Similarity reduction of the modified Yajima-Oikawa equation

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Abstract

We study a similarity reduction of the modified Yajima-Oikawa hierarchy. The hierarchy is associated with a non-standard Heisenberg subalgebra in the affine Lie algebra of type $A_{2}^{(1)}$. The system of equations for self-similar solutions is presented as a Hamiltonian system of degree of freedom two, and admits a group of Bäcklund transformations isomorphic to the affine Weyl group of type $A_{2}^{(1)}$. We show that the system is equivalent to a two-parameter family of the fifth Painlevé equation.

1 Introduction

In applications of the theory of affine Lie algebras to integrable hierarchies, the Heisenberg subalgebras play important roles, since they correspond to the varieties of time-evolutions. Let $\hat{\mathfrak{g}}$ be the untwisted affine Lie algebra associated with a finite-dimensional simple Lie algebra $\mathfrak{g}$. Up to conjugacy, the Heisenberg subalgebras in $\hat{\mathfrak{g}}$ are in one-to-one correspondence with the conjugacy classes of the Weyl group of $\mathfrak{g}$\textsuperscript{[3]}. In particular, the conjugacy class containing the Coxeter element, to which the principal Heisenberg subalgebra of $\hat{\mathfrak{g}}$ is associated, leads to the Drinfel’d-Sokolov hierarchy\textsuperscript{[2]}, whereas the class of the identity element corresponds to the homogeneous Heisenberg subalgebra. Associated with arbitrary conjugacy class, M. F. de Groot, T. J. Hollowood, J. L. Miramontes\textsuperscript{[1]} developed the theory of integrable systems called generalized Drinfel’d-Sokolov hierarchies.

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When $\mathfrak{g}$ is of type $A_{n-1}$, the conjugacy classes are parametrized by the partitions of $n$. In this paper we consider the modified Yajima-Oikawa hierarchy, which turns out to be a hierarchy related to the affine Lie algebra of type $A_2^{(1)}$ and its non-standard Heisenberg subalgebra associated with the partition $(2,1)$, while the principal (resp. homogeneous) case corresponds to the partition $(3)$ (resp. $(1,1,1)$).

Among the issues on integrable hierarchies, the study of similarity reduction is important. For example, M. Noumi and Y. Yamada introduced a higher order Painlevé system associated with the affine root system of type $A_n^{(1)}$ and now the system is known to be equivalent to a similarity reduction of the system associated with the Coxeter class $(n)$ of $A_{n-1}$. The aim of this paper is to investigate a similarity reduction of the modified Yajima-Oikawa hierarchy. Starting with universal viewpoints, we derive a system of ordinary differential equations for unknown functions $f_0, f_1, f_2, u_0, u_1, u_2, g, q, r$ and complex parameters $\alpha_0, \alpha_1, \alpha_2$:

\begin{align}
\alpha_0' &= \alpha_0 = \alpha_2' = 0, \\
f_0' &= f_0(u_2 - u_0) - \alpha_0, \quad g' = g(u_0 - u_2) - qf_1 + rf_2 + \alpha_0 + 4, \\
f_1' &= f_1(u_0 - u_1) - r\alpha_1, \quad 3q' = 3q(u_1 - u_0) + qf_0 - f_2, \\
f_2' &= f_2(u_1 - u_2) - q\alpha_2, \quad 3r' = 3r(u_2 - u_1) - rf_0 + f_1.
\end{align}

where $' = d/dx$ denote the derivative with respect to the independent variable $x$. Under the algebraic relations

\begin{align}
\alpha_0 + \alpha_1 + \alpha_2 &= -4, \quad g = f_0 + 3qr, \quad u_0 + u_1 + u_2 = 0, \quad u_1 = qr, \\
2gu_0 &= qf_1 - rf_2 - gqr - \alpha_0 - 2,
\end{align}

the system (1.1) turns out to be equivalent to the fifth Painlevé equation for $y = -f_0/(3u_1)$:

\[ y'' = \left( \frac{1}{2y} + \frac{1}{y - 1} \right) (y')^2 - \frac{y'}{x} + \frac{(y - 1)^2}{x^2} \left( Ay + B \frac{y}{y} \right) + \frac{C}{x} y + D \frac{y(y + 1)}{(y - 1)}, \]

where the change of variable $x \to x^2$ is employed and the parameters are given by

\[ A = \frac{1}{2} \left( \frac{\alpha_0 - \alpha_1}{12} \right)^2, \quad B = -\frac{1}{2} \left( \frac{\alpha_0}{4} \right)^2, \quad C = -\frac{\alpha_2 - \alpha_1}{18}, \quad D = -\frac{1}{18}. \]

On introducing the system (1.1), we shall describe the system in three ways:

1. Compatibility condition for a system of linear differential equations (Section 3),
2. A Hamiltonian system whose degree of freedom is two (Theorem 2),
3. Hirota bilinear equations for $\tau$-functions (Theorem 3).

The system (1.1) has a symmetry of the affine Weyl group of type $A_2^{(1)}$ as a group of Bäcklund transformations. First we give the symmetry as the compatibility of gauge transformations of linear differential equations and state it in the automorphism of the differential field

\[ K = C(\alpha_0, \alpha_1, \alpha_2, f_0, f_1, f_2, g, q, r, u_0, u_1, u_2) \]
Then we extend the action of affine Weyl group on $K$ to the extended field $\hat{F}$ of $K$:

$$\hat{F} = C(\alpha_0, \alpha_1, \alpha_2, x; \tau_0, \tau_1, \tau_2, \sigma_1, \sigma_2, \tau_0', \tau_1', \tau_2', \sigma_1', \sigma_2')$$

as a Bäcklund transformations, which is discussed in section 11 (Theorem 1).

The paper is organized as follows. In Sect.2 we review the notation related to the affine Lie algebra of type $A_2^{(1)}$. On the basis of the affine Lie algebra, we introduce the modified Yajima-Oikawa hierarchy in Sect.3. In Sect.4 we consider a condition of self-similarity on the solutions of the hierarchy. This condition yields a system of ordinary differential equations, which is a main object in this paper. In Sect.5 the condition of self-similarity is also presented as a Lax-type equation. In Sect.6, we give a Weyl group action on the solutions of the hierarchy. This condition yields a system of ordinary differential equations, which is a main object in this paper. In Sect.7 a Hamiltonian structure is introduced (Theorem 2). In Sect.8 we prove that our system is equivalent to a two-parameter family of the fifth Painlevé equation. In Sect.9 we introduce a set of $\tau$-functions and give a bilinear form of differential system (Theorem 3). In Sect.10 we prove that the Weyl group action on the $\tau$-functions commute with the derivation $' = d/dx$.

### 2 Preliminaries on the affine Lie algebra of type $A_2^{(1)}$

In this section, we collect necessary notions about the affine Lie algebra of type $A_2^{(1)}$. We mainly follow the notation used in [4], to which one should refer for further details.

