DENSITY OF SCHRODINGER TITCHMARSH-WEYL M FUNCTIONS ON HERGLOTZ FUNCTIONS

INJO HUR

ABSTRACT. We will show that all Titchmarsh-Weyl m functions corresponding to Schrödinger operators are dense in all Herglotz functions. To show this, we first discuss a topology on canonical systems which interacts with the uniform convergence of Herglotz functions on compact subsets of the upper half plane. We then characterize canonical systems which can be written in a Schrödinger (eigenvalue) equation and vice versa, which gives us an easy way to construct m functions corresponding to Schrödinger equations in terms of canonical systems. Finally we approximate the canonical system whose m function is a given Herglotz function by canonical systems which can be written in Schrödinger equations such that their m functions converges to a given Herglotz function.

1. Introduction

We consider one-dimensional Schrödinger eigenvalue equation
\begin{equation}
- y''(x, z) + V(x)y(x, z) = zy(x, z), \quad x \in (0, b),
\end{equation}
with \( V \in L_1[0, b] \) when \( b < \infty \), and \( V \in L_{1,loc}[0, b] \) when \( b = \infty \). Here \( z \) is a spectral parameter.

To specify a self-adjoint operator corresponding to (1.1), we need boundary conditions at 0 and possibly at \( b \). We first put a boundary condition at 0, namely
\begin{equation}
y(0) \cos \alpha - y'(0) \sin \alpha = 0
\end{equation}
for some number \( \alpha \in [0, \pi) \). When \( b < \infty \), we place another boundary condition at \( b \),
\begin{equation}
y(b) \cos \beta + y'(b) \sin \beta = 0
\end{equation}
for \( \beta \in [0, \pi) \). By Weyl theory it is well known that if \( b = \infty \) and (1.1) is a limit-point case at \( \infty \), no more boundary condition except (1.2) is needed to specify the corresponding self-adjoint operator. However, if
$b = \infty$ and (1.1) is a limit-circle case at $\infty$, then it turns out that each point on the limit circle corresponds to some self-adjoint operator and hence we need a limit-type boundary condition at $\infty$. We will briefly review this in section 3.

It is well known that each equation (with boundary conditions) has a unique Titchmarsh-Weyl $m$ function denoted by $m_{\alpha,\beta}$. Here we use $\beta$ as a parameter for the boundary condition at $b$. It turns out that $m_{\alpha,\beta}$ can be expressed by the following:

\begin{equation}
(1.4) \quad m_{0,\beta}^S(z) = \frac{y'(0, z)}{y(0, z)}, \quad m_{\alpha,\beta}^S(z) = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \cdot m_{0,\beta}^S(z)
\end{equation}

where $y$ is a solution to (1.1) which is square-integrable near $b$ (when $b = \infty$ and (1.1) is a limit-point case), or satisfying either (1.3) (when $b < \infty$) or some limit-type boundary condition at $b$ (when $b = \infty$ and (1.1) is a limit-circle case). Here $\cdot$ means the action of $2 \times 2$ matrix as a linear fractional transformation. All $m$ functions corresponding to (1.1) are called Schrödinger $m$ functions for convenience. It is well known that Schrödinger $m$ functions are Herglotz functions, that is, they map the upper half plane $\mathbb{C}^+$ holomorphically to itself.

However, not all Herglotz functions are Schrödinger $m$ functions. The way to see is to discuss the asymptotic behavior of Schrödinger $m$ functions near $\infty$. Everitt [5] showed that, for $\alpha \in (0, \pi)$, $m_{\alpha,\beta}^S$ satisfy the asymptotic behavior

\begin{equation}
(1.5) \quad m_{\alpha,\beta}^S(z) = \frac{\cos \alpha}{\sin \alpha} + \frac{1}{\sin^2 \alpha} \frac{i}{\sqrt{z}} + O(|z|^{-1})
\end{equation}

or for $\alpha = 0$,

\begin{equation}
(1.6) \quad m_{0,\beta}^S(z) = i\sqrt{z} + o(1)
\end{equation}

when $z$ is large enough. We can now see that all Herglotz functions cannot be Schrödinger $m$ functions because of Herglotz representation: given a Herglotz function $F$

\begin{equation}
(1.7) \quad F(z) = A + \int_{\mathbb{R}_\infty} \frac{1 + tz}{t - z} d\rho(t)
\end{equation}

where $A \in \mathbb{R}$ (i.e., $A$ is a real number) and $d\rho$ is a finite positive Borel measure on $\mathbb{R}_\infty$ where $\mathbb{R}_\infty$ is one point compactification of $\mathbb{R}$. From the Herglotz representation it is easy to see that not all Herglotz function can satisfy the asymptotic behavior (1.5) or (1.6). For example, any Herglotz function which has any measure $\rho$ having a positive point mass at $\infty$ cannot satisfy (1.5) or (1.6). However a more issue is on the asymptotic behavior of the measure $\rho$ near $\infty$. See two sections 17
and 19 in [10] for more details.

To see a more general connection between Herglotz functions and equations we consider a canonical system,

\[(1.8) \quad Ju'(x, z) = zH(x)u(x, z), \quad x \in (0, \infty)\]

where \(H(x)\) is a positive semidefinite \(2 \times 2\) matrix whose entries are real-valued, locally integrable functions and \(J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\). We call (1.8) a trace-normed canonical system if \(\text{Tr } H(x) = 1\) for all \(x \in (0, \infty)\). For a canonical system we always place a boundary condition at 0

\[(1.9) \quad u_1(0, z) = 0.\]

Similar to (1.4), its \(m\) function \(m_H\) can be expressed by

\[(1.10) \quad m_H(z) = \frac{\bar{u}_2(0)}{\bar{u}_1(0)}\]

where \(\bar{u} = (\bar{u}_1, \bar{u}_2)^t\) is a solution to (1.8) satisfying

\[(1.11) \quad \int_{[0, \infty)} \bar{u}^*(x)H(x)\bar{u}(x)dx < \infty\]

where * means the Hermitian adjoint. (Such a function satisfying (1.11) is called \(H\)-integrable.) This is working because a trace-normed canonical system is always a limit-point case at \(\infty\). In other words, there is only one \(H\)-integrable solution up to a multiplicative constant. See the original argument by de Branges [2] or an alternative proof by Acharya [1] for more details. Then de Branges [3] and Winkler [12] showed that for a given Herglotz function, there exists a unique trace-normed canonical system (1.8) such that \(m_H\) is the given Herglotz function.

