SURFACES WITH \( K^2 = 2g - 2 \) AND \( p_g \geq 5 \)

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Abstract. This note describes minimal surfaces \( S \) of general type satisf-
ying \( p_g \geq 5 \) and \( K^2 = 2p_g \). For \( p_g \geq 8 \) the canonical map of such
surfaces is generically finite of degree 2 and the bulk of the paper is a
complete characterization of such surfaces with non birational canonical
map. It turns out that if \( p_g \geq 13 \), \( S \) has always an (unique) genus 2
fibration, whose non 2-connected fibres can be characterized, whilst for
\( p_g \leq 12 \) there are two other classes of such surfaces with non birational
canonical map.

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1. Introduction

Noether’s well known inequality states that a minimal surface of general
type satisfies \( K^2 \geq 2\chi - 6 \). Surfaces with \( K^2 < 2\chi \) are always regular
and Horikawa completely classified minimal surfaces satisfying \( K^2 = 2\chi -
6, 2\chi - 5 \) and \( 2\chi - 4 \) (\[16\], \[17\], \[18\], \[19\]). Some aspects of surfaces with
\( K^2 = 2\chi - 3 \) have been studied by other authors (e.g. \[29\]).

In this paper we characterize minimal surfaces satisfying \( K^2 = 2\chi - 2 \)
and \( p_g \geq 5 \). Note that the case \( p_g = 4 \) has been studied in \[1\]. Let us point
out that with our methods we could also recover the classification of \[1\].

We start with an overview of the case. From the results of \[18\] the cano-
nical map is not composed with a pencil. Also, by \[22\], the canonical map
has always degree \( \leq 2 \). If the canonical map is birational, then \( p_g \leq 7 \). The
bulk of our analysis is the case when the canonical map has degree 2. In this
case the canonical image is always a rational surface and we consider the
number \( t \) of isolated fixed points of the involution induced by the canonical
map. The main results obtained are:

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Theorem 3.6  Let \( S \) be a minimal surface with \( K_S^2 = 2\chi - 2 \) and \( p_g \geq 5 \). Then \( S \) satisfies exactly one of the following:

(I) the canonical map \( \phi_{K_S} \) is birational and
   (Ia) \( |K_S| \) is free from base points and \( p_g \leq 7 \), or
   (Ib) \( |K_S| \) has exactly one (simple) base point and \( p_g = 5 \);

(II) the canonical map factors through an involution \( i \) and the number \( t \) of isolated fixed points of \( i \) is:
   (IIa) \( t = 0 \), or
   (IIb) \( t = 2 \), or
   (IIc) \( t = 4 \).

Furthermore

• Proposition 4.3: If \( t = 0 \), then \( p_g \leq 12 \) and \( S \) is the minimal resolution of a double cover of \( \mathbb{F}_r \), \( r \leq 3 \), branched on a curve in \( |8C_0 + 2(5 + 2r)f| \) having 12 – \( p_g \) singular points of multiplicity 4 as only essential singularities.

• Theorem 5.2: If \( t = 2 \), then \( p_g \leq 8 \) and one of the following occurs:
   (i) \( S \) is the minimal resolution of a double cover of a weak Del Pezzo surface \( T \) of degree \( p_g + 1 \) branched on an effective divisor in \( | - 4K_T| \) having exactly two (3,3)-points as essential singularities.
   (ii) \( S \) is the minimal resolution of a double cover of \( \mathbb{F}_r \), \( r \leq 2 \), whose branch curve is the union of a curve in \( |8C_0 + (9 + 4r)f| \) with a fibre. The curve has \( 8 - p_g \) singular points of multiplicity 4 and another of type (4,4) and the fibre is tangent to the curve at the (4,4)-point.

• If \( t = 4 \), the surface has a unique genus 2 pencil and in Proposition 6.2 we see the different possibilities for the singularities of the branch locus.

All these types of surfaces, except possibly type Ib, do exist. For surfaces of type I we refer to Remark 4 and Proposition 3.7. Surfaces of type IIa, IIb and IIc are easily seen to exist using the descriptions as double covers given in Proposition 4.3, Theorem 5.2 and Proposition 6.2.

The paper is organized as follows. In Section 2 some general properties of involutions are recalled. In Section 3 the canonical map of these surfaces is studied, yielding a first division into cases. In the remaining sections each of these cases is described.

Notation. We work over the complex numbers. All varieties are projective algebraic. All the notation we use is standard in algebraic geometry. We just recall the definition of the numerical invariants of a smooth surface \( X \): the self-intersection number \( K_X^2 \) of the canonical divisor \( K_X \), the geometric genus \( p_g(X) := h^0(K_X) = h^2(\mathcal{O}_X) \), the irregularity \( q(X) := h^0(\Omega_X^1) = h^1(\mathcal{O}_X) \) and the holomorphic Euler characteristic \( \chi(X) := 1 + p_g(X) - q(X) \).
An involution of a surface $S$ is an automorphism of $S$ of order 2. We say that a curve singularity is nonessential if it is either a double point or a triple point which resolves to at most a double point after one blow-up. Other curve singularities are said to be essential. A $(m, k)$-point of a curve is a point of multiplicity $m$, which resolves to an ordinary point of multiplicity $k$ after one blow-up. We say that a map is composed with an involution $i$ of $S$ if it factors through the double cover $S \rightarrow S/i$.

We do not distinguish between line bundles and divisors on a smooth variety. Linear equivalence is denoted by $\equiv$ and numerical equivalence by $\sim$.

