Cyclic Proofs, Hypersequents, and Transitive Closure Logic

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Abstract
We propose a cut-free cyclic system for Transitive Closure Logic (TCL) based on a form of hypersequents, suitable for automated reasoning via proof search. We show that previously proposed sequent systems are cut-free incomplete for basic validities from Kleene Algebra (KA) and Propositional Dynamic Logic (PDL), over standard translations. On the other hand, our system faithfully simulates known cyclic systems for KA and PDL, thereby inheriting their completeness results. A peculiarity of our system is its richer correctness criterion, exhibiting ‘alternating traces’ and necessitating a more intricate soundness argument than for traditional cyclic proofs.

1 Introduction

Transitive Closure Logic (TCL) is the extension of first-order logic by an operator computing the transitive closure of definable binary relations. It has been studied by numerous authors, e.g. [16, 15, 14], and in particular has been proposed as a foundation for the mechanisation and automation of mathematics [1].

Recently, Cohen and Rowe have proposed non-wellfounded and cyclic systems for TCL [8][10]. These systems differ from usual ones by allowing proofs to be infinite (finitely branching) trees, rather than finite ones, under some appropriate global correctness condition (the ‘progressing criterion’). One particular feature of the cyclic approach to proof theory is the facilitation of automation, since complexity of inductive invariants is effectively traded off for a richer proof structure. In fact this trade off has recently been made formal, cf. [2][11], and has led to successful applications to automated reasoning, e.g. [6][5][23][26][27].

In this work we investigate the capacity of cyclic systems to automate reasoning in TCL. Our starting point is the demonstration of a key shortfall of Cohen and Rowe’s system: its cut-free fragment, here called TC_G, is unable to cyclically prove even standard theorems of relational algebra, e.g. \((a \cup b)^* = a^* (ba^*)^*\) and \((aa \cup aba)^+ \leq a^+ ((ba^+ \cup a)^+)\) (Thm. 5.5). An immediate consequence of this is that cyclic proofs of TC_G do not enjoy cut-admissibility (Cor. 5.12). On the other hand, these (in)equations are theorems of Kleene Algebra (KA) [17][15], a
decidable theory which admits automation-via-proof-search thanks to the recent cyclic system of Das and Pous [12].

What is more, TCL is well-known to interpret Propositional Dynamic Logic (PDL), a modal logic whose modalities are just terms of KA, by a natural extension of the ‘standard translation’ from (multi)modal logic to first-order logic (see, e.g., [4, 3]). Incompleteness of cyclic-TC for PDL over this translation is inherited from its incompleteness for KA. This is in stark contrast to the situation for modal logics without fixed points: the standard translation from K (and, indeed, all logics in the ‘modal cube’) to first-order logic actually lifts to cut-free proofs for a wide range of modal logic systems, cf. [21, 20].

A closer inspection of the systems for KA and PDL reveals the stumbling block to any simulation: these systems implicitly conduct a form of ‘deep inference’, by essentially reasoning underneath $\exists$ and $\land$. Inspired by this observation, we propose a form of hypersequents for predicate logic, with extra structure admitting the deep reasoning required. We present the cut-free system HTC and a novel notion of cyclic proof for these hypersequents. In particular, the incorporation of some deep inference at the level of the rules necessitates an ‘alternating’ trace condition corresponding to alternation in automata theory.

Our first main result is the Soundness Theorem (Thm. 5.1): non-wellfounded proofs of HTC are sound for standard semantics. The proof is rather more involved than usual soundness arguments in cyclic proof theory, due to the richer structure of hypersequents and the corresponding progress criterion. Our second main result is the Simulation Theorem (Thm. 6.1): HTC is complete for PDL over the standard translation, by simulating a cut-free cyclic system for the latter. This result can be seen as a formal interpretation of cyclic modal proof theory within cyclic predicate proof theory, in the spirit of [21, 20].

To simplify the exposition, we shall mostly focus on equality-free TCL and ‘identity-free’ PDL in this extended abstract, though all our results hold also for the ‘reflexive’ extensions of both logics. We discuss these extensions in Sec. 7 and present further insights and conclusions in Sec. 8.

2 Preliminaries

We shall work with a fixed first-order vocabulary consisting of a countable set $Pr$ of unary predicate symbols, written $p, q$, etc., and of a countable set $Rel$ of binary relation symbols, written $a, b$, etc. We build formulas from this language differently in the modal and predicate settings, but all our formulas may be formally evaluated within structures:

Definition 2.1 (Structures). A structure $\mathcal{M}$ consists of a set $D$, called the domain of $\mathcal{M}$, which we sometimes denote by $|\mathcal{M}|$; a subset $p^\mathcal{M} \subseteq D$ for each $p \in Pr$; and a subset $a^\mathcal{M} \subseteq D \times D$ for each $a \in Rel$.

As above, we shall generally distinguish the words ‘predicate’ (unary) and ‘relation’ (binary). We could include further relational symbols too, of higher arity, but choose not to in order to calibrate the semantics of both our modal and predicate settings.


2.1 Transitive Closure Logic

In addition to the language introduced at the beginning of this section, in the predicate setting we further make use of a countable set of function symbols, written $f^i, g^j$, etc., where the superscripts $i, j \in \mathbb{N}$ indicate the arity of the function symbol and may be omitted when it is not ambiguous. Nullary function symbols (aka constant symbols), are written $c, d$ etc. We shall also make use of variables, written $x, y$, etc., typically bound by quantifiers. Terms, written $s, t$, etc., are generated as usual from variables and function symbols by function application. A term is closed if it has no variables.

We consider the usual syntax for first-order logic formulas over our language, with an additional operator for transitive closure (and its dual). Formally, TCL formulas, written $A, B$, etc., are defined in a way that is compatible with both notions of formula.

We are using the same metavariables $A, B$, etc. to vary over both PDL+ and TCL formulas. This should never cause confusion due to the context in which they appear. Moreover, this coincidence is suggestive, since many notions we consider, such as duality and satisfaction, are defined in a way that is compatible with both notions of formula.

Definition 2.3 (Duality). For a formula $A$ we define its complement, $\overline{A}$, by:

\[
\begin{align*}
\overline{\overline{A}} & := A \\
\overline{A} & := \forall x \overline{A} \\
\overline{x}A & := \exists x A \\
\overline{TC(A)(s)} & := \overline{TC(A)}(s, t)
\end{align*}
\]

We shall employ standard logical abbreviations, e.g. $A \supset B$ for $\overline{A} \lor B$. We may evaluate formulas with respect to a structure, but we need additional data for interpreting function symbols:

Definition 2.4 (Interpreting function symbols). Let $\mathcal{M}$ be a structure with domain $D$. An interpretation is a map $\rho$ that assigns to each function symbol $f^n$ a function $D^n \to D$. We may extend any interpretation $\rho$ to an action on (closed) terms by setting recursively $\rho(f(t_1, \ldots, t_n)) := \rho(f(\rho(t_1), \ldots, \rho(t_n)))$.

We only consider standard semantics in this work: $TC$ (and $\overline{TC}$) is always interpreted as the real transitive closure (and its dual) in a structure, rather than being axiomatised by some induction (and coinduction) principle.

In order to facilitate the formal definition of satisfaction, namely for the quantifier and reflexive transitive closure cases, we shall adopt a standard convention of assuming among our constant symbols arbitrary parameters from the model $\mathcal{M}$. Formally this means that we construe each $v \in D$ as a constant symbol for which we shall always set $\rho(v) = v$. 

3
Definition 2.5 (Semantics). Given a structure \( \mathcal{M} \) with domain \( D \) and an interpretation \( \rho \), the judgement \( \mathcal{M}, \rho \models A \) is defined as follows:

- \( \mathcal{M}, \rho \models p(t) \) if \( \rho(t) \in p^{\mathcal{M}} \).
- \( \mathcal{M}, \rho \models \lnot p(t) \) if \( \rho(t) \notin p^{\mathcal{M}} \).
- \( \mathcal{M}, \rho \models a(s, t) \) if \( (\rho(s), \rho(t)) \in a^{\mathcal{M}} \).
- \( \mathcal{M}, \rho \models \lnot a(s, t) \) if \( (\rho(s), \rho(t)) \notin a^{\mathcal{M}} \).
- \( \mathcal{M}, \rho \models A \land B \) if \( \mathcal{M}, \rho \models A \) and \( \mathcal{M}, \rho \models B \).
- \( \mathcal{M}, \rho \models A \lor B \) if \( \mathcal{M}, \rho \models A \) or \( \mathcal{M}, \rho \models B \).
- \( \mathcal{M}, \rho \models \forall x A \) if, for every \( v \in D \), we have \( \mathcal{M}, \rho \models A[v/x] \).
- \( \mathcal{M}, \rho \models \exists x A \) if, for some \( v \in D \), we have \( \mathcal{M}, \rho \models A[d/x] \).

\( \mathcal{M}, \rho \models \text{TC}(A)(s, t) \) if there are \( v_0, \ldots, v_{n+1} \in D \) with \( \rho(s) = v_0, \rho(t) = v_{n+1} \), such that for every \( i \leq n \) we have \( \mathcal{M}, \rho \models A(v_i, v_{i+1}) \).

\( \mathcal{M}, \rho \models \overline{\text{TC}}(A)(s, t) \) if for all \( v_0, \ldots, v_{n+1} \in D \) with \( \rho(s) = v_0 \) and \( \rho(t) = v_{n+1} \), there is some \( i \leq n \) such that \( \mathcal{M}, \rho \models A(v_i, v_{i+1}) \).

If \( \mathcal{M}, \rho \models A \) for all \( \mathcal{M} \) and \( \rho \), we simply write \( \models A \).

Remark 2.6 (\( \text{TC} \) and \( \overline{\text{TC}} \) as least and greatest fixed points). As expected, we have \( \mathcal{M}, \rho \not\models \text{TC}(A)(s, t) \) just if \( \mathcal{M}, \rho \models \overline{\text{TC}}(A)(s, t) \), and so the two operators are semantically dual. Thus, \( \text{TC} \) and \( \overline{\text{TC}} \) duly correspond to least and greatest fixed points, respectively, satisfying in any model:

\[
\text{TC}(A)(s, t) \iff A(s, t) \lor \exists x(A(s, x) \land \text{TC}(A)(x, t)) \tag{1}
\]

\[
\overline{\text{TC}}(A)(s, t) \iff A(s, t) \land \forall x(A(s, x) \lor \overline{\text{TC}}(A)(x, t)) \tag{2}
\]

We have included both operators as primitive so that we can reduce negation to atomic formulas, allowing a one-sided formulation of proofs.

Let us point out that our \( \overline{\text{TC}} \) operator is not the same as Cohen and Rowe’s transitive ‘co-closure’ operator \( \text{TC}^{op} \) in \cite{3}. As they already note there, \( \text{TC}^{op} \) cannot be defined in terms of \( \text{TC} \) (using negations), whereas \( \text{TC} \) is the formal De Morgan dual of \( \text{TC} \) and, in the presence of negation, are indeed interdefinable, cf. Definition 2.7.

2.2 Cohen-Rowe cyclic system for TCL

Cohen and Rowe proposed in \cite{3} a non-wellfounded sequent system for TCL, which we call \( \text{TCL}_G \), extending a standard sequent calculus \( \text{LK} \) for first-order logic with equality and substitution by rules for \( \text{TC} \) inspired by its characterisation as a least fixed point, cf. \cite{1}.

Definition 2.7 (System). A sequent, written \( \Gamma, \Delta \) etc., is a set of formulas. The rules of \( \text{TCL}_G \) are shown in Figure \cite{1}. \( \text{TCL}_G \)-preproofs are possibly infinite trees of sequents generated by the rules of \( \text{TCL}_G \). A preproof is regular if it has only finitely many distinct sub-preproofs.
\[ \frac{\text{id}}{\Gamma, p(t), \bar{p}(t)} \quad \forall c \frac{\Gamma, A_i}{\Gamma, A_0 \lor A_1} \quad \forall \frac{\Gamma, A \land B}{\Gamma, \exists x A(x)} \quad \forall \frac{\Gamma, A[c/x]}{\Gamma, \forall x A} \quad \text{c fresh} \]

\[ \frac{\text{wk}}{\Gamma, \Gamma' = \Gamma, t = t} \quad \frac{\Gamma, A(t)}{\Gamma, s \neq t, A(s)} \quad \frac{\Gamma, A(s)}{\Gamma, s \neq t, A(t)} \quad \frac{\Gamma}{\sigma(\Gamma)} \]

\[ \frac{\text{TC}_0}{\Gamma, A(s, t)} \quad \frac{\text{TC}_1}{\text{TC}(A)(s, t)} \quad \frac{\text{TC}_0}{\text{TC}(A)(r, t)} \quad \frac{\text{TC}_1}{\text{TC}(A)(s, t)} \quad \frac{\text{c fresh}}{c} \]

Figure 1: Sequent calculus $\text{TC}_G$. The first two lines of the Figure contain the rules of the Tait-style sequent system $\text{LK}_\omega$ for first-order predicate logic with equality.

In Figure 1, $\sigma$ is a map ("substitution") from constants to terms and other function symbols to function symbols of the same arity, extended to terms, formulas and sequents in the natural way. The substitution rule is redundant for usual provability, but facilitates the definition of ‘regularity’ in predicate cyclic proof theory.

The notions of non-wellfounded and cyclic proofs for $\text{TC}_G$ are formulated similarly to those for first-order logic with (ordinary) inductive definitions [7]:

Definition 2.8 (Traces and proofs). Given a $\text{TC}_G$ preproof $\mathcal{D}$ and a branch $\mathcal{B} = (\tau_i)_{i \in \omega}$ of inference steps, a trace is a sequence of formulas of the form $(\text{TC}(A)(s_i, t_i))_{i \geq k}$ such that for all $i \geq k$ either:

- $\tau_i$ is not a substitution step and $(s_{i+1}, t_{i+1}) = (s_i, t_i)$; or,
- $\tau_i$ is a $\text{TC}$ step with principal formula $\text{TC}(A)(s_i, t_i)$ and $(s_{i+1}, t_{i+1}) = (c, t_i)$, where $c$ is the eigenvariable of $\tau_i$; or,
- $\tau_i$ is a substitution step with respect to $\sigma$ and $(\sigma(s_{i+1}), \sigma(t_{i+1})) = (s_i, t_i)$.

We say that the trace is progressing if the second case above happens infinitely often. A $\text{TC}_G$-preproof $\mathcal{D}$ is a proof if each of its infinite branches has a progressing trace. If $\mathcal{D}$ is regular we call it a cyclic proof. As in the main text, we write $\text{TC}_G \vdash_{\text{cyc}} A$ if there is a cyclic proof in $\text{TC}_G$ of $A$.

Proposition 2.9 (Soundness, [8, 10]). If $\text{TC}_G \vdash_{\text{cyc}} A$ then $\models A$.

