Analysis of complete positivity conditions for quantum qutrit channels

Agata Chęcińska

Instytut Fizyki Teoretycznej, Uniwersytet Warszawski, Warszawa 00–681, Poland and
ICFO-Institut de Ciencies Fotoniques, Mediterranean Technology Park, 08860 Castelldefels (Barcelona), Spain

Krzysztof Wódkiewicz

Instytut Fizyki Teoretycznej, Uniwersytet Warszawski, Warszawa 00–681, Poland and
Department of Physics and Astronomy, University of New Mexico, Albuquerque, NM 87131-1156, USA

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We present an analysis of complete positivity (CP) constraints on qutrit quantum channels that have a form of affine transformations of generalized Bloch vector. For diagonal (damping) channels we derive conditions analogous to the ones that in qubit case produce tetrahedron structure in the channel parameter space.

I. INTRODUCTION

The analysis of quantum channels - completely positive trace preserving maps - is one of the crucial points for quantum information theory. Quantum channels correspond to processes that physically may take place and lead to evolution of a quantum state [1, 2]. For qubit case, we already know the analysis of quantum qubit channels [3]. This analysis gives us a connection between the physical evolution of the system written via Bloch equations and quantum channel formalism [3]. Hence one can derive mathematical conditions on parameters that appear in Bloch formalism. These mathematical conditions are the consequence of the fact that each physical process is a completely positive map.

In our work, we aim at presenting similar analysis for qutrit channels. Qutrit states are states belonging to three dimensional Hilbert space and, in analogy to qubit case, one can use generalized Bloch formalism to describe their evolution [4]. Generalized Bloch equations, that describe for instance three level atoms, may present the physical context for the evolution of qutrit quantum state. In analogy to qubit case, we investigate the evolution of generalized Bloch vector that evolves within a Bloch ball. As a natural choice, we investigate qutrit channels of the general form of linear transformation on the qutrit Bloch vector - studying affine transformations on qutrit Bloch vectors. We parameterize qutrit channels and then derive conditions for channel parameters in order to obtain physical transformations: completely positive maps (CPM). In qubit channel analysis a tetrahedron structure
of completely positive maps appears (for qubit unital channels), we show analogous analysis for qutrit channels, for which more sophisticated channel geometry emerge. This new result on qutrit channels can be linked with the analysis of bipartite qutrit states via Jamiolkowski isomorphism.

II. STATE DESCRIPTION

Let us first recall the idea behind the Bloch formalism. This, in qubit case, corresponds to the choice of representation of the qubit state density operator: the basis of Pauli matrices $\sigma_i$:

$$
\rho_{qb} = \frac{1}{2}(I + \vec{b} \cdot \vec{\sigma}),
$$

(1)

where $\vec{b}$ is a three dimensional, real Bloch vector, describing the qubit state and satisfying $\vec{b}^2 \leq 1$ (equality for pure states). Qubit states occupy entirely the Bloch ball. In a similar way we can represent a qutrit state, a state belonging to three dimensional Hilbert space. In qutrit case the choice of representation is set to be the basis of Gell-Mann matrices $\lambda_i$, the generators of $SU(3)$ group:

$$
\rho_{qt} = \frac{1}{3}(I + \sqrt{3} \vec{n} \cdot \vec{\lambda}).
$$

(2)

Here $\vec{n}$ is a generalized Bloch vector, real and eight dimensional. Qutrit states can be characterized by condition $\vec{n} \leq 1$. However, qutrit case is more sophisticated: pure states are states for which two conditions are satisfied:

$$
\vec{n}^2 = 1, \quad \vec{n} \ast \vec{n} = \vec{n},
$$

(3)

where $\ast$-product is defined as $(\vec{A} \ast \vec{B})_i = d_{ijk} A_j B_k$, with $d_{ijk}$ being totally symmetric tensor [1]. Qutrit states belong to a generalized Bloch ball, qutrit pure states belong to the unit sphere $S^7 = \{ \vec{n} \in R^8 : \vec{n}^2 = 1 \}$. However, physical qutrit states do not occupy entirely the generalized Bloch ball. The pure qutrit states (states satisfying conditions (3)) form a subset of the unit sphere. They can be parametrized with 4 parameters [6] as follows

$$
|\Psi\rangle = e^{i\chi_1} \sin \theta \cos \phi |0\rangle + e^{i\chi_2} \sin \theta \sin \phi |1\rangle + \cos \theta |2\rangle,
$$

