Tunneling measurement of quantum spin oscillations

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We consider the problem of tunneling between two leads via a localized spin 1/2 or any other microscopic system (e.g., a quantum dot) which can be modeled by a two-level Hamiltonian. We assume that a constant magnetic field $B_0$ acts on the spin, that electrons in the leads are in a voltage driven thermal equilibrium and that the tunneling electrons are coupled to the spin through exchange and spin-orbit interactions. Using the non-equilibrium Keldysh formalism we find the dependence of the spin-spin and current-current correlation functions on the applied voltage between leads $V$, temperature $T$, $B_0$, and on the degree and orientation $m_\alpha$ of spin polarization of the electrons in the right ($\alpha =$R) and left ($\alpha =$L) leads. We show that a) The spin-spin correlation function exhibits a peak at the Larmor frequency, $\omega_L$, corresponding to the effective magnetic field $B$ acting upon the spin as determined by $B_0$ and the exchange field induced by tunneling of spin polarized electrons, b) If the $m_\alpha$’s are not parallel to $B$ the second order derivative of the average tunneling current $I(V)$ with respect to $V$ is proportional to the spectral density of the spin-spin correlation function, i.e., exhibits a peak at the voltage $V = h\omega_L/e$, c) In the same situation when $V > B$ the current-current correlation function exhibits a peak at the same frequency, d) The signal-to-noise (shot noise) ratio $R$ for this peak reaches a maximum value of order unity, $R \leq 4$, at large $V$ when the spin is decoupled from the environment and the electrons in both leads are fully polarized in the direction perpendicular to $B$, and e) $R \ll 1$ if the electrons are weakly polarized, or if they are polarized in a direction close to $B_0$, or if the spin interacts with the environment stronger than with the tunneling electrons. Our results of a full quantum-mechanical treatment of the tunneling-via-spin model when $V \gg B$ are in agreement with those previously obtained in the quasi-classical approach. We discuss also the experimental results observed using STM dynamic probes of the localized spin.

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I. INTRODUCTION

Interest in quantum information processing has brought significant attention to the problem of measurement of tunneling currents via a microscopic system that can be modeled by a two-level Hamiltonian (such as a quantum dot, or a molecule or an atom with a localized spin $^{1,2,3,4}$). In the case of a single spin, measurements of the tunneling current in such a system provide information on spin orientation and its dynamics, and they constitute an example of indirect-continuous quantum measurement $^{1}$. A fundamental question that arises is what signatures of the spin dynamics are encoded in the tunneling current and how this current affects the spin dynamics.

Scanning tunneling microscopy (STM) experiments $^{5,6}$ on a single molecule with a spin, in the presence of a magnetic field $B_0$, have reported a peak in the current noise power spectrum (i.e., current-current correlation function) $P(\omega)$ at the Larmor frequency $\omega_L = \gamma B_0$, where $\gamma$ is the gyromagnetic ratio. Experiments were done at room temperature and the authors found that the signal-to-noise ratio $R$ (ratio of the power at the peak frequency to the shot noise power) exceeded unity and was almost independent of the orientation of the applied magnetic field $B_0$. In the non-relativistic approach the tunneling electrons couple to the spin by exchange interaction. In this case, electrons with a spin polarization along $B_0$ do not couple with the oscillatory components of the spin (which are perpendicular to $B_0$). The experimental results $^{5,6}$ are difficult to explain in the framework of a single-spin non-relativistic model since electrons in the leads were polarized by the same magnetic field which acted on the spin. Possible relevance of spin-orbit interaction to explain these data was discussed by Shchul and Manassen and later on by Balatsky and Martin $^7$. Recently, Levitov and Rashba $^8$ noticed that in systems with low space symmetry (as in dots or a molecule near a surface) the nonvanishing orbital moment of electrons near the spin (in the dot or molecule) provides a strong coupling of the tunneling electrons to the spin, and they speculated that this mechanism may lead to a significant effect of the spin oscillatory component on the tunneling current. In fact, in order to fit the experimental data $^5,^6$ one needs a model of electron tunneling via a single spin which is able to explain not only the existence of a peak in the current power spectrum, but also a significant signal-to-noise ratio $R > 1$ almost independent of the orientation of $B_0$.

In the following we consider a model with exchange (non-relativistic) coupling of a single spin 1/2 and the tunneling electrons, including spin-orbit coupling. In the framework of this model we analyze the dependence of $R$ and linewidth $\Gamma$ on the applied voltage $V$ between leads, the applied magnetic field $B_0$, the temperature $T$, and on the degree and orientation $m_{\alpha_0}$ of electron polarization in the right ($\alpha =$R) and left ($\alpha =$L) leads in the steady state. This state establishes in a transient time after the voltage or tunneling matrix elements are switched on.

The problem of tunneling via a single localized spin 1/2 is similar to the problem of tunneling via a double quantum dot (two-level system) extensively studied by Korotkov and Averin $^3$ and by Ruskov and Korotkov $^4$. Indeed, they were interested in the question of how quantum oscillations in the two-level system, representing a qubit, may be detected by a tunneling current. Such a system can be described by a model of electron tunneling via a spin when leads ele-
trons are fully polarized. Using the Bloch equations describing the ensemble-averaged evolution of the density matrix for the coupled qubit-detector system, they obtained oscillations at the Larmor frequency in the noise power spectrum $\mathcal{P}(\omega)$, with $R \leq 4$ in the case of weak interaction between the qubit and the detector. A similar quasi-classical approach for tunneling of electrons via a single spin was used in Ref. [1].

In such an approximation, the action of the electrons on the spin is replaced by the action of an effective classical magnetic field with a shot noise spectrum depending on the tunneling current between the leads. This approach has sense at high currents (voltages), while at low currents it does not account for the quantum nature of the tunneling electrons and hence a more elaborate solution of the quantum transport equation for both spin and electrons is desired. Here the situation is formally similar to the case of the photo-electric effect where in some circumstances a full quantum mechanical treatment of the problem provides the same information as a semi-classical approach, while in other cases, such as the measurement of the statistics of the photo-current fluctuations, the predictions of the two approaches differ qualitatively [2].

An important step toward a full quantum mechanical treatment of electron tunneling via a spin was recently given by Parcollet and Hooley [2]. Assuming that the spin interacts only with the tunneling electrons, and electrons in the leads are in a voltage driven thermal equilibrium, they computed the spin magnetization of a quantum dot in the two-leads Kondo model as a function of the temperature of the electrons, magnetic field, and voltage between the leads (which exceeds the Kondo temperature $T_K$) in the steady state, i.e., in the long time limit. They considered the situation where electrons in the leads are polarized in the same direction as the magnetic field acting on the spin, $\mathbf{m}_{\text{R}0} = \mathbf{m}_{\text{L}0} = \mathbf{m}_0$, $\mathbf{m}_0 \parallel \mathbf{B}_0$. Their calculations confirmed previously obtained quasi-classical result [10] that, in the steady state even at zero order in the spin-leads coupling, the magnetization is not given by the thermal equilibrium expression $M_{eq} = (1/2) \tanh(B_0/2T)$ when the spin is more strongly coupled to the leads than to any other thermal bath. At long times, the state of the spin is completely determined by the characteristics of the tunneling electrons, and since the system is out of equilibrium, the steady state thus achieved is described by a steady state distribution function which differs from the Gibbs distribution. It turns out that this distribution function depends upon the voltage between the leads. They also showed that the correct way to calculate the perturbative corrections to the spin distribution function and the spin decoherence rate $\Gamma$ in the steady state is by solving a quantum transport equation self-consistently. We will extend the treatment of Parcollet and Hooley to arbitrary orientations of $\mathbf{m}_{\text{R}0}$, and $\mathbf{m}_{\text{L}0}$ with respect to $\mathbf{B}_0$, and account for a direct tunneling of electrons between leads.

The main goals in this article are (i) to examine under what tunneling conditions the current-current correlation function $\mathcal{P}(\omega)$ exhibits a peak at the Larmor frequency in the steady state, and (ii) to calculate $\Gamma(V, T, \mathbf{B}_0, \mathbf{m}_{\text{id}})$ and $R(V, T, \mathbf{B}_0, \mathbf{m}_{\text{id}})$ in the regime where $eV \gg T_K$, with $e$ denoting the charge of the electron. As in Ref. [2] we use the Keldysh formalism [11,12] and the Majorana-fermion representation [13] for the spin to find the spin distribution function and the current-current correlation function.

We note, from a more general perspective, that the problem of electron tunneling via a spin represents an example of indirect quantum measurement. The spin is probed by the tunneling electrons whose correlation function is measured in a continuous fashion with a classical apparatus (which is not included). The goal of quantum measurement is to determine the frequency of precession of the isolated spin (energy separation in the isolated two-level system) and to obtain information on the initial state of the system. However, when the spin is decoupled from the environment, tunneling via the spin changes its state as the measurement goes on. After switching on the voltage, or the tunneling matrix elements, there is an initial period of time where the tunneling current depends upon the initial state of the spin. After some transient time the steady state, which is in general independent of the initial conditions, establishes. In the steady state the frequency of precession is renormalized by the tunneling electrons, and the width of the peak (at the precession frequency) and corresponding signal-to-noise ratio depend upon the precession frequency and characteristics of the tunneling electrons. To extract information about the precession of the isolated spin one needs to have complete information on the tunneling electrons. We show how to relate the precession frequency of the isolated spin to results of current measurements in the steady state (peak frequency, width of the peak and signal-to-noise ratio). To obtain information on the initial state of the spin, measurements of the current during the initial transient period are needed. Here we do not consider the spin dynamics in this short time interval.

The plan of the article is the following. In the next section we present the model for tunneling via a single spin 1/2, with and without spin-orbit coupling, and a model describing tunneling via a two-level quantum dot. Then we introduce Majorana fermions and the Keldysh technique. In the subsequent section we will compute the spin distribution function, broadening of the spin precession due to the tunneling of electrons, and the spin-spin correlation function. Then we will calculate the dependence of the tunneling current on the spin and determine the current-current correlation function $\mathcal{P}(\omega)$. Finally, we discuss the STM experimental results and compare our results to those obtained in the quasiclassical approach [1,2,4] and other theoretical works [7,8,14].