Let $\mathfrak{g} = \mathfrak{sl}_3$. The affine Lie algebra $\hat{\mathfrak{g}}$ is realized as a central extension of the loop algebra $L\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C}[z, z^{-1}])$, together with the derivation $d = z\partial_z$

$$\hat{\mathfrak{g}} = \mathfrak{sl}_3(\mathbb{C}[z, z^{-1}]) \oplus \mathbb{C}c \oplus Cd,$$

where $c$ denotes the canonical central element. Let us define the Chevalley generators $E_i, F_i, H_i (i = 0, 1, 2)$ for the affine Lie algebra $\hat{\mathfrak{g}}$ by

$$E_0 = zE_{3,1}, \quad E_1 = E_{1,2}, \quad E_2 = E_{2,3}, \quad F_0 = z^{-1}E_{1,3}, \quad F_1 = E_{2,1}, \quad F_2 = E_{3,2};$$

$$H_0 = c + E_{3,3} - E_{1,1}, \quad H_1 = E_{1,1} - E_{2,2}, \quad H_2 = E_{2,2} - E_{3,3},$$

where $E_{i,j}$ is the matrix unit $E_{i,j} = (\delta_{ia}\delta_{jb})^3_{a,b=1}$. The Cartan subalgebra of $\hat{\mathfrak{g}}$ is defined as $\mathfrak{h} = \bigoplus_{i=0}^2 \mathbb{C}H_i \oplus Cd$. We introduce the simple roots $\alpha_j$ and the fundamental weights $\Lambda_j$ as the following linear functionals on the Cartan subalgebra $\mathfrak{h}$:

$$\langle H_i, \alpha_j \rangle = a_{ij}, \quad \langle H_i, \Lambda_j \rangle = \delta_{ij} \quad (i = 0, 1, 2), \quad \langle d, \alpha_j \rangle = \delta_{0j}, \quad \langle d, \Lambda_j \rangle = 0$$

for $j = 0, 1, 2$, where $(a_{ij})_{i=0}^3$ is the generalized Cartan matrix of type $A_2^{(1)}$ defined by

$$\begin{bmatrix}
2 & -1 & -1 \\
-1 & 2 & -1 \\
-1 & -1 & 2
\end{bmatrix}.$$
We define a non degenerate symmetric bilinear form \((\cdot | \cdot)\) on \(V = \hat{\mathfrak{h}}^*\) as follows:

\[
(\alpha_i | \alpha_j) = a_{ij}, \quad (\alpha_i | \Lambda_0) = \delta_{i0}, \quad (\Lambda_0 | \Lambda_0) = 0.
\]

We define simple reflections \(s_i\) \((i = 0, 1, 2)\) by

\[
s_i(\lambda) = \lambda - \langle H_i, \lambda \rangle \alpha_i, \quad \lambda \in V.
\]

They satisfy the fundamental relations

\[
s_i^2 = 1, \quad s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \quad (i = 0, 1, 2),
\]

where the indices are understood as elements of \(\mathbb{Z}/3\mathbb{Z}\). Consider the group

\[
W = \langle s_0, s_1, s_2 \rangle \subset \text{GL}(V), \quad (2.2)
\]

generated by the simple reflections. The group \(W\) is called the affine Weyl group of type \(A_2^{(1)}\).

## 3 Modified Yajima-Oikawa hierarchy

In this section we introduce the modified Yajima-Oikawa hierarchy as generalized Drinfel’d-Sokolov reduction associated to the loop algebra \(L\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C}[z, z^{-1}])\), following [1]. Let us introduce the following derivation on \(L\mathfrak{g}\):

\[
D = 4z \frac{\partial}{\partial z} - \text{diag}(-1, 0, 1). \quad (3.1)
\]

Set

\[
L\mathfrak{g}_j = \{ A \in L\mathfrak{g} \mid [D, A] = jA \}.
\]

Then we have a \(\mathbb{Z}\)-gradation \(L\mathfrak{g} = \bigoplus_j L\mathfrak{g}_j\). Note that

\[
\deg(E_0) = -\deg(F_0) = 2, \quad \deg(E_j) = -\deg(F_j) = 1 \quad (j = 1, 2).
\]

Consider the particular element

\[
\gamma = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
z & 0 & 0
\end{bmatrix}
\]

and let \(\mathfrak{s}\) be the centralizer of \(\gamma\) in \(L\mathfrak{g}\)

\[
\mathfrak{s} = \text{Ker} (\text{ad}\gamma) = \{ A \in L\mathfrak{g} \mid [\gamma, A] = 0 \}.
\]

The subalgebra \(\mathfrak{s}\) is a maximal commutative subalgebra in \(\mathfrak{g}\), which has the following basis:

\[
\gamma_{4j+2} = z^j\gamma, \quad \gamma_{4j} = z^j\text{diag}(1, -2, 1) \quad (j \in \mathbb{Z}).
\]
Then $\mathfrak{s}$ is a graded subalgebra of $L\mathfrak{g}$ with respect to the gradation. We have $\gamma_{2j} \in L\mathfrak{g}_{2j}$. The commutative subalgebra $\mathfrak{s}$ is the image of a Heisenberg subalgebra in $\hat{\mathfrak{g}}$ associated with the conjugacy class $(2, 1)$ (3), see also [10] and [4]. We put $\mathfrak{b} := \oplus_{j \geq 0} L\mathfrak{g}_j$.

To introduce our hierarchy, we begin with the differential operator

$$L := \frac{\partial}{\partial x} - \gamma - Q,$$

where $Q$ is an $x$-dependent element of $\mathfrak{b}_{<2}$. We set $\mathfrak{s}^\perp := \text{Im} (\text{ad}_\gamma)$. It is clear that $\mathfrak{s}^\perp = \oplus_s \mathfrak{s}^\perp_s \cap L\mathfrak{g}_s$. There is a unique formal series $U = \sum_{j=1}^{\infty} U_{-j} (U_{-j} \in \mathfrak{s}^\perp_{-j})$ such that the operator $L_0 := e^{\text{ad}U}(L)$ has the form

$$L_0 = \frac{\partial}{\partial x} - \gamma - \sum_{j=0}^{\infty} h_{-2j}, \quad h_{-2j} \in \mathfrak{s}_{-2j}.$$\n
Moreover $U_{-j}$ and $h_{-2j}$ are polynomials in the components of $Q$ and their $x$ derivatives. For any $j > 0$ we set

$$B_{2j} = (e^{-\text{ad}U\gamma_{2j}})_{>0}.$$

The modified Yajima-Oikawa hierarchy is defined by the Lax equations

$$\frac{\partial L}{\partial t_{2j}} = [B_{2j}, L] \quad (j = 1, 2, \ldots).$$

We describe the above construction concretely. First we set

$$Q = \begin{bmatrix} u_0 & r & 0 \\ 0 & u_1 & q \\ 0 & 0 & u_2 \end{bmatrix}, \quad u_0 + u_1 + u_2 = 0$$

and solve for the first few terms of $U_j$ and $h_j$:

$$U_{-1} = -q E_{2,1} + r E_{3,2},$$

$$U_{-2} = \frac{u_2 - u_0}{4} (x^{-1} E_{1,3} - E_{3,1}),$$

$$U_{-3} = \left[ \left( \frac{3u_0}{8} + \frac{3u_1}{2} - \frac{3u_2}{8} - qr \right) r + r' \right] E_{1,2}$$

$$+ \left[ \left( \frac{7u_0}{8} - u_1 + \frac{u_2}{8} + qr \right) q + q' \right] E_{2,3},$$

$$U_{-4} = \left[ \frac{u_0' - u_2'}{8} + \frac{q'r + 3qr'}{8} + \left( \frac{u_0}{16} - \frac{5u_1}{16} + \frac{u_2}{16} + \frac{5}{16} qr \right) q \right] (E_{1,1} - E_{3,3}),$$

