Computationally efficient sparse clustering

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Abstract: We study statistical and computational limits of clustering when the means of the centres are sparse and their dimension is possibly much larger than the sample size. Our theoretical analysis focuses on the simple model $X_i = z_i \theta + \varepsilon_i$, $z_i \in \{-1, 1\}$, $\varepsilon_i \sim \mathcal{N}(0, I)$, which has two clusters with centres $\theta$ and $-\theta$.

We provide a finite sample analysis of a new sparse clustering algorithm based on sparse PCA and show that it achieves the minimax optimal misclustering rate in the regime $\|\theta\| \to \infty$, matching asymptotically the Bayes error.

Our results require the sparsity to grow slower than the square root of the sample size. Using a recent framework for computational lower bounds—the low-degree likelihood ratio—we give evidence that this condition is necessary for any polynomial-time clustering algorithm to succeed below the BBP threshold. This complements existing evidence based on reductions and statistical query lower bounds. Compared to these existing results, we cover a wider set of parameter regimes and give a more precise understanding of the runtime required and the misclustering error achievable.

We also discuss extensions of our results to more than two clusters.

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1. Introduction

Clustering data points $(X_1, \ldots, X_n)$ into homogeneous groups is a fundamental and important data processing step in statistics and machine learning. In recent years, clustering in the high-dimensional settings has seen an increasing influx of attention; see for example [BBS14] for a recent review.

When the dimensionality of the data points, $p$, is large compared to the number of samples, $n$, consistent clustering is, in general, information theoretically impossible [Nda19]. Consequently, traditional clustering algorithms such as Lloyd’s algorithm [Llo82], spectral clustering [VW04, VL07], SDP relaxations of

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$k$-means \cite{PW07}, the EM algorithm \cite{DLR77} and approximate message passing \cite{LDBB+16} are poised to fail in this regime. To circumvent this issue, additional parsimony assumptions have to be taken into account. The prevalent approach is to assume sparsity of the cluster centres. For instance, \cite{FM04, WT10} propose to use weighted versions of $k$-means where only a small number of features are considered and chosen by $\ell^1$-penalization. Similarly, \cite{PS07} and \cite{WZ08} propose to use $\ell^1$-penalized versions of $k$-means which are computed by iterative update steps. These algorithms work well empirically, but a sound theoretical treatment of them is lacking so far.

Only more recently has there been a series of papers which provide algorithms with some theoretical guarantees, considering a sparse Gaussian mixture model

$$X_i = \theta_{z_i} + \varepsilon_i, \quad \varepsilon_i \sim \mathcal{N}(0, I_p), \quad z_i \in \{1, \ldots, k\}, \quad \left| \bigcup_{j=1}^k \text{supp}(\theta_j) \right| \leq s$$

and extensions of it.

Above the BBP transition \cite{BBP05}, i.e. when $p = o(n \min_{j \neq l} || \theta_j - \theta_l ||^4)$, it is not necessary to use the sparsity assumption. In particular, an SDP relaxation of $k$-means \cite{GV19} and Lloyd’s algorithm \cite{LZ16, Nda19} have been shown to achieve minimax optimal misclustering rates as $||\theta|| \to \infty$.

In contrast, below the BBP transition it is necessary to take the sparsity assumption into account and use modified algorithms \cite{Nda19}. For instance, \cite{CMZ19} propose a high-dimensional modification of the EM-algorithm which takes sparsity into account. They show that the cluster centres are estimated at the optimal rate and give upper and lower bounds for the accuracy of predicting the label of a new observation. The same loss function is also considered in \cite{ASW13} and \cite{ASLW15}.

Another possibility is to select the relevant features first and afterwards use a vanilla clustering algorithm such as Lloyd’s algorithm or $k$-means. This approach has been analyzed in \cite{ASW13} and \cite{JW16, JKW17}. Particularly, Jin et al. \cite{JKW17} develop a precise theory for the simplified model

$$X_i = z_i \theta + \varepsilon_i, \quad z_i \in \{-1, 1\}, \quad \theta_j \sim \mathcal{N}(0, \delta_0 + \epsilon \delta_a), \quad a \in \mathbb{R}$$

and show in which asymptotic regimes of $(p, n, \epsilon, a)$ consistent clustering is possible when considering the misclustering error

$$\ell(\hat{z}, z) := \min_{\pi \in \{-1, 1\}} \frac{1}{n} \sum_{i=1}^n \mathbf{1}(\pi \hat{z}_i \neq z_i). \quad (1)$$

Moreover, Jin et al. \cite{JKW17} conjecture the existence of a computational barrier, meaning a region of parameters $(p, n, \epsilon, a)$ where consistent clustering is possible only when using algorithms that are not computable in polynomial time.

The phenomena of such computational barriers has been recently discovered in other sparse problems too, such as sparse PCA \cite{BR13b, WBS16, HKP+17,
BB19b, DKWB19], sparse CCA [GMZ17], sub-matrix detection [MW15b], bi-clustering [CLR17, BKR+11] and robust sparse mean detection [BB19a, BB20]. Most relevant to the present article, [FLWY18, BB19a] give evidence that detection in the sparse Gaussian mixture model is impossible in polynomial time under certain conditions which include $p = \omega(n\|\theta\|^4)$, $s^2 = \omega(n\|\theta\|^4)$, and $n = o(p)$.

It seems plausible that as a consequence, clustering better than with a random guess should also be hard in this regime, but a formal proof of this intuition is missing in the literature so far. Giving some evidence in this direction, [FLWY18] present a reduction which shows that, assuming hardness of the detection problem, no polynomial-time classification rule can near-perfectly match the output of Fisher’s linear discriminant (the statistically optimal classifier) when predicting the label of a new observation.

In this article we further investigate statistical and computational limits of clustering in the high-dimensional limit, extending and building on the results of [JKW17, FLWY18, BB19a, BB20]. For our theoretical results we focus on the symmetric two-cluster model

$$X_i = z_i\theta + \varepsilon_i, \quad \varepsilon_i \sim_d N(0, I_p), \quad z_i \in \{-1, 1\}, \quad \|\theta\|_0 \leq s, \quad i = 1, \ldots, n.$$ (2)

Particularly, we show that when $s^2 \log(p) = o(n\|\theta\|^2)$, a simple polynomial-time algorithm based on sparse PCA, in the spirit of [ASLW15, JW16, JKW17], achieves the sharp exponential minimax optimal misclustering rate as $\|\theta\| \to \infty$. Notably, this rate coincides with the low-dimensional minimax rate without assuming sparsity [LZ16]. Under the additional assumption $\|\theta\|_\infty = O(1)$, our algorithm succeeds under the weaker condition $s^2 \log(p) = o(n\|\theta\|^4)$.

We give evidence based on the recently proposed low-degree likelihood ratio [HS17, HKP+17, Hop18] that when $p \geq (1 + \Omega(1))n\|\theta\|^4$ (i.e., below the BBP transition) and $s^2 = \omega(n\|\theta\|^4 \log n)$, no polynomial-time algorithm can distinguish the model (2) from i.i.d. Gaussian samples, corroborating and extending the existing computational lower bounds [FLWY18, BB19a]. We furthermore give a reduction showing that if this detection problem is indeed hard, then no polynomial-time algorithm can cluster better than a random guess. A similar reduction has also appeared in recent independent work [BB20]. Hence, the sample size requirement for our algorithm appears to be almost computationally optimal.

In addition, we discuss how to extend our algorithm to more than two clusters.

**Notation**

We use standard linear algebra notation. $\|v\|_p$ denotes the usual $\ell^p$-norm of a vector $v$ and $\|v\| := \|v\|_2$ denotes the Euclidean norm. If $A$ is a matrix, $\|A\|$ denotes the spectral norm, $\|A\|_F$ the Frobenius (Hilbert-Schmidt) norm, $\|A\|_1 := \sum_{i,j} |A_{ij}|$ the entrywise $\ell^1$-norm and $\|A\|_\infty := \max_{i,j} |A_{ij}|$ the entrywise $\ell_\infty$-norm. We use the notation $x \preceq y$ to denote that there exists a universal constant $c$, not dependent on $p, n, s$ or $\theta$ such that $x \leq cy$. Moreover, for matrices $A, B$ we write $A \preceq B$ if $B - A$ is positive semi-definite. For a vector $a$ we
denote by \( a_S \) the restriction of \( a \) to the set \( S \subset \{1, \ldots, p\} \), i.e. \( a_S := (a_i)_{i \in S} \) and similarly for a matrix \( A \), \( A_S := (A_{ij})_{i \in S, j \in S} \). For a projection matrix \( P \) we denote \( \text{supp}(P) := \{i : P_{ii} \neq 0\} \); equivalently, \( \text{supp}(P) \) is the union of supports of all vectors in the image of \( P \). By \( \mathcal{F}_k \) we denote the \( k \)-Fantope, \( \mathcal{F}_k := \{ P : \text{tr}(P) = k, 0 \preceq P \preceq I, P^T = P \} \). We denote the sign function by \( \text{sgn}(x) := 1(x > 0) - 1(x < 0) \). The notation \( E(\theta, z) \) (or \( P(\theta, z) \)) denotes expectation (or probability, respectively) over samples \( \{X_i\} \) drawn from the model (2) with parameters \( \theta \) and \( z \). Our asymptotic notation (e.g. \( o(1) \)) assumes a scaling regime where the parameters \( n, p, s \) all tend to infinity.

2. Symmetric two-cluster setting

To illustrate our ideas and with the goal of providing concise and clear proofs, we restrict our main considerations to the symmetric two-cluster sparse Gaussian mixture model

\[
X_i = z_i \theta + \varepsilon_i, \quad \varepsilon_i \sim i.i.d. \mathcal{N}(0, I_p), \quad z_i \in \{-1, 1\}, \quad \|\theta\|_0 \leq s, \quad i = 1, \ldots, n.
\]

This model has been thoroughly investigated and serves as a toy model for the analysis of more complex (sparse) Gaussian mixture models. For instance, Verzelen and Arias-Castro [VAC17] study detection limits, Fan et al. [FLWY18] computational detection limits, and Jin et al. [JKW17] the possibility of consistent clustering. Moreover, in the non-sparse setting, [BWY17] and [WZ19] analyze the performance of the EM algorithm in this model.