II. THE MODEL

For the system consisting of a spin coupled to the leads by an exchange mechanism (in the non-relativistic approximation, i.e., neglecting the spin-orbit interaction), we use the Hamiltonian of the two-leads Kondo model [1,2] with a direct tunneling term included (which we call tunneling-via-
spin (TvS) model

\[ \mathcal{H} = \mathcal{H}_e + \mathcal{H}_s + \mathcal{H}_T, \quad \mathcal{H}_T = \mathcal{H}_{\text{ref}} + \mathcal{H}_{\text{tr}}, \]

\[ \mathcal{H}_e = \sum_{\alpha n, \sigma \sigma'} \left[ \epsilon_{n\alpha} \delta_{\sigma \sigma'} - \frac{1}{2} \mathbf{B}_\alpha \cdot \mathbf{\hat{\sigma}}_{\sigma \sigma'} \right] c_{\alpha n\sigma}^\dagger c_{\alpha n\sigma'}, \]

\[ \mathcal{H}_s = -g\mu_B \mathbf{B}_0 \cdot \mathbf{S}, \]

\[ \mathcal{H}_{\text{ref}} = \sum_{\alpha n, \sigma \sigma'} c_{\alpha n\sigma}^\dagger (\hat{T}_{\alpha\alpha})_{\sigma \sigma'} c_{\alpha n\sigma'}, \quad \hat{T}_{\alpha\alpha} = \hat{T}^{(\text{ex})}_{\alpha\alpha} \mathbf{S} \cdot \mathbf{\hat{\sigma}}_{\sigma \sigma'}, \]

\[ \mathcal{H}_{\text{tr}} = \sum_{n, \sigma \sigma'} c_{\alpha n\sigma}^\dagger (\hat{\mathcal{T}}_{\text{RL}})_{\sigma \sigma'} c_{\alpha n\sigma'} + H.c., \]

\[ (\hat{T}_{\text{RL}})_{\sigma \sigma'} = T_0 \delta_{\sigma \sigma'} + \hat{T}^{(\text{ex})}_{\text{RL}} \mathbf{S} \cdot \mathbf{\hat{\sigma}}_{\sigma \sigma'}, \]

where \( c_{\alpha n\sigma}^\dagger (c_{\alpha n\sigma}) \) creates (annihilates) an electron in the left or right lead (depending on \( \alpha \in \{L, R\} \)) in the eigenstate \( n \), and with spin \( \sigma \). Further, \( \epsilon_{n\alpha} = \epsilon_n - \mu_\alpha \), where \( \epsilon_n \) is the energy in the state \( n \) and \( \mu_\alpha \) is the chemical potential in the lead \( \alpha \), while \( \mathbf{\hat{\sigma}} \) represents the three Pauli matrices. \( T^{(\text{ex})}_{\alpha\alpha}, T^{(\text{ex})}_{\alpha\beta} \) and \( T^{(\text{ex})}_{\beta\alpha} \) are tunneling matrix elements due to the exchange interaction for the electron tunneling from the leads to the molecule with the spin 1/2, while \( T_0 \) is the direct tunneling matrix element. We take them as real numbers. The spin localized in the molecule is described by the operator \( \mathbf{S} = (S_x, S_y, S_z) \). Figure 1 sketches the physical setup we want to study and which basically represents the model Hamiltonian \( \mathcal{H} \).

In the following we use energy units for the bias voltage \( V \), i.e., we denote \( eV \) by \( V \). Also, by \( B \) we mean \( g\mu_B B \), write \( T \) instead of \( k_B T \), and \( \omega \) represents \( \hbar \omega \).

We describe the leads by a free electron gas with a density of states \( \rho(\epsilon) \) of bandwidth \( D \), so that \( \epsilon_n \) is the bare energy of the electron in the eigenstate \( n \), the same in both leads. We assume weak tunneling, \( T_0^2 \rho_0^2 / |T_{\text{RL}}^{(\text{ex})}|^2 \rho_0^2 \ll 1 \), where \( \rho_0 \) is the density of states per spin (DOS) of the leads at the Fermi level (when the leads are different \( \rho_0^L = \rho_0^R \) where \( \rho_0^L \) is the DOS in the lead \( \alpha \)). The applied voltage is represented by the difference in chemical potentials, \( \mu_L - \mu_R = V \). We also assume \( V \gg T_K \approx |T_{\text{RL}}^{(\text{ex})}| \exp(-|T_{\text{RL}}^{(\text{ex})}|^2 \rho_0^2), \) and \( T_0 \gg T_{\alpha\beta}^{(\text{ex})} \) (\( \alpha, \beta \in \{L, R\} \)). We assume in the following that \( B\rho_0 \ll 1 \) and \( V \rho_0 \ll 1 \); in other words, the lead-electron bandwidth \( D \sim 1/\rho_0 \) is the largest energy scale. The leads are supposed to be in a voltage driven thermal equilibrium at temperature \( T \). \( \mathcal{H}_{\text{ref}} \) describes spin-flip scattering of an electron from one lead back into the same lead, and \( \mathcal{H}_T \) represents both direct and spin-assisted tunneling between leads. In the case of fully polarized leads, the summation in the Hamiltonian is taken only over one type of electron level.

In the model with spin-orbit interaction proposed by Levitov and Rashba in Ref. 3 the spin precession induces electron density oscillations inside the dot, and thus an electric field inside and around the dot, \( \mathbf{E} = (\mathbf{n} \times (\mathbf{S} \times \mathbf{L})) \), where \( \mathbf{L} \) is the orbital momentum, while \( \mathbf{n} \) is a polar vector allowed by the symmetry of the system. This electric field modulates the tunneling barrier between the leads and the dot. Hence, the tunneling matrix element acquires, in addition to the non-relativistic exchange term, a dependence on the localized spin \( \mathbf{S} \) which is similar for both spin components \( \sigma \) of the tunneling electrons.

![FIG. 1: Schematics of the physical systems represented by the Hamiltonian \( \mathcal{H} \) and measurement process involved in determining the current \( \langle I(t) \rangle \) and current-current correlation function \( \langle I(t)I(0) \rangle \) which contain encoded information about the spin \( \mathbf{S} \). The electronic tunneling current is established by a dc voltage \( V \).](image)

To describe such relativistic corrections we need to add to the transfer matrix elements in the Hamiltonian, Eq. (1), the spin-orbit terms. The compound matrix elements are finally given as

\[ (\hat{T}_{\alpha\alpha})_{\sigma \sigma'} = T^{(\text{ex})}_{\alpha\alpha} \mathbf{S} \cdot \mathbf{\hat{\sigma}}_{\sigma \sigma'} + T^{(\text{so})}_{\alpha\alpha} \mathbf{S} \cdot \mathbf{\hat{\sigma}}_{\sigma \sigma'}, \]

\[ (\hat{T}_{\text{RL}})_{\sigma \sigma'} = (T_0 + T^{(\text{so})}_{\text{RL}} \mathbf{S} \cdot I) \delta_{\sigma \sigma'} + T^{(\text{ex})}_{\text{RL}} \mathbf{S} \cdot \mathbf{\hat{\sigma}}_{\sigma \sigma'}. \]

Here the unit pseudo-vector \( l \) indicates what projections of the localized spin are modulating the tunneling matrix elements depending upon the geometry of the system. We can estimate \( T^{(\text{ex})}_{\alpha\beta} \sim r^2 E_0 \), where \( E_0 \) is an energy of the order of an atomic energy, while \( r \) is a small dimensionless parameter which describes the weak overlap of the lead-electron wave-function to that of the molecule with the spin. For spin-orbit tunneling we have an additional small relativistic multiplicative factor \( \beta^2 = v^2/c^2 \), i.e., \( T^{(\text{so})}_{\alpha\beta} \sim \beta^2 T^{(\text{ex})}_{\alpha\beta} \), where \( v \) is a typical electron velocity.

Finally, we presume that the spin also couples to the environment due to the spin-phonon coupling, the coupling with nuclear spins, etc. To account for this spin relaxation mechanism we introduce the relaxation rate \( \Gamma_{\text{env}} \).

In order to put our model in perspective let us compare it to other models used in the literature. For the quantum dot studied in Refs. 2, 3 the Hamiltonian for spinless fermion
tunneling is given by

\[ H = \sum_{\sigma, m} c_{m \sigma} c_{m \sigma}^\dagger + \frac{b_z}{2} (d_1^\dagger d_1 - d_2^\dagger d_2) + \frac{b_z}{2} (d_1^\dagger d_2 + d_2^\dagger d_1) + H_{tr}, \]

where \( d_{1,2}^\dagger (d_{1,2}) \) creates (annihilates) an electron in the state 1 or 2 of the dot. Here \( b_z/2 \) is the energy splitting between states 1 and 2, \( b_z/2 \) is the tunneling matrix element connecting them, and tunneling of electrons between the leads depends on the population of the states 1 and 2. The difference between this and the TVS model (with \( H_{tr}=0 \)) is that the couplings to the leads only depend upon the difference of fillings of states 1 and 2 or, making correspondence with the spin operators, the Hamiltonian \( \mathcal{H}_t \) is

\[
\begin{align*}
\mathcal{H}_s &= b_x S_x + b_z S_z, \\
S_x &= (1/2)(d_1^\dagger d_2 + d_2^\dagger d_1), \\
S_z &= (1/2)(d_1^\dagger d_1 - d_2^\dagger d_2),
\end{align*}
\]

while the coupling, \( \mathcal{H}_{tr} \), depends only on one spin component, \( S_z \) of the probed system, instead of all of them as in the exchange term \( S \cdot \mathbf{\sigma} \). Moreover, the leads electrons are fully polarized along the \( z \)-axis.

### III. MAJORANA-KELDYSH FORMULATION OF TUNNELING VIA A SINGLE SPIN

It is assumed that for sufficiently long times the composite (spin-elecrons) system reaches a dc (non-equilibrium) steady state (which does not depend upon the initial conditions). We use the Majorana fermion representation for the spin and the Keldysh diagrammatic technique \[13\] to describe this technique is nowadays a standard method used in non-equilibrium phenomena and its basic description can be found in standard textbooks such as Ref. \[15\]. Its application to Kondo-like problems is described in Ref. \[2\].

Since spin operators are not amenable to the application of the Wick’s theorem, and hence are not appropriate for diagrammatic techniques, we represent spin-1/2 operators \( S_\mu \) in terms of three Majorana fermions \( \eta_\mu \), where \( \mu \in \{x, y, z\} \), satisfying the following mapping and relations

\[
\begin{align*}
S_x &= -i \eta_y \eta_z, \\
S_y &= -i \eta_z \eta_x, \\
S_z &= -i \eta_x \eta_y, \\
(\eta_\mu)^\dagger &= \eta_\mu, \\
(\eta_\mu, \eta_\nu) &= \delta_{\mu \nu}.
\end{align*}
\]