$$h_0 = \frac{qr - u_1}{2} \gamma_0,$$

$$h_{-2} = \left[ \frac{u_0^2 + u_2^2}{8} - \frac{u_0 u_2}{4} - \frac{q'r + 3qr'}{4} - \left( \frac{u_0}{8} - \frac{5u_1}{8} + \frac{u_2}{8} + \frac{3}{8} qr \right) q \right] \gamma_{-2}.$$
Here $'$ means $\partial/\partial x$. In fact, $h_0$ is a constant along all the flows and we can put $h_0 = 0$ (see [1]). So we fix

$$u_1 = qr \quad \text{(3.2)}$$

from now on. By using $U_j$’s and condition (3.2) we have

$$B_2 = \gamma_2 + \begin{bmatrix} u_0 & r & 0 \\ 0 & u_1 & q \\ 0 & 0 & u_2 \end{bmatrix}, \quad \text{(3.3)}$$

$$B_4 = \gamma_4 + 3 \begin{bmatrix} -qr' + qru_2 & r' - ru_2 \\ qz & qr' - qr + qru_1 \\ -qrz & rz \end{bmatrix} \quad \text{(3.4)}$$

The modified Yajima-Oikawa equation is obtained by the following zero-curvature condition:

$$\frac{\partial B_2}{\partial t_4} = \frac{\partial B_4}{\partial t_2} - [B_2, B_4]. \quad \text{(3.5)}$$

In fact this yields the following system of differential equations:

$$q_t + 3 \left(q'' + q(-qr' + u_0' + qru_2 + u_2^2)\right) = 0, \quad \text{(3.6)}$$

$$r_t - 3 \left(r'' - r(-q'r + u_2' - qru_0 + u_0^2)\right) = 0, \quad \text{(3.7)}$$

$$(u_0)_t = 3(-qr' + qru_2)', \quad (u_1)_t = 3(qr' - qr + qru_1)' , \quad (u_2)_t = 3(q' r + qru_0)'. \quad \text{(3.8)}$$

Here we identify $x$ and $t_2$, and put $t = t_4$.

**Remark:** This system of equations is related to the Yajima-Oikawa equation [11]:

$$\Psi_t + 3 (\Psi'' + u\Psi) = 0, \quad \text{(3.9)}$$

$$\Phi_t - 3 (\Phi'' + u\Phi) = 0, \quad \text{(3.10)}$$

$$u_t + 6(\Psi\Phi)' = 0. \quad \text{(3.11)}$$

The relation is established by the following map, which takes a solution $q, r, u_j$ ($j = 0, 1, 2$) of (3.6), (3.7), (3.8) into a solution $\Psi, \Phi, u$ of (3.9), (3.10), (3.11) and is an analog of the Miura map in the case of KdV and mKdV equations:

$$\Psi = -q' - qu_0, \quad \Phi = r' - ru_2, \quad -u = u_0^2 + u_2^2 + u_0u_2 + u_0' + qr'.$
4 Similarity reduction

In this section we consider a self-similarity condition on the solutions of the modified Yajima-Oikawa equation (3.6), (3.7), (3.8). These are the main object of this paper. A solution \( q(x, t), r(x, t), u_j(x, t) \) \( (j = 0, 1, 2) \) is said to be self-similar if

\[
q(\lambda^2 x, \lambda^4 t) = \lambda^{-1} q(x, t), \quad r(\lambda^2 x, \lambda^4 t) = \lambda^{-1} r(x, t), \quad u_j(\lambda^2 x, \lambda^4 t) = \lambda^{-2} u_j(x, t). \tag{4.1}
\]

Here we count a degree of variables by \( \text{deg} x = \text{deg} t = -2, \text{deg} t = \text{deg} t = -4 \). Note that such functions are uniquely determined by its values at fixed \( t \), say at \( t = 1/4 \). Differentiating (4.1) with respect to \( \lambda \) at \( \lambda = 1 \), we obtain the Euler equations

\[
2x \frac{\partial q}{\partial x} + 4t \frac{\partial q}{\partial t} = -q, \quad 2x \frac{\partial r}{\partial x} + 4t \frac{\partial r}{\partial t} = -r, \quad 2x \frac{\partial u_j}{\partial x} + 4t \frac{\partial u_j}{\partial t} = -2u_j.
\]

At \( t = 1/4 \) these identities become

\[
\frac{\partial q}{\partial t} = -2 \frac{\partial (xq)}{\partial x} + q, \quad \frac{\partial r}{\partial t} = -2 \frac{\partial (xr)}{\partial x} + r, \quad \frac{\partial u_j}{\partial t} = -2 \frac{\partial (xu_j)}{\partial x}.
\]

This can be written in the matrix form

\[
\frac{\partial B_2}{\partial t} = -2 \frac{\partial (xB_2)}{\partial x} + [D, B_2],
\]

where \( D \) is the derivation defined in (3.1). Substituting this last identity into the zero-curvature equation (3.5), we obtain

\[
\frac{\partial M}{\partial x} = \left[ 4z \frac{\partial}{\partial z} - M, B_2 \right], \tag{4.2}
\]

where we set

\[
M = \begin{bmatrix}
\varepsilon_1 & f_1 & g \\
0 & \varepsilon_2 & f_2 \\
0 & 0 & \varepsilon_3
\end{bmatrix}
+ z \begin{bmatrix}
1 & 0 & 0 \\
3q & -2 & 0 \\
f_0 & 3r & 1
\end{bmatrix}
:= \text{diag}(-1, 0, 1) + 2xB_2 + B_4. \tag{4.3}
\]

The correspondence of variables is given as follows: \( v \)

\[
\varepsilon_1 = -1 + 2xu_0 - 3q(r' - ru_2), \tag{4.4}
\]
\[
\varepsilon_2 = 2xu_1 + 3(qr' - q' r + qu_1), \tag{4.5}
\]
\[
\varepsilon_3 = 1 + 2xu_2 + 3r(q' + qu_0) \tag{4.6}
\]

and \( g = 2x \),

\[
f_0 = 2x - 3qr, \quad f_1 = 2xr + 3(r' - ru_2), \quad f_2 = 2xq - 3(q' + qu_0). \tag{4.7}
\]

Here we regard the variables \( q = q(x, 1/4), r = r(x, 1/4), u_j = u_j(x, 1/4) \) \( (j = 0, 1, 2) \) are functions only in \( x \). Note that the definition of \( M \) has a freedom of adding a constant diagonal matrix and here we normalize

\[
\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0. \tag{4.8}
\]
5 Lax pair formalism

Consider the following system of linear differential equations for the column vector \( \vec{\psi} = \psi_1, \psi_2, \psi_3 \) of three unknown functions \( \psi_i = \psi_i(z, x) \) \((i = 1, 2, 3)\):

\[
4z \frac{\partial \vec{\psi}}{\partial z} = M \vec{\psi}, \quad \frac{\partial \vec{\psi}}{\partial x} = B \vec{\psi}.
\] (5.1)

