Intersection times for critical branching random walk

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Abstract

We present results relating mixing times to the intersection time of branching random walk (BRW) in which the logarithm of the expected number of particles at time \( t \) grows like \( \text{gap} \cdot t \). This is a finite state space analog of a critical branching process. Namely, we show that the maximal expected hitting time of a state by such a BRW is up to a universal constant larger than the \( L_\infty \) mixing-time, whereas under transitivity the same is true for the intersection time of two independent such BRWs.

Using the same methodology, we show that for a sequence of reversible Markov chains, the \( L_\infty \) mixing-times \( t_{\text{mix}}^{(\infty)} \) are of smaller order than the maximal hitting times \( t_{\text{hit}} \) iff the product of the spectral-gap \( \text{gap} \) and \( t_{\text{hit}} \) diverges, by establishing the inequality \( t_{\text{mix}}^{(\infty)} \lesssim \frac{1}{\text{gap}} \log(1 + t_{\text{hit}} \cdot \text{gap}) \). This resolves a conjecture of Aldous and Fill [5] Open Problem 14.12 asserting that under transitivity the condition that \( t_{\text{hit}} \gg \frac{1}{\text{gap}} \) implies mean-field behavior for the coalescing time of coalescing random walks.

Keywords: Mixing times, hitting times, vertex-transitive graphs, intersection times, branching random walk, spectral-gap, coalescing random walk.

1 Introduction

Spectral conditions play an important role in the modern theory of Markov chains. A common theme is that for a sequence of reversible Markov chains with finite state spaces of diverging sizes, certain phenomena can be understood in terms of the simple condition that the product of the spectral-gap and some other natural quantity diverges. One instance is the cutoff phenomenon and the well-known product condition [13, Proposition 18.4].[1]

Another such example is given in [8], where it is shown that for a sequence of reversible

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[1]This is the condition that the product of the mixing time and the gap diverges. It is a necessary condition for precutoff in total-variation and a necessary and sufficient condition for cutoff in \( L_2 \) [7].
transitive Markov chains (or more generally, ones for which the average and maximal hitting times of states are of the same order) the cover time is concentrated around its mean (for all initial states) iff the product of the spectral-gap and the (expected) cover time diverges (this refines a classical result of Aldous [4]).

Our first result concerns the condition that the product of the spectral-gap and the maximal hitting time diverges. This condition was first studied in the context of hitting times in transitive reversible chains by Aldous [2], where it is shown to imply that the maximal and the average hitting time differ only by a smaller order term, and that the law of the hitting time of a vertex is close to an exponential distribution for most initial states.

Let \((X_t)_{t \geq 0}\) be an irreducible Markov chain on a finite state space \(V\) with transition matrix \(P\) and stationary distribution \(\pi\) (we use this notation throughout the paper). We consider the continuous-time rate 1 version of the chain. Let \(H_t := e^{-t(I-P)}\) be its heat kernel (so that \(H_t(\cdot, \cdot)\) are the time \(t\) transition probabilities). We note that our results are valid also in the discrete-time setup when \(\min_{x \in V} P(x, x)\) is bounded away from 0. We denote the eigenvalues of the Laplacian \(I - P\) by \(0 = \lambda_1 < \lambda_2 \leq \ldots \lambda_{|V|} \leq 2\). The spectral-gap is defined as \(\lambda_2\) and the relaxation-time as \(t_{\text{rel}} := \frac{1}{\lambda_2}\).

The maximal expected hitting time of a state is given by
\[
 t_{\text{hit}} := \max_{x, y \in V} \mathbb{E}_x[T_y], \quad \text{where} \quad T_y := \inf\{t : X_t = y\}.
\]
The average hitting time \(\alpha := \sum_{x, y} \pi(y) \mathbb{E}_x[T_y]\) is independent of \(x\) (see §2.2) and so
\[
\alpha = \sum_y \pi(y) \mathbb{E}_x[T_y].
\]

We denote the \(L_\infty\) mixing time and the average \(L_2\) mixing time, respectively, by
\[
 t_{\text{mix}}^{(\infty)} := \inf\{t : \max_{a,b} |\frac{H_t(a,b)}{\pi(b)} - 1| \leq \frac{1}{2}\} = \inf\{t : \max_a |\frac{H_t(a,a)}{\pi(a)} - 1| \leq \frac{1}{2}\},
\]
\[
 t_{\text{ave- \text{mix}}}^{(2)} := \inf\{t : \sum_x H_{2t}(x, x) \leq \frac{2}{t}\}
\]
(see (2.2)-(2.3)). Throughout the superscript ‘(n)’ indicates that we are considering the \(n\)th Markov chain in the sequence.

**Theorem 1.** For an irreducible reversible Markov chain with a finite state space we have\(^{[2]}\)
\[
 t_{\text{mix}}^{(\infty)} \lesssim t_{\text{rel}} \log(1 + t_{\text{hit}}/t_{\text{rel}}),
\]
\[
 t_{\text{ave- \text{mix}}}^{(2)} \lesssim t_{\text{rel}} \log(1 + \alpha/t_{\text{rel}}).
\]

Hence for a sequence of such Markov chains the following are equivalent:

\(^{[2]}\)We write \(o(1)\) for terms which vanish as \(n \to \infty\). We write \(f_n = o(g_n)\) or \(f_n \ll g_n\) if \(f_n/g_n = o(1)\). We write \(f_n = O(g_n)\) and \(f_n \lesssim g_n\) (and also \(g_n = \Omega(f_n)\) and \(g_n \gtrsim f_n\) if there exists a constant \(C > 0\) such that \(|f_n| \leq C|g_n|\) for all \(n\). We write \(f_n = \Theta(g_n)\) or \(f_n \asymp g_n\) if \(f_n = O(g_n)\) and \(g_n = O(f_n)\).
Also \((t_{\text{mix}}^{(\infty)})^{(n)} \approx t_{\text{hit}}^{(n)}\) respectively, \((t_{\text{mix}}^{(\infty)})^{(n)} \ll t_{\text{hit}}^{(n)}\),

(ii) \(t_{\text{rel}}^{(n)} \ll t_{\text{hit}}^{(n)}\) respectively, \(t_{\text{rel}}^{(n)} \ll t_{\text{hit}}^{(n)}\),

Also \((t_{\text{ave-mix}}^{(2)})^{(n)} \ll \alpha^{(n)}\) iff \(t_{\text{rel}}^{(n)} \ll \alpha^{(n)}\), while \((t_{\text{ave-mix}}^{(2)})^{(n)} \ll \alpha^{(n)}\) iff \(t_{\text{rel}}^{(n)} \ll \alpha^{(n)}\).

The total-variation mixing time is given by \(t_{\text{mix}}^{TV} := \inf\{t : \max_{x} \sum_{y} |H_{t}(x, y) - \pi(y)| \leq \frac{1}{2}\}\).

It follows from Theorem 1 that the condition \((t_{\text{mix}}^{(\infty)})^{(n)} \ll t_{\text{hit}}^{(n)}\) as well as the condition \((t_{\text{mix}}^{TV})^{(n)} \ll t_{\text{hit}}^{(n)}\) are robust under rough-isometries, a fact which a-priori is entirely non-obvious.\(^{[3]}\) This is in contrast with the spectral condition \(t_{\text{rel}}^{(n)} \ll (t_{\text{mix}}^{TV})^{(n)}\) [11, Thm. 3].

As we now explain, Theorem 1 resolves a conjecture of Aldous and Fill [5, Open Problem 14.12] (re-iterated more recently by Aldous [1, Open Problem 5]). The conjecture asserts that for a sequence of vertex-transitive graphs, the condition that \(t_{\text{rel}}^{(n)} \ll t_{\text{hit}}^{(n)}\) implies mean-field behavior for the coalescing time \(\tau_{\text{coal}}\) of coalescing random walks. The term mean-field behavior here means that if \(t_{\text{meet}}\) is the “meeting-time”, which is defined as the expected collision time of two independent walks started each at equilibrium, then the law of \(\tau_{\text{coal}}/t_{\text{meet}}\) converges in distribution (as the index of the graph diverges) to the corresponding limit for the complete graph on \(n\) vertices, which is the law of the coalescing time in Kingman’s coalescence.

Oliveira [15] verified this (for vertex-transitive graphs) under the seemingly stronger condition \((t_{\text{mix}}^{TV})^{(n)} \ll t_{\text{hit}}^{(n)}\) (see the two comments at the top of p. 3423 in [15]). However, Theorem 3 asserts that this condition is in fact equivalent to the condition \(t_{\text{rel}}^{(n)} \ll t_{\text{hit}}^{(n)}\). We strongly believe that by combining Oliveira’s [15] methodology with the one from [8] it is possible to show that for a sequence of reversible chains on finite state spaces \(\Omega_{n}\) (of diverging sizes) with stationary distributions \(\pi_{n}\) satisfying that \(\max_{x \in \Omega_{n}} \pi_{n}(x) \approx \min_{x \in \Omega_{n}} \pi_{n}(x)\) and \(\min_{x \in \Omega_{n}} \mathbb{E}_{\pi_{n}}[T_{x}] \approx \max_{x \in \Omega_{n}} \mathbb{E}_{\pi_{n}}[T_{x}]\), the condition that \(t_{\text{rel}}^{(n)} \ll t_{\text{hit}}^{(n)}\) implies mean-field behavior for the coalescing time of coalescing random walks. (Crucially, one can show that such a sequence satisfies \(t_{\text{meet}}^{(n)} \ll t_{\text{hit}}^{(n)}\).

Theorem 1 is a consequence of a more quantitative result (Proposition 3.1). The idea of the proof is to study the mixing time as an optimization problem, with the variables substituting the eigenvalues of \(I - P\). The variables are thus constrained to be as large as the spectral-gap and satisfy other constraints coming from expressing hitting times in terms of the eigenvalues. The same is done in the proofs of the results in the next section.