In fact, this result is subsumed by our main soundness result for $\text{HTC}$ (Thm. 5.1) and its simulation of $\text{TC}_G$ (Thm. 4.12). In the presence of cut, a form of converse of Prop. 2.9 holds: cyclic $\text{TC}_G$ proofs are ‘Henkin complete’, i.e. complete for all models of a particular axiomatisation of TCL based on (co)induction principles [8, 10]. However, the counterexample we present in the next section implies that cut is not eliminable (Cor. 3.12).
2.3 Differences to \cite{8, 10}

Our formulation of $\text{TC}_G$ differs slightly from the original presentation in \cite{8, 10}, but in no essential way. Nonetheless, let us survey these differences now.

2.3.1 One-sided vs. two-sided.

Cohen and Rowe employ a two-sided calculus as opposed to our one-sided one, but the difference is purely cosmetic. Sequents in their calculus are written $A_1, \ldots, A_m \Rightarrow B_1, \ldots, B_n$, which may be duly interpreted in our calculus as $\bar{A}_1, \ldots, \bar{A}_m, B_1, \ldots, B_n$. Indeed we may write sequents in this two-sided notation at times in order to facilitate the reading of a sequent and to distinguish left and right formulas. For this reason, Cohen and Rowe do not include a $\text{TC}$ operator in their calculus, but are able to recover it thanks to a formal negation symbol, cf. Dfn. 2.3.

2.3.2 $\text{TC}$ vs. $\text{RTC}$.

Cohen and Rowe’s system is originally called $\text{RTC}_G$, rather using a ‘reflexive’ version $\text{RTC}$ of the $\text{TC}$ operator. As they mention, this makes no difference in the presence of equality. Semantically we have $\text{RTC}(A)(s, t) \iff s = t \lor \text{TC}(A)(s, t)$, but this encoding does not lift to proofs, i.e. the $\text{RTC}$ rules of \cite{8} are not locally derived in $\text{TC}_G$ modulo this encoding. However, the encoding $\text{RTC}(A)(s, t) := \text{TC}((x = y \lor A))(s, t)$ suffices for this purpose.

2.3.3 Alternative rules and fixed point characterisations.

Cohen and Rowe use a slightly different fixed point formula to induce rules for $\text{RTC}$ and $\text{RTC}$ (i.e. $\text{RTC}$ on the left) based on the fixed point characterisation,

\[
\text{RTC}(A)(s, t) \iff s = t \lor \exists x(\text{RTC}(A)(s, x) \land A(x, t))
\]

decomposing paths ‘from the right’ rather than the left. These alternative rules induce analogous notions of trace and progress for preproofs such that progressing preproofs enjoy a similar soundness theorem, cf. Proposition 2.9.

The reason we employ a slight variation of Cohen and Rowe’s system is to remain consistent with how the rules of LPD and HTC are devised later. To the extent that we prove things about $\text{TC}_G$, namely its (cut-free) regular incompleteness in Theorem 3.5, the particular choice of rules turns out to be unimportant. The counterexample we present there is robust: it applies to systems with any (and indeed all) of the above rules.

3 Interlude: motivation from PDL and Kleene Algebra

Given the TCL sequent system proposed by Cohen and Rowe, why do we propose a hypersequential system? Our main argument is that proof search in $\text{TC}_G$ is rather weak, to the extent that cut-free cyclic proofs are unable to simulate a basic (cut-free) system for modal logic PDL (regardless of proof search strategy). At least one motivation here is to ‘lift’ the standard translation from cut-free cyclic proofs for PDL to cut-free cyclic proofs in an adequate system for TCL.
3.1 Identity-free PDL

Identity-free propositional dynamic logic (PDL\(^+\)) is a version of the modal logic PDL without tests or identity, thereby admitting an 'equality-free' standard translation into predicate logic. Formally, PDL\(^+\) formulas, written \(A, B, \ldots\), and programs, written \(\alpha, \beta, \ldots\), are generated by the following grammars:

\[
A, B ::= p | \top | A \land B | A \lor B | [\alpha]A | \langle \alpha \rangle A
\]

\[
\alpha, \beta ::= a | \alpha; \beta | \alpha \cup \beta | \alpha^+\]

We sometimes simply write \(\alpha\beta\) instead of \(\alpha; \beta\), and \((\alpha)A\) for a formula that is either \(\langle \alpha \rangle A\) or \([\alpha]A\).

**Definition 3.1** (Duality). For a formula \(A\) we define its complement, \(\bar{A}\), by:

\[
\bar{p} ::= p \quad \bar{A \lor B} ::= \bar{A} \land \bar{B} \quad \bar{[\alpha]A} ::= \langle \alpha \rangle \bar{A} \quad \bar{\langle \alpha \rangle A} ::= [\alpha] \bar{A}
\]

We evaluate PDL\(^+\) formulas using the traditional relational semantics of modal logic, by associating each program with a binary relation in a structure. Again, we only consider standard semantics:

**Definition 3.2** (Semantics). For structures \(\mathcal{M}\) with domain \(D\), elements \(v \in D\), programs \(\alpha\) and formulas \(A\), we define \(\alpha^M \subseteq D \times D\) as follows:

- \((a^M\) is already given in the specification of \(\mathcal{M}\), cf. Dfn. 2.1).
- \((\alpha; \beta)^M := \{(u, v) : \text{there is } w \in D \text{ s.t. } (u, w) \in \alpha^M \text{ and } (w, v) \in \beta^M\}\).
- \((\alpha \cup \beta)^M := \{(u, v) : (u, v) \in \alpha^M \text{ or } (u, v) \in \beta^M\}\).
- \((\alpha^+)^M := \{(u, v) : \text{there are } w_0, \ldots, w_{n+1} \in D \text{ s.t. } u = w_0, v = w_{n+1} \text{ and for every } i \leq n(w_i, w_{i+1}) \in \alpha^M\}\).

and:

- \(\mathcal{M}, v \models p\) if \(v \in p^M\).
- \(\mathcal{M}, v \models \top\) if \(v \notin p^M\).
- \(\mathcal{M}, v \models A \land B\) if \(\mathcal{M}, v \models A\) and \(\mathcal{M}, v \models B\).
- \(\mathcal{M}, v \models A \lor B\) if \(\mathcal{M}, v \models A\) or \(\mathcal{M}, v \models B\).
- \(\mathcal{M}, v \models [\alpha]A\) if \(\forall (v, w) \in \alpha^M\) we have \(\mathcal{M}, w \models A\).
- \(\mathcal{M}, v \models \langle \alpha \rangle A\) if \(\exists (v, w) \in \alpha^M\) with \(\mathcal{M}, w \models A\).

If \(\mathcal{M}, v \models A\) for all \(\mathcal{M}\) and \(v \in D\), then we write \(\models A\).

Note that we are overloading the satisfaction symbol \(\models\) here, for both PDL\(^+\) and TCL. This should never cause confusion, in particular since the two notions of satisfaction are 'compatible', given that we employ the same underlying language and structures. In fact such overloading is convenient for relating the two logics, as we shall now see.
3.2 The standard translation

The so-called “standard translation” of modal logic into predicate logic is induced by reading the semantics of modal logic as first-order formulas. We now give a natural extension of this that interprets PDL$^+$ into TCL. At the logical level our translation coincides with the usual one for basic modal logic; our translation of programs, as expected, requires the TC operator to interpret the $+$ of PDL$^+$.

Definition 3.3. For a PDL$^+$ formula $A$ and program $\alpha$, we define the standard translations $\text{ST}(A)(x)$ and $\text{ST}(\alpha)(x, y)$ as TCL-formulas with free variables $x$ and $x, y$, resp., mutually inductively as follows,

$$\begin{align*}
\text{ST}(p)(x) & := p(x) \\
\text{ST}(\overline{p})(x) & := \overline{p}(x) \\
\text{ST}(A \lor B)(x) & := \text{ST}(A)(x) \lor \text{ST}(B)(x) \\
\text{ST}(A \land B)(x) & := \text{ST}(A)(x) \land \text{ST}(B)(x) \\
\text{ST}(\alpha)(x, y) & := \text{ST}(\alpha)(x, y) \\
\text{ST}(\overline{\alpha})(x, y) & := \overline{\text{ST}(\alpha)(x, y)} \\
\text{ST}((\alpha; \beta))(x, y) & := \exists x \exists y (\text{ST}(\alpha)(x, z) \land \text{ST}(\beta)(z, y)) \\
\text{ST}(\lambda x. \alpha)(x) & := \forall y (\text{ST}(\alpha)(x, y) \lor \text{ST}(A)(y)) \\
\text{ST}((\alpha; \beta))(x) & := \lambda x, y. \text{ST}(\alpha)(x, y) \lor \text{ST}(A)(y)
\end{align*}$$

where we have written simply $TC(\text{ST}(\alpha))$ instead of $TC(\lambda x. y. \text{ST}(\alpha)(x, y))$.

It is routine to show that $\text{ST}(A)(x) = \text{ST}(\overline{A})(x)$, by structural induction on $A$, justifying our overloading of the notation $\overline{A}$, in both TCL and PDL$^+$. Yet another advantage of using the same underlying language for both the modal and predicate settings is that we can state the following (expected) result without the need for encodings, following by a routine structural induction (see, e.g., [3]):

Theorem 3.4. For PDL$^+$ formulas $A$, we have $\mathcal{M}, v \models A$ iff $\mathcal{M} \models \text{ST}(A)(v)$.

3.3 Cohen-Rowe system is not complete for PDL$^+$

PDL$^+$ admits a standard cut-free cyclic proof system LPD$^+$ (see Sec. 6.1) which is both sound and complete (cf. Thm. 6.4). However, a shortfall of TC$^*_G$ is that it is unable to cut-free simulate LPD$^+$. In fact, we can say something stronger:

Theorem 3.5 (Incompleteness). There exist a PDL$^+$ formula $A$ such that $\models A$ but $\not\models_{cyc} \text{ST}(A)(x)$ (in the absence of cut).

This means not only that TC$^*_G$ is unable to locally cut-free simulate the rules of LPD$^+$, but also that there are some validities for which there are no cut-free cyclic proofs at all in TC$^*_G$. One example of such a formula is:

$$((aa \cup aba)^+ \cup (ba)^+ \cup a)p \supset (a^+((ba)^+ \cup a))p$$

(4)

This formula is derived from the well-known PDL validity $((a \cup b)^*)p \supset (a^+(ba)^*)p$ by identity-elimination. This in turn is essentially a theorem of relational algebra, namely $(a \cup b)^* \leq a^+(ba)^*$, which is often used to eliminate $\cup$ in (sums of) regular expressions. The same equation was (one of those) used by Das and Pous in [12] to show that the sequent system LKA for Kleene Algebra is cut-free cyclic incomplete.

In the remainder of this Section, we shall give a proof of Thm. 3.5. The argument is much more involved than the one from [12], due to the fact we are
working in predicate logic, but the underlying basic idea is similar. At a very high level, the RHS of \( \text{(4)} \) (viewed as a relational inequality) is translated to an existential formula
\[
\exists z \left( \text{ST}(a^+)(x, z) \land \text{ST}((ba^+) + \cup a)(z, y) \right)
\]
that, along some branch (namely the one that always chooses \( aa \) when decomposing the LHS of \( \text{(4)} \)) can never be instantiated while remaining valid. This branch witnesses the non-regularity of any proof.

### 3.3.1 Some closure properties for cyclic proofs.

Demonstrating that certain formulas do not have (cut-free) cyclic proofs is a delicate task, made more so by the lack of a suitable model-theoretic account (indeed, cf. Corollary 3.12). In order to do so formally, we first develop some closure properties of cut-free cyclic provability.

**Proposition 3.6 (Inversions).** We have the following:

1. If \( \text{TC}_G \vdash_{cyc} \Gamma, A \lor B \) then \( \text{TC}_G \vdash_{cyc} \Gamma, A, B \).
2. If \( \text{TC}_G \vdash_{cyc} \Gamma, A \land B \) then \( \text{TC}_G \vdash_{cyc} \Gamma, A \) and \( \text{TC}_G \vdash_{cyc} \Gamma, B \).
3. If \( \text{TC}_G \vdash_{cyc} \Gamma, \forall x A(x) \) then \( \text{TC}_G \vdash_{cyc} \Gamma, A(c) \), as long as \( c \) is fresh.

**sketch.** All three statements are proved similarly.

For Item 1, replace every direct ancestor of \( A \lor B \) with \( A, B \). The only critical steps are when \( A \lor B \) is principal, in which case we delete the step, or is weakened, in which case we apply two weakenings, one on \( A \) and one on \( B \). If the starting proof had only finitely many distinct subproofs (up to substitution), say \( n \), then the one obtained by this procedure has at most \( 2^n \) distinct subproofs (up to substitution), since we simulate a weakening on \( A \lor B \) by two weakenings.

For Item 2, replace every direct ancestor of \( A \land B \) with \( A \) or \( B \), respectively. The only critical steps are when \( A \land B \) is principal, in which case we delete the step and take the left or right subproof, respectively, or is weakened, in which case we simply apply a weakening on \( A \) or \( B \), respectively. The proof we obtain has at most the same number of distinct subproofs (up to substitution) as the original one.

For Item 3, replace every direct ancestor of \( \forall x A(x) \) with \( A(c) \). The only critical steps are when \( \forall x A(x) \) is principal, in which case we delete the step and rename the eigenvariable in the remaining subproof everywhere with \( c \), or is weakened, in which case we simply apply a weakening on \( A(c) \). The proof we obtain has at most the same number of distinct subproofs (up to substitution) as the original one.

**Proposition 3.7 (Predicate admissibility).** Suppose \( \text{TC}_G \vdash_{cyc} \Gamma, p(t) \) or \( \text{TC}_G \vdash_{cyc} \Gamma, \bar{p}(t) \), where \( \bar{p} \) or \( p \) (respectively) does not occur in \( \Gamma \). Then it holds that \( \text{TC}_G \vdash_{cyc} \Gamma \).

**sketch.** Delete every ancestor of \( p(t) \) or \( \bar{p}(t) \), respectively. The only critical case is when one of the formulas is weakened, in which case we omit the step. Note that there cannot be any identity on \( p \), due to the assumption on \( \Gamma \), and by the subformula property.
3.3.2 Reducing to a relational tautology.

Here, and for the remainder of this section, we shall simply construe PDL+ programs $\alpha$ and formulas $A$ as TCL formulas with two free variables and one free variable, respectively, by identifying them with their standard translations $ST(\alpha)(x, y)$ and $ST(A)(x)$, respectively. This modest abuse of notation will help suppress much of the notation in what follows.

**Lemma 3.8.** If $TC_G \vdash cyc (((aa \cup aba)^+)p \supset (a^+((ba)^+ \cup a))p(c)$ then also $TC_G \vdash cyc (aa \cup aba)^+(c, d) \supset (a^+((ba)^+ \cup a))(c, d)$.