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2. Involution on surfaces

Let $S$ be a minimal surface of general type. Given an involution $i$ on $S$, its fixed locus is the union of a smooth curve $R$ (possibly empty) and of $t \geq 0$ isolated points $P_1, ..., P_t$. Let $\pi' : S \rightarrow S/i$ be the quotient map and set $B'' := \pi'(R)$. The surface $S/i$ is normal and $Q_1 := \pi'(P_1), ..., Q_t := \pi'(P_t)$ are ordinary double points, which are the only singularities of $S/i$. Resolving these singularities we get a commutative diagram

$$
\begin{array}{ccc}
V & \xrightarrow{h} & S \\
\downarrow \pi & & \downarrow \pi' \\
W & \xrightarrow{g} & S/i
\end{array}
$$

where $g$ is the minimal desingularization map, $h$ is the blow up of $S$ at $P_1, ..., P_t$ and $V$ is obtained by base change and normalization. Notice that the curves $A_i := g^{-1}(Q_i)$ are $(-2)$-curves. Setting $B := g^*(B'')$, $\pi$ is a double cover whose branch locus $B'$ is given by:

$$2L \equiv B' := B + \sum_{i=1}^{t} A_i.
$$

We recall the well known formulas (cf. [3], Chapter V, Section 22):

$$K_S^2 - t = K_V^2 = 2(K_W + L)^2,
$$
\[(2.3) \quad \chi(O_S) = \chi(O_V) = 2\chi(O_W) + \frac{1}{2} L(K_W + L).\]

Since \(\pi^*(2K_W + B) = h^*(2K_S)\) and \(S\) is a minimal surface of general type, \(2K_W + B\) is a nef and big divisor and (see [8], [23], also (28)):

\[(2.4) \quad (2K_W + B)^2 = 2K_S^2,\]

\[(2.5) \quad K_W(K_W + L) = \frac{1}{2} K_W(2K_W + B) = \frac{1}{2}(K_S^2 - t) - 2\chi(O_S) + 4\chi(O_W),\]

\[(2.6) \quad h^1(2K_W + L) = h^2(2K_W + L) = 0,\]

\[(2.7) \quad t = K_S^2 + 6\chi(O_W) - 2\chi(O_S) - 2h^0(W, O_W (2K_W + L)),\]

Note that the bicanonical map of \(S\) factors through \(i\) if and only if \(h^0(W, O_W (2K_W + L)) = 0\) (see e.g. proof of Proposition 2.1 of [23]).

It will be important in what follows to study the divisor \(3K_W + B\). This divisor is not necessarily nef but, as shown in Proposition 3.9 of [8], it is possible to assume it is nef.

More precisely, from this proposition and its proof we obtain:

**Proposition 2.1** ([8]). Suppose \(h^0(3K_W + B) \neq 0\). There exists a birational morphism \(p : W \to P\) where \(P\) is a smooth surface and an effective divisor \(\bar{B}\) on \(P\) with the following properties:

- there are \(t\) \((-2)\)-curves \(C_i\) on \(P\) such that \(p^*(C_i) = A_i, i = 1, ..., t\) and \(\bar{B}\) is disjoint from the union of the curves \(C_i\);
- there is \(\bar{L}\) in \(\text{Pic} (P)\) such that \(\bar{B} + \sum_i C_i = 2\bar{L} = \bar{B}'\) and \(p^*(K_P + \bar{L}) = K_W + L\);
- the double cover \(\bar{V}\) of \(P\) defined by \(\bar{B} + \sum_i C_i = 2\bar{L}\) is a surface with at most Du Val singularities and such that \(V\) is the minimal desingularization of \(\bar{V}\);
- \(p^*(2K_P + \bar{B}) = 2K_W + B\);
- \(3K_P + \bar{B}\) is nef.

**Proof.** All the above follows easily from the statement and the proof of Proposition 3.9 of [8]. We remark that although Proposition 3.9 of [8] is stated for surfaces \(S\) with \(p_g = 0\), the proof is valid for any double cover as above. \(\square\)

**Remark 1.** It is easily seen that the formulas and properties given above will still hold if we substitute \(W\) with \(P\), \(B\) with \(\bar{B}\) and \(L\) with \(\bar{L}\). Note also that, for \(m \geq 3\), \(mK_W + B = p^*(mK_P + \bar{B}) + (m - 2)E\), where \(E\) is the exceptional divisor of \(p\). Hence \(|mK_W + B| = p^*(|mK_P + \bar{B}|) + (m - 2)E\) and thus in particular \(|mK_W + B|\) is composed with a pencil if and only if \(|mK_P + \bar{B}|\) is.
Remark 2. In the case where the involution $i$ has no isolated fixed points and $H^0(W, 2K_W + L) \neq 0$, we notice that also $2K_P + \bar{L}$ will be nef. In fact in that case, $V = S$ and $\pi^*(K_W + L) = K_S$, yielding in particular that $K_W + L$ is nef and thus also that $K_P + L$ is nef. Suppose, for contradiction, that $2K_P + \bar{L}$ is not nef. Then there exists an irreducible curve $E$ such that $(2K_P + \bar{L})E < 0$. Since $2K_P + L$ is a nonzero effective divisor, one must have that $E^2 < 0$ (see [26], pg 34). On the other hand, since $K_P + L$ is nef, necessarily $K_P E < 0$. So $E$ is a $(-1)$-curve. Then from $E(2K_P + \bar{L}) < 0$, we conclude $E(K_P + \bar{L}) = 0$ and so $E(3K_P + \bar{B}) < 0$. This contradicts the construction of $P$ as above. So $2K_P + \bar{L}$ is nef.

3. The canonical map

In this section $S$ denotes a minimal surface of general type with $K_S^2 = 2\chi - 2$ and $p_g \geq 5$. First:

**Lemma 3.1.** Let $S$ be a minimal surface of general type with $K_S^2 = 2\chi - 2$ and $p_g \geq 5$. Then:

- $S$ is regular;
- $K_S^2 = 2p_g$;
- $S$ has no torsion divisors.

**Proof.** From ([7], Lemma 14), one has that $q = 0$ and thus $K_S^2 = 2p_g$. Also, by ([12], Theorem A), surfaces satisfying $K = 2\chi - 2$ can only have torsion if $p_g \leq 4$. □

We write $|K_S| = |H| + F$, where $|H|$ is the moving part of $|K_S|$ and $F$ the fixed part and let $\rho : \tilde{S} \rightarrow S$ be a composition of blow-ups such that the variable part $|\tilde{H}|$ of $|\rho^*K_S|$ is free from base points. We denote by $E$ the exceptional divisor of $\rho$, by $F'$ the fixed part of $|\rho^*K_S|$ and by $\Sigma$ the canonical image of $S$.

