MITTAG-LEFFLER TYPE SUMS ASSOCIATED WITH
ROOT SYSTEMS

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Abstract. This is a largely expository note which applies standard
techniques of the theory of Duijstermaat-Heckman measures for compact
Lie groups and results of P. Littelmann to prove a generalization of a
conjecture of Coquereaux and Zuber.

1. The main theorem

Let $G$ be a simply connected simple algebraic group over $\mathbb{C}$. Let $\mathfrak{g} = \text{Lie } G$, $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, $W$ the Weyl group, $Q, P, Q^\vee, P^\vee$ the root, weight, coroot and coweight lattices, $\rho \in \mathfrak{h}^*$ the half-sum of positive
roots, $P_+ \subset P$ the set of dominant integral weights, $V_\lambda$ the irreducible representation of $G$ with highest weight $\lambda \in P_+$, and $k$ a positive integer.

Consider the function on $\mathfrak{h}$ given by

$$f(x) = \prod_{\alpha > 0} \frac{\sin \pi \alpha(x)}{\pi \alpha(x)},$$

where the product is taken over positive roots of $G$.

Let $Z \subset G$ be the center and $\xi : Z \to \mathbb{C}^*$ a character. Since $Z = P^\vee/Q^\vee$, we may view $\xi$ as a character of $P^\vee$ which is trivial on $Q^\vee$. Define the function

$$F_{k, \xi}(x) := \sum_{a \in P^\vee} \xi(a) f^k(x + a).$$

(If $G = SL_2$ and $k = 1$ then the sum is not absolutely convergent, and should be understood in the sense of principal value). Thus, the meromorphic
function

$$M_{k, \xi}(x) := \frac{F_{k, \xi}(x)}{\prod_{\alpha > 0} \pi^{-k} \sin^k \pi \alpha(x)}$$

has a Mittag-Leffler type decomposition

$$M_{k, \xi}(x) = \sum_{a \in P^\vee} (-1)^{2k\rho(a)} \xi(a) \frac{\prod_{\alpha > 0} \alpha^k(x)}{\prod_{\alpha > 0} \alpha^k(x)}.$$

For example, for $G = SL_2$, we have

$$F_{1,1}(x) = 1, F_{1,-1}(x) = \cos \pi x,$$

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which gives the classical Mittag-Leffler decompositions
\[\pi \cot \pi x = \sum_{n \in \mathbb{Z}} \frac{1}{x + n}, \quad \pi \sin \pi x = \sum_{n \in \mathbb{Z}} \frac{(-1)^n}{x + n}.\]

The goal of this note is to prove the following theorem.

**Theorem 1.1.** The function \(F_{k,\xi}\) is a \(W\)-invariant trigonometric polynomial on the maximal torus \(T = \mathfrak{h}/Q^\vee\) of \(G\), which is a nonnegative rational linear combination of irreducible characters of \(G\).

For \(G = SL_n\) and \(\xi\) being a character of order 2, this theorem was conjectured by R. Coquereaux and J.-B. Zuber ([CZ], Conjecture 1 in Subsection 2.2).

Since all the techniques and ideas we use are well known, this note should be viewed as largely expository.

2. **Proof of the main theorem**

2.1. **Contraction of representations.** We start with the following general fact.

**Proposition 2.1.** ([Li1], Proposition 3) Let \((V, \rho_V)\) be a rational representation of \(G\), and \(N\) a positive integer. Let \(V_N\) be the direct sum of all the weight subspaces of \(V\) of weights divisible by \(N\). Then the action of \(T\) on \(V_N\) given by \(t \circ v := \rho_V((1/N)t)v\) extends to an action of \(G\). In other words, the function
\[\chi_{V,N}(x) := \sum_{\lambda \in P} \dim V[N\lambda] e^{2\pi i \lambda(x)}\]
is a nonnegative linear combination of irreducible characters of \(G\). Namely, the multiplicity of \(\chi_{\lambda}\) in \(\chi_{V,N}(x)\) equals the multiplicity of \(V_N\lambda + (N-1)\rho\) in \(V \otimes V_{(N-1)\rho}\).

**Proof.** Littelmann proves this proposition via his path model as an illustration of its use, but we give a more classical proof using the Weyl character formula. We have to show that the integral
\[I := \int_{\mathfrak{h}/Q^\vee} \sum_{\lambda \in P} \dim V[N\lambda] e^{-2\pi i \lambda(x)} |\chi_{\lambda}(x)|^2 dx\]
is nonnegative, where \(\Delta(x)\) is the Weyl denominator, since the multiplicity in question is \(I/|W|\).

