HIGHER ORDER TWO-POINT BOUNDARY VALUE PROBLEMS
WITH ASYMMETRIC GROWTH

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Abstract. In this work it is studied the higher order nonlinear equation
\[ u^{(n)}(x) = f(x, u(x), u'(x), \ldots, u^{(n-1)}(x)) \]
with \( n \in \mathbb{N} \) such that \( n \geq 2 \), \( f : [a, b] \times \mathbb{R} \rightarrow \mathbb{R} \) a continuous function, and the two-point boundary conditions
\[ u^{(i)}(a) = A_i, \quad A_i \in \mathbb{R}, \quad 0 \leq i \leq n - 3, \]
\[ u^{(n-1)}(a) = u^{(n-1)}(b) = 0. \]

From one-sided Nagumo-type condition, allowing that \( f \) can be unbounded, it is obtained an existence and location result, that is, besides the existence, given by Leray-Schauder topological degree, some bounds on the solution and its derivatives till order \( (n-2) \) are given by well ordered lower and upper solutions.

An application to a continuous model of human-spine, via beam theory, will be presented.

1. Introduction. In this paper it is studied the \( n^{th} \)-order differential equation
\[ u^{(n)}(x) = f(x, u(x), u'(x), \ldots, u^{(n-1)}(x)), \quad (1) \]
for \( n \geq 2, x \in [a, b] \) and \( f : [a, b] \times \mathbb{R}^n \rightarrow \mathbb{R} \) a continuous function, with the boundary conditions
\[ u^{(i)}(a) = A_i, \quad A_i \in \mathbb{R}, \quad 0 \leq i \leq n - 3 \]
\[ u^{(n-1)}(a) = 0, \quad u^{(n-1)}(b) = 0. \quad (2) \]

These higher order two-point boundary value problems have been studied in [12], in [6, 7], for even-order differential equations and in [1], for multi-point boundary conditions, among others. In the referred papers it is not considered a nonlinear fully differential equation, that is, the nonlinear part cannot depend on every derivatives till order \( n - 1 \). This dependence is overcome assuming a Nagumo-type growth condition on the nonlinear part (see [2]). In short it is assumed the existence of continuous functions \( \gamma_i, \Gamma_i(x) : [a, b] \rightarrow \mathbb{R} \), and a set
\[ E = \{(x, y_0, \ldots, y_{n-1}) \in [0, 1] \times \mathbb{R}^n : \gamma_i(x) \leq y_i \leq \Gamma_i(x), \quad i = 0, \ldots, n - 2 \}, \quad (3) \]
such that the nonlinearity \( f : E \rightarrow \mathbb{R} \) verifies
\[ |f(x, y_0, \ldots, y_{n-1})| \leq \varphi_E(\|y_{n-1}\|), \quad \forall (x, y_0, \ldots, y_{n-1}) \in E, \quad (4) \]

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for some positive continuous function $\varphi_E$ satisfying
\[ \int_0^{+\infty} \frac{\xi}{\varphi_E(\xi)} \, d\xi = +\infty. \tag{5} \]
As it can be seen in [5], for some second order two-point boundary value problems this type of restriction is a necessary condition to obtain a solution. In this work the bound restriction defined by the bilateral condition (4) is replaced by a weaker one-sided Nagumo condition,
\[ f(x, y_0, \ldots, y_{n-1}) \leq \varphi_E(|y_{n-1}|), \tag{6} \]
or
\[ f(x, y_0, \ldots, y_{n-1}) \geq -\varphi_E(|y_{n-1}|), \tag{7} \]
for every $(x, y_0, \ldots, y_{n-1}) \in E$, allowing an unbounded behaviour on $f$ as it is suggested, for third order problems, in [3, 4]. As an example, it is mentioned that function $f : [0, 1] \times \mathbb{R}^n \to \mathbb{R}$, $n \geq 2$, given by
\[ f(x, y_0, \ldots, y_{n-1}) = -(y_0 + 1)[(n-2)! - y_{n-2}]^2 - y_{n-1}^{2p+4}, \]
with $p \in \mathbb{N}_0$, verifies (6) in $E_* = \left\{ (x, y_0, \ldots, y_{n-1}) \in [0, 1] \times \mathbb{R}^n : \frac{d}{dx^j} (-x^n) \leq y_j \leq \frac{d}{dx^j} (x + 1)^{n-2}, \right. \]
\[ j = 0, \ldots, n-3, -\frac{n^2}{2} x^2 \leq y_{n-2} \leq (n-2)! \]
with $\varphi_{E_*} (x_{n-1}) := k > 0$, but condition (4) does not hold in $E_*$. In fact, if there was a function $\varphi_{E_*} \in C(\mathbb{R}_0^+, \mathbb{R}^+)$ verifying (4) and (5) then, in particular,
\[ -f(x, y_0, \ldots, y_{n-1}) \leq \varphi_{E_*}(|y_{n-1}|), \quad \forall (x, y_0, \ldots, y_{n-1}) \in E_* \]
and, for $x \in [0, 1], y_0 = 0, y_{n-2} = 0$ and $y_{n-1} \in \mathbb{R}$,
\[ -f(x, 0, y_1, \ldots, y_{n-3}, 0, y_{n-1}) \]
\[ = [(n-2)!]^2 + y_{n-1}^{2p+4} \leq \varphi_{E_*}(|y_{n-1}|) \]
and the following contradiction is obtained
\[ +\infty > \int_0^{+\infty} \frac{\xi}{[(n-2)!]^2 + \xi^{2p+4}} \, d\xi \]
\[ \geq \int_0^{+\infty} \frac{\xi}{\varphi_{E_*}(|\xi|)} \, d\xi \]
\[ = +\infty. \]

