STABILITY CONDITIONS, BOGOMOLOV-GIESEKER TYPE INEQUALITIES AND FANO 3-FOLDS

DULIP PIYARATNE

Abstract. We develop a framework to modify the Bogomolov-Gieseker type inequality conjecture introduced by Bayer-Macrì-Toda, in order to construct a family of geometric Bridgeland stability conditions on any smooth projective 3-fold. We show that it is enough to check these modified inequalities on a small class of tilt stable objects. We extend some of the techniques in the works by Li and Bernardara-Macrì-Schmidt-Zhao to formulate a strong form of Bogomolov-Gieseker inequality for tilt stable objects on Fano 3-folds. Consequently, we establish our modified Bogomolov-Gieseker type inequality conjecture for general Fano 3-folds, including an optimal inequality for the blow-up of \( \mathbb{P}^3 \) at a point.

1. Introduction

1.1. Motivation and background. The notion of stability conditions on triangulated categories was introduced by Bridgeland (see [Bri]). Such a stability condition on a triangulated category is defined by giving a bounded t-structure together with a stability function on its heart satisfying the Harder-Narasimhan property. This can be interpreted essentially as an abstraction of the usual slope stability for sheaves. Construction of Bridgeland stability conditions on the bounded derived category of a given projective threefold is an important problem. However, unlike for a projective surface, there is no known construction which gives stability conditions for all projective threefolds. See [Huy2, MS] for further details.

In [BMT], Bayer, Macrì and Toda introduced a conjectural construction of Bridgeland stability conditions for any projective threefold. Here the problem was reduced to proving so-called Bogomolov-Gieseker type inequality holds for certain tilt stable objects. It has been shown to hold for all Fano 3-folds with Picard rank one (see [BMT, Mac, Sch1, Li]), abelian 3-folds (see [MP1, MP2, Piy1, Piy2, BMS]), ´etale quotients of abelian 3-folds (see [BMS]), some toric 3-folds (see [BMSZ]) and 3-folds which are products of projective spaces and abelian varieties (see [Kos]). Recently, Schmidt found a counterexample to the original Bogomolov-Gieseker type inequality conjecture when \( X \) is the blowup at a point of \( \mathbb{P}^3 \) (see [Sch2]). Therefore, this inequality needs some modifications in general setting and this is one of the main goals of this paper.

1.2. Modification of Bogomolov-Gieseker type inequality conjecture. Let \( X \) be a smooth projective 3-fold, and let \( H \in \text{NS}(X) \) and \( B \in \text{NS}_R(X) \) are some fixed classes such that \( H \) is ample. Let \( \beta \in \mathbb{R} \) and \( \alpha \in \mathbb{R}_{>0} \). In [BMT], the authors tilted the abelian category of coherent sheaves on \( X \) with respect to a torsion pair coming from usual slope stability, to...
get the abelian category $\mathcal{B}_{H, B + \beta H}$. Moreover, they introduced the notion of tilt stability by defining the $\mathcal{V}_{H, B + \beta H, \alpha}$ tilt slope on $\mathcal{B}_{H, B + \beta H}$ by

$$\mathcal{V}_{H, B + \beta H, \alpha} : \mathcal{B}_{H, B + \beta H} \ni E \mapsto \frac{\text{ch}_2^{B + \beta H}(E) - (\alpha^2/2)\text{ch}_0(E)}{H^2 \text{ch}_1^{B + \beta H}(E)} \in \mathbb{R} \cup \{+\infty\}.$$  

We modify the expression in the Bogomolov-Gieseker type inequality conjecture by introducing an extra term $\xi \in \mathbb{R}_{\geq 0}$ together with a class $\Lambda \in H^4(X, \mathbb{Q})$ satisfying $\Lambda \cdot H = 0$:

$$D_{\alpha, \beta}^{B, \xi}(E) = \text{ch}_3^{B + \beta H}(E) + \left(\Lambda - \left(\xi + \frac{1}{6}\alpha^2\right)H^2\right)\text{ch}_1^{B + \beta H}(E).$$

More precisely, we conjecture the following for a family of stability parameters. Let $A : B + \mathbb{R}(H) \to \mathbb{R}_{\geq 0}$ be a continuous function.

**Conjecture 1.1** ($=\text{[4.5]}$). There exist $\Lambda \in H^4(X, \mathbb{Q})$ satisfying $\Lambda \cdot H = 0$, and a constant $\xi(A) \in \mathbb{R}_{\geq 0}$ such that for any $(\beta, \alpha) \in \mathbb{R} \times \mathbb{R}_{\geq 0}$, with $\alpha \geq \Lambda(B + \beta H)$, all $\mathcal{V}_{H, B + \beta H, \alpha}$ tilt slope stable objects $E \in \mathcal{B}_{H, B + \beta H}$ with $\mathcal{V}_{H, B + \beta H, \alpha}(E) = 0$ satisfy the inequality:

$$D_{\alpha, \beta}^{B, \xi(A)}(E) \leq 0.$$

Hence, for any $\xi \geq \xi(A)$, we have $D_{\alpha, \beta}^{B, \xi}(E) \leq 0$.

As similar to [BMT][BMS], the above modification conjecturally gives us a family of Bridgeland stability conditions. More specifically, when our modified Conjecture holds for $X$, the tilt of $\mathcal{B}_{H, B + \beta H}$ as in the construction of [BMT] together with some central charge functions define those stability conditions. See Theorem 4.12 for further details.

Our modified conjectural inequality coincides with the Bogomolov-Gieseker type inequality in [BMT] when $\Lambda = 0$, $\Lambda = 0$ and $\xi(A) = 0$. In this paper, we are mostly interested in the following choice:

$$\Lambda = \frac{c_2(X)}{12} - \frac{c_2(X) \cdot H}{12H^3}H^2.$$  

Furthermore, this $\Lambda$ vanishes for many 3-folds where the Bogomolov-Gieseker type inequality conjecture in [BMT] holds. For example, when $X$ is an abelian 3-fold ($c_2(X) = 0$), or a Fano 3-fold with Picard rank one ($c_2(X)$ is proportional to $H^2$).

In Section 4.2, we extend the notion of $\mathcal{B}$-stability in [Li][BMS]. For the continuous function $A : B + \mathbb{R}(H) \to \mathbb{R}_{\geq 0}$, we define $\mathcal{B}_A(E)$ to be the set of roots $\beta$ of

$$H\text{ch}_2^{B + \beta H}(E) - \frac{1}{2}(A(B + \beta H))^2\text{ch}_0(E) = 0.$$

We call an object $E$ in $\mathcal{D}^b(X)$ $\mathcal{B}_A$-stable if for any $\beta \in \mathcal{B}_A(E)$ there is an open neighbourhood $U \subset \mathbb{R}^2$ containing $(\beta, A(B + \beta H))$ such that for any $(\beta, \alpha) \in U$ with $\alpha > 0$, $E \in \mathcal{B}_{H, B + \beta H}$ is $\mathcal{V}_{H, B + \beta H, \alpha}$-stable.

In this paper we reduce the requirement of our modified Bogomolov-Gieseker type inequalities to $\mathcal{B}_A$ stable objects. More precisely, we show that Conjecture 1.1 is equivalent to the following. See Theorem 4.19 for further details.

**Conjecture 1.2** ($=\text{[4.19]}$). There exist $\Lambda \in H^4(X, \mathbb{Q})$ satisfying $\Lambda \cdot H = 0$, and a constant $\xi(A) \in \mathbb{R}_{\geq 0}$ such that any $\mathcal{B}_A$-stable object $E$ in $\mathcal{D}^b(X)$ satisfies the inequality:

$$D_{A(B + \beta H), \beta}^{B, \xi(A)}(E) \leq 0, \text{ for each } \beta \in \mathcal{B}_A(E).$$
1.3. Bridgeland stability conditions on Fano 3-folds. In this paper we extend the works of [Li, BMSZ] to establish Conjecture 1.2 for Fano 3-folds in optimal sense, that is having a minimal $\xi(A)$.

**Theorem 1.3** ([Li], Picard rank one case; [BMSZ], general Fano 3-folds; Theorem 7.4 for an optimal inequality on general Fano 3-folds).

Let $X$ be a smooth Fano 3-fold and $B$, $H$ are proportional to $-K_X$. Then Conjecture 1.2 holds on $X$ with respect to $A = 0$ for some finite $\xi(A) \geq 0$ and $\Lambda$ as in (1).

Moreover, for the blow-up of $\mathbb{P}^3$ at a point we show that an optimal modified Bogomolov-Gieseker type inequality holds.

**Theorem 1.4** (8.2).

Let $X$ be the blow-up of $\mathbb{P}^3$ at a point, and let $H = -K_X/2$. Let $A : \mathbb{R}(H) \to \mathbb{R}$ be the continuous function defined by, for $\beta \in [-1/2, 1/2]$

$$A(\beta H) = \begin{cases} 
(1 - \beta) & \text{if } \beta \in [-1/2, 0) \\
(1 + \beta) & \text{if } \beta \in [0, 1/2]
\end{cases}$$

together with the relation $A((\beta + 1)H) = A(\beta H)$. Then the modified Bogomolov-Gieseker type inequality in Conjecture 1.2 holds for $X$, with $\xi(A) = 0$ and $\Lambda$ as in (1).

The main ideas of the proofs of above results for Fano 3-folds are similar to the work of Li and Bernardara-Macrì-Schmidt-Zhao in [Li, BMSZ]. More precisely, by dualizing and tensoring by a line bundle, we reduce the requirement of the Bogomolov-Gieseker type inequalities to $\beta_A$ stable objects having $\beta_A$ values in a unit interval in $\mathbb{R}$ such that $c_0 \geq 0$. Then we compare the tilt slopes of such objects with the tilt slopes of certain tilt stable line bundles and their shift by [1]. In this way we get certain Hom vanishings, and by using the Serre duality we obtain a bound for the Euler characteristic involving our $\beta_A$ stable objects. Conclusively, the Hirzebruch-Riemann-Roch formula gives us the modified Bogomolov-Gieseker type inequalities for those restricted class of tilt stable objects. Particularly, the following strong form of Bogomolov-Gieseker inequality for tilt stable objects on Fano 3-folds, is crucial for us. This generalizes the earliest results [Li, Proposition 3.2] and [BMSZ, Theorem 3.1], and see Remark 6.6 for further details.

**Theorem 1.5** (6.5).

Let $X$ be a Fano 3-fold and let $H$ and $B$ be classes proportional to $-K_X$. Let $E$ be a tilt stable object with finite tilt slope and non-isomorphic to $\mathcal{O}_X(mH)[1]$ or $\mathcal{I}_Z(mH)$ for any $m \in \mathbb{Z}$ and 0-subscheme $Z \subset X$. Further we assume $c_0(E) \neq 0$, and $E$ satisfies certain conditions, namely (12), (13), and (14). Then

$$\frac{\Delta_H(E)}{(H^3 c_0(E))^2} \geq \kappa(X).$$

Here $\Delta_H$ is the discriminant as in Definition 3.3, and $\kappa(X)$ is a constant as in Definition 6.2.

1.4. Relation to the existing works. This paper supersedes the author’s unpublished work [Piy3].

A modification of the Bogomolov-Gieseker type inequality conjecture for Fano 3-folds appeared in [BMSZ] and the author’s unpublished preprint [Piy3] almost at the same time. One of the key ingredients in those works is the formulation of a strong form of Bogomolov-Gieseker inequality for tilt stable objects on Fano 3-folds having higher Picard ranks, extending the previous work of Li in the Picard rank one case (see [Li]). However, both formulations had
a gap and it was fixed by the authors in [BMSZ]. In this paper we further strengthen their inequality as presented in Theorem 1.5 above. See Remark 6.6 for further details.

Following similar ideas in [Li, BMSZ], we establish Conjecture 1.2 or equivalently Conjecture 1.1 for Fano 3-folds as stated in Theorems 1.3 and 1.4. The class \( \Gamma \) in Theorem 1.1 of [BMSZ] is exactly equal to the class \(-\Lambda + \xi(A)H^2\) with \(A = 0\) in our paper. However, the modified Bogomolov-Gieseker type inequality that we propose in this paper is rather general, and also, the following are some significant points relevant to Fano 3-folds.

(i) According to our notation, in [BMSZ] the authors only consider the modified Bogomolov-Gieseker type inequalities for Fano 3-folds with respect to \(A = 0\).

(ii) Since Theorem 1.5 further generalizes [BMSZ, Theorem 3.1], our modified Bogomolov-Gieseker type inequalities for Fano 3-folds become stronger, specifically for the Fano 3-folds having \(\xi(A) > 0\) in Theorem 1.3.

(iii) In [BMSZ], the authors did not optimize the modified Bogomolov-Gieseker type inequalities for the blow-up of \(\mathbb{P}^3\) at a point. Specifically, \(\Gamma\) in [BMSZ, Theorem 1.1] is a class with \(\Gamma \cdot H > 0\) (see Sections 4.B and 6 of [BMSZ]). However, in Section 8 we show that \(\xi(A) = 0\) for some non-zero \(A\) as stated in Theorem 1.4 above.

1.5. Plan of the paper. In Section 2, we briefly recall the notion of tilt stability and some important results associated to sheaves and Fano 3-folds. In Section 3, we discuss properties of tilt stable objects in detail. In particular, we see that objects in the first tilted category of 3-folds behave somewhat similar to coherent sheaves on a projective surface under the dualizing. In Section 4, we develop the framework to modify the Bogomolov-Gieseker type inequality introduced by Bayer, Macrì and Toda, in order to construct a family of geometric Bridgeland stability conditions on any smooth projective 3-fold. Moreover, we introduce the notion of \(\beta_A\) stability, and reduce the requirement of modified Bogomolov-Gieseker type inequality conjecture to those stable objects. In Section 5, we get certain Hom vanishing results for \(\beta_A\) stable objects with respect to some line bundles. We prove a strong from of Bogomolov-Gieseker inequality for Fano 3-folds in Section 6. In Section 7, we establish the modified Bogomolov-Gieseker type inequalities for Fano 3-folds, and in Section 8 an optimal inequality for the blow-up of \(\mathbb{P}^3\) at a point.

1.6. Notation. Let us collect some of the important notations that we use in this paper as follows:

- When \(A\) is the heart of a bounded t-structure on a triangulated category \(D\), by \(H^i_A(\cdot)\) we denote the corresponding \(i\)-th cohomology functor.
- For a set of objects \(S \subset D\) in a triangulated category \(D\), by \(\langle S \rangle \subset D\) we denote its extension closure, that is the smallest extension closed subcategory of \(D\) which contains \(S\).
- Unless otherwise stated, throughout this paper, all the varieties are smooth projective and defined over \(\mathbb{C}\). For a variety \(X\), by \(\text{Coh}(X)\) we denote the category of coherent sheaves on \(X\), and by \(\text{D}^b(X)\) we denote the bounded derived category of \(\text{Coh}(X)\).
- For a variety \(X\), by \(\omega_X\) we denote its canonical line bundle, and let \(K_X = c_1(\omega_X)\).
- For \(M = \mathbb{Q}, \mathbb{R}\), or \(\mathbb{C}\) we write \(\text{NS}_M(X) = \text{NS}(X) \otimes_{\mathbb{Z}} M\).
- For \(E, F \in \text{D}^b(X)\), denote \(\text{hom}_X(E, F) = \dim \Hom_X(E, F)\), and when \(E\) is a sheaf, \(h^i(E) = \dim H^i(E, X)\).
- For the bounded derived category of a variety \(X\), we simply write \(\mathcal{F}^i(\cdot)\) for \(H^i_{\text{Coh}(X)}(\cdot)\).
• For $0 \leq i \leq \dim X$, $\operatorname{Coh}_{\leq i}(X) = \{ E \in \operatorname{Coh}(X) : \dim \operatorname{Supp}(E) \leq i \}$, $\operatorname{Coh}_{> i}(X) = \{ E \in \operatorname{Coh}(X) : 0 \neq F \subset E, \dim \operatorname{Supp}(F) > i \}$ and $\operatorname{Coh}_i(X) = \operatorname{Coh}_{\leq i}(X) \cap \operatorname{Coh}_{> i}(X)$.

• For $E \in D^b(X)$, $E^\vee = R\mathcal{H}om(E, \mathcal{O}_X)$. When $E$ is a sheaf we write its dual sheaf $\mathcal{F}^0(E^\vee)$ by $E^\vee$.

• The skyscraper sheaf of a closed point $x \in X$ is denoted by $\mathcal{O}_x$.

• For $B \in \text{NS}_B(X)$, the twisted Chern character $\text{ch}^B(\cdot) = e^{-B} \cdot \text{ch}(\cdot)$. For ample $H \in \text{NS}(X)$ and, $\mu_{H,B}(E) = (H^2 ch_1^B(E))/(H^3 ch_0(E))$. We write $\nu_H = \mu_{H,0}$ and $\mu_{H,\beta} = \mu_{H,0} - \beta H$.

• Tilt slope on $\mathcal{B}_{H,B}$ is defined by $\nu_{H,B,(\alpha,\beta)}(E) = H \text{ch}_2^B(E) - (\alpha^2/2)H^3 \text{ch}_0(E)/H^2 \text{ch}_1^B(E)$. Sometimes we write $\nu_{B,\alpha,\beta} = \nu_{H,B,(\alpha,\beta)}$ and $\mathcal{B}_B = \mathcal{B}_{H,B}$.

• $H_{H,B}(I) = \langle E \in \operatorname{Coh}(X) : E \text{ is } \mu_{H,B}\text{-semistable with } \mu_{H,B}(E) \in I \rangle$. Similarly, we define $H_{H,B}^1(I) \subset \mathcal{B}_{H,B}$.

• For $E \in \mathcal{B}_{H,B}$ we write $E^i = H^i_{\mathcal{B}_{H,B}}(E^\vee)$. So for example $E^{12} = H^2_{\mathcal{B}_{H,B}}\left(\left(H^1_{\mathcal{B}_{H,B}}(E^\vee)\right)^\vee\right)$.

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2. **Preliminaries**

2.1. **Tilt stability on 3-folds.** Let us briefly recall the notions of slope and tilt stabilities for a given smooth projective threefold $X$ as introduced in [BMT].

Let $H \in \text{NS}(X)$ be an ample divisor class, and $B \in \text{NS}_B(X)$. The twisted Chern character with respect to $B$ is defined by $\text{ch}^B(\cdot) = e^{-B} \cdot \text{ch}(\cdot)$. The twisted slope $\mu_{H,B}$ on $\operatorname{Coh}(X)$ is defined by, for $E \in \operatorname{Coh}(X)$

$$
\mu_{H,B}(E) = \begin{cases} 
+\infty & \text{if } E \text{ is a torsion sheaf} \\
\frac{H^2 \text{ch}_1^B(E)}{H^3 \text{ch}_0(E)} & \text{otherwise.}
\end{cases}
$$

For simplicity we write $\mu_H = \mu_{H,0}$.

So we have $\mu_{H,B} = \mu_H - (BH^2)/(H^3)$.