We assume that the matrix \( M \) and \( g \) is equivalent to the relations

\[
4z \frac{\partial \vec{\psi}}{\partial z} = M \vec{\psi}, \quad \frac{\partial \vec{\psi}}{\partial x} = B \vec{\psi}.
\] (5.2)

is equivalent to the relations

\[
\begin{align*}
\varepsilon'_1 &= \varepsilon'_2 = \varepsilon'_3 = 0, \\
f'_0 &= f_0(u_2 - u_0) - (\varepsilon_3 - \varepsilon_1 - 4), \\
f'_1 &= f_1(u_0 - u_1) - r(\varepsilon_1 - \varepsilon_2), \\
f'_2 &= f_2(u_1 - u_2) - q(\varepsilon_2 - \varepsilon_3),
\end{align*}
\] (5.3)

If we forget the relation \((4.3)\) of \( M \) and \( B_1, B_2 \) and start from the Lax equation \((5.3)\), we can recover some of the relations of variables. For instance, differentiating both-hand side of \( g = f_0 + 3qr \) and eliminate the variables except \( g' \) by means of \((5.3)\), we get \( g' = 2 \) and therefore assume

\[
g = 2x.
\]

In what follows we shall impose the following constraint on the variables:

\[
3u_0 + u_1 + u_2 = 0, \quad u_1 = qr.
\] (5.4)

The joint system \((5.2)\) and \((5.3)\) is the main object that we investigate in this paper. Using system \((5.3)\) together with the constraint, we can derive the following equation:

\[
2gu_0 = qf_1 - rf_2 - gqr - \varepsilon_3 + \varepsilon_1 + 2.
\] (5.5)

After the elimination of the variables \( f_0, u_0, u_1, u_2 \) by \((5.3), (5.4) \) and \((5.5)\), we obtain a system of ODE for the unknown functions \( f_1, f_2, q, r \) with the parameters \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \). We can obtain the set of explicit formulae of \( f'_1, f'_2, q', r' \) in terms of \( f_1, f_2, q, r \) and \( g \), and the results are

\[
\begin{align*}
f'_1 &= \frac{f_1}{2g}(f_1q - f_2r) - \beta \frac{f_1}{2g} + (\varepsilon_1 - \varepsilon_3)\frac{f_1}{2g} - (\varepsilon_1 - \varepsilon_2)r + \frac{f_1}{g}, \\
f'_2 &= \frac{f_2}{2g}(f_1q - f_2r) + \beta \frac{f_2}{2g} + (\varepsilon_1 - \varepsilon_3)\frac{f_2}{2g} - (\varepsilon_2 - \varepsilon_3)q + \frac{f_2}{g}, \\
q' &= -\frac{q}{2g}(f_1q - f_2r) + \beta \frac{q}{2g} - (\varepsilon_1 - \varepsilon_3)\frac{q}{2g} + \frac{gq - f_2}{3} - \frac{q}{g}, \\
r' &= -\frac{r}{2g}(f_1q - f_2r) - \beta \frac{r}{2g} - (\varepsilon_1 - \varepsilon_2)\frac{r}{2g} - \frac{gr - f_1}{3} - \frac{r}{g}.
\end{align*}
\] (5.6-5.9)
In Sect.7 we present the system of ODE in the Hamiltonian form.

Remark. Using (5.5) and (5.3), we can also derive the following differential equation:

$$gu' = (\varepsilon_2 - \varepsilon_3)qr + \frac{f_2}{3}(rf_0 - f_1) - 2u_0.$$  \hspace{1cm} (5.10)

6 Bäcklund transformations

Let us pass to the investigation of a group of Bäcklund transformations. For this purpose, it is convenient to introduce the following set of parameters:

$$\alpha_0 = \varepsilon_3 - \varepsilon_1 - 4, \quad \alpha_1 = \varepsilon_1 - \varepsilon_2, \quad \alpha_2 = \varepsilon_2 - \varepsilon_3.$$  \hspace{1cm} (6.1)

They are identified with the simple roots of the affine root system of type $A_2^{(1)}$.

We define the Bäcklund transformations for the system by considering the gauge transformations of the linear system (5.1)

$$s_i \psi = G_i \psi \quad (i = 0, 1, 2).$$  \hspace{1cm} (6.2)

The matrices $G_i$ are given as follows:

$$G_i = 1 + \frac{\alpha_i}{f_i} F_i \quad (i = 0, 1, 2),$$  \hspace{1cm} (6.3)

where $F_0, F_1, F_2$ are Chevalley generators (2.1) of the loop algebra $sl_3(\mathbb{C}[z, z^{-1}])$. The compatibility condition of (5.1) and (6.2) is

$$s_i(M) = G_iMG_i^{-1} + 4z \frac{\partial G_i}{\partial z} G_i^{-1}, \quad s_i(B) = G_iBG_i^{-1} + \frac{\partial G_i}{\partial x} G_i^{-1}.$$  \hspace{1cm} (6.4)

On the components of the matrices $M, B$, the actions of $s_i(i = 0, 1, 2)$ are given explicitly as in the following tables:

|   | $f_0$         | $f_1$         | $f_2$         | $g$  | $q$  | $r$  |
|---|--------------|--------------|--------------|-----|-----|-----|
| $s_0$ | $f_0$       | $f_0 + 3r\frac{\alpha_0}{f_0}$ | $f_2 - 3q\frac{\alpha_0}{f_0}$ | $g$  | $q$  | $r$  |
| $s_1$ | $f_0 - 3r\frac{\alpha_1}{f_1}$ | $f_1$         | $f_2 + g\frac{\alpha_1}{f_1}$ | $g$  | $q + \frac{\alpha_1}{f_1}$ | $r$  |
| $s_2$ | $f_0 + 3q\frac{\alpha_2}{f_2}$ | $f_1 - g\frac{\alpha_2}{f_2}$ | $f_2$         | $g$  | $q$  | $r - \frac{\alpha_2}{f_2}$ |

|   | $\alpha_0$ | $\alpha_1$ | $\alpha_2$ | $u_0$ | $u_1$ | $u_2$ |
|---|------------|------------|------------|------|------|------|
| $s_0$ | $-\alpha_0$ | $\alpha_1 + \alpha_0$ | $\alpha_2 + \alpha_0$ | $u_0 + \frac{\alpha_0}{f_0}$ | $u_1$ | $u_2 - \frac{\alpha_0}{f_0}$ |
| $s_1$ | $\alpha_0 + \alpha_1$ | $-\alpha_1$ | $\alpha_2 + \alpha_1$ | $u_0 - r\frac{\alpha_1}{f_1}$ | $u_1 + r\frac{\alpha_1}{f_1}$ | $u_2$ |
| $s_2$ | $\alpha_0 + \alpha_2$ | $\alpha_1 + \alpha_2$ | $-\alpha_2$ | $u_0$ | $u_1 - q\frac{\alpha_2}{f_2}$ | $u_2 + q\frac{\alpha_2}{f_2}$ |
The automorphisms $s_i (i = 0, 1, 2)$ generate a group of Bäcklund transformations for our differential system. To state this fact clearly, it is convenient to introduce the field

$$K = \mathbb{C}(\alpha_0, \alpha_1, \alpha_2, f_0, f_1, f_2, g, q, r, u_0, u_1, u_2),$$

(6.5)

where the generators satisfy the following algebraic relations:

$$\alpha_0 + \alpha_1 + \alpha_2 = -4, \quad f_0 = g - 3qr, \quad u_0 + u_1 + u_2 = 0, \quad u_1 = qr,$$

$$2gu_0 = qf_1 - rf_2 - gqr - \varepsilon_3 + \varepsilon_1 + 2.$$

We have the automorphisms $s_i (i = 0, 1, 2)$ of the field $K$ defined by the above table. Note that the field $K$ is thought to be a differential field with the derivation $' : K \to K$ defined by (5.3).

