Cubic B-spline solution for two-point boundary value problem with AOR iterative method

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Abstract. In this study, the cubic B-spline approximation equation has been derived by using the cubic B-spline discretization scheme to solve two-point boundary value problems. In addition to that, system of cubic B-spline approximation equations is generated from this spline approximation equation in order to get the numerical solutions. To do this, the Accelerated Over Relaxation (AOR) iterative method has been used to solve the generated linear system. For the purpose of comparison, the GS iterative method is designated as a control method to compare between SOR and AOR iterative methods. There are two examples of proposed problems that have been considered to examine the efficiency of these proposed iterative methods via three parameters such as their number of iterations, computational time and maximum absolute error. The numerical results are obtained from these iterative methods, it can be concluded that the AOR iterative method is slightly efficient as compared with SOR iterative method.

1. Introduction
Boundary value problems plays an important roles to apply and solve many application of science and engineering phenomenon. Researchers in science, physics and engineering get more advantages from the numerical solutions which obtained from the two-point boundary value problems. Actually there are various methods are used to get the solutions in boundary value problems such as families of Galerkin methods namely Sinc-Galerkin method [1] and hybrid Galerkin method [2], Adomain decomposition method [3] and shooting method [4]. The other researchers also have been used the families of B-spline [5,6,7] to solve the same problems. B-spline scheme was considered in this paper to apply with the aim of discretizing the proposed problem.

To obtain the cubic B-spline approximation equation of the proposed problem, firstly, the proposed problem needs to be discretized via family of spline or more specifically by imposing cubic B-spline discretization scheme. Then the approximation equation can be derived and used to construct a linear system. Next, in this paper, the linear system can be solved via the iterative methods. This is because of motivation given by Young [8], Hackbush [9] and Saad [10] in which they have been proposed and discussed about the various iterative methods to solve any linear system.

As mentioned in the second paragraph, the utmost objective of this paper is used the cubic B-spline approximation equation to solve two-point boundary value problem with attention to investigate the application of AOR iterative method. To analyze the performance of AOR, the implementation of the SOR iteration family method has been used as control methods.
Before investigation started, let two-point boundary value problems be defined as

$$y'' + f(s)y' + g(s)y = r(s), s \in [s_0, s_N]$$

(1)

with the boundary conditions

$$y(s_0) = v, \quad y(s_N) = x$$

(2)

where \(v\) and \(x\) in equation (2) stand for the left and the right boundary condition of two-point boundary value problem [5].

Now, constructing the approximation equation of problem (1), the B-spline curve can be defined as [11]

$$y(s) = \sum_{h=0}^{N} C_h \cdot \beta_{h,d}(s), 0 \leq s \leq 1$$

(3)

where the \(C_h\) stand for control point. The general formula of \(\beta_{h,d}(s)\) in equation (3) which represent as B-spline basis functions as follows [12]

$$\beta_{h,x}(s) = \frac{s-s_h}{s_{h+x-1}-s_h} \beta_{h,x-1}(s) + \frac{s_{h+x}-s_h}{s_{h+x}-s_{h+1}} \beta_{h+1,x-1}(s)$$

(4)

with condition,

$$\beta_{h,0}(s) = \begin{cases} 1, & s \in [s_h, s_{h+1}] \\ 0, & otherwise \end{cases}$$

(5)

2. Cubic B-Spline approximation Equations

In this section, the ways to construct the linear system from the discretization of the proposed problem to drive B-spline approximation equation can be obtained via the cubic B-spline descretization scheme. By referring the B-spline function in equation (4) and taking \(z = 3\), we get the function of cubic B-spline as [13]

$$\beta_{h,3}(s) = \frac{s-s_h}{s_{h+3}-s_h} \left[ \frac{s-s_h}{s_{h+2}-s_h} \beta_{h,0}(s) + \frac{s-s_h}{s_{h+2}-s_h} \beta_{h+1,0}(s) \right] + \frac{s_{h+3}-s_h}{s_{h+4}-s_{h+1}} \left[ \frac{s-s_h}{s_{h+2}-s_h} \beta_{h+1,0}(s) + \frac{s-s_h}{s_{h+2}-s_h} \beta_{h+1,0}(s) \right]$$

(6)