1.1 Hitting and intersection times for “critical” branching random walk

A branching random walk (BRW) with rate \(\gamma\) (think of \(\gamma\) as the spectral-gap) is a continuous-time process in which each particle splits into two particles at rate \(\gamma\), inde-

\(^{[3]}\) The fact that the maximal (expected) hitting time can change only by a bounded factor under a quasi isometry can be seen from the commute-time identity (e.g. [13, Eq. (10.14)]) combined with the robustness of the effective-resistance under quasi isometries (cf. the proof of Theorem 2.17 in [14]).
pendently of the rest of the particles. Each particle performs a rate 1 continuous-time random walk corresponding to some transition matrix $P$, which we assume to be reversible w.r.t. $\pi$, independently of the rest of the particles. We consider the case that initially there is a single particle whose initial distribution is the stationary distribution $\pi$.

Theorem 1 and Proposition 3.1 explore the relation between hitting-times, mixing-times and the relaxation-time. The hitting-times of a single state are often much larger than the mixing-time (mixing-times are in fact equivalent to hitting-times of large sets [3, 6, 18, 16, 9, 10]), and so it is interesting to relate hitting times of a BRW with $\gamma = \text{gap}$ to mixing-times. As we now explain, the choice $\gamma = \text{gap}$ is natural. With this choice, the number of particles grow by a constant factor every $1/\gamma$ time units. The analog of $1 - \gamma$ for infinite irreducible reversible chains on a countable state space is the spectral-radius $\rho$ (see e.g., [14, §6.2]). It is classical that $\rho \leq 1$ and that when $\rho = 1$ a branching random walk with offspring distribution of mean $\mu > 1$ is recurrent, while when $\rho < 1$ the critical mean offspring distribution for recurrence (on survival) of a branching random walk is $\mu_c = 1/\rho$ (e.g. [12]). At $\mu_c$ the number of particle grows by a constant factor every $1/(1 - \rho)$ time units. Since $1/\gamma$ is the finite setup analog of $1/(1 - \rho)$, we may interpret our BRW as a “critical BRW”. It is thus less surprising that such a BRW has interesting connections with the mixing time of the chain.

Let $T_x$ be the first time at which state $x$ is visited by a particle. The intersection-time of two independent BRWs as above (with independent initial distributions, sampled from $\pi$), denoted by $\mathcal{I}$, is defined to be the first time $t$ at which a particle from one of the two processes visits a state which was previously visited by a particle from the other process.

The $L_2$ mixing time, started from initial state $x$, is defined to be $t_{\text{mix}}(x) := t_{\text{mix}}^2(x, 1/2)$, where $t_{\text{mix}}^2(\varepsilon) := \inf\{t : H_{2t}(x, x) - \pi(x) - \varepsilon^2 \}$ (see (2.1)-(2.2)). We write

$$
\rho_x := \int_0^{t_{\text{vel}}^2 + 2t_{\text{mix}}^2} \frac{s(H_t(x, x) - \pi(x))}{\pi(x)} ds,
$$

where $a \wedge b := \min\{a, b\}$, and $\rho_{\max} := \max_x \rho_x$ and $\rho_{\min} := \min_x \rho_x$.

We denote the size of the state space by $n$ and write

$$
Q := \sum_{i=2}^{n} \frac{1}{\lambda_i^2},
$$

where as above $0 = \lambda_1 < \lambda_2 = \text{gap} \leq \cdots \leq \lambda_n \leq 2$ are the eigenvalues of $I - P$.

Lastly, we say a Markov chain on a finite state space $V$ with transition matrix $P$ is transitive if for every $x, y \in V$ there is a bijection $f : V \rightarrow V$ such that $f(x) = y$ and $P(x, z) = P(y, f(z))$ for all $z \in V$. The following theorem and Corollary 1.1 refine Theorem 1. The implicit constants below are all independent of the choice of $P$.

[4] We can treat other variants, including working in discrete-time and/or having a random offspring distribution supported on $\mathbb{N} := \{1, 2, \ldots\}$. The important thing is that the number of particles at time $t$ grows like $\exp(\Theta(t \cdot \text{gap}))$. See Remark 1.1 for more details.
**Theorem 2.** In the above setup, with $\gamma$ taken to be the spectral-gap of $P$, we have that

$$t_{\text{mix}}^{(\infty)} \lesssim t_{\text{rel}} \log(1 + t_{\text{hit}}/t_{\text{rel}}) \simeq \max_x E_{\text{BRW}}[T_x],$$

(1.1)

Moreover, there exist absolute constants $C_0, C_1$ such that (uniformly in $x$ and $P$)

$$t_{\text{rel}} \log(E_{\pi}[T_x]/t_{\text{rel}}) 1_{\{E_{\pi}[T_x] \geq C_0 t_{\text{rel}}\}} \lesssim E_{\text{BRW}}[T_x] \lesssim t_{\text{mix}} + t_{\text{rel}} \log(1 + E_{\pi}[T_x]/t_{\text{rel}}),$$

(1.2)

$$E_{\text{BRW}}[\mathcal{I}] \lesssim t_{\text{rel}} \log(1 + \text{gap} \sqrt{\rho_{\max}}),$$

(1.3)

and

$$E_{\text{BRW}}[\mathcal{I}] \gtrsim t_{\text{rel}} \log(1 + \text{gap} \sqrt{\rho_{\min}}) 1_{\{\rho_{\min} \geq C_1 t_{\text{rel}}^2\}}.$$  

(1.4)

If $P$ is also transitive we have that

$$t_{\text{mix}}^{(\infty)} \lesssim t_{\text{rel}} \log(1 + \sqrt{Q}/t_{\text{rel}}) \simeq E_{\text{BRW}}[\mathcal{I}].$$

(1.5)

**Remark 1.1.** We note that we could have assumed that at rate $\gamma \geq \text{gap}$ each particle splits into a random number of particles with mean $\mu \geq 1$ such that $\mu - 1 \simeq \text{gap}/\gamma$ and with a finite second moment $\hat{\mu}$. The above bounds still hold, with the implicit constants depending only on $(\mu - 1)\gamma/\text{gap}$ and $\hat{\mu}$. Similarly, we could have worked in discrete-time and make the offspring distribution $\nu$ of each particle be supported on $\mathbb{N} := \{1, 2, \ldots\}$. In this setup, at each step each particle makes a step according to $P$ (independently of the rest of the particles), then gives birth to a random number of offspring (with law $\nu$, independently of the rest of the particles) and then vanishes. If the mean of $\nu$ is $1 + \Theta(\text{gap})$ then the same bounds as above hold (up to a constant factor), with $t_{\text{rel}}$ replaced by the absolute relaxation-time, which is $\max\{1/|\lambda| : \lambda \neq 1 \text{ is an eigenvalue of } P\}$.

**Remark 1.2.** It is natural to consider the case where for $T_x$ the starting point of the BRW is a worst-case starting state, rather than stationary. Likewise, for $\mathcal{I}$ it is natural to consider the case that the two BRWs start from a worst pair of initial states. It is easy to reduce the setup of worst-case starting point(s) to the setup of stationary starting point(s), by starting with a burn-in period of duration $\Omega(t_{\text{mix}}^{TV})$. Indeed, by the following proposition the upper bounds in Theorem 2 are all $\Omega(t_{\text{mix}}^{TV})$, so allowing such a burn-in period does not increase their order.

We believe that $E_{\text{BRW}}[\mathcal{I}] \simeq t_{\text{rel}} \log(1 + \sqrt{Q}/t_{\text{rel}})$ whenever $Q \simeq \rho_{\max}$. A weaker statement that appears to not require much additional work is that this holds whenever $\rho_{\min} \simeq \rho_{\max}$.

**Proposition 1.1.** In the above notation and setup (where $P$ is reversible) we have that

$$\forall x, \quad t_{\text{mix}}^{(2),x} \lesssim t_{\text{rel}} \log (1 + \text{gap} \sqrt{\rho_x}),$$

(1.6)

$$t_{\text{ave-mix}}^{(2)} \leq t_{\text{rel}} \log (1 + e^{c-1} \sqrt{Q}/t_{\text{rel}}),$$

(1.7)

$$Q \simeq \sum_x \pi(x) \rho_x.$$  

(1.8)
Let \((X_t)_{t \geq 0}\) and \((Y_t)_{t \geq 0}\) be two independent realizations of the rate 1 continuous-time Markov chain corresponding to transition matrix \(P\). The intersection-time is defined as
\[
I := \inf \{ t : X_t \in \{ Y_s : s \in [0, t] \} \text{ or } Y_t \in \{ X_s : s \in [0, t] \} \}.
\]
It is shown in [19] that under reversibility\(^5\)
\[
t^{TV}_{mix} \leq t_1 := \max_{x,y} \mathbb{E}[I \mid X_0 = x, Y_0 = y],
\]
and that if in addition \(P\) is also transitive then
\[
t_1 \simeq \mathbb{E}_{\pi,\pi}[I] \simeq \sqrt{Q}, \tag{1.9}
\]
where \(\mathbb{E}_{\pi,\pi}\) is the expectation when \(X_0 \sim \pi\) and \(Y_0 \sim \pi\) (independently). It is shown in [19] (Lemma 3.7) that in the transitive reversible setup \(t^{(\infty)}_{mix}(1/4) \leq 2\sqrt{Q}\). Theorem 2 (namely (1.5)) offers a substantial improvement in the transitive reversible setup.

Using (1.9), the following corollary is an immediate consequence of (1.5).

**Corollary 1.1.** *In the above setup, for a sequence of reversible transitive chains:*
\[
t_{rel}^{(n)} \ll t_1^{(n)} \text{ if and only if } (t^{(\infty)}_{mix})^{(n)} \ll t_1^{(n)}.
\]

### 1.2 Organization of this note

In §2 we introduce notation and definitions. In §3 we prove a refined version of Theorem 1 (Proposition 3.1) and Proposition 1.1 (whose proof is similar to that of Proposition 3.1). In §4 we prove Theorem 2.