The Hilbert space for Majorana fermions is eight-dimensional (8d) and the transformation from the spin 1/2 to the Majorana fermion representation may be performed in the following way. In general, the spin expectation values that we need to calculate are of the form of a trace, \( \text{Tr}[F(S_\mu)] \), over the two-dimensional (2d) spin Hilbert space. Here \( F \) is some functional of the spin operators. We replace the spin operators by Majorana fermion operators using Eq. \[3\] and we represent the Majorana fermion operators as

\[ \eta_\mu = (c_\mu^\dagger + c_\mu)/\sqrt{2}, \]

where \( c_\mu^\dagger \) and \( c_\mu \) are the creation and annihilation operators of a type \( \mu \in \{x, y, z\} \) fermion satisfying the canonical anti-commutation relation \( \{c_\mu^\dagger, c_\nu\} = \delta_{\mu \nu} \). This representation guarantees that Eq. \[4\] for the Majorana fermion operators are satisfied. A possible orthonormal basis for the 8d Hilbert space \( \mathcal{H}_M \) is \( \varphi_1 = \Phi_0 \), where \( \Phi_0 \) is the vacuum, \( \varphi_2 = c_x^\dagger \Phi_0 \), \( \varphi_3 = c_y^\dagger \Phi_0 \), \( \varphi_4 = c_z^\dagger \Phi_0 \), \( \varphi_5 = c_x^\dagger c_y^\dagger \Phi_0 \), \( \varphi_6 = c_x^\dagger c_z^\dagger \Phi_0 \), \( \varphi_7 = c_y^\dagger c_z^\dagger \Phi_0 \), and \( \varphi_8 = c_x^\dagger c_y^\dagger c_z^\dagger \Phi_0 \). \( \mathcal{H}_M \) may be written as a direct sum of four orthogonal 2d subspaces, \( \mathcal{H}_M = \mathcal{H}_{1+} \oplus \mathcal{H}_{1-} \oplus \mathcal{H}_{2+} \oplus \mathcal{H}_{2-} \), which are invariant under the action of the spin-1/2 operators \( S_\mu \). Here, basis sets for \( \mathcal{H}_{1+} \), \( \mathcal{H}_{1-} \), \( \mathcal{H}_{2+} \), and \( \mathcal{H}_{2-} \) are:

\[
\begin{align*}
\{ (\varphi_0 + i \varphi_7) / \sqrt{2}, (\varphi_1 + i \varphi_8) / \sqrt{2} \}, \\
\{ (\varphi_1 - i \varphi_5) / \sqrt{2}, (\varphi_2 - i \varphi_6) / \sqrt{2} \}, \\
\{ (\varphi_2 - i \varphi_3) / \sqrt{2}, (\varphi_4 - i \varphi_5) / \sqrt{2} \}, \\
\{ (\varphi_4 + i \varphi_2) / \sqrt{2}, (\varphi_5 - i \varphi_6) / \sqrt{2} \}.
\end{align*}
\]

In other words, the Majorana fermion representation of the spin-1/2 operator \( S_\mu \) is an irreducible one in any of the four mentioned subspaces. Furthermore, these representations are equivalent to each other. Therefore, we can write down

\[
\text{Tr}[F(S_\mu)] = \frac{1}{4} \text{Tr}[\tilde{F}(\eta_\mu)],
\]

where the second trace is computed in the extended Hilbert space \( \mathcal{H}_M \). (The trace is invariant under a change of basis in \( \mathcal{H}_M \).) Now one can prove Wick’s theorem in the same way as it was proved for the free Fermi and Bose fields.

The time-ordered averages in the Keldysh technique are taken with the help of the evolution operator \( \Phi_c \) along a closed time contour (its part running from \(-\infty \) to \(+\infty \) denoted by + and the part from \(+\infty \) to \(-\infty \) by –). One defines the four (non-independent) real-time Green’s functions

\[
\begin{align*}
G^\psi_\psi^{+}(t, t') &= -i(T \psi(t) \psi^\dagger(t')), \\
G^\psi_\psi^{-}(t, t') &= -i(\psi(t) \psi^\dagger(t')), \\
G^\psi_\psi_0^{+}(t, t') &= i(\psi(t) \psi^\dagger(t')), \\
G^\psi_\psi_0^{-}(t, t') &= -i(\tilde{T} \psi(t) \psi^\dagger(t')), \quad \text{(10)}
\end{align*}
\]

which can be compactly written as a matrix \( G_\psi(t, t') = -i(\langle \psi(t), \psi^\dagger(t') \rangle) \). Here \( T \) and \( \tilde{T} \) are the time- and anti-time-ordering operators on the + and – parts of the Keldysh contour and \( \psi \) represents an arbitrary fermionic field. They are related through:

\[ G_\psi^\dagger + G_\psi^\dagger = G_\psi^{-} + G_\psi^+, \]

and the brackets \( \langle \rangle \) have the meaning of either a pure state or a distribution of the available phase space of the interacting system.

The four Green’s functions (the “+ basis”) can be expressed in terms of the retarded, advanced and Keldysh Green’s functions (“Larkin-Ovchinnikov (LO) basis”)

\[
\begin{align*}
G^R_\psi(t, t') &= -i \theta(t - t') \langle \psi(t), \psi^\dagger(t') \rangle, \\
G^A_\psi(t, t') &= i \theta(t - t') \langle \psi(t), \psi^\dagger(t') \rangle, \\
G^K_\psi(t, t') &= -i \langle \psi(t), \psi^\dagger(t') \rangle, \quad \text{(11)}
\end{align*}
\]
by use of the transformation
\[
\begin{pmatrix}
G^R_{\psi} & G^K_{\psi} \\
0 & G^A_{\psi}
\end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} G_{\psi}^{\alpha+\alpha} & G_{\psi}^{\alpha-\alpha} \\ G_{\psi}^{\alpha+\alpha} & G_{\psi}^{\alpha-\alpha} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.
\]

In general, the retarded and advanced functions are Hermitian conjugates of each other, i.e., \(G^\alpha_{\psi}(t,t') = (G^\alpha_{\psi}(t',t))^*\). Furthermore, in thermal equilibrium the fluctuation-dissipation theorem connects them through
\[
G^K_{\psi}(\omega) = h_{eq}(\omega) [G^\alpha_{\psi}(\omega) - G^R_{\psi}(\omega)],
\]
with the thermal distribution function of fermions \(h_{eq}(\omega) = -\tanh(\omega/2T)\). As in Ref. \(\text{[2]}\) we assume that for sufficiently long times the spin reaches a dc (non-equilibrium) steady state which does not depend on the initial conditions. In such dc steady state correlation functions depend upon the time difference and the new distribution function \(h(\omega)\) differs from \(h_{eq}(\omega)\). It should be determined as a stationary solution of the quantum kinetic equation.

Firstly, we introduce the bare Green functions for electrons in the presence of an applied magnetic field (or internal exchange field in the case of magnetically ordered leads) \(B_0\) which induces a spin polarization of electrons in the directions \(m_{e0} = B_0/B_0\). Since the leads are assumed to be coupled to a thermal bath, the bare lead-electron Green’s functions are in a voltage driven thermal equilibrium. They have the matrix form in the space of electron spin \((G_{\alpha}(t,t'),\sigma) = -i\langle t|\sigma^\alpha (t)\sigma_{\alpha}(t')\rangle\), where \(c_{\sigma\sigma} = \sum_{\alpha} c_{\alpha\sigma}\)
\[
\begin{align*}
G_{\alpha}(\omega) &= G_{\alpha}(\omega) I + G_{\alpha+}(\omega) m_{e0} \cdot \sigma, \quad (13) \\
G_{\alpha+}(\omega) &= \frac{1}{2} [G_{\alpha+}(\omega) + G_{\alpha-}(\omega)], \\
G^R_{0\alpha}(\omega) &= \int \frac{d\epsilon}{2\pi} \rho_{\alpha}(\epsilon) \omega - \epsilon + i0^+, \\
G^K_{0\alpha}(\omega) &= 2\pi i \hbar_{0\alpha}(\omega) \rho_{\alpha}(\omega), \\
h_{0\alpha}(\omega) &= h_{eq}(\omega - \mu_\alpha) = -\tanh(\omega - \mu_\alpha) / 2T, \\
\rho_{0\alpha}(\epsilon) &= \rho(\epsilon - \kappa B_0 - \mu_\alpha),
\end{align*}
\]

where \(\kappa = \pm \) and \(\rho(\omega)\) is the bare electron density states of (spin) which we assume to be similar in both leads. We consider here only local Green’s functions, so the eigenstate index \(n\) can be dropped. Note that for fully polarized electrons \((B_0 \gg D)\) we obtain \(G_{\alpha}(\omega) = G_{\alpha+}(\omega)\), while for weakly spin polarized electrons \((B_0 \ll D)\) \(G_{\alpha}(\omega) \ll G_{\alpha+}(\omega)\).

The propagators for the Majorana fields can be compactly written as \(G_{\alpha}(t,t') = -i\langle \eta_{\alpha}(t)\eta_{\alpha}(t')\rangle\). The bare propagators satisfy \(G^{K}_{x^2} = G_{yy}^R, G^{R}_{y^2} = -G_{yy}^R, G^{R}_{x^2} = G_{yy}^R, G^{K}_{x^2} = 0\), due to the symmetry under rotations about the z-axis. A useful property holds for the Keldysh components of the spin Green’s functions: Since Majorana fermion operators are Hermitian, from the properties of the commutator, it turns out that \(G^{K}_{\mu\nu}(t,t') = -G^{R}_{\mu\nu}(t',t)\). In the frequency \(\omega\) representation, for time-translation invariant solutions, this means that \(G^{K}_{\mu\nu}(\omega) = -G^{R}_{\mu\nu}(\omega)\), and therefore \(G^{R}_{\mu\nu}(\omega)\) is an odd function of \(\omega\).

To write down Dyson’s equation, it is convenient to use a basis in which the bare propagator is diagonal. Defining the canonical fermion operators \(f = (\eta_x - i\eta_y)/\sqrt{2}\) and \(f^\dagger = (\eta_x + i\eta_y)/\sqrt{2}\), with the associated Green’s functions \(G_{ff}(t) = -i\langle f(t) f^\dagger(0)\rangle\) and \(G_{ff^\dagger}(t) = -i\langle f^\dagger(t) f(0)\rangle\), one has the following expressions for the bare propagators in the \((f, f^\dagger, \eta_z)\) basis in the steady state in the lowest order of perturbation theory
\[
\begin{align*}
G_{zz}(\omega) &= \frac{1}{\omega + i\delta}, \\
G_{zz}(\omega) &= \frac{2i\hbar z(\omega)\delta}{\omega^2 + \delta^2}, \\
G_{ff}(\omega) &= \frac{1}{\omega - B + i\delta}, \\
G_{ff^\dagger}(\omega) &= \frac{2i\hbar f(\omega)\delta}{(\omega - B)^2 + \delta^2}, \\
G_{ff^\dagger}(\omega) &= \frac{1}{\omega + B + i\delta}, \\
G_{ff^\dagger}(\omega) &= \frac{2i\hbar f(\omega)\delta}{(\omega + B)^2 + \delta^2},
\end{align*}
\]

where \(\delta \rightarrow 0^+\) is a small regulator (a width due to an infinitesimally small coupling to a thermal bath), and \(B\) is the total effective magnetic field acting on the spin (see next section). In the thermal equilibrium state \(h_z(\omega) = h_f(\omega) = h_{f^\dagger}(\omega) = -\tanh(\omega/2T)\).

The perturbative calculations of various physical quantities are very conveniently performed by using the matrix form of the Green’s functions in the Keldysh space. Graphical representations can then be done by Feynman diagrams with lines representing matrices of Green’s functions, where at each vertex of an internal point is assigned an additional + or − factor (due to the opposite direction of time integration for the points on the − part of the Schwinger-Keldysh contour). The interaction vertices of Majorana fermions with electrons follow from the form of the tunneling Hamiltonian \(\text{[1]}\)

\[
T^{(ex)}_{\alpha\beta} g \cdot \sigma = T^{(ex)}_{\alpha\beta} \frac{\epsilon}{\sqrt{2}} \sigma_z (f^\dagger f - f f^\dagger) + T^{(ex)}_{\alpha\beta} \frac{\epsilon}{\sqrt{2}} \sigma^+ f^\dagger \sigma^- \eta_z, \\
T^{(so)}_{\alpha\beta} g \cdot \sigma = \frac{T^{(so)}_{\alpha\beta}}{\sqrt{2}} \sigma_z (f^\dagger f - f f^\dagger) + \frac{T^{(so)}_{\alpha\beta}}{\sqrt{2}} (f^\dagger f - f f^\dagger) \eta_z, \\
\sigma^\pm = \sigma_x \pm i\sigma_y, \\
l^\pm = l_x \pm il_y
\]

accounting for the relation \(f^\dagger f = f f^\dagger = 1\). We represent the electron and spin Green’s functions as well as the vertices in Fig. \(\text{[3]}\).