Then, the equation (6) will be simplified to get the piecewise of the cubic B-spline functions at the several different intervals and defined as follows

$$\beta_{h,3}(s) = \frac{1}{6k^3} \begin{cases} (s - s_h)^3, & s \in [s_h, s_{h+1}] \\ k^3 + 3k^2(s - s_{h+1}) + 3k(s - s_{h+1})^2 + 3(s - s_{h+1})^3, & s \in [s_{h+1}, s_{h+2}] \\ k^3 + 3k^2(s_{h+3} - s) + 3k(s_{h+3} - s)^2 + 3(s_{h+3} - s)^3, & s \in [s_{h+2}, s_{h+3}] \\ (s_{h+4} - s)^3, & s \in [s_{h+3}, s_{h+4}] \end{cases}$$

(7)
where equation (1). Therefore, the general equation of the approximation equation in equation (14), we get

\[
\beta_{h-1,3}(s) = \frac{1}{6k^3} \begin{cases} 
(s - s_{h-1})^3, & s \in [s_{h-1}, s_h] \\
3k^3 (s - s_h) + 3k(s - s_h)^2 + 3(s - s_h)^3, & s \in [s_h, s_{h+1}] \\
k^3 + 3k^2 (s_{h+1} - s) + 3k(s_{h+1} - s)^2 + 3(s_{h+1} - s)^3, & s \in [s_{h+1}, s_{h+2}] \\
(s_{h+2} - s)^3, & s \in [s_{h+2}, s_{h+3}]
\end{cases}
\]  

(8)

where

\[
\beta_{h-2,3}(s) = \frac{1}{6k^3} \begin{cases} 
(s - s_{h-2})^3, & s \in [s_{h-2}, s_{h-1}] \\
k^3 + 3k^2 (s - s_{h-1}) + 3k(s - s_{h-1})^2 + 3(s - s_{h-1})^3, & s \in [s_{h-1}, s_{h}] \\
k^3 + 3k^2 (s_{h+1} - s) + 3k(s_{h+1} - s)^2 + 3(s_{h+1} - s)^3, & s \in [s_{h}, s_{h+1}] \\
(s_{h+2} - s)^3, & s \in [s_{h+1}, s_{h+2}]
\end{cases}
\]  

(9)

where

\[
\beta_{h-3,3}(s) = \frac{1}{6k^3} \begin{cases} 
(s - s_{h-3})^3, & s \in [s_{h-3}, s_{h-2}] \\
k^3 + 3k^2 (s - s_{h-2}) + 3k(s - s_{h-2})^2 + 3(s - s_{h-2})^3, & s \in [s_{h-2}, s_{h-1}] \\
k^3 + 3k^2 (s_{h} - s) + 3k(s_{h} - s)^2 + 3(s_{h} - s)^3, & s \in [s_{h-1}, s_{h}] \\
(s_{h+1} - s)^3, & s \in [s_{h}, s_{h+1}]
\end{cases}
\]  

(10)

After that, the formulation in equations (7)-(10) are considered with \( s = s_h \), it can be shown that we have

\[
\beta_{h,3}(s_h) = 0 \quad \beta_{h-1,3}(s_h) = \frac{1}{6} \\
\beta_{h-2,3}(s_h) = \frac{4}{6} \quad \beta_{h-3,3}(s_h) = \frac{1}{6}
\]

(11)

Again, by considering the first derivative of the functions (7)-(10) and substituting \( s = s_h \), we can show

\[
\beta'_{h,3}(s_p) = 0 \quad \beta'_{h-1,3}(s_p) = \frac{1}{2k} \\
\beta'_{h-2,3}(s_p) = \frac{1}{6k} \quad \beta'_{h-3,3}(s_p) = -\frac{1}{2k}
\]

(12)

Based on the same steps for equation (12), it can be stated that

\[
\beta''_{h,3}(s_h) = 0 \quad \beta''_{h-1,3}(s_h) = \frac{1}{k^3} \\
\beta''_{h-2,3}(s_h) = -\frac{2}{k^3} \quad \beta''_{h-3,3}(s_h) = \frac{1}{k^2}
\]

(13)

For simplicity and taking \( N = 8 \), the approximate solution of \( y(x) \) in equation (3) shows as

\[
y(s) = C_{-3} \cdot \beta_{-3,3}(s) + C_{-2} \cdot \beta_{-2,3}(s) + C_{-1} \cdot \beta_{-1,3}(s) + \cdots + C_6 \cdot \beta_{6,3}(s) + C_7 \cdot \beta_{7,3}(s)
\]