### 2 Preliminaries and notation

Let \((X_t)_{t=0}^{\infty}\) be an irreducible reversible Markov chain on a finite state space \(V\) with transition matrix \(P\) and stationary distribution \(\pi\). Denote the law of the continuous-time rate 1 version of the chain starting from vertex \(x\) (resp. initial distribution \(\mu\)) by \(P_x\) (respectively, \(P_\mu\)). Denote the corresponding expectation by \(E_x\) (respectively, \(E_\mu\)). For further background on mixing and hitting times see [5, 13].

\(^5\)In [19] discrete-time lazy chains are considered. However their analysis extends to the continuous-time setup.
2.1 Mixing times and $L_p$ norms

The $L_p$ norm and variance of a function $f \in \mathbb{R}^V$ are $\|f\|_p := (\mathbb{E}_x[|f|^p])^{1/p}$ for $1 \leq p < \infty$ (where $\mathbb{E}_x[h] := \sum_x \pi(x)h(x)$ for $h \in \mathbb{R}^V$) and $\|f\|_\infty := \max_x |f(x)|$. The $L_p$ norm of a signed measure $\sigma$ on $V$ is

$$\|\sigma\|_{p,\pi} := \|\sigma/\pi\|_p, \text{ where } (\sigma/\pi)(x) = \sigma(x)/\pi(x).$$

We note that the worst-case $L_p$ distance at time $t$ by $d_p(t) := \max_x d_{p,x}(t)$, where $d_{p,x}(t) := \|P_t^x - \pi\|_{p,\pi}$. Under reversibility for all $x \in V$ and $t \geq 0$ (e.g. [13, Prop. 4.15]) we have that
d

$$d_{2,x}^2(t) = \frac{H_v(x,x)}{\pi(x)} - 1, \quad d_{\infty}(t) := \max_{x,y} \left| \frac{H_v(x,y)}{\pi(y)} - 1 \right| = \max_y \frac{H_v(y,y)}{\pi(y)} - 1. \quad (2.1)$$

The $\varepsilon L_p$ mixing time of the chain (respectively, for initial state $x$) is defined as

$$t_{\text{mix}}^{(p)}(\varepsilon) := \min \{ t : d_p(t) \leq \varepsilon \},$$

$$t_{\text{mix}}^{(p,x)}(\varepsilon) := \min \{ t : d_{p,x}(t) \leq \varepsilon \}. \quad (2.2)$$

When $\varepsilon = 1/2$ we omit the above notation. The $\varepsilon$ total variation mixing time is defined as $t_{\text{mix}}^{(\text{TV})}(\varepsilon) := t_{\text{mix}}^{(1)}(2\varepsilon)$. We write $t_{\text{mix}}^{(\text{TV})} := t_{\text{mix}}^{(1)}(\varepsilon)$. Clearly, $t_{\text{mix}}^{(p)}$ is non-decreasing in $p$. Finally, we define the average $\varepsilon L_2$ mixing time as

$$t_{\text{ave-mix}}^{(\varepsilon)}(\varepsilon) := \min \{ t : \sum_{v \in V} \pi(v)d_{2,v}^2(t) \leq \varepsilon^2 \}$$

(by (2.1))

$$= \min \{ t : \sum_{v \in V} H_v(v, v) \leq 1 + \varepsilon^2 \} = \min \{ t : \sum_{i=2}^{|V|} e^{-2\lambda_i t} \leq \varepsilon^2 \}. \quad (2.3)$$

We now recall the hierarchy between the various quantities considered above. Under reversibility we have that (e.g. [13, Theorems 10.22, 12.4, 12.5 and Lemmas 4.18 and 20.5])

$$\forall \varepsilon \in (0, 1), \quad \frac{1}{\lambda_2^2} \log \varepsilon \leq t_{\text{mix}}^{(\text{TV})}(\varepsilon/2) \leq t_{\text{mix}}^{(2)}(\varepsilon) = \frac{1}{2} t_{\text{mix}}^{(\text{TV})}(\sqrt{\varepsilon}) \leq \frac{1}{2\lambda_2} \log(\varepsilon \min_x \pi(x)), \quad (2.4)$$

$$t_{\text{mix}}^{(\infty)} \leq 9t_{\text{hit}}. \quad (2.5)$$

2.2 Hitting-times

We now present some background on hitting times. The random target identity (e.g. [13, Lemma 10.1]) asserts that $\alpha := \sum_y \pi(y)\mathbb{E}_x[T_y]$ is independent of $x$, while for all $x \in V$ we have that (e.g. [13, Proposition 10.26])

$$\alpha_x := \mathbb{E}_x[T_x] = \frac{1}{\pi(x)} \sum_{i=0}^{\infty} (P^i(x,x) - \pi(x)) = \frac{1}{\pi(x)} \int_0^{\infty} (H_t(x,x) - \pi(x)) \, dt. \quad (2.6)$$

[6]Recall that the total-variation distance is $\|\mu - \nu\|_{\text{TV}} := \frac{1}{2} \|\mu - \nu\|_{1,\pi}$. 

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Averaging over $x$ yields the eigentime identity ([5, Proposition 3.13])
\[
\alpha = \sum_{x,y} \pi(x)\pi(y)\mathbb{E}_x[T_y] = \sum_{y} \sum_{i=0}^{\infty} (P^i(y, y) - \pi(y)) = \sum_{i=0}^{\infty} [\text{Trace}(P^i) - 1] = \sum_{i \geq 2} \frac{1}{\lambda_i}. \tag{2.7}
\]

Let $U \sim \pi$ be independent of the chain. Noting that $T_x \leq T_U + \inf\{t : X_{t+T_U} = x\}$ and using the random target lemma to argue $\mathbb{E}[T_U] = \alpha$, as well as the strong Markov property to argue $\mathbb{E}[\inf\{t : X_{t+T_U} = x\}] = \mathbb{E}_x[T_x] =: \alpha_x$, yields:

**Fact 2.1** ([13] Lemma 10.2), $\max_x \alpha_x \leq t_{\text{hit}} \leq \alpha + \max_x \alpha_x \leq 2 \max_x \alpha_x$.

The following material can be found at [5, §3.5]. Under reversibility, for any set $A$ the law of its hitting time $T_A := \inf\{t : X_t \in A\}$ under initial distribution $\pi$ conditioned on $A^c$, is a mixture of Exponential distributions, whose minimal parameter $\lambda(A)$ is the Dirichlet eigenvalue of the set $A^c$. There exists a distribution $\mu_A$, known as the quasi-stationary distribution of $A^c$, under which $T_A$ has an Exponential distribution of parameter $\lambda(A)$. It follows that $\lambda(A) \geq \frac{1}{\max_x \mathbb{E}_x[T_A]}$. We see that for all $t \geq 0$,
\[
P_{\pi}[T_y > t] \leq \exp(-t/t_{\text{hit}}), \quad \text{and so} \quad \mathbb{E}_{\pi}[T_y^2] \leq 2t_{\text{hit}}^2. \tag{2.8}
\]

Using the above description of the law of $T_A$ it is not hard to show [5, p. 86] that
\[
\forall s, t \geq 0, \quad P_{\pi}[T_y > t + s \mid T_y \geq s] \geq P_{\pi}[T_y > t]. \tag{2.9}
\]

It follows from the spectral decomposition (e.g., [13, §12.1]) that for all $x$ and all $s, t \geq 0$ we have that
\[
0 < H_{t+s}(x, x) - \pi(x) \leq e^{-s/t_{\text{hit}}}(H_t(x, x) - \pi(x)). \tag{2.10}
\]

This easily implies the following lemma:

**Lemma 2.1.** For every irreducible, reversible Markov chain on a finite state space with a stationary distribution $\pi$, for every state $x$ and all $M > 0$ we have that
\[
\int_0^{\infty} (H_s(x, x) - \pi(x))ds \leq \frac{e^M}{e^M - 1} \int_0^{M_{\text{rel}}} (H_s(x, x) - \pi(x))ds. \tag{2.11}
\]
\[
\int_0^{\infty} s(H_s(x, x) - \pi(x))ds \leq \left(1 + \sum_{i=1}^{\infty} (i + 1)e^{-iM}\right) \int_0^{2M_{\text{rel}}} s(H_s(x, x) - \pi(x))ds. \tag{2.12}
\]

**Proof.** By (2.10) $\frac{f(i)}{\int_0^i} \leq e^{-M_i}$, where $f(i) := \int_{iM_{\text{rel}}}^{(i+1)M_{\text{rel}}} (H_s(x, x) - \pi(x))ds$. This easily implies (2.11). Likewise writing $g(i) := \int_{iM_{\text{rel}}}^{(i+1)M_{\text{rel}}} s(H_s(x, x) - \pi(x))ds$, (2.12) follows from $\frac{g(i)}{g(1)} \leq (i + 1)e^{-M_i}$ (which again follows from (2.10)).
3 Proof of Theorem 1 and Proposition 1.1

Recall that \( \alpha_x := \mathbb{E}_\pi[T_x] \).

**Proposition 3.1.** For every irreducible reversible Markov chain on a finite state space \( V \)
\[
\forall x \in V, \varepsilon \in (0, \frac{1}{2}], \quad t^{(2),x}_{\text{ave-mix}}(\varepsilon) \leq \begin{cases} \frac{1}{2\text{gap}} \log[\varepsilon^{-2}\alpha_x \text{ gap}] & \text{if } \alpha_x \text{ gap} \geq e\varepsilon^2 \\ \min\left\{ \frac{1}{2\text{gap}} \log[\varepsilon^{-2}\alpha_x \text{ gap}], \frac{\alpha_x \text{ gap}}{2\varepsilon^2} \right\} & \text{otherwise} \end{cases}
\]

The assertion of Theorem 1 follows at once from Proposition 3.1 by considering \( x \) such that \( t^{(2),x}_{\text{ave-mix}}(\frac{1}{2}) = \frac{1}{2} t^{(\infty)}_{\text{ave-mix}}(\frac{1}{2}) \) (such \( x \) exists by (2.1)).