We say $E \in \operatorname{Coh}(X)$ is $\mu_{H,B}$-(semi)stable, if for any $0 \neq F \subsetneq E$, $\mu_{H,B}(F) < (\leq) \mu_{H,B}(E/F)$. The Harder-Narasimhan property holds for $\operatorname{Coh}(X)$, and for a given interval $I \subset \mathbb{R} \cup (+\infty)$, we define the subcategory $H_{H,B}^\mu(I) \subset \operatorname{Coh}(X)$ by

$$
(2) \quad H_{H,B}^\mu(I) = \langle E \in \operatorname{Coh}(X) : E \text{ is } \mu_{H,B}\text{-semistable with } \mu_{H,B}(E) \in I \rangle.
$$

The subcategories $\mathcal{F}_{H,B}$ and $\mathcal{J}_{H,B}$ of $\operatorname{Coh}(X)$ are defined by

$$
\mathcal{F}_{H,B} = H_{H,B}^\mu((0, +\infty)), \quad \mathcal{J}_{H,B} = H_{H,B}^\mu((-\infty, 0]).
$$

Now $(\mathcal{J}_{H,B}, \mathcal{F}_{H,B})$ forms a torsion pair on $\operatorname{Coh}(X)$ and let the abelian category $\mathcal{B}_{H,B} = \langle \mathcal{J}_{H,B}[1], \mathcal{J}_{H,B} \rangle \subset D^b(X)$ be the corresponding tilt of $\operatorname{Coh}(X)$.
Let \( \alpha \in \mathbb{R}_{>0} \). Following [BMT], the tilt-slope \( \nu_{H,B,\alpha} \) on \( B_{H,B} \) is defined by, for \( E \in B_{H,B} \)

\[
\nu_{H,B,\alpha}(E) = \begin{cases} 
+\infty & \text{if } H^2 ch^1_B(E) = 0 \\
\frac{H ch^2_B(E) - (\alpha^2/2)H^3 ch_0(B)}{H^2 ch^1_B(E)} & \text{otherwise.}
\end{cases}
\]

In [BMT] the notion of \( \nu_{H,B,\alpha} \)-stability for objects in \( B_{H,B} \) is introduced in a similar way to \( \mu_{H,B} \)-stability for \( \text{Coh}(X) \). Also it is proved that the abelian category \( B_{H,B} \) satisfies the Harder-Narasimhan property with respect to \( \nu_{H,B,\alpha} \)-stability. Then similar to (2) we define the subcategory \( HN_{H,B,\alpha}(I) \subset B_{H,B} \) for an interval \( I \subset \mathbb{R} \cup \{+\infty\} \). The subcategories \( \mathcal{T}'_{H,B,\alpha} \) and \( \mathcal{T}'_{H,B,\alpha} \) of \( B_{H,B} \) are defined by \( \mathcal{T}'_{H,B,\alpha} = HN_{H,B,\alpha}([0,+\infty]) \) and \( \mathcal{T}'_{H,B} = HN_{H,B,\alpha}((-\infty,0]) \). Then the pair \( (\mathcal{T}'_{H,B,\alpha}, \mathcal{T}'_{H,B,\alpha}) \) forms a torsion pair on \( B_{H,B} \) and let the abelian category

\[
A_{H,B,\alpha} = \langle \mathcal{T}'_{H,B,\alpha}[1], \mathcal{T}'_{H,B,\alpha} \rangle \subset D^b(X)
\]

be the corresponding tilt.

2.2. Some homological algebraic results. An object of an abelian category is called minimal when it has no proper subobjects or equivalently no nontrivial quotients in the category. For example skyscraper sheaves of closed points are the only minimal objects of the abelian category of coherent sheaves on a scheme. Moreover, for the abelian category \( B_{H,B} \) of a 3-fold, we have the following:

**Proposition 2.1.** The objects which are isomorphic to the following types are minimal in \( B_{H,B} \):

(i) skyscraper sheaves \( O_x \) of \( x \in X \).
(ii) \( E[I] \), where \( E \) is a \( \mu_{H,B} \)-stable reflexive sheaf with \( \mu_{H,B}(E) = 0 \).

**Proof.** Similar to the proof of [Huy] Proposition 2.2. (Also one can see these objects as examples of the class of minimal objects considered abstractly in [PT] Aside 2.12.) \( \square \)

Let \( E,F \) be two objects in the derived category \( D^b(X) \) of a smooth projective variety \( X \). The Euler characteristic \( \chi(E,F) \) is defined by

\[
\chi(E,F) = \sum_{i \in \mathbb{Z}} \hom_X(E,F[i]).
\]

We write \( \chi(O_X,E) \) by \( \chi(E) \), and so \( \chi(E,F) = \chi(E^\vee \otimes F) \). The Hirzebruch-Riemann-Roch theorem says,

\[
\chi(E) = \int_X ch(E) \cdot \text{td}(X).
\]

Here \( \text{td}(X) \) is the Todd class \( \text{td}(T_X) \) of the tangent bundle \( T_X \) of \( X \). When \( X \) is 3-dimensional, from [Har] Section 4, Appendix A]

\[
\text{td}(X) = 1 + \frac{1}{2}c_1(X) + \frac{1}{12}(c_1(X)^2 + c_2(X)) + \frac{1}{24}c_1(X)c_2(X).
\]

Here \( c_1(X) \) denotes the i-th Chern class \( c_i(T_X) \) of the tangent bundle \( T_X \).
2.3. Some sheaf theory. Let us recall some useful results for coherent sheaves.

Proposition 2.2 ([OSS [HI]]). Let $X$ be an $n$-dimensional smooth projective variety. Then we have the following for $E \in \text{Coh}(X)$:

(i) If $E \in \text{Coh}_{\leq d}(X)$ then it fits into the short exact sequence

$$0 \to E_{\leq d-1} \to E \to \mathcal{E}xt^d(E, O_X) \to 0$$

in $\text{Coh}(X)$ for some $E_{\leq d-1} \in \text{Coh}_{\leq d-1}(X)$ and $E_d \in \text{Coh}_d(X)$.

(ii) $\mathcal{E}xt^d(E, O_X) \in \text{Coh}_{\leq n-1}(X)$.

(iii) If $E \in \text{Coh}_d(X)$ then it fits into the short exact sequence

$$0 \to E \to \mathcal{E}xt^{n-d}(\mathcal{E}xt^{n-d}(E)) \to Q \to 0$$

in $\text{Coh}(X)$ for some $Q \in \text{Coh}_{\leq d-2}(X)$.

Lemma 2.3 ([HI] Theorem 7.3.1, [Sim] Theorem 2). Let $X$ be a smooth projective variety of dimension $n \geq 3$ and let $H \in \text{NS}(X)$ be an ample divisor class. Let $E$ be a $\mu_H$ slope semistable torsion free sheaf on $X$. Then we have the following:

(i) Sheaf $E$ satisfies the so called Bogomolov-Gieseker inequality:

$$H^{n-2}\Delta(E) \geq 0, \text{ where } \Delta(E) = (ch_1(E))^2 - 2 \cdot ch_0(E) \cdot ch_2(E).$$

(ii) If $H^{n-1} ch_1(E^{**}) = 0$ and $H^{n-2} ch_2(E^{**}) = 0$, then all Jordan-H"{o}lder slope stable factors of $E^{**}$ are locally free sheaves which have vanishing Chern classes.

(iii) If $E$ is a $\mu_H$ semistable reflexive sheaf with $H^{n-2}\Delta(E) = 0$, then $E$ is a locally free sheaf with $ch_i(E \otimes E^*) = 0$ for $i \geq 1$; in particular $\Delta(E) = 0$.

2.4. Fano 3-folds. Let us recall some important notions associated to Fano varieties. A Fano variety $X$ is a smooth projective variety whose anti-canonical divisor $-K_X$ is ample. A basic invariant of $X$ is its index, this is the maximal integer $r(X)$ such that $K_X$ is divisible by $r(X)$ in $\text{NS}(X)$. So $-K_X = r(X) \cdot H$ for an ample divisor class $H$ in $\text{NS}(X)$. The number $d(X) = H^{\dim X}$ is usually called the degree of $X$.

If $X$ is an $n$-dimensional Fano variety then $r(X) \leq n + 1$. Moreover, if $r(X) = n + 1$ then $X \cong \mathbb{P}^n$, and if $r(X) = n$ then $X$ is a quadric. For Fano 3-folds there is an explicit Iskovskikh-Mori-Mukai classification. See [IP] Chapter 12 or [MM] for further details.

Let us collect some basic properties for Fano 3-folds, that we will need in the proceeding sections.

Proposition 2.4. Let $X$ be a Fano 3-fold of index $r(X) = r$ and degree $d(X) = d$. Then we have the following:

(i) $h^i(\mathcal{O}_X) = 0$ for all $i > 0$, and $\chi(\mathcal{O}_X) = 1$.

(ii) $H \cdot c_2(X) = 24/r$.

(iii) $\text{td}(X) = (1, \frac{4}{3}rH, \frac{1}{12}(r^2H^2 + c_2(X)), 1)$.

Proof. Since $-K_X$ is ample, from the Kodaira's vanishing theorem $H^i(\mathcal{O}_X, X) = 0$ for all $i > 0$. So we have $\chi(\mathcal{O}_X) = h^0(\mathcal{O}_X) = 1$.

Let us compute the Todd class of the tangent bundle $T_X$ of $X$. Since the cotangent bundle is $\Omega_X \cong T_X$, $c_1(\Omega_X) = -c_1(\mathcal{O}_X)$. Also $\omega_X = \text{det}(\Omega_X)$ and so $c_1(\omega_X) = -c_1(\omega_X) = -K_X = rH$. From the Hirzebruch-Riemann-Roch theorem [HI], $\chi(\mathcal{O}_X) = \int_X ch_1(\mathcal{O}_X) \cdot \text{td}(X)$, and so $\frac{1}{24}c_1(X)c_2(X) = 1$. The required expression for Todd class follows from [5].
3. SOME PROPERTIES OF TILT STABLE OBJECTS

3.1. Some slope bounds for tilt stable objects. Let $X$ be a smooth projective 3-fold. We follow the same notations for tilt stability introduced in Section 2.1 for $X$.

By construction, $\text{Coh}_{\leq 2}(X) \subset \mathcal{B}_{H,B}$. Moreover, we have the following for its subcategory $\text{Coh}_{\leq 1}(X)$.

**Proposition 3.1.** We have $\text{Coh}_{\leq 1}(X) \subset H^\vee_{H,B,\alpha}(+\infty)$.

**Proof.** Let $E \in \text{Coh}_{\leq 1}(X)$. Assume the opposite for a contradiction; so that $0 \to E \to E \to 0$ is a short exact sequence on $\mathcal{B}_{H,B}$ with $\nu_{H,B,\alpha}(Q) < +\infty$. By considering the long exact sequence of Cohomology we have $\mathcal{H}^{-1}(P) = 0$, and since $\text{ch}_1(E) = 0$, $\text{ch}_1(\mathcal{H}^{-1}(Q)) = \text{ch}_1(\mathcal{H}^0(P))$. Since $\mathcal{H}^{-1}(Q) \in H^\mu_{H,B}((-\infty, 0])$ and $\mathcal{H}^0(P) \in H^\mu_{H,B}((0, +\infty))$, we have $H^2 \text{ch}_1^B(\mathcal{H}^0(P)) = 0$. So $\mathcal{H}^0(P) \in \text{Coh}_{\leq 1}(X)$, and $\mathcal{H}^{-1}(Q) = 0$. Hence, $Q \cong \mathcal{H}^0(Q)$ is a quotient sheaf of $E \in \text{Coh}_{\leq 1}(X)$; that is $Q \in \text{Coh}_{\leq 1}(X)$. Therefore, $\nu_{H,B,\alpha}(Q) = +\infty$. This is the required contradiction. □

**Proposition 3.2.** Let $E \in H^\vee_{H,B,\alpha}((-\infty, +\infty))$. Then $\mathcal{H}^{-1}(E)$ is a reflexive sheaf.

**Proof.** For $E \in H^\vee_{H,B,\alpha}((-\infty, +\infty))$, let us denote $E_j = \mathcal{H}^j(E)$. Object $E$ fits into the short exact sequence $0 \to E_{-1}[1] \to E \to E_0 \to 0$ in $\mathcal{B}_{H,B}$. Here $E_{-1}$ is torsion free and so it fits into the short exact sequence $0 \to E_{-1} \to E_{-1}^* \to Q \to 0$ in Cohomology for some $Q \in \text{Coh}_{\leq 1}(X)$. Therefore, $0 \to Q \to E_{-1}[1] \to E_{-1}^*[1] \to 0$ is a short exact sequence in $\mathcal{B}_{H,B}$. Hence $Q$ is a subobject of $E \in H^\vee_{H,B,\alpha}((-\infty, +\infty))$. By Proposition 3.1, $Q \in H^\vee_{H,B,\alpha}(+\infty)$, and so $Q = 0$. That is $E_{-1}$ is reflexive. □

**Definition 3.3.** For $E \in D^B(X)$ we define

$$\Delta(E) = (\text{ch}_1(E))^2 - 2 \text{ch}_0(E) \text{ch}_2(E) \in H^4(X, \mathbb{Z}),$$

$$\overline{\Delta}_{H,B}(E) = (H^2 \text{ch}_1^B(E))^2 - 2H^3 \text{ch}_0(E)H \text{ch}_2^B(E).$$

We simply write $\overline{\Delta}_H = \overline{\Delta}_{H,0}$.

We have $\overline{\Delta}_{H,B} = H^3 \cdot \Delta(E) + (H^2 \text{ch}_1^B(E))^2 - H^3 \text{ch}_0^B(E)^2$. From the Hodge index theorem, $(H^2 \text{ch}_1^B(E))^2 - H^3 \text{ch}_0^B(E)^2 \geq 0$, and so

$$\overline{\Delta}_{H,B}(E) \geq H^3 \cdot \Delta(E).$$

For any $t \in \mathbb{R}$ we have

$$\nu_{H,B,\alpha} - t = \frac{H \text{ch}_2^B - (1/2)(a^2)H^3 \text{ch}_0}{H^2 \text{ch}_1^B} - t$$

$$= \frac{H \text{ch}_2^B + tH - (1/2)(t^2 + \alpha^2)H^3 \text{ch}_0}{H^2 \text{ch}_1^B}.$$

Let us recall the following slope bounds from [PT] for cohomology sheaves of complexes in the abelian category $\mathcal{B}_{H,B}$.

**Proposition 3.4.** ([PT] Proposition 3.13). Let $E \in \mathcal{B}_{H,B}$ and $E_1 = \mathcal{H}^i(E)$. Then we have the following:

(i) if $E \in H^\mu_{H,B,\alpha}((-\infty, t])$, then $E_{-1} \in H^\mu_{H,B}((-\infty, t - \sqrt{t^2 + \alpha^2}))$;

(ii) if $E \in H^\mu_{H,B}((t, +\infty))$, then $E_0 \in H^\mu_{H,B}((t + \sqrt{t^2 + \alpha^2}, +\infty))$; and
(iii) if \( E \) is tilt semistable with \( \nu_{H,B,\alpha}(E) = t \), then
(a) \( E_{-1} \in HN^H_{H,B}((\infty, t - \sqrt{t^2 + \alpha^2}) \) with equality \( \nu_{H,B}(E_{-1}) = t - \sqrt{t^2 + \alpha^2} \) holds
if and only if \( H^2 \text{ch}^2(E_{-1}) = 0 \), that is when \( \overline{\Delta}_{H,B}(E_{-1}) = 0 \), and
(b) when \( E_0 \) is torsion free \( E_0 \in HN^H_{H,B}([t + \sqrt{t^2 + \alpha^2}, +\infty)) \) with equality \( \nu_{H,B}(E_0) = t + \sqrt{t^2 + \alpha^2} \) holds if and only if \( H^2 \text{ch}^2(E_0) = 0 \), that is when \( \Delta_{H,B}(E_0) = 0 \).

**Definition 3.5.** For an object \( E \) and \( \delta \in \mathbb{R}_{\geq 0} \), we define
\[
\Psi^+_{H,B,\alpha,\delta}(E) = \nu_{H,B,\alpha}(E) \pm \sqrt{\nu_{H,B,\alpha}(E)^2 + \alpha^2 + \delta}.
\]

**Proposition 3.6.** Let \( E \in B_{H,B} \) be a tilt stable object with \( \nu_{H,B,\alpha}(E) = t < +\infty \). Then we have the following:

(i) \( H^2 \text{ch}^2(E) - \sqrt{t^2 + \alpha^2} \overline{\Delta}_{H,B}(E) \geq 0 \), with equality holds when \( \mathcal{H}^{-1}(E) = 0 \) and
\( (i) \mathcal{H}^{-1}(E) = 0 \) and \( \mathcal{H}^0(E) \) is a slope stable torsion free sheaf such that \( \mathcal{H}^0(E)^{**} \) is locally free with \( \overline{\Delta}_{H,B} = 0 \). In particular, when \( \nu_{H,B}(E) > 0 \),
\[
\mu_{H,B}(E) \geq t + \sqrt{t^2 + \alpha^2} = \Psi^+_{H,B,\alpha,0}(E).
\]

(ii) \( H^2 \text{ch}^2(E) + \sqrt{t^2 + \alpha^2} \overline{\Delta}_{H,B}(E) \geq 0 \), with equality holds when \( \mathcal{H}^0(E) = 0 \) and
\( \mathcal{H}^{-1}(E) = 0 \) and \( \mathcal{H}^{-1}(E) = 0 \) and \( \mathcal{H}^0(E) \) is a slope stable locally free sheaf with \( \overline{\Delta}_{H,B} = 0 \). In particular, when \( \nu_{H,B}(E) < 0 \),
\[
\mu_{H,B}(E) \leq t - \sqrt{t^2 + \alpha^2} = \Psi^-_{H,B,\alpha,0}(E).
\]

(iii) \( \overline{\Delta}_{H,B}(E) \geq 0 \); if \( \overline{\Delta}_{H,B}(E) = 0 \) then \( E \) is isomorphic to either \( E_{-1}[1] \) for some \( \mu_H \)

stable locally free sheaf \( E_{-1} \), or \( \mu_H \) stable torsion free sheaf \( E_0 \) such that \( E_0^{**} \) is locally free with \( E_0^{**}/E_0 \in \operatorname{Coh}_0(X) \).

**Proof.** We have \( H \text{ch}^2(B^{t+H})(E) = (1/2)(t^2 + \alpha^2) \text{ch}^2(E) \). Let us denote \( E_1 = \mathcal{H}^1(E) \) and
\[
D_1 = H^2 \text{ch}^2(B^{t+H})(E_1), \quad R_1 = \sqrt{t^2 + \alpha^2} \text{ch}^2(E_i).
\]

From Proposition 3.4, \( D_0 \geq R_0 \) and \( D_{-1} \leq -R_{-1} \).

(i) We have
\[
H^2 \text{ch}^2(B^{t+H})(E) - \sqrt{t^2 + \alpha^2} \text{ch}^2(E_0) = (D_0 - D_{-1}) - (R_0 - R_{-1})
\]
\[
= (D_0 - R_0) + (-D_{-1} + R_{-1})
\]
\[
\geq (D_0 - R_0) + 2R_{-1} \geq 0.
\]

Here the equality in the last “\( \geq \)” holds when \( D_0 = R_0 \) and \( R_{-1} = 0 \). Let us consider this case. We have \( E_{-1} = 0 \), that is \( E \cong E_0 \). Let us prove \( E_0 \) is a slope stable torsion free sheaf. Assume the opposite; so there exists a slope stable quotient sheaf \( G \) of \( E_0 \) with \( \mu_{H,B}(G) \leq t + \sqrt{t^2 + \alpha^2} \). Moreover, \( E_0 \to G \) is also a surjection in \( B_{H,B} \), and since \( E_0 \) is tilt stable, \( G \in HN^H_{H,B,\alpha}(t, +\infty) \). From (ii) of Proposition 3.4 \( \mu_{H,B}(G) > t + \sqrt{t^2 + \alpha^2} \); this is not possible. Hence \( E_0 \) is slope stable.

So \( \overline{\Delta}_{H,B}(E_0) = 0 \). From Lemma 2.3 slope stable reflexive sheaf \( E_0^{**} \) is locally free with \( \overline{\Delta}_{H,B} = 0 \); hence, \( E_0^{**}/E_0 \in \operatorname{Coh}_0(X) \).
Moreover, if we have one of the above equivalent inequalities for one can get the required inequalities in both directions.

