( \varphi^{GD} \).

The structure of a \( G \)-Drazin inverse of a finite potent endomorphism is given by the following proposition:

**Proposition 3.4:** Given a finite potent endomorphism \( \varphi \in \text{End}_k(V) \) with \( i(\varphi) = r \) and \( \text{AST-decomposition} \ V = W_\varphi \oplus U_\varphi \), one has that \( \varphi^{GD} \in \text{End}_k(V) \) is a \( G \)-Drazin inverse of \( \varphi \) if and only if \( W_\varphi \) and \( U_\varphi \) are invariant under the action of \( \varphi^{GD} \). Also, since \( \varphi \circ \varphi^{GD} \circ \varphi = \varphi \), it is clear that \( (\varphi^{GD})|_{W_\varphi} = (\varphi|_{W_\varphi})^{-1} \) and \( (\varphi^{GD})|_{U_\varphi} = (\varphi|_{U_\varphi})^{-1} \), where \( (\varphi|_{U_\varphi})^{-1} \) is a generalized inverse of \( \varphi|_{U_\varphi} \).

**Proof:** Let \( \varphi^{GD} \in \text{End}_k(V) \) be a \( G \)-Drazin inverse of \( \varphi \). It follows from Corollary 3.3 that \( W_\varphi \) and \( U_\varphi \) are invariant under the action of \( \varphi^{GD} \). Also, since \( \varphi \circ \varphi^{GD} \circ \varphi = \varphi \), it is easy to check that \( \varphi \circ \varphi^{GD} \circ \varphi = \varphi \). Moreover, since \( (\varphi^r)|_{U_\varphi} = 0 \), then it follows from the properties of \( \psi \) that
\[
(\varphi \circ \varphi^r)|_{W_\varphi} = (\varphi^r)^{-1}|_{W_\varphi} = (\varphi \circ \psi)|_{W_\varphi}
\]
from where we deduce that
\[
\varphi \circ \varphi^r = \varphi^r \circ \varphi
\]
and the statement is proved.

Direct consequences of Proposition 3.4 are:

**Corollary 3.5:** Given a finite potent endomorphism \( \varphi \in \text{End}_k(V) \) with \( \text{CN-decomposition} \ \varphi = \varphi_1 + \varphi_2 \), if \( \varphi^{GD} \in \text{End}_k(V) \) is a \( G \)-Drazin inverse of \( \varphi \), one has that \( \varphi^{GD} = \varphi^{D} + \varphi_2^{GD} \), where \( \varphi^{D} \) is the Drazin inverse of \( \varphi \), and \( \varphi_2^{GD} \in \text{End}_k(V) \) is the unique linear map satisfying that
\[
\varphi_2^{GD}(v) = \begin{cases} 
0 & \text{if } v \in W_\varphi \\
(\varphi^{GD})|_{U_\varphi} & \text{if } v \in U_\varphi 
\end{cases}
\]
which is a \( G \)-Drazin inverse of \( \varphi_2 \).

**Corollary 3.6:** If \( \varphi = \varphi_1 + \varphi_2 \) is the \( \text{CN-decomposition} \) of a finite potent endomorphism \( \varphi \in \text{End}_k(V) \), then the Drazin inverse \( \varphi^{D} \) is a \( G \)-Drazin inverse of \( \varphi_1 \).
We shall now characterize all of the G-Drazin inverses of a finite potent endomorphism.

**Lemma 3.7:** Let $E$ be a $k$-vector space of dimension $n$ and let $f \in \text{End}_k(E)$ be an endomorphism with annihilating polynomial $a_f(x) = x^n$. If $e \in E$ is a vector such that $f^{n-1}(e) \neq 0$, then every generalized inverse $f^- \in \text{End}_k(E)$ is determined by the expressions

$$f^-(f^i(e)) = \begin{cases} f^{i-1}(e) + \lambda_if^{n-1}(e) & \text{if } i \geq 1 \\ \tilde{e}_i & \text{if } i = 0 \end{cases},$$

with $i \in \{0, 1, \ldots, n-1\}$, $\lambda_i \in k$ for every $i \in \{1, \ldots, n-1\}$ and $\tilde{e} \in E$ being an arbitrary vector.

**Proof:** Since $f \circ f^- \circ f = f$, we have that $f^-$ is a generalized inverse of $f$ if and only if $(f \circ f^-)|_{\text{Im}f} = \text{Id}|_{\text{Im}f}$. Hence, since $e \notin \text{Im}f$ we have that $f^-(e) = \tilde{e}$, where $\tilde{e} \in E$ is an arbitrary vector.

Moreover, bearing in mind that $(f \circ f^-)(f^i(e)) = f^i(e)$ for all $i \geq 1$, one has that

$$f^-(f^i(e)) \in f^{-1}(f^i(e)) + \text{Ker}f,$$

and we get that

$$f^-(f^i(e)) = f^{i-1}(e) + \lambda_if^{n-1}(e)$$

for all $i \in \{1, \ldots, n-1\}$.

Accordingly, since $\{e, f(e), \ldots, f^{n-1}(e)\}$ is a Jordan basis of $E$ induced by $f$, the claim is deduced. \[\blacksquare\]

We can reformulate the statement of Lemma 3.9 as follows:

**Corollary 3.8:** Let $E$ be a $k$-vector space of dimension $n$ and let $f \in \text{End}_k(E)$ be an endomorphism with annihilating polynomial $a_f(x) = x^n$. If $e \in E$ is a vector such that $f^{n-1}(e) \neq 0$, then every generalized inverse $f^- \in \text{End}_k(E)$ is determined by the expressions

$$f^-(f^i(e)) = \begin{cases} f^{i-1}(e) + \lambda_if^{n-1}(e) & \text{if } i \geq 1 \\ \sum_{h=0}^{n-1} \alpha_hf^h(e) & \text{if } i = 0 \end{cases},$$

with $i \in \{0, \ldots, n-1\}$, $\lambda_i, \alpha_h \in k$ for every $i \in \{1, \ldots, n-1\}$ and $h \in \{0, \ldots, n-1\}$.