The interacting Green’s function for the spin, \(G\), satisfies the Dyson’s equation
\[
G^{-1}(\omega) = G_0^{-1}(\omega) - \Sigma(\omega),
\]
for the \(6 \times 6\) matrices (in the tensor product of the \(x, y, z\) space and the Keldysh space), where the free propagator is given by Eqs. \(\text{[4]}\). To lowest order, the imaginary part of the self-energy \(\Sigma(\omega)\) is of order \(T^2_{\alpha\beta} P_0^\alpha\) (its ω-dependent part produces a trivial small energy shift that we neglect in the following). Calculating \(G(\omega)\) and neglecting terms of order \(T^2_{\alpha\beta} P_0^\alpha\) we can take the \(2 \times 2\) blocks \(\Sigma_{ff^\dagger}, \Sigma_{ff^\dagger}, \Sigma_{ff}, \Sigma_{ff^\dagger}, \Sigma_{zz}, \Sigma_{zf^\dagger}\) as zeroes. Then, \(G^{-1}(\omega)\) has nonvanishing off-diagonal blocks \(f f^\dagger\) and \(f^\dagger f\),
\[
G^{-1}_{ff^\dagger} = \begin{pmatrix}
\omega - B - \Sigma_{ff^\dagger}(\omega) & -\Sigma_{K}(\omega) \\
-\Sigma_{ff}(\omega) & \omega - B - \Sigma_{ff}(\omega)
\end{pmatrix},
\]
In a steady state, we can define a distribution function.

The perturbation theory have to be the same as the zeroth-order distribution functions. We replace the perturbations with correct zeroth-order distribution functions. One can, nevertheless, use the perturbation theory built upon the appropriate bare Green's functions calculated perturbatively. One can, nevertheless, use the perturbation theory built upon the appropriate bare Green's functions calculated perturbatively.

Parcollet and Hooley [2] have found that if one starts from the equilibrium distribution functions for Majorana fermions, while for \( h_{f,f} \) we replace \( f \rightarrow f^\dagger \) and \( B \rightarrow -B \). For \( G_{zz} \) we replace \( zz \) for \( f^\dagger f \) and put \( B = 0 \).

Parcollet and Hooley [2] have found that if one starts from the equilibrium distribution functions for Majorana fermions, the perturbation theory breaks down when one takes the limit of zero coupling to the leads prior to taking the limit of zero coupling to the thermal bath. This is not the non-commutativity of the limits is that in a non-equilibrium steady state the distribution function of the system can deviate from the equilibrium one significantly in the long-time limit even for very weak tunneling. Hence, at long times it cannot be calculated perturbatively. One can, nevertheless, use the perturbation theory built upon the appropriate bare Green's functions with correct zeroth-order distribution functions \( h_z(\omega) \), and \( h_{f,f}(\omega) \) which are stable with respect to weak perturbations.

These distributions should be obtained self-consistently, i.e., \( h_z(\omega) \) and \( h_{f,f}(\omega) \) computed using second order perturbation theory have to be the same as the zero order distribution function. Hence, assuming that after a long time the system is in a steady state, we can define a distribution function \( h_f(\omega) \) obeying the self-consistency equations

\[
G^K_{f,f}(\omega) = h_f(\omega)[G^A_{f,f}(\omega) - G^R_{f,f}(\omega)],
\]

\[
h_f(\omega) = \frac{\Sigma^K_{f,f}(\omega)}{\Sigma^A_{f,f}(\omega) - \Sigma^K_{f,f}(\omega)},
\]

where \( \Sigma_{f,f}(\omega) \) is calculated in the second order perturbation theory with respect to the electron-spin interaction. \( h_f(\omega) \) and \( h_z(\omega) \) have to be obtained in a similar way. Therefore, one needs to take the zeroth-order Green’s functions for the spin in the form of Eq. (19), calculate the second order self-energies \( \Sigma^K_{f,f}(\omega) \) and \( \Sigma^A_{f,f}(\omega) \), and solve Eq. (19). This procedure, as well as taking the limits \( s \rightarrow 0^+ \) and \( T_{\alpha\beta} \rightarrow 0 \), was carefully outlined by Parcollet and Hooley [2].

A useful relation between the distribution functions \( h_f(\omega) \) and \( h_{f,f}(\omega) \) can be derived. Using the definitions in Eq. (11) and the commutator and anticommutator properties one can show that

\[
G^K_{f,f}(-\omega) = -G^K_{f,f}(\omega),
\]

\[
G^A_{f,f}(-\omega) - G^R_{f,f}(-\omega) = G^A_{f,f}(\omega) - G^R_{f,f}(\omega),
\]

and using the definition of Eq. (19) one obtains that

\[
h_f(\omega) = -h_f(-\omega), \quad h_z(\omega) = -h_z(-\omega),
\]

so that \( h_z(0) = 0 \).

**IV. EFFECT OF THE TUNNELING ELECTRONS ON THE SPIN DYNAMICS**

**A. Effective dc magnetic field acting on the spin**

The diagrams shown in Fig. 3 lead to a renormalization of the effective dc magnetic field acting on the spin when the electrons are polarized. To first order in the electron-spin interaction (the first diagram in Fig. 3) the additional dc magnetic field caused by the tunneling electrons is given by

\[
\mu B_T = 2\text{Re} \sum_{\alpha,\sigma} T^{(ex)}_{\alpha\sigma}(\bar{c}_{\alpha\sigma}\bar{\sigma}\sigma c_{\alpha\sigma}'),
\]

\[
= 4 \sum_{\alpha} s_{\alpha} m_{\alpha} T^{(ex)}_{\alpha\alpha},
\]

neglecting spin-orbit corrections. Here \( s_{\alpha} \) is the degree of the electron spin polarization in the lead \( \alpha \), i.e., the ratio of the electron magnetization to the optimal magnetization. This additional magnetic field originates due to the exchange interaction between the spin and the electrons in the leads, and it is voltage independent.

We may estimate this contribution using the expression for the tunneling resistance \( R_T = h/(4\pi e^2 T_0 \rho_0) \). For \( R_T = 100 \)
Mohm we obtain $T_0\rho_0 \approx 1.6 \cdot 10^{-3}$ and taking $T_{\alpha\alpha}^{(ex)} \approx 0.1T_0$ we estimate at $\rho_0 \approx eV^{-1}$ the additional magnetic field $B_T \approx 16\psi$, T. When electrons are polarized by the magnetic (or exchange) field $B_\alpha$ such that $B_\alpha\rho_0 \ll 1$, we estimate $B_T \approx 10^{-3}B_\alpha$. The second diagram in Fig. 4 is smaller by a factor $T_0\rho_0$ and, thus, may be neglected.

In the following we choose the $z$-axis along the total field $B = B_0 + B_T$. By $m_\alpha$ we denote the direction of electron spin polarization in this new coordinate system. In the case that $m_{\alpha B_0} = m_{10} = m_0$, and $m_0$ has the components $m_{\alpha x}, m_{\alpha z}$ in the coordinate system with the $z$-axis aligned along the bare field $B_0$, we get that $m$ has coordinates $m_x, m_z$ in the coordinate system where the $z$-axis is aligned along $B$ with

$$m_x = \frac{\sqrt{B_0^2 - (B_0 \cdot m_0)^2}}{\sqrt{B_0^2 + B_T^2 + 2B_TB_0 \cdot m_0}}$$

$$m_z = \frac{(B_0 \cdot m_0) + B_T}{\sqrt{B_0^2 + B_T^2 + 2B_TB_0 \cdot m_0}}$$

B. Self-energy of Majorana fermions and steady state distribution function

The diagrams shown in Figs. 4 determine the imaginary part of the Majorana fermion self-energies which depend upon $T$, $V$, $m_\alpha$ and $B$. We start by considering only the interaction of the spin with the tunneling electrons, assuming $T_{env} = 0$. In this case, the spin is precessing freely between consecutive passings of tunneling electrons which change the phase of precession randomly, preserving time-translation invariance on the average in the steady state. In the quasi-classical approach these passings are replaced by an effective classical magnetic field representing white noise. On the other hand, in the quantum description the electrons are treated as tunneling particles with initial energy $\epsilon_n + V$ and final energy $\epsilon_{n'}$. The tunneling process now has similarities with the scattering of an individual particle by the spin. The difference is that both the initial and final energy of the tunneling electrons are not fixed, in our model being restricted only by the bands of the leads-electrons.

The diagrams shown in Fig. 4 lead to the self-energy

$$\Sigma_{f1}^{-\omega} = -\int d\omega \frac{d\nu}{(4\pi)^2} \delta(\omega + v - u - \epsilon) \times$$

$$\{4\text{Tr}[\sigma_\gamma G^{\omega}_{\alpha\gamma}(u)\sigma_\gamma G^{\omega}_{\beta\gamma}(v)]G_{f1f}^{\omega}\gamma(e) +$$

$$2\text{Tr}[\sigma_\gamma G^{\omega}_{\alpha\gamma}(u)\sigma_\gamma G^{\omega}_{\beta\gamma}(v)]G_{z\gamma}^{\omega}\gamma(e)\}.$$ (24)

The expression for $\Sigma_{f2}^{-\omega}$ can be similarly obtained. Note, that $\Sigma_{f2}^{-\omega}$ depends on $h_{f1}(-B)$ via $G_{f1}^{\omega}$, while $\Sigma_{f1}^{-\omega}$ depends on $h_{f1}(B)$. Then, Eq. (12) is a closed equation for $h_{f1}(\omega)$. The other diagrams in Fig. 4 give

$$\Sigma_{zz}^{-\omega} = -\int d\omega \frac{d\nu}{8\pi^2} \delta(\omega + v - u - \epsilon) \times$$

$$\{\text{Tr}[\sigma_\gamma G^{\omega}_{\alpha\gamma}(u)\sigma_\gamma G^{\omega}_{\beta\gamma}(v)]G_{f1f}^{\omega}\gamma(e) +$$

$$\text{Tr}[\sigma_\gamma G^{\omega}_{\alpha\gamma}(u)\sigma_\gamma G^{\omega}_{\beta\gamma}(v)]G_{zz}^{\omega}\gamma(e)\}. $$ (25)

The traces for the spin matrices in the expressions for the self-energy can be performed with the help of the relations

$$\frac{1}{2} \sum_{\gamma\delta} m_{\alpha\gamma} m_{\beta\delta} \text{Tr}[\sigma_\gamma \sigma_\delta] =$$

$$-m_{\alpha\mu} m_{\beta\nu} \delta_{\mu\nu} + m_{\alpha\mu} m_{\beta\nu} + m_{\alpha\mu} m_{\beta\nu},$$

$$\sum_{\nu} m_{\alpha\nu} \text{Tr}[\sigma_\nu \sigma_\nu] = 2m_{\alpha\nu}, \quad \text{Tr}[\sigma_\nu \sigma_\nu \sigma_\nu] = 2\iota.$$ (26)

The transformation from the ”± basis” to the “LO basis” for the self-energy is the same as for the inverse Green’s function

$$\begin{pmatrix} \Sigma_{\omega}^{R} & \Sigma_{\omega}^{K} \\ 0 & \Sigma_{\omega}^{A} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \Sigma_{\omega}^{++} & \Sigma_{\omega}^{+-} \\ \Sigma_{\omega}^{+-} & \Sigma_{\omega}^{-+} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

which implies that $\Sigma_{\omega}^{++} + \Sigma_{\omega}^{+-} + \Sigma_{\omega}^{-+} + \Sigma_{\omega}^{--} = 0$ and, therefore, $\Sigma^{K} = -(\Sigma^{++} + \Sigma^{-+})$ and $\Sigma^{A} - \Sigma^{R} = \Sigma^{--} - \Sigma^{+-}$.