As in the proof of (iii) of Proposition 3.6, we have

\[ \Delta_{H,B}(E) = (H^2 \chi_1^{B+tH}(E))^2 - (t^2 + \alpha^2)(H^3 \chi_0(E))^2 \]

\[ = \left\{ H^2 \chi_1^{B+tH}(E) - \sqrt{t^2 + \alpha^2} H^3 \chi_0(E) \right\} \times \left\{ H^2 \chi_1^{B+tH}(E) + \sqrt{t^2 + \alpha^2} H^3 \chi_0(E) \right\}. \]

From (i) and (ii), \( \Delta_{H,B}(E) \geq 0 \); and the equality holds when we have the equalities in either (i) or (ii).

**Proposition 3.7.** Let \( E \) be an \( \nu_{H,B,\alpha} \) semistable object in \( \mathcal{B}_{H,B} \) with \( \nu_{H,B,\alpha}(E) < +\infty \), \( \chi_0(E) > 0 \), and let \( \lambda_1, \lambda_2 \) be some non-negative constants. We have

\[ \lambda_1 < \frac{\Delta_{H,B}}{(H^3 \chi_0(E))^2} < \lambda_2 \]

if and only if

\[ \Psi_{H,B,\alpha,\lambda_1}^+(E) < \mu_{H,B}(E) < \Psi_{H,B,\alpha,\lambda_2}^+(E). \]

Moreover, if we have one of the above equivalent inequalities for \( E \), then

\[ \frac{\partial \Psi_{H,B,\alpha,\lambda_1}^+(E)}{\partial \alpha} > 0 > \frac{\partial \Psi_{H,B,\alpha,\lambda_2}^+(E)}{\partial \alpha}. \]

**Proof.** As in the proof of (iii) of Proposition 3.6 we have

\[ \Delta_{H,B}(E) = \left( H^2 \chi_1^{B+\nu_{H,B,\alpha}(E)H}(E) \right)^2 - \left( (\nu_{H,B,\alpha}(E))^2 + \alpha^2 \right) (H^3 \chi_0(E))^2. \]

From Proposition 3.6, \( H^2 \chi_1^{B+\nu_{H,B,\alpha}(E)H}(E) \geq 0 \) and since \( \chi_0(E) > 0 \), by direct computation one can get the required inequalities in both directions.

By differentiating \( \nu_{H,B,\alpha}(E) \) with respect to \( \alpha \) we get

\[ \frac{\partial \nu_{H,B,\alpha}(E)}{\partial \alpha} = -\frac{\alpha}{\mu_{H,B}(E)}. \]

By differentiating \( \Psi_{H,B,\alpha,\lambda_1}^+(E) \) with respect to \( \alpha \) we get

\[ \frac{\partial \Psi_{H,B,\alpha,\lambda_1}^+(E)}{\partial \alpha} = \frac{\partial \nu_{H,B,\alpha}(E)}{\partial \alpha} + \frac{1}{\sqrt{(\nu_{H,B,\alpha}(E))^2 + \alpha^2 + \lambda_1}} \left( \alpha + \nu_{H,B,\alpha}(E) \cdot \frac{\partial \nu_{H,B,\alpha}(E)}{\partial \alpha} \right) \]

\[ = \frac{\mu_{H,B}(E) \sqrt{(\nu_{H,B,\alpha}(E))^2 + \alpha^2 + \lambda_1}}{(\nu_{H,B,\alpha}(E))^2 + \alpha^2 + \lambda_1} \left( \mu_{H,B}(E) - \Psi_{H,B,\alpha,\lambda_1}^+(E) > 0 \right) \]

as required. Similarly one can get the other inequality.

**Remark 3.8.** One can have a similar Proposition considering \( \chi_0(E) < 0 \) case involving \( \Psi_{H,B,\alpha,\lambda_1}^-(E) \).

Recall the following result about the walls for tilt stable objects from [PT]:

**Proposition 3.9 ([PT] Lemma 3.15).** Let \( E \in \mathcal{B}_{H,B} \) be a tilt stable object with \( \nu_{H,B,\alpha}(E) < +\infty \). Then \( E \in \mathcal{B}_{H,B+bH} \) is \( \nu_{H,B+bH,\alpha} \)-stable for all \( a \in \mathbb{R}_{>0} \) and \( b \in \mathbb{R} \) such that

\[ a^2 + (b - \nu_{H,B,\alpha}(E))^2 = (\nu_{H,B,\alpha}(E))^2 + \alpha^2. \]

The following results are crucial for us.
Proposition 3.10 ([BMS] Lemma 2.7). Let $E \in \mathcal{B}_{H,B}$ be $\nu_{H,B,\alpha}$ tilt stable for all $\alpha > \alpha_0$ for some $\alpha_0 > 0$ with $\nu_{H,B,\alpha_0}(E) < +\infty$. Then we have the following:

(i) If $\text{ch}_0(E) > 0$ then $E$ is a slope semistable torsion free sheaf.

(ii) If $\text{ch}_0(E) = 0$ then $E$ is a slope semistable pure torsion sheaf in $\text{Coh}_{\leq 2}(X)$.

(iii) If $\text{ch}_0(E) < 0$ then $\mathcal{H}^{-1}(E)$ is a slope semistable reflexive sheaf and $\mathcal{H}^0(E) \subset \text{Coh}_{\leq 1}(X)$.

Proposition 3.11 ([BMT] Proposition 7.4.1). If $E$ is a $\mu_{H,B}$-stable sheaf with $\overline{\Delta}_{H,B}(E) = 0$, then $E$ or $E^{**}[1]$ in $\mathcal{B}_{H,B}$ is $\nu_{H,B,\alpha}$-(semi)stable.

We have following result for certain short exact sequences in $\mathcal{B}_{H,B}$.

Proposition 3.12. Let $0 \to E_1 \to E \to E_2 \to 0$ be a short exact sequence in $\mathcal{B}_{H,B}$ such that $E, E_1, E_2$ are $\nu_{H,B,\alpha}$-semistable with $\nu_{H,B,\alpha}(E_1) = \nu_{H,B,\alpha}(E_2) < +\infty$. Then

$$\overline{\Delta}_{H,B}(E) \geq \overline{\Delta}_{H,B}(E_1) + \overline{\Delta}_{H,B}(E_2),$$

where the equality holds when $\overline{\Delta}_{H,B} = 0$ for $E, E_1, E_2$.

Moreover, when $B \in \mathbb{R}(H)$, if $\overline{\Delta}_{H,B}(E_1) = 0$ and $\overline{\Delta}_{H,B}(E_j) > 0$ for $i \neq j$ with $i, j \in \{1, 2\}$, then

$$\overline{\Delta}_{H,B}(E) \geq \overline{\Delta}_{H,B}(E_j) + 1.$$
Proof. Let $E$ be a $\mu_H$ stable reflexive sheaf with $\overline{A}_H = 0$. Let $k$ be the rational number defined by

$$k = \frac{H^2 \chi_1(E)}{H^0 \chi_0(E)}.$$ 

From Proposition 3.11 $E$ and $E[1]$ are tilt stable. In particular, let us consider the tilt slopes with respect to the stability parameters

$$\beta = k - (r/2),$$

$$\alpha = r/2 - \varepsilon$$

for some $\varepsilon \in (0, r/2)$. We have $E, E(-rH)[1] \in B_{H, \beta H}$, and by direct computation,

$$\nu_{H, \beta H, \alpha}(E) > 0 > \nu_{H, \beta H, \alpha}(E(-rH)[1]).$$

So we have

$$\text{Hom}_X(E, E(-rH)[1]) = 0.$$ 

Since $\omega_X = O_X(-rH)$, from the Serre duality, $\text{Hom}_X(E, E[2]) = 0$. Since $E$ is tilt stable $\text{Hom}_X(E, E) \cong \mathbb{C}$; therefore

$$\chi(E, E) = \sum_i \text{hom}_X(E, E[i]) \leq \text{hom}_X(E, E) + \text{hom}_X(E, E[2]) = \text{hom}_X(E, E) = 1.$$ 

On the other hand from the Riemann-Roch formula \[4\]

$$\chi(E, E) = \int_X \chi(E) \chi(E) \cdot H = -\frac{r}{2} H(ch_1(E)^2 - 2ch_0(E)ch_2(E)) + ch_0(E)^2.$$ 

Therefore, $\chi(E, E) \leq 1$ implies

$$ch_0(E)^2 \leq 1 + \frac{r}{2} H\overline{A}(E).$$ 

Since $0 = \overline{A}_H(E) \geq H^3H\overline{A}(E) \geq 0$, we have $H\overline{A}(E) = 0$. Therefore, $ch_0(E) \leq 1$. Since $E$ is torsion free $ch_0(E)$ is integral, and so we have $ch_0(E) = 1$. Also the reflexivity of $E$ implies it is a line bundle. So $ch(E) = e^D$ for some $D \in NS(X)$. Since $\overline{A}_H(E) = 0$, we have $(H^2D)^2 = H^3HD^2$, that is $D \in \mathbb{Z}(H)$ as required. This completes the proof. \hfill $\Box$

3.2. Tilt stability under dualizing.

Notation 3.14. For $E \in B_{H, B}$ we write

$$E^i = H^i_{B_{H, B}}(E^\vee).$$

So for example

$$E^{12} = H^2_{B_{H, B}} \left( \left( H^4_{B_{H, B}}(E^\vee) \right)^\vee \right).$$

We have the following.

Proposition 3.15. Let $E \in \text{HN}_{H, B, \alpha}((-\infty, +\infty))$. Then $E^i = 0$ for $i \neq 1, 2$ with $E^2 \in \text{Coh}_0(X)$.

Proof. For $E \in \text{HN}_{H, B, \alpha}((-\infty, +\infty))$, let us denote $E_j = J^i_j(E)$. Object $E$ fits into the short exact sequence

$$0 \to E_{-1}[1] \to E \to E_0 \to 0$$

in $B_{H, B}$. From Proposition 3.2 $E_{-1}$ is a reflexive sheaf.
By dualizing the above short exact sequence, we have the following distinguished triangle (7)

\[ E_0^i \to E^\vee \to E_{-1}^\vee [-1] \to E_0^i [1]. \]

By considering \( t \to +\infty \) in (i) of Proposition 3.15, we have \( E_{-1} \in \text{HN}^\mu_{H,B}((-\infty,0)) \). So \( E_{-1} \in \text{HN}^\mu_{H,B}((0, +\infty)) \). Since \( E_{-1} \) is reflexive, \( \mathcal{E}xt^1(E_{-1}, \mathcal{O}_X) = \mathcal{H}^1(E_{-1}^\vee) \in \text{Coh}_0(X) \) and \( \mathcal{E}xt^1(E_{-1}, \mathcal{O}_X) = \mathcal{H}^1(E_{-1}^\vee) = 0 \) for \( i \geq 2 \). Therefore, \( (E_{-1}[1])^1 = 0 \) for \( i \neq 1,2 \).

The sheaf \( E_0 \in \text{HN}^\mu_{H,B}((0, +\infty)) \) and so \( E_0^i \in \text{HN}^\mu_{H,B}((-\infty,0)) \subset \mathcal{B}_{H,B}[-1] \). Moreover, for \( i \geq 1 \), \( \mathcal{E}xt^1(E_0, \mathcal{O}_X) = \mathcal{H}^1(E_0^\vee) \in \text{Coh}_{\leq 1}(X) \in \mathcal{B}_{H,B} \). So \( E_0^i = 0 \) for \( i \neq 1,2,3 \) with \( E_0^2 \in \text{Coh}_{\leq 1}(X) \) and \( E_0^3 \in \text{Coh}_0(X) \). Therefore, by considering the long exact sequence of \( \mathcal{B}_{H,B} \)-cohomologies associated to the triangle (7), we have \( E^1 = 0 \) for \( i \neq 1,2,3 \) with \( E^2 \in \text{Coh}_{\leq 1}(X) \) and \( E^3 \in \text{Coh}_0(X) \).

For any \( x \in X \),

\[
\text{Hom}_x(E^3, \mathcal{O}_x) \cong \text{Hom}_x(E^\vee[3], \mathcal{O}_x)
\]

\[
\cong \text{Hom}_x(E^\vee, \mathcal{O}_x[-3])
\]

\[
\cong \text{Hom}_x((\mathcal{O}_x[-3])^\vee, E)
\]

\[
\cong \text{Hom}_x(\mathcal{O}_x, E) = 0,
\]

as the skyscraper sheaf \( \mathcal{O}_x \in \text{Coh}_0(X) \subset \text{HN}^\nu_{H,B,\alpha}(+\infty) \) and \( E \in \text{HN}^\nu_{H,B,\alpha}((-\infty, +\infty)). \) Therefore, \( E^3 = 0 \).

For any \( T \in \text{Coh}_1(X) \),

\[
\text{Hom}_x(E^2, T) \cong \text{Hom}_x(E^\vee[2], T)
\]

\[
\cong \text{Hom}_x(E^\vee, T[-2])
\]

\[
\cong \text{Hom}_x((T[-2])^\vee, E)
\]

\[
\cong \text{Hom}_x(\mathcal{E}xt^2(T, \mathcal{O}_X), E) = 0,
\]

as \( \mathcal{E}xt^2(T, \mathcal{O}_X) \in \text{Coh}_1(X) \subset \text{HN}^\nu_{H,B,\alpha}(+\infty) \) and \( E \in \text{HN}^\nu_{H,B,\alpha}((-\infty, +\infty)). \) Therefore, \( E^2 \in \text{Coh}_0(X) \). This completes the proof. \( \square \)

**Proposition 3.16.** We have the following for \( E \in \text{HN}^\nu_{H,B,\alpha}((-\infty, +\infty)) \):

(i) \( E \) fits into the short exact sequence

\[ 0 \to E \to E^{11} \to E^{23} \to 0 \]

in \( \mathcal{B}_{H,B} \), where \( E^{23} \in \text{Coh}_0(X) \),

(ii) \( E^{1,k} = 0 \) for \( k \neq 1 \),

(iii) \( \text{Hom}_x(\text{Coh}_{\leq 1}(X), E^1) = 0 \), and

(iv) \( \text{Hom}_x(\text{Coh}_0(X), E^1[1]) = 0 \).

**Proof.** By Proposition 3.15, \( E^1 = 0 \) for \( i \neq 1,2 \) and \( E^2 \in \text{Coh}_0(X) \). So \( (E^2)^\vee \cong E^{23}[3] \).

Since \( E^{\vee\vee} \cong E \), we have the spectral sequence:

\[
H^p_{\mathcal{B}_{H,B}} \left( \left( H^q_{\mathcal{B}_{H,B}}(E^{\vee}) \right)^\vee \right) \to H^{p+q}_{\mathcal{B}_{H,B}}(E).
\]

Consider the convergence of this spectral sequence for \( E \in \text{HN}^\nu_{H,B,\alpha}((-\infty, +\infty)). \) From the convergence we get \( E^{1,k} = 0 \) for \( k \neq 1 \) and also we have the short exact sequence \( 0 \to E \to E^{11} \to E^{23} \to 0 \) in \( \mathcal{B}_{H,B} \).
For $T \in \text{Coh}_{\leq 1}(X)$, $\text{Ext}^i(T, O_X) \in \text{Coh}_i(X)$; and so $T^\vee \in (B_{H,B}[-2], B_{H,B}[-3])$. On the other hand $(E^1)^\vee \in B_{H,B}[-1]$. Hence,

$$\text{Hom}_X(T, E^1) \cong \text{Hom}_X((E^1)^\vee, T^\vee) = 0$$

as required in part (iii).

For any skyscraper sheaf $O_x$ of $x \in X$, we have

$$\text{Hom}_X(O_x, E^1[1]) \cong \text{Hom}_X((E^1[1])^\vee, O_x^\vee) \cong \text{Hom}_X(E^{11}[2], O_x[-3]) = 0$$

as required in part (iv).

\[\square\]

**Proposition 3.17.** Let $E \in \text{HN}_{H,B,\alpha}((\infty, +\infty))$. Then

(i) $E$ is $\nu_{H,B,\alpha}$-stable (resp. $\nu_{H,B,\alpha}$-semistable) if and only if $E^{11}$ is $\nu_{H,B,\alpha}$-stable (resp. $\nu_{H,B,\alpha}$-semistable),

(ii) $\nu_{H,\alpha}(E^1) = -\nu_{H,B,\alpha}(E),

(iii) $E$ is $\nu_{H,B,\alpha}$-stable (resp. $\nu_{H,B,\alpha}$-semistable) if and only if $E^1$ is $\nu_{H,-B,\alpha}$-stable (resp. $\nu_{H,B,\alpha}$-semistable), and

(iv) $E^1 \in \text{HN}_{H,-B,\alpha}((\infty, +\infty))$.

**Proof.** From part (2) of Proposition 3.5 in [LM], we have (i).

By Proposition 3.15 and from definition of the twisted Chern character we have

$$-\text{ch}^B(E^1) + \text{ch}^B(E^2) = \text{ch}^B(E^\vee) = e^B \text{ch}(E^\vee) = (e^{-B} \text{ch}(E))^\vee = (\text{ch}^B(E))^\vee = (\text{ch}_0^B(E), -\text{ch}_1^B(E), \text{ch}_2^B(E), -\text{ch}_3^B(E)).$$

Since $E^2 \in \text{Coh}_0(X)$, we have $\nu_{H,-B,\alpha}(E^1) = -\nu_{H,B,\alpha}(E)$.

Let $E \in B_{H,B}$ be a $\nu_{H,B,\alpha}$-semistable object. Assume $E^1 \in B_{H,-B}$ is $\nu_{H,-B,\alpha}$-unstable. From the Harder-Narasimhan filtration there exists a quotient $E^1 \rightarrow Q$ in $B_{H,-B}$, where $Q$ is the lowest $\nu_{H,-B,\alpha}$-semistable Harder-Narasimhan factor. Since $\nu_{H,-B,\alpha}(E^1) = -\nu_{H,B,\alpha}(E)$, $\nu_{H,-B,\alpha}(Q) < \nu_{H,-B,\alpha}(E^1) < +\infty$. By (ii), $\nu_{H,B,\alpha}(Q^1) > \nu_{H,B,\alpha}(E^{11})$ with $Q^1 \hookrightarrow E^{11}$ in $B_{H,B}$; this is not possible as $E^{11}$ is $\nu_{H,B,\alpha}$-semistable by (i).

Part (iv) is a direct consequence of (iii).

Consequently, we have the following:

**Corollary 3.18.** We only need to check Bogomolov-Gieseker type inequalities in \[\text{BMT}, BMS, PT\], Conjectures 4.10 and 4.12 for tilt stable objects $E$ satisfying

- $E \cong E^{11}$
- $\text{ch}_0(E) \geq 0$ (or $\text{ch}_0(E) \leq 0$).

**Aside 3.19.** Let $E$ be an $\nu_{H,B,\alpha}$-stable object in $B_{H,B}$ with $\nu_{H,B,\alpha}(E) = 0$. By Proposition 3.16 it fits into the short exact sequence $0 \rightarrow E \rightarrow E^{11} \rightarrow E^{23} \rightarrow 0$ in $B_{H,B}$ with $E^{23} \in \text{Coh}_0(X)$. Moreover, by Proposition 3.17 $E^{11} \in B_{H,B}$ is $\nu_{H,B,\alpha}$-stable with $\nu_{H,B,\alpha}(E^{11}) = 0$.

Also by Proposition 3.16 $\text{Hom}_X(\text{Coh}_0(X), E^{11}[1]) = 0$. Hence by [MPT, Lemma 2.3] or [PT, Aside 2.12], $E^{11}[1] \in A_{H,B,\alpha}$ is a minimal object.