**Sketch.** Suppose $TC_G \vdash cyc (((aa \cup aba)^+)p \supset (a^+((ba)^+ \cup a))p(c)$ so, by unwinding the definition of $ST$ and since duality commutes with the standard translation, cf. Sec. 3.2, we have that $TC_G \vdash cyc (((aa \cup aba)^+)\overline{p})(c) \lor (a^+((ba)^+ \cup a))p(c)$. By $\lor$-inversion (Prop. 3.6.1,1) we have:

$$TC_G \vdash cyc (((aa \cup aba)^+)\overline{p})(c), ((a^+((ba)^+ \cup a))p(c)$$

Again unwinding the definition of $ST$, and by the definition of duality, we thus have:

$$TC_G \vdash cyc \forall x((aa \cup aba)^+\overline{p}(c, x)) \lor \overline{p}(x)), \exists y((a^+((ba)^+ \cup a)(c, y) \land p(y))$$

Now, by $\lor$-inversion and $\lor$-inversion, Prop. 3.6.11 we have:

$$TC_G \vdash cyc ((aa \cup aba)^+\overline{p}(c, d), \overline{p}(d), \exists y((a^+((ba)^+ \cup a)(c, y) \land p(y))$$

Without loss of generality we may instantiate the $\exists y$ by $d$ and so by $\land$-inversion, Prop. 3.6.11 we have:

$$TC_G \vdash cyc (aa \cup aba)^+\overline{p}(c, d), \overline{p}(d), a^+((ba)^+ \cup a)(c, d)$$

Since there is no occurrence of $p$ above, by Prop. 3.7 we conclude

$$TC_G \vdash cyc (aa \cup aba)^+\overline{p}(c, d), a^+((ba)^+ \cup a)(c, d)$$

as required.

\[\square\]

3.3.3 Irregularity via an adversarial model.

In the previous subsubsection we reduced the incompleteness of cut-free cyclic sequent proofs for TCL over the image of the standard translation on PDL+ to the non-regular cut-free provability of a particular relational validity. Unwinding this a little, the sequent that we shall show has no (cut-free) cyclic proof in $TC_G$ can be written in ‘two-sided notation’ as follows:

$$TC(aa \lor aba)(c, d) \Rightarrow \exists z(a^+(c, z) \land ((ba)^+ \cup a)(z, d)) \quad (5)$$

This two-sided presentation is simply a notational variant that allows us to more easily reason about the proof search space (e.g. referring to ‘LHS’ and ‘RHS’). Formally:

**Remark 3.9** (Two-sided notation). We may write $\Gamma \Rightarrow \Delta$ as shorthand for the sequent $\Gamma, \Delta$, where $\Gamma = \{ A : A \in \Gamma \}$. References to the ‘left-hand side (LHS)’ and ‘right-hand side (RHS)’ have the obvious meaning, always with respect to the delimiter $\Rightarrow$. 

10
To facilitate our argument, we shall only distinguish sequents ‘modulo substitution’ rather than allowing explicit substitution steps when reasoning about (ir)regularity of a proof.

We shall design a family of ‘adversarial’ models, and instantiate proof search to just these models. In this way, we shall show that any non-wellfounded TC\textsubscript{G} proof of the sequent \([5]\) must have arbitrarily long branches without a repetition (up to substitution). Since TC\textsubscript{G} is finitely branching, by König’s Lemma this means that any non-wellfounded TC\textsubscript{G} proof of \([5]\) has an infinite branch with no repetitions (up to substitution), as required.

**Definition 3.10** (An adversarial model). For \(n \in \mathbb{N}\), define the structure \(A_n\) as follows:

- The domain of \(A_n\) is \(\{u_0, u'_0, \ldots, u'_{n-1}, u_n, v\}\).
- \(a^{A_n} = \{(u_i, u'_i), (u'_i, u_{i+1})\}_{i<n}\).
- \(b^{A_n} = \{(u_n, v)\}\).

Note that, since the sequent \([5]\) that we are considering is purely relational, it does not matter what sets \(A_n\) assigns to the predicate symbols.

**Lemma 3.11.** Let \(n \in \mathbb{N}\). Any proof \(D\) of \([5]\) has a branch with no repetitions (up to substitutions) among its first \(n\) sequents.

**Proof.** Set \(c_0 = c\). Consider some (possibly finite, but maximal) branch \(B = (r_i)_{i \leq \nu}\) (with \(\nu \leq \omega\)) of \(D\) satisfying:

- whenever TC on the LHS is principal, the right premiss is followed; and,
- whenever \((aa)(s, t)\lor(aba)(s, t)\) is principal (for any \(s\) and \(t\)) the left premiss (corresponding to \((aa)(s, t)\)) is followed.

Let \(k \leq n\) be maximal such that, for each \(i \leq k\), \(r_i\) has principal formula on the LHS. Now:

1. For \(i \leq k\), each \(r_i\) has conclusion with LHS of the form:
   \[
   \Gamma_j(c_{j-1}, c_j), \ TC(aa \lor aba)(c_l, d)
   \]  
   where \(c = c_0, \ldots, c_{l-1}\) for some \(l \leq i\) and each \(\Gamma_j(c_{j-1}, c_j)\) has the form
   \(a(c_{j-1}, c'_{j-1}), a(c'_{j-1}, c_j)\) or \(a(c_{j-1}, c'_{j-1}) \land a(c'_{j-1}, c_j)\) or \((aa)(c_{j-1}, c_j)\) or \((aa)(c_{j-1}, c_j) \lor (aba)(c_{j-1}, c_j)\) To see this, proceed by induction on \(i \leq k\):
   - The base case is immediate, by setting \(l = 0\).
   - For the inductive step, note that the principal formula of \(r_i\) must be on the LHS, since \(i \leq k\). Thus by the inductive hypothesis the principal formula of \(r_i\) must have the form:
     - \(a(c_{j-1}, c'_{j}) \land a(c'_{j}, c_j)\) (on the LHS), in which case the premiss of \(r_i\) (which is a left-\(\land\) step) replaces it by \(a(c_{j-1}, c'_{j}), a(c'_{j}, c_j)\);
     - \((aa)(c_{j-1}, c_j)\) (on the LHS), in which case the premiss of \(r_i\) (which is a left-\(\lor\) step) replaces it by \(a(c_{j-1}, c'_{j}) \land a(c'_{j}, c_j)\);
     - \((aa)(c_{j-1}, c_j) \lor (aba)(c_{j-1}, c_j)\) (on the LHS), in which case, by definition of \(B\), the \(\mathcal{B}\)-premiss of \(r_i\) (which is a left-\(\lor\) step) replaces this formula by some \(a(c_{j-1}, c_j)\); or,
2. Moreover, for $i < j \leq k$, the conclusion of $r_i$ and $r_j$ are not equal (up to substitution). To see this, note that any rule principal on an LHS of form in (6) either decreases the size of some $\Gamma_j(c_{j-1}, c_j)$ (when it is a left $\land$, $\exists$ or $\lor$ step) or increases the number of eigenvariables in the sequent (when it is a left TC step), in particular the $l$ such that $TC(aa \lor aba)(c_l, d)$ appears.

3. Since proofs must be sound for all models (by soundness), we shall work in $A_n$ with respect to an interpretation $\rho_n$ satisfying $c_i \mapsto u_i$ for $i \leq n$ and $c_i' \mapsto u_i'$ for $i < n$ and $d \mapsto v$. It follows by inspection of (6) that, for $i \leq k$, each formula on the LHS of the conclusion of $r_i$ is true in $(A_n, \rho_n)$.

4. Along $B$, the RHS cannot be principal until $l = n$ in (6), so in particular $k \geq n$. To see this:

   - Recall that the interpretation $\rho_n$ assigns to $c_0, c_0', \ldots, c_{n-1}, c_n$ the worlds $u_0, u_0', \ldots, u_{n-1}, u_n$ respectively.
   - If the existential formula on the RHS is instantiated by some $c_i$ with $i \neq n$ or $c_i'$ with $i < n$ then the resulting sequent is false in $(A_n, \rho_n)$ (recall that, by Item 3, every formula on the LHS is true, so we require the RHS to be true too). To see this, note that the RHS in particular would imply $(ba^+)(c_i, d)$ or $a(c_i, d)$ or $(ba^+)^+(c_i', d)$ or $a(c_i', d)$. However when $i \leq n$, or $i < n$ respectively, this is not true with respect to $(A_n, \rho_n)$.

   By Item 3 we have that $k \geq n$ and so $k = n$. Thus, by Item 1 and Item 2 there are no repeated sequents (up to substitution) in $(r_i)_{i \leq n}$, as required.

### 3.3.4 Putting it all together.

We are now ready to give the proof of the main result of this section.

**Proof of Thm. 3.5 sketch.** Since the choice of $n$ in Lemma 3.11 was arbitrary, any $TC_G$ proof $D$ of (5) must have branches with arbitrarily long initial segments without any repetition (up to substitution). Since the system is finitely branching, by König’s Lemma we have that there is an infinite branch through $D$ without any repetition (up to substitution), and thus $D$ is not regular. Thus $TC_G \nvdash cyc(5)$. Finally, by contraposition of Lemma 3.8 we have, as required:

$$TC_G \nvdash cyc(\langle ((aa \lor aba)^+)^+ p \supset \langle a^+ ((ba^+)^+ \cup a)\rangle p \rangle(c))$$

An immediate consequence of Thm. 3.5 is:

**Corollary 3.12.** The class of cyclic proofs of $TC_G$ does not enjoy cut-admissibility.
4 Hypersequent calculus for TCL

Let us take a moment to examine why any ‘local’ simulation of $\text{LPD}^+$ by $\text{TC}_G$ fails, in order to motivate the main system that we shall present. The program rules, in particular the $\langle \rangle$-rules, require a form of deep inference to be correctly simulated, over the standard translation. For instance, let us consider the action of the standard translation on two rules we shall see later in $\text{LPD}^+$ (cf. Sec. 4.1):

\[
\begin{align*}
\langle \rangle & \quad \Gamma, \langle a \rangle \langle \rangle p & \quad \rightsquigarrow & \quad ST(\langle \rangle, \exists x(a(x) \land p(x))) \\
\langle \rangle & \quad \Gamma, \langle a \rangle \langle a \rangle p & \quad \rightsquigarrow & \quad ST(\langle \rangle, \exists y(a(y) \land \exists x(b(y, x) \land p(x)))) \\
\langle \rangle & \quad \Gamma, \langle a \rangle \langle b \rangle p & \quad \rightsquigarrow & \quad ST(\langle \rangle, \exists x(a(x) \land \exists y(b(x, y) \land p(x))))
\end{align*}
\]

The first case above suggests that any system to which the standard translation lifts must be able to reason underneath $\exists$ and $\wedge$, so that the inference indicated in blue is ‘accessible’ to the prover. The second case above suggests that the existential-conjunctive meta-structure necessitated by the first case should admit basic equivalences, in particular certain prenexing. This section is devoted to the incorporation of these ideas (and necessities) into a bona fide proof system.

4.1 Annotated hypersequents

An annotated cedent, or simply cedent, written $S, S'$ etc., is an expression $\{\Gamma\}^x$, where $\Gamma$ is a set of formulas and the annotation $x$ is a set of variables. We sometimes construe annotations as lists rather than sets when it is convenient, e.g. when taking them as inputs to a function.

Each cedent may be intuitively read as a TCL formula, under the following interpretation: $\text{fm}(\{\Gamma\}^x_1, \ldots, \{\Gamma_n\}^x_n) := \exists x_1 \ldots \exists x_n \wedge \Gamma$. When $x = \emptyset$ then there are no existential quantifiers above, and when $\Gamma = \emptyset$ we simply identify $\wedge \Gamma$ with $\top$. We also sometimes write simply $A$ for the annotated cedent $\{A\}^\emptyset$.

A hypersequent, written $S, S'$ etc., is a set of annotated cedents. Each hypersequent may be intuitively read as the disjunction of its cedents. Namely we set: $\text{fm}(\{\Gamma_1\}^{x_1}, \ldots, \{\Gamma_n\}^{x_n}) := \text{fm}(\{\Gamma_1\}^{x_1}) \lor \ldots \lor \text{fm}(\{\Gamma_n\}^{x_n})$.

4.2 Non-wellfounded hypersequent proofs

We now present our hypersequential system for TCL and its corresponding notion of ‘non-wellfounded proof’.

**Definition 4.1** (System). The rules of $\text{HTC}$ are given in Fig. 2. A $\text{HTC}$ preproof is a (possibly infinite) derivation tree generated by the rules of $\text{HTC}$. A preproof is regular if it has only finitely many distinct subproofs.

**Remark 4.2** (Herbrand constants). Our rules for $\text{TC}$ and $\text{TCl}$ are induced by the characterisation of $\text{TC}$ as a least fixed point in $\mathbb{H}$. Note that the rules $\text{TCl}$ and $\forall$ introduce, bottom-up, the fresh function symbol $f$, which plays the role of the Herbrand function of the corresponding $\forall$ quantifier: just as $\forall x \exists y A(x)$ is equisatisfiable with $\forall x A(f(x))$, when $f$ is fresh, by Skolemisation, by duality $\exists x \forall y A(x)$ is equivalent with $\exists x A(f(x))$, when $f$ is fresh, by Herbrandisation. Note that the usual $\forall$ rule of the sequent calculus is just a special case of this, when $x = \emptyset$, and so $f$ is a constant symbol.
Figure 2: Hypersequent calculus HTC. $\sigma$ is a ‘substitution’ map from constants to terms and a renaming of other function symbols and variables.

Our notion of ancestry, as compared to traditional sequent systems, must account for the richer structure of hypersequents. Informally, referring to Figure 2, a formula $C$ in the premiss is an immediate ancestor of a formula $C'$ in the conclusion if they have the same colour; if $C, C' \in \Gamma$ then we further require $C = C'$, and if $C, C'$ occur in $S$ then $C = C'$ occur in the same cedent. A cedent $S$ in the premiss is an immediate ancestor of a cedent $S'$ in the conclusion if some formula in $S$ is an immediate ancestor of some formula in $S'$. The following definitions formally explain these notions independently of of the colouring in Figure 2.

**Definition 4.3** (Ancestry for cedents). Fix an inference step $r$, as typeset in Fig. 2. We say that a cedent $S$ in a premiss of $r$ is an immediate ancestor of a cedent $S'$ in the conclusion of $r$ if either:

- $S = S' \in S$, i.e. $S$ and $S'$ are identical ‘side’ cedents of $r$; or,
- $r \neq \text{id}$, and $S$ is the (unique) cedent indicated in the conclusion of $r$, and $S'$ is a cedent indicated in a premiss of $r$; or,
- $r = \text{id}$ and $S$ is the (unique) cedent indicated in the premiss of $\text{id}$ and $S'$ is the cedent $\{\Gamma, A\}^\times$ indicated in the conclusion of $\text{id}$.

Note in particular that in $\text{id}$, as typeset in Figure 2 \{\Gamma\}^\times is not an immediate ancestor of $\{A\}^\times$.

**Definition 4.4** (Ancestry for formulas). Fix an inference step $r$, as typeset in Fig. 2. We say that a formula $S$ in a premiss of $r$ is an immediate ancestor of a formula $S'$ in the conclusion of $r$ if either:

- $F = F' \subseteq S$, i.e., $F$ and $F'$ are formulas occurring in some cedent $S \in S$, and are identical modulo substitution;
- $F = F' \in \Gamma$, i.e., $F$ and $F'$ are formulas occurring in $\Gamma$, and are identical modulo substitution; or,
• $F$ is one of the formulas explicitly indicated in the premiss of $r$ and $F'$ is the formula explicitly indicated in the conclusion of $r$.

