A Survey on Methods and Systems for Graph Compression

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Abstract. We present an informal survey (meant to accompany another paper) on graph compression methods. We focus on lossless methods, briefly list available approaches, and compare them where possible or give some indicators on their compression ratios. We also mention some relevant results from the field of lossy compression and algorithms specialized for the use on large graphs. — Note: The comparison is by no means complete. This document is a first draft and will be updated and extended.

1 Introduction

Graphs are an important data structure in computer science. A graph consists of a set of nodes $V$ and a set of edges $E$ connecting the nodes. We say that two nodes are adjacent, if they are connected by an edge. The nodes and edges may be labeled. The edges may be directed or undirected. Furthermore sometimes multigraphs are considered, which allow for more than one edge between two nodes (called multiedges). Here we want to consider how graphs are represented in memory and how they are stored in external memory. This is of renewed interest, as many recent graph data is too large to fit into memory. Compression can be applied to fit large graphs in memory and to store them efficiently on secondary memory.

Let $n = |V|$ and $m = |E|$. The most basic model of a graph, is one without edge labels and multiedges. Such graphs can be represented as:

**Adjacency matrix:** a square 0/1-matrix with one row/column for every node.

A 1-value at position $i,j$ means that there is an edge from node $i$ to node $j$. Uses $O(n^2)$ bits of space and deciding adjacency is possible in constant time.

**Adjacency list:** Represents just the edges by listing the neighbors of every nodes in a list. Can be imagined as an adjacency matrix that without the 0-values. Uses $O((n + m) \log_2 n)$ space and deciding adjacency takes $O(\log_2 n)$ time.

Note that both representations automatically provide node IDs, i.e. every node can be uniquely identified by an integer assigned to it. For us, a compressed graph is a representation that uses less space than the above two. Compression
methods are often compared by the amount of bits they need on average to store one edge (bits per edge, \(bpe\)). To give a comparison: an adjacency list representation of a graph with at least as many edges as nodes and node IDs represented by 32 bit integers would roughly need 32 to 64 \(bpe\).

Most compression approaches we mention support simple queries on the compressed graph, without the need for decompression. The most common such queries are the sets of out-neighbors or in-neighbors of a node \(v\). If there is an edge from \(u\) to \(v\) then \(v\) is an out-neighbor of \(u\) and \(u\) is an in-neighbor of \(v\). Common traversal mechanisms like BFS or DFS can be implemented using in- or out-neighbors.

Note: If not stated otherwise, the methods discussed in this survey assume directed graphs with unique node IDs and without edge-labels or multiedges.

2 Lossless Graph Compression

We begin with compression methods that can be reversed. We divide them into two categories: (1) succinct representations encode the graph in a succinct bit-string, but do not make structural changes that have to be reversed for decompression, (2) Structural approaches replace repeating structures by a short identifier. Note that these two approaches are often combined.

2.1 Succinct Representation

Succinct data structures typically mean encodings that represent the data within the information theoretic lower bound plus \(o(n)\) additional bits. We relax this definition in the second subsection, where such proofs of optimality are not given.

Without Implementation To our knowledge the following methods are investigated theoretically and have not been implemented. Most of them work only on certain families of graphs, not arbitrary ones.

Galperin and Wigderson \[GW83\] mention succinct graph representations are mentioned by, where they examine the complexity of certain graph problems under the assumption that the input is given in a succinct way. In this context that means, that the input size is not polynomial in \(|V|\), but polylog in \(|V|\). They show that every nontrivial problem they examine becomes \(NP\)-hard in this setting, even the ones that were in \(P\) for standard representations. How to achieve a succinct representation is not mentioned in the paper.

Turán \[Tur84\] presents a particular succinct representation. It encodes undirected planar graphs without loops or multiple edges into a bitstring of length \(\leq 12n\). Furthermore an undirected planar graph with node labels can be represented by \(n[\log_2 n] + 12n\) bits. This is achieved by first choosing a spanning tree of the graph. The nodes of the graph are then arranged into a cyclic sequence according to a post-order traversal of the spanning tree (length: \(2n – 2\)). The remaining edges (i.e. the ones not part of the spanning tree) are included as diagonals of the cycle. This is where planarity is necessary, as otherwise these
diagonals might cross. The sequence is then encoded as a string consisting of the symbols $+, -, (, )$, which can be encoded by 2 bits each, leading to an encoding of at most $12n - 24$ bits. In the labeled case the same encoding is used, padded to the maximum amount of bits, followed by a dictionary for the labels. Access to the compressed structure is not considered.

Jacobson’s approach \cite{Jac89} works on graphs of bounded pagenumber. This means that the graph can be represented by a bounded number of pages, which is a linear drawing of a subset of nodes with the edges connecting them drawn above so that they do not intersect. Such a page can be encoded as a string of parentheses, leading to an encoding for the full graph by combining the encodings of the single pages. He proves it to be optimal for trees and within a logarithmic factor of the information theoretic lower bound for graphs of bounded pagenumber. Jacobson also considers accessing the succinct structure and shows that an algorithm can test adjacency of two nodes in $O(\log n)$ bit inspections for graphs with one page. For graphs with $k$ pages, Jacobson’s approach uses $O(kn)$ bits and tests adjacency in $O(k \log n)$ time.