The arguments make use of lower and upper solutions technique and topological degree theory to obtain existence and location results, meaning that it is proved not only the existence of solutions, but some information about its localization and of some derivatives is also attained. The existence of positive (negative) solutions is a particular case of the above results because it is enough to consider a nonnegative lower solution or a non positive upper one.

Last section provides an application of these kind of problems, for $n = 4$, to continuous models of the human-spine to estimate, by beam theory, its lateral displacement under some loading forces. It is pointed out that, in these models, lower and upper solutions method is highly recommended due to the location part. In
fact, it is estimated not only the total displacement of the column but also its torsion and curvature.

2. Definitions and a priori bound. This section will provide some definitions to be used forward and an a priori estimation for the derivative $u^{(n-1)}$, which is one of the key points of the Nagumo-type condition. In fact, the growth restriction on the nonlinear part is given by the one sided Nagumo-type assumption, in spite of the usual bilateral one, as it is precisely in the following definition:

**Definition 1.** Given a subset $E \subset [a, b] \times \mathbb{R}^n$, a function $g : [a, b] \times \mathbb{R}^n \rightarrow \mathbb{R}$ is said to satisfy an one-sided Nagumo-type condition in $E$ if there exists $\varphi \in C \left( \mathbb{R}^+_0, [k, +\infty) \right)$, with $k > 0$, verifying (6) and (5).

The a priori bound on the $(n-1)^{th}$ derivative is obtained in the following way:

**Lemma 1.** Assume that $\gamma_j, \Gamma_j \in C \left( [a, b], \mathbb{R} \right)$ are such that

$$
\gamma_j(x) \leq \Gamma_j(x), \ \forall x \in [a, b], \ j = 0, \ldots, n-2,
$$

and define $r \geq 0$ such that

$$
r := \max \left\{ \frac{\Gamma_{n-2}(b) - \gamma_{n-2}(a)}{b - a}, \frac{\Gamma_{n-2}(a) - \gamma_{n-2}(b)}{b - a} \right\}.
$$

Let $f : [a, b] \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous function verifying (6) and

$$
\int_r^{+\infty} \frac{s}{\varphi(s)} ds > \max_{x \in [a, b]} \Gamma_{n-2}(x) - \min_{x \in [a, b]} \gamma_{n-2}(x)
$$

in

$$
E = \{(x, y_0, \ldots, y_{n-1}) \in [a, b] \times \mathbb{R}^n : \gamma_j(x) \leq y_j \leq \Gamma_j(x), \ j = 0, \ldots, n-2\}.
$$

Then, there exists $R > 0$ such that every $u(x)$ solution of (1) verifying

$$
u^{(n-1)}(a) \leq r, \ u^{(n-1)}(b) \geq -r
$$

and

$$
\gamma_j(x) \leq u^{(j)}(x) \leq \Gamma_j(x), \ \forall x \in [a, b], \ j = 0, \ldots, n-2,
$$

satisfies $\|u^{(n-1)}\|_\infty < R$.