4. **Bogomolov-Gieseker Type Inequality Conjecture for 3-folds**

In this section we let $X$ be a smooth projective 3-fold.
4.1. Modified conjectural inequality. Let us modify the Bogomolov-Gieseker type inequality conjecture for our smooth projective 3-fold $X$.

First we introduce the expression of the inequality as follows:

**Definition 4.1.** Let us fix classes $H \in \text{NS}(X)$, $B \in \text{NS}_R(X)$ such that $H$ is ample, and $\Lambda \in H^4(X, \mathbb{Q})$ satisfying $\Lambda \cdot H = 0$. For $\xi, \alpha \in \mathbb{R}_{>0}$ and $\beta \in \mathbb{R}$, we define

$$D_{\alpha, \beta}^{B, \xi}(E) = \text{ch}_3^{B+\beta H}(E) + \left(\Lambda - \left(\xi + \frac{1}{6} \alpha^2\right) H^2\right) \text{ch}_1^{B+\beta H}(E).$$

**Remark 4.2.** In the next sections we are mostly interested in the following choice for $\Lambda$:

$$\Lambda = \frac{c_2(X)}{12} - \frac{c_2(X) \cdot H^3}{12 H^2}.$$ 

Since $\Lambda \cdot H = 0$, we can write

$$(8) \quad D_{\alpha, \beta}^{B, \xi}(E) = \text{ch}_3^{B+\beta H}(E) + \Lambda \text{ch}_1^{B}(E) - \left(\xi + \frac{1}{6} \alpha^2\right) H^2 \text{ch}_1^{B+\beta H}(E).$$

Moreover, $D_{\alpha, \beta}^{B, \xi}(E) = D_{\alpha, 0}^{B, \beta H, \xi}.$

**Conjecture 4.3.** Let us fix classes $H \in \text{NS}(X)$, $B \in \text{NS}_R(X)$ such that $H$ is ample. Then there exist $\Lambda \in H^4(X, \mathbb{Q})$ satisfying $\Lambda \cdot H = 0$, and for any $\alpha \in \mathbb{R}_{>0}$, $\beta \in \mathbb{R}$, there is a minimal constant $\xi(\alpha, \beta) \in \mathbb{R}_{\geq 0}$ such that all tilt slope $\nu_{H, B+\beta H, \alpha}$-stable objects $E \in \mathbb{P}_{H, B+\beta H}$ with $\nu_{H, B+\beta H, \alpha}(E) = 0$ satisfy the inequality:

$$D_{\alpha, \beta}^{\nu_{H, B+\beta H, \alpha}}(E) \leq 0.$$ 

Hence, for any $\xi \geq \xi(\alpha, \beta)$, we have $D_{\alpha, \beta}^{B, \xi}(E) \leq 0$.

**Remark 4.4.** Let

$$A : B + \mathbb{R}(H) \rightarrow \mathbb{R}_{\geq 0}$$

be a continuous function. Assume Conjecture 4.3 holds for all $\alpha \in \mathbb{R}_{>0}, \beta \in \mathbb{R}$ such that $\alpha \geq A(B + \beta H)$. Then we can define the following non-negative constant

$$\xi(A) = \max\{\xi(\alpha, \beta) : \alpha \in \mathbb{R}_{>0}, \beta \in \mathbb{R}, \alpha \geq A(B + \beta H)\}.$$ 

Therefore, in addition to Conjecture 4.3 we can conjecture the following for a family of stability parameters.

**Conjecture 4.5.** Let us fix classes $H \in \text{NS}(X)$, $B \in \text{NS}_R(X)$ such that $H$ is ample. Let $A : B + \mathbb{R}(H) \rightarrow \mathbb{R}_{\geq 0}$ be a continuous function. There exist $\Lambda \in H^4(X, \mathbb{Q})$ satisfying $\Lambda \cdot H = 0$, and a constant $\xi(A) \in \mathbb{R}_{\geq 0}$ such that for any $\alpha \in \mathbb{R}_{>0}, \beta \in \mathbb{R}$ with $\alpha \geq A(B + \beta H)$, all tilt slope $\nu_{H, B+\beta H, \alpha}$-stable objects $E \in \mathbb{P}_{H, B+\beta H}$ with $\nu_{H, B+\beta H, \alpha}(E) = 0$ satisfy the inequality:

$$D_{\alpha, \beta}^{B, \xi(A)}(E) \leq 0.$$ 

**Remark 4.6.** This modified conjectural inequality coincides with Bogomolov-Gieseker type inequality in [BMT] when

$$(9) \quad A = 0, \ \Lambda = 0, \ \text{and} \ \xi(A) = 0.$$ 

In this paper, we are mostly interested in the class $\Lambda$ as defined in Remark 4.2. Many 3-folds where the Bogomolov-Gieseker type inequality conjecture in [BMT] holds satisfy (9) for $\Lambda$...
We adapt some methods from [BMS, Section 5] and [Mac]. Conjecture 5.3 and in the next subsection we show that it is equivalent to Conjecture 4.5.

We verify the above conjecture for the blow-up of $P^3$ at a point in Section 8.

This modification of the conjectural inequalities does not affect the corresponding constructions of Bridgeland stability conditions. In particular, similar to [BMSZ, Lemma 8.3] we have the following:

\begin{equation}
A(B + (\beta + 1)H) = A(B + \beta H),
\end{equation}

and when $B \in \mathbb{R}\langle H \rangle$, $A(-B - \beta H) = A(B + \beta H)$.

The following is a straightforward expectation from the formulation of the modified Bogomolov-Gieseker type inequalities.

\begin{equation}
\xi(A') \leq \xi(A).
\end{equation}

In particular, we conjecture the following:

**Conjecture 4.10.** Suppose Conjecture 4.5 holds for $X$ with respect to some continuous function $A : B + \mathbb{R}\langle H \rangle \to \mathbb{R}_{\geq 0}$, such that $A'(B + \beta H) \geq A(B + \beta H)$ for all $\beta \in \mathbb{R}$. Then for minimal possible $\xi(A)$ and $\xi(A')$, we have $\xi(A') \leq \xi(A)$.

This modification of the conjectural inequalities does not affect the corresponding constructions of Bridgeland stability conditions. In particular, similar to [BMS, Section 5] we have the following:

**Theorem 4.12.** If Conjecture 4.5 holds for $X$ with respect to some $\alpha, \beta$ then the pair

\begin{equation}
(A_{H,B+\beta H,\alpha}, Z_{H,B+\beta H,\alpha})
\end{equation}

defines a Bridgeland stability condition on $X$. Here $A_{H,B+\beta H,\alpha}$ is the heart of a bounded t-structure as constructed in (3) of Section 2.1 and

\begin{equation}
Z_{H,B+\beta H,\alpha} = \left(-c_3^{B+\beta H} + bHc_2^{B+\beta H} + (-\Lambda + \alpha H^2)ch_1^{B+\beta H} + \sqrt{-1} \left(Hh_2^{B+\beta H} - \frac{\alpha^2}{2} H^3 ch_0 \right) \right)
\end{equation}

with $a, b \in \mathbb{R}$ satisfying $a > \xi(\alpha, \beta) + (\alpha^2/6) + (\alpha|b|/2)$.

\section*{4.2. Equivalent form of the conjecture.} In this subsection we formulate an equivalent form of Conjecture 4.5 which only considers the modified Bogomolov-Gieseker type inequalities for a small class of tilt stable objects. This can be considered as a modification of [BMS, Conjecture 5.3] and in the next subsection we show that it is equivalent to Conjecture 4.5.

We adapt some methods from [BMS, Section 5] and [Mac].

Let us consider the $v_{H,B+\beta H,\alpha}$ tilt stability parametrized by $\alpha \in \mathbb{R}_{> 0}$ and $\beta \in \mathbb{R}$. By definition

\begin{equation}
v_{H,B+\beta H,\alpha} = \frac{Hch_2^{B+\beta H} - (\alpha^2/2)H^3 ch_0}{H^2 ch_1^{B+\beta H}}.
\end{equation}
Hence, we consider
\[ Z_{\alpha, \beta}^\nu = -\left( H \text{ch}_2^{B+\beta H} - \frac{\alpha^2}{2} H^3 \text{ch}_0 \right) + \sqrt{-1} H^2 \text{ch}_1^{B+\beta H} \]
as the associated group homomorphism, more precisely, the weak stability function as introduced in [PT] of the corresponding tilt stability.

In the rest of this section, let us fix some \( \alpha_0 \in \mathbb{R} > 0 \).

**Definition 4.13.** Let \( E \) be an object in \( \mathcal{B}_{H, B} \) with \( \nu_{H, B, \alpha_0}(E) = 0 \).

\[ C(E) = \left\{ (\beta, \alpha) : \text{Re} \ Z_{\alpha, \beta}^\nu(E) = H \text{ch}_2^{B+\beta H}(E) - \frac{\alpha^2}{2} H^3 \text{ch}_0(E) = 0, \text{ and } 0 \leq \alpha \leq \alpha_0 \right\}. \]

Hence, \((0, \alpha_0) \in C(E)\). See Figures 1 and 2.

![Figure 1](image1.png)  
**Figure 1.** \( C(E) \) when \( \text{ch}_0(E) > 0 \)  

**Figure 2.** \( C(E) \) when \( \text{ch}_0(E) < 0 \)

Let \( A : B + \mathbb{R}(H) \to \mathbb{R}_{\geq 0} \) be a continuous function. For a given object \( E \), if we have
\[ \lim_{\alpha \to A(B+\beta H)^+} -\text{Re} \ Z_{\alpha, \beta}^\nu(E) = 0, \]
when \( \beta \to \overline{\beta} \), then \( \overline{\beta} \) satisfies
\[ (10) \quad H \text{ch}_2^{B+\overline{\beta} H}(E) - \frac{(A(B+\overline{\beta} H))^2}{2} H^3 \text{ch}_0(E) = 0. \]

That is, \( \overline{\beta}^2 (H^3 \text{ch}_0(E)) - 2\overline{\beta} (H^2 \text{ch}_1^B(E)) - (A(B+\overline{\beta} H))^2 (H^3 \text{ch}_0(E)) + 2H \text{ch}_2^B = 0. \)

**Definition 4.14.** We define \( \overline{A}_A(E) \) to be the set of roots \( \overline{\beta} \) of \( (10) \); so that \( (\overline{\beta}, A(B+\overline{\beta} H)) \in C(E) \) for each \( \overline{\beta} \in \overline{A}_A(E) \). See Figure 3.
Example 4.15. Unlike the case in Figure 3, the set $\overline{\beta}(E)$ can have many points. The following is such an example, and it appears in Section 8. For any $m \in \mathbb{Z}$, $O_X(mH)$ and $O_X(mH)[1]$ are tilt stable. Let us consider the continuous function $A : \mathbb{R}(H) \to \mathbb{R}_{\geq 0}$, defined by, for $\beta \in [-1/2, 0)$, $A(\beta H) = 1 + \beta$; for $\beta \in [0, 1/2)$, $A(\beta H) = 1 - \beta$; together with the relation $A((\beta + 1)H) = A(\beta H)$. One can check that $\overline{\beta}(O_X(mH)) = [m - 1, m - (1/2)]$. See Figure 4 for $\overline{\beta}(O_X(2H))$. 

We need the following definition extending the similar notation in [Li].

Definition 4.16. An object $E \in \mathbb{D}^b(X)$ is called $\overline{\beta}_A$-stable if for any $\overline{\beta} \in \overline{\beta}_A(E)$ there is an open neighbourhood $U \subset \mathbb{R}^2_{\beta, \alpha}$ containing $\overline{\beta}, A(B + \beta H)$ such that for any $(\beta, \alpha) \in U$ with $\alpha > 0$, $E \in B_{H, B + \beta H}$ is $\gamma_{H, B + \beta H, \alpha}$-stable.

Remark 4.17. When $A = 0$, the above notion of $\overline{\beta}_A$ stability is exactly the same notion of $\overline{\beta}$ stability in [Li].

From the definition of $\overline{\beta}_A$-stability and Proposition 3.17 we have the following:
Proposition 4.18. Let \( E \) be an object in \( D^b(X) \). Then \( E \) is \( \overline{\beta}_A \)-stable with respect to the stability parameters \( B \in \text{NS}_R(X) \), and some continuous function \( \Lambda : B + \mathbb{R} \langle H \rangle \to \mathbb{R}_{\geq 0} \) if and only if \( E^1 = H^1_{\overline{\beta}_A} (E^\vee) \) is \( \overline{\beta}_A \)-stable with respect to the stability parameters \( -B \in \text{NS}_R(X) \), and the continuous function \( \hat{\Lambda} : -B + \mathbb{R} \langle H \rangle \to \mathbb{R}_{\geq 0} \) defined by \( \hat{\Lambda}(-B - \beta H) = \Lambda(B + \beta H) \).

For a small class of tilt stable objects, Conjecture 4.19 reads as follows:

Conjecture 4.19. Let us fix classes \( H \in \text{NS}(X) \), \( B \in \text{NS}_R(X) \) such that \( H \) is ample. Let \( \Lambda : B + \mathbb{R} \langle H \rangle \to \mathbb{R}_{\geq 0} \) be a continuous function. There exist \( \Lambda \in H^4(X, \mathbb{Q}) \) satisfying \( \Lambda \cdot H = 0 \), and a constant \( \xi(A) \in \mathbb{R}_{\geq 0} \) such that any \( \overline{\beta}_A \)-stable object \( E \in D^b(X) \) satisfies the inequality

\[
D^{B, \xi(A)}_{A(B + \beta H), \overline{\beta}(E)} \leq 0, \quad \text{for each } \overline{\beta} \in \overline{\beta}_A(E).
\]

The following is the key theorem for us.

Theorem 4.20. Conjectures 4.18 and 4.19 are equivalent.

4.3. Proof of the equivalences of the conjectures. We need few results to prove Theorem 4.20.

Let \( \xi \in \mathbb{R}_{\geq 0} \) be some fixed constant.

Lemma 4.21. Let \( E \) be an object in \( \mathcal{P}_{H, B} \) with \( \nu_{H, B, \alpha_0}(E) = 0 \). Then along \( C(E) \) we have

\[
\frac{d}{d\alpha} \left( D^{B, \xi}_{\alpha, \overline{\beta}}(E) \right) = -\frac{\alpha \Delta_{H, B}(E) - 3\xi (H^3 \text{ch}_0(E))^2}{3 H^2 \text{ch}_1^{B+\beta H}(E)}.
\]

Proof. For \( (\beta, \alpha) \in C(E) \), we have \( H \text{ch}_2^{B+\beta H}(E) - (\alpha^2/2) H^3 \text{ch}_0(E) = 0 \). By differentiating both sides with respect to \( \alpha \) we get

\[
\frac{d\beta}{d\alpha} = -\frac{\alpha H^3 \text{ch}_0(E)}{H^2 \text{ch}_1^{B+\beta H}(E)}.
\]

By differentiating the expression of \( D^{B, \xi}_{\alpha, \overline{\beta}}(E) \) in (9) with respect to \( \alpha \), we get

\[
\frac{d}{d\alpha} \left( D^{B, \xi}_{\alpha, \overline{\beta}}(E) \right) = -H \text{ch}_2^{B+\beta H}(E) \frac{d\beta}{d\alpha} = \frac{\alpha H^3 \text{ch}_0(E)}{H^2 \text{ch}_1^{B+\beta H}(E)} \left( \xi + \frac{\alpha^2}{6} \right) H^3 \text{ch}_0(E) \frac{d\beta}{d\alpha}.
\]

Since \( H \text{ch}_2^{B+\beta H}(E) = (\alpha^2/2) H^3 \text{ch}_0(E) \) and by substituting the expression of \( d\beta/d\alpha \), we obtain the required expression. \( \square \)

Note 4.22. Let \( E \) be an object satisfying the conditions in above lemma. So \( H \text{ch}_2^{B}(E) = (\alpha_0^2/2) H^3 \text{ch}_0(E) \), and for \( (\beta, \alpha) \in C(E) \) we have

\[
(\beta^2 - \alpha^2) H^3 \text{ch}_0(E) - 2\beta H^2 \text{ch}_1^{B}(E) + \alpha^2 H^3 \text{ch}_0(E) = 0.
\]

Moreover, by Proposition 3.6 we have

\[
\Delta_{H, B}(E) = (H^2 \text{ch}_1^{B}(E))^2 - \alpha_0^2 (H^3 \text{ch}_0(E))^2 \geq 0.
\]

When \( \text{ch}_0(E) = 0 \), \( C(E) \) is a vertical line at \( \beta = 0 \) from \( \alpha = 0 \) to \( \alpha_0 \) in \( (\beta, \alpha) \)-plane.
Let us consider the case \( \text{ch}_0(E) \neq 0 \). By \( \| \) in Lemma 4.21 along \( C(E) \) at \((\beta, \alpha)\) we have
\[
\left( \frac{d \alpha}{d \beta} \right)^2 = \left( \frac{H^2 \text{ch}_0^B(E)}{\alpha H^3 \text{ch}_0(E)} \right)^2 = \frac{\overline{\Lambda}_{H,B}(E)}{\alpha^2(H^3 \text{ch}_0(E))^2} + 1 \\
\geq \frac{\overline{\Lambda}_{H,B}(E)}{\alpha^2(H^3 \text{ch}_0(E))^2} + 1 \geq 1.
\]

**Proposition 4.23.** Let \( E \in \mathcal{B}_{H,B} \) be a tilt stable object with \( \nu_{H,B,\alpha_0}(E) = 0 \). Then \( E \in \mathcal{B}_{H,B + \beta H} \) for \( \beta \in [-\alpha_0, \alpha_0] \); in particular \( E \in \mathcal{B}_{H,B + \beta H} \) for all \((\alpha, \beta) \in C(E)\).

**Proof.** From Proposition 3.4, we have \( B \) implies using contradiction method. Proof of Theorem 4.20. Let us consider the tilt stability of \( E \) with \( \alpha = 0 \) and \( \beta = \beta_H \) from the discussion in Note 4.22, for any \((\beta, \alpha)\) on \( C(E) \).

Let us prove the key theorem.

**Proof of Theorem 4.20.** One implication in the theorem is obvious. Let us prove the other implication using contradiction method.

Assume Conjecture 4.19 holds for our 3-fold \( X \), and there is a counterexample for Conjecture 4.3. Let \( E \in \mathcal{B}_{H,B} \) be a \( \nu_{H,B,\alpha_0} \) tilt stable object with \( \nu_{H,B,\alpha_0}(E) = 0 \). By deforming tilt stability parameters appropriately in a small neighbourhood, we can assume \( B \) is a rational class.

Suppose \( \text{D}^B_{\alpha_0,0}(E) > 0 \) for a contradiction. By Proposition 4.23 \( E \) stays in the same tilt category for all \((\beta, \alpha) \in C(E)\).

Let us consider the tilt stability of \( E \) along \( C(E) \) when \( \alpha \) is decreasing from \( P_0 = (0, \alpha_0) \).