Deo and Litow \cite{DL98} show that graphs of bounded genus $g$ (which is the same as bounded pagenumber) can be represented using $O(g+n)$ bits and $O(g+n)$ time. However, finding a minimal genus embedding for a graph $G$ is NP-complete, but fixed-parameter tractable, therefore finding an embedding for a fixed genus $g$ is possible in P. They furthermore show separator theorems for graphs defined by an excluded minor. These and similar separator techniques are later used for compression, as further mentioned below.

The approach by Farzan and Munro \cite{FM13} works for arbitrary graphs. They first prove that it is not possible to achieve a representation within the information theoretic lower bound up to lower order terms unless the graph is either too sparse (i.e. $m = o(n^\delta)$ for any constant $\delta > 0$) or too dense (i.e. $m = \omega(n^{2-\delta})$ for any constant $\delta > 0$). They then present a succinct encoding supporting adjacency and neighborhood queries in constant time. The space needed for the encoding is within a $(1 + \varepsilon)$-factor of the information theoretic lower bound for any arbitrarily small constant $\varepsilon > 0$.

Lu \cite{Lu14} gives a compression scheme for hereditary classes of graphs. A class of graphs is hereditary, if it is closed under taking subgraphs. Let $\mathcal{G}$ be such a class of graphs, and let $\text{num}(\mathcal{G}, n)$ be the amount of graphs with $n$ nodes in $\mathcal{G}$. They prove that, if $\log \text{num}(\mathcal{G}, n) = O(n)$, given a graph $G \in \mathcal{G}$ with $n$ nodes and a genus-$o(\frac{n}{\log n})$ embedding, it is possible to represent the graph using $\beta n + o(n)$ bits, where $\beta$ is any positive constant such that $\log \text{num}(\mathcal{G}, n) \leq \beta n + o(n)$. This is achieved by using graph separators as above.

**With Implementation** The representations in this section are implemented and experimentally evaluated. Most of them focus on compression of web graphs, which are graphs where the nodes represent websites (labeled by the URL), and a directed edge from node $v$ to node $u$ means that $v$ has a hyperlink to $u$. Compression methods focusing on web graphs often rely on properties specific to web graphs, such as:
**Locality:** most links lead to pages within the same host.

**Similarity:** pages on the same host often share the same links.

One way to work with these properties is to order the nodes lexicographically by their URLs. This leads to an order where the source and target of the edges tend to be close to each other with respect to that order. Once this order is computed the node labels can be replaced by IDs indicating their position in the order. The *WebGraph framework* [BV04] by Boldi and Vigna is based on this order. They compress the adjacency lists using several methods. Among those methods are the representation of adjacent nodes by the difference to the previous value, instead of the nodes’ ID, which reduces the amount of different values to be stored due to the high locality of web graphs. This technique is further improved by considering integer intervals appearing in the adjacency lists, which allows for even less variance. The high similarity can be utilized by using a reference encoding. Instead of storing the (mostly) same lists many times, a reference number and a copy instruction is stored with an adjacency list $l$, specifying which IDs from which list are to be copied into $l$. Finally a $\zeta$-code is applied to the adjacency lists to store them in small space. The methods employed to transform the adjacency lists are influenced by the node ordering.

In the first version a lexicographical ordering is used, but they later improve the framework further by proposing a different ordering [BSV09]. Using these improvements they report compression for some web graphs as low as 2.6 bpe. Furthermore the resulting structure can be used as an in-memory data structure, preserving the ability to query the out-neighborhood of a node. However, the time this querying takes depends on the quality of the compression. With the highest compression ratio, the answer to queries is in the realm of milliseconds instead of microseconds. It should be noted that network graphs that are not web graphs do not necessarily have the same locality and similarity properties and therefore may not compress as well, using their methods.

Apostolico and Drovandi [AD09] propose a different encoding which has the advantage of not relying on an the availability of some natural ordering of the nodes and is therefore more suitable to compress general graphs, though they only evaluate it on web graphs. They use a BFS-order of the nodes and store a succinct representation of the adjacency list using an entropy-based encoding. The adjacency list can still be queried, achieving access times which are competitive to the ones in the WebGraph framework. At the same time they report better compression ratios as low as 1.83 bpe for the UK2002 graph with 18 million nodes and 298 million edges, whereas the WebGraph framework achieves 2.22 bpe on the same graph. Both of these numbers are for highest compression, disregarding access times. The best compromise of compression and access times is 3.00 bpe with the WebGraph framework and 2.62 bpe using this method, which is then still slightly faster than the WebGraph framework.