**Remark 1.** Observe that:

(i) condition (5) implies (8).

(ii) $R$ depends only on $\Gamma_{n-2}, \gamma_{n-2}$ and $\varphi$.

(iii) If condition (6) is replaced by (7) the a priori estimation given by Lemma 1 still holds if it is assumed, instead (9),

$$
u^{(n-1)}(a) \geq -r \text{ and } u^{(n-1)}(b) \leq r.
$$

**Proof.** Let $u$ be a solution of the differential equation (1) satisfying (9) and (10).

Suppose that $|u^{(n-1)}(x)| > r$, for every $x \in [a, b]$. In the case $u^{(n-1)}(x) > r$, for all $x \in [a, b]$, the following contradiction is obtained

$$
\Gamma_{n-2}(b) - \gamma_{n-2}(a) \geq u^{(n-2)}(b) - u^{(n-2)}(a) = \int_a^b u^{(n-1)}(\tau) \ d\tau
$$

$$
> \int_a^b r \ d\tau \geq \Gamma_{n-2}(b) - \gamma_{n-2}(a).
$$
If \( u^{(n-1)}(x) < -r \), for every \( x \in [a, b] \), a similar contradiction is achieved. So, there is \( x \in [a, b] \) such that \( |u^{(n-1)}(x)| \leq r \).

By (8), we can take \( R \geq r \) such that

\[
\int_r^R \frac{s}{\varphi(s)} \, ds > \max_{x \in [a, b]} \Gamma_{n-2}(x) - \min_{x \in [a, b]} \gamma_{n-2}(x). \tag{11}
\]

If \( |u^{(n-1)}(x)| \leq r \), for every \( x \in [a, b] \), then \( u^{(n-1)}(x) < R \) and the proof would be finished. If not, take \( x_1 \in [a, b] \) such that \( u^{(n-1)}(x_1) > r \) or \( u^{(n-1)}(x) < -r \).

In the first case consider, by (9), an interval \( J = [x_0, x_1] \) such that \( u^{(n-1)}(x_0) = r \), \( u^{(n-1)}(x) > r, \forall \ x \in J \setminus \{x_0\} \).

Then, with a convenient change of variable, by (6) and (11), it is obtained, for arbitrary \( x_2 \in J \setminus \{x_0\} \),

\[
\int_{u^{(n-1)}(x_0)}^{u^{(n-1)}(x_2)} \frac{s}{\varphi(s)} \, ds = \int_{x_0}^{x_2} \frac{u^{(n-1)}(x)}{\varphi(u^{(n-1)}(x))} u^{(n)}(x) \, dx
\]
\[
= \int_{x_0}^{x_2} f(x, u(x), u'(x), \ldots, u^{(n-1)}(x)) \frac{u^{(n-1)}(x)}{\varphi(u^{(n-1)}(x))} u^{(n)}(x) \, dx
\]
\[
\leq \int_{x_0}^{x_2} u^{(n-1)}(x) \, dx = u^{(n-2)}(x_2) - u^{(n-2)}(x_0)
\]
\[
\leq \max_{x \in [a, b]} \Gamma_{n-2}(x) - \min_{x \in [a, b]} \gamma_{n-2}(x)
\]
\[
< \int_r^R \frac{s}{\varphi(s)} \, ds.
\]

Therefore, \( u^{(n-1)}(x_2) < R \) and, as \( x_1 \) is arbitrary as long as \( u^{(n-1)}(x) > r \), then \( u^{(n-1)}(x) < R \), for every \( x \in J \).

By a similar way, it can be proved that \( u^{(n-1)}(x) > -R \), for every \( x \in [a, b] \) such that \( u^{(n-1)}(x) < -r \).

The method of lower and upper solutions is an important tool to obtain existence and location results in next sections. The functions used in this technique are defined as follows:

**Definition 2.** Consider \( n \geq 2 \) and \( A_i \in \mathbb{R} \), for \( i = 0, \ldots, n-3 \).

A function \( \alpha \in C^n ([a, b]) \) is said to be a lower solution of problem (1)-(2) if

\[
\alpha^{(n)}(x) \geq f(x, \alpha(x), \alpha'(x), \ldots, \alpha^{(n-1)}(x)), \tag{12}
\]

for \( x \in [a, b] \), and

\[
\alpha^{(i)}(a) \leq A_i, \quad i = 0, \ldots, n-3, \quad \alpha^{(n-1)}(a) \geq 0, \quad \alpha^{(n-1)}(b) \leq 0. \tag{13}
\]

A function \( \beta \in C^n ([a, b]) \) is said to be an upper solution of problem (1)-(2), if it verifies the reversed inequalities.
3. Existence and location result. The main result is an existence and location theorem, as is usual in lower and upper solutions technique. In this case some data on the location of the derivatives until \((n-2)\)th-order are also given.