In this way, for the fully spin-polarized case, $m_\alpha = m_{\beta} = 1$, we obtain

$$\Sigma_{f1f1}^{K}(\omega) = \frac{i\pi}{2} \rho_0^2 \sum_{\alpha\beta} T_{\alpha\beta}^2 \{(1 + 2m_{\alpha x}m_{\beta z} - m_{\alpha} \cdot m_{\beta})$$

$$[(\mu_\alpha - \mu_\beta - \omega + B) \mp h_{f1}(B) T \phi(\mu_\beta - \mu_\alpha + \omega \pm B)$$

$$- (1 - m_{\alpha z}m_{\beta x} + m_{\alpha z} + m_{\beta x})(\mu_\beta - \mu_\alpha + \omega)\}\}.$$ (27)

and

$$\Sigma_{f1f1}^{A} = \Sigma_{f1f1}^{K}(\omega) \mp \frac{i\pi}{2} \rho_0^2 \sum_{\alpha\beta} T_{\alpha\beta}^2 \times$$

$$\{(1 + 2m_{\alpha x}m_{\beta z} - m_{\alpha} \cdot m_{\beta})$$

$$[T \phi(\mu_\beta - \mu_\alpha + \omega \pm B) \mp h_{f1}(B)(\mu_\beta - \mu_\alpha + \omega \pm B)$$

$$+(1 - m_{\alpha z}m_{\beta x} + m_{\alpha z} + m_{\beta x})T \phi(\mu_\beta - \mu_\alpha + \omega \pm B)]\},$$
In the next section we show that the imaginary part of the self-energy \( \Sigma_{ff \uparrow}(\omega) \), \( \Gamma_\perp = \text{Im} \Sigma_{ff \uparrow}(\omega) \), describes dephasing of the spin precession caused by the tunneling electrons. It determines the current noise power as shown later.

Therefore, at frequency \( \omega = \pm B \), Eq. (12) gives us a self-consistent equation for \( h_f(B) = -h_f(-B) \) in the case of full spin polarization. This equation has a unique solution for the distribution function \( b = B/T, v = V/T \)

\[
h_f(B) = \frac{2b(1 - m_R m_L) - 2\nu(m_R m_L) + b \theta}{\phi^+(1 - m_R m_L) - \phi^-(m_R m_L) + \phi(b) \theta},
\]

\[
\theta = \frac{T_{2RR}^2 (1 - m_{RL}^2) + T_{2LL}^2 (1 - m_{LR}^2)}{T_{2RL}^2},
\]

\[
\phi^\pm = \phi(v \pm b) \pm \phi(v - b),
\]

provided that we do not have one of the cases \( m_{RL} = m_{LR} = \pm 1 \). In the latter two cases, the self-consistency equation is an identity and the spin steady state can be any. When \( m_\alpha = 0 \) we reproduce the results of Ref. 2 for unpolarized electrons (see Fig. 5). If we account for environment-driven relaxation, \( \Gamma_{\text{env}} \neq 0 \), the distribution function will be closer to that in thermal equilibrium. When such relaxation dominates, the distribution function \( h_f(\omega) \) coincides with the equilibrium one.

### C. Average spin magnetization and the spin-spin correlation function

We compute first the average spin magnetization \( \langle S_\mu(t) \rangle = -i/(2) e^{i\mu \gamma} \langle \nu_\mu(t) \eta_\gamma(t) \rangle \) (time-independent in the dc steady state), where \( e^{i\mu \gamma} \) is the antisymmetric unit tensor. To lowest order, the values \( \langle S_x \rangle \) and \( \langle S_y \rangle \) are proportional to the small parameters \( T_{\alpha \beta}^2 \theta_0^2 \) and can be neglected. For \( \langle S_z \rangle \), we get (to zeroth order)

\[
\langle S_z \rangle = -\frac{1}{2} h_f(B),
\]

When \( m_R = m_L \) (except \( m_\alpha = \pm 1 \)), we obtain the same result as for unpolarized electrons found by Parcollet and Hookey

\[
\langle S_z \rangle = \frac{1}{2} \tanh \left( \frac{b}{2} \right) \frac{\phi(b)(2 + \theta)}{\phi^+ + \phi(b) \theta}.
\]

Here \( \theta = (T_{2LL}^2 + T_{2RR}^2)/T_{2RL}^2 \). At \( T = 0 \), and \( V < B \) we get \( \langle S_z \rangle = 1/2 \), i.e., the spin is in the ground state because electrons do not have enough energy to flip the spin. When \( V > B \) the electrons can flip the spin, thus reducing the spin magnetization to \( \langle S_z \rangle = (1 + \theta/2)/(2x + \theta) \), where \( x = V/B \). It drops as \( 1/V \) for large \( V \).

As noticed by Shirman and Malkhlin [16] the spin-spin correlation function \( S_\alpha(t)S_\alpha(0) \) may be expressed via the \( f \)-fermion Green’s functions. We rewrite the spin operators in the following fashion

\[
S_x = -(f^t + f) \hat{\tau}^x/2 = -\hat{\tau}^x(f^t + f)/2,
\]

\[
S_y = i(f^t - f) \hat{\tau}^y/2 = i\hat{\tau}^y(f^t - f)/2,
\]

\[
S_z = -\eta_z \hat{\tau}^z/2 = -\hat{\tau}^z \eta_z/2.
\]
The operator $\hat{\tau} = \sqrt{2}(1 - 2f^\dagger f)\eta_z = i2\sqrt{2}n_x\eta_y\eta_z$ commutes with the spin operators $S_\mu$, and hence with the Hamiltonian, while $\hat{\tau}^2 = I$. (Notice that the operator $\hat{\tau}$ maps the orthogonal subspaces $\mathcal{H}_{1,\pm} \leftrightarrow \mathcal{H}_{2,\pm}$ of section III among themselves.)

As a result,

$$S_{xx}(t) = \langle S_x(t)S_x(0) \rangle \quad (34)$$

$$= \frac{i}{4}([f^\dagger(t) + f(t)][f^\dagger(0) + f(0)])$$

$$= \frac{i}{4}[G_{ff}^+(t) + G_{ff}^+(t) + G_{ff}^+(t) + G_{ff}^-(t)].$$

Similarly,

$$S_{yy}(t) = -\frac{i}{4}([f^\dagger(t) - f(t)][f^\dagger(0) - f(0)])$$

$$= \frac{i}{4}[G_{ff}^+(t) + G_{ff}^+(t) - G_{ff}^+(t) - G_{ff}^-(t)].$$

$$S_{zz}(t) = \frac{i}{2}G_{zz}^+(t).$$

To zeroth order in perturbation theory we obtain

$$S_{xx}(\omega) = S_{yy}(\omega) = \frac{i}{4}[G_{ff}^+(\omega) + G_{ff}^-(\omega)]$$

$$= \frac{\pi}{4}([1 - h_f(B)]\delta(\omega - B) + [1 + h_f(B)]\delta(\omega + B)],$$

while $S_{zz}(\omega) = (\pi/2)\delta(\omega)$.

To describe the contribution which displays a peak at the Larmor frequency in the correlation function $S_{xx}(\omega)$ we need to account for the imaginary part of the self-energy. For $G_{ff}^+, f^\dagger f$ we have

$$G_{ff}^{R,A}(\omega) = \frac{1}{\omega - B - \Sigma_{ff}^{R,A}}, \quad (37)$$

$$G_{ff}^{R,A}(\omega) = \frac{1}{\omega + B - \Sigma_{ff}^{R,A}}.$$  

At the weak tunneling condition, $|t_{RL}|^2\rho_0 < 1$, when the relaxation due to the environment is negligible, we can also neglect the contributions from $G_{ff}$ and $G_{f f^\dagger}$ to $S_{xx}$ with respect to those from $G_{ff}^+$ and $G_{ff}^-$. We obtain

$$S_{xx}(\omega) = \frac{1}{4}\left[\left(1 - h_f(B)\right)\Gamma_\perp + \left(1 + h_f(B)\right)\Gamma_\perp\right]$$

$$\left[\omega - B + \Sigma_{ff}^{R,A}\right] + \left[\omega + B + \Sigma_{ff}^{R,A}\right],$$

where, at $T = 0$, for fully polarized electrons along the $x$-axis we get

$$\Gamma_\perp(B, V) = \text{Im}\left[\Sigma_{ff}^{R,A}(B)\right]$$

$$= \frac{\pi}{4}T_{RL}^2\rho_0^2[|V + B| + |V - B| + \theta B].$$

When $V < B$ this gives the width of the precession resonance due to quantum fluctuations of the current (the current operator does not commute with the Hamiltonian, and the current fluctuates between the leads and the spin),

$$\Gamma_{xx}^{(0)}(B) = \frac{\pi}{4}T_{RL}^2\rho_0 B(2 + \theta), \quad (40)$$

while for $V > B$ the decoherence rate has an additional contribution due to the voltage induced transport current

$$\Gamma_\perp(B, V) = \Gamma_{xx}^{(0)}(B) + \frac{\pi}{2}T_{RL}^2\rho_0^2(V - B)\Theta(V - B). \quad (41)$$

We estimate $\Gamma_\perp \approx 10^{-5}V$, in a junction with $T_{RL} = 0.1T_0$ and tunneling resistance $R_t = 100$ Mohm. To determine $\Gamma_\perp$ in the unpolarized case we need to put $\mathbf{m}_n = 0$ and multiply by 4 in Eq. (39).

Thus, we see that the spin-spin correlation functions $S_{xx}(\omega)$ and $S_{yy}(\omega)$ exhibit oscillations at a renormalized Larmor frequency $\omega_L$, with a peak width depending on $V$, $B$ and $T$. When the relaxation due to the environment dominates, $B \gg \Gamma_{env} \gg \Gamma_\perp$, we need to replace $\Gamma_\perp$ by $\Gamma_{env}$ in Eq. (38). The important point is that $\Gamma_\perp$ provides a lower bound for the decoherence rate $\Gamma$, i.e., for the width of the peak at $\omega_L$ in the spin-spin correlation function.