In this paper, we will show the density of Schrödinger \(m\) functions on all Herglotz functions or, equivalently, all \(m\) functions \(m_H\) to (1.8). Moreover, we will see that we can choose \(\alpha\) in (1.2) at our disposal.

**Theorem 1.1.** The set of Schrödinger \(m\) functions with some fixed number \(\alpha \in [0, \pi)\) in (1.2) are dense in the set of all Herglotz functions.

By Theorem 1.1 we cannot expect that a sequence of Schrödinger operators converges to a Schrödinger operator in general in the sense that this convergence is equivalent to the uniform convergence on compact subsets of \(\mathbb{C}^+\) of their \(m\) functions. That is why only subclasses of Schrödinger operators are considered in many applications to make them compact.

In (1.7) we see that the uniform convergence of Herglotz functions on compact subsets of \(\mathbb{C}^+\) is equivalent to both the weak-* convergence
of the corresponding measures $d\rho$ and the convergence of the constants $A$. Any finite positive Borel measure on $\mathbb{R}_\infty$ can be approximated by spectral measures corresponding to (1.1) in the weak-$^*-$sense: let $d\rho$ be a finite positive Borel measure on $\mathbb{R}_\infty$ and let $d\rho_{free}$ be the spectral measure for (1.1) with $V \equiv 0$. Construct a sequence of measures $d\rho_n$ by

$$d\rho_n(t) = \chi_{(-n,n)}(t)d\rho(t) + \chi_{\mathbb{R}_\infty \backslash (-n,n)}(t)d\rho_{free}(t) + \rho\{\infty\}\delta_n(t)$$

where $\delta_n$ is a Dirac measure at $n$. In other words, this is a sequence of truncated measures having the tail of the measure $d\rho_{free}$, which implies that $\rho_n$’s are spectral measures of (1.1). It is then easy to show that $d\rho_n \to d\rho$, as $n \to \infty$, in the weak-$^*$-sense. However, the weak-$^*$-convergence of spectral measures does not imply the convergence of $m$ functions corresponding to $d\rho_n$. This is because any spectral measures of (1.1) determine their $m$ functions: the error term is at least $o(1)$ in (1.5) or (1.6), which means that $d\rho_n$’s determine the corresponding constants $A_n$ in (1.7). Therefore it is unclear if $A_n$ converges to the constant $A$ which was given to us.

In section 2 we will discuss a topology on canonical systems (1.8) which works well with their $m$ functions. We will then characterize all canonical systems which can be written in Schrödinger equations (1.1) in section 3. We will not prove Theorem 1.1 in terms of $m$ functions directly. Instead, by using one-to-one correspondences between Herglotz functions and canonical systems and between Schrödinger equations and these characterized canonical systems (which will be (3.2)), in section 4 we will approximate a canonical system whose $m$ function is a given Herglotz function by these characterized canonical systems whose $m$ functions are Schrödinger $m$ functions in the sense of the topology discussed in section 2.

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2. Topology on canonical systems

We need a topology on the set of canonical systems, which will interact well with the convergence on their $m$ functions. Let $\mathbb{V}_+$ be the set of the matrices $H$ on trace-normed canonical systems, that is,

$$\mathbb{V}_+ = \{H(x) \text{ in (1.8)} : \operatorname{Tr} H(x) = 1 \text{ for all } x\}.$$
We say that $H_n$ converges to $H$ weak-$\ast$ if
\[ \int_0^\infty f^* H_n f \to \int_0^\infty f^* H f \]
for all continuous functions $f = (f_1, f_2)^t$ with compact support of $[0, \infty)$. Observe that, for given such a function $f$, two convergences
\[ \int_0^\infty f^* H_n f \to \int_0^\infty f^* H f \quad \text{and} \quad \int_0^\infty H_n f \to \int_0^\infty H f \]
are equivalent.

We can proceed to define a metric on $\mathbb{V}_+$: pick a countable dense (with respect to $\|\cdot\|_\infty$) subset $\{f_n : n \in \mathbb{N}\}$, continuous functions of compact support, and put
\[ d_n(H_1, H_2) = \left| \int_{(0,\infty)} f_n^*(x) (H_1 - H_2)(x) f_n(x) \, dx \right|. \]
Then define a metric $d$ as
\[ d(H_1, H_2) = \sum_{n=1}^\infty 2^{-n} \frac{d_n(H_1, H_2)}{1 + d_n(H_1, H_2)}. \]
Clearly, $d(H_n, H) \to 0$ if and only if $H_n$ converges to $H$ weak-$\ast$. Moreover, $(\mathbb{V}_+, d)$ is compact. To prove this, let $H_n \in \mathbb{V}_+$. By the Banach-Alaoglu Theorem (on finite intervals $[0, L]$ for some positive number $L$) and a diagonal process (for the whole half line $[0, \infty)$) we can find a subsequence $H_n_j$ with the property that $H_{n_j}(t) dt$ converges to some matrix-valued measure $d\mu$ in the weak-$\ast$ sense. The proof can now be completed by noting that the trace-normed condition $\text{Tr} H(x) = 1$ is preserved in the limiting process, which implies that the limit measure $d\mu$ is absolutely continuous with respect to the Lebesgue measure and it can be expressed by $H(t) dt$ for some $H \in \mathbb{V}_+$.