(14)

where \( C_h \) are unknown coefficients. Now, to solve two-point boundary value problem via the cubic B-spline approximation equation can be done by substituting the function in equations (11)-(13) into equation (1). Therefore, the general equation of the approximation equation in equation (14), we get

\[
A_{1h} \cdot C_{h-3} + B_{1h} \cdot C_{h-2} + C_{1} \cdot C_{h-1} = R_h
\]

(15)

where

\[
A_{1h} = \frac{1}{k^2} - \frac{f_h}{2k} + \frac{g_h}{6}, \quad B_{1h} = -\frac{2}{k^2} + \frac{f_h}{6k} + \frac{4g_h}{6}, \quad C_{1} = \frac{1}{k^2} + \frac{f_h}{2k} + \frac{g_h}{6}
\]

for \( h = 1,2,3,...,8 \). In addition to that, the approximation equation (15) can be used to construct the tridiagonal matrix form of linear system which is given as

\[
AJ = M
\]

(16)

where
\[ A = \begin{bmatrix}
A_{10} & B_{10} & C_{10} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & A_{11} & B_{11} & C_{11} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & A_{12} & B_{12} & C_{12} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & A_{13} & B_{13} & C_{13} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & A_{14} & B_{14} & C_{14} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & A_{15} & B_{15} & C_{15} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A_{16} & B_{16} & C_{16} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{17} & B_{17} & C_{17} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{18} & B_{18} & C_{18}
\end{bmatrix}, 
\]

\[ J = [j_{-2}, j_{-1}, j_0, j_1, j_2, j_3, j_4, j_5, j_6]^T, \]

\[ M = [m_0 - \alpha, m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8 - \beta]^T. \]

In fact, \( A \) is the coefficient matrix, \( J \) and \( M \) represent as an unknown vector and a known vector respectively. To get the numerical solution of the linear system (16) iteratively, the coefficient matrix, \( A \) must be the positive definite, \( [A_{1_{hh}}] \geq \sum_{h \neq 1} [A_{1_{hh}}] \). This is called as the sufficient condition for solving any linear system [8].

### 3. AOR Iterative Method

Refer to a linear system in equation (16), iterative methods are chosen which is AOR iterative method is considered as linear solver to solve the linear system. By referring the Young [8], Hackbush [9] and Saad [10] study, iterative methods the best way to solve the linear system which have the coefficient matrix, \( A \) is large scale and sparse. Let the coefficient matrix, \( A \) be defined as

\[ A = L + D + V \quad (17) \]

where \( D \) is diagonal matrix of matrix \( A \) and \( L \) is strictly lower matrix and \( V \) is strictly upper matrix respectively. Then substitute equation (17) into equation (16), we have the following linear system as

\[ (L + D + V)J = M \quad (18) \]

In 1978, AOR iterative method was introduced by Hadjidimos [14] as two-parameter generalization of the SOR method. According to equation (18), the general form of AOR iterative method can be defined as

\[ J^{r+1} = (1 - \omega_1)^{-1} [(1 - \omega_2)D + (\omega_2 - \omega_1)L + \omega_2V]J^r + \omega_2(D - \omega_1L)^{-1}M \quad (19) \]

where \( J^{r+1} \) represent as an unknown vector at \( r^{th} \) iteration, while \( \omega_1 \) and \( \omega_2 \) are overrelaxation parameter and accelerated parameter respectively. The AOR iterative method can be reduced into another iterative methods if the parameters \( \omega_1 \) and \( \omega_2 \) are set up in specific value [14,15]. For example, when \( (\omega_2, \omega_1) \) takes \( (1, 0), (1, 1), (\omega_2, 0) \) or \( (\omega_1, \omega_1) \), the AOR will reduces into Jacobi, Gauss Seidel, Simultaneous Overrelaxation and Successive Overrelaxation respectively.

### 4. Numerical Experiment

With attention to analyze the performance of the proposed iterative method, two examples problems are taken from the two-point boundary value problems. The number of iterations (Iter), computational time in second (Time) and maximum absolute error (Error) are considered as parameter of comparision to investigate the iterative methods. The value of tolerance error set up as constant which is \( \epsilon = 10^{-10} \) at different sizes grid.