**Theorem 1** The automorphism $s_0, s_1, s_2$ of $K$ define a representation of the affine Weyl group $W$ (2.2) on the field $K$ such that the action of the each element $w \in W$ commutes with the derivation of the differential field $K$.

Theorem 1 is proved by straightforward computations. Note that the independent variable $x = g/2$ is fixed under the action of $W$.

### 7 Hamiltonian structure

We shall equip $K$ (6.5) with the Poisson algebra structure $\{ , \} : K \times K \to K$ defined as follows:

|   | $f_1$ | $f_2$ | $q$ | $r$ |
|---|---|---|---|---|
| $f_1$ | 0 | $g$ | 1 | 0 |
| $f_2$ | $-g$ | 0 | 0 | $-1$ |
| $q$ | $-1$ | 0 | 0 | 0 |
| $r$ | 0 | 1 | 0 | 0 |

That is, $\{f_1, f_2\} = g$ and so on. Note that the Poisson structure comes from the Lie algebra structure of $\hat{\mathfrak{g}}$ (see [9] for an exposition). We can describe the action of $s_i (i = 0, 1, 2)$ on the generators $f = f_j, u_j, q, r, g (j = 0, 1, 2)$ of $K$ by

$$s_i(f) = f + \frac{\alpha_i}{f_i}\{f_i, f\}.$$

We introduce the function $h$ by

$$h := \frac{1}{2}(f_1q^2r + f_2qr^2) - \frac{1}{4g}(f_1^2q^2 + f_2^2r^2 + q^2r^2g^2) + \left(\frac{qr}{2g} - \frac{1}{3}\right)f_1f_2$$

$$+ \left(\frac{g}{3} - \frac{\alpha_1 + \alpha_2}{2g}\right)f_1q + \left(\frac{g}{3} + \frac{\alpha_1 + \alpha_2}{2g}\right)f_2r - \left(\frac{g}{3} - \frac{\alpha_1 - \alpha_2}{2g}\right)qrg.$$
Then the differential system (5.6)–(5.9) can be expressed

\[ f'_1 = \{h, f_1\} + \frac{f_1}{g}, \quad q' = \{h, q\} - \frac{q}{g}, \tag{7.1} \]

\[ f'_2 = \{h, f_2\} + \frac{f_2}{g}, \quad r' = \{h, r\} - \frac{r}{g}. \]

Let us introduce the variables

\[ p_1 = f_1, \quad q_1 = q, \quad p_2 = \frac{f_2}{g} - q, \quad q_2 = -gr. \]

It is easy to show that

\[ \{p_i, q_j\} = \delta_{ij}, \quad \{p_i, p_j\} = \{q_i, q_j\} = 0 \quad (i, j = 1, 2). \]

**Theorem 2** Let \( H \) be the function defined as

\[
xH = -\frac{1}{4}p_1p_2q_1q_2 - \frac{1}{8}(p_1^2q_1^2 + p_2^2q_2^2) - \frac{1}{2}p_1q_1^2q_2 \]

\[-\frac{1}{4}(\alpha_1 + \alpha_2 + 2)p_1q_1 - \frac{1}{4}(\alpha_1 + \alpha_2 - 2)p_2q_2 - \frac{\alpha_1}{2}q_1q_2 - \frac{2x^2}{3}(q_2 + p_1)p_2 \]

Then the system of ODEs (5.6), (5.7), (5.8), (5.9) is equivalent to the Hamiltonian system

\[
\frac{dq_1}{dx} = \frac{\partial H}{\partial p_1}, \quad \frac{dq_2}{dx} = \frac{\partial H}{\partial p_2}, \quad \frac{dp_1}{dx} = -\frac{\partial H}{\partial q_1}, \quad \frac{dp_2}{dx} = -\frac{\partial H}{\partial q_2}. \tag{7.2} \]

**Proof.** We define

\[ H = h - \frac{f_1q + f_2r}{g} + qr \]

and rewrite this in the coordinate \( p_j, q_j \) \((j = 1, 2)\). Then the equations (7.1) can be expressed as (7.2). \( \square \)

The behavior of the Hamiltonian under the Bäcklund transformations is given by the simple formulae

\[ s_0(\tilde{H}) = \tilde{H} + 6qr \frac{\alpha_0}{f_0}, \quad s_j(\tilde{H}) = \tilde{H} (j = 1, 2), \]

where we set \( \tilde{H} = xH + a \) with the correction term

\[ a = \frac{1}{24}(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_2 - 4). \]
8 Reduction to the fifth Painlevé equation

In this section, we show the system (5.3) is equivalent to a two-parameter family of the fifth Painlevé equation. By linear change of the independent variable, we ensure the normalization

\[ f_0 + \frac{f_1}{r} + \frac{f_2}{q} + 3 \left( \frac{q'}{q} - \frac{r'}{r} \right) = 3g = 6x \]  

(8.1)

holds. After the elimination of \( u_0 \) and \( u_2 \), we have

\[ f_0' = -\frac{f_0}{3} \left( \frac{f_1}{r} - \frac{f_2}{q} \right) + \frac{f_0 u_1'}{u_1} - \alpha_0 \]  

(8.2)

\[ \left( \frac{f_1}{r} \right)' = -\frac{f_1}{3r} \left( \frac{f_2}{q} - f_0 + \frac{3u_1'}{u_1} \right) - \alpha_1, \]  

(8.3)

\[ \left( \frac{f_2}{q} \right)' = -\frac{f_2}{3q} \left( f_0 - \frac{f_1}{r} + \frac{3u_1'}{u_1} \right) - \alpha_2. \]  

(8.4)

Here we introduce a new variable

\[ y := -\frac{f_0}{3u_1}. \]

Notice the relations

\[ y - 1 = -\frac{2x}{3u_1}, \quad \frac{y'}{y - 1} = \frac{1}{x} - \frac{u_1'}{u_1} \]  

(8.5)

holds by \( f_0 = g - 3qr = 2x - 3u_1 \). Then we rewrite (8.2) as

\[ y' = -\frac{y}{3} \left( \frac{f_1}{r} - \frac{f_2}{q} \right) + \frac{\alpha_0}{3u_1}, \]  

(8.6)

After differentiating (8.6), elimination of the variables \( f_1, f_2, q, r, u_1 \) by (8.1), (8.3), (8.4), (8.5), (8.6) and the definition of the constant \( \varepsilon_2 \) (4.5) leads to the following equation of \( y \):

\[ y'' = \left( \frac{1}{2y} + \frac{1}{y - 1} \right) (y')^2 - \frac{y'}{x} + \frac{(y - 1)^2}{8x^2} \left( \varepsilon_2 y - \frac{\alpha_0^2}{y} \right) \]

\[ - \frac{2x^2 y}{9} - \frac{4x^2 y}{9(y - 1)} - \frac{(\alpha_2 - \alpha_1)y}{3} + \frac{\varepsilon_2 y}{3}. \]  

(8.7)