This gives us a topology on $\mathbb{V}_+$, which will work with $m$ functions by the following proposition.

**Proposition 2.1.** The map from $\mathbb{V}_+$ to $\mathbb{H}$, defined by $H \mapsto m_H$, is a homeomorphism where $\mathbb{H} = \mathbb{H} \cup \mathbb{R} \cup \{\infty\}$ and $\mathbb{H}$ is the set of all (genuine) Herglotz functions.

It is well known that $\mathbb{H}$ is compact with the uniform convergence on compact subsets of $\mathbb{C}^+$ which gives us a natural topology for holomorphic functions.

As we discussed in the introduction, this map is a bijection by Winkler \[12\]. Since $\mathbb{V}_+$ is compact, it suffices to show that this map is continuous. Roughly speaking, this map would be continuous because
of Weyl theory and the fact that $\text{Tr } H = 1$ implies that (1.8) is a limit-point case at $\infty$. In other words, $H$ on $(0, L)$ for a sufficiently large number $L > 0$ almost determines its $m$ function $m_H$.

To see the continuity we first show that solutions converge as a subsequence uniformly on any compact subsets of $[0, \infty)$.

**Lemma 2.2.** Assume that $H_n$ converges to $H$ weak-$\ast$, as $n \to \infty$, and let $u_n$ be solutions to (1.8) with $H_n$ such that the initial values $u_n(0)$ are the same for all $n$. Then the sequence $u_n$ has a subsequence which converges uniformly on any compact subsets of $[0, \infty)$. If $u$ is such a limit, then $u$ satisfies (1.8) with $H$.

**Proof.** We first show that $u_n$'s are uniformly bounded on any bounded interval in $[0, \infty)$, and the sequence $u_n$ converges in a subsequence uniformly on any compact subsets. Given a positive number $R$, assume that $|z| < R$ and put $\eta = \frac{1}{2R}$. We define an operator $T_n$ from $C[0, \eta]$, the set of continuous vector-valued functions on $[0, \eta]$, to itself by

$$(T_n u)(x) = -z \int_0^x J H_n(t) u(t) \, dt.$$  

Then $||T_n|| \leq 1/2$. Indeed, since $J$ is unitary, the trace-normed condition $\text{Tr } H = 1$ and positive semidefiniteness imply that $||J H_n|| \leq 1$ and so

$$(2.1) \quad ||T_n|| = \sup_{||u||_\infty = 1} ||T_n u(x)|| \leq R |x| = 1/2$$

for $x \in [0, \eta]$. In other words, $T_n$'s are uniformly bounded for $n$. Since the Neumann series converges, we have

$$(2.2) \quad (I - T_n)^{-1} = \sum_{k=0}^{\infty} T_n^k,$$

and

$$(2.3) \quad u_n(x) - u_n(0) = (T_n u_n)(x), \quad u_n(x) = (I - T_n)^{-1} u_n(0)$$

for solutions $u_n$ to (1.8) with $H_n$. Because of (2.2) and (2.1), we have $||(I - T_n)^{-1}|| \leq 2$. Therefore the solutions $u_n$ are uniformly bounded in $n$ on any bounded subsets of $[0, \infty)$. The expression (2.3) implies that $u_n$'s are equicontinuous. By Arzela-Ascoli Theorem, the sequence $u_n$ has a subsequence which converges uniformly on any compact subset of $[0, \infty)$, say, $u_n \to u$ as $n \to \infty$ (for convenience, we keep the same notation $u_n$ for the subsequence).
Let’s see that \( u \) satisfies (1.8) with \( H \). Since \( u_n \)'s are solutions for \( H_n \), we have
\[
u_n(x) - u_n(0) = -z \int_0^x JH_n(t)u_n(t)dt.
\]
The left-hand side goes to \( u(x) - u(0) \) as \( n \to \infty \) by continuity. By splitting the right-hand side by
\[
\begin{align*}
\int_0^x JH_n(t)u_n(t)dt &= \int_0^x JH_n(t)(u_n(t) - u(t))dt + \int_0^x J(H_n(t) - H(t))u(t)dt \\
&= : I + II
\end{align*}
\]
it suffices to show that \( I \) and \( II \) go to zero as \( n \to \infty \). Then \( I \to 0 \) because \( JH_n(t)dt \) are finite measures on \([0, x]\), and \( u_n \) converges to \( u \) uniformly on \([0, x]\). To see that \( II \to 0 \), we first recognize our test function through
\[
II = \int_0^\infty J(H_n(t) - H(t))\chi_{[0, x]}(t)u(t)dt.
\]
Since \( H(t)dt \) is absolutely continuous with respect to the Lebesgue measure (in particular, \( H(t)dt \) does not have point masses), the weak-* convergence works with any characteristic functions \( \chi_I \) for any bounded interval \( I \). Then \( II \to 0 \) as \( n \to \infty \). Hence \( u \) satisfies the equation
\[
u(x) - u(0) = -z \int_0^x JH(t)u(t)dt,
\]
in other words, \( u \) is a solution to (1.8) with \( H \).

\[\square\]

We now prove Proposition 2.1.

**Proof of Proposition 2.1.** Choose a sequence \( H_n \) which converges to \( H \) weak-*, as \( n \to \infty \). We first assume that \( H \) not be the same as the constant matrix \( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) on \((0, \infty)\) and that \( m_{H_n} \) not converge to \( \infty \), even in a subsequence. (Later we will show that \( H_n \) converges weak-* to the constant matrix \( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) on \((0, \infty)\) if and only if \( m_{H_n} \) converges to \( \infty \). Therefore it would be enough to assume the first condition only.)