**Lemma 3.2.** Let $S$ be a minimal surface of general type with $K_S^2 = 2\chi - 2$ and $p_g \geq 5$. Then one of the following occurs:

- $\phi_{K_S}$ is a birational map;
- $\phi_{K_S}$ is a rational map of degree 2 onto a rational surface.

**Proof.** Note first that $\phi_{K_S}$ is generically finite because from ([18], Theorem 1.2) surfaces with $p_g \geq 5$ and $|K|$ composed with a pencil must satisfy $K^2 > 4p_g - 7$. So, the canonical image $\Sigma$ is an irreducible and nondegenerate surface in $\mathbb{P}^{p_g-1}$,

$$2p_g = K_S^2 \geq (\deg \phi_{K_S})(\deg \Sigma) \geq (\deg \phi_{K_S})(p_g - 2).$$

So $\deg \phi_{K_S} \leq 3$ and if $\deg \phi_{K_S} = 3$, then $p_g = 5$ or $p_g = 6$.

Assume that $\deg \phi_{K_S} = 3$. Then a general curve in $|K_S|$ is smooth, because $|K_S|$ has no base points if $p_g = 6$, or a unique base point if $p_g = 5$. Then $\phi_{K_S}$ is a birational map by the adjunction formula.

$$\phi_{K_S}^* K_S = 3K_S + 3E \geq 3(3K_S + 3E) \geq (\deg \phi_{K_S})(3K_S + 3E).$$

So $\deg \phi_{K_S} \geq 3$.

If $\deg \phi_{K_S} = 2$, then $S$ is a minimal surface with $K_S^2 = 2\chi - 2$ and $p_g \geq 5$ such that $\phi_{K_S}$ is not an isomorphism, but $\phi_{K_S}$ is a finite map of degree 2 onto a surface $\Sigma$ with $|\Sigma| = |K_S| = |H| + F$.

If $\deg \phi_{K_S} = 1$, then $S$ is a minimal surface with $K_S^2 = 2\chi - 2$ and $p_g \geq 5$ such that $\phi_{K_S}$ is a composition of blow-ups such that the canonical image $\Sigma$ is an irreducible and nondegenerate surface in $\mathbb{P}^{p_g-1}$.

So $\phi_{K_S}$ is a rational map of degree 2 onto a rational surface.
5. Since $\Sigma$ is a surface of minimal degree $p_g - 2$ in $\mathbb{P}^{p_g-1}$ we obtain a contradiction to ([22], Theorem 2.1) where it is shown that if the general curve in $|K_S|$ is smooth and the canonical map is of degree 3 onto a surface of minimal degree $p_g - 2$ in $\mathbb{P}^{p_g-1}$, then $p_g \leq 5$ and $K_S^2 \leq 9$. Therefore, we have $\deg \phi_K \leq 2$.

If $\deg \phi_K = 2$, then $\Sigma$ is a surface of degree $\leq p_g$ in $\mathbb{P}^{p_g-1}$. From $p_g \geq 5$, one obtains $\deg \Sigma \leq p_g < 2p_g - 4$ and thus, by ([6], Lemme 1.4), $\Sigma$ has Kodaira dimension $-\infty$. Since $S$ is regular we conclude that $\Sigma$ is a rational surface.

If $\phi_K$ has degree 2, then the canonical map factors through an involution $i$. In this case, we recall the diagram (2.1)

\[
\begin{array}{ccc}
V & \xrightarrow{h} & S \\
\pi & & \pi' \\
W & \xrightarrow{g} & S/i
\end{array}
\]

where $\pi$ is a double cover with branch locus $2L \equiv B' = B + \sum_i A_i$. We consider the $\mathbb{Q}$-divisor $B/2$ and we keep the notation of Section 2.

Remark 3. Note that if the canonical map factors through an involution, since $\chi(S/i) = 1$ by Lemma 3.2, we have

\[t = 4 - 2h^0(2K_W + L)\]

so the number of isolated fixed points of the involution is $t = 0, 2$ or 4.

Also, as a consequence of the double cover formulas

**Proposition 3.3.** Let $S$ be a minimal surface of general type with $K_S^2 = 2\chi - 2$ such that the canonical map factors through an involution with rational quotient. Then, with the notation above:

(i) $K_W(K_W + L) = -p_g + 2 - \frac{t}{2}$;
(ii) $h^0(2K_W + L) = 2 - t/2$;
(iii) $(2K_W + L)^2 = K_W^2 - p_g - \frac{3}{2}t + 4$;
(iv) $(2K_W + B)^2 = 4p_g$;
(v) $(2K_W + B)(2K_W + L) = 4\epsilon + 4 - t$;
(vi) $K_WL = K_W(B/2) = 2 - p_g - t/2 - K_W^2$;
(vii) $(B/2)^2 = L^2 + \frac{t}{2} = 3p_g + t + K_W^2 - 4$;
(viii) $p_a(2K_W + B) = p_g + 3 - \frac{t}{2}$;
(ix) $h^0(3K_W + B) = p_a(2K_W + B)$;
(x) if $3K_W + B$ is nef and big then $h^0(4K_W + B) = p_a(3K_W + B)$.

**Proof.** All the equalities above, except the two last, are a direct consequence of the formulas in the previous section. In fact, since $S$ is regular, it satisfies $\chi(O_S) = p_g + 1$ and $W$ being a rational surface satisfies $\chi(O_W) = 1$. 

The two last equalities come from the Riemann-Roch theorem since $2K_W + B$ is nef and big. □

Let us recall the next result due to Castelnuovo (cf. [1]):

**Lemma 3.4 (Castelnuovo’s Bound).** Let $C$ be a smooth curve of genus $g$ that admits a birational mapping onto a nondegenerate curve of degree $d$ in $\mathbb{P}^r$. Let $m = [(d - 1)/(r - 1)]$ and $\varepsilon = (d - 1) - m(r - 1)$. Then $g \leq \pi(d, r)$, where $
abla\pi(d, r) = m(m - 1)(r - 1)/2 + m \varepsilon$.