(4)

where $0 \leq \theta, \phi < \frac{\pi}{2}$, $0 \leq \chi_1, \chi_2 < 2\pi$ and overall phase was omitted. Hence, the set of pure qutrit states is a 4 dimensional subset of 7 dimensional sphere.

III. COMPLETELY POSITIVE TRACE PRESERVING MAPS

It was shown in [7] that physical transformations must not only be positivity preserving but there exist more subtle conditions to satisfy, these are called complete positivity (CP) conditions [2, 7]. The classification of qubit channels according to complete positivity is well known [3]. We want to present similar analysis for qutrit channels.

We represent physical system with Hilbert space $\mathcal{H}$. $\mathcal{B}(\mathcal{H})$ is the algebra of all bounded operators on $\mathcal{H}$, a linear map $\Phi : \mathcal{B}(\mathcal{H}) \mapsto \mathcal{B}(\mathcal{H})$ is completely positive if for every positive inte-
ger $m$ the map:

$$\Phi^{(m)} = \Phi \otimes I^{(m)} : \mathcal{B}(\mathcal{H}) \otimes \mathcal{M}^{(m)} \to \mathcal{B}(\mathcal{H}) \otimes \mathcal{M}^{(m)},$$

(5)

is positive (where $I^{(m)}$ is the identity operator on the algebra $\mathcal{M}^{(m)}$ of $m \times m$ complex matrices) [7]. Clearly, this amounts to saying that $\Phi$ acts on a subsystem $A$ of a larger Hilbert space and there is a reservoir (or subsystem $B$) on which we act with unit operator $I^{(m)}$. Here, we do not know the dimension of the reservoir and therefore $\Phi^{(m)}$ must be positive for any $m$.

It was shown that every CPM has an operator-sum or Kraus representation [8]

$$\Phi^{CPM}(\rho) = \sum_i K_i \rho K_i^\dagger,$$

(6)

with $K_i$ being a set of Kraus operators satisfying $\sum_i K_i^\dagger K_i = I$.

To evaluate whether a given transformation $\Phi$ (a linear map) is completely positive we need to construct the so called dynamical (or Choi) matrix of the size $N^2 \times N^2$ ($N$ is the dimension of the system of interest). We will denote the dynamical matrix with $D_\Phi$ [2, 7]. Dynamical matrix represents uniquely channel action. We denote with $E_{jk}$ $N \times N$ matrix with 1 at position $(j,k)$ and zeros elsewhere. The map $\Phi$ is CPM iff

$$D_\Phi = \sum_{i,j=1}^{N} \Phi(E_{ij}) \otimes E_{ij},$$

(7)

is positive semi-definite ($D_\Phi \geq 0$).

Channel $\Phi$ must preserve hermiticity of density matrix and therefore its dynamical matrix must be hermitian: $D_\Phi = D_\Phi^\dagger$. Trace preserving of the density operators means that the partial trace of $D_\Phi$ with respect to the first subsystem $(A)$ gives the unit operator for the second subsystem: $Tr_A D_\Phi = I$. To evaluate the entries of dynamical matrix, we need to compute the action of the channel $\Phi$ on $E_{jk}$. Once again, when its action is rewritten as

$$\Phi(\rho)_{\mu\nu} = \sum_{\sigma,\tau=1}^{N} \Phi_{\mu\nu,\sigma\tau} \rho_{\sigma\tau},$$

(8)

where coefficients $\Phi_{\mu\nu,\sigma\tau}$ characterize channel action, we see that we work with $N^4$ numbers.