From Weyl theory for a canonical system (which is similar to that for Schrödinger operators) it is well known that we can choose \( f_n = \)
(\(f_{n,1}, f_{n,2}\))^t which are \(H_n\)-integrable solutions to (1.8) with \(H_n\) (i.e., \(f_n\)'s satisfy \(\int_0^\infty f_n^* H_n f_n < \infty\)) such that, for \(z \in \mathbb{C}^+\),

\[(2.4) \quad f_n(x, z) = u_n(x, z) + m_{H_n}(z)v_n(x, z)\]

where \(m_{H_n}\) are the \(m\) functions of (1.8) with \(H_n\), and \(u_n\) and \(v_n\) are the solutions for \(H_n\) with \(u_n(0) = (1_0)\) and \(v_n(0) = (0_1)\), and

\[(2.5) \quad \frac{\text{Im} m_{H_n}(z)}{\text{Im} z} = \int_0^\infty f_n^*(x, z) H_n(x) f_n(x, z) dx.\]

Then due to (2.4) we have

\[(2.6) \quad m_{H_n}(z) = \frac{f_{n,2}(0, z)}{f_{n,1}(0, z)}.\]

We first show that the sequence \(f_n\) has a convergent subsequence. The compactness of \(H\) implies that \(m_{H_n}\) has a convergent subsequence, say, \(m_{H_n}(z) \to m(z) \neq \infty \in \mathbb{H}\), locally uniformly in \(z\) (for convenience we use the same notation for a subsequence). Since \(H_n\) converges to \(H\) weak-\(*\), and by Lemma 2.2, we see that

\[u_n(x, z) \to u(x, z), \quad v_n(x, z) \to v(x, z)\]

(in subsequence) uniformly on any compact subset (in \(x\) and \(z\)) with \(u(0) = (1_0)\) and \(v(0) = (0_1)\). Hence, by (2.4), \(f_n\) converges in a subsequence, and we put

\[(2.7) \quad f(x, z) := u(x, z) + m(z)v(x, z).\]

It is sufficient to show that \(f\) is \(H\)-integrable. If \(f\) is \(H\)-integrable, then, by the similar relation to (2.6) for \(f\) and \(m_H\) and (2.7), we have

\[m(z) = \frac{f_2(0, z)}{f_1(0, z)} = m_H(z),\]

which implies that \(m_H\) is the only possible limit. This is because of the fact that a trace-normed canonical system is always a limit-point case at \(\infty\). Therefore \(m_{H_n}\) converges to \(m_H\) uniformly on compact subsets of \(\mathbb{C}^+\).

Let’s see \(H\)-integrability of \(f\). Since \(m \neq \infty\), due to (2.5), the sequence \(\int_0^\infty f_n^* H_n f_n\) converges to some nonnegative number as \(n \to \infty\). In particular, the quantities \(\int_0^\infty f_n^* H_n f_n\) are uniformly bounded for \(n\),

\[\int_0^\infty f_n^* H_n f_n < \infty.\]
say, by $M$. Then we have that, for all $n$,

$$M \geq \int_0^\infty f_n^* H_n f_n$$

$$\geq \int_0^L f_n^* H_n f_n \quad \text{for all positive } L$$

$$= \int_0^L (f_n - f)^* H_n f_n + \int_0^L f_n^* H_n (f_n - f) + \int_0^L f^* H_n f$$

Since $f_n$ converges to $f$ locally uniformly and $H(x) dx$ is absolutely continuous with respect to the Lebesgue measure, the weak-* convergence of $H_n$ implies that the first and second integrals go to zero and the third converges to $\int_0^L f^* H f$, as $n \to \infty$. By taking $L \to \infty$, we see that $f$ is $H$-integrable, for these inequalities are uniform in both $n$ and $L$.

We now talk about the special case when $H = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ on $(0, \infty)$, denoted by $H_\infty$. This is because the $m$ function of the canonical system with $H_\infty$ is $\infty$. Here we will show that $H_n$ converges to $H_\infty$ weak-* if and only if $m_{H_n}$ converges to $\infty$. Before proving, note that, because of the previous statement, it was enough to assume that $H_n$ did converge to $H$ which was not $H_\infty$ in the previous case as we talked before.

Observe that

$$\int_0^L v^* H v \ dx = 0 \quad \text{if and only if} \quad H = H_\infty \text{ on } (0, L).$$

Indeed, for any open interval $I$ in $(0, \infty)$ we have either that

$$\int_I e^* He \ dx = 0, \ e \in C^2$$

(i.e., $I$ is of positive type in $\mathbb{R}$), or that $I$ is a singular interval of type $\theta$, in other words, $H$ is the constant matrix

$$\begin{pmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{pmatrix}$$

almost everywhere on $I$ for some $\theta$ in $[0, \pi)$ (i.e., $I$ is $H$-invisible of type $\theta$ in $\mathbb{R}$). See Lemma 3.1 in [8] for more details. Because of the continuity of $v$ and the initial condition $v(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we can see that if $\int_0^L v^* H v \ dx = 0$, then $\theta = 0$ on $(0, L)$, which shows the sufficiency of (2.8). Since $v_\infty(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ on $(0, L)$ where $v_\infty$ is the solution to (1.8) with $H_\infty$ on $(0, L)$, we have the necessity.