We put \( \xi = x^2 \), then the equation (8.7) can be brought into the fifth Painlevé equation

\[ y_{\xi\xi} = \left( \frac{1}{2y} + \frac{1}{y - 1} \right) (y_{\xi})^2 - \frac{1}{\xi} y_{\xi} + \frac{(y - 1)^2}{\xi^2} \left( Ay + \frac{B}{y} \right) + C \frac{\xi y - y(y + 1)}{18(y - 1)}. \]

where

\[ A = \frac{\varepsilon_2^2}{32}, \quad B = \frac{\alpha_0^2}{32}, \quad C = -\frac{\varepsilon_2}{6}. \]

Note that \( \varepsilon_2 = (\alpha_2 - \alpha_1)/3 \) holds by (4.8) and (6.4).
We introduce the \( \tau \)-functions \( \tau_0, \tau_1, \tau_2, \sigma_1 \) and \( \sigma_2 \) to be the dependent variables satisfying the following equations:

\[
\frac{f_1}{r} = 2x + 3 \left( \frac{\sigma'_2}{\sigma_2} - \frac{\sigma'_0}{\tau_0} \right), \quad \frac{f_2}{q} = 2x - 3 \left( \frac{\sigma'_1}{\sigma_1} - \frac{\sigma'_0}{\tau_0} \right), \quad q = -\frac{\sigma_1}{\tau_1}, \quad r = \frac{\sigma_2}{\tau_2}.
\tag{9.1}
\]

To fix the freedom of overall multiplication by a function in the defining equation \(9.1\) for \( \tau_0, \tau_1, \tau_2, \sigma_1 \) and \( \sigma_2 \), we impose the equation

\[
\left( \log \tau_0^2 \tau_1^2 \tau_2^2 \sigma_1 \sigma_2 \right)'' + u_0^2 + u_2^2 + \left( u_0 - \frac{f_1}{3r} + \frac{2x}{3} \right)^2 + \left( u_2 - \frac{f_2}{3q} + \frac{2x}{3} \right)^2 - \frac{2x}{9} \left( 4x - \frac{f_1}{r} - \frac{f_2}{q} \right) - \frac{\alpha_1 - \alpha_2}{9} = 0.
\tag{9.2}
\]

The differential equations for the variables \( q \) and \( r \) in the system \(5.3\) lead to

\[
u_0 = \frac{\tau'_1}{\tau_1} - \frac{\tau'_0}{\tau_0}, \quad u_2 = \frac{\tau'_0}{\tau_0} - \frac{\tau'_2}{\tau_2} \tag{9.3}
\]
respectively. Here we have used the relations

\[
u_1 = qr = -\frac{\sigma_1 \sigma_2}{\tau_1 \tau_2}, \quad f_0 = 2x - 3qr = 2x + 3\sigma_1 \sigma_2.
\]

If the equations \(9.3\) are satisfied, we have

\[
u_1 = \frac{\tau'_1}{\tau_1} - \frac{\tau'_2}{\tau_2}, \tag{9.4}
\]
by \( u_0 + u_1 + u_2 = 0 \) and therefore have the following formula of the variable \( f_0 \) in terms of the \( \tau \)-functions:

\[
f_0 = 2x + 3 \left( \frac{\tau'_1}{\tau_1} - \frac{\tau'_2}{\tau_2} \right).
\tag{9.5}
\]

Let \( D_x \) and \( D_x^2 \) be Hirota’s bilinear operators:

\[
D_x F \cdot G := F'G - FG', \quad D_x^2 F \cdot G := F''G - 2F'G' + FG''.
\]

In this notation, the relation \( u_1 = qr \), for example, can be written in

\[
D_x \tau_1 \cdot \tau_2 = \sigma_1 \sigma_2. \tag{9.6}
\]

We introduce a system of bilinear equations that leads to our differential system \(5.3\).
Theorem 3 Let $\tau_0, \tau_1, \tau_2, \sigma_1, \sigma_2$ be a set of functions that satisfies the following system of Hirota bilinear equations:

\[
\begin{align*}
(3D_x^2 - 2xD_x + \frac{1}{6}(\alpha_0 - 4\alpha_1 - 2))\tau_0 \cdot \tau_1 &= 0, \\
(3D_x^2 - 2xD_x + \frac{1}{6}(\alpha_0 - 4\alpha_2 - 2))\tau_2 \cdot \tau_0 &= 0, \\
(3D_x^2 - 2xD_x + \frac{1}{6}(\alpha_1 - \alpha_2 + 6))\tau_1 \cdot \sigma_2 &= 0, \\
(3D_x^2 - 2xD_x + \frac{1}{6}(\alpha_1 - \alpha_2 - 6))\sigma_1 \cdot \tau_2 &= 0,
\end{align*}
\]

together with (9.6). If we define the functions $f_0, f_1, f_2, q, r, u_0, u_1$ and $u_2$ by the formulae (9.1), (9.3), (9.4) then this set of functions satisfies our ODE system (5.3) together with algebraic equations (5.4).

Proof. We can verify that the differential equations for $q$ and $r$ are satisfied if we assume the existence of the $\tau$-functions such that equations (9.1), (9.3) holds. The differential equations for $f_0$ is written as

\[
3(g''_1 - g''_2) + 2 = (3(g'_1 - g'_2) + 2x)(2g'_0 - g'_1 - g'_2) - \alpha_0,
\]

where $g_j = \log \tau_j, (j = 0, 1, 2)$. This equation is obtained if we subtract (9.7) from (9.8).

The differential equations for $f_1$ and $f_2$ can be rewritten as

\[
\left(\frac{f_1}{r}\right)' = \frac{f_1}{r} \left( u_0 - u_1 - \frac{r'}{r} - \alpha_1 \right), \quad \left(\frac{f_2}{q}\right)' = \frac{f_2}{q} \left( u_1 - u_2 - \frac{q'}{q} - \alpha_2 \right),
\]

respectively. In terms of the $\tau$-functions, these equations read

\[
\begin{align*}
3(h''_2 - g''_0) + 2 &= (3(h'_2 - g'_0) + 2x)(2g'_1 - g'_0 - h'_2) - \alpha_1, \\
3(g''_0 - h''_1) + 2 &= (3(g'_0 - h'_1) + 2x)(2g'_2 - g'_0 - g'_1) - \alpha_2,
\end{align*}
\]

where $h_1 = \log \sigma_1, h_2 = \log \sigma_2$. In fact, from (9.7) and (9.9) we can eliminate $g''_1$ to obtain (9.13). In the similar way from (9.8) and (9.10), we can eliminate $g''_2$ to obtain (9.14).

We remark that the normalization of $\tau$-functions (9.2) is obtained by taking the sum of four equations in this theorem.

10 Jacobi-Trudi type formula

In this section we lift the action of $W$ to the $\tau$-functions. Consider the field extension $\tilde{K} = K(\tau_0, \tau_1, \tau_2, \sigma_1, \sigma_2)$. Then we can prove the next Theorem by a direct computation.
Theorem 4 We extend each automorphism $s_i$ of $K$ to an automorphism of the field $\bar{K} = K(\tau_0, \tau_1, \tau_2, \sigma_1, \sigma_2)$ by the formulae $s_i(\tau_j) = \tau_j (i \neq j)$, $s_i(\sigma_k) = \sigma_k (i \neq k)$ and

$$
\begin{align*}
    s_0(\tau_0) &= f_0 \frac{\tau_2 \tau_1}{\tau_0}, & s_1(\tau_1) &= f_1 \frac{\tau_0 \tau_2}{\tau_1}, & s_1(\sigma_1) &= -(f_1 q + \alpha_1) \frac{\tau_0 \tau_2}{\tau_1}, \\
    s_2(\tau_2) &= f_2 \frac{\tau_1 \tau_0}{\tau_2}, & s_2(\sigma_2) &= (f_2 r - \alpha_2) \frac{\tau_1 \tau_0}{\tau_2}.
\end{align*}
$$

(10.1) (10.2)

Then these automorphisms define a representation of $W$ on $\bar{K}$.