Channels versus states - Jamiolkowski isomorphism

On the other hand, $N^2 \times N^2$, positive and hermitian dynamical matrix $D_\Phi$ must correspond to a density operator acting on an $N^2$-dimensional Hilbert space. This correspondence is up to the normalization factor, since $Tr D_\Phi = N$. Hence $\rho_\Phi = \frac{1}{N} D_\Phi$ is a proper density matrix that we can write as

$$\rho_\Phi = \frac{1}{N} D_\Phi = \frac{1}{N} \sum_{i,j=1}^{N} \Phi(|i\rangle\langle j|) \otimes |i\rangle\langle j|$$

$$= \frac{1}{N} D_\Phi = \frac{1}{N} \sum_{i,j=1}^{N} \Phi(E_{ij}) \otimes E_{ij}.$$

(9)

The set of density operators defined by the dynamical matrices is only a subset of density matrices in $N^2$-dimensional Hilbert space, since dynamical matrices must satisfy $Tr_A D_\Phi = I$. The fact that completely positive maps $\Phi^{CPM}$, which are represented uniquely by they dynamical matrices, correspond to states is known as the Jamiolkowski isomorphism [9]. Therefore, when analyzing quantum qutrit
channels we can reinterpret it as an analysis of two qutrit quantum states, in qutrit case $N = 3$, therefore $\rho_{\Phi}$ is a $9 \times 9$ matrix.

IV. QUANTUM QUBIT CHANNELS

A. Bloch equations

To recall the qubit case analysis, we can start with a two-level quantum system and its evolution. The latter can be written by means of Bloch equations that are equations for components of Bloch vector $\vec{b} = (u, v, w)$. If we take, for instance, the decoherence of a two-level atom, these equations read

$$
\begin{align*}
\dot{u} &= -\frac{1}{T_u}u - \Delta v, \\
\dot{v} &= -\frac{1}{T_v}v + \Delta u + \Omega w, \\
\dot{w} &= -\frac{1}{T_w}(w - w_{eq}) - \Omega w,
\end{align*}
$$

and represent the evolution of the system. Here $\Omega, \Delta$ are Rabi frequency and detuning respectively. $\frac{1}{T_i}$ stand for decay rates for the atomic dipole ($i = u, v$) and decay rate of the atomic inversion ($i = w$). These equations, when put together, give rise to an affine transformation of the qubit Bloch vector that is governed by the parameters listed above. Any physical process amounts to a transformation of the qubit state that is already a completely positive map. However, not every affine transformation of the (qubit) Bloch vector will be a completely positive map.

B. Affine transformations on qubit Bloch vectors

The analysis of completely positive trace preserving maps on $M_2$ (complex two dimensional matrices) has been studied extensively and gives the answer to the problem. Without loss of generality one can analyze qubit channels that transform qubit Bloch vector according to

$$
\Phi^{qb} : \vec{b} \mapsto \vec{b}' = \Lambda^{qb}\vec{b} + \vec{t}^{qb},
$$

where matrix $\Lambda^{qb} = diag\{\Lambda_1^{qb}, \Lambda_2^{qb}, \Lambda_3^{qb}\}$ consists of damping eigenvalues $\Lambda_i^{qb}$ and $\vec{t}^{qb} = (t_1^{qb}, t_2^{qb}, t_3^{qb})$ is a translation. The image of the set of pure states ($\vec{b}^2 = 1$, Bloch sphere) under such transformation is the ellipsoid

$$
\left(\frac{u' - t_1^{qb}}{\Lambda_1^{qb}}\right)^2 + \left(\frac{v' - t_2^{qb}}{\Lambda_2^{qb}}\right)^2 + \left(\frac{w' - t_3^{qb}}{\Lambda_3^{qb}}\right)^2 = 1,
$$

with its center defined by $\vec{t}$ and its axes by $\Lambda_i$. The set of conditions on both $\Lambda_i^{qb}$ and $t_i^{qb}$ can be found in [3, 5]. When we limit ourselves just to diagonal qubit channels (meaning $\vec{t}^{qb} = 0$), then the set of allowed $\Lambda_i^{qb}$ forms a tetrahedron structure [2, 3]. This structure reappears also in the space of two qubit states.