The above algorithms only support out-neighborhood queries. To also query for incoming neighbors, they need to store the graph twice, once with inverted edges. Maserrat [Mas12] presents lossless and lossy compression schemes which support both, in- and out-neighborhood queries, and also more efficient incre-
mental updates than the above. These schemes are evaluated on social networks as well as web graphs, achieving between 5 and 30 bpe. Social networks tend to be more difficult to compress than web graphs, because there are no obvious properties such as locality or similarity to exploit for a good ordering of the nodes. A specific example is a lifejournal graph with about 4.8 million nodes and 68.5 million edges, where this algorithm achieves 13.9 bpe, while the WebGraph framework on the same graph achieves 14.3 bpe, but only supporting out-neighbor queries.

So far every method discussed is based on adjacency lists. An adjacency matrix based is proposed by Brisaboa et al. [BLN09] who consider $k^2$-trees, a compact tree structure taking advantage of large empty areas of the adjacency matrix of the graph. The general idea is, that the adjacency matrix is first partitioned into rectangles. Rectangles that only include 0-values are represented by a leaf with value 0. The other ones are recursively further partitioned, until a tree is produced which represents the full adjacency matrix. Their encoding is a succinct representation of this tree. With this method, they can provide full support for both, in- and out-neighbor queries. They can show that their method achieves better compression ratios in this setting: they achieve 4.2 bpe for the UK2002 graph, where the WebGraph framework needs more than 6 bpe to store two copies of the same graph. The access times are slightly slower however, especially for in-neighbors.

An interesting aspect of the $k^2$-tree method is that it can not only be used to represent adjacency matrices, but is also applicable to basically any binary relation. This aspect is used by Álvarez et al. [ABLP10] to extend the $k^2$-tree representation to general graph databases instead of web graphs. Specifically they represent graphs, which have uniquely labeled nodes, while the edges are labeled from an alphabet, allowing multiedges if differently labeled. To achieve this, they represent a labeled, directed, attributed, multigraph $G$ using a new data structure called Compact Graph Database. Essentially, this consists of three $k^2$-tree representations, encoding three different relations, which are:

- The binary relation between nodes and their attribute values, where the nodes are the rows and all possible values the columns. A “1” signifies that the node has the corresponding columns attribute value.
- Analogously the relation between edges and their labels can be represented with a $k^2$-tree in this way.
- The relation between nodes is mostly a standard $k^2$-tree representation. The difference is, that they extend the standard representation to deal with multiple edges between nodes.

They evaluate their method on two datasets: Youtube has 1.2 million nodes and 5.2 million edges and is compressed to about 47%; Wikipedia has 11.1 million nodes and 90.1 million edges and is compressed to about 43% of the file size a plain representation needs. We do not know how many bpe this corresponds to. They compare the method to a graph database (DEX [MMG+07]) that does not aim to achieve a succinct representation (in fact, it needs 2 to 4 times more space than the plain file size on the evaluated graphs), but at providing fast
access times. Expectedly, the access times of DEX are 30 to 100 times faster than the ones of this method, but they conclude that “taking into account the difference between both methods in terms of space requirements, the difference in navigation times is not too large”.

2.2 Structural Approaches

Contrary to the previous section, these approaches analyse and in some way change the graph structure to achieve a compressed representation. These structural changes are done so that they are reversible. This is usually achieved by using a dictionary or grammar to describe rules on how to reverse the changes. All of the methods discussed here have been implemented. The best results are generally achieved by combining a structural approach with one of the succinct representations above to represent the remaining graph.

Buehrer and Chellapilla [BC08] compress by searching for bi-cliques (complete bipartite graphs) and replacing them with virtual nodes. Their method achieves 1.95 bpe on the UK2002 graph, a graph with 18 million nodes and 298 million edges. The method retains querying capabilities but no experiments on access times are described. Furthermore it is not clearly specified if this includes in-neighbors or only out-neighbors. As finding bi-cliques is a hard problem they present several heuristics to do so.

Even better results are reported by Asano et al. [AMN08] by using a method exploiting frequent patterns in the adjacency matrix. The examples they give for experimental evaluation are rather small (at most 19 million edges), but on these they achieve compression ratios of 1.7 to 2.7 bpe, whereas the WebGraph framework needs 2.1 to 4.38 bpe on the same graphs. According to [CN10] the retrieval of edges is not efficiently possible in this method.

The RePair-compression scheme is a popular approximation of the smallest grammar generating a given string. Claude and Navarro [CN10] apply it to graphs by compressing a string representation of the adjacency list. Their input file format is a binary representation of the adjacency list. The resulting grammar uses the same format. The method allows for partial decompression and thus the answering of out-neighborhood queries. They obtain 4.23 bpe on the UK2002 graph, which is slightly worse than WebGraph, but their access times are better.

Claude and Ladra [CL11] then combine the previous method and the \( k^2 \)-tree representation. They achieve a compression of 2.27 bpe on the UK2002-graph, while retaining the ability to query for both, in- and out-neighbors.