2.1. A computationally feasible, minimax optimal algorithm

Below the BBP threshold [BBP05], i.e., when \( p > n\|\theta\|_4^4 \), consistent clustering is, in general, information theoretically impossible [Nda19]. To circumvent this issue it is necessary to take the additional information that \( \theta \) is sparse into account.

We modify the low-dimensional clustering algorithm proposed by Vempala and Wang [VW04]. Their approach consists of computing the SVD of the data matrix, projecting the data onto the low-dimensional space spanned by the first left singular vector and then running a clustering algorithm such as \( k \)-means.

We propose to modify the first step of this algorithm by running a sparse PCA algorithm. In particular, we use a variant of a semidefinite program which was initially proposed by d’Aspremont et al. [dEJL07] and further developed and analyzed in [AW09, VCLR13, LV15]. Other theoretically investigated approaches to solve the sparse PCA problem include diagonal thresholding [JL07], iterative thresholding [Ma13], covariance thresholding [KNV15, DM16] and axis aligned random projections [GWS20].

In spirit, Algorithm 1 is similar to the two-stage selection methods proposed in [ASW13, JW16, JKW17], and we discuss differences below.

In the following theorem we show that Algorithm 1 achieves an exponentially-small misclustering error when the squared sparsity is of smaller order than the
Algorithm 1: Sparse spectral clustering

Input: Data matrix $X = [X_1, \ldots, X_n] \in \mathbb{R}^{p \times n}$, tuning parameter $\lambda$
Output: Clustering label vector $\hat{z} \in \{-1, 1\}^n$

1. Compute estimator for the projector onto first sparse principal component via SDP

\[ \hat{P} := \arg\max_{P \in F^1} \left( \frac{X X^T}{n}, P \right) - \lambda \| P \|_1, \]

where $F^1 := \{ P : P^T = P, \text{tr}(P) = 1, 0 \preceq P \preceq I \}$.

2. Perform an eigendecomposition of $\hat{P}$ and compute the leading eigenvector $\hat{u}$.

3. Define $\hat{Y} = \hat{u}^T X \in \mathbb{R}^n$ and return

\[ \hat{z}_i = \text{sgn}(\hat{Y}_i). \]

sample size. We emphasize that Algorithm 1 does not require an impractical sample splitting step. This makes the proof of Theorem 2.1 more difficult as $\hat{u}$ and each $X_i$ are not independent. We overcome this difficulty by using the leave-one-out method combined with a careful analysis of the KKT conditions of the SDP estimator $\hat{P}$.

Theorem 2.1. Suppose that $\log(p) \leq n$ and that for some large enough constant $C > 0$, $\lambda = C \sqrt{\log(p)} \| \theta \|$ and that for some small enough constant $c > 0$

\[ \frac{s \sqrt{\log(p)}}{\sqrt{n \| \theta \|^2}} =: \tau_n \leq c. \]

Then the output of Algorithm 1 satisfies with probability at least $1 - 6p^{-1} - e^{-\| \theta \|^2/2}$ that for another constant $c' > 0$,

\[ \ell(\hat{z}, z) := \min_{\pi \in \{-1, 1\}} \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}(\pi\hat{z}_i \neq z_i) \leq 3 \exp \left( -\frac{\| \theta \|^2}{2} (1 - c' \tau_n - \| \theta \|^{-1/2}) \right). \]

(5)

Remark 2.1. When assuming $\| \theta \|_\infty = O(1)$ we can improve the signal strength condition (4) to $s^2 \log(p) \leq cn \| \theta \|^4$ when choosing $\lambda = C \sqrt{\log(p)}$. This is due to the fact that in the current proof of Theorem 2.1 we use the crude variance bound $\text{Var}(\sum_{i=1}^{n} z_i \theta_l x_i) \leq n \| \theta \|^2$ for $1 \leq l \leq p$.

Lu and Zhou [LZ16] prove a minimax lower bound in the analogous setting without sparsity. Their result implies the following minimax lower bound in our setting (even if $\theta$ is known to the algorithm):

\[ \inf_{\hat{z}} \sup_{(\theta, z) : \| \theta \|_0 \leq s, \| \theta \| \geq \Delta} \mathbb{E}(\theta, z) \ell(\hat{z}, z) \geq \exp \left( -\frac{\Delta^2}{2} (1 - o(1)) \right) \text{ as } \Delta \to \infty. \]

\[ Sample splitting \] would split the data into two copies with independent noise (as in the proof of Theorem 2.3) and use one for each of the steps 1 and 3 of Algorithm 1. This would make the analysis easier but yields an algorithm that would be less natural to use in practice.
Hence, when $\|\theta\| = \omega(1)$ and $\tau_n = o(1)$, the convergence rate in (5) is minimax optimal. It was previously shown by [LZ16, Nda19] that this rate is achievable when $p = o(n\|\theta\|^2)$ (which is above the BBP transition [BBP05], where the sparsity assumption is not needed). Algorithm 1 is the first clustering procedure to provably achieve this minimax optimal misclustering error in the regime where sparsity must be exploited.

We now discuss existing theoretical results in high-dimensional sparse clustering. The articles [ASW13, ASLW15, CMZ19] focus on estimating classification rules from unlabeled data and study the risk of misclassifying a new observation. This risk measure is easier to analyze because the classification rule is independent of the new observation. Moreover, their setting is slightly different. For instance, Cai et al. [CMZ19] assume that $\|\theta\| = O(1)$ and consider the more general case with arbitrary covariance matrix $\Sigma$ and sparsity of $\Sigma^{-1}\theta$. They propose a sparse high-dimensional EM-algorithm and prove sharp bounds on the excess prediction risk, provided that they have a sufficiently good initializer. They (and Azizyan et al. [ASLW15], too) obtain this initializer by using the Hardt-Price algorithm [HP15] and penalized estimation. Ultimately this requires that $s^{12} = o(n)$.

Similarly to the present article, [ASW13, JW16, JKW17] propose two-stage algorithms, selecting first the relevant coordinates and then performing clustering. In [JW16] and [JKW17], suboptimal polynomial rates of convergence are shown for the misclustering error when $\theta$ is sampled from a specific prior. In particular, Jin et al. [JKW17] assume that $\theta_i \overset{i.i.d.}{\sim} (1 - \epsilon)\delta_0 + \epsilon\delta_a$ where $a = o(1), \epsilon = o(1)$ and give precise bounds for which regimes of $(p, n, a, \epsilon)$ consistent clustering is possible and for which it is not. Most relevant to the present work, they prove, ignoring log-factors, that $\|\theta\|^2 = \omega(1)$ is a necessary and sufficient condition for consistent clustering when $s = o(n)$. However, the algorithm that achieves this performance bound is based on exhaustive search which is not computationally efficient. In contrast, ignoring log-factors and in view of remark 2.1, we require in addition that $\|\theta\|^4 = \omega(s^2/n)$ for consistent recovery when $\theta$ is sampled from their prior, matching the requirements of a polynomial-time algorithm they propose. Observing this discrepancy between the performance of their polynomial-time algorithm and the exhaustive search algorithm, they conjecture computational gaps, which we discuss in Section 2.2.

In contrast to the above work, our results apply for arbitrary $\theta$ and $z$ from a given parameter parameter space (instead of a specific prior), and we achieve optimal exponential convergence rates.

2.2. Computational lower bounds

Jin et. al. [JKW17] conjecture a computational gap. In particular, in analogy to sparse PCA, they suggest that a polynomial-time algorithm can have expected misclustering error better than $1/2$ in the regime $p = \omega(n\|\theta\|^4)$ (below BBP) only if the additional condition $n\|\theta\|^4 = \Omega(s^2)$ is fulfilled. This is nontrivial when $s^2 \geq n$ and suggests that there is a range of parameters where from a sta-
tistical point of view consistent estimation is possible, but from a computational perspective it is not.

Starting with the seminal work of Berthet and Rigollet on sparse PCA [BR13a, BR13b] there has been a huge influx of works studying such computational gaps in sparse PCA and related problems. Various forms of rigorous evidence for computational hardness have been proposed, including reductions from the conjectured-hard planted clique problem [BR13a, GMZ17, BB19b, BB19a, BB20], statistical query lower bounds [Kea98, FGR+17, DKS17, FLWY18, NWR19], sum-of-squares lower bounds [MW15a, BHK+19, HKP+17], and analysis of the low-degree likelihood ratio [HS17, HKP+17, Hop18, KWB19, DKWB19].

In the Gaussian mixture model we consider, Fan et al. [FLWY18] have shown that in the statistical query model, testing

\[ H_0 : \theta = 0 \quad \text{vs.} \quad H_1 : \|\theta\| \geq \Delta \]  

(6)
is not possible in polynomial time under certain conditions which include \( p = \omega(n\Delta^4) \), \( s^2 = \omega(n\Delta^4) \), and \( n = o(p) \). A similar result has been proven when assuming the planted clique conjecture [BB19a]. In addition, Fan et al. [FLWY18] show that if the above testing problem is indeed hard, then as a consequence, no polynomial-time classification rule can near-perfectly match (with error probability \( o(1) \)) the output of Fisher’s linear discriminant (the statistically optimal classifier) when predicting the label of a new observation.

We now complement these results two-fold: first, we give evidence based on the low-degree likelihood ratio that the above testing problem is computationally hard when \( p \geq (1 + \Omega(1))n\Delta^4 \) (i.e. below BBP) and \( s^2 = \omega(n\Delta^4 \log \Delta) \), including a precise lower bound on the conjectured runtime. Our result covers a wider regime of parameters than prior work: we do not require \( n = o(p) \), and capture the sharp BBP transition. We also give a reduction showing that if the testing problem is indeed hard then this implies that even “weak” clustering (better than random guessing) cannot be achieved in polynomial time. Recent independent work [BB20] has used a similar reduction in the settings of community detection and signed sparse linear regression.