**Proof.** We first prove (3.2). The inequality \( t^{(2),x}_{\text{mix}}(\varepsilon) \leq \frac{1}{2x^2} \alpha_x \) holds for all \( x \) for every irreducible reversible Markov chain, as a consequence of the general inequality
\[
\frac{\alpha_x}{t} \geq \frac{1}{t} \int_0^t \frac{H_s(x,x) - \pi(x)}{\pi(x)}ds \geq \frac{H_t(x,x)}{\pi(x)} - 1 = d^2_{2,x}(t/2),
\]
see [13, p. 144]. We now show that \( t^{(2),x}_{\text{mix}}(\varepsilon) \leq \frac{1}{2\text{gap}} \log[\varepsilon^{-2}\alpha_x \text{ gap}] \) when \( \alpha_x \lambda_2 \geq e\varepsilon^2 \). Let \( f_1 = 1, f_2, \ldots, f_n \) be an orthonormal basis of \( \mathbb{R}^V \) w.r.t. \( I - P \) corresponding to \( \lambda_1 = 0 < \lambda_2 \leq \cdots \lambda_n \leq 2 \). By (2.1) and the spectral decomposition (e.g. [13, §12.1])
\[
d^2_{2,x}(t) = \frac{H_{2t}(x,x)}{\pi(x)} - 1 = \sum_{i=2}^n f_i^2(x)e^{-2\lambda_i t}. \tag{3.3}
\]

Recall that by (2.6) \( \sum_{i=2}^n \frac{f_i(x)}{\lambda_i} = \int_0^\infty (\frac{H_t(x,x)}{\pi(x)} - 1)dt = \alpha_x \). We now fix \( t := \frac{1}{2\lambda_2} \log(\varepsilon^{-2}\alpha_x \lambda_2) \). The r.h.s. of (3.3) is clearly bounded by the value of the solution to the optimization problem:
\[
\max \sum_{i=1}^{n-1} a_i e^{-2\beta_i t},
\]
subject to the conditions
\[
\begin{align*}
(1) \quad & \sum_{i=1}^{n-1} \frac{a_i}{\beta_i} = \alpha_x, \\
(2) \quad & a_i \geq 0 \text{ for all } i, \text{ and} \\
(3) \quad & \lambda_2 \leq \beta_i \leq \infty \text{ for all } i.
\end{align*}
\]
Observe that we may restrict to solutions of the form $\beta_1 < \beta_2 < \cdots < \beta_i < \infty = \beta_{i+1} = \cdots = \beta_{n-1}$ (for some $1 \leq i \leq n-2$), as if $\beta_i = \beta_j$ for some $i < j$ we can set $a'_i = a_i + a_j$ and $\beta'_i = \beta_i$, while $\beta'_j = \infty$. Doing so repeatedly, we eventually obtain a solution of the desired form (up to a permutation of the indices) with the same value as the original one.

We argue that the maximum is attained when $\beta_1 = \lambda_2$ and $a_1 = \alpha_x \lambda_2$, while $a_2/\beta_2 = \cdots = a_{n-1}/\beta_{n-1} = 0$. To see this, first note that by a simple Lagrange multipliers calculation one gets that for any maximizer if $\beta_i, \beta_j \notin \{\lambda_2, \infty\}$ and $\min\{a_i, a_j\} > 0$ then $\beta_i = \beta_j$ (as can be seen by considering the derivatives w.r.t. $a_i$ and $a_j$).

We now argue that if $A \geq 0$, $B > 0$ and $b > \lambda_2$ satisfy that $\frac{A}{\lambda_2} + \frac{B}{b} =: D \leq \alpha_x$, then (for $t$ as above) $A e^{-2\lambda_2 t} + Be^{-2bt} < \lambda_2 De^{-2\lambda_2 t}$ (the r.h.s. corresponds to the case $B = 0$ and $A = D \lambda_2$). Indeed, after rearranging and simplifying this is equivalent to the inequality $be^{-2bt} < \lambda_2 e^{-2\lambda_2 t}$, which indeed holds as $h(x) = x e^{-2tx}$ is decreasing in $[\frac{1}{2}, \infty)$ and $b > \lambda_2 \geq \frac{1}{2t}$ (by the assumption $\alpha_x \lambda_2 \geq e\varepsilon^2$). This shows that in any optimal solution (such that $\beta_i < \beta_j$ for all $i < j$ such that $\beta_i < \infty$), we must have $\beta_1 = \lambda_2$, $a_1 = \alpha_x \lambda_1$ and $a_2/\beta_2 = \cdots = a_{n-1}/\beta_{n-1} = 0$. (Indeed, by the above if $\beta_i \in (\lambda_2, \infty)$ and $a_i > 0$, setting $\beta'_i = \lambda_2$ and $a'_i = a_i \lambda_2 / \beta_i$ while keeping the values of the rest of the $a$’s and $\beta$’s the same, yields a new solution with a larger value.) Substituting our choice of $t$ we see that the maximum is at most $\varepsilon^2$, as desired.

Finally, if $\alpha_x \varepsilon \leq \varepsilon^2$ then $\alpha_x < t_{rel} < \alpha$, and so the inequality

$$t^{(2),x}_{\text{mix}}(\varepsilon) \leq t^{(2),x}_{\text{mix}}(\varepsilon^{\sqrt{\frac{\alpha}{\alpha_x}}}) \leq \frac{1}{2 \varepsilon} \log(\varepsilon^{-2} \alpha \varepsilon^2) =: t'$$

follows by considering the same optimization problem as above, with $t'$ instead of $t$ (noting that $\lambda_2 \geq \frac{1}{2t}$).

We now prove (3.1). By (2.3) we need to verify that $\sum_{i=2}^{n} e^{-2\lambda_2 i} \leq \frac{1}{4}$ for $t := \frac{1}{2\lambda_2} \log(4 \alpha \lambda_2)$. Recall that $\sum_{i=2}^{n} \frac{1}{\lambda_i} = \alpha$ and so as above $\lambda_2 \geq (2t)^{-1}$. The proof is almost identical to that of (3.2). Namely, we consider the same optimization problem as above with $\alpha_x$ in constraint (1) replaced with $\alpha$ and with $t$ replaced by $t$. We leave the details as an exercise. □

**Proof of Proposition 1.1.** We first prove that

$$\frac{2}{e} \leq Q/\sum_{x} \pi(x) \rho_x \leq 1 + \sum_{i=1}^{\infty} (i+1) e^{-i/2} =: x.$$

Using $\int_{0}^{\infty} \lambda s e^{-\lambda s} ds = \frac{1}{\lambda}$ for $\lambda > 0$, and changing the order of summation and integration (twice) yields that

$$Q = \sum_{i=2}^{n} \int_{0}^{\infty} s e^{-\lambda s} ds = \int_{0}^{\infty} s (\text{Trace}(H_s) - 1) ds = \sum_{x} \int_{0}^{\infty} s (H_s(x, x) - \pi(x)) ds.$$

We write $t_x := t^{(2),x}_{\text{mix}}$. By (2.12)

$$\frac{1}{x} \int_{0}^{\infty} s (H_s(x, x) - \pi(x)) ds \leq \int_{t_{\text{rel}}}^{t_{x}} s (H_s(x, x) - \pi(x)) ds = \pi(x) \left( \rho_x - \frac{t_{x}^{2} - 4t_{x}^{2}}{2} 1\{2t_{x} < t_{\text{rel}}\} \right).$$
It follows that $Q \leq \int_0^\infty s(H(s, x, x) - \pi(x)) ds \geq \int_0^{t_{rel}} s(H(s, x, x) - \pi(x)) ds$, by the above $Q \geq \sum_x \pi(x) \rho_x - t_{rel}^2/2$. Hence $Q \geq \frac{2}{3} \sum_x \pi(x) \rho_x$ (as $Q \geq t_{rel}^2$).

We now prove (1.6), i.e., that $t_{mix}^{(2), x} \lesssim t_{rel} \log(1 + \sqrt{\rho_x} / t_{rel})$. We begin by noting that the claim is trivial if $\max\{2t_{mix}^{(2), x}, \sqrt{\rho_x}\} < t_{rel}$, as by the definition of $\rho_x$ (and the fact that $\frac{(H(s, x, x) - \pi(x))}{\pi(x)} \geq 1/4$ for all $s \leq 2t_{mix}^{(2), x}$) we have that $\rho_x \geq \frac{1}{4} \int_0^{t_{rel}} \sqrt{2t_{mix}^{(2), x}} s ds = \frac{1}{8} (t_{rel} \wedge 2t_{mix}^{(2), x})^2$.

Thus if $\max\{2t_{mix}^{(2), x}, \sqrt{\rho_x}\} < t_{rel}$ we have that $t_{rel} \log(1 + \sqrt{\rho_x} / t_{rel}) \leq \sqrt{\rho_x} \geq \frac{1}{2} t_{mix}^{(2), x}$, as desired. For the remainder of the proof we consider the case that $t_{rel} < \max\{2t_{mix}^{(2), x}, \sqrt{\rho_x}\}$.