Grabowski and Bieniecki [GB11] propose a merging contiguous blocks of adjacency lists into a single ordered list. Again on the UK2002 graph they reach a compression ratio of 1.92 bpe. As usual this is sacrificing access time, the degree to which this is happening can be given as a parameter. A good compromise for the UK2002 graph is 2.73 bpe, which is still better than the WebGraph frameworks compression, but provides roughly the same access time. Only out-neighbors are supported.

As far as we know, the current state of the art for methods supporting in- and out-neighbor queries is by Hernández and Navarro [HN14] who improve
the method by Buehrer and Chellapilla by expanding it to a general search for dense subgraphs instead of just bicliques. They then combine this with the BFS-representation by Apostolico and Drovandi and with the $k^2$-tree representation, where the $k^2$-representation works best regarding compression, but makes the access times slower. This achieves representations of 0.9 to 1.5 bpe on web graphs (UK2002: 1.53 bpe). When considering queries for both incoming and outgoing edges this is the current state of the art regarding compression. Access is possible, but comparably slow: queries take approximately 2 to 3 times longer than with the $k^2$-tree method. If only out-neighbor queries are considered the best space/time trade-off is not at the best compression. In this case they compress the UK2002 graph to about 2 bpe achieving comparable access times to the method by Grabowski and Bienicki above. However, it achieves about 1 bpe better compression when considering bidirectional queries at comparable access times and they improve social network compression by up to 10 bpe.

Navlakha et al. [NRS08] consider graph summarization with bounded error using the minimum description length principle. A graph summary is a graph consisting of “supernodes”, a mapping that maps the supernodes back to actual nodes and a list of corrections. The idea is that if there is an edge from one supernode to another, then there is an edge from every actual node the first supernode represents to every actual node the second represents. The list of corrections is then used to remove surplus edges this creates, or to add edges that are missing from this method. The user can further specify an error bound, to achieve lossy compression. If this is done, the decompressed graph can have additional edges, or missing edges compared to the original, up to a fraction given by the bound. They define the cost of their representation as the sum of the number of edges in the summary graph and the number of corrections. The mapping of the supernodes are ignored, because they will generally be small. Several algorithms to compute such a representation are proposed, of which a greedy method achieves the best result. Comparing the cost percentage (how much lower the cost of the representation is to the original) they achieve compression of 30% to 80%, consistently better by about 10% to 20% than WebGraph. Access to the compressed structure is not considered. Note that, while graph summarization is compression, the focus is not on finding the smallest possible representation. Instead a graph summary is an effort to simplify the complex structure such that relations between some represented information is revealed.

Based on the separator technique mentioned by Deo and Litow [DL98], Blandford et al. [BBK03,Bla06] design a compressed data structure. The idea is to find graph components that can be disconnected from the rest of the graph by only removing a small number of edges. According to [CN10] the improvements this achieves over a plain representation of the adjacency list mainly rely on smart caching and it is therefore unclear if it would be useful on large web graphs. They do evaluate their technique on several different graphs, including 3D meshes, circuits, street maps, routing maps and two version of a Google web graph (once with reversed edges). The latter is a graph with 916 thousand nodes and 5.1 million edges, of which they report compression of 9.9 bpe and 7.86 bpe
with reversed edges. As this method predates every other implementation discussed here, a comparison with other methods is not provided.

### 2.3 RDF Compression

RDF graphs are a fairly recent graph data format. It is often used to represent semantic information. The idea is to provide the semantic information in a format that can be interpreted by machines, ultimately aiding knowledge discovery. RDF in its purest form is a set of triples of subject, predicate and object.

All three – subject, predicate, and object – allow for strings as values. Explicitly storing the triples is therefore very memory intensive. Thus it is common practice to replace these concrete values by some short encoding and keep a dictionary mapping the code to the concrete value. This leads to two common approaches to compress RDF files: one is to compress the dictionary, the other to compress the underlying graph structure.

Fernández et al. [FGM10] compare four basic approaches to compressing RDF data on three data sets: Billion Triples (541.5 MB), Uniprot (239.4 MB) and US Census data (148.2 MB). The compression ratios compare the original uncompressed file size with the compressed one. Lower percentages mean better compression. The approaches are:

- Directly compressing the original file using standard methods (gzip, bzip2 and ppmidi-6). Of those, ppmidi-6 gave the best results with compression ratios of 3.1% to 7%.
- Representing the RDF-file as an adjacency list can be done in three ways: each the subject, the predicate or the object can be used as the head of the list. A triple \((s, p, o)\) can therefore be represented by the adjacency lists \(s \rightarrow (p, o)\), \(p \rightarrow (s, o)\), and \(o \rightarrow (s, p)\). They evaluated all three representations and again compressed the resulting adjacency list using ppmidi-6. Using the subject as head worked best, achieving compression ratios of 2.2% to 6.6%
- The final two methods implement the same approach using two different techniques. In both cases a dictionary is built from the triples as explained above. The triples are then again represented as an adjacency list which is Huffman-encoded. They evaluate two different methods of representing the dictionary: (1) delta-codes and (2) a succinct tree structure, where the tree is represented as a string of balanced parentheses. They both are then again compressed using standard compression methods. The delta-codes work best achieving 3.8% to 7% compression ratios.