We now describe the low-degree framework upon which our first result is based, referring the reader to [Hop18, KWB19] for more details. Suppose \( P_n \) and \( Q_n \) are probability distributions on \( \mathbb{R}^N \) for some \( N = N_n \). We will be interested in how well a (multivariate) low-degree polynomial \( f : \mathbb{R}^N \rightarrow \mathbb{R} \) can distinguish \( P_n \) from \( Q_n \) in the sense that \( f \) outputs a large value when the input is drawn from \( P_n \) and a small value when the input is drawn from \( Q_n \). Specifically, we will be interested in the quantity

\[ \| L_{\leq D}^n \| := \max_{f \ \text{deg} \leq D} \frac{\mathbb{E}_{Y \sim P_n} [f(Y)]}{\sqrt{\mathbb{E}_{Y \sim Q_n} [f(Y)^2]}} \]  

(7)

where the maximization is over polynomials \( f : \mathbb{R}^N \rightarrow \mathbb{R} \) of degree (at most) \( D \). This value is large if there exists a “good” distinguishing polynomial (in a certain \( L^2 \) sense). The notation \( \| L_{\leq D}^n \| \) comes from the fact that an equivalent
characterization of this value is the $L^2(Q_n)$-norm of the low-degree likelihood ratio $L^D_n$, which is the orthogonal projection of the likelihood ratio $L_n = \frac{P_n}{Q_n}$ onto the subspace of degree-$D$ polynomials [HS17, HKP+17, Hop18]. For many “natural” high-dimensional testing problems—including planted clique, sparse PCA, community detection, tensor PCA, and others—the performance of the best known polynomial-time algorithms can be matched by an $O(\log n)$-degree polynomial (where $n$ is the natural notion of problem size). More specifically, state-of-the-art performance can often be achieved by some spectral method that thresholds the leading eigenvalue of a matrix whose entries are constant-degree polynomials of the input, and such a spectral method can be approximated by an $O(\log n)$-degree polynomial (under certain mild conditions, including a spectral gap; see Theorem 4.4 of [KWB19]). It is also believed that the class of $O(\log n)$-degree polynomials is as powerful as the sum-of-squares hierarchy; see [HKP+17, Hop18]. In light of the above, failure of low-degree polynomials to solve a testing problem suggests inherent computational hardness. The following informal conjecture based on [HS17, HKP+17, Hop18] makes this precise in terms of $\|L^D_n\|$.

**Conjecture 2.1** (Low-degree conjecture, informal). Let $P_n$ and $Q_n$ be “natural” distributions and let $r_n$ be a sequence of positive real numbers tending to $\infty$.

- If there exists $D = D_n$ satisfying $D = \omega(\log r_n)$ and $\|L^D_n\| = O(1)$ then $P_n$ and $Q_n$ cannot be consistently distinguished in time $O(r_n)$, i.e., there is no $O(r_n)$-time test $t_n : \mathbb{R}^N \to \{0, 1\}$ satisfying
  $$\mathbb{E}[t_n(Y)|Y \sim P_n] + \mathbb{E}[1 - t_n(Y)|Y \sim Q_n] = o(1).$$

- If there exists $D = D_n$ satisfying $D = \omega(\log r_n)$ and $\|L^D_n\| = 1 + o(1)$ then $P_n$ and $Q_n$ cannot be weakly distinguished in time $O(r_n)$, i.e., there is no $O(r_n)$-time test $t_n : \mathbb{R}^N \to \{0, 1\}$ satisfying
  $$\mathbb{E}[t_n(Y)|Y \sim P_n] + \mathbb{E}[1 - t_n(Y)|Y \sim Q_n] = 1 - \Omega(1).$$

Here we do not attempt to formalize the meaning of “natural”, which is intended to capture high-dimensional testing problems that are sufficiently symmetric and noisy like the ones we consider here; see [Hop18, KWB19, HW20] for further discussion.\footnote{Strictly speaking, the conclusion of Conjecture 2.1 should be indistinguishability of $T_\delta P_n$ from $Q_n$ for any constant $\delta > 0$, where $T_\delta$ is a noise operator as in [Hop18] (see also [HW20]). This rules out issues that occur e.g. when approaching the critical value for the BBP transition [BBP06]. For our purposes, $T_\delta P_n$ can be thought of as replacing $\theta$ with $(1 - \delta)\theta$, which is immaterial for our results.}

Conjecture 2.1 is a refined version of existing ideas [HS17, HKP+17, Hop18] but has not appeared in this precise form before. For instance, the second statement regarding weak distinguishing has not (to our knowledge) been stated before, although it is a natural extension of the first statement which can be justified by recalling the variational formula (7) for $\|L^D_n\|$. The correspondence $D = \omega(\log r_n)$ between degree and runtime is slightly more precise than is often
stated. This correspondence is justified by the belief that if a $O(r_n)$-time test exists then there should also exist a spectral method of dimension $\text{poly}(r_n)$ with entries of constant degree and a spectral gap of constant size (see Theorem 4.4 of [KWB19]).

In our setting, we index the distributions $P_n$ and $Q_n$ by the sample size $n$, and allow the other parameters to scale with $n$: $p = p_n$, $\Delta = \Delta_n$, $s = s_n$, $D = D_n$. We take $N = pn$ and define $Q_n$ and $P_n$ as follows. Under $P_n$, first $(\theta, z)$ is drawn from some prior $\pi = \pi_n$ and then we observe $n$ samples from the model (2). Under $Q_n$, we set $\theta = 0$ (which yields $z$ immaterial) and again observe $n$ samples of the form (2). We let $L_\pi$ denote the associated likelihood ratio $\frac{dP_n}{dQ_n}$. We now state our low-degree lower bound for the associated testing problem.

**Theorem 2.2.** In the above setting, suppose that

$$\limsup_n \left[ \sqrt{\frac{n\Delta^4}{p}} + \sqrt{\frac{4n\Delta^4D}{s^2}} \right] < 1. \tag{8}$$

Then there exists a prior $\pi$ taking values in $P_\Delta = \{ (\theta, z) : z \in \{-1, 1\}^n, \|\theta\|_0 \leq s, \|\theta\| \geq \Delta, \theta \in \mathbb{R}^p \}$, such that

$$\|L_{\pi}^D\|^2 = 1 + O\left( \frac{n\Delta^4}{p} + \frac{n\Delta^4D}{s^2} \right).$$

Our primary regime of interest is $p \geq (1 + \Omega(1))n\Delta^4$, i.e., below the BBP transition [BBP05]. Otherwise the sparsity assumption is not needed and a polynomial-time test based on the top eigenvalue of $XX^T$ solves the testing problem (6) [BBP05, VAC17], and clustering better than random guess in polynomial time is provably possible [Nda19]. In this case, to satisfy (8), it is sufficient to have $s^2/(n\Delta^4) = \omega(D)$. Thus, Conjecture 2.1 posits that if $s^2/(n\Delta^4) = \omega(\log n)$ then consistent testing cannot be achieved in polynomial time, and that furthermore the runtime required is at least $\exp(O(s^2/(n\Delta^4)))$. (This is in fact tight due to the subexponential-time algorithms for sparse PCA [DKWB19, HSV19].) In contrast, reductions from planted clique only suggest a lower bound of $n^{O(\log n)}$ on the runtime, because this runtime is sufficient to solve planted clique.

A related analysis of the low-degree likelihood ratio has been given in the setting of sparse PCA [DKWB19]. In contrast, our approach is simpler, avoiding direct moment calculations by using Bernstein’s inequality.

Our next result is a reduction showing that if the above testing problem is indeed hard (as suggested by Theorem 2.2) then clustering better than with a random guess is at least as hard.

**Theorem 2.3.** Suppose there exists a clustering algorithm $\hat{z}$ with runtime $O(r_n)$ such that for some $0 \leq \delta < 1/2$ and $0 \leq \alpha \leq 1$,

$$\sup_{(\theta, z) \in P_\Delta} \mathbb{P} (\ell(\hat{z}, z) \leq \delta) \geq \alpha, \tag{9}$$
for some

\[ \Delta > 4\sqrt{2}(1 - 2\delta)^{-1}\sqrt{1 + \epsilon^{-1}\sqrt{s\log(2p)/n}} \]  

(10)

and where \( \epsilon \in (0,1) \). Then there exists a test \( t_n = t_n(X) \) with runtime \( O(r_n + np) \) such that

\[ E_0[t_n] + \sup_{(\theta, z) \in \mathcal{P}_\Delta} \mathbb{E}_{\theta, z}[1 - t_n] \leq 1 - \alpha + p^{-1}. \]  

(11)

The condition (10) is essentially the condition under which the detection problem is information-theoretically possible [VAC17] when \( \Delta \to \infty \). This condition is not restrictive in our setting: since our aim is to show hardness whenever \( s^2 = \omega(n\Delta^4 \log n) \), it is sufficient to restrict to the “boundary” case, say, \( s^2 = o(n\Delta^4 \log^2 n) \), in which case (10) is satisfied under the very mild condition \( \log(p) = o(\sqrt{n}/\log n) \).

Combining Theorems 2.2 and 2.3, and assuming the low degree conjecture (Conjecture 2.1), we can conclude the following. Below the BBP transition, i.e., when \( p \geq (1 + \Omega(1))n\Delta^4 \), if \( s^2 = \omega(n\Delta^4 \log n) \) and (10) holds then any polynomial-time algorithm \( \hat{z} \) fails with an asymptotically strictly positive probability in the sense that for any fixed \( 0 \leq \delta < 1/2 \),

\[ \sup_{(\theta, z) \in \mathcal{P}_\Delta} \mathbb{P}_\theta(\ell(\hat{z}, z) > \delta) = \Omega(1). \]

Moreover, if we additionally assume that we are strictly below the BBP transition in the sense that \( p = \omega(n\Delta^4) \), we can conclude the stronger statement that any polynomial-time algorithm \( \hat{z} \) fails with probability approaching one, i.e., for any fixed \( 0 \leq \delta < 1/2 \),

\[ \sup_{(\theta, z) \in \mathcal{P}_\Delta} \mathbb{P}_\theta(\ell(\hat{z}, z) > \delta) = 1 - o(1). \]

Up to logarithmic factors these bounds match the conjecture by Jin et al. [JKW17].

The condition \( s^2 \log(p) \leq cn\|\theta\|^2 \) in Theorem 2.1 is thus likely to be suboptimal by a factor of \( \|\theta\|^2 \log(p) \). As discussed in Remark 2.1 the suboptimal factor of \( \|\theta\|^2 \) may be dealt with by assuming additionally that \( \|\theta\|_\infty = O(1) \) and hence Algorithm 1 is, up to the logarithmic factor in \( p \), computationally optimal.

Deshpande and Montanari [DM16] show how to remove the \( \sqrt{\log(p)} \)-factor in sparse PCA using the covariance thresholding algorithm. We leave it open as an interesting question how to do the same in our clustering situation.