Denoting the character of \(V\) by \(\chi_V\), we have
\[I = \int_{\mathfrak{h}/Q^\vee} \int_{\mathfrak{h}/Q^\vee} \sum_{\lambda \in P} \chi_V(y) e^{2\pi i N\lambda(y)} e^{-2\pi i \lambda(x)} |\chi_{\lambda}(x)|^2 dydx = \int_{\mathfrak{h}/Q^\vee} \int_{\mathfrak{h}/Q^\vee} \chi_V(y) e^{2\pi i \lambda(x)} |\chi_{\lambda}(x)|^2 dydx = \]

\footnote{Note that \(\rho_V((1/N)t)v\) is independent on the choice of the \(N\)-th root \(t^{1/N}\).}
\[ \int_{\mathfrak{h}_R/Q^\vee} \chi_V(y) \chi(\lambda(Ny)) \Delta(Ny) \, dy. \]

Using the Weyl character formula, we then have

\[ I = \int_{\mathfrak{h}_R/Q^\vee} \chi_V(y) \left( \sum_{w \in W} (-1)^w e^{2\pi i (w(\lambda+\rho),Ny)} \right) \Delta(Ny) \, dy = \]

\[ \int_{\mathfrak{h}_R/Q^\vee} \chi_V(y) \sum_{w \in W} (-1)^w e^{2\pi i (w(\lambda+\rho),Ny)} \frac{\Delta(Ny)}{\Delta(y)} \Delta(y) \, dy = \]

\[ \int_{\mathfrak{h}_R/Q^\vee} \chi_V(y) \chi_{N\lambda+(N-1)\rho}(y) \left( \frac{\Delta(Ny)}{\Delta(y)} \right) \Delta(y) \, dy. \]

Now recall that \( \frac{\Delta(Ny)}{\Delta(y)} = \chi(N-1)\rho(y) \). Thus we get

\[ I = \int_{\mathfrak{h}_R/Q^\vee} \chi_V(y) \chi(N-1)\rho(y) \chi_{N\lambda+(N-1)\rho}(y) \Delta(y) \, dy, \]

i.e., \( I/|W| \) is the multiplicity of \( V_{N\lambda+(N-1)\rho} \) in \( V \otimes V_{(N-1)\rho} \), as desired. \( \square \)

**Remark 2.2.**

1. If \( N \) is odd (and coprime to 3 for \( G \) of type G2), then Proposition 2.1 has a nice representation-theoretic interpretation. Namely, if \( q \) is a root of unity of order \( N \) and \( G_q \) the corresponding Lusztig quantum group, then there is an exact contraction functor \( F : \text{Rep} G_q \to \text{Rep} G \) which at the level of \( P \)-graded vector spaces transforms \( V \) into \( V_N \) with weights divided by \( N \) (see [GK] and references therein). Proposition 2.1 is then obtained by applying the functor \( F \) to a Weyl module.

2. Suppose that \( G \) is not simply laced. Normalize the inner product on \( \mathfrak{h}^* \) so that long roots have squared length 2. This inner product identifies \( \mathfrak{h} \) with \( \mathfrak{h}^* \) so that \( \alpha_i^v \) map to \( 2\alpha_i/\langle \alpha_i,\alpha_i \rangle \). Note that \( 2/\langle \alpha_i,\alpha_i \rangle \) is an integer, so under this identification \( Q^v \subset Q \), hence \( P^v \subset P \). Let \( V'_N \subset V_N \) be the span of the weight subspaces of \( V \) of weights belonging to \( NP^v \) with weights divided by \( N \). Then, analogously to Proposition 2.1, \( V'_N \) extends to a representation of the Langlands dual Lie algebra \( g^L \), with a similar description of multiplicities ([Li1], Proposition 4). Note that this statement is nontrivial even if \( g^L \cong g \), since the arrow on the Dynkin diagram is reversed. This also has a representation-theoretic interpretation similar to (1), see [Li1], Section 3, [GK].

3. As explained in [Li1], Proposition 2.1 generalizes to symmetrizable Kac-Moody algebras (both our proof and that of [Li1] can be straightforwardly extended to this case). So does the non-simply laced version of Proposition 2.1 given in (2) and the above representation-theoretic interpretations, see [Li2].
2.2. Duistermaat-Heckman measures and proof of Theorem 1.1

Now recall ([GLS]) that for each dominant $\lambda \in \mathfrak{h}^*_R$, we can define the Duistermaat-Heckman measure $DH_\lambda(\mu) d\mu$ on $\mathfrak{h}^*_R$, which is the direct image of the Liouville measure on the coadjoint orbit of $\lambda$. This measure is supported on the convex hull of the Weyl group orbit $W\lambda$, and its Fourier transform is given by the formula

$$ F(DH_\lambda)(x) = \sum_{w \in W} (-1)^w e^{2\pi i (w\langle x, \lambda \rangle)} \prod_{\alpha > 0} 2\pi i \alpha(x). $$

For simplicity assume that $\lambda$ is regular and $G \neq SL_2$. Then $DH_\lambda$ is absolutely continuous with respect to the Lebesgue measure (i.e., the density function $DH_\lambda(\mu)$ is continuous). Then it is known ([GLS]) that if $\mu_N \in P$, $\lambda_N \in P_+$ are sequences such that $\mu_N/N \to \mu$, $\lambda_N/N \to \lambda$ as $N \to \infty$ and $\lambda_N - \mu_N \in Q$ then

$$ \lim_{N \to \infty} \frac{\dim V_{\lambda_N}[\mu_N]}{N^{\frac{1}{|R_+|}}} = DH_\lambda(\mu), $$

where $R_+$ is the set of positive roots. Note that equation (1) follows immediately from equation (2) and the Weyl character formula.