V. QUANTUM QUTRIT CHANNELS

A. Qutrit Bloch equations

As before, we can start the analysis of transformations on qutrit quantum states with the analysis of a three level atom for which we can
write down Bloch equations. Three level atom is not the only possible physical realization but it is very illustrative. The analog of the Bloch vector for the case of a three level atom was in the beginning introduced as a (eight dimensional, real) coherent vector $\vec{S}$, which components (denoted as $u, v, w$) were defined as

$$u_{jk} = \rho_{jk} + \rho_{kj},$$

$$v_{jk} = i(\rho_{jk} - \rho_{kj}),$$

$$w_{jk} = -\sqrt{\frac{2}{l(l+1)}} \times (\rho_{11} + \rho_{22} + ... + \rho_{ll} - l \rho_{l+1,l+1}),$$

with $1 \leq j < k \leq 3$ and $1 \leq l \leq 2$. Now, as an example of the physical system we can take a three level atom for which nonzero dipole moments are between levels 1 and 2, and 2 and 3. The atom interacts with the electric field (two electromagnetic waves incident on the atom) and we assume that detunings are the same ($\Delta_{12} = -\Delta_{23} = \Delta$). The corresponding Bloch equations for coherent vector $\vec{S}$ are

$$\dot{u}_{12} = \Delta v_{12} + \beta v_{13},$$

$$\dot{u}_{23} = -\Delta v_{23} - \alpha v_{13},$$

$$\dot{u}_{13} = \beta v_{12} - \alpha v_{23},$$

$$\dot{v}_{12} = -\Delta u_{12} + \beta u_{13} + 2\alpha w_{1},$$

$$\dot{v}_{23} = \Delta u_{23} + \alpha u_{13} - \beta w_{1} + \sqrt{3}\beta w_{2},$$

$$\dot{v}_{13} = -\beta u_{12} + \alpha u_{23},$$

$$\dot{w}_{1} = -2\alpha v_{12} + \beta v_{23},$$

$$\dot{w}_{2} = -\sqrt{3}\beta v_{23},$$

(14)

where $\alpha, \beta$ are related to two Rabi frequencies [12]. These equations are a generalization of the equations we have seen in the qubit case. In this work we use slightly different notation for the qutrit vector - we already have introduced qutrit Bloch vector $\vec{n}$ related to the choice of Gell-Mann matrices basis (in some works generalized Bloch vectors are also called coherent vectors [4]). These two vectors ($\vec{S}$ and $\vec{n}$) are of course equivalent.

Parameters that appear in the qutrit Bloch equations have physical background, therefore the resulting affine transformation is a completely positive map. However, we can as well ask the opposite question: given an arbitrary affine transformation on qutrit Bloch vector what are the conditions on its parameters which guarantee complete positivity?

### B. Affine transformations of qutrit Bloch vectors

Having in mind the question stated above, we will look at transformations of qutrit Bloch vector that have a form

$$\Phi : \vec{n} \mapsto \vec{n}' = \Lambda \vec{n} + \vec{t},$$

(15)

where $\Lambda = diag\{\Lambda_1, ..., \Lambda_8\}$ consists of 8 damping coefficients and $\vec{t}$ is an eight dimensional translation. The image of the set of pure states under this transformation is

$$\sum_{i=1}^{8} \left(\frac{n_i' - t_i}{\Lambda_i}\right)^2 = 1,$$

(16)

together with the condition for $*$-product $\vec{n} * \vec{n}' = \vec{n}$

$$\frac{n_i' - t_i}{\Lambda_i} = d_{ijk} \frac{n_j' - t_j n_k' - t_k}{\Lambda_j \Lambda_k}.$$  

(17)
On the other hand, parameters $\Lambda_i$, $t_i$ must satisfy
\[
\sum_i (\Lambda_i n_i + t_i)^2 \leq 1, \quad \text{(18)}
\]
according to the requirement $\vec{n}'^2 \leq 1$. However, complete positivity is a much stronger condition than condition saying that we cannot exceed value 1 for the length of Bloch vector. The latter, in qubit case, amounts only to statement that the density operator must be a positive definite operator. In qutrit case however, it is even less than that - since not every point within the $S^7$ sphere corresponds to density operator.