**Theorem 1.** Assume that \(\alpha, \beta \in C^n([a, b])\) are lower and upper solutions of problem \((1)-(2)\), respectively, such that
\[
\alpha^{(n-2)}(x) \leq \beta^{(n-2)}(x), \quad \forall x \in [a, b],
\]
and define the set
\[
E_\ast = \{(x, y_0, \ldots, y_{n-1}) \in [a, b] \times \mathbb{R}^n : \alpha^{(i)}(x) \leq y_i \leq \beta^{(i)}(x), \quad i = 0, \ldots, n-2\}.
\]
Let \(f : [a, b] \times \mathbb{R}^n \to \mathbb{R}\) be a continuous function verifying the one-sided Nagumo-type condition \((6)\) and
\[
f(x, \alpha(x), \ldots, \alpha^{(n-3)}(x), y_{n-2}, y_{n-1}) \\
\geq f(x, y_0, \ldots, y_{n-1}) \\
\geq f(x, \beta(x), \ldots, \beta^{(n-3)}(x), y_{n-2}, y_{n-1}),
\]
for fixed \(x, y_{n-2}, y_{n-1}\) and
\[
\alpha^{(i)}(x) \leq y_i \leq \beta^{(i)}(x), \quad i = 0, \ldots, n-3, \forall x \in [a, b].
\]
Then problem \((1)-(2)\) has at least one solution \(u \in C^n([a, b])\) such that
\[
\alpha^{(i)}(x) \leq u^{(i)}(x) \leq \beta^{(i)}(x),
\]
for every \(x \in [a, b]\) and \(i = 0, \ldots, n-2\).

**Remark 2.** The relations \(\alpha^{(j)}(x) \leq \beta^{(j)}(x), \quad j = 0, \ldots, n-3\) are obtained from \((14)\) by integration and boundary conditions given by Definition 2.

**Proof.** For \(i = 0, \ldots, n-2\) define the continuous functions \(\xi_i : [a, b] \times \mathbb{R} \to \mathbb{R}\) given by
\[
\xi_i(x, y_i) = \begin{cases} 
\beta^{(i)}(x) & \text{if } y_i > \beta^{(i)}(x), \\
y_i, & \text{if } \alpha^{(i)}(x) \leq y_i \leq \beta^{(i)}(x), \\
\alpha^{(i)}(x), & \text{if } y_i < \alpha^{(i)}(x),
\end{cases}
\]
and, for \(\lambda \in [0, 1]\), the auxiliary problem composed by the equation
\[
u^{(n-1)}(x) \\
= \lambda f(x, \xi_0(x, u), \ldots, \xi_{n-2}(x, u^{(n-2)}), u^{(n-1)}),
\]
and the boundary conditions
\[
u^{(i)}(a) = \lambda A_i, \quad \text{for } i = 0, \ldots, n-3, \\
u^{(n-1)}(a) = 0, \quad u^{(n-1)}(b) = 0.
\]
Take \(r_\ast > 0\) such that, for every \(x \in [a, b]\),
\[
r_\ast < \alpha^{(n-2)}(x) \leq \beta^{(n-2)}(x) < r_\ast, \\
f(x, \alpha(x), \ldots, \alpha^{(n-2)}(x), 0) - r_\ast - \alpha^{(n-2)}(x) < 0, \\
f(x, \beta(x), \ldots, \beta^{(n-2)}(x), 0) + r_\ast - \beta^{(n-2)}(x) > 0,
\]
Step 1: Every solution $u$ of problem (17)-(18) verifies $|u^{(n-2)}(x)| < r_*$ and $|u^{(i)}(x)| < r_i$, in $[a,b]$, with $r_*$ given above and

$$r_i := \sum_{k=i}^{n-3} |A_k| (b-a)^{k-i} + r_*(b-a)^{n-2-i},$$

for $i = 0, \ldots, n-3$, independently of $\lambda \in [0,1]$.