Let us show that if $H_n$ converges to $H_\infty$ weak-*, then $m_{H_n}$ converges to $\infty$. Assume that $m_{H_n}$ does not converge to $\infty$. By the compactness
we can choose \( m \in \mathbb{H} \setminus \{\infty\} \) such that \( m_{H_n} \) converges to \( m \) at least in a subsequence. (For convenience we keep the same notation for the subsequence.) Because of (2.4) and (2.5) we have that, for any finite number \( L > 0 \),

\[
\frac{\text{Im } m_{H_n}}{\text{Im } z} = \int_0^\infty f^*_n H_n f_n
\]

\[
\geq \int_0^L f^*_n H_n f_n
\]

\[
= \int_0^L u^*_n H_n u_n + m_{H_n} \int_0^L u^*_n H_n v_n
\]

\[
+ \bar{m}_{H_n} \int_0^L v^*_n H_n u_n + |m_{H_n}|^2 \int_0^L v^*_n H_n v_n.
\]

By taking \( n \to \infty \) and Lemma 2.2 at least in a subsequence,

\[
\frac{\text{Im } m}{\text{Im } z} \geq \int_0^L u^*_\infty H_\infty u_\infty + m \int_0^L u^*_\infty H_\infty v_\infty
\]

\[
+ \bar{m} \int_0^L v^*_\infty H_\infty u_\infty + |m|^2 \int_0^L v^*_\infty H_\infty v_\infty
\]

\[
= \int_0^L |1 - zx|^2 dx
\]

for any finite number \( L > 0 \) where \( u_\infty \) and \( v_\infty \) are the solutions for \( H_\infty \) satisfying \( u_\infty(0) = \binom{1}{0} \) and \( v_\infty(0) = \binom{0}{1} \). By direct computation we can see that \( u_\infty(x) = \binom{1 - 2x}{0} \) and \( v_\infty(x) = \binom{0}{1} \) on \( x \in (0, \infty) \), which implies the last equality. Since we can choose \( L \) arbitrarily, the last integral \( \int_0^L |1 - zx|^2 dx \) is arbitrarily large, which contradicts that \( \frac{\text{Im } m}{\text{Im } z} \) is a number (which is independent of \( L \)). Therefore, if \( H_n \) converges to \( H_\infty \) weak-*, then \( m_{H_n} \) converges to \( \infty \).

The remaining is to show that if \( m_{H_n} \) converges to \( \infty \), then \( H_n \) converges to \( H_\infty \) weak-*. Put \( \tilde{f}_n = -\bar{m}_{H_n} u_n + v_n \). In other words, since \( m_{H_n} \to \infty \), we cannot use (2.4) directly. Instead we consider the negative reciprocals of \( m_{H_n} \). Indeed, similar to (2.5), we can see that

\[
\begin{equation}
(2.10) \quad \frac{\text{Im } \bar{m}_{H_n}(z)}{\text{Im } z} = \int_0^\infty \tilde{f}_n^*(x, z) H_n(x) \tilde{f}_n(x, z) dx
\end{equation}
\]
where $\tilde{m}_{H_n}(z) = -\frac{1}{m_{H_n}(z)}$. Therefore, for any finite number $L > 0$,

$$
\frac{\text{Im } \tilde{m}_{H_n}}{\text{Im } z} = \int_0^\infty \bar{f}_n^* H_n f_n \\
\geq \int_0^L \bar{f}_n^* H_n f_n \\
= |\tilde{m}_{H_n}|^2 \int_0^L u_n^* H_n u_n - \tilde{m}_{H_n} \int_0^L u_n^* H_n v_n \\
- \tilde{m}_{H_n} \int_0^L v_n^* H_n u_n + \int_0^L v_n^* H_n v_n
$$

Since $\tilde{m}_{H_n} \to 0$, the left hand side on (2.10) converges to 0. By the compactness of $\mathbb{V}_+$ and Lemma 2.2 for $L > 0$, we have that

$$
\int_0^L \bar{f}_n^* H_n f_n \to \int_0^L v^* H v
$$

as $n \to \infty$ (at least in a subsequence). Hence we have showed that, whenever $H_n$ converges to $H$ weak-$*$ in a subsequence,

$$
\int_0^L v^* H v = 0
$$

for any $L > 0$, which, by [2.8], implies that $H = H_\infty$ on $(0, \infty)$. Therefore $H_n$ converges to $H_\infty$ since $H_\infty$ is the only possible limit.

\[ \square \]

3. Schrödinger equations in canonical systems

In this section, we will show that Schrödinger equations have a special form in canonical systems. Before continuing, we need to specify the boundary condition at $\infty$ when (1.1) is a limit-circle case. Let $f(x, z) := u_\alpha(x, z) + m(z)v_\alpha(x, z)$ where $u_\alpha$ and $v_\alpha$ are the solutions to (1.1) satisfying $u_\alpha(0, z) = v'_\alpha(0, z) = \cos \alpha$ and $-u'_\alpha(0, z) = v_\alpha(0, z) = \sin \alpha$. Then $m$ is on the limit circle if and only if

$$
\lim_{N \to \infty} W_N(\bar{f}, f) = 0
$$

where $W_N$ is the Wronskian at $N$, that is, $W_N(f, g) = f(N)g'(N) - f'(N)g(N)$ and $\bar{f}$ is the complex conjugate of $f$. Similarly we use $\beta$ as a parameter for boundary conditions at $\infty$. In other words, $m_{\alpha, \beta}$ will be the $m$ function of (1.1) with (1.2) and some boundary condition at $\infty$ corresponding to $\beta$. See [4] or [11] for more details.

We now talk about the representation of Schrödinger equations in canonical systems.
Proposition 3.1. A Schrödinger equation (1.1) with boundary conditions (1.2) and, if necessary, (1.3)/(3.1) can be expressed as the following half-line trace-normed canonical system such that both the given Schrödinger equation and the following canonical system have the same Titchmarsh-Weyl $m$ functions:

$$J \frac{d}{dt} u(t, z) = zH(t)u(t, z), \quad t \in (0, \infty),$$

where

$$H(t) = P_{\varphi(t)} := \begin{bmatrix} \cos^2 \varphi(t) & \cos \varphi(t) \sin \varphi(t) \\ \cos \varphi(t) \sin \varphi(t) & \sin^2 \varphi(t) \end{bmatrix}.$$

Here a new variable $t$ is defined by

$$t(x) = \int_0^x (u_0^2(s) + v_0^2(s)) \, ds$$

where $u_0$ and $v_0$ are the solutions to the given Schrödinger equation for $z = 0$ with $u_0(0) = v_0' (0) = \cos \alpha$ and $-u_0'(0) = v_0(0) = \sin \alpha$.