If $2t_{mix}^{(2), x} < t_{rel} \leq \sqrt{\rho_x}$, then using the fact that $H_x(x, x)$ is non-decreasing in $s$ (and so $\frac{(H(s, x, x) - \pi(x))}{\pi(x)} \leq 1/4$ for $s \geq 2t_{mix}^{(2), x}$)

$$
\sigma_x := \int_0^{t_{rel}} \frac{s(H(s, x, x) - \pi(x))}{\pi(x)} ds \leq \rho_x + \int_0^{t_{rel}} \frac{s ds}{4} \leq \frac{9}{8} \rho_x,
$$

whereas if $t_{rel} \leq 2t_{mix}^{(2), x}$, then $\rho_x = \sigma_x$. Hence, it suffices to show that

$$
t_{mix}^{(2), x} \lesssim t_{rel} \log(1 + \sqrt{\sigma_x} / t_{rel}).
$$

Next, we argue that it suffices to consider the case that $\sigma_x \leq \min\{\alpha_x^2, \alpha_x t_{rel}\}$. First consider the case that $\sqrt{\sigma_x} < t_{rel}$, in which case we need to show that $t_{mix}^{(2), x} \lesssim \sqrt{\sigma_x}$. If $\alpha_x t_{rel} < \sigma_x$ and $\alpha_x < et_{rel}$ then (3.2) implies that $t_{mix}^{(2), x} \lesssim \alpha_x < \sqrt{e \alpha_x t_{rel}} < \sqrt{e \sigma_x}$, while if $\alpha_x t_{rel} < \sigma_x$ and $\alpha_x \geq et_{rel}$ then (3.2) together with the fact that $\log(1 + x) \leq x$ for $x > -1$, imply that

$$
t_{mix}^{(2), x} \lesssim t_{rel} \log(\alpha_x / t_{rel}) = 2t_{rel} \log(\sqrt{\alpha_x / t_{rel}}) \leq 2 \sqrt{\alpha_x t_{rel}} < 2 \sqrt{\sigma_x}.
$$

If $\alpha_x^2 < \sigma_x$ and $\alpha_x < et_{rel}$ then (3.2) implies that $t_{mix}^{(2), x} \lesssim \alpha_x < \sqrt{\sigma_x}$, while if $\alpha_x^2 < \sigma_x$ and $\alpha_x \geq et_{rel}$ then (3.2), together with the fact that $\log(1 + x) \leq x$ for $x > -1$, imply that

$$
t_{mix}^{(2), x} \lesssim t_{rel} \log(\alpha_x / t_{rel}) \leq \alpha_x < \sqrt{\sigma_x}.
$$

Likewise, when $\sqrt{\sigma_x} \geq t_{rel}$ (in which case, we need to show that $t_{mix}^{(2), x} \lesssim t_{rel} \log(e \sqrt{\sigma_x} / t_{rel}))$ we may again assume that $\sigma_x \leq \alpha_x t_{rel}$ (which as $\sqrt{\sigma_x} \geq t_{rel}$ implies that $\sigma_x \leq \alpha_x^2$). Indeed, if $\sqrt{\sigma_x} \geq t_{rel}$, $\alpha_x t_{rel} < \sigma_x$ and $\alpha_x \geq et_{rel}$ then (3.2) implies that

$$
t_{mix}^{(2), x} \lesssim t_{rel} \log(\alpha_x / t_{rel}) < t_{rel} \log(\sigma_x / t_{rel}^2) = 2t_{rel} \log(\sqrt{\sigma_x} / t_{rel}),
$$

while if $\sqrt{\sigma_x} \geq t_{rel}$, $\alpha_x t_{rel} < \sigma_x$ and $\alpha_x < et_{rel}$ then (3.2) implies that

$$
t_{mix}^{(2), x} \lesssim \alpha_x < et_{rel} \leq et_{rel} \log(e \sqrt{\sigma_x} / t_{rel}).
$$

This concludes the proof of the fact that we may assume that $\sigma_x \leq \min\{\alpha_x^2, \alpha_x t_{rel}\}$. 

11
Recall that by the spectral-decomposition we have
\[ d_{2,x}^2(t) = H_{2t}(x, x)/\pi(x) - 1 = \sum_{i=2}^{n} f_i^2(x)e^{-2\lambda_i t}. \]

Recall that \( \kappa = 1 + \sum_{i=1}^{\infty} (i + 1)e^{-i/2} \). By (2.12) we have that
\[ \kappa\sigma_x = \kappa \int_{0}^{t_{rel}} s(H_s(x, x)/\pi(x) - 1)ds \geq \int_{0}^{\infty} s(H_s(x, x)/\pi(x) - 1)ds = \sum_{i=2}^{n} \frac{f_i^2(x)}{\lambda_i^2}. \]

Set \( t := C\left[ t_{rel}\log(e\sqrt{\sigma_x/t_{rel}})1_{(\sigma_x > t_{rel}^2)} + 1_{(\sigma_x < t_{rel}^2)}\sqrt{\sigma_x} \right] \), for some \( C > 0 \) to be determined later. It suffices to show that \( d_{2,x}^2(t) \leq 1/4 \), provided that \( C \) is sufficiently large. Observe that \( d_{2,x}^2(t) \) is bounded by the value of the solution to the optimization problem:
\[ \max \sum_{i=1}^{n-1} a_i e^{-2\beta_i t}, \]
subject to the conditions
\begin{enumerate}
  \item \( \sum_{i=1}^{n-1} \frac{a_i}{\beta_i} = \alpha_x \),
  \item \( a_i \geq 0 \) for all \( i \),
  \item \( \lambda_2 < \beta_i \leq \infty \) for all \( i \), and
  \item \( \sum_{i=1}^{n-1} \frac{a_i}{\beta_i} \leq \kappa\sigma_x \).
\end{enumerate}

As in the proof of Proposition 3.1, we may restrict to solutions of the form \( \beta_1 < \beta_2 < \cdots < \beta_i < \infty = \beta_{i+1} = \cdots = \beta_{n-1} \) (for some \( 1 \leq i \leq n - 2 \)).

We argue that the maximum is attained when
\[ \beta_1 = \max \{ \lambda_2, \kappa \}, \quad \text{where } \kappa := \frac{\alpha_x}{\kappa\sigma_x} \quad \text{and} \quad a_1 = \alpha_x \beta_1, \quad \text{while } a_2/\beta_2 = \cdots = a_{n-1}/\beta_{n-1} = 0. \]

First consider the case that \( \lambda_2 \geq \kappa \). By a simple Lagrange multipliers calculation one gets that for any maximizer if \( \beta_i, \beta_j \notin \{ \lambda_2, \infty \} \) and \( \min\{a_i, a_j\} > 0 \) then \( \beta_i = \beta_j \) (as can be seen by considering the derivatives w.r.t. \( a_i \) and \( a_j \); crucially by adjusting the values of \( \beta_i \) and \( \beta_j \), such \( (a_i, a_j) \) can be perturbed in all directions while keeping the conditions (1)-(4) and keeping the values of the rest of the \( a \)’s and the \( \beta \)’s unchanged).

We now argue that if \( A > 0, B > 0 \) and \( b > \lambda_2 \) satisfy that \( \frac{A}{\lambda_2} + \frac{B}{b} =: D \leq \alpha_x \), then (for \( t \) as above) \( Ae^{-2\lambda_2 t} + Be^{-2bt} < \lambda_2 De^{-2\lambda_2 t} \) (the r.h.s. corresponds to the case \( A = \lambda_2 D \) and \( B = 0 \)). Indeed, after rearranging and simplifying, this is equivalent to the
We now argue that if \( \lambda_2 \geq \kappa \), the optimal value \( \beta \) in any optimal solution (such that \( \beta \leq \lambda_2 \)) is at most \( \frac{\alpha_x}{\kappa \sigma_x} \) (as this is true also for \( \sigma_x \)).

The value of the maximum of the optimization problem (when \( \lambda_2 \geq \kappa \)) is thus \( \alpha_x \lambda_2 e^{-2\lambda_2 t} \).

First consider the case that \( \sigma_x \leq \frac{t_{rel}^2}{4\pi} \) and so \( t = C \sqrt{\sigma_x} \). Then as \( \lambda_2 \geq \kappa = \frac{\alpha_x}{\kappa \sigma_x} \) we have that
\[
\alpha_x \lambda_2 e^{-2\lambda_2 t} \leq \frac{\lambda_x^2 \sigma_x}{\pi} \exp \left( -\frac{1}{\Delta^2} C \log(8 \sigma_x \lambda_2^2) \right) \leq 1/4,
\]
provided \( C \) is sufficiently large.

We now deal with the case that \( \lambda_2 < \kappa = \frac{\alpha_x}{\kappa \sigma_x} \). By a simple Lagrange multipliers calculation one gets that for any maximizer, if \( \beta_i, \beta_j \neq \infty \), \( \min\{\beta_i, \beta_j\} > \kappa \) and \( \min\{a_i, a_j\} > 0 \), then \( \beta_i = \beta_j \). As (by assumption) the values of \( \beta_i \) and \( \beta_j \), such \( \alpha_x \)'s are unchanged.

We now argue that if \( A \geq 0, B > 0 \) and \( b > \kappa \) satisfy that \( \frac{A}{\kappa} + \frac{B}{\kappa} =: D \leq \alpha_x \), then (for \( t \) as above) \( Ae^{-2nt} \leq B \kappa e^{-2kt} \) (the l.h.s. corresponds to the case \( B = 0 \) and \( A = D \kappa \)). Indeed, after rearranging and simplifying, this is equivalent to the inequality \( be^{-2nt} < B \kappa e^{-2kt} \), which indeed holds if \( h(x) = xe^{-2tx} \) is decreasing in \( \left[ \frac{1}{2t}, \infty \right) \) and \( b > \kappa \geq \frac{1}{2t} \), provided \( C \) is sufficiently large. Indeed \( \kappa^2 = \left( \frac{\alpha_x}{\kappa \sigma_x} \right)^2 \geq \frac{1}{4e^2 \sigma_x} \), provided \( C \) is sufficiently large, as (by assumption) \( \alpha_x^2 \geq \sigma_x \), and so \( \kappa \geq \frac{1}{2t} \) when \( \sigma_x < t_{rel}^2 \) (in which case \( t = C \sqrt{\sigma_x} \)).

Likewise, if \( \sigma_x > t_{rel}^2 \), then \( t = \frac{1}{2} C t_{rel} \log(e^2 \sigma_x / t_{rel}^2) \) and
\[
\frac{1}{C t_{rel} \log(e^2 \sigma_x / t_{rel}^2)} \leq \frac{1}{2e C \sqrt{\sigma_x}} \leq \frac{\alpha_x}{\kappa \sigma_x} = \kappa,
\]
for sufficiently large \( C \), where in the last inequality we used the assumption that \( \sigma_x < \alpha_x^2 \).

This shows that in any optimal solution (such that \( \beta_i < \beta_j \) for all \( i < j \) such that \( \beta_i \geq \kappa \)) we must have \( a_i / \beta_i = 0 \), if \( \beta_i > \kappa \). The only solution to constraints (1)-(4) satisfying this, as well as \( \beta_1 < \beta_j \) for all \( i < j \) such that \( \beta_i \geq \kappa \), is \( \beta_1 = \kappa \), \( a_1 = \beta_1 \alpha_x \) and \( a_2 / \beta_2 = \cdots = a_{n-1} / \beta_{n-1} = 0 \) (for this solution we have \( a_1 / \beta_1 = \alpha_x \) and \( a_1 / \beta_1 \)).