Let $u$ be a solution of the problem (17)-(18).

Assume, by contradiction, that there exists $x \in [a,b]$ such that $u^{(n-2)}(x) \geq r_*$ or $u^{(n-2)}(x) \leq -r_*$. In the first case define

$$u^{(n-2)}(x_0) := \max_{x \in [a,b]} u^{(n-2)}(x) \quad (\geq r_* > 0).$$

If $x_0 \in [a,b]$, then $u^{(n-1)}(x_0) = 0$ and $u^{(n)}(x_0) \leq 0$. For $\lambda \in [0,1]$, by (15), (19) and (20), it is obtained the following contradiction

$$0 \geq u^{(n)}(x_0)$$

$$= \lambda f(x_0, \xi_0(x_0, u(x_0)), \ldots, \xi_{n-2}(x_0, u^{(n-2)}(x_0)), u^{(n-1)}(x_0)) + u^{(n-2)}(x_0)$$

$$- \lambda \xi_{n-2}(x_0, u^{(n-2)}(x_0))$$

$$= \lambda f(x_0, \xi_0(x_0, u(x_0)), \ldots, \xi_{n-3}(x_0, u^{(n-3)}(x_0)), \beta^{(n-2)}(x_0), 0)$$

$$+ u^{(n-2)}(x_0) - \lambda \beta^{(n-2)}(x_0)$$

$$\geq \lambda f(x_0, \beta(x_0), \ldots, \beta^{(n-2)}(x_0), 0) + u^{(n-2)}(x_0) - \lambda \beta^{(n-2)}(x_0)$$

$$= \lambda \left[ f(x_0, \beta(x_0), \ldots, \beta^{(n-2)}(x_0), 0) + r_* - \beta^{(n-2)}(x_0) \right]$$

$$+ u^{(n-2)}(x_0) - \lambda r_* > 0.$$

If $x_0 = a$ then $\max_{x \in [a,b]} u^{(n-2)}(x) = u^{(n-2)}(a) \quad (\geq r_* > 0)$. By (18), $u^{(n)}(a) \leq 0$ and applying the above arguments with $x_0$ replaced by $a$, a similar contradiction holds.

In the case $x_0 = b$ the arguments are analogous. Thus, $u^{(n-2)}(x) < -r_*$, for every $x \in [a,b]$. In a similar way it can be proved that $u^{(n-2)}(x) > -r_*$, for every $x \in [a,b]$.

Furthermore, since $u^{(i)}(a) = \lambda A_i$, for $i = 0, \ldots, n-3$, the estimates $|u^{(i)}(x)| < r_i$, for every $x \in [a,b]$, where $r_i$ is defined by $r_i := \sum_{k=i}^{n-3} |A_k| (b-a)^{k-i} + r_*(b-a)^{n-2-i}$, are easily obtained by integration.

Step 2: There exists $R > 0$ such that every solution $u$ of problem (17)-(18) satisfies $|u^{(n-1)}(x)| < R$, in $[a,b]$, independently of $\lambda \in [0,1]$.

If $u$ is a solution of problem (17)-(18) then

$$u^{(n)}(x) = \lambda f(x, \xi_0(x, u(x)), \ldots, \xi_{n-2}(x, u^{(n-2)}(x)), u^{(n-1)}(x))$$

$$+ u^{(n-1)}(x) - \lambda \xi_{n-2}(x, u^{(n-2)}(x)).$$

Consider the set

$$E_\ast := \{(x, y_0, \ldots, y_{n-1}) \in [a,b] \times \mathbb{R}^n : -r_i \leq y_i \leq r_i, \ i = 0, \ldots, n-3,$$

$$-r_* \leq y_{n-2} \leq r_* \}.$$

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and the function $F_{\lambda} : E_{**} \rightarrow \mathbb{R}$ defined by

$$F_{\lambda}(x, y_0, \ldots, y_{n-1}) = \lambda f(x, \xi_0(x, y_0), \ldots, \xi_{n-2}(x, y_{n-2}), y_{n-1}) + y_{n-2} - \lambda \xi_{n-2}(x, y_{n-2}).$$