Furthermore, similar to Lemma 3.9 one can prove that
Lemma 3.9: If $E$ is a finite-dimensional $k$-vector space, $f \in \text{End}_k(E)$ is an endomorphism with annihilating polynomial $a_f(x) = x^n$ and

$$\bigcup_{j=1}^{r}\{e_j, f(e_j), \ldots, f^{n_j-1}(e_j)\}$$

is a Jordan basis of $E$ induced by $f$, then every generalized inverse $f^- \in \text{End}_k(E)$ is determined by the expressions

$$f^-(f^i(e_j)) = \begin{cases} f^{i-1}(e_j) + \sum_{s=1}^{r} \lambda_{j,s}^{i} f^{n_s-1}(e_s) & \text{if } i \geq 1 \\ \sum_{s=1}^{r} \left[ \sum_{h=0}^{n_s-1} \alpha_{j,s}^{i,h} f^{h}(e_s) \right] & \text{if } i = 0 \end{cases},$$

with $\lambda_{j,s}^{i} \in k$ for each $j, s \in \{1, \ldots, r\}$, $i \in \{0, \ldots, n_j - 1\}$ and $h \in \{0, \ldots, n_s - 1\}$.

Let us again consider a finite potent endomorphism $\varphi \in \text{End}_k(V)$ of an arbitrary $k$-vector space $V$ with $i(\varphi) = r$. If $V = W_\varphi \oplus U_\varphi$ is the AST-decomposition induced by $\varphi$, let

$$B_\varphi = \bigcup_{1 \leq h \leq r} \{v_{sh}, \varphi(v_{sh}), \ldots, \varphi^{h-1}(v_{sh})\}$$

be a Jordan basis of $U_\varphi$ induced by $\varphi|_{U_\varphi}$ (see Subsection 2.2).

Let us now denote

$$\mathfrak{S}_{\alpha,\varphi} = S_{\alpha_1}(V,\varphi|_{U_\varphi}) \cup \ldots \cup S_{\alpha_r}(V,\varphi|_{U_\varphi})$$

and $\beta(V, \varphi) = \#\mathfrak{S}_{\alpha,\varphi}$.

Accordingly, similar to Lemma 3.9, it is easy to check that

Lemma 3.10: Given an arbitrary $k$-vector space $V$ and a finite potent endomorphism $\varphi \in \text{End}_k(V)$ with $i(\varphi) = r$ and AST-decomposition $V = W_\varphi \oplus U_\varphi$, fixing a Jordan basis $B_\varphi$ of $U_\varphi$ as in (12), then every generalized inverse $(\varphi|_{U_\varphi})^- \in \text{End}_k(U_\varphi)$ is determined by the expressions

$$(\varphi|_{U_\varphi})^-(\varphi^i(v_{sh})) = \begin{cases} \varphi^{i-1}(v_{sh}) + \sum_{s,t \in \mathfrak{S}_{\alpha,\varphi}} \lambda_{s,t}^{i} \varphi^{t-1}(v_{st}) & \text{if } 1 \leq i \leq h - 1 \\ \varphi^i(v_{sh}) & \text{if } i = 0 \end{cases},$$

with $v \in U_\varphi$, $1 \leq h \leq r$, $i \in \{0, 1, \ldots, h - 1\}$, $\lambda_{s,t}^{i} \in k$ and $\lambda_{s,t}^{i} = 0$ for almost all $s, t \in \mathfrak{S}_{\alpha,\varphi}$ (for each $s_h \in S_{\alpha_h}(V,\varphi|_{U_\varphi})$).

Writing

$$(\varphi|_{U_\varphi})^-(v_{sh}) = \sum_{s,t \in \mathfrak{S}_{\alpha,\varphi}} \alpha_{s,t}^{i} \varphi^{i}(v_{st}) \in U_\varphi,$$

one has that
Corollary 3.11: Given an arbitrary $k$-vector space $V$ and a finite potent endomorphism $\varphi \in \text{End}_k(V)$ with $i(\varphi) = r$ and AST-decomposition $V = W_\varphi \oplus U_\varphi$, fixing a Jordan basis $B_\varphi$ of $U_\varphi$ as in (12), then every generalized inverse $(\varphi |_{U_\varphi})^− \in \text{End}_k(U_\varphi)$ is determined by the expressions

$$(\varphi |_{U_\varphi})^−(\varphi^i(v_{sh})) = \begin{cases} 
\varphi^{i−1}(v_{sh}) + \sum_{s_t \in \mathcal{S}_{\alpha,\varphi}} \lambda_{s_h,s_t}^i \varphi^{i−1}(v_{st}) & \text{if } 1 \leq i \leq h − 1 \\
\sum_{s_l \in \mathcal{S}_{\alpha,\varphi}} \alpha_{s_h,s_l}^j \varphi^j(v_{sl}) & \text{if } i = 0 
\end{cases}$$

with $1 \leq h \leq r$, $i \in \{0, 1, \ldots, h − 1\}$, $\lambda_{s_h,s_t}^i, \alpha_{s_h,s_l}^j \in k$, and $\lambda_{s_h,s_t}^i = 0 = \alpha_{s_h,s_l}^j$ for almost all $s_t, s_l \in \mathcal{S}_{\alpha,\varphi}$ and $0 \leq j \leq l − 1$.

Corollary 3.12: With the notation of Corollary 3.5, $\varphi_2^{GD} = 0$ if and only if $i(\varphi) \leq 1$.

Proof: It follows from Lemma 3.10 that $\varphi_2^{GD} = 0$ if and only if $\text{Ker} \varphi = U_\varphi$, from where the claim is proved.

Corollary 3.13: Given a finite potent endomorphism $\varphi \in \text{End}_k(V)$, one has that the Drazin inverse $\varphi^D \in \text{End}_k(V)$ is a G-Drazin inverse of $\varphi$ if and only if $i(\varphi) \leq 1$.

Proof: If $\varphi = \varphi_1 + \varphi_2$ is the CN-decomposition, then according to Corollary 3.5 one has that $\varphi^D \in \text{End}_k(V)$ is a G-Drazin inverse of $\varphi$ if and only if $0$ is a G-Drazin inverse of $\varphi_2$ and, bearing in mind Corollary 3.12, the statement is deduced.

Accordingly, from Proposition 3.4 and Lemma 3.10, we have characterized all the G-Drazin inverses of a finite potent endomorphism $\varphi \in \text{End}_k(V)$. Note that, in general, a G-Drazin inverse $\varphi^{GD} \in \text{End}_k(V)$ is not a finite potent endomorphism.