**Notation.** For a sequence of pairs \( P_j = (\beta_j, \alpha_j), j \geq 0 \) in \( \mathbb{R}^2 \) we simply write
\[
\mathcal{B}_{P_j} = \mathcal{B}_{H,B + (\beta_1, \ldots, \beta_j)H}, \\
\nu_{P_j} = \nu_{H,B + (\beta_1, \ldots, \beta_j)H, \alpha_j}.
\]

By Proposition 3.6 \( \overline{\Lambda}_{H,B}(E) \geq 0 \). When \( \overline{\Lambda}_{H,B}(E) > 0 \), there might be a point \( P_1 = (\beta_1, \alpha_1) \in C(E) \) such that \( E \in \mathcal{B}_{P_1} \) becomes strictly \( \nu_{P_1} \)-semistable. From Lemma 4.21 we have
\[
0 < \text{D}^B_{\alpha_0,0} \nu_{\beta_1}(E) < \text{D}^B_{\alpha_1,0} \nu_{\beta_1}(E) = \text{D}^B_{\alpha_1,0}(E).
\]

From the Jordan-Hölder filtration of \( E \), there exists \( \nu_{P_1} \)-stable factor \( E_1 \in \mathcal{B}_{P_1} \) of \( E \) with \( \text{D}^B_{\alpha_1,0}(E_1) > 0 \). Moreover, from Proposition 3.12
\[
\overline{\Lambda}_{H,B}(E) > \overline{\Lambda}_{H,B}(E_1).
\]

Now we take \( E_1 \in \mathcal{B}_{P_1} \) and consider the tilt stability along \( C(E_1) \) in \( \alpha \) decreasing direction from \((0, \alpha_1) \in C(E_1)\). In this way there exists a sequence of points \( P_j = (\beta_j, \alpha_j) \in C(E_{j-1}) \) with
\[
\alpha_0 > \alpha_1 > \alpha_2 > \cdots > \alpha_j > \cdots
\]
\[
\text{D}^B_{\alpha_0,0} \nu_{\beta_1, \ldots, \beta_j}(E_j) > 0 \text{ for all } j, \text{ and}
\]
\[
\overline{\Lambda}_{H,B}(E) > \overline{\Lambda}_{H,B}(E_1) > \cdots > \overline{\Lambda}_{H,B}(E_j) > \cdots \geq 0.
\]

Since \( B \) is chosen to be rational, the image of \( \overline{\Lambda}_{H,B} \) forms a discrete set in \( \mathbb{R} \); hence, this sequence terminates. That is there exists \( E_j \in \mathcal{B}_{P_j} \) which is \( \nu_{P_j} \)-stable, with \( \text{D}^B_{\alpha_0,0} \nu_{\beta_1, \ldots, \beta_j}(E_j) > 0 \), and
(i) either \( \overline{\chi}_{H, \beta}(E_j) = 0, \)
(ii) or \( E_j \) is \( \nu_{H, B + (\beta_1 + \cdots + \beta_j + \beta)}(\beta, \alpha) \)-stable for all \( (\beta, \alpha) \in C(E_j). \)

From Propositions 5.6 and 5.11 in case (i) we have \( E_j \) is \( \nu_{H, B + (\beta_1 + \cdots + \beta_j + \beta)}(\beta, \alpha) \)-stable for all \( (\beta, \alpha) \in C(E_j). \) From Lemma 4.21 we have
\[
0 < D_{\alpha_i, 0}^{B_1 + \cdots + \beta_1} H, \xi(E_j) \leq D_{\alpha_i, \beta}^{B_1 + \cdots + \beta_1} H, \xi(E_j),
\]
where \( (\beta, \alpha) \in C(E_j) \), such that \( \overline{\chi} = \chi(A(B + (\beta_1 + \cdots + \beta_j + \beta)) H) \); that is \( \overline{\chi} \in \overline{\chi}_A(E_j) \). But this is not possible as we already assume Conjecture 4.19 holds for \( X \). This completes the proof.

5. Some Hom Vanishing Results for \( \overline{\chi}_A \) Stable Objects

We follow the same notation in Section 4 for our smooth projective 3-fold \( X \). Let \( H \in \text{NS}(X) \) be an ample divisor class. Let \( B \) be a class proportional to \( H \).

We have the following vanishing results for \( \overline{\chi}_A \)-stable objects.

Proposition 5.1. Let \( A : B + \mathbb{R}(H) \to \mathbb{R}_{\geq 0} \) be a continuous function. Let \( E \in \mathcal{D}^b(X) \) be a \( \overline{\chi}_A \)-stable object. Let \( (\overline{\chi}, B, A) \in C(E) \), such that \( \overline{\chi} = A(B \chi + (\beta_1 + \cdots + \beta_j + \beta)(H)) \). In other words, there is a small neighbourhood \( U \subset \mathbb{R}_{\beta, \alpha}^2 \) containing \( (\overline{\chi}, \beta, A) \), such that for any \( (\beta, \alpha) \in U \) with \( \alpha > 0 \), \( E_{H, \beta H} \alpha \)

is \( \nu_{H, \beta H, \alpha} \) tilt stable, satisfying \( H \chi_2^2 - (\overline{\chi}^2 / 2) H^3 \chi_0 = 0 \). Suppose \( H^2 \chi_1^2 H(E) > 0 \). For any \( k \in \mathbb{Z} \) we have the following:

(i) If \( k < \overline{\chi} - \overline{\chi} \) then for all \( j \leq 0 \)
\[
\text{Hom}_X(E, \mathcal{O}_X(k H)[1 + j]) = 0.
\]

(ii) If \( k = \overline{\chi} - \overline{\chi} \), with \( \overline{\chi}_H(E) > 0 \), and \( \overline{\chi} > 0 \), then for all \( j \leq 0 \)
\[
\text{Hom}_X(E, \mathcal{O}_X(k H)[1 + j]) = 0.
\]

(iii) If \( k > \overline{\chi} + \overline{\chi} \) then for all \( j \leq 0 \)
\[
\text{Hom}_X(\mathcal{O}(k H), E[j]) = 0.
\]

(iv) If \( k = \overline{\chi} + \overline{\chi} \), with \( \overline{\chi}_H(E) > 0 \), and \( \overline{\chi} > 0 \), then for all \( j \leq 0 \)
\[
\text{Hom}_X(\mathcal{O}(k H), E[j]) = 0.
\]

Proof. (i) Let \( k \) be an integer such that \( k < \overline{\chi} - \overline{\chi} \). Since \( E, \mathcal{O}_X(k H)[1] \in \mathcal{B}_{H, \beta H} \), we have \( \text{Hom}_X(E, \mathcal{O}_X(k H)[1 + j]) = 0 \) for all \( j \leq 1 \). Let us prove the Hom vanishing for \( j = 0 \) case. Let
\[
0 < \varepsilon < (\overline{\chi} - \overline{\chi} - k)/2.
\]
By Proposition 3.11
\[
\mathcal{O}_X(k H)[1] \in \mathcal{B}_{H, (\overline{\chi} - k - 2 \varepsilon) H}
\]
is \( \nu_{H, (\overline{\chi} - k - \varepsilon) H, \beta, \alpha} \)-stable with
\[
\nu_{H, (\overline{\chi} - k - \varepsilon) H, \beta, \alpha}(\mathcal{O}_X(k H)[1]) = - \frac{(\overline{\chi} - \overline{\chi} - k - 2 \varepsilon)(\overline{\chi} + \overline{\chi} - k)}{(\overline{\chi} - k - \varepsilon)} < 0.
\]


Since \( \text{ch}_2(\mathcal{P}_H^j(E)) - (\mathcal{A}^2/2)H^3 \text{ch}_0(E) = 0 \),
\[
\nu_{H,(\mathcal{P} - \epsilon)H,(\mathcal{A} + \epsilon)}(E) = \frac{\epsilon H^2 \text{ch}_1(\mathcal{P} - \mathcal{A})H^j(E)}{H^2 \text{ch}_1(\mathcal{P} - \epsilon)H^j(E)}.
\]

Since \( E \) is \( \beta_A \)-stable with \( H^2 \text{ch}_1(\mathcal{P}_H^j(E)) > 0 \), so for small enough \( \epsilon > 0 \), \( H^2 \text{ch}_1(\mathcal{P} - \epsilon)H^j(E) > 0 \). Also by (ii) of Proposition 3.6 \( H^2 \text{ch}_1(\mathcal{P} - \mathcal{A})H^j(E) \geq 0 \). Therefore,
\[
\nu_{H,(\mathcal{P} - \epsilon)H,(\mathcal{A} + \epsilon)}(E) \geq 0,
\]
and hence, we have \( \text{Hom}_X(E, \mathcal{O}_X(kH)[1]) = 0 \) as required.

(ii) For \( \epsilon > 0 \), by direct computation,
\[
\nu_{H,(\mathcal{P} - \epsilon)H,(\mathcal{A} - \epsilon)}(\mathcal{O}_X(kH)[1]) = 0, \quad \text{and}
\nu_{H,(\mathcal{P} - \epsilon)H,(\mathcal{A} - \epsilon)}(E) = \frac{\epsilon H^2 \text{ch}_1(\mathcal{P} - \mathcal{A})H^j(E)}{H^2 \text{ch}_1(\mathcal{P} - \epsilon)H^j(E)}.
\]

From (i) of Proposition 3.6 we have \( H^2 \text{ch}_1(\mathcal{P} - \mathcal{A})H^j(E) > 0 \), and so for small enough \( \epsilon > 0 \)
\[
\nu_{H,(\mathcal{P} - \epsilon)H,(\mathcal{A} - \epsilon)}(E) > 0.
\]

Therefore, we get the required \( \text{Hom} \) vanishing by comparing the tilt slopes of tilt stable objects \( \mathcal{O}_X(kH)[1] \) and \( E \) for small enough \( \epsilon > 0 \).

(iii) Let \( k \) be an integer such that \( k > \beta_A + \mathcal{A} \). Since \( E, \mathcal{O}_X(kH) \in \mathcal{B}_{H, \mathcal{P}_H} \), we have \( \text{Hom}_X(\mathcal{O}_X(kH), \mathcal{E}[j]) = 0 \) for all \( j \leq -1 \). Let us prove the vanishing for \( j = 0 \) case.

From Proposition 4.18 \( E^1 \in D^b(X) \) is \( \beta_A \)-stable where \( \mathcal{A} \) is defined by \( \hat{\mathcal{A}}(\beta_H) = \mathcal{A}(-\beta_H) \). Hence, \( \beta_A(E^1) = -\beta_A(E) \). So from part (i), for \( -k < -\beta - \mathcal{A} \), we have \( \text{Hom}_X(E^1, \mathcal{O}_X(-kH)[1]) = 0 \). By Proposition 3.16 \( E \) fits into the short exact sequence:
\[
0 \to E \to E^{11} \to E^{23} \to 0
\]
in \( \mathcal{B}_{H, \mathcal{P}_H} \) with \( E^{23} \in \text{Coh}_0(X) \). By applying the functor \( \text{Hom}_X(\mathcal{O}(kH), -) \) we get
\[
\text{Hom}_X(\mathcal{O}(kH), E) \hookrightarrow \text{Hom}_X(\mathcal{O}(kH), E^{11}) = \text{Hom}_X(E^1, \mathcal{O}_X(-kH)[1]) = 0.
\]

So we have \( \text{Hom}_X(\mathcal{O}(kH), E) = 0 \) as required.

(iv) For \( \epsilon > 0 \), by direct computation,
\[
\nu_{H,(\mathcal{P} + \epsilon)H,(\mathcal{A} - \epsilon)}(\mathcal{O}_X(kH)) = 0, \quad \text{and}
\nu_{H,(\mathcal{P} + \epsilon)H,(\mathcal{A} - \epsilon)}(E) = \frac{-\epsilon H^2 \text{ch}_1(\mathcal{P} + \mathcal{A})H^j(E)}{H^2 \text{ch}_1(\mathcal{P} + \epsilon)H^j(E)}.
\]

From (i) of Proposition 3.6 we have \( H^2 \text{ch}_1(\mathcal{P} + \mathcal{A})H^j(E) > 0 \), and so for small enough \( \epsilon > 0 \)
\[
\nu_{H,(\mathcal{P} + \epsilon)H,(\mathcal{A} - \epsilon)}(E) > 0.
\]

Therefore, we get the required \( \text{Hom} \) vanishing by comparing the tilt slopes of tilt stable objects \( \mathcal{O}_X(kH) \) and \( E \) for small enough \( \epsilon > 0 \). \( \square \)
6. Strong Form of Bogomolov-Gieseker Inequality for Tilt Stable Objects

6.1. Formulation of the Inequality. This section discusses a strong form of Bogomolov-Gieseker inequality for tilt stable objects. In the earliest preprint [Piy3] of this work this generalized formulation appeared, and that was somewhat similar to the one appeared in the previous preprints of [BMSZ] by Bernardara-Macrì-Schmidt-Zhao. However, there were some issues in those formulations, and the authors of [BMSZ] fixed the problem in their published work. In particular, they formed a strong form of Bogomolov-Gieseker inequality for tilt stable objects generalizing the previous work of Li in [Li]. In this section we further generalize [BMSZ] Theorem 3.1.

First we need some notions for Fano 3-folds. Suppose $X$ be a Fano 3-fold of index $r$. So $-K_X = rH$ for some ample divisor class $H$, where $r \in \{1, 2, 3, 4\}$. Let $d = H^3$ be the degree of $X$. Let $\rho(X) = \text{rk NS}(X)$ be its Picard rank.

**Notation 6.1.** We use the following notation in the rest of this paper:

- $\mu_{H, \beta} = \mu_{H, \beta H}$ and $\mu_H = \mu_{H, 0}$.
- $\mathcal{B}_\beta = \mathcal{B}_{H, \beta H}$.
- $\nu_{\beta, \alpha} = \nu_{H, \beta H, \alpha}$.
- We say $E \in \mathcal{B}_{H, \beta H}$ is $\nu_{H, \beta H, \alpha}$-(semi)stable simply by $E$ is tilt (semi)stable with respect to the stability parameter $(\beta, \alpha)$.
- $\overline{X}_H = \overline{X}_{H, \text{th}} = (H^2 ch_1)^2 - 2H^3 ch_0 H ch_2$ (see Definition 3.3).

**Definition 6.2.** If $\rho(X) > 1$ then, we define:

$$e_1(X) = \min \{ (H^2 D)^2 - H^3 (HD^2) > 0 : D \in \text{NS}(X) \},$$

$$e_2(X) = \min \{ (H^2 D)^2 + 1 : D \in \text{NS}(X) \text{ is effective} \},$$

$$\kappa(X) = \min \left\{ \frac{e_1(X)}{d^2}, \frac{e_2(X)}{d^2}, \frac{3}{2rd} \right\}.$$  

Otherwise, that is, for $\rho(X) = 1$ we set

$$\kappa(X) = \frac{3}{2rd}.$$

**Example 6.3.** Let $X$ be the blowup of $\mathbb{P}^3$ at a point. Let $f : X \rightarrow \mathbb{P}^3$ be the blow up morphism. Let $L = c_1(f^* O_{\mathbb{P}^3}(1))$ and let $E$ be the exceptional divisor class. We have $L^3 = E^3 = 1$, and $L^i E^j = 0$ for $i, j \neq 0$. Also the group $\text{NS}(X) = \mathbb{Z}(L, E)$. From the blowup formula, $-K_X = 4L - 2E$ and so $X$ is a index 2 Fano 3-fold. Therefore, in the above notation $H = 2L - E$.

By direct computation, the degree of $X$ is $d = H^3 = 7$.

Let $D = aL + bE$ for some $a, b \in \mathbb{Z}$. Then we have $H^2 D = (4a + b)$, $HD^2 = (2a^2 - b^2)$, and so $(H^2 D)^2 - H^3 (HD^2) = 2(a + 2b)^2$. Hence we have

$$e_1(X) = 2, \quad e_2(X) = 2, \quad \text{and} \quad 3/(2rd) = 1/21.$$

Therefore,

$$\kappa(X) = \frac{2}{49}.$$

**Example 6.4.** Let us consider the Fano 3-fold $X = \mathbb{P}^2 \times \mathbb{P}^1$ which is of index one. Let $p_1 : X \rightarrow \mathbb{P}^2$, $p_2 : X \rightarrow \mathbb{P}^1$ be the corresponding projections. Denote $L_1 = c_1(p_1^* O_{\mathbb{P}^2}(1))$ and
L_2 = c_1(p_1^*O_p(1)). Then NS(X) = Z(L_1, L_2), and L_1^2L_2 = 1. Also \(-K_X = H = 3L_1 + 2L_2\) and the degree of X is \(d = H^3 = 54\).

Let \(D = aL_1 + bL_2\) for some \(a, b \in \mathbb{Z}\). Then we have \(H^2D = (12a + 9b), HD^2 = (2a^2 + 6ab)\), and so \((H^2D)^2 - H^3(HD^2) = 9(2a - 3b)^2\). Hence we have

\[
\begin{align*}
\epsilon_1(X) &= 9, \\
\epsilon_2(X) &= 9 + 12 + 1 = 22, \quad \text{and} \quad 3/(2\tau d) = 1/36.
\end{align*}
\]

Therefore,

\[
\kappa(X) = \frac{\epsilon_1(X)}{d^2} = \frac{1}{324}.
\]

The aim of the rest of this section is to prove the following, which generalizes [BMSZ, Theorem 3.1].

**Theorem 6.5.** Let \(E\) be a \(\nu_{\beta_0, \alpha_0}\)-tilt stable object with finite tilt slope and non-isomorphic to \(\mathcal{O}_X(mH)[1]\) or \(\mathcal{O}_Z(mH)\) for any \(m \in \mathbb{Z}\) and 0-subscheme \(Z \subset X\). Let us suppose \(\text{ch}_0(E) \neq 0\),

\[
\begin{align*}
(12) & \quad \text{if } \text{ch}_0(E) > 0, \text{ there exist no integers between} \\
& \quad \beta_0 + \nu_{\beta_0, \alpha_0}(E) + \sqrt{(\nu_{\beta_0, \alpha_0}(E))^2 + \alpha_0^2}, \text{ and} \\
& \quad 2\mu_H(E) - \beta_0 - \nu_{\beta_0, \alpha_0}(E) - \sqrt{(\nu_{\beta_0, \alpha_0}(E))^2 + \alpha_0^2}, \\
(13) & \quad \text{if } \text{ch}_0(E) < 0, \text{ there exist no integers between} \\
& \quad -\beta_0 - \nu_{\beta_0, \alpha_0}(E) + \sqrt{(\nu_{\beta_0, \alpha_0}(E))^2 + \alpha_0^2}, \text{ and} \\
& \quad -2\mu_H(E) + \beta_0 + \nu_{\beta_0, \alpha_0}(E) - \sqrt{(\nu_{\beta_0, \alpha_0}(E))^2 + \alpha_0^2}, \\
& \quad \text{and} \\
(14) & \quad 2\mu_H - (\beta_0 + \nu_{\beta_0, \alpha_0}(E)) < \tau.
\end{align*}
\]

Then

\[
\frac{\overline{\Delta}_H(E)}{(H^3\text{ch}_0(E))^2} \geq \kappa(X).
\]

The proof of this theorem is similar to that of [BMSZ, Theorem 3.1], and we discuss it in the next subsections.