#### i. Problem 1 [16]

Consider the problem was given as

\[ y'' - y' = -e^{(s-1)-1}, \quad s \in [0,1] \quad (20) \]
Then, the analytical solution for this problem as follows

\[ y(s) = s \left( 1 - e^{(s-1)} \right), \quad s \in [0,1] \]

\[ y'' - 4y = \cosh(1), \quad s \in [0,1] \]  

(21)

Then, the analytical solution for this problem as follows

\[ y(s) = \cosh(2s - 1) - \cosh(1), \quad s \in [0,1] \]

Table 1 shows that the numerical results for the proposed problems in equations (20) and (21) were obtained after the proposed iterative methods have been tested at different sizes grid. The reduction percentage of SOR and AOR iterative methods are obtained in Table 2. The GS iterative method is set up as a controlling method for SOR and AOR iterative methods.

| \( M \) | Method | Problem 1 | Problem 2 |
|---|---|---|---|
| | | Iter | Time(second) | Error | Iter | Time(second) | Error |
| 1024 | GS | 1025490 | 109.60 | 1.03e-05 | 848604 | 96.58 | 7.44e-06 |
| | SOR | 2946 | 0.57 | 3.05e-08 | 2613 | 0.54 | 1.35e-07 |
| | AOR | 2946 | 0.56 | 3.02e-08 | 2611 | 0.53 | 1.35e-07 |
| 2048 | GS | 3527433 | 501.08 | 4.14e-05 | 2975185 | 466.09 | 3.02e-05 |
| | SOR | 5792 | 1.14 | 1.49e-08 | 5218 | 1.33 | 3.44e-08 |
| | AOR | 5792 | 1.10 | 1.54e-08 | 5214 | 1.11 | 3.44e-08 |
| 4096 | GS | 11811520 | 3214.48 | 1.66e-04 | 10223821 | 2999.24 | 1.21e-04 |
| | SOR | 10245 | 2.78 | 9.42e-08 | 9886 | 3.55 | 2.66e-08 |
| | AOR | 10199 | 2.59 | 1.03e-07 | 9881 | 2.94 | 2.81e-08 |
| 8192 | GS | 38052999 | 15288.39 | 6.63e-04 | 34187618 | 15930.80 | 4.84e-04 |
| | SOR | 19073 | 8.41 | 1.77e-07 | 17413 | 9.54 | 2.16e-07 |
| | AOR | 19036 | 7.69 | 1.82e-07 | 17379 | 8.99 | 2.34e-07 |
| 16384 | GS | 115439220 | 58347.49 | 2.65e-03 | 109919813 | 57115.58 | 1.94e-03 |
| | SOR | 36021 | 27.47 | 2.90e-07 | 36776 | 33.21 | 3.63e-07 |
| | AOR | 35740 | 24.71 | 3.61e-07 | 36776 | 32.20 | 3.63e-07 |

Table 2. Reduction percentage for the SOR and AOR iterative methods.

| Problem | Iter | SOR | AOR |
|---|---|---|---|
| 1 | 99.71-99.97% | 99.71-99.97% |
| 2 | 99.69-99.97% | 99.69-99.97% |

Based on the numerical result recorded in Table 1, clearly, it can be concluded that AOR has slightly more efficient than SOR iterative method but both of these method efficient than GS iterative method in term of number of iteration. Similar to computational time, AOR iterative methods faster than the other iterative methods has been considered. From the Table 2, the reduction percentage of AOR iterative method higher than SOR iterative method which is GS iterative method acts as control method.

5. Conclusion
This paper deals with the cubic B-spline approach to generate the cubic B-spline approximation equation. The approximate solution will be constructing the linear system which has been solved via
three iterative methods. The GS, SOR and AOR methods are chosen iterative methods to solve two-point boundary value problems. Throughout the numerical implementation of the proposed iterative methods are recorded in Table 1, it can be pointed out that number of iterations for SOR and AOR iterative method has seen drastically reduces as compared with GS iterative method. In term computational time, AOR iterative method is the most faster than the other iterative methods are considered. For the further work, the proposed scheme of this paper can be extended into the high order problems such as fourth-order and sixth-order two-point boundary value problems [17,18].

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