Transformation (15) can be rewritten to give channel coefficients $\Phi_{\mu\nu,\sigma\tau}$. To construct dynamical matrix $D_\Phi$, we apply the channel action to $E_{jk} \mapsto \Phi (E_{jk})$, representing it in the basis of Gell-Mann matrices: $E_{jk} = \frac{1}{\sqrt{3}} n_{a}^{jk} \lambda_{a}$ (where $\alpha \in \{0, \ldots, 8\}$, $\lambda_0 = \sqrt{\frac{2}{3}}1$ and $n_{a}^{jk}$ can be interpreted as an analog of Bloch vector).

We will first look at the channels that consist only of damping matrix (diagonal channels) and do not have a translation. These channels are in fact unital, since they leave maximally mixed state unchanged (they are called bistochastic maps \([2]\)). Later on, we will look at channels that include also translations of Bloch vector.

### C. CPM conditions for diagonal channels

In qubit case, the action of the diagonal channel can be written as
\[
\vec{b} \to \vec{b}' = \Lambda^{qb} \vec{b}, \quad \Lambda^{qb} = \text{diag}\{\Lambda_{1}^{qb}, \Lambda_{2}^{qb}, \Lambda_{3}^{qb}\}, \quad \text{(19)}
\]
whereas for qutrits we have
\[
\vec{n} \to \vec{n}' = \Lambda \vec{n}, \quad \Lambda = \text{diag}\{\Lambda_{1}, \ldots, \Lambda_{8}\}. \quad \text{(20)}
\]
In both cases, we assume that the nature of $\Lambda_i$ parameters is quasi-damping, hence $|\Lambda_i| \leq 1$. This comes from the fact that $\vec{n}'^2 \leq 1$ at all times, therefore, the change of any initial Bloch vector will lead to a vector within the (generalized) Bloch ball. Dynamical matrix $D_\Phi$ for a qutrit channel of the form (15) must be positive semi-definite in order to correspond to CPM. There are nine eigenvalues $d_i$ that must be non-negative to satisfy positivity of $D_\Phi$. The first six eigenvalues give rise to conditions that can be written as
\[
\begin{align*}
1 - \Lambda_8 + \frac{3}{2}(\Lambda_1 - \Lambda_5) & \geq 0, \\
1 - \Lambda_8 - \frac{3}{2}(\Lambda_1 - \Lambda_5) & \geq 0, \\
1 - \Lambda_8 + \frac{3}{2}(\Lambda_6 - \Lambda_7) & \geq 0, \\
1 - \Lambda_8 - \frac{3}{2}(\Lambda_6 - \Lambda_7) & \geq 0, \\
1 - \Lambda_8 + \frac{3}{2}(\Lambda_1 - \Lambda_2) + \frac{3}{2}(\Lambda_8 - \Lambda_3) & \geq 0, \\
1 - \Lambda_8 - \frac{3}{2}(\Lambda_1 - \Lambda_2) + \frac{3}{2}(\Lambda_8 - \Lambda_3) & \geq 0.
\end{align*}
\]
These conditions alone lead to the set of allowed $\Lambda_i$ that has a polyhedron like structure. In qubit case we have similar set of equations for $\Lambda^{qb}$ that define the tetrahedron structure. However, in qutrit case there are three remaining inequalities (given by eigenvalues $d_7$, $d_8$, $d_9 \geq 0$) which reveal coupling between all the parameters.

\[
d_7(\Lambda_1, \ldots, \Lambda_8) \geq 0,
\]
Because of their numerical complexity they are discussed in Appendix. Matrix $D_\Phi$ is hermitian, therefore eigenvalues $d_7, d_8, d_9$ must be real.