### 3. Extension to a general number of clusters

Generalizing Algorithm 1 to a setting with \( k \) clusters is possible by combining sparse PCA again with the approach proposed by Vempala and Wang [VW04].
Indeed, similar to before, we first perform sparse PCA via SDP, project the data onto the \( k \)-dimensional subspace, and then use an approximation to the \( k \)-means algorithm. Computing the exact solution to the \( k \)-means algorithm is known to be NP-hard \([ADHP09]\). However there exist algorithms that compute a \((1 + \epsilon)\) solution in time \( O(2^{C(k/\epsilon)n}) \) \([KSS04]\). As long as \( k \) grows slowly enough or stays fixed this yields a polynomial-time algorithm.

Algorithm 2: Sparse spectral clustering for general \( k \)

**Input:** Data matrix \( X = [X_1, \ldots, X_n] \in \mathbb{R}^{p \times n} \), number of clusters \( k \), tuning parameter \( \lambda \), \( k \)-means accuracy parameter \( \epsilon \)

**Output:** Clustering label vector \( \hat{z} \in [k]^n \)

1. Compute estimator for the projector onto first \( k \) sparse principal components via SDP

\[
\hat{P} := \arg \max_{P \in \mathcal{F}_k} \frac{XX^T}{n} P - \lambda \|P\|_1,
\]

where \( \mathcal{F}_k := \{P : P^T = P, \tr(P) = k, 0 \preceq P \preceq I_p\} \).

2. Perform an eigendecomposition of \( \hat{P} \) and compute the leading \( k \) eigenvectors

\((\hat{u}_1, \ldots, \hat{u}_k) = \hat{U} \in \mathbb{R}^{p \times k}\).

3. Define

\[
\hat{Y} = \hat{U}^T X \in \mathbb{R}^{k \times n}.
\]

Denote \( \hat{Y}_i \in \mathbb{R}^k \) as the \( i \)th column of \( \hat{Y} \). Compute \( \hat{z} \) as a \((1 + \epsilon)\)-solution to the \( k \)-means objective

\[
\arg \min_{z \in \{1, \ldots, k\}^n} \min_{c_j \in \mathbb{R}^k} \sum_{i=1}^n \|\hat{Y}_i - c_{z_i}\|^2.
\]

In the low-dimensional case without sparsity and when performing vanilla PCA this algorithm has recently been shown to achieve the minimax optimal misclustering rate \([LZZ19]\) without assumptions on the eigenvalues of the population matrix \( \mathbb{E}XX^T \). In contrast, we are not able to prove such a result here, as the analysis of the sparse PCA algorithm we use relies crucially on a large enough eigengap of the \( k \)-th eigenvalue. Such eigengap assumptions are widely spread in the spectral clustering literature \([LR15, VL07]\). Assuming an eigengap condition, and combining the proof of Theorem 2.1 with the analysis of the \( k \)-means algorithm in \([LZZ19]\), it is possible to extend the result of Theorem 2.1 to the \( k \)-cluster case. However, ultimately, this is not fully satisfactory as the performance of a good clustering algorithm should only depend on the distance between the clusters and eigenvalues should not matter. Hence, we do not provide a proof here and leave it open as an interesting future research question how to extend Theorem 2.1 to the \( k \)-cluster case without an eigengap condition.
4. Proofs

4.1. Proof of Theorem 2.1

We first note that the estimator $\hat{P}$ computed in step 1 of Algorithm (1) fulfills the following identity

$$\hat{P} = \arg \max_{P \in \mathcal{F}_1} \left\langle \frac{XX^T}{n}, P \right\rangle - \lambda \|P\|_1 = \arg \max_{P \in \mathcal{F}_1} \left\langle \frac{XX^T}{n} - I_p, P \right\rangle - \lambda \|P\|_1,$$

which follows from the trace-constraint in $\mathcal{F}_1$. Henceforth we will always work with the last representation of $\hat{P}$. We define $P = \theta \theta^T / \|\theta\|^2$ and start with the following preliminary lemma which applies the results of [VCLR13] and [LV15] to obtain rates for $\|\hat{P} - P\|_F$ and false positive control. For completeness we provide a proof in the appendix.

Lemma 4.1. Suppose that $\log(p) \leq n$ and that for some large enough constant $C > 0$, $\lambda = C \|\theta\| \sqrt{\log(p)/n}$. Denote $\text{supp}(P) = S$ and $\text{supp}(\hat{P}) = \hat{S}$. Then, with probability at least $1 - 2p^{-2}$ we have that

$$\|\hat{P} - P\|_F \lesssim \frac{s^2 \log(p)}{n \|\theta\|^2} \quad \text{and} \quad \hat{S} \subset S. \quad (13)$$

Next we prove a key-technical lemma for the deviation of the leave-one-out estimator $\hat{P}^{(i)}$ which is defined analogous to $\hat{P}$ with $X_i$ being replaced by an independent copy of it, $X'_i = \theta z_i + \epsilon'_i$. Denote $X^{(i)} = [X_1, \ldots, X_{i-1}, X'_i, X_{i+1}, \ldots, X_n]$. The proof uses the KKT-conditions and, crucially, false positive control of $\hat{P}$ and $\hat{P}^{(i)}$.

Lemma 4.2. Suppose $\log(p) \leq n$ for some large enough $c > 0$ and that for another large enough constant $C > 0$, $\|\theta\| \geq 3Cs \sqrt{\log(p)/n}$ and $\lambda = C \sqrt{\log(p)/\|\theta\|}$. Denote by $\hat{P}^{(i)}$ the leave-one-out estimator defined analogous to $\hat{P}$ with $X$ replaced by $X^{(i)}$. Then we have with probability at least $1 - 6p^{-2}$ that

$$\|\hat{P} - \hat{P}^{(i)}\|_F \lesssim \frac{s^2 \log(p)}{n \|\theta\|^2} \left( \sqrt{\log(p)} + \|\theta\| \right). \quad (14)$$

Moreover, on the same event, $\hat{P}$ and $\hat{P}^{(i)}$ are rank 1 projection matrices.

Proof. Throughout we work on the event where the results of Lemma 4.1 apply to both $\hat{P}$ and $\hat{P}^{(i)}$, which, by a union bound, occurs with probability at least $1 - 4p^{-2}$. We denote

$$\tilde{M} = \frac{XX^T}{n} - I_p, \quad \tilde{M}^{(i)} = \frac{X^{(i)}(X^{(i)})^T}{n} - I_p \quad \text{and} \quad M = \mathbb{E}\tilde{M}.$$

By strong duality we have that

$$\max_{P \in \mathcal{F}_1} \langle \tilde{M}, P \rangle - \lambda \|P\|_1 \iff \max_{P \in \mathcal{F}_1} \min_{Z \in B} \langle \tilde{M} - \lambda Z, P \rangle.$$
where $B := \{ Z : \text{diag}(Z) = 0, Z = Z^T, \|Z\|_\infty \leq 1 \}$ and by the KKT-condition a pair $(\hat{P}, \hat{Z})$ is an optimal solution if and only if $\hat{Z} \in B$ and

$$\hat{Z}_{ij} = \text{sgn}(\hat{P}_{ij}), \quad (i, j) \in \hat{S} \times \hat{S}, \quad \hat{P} = \arg \max_{P \in F_1} \langle \hat{M} - \lambda \hat{Z}, P \rangle.$$  

Since $\text{supp}(\hat{P}) \subset S$ for any solution $(\hat{P}, \hat{Z})$ we may define $\tilde{Z}_{ij} = \frac{1}{\lambda} \hat{M}_{ij}$ for $(i, j) \notin S \times S$, $i \neq j$ and $\tilde{Z}_{ij} = \hat{Z}_{ij}$ otherwise and observe that the pair $(\hat{P}, \hat{Z})$ also fulfills the KKT-conditions and is therefore also an optimal solution. We have that

$$\hat{M} - \lambda \hat{Z} = \begin{cases} \hat{M}_{ij} - \lambda \hat{Z}_{ij} & (i, j) \in \hat{S}, \\ 0 & \text{else} \end{cases}$$

and particularly, by Lidski’s inequality the first eigenvalue of $\hat{M} - \lambda \hat{Z}$ is bounded by

$$\|M\| - \|(\hat{M} - \lambda \hat{Z})_S\| \geq \|\theta\|^2 - \lambda - s\|\hat{M} - M\|_\infty$$

$$\geq \|\theta\|^2 - \lambda(s + 1) > \lambda(s + 1) > 0,$$

and by the same argument all other eigenvalues of $\hat{M}$ are bounded by $\lambda(s + 1)$. Hence, and since in addition $F_1$ is the convex hull of its extremal points, the set of rank 1 projection matrices, $\hat{P}$ coincides with projection matrix onto the first eigenspace of $\hat{M} - \lambda \hat{Z}$ (see also Lemma 1 in [LV15]). Likewise, we see that $\hat{P}^{(i)}$ is also a spectral projector of rank 1. Moreover, the first spectral gap of $\hat{M} - \lambda \hat{Z}$ is, by the above reasoning, lower bounded by $\|\theta\|^2 - 2\lambda(s + 1) \geq \|\theta\|^2/2$. We now apply the curvature Lemma 3.1 in [VCLR13] to obtain that

$$\|\hat{P} - \hat{P}^{(i)}\|_F^2 \leq \frac{2}{\|\theta\|^2 - 2\lambda(s + 1)} \left\langle \hat{M} - \lambda \hat{Z}, \hat{P} - \hat{P}^{(i)} \right\rangle.$$  

Moreover, by definition of $\hat{P}^{(i)}$ we have that,

$$\left\langle \hat{M}^{(i)}, \hat{P} - \hat{P}^{(i)} \right\rangle - \lambda \|\hat{P}\|_1 + \lambda \|\hat{P}^{(i)}\|_1 \leq 0$$

Hence, we obtain that

$$\|\hat{P} - \hat{P}^{(i)}\|_2^2$$

$$\leq \frac{4}{\|\theta\|^2} \left[ \left\langle \hat{M} - \hat{M}^{(i)}, \hat{P} - \hat{P}^{(i)} \right\rangle + \lambda \|\hat{P}\|_1 - \lambda \|\hat{P}^{(i)}\|_1 - \lambda(\hat{Z}, \hat{P} - \hat{P}^{(i)}) \right]. \quad (15)$$