Immediate ancestry on both formulas and cedents is a binary relation, inducing a directed graph whose paths form the basis of our correctness condition:

**Definition 4.5** ((Hyper)traces). A *hypertrace* is a maximal path in the graph of immediate ancestry on cedents. A *trace* is a maximal path in the graph of immediate ancestry on formulas.

Thus, in the id rule, as typeset in Figure 2, no (infinite) trace can include the indicated $A$ or $\bar{A}$. From the above definitions it follows that whenever a cedent $S$ in the premiss of a rule $r$ is an immediate ancestor of a cedent $S'$ in the conclusion, then some formula in $S$ is an immediate ancestor of some formula in $S'$, and vice-versa. Thus, for a hypertrace $(S_i)_{i<\omega}$, there is at least one trace $(F_i)_{i<\omega}$ which is ‘within’ or ‘along’ the hypertrace, i.e., such that $F_i \in S_i$ for all $i$.

**Definition 4.6** (Progress and proofs). Fix a preproof $D$. A (infinite) trace $(F_i)_{i<\omega}$ is *progressing* if there is $k$ such that, for all $i > k$, $F_i$ has the form $\text{TC}(A)(s_i, t_i)$ and is infinitely often principal. A (infinite) hypertrace $H$ is *progressing* if every infinite trace along it is progressing. A (infinite) branch is progressing if it has a progressing hypertrace. $D$ is a *proof* if every infinite branch is progressing. If, furthermore, $D$ is regular, we call it a *cyclic proof*.

We write $\text{HTC} \vdash_{\text{nwf}} S$ (or $\text{HTC} \vdash_{\text{cyc}} S$) if there is a proof (or cyclic proof, respectively) of $\text{HTC}$ of the hypersequent $S$.

### 4.3 Some examples

Let us consider some examples of cyclic proofs in $\text{HTC}$ and compare the system to $\text{TC}_G$.

**Example 4.7** (Fixed point identity). Here follows a cyclic proof in $\text{TC}_G$ of sequent $\{\text{TC}(a)(c, d)\}^\varnothing, \{\text{TC}(\bar{a})(c, d)\}^\varnothing$:

\[
\begin{array}{c}
\frac{\text{id} \quad a(c, d), \overline{a}(c, d)}{\text{TC} \quad a(c, d), \text{TC}(\overline{a})(c, d)}
\end{array}
\]

\[
\begin{array}{c}
\frac{\text{id} \quad a(c, e), \overline{a}(c, e) \quad \text{TC} \quad a(c, e), \text{TC}(\overline{a})(e, d), \text{TC}(\overline{a})(c, d)}{\text{TC} \quad a(c, e), \text{TC}(\overline{a})(e, d), \text{TC}(\overline{a})(c, d)}
\end{array}
\]

There is not much choice in the construction of this cyclic proof, bottom-up: we must apply $\text{TC}$ first and branch before applying $\text{TC}$ differently on each branch. This cyclic proof is naturally simulated by the following HTC one, where the progressing hypertrace is marked in blue:

\[
\begin{array}{c}
\frac{\text{id} \quad a(c, d), \overline{a}(c, d)}{\text{TC} \quad a(c, d), \text{TC}(\overline{a})(c, d)}
\end{array}
\]
we actually have some liberty in how we implement such a derivation. E.g., the previous example we may mimic that proof line by line, but we give a slightly looping on \( \bullet \), is progressing by the blue hypertrace.

The sequent \( \{ \text{TC}(a)(c, d) \}^\omega, \{ \text{TC}(a)(c, d) \}^\omega \) is finitely derivable using rule init on \( \text{TC}(a)(c, d) \) and the init rule. However, we can also cyclically reduce it to a simpler instance of init. Due to the granularity of the inference rules of \text{TC}, we actually have some liberty in how we implement such a derivation. E.g., the HTP-proof below applies \text{TC} rules below \text{TC} ones, and delays branching until the ‘end’ of proof search, which is impossible in \text{TC}_G. The only infinite branch, looping on \( \bullet \), is progressing by the blue hypertrace.

This is an example of the more general ‘rule permutations’ available in HTP, hinting at a more flexible proof theory (we discuss this further in Sec. 5).

**Example 4.8** (Transitivity). \text{TC} can be proved transitive by way of a cyclic proof in \text{TC}_G of the sequent \( \text{TC}_G(a)(c, d), \text{TC}_G(a)(d, e), \text{TC}_G(a)(e, c) \). As in the previous example we may mimic that proof line by line, but we give a slightly different one that cannot directly be interpreted as a \text{TC}_G proof:

The only infinite branch (except for that from Ex. 4.7), looping on \( \circ \), is progressing by the red trace.

**Example 4.9.** We show a cyclic proof of the following hypersequent:

\[ \{ \text{TC}(\alpha)(c, d) \}^\omega, \{ \text{TC}(a)(c, d) \}^\omega, \{ \text{TC}(\beta)(c, d) \}^\omega, \{ \text{TC}(a)(c, y), \text{TC}(\beta)(y, d) \}^\omega \]

where \( \alpha(c, d) = \text{ST}(aa \cup aba)(c, d) \) and \( \gamma(c, d) = \text{ST}(ba^+)(c, d) \). The progressing hypertraces of the two infinite branches of the proof are highlighted in red.
follows the infinite derivation branch starting at \( Q_3 \), but we shall present it.

\[
Q_1 = (\{c(d)\})^a, (\{c,d\}), (T(c,d)) (T(c,y), (T(c,y))(y,d))^b
\]

\[
Q_2 = (\{c,e\})^a, (\{c,e\}), (T(c,y)) (T(c,y))(y,d)^b
\]

\[
Q_3 = (\{a(c,e)\})^a, (T(c,e)) (T(c,y))(y,d)^b
\]

Finally, it is pertinent to revisit the 'counterexample' \( \mathfrak{H} \) that witnessed incompleteness of \( T_G \) for PDL+.

We do not show the finite derivations of hypersequences \( Q_1 \) and \( Q_2 \) here, but the infinite derivation branch starting at \( Q_3 \),

\[
\text{Proposition 4.10.} \ T \vdash_{cyc} ST((aa \cup aba)^+)(c,d) \supset ST(a^+(ba^+ \cup a))(c,d).
\]

\text{Proof.} We use the following abbreviations: \( a(c,d) = ST(aa \cup aba)(c,d) \) and \( \beta(c,d) = ST((ba^+ \cup a))(c,d) \). The progressing hypertrace is marked in blue.
In the above derivation, ω is decidable by straightforward reduction to the universality of nondeterministic R-word-automata, with runs ‘guessing’ a progressing thread along an infinite branch.

In the above derivation, R, R' and P are the following hypersequents:

\[
R = \{\text{TC}(c, d)\}^\omega, \{\text{TC}(c, d), \text{TC}(\neg(c, d))\}^\omega, \{\text{TC}(a(c, y), \beta(y, d))\}^\omega
\]

\[
R' = \{\text{TC}(c, d)\}^\omega, \{\text{TC}(c, d), \text{TC}(\neg(c, d))\}^\omega, \{\text{TC}(a(c, y), \beta(y, d))\}^\omega
\]

\[
P = \{aba(c, e)\}^\omega, \{\text{TC}(\neg(c, d), \alpha)\}^\omega, \{\text{TC}(a(c, y), \beta(y, d))\}^\omega
\]

In Example 19 above, we use the following abbreviations:

\[
\begin{align*}
\alpha(c, d) & = \text{ST}(aa \cup aba)(c, d) \\
\beta(c, d) & = \text{ST}((ba^+) \cup a)(c, d) \\
\gamma(c, d) & = \text{ST}(ba^+)(c, d)
\end{align*}
\]

\[
\begin{align*}
\text{Example 19:} & \\
\text{inst:} & \text{TC}(c, d)\}^\omega, \{\text{TC}(\neg(c, d))\}^\omega, \{\text{TC}(\neg(c, d))\}^\omega, \{\text{TC}(a(c, y), \beta(y, d))\}^\omega
\end{align*}
\]

\[
\begin{align*}
\text{4.4 On cyclic-proof checking} & \\
\text{In usual cyclic systems, checking that a regular preproof is progressing is decidable by straightforward reduction to the universality of nondeterministic ω-word-automata, with runs ‘guessing’ a progressing thread along an infinite branch. Our notion of progress exhibits an extra quantifier alternation: we must guess an infinite hypertrace in which every trace is progressing. Nonetheless, by}
\end{align*}
\]
appealing to determinisation or alternation, we can still decide our progressing condition:

**Proposition 4.11.** Checking whether a HTC preproof is a proof is decidable by reduction to universality (or emptiness) of $\omega$-regular languages.

**Proof sketch.** The result is proved using using automata-theoretic techniques. Fix a cyclic HTC preproof $D$. First, using standard methods from cyclic proof theory, it is routine to construct a nondeterministic Büchi automaton recognising non-progressing hypertraces of $D$. The construction is similar to that recognising progressing branches in cyclic sequent calculi, e.g. as found in [13] or [24][11], since we are asking that there exists a non-progressing trace within a hypertrace. By Büchi’s complementation theorem and McNaughton’s determinisation theorem (see, e.g., [28] for details), we can thus construct a deterministic parity automaton $P_H$ recognising progressing hypertraces. (This is overkill, but allows us to easily deal with the issue of alternation in the progressing condition.)

Now we can construct a nondeterministic parity automaton $P$ recognising progressing branches of $D$ similarly to the previous construction, but further keeping track of states in $P_H$:

- $P$ essentially guesses a ‘progressing’ hypertrace along the branch input;
- at the same time, $P$ runs the hypertrace-in-construction along $P_H$ and keeps track of the state therein;
- acceptance for $P$ is inherited directly from $P_H$, i.e. the hypertrace guessed is accepting for $P$ just if it is accepted by $P_H$.

Now it is clear that $P$ accepts a branch of $D$ if and only if it is progressing. Assuming that $P$ also accepts any $\omega$-words over the underlying alphabet that are not branches of $D$ (by adding junk states), we have that $D$ is a proof (i.e. is progressing) if and only if $P$ is universal.

**4.5 Simulating Cohen-Rowe**

As we mentioned earlier, cyclic proofs of HTC indeed are at least as expressive as those of Cohen and Rowe’s system by a routine local simulation of rules:

**Theorem 4.12.** If $TC_G \vdash cyc A$ then $HTC \vdash cyc A$.

**Proof sketch.** Let $D$ be a $TC_G$ cyclic proof. We can convert it to a HTC cyclic proof by simply replacing each sequent $A_1, \ldots, A_n$ by the hypersequent $\{A_1\}^{\omega}, \ldots, \{A_n\}^{\omega}$ and applying some local corrections. In what follows, if $\Gamma = A_1, \ldots, A_n$, let us simply write $S_{\Gamma}$ for $\{A_1\}^{\omega}, \ldots, \{A_n\}^{\omega}$.

- Any id step of $D$ must be amended as follows:

\[
\text{id} \quad \frac{\Gamma, p(t), \bar{p}(t)}{\text{init} \quad \{p(t)\}^{\omega}, \{\bar{p}(t)\}^{\omega}} \quad \text{wk} \quad S_{\Gamma}, \{p(t)\}^{\omega}, \{\bar{p}(t)\}^{\omega}
\]
• Any $\lor$ step of $\mathcal{D}$ becomes a correct $\lor$ step of $\text{HTC}$ or $\text{HTC}_=$.

• Any $\land$ step of $\mathcal{D}$ must be amended as follows:

$$\Gamma, A, \Gamma, B \quad \Gamma, A \land B \quad \Rightarrow \quad S_{\Gamma, \{A, B\}^\sigma} \cup \{S_{\Gamma, \{A\}^\sigma}, S_{\Gamma, \{B\}^\sigma}\}$$

• Any $\exists$ step of $\mathcal{D}$ must be amended as follows:

$$\exists \Gamma, A(t) \quad \Gamma, \exists x A(x) \quad \Rightarrow \quad S_{\Gamma, \{A(x)\}^\sigma} \quad \text{inst} \quad S_{\Gamma, \{A(t)\}^\sigma}$$

• Any $\forall$ step of $\mathcal{D}$ becomes a correct $\forall$ step of $\text{HTC}$ or $\text{HTC}_=$.

• Any $\text{TC}_0$ step of $\mathcal{D}$ becomes a correct $\text{TC}_0$ step of $\text{HTC}$.

• Any $\text{TC}_1$ step of $\mathcal{D}$ must be amended as follows:

$$\text{TC}_1 \quad \Gamma, A(s, r) \quad \Gamma, \text{TC}(A)(r, t) \quad \Gamma, \text{TC}(A)(s, t) \quad \Rightarrow \quad S_{\Gamma, \{A(s, r)\}^\sigma} \cup \{S_{\Gamma, \{A(s, c)\}^\sigma}, S_{\Gamma, \{\text{TC}(A)(s, t)\}^\sigma}\}$$

• Any $\overline{\text{TC}}$ step of $\mathcal{D}$ must be amended as follows:

$$\overline{\text{TC}} \quad \Gamma, A(s, t) \quad \Gamma, A(s, t), \overline{\text{TC}}(A)(c, t) \quad \Gamma, \overline{\text{TC}}(A)(s, t) \quad \Rightarrow \quad S_{\Gamma, \{A(s, t)\}^\sigma} \cup \{S_{\Gamma, \{A(s, c)\}^\sigma}, S_{\Gamma, \{\overline{\text{TC}}(A)(s, t)\}^\sigma}\}$$

Particular inspection of the $\overline{\text{TC}}$ case shows that progressing traces of $\text{TC}_G$ induce progressing hypertraces of $\text{HTC}$.

\section{5 Soundness of HTC}

This section is devoted to the proof of the first of our main results:

\textbf{Theorem 5.1} (Soundness). \textit{If $\text{HTC} \vdash_{\text{wbf}} S$ then $\models S$.}

The argument is quite technical due to the alternating nature of our progress condition. In particular the treatment of traces within hypertraces requires a more fine grained argument than usual, bespoke to our hypersequential structure.

Throughout this section, we shall fix a $\text{HTC}$ preproof $\mathcal{D}$ of a hypersequent $S$. We start by introducing some additional definitions and propositions.
5.1 Some conventions on (pre)proofs and semantics

First, we work with proofs without substitution, in order to control the various symbols occurring in a proof.

**Proposition 5.2.** If $\text{HTC} \vdash \text{nwf} S$ then there is also a HTC proof of $S$ that does not use the substitution rule.

**Proof sketch.** We proceed by a coinductive argument, applying a meta-level substitution operation on proofs to admit each substitution step. Productivity of the translation is guaranteed by the progressing condition: each infinite branch must, at the very least, have infinitely many $\text{TC}$ steps.

The utility of this is that we can now carefully control the occurrences of eigenfunctions in a proof so that, bottom-up, they are never ‘re-introduced’, thus facilitating the definition of the interpretation $\delta_H$ on them.

Throughout this section, we shall allow interpretations to be only partially defined, i.e. they are now partial maps from the set of function symbols of our language to appropriately typed functions in the structure at hand. Typically our interpretations will indeed interpret the function symbols in the context in which they appear, but as we consider further function symbols it will be convenient to extend an interpretation ‘on the fly’. This idea is formalised in the following definition:

**Definition 5.3** (Interpretation extension). Let $M$ be a structure and $\rho, \rho'$ be two interpretations over $|M|$. We say that $\rho'$ is an extension of $\rho$, written $\rho \subseteq \rho'$, if $\rho'(f) = \rho(f)$, for all $f$ in the domain of $\rho$.