Following [4], we will describe the Weyl group orbit of the $\tau$-functions (see also [9]). For any $w \in W$ and $k = 0, 1, 2$, there exists a rational function $\phi_w^{(k)} \in K$ such that

$$
    w(\tau_k) = \phi_w^{(k)} \prod_{i=0,1,2} \tau_i^{(\alpha_i | w(\Lambda_i))}.
$$

(10.3)

We shall give an expression of $\phi_w^{(k)}$ in terms of the Jacobi-Trudi type determinant.

A subset $M$ of $\mathbb{Z}$ is called a Maya diagram if $M \cap \mathbb{Z}_{\geq 0}$ and $M^c \cap \mathbb{Z}_{< 0}$ are finite sets. We define an integer

$$
    c(M) := \# (M \cap \mathbb{Z}_{\geq 0}) - \# (M^c \cap \mathbb{Z}_{< 0})
$$

called the charge of $M$. If $c(M) = r$, we can express $M$ as $\{i_k | k < r\}$ by using an strictly increasing sequence $i_k (k < r)$ such that $i_k = k$ for $k \ll r$. Then we associate a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ given by

$$
    \lambda_j = i_{r-j+1} - (r - j + 1), \quad (j = 1, 2, \ldots).
$$

The Weyl group $W = \langle s_0, s_1, s_2 \rangle$ can be realized as a subgroup of the group of bijections $w : \mathbb{Z} \to \mathbb{Z}$ by setting

$$
    s_k = \prod_{j \in \mathbb{Z}} \sigma_{\delta j + k-1} \quad (k = 0, 1, 2),
$$

where $\sigma_i (i \in \mathbb{Z})$ is the adjacent transposition $(i, i + 1)$. For a Maya diagram $M$ and $w \in W$, we see that $w(M) \subset \mathbb{Z}$ is also a Maya diagram of the same charge.

For any $w \in W$ and $k = 0, 1, 2$, let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be the partition corresponding to the Maya diagram $M = w(\mathbb{Z}_{< k})$. We set

$$
    N_{\lambda}^{(k)} = \prod_{i<j \in M^c, j \in M} (\varepsilon_i - \varepsilon_j),
$$

where we impose the relation $\varepsilon_i - \varepsilon_{i+3} = -4 (i \in \mathbb{Z})$, so we have $N_{\lambda}^{(k)} \in \mathbb{C}[\alpha_0, \alpha_1, \alpha_2]$. We can apply the following formula due to Y. Yamada [12]:

$$
    \phi_w(\Lambda_k) = N_{\lambda}^{(k)} \det \left( g_{\lambda_j-j+i}^{(k-i+1)} \right)_{1 \leq i, j \leq r}.
$$

(10.4)
Here \( g_p^{(k)} \) (\( k \in \mathbb{Z}/3\mathbb{Z}, p \in \mathbb{Z}_{>0} \)) are the determinant of \( p \times p \) matrix described as follows. First we define \( g_p^{(0)} \) by

\[
g_p^{(0)} := \frac{1}{N_p^{(0)}} \begin{vmatrix} f_{00} & f_{01} & f_{02} & 0 \\ \beta_1 & f_{11} & f_{12} & f_{13} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \beta_{p-2} & f_{p-2,p-2} & f_{p-2,p-1} \\ & & & \beta_{p-1} & f_{p-1,p-1} \end{vmatrix},
\]

where the components are

\[
f_{i,i} = f_i \quad (f_{i+3} = f_i),
\]

\[
f_{i,i+1} = g \quad (i \equiv 1), \quad 3q \quad (i \equiv 2), \quad 3r \quad (i \equiv 0),
\]

\[
f_{i,i+2} = 1 \quad (i \equiv 1, 0), \quad -2 \quad (i \equiv 2),
\]

and \( \beta_j = \sum_{i=j}^{p-1} \alpha_i = \varepsilon_j - \varepsilon_p \). Then we put \( g_p^{(1)} = \pi(g_p^{(0)}) \) and \( g_p^{(2)} = \pi^2(g_p^{(0)}) \) by the automorphism \( \pi \):

\[
\pi(f_{ij}) = f_{i+1,j+1}, \quad \pi(\varepsilon_j) = \varepsilon_{j+1}.
\]

The formula (10.4) is valid since the action of \( W = \langle s_0, s_1, s_2 \rangle \) in our setting is reduced from the action of \( A_\infty \) (cf. [9]):

\[
s_i(\alpha_i) = -\alpha_i, \quad s_i(\alpha_{i+1}) = \alpha_{i+1} + \alpha_i, \quad s_i(\alpha_j) = \alpha_j \quad (j \neq i, i \pm 1),
\]

where \( \alpha_j := \varepsilon_j - \varepsilon_{j+1} \quad (j \in \mathbb{Z}) \) and

\[
s_k(\varepsilon_{i,j}) = f_{i,j} + (\delta_{k+1,i} f_{k,j} - \delta_{j,k} f_{i,k+1}) \frac{\alpha_k}{f_k}.
\]

\section{11 Differential field of \( \tau \)-functions}

In this section we give supplementary discussions on the affine Weyl group action. In particular, we consider a differential field of \( \tau \)-functions that naturally contains the fields \( K \) and \( \tilde{K} \). The field \( \hat{F} \) we consider can be presented as

\[
\mathbb{C} \langle \alpha_0, \alpha_1, \alpha_2, x; \tau_0, \tau_1, \tau_2, \sigma_1, \sigma_2, \sigma_1', \sigma_2' \rangle
\]

with some relations discussed below. Then the set of bilinear equations in Theorem 3 makes \( \hat{F} \) into the differential field. To show some basic facts on \( \hat{F} \), we introduce some intermediate fields.