To bound the last three terms above, we observe that

$$\langle \hat{Z}, \hat{P} - \hat{P}^{(i)} \rangle = \sum_{k,j} \hat{Z}_{kj} \hat{P}_{kj} - \sum_{k,j} \hat{Z}_{kj} \hat{P}^{(i)}_{kj}$$

$$= \sum_{(k,j) \in \hat{S} \times \hat{S}} \text{sgn}(\hat{P}_{kj}) \hat{P}_{kj} - \sum_{k,j} \hat{Z}_{kj} \hat{P}^{(i)}_{kj}$$

$$\geq \|\hat{P}\|_1 - \|\hat{P}^{(i)}\|_1.$$
Hence the last three terms in (15) are bounded above by 0 and we estimate

\[ \| \hat{P} - \hat{P}^{(i)} \|_2^2 \leq \frac{4}{\| \theta \|^2} \left\langle \hat{M} - \hat{M}^{(i)}, \hat{P} - \hat{P}^{(i)} \right\rangle \]

\[ \leq \frac{4}{\| \theta \|^2} \| \hat{M} - \hat{M}^{(i)} \|_\infty \| \hat{P} - \hat{P}^{(i)} \|_1 \]

\[ \leq \frac{4}{\| \theta \|^2} \left( \left\| \frac{\varepsilon_i \varepsilon_i^T - \varepsilon_i^{(i)}(\varepsilon_i^{(i)})^T}{n} \right\|_\infty + \left\| \theta z_i (\varepsilon_i - \varepsilon_i^{(i)})^T \right\|_\infty \right) \| \hat{P} - \hat{P}^{(i)} \|_1 \]

Since by Lemma 4.1 we have on the event we are working on that \( \hat{S} \subset S \) and \( \hat{S}^{(i)} \subset S \) we further bound \( \| \hat{P} - \hat{P}^{(i)} \|_1 \leq s \| \hat{P} - \hat{P}^{(i)} \|_F \leq s \| \hat{P} - P \|_F + s \| \hat{P}^{(i)} - P \|_F \). Moreover, by Gaussian concentration, we have with probability at least \( 1 - p^{-2} \) that \( \| \varepsilon_i \varepsilon_i^T - \varepsilon_i^{(i)}(\varepsilon_i^{(i)})^T \|_\infty \leq 4 \log(p) \). Applying Gaussian concentration again we bound with probability at least \( 1 - p^{-2} \)

\[ \| (\theta z_i (\varepsilon_i - \varepsilon_i^{(i)})^T) \|_\infty \lesssim \sqrt{\log(p)} \| \theta \|. \]

Thus, concluding, using another union bound, to incorporate the two events we have on an event of probability at least \( 1 - 6p^{-1} \) that

\[ \| \hat{P} - \hat{P}^{(i)} \|_F \lesssim \frac{s \log(p) + s \sqrt{\log(p)} \| \theta \|}{n \| \theta \|^2} \left( \| P - \hat{P} \|_2 + \| \hat{P}^{(i)} - P \|_2 \right) \]

\[ \lesssim \frac{s^2 \log(p)}{n^{3/2} \| \theta \|^3} \left( \sqrt{\log(p)} + \| \theta \| \right). \]

\[ \square \]

We are now ready to prove Theorem 2.1.

**Proof.** Recall that

\[ \ell(\hat{z}, z) = \inf_{\pi \in \{-1, 1\}} \frac{1}{n} \sum_{i=1}^n 1(\pi \hat{z}_i \neq z_i). \]

Throughout we work on the global event \( \Omega \) where the statement of Lemma 4.1 and for all \( i \) the statement of Lemma 4.2 hold which occurs with probability at least \( 1 - 6p^{-1} \). We denote this event by \( \Omega \). Fix one particular \( z_i \) and suppose without loss of generality that \( z_i = 1 \). We have that

\[ 1(\hat{z}_i \neq \pi \hat{z}_i) = 1(\pi \hat{u}^T X_i \leq 0). \]

We define \( u = \theta / \| \theta \| \) such that \( P = uu^T \). Since \( \pi \) is either \(-1\) or \(+1\) for all \( i \) we may choose \( \pi \) such that \( \langle u, \pi \hat{u} \rangle = \frac{1}{\| \theta \|} \langle \theta, \pi \hat{u} \rangle \geq 0 \). Without loss of generality and to simplify notation assume now that \( \pi = 1 \). Hence, we obtain that

\[ 1(\hat{z}_i \neq \pi \hat{z}_i) = 1(\hat{u}^T \theta + \hat{u}^T \varepsilon_i \leq 0) \]

\[ \leq 1 \left( u^T \theta - \| u - \hat{u} \| \| \theta \| + (\hat{u}^{(i)})^T \varepsilon_i + (\hat{u} - \hat{u}^{(i)})^T \varepsilon_i \leq 0 \right). \]
where we pick $\hat{u}^{(i)}$ such that $\hat{P}^{(i)} = \hat{u}^{(i)}(\hat{u}^{(i)})^T$ and $\langle \hat{u}^{(i)}, \hat{u} \rangle \geq 0$. Since $\langle u, \hat{u} \rangle \geq 0$ we have that 
\[
\|\hat{u} - u\| \leq \sqrt{2}\|\hat{P} - P\|_F \lesssim \frac{s\sqrt{\log(p)}}{\sqrt{n}\|\theta\|^2} \lesssim \tau_n,
\]
per our assumptions. Moreover, since $\text{supp}(\hat{P}) \subset S$ and $\text{supp}(\hat{P}^{(i)}) \subset S$, we see that $\hat{u}$ and $\hat{u}^{(i)}$ are non-zero only at coordinates $i \in S$. Hence, we obtain 
\[
|\langle \hat{u} - \hat{u}^{(i)} \rangle^T\varepsilon_i| = |\langle \hat{u} - \hat{u}^{(i)} \rangle^T\varepsilon_i|_S \leq \|\varepsilon_i\|\|\hat{u} - \hat{u}^{(i)}\|.
\]
By Jensen’s inequality we have that 
\[
\mathbb{E}\|\varepsilon_i\|_S \leq \sqrt{s}
\]
and using furthermore Borell’s inequality (e.g. Theorem 2.2.7 in [GN16]), recognizing that $\|\varepsilon\| = \sup_{g \in \mathbb{R}^n : \|g\|_1 = 1} \langle \varepsilon, g \rangle$, implies that with probability at least $1 - e^{-\|\theta\|^2}$ 
\[
\|\varepsilon_i\| \leq \sqrt{s + \sqrt{2}\|\theta\|}.
\]
Hence, conditionally on the event $\Omega$ where Lemma 4.2 holds and since $\langle \hat{u}, \hat{u}^{(i)} \rangle \geq 0$ we obtain that with probability at least $1 - e^{-\|\theta\|^2}$ 
\[
|\langle \hat{u} - \hat{u}^{(i)} \rangle^T\varepsilon_i| \leq \sqrt{2}\|\hat{P} - \hat{P}^{(i)}\|_F\|\varepsilon_i\|_S \lesssim \frac{s^2\log(p)(\sqrt{s} + \|\theta\|)\left(\sqrt{\log(p)} + \|\theta\|\right)}{n^{3/2}\|\theta\|^3} \lesssim \tau_n\|\theta\|.
\]
Hence, on the intersection of these events we can further bound for some constant $c > 0$ 
\[
1 (z_i \neq r\hat{\varepsilon}_i) \leq 1 \left(\|\theta\|(1 - c\tau_n) + (\hat{u}^{(i)})^T\varepsilon_i \leq 0\right).
\]
By construction of $\hat{u}^{(i)}$, $\hat{u}^{(i)}$ and $\varepsilon_i$ are independent and hence $(\hat{u}^{(i)})^T\varepsilon_i$ is univariate standard Gaussian distributed. Hence, by a standard tailbound for Gaussian random variables we obtain that 
\[
\mathbb{E}1 \left(\|\theta\|(1 - c\tau_n) + (\hat{u}^{(i)})^T\varepsilon_i \leq 0\right) = \Phi(-\|\theta\|(1 - c\tau_n)) \leq e^{-\|\theta\|^2(1 - c\tau_n)^2/2}
\]
where $\Phi$ denotes the C.D.F. of a standard Gaussian random variable. Summarizing, conditionally on the global event $\Omega$ where Lemma 4.2 holds, we have, summing over each $i$ (after proper global permutation) that 
\[
\mathbb{E}\ell(\hat{z}, z) | \Omega = \frac{1}{n} \sum_{i=1}^n \mathbb{E}1 (z_i \neq r\hat{\varepsilon}_i) | \Omega \leq e^{-\|\theta\|^2} + e^{-\|\theta\|^2(1 - c\tau_n)^2/2} \leq 2e^{-\|\theta\|^2(1 - c\tau_n)^2/2}.
\]
Therefore, applying Markov’s inequality, we have with probability at least $1 - e^{-\|\theta\|^2/2 - 6p^{-1}}$ that 
\[
\ell(\hat{z}, z) \leq 3e^{-\|\theta\|^2(1 - c\tau_n - \|\theta\|^{-1})^2/2}.
\]
4.2. Proof of Theorem 2.2

Proof. We construct a prior $\pi$ by putting independent priors on $z$ and $\theta$ as follows:

$$z_i \overset{i.i.d.}{\sim} \mathcal{R}$$

where $\mathcal{R}$ denotes the Rademacher distribution. Moreover, we sample $\theta$ as follows:

$$S \overset{\text{unif.}}{\sim} \{S \subset \{1, \ldots, p\}, |S| = s\}$$

$$\theta_i | S \overset{\text{independently}}{\sim} \begin{cases} \frac{\Delta}{\sqrt{s}} \mathcal{R} & i \in S \\ 0 & \text{else.} \end{cases}$$

Observe that $(\theta, z) \in \mathcal{P}_\Delta$. By Theorem 2.6. in [KWB19] we have that

$$\|L_ \pi^{\leq D}\|^2 = \mathbb{E}_{(\theta, z), (\tilde{\theta}, \tilde{z}) \sim \pi} \sum_{d=0}^{D} \frac{1}{d!} (z, \tilde{z})^d (\theta, \tilde{\theta})^d$$

$$= 1 + \mathbb{E}_{(\theta, z), (\tilde{\theta}, \tilde{z}) \sim \pi} \sum_{d=1}^{\lfloor D/2 \rfloor} \frac{1}{(2d)!} (z, \tilde{z})^{2d} (\theta, \tilde{\theta})^{2d}.$$