If we denote by $X_{\varphi}^{[GD]}$ the set of all G-Drazin inverses of a finite potent endomorphism $\varphi \in \text{End}_k(V)$, fixing a Jordan basis $B_\varphi$ of $U_\varphi$ as in (12), with the notation of Subsection 2.2, $i(\varphi) = r$ and we have a bijection

$$X_{\varphi}^{[GD]} \sim \prod_{h=1}^r \left( \prod_{s_t \in \mathcal{S}_{\alpha,\varphi}} \left[ \left( U_\varphi \times \bigoplus_{i=1}^{h−1} \left( \bigoplus_{\beta(V,\varphi)} k \right) \right) \right] \right).$$

(13)

Example 3.1: Let $k$ be an arbitrary ground field, let $V$ be a $k$-vector space of countable dimension over $k$ and let $\{v_1, v_2, v_3, \ldots\}$ be a basis of $V$ indexed by the natural numbers.
Let $\varphi \in \text{End}_k(V)$ the finite potent endomorphism defined as follows:

\[
\varphi(v_i) = \begin{cases} 
  v_2 + v_5 + v_7 & \text{if } i = 1 \\
  v_1 + 3v_2 & \text{if } i = 2 \\
  v_4 & \text{if } i = 3 \\
  v_1 - v_3 & \text{if } i = 4 \\
  -v_3 + 2v_5 + 2v_7 & \text{if } i = 5 \\
  3v_{i+1} & \text{if } i = 5h + 1 \\
  0 & \text{if } i = 5h + 2 \\
  -v_{i-2} + 2v_{i+1} & \text{if } i = 5h + 3 \\
  v_{i-2} + v_{i+1} & \text{if } i = 5h + 4 \\
  -v_{i-4} + 5v_{i-3} & \text{if } i = 5h + 5 
\end{cases}
\]

for all $h \geq 1$.

We have that the AST-decomposition $V = U_\varphi \oplus W_\varphi$ is determined by the subspaces

\[
W_\varphi = \langle v_1, v_2, v_3, v_4, v_5 + v_7 \rangle \quad \text{and} \quad U_\varphi = \langle v_j \rangle_{j \geq 6}.
\]

In this basis of $W_\varphi$ one has that

\[
\varphi|_{W_\varphi} \equiv A_{W_\varphi} = \begin{pmatrix} 
  0 & 1 & 0 & 1 & 0 \\
  1 & 3 & 0 & 0 & 0 \\
  0 & 0 & 0 & -1 & -1 \\
  0 & 0 & 1 & 0 & 0 \\
  1 & 0 & 0 & 0 & 2 
\end{pmatrix},
\]

and, computing $A_{W_\varphi}^{-1}$, we obtain that the Drazin inverse of $\varphi$ is

\[
\varphi^D(v_i) = \begin{cases} 
  6v_1 - 2v_2 + 3v_4 - 3v_5 & \text{if } i = 1 \\
  -2v_1 + v_2 - v_4 + v_5 & \text{if } i = 2 \\
  6v_1 - 2v_2 + 2v_4 - 3v_5 & \text{if } i = 3 \\
  v_3 & \text{if } i = 4 \\
  3v_1 - v_2 + v_4 - v_5 & \text{if } i = 5 \\
  0 & \text{if } i \geq 6 
\end{cases}.
\]

Moreover, we can write $U_\varphi = \bigoplus_{i \geq 2} H_i$ with $H_i = \langle v_{5i-4}, v_{5i-3}, v_{5i-2}, v_{5i-1}, v_{5i} \rangle$ for all $i \geq 2$, and in the same bases we have that

\[
\varphi|_{H_i} \equiv A_{H_i} = \begin{pmatrix} 
  0 & 0 & -1 & 0 & -1 \\
  3 & 0 & 0 & 1 & 5 \\
  0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 2 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 
\end{pmatrix}
\]

for all $i \geq 2$. 
For every \( i \geq 2 \), one has that
\[
\{v_{5i-2}, -v_{5i-4} + 2v_{5i-1}, -v_{5i-3} + 2v_{5i}, -2v_{5i-4} + 10v_{5i-3}, -6v_{5i-3}\}
\]
is a Jordan basis of \( H_i \) induced by \( \varphi|_{H_i} \) and, therefore,
\[
\varphi|_{H_i} \equiv P \cdot \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix} \cdot P^{-1}
\]
with
\[
P = \begin{pmatrix}
0 & -1 & 0 & -2 & 0 \\
0 & 0 & -1 & 10 & -6 \\
1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0
\end{pmatrix}.
\]
Accordingly, it follows from Corollaries 3.5 and 3.11, that \( \varphi^{GD}_2 \in \text{End}_k(V) \) is the unique linear map such that:

- \( \varphi^{GD}_2(v_i) = 0 \) for \( i \in \{1, 2, 3, 4\} \);
- \( \varphi^{GD}_2(v_5 + v_7) = 0 \);
- \( \varphi^{GD}_2(v_{5i-2}) = \sum_{j=6}^\infty \alpha_{ij} v_j \)
- \( \varphi^{GD}_2(-v_{5i-4} + 2v_{5i-1}) = v_{5i-2} + \sum_{h \geq 2} \lambda_{i,5h-3}^1 v_{5h-3} \)
- \( \varphi^{GD}_2(-v_{5i-3} + 2v_{5i}) = -v_{5i-4} + 2v_{5i-1} + \sum_{h \geq 2} \lambda_{i,5h-3}^2 v_{5h-3} \)
- \( \varphi^{GD}_2(-2v_{5i-4} + 10v_{5i-3}) = -v_{5i-3} + 2v_{5i} + \sum_{h \geq 2} \lambda_{i,5h-3}^3 v_{5h-3} \)
- \( \varphi^{GD}_2(-6v_{5i-3}) = -2v_{5i-4} + 10v_{5i-3} + \sum_{h \geq 2} \lambda_{i,5h-3}^4 v_{5h-3} \)

for every \( i \geq 2 \), and with \( \alpha_{ij} = 0 \) for almost all \( j \geq 6 \) (for each \( i \geq 2 \)) and \( \lambda_{5h-3,s}^i = 0 \) for almost all \( h \geq 2 \) (for every \( i \geq 2 \) and for every \( s \in \{2, 3, 4, 5\} \)).