Finally, we assume that each quantifier in $S$ binds a distinct variable. Note that this convention means we can simply take $y = x$ in the $\exists$ rule in Fig. 2.

5.2 Constructing a ‘countermodel’ branch

Recall that we have fixed at the beginning of this section a HTC preproof $D$ of a hypersequent $S$. Let us fix some structure $M^*$ and an interpretation $\rho_0$ such that $\rho_0 \not\models S$ (within $M^*$). Since each rule is locally sound, by contraposition we can continually choose ‘false premisses’ to construct an infinite ‘false branch’:

**Lemma 5.4** (Countermodel branch). There is a branch $B^* = (S_i)_{i<\omega}$ of $D$ and an interpretation $\rho^*$ such that, with respect to $M^*$:

1. $\rho^* \not\models S_i$, for all $i < \omega$;
2. Suppose that $S_i$ concludes a $\text{TC}$ step, as typeset in Fig. 3, and $\rho^* \models \text{TC}(\bar{A})(s, t)[d/x]$. If $n$ is minimal such that $\rho^* \models A(d_i, d_{i+1})$ for all $i \leq n$, $\rho^*(s) = d_0$ and $\rho^*(t) = d_n$, and $n > 1$, then $\rho^*(f)(d) = d'$ so that $\rho_{i+1} \models A(s, f(x))[d/x]$ and $\rho^* \models \text{TC}(\bar{A})(f(x), t)[d/x]$.

Intuitively, our interpretation $\rho^*$ is going to be defined at the end of the construction as the limit of a chain of ‘partial’ interpretations $(\rho_i)_{i<\omega}$, with each $\rho_i \not\models S_i$ (within $M^*$). Note in particular that, by Item 2 whenever some $\text{TC}$-formula is principal, we choose $\rho_{i+1}$ to always assign to it a falsifying

---

2To be clear, we here choose an arbitrary such minimal ‘$\bar{A}$-path’.
path of minimal length (if one exists at all), with respect to the assignment to variables in its annotation. It is crucial at this point that our definition of $\rho^x$ is parametrised by such assignments.

**Proof of Lemma 5.4.** To construct $\rho^x$, we extend the interpretation $\rho_0$ at every step of the derivation. Thus, we shall define a chain of interpretations $\rho_0 \subseteq \rho_1 \subseteq \rho_2 \subseteq \cdot \cdot \cdot$ such that, for each $i$, $\mathcal{M}^x, \rho_i \models S_i$. We will define $\rho^x$ as the limit of this chain. In the following, we shall write $\rho \models S$ instead of $\rho \models \text{fr}n(S)$.

We distinguish cases according to the last rule $r_i$ applied to $S_i$. For the case of weakening, $S_{i+1}$ is the unique premiss of the rule, and $\rho_{i+1} = \rho_i$.

**Case** ($\cup$)

\[
\begin{align*}
S_1 &= Q, \{\Gamma_1\}^{x_1} \quad S_2 = Q, \{\Gamma_2\}^{x_2} \\
S_i &= Q, \{\Gamma_1, \Gamma_2\}^{x_1, x_2}
\end{align*}
\]

By assumption, $\rho_i \not\models Q$ and $\rho_i \models \forall x_1 \forall x_2 (\sqrt{T_1} \lor \sqrt{T_2})$. Set $\rho_{i+1} = \rho_i$. By the truth condition associated to $\lor$, we have that for all $n$-tuples $d_1 \in |\mathcal{M}^x|$ and $n$-tuples $d_2 \in |\mathcal{M}^x|$, for $n = |x_1|, m = |x_2|$, it holds that 3:

\[
\rho_{i+1} \models \sqrt{T_1} \lor \sqrt{T_2} [d_1/x_1][d_2/x_2]
\]

By the truth condition associated to $\lor$ we can conclude that, for all $d_1, d_2$,

\[
\rho_{i+1} \models \sqrt{T_1} [d_1/x_1][d_2/x_2] \quad \text{or} \quad \rho_{i+1} \models \sqrt{T_2} [d_1/V_1][d_2/V_2]
\]

Since $x_1 \cap \text{fv}(\Gamma_2) = \emptyset$ and $x_2 \cap \text{fv}(\Gamma_1) = \emptyset$, the above is equivalent to:

\[
\rho_{i+1} \models \sqrt{T_1} [d_1/x_1] \quad \text{or} \quad \rho_{i+1} \models \sqrt{T_2} [d_2/x_2]
\]

And, since this holds for all choices of $d_1$ and $d_2$, we can conclude that:

\[
\rho_{i+1} \models \forall x_1(\sqrt{T_1}) \quad \text{or} \quad \rho_{i+1} \models \forall x_2(\sqrt{T_2})
\]

Take $S_{i+1}$ to be the $S^k$ such that $\rho_{i+1} \models \forall x_k(\sqrt{T_k})$, for $k = \{1, 2\}$.

For all the remaining cases, $S_{i+1}$ is the unique premiss of the rule $r_i$. Moreover, for $x$ the unique (possibly empty) annotation explicitly reported in the remaining rules, let $n = |x|$ and $d \in |\mathcal{M}^x|$ be a $n$-tuple of elements of the domain.

**Cases** ($\land$, ($\lor$), ($\exists$), (id) and (TC))

\[
\begin{align*}
\begin{array}{c}
\land \quad S_{i+1} = Q, \{\Gamma, A, B\}^x \\
S_i = Q, \{\Gamma, A \land B\}^x
\end{array} & \quad \begin{array}{c}
\lor \quad S_{i+1} = Q, \{\Gamma, A\}^x, \{\Gamma, B\}^x \\
S_i = Q, \{\Gamma, A \lor B\}^x
\end{array} & \quad \begin{array}{c}
\exists \quad S_{i+1} = Q, \{\Gamma, A(x)\}^x \\
S_i = Q, \{\Gamma\}^x
\end{array} & \quad \begin{array}{c}
\text{id} \quad S_{i+1} = Q, \{\Gamma\}^x \\
S_i = Q, \{\Gamma, A\}^x
\end{array} & \quad \begin{array}{c}
\text{TC} \quad S_{i+1} = Q, \{\Gamma, A(s, t)\}^x, \{\Gamma, A(s, z), TCA(\Gamma)(z, t)\}^x \\
S_i = Q, \{\Gamma, TCA(\Gamma)(x, y)\}^x
\end{array}
\end{align*}
\]

3Recall that $x_1, x_2$ are defined as sets of variables, but we are here considering them as lists, assuming their elements to be ordered according to some fixed canonical ordering of variables.
For all these cases set $\rho_{i+1} = \rho_i$. In all these cases, the formula interpretation of the conclusion logically implies the formula interpretation of the premise. Thus, form $\mathcal{M}^x, \rho_i \not\models S_i$ we have that $\mathcal{M}^x, \rho_{i+1} \not\models S_{i+1}$. We explicitly show the construction for (3), (id) and (tc).

(3) By assumption, $\rho_i \not\models Q$ and $\rho_i \models \forall x(\sqrt{T} \lor \forall x(\overline{A}(x)))$. We have that $\rho_{i+1} \not\models Q$. Moreover, by prenexing the quantifier we obtain that $\rho_{i+1} = \forall x \forall \overline{x}(\sqrt{T} \lor \overline{A}(x))$.

(id) By assumption, $\rho_i \not\models Q$ and $\rho_i \models \forall x(\sqrt{T} \lor \overline{A})$ and $\rho_i \models A$. By the forcing condition associated to $\forall$, we have that, for all choices of $d$, it holds that:

$$\rho_{i+1} = \forall x(\sqrt{T} \lor \overline{A}) [d/x]$$

By the truth condition associated to $\lor$, for every choice of $d$, it holds that either:

$$\rho_{i+1} = \forall x(\sqrt{T} [d/x]) \quad \text{or} \quad \rho_{i+1} = \overline{A} [d/x]$$

Since $\text{fv}(A) \cap x = \emptyset$, the above is equivalent to:

$$\rho_{i+1} = \forall x(\sqrt{T} [d/x]) \quad \text{or} \quad \rho_{i+1} = \overline{A}$$

By assumption, $\rho_{i+1} = A$. Thus, the second disjunct cannot hold, and we have that $\rho_{i+1} = \forall x(\sqrt{T} [d/x])$. Since this holds for all choices of $d$, we conclude that $\rho_{i+1} = \forall x(\sqrt{T})$.

(tc) By assumption, $\rho_i \not\models Q$ and $\rho_i \models \forall x(\sqrt{T} \lor \overline{TC}(A)(s,t))$. Recall that $\overline{TC}(A)(s,t) := \overline{TC}(\overline{A})(s,t)$. We reason as follows:

$$\rho_{i+1} = \forall x(\sqrt{T} \lor \overline{TC}(\overline{A})(s,t))$$

$$\rho_{i+1} = \forall x(\sqrt{T} \lor (\overline{A}(s,t) \lor \forall z(\overline{A}(s,z) \lor \overline{TC}(\overline{A})(z,t))))$$

$$\rho_{i+1} = \forall x((\sqrt{T} \lor \overline{A}(s,t)) \lor (\sqrt{T} \lor \forall z(\overline{A}(s,z) \lor \overline{TC}(\overline{A})(z,t))))$$

$$\rho_{i+1} = \forall x((\sqrt{T} \lor \overline{A}(s,t)) \lor \forall z((\sqrt{T} \lor \overline{A}(s,z) \lor \overline{TC}(\overline{A})(z,t))))$$

$$\rho_{i+1} = \forall x((\sqrt{T} \lor \overline{A}(s,t)) \lor \forall z((\sqrt{T} \lor \overline{A}(s,z) \lor \overline{TC}(\overline{A})(z,t))))$$

In the above, step (*) follows from the inductive definition of $\overline{TC}$, and step (*) is obtained by distributing $\forall$ over $\lor$, i.e., by means of the classical theorem $\forall x(A \lor B) \lor (\forall x(A) \lor \forall x(B))$. The other steps are standard theorems or follows from the truth conditions of the operators.

For the three remaining case of (inst), (\forall) and (\overline{c}), $\rho_{i+1}$ extends $\rho_i$ by interpreting the function symbols introduced, bottom-up, by the rules.

> **Case** (inst)

$$\infer[\text{inst}] {S_i = Q, \{\Gamma(y)} \overline{\{x\gamma\}}} {S_{i+1} = Q, \{\Gamma(y)} \overline{\{x\gamma\}}}$$
By assumption, $\rho_1 \not\models \mathbb{Q}$ and $\rho_i \models \forall x (\Gamma_1(x))$. Thus, for all choices of $d$, we have that $\rho_i \models \forall x (\Gamma_1(x))[d/x]$. By the truth condition associated to $\forall$, this means that, for all $d \in |M^x|$, $\rho_i \models \Gamma_1(x)[d/x]$. Take $\rho_{i+1}$ to be any extension of $\rho_i$ that is defined on the language of $S_{i+1}$. That is, if $f$ is a function symbol in $t$ to which $\rho_i$ already assigns a map, then $\rho_{i+1}$ assigns to it that same map. Otherwise, $\rho_{i+1}$ assigns an arbitrary map to $f$. It follows that $\rho_{i+1} \not\models fm(\mathbb{Q})$ and $\rho_{i+1} \models \Gamma_1(t)[d/x]$ and, since this holds for all $d$, we have that $\rho_{i+1} \models \forall x (\Gamma_1(t))$. Thus $\rho_{i+1} \not\models S_{i+1}$.

\begin{itemize}
  \item \textbf{Case ($\forall$)}
    \begin{align*}
      S_{i+1} & = Q, \{\Gamma, A(f(x))\}^x \\
      S_i & = Q, \{\Gamma, \forall x A(x)\}^x
    \end{align*}
  \\
  By assumption, $\rho_1 \not\models Q$ and $\rho_i \models \forall x (\Gamma \vee \exists x (\overline{A}(x)))$. By the truth condition associated to $\forall$ and to $\vee$, for all choices of $d$ we have that:
  
  \begin{align*}
    \rho_i \models \exists x (\overline{A}(x))[d/x] \quad \text{or} \quad \rho_i \models \forall x (\Gamma)[d/x]
  \end{align*}
  
  We define $\rho_{i+1}$ to extend $\rho_i$ by defining $\rho_{i+1}(f)$ as follows. Let $d \subseteq |M^x|$. If $\rho_i \models \exists x (\Gamma_1)[d/x]$ then we may set $\rho_{i+1}(f)(d)$ to be arbitrary. We still have $\rho_{i+1} \models \Gamma_1(t)[d/x]$, as required. Otherwise, it holds that $\rho_i \models \exists x (\overline{A}(x))[d/x]$. By the truth condition associated to $\exists$, there is a $d \in |M^x|$ such that $\rho_i \models \overline{A}(x)[d/x]$. In this case, we define $\rho_{i+1}(f)(d) = d$, so that $\rho_{i+1} \models \overline{A}(f(x))[d/x]$. So, for all $d$, we have that $\rho_{i+1} \models \forall x (\Gamma \vee \overline{A}(f(x)))[d/x]$, and so $\rho_{i+1} \models \forall x (\Gamma \vee \overline{A}(f(x)))$. Thus, $\rho_{i+1} \not\models S_{i+1}$, as required.
  
  \item \textbf{Case ($\text{TC}$)}
    \begin{align*}
      \text{TC} \quad S_{i+1} & = Q, \{\Gamma, A(s, t), A(s, f(x))\}^x, \{\Gamma, A(s, t), \text{TC}(A)(f(x), t)\}^x \\
      S_i & = Q, \{\Gamma, \text{TC}(A)(s, t)\}^x
    \end{align*}
  \\
  By assumption, $\rho_1 \not\models Q$ and $\rho_i \models \forall x (\Gamma \vee \text{TC}(A)(s, t))$ which, by definition of duality, means $\rho_i \models \forall x (\Gamma \vee \overline{\text{TC}(A)}(s, t))$. By the truth conditions for $\vee$ we have, for all $d$:
  
  \begin{enumerate}
    \item $\rho_i \models \Gamma_1[d/x]$ or
    \item $\rho_i \models \text{TC}(\overline{A})(s, t)[d/x]$
  \end{enumerate}
  
  We define $\rho_{i+1}$ to extend $\rho_i$ by defining $\rho_{i+1}(f)$ as follows. Let $d \subseteq |M^x|$. If 1) holds, then we may set $\rho_{i+1}(d)$ to be an arbitrary element of $|M^x|$. Otherwise, 2) must hold, so by the truth conditions for $\text{TC}$ there is a $\overline{\Gamma}$-path between $\rho_i(s)$ and $\rho_i(t)$ of length greater or equal than 1, i.e. there are elements $d_0, \ldots, d_n$, with $n > 0$ and $\rho_i(s) = d_0$ and $\rho_i(t) = d_n$, such that $\rho_i \models \overline{A}(d_i, d_{i+1})$ for all $i < n$. We select a \textit{shortest} such path, i.e. one with smallest possible $n > 0$. There are two cases:
  
  \begin{enumerate}
    \item if $n = 1$, then already $\rho_i \models \overline{A}(s, t)[d/x]$, so we may set $\rho_{i+1}(f)(d)$ to be arbitrary;
    \item otherwise $n > 1$ and we set $\rho_{i+1}(f)(d) = d_1$, so that $\rho_{i+1} \models \overline{A}(s, f(x))[d/x]$ and $\rho_{i+1} \models \overline{\text{TC}}(\overline{A})(f(x), t)[d/x]$.
  \end{enumerate}
We have considered all the rules; the construction of $B^\times$ is complete. From here, note that we have $\rho_i \subseteq \rho_{i+1}$, for all $i < \omega$. Thus we can construct the limit $\rho^\times = \bigcup_{i<\omega} \rho_i$, that we shall call the interpretation induced by $\mathcal{M}^\times$ and $\rho_0$.

