Vortex waves and the onset of turbulence in $^3$He-B

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Abstract. – In a recent experiment Finne et al. discovered an intrinsic condition for the onset of quantum turbulence in $^3$He-B, that $q = \alpha/(1 - \alpha') < 1$, where $\alpha$ and $\alpha'$ are mutual friction parameters. The authors argued that this condition corresponds to Kelvin waves which are marginally damped, so for $q > 1$ Kelvin waves cannot grow in amplitude and trigger vortex reconnections and turbulence. By analysing both axisymmetric and non-axisymmetric modes of oscillations of a rotating superfluid, we confirm that in the long axial wavelength limit the simple condition $q = 1$ is indeed the crossover between damped and propagating Kelvin waves.

Introduction. – A striking feature of superfluidity is the existence of quantised vortex filaments. Quantised vortices are particularly interesting because, unlike vortices in a classical turbulent fluid, they are discrete, stable topological defects. Until recently the study of quantised vorticity and quantum turbulence [1] was mainly confined to $^4$He, but in the last few years there has been progress in other contexts [2], ranging from atomic Bose–Einstein condensates to $^3$He, which is a fermionic superfluid.

Our work is motivated by the recent discovery by Finne et al. [3] of a sharp transition at the temperature $T = 0.6 T_c$ between two distinct hydrodynamical regimes in $^3$He-B, where $T_c$ is the critical temperature. In their experiment Finne et al. injected one or a few seeding vortex loops into rotating $^3$He-B. Using a non-invasive NMR measurement technique, they measured the total length of the quantised vortices as a function of time. They found that if $T > 0.6 T_c$ the initial loops expand and straighten into rectilinear vortices which align along the axis of rotation. However, if $T < 0.6 T_c$, the seeding loops evolve into a turbulent vortex tangle. The effect is independent of the fluid’s velocity and the details of the initial condition, which suggests the existence of a remarkable intrinsic criterion for the onset of quantum turbulence. On the contrary, the condition for the onset of turbulence in a classical fluid, that the Reynolds number is large, is extrinsic, because the Reynolds number depends on the geometry and the scale of the flow, not only on properties of the fluid.

Finne et al. interpreted their experimental result in terms of Kelvin waves on the vortex filaments - helical perturbations of the position of a vortex core away from the unperturbed straight shape (see Fig. 1). They argued that the superfluid vorticity is determined by the
ratio $q$ of dissipative and inertial forces in the superfluid, in analogy with the classical Reynolds number, which expresses the ratio of inertial and viscous forces in an ordinary Navier–Stokes fluid. The quantity $q$ is defined as

$$q = \frac{\alpha}{1 - \alpha'},$$

where $\alpha$ and $\alpha'$ are known [4] mutual friction coefficients, which determine the strength of the interaction between the superfluid vortices and the normal fluid. Finne et al. noticed that, for a single superfluid vortex, it was predicted [2] that $q = 1$ is the crossover from Kelvin waves which propagate ($q < 1$) and Kelvin waves which are overdamped ($q > 1$). By performing enlightening numerical simulations, Finne et al. confirmed that, if $q > 1$ an injected vortex loop straightens and becomes rectilinear, whereas if $q < 1$ Kelvin waves propagate freely, as predicted. In the latter case, the amplitude of the Kelvin wave grows, driven by the local difference between the normal fluid velocity and the velocity of the vortex line [5]. When the waves’ amplitude becomes of the same order of the average intervortex spacing, the vortices reconnect with each other, quickly forming a disordered turbulent tangle. The value of $q$ depends on $T$, and the observed temperature $0.6 T_c$ corresponds to $q = 1.3$, which is indeed close to unity.