Martínez-Prieto et al. [MFC12] focus on compressing the RDF dictionary. They develop a compressor which can be optimized for either compression or for querying. Among others they also evaluate on a Uniprot dataset (9.11 GB file size, however using an uncompressed dictionary already reduces this to 1.21 GB) and achieve a compression ratio of 27.61% on it with the compressor optimized for compression. In this case compression ratio is calculated by dividing the compressed dictionary size by the original dictionary size. Optimizing the compressor
for querying leads to compression that takes approximately twice the space but has two to three times faster access times. Another approach on dictionary compression is by Urbani et al. [UMD+13]. They implement a parallel approach using the MapReduce method. On a Uniprot dataset (230.9 GB) they achieve a compression ratio of about 12%. Note that this number is not directly comparable to the previous approach as it represents a comparison of the compressed data structure to the raw file, while the compression ratio in the previous method was comparing the compressed dictionary with an uncompressed dictionary that already uses much less space than the raw file. Another parallel approach is developed by Cheng et al. [CMK+14]. They present a scalable solution for compressing RDF data. Instead of using a single dictionary they encode triples in parallel with multiple dictionaries achieving a compression ratio of about 22% on a Uniprot dataset (797 GB).

Contrary to the previous works, Jiang et al. [JZG+13] focus on compressing the underlying graph structure. They propose two compression schemes ("equivalent" and "dependent" compression) to compress annotated graphs called Typed Object Graphs. These are RDF-graphs with type information attached to each node. Thus the edges are quintuples in this model instead of the usual RDF-triples, as subject and object also get a type attached. The equivalent compression aims to combine nodes that have the same type and the same family (i.e. the same neighbors and the same relations to these) into one node. Dependent compression searches for nodes that have only one neighbor and combines the information stored in this node with the information stored in the node of their only neighbor. They define two metrics for compression: \( \text{CR}_q \) is the compression ratio of the quintuple compression. This is calculated by dividing the number of quintuples in the original graph by the number of quintuples in the compressed graph. Analogously \( \text{CR}_i \) is the node compression. They evaluate both compression approaches on 5 datasets with 100 thousand to 6 million edges. The dependent compression consistently yields better results, achieving quintuple compression ratios of 0.83 to 0.06 and node compression ratios of 0.95 to 0.18. As a specific example they achieve 40% quintuple compression and 20% node compression on the Jamendo dataset, which has 373 thousand quintuples and 281 thousand nodes.

Similarly, Pan et al. [PGR+14] also focus on compressing the graph structure by mining for redundant graph patterns. They describe two methods of removing redundancies:

- By logical compression they mean the search for patterns that appear often and can be replaced by a generalized triple. For example, if the pattern

\[
<?x, a, \text{foaf: Person}> , <?x, a, \text{dbp: Person}>
\]

appears often, a type \( T \) could be introduced, along with a rule to expand \( T \) to \( \text{foaf: Person} \) and \( \text{dbp: Person} \). Then the single triple \( <?x, a, T> \) would represent the above pattern.

- RDF files are a textual representation of the graph structure. As there are many ways of serializing such a graph, some of these serializations are more
concise than others. They therefore describe a way of using graph patterns to group triples in such a way that they can then be serialized more concisely, resulting in shorter files.

The two methods are joined by a pattern mining phase at the beginning to find the patterns which are used to achieve the compression. Then all three steps (pattern mining, logical compression, serialization) can be iterated to achieve compression. The results they give are achieved by only one iteration. They evaluate their method on four data sets with 431 thousand to 94 million RDF triples, achieving compression rates (original number of triples/compressed number of triples) of 1.7 to 2.8. On the Jamendo dataset, which has 1 million triples, they achieve a compression ratio of 1.72. Again, they seem to use a different version of this (and other) dataset than the previous work, as the number of triples is not the same.

A different problem is considered by Fernández et al. \cite{FAS14} in the compression of streaming data. They propose a method based on Zlib to achieve lossless compression of RDF streams. The idea is to use the structural similarities among items in an RDF stream by combining a differential item encoding with the general purpose compression of Zlib. An evaluation of their method achieves 10 to 30% smaller streams than would be achieved by directly storing the stream in a Zlib-compressed file.

3 Lossy Compression and Reachability

Another approach to compression is based on the assumption that only certain queries are of interest. The idea is, to remove redundant information irreversibly, leaving just enough information to answer the queries in question.

Fan et al. \cite{FLWW12} propose such query-preserving lossy compression for reachability and graph pattern queries. They also consider algorithms that preserve the compressed representation in the case of incremental changes to the graph. In their experimental analysis they show that this method can reduce the size of a graph for reachability queries by 95% and for graph pattern queries (a variant of subgraph matching queries, matching graph patterns in terms of bounded simulation) by 56%. Zhang et al. \cite{ZDY14} apply the concept to RDF data. They are able to achieve compression ratios of 5% to 30%, considering subgraph matching as queries.