Hence the value of the maximum of the optimization problem is
\[
\alpha_x \kappa e^{-2\kappa t} = \frac{\alpha_x^2}{\kappa \sigma_x} \exp \left( -\frac{2\alpha_x}{\kappa \sigma_x} t \right).
\]
If \( \sigma_x \geq t_{rel}^2 \), then \( t = C \sqrt{\sigma_x} \), and so using the assumption \( \alpha_x^2 \geq \sigma_x \) we see that the r.h.s.

is at most 1/4, provided \( C \) is sufficiently large. If \( \sigma_x \geq t_{rel}^2 \) then \( t = C t_{rel} \log(e \sqrt{\sigma_x} / t_{rel}) \),

and so using the assumption \( \alpha_x t_{rel} \geq \sigma_x \), as well as \( \sigma_x \geq t_{rel}^2 \), we see that, provided \( C \) is sufficiently large,

\[
\exp \left( \frac{-2\alpha_x}{\kappa \sigma_x} t \right) \leq \exp \left( -\frac{C \alpha_x t_{rel}}{\kappa \sigma_x} \log(e^2 \sigma_x / t_{rel}^2) \right)
\]

(using \( x \log y \leq \log x + \log y \) for \( y \geq e \) and \( x \geq 1 \), with \( x = \frac{\sqrt{C} \alpha_x t_{rel}}{\sigma_x} \) and \( y = \frac{e^2 \sigma_x}{t_{rel}^2} \))

\[
\leq \exp \left( -\frac{\sqrt{C}}{2\kappa} \left[ \log \left( \frac{\sqrt{C} \alpha_x t_{rel}}{\sigma_x} \right) + \log \left( e^2 \sigma_x / t_{rel}^2 \right) \right] \right)
\]

\[
\leq \left( t_{rel} / (e^2 \alpha_x) \right) \sigma_x \leq \left( t_{rel} / \alpha_x \right)^2 \times \frac{\kappa}{4}.
\]

As \( \sigma_x \geq t_{rel}^2 \), we see that \( (t_{rel} / \alpha_x)^2 \leq \frac{\sigma_x}{\alpha_x^2} \), and so \( \frac{\alpha_x^2}{\kappa \sigma_x} \exp \left( -\frac{2\alpha_x}{\kappa \sigma_x} t \right) \leq \frac{1}{4} \), as desired.

We now prove that \( t_{ave-mix}^{(2)} \leq \hat{t} := t_{rel} \log(1 + e^{e-1} \sqrt{Q} / t_{rel}) \). If \( \sqrt{Q} \leq et_{rel} \) then \( \hat{t} \geq \sqrt{Q} \)

(as \( \frac{\log(1+e^{e-1}x)}{x} \) is decreasing on \([1, e]\)) and so it suffices to show that \( t_{ave-mix}^{(2)} \leq \sqrt{Q} \). This follows from the proof of Lemma 3.7 in [19].\(^7\)

We may thus consider the case that \( \sqrt{Q} \geq et_{rel} \). Recall that \( \sum_{v \in V} \pi(v) d_{2,v}(\hat{t}) = \sum_{i=2}^{n} e^{-2\lambda_i \hat{t}} \).

Hence \( \sum_{v \in V} \pi(v) d_{2,v}(\hat{t}) \) is bounded by the value of the solution to the optimization problem:

\[
\max \sum_{i=1}^{n-1} a_i e^{-2\beta_i \hat{t}},
\]

subject to the conditions

1. \( \sum_{i=1}^{n-1} \frac{a_i}{\beta_i} = Q \),
2. \( a_i \geq 0 \) for all \( i \), and
3. \( \lambda_2 \leq \beta_i \leq \infty \) for all \( i \).

Using the same reasoning as in the proof of Proposition 3.1, using the fact that \( \lambda_2 \geq \frac{1}{2} \geq \frac{\lambda_2}{2 \log(1+e^e)} \) whenever \( \sqrt{Q} \geq et_{rel} \), it is not hard to verify that the maximum is attained when

\[
\beta_1 = \lambda_2, \quad \text{and} \quad a_1 = Q \lambda_2^2, \quad \text{while} \quad a_2 / \beta_2 = \cdots = a_{n-1} / \beta_{n-1} = 0.
\]

Thus \( \sum_{v \in V} \pi(v) d_{2,v}(\hat{t}) \leq Q \lambda_2^2 \exp(-2\lambda_2 \hat{t}) \). Substituting the value of \( \hat{t} \) concludes the proof. \( \square \)

\(^7\)While the assertion of Lemma 3.7 in [19] is that for transitive reversible irreducible Markov chains \( t_{mix}^{(\infty)}(1/4) \leq 2 \sqrt{Q} \), the proof effectively shows that \( t_{ave-mix}^{(2)} \leq \sqrt{Q} \) even without transitivity. As mentioned in footnote 5, their analysis extends to the continuous-time setup.
4 Hitting and intersection times for branching random walk -
Proof of Theorem 2

Proof. We first note that the first inequality in (1.1) follows from Proposition 3.1. We now
argue that the relation \( t_{\text{rel}} \log(1 + t_{\text{hit}}/t_{\text{rel}}) \preceq \max_x \mathbb{E}_{\text{BRW}}[\mathcal{T}_x] \) in (1.1) follows from (1.2):
\[
J_x \mathbf{1}_{\{\alpha_x \geq C_0 t_{\text{rel}}\}} \leq \mathbb{E}_{\text{BRW}}[\mathcal{T}_x] \preceq t_{\text{mix}}^\text{TV} + J_x,
\]
where \( J_x := t_{\text{rel}} \log(1 + \alpha_x/t_{\text{rel}}) \) and \( \alpha_x := \mathbb{E}_\pi[\mathcal{T}_x] \). Indeed, if there exists \( x \) such that \( \alpha_x \geq C_0 t_{\text{rel}} \), then by Proposition 3.1 we can also find \( x \) such that
\[
t_{\text{rel}} \log(\alpha_x/t_{\text{rel}}) \geq t_{\text{mix}}^\text{TV}/32,
\]
and so by (1.2) \( \mathbb{E}_{\text{BRW}}[\mathcal{T}_x] \succeq J_x \succeq t_{\text{mix}}^\text{TV} \), from which it is easy to see that \( \max_x \mathbb{E}_{\text{BRW}}[\mathcal{T}_x] \succeq \max_x J_x \succeq t_{\text{rel}} \log(1 + t_{\text{hit}}/t_{\text{rel}}) \) (as \( \max_x \alpha_x \asymp t_{\text{hit}} \) by Fact 2.1).

If \( \max_x \alpha_x < C_0 t_{\text{rel}} \), then again using \( \max_x \alpha_x \asymp t_{\text{hit}} \) we see that \( t_{\text{hit}} \asymp t_{\text{rel}} \), and so \( t_{\text{rel}} \log(1 + t_{\text{hit}}/t_{\text{rel}}) \asymp t_{\text{hit}} \), and so \( \max_x \mathbb{E}_{\text{BRW}}[\mathcal{T}_x] \succeq \max_x \alpha_x \succeq t_{\text{hit}} \asymp t_{\text{rel}} \log(1 + t_{\text{hit}}/t_{\text{rel}}) \) (the first inequality follows by considering the hitting time of \( x \) by the first particle). Moreover, if \( t_{\text{hit}} \asymp t_{\text{rel}} \) then with a positive probability we have that the BRW has a single particle by time \( \alpha_x/2 \), where \( x \) is picked so that \( \alpha_x = \max_x \alpha_x \). By the Paley-Zygmund inequality we have that \( \mathbb{P}_\pi[\mathcal{T}_x > \alpha_x/2] \geq \frac{1}{2} \frac{\alpha_x^2}{\mathbb{E}_\pi[\mathcal{T}_x^2]} \geq \frac{1}{32} \), where we have used \( \alpha_x \geq \frac{1}{2} t_{\text{hit}} \) (Fact 2.1) and \( \mathbb{E}_\pi[\mathcal{T}_x^2] \leq 2 t_{\text{hit}}^2 \) (2.8).

It follows that when \( \max_x \alpha_x < C_0 t_{\text{rel}} \) we have that for some \( x \) with probability bounded from below (uniformly in \( P \)) \( \mathcal{T}_x \geq \frac{\alpha_x}{2} \asymp t_{\text{hit}} \asymp t_{\text{rel}} \log(1 + t_{\text{hit}}/t_{\text{rel}}) \). This concludes the derivation of (1.1) from (1.2).

We now prove (1.2). We first show that \( \mathbb{E}_{\text{BRW}}[\mathcal{T}_x] \preceq t_{\text{mix}}^\text{TV} + J_x \). If \( \alpha_x < c t_{\text{rel}} \) then we are done as \( \mathbb{E}_{\text{BRW}}[\mathcal{T}_x] \preceq \alpha_x \preceq J_x \). Now assume that \( \alpha_x \geq c t_{\text{rel}} \). At time \( 8 J_x \) the number of particles is at least \( \alpha_x/t_{\text{rel}} \) with probability bounded from below (uniformly in \( x \) and \( P \)).