V. EFFECT OF THE SPIN ON THE TUNNELING CURRENT

A. The average current

Let us now turn to the evaluation of the average current and current power spectrum. The current operator is given as

$$\hat{I}(t) = -ie\sum_{n\sigma} [c_{RLn\sigma}(t)\langle \hat{T}_{RL}\rangle_{\sigma\sigma'}c_{Ln\sigma'}(t) - \text{H.c.}]. \quad (42)$$

The average current (involving the complete spin-electrons Hamiltonian $\mathcal{H}$) is determined from

$$I = \langle \hat{I}(t) \rangle = 2e\text{Im}\sum_{n\sigma} \langle c_{RLn\sigma}(t)\langle \hat{T}_{RL}\rangle_{\sigma\sigma'}c_{Ln\sigma'}(t) \rangle. \quad (43)$$

In the following, we will assume that $\mathcal{H}_T$ is turned on adiabatically and we are interested only in the average current up to second order in $\mathcal{H}_T$. (Remember that we are only interested in the dc steady state solution.) The following derivation is

FIG. 6: A diagram for the current-current correlation function that does not contribute to the peak at the renormalized Larmor frequency but gives a spin-dependent correction to the shot noise.
similar to that explained in Ref. [15] for the quantum dot tunneling. Denoting

\[
\hat{A}(t) = \sum_{n,m,\sigma,\sigma'} c^\dagger_{Rn\sigma}(t) \langle \hat{T}_{RL} \rangle_{\sigma\sigma'} c_{Lm'\sigma'}(t),
\]

we obtain (assuming that the leads are not superconducting)

\[
\langle \dot{I}(V) \rangle = -2e \operatorname{Im}\{U_R(-V)\},
\]

\[
U_R(\omega) = -i \int_{-\infty}^{\infty} dt \ e^{i\omega t} \langle [\hat{A}(t), \hat{A}^\dagger(0)] \rangle.
\]

Assuming the dependence \(\rho(\epsilon) = \rho_0\) near the Fermi energy we obtain for fully polarized electrons, \(B_o \gg D\),

\[
\begin{align*}
I(V) &= I_0(V) + I_s(V) \langle \mathcal{S} \rangle, \\
I_0(V) &= \pi e(1 + \mathbf{m}_R \cdot \mathbf{m}_L) T_0^{(ex)} \rho_0^2 V, \\
I_s^{(1)}(V) &= 2\pi e(\mathbf{m}_R + \mathbf{m}_L) T_0^{(ex)} \rho_0^2 V.
\end{align*}
\]

For weakly polarized electrons, \(B_o = B \ll D\), approximating \(\rho(\epsilon) \approx \rho_0(1 + \epsilon/D)\) near the Fermi energy, we get \(\mathcal{I}_I \propto e T_0^{(ex)} \rho_0^2 (B/D) V\), while \(I_0 = 4\pi e T_0^2 \rho_0^2 V\).

When \((\mathbf{m}_R, \mathbf{m}_L) \perp B\) the term of order \(T_0^{(ex)}\) vanishes because \(\langle \mathcal{S} \rangle\) is parallel to \(B\). The next spin-dependent term in the average current, \(I_s^{(2)}\), is of order \((T_0^{(ex)})^2\) and, in the fully polarized case along the \(x\)-axis, is given by

\[
I_s^{(2)} = 2 e (T_0^{(ex)})^2 \text{Re} \left\{ \int_{-\infty}^{\infty} dt e^{-iV(t-t')} \right. \\
&\left. \left[ G_R(t'-t) G_L(t-t') S_{xx}(t-t') - G_R'(t'-t) G_L'(t-t') S_{xx}(t-t') \right] \right\}.
\]

Here \(G_n(t)\) is the causal Green’s function. Using Fourier transforms and Eq. (35) the result, at \(T = 0\), is

\[
I_s^{(2)} = \frac{\pi}{2} e (T_0^{(ex)})^2 \rho_0^2 [V + h_f(B)B] \Theta(V - B).
\]

Hence, when \(V < B\) the average current is not affected by the spin because electrons have not enough energy to flip it. The spin remains in the ground state and thus cannot affect tunneling electrons and be probed. In the case that there is no reflection, i.e., \(\theta = 0\), the spin-dependent contribution is related to the spin-flip rate \(\Gamma_{\parallel}\),

\[
I_s^{(2)} = e\Gamma_{\parallel},
\]

because the spin-dependent contribution to the current is determined by the matrix element of the operator \(\hat{S}_x\) between states with opposite spin projections. Hence, each tunneling electron needs to flip the localized spin in order to go through. In the large \(V\) limit we obtain \(\Gamma_{\parallel} \to \Gamma_{\perp}\) in agreement with the quasiclassical results [13].

At \(T = 0\), the contribution \(I_s^{(2)}\) leads to a discontinuous derivative of \(I(V)\) at \(V = B\) due to the opening of a new channel corresponding to the tunneling of electrons via the spin. The change is more pronounced in the second order derivative of \(I(V)\) with respect to \(V\). Accounting for broadening of the spin energy, \(\Gamma_{\perp}\), we derive

\[
\frac{d^2 I}{dV^2} = 2e(T_0^{(ex)})^2 \rho_0 \sum_{\mu = x, y} S_{\mu \mu}(V) (m_{RL\mu} + m_{L\mu})^2.
\]

We see that this second order derivative of the average tunneling current shows a peak (see Eq. (50)) at \(V = B\). Hence, measurements of the \(I-V\) tunneling characteristics provide information on the spectral density of the spin-spin correlation function. Similar behavior for the average current with respect to bias voltage was used previously for inelastic tunneling spectroscopy of phonons [17]. It is based on an equation similar to Eq. (50) relating \(d^2 I/dV^2\) to second order derivative of states.

Note, that in the case of unpolarized electrons \(I_s\) is small, but the second derivative of \(I(V)\) also shows a similar peak \[18\]. However, now its amplitude has an additional small factor \(T_0^2 \rho_0^2\).

### B. The shot noise power

Next we calculate the current-current correlation function. The particular fluctuation operator one needs to evaluate depends upon the nature of the measurement itself [19], see also [20]. In the general case the result of measurements depends on the correlation functions describing emission, (+), and absorption, (-), by the tunneling contact

\[
P_\pm(\omega) = \int_{-\infty}^{\infty} dt e^{\pm i\omega t} P(t), \quad P(t) = \langle \hat{I}(0) \hat{I}(t) \rangle.
\]

We obtain

\[
P_\pm(\omega) = P_{s\pm}(\omega) + P_{p\pm}(\omega),
\]

\[
P_s(t) = e^2 \sum_{n,m,m',\sigma,\sigma',\gamma,\gamma'} \langle \langle \langle c^\dagger_{Rn\sigma}(0) c_{Rm'\gamma}(0) \rangle \langle \hat{T}_{RL} \rangle_{\sigma\sigma'}(0) \rangle \langle \langle c_{Lm'\gamma'}(0) c^\dagger_{Lm\sigma'}(0) \rangle \langle \hat{T}_{RL} \rangle_{\sigma'\gamma'}(0) \rangle_s + (R \leftrightarrow L) \rangle,
\]

where the term \(P_s(t)\) describes the shot noise to second order in the matrix elements \(T_0\) and \(T_{RL}\), while \(P_p(t)\) represents the contribution which displays a peak at the renormalized Landau frequency of order \((T_0 T_{RL})^2\) (see next subsection). We calculate \(P_s(t)\) in the framework of standard time-dependent perturbation theory (as we did to calculate the average current). In the equation above, \((\ldots)_s\) means average over electron degrees of freedom, after the perturbation due to the spin-electron interaction is accounted for, while \((\ldots)_p\) means average over the localized spin degrees of freedom. Averages over spin operators are performed using the known spin correlation functions determined in previous sections.

Contributions proportional to \(T_0^2\) and \(T_0 T_{RL}\) give the standard expression [21] modified by the presence of the localized spin, in the same way as the average current Eq. (46). For fully polarized electrons along the \(x\)-axis we obtain, at \(T = 0\),

\[
P_{s+}^{(0)}(\omega) = 2\pi e^2 (T_0^2 + 2T_0 T_{RL}) \rho_0^2 |V - \omega|.
\]
For unpolarized electrons we need to multiply this result by a factor 4. We obtain for the symmetrized shot noise power ($\omega < V$)

$$[\mathcal{P}^{(0)}_{s+} + \mathcal{P}^{(0)}_{s-}] / 2 = e (I_0 + I_s^{(1)}).$$

The terms proportional to $T_{RL}$ describe the effect of the spin dynamics on the shot noise, $\mathcal{P}_s^{(1)}$. A diagram contributing to $\mathcal{P}_s^{(1)}$ is shown in Fig. [4]. For electrons fully polarized along the $x$-direction we derive for the emission part at $T = 0$ and $\Gamma_\perp = 0$

$$\mathcal{P}_{s+}^{(1)}(\omega) = \frac{\pi}{4} e^2 (T_{RL}^{(ex)})^2 \rho_0^2 \Theta(V - B) \left[ (1 - h_f(B)) \right.
\times F(-\omega + V - B) + (1 + h_f(B)) F(-\omega + V + B) \left. \right],$$

where

$$F(x) = x \Theta(x).$$

Emission is possible at frequencies $\omega < V - B$, when the spin is in the ground state, and frequencies $\omega < V + B$, when the spin is in an excited state. The second order derivative of $\mathcal{P}_{s+}^{(1)}(\omega)$ with respect to $\omega$, when $V > B$, has peaks with amplitudes $\pi e^2 (T_{RL}^{(ex)})^2 \rho_0^2 (1 \pm h_f(B))/2$ and widths $\Gamma_\perp$ at frequencies $\omega = V \pm B$. This provides a way to detect the presence of the localized spin.

The absorption part at $T = 0$ and $\Gamma_\perp = 0$ is given as

$$\mathcal{P}_{s-}^{(1)}(\omega) = \frac{\pi}{4} e^2 (T_{RL}^{(ex)})^2 \rho_0^2 \{ \Theta(V - B) \left[ (1 - h_f(B)) (\omega + V - B + F(\omega - V - B)) + (1 + h_f(B)) (\omega + V + B + F(\omega - V + B)) \right] \
+ 2 \Theta(B - V) \left[ F(\omega + V - B) + F(\omega - V - B) \right] \},$$

with the result

$$[\mathcal{P}_{s+}^{(1)} + \mathcal{P}_{s-}^{(1)}] / 2 = e I_{s\perp}^{(2)}$$

for $\omega < |V - B|$. Figure [4] displays the emission and absorption contributions.

For unpolarized electrons we derive

$$\mathcal{P}_{s+}^{(un)}(\omega) = 4 \mathcal{P}_{s+}^{(1)}(\omega) + 2 \pi e^2 (T_{RL}^{(ex)})^2 \rho_0^2 (V - B),$$

where $\mathcal{P}_{s+}^{(1)}(\omega)$ is given by Eq. (55) with the same $h_f(B)$ as for fully polarized electrons. This noise power also exhibits anomalies at frequencies $\omega = V \pm B$. We see that $\mathcal{P}_{s+}^{(1)}(\omega)$ has no peak at the Larmor frequency neither for polarized nor for unpolarized electrons contrary to the results of Ref. [14].