We bound the quantities in the sum above one after the other. Observe that $(z, \tilde{z})$ is a sum of $n$ i.i.d. Rademacher random variables $R_i$, $i = 1, \ldots, n$. We argue as in the proof of the Khinchine-inequality. Indeed, denoting by $g_1, \ldots, g_n$ i.i.d. Gaussians we have by Jensen’s inequality for $d_i \geq 2$, $d_i$ being a multiple of 2, that $\mathbb{E} g_i^{d_i} \geq (\mathbb{E} g_i^2)^{d_i/2} = 1 = \mathbb{E} R_i^{d_i}$. Hence, we obtain that

$$\mathbb{E} \left( \sum_{i=1}^{n} R_i \right)^{2d} = \sum_{2d_1 + \ldots + 2d_n = 2d} \mathbb{E} R_1^{2d_1} \ldots R_n^{2d_n} \leq \sum_{2d_1 + \ldots + 2d_n = 2d} \mathbb{E} g_1^{2d_1} \ldots g_n^{2d_n}$$

$$= \mathbb{E} \left( \sum_{i=1}^{n} g_i \right)^{2d} = n^d \mathbb{E} g_1^{2d} = n^d (2d - 1)!!,$$

where $(d - 1)!! := (d - 1)(d - 3) \cdots 3 \cdot 1$. Next, given $S$ and $\tilde{S}$ observe that

$$(\theta, \tilde{\theta}) | S, \tilde{S} \overset{d}{=} \frac{\Delta^2}{s} \sum_{i \in S \cap \tilde{S}} R_i,$$

where $R_i$ are i.i.d. Rademacher random variables again. Hence, arguing as before, using comparison to Gaussian random variables again we obtain

$$\mathbb{E} (\theta, \tilde{\theta})^{2d} = \left( \frac{\Delta^2}{s} \right)^{2d} \mathbb{E} \left[ \mathbb{E} \left( \sum_{i \in S \cap \tilde{S}} R_i \right)^{2d} \right] \leq (2d - 1)!! \left( \frac{\Delta^2}{s} \right)^{2d} \mathbb{E} |S \cap \tilde{S}|^d.$$
Define $Z_i = 1(i \in S)$, $\tilde{Z}_i = 1(i \in \tilde{S})$ and observe that
\[ |S \cap \tilde{S}| = \sum_{i=1}^{p} Z_i \tilde{Z}_i. \]

We have that $EZ_i = \mathbb{E}\tilde{Z}_i = \frac{s}{p}$ and hence the $Z_i$ are Bernoulli random variables with success probability $s/p$ (but not independent). Moreover, the sequence of the $Z_i$ is drawn without replacement and hence negatively associated (see e.g. [JDP83]). Then, we obtain that
\[ \mathbb{E} \left| \sum_{i=1}^{p} Z_i \tilde{Z}_i \right|^d = \mathbb{E} \left[ \mathbb{E} \left| \sum_{i \in S} Z_i \right|^d \mid \tilde{S} \right]. \]

Since $\{Z_i\}_{i=1}^{p}$ are negatively associated and since $S$ and $\tilde{S}$ are independent, $\{Z_i\}_{i \in S}$ are also negatively associated by P4 in [JDP83].

Let $B_i$ and $\tilde{B}_i$ be independent Bernoulli random variables with success probability $s/p$ each and observe that $|\cdot|^d$ is convex. Hence, by Theorem 2 in [Sha00], we obtain that
\[ \mathbb{E} \left[ \mathbb{E} \left| \sum_{i \in S} Z_i \right|^d \right] \leq \mathbb{E} \left[ \mathbb{E} \left| \sum_{i \in S} B_i \right|^d \mid \tilde{S} \right] = \mathbb{E} \left| \sum_{i} B_i \tilde{Z}_i \right|^d. \]

Similarly, conditioning on $S_B := \{i : B_i = 1\}$, since the $\tilde{Z}_i$ are also negatively associated and applying Theorem 2 in [Sha00] again, we obtain further
\[ \mathbb{E} \left| \sum_{i} B_i \tilde{Z}_i \right|^d = \mathbb{E} \left[ \mathbb{E} \left| \sum_{i \in S_B} \tilde{Z}_i \right|^d \mid S_B \right] \leq \mathbb{E} \left[ \mathbb{E} \left| \sum_{i \in S_B} B_i \tilde{B}_i \right|^d \mid S_B \right] = \mathbb{E} \left[ \sum_{i=1}^{p} B_i \tilde{B}_i \right]^d = \mathbb{E} \left| \sum_{i=1}^{p} \tilde{B}_i \right|^d, \]

where $\tilde{B}_i$ are independent Bernoulli random variables with success probability $s^2/p^2$ each. In particular, $\mathbb{E} \sum_{i=1}^{p} \tilde{B}_i = \frac{s^2}{p}$. Since the $\tilde{B}_i$ are independent, bounded by 1 and have variance bounded by $s^2/p^2$ we obtain by Bernstein’s inequality that
\[ \mathbb{P} \left( \left| \sum_{i=1}^{p} \tilde{B}_i - \mathbb{E}\tilde{B}_i \right| > t \right) \leq 2 \exp \left( -\frac{t^2 s^2}{2p} + \frac{4t^2 s^2}{3p} \right). \]
Hence, by the triangle inequality for the $\ell_d$-norm, we obtain that

\[
\left[\mathbb{E}\left|\sum_{i=1}^{p} \tilde{B}_i\right|^d\right]^{1/d} \leq \frac{s^2}{p} + \left[\mathbb{E}\left|\sum_{i=1}^{p} \tilde{B}_i - \mathbb{E}\tilde{B}_i\right|^d\right]^{1/d}
\]

\[
= \frac{s^2}{p} + \left[\int_0^\infty \mathbb{P}\left(\left|\sum_{i=1}^{p} \tilde{B}_i - \mathbb{E}\tilde{B}_i\right| > t^{1/d}\right) dt\right]^{1/d}
\]

\[
\leq \frac{s^2}{p} + 2^{1/d} \left[\int_0^\infty \exp\left(-\frac{t^{2/d}}{2\frac{s^2}{p} + \frac{2}{\delta^2} t^{1/d}}\right) dt\right]^{1/d}.
\]

We bound the integral above. Indeed, we have that

\[
\int_0^\infty \exp\left(-\frac{t^{2/d}}{2\frac{s^2}{p} + \frac{2}{\delta^2} t^{1/d}}\right) dt \leq \int_0^\infty \exp\left(-\frac{t^{2/d}}{4\frac{s^2}{p}}\right) dt + \int_0^\infty \exp\left(-\frac{3t^{1/d}}{4}\right) dt
\]

\[
\leq d^{d/4}d^{d-1}\left(\frac{s^2}{p}\right)^{d/2} + \left(\frac{4}{3}\right)^d.
\]

Hence, overall we obtain that

\[
\|L\|^{\leq D}_\pi^2 - 1 = \mathbb{E}_{(\theta,z),(\tilde{\theta},\tilde{z}) \sim \pi}^{|D/2|} \sum_{d=1}^{n} \frac{1}{(2d)!} (\tilde{z}, \tilde{z})^{2d} (\tilde{\theta}, \tilde{\theta})^{2d}
\]

\[
\leq \sum_{d=1}^{|D/2|} \left(\frac{n\Delta^4}{p} + \frac{4n\Delta^4 D}{s^2} + \frac{4n\Delta^4 D^{1/2}}{p^{1/2} s}\right)^d
\]

\[
\leq \sum_{d=1}^{|D/2|} \left(\sqrt{\frac{n\Delta^4}{p}} + \sqrt{\frac{4n\Delta^4 D}{s^2}}\right)^{2d} = O\left(\frac{n\Delta^4}{p} + \frac{4n\Delta^4 D}{s^2}\right)
\]

per our assumptions and where we used that $((2d-1)!!)^2/(2d)! \leq 1$.

### 4.3. Proof of Theorem 2.3

Proof. Assuming that (9) holds and given data $[x_1, \ldots, x_n] := X$ either generated from $\mathbb{P}_0$ or $\mathbb{P}_{\theta,z}$, $\theta(1-\epsilon), z \in \mathcal{P}_\Delta$, we perform the following sample splitting trick: we denote $E = [\varepsilon_1, \ldots, \varepsilon_n]$ and generate $\tilde{E}$ such that $\tilde{E} \overset{d}{=} E$ and $\tilde{E}$ and $E$ are independent and for $\epsilon \in (0,1)$ define

\[
\tilde{X} = \frac{X + \frac{1}{\epsilon} \tilde{E}}{\sqrt{1 + \epsilon}}, \quad \tilde{X}(-) = \frac{X - \epsilon \tilde{E}}{\sqrt{1 + \epsilon}}.
\]

Since for fixed $(\theta, z)$

\[
\operatorname{Cov}\left(\tilde{X}, \tilde{X}(-)\right) = \frac{1}{\sqrt{2 + \epsilon + \frac{1}{\epsilon}}} \operatorname{Cov}\left(E + \frac{1}{\epsilon} \tilde{E}, E - \epsilon \tilde{E}\right) = 0,
\]

per our assumptions
and $E$ and $\tilde{E}$ are Gaussian, we obtain that $\hat{X}$ and $\tilde{X}^{(-)}$ are independent. Moreover, since $\tilde{X}^{(-)} = (\theta/\sqrt{1+\epsilon})z^{T} + (E - \epsilon\tilde{E})/\sqrt{1+\epsilon}$, $(\theta/\sqrt{1+\epsilon}, z) \in \mathcal{P}_{\Delta}$ (as $1 - \epsilon < 1/\sqrt{1+\epsilon}$) and $(E - \epsilon\tilde{E})/\sqrt{1+\epsilon} \overset{d}{=} E$ we see that $\tilde{X}^{(-)}$ can be viewed as generated from a parameter in $\mathcal{P}_{\Delta}$. Hence, by assumption in (9), we have that

$$\mathbb{P}_{\theta, z} \left( \ell(\hat{z}(\tilde{X}^{(-)}), z) > \delta \right) \leq 1 - \alpha$$

and by construction $\hat{z} = \hat{z}(\tilde{X}^{(-)})$ is independent of $\tilde{X}$. Next define a test statistic

$$T_{n}^{2} := \sum_{i=1}^{s} \left[ \left( \frac{\hat{X}^{(i)}}{n} \right) \right]^{2},$$

where $a_{(i)}$ denotes the $i$-th largest (in absolute value) element of $a$. Note that $(E + \frac{1}{\epsilon}\tilde{E})\hat{z}/\sqrt{n + \frac{1}{\epsilon}} \sim N(0, nI_{p})$ and that by Borell’s inequality we have the tail inequality

$$\left\| \frac{(E + \frac{1}{\epsilon}\tilde{E})\hat{z}/\sqrt{n + \frac{1}{\epsilon}}} \right\|_{\infty} \leq \sqrt{2\log(2p)} + t$$

with probability at least $1 - e^{-t^{2}/2}$. Hence, under $H_{0} : \theta = 0$ we bound with probability at least $1 - (2p)^{-1}$

$$T_{n}^{2} = \sum_{i=1}^{s} \left[ \left( \frac{E + \frac{1}{\epsilon}\tilde{E}}{n\sqrt{1 + \frac{1}{\epsilon}}} \hat{z} \right) \right]^{2} \leq \frac{8s \log(2p)}{n}.$$