The \( \delta \)-separation mixing-time is defined as
\[
t_{\text{sep}}(\delta) := \inf\{ t : \min_{x,y} H_t(x,y)/\pi(y) \geq 1 - \delta \}.
\]

For reversible Markov chains we have that \( t_{\text{mix}}^\text{TV}(\varepsilon) \leq t_{\text{sep}}(\varepsilon) \leq 2 t_{\text{mix}}^\text{TV}(\varepsilon/4) \) for all \( \varepsilon \in (0, \frac{1}{4}) \) (e.g., [13, Lemmas 6.16 and 6.17]). As \( \min_{x,y} H_t(x,y)/\pi(y) \) is non-decreasing ([13, Exercise 6.4]), the law of the chain at time \( s \geq t_{\text{sep}}(\varepsilon) \) can be generated by taking a sample of \( \pi \) with probability \( 1 - \varepsilon \) and with probability \( \varepsilon \) taking a sample from some other distribution. It follows that on the last event (i.e., having at least \( \alpha_x/t_{\text{rel}} \) particles at time \( 8 J_x \)), by taking a burn-in period of \( 10 t_{\text{mix}}^\text{TV} \geq 2 t_{\text{mix}}(2^{-4}) \geq t_{\text{sep}}(\frac{1}{16}) \) time units, we can assume that at least half of the particles are distributed as \( \pi \), independently (cf. the proof of Proposition 4.3 in [17], where such an argument is carried out in detail). (The argument says that on a large probability event, we may assume this is the case. If this event fails, we may use the Markov property and repeat until the first success. As we are interesting in bounding a certain expectation, this is not a problem, since the number of trials until the first success is stochastically dominated by a Geometric distribution. We omit the details as this is routine).
Using the Markov property, it is thus suffices to show that \([\alpha_x/2t_{rel}]\) particles, each evolving independently according to \(P\), with independent initial positions distributed as \(\pi\), satisfy that the first time \(\tau_x\) at which one of them hits \(x\) is at most \(e t_{rel}\), with some probability bounded from below (uniformly in \(x\) and \(P\)). Note that there are no branching here. Indeed, by (2.9) (used in the first inequality) and Markov’s inequality we have that
\[
P[\tau_x > e t_{rel}] = (P_x[T_x > e t_{rel}])^{\alpha_x/2t_{rel}} \leq P_x[T_x > e_t_{rel} | \alpha_x/2t_{rel}] \leq 2/e.
\]
We now show that \(E_{BRW}[T_x] \geq J_x\) when \(\alpha_x \geq C_0 t_{rel}\), provided \(C_0\) is sufficiently large. Write \(N_x(t) := \int_0^t\) (number of particles at \(x\) at time \(s\))\(ds\). Then
\[
P_{BRW}[T_x \leq \frac{1}{4}J_x] = P_{BRW}[N_x(\frac{1}{4}J_x) > 0] \leq \frac{E_{BRW}[N_x(\frac{1}{4}J_x)]}{E_{BRW}[N_x(\frac{1}{4}J_x) | N_x(\frac{1}{4}J_x) > 0]}.
\]
(4.1)
By the Markov property, and using (2.11) and the fact that \(J_x \geq 16t_{rel}\), provided that \(C_0\) is sufficiently large, we have that
\[
E_{BRW}[N_x(\frac{1}{4}J_x) | N_x(\frac{1}{4}J_x) > 0] \geq \int_{\frac{1}{4}J_x}^{J_x} H_s(x, x)ds \geq \frac{7}{8} \int_0^\infty (H_s(x, x) - \pi(x))ds.
\]
Recall that \(\alpha_x = \frac{1}{\pi(x)} \int_0^\infty (H_s(x, x) - \pi(x))ds\) ((2.6)) and so
\[
E_{BRW}[N_x(\frac{1}{4}J_x) | N_x(\frac{1}{4}J_x) > 0] \geq \frac{7}{8} \alpha_x \pi(x).
\]
On the other hand, by stationarity, and using the fact that the expected number of particles at time \(t\) is \(2^{t/t_{rel}}\), we have that
\[
E_{BRW}[N_x(\frac{1}{4}J_x)] = \pi(x) \int_{0}^{\frac{1}{4}J_x} 2^{s/t_{rel}} ds \leq \frac{\pi(x) t_{rel}}{2^{1/2t_{rel}}} \leq \pi(x) \alpha_x/2,
\]
where the last inequality follows by the definition of \(J_x\), provided \(C\) is sufficiently large. Plugging the last two estimates into (4.1), we see that \(P_{BRW}[T_x \leq \frac{1}{4}J_x] \leq 4/7\), which implies that \(E_{BRW}[T_x] \geq J_x\), as desired.

We now prove (1.4). In fact, we prove a slightly stronger statement. Before stating it we need to introduce a new quantity (we expect that typically \(\chi(2) \approx \rho_{min})\):
\[
\chi(a) := \min_x \chi_x(a), \quad \text{where} \quad \chi_x(a) := \inf \left\{ t : \int_0^t \frac{sH_s(x, x)}{\pi(x)} ds \leq a t^2 \right\}.
\]
We show that \(L \leq E_{BRW}[\mathcal{T}]\), where
\[
L := \begin{cases} c t_{rel} \log \left(1 + \text{gap} \sqrt{\rho_{min}}\right) & \text{if} \ \rho_{min} \geq e^{2/c^2 t_{rel}} \text{or} \ \chi(2^{10}) \geq t_{rel} \\ \chi(2^{10}) & \text{if} \ \rho_{min} < e^{2/c^2 t_{rel}} \text{and} \ \chi(2^{10}) < t_{rel} \end{cases}
\]
[8]By the spectral-decomposition \(H_s(x, x)\) is non-decreasing in \(s\) (and \(H_s(x, x) \geq \pi(x)\) as \(s \to \infty\)), which implies that \(\int_0^t \frac{sH_s(x, x)}{\pi(x)} ds \geq \frac{H_s(x, x)}{2s(x)} t^2\). This, together with \(H_s(x, x) \geq 1/e\) implies that for some absolute constant \(c_0 > 0\), for all \(a > 1\) we have that \(\chi_x(a/2) \geq \max\{2t_{mix}^2 (\sqrt{a-1}), c_0(a\pi(x))^{-1/2}\}.\)
for some absolute constant $c \in (0,1)$ to be determined later.

We write $U(x,s)$ (respectively, $V(x,s)$) for the number of particles from the first (respectively, second) BRW which occupy state $x$ at time $s$. Then

$$M(t) := \sum_x \int_0^t \int_0^t U(x,s)V(x,r) \frac{\pi(x)}{\pi(x)} ds dr.$$ 

Then

$$\mathbb{P}_{\text{BRW}}[I \leq L] = \mathbb{P}_{\text{BRW}}[M(L) > 0] \leq \frac{\mathbb{E}_{\text{BRW}}[M(2L)]}{\mathbb{E}_{\text{BRW}}[M(2L) \mid M(L) > 0]}.$$ (4.2)

By the Markov property and the independence between the two BRWs, we have that

$$\mathbb{E}_{\text{BRW}}[M(2L) \mid M(L) > 0] \geq \min_x \int_0^L \int_0^L \sum_y \frac{H_s(x,y)H_r(x,y)}{\pi(y)} \pi(x) ds dr.$$ (4.3)

By reversibility $\sum_y \frac{H_s(x,y)H_r(x,y)}{\pi(y)} = H_{s+r}(x,x)$ and so by the definition of $z$ we have that

$$\mathbb{E}_{\text{BRW}}[M(2L) \mid M(L) > 0] \geq \min_x \int_0^L \frac{sH_s(x,x)}{\pi(x)} ds.$$ (4.4)

On the other hand, by stationarity and independence of the two BRWs, and using the fact that for each of the BRWs the expected number of particles at time $t$ is $2^{t/t_{\text{rel}}}$, we have that

$$\mathbb{E}_{\text{BRW}}[M(2L)] = \sum_x \pi(x) \int_0^L \int_0^L 2^{(s+r)/t_{\text{rel}}} ds dr = \frac{t_{\text{rel}}^2}{(\log 2)^2} (2^{L/t_{\text{rel}}} - 1)^2.$$ (4.5)

First consider the case that $\rho_{\text{min}} \geq e^{2/c L_{\text{rel}}}$. Then $L \geq t_{\text{rel}}$, and so by the definition of $\rho_{\text{min}}$ we have that

$$\mathbb{E}_{\text{BRW}}[M(2L) \mid M(L) > 0] \geq \min_x \int_0^L \frac{sH_s(x,x)}{\pi(x)} ds \geq \rho_{\text{min}}.$$ 

By (4.5), if $c$ is sufficiently small we have that

$$\mathbb{E}_{\text{BRW}}[M(2L)] \leq \frac{\rho_{\text{min}}}{2}.$$ 

Plugging the last two estimates into (4.2) concludes the proof in this case.

Next consider the case that $\rho_{\text{min}} < e^{2/c L_{\text{rel}}}$ and $\chi(2^{10}) < t_{\text{rel}}$. Then $L = \chi(2^{10})$. Using the fact that $H_s(x,x)$ is non-decreasing, and using the definition of $\chi(\cdot)$, we see that

$$\min_x \int_0^L \frac{sH_s(x,x)}{\pi(x)} ds \geq \min_x \int_0^{\chi(2^{10})} \frac{sH_s(x,x)}{\pi(x)} ds = 2^{10} \chi(2^{10})^2 = 2^{10} L^2.$$ 

By (4.5), and the fact that $L = \chi(2^{10}) < t_{\text{rel}}$, we have that

$$\mathbb{E}_{\text{BRW}}[M(2L)] \leq 16L^2.$$
Using (4.2) concludes the proof in this case.

Finally, consider the case that $\rho_{\text{min}} < e^{2\varepsilon/t_{\text{rel}}^2}$ and $\chi(2^{10}) \geq t_{\text{rel}}$. Then $\frac{1}{2}L \leq t_{\text{rel}} \leq \chi(2^{10})$. Hence $\min_x \int_0^L sH_s(x,x) \pi(x) ds \geq \min_x \int_0^{L/2} sH_s(x,x) \pi(x) ds \geq 2^{10}(L/2)^2$. Using $L \leq 2t_{\text{rel}}$ again, by (4.5) we have that $E_{\text{BRW}}[M(2L)] \leq 100L^2$. The proof is hence concluded using (4.2).