First it will be proved that $F_{\lambda}$ satisfies one-sided Nagumo-type condition in $E_{**}$. In fact, as $f$ fulfills (6) in $E_*$, by (16) and (19),

$$F_{\lambda}(x, y_0, \ldots, y_{n-1}) = \lambda f(x, \xi_0(x, y_0), \ldots, \xi_{n-2}(x, y_{n-2}), y_{n-1}) + y_{n-2} - \lambda \xi_{n-2}(x, y_{n-2}) \leq \lambda \varphi(|y_{n-1}|) + r_\ast - \lambda \alpha^{(n-2)}(x) \leq \varphi(|y_{n-1}|) + 2 r_\ast.$$

Therefore, defining $\varphi(y_{n-1}) := \varphi(|y_{n-1}|) + 2r_\ast$, $F_{\lambda}$ verifies condition (6) in $E_{**}$ with $\varphi(y_{n-1})$ replaced by $\varphi(y_{n-1})$ and as

$$\int_{r}^{+\infty} \frac{s}{\varphi(|s|)} ds = \int_{r}^{+\infty} \frac{s}{\varphi(|s|) + 2r_\ast} ds \geq \frac{1}{1 + \frac{2r_\ast}{r}} \int_{r}^{+\infty} \frac{s}{\varphi(|s|)} ds = +\infty,$$

then $F_{\lambda}$ satisfies (6) and, by Remark 1, condition (8) holds. Taking Step 1 into account and considering,

$$\gamma_i(x) = -r_i, \quad \Gamma_i(x) = r_i, \text{ for } i = 0, \ldots, n - 3,$$

$$\gamma_{n-2}(x) = -r_\ast, \quad \Gamma_{n-2}(x) = r_\ast, \quad r_\ast = \frac{2r_\ast}{b - a} (> 0),$$

Lemma 1 can be applied with $E$ replaced by $E_{**}$ and $r$ by $r_\ast$, since every solution $u$ of (17)-(18) satisfy $u^{(n-1)}(a) = 0 \leq r_\ast$ and $u^{(n-1)}(b) = 0 \geq -r_\ast$. So, there exists $R > 0$, depending on $r_\ast$ and $\varphi$, such that $|u^{(n-1)}(x)| < R$, for every $x \in [a, b]$. As $r_\ast$ and $\varphi$ do not depend on $\lambda$, the estimate $|u^{(n-1)}(x)| < R$, is independent of $\lambda$.

**Step 3:** For $\lambda = 1$, problem (17)-(18) has at least a solution $u_1(x)$.

Define the operators $L : C^n([a, b]) \subseteq C^{n-1}([a, b]) \rightarrow C([a, b]) \times \mathbb{R}^n$ by

$$Lu = (u^{(n)} - u^{(n-2)}, u(a), u'(a), \ldots, u^{(n-3)}(a), u^{(n-1)}(a), u^{(n-1)}(b))$$

and $N_{\lambda} : C^{n-1}([a, b]) \rightarrow C([a, b]) \times \mathbb{R}^n$ by

$$N_{\lambda} u = (\lambda f(x, \xi_0(x, u(x)), \ldots, \xi_{n-2}(x, u^{(n-2)}(x)), u^{(n-1)}(x)) - \lambda \xi_{n-2}(x, u^{(n-2)}(x)), \lambda A_0, \ldots, \lambda A_{n-3}, 0, 0)$$

Since $L^{-1}$ is compact, it can be defined the completely continuous operator $T_{\lambda} : (C^{n-1}([a, b]), \mathbb{R}) \rightarrow (C^n([a, b]), \mathbb{R})$ by $T_{\lambda}(u) = L^{-1}N_{\lambda}(u)$.

For $R$ given in Step 2, consider the set

$$\Omega = \left\{ y \in C^{n-1}([a, b]) : \|y^{(i)}\|_\infty < r_i, \; i = 0, \ldots, n - 3, \quad \|y^{(n-2)}\|_\infty < r_\ast, \quad \|y^{(n-1)}\|_\infty < R \right\}.$$

By Steps 1 and 2, for every solution $u$ of (17)-(18), $u \not\in \partial \Omega$ and so the degree $d(I - T_{\lambda}, \Omega, 0)$ is well defined for every $\lambda \in [0, 1]$. Then, by the invariance under homotopy, $d(I - T_0, \Omega, 0) = d(I - T_1, \Omega, 0)$.