Now consider the alternative $H_{1} : (\theta/\sqrt{1+\epsilon}, z) \in \mathcal{P}_{\Delta}$. Denote by $i(\theta)$ the index that corresponds to the $i$-th largest element of $\theta$. Hence, conditionally on the event

$$\{ \ell(\hat{z}(\tilde{X}^{(-)}), z) \leq \delta \} \cap \left\{ \left\| \frac{(E + \frac{1}{\epsilon}\tilde{E})\hat{z}}{\sqrt{n + \frac{1}{\epsilon}}} \right\|_{\infty}^{2} \leq 8\log(2p) \right\}$$
we obtain that
\[
T_n \geq \left( \sum_{i=1}^{s} \left( \frac{\tilde{X} \hat{z}}{n} \right)_{i(\theta)}^2 \right)^{1/2}
\]
\[
\geq \frac{\|\theta z^T \hat{z}\|}{n\sqrt{1 + \frac{1}{\epsilon}}} - \left( \sum_{i=1}^{s} \left( \frac{(E + \frac{1}{n} \tilde{E}) \hat{z}}{n\sqrt{1 + \frac{1}{\epsilon}}} \right)_{i(\theta)}^2 \right)^{1/2}
\]
\[
\geq \frac{\|\theta\|(1 - 2\ell(\hat{z}, z))}{\sqrt{1 + \frac{1}{\epsilon}}} - \frac{8s \log(2p)}{n}
\]
\[
\geq \Delta \frac{1}{(1 - \epsilon)\sqrt{1 + \frac{1}{\epsilon}}} (1 - 2\delta) - \frac{8s \log(2p)}{n}.
\]

Hence, for
\[
\Delta > 4\sqrt{2}(1 - 2\delta)^{-1} \sqrt{1 + \frac{1}{\epsilon} \sqrt{s \log(2p)}}
\]
defining the test
\[
t_n(X) := \begin{cases} 
1 & \text{if } T_n^2 > \frac{8s \log(2p)}{n} \\
0 & \text{else}
\end{cases}
\]
we have that
\[
E_0[t_n(X)] + \sup_{\|\theta\|/\sqrt{1 + \epsilon}, z} \mathbb{E}_{\theta, z} [1 - t_n(X)] \leq p^{-1} + 1 - \alpha.
\]

Finally, after having obtained \( \hat{z} \) which has complexity \( O(r_n) \) by assumption, the complexity of calculating \( t_n \) is dominated by the matrix-vector multiplication \( \tilde{X}\hat{z} \) which has complexity \( O(np) \). \( \square \)

**Appendix A: Proof of Lemma 4.1**

**Proof.** Note that
\[
\mathbb{E} \left[ \frac{XX^T}{n} - I_p \right] = \theta \theta^T P.
\]
As usual in sparse PCA we next control the deviations from the mean in \( \ell_\infty \)-norm. Indeed, denoting \( X = [X_1, \ldots, X_n] \) and \( E = [\varepsilon_1, \ldots, \varepsilon_n] \) and \( z = (z_1, \ldots, z_n)^T \), we have that
\[
\left\| \frac{XX^T}{n} - I_p - \theta \theta^T \right\|_\infty \leq \left\| \frac{EE^T}{n} - I \right\|_\infty + 2 \left\| \frac{\theta z^T E^T}{n} \right\|_\infty =: T_1 + T_2
\]
The first term $T_1$ can be bound using Bernstein’s inequality and a union bound with probability $1 - p^{-2}$ by $c\sqrt{\log(p)/n}$, compare e.g. the proof of Lemma D.1 in [LV13], noting that the assumption $\log(p) \leq n$ ensures that the sub-Gaussian component in the exponential inequality dominates. For the second term $T_2$ we use sub-Gaussian concentration directly. Indeed, we have that

$$(\theta z^T E^T)_{jl} = \sum_{i=1}^{n} z_i \theta_j e_{il} \sim \mathcal{N}(0, n\theta_j^2),$$

and hence, using a union bound and sub-Gaussian concentration we obtain that with probability at least $1 - p^{-2}$

$$T_2 \leq c'\|\theta\|\sqrt{\log(p)/n}.$$

Denote $M = \theta\theta^T$ and $\hat{M} = XX^T/n - I_p$. Hence, overall, with probability at least $1 - 2p^{-2}$ the following event occurs

$$\Omega := \left\{ \left\| \frac{XX^T}{n} - I_p - \theta\theta^T \right\|_\infty < C\|\theta\|\sqrt{\log(p)/n} \right\} = \{\|M - \hat{M}\|_\infty < \lambda\}.$$

For the rest of the proof suppose that we work on $\Omega$. Since $P \in \mathcal{F}^1$ we have by definition of the objective function that

$$\langle \hat{M}, \hat{P} \rangle - \lambda\|\hat{P}\|_1 \geq \langle \hat{M}, P \rangle - \lambda\|P\|_1.$$

Using curvature Lemma 3.1 from [LV13] for the first inequality and afterwards the above inequality we obtain that

$$\|\hat{P} - P\|_F^2 \leq \frac{2}{\|\theta\|^2} \langle \hat{M}, P - \hat{P} \rangle = \frac{2}{\|\theta\|^2} \left[ \langle \hat{M}, P - \hat{P} \rangle + \langle M - \hat{M}, P - \hat{P} \rangle \right]$$

$$\leq \frac{2}{\|\theta\|^2} \left[ \lambda\|P\|_1 - \lambda\|\hat{P}\|_1 + \|M - \hat{M}\|_\infty\|P - \hat{P}\|_1 \right]$$

$$\leq \frac{2\lambda}{\|\theta\|^2} \left[ \|P_S\|_1 - \|\hat{P}_S\|_1 + \|(P - \hat{P})_S\|_1 \right]$$

$$\leq \frac{4\lambda s\|P - \hat{P}\|_F}{\|\theta\|^2}.$$

This shows the first assertion. We next prove false positive control. Recall that

$$\hat{P} = \arg\max_{\hat{P} \in \mathcal{F}^1} \langle \hat{M}, \hat{P} \rangle - \lambda\|\hat{P}\|_1. \quad (16)$$

We show that any solution to this objective which has support outside $S$ yields a strictly smaller objective and is thus not a maximizer. Indeed, we have that

$$\langle \hat{M}, P \rangle - \lambda\|P\|_1 = \langle \hat{M}_S, P_S \rangle - \lambda\|P_S\|_1 + \sum_{(i,j) \notin S \times S} M_{ji} P_{ij} - \lambda|P_{ji}|,$$
where $\hat{M}_S$ denotes the restriction of $\hat{M}$ to $S$, i.e. $\hat{M} = (\hat{M})_{(i,j) \in S \times S}$. We have that

$$\sum_{(i,j) \notin S \times S} M_{ji} P_{ij} - \lambda |P_{ji}| \leq \sum_{(i,j) \notin S \times S} (|\hat{M}_{ij}| - \lambda) |\hat{P}_{ij}|$$

$$\leq \sum_{(i,j) \notin S \times S} (||\hat{M} - M||_{\infty} + |M_{ij}| - \lambda) |\hat{P}_{ij}| < 0.$$ 

Hence, any maximizer of (16) must have support in $S$ and thus $\hat{S} \subset S$. 

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References

[ADHP09] D. Aloise, A. Deshpande, P. Hansen, and P. Popat. NP-hardness of euclidean sum-of-squares clustering. Mach. Learn., 75:245–248, 2009.

[ASLW15] M. Azizyan, A. Singh, and L. L. Wasserman. Efficient sparse clustering of high-dimensional non-spherical gaussian mixtures. In Proceedings of the Eighteenth International Conference on Artificial Intelligence and Statistics, volume 38, pages 37–45. PMLR, 2015.

[ASW13] M. Azizyan, A. Singh, and L. Wasserman. Minimax theory for high-dimensional gaussian mixtures with sparse mean separation. Advances in Neural Information Processing Systems (NIPS), pages 2139–2147, 2013.

[AW09] A.A. Amini and M.J. Wainwright. High-dimensional analysis of semidefinite relaxations for sparse principal components. Ann. Statist., 37:2877–2921, 2009.

[BB19a] M. Brennan and G. Bresler. Average-Case Lower Bounds for Learning Sparse Mixtures, Robust Estimation and Semirandom Adversaries. arXiv preprint, 2019.

[BB19b] M. Brennan and G. Bresler. Optimal Average-Case Reductions to Sparse PCA: From Weak Assumptions to Strong Hardness. Conference on Learning Theory (COLT), 2019.

[BB20] M. Brennan and G. Bresler. Reducibility and statistical-computational gaps from secret leakage. arXiv preprint, 2020.

[BBP05] J. Baik, G. Ben Arous, and S. Péché. Phase transition of the largest eigenvalue for non-null complex sample covariance matrices. Ann. Probab., 33(5):1634–1697, 2005.

[BBS14] C. Bouveyron and C. Brunet-Saumard. Model-based clustering of high-dimensional data: A review. Comput. Statist. Data Anal., 71:52–78, 2014.

[BHK+19] B. Barak, S. Hopkins, J. Kelner, P.K. Kothari, A. Moitra, and A. Potechin. A nearly tight sum-of-squares lower bound for the planted clique problem. SIAM J. Comput., 48(2):687–735, 2019.
[BKR+11] S. Balakrishnan, M. Kolar, A. Rinaldo, A. Singh, and L. Wasserman. Statistical and computational tradeoffs in biclustering. In *NIPS 2011 Workshop on Computational Trade-offs in Statistical Learning*, 2011.