For some cases, three conditions (22) reduce to just two (see Appendix). All the inequalities characterize the set of allowed $\{\Lambda_{CPM}\}$, in other words, channel parameters $\Lambda_i$ for which $\Phi$ is a CPM. The boundaries of the set $\{\Lambda_{CPM}\}$ can be computed by analyzing values of $\Lambda_i$ satisfying equations instead of inequalities given by (21) and (22). In principle, parameters $\Lambda_i$ can be time dependant, still, conditions (21) and (22) must be satisfied for any time $t$ to have a CPM.

If we assume, for example, that

$$\frac{dn_i(t)}{dt} = \gamma_i n_i(t),$$  \hspace{1cm} (23)$$

then time evolution of the Bloch vector $\vec{n}(t)$ is given by:

$$n_i(t) = e^{\gamma t} n_i(0).$$  \hspace{1cm} (24)$$

We can then identify $\Lambda_i = e^{\gamma_i t}$ and conditions on $\Lambda_i$ will impose conditions on $\gamma_i$. For this type of evolution, one can write the Lindblad equation for qutrit density operator $\rho(t)$, corresponding to the channel action. Some more details on relation between complete positivity and master equation and Lindblad operators one can find in [3, 13, 14].

For qubit case, the allowed values of damping parameters $\{\Lambda_{CPM}^{qb,i}\}_{i=1,2,3}$ form a characteristic structure (tetrahedron, [13]). We are interested in the structure that appears in qutrit case. The main obstacle here is the size of parameter space. We have eight parameters on which we impose our CPM constraints. We can investigate the $\{\Lambda_{CPM}\}$ set projecting it onto subspaces. It is easier to work with parameters paired according to $(\Lambda_1, \Lambda_2), (\Lambda_4, \Lambda_5), (\Lambda_6, \Lambda_7)$ (this pairing refers to the form of Gell-Mann matrices). We can also put together $(\Lambda_3, \Lambda_8)$ (two diagonal Gell-Mann matrices) though not necessarily, since the equations are not symmetric in these two.

On Fig.1-Fig.4 we show projections of $\{\Lambda_{CPM}\}$ onto the various subspaces in 8 dimensional space of parameters $\Lambda_1, \ldots, \Lambda_8$. The dark regions in these figures correspond to these combinations of $\Lambda_i$ which are satisfying CP conditions. There are many such projections that can be obtained from the conditions that we have derived. In principle, the structure of the set of $\{\Lambda_{CPM}\}$ is not simply a generalization of a tetrahedron. Since we have three (or two) conditions for $\Lambda_i$ that couple all the parameters (in a nonlinear way), the simple polyhedron type structure (emerging from inequalities that are linear in $\Lambda_i$, (21)) is altered. In the figures, one can see combination of almost rough edges with smooth behavior elsewhere.
FIG. 1: The dark region in the figure shows these values of parameters $\Lambda_1 = \Lambda_2 = Y$, $\Lambda_j \neq 1,2 = X$ that satisfy CP conditions. The qutrit channel has only the diagonal (damping) part.

FIG. 2: The dark region in the figure shows these values of parameters $\Lambda_3 = \Lambda_8 = Y$, $\Lambda_j \neq 3,8 = X$ that satisfy CP conditions. The qutrit channel has only the diagonal (damping) part.

FIG. 3: The dark region in the figure shows these values of parameters $\Lambda_3 = X$, $\Lambda_8 = Y$, $\Lambda_j \neq 3,8 = XY$ that satisfy CP conditions. The qutrit channel has only the diagonal (damping) part.

FIG. 4: The dark region in the figure shows these values of parameters $\Lambda_1 = \Lambda_2 = X$, $\Lambda_3 = \Lambda_8 = XY$, $\Lambda_j \neq 1,2,3,8 = Y$ that satisfy CP conditions. The qutrit channel has only the diagonal (damping) part.