5.3 Canonical assignments along countermodel branches

Let us now fix $B^\times$ and $\rho^\times$ as provided by Lemma 5.4 above. Moreover, let us henceforth assume that $D$ is a proof, i.e. it is progressing, and fix a progressing hypertrace $H = \{ \{ \Gamma_i \}^{\times_i} \}_{i<\omega}$ along $B^\times$. In order to carry out an infinite descent argument, we will need to define a particular trace along this hypertrace that ‘preserves’ falsity, bottom-up. This is delicate since the truth values of formulas in a trace depend on the assignment of elements to variables in the annotations. A particular issue here is the instantiation rule inst, which requires us to ‘revise’ whatever assignment of $y$ we may have defined until that point. Thankfully, our earlier convention on substitution-freeness and uniqueness of bound variables in $D$ facilitates the convergence of this process to a canonical such assignment:

Definition 5.5 (Assignment). We define $\delta_H : \bigcup_{i<\omega} \mathcal{M}^\times |_{\mathcal{X}_i} \rightarrow |\mathcal{M}^\times|$ by $\delta_H(x) := \rho(t)$ if $x$ is instantiated by $t$ in $H$; otherwise $\delta_H(x)$ is some arbitrary $d \in |\mathcal{M}^\times|$.

Note that $\delta_H$ is indeed well-defined, thanks to the convention that each quantifier in $S$ binds a distinct variable. In particular we have that each variable $x$ is instantiated at most once along a hypertrace. Henceforth we shall simply write $\rho, \delta_H \models A(x)$ instead of $\rho \models A(\delta_H(x))$. Working with such an assignment ensures that false formulas along $H$ always have a false immediate ancestor:

Lemma 5.6 (Falsity through $H$). If $\rho^\times, \delta_H \not\models F$ for some $F \in \Gamma_i$, then $F$ has an immediate ancestor $F' \in \Gamma_{i+1}$ with $\rho^\times, \delta_H \not\models F'$.

Proof. Suppose that $F$ is a formula occurring in some cedent $\{ \Gamma_i \}^{\times_i}$ in $S$, such that $\rho^\times, \delta_H \not\models F$. We show how to choose a $F'$ satisfying the conditions of the Lemma. We distinguish cases according to the rule $rr_i$ applied. Propositional cases are immediate, by setting $F' = F$, as well as $\cup$, since the failed branch has been chosen during the construction of $B^\times$. For the rule of weakening, observe that we could not have chosen a hypertrace going through the structure which gets weakened, as by assumption the hypertrace is infinite. We show the remaining cases. It can be easily checked that, given a formula $F$ such that $\mathcal{M}^\times, \rho^\times, \delta_H \not\models F$, the formula $F'$ is an immediate ancestor of $F$.

Case ($\exists$). Suppose $\{ \Gamma_i \}^{\times_i} = \{ \Gamma, \exists x(A(x)) \}^\times$ and $\{ \Gamma_{i+1} \}^{\times_{i+1}} = \{ \Gamma, A(x) \}^{\times_{i+1}}$. By assumption, $\rho^\times \models \forall x(\bigvee \Gamma \lor \forall x(\overline{A(x)})$). Applying the truth condition associated to $\forall$ we obtain that, for all $n$-tuples $d$ of elements of $|\mathcal{M}^\times|$, for $n = |x|$, it holds that:

$$\rho^\times \models \bigvee \Gamma \lor \forall x(\overline{A(x)}) \quad [d/x]$$

By definition, $\delta_H$ assigns a value in the domain for all the variables in $\text{ann}(H)$, and $x \subseteq \text{ann}(H)$. From (7) and from the truth condition associated to $\lor$ it follows that either:

$$\rho^\times, \delta_H \models \bigvee \Gamma \quad \text{or} \quad \rho^\times, \delta_H \models \forall x(\overline{A(x)})$$

By definition, $\delta_H$ assigns a value in the domain for all the variables in $\text{ann}(H)$, and $x \subseteq \text{ann}(H)$. From (7) and from the truth condition associated to $\lor$ it follows that either:

$$\rho^\times, \delta_H \models \bigvee \Gamma \quad \text{or} \quad \rho^\times, \delta_H \models \forall x(\overline{A(x)})$$
Case (inst). Suppose \( \{\Gamma_i\}^{x_i} = \{\Gamma(y)\}^{x,y} \) and \( \{\Gamma_i+1\}^{x_i+1} = \{\Gamma(t)\}^{x} \). By construction, \( \rho^x \models \forall y(\bigvee \Gamma(y)) \). Reasoning as in the previous case, from the truth condition associated to \( \forall \) it follows that, for all choices of \( d \in \mathcal{M}^x \), and thus also for the \( n \)-tuple selected by \( \delta_H \), \( \rho^x, \delta_H \models \bigvee \Gamma(y) \). If \( F \) does not contain \( y \), then \( \rho^x, \delta_H \models \mathcal{F} \), and we set \( F' = F \). Otherwise, if \( F \) contains \( y \), then \( \rho^x, \delta_H \models \mathcal{F}(y) \). By Lemma 5.4, \( \rho^x \) assigns a value to \( t \), and by Definition 5.5, since \( y \) is instantiated with \( t \) along \( \mathcal{H} \), \( \delta_H(y) = \rho^x(t) \). Therefore, set \( F' = (F(t)) \) and conclude that \( \rho^x, \delta_H \models F' \).

Case (id). Suppose \( \{\Gamma_i\}^{x_i} = \{\Gamma, A\}^{x} \) and \( \{\Gamma_i+1\}^{x_i+1} = \{\Gamma\}^{x} \). Observe that the hypertrace \( \mathcal{H} \) could not have gone through the structure \( \{\mathcal{A}\}^{\mathcal{S}} \) occurring in the conclusion of the rule, because by assumption \( \mathcal{H} \) is infinite. Moreover, by construction the formula interpretation of all the cedents along \( B^x \) is not valid, and thus \( \rho^x \models t \) is not valid. This implies that \( \rho^x \models A \) and so:

\[
\rho^x, \delta_H \models \bigvee \Gamma
\]

So we have that \( F \in \Gamma \) and \( \rho^x, \delta_H \models \mathcal{F} \). Set \( F' = F \).

Case (TC). Suppose \( \{\Gamma_i\}^{x_i} = \{\Gamma, TC(A)(s, t)\}^{x} \). By assumption, \( \rho^x \models \forall x(\bigvee \Gamma \lor TC(A)(s, t)) \). Thus, we have that:

\[
\rho^x, \delta_H \models \bigvee \Gamma \lor \rho^x, \delta_H \models TC(A)(s, t)
\]

From the inductive definition of \( TC \) and the truth condition for \( \land \), the second disjunct is equivalent to:

\[
\rho^x, \delta_H \models \bigvee \Gamma \lor (\mathcal{A}(s, z) \lor \bigvee \mathcal{C}(\mathcal{A})(z, t)) \quad (8)
\]

There are two cases to consider, since the premiss of the rule has two cedents that the hypertrace \( \mathcal{H} \) could follow:

i) \( \{\Gamma_i+1\}^{x_i+1} = \{\Gamma, A(s, t)\}^{x} \). By construction, \( \rho^x \models \forall x(\bigvee \Gamma \lor \mathcal{A}(s, t)) \), and thus \( \rho^x, \delta_H \models \bigvee \Gamma \lor \mathcal{A}(s, t) \). If \( F \in \Gamma \), then \( \rho^x, \delta_H \models \mathcal{F} \). Set \( F' = F \). Otherwise, \( F = TC(A)(s, t) \) and \( \rho^x, \delta_H \models \mathcal{F} \). By (8) we have that \( \rho^x, \delta_H \models \mathcal{A}(s, t) \). Set \( F' = A(s, t) \).

ii) \( \{\Gamma_i+1\}^{x_i+1} = \{\Gamma, A(z, s), TC(A)(z, t)\}^{x,z} \). By construction, we have that \( \rho^x \models \forall x \forall z(\bigvee \Gamma \lor \mathcal{A}(s, z) \lor \bigvee \mathcal{C}(\mathcal{A})(z, t)) \). Since \( z \) does not occur free in \( \Gamma \), this is equivalent to \( \rho^x, \delta_H \models \bigvee \Gamma \lor (\mathcal{A}(s, z) \lor \bigvee \mathcal{C}(\mathcal{A})(z, t)) \). If \( F \in \Gamma \) and \( \rho^x, \delta_H \models \mathcal{F} \), set \( F' = F \). Suppose \( F = TC(A)(s, t) \) and \( \rho^x, \delta_H \models \mathcal{F} \). From (8) and since \( z \in \text{ann}(\mathcal{H}) \) we have that:

\[
\rho^x, \delta_H \models \mathcal{A}(s, z) \lor \bigvee \mathcal{C}(\mathcal{A})(z, t)
\]

If \( \rho^x, \delta_H \models \mathcal{A}(s, z) \), set \( F' = A(s, z) \). Otherwise, if \( \rho^x, \delta_H \models \bigvee \mathcal{C}(\mathcal{A})(z, t) \), set \( F' = TC(A)(z, t) \).
\[\text{\textcircled{\#} Case } (\forall). \text{ Suppose } \{\Gamma_i\}_i^{x_i} = \{\Gamma, \forall x(A(x))\}^x \text{ and } \{\Gamma_i+1\}_i^{x_i+1} = \{\Gamma, A(f(x))\}^x.\]

By assumption, \(\rho^x, \delta_H \models \forall x(\sqrt{T} \lor \exists x(A(x)))\) and, from the truth conditions associated to \(\forall \) and \(\lor\):

\[\rho^x, \delta_H \models \sqrt{T} \text{ or } \rho^x, \delta_H \models \exists x(A(x))\]

If \(F \in \Gamma \) and \(\rho^x, \delta_H \models \sqrt{T}\), set \(F' = F\), since \(x\) does not occur in \(\Gamma\). Otherwise, \(F = \forall x(A(x))\) and \(\rho^x, \delta_H \models \exists x(A(x))\). Suppose \(\delta_H(x) = d \subseteq |M^x|\). By definition of \(\rho^x\) at step (\(\forall\)), we have that \(\rho^x(f)(d)\) selects a \(d \in |M^x|\) such that \(\rho^x, \delta_H \models \overline{A}(f(x))\). Set \(F' = A(f(x))\).

\[\text{\textcircled{\#} Case } (\overline{TC}). \text{ Suppose } \{\Gamma_i\}_i^{x_i} = \{\Gamma, \overline{TC}(A)(s, t)\}^x. \text{ By assumption, } \rho^x \models \forall x(\sqrt{T} \lor \overline{TC}(A)(s, t)), \text{ that is, } \rho^x \models \forall x(\sqrt{T} \lor \overline{TC}(A)(s, t)). \text{ Thus, we have that:}\]

\[\rho^x, \delta_H \models \sqrt{T} \text{ or } \rho^x, \delta_H \models \overline{TC}(A)(s, t) \quad (9)\]

We need to consider two cases, depending on which cedent the hypertrace \(H\) follows:

\[i) \{\Gamma_i+1\}_i^{x_i+1} = \{\Gamma, A(s, t), A(s, f(x))\}^x. \text{ By construction, } \rho^x \models \forall x(\sqrt{T} \lor (\overline{A}(s, t) \lor \overline{A}(s, f(x))), \text{ that is,}\]

\[\rho^x, \delta_H \models \sqrt{T} \text{ or } \rho^x, \delta_H \models \overline{A}(s, t) \text{ or } \rho^x, \delta_H \models \overline{A}(s, f(x)) \quad (10)\]

Consider (10). If \(F \in \Gamma \) and \(\rho^x, \delta_H \models \sqrt{T}\), then set \(F = F'\). Otherwise, \(F = \overline{TC}(A)(s, t)\) and it holds that \(\rho^x, \delta_H \models \overline{TC}(A)(s, t)\). By the inductive definition of \(TC\) and the truth condition associated to \(\forall\), this is equivalent to:

\[\rho^x, \delta_H \models \overline{A}(s, t) \text{ or } \rho^x, \delta_H \models \exists z(\overline{A}(s, z) \land \overline{TC}(A)(z, t))\]

First check if \(\rho^x, \delta_H \models \overline{A}(s, t)\). If this is the case, set \(F' = A(s, t)\) and conclude by (10) that \(\rho^x, \delta_H \models \overline{A}(s, t)\). Otherwise, \(\rho^x, \delta_H \models \exists z(\overline{A}(s, z) \land \overline{TC}(A)(z, t))\). Let \(\delta_H(x) = d\). According to the definition of \(\rho^x\) at the (\(\overline{TC}\)) step, since \(\rho^x, \delta_H \not\models \sqrt{T}\) and \(\rho^x, \delta_H \not\models \overline{A}(s, t)\), then \(\rho^x(f)(d)\) is defined as in case 2), subcase ii). Thus, \(\rho^x(f)(d)\) is an element \(d \in |M^x|\) such that \(\rho^x, \delta_H \models \overline{A}(s, f(x))\) and \(\rho^x, \delta_H \models \overline{TC}(A)(f(x), t)\). Set \(F' = A(s, f(x))\) and conclude, by (10), that \(\rho^x+1, \delta_H \models \overline{A}(s, f(x))\).

\[ii) \{\Gamma_i+1\}_i^{x_i+1} = \{\Gamma, A(s, t), \overline{TC}(A)(f(x), t)\}^x. \text{ By construction, it holds that either:}\]

\[\rho^x, \delta_H \models \sqrt{T} \text{ or } \rho^x, \delta_H \models \overline{A}(s, t) \text{ or } \rho^x, \delta_H \models \overline{TC}(A)(f(x), t) \quad (11)\]

At this point the proof proceeds exactly as in the previous case except for the very last step, where \(F'\) is set to be \(\overline{TC}(A)(f(x), t)\), and we conclude by (10) that \(\rho^x, \delta_H \models \overline{TC}(A)(f(x), t)\).
5.4 Putting it all together

Note how the inst of Fig. 2 is handled in the proof above: if $F \in \Gamma(y)$ then we can choose $F' = F[t/y]$ which, by definition of $\delta_H$, has the same truth value.

By repeatedly applying Lemma 5.6 we obtain:

**Proposition 5.7** (False trace). There exists an infinite trace $\tau^x = (F_i)_{i<\omega}$ through $H$ such that, for all $i$, it holds that $M^x, \rho^x, \delta_H \not\models F_i$.