[BR13a] Q. Berthet and P. Rigollet. Complexity theoretic lower bounds for sparse principal component detection. *Conference on Learning Theory*, pages 1046–1066, 2013.

[BR13b] Q. Berthet and P. Rigollet. Optimal detection of sparse principal components in high dimension. *Ann. Statist.*, 41(4):1780–1815, 2013.

[BWY17] S. Balakrishnan, M.J. Wainwright, and B. Yu. Statistical guarantees for the EM algorithm: From population to sample-based analysis. *Ann. Statist.*, 45(1):77–120, 2017.

[CLR17] T.T. Cai, T. Liang, and A. Rakhlin. Computational and statistical boundaries for submatrix localization in a large noisy matrix. *Ann. Statist.*, 45(4):1780–1815, 2017.

[CMZ19] T.T. Cai, J. Ma, and L. Zhang. CHIME: Clustering of high-dimensional gaussian mixtures with EM algorithm and its optimality. *Ann. Statist.*, 47(3):1234–1267, 2019.

[dEJL07] A. d’Aspremont, L. El Ghaoui, M.I. Jordan, and G.R.G. Lanckriet. A direct formulation of sparse PCA using semidefinite programming. *SIAM Review*, 49(3):434–448, 2007.

[DKS17] I. Diakonikolas, D.M. Kane, and A. Stewart. Statistical query lower bounds for robust estimation of high-dimensional gaussians and gaussian mixtures. In *2017 IEEE 58th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 73–84. IEEE, 2017.

[DKWB19] Y. Ding, D. Kunisky, A.S. Wein, and A.S. Bandeira. Subexponential-Time Algorithms for Sparse PCA. *arXiv preprint*, 2019.

[DLR77] A. Dempster, N. Laird, and D. Rubin. Maximum likelihood from incomplete data via the EM algorithm (with discussion). *J. R. Statist. Soc. B*, 39:1–38, 1977.

[DM16] Y. Deshpande and A. Montanari. Sparse PCA via covariance thresholding. *J Mach Learn Res*, 17(1):1–41, 2016.

[FGR+17] V. Feldman, E. Grigorescu, L. Reyzin, S. Vempala, and Y. Xiao. Statistical algorithms and a lower bound for detecting planted cliques. *J. ACM*, 64(2):1–37, 2017.

[FLWY18] J. Fan, H. Liu, Z. Wang, and Z. Yang. Curse of heterogeneity: Computational barriers in sparse mixture models and phase retrieval. *arXiv preprint*, 2018.

[FM04] J. Friedman and J. Meulman. Clustering objects on subsets of attributes. *J. Roy. Statist. Soc. Ser. B*, 66:815–849, 2004.

[GMZ17] C. Gao, Z. Ma, and H.H. Zhou. Sparse CCA: Adaptive Estimation and Computational Barriers. *Ann. Statist.*, 45(5):2074–2101, 2017.

[GN16] E. Giné and R. Nickl. *Mathematical Foundations of Infinite-
Löffler, Wein, Bandeira/Sparse clustering

Dimensional Statistical Methods. Cambridge University Press, 2016.

[GV19] C. Giraud and N. Verzelen. Partial recovery bounds for clustering with the relaxed k-means. Mathematical Statistics and Learning, 1(3/4):317–374, 2019.

[GWS20] M. Gataric, T. Wang, and R.J. Samworth. Sparse principal component analysis via axis-aligned random projections. J R Stat Soc B, to appear, 2020.

[HKP+17] S.B. Hopkins, P.K. Kothari, A. Potechin, P. Raghavendra, T. Schramm, and D. Steurer. The power of sum-of-squares for detecting hidden structures. In 2017 IEEE 58th Annual Symposium on Foundations of Computer Science. IEEE, 2017.

[Hop18] S.B. Hopkins. Statistical Inference and the Sum of Squares Method. PhD thesis, Cornell University, 7 2018.

[HP15] M. Hardt and E. Price. Tight bounds for learning a mixture of two Gaussians. In STOC’15Proceedings of the 2015 ACM Symposium on Theory of Computing, pages 753–760. ACM, New York, 2015.

[HS17] S.B. Hopkins and D. Steurer. Efficient bayesian estimation from few samples: community detection and related problems. In 58th Annual IEEE Symposium on Foundations of Computer Science, volume 1, pages 379–390. IEEE, 2017.

[HSV19] Guy Holtzman, Adam Soffer, and Dan Vilenchik. A greedy anytime algorithm for sparse PCA. arXiv preprint arXiv:1910.06846, 2019.

[HW20] J. Holmgren and A.S. Wein. Counterexamples to the low-degree conjecture. arXiv preprint arXiv:2004.08454, 2020.

[JDP83] K. Joag-Dev and F. Proschran. Negative Association of Random Variables with Applications. Ann. Statist., 11(1):286–295, 1983.

[JKW17] J. Jin, Z.T. Ke, and W. Wang. Phase transitions for high dimensional clustering and related problems. Ann. Statist., 45(5):2151–2189, 2017.

[JL07] I.M. Johnstone and A.Y. Lu. On Consistency and Sparsity for Principal Components Analysis in High Dimensions. J Am Stat Assoc, 104(486):682–693, 2007.

[JW16] J. Jin and W. Wang. Influential features PCA for high-dimensional clustering. Ann. Statist., 44(6):2323–2359, 2016.

[Kea98] M. Kearns. Efficient noise-tolerant learning from statistical queries. J. ACM, 45(6):983–1006, 1998.

[KNV15] R. Krauthgamer, B. Nadler, and D. Vilenchik. Do semidefinite relaxations solve sparse PCA up to the information limit? Ann. Statist., 43(3):1300–1322, 2015.

[KSS04] A. Kumar, Y. Sabharwal, and S. Sen. A simple linear time (1 + ε)-approximation algorithm for k-means clustering in any dimensions. In Proceedings of the 45th Annual IEEE Symposium on Foundations of Computer Science, pages 454–462. IEEE Computer Society, Washington, DC, 2004.

[KWB19] D. Kunisky, A.S. Wein, and A.S. Bandeira. Notes on Computa-
tional Hardness of Hypothesis Testing: Predictions using the Low-Degree Likelihood Ratio. *arXiv preprint*, 2019.

[LDBB+16] T. Lesieur, C. De Bacco, J. Banks, F. Krzakala, C. Moore, and L. Zdeborová. Phase transitions and optimal algorithms in high-dimensional Gaussian mixture clustering. *2016 54th Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, pages 601–608, 2016.

[Llo82] S. Lloyd. Least squares quantization in PCM. *IEEE Trans. Inf. Theor.*, 28(2):129–137, 1982.

[LR15] J. Lei and A. Rinaldo. Consistency of spectral clustering in stochastic block models. *Ann. Statist.*, 43(1):215–237, 2015.

[LV13] J. Lei and V.Q. Vu. Minimax sparse principal subspace estimation in high dimensions. *Ann. Statist.*, 41(6):2905–2947, 2013.

[LV15] J. Lei and V.Q. Vu. Sparsistency and agnostic inference in sparse PCA. *Ann. Statist.*, 43(1):299–322, 2015.

[LZ16] Y. Lu and H.H. Zhou. Statistical and Computational Guarantees of Lloyd’s Algorithm and its Variants. *arXiv preprint*, 2016.

[LZZ19] M. Löffler, A.Y. Zhang, and H.H. Zhou. Optimality of spectral clustering for Gaussian mixture model. *arXiv preprint*, 2019.

[Ma13] Z. Ma. Sparse principal component analysis and iterative thresholding. *Ann. Statist.*, 41(2):772–801, 2013.

[MW15a] T. Ma and A. Wigderson. Sum-of-squares lower bounds for sparse PCA. In *Advances in Neural Information Processing Systems (NIPS)*, pages 1612–1620, 2015.

[MW15b] Z. Ma and Y. Wu. Computational barriers in minimax submatrix detection. *Ann. Statist.*, 43(3):1089–1116, 2015.

[Nda19] M. Ndaoud. Sharp optimal recovery in the two component gaussian mixture model. *arXiv preprint*, 2019.

[NWR19] J. Niles-Weed and P. Rigollet. Estimation of wasserstein distances in the spiked transport model. *arXiv preprint*, 2019.

[PS07] W. Pan and X. Shen. Penalized model-based clustering with application to variable selection. *J. Mach. Learn. Res.*, 8:1145–1164, 2007.

[PW07] J. Peng and Y. Wei. Approximating k-means-type clustering via semidefinite programming. *SIAM J. on Optimization*, 18(1):186–205, 2007.

[Sha00] Q. Shao. A Comparison Theorem on Moment Inequalities Between Negatively Associated and Independent Random Variables. *J. Theor. Probab.*, 13(2), 2000.

[VAC17] N. Verzelen and E. Arias-Castro. Detection and feature selection in sparse mixture models. *Ann. Statist.*, 45(5):1920–1950, 2017.

[VCLR13] V.Q. Vu, J. Cho, J. Lei, and K. Rohe. Fantope Projection and Selection: A near-optimal convex relaxation of Sparse PCA. *Advances in Neural Information Processing Systems (NIPS)*, 26, 2013.

[VL07] U. Von Luxburg. A tutorial on spectral clustering. *Stat. Comput.*, 17(4):395–416, 2007.
[VW04] S. Vempala and G. Wang. A spectral algorithm for learning mixture models. *J. Comput. Syst. Sci.*, 68(4):841–860, 2004.

[WBS16] T. Wang, Q. Berthet, and R.J. Samworth. Statistical and computational trade-offs in estimation of sparse principal components. *Ann. Statist.*, 44(5):1896–1930, 2016.

[WT10] D.M. Witten and R. Tibshirani. A framework for feature selection in clustering. *J Am Stat Assoc*, 105(490):713–726, 2010.

[WZ08] S. Wang and J. Zhu. Variable selection for model-based high-dimensional clustering and its application to microarray data. *Biometrics*, 64:440–448, 2008.

[WZ19] Y. Wu and H.H. Zhou. Randomly initialized EM algorithm for two-component Gaussian mixture achieves near optimality in $O(\sqrt{n})$ iterations. *arXiv preprint*, 2019.