The fourth order terms, $T_{RL}^2 T_{RL}^2$, in $\mathcal{P}(\omega)$ correspond to 10 skeleton diagrams some of which are shown in Figs. [8] and [9]. In these 4-vertex diagrams the external frequency $\omega$ (via the current vertex) may enter in any of the two vertices. Diagrams in Fig. [8] (all six), as well as the diagram shown in Fig. [9] contain integration over the energy $\epsilon$ entering in all the electron and spin lines of the diagram. Hence, these contributions change at least on the frequency scale of order $V, B$ as the term $\mathcal{P}_{s+}^{(1)}(\omega)$. They are small in comparison to $\mathcal{P}_{s+}^{(1)}(\omega)$, and may be neglected.

**FIG. 7:** Emission and absorption contributions to the shot noise power to order $T_{RL}^2$, at $T = 0$, for similar electron polarizations in the leads $\mathbf{m} \perp B$. $\mathcal{P}_{s\perp}^{(1)}$ is in units $(\pi Be^2 (T_{RL}^2 \rho_0^2)/4)$. For $x < 1$ the emission contribution vanishes. Notice the different regimes, characterized by a change in the slope, that appear because of the presence of the localized spin.

**FIG. 8:** Two of the six diagrams for the current-current correlation function that do not contribute to the peak at the renormalized Larmor frequency to order $T_{RL}^2 T_{RL}^2$. The external vertices can be any two of the four vertices in the diagram.

**C. The peak at the renormalized Larmor frequency**

To derive this peak we consider the 4 diagrams of the type shown in Fig. [8] which give contributions of order $T_{RL}^2 T_{RL}^2$. In these diagrams integration over energy $\epsilon$ in the spin lines is independent of the integrations in the electron lines and they provide a contribution to the current-current correlation func-
tion $P_p (\omega)$ which has the same peaks at $\omega = \pm B$ as the spin-spin correlation function. To calculate this contribution we use the Keldysh technique as done by Shnirman et al. We will assume that to study the peak at the renormalized Larmor frequency experimentally one can measure the instantaneous values of the current over a long period of time and then obtain the symmetrized current power spectrum defined as

$$P (\omega) = \frac{1}{2} [P_+ (\omega) + P_- (\omega)]. \quad (60)$$

To calculate the first term in Eq. (51), we put $t = t_-$ and $0 = 0_+$. For electrons fully polarized along the $x$-axis the first term is given as

$$\langle I (t) I (0) \rangle = (-T_0^2 \hat{B} (t) \hat{B} (0) - T_0 T_{RL} \hat{A} (t) S_z (t) \hat{B} (0) + \hat{B} (t) \hat{A} (0) S_z (0) - T_{RL}^2 \hat{A} (t) S_z (t) \hat{B} (0) S_z (0)), \quad (61)$$

$$\hat{B} (t) = \sum_n c_{t, n x}^c c_{n z L x} (t),$$

$$\hat{A} (t) = \sum_n c_{t, n x}^c \sigma_y c_{n z L x} (t).$$

Now, in the framework of the Keldysh technique, we expand $B (t) B (0)$ up to $(T_{RL})^2$ order in the first term, up to $T_0 T_{RL}$ in the second and third terms, and up to $T_0^2$ in the last term. Let us consider the first term which corresponds to the second diagram in Fig. 2. This diagram leads to the contribution

$$-(T_0 T_{RL})^2 \sum_{\mu, \nu, \rho, \sigma} \Lambda^{-\mu}_{RL} (-\omega) \mu \nu S^{\mu \nu} (\omega) \Lambda^{\nu \rho} (\omega).$$

Here the product $\mu \nu$ is the sign factor $+1$ if $\mu = \nu$ and $-1$ otherwise, while

$$\Lambda^{\mu \nu}_{RL} (\omega) = \int \frac{d \epsilon}{2 \pi} \text{Tr} [G^{\mu \nu}_{RL} (\epsilon) \sigma_y G^{\nu \mu}_{RL} (\epsilon - \omega)], \quad (62)$$

$$\Lambda^{\mu \nu}_{LR} (\omega) = [\Lambda^{\mu \nu}_{RL} (\omega) - \Lambda^{\nu \mu}_{LR} (\omega)], \quad (63)$$

where the trace is taken over spin variables. Further, we denote by $S^{\mu \nu} (\omega)$ the combination of Majorana fermion Green’s functions which are represented by broken lines in the diagrams in Fig. 2.

$$S^{\mu \nu} (\omega) = \int \frac{d \epsilon}{2 \pi} G^{\mu \nu}_{yy} (\epsilon + \omega) G^{\nu \mu}_{zz} (\epsilon) = \int \frac{d \epsilon}{2 \pi} G^{\mu \nu}_{zz} (\epsilon + \omega) G^{\nu \mu}_{yy} (\epsilon), \quad (64)$$

$$\int \frac{d \epsilon}{2 \pi} G^{\mu \nu}_{zz} (\epsilon + \omega) G^{\nu \mu}_{yy} (\epsilon) = \int \frac{d \epsilon}{2 \pi} G^{\mu \nu}_{yy} (\epsilon + \omega) G^{\nu \mu}_{zz} (\epsilon),$$

with the property

$$S^{\mu \nu} (\omega) = S^{\nu \mu} (-\omega). \quad (65)$$

Adding the term with interchanged $R$ and $L$ and symmetrizing with respect to time we obtain for the first term

$$P^{(1)}_p (\omega) = -\frac{1}{2} (T_0 T_{RL})^2 \sum_{\mu, \nu} \mu \nu [\Lambda^{-\mu} (\omega) S^{\mu \nu} (\omega) \Lambda^{\nu \rho} (\omega) + \Lambda^{\mu \nu} (\omega) S^{\rho \mu} (\omega) \Lambda^{\nu \rho} (\omega)] \quad (66)$$

Similarly we calculate the other three terms

$$P^{(2)}_p (\omega) = -\frac{1}{2} (T_0 T_{RL})^2 \sum_{\mu, \nu} \mu \nu [\Lambda^{-\mu} (\omega) S^{\mu \nu} (\omega) \Lambda^{\nu \rho} (\omega) + \Lambda^{\mu \nu} (\omega) S^{\rho \mu} (\omega) \Lambda^{\nu \rho} (\omega)], \quad (67)$$

$$P^{(3)}_p (\omega) = -\frac{1}{2} (T_0 T_{RL})^2 \sum_{\mu, \nu} \mu \nu [\Lambda^{-\mu} (\omega) S^{\mu \nu} (\omega) \Lambda^{\nu \rho} (\omega)], \quad (68)$$

$$P^{(4)}_p (\omega) = -\frac{1}{2} (T_0 T_{RL})^2 \sum_{\mu, \nu} \mu \nu [\Lambda^{-\mu} (\omega) S^{\mu \nu} (\omega) \Lambda^{\nu \rho} (\omega)], \quad (69)$$

Let us calculate $S^{\mu \nu} (\omega)$ using the expressions for $G^{-\mu}$ and $G^{\mu}$ in terms of the zeroth-order functions $G^R$, $G^A$ and $G^K$

$$S^{\mu \nu} (\omega) = \frac{1}{4} \int \frac{d \epsilon}{2 \pi} \{G_{yy}^R (\epsilon + \omega) G_{zz}^K (\epsilon) + \overline{G_{yy}^R (\epsilon + \omega) G_{zz}^K (\epsilon)} \} \quad (70)$$

$$G_{yy}^A (\epsilon + \omega) G_{zz}^R (\epsilon) + G_{zz}^R (\epsilon) G_{yy}^A (\epsilon) + G_{zz}^K (\epsilon + \omega) G_{yy}^A (\epsilon) + G_{yy}^A (\epsilon) G_{zz}^K (\epsilon)],$$

Since $h_+ (\omega) = 0$, we have $G_{zz}^K (\omega) = 0$. Therefore, the contribution to $S^{\mu \nu} (\omega)$ from the terms containing $G_{zz}^K$ is negligible, but the last term does contribute. We get

$$S^{\mu \nu} (\omega) = S^{\mu \nu} (\omega) \quad (71)$$

where $\delta^+ (\omega) = \delta (\omega - B) + \delta (\omega + B)$. Next, we take into account the broadening of the resonances at $\omega = \pm B$ by replacing

$$\pi \delta (\omega \pm B) \Rightarrow \frac{\Gamma_+}{(\omega \pm B)^2 + \Gamma_+^2} \quad (74)$$

Such procedure follows from the fact that accounting for higher order terms of perturbation theory in $T_{RL}$ we need to replace the product $G_{yy}^{\mu \nu} (t) G_{zz}^{\nu \rho} (t)$ in Eq. (64) by

$$\langle \eta_{\mu} (t_\nu) \eta_{\nu} (0_\nu) \eta_{\rho} (0_\nu) \eta_{\rho} (t_\mu) \rangle. \quad (75)$$

while from the other this spin-spin correlation function is determined by Eq. (33). At $T = 0$ we obtain, neglecting terms of order $\omega / D, V / D$,

$$\Lambda^{\mu \nu}_{RL} (\omega, V) = \Lambda^{\mu \nu}_{LR} (\omega, V) = 2 \pi (\omega + V) \Theta (\omega + V) \rho_0^0,$$

$$\Lambda^{\mu \nu}_{RL} (\omega, -V) = \Lambda^{\mu \nu}_{LR} (\omega, -V) = -2 \pi (\omega + V) \Theta (\omega - V) \rho_0^0,$$

$$\Lambda^{\mu \nu}_{RL} (\omega, -V) = \Lambda^{\mu \nu}_{LR} (\omega, -V) = \pi |\omega + V| \rho_0^2.$$
Summing up all terms for $P_p(\omega)$ we obtain zero for the amplitude of the peak at $V < B$. This same result was obtained by Shnirman et al. \cite{Shnirman2000}. It means that quantum fluctuations of the spin in the ground state which are present in the spin-spin correlation function cannot be probed by tunneling electrons.

When $V > B$ we obtain, taking into account the broadening of the resonance,

$$
P_p(\omega, V, B) = P(V, B) \frac{\Gamma_{\perp}}{\Gamma_{\perp}^2 + (\omega - B)^2},
$$

$$
P(V, B) = 4(\pi T_0 T_{RL} \rho_0^2)[V^2 + V B h_f(B)],
$$

(77)

More generally, the coefficient 4 is substituted by $|m_{RL\perp} + m_{L\perp}|^2$ for fully polarized electrons in the leads, while for weakly polarized electrons we get the factor $|m_{RL\perp} s_R + m_{L\perp} s_l|^2$. Accounting also for spin-orbit interaction we obtain an additional term in $P(V, B)$ which is proportional to the small factor $[(2B_\alpha/D)^2 + \beta^4(l_x^2 + l_y^2)]$.

We see that $P_p(\omega)$ exhibits a peak at the renormalized Larmor frequency with a width $\Gamma_{\perp}$ only when the electrons are polarized in a direction which is not parallel to $B_0$. This peak is caused by precession of the spin excited by tunneling electrons. Tunneling of electrons excites the spin, resulting in the modulation of the tunneling current, but they also cause decoherence of the spin precession with a rate $\Gamma_{\perp}$.