We will need also:

**Lemma 3.5.** Let $S$ be a minimal surface of general type such that $K_S^2 = 3pg - 5$ and $q = 0$. If the canonical map is birational, then $|K_S|$ does not have a fixed component and has at most one (simple) base point.

**Proof.** We use the notation introduced in the beginning of this section. Since we are assuming $\phi_{K_S}$ birational, by [17], $\tilde{H}^2 \geq 3pg - 7$. Thus we have

$\tilde{H}^2 = 3pg - 5, 3pg - 6,$ or $3pg - 7$.

If $\tilde{H}^2 = 3pg - 5$, the $|K_S|$ has no base points. If $\tilde{H}^2 = 3pg - 6$, since $K_S = H + F$ and $K_S$ is nef and 2-connected, then $F = 0$ and we have at most one (simple) base point. So, if the statement is not true, then necessarily $\tilde{H}^2 = 3pg - 7$ and the general curve $C$ in $|\tilde{H}|$ is nonsingular. Note that $|\rho^*K_S| = |\tilde{H}| + E + F'$, where $E$ is the exceptional divisor of $\rho$ and $F'$ the fixed part of $\rho^*K_S$, then $\tilde{H}(E + F') > 0$. So, by the adjunction formula, $C$ is of genus

$3pg - 6 + \frac{1}{2} \tilde{H}(2E + F')$.

On the other hand, from Lemma [3.4] we obtain $g(C) \leq 3pg - 6$, a contradiction. □

We can now give a rough classification:

**Theorem 3.6.** Let $S$ be a minimal surface with $K_S^2 = 2x - 2$ and $pg \geq 5$. Then $S$ satisfies exactly one of the following:

(I) the canonical map $\phi_{K_S}$ is birational and
   (Ia) $|K_S|$ is free from base points and $pg \leq 7$, or
   (Ib) $|K_S|$ has exactly one (simple) base point and $pg = 5$;

(II) the canonical map factors through an involution $i$ and the number $t$ of isolated fixed points of $i$ is:
   (IIa) $t = 0$, or
   (IIb) $t = 2$, or
   (IIc) $t = 4$.

**Proof.** If $\deg \phi_{K_S} = 1$, by the Castelnuovo inequality, one has $pg \leq 7$. Now, using ([21], Lemma 1.3) and ([2], Lemma 1.1) for $pg = 6$ and 7 respectively, we obtain that $|K_S|$ is free from base points. Finally, by Lemma [3.5] for
$p_g = 5$, the canonical system $|K_S|$ does not have fixed components and has at most one (simple) base point.

If $\deg \phi_{K_S} = 2$, the result follows from Remark 3.

Remark 4. Surfaces of type $(Ia)$ with $p_g = 7$, have been studied, among others, by Ashikaga and Konno ([2]) and Miranda ([24]). In particular, Miranda has proved that $\phi_{K_S}$ maps $S$ into the Veronese cone or into a rational normal scroll.

For surfaces of type $(Ia)$ with $p_g = 6$, we refer to [21]. For this case Konno has shown that the canonical image is contained in a threefold $W$ of $\Delta$-genus $\leq 1$ which is cut out by all quadrics through the canonical image.

Ciliberto in [11], proves the existence of surfaces of type $(Ia)$ with $p_g = 5$. He has studied the moduli space of such surfaces, its dimension and its unirationality. Furthermore he has shown that the canonical image of a generic such surface has only isolated singularities and cannot lie in a quadric.

Surfaces of type $(Ib)$, as far as we know, have not been studied yet and we do not know whether they exist. Next we give some of their properties.

**Proposition 3.7.** If $S$ is a surface of type $(Ib)$, then:

- a general canonical curve $D$ is smooth and non hyperelliptic of genus 11;
- the image of $D$ via the canonical map of $S$ is a curve $D_0$ of degree 9 in $\mathbb{P}^3$ with one double point;
- the canonical image of $S$ is contained in a singular quadric of $\mathbb{P}^4$.

**Proof.** Since by Theorem 3.6 $|K_S|$ has only one simple base point, a general member $D \in |K_S|$ is irreducible and nonsingular. Thus, from $K_S^2 = 10$ and the adjunction formula we obtain that the geometrical genus of $D$, $g(D)$, is 11. Also $D$ is nonhyperelliptic because the canonical map of $S$ is birational. The image $D_0$ of $D$ by $\phi_{K_S}$ is an irreducible nondegenerate curve of degree 9 in $\mathbb{P}^3$. Since $g(D) = 11$, $11 \leq p_a(D_0)$. On the other hand, applying the main theorem of [10] we obtain $p_a(D_0) \leq 12$. If $p_a(D_0) = 11$, $D_0$ is a nonsingular curve with degree 9 and $g = 11$ in $\mathbb{P}^3$, but this is not possible, (cf. [14], Exercise 6.4 and Remark 6.4.1). Hence $p_a(D_0) = 12$ and so $D_0$ has exactly one singular double point.