**Remark 6.6.** Suppose \(E\) be an object as in Theorem 6.5 with \(\text{ch}_0(E) > 0\). Let us write \(\tilde{\alpha}_0 = \sqrt{(\nu_{\beta_0, \alpha_0}(E))^2 + \alpha_0^2}\), and \(\tilde{\beta}_0 = \beta_0 + \nu_{\beta_0, \alpha_0}(E)\). We have \(\overline{\Delta}_H(E) = (H^2\text{ch}_1(\tilde{\beta}_0H(E))^2 - \tilde{\alpha}_0^2(H^3\text{ch}_0(E))^2)\). So

\[
\frac{\overline{\Delta}_H(E)}{(H^3\text{ch}_0(E))^2} = \frac{H^2\text{ch}_1(\tilde{\beta}_0-\tilde{\alpha}_0)H(E) + H^2\text{ch}_1(\tilde{\beta}_0+\tilde{\alpha}_0)H(E)}{(H^3\text{ch}_0(E))^2} = \frac{(H^2\text{ch}_1(\tilde{\beta}_0+\tilde{\alpha}_0)H(E))^2 + 2\tilde{\alpha}_0H^3 \cdot H^2\text{ch}_1(\tilde{\beta}_0+\tilde{\alpha}_0)H(E)}{(H^3\text{ch}_0(E))^2}.
\]

Therefore,

\[
\frac{(H^2\text{ch}_1(\tilde{\beta}_0+\tilde{\alpha}_0)H(E))^2}{(H^3\text{ch}_0(E))^2} < \frac{\overline{\Delta}_H(E)}{(H^3\text{ch}_0(E))^2}.
\]
Hence,
\[ \mu_H - \tilde{\beta}_0 - \tilde{\alpha}_0 < \sqrt{\frac{\Delta_H(E)}{(H^3 ch_0(E))^2}}. \]
That is,
\[ \mu_H - \sqrt{\frac{\Delta_H(E)}{(H^3 ch_0(E))^2}} < \tilde{\beta}_0 + \tilde{\alpha}_0, \]
\[ 2\mu_H - (\tilde{\beta} + \tilde{\alpha}_0) < \mu_H + \sqrt{\frac{\Delta_H(E)}{(H^3 ch_0(E))^2}}. \]
So we have (12) of Theorem 6.5 when there are no integers in the interval
\[ \left( \mu_H - \sqrt{\frac{\Delta_H(E)}{(H^3 ch_0(E))^2}}, \mu_H + \sqrt{\frac{\Delta_H(E)}{(H^3 ch_0(E))^2}} \right). \]
This is the interval that the authors used in the formulation of Theorem 3.1 in [BMSZ]. Similarly one can consider the case \( ch_0(E) < 0 \).
Clearly we have \( \kappa(X) \geq \min \left\{ \frac{1}{d^2}, \frac{3}{(2\text{rd})} \right\} \); where the later constant was considered in [BMSZ] Theorem 3.1. In particular, for Examples 6.3 and 6.4 we have \( \kappa(X) > \min \left\{ \frac{1}{d^2}, \frac{3}{(2\text{rd})} \right\} \).

6.2. Strong form of Bogomolov-Gieseker inequality for \(|ch_0| = 1\) case.

**Proposition 6.7.** Let \( E \) be a tilt stable object with a finite slope, \( \Delta_H(E) > 0 \) and \(|ch_0(E)| = 1\). Then
\[ \frac{\Delta_H(E)}{d^2} \geq \kappa(X). \]

**Proof.** Let \( E \in B_{\beta_0} \) be a \( \nu_{\beta_0, \alpha_0} \) tilt stable object with a finite tilt slope. Since tilt stability is preserved under small deformation of the numerical parameters, we can choose \( \beta_0 \in \mathbb{Q} \).

Now consider the stability of \( E \) along the line \( \beta = \beta_0 \) from \( \alpha = \alpha_0 \) in the \( \alpha \) increasing direction on \( \mathbb{R}^2_{\beta, \alpha} \) plane. There might be a point \( P = (\beta_0, \alpha_0) \) where \( E \) becomes strictly semistable. Let \( E_i \) be the Jordan-Hölder tilt stable factors of \( E \) at \( P \). Since \( E \) has a finite slope, all \( E_i \)'s have finite tilt slopes.

Now consider the tilt stability of each \( E_i \) in the \( \alpha \) increasing direction from \( P \) along \( \beta = \beta_0 \). So each \( E_i \) has Jordan-Hölder tilt stable factors \( E_{i,j} \) at a point \( P_i \). In this way we can find a sequence of tilt stable objects \( \{E_{i,j,k,...}\} \), with finite tilt slopes. So
\[ H^2 ch^0_{H}(E) = \sum_{i,j,k,...} H^2 ch^0_{H}(E_{i,j,k,...}), \text{ and } 0 < H^2 ch^0_{H}(E_{i,j,k,...}) < \infty. \]
Since \( \beta_0 \in \mathbb{Q} \), this sequence terminates. That is we have a finite collection of objects \( \{F_s\} = \{E_{i,j,k,...}\} \).
in $\mathcal{B}_{\beta_0}$ which are tilt stable for $\alpha \geq R$ for some finite $R > 0$. Moreover, by repeatedly applying Proposition 3.12, we have

$$\Delta_H(E) \geq \sum_s \Delta_H(F_s),$$

where the equality holds only when all $\Delta_H(F_s) = 0$.

If all $F_s$ have $\text{ch}_1(F_s) \in \mathbb{Z}(H)$, then by definition $\Delta_H(E) = H^3 H(\text{ch}_1(E)^2 - 2 \text{ch}_2(E)) = 2H^3 H \cdot C$ for some $C \in H_2(X, \mathbb{Z})$. So $\Delta_H(E) \geq 2d > 3d/(2r) \geq d^2 \cdot \kappa(X)$ as required.

Otherwise, there exists at least one $F_s$ with $\text{ch}_1(F_s) \not\in \mathbb{Z}(H)$, and we have one of the following cases:

- If $\text{ch}_0(F_s) > 0$ then from Proposition 3.10, $F_s \equiv H^0(F_s)$ is a slope semistable torsion free sheaf and so
  
  $$\Delta_H(E) \geq \Delta_H(F_s) = H^3 H \cdot \Delta(F_s) + (H^2 \text{ch}_1(F_s))^2 - H^3 H \cdot (\text{ch}_1(F_s))^2 \geq (H^2 \text{ch}_1(F_s))^2 - H^3 H \cdot (\text{ch}_1(F_s))^2 \geq e_1(X) \geq d^2 \cdot \kappa(X),$$

  as required.

- If $\text{ch}_0(F_s) < 0$ then from Proposition 3.12, $\mathcal{K}^{-1}(F_s)$ is a slope semistable reflexive sheaf and $\mathcal{K}^0(F_s) \in \text{Coh}_{\leq 1}(X)$; so

  $$\Delta_H(E) \geq \Delta_H(F_s) = \Delta_H(\mathcal{K}^{-1}(F_s)) + 2H^3 \text{ch}_0(\mathcal{K}^{-1}(F_s)) \text{ch}_2(\mathcal{K}^0(F_s)) \geq \Delta_H(\mathcal{K}^{-1}(F_s))$$

  $$= H^3 H \cdot \Delta(\mathcal{K}^{-1}(F_s)) + (H^2 \text{ch}_1(\mathcal{K}^{-1}(F_s))^2 - H^3 H \cdot (\text{ch}_1(\mathcal{K}^{-1}(F_s))^2)$$

  $$\geq (H^2 \text{ch}_1(\mathcal{K}^{-1}(F_s))^2 - H^3 H \cdot (\text{ch}_1(\mathcal{K}^{-1}(F_s))^2 \geq e_1(X) \geq d^2 \cdot \kappa(X),$$

  as required.

- If $\text{ch}_0(F_s) = 0$, then from Proposition 3.10, $F_s$ is a tilt stable sheaf in $\text{Coh}_{\leq 2}(X) \setminus \text{Coh}_{\leq 1}(X)$. So from Proposition 3.12,

  $$\Delta_H(E) \geq \Delta_H(F_s) + 1 = (H^2 \text{ch}_1(F_s))^2 + 1 \geq e_2(X) \geq d^2 \cdot \kappa(X),$$

  as required.

This completes the proof. \qed

6.3. Proof of the strong form of Bogomolov-Gieseker inequality. Assume there is a counter example to Theorem 6.3. From Proposition 6.7 it has $|\text{ch}_0| \geq 2$. Also from Proposition 3.17 we can assume $\text{ch}_0 \geq 2$. Let $E \in \mathcal{B}_{\beta_0}$ be such a $\nu_{\beta_0, \alpha}$ tilt stable object with minimum $\Delta_H \geq 0$. From Lemma 3.13 $\Delta_H(E) \geq 0$. Also from Proposition 3.17 we can assume

$$E \equiv E^{11} = H_{\beta_0}^1 (H_{\beta_0}^1 (E^\vee)) \vee.$$

So we have

$$0 < \frac{\Delta_H(E)}{(H^3 \text{ch}_0(E))^2} < \kappa(X).$$

Recall, Notation 3.5

$$\Psi_{\beta_0, \alpha_0, \lambda}(E) := \Psi_{H, \beta_0, H, \alpha_0, \lambda}(E) = \nu_{\beta_0, \alpha_0}(E) + \sqrt{\alpha_0^2 + (\nu_{\beta_0, \alpha_0}(E))^2 + \lambda}.$$

**Proposition 6.8.** We have $E \equiv H^0(E)$ and it is $\nu_{\beta_0, \alpha}$ tilt stable for all $\alpha \geq \alpha_0$. In particular, from Proposition 3.10 $E$ is a $H$-slope semistable reflexive sheaf.
Proof. Consider the stability of $E$ along the line $\beta = \beta_0$ from $\alpha = \alpha_0$ in the $\alpha$ increasing direction on $\mathbb{R}^2_{\beta, \alpha}$ plane. Then there might be a point $P_1 = (\beta_0, \alpha_1)$ where $E$ becomes strictly semistable.

Let 
\[
E_i, \quad i = 1, \ldots, N,
\]
be the Jordan-Hölder tilt stable factors of $E$ at $P$. Since $E$ has a finite tilt slope, all $E_i$'s have finite tilt slopes. Therefore, $H^2 \cdot ch^{\beta_0 H}(E_i) > 0$ for all $i$.

There exist $\theta$ for $E$, and $\theta_i$ for each $E_i$ such that 
\[
H^3 \cdot ch_0(E_i) + \sqrt{-1} H^2 \cdot ch^1_0(E_i) \in \mathbb{R}_{>0} e^{\sqrt{-1} \theta_i},
\]
and 
\[
H^3 \cdot ch_0(E) + \sqrt{-1} H^2 \cdot ch^1_0(E) \in \mathbb{R}_{>0} e^{\sqrt{-1} \theta},
\]
satisfying $\theta \in (0, \pi/2)$ and $\theta_i \in (0, \pi)$. So we have 
\[
H^3 \cdot ch_0(E + \sqrt{-1} H^2 \cdot ch^1_0(E_i)) = \sum_i \left( H^3 \cdot ch_0(E_i) + \sqrt{-1} H^2 \cdot ch^1_0(E_i) \right).
\]

There exists an object $E_k$ such that 
\[
0 < \theta_k \leq \theta;
\]
because, otherwise all $\theta_k \in (\theta, \pi)$, and so from (14), $\theta \in (\theta, \pi)$; but this is not possible.

That is $ch_0(E_k) > 0$ and $0 < \mu_{H, \beta_0}(E_k) \leq \mu_{H, \beta_0}(E)$. From Proposition 3.7 we have 
\[
\psi_{\beta_0, \alpha_1, 0}^+(E_k) \leq \mu_{H, \beta_0}(E_k).
\]
Here the equality holds when $\Lambda_H(E_k) = 0$, and from Lemma 3.13 in this case we have 
\[
ch_0(E_k) = 1, \quad ch_1(E_k) \in \mathbb{Z}(H), \quad \text{and so}
\]
\[
\mu_H(E_k) = \beta_0 + \psi_{\beta_0, \alpha_1, 0}^+(E_k) \in \mathbb{Z}.
\]

From Proposition 3.7, 
\[
\psi_{\beta_0, \alpha_0, 0}^+(E) < \psi_{\beta_0, \alpha_1, 0}^+(E) < \mu_{H, \beta_0}(E) < \psi_{\beta_0, \alpha_1, \kappa(X)}^+(E) < \psi_{\beta_0, \alpha_0, \kappa(X)}^+(E).
\]
Since $\nu_{\beta_0, \alpha_1}(E_k) = \nu_{\beta_0, \alpha_1}(E)$, we have 
\[
\beta_0 + \psi_{\beta_0, \alpha_0, 0}^+(E) < \beta_0 + \psi_{\beta_0, \alpha_1, 0}^+(E) = \beta_0 + \psi_{\beta_0, \alpha_1, 0}^+(E_k) \leq \mu_H(E_k)
\]
\[
\leq \mu_H(E) < \beta_0 + \psi_{\beta_0, \alpha_1, \kappa(X)}^+(E) = \beta_0 + \psi_{\beta_0, \alpha_1, \kappa(X)}(E_k) < \beta_0 + \psi_{\beta_0, \alpha_0, \kappa(X)}^+(E).
\]
From assumption (12), there are no integers in the interval 
\[
(\beta_0 + \psi_{\beta_0, \alpha_0, 0}^+(E), \quad \mu_H(E)]
\]
So we have $\beta_0 + \psi_{\beta_0, \alpha_1, 0}^+(E) \not\in \mathbb{Z}$. That is, $\mu_H(E_k) \not\in \mathbb{Z}$, and so 
\[
\Lambda_H(E_k) > 0.
\]

On the other hand, from Proposition 3.12 
\[
\Lambda_H(E) \geq \sum_{1 \leq i \leq N} \Lambda_H(E_i) \geq \Lambda_H(E_k),
\]
where all the equalities hold only when $\Lambda_H(E) = 0$. Since $\Lambda_H(E), \Lambda_H(E_k) > 0$ we have 
\[
\Lambda_H(E) > \Lambda_H(E_k) > 0.
\]
Since $\beta_0 + \lambda_{\beta_0, \alpha_0, 0}(E_k) < \mu_H(E_k) < \beta_0 + \lambda_{\beta_0, \alpha_1, 0}(E_k)$, from Proposition 3.7, $E_k$ is also a tilt stable object which contradicts Theorem 6.5. Since $E$ is chosen with minimal $\Delta_H$, this is not possible; so $E$ stays tilt stable for all $(\beta_0, \alpha)$, $\alpha \geq \alpha_0$. Since $ch_0(E) > 0$, from Proposition 3.10

$$E \cong \mathcal{O}(E)$$

is a slope semistable torsion free sheaf. Since $E \cong E^1$, we have $E \cong E^{* *}$; that is $E$ is a reflexive sheaf. \hfill \square

**Definition 6.9.** For any object $F$ in $D^b(X)$

$$Z(F) = \{(\beta, \alpha) \in \mathbb{R}^2 : H ch_2^{\beta H}(F) - (\alpha^2/2)H^3 ch_0(F) = 0\}.$$ 

So $Z(F[1]) = Z(F)$.

Let us define

$$\tilde{\beta}_0 = \beta_0 + \lambda_{\beta_0, \alpha_0}(E), \quad \tilde{\alpha}_0 = \sqrt{\alpha_0^2 + \lambda_{\beta_0, \alpha_0}(E)}^2.$$ 

Then by direct computation one can verify that $(\tilde{\beta}_0, \tilde{\alpha}_0) \in Z(E)$ (also see (6) in Section 3).

From Proposition 3.9, $E$ is tilt stable, with zero tilt slope, with respect to $(\beta, \alpha) \in Z(E)$ such that $\beta < \tilde{\beta}_0$ and $\alpha \geq \tilde{\alpha}_0$.

Moreover, by solving $H ch_2^{\beta H}(E) - (\alpha^2/2)H^3 ch_0(E) = 0$, we have

$$(-\tilde{\beta}_0 + 2\mu_H(E), \tilde{\alpha}_0) \in Z(E).$$

Since $\mu_{H, \beta_0}(E) > \lambda_{\beta_0, \alpha_0, 0}(E) = (\tilde{\beta}_0 - \beta_0) + \tilde{\alpha}_0 > \tilde{\beta}_0 - \beta_0$, we have

$$\mu_H(E) < -\tilde{\beta}_0 + 2\mu_H(E).$$

As $E$ is a slope semistable reflexive sheaf $E[1] \in \mathcal{B}_{-\tilde{\beta}_0 + 2\mu_H(E)}$.

**Proposition 6.10.** The object $E[1] \in \mathcal{B}_{-\tilde{\beta}_0 + 2\mu_H(E)}$ is tilt stable with respect to

$$(-\tilde{\beta}_0 + 2\mu_H(E), \alpha), \text{ for any } \alpha \geq \tilde{\alpha}_0.$$ 

**Proof.** Since $E$ is a reflexive sheaf, from Proposition 3.17 the claim is equivalent to $E^* \in \mathcal{B}_{\tilde{\beta}_0 - 2\mu_H(E)}$ is tilt stable with respect to

$$(\tilde{\beta}_0 - 2\mu_H(E), \alpha), \text{ for any } \alpha \geq \tilde{\alpha}_0.$$ 

Let us denote

$$\hat{\beta}_0 = \tilde{\beta}_0 - 2\mu_H(E), \quad \hat{\alpha}_0 = \tilde{\alpha}_0.$$ 

Since $E \in \mathcal{B}_{\hat{\beta}_0}$ is $\lambda_{\hat{\beta}_0, \alpha}$ tilt stable for all $\alpha \geq \hat{\alpha}_0$, there exists large enough $\alpha > 0$ such that $E^* \in \mathcal{B}_{\hat{\beta}_0}$ is $\lambda_{\hat{\beta}_0, \alpha}$ tilt stable for all $\alpha \geq \alpha$.

Consider the tilt stability of $E^*$ along the line $\beta = \hat{\beta}_0$ from $\alpha = \alpha$ in the $\alpha$ decreasing direction on $\mathbb{R}^2_{\hat{\beta}_0, \alpha}$ plane. Assume for a contradiction there is a point $(\hat{\beta}_0, \hat{\alpha}_1)$ where $E^*$ becomes strictly semistable for some $\hat{\alpha}_1 > \hat{\alpha}_0$. Let $F_i$, $i = 1, \ldots, N'$, be the Jordan-Hölder tilt stable factors of $E^*$ with respect to $(\hat{\beta}_0, \hat{\alpha}_1)$. Since $E^*$ has a finite tilt slope, all $F_i$'s have finite tilt slopes. Therefore, $H^2 ch_1^{\hat{\beta}_0 H}(F_i) > 0$ for all $i$. 
There exist $\varphi$ for $E^*$ and $\varphi_i$ for each $F_i$, such that
\[
H^3 \operatorname{ch}_0(F_i) + \sqrt{-1} H^2 \operatorname{ch}_1 \hat{\beta}_0 H(F_i) \in \mathbb{R}_{>0} e^{\sqrt{-1} \varphi_i},
\]
and
\[
H^3 \operatorname{ch}_0(E^*) + \sqrt{-1} H^2 \operatorname{ch}_1 \hat{\beta}_0 H(E^*) \in \mathbb{R}_{>0} e^{\sqrt{-1} \varphi},
\]
satisfying $\varphi \in (0, \pi/2)$ and $\varphi_i \in (0, \pi)$. We have
\[
\tag{16} H^3 \operatorname{ch}_0(E^*) + \sqrt{-1} H^2 \operatorname{ch}_1 \hat{\beta}_0 H(E^*) = \sum_i \left( H^3 \operatorname{ch}_0(F_i) + \sqrt{-1} H^2 \operatorname{ch}_1 \hat{\beta}_0 H(F_i) \right).
\]
There exists an object $F_k$ such that
\[
0 < \varphi_k \leq \varphi,
\]
because, otherwise all $\varphi_k \in (\varphi, \pi)$, and so from \([10]\), $\varphi \in (\varphi, \pi)$; but this is not possible. That is $\operatorname{ch}_0(F_k) > 0$ and $0 < \mu_{H, \beta_0}(F_k) \leq \mu_{H, \beta_0}(E^*)$.

From Proposition 3.7, we have
\[
\Psi_{\hat{\beta}_0, \hat{\alpha}_i, 0}^+(F_k) \leq \mu_{H, \beta_0}(F_k).
\]
Here the equality holds when $\overline{\Delta}_H(F_k) = 0$, and in this case from Lemma 3.13 we have $\operatorname{ch}_0(F_k) = 1$, $\operatorname{ch}_1(F_k) \in \mathbb{Z}(H)$, and so
\[
\mu_H(F_k) = \hat{\beta}_0 + \Psi_{\hat{\beta}_0, \hat{\alpha}_i, 0}^+(F_k) \in \mathbb{Z}.
\]
From Proposition 3.7,
\[
\Psi_{\hat{\beta}_0, \hat{\alpha}_0, 0}^+(E^*) < \Psi_{\hat{\beta}_0, \hat{\alpha}_i, 0}^+(E^*) < \mu_{H, \beta_0}(E^*) < \Psi_{\hat{\beta}_0, \hat{\alpha}_0, \kappa(X)}^+(E^*) < \Psi_{\hat{\beta}_0, \hat{\alpha}_0, \kappa(X)}^+(E^*).
\]
Here $\hat{\alpha}_0 = \Psi_{\hat{\beta}_0, \hat{\alpha}_0, 0}^+(E^*)$ and $\Psi_{\hat{\beta}_0, \hat{\alpha}_0, \kappa(X)}^+(E^*) = \sqrt{\alpha_0^2 + \kappa(X)}$.