D. CPM conditions for channels based only on translations

In this section we will analyze shortly the constraints of complete positivity on the possible translations. The change of qutrit Bloch vector in this case will be of the form:

$$\vec{n} \mapsto \vec{n}' = \vec{n} + \vec{t},$$

where $\vec{t} = (t_1, ..., t_8)$. This type of channel is nonunital. Below we show some examples of channels, for which we choose just two free parameters. First, let us look at the translation of the form

$$T_1 = (X, X, Y, 0, 0, 0, 0, 0).$$

It turns out that effectively, parameters $X, Y$ must satisfy

$$1 - 3\sqrt{3}\sqrt{2X^2 + Y^2} \geq 0,$$

what graphically is represented on Fig.5 - the dark region corresponding to CP-allowed pa-
rameter values has an ellipsoid form.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure5.png}
\caption{The dark region shows these values of parameters $X, Y$ that satisfy CP conditions, when translation acting on qutrit Bloch vector has the form $T_1 = (X, X, Y, 0, 0, 0, 0)$.}
\end{figure}

$X, Y$ must satisfy

$$1 + 3Y - 3\sqrt{3}\sqrt{2X^2 + Y^2} \geq 0. \quad (29)$$

This is shown on Fig.5. The ellipsoid shape from Fig.5 is now reshaped.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure6.png}
\caption{The dark region shows these values of parameters $X, Y$ that satisfy CP conditions, when translation acting on qutrit Bloch vector has the form $T_2 = (X, X, Y, 0, 0, 0, Y)$.}
\end{figure}

On the other hand, if we let the translation to shift also the 8th component by the same amount as the 3rd component, therefore translation having a form

$$T_2 = (X, X, Y, 0, 0, 0, Y), \quad (28)$$

then the allowed set of parameters $X, Y$ is further limited with respect to $X$ - this parameter must satisfy $Y \leq 1/6$. And both parameters

We have seen what are the CP conditions for diagonal channels and investigated some examples of channels built up only with translations. What occurs when these two effects combine? Let us take the channel that changes the Bloch vector according to

$$\vec{n} \mapsto \vec{n}' = \Lambda \vec{n} + \vec{t}, \quad \rho \mapsto \Phi(\rho) = \frac{1}{3}(I + \sqrt{3}(\Lambda \vec{n} + \vec{t}) \cdot \vec{\lambda}). \quad (30)$$

In this case, to evaluate positivity of $D_\Phi$ we evaluated the principal minors of the matrix. In principle, the matrix is positive when all the principal minors are positive, and it is negative, when the principle minors have alternating signs. However, it may happen that for some parameter values the principal minors equal to zero and the method do not detect all possible parametrization allowed by CP conditions. Nevertheless, we use this method to analyze some cases and detect possible regions of positivity of dynamical matrix $D_\Phi$. We do not present here the list of inequalities corresponding to CP conditions because their complexity would unable any insight into the problem. We project the set of $\{\Lambda_i, t_i\}^{CPM}$ onto some subspaces to gain a geometrical picture. Below,
we can see two examples. On Fig.7 the dark region corresponds to the CP-allowed values of $X, Y$, for which we assume that all $\Lambda_i = X$ and $t_i = Y$. Fig.8 shows a different choice of parametrization and relation between damping parameters and translation. We let $\Lambda_{1,2}$ have independent value (X) from the rest of $\Lambda_i (Y)$, and we assume that translation is equal to the product of these two (XY).

**F. Two-qutrit states and affine transformations of qutrit Bloch vectors**

As already said, the dynamical matrix $D_{\Phi}$ corresponds to a density matrix via $\rho_{\Phi} = \frac{1}{N} D_{\Phi}$. The latter, in our case ($N = 3$) describes a class of two-qutrit states that can be parameterized by $\{\Lambda_i, t_i\}_{C^P M}$. The two qutrit state space is being investigated. Especially, the so called magic simplex which can be considered an analog of the magic tetrahedron of bipartite qubits [15]. The magic simplex of bipartite qutrits is only embedded in the space of all bipartite qutrits. As an example, it does not contain a state given by the density operator