A wide range of algorithms is concerned with the answer to reachability queries on large graphs. Commonly the procedure is to build an index structure, which is sufficient to answer reachability queries, but can not be used to rebuild the original graph. A frequently used preprocessing step is to construct a DAG in which every strongly connected component \( c \) of the original graph is replaced by a node, as every node in \( c \) can reach any other node in \( c \). Therefore reachability on DAGs is equivalent to reachability on arbitrary graphs and it is often assumed that the input structures are DAGs. Yu and Cheng \cite{YC10} provide a survey on reachability queries in large graphs, we will further only mention a few newer results and otherwise point to this survey. Yildirim et al. \cite{YCD12} present
GRAIL, an indexing structure which works well on very large graphs, while not being the best approach for smaller graphs. They use an interval labeling, which effectively solves reachability for trees. The idea is that the nodes are labeled by intervals, such that the question if v can reach u can be answered by testing for interval containment, i.e. reachability is given if and only if the interval of u is contained in the interval of v. This method is often extended to DAGs by labeling a subtree of the DAG and then supplementing the index structure to cover the non-tree edges. GRAIL on the other hand is an effort to directly label the DAG instead of a subtree, which has the opposite problem: it can introduce false positives, which have to be handled separately. One way they pursue this problem is by introducing multiple intervals on every node instead of only one. These are created by taking random walks through the graph, and drastically reduce the number of false positives. For the remaining exceptions they propose the use of exception lists, but conclude that they do not scale well to large graphs. Instead they use a smart DFS based recursive pruning approach. If an exception is possible, they check if the neighborhood can reach the target and return false if this is false for the entire neighborhood. The authors mention that more sophisticated approaches work better on small graphs, their approach scales better to graphs with millions of nodes and edges. They experimentally show that both the index construction as well as the query time are much faster (up to a factor of 40 for construction and over 200 for querying) than the closest competitors on large graphs, however their index tends to be larger by a factor of 2 to 3 as well. The same authors extend the system to one for dynamic graphs [YCZ13]. Some of the techniques used in GRAIL (particularly pruning of labels) are also extended into a data structure called PReaCH by Merz and Sanders [MS14], who combine it with the idea of contraction hierarchies to achieve a new data structure for the answer of reachability queries. Contraction hierarchies are a technique originally used to speed up shortest path queries. Reachability contraction hierarchies repeatedly remove nodes of in- or out-degree 0. This makes it possible to mark the incident edges for forward (or backward when the out-degree is 0) exploration. If no bidirectional exploration is done, this achieves that only a subset of the edges have to be considered. Their experiments show better results than GRAIL (factor 3 to 13 faster index construction and factor 2 to 24 for querying), again particularly on large graphs. Both GRAIL and PReaCH also guarantee linear preprocessing time and space. PReaCH seems to be the current state of the art for reachability queries on large directed graphs.

A generalization of reachability queries is studied by Zou et al. [ZXY+14]. They study label-constraint reachability queries, where a set of labels is given and only edges with a label from this set may be used to determine reachability. They propose a transformation of an edge-labeled directed graph into an augmented DAG replacing the maximal strongly connected components as bipartite graphs. Then a Dijkstra-like algorithm can used to answer reachability by assigning weights to the labels. As the partition-based method for transforming the graph is NP-hard, they also propose a sampling based approximation. Their evaluation is done on 2 rather small real world graphs (at most 5 thousand nodes and 66 edge
Wei et al. \cite{WYLJ14} propose the use of a different node labeling approach that can be used to test reachability by checking for set containment. As this is time consuming for large sets, they use a randomized approach to check set containment. If this does not lead to an answer that is correct with 100% probability, a DFS search is started to answer the query. Their approach can not give the same worst-case guarantees that GRAIL provides, but they show it to work faster in practice at both index construction and query answering time.

4 Other related Works

Almost every implemented lossless compression method we discuss above considers querying the compressed graph in some way and is therefore suitable as an in-memory representation of the graph. The method by Kang and Faloutsos \cite{KF11} in contrast is does not consider access to the graph. The method recursively removes the nodes of highest degree and uses the resulting structure to reorder the adjacency matrix aiming for a representation that groups all the 0-values into the same regions. Due to their recursive structure, this works particularly well on power-law graphs. The final compression then relies on standard gzip-compression (which cannot be queried). They achieve results of about 7 to 16 bpe. One advantage of this method is its suitability for for block based matrix multiplication, which can be used to execute graph mining algorithms over MapReduce.

Nourbakhsh et al. \cite{NRBP14} also propose a compression method but do not consider access to the compressed structure. They formulate graph compression as a matrix factorization problem and prove that solving a continuous relaxation of this discrete optimization problem, yields an optimal result for the original problem. They then present a novel algorithm to approximate this solution and use it to compress the adjacency matrix representation of undirected, edge-weighted graphs with \( n \) nodes (thus a matrix of size \( n^2 \)) to a representation of size \( (n + k^2) \) for some \( k < n \). The method is not necessarily lossless, as not every matrix can be represented by a smaller factorization.