As the equation $u = T_0(u)$ that is, the problem

$$\begin{cases}
  u^{(n)} - u^{(n-2)} = 0, \\
  u^{(i)}(a) = u^{(n-1)}(a) = u^{(n-1)}(b) = 0, \; \text{for } i = 0, \ldots, n - 3,
\end{cases}$$

has only the trivial solution, by degree theory, $d(I - T_0, \Omega, 0) = \pm 1$ and, therefore, equation $u = T_1(u)$ has at least one solution $u_1(x)$. This means that the problem composed by the equation
\[
u^{(n-1)}(x) = f(x, \xi_0(x, u(x)), \ldots, \xi_{n-2}(x, u^{(n-2)}(x)), u^{(n-1)}(x)) + u^{(n-2)}(x) - \xi_{n-2}(x, u^{(n-2)}(x))
\]
with boundary conditions (2) has at least a solution $u_1(x)$ in $\Omega$.

**Step 4:** The function $u_1(x)$ is also a solution of the initial problem.

In fact, this solution $u_1(x)$ will be solution of the initial problem (1)-(2) if it verifies on $[a, b]$
\[
a^{(i)}(x) \leq u_1^{(i)}(x) \leq \beta^{(i)}(x), \quad \text{for } i = 0, \ldots, n - 2.
\]
Suppose, by contradiction, that there exists $x \in [a, b]$ such that
\[
\alpha^{(n-2)}(x) > u_1^{(n-2)}(x)
\]
and define
\[
\min_{x \in [a, b]} \left[ u_1^{(n-2)}(x) - \alpha^{(n-2)}(x) \right] := u_1^{(n-2)}(x_1) - \alpha^{(n-2)}(x_1) < 0.
\]
If $x_1 \in ]a, b[$ then $u_1^{(n-1)}(x_1) = \alpha^{(n-1)}(x_1)$ and $u_1^{(n)}(x_1) - \alpha^{(n)}(x_1) \geq 0$. Therefore, by (12) and (15), the following contradiction is obtained
\[
0 \leq u_1^{(n)}(x_1) - \alpha^{(n)}(x_1)
\]
\[
\leq f(x_1, \xi_0(x_1, u_1(x_1)), \ldots, \xi_{n-3}(x_1, u_1^{(n-3)}(x_1)), \alpha^{(n-2)}(x_1), \alpha^{(n-1)}(x_1))
\]
\[
+ u_1^{(n-2)}(x_1) - \alpha^{(n-2)}(x_1) - f(x_1, \alpha(x_1), \ldots, \alpha^{(n-1)}(x_1))
\]
\[
< f(x_1, \xi_0(x_1, u_1(x_1)), \ldots, \xi_{n-3}(x_1, u_1^{(n-3)}(x_1)), \alpha^{(n-2)}(x_1), \alpha^{(n-1)}(x_1))
\]
\[
- f(x_1, \alpha(x_1), \ldots, \alpha^{(n-3)}(x_1), \alpha^{(n-2)}(x_1), \alpha^{(n-1)}(x_1)) \leq 0.
\]
If $x_1 = a$ then
\[
\min_{x \in [a, b]} \left[ u_1^{(n-2)}(x) - \alpha^{(n-2)}(x) \right] := u_1^{(n-2)}(a) - \alpha^{(n-2)}(a) < 0.
\]
and
\[
u_1^{(n-1)}(a^+) - \alpha^{(n-1)}(a^+) = u_1^{(n-1)}(a) - \alpha^{(n-1)}(a) \geq 0.
\]
So, by (13) and (18), $u_1^{(n-1)}(a) - \alpha^{(n-1)}(a) = 0$ and $u_1^{(n)}(a) - \alpha^{(n)}(a) \geq 0$. Therefore, applying similar arguments, with $x_1$ replacing by $a$ or $b$, contradictions will be obtained proving that $x_1 \neq a$ and $x_1 \neq b$. Then
\[
\alpha^{(n-2)}(x) \leq u_1^{(n-2)}(x), \quad \forall x \in [a, b].
\]

Using an analogous technique, it can be shown that $u_1^{(n-2)}(x) \leq \beta^{(n-2)}(x)$, for all $x \in [a, b]$ and, therefore,
\[
\alpha^{(n-2)}(x) \leq u_1^{(n-2)}(x) \leq \beta^{(n-2)}(x), \quad \forall x \in [a, b].
\]