Since $\nu_{\hat{\beta}_0, \hat{\alpha}_i}(F_k) = \nu_{\hat{\beta}_0, \hat{\alpha}_0}(E^*)$ we have
\[
\hat{\beta}_0 + \hat{\alpha}_0 < \hat{\beta}_0 + \Psi_{\hat{\beta}_0, \hat{\alpha}_i, 0}^+(E^*) = \hat{\beta}_0 + \Psi_{\hat{\beta}_0, \hat{\alpha}_i, 0}^+(F_k) \leq \mu_H(F_k)
\]
\[
\leq \mu_H(E^*) < \hat{\beta}_0 + \Psi_{\hat{\beta}_0, \hat{\alpha}_i, \kappa(X)}^+(E^*) = \hat{\beta}_0 + \Psi_{\hat{\beta}_0, \hat{\alpha}_i, \kappa(X)}^+(F_k) < \hat{\beta}_0 + \sqrt{\alpha_0^2 + \kappa(X)}.
\]
We have $\hat{\beta}_0 + \hat{\alpha}_0 = \hat{\beta}_0 + \nu_{\beta_0, \alpha_0}(E) - 2 \mu_H(E) + \sqrt{(\nu_{\beta_0, \alpha_0}(E))^2 + \alpha_0^2}$, and $\mu_H(E^*) = -\mu_H(E)$.

Therefore, from the assumption \([12]\) in Theorem 6.5 there are no integers in the interval
\[
(\hat{\beta}_0 + \hat{\alpha}_0, \mu_H(E^*)].
\]
So $\mu_H(F_k) \not\in \mathbb{Z}$, and hence, $\overline{\Delta}_H(F_k) > 0$.

On the other hand, from Proposition 3.12
\[
\overline{\Delta}_H(E^*) \geq \sum_{1 \leq i \leq N^*} \overline{\Delta}_H(F_i) \geq \overline{\Delta}_H(F_k),
\]
where all the equalities hold only when, $\overline{\Delta}_H(E^*) = 0$. Since $\overline{\Delta}_H(E) = \overline{\Delta}(E^*), \overline{\Delta}_H(F_k) > 0$ we have
\[
\overline{\Delta}_H(E) > \overline{\Delta}_H(F_k) > 0.
\]
Since $\hat{\beta}_0 + \Psi_{\hat{\beta}_0, \hat{\alpha}_i, 0}^+(F_k) < \mu_H(F_k) < \hat{\beta}_0 + \Psi_{\hat{\beta}_0, \hat{\alpha}_i, \kappa(X)}^+(F_k)$, from Proposition 3.7 $F_k$ is also a tilt stable object which contradicts Theorem 6.5. Since $E$ is chosen with minimal $\overline{\Delta}_H$ this is not possible; so $E^*$ stays tilt stable for all $(\hat{\beta}_0, \hat{\alpha}), \alpha \geq \hat{\alpha}_0 = \tilde{\alpha}_0$ as required. \(\square\)
Proof of Theorem 6.5. Let us define

\[ C' = Z(E) \cap \{ \beta \leq \tilde{\beta}_0 \}, \]
\[ C'' = Z(E(-rH)[1]) \cap \{ \beta \geq 2\mu_H(E) - \tilde{\beta}_0 - r \}. \]

The objects \( E \) and \( E(-rH)[1] \) are tilt stable along the paths \( C' \) and \( C'' \) respectively, with zero tilt slopes.

Assumption (14) gives \( 2\mu_H(E) - \tilde{\beta}_0 - r < \tilde{\beta}_0 \). So \( C' \) and \( C'' \) intersect each other at \( Q = (\beta^0, \alpha^0) \). See Figure 5.

There exists a point \( Q_\epsilon = (\beta^0, \alpha^0 - \epsilon) \) close to \( Q \) such that \( E \) and \( E(-rH)[1] \) are both tilt stable at \( Q_\epsilon \) with \( \nu_{Q_\epsilon}(E(-rH)[1]) < 0 < \nu_{Q_\epsilon}(E) \).

So we have
\[ \text{Hom}_X(E, E(-rH)[1]) = 0. \]

From the Serre duality, \( \text{Hom}_X(E, E[2]) = 0. \) Since \( E \) is tilt stable, \( \text{Hom}_X(E, E) \cong \mathbb{C} \), Therefore,
\[ \chi(E, E) = \sum_{i \in \mathbb{Z}} \text{hom}_X(E, E[i]) \leq \text{hom}_X(E, E) + \text{hom}_X(E, E[2]) = \text{hom}_X(E, E) = 1. \]

On the other hand from the Riemann-Roch formula (4)
\[ \chi(E, E) = \int_X \text{ch}(E) \text{ch}(E^\vee) \text{td}_X = -(rH/2)(\text{ch}_1(E)^2 - 2\text{ch}_0(E)\text{ch}_2(E)) + \text{ch}_0(E)^2. \]

Therefore,
\[ H \cdot \Delta(E) \geq \frac{2}{r}(\text{ch}_0(E)^2 - 1). \]

Since \( \overline{\Delta}_H(E) \geq H^3H \cdot \Delta(E) \) and \( \text{ch}_0(E) \geq 2 \), we have
\[ \frac{\overline{\Delta}_H(E)}{(H^3 \text{ch}_0(E))^2} \geq \frac{2(\text{ch}_0(E))^2 - 1}{rH^3 \text{ch}_0(E)^2} = \frac{2}{rd} \left( 1 - \frac{1}{\text{ch}_0(E)^2} \right) \geq \frac{2}{rd} \left( 1 - \frac{1}{2^2} \right) = \frac{3}{2rd}. \]

But this is not possible as we have chosen \( \frac{\overline{\Delta}_H(E)}{(H^3 \text{ch}_0(E))^2} < \kappa(X) \leq \frac{3}{2rd} \). This is the required contradiction to complete the proof of Theorem 6.5. \( \square \)
7. Bogomolov-Gieseker Type Inequality for Fano 3-folds

Let $X$ be a Fano 3-fold of index $r$. So

$$-K_X = rH$$

for some ample divisor class $H$. Let the degree of $X$ be

$$d = H^3.$$  

We carry the same notation in Section 4 for our Fano 3-fold $X$. We only consider our modified Bogomolov-Gieseker type conjecture on $X$ when $B$ is proportional to $H$.

From Proposition 2.4, the Todd class of $X$ is

$$td(X) = \left(1, \frac{rH}{2}, \frac{r^2H^2}{12} + \frac{c_2(X)}{12}, 1\right).$$

**Proposition 7.1.** Let $E \in D^b(X)$. Then for any $\beta \in \mathbb{R}$, we have

$$\chi(E(-H)) = ch_3^H(E) + f_2(\beta)Hch_2^H(E) + \Lambda ch_1^H(E) + f_1(\beta)H^2ch_1^H(E) + f_0(\beta)H^3ch_0(E),$$

where

$$\Lambda = \frac{c_2(X)}{12} - \frac{2}{rd}H^2,$$

$$f_2(\beta) = \beta + \left(\frac{r}{2} - 1\right),$$

$$f_1(\beta) = \frac{\beta^2}{2} + \left(\frac{r}{2} - 1\right)\beta + \left(\frac{1}{2} - \frac{r}{2} + \frac{r^2}{12} + \frac{2}{rd}\right),$$

$$f_0(\beta) = \frac{\beta^3}{6} + \left(\frac{r}{2} - 1\right)\frac{\beta^2}{2} + \left(\frac{1}{2} - \frac{r}{2} + \frac{r^2}{12} + \frac{2}{rd}\right)\beta + \left(-\frac{1}{6} + \frac{r}{4} - \frac{r^2}{12} - \frac{2}{rd} + \frac{1}{d}\right).$$

**Proof.** From the Hirzebruch-Riemann-Roch theorem [1], we have

$$\chi(E(-H)) = ch_3(E(-H)) + \frac{rH}{2}ch_2(E(-H)) + \left(\frac{r^2H^2}{12} + \frac{c_2(X)}{12}\right)ch_1(E(-H)) + ch_0(E(-H)).$$

From Proposition 2.4, we have $(c_2(X),H)/(12H^3) = 2/(rd)$. Since $ch(E(-H)) = ch(E) \cdot e^{-H}$, and $ch(E) = ch_1^H(E) \cdot e^{H}$ one can get the required expression in terms of $ch_1^H$'s.  

**Note 7.2.** Let us use the Iskovskikh-Mori-Mukai classification of smooth Fano 3-folds, to find certain inequalities involving $f_0(\beta)$ and $f_1(\beta)$ in Proposition 7.1 when $\beta \in [0,1)$ according to the index $r$ of the Fano 3-fold $X$.

(i) When $r = 4$, $X$ is $\mathbb{P}^3$; so $d = H^3 = 1$. Hence, $f_0(\beta) = (\beta^3/6) + (\beta^2/2) + (\beta/3) \geq 0$, and $f_1(\beta) = (\beta^2/2) + \beta + (1/3) > 0$, for $\beta \in [0,1)$.

(ii) When $r = 3$, $X$ is a quadric; so $d = H^3 = 1$. Hence, $f_0(\beta) = (\beta^3/6) + (\beta^2/4) + (5\beta/12) + (1/6) > 0$, and $f_1(\beta) = (\beta^2/2) + (\beta/2) + (5/12) > 0$, for $\beta \in [0,1)$.

(iii) When $r = 2$, $1 \leq d \leq 7$. By simplifying $f_0(\beta) = (\beta^2 + ((6 - d)\beta)/(6d))$, and $f_1 = (\beta^2/2) + ((6 - d)/(6d))$. Hence, when $1 \leq d \leq 6$, we have $f_0(\beta), f_1(\beta) \geq 0$ for $\beta \in [0,1)$.  


(iv) When \( r = 1, 1 \leq d \leq 62 \). By simplifying
\[
\begin{align*}
\beta_0(\beta) &= \frac{1}{6} \left( \beta - \frac{1}{2} \right)^3 + \left( \frac{48 - d}{24d} \right) \left( \beta - \frac{1}{2} \right), \\
\beta_1(\beta) &= \frac{1}{2} \left( \beta - \frac{1}{2} \right)^2 + \left( \frac{48 - d}{24d} \right).
\end{align*}
\]

We have the following:

**Proposition 7.3.** There exists a minimum \( \xi \geq 0 \) satisfying
\[
\begin{align*}
(i) & \, \beta_1(\beta) + \xi \geq 0, \\
(ii) & \, (1 - \beta) \beta_1(\beta) + \xi + 2\beta_0(\beta) \geq 0, \text{ and} \\
(iii) & \, \sqrt{\kappa(X)} (\beta_1(\beta) + \xi) + \beta_0(\beta) \geq 0,
\end{align*}
\]
for all \( \beta \in [0, 1] \). Here \( \kappa(X) \) is the constant as in Definition 6.2.

**Proof.** From Note 7.2, for large enough \( \xi \), we have \( \beta_1(\beta) + \xi \geq 0, (1 - \beta) \beta_1(\beta) + \xi + 2\beta_0(\beta) \geq 0, \) and \( \sqrt{\kappa(X)} (\beta_1(\beta) + \xi) + \beta_0(\beta) \geq 0, \) for all \( \beta \in [0, 1] \). Therefore, there is a minimum \( \xi \) satisfying those inequalities. \( \square \)

We prove that Conjecture 4.19 holds for \( X \) with respect to the zero function \( A = 0 \) and for some \( 0 \leq \xi(A) \leq \xi \).

**Theorem 7.4.** Let \( A \) be the zero function: \( A : \mathbb{R}(H) \to \mathbb{R}_{\geq 0}, \beta H \mapsto 0 \). Let \( E \in D^b(X) \) be a \( \beta_A \)-stable object. Then we have \( D^{0,\xi}(E) \leq 0 \) for \( \beta \in \beta_A(E) \).

**Remark 7.5.** For the following cases, the constant \( \xi(A) = 0 \):

- \( r = 4 \)
- \( r = 3 \)
- \( r = 2 \) with \( 1 \leq d \leq 6 \)
- From Proposition 7.7, Fano 3-folds \( X \) with index \( r = 1 \) having degree \( 1 \leq d \leq 48 \) and \( \kappa(X) = 3/(2rd) \); in particular, Fano 3-folds of index one with Picard rank one.

**Note 7.6.** By using the functions \( \beta_0 \) and \( \beta_1 \) in Note 7.4, one can show that for \( \xi \), defined in Proposition 7.3,
\[
\beta_1(\beta) + \xi > 0, \text{ for } \beta \in [0, 1].
\]

We adapt some of the techniques from [Li, BMSZ] to prove Theorem 7.4. First we need the following:

**Proposition 7.7.** Let \( E \in D^b(X) \) be a \( \beta_A \)-stable object, with \( \alpha_H(E) > 0, \chi_0(E) \geq 0, \beta_A(E) \subset [0, 1) \) and \( \chi(E(-H)) \leq 0 \). Then we have \( \beta^{0,\xi}(E) \leq 0 \) for each \( \beta \in \beta_A(E) \).

**Proof.** Let \( \beta \in \beta_A(E) \). We have \( H \chi_1^{\beta_H}(E) = 0 \), and so \( \alpha_H(E) = (H^2 \chi_1^{\beta_H}(E))^2 \). Since \( \alpha_H(E) > 0, H^2 \chi_1^{\beta_H}(E) > 0 \).

From Proposition 7.1, we have
\[
0 \geq \chi(E(-H)) = D^{0,\xi}_{0,\beta}(E) + \left( \beta_1(\beta) + \xi \right) H^2 \chi_1^{\beta_H}(E) + f_0(\beta)H^3 \chi_0(E).
\]

Since \( H^2 \chi_1^{\beta_H}(E) \left( \beta_1(\beta) + \xi \right) \geq 0 \), when \( \chi_0(E) = 0 \), we have \( 0 \geq \chi(E(-H)) \geq D^{0,\xi}(E) \) as required. So let us assume \( \chi_0(E) > 0 \).
Now suppose $D^{0,ξ}_0(E) > 0$ for a contradiction. Therefore, we have $(f_1(\overline{β}) + ξ) H^2 ch^1(E) + f_0(\overline{β}) H^3 ch_0(E) < 0$. Hence

$$\left(2\mu_H(E) - \overline{β}\right) - 1 < \frac{(-(1 - \overline{β})(f_1(\overline{β}) + ξ) + 2f_0(\overline{β}))}{(f_1(\overline{β}) + ξ)} \leq 0.$$ 

Here the last inequality follows from the definition of $ξ$ in Proposition 7.3 together with Note 7.6. Since $\overline{β} \in [0, 1)$, there are no integers in the interval $(\overline{β}, 2\mu_H(E) - \overline{β}) \subset (0, 1)$. Therefore, from Theorem 6.5, we get

$$H^2 ch^1(E) + f_0(\overline{β}) H^3 ch_0(E) \geq \kappa(X).$$

So we have

$$(f_1(\overline{β}) + ξ) H^2 ch^1(E) + f_0(\overline{β}) H^3 ch_0(E) \geq \left(\sqrt{\kappa(X)}(f_1(\overline{β}) + ξ) + f_0(\overline{β})\right) H^3 ch_0(E) \geq 0.$$ 

Here the last inequality follows from the definition of $ξ$ in Proposition 7.3. Hence, we have $0 \geq \chi(E(-H)) \geq D^{0,ξ}_0(E)$. This is the required contradiction. □

Proof of Theorem 7.4. Let $\overline{β} \in \overline{β}_A(E)$.

If $\overline{A}_H(E) = 0$ then from Lemma 3.13 such objects are isomorphic to $O_X(\mathbb{m}H)[1]$ or $I_Z(\mathbb{m}H)$ for some $\mathbb{m} \in \mathbb{Z}$ and 0-subscheme $Z \subset X$. By direct computation one can check the Bogomolov-Gieseker type inequalities hold for such objects. So we can assume $\overline{A}_H(E) > 0$.

By using Proposition 3.17 and since tilt stability is preserved under the tensoring by a line bundle, further we can assume

$$ch_0(E) \geq 0 \text{ and } \overline{β} \in [0, 1).$$

(i) **When we do not have the case** $r = 1$ **with** $\overline{β} = 0$. From Proposition 5.1 for any $j \leq 0$ we have

$$\text{Hom}_X(O_X(H), E[j]) = 0, \text{ and } \text{Hom}_X(E, O_X((1 - r)H)[1 + j]) = 0.$$ 

Since $ω_X = O_X(-rH)$, from the Serre duality, $\text{Hom}_X(E, O_X((1 - r)H)[1 + j]) \cong \text{Hom}_X(O_X(H), E[2 - j])^*$. Therefore, from the Hirzebruch-Riemann-Roch theorem

$$\chi(O_X(H), E) = \sum_{i \in \mathbb{Z}} (-1)^i \text{Hom}_X(O_X(H), E[i]) = \text{Hom}_X(O_X(H), E[1]) \leq 0.$$ 

That is $\chi(E(-H)) \leq 0$. So from Proposition 7.7 we have the required inequality.
(ii) When we have the case \( r = 1 \) with \( \beta = 0 \). Suppose for a contradiction there exists a counterexample \( E \); so \( D_{0,0}^{0,\xi}(E) > 0 \). Since \( \text{Hch}_2(E) = 0, \Delta_H(E) = (H^2 \text{ch}_1(E))^2 > 0 \). Let \( E \) be such one example with minimum \( H^2 \text{ch}_1 \).

If \( \chi(E(-H)) \leq 0 \) then from Proposition 7.7 we have \( D_{0,0}^{0,\xi}(E) \leq 0 \); however, since we already assumed \( D_{0,0}^{0,\xi}(E) > 0 \), we have \( \chi(E(-H)) > 0 \).