Different approaches are needed to deal with large graphs in external memory. Working on graphs with external memory is quite difficult, as there is no obvious way of partitioning them into the block structure of external memory. Kyrola et al. \cite{KBG12} develop GraphChi, a library to do large-scale graph computation on external memory efficiently. For example computing 100 iterations of the PageRank algorithm on a graph with 3.8 billion edges takes 581 minutes on a machine with an SSD, which is only about 4 times slower than the same computation on a cluster of 30 machines. It is essentially an external memory implementation of GraphLab \cite{LGK+10}. Kyrola and Guestrin \cite{KG14} improve this through a new data structure – Parallel Adjacency Lists – which is based on the graph storage model of GraphChi.

Algorithms optimized on I/O-efficiency also exist. Traversing large graphs breadth-first in external memory is considered by Beckmann et al. \cite{BMV13} and
they present an I/O-efficient implementation. Their algorithm uses a clustering method grouping nodes into a hierarchy. Furthermore they consider dynamic data and are able to update the BFS levels without recomputing the whole traversal from scratch. Closely related to reachability but generally more difficult is the question of finding a shortest path between two nodes in a weighted graph. Using external memory is difficult for several reasons (cf. Meyer [Mey09]), but Meyer and Zeh achieve some results on I/O-efficient shortest path algorithms, first for undirected graphs with bounded edge lengths [MZ12], which they then extend to graphs with unbounded edge lengths [MZ06]. These algorithms are based on I/O-efficient BFS traversals by Meyer and Mehlhorn [MM02].

References

ABLP10. S. Álvarez, N. R. Brisaboa, S. Ladra, and Ó. Pedreira. A compact representation of graph databases. In Proceedings of the Eighth Workshop on Mining and Learning with Graphs, pages 18–25, 2010.

AD09. A. Apostolico and G. Drovandi. Graph Compression by BFS. Algorithms, pages 1031–1044, 2009.

AMN08. Y. Asano, Y. Miyawaki, and T. Nishizeki. Efficient compression of web graphs. In Computing and Combinatorics, pages 1–11. 2008.

BBK03. D. K. Blandford, G. E. Blelloch, and I. A. Kash. Compact representations of separable graphs. In ACM-SIAM SODA, pages 679–688, 2003.

BC08. G. Buehrer and K. Chellapilla. A Scalable Pattern Mining Approach to Web Graph Compression with Communities. In WSDM, pages 95–106, 2008.

Bla06. Daniel K Blandford. Compact data structures with fast queries. PhD thesis, Carnegie Mellon University Pittsburgh, PA, 2006.

BLN09. N. Brisaboa, S. Ladra, and G. Navarro. k2-trees for compact web graph representation. In Computing and Combinatorics, pages 18–30. 2009.

BMV13. A. Beckmann, U. Meyer, and D. Veith. An implementation of I/O-efficient dynamic breadth-first search using level-aligned hierarchical clustering. In ESA, pages 121–132, 2013.

BSV09. P. Boldi, M. Santini, and S. Vigna. Permuting web graphs. In Algorithms and Models for the Web-Graph, pages 116–126. 2009.

BV04. P. Boldi and S. Vigna. The webgraph framework I: compression techniques. In WWW, pages 595–602, 2004.

CL11. F. Claude and S. Ladra. Practical representations for web and social graphs. In ACM CIKM, pages 1185–1190, 2011.

CMK14. L. Cheng, A. Malik, S. Kotoulas, T. E. Ward, and G. Theodoropoulos. Scalable RDF data compression using X10. CoRR, abs/1403.2404, 2014.

CN10. F. Claude and G. Navarro. Fast and Compact Web Graph Representations. ACM Trans. Web, pages 16:1–16:31, 2010.

DL98. N. Deo and B. Litow. A structural approach to graph compression. In MFCS Workshop on Communications, pages 91–101, 1998.

FAS14. N. Fernández, J. Arias-Fisteus, L. Sánchez, D. Fuentes-Lorenzo, and Ó. Corcho. RDSZ: an approach for lossless RDF stream compression. In ESWC, pages 52–67, 2014.
FGM10. J. D. Fernández, C. Gutierrez, and M. A. Martínez-Prieto. RDF compression: basic approaches. In *WWW*, pages 1091–1092, 2010.

FLWW12. W. Fan, J. Li, X. Wang, and Y. Wu. Query preserving graph compression. In *SIGMOD*, pages 157–168, 2012.

FM13. A. Farzan and J. I. Munro. Succinct encoding of arbitrary graphs. *Theor. Comput. Sci.*, pages 38–52, 2013.

GB11. S. Grabowski and W. Bieniecki. Merging adjacency lists for efficient web graph compression. In *ICMII*, pages 385–392, 2011.

GW83. H. Galperin and A. Wigderson. Succinct representations of graphs. *Information and Control*, pages 183–198, 1983.

HN14. C. Hernández and G. Navarro. Compressed representations for web and social graphs. *Knowl. Inf. Syst.*, pages 279–313, 2014.