Integrating both inequalities on $[a, x]$
\[
\alpha^{(n-3)}(x) - \alpha^{(n-3)}(a) + A \leq u_1^{(n-3)}(x) \leq \beta^{(n-3)}(x) - \beta^{(n-3)}(a) + A,
\]
and, by (13),
\[
\alpha^{(n-3)}(x) \leq u_1^{(n-3)}(x) \leq \beta^{(n-3)}(x), \quad \forall x \in [a, b].
\]
Iterating this integration
\[
\alpha^{(i)}(x) \leq u_1^{(i)}(x) \leq \beta^{(i)}(x), \forall x \in [a, b], \text{ for } i = 0, \ldots, n - 3
\]
and so, \( u_1(x) \) is a solution of (1)-(2).

4. Applications. The case \( n = 4 \) will be studied in this section. A “natural” application to a fourth order differential equation is in beam theory, where two-point boundary conditions mean the different types of support at the endpoints. For example, the problem
\[
\begin{align*}
\frac{d^4}{dx^4}y(x) &= f(x, u(x), u'(x), u''(x), u'''(x)) \\
u(a) &= u'(a) = u''(a) = u'''(b) = 0
\end{align*}
\]
models the bending of an elastic beam, cantilevered at the left endpoint and with null shear force (vertical) at both endpoints. A not so common application is the study of some mechanical properties of the human spine by a continuous beam model, illustrated in Fig 1, as it is done in [10, 11] and the references therein.

In short, the total lateral displacement, \( y(x) \), of the beam-column is the sum of the initial displacement, \( y_0(x) \), (known) and the lateral displacement due to the axial and transverse loads, \( y_1(x) \), i.e.,
\[
y(x) = y_0(x) + y_1(x).
\]
This unknown displacement \( y_1(x) \) is modelled (see [11]) by the problem composed by the differential equation
\[
EI \frac{d^4}{dx^4}y_1(x) - P y_1''(x) = Q(x) + P y_0''(x)
\]
where \( EI \) is the flexural rigidity of the beam-column, \( P \) the axial load and \( Q(x) \) the transverse load (Figure 1), and the boundary conditions
\[
y_1 \left( -\frac{L}{2} \right) = y_1' \left( -\frac{L}{2} \right) = y_1'' \left( -\frac{L}{2} \right) = y_1''' \left( \frac{L}{2} \right) = 0.
\]
Defining $J = [-L/2, L/2]$, the function $f$ by
\[ f(x, z_1, z_2, z_3, z_4) = \frac{P}{EI} z_3 + \frac{P}{EI} y_0''(x) + Q(x), \]
assuming that the initial lateral displacement $y_0 \in C^2(J)$ and the function $Q \in C(J)$ verify, for $x \in J$,
\[ b \left( EI - P \frac{x^2}{2} - PL^2 \right) \leq P y_0''(x) + Q(x) \leq a \left( EI - P \frac{x^2}{2} + PL^2 \right), \]
then, for some $0 < -a < b$, the functions
\[ \alpha(x) = a \left( \frac{x^4}{24} - \frac{L^2}{2} x^2 + \frac{47}{384} L^4 \right) \]
and
\[ \beta(x) = b \left( \frac{x^4}{24} + \frac{L^2}{2} x^2 + 2L^3 x + L^4 \right) \]
are, respectively, lower and upper solutions of problem (21)-(22). As $f$ verifies a Nagumo-type condition in every set
\[ E = \left\{ (x, z) \in J \times \mathbb{R} : \frac{a}{2} x^2 - a \ L^2 \leq z \leq \frac{b}{2} x^2 + b \ L^2 \right\} \]
then, by Theorem 1, there is a solution of problem (21)-(22) such that the lateral displacement due to the axial and transverse loads verifies
\[ a \left( \frac{x^4}{24} - \frac{L^2}{2} x^2 + \frac{47}{384} L^4 \right) \leq y_1(x) \leq b \left( \frac{x^4}{24} + \frac{L^2}{2} x^2 + 2L^3 x + L^4 \right) \]
\[ a \left( \frac{x^3}{6} - L^2 x \right) \leq y_1'(x) \leq b \left( \frac{x^3}{6} + L^2 x + 2L^3 \right) \]
and
\[ a \left( \frac{x^2}{2} - L^2 \right) \leq y''_1(x) \leq b \left( \frac{x^2}{2} + L^2 \right) \]
for every $x \in J$.

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