From Proposition 5.1, for any \( j \leq 0 \) we have
\[
\text{Hom}_X(O_X[H], E[j]) = 0, \text{ Hom}_X(E, O_X[j]) = 0.
\]
By using the Serre duality we get
\[
\chi(O_X(H), E) = \sum_{i \in \mathbb{Z}} (-1)^i \text{hom}_X(O_X(H), E[i]) = -\text{hom}_X(O_X(H), E[1]) + \text{hom}_X(O_X(H), E[2]).
\]
Since \( \chi(E(-H)) > 0 \), from the Serre duality we have \( \text{hom}_X(O_X(H), E[2]) \cong \text{hom}_X(E, O_X[1]) \neq 0 \). So there is a non-trivial map \( E \to O_X[1] \) in \( D^b(X) \) and hence, we have the distinguished triangle
\[
O_X \to E_1 \to E \to O_X[1]
\]
for some \( E_1 \in D^b(X) \). Here \( E, O_X[1] \in \mathbb{B}_0 \), and also from Proposition 2.1, \( O_X[1] \) is a minimal object. Therefore by considering the long exact sequence of \( \mathbb{B}_0 \)-cohomologies we get \( E_1 \in \mathbb{B}_0 \) and the following non-splitting short exact sequence in \( \mathbb{B}_0 \):}
\[
0 \to E_1 \to E \to O_X[1] \to 0.
\]
By applying the functor \( \text{Hom}_X(\cdot, O_X[1]) \) to the above short exact sequence, we get
\[
\text{hom}_X(E_1, O_X[1]) = \text{hom}_X(E, O_X[1]) - \text{hom}_X(O_X[1], O_X[1]) < \text{hom}_X(E, O_X[1]).
\]
We have \( \text{ch}_0(E_1) = \text{ch}_0(E) + 1 \geq 0 \), and
\[
\chi(E_1(-H)) = \chi(E(-H)) - \chi(O_X(-H)[1]) = \chi(E(-H)) - 1.
\]
If \( \chi(E_1(-H)) > 0 \) then we can repeat the above process for \( E_1 \). In this way we get a sequence of non-splitting short exact sequences
\[
0 \to E_{i+1} \to E_i \to O_X[1] \to 0
\]
in \( \mathbb{B}_0 \) for some \( E_i \in \mathbb{B}_0 \), with \( E_0 = E \), and
\[
\cdots < \chi(E_1(-H)) < \cdots < \chi(E_i(-H)) < \chi(E(-H)).
\]
Therefore, for some \( m \geq 1 \) we have the short exact sequence
\[
0 \to F := E_m \to E \to O^\oplus_X[1] \to 0
\]
in \( \mathbb{B}_0 \) with \( \chi(F(-H)) \leq 0 \). Also \( \text{ch}_0(F) = \text{ch}_0(E) + m \), and for \( i \geq 1 \), \( \text{ch}_i(F) = \text{ch}_i(E) \).

Let us prove \( F \in \mathbb{B}_0 \) is \( A \)-stable with \( F_A(F) = \{0\} \). Consider \( \nu_{0,\alpha} \) tilt stability of \( F \) for sufficiently small \( \alpha > 0 \). Since \( \Delta_H(E) > 0 \), \( H^2 \text{ch}_1(E) = H^2 \text{ch}_1(F) > 0 \). Hence, from the properties of Harder-Narasimhan filtrations and Jordan-Hölder filtrations, there is a filtration:
\[
0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_k = F
\]
with \( G_i := F_i/F_{i-1} \in \mathbb{B}_0 \) are \( \nu_{0,\alpha} \) stable with
\[
\nu_{0,\alpha}(G_i) \geq \nu_{0,\alpha}(G_{i+1})
\]
for sufficiently small enough $\alpha > 0$. Since $H^2 ch_1(F) > 0$, $\nu_{0,\alpha}$ slope of each $G_i$'s are finite. So $H^2 ch_1(G_i) \geq 1$. Moreover, $G_1 = F_1$ is a subobject of $E$. So
\[
\nu_{0,\alpha}(E) > \nu_{0,\alpha}(F_1) \geq \cdots \geq \nu_{0,\alpha}(F_{k-1}) \geq \nu_{0,\alpha}(F_k) = \nu_{0,\alpha}(F)
\]
for sufficiently small enough $\alpha > 0$. Therefore, by considering the limit $\alpha \to 0^+$, we get $H ch_2(F_i) = 0$ for all $i$. Since $0 < D_{0,0}^{1,\xi}(E) = D_{0,0}^{1,\xi}(F) = \sum_i D_{0,0}^{1,\xi}(G_i)$, there is at least one $G_i$ such that $D_{0,0}^{1,\xi}(G_i) > 0$. Therefore $\bar{A}$ stable object $G_i := H_{B_0}^{1,\xi}(G_i)$ also satisfies $D_{0,0}^{1,\xi} > 0$. Since $ch_0(G_i) = -ch_0(G_i^1)$, one of $G_1, G_i^1$ have $ch_0 \geq 0$. That is one of $G_i, G_i^1$ is also a counterexample like $E$. Since we assumed $E$ to be a counterexample with minimal $H^2 ch_1$, we have $H^2 ch_1(G_i) = H^2 ch_1(E)$. So $F$ is also $\bar{A}$ stable with $\bar{A}_A(F) = \{0\}$.

Since $\chi(F(-H)) < 0$, from Proposition we get $D_{0,0}^{1,\xi}(F) \leq 0$; this is the required contradiction. This completes the proof. \hfill \Box

8. Optimal Bogomolov-Gieseker Type Inequality for Blow-up of $\mathbb{P}^3$ at a Point

Let $X$ be the blow-up of $\mathbb{P}^3$ at a point. From Iskovskikh-Mori-Mukai classification of smooth Fano 3-folds $X$ is the only Fano 3-fold of index $r = 2$ having degree $d > 6$. In particular, the degree of $X$ is $d = 7$. In this section we optimize the Bogomolov-Gieseker type inequality for $X$. More precisely, we show that the modified inequality in Conjecture holds with $\xi(A) = 0$ for some $A \neq 0$.

**Definition 8.1.** The continuous function $A : \mathbb{R} \langle H \rangle \to \mathbb{R}_{\geq 0}$ is defined by
\[
\begin{aligned}
&\text{for } \beta \in [-1/2, 0), \quad A(\beta H) = 1 + \beta; \\
&\text{for } \beta \in [0, 1/2), \quad A(\beta H) = 1 - \beta;
\end{aligned}
\]

Together with the relation $A((\beta + 1)H) = A(\beta H)$.

See Figure 6.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig6}
\caption{Graph of $\alpha = A(\beta H)$ in $(\beta, \alpha)$-plane}
\end{figure}

We prove Conjecture or equivalently Conjecture holds for $X$ with respect to the function $A$ and constant $\xi(A) = 0$. 
Theorem 8.2. Let $E \in D^b(X)$ be a $\overline{\beta}_A$-stable object. Then $D_{A([\beta]_E)}^{0,0}(E) \leq 0$ for each $\overline{\beta} \in \overline{\beta}_A(E)$. In particular, Conjecture 4.10 holds for $X$.

We need the following:

Proposition 8.3. Let $E \in D^b(X)$ be a $\overline{\beta}_A$-stable object with $\overline{\beta}_H(E) > 0$, $\chi_0(E) > 0$, $\overline{\beta}_A(E) \in [-1/2, 1/2)$, and $\chi(E(-H)) \leq 0$. Then we have $D_{A([\overline{\beta}]_E)}^{0,0}(E) = 0$ for each $\overline{\beta} \in \overline{\beta}_A(E)$.

Proof. Let $\overline{\beta} \in \overline{\beta}_A(E)$ and $\overline{\alpha} = A(\overline{\beta} H)$. So $H \cdot \overline{\beta}_2(E) = ([\overline{\alpha}_2]/2)H^3 \chi_0(E)$, and from Proposition 8.1,

$$\chi(E(-H)) = D_{A,\overline{\beta}}^{0,0}(E) + pH^2 \cdot \overline{\beta}_1(E) + qH^3 \chi_0(E),$$

where

$$p = \frac{1}{6} \overline{\alpha}^2 + \frac{1}{2} \overline{\beta}^2 - \frac{1}{42},$$

$$q = \overline{\beta} \left( \frac{1}{2} \overline{\alpha}^2 + \frac{1}{6} \overline{\beta}^2 - \frac{1}{42} \right).$$

By using the definition of $A$ and simplifying $p$ we obtain, for $\overline{\beta} \in [-1/2, 0)$, $p = (1/24)(4\overline{\beta} + 1)^2 + (17/168)$, and for $\overline{\beta} \in [0, 1/2)$, $p = (1/24)(4\overline{\beta} - 1)^2 + (17/168).$ Therefore,

$$p > 0.$$ 

Hence, since $H^2 \cdot \overline{\beta}_1(E) \geq 0$, when $\chi_0(E) = 0$ we have $0 \geq \chi(E(-H)) \geq D_{A,\overline{\beta}}^{0,0}(E)$ as required.

So we assume $\chi_0(E) > 0$.

(i) Case $\overline{\beta} \in [0, 1/2)$: We have $\overline{\alpha} = 1 - \overline{\beta}$ and so

$$q = \overline{\beta} \left( \frac{(1 - \overline{\beta})^2}{2} + \frac{\overline{\beta}^2}{6} - \frac{1}{42} \right) = \frac{2}{3} \overline{\beta} \left( \left( \frac{1 - \overline{\beta}}{2} \right)^2 + \frac{17}{112} \right) \geq 0.$$ 

Since $H^2 \cdot \overline{\beta}_1(E), H^3 \chi_0(E) \geq 0$, we have $0 \geq \chi(E(-H)) \geq D_{A,\overline{\beta}}^{0,0}(E)$ as required.

(ii) Case $\overline{\beta} \in [-1/2, 0)$: We have $\overline{\alpha} = 1 + \overline{\beta}$.

Suppose for a contradiction $\overline{\beta}$ is a counterexample; that is $D_{A,\overline{\beta}}^{0,0}(E) > 0$. Since $\chi(E(-H)) \leq 0$ and $D_{A,\overline{\beta}}^{0,0}(E) > 0$, we have $pH^2 \cdot \overline{\beta}_1(E) + qH^3 \chi_0(E) < 0$. Therefore,

$$(2\mu_H(E) - \overline{\beta} - \overline{\alpha}) - 1 < \frac{-2p - 2q}{p} < 0.$$ 

Here the last inequality follows from (i) of Proposition 8.2. On the other hand $\overline{\beta} + \overline{\alpha} = 1 + 2\overline{\beta} \in [0, 1)$. Therefore, there are no integers in the interval $(\overline{\beta} + \overline{\alpha}, 2\mu_H(E) - \overline{\beta} - \overline{\alpha}) \subset (0, 1)$.

Hence, from Theorem 6.5 we have

$$\frac{\overline{\beta}_H(E)}{(H^3 \chi_0(E))^2} \geq \kappa(X) = \frac{2}{49}.$$
Therefore,
\[
\frac{H^2 \text{ch}_1^3 E}{H^3 \text{ch}_0(E)} \geq \sqrt{\mathcal{A}^2 + \kappa(X)}.
\]
So
\[
pH^2 \text{ch}_1^3 E + qH^3 \text{ch}_0(E) \geq \left( p \sqrt{\mathcal{A}^2 + \kappa(X) + q} \right) H^3 \text{ch}_0(E) \geq 0.
\]
Here the last inequality follows from (ii) of Proposition A.2. Therefore, \(0 \geq \chi(E(-H)) \geq D_0^0(\mathcal{A}, \mathcal{P})(E)\); this is the required contradiction.

This completes the proof. \(\square\)

Proof of Theorem 3.2. Let \(\beta \in \beta(A)(E)\) and \(\mathcal{A} = A(\beta H)\).

If \(\Delta_H(E) = 0\) then from Lemma 3.13 such objects are isomorphic to \(\mathcal{O}_X(mH)[1]\) or \(\mathcal{I}_Z(mH)\) for some \(m \in \mathbb{Z}\) and 0-subscheme \(Z \subset X\). By direct computation one can check the Bogomolov-Gieseker type inequalities hold for such objects. So we can assume \(\Delta_H(E) > 0\).

Since tilt stability is preserved under tensoring by a line bundle, and also from Proposition 3.17 we further can assume
\[
\text{ch}_0(E) \geq 0, \text{ and } \beta \in [-1/2, 1/2).
\]
From Proposition 5.1 for any \(j \leq 0\) we have
\[
\text{Hom}_X(\mathcal{O}_X(H), E[j]) = 0, \text{ and } \text{Hom}_X(\mathcal{O}_X(-H)[1+j]) = 0.
\]
By the Serre duality, \(\text{Hom}_X(\mathcal{O}_X(-H)[1+j]) \cong \text{Hom}_X(\mathcal{O}_X(H), E[2-j])^*. \) Hence,
\[
\chi(\mathcal{O}_X(H), E) = \sum_{i \in \mathbb{Z}} (-1)^i \text{hom}_X(\mathcal{O}_X(H), E[i]) = -\text{hom}_X(\mathcal{O}_X(H), E[1]) \leq 0.
\]
From Proposition 5.3 we have the required inequality. \(\square\)

APPENDIX A.

Proposition A.1. Let \(d\) be a positive integer such that \(1 \leq d \leq 48\). Let \(f : \mathbb{R} \to \mathbb{R}\) be the function defined by
\[
f(x) = \frac{1}{6} \left( x - \frac{1}{2} \right)^3 + \frac{(48 - d)}{24} \left( x - \frac{1}{2} \right).
\]
Let \(f'(x)\) be the derivative of \(f(x)\) with respect to \(x\). Then for \(x \in [0, 1)\) we have
(i) \(\sqrt{3/(2d)} f'(x) + f(x) \geq 0\).
(ii) \((1-x) f'(x) + 2f(x) \geq 0\).

Proof. (i) Let \(g\) be the function defined by, \(g(x) = \sqrt{3/(2d)} f'(x) + f(x)\). Here \(f'(x) = (1/2)(x - (1/2))^2 + (48 - d)/(24d)\).

By differentiating \(g(x)\) we get
\[
g'(x) = \frac{1}{2} \left( x - \frac{1}{2} \right)^2 + \sqrt{\frac{3}{2d}} \left( x - \frac{1}{2} \right) + \frac{(48 - d)}{24d}
\]
\[
= \frac{1}{2} \left( x - \frac{1}{2} + \sqrt{\frac{3}{2d}} \right)^2 - \frac{(d - 30)}{12d}.
\]
By evaluating \( g(x) \) at \( x = 0 \):

\[
g(0) = \sqrt[3]{\frac{6}{d}} \left( \frac{1}{\sqrt{d}} - \frac{1}{\sqrt[3]{24}} \right)^2 \geq 0.
\]

When \( d \leq 30 \) we have \( g'(x) \geq 0 \) for all \( x \in \mathbb{R} \) with \( g(0) > 0 \). Hence for \( x \in [0,1) \), \( g(x) \geq 0 \).

Let us consider the case \( d > 30 \). The derivative \( g'(x) \) is vanishing at \( x = \lambda_1, \lambda_2 \) with \( \lambda_1 < \lambda_2 \). Here

\[
\lambda_2 = \frac{1}{2} - \sqrt{\frac{3}{2d}} + \sqrt{\frac{(d-30)}{12d}} = \frac{1}{2} - \frac{(48-d)}{\sqrt{12d(\sqrt{18} + \sqrt{d-30})}}.
\]

One can rearrange \( g \) as

\[
g(x) = \frac{1}{6} \left( x - \frac{1}{2} + \sqrt{\frac{3}{2d}} \right)^3 + \frac{(30-d)}{24d} \left( x - \frac{1}{2} \right) + \sqrt{\frac{3}{2d}} \left( \frac{42-d}{24d} \right).
\]

Let us consider the remaining case \( 30 < d \leq 48 \). The local minimum value of \( g(x) \) at \( x = \lambda_2 \) is

\[
g(\lambda_2) = -\frac{1}{3} \left( \frac{d-30}{12d} \right)^{3/2} + \frac{1}{2} \sqrt{\frac{3}{2d}} = \frac{18^{3/2} - (d-30)^{3/2}}{72\sqrt{3d^{3/2}}} \geq 0,
\]

with equality when \( d = 48 \). Moreover, \( \lambda_2 = 1/2 \) when \( d = 48 \). Since \( g(0) \geq 0 \), and so for \( x \in [0,1) \) we have \( g(x) \geq 0 \).

(ii) Let \( h(x) = (1-x)f'(x) + 2f(x) \). By simplifying we get

\[
h(x) = \frac{1}{24} (2-x)(1-2x)^2 + \frac{2(48-d)}{48d}.
\]

Since \( 1 \leq d \leq 48 \), for all \( x \in [0,1) \) we have \( h(x) \geq 0 \) as required. \( \square \)

**Proposition A.2.** Define the functions \( p(x), q(x) \) by

\[
p(x) = \frac{1}{6} (1+x)^2 + \frac{1}{2}x^2 - \frac{1}{42},
\]

\[
q(x) = x \left( \frac{1}{2} (1+x)^2 + \frac{1}{6} x^2 - \frac{1}{42} \right).
\]

For \( x \in [-1/2,0] \), we have the following:

(i) \( p(x) + q(x) > 0 \).

(ii) \( p(x) \sqrt{(1+x)^2 + \frac{2}{21}} + q(x) > 0 \).

**Proof.** (i) By simplifying, we have

\[
p(x) + q(x) = \frac{2}{3}x^3 + \frac{5}{3} x^2 + \frac{17}{21} x + \frac{1}{7}.
\]

Let us find the critical points of \( p(x) + q(x) \). The derivative of \( p(x) + q(x) \) with respect to \( x \) is

\[
2(x+(5/6))^2-(73/126),
\]

and so its roots are \( -(5/6) \pm \sqrt{73/252} \). Therefore, the critical values of \( p(x) + q(x) \) are \( (1904 \pm 73\sqrt{511})/7938 > 0 \). Hence, since \( -(5/6) - \sqrt{73/252} < -1/2 < -(5/6) + \sqrt{73/252} < 0 \), we get the required inequality for all \( x \in [-1/2,0] \).
(ii) For \(x \in [-1/2, 0]\),

\[
p(x) = \frac{1}{24}(4x + 1)^2 + \frac{17}{168} > 0, \quad \text{and} \quad q(x) = x \left( \frac{1}{24}(4x + 3)^2 + \frac{17}{168} \right) < 0.
\]

So it is enough to show that for \(x \in [-1/2, 0]\),

\[
F(x) := \left( (1 + x)^2 + \frac{2}{49} \right) p(x)^2 - q(x)^2 > 0.
\]

Let us find the critical points of \(F\). By differentiating \(F\) we get

\[
F'(x) := \frac{dF}{dx} = \frac{4}{3087} (56x^3 + 483x^2 + 460x + 108).
\]

The equation \(F'(x) = 0\) has three different real roots; so \(F\) has three critical points. They are

\[
\lambda_1 = \frac{23}{8} + \frac{1}{8} \left( \frac{\zeta}{\sqrt[3]{441}} + \frac{7429}{\sqrt[3]{21\zeta}} \right),
\]

\[
\lambda_2 = \frac{23}{8} - \frac{1}{16} \frac{(1 + \sqrt[3]{3})\zeta}{\sqrt[3]{441}} - \frac{7429(1 - \sqrt[3]{3})}{16\zeta},
\]

\[
\lambda_3 = \frac{23}{8} - \frac{1}{16} \frac{(1 - \sqrt[3]{3})\zeta}{\sqrt[3]{441}} + \frac{7429(1 + \sqrt[3]{3})}{16\zeta}.
\]

Here

\[
\zeta = \left( -2917215 + 32i\sqrt{97656006} \right)^{1/3} \in \mathbb{R}e^{(0, \pi/3)}.
\]

The critical points satisfy

\[
\lambda_3 < \lambda_2 < -1/2 < \lambda_1 < 0.
\]

Since \(F\) is a degree 4 polynomial of \(x\) with a positive leading coefficient, we have for all \(x \in [-1/2, 0]\),

\[
F(x) \geq F(\lambda_1).
\]

On the other hand, by direct computation we have

\[
F(\lambda_1) = -\frac{164676683}{33191424} + \frac{304594062581681}{4741632\sqrt[4]{21} \zeta^4} - \frac{410006814589}{2370816\sqrt[4]{441} \zeta^2} + \frac{i\sqrt{2325143} \zeta}{1555848\sqrt[4]{21}} - \frac{7429 \zeta^2}{49787136\sqrt[4]{21}} + \frac{37145 \sqrt[4]{9}}{64\sqrt[4]{7}\zeta} + \frac{5\sqrt[4]{21} \zeta}{512} > 0.
\]

In fact, by direct computation one can check that

\[
0.000028 < F(\lambda_1) < 0.000029.
\]

This completes the proof. \(\square\)
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Kavli Institute for the Physics and Mathematics of the Universe (WPI), The University of Tokyo Institutes for Advanced Study, The University of Tokyo, Kashiwa, Chiba 277-8583, Japan.

E-mail address: dulip.piyaratne@ipmu.jp