Jac89. G. Jacobson. Space-efficient static trees and graphs. In *FOCS*, pages 549–554, 1989.

JZG+13. X. Jiang, X. Zhang, F. Gao, C. Pu, and P. Wang. Graph compression strategies for instance-focused semantic mining. In *Linked Data and Knowledge Graph*, Communications in Computer and Information Science, pages 50–61, 2013.

KBG12. A. Kyrola, G. E. Blelloch, and C. Guestrin. Graphchi: Large-scale graph computation on just a PC. In *OSDI*, pages 31–46, 2012.

KF11. U. Kang and C. Faloutsos. Beyond ‘Caveman Communities’: Hubs and Spokes for Graph Compression and Mining. In *ICDM*, pages 300–309, 2011.

KG14. A. Kyrola and C. Guestrin. Graphchi-db: Simple design for a scalable graph database system - on just a PC. *CoRR*, 2014.

LGK+10. Y. Low, J. Gonzalez, A. Kyrola, D. Bickson, C. Guestrin, and J. M. Hellerstein. GraphLab: A new framework for parallel machine learning. In *UAI*, pages 340–349, 2010.

Lu14. H. Lu. Linear-time compression of bounded-genus graphs into information-theoretically optimal number of bits. *SIAM J. Comput.*, pages 477–496, 2014.

Mas12. H. Maserrat. *Compression of social networks*. PhD thesis, Simon Fraser University, 2012.

Mey09. U. Meyer. Via detours to i/o-efficient shortest paths. In *Efficient Algorithms, Essays Dedicated to Kurt Mehlhorn on the Occasion of His 60th Birthday*, pages 219–232, 2009.

MFC12. M. A. Martínez-Prieto, J. D. Fernández, and R. Cánovas. Compression of RDF dictionaries. In *ACM SAC*, pages 340–347, 2012.

MM02. K. Mehlhorn and U. Meyer. External-memory breadth-first search with sublinear I/O. In *ESA*, pages 723–735, 2002.

MMG+07. N. Martínez-Bazan, V. Muntés-Mulero, S. Gómez-Villamor, J. Nin, M. A. Sánchez-Martínez, and J. L. Larriba-Pey. Dex: high-performance exploration on large graphs for information retrieval. In *CIKM*, pages 573–582, 2007.

MS14. F. Merz and P. Sanders. PReaCH: A fast lightweight reachability index using pruning and contraction hierarchies. In *ESA*, pages 701–712, 2014.

MZ06. U. Meyer and N. Zeh. I/O-efficient undirected shortest paths with unbounded edge lengths. In *ESA*, pages 540–551, 2006.

MZ12. U. Meyer and N. Zeh. I/O-efficient shortest path algorithms for undirected graphs with random or bounded edge lengths. *ACM Transactions on Algorithms*, pages 22:1–22:28, 2012.
NRBP14. F. Nourbakhsh, S. R. Bulò, and M. Pelillo. A matrix factorization approach to graph compression with partial information. *International Journal of Machine Learning and Cybernetics*, pages 1–14, 2014.

NRS08. S. Navlakha, R. Rastogi, and N. Shrivastava. Graph summarization with bounded error. In *SIGMOD*, pages 419–432, 2008.

PGR+14. J. Z. Pan, J. M. Gómez-Pérez, Y. Ren, H. Wu, H. Wang, and M. Zhu. Graph pattern based RDF data compression. In *JIST*, pages 239–256, 2014.

Tur84. G. Turán. On the succinct representation of graphs. *Discrete Applied Mathematics*, pages 289–294, 1984.

UMD+13. J. Urbani, J. Maassen, N. Drost, F. J. Seinstra, and H. E. Bal. Scalable RDF data compression with mapreduce. *Concurrency and Computation: Practice and Experience*, pages 24–39, 2013.

WYLJ14. H. Wei, J. X. Yu, C. Lu, and R. Jin. Reachability querying: An independent permutation labeling approach. *PVLDB*, pages 1191–1202, 2014.

YC10. J. X. Yu and J. Cheng. Graph reachability queries: A survey. In *Managing and Mining Graph Data*, pages 181–215, 2010.

YCY12. H. Yildirim, V. Chaoji, and M. J. Zaki. GRAIL: A scalable index for reachability queries in very large graphs. *The VLDB Journal*, pages 509–534, 2012.

YCY13. H. Yildirim, V. Chaoji, and M. J. Zaki. DAGGER: A scalable index for reachability queries in large dynamic graphs. *CoRR*, abs/1301.0977, 2013.

ZDY14. H. Zhang, Y. Duan, X. Yuan, and Y. Zhang. ASSG: adaptive structural summary for RDF graph data. In *ISWC*, pages 233–236, 2014.

ZXY+14. L. Zou, K. Xu, J. X. Yu, L. Chen, Y. Xiao, and D. Zhao. Efficient processing of label-constraint reachability queries in large graphs. *Inf. Syst.*, pages 47–66, 2014.