The aim of our work is to reconsider the condition $q = 1$ (which for a single vortex is the crossover from free to overdamped motion) for a large density of vortices, such as the vortex lattice in the rotating helium experiment of Finne et al. In this the context the behaviour of the superfluid is described by the Hall-Vinen equations [6]. Thus, rather than individual vortex lines, we consider a continuum of vortex lines. The advantage of this model is that it allows us to explore effects which the theory of a single vortex filament cannot describe, notably the presence of boundaries and the degrees of freedom represented by the coherent oscillatory motion of many vortices. Mathematically, this corresponds to the fact that for a single vortex, since the core radius is fixed, only $m = 1$ modes (sideway displacements of the vortex core) are possible, whereas for a bundle of vortices any azimuthal $m$ symmetry is permitted. The simple question which we address is therefore the following: given a superfluid in a cylindrical container of radius $a$ rotating at constant angular velocity $\Omega$, thus forming a vortex lattice with a large density of vortex lines, does the condition $q = 1$ represent the crossover from propagating Kelvin waves to overdamped Kelvin waves for all possible modes of oscillations?

The governing equations and the basic state. – The equation of motion of the superfluid in a coordinate system rotating with angular velocity $\Omega = \Omega e_z$ may be written as

$$\frac{\partial v^s}{\partial t} + (v^s \cdot \nabla)v^s = \nabla \Psi + 2v^s \times \Omega + \alpha \hat{\lambda} \times [\lambda \times (v^s - v^n)] + \alpha' \lambda \times (v^s - v^n) - \alpha \nu \hat{\lambda} \times (\lambda \cdot \nabla)\hat{\lambda} + \nu(1 - \alpha')(\lambda \cdot \nabla)\hat{\lambda}$$

(2)

where $v^s$ and $v^n$ are the superfluid and normal fluid velocities in the rotating frame, $\lambda = \nabla \times v^s + 2\Omega$, $\hat{\lambda} = \lambda / |\lambda|$ is the unit vector in the direction of $\lambda$ and $\Psi$ is a collection of scalar terms. Given the high viscosity of $^3$He-B, we assume that the normal fluid is in solid body rotation around the z-axis, thus in the rotating frame $v^n = 0$. Eq. (2) must be solved under the condition that $\nabla \cdot v^s = 0$. The quantity $\nu = (\Gamma/4\pi)\log(b_0/a_0)$ is the vortex tension parameter, $\Gamma$ is the quantum of circulation, $a_0$ is the vortex core radius and $b_0 = (|\lambda|/\Gamma)^{-1/2}$ is the average distance between vortices. The unperturbed vortex lattice corresponds to the basic state $v^s_0 = v^n_0 = 0, \nabla \Psi_0 = 0$ for which $\lambda = 2\Omega e_z$, working in cylindrical coordinates $(r, \phi, z)$.
The equations of the perturbations. – We perturb the basic state by letting \( \mathbf{v}^* = \mathbf{u} = (u_r, u_\phi, u_z) \), \( \Psi = \Psi_0 + \psi \), where \(|\mathbf{u}| \ll 1 \) and \(|\psi| \ll 1 \) are small perturbations. We assume normal modes of the form \( \exp(\mathrm{i}(\alpha \phi + \beta z)) \), where \( \alpha \) and \( \beta \) are the azimuthal and axial wavenumbers and \( \sigma \) is the growth rate. The aim of our calculation is to determine the real and imaginary parts of \( \sigma \), namely \( \Re(\sigma) \) and \( \Im(\sigma) \). The resulting linearised equations for the perturbations are

\[
(\sigma + \alpha \eta)u_r - \eta(1 - \alpha')u_\phi &= \frac{d\psi}{dr} - \frac{m\kappa \nu^s}{r}(1 - \alpha')u_z - \kappa \nu^s \alpha \frac{du_z}{dr} = 0, \\
\eta(1 - \alpha')u_r + (\sigma + \alpha \eta)u_\phi &= \frac{m}{r} \psi + \frac{\alpha m \kappa \nu^s}{r}u_z - \kappa \nu^s (1 - \alpha') \frac{du_z}{dr} = 0, \\
\sigma u_z &= \kappa \psi, \\
\frac{1}{r} \frac{d}{dr} \left( r \frac{du_z}{dr} \right) + \frac{m}{r} u_\phi + \kappa u_z &= 0,
\]