**Proof.** We inductively define $\tau^x$ as follows. By assumption, $M^x, \rho^x \not\models S_0$, and thus in particular $\rho^x \models x_0(\sqrt{\Gamma_0})$. Thus, for all $n$-tuples $d$ of elements of $|M^x|$, for $n = |x_0|$, we have that $\rho^x \models \sqrt{\Gamma_0}(d/x_0)$. Since $\delta_H$ assigns a value to all variables occurring in annotations of cedents in $H$, $\rho^x, \delta_H \models \sqrt{\Gamma_0}$. Take $F_0 \in S_0$ such that $M^x, \rho^x, \delta_H \not\models F_0$.

For the inductive step, we define $F_{i+1}$ by inspecting the rule $r_i$ applied and at the corresponding case in Lemma 5.6. Since by assumption $H$ is infinite, and since Lemma 5.6 ensures that for every formula $F_i$ such that $M^x, \rho^x, \delta_H \not\models F_i$, it is possible to find a formula $F_{i+1}$ which is an immediate ancestor of $F_i$ and such that $M^x, \rho^x, \delta_H \not\models F_{i+1}$.

We are now ready to prove our main soundness result.

**Proof of Thm. 5.1.** Fix the infinite trace $\tau^x = (F_i)_{i<\omega}$ through $H$ obtained by Prop. 5.7. Since $\tau^x$ is infinite, by definition of HTC proofs, it needs to be progressing, i.e., it is infinitely often $\overline{TC}$-principal and there is some $k \in \mathbb{N}$ s.t. for $i > k$ we have that $F_i = \overline{TC}(A)(s_i, t_i)$ for some terms $s_i, t_i$.

To each $F_i$, for $i > k$, we associate the natural number $n_i$ measuring the $\overline{A}$-distance between $s_i$ and $t_i$. Formally, $n_i \in \mathbb{N}$ is least such that there are $d_0, \ldots, d_{n_i} \in |M^x|$ with $\rho^x(s) = d_0, \rho^x(t) = d_{n_i}$ and, for all $i < n_i$, $\rho^x, \delta_H \models A(d_i, d_{i+1})$. Our aim is to show that $(n_i)_{i<k}$ has no minimal element, contradicting wellfoundness of $N$. For this, we establish the following two local properties:

1. $(n_i)_{i<k}$ is monotone decreasing, i.e., for all $i > k$, we have $n_{i+1} \leq n_i$;

2. Whenever $F_i$ is principal, we have $n_{i+1} < n_i$.

We proceed by inspection on HTC rules. We start with item 2. Suppose $F_i = \overline{TC}(A)(s, t)$ is the principal formula in an occurrence of $\overline{TC}$ so $F_{i+1} = \overline{TC}(A)(f(x), t)$, for some $x$. Moreover, by construction $\rho^x, \delta_H \models TC(A)(s, t)$ and $\rho^x, \delta_H \models TC(A)(f(x), t)$. We have to show that $n_i$, the $\overline{A}$-distance between $f(x)$ and $t$, is strictly smaller than $n_{i+1}$, the $\overline{A}$-distance between $s$ and $t$, under $\rho^x$ and $\delta_H$.

Let $\delta_H(x) = d$. By case $\overline{TC}$ of Lem. 5.3 and since $\rho^x, \delta_H \models TC(A)(f(x), t)$, there is a shortest $\overline{A}$-path between $\rho^x(s)$ and $\rho^x(t)$, composed of $n > 1$ elements $d_0, \ldots, d_n$, with $\rho^x(s) = d_0$ and $\rho^x(t) = d_{n}$, and $\rho^x(f)(d) = d_1$. Consequently, it holds that $\rho^x, \delta_H \models \overline{A}(s, f(x))$ and $\rho^x, \delta_H \models TC(A)(f(x), t)$. Thus, there is an $\overline{A}$-edge between $\rho^x(s)$ and $\rho^x(f)(d)$, and $\rho^x(f)(d)$ is one edge closer to $\rho^x(t)$ in one of the shortest $\overline{A}$-paths between $\rho^x(s)$ and $\rho^x(t)$. We conclude that $n_{i+1}$ is exactly $n_i - 1$, and $n_{i+1} < n_i$.

To prove item 1, suppose that $F_i$ is not principal in the occurrence $r_i$ of a HTC rule. Suppose $r_i$ is inst, $F_i = \overline{TC}(A)(s, x)$, $F_{i+1} = \overline{TC}(A)(s, t)$, and $x$ gets instantiated with $t$ by inst. By construction, $\rho^x, \delta_H \models TC(A)(s, x)$. Let
Figure 3: Rules of LPD$^+$. 

$n_i$ be the distance between $\rho^s(x)$ and $\delta H(x)$. By definition, $\delta H(x) = \rho^s(t)$. Thus, the distance between $\rho^s(x)$ and $\rho^s(t)$ is $n_i+1 = n_i$. In all the other cases, $F_i = TC(A)(s, t) = F_{i+1}$, and thus $n_{i+1} = n_i$.

So $(n_i)_{i \leq k}$ is monotone decreasing (by point 1) but cannot converge by point 2, and by definition of progressing trace. Thus $(n_i)_{k < i}$ has no minimal element, yielding the required contradiction.

6 Completeness for PDL$^+$, over the standard translation

In this section we give our next main result:

**Theorem 6.1** (Completeness for PDL$^+$). For a PDL$^+$ formula $A$, if $\models A$ then $HTC \vdash cyc ST(A)(c)$.

The proof is by a direct simulation of a cut-free cyclic system for PDL$^+$ that is complete. We shall briefly sketch this system below.

6.1 Circular system for PDL$^+$

The system LPD$^+$, shown in Figure 3, is the natural extension of the usual sequent calculus for basic multimodal logic $K$ by rules for programs. In Figure 3, $\Gamma, \Delta$ etc. range over sets of PDL$^+$ formulas, and we write $\langle a \rangle \Gamma$ is shorthand for $\{\langle a \rangle B : B \in \Gamma\}$. (Regular) preproofs for this system are defined just like for HTC or TC$^G$.

The notion of ancestry for formulas is colour-coded in Figure 3 as before: a formula $C$ in a premiss is an immediate ancestor of a formula $C'$ in the conclusion if they have the same colour; if $C, C' \in \Gamma$ then we furthermore require $C = C'$. More formally (and without relying on colours):

**Definition 6.2** (Immediate ancestry). Fix a preproof $D$. We say that a formula occurrence $C$ is an immediate ancestor of a formula occurrence $D$ in $D$ if $C$ and $D$ occur in a premiss and the conclusion, respectively, of an inference step $r$ of $D$ and:

- If $r$ is a $k$ step then, as typeset in Figure 3.
\[ D \text{ is } \langle a \rangle B \text{ for some } B \in \Gamma \text{ and } C \text{ is } B; \text{ or,} \]
\[ D \text{ is } [a]A \text{ and } C \text{ is } A. \]

- If \( r \) is not a \( k \)-step then:
  - \( C \) and \( D \) are occurrences of the same formula; or,
  - \( D \) is principal and \( C \) is auxiliary in \( r \), i.e. as typeset in Figure 3, \( C \) and \( D \) are the (uniquely) distinguished formulas in a premiss and conclusion, respectively;

**Definition 6.3** (Non-wellfounded proofs). Fix a preproof \( D \) of a sequent \( \Gamma \). A thread is a maximal path in its graph of immediate ancestry. We say a thread is **progressing** if it has a smallest infinitely often principal formula of the form \([\alpha^+]A \). \( D \) is a proof if every infinite branch has a progressing thread. If \( D \) is regular, we call it a cyclic proof and we may write \( \text{LPD}^+ \vdash_{\text{cyc}} \Gamma \).

Soundness of cyclic-LPD\(^+\) is established by a standard infinite descent argument, but is also implied by the soundness of cyclic-HTC (Thm. \ref{thm:htc_sound}) and the simulation we are about to give (Thm. \ref{thm:lpd_mock_sound}), though this is somewhat overkill. Completeness may be established by the game theoretic approach of Niwinski and Walukiewicz [22], as carried out by [19], or by the purely proof theoretic techniques of Studer [25]. Either way, both results follow immediately by a standard embedding of PDL\(^+\) into the (guarded) \( \mu \)-calculus and its known completeness results [22, 25], by way of a standard ‘proof reflection’ argument: \( \mu \)-calculus proofs of the embedding are ‘just’ step-wise embeddings of LPD\(^+\) proofs.

**Theorem 6.4** (Soundness and completeness, [19]). Let \( A \) be a PDL\(^+\) formula.
\[ \models A \text{ iff } \text{LPD}^+ \vdash_{\text{cyc}} A. \]

### 6.2 Examples of cyclic proofs in LPD\(^+\)

Before giving our main simulation result, let us first see some examples of proofs in LPD\(^+\), in particular addressing the ‘counterexample’ from Section 3.3.

**Example 6.5.** We show a cyclic LPD\(^+\) proof of the LPD\(^+\) sequent:
\[ [(aa \cup aba)^+]p, \langle a^+ \rangle p, (ba^+)p, \langle a^+ \rangle (ba^+)p \]

We use the following abbreviations: \( \alpha = (aa \cup aba)^+ \) and \( \beta = (ba^+) \). Moreover, we sometimes use rule \( \langle + \rangle \), which is derivable from rules \( \langle + \rangle_0 \) and \( \langle + \rangle_1 \), keeping in mind that LPD\(^+\) sequents are sets of formulas:

\[
\begin{array}{c}
\text{Rule } \langle + \rangle
\end{array}
\]

\[
\frac{\Gamma, \langle \alpha \rangle A, \langle \alpha^+ \rangle A}{\Gamma, \langle \alpha^+ \rangle A}
\]

Similarly for rule \( \lor \). The progressing threads (one for each infinite branch) are highlighted in blue. \( \Gamma \) is sequent \([aa \cup aba]^p, \langle a^+ \rangle p, \langle \beta \rangle p, \langle a^+ \rangle \langle \beta \rangle p\), derivable by means of a finite derivation (which we do not show).
Example 6.6. We show a cyclic LPD* proof of formula (11), which witnesses the incompleteness of TC_G without cut:

\((aa \cup aba)^+ \supset (a^+ ((ba^+)^+ \cup a))p\)

We employ the same shorthands as in the previous example, i.e., \(\alpha = (aa \cup aba)^+\) and \(\beta = (ba^+)^+\). The progressing threads are highlighted in blue; in the rightmost branch, the thread continues into the thread shown in Example 6.5 where progress can be observed.
Here follows the finite derivation of $\Gamma'$, i.e., sequent $[aa \cup aba] \overline{p}, \langle a^+ \rangle \langle \beta \cup a \rangle p$.
1. There is a finite cut-free HTC-derivation from \( S, HT(A)(c) \) to \( S, \{CT(A)(c)\}^{x_a} \); and,

2. There is a finite cut-free derivation from \( S, \{CT(A)(c)\}^{x_a} \) to \( S, \{ST(A)(c)\}^{\varphi} \).

Proof. By induction on the complexity of \( A \). If \( A \) is atomic, a conjunction or a disjunction, then \( HT(A)(c) = \{CT(A)(c)\}^{x_a} = \{ST(A)(c)\}^{\varphi} \). The remaining cases are shown below. If \( A = \langle \alpha_1 \rangle \ldots \langle \alpha_n \rangle B \), for \( 1 \leq n \), then \( \{CT(A)(y)\}^{x_a} = HT(A)(c) \), and \( x_A = \{y_1, \ldots, y_n\} \), for \( y_1, \ldots, y_n \) variables that do not occur in \( B \). The double line in the derivation below denotes \( n - 1 \) occurrences of rules \( \exists, \land \).

\[
\]
where (omitted) left-premises of $\cup$ steps are simply proved by wk, id, init.

\[
\begin{array}{c}
\frac{\Gamma; [\alpha]A \quad \Gamma; [\beta]A}{\Gamma; [\alpha \lor \beta]A} \quad \Rightarrow \\
\hline
\begin{array}{c}
\text{HT}(\Gamma')(c), \text{HT}([\alpha]A(c)) \\
\cup \\
\text{HT}(\Gamma')(c), \{\text{ST}(\alpha(c,d))\}^{\beta}, \text{HT}(A(c)) \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\frac{\Gamma; [\alpha_0]A \quad I \in \{0, 1\}}{\Gamma; [\alpha_0 \lor \alpha_1]A} \quad \Rightarrow \\
\hline
\begin{array}{c}
\text{HT}(\Gamma')(c), \text{HT}([\alpha_0]A(c)) \\
\Rightarrow \\
\text{HT}(\Gamma')(c), \{\text{ST}(\alpha_0(c,y), \text{CT}(A)(y))\}^{\beta, y} \\
\text{HT}(\Gamma')(c), \{\text{ST}(\alpha_0(c,y) \lor \text{ST}(\alpha_1(c,y), \text{CT}(A)(y))\}^{\beta, y} \\
\text{HT}(\Gamma')(c), \text{HT}([\alpha_0 \lor \alpha_1]A(c)) \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\frac{\Gamma; [\alpha][\beta]A \quad \Gamma; [\alpha; \beta]A}{\Gamma; [\alpha \lor \beta]A} \quad \Rightarrow \\
\hline
\begin{array}{c}
\text{HT}(\Gamma')(c), \{\text{ST}(\alpha(c,e))\}^{\beta}, \{\text{ST}(\beta(c,d))\}^{\beta}, \text{HT}(A(c)) \\
\Rightarrow \\
\text{HT}(\Gamma')(c), \{\text{ST}(\alpha(c,e) \lor \text{ST}(\beta(c,d))\}^{\beta}, \text{HT}(A(c)) \\
\text{HT}(\Gamma')(c), \{\forall z(\text{ST}(\alpha(c,z) \lor \text{ST}(\beta(z,d))\}^{\beta}, \text{HT}(A(c)) \\
\text{HT}(\Gamma')(c), \text{HT}([\alpha; \beta]A(c)) \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\frac{\Gamma; [\alpha](\beta)A \quad \Gamma; [\alpha; \beta]A}{\Gamma; [\alpha; \beta]A} \quad \Rightarrow \\
\hline
\begin{array}{c}
\text{HT}(\Gamma')(c), \text{HT}((\alpha)(\beta)A(c)) \\
\Rightarrow \\
\text{HT}(\Gamma')(c), \{\text{ST}(\alpha(c,z), \text{ST}(\alpha(z,y), \text{CT}(A)(y))\}^{\beta, y, z} \\
\text{HT}(\Gamma')(c), \{\exists z(\text{ST}(\alpha(c,z) \land \text{ST}(\alpha(z,y)), \text{CT}(A)(y))\}^{\beta, y, z} \\
\text{HT}(\Gamma')(c), \text{HT}((\alpha; \beta)A(c)) \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\frac{\Gamma; [\alpha](\alpha^+)A \quad \Gamma; [\alpha^+]A}{\text{HT}(\Gamma')(c), \text{HT}((\alpha^+)A(c))} \quad \Rightarrow \\
\hline
\begin{array}{c}
\text{HT}(\Gamma')(c), \text{HT}([\alpha]A(c)) \\
\text{HT}(\Gamma')(c), \{\text{ST}(\alpha(c,y), \text{CT}(A)(y))\}^{\beta, y} \\
\text{HT}(\Gamma')(c), \{\text{ST}(\alpha(c,y), \text{CT}(A)(y))\}^{\beta, y} \\
\text{HT}(\Gamma')(c), \{\text{TC}(\text{ST}(\alpha(c,y), \text{CT}(A)(y))\}^{\beta, y} \\
\text{HT}(\Gamma')(c), \text{HT}((\alpha^+)A(c)) \\
\end{array}
\end{array}
\]

34
Thus, if only finitely many distinct subproofs.

Notice that each rule in Proof.

\[ \text{LPD} \]

6.4 Justifying regularity and progress

Proposition 6.11. If \( D \) is regular, then so is \( \text{HT}(D)(c) \).