For the last assertion consider the subspace $V$ of $H^0(2K_S)$ generated by products of sections of $K_S$. We claim that $V$ has dimension 14. By [13] Prop. 3.1 $\dim V \geq 14$. Assume for contradiction that $\dim V \geq 15$. Since the kernel of the restriction map $H^0(S, 2K_S) \to H^0(D, K_D)$ is isomorphic to $H^0(S, K_S)$ and so 5-dimensional, the image of the restriction of $V$ to $D$ is at least 10-dimensional. Let $x$ be the unique base point of $|K_S|$, then every section in $V$ vanishes at least twice in $x$. So we conclude that $h^0(D, K_D - 2x) \geq 10$ and hence, by the Riemann-Roch theorem and $g(D) = 11$, $h^0(D, 2x) \geq 2$, a contradiction because $D$ is nonhyperelliptic.
Lemma 4.1. Let \( P \) be a quadric of \( S \) be singular because the degree of \( \Sigma \) is 9 and any surface on a non-singular part. Since \( D \) is nef and \( M \), hence, \( \phi(M) \) is nef and \( h^0(D) = a \) with \( h^0(M) = a + 1 \), we conclude that \( KSN < -2 \) and \( KN < 0 \) and so, by adjunction, \( KN = -2 \) and \( KM \geq KD \). Since \(-2a = KM \leq KD = -2r, Z \) must be zero and \( a = r \), i.e. \( h^0(M) = r+1 \). □

4. Surfaces of type (IIa)

We start by stating a general fact:

Lemma 4.2. Let \( S \) be as in Assumption 5, then the linear system \( |2K_P + L| \) is a base point free pencil of rational curves and \( a \) is a positive rational number. From \( 2K_P + L \) is nef, \( 0 = D^2 \geq DM = M^2 + MZ \geq MZ \geq 0 \).

Throughout this section, we make the following assumption:

Assumption 5. \( S \) is a minimal surface with \( K_S^2 = 2p_g, q = 0, p_g \geq 5 \), and such that \( \phi_{K_S} \) has degree 2 with \( t = 0 \).

We keep the notation of Section 2, and in particular \( p : S/i \to P \) is the birational morphism such that \( 2K_P + L \) is nef (see Remark 2). Then:

Lemma 4.3. Let \( S \) be as in Assumption 5, then the linear system \( |2K_P + L| \) is a rational pencil without base points. Moreover, \( |\pi^*(2K_P + L)| \) is a hyperelliptic pencil of genus 3 in \( S \).

Proof. Remark first that \( (2K_P + L)^2 = \alpha \geq 0 \), since \( 2K_P + L \) is nef. Next, by Proposition 3.3, we get \( (K_P + L)^2 = p_g \) and \( (K_P + L)(2K_P + L) = 2 \). It follows immediately that \( (K_P + L + 2K_P + L)^2 = p_g + \alpha + 4 > 0 \), so as a consequence of the Index theorem \( p_g \cdot \alpha \leq 4 \) and hence \( (2K_P + L)^2 = 0 \). Also, from Proposition 3.3 we see \( (2K_P + L)K_P = -2 \). So, as a consequence of Lemma 4.1, we conclude that \( |2K_P + L| \) is a base point free pencil of rational curves and, since \( (2K_P + L)(K_P + L) = 2 \), we finish the proof. □

Proposition 4.3. Let \( S \) be a surface as in Assumption 2. Then \( p_g \leq 12 \) and \( S \) is the minimal resolution of a double cover of \( \mathbb{P}_r, r \leq 3 \), branched
on a curve in \( |8C_0 + 2(5 + 2r)f| \) having 12 \(- p_g\) singular points (possibly infinitely near) of multiplicity 4 as only essential singularities.

**Proof.** Let \( S \to P \) be the map of degree 2 with branch curve \( B = 2L \), with possibly inessential singularities. Using Lemma 4.2, we know that \( |2K_P + L| \) is a genus 0 pencil without base points. Then, we have \( P \not\equiv \mathbb{P}^2 \) and, from Proposition 3.3, \( K_P^2 = p_g - 4 \). So contracting \( 12 - p_g \) \((-1)\)-curves contained in the fibres of \( |2K_P + L| \) we get a birational morphism \( \gamma : P \to \mathbb{F}_r \).

Let \( f \) be a fibre of \( \mathbb{F}_r \) and \( C_0 \) a section with \( C_0^2 = -r \). Denote \( \bar{L} = \gamma^*(aC_0 + bf) - \sum c_iE_i \), since \( 2K_P + \bar{L} = \gamma^*(f) \), then

\[
(a - 4)C_0 + (b - 2(2 + r))f + \sum (2 - c_i)E_i = f
\]

and we obtain \( a = 4 \), \( c_i = 2 \) and \( b = +5 + 2r \).

Note that \( c_i = 2 \) for every \( i \) means that the essential singularities are quadruple points.

In the end, we can write \( \gamma^*(C_0) = B_0 + \sum \xi_i E_i \), where \( B_0 \) is the strict transform of \( C_0 \). Then \( 0 \leq (K_P + \bar{L})B_0 = (2C_0 + (3 + r)f)C_0 - \sum \xi_i \leq 3 - r \) which implies \( r \leq 3 \).

**Remark 6.** In Proposition 4.3, if \( p_g \leq 11 \) we have also that \( r \leq 2 \). Indeed, if \( r = 3 \) the image of \( \bar{B} \) is in \( |8C_0 + 22f| \). We see that \( C_0 \) is in the fixed part of \( |8C_0 + 22f| \) and \( C_0(7C_0 + 22f) = 1 \), so the essential singularities are not contained in \( C_0 \). Since \( p_g \leq 11 \) there exists at least one essential singularity on a fibre of the ruling. Blowing up this point and next contracting the strict transform of the fibre, we obtain a new birational morphism from \( P \) onto \( \mathbb{F}_2 \) with a quadruple point on the infinity section.

**Corollary 4.4.** Let \( S \) be as in Proposition 4.3. Assume also that \( p_g \leq 11 \) and \( C_0 \) is not contained in \( \bar{B} \). Then there exists a rational map such that the image of \( \bar{B} \) in \( \mathbb{P}^2 \) is a curve of degree 14 with \( 12 - p_g \) points of multiplicity 4 and one point of multiplicity 6 as unique essential singularities. For \( r = 2 \), the singular point of multiplicity 6 is infinitely near to, at least, a point of multiplicity 4.

**Proof.** From Proposition 4.3 and Remark 6, up to an elementary transformation of \( F_r \) centered in a quadruple point, we can assume that \( r = 0 \) or 2.