Therefore, the spin precession cannot be seen in the current-current correlation function without spin-orbit interaction when the leads-electron polarization is parallel to $B_0$, or the electrons are unpolarized, despite the fact that the spin-spin correlation functions for the $S_x$ and $S_y$ spin components exhibit oscillations at $\omega_L$. The reason for this is that to observe oscillations in $P(\omega)$ one needs to probe with the current a component of the spin perpendicular to $B$. For that the electrons need to have a component of polarization $m_R$ or $m_L$ that couples to $S_x$ or $S_y$. Spin-orbit interaction leads to a signal when the electrons are unpolarized or $B_0 \parallel m_{R,0}$ if $B_0$ is not parallel to $I$. However, the signal in that case is much weaker than in the case of fully polarized electrons.

D. The signal-to-noise ratio $R$

For the symmetrized signal the signal-to-noise ratio reaches its maximum at low temperatures when $\Gamma_{env} \ll \Gamma_{\perp}$, while $m_R \parallel m_L$, and both are perpendicular to $B$. Indeed, electrons with components of the spin polarization parallel to $B$ do not contribute to the signal but they do enhance $\Gamma_{\perp}$. When $m_{RL\perp} = m_{L\perp} = 1$ the relaxation rate $\Gamma_{\perp}(V, B)$ (neglecting relaxation due to the environment) is determined by Eq. (69). The signal-to-noise ratio at the peak position, for the case of perpendicular spin polarization at $T = 0$ and $V > B$, is given by the expression

$$
R = \frac{P_p(B)}{P_s^{(0)}(B)} = \frac{P(V, B)}{2\pi V T_0^2 \rho_0^2 \Gamma_{\perp}(V, B)} = \frac{4}{V + h_f(B)B}. \quad (78)
$$

Here $P_p(B)$ is the height of the peak at the renormalized Larmor frequency in the symmetrized current-current correlation function, while $P_s^{(0)}$ is the current power spectrum for the shot noise neglecting the smaller contribution $P_s^{(1)}$.

$R$ reaches its maximum value $R_{\text{max}} = 4$ for $V \gg B$. In the case of symmetrical leads, $T_{RR} = T_{LL} = T_{RL}$ we get $\theta = 2$, and the function $R(V/B)$ for $\theta = 0, 2$ is shown in Fig. 11.

When the spin relaxation due to the environment becomes dominant, $\Gamma_{env} \gg \Gamma_{\perp}$, the signal-to-noise ratio for the case of fully polarized electrons is smaller by the factor $\Gamma_{\perp}/\Gamma_{env} \ll 1$. If the perpendicular components of $m_s$ were absent, we see that $R$ becomes as small as $\beta^4$, while for weak polarization in the direction perpendicular to $B$ it is as small as $(B_\alpha/D)^2$.

VI. DISCUSSION

We start summarizing our main results for the model of a localized spin interacting with the tunneling electrons only via the exchange interaction, i.e., when the tunneling matrix element contains the term $T_{RL}^{(ex)} \sigma \cdot S$ in addition to the spin-independent term $T_0$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig9.png}
\caption{Two of the four diagrams contributing to the peak at the renormalized Larmor frequency in the current-current correlation function to order $T_0^2 T_{RL}^2$. There are two other diagrams with different current vertices.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig10.png}
\caption{Width of the peak as a function of voltage for the case where electrons are fully polarized in the direction perpendicular to the effective magnetic field acting on the spin.}
\end{figure}
FIG. 11: Signal-to-noise ratio $R(x)$ for fully polarized electrons.

FIG. 12: Comparison between the dependencies of $\Gamma_{\perp}$, $\bar{T} - I_0$ and $R$ vs. $x = V/B$ at $T = 0$ and $\theta = 0$, explaining why $R$ reaches a maximum at large $V$.

1. The Larmor frequency is renormalized from the one determined by the external magnetic field $B_0$ to $B = \sqrt{B_0^2 + B_B^2}$ (see Eq. 22). An estimate for $B_T$ is given by Eq. 22.

2. In the steady state the $I$-$V$ characteristics, namely $d^2I(V)/dV^2$, exhibits a peak at $V = B$ when the electrons are polarized in a direction different from $B$. This peak is due to the opening of a new channel for electron tunneling via the localized spin (spin-flip tunneling). The width of this peak is determined by the spin decoherence rate $\Gamma_{\perp}$, see Eq. 39.

3. When the electrons are polarized and $V > B$, on the background of frequency-independent shot noise caused by the discrete nature of the electrons, there is a peak in the current noise power $|P_+ (\omega) + P_- (\omega)|/2$ at the renormalized Larmor frequency, caused by the precession of the spin excited by the tunneling electrons. This peak is absent when $V < B$ because the spin is in the ground state since electrons do not have enough energy to flip it.

4. The signal-to-noise ratio for this peak, $R = P_p(\omega_L)/P_p^{(0)}$, strongly depends upon the degree of spin polarization of the leads-electrons and the orientation of this polarization with respect to the external magnetic field $B_0$ acting on the localized spin. The maximum value of $R$ is reached when the electrons are polarized in a direction perpendicular to the effective magnetic field acting on the spin because in this case they probe in an optimal way the precessing spin components. When the relaxation of the localized spin with the environment is weak, $R$ reaches the values of order unity only if the leads are almost fully polarized in a direction close to perpendicular to $B$.

5. In the case of full polarization and $m_{R0} = m_{L0} \perp B$ at low temperatures $T \ll B_{\text{eff}}/V$, the dependence of $R$ on $x = V/B$ is given by Eq. 75, where $\theta$ comes from the leads-electron reflection via the localized spin (see Eqs. 30). The maximum value, $R_{\text{max}} = 4$, is reached when $V \gg B$, see Figs. 11 and 12. The signal-to-noise ratio is reduced by the presence of the electron reflection couplings ($\theta \neq 0$) as expected, since this process does not contribute to the current but does affect the width of the peak.

6. When the spin relaxation due to the environment is negligible, the peak width in the noise power spectrum (at the renormalized Larmor frequency) depends weakly on the orientation of the polarization. It increases linearly with $V$, when $V \gg B \gg T$ and the tunneling electrons cause spin flips, see Fig. 10. At high temperatures $T \gg B, V$, the width of the peak is determined by $T (\Gamma_{\parallel} \propto T)$.

7. In points 2 through 4 we assumed that the spin relaxation with the environment, $\Gamma_{\text{env}}$, was negligible in comparison to the relaxation due to the leads-electrons, $\Gamma_{\perp}$. This assumption is crucial to obtain a maximum $R$ of order unity because in this case $\Gamma_{\parallel} \propto |T_{RL}^{(e)}(\theta, \beta)|^2 \rho_0$, and $\mathbb{P}_p(B) \propto (T_{RL}^{(e)}(\theta, \beta))^2 \rho_0 \Gamma_{\perp}$, so that $R$ does not depend on the small parameter $(T_{RL}^{(e)}(\theta, \beta))^2 \rho_0$, see Eq. 75. If the spin relaxation rate due to the environment dominates we get $R \propto \Gamma_{\parallel}/T_{\text{env}} \ll 1$.

8. For weak polarization of the electrons, $R$ becomes proportional to the degree of polarization, $R \sim (B_0/D)^2$, where $B_0$ is the magnetic or exchange field acting on the leads-electrons and $D$ represents their bandwidth. This is because only polarized tunneling electrons contribute to the signal at the Larmor frequency, while all tunneling electrons contribute to the broadening of this signal. We estimate $B_0/D \sim 10^{-5}$ for $B_0 = 100 \text{ G}$ and $D = 0.1 \text{ eV}$ and this estimate leads to $R \sim (B_0/D)^2 < 10^{-8}$.

9. Spin-orbit (i.e., relativistic $\beta^2$) corrections of any type can only provide a very small signal-to-noise ratio, $R \sim \beta^4 \sim 10^{-8}$, in the situation where the exchange coupling leads to a vanishing signal (i.e., the electrons are unpolarized or they are polarized parallel to $B_0$). The signal in this case is determined by the relativistic coupling between the spin and the tunneling electrons, while the broadening is determined by the dominant non-relativistic exchange coupling. Hence, for a tunneling process via a single localized spin in the steady state, the signal-to-noise ratio becomes of order unity only when

a) the spin is decoupled from the environment, and
b) the electrons are almost fully polarized in a direction perpendicular to the effective field acting on the localized spin.

10. We derived the correction to the standard shot noise power caused by the presence of the localized spin when the
leads—electrons are unpolarized. This correction is small when $T_{RL} \ll T_0$, but it changes behavior at $V = B$, see Eqs. (35), and (52). On the other hand, the standard shot noise power increases linearly with $V$. This new result can be observed, in principle, experimentally.

We are now in position to compare our results to the experimental data obtained by Manassen et al. [5] and Durkan and Welland [6], and decide whether the model of tunneling via a single localized spin can explain these data. In these experiments the polarization of the electrodes was weak, $R$ was of order unity or larger almost independently of the orientation of the applied magnetic field, and the position of the peak in the noise power spectrum $\mathcal{P}(\omega)$ was found at the Larmor frequency corresponding to the applied magnetic field. Our results show that a combination of weak polarization and a signal-to-noise ratio of order unity are incompatible for the dc steady state described by our model, which accounts for both exchange and spin-orbit coupling of the tunneling electrons and the localized spin.

Next, at high voltages we reproduced the quasi-classical results [1, 3] which are valid in the limit $V \gg B$.

From the point of view of the theory of quantum measurement we see that after a long time, any measurement usually leads to a steady state where information on the initial state of the quantum system is probably lost. Hence, an important question is what is the transient time for a given measurement.

We note that $d^2 I / dV^2$ and the contribution $\mathcal{P}_p(\omega)$ in the current-current correlation function are proportional to the spin-spin correlation function $S_{xx}(V)$ and $S_{xx}(\omega)$, respectively (if the electrons are spin polarized along the $x$-axis). Suppose that one wants to probe a single spin, say A, which is part of an ensemble of other spins interacting with each other. Measurement of the $I$-$V$ characteristics, namely $d^2 I / dV^2$, and current noise power, $\mathcal{P}(\omega)$, at low voltages will carry information on the dynamics of that particular (coupled) spin A, whenever tunneling occurs via that single spin. Effectively, because of the coupling, that measurement provides information on the dynamics of the whole system and Eqs. (36) and (37) form the background for such tunneling spectroscopy of quantum systems. Note, however, that the subject of study, the spin-spin correlation function, is affected by tunneling measurements (see the way the bare spin-spin correlation function, Eq. (36), was modified in the case of a single-spin system, Eq. (39)). We have shown how this modification may be accounted for in the case of steady-state measurements of a single-spin system. In a similar way it may be accounted for in more complicated systems, and corrections to obtain the bare correlation function of the system studied can, in principle, be made.

We note that we presented here a theoretical description of the tunneling spectroscopy of a single localized spin-1/2 system. However, such a description is also valid for any two-level system (quantum dot), and Eq. (3) describes the mapping between the two-level system and the spin-1/2 system.

In conclusion, we described the characteristics of the tunneling current via an isolated single spin 1/2 (two-level system) in the steady state, i.e., in the long time limit after switching on the voltage or the tunneling matrix elements. We found optimal conditions when the tunneling current carries maximum information about dynamic quantum fluctuations of the localized spin. We showed what type of information about the isolated spin may be extracted from measurements of the tunneling current in the steady state.

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