Proof. Notice that each rule in \( D \) is translated to a finite derivation in \( \text{HT}(D)(c) \). Thus, if \( D \) has only finitely many distinct subproofs, then also \( \text{HT}(D)(c) \) has only finitely many distinct subproofs.

Remark 6.10 (Deeper inference). Observe that \( \text{HTC} \) can also simulate ‘deeper’ program rules than are available in \( \text{LPD}^+ \). E.g. a rule \( \Gamma, (\alpha \langle \alpha^+ \rangle A) \rightarrow_\text{HTC} \Gamma, (\langle \alpha \rangle \langle \alpha^+ \rangle A)(c) \) may be simulated too (similarly for \( \| \)). Thus \( \langle \alpha^+ \rangle \langle b \rangle p \supset_\text{HTC} (\langle \alpha^+ \rangle \langle b \cup c \rangle p \).

6.4 Justifying regularity and progress

Proposition 6.12. If \( D \) is progressing, then so is \( \text{HT}(D)(c) \).

We need to show that every infinite branch of \( \text{HT}(D)(c) \) has a progressing hypertrace. Since the \( \text{HT} \) translation is defined stepwise on the individual steps of \( D \), we can associate to each infinite branch \( B \) of \( \text{HT}(D)(c) \) a unique infinite branch \( B' \) of \( D \). Since \( D \) is progressing, let \( \tau = (F_i)_{i < \omega} \) be a progressing thread along \( B' \). By inspecting the rules of \( \text{LPD}^+ \) (and by definition of progressing thread), for some \( k \in \mathbb{N} \), each \( F_i \) for \( i > k \) has the

\[
\begin{align*}
\Gamma', (\alpha)(\langle \alpha^+ \rangle A) &\rightarrow \text{HT}(\Gamma')(c), \text{HT}(\langle \alpha \rangle \langle \alpha^+ \rangle A)(c) \\
&\vdash \text{HT}(\Gamma')(c), \text{HT}(\langle \alpha \rangle \langle \alpha^+ \rangle A)(c)
\end{align*}
\]
form: \([\alpha_1, \ldots, \alpha_n] A\), for some \(n_i \geq 0\). So, for \(i > k\), \(HT(F_i)(d_i)\) has the form:
\[
\{ST(\alpha_1)(c, d_{i,1})\}^\circ, \ldots, \{ST(\alpha_{n_i})(d_{i, n_i-1}, d_{i, n_i})\}^\circ, \{TC(ST(\alpha))(d_{i, n_i}, d_{i+1})\}^\circ, HT(A)(d_{i+1})
\]

By inspection of the HT-translation (Dfn. 6.10), whenever \(F_{i+1}\) is an immediate ancestor of \(F_i\) in \(B'\), there is a path from the cedent \(\{TC(ST(\alpha))(d_{i+1, m_i+1}, d_{i+1})\}^\circ\) to the cedent \(\{TC(ST(\alpha))(d_{i, n_i}, d_{i+1})\}^\circ\) in the graph of immediate ancestry along \(B\). Thus, since \(\tau = (F_i)_{i<\omega}\) is a trace along \(B'\), we have a (infinite) hypertrace of the form \(H_\tau := (\{\Delta_i \tau \cup TC(ST(\alpha))(d_{i, n_i}, d_{i+1})\}^\circ)_{i>k}\) along \(B\). By construction \(\Delta_i = \emptyset\) for infinitely many \(i > k'\), and so \(H_\tau\) has just one infinite trace. Moreover, by inspection of the \([\ast]\) step in Dfn. 6.9, this trace progresses in \(B\) every time \(\tau\) does in \(B'\), and so progresses infinitely often. Thus, \(H\) is a progressing hypertrace. Since the choice of the branch \(B\) of \(D\) was arbitrary, we are done.

\[\Box\]

6.5 Putting it all together

We can now finally conclude our main simulation theorem:

**Proof of Thm. 6.1 sketch.** Let \(A\) be a PDL\(^+\) formula s.t. \(A\). By the completeness result for LPD\(^+\), Thm. 6.4, we have that LPD\(^+\) \(\vdash \text{cyc} \ A\), say by a cyclic proof \(D\). From here we construct the HTC preproof \(HT(D)(c)\) which, by Props. 6.11 and 6.12, is in fact a cyclic proof of \(HT(A)(c)\). Finally, we apply some basic \(\lor, \land, \exists, \forall\) steps to obtain a cyclic HTC proof of \(ST(A)(c)\).

\[\Box\]

7 Extension by equality and simulating full PDL

We now briefly explain how our main results are extended to the ‘reflexive’ version of TCL.

7.1 Hypersequent system with equality

The language of HTC\(_\omega\) allows further atomic formulas of the form \(s = t\) and \(s \neq t\). The calculus HTC\(_\omega\) extends HTC by the rules:

\[
\frac{}{S, \{\Gamma\}^x} \hspace{2cm} \frac{}{S, \{\Gamma(s), \Delta(s)\}^x} \neq \frac{}{S, \{\Gamma(s), s \neq t\}^x, \{\Delta(t)\}^x} \tag{12}
\]

The notion of immediate ancestry for formulas and cedents is colour-coded in \(\{\\}\) just as we did for HTC in Section 6.2. The resulting notions of (pre)proof, (hyper)trace and progress are as in Dfn. 4.6. Specifically, we have that in rule = as typeset in \(\{\\}\) no infinite trace can include formula \(t = t\). Moreover, in rule \(\neq\) no infinite trace can include formulas in \(\Delta(t)\), while all formulas occurring in \(\Delta(s)\) belong to a trace where \(s \neq t\) belongs.

The simulation of Cohen and Rowe’s system TC\(_G\) extends to their reflexive system, RTC\(_G\), by the definition of their operator \(RTC(\forall x, y. A)(s, t) := TC(\lambda x, y. x = y \lor A)(s, t)\). Semantically, it is correct to set \(RTC(A)(s, t)\) to be \(s = t \lor TC(A)(s, t)\), but this encoding does not lift to the Cohen-Rowe rules for RTC.
7.2 Extending the soundness argument

Understanding that structures interpret $=$ as true equality, a modular adaptation of the soundness argument for HTC, cf. Sec. 5 yields:

**Theorem 7.1** (Soundness of HTC$_=$). If HTC$_= \vdash_{nf} S$ then $\models S$.

*Proof sketch.* In the soundness argument for HTC, in Lem. 5.4 we must further consider cases for equality as follows:

> **Case $(\equiv)$**

\[
\frac{S_{t+1} = Q_s[\Gamma]^x}{S_i = Q_s[t = t, \Gamma]^x}
\]

By assumption, $\rho_i \not\models Q_s[\Gamma] \land \rho_i \not\models \{t = t\}^x$, i.e., $\rho_i \models \forall x(\forall \Gamma \lor t \neq t)$. Since $fv(\rho) \cap x = \emptyset$, this is equivalent to $\rho_i \models \forall x(\forall \Gamma)$. We set $\rho_{i+1} = \rho_i$ and conclude that $\rho_{i+1} \not\models \{\Gamma\}^x$.

> **Case $(\neq)$**

\[
\frac{S \models Q_s[\Gamma(s), \Delta(s)]^x}{S_r \models Q_s[\Gamma(s), s \neq t]^x \land \{\Delta(t)\}^x}
\]

By assumption, $\rho_i \not\models Q_s[\Gamma(s), s \neq t]^x$ and $\rho_i \not\models \{\Delta(t)\}^x$. Thus,

\[
\rho_i \models \forall x(\forall \Gamma(s) \lor s = t) \quad \text{and} \quad \rho_i \models \forall x(\forall \Delta(t))
\]

Set $\rho_{i+1} = \rho_i$. If $\rho_i \models \forall x(\forall \Gamma(s))$, then $\rho_{i+1} \models \forall x(\forall \Gamma(s) \lor \forall \Delta(s))$, and thus $\rho_{i+1} \not\models \{\Gamma(s) \lor \Delta(s)\}^x$. Otherwise, if $\rho_i \models \forall x(s = t)$, we can safely substitute term $t$ with term $s$ in the second conjunct, obtaining $\rho_{i+1} \models \forall x(\forall \Delta(s))$. Thus, $\rho_{i+1} \models \forall x(\forall \Gamma(s) \lor \forall \Delta(s))$, and $\rho_{i+1} \not\models \{\Gamma(s) \lor \Delta(s)\}^x$.

For the construction of the ‘false trace’ in Lem. 5.6 we add the following cases for equality:

> **Case $(\equiv)$**. Suppose $[\Gamma_i]^x = \{t = t, \Gamma\}^x$. By assumption, $\rho^x \models \forall x(t \neq t \lor \forall \Gamma)$ and thus $\rho^x, \delta_H \models t \neq t$ or $\rho^x, \delta_H \models \forall \Gamma$. Since the first disjunct cannot hold, the trace cannot follow the formula $t = t$. Thus, if $F = C$ for some $C$ occurring in $\Gamma$ and $\rho^x, \delta_H \models \Gamma$, set $F' = F$ and conclude that $\rho^x, \delta_H \models \Gamma$.

> **Case $(\neq)$**. The hypertrace $H$ cannot go through the structure $\{\Delta(t)\}^x$, because by hypothesis $H$ is infinite. Thus, $[\Gamma_i]^x = \{t = t, s \neq t\}^x$. By assumption, we have that either:

\[
\rho^x, \delta_H \models \forall \Gamma(s) \quad \text{or} \quad \rho^x, \delta_H \models s = t
\]

If $F = C(s)$ for some $C(s)$ in $\Gamma(s)$ and $\rho^x, \delta_H \models C$, set $F' = F$. Otherwise, if $\rho^x, \delta_H \not\models \forall \Gamma(s)$, then $\rho^x, \delta_H \models s = t$ and $F$ is $s \neq t$. At every occurrence of rule $\neq$ all the formulas occurring in $\Delta(s)$ are immediate ancestors of $s \neq t$. Moreover, by assumption $\rho^x \models \forall x(\forall \Delta(t))$. By the truth condition associated to $\forall$ and since $x$ is contained in the domain of $\delta_H$, we have that $\rho^x, \delta_H \models \forall \Delta(t)$. Since $\rho^x, \delta_H \models s = t$, we conclude that $\rho^x, \delta_H \models \forall \Delta(s)$. Thus, there exists a $D(s) \in \Delta(s)$ such that $\rho^x, \delta_H \models \forall D(s)$. Set $F' = D(s)$. 

37
7.3 Completeness for PDL (with tests)

Turning to the modal setting, PDL may be defined as the extension of $PDL^+$ by including a program $A^?$ for each formula $A$. Semantically, we have $A^?_{\mathcal{M}} = \{(v, v) : \mathcal{M}, v \models A \}$. From here we may define $\varepsilon := \top$ and $\alpha^* := (\varepsilon \cup \alpha)^+$. Again, while it is semantically correct to set $\alpha^* = \varepsilon \cup \alpha^+$, this encoding does not lift to the standard sequent rules for $\ast$.

The system LPD is obtained from $LPD^+$ by including the rules:

\[ \frac{\Gamma, A \Delta B}{\Gamma, (A^?)B} \quad \frac{\Gamma, \bar{A}, B}{\Gamma, [A^?]B} \]

The notion of ancestry for formulas is defined as for $LPD^+$ (Def. 6.3) and colour-coded in the rules. The resulting notions of (pre)proof, thread and progress are as in Def. 6.3. We write $LPD \vdash_{cyc} A$ meaning that there is a cyclic proof of $A$ in LPD. Just like for $LPD^+$, a standard encoding of LPD into the $\mu$-calculus yields its soundness and completeness, thanks to known sequent systems for the latter, cf. [25, 22, 19].

**Theorem 7.2 (Soundness and completeness, [19]).** Let $A$ be a PDL formula. $\models A$ iff $LPD \vdash_{cyc} A$.

Again, a modular adaptation of the simulation of $LPD^+$ by HTC, cf. Sec. 6 yields:

**Theorem 7.3 (Completeness for PDL).** Let $A$ be a PDL formula. If $\models A$ then $HTC_{\ast} \vdash_{cyc} ST(A)(c)$.

**Proof sketch.** For the Simulation Theorem, Thm. 7.3 we must add the following cases for the test rules:

\[ \frac{\Gamma, A \Delta B}{\Gamma, (A^?)B} \quad \frac{\Gamma, A, B}{\Gamma, (A^?)B} \quad \frac{\Gamma, A, B}{\Gamma, (A^?)B} \]

\[ \frac{\Gamma, A}{\Gamma, [A^?]B} \quad \frac{\Gamma, A}{\Gamma, [A^?]B} \quad \frac{\Gamma, A}{\Gamma, [A^?]B} \]

\[ \frac{\Gamma, A}{\Gamma, [A^?]B} \quad \frac{\Gamma, A}{\Gamma, [A^?]B} \quad \frac{\Gamma, A}{\Gamma, [A^?]B} \]

\[ \frac{\Gamma, A}{\Gamma, [A^?]B} \quad \frac{\Gamma, A}{\Gamma, [A^?]B} \quad \frac{\Gamma, A}{\Gamma, [A^?]B} \]

38
8 Conclusions

In this work we proposed a novel cyclic system HTC for Transitive Closure Logic (TCL) based on a form of hypersequents. We showed a soundness theorem for standard semantics, requiring an argument bespoke to our hypersequents. Our system is cut-free, rendering it suitable for automated reasoning via proof search. We showcased its expressivity by demonstrating completeness for PDL, over the standard translation. In particular, we demonstrated formally that such expressivity is not available in the previously proposed system TC\textsubscript{G} of Cohen and Rowe (Thm. 4.5). Our system HTC locally simulates TC\textsubscript{G} too (Thm. 4.12).

As far as we know, HTC is the first cyclic system employing a form of deep inference resembling alternation in automata theory, e.g. wrt. proof checking, cf. Prop. 4.11. It would be interesting to investigate the structural proof theory that emerges from our notion of hypersequent. As hinted at in Exs. 4.7 and 4.8 HTC exhibits more liberal rule permutations than usual sequents, so we expect their focussing and cut-elimination behaviours to similarly be richer, cf. [21, 20].

Finally, our work bridges the cyclic proof theories of (identity-free) PDL and (reflexive) TCL. With increasing interest in both modal and predicate cyclic proof theory, it would be interesting to further develop such correspondences.

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