We now show that $E_{\text{BRW}}[I] \lesssim t_{\text{rel}} \log (1 + \sqrt{\rho_{\text{max}}})$. By (1.8) $t_{\text{rel}}^2 \leq Q \simeq \sum_{x} \pi(x) \rho_x$, and so $\sqrt{\rho_{\text{max}}} \gtrsim 1$. Hence at time $t = C_2 t_{\text{rel}} \log (1 + \sqrt{\rho_{\text{max}}})$ both BRWs have at least $2C_3(\sqrt{\rho_{\text{max}}})^2$ particles with probability bounded from below (uniformly in $P$), provided $C_2$ is sufficiently large. Similarly as in the proof above of $E_{\text{BRW}}[T_x] \lesssim t_{\text{mix}}^+ + J_x$, taking a burn period of duration $t_{\text{sep}}(1/4)$ (by Proposition 1.1 $t_{\text{sep}}(1/4) \lesssim t_{\text{rel}} \log (1 + \sqrt{\rho_{\text{max}}})$ and so the duration of such a burn in period can be absorbed into the implicit constant), we may assume that each of BRW has $[C_3(\sqrt{\rho_{\text{max}}})^2]$ particles whose positions are at equilibrium, independently.

We can now label the particles in the two processes by $1, \ldots, S := [C_3(\sqrt{\rho_{\text{max}}})^2]$ and $1', \ldots, S'$, respectively. As by adjusting $C_2$ we may ensure $C_3$ is arbitrarily large, using independence and by only considering intersections of particle $i$ with particle $i'$, it suffices to show that,

$$P_{\pi,\pi}[I < t_{\text{rel}}] \gtrsim \frac{1}{\sqrt{\rho_{\text{max}}}},$$

(4.6)

where as in §1.1, $I$ is the intersection time of two independent realizations $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ of the Markov chain (with no branching), and $P_\pi,\pi$ indicates that $X_0 \sim \pi$ and $Y_0 \sim \pi$, independently.

Consider $R(t) := \sum_x \frac{1}{\pi(x)} \int_0^t \int_0^t 1_{\{X_s=x,Y_r\}} ds dr$. Then $E_{\pi,\pi}[R(t)] = t^2$ and

$$R(t)^2 = \sum_{x,y} \frac{1}{\pi(x)\pi(y)} \int_0^t \int_0^t \int_0^t \int_0^t 1_{\{X_s=x,Y_r=x,Y_a=y\}} ds dr da db.$$

Taking expectation, and using reversibility, we see that

$$E_{\pi,\pi}[R(t)^2] \lesssim \sum_x \pi(x) t^2 \int_0^t \int_0^t \int_0^t \int_0^t \frac{H_a(x,y) H_b(x,y)}{\pi(y)} da db$$

$$= \sum_x \pi(x)^2 \int_0^t \int_0^t \frac{H_{a+b}(x,x)}{\pi(x)} da db \leq t^2 \max_x \int_0^{2t} \frac{sH_s(x,x)}{\pi(x)} ds \lesssim t^2 \max_x \int_0^{2t} \frac{sH_s(x,x)}{\pi(x)} ds,$$

(whence we have used the fact that $H_s(x,x)$ is non-decreasing in $s$ to argue that $\int_0^{2t} \frac{sH_s(x,x)}{\pi(x)} ds \leq 16 \max_x \int_{t/2}^t \frac{sH_{s+}(x,x)}{\pi(x)} ds$). Now, for $t = t_{\text{rel}}$ we have that

$$\max_x \int_0^t \frac{sH_s(x,x)}{\pi(x)} \leq \rho_{\text{max}} + \int_0^t s ds = \rho_{\text{max}} + \frac{1}{2}t_{\text{rel}}^2 \lesssim \rho_{\text{max}},$$
where we have used the fact that by (1.8) \( t_{\text{rel}} \leq \sqrt{Q} \lesssim \sum_x \pi(x) \rho_x \leq \rho_{\text{max}} \). Thus by the Payley-Zygmund inequality

\[
P_{\pi,\pi}[I < t_{\text{rel}}] = P_{\pi,\pi}[R(t_{\text{rel}}) > 0] \geq \frac{(E_{\pi,\pi}[R(t_{\text{rel}})])^2}{E_{\pi,\pi}[R(t_{\text{rel}})^2]} \gtrsim t_{\text{rel}}^2 / \rho_{\text{max}},
\]

and so (4.6) holds.

We now prove that if \( P \) is transitive we have that

\[
t_{\text{mix}}^{(\infty)} \lesssim t_{\text{rel}} \log(1 + \sqrt{Q} / t_{\text{rel}}) \asymp E_{\text{BRW}}[I].
\]

The first inequality follows from (1.7). The inequality \( E_{\text{BRW}}[I] \lesssim t_{\text{rel}} \log(1 + \sqrt{\rho_{\text{max}} / t_{\text{rel}}}) \) follows from \( E_{\text{BRW}}[I] \lesssim t_{\text{rel}} \log(1 + \sqrt{\rho_{\text{max}} / t_{\text{rel}}}) \) and (1.8). The proof of (1.8) gives that \( \frac{Q}{\pi(x) \rho_x} \leq \kappa := 1 + \sum_{i=1}^{\infty} \frac{i}{1 + \frac{i}{2} e^{-i/2}} \). When \( Q \geq C_1 \kappa t_{\text{rel}}^2 \) by (1.4) we have that \( E_{\text{BRW}}[I] \gtrsim t_{\text{rel}} \log(1 + \sqrt{Q} / t_{\text{rel}}) \). If \( Q < C_1 \kappa t_{\text{rel}}^2 \), then as \( Q \geq t_{\text{rel}}^2 \) we have that \( t_{\text{rel}} \log(1 + \sqrt{Q} / t_{\text{rel}}) \asymp t_{\text{rel}} \), and so to prove \( E_{\text{BRW}}[I] \gtrsim t_{\text{rel}} \log(1 + \sqrt{Q} / t_{\text{rel}}) \), it suffices to show that for some \( c_1, c_2 \in (0, 1) \) we have that

\[
P_{\pi,\pi}[I > c_1 t_{\text{rel}}] \gtrsim c_2.
\]

By (1.9) [19, Lemma 3.7] \( t_{\text{mix}}^{(\infty)}(1/4) \leq 2\sqrt{Q} \asymp E_{\pi,\pi}[I] \). Applying Markov’s inequality and the Markov property inductively, we have that \( P_{\pi,\pi}[I > j \left( 2E_{\pi,\pi}[I] + t_{\text{mix}}^{(\infty)}(1/4) \right) \] decays exponentially in \( j \) (we omit the details, as they are routine). Hence if \( Q < C_1 \kappa t_{\text{rel}}^2 \) we have that \( E_{\pi,\pi}[I^2] \lesssim (E_{\pi,\pi}[I])^2 \), and so \( P_{\pi,\pi}[I > c_1 t_{\text{rel}}] \gtrsim c_2 \) follows from the Paley-Zygmund inequality, and \( t_{\text{rel}} \leq \sqrt{Q} \asymp E_{\pi,\pi}[I] \). This concludes the proof.

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References

[1] Aldous, D., Mixing times and hitting times. (2010). Available at http://www.stat.berkeley.edu/aldous/Talks/slides.html

[2] Aldous, D., Hitting times for random walks on vertex-transitive graphs. Math. Proc. Cambridge Philos. Soc. 106 (1989), no. 1, 179–191. MR0994089.

[3] Aldous, D., Some inequalities for reversible Markov chains. Journal of the London Mathematical Society, 2(3):564–576, 1982. MR657512

[4] Aldous, D., Threshold limits for cover times. J. Theoret. Probab. 4 (1991), no. 1, 197–211. MR1088401.

[5] Aldous, D., and Fill, J., Reversible Markov chains and random walks on graphs. Unfinished manuscript. Available at the first author’s website.
[6] Basu, R., Hermon, J., Peres, Y., Characterization of cutoff for reversible Markov chains. *Ann. Probab.* 45 (2017), no. 3, 1448–1487. MR3650406

[7] Chen, G. Y., and Saloff-Coste, L., The cutoff phenomenon for ergodic Markov processes. *Electron. J. Probab.* 13 (2008), no. 3, 26–78. MR2375599

[8] Hermon, J., A spectral characterization for concentration of the cover time. (2019) Arxiv preprint arXiv:1809.00145

[9] Hermon, J., A technical report on hitting times, mixing and cutoff. *ALEA Lat. Am. J. Probab. Math. Stat.* 15 (2018), no. 1, 101–120. MR3765366

[10] Hermon, J., and Peres, Y., A characterization of $L_2$ mixing and hypercontractivity via hitting times and maximal inequalities. *Probab. Theory Related Fields* 170 (2018), no. 3-4, 769–800. MR3773799

[11] Hermon, J., and Peres, Y., On sensitivity of mixing times and cutoff. *Electron. J. Probab.* 23 (2018), Paper No. 25, 34 pp. MR3779818

[12] Gantert, N., and Müller, S., The critical branching Markov chain is transient. *Markov Process. Related Fields* 12 (2006), no. 4, 805–814. MR2284404

[13] Levin, D., and Peres, Y., (2017). *Markov chains and mixing times*. American Mathematical Society, Providence, RI. With contributions by Elizabeth L. Wilmer and a chapter by James G. Propp and David B. Wilson. MR3726904

[14] Lyons, R., and Peres, Y., *Probability on trees and networks*. Cambridge Series in Statistical and Probabilistic Mathematics, 42. Cambridge University Press, New York, 2016. MR3616205

[15] Oliveira, R., Mean field conditions for coalescing random walks. *Ann. Probab.* 41 (2013), no. 5, 3420–3461. MR3127887

[16] Oliveira, R., Mixing and hitting times for finite Markov chains. *Electron. J. Probab.* 17 (2012), no. 70, 12 pp. MR2968677

[17] Oliveira, R., On the coalescence time of reversible random walks. *Trans. Amer. Math. Soc.* 364 (2012), no. 4, 2109–2128. MR2869200

[18] Peres, Y., and Sousi, P., Mixing times are hitting times of large sets. *J. Theoret. Probab.* 28 (2015), no. 2, 488–519. MR3370663

[19] Peres, Y., Sauerwald, T., Sousi, P., and Stauffer, A., Intersection and mixing times for reversible chains. *Electron. J. Probab.* 22 (2017), No. 12, 16 pp. MR3613705