**Proposition 2.3.** Let $\lambda_1, ..., \lambda_k \in \mathfrak{h}^*_R$ be regular dominant weights. Then the trigonometric polynomial

$$ \sum_{\mu \in P} (DH_{\lambda_1} * ... * DH_{\lambda_k})(\mu)e^{2\pi i \mu(x)} $$

(where $*$ denotes convolution of measures) is a linear combination of irreducible characters of $G$ with nonnegative real coefficients.

**Proof.** First assume that $\lambda_i$ are rational, and let $d$ be their common denominator. Then, taking the limit as $N \to \infty$ in Proposition 2.1 with $V = V_{\lambda_1} \otimes ... \otimes V_{\lambda_k}$ and $N$ divisible by $d$, we obtain the desired statement. Now the general case follows from the facts that rational weights are dense in $\mathfrak{h}_R$ and $DH_\lambda(\mu)$ is continuous in $\lambda$. $\square$

Now Theorem 1.1 follows from equation (1) and Proposition 2.3 by taking $\lambda_1, ..., \lambda_k = \rho$ and noting that by the Weyl denominator formula

$$ \sum_{w \in W} (-1)^w e^{2\pi i (w\rho, x)} \prod_{\alpha > 0} 2\pi i \alpha(x) = \prod_{\alpha > 0} \frac{\sin \pi \alpha(x)}{\pi \alpha(x)} = f(x). $$

The rationality of the coefficients follows from the rationality of the values of the convolution power $(DH_\rho)^k$ at rational points.

**Remark 2.4.** It follows from (3) that the measure $DH_\rho$ is the convolution of uniform measures on $[-\alpha/2, \alpha/2]$ over all positive roots $\alpha$. 

2.3. The characters occurring in $F_{k,\xi}$. Let us now discuss which irreducible characters can occur in the decomposition of $F_{k,\xi}$. Let us view $\xi$ as an element of $P/Q$. Clearly, if $\chi_\lambda$ occurs in $F_{k,\xi}$ then the central character of the representation $V_\lambda$ must be $\xi = k\rho - \lambda \mod Q$. If so, then, as shown above, the multiplicity of $\chi_\lambda$ in $F_{k,\xi}$ is $(DH_\rho)^k(\lambda)$. Since this density is continuous and supported on the $k$ times dilated convex hull $B$ of the orbit $W_\rho$, we see that if $\chi_\lambda$ occurs then $\lambda$ must be strictly in the interior of $B$.

Let $m_i(\xi)$ be the smallest strictly positive number such that $m_i(\xi) \equiv (\xi, \omega_i^\vee) \mod \mathbb{Z}$, where $\omega_i^\vee$ are the fundamental coweights, and let $\beta_\xi := \sum_i m_i(\xi) \alpha_i \in P$. Then we get

**Proposition 2.5.** The character $\chi_\lambda$ occurs in $F_{k,\xi}$ if and only if $\xi = k\rho - \lambda \mod Q$ and $(\lambda, \omega_i^\vee) < k(\rho, \omega_i^\vee)$ for all $i$. Moreover, in presence of the first condition, the second condition is equivalent to the inequality $\lambda \leq k\rho - \beta_\xi$.

We also have

**Proposition 2.6.** The weight $\rho - \beta_\xi$ (and hence $k\rho - \beta_\xi$ for all $k \geq 1$) is dominant.

**Proof.** We need to show that for all $i$ we have $(\rho - \beta_\xi, \alpha_i^\vee) \geq 0$. Since $\rho - \beta_\xi$ is integral, it suffices to show that $(\rho - \beta_\xi, \alpha_i^\vee) \geq 0$, i.e., $(\beta_\xi, \alpha_i^\vee) < 2$. But

$(\beta_\xi, \alpha_i^\vee) = 2m_i + \sum_{j \neq i} m_j(\alpha_j, \alpha_i^\vee) < 2$, since $0 < m_j \leq 1$ and $(\alpha_j, \alpha_i^\vee) \leq 0$ for all $j \neq i$ and is strictly negative for some $j$. \hfill \Box

**Corollary 2.7.** $F_{k,\xi} = \sum_{\mu \leq k\rho - \beta_\xi} C_{k,\xi}(\mu) \chi_\mu$, where $C_{k,\xi}(\mu) \in \mathbb{Q}_{>0}$. In particular, the leading term is a multiple of $\chi_{k\rho - \beta_\xi}$.

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