Moreover, define $t_b := \lim_{x \to b} t(x)$. Then $\varphi$ is a strictly increasing function on $(0, t_b)$, which has a locally integrable third derivative on $(0, t_b)$, satisfying $\varphi(0) = \alpha$, $\varphi'(0) = 1$, and $\varphi''(0) = 0$. If $t_b < \infty$, then $\varphi(t) = \tilde{\beta}$ on $(t_b, \infty)$ for some real number $\tilde{\beta} \in [0, \pi]$.

Conversely, any canonical system (3.2) above can be written as (1.1) with some locally integrable potential $V$.

The above proposition says that Schrödinger equations (1.1) are the canonical systems (1.8) with projection-type matrices $P_{\varphi(t)}$ (as their $H$) such that $\varphi$'s are strictly increasing functions having the third derivatives (which have the same regularity with potentials $V$), and it behaves as a linear function with slope 1 near 0. Moreover its function value at 0, $\varphi(0)$, is $\alpha$ in the boundary condition (1.2) up to multiples of $\pi$.

We have projection-type matrices $P_{\varphi}$, because of the asymptotic behavior of the solutions of (1.1). De Branges [2] showed that solutions to (1.8) belong to Cartwright class of the exponential type $h$

$$h = \int_0^x \sqrt{\det H(t)} \, dt$$

for fixed $x$. An entire function $F$ belongs to Cartwright class of the exponential type $h$ if

$$h := \limsup_{|z| \to \infty} \frac{\ln |F(z)|}{|z|} \text{ is finite},$$
and
\[ \int_{-\infty}^{\infty} \frac{|\ln |F(x)||}{1 + x^2} < \infty. \]

Pöschel and Trubowitz \cite{9} estimated solutions to (1.1) and showed that they are of order 1/2 as entire functions with respect to \(z\) for fixed \(x\). (See also (4.3) in \cite{10}.) In particular, they are of exponential type 0. Hence \(\det H = 0\) for Schrödinger equations. Since \(H\) is symmetric, the two conditions, \(\text{Tr} H = 1\) and \(\det H(x) = 0\) for all \(x\), imply that we have projection-type matrices \(P_{\varphi(t)}\) after some change of variables.

Proposition 3.1 gives us an easy way to construct trace-normed canonical systems corresponding to (1.1). As we can see in the inverse spectral theories by Gelfand and Levitan \cite{6}, Gesztesy and Simon \cite{7} or Remling \cite{10}, given a Herglotz function, it is not easy to see if it is a Schrödinger \(m\) function. We will use this easy construction to prove Theorem 1.1.

We now prove Proposition 3.1.

\textbf{Proof of Proposition 3.1} \ Let \(y\) be a solution to the given Schrödinger equation (1.1). Define \(u = u(x, z) = (u_1(x, z), u_2(x, z))^t\) by
\begin{equation}
\begin{pmatrix}
 u_1 \\
 u_2
\end{pmatrix} = \begin{pmatrix}
 u_0(x) & v_0(x) \\
 v_0'(x) & u_0'(x)
\end{pmatrix}^{-1} \begin{pmatrix}
 y \\
 y'
\end{pmatrix}.
\end{equation}

Note that this definition is well defined because the determinant of the \(2 \times 2\) matrix inside (3.5) is the Wronskian of \(u_0\) and \(v_0\), \(W(u_0, v_0)\), which is 1 for all \(x\); in particular, this matrix is invertible. Then \(u\) solves (1.8) with \(H(x) = H_0(x) := \begin{pmatrix}
 u_0^2(x) & u_0(x)v_0(x) \\
 u_0(x)v_0(x) & v_0^2(x)
\end{pmatrix}\).

This is shown by direct computation, which is left to the reader.

To have a trace-normed canonical system, we define \(R\) and \(\varphi\) by
\begin{equation}
\begin{pmatrix}
 u_0(x) + iv_0(x) := R(x)(\cos \varphi(x) + i \sin \varphi(x))
\end{pmatrix},
\end{equation}
and \(t\) by (3.4). Then we have (3.2) with (3.3), but only on \((0, t_b)\).

Directly computing the Wronskian from (3.6), we see that
\begin{equation}
(1 =) W(u_0, v_0)|_x = R^2(x)\varphi'(x), \quad x \in (0, b).
\end{equation}

Hence \(\varphi(x)\) is strictly increasing on \((0, b)\) with respect to \(x\). Since \(dt/dx = R^2\), so is \(\varphi\) on \((0, t_b)\) with respect to \(t\). With the aid of (3.7) the definition of \(u_0\) implies that the functions \(V, u_0^2, \) and \(\varphi''\) have the same regularity. Because of the initial values of \(u_0\) and \(v_0\), we
have $\varphi(0) = \alpha$, $R(0) = 1$, and $R'(0) = 0$, or equivalently, $\varphi(0) = \alpha$, $\varphi'(0) = 1$, and $\varphi''(0) = 0$ because $\varphi' = 1/R^2$ (note that $t = 0$ when $x = 0$).

When $t_b < \infty$, it is easy to see that the boundary condition (1.3) or (3.1) is transformed to a similar boundary condition for $u$

$$u_1(t_b) \cos(\tilde{\beta}) + u_2(t_b) \sin(\tilde{\beta}) = 0$$

for some number $\tilde{\beta}$. To have a half-line canonical system we change (3.8) to a canonical system on the singular interval $(t_b, \infty)$ of type $\tilde{\beta}$. (See (2.9) for the definition of a singular interval.) We then see that, for all $t \in (t_b, \infty)$, $u(t) = u(t_b)$ and

$$u^*(t)H(t)u(t) = u^*(t)P_{\tilde{\beta}} u(t) = 0.$$  

Indeed, since $I - zLH$ is the transfer matrix on $(t_b, \infty)$ for a nonnegative number $L$, $u(t_b + L) = (I - zLH)u(t_b) = (I - zLP_{\tilde{\beta}})u(t_b) = u(t_b)$ because of the boundary condition (3.8). Here $I$ is the $2 \times 2$ identity matrix.