$$\rho = \frac{1}{3} (|\Psi\rangle\langle\Psi| + 2|\Phi\rangle\langle\Phi|),$$

where $|\Psi\rangle = |0,0\rangle$ and $|\phi\rangle = \frac{1}{\sqrt{2}}(|1,1\rangle + |2,2\rangle)$. Interestingly, this state can be obtained from $\rho_{\Phi} = \frac{1}{N} D_{\Phi}$ (N=3) with a proper choice of parameters: $\Lambda_{3,6,7,8} = 1$ and the rest equal to 0. A diagonal channel with such parameter values will transform any qutrit Bloch vector according to

$$\vec{n} \rightarrow \vec{n}' = \{0,0,n_3,0,0,n_6,n_7,n_8\}. \quad (32)$$

Geometrically, the channel projects the Bloch vector onto the 3-6-7-8 subspace and the other components of $\vec{n}$ are lost. It resembles therefore a phase flip type channel [1].

**VI. SUMMARY**

Basing on the generalized Bloch formalism for qutrit quantum states we have analyzed
quantum qutrit channels that have a form of affine transformations on Bloch vectors. The aim was to derive complete positivity conditions on channel parameters that may appear in equations of evolution of the Bloch vector. We analyzed diagonal channels (only with damping coefficients), for which we obtained CP conditions on parameters in form of inequalities. Analogous inequalities appear in qubit case - for which CP-allowed channel parameters form a tetrahedron structure. The structure of the corresponding set in qutrit case is more sophisticated, and reveals not only polyhedron like characteristics. We analyzed also channels which allow only shifting the Bloch vector, in which case we investigated some specific examples of translations and the corresponding CP constraints. The combined effect of damping and shifting qutrit Bloch vector (diagonal channels with translation) was also presented by projecting the CP-allowed set of channel parameters onto specific subspaces. At the end we looked at the two qutrit states that correspond to qutrit channels we investigated, via Jamiolkowski isomorphism. As an example we give a channel that corresponds to a state which does not belong to the so called magic simplex.

One of the interesting points would be to establish the relation between the set of two qutrit states given by the dynamical matrix that we analyze and the magic simplex for qutrits. Also the analysis of the structure of CP-allowed channel parameters with respect to entanglement breaking properties could reveal some intriguing results.

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APPENDIX

In the discussion of the diagonal (damping) qutrit channels we presented a set of conditions (21, 22) that come from imposing positivity condition on $D_\Phi$. They correspond to eigenvalues of the matrix, and three of them have sophisticated form. In qutrit case, the dynamical matrix $D_\Phi$ is nine dimensional. We already showed 6 eigenvalues. The remaining three correspond to finding roots of polynomial of the 3rd order. Since $D_\Phi$ is hermitian, all its eigenvalues must be real, to guarantee complete positivity of the channel, they must also be nonnegative. The polynomial

$$P(x) = Ax^3 + Bx^2 + Cx + D$$

which we analyze in order to obtain the rest of CP conditions (eigenvalues $d_{7,8,9}$) has coefficients

$$A = 8, \quad B = -24(1 + \lambda_3 + \lambda_8),$$
$$C = 18((\lambda_3 + \lambda_8)^2 - (\lambda_1 + \lambda_2)^2 - (\lambda_4 + \lambda_5)^2 - (\lambda_6 + \lambda_7)^2) +$$
$$+24(1 + 2\lambda_3 + 2\lambda_8 + 3\lambda_1\lambda_8),$$
$$D = -8 - 18((\lambda_3 + \lambda_8)^2 - (\lambda_1 + \lambda_2)^2 - (\lambda_4 + \lambda_5)^2 +$$
$$- (\lambda_6 + \lambda_7)^2) + 27\lambda_1((\lambda_4 + \lambda_5)^2 + (\lambda_6 + \lambda_7)^2) +$$
and evaluate it at $f(B/8, C/8, D/8)$ (where $B, C, D$ are polynomial coefficients given above). For the dynamical matrix $D_\Phi$ this function is always nonpositive (and indicates therefore real roots). When $f(B/8, C/8, D/8) = 0$ the polynomial has three real roots and at least two are equal - then the number of CP conditions reduces. This occurs for parametrization shown in Fig 2.

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