Let \( F \) denote the extended field of \( \mathbb{C} \langle \alpha_0, \alpha_1, \alpha_2, x \rangle \) obtained by adjoining the variables \( g'_0, g'_1, g'_2, h'_1, h'_2 \) with the following relations:

\[
3(g'_0 - 2h'_2 + h'_1)(g'_1 - g'_2) + 2x(g'_0 - 2g'_1 + g'_2) + \alpha_1 + 1 = 0, \quad (11.2)
\]

\[
3(h'_2 - 2h'_1 + g'_0)(g'_1 - g'_2) + 2x(g'_1 - 2g'_2 + g'_0) + \alpha_2 + 1 = 0. \quad (11.3)
\]

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As in the proof of Theorem \textup{3} we will identify \(g_j^\prime\) with \((\log \tau_j)^\prime\) and \(h_1^\prime, h_2^\prime\) with \((\log \sigma_1)^\prime, (\log \sigma_2)^\prime\) respectively. Note that the relations \textup{(11.2)}, \textup{(11.3)} correspond to \textup{(4.4)}, \textup{(4.5)}, \textup{(4.6)}. It is easy to see \(F = \mathbb{C}(\alpha_0, \alpha_1, \alpha_2, x)(g_0^\prime, g_1^\prime, g_2^\prime)\), and \(g_0^\prime, g_1^\prime, g_2^\prime\) are algebraically independent over \(\mathbb{C}(\alpha_0, \alpha_1, \alpha_2, x)\). So if we fix \(g_j'' \in F (j = 0, 1, 2)\) in an arbitrary way, then we have a derivation on \(F\). Now we want to introduce a derivation on \(F\) in such a way that is consistent with the bilinear equations. Actually we can prove the following lemma by lengthy but straightforward computations:

\textbf{Lemma 1} \textit{There exists a unique derivation on \(F\) such that the set of bilinear equations in Theorem \textup{3} holds.}

Consider the extended field \(\tilde{F} := F(\tau_0, \tau_1, \tau_2, \sigma_1, \sigma_2)\) with a relation
\[
\tau_1^\prime \tau_2 - \tau_2 \tau_1^\prime = \sigma_1 \sigma_2.
\]
We can naturally extend the derivation by \(\tau_j^\prime = g_j^\prime \tau_j, \ \sigma_k^\prime = h_k^\prime \sigma_k (j = 0, 1, 2, k = 1, 2)\). Then we have the previous presentation \textup{(11.1)}. Now the next lemma is a direct consequence of Theorem \textup{3}.

\textbf{Lemma 2} \textit{We have a natural embedding of the differential fields}
\[
K \subset \tilde{F}.
\]

Our next task is to extend the affine Weyl group action on \(\tilde{K} = K(\tau_0, \tau_1, \tau_2, \sigma_1, \sigma_2)\) (Theorem \textup{4}) to \(\tilde{F}\). The following two lemmas can be easily verified.

\textbf{Lemma 3} \textit{By the following formulae, we can introduce an action of the affine Weyl group \(W\) on \(\tilde{F}\) as a group of automorphisms:}
\[
\begin{align*}
s_0(\tau_0') &= \tau_0' - \alpha_0, \\
s_0(\tau_1') &= \tau_1' - \alpha_1 \sigma_2, \\
s_0(\tau_2') &= \tau_2' + \alpha_2 \sigma_1, \\
n_1(\tau_1') &= \tau_1' - \alpha_1 \tau_2, \\
n_1(\tau_2') &= \tau_2' - \alpha_1 \tau_1, \\
n_2(\tau_1') &= \tau_1' - \alpha_2 \tau_2, \\
n_2(\tau_2') &= \tau_2' - \alpha_2 \tau_1,
\end{align*}
\]
and \(s_i(\tau_j) = \tau_j' (i \neq j), \ s_i(\sigma_k) = \sigma_k' (i \neq k)\). Moreover this action is an extension of the action of \(W\) on \(\tilde{K}\).

\textbf{Lemma 4} \textit{For} \(i, j = 0, 1, 2\) \textit{and} \(k = 1, 2\) \textit{we have}
\[
s_i(\tau_j') = s_i(\tau_j)', \quad s_i(\sigma_k') = s_i(\sigma_k)'.
\]

\textbf{Remark.} Although we have introduced the Weyl group action on the \(\tau\)-functions in an ad hoc manner, these formulae can be derived systematically by using the gauge matrices \(G_i\) \textup{(0.3)}, if we identify the \(\tau\)-functions with the components of a \textit{dressing matrix}. We will give an explanation of this point in a separate article.

The goal of this section is the following fact:
Theorem 5  The derivation of $\hat{F}$ commutes with the action of $W$ on $\hat{F}$.

A straightforward verification of this fact may require quite a bit of calculations, because the second derivatives of $\tau$-functions are determined implicitly by the bilinear equations. To avoid the complexity, we make use of the fact $\hat{F} = \tilde{K}(k)$, which is easily seen from (9.1), (9.3), and (9.4), where we set

$$k = 2 \left( \frac{\tau'_0}{\tau_0} + \frac{\tau'_1}{\tau_1} + \frac{\tau'_2}{\tau_2} \right) + \frac{\sigma'_1}{\sigma_1} + \frac{\sigma'_2}{\sigma_2}.$$  

As for the first derivatives of $\tau$-functions, we have already lemma 4. Therefore, in order to prove Theorem 5, it suffices to show the next lemma.

Lemma 5

$$s_i(k') = s_i(k)' \quad (i = 0, 1, 2). \quad (11.4)$$

Proof. By Lemma 3 we have

$$s_0(k) - k = 2 \left( \frac{s_0(\tau'_0)}{s_0(\tau_0)} - \frac{\tau'_0}{\tau_0} \right) = -2 \frac{\alpha_0}{f_0},$$

$$s_1(k) - k = 2 \left( \frac{s_1(\tau'_1)}{s_1(\tau_1)} - \frac{\tau'_1}{\tau_1} \right) + \left( \frac{s_1(\sigma'_1)}{s_1(\sigma_1)} - \frac{\sigma'_1}{\sigma_1} \right), \quad (11.5)$$

$$s_2(k) - k = 2 \left( \frac{s_2(\tau'_2)}{s_2(\tau_2)} - \frac{\tau'_2}{\tau_2} \right) + \left( \frac{s_2(\sigma'_2)}{s_2(\sigma_2)} - \frac{\sigma'_1}{\sigma_1} \right). \quad (11.6)$$

We can rewrite the right hand sides of (11.5) and (11.6) into

$$s_1(k) - k = -2 \frac{\alpha_1}{f_1} r - \frac{\alpha_1(2r - f_2)}{3q(r + f_1 + \alpha_1)},$$

$$s_2(k) - k = -2 \frac{\alpha_2}{f_2} q - \frac{\alpha_2(2r - f_1)}{3r(f_2 - \alpha_2)}$$

by using (9.1), (10.1) and (10.2). On the other hand, the normalization condition (9.2) reads

$$k' = -u_0^2 - u_2^2 - \left( u_0 - \frac{f_1}{3r} + \frac{2x}{3} \right)^2 - \left( u_2 - \frac{f_2}{3q} + \frac{2x}{3} \right)^2$$

$$+ \frac{2x}{9} \left( 4x - \frac{f_1}{r} - \frac{f_2}{q} \right) + \frac{\alpha_1 - \alpha_2}{9}.$$  

Then we can verify (11.4) by applying (6.4) to $s_i(k')$ and the ODE (5.3) to $s_i(k)'$. □
12 Discussion

We have derived a two-parameter family of the fifth Painlevé equation as a similarity reduction of the modified Yajima-Oikawa hierarchy, which is related to a non-standard Heisenberg subalgebra of $A_2^{(1)}$. The system admits a group of Bäcklund transformations of type $W(A_2^{(1)})$. By a suitable modification of our construction, it may be possible to recover a missing parameter and get the fifth Painlevé with the full symmetry of type $W(A_3^{(1)})$. Combinatorial and/or representation theoretical structure of the hierarchy is also deserves to be investigated. A combinatorial aspect of representation associated with the Yajima-Oikawa hierarchy is studied by S. Leidwanger in [5]. It seems that the work is closely related some family of polynomial solutions of the fifth Painlevé equation. We hope that we discuss these issues in future publications.

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