It remains to show that both (1.1) and (3.2) above have the same $m$ function. To see this we compare their solutions. Let $\tilde{y}$ be a solution to (1.1) which is square-integrable near $b$ or satisfying either (1.3) or (3.1), and let $\tilde{u}$ be the solution to (3.2) corresponding to $\tilde{y}$ through (3.5). Then $\tilde{u}$ is $H$-integrable or satisfying (3.8). Indeed, it is clear that we can fine some $\tilde{\beta}$ such that $\tilde{u}$ satisfies (3.8) when $\tilde{y}$ does (1.3) or (3.1). The condition $\tilde{y} \in L^2[0, b)$ means

$$\int_{[0, b)} \tilde{y}(x)^* \tilde{y}(x) dx < \infty.$$  

By (3.5) the above is equivalent to

$$\int_{[0, b)} \tilde{u}(x)^* H_0(x) \tilde{u}(x) dx < \infty.$$  

Then, by change of variable to a new variable $t$, we have

$$\int_{[0, t_b)} \tilde{u}(t)^* H(t) \tilde{u}(t) dt < \infty.$$  

If $t_b = \infty$, then $\tilde{u}$ is $H$-integrable. When $t_b < \infty$, $\tilde{u}$ is still $H$-integrable because of (3.9). In other words, if $\tilde{u}$ satisfies (3.8), then its trivial extension $\hat{u}$ is an $H$-integrable solution to (3.2) with (3.3).

We now compare their $m$ functions. Let $m^S_{\alpha, \tilde{\beta}}(z)$ be the $m$ function for (1.1) with (1.2) and, if necessary, (1.3)/(3.1), and $m_H$ the $m$
We have constructed (3.2) from (1.1) as desired. For the converse, we go through the previous process, but in reverse. Assume that we have (3.2) with (3.3) such that \( \varphi \) has the same properties in Proposition 3.1. Since \( \frac{d\varphi}{dt} > 0 \), we can recognize a variable \( x \) corresponding to (1.1) by solving the integral equation
\[
t(x) = \int_0^x \left[ \frac{d}{dt} \varphi(t(s)) \right]^{-1/2} ds.
\]
Put \( R(x) \equiv \left[ \frac{d}{dx} \varphi(t(x)) \right]^{-1/4} \). Then \( t'(x) = R^2(x) \) and \( R^2(x)\varphi'(x) = 1 \) (here ‘ means \( \frac{d}{dx} \)). We now define \( u_0 \) and \( v_0 \) by \( u_0(x) = R(x) \cos(\varphi(t(x))) \) and \( v_0(x) = R(x) \sin(\varphi(t(x))) \). By direct computation we see that \( R^2(x)\varphi'(x) = W(u_0(x), v_0(x)) \) for all \( x \). Then \( u \) satisfies (1.8) with \( H_0(x) \). Due to the Wronskian condition we can define \( y \) and \( y' \) through (3.5). Then \( y \) satisfies (1.1) with the potential \( V \)
\[
V = \frac{7}{16} \left( \varphi''(x) \right)^2 - \frac{1}{4} \left( \varphi'(x) \right)^2 - \varphi'(x)
\]
\[
= \frac{R''}{R} - \frac{1}{R^4}.
\]
This is shown by direct computation, which is left to readers.

Similar to the previous argument, we can put a suitable boundary condition, when \( \varphi \) is constant on a unbounded interval \( (c, \infty) \) for some \( c \), and we can show that two \( m \) functions are the same. Thus we have proven Proposition 3.1.

4. Proof of Theorem 1.1

In this section we prove Theorem 1.1. To do this, we will show that since a symmetric matrix is expressed by the sum of projections, it is at
least locally and averagely that \( H \) is the sum of projection-type matrices. In other words \( H \) is almost \( P_\varphi \) with a nondecreasing step function \( \varphi \). We will then approximate such a function \( \varphi \) by strictly increasing smooth functions in the \( L_1 \)-sense. Due to this \( L_1 \) approximation we can choose the boundary condition at 0 freely.

**Proof of Theorem 1.1.** Choose any \( H \) in \( V_+ \). We first approximate \( H \) by \( H_n \) such that they are projection-type \( P_\varphi \) whose \( \varphi \)'s are nondecreasing step functions. Given \( n \in \mathbb{N} \) we put \( I_{j,n} = [j2^n, (j+1)2^n] \) where \( j = 0, 1, 2, \ldots \). Let \( H_{j,n} := 2^n \int_{I_{j,n}} H(x) \, dx \). Then \( H_{j,n} \) is a constant \( 2 \times 2 \) matrix which is positive semidefinite with \( \text{Tr} (H_{j,n}) = 1 \). Since \( H_{j,n} \) is symmetric, by Spectral Theorem, there is some real number \( \lambda_{j,n} \) such that

\[
H_{j,n} = \lambda_{j,n} P_{\varphi_{j,n}} + (1 - \lambda_{j,n}) P_{\varphi_{j,n} + \pi/2}
\]

where \( \lambda_{j,n} \) is an eigenvalue of \( H_{j,n} \) and \( P_{\varphi_{j,n}} \) (which is (3.3) with \( \varphi_{j,n} \)) is an orthogonal projection onto the eigenspace for \( \lambda_{j,n} \). The projection for the other eigenvalue is \( P_{\varphi_{j,n} + \pi/2} \) because of the orthogonality of eigenspaces of two eigenvalues. Construct \( \varphi_n \) by

\[
\varphi_n(x) := \begin{cases} 
\varphi_{j,n} & x \in \left[ j2^n, j+\lambda_{j,n}/2^n \right] \\
\varphi_{j,n} + \pi/2 & x \in \left[ j+\lambda_{j,n}/2^n, j+1/2^n \right]
\end{cases}
\]

in such a way that \( \varphi_{j+1,n} \geq \varphi_{j,n} + \pi/2 \) for all \( j \). Indeed, if \( \varphi_{j+1,n} < \varphi_{j,n} + \pi/2 \) for some \( j \), then we add the smallest multiple of \( \pi \) to \( \varphi_{j+1,n} \) in order to make \( \varphi_n \) nondecreasing. We do this process from \( j = 1 \), inductively.