If \( r = 0 \), let \( f_1 \) and \( f_2 \) be the two rulings of \( \mathbb{F}_0 \). Then the image of \( \bar{B} \) in \( \mathbb{F}_0 \) is a curve \( \bar{B}_{\mathbb{F}_0} \in |8f_1 + 10f_2| \). Since \( p_g \leq 11 \), there exists at least one point of multiplicity 4. We blow up one of these points of \( \mathbb{F}_0 \) and obtain a new line, a \((-1)\)-curve; next blow down the two fibres passing through the point. We obtain two singularities of multiplicity 4 and 6; the image of \( \bar{B}_{\mathbb{F}_0} \) meets the new line in \( 6 + 4 + 4 = 14 \) points. In sum, there exists a birational map \( \mathbb{F}_0 \dashrightarrow \mathbb{P}^2 \) such that the image of \( \bar{B}_{\mathbb{F}_0} \) is a curve of degree 14 with two (distinct) points of multiplicity 6 and 4 plus \( 11 - p_g \) points of multiplicity 4. Note that some of the essential singular points are possibly infinitely near.
Similarly, if \( r = 2 \), then \( \bar{B}_{\mathbb{P}^2} \in |8C_0 + 18f |. \) Since by assumption \( C_0 \) is not contained in \( \bar{B} \), as before, there exists at least one point of multiplicity 4 not contained in the infinity section, so there exists a birational map \( \mathbb{P}_2 \rightarrow \mathbb{P}^2 \) such that the image of \( \bar{B}_{\mathbb{P}^2} \) in \( \mathbb{P}^2 \) is a curve of degree 14 with a point of multiplicity \((6, 4)\) plus \(11 - p_g \) points of multiplicity 4. As in the case \( r = 0 \) we note that apart from the singular point of multiplicity \((6, 4)\), some of the other essential singularities are possibly infinitely near. \( \square \)

5. Surfaces of type \((IIb)\)

We recall that \( S \) is a surface of type \((IIb)\) if the canonical map factors through an involution with \( t = 2 \). We can then write the branch curve of the double cover \( V \rightarrow W \) as \( 2L \equiv B' = B + A_1 + A_2 \), where \( A_1, A_2 \) are \((-2)\)-curves.

From Proposition 3.3 we have \( h^0(3K_W + B) = p_g + 2 \), so if \( P \) and \( \bar{B} \) are as in Proposition 2.1, the effective divisor \( 3K_P + \bar{B} \) is nef.

Also, from Proposition 3.3:

**Lemma 5.1.** Let \( S \) be a minimal surface with \( K_S^2 = 2p_g, q = 0 \) and \( p_g \geq 5 \), and \( i \) an involution on \( S \) such that \( t = 2 \). Then \( K_P(\bar{B}/2) = 1 - p_g - K_P^2 \) and \( (\bar{B}/2)^2 = \tilde{L}^2 + 1 = 3p_g - 2 + K_P^2 \);

Throughout this section we will prove the following theorem:

**Theorem 5.2.** Let \( S \) be a minimal surface with \( K_S^2 = 2p_g, q = 0 \) and \( p_g \geq 5 \), such that the canonical map factors through an involution with \( t = 2 \). Then \( p_g \leq 8 \) and one of the following occurs:

(i) \( S \) is the minimal resolution of a double cover of a weak Del Pezzo surface \( T \) of degree \( p_g + 1 \) branched on an effective divisor in \( |-4K_T| \) having exactly two \((3, 3)\)-points as essential singularities.

(ii) \( S \) is the minimal resolution of a double cover of \( \mathbb{F}_r, r \leq 2 \), whose branch curve is the union of a curve in \( |8C_0 + (9 + 4r)f| \) with a fibre. The curve has \( 8 - p_g \) singular points (possibly infinitely near) of multiplicity 4 and another of type \((4, 4)\) and the fibre is tangent to the curve at the \((4, 4)\)-point.

**Proof.** We divide the proof into steps.

**Step 1:** With the usual notation \( K_P^2 = p_g - 2 \) or \( K_P^2 = p_g - 3 \).

From Proposition 3.3 \( h^0(2K_W + L) = 1 \) and also \( 0 \leq (3K_P + \bar{B})(2K_P + \tilde{L}) = K_P^2 + 3 - p_g \), therefore \( K_P^2 \geq p_g - 3 \). On the other hand, by the Index theorem \( K_P^2(\bar{B}/2)^2 \leq (K_P(\bar{B}/2))^2 \) and hence, from Lemma 5.1 we get \( K_P^2 \leq \frac{(p_g - 1)^2}{p_g} \) and the assertion follows.
Step 2: If $4K_P + \bar{B}$ is not nef, there exists a birational morphism $p_1 : P \to P_1$ such the divisor $4K_{P_1} + B_{P_1}$ is nef.

From Proposition [3] and Step 1, $h^0(4K_P + \bar{B}) > 0$. So, if $4K_P + \bar{B}$ is not nef, there is an irreducible curve $E_1$ such that $E_1(4K_P + \bar{B}) < 0$ and so as in Remark [2] we can see that $E_1$ is a $(-1)$-curve with $E_1(3K_P + \bar{B}) = 0$ and so $E_1\bar{B} = 3$. Since $\bar{B}'$ is an even divisor, it is clear that $E_1(C_1 + C_2) > 0$ and it is an odd number. Since $(E_1 + C_1 + C_2)(3K_P + \bar{B}) = 0$, $(E_1 + C_1 + C_2)^2 < 0$, by the Index theorem. As $(E_1 + C_1 + C_2)^2 = -1 + 2E_1(C_1 + C_2) + -4$ the only possibility is $E(C_1 + C_2) = 1$. If, say, $E_1C_1 = 1$, when $E_1$ is contracted the image of $C_1$ is a $(-1)$-curve that is in the branch locus and intersects the image of $\bar{B}$ at a triple point. So at this point, the image of $4K_P + \bar{B}$ is not nef any more. Therefore, we have to contract the image of $C_1$ also, so the inductive step consists in contracting twice. If necessary, we can repeat the same argument for another $(-1)$-curve $E_2$ with $E_2C_2 = 1$, obtaining the result.

To continue with the proof, we analyse the two values of $K^2_P$.