Thus, a non-difficult computation shows that \( \varphi^{GD}_2 \in \text{End}_k(V) \) is the unique linear map such that:

- \( \varphi^{GD}_2(v_i) = 0 \) for \( i \in \{1, 2, 3, 4\} \);
- \( \varphi^{GD}_2(v_5) = -\frac{1}{3}v_6 + \frac{5}{3}v_7 + \frac{1}{6} \sum_{h \geq 2} \lambda_{5h-3,5}^2 v_{5h-3} \)
- \( \varphi^{GD}_2(v_{5i-4}) = -\frac{5}{3}v_{5i-4} - \frac{47}{6}v_{5i-3} - v_{5i} - \sum_{h \geq 2} \left(\frac{1}{2} \lambda_{i,5h-3}^3 + \frac{5}{6} \lambda_{i,5h-3}^4\right) v_{5h-3} \)
- \( \varphi^{GD}_2(v_{5i-3}) = \frac{1}{3}v_{5i-4} - \frac{5}{3}v_{5i-3} + \frac{1}{6} \sum_{h \geq 2} \lambda_{i,5h-3}^4 v_{5h-3} \)
- \( \varphi^{GD}_2(v_{5i-2}) = \sum_{j=6}^\infty \alpha_{ij} v_j \)
- \( \varphi^{GD}_2(v_{5i-1}) = \frac{5}{6}v_{5i-4} - \frac{47}{12}v_{5i-3} + \frac{1}{2}v_{5i-2} - \frac{1}{2}v_{5i} + \sum_{h \geq 2} \left(\frac{1}{2} \lambda_{i,5h-3}^2 - \frac{1}{4} \lambda_{i,5h-3}^3 v_{5h-3} - \frac{5}{12} \lambda_{i,5h-3}^4\right) \)
- \( \varphi^{GD}_2(v_{5i}) = -\frac{1}{3}v_{5i-4} - \frac{5}{6}v_{5i-3} + v_{5i-1} - \frac{5}{6}v_{5i} + \sum_{h \geq 2} \left(\frac{1}{2} \lambda_{i,5h-3}^2 - \frac{1}{4} \lambda_{i,5h-3}^3 v_{5h-3} \right) \)

for every \( i \geq 2 \), and with \( \alpha_{ij} = 0 \) for almost all \( j \geq 6 \) (for each \( i \geq 2 \)) and \( \lambda_{5h-3,s}^i = 0 \) for almost all \( h \geq 2 \) (for every \( i \geq 2 \) and for every \( s \in \{2, 3, 4, 5\} \)).
Hence, bearing in mind that $\varphi^{GD} = \varphi^D + \varphi_2^{GD}$ (Corollary 3.5), one has that a G-Drazin inverse $\varphi^{GD}$ is determined by:

- $\varphi^{GD}(v_1) = 6v_1 - 2v_2 + 3v_4 - 3v_5$;
- $\varphi^{GD}(v_2) = -2v_1 + v_2 - v_4 + v_5$;
- $\varphi^{GD}(v_3) = 6v_1 - 2v_2 - 2v_4 - 3v_5$;
- $\varphi^{GD}(v_4) = v_3$;
- $\varphi^{GD}(v_5) = 3v_1 - v_2 + v_4 - v_5 - \frac{1}{3}v_6 + \frac{5}{3}v_7 + \frac{1}{6}\sum_{h \geq 2} \lambda_{5h-3,5}^2v_{5h-3}$;
- $\varphi^{GD}(v_{5i-4}) = -\frac{5}{6}v_{5i-4} - \frac{1}{2}v_{5i-2} - \frac{1}{2}v_{5i} + \sum_{h \geq 2} \left(\frac{1}{2}\lambda_{i,5h-3}^1 - \lambda_{i,5h-3}^3 \right)v_{5h-3}$;
- $\varphi^{GD}(v_{5i-3}) = \frac{1}{3}v_{5i-4} - \frac{1}{6}\sum_{h \geq 2} \lambda_{i,5h-3}^4v_{5h-3}$;
- $\varphi^{GD}(v_{5i-2}) = \sum_{j \geq 6} \alpha_{i,j}v_j$;
- $\varphi^{GD}(v_{5i-1}) = \frac{5}{6}v_{5i-4} - \frac{1}{2}v_{5i-3} - \frac{1}{2}v_{5i} + \sum_{h \geq 2} \left(\frac{1}{2}\lambda_{i,5h-3}^1 - \lambda_{i,5h-3}^3 \right)v_{5h-3}$;
- $\varphi^{GD}(v_{5i}) = -\frac{1}{3}v_{5i-4} - \frac{1}{6}v_{5i-3} + v_{5i-1} - \frac{1}{6}v_{5i} + \sum_{h \geq 2} \left(\frac{1}{2}\lambda_{i,5h-3}^2 - \lambda_{i,5h-3}^4 \right)v_{5h-3}$;

for every $i \geq 2$, and with $\alpha_{i,j} = 0$ for almost all $j \geq 6$ (for each $i \geq 2$) and $\lambda_{i,5h-3,s}^j = 0$ for almost all $h \geq 2$ (for every $i \geq 2$ and for every $s \in \{2, 3, 4, 5\}$).

**Remark 3.14:** Note that the explicit expression of the bijection (13) for the finite potent endomorphism $\varphi$ of Example 3.1 is

$$X^{GD}_{\varphi} \sim \prod_{i \in \mathbb{N}} \left( U_{\varphi} \times \bigoplus_{j=1}^{4} \bigoplus_{h \in \mathbb{N}} k \right).$$

$$\varphi^{GD} \mapsto (\varphi^{GD}(v_{5i-2}), ((\lambda_{i,5h-3}^j)_{h \in \mathbb{N}})_{1 \leq j \leq 4})_{i \in \mathbb{N}}$$

To finish this section, we shall briefly study the G-Drazin inverses of a finite potent endomorphism that also are finite potent.

Let $V$ be an arbitrary $k$-vector space and let $\varphi \in \text{End}_k(V)$ be a finite potent endomorphism.

With the above notation, if we denote by $(\varphi^{GD})_{\{\lambda_{i,j}, \alpha_{i,j}^h\}}$ to the unique linear map of $V$ such that

$$(\varphi^{GD})_{\{\lambda_{i,j}, \alpha_{i,j}^h\}}(v) = \begin{cases} (\varphi|_W)^{-1}(v) & \text{if } v \in W_{\varphi} \\ (\varphi|_{U_{\varphi}})^{-1}(v) & \text{if } v \in U_{\varphi} \end{cases},$$

where $(\varphi|_{U_{\varphi}})^{-1}$ is the generalized inverse of $\varphi|_{U_{\varphi}} \in \text{End}_k(U_{\varphi})$ characterized in Corollary 3.11, it is clear that $(\varphi^{GD})_{\{\lambda_{i,j}, \alpha_{i,j}^h\}}$ is finite potent when

$$\lambda_{i,j}^{i,j} = 0 = \alpha_{i,j}^{i,j} \quad \text{for almost all } i, j \text{ and } s_h.$$  \hfill (14)

It is clear that the condition (14) is sufficient for determining that $(\varphi^{GD})_{\{\lambda_{i,j}, \alpha_{i,j}^h\}}$ is a finite potent endomorphism, but this condition is not necessary for this fact as it is immediately deduced from the following counter-example: given a countable $k$-vector space $V$
with a basis \( \{v_1, v_2, v_3, \ldots \} \) indexed by the natural numbers, if we consider the finite potent endomorphism \( \varphi \in \text{End}_k(V) \) defined as

\[
\varphi(v_i) = \begin{cases} 
  v_1 + v_2 & \text{if } i = 1 \\
  v_1 + 2v_2 & \text{if } i = 2 \\
  v_{i+1} & \text{if } i = 2j + 1 \\
  0 & \text{if } i = 2j + 2
\end{cases}
\]

for every \( j \geq 1 \), then it is clear that

\[
\varphi^{GD}(v_i) = \begin{cases} 
  2v_1 - v_2 & \text{if } i = 1 \\
  -v_1 + v_2 & \text{if } i = 2 \\
  -v_i - v_{i+1} & \text{if } i = 2j + 1 \\
  v_{i-1} + v_i & \text{if } i = 2j + 2
\end{cases}
\]

for every \( j \geq 1 \), is a G-Drazin inverse of \( \varphi \) that does not satisfy the condition (14).

**Remark 3.15:** A remaining problem is obtaining a computable method for determining when a G-Drazin inverse \( \varphi^{GD} \) of a finite potent endomorphism \( \varphi \) is also finite potent.

**Remark 3.16:** If \( \varphi^{GD} \) is a finite potent G-Drazin inverse of a finite potent endomorphism \( \varphi \) such that \( W_{\varphi^{GD}} = W_\varphi \), then

\[
\text{tr}_V\varphi^{GD} = \text{tr}_V\varphi^D \quad \text{and} \quad \det_k^k\varphi^{GD} = \det_k^k\varphi^D.
\]

Indeed, in this case, if \( \{\lambda_1, \ldots, \lambda_n\} \) are the eigenvalues of \( \varphi|_{W_\varphi} \) in the algebraic closure of \( k \) (with their multiplicity), one has that:

- \( \text{tr}_V(\varphi^{GD}) = \lambda_1^{-1} + \cdots + \lambda_n^{-1} \);
- \( \det_k^k(1 + \varphi^{GD}) = \prod_{i=1}^n (1 + \lambda_i^{-1}) \).

### 4. Explicit computation of the G-Drazin inverses of a square matrix

The final section of this work is devoted to offer a method for computing explicitly all the G-Drazin inverses of a square matrix.

If \( k \) is an arbitrary ground field, let us consider a square matrix \( A \in \text{Mat}_{n \times n}(k) \) with \( i(A) = r \).

Fixing a \( k \)-vector space \( E \) with dimension \( n \), a basis \( B = \{e_1, \ldots, e_n\} \) of \( E \) and an endomorphism \( \varphi \in \text{End}_k(E) \) associated with \( A \) in the basis \( B \), from the AST-decomposition \( E = W_\varphi \oplus U_\varphi \) one has that

\[
A = P \begin{pmatrix} A_W & 0 \\ 0 & J_U \end{pmatrix} P^{-1},
\]

where \( \varphi|_{W_\varphi} = A_W, J_U \) is the Jordan matrix determined by \( \varphi|_{U_\varphi} \) and \( P \) is the corresponding basis change matrix.
If $A \in \text{Mat}_{n \times n}(k)$ is again a square matrix with $i(A) = r$ and $\text{rk}(A^i)$ is the rank of $A^i$, we can write $v_i(A) = n - \text{rk}(A^i)$ for all $i \in \{1, \ldots, r\}$ and we can consider the non-negative integers $\{\delta_1(A), \ldots, \delta_r(A)\}$ defined from the equations:

\[
\delta_r(A) = v_r(A) - v_{r-1}(A) \\
2\delta_r(A) + \delta_{r-1}(A) = v_r(A) - v_{r-2}(A) \\
\vdots \\
(r-1)\delta_r(A) + \cdots + 2\delta_3(A) + \delta_2(A) = v_r(A) - v_1(A) \\
r\delta_r(A) + (r-1)\delta_{r-1}(A) + \cdots + 2\delta_2(A) + \delta_1(A) = v_r(A)
\]

From these relations it is clear that

\[
v_1(A) = \sum_{i=1}^{r} \delta_i(A) \quad \text{and} \quad \sum_{j=1}^{r} \delta_j(A)(v_r(A) - j) = [v_1(A) - 1]v_r(A).
\]

Accordingly, the explicit expression of the matrix $J_U$ is

\[
J_U = \begin{pmatrix}
A_1^1 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\
0 & \ddots & \ddots & \cdots & \cdots & \vdots & 0 \\
\vdots & \ddots & A_1^{\delta_1(A)} & \cdots & \cdots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & A_r^1 & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \vdots & A_r^{\delta_r(A)}
\end{pmatrix} \in \text{Mat}_{\nu_r \times \nu_r}(k),
\]

where

\[
A_j^s = \begin{pmatrix}
0 & 0 & \cdots & \cdots & \cdots & 0 \\
1 & 0 & \ddots & \cdots & \cdots & \vdots \\
0 & 1 & 0 & \ddots & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & 0 & 0 & 0 \\
0 & \cdots & 0 & 1 & 0 & 0
\end{pmatrix} \in \text{Mat}_{j \times j}(k)
\]

for every $j \in \{1, \ldots, r\}$ and $1 \leq s \leq \delta_j(A)$.

With the above notation, if $A(GD)$ is the set of the G-Drazin inverses of $A$, it follows from Lemma 3.9 that there exists a bijection

\[
k^{v_1(A)-v_r(A)} \times k^{[v_r(A)-v_1(A)]} \times k^{[v_1(A)-1][v_r(A)-v_1(A)]} \rightarrow A(GD)
\]

\[
((\alpha_{j,h}^s, (\alpha_{j,j';z}^s)), (\lambda_{j,t}^s), (\lambda_{j';x}^s)) \mapsto (A^{GD})^{((\alpha_{j,h}^s, (\lambda_{j,t}^s)), (\alpha_{j,j';z}^s), (\lambda_{j';x}^s))},
\]

(16)
where \( j, j' \in \{1, \ldots, r\}; j \neq j' \); \( 1 \leq h \leq j; \ z \in \{1, \ldots, f\}; t \in \{2, \ldots, j\}; x \in \{2, \ldots, f\} \); \( s \in \{1, \ldots, \delta_j(A)\} \) and

\[
(A^{GD})_{((\alpha_{j,h}^s, \lambda_{j,h}^z), (\alpha_{j,h}^x, \lambda_{j,h}^t))} = P \cdot \begin{pmatrix}
(A_W)^{-1} & 0 \\
0 & (J_U)^{((\alpha_{j,h}^x, \lambda_{j,h}^t), (\alpha_{j,h}^s, \lambda_{j,h}^z))}
\end{pmatrix} \cdot P^{-1}
\]

with

\[
(J_U)^{((\alpha_{j,h}^x, \lambda_{j,h}^t), (\alpha_{j,h}^s, \lambda_{j,h}^z))} = (J_U)^{((\alpha_{j,h}^0, \lambda_{j,h}^{0}), (\alpha_{j,h}^0, \lambda_{j,h}^{0}))}_{lm, 1 \leq l, m \leq v_1} \in \text{Mat}_{v_1 \times v_1}(k)
\]

such that

- if \( j \in \{1, \ldots, r\} \) and \( 1 \leq s \leq \delta_j(A) \) are such that \( l = (\sum_{i=1}^{j-1} \delta_i) + s \), then \((J_U)_{ll} \in \text{Mat}_{j \times j}(k)\) with

\[
(J_U)^{((\alpha_{j,h}^x, \lambda_{j,h}^t), (\alpha_{j,h}^s, \lambda_{j,h}^z))}_{ll} = (A_{((\alpha_{j,h}^x, \lambda_{j,h}^t), (\alpha_{j,h}^s, \lambda_{j,h}^z))})_{ll}
\]

\[
= \begin{pmatrix}
\alpha_{j,1}^s & 1 & 0 & \ldots & \ldots & 0 \\
\alpha_{j,2}^s & 0 & 1 & 0 & \ldots & \ldots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\alpha_{j,j-2}^s & 0 & \ldots & \ddots & 1 & 0 \\
\alpha_{j,j-1}^s & 0 & \ldots & 0 & 1 & \alpha_{j,j}^s \\
\alpha_{j,j}^s & \lambda_{j,j}^z & 0 & \ldots & \lambda_{j,j}^z & \lambda_{j,j}^z
\end{pmatrix}
\]

for all \( \alpha_{j,h}^s, \lambda_{j,t}^z \in k, h \in \{1, \ldots, j\}, t \in \{2, \ldots, j\}, j \in \{1, \ldots, r\} \) and \( s \in \{1, \ldots, \delta_j(A)\} \);

- if \( j, j' \in \{1, \ldots, r\}, 1 \leq s \leq \delta_j(A) \) and \( 1 \leq s' \leq \delta_j(A) \) are such that

\[
l = \left( \sum_{i=1}^{j-1} \delta_i \right) + s \quad \text{and} \quad m = \left( \sum_{i=1}^{j'-1} \delta_i \right) + s',
\]

with \( l \neq m \), then \((J_U)_{lm} \in \text{Mat}_{j \times j}(k)\) where

\[
(J_U)^{((\alpha_{j,h}^x, \lambda_{j,h}^t), (\alpha_{j,h}^s, \lambda_{j,h}^z))}_{lm} = (A_{((\alpha_{j,h}^x, \lambda_{j,h}^t), (\alpha_{j,h}^s, \lambda_{j,h}^z))})_{lm}
\]

\[
= \begin{pmatrix}
\alpha_{j,j'}^z & 0 & \ldots & \ldots & 0 \\
\alpha_{j,j'}^z & \ldots & \ldots & \ldots & \ldots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\alpha_{j,j'}^z & 0 & \ldots & 0 & \alpha_{j,j'}^z \\
\alpha_{j,j'}^z & \lambda_{j,j'}^{x, s} & \ldots & \lambda_{j,j'}^{x, s} & \lambda_{j,j'}^{x, s}
\end{pmatrix}
\]

for every \( \alpha_{j,h}^s, \lambda_{j,t}^z \in k, j \neq j', 1 \leq j, j' \leq r, z \in \{1, \ldots, j\}, x \in \{2, \ldots, j'\} \) and \( s \in \{1, \ldots, \delta_j(A)\} \).
If \( \tilde{A} \in \text{Mat}_{n \times n}(k) \) with \( i(\tilde{A}) = 1 \), it follows from (16) that there exists a bijection \( \tilde{A}^{GD} \sim k^{[\nu_1(\tilde{A})^2]} \).

From the results of this work, we can finally offer the following algorithm for computing the G-Drazin inverses of \( A \in \text{Mat}_{n \times n}(k) \).

1. Fix a \( k \)-vector space \( E \) with dimension \( n \), a basis \( B = \{e_1, \ldots, e_n\} \) of \( E \) and an endomorphism \( \varphi \in \text{End}_k(E) \) associated with \( A \) in the basis \( B \), to facilitate the computations.
2. Compute the AST-decomposition \( E = W_\varphi \oplus U_\varphi \) and the matrix expression (15) for \( A \).
3. Calculate the non-negative integer numbers \( \{v_1(A), \ldots, v_r(A)\} \) and \( \{\delta_1(A), \ldots, \delta_r(A)\} \).
4. Construct the matrices \( (J_U^{-1})^{((\alpha^i_j)_{j \neq s},(\lambda^i_j)_{j \neq t})} \) and compute \( (A_W)^{-1} \).
5. Get all the G-Drazin inverses \( A^{GD} \) of \( A \).

**Remark 4.1:** We wish remark that is not necessary to compute the characteristic polynomial \( c_A(x) \) in the method offered in this paper for calculate all the G-Drazin polynomials of a square matrix \( A \) with \( i(A) = r \), because we can obtain the matrices \( A_W \) and \( A_U \) by computing \( R(A^t) \) and \( N(A^t) \), where \( R(B) \) and \( N(B) \) are the range and the nullspace of a matrix \( B \) respectively.

**Example 4.1:** Let us consider an arbitrary field \( k \) and the matrix

\[
A = \begin{pmatrix}
-9 & -7 & 11 & -3 & -6 & -4 & -2 \\
-3 & 1 & 2 & 1 & 1 & 0 & 1 \\
-13 & -8 & 15 & -3 & -7 & -5 & -2 \\
-4 & -3 & 5 & -1 & -3 & -2 & -1 \\
-13 & -12 & 17 & -6 & -11 & -7 & -4 \\
11 & 10 & -14 & 5 & 9 & 6 & 3 \\
8 & 6 & -10 & 3 & 6 & 4 & 2
\end{pmatrix} \in \text{Mat}_{7 \times 7}(k).
\]

We shall compute all the G-Drazin inverses \( A^{GD} \) of \( A \).

Let us now fix a \( k \) vector space \( E \) with basis \( \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\} \) and an endomorphism \( \varphi \in \text{End}_k(E) \) such that \( \varphi \equiv A \) in this basis.

It is easy to check that \( i(A) = 3 \), \( \text{rk}(A) = 5 \), \( \text{rk}(A^2) = 3 \) and \( \text{rk}(A^3) = 2 \). Accordingly, \( v_1(A) = 2 \), \( v_2(A) = 4 \), \( v_3(A) = 5 \), \( \delta_1(A) = 0 \), \( \delta_2(A) = 1 \) and \( \delta_3(A) = 1 \).

Now, a non-difficult computation shows that \( W_\varphi = \langle e_1 + e_2 + 2e_3 + e_5 - e_7, e_4 - e_6 \rangle \), \( U_\varphi = \langle e_1 + e_3, e_2 - e_5, -e_5 + e_6, e_4 - e_7, e_1 + e_3 - e_7 \rangle \),

\[
A_W = \begin{pmatrix}
2 & 1 \\
1 & 1
\end{pmatrix}
\]
and

\[
A_U = \begin{pmatrix}
3 & -1 & 3 & -2 & 0 \\
-1 & 0 & -1 & 0 & 0 \\
-3 & 1 & -3 & 2 & 0 \\
1 & 0 & 1 & 0 & 0 \\
-1 & 0 & -1 & 1 & 0
\end{pmatrix}.
\]

Moreover, from the Jordan basis

\[
\{-e_1 + e_2 - e_3 - e_5 - e_7, -e_1 - e_3 - e_5 + e_6, e_1 + e_2 + e_3 + e_4 - e_5 - e_7, \\
-e_2 + e_4 + e_5 - e_7, -e_5 + e_6 + e_7\}
\]

of \(U_\phi\) induced by \(\varphi_{|_{U_\phi}}\), one gets that

\[
A = P \cdot \begin{pmatrix}
2 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \cdot P^{-1},
\]

with

\[
P = \begin{pmatrix}
1 & 0 & -1 & -1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & -1 & 0 \\
2 & 0 & -1 & -1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & -1 & -1 & -1 & 1 & -1 \\
0 & -1 & 0 & 1 & 0 & 0 & 1 \\
-1 & 0 & -1 & 0 & -1 & -1 & 1
\end{pmatrix},
\]

and

\[
P^{-1} = \begin{pmatrix}
-1 & 0 & 1 & 0 & 0 & 0 & 0 \\
-2 & -1 & 2 & 0 & -1 & -1 & 0 \\
5 & 6 & -7 & 3 & 5 & 3 & 2 \\
-8 & -8 & 10 & -4 & -7 & -4 & -3 \\
-1 & -2 & 2 & -1 & -2 & -1 & -1 \\
3 & 3 & -4 & 2 & 3 & 2 & 1 \\
6 & 7 & -8 & 4 & 6 & 4 & 3
\end{pmatrix}.
\]

Thus, it follows from the method described above that a G-Drazin inverse of \(A\) has the explicit expression

\[
\left(A^{GD}_G\right)_{((\gamma_{jk}, (\alpha_{jk}^1,\lambda_{jk}^1)))} = P \cdot \begin{pmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha_{2,1}^1 & 1 & \alpha_{2,3,1}^1 & 0 & 0 \\
0 & 0 & \alpha_{2,2}^1 & \lambda_{2,2}^1 & \alpha_{2,3,2}^1 & \lambda_{2,3,2}^1 & \lambda_{2,3,3}^1 \\
0 & 0 & \alpha_{3,2,1}^1 & 0 & \alpha_{3,1}^1 & 1 & 0 \\
0 & 0 & \alpha_{3,2,2}^1 & 0 & \alpha_{3,2}^1 & 0 & 1 \\
0 & 0 & \alpha_{3,2,3}^1 & \lambda_{3,2,2}^1 & \alpha_{3,3}^1 & \lambda_{3,2}^1 & \lambda_{3,3}^1
\end{pmatrix} \cdot P^{-1},
\]
with \( \alpha_{j,h}, \lambda_{j,t}, \alpha_{j',z}, \lambda_{j',x} \in k \) for every \( j \neq j', 2 \leq j, j' \leq 3, h \in \{1, \ldots, j\}, t \in \{2, \ldots, j\}, z \in \{1, \ldots, j\} \) and \( x \in \{2, \ldots, j'\} \).

Hence, with the data of this example, we have the following bijection that determines all the G-Drazin inverses of \( A \):

\[
k^{10} \times k^3 \times k^3 \sim A\{GD\}
\]

\[
((\alpha_{j,h}^1), (\alpha_{j',z}^1))_{(j, j', h, z)}, (\lambda_{j,t}^1), (\lambda_{j',x}^1))_{(j, j', t, x)} \mapsto (A^{GD})_{(\alpha_{j,h}^1), (\lambda_{j,t}^1), (\alpha_{j',z}^1), (\lambda_{j',x}^1)}
\]

with \( j \neq j', 2 \leq j, j' \leq 3, h \in \{1, \ldots, j\}, t \in \{2, \ldots, j\}, z \in \{1, \ldots, j\} \) and \( x \in \{2, \ldots, j'\} \).

Finally, we shall study the relationships that exist between G-Drazin inverses and the core-nilpotent decomposition of a matrix \( A \).

**Remark 4.2:** If \( k \) is an arbitrary ground field, \( A \in \text{Mat}_{n \times n}(k) \), \( A = A_1 + A_2 \) is its core-nilpotent decomposition and \( A^{GD} \) is a G-Drazin inverse of \( A \) with core-nilpotent decomposition \( A^{GD} = (A^{GD})_1 + (A^{GD})_2 \), Example 4.1 shows that, in general, \((A^{GD})_1\) is not a G-Drazin inverse of \( A_1 \) and \((A^{GD})_2\) is not a G-Drazin inverse of \( A_2 \).

As a counterexample of this fact we offer the following: if \( A \) is the matrix studied in Example 4.1, an easy computation shows that its core-nilpotent decomposition is \( A = A_1 + A_2 \) with

\[
A_1 = \begin{pmatrix}
-4 & -1 & 4 & 0 & -1 & -1 & 0 \\
-4 & -1 & 4 & 0 & -1 & -1 & 0 \\
-8 & -2 & 8 & 0 & -2 & -2 & 0 \\
-3 & -1 & 3 & 0 & -1 & -1 & 0 \\
-4 & -1 & 4 & 0 & -1 & -1 & 0 \\
3 & 1 & -3 & 0 & 1 & 1 & 0 \\
4 & 1 & -4 & 0 & 1 & 1 & 0
\end{pmatrix}
\]

and

\[
A_2 = \begin{pmatrix}
-5 & -6 & 7 & -3 & -5 & -3 & -2 \\
1 & 2 & -2 & 1 & 2 & 1 & 1 \\
-5 & -6 & 7 & -3 & -5 & -3 & -2 \\
-1 & -2 & 2 & -1 & -2 & -1 & -1 \\
-9 & -11 & 13 & -6 & -10 & -6 & -4 \\
8 & 9 & -11 & 5 & 8 & 5 & 3 \\
4 & 5 & -6 & 3 & 5 & 3 & 2
\end{pmatrix}.
\]

If we now consider

\[
A^{GD} = \begin{pmatrix}
7 & 6 & -8 & 3 & 6 & 4 & 2 \\
-10 & -11 & 13 & -6 & -9 & -5 & -5 \\
8 & 7 & -9 & 3 & 7 & 5 & 2 \\
6 & 8 & -9 & 6 & 7 & 4 & 4 \\
8 & 9 & -10 & 4 & 8 & 5 & 4 \\
7 & 6 & -8 & 2 & 5 & 4 & 1 \\
-3 & -5 & 5 & -3 & -5 & -4 & 2
\end{pmatrix}
\]
which is the G-Drazin inverse of $A$ determined by $\lambda_{2,1}^1 = \lambda_{3,1}^1 = 1$ and otherwise $a_{j,h}^1 = \lambda_{j,t}^1 = \gamma_{j',z}^1 = 0$, one has that $(A^{GD})_1 = (A^{GD})_1^{-1}$ and $(A^{GD})_2 = 0$, and we can immediately check that $(A^{GD})_1$ is not a G-Drazin inverse of $A_1$ and $(A^{GD})_2$ is not a G-Drazin inverse of $A_2$.

Furthermore, if $(A_1^{GD})$ is a G-Drazin inverse of $A_1$ and $(A_2^{GD})$ is a G-Drazin inverse of $A_2$, in general, one has that $\tilde{A}^{GD} = (A_1^{GD}) + (A_2^{GD})$ is not a G-Drazin inverse of $A$, as can be deduced from this counterexample: keeping again the data of Example 4.1, if we consider

$$
(A_1^{GD}) = P \cdot \begin{pmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \cdot P^{-1}
$$

and

$$
(A_2^{GD}) = P \cdot \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \cdot P^{-1},
$$

then it is clear that

$$
\tilde{A}^{GD} = P \cdot \begin{pmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \cdot P^{-1}
$$

is not a G-Drazin inverse of $A$.

**Acknowledgements**

The author would like to thank the anonymous reviewer for his/her valuable comments to improve the quality of the paper.

**Disclosure statement**

No potential conflict of interest was reported by the author(s).
**Funding**

This work is partially supported by the Ministerio de Ciencia e Innovación – Spanish Government research projects nos. MTM2015-66760-P and PGC2018-099599-B-I00 and the Consejería de Educación, Junta de Castilla y León – Regional Government of Castile and Leon research project no. J416/463AC03.

**ORCID**

Fernando Pablos Romo  http://orcid.org/0000-0003-3444-1800

**References**

[1] Wang X, Liu H. Partial orders based on core-nilpotent decomposition. Linear Algebra Appl. 2016;488:235–248.

[2] Coll C, Lattanzi M, Thome N. Weighted G-Drazin inverses and a new pre-order on rectangular matrices. Appl Math Comput. 2018;317:12–24.

[3] Tate J. Residues of differentials on curves. Ann Sci École Norm Sup 4. 1968;1:149–159.

[4] Pablos Romo F. Core-Nilpotent decomposition and new generalized inverses of finite potent endomorphisms. Linear Multilinear Algebra. 2019. doi:10.1080/03081087.2019.1578332

[5] Pablos Romo F. On Drazin-Moore-Penrose inverses of finite potent endomorphisms. Linear Multilinear Algebra. 2019. doi:10.1080/03081087.2019.1612834

[6] Pablos Romo F. On the Drazin inverse of finite potent endomorphisms. Linear Multilinear Algebra. 2019;67(10):2135–2146.

[7] Argerami M, Szechtman F, Tifenbach R. On Tate’s trace. Linear Multilinear Algebra. 2007;55(6):515–520.

[8] Hernández Serrano D, Pablos Romo F. Determinants of finite potent endomorphisms, symbols and reciprocity laws. Linear Algebra Appl. 2013;439:239–261.

[9] Pablos Romo F. On the linearity property of Tate’s trace. Linear Multilinear Algebra. 2007;55(4):323–326.

[10] Ramos González J, Pablos Romo F. A negative answer to the question of the linearity of Tate’s trace for the sum of two endomorphisms. Linear Multilinear Algebra. 2014;62(4):548–552.

[11] Pablos Romo F. Classification of finite potent endomorphisms. Linear Algebra Appl. 2014;440:266–277.

[12] López-Pellicer M, Bru R. Jordan basis for an infinite-dimensional space. Port Math. 1985/86;43(1):153–156.

[13] Drazin MP. Pseudo-inverses in associative rings and semigroups. Amer Math Monthly. 1958;65(7):506–514.