(For convenience we denote a new value by \( \varphi_{j,n} \) again, because we later deal only with \( \cos^2 \varphi_n, \sin^2 \varphi_n \) and \( \cos \varphi_n \sin \varphi_n \) which are periodic with the period \( \pi \).) Then \( \varphi_n \) is a nondecreasing step function. (See the figure 1 below.)

Denote \( H_n \) by \( H_n(x) = P_{\varphi_n(x)} \) which is (3.3) with \( \varphi_n \). By the definition of \( H_n \) we see that, for given a natural integer \( m \),

\[
\int_{I_{j,m}} H_n = \int_{I_{j,m}} H
\]

for all \( n \geq m \).

Next we show that \( H_n \) converges weak-* to \( H \). Let \( f \) be a continuous function with support contained in \([0, L]\) for some positive number \( L \). Then, by the Lebesgue lemma, for given \( \epsilon > 0 \) there are numbers \( M, n_0 \) and \( f_{j,n_0} \) such that for all \( x \in [0, L] \)

\[
\sup \|f(x)\| \leq M
\]
and for all \( n \geq n_0 \)

\[
\sup \| f(x) - f_{j,n_0} \| \leq \frac{\epsilon}{2ML} \quad \text{for all } x, y \in I_{j,n}.
\]

Then we estimate \( H_n - H \) on each small interval as follows:

\[
\int_{I_{j,n_0}} f^*(H_n - H)f = \int_{I_{j,n_0}} (f - f_{j,n_0})^*(H_n - H)f + \int_{I_{j,n_0}} f^*_{j,n_0}(H_n - H)(f - f_{j,n_0}) + \int_{I_{j,n_0}} f^*_{j,n_0}(H_n - H)f_{j,n_0}.
\]

By (4.1) the third integral is zero for all \( n \geq n_0 \). We also see that the absolute values of the first and second integrals are bounded by \( \frac{\epsilon}{L^2n_0} \), since the operator norm of \( H_n - H \) is bounded by 2 (each \( H \) is bounded by 1 due to \( \text{Tr} H = 1 \) and positive semidefiniteness of \( H \)). Therefore, the integral \( \int_0^\infty f^*(H_n - H)f \leq \epsilon \) and then, since \( \epsilon \) is arbitrary, \( H_n \) converges to \( H \) weak-\(*\).

So far, we have constructed nondecreasing step functions \( \varphi_n \) such that \( H_n (= P_{\varphi_n}) \) converges to \( H \) weak-\(*\). However, this \( \varphi_n \) is not the one corresponding to a Schrödinger equation, since \( \varphi_n \) is not smooth enough (it is not discontinuous unless it is constant), not linear near 0, and not strictly increasing.

To overcome these, for each \( n \) we construct a new function \( \tilde{\varphi}_n \) in the following way. (In this paragraph we drop the subscript \( n \) for convenience.) Assume that all the steps are bounded. This is OK because if we have an unbounded step, then it would be the last step. In this case we can change this singular interval to some boundary condition at the starting point of the unbounded interval as we talked before.

Except for the first step and an unbounded step (which was taken care of by some boundary condition), we approximate all other steps (i.e., \( \varphi \)) by a piecewise linear, strictly increasing, and continuous function \( \tilde{\varphi} \) so that \( \varphi \) and \( \tilde{\varphi} \) are very close in the \( L_1 \)-sense. Look at the figure 1 for more details. For the first (bounded) step we assume that \( \varphi(0) > \alpha \) where \( \alpha \) will be in the boundary condition at 0 of Schrödinger equation we want. This works because we can shift \( \varphi \) up by \( \pi \). The function \( \tilde{\varphi} \) starts at \((0, \alpha)\) and then slightly moves up linearly with slope 1. We then do the similar procedure for its remaining part with other steps. This is OK because we approximate \( \varphi \) by \( \tilde{\varphi} \) in the \( L_1 \)-sense. Note that \( \tilde{\varphi} \) is a strictly increasing, piecewise-linear, continuous
function which looks linear with slope 1 near 0 with $\bar{\varphi}(0) = \alpha$, but it is not differentiable.

By using the mollifier we can make $\bar{\varphi}$ smooth functions (and we denote it by $\bar{\varphi}$ again). Then the constructed function $\bar{\varphi}$ is a smooth, strictly increasing function which is linear with slope 1 near 0. Therefore $\bar{\varphi}$ corresponds to some Schrödinger equation by Proposition 3.1.

In our construction we can choose a sequence $\bar{\varphi}_{n,j}$ such that

\[
\|\bar{\varphi}_{n,j} - \varphi_n\|_{L_1} \to 0
\]

as $j \to \infty$. Since sine and cosine are uniformly continuous, we can see that (4.2) implies the weak-* convergence of $\bar{H}_{n,j}$ to $H_n$ where $\bar{H}_{n,j} = P_{\bar{\varphi}_{n,j}}$. Since $H_n$ converges to $H$ weak-*, by Proposition 2.1, $m_{\bar{H}_{n,j}}$ converges to $m_H$ uniformly on any compact subsets of $\mathbb{C}^+$ (for suitably chosen $n$ and $j$). Thus we have shown Theorem 1.1 because $m_{\bar{H}_{n,j}}$’s are some Schrödinger $m$ functions.

\[\square\]

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Mathematics Department, University of Oklahoma, Norman, OK 73019

E-mail address: ihur@math.ou.edu