Step 3: If $K^2_P = p_g - 3$, then $S$ is the minimal resolution of a double cover of a weak Del Pezzo surface $T$ of degree $p_g + 1$ branched on a divisor in $| - 4K_T|$ having two $(3,3)$-points.

Let $p_1 : P \to P_1$ be the birational morphism such that $4K_{P_1} + B_{P_1}$ is nef. If $s$ is the number of $(-1)$-curves contracted by $p_1$, from Lemma [5.1], $(4K_P + \bar{B})^2 = -4$ and one has $s \geq 4$. Besides, note that $K^2_{P_1} = K^2_P + s$ and $(B_{P_1}/2)^2 = (\bar{B}/2)^2 + 2s$ by Lemma [5.1], so

$$0 \leq (2K_{P_1} + B_{P_1})(4K_{P_1} + B_{P_1}) = 4 - s$$

hence, $s = 4$; since $4K_{P_1} + B_{P_1}$ is an effective divisor and by the Index theorem we have that $2K_{P_1} + B_{P_1}/2$ is a trivial divisor. Therefore $-K^2_{P_1} = K^2_{P_1} + B_{P_1}/2$ gives $-K_{P_1}$ nef and big, so $P_1$ is a weak Del Pezzo surface of degree $K^2_{P_1} = p_g + 1$. Finally, let us analyse the image of $C_1$ and $C_2$ in $P_1$. As we have seen, $C_1$ and $C_2$ are $(-2)$-curves in $P$, however, if $E_1$ is a $(-1)$-curve contracted by $p_1$, by Step 2, we can suppose that $E_1C_1 = 1$ and $C_1$ becomes a $(-1)$-curve, whose intersection with the image of $\bar{B}$ is equal to 3 and it will be contracted as well. Since $s = 4$, we can repeat the same argument for another $(-1)$-curve $E_2$ with $E_2C_2 = 1$, obtaining the two singular $(3,3)$-points.

Remark 7. Notice that Theorem 4.2 of [4] gives the same result as in Step 3 for the case $p_g = 4$.

Step 4: If $K^2_P = p_g - 2$, then $|4K_P + \bar{B}|$ is a rational pencil without base points.

Keeping the same notation as in the proof of Step 3, and using the Index theorem we obtain that $K^2_{P_1}(B_{P_1}/2)^2 \leq (K_{P_1}(B_{P_1}/2))^2$, which implies $s \leq$
Remark 6. Surfaces with $K^2 = 2X - 2$ and $p_g \geq 5$

By Lemma [5.1] hence $s = 0$ and we conclude that $4K_P + \tilde{B}$ is nef. From Proposition [3.3] we see that $(4K_P + B)^2 = 0$; besides, from Lemma [5.2] we have $K_P(4K_P + B) = -2$. So, applying Lemma [4.1] the result follows.

**Step 5:** If $K_P^2 = p_g - 2$, then $S$ is the minimal resolution of a double cover of $\mathbb{F}_r$, $r \leq 2$, whose branch curve is the union of a curve in $|8C_0 + (9 + 4r)f|$ with a fibre. The curve has $8 - p_g$ singular points of multiplicity 4 and another of type $(4, 4)$ and the fibre is tangent to the curve at the $(4, 4)$-point.

From Step 4, $|4K_P + \tilde{B}|$ is a genus 0 pencil without base points, hence $P \neq \mathbb{P}^2$ and contracting $10 - p_g$ exceptional curves, we get a birational morphism $\gamma : P \to \mathbb{F}_r$.

Let $f$ be a fibre of $\mathbb{F}_r$ and $C_0$ a section with $C_0^2 = -r$. Write $\tilde{B} = \gamma^*(aC_o + bf) - \sum c_iE_i$, since

$$4K_P + \tilde{B} = \gamma^*((a - 8)C_0 + [b - 4(2 + r)]f) + \sum (4 - c_i)E_i = \gamma^*(f)$$

we obtain $\tilde{B} = \gamma^*(8C_0 + (9 + 4r)f) - 4 \sum E_i$.

Also, from $0 \leq \gamma^*(C_0)(2K_P + \tilde{B})$, we have $r \leq 2$.

By Proposition [2.1] we know that $C_1$ and $C_2$ are $(-2)$-curves on $P$ and hence $C_l(4K_P + \tilde{B}) = 0$, so they are contained in the fibres. More precisely, since $C_l(2K_P + \tilde{L}) = -1$, we can write $2K_P + \tilde{L} = D + C_1 + C_2$ where $D$ is an effective divisor with $h^0(D) = 1$. Hence, $4K_P + \tilde{B} = 2D + C_1 + C_2$, so $C_1$ and $C_2$ are in the same fibre. By easy calculations we have $D^2 = -1$ and $K_PD = -1$. Then it is easy to see that contracting $D$ and then, say, the image of $C_1$, we obtain a singularity of multiplicity $(4, 4)$ of the image of $\tilde{B}$, such that the fibre passing through this point is contained in the branch locus and it is tangent to $\tilde{B}$ at the $(4, 4)$-point. Finally, since there are $10 - p_g$ singular points of multiplicity 4, then $p_g \leq 8$. \hfill $\square$

To end this section we make several remarks.

**Remark 8.** If $S$ is a surface as in Step 5, we can proceed as in the proof of Corollary [4.4]. First, by Step 5 we have $r \leq 2$, so we can suppose that $r = 0, 2$. If $r = 0$, there exists a birational map $\mathbb{F}_0 \dashrightarrow \mathbb{P}^2$ such that we can see the image of the branch $B'$ on $\mathbb{P}^2$ as a curve of degree 14 of type $C + l$, where $l$ is a line and $C$ is a curve of degree 13 having two (distinct) points $P_1$ and $P_2$ of multiplicity 5 and $(4, 4)$ respectively at the intersection with $l$, plus $8 - p_g$ points of multiplicity 4, and no further essential singularities. For the case $r = 2$ it is easily seen that the infinity section $C_0$ cannot contain the $(4, 4)$-point. Then with the same procedure the case $r = 2$ can be expressed as a degeneration of the case $r = 0$. The degeneration consists of having $P_1$ infinitely near to $P_2$. Note also that $l$ is tangent to $C$ at $P_2$.