where \( \eta = 2\Omega + \nu^s k^2 \). Eliminating \( u_r, u_\phi \) and \( \psi \) we obtain the following differential equation for \( u_z \):

\[
\frac{1}{r} \frac{d}{dr} \left( r \frac{du_z}{dr} \right) - \frac{m^2}{r^2} u_z + \beta^2 u_z = 0,
\]

where

\[
\beta^2 = \frac{-k^2[(\sigma + \alpha \eta)^2 + (1 - \alpha')^2 \eta^2]}{[(\sigma + \alpha \eta)(\sigma + k^2 \nu^s \alpha) + (1 - \alpha')^2 \nu^s k^2 \eta]}
\]

The solution of Eq. (11) which is regular as \( r \to 0 \) is the Bessel function of the first kind of order \( m, J_m(\beta r) \). To determine \( \beta \) we enforce the boundary condition \( u_r = 0 \) on the wall of the container \( r = a \), which yields the secular equation

\[
\frac{(ka)^2 J_m'(\beta a)}{(\beta a) J_m(\beta a)} [(\sigma + \alpha \eta)^2 + (1 - \alpha')^2 \eta^2] + 2m\Omega \sigma (1 - \alpha') = 0,
\]

where the prime denotes the derivative of the Bessel function with respect to its argument.

Limiting cases. – The dispersion law which we have obtained, represented by Eq.’s (8,9), has two limiting cases which have already appeared in the literature. Firstly, if we set \( \alpha = \alpha' = \nu^s = 0 \), we recover Chandrasekhar’s classical result for the modes of vibrations of a rotating column of liquid [7]: \( \sigma \) is determined by

\[
\beta^2 = k^2 \left( \frac{4\Omega^2}{\sigma^2} - 1 \right),
\]

where the values of \( \beta \) are given by the roots of

\[
\sigma \beta a J_m'(\beta a) + 2m\Omega J_m(\beta a) = 0.
\]

Secondly, if we set \( \alpha = \alpha' = m = 0 \), Eq. (12) reduces to \( J_0(\beta a) = 0 \). Let \( \xi_j \) be the \( j^{th} \) zero of \( J_1(\xi) = J_0(\xi) \) (that is \( \xi_1 = 3.83171, \xi_2 = 7.01559 \) etc); then we have

\[
\sigma^2 = \frac{(k^2 a^2 \eta^2 + \nu^s \eta k^2 \xi_j^2)}{(\xi_j^2 + k^2 a^2)}.
\]
If the wavelength of the perturbations is larger than the radius of the cylinder we may neglect edge effects, \((ka)^2 \gg \xi_j^2\), and we recover Hall’s result [8] for the axisymmetric oscillations of a rotating superfluid:

\[
\sigma^2 \approx \eta^2 = 2\Omega + \nu^*k^2.
\] (13)

Eq. (13) generalises to a vortex lattice the dispersion relation \(\sigma = \nu^*k^2\) of an individual vortex line [2].

**The axisymmetric case.** – Eq.’s (8,9) simplify in the axisymmetric case, \(m = 0\). We have again \(J_0(\beta a) = 0\) for which \(\beta a = \xi\) and Eq. (9) becomes

\[
\frac{\xi_j^2}{(ka)^2} = -\frac{[\iota(\sigma + \alpha\eta)^2 + (1 - \alpha')^2\eta^2]}{[\iota(\sigma + \alpha\eta)(\sigma + \alpha\nu^*k^2) + (1 - \alpha')^2\nu^*\eta k^2]}.
\] (14)

This quadratic equation has solutions

\[
((ak)^2 + \xi_j^2)\sigma = \iota\alpha(2\Omega(ak)^2 + \Omega\xi_j^2 + \nu^*k^2((ak)^2 + \xi_j^2))
\]

\[
\pm \sqrt{\{(1 - \alpha')^2k^2((ak)^2 + \xi_j^2)(2\Omega + \nu^*k^2)(2\Omega a^2 + \nu^*((ak)^2 + \xi_j^2)) - \alpha^2\Omega^2\xi_j^4\}}
\] (15)

This result is identical to that derived by Glaberson et al. [9] who expanded in normal modes of the form \(\exp(\iota k_1x + \iota k_2y + \iota k_3z + \iota \sigma t)\) where \(\xi_j^2/\alpha^2 = k_1^2 + k_2^2\). Expanding in normal modes of this form means that the boundary condition at \(r = a\) is not enforced.

From Eq. (15) it can be seen that \(\Im(\sigma)\) is non-negative, so the system is always stable to infinitesimal disturbances. If we neglect perturbations in the non-axial direction (setting \(\beta = \xi = k_1 = k_2 = 0\)) we obtain

\[
\sigma = \iota\alpha(2\Omega + \nu^*k^2) \pm (1 - \alpha')(2\Omega + \nu^*k^2),
\] (16)

as quoted by Hall [8]. This solution is illustrated in Fig. 2 by a dashed line and satisfies \(|\Im(\sigma)/\Re(\sigma)| = q\). However, if we do not neglect perturbations in the non-axial direction, Eq. (16) has an infinite number of solutions, according to the value of \(\xi_j\) considered. The least stable mode is the one for which \(\Im(\sigma)\) is minimum and is the mode which we would expect to be observed provided the initial state is sufficiently random. \(\Im(\sigma)\) decreases monotonically as \(k \to 0\) and \(\xi_j \to \infty\). For small \(k\) the term inside the square root of Eq. (15) is negative, resulting in \(\Re(\sigma)\) being zero and the mode decays monotonically with time. As \(k\) increases, the term inside the square root is positive and the mode displays oscillatory decay with time. The solution which has the smallest decay rate is

\[
\sigma = \iota\alpha(\Omega + \nu^*k^2) \pm \sqrt{\{(1 - \alpha')^2\nu^*k^2(2\Omega + \nu^*k^2) - \alpha^2\Omega^2\}}
\] (17)

which corresponds to \(\xi_j \to \infty\). Plots of the decay rate, \(-\Im(\sigma)\) and the ratio of the decay rate of the wave and its angular frequency, \(|\Im(\sigma)/\Re(\sigma)|\) can be found in Fig. 2 where the mutual friction parameters correspond to \(T = 0.4T_c\), for which \(\alpha = 0.1125\) and \(\alpha' = 0.1\), and we have taken \(\nu^*/\alpha^2 = 0.001\) and \(\Omega = 1\). For large \((ak)\) we find

\[
|\Im(\sigma)/\Re(\sigma)| \to \frac{\alpha}{(1 - \alpha')} = q.
\] (18)

The minimum permitted value of \(k\) will be governed by the height, \(h\) of the apparatus by the relation \(k_{\text{min}} = 2\pi/h\), so provided the aspect ratio \(h/a\) is small enough we find agreement with the argument of Finne et al. [3].
Non-axisymmetric case. – In order to consider the non-axisymmetric modes we must solve the coupled equations \(\text{(8,9)}\). For given temperature, \(T\), wavenumbers \(m, ak\) and parameters \(\nu'/a^2, \Omega\), Eq.‘s \(\text{(8,9)}\) were solved using NAG routine C05NDF on the real and imaginary parts of \(\sigma\) and \((a\beta)^2\). As for the axisymmetric case, there will be an infinite number of solutions for each mode, \(m\) considered.

In Fig. 3 we illustrate some of the solutions computed for the \(m = 1\) mode. The quantities \(-I_m(\sigma)\) and \(|I_m(\sigma)/\Re(\sigma)|\) are plotted against \(ak\) using the same parameters as for Fig. 2. The dotted line represents the least stable axisymmetric mode. We can see that the least stable \(m = 1\) mode is bound by the least stable axisymmetric mode and that, as for the \(m = 0\) case, all the computed solutions show \(|I_m(\sigma)/\Re(\sigma)| \to q\) at large values of \(ak\).

In Fig. 4 we plot \(-I_m(\sigma)\) and \(|I_m(\sigma)/\Re(\sigma)|\) against \(ak\) of the least stable computed modes for \(m = 0, 2, 4, 8, 11, \infty\) using the same parameters as for Fig. 2. The arrow shows the direction of increasing \(m\). From the values of \(I_m(\sigma)\) it can be seen that all the modes are stable and that the least stable mode occurs as \(m \to \infty\). For all the modes we find that \(|I_m(\sigma)/\Re(\sigma)| \to q\) for large values of \(ak\) and using the same argument put forward for the axisymmetric case, we conclude that we find agreement with the argument of Finne provided the aspect ratio of the apparatus is small enough.

As \(m \to \infty\) we are able to make a simplification to Eq. \(\text{(9)}\). From Abromavitch & Stegun [10] we find

\[
J_m(z) \approx \frac{1}{\sqrt{2\pi m}} \left( \frac{e^z}{2m} \right)^m \quad \text{as} \quad m \to \infty,
\]

so, in this limit, we may take

\[
\frac{J'_m(a\beta)}{(a\beta)J_m(a\beta)} \approx \frac{m}{(a\beta)^2},
\]

Substituting into Eq. \(\text{(9)}\) we find that the resulting equation is independent of \(m\). This equation may be solved analytically and yields two solutions for \(\sigma\), namely \(\sigma = (1 - \alpha')\nu^*k^2 + i\nu^*k^2\) and \(\sigma = -(1 - \alpha')\eta + i\eta\). The first solution will be the least stable and this is the one that has been plotted in Fig. 4. Note that both of these solutions are such that \(|I_m(\sigma)/\Re(\sigma)| = q\).

Conclusion. – In conclusion, our analysis shows that in the case of both axisymmetric (\(m = 0\)) and non-axisymmetric (\(m \neq 0\)) perturbations of a vortex lattice, provided that the aspect ratio is small enough \((h/a \gg 1)\), Kelvin waves propagate if \(q < 1\) and are damped if \(q > 1\). This result confirms the argument put forward by Finne et al. Finally, the exact dispersion relation which we have found (Eq. \(\text{5}\) and \(\text{8}\)) can be used to study with precision the spectrum of Kelvin waves on a rotating vortex lattice for any height and radius of experimental apparatus.

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Fig. 1 – (a): unperturbed vortex; (b): Kelvin wave.

Fig. 2 – Plots of (a) the decay rate, $I m(\sigma)$ (b) $|I m(\sigma)/R e(\sigma)|$ of the axisymmetric mode ($m = 0$) against $ak$ for various values of $\xi_j$. The arrows show the direction of increasing $\xi_j$, where $j = 1, 2, 5, 10, 20, 100, \infty$. The dashed line shows the result for $\xi_j = 0$.

Fig. 3 – Plots of (a) the decay rate, $I m(\sigma)$ (b) $|I m(\sigma)/R e(\sigma)|$ against $ak$ of the $m = 1$ mode. The dotted line represents the least stable axisymmetric mode.
Fig. 4 – Plots of (a) the decay rate, $\Im(\sigma)$ and (b) $|\Im(\sigma)/\Re(\sigma)|$ against $ak$ for the $m = 2, 4, 8, 11$ mode. The dotted line represents the least stable axisymmetric mode. The solid line represents the $m = \infty$ mode. The arrow shows the direction of increasing $m$.

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