6. **Surfaces of type (IIc)**
Finally, we are going to study surfaces with $K^2_S = 2p_g$, $q = 0$ and $p_g \geq 5$, such that the canonical map factors through an involution with $t = 4$.

For these surfaces we have that $h^0(2K_W + L) = 0$. Thus the bicanonical map of $S$ is composed with $i$, hence not birational. Since by hypothesis $K^2_S \geq 10$, using ([27], Proposition 3), $S$ has a pencil of curves of genus 2, necessarily rational because $q = 0$. We remark that the existence of the genus 2 pencil can be also checked directly by considering the linear system $|3K_W + B|$.

**Remark 9.** It is easy to see that the rational pencil of curves of genus 2 is unique. Otherwise, let $|G_1|$ and $|G_2|$ be two pencils of genus 2 without base points; since $G_1G_2 \geq 2$, $(G_1 + G_2)^2 \geq 4$. Since $K_S(G_1 + G_2) = 4$, we obtain $K^2_S(G_1 + G_2)^2 - (K_S(G_1 + G_2))^2 > 0$, a contradiction to the Index theorem, because $K^2_S \geq 10$.

**Remark 10.** Recall that if a surface has a pencil of genus 2, there exists a map of degree 2 onto a ruled surface, mapping each genus 2 fibre by its canonical map onto a fibre of the ruling. Horikawa in [15] proved that with elementary transformations it is possible to obtain a minimal model whose branch locus has only singularities of the following types: $(0)$, $(I_k)$, $(II_k)$, $(III_k)$, $(IV_k)$ and $(V)$ (in Horikawa’s notation).

In what follows, we will say that a rational fibre of Horikawa’s model is a singular fibre of type $(I_k)$, $(II_k)$, $(III_k)$, $(IV_k)$ or $(V)$ if its corresponding rational fibre is of this type.

The next result is well known but for completeness we include its proof.

**Lemma 6.1.** Let $S$ be a minimal algebraic surface of general type with $p_g \geq 4$, $q = 0$ and canonical map not composed with a pencil. If $S$ has a pencil of genus 2, then the canonical map has degree 2.

**Proof.** Let $|G|$ be the genus 2 pencil on $S$ and note that $K_S|_G \simeq \omega_G$. Since $\omega_G$ is a $g^1_2$ on $G$, $\phi_{K_S}$ has even degree.

Since $h^0(S, K_S) \geq 4$ and $h^0(G, \omega_G) = 2$, from the long exact sequence:

$$0 \rightarrow \mathcal{O}_S(K_S - G) \rightarrow \mathcal{O}_S(K_S) \rightarrow \mathcal{O}_S(K_S)|_G \rightarrow 0,$$

we have that $h^0(S, K_S - G) \geq 2$. Therefore, $\phi_{K_S}$ separates the fibres and we obtain the result. \hfill $\square$

**Proposition 6.2.** The pencil of genus 2 in $S$ is the pull-back of a ruling of the canonical image $\Sigma$ of $S$. Moreover, the essential singularities of the branch locus in Horikawa’s model are of type: $(I_k)$, $(II_k)$, $(III_k)$, $(IV_k)$, with $k = 1, 2$, and $(V)$.

**Proof:** Since $K_S|_G = \omega_G$ and $|K_S|$ is not composed with a pencil, the image of a general element of $|G|$ is a line, and so a ruling of $\Sigma$. Using ([15],
Theorem 3) we get

\begin{equation}
4 = \sum_k \{ (2k-1)(\nu(I_k) + \nu(III_k)) + 2k(\nu(II_k) + \nu(IV_k)) \} + \nu(V),
\end{equation}

where \( \nu(*) \) denotes the number of singularities of type \((*)\). So we immediately obtain that \( k = 1, 2 \).

**Remark 11.** Looking carefully at the resolution of the singular fibres as in Proposition 6.2, we obtain that:

(i) each singular fibre of type \((I_1)\) or \((III_1)\) or \((V)\) corresponds to one base point of \(|K_S|\) and one isolated fixed point of the involution;
(ii) each singular fibre of type \((I_2)\) or \((III_2)\) corresponds to a fixed component plus one base point of \(|K_S|\), and three isolated fixed points of the involution;
(iii) each singular fibre of type \((II_1)\) or \((IV_1)\) corresponds to a fixed component of \(|K_S|\), and there are two isolated fixed points of the involution;
(iv) finally, each singular fibre of type \((II_2)\) or \((IV_2)\) corresponds to a fixed component plus two base points of \(|K_S|\), and four isolated fixed points of the involution.

We point out that all the fixed components of \(|K_S|\) are \((-2)\)-curves.

Using double or bidouble covers, it is not difficult to find examples, for instance,

**Example 6.1.** In \( \mathbb{F}_0 \), let \( f_1 \) and \( f_2 \) denote general fibres of each ruling. Consider the bidouble cover \( \pi : S \to \mathbb{F}_0 \) with smooth branch curves \( D_1 \in |f_1 + f_2|, \ D_2 \in |f_1 + 3f_2| \) and \( D_3 \in |3f_1 + (2a+1)f_2|, a \geq 2 \). Using the bidouble cover formulas (see [9], [25]), the surface \( S \) has the invariants \( K^2 = 4a + 2, \ p_g = 2a + 1 \) and \( q = 0 \). Now, we analyse the double cover as a composition of two double covers. First, the double cover with branch curve \( D_1 + D_2 \), this is \( Y \to \mathbb{F}_0 \), where \( Y \) is a rational surface with four \((-2)\)-curves coming from the intersection points of \( D_1 \) and \( D_2 \). The linear system \( |\pi^* f_2| \) is the pencil of genus 2, whose fibres in the general case will be of type \((I_1)\). It is easy to see that a mild degeneration of the construction will yield fibres of type \((